THE CARLITZ SHTUKA

LENNY T AELMAN

Abstract. Recently we have used the Carlitz exponential map to define a finitely generated submodule of the Carlitz module having the right properties to be a function field analogue of the group of units in a number field. Similarly, we constructed a finite module analogous to the class group of a number field.

In this short note more algebraic constructions of these “unit” and “class” modules are given and they are related to Ext modules in the category of shtukas.

1. Introduction and statement of the main results

1.1. Notation. Let \( k \) be a finite field of \( q \) elements. Without mention to the contrary schemes are understood to be over \( \text{Spec} \ k \) and tensor products over \( k \).

Let \( t \) be the standard coordinate on the projective line \( \mathbb{P}^1 \) over \( k \), let \( F = k(t) \) the function field of \( \mathbb{P}^1 / k \) and let \( A = k[t] \) the ring of functions regular away from the “point at infinity” \( \infty \in \mathbb{P}^1 \).

Let \( X \) be a smooth projective geometrically connected curve over \( k \) and \( X \to \mathbb{P}^1 \) a surjective map. Denote the function field of \( X \) by \( K \). Let \( Y \subset X \) be the inverse image of \( \text{Spec} \ A = \mathbb{P}^1 - \infty \).

1.2. The Carlitz module.

Definition 1. The Carlitz module is the functor

\[ C_0 : \{ \text{Spec} \ A\text{-schemes} \} \to \{ A\text{-modules} \} \]

which associates to a scheme \( S \) over \( \text{Spec} \ A \) the \( A \)-module \( C(S) \) given by \( C(S) = \Gamma(S, \mathcal{O}_S) \) as a \( k \)-vector space, with \( A \)-module structure

\[ \varphi : A \to \text{End}_k \Gamma(S, \mathcal{O}_S) : t \mapsto (c \mapsto tc + c^q) \, . \]

The functor \( C_0 \) is in many ways an analogue of the multiplicative group

\[ \mathbb{G}_m : \{ \text{Spec} \ \mathbb{Z}\text{-schemes} \} \to \{ \mathbb{Z}\text{-modules} \} : S \mapsto \Gamma(S, \mathcal{O}_S)^\times \, . \]

Yet, in contrast with Dirichlet’s unit theorem we have the following negative result:

Proposition 1 (Poonen [8]). The \( A \)-module \( C_0(Y) \) is not finitely generated. \( \square \)
1.3. A construction using the Carlitz exponential. In [9] we have used the Carlitz exponential map to cut out a canonical finitely generated sub-$A$-module from $C_0(Y)$. We recall and reformulate this construction.

A simple recursion shows that there is a unique power series $\exp x$ in $F[[x]]$ which is of the form

$$\exp x = x + e_1 x^q + e_2 x^{q^2} + \cdots$$

and which satisfies

$$(1) \quad \exp(tx) = t \exp x + (\exp x)^q.$$  

This power series is called the Carlitz exponential. It is entire and for every point $z$ of $X \setminus Y$ it defines an $A$-linear map $\exp : K_z \to C_0(K_z)$. We extend it to a sheaf $C$ on $X$ as follows:

$$C(U) := \left\{ (c, (\gamma_z)_z) \in C_0(U \cap Y) \times \prod_{z \in U \setminus Y} K_z \mid \forall z \exp\gamma_z = c \right\}.$$  

One verifies easily that this indeed defines a sheaf on $X$. The main result of [9] can be restated as follows:

**Proposition 2.**

(1) $H^0(X, C)$ is a finitely generated $A$-module;

(2) $H^1(X, C)$ is a finite $A$-module.

In §2 we will show how to deduce this result from [9].

In particular, the image of the restriction map $C(X) \to C(Y) = C_0(Y)$ is a canonical finitely generated submodule of $C_0(Y)$, it is a Carlitz analogue of the group of units in a number field. Similarly, $H^1(X, C)$ is a Carlitz analogue of the class group of a number field (this should of course be compared with the isomorphism $H^1(O_L, G_m) = \text{Pic } O_L$).

