Kontsevich’s Swiss cheese conjecture

JUSTIN THOMAS

We prove a conjecture of Kontsevich, which states that if $A$ is an $E_d - 1$ algebra then the Hochschild cochain object of $A$ is the universal $E_d$ algebra acting on $A$. The notion of an $E_d$ algebra acting on an $E_{d-1}$ algebra was defined by Kontsevich using the Swiss cheese operad of Voronov. The degree 0 and 1 pieces of the Swiss cheese operad can be used to build a cofibrant model for $A$ as an $E_{d-1} - A$–module. The theorem amounts to the fact that the Swiss cheese operad is generated up to homotopy by its degree 0 and 1 pieces.

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1 Introduction

Gerstenhaber [10] showed that the Hochschild cohomology $\text{HH}^*(A)$ of an associative algebra $A$ is a graded Lie algebra and a graded commutative algebra, and the two structures are compatible. Any graded vector space with this algebraic structure is now called a Gerstenhaber algebra. Cohen [7] showed that the homology of the little disks operad, $H_*(E_2)$, is the Gerstenhaber operad. Sinha [20] has also shown this. Deligne later asked if the action of $H_*(E_2)$ on $\text{HH}^*(A)$ descends from a natural action at the level of chains. In other words, is there a natural algebra structure on $\text{CH}^*(A)$ of $\text{Chains}(E_2)$ which recovers the structure discovered by Gerstenhaber after passing to (co)homology?

Already, this question is evidently in the realm of homotopy theory. So let us replace the associative algebras by $E_1$ algebras. This makes it clear that the question is fundamentally one about the relationship between the operads $E_1$ and $E_2$. Indeed, we can generalize to consider the relationship between $E_d$ and $E_{d-1}$ algebras. For any $E_{d-1}$ algebra in a sufficiently rich homotopical category $C$ we can make sense of its Hochschild cochains as an object of $C$. The Hochschild cochain object of $A$ is denoted by $\text{Hoch}(A)$ and is an object of the same category to which $A$ belongs. This terminology is based on the case where $C$ is the category of differential graded vector spaces and $A$ is an associative algebra. In that case, $\text{Hoch}(A)$ is the usual Hochschild cochain complex of $A$. 

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The original Deligne conjecture, where $A$ is an $E_1$ algebra in the category of chain complexes, has been solved by Tamarkin [22], Kontsevich and Soibelman [17], Voronov [26], McClure and Smith [19], Berger and Fresse [2] and Kaufmann and Schwell [15]. A cyclic version is also due to Kaufmann [14]. Both cyclic and non-cyclic versions can be found in Batanin and Berger [1]. Vallette [24] generalized the theorem to include certain other Koszul operads. The generalized version, where $A$ is an $E_d$ algebra in a general category like $C$, has been proven in the $\infty$–operad setting by Lurie [18]. We show here that Hoch$_1(A)$ is not just an $E_d$ algebra, but comes equipped with a universal property. It is the universal $E_d$ algebra acting on the $E_{d-1}$ algebra $A$. A chain-level realization of the Swiss cheese action is constructed in Dolgushev, Tamarkin and Tsygan [9].

The notion of an $E_d$ algebra acting on an $E_{d-1}$ algebra was introduced by Kontsevich [16]. This notion uses the Swiss cheese operad $SC_d$ of Voronov [25]. This is a 2–colored operad which interpolates between $E_d$ and $E_{d-1}$. A Swiss cheese algebra is a pair $(B, A)$ where $B$ is an $E_d$ algebra, $A$ is an $E_{d-1}$ algebra and there is some extra structure compatible with these, as seen in Definition 2.1.9. We refer to this extra structure as an action of $B$ on $A$.

The case $d = 1$ is enlightening. For simplicity, let us work in the category of vector spaces. A (non-unital) $E_0$ algebra $A$ in vector spaces is just a vector space with no extra data. The Hochschild object in this case is hom$_1(A, A)$. If $B$ is an associative algebra, it is in particular an $E_1$ algebra. An $SC_1$ structure on the pair $(B, A)$ then amounts to the choice of a $B$–module structure on $A$.

In this case, the Swiss cheese conjecture merely states that hom$_1(A, A)$ is an associative algebra and giving an $SC_1$ structure on $(B, A)$ is equivalent to giving a map of associative algebras $B \to$ Hoch$_1(A) =$ hom$_1(A, A)$. We prove the analog of this when $B$ is an $E_d$ algebra and $A$ is an $E_{d-1}$ algebra, $d \geq 1$.

Joseph Hirsh brought to the attention of the author the following helpful characterization of the results of this paper. Given a bifibrant $E_{d-1}$ algebra $A$ the functor from $E_d$ algebras to spaces

$$B \mapsto \{\text{the space of Swiss cheese actions of } B \text{ on } A\}$$

is represented by the Hochschild cochain object of $A$.

1.1 Outline of the paper

In Section 2 we define the $E_d$ and $SC_d$ operads and give both an imprecise and a precise statement of the theorem we will prove. We also outline the idea of the proof.
In Section 3 we define the Hochschild cochain object for $E_{d-1}$ algebras and show that we can use the Swiss cheese operad to construct a model for the Hochschild cochain object. We use this model in Section 4 to prove a “universal cheese” theorem which applies to an arbitrary operad acting on Hochschild cochains. In Section 5 we show that $E_d$ does indeed act, up to homotopy, on Hochschild cochains, and the universal cheese theorem specializes to the main theorem of this paper. Finally, in Section 6 we prove the main theorem that allows the homotopy $E_d$ action: the Swiss cheese operad is freely generated up to homotopy by its degree 0 and 1 pieces.

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2 The Swiss cheese operad

We will define $K$–colored operads in general and the Swiss cheese operad in particular. We also describe algebras over the Swiss cheese operad and state the main theorem of this paper.

2.1 The colored operad Swiss cheese

Fix a set $K$. A $K$–colored set is a pair $(I, \text{col})$ where $I$ is a set and col: $I \to K$ is a map of sets, called the coloring. We will often denote such a colored set simply by $I$, leaving the coloring implicit. Let aut$I$ be the group of bijections on the set $I$ which preserve its coloring. Since we have left the coloring implicit, we use the notation $I^#$ to refer to the underlying uncolored set.

Let $(S, \otimes)$ be a symmetric monoidal category. We can speak of categories enriched over $S$. In particular, suppose $O$ is a category enriched over $S$ and suppose the objects of $O$ are finite $K$–colored sets. We let $O(I; J)$ denote the $S$ object of morphisms in $O$ from $I$ to $J$. If we further suppose that disjoint union of finite $K$–colored sets extends to an $S$–enriched symmetric monoidal structure on $O$, then each $O(I; J)$ is a right aut$(I)$– and left aut$(J)$–module in a natural way. In addition, if $S$ contains finite coproducts, the symmetric monoidal structure on morphisms in $O$ is specified by equivariant maps

\[
\bigotimes_{f: I^\# \to J^\#} O(f^{-1}\{j\}; \{j\}) \to O(I; J).
\]

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The following definition follows Boardman and Vogt [6].

