Blow-up profile for spinorial Yamabe type equation on $S^m$

Tian Xu

Abstract

Motivated by recent progress on a spinorial analogue of the Yamabe problem in the geometric literature, we study a conformally invariant spinor field equation on the $m$-sphere, $m \geq 2$. Via variational methods, we study analytic aspects of the associated energy functional, culminating in a blow-up analysis.

Keywords. Dirac equations, Conformal geometry, Blow-up

1 Introduction

Within the framework of Spin Geometry, a problem analogous to the Yamabe problem has received increasing attention in recent years. Several works of Ammann [2, 3] and Ammann, Humbert and others [4–6] provide a brief picture of how variational method may be employed to the investigation. Their starting point was the Hijazi inequality [16] which links the first eigenvalue of two important elliptic differential operators: the conformal Laplacian and the Dirac operator.

Let $(M, g, \sigma)$ be an $m$-dimensional closed spin manifold with a metric $g$, a spin structure $\sigma : P_{\text{spin}}(M) \to P_{\text{SO}}(M)$ and a spin representation $\rho : \text{Spin}(m) \to \text{End}(S_m)$. Let us denote by $\mathcal{S}(M) = P_{\text{spin}}(M) \times _\rho S_m$ the spinor bundle on $M$ and $D_g : C^\infty(M, \mathcal{S}(M)) \to C^\infty(M, \mathcal{S}(M))$ the Dirac operator (see Section 2 for more details). A spin conformal invariant is defined as

$$\lambda^+_{\text{min}}(M, [g], \sigma) := \inf_{\tilde{g} \in [g]} \frac{\lambda^+_{\text{min}}(\tilde{g}) \text{Vol}(M, \tilde{g})^\frac{1}{m+1}}{\text{Vol}(M, \tilde{g})^\frac{m}{m+1}}$$

(1.1)

where $\lambda^+_{\text{min}}(\tilde{g})$ denotes the smallest positive eigenvalue of the Dirac operator $D_{\tilde{g}}$ with respect to the conformal metric $\tilde{g} \in [g] := \{ f^2 g : f \in C^\infty(M), f > 0 \}$. Ammann points out in [2, 3] that studying critical metrics for this invariant involves similar analytic problems to those appearing in the Yamabe problem. It follows that finding a critical metric of (1.1) is equivalent to prove the existence of a spinor field $\psi \in C^\infty(M, \mathcal{S}(M))$ minimizing the functional defined by

$$J_{\tilde{g}}(\phi) = \frac{\int_M |D_{\tilde{g}}\phi|^{2m} d\text{vol}_{\tilde{g}}}{\left| \int_M (D_{\tilde{g}}\phi, \phi) d\text{vol}_{\tilde{g}} \right|^{\frac{m+1}{m}}}$$

(1.2)

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with the Euler-Lagrange equation

\[ D_g \psi = \lambda^{+}_{\min}(M, [g], \sigma)|\psi|^{\frac{2}{m+1}} \psi. \tag{1.3} \]

As was pointed out in [2], standard variational method does not imply the existence of minimizers for \( J_g \) directly. This is due to the criticality of the non-quadratic term in (1.2). Indeed, the exponent \( \frac{2m}{m+1} \) is critical in the sense that Sobolev embedding involved is precisely the one for which the compactness is lost in the Rellich-Kondrakov theorem. Similar to the argument in solving the Yamabe problem, one might be able to find a criterion which recovers the compactness. And it is crucial to note that a spinorial analogue of Aubin’s inequality holds (see [4])

\[ \lambda^{+}_{\min}(M, [g], \sigma) \leq \lambda^{+}_{\min}(S^m, [g_{S^m}], \sigma_{S^m}) = \frac{m}{2} \omega_m \tag{1.4} \]

where \((S^m, g_{S^m}, \sigma_{S^m})\) is the \(m\)-dimensional sphere equipped with its canonical metric \(g_{S^m}\) and its standard spin structure \(\sigma_{S^m}\), and \(\omega_m\) is the standard volume of \((S^m, g_{S^m})\). The criterion obtained in [2] shows that if inequality (1.4) is strict then the spinorial Yamabe problem (1.3) has a nontrivial solution minimizing the functional \(J_g\).

Tightly related to geometric data, the nonlinear problem (1.3) provides a strong tool for showing the existence of constant mean curvature hypersurfaces in Euclidean spaces. This is one of the most attractive features of the spinorial Yamabe problem that unseals new researches in both PDE theory and Riemann geometry. In this paper, we are concerned with a more general form of (1.3):

\[ D_g \psi = H(\xi)|\psi|^{\frac{2}{m+1}} \psi \quad \text{on} \quad M \tag{1.5} \]

where \(H : M \to \mathbb{R}\) is a given function on \(M\). As was observed in [14][20][27,29], the function \(H\) plays the role of the mean curvature for a conformal immersion \(M \to \mathbb{R}^{m+1}\). Indeed, if there exists a conformal immersion of \(M\) into \(\mathbb{R}^{m+1}\) with mean curvature \(H\), then from the restriction to \(M\) of any parallel spinor on \(\mathbb{R}^{m+1}\), one obtains a spinor field which satisfies the equation (1.5). The converse is also true if \(m = 2\). Particularly, if \(M\) is a simply connected surface, i.e. a 2-sphere, then for any non-trivial solution to (1.5) there corresponds to a conformal immersion (possibly with branching points) \(S^2 \to \mathbb{R}^3\) with mean curvature \(H\). This one-to-one correspondence is the so-called Spinorial Weierstraß representation, see [14] for a nice explanation and [20] for a rich source of results on Dirac operators on compact surfaces.

Raulot [25] obtained an existence criterion for the problem (1.5) which is similar to the Aubin’s inequality. One of his results shows that if the Dirac operator \(D_g\) is invertible, \(H\) is positive and satisfies

\[ \lambda_{\min,H} \cdot \left( \max_{\xi \in M} H(\xi) \right)^{\frac{m-1}{m}} < \frac{m}{2} \omega_m, \quad \lambda_{\min,H} := \inf_{\phi \neq 0} \left( \frac{\int_M H^{\frac{m-1}{m+1}}|D_g \phi|^{\frac{2m}{m+1}} \text{dvol}_g}{\left| \int_M (D_g \phi, \phi) \text{dvol}_g \right|} \right)^{\frac{m+1}{m}} \tag{1.6} \]

then there exists a solution to the equation

\[ D_g \psi = \lambda_{\min, H}(\xi)|\psi|^{\frac{2}{m+1}} \psi \quad \text{and} \quad \int_M H(\xi)|\psi|^{\frac{2m}{m+1}} \text{dvol}_g = 1. \]
However, condition (1.6) is only verified for some special cases and general existence result is still lacking (cf. [6, 15]). In particular, the existence results in [25] does not apply for $M = S^m$ since the strict inequality of (1.6) is not valid on the spheres in any circumstance.

A different point of view was introduced by Isobe, who suggested to consider the geometric property of the function $H$ for the spinorial Yamabe equation (1.5). In his paper [19], Isobe established existence results for $H$ close to a constant function on $S^m$. The idea was to count Morse index at critical points of $H$ and to pose an index counting condition. Similar idea has been employed in the study of scalar curvature equations, see for instance Bahri and Coron [10] and Chang et al. [12, 13].

In this paper, we will also consider the spinorial Yamabe equation (1.5) on the $m$-sphere, i.e. we take $M = S^m$, $m \geq 2$. We aim to provide an analytic foundation for Eq. (1.5) with general functions $H$ (the case $H$ is "far" from a constant function is allowed). Inspired by Yamabe [30] and Aubin [7], we start with basic points like the compactness of subcritical approximation argument. As was established in the Yamabe problem, the key analytical points are the singularities in solutions of subcritical approximation equations which can appear at isolated points. For Eq. (1.5), at those isolated singularities, rescaling produces an entire solution of the spinorial Yamabe equation in the Euclidean space which can be compactified to a spherical "bubble". These entire solutions or "bubbles" can be viewed as obstructions to the compactness of Eq. (1.5). Hence, another important point will be the asymptotic behavior of such entire solutions. In this paper, we therefore perform such an analysis for the spinorial Yamabe equation. As in the classical cases, this provides a basic analytical picture. A general existence result and the geometric applications of Eq. (1.5) will be established in a forthcoming paper.

The subcritical approximate point of view is to consider the problem of the form

$$D_g \psi = H(\xi) |\psi|^{2^* - \varepsilon} \psi \quad \text{on } M$$

where $\varepsilon > 0$ is small. The above equation has a variational structure. In fact, $\psi \in C^1(M, S(M))$ is a solution to Eq. (1.7) if and only if $\psi$ is a critical point of the functional

$$\mathcal{L}_\varepsilon(\psi) := \frac{1}{2} \int_M (D_g \psi, \psi) d\text{vol}_g - \frac{1}{2^* - \varepsilon} \int_M H(\xi) |\psi|^{2^* - \varepsilon} d\text{vol}_g$$

where $2^* := \frac{2m}{m-1}$. Assume that $\{\psi_\varepsilon\}$ is a sequence of solutions of (1.7) with $\mathcal{L}_\varepsilon(\psi_\varepsilon) \leq C_0$ for some positive constant $C_0$, Raulot’s condition (1.6) can be interpreted as (via the dual variational principle) if $C_0 < C_{\text{crit}} := \frac{1}{2m(\max H)^m} \left(\frac{m}{2}\right)^m \omega_m$ then $\{\psi_\varepsilon\}$ admits a subsequence converging to a smooth solution of (1.5).

When $C_0$ is big (i.e. the condition (1.6) fails), the singularities or the so-called "blow-up" phenomenon may occur. To simplify the presentations of our main results, we set $M = S^m$ in (1.7) and (1.8) and denote $\text{Crit}[H] = \{\xi \in S^m : \nabla H(\xi) = 0\}$ for a fixed function $H \in C^1(S^m)$ and $H > 0$. Assume that $\{\psi_\varepsilon\}$ is a sequence of solutions of (1.7) and satisfying

$$C_{\text{crit}} \leq \mathcal{L}_\varepsilon(\psi_\varepsilon) \leq 2C_{\text{crit}} - \theta$$

(1.9)
for some small constant $\theta > 0$. Define

$$\Sigma = \{ a \in S^m : \text{there exists } a_\varepsilon \to a \text{ such that } |\psi_\varepsilon(a_\varepsilon)| \to +\infty \text{ as } \varepsilon \to 0 \}.$$

Then we show that $\Sigma \subset \text{Crit}[H]$ and $\{ \psi_\varepsilon \}$ admits a subsequence, still denoted by $\{ \psi_\varepsilon \}$, satisfying one of the following alternatives

1. (compactness) $\Sigma = \emptyset$, $\{ \psi_\varepsilon \}$ is compact in $C^1(S^m, S(S^m))$ and converges to a solution $\psi$ of Eq. (1.5) as $\varepsilon \to 0$;

2. (concentration) $\Sigma \neq \emptyset$ contains a single point and $|\psi_\varepsilon| \to 0$ as $\varepsilon \to 0$ uniformly on compact subsets of $S^m \setminus \Sigma$.

The conclusion described above is only a rough summation of the results in Proposition 4.1 and 4.8-4.10.

Observe that condition (1.9) performs a key ingredient in the blow-up profile, as it exhibits a unique microscopic bubbling pattern. By virtue of Raulot’s result, it can be seen that $C_{\text{crit}}$ is the threshold for bubbling. The alternative blow-up profile obtained in the present paper can be viewed as a local version of concentration-compactness type result above the bubbling threshold, i.e. in the range $[C_{\text{crit}}, 2C_{\text{crit}})$. A geometric motivation lying behind our interest in this $2C_{\text{crit}}$ upper bound is, in dimension 2, it would give an upper bound estimate for the Willmore energy provided that one can rule out the formation of bubbles. Indeed, in the setting of Spinorial Weierstraß representation, if $\psi \in C^1(S^2, S(S^2))$ is a nontrivial solution of (1.5) then $(S^2, |\psi|^4 g_{S^2})$ can be isometrically immersed into $\mathbb{R}^3$ with mean curvature being prescribed as $H$. Then the Willmore energy of such an immersion is defined by

$$W(\psi) = \int_{S^2} H^2(\xi) d\text{vol}_{|\psi|^4 g_{S^2}} = \int_{S^2} H^2(\xi) |\psi|^4 d\text{vol}_{g_{S^2}}.$$

Once the blow-up is ruled out, (1.9) and the compactness of the subcritical approximating sequence $\{ \psi_\varepsilon \}$ gives a solution $\psi$ to Eq. (1.5) such that

$$W(\psi) \leq \max_{\xi \in S^2} H \int_{S^2} H(\xi) |\psi|^4 d\text{vol}_{g_{S^2}} = \max_{\xi \in S^2} H \cdot \lim_{\varepsilon \to 0} \left( 4L_\varepsilon(\psi_\varepsilon) - 2L_\varepsilon'(\psi_\varepsilon)[\psi_\varepsilon] \right) < 8\pi.$$

Therefore, by Li and Yau’s inequality [22 Theorem 6], we obtain the immersion of $(S^2, |\psi|^4 g_{S^2})$ into $\mathbb{R}^3$ has no self-intersection, i.e. it is a global embedding. This will be of particular geometric interests and will appear in our future attempt on the existence issues.

The paper is organized as follows: Section 2 contains the basic geometric backgrounds and functional settings; in Section 3 the continuity of the energy functionals with respect to the subcritical approximation is established; Section 4 contains the detailed proofs of the alternative blow-up profile.
2 Preliminaries

2.1 Spin structure and the Dirac operator

Let \((M, g)\) be an \(m\)-dimensional Riemannian manifold with a chosen orientation. Let \(P_{SO}(M)\) be the set of positively oriented orthonormal frames on \((M, g)\). This is an \(SO(m)\)-principal bundle over \(M\). A spin structure on \(M\) is a pair \(\sigma = (P_{Spin}(M), \vartheta)\) where \(P_{Spin}(M)\) is a \(Spin(m)\)-principal bundle over \(M\) and \(\vartheta : P_{Spin}(M) \to P_{SO}(M)\) is a map such that

\[
P_{Spin}(M) \times Spin(m) \xrightarrow{\vartheta \times \Theta} M
\]

commutes, where \(\Theta : Spin(m) \to SO(m)\) is the nontrivial double covering of \(SO(m)\). There is a topological condition for the existence of a spin structure, that is, the vanishing of the second Stiefel-Whitney class \(\omega_2(M) \in H^2(M, \mathbb{Z}_2)\). Furthermore, if a spin structure exists, it need not to be unique. For these, we refer to \([14, 21]\).

To introduce the spinor bundle, we recall that the Clifford algebra \(Cl(\mathbb{R}^m)\) is the associative \(\mathbb{R}\)-algebra with unit, generated by \(\mathbb{R}^m\) satisfying the relation \(x \cdot y - y \cdot x = -2 (x, y)\) for \(x, y \in \mathbb{R}^m\) (here \((\cdot, \cdot)\) is the Euclidean scalar product on \(\mathbb{R}^m\)). It turns out that \(Cl(\mathbb{R}^m)\) has a smallest representation \(\rho : Spin(m) \to End(S_m)\) of dimension \(\dim \mathbb{C}(S_m) = 2^{|m|}\) such that \(Cl(\mathbb{R}^m) := Cl(\mathbb{R}^m) \otimes \mathbb{C} \cong End(S_m)\) as \(\mathbb{C}\)-algebra. The spinor bundle is then defined as the associated vector bundle

\[
S(M) := P_{Spin}(M) \times_{\rho} S_m.
\]

Note that the spinor bundle carries a natural Clifford multiplication, a natural hermitian metric and a metric connection induced from the Levi-Civita connection on \(TM\) (see \([14, 21]\)), this bundle satisfies the axioms of Dirac bundle in the sense that

1. for any \(x \in M\), \(X,Y \in T_xM\) and \(\psi \in S_x(M)\)
   \[
   X \cdot Y \cdot \psi + Y \cdot X \cdot \psi + 2g(X,Y)\psi = 0;
   \]

2. for any \(X \in T_xM\) and \(\psi_1, \psi_2 \in S_x(M)\),
   \[
   (X \cdot \psi_1, \psi_2) = -(\psi_1, X \cdot \psi_2),
   \]
   where \((\cdot, \cdot)\) is the hermitian metric on \(S(M)\);

3. for any \(X,Y \in \Gamma(TM)\) and \(\psi \in \Gamma(S(M))\),
   \[
   \nabla^S_X(Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot \nabla^S_X \psi,
   \]
   where \(\nabla^S\) is the metric connection on \(S(M)\).
On the spinor bundle $S(M)$, the Dirac operator is then defined as the composition

$$\begin{align*}
D : \Gamma(S(M)) \xrightarrow{\nabla} \Gamma(T^*M \otimes S(M)) \xrightarrow{\nabla} \Gamma(TM \otimes S(M)) \xrightarrow{m} \Gamma(S(M))
\end{align*}$$

where $m$ denotes the Clifford multiplication $m : X \otimes \psi \mapsto X \cdot \psi$.

The Dirac operator behaves very nicely under conformal changes in the following sense (see [16, 17]):

**Proposition 2.1.** Let $g_0$ and $g = f^2 g_0$ be two conformal metrics on a Riemannian spin manifold $M$. Then, there exists an isomorphism of vector bundles $\iota : S(M, g_0) \rightarrow S(M, g)$ which is a fiberwise isometry such that

$$D_g(\iota(\psi)) = \iota \left( f^{-\frac{m+1}{2}} D_{g_0} \left( f^{\frac{m-1}{2}} \psi \right) \right),$$

where $S(M, g_0)$ and $S(M, g)$ are spinor bundles on $M$ with respect to the metrics $g_0$ and $g$, respectively, and $D_{g_0}$ and $D_g$ are the associated Dirac operators.

### 2.2 Bourguignon-Gauduchon trivialization

In what follows, let us introduce briefly a local trivialization of the spinor bundle $S(M)$ constructed by Bourguignon and Gauduchon [9].