We do need to pass to the completed curve $X$ to get something interesting: By Poonen’s theorem $H^0(Y, C_0)$ is not finitely generated, and since $C_0 \cong O_Y$ as sheaves of abelian groups we have that $H^1(Y, C_0) = 0$.

Unfortunately the above definition of the sheaf $C$ is analytic in nature, and it would be desirable to have a purely algebraic description of $C$. The aim of this paper is to provide such a description, as well as a more “motivic” interpretation of it.

1.4. An algebraic description of the sheaf $C$. For an integer $n$, denote by $\mathcal{O}_{P^1}(n\infty)$ the sheaf of functions on $P^1$ that have a pole of order at most $n$ at $\infty$ and by $\mathcal{O}_X(n\infty)$ its pullback over $X \to P^1$. 
Theorem 1. There is a short exact sequence of sheaves of $A$-modules on $X$
\begin{equation}
0 \longrightarrow O_X \otimes A \xrightarrow{\partial} O_X(\infty) \otimes A \longrightarrow C \longrightarrow 0
\end{equation}
where
\[ \partial: f \otimes a \mapsto f \otimes t a - (t f + f^q) \otimes a. \]

The proof of this theorem will be given in section §3.

1.5. Interpretation in terms of shtukas. The short exact sequence of Theorem 1 can be reinterpreted in terms of shtukas.

For any $k$-scheme $S$ denote by $S_A$ the base change of $S$ to $\text{Spec} A$ and by $\tau_A: S_A \to S_A$ the base change of the $q$-th power Frobenius endomorphism $\tau: S \to S$.

Definition 2. A (right) $A$-shtuka on a $k$-scheme $S$ is a diagram
\[ \mathcal{M} = \left[ \mathcal{M} \xrightarrow{\sigma^j} \mathcal{M}' \xleftarrow{\tau^*_A \mathcal{M}} \right] \]
of quasi-coherent $O_{S_A}$-modules.

With the obvious notion of morphism, the shtukas on $S$ form an $A$-linear abelian category. In particular, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two shtukas on $S$ then the Yoneda extension groups $\text{Ext}^i(\mathcal{M}_1, \mathcal{M}_2)$ are $A$-modules.

We have a natural isomorphism of sheaves of $O_X$-modules
\[ \tau^* O_X \xrightarrow{\sim} O_X, \]
and will identify source and target in what follows.

If $\mathcal{F}$ is a coherent sheaf of $O_X$-modules and $M$ an $A$-module we denote by $\mathcal{F} \boxtimes M$ the coherent sheaf of $O_{X \times \text{Spec}(A)}$-modules
\[ \mathcal{F} \boxtimes M = \text{pr}_1^* \mathcal{F} \otimes_{O_{X \times \text{Spec}(A)}} \text{pr}_2^* M \]
where $\text{pr}_1$ and $\text{pr}_2$ denote the projections from $X \times \text{Spec} A$ to $X$ and $\text{Spec} A$ respectively.

Definition 3. The unit shtuka on $X$ is defined to be the shtuka
\[ \mathbf{1} = \left[ O_X \boxtimes A \xrightarrow{1} O_X \boxtimes A \xleftarrow{\tau^* O_X \boxtimes A} \right]. \]

Definition 4. We define the Carlitz shtuka on $X$ to be the shtuka
\[ \mathcal{C} = \left[ O_X \boxtimes A \xrightarrow{\sigma} O_X(\infty) \boxtimes A \xleftarrow{\tau^* O_X \boxtimes A} \right] \]
with
\[ \sigma = 1 \otimes t - t \otimes 1. \]

The following is essentially a formal consequence of Theorem 1.
**Theorem 2.** There are natural isomorphisms

\[
\text{Ext}^i(1, C) \xrightarrow{\sim} H^{i-1}(X, C)
\]

for all \( i \).

