An Improved Sample Complexity Lower Bound for Quantum State Tomography

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Abstract

We show that $\Omega(\frac{rd}{\epsilon})$ copies of an unknown rank-$r$, dimension-$d$ quantum mixed state are necessary in order to learn a classical description with $1 - \epsilon$ fidelity. This improves upon the tomography lower bounds obtained by Haah, et al. and Wright.

1 Background

We consider general quantum state tomography: given $n$ copies of a mixed state $\rho$, output a classical description of a state $\sigma$ that is close to $\rho$. In this note, we measure closeness between $\rho$ and $\sigma$ via their fidelity $F(\rho, \sigma)$, defined as the supremum of $|\langle \varphi | \psi \rangle|^2$ over all purifications $|\psi\rangle$, $|\varphi\rangle$ of $\rho$, $\sigma$ respectively.\(^1\) Haah, et al. [Haa+17] showed that $n = O(rd \log(d/\epsilon)/\epsilon)$ is sufficient for tomography where $r$ is the rank of the density matrix $\rho$, and they showed $n = \Omega(\frac{rd}{\epsilon \log(d/\epsilon)})$ is necessary; thus their upper and lower bounds are tight up to logarithmic factors.

O'Donnell and Wright [OW16] studied “trace distance tomography”; that is, the task of tomography where the closeness measure is the trace distance $\|\rho - \sigma\|_1 = \frac{1}{2} \text{Tr}(|\rho - \sigma|)$. They showed that $n = O(rd/\delta^2)$ input samples are sufficient to achieve trace distance error $\delta$ in the resulting estimate. In his PhD thesis, Wright [Wri16] showed that $n = \Omega(rd)$ samples are necessary when the desired trace distance error is fixed to some constant, but does not establish a matching lower bound of $\Omega(rd/\delta^2)$.

In this note we improve the lower bound of Haah, et al. [Haa+17] and show that $n = \Omega(rd/\epsilon)$ input samples are needed to perform tomography with the fidelity closeness measure. We believe that this should be optimal, i.e., $O(rd/\epsilon)$ input samples are sufficient for fidelity tomography; however we leave this for future work.

2 The argument

We prove our lower bound via reduction to the pure state tomography scenario, in which the input samples $\rho$ are guaranteed to be pure states (in other words, the rank of the density matrix is 1). It was proved by Bruß and Macchiavello [BM99] that $n = \Theta(d/\epsilon)$ samples are necessary and sufficient to achieve fidelity $1 - \epsilon$; this was based on a tight connection between optimal pure state tomography and optimal pure state cloning [Wer98].

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\(^1\)We note that there are two versions of fidelity considered in the literature; this is the squared one.
Suppose there is an algorithm $A$ that, on input $n$ copies of a rank-$r$, dimension-$d$ mixed state $\rho$, outputs with high probability a classical description of a state $\sigma$ that has fidelity $1 - \epsilon$ with $\rho$. Then we use this algorithm to construct another algorithm $B$ that performs tomography on pure, dimension-$rd$ states using $O\left(n + \frac{r^2}{\epsilon} \right)$ input samples and achieves $1 - O(\epsilon)$ fidelity. The performance of algorithm $B$ is subject to the bounds of Bruß and Macchiavello [BM99] – in other words, $B$ must use at least $\Omega(rd/\epsilon)$ input samples. Thus it must be that

$$n = \Omega(rd/\epsilon) - O(r^2/\epsilon) = \Omega(rd/\epsilon) ,$$

as desired.

The algorithm $B$ works as follows. Let $|\psi\rangle_{XY}$ denote the $rd$-dimensional pure input sample where $X$ denotes an $r$-dimensional register and $Y$ denotes a $d$-dimensional register.

1. The algorithm $B$ takes $n$ input samples $|\psi\rangle_{XY}$ and traces out the $X$ register in each copy to obtain $n$ copies of a mixed state $\rho \in \mathbb{C}^{d \times d}$.

2. Run algorithm $A$ on the $n$ copies of $\rho$ to obtain (with high probability) a classical description of a rank-$r$, dimension-$d$ state $\sigma$ that has fidelity $1 - \epsilon$ with $\rho$.

3. Compute a classical description of the rank-$r$ projector $\Pi$ onto the support of $\sigma$.

4. Take $O(r^2/\epsilon)$ additional copies of the input state $|\psi\rangle_{XY}$ and measure the $Y$ register of each copy using the projective measurement $\{\Pi, I - \Pi\}$, and keep the post-measurement states $|\tilde{\psi}\rangle$ of the copies where the $\Pi$ outcome was obtained.

