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ARITHMETIC PROGRESSIONS IN
BINARY QUADRATIC FORMS AND NORM FORMS

CHRISTIAN ELSHOLTZ AND CHRISTOPHER FREI

Abstract. We prove an upper bound for the length of an arithmetic progression represented
by an irreducible integral binary quadratic form or a norm form, which depends only on the
form and the progression’s common difference. For quadratic forms, this improves significantly
upon an earlier result of Dey and Thangadurai.

1. Introduction

1.1. Binary quadratic forms. De la Vallée Poussin \[5, 6\] proved that the set of primes repre-
sested by an irreducible primitive integral binary quadratic form \(F\) that is not negative definite
has positive relative density in the primes. This fact, which is usually seen today as a well-known
consequence of the Chebotarev density theorem, implies together with the Green-Tao theorem \[8\]
that each such quadratic form represents arithmetic progressions of arbitrary length. Our first
main result provides an upper bound for the length of an arithmetic progression represented by
\(F\) in terms of \(F\) and the progression’s modulus.

Recall that the integral binary quadratic form \(F = ax^2 + bxy + cy^2\) is irreducible, if it is
irreducible over \(\mathbb{Q}\), and primitive, if \(\gcd(a, b, c) = 1\). Its discriminant is \(d = b^2 - 4ac\).
Assume now that \(F\) is irreducible and primitive, and that it is positive definite in case \(d < 0\). Let \(\Delta\) be the
discriminant of the quadratic number field over which \(F\) splits into linear factors. The conductor
of \(F\) is the positive integer \(f\) for which \(d = f^2 \Delta\).

Theorem 1.1. Let \(F\) be an irreducible primitive integral binary quadratic form and let \(\Delta\) be the
discriminant of its splitting field. Assume that \(F\) is positive definite if \(\Delta < 0\). Let

\[A = \{\ell + rg \mid 0 \leq r < k\}, \ \ell \in \mathbb{Z}, \ g, k \in \mathbb{N},\]

be an arithmetic progression represented by \(F\), i.e. \(A \subset F(\mathbb{Z}^2)\). Then

\[k \leq C \left(\sqrt{|\Delta|} \log g + |\Delta|^{L}\right),\]

with absolute constants \(C\) and \(L\). Moreover, the value \(L = 7.999\) is admissible.

The conditions that \(F\) be primitive and positive definite if \(\Delta < 0\) are no substantial restrictions.
If \(F\) is negative definite, replace it by \(-F\), which represents arithmetic progressions of the same
modulus and length. If \(F\) is not primitive, replace it by \(F/\gcd(a, b, c)\). This does not change the
length of a progression, and it only decreases the modulus \(g\). Some slight improvements to the
bound in Theorem 1.1 are recorded in Remark 3.3.

Theorem 1.1 improves in all aspects upon an earlier result of Dey and Thangadurai \[7\], which
yields

\[k < C\ell(g^2|d|)^{L_1},\] (1.1)

with a constant \(L_1\), for which the value 15.6 is admissible. The improvement is particularly visible
when we regard \(F\) as fixed, in which case, even if the dependence on \(\ell\) in (1.1) is ignored, our
bound is essentially the logarithm of (1.1), or when \(\Delta\) and \(g\) are fixed, in which case our bound
is constant whereas \(1.1\) grows polynomially in \(f\). Our proof of Theorem 1.1 is not longer or less elementary than 7.

1.2. Norm forms. Every irreducible integral binary quadratic form is a constant multiple of a norm form of a quadratic field (see \[2, §2.7\]). Our second main result is a far-reaching generalisation of Theorem 1.1 to arbitrary norm forms

\[
N(x_1, \ldots, x_n) := N(x_1 \omega_1 + \cdots + x_n \omega_n), \tag{1.2}
\]

where \(\omega_1, \ldots, \omega_n\) is a basis of a number field \(K\). Since the length of an arithmetic progression is invariant under multiplication by constants, we rescale our form \(N\) in an analogous way to how quadratic norm forms are rescaled to yield primitive integral binary quadratic forms. Denote the \(\mathbb{Z}\)-module generated by the basis \(\omega_1, \ldots, \omega_n\) by

\[
M = \langle \omega_1, \ldots, \omega_n \rangle,
\]