The proof will be given in §4.

1.6. **Remarks.**

**Remark 1.** Our notion of shtuka is the same as the one in V. Lafforgue [5]. It is similar to the one used by Drinfeld [3] and L. Lafforgue [4], but of a more arithmetic nature. Rather than compactifying the “coefficients” \( \text{Spec} A \) to a complete curve, we compactify the “base” \( A^1 \) to \( \mathbb{P}_k^1 \).

**Remark 2.** Shtukas are function field toy models for (conjectural) mixed motives. The Carlitz shtuka \( C \) is an analogue of the Tate motive \( Z(1) \) and Theorem 2 should be compared with the isomorphisms

\[
\text{Ext}^1_X(1, Z(1)) = \Gamma(X, \mathcal{O}_X^\times) \quad \text{and} \quad \text{Ext}^2_X(1, Z(1)) = \text{Pic} X
\]

from motivic cohomology, see for example [6, p. 25].

**Remark 3.** In the \((\infty\text{-adic})\) “class number formula” proven in [10], the \( A \)-modules \( H^0(X, C) \) and \( H^1(X, C) \) play a role analogous to the groups of units and the class group in the classical class number formula. In the guise of \( \text{Ext}^1(1, C) \) and \( \text{Ext}^2(1, C) \) they play a similar role in V. Lafforgue’s result [5] on \((v\text{-adic, } v \neq \infty)\) special values.

**Remark 4.** For any \( m \) there is a natural isomorphism \( \tau^* \mathcal{O}_X(m \infty) \xrightarrow{\sim} \mathcal{O}_X(qm \infty) \). So in the definition of the Carlitz shtuka one could twist both line bundles with \( \mathcal{O}_X(-n \infty) \) for some \( n \geq 0 \) to obtain

\[
\mathcal{O}_X(-n \infty) \boxtimes A \xrightarrow{\tau^*} \mathcal{O}_X((1 - n) \infty) \boxtimes A \xleftarrow{\tau^*} \mathcal{O}_X(-n \infty) \boxtimes A.
\]

The same results with the same proofs hold for this shtuka. We have chosen \( n = 0 \) in our definition somewhat arbitrarily, distinguishing it from the other choices only by its minimality.

**Remark 5.** We have treated in this note only a very special case. One should try to obtain similar results for higher rank Drinfeld modules over general Drinfeld rings \( A \), and even for the abelian \( t \)-modules of Anderson [11]. Unfortunately it seems that these generalizations are not without difficulty, and even for higher rank Drinfeld modules it is not clear to me what the precise statement should be.
1.7. Acknowledgements. This work has been inspired by work of Anderson and Thakur [2], Woo [11], Papanikolas and Ramachandran [7], and V. Lafforgue [5]. The author is grateful to David Goss for his feedback and constant encouragement, and to the referee for several useful suggestions.

The author is supported by a VENI Grant from the Netherlands Organization for Scientific Research (NWO). Part of the research leading to this paper was carried out at the Ecole Polytechnique Fédérale de Lausanne.

2. The cohomology of the sheaf $C$

In this section we show how the modules $H^0(X, C)$ and $H^1(X, C)$ compare with the modules studied in [9]. We recall the main result of loc. cit.

Consider the map

$$\delta: C_0(Y) \times \prod_{z \in X \setminus Y} K_z \to \prod_{z \in X \setminus Y} C_0(K_z): (c, (\gamma_z)_z) \mapsto (c - \exp \gamma_z)_z.$$ 

**Theorem 3** ([9]). $\ker \delta$ is a finitely generated $A$-module and $\coker \delta$ is a finite $A$-module.

We now show that $H^0(X, C)$ and $H^1(X, C)$ coincide with the modules $\ker \delta$ (“$\exp^{-1} C(R)$” in the notation of loc. cit.) and $\coker \delta$ (“$H^1_R$”) above, and hence that Proposition 2 follows from Theorem 5.

