A fully implicit, scalable, nonlinear, conservative, relativistic Fokker-Planck solver for runaway electrons

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Abstract

Upon application of a sufficiently strong electric field, electrons break away from thermal equilibrium and approach relativistic speeds. These highly energetic ‘runaway’ electrons (∼ MeV) play a crucial role in understanding tokamak disruption events, and therefore their accurate understanding is essential to develop reliable mitigation strategies. For this purpose, we have developed a fully implicit, scalable relativistic Fokker-Planck kinetic electron solver. Energy and momentum conservation are ensured for the electron-electron relativistic collisional interactions. Electron-ion interactions are modeled using the Lorentz operator, and synchrotron damping using the Abraham-Lorentz-Dirac reaction term. We use a positivity-preserving finite-difference scheme for both advection and tensor-diffusion terms. The proposed numerical treatment allows us to investigate accurately phenomena spanning a wide range of temporal scales. We demonstrate the proposed scheme with numerical results ranging from small electric-field electrical conductivity measurements, to the accurate reproduction of runaway tail dynamics when strong electric fields are applied.

1 Introduction

The relativistic Fokker-Planck collision operator models Coulomb collisional effects between species under the assumption of small-angle binary collisions [3, 4]. Similarly to its non-relativistic counterpart [15], the operator is well-posed, and features strict conservation of number density, total momentum and total energy [5]. Understanding the evolution of relativistic electron dynamics under weak and strong electric fields is crucial to understand plasma disruptions in fusion devices such as tokamaks.

In tokamak disruptions, the plasma is unable to sustain itself due to rising instabilities. The subsequent plasma cooling results in a significant induced loop voltage [21]. Because of the induced electric field, some of the electrons break away from the thermal bulk. These electrons are referred to as ‘runaway’ electrons, and the resulting large electron current can severely damage the plasma facing materials in tokamaks. Runaway current can also be amplified by secondary mechanisms such as the transfer of energy from the primary runaway electron current to the thermal ‘bulk’ electrons through knock-on (large-angle) collisions. Understanding these nonlinear mechanisms is crucial to develop avoidance and mitigation strategies of runaway electrons in tokamaks. In this study, as a first step, we primarily focus on capturing accurately runaway electrons produced by the primary mechanism of a large induced loop voltage.

A solver designed to capture runaway-electron dynamics requires certain features. For example, capturing small amplitude tails necessitates strict positivity-preservation. Runaway-electron generation time may be large: a sizeable runaway tail length may take $\mathcal{O}(10^3)$ electron-electron collisional thermal time scales to develop. Therefore, one requires an implicit solver that can step over stiff thermal collisional time scales. The ability to use large time steps also demands that the scheme be asymptotic preserving, which in turn requires enforcing strict conservation properties [18]. It is also essential that the solver is

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optimal and scales with the number of mesh points, as capturing small-scale features may require fine grids. The relativistic Fokker-Planck operator can be expressed either in integral form [3] or in differential form [4]. We employ the differential form, in which the collisional coefficients are expressed in terms of relativistic potentials. This form is more conductive to an optimal $O(N)$ solver, where $N$ is the number of grid points, as the integral form inevitably produces an $O(N^2)$ scaling.

With regard to time-stepping, a fully implicit time stepping scheme method is more suitable than either linearly implicit or explicit schemes, as the former are not asymptotic preserving (and therefore feature time-step constraints for accuracy reasons), and the latter can be expensive (one needs to resolve stiff collisional time scales). NORSE [17], a recently developed relativistic electron solver, employs a linearly implicit method. The potentials are expressed in terms of Legendre modes, which are individually solved for using the distribution at the beginning of a time step. CQL3D [12], a bounce-averaged Fokker-Planck code, also solves for the potentials in a similar manner [14] but does so fully implicitly. However, neither of these codes are asymptotic preserving, because of the linearizations involved and/or the lack of discrete conservation properties. In this study, we propose a fully nonlinear implicit solver for solving the relativistic Fokker-Planck equation that has discrete conservation properties and is also optimally preconditioned and scalable in parallel.

The paper is organized as follows. In §2 we discuss the full relativistic electron-electron operator, the Lorentz operator for electron-ion interactions, and the Abraham-Lorentz-Dirac reaction term for modeling losses due to synchrotron damping. Then in §3 we discuss the algorithmic aspects with regard to the discrete conservation strategy, positivity preservation, and our optimal strategy for determination of the potentials. In §4 we briefly describe our fully implicit nonlinear solver using an Anderson Acceleration scheme. In §5 we discuss the numerical results that demonstrate the correctness of our implementation, and finally in §6 we list the conclusions and scope for future work.

2 Formulation

We model a homogeneous quasi-neutral plasma. We evolve the electron species with the relativistic Fokker-Planck equation for the electron distribution function, $f_e$, in the presence of background species $\beta$,

$$
\frac{\partial_t f_e}{\tau_{ee}^{\text{relativistic}}} + \mathbf{p} \cdot \nabla f_e = \sum_{\beta=1,e} C(f_{\beta}, f_e),
$$

where $t$ is time normalized with the relativistic electron collision time,

$$
\tau_{ee}^{\text{relativistic}} = \frac{4\pi q_e^2 m_e^2 c^3}{\epsilon_0 n_e \ln \Lambda_{ee}},
$$

$p$ is the momentum vector normalized with $m_e c$, $m_e$ is the electron mass, $c$ is the speed of light, $q_e$ is the electron charge, $\vec{E}$ is the electric field normalized with the critical value for runaway electron generation [7], $E_c = n_e q_e^2 \ln \Lambda_{ee}/4\pi\epsilon_0^2 m_e c^2$, $n_e$ is the electron number density, $\epsilon_0$ is the electrical permittivity, $\ln \Lambda_{ee}$ is the Coulomb logarithm, $\vec{F}_S$ refers to the electron friction coefficients associated with synchrotron radiation damping effects, and $C$ is the collision operator. Though the equation in principle may be used for multiple species, here we only consider the evolution of electrons interacting with themselves, ions and external electric fields.

The distribution function is described in a two-dimensional cylindrical domain $(p_{\parallel}, p_{\perp})$, with the subscripts $\parallel$ and $\perp$ referring to directions parallel and perpendicular to the magnetic field, respectively, see Fig. 1. The azimuthal direction is ignored because the distribution is axisymmetric. The electron-electron interactions are described using the full form of the collision operator, while the electron-ion interaction is modeled with the Lorentz operator (which assumes the ions to be cold and infinitely massive, $m_i >> \gamma m_e$ with $\gamma$ the Lorentz factor $\gamma = \sqrt{1+p^2}$).

The collision operator $C$ can be expressed as:

$$
C(f_{\beta}, f_e) = \partial_{\mathbf{p}} \cdot \left[ \nabla f_{\beta} - \frac{m_e}{m_\beta} \vec{F}_S f_e \right],
$$

2
Figure 1: We consider a cylindrical geometry representation \((p_\parallel, p_\perp)\) with azimuthal symmetry (left). Following a finite volume formulation, we define the distribution function on cell centers (crosses) and fluxes on edges (arrows). The ghost cells (circles) are exterior to domain boundaries. A typical stencil is shown on the right. The discrete volume for cell \((j, k)\) is computed as \(\Delta V_{j,k} = 2\pi p_\perp \Delta p_\parallel \Delta p_\perp\), where \(\Delta p_\parallel,j\) and \(\Delta p_\perp,k\) are the discrete momentum space cell sizes in the parallel and perpendicular directions.

where \(\overline{D}_\beta\) represents the collisional diffusion tensor coefficients and \(\overline{F}_\beta\) represents the collisional friction vector coefficients (computed based on the appropriate background species).

### 2.1 Electron-electron collisions

The collisional coefficients, \(\overline{D}_\beta\) and \(\overline{F}_\beta\), for electrons are expressed in a purely differential form in terms of potentials. We adopt the differential form for algorithmic scalability reasons \([\mathcal{O}(N)\) versus \(\mathcal{O}(N^2)\)]. The potentials are obtained by inverting a set of elliptic equations. This elliptic solve is performed optimally, \(\mathcal{O}(N)\), with multigrid techniques. The collisional coefficients are given by [4]:

\[
\overline{D}_e = -\frac{4\pi}{n_\beta} \gamma^{-1} [\overline{L} + \overline{P}] h_1 + 4\gamma^{-1} [\overline{L} - \overline{P}] h_2, \tag{3}
\]

\[
\overline{F}_e = -\frac{4\pi}{n_\beta} \gamma^{-1} \overline{K}(g_0 - 2g_1), \tag{4}
\]

where the operators \(\overline{L}, \overline{K},\) and \(\overline{P}\) are defined as:

\[
\overline{L}\psi = \overline{P} \cdot \frac{\partial^2 \psi}{\partial \phi \partial \rho} \cdot \overline{P} - \overline{P} \left( \overline{\rho} \cdot \frac{\partial \psi}{\partial \rho} \right),
\]

\[
\overline{K}\psi = \overline{P} \cdot \frac{\partial \psi}{\partial \rho},
\]

\[
\overline{P} = \overline{I} + \rho \rho.
\]

To obtain the transport coefficients, we first compute the \(h\) potentials by solving the partial differential equations,

\[
[L + 1]h_0 = f_e, \quad [L - 3]h_1 = h_0, \quad [L - 3]h_2 = h_1,
\]

and then the \(g\) potentials by solving:

\[
Lg_0 = f_e, \quad Lg_1 = g_0.
\]

3
Here, the operator $L$ is defined as:

$$L \psi = \vec{P} : \nabla^2 \psi + 3\vec{p} \cdot \nabla \psi.$$  \hspace{1cm} (7)

To solve these linear potential equations, we require boundary conditions. They are determined from the Green’s function solution of the elliptic equations, Eqs. (5,6) [3].

\begin{align*}
 h_0 &= -\frac{1}{4\pi} \int (r^2 - 1)^{-1/2} f_\beta(\vec{p}) \frac{1}{\gamma'} d^3\gamma', \\
 h_1 &= -\frac{1}{8\pi} \int \sqrt{r^2 - 1} f_\beta(\vec{p}) \frac{1}{\gamma'} d^3\gamma', \\
 h_2 &= -\frac{1}{32\pi} \int (r \cosh^{-1} r - \sqrt{r^2 - 1}) f_\beta(\vec{p}) \frac{1}{\gamma'} d^3\gamma', \\
 g_0 &= -\frac{1}{4\pi} \int r(r^2 - 1)^{-1/2} f_\beta(\vec{p}) \frac{1}{\gamma'} d^3\gamma', \\
 g_1 &= -\frac{1}{8\pi} \int \cosh^{-1} r f_\beta(\vec{p}) \frac{1}{\gamma'} d^3\gamma',
\end{align*}  \hspace{1cm} (8)

where $r = \gamma' - \vec{p} \cdot \vec{p}$. Note that the integral kernels of $h_0$ and $g_0$ are singular when $r \to 1 (\vec{p} \to \vec{p}')$, which requires a specialized numerical treatment, as described later in §3.3.