Then the \(\mathcal{O}_K\)-module \(\mathcal{O}_K M\) generated by \(\omega_1, \ldots, \omega_n\) is a fractional ideal of \(K\) that contains \(M\). The norm \(N(\mathcal{O}_K M)\) is defined as the absolute value of the determinant of the matrix of basis change from a basis of \(\mathcal{O}_K\) to a basis of \(\mathcal{O}_K M\). It is independent of the choice of these bases. To the basis \(\omega_1, \ldots, \omega_n\), we attach the form

\[
F(x_1, \ldots, x_n) := \frac{N(x_1 \omega_1 + \cdots + x_n \omega_n)}{N(\mathcal{O}_K M)} \in \mathbb{Q}[x_1, \ldots, x_n]. \tag{1.3}
\]

If the basis \(\omega_1, \ldots, \omega_n\) is replaced by the basis \(a \omega_1, \ldots, a \omega_n\) for some \(a \in \mathbb{Q}, a > 0\), this does not change the form \(F\), as then both numerator and denominator are multiplied by \(a^n\). We may thus assume without loss of generality that \(M \subset \mathcal{O}_K\), so \(\mathcal{O}_K M\) is an ideal of \(\mathcal{O}_K\) and 

\[
N(\mathcal{O}_K M) = |\mathcal{O}_K : \mathcal{O}_K M| \text{ is its absolute norm.}
\]

This makes it clear that \(F(\mathbb{Z}^n) \subset \mathbb{Z}\), since for all \(\alpha \in M\) we have \(\alpha \mathcal{O}_K \subset \mathcal{O}_K M\) and thus \(N(\mathcal{O}_K M) | N(\alpha \mathcal{O}_K) = | N(\alpha) |\). The following result is our generalisation of Theorem 1.1 to arbitrary norm forms.

Theorem 1.2. Let \(K\) be a number field of degree \(n \geq 2\) with basis \(\omega_1, \ldots, \omega_n\) and consider a form \(F\) as in 1.3. Let \(K\) be a normal closure of \(K\) and denote its discriminant by \(\Delta_K\). Let

\[
A = \{f + rg \mid 0 \leq r < k\}, \quad f \in \mathbb{Z}, \; g, k \in \mathbb{N},
\]

be an arithmetic progression represented by \(F\), i.e. \(A \subset F(\mathbb{Z}^n)\). Then

\[
k \leq C_n \left( |\Delta_K|^{15} \log g + |\Delta_K|^{16} \right), \tag{1.4}
\]

with a constant \(C_n\) depending at most on \(n\) and an absolute constant \(L\), for which one may take \(L = 694\).

Note that, up to the quality of the exponents on \(|\Delta|\), Theorem 1.1 is a special case of Theorem 1.2. Theorem 1.2 also implies the same bound 1.4 for the length of an arithmetic progression represented by a usual norm form as in 1.2 coming from a basis \(\omega_1, \ldots, \omega_n\) with \(\omega_i \in \mathcal{O}_K\). This is clear, since then \(N(\mathcal{O}_K M) \geq 1\), so the re-scaling in 1.3 only decreases the modulus \(g\) of an arithmetic progression, while keeping the length intact.

Our next result shows that, as in the case of binary quadratic forms, Theorem 1.2 is non-trivial, in the sense that there is no upper bound depending only on \(F\) for the length of an arithmetic progression represented by \(F\).

Theorem 1.3. The form \(F\), as in 1.3, represents arithmetic progressions of arbitrary length.

It is not hard to see that Theorem 1.3 is implied by its special case where \(\omega_1, \ldots, \omega_n\) is an integral basis of \(K\). Moreover, in this case one can deduce Theorem 1.3 from 3 Theorem 5.2, which, with the linear forms \(f_i(u, v) = u + (i - 1)v\) for \(1 \leq i \leq v\), provides an asymptotic formula for the number of suitably constrained norms representing \(k\)-term progressions.