### 2.1.1 Definition
Let \( (S, \otimes) \) be a symmetric monoidal category containing all finite coproducts where \( \otimes \) distributes over finite coproducts. Let \( K \) be a set. The data of a \( K \)-colored operad \( O \) in the symmetric monoidal category \( S \) is a symmetric monoidal category, denoted by \( \mathcal{O} \), which is enriched over \( S \) and whose objects are \( K \)-colored finite sets \( I \to K \). This data must satisfy two conditions. First, on objects, the symmetric monoidal structure of \( \mathcal{O} \) is the disjoint union of sets over \( K \). Second, the map in (1) must be an isomorphism for every \( I \) and \( J \).

### 2.1.2 Remark
We will use the unmodified noun operad to mean \( K \)-colored operad when the coloring set \( K \) is clear from context. The reader should note that this differs from an equally plausible convention, where operad is always used to denote \( \{*\} \)-colored operads.

### 2.1.3 Notation
Let \( n \) denote the finite set \( \{1, \ldots, n\} \). Typically \( K \) will be \( K = \{f, h\} \), where \( f \) stands for full-disk and \( h \) stands for half-disk. In this case we use \( (n, m) \) to denote the \( K \)-colored set where \( \{1, \ldots, n\} \) has color \( f \) and \( \{n+1, \ldots, m\} \) has color \( h \). If \( K \simeq \{*, \} \), then a \( K \)-colored operad will simply be called a 1-colored operad. Any \( \{f, h\} \)-colored operad \( O \) gives, for each \( n, m, n' \in \mathbb{Z}_{\geq 0} \), objects \( O(n', m'; n, m) := \mathcal{O}((n', m'); (n, m)) \in S \). We denote \( \mathcal{O}(n'; m) \) simply by \( \mathcal{O}(m'; m) \) and \( \mathcal{O}(m; 1) \) simply by \( \mathcal{O}(m) \).

### 2.1.4 Definition
Let \( \mathcal{O}(\{f, h\}; (n, m)) \) denote the symmetric monoidal category of compactly generated Hausdorff topological spaces with the cartesian product.

### 2.1.5 Example
Let \( K \) be the one-point set \( \{f\} \) and let \( (S, \otimes) = (\text{Top}, \times) \). The operad \( E_d \) is an \( \{f\} \)-colored operad in the category \( \text{Top} \). Let \( \overline{D}^d \) be the closed unit disk inside \( \mathbb{R}^d \). Call a map \( f : \overline{D}^d \to \overline{D}^d \) a little full-disk (or little \( d \) disk or simply little disk) if \( f \) is of the form \( f(x) = r x + c \) for some \( 0 < r \leq 1 \) and \( c \in \mathbb{R}^d \). Given a finite set \( I \), the underlying set of \( E_d(I) := E_d^f(I) \) is the set of embeddings

\[
f : \bigsqcup_{i \in I} \overline{D}^d \to \overline{D}^d,
\]

where each restriction \( f_i : \overline{D}^d \to \overline{D}^d \) is a little full-disk. Using Notation 2.1.3, any isomorphism \( I \to \mathbb{N} \) allows us to identify \( E_d(I) \) with a subset of \( \mathbb{R}^{n+d} \). This gives...
each $E_d(I)$ a topology. The operadic structure is given by composing little $d$ disks as maps $\overline{D}^d \to \overline{D}^d$. The identity of $E_d$ is the little full-disk id: $\overline{D}^d \to \overline{D}^d$. This is the unital version of $E_d$, so $E_d(0) = \ast$ and $E_d(1)$ consists of more than just the identity.

2.1.6 Example Let $K = \{f, h\}$. The $K$–colored operad $SC_d$ is called the (d–dimensional) Swiss cheese operad and is the principal subject of this paper. Like Example 2.1.5, it is an operad in $(\text{Top}, \times)$.

By Definition 2.1.1 and formula (1), to define $SC_d$ as a $K$–colored operad we only need to define the spaces $SC^f_d(I)$ and $SC^h_d(I)$ for every $K$–colored set $(I, \text{col}: I \to K)$. First, we define the “full-disk output” part of $SC_d$,

$$SC^f_d(I) = \begin{cases} E_d(I) & \text{if } \text{col}^{-1}(f) = I, \\ \emptyset & \text{else.} \end{cases}$$

To define the “half-disk output” part of $SC_d$, that is $SC^h_d(I)$, we first need the notion of little half-disks. Let $\overline{D}^d_+$ be the closed $d$–dimensional half-disk,

$$\overline{D}^d_+ = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid |x| \leq 1 \text{ and } x_d \geq 0\}.$$

A little half-disk is defined to be a map $f: \overline{D}^d_+ \to \overline{D}^d_+$ of the form $f(x) = rx + c$ for some $0 < r \leq 1$ and $c \in \mathbb{R}^{d-1} \times \{0\}$. As a set, we define $SC^h_d(I)$ to consist of embeddings

$$f: \coprod_{i \in I} \overline{D}^d \to \overline{D}^d_+,$$

where each restriction $f_i: \overline{D}^d \to \overline{D}^d_+$ is a little full-disk (Example 2.1.5) if $\text{col}(i) = f$ or a little half-disk if $\text{col}(i) = h$. It is clear that if $|\text{col}^{-1}(f)| = n$ and $|\text{col}^{-1}(h)| = m$, then $SC^h_d(I)$ can be naturally embedded inside $\mathbb{R}^N$, where $N = (d + 1)n + dm$. We give $SC^h_d(I)$ the subspace topology inherited from such an embedding.

Following Notation 2.1.3, a point in $SC^h_d(n, m)$ is given by $n$ labeled full-disks and $m$ labeled half-disks in the unit half-disk, where none of the disks intersect and the half-disks all lie on the bottom. We allow the degenerate configuration when $(n, m) = (0, 1)$, which is the unit half-disk contained in itself. Note that we have $SC^h_d(0, 0) = \ast$ and $SC^h_d(1, 0)$ contains more than one point. Thus we are using the unital Swiss cheese operad. This differs from [16; 25].