### 2.2 Modeling external effects

We consider several external effects, including an imposed electric field, $\vec{E} = (E_\parallel, 0)$, ions, and synchrotron radiation.

Electron-ion scattering is modeled with the Lorentz or pitch-angle scattering operator [8], which assumes ions are cold and infinitely massive. The operator causes scattering of the electrons in the pitch angle (arccos($p_\parallel / p$)) direction and, in this simplified form, it preserves kinetic energy. It has finite diffusion coefficients and zero friction coefficients, given by:

$$D_{i, \parallel} = \frac{Z_{\text{eff}} p_\parallel^2}{2v}, \quad D_{i, \perp} = \frac{Z_{\text{eff}} p_\perp p_\parallel}{2v}, \quad D_{i, \parallel} = \frac{Z_{\text{eff}} p_\parallel^2}{2v}, \quad \vec{F}_i = \vec{0},$$  \hspace{1cm} (9)

where $p^2 = p_\perp^2 + p_\parallel^2$, $v$ is the velocity magnitude (normalized with $c$), and $Z_{\text{eff}} = \sum n_i Z_i^2 / \sum n_i Z_i$ is the effective ion charge state ($n_i$ and $Z_i$ refers to ion densities and charges). For a quasi-neutral plasma, $\sum n_i Z_i = n_e$. Note that the electron-ion collisional operator becomes singular at the origin $v \to 0$. We modify this singularity by reformulating the singular part, as:

$$\frac{1}{v} \approx \frac{1}{\sqrt{v^2 + v_{\text{cut}}^2}},$$

where $v_{\text{cut}} = p_{\text{cut}} / \sqrt{1 + p_{\text{cut}}^2}$ is the velocity cut-off, with $p_{\text{cut}} = 2\Delta p$. Note this approximation of the singular term in the cylindrical space introduces a finite but small amount of heating as $p \to 0$.

Finally, we consider synchrotron radiation, which results in loss of momentum for the electrons. We model this with the Abraham-Lorentz-Dirac reaction term [8]. The reaction term has finite friction coefficients, given by:

$$F_{S, \perp} = -S \frac{p_\perp}{\gamma} (1 + p_\perp^2), \quad F_{S, \parallel} = -S \frac{p_\parallel}{\gamma} p_\perp^2,$$  \hspace{1cm} (10)

where $S = \tau_{\text{re}}^{\text{relativistic}} / \tau_r$ controls the time scale of the synchrotron radiation damping $\tau_r$ to that of relativistic electron-electron collisions $\tau_{ee}^{\text{relativistic}}$.  

4
3 Algorithm

3.1 General discretization strategy

We employ conservative finite differences. The distribution is evaluated at cell centers, while the friction and diffusion fluxes are evaluated at cell faces. The distribution function has Neumann boundary conditions, \( \vec{n}_{\text{surface}} \cdot \nabla_p f_e = 0 \), where \( \vec{n}_{\text{surface}} \) is the unit normal to the surface. Recall the electron-electron collision operator is the divergence of a collisional flux,

\[
C(f_e,f_e) = \partial_{\vec{p}} \cdot (\vec{D} \cdot \nabla_p f_e - \vec{F}_e f_e) \approx \delta^D \cdot (\vec{R}_D - \vec{R}_F) = \delta^D \cdot \vec{R},
\]

where \( \vec{R}_D \) and \( \vec{R}_F \) are the diffusion and friction fluxes, and \( \delta^D \) denotes the discrete form of the divergence operator. We discretize Eq. (11) in cylindrical-momentum space as:

\[
(\delta^D \cdot \vec{R})_{j,k} = \left( \frac{R_{\parallel,j+1/2,k} - R_{\parallel,j-1/2,k}}{\Delta p_{\parallel,j}} + \frac{p_{\perp,k+1/2}R_{\perp,j,k+1/2} - p_{\perp,k-1/2}R_{\perp,j,k-1/2}}{p_{\perp,k} \Delta p_{\perp,k}} \right) \tag{12}
\]

Fluxes at cell faces are given by:

\[
R_{D,\parallel,j+\frac{1}{2},k} = (D_{\parallel} \partial_{p_{\parallel}} f_e + D_{\perp} \partial_{p_{\perp}} f_e)_{j+\frac{1}{2},k}, \quad R_{D,\perp,j,k+\frac{1}{2}} = (D_{\perp} \partial_{p_{\parallel}} f_e + D_{\perp} \partial_{p_{\perp}} f_e)_{j,k+\frac{1}{2}},
\]

\[
R_{F,\parallel,j+\frac{1}{2},k} = F_{\parallel,j+\frac{1}{2},k} f_e,j+\frac{1}{2},k, \quad R_{F,\perp,j,k+\frac{1}{2}} = F_{\perp,j,k+\frac{1}{2}} f_e,j,k+\frac{1}{2}.
\]

The potential operator \( L \) (Eq. (7)) is discretized using central differences (see App. A1 for details). The potentials are evaluated at cell centers and their boundary conditions are specified at ghost cells. For the potential Eqs. (7, 8), we apply Dirichlet boundary conditions using Eqs. (8) as we discuss in §3.5. The collisional coefficients, \( \vec{D}_\beta \) and \( \vec{F}_\beta \), are also evaluated at cell centers using the computed potentials (see App. A2 for discretization details). The collisional coefficients at the ghost cells are evaluated by linearly extrapolating the values from adjacent cell-centered values.

The external effects on the electron system (such as scattering due to ion interactions, Eq. (9), synchrotron damping effects, Eq. (10), and electric field acceleration terms) are expressed in terms of momentum coordinates \( (p_{\parallel}, p_{\perp}) \). The coefficients are evaluated at cell-centers. Where needed, values at cell faces are found by linear averaging of two adjacent cell-centered values within the computational domain.

3.2 Discrete conservation strategy

The relativistic electron-electron collisions conserve the number density, \( n_e \), momentum, \( \vec{p} = \gamma \vec{v} \), and energy \( E = \gamma \), as the moments of the collision operator satisfy:

\[
\langle 1, C(f_e,f_e) \rangle_p = 0, \tag{13}
\]

\[
\langle p_{\parallel}, C(f_e,f_e) \rangle_p = 0, \tag{14}
\]

\[
\langle \gamma, C(f_e,f_e) \rangle_p = 0, \tag{15}
\]

where \( \langle a,b \rangle_p = \int_p ab2\pi p_{\perp} dp_{\parallel} dp_{\perp} \). Discretely, these inner products may approximated via mid-point quadrature rule as:

\[
\langle A, B \rangle_p^D \approx 2\pi \sum_{j=1}^{N_{\parallel}} \sum_{k=1}^{N_{\perp}} A_{j,k} B_{j,k} p_{\parallel,k} \Delta p_{\parallel,j} \Delta p_{\perp,k},
\]

where the superscript \( D \) refers to the discrete representation of the summation operator, and \( \Delta p_{\parallel,j} \) and \( \Delta p_{\perp,k} \) are the width and height of a rectangular cell located at \((j,k)\). The conservation of the moments of the collisional operator will be disrupted due to numerical errors. Discrete mass conservation (Eq.
where constants backward time discretization scheme at time step where the magnitudes of
We employ a similar methodology here. Firstly, we multiply the diffusion flux by a factor

\[
\eta = 1 + \eta_0 + \eta_1 (p_\parallel - \bar{p}_\parallel),
\]

where the magnitudes of \(\eta_0\) and \(\eta_1\) are expected to be of the order of truncation error, and \(\bar{p}_\parallel = (f_e, p_\parallel)_p / (1, f_e)_p\) is the mean momentum. Thus, the discrete collisional operator is of the form,

\[
C^D (f_e, f_e) = \delta F \cdot (\eta \bar{R}_D - \bar{R}_F),
\]

Integrating over the cylindrical-momentum domain,

\[
\begin{align*}
\langle p_\parallel, C^D (f_e, f_e) \rangle_p^D &= 0, \\
\langle \gamma, C^D (f_e, f_e) \rangle_p^D &= 0,
\end{align*}
\]

we obtain a system of two equations,

\[
\begin{bmatrix}
\langle \gamma_\delta F \cdot \bar{R}_D^D \rangle & \langle \gamma (p_\parallel - \bar{p}_\parallel) \delta F \cdot \bar{R}_D^D \rangle \\
\langle p_\parallel \delta F \cdot \bar{R}_D^D \rangle & \langle p_\parallel (p_\parallel - \bar{p}_\parallel) \delta F \cdot \bar{R}_D^D \rangle
\end{bmatrix}
\begin{bmatrix}
\eta_0 \\
\eta_1
\end{bmatrix}
= \begin{bmatrix}
\langle \gamma_\delta F \cdot (\bar{R}_F - \bar{R}_D) \rangle_p^D \\
\langle p_\parallel \delta F \cdot (\bar{R}_F - \bar{R}_D) \rangle_p^D
\end{bmatrix},
\]

for unknowns \([\eta_0, \eta_1]\), which can be inverted straightforwardly. This strategy conserves momentum and energy at the discrete level for electron-electron collisions. Note that, because we assume the ions to be cold and infinitely massive, there are no conservation properties associated with electron-ion collisions (i.e. electron energy and momentum transferred to ions is a net loss).