In this note we follow a different approach to proving Theorem 1.3. It will follow from the Green-Tao theorem 8 combined with our next result, which states that a large class of norm forms as in 1.3 represent a positive proportion of the primes. This is certainly well known if the basis \(\omega_1, \ldots, \omega_n\) is an integral basis of \(K\), so \(M = \mathcal{O}_K\), and also when \(|K : \mathbb{Q}| = 2\), in which case it states that an irreducible primitive integral binary quadratic form that is not negative definite
represents a positive density of primes. We are not aware of a published result in our situation, which generalises both of these special cases simultaneously. Recall that the ring of multipliers \( \mathcal{O} = \{ \alpha \in K \mid \alpha M \subseteq M \} \) of \( M \) is an order of the number field \( K \).

**Theorem 1.4.** Suppose that \( M \) is an invertible ideal of its ring of multipliers \( \mathcal{O} \). Then the set of primes represented by the form \( F \) defined in (1.3) has positive relative density.

If \( [K : \mathbb{Q}] = 2 \), then every ideal of \( \mathcal{O} \) with ring of multipliers \( \mathcal{O} \) is invertible, see [4] Corollary 4.4. However, this can fail in degree 3. In some situations, we can extend the conclusion of Theorem 1.4 to ideals that are not invertible. Our choice of the number field, order and ideal in the following example was motivated by [4] Example 3.5.

**Example 1.5.** Let \( K = \mathbb{Q}(\sqrt{2}) \) and choose the basis \( 8, 2\sqrt{2}, 2\sqrt{2}^2 \). Then \( M = \langle 8, 2\sqrt{2}, 2\sqrt{2}^2 \rangle \). The ring of multipliers of \( M \) is \( \mathcal{O} = \mathbb{Z} + 2\mathcal{O}_K = \langle 1, 2\sqrt{2}, 2\sqrt{2}^2 \rangle \), and \( M \) is not an invertible ideal of \( \mathcal{O} \). We have \( \mathcal{O}_K M = \langle 4, 2\sqrt{2}, 2\sqrt{2}^2 \rangle \) with \( N(\mathcal{O}_K M) = 16 \), so the corresponding norm form is

\[
F(x, y, z) = \frac{N(8x + 2\sqrt{2}y + 2\sqrt{2}^2 z)}{16} = 32x^3 + y^3 + 2z^3 - 12xyz.
\]

Even though \( M \) is not invertible, we can use Theorem 1.4 to show that \( F \) represents a positive density of the primes.

**Corollary 1.6.** The integral form \( 32x^3 + y^3 + 2z^3 - 12xyz \) represents a positive density of primes.

It would be interesting to see how far this can be generalised, and whether one can eliminate the restriction that \( M \) be invertible from Theorem 1.4.

2. A congruence argument modulo \( p^2 \)

Let \( S \subseteq \mathbb{N} \) be a subset and let \( \mathcal{P}_S \) be the set of primes \( p \) that satisfy

for all \( n \in S \): if \( p \mid n \) then \( p^2 \mid n \). \hfill (2.1)

**Lemma 2.1.** Let \( \ell \in \mathbb{Z} \), \( k, g \in \mathbb{N} \), and suppose that \( \ell + rg \in S \) holds for all \( r \in \{0, \ldots, k-1\} \). Then any \( p \in \mathcal{P}_S \) with \( 2p \leq k \) satisfies \( p \mid g \).

**Proof.** If \( p \mid g \), then the progression \( \ell, \ell + g, \ell + 2g, \ldots, \ell + (2p-1)g \) will cover every residue class modulo \( p \) exactly twice. Hence there exists \( 0 \leq i \leq p-1 \) such that \( \ell + ig \equiv \ell + (i+p)g \equiv 0 \mod p \), but that at least one of \( \ell + ig \) or \( \ell + (i+p)g \) is not congruent to \( 0 \mod p^2 \), a contradiction to \( p \in \mathcal{P}_S \). \( \square \)

**Lemma 2.2.** Assume that for \( C_1, C_2 > 0 \) we have an inequality

\[
\sum_{\substack{p \in \mathcal{P}_S \atop p \leq X}} \log p \geq \frac{X}{C_1}, \quad \text{for all} \ X \geq C_2. \hfill (2.2)
\]

Then any arithmetic progression \( \{ \ell + rg \mid 0 \leq r < k \} \subseteq S \) satisfies

\[
k \leq 2 \max \{ C_1 \log g, C_2 \}.
\]