Composition in $SC_d$ is given by substituting full-disks and half-disks into each other. More precisely, we have maps

$$(2) \quad E_d(n) \times E_d(k_1) \times \cdots \times E_d(k_n) \to E_d(k_1 + \cdots + k_n)$$
and

\[ SC^h_d(n, m) \times E_d(k_1) \times \cdots \times E_d(k_n) \times SC^h_d(k_{n+1}, \ell_1) \times \cdots \times SC^h_d(k_{n+m}, \ell_m) \rightarrow SC^h_d(k_1 + \cdots + k_{n+m}, \ell_1 + \cdots + \ell_m). \]

Notice that we can identify \( SC^h_d(0, m) \) with \( E_d(1, m) \) so that the restriction of \( SC_d \) to the spaces \( SC^h_d(0, \bullet) \) is isomorphic to the operad \( E_d-1 \). We say that \( E_{d-1} \) is the \( h \) color of \( SC_d \) and \( E_d \) is the \( f \) color of \( SC_d \). We think of \( SC_d \) as interpolating between \( E_d \) and \( E_d-1 \).

2.1.7 Definition Suppose \( O \) is a \( K \)–colored operad in \( S \) and \( C \) is a symmetric monoidal category enriched over \( S \). An algebra over \( O \) in the category \( C \) is a strong symmetric monoidal functor \( O \rightarrow C \). A morphism of \( O \) algebras is a monoidal natural transformation. The category of \( O \) algebras in \( C \) will be denoted by \( \text{Alg}_O(C) \).

2.1.8 Example If \( C \) is a symmetric monoidal category enriched over \( \text{Top} \), we can consider algebras over \( SC_d \) in \( C \). Such an algebra gives the data of a pair \((B, A)\) of objects in \( C \) together with maps of topological spaces

\[ E_d(n) \rightarrow \text{map}_C(B^{\otimes n}, B) \]

and

\[ SC^h_d(n, m) \rightarrow \text{map}_C(B^{\otimes n} \otimes A^{\otimes m}, A), \]

where \( \text{map}_C(C, C') \) is the topological space of maps between two objects \( C \) and \( C' \) in \( C \). These data must satisfy conditions guaranteeing they assemble into a strong symmetric monoidal functor \( SC_d \rightarrow C \).

The object \( B \) corresponds to the object \( \{f\} \) of \( SC_d \) and the object \( A \) corresponds to \( \{h\} \). Together these form a \( K \)–colored operad \( \text{End}(B, A) \) in \( \text{Top} \), where, using Notation 2.1.3,

\[ \text{End}^f(B, A)(n, m) = \text{map}(B^{\otimes n} \otimes A^{\otimes m}, B), \]

\[ \text{End}^h(B, A)(n, m) = \text{map}(B^{\otimes n} \otimes A^{\otimes m}, A). \]

The definition of the operad morphisms on \( \text{End}(B, A) \) are left to the reader. The collection of \( SC_d \) algebra structures on a fixed pair \((B, A)\) is the collection of strong symmetric monoidal functors

\[ SC_d \rightarrow \text{End}(B, A), \]

which restrict to the identity on the set of objects. Simply put, a Swiss cheese algebra \((B, A)\) is an \( E_d \) algebra \( B \), an \( E_{d-1} \) algebra \( A \) and some chosen mixing of these structures. We refer to this mixing as an action of \( B \) on \( A \).
2.1.9 Definition (Kontsevich [16]) Let $B$ be an $E_d$ algebra and $A$ an $E_{d-1}$ algebra. A Swiss cheese action of $B$ on $A$ is the structure of a Swiss cheese algebra on the pair $(B, A)$ extending the given $E_d$ and $E_{d-1}$ structures. We may also simply call this “an action of $B$ on $A$”.

2.2 Statement of main theorem

Now we can informally state the conjecture proven in this paper.

Informal statement of Swiss cheese conjecture The Hochschild cochain object $\text{Hoch}(A)$ of an $E_{d-1}$ algebra $A$ is the universal $E_d$ algebra acting on $A$. In other words, for any $E_{d-1}$ algebra $A$, there is an $E_d$ algebra structure on $\text{Hoch}(A)$ such that, for any $E_d$ algebra $B$, giving a map of $E_d$ algebras $B \rightarrow \text{Hoch}(A)$ is equivalent to giving the structure of an $SC_d$ algebra on the pair $(B, A)$ extending the given $E_d$ and $E_{d-1}$ structures.

The basic structure of the proof is outlined in this diagram:

\[
\begin{align*}
\text{Op}_K & \quad \rightarrow \quad \text{Op}(\text{Top}^\Sigma) & \rightarrow & \quad \text{Op}(\text{Top}^{\Sigma \leq 1}) \\
SC_d & \quad \rightarrow \quad SC_d^h & \rightarrow & \quad SC_d^{h1} \\
SC_d^{h\infty} \times E & \quad \leftrightarrow \quad SC_d^{h\infty} & \leftrightarrow & \quad SC_d^{h1}
\end{align*}
\]

The categories of operads shown in the diagram are defined precisely in Section 2.3. The category $\text{Top}^\Sigma$ consists of symmetric sequences of topological spaces. Justification will be given later, but for now think of $\text{Op}(\text{Top}^\Sigma)$ as consisting of $K$–colored operads whose $f$–colored output is trivial. Further, we will introduce a category $\text{Op}(\text{Top}^{\Sigma \leq 1})$, consisting of $K$–colored operads whose $f$–colored inputs total 0 or 1. The forgetful functors are presented below as straight arrows; there is a left adjoint shown as a bent arrow.

Boardman and Vogt’s $W$ construction [6] is an explicit cofibrant replacement functor, which we apply to $SC_d$, to get an equivalent cofibrant operad $SC_d := WSC_d$. The $W$ construction does not strictly commute with the forgetful functors in (6). In particular, $W(SC_d^h)$ is not isomorphic to $(WSC_d)^h$, but they are homotopy equivalent. We will let $SC_d^h$ denote $W(SC_d^h)$ and $SC_d^{h1} = W(SC_d^{h1})$.

Proposition 3.2.7 shows that we can use $SC_d^{h1}$ to construct a model for the Hochschild cochain object. This allows us to prove a weak version of the Swiss cheese theorem in Proposition 4.1.2, taking place in the context of $\text{Op}(\text{Top}^{\Sigma \leq 1})$. Next, we will take the free extension of $SC_d^{h1} \in \text{Op}(\text{Top}^{\Sigma \leq 1})$ to an operad in $\text{Op}(\text{Top}^\Sigma)$, to get $SC_d^{h\infty}$. This
immediately gives a version of the Swiss cheese theorem in the context of \( \text{Op}(\text{Top}^\Sigma) \); see Corollary 4.1.4. Then we use the fact that \( \text{Sc}^{\text{h}\infty}_d \) is freely generated by its degree 0 and 1 pieces to prove a version of the Swiss cheese theorem in the context of \( \text{Op}_K \). None of these three versions of the Swiss cheese theorem make any use of \( E_d \). One can think of this last “universal cheese” theorem (Proposition 4.1.9) as a construction of the universal \( K \)-colored operad built from \( E_{d-1} \) and controlling \( E_{d-1} \)-linear actions on \( A \).

