An Extension of Mok’s Theorem on the Generalized Frankel Conjecture

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Abstract In this paper, we will give an extension of Mok’s theorem on the generalized Frankel conjecture under the condition of the orthogonal holomorphic bisectional curvature.

1. Introduction

Let $M^n$ be a complex $n$-dimensional compact Kähler manifold. One of the interesting problems is to give the classification of the manifolds under certain curvature conditions. Corresponding to the sectional curvature condition in Riemannian geometry, one usually considers the holomorphic bisectional curvature in complex differential geometry. In 1979 Mori [10] and in 1980 Siu-Yau [12] independently proved the famous Frankel conjecture by using different methods. They proved that: any compact Kähler manifold with positive holomorphic bisectional curvature must be biholomorphic to the complex projective space. After the work of Mori and Siu-Yau, in 1988, Mok [9] generalized the Frankel conjecture to the
nonnegative case, usually we call it the generalized Franke conjecture which states that: *any compact irreducible Kähler manifold with nonnegative bisectional curvature must be either a Hermitian symmetric manifold or biholomorphic to the complex projective space.* Recently, based on the work of Brendle-Schoen [2], the first author [5] gave a simple and completely transcendental proof to Mok’s theorem on the generalized Franke conjecture. In the late 80’s, Cao and Hamilton [3] introduced the concept of orthogonal holomorphic bisectional curvature and observed that the nonnegativity of the orthogonal holomorphic bisectional curvature is preserved under the Kähler-Ricci flow. (For the definition of the orthogonal holomorphic bisectional curvature we will give in the following.) In 2006, X.X.Chen [4] generalized the Frankel conjecture in another aspect with the orthogonal holomorphic bisectional curvature but under some additional condition. He proved that: *any compact irreducible Kähler manifold with positive orthogonal holomorphic bisectional curvature and $c_1 > 0$ must be biholomorphic to the complex projective space.*

**Definition 1.1** A complex $n$-dimensional ($n \geq 2$) Kähler manifold $(M^n, h)$ is said to have nonnegative orthogonal holomorphic bisectional curvature if for any orthonormal basis $\{e_\alpha\}$, the following holds:

$$R(e_\alpha, \overline{e_\alpha}, e_\beta, \overline{e_\beta}) = R_{\alpha\overline{\alpha}\beta\overline{\beta}} \geq 0,$$

for any $\alpha \neq \beta$.

If we consider the Kähler manifold as a Riemannian manifold, then we define the manifold has nonnegative orthogonal holomorphic bisectional curvature by

$$R(u_i, Ju_i, Ju_j, u_j) \geq 0,$$

for any $< u_i, u_j > = < u_i, Ju_j > = 0$, where $J$ is the complex structure of $M$. The above Definition 1.1 is equivalent to that in the Riemannian case. Indeed, we can choose an orthonormal basis $\{u_1, u_2, \cdots, u_{2n}\}$ such that $Ju_i = u_{n+i}$ for $i = 1, 2, \cdots, n$. Set $e_i = \frac{1}{\sqrt{2}}(u_i - \sqrt{-1}Ju_i)$, then $\{e_i\}$ is an orthonormal basis. It follows that

$$R(e_i, \overline{e_i}, e_j, \overline{e_j}) = R_{ij\overline{j}i\overline{j}} = R(u_i, Ju_i, Ju_j, u_j),$$

for any $i \neq j$. This implies the two definitions are equivalent.

Recently, Seshadri [11] gives the classification of manifolds with nonnegative isotropic curvature. He proved that: *any compact irreducible Kähler manifold with*
nonnegative isotropic curvature must be either a Hermitian symmetric manifold or biholomorphic to the complex projective space. From the computation in Lemma 2.1 in [11], we can see that nonnegative isotropic curvature implies the nonnegative orthogonal holomorphic bisectional curvature. However, the converse is not true. Following we give an example and other examples can be given in a similar way:

**Example 1.2** Let 

\[(M, h) = (\Sigma, g) \times (CP^n, g_0),\]

where \(\Sigma\) is a Riemann surface with Gauss curvature \(\kappa(\Sigma) \geq -4\) and \(\min(\kappa(\Sigma)) = -4\) and \(g_0\) is the standard Fubini-Study metric such that the sectional curvature of \(CP^n\) satisfies \(1 \leq K(p) \leq 4\). In the following, we want to show that \(M\) has nonnegative orthogonal holomorphic bisectional curvature but the isotropic curvature is not nonnegative.

Indeed, suppose \(\tilde{e}_0\) and \(\{\tilde{e}_i\}, (1 \leq i \leq n)\), are the orthonormal basis of \(T_{p,0}^{1,0}(\Sigma)\) and \(T_{q,0}^{1,0}(CP^n)\) respectively. Then we can naturally extend them to be the orthonormal basis \(\{e_i\}, (0 \leq i \leq n)\), of \(T_{x,0}^{1,0}(M)\) at the point \(x = (p, q) \in M\), such that \(pr_1(e_0) = \tilde{e}_0\), and \(pr_2(e_i) = \tilde{e}_i\),

where \(pr_1, pr_2\) denote the canonical projection onto \(\Sigma\) and \(CP^n\) respectively.

Now for any two orthogonal vectors \(X, Y\) on \(M\), we assume that:

\[X = \sum_{i=0}^{n} a_i e_i, \text{ and } Y = \sum_{i=0}^{n} b_i e_i,\]

where \(a_i, b_i\) are complex numbers satisfy \(\sum_{i=0}^{n} a_i \overline{b_i} = 0\).

Then by direct computation we can get that

\[R(X, \overline{X}, Y, \overline{Y})\]

\[= |a_0|^2 |b_0|^2 R_{0000} + 4 \sum_{i=1}^{n} (|a_i|^2 |b_i|^2) + 2 \sum_{i=1}^{n} \sum_{j \neq i} (|a_i|^2 |b_j|^2 + a_i \overline{a_j} b_j \overline{b_i})\]

\[\geq -4 \sum_{i=1}^{n} |a_i|^2 |b_i|^2 + 4 \sum_{i=1}^{n} (|a_i|^2 |b_i|^2) + 2 \sum_{i=1}^{n} \sum_{j \neq i} (|a_i|^2 |b_j|^2 + a_i \overline{a_j} b_j \overline{b_i})\]

\[= 2 \sum_{1 \leq i < j \leq n} |a_i b_j - a_j b_i|^2\]

\[\geq 0.\]
This implies that the orthogonal holomorphic bisectional curvature of \( M \) is non-negative. On the other hand, by direct computation or the result of [8], it is easy to see that the isotropic curvature is not nonnegative.