**Proof.** We may suppose \( k \geq 2C_2 \), as otherwise there is nothing to prove. We define

\[
P := \prod_{\substack{p \in \mathcal{P}_S \atop p \leq k/2}} p.
\]

Then \( P \mid g \) by Lemma 2.1. Hence,

\[
g \geq P = \exp \left( \sum_{\substack{p \in \mathcal{P}_S \atop p \leq k/2}} \log p \right) \geq \exp \left( \frac{k}{2C_1} \right),
\]

and thus \( k \leq 2C_1 \log g \). \( \square \)
3. Binary quadratic forms

Let $F = ax^2 + bxy + cy^2$ be a primitive irreducible integral binary quadratic form that is not negative definite, and write $d = F^2\Delta$, where $d = b^2 - 4ac$ is the discriminant of $F$, $f$ is its conductor, and $\Delta$ is the discriminant of its splitting field. We are interested in the value set $S = F(\mathbb{Z}^2)$ and the corresponding set $\mathcal{P}_S$ of primes $p$ defined by (2.1), i.e. for all $n \in S$: if $p \mid n$ then $p^2 \mid n$.

**Lemma 3.1.** There is a subset $T \subset (\mathbb{Z}/\Delta\mathbb{Z})^\times$ of cardinality $\phi(|\Delta|)/2$, such that $\mathcal{P}_S$ contains every odd prime $p$ that satisfies $(p \bmod \Delta) \in T$.

**Proof.** For every odd prime $p \mid d$, the quadratic form $F$ is degenerate modulo $p$ and thus of the form $F(x, y) \equiv A L(x, y)^2 \pmod{p}$, for some $A \in \mathbb{Z}$ and a linear form $L(x, y)$ with integral coefficients. Since $p$ cannot divide all coefficients of $F$, we get $A \not\equiv 0 \pmod{p}$. Hence, for $x, y \in \mathbb{Z}$,

$$p \mid F(x, y) \iff p \mid L(x, y) \iff p^2 \mid F(x, y),$$

and thus $p \in \mathcal{P}_S$. Assume now that $p \nmid d$. Then, with $(\cdot)$ denoting the Kronecker symbol,

$$(\frac{d}{p}) = (\frac{f^2\Delta}{p}) = (\frac{\Delta}{p}).$$

Since the Kronecker symbol $(\frac{\cdot}{\Delta})$ is a non-principal quadratic Dirichlet character modulo $\Delta$ for every fundamental discriminant $\Delta$, the complement $T \subset (\mathbb{Z}/\Delta\mathbb{Z})^\times$ of its kernel has cardinality $\phi(|\Delta|)/2$. If $(p \bmod \Delta) \in T$, then $d$ is not a square modulo $p$, so $F$ is not isotropic modulo $p$, which means exactly that $F(x, y) \equiv 0 \pmod{p}$ implies $x, y \equiv 0 \pmod{p}$, and thus $F(x, y) \equiv 0 \pmod{p^2}$. \hfill $\square$

**Lemma 3.2.** The set $S = F(\mathbb{Z}^2)$ satisfies (2.2) with $C_1 \ll \sqrt{\Delta}$ and $C_2 \ll |\Delta|^L$, for some $L > 0$.

**Proof.** Using Lemma 3.1 it is enough to show that

$$\sum_{p \leq X, p \equiv c \bmod \Delta} \log p \gg \frac{X}{\phi(|\Delta|)\sqrt{\Delta}}$$

holds for all $c$ with $\gcd(c, \Delta) = 1$, whenever $X \gg |\Delta|^{1/4}$. For $|\Delta| \ll 1$, this follows from the prime number theorem in arithmetic progressions. For all large enough $|\Delta|$, on the other hand, this is a quantitative version of Linnik’s theorem on the least prime in an arithmetic progression, see [9 Corollary 18.8]. \hfill $\square$

**Proof of Theorem 1.1.** To prove the upper bound on $k$ apply Lemma 2.2 with $C_1, C_2$ from Lemma 3.2. We now show that $L = 7.999$ is admissible. For large enough $|\Delta|$ and $x \geq |\Delta|^{7.999}$, Proposition 5 implies that

$$\sum_{p \leq X, p \equiv c \bmod \Delta} \log p \gg \frac{\lambda X}{\phi(|\Delta|)^{1/2}}.$$ 