To bring \( E_d \) back into the story, we use a technical result, Theorem 5.1.1, which shows that the canonical map \( \text{Sc}^{\text{h}\infty}_d \to \text{Sc}^h_d \) is an equivalence. Observe that one can view \( \text{Sc}_d \) as \( \text{Sc}^h_d \) equipped with the extra structure of a right action of \( E_d \). Now use a transfer of structure argument to construct an operad \( E \) which is equivalent to \( E_d \) and which acts on the right on \( \text{Sc}^{\text{h}\infty}_d \). This allows us to define \( \text{Sc}^{\text{h}\infty}_d \times E \), which we show is equivalent to \( \text{Sc}_d \) in Lemma 5.1.6. The universal property of \( \text{Sc}^{\text{h}\infty}_d \times E \) with respect to the Hochschild cochain object is stated in Theorem 2.2.1 and follows from Proposition 4.1.9.

It should be noted that the theorem rests on the equivalence between \( \text{Sc}^{\text{h}\infty}_d \) and \( \text{Sc}^h_d \). At first glance, these spaces seem very different. The former has its \( d \)-dimensional disks very spread out. The latter allows its \( d \)-dimensional disks to be arbitrarily close together. Each element of \( \text{Sc}^{\text{h}\infty}_d \) can be broken into a tree where each piece has at most one \( d \)-dimensional disk. Thus, the only interesting topology on \( \text{Sc}^{\text{h}\infty}_d \), aside from the topology of \( E_{d-1} \), is given by the combinatorics of these trees. The surprising fact is that these combinatorics are rich enough to account for the topology of configuration spaces of \( d \)-dimensional disks. The best summary explanation for this equivalence was revealed to me in a conversation with Ralph Kaufmann, where he pointed out that \( \text{Sc}^{\text{h}\infty}_d \) resembles a kind of cellular structure relative to \( E_{d-1} \) on \( \text{Sc}^h_d \).

\[ \textbf{2.2.1 Theorem} \quad \text{(Precise version of Kontsevich’s Swiss cheese conjecture)} \quad \text{Let} \ C \text{ be a symmetric monoidal model category tensored over} \ \text{Top} \ \text{and satisfying the conditions in Notation 3.1.9. Let} \ A \in \text{Alg}_{E_{d-1}}(C) \ \text{be cofibrant and fibrant using the projective model structure (Definition 3.1.8). There is a model of} \ \text{Sc}_d, \ \text{called} \ \text{Sc}^{\text{h}\infty}_d \times E, \ \text{where} \ E \simeq E_d. \ \text{There is also a model for the Hochschild cochain object of} \ A, \ \text{called} \ \text{Hoch}(A), \ \text{such that} \ \text{Hoch}(A) \ \text{is the universal} \ E \ \text{algebra acting on} \ A \ \text{through} \ \text{Sc}^{\text{h}\infty}_d \times E. \ \text{That is,} \ \text{Hoch}(A) \ \text{is an} \ E \ \text{algebra and this structure, together with the} E_{d-1} \ \text{algebra structure on} \ A, \ \text{can be extended to an} \ \text{Sc}^{\text{h}\infty}_d \times E \ \text{algebra structure on} \ (\text{Hoch}(A), A) \ \text{in such a way that there is an equivalence of relative categories} \]

\[ \text{Alg}_{E_{d-1}}^A(C) \cong \text{Alg}_E(C)_{/\text{Hoch}(A)}. \]
The category on the left consists of $E$ algebras $B$ together with an action of $B$ on $A$. The category on the right consists of $E$ algebras $B$ together with an $E$ algebra map $B \to \text{Hoch}(A)$.

Lemma 5.1.6 shows that $SC_{d}^{\text{h}\infty} \times E$ is equivalent to $SC_{d}$. Lemma 5.1.5 shows that $E$ is equivalent to $E_{d}$. Proposition 3.2.7 sets up our choice of model of $\text{Hoch}(A)$. Proposition 4.1.9, together with the construction of the operad $SC_{d}^{\text{h}\infty} \times E$ in Section 5, shows the desired equivalence of relative categories.

2.3 Defining $SC_{d}^{h}$ and $SC_{d}^{h1}$

Recall that $C$ is a symmetric monoidal category enriched over $S$, our basic category in which our operads live. We will assume that both $C$ and $S$ have all coproducts and that tensor products distribute over finite coproducts. In the case of operads from $\text{Op}_{K}$ we have $(S, \otimes) = (\text{Top}, \times)$, the symmetric monoidal category of compactly generated topological spaces with cartesian product. In the case of operads such as $SC_{d}^{h}$ and $SC_{d}^{h0}$ from $\text{Op}(\text{Top}^{\Sigma})$, we have $(S, \otimes) = (\text{Top}^{\Sigma}, \otimes)$ from Definition 2.3.1. Finally, for $SC_{d}^{h1}$ and $SC_{d}^{h1}$ we use $S = \text{Top}^{\Sigma_{\leq 1}}$ as in Definition 2.3.2.

2.3.1 Definition Let $\Sigma$ denote the category of finite sets with morphisms given by bijections and let $(D, \otimes_{D})$ be any symmetric monoidal category. The category of functors $\Sigma^{\text{op}} \to D$, denoted by $D^{\Sigma}$, is usually called the category of symmetric sequences in $D$. Such a functor is a sequence $(X(n))_{n \geq 0}$ of objects in $D$ together with a right aut $n$ action on $X(n)$ for each $n \geq 0$. We endow $D^{\Sigma}$ with the usual symmetric monoidal structure given by left Kan extension of $\Sigma \times \Sigma \to D \times D \to D$ along disjoint unions of sets $\Sigma \times \Sigma \to \Sigma$. Specifically, if $X, Y \in D^{\Sigma}$, then $X \otimes Y \in D^{\Sigma}$ satisfies

$$(X \otimes Y)(\eta) = \bigsqcup_{n_{1} + n_{2} = n} \text{Ind}_{S_{n_{1}} \times S_{n_{2}}}^{S_{n}} (X(\eta_{1}) \otimes_{D} Y(\eta_{2})).$$

where $S_{n} = \text{aut}_{n}$. See Harper [12, Definition 3.3] for more details.

2.3.2 Definition Let $\Sigma_{\leq 1}$ denote the full subcategory of $\Sigma$ consisting of finite sets of size 0 or 1 together with bijections as morphisms. Let $D^{\Sigma_{\leq 1}}$ denote the symmetric monoidal category of functors $\Sigma_{\leq 1} \to D$, with monoidal structure inherited from $D^{\Sigma}$. Call these the degree 0–1 symmetric sequences in $D$. Concretely, $D^{\Sigma_{\leq 1}}$ is just the category $D \times D$ endowed with the symmetric monoidal structure

$$(7) \quad (C_{0}, C_{1}) \otimes (D_{0}, D_{1}) = (C_{0} \otimes_{D} D_{0}, (C_{0} \otimes_{D} D_{1}) \sqcup (C_{1} \otimes_{D} D_{0})).$$

The braiding isomorphism $(C_{0}, C_{1}) \otimes (D_{0}, D_{1}) \to (D_{0}, D_{1}) \otimes (C_{0}, C_{1})$ is induced from the braiding isomorphism on $D$. 