Let $a \in V \subset M$ be an arbitrary point and let $(x_1, \ldots, x_m)$ be the normal coordinates given by the exponential map $\exp_a : U \subset T_a M \cong \mathbb{R}^m \rightarrow V, x = (x_1, \ldots, x_n) \mapsto y = \exp_a x$. Then, we have $(\exp_a)_y : (T_{\exp^{-1} y}U \cong \mathbb{R}^m, \exp^*_y g_y) \rightarrow (T_y M, g_y)$ is an isometry for each $y \in V$. Thus, we can obtain an isomorphism of $SO(m)$-principal bundles:

$$P_{SO}(U, \exp^*_a g) \xrightarrow{(\exp_a)_*} P_{SO}(V, g)$$

where $(\exp_a)_* \{z_1, \ldots, z_m\} = \{(\exp_a)_* z_1, \ldots, (\exp_a)_* z_m\}$ for an oriented frame $\{z_1, \ldots, z_m\}$ on $U$. Notice that $(\exp_a)_*$ commutes with the right action of $SO(m)$, then we infer that $(\exp_a)_*$ induces an isomorphism of spin structures:

$$Spin(m) \times U = P_{Spin}(U, g_{\mathbb{R}^m}) \xrightarrow{(\exp_a)_*} P_{Spin}(V, g) \subset P_{Spin}(M)$$

Hence, we can obtain an isomorphism between the spinor bundles $S(U)$ and $S(V)$ by

$$S(U) := P_{Spin}(U, g_{\mathbb{R}^m}) \times_{\rho} S_m \rightarrow S(V) := P_{Spin}(V, g) \times_{\rho} S_m \subset S(M)$$

$$\psi = [s, \varphi] \mapsto \tilde{\psi} := [(\exp_a)_*(s), \varphi]$$
where \((\rho, S_m)\) is the complex spinor representation, and \([s, \varphi]\) and \([\exp_a(s), \varphi]\) denote the equivalence classes of \((s, \varphi) \in P_{Spin(U, g_{\mathbb{R}^m})} \times S_m\) and \((\exp_a(s), \varphi) \in P_{Spin(V, g)} \times S_m\) under the action of \(Spin(m)\), respectively.

For more background material on the Bourguignon-Gauduchon trivialization we refer the reader to \([4, 9]\).

### 2.3 \(H^{\frac{1}{2}}\)-spinors on \(S^m\)

Let us consider the case \(M = S^m\) with the standard metric \(g_{S_m}\). Let \(Spec(D)\) denote the spectrum of the Dirac operator \(D\). It is well-known that \(D\) is essentially self-adjoint in \(L^2(S^m, S(S^m))\) and has compact resolvents (see \([14, 15, 21]\)). Particularly, we have

\[
Spec(D) = \left\{ \pm \left( \frac{m}{2} + j \right) : j = 0, 1, 2, \ldots \right\}
\]

and, for each \(j\), the eigenvalues \(\pm \left( \frac{m}{2} + j \right)\) have the multiplicity

\[
2^{[\frac{m}{2}]} \binom{n + j - 1}{j}
\]

(see \([26]\)). For notation convenience, we can write \(Spec(D) = \{\lambda_k\}_{k \in \mathbb{Z} \setminus \{0\}}\) with

\[
\lambda_k = \frac{k}{|k|} \left( \frac{m}{2} + |k| - 1 \right).
\]

The eigenspaces of \(D\) form a complete orthonormal decomposition of \(L^2(S^m, S(S^m))\), that is,

\[
L^2(S^m, S(S^m)) = \bigoplus_{\lambda \in Spec(D)} \ker(D - \lambda I).
\]

Now let us denote by \(\{\eta_k\}_{k \in \mathbb{Z} \setminus \{0\}}\) the complete orthonormal basis of \(L^2(S^m, S(S^m))\) consisting of the smooth eigenspinors of the Dirac operator \(D\), i.e., \(D\eta_k = \lambda_k \eta_k\). We then define the operator \(|D|^{\frac{1}{2}} : L^2(S^m, S(S^m)) \to L^2(S^m, S(S^m))\) by

\[
|D|^{\frac{1}{2}} \psi = \sum_{k \in \mathbb{Z} \setminus \{0\}} |\lambda_k|^{\frac{1}{2}} a_k \eta_k,
\]

where \(\psi = \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k \eta_k \in L^2(S^m, S(S^m))\). Let us set

\[
H^{\frac{1}{2}}(S^m, S(S^m)) := \left\{ \psi = \sum_{k \in \mathbb{Z}} a_k \eta_k \in L^2(S^m, S(S^m)) : \sum_{k \in \mathbb{Z} \setminus \{0\}} |\lambda_k|^{\frac{1}{2}} |a_k|^2 < \infty \right\}.
\]

We have \(H^{\frac{1}{2}}(S^m, S(S^m))\) coincides with the Sobolev space \(W^{\frac{1}{2}, 2}(S^m, S(S^m))\) (see \([1, 2]\)). We could now endow \(H^{\frac{1}{2}}(S^m, S(S^m))\) with the inner product

\[
\langle \psi, \varphi \rangle = \text{Re}\left(|D|^{\frac{1}{2}} \psi, |D|^{\frac{1}{2}} \varphi\right)_2
\]
and the induced norm $\| \cdot \| = \langle \cdot , \cdot \rangle^{\frac{1}{2}}$, where $(\psi , \varphi)_2 = \int_{S^m} (\psi , \varphi) d\text{vol}_{g_{S^m}}$ is the $L^2$-inner product on spinors. In particular, $E := H^{\frac{1}{2}}(S^m, \mathbb{S}(S^m))$ induces a splitting $E = E^+ \oplus E^-$ with

$$E^+ := \text{span}\{ \eta_k \}_{k > 0} \quad \text{and} \quad E^- := \text{span}\{ \eta_k \}_{k < 0},$$

where the closure is taken in the $\| \cdot \|$-topology. It is then clear that these are orthogonal subspaces of $E$ on which the action

$$\int_{S^m} (D\psi , \psi) d\text{vol}_{g_{S^m}}$$

is positive or negative. In the sequel, with respect to this decomposition, we will write $\psi = \psi^+ + \psi^-$ for any $\psi \in E$. The dual space of $E$ will be denoted by $E^* := H^{-\frac{1}{2}}(S^m, \mathbb{S}(S^m))$.

3 Perturbation from subcritical

For $p \in (2 , 2^*]$, let us consider

$$D\psi = H(\xi)|\psi|^{p-2}\psi \quad \text{on } S^m$$

(3.1)

where $H \in C^1(S^m)$ and $H > 0$. The corresponding energy functional will be

$$\mathcal{L}_p(\psi) = \frac{1}{2} \int_{S^m} (D\psi , \psi) d\text{vol}_{g_{S^m}} - \frac{1}{p} \int_{S^m} H(\xi)|\psi|^p d\text{vol}_{g_{S^m}}$$

$$= \frac{1}{2} (\| \psi^+ \|^2 - \| \psi^- \|^2) - \frac{1}{p} \int_{S^m} H(\xi)|\psi|^p d\text{vol}_{g_{S^m}}.$$

For $p = 2^*$ we will simply use the notation $\mathcal{L}$ instead of $\mathcal{L}_{2^*}$. For $p < 2^*$, the problem is subcritical and, due to the compact embedding of $E \hookrightarrow L^p(S^m, \mathbb{S}(S^m))$, it is not difficult to see that there always exist nontrivial weak solutions of (3.1). So, it remains to find conditions under which solutions of the subcritical problem will converge to a nontrivial solution of the critical problem. In what follows, we will first study the behavior of the energy functional $\mathcal{L}_p$, $p \in (2 , 2^*]$.

For notation convenience, in the sequel, by $L^p$ we denote the Banach space $L^p(S^m, \mathbb{S}(S^m))$ for $p \geq 1$ and by $| \cdot |_p$ we denote the usual $L^p$-norm. We also denote $H_{\max} = \max_{S^m} H$ and $H_{\min} = \min_{S^m} H$. And without loss of generality, we assume $\int_{S^m} H(\xi) d\text{vol}_{g_{S^m}} = 1$.

3.1 A reduction argument

Given $u \in E^+ \setminus \{ 0 \}$, we set

$$W(u) = \text{span}\{ u \} \oplus E^- = \{ \psi \in E : \psi = tu + v , \ v \in E^- , \ t \in \mathbb{R} \}.$$

Lemma 3.1. For $u \in E^+ \setminus \{ 0 \}$, $\mathcal{L}_p$ is anti-coercive on $W(u)$, that is,

$$\mathcal{L}_p(\psi) \to -\infty \text{ as } \| \psi \| \to \infty , \ \psi \in W(u).$$
Proof. To begin with, for $\psi \in W(u)$, let us write $\psi = tu + v$ with $v \in E^-$. We may then fix a constant $C_p > 0$ such that

$$|\psi|_p \geq C_p(|tu|_p + |v|_p) \text{ for all } \psi \in W(u).$$

Therefore, we can infer

$$L_p(\psi) \leq \frac{1}{2}(||\psi^+||^2 - ||\psi^-||^2) - \frac{H_{\min}}{p} |\psi|_p^p \leq \frac{t^2}{2} \left(||u||^2 - \frac{2H_{\min}}{p} \cdot C \cdot t^{p-2} |u|_p^p\right) = \frac{1}{2} ||v||^2.$$

And thus the conclusion follows. \qed

For a fixed $u \in E^+ \setminus \{0\}$, we define $\phi_u : E^- \to \mathbb{R}$ by

$$\phi_u(v) := L_p(u + v) = \frac{1}{2} \int_{S^m} (D(u + v), u + v) d\text{vol}_{g_{Sm}} - \frac{1}{p} \int_{S^m} H(x)|u + v|^pd\text{vol}_{g_{Sm}}.$$

From the convexity of the map $\psi \mapsto \int_{S^m} H(x)|\psi|^pd\text{vol}_{g_{Sm}}$, it can be straightforwardly verified that

$$\phi_u''(v)[w, w] \leq \int_{S^m} (Dw, w) d\text{vol}_{g_{Sm}} - \int_{S^m} H(x)|u + v|^{p-2} : |w|^2 d\text{vol}_{g_{Sm}} \leq -\|w\|^2 \quad (3.2)$$

for all $v, w \in E^-$. This suggests that $\phi_u$ is concave. Moreover, we have

**Proposition 3.2.** There exists a $C^1$ map $h_p : E^+ \to E^-$ such that

$$||h_p(u)||^2 \leq \frac{2}{p} \int_{S^m} H(x)|u|^pd\text{vol}_{g_{Sm}}$$

and

$$L_p'(u + h_p(u))[v] = 0 \quad \forall v \in E^- \quad (3.3)$$

Furthermore, if denoted by $I_p(u) = L_p(u + h_p(u))$, the function $t \mapsto I_p(tu)$ is $C^2$ and, for $u \in E^+ \setminus \{0\}$ and $t > 0$,

$$I_p'(tu)[u] = 0 \Rightarrow I_p''(tu)[u, u] < 0 \quad (3.4)$$

**Proof.** We sketch the proof as follows. First of all, by Lemma 3.1, we have $\lim_{\|v\| \to \infty} \phi_u(v) = -\infty$ which implies $\phi_u$ is anti-coercive. Then it follows from (3.2) and the weak sequential upper semi-continuity of $\phi_u$ that there exists a unique strict maximum point $h_p(u)$ for $\phi_u$, which is also the only critical point of $\phi_u$ on $E^+_\chi$.

Notice that

$$0 \leq L_p(u + h_p(u)) - L_p(u) = \frac{1}{2}(||u||^2 - ||h_p(u)||^2) - \frac{1}{p} \int_{S^m} H(x)|u + h_p(u)|^pd\text{vol}_{g_{Sm}} - \frac{1}{2} ||u||^2$$

$$+ \frac{1}{p} \int_{S^m} H(x)|u|^pd\text{vol}_{g_{Sm}}.$$
we therefore have

\[ ||h_p(u)||^2 \leq \frac{2}{p} \int_{S^m} H(x)|u|^p d\text{vol}_{g_{sm}}. \]

We define \( \beta : E^+ \times E^- \to E^- \) by

\[ \beta(u, v) = \phi'_u(v) = \mathcal{L}'_p(u + v)\big|_{E^-}, \]

where we have identified \( E^- \) with its dual space. Observe that, for every \( u \in E^+ \), we have

\[ \phi'_u(h_p(u))[w] = \mathcal{L}'_p(u + h_p(u))[w] = 0, \quad \forall w \in E^- . \]

This implies \( \beta(u, h_p(u)) = 0 \) for all \( u \in E^+ \). Notice that \( \partial_v \beta(u, h_p(u)) = \phi''_u(h_p(u)) \) is a bilinear form on \( E^- \) which is bounded and anti-coercive. Hence \( \partial_v \beta(u, h_p(u)) \) is an isomorphism. And therefore, by the implicit function theorem, we can infer that the uniquely determined map \( h_p : E^+ \to E^- \) is of \( C^1 \) smooth with its derivative given by

\[ h'_p(u) = -\partial_v \beta(u, h_p(u))^{-1} \circ \partial_u \beta(u, h_p(u)), \quad (3.5) \]

this completes the proof of the first statement.

To prove \( (3.4) \), it sufficient to show that

\[ \text{If } u \in E^+ \setminus \{0\} \text{ satisfies } I'_p(u)[u] = 0, \text{ then } I''_p(u)[u, u] < 0. \quad (3.6) \]

Now, for simplicity, let us denote \( \mathcal{G}_p : E \to \mathbb{R} \) by \( \mathcal{G}_p(\psi) = \frac{1}{p} \int_{S^m} H(\xi)|\psi|^p d\text{vol}_{g_{sm}} \) and set \( z = u + h_p(u) \) and \( w = h'_p(u)[u] - h_p(u) \). By using \( \mathcal{L}'_p(u + h_p(u))\big|_{E^-} \equiv 0 \), we have \( (3.6) \) is a consequence of the following computation

\[
\begin{align*}
I''_p(u)[u, u] &= \mathcal{L}''_p(z)[u + h'_p(u)[u], u] = \mathcal{L}''_p(z)[z + w, z + w] \\
&= \mathcal{L}''_p(z)[z, z] + 2\mathcal{L}'_p(z)[z, w] + \mathcal{L}'_p(z)[w, w] \\
&= I'_p(u)[u] + (\mathcal{G}_p(z)[z] - \mathcal{G}_p(z)[z, z]) + 2(\mathcal{G}_p(z)[w] - \mathcal{G}_p(z)[z, w]) \\
&\quad - \mathcal{G}_p(z)[w, w] + \int_{S^m} (Dw, w) d\text{vol}_{g_{sm}} \\
&\leq I'_p(u)[u] - \frac{p-2}{p-1} \int_{S^m} H(\xi)|z|^p d\text{vol}_{g_{sm}} - ||w||^2. \quad (3.7)
\end{align*}
\]

\[ \square \]

In the sequel, we shall call \( (h_p, I_p) \) the reduction couple for \( \mathcal{L}_p \) on \( E^+ \). It is all clear that critical points of \( I_p \) and \( \mathcal{L}_p \) are in one-to-one correspondence via the injective map \( u \mapsto u + h_p(u) \) from \( E^+ \) to \( E \). Let us set

\[ \mathcal{N}_p = \{ u \in E^+ \setminus \{0\} : I'_p(u)[u] = 0 \}. \quad (3.8) \]

By Proposition \( 3.2 \) we have \( \mathcal{N}_p \) is a smooth manifold of codimension 1 in \( E^+ \) and is a natural constraint for the problem of finding non-trivial critical points of \( I_p \). Furthermore, the function
$t \mapsto I_p(tu)$ attains its unique critical point at $t = t(u) > 0$ (such that $t(u)u \in \mathcal{M}^\prime_u$) which realizes its maximum. And at this point, we have
\[
\max_{t>0} I_p(tu) = \max_{\psi \in W(u)} \mathcal{L}_p(\psi).
\]

### 3.2 Continuity with respect to the perturbation

Now we will work with a convergent sequence $\{p_n\}$ in $(2, 2^\ast]$ and we allow the case $p_n \equiv p$. The main result here is the following:

**Proposition 3.3.** Let $p_n \to p$ in $(2, 2^\ast]$ as $n \to \infty$ and $c_1, c_2 > 0$. For any $\theta > 0$, there exists $\alpha > 0$ such that for all large $n$ and $\psi \in E$ satisfying
\[
c_1 \leq \mathcal{L}_{p_n}(\psi) \leq c_2 \quad \text{and} \quad \|\mathcal{L}'_{p_n}(\psi)\|_{E^*} \leq \alpha
\]
we have
\[
\max_{t>0} I_{p_n}(t\psi^+) \leq \mathcal{L}_{p_n}(\psi) + \theta.
\]

This result provides a kind of continuity of energy levels with respect to the exponents $p_n$. This is due to the fact, by Proposition 3.2, we always have $\mathcal{L}_{p_n}(\psi) \leq \max_{t>0} I_{p_n}(t\psi^+)$. The proof of Proposition 3.3 is divided into several steps. First of all, let $p_n \to p$ in $(2, 2^\ast]$ and suppose that there exists $\{\psi_n\} \subset E$ satisfying
\[
c_1 \leq \mathcal{L}_{p_n}(\psi_n) \leq c_2 \quad \text{and} \quad \mathcal{L}'_{p_n}(\psi_n) \to 0
\]
for some constants $c_1, c_2 > 0$

**Lemma 3.4.** Under (3.9), $\{\psi_n\}$ is bounded in $E$.

**Proof.** Since $\mathcal{L}'_{p_n}(\psi_n) \to 0$, we have
\[
c_2 + o(\|\psi_n\|) \geq \mathcal{L}_{p_n}(\psi_n) - \frac{1}{2} \mathcal{L}'_{p_n}(\psi_n)[\psi_n] = \left(\frac{1}{2} - \frac{1}{p_n}\right) \int_{S^m} H(\xi)|\psi_n|^{p_n}d\text{vol}_{g_{S^m}}. \tag{3.10}
\]
We also have
\[
o(\|\psi_n\|) = \mathcal{L}'_{p_n}(\psi_n)[\psi_n^+ - \psi_n^-] = \|\psi_n\|^2 - \text{Re} \int_{S^m} H(\xi)|\psi_n|^{p_n-2}(\psi_n^+ - \psi_n^-)d\text{vol}_{g_{S^m}}.
\]
From this and the Hölder and Sobolev inequalities, we obtain
\[
\|\psi_n\|^2 \leq \int_{S^m} H(\xi)|\psi_n|^{p_n-1}|\psi_n^+ - \psi_n^-|d\text{vol}_{g_{S^m}} + o(\|\psi_n\|)
\]
\[
\leq \left(\int_{S^m} H(\xi)|\psi_n|^{p_n}d\text{vol}_{g_{S^m}}\right)^{p_n-1} \left(\int_{S^m} H(\xi)|\psi_n^+ - \psi_n^-|^{p_n}d\text{vol}_{g_{S^m}}\right)^{\frac{1}{p_n}} + o(\|\psi_n\|)
\]
Then we have

\[ \text{Lemma 3.5.} \]

\[ \text{Let} \quad \psi_n \]

In particular, if \( \{ \psi_n \} \) show that \( \psi_n \to 0 \) if and only if \( \mathcal{L}_{p_n}(\psi_n) \to 0 \), and hence \( \|\psi_n\| \) should be bounded away from zero under the assumption \( (3.9) \).

\[ \square \]

**Lemma 3.5.** Let \( \{ \psi_n \} \subset E \) be a sequence such that \( \mathcal{L}_{p_n}(\psi_n)|_{E^-} = o_n(1) \), i.e.,

\[ \sup_{v \in E^- : \|v\| = 1} \mathcal{L}'_{p_n}(\psi_n)[v] = o_n(1). \] (3.12)

Then

\[ \|\psi^-_n - h_{p_n}(\psi^+_n)\| \leq O \left( \|\mathcal{L}'_{p_n}(\psi_n)|_{E^-}\| \right). \]

In particular, if \( \{ \psi_n \} \) is such that \( \mathcal{L}'_{p_n}(\psi_n) = o_n(1) \), then \( I'_{p_n}(\psi^+_n) = o_n(1) \).