### 3.3 Time stepping strategy

A huge separation in time scales exists in runaway-electron dynamics. Long time-scales are of the order of the relativistic collision times, \(\mathcal{O}(\tau_{ee}^{\text{relativistic}})\). For typical bulk temperatures \(\Theta = T/\nu_c e^2 \sim 10^{-4}\), this implies a time-scale separation of six orders of magnitude between thermal and relativistic time scales (as \(\tau_{ee}^{\text{thermal}} = \Theta^{3/2} \tau_{ee}^{\text{relativistic}}\)). Stepping over fast time scales demands a fully implicit temporal scheme with strict conservation and positivity preservation properties. We describe our approach next.

The discrete system of equations representing the effects of electron-electron collisional interactions \(C\) and external effects \(\mathcal{E}\) on electron evolution can be written as:

\[
\delta_t f^e_n = C^D (f^e_n, f^e_n) + \underbrace{C^D (f^a_n, f^e_n) - \delta_F \left[ (\bar{F}_S + \bar{E}) f^e_n \right]}_{\mathcal{E}(f^e_n)},
\]

where the superscript \(D\) represents the appropriate discrete form defined in \(3.2\). For a general implicit backward time discretization scheme at time step \(n\), we have,

\[
\delta_t f^e_n = \sum_{i=0,1,2,\ldots} b_i f^{e,n-i}_n / \Delta t,
\]

where constants \(b_i\) satisfy \(\sum_i b_i = 0\). We use both first-order (Euler, BDF1) and second-order (BDF2) schemes for time advancing. For BDF1, \(b_0 = -1\) and \(b_1 = 1\) and for BDF2 with constant time steps, \(b_0 = 3/2, b_1 = -2, b_2 = 1/2\). The coefficients can be generalized for non-constant time steps.

Discrete conservation properties are crucial when investigating long-time dynamics, \(\tau \gg \tau_{ee}^{\text{thermal}}\). Multiplying Eq. \(17\) with \(\bar{c} = (1, p_\parallel, \gamma)\), and averaging over the momentum space, we obtain:

\[
\sum_{i=0,1,2,\ldots} \frac{b_i (\bar{c} f^{e,n-i}_n)^D}{\Delta t} = \langle \bar{c}, C^D (f^e_n, f^e_n) \rangle_p^D + \langle \bar{c}, \mathcal{E}(f^e_n) \rangle_p^D.
\]
Because of discrete conservation properties of the electron-electron collisional operator, the first term in the right hand side vanishes. Therefore, any overall change in momentum or energy of electrons can only be due to external effects such as ion-electron collisions, synchrotron radiation, and electric field acceleration.

3.4 Positivity-preserving strategy

Positivity-preserving schemes are essential to capture small-amplitude runaway tails. Our strategy is to leverage the structure of the differential operators (advection-diffusion), and use existing positivity-preserving discretizations for these terms.

For all advective terms in the relativistic kinetic equation, we use the positivity-preserving SMART flux limiter \cite{10} to construct the associated fluxes. For the diagonal components of the tensor diffusion term, $\vec{D} \cdot \nabla p f_e |_{\|\|}$ and $\vec{D} \cdot \nabla p f_e |_{\perp\perp}$, we employ a standard second-order discretization:

$$
(\vec{D} \cdot \nabla p f_e)_{\|| j + \frac{1}{2}, k} = (D_{||, j + 1, k} + D_{||, j, k} f_{e,j+k} - f_{e,j,k}) \frac{2}{\Delta p_{||, j + \frac{1}{2}}},
$$

$$
(\vec{D} \cdot \nabla p f_e)_{\perp\perp, j + \frac{1}{2}, k} = (D_{\perp\perp, j, k + 1} + D_{\perp\perp, j, k} f_{e,j+k} - f_{e,j,k}) \frac{2}{\Delta p_{\perp\perp, j + \frac{1}{2}}},
$$

which is numerically well-posed (does not feature a null space and features a maximum principle). However, the off-diagonal diffusion tensor terms do not feature a discrete maximum principle, resulting in loss of boundedness, unless care is taken. To address this issue, we reformulate the off-diagonal components as effective friction forces as proposed in Ref. \cite{9}:

$$
(\vec{D} \cdot \nabla p f_e)_{\|\|} = D_{\|\|, \perp} \partial_{\perp} f_e = f_e \frac{D_{\|\|, \perp}}{f_{\text{eff}}} \ln f_e = f_e F_{\|, \|} = R_{\|, \|}^{\text{eff}},
$$

$$
(\vec{D} \cdot \nabla p f_e)_{\perp\perp} = f_e \frac{D_{\perp\perp, \parallel}}{f_{\text{eff}}} \ln f_e = f_e F_{\perp, \perp}^{\text{eff}} = R_{\perp, \perp}^{\text{eff}}.
$$

Once formulated as advective terms, we use flux-limiting advective schemes (similar to the collisional friction terms) to calculate the effective flux. Discretization details can be found in App. A3.

3.5 Strategy for evaluating boundary conditions of collision potentials

The boundary conditions for the relativistic potential equations for $h$ and $g$ are found using the integral formulations \cite{5}. For the $h_3, h_2, g_1$ relativistic potentials, we use a trapezoidal-rule numerical integration with 24 discrete points in the $\phi$ angle. However, the kernels in $g_0, h_0$ become singular in the limit of $p' \to p \implies r \to 1$, complicating direct numerical integration. However, these complexities can be eliminated by reformulating these integrals in terms of complete elliptic integrals. We begin by noting that, because the distribution is axisymmetric, the 3D momentum-space integration can be rewritten as a 2D momentum space integration over the PDF and a 1D azimuthal angle integration as:

$$
h_{\beta,0} = \frac{1}{4\pi} \int \frac{f_{\beta}(p_{\parallel}, p_{\perp})}{\gamma'} p_{\perp} dp_{\parallel} dp_{\perp}' \int_0^{\pi} \frac{1}{(r^2 - 1)^{1/2}} d\phi = \frac{1}{4\pi} \int \frac{f_{\beta}(p_{\parallel}, p_{\perp})}{\gamma'} I(p_{\parallel}, p_{\perp}, p_{\parallel}', p_{\perp}') p_{\perp} dp_{\parallel} dp_{\perp}',
$$

$$
g_{\beta,0} = \frac{1}{4\pi} \int \frac{f_{\beta}(p_{\parallel}, p_{\perp})}{\gamma'} p_{\perp} dp_{\parallel} dp_{\perp}' \int_0^{\pi} \frac{r}{(r^2 - 1)^{1/2}} d\phi = \frac{1}{4\pi} \int \frac{f_{\beta}(p_{\parallel}, p_{\perp})}{\gamma'} H(p_{\parallel}, p_{\perp}, p_{\parallel}', p_{\perp}') p_{\perp} dp_{\parallel} dp_{\perp}',
$$

The segregated integrals $I$ and $H$ are written in terms of complete integrals of the first and third kind (see App. B).
Algorithm 1: Adaptive spline based potential boundary treatment

1. Initialize a set of knots
2. Evaluate potential integrals and create cubic spline
3. Bisect original knots to create new knots
4. Evaluate potential integrals at each knot and check error using Eq. (19): $|\phi_I - \phi_S|$
5. Where error is small, stop local bisection. Where error is large, go to step 3.

Figure 2: Illustration of adaptive spline technique.

However, evaluating potentials at all ghost points in the boundary is expensive. There are approximately $O(N^{1/2})$ ghost cell boundary points, each point requiring $O(N)$ integrals when using Eqs. (8). This makes the potential boundary evaluations scale poorly with the number of mesh points $N$, $O(N^{3/2})$. To recover optimal scaling $O(N)$ for the boundary condition treatment, we select a small number of boundary points for the potential evaluations, with the remaining ghost points found by interpolation using a cubic spline. It is not a priori clear what the optimal number of spline knots is, or what their optimal locations are. To overcome this, we have devised an algorithm to find the minimum number of spline knots needed for a given tolerance, as outlined in Algorithm 1, and illustrated in Fig. 2. We begin with a set of uniformly distributed ghost points at the boundary, for example four points (black crosses in first row) and evaluate the values of the potential integral. We fit a cubic spline through these values (blue crosses in second row). New knots are then created (black crosses in third row) where integrals are evaluated, and then the absolute error is computed as the difference between the value given by the spline interpolation, $\phi_S$, and the actual value of the potential integral at a given point, $\phi_I$:

$$a_b = |\phi_I - \phi_S|.$$ (19)

This results in a set of knots which do not satisfy the prescribed tolerance (red knot in fourth row), and need to be bisected further. This process is continued until a spline fit of desired accuracy is obtained.