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2.3.3 Definition  Let $\text{Op}_K$ denote the category of $K = \{f, h\}$–colored operads in $\text{Top}$. There is a forgetful functor $\text{Op}_K \rightarrow \text{Op}(\text{Top}^\Sigma)$. This functor takes $\mathcal{O} \in \text{Op}_K$ to the operad $\mathcal{O}^h$ whose arity-$m$ component is the symmetric sequence $n \mapsto \mathcal{O}^h(n, m)$ (see Notation 2.1.3). We forget the spaces $\mathcal{O}^f(n, m)$ and think of elements of $\mathcal{O}^h(n, m)$ as degree-$n$, arity-$m$ elements of $\mathcal{O}^h$.

The functor $\text{Op}(\text{Top}^\Sigma) \rightarrow \text{Op}(\text{Top}^\Sigma_{\leq 1})$ is induced by the symmetric monoidal forgetful functor $\text{Top}^\Sigma \rightarrow \text{Top}^\Sigma_{\leq 1}$. Write the image of $\mathcal{O}$ in $\text{Op}(\text{Top}^\Sigma_{\leq 1})$ as $\mathcal{O}^h$. Furthermore, the functor $\text{Op}(\text{Top}^\Sigma_{\leq 1}) \rightarrow \text{Op}(\text{Top})$ is induced by the symmetric monoidal functor $\text{Top}^\Sigma_{\leq 1} \rightarrow \text{Top}$, $(X, Y) \mapsto X$. We denote this functor by $\mathcal{O} \mapsto \mathcal{O}^0$.

Observe that if $\mathcal{O} \in \text{Op}(\text{Top}^\Sigma_{\leq 1})$ then $\mathcal{O}^0$ has commuting left and right actions on $\mathcal{O}$. By this we mean we have maps

$\mathcal{O}(0, m') \times \mathcal{O}(n, m'') \times \mathcal{O}(0, m'; 0, m') \rightarrow \mathcal{O}(n, m)$,

satisfying straightforward conditions which we leave to the reader.

2.3.4 Example  We outline the structure of $\text{SC}^h_d$ as an operad in $\text{Op}(\text{Top}^\Sigma)$. Think of $\text{SC}^h_d(\bullet, m)$ as the symmetric sequence $n \mapsto \text{SC}^h_d(n, m)$. In (8), $\otimes$ is the tensor product of symmetric sequences. The operad composition law is

\[
\text{SC}^h_d(\bullet, m) \otimes \text{SC}^h_d(\bullet, \ell_1) \otimes \cdots \otimes \text{SC}^h_d(\bullet, \ell_m) \rightarrow \text{SC}^h_d(\bullet, \ell),
\]

where $\ell = \sum_i \ell_i$. The degree-$n$ component of the right-hand symmetric sequence is $\text{SC}^h_d(n, \ell)$. The degree-$n$ component of the left-hand symmetric sequence is

\[
\bigoplus_{n_0 + \cdots + n_m = n} \text{Ind} \text{SC}^h_d(n_0, m) \times \text{SC}^h_d(n_1, \ell_1) \times \cdots \times \text{SC}^h_d(n_m, \ell_m),
\]

where $\text{Ind}$ is the induction functor giving the correct symmetric group action. The point is that if we delete all appearances of $E_d$ from (3) then it provides exactly the data of (8).

2.3.5 Example  We outline the structure of $\text{SC}^{h1}_d$ as a $1$–colored operad in $\text{Top}^\Sigma_{\leq 1}$. For each $m'$ and $m$, define an object $\text{SC}^{h1}_d(m'; m) \in \text{Top}^\Sigma_{\leq 1}$,

\[
\text{SC}^{h1}_d(m'; m) = (\text{SC}^h_d(0, m'; 0, m), \text{SC}^h_d(1, m'; 0, m))
\]

\[
\cong (E_{d-1}(m'; m), \text{SC}^h_d(1, m'; 0, m)).
\]

Using the symmetric monoidal structure from Definition 2.3.2 and the identification in (9), we can write the operad structure maps on $\text{SC}^{h1}_d$ as a triple of morphisms: the map of degree-0 pieces

$E_{d-1}(m''; m) \times E_{d-1}(m'; m'') \rightarrow E_{d-1}(m'; m)$.
and the maps of degree-1 pieces
\[ E_{d-1}(m''; m) \times SC^h_d(1, m'; 0, m'') \to SC^h_d(1, m'; 0, m) \]
and
\[ SC^h_d(1, m''; 0, m) \times E_{d-1}(m'; m'') \to SC^h_d(1, m'; 0, m). \]

3 Hochschild cohomology from Swiss cheese

For the remainder of the paper we replace \( E_{d-1}, E_d \) and \( SC_d \) by cofibrant models given by the Boardman–Vogt \( W \) construction [6]. A proof that this gives a cofibrant replacement for certain operads can be found in Berger and Moerdijk [4]. Recall that we will denote these cofibrant replacements by \( E_{d-1}, E_d \) and \( SC_d \). We also want to restrict our attention to Swiss cheese algebras in categories where we can do homotopy theory. In the proper context the Hochschild cochain object of an \( E_{d-1} \) algebra \( A \) has a natural model constructed from \( A \) and the degree 0–1 parts of \( SC^h_d \). This is the content of Proposition 3.2.7.

3.1 Homotopy-theoretic context

3.1.1 Definition (Hovey [13, Definition 4.2.6]) A symmetric monoidal model category \( S \) is a closed symmetric monoidal category whose monoidal structure \( \otimes: S \times S \to S \) is a Quillen bifunctor and where the cofibrant replacement \( Q1 \to 1 \) of the monoidal unit induces weak equivalences \( Q1 \otimes X \to X \) for every cofibrant \( X \).

3.1.2 Example The category \((\text{Top}, \times)\) of compactly generated spaces with the cartesian product and Serre model structure is a symmetric monoidal model category.

3.1.3 Definition Let \( S \) be a symmetric monoidal model category. A symmetric monoidal model category tensored over \( S \) is a closed symmetric monoidal model category \( C \), together with a symmetric monoidal Quillen functor \( S \to C \). For more details see [13, Definition 4.2.20].