**Lemma 1.** There is an exact sequence of $A$-modules

$$0 \to C(X) \to C_0(Y) \times \prod_{z \in X \setminus Y} K_z \xrightarrow{\delta} \prod_{z \in X \setminus Y} C_0(K_z) \to H^1(X, C) \to 0.$$

**Proof.** Denote by $i: Y \to X$ and by $i_z: \{z\} \to X$ the inclusions of $Y$ and the points $z$ in $X$. Then the following sequence of sheafs on $X$ is exact:

$$0 \to C \to i_* C_0 \times \prod_{z \in X \setminus Y} i_{z,*} K_z \to \prod_{z \in X \setminus Y} i_{z,*} C_0(K_z) \to 0.$$

(Here the middle map is the difference of the natural map and the map induced by exp.) Left exactness follows from the definition of $C$. For right exactness, one uses the fact that for all $z \in X \setminus Y$ we have $C_0(K_z) = C_0(K) + \exp K_z$ (which follows, for example, from Corollary 11 below).

Note that $H^1(X, i_* C_0) = H^1(Y, C_0) = 0$ so that the desired exact sequence is precisely the long exact sequence of cohomology obtained from taking global sections in (3).
3. Proof of Theorem \[ \text{1} \]

3.1. Away from \(\infty\). Let \(R\) be an \(A\)-algebra. Denote by \(\alpha\) the \(A\)-linear map

\[
\alpha: R \otimes A \to C_0(R): r \otimes a \mapsto \varphi(a)(r).
\]

**Proposition 3.** The sequence of \(A\)-modules

\[
0 \rightarrow R \otimes A \overset{\partial}{\rightarrow} R \otimes A \overset{\alpha}{\rightarrow} C_0(R) \rightarrow 0
\]

is exact.

**Proof.** Straightforward. \(\square\)

In particular this provides the desired short exact sequence of sheaves \((2)\) on the affine curve \(Y \subset X\). In the following paragraphs we will extend it to the whole of \(X\).

3.2. Inversion of the exponential map. Let \(z \in X \setminus Y\) and let \(|\cdot|\) be an absolute value on \(K_z\) inducing the \(z\)-adic topology, so in particular \(|t| > 1\).

**Lemma 2.** For all \(x \in K_z\) with \(|x| < |t|^{q/(q-1)}\) we have \(|\exp x - x| < |x|\).

**Proof.** Write \(\exp x = \sum_{i=0}^{\infty} e_i x^q\). It follows from \((1)\) and from \(e_0 = 1\) that for all \(i\) we have \(|e_i| = |t|^{-iq}\). From this one deduces that for all \(i > 0\) and all \(x\) with \(|x| < |t|^{q/(q-1)}\) the inequality \(|e_i x^q| < |x|\) holds. Hence \(|\exp x - x| = |\sum_{i>0} e_i x^q| < |x|\), as claimed. \(\square\)

**Corollary 1.** For all \(m \leq 1\) the exponential map restricts to a \(k\)-linear isomorphism \(t^m \mathcal{O}_{X,z}^\wedge \rightarrow t^m \mathcal{O}_{X,z}^\wedge\).

We denote its inverse by \(\log\).

3.3. Near \(\infty\). Let \(z \in X \setminus Y\). Consider the \(A\)-linear map

\[
\lambda: t \mathcal{O}_{X,z}^\wedge \otimes A \rightarrow K_z: f \otimes a \mapsto a \log f.
\]

**Proposition 4.** The sequence of \(A\)-modules

\[
0 \rightarrow \mathcal{O}_{X,z}^\wedge \otimes A \overset{\partial}{\rightarrow} t \mathcal{O}_{X,z}^\wedge \otimes A \overset{\lambda}{\rightarrow} K_z \rightarrow 0
\]

is exact.