To ensure the spline error is commensurate with other sources of error in the algorithm, the absolute tolerance criteria is chosen to be a function of the momentum mesh spacing as:

$$a_b \sim 0.1 \Delta p_\perp \Delta p_\parallel.$$ 

Figure 3 illustrates the adaptive knots generated with the adaptive spline algorithm for the $g_0$ potential for a Maxwell-Juttner distribution of $\Theta = 10^{-4}$ in a mesh of $N_\parallel = 2048$ and $N_\perp = 64$. The red dots along the left, top, and right boundaries point to the location of the spline knots generated using Algorithm 1. The contour of $g_0$ is also illustrated to demonstrate the variation of $g_0$ at points close and far away from the distribution. The algorithm generates more spline knots when the function varies significantly. At the far right boundary, the points are few and equally spaced, as the function variation is small. A clear
43 non-uniform spline knots
27 non-uniform spline knots
15 uniform spline knots

Figure 3: Adaptive spline knots in a uniform computational domain with $N_{\parallel} = 2048$ and $N_{\perp} = 64$. The figure illustrates the $g_0$ potential for a Maxwell-Juttner distribution of $\Theta = 10^{-4}$. The red dots represent the location of the spline knots.

benefit of the adaptive spline approach can be seen at the top boundary, where it is determined that only 43 functional evaluations are needed for an accurate estimate of the potential along the entire boundary, which spans a total of 2048 mesh points.

4 Nonlinear solver

The spatial and temporal discretization techniques prescribed in §3 lead to a coupled nonlinear system of equations, which requires an iterative nonlinear solver for the distribution function. We use an Anderson Acceleration scheme [2] to converge iteratively the system, which we briefly summarize next.

Given a fixed point map based Picard iteration,

$$f^{k+1} = G(f^k),$$

where the superscript $k$ denotes the iteration step, Anderson Acceleration scheme [19] accelerates the convergence of the Picard iteration by using the history of past nonlinear solutions via:

$$f^{k+1} = \sum_{i=0}^{m_k} \alpha^k_i G(f^{k-m_k+i}),$$

(20)

where in this study $m_k = \min(5, k)$. The coefficients $\alpha^k_i$ are determined via an optimization procedure that minimizes,

$$\left\| \sum_{i=0}^{m_k} \alpha^k_i (G(f^{k-m_k+i}) - f^{k-m_k+i}) \right\|,$$

subject to $\sum_{i=0}^{m_k} \alpha^k_i = 1$.

We use a quasi-Newton fixed-point iteration map where:

$$f^{k+1} = G(f^k) = f^k + \delta f^k = f^k - (P^k)^{-1} \mathcal{R}^k,$$

(21)

with $P^k$ the preconditioner, $\delta f^k$ the nonlinear increment, and $\mathcal{R}^k$ the nonlinear residual. Given an electron distribution, $f_e$, the residual for the nonlinear system is evaluated as outlined in Algorithm 2. Note that if $P$ is the Jacobian, i.e. $P^k = (\partial \mathcal{R}/\partial f_e)^k$, then Eq. (21) becomes a Newton iteration.

The residual contribution from electron-electron collisions requires the solutions of five potentials, which require inversions of the linear equations in Eqs. (5,6). These are inverted for each nonlinear
**Algorithm 2** Evaluating nonlinear residual, $\mathcal{R}$

1. Initialize residual by calculating time derivative using given electron distribution $f_e$
2. Compute $\delta_t f_e$ and boundary conditions for potentials
3. Invert potential equations for $h_0, h_1, h_2, g_0, g_1$ using Eqs. (5-6) and evaluate collisional coefficients
4. Compute collisional operator and enforce conservation symmetries, $C(f_e, f_e)$
5. Compute external physics: electron-ion scattering operator $C(f_i, f_e)$, synchrotron damping radiation and parallel electric field acceleration, $\delta_p \cdot \left[ (\vec{F}_S + \vec{E})f_e \right]$
6. Assemble nonlinear residual:

$$\mathcal{R}(f_e) = \delta_t f_e - C(f_e, f_e) - C(f_i, f_e) + \delta_p \cdot \left[ (\vec{F}_S + \vec{E})f_e \right]$$

iteration at flux-assembly time along with the computation of conservation constraints $\eta$. The nonlinear elimination from the residuals of the potentials and conservation constraints follows from previous studies [5,15], and enables a conservative, optimal $O(N)$ solver when the Poisson operators are inverted optimally and scalably. Here, the linear potential equations are solved using a multigrid-preconditioned GMRES [16] solver. The multigrid preconditioner features 1 V cycle with 4 passes of damped Jacobi (damping factor of 0.7), along with agglomeration for restriction and a second-order prolongation. At the beginning of the solve, the five potentials are solved using a tighter relative tolerance criteria of $10^{-8}$ and then followed by a looser relative tolerance criteria of $10^{-5} - 10^{-7}$ during each nonlinear solve, depending on the problem.

The preconditioner in Eq. (21) is obtained by Picard linearization of the potentials and subsequent discretization of the full system,

$$P^k = \delta_t f_e^k - C(f_e^{k-1}, f_e^k) - \mathcal{E}(f_e^k),$$

where $\mathcal{E}$ represents the net external effects on electrons, see Eq. (17). The transport coefficients in the electron-electron collision operator $C$ are Picard-linearized and computed at the previous nonlinear iteration, $k-1$. All advective terms in the preconditioner are discretized using a linear upwinding scheme. During each nonlinear step $k$, the linear system $P^k \delta f^k = -R^k$ is solved with one multigrid V-cycle and 3 passes of damped Jacobi (with damping constant 0.7). We use agglomeration for restriction and second-order prolongation.

The nonlinear iteration ends when the desired relative nonlinear residual convergence ratio $r_{NL}$ is reached,

$$r_{NL} = \frac{\| \mathcal{R} \|_t}{\| \mathcal{R} \|_{t=0}}.$$  

Cases with large disparities in signal amplitudes, for example a Maxwellian thermal bulk along with small amplitude runaway tail, may require a tighter convergence ratio, $r_{NL} = 10^{-7}$. In contrast, a single deforming electron thermal bulk requires a significantly looser nonlinear convergence, $r_{NL} = 10^{-4}$.

5 Results

We begin this section with some verification studies, and finish it with scalability and accuracy studies to assess the performance of the algorithm.
5.1 Verification

5.1.1 Preservation of stationary and boosted Maxwell-Jüttner distributions.

The computational domain is uniform with $N_\parallel = 256$ and $N_\perp = 128$. The nonlinear residual is converged to a relative tolerance of $10^{-4}$ unless otherwise specified. The discrete conservation properties are satisfied to nonlinear tolerance and are independent of the time step used. The electron density is normalized, $n_e = 1$. The domain is chosen such that the distribution function is sufficiently small at boundaries. The entire domain is shown in the figures illustrating the distribution function. Figure 4(a) illustrates a static Maxwell-Jüttner (MJ):

$$f_{\text{MJ}} = \frac{n_e}{4\pi\Theta K_2(1/\Theta)} \exp \left[ -\frac{\gamma p}{\Theta} \right]$$

in log scale with normalized temperature, $\Theta = T/m_e c^2 = 1$ and $K_2$ is the modified Bessel function of the second kind. We observe that the distribution retains its initial shape for the whole simulation.

Figure 4(b) demonstrates the evolution of relative errors in number density, momentum and energy for 200 electron-electron collision times. The relative errors are measured as,

$$\text{relative error} = \frac{|g(t) - g(0)|}{g(0)}.$$

where $g$ is either the number density, momentum or energy. The figure shows that the relative errors in number density are one part in $10^{11}$, while errors in relativistic momentum and energy remain smaller than one part in $10^8$.

In a boosted frame, the equilibrium distribution appears deformed and is given by:

$$f_{\text{BMJ}} = \frac{n_e}{4\pi\Theta_b K_2(1/\Theta_b)} \exp \left[ -\frac{\gamma_b \gamma - p_\parallel}{\Theta_b} \right],$$

where the subscript $b$ stands for the boosted values. Fig. 5 illustrates a boosted MJ equilibrium distribution with $\Theta_b = 0.15$, in a frame boosted by 2 units, $p_\parallel = 2$ and with $\gamma_b = \sqrt{1 + p_\parallel^2}$. The relative errors are shown for 10 relativistic collision times, demonstrating identical behavior as in the static MJ case.

5.1.2 Conservation properties during collisional relaxation dynamics

We consider first the case of two boosted MJ electron distributions which collide and result in a single MJ distribution. Figure 6(a) illustrates the collisional relaxation of two MJ distributions boosted by 2
Figure 5: Preservation of a boosted Maxwell-Jüttner distribution for \( n_e = 1, \Theta_b = 0.15, p_b = 2, N_\parallel = 256, N_\perp = 128 \). (a) Log contour of electron distribution, \( f_e \). (b) Time evolution of relative errors in number density, momentum and energy.

units in opposite directions. Figure 6(b) demonstrates the relative errors in number density, momentum and energy. After an initial transient stage \( t \in (0, 300) \), the relative errors in momentum and number density remain small and asymptotes in time.

To demonstrate that the discrete conservation strategy works in more complicated cases, we consider the thermalization of a random distribution, see Fig. 7, of the form:

\[
f_{\text{rand}}^e = \frac{\mathcal{P}}{(p, \mathbb{1})^2}, \quad \text{where} \quad \mathcal{P}(p_\parallel, p_\perp) = \frac{J(p_\parallel, p_\perp)}{4\pi \Theta K_2(1/\Theta)} \exp \left[ -\frac{\gamma(p)}{\Theta} \right],
\]

where \( \Theta = 1 \) and \( J \) is a random number function with a uniform distribution in the range \([0, 0.2]\). The presence of large gradients in the distribution and small tails makes this an excellent problem to test discrete conservation errors and positivity preservation. For a nonlinear relative tolerance of \( 10^{-4} \), the relative errors in momentum are larger than in previous cases, 1 part in \( 10^4 \). Upon tightening the relative tolerance to \( 10^{-6} \), we find that we can decrease the errors to 1 part in \( 10^6 \). We also observed a further decrease in the relative errors of momentum when tightening the relative tolerance to \( 10^{-8} \) (not shown).