Clearly nonnegative holomorphic bisectional curvature also implies the nonnegative orthogonal holomorphic bisectional curvature, naturally we want to know the relations between holomorphic bisectional curvature and isotropic curvature. By the work of Ivey [7], we know that in the complex 2-dimensional case, nonnegative holomorphic bisectional curvature implies nonnegative isotropic curvature. So the result of Seshadri [11] can be viewed as a generalization of Mok's theorem on the generalized Frankel conjecture in complex 2-dimension. But in higher dimensional case, we do not know whether this is also true, since the nonnegative holomorphic bisectional curvature means the bisectional curvature is nonnegative on any holomorphic complex plane, while nonnegative isotropic curvature requires on any 2-dimensional isotropic plane. Even though, we know that both holomorphic bisectional curvature and isotropic curvature imply orthogonal holomorphic bisectional curvature. So orthogonal holomorphic bisectional curvature is the weakest one among the three curvature conditions. In [4], X.X. Chen asked a question: whether a compact Kähler manifold with positive orthogonal holomorphic bisectional curvature necessary has \( c_1 > 0 \). In this paper, we give an affirmative answer to this question and hence solve the Question/Conjecture 1.6 in [4]. Moreover, we will also give a complete classification of manifolds with nonnegative orthogonal holomorphic bisectional curvature. This can be considered as an extension of the generalized Frankel conjecture. Our main result is the following:

**Theorem 1.3** Suppose \((M^n, h)\) is an \( n \)-dimensional \( (n \geq 2) \) compact Kähler manifold of nonnegative orthogonal holomorphic bisectional curvature. Let \((\tilde{M}^n, \tilde{h})\) be its universal covering space. Then \((\tilde{M}^n, \tilde{h})\) is isometrically biholomorphic to one of the following two cases:

1. \((C^k, h_0) \times (M_1, h_1) \times \cdots \times (M_l, h_l) \times (CP^{n_1}, \theta_1) \times \cdots \times (CP^{n_r}, \theta_r),\)

where \(h_0\) denotes the Euclidean metric on \(C^k, h_i(1 \leq i \leq l)\) are canonical metrics on the irreducible compact Hermitian symmetric spaces \(M_i\) of rank \( \geq 2 \), and \(\theta_j(1 \leq j \leq r)\) is a Kähler metric on \(CP^{n_j}\) carrying nonnegative orthogonal holomorphic bisectional curvature;

2. \((Y, g_0) \times (M_1, h_1) \times \cdots \times (M_l, h_l) \times (CP^{n_1}, \theta_1) \times \cdots \times (CP^{n_r}, \theta_r),\)
where $Y$ is a simply connected Riemann surface with Gauss curvature negative somewhere or a simply connected noncompact Kähler manifold with $\dim(Y) \geq 2$ and has nonnegative orthogonal holomorphic bisectional curvature and the minimum of the holomorphic sectional curvature $< 0$ somewhere, $M_i, \mathbb{CP}^n_j (1 \leq i \leq l, 1 \leq j \leq r)$ are the same as in case (1). Moreover, we have the holomorphic sectional curvatures of $M_i$ and $\mathbb{CP}^n_j$ are $\geq -\min\{\text{holomorphic sectional curvature of } Y\} > 0$.

This paper contains three sections and the organization is as follows. In section 2, we will prove the positivity of the first Chern class under the positive orthogonal holomorphic bisectional curvature condition and give some results on the irreducible manifolds which will be used in the proof of our main theorem. In section 3, we will complete the proof of the Theorem 1.3.

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2. Some Results on Irreducible Manifolds

In the following we first give a similar result to [8] in terms of the orthogonal holomorphic bisectional curvature in the Kähler manifolds. We will show that the curvature term in the Weitzenböck formula on $(1,1)$-forms involves only the orthogonal holomorphic bisectional curvature. This also gives the answer to the positivity of the first Chern class under the positive orthogonal holomorphic bisectional curvature condition. In this section we always assume that the complex dimension $n$ of the Kähler manifold $M^n$ satisfies $n \geq 2$.

Theorem 2.1 Let $(M^n, h)$ be a compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature. Then all real harmonic $(1,1)$-forms are parallel. Furthermore, we have

(i) If $b_{1,1}(M) = \dim H^{1,1}(M) = 1$, then $c_1(M) > 0$;

(ii) If in addition $M$ is locally irreducible, then we have $b_{1,1}(M) = \dim H^{1,1}(M) = 1$ and hence by (i) we have $c_1(M) > 0$. 


Proof. Suppose \((M^n, h)\) is a compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature and \(J\) is the complex structure. Let \(\eta\) be a nontrivial harmonic \((1,1)\)-form on \(M\).

In the following, we want to show that \(\eta\) is parallel. Indeed, the parallelity of \(\eta\) was already obtained by [13] under the condition of nonnegative holomorphic bisectional curvature. For the completeness of our paper, we will adapt the argument in [13] and [8] to show that \(\eta\) is parallel under the condition of nonnegative orthogonal holomorphic bisectional curvature.

Now we can choose an orthonormal basis \(\{e_\beta\}_{\beta=1}^n\) such that under this basis
\[
\eta = \frac{\sqrt{-1}}{2} \sum_{\beta=1}^n 2a_\beta \cdot e_\beta \wedge \overline{e_\beta}.
\]

Set
\[
e_\beta = \frac{1}{\sqrt{2}} (u_\beta - \sqrt{-1}Ju_\beta), \quad \text{for} \quad 1 \leq \beta \leq n,
\]
where \(\{u_1, Ju_1, \ldots, u_n, Ju_n\}\) is an orthonormal basis of \(M\) in the sense of considering \(M\) as a Riemannian manifold. So in the basis \(\{u_1, Ju_1, \ldots, u_n, Ju_n\}\), \(\eta\) becomes
\[
\eta = -\sum_{i=1}^n a_i \cdot u_i \wedge Ju_i.
\]

By the Bochner formula we have
\[
\Delta \eta = \nabla^\ast \nabla \eta + \mathcal{L}(\eta),
\]
where \(\mathcal{L}(\eta) = -\frac{1}{4} \sum_{i,j} (R(u_i), \eta_j)[\eta_i, [\eta_j, \eta]]\) and \((\cdot, \cdot)\) denotes the corresponding Riemannian metric. Then by the same argument as in [8], we know that
\[
(\mathcal{L}(\eta), \eta) = \frac{1}{2} \sum_{\alpha > \theta} (R(X_\alpha), X_{-\alpha})(-\alpha(\eta)X_{-\alpha}, \alpha(\eta)X_\alpha)
= \frac{1}{2} \sum_{\alpha > \theta} -\alpha(\eta)^2 (R(X_\alpha), X_{-\alpha}),
\]
where \(\alpha(\eta)\) satisfies \([X_\alpha, \eta] = -[\eta, X_\alpha] = -\alpha(\eta)X_\alpha\) and \(-\alpha(\eta)^2\) is nonnegative and the symbols are the same as in [4]. Now for the positive roots \(x_i + x_j, (1 \leq i < j \leq n)\), we have
\[
X_\alpha = \frac{1}{2} (u_i + \sqrt{-1}Ju_i) \wedge (u_j + \sqrt{-1}Ju_j) = \overline{e_i} \wedge \overline{e_j},
\]
and
\[ X_{-\alpha} = \frac{1}{2}(u_i - \sqrt{-1} Ju_i) \wedge (u_j - \sqrt{-1} Ju_j) = e_i \wedge e_j. \]

For the positive roots \( x_i - x_j, (1 \leq i < j \leq n) \), we have
\[ X_\alpha = \frac{1}{2}(u_i + \sqrt{-1} Ju_i) \wedge (u_j - \sqrt{-1} Ju_j) = \overline{e_i} \wedge e_j, \]

and
\[ X_{-\alpha} = \frac{1}{2}(u_i - \sqrt{-1} Ju_i) \wedge (u_j + \sqrt{-1} Ju_j) = e_i \wedge \overline{e_j}. \]

So \((R(X_\alpha), X_{-\alpha}) = 0\) for the previous case and for the other case, we have
\[ (R(X_\alpha), X_{-\alpha}) = R(\overline{e_i} \wedge e_j, e_i \wedge \overline{e_j}) = R(e_i, \overline{e_j}, e_i, \overline{e_j}) = R(e_i, e_i, e_j, \overline{e_j}) \geq 0, \]

since the orthogonal holomorphic bisectional curvature is nonnegative. Then by the standard Bochner argument we can obtain that all real harmonic \((1,1)\)-forms are parallel.