**Proof.** For simplicity of notation, let us denote \( z_n = \psi^+_n + h_{p_n}(\psi^+_n) \) and \( v_n = \psi^-_n - h_{p_n}(\psi^+_n) \).

Then we have \( v_n \in E^- \) and, by definition of \( h_{p_n} \),

\[ 0 = \mathcal{L}'_{p_n}(z_n)[v_n] = \text{Re} \int_{S^m} (Dz_n, v_n) d\text{vol}_{g_{sm}} - \text{Re} \int_{S^m} H(\xi)|z_n|^{p_n-2}(z_n, v_n) d\text{vol}_{g_{sm}}. \]

By (3.12), it follows that

\[ o(\|v_n\|) = \mathcal{L}'_{p_n}(\psi_n)[v_n] = \text{Re} \int_{S^m} (D\psi_n, v_n) d\text{vol}_{g_{sm}} - \text{Re} \int_{S^m} H(\xi)|\psi_n|^{p_n-2}(\psi_n, v_n) d\text{vol}_{g_{sm}}. \]

And hence we have

\[ o(\|v_n\|) = - \int_{S^m} (Dv_n, v_n) d\text{vol}_{g_{sm}} + \text{Re} \int_{S^m} H(\xi)|\psi_n|^{p_n-2}(\psi_n, v_n) d\text{vol}_{g_{sm}} \]

\[ - \text{Re} \int_{S^m} H(\xi)|z_n|^{p_n-2}(z_n, v_n) d\text{vol}_{g_{sm}}. \] (3.13)

Remark that the functional \( \psi \mapsto |\psi|^p_p \) is convex for any \( p [2, 2^*) \), we have

\[ \text{Re} \int_{S^m} H(\xi)|\psi_n|^{p_n-2}(\psi_n, v_n) d\text{vol}_{g_{sm}} - \text{Re} \int_{S^m} H(\xi)|z_n|^{p_n-2}(z_n, v_n) d\text{vol}_{g_{sm}} \geq 0. \]

Thus, from (3.13) and \( v_n \in E^- \), we can infer that

\[ |\mathcal{L}'_{p_n}(\psi_n)[v_n]| \geq - \int_{S^m} (Dv_n, v_n) d\text{vol}_{g_{sm}} = \|v_n\|^2. \] (3.14)

And therefore we conclude \( \|v_n\| \leq O(\|\mathcal{L}'_{p_n}(\psi_n)|_{E^-}\|) \).

If \( \{ \psi_n \} \) is such that \( \mathcal{L}'_{p_n}(\psi_n) = o_n(1) \), then estimate (3.14) implies that \( \{ z_n \} \) is also satisfying \( \mathcal{L}'_{p_n}(z_n) = o_n(1) \). Thus, we have \( I'_{p_n}(\psi^+_n) = o_n(1) \).

\[ \square \]
Now, let us introduce the functional $\mathcal{K}_p : E^+ \to \mathbb{R}$ by
\[
\mathcal{K}_p(u) = I'_p(u)[u], \quad u \in E^+.
\]
It is clear that $\mathcal{K}_p$ is $C^1$ and its derivative is given by the formula
\[
\mathcal{K}_p'(u)[w] = I'_p(u)[w] + I''_p(u)[u,w]
\]
for all $u, w \in E^+$. We also have $\mathcal{N}_p = \mathcal{K}_p^{-1}(0) \setminus \{0\}$.

**Lemma 3.6.** For any $u \in E^+$ and $p \in (2, 2^*)$, we have
\[
\mathcal{K}_p'(u)[u] \leq 2\mathcal{K}_p(u) - \frac{p-2}{p-1} \int_{S^m} H(\xi)|u + h_p(u)|^p d\nu_{g_{sm}}.
\]

**Proof.** This estimate follows immediately from a similar estimate as (3.7).

We next have:

**Lemma 3.7.** Let $p_n \to p$ in $(2, 2^*)$ as $n \to \infty$, if $\{u_n\} \subset E^+$ is bounded, $\liminf_{n \to \infty} I_{p_n}(u_n) > 0$ and $I'_{p_n}(u_n) \to 0$ as $n \to \infty$, then there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $t_nu_n \in \mathcal{N}_{p_n}$ and $|t_n - 1| \leq O(\|I'_{p_n}(u_n)\|)$.

**Proof.** Since $\liminf_{n \to \infty} I_{p_n}(u_n) > 0$, we have
\[
\liminf_{n \to \infty} \int_{S^m} H(\xi)|u_n + h_{p_n}(u_n)|^{p_n} d\nu_{g_{sm}} \geq c_0
\]
for some constant $c_0 > 0$. Now, for $n \in \mathbb{N}$, let us set $g_n : (0, +\infty) \to \mathbb{R}$ by
\[
g_n(t) = \mathcal{K}_{p_n}(tu_n).
\]
We then have $tg_n(t) = \mathcal{K}_{p_n}'(tu_n)[tu_n]$ for all $t > 0$ and $n \in \mathbb{N}$. Hence, by Lemma 3.6, Taylor’s formula and the uniform boundedness of $g_n(t)$ on bounded intervals, we get
\[
tg_n'(t) \leq 2g_n(1) - \frac{p_n - 2}{p_n - 1} \int_{S^m} H(\xi)|u_n + h_{p_n}(u_n)|^{p_n} d\nu_{g_{sm}} + C|t - 1|
\]
for $t$ close to 1 with $C > 0$ independent of $n$. Notice that $g_n(1) = I'_{p_n}(u_n)[u_n] \to 0$, thus there exists a small constant $\delta > 0$ such that
\[
g'(t) < -\delta \text{ for all } t \in (1 - \delta, 1 + \delta) \text{ and } n \text{ large enough}.
\]
Moreover, we have $g_n(1 - \delta) > 0$ and $g_n(1 + \delta) < 0$. Then, by Inverse Function Theorem,
\[
\tilde{u}_n := g_n^{-1}(0)u_n \in \mathcal{N}_{p_n} \cap \text{span}\{u_n\}
\]
is well-defined for all $n$ large enough. Furthermore, since $|g_n'(t)|^{-1}$ is bounded by a constant, say $c_\delta > 0$, on $(1 - \delta, 1 + \delta)$, we consequently get
\[
\|u_n - \tilde{u}_n\| = |g_n^{-1}(0) - g_n^{-1}(\mathcal{K}_{p_n}(u_n))| \cdot \|u_n\| \leq c_\delta |\mathcal{K}_{p_n}(u_n)| \cdot \|u_n\|
\]
Now the conclusion follows from $\mathcal{K}_{p_n}(u_n) \leq O(\|I'_{p_n}(u_n)\|)$.
\qed
Corollary 3.8. Let \( \{ \psi_n \} \) satisfies (3.9), then there exists \( \{ \tilde{u}_n \} \subset \mathcal{N}_{p_n} \) such that \( \| \psi_n - \tilde{u}_n - h_{p_n}(\tilde{u}_n) \| \leq O(\| \mathcal{L}'_{p_n}(\psi_n) \|) \). In particular

\[
\max_{t > 0} I_{p_n}(t \psi_n^+) = I_{p_n}(\tilde{u}_n) \leq \mathcal{L}_{p_n}(\psi_n) + O(\| \mathcal{L}'_{p_n}(\psi_n) \|_{E^*}).
\]

Proof. According to Lemma 3.5, by setting \( z_n = \psi_n^+ + h_{p_n}(\psi_n^+) \), we have

\[
\| \psi_n - z_n \| \leq O(\| \mathcal{L}'_{p_n}(\psi_n) \|_{E^*})
\]

and \( \{ \psi_n^+ \} \subset E^+ \) is a sequence satisfying the assumptions of Lemma 3.7. Hence, there exists \( t_n > 0 \) such that \( \tilde{u}_n := t_n \psi_n^+ \in \mathcal{N}_{p_n} \) and

\[
\| \psi_n - \tilde{u}_n - h_{p_n}(\tilde{u}_n) \| \leq \| \psi_n - z_n \| + |t_n - 1| \cdot \| \psi_n^+ \| + \| h_{p_n}(\tilde{u}_n) - h_{p_n}(\psi_n^+) \|
\]

\[
\leq O(\| \mathcal{L}'_{p_n}(\psi_n) \|_{E^*}) + O(\| I_{p_n}(\psi_n^+) \|)
\]

(3.15)

where we have used an easily checked inequality

\[
\| h_{p_n}(\tilde{u}_n) - h_{p_n}(\psi_n^+) \| \leq \| h_{p_n}(\tau u_n) \| \cdot \| \tilde{u}_n - \psi_n^+ \| = O(\| \tilde{u}_n - \psi_n^+ \|).
\]

for some \( \tau \) between \( t_n \) and 1. Remark that \( \| I_{p_n}(\psi_n^+) \| = \| \mathcal{L}'_{p_n}(z_n) \|_{E^*} \) and, by using the \( C^2 \) smoothness of \( \mathcal{L}_{p_n} \), we have

\[
\| I_{p_n}(\psi_n^+) \| = \| \mathcal{L}'_{p_n}(z_n) \| \leq \| \mathcal{L}'_{p_n}(\psi_n) \| + O(\| \psi_n - z_n \|) = O(\| \mathcal{L}'_{p_n}(\psi_n) \|_{E^*})
\]

This together with (3.15) implies

\[
\| \psi_n - \tilde{u}_n - h_{p_n}(\tilde{u}_n) \| \leq O(\| \mathcal{L}'_{p_n}(\psi_n) \|_{E^*}).
\]

We are now in a position to complete the proof of Proposition 3.3.

Proof of Proposition 3.3. We proceed by contradiction. Assume to the contrary that there exist \( \theta_0 > 0, \alpha_n \rightarrow 0 \) and \( \{ \psi_n \} \subset E \) such that

\[
c_1 \leq \mathcal{L}_{p_n}(\psi_n) \leq c_2 \quad \text{and} \quad \| \mathcal{L}'_{p_n}(\psi_n) \|_{E^*} \leq \alpha_n
\]

and

\[
\max_{t > 0} I_{p_n}(t \psi_n^+) > \mathcal{L}_{p_n}(\psi_n) + \theta_0.
\]

(3.16)

Then it is clear that \( \{ \psi_n \} \) satisfies (3.9). Therefore, by Corollary 3.8, we should have that

\[
\max_{t > 0} I_{p_n}(t \psi_n^+) = I_{p_n}(\tilde{u}_n) \leq \mathcal{L}_{p_n}(\psi_n) + O(\| \mathcal{L}'_{p_n}(\psi_n) \|_{E^*}).
\]

This contradicts (3.16).
3.3 A Rayleigh type quotient

For any $p \in (2, 2^*)$ we define the functional

$$\mathcal{R}_p : E \setminus \{0\} \to \mathbb{R}, \quad \psi \mapsto \frac{\int_{S^m} (D\psi, \psi) \, d\vol_{g_{S^m}}}{\left(\int_{S^m} H(\xi) |\psi|^p \, d\vol_{g_{S^m}}\right)^{\frac{2}{p}}}.$$  

Here we remark that $\mathcal{R}_p$ is a differentiable functional and its derivation is given by

$$\mathcal{R}_p'(\psi)[\varphi] = \frac{2}{2^*} \left[ \Re \int_{S^m} (D\psi, \varphi) \, d\vol_{g_{S^m}} - \frac{\mathcal{R}_p(\psi)}{p} A(\psi) \frac{2-p}{p} \cdot A'(\psi)[\varphi] \right]$$

where (for simplicity) we have used the notations

$$A(\psi) := \int_{S^m} H(\xi) |\psi|^p \, d\vol_{g_{S^m}} \quad \text{and} \quad A'(\psi)[\varphi] = \frac{2}{p} \Re \int_{S^m} H(\xi) |\psi|^{p-2} (\psi, \varphi) \, d\vol_{g_{S^m}}.$$

Let $u \in E^+ \setminus \{0\}$, we define

$$\pi_u : E^- \to \mathbb{R}, \quad v \mapsto \mathcal{R}_p(u + v) = \frac{\|u\|^2 - \|v\|^2}{A(u + v)^{\frac{2}{p}}}.$$  

Then we see that $\sup_{v \in E^-} \pi_u(v) > 0$ is attained by some $v_u \in E^-$. Notice that for any positive $c > 0$ the set $\{v \in E^- : \pi_u(v) \geq c\}$ is strictly convex because the map

$$v \mapsto \|u\|^2 - \|v\|^2 - cA(u + v)^{\frac{2}{p}}$$

is strictly concave on $E^-$. Hence the maximum point $v_u \in E^-$ is uniquely determined.

Now, let’s define the map

$$\mathcal{J}_p : E^+ \setminus \{0\} \to E^-, \quad u \mapsto v_u \text{ which is the maximum point of } \pi_u$$

and the functional $\mathcal{F}_p : E^+ \setminus \{0\} \to \mathbb{R}$

$$\mathcal{F}_p(u) = \mathcal{R}_p(u + \mathcal{J}_p(u)) = \max_{v \in E^-} \mathcal{R}_p(u + v).$$

Remark that $\mathcal{R}_p(t\psi) = \mathcal{R}_p(\psi)$ for all $t > 0$, thus we have $\mathcal{J}_p(tu) = t \mathcal{J}_p(u)$. Moreover, since $\mathcal{F}_p(u) > 0$, we must have $\|\mathcal{J}_p(u)\| < \|u\|$ for all $u \in E^+ \setminus \{0\}$.

Lemma 3.9. $\mathcal{F}_p(u) = \left(\frac{2}{p-2} I_p(u)\right)^{\frac{p-2}{p}}$ for $u \in \mathcal{N}_p$.

Proof. Let $u \in \mathcal{N}_p$, then

$$0 = I_p(u)[u] = \int_{S^m} (D(u + h_p(u)), u + h_p(u)) \, d\vol_{g_{S^m}} - \int_{S^m} H(\xi) |u + h_p(u)|^p \, d\vol_{g_{S^m}}.$$  

Hence $I_p(u) = I_p(u) - \frac{1}{2} I_p(u)[u] = \frac{p-2}{2p} \int_{S^m} H(\xi) |u + h_p(u)|^p \, d\vol_{g_{S^m}}$.  

On the other hand, by (3.17), we have
\[ R_p'(u + h_p(u)) \equiv 0 \quad \forall v \in E^-. \]
This, together with the fact \( R_p(u + h_p(u)) > 0 \), suggests that \( J_p(u) = h_p(u) \) for \( u \in \mathcal{N}_p \). And therefore we get
\[ \mathcal{F}_p(u) = \left( \int_{S^m} H(\xi) |u + J_p(u)|^p d\text{vol}_{g_{S^m}} \right)^{1 - \frac{2}{p}} = \left( \frac{2p}{p - 2} I_p(u) \right)^{\frac{p - 2}{p}}. \]

By Lemma 3.9 and the fact \( \forall u \in E^+ \setminus \{0\} \) there uniquely exists \( t = t(u) > 0 \) such that \( t(u)u \in \mathcal{N}_p \), we have critical points of \( \mathcal{F}_p \) and \( I_p \) are in one-to-one correspondence via the map \( u \mapsto t(u)u \) from \( E^+ \setminus \{0\} \) to \( \mathcal{N}_p \).

Next, for any \( p \in (2, 2^*) \), we define
\[ \tau_p = \inf_{u \in E^+ \setminus \{0\}} \mathcal{F}_p(u). \] (3.18)

By Lemma 3.9, we have \( \tau_p \in (0, +\infty) \). In order to show properties of the map \( p \mapsto \tau_p \), we shall first prove the following:

**Lemma 3.10.** Let \( \{p_n\} \subset (2, 2^*) \) be an increasing sequence that converges to \( p \leq 2^* \). For each \( u \in E^+ \setminus \{0\} \), we have \( J_n(u) := J_{p_n}(u) \to J_p(u) \) as \( n \to \infty \).

**Proof.** We fix \( u \in E^+ \setminus \{0\} \) and, by noting that \( \| J_n(u) \| < \| u \| \), we can assume without loss of generality that \( J_n(u) \to v \in E^- \) as \( n \to \infty \). Moreover, up to a subsequence, we may have
\[ \lim_{n \to \infty} \left( \int_{S^m} H(\xi) |u + J_n(u)|^{p_n} d\text{vol}_{g_{S^m}} \right)^{\frac{1}{p_n}} = \ell > 0 \]

Remark that for each \( \psi \in E \)
\[ q \mapsto \left( \int_{S^m} H(\xi) |\psi|^q d\text{vol}_{g_{S^m}} \right)^{\frac{1}{q}} \quad \text{is nondecreasing} \]
as we have assumed \( \int_{S^m} H(\xi) d\text{vol}_{g_{S^m}} = 1 \). Then
\[ \mathcal{F}_{p_{n+1}}(u) = R_{p_{n+1}}(u + J_{n+1}(u)) \leq R_{p_n}(u + J_{n+1}(u)) \leq R_{p_n}(u + J_n(u)) = \mathcal{F}_{p_n}(u). \] (3.19)

Hence we have \( \{\mathcal{F}_{p_n}(u)\} \) is a non-increasing sequence and \( \mathcal{F}_{p_n}(u) \to \tau > 0 \) as \( n \to \infty \).
Choose $t_n > 0$ such that $t_n u \in \mathcal{N}_{p_n}$, as was argued in Lemma 3.9, we shall have $t_n \mathcal{J}_n(u) = \mathcal{J}_n(t_n u) = h_{p_n}(t_n u)$. And hence, we can deduce

$$
\mathcal{F}_{p_n}(u) = R_{p_n}(t_n u + t_n \mathcal{J}_n(u)) = \left( \int_{S^m} H(\xi)|t_n u + t_n \mathcal{J}_n(u)|^{p_n} d\text{vol}_{g_{S^m}} \right)^{\frac{1}{p_n}}.
$$

And therefore we have

$$
\lim_{n \to \infty} t_n = t_0 := \frac{\tau}{\ell} > 0
$$

and $t_n \mathcal{J}_n(u) \to t_0 v$ as $n \to \infty$.