In particular, \( C \) comes equipped with functors
\[ \otimes: C \times C \to C, \quad \otimes: S \times C \to C, \]
\[ \hom_C: C^{\text{op}} \times C \to C, \quad \map_C: C^{\text{op}} \times C \to S. \]

The mapping spaces \( \map_C(A, B) \) give \( C \) the structure of a category enriched over \( S \), so we can speak of \( E_{d-1}, E_d \) and \( SC_d \) algebras in \( C \).

For any object \( A \) of \( C \), the functor \(- \otimes A\) has right adjoints \( \hom_C(A, -): C \to C \) and \( \map_C(A, -): C \to S \). This data satisfies Quillen’s SM7 axiom [13, Section 4.2].
3.1.4 Example  The category $C^{\Sigma \leq 1}$ from Definition 2.3.2 is tensored over the symmetric monoidal category $\text{Top}^{\Sigma \leq 1}$ with

$$\text{Top}^{\Sigma \leq 1} \otimes C^{\Sigma \leq 1} \to C^{\Sigma \leq 1}$$

given by the analogue of (7),

$$(X_0, X_1) \otimes (C_0, C_1) = (X_0 \otimes C_0, X_0 \otimes C_1 \boxtimes X_1 \otimes C_0),$$

where $X_i \in \text{Top}$ and $C_i \in C$.

3.1.5 Example  The symmetric monoidal functor $\text{Top} \to \text{Top}^{\Sigma \leq 1}$ sending $X$ to $(X, \emptyset)$ makes both $\text{Top}^{\Sigma \leq 1}$ and $C^{\Sigma \leq 1}$ into symmetric monoidal model categories tensored over $\text{Top}$. Note that for a topological 1–colored operad $\mathcal{O}$, we can consider algebras over $\mathcal{O}$ in $C$ as well as algebras over $\mathcal{O}$ in $C^{\Sigma \leq 1}$.

The category of degree 0–1 symmetric sequences is naturally home to $\mathcal{O}$–algebras $A$ and $\mathcal{O}$–$A$–modules $M$.

3.1.6 Definition  Suppose $\mathcal{O}$ is a 1–colored operad in $\text{Top}$ and let $(A, M)$ be an object of $C^{\Sigma \leq 1}$. The structure of an $\mathcal{O}$ algebra on the degree 0–1 symmetric sequence $(A, M)$ is the structure of an $\mathcal{O}$ algebra on $A$ together with the data of maps

$$\mathcal{O}(m) \otimes M \otimes A^\otimes m^{-1} \to M,$$

satisfying certain conditions (see diagram (16)). We call this data the structure of an $\mathcal{O}$–$A$–module on $M$. Given a fixed $\mathcal{O}$ algebra $A \in C$, the category $\text{Mod}^A_{\mathcal{O}}(C)$ of $\mathcal{O}$–$A$–modules has objects $M \in C$, together with the structure of an $\mathcal{O}$ algebra on $(A, M) \in C^{\Sigma \leq 1}$ extending the given $\mathcal{O}$ algebra structure on $A$.

The author was shown the following quick way to see that the category of $\mathcal{O}$–$A$–modules is naturally enriched over $\text{Top}$. First, one can show that this category is equivalent to the category of algebras (classical associative algebras, not $E_1$) over the universal enveloping monoid of $A$. Then this implies that the free $A$–module monad is a strong and enriched monad, from which it follows that $\text{Mod}^A_{\mathcal{O}}(C)$ is enriched.

3.1.7 Example  Let $\mathcal{O}$ be a 1–colored topological operad and let $A \in C$ be an $\mathcal{O}$ algebra; then the degree 0–1 symmetric sequence $(A, A)$ is naturally an $\mathcal{O}$ algebra. That is, $A$ is naturally an $\mathcal{O}$–$A$–module.

3.1.8 Definition  Let $\mathcal{O}$ be an operad in $\text{Top}^{\Sigma \leq 1}$ and let $\mathcal{P}$ be the image of $\mathcal{O}$ under the functor $\text{Op}(\text{Top}^{\Sigma \leq 1}) \to \text{Op}(\text{Top})$ (for example, if $\mathcal{O}$ is $SC_{d+1}$ then $\mathcal{P}$ is $E_{d-1}$). Since $C^{\Sigma \leq 1}$ is enriched over $\text{Top}^{\Sigma \leq 1}$ by Example 3.1.4, we can consider $\mathcal{O}$ algebras
in \( C_{\Sigma}^{\leq 1} \). In addition, by Example 3.1.5 we can consider \( \mathcal{P} \) algebras in \( C_{\Sigma}^{\leq 1} \). There are adjunctions

\[
\mathcal{C} \leftrightarrows \text{Alg}_{\mathcal{P}}(\mathcal{C}), \quad \mathcal{C} \leftrightarrows \text{Mod}^A_{\mathcal{P}}(\mathcal{C}) \quad \text{and} \quad \mathcal{C} \times \mathcal{C} \leftrightarrows \text{Alg}_{\mathcal{P}}(C_{\Sigma}^{\leq 1}) \leftrightarrows \text{Alg}_{O}(C_{\Sigma}^{\leq 1}).
\]

We describe the right adjoints only. On the top left the \( \mathcal{P} \) algebra \( A \) is sent to the underlying object \( A \) of \( \mathcal{C} \). The top right functor sends the \( \mathcal{P} \)–\( A \)–module \( M \) to the underlying object \( M \) of \( \mathcal{C} \). In the pair of composable adjunctions, a \( \mathcal{P} \) algebra \( (A, M) \) can be considered as a \( \mathcal{P} \) algebra by forgetting the structure maps in (13). For the final adjunction, any \( \mathcal{P} \) algebra in \( C_{\Sigma}^{\leq 1} \) has an underlying pair of \( \mathcal{C} \)–objects \( (A, M) \).

3.1.9 Notation  Throughout the remainder of this paper \( C \) will be a cofibrantly generated symmetric monoidal model category tensored over \( \text{Top} \) such that the adjunctions in Definition 3.1.8 are Quillen adjunctions.