In order to prove the left conclusions (i) and (ii), we evolve the metric by the Kähler Ricci flow:
\[
\begin{cases}
\frac{\partial}{\partial t} g_{ij}(x, t) = -R_{i\overline{j}}(x, t), \\
g_{ij}(x, 0) = h_{ij}(x).
\end{cases}
\]

Then by Shi’s short-time existence theorem, we know that there is a \( T > 0 \) such that the Ricci flow has a smooth bounded curvature solution \((M, g_{ij}(t))\) for \( t \in [0, T) \). It is due to Cao-Hamilton [3] that the solution \( g_{ij}(t) \) still has nonnegative orthogonal holomorphic bisectional curvature. Suppose \( \{e_\alpha\} \) is an orthonormal basis, then for any \( \alpha \neq \beta \), we have:
\[
R(e_\alpha - e_\beta, \overline{e_\alpha} - \overline{e_\beta}, e_\alpha + e_\beta, \overline{e_\alpha} + \overline{e_\beta}) = R_{\alpha\overline{\alpha}\beta\overline{\beta}} + R_{\beta\overline{\beta}\alpha\overline{\alpha}} - R_{\alpha\overline{\beta}\beta\overline{\alpha}} - R_{\beta\overline{\alpha}\alpha\overline{\beta}} \geq 0, \tag{2.1}
\]

where we have used the assumption and the result that the nonnegativity of the orthogonal holomorphic bisectional curvature is preserved under the Ricci flow due to Cao-Hamilton [3]. Similarly change \( e_\beta \) by \( \sqrt{-1} e_\beta \), we have
\[
R_{\alpha\overline{\alpha}\beta\overline{\beta}} + R_{\beta\overline{\beta}\alpha\overline{\alpha}} + R_{\alpha\overline{\beta}\beta\overline{\alpha}} + R_{\beta\overline{\alpha}\alpha\overline{\beta}} \geq 0. \tag{2.2}
\]
By (2.1) and (2.2) we obtain that

\[ R_{\bar{\alpha}\alpha\bar{\alpha}\alpha} + R_{\bar{\beta}\beta\bar{\beta}\beta} \geq 0 \]  

(2.3)

for any orthonormal 2-frames \( \{e_\alpha, e_\beta\} \). So by the assumption and (2.3), we have

\[ R = \sum_{\alpha, \beta} R_{\bar{\alpha}\alpha\bar{\beta}\beta} = \sum_\alpha \sum_{\beta \neq \alpha} R_{\bar{\alpha}\alpha\bar{\beta}\beta} + \sum_\alpha R_{\bar{\alpha}\alpha\bar{\alpha}\alpha} \geq 0. \]  

(2.4)

If \( b_{1,1}(M) = \dim H^{1,1}(M) = 1 \), let \( \rho \) and \( \omega \) denote the Ricci form and Kähler form respectively, then by the Hodge theory, we have \( \rho = \lambda \omega + \eta \), where \( \lambda \) is a real number and \( \int_M \langle \omega, \eta \rangle = 0 \). On the other hand, we have

\[ \int_M \langle \rho, \omega \rangle = \frac{1}{4} \int_M R = \lambda \| \omega \|^2 \geq 0, \]

since the scalar curvature \( R \geq 0 \) by (2.4). Hence we have \( c_1(M) \geq 0 \). Moreover if the scalar curvature \( R \) at some point is positive, then \( c_1(M) > 0 \). So now we can assume that the scalar curvature \( R(t) \equiv 0 \) for all sufficiently small \( t \). Then by the evolution equation of the scalar curvature

\[ \frac{\partial R}{\partial t} = \Delta R + |Ric|^2 \]

we know that for all sufficiently small \( t \),

\[ Ric(t) \equiv 0. \]  

(2.5)

We claim that for all \( \alpha \), the holomorphic sectional curvature \( R_{\bar{\alpha}\alpha\bar{\alpha}\alpha} = 0 \).

Indeed, by (2.3)-(2.5), we know that for any \( \alpha \neq \beta \):

\[ R_{\bar{\alpha}\alpha\bar{\beta}\beta} = 0, \quad \text{and} \quad R_{\bar{\alpha}\alpha\bar{\alpha}\alpha} + R_{\bar{\beta}\beta\bar{\beta}\beta} = 0. \]

Suppose there exists \( 1 \leq \alpha \leq n \) such that \( R_{\bar{\alpha}\alpha\bar{\alpha}\alpha} \neq 0 \), then

\[ R_{\bar{\alpha}\alpha} = \sum_{\beta \neq \alpha} R_{\bar{\alpha}\alpha\bar{\beta}\beta} + R_{\bar{\alpha}\alpha\bar{\alpha}\alpha} \neq 0. \]

And this contradicts with (2.5). So we have proved the claim and hence the curvature operator is equal to zero. Therefore \((M^n, h)\) is flat. However, note that \( n \geq 2 \), we know that there exists no compact and flat Kähler manifold satisfying
$b_{1,1}(M) = \dim H^{1,1}(M) = 1$. Thus the scalar curvature must be positive at some point. Hence $c_1(M) > 0$. This completes the proof of (i).

In the following we will give the proof of (ii). We argue by contradiction. Suppose $b_{1,1}(M) = \dim H^{1,1}(M) > 1$, then by the same argument as in [8] in the proof of Theorem 2.1 (b) and note that $M$ is locally irreducible, we know that $h$ is hyper-Kähler and hence Ricci flat. So by the argument above, we know that $M$ is flat. And this is a contradiction with the local irreducibility of $M$. So $b_{1,1}(M) = \dim H^{1,1}(M) = 1$. Then by (i) we know that $c_1(M) > 0$. This completes the proof of (ii).

Therefore we complete the proof of Theorem 2.1.

From Theorem 2.1 and the result of [4], we immediately obtain:

**Corollary 2.2** Let $(M^n, h)$ be a compact Kähler manifold with positive orthogonal holomorphic bisectional curvature. Then the first Chern class $c_1(M) > 0$. Moreover, the underlying manifold is biholomorphic to $CP^n$.

**Proof.** Since $(M^n, h)$ has positive orthogonal holomorphic bisectional curvature, we get that $M$ is locally irreducible. Then by Theorem 2.1 (ii) we know that $c_1(M) > 0$. Combining the result of [4], we obtain that $M$ is biholomorphic to the complex projective space $CP^n$.