Now, let us take arbitrarily $w \in W(u) = \text{span}\{u\} \oplus E^\perp$. Since $t_n u \in \mathcal{N}_{p_n}$, we get

$$
\int_{S^m} (D(t_0 u + t_0 v), w) d\text{vol}_{g_{S^m}} - \text{Re} \int_{S^m} H(\xi)|t_0 u + t_0 v|^{p-2}(t_0 u + t_0 v, w) d\text{vol}_{g_{S^m}}
= \lim_{n \to \infty} \left[ \int_{S^m} (D(t_n u + t_n \mathcal{J}_n(u)), w) d\text{vol}_{g_{S^m}}
- \text{Re} \int_{S^m} H(\xi)|t_n u + t_n \mathcal{J}_n(u)|^{p_n-2}(t_n u + t_n \mathcal{J}_n(u), w) d\text{vol}_{g_{S^m}} \right]
= \lim_{n \to \infty} I_{p_n}(t_n u)[w] = 0
$$

This implies, by using the fact $t_0 > 0$, $t_0 v = h_{p}(t_0 u)$ and $t_0 u \in \mathcal{N}_p$. And thus $v = \mathcal{J}_p(u)$ and $\mathcal{J}_n(u) \to \mathcal{J}_p(u)$ as $n \to \infty$.

To complete the proof, let us now assume to the contrary that $\mathcal{J}_n(u) \not\to \mathcal{J}_p(u)$ (up to any subsequence). Then we must have $\lim_{n \to \infty} \|\mathcal{J}_n(u)\| > \|\mathcal{J}_p(u)\|$. And hence we get

$$
\int_{S^m} H(\xi)|t_0 u + t_0 \mathcal{J}_p(u)|^{p_n} d\text{vol}_{g_{S^m}} = \int_{S^m} (D(t_0 u + t_0 \mathcal{J}_p(u)), t_0 u + t_0 \mathcal{J}_p(u)) d\text{vol}_{g_{S^m}}
= \|t_0 u\|^2 - \|t_0 \mathcal{J}_p(u)\|^2
> \lim_{n \to \infty} (\|t_n u\|^2 - \|t_n \mathcal{J}_n(u)\|^2)
= \lim_{n \to \infty} \int_{S^m} H(\xi)|t_n u + t_n \mathcal{J}_n(u)|^{p_n} d\text{vol}_{g_{S^m}}
\geq \int_{S^m} H(\xi)|t_0 u + t_0 \mathcal{J}_p(u)|^{p_n} d\text{vol}_{g_{S^m}}
$$

where the last inequality follows from Fatou’s lemma. And this estimate is obviously impossible.

\[\square\]

**Proposition 3.11.** The function $(2, 2^*] \to (0, +\infty)$, $p \mapsto \tau_p$ is

1. non-increasing,
2. continuous from the left.
Proof. Since (1) is evident as was already shown in (3.19), we only need to prove (2).

Given \( p \in (2, 2^* \rangle \), we choose \( u \in E^+ \setminus \{0\} \) such that \( F_p(u) \leq \tau_p + \epsilon \). Observe that for all \( p' \leq p \)

\[
F_{p'}(u) = R_{p'}(u + J_{p'}(u)) = \left( \frac{\int_{S^m} H(\xi)|u + J_{p'}(u)|p' dvol_{S^m}}{\left( \int_{S^m} H(\xi)|u + J_{p'}(u)|p dvol_{S^m} \right)^{\frac{p'}{p}}} \right)^{\frac{p'}{p}} R_p(u + J_{p'}(u)).
\]

Thanks to Lemma 3.10, the function

\[
p' \mapsto \left( \frac{\int_{S^m} H(\xi)|u + J_{p'}(u)|p' dvol_{S^m}}{\left( \int_{S^m} H(\xi)|u + J_{p'}(u)|p dvol_{S^m} \right)^{\frac{p'}{p}}} \right)^{\frac{p'}{p}}
\]
is continuous from the left. Hence, if \( p' \) is sufficiently close to \( p \), then

\[
\tau_{p'} \leq F_{p'}(u) \leq R_p(u + J_{p'}(u)) + \epsilon \leq \tau_p + 2\epsilon.
\]

Because \( p \mapsto \tau_p \) is non-increasing, the statement follows.

\[ \square \]

Remark 3.12. Recall that the sphere of constant sectional curvature 1 carries a Killing spinor \( \psi^* \) with length 1 to the constant \(-\frac{1}{2}\), that is, \( \psi^* \) satisfies \( |\psi^*| = 1 \) and

\[ \nabla_X \psi^* = -\frac{1}{2} X \cdot \psi^*, \quad \forall X \in \Gamma(TM) \]

where \( \cdot \) denotes the Clifford multiplication. Therefore we have \( D\psi^* = \frac{m}{2} \psi^* \). And it follows from [3, Section 4] that

\[
\inf_{u \in E^+ \setminus \{0\}} \max_{v \in E^-} \frac{\int_{S^m} (D(u + v), u + v) dvol_{S^m}}{\left( \int_{S^m} |u + v|^{2*} dvol_{S^m} \right)^{\frac{1}{2*}}} = \frac{\int_{S^m} (D\psi^*, \psi^*) dvol_{S^m}}{\left( \int_{S^m} |\psi^*|^{2*} dvol_{S^m} \right)^{\frac{1}{2*}}} = \left( \frac{m}{2} \right) \omega_{\frac{m}{2}}.
\]

Then, we can derive that \( \tau_{2^*} \geq \left( \frac{1}{H_{\text{max}}} \right) \left( \frac{m}{2} \right) \omega_{\frac{m}{2}} \). Thanks to the technical argument in [4], we know that \( \tau_{2^*} \leq \left( \frac{1}{H_{\text{max}}} \right) \left( \frac{m}{2} \right) \omega_{\frac{m}{2}} \). Hence we have

\[
\tau_{2^*} = \left( \frac{1}{H_{\text{max}}} \right) \left( \frac{m}{2} \right) \omega_{\frac{m}{2}}.
\]

(3.20)

4 Blow-up analysis

In this section we choose \( \{p_n\} \) to be an strictly increasing sequence such that \( \lim_{n \to \infty} p_n = 2^* \). In what follows, we shall investigate the possible convergent properties of solutions to the equation

\[ D\psi = H(\xi)|\psi|^{p_n-2}\psi \quad \text{on } S^m \]

(4.1)

with some specific energy constraints. Our arguments will be divided into three parts. In Subsection 4.1 we establish a kind of alternative behavior for solutions of (4.1), which shows either compactness or blow-up phenomenon. In Subsection 4.2, we describe the specific blow-up phenomenon which appears if we exclude the compactness. And in Subsection 4.3 we deal with the stereographic projected view of the blow-up behavior.
4.1 An alternative property

As before, we will denote $L_{p_n}$ the energy functional associated to Equation (4.1) and $(h_{p_n}, I_{p_n})$ the reduction couple for $L_{p_n}$. Our alternative result comes as follows:

**Proposition 4.1.** Suppose $\{\psi_n\} \subset E$ is a sequence such that

\[ \frac{1}{2m}(\tau_{2*})^m \leq L_{p_n}(\psi_n) \leq \frac{1}{m}(\tau_{2*})^m - \theta \quad \text{and} \quad L'_{p_n}(\psi_n) \to 0 \quad (4.2) \]

for some constant $\theta > 0$. Then, up to a subsequence, either $\psi_n \to 0$ or $\psi_n \to \psi_0 \neq 0$ in $E$.

**Proof.** Notice that $\{\psi_n\}$ is bounded in $E$ (by Lemma 3.4), and we may then assume that $\psi_n \rightharpoonup \psi_0$ in $E$ as $n \to \infty$ with some $\psi_0$ satisfying the equation

\[ D\psi_0 = H(\xi)|\psi_0|^{2∗-2}\psi_0 \quad \text{on } S^m. \quad (4.3) \]

Set $\bar{\psi}_n = \psi_n - \psi_0$. Then we have $\bar{\psi}_n$ satisfy

\[
\begin{align*}
D\bar{\psi}_n &= H(\xi)|\psi_n|^{p_n-2}\psi_n - H(\xi)|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n - H(\xi)|\psi_0|^{p_n-2}\psi_0 \\
&\quad + H(\xi)|\psi_0|^{p_n-2}\psi_0 - H(\xi)|\psi_0|^{2∗-2}\psi_0 \\
&\quad + H(\xi)|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n + o_n(1)
\end{align*}
\]

where $o_n(1) \to 0$ as $n \to \infty$ in $E^*$.

To proceed, we set

\[ \Phi_n = H(\xi)|\psi_n|^{p_n-2}\psi_n - H(\xi)|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n - H(\xi)|\psi_0|^{p_n-2}\psi_0 \]

It is easy to see that there exists $C > 0$ (independent of $n$) such that

\[ |\Phi_n| \leq C|\bar{\psi}_n|^{p_n-2}|\psi_0| + C|\psi_0|^{p_n-2}|\bar{\psi}_n|. \quad (4.4) \]

Thanks to Egorov theorem, for any $\epsilon > 0$, there exists $\Omega_\epsilon \subset S^m$ such that $\text{meas}\{S^m \setminus \Omega_\epsilon\} < \epsilon$ and $\bar{\psi}_n \to 0$ uniformly on $\Omega_\epsilon$ as $n \to \infty$. Therefore, by (4.4) and the Hölder inequality, we have

\[
\text{Re} \int_{S^m} (\Phi_n, \varphi) d\text{vol}_{g_{S^m}} = \text{Re} \int_{S^m \setminus \Omega_\epsilon} (\Phi_n, \varphi) d\text{vol}_{g_{S^m}} + \text{Re} \int_{\Omega_\epsilon} (\Phi_n, \varphi) d\text{vol}_{g_{S^m}} \\
\leq C\left(\int_{S^m \setminus \Omega_\epsilon} |\bar{\psi}_n|^{2∗} d\text{vol}_{g_{S^m}}\right)^{\frac{p_n-2}{2∗}} \left(\int_{S^m \setminus \Omega_\epsilon} |\psi_0|^{2∗} d\text{vol}_{g_{S^m}}\right)^{\frac{1}{2∗}} ||\varphi|| \\
+ C\left(\int_{S^m \setminus \Omega_\epsilon} |\psi_0|^{2∗} d\text{vol}_{g_{S^m}}\right)^{\frac{p_n-2}{2∗}} \left(\int_{S^m \setminus \Omega_\epsilon} |\bar{\psi}_n|^{2∗} d\text{vol}_{g_{S^m}}\right)^{\frac{1}{2∗}} ||\varphi|| \\
+ \int_{\Omega_\epsilon} |\Phi_n| \cdot |\varphi| d\text{vol}_{g_{S^m}}, \quad (4.5)
\]
for arbitrary $\varphi \in E$ with $\|\varphi\| \leq 1$. It is evident that the last integral in (4.5) converges to 0 as $n \to \infty$ and the remaining integrals tends to 0 uniformly in $n$ as $\epsilon \to 0$. Thus, we get $\Phi_n = o_n(1)$ in $E^*$. Noting that $q \mapsto H(\cdot)|\psi_0|^{q-2}\psi_0$ is continuous in $E^*$, hence we have

$$D\bar{\psi}_n = H(\xi)|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n + o_n(1) \quad \text{in } E^*. \quad (4.6)$$

Now assume $\psi_0 \neq 0$. If there exists a subsequence such that $L_{p_n}(\bar{\psi}_n) \to 0$, then it follows from the proof of Lemma 3.4 that we must have $\bar{\psi}_n \to 0$. So we now assume that, up to any subsequence, $L_{p_n}(\bar{\psi}_n) \not\to 0$.

Since $\psi_0$ is a non-trivial solution to (4.3), by Lemma 3.9 and the definition of $\tau_{2*}$ (c.f. (3.18)), we have

$$\tau_{2*} \left( \int_{S^m} H(\xi)|\psi_0|^{2*} \, d\text{vol}_{g_{Sm}} \right)^{\frac{1}{2*}} \leq \int_{S^m} (D\psi_0, \psi_0) \, d\text{vol}_{g_{Sm}} = \int_{S^m} H(\xi)|\psi_0|^{2*} \, d\text{vol}_{g_{Sm}}$$

and thus

$$\int_{S^m} (D\psi_0, \psi_0) \, d\text{vol}_{g_{Sm}} \geq \left( \tau_{2*} \right)^{\frac{2}{2*}} \quad (4.7)$$

On the other hand, by (4.6) and $L_{p_n}(\bar{\psi}_n) \not\to 0$, we have

$$c_1 \leq L_{p_n}(\bar{\psi}_n) \leq c_2 \quad \text{and} \quad L'_{p_n}(\bar{\psi}_n) \to 0$$

for some $c_1, c_2 > 0$. Therefore, by Corollary 3.8, Lemma 3.9 and the uniform boundedness of the second derivatives of $L_{p_n}$ near $\bar{\psi}_n$, we can conclude

$$\tau_{p_n} \leq \mathcal{F}_{p_n}(\bar{\psi}_n) = \max_{t > 0} \left( \frac{2p_n}{p_n - 2} I_{p_n}(t\bar{\psi}_n) \right)^{\frac{p_n-2}{p_n}} \leq \left( \frac{2p_n}{p_n - 2} L_{p_n}(\bar{\psi}_n) + o_n(1) \right)^{\frac{p_n-2}{p_n}}.$$

This, together with Proposition 3.11, implies

$$\int_{S^m} (D\bar{\psi}_n, \bar{\psi}_n) \, d\text{vol}_{g_{Sm}} \geq \left( \tau_{p_n} \right)^{\frac{p_n}{p_n-2}} + o_n(1) = \left( \tau_{2*} \right)^{\frac{2}{2*}} + o_n(1). \quad (4.8)$$

And we thus have

$$L_{p_n}(\psi_n) = \frac{p_n - 2}{2p_n} \int_{S^m} (D\psi_n, \psi_n) \, d\text{vol}_{g_{Sm}} + o_n(1)$$

$$= \frac{p_n - 2}{2p_n} \int_{S^m} (D\bar{\psi}_n, \bar{\psi}_n) \, d\text{vol}_{g_{Sm}} + \frac{p_n - 2}{2p_n} \int_{S^m} (D\psi_0, \psi_0) \, d\text{vol}_{g_{Sm}} + o_n(1)$$

$$\geq \frac{1}{m} (\tau_{2*})^m + o_n(1)$$

where the last inequality follows from (4.7) and (4.8). This contradicts (4.2). $\square$
4.2 Blow-up phenomenon

Let \( \{ \psi_n \} \subset E \) fulfill the assumption of Proposition 4.1 that is (4.2). If \( \{ \psi_n \} \) has a subsequence which is compact in \( E \), then the same subsequence converges and the limit spinor \( \psi_0 \) is a non-trivial solution to (4.3). Thus we are interested in the case where any subsequence of \( \{ \psi_n \} \) does not converge. From now on, by Proposition 4.1, we may assume \( \psi_n \to 0 \) in \( E \) as \( n \to \infty \).

To begin with, we shall first introduce an useful concept of blow-up set of \( \{ \psi_n \} \):

\[
\Gamma := \left\{ a \in M : \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(a)} |\psi_n|^{p_n} dvol_{g_{Sm}} \geq \delta_0 \right\}
\]

where \( B_r(a) \subset S^m \) is the distance ball of radius \( r \) with respect to the metric \( g_{Sm} \) and \( \delta_0 > 0 \) is a positive constant. The value of \( \delta_0 \) can be determined in the sense of the following lemma:

**Lemma 4.2.** Let \( \{ \psi_n \} \) be as above. Then there exists \( \delta_0 > 0 \) such that \( \Gamma \neq \emptyset \).

**Proof.** Assume to the contrary that \( \Gamma = \emptyset \) for any choice of \( \delta_0 \) (up to any subsequence of \( \{ \psi_n \} \)). Then we may fix \( \delta_0 \) arbitrary small. And we have, for any \( a \in M \), there exists \( r_0 > 0 \) such that

\[
\int_{B_{2r_0}(a)} |\psi_n|^{p_n} dvol_{g_{Sm}} < \delta_0.
\]

for all \( n \) large.

Let us take \( \eta \in C^\infty(S^m) \) such that \( \eta \equiv 1 \) on \( B_{r_0}(a) \) and \( \eta \equiv 0 \) on \( S^m \setminus B_{2r_0}(a) \). Since \( \mathcal{L}_{p_n} \psi_n = o_n(1) \) in \( E^* \), we can obtain

\[
D(\eta \psi_n) = \eta D\psi_n + \nabla \eta \cdot \psi_n = \eta(|\xi| H(\xi)|\psi_n|^{p_n-2} \psi_n + \nabla \eta \cdot \psi_n + o_n(1)
\]

where \( \cdot \) denotes the Clifford multiplication and \( o_n(1) \to 0 \) in \( E^* \) as \( n \to \infty \).

Noting that there exists \( C > 0 \) such that

\[
\|\psi\| \leq C \|D\psi\|_{E^*} + C |\psi|_2 \quad \forall \psi \in E.
\]

Thus, we have

\[
\|\eta \psi_n\| \leq C \|D(\eta \psi_n)\|_{E^*} + C |\eta \psi_n|_2
\]

\[
\leq C \|\eta(|\xi| H(\xi)|\psi_n|^{p_n-2} \psi_n + \nabla \eta \cdot \psi_n\|_{E^*} + C |\eta \psi_n|_2 + o_n(1)
\]

\[
\leq C \|\eta(|\xi| H(\xi)|\psi_n|^{p_n-2} \psi_n\|_{E^*} + C \|\nabla \eta \cdot \psi_n\|_{E^*} + C |\eta \psi_n|_2 + o_n(1). \quad (4.10)
\]

Remark that, by the Sobolev embedding \( L^2 \hookrightarrow E^* \), we have \( \|\nabla \eta \cdot \psi_n\|_{E^*} \leq C' \|\nabla \eta \cdot \psi_n\|_2 \) for some constant \( C' > 0 \). Moreover, by the Sobolev embedding \( E \hookrightarrow L^{p_n} \) and the Hölder inequality, there holds

\[
\text{Re} \int_{S^m} \eta(|\xi| H(\xi)|\psi_n|^{p_n-2} \psi_n \varphi) dvol_{g_{Sm}} \leq |H|_{\infty} |\varphi|_{p_n} |\eta \psi_n|_{p_n} \left( \int_{B_{2r_0}(a)} |\psi_n|^{p_n} dvol_{g_{Sm}} \right)^{\frac{p_n-2}{p_n}}
\]
for all \( \varphi \in E \), where \( C'' > 0 \) depends only on \( S^m \) and, in the last inequality, we have used (4.9).