5.1.3 Electrical conductivity in weak and strong electric fields

We consider next the case where collisional friction balances an externally imposed electric field, leading to finite electrical conductivity. We have verified the code for a wide range of initial temperatures for a Maxwell-Jüttner distribution by comparing the electrical conductivity to results provided by Braams and Karney [5]. To measure conductivity, we apply a small electric field, \( \hat{E}_\parallel = 10^{-3} E_D \), where \( E_D \) is the Dreicer field, \( E_D = E_c/\Theta \), with \( E_c \) the Connor-Hastie critical electric field [7]. The electron distribution is initialized using the Maxwell-Jüttner distribution at various temperatures \( \Theta \) and the normalized electrical conductivity is computed as:

\[
\bar{\sigma} = \frac{Z_{\text{eff}}}{n_e q_e \Theta^{3/2} \frac{j}{E_\parallel}}, \quad \text{where} \quad j = -n_e q_e v_\parallel,
\]

where \( v_\parallel = p_\parallel/\gamma \). A small electric field deforms the Maxwellian slightly to produce a net electron flow in the positive \( p_\parallel \) direction. Figure 8(a) depicts \( \bar{\sigma}/Z_{\text{eff}} \) at various temperatures. The numerical simulation results (circles) are in excellent agreement with the analytical results (lines) from Ref. [5].

Fig. 8(b) illustrates the time evolution of electrical conductivity for various electric-field strengths in the non-relativistic limit. Results of NORSE [17] (circles) and Weng et. al [20] (squares) are also shown. The electron distribution is initialized with Maxwellian corresponding to an initial temperature of \( \Theta = 10^{-4} \) and \( n_e = 1 \). Note that the study in Ref. [20] uses the nonrelativistic Fokker-Planck
Figure 6: Collisional relaxation of two boosted Maxwell-Jüttner distributions with $n_e = 1$, $p_b = -2, 2$, $\Theta_b = 0.15$, $N_\parallel = 256$, $N_\perp = 128$. (a) Evolution of electron distribution contours from two distinct distributions at initial time (top) to a single Maxwell-Jüttner (bottom) (b) Time evolution of relative errors measured during the collisional relaxation process.

Figure 7: Thermal relaxation of a random perturbed Maxwell-Jüttner distribution for $N_\parallel = 256$, $N_\perp = 128$. (a) Initial random electron distribution, see Eq. (22). (b)-(c) Evolution of relative errors in discrete conservation properties for a nonlinear relative tolerance of $10^{-4}$ in (b) and $10^{-6}$ in (c).
operator. For all values of electric field, we have good agreement with previous methods. For the case of \( \hat{E}_∥ = 0.01E_D \), we have better agreement with NORSE than Weng et al., as expected.

5.1.4 Reproducing runaway-electron tail dynamics

To verify runaway dynamics with existing studies, we performed two linearized numerical simulations where we kept the collisional coefficients fixed with time and used an initial Maxwell-Jüttner distribution of \( n_e = 1 \) and \( \Theta = 1 \). In the first simulation, we applied an electric field that was 2.25 times the critical value, i.e. \( E_∥ = 2.25 \). This caused some electrons to overcome the frictional force and accelerate to high speeds. Figure 9 demonstrates the evolution of the runaway tail at 42000 \( \tau_{ee} \) collision times. The asymptotic slope of the runaway tail as predicted by Connor-Hastie [7] is:

\[
f_e^{tail} \propto \frac{1}{p_∥} \exp \left( -\frac{(E_∥ + 1)p_∥^2}{2(1 + Z_{eff})p_∥} \right).
\]

As can be seen in the figure, the runaway tail produced by the algorithm is in excellent agreement with the asymptotic theoretical results.

In the second simulation, we verified runaway electron dynamics in the presence of the synchrotron radiation damping term in Eq. (10). Figure 10 shows a steady-state contour of the electron distribution function when we use a damping coefficient of \( S = 0.1 \). Other parameters are \( E_∥ = 2.25, \Theta = 0.01, Z_{eff} = 1 \). As can be seen, the electrons accumulate in the momentum space around \( p_∥ \approx 18 \), forming a second maximum, and in good agreement with Ref. [11].

5.2 Solver performance

5.2.1 Algorithmic and parallel scalability.

Table 1 lists weak parallel scalability results for the thermalization of a random electron distribution of unit number density. These results include the wall clock time (WCT) per time step, the number of nonlinear iterations (NI) per implicit time step \( \Delta t \), the ratio WCT/NI (which is an indirect measure of
Figure 9: Electron distribution function vs. $p_\parallel$ for $p_\perp = \Delta p_\perp/2$, $E_\parallel = 2.25$, $\Theta = 0.01$ and $Z_{\text{eff}} = 1$. The computational domain is uniform with $N_\parallel = 2048$ and $N_\perp = 512$ with $p_\perp \in (0, 10)$ and $p_\parallel \in (-10, 60)$. The initial Maxwellian is represented by the blue line concentrated at the origin, $p_\parallel = 0$. At $7000 \tau_{\text{ee}}^{\text{thermal}}$ (green line), we see a finite tail develop from the Maxwellian. This tail grows steadily as time increases. Time step $\Delta t = 0.01$ i.e. 10 thermal collision time scales, and the tolerances are $r_{NL} = 10^{-6}$.

Figure 10: Electron distribution function in the $(p_\parallel, p_\perp)$ space for $E_\parallel = 2.25$, $\Theta = 0.01$, $Z_{\text{eff}} = 1$ and synchrotron damping factor $S = 0.1$. The second maximum is located at $p \approx 18$. The computational domain is uniform with $N_\parallel = 2048$ and $N_\perp = 512$. The contour is plotted at time $t = 777$. Time step, $\Delta t = 0.5$ i.e. $500 \tau_{\text{ee}}^{\text{thermal}}$, and the tolerances are $r_{NL} = 10^{-6}$. 
Table 1: Parallel and algorithmic scaling: Thermalization of a random distribution in domain with $p_\parallel \in (-10, 10)$ and $p_\perp \in (0, 10)$. The results are averaged over 2 time steps with nonlinear relative convergence tolerance of $r_{NL} = 10^{-5}$ (considering more time steps is not useful, as the solution has settled into a MJ distribution).

| $N_\parallel$ | $N_\perp$ | np | NI per $\Delta t$ | WCT (sec) per $\Delta t$ | WCT/NI | $\Delta t/\Delta t_{\text{explicit}}$ | PLI |
|----------------|----------------|-----|---------------------|--------------------------|--------|-------------------------------------|-----|
| 128            | 64             | 4   | 5                   | 14.55                    | 2.918  | 82                                  | 7.8 |
| 256            | 128            | 16  | 4                   | 14.95                    | 3.7    | 329                                 | 9.14|
| 512            | 256            | 64  | 4                   | 17.68                    | 4.42   | 1318                                | 9.475|
| 1024           | 512            | 256 | 4.5                 | 20.83                    | 4.628  | 5270                                | 9.088|
| 2048           | 1024           | 1024| 6.5                 | 29.03                    | 4.4669 | 21083                               | 8.969|
| 4096           | 2048           | 4096| 10.5                | 46.3                     | 4.409  | 84331                               | 9.0  |

Table 2: Parallel and algorithmic scaling: Collisional relaxation of two boosted Maxwell-Jüttner distributions in domain with $p_\parallel \in (-15, 15)$ and $p_\perp \in (0, 15)$. The results are averaged over 10 time steps with $r_{NL} = 10^{-5}$.

| $N_\parallel$ | $N_\perp$ | np | NI per $\Delta t$ | WCT (sec) per $\Delta t$ | WCT/NI | $\Delta t/\Delta t_{\text{explicit}}$ | PLI |
|----------------|----------------|-----|---------------------|--------------------------|--------|-------------------------------------|-----|
| 128            | 64             | 4   | 3.3                 | 15.1                     | 4.57   | 0.45                                | 8.5 |
| 256            | 128            | 16  | 3.3                 | 17.3                     | 5.24   | 1.8                                 | 8.6 |
| 512            | 256            | 64  | 3                   | 18.6                     | 6.2    | 29                                  | 10.8|
| 1024           | 512            | 256 | 3                   | 19.1                     | 6.4    | 29                                  | 11.6|
| 2048           | 1024           | 1024| 3                   | 23.1                     | 7.7    | 115                                 | 12.4|
| 4096           | 2048           | 4096| 4                   | 30.6                     | 7.65   | 460                                 | 13.55|

The parallel performance), and the ratio between implicit and explicit time steps (which is a measure of numerical stiffness). The potential linear iterations (PLI) is the average number of GMRES iterations required for each potential solve. The absolute tolerances are set to $10^{-8}$. For the initial solve, the potentials are converged tightly to a relative tolerance of $10^{-8}$, with a looser tolerance of $10^{-6}$ used for subsequent nonlinear iterations. The explicit time step is calculated as:

$$\Delta t_{\text{explicit}} = 0.25 \min_{\beta=\parallel,\perp} \left( \frac{\Delta p_\parallel^2}{\max(D_{\beta,\parallel})}, \frac{\Delta p_\perp^2}{\max(D_{\beta,\perp})}, \frac{\Delta p_\perp}{\max(A_{\beta,\perp})}, \frac{\Delta p_\parallel}{\max(A_{\beta,\parallel})} \right).$$

Table 1 demonstrates excellent parallel scalability (WCT/NI) up to 4096 processors. When increasing the number of processors while keeping the problem size per processor constant, we are effectively increasing the resolution of the problem, thus making the problem harder to solve for a fixed time step (as evidenced by the increasing implicit-to-explicit timestep ratio). This manifests in a mild growth of the number of nonlinear iterations (NLI) as we increase the number of cores.