**Proof.** Denote by \(\mu\) the multiplication map

\[
\mu: t \mathcal{O}_{X,z}^\wedge \otimes A \rightarrow K_z: f \otimes a \mapsto af.
\]
Using the identity (1) one verifies that the diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & O_{X,z}^\wedge \otimes A & \stackrel{1 \otimes t \otimes 1}{\longrightarrow} & tO_{X,z}^\wedge \otimes A & \stackrel{\mu}{\longrightarrow} & K_z & \longrightarrow & 0 \\
\exp \otimes \text{id} & \downarrow & \exp \otimes \text{id} & \downarrow & \text{id} & & & & \\
0 & \longrightarrow & O_{X,z}^\wedge \otimes A & \stackrel{\partial}{\longrightarrow} & tO_{X,z}^\wedge \otimes A & \stackrel{\lambda}{\longrightarrow} & K_z & \longrightarrow & 0
\end{array}
\]
commutes. The vertical arrows are isomorphisms by Corollary and since the top sequence is exact the same holds for the bottom sequence. □

3.4. Conclusion. It is now a purely formal matter to deduce Theorem 1 from Propositions 3 and 4

Proof of Theorem 4. Clearly the map
\[
O_X \otimes A \longrightarrow O_X(\infty) \otimes A
\]
is injective. We need to construct an isomorphism \( \text{coker} \partial \rightarrow C \).

For every open \( U \subset X \) and every integer \( m \) we have an exact sequence
\[
0 \longrightarrow O_X(m \infty)(U) \longrightarrow O_X(U \cap Y) \times \prod_z O_{X,z}^\wedge \delta \longrightarrow \prod_z K_z
\]
where the products range over \( z \in U \setminus Y \) and where \( \delta(f, g) := f - g \). If moreover \( U \) is affine then \( \delta \) is surjective and we obtain a short exact sequence which we denote by \( E(m) \).

Now, for an affine \( U \), consider the map of exact sequences
\[
\partial: E(0) \otimes A \rightarrow E(1) \otimes A.
\]
It is injective in all three positions. Using (4) and (5) one sees that the quotient is isomorphic with a short exact sequence
\[
0 \longrightarrow (\text{coker} \partial)(U) \longrightarrow C_0(U \cap Y) \times \prod_z K_z \longrightarrow \prod_z C_0(K_z) \longrightarrow 0,
\]
the last map being \( (c, f) \mapsto c - \exp f \). This provides an isomorphism \( (\text{coker} \partial)(U) \rightarrow C(U) \) for every affine open \( U \subset X \), and clearly these glue to an isomorphism of sheaves. This proves Theorem 11. □

4. Proof of Theorem 2

Let \( S \) be a \( k \)-scheme. For any \( O_S \_A \)-module \( F \) we denote by \( \tau \) the canonical isomorphism of \( S_A \)-sheaves
\[
\tau: F \longrightarrow \tau^*_A F
\]
which is \( A \)-linear but generally not \( O_S \_A \)-linear. If
\[
\mathcal{M} = \begin{bmatrix} \mathcal{M} \twoheadrightarrow \mathcal{M}' \xleftarrow{i} \tau^* \mathcal{M} \end{bmatrix}
\]
is a shtuka on $S$ then we denote by $\mathcal{M}^\bullet$ the complex of $S_A$-sheaves

$$\mathcal{M} \xrightarrow{\partial} \mathcal{M}'$$

in degrees 0 and 1, with $\partial = \sigma - j \circ \tau$.

The following Proposition can be found implicitly in [5].

**Proposition 5.** For all $i$ and all $\mathcal{M}$ there are natural isomorphisms

$$\text{Ext}^i(1, \mathcal{M}) = \mathbb{H}^i(S_A, \mathcal{M}^\bullet),$$

functorial in $\mathcal{M}$ and in $S$.

Before giving a proof, we first deduce Theorem 2 from this proposition.