Recall that we have \( \{p_n\} \) is a strictly increasing sequence such that \( p_n \to 2^* \) as \( n \to \infty \). Therefore, we may choose \( \delta_0 \) so small that \( C C'' |H|_\infty \delta_0^{\frac{m-2}{m}} < \frac{1}{2} \). Then by (4.10) and (4.11), we get

\[
\|\eta \psi_n\| \leq C C' |\nabla \cdot \psi_n|_2 + C |\eta \psi_n|_2 + o_n(1).
\]

Let us mention that we have assumed \( \psi_n \rightharpoonup 0 \). Hence, by the compact embedding \( E \hookrightarrow L^2 \), we are arrived at \( \|\eta \psi_n\| = o_n(1) \) as \( n \to \infty \).

Since \( a \in S^m \) is arbitrary and \( S^m \) is compact, we can conclude \( \psi_n \to 0 \) in \( E \) which contradicts (4.2).

Another useful concept in this context is the concept of the concentration function introduced in [11, 23, 24]. For \( r \geq 0 \), let us define

\[
\Theta_n(r) = \sup_{a \in S^m} \int_{B_r(a)} |\psi_n|^{p_n} d\text{vol}_{g_{S^m}}.
\]

Choose \( \bar{\delta} > 0 \) small, say \( \bar{\delta} < \delta_0 \) where \( \delta_0 \) is as in Lemma 4.2. Then there exist a decreasing sequence \( \{R_n\} \subset \mathbb{R}, R_n \to 0 \) as \( n \to \infty \) and \( \{a_n\} \subset S^m \) such that

\[
\Theta_n(R_n) = \int_{B_{R_n}(a_n)} |\psi_n|^{p_n} d\text{vol}_{g_{S^m}} = \bar{\delta}.
\]

Up to a subsequence if necessary, we assume that \( a_n \to a \in S^m \) as \( n \to \infty \).

Now, let us define the rescaled geodesic normal coordinates near each \( a_n \) via the formula

\[
\mu_n(x) = \exp_{a_n}(R_n \cdot x).
\]

Denoting \( B_R^0 = \{x \in \mathbb{R}^m : |x| < R\} \), where \( |\cdot| \) is the Euclidean norm in \( \mathbb{R}^m \), we have a conformal equivalence \( (B_R^0, R_n^{-2} \mu_n^* g_{S^m}) \cong (B_{R_n R(a_n)}, g_{S^m}) \subset S^m \) for all large \( n \).

For ease of notation, we set \( g_n = R_n^{-2} \mu_n^* g_{S^m} \). Writing the metric \( g_{S^m} \) in geodesic normal coordinates centered in \( a \), one immediately sees that, for any \( R > 0 \), \( g_n \) converges to the Euclidean metric in \( C^\infty(B_R^0) \) as \( n \to \infty \).

Now, following Proposition 2.1 and the idea of local trivialization introduced in Subsection 2.2, we can conclude that the coordinate map \( \mu_n \) induces a spinor identification \( (\mu_n)_* : \mathbb{S}_n(B_R^0, g_n) \to \mathbb{S}_{\mu_n}(B_{R_n R(a_n)}, g_{S^m}) \). If we define spinors \( \phi_n \) on \( B_R^0 \) by

\[
\phi_n = R_n^{m-1} (\mu_n)_*^{-1} \circ \psi_n \circ \mu_n,
\]

then a straightforward calculation shows that

\[
D_{g_n} \phi_n = R_n^{m+1} (\mu_n)_*^{-1} \circ (D \psi_n) \circ \mu_n,
\]
\[
\int_{B_R^0} (D_{g_n} \phi_n, \phi_n) d\text{vol}_{g_n} = \int_{B_{R_nR}(a_n)} (D\psi_n, \psi_n) d\text{vol}_{g_{Sm}},
\] (4.15)

\[
\int_{B_R^0} |\phi_n|^2 d\text{vol}_{g_n} = \int_{B_{R_nR}(a_n)} |\psi_n|^2 d\text{vol}_{g_{Sm}},
\] (4.16)

\[
\int_{B_R^0} |\phi_n|^{p_n} d\text{vol}_{g_n} = R_n^{- \frac{m-1}{2} (2^*-p_n)} \int_{B_{R_nR}(a_n)} |\psi_n|^{p_n} d\text{vol}_{g_{Sm}}.
\] (4.17)

Moreover, since \( \{\psi_n\} \) is bounded in \( E \), we have

\[
\sup_{n \geq 1} \int_{B_R^0} |\phi_n|^2 d\text{vol}_{g_n} \leq \sup_{n \geq 1} \int_{S^m} |\psi_n|^2 d\text{vol}_{g_{Sm}} < +\infty
\] (4.18)

for any \( R > 0 \).

**Lemma 4.3.** There is \( \bar{\lambda} > 0 \) such that

\[
\bar{\lambda} \leq \lim_{n \to \infty} R_n^{- \frac{m-1}{2} (2^*-p_n)} \leq \lim_{n \to \infty} R_n^{- \frac{m-1}{2} (2^*-p_n)} \leq 1.
\]

**Proof.** Since we have

\[
\int_{B_{R_n}(a_n)} |\psi_n|^{p_n} d\text{vol}_{g_{Sm}} = \bar{\delta},
\]

it follows from (4.17) and Hölder inequality that

\[
\bar{\delta} = \int_{B_{R_n}(a_n)} |\psi_n|^{p_n} d\text{vol}_{g_{Sm}} \leq \left( \int_{B_R^0} |\phi_n|^2 d\text{vol}_{g_n} \right)^{\frac{p_n}{2}} \left( \int_{B_R^0} d\text{vol}_{g_n} \right)^{\frac{2^*-p_n}{2}} R_n^{- \frac{m-1}{2} (2^*-p_n)}.
\]

Noting that \( g_n \) converges to the Euclidean metric in the \( C^\infty \)-topology on \( B_1^0 \), we can conclude immediately from \( p_n \to 2^* \) and (4.18) that

\[
\bar{\delta} \leq C \cdot R_n^{- \frac{m-1}{2} (2^*-p_n)}
\]

for some constant \( C > 0 \).

On the other hand, suppose there exists some \( \delta > 0 \) such that \( R_n^{- \frac{m-1}{2} (2^*-p_n)} \geq 1 + \delta \) for all large \( n \). Then, we must have

\[
\ln R_n \geq \frac{2 \ln(1+\delta)}{(m-1)(2^*-p_n)} \to +\infty \quad \text{as} \quad n \to \infty.
\]

This implies \( R_n \to +\infty \) which is absurd. \( \square \)

Moreover, we have

**Lemma 4.4.** Let \( \{\phi_n\} \) be defined in (4.13). Define

\[
\tilde{L}_n = D_{g_n} \phi_n - R_n^{- \frac{m-1}{2} (2^*-p_n)} H \circ \mu_n(\cdot) |\phi_n|^{p_n-2} \phi_n \in H_{loc}^{\frac{1}{2}}(\mathbb{R}^m, \mathcal{S}_m).
\]

Then \( \tilde{L}_n \to 0 \) in \( H_{loc}^{\frac{1}{2}}(\mathbb{R}^m, \mathcal{S}_m) \) in the sense that, for any \( R > 0 \), there holds

\[
\sup \left\{ \langle \tilde{L}_n, \varphi \rangle : \varphi \in H^{\frac{1}{2}}(\mathbb{R}^m, \mathcal{S}_m), \text{ supp } \varphi \subset B_R^0, \|\varphi\|_{H^{\frac{1}{2}}} \leq 1 \right\} \to 0
\]

as \( n \to \infty \).
Lemma 4.5. Furthermore, we have
\[ D_{g_n} \phi_n - R_n^{m-1}(2^* - p_n)H \circ \mu_n(\cdot)|\phi_n|^{p_n-2} \phi_n = R_n^{m-1}(\mu_n)^{-1} \circ (D\psi_n - H(\cdot)|\psi_n|^{p_n-2}\psi_n) \circ \mu_n \]
\[ = R_n^{m-1}(\mu_n)^{-1} \circ L_n \circ \mu_n \]
where \( L_n = D\psi_n - H(\cdot)|\psi_n|^{p_n-2}\psi_n \in E^* \).

Let \( \varphi \in H^{1,2}(\mathbb{R}^m, S_m) \) be such that \( \text{supp} \ \varphi \subset B_R^0 \) and \( \| \varphi \|_{H^{1,2}} \leq 1 \). Then, for all large \( n \), we get \( d\text{vol}_{g_n} = (1 + o_n(1))d\text{vol}_{g_{S_m}} \) and

\[
(1 + o_n(1)) \langle \tilde{L}_n, \varphi \rangle = \text{Re} \int_{B_R^0} (\tilde{L}_n, \varphi) d\text{vol}_{g_n} = \left( \frac{1}{n} \right) \int_{B_{1/(R)}^0} (\tilde{L}_n, \varphi) d\text{vol}_{g_n} \]
\[ = \text{Re} \int_{B_R^0} \left( \left( \mu_n \right)^{-1} \circ L_n \circ \mu_n, R_n^{m-1} \varphi \right) d\text{vol}_{g_n} \]
\[ = \text{Re} \int_{B_R^0} \left( \left( \mu_n \right)^{-1} \circ L_n \circ \mu_n, R_n^{m-1} \varphi \right) d\text{vol}_{g_{S_m}} \]
\[ = \text{Re} \int_{B_{1/(R)}^0} \left( L_n, R_n^{-m-1} \left( \mu_n \right)^{-1} \circ \varphi \circ \mu_n^{-1} \right) d\text{vol}_{g_{S_m}}. \quad (4.19) \]

Noting that \( \text{supp} \ \varphi \subset B_R^0 \) and \( \| \varphi \|_{H^{1,2}} \leq 1 \), we can find a constant \( C > 0 \) independent of \( n \) and \( \varphi \) such that \( \| R_n^{m-1}(\mu_n)^{-1} \circ \varphi \circ \mu_n^{-1} \| \leq C \). Thus, by (4.2) and (4.19), we can obtain the desired assertion. \qed

In what follows, by Lemma 4.3, we may assume that, after taking a subsequence if necessary,
\[ R_n^{m-1}(2^* - p_n) \to \lambda \in [\bar{\lambda}, 1]. \]

Since \( \{ \phi_n \} \) is bounded in \( H^{1,2}_{\text{loc}}(\mathbb{R}^m, S_m) \), we can assume (up to a subsequence) \( \phi_n \to \phi_0 \) in \( H^{1,2}_{\text{loc}}(\mathbb{R}^m, S_m) \). Thanks to the compact embedding \( H^{1,2}_{\text{loc}}(\mathbb{R}^m, S_m) \hookrightarrow L^q(\mathbb{R}^m, S_m) \) for \( 1 \leq q < 2^* \), it is easy to see that \( \phi_0 \in L^{2^*}(\mathbb{R}^m, S_m) \) satisfies
\[ D_{g_{S_m}} \phi_0 = \lambda H(a)|\phi_0|^{2^* - 2}\phi_0 \quad \text{on} \ \mathbb{R}^m. \]

Furthermore, we have

**Lemma 4.5.** \( \phi_n \to \phi_0 \) in \( H^{1,2}_{\text{loc}}(\mathbb{R}^m, S_m) \) as \( n \to \infty \).

**Proof.** For ease of notation, we shall set \( z_n = \phi_n - \phi_0 \). Let us fix \( y \in \mathbb{R}^m \) arbitrarily, then it follows from (4.12) and (4.17) that
\[ R_n^{m-1}(2^* - p_n) \int_{B_R^0(y)} |\phi_n|^{p_n} d\text{vol}_{g_n} \leq \tilde{c} \quad \text{for all} \ n \ \text{large}, \quad (4.20) \]
where $B_R^0(y) = \{x \in \mathbb{R}^m : |x - y| < R\}$ is the Euclidean ball centered at $y$ for any $R > 0$. And we can also conclude from Fatou lemma that

$$
\lambda \int_{B_R^0(y)} |\phi_0|^2 \, d\text{vol}_{g_{\mathbb{R}^m}} \leq \delta. \tag{4.21}
$$

Taking a smooth function $\beta : \mathbb{R}^m \to [0, 1]$ such that $\text{supp} \beta \subset B_R^0(y)$. Since, for any $\phi \in H^1(\mathbb{R}^m, S_m)$, we have the estimate

$$
\|\phi\|_{H^{1/2}} \leq C\|D_{g_{\mathbb{R}^m}} \phi\|_{H^{-1/2} + C|\phi|_2} \tag{4.22}
$$

for some constant $C > 0$ depends only on $m$, we soon get the estimate for $\beta^2 z_n$ as

$$
\|\beta^2 z_n\|_{H^{1/2}} \leq C\|D_{g_{\mathbb{R}^m}} (\beta^2 z_n)\|_{H^{-1/2}} + C|\beta^2 z_n|_2
$$

where $D_{g_{\mathbb{R}^m}}$ is the adjoint of $D_{g_n}$ with respect to the metric $g_{\mathbb{R}^m}$.

By recalling that $g_n$ converges to $g_{\mathbb{R}^m}$ in $C^\infty$-topology on bounded domains in $\mathbb{R}^m$, we get

$$
\beta(D_{g_{\mathbb{R}^m}} - D_{g_n}^*) : H^1(\mathbb{R}^m, S_m) \to L^2(\mathbb{R}^m, S_m)
$$

satisfies

$$
\|\beta(D_{g_{\mathbb{R}^m}} - D_{g_n}^*)\|_{H^1 \to L^2} \to 0 \tag{4.23}
$$

and

$$
(D_{g_{\mathbb{R}^m}} - D_{g_n}) \beta : H^1(\mathbb{R}^m, S_m) \to L^2(\mathbb{R}^m, S_m)
$$

satisfies

$$
\|\beta(D_{g_{\mathbb{R}^m}} - D_{g_n})\|_{H^1 \to L^2} \to 0 \tag{4.24}
$$

as $n \to \infty$. Then, by taking the dual of (4.24), we get $\beta(D_{g_{\mathbb{R}^m}} - D_{g_n}^*) : L^2(\mathbb{R}^m, S_m) \to H^{-1}(\mathbb{R}^m, S_m)$ satisfies

$$
\|\beta(D_{g_{\mathbb{R}^m}} - D_{g_n}^*)\|_{L^2 \to H^{-1}} \to 0 \tag{4.25}
$$

as $n \to \infty$.

Therefore, interpolating (4.23) and (4.25), we see that

$$
\|\beta(D_{g_{\mathbb{R}^m}} - D_{g_n}^*)\|_{H^{1/2} \to H^{-1/2} \to H^{-1/2}} \to 0
$$

and

$$
\|\beta(D_{g_{\mathbb{R}^m}} - D_{g_n})\|_{H^{-1/2} \to H^{1/2} \to H^{1/2} \to H^{-1/2}} \to 0
$$

Noting that $z_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^m, S_m)$, we immediately have $|\beta^2 z_n|_2 = o_n(1)$ as $n \to \infty$. To estimate the second term, we employ an argument of Isobe [18, Lemma 5.5]: we first observe that

$$
\langle (D_{g_{\mathbb{R}^m}} - D_{g_n})(\beta^2 \phi_n), \varphi \rangle = \langle \beta \phi_n, \beta(D_{g_{\mathbb{R}^m}} - D_{g_n}^*) \varphi \rangle
$$

for any $\varphi \in H^1(\mathbb{R}^m, S_m)$, where $D_{g_n}^*$ is the adjoint of $D_{g_n}$ with respect to the metric $g_{\mathbb{R}^m}$. 
as \( n \to \infty \).

To complete the proof, it remains to estimate the first term in (4.22). Recall that, by Lemma 4.4, we have

\[
D_{g_n} \phi_n = R_{n}^{m-1} (2^* - p_n) H \circ \mu_n (\cdot) |\phi_n|^{p_n-2} \phi_n + L_n
\]

and \( \bar{L}_n \to 0 \) in \( H^1_{\text{loc}} (\mathbb{R}^m, \mathbb{S}_m) \) as \( n \to \infty \). Hence, we deduce

\[
\|D_{g_n} (\beta^2 \phi_n) - D_{g_{\mathbb{R}^m}} (\beta^2 \phi_0)\|_{H^{-1/2}} \\
\leq \|\beta^2 R_{n}^{m-1} (2^* - p_n) H \circ \mu_n |\phi_n|^{p_n-2} \phi_n - \beta^2 \lambda H(a) |\phi_0|^{2^*-2} \phi_0\|_{H^{-1/2}} \\
+ \|\nabla (\beta^2) \cdot_{g_n} \phi_n - \nabla (\beta^2) \cdot_{g_{\mathbb{R}^m}} \phi_0\|_{H^{-1/2}} + o_n(1),
\]

(4.26)

where \( \cdot_{g_n} \) and \( \cdot_{g_{\mathbb{R}^m}} \) are Clifford multiplication with respect to the metrics \( g_n \) and \( g_{\mathbb{R}^m} \), respectively.

Remark that \( L_{\mathbb{R}^{m+1}}^2 (\mathbb{R}^m, \mathbb{S}_m) \hookrightarrow H^{-\frac{1}{2}} (\mathbb{R}^m, \mathbb{S}_m) \), we have

\[
\|\nabla (\beta^2) \cdot_{g_n} \phi_n - \nabla (\beta^2) \cdot_{g_{\mathbb{R}^m}} \phi_0\|_{H^{-1/2}} \leq \|\nabla (\beta^2) \cdot_{g_n} \phi_n - \nabla (\beta^2) \cdot_{g_{\mathbb{R}^m}} \phi_0\|_{H^{-1/2}} \to 0
\]
as \( n \to \infty \).

On the other hand, since we are working on the bounded domain \( B_1^0 (y) \subset \mathbb{R}^m \), we can argue as (4.3) to obtain

\[
\|\beta^2 R_{n}^{m-1} (2^* - p_n) H \circ \mu_n |\phi_n|^{p_n-2} \phi_n - \beta^2 \lambda H(a) |\phi_0|^{2^*-2} \phi_0\|_{H^{-1/2}} \\
= \|\beta^2 R_{n}^{m-1} (2^* - p_n) H \circ \mu_n |z_n|^{p_n-2} z_n\|_{H^{-1/2}} + o_n(1)
\]
as \( n \to \infty \). And thus, by Sobolev embedding and Hölder inequality, we have

\[
\|\beta^2 R_{n}^{m-1} (2^* - p_n) H \circ \mu_n |\phi_n|^{p_n-2} \phi_n - \beta^2 \lambda H(a) |\phi_0|^{2^*-2} \phi_0\|_{H^{-1/2}} \\
\leq CR_{n}^{m-1} (2^* - p_n) \left( \int_{B_1^0 (y)} |z_n|^{p_n} d\text{vol}_{g_n} \right)^{\frac{p_n-2}{p_n}} \|\beta^2 z_n\|_{H^{1/2}} + o_n(1)
\]

(4.27)

for some constant \( C > 0 \). Moreover, by (4.20) and (4.21), we can infer that

\[
R_{n}^{m-1} (2^* - p_n) \frac{1}{p_n} \left( \int_{B_1^0 (y)} |z_n|^{p_n} d\text{vol}_{g_n} \right)^{\frac{1}{p_n}} \leq 2\delta_{p_n} + o_n(1).
\]

(4.28)

Therefore, combining (4.22), (4.26), (4.27) and (4.28), we can get

\[
\|\beta^2 z_n\|_{H^{1/2}} \leq C \|\beta^2 R_{n}^{m-1} (2^* - p_n) H \circ \mu_n |z_n|^{p_n-2} z_n\|_{H^{-1/2}} + o_n(1)
\]

\[
\leq CR_{n}^{m-1} (2^* - p_n) \delta_{p_n} \|\beta^2 z_n\|_{H^{1/2}} + o_n(1)
\]

\[
\leq C\delta_{p_n} \|\beta^2 z_n\|_{H^{1/2}} + o_n(1)
\]
as \( n \to \infty \), where we have used \( R_m^{m-1/2(2^* - p_n)} \to \lambda \leq 1 \) in the last inequality. And if we fix \( \tilde{\delta} \) small such that \( C \delta \tilde{\delta}^{1/2} < \frac{1}{2} \) (recall that \( p_n \to 2^* = \frac{2m}{m-1} \)), the above estimate implies that \( \beta z_n \to 0 \) in \( H^{1/2}(\mathbb{R}^m, S_m) \). Since \( y \in \mathbb{R}^m \) and \( \beta \in C_c^\infty(\mathbb{R}^m) \) with \( \text{supp} \beta \subset B_1(y) \) are arbitrary, the conclusion follows directly.