Here, $\lambda \gg 1$ if no Dirichlet $L$-function $L(s, \chi)$, for any Dirichlet character $\chi$ modulo $\Delta$, has a real zero in the range $1 - \eta/(\log |\Delta|) \leq \text{Re} z \leq 1$, for some sufficiently small absolute constant $\eta > 0$. Otherwise, if such a zero $z$ exists, then it is necessarily unique, and we take $\lambda \in (0, \eta]$ such that $z = 1 - \lambda/(\log |\Delta|)$. In any case, we have $\lambda \gg 1/\sqrt{\Delta}$ (see [13 Theorem 3]), and thus (3.1). Use this instead of [9 Corollary 18.8] in the proof of Lemma 3.2.

**Remark 3.3.** Pintz’s result [13 Theorem 3] provides in fact the bound $\lambda \gg \log(|\Delta|)/\sqrt{\Delta}$. Moreover, one can use Siegel’s result $\lambda \gg |\Delta|^{-\epsilon}$, which holds for $\epsilon > 0$ with an ineffective implied constant (see [9 Theorem 5.28]). These lead to slight improvements in Theorem 1.1 giving the bounds

$$k \ll \frac{\sqrt{|\Delta|}}{\log |\Delta|} \log g + |\Delta|^L$$

and

$$k \ll \epsilon |\Delta|^\epsilon \log g + |\Delta|^L,$$

respectively, where the implied constant in the second bound is ineffective for $\epsilon < 1/2$. 


4. Arbitrary norm forms

In this section we prove Theorem 1.2. Let $K$ be a number field of degree $n \geq 2$ and $F$ a form as in (1.3). Consider the value set

$$S = F(\mathbb{Z}^n) = \{N(\alpha)/N(O_K M) \mid \alpha \in M\}$$

and the corresponding set of primes $P_S$ defined in (2.1). As explained before the statement of Theorem 1.2, we may assume that $M \subset O_K$, so $O_K M$ is an ideal of $O_K$.

**Lemma 4.1.** Let $p$ be a prime such that all prime ideals $\mathfrak{p}$ of $O_K$ above $p$ satisfy $p^2 \mid N(p)$. Then $p \in P_S$.

**Proof.** Let $p$ be a prime satisfying the hypothesis of the lemma and $m \in S$ with $p \mid m$. Then $m = N(\alpha)/N(O_K M) = N(\alpha(O_K M)^{-1})$ for some $\alpha \in M \subset O_K M$. Since $p \mid m$, there is a prime ideal $\mathfrak{p}$ of $O_K$ above $p$ that divides the ideal $\alpha(O_K M)^{-1}$, and thus $p^2 \mid N(p) \mid N(\alpha(O_K M)^{-1}) = m$. □

Recall that $\tilde{K}$ is a normal closure of $K$. For every prime $p$ that is unramified in $\tilde{K}$, we write $\text{Frob}_p \subset \text{Gal}(\tilde{K}/\mathbb{Q})$ for the conjugacy class of Frobenius automorphisms of prime ideals above $p$ in $O_{\tilde{K}}$.

**Lemma 4.2.** There is a non-empty union $U$ of conjugacy classes of $\text{Gal}(\tilde{K}/\mathbb{Q})$, such that $P_S$ contains every prime $p$ that is unramified in $\tilde{K}$ and satisfies $\text{Frob}_p \subset U$.

**Proof.** Write $G = \text{Gal}(\tilde{K}/Q)$ and $H = \text{Gal}(\tilde{K}/K) \subset G$. Let $U$ be the union of all conjugacy classes $C$ of $G$ with $C \cap H = \emptyset$. By [12, Lemma 13.5], all unramified primes $p$ with $\text{Frob}_p \subset U$ satisfy the hypothesis of Lemma 4.1 and are thus in $P_S$. Since $H$ is a proper subgroup of $G$, its conjugate subgroups do not cover $G$. Indeed, the number of conjugate subgroups of $H$ is at most $[G : H]$ by the orbit-stabiliser theorem applied to the action of $G$ on its subgroups by conjugation. On the other hand, the union of these conjugate subgroups is not disjoint. Hence, there is $\sigma \in G$ all of whose conjugates are in $G \setminus H$, which shows that $U \neq \emptyset$. □