**Proof of Theorem 2.** Applying the Proposition to $S = X$ and $\mathcal{M} = C$ we find

$$\text{Ext}^i(1, C) = \mathbb{H}^i(X_A, O_X \otimes A \xrightarrow{\partial} O_X(\infty) \otimes A).$$

The latter is isomorphic with

$$\mathbb{H}^i(X, O_X \otimes A \xrightarrow{\partial} O_X(\infty) \otimes A)$$

which by Theorem 1 is isomorphic with $H^{i-1}(X, C)$. \hfill $\Box$

**Proof of Proposition 5.** We will first establish a canonical isomorphism for $i = 0$, and then conclude the general case by a purely formal argument.

A homomorphism $1 \to \mathcal{M}$ of shtukas on $S$ is a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{S_A} & \xrightarrow{1} & \mathcal{O}_{S_A} \\
\downarrow f & & \downarrow f' \\
\mathcal{M} & \xrightarrow{\sigma} & \mathcal{M}' \\
\uparrow \tau f & & \uparrow \tau f' \\
\end{array}$$

Clearly the homomorphism is uniquely determined by $f \in \Gamma(S_A, \mathcal{M})$, and an $f \in \Gamma(S_A, \mathcal{M})$ extends to a homomorphism of shtukas if and only if $\partial f = 0$. So we obtain an exact sequence

$$0 \to \text{Hom}(1, \mathcal{M}) \to \Gamma(S_A, \mathcal{M}) \xrightarrow{\partial} \Gamma(S_A, \mathcal{M}')$$

and hence an isomorphism

$$\text{Hom}(1, \mathcal{M}) = \mathbb{H}^0(S_A, \mathcal{M}^\bullet).$$

Now any shtuka

$$\mathcal{I} = \begin{bmatrix} \mathcal{I} \xrightarrow{\sigma} \mathcal{I}' \xleftarrow{i} \tau^* \mathcal{I} \end{bmatrix}$$

with $\mathcal{I}$ and $\mathcal{I}'$ injective $\mathcal{O}_{S_A}$-modules is an injective object in the category of shtukas on $S$. So we can find an injective resolution $\mathcal{I}^\bullet$ of the shtuka
such that the resulting double complex $\mathcal{I}^\bullet$ is an injective resolution of the complex $\mathcal{M}^\bullet$. We obtain a canonical isomorphism

$$\text{Ext}^i(1, \mathcal{M}) = \mathcal{H}^i(S_A, \mathcal{M}^\bullet)$$

for all $i$. □

References

[1] Greg W. Anderson. t-motives. *Duke Math. J.*, 53(2):457–502, 1986. MR850546
[2] Greg W. Anderson and Dinesh S. Thakur. Tensor powers of the Carlitz module and zeta values. *Ann. of Math. (2)*, 132(1):159–191, 1990. MR1059938
[3] V. G. Drinfel’d. Commutative subrings of certain noncommutative rings. *Funkcional. Anal. i Prilozhen.*, 11(1):11–14, 96, 1977. MR0476732
[4] L. Lafforgue. Chtoucas de Drinfeld et applications. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 563–570 (electronic), 1998. MR1648105
[5] Vincent Lafforgue. Valeurs spéciales des fonctions L en caractéristique p. *J. Number Theory*, 129(10):2600–2634, 2009. MR2541033
[6] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2006. MR2242284
[7] Matthew A. Papanikolas and Niranjan Ramachandran. A Weil-Barsotti formula for Drinfeld modules. *J. Number Theory*, 98(2):407–431, 2003. MR1955425
[8] Bjorn Poonen. Local height functions and the Mordell-Weil theorem for Drinfeld modules. *Compositio Math.*, 97(3):349–368, 1995. MR1353279
[9] Lenny Taelman. A Dirichlet unit theorem for Drinfeld modules. *Math. Ann.*, 2010.
[10] Lenny Taelman. Special $L$-values of Drinfeld modules. *preprint*, 2010.
[11] Sung Sik Woo. Extensions of Drinfeld modules of rank 2 by the Carlitz module. *Bull. Korean Math. Soc.*, 32(2):251–257, 1995. MR1356079