By Lemma 4.5 (4.12) and (4.17), we have

\[
\lambda \int_{B_1^0} |\phi_0|^{2^*} \, d\text{vol}_{g_{\mathbb{R}^m}} = \tilde{\delta}.
\]

This implies \( \phi_0 \) is a non-trivial solution of

\[
D_{g_{\mathbb{R}^m}} \phi_0 = \lambda H(a)|\phi_0|^{2^*-2}\phi_0 \quad \text{on } \mathbb{R}^m.
\]

(4.29)

By the regularity results (see [2, 18]), we have \( \phi_0 \in C^{1,\alpha}(\mathbb{R}^m, S_m) \) for some \( 0 < \alpha < 1 \). Since \( \phi_0 \in L^{2^*}(\mathbb{R}^m, S_m) \) and \( \mathbb{R}^m \) is conformal equivalent to \( S^m \setminus \{N\} \) (where \( N \in S^m \) is the north pole), it is already known that \( \phi_0 \) extends to a non-trivial solution \( \bar{\phi}_0 \) to the equation

\[
D \bar{\phi}_0 = \lambda H(a)|\bar{\phi}_0|^{2^*-2}\bar{\phi}_0 \quad \text{on } S^m
\]

(cf. [3, Theorem 5.1], see also [2]). Recall that \( \frac{m}{2} \omega_m \) is the smallest positive eigenvalue of \( D \) on \( (S^m, g_{S^m}) \), and it can be characterized variationally as (see for instance [2, 4, 15])

\[
\frac{m}{2} \omega_m = \frac{m}{2} \text{vol}(S^m, g_{S^m}) \frac{1}{m} = \inf_{\psi} \left( \frac{\int_{S^m} |D\psi|^{2m} \, d\text{vol}_{g_{S^m}}}{\int_{S^m} (D\psi, \psi) \, d\text{vol}_{g_{S^m}}} \right)^{\frac{m+1}{m}}
\]

where the infimum is taken over the set of all smooth spinor fields for which

\[
\int_{S^m} (D\psi, \psi) \, d\text{vol}_{g_{S^m}} > 0.
\]

Then we can conclude from the conformal transformation that

\[
\int_{S^m} |\bar{\phi}_0|^{2^*} \, d\text{vol}_{g_{S^m}} = \int_{\mathbb{R}^m} |\phi_0|^{2^*} \, dx \geq \frac{1}{\lambda H(a)^m} \frac{m}{2} \omega_m.
\]

(4.30)

With these preparations out of the way, we may now choose \( \eta \in C^\infty(S^m) \) be such that \( \eta \equiv 1 \) on \( B_r(a) \) and \( \text{supp} \eta \subset B_{2r}(a) \) for some \( r > 0 \) (for sure \( r \) should not be large in the sense that we shall assume \( 3r < \inf r_{S^m} \) where \( \inf r_{S^m} \) denotes the injective radius) and define a spinor field \( z_n \in C^\infty(S^m, \mathbb{S}(S^m)) \) by

\[
z_n = R_n^{m-1} \eta(\cdot)(\mu_n)_* \circ \phi_0 \circ \mu_n^{-1}.
\]

Setting \( \varphi_n = \psi_n - z_n \), we have

**Lemma 4.6.** \( \varphi_n \to 0 \) in \( E \) as \( n \to \infty \).
Proof. Since we have assumed \( \psi_n \to 0 \) in \( E \) as \( n \to \infty \), we only need to show that \( z_n \to 0 \) in \( E \). Remark that, through the conformal transformation and the local trivialization, it is easy to check that \( \{z_n\} \) is bounded. And hence, by the Sobolev embedding, this sequence is weakly compact in \( E \) and compact in \( L^2 \). So, it suffices to prove

\[
\int_{S^m} |z_n|^2 d\text{vol}_g \to 0
\]
as \( n \to \infty \).

Noting that, for arbitrary \( R > 0 \), we have

\[
\int_{B_R(a_n)} |z_n|^2 d\text{vol}_g = R^{-m+1} \int_{B_R^0} |\phi_0|^2 d\text{vol}_{\mu^* g} = R_n \int_{B_R^0} |\phi_0|^2 d\text{vol}_{g_n}. \tag{4.31}
\]

And on the other hand, for all large \( n \),

\[
\int_{S^m \setminus B_{R_n}(a_n)} |z_n|^2 d\text{vol}_g \leq \int_{B_{3R_n}(a_n) \setminus B_{R_n}(a_n)} |z_n|^2 d\text{vol}_g = CR_n \int_{B_{3R_n/R_n}^0 \setminus B_{R_n}^0} |\phi_0|^2 d\text{vol}_g
\]
\[
\leq CR_n \left( \int_{B_{3R_n/R_n}^0 \setminus B_{R_n}^0} |\phi_0|^2 d\text{vol}_g \right) \frac{2}{1+2} \left( \left( \frac{3r}{R_n} \right)^m - R^m \right)^{\frac{2}{m-1}},
\]

where we used \( d\text{vol}_{g_n} \leq C d\text{vol}_{g_{2m}} \) on \( B_{3R_n/R_n}^0 \) for some constant \( C > 0 \) (since \( a_n \to a \) in \( S^m \)).

Recall that \( 2^* = \frac{2n}{m-1} \), it follows from the above inequality that

\[
\int_{S^m \setminus B_{R_n}(a_n)} |z_n|^2 d\text{vol}_g \leq C \left( \int_{B_{3R_n/R_n}^0 \setminus B_{R_n}^0} |\phi_0|^2 d\text{vol}_g \right)^{\frac{2}{1+2}} \left( (3r)^m - (R_n R)^m \right). \tag{4.32}
\]

Combining (4.31) and (4.32), we can infer that

\[
\int_{S^m} |z_n|^2 d\text{vol}_g \leq R_n \int_{B_R^0} |\phi_0|^2 d\text{vol}_{g_n}
\]
\[
+ C \left( \int_{B_{3R_n/R_n}^0 \setminus B_{R_n}^0} |\phi_0|^2 d\text{vol}_g \right)^{\frac{2}{1+2}} \left( (3r)^m - (R_n R)^m \right),
\]

which shows \( |z_n|_2 \to 0 \) as \( n \to \infty \). This completes the proof.

Focusing on the description of the new sequence \( \{\varphi_n\} \), we have the following result which yields the limiting behavior.

**Lemma 4.7.** \( \mathcal{L}_{p_n} \varphi_n \to 0 \) and \( \mathcal{L}_{p_n} \varphi_n \to 0 \) as \( n \to \infty \).

**Proof.** Let \( \varphi \in E \) be an arbitrary test spinor, it follows that

\[
\mathcal{L}_{p_n} \varphi = \text{Re} \int_{S^m} (Dz_n, \varphi) d\text{vol}_g - \text{Re} \int_{S^m} H(\xi)|z_n|^{p_n-2}(z_n, \varphi) d\text{vol}_g. \tag{4.33}
\]
On the other hand, since \( z_n = R_n^{-\frac{m-1}{2}} \eta(\cdot)(\mu_n)_* \circ \phi_0 \circ \mu_n^{-1} \), we have

\[
Dz_n = R_n^{-\frac{m-1}{2}} \nabla \eta \cdot g_{Sm} (\mu_n)_* \circ \phi_0 \circ \mu_n^{-1} + R_n^{-\frac{m+1}{2}} \eta(\cdot)(\mu_n)_* \circ (Dg_{n0} \phi_0) \circ \mu_n^{-1},
\]

where \( \cdot_{g_{Sm}} \) is the Clifford multiplication with respect to the metric \( g_{Sm} \). Substituting this into (4.33), we have

\[
\mathcal{L}_{p_n}'(z_n)[\varphi] = l_1 + l_2 + l_3 - l_4
\]

where (through the conformal transformation)

\[
l_1 = R_n^{-\frac{m+1}{2}} \text{Re} \int_{S^m} (\nabla \eta \cdot g_{Sm} (\mu_n)_* \circ \phi_0 \circ \mu_n^{-1}, \varphi) d\text{vol}_{g_{Sm}}
\]

\[
= R_n^{-\frac{m+1}{2}} \text{Re} \int_{B^0_{3^r/R_n}} ((\nabla \circ \mu_n) \cdot_{g_n} \phi_0, (\mu_n)_*^{-1} \circ \varphi \circ \mu_n) d\text{vol}_{g_n},
\]

\[
l_2 = R_n^{-\frac{m+1}{2}} \text{Re} \int_{S^m} (\eta(\cdot)(\mu_n)_* (Dg_{n0} \phi_0 - Dg_{Sm} \phi_0) \circ \mu_n^{-1}, \varphi) d\text{vol}_{g_{Sm}}
\]

\[
= R_n^{-\frac{m+1}{2}} \text{Re} \int_{B^0_{3^r/R_n}} (\eta \circ \mu_n) (Dg_{Sm} \phi_0, (\mu_n)_*^{-1} \circ \varphi \circ \mu_n) d\text{vol}_{g_n},
\]

\[
l_3 = R_n^{-\frac{m+1}{2}} \text{Re} \int_{S^m} (\eta(\cdot)(\mu_n)_* (Dg_{Sm} \phi_0) \circ \mu_n^{-1}, \varphi) d\text{vol}_{g_{Sm}}
\]

\[
= R_n^{-\frac{m+1}{2}} \text{Re} \int_{B^0_{3^r/R_n}} (\eta \circ \mu_n) (Dg_{Sm} \phi_0, (\mu_n)_*^{-1} \circ \varphi \circ \mu_n) d\text{vol}_{g_n},
\]

and

\[
l_4 = R_n^{-\frac{m-1}{2}(p_n-1)} \text{Re} \int_{S^m} H \cdot \eta^{-\frac{m+1}{2}} \cdot ((\mu_n)_* \circ \phi_0 \circ \mu_n^{-1})^{p_n-2} ((\mu_n)_* \circ \phi_0 \circ \mu_n^{-1}, \varphi) d\text{vol}_{g_{Sm}}
\]

\[
= R_n^{-\frac{m-1}{2}(2^r+1-p_n)} \text{Re} \int_{B^0_{3^r/R_n}} (H \circ \mu_n)(\eta \circ \mu_n)^{\frac{m+1}{2}} |\phi_0|^{p_n-2}(\mu_n)_*^{-1} \circ \varphi \circ \mu_n) d\text{vol}_{g_n}.
\]

We point out here that \( l_1 \) can be estimated similarly as we have done in Lemma 4.6. Indeed, by Hölder inequality, we observe that

\[
|l_1| \leq R_n \int_{B^0_{3^r/R_n}} |(\nabla \eta \circ \mu_n) \cdot_{g_n} \phi_0| \cdot |R_n^{-\frac{m+1}{2}} (\mu_n)_*^{-1} \circ \varphi \circ \mu_n| d\text{vol}_{g_n}
\]

\[
\leq CR_n \left( \int_{B^0_{3^r/R_n} \backslash \overline{B^0_{3^r/2R_n}}} d\text{vol}_{g_{Sm}} \right)^{\frac{2^r-2}{2^r-2}} \left( \int_{B^0_{3^r/R_n} \backslash \overline{B^0_{3^r/2R_n}}} |\phi_0|^{2^r} d\text{vol}_{g_{Sm}} \right)^{\frac{1}{2^r}} |\varphi|_{2^r}
\]

\[
\leq C \tau^m \left( \int_{B^0_{3^r/R_n} \backslash \overline{B^0_{3^r/2R_n}}} |\phi_0|^{2^r} d\text{vol}_{g_{Sm}} \right)^{\frac{1}{2^r}} \|\varphi\|.
\]
where we have used the estimate
\[
\int_{B^0_{3r/R}} |R_n^{\frac{m-1}{2}} (\mu_n)^{-1} \circ \varphi \circ \mu_n |^2 d\nu_{g_n} = \int_{B_{3r}(a_n)} |\varphi|^2 d\nu_{g_{SM}} \leq |\varphi|_2^2.
\]
Since \( \phi_0 \in L^2 (\mathbb{R}^m, S_m) \), we obtain from (4.35)
\[
|l_1| \leq o_n(1)\|\varphi\| \quad \text{as} \quad n \to \infty.
\] (4.36)

For \( l_2 \), by H"older inequality again, we have
\[
|l_2| \leq \int_{B^0_{3r/R}} |D_{g_n} \phi_0 - D_{g_{SM}} \phi_0| \cdot |R_n^{-\frac{m-1}{2}} (\mu_n)^{-1} \circ \varphi \circ \mu_n| d\nu_{g_n}
\]
\[
\leq C \left( \int_{B^0_{3r/R}} \left| D_{g_n} \phi_0 - D_{g_{SM}} \phi_0 \right|\frac{2m}{m+1} d\nu_{g_{SM}} \right)^\frac{m+1}{2m} \|\varphi\|. \quad (4.37)
\]

Fix \( R > 0 \) arbitrarily, we deduce that
\[
\int_{B^0_{3r/R}} |D_{g_n} \phi_0 - D_{g_{SM}} \phi_0|\frac{2m}{m+1} d\nu_{g_{SM}}
\]
\[
= \int_{B^0_{3r/R} \setminus B^0_R} |D_{g_n} \phi_0 - D_{g_{SM}} \phi_0|\frac{2m}{m+1} d\nu_{g_{SM}} + \int_{B^0_R} |D_{g_n} \phi_0 - D_{g_{SM}} \phi_0|\frac{2m}{m+1} d\nu_{g_{SM}}
\]
and, since \( \nabla \phi_0 \in L^{2m/(m+1)} (\mathbb{R}^m, S_m) \) and \( g_n \to g_{SM} \) in \( C^\infty (B^0_R) \) as \( n \to \infty \), we can get further from (4.37) that
\[
|l_2| \leq o_n(1)\|\varphi\| \quad \text{as} \quad n \to \infty.
\] (4.38)

Now, it remains to estimate \( |l_3 - l_4| \). Noting that \( \phi_0 \) satisfies Eq. (4.29), we soon have
\[
l_3 = \lambda H(a) \Re \int_{B^0_{3r/R}} (\eta \circ \mu_n) |\phi_0|^{2^* - 2} \phi_0 \cdot R_n^{-\frac{m-1}{2}} (\mu_n)^{-1} \circ \varphi \circ \mu_n d\nu_{g_n}. \quad (4.39)
\]

Since the “blow-up points” \( a_n \to a \) in \( S^m \) and \( \eta \equiv 1 \) on \( B_r(a) \), we have \( \eta \circ \mu_n \equiv 1 \) on \( B^0_R \) for all large \( n \) where \( R > 0 \) is fixed. Therefore, by \( \lim_{n \to \infty} R_n^{\frac{m-1}{2}(2^*-p_n)} = \lambda \) and \( H \circ \mu_n \to H(a) \) uniformly on \( B^0_R \) as \( n \to \infty \), we have
\[
R_n^{-\frac{m-1}{2}(2^*-p_n)} \Re \int_{B^0_R} (H \circ \mu_n) |\phi_0|^{p_n-2} \phi_0 \cdot R_n^{-\frac{m-1}{2}} (\mu_n)^{-1} \circ \varphi \circ \mu_n d\nu_{g_n}
\]
\[
= \lambda H(a) \Re \int_{B^0_R} |\phi_0|^{2^* - 2} \phi_0 \cdot R_n^{-\frac{m-1}{2}} (\mu_n)^{-1} \circ \varphi \circ \mu_n d\nu_{g_n} + o_n(1)\|\varphi\|. \quad (4.40)
\]

On the other hand, since \( \phi_0 \in L^2(\mathbb{R}^m, S) \), it follows that
\[
\int_{B^0_{3r/R} \setminus B^0_R} (\eta \circ \mu_n) |\phi_0|^{2^* - 2} \phi_0 \cdot R_n^{\frac{m-1}{2}} (\mu_n)^{-1} \circ \varphi \circ \mu_n d\nu_{g_n}
\]
\[
\leq C \left( \int_{B^0_{3r/R} \setminus B^0_R} |\phi_0|^2 d\nu_{g_{SM}} \right)^{\frac{2}{2^* - 2}} \left( \int_{B_{3r}(a_n)} |\varphi|^2 d\nu_{g_{SM}} \right)^{\frac{1}{2^*}} \quad (4.41)
\]
Thus, combining (4.39), (4.40) and the above two estimates, we can conclude

\[ |l_3 - l_4| \leq o_n(1)\|\varphi\| \quad \text{as } n \to \infty. \quad (4.41) \]

And then, it follows from (4.36), (4.38) and (4.41) that \( L'_{\varphi_n}(z_n) \to 0 \) as \( n \to \infty. \)

Now we turn to prove \( L'_{\varphi_n}(\varphi_n) \to 0 \) as \( n \to \infty. \)

Again, we choose \( \varphi \in E \) be an arbitrary test spinor. We then have

\[
L'_{\varphi_n}(\varphi) \left[ \varphi \right] = \text{Re} \int_{S_m} (D \varphi_n, \varphi) d\text{vol}_{g_{S_m}} - \int_{S_m} H(\xi)|\varphi_n|^p \varphi(\varphi_n, \varphi) d\text{vol}_{g_{S_m}}
\]

\[ = L'_{\varphi_n}(\varphi) - L'_{\varphi_n}(z_n)[\varphi] + \text{Re} \int_{S_m} (\Psi_n, \varphi) d\text{vol}_{g_{S_m}}, \quad (4.42) \]

where

\[ \Psi_n = H(\xi)|\psi_n|^p \varphi_n - H(\xi)|z_n|^p \varphi_n - H(\xi)|\varphi_n|^p \varphi_n. \]

Since we have assumed \( \{\psi_n\} \) satisfies (4.2), it follows that we only need to show that \( \|\Psi_n\|_{E^*} \to 0 \) as \( n \to \infty. \) Similarly as was argued in (4.5), we will use the fact that there exists \( C > 0 \) (independent of \( n \)) such that

\[ |\Psi_n| \leq C |z_n|^p |\varphi_n| + C |\varphi_n|^p |z_n|. \]

For any \( R > 0 \), we first observe that for all \( n \) large

\[
\int_{S_m \setminus B_{Rn}(a_n)} |z_n|^p |\varphi_n| |\varphi| d\text{vol}_{g_{S_m}}
\]

\[ \leq \omega_m^2 \int_{B_{Rn}(a_n)} |z_n|^2 d\text{vol}_{g_{S_m}} \left( \int_{S_m \setminus B_{Rn}(a_n)} |\varphi_n|^2 d\text{vol}_{g_{S_m}} \right)^{\frac{p}{2}} \left( \int_{S_m \setminus B_{Rn}(a_n)} |\varphi|^2 d\text{vol}_{g_{S_m}} \right)^{\frac{1}{2}} |\varphi|^2
\]

\[ \leq C \left( \int_{B_{3r/Rn}(a_n)} |\varphi_0|^2 d\text{vol}_{g_{S_m}} \right)^{\frac{p}{2}} \|\varphi_n\| \cdot \|\varphi\| = o_R(1)\|\varphi\|, \]
and
\[
\int_{S^m \setminus B_{R_n}(a_n)} |\varphi_n|^2 \cdot |z_n| \cdot |\varphi| \, d\text{vol}_{g_{S^m}} \\
\leq \omega_m \frac{\mu_{n-2}}{2^m} \left( \int_{B_{R_n}(a_n)} |\varphi_n|^2 \, d\text{vol}_{g_{S^m}} \right)^{\frac{\mu_{n-2}}{2}} \left( \int_{B_{R_n}(a_n)} |z_n|^2 \, d\text{vol}_{g_{S^m}} \right)^{\frac{1}{2^m}} |\varphi|^2 \\
\leq C \left( \int_{B_R^o} |\phi_0|^2 \, d\text{vol}_{g_R} \right)^{\frac{\mu_{n-2}}{2}} \left( \int_{B_R^o} |\phi_0|^2 \, d\text{vol}_{g_R} \right)^{\frac{1}{2^m}} \|\varphi\| = o_R(1)\|\varphi\|, 
\]
where \(\omega_m\) stands for the volume of \((S^m, g_{S^m})\) and \(o_R(1) \to 0\) as \(R \to \infty\).