3.1.10 Remark  Spitzweck [21, Theorem 6] shows that \( \text{Mod}^A_{\mathcal{P}}(\mathcal{C}) \) is a Quillen adjunction if \( A \) is cofibrant in \( \mathcal{C} \). In addition, Berger and Moerdijk [3, Proposition 4.1] show that, if \( \mathcal{D} \) is a symmetric monoidal model category which is cofibrantly generated and has a cofibrant unit and a symmetric monoidal fibrant replacement functor, then the category \( \text{Alg}_{O}(\mathcal{D}) \) has the projective model structure induced from the forgetful functor to \( \mathcal{D} \). The operad \( O \) in this theorem is an operad in \( \mathcal{D} \). However, their result is more general, as seen in [3, Remark 4.6.4]. Their argument extends without change to show that \( \text{Alg}_{O}(\mathcal{D}) \) has the desired model structure in the case that \( O \) is an operad in \( \text{Top} \) and \( \mathcal{D} \) is tensored over \( \text{Top} \) as in Definition 3.1.1, and the generating trivial cofibrations of \( \mathcal{D} \) have cofibrant domains. The condition that the monoidal unit of \( \mathcal{D} \) is cofibrant is not necessary in this situation. The cofibrance of the monoidal unit in \( \text{Top} \) is enough. Taking \( O = E_{d-1} \) or \( O = SC^h_d \) and \( \mathcal{D} = \mathcal{C} \) or \( \mathcal{D} = C_{\Sigma}^{\leq 1} \), we conclude that the assumptions in Notation 3.1.9, and in the main theorem of this paper, hold when \( \mathcal{C} \) is the category of compactly generated Hausdorff spaces (\( \text{Top} \)), or spectra, or chain complexes.

We use the model structure on the category of \( \mathcal{P} \)–\( A \)–modules to define the Hochschild cochain object.

3.1.11 Definition  Given a 1–colored topological operad \( \mathcal{P} \) and a \( \mathcal{P} \) algebra \( A \in \mathcal{C} \), let the Hochschild cochain object of \( A \) be

\[
\text{Hoch}(A) = \text{hom}^A_{\mathcal{P}}(A^c, A^f),
\]

where \( \text{hom}^A_{\mathcal{P}} \) is given by the equalizer in Definition 3.1.6. The \( \mathcal{P} \)–\( A \)–modules \( A^c \) and \( A^f \) are cofibrant and fibrant replacements for \( A \), respectively. Note that \( \text{Hoch}(A) \) is an object of \( \mathcal{C} \).
3.2 Swiss cheese in degrees zero and one

Let \( \mathcal{O} \) be a \((1\text{–colored})\) operad in \( \text{Top}^{\Sigma_{\leq 1}} \). A \((1\text{–colored})\) \( \mathcal{O} \) algebra (see Definition 3.1.8 and Example 2.3.5) is a pair \((A, M)\) of objects of \( C \) together with maps in \( C^{\Sigma_{\leq 1}} \), for every \( m \),

\[
(\mathcal{O}(0, m), \mathcal{O}(1, m)) \otimes (A, M)^{\otimes m} \to (A, M).
\]

Alternatively, we can view the morphism in (10) as three separate maps in \( C \):

\[
\begin{align*}
\mathcal{O}(0, m) & \otimes A^{\otimes m} \to A, \\
\mathcal{O}(0, m) & \otimes M \otimes A^{\otimes m-1} \to M, \\
\mathcal{O}(1, m) & \otimes A^{\otimes m} \to M.
\end{align*}
\]

The condition that the maps in (10) define an \( \mathcal{O} \) structure on the pair \((A, M)\) is the condition that the diagram (14) commutes in the \( \text{Top}^{\Sigma_{\leq 1}} \)-enriched category \( C^{\Sigma_{\leq 1}} \):

\[
\begin{array}{ccc}
\mathcal{O}(m') \otimes \mathcal{O}(m'; m') \otimes (A, M)^{\otimes m} & \to & \mathcal{O}(m) \otimes (A, M)^{\otimes m} \\
\mathcal{O}(m') \otimes (A, M)^{\otimes m'} & \to & (A, M)
\end{array}
\]

In terms of equations (11), (12) and (13), the \( C^{\Sigma_{\leq 1}} \) diagram (14) splits into four diagrams in \( C \). Each diagram is determined by the degrees of the three tensor factors in the upper left-hand corner of (14). In the first the degrees are 0, 0, 0; in the second the degrees are 0, 0, 1; in the third, 1, 0, 0; and in the fourth, 0, 1, 0.

The following diagram commutes for all \( m' \) and \( m \) if and only if \( A \) is an \( \mathcal{O}^0 \) algebra (see Definition 2.3.3):

\[
\begin{array}{ccc}
\mathcal{O}(0, m') \otimes \mathcal{O}(0, m; 0, m') \otimes A^{\otimes m} & \to & \mathcal{O}(0, m) \otimes A^{\otimes m} \\
\mathcal{O}(0, m') \otimes A^{\otimes m'} & \to & A
\end{array}
\]

Diagram (16) commutes if and only if \( M \) is an \( \mathcal{O}^0-A \)–module:

\[
\begin{array}{ccc}
\mathcal{O}(0, m') \otimes \mathcal{O}(0, m; 0, m') \otimes M \otimes A^{\otimes m-1} & \to & \mathcal{O}(0, m) \otimes M \otimes A^{\otimes m'-1} \\
\mathcal{O}(0, m) \otimes M \otimes A^{\otimes m-1} & \to & M
\end{array}
\]
Diagram (17) shows a compatibility condition between the degree 0 and degree 1 structures:

\[
\begin{array}{ccc}
\mathcal{O}(1, m') \otimes \mathcal{O}(0, m; 0, m') \otimes A^\otimes m & \longrightarrow & \mathcal{O}(1, m) \otimes A^\otimes m \\
\downarrow & & \downarrow \\
\mathcal{O}(1, m') \otimes A^\otimes m' & \longrightarrow & M
\end{array}
\]

Diagram (18) presents another compatibility condition between the degree 0 and degree 1 structures:

\[
\begin{array}{ccc}
\mathcal{O}(0, m') \otimes \mathcal{O}(1, m; 0, m') \otimes A^\otimes m & \longrightarrow & \mathcal{O}(1, m) \otimes A^\otimes m \\
\downarrow & & \downarrow \\
\mathcal{O}(0, m') \otimes M \otimes A^\otimes m' - 1 & \longrightarrow & M
\end{array}
\]

3.2.1 Example  In this example we construct the universal extension of an \( \mathcal{O}^0 \) algebra \( A \) to an \( \mathcal{O} \) algebra. We denote this universal pair by \( (A, A^o) \). The composite forgetful functor

\[ \text{Alg}_{\mathcal{O}}(C^{\Sigma_{\leq 1}}) \to \text{Alg}_{\mathcal{O}^0}(C^{\Sigma_{\leq 1}}) \to \text{Alg}_{\mathcal{O}^0}(C), \]

has a left adjoint, which sends the \( \mathcal{O}^0 \) algebra \( A \) to the pair \( (A, A^o) \), where \( A^o \) is a quotient of

\[ \tilde{A}^o = \bigsqcup_{m \geq 0} \mathcal{O}(1, m) \otimes S_m A^\otimes m. \]

We can think of \( \tilde{A}^o \) heuristically as \( \mathcal{O}(1, \bullet) \otimes A^\otimes \bullet \). Now, both \( A \) and \( \mathcal{O}(1, \bullet) \) carry an action of \( \mathcal{O}^0 \), so we can form the quotient \( A^o := \mathcal{O}(1, \bullet) \otimes_{\mathcal{O}^0} A^\otimes \bullet \). More precisely, \( A^o \) is defined as the coequalizer

\[ \bigsqcup_{m, m'} \mathcal{O}(1, m) \otimes \mathcal{O}(0, m'; 0, m) \otimes A^\otimes m' \Rightarrow \tilde{A}^o \xrightarrow{\text{coeq}} A^o, \]

where one of the arrows is given by the operadic composition on Swiss cheese and the other by the \( \mathcal{O}^0 \) structure on \( A \).