Suppose $(M^n, h)$ is a compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature and $g_{ij}(t), 0 \leq t \leq \delta$, is the solution to the Kähler Ricci flow with the initial data $h$. Let $P$ be the bundle with the fixed metric $h$ and the fibre over $p \in M$ consists of all orthogonal 2-vectors $\{X, Y\} \subset T^1_p(M)$. We define a function on $P \times (0, \delta)$ by

$$u(\{X, Y\}, t) = R(X, \overline{X}, Y, \overline{Y}),$$

where $R$ denotes the pull-back of the curvature tensor of $g_{ij}(t)$.

**Proposition 2.3** There exists $c > 0$ such that

$$\frac{\partial u}{\partial t} \geq Lu + c \min \left\{0, \inf_{\xi \in V, |\xi| = 1} D^2u(\xi, \xi)\right\} - c \sup_{\xi \in V, |\xi| = 1} Du(\xi) - cu,$$
where \( L \) is the horizontal Laplacian on \( P \) and \( V \) denotes the vertical subspace of the bundle.

**Proof.** According to Hamilton [6], under the evolving orthonormal frame \( \{ e_\alpha \} \), we have

\[
\frac{\partial}{\partial t} R_{\alpha\beta\bar{\alpha}\bar{\beta}} = \Delta R_{\alpha\beta\bar{\alpha}\bar{\beta}} + \sum_{\mu,\nu} \left( R_{\alpha\mu\bar{\nu}} R_{\nu\beta\bar{\mu}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 \right)
\]

\[
= \Delta R_{\alpha\beta\bar{\alpha}\bar{\beta}} + \sum_{\mu,\nu=\alpha,\beta} \left( R_{\alpha\mu\bar{\nu}} R_{\nu\beta\bar{\mu}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 \right)
\]

\[
+ \left( \sum_{\mu=\alpha,\beta} + \sum_{\nu=\alpha,\beta} \right) \left( R_{\alpha\mu\bar{\nu}} R_{\nu\beta\bar{\mu}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 \right)
\]

\[
+ \sum_{\mu,\nu \neq \alpha,\beta} \left( R_{\alpha\mu\bar{\nu}} R_{\nu\beta\bar{\mu}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 \right)
\]

\[
\triangle R_{\alpha\beta\bar{\alpha}\bar{\beta}} = \Delta R_{\alpha\beta\bar{\alpha}\bar{\beta}} + (I) + (II) + (III).
\]

During the following proof, we assume that \( c \) denotes the various positive constants which depend on the bound of the curvature and its derivatives.

**Claim 1.** There exist constants \( c_1 > 0, c_2 > 0 \) such that

\[
I \geq -c_1 \cdot u(\{ e_\alpha, e_\beta \}, t) - c_2 \sup_{\xi \in V, |\xi| = 1} Du(\{ e_\alpha, e_\beta \}, t)(\xi).
\]

Indeed: by definition and direct computation, we have

\[
I = \sum_{\mu,\nu=\alpha,\beta} \left( R_{\alpha\mu\bar{\nu}} R_{\nu\beta\bar{\mu}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 \right)
\]

\[
= R_{\alpha\beta\bar{\alpha}\bar{\beta}}(R_{\alpha\alpha\alpha\beta} + R_{\beta\beta\beta\bar{\alpha}} - R_{\alpha\bar{\alpha}\beta\bar{\beta}}) + |R_{\alpha\beta\bar{\alpha}\bar{\beta}}|^2 + 2Re(R_{\alpha\alpha\bar{\alpha}\bar{\beta}} R_{\alpha\bar{\alpha}\beta\bar{\beta}})
\]

(2.7)

Now we consider the orthogonal 2-frames \( \{ \cos \theta e_\alpha + \sin \theta e_\beta, -\sin \theta e_\alpha + \cos \theta e_\beta \} \), we have

\[
u(\{ \cos \theta e_\alpha + \sin \theta e_\beta, -\sin \theta e_\alpha + \cos \theta e_\beta \}, t)
\]

\[
= R(\cos \theta e_\alpha + \sin \theta e_\beta, \cos \theta e_\alpha + \sin \theta e_\beta, -\sin \theta e_\alpha + \cos \theta e_\beta, -\sin \theta e_\alpha + \cos \theta e_\beta).
\]

Then

\[
\frac{du}{d\theta} \bigg|_{\theta=0} = 2Re(R_{\alpha\beta\bar{\alpha}\bar{\beta}} - R_{\alpha\alpha\bar{\alpha}\bar{\beta}}).
\]
So
\[ |Re(R_{\alpha\bar{\beta}\bar{\beta}} - R_{a\bar{a}\bar{a}\bar{\beta}}})| \leq c \cdot \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi), \tag{2.8} \]
for some constant \( c > 0 \).

Similarly, if we change \( e_\beta \) by \( \sqrt{-1}e_\beta \), and consider the orthogonal 2-frames \( \{\cos \theta e_\alpha + \sin \theta \sqrt{-1}e_\beta, -\sin \theta e_\alpha + \cos \theta \sqrt{-1}e_\beta\} \), note that
\[
R(\cos \theta e_\alpha + \sin \theta \sqrt{-1}e_\beta, \cos \theta e_\alpha - \sin \theta \sqrt{-1}e_\beta, \\
- \sin \theta e_\alpha + \cos \theta \sqrt{-1}e_\beta, -\sin \theta e_\alpha - \cos \theta \sqrt{-1}e_\beta) \\
= R(\cos \theta e_\alpha + \sin \theta \sqrt{-1}e_\beta, \cos \theta e_\alpha - \sin \theta \sqrt{-1}e_\beta, \\
\sqrt{-1} \sin \theta e_\alpha + \cos \theta e_\beta, -\sqrt{-1} \sin \theta e_\alpha + \cos \theta e_\beta),
\]
we can obtain that
\[ |Im(R_{\alpha\bar{\beta}\bar{\beta}} - R_{a\bar{a}\bar{a}\bar{\beta}}})| \leq c \cdot \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi). \tag{2.9} \]

By (2.8) and (2.9) we get that
\[ |R_{\alpha\bar{\beta}\bar{\beta}} - R_{a\bar{a}\bar{a}\bar{\beta}}|^2 \leq c \cdot \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi) \]
i.e.,
\[
|R_{\alpha\bar{\beta}\bar{\beta}}|^2 + |R_{a\bar{a}\bar{a}\bar{\beta}}|^2 - 2Re(R_{a\bar{a}\bar{a}\bar{\beta}}R_{\alpha\bar{\beta}\bar{\beta}}) \leq c \cdot \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi).
\]
So we have
\[ 2Re(R_{a\bar{a}\bar{a}\bar{\beta}}R_{\alpha\bar{\beta}\bar{\beta}}) \geq -c \cdot \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi), \tag{2.10} \]
for some constant \( c > 0 \).