**Proof of Theorem 1.2.** A result of Thorner and Zaman [14] (3.2) guarantees that

$$\sum_{p \leq X \atop \text{Frob}_p \subset U} 1 \geq \delta \frac{X}{\log X},$$

where $\delta \gg_n \frac{1}{|\Delta_K|^L}$, under the conditions that $|\Delta_K|$ is large enough and $X \gg_n |\Delta_K|^L$, for a positive constant $L$ (where $L = 694$ is admissible). For $|\Delta_K| \ll_n 1$, the same bound follows from the Chebotarev density theorem. Using this and Lemma 4.2 we see that

$$\sum_{p \leq X} \log p \geq \sum_{p \leq X \atop \text{Frob}_p \subset U} \log p \geq \sum_{\text{the first } \delta X/\log X \text{ primes } p} \log p \gg_n \frac{X}{|\Delta_K|^L}.$$ 

Here the estimate of the prime number theorem that the $n$-th prime has size $p_n \sim n \log n$ was used. Hence, we can apply Lemma 2.2 with $C_1 \ll_n |\Delta_K|^L$, $C_2 \ll_n |\Delta_K|^L$.

5. Class field theory

In this section, we prove Theorem 1.3 and Theorem 1.4 as well as Corollary 1.6. Let $K$ be a number field and $\mathcal{O}$ an order of $K$ of conductor $f$. We construct a class field similar to the ring class field of $\mathcal{O}$ studied in [10], except that our congruence subgroup also incorporates the condition that $N(\alpha) > 0$. We consider the modulus $f \mathcal{O}$ of $\mathcal{O}_K$, where $f$ is the product of all real places of $K$. Let $\mathcal{I}_K^f$ be the subgroup of fractional $\mathcal{O}_K$-ideals of $K$ generated by the prime ideals not dividing $f$, and let $P_{K,\mathcal{O}}^{f,+}$ be the subgroup generated by all principal ideals

$$\alpha \mathcal{O}_K \quad \text{with} \quad \alpha \in \mathcal{O}, \alpha \mathcal{O} + f = \mathcal{O}, \quad N(\alpha) > 0.$$ 

We moreover let $I(\mathcal{O}, f)$ denote the subgroup of invertible ideals of $\mathcal{O}$ that is generated by all prime ideals $P$ with $P + f = \mathcal{O}$, and $P(\mathcal{O}, f)^+$ the subgroup generated by all principal ideals

$$\alpha \mathcal{O} \quad \text{with} \quad \alpha \in \mathcal{O}, \alpha \mathcal{O} + f = \mathcal{O}, \quad N(\alpha) > 0.$$
Lemma 5.1.

1. The group $P_{K, O}^{f, +}$ is a congruence subgroup modulo $f, x$. 
2. The map $I(O, \mathfrak{f}) \rightarrow I_{K, O}^{f}$ defined by $I \mapsto IO_{K}$ is an isomorphism.
3. The isomorphism from (2) induces an isomorphism $I(O, \mathfrak{f})/P(O, \mathfrak{f})^{+} \rightarrow I_{K, O}^{f}/P_{K, O}^{f, +}$.

Proof. (1): it is obvious that $P_{K, O}^{f, +}$ contains all principal ideals $\alpha O_{K}$ with $\alpha \in O_{K}, \alpha \equiv 1 \mod f$ and $\sigma(\alpha) > 0$ for all real embeddings $\sigma$ of $K$. Any such $\alpha$ clearly satisfies $N(\alpha) > 0$ and, moreover, $\alpha \equiv 1 \mod f$ implies $\alpha \in O_{K}$ and $\alpha O_{K} + \mathfrak{f} = O_{K}$. Hence, $\alpha \in P_{K, O}^{f, +}$.

(2): this is [10] Proposition 3.4.

(3): the isomorphism from (2) maps $P(O, \mathfrak{f})^{+}$ to $P_{K, O}^{f, +}$.

By the existence theorem of class field theory, there is a unique Abelian extension $K_{O}/O_{K}$, all of whose ramified primes divide $f, x$, for which the Artin map

$$\psi : I_{K}^{f} \rightarrow \Gal(K_{O}^{f}/K)$$

is surjective and has kernel $P_{K, O}^{f, +}$. Let $L$ be the normal closure of $K_{O}^{f}$. For any prime $p$ unramified in $L/Q$, we denote its Frobenius class in $\Gal(L/Q)$ by $\Frob_{L/Q, p}$.