And on the other hand, inside \(B_{R_n}(a_n)\), we have
\[
\int_{B_{R_n}(a_n)} |z_n|^2 \cdot |\varphi_n| \cdot |\varphi| \, d\text{vol}_{g_{S^m}} \\
\leq \omega_m \frac{\mu_{n-2}}{2^m} \left( \int_{B_{R_n}(a_n)} |\varphi_n|^2 \, d\text{vol}_{g_{S^m}} \right)^{\frac{\mu_{n-2}}{2}} \left( \int_{B_{R_n}(a_n)} |z_n|^2 \, d\text{vol}_{g_{S^m}} \right)^{\frac{1}{2^m}} |\varphi|^2 \\
\leq C \left( \int_{\mathbb{R}^m} |\phi_0|^2 \, d\text{vol}_{g_R} \right)^{\frac{\mu_{n-2}}{2}} \left( \int_{\mathbb{R}^m} |\phi_0|^2 \, d\text{vol}_{g_R} \right)^{\frac{1}{2^m}} \|\varphi\| = o_n(1)\|\varphi\|
\]
and
\[
\int_{B_{R_n}(a_n)} |\varphi_n|^2 \cdot |z_n| \cdot |\varphi| \, d\text{vol}_{g_{S^m}} \\
\leq \omega_m \frac{\mu_{n-2}}{2^m} \left( \int_{B_{R_n}(a_n)} |\varphi_n|^2 \, d\text{vol}_{g_{S^m}} \right)^{\frac{\mu_{n-2}}{2}} \left( \int_{B_{R_n}(a_n)} |z_n|^2 \, d\text{vol}_{g_{S^m}} \right)^{\frac{1}{2^m}} |\varphi|^2 \\
\leq C \left( \int_{B_R^o} |\phi_0|^2 \, d\text{vol}_{g_R} \right)^{\frac{\mu_{n-2}}{2}} \left( \int_{B_R^o} |\phi_0|^2 \, d\text{vol}_{g_R} \right)^{\frac{1}{2^m}} \|\varphi\| = o_n(1)\|\varphi\|
\]
as \(n \to \infty\), where we have used the fact \(\phi_n \to \phi_0\) in \(H^2_{\text{loc}}(\mathbb{R}^m, S^m)\) (see Lemma 4.5). Therefore, we can conclude that \(\Psi_n \to 0\) in \(E^*\) as \(n \to \infty\) which completes the proof. \(\square\)

At this point we have the following result which summarizes the blow-up phenomenon.

**Proposition 4.8.** Let \(\{\psi_n\} \subset E\) fulfill the assumption of Proposition 4.1. If \(\{\psi_n\}\) does not contain any compact subsequence. Then, up to a subsequence if necessary, there exist a convergent sequence \(\{a_n\} \subset S^m\), \(a_n \to a\) as \(n \to \infty\), a sequence of radius \(\{R_n\}\) converging to 0, a real number \(\lambda \in (2^{m-2} - 2^{-m}, 1]\) and a non-trivial solution \(\phi_0\) of Eq. (4.29) such that
\[
R_n^{\frac{m-1}{2} (2^* - p_n)} = \lambda + o_n(1)
\]
and
\[
\psi_n = R_n^{\frac{m-1}{2}} \eta(\cdot)(\mu_n) \circ \phi_0 \circ \mu_n^{-1} + o_n(1) \quad \text{in} \quad E
\]
as \( n \to \infty \), where \( \mu_n(x) = \exp_{an}(R_n x) \) and \( \eta \in C^\infty(S^m) \) is a cut-off function such that \( \eta(\xi) = 1 \) on \( B_r(a) \) and \( \text{supp } \eta \subseteq B_{2r}(a) \), some \( r > 0 \). Moreover, we have

\[
\mathcal{L}_{p_n}(\psi_n) \geq \frac{1}{2m(\lambda H(a))^{m-1}} \left( \frac{m}{2} \right)^m \omega_m + o_n(1)
\]
as \( n \to \infty \).

**Proof.** Inherit from the previous lemmas, let us first set 

\[
L \phi_n \rightarrow \infty.
\]

Hence, by the left continuity of \( p \), we have

\[
\mathcal{L}_{p_n}(z_n) + o_n(1) = \mathcal{L}_{p_n}(z_n) - \frac{1}{2} \mathcal{L}_{p_n}(z_n)[z_n] = \frac{p_n - 2}{2p_n} \int_{S^m} H(\xi)|z_n|^{p_n} \text{dvol}_{S^m} \geq 0
\]
and

\[
\mathcal{L}_{p_n}(\varphi_n) + o_n(1) = \mathcal{L}_{p_n}(\varphi_n) - \frac{1}{2} \mathcal{L}_{p_n}(\varphi_n)[\varphi_n] = \frac{p_n - 2}{2p_n} \int_{S^m} H(\xi)|\varphi_n|^{p_n} \text{dvol}_{S^m} \geq 0
\]

We claim that

**Claim 4.1.** \( \mathcal{L}_{p_n}(\psi_n) = \mathcal{L}_{p_n}(z_n) + \mathcal{L}_{p_n}(\varphi_n) + o_n(1) \) as \( n \to \infty \).

Assuming Claim 4.1 for the moment, then we shall get \( \mathcal{L}_{p_n}(\varphi_n) \rightarrow 0 \) as \( n \to \infty \). Indeed, suppose to the contrary that (up to a subsequence) \( \mathcal{L}_{p_n}(\varphi_n) \geq c > 0 \), it follows from the boundedness of \( \{\varphi_n\} \) in \( E \), Corollary 3.8 and Lemma 3.9 that

\[
\tau_{p_n} \leq \mathcal{F}_{p_n}(\varphi_n^+) = \max_{t > 0} \left( \frac{2p_n}{p_n - 2} I_{p_n}(t \varphi_n^+) \right)^{p_n/2} = \left( \frac{2p_n}{p_n - 2} \mathcal{L}_{p_n}(\varphi_n) + o_n(1) \right)^{p_n/2}.
\]

Hence, by the left continuity of \( p \) (see Proposition 3.11), we get

\[
\mathcal{L}_{p_n}(\varphi_n) \geq \frac{p_n - 2}{2p_n} (\tau_{p_n})^{p_n/2} + o_n(1) = \frac{1}{2m} (\tau_{2^*})^m + o_n(1). \tag{4.43}
\]

On the other hand, we have

\[
\mathcal{L}_{p_n}(z_n) = \frac{p_n - 2}{2p_n} \int_{S^m} H(\xi)|z_n|^{p_n} \text{dvol}_{S^m} + o_n(1)
\]

\[
= \frac{p_n - 2}{2p_n} \int_{B_R^0} (H \circ \mu_n)|\phi_0|^{p_n} \text{dvol}_{g_{S^m}} + o_n(1) + o_R(1)
\]

\[
= \frac{1}{2m} \lambda H(a) \int_{B_R^0} |\phi_0|^{2^*} \text{dvol}_{g_{S^m}} + o_n(1) + o_R(1)
\]

for \( R > 0 \) large. Thus, by (3.20), (4.30), \( \lambda \in (0, 1) \) and \( H(a) \leq H_{\max} \), we obtain

\[
\mathcal{L}_{p_n}(z_n) \geq \frac{1}{2m(\lambda H(a))^{m-1}} \left( \frac{m}{2} \right)^m \omega_m + o_n(1) \geq \frac{1}{2m} (\tau_{2^*})^m + o_n(1). \tag{4.44}
\]
Combining Claim 4.1, (4.43) and (4.44), we have \( \mathcal{L}_{p_n}(\psi_n) \geq \frac{1}{m}(\tau_2^*)^m + o_n(1) \) as \( n \to \infty \) which contradicts to (4.2). Therefore, we have \( \mathcal{L}_{p_n}(\varphi_n) \to 0 \) as \( n \to \infty \) and this, together with \( \mathcal{L}_{p_n}'(\varphi_n) \to 0 \), implies \( \varphi_n \to 0 \) in \( E \) as \( n \to \infty \). Moreover, we can get a lower bound for \( \lambda \) since \( H(a) \leq H_{max} \) and \( \mathcal{L}_{p_n}(\psi_n) < \frac{1}{m}(\tau_2^*)^m \), i.e. \( \lambda > 2 \frac{1}{m-1} \).

Now it remains to prove Claim 4.1. We would like to point out here that (thanks to Lemma 4.7) this is equivalent to show

\[
\int_{S^m} (D\psi_n, \psi_n) dvol_{gSm} = \int_{S^m} (Dz_n, z_n) dvol_{gSm} + \int_{S^m} (D\varphi_n, \varphi_n) dvol_{gSm} + o_n(1). \tag{4.45}
\]

And since \( \varphi_n = \psi_n - z_n \), it suffices to prove \( \int_{S^m} (Dz_n, \varphi_n) dvol_{gSm} = o_n(1) \) as \( n \to \infty \). In fact, for arbitrary \( R > 0 \), we have

\[
\int_{S^m} (Dz_n, \varphi_n) dvol_{gSm} = \int_{B_{R_n}(a_n)} (Dz_n, \varphi_n) dvol_{gSm} + \int_{B_R(0)} \left( Dz_n, \varphi_n \right) dvol_{gSm} = \int_{B_R(0)} (D_{g_n} \phi_0, \phi_n - \phi_0) dvol_{g_n} + \int_{B_R(0)} (Dz_n, \varphi_n) dvol_{gSm}.
\]

And for the first integral, by Lemma 4.5 we can get

\[
\left| \int_{B_R(0)} (D_{g_n} \phi_0, \phi_n - \phi_0) dvol_{g_n} \right| \leq C |\nabla \phi_0| \frac{z_n}{m+2} \cdot \| \phi_n - \phi_0 \|_{B^{1/2}} \to 0 \tag{4.46}
\]

as \( n \to \infty \). Meanwhile to estimate the second integral, we first observe that (through the conformal transformation)

\[
\sup_n \int_{B_R(0)} |\phi_n - \phi_0|^2 dvol_{gSm} \leq C \sup_n \int_{B_R(0)} |\psi_n - z_n|^2 dvol_{gSm} < +\infty
\]

for some \( C > 0 \). Thus, by \( dvol_{g_n} \leq C dvol_{gSm} \), we have

\[
\left| \int_{B_R(0)} (D_{g_n} \phi_0, \phi_n - \phi_0) dvol_{g_n} \right| \leq C \left( \int_{B_R(0)} |\nabla \phi_0| \frac{z_n}{m+2} dvol_{gSm} \right)^{\frac{m+1}{2m}} \to 0 \tag{4.47}
\]

as \( R \to \infty \). Therefore, by (4.46) and (4.47), we obtain (4.45) is valid and the proof is hereby completed.

\[ \Box \]

### 4.3 Using the stereographic projection

According to Proposition 4.8 any non-compact sequence \( \{\psi_n\} \) which satisfies (4.2), blows up around a point \( a \in S^m \). And due to the statement, it is natural to ask further questions:

1. Where the blow-up point \( a \) locates or whether \( a \) has any relation with the function \( H \) particularly when \( H \neq \text{constant} \)?
2. Whether or not the value of $\lambda$ can be fixed precisely?

We will show now that, if blow-up happens, such $a \in S^m$ must be a critical point of $H$ and $\lambda \equiv 1$. Before proving the results, we begin with some elementary materials on stereographic projection.

First of all, for arbitrary $\xi \in S^m$, we can always embed $\mathbb{S}^m$ into $\mathbb{R}^{m+1}$ in the way that $\xi$ has the coordinate $\xi = (0, \ldots, 0, -1) \in \mathbb{R}^{m+1}$, i.e. $\xi$ is the South pole. Denoting $S_\xi : S^m \setminus \{\xi\} \to \mathbb{S}^m$ the stereographic projection from the new North pole $-\xi$, we have $S_\xi(\xi) = 0$. Moreover, $S^m \setminus \{\xi\}$ and $\mathbb{S}^m$ are conformally equivalent due to the fact $(S_\xi^{-1})^* g_{S^m} = f^2 g_{\mathbb{S}^m}$ with $f(x) = \frac{2}{1+|x|^2}$.

Recall the conformal transformation formula mentioned in Proposition 2.1, there is an isomorphism of vector bundles $\iota : \mathbb{S}(\mathbb{S}^m, (S_\xi^{-1})^* g_{S^m}) \to \mathbb{S}(\mathbb{S}^m, g_{\mathbb{S}^m})$ such that

$$D_{g_{\mathbb{S}^m}}(\iota(\varphi)) = \iota\left(f^{\frac{m+1}{2}} D_{(S_\xi^{-1})^* g_{S^m}}(f^{-\frac{m+1}{2}} \varphi)\right),$$

where $D_{(S_\xi^{-1})^* g_{S^m}}$ is the Dirac operator on $\mathbb{S}^m$ with respect to the metric $(S_\xi^{-1})^* g_{S^m}$. Thus when $\psi$ is a solution to the equation $D \psi = H(\xi) |\psi|^{p-2} \psi$ on $(S^m, g_{S^m})$ for some $p \in (2, 2^*)$, then $\phi := \iota(f^{\frac{m+1}{2}} \psi \circ S_\xi^{-1})$ will satisfy the transformed equation

$$D_{g_{\mathbb{S}^m}} \phi = f^{\frac{m+1}{2}} (2^* - p) (H \circ S_\xi^{-1}) |\phi|^{p-2} \phi \quad \text{on} \quad (\mathbb{S}^m, g_{\mathbb{S}^m}).$$

Moreover, since $d\text{vol}_{(S_\xi^{-1})^* g_{S^m}} = f^m d\text{vol}_{g_{S^m}}$, we have

$$\int_{\mathbb{S}^m} (D_{g_{\mathbb{S}^m}} \phi, \phi) d\text{vol}_{g_{\mathbb{S}^m}} = \int_{S^m} (D \psi, \psi) d\text{vol}_{g_{S^m}},$$

$$\int_{\mathbb{S}^m} f^{\frac{m+1}{2}} (2^* - p) |\phi|^p d\text{vol}_{g_{\mathbb{S}^m}} = \int_{S^m} |\psi|^p d\text{vol}_{g_{S^m}},$$

and

$$\int_{\mathbb{S}^m} |\phi|^{2^*} d\text{vol}_{g_{\mathbb{S}^m}} = \int_{S^m} |\psi|^{2^*} d\text{vol}_{g_{S^m}}.$$

Returning to our case, let us assume $\{\psi_n\} \subset E$ be a sequence of solutions to the equations

$$D \psi_n = H(\xi) |\psi_n|^{p_n-2} \psi_n \quad \text{on} \quad S^m, \quad n = 1, 2, \ldots$$ (4.48)

and satisfying

$$\frac{1}{2m} (\tau_{2^*})^m \leq L_{p_n}(\psi_n) \leq \frac{1}{m} (\tau_{2^*})^m - \theta$$ (4.49)

for all $n$ large and some $\theta > 0$. Then, it is clear that $L_{p_n}'(\psi_n) \equiv 0$ for all $n$. And hence $\{\psi_n\}$ fulfills the assumption of Proposition 4.1. Moreover, by the regularity results proved in [2], these solutions are in fact $C^{1, \alpha}$ for some $\alpha \in (0, 1)$ and are classical solutions to (4.48).
Proposition 4.9. Suppose \( \{ \psi_n \} \) satisfies (4.48) and (4.49) and does not contain any compact subsequence. Let \( a \in S^m \) be the associate blow-up point found in Proposition 4.8 (up to a subsequence if necessary). Then \( \nabla H(a) = 0 \).