By (2.7) and (2.10), we know that
\[ I \geq -c_1 \cdot u(\{e_\alpha, e_\beta\}, t) - c_2 \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi), \]
for some constants \( c_1 > 0, c_2 > 0 \). So we have proved Claim 1.
Claim 2. There exists constant $c_3 > 0$, such that

$$II \geq -c_3 \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi).$$

Indeed: by definition and direct computation, we have

$$II = \left( \sum_{\mu \neq \alpha, \beta} + \sum_{\mu \neq \alpha, \beta} \right) \left( R_{\alpha\bar{\alpha}\mu\bar{\beta}} R_{\nu\bar{\beta}\bar{\beta}} - |R_{\alpha\bar{\alpha}\beta\bar{\beta}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\beta}}|^2 \right)$$

$$= \sum_{\mu \neq \alpha, \beta} 2Re \left( R_{\alpha\bar{\alpha}\mu\bar{\beta}} R_{\mu\bar{\beta}\bar{\beta}} + R_{\alpha\bar{\beta}\mu\bar{\beta}} R_{\mu\bar{\beta}\bar{\beta}} \right)$$

$$+ \sum_{\mu \neq \alpha, \beta} \left( |R_{\alpha\bar{\beta}\mu\bar{\beta}}|^2 - |R_{\alpha\bar{\beta}\mu\bar{\beta}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\beta}}|^2 - |R_{\alpha\bar{\beta}\mu\bar{\beta}}|^2 \right).$$

(2.11)

Now for $\mu \neq \alpha, \beta$, we consider the orthogonal 2-vectors $\{e_\alpha + s e_\mu, e_\beta\}$, we have

$$u(\{e_\alpha + s e_\mu, e_\beta\}, t) = R(e_\alpha + s e_\mu, e_\alpha + s e_\mu, e_\alpha + s e_\mu, e_\beta).$$

Then

$$\left. \frac{du}{ds} \right|_{s=0} = R_{\mu\bar{\beta}\mu} + R_{\alpha\bar{\beta}\mu} = 2Re(R_{\alpha\bar{\beta}\mu}).$$

So we have

$$|Re(R_{\alpha\bar{\beta}\mu})| \leq c \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi),$$

(2.12)

for some constant $c > 0$.

Change $e_\mu$ by $\sqrt{-1} e_\mu$, we can obtain

$$|Im(R_{\alpha\bar{\beta}\mu})| \leq c \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi).$$

(2.13)

By (2.12) and (2.13) we get

$$|R_{\alpha\bar{\beta}\mu}| \leq c \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi).$$

(2.14)

Similarly, we can obtain that

$$|R_{\alpha\bar{\beta}\mu}| \leq c \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi).$$

(2.15)

By (2.11), (2.14) and (2.15) we know that

$$II \geq -c \sup_{\xi \in V, |\xi| = 1} Du(\{e_\alpha, e_\beta\}, t)(\xi),$$

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for some constant $c > 0$. Hence we proved Claim 2.

**Claim 3.** There exists constant $c_4 > 0$, such that

$$III \geq c_4 \cdot \min \left\{ 0, \inf_{\xi \in V, ||\xi||=1} D^2u(\{e_\alpha, e_\beta\}, t)(\xi, \xi) \right\}.$$ 

Indeed: in the following we will prove that

$$\sum_{\mu, \nu \neq \alpha, \beta} \left( R_{\alpha \bar{\mu} \beta} R_{\nu \bar{\beta} \bar{\mu}} - |R_{\alpha \bar{\beta} \bar{\mu}}|^2 \right) \geq c_4 \cdot \min \left\{ 0, \inf_{\xi \in V, ||\xi||=1} D^2u(\{e_\alpha, e_\beta\}, t)(\xi, \xi) \right\},$$

for some constant $c_4 > 0$.

For any vectors $\omega_\alpha, \omega_\beta$ orthogonal to $e_\alpha, e_\beta$, we define an orthogonal 2-vectors \( \{v_\alpha(s), v_\beta(s)\} \) by:

$$v_\alpha(s) = e_\alpha + s\omega_\alpha - \frac{1}{2}s^2 \sum_{j=\alpha, \beta} <\omega_\alpha, \omega_j > e_j + O(s^3),$$

$$v_\beta(s) = e_\beta + s\omega_\beta - \frac{1}{2}s^2 \sum_{j=\alpha, \beta} <\omega_\beta, \omega_j > e_j + O(s^3).$$

Then consider

$$u(\{v_\alpha(s), v_\beta(s)\}, t) = R(v_\alpha, v_\alpha, v_\beta, v_\beta).$$

By direct computation we have

$$\frac{1}{2} \frac{d^2u(s)}{ds^2} \big|_{s=0} = R(\omega_\alpha, \omega_\alpha, e_\beta, e_\beta) + R(e_\alpha, e_\alpha, \omega_\beta, \omega_\beta) + 2Re(R(\omega_\alpha, e_\alpha, e_\beta, \omega_\beta))$$

$$+ 2Re(R(e_\alpha, \omega_\alpha, e_\beta, \omega_\beta)) - ( <\omega_\alpha, \omega_\alpha > + <\omega_\beta, \omega_\beta > )R_{\alpha \bar{\beta} \bar{\alpha}}$$

$$- Re( <\omega_\alpha, \omega_\beta > R_{\beta \bar{\alpha} \bar{\beta}} + <\omega_\beta, \omega_\alpha > R_{\alpha \bar{\alpha} \beta} ).$$

So we have

$$R(\omega_\alpha, \omega_\alpha, e_\beta, e_\beta) + R(e_\alpha, e_\alpha, \omega_\beta, \omega_\beta) + 2Re(R(\omega_\alpha, e_\alpha, e_\beta, \omega_\beta))$$

$$+ 2Re(R(e_\alpha, \omega_\alpha, e_\beta, \omega_\beta)) - ( <\omega_\alpha, \omega_\alpha > + <\omega_\beta, \omega_\beta > )R_{\alpha \bar{\beta} \bar{\alpha}}$$

$$- Re( <\omega_\alpha, \omega_\beta > R_{\beta \bar{\alpha} \bar{\beta}} + <\omega_\beta, \omega_\alpha > R_{\alpha \bar{\alpha} \beta} )$$

$$\geq c \cdot \min \left\{ 0, \inf_{\xi \in V, ||\xi||=1} D^2u(\{e_\alpha, e_\beta\}, t)(\xi, \xi) \right\},$$

(2.16)
for some constant \( c > 0 \). If we change \( e_{\alpha} \) by \(-\sqrt{-1}e_{\alpha}\) and \( e_{\beta} \) by \( \sqrt{-1}e_{\beta} \), we can obtain that

\[
R(\omega_{\alpha}, \overline{\omega}_{\alpha}, e_{\beta}, \overline{e}_{\beta}) + R(e_{\alpha}, \overline{e}_{\alpha}, \omega_{\beta}, \overline{\omega}_{\beta}) - 2Re(R(e_{\alpha}, \overline{e}_{\alpha}, e_{\beta}, \overline{\omega}_{\beta} - (\omega_{\alpha}, \omega_{\alpha}) + (\omega_{\beta}, \omega_{\beta})) R_{\alpha\beta\beta} \\
+ 2Re(R(e_{\alpha}, \overline{e}_{\alpha}, e_{\beta}, \overline{\omega}_{\beta}) - (\omega_{\alpha}, \omega_{\alpha}) + (\omega_{\beta}, \omega_{\beta})) R_{\alpha\beta\beta} \\
+ RE(<\omega_{\alpha}, \omega_{\alpha}>) R_{\alpha\alpha\beta} + <\omega_{\beta}, \omega_{\beta}>) R_{\alpha\alpha\beta}) \\
\geq c \cdot \min \left\{ 0, \inf_{\xi \in V, |\xi|=1} D^2 u(\{e_{\alpha}, e_{\beta}\}, t)(\xi, \xi) \right\},
\]