5. Use the tomography procedure of [BM99] for dimension-$r^2$ pure states on the copies of $|\tilde{\psi}\rangle$ where we treat the states as residing in the dimension-$r^2$ subspace

$$\mathbb{C}^r \otimes \text{supp(\Pi)} \subseteq \mathbb{C}^r \otimes \mathbb{C}^d .$$

Let $|\varphi\rangle \in \mathbb{C}^r \otimes \text{supp(\Pi)}$ denote the result of this pure state tomography procedure. The algorithm $B$ then outputs the classical description of $|\varphi\rangle$ as its estimation of $|\psi\rangle$.

We analyze the algorithm $B$. Let

$$|\psi\rangle_{XY} = \sum_{i=1}^{r} \lambda_i |u_i\rangle \otimes |v_i\rangle$$

denote the Schmidt decomposition of $|\psi\rangle$ where $\{|u_1\rangle, \ldots, |u_r\rangle\}$ is an orthonormal basis for $\mathbb{C}^r$ and $\{|v_1\rangle, \ldots, |v_r\rangle\}$ is an orthogonal set of vectors in $\mathbb{C}^d$. We can then write $\rho$ as

$$\rho = \text{Tr}_X(|\psi\rangle\langle\psi|) = \sum_{i=1}^{r} \lambda_i^2 |v_i\rangle\langle v_i| .$$

Note that $\rho$ is a rank-$r$ density matrix.

Let $\sigma = \sum_{i=1}^{r} \mu_i |w_i\rangle\langle w_i|$ denote the estimate produced by Step 2. By the guarantees of algorithm $A$, we have that (with high probability) $F(\rho, \sigma) \geq 1 - \epsilon$. Let $\Pi = \sum_{i=1}^{r} |w_i\rangle\langle w_i|$ denote the projector onto the support of $\sigma$. By the definition of fidelity, there exists a purification $|\varphi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^d$ of $\sigma$ such that

$$F(\rho, \sigma) = |\langle \psi \mid \varphi \rangle|^2 \geq 1 - \epsilon .$$
On the other hand,

\[ |\langle \psi | \varphi \rangle|^2 = |\langle \psi | (I \otimes \Pi) | \varphi \rangle|^2 \leq \langle \psi | I \otimes \Pi | \psi \rangle \cdot \langle \varphi | \varphi \rangle = \langle \psi | I \otimes \Pi | \psi \rangle \]

where the inequality uses Cauchy-Schwarz. Let |\tilde{\psi}\rangle = (I \otimes \Pi) |\psi\rangle / \|I \otimes \Pi | \psi\rangle \|, and observe that

\[ |\langle \tilde{\psi} | \psi \rangle|^2 \geq \langle \psi | I \otimes \Pi | \psi \rangle^2 \geq (1 - \epsilon)^2 \geq 1 - 2\epsilon. \]

The number of copies of |\tilde{\psi}\rangle available in Step 5 is, with high probability, at least \( \Omega(\tau^2/\epsilon) \). Thus the estimate |\varphi\rangle computed by Step 5 will satisfy

\[ |\langle \varphi | \tilde{\psi} \rangle|^2 \geq 1 - \epsilon \]

with high probability.

It is easy to see that if |\langle \tilde{\psi} | \psi \rangle| \geq 1 - \eta and |\langle \varphi | \tilde{\psi} \rangle| \geq 1 - \eta, then |\langle \varphi | \psi \rangle| \geq 1 - 4\eta. Therefore with high probability, the estimate |\varphi\rangle produced by algorithm \( \mathcal{B} \) satisfies

\[ |\langle \varphi | \psi \rangle|^2 \geq (1 - 8\epsilon)^2 \geq 1 - 16\epsilon. \]

3 Conclusion

We proved a \( \Omega(rd/\epsilon) \) sample complexity lower bound for fidelity tomography for rank-\( r \), dimension-\( d \) mixed states where \( 1 - \epsilon \) is the fidelity of the resulting estimate. This is proved via reduction to the \( \Omega(d/\epsilon) \) lower bound for pure state tomography established by [BM99]. In contrast, the lower bounds of [Haa+17] and [Wri16] are based on communication complexity arguments. Natural questions include: (a) whether the upper bound for fidelity tomography can be improved to \( O(rd/\epsilon) \), and (b) whether a \( \Omega(rd/\delta^2) \) lower bound can be established for trace distance tomography. One obstacle to extending our argument to the trace distance setting is that we do not know whether applying the projection \( \Pi \) to the state |\psi\rangle (if \( \Pi \) is the projector onto the support of a state \( \sigma \) that is \( \delta \)-close to \( \rho \) in trace distance) yields a state that is \( O(\delta) \)-close to |\psi\rangle in trace distance. The Gentle Measurement Lemma [Win99] implies that the post-measurement state is \( O(\sqrt{\delta}) \)-close to |\psi\rangle; this ultimately yields a \( \Omega(rd/\delta) \) lower bound on trace distance tomography, which we believe is not optimal.

References

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