Lemma 5.2. Let $I \subset O$ be an ideal with $I + \mathfrak{f} = O$. Suppose that the prime $p$ is unramified in $L$, satisfies $pO_{K} + \mathfrak{f} = O_{K}$, and moreover there is $\sigma \in \Frob_{L/Q, p}$ with

$$\sigma |_{K_{O}^{f}} = \psi((O_{K}I)^{-1}) \in \Gal(K_{O}^{f}/K).$$

Then there is $\alpha \in I$ with $N(\alpha) = N(O_{K}I)p$.

Proof. Let $\mathfrak{P}$ be a prime ideal of $O_{L}$ with $\Frob_{\mathfrak{P}}/p = \sigma$. Then $x = x^{p} \mod \mathfrak{P}$ for all $x \in O_{L}$. Let $p = \mathfrak{P} \cap K$. Since $\sigma \in \Gal(L/K)$, we get $x = \sigma(x) \equiv x^{p} \mod p$ for all $x \in O_{K}$, and thus $p$ is a degree-1 prime ideal of $O_{K}$ above $p$. Hence, $N(p) = p$, which shows moreover that $\sigma(x) \equiv x^{N(p)} \mod \mathfrak{P}$ for all $x \in O_{K}$, so $\sigma = \Frob_{\mathfrak{P}}/p \in \Gal(L/K)$. Hence, $\psi(p) = \Frob_{K_{O}^{f}/K, p} = \sigma |_{K_{O}^{f}} = \psi((O_{K}I)^{-1})$, which shows that the classes of $p$ and $O_{K}I^{-1}$ in $I_{K}^{f}/P_{K, O}^{f, +}$ coincide. By Lemma 5.1, the classes of $P = p \cap O$ and $I^{-1}$ in $I(O, \mathfrak{f})/P(O, \mathfrak{f})^{+}$ coincide as well, and thus $\alpha I^{-1} = P$ for some $\alpha \in P(O, \mathfrak{f})^{+}$. In particular, $\alpha \in \alpha O_{K} = \alpha I^{-1} = P I \subset I$. Since $N(\alpha) > 0$, we get $N(\alpha) = N(\alpha O_{K}) = N(\alpha O_{K}I^{-1}O_{K}I) = N(\alpha O_{K}I^{-1})N(O_{K}I) = N(p)N(O_{K}I) = pN(O_{K}I)$, as desired.

Proof of Theorem 1.4. Assume that $M$ is an invertible ideal of its ring of multipliers, and let $\mathfrak{f}$ be the conductor ideal of the order $O$. By [10] Theorem 3.11, the invertible ideal $M$ is of the form $M = \gamma I$, where $\gamma \in K \setminus \{0\}$ and $I$ is an ideal of $O$ with $I + \mathfrak{f} = O$. By Lemma 5.2 and the Chebotarev density theorem, a positive density of the classes of $\gamma I$ in $I_{K}^{f}/P_{K, O}^{f, +}$ coincide. By the Green-Tao theorem [8], the set of values $N(\alpha)/N(\gamma O_{K})$ for $\alpha \in \gamma O \subset M$. A fortiori, there is a positive density of primes of the form $N(\alpha)/16$, for $\alpha \in \gamma O \subset M$. A fortiori, there is a positive density of primes of the form $N(\alpha)/16$ for $\alpha \in M$.

Proof of Corollary 1.6. Let $M$ and $O$ be as in Example 1.5. The element $\gamma = 2\sqrt{2} \in M$ satisfies $N(\gamma) = 16$. Since $\gamma O$ is invertible, Theorem 1.4 yields a positive density of primes of the form $N(\alpha)/16$, for $\alpha \in \gamma O \subset M$. A fortiori, there is a positive density of primes of the form $N(\alpha)/16$ for $\alpha \in M$.

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Technische Universität Graz, Institut für Analysis und Zahlentheorie, Kopernikusgasse 24/II, A-8010 Graz, Austria
Email address: elsholtz@math.tugraz.at

School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom
Email address: christopher.frei@manchester.ac.uk