By (2.16) and (2.17) we have:

\[
R(\omega_{\alpha}, \overline{\omega}_{\alpha}, e_{\beta}, \overline{e}_{\beta}) + R(e_{\alpha}, \overline{e}_{\alpha}, \omega_{\beta}, \overline{\omega}_{\beta}) + 2Re(R(e_{\alpha}, \overline{e}_{\alpha}, e_{\beta}, \overline{\omega}_{\beta}) \\
\geq c \cdot \min \left\{ 0, \inf_{\xi \in V, |\xi|=1} D^2 u(\{e_{\alpha}, e_{\beta}\}, t)(\xi, \xi) \right\}. \tag{2.18}
\]

If we set

\[
A(X, Y) = R(X, Y, e_{\beta}, \overline{e}_{\beta}),
\]

\[
B(X, Y) = R(\overline{e}_{\alpha}, X, e_{\beta}, Y),
\]

\[
C(X, Y) = R(e_{\alpha}, \overline{e}_{\alpha}, X, Y).
\]

Then by (2.18) we know that

\[
\begin{pmatrix}
A & B \\
B^T & C
\end{pmatrix} \geq c \cdot \min \left\{ 0, \inf_{\xi \in V, |\xi|=1} D^2 u(\{e_{\alpha}, e_{\beta}\}, t)(\xi, \xi) \right\}.
\]

Hence we have

\[
tr(AC) - tr(BB) \geq c \cdot \min \left\{ 0, \inf_{\xi \in V, |\xi|=1} D^2 u(\{e_{\alpha}, e_{\beta}\}, t)(\xi, \xi) \right\},
\]

where \( c > 0 \) is a constant depending on the bound of the curvature and its derivatives. i.e.,

\[
\sum_{\mu, \nu \neq \alpha, \beta} \left( R_{\alpha\mu\beta\nu} R_{\nu\mu\beta\beta} - |R_{\alpha\mu\beta\nu}|^2 \right) \geq c \cdot \min \left\{ 0, \inf_{\xi \in V, |\xi|=1} D^2 u(\{e_{\alpha}, e_{\beta}\}, t)(\xi, \xi) \right\}, \tag{2.19}
\]

for some constant \( c > 0 \).
By the definition of $III$ and (2.19), we get

$$III \geq c \cdot \min \left\{ 0, \inf_{\xi \in V, |\xi|=1} D^2 u(\{e_\alpha, e_\beta\}, t)(\xi, \xi) \right\},$$

for some constant $c > 0$. Therefore we have proved Claim 3.

By (2.6), Claim 1, Claim 2 and Claim 3, we can get that

$$\frac{\partial u}{\partial t} \geq Lu + c \min \left\{ 0, \inf_{\xi \in V, |\xi|=1} D^2 u(\xi, \xi) \right\} - c \sup_{\xi \in V, |\xi|=1} Du(\xi) - cu,$$

for some constant $c > 0$, where $L$ is the horizontal Laplacian on $P$ and $V$ denotes the vertical subspace of the bundle.

This completes the proof of Proposition 2.3.

Remark 2.4 In our proof, we have used the result that the nonnegativity of the orthogonal holomorphic bisectional curvature is preserved under the Kähler Ricci flow, which is due to Cao-Hamilton [3] in an unpublished work. However, we only used this result for the first term of (2.7) and for obtaining of (2.18). So if we assume $R_{\alpha\bar{\beta}\beta\bar{\alpha}} = 0$, then the first term of (2.7) is equal to zero and (2.18) is also true. Then combining $Du(\{e_\alpha, e_\beta\}, t) = 0$ and $D^2 u(\{e_\alpha, e_\beta\}, t) \geq 0$, we can see that the argument of Proposition 2.3 has already given a proof to this result. Also it is not hard to see that the positivity of the orthogonal holomorphic bisectional curvature is preserved under the Kähler Ricci flow.

In the following we will give a result on the irreducible compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature.

Proposition 2.5 Let $(M^n, h)$ be a compact irreducible Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature. Then either $M$ is biholomorphic to the complex projective space or $(M, h)$ is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geq 2$.

Proof. Suppose $(M^n, h)$ is a compact irreducible Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature, then by Theorem 2.1 (ii), we know that $c_1(M) > 0$. 

#
First we evolve the metric by the Kähler Ricci flow:

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij}(x, t) &= -R_{ij}(x, t), \\
g_{ij}(x, 0) &= h_{ij}(x).
\end{align*}
\]

According to Bando [1], we know that the evolved metric \(g_{ij}(t), t \in (0, T)\), remains Kähler. Then by the result due to Cao-Hamilton [3], we know that for \(t \in (0, T)\), \(g_{ij}(t)\) has nonnegative orthogonal holomorphic bisectional curvature. Moreover, according to Hamilton [6], under the evolving orthonormal frame \(\{e_\alpha\}\), we have

\[
\frac{\partial}{\partial t} R^{\alpha\bar{\alpha}\beta\bar{\beta}} = \Delta R^{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_{\mu, \nu} \left( R^{\alpha\bar{\alpha}\mu\nu} R^{\nu\bar{\beta}\beta\bar{\beta}} - |R^{\alpha\bar{\alpha}\beta\bar{\beta}}|^2 + |R^{\alpha\bar{\beta}\beta\bar{\alpha}}|^2 \right).
\]

Suppose \((M, h)\) is not locally symmetric. In the following, we want to show that \(M\) is biholomorphic to the complex projective space \(CP^n\).

Since the smooth limit of locally symmetric space is also locally symmetric, we can obtain that there exists \(\delta \in (0, T)\) such that \((M, g_{ij}(t))\) is not locally symmetric for \(t \in (0, \delta)\). Combining the Kählerity of \(g_{ij}(t)\) and Berger’s holonomy theorem and note that \(c_1(M) > 0\), we know that the holonomy group Hol\((g(t)) = U(n)\).