Table 2 lists parallel and algorithmic scalability results for the case of collisional relaxation of two Maxwell-Jüttner distributions, boosted by one momentum unit in opposite directions and a normalized temperature of $10^{-1}$ in their frames of reference. We observe good parallel (WCT/NI) and algorithmic scalability (NI) up to 4096 processors with $\Delta t/\Delta t_{\text{explicit}} \sim 460$ for the high resolution case of $4096 \times 2048$. Figure 11 illustrates weak scaling results of the wall clock time per nonlinear iteration (WCT/NI) vs. the number of cores for the random electron thermalization (red line, Table 1) and the boosted MJ relaxation (blue line, Table 2). We observe excellent parallel scalability in both cases.

5.2.2 Spatial and temporal accuracy

Figure 12 illustrates the spatial accuracy of the proposed scheme with the boosted MJ problem described in 5.2.1. The 'exact' electron distribution $f_{\text{exact}}^e$ is obtained at $t = 0.15$ with $N_\parallel = 1024$, $N_\perp = 512$ and $\Delta t = 0.01$, while the three data points (solid dots) correspond to coarser grids of $128 \times 64$, $256 \times 128$ and $512 \times 256$. The blue dashed line corresponds to a second-order decrease in the error. The $\ell_2$-norm of the error between the 'exact' and numerical electron distributions is computed as:
Figure 11: Weak scaling test with local processor size of $64 \times 32$.

Figure 12: Spatial and temporal accuracy measurement of the proposed scheme using the two boosted MJ configuration.
We confirm that the proposed implementation is second-order accurate in space.

Figure 12(b) illustrates the temporal accuracy of the implementation. In this case, the 'exact' electron distribution \( f^\text{exact}_e \) is obtained in a \( 256 \times 128 \) grid using the BDF2 time advancement scheme with \( \Delta t = 5 \times 10^{-4} \), see description in §3.3. The four data points correspond to larger time steps of \( \Delta t = 10^{-4}, 2 \times 10^{-4}, 4 \times 10^{-4}, \) and \( 10^{-3} \). As expected, the proposed implementation is second-order accurate in time.

6 Conclusions and future work

We have developed a fully implicit, optimal \( O(N) \) relativistic nonlinear Fokker-Plank solver for runaway electrons, where \( N \) is the number of grid points. We considered a 0D2P cylindrical momentum space representation. Our implementation is parallel and scalable. Our solver employs the differential form of the Fokker-Planck equation, which requires the solution of five relativistic potentials in momentum space to obtain the collisional coefficients. To ensure optimal scaling with mesh points, we have developed an adaptive spline technique for filling potential ghost cells at boundaries. Using multi-grid preconditioned GMRES solve for the potentials, and a robust Anderson Acceleration fixed-point iteration scheme for our nonlinear solves, we are able to obtain \( O(N) \) scaling for the overall algorithm. Our solver has been implemented with both first-order and second-order backward differentiation formulas. We verified our solver by comparing with previous results for electrical conductivity measurements in the weak and strong electric field limits. We demonstrated the accuracy of conserved quantities in electron-electron collisions, with small relative errors, in number density, relativistic momentum, and energy. In addition, we have examined runaway dynamics and verified it by comparing to known results \[7, 11\]. In future work, we will extend this method to study inhomogeneous plasmas by considering the spatial dependence of the electron distribution function.

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Appendix A. Discretization of operators in potential equations and collisional coefficients

A1. Linear potential equations.

The potential operator \( L \) consists of Hessian and advective terms,

\[
L \psi = (I + \vec{\rho} \vec{p}) : \frac{\partial^2 \psi}{\partial \vec{\rho} \partial \vec{p}} + 3 \vec{p} \frac{\partial \psi}{\partial \vec{\rho}}.
\]

The terms are discretized using central differencing. The Laplacian piece is computed as:
Once the potentials are determined, the friction coefficients are evaluated using Eq. (4). The components

$A_2$. Collisional coefficients.

where,

$X_{j+\frac{1}{2},k} = \frac{\psi_{j+1,k} - \psi_{j,k}}{\Delta p_{\parallel,j + \frac{1}{2}}}$, \hspace{1cm} Y_{j,k+\frac{1}{2}} = \frac{(\psi_{j,k+1} - \psi_{j,k})}{\Delta p_{\perp,k+\frac{1}{2}}}$.

The remaining Hessian piece is computed as:

$\left( \tilde{T} : \frac{\partial^2 \psi}{\partial \tilde{p} \partial \tilde{p}} \right)_{j,k} = \left( \frac{\partial^2 \psi}{\partial p_{\parallel}^2} + \frac{\partial^2 \psi}{\partial p_{\perp}^2} \right)_{j,k} = \frac{X_{j+\frac{1}{2},k} - X_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} + \frac{P_{\perp,k+\frac{1}{2}}Y_{j,k+\frac{1}{2}} - P_{\perp,k-\frac{1}{2}}Y_{j,k-\frac{1}{2}}}{P_{\perp,k}\Delta p_{\perp,k}}$,

where,

$Q_{j+\frac{1}{2},k} = \frac{\psi_{j+1,k} - \psi_{j,k}}{\Delta p_{\parallel,j + \frac{1}{2}}}$, \hspace{1cm} R_{j,k+\frac{1}{2}} = \frac{\psi_{j,k+1} - \psi_{j,k}}{\Delta p_{\perp,k + \frac{1}{2}}}$, \hspace{1cm} $T_{j,k+\frac{1}{2}} = \frac{1}{2} \left( \frac{\psi_{j+\frac{1}{2},k+1} - \psi_{j-\frac{1}{2},k+1}}{\Delta p_{\parallel,j}} + \frac{\psi_{j+\frac{1}{2},k} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} \right) + \frac{1}{2} \left( \frac{\psi_{j+\frac{1}{2},k+1} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} + \frac{\psi_{j+\frac{1}{2},k} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} \right) + \frac{1}{2} \left( \frac{\psi_{j+\frac{1}{2},k+1} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} + \frac{\psi_{j+\frac{1}{2},k} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} \right)$.

The advective piece is computed as:

$\left( \tilde{p} \frac{\partial \psi}{\partial \tilde{p}} \right)_{j,k} = 3 \left( p_{\parallel,j} \frac{\psi_{j+\frac{1}{2},k} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} + p_{\perp,k} \frac{\psi_{j,k+\frac{1}{2}} - \psi_{j,k-\frac{1}{2}}}{\Delta p_{\perp,k}} \right)$.

Note the cell faced values of $\psi$ are found by linear averaging across cell centered values, for example $\psi_{j+1/2,k} = 0.5(\psi_{j,k} + \psi_{j+1,k})$ and $\psi_{j,k+1/2} = 0.5(\psi_{j,k} + \psi_{j,k+1})$.

$A_2$. Collisional coefficients.

Once the potentials are determined, the friction coefficients are evaluated using Eq. (4). The components

$K\psi$ at the cell center are defined as:

$\left( K\psi \right)_{j,k} = \left( \tilde{T} + \tilde{p} \frac{\partial \psi}{\partial \tilde{p}} \right)_{j,k} \cdot \frac{\partial \psi}{\partial \tilde{p}}_{j,k}$,

where,$$
\begin{align*}
(1 + p_{\parallel,j}P_{\parallel,j}) \left( \frac{\psi_{j+\frac{1}{2},k} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\parallel,j}} \right) + p_{\parallel,k}P_{\perp,k} \left( \frac{\psi_{j,k+\frac{1}{2}} - \psi_{j,k-\frac{1}{2}}}{\Delta p_{\perp,k}} \right) \\
(1 + p_{\perp,k}P_{\perp,k}) \left( \frac{\psi_{j,k+\frac{1}{2}} - \psi_{j,k-\frac{1}{2}}}{\Delta p_{\parallel,k}} \right) + p_{\parallel,j}P_{\perp,k} \left( \frac{\psi_{j+\frac{1}{2},k} - \psi_{j-\frac{1}{2},k}}{\Delta p_{\perp,k}} \right)
\end{align*}
$$

The cell face values of $\psi$ are found by taking the average of cell-centered values. A similar discretization approach is used when evaluating the diffusion coefficient, Eq. (3).
A3. Reformulated off-diagonal tensor diffusion terms (effective friction coefficients).

The off-diagonal diffusion coefficients are expressed as effective friction coefficients of the form \( D_{\perp \parallel} \partial \ln f / \partial p_\perp \) and \( D_{\perp \parallel} \partial \ln f / \partial p_\parallel \), see Eq. (18). The momentum-space derivatives of \( \ln f \) at cell centers are evaluated by averaging the cell vertex values, for example:

\[
\frac{\partial \ln f}{\partial p_\perp} = \frac{1}{4} \left( \frac{\partial \ln f}{\partial p_\perp} \bigg|_{j+\frac{1}{2},k+\frac{1}{2}} + \frac{\partial \ln f}{\partial p_\perp} \bigg|_{j-\frac{1}{2},k+\frac{1}{2}} + \frac{\partial \ln f}{\partial p_\perp} \bigg|_{j-\frac{1}{2},k-\frac{1}{2}} + \frac{\partial \ln f}{\partial p_\perp} \bigg|_{j+\frac{1}{2},k-\frac{1}{2}} \right),
\]

where the cell vertex value is obtained by averaging over adjacent face-centered values:

\[
\frac{\partial \ln f}{\partial p_\perp} \bigg|_{j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{2} \left( \frac{\ln(|f_{j+1,k+1}| + \epsilon) - \ln(|f_{j,k+1}| + \epsilon)}{\Delta p_{\perp,k+\frac{1}{2}}} + \frac{\ln(|f_{j+1,k+1}| + \epsilon) - \ln(|f_{j+1,k+1}| + \epsilon)}{\Delta p_{\perp,k+\frac{1}{2}}} \right),
\]