**Proof.** Let us consider the stereographic projection \( S_a : S^m \setminus \{ N \} \to \mathbb{R}^m \) and the associated bundle isomorphism \( \iota : S(\mathbb{R}^m, (S_a^{-1})^* g_{S^m}) \to S(\mathbb{R}^m, g_{\mathbb{R}^m}). \) Denoted by \( \tilde{\phi}_n = \iota(f^{\frac{1}{m-1}} \psi_n \circ S_a^{-1}) \), we have \( \tilde{\phi}_n \) satisfies

\[
D_{g_{\mathbb{R}^m}} \tilde{\phi}_n = f^{\frac{m-1}{2}}(H \circ S_a^{-1})|\tilde{\phi}_n|^{m-2}\tilde{\phi}_n \quad \text{on} \quad (\mathbb{R}^m, g_{\mathbb{R}^m}). \quad (4.50)
\]

Take \( \beta \in C_c^\infty(S^m) \) be a cut-off function on \( S^m \) such that \( \beta \equiv 1 \) on \( B_{2r}(a) \) and \( \text{supp} \beta \subset B_{3r}(a) \) where \( r > 0 \) comes from Proposition 4.8. Then we are allowed to multiply (4.50) by \( \partial_k ((\beta \circ S_a^{-1}) \tilde{\phi}_n) \) as a test spinor for each \( k = 1, 2, \ldots, m \), and consequently we have

\[
\begin{align*}
\text{Re} \int_{\mathbb{R}^m} \left( D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \partial_k ((\beta \circ S_a^{-1}) \tilde{\phi}_n) \right) d\text{vol}_{g_{\mathbb{R}^m}} &= \text{Re} \int_{\mathbb{R}^m} f^{\frac{m-1}{2}}(H \circ S_a^{-1})|\tilde{\phi}_n|^{m-2}\tilde{\phi}_n \partial_k ((\beta \circ S_a^{-1}) \tilde{\phi}_n) d\text{vol}_{g_{\mathbb{R}^m}}. \quad (4.51)
\end{align*}
\]

Remark that \( (\beta \circ S_a^{-1}) \tilde{\phi}_n \) has a compact support, we may integrate by parts to get

\[
0 = \text{Re} \int_{\mathbb{R}^m} \partial_k (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, (\beta \circ S_a^{-1}) \tilde{\phi}_n) d\text{vol}_{g_{\mathbb{R}^m}}
= 2 \text{Re} \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \partial_k ((\beta \circ S_a^{-1}) \tilde{\phi}_n)) d\text{vol}_{g_{\mathbb{R}^m}}
+ \text{Re} \int_{\mathbb{R}^m} (\partial_k \tilde{\phi}_n, \nabla ((\beta \circ S_a^{-1}) \tilde{\phi}_n)) d\text{vol}_{g_{\mathbb{R}^m}}
- \text{Re} \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \partial_k (\beta \circ S_a^{-1}) \tilde{\phi}_n) d\text{vol}_{g_{\mathbb{R}^m}}, \quad (4.52)
\]

where \( \cdot_{g_{\mathbb{R}^m}} \) denotes the Clifford multiplication with respect to \( g_{\mathbb{R}^m} \). Now let us evaluate the last two integrals of the previous equality. First of all, by noting that \( \{ \psi_n \} \) is bounded in \( E \), we can see from the conformal transformation and the regularity results (see [2]) that \( \{ \nabla \tilde{\phi}_n \} \) is uniformly bounded in \( L^2(\mathbb{R}^m, g_{S^m}) \). And so, by Proposition 4.8,

\[
\left| \int_{\mathbb{R}^m} (\partial_k \tilde{\phi}_n, \nabla ((\beta \circ S_a^{-1}) \tilde{\phi}_n)) d\text{vol}_{g_{\mathbb{R}^m}} \right| \leq C \left( \int_{B_{3r}(a) \setminus B_{2r}(a)} |\psi_n|^2 d\text{vol}_{g_{S^m}} \right)^{\frac{1}{2}} \to 0
\]

as \( n \to \infty \). Analogously, we have

\[
\left| \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \partial_k ((\beta \circ S_a^{-1}) \tilde{\phi}_n)) d\text{vol}_{g_{\mathbb{R}^m}} \right| \to 0
\]

as \( n \to \infty \). And thus, we conclude from (4.52) that

\[
\text{Re} \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \partial_k ((\beta \circ S_a^{-1}) \tilde{\phi}_n)) d\text{vol}_{g_{\mathbb{R}^m}} = o_n(1) \quad \text{as} \quad n \to \infty. \quad (4.53)
\]
On the other hand, to evaluate the second integral of (4.51), we have

\[
0 = \int_{\mathbb{R}^m} \partial_k \left[ f^{m-1}(2^*-p_n) (H \circ S_{a}^{-1})(\beta \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n} \right] d\text{vol}_{g_{\mathbb{R}^m}} \\
= \frac{m-1}{2} (2^* - p_n) \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n)^{-1} \partial_k f \cdot (H \circ S_{a}^{-1})(\beta \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \\
+ \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n) \partial_k (H \circ S_{a}^{-1})(\beta \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \\
+ p_n \text{Re} \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n) (H \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n-2} \partial_k (\beta \circ S_{a}^{-1})\tilde{\phi}_n \partial_k (\beta \circ S_{a}^{-1})\tilde{\phi}_n d\text{vol}_{g_{\mathbb{R}^m}} \\
- (p_n - 1) \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n) (H \circ S_{a}^{-1})\partial_k (\beta \circ S_{a}^{-1})|\ sol_{g_{\mathbb{R}^m}} |
\]

(4.54)

It is evident that the last integral converges to 0 as \( n \to \infty \), and we only need to estimate the remaining terms. Notice that \( f(x) = \frac{2}{1+|x|^2} \) and \( \beta \circ S_{a}^{-1} \) has a compact support on \( \mathbb{R}^m \), hence \( f, f^{-1} \) and \( \nabla f \) are bounded uniformly on \( \text{supp}(\beta \circ S_{a}^{-1}) \) and

\[
\left| \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n) \partial_k (H \circ S_{a}^{-1})(\beta \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \right| \\
\leq C \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n)|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \leq C \int_{S^m \setminus B_R(a)} |\psi|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \to 0
\]

as \( n \to \infty \). For the second integral, take arbitrarily \( R > 0 \) small, we deduce that

\[
\left| \int_{B_R^0} f^{m-1}(2^*-p_n) \partial_k (H \circ S_{a}^{-1})(\beta \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \right| \\
\leq C \int_{B_R^0} f^{m-1}(2^*-p_n)|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \leq C \int_{B_R^0} \left| \frac{1}{(2^*-p_n)} \right|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \to 0
\]

as \( n \to \infty \). And inside \( B_R^0 \), we have

\[
\int_{B_R^0} f^{m-1}(2^*-p_n) \partial_k (H \circ S_{a}^{-1})(\beta \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \\
= \partial_k (H \circ S_{a}^{-1})(0) \int_{B_R^0} f^{m-1}(2^*-p_n)|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} + O \left( \int_{B_R^0} |x| \cdot |\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} \right) + o_n(1) \\
= \partial_k (H \circ S_{a}^{-1})(0) \int_{B_R^0} f^{m-1}(2^*-p_n)|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} + O(R) + o_n(1)
\]

as \( n \to \infty \) and \( R \to 0 \). Thus by (4.54), for arbitrarily small \( R > 0 \), we get

\[
\text{Re} \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n) (H \circ S_{a}^{-1})|\tilde{\phi}_n|^{p_n-2} \partial_k (\beta \circ S_{a}^{-1})\tilde{\phi}_n \partial_k (\beta \circ S_{a}^{-1})\tilde{\phi}_n d\text{vol}_{g_{\mathbb{R}^m}} \\
= -\frac{1}{p_n} \partial_k (H \circ S_{a}^{-1})(0) \int_{\mathbb{R}^m} f^{m-1}(2^*-p_n)|\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} + O(R) + o_n(1).
\]

(4.55)

Combining (4.51), (4.55) and (4.55), we conclude that

\[
\partial_k (H \circ S_{a}^{-1})(0) \int_{\mathbb{R}^m} |\tilde{\phi}_n|^{p_n} d\text{vol}_{g_{\mathbb{R}^m}} = O(R) + o_n(1)
\]

(4.56)
as \( n \to \infty \) and \( R \) can be fixed arbitrarily small. Since we already know from the blow-up analysis that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^m} f_m^{m-1}(2^* - p_n) |\tilde{\phi}_n|^{p_n} d\text{vol}_{g_R} = \lim_{n \to \infty} \int_{S^m} |\psi_n|^{p_n} d\text{vol}_{g_S} > 0,
\]

(4.56) gives us nothing but \( \partial_k (H \circ S_a^{-1})(0) \equiv 0 \). Notice that \( k \) can be varying from 1 to \( m \), we have \( \nabla (H \circ S_a^{-1})(0) = 0 \), i.e. \( \nabla H(a) = 0 \) which completes the proof.

\[ \Box \]

**Proposition 4.10.** Suppose \( \{\psi_n\} \) satisfies (4.48) and (4.49) and does not contain any compact subsequence. Let \( \{R_n\} \) be the associated radius found in Proposition 4.8. Then

\[
\lim_{n \to \infty} R_n^{-\frac{m-1}{2}(2^* - p_n)} = 1.
\]

**Proof.** Let us recall the equation under stereographic projection (4.50) and consider the conformal change of \( \tilde{\phi}_n \) as

\[
\tilde{\phi}_{n,R}(x) = R^{\frac{m-1}{2}} \tilde{\phi}_n(Rx) \quad \text{for } R > 0.
\]

Then we have

\[
D_{g_R} \tilde{\phi}_{n,R} = R^{\frac{m-1}{2}(2^* - p_n)} \tilde{H}_{n,R} \tilde{\phi}_{n,R}^{p_n - 2} \tilde{\phi}_{n,R} \quad \text{on } \mathbb{R}^m
\]

(4.57)

where, for ease of notations, we have denoted \( \tilde{H}_{n,R}(x) = f_{m-\frac{1}{2}(2^* - p_n)}(Rx) \cdot H \circ S_a^{-1}(Rx) \).

Let \( \beta \in C_c^\infty(S^m) \) be the same cut-off function as in (4.51), we set

\[
\hat{\phi}_{n,R}(x) = \beta \circ S_a^{-1}(Rx) \cdot \tilde{\phi}_{n,R}(x).
\]

Then a direct calculation shows that

\[
\int_{\mathbb{R}^m} (D_{g_R} \hat{\phi}_{n,R} \cdot \hat{\phi}_{n,R}) d\text{vol}_{g_R} = \int_{S^m} (D\psi_n, \beta \psi_n) d\text{vol}_{g_S} \quad (4.58)
\]

and

\[
\int_{\mathbb{R}^m} (\beta \circ S_a^{-1})(Rx) \cdot \tilde{H}_{n,R} |\tilde{\phi}_{n,R}|^{p_n} d\text{vol}_{g_R} = R^\frac{m-1}{2}(p_n - 2) \int_{S^m} \beta H |\psi_n|^{p_n} d\text{vol}_{g_S} \quad (4.59)
\]

Hence, take derivative with respect to \( R \) in (4.58), we have

\[
0 = \frac{d}{dR} \bigg|_{R=R_n} \int_{\mathbb{R}^m} (D_{g_R} \hat{\phi}_{n,R} \cdot \hat{\phi}_{n,R}) d\text{vol}_{g_R} = 2\Re \int_{\mathbb{R}^m} \left( D_{g_R} \hat{\phi}_{n,R}, \frac{d}{dR} \bigg|_{R=R_n} \hat{\phi}_{n,R} \right) d\text{vol}_{g_R}
\]

\[+ \Re \int_{\mathbb{R}^m} \left( \frac{d}{dR} \bigg|_{R=R_n} \hat{\phi}_{n,R}, R_n \nabla (\beta \circ S_a^{-1})(R_n x) \cdot g_R \tilde{\phi}_{n,R} \right) d\text{vol}_{g_R}
\]

\[- \Re \int_{\mathbb{R}^m} \left( D_{g_R} \hat{\phi}_{n,R}, \frac{d}{dR} \bigg|_{R=R_n} \left[ (\beta \circ S_a^{-1})(R_n x) \right] \tilde{\phi}_{n,R} \right) d\text{vol}_{g_R}. \quad (4.60)
\]
To evaluate the last two integrals above, we first notice that
\[
\frac{d}{dR} \bigg|_{R=R_n} \tilde{\phi}_{n,R}(x) = \frac{m-1}{2} R_n \tilde{\phi}_n(R_n x) + R_n^{m+1} \nabla \tilde{\phi}_n(R_n x) \cdot x,
\]
and by the property of the hermitian product on \( \mathbb{S}(S^m) \) (see the second axiom of the Dirac bundle) we have
\[
\text{Re} \int_{\mathbb{R}^m} \left( \tilde{\phi}_n(R_n x), \nabla (\beta \circ S^{-1}_a)(R_n x) \cdot g_{S^m} \tilde{\phi}_{n,R_n} \right) d\text{vol}_{g_{S^m}} = 0.
\]
Moreover, using the fact \( \{ \nabla \tilde{\phi}_n \} \) is uniformly bounded in \( L^{\frac{2m}{m+1}}(\mathbb{R}^m, S_m) \) and \( \beta \) has a compact support, we obtain
\[
\left| \int_{\mathbb{R}^m} \left( R_n \frac{d}{dR} \bigg|_{R=R_n} \tilde{\phi}_{n,R}, R_n \nabla (\beta \circ S^{-1}_a)(R_n x) \cdot g_{S^m} \tilde{\phi}_{n,R_n} \right) d\text{vol}_{g_{S^m}} \right| = o(n)
\]
as \( n \to \infty \). Hence
\[
\left| \int_{\mathbb{R}^m} \left( \frac{d}{dR} \bigg|_{R=R_n} \tilde{\phi}_{n,R}, R_n \nabla (\beta \circ S^{-1}_a)(R_n x) \cdot g_{S^m} \tilde{\phi}_{n,R_n} \right) d\text{vol}_{g_{S^m}} \right| = o(n)
\]
as \( n \to \infty \). Analogously, it follows that
\[
\left| \int_{\mathbb{R}^m} \left( D_{g_{S^m}} \tilde{\phi}_{n,R_n}, \frac{d}{dR} \bigg|_{R=R_n} \tilde{\phi}_{n,R} \right) d\text{vol}_{g_{S^m}} \right| = o(n)
\]
as \( n \to \infty \). And thus, from (4.60), we find
\[
\text{Re} \int_{\mathbb{R}^m} \left( D_{g_{S^m}} \tilde{\phi}_{n,R_n}, \frac{d}{dR} \bigg|_{R=R_n} \tilde{\phi}_{n,R} \right) d\text{vol}_{g_{S^m}} = o(n)
\]
as \( n \to \infty \).

To proceed, we use (4.59) to obtain
\[
\frac{d}{dR} \bigg|_{R=R_n} \int_{\mathbb{R}^m} (\beta \circ S^{-1}_a)(R x) \cdot \tilde{H}_{n,R} \tilde{\phi}_{n,R} |^{p_n} d\text{vol}_{g_{S^m}}
\]
\[
= \frac{m-1}{2} (p_n - 2^*) R_n \frac{m+1}{2} (p_n - 2^* - 1) \int_{S^m} \beta H |\psi_n|^{p_n} d\text{vol}_{g_{S^m}}.
\]
On the other hand,
\[
\frac{d}{dR} \bigg|_{R=R_n} \int_{\mathbb{R}^m} (\beta \circ S^{-1}_a)(R x) \cdot \tilde{H}_{n,R} \tilde{\phi}_{n,R} |^{p_n} d\text{vol}_{g_{S^m}}
\]
\[
= \int_{\mathbb{R}^m} \frac{d}{dR} \bigg|_{R=R_n} [(\beta \circ S^{-1}_a)(R x) \cdot \tilde{H}_{n,R} ] \cdot |\tilde{\phi}_{n,R_n}|^{p_n} d\text{vol}_{g_{S^m}}
\]
\[
+ p_n \text{Re} \int_{\mathbb{R}^m} (\beta \circ S^{-1}_a)(R x) \cdot \tilde{H}_{n,R_n} |\tilde{\phi}_{n,R_n}|^{p_n} \left( \frac{d}{dR} \bigg|_{R=R_n} \tilde{\phi}_{n,R} \right) d\text{vol}_{g_{S^m}}
\]
\[
- p_n \int_{\mathbb{R}^m} \frac{d}{dR} \bigg|_{R=R_n} [(\beta \circ S^{-1}_a)(R x) \cdot \tilde{H}_{n,R_n} |\tilde{\phi}_{n,R_n}|^{p_n} d\text{vol}_{g_{S^m}}.
\]
and by using $f(x) = \frac{2}{1+|x|^2}$ and $\beta$ has compact support, we have

$$\int_{\mathbb{R}^n} \frac{d}{dR} \bigg|_{R=R_n} \left[ (\beta \circ S^{-1}_a)(Rx) \cdot \tilde{H}_{n,R} \right] \cdot |\tilde{\phi}_{n,R_n}|^p \, d\text{vol}_{\mathbb{R}^m} = O_n(1)$$

and

$$\left| \int_{\mathbb{R}^n} \frac{d}{dR} \bigg|_{R=R_n} \left[ (\beta \circ S^{-1}_a)(Rx) \right] \cdot \tilde{H}_{n,R_n} |\tilde{\phi}_{n,R_n}|^p \, d\text{vol}_{g_{sm}} \right| \leq C \int_{B_{3r}(a) \setminus B_{2r}(a)} |\psi_n|^p \, d\text{vol}_{g_{sm}} = o_n(1)$$

as $n \to \infty$. Thus, by virtue of (4.62), we infer that

$$\text{Re} \int_{\mathbb{R}^n} (\beta \circ S^{-1}_a)(Rx) \cdot \tilde{H}_{n,R_n} |\tilde{\phi}_{n,R_n}|^{p-2} \left( \tilde{\phi}_{n,R_n} \cdot \frac{d}{dR} \bigg|_{R=R_n} \tilde{\phi}_{n,R} \right) \, d\text{vol}_{g_{sm}}$$

$$= \frac{m-1}{2} (p_n - 2^*) R_n^{m-1} (p_n-2^*)^{-1} \int_{S^m} \beta H |\psi_n|^{p_n} \, d\text{vol}_{g_{sm}} + O_n(1)$$

(4.63)

as $n \to \infty$. Combining (4.57), (4.61), (4.63) and Lemma 4.3 we can conclude

$$(2^* - p_n) R_n^{-1} \int_{S^m} \beta H |\psi_n|^{p_n} \, d\text{vol}_{g_{sm}} = O_n(1) \quad \text{as } n \to \infty.$$ 

Since the blow-up phenomenon suggests $\lim_{n \to \infty} \int_{S^m} \beta H |\psi_n|^{p_n} \, d\text{vol}_{g_{sm}} > 0$, we find $2^* - p_n = O(R_n)$ as $n \to \infty$. Therefore

$$\lim_{n \to \infty} R_n^m (2^* - p_n) = \lim_{n \to \infty} c^{O(1) R_n \ln R_n} = 1.$$ 

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TIAN XU
CENTER FOR APPLIED MATHEMATICS, TIANJIN UNIVERSITY
TIANJIN, 300072, CHINA
xutian@amss.ac.cn