As above, let \(P\) be the fiber bundle with the fixed metric \(h\) and the fiber over \(p \in M\) consists of all orthogonal 2-vectors \(\{X, Y\} \subset T^1,0_p(M)\). We define a function \(u\) on \(P \times (0, \delta)\) by

\[
u(\{X, Y\}, t) = R(X, \overline{\nabla}, Y, \overline{\nabla}),
\]

where \(R\) denotes the pull-back of the curvature tensor of \(g_{ij}(t)\). Clearly we have \(u \geq 0\), since \((M, g_{ij}(t))\) has nonnegative orthogonal holomorphic bisectional curvature due to Cao-Hamilton [3]. Denote \(F = \{(\{X, Y\}, t)|u(\{X, Y\}, t) = 0, X \neq 0, Y \neq 0\} \subset P \times (0, \delta)\) consists of all pairs \((\{X, Y\}, t)\) such that \(\{X, Y\}\) has zero orthogonal holomorphic bisectional curvature with respect to \(g_{ij}(t)\). By Proposition 2.3, we know that

\[
\frac{\partial u}{\partial t} \geq Lu + c \min \left\{ 0, \inf_{\xi \in V, ||\xi||=1} D^2 u(\xi, \xi) \right\} - c \sup_{\xi \in V, ||\xi||=1} Du(\xi) - cu,
\]

for some constant \(c > 0\), where \(L\) is the horizontal Laplacian on \(P\) and \(V\) denotes the vertical subspace of the bundle. By Proposition 2 in [2], we know that the set

\[
F = \left\{ (\{X, Y\}, t)|u(\{X, Y\}, t) = 0, X \neq 0, Y \neq 0\right\} \subset P \times (0, \delta)
\]
is invariant under parallel transport.

Next, by adapting the argument in [5], we claim that $R_{\alpha\beta\bar{\beta}} > 0$ for all $t \in (0, \delta)$ and all $\alpha \neq \beta$.

Indeed, suppose not. Then $R_{\alpha\beta\bar{\beta}} = 0$ for some $t \in (0, \delta)$ and some $\alpha \neq \beta$. Therefore

$$((e_\alpha, e_\beta), t) \in F.$$ Combining $R_{\alpha\beta\bar{\beta}} = 0$ and the computation for (2.7), (2.11) and (2.19) in Proposition 2.3, it is not hard to obtain that:

$$\sum_{\mu, \nu} (R_{\alpha \mu \bar{\alpha}} R_{\nu \bar{\mu} \bar{\beta}} - |R_{\alpha \bar{\mu} \bar{\beta} \nu}|^2) = 0, \quad R_{\alpha \beta \mu \nu} = 0, \quad \forall \mu, \nu,$$

$$R_{\alpha \beta \mu \bar{\beta}} = R_{\bar{\beta} \bar{\beta} \mu \alpha} = 0, \quad \forall \mu. \quad (2.20)$$

We define an orthonormal 2-frames $\{\tilde{e}_\alpha, \tilde{e}_\beta\} \subset T^1_p(M)$ by

$$\tilde{e}_\alpha = \sin \theta \cdot e_\alpha - \cos \theta \cdot e_\beta,$$

$$\tilde{e}_\beta = \cos \theta \cdot e_\alpha + \sin \theta \cdot e_\beta.$$ Then

$$\overline{e}_\alpha = \sin \theta \cdot \overline{e}_\alpha - \cos \theta \cdot \overline{e}_\beta,$$

$$\overline{e}_\beta = \cos \theta \cdot \overline{e}_\alpha + \sin \theta \cdot \overline{e}_\beta.$$ Since $F$ is invariant under parallel transport and $(M, g_{ij}(t))$ has holonomy group $U(n)$, we obtain that

$$((\overline{e}_\alpha, \overline{e}_\beta), t) \in F,$$

that is,

$$R(\tilde{e}_\alpha, \overline{e}_\alpha; \tilde{e}_\beta, \overline{e}_\beta) = 0.$$
On the other hand,

\[
R(\tilde{e}_\alpha, \overline{\tilde{e}_\alpha}, \tilde{e}_\beta, \overline{\tilde{e}_\beta}) = \sin^2 \theta \cos^2 \theta R_{\bar{\alpha}\bar{\alpha}\bar{\alpha}\bar{\alpha}} + \sin^3 \theta \cos \theta R_{\bar{\alpha}\bar{\alpha}\bar{\beta}\bar{\beta}} + \sin^3 \theta \cos \theta R_{\alpha\alpha\beta\beta}

+ \sin^4 \theta R_{\bar{\alpha}\bar{\beta}\bar{\beta}\bar{\beta}} - \sin \theta \cos^3 \theta R_{\alpha\beta\bar{\alpha}\bar{\beta}} - \sin^2 \theta \cos^2 \theta R_{\bar{\alpha}\bar{\beta}\alpha\beta}

- \sin^2 \theta \cos^2 \theta R_{\bar{\alpha}\beta\bar{\alpha}\bar{\beta}} - \sin^3 \theta \cos \theta R_{\alpha\beta\bar{\alpha}\bar{\beta}} - \cos \sin^3 \theta R_{\beta\alpha\bar{\beta}\bar{\beta}}

+ \cos^4 \theta R_{\bar{\beta}\bar{\alpha}\bar{\alpha}\bar{\alpha}} + \cos^3 \theta \sin \theta R_{\beta\alpha\bar{\beta}\bar{\beta}} + \cos^3 \theta \sin \theta R_{\beta\bar{\beta}\bar{\alpha}\bar{\alpha}}

+ \cos^2 \theta \sin^2 \theta R_{\beta\bar{\beta}\bar{\beta}}

= \cos^2 \theta \sin^2 \theta (R_{\alpha\alpha\alpha\alpha} + R_{\beta\beta\beta\beta}).
\]

where in the last equality we have used (2.20). So we have

\[
R_{\beta\beta\beta\beta} + R_{\bar{\alpha}\bar{\alpha}\bar{\alpha}\bar{\alpha}} = 0,
\]
if we choose \(\theta\) such that \(\cos^2 \theta \sin^2 \theta \neq 0\).

Clearly we can find an element of \(U(n)\) such that it changes \(e_\alpha\) to \(e_\mu\) and fixed \(e_\beta\). Then we can see that

\[
(\{e_\mu, e_\beta\}, t) \in F.
\]

By the same argument as above, we get

\[
R_{\beta\beta\mu\bar{\mu}} = R_{\beta\beta\bar{\beta}\bar{\beta}} + R_{\mu\bar{\mu}\mu\bar{\mu}} = 0.
\]

Similarly we can obtain that for any \(e_\mu\) and \(e_\nu\) with \(\mu \neq \nu\), the following holds:

\[
R_{\mu\nu\nu\bar{\nu}} = R_{\nu\nu\bar{\nu}\bar{\nu}} + R_{\mu\bar{\mu}\mu\bar{\mu}} = 0. \tag{2.21}
\]

So we have the scalar curvature

\[
R = \sum_\alpha R_{\alpha\alpha} = \sum_\alpha \sum_{\beta \neq \alpha} R_{\alpha\alpha\beta\bar{\beta}} + \sum_\alpha R_{\alpha\alpha\alpha\alpha} = 0. \tag{2.22}
\]

Then by the same argument as in Theorem 2.1, we can obtain that the manifold is flat, and this contradicts with the irreducibility of \(M\). Hence we prove that \(R_{\alpha\alpha\beta\bar{\beta}} > 0\), for all \(t \in (0, \delta)\) and all \(\alpha \neq \beta\).
Therefore note that $c_1(M) > 0$ and then using the result of [4], we can get $M$ is biholomorphic to the complex projective space $CP^n$.

This completes the proof of Proposition 2.5.