\[
\frac{\partial \ln f}{\partial p_\perp} \bigg|_{j+\frac{1}{2},k-\frac{1}{2}} = \frac{1}{2} \left( \frac{\ln(|f_{j+1,k-1}| + \epsilon) - \ln(|f_{j,k-1}| + \epsilon)}{\Delta p_{\perp,k-\frac{1}{2}}} + \frac{\ln(|f_{j+1,k-1}| + \epsilon) - \ln(|f_{j+1,k-1}| + \epsilon)}{\Delta p_{\perp,k-\frac{1}{2}}} \right),
\]

\[
\frac{\partial \ln f}{\partial p_\perp} \bigg|_{j-\frac{1}{2},k+\frac{1}{2}} = \frac{1}{2} \left( \frac{\ln(|f_{j-1,k+1}| + \epsilon) - \ln(|f_{j-1,k+1}| + \epsilon)}{\Delta p_{\perp,k+\frac{1}{2}}} + \frac{\ln(|f_{j-1,k+1}| + \epsilon) - \ln(|f_{j-1,k+1}| + \epsilon)}{\Delta p_{\perp,k+\frac{1}{2}}} \right),
\]

\[
\frac{\partial \ln f}{\partial p_\perp} \bigg|_{j-\frac{1}{2},k-\frac{1}{2}} = \frac{1}{2} \left( \frac{\ln(|f_{j-1,k-1}| + \epsilon) - \ln(|f_{j-1,k-1}| + \epsilon)}{\Delta p_{\perp,k-\frac{1}{2}}} + \frac{\ln(|f_{j-1,k-1}| + \epsilon) - \ln(|f_{j-1,k-1}| + \epsilon)}{\Delta p_{\perp,k-\frac{1}{2}}} \right),
\]

where \( \epsilon_i = 10^{-30} \) is added to mollify singularities. Once computed at the cell centers, the friction coefficients at the cell faces are found by linear averaging.

Appendix B: Solution of singular integrals in relativistic potentials

B1. Solution of first singular integral

We seek a solution of the integral:

\[
I = \int_0^{2\pi} \frac{d\phi}{\sqrt{r^2 - 1}}, \quad (23)
\]

with:

\[
r = \sqrt{(1 + p^2)(1 + (p')^2) - \mathbf{p} \cdot \mathbf{p}'} = \sqrt{(1 + p^2)(1 + (p')^2) - p_\parallel p'_\parallel b^2 - p_\perp p'_\perp b^2 \cos \phi} = a^2 - b^2 \cos \Phi.
\]

We consider the case of \( b^2 > 0 \) (strictly positive; this is the case of interest numerically). Note \( r^2 - 1 = (r + 1)(r - 1) \). Since \( r \geq 1 \), it follows that:

\[
a^2 \geq b^2 + 1. \quad (24)
\]

To begin, we consider the change of variable \( t = \cos \phi \). The integration limits for this change of variable needs to be treated carefully, as otherwise one may conclude the integral \( I \) is zero! We consider the following cases:

\[
t = \cos \phi \quad , \quad \phi \in \left[0, \frac{\pi}{2}\right], \phi \in \left[\frac{3\pi}{2}, 2\pi\right] ; \quad d\phi = \frac{-dt}{\sqrt{1 - t^2}} \]

\[
t = -\cos \phi \quad , \quad \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] ; \quad d\phi = \frac{dt}{\sqrt{1 - t^2}}.
\]
This gives:

\[ I = 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(a^2+1-b^2t)(a^2-1-b^2t)}} + 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(a^2+1+b^2t)(a^2-1+b^2t)}}. \]

**Solution of \( I_2 \) integral**

We begin with the integral \( I_2 \). We follow Abramowitz & Stegun [1], and consider the polynomials:

\[
Q_1 = 1 - t^2, \\
Q_2 = (a^2 + 1 + b^2t)(a^2 - 1 - b^2t).
\]

These polynomials have real roots \( \pm 1, -\frac{a^2+1}{b^2}, -\frac{a^2-1}{b^2} \). Because of Eq. (24), it is apparent that the last two roots are \( \leq -1 \), and hence \( Q_1 \) and \( Q_2 \) do not have nested roots. In this case, one can consider the transformation to the canonical forms of the elliptic integrals by constructing the polynomial:

\[
Q_1 - \lambda Q_2 = -\left(1 + \lambda b^4\right)t^2 - 2b^2a^2\lambda t + 1 - \lambda(a^4 - 1). \tag{25}
\]

Seeking a zero discriminant for the quadratic form in \( t \) gives the following value for \( \lambda \):

\[
b^4a^4\lambda^2 = \left(\lambda(a^4 - 1) - 1\right)(1 + \lambda b^4) \Rightarrow \lambda^2b^4 - \lambda(a^4 - b^4 - 1) + 1 = 0
\]

\[
\Rightarrow \lambda_+ = \frac{(a^4 - b^4 - 1) \pm \sqrt{(a^4 - b^4 - 1)^2 - 4b^4}}{2b^4}. \tag{26}
\]

Note that these roots are real and semi-positive, since, by Eq. (24):

\[(a^4 - b^4 - 1) \geq 2b^2. \]

Also, it is obvious that

\[\lambda_+ > \lambda_- > 0,\]

and that:

\[\lambda_+\lambda_- = \frac{1}{b^4}. \tag{27}\]

Since the discriminant for Eq. (25) vanishes, it follows that the roots of \( Q_1 - \lambda Q_2 \) are perfect squares and are given by:

\[t = -t_\pm; t_\pm = \frac{\lambda_\pm b^2a^2}{1 + \lambda_\pm b^4}. \tag{29}\]

Therefore:

\[
Q_1 - \lambda_+ Q_2 = -\left(1 + \lambda b^4\right)(t + t_+)^2, \tag{30}
\]

\[
Q_1 - \lambda_- Q_2 = -\left(1 + \lambda b^4\right)(t + t_-)^2. \tag{31}\]

At this point, it is useful to point out a few properties of the roots \( t_\pm \) in Eq. (29). Firstly, from Eq. (27) it follows that:

\[t_+ > t_- \tag{32}\]

Secondly, from the polynomial in Eq. (25) and the properties of the quadratic equations, we can write:

\[
t_\pm^2 = \frac{\lambda_\pm(a^4 - 1) - 1}{1 + \lambda_\pm b^4}, \tag{33}\]

which can be used to prove that:

\[t_- \leq 1 \tag{34}\]
Hence:
\[ t_-^2 \leq 1 \Leftrightarrow \lambda_- (a_-^4 - b_-^4 - 1) < 2 \Leftrightarrow \lambda_- \leq 1/b^2, \]  
(35)
which can be shown to be true when noting that:
\[ (a_-^4 - b_-^4 - 1)^2 - 4b_-^4 = (a_-^4 - b_-^4 - 1 - 2b_-^2)(a_-^4 - b_-^4 - 1 + 2b_-^2) \geq (a_-^4 - b_-^4 - 1 - 2b_-^2)^2. \]
The inequality follows from Eq. (26). Finally, using Eq. (35) and the inequality above, we can also readily prove that:
\[ t_+ = \frac{\lambda_+ b_-^2 a_-^2}{1 + \lambda_+ b_-^2 a_-^2} \leq \frac{a_-^2}{b_-^2 + 1} \geq \frac{a_-^2}{b_-^2} \geq 1, \]  
(36)
which will be important later.
Eqs (30, 31) can be solved for \( Q_1 \) and \( Q_2 \) as follows:
\[ Q_2 = a_2_+ (t + t_+)^2 - a_2_- (t + t_-)^2, \]
\[ Q_1 = a_1_+ (t + t_+)^2 - a_1_- (t + t_-)^2. \]
Here:
\[ a_2 \pm = \frac{1 + \lambda_+ b_-^4}{\lambda_+ - \lambda_-}; \quad a_1 \pm = \frac{\lambda_\pm (1 + \lambda_\pm b_-^4)}{\lambda_+ - \lambda_-}. \]
Note that:
- \( a_2_+ > a_2_- \) (from Eq. 27).
- \( a_1_+/a_1_- = t_-/t_+ < 1 \) (from Eq. 32).

From the expressions of \( Q_1, Q_2 \), one can write:
\[ Q_1 Q_2 = (t + t_+)^4 \left[ a_2_+ - a_2_- \left( \frac{(t + t_-)^2}{(t + t_+)^2} \right) \right] \left[ a_1_+ - a_1_- \left( \frac{(t + t_-)^2}{(t + t_+)^2} \right) \right]. \]

Following Ref. [11], we postulate the change of variables:
\[ w = \frac{(t + t_-)}{(t + t_+)} \Rightarrow dw = \frac{t_- - t_-}{(t + t_+)^2} dt. \]

Hence:
\[ I_2 = \int_0^1 \frac{dt}{\sqrt{Q_1 Q_2}} = \frac{1}{w_0} \int_{w_0}^{w_1} \frac{dw}{\sqrt{a_2_+ - a_2_- w^2 [a_1_+ - a_1_- w^2]}}. \]  
(37)
Here:
\[ w_0 = \frac{t_-}{t_+} < 1; \quad w_1 = \frac{1 + t_-}{1 + t_+}; \quad w_0 < w_1 < 1. \]
The result in Eq. (37) can be written as a canonical elliptic integral by considering:
\[ \frac{a_1_+}{a_1_-} = \frac{t_-}{t_+} = w_0 = e^2 < 1; \quad \frac{a_2_+}{a_2_-} = \frac{1 + \lambda_+ b_-^4}{1 + \lambda_- b_-^4} = d^2 > 1, \]
(38)
to find:
\[ I_2 = \int_0^1 \frac{dt}{\sqrt{Q_1 Q_2}} = \frac{1}{(t_- + t_+ - 1)\sqrt{a_2_+ - a_2_-}} \int_{e^2}^e \frac{dw}{\sqrt{d^2 - w^2}}. \]
Here, we have used the surprising property that:
\[ w_1^2 = \left( \frac{1 + t_-}{1 + t_+} \right)^2 = \frac{t_-}{t_+} = e^2 \Rightarrow w_1 = e, \]
which can be demonstrated by using the definition of \( t_\pm \) (Eq. 29) and \( t_\pm^2 \) (Eq. 33).
Solution of $I_1$ integral

The solution of the integral $I_1$ follows a similar development, except now:

$$
Q_1 = 1 - t^2,
Q_2 = (a^2 + 1 - b^2 t)(a^2 - 1 - b^2 t).
$$

With these definitions, it can be shown that the discriminant of the combination $Q_1 - \lambda Q_2$ is exactly the same, and therefore so are the solutions $\lambda_{\pm}$. However the roots in $t$ now have opposite signs:

$$
t = t_{\pm} \quad t_{\pm} = \frac{\lambda_{\pm} b^2 a^2}{1 + \lambda_{\pm} b^4},
$$

and the factorization of $Q_{1,2}$ reads:

$$
Q_2 = a_{2+}(t-t_+)^2 - a_{2-}(t-t_-)^2,
Q_1 = a_{1+}(t-t_+)^2 - a_{1-}(t-t_-)^2.
$$

From the expressions of $Q_1$, $Q_2$, one can write:

$$
Q_1Q_2 = (t-t_+)^4 \left[ a_{2+} - a_{2-} \frac{(t-t_-)^2}{(t-t_+)^2} \right] \left[ a_{1+} - a_{1-} \frac{(t-t_-)^2}{(t-t_+)^2} \right].
$$

Following Ref. [1], we postulate the change of variables:

$$
w = \frac{(t-t_-)}{(t_+-t)} \Rightarrow dw = \frac{t_+-t_-}{(t_+-t)^2} dt.
$$

When postulating this change of variables, we have taken into account the fact that $t \leq 1 < t_+$ (Eq. [36]), and that $t_- < 1$ (Eq. [34]). It follows that:

$$
I_1 = \int_0^1 \frac{dt}{\sqrt{Q_1Q_2}} = \frac{1}{t_+-t_-} \int_{w_0}^{w_1} \frac{dw}{\sqrt{[a_{2+}-a_{2-}w^2][a_{1+}-a_{1-}w^2]}},
$$

where:

$$
w_0 = -\frac{t_-}{t_+} = -e^2 < 0 \quad w_1 = \frac{1-t_-}{t_+} = 1 > 0.
$$

As before, one can readily prove that:

$$
w_1^2 = \left( \frac{1-t_-}{t_+} \right)^2 = \frac{t_-}{t_+} = e^2,
$$

and therefore $w_1 = e$. There results:

$$
I_1 = \int_0^1 \frac{dt}{\sqrt{Q_1Q_2}} = \frac{1}{(t_+-t_-)\sqrt{a_{2-}a_1-}} \int_{-e^2}^{e} \frac{dw}{\sqrt{[d^2-w^2][e^2-w^2]}},
$$

Solution of total integral $I$

When combining these two solutions, we find:

$$
I = 2(I_1 + I_2) = \frac{2}{(t_+-t_-)\sqrt{a_{2-}a_1-}} \left[ \int_{-e^2}^{e} \frac{dw}{\sqrt{[d^2-w^2][e^2-w^2]}} + \int_{-e^2}^{e} \frac{dw}{\sqrt{[d^2-w^2][e^2-w^2]]} \right]
$$

$$
= \frac{2}{(t_+-t_-)\sqrt{a_{2-}a_1-}} \left[ \int_{-e^2}^{e} + \int_{0}^{e} + \int_{0}^{e} \right] = \frac{2}{(t_+-t_-)\sqrt{a_{2-}a_1-}} \left[ \int_{-e^2}^{e} + \int_{0}^{e} + \int_{0}^{e} \right]
$$

$$
= \frac{4}{(t_+-t_-)\sqrt{a_{2-}a_1-}} \int_{0}^{e} \frac{dw}{\sqrt{[d^2-w^2][e^2-w^2]}}.
$$
which can be written in terms of the complete elliptic integral of the second kind as [1]:

\[ I = \frac{4K(m)}{d(t_+ - t_-)\sqrt{a_2 - a_1}} = \frac{4K(m)}{(t_+ - t_-)\sqrt{a_2 + a_1}}, \]

where in the last step we have used the definition of \( d \) (Eq. 38), and where:

\[ m = c^2/d^2. \]

**B2. Solution of second singular integral**

In the previous section, we determined the root structure of the radicand and removed the odd terms in the radicand. We employ this approach and also use ideas from Ref. [13] to express the following elliptic integral,

\[ H = \int_0^{2\pi} \frac{r \, d\phi}{\sqrt{r^2 - 1}} \]

in terms of Legendre’s elliptic functions. Recall that

\[ r = a^2 - b^2 \cos \phi. \]

The integral can thus be expressed as,

\[ H = 2 \int_0^1 \frac{(a^2 - b^2) \, dt}{\sqrt{(1 - t^2)(a^2 + 1 - b^2 t)(a^2 - 1 - b^2 t)}} + 2 \int_0^1 \frac{(a^2 + b^2) \, dt}{\sqrt{(1 - t^2)(a^2 + 1 + b^2 t)(a^2 - 1 + b^2 t)}}. \]

We can regroup as we know from §B1 the solution when the numerator is unity,

\[ H = a^2 I + 2 \left( \int_0^1 \frac{-b^2 \, dt}{H_1} \right) + 2 \left( \int_0^1 \frac{b^2 \, dt}{H_2} \right), \]

(42)

Removing the odd terms in the radicand, we obtain,

\[ H_1 = \int_0^1 \frac{-b^2 \, dt}{\sqrt{Q_1Q_2}} = \frac{-b^2}{(t_+ - t_-)\sqrt{a_2 - a_1}} \int_{-e^2}^{e^2} \frac{(wt_+ + t_-)/(1 + w) \, dw}{\sqrt{[d^2 - w^2][e^2 - w^2]}} \]

\[ = \frac{-b^2}{(t_+ - t_-)\sqrt{a_2 - a_1}} \int_{-e^2}^{e^2} R_1(w) \, dw \]

and

\[ H_2 = \int_0^1 \frac{b^2 \, dt}{\sqrt{Q_1Q_2}} = \frac{b^2}{(t_+ - t_-)\sqrt{a_2 - a_1}} \int_{-e^2}^{e^2} \frac{(-wt_+ + t_-)/(1 - w) \, dw}{\sqrt{[d^2 - w^2][e^2 - w^2]}} \]

\[ = \frac{b^2}{(t_+ - t_-)\sqrt{a_2 - a_1}} \int_{-e^2}^{e^2} R_2(w) \, dw \]

The rational functions of \( w, R_1, \) and \( R_2, \) can be expressed in terms of odd and even functions. This is because the odd term can be simplified into elementary functions via trigonometric substitutions, see Ref. [13].

\[ R_1(w) = \frac{w}{1 - w^2}(t_+ - t_-) + \frac{t_- - w^2 t_+}{1 - w^2} \]
\[ R_2(w) = \frac{w}{1 - w^2}(t_+ - t_-) - \frac{t_- - w^2t_+}{1 - w^2} \]

However, in our case we observe that these terms cancel each other. Examining the odd terms in \( R_1 \) and \( R_2 \), and adding together their contribution to \( H \), we find:

\[ H_1^{\text{odd}} + H_2^{\text{odd}} = - \int_{-e^2}^{e^2} + \int_{e^2}^{e^2} = - \int_{-e^2}^{e^2} = 0, \]

as the odd function is asymmetric about the origin. The even term can be further factorized into:

\[ \frac{t_- - w^2t_+}{1 - w^2} = t_+ - \frac{t_+ - t_-}{1 - w^2}. \]

Putting the above expression into \( H_1 \) and \( H_2 \), the contributions of the even terms may be expressed as,

\[ H_1 + H_2 = - \frac{b^2t_+}{(t_+ - t_-)\sqrt{a_2 - a_1}} \left( \int_{-e^2}^{e^2} \frac{dw}{\sqrt{[d^2 - w^2][e^2 - w^2]}} + \int_{e^2}^{e^2} \frac{dw}{\sqrt{[d^2 - w^2][e^2 - w^2]}} \right) + \frac{b^2}{\sqrt{a_2 - a_1}} \left( \int_{-e^2}^{e^2} \frac{dw}{(1 - w^2)\sqrt{[d^2 - w^2][e^2 - w^2]}} + \int_{e^2}^{e^2} (1 - w^2)\sqrt{[d^2 - w^2][e^2 - w^2]} \right) \]

Simplifying and regrouping, we obtain the expression for \( H \), see Eq. (42) as,

\[ H = (a^2 - b^2t_+)I + \frac{4b^2}{\sqrt{a_2 + a_1}} \Pi(e^2, m) \quad (43) \]

where \( \Pi \) is the complete elliptic integral of the third kind, with \( e^2 < 1 \). This formula has been verified numerically. Also in the above, we made use of the following step which was described earlier in the previous section. It is as follows,

\[ \int_{-e^2}^{e^2} + \int_{e^2}^{e^2} = \int_{-e^2}^{e^2} + 2\int_{0}^{e^2} - \int_{0}^{e^2} \]

Then substituting \( t = w/e \) to get the final form (43). The complete elliptic integral of the third kind is given by,

\[ \Pi(e^2, m) = \int_{0}^{1} \frac{dt}{(1 - e^2t)\sqrt{(1 - t^2)(1 - mt^2)}}. \]

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