#

3. The Proof of the Main Theorem

**Proof of Theorem 1.3.** Suppose $(M^n, h)$ is an $n$-dimensional ($n \geq 2$) compact Kähler manifold of nonnegative orthogonal holomorphic bisectional curvature. By applying the standard de Rham decomposition theorem, we know that the universal cover $(\tilde{M}, \tilde{h})$ can be isometrically and holomorphically splitted as

$$(C^k, h_0) \times (M_1^{n_1}, h_1) \times \cdots \times (M_l^{n_l}, h_l)$$

where each $(M_i^{n_i}, h_i), 1 \leq i \leq l$, is irreducible and non-flat, $h_0$ is the standard flat metric on $C^k$ and $k, n_1, \cdots, n_l$ are nonnegative integers.

In the following we divide it into three cases:

**Case 1.** $k = 0$ and in the de Rham decomposition there exists a complex 1-dimensional irreducible factor $\Sigma = M_1$ with Gauss curvature $\kappa(\Sigma)$ negative somewhere.

In this case, let $e_1$ be the unit basis of $T_p^{1,0}(\Sigma)$ and $\{e_j^i\}, (1 \leq j \leq n_i, 2 \leq i \leq l)$, be the orthonormal basis of $T_q^{1,0}(\tilde{M}_i)$ for arbitrary points $p \in \Sigma, q_i \in M_i$. Naturally we can extend $e_1$ and $\{e_j^i\}$ to be an orthonormal basis of $T_x^{1,0}(\tilde{M})$ for $x = (p, q_2, \cdots, q_l) \in \tilde{M}$, still we denote by $e_1$ and $e_j^i, (1 \leq j \leq n_i, 2 \leq i \leq l)$. Since $M$ has nonnegative orthogonal holomorphic bisectional curvature, we obtain

$$R(e_1 - e_j^i, \overline{e_1} - \overline{e_j^i}, e_1 + \overline{e_j^i}, \overline{e_1} + \overline{e_j^i}) = R^{(1)}_{1111} + R^{(i)}_{jjjj} = \kappa(p) + R^{(i)}_{jjjj} \geq 0,$$

where $R^{(i)}$ denotes the curvature on $M_i$. So for each $i \neq 1$ we have

$$R^{(i)}_{jjjj} \geq -\kappa(p).$$

By the arbitrariness of $p, q_i$, we know that

$$\min\{\text{holomorphic sectional curvature of } M_i\} \geq -\min\{\kappa(\Sigma)\} > 0.$$
So we have proved that all $M_i, (i \neq 1)$, have nonnegative holomorphic bisectional curvature. If $\dim(M_i) = n_i \geq 2$, then we know that it also has nonnegative Ricci curvature. So $M_i$ is compact, otherwise, it will split off a line and we can obtain a contradiction with the irreducibility of $M_i$. Then by Proposition 2.5 we obtain that either $M_i$ is biholomorphic to the complex projective space $CP^{n_i}$ or $M_i$ is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geq 2$. If $\dim(M_i) = n_i = 1, (i \neq 1)$, then by the Gauss-Bonnet Theorem, we know that $M_i$ is $S^2(= CP^1)$ with a nonnegatively curved metric. Hence this case is contained in (2).

**Case 2.** $k = 0$ and in the de Rham decomposition there exists no complex 1-dimensional irreducible factor or there exist complex 1-dimensional irreducible factors and all these complex 1-dimensional irreducible factors have nonnegatively curved metric.

In this case, we know that all the complex 1-dimensional irreducible factors, if exists, are compact by the Gauss-Bonnet Theorem and are $S^2(= CP^1)$.

If all the irreducible factors $M_i$ with $\dim(M_i) \geq 2$ are compact, then by Proposition 2.5 we obtain that either $M_i$ is biholomorphic to the complex projective space $CP^{n_i}$ or $M_i$ is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geq 2$. Hence this is contained in (1).

If there exists an irreducible factor, without loss of generality, denoted by $M_1$, is noncompact, then we claim that the minimal of the holomorphic sectional curvature of $M_1 < 0$ somewhere. Otherwise, suppose the holomorphic bisectional curvature of $M_1 \geq 0$ and hence it has nonnegative Ricci curvature, so it is compact which contradicts to the noncompactness of $M_1$. So we have proved the claim. Then by the nonnegativity of the orthogonal holomorphic bisectional curvature and the same argument as in Case 1, we know that all the other irreducible factors $M_i, (i \neq 1)$, have nonnegative holomorphic bisectional curvature and hence are compact. Therefore as above, by Proposition 2.5 we obtain that either $M_i, (i \neq 1)$, is biholomorphic to the complex projective space $CP^{n_i}$ or $M_i, (i \neq 1)$, is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geq 2$. This is contained in (2).

**Case 3.** $k \geq 1$.

In this case, by the nonnegativity of the orthogonal holomorphic bisectional
curvature of $\tilde{M}$ and the same argument as in Case 1, we know that all the other irreducible factors $M_i$ have nonnegative holomorphic bisectional curvature. Again by the same argument as in Case 1, we can obtain that if $\dim(M_i) = n_i \geq 2$, then either $M_i$ is biholomorphic to the complex projective space $CP^{n_i}$ or $M_i$ is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geq 2$. If $\dim(M_i) = n_i = 1$, then by the Gauss-Bonnet Theorem, we know that $M_i$ is $S^2(=CP^1)$ with a nonnegatively curved metric. This case is contained in (1).

Hence from above argument, we have proved the Theorem 1.3.

#

References

[1] S. Bando, *On three-dimensional compact Kähler manifolds of nonnegative bisectional curvature*, J. Diff. Geom. 19, (1984), 283-297.

[2] S. Brendle, and R. Schoen, *Classification of manifolds with weakly $1/4$-pinched curvatures*, arXiv:math.DG/0705.3963 v1 May 2007.

[3] H. D. Cao, and R. S. Hamilton, unpublished work.

[4] X. X. Chen, *On Kähler manifolds with positive orthogonal bisectional curvature*, arXiv:math.DG/0606229 v1 June 2006.

[5] H. L. Gu, *A simple proof for the generalized Frankel conjecture*, arXiv: math. DG/0707.0035 v2 Aug 2007.

[6] R. S. Hamilton, *Four–manifolds with positive curvature operator*, J. Differential Geom. 24 (1986), 153-179.

[7] T. Ivey, *Ricci solitons on compact Kähler surfaces*, Proc. Amer. Math. Soc. 125, (1997), no.4, 1203-1208.

[8] M. Micallef, and M. Wang, *Metrics with nonnegative isotropic curvature*, Duke Math. J. 72, (1993), no.3, 649-672.

[9] N. Mok, *The uniformization theorem for compact Kähler manifolds of non-negative bisectional curvature*, J. Diff. Geom. 27, (1988), 179-214.
[10] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) 110 (1979), 593-606.

[11] H. Seshadri, *Manifolds with nonnegative isotropic curvature*, arXiv:math.DG/0707.3894 v1 July 2007.

[12] Y. T. Siu, and S. T. Yau, *Complex Kähler manifolds of positive bisectional curvature*, Invent. Math. 59 (1980), 189-204.

[13] H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature II*, Acta Math. 147 (1981), 57-70.