Verifying that \( (A, A^o) \) is a 1–colored \( \mathcal{O} \) algebra is a matter of using the commuting left and right actions of \( \mathcal{O}^0 \) on \( \mathcal{O} \). By this we mean the morphism below uses both left and right actions, and can be obtained by performing the left action first then the right, or vice versa,

\[ \mathcal{O}(0, m'') \times \mathcal{O}(1, m'; 0, m'') \times \mathcal{O}(0, m; 0, m) \to \mathcal{O}(1, m). \]
The left action defines a map
\begin{equation}
\mathcal{O}(0, m) \otimes \tilde{A}^o \otimes A^\otimes m \rightarrow \tilde{A}^o,
\end{equation}
using the \( \circ_1 \) operad composition. Since the left and right \( \mathcal{O}^0 \) actions on \( \mathcal{O} \) commute, the arrow in (21) descends to give the data of an \( \mathcal{O}^0 - A \)–module structure on \( A^o \), ie (12) for \( M = A^o \). Of course, the maps from (13) with \( M = A^o \) are simply given by \( \tilde{A}^o \rightarrow A^o \).

Now let us observe that the four diagrams (15)–(18) commute for \( (A, A^o) \). The first diagram, (15), is trivial since \( A \) is an \( \mathcal{O}^0 \) algebra. The second diagram, (16), commutes since the left action of \( \mathcal{O}^0 \) on \( \mathcal{O} \) is indeed an action. That is, it is compatible with composition in \( \mathcal{O}^0 \). The third diagram, (17), clearly commutes. Indeed, this diagram is the reason the coequalizer \( \tilde{A}^o \rightarrow A^o \) in (20) was defined in the first place. Finally, the fourth diagram, (18), certainly commutes if \( M \in Mod_{\mathcal{O}^0}(C) \). In this case, note that the bottom map in diagram (18) corresponds to (21). Thus, if we pass from \( \tilde{A}^o \) to \( A^o \) this last diagram still commutes since, by definition, the \( \mathcal{O}^0 - A \)–module structure on \( A^o \) is defined using the quotient map \( \tilde{A}^o \rightarrow A^o \) together with (21).

### 3.2.2 Lemma
Fix an \( \mathcal{O}^0 \) algebra \( A \) and consider \( A^o \) as an \( \mathcal{O}^0 - A \) module. There is an equivalence of relative categories
\[ Mod^A_{\mathcal{O}^0}(C) \cong Mod^A_{\mathcal{O}^0 \mathcal{O}^0}(C)_{A^o/}, \]
where \( Mod^A_{\mathcal{O}^0}(C) \) is the fiber over \( A \) of the forgetful functor \( \text{Alg}_{\mathcal{O}^0}(C \Sigma^{\leq 1}) \rightarrow \text{Alg}_{\mathcal{O}^0}(C) \).

**Proof** Let \( (A, M) \) be an \( SC^h_d \) algebra extending the existing \( \mathcal{O}^0 \) algebra structure on \( A \). Then \( M \in Mod^A_{\mathcal{O}^0}(C) \) and the structure maps in (13), when combined for all \( m \), give a map \( \tilde{A}^o \rightarrow M \). This descends to an \( \mathcal{O}^0 - A \)–module map \( A^o \rightarrow M \).

On the other hand, if \( M \) is an \( \mathcal{O} - A \)–module, then \( M \) is already equipped with the data of (12). If \( A^o \rightarrow M \) is a morphism of \( \mathcal{O} - A \)–modules, then \( M \) is equipped with the data of (13). The diagram (14) commutes because of the relation defining \( A^o \) in (20) and because \( A^o \rightarrow M \) is a morphism of \( \mathcal{O}^0 - A \)–modules.

### 3.2.3 Corollary
Take \( \mathcal{O} = SC^h_d \) and denote \( A^o \) by \( A^{sc} \). Let \( A \) be a cofibrant \( E_{d-1} \) algebra. Then \( A^{sc} \) is a cofibrant \( E_{d-1} - A \)–module.

**Proof** The forgetful functor \( \text{Alg}_{SC^h_d}(C) \rightarrow \text{Alg}_{E_{d-1}}(C) \) preserves fibrations (Definition 3.1.8). Thus the left adjoint of this forgetful functor, applied to the cofibrant object \( A \), gives a cofibrant \( SC^h_d \) algebra \( (A, A^{sc}) \). Thus \( A^{sc} \) is cofibrant in \( Mod^A_{SC^h_d}(C) \). Lemma 3.2.2 shows that the forgetful functor \( Mod^A_{SC^h_d}(C) \rightarrow Mod^A_{E_{d-1}}(C) \) preserves pushouts. The model structures here are cofibrantly generated, so \( A^{sc} \) is also cofibrant as an object of \( Mod^A_{E_{d-1}}(C) \).
3.2.4 Definition Let $p(m) : \text{SC}_d^h(1, m) \to \text{SC}_d^h(0, m) \simeq \text{E}_{d-1}(m)$ be the projection which forgets the single full-disk. Let $p$ be the morphism $(p(m))_{m \geq 0}$ of operads in $\text{Top}^\Sigma_{\leq 1}$. For each $m$ consider the degree $0$–1 symmetric sequence $(\text{E}_{d-1})_{\leq 1}^\Sigma(m) := (\text{E}_{d-1}(m), \text{E}_{d-1}(m))$. The structure of $\text{E}_{d-1}$ as an operad in $\text{Top}$ can be used to make $(\text{E}_{d-1})_{\leq 1}^\Sigma$ an operad in $\text{Top}^\Sigma_{\leq 1}$. This makes $(\text{id}, p): \text{SC}_d^h \to (\text{E}_{d-1})_{\leq 1}^\Sigma$ into a morphism of operads.

If $A$ is an $\text{E}_{d-1}$ algebra we can define a morphism in $C$,

$$\tilde{A}^{sc} = \bigoplus_{m \geq 0} \text{SC}_d^h(1, m) \otimes_{\mathbb{S}} A^{\otimes m} \to \bigoplus_{m \geq 0} \text{E}_{d-1}(m) \otimes_{\mathbb{S}} A^{\otimes m} \to A,$$

where the first arrow uses $p$ and the second arrow uses the $\text{E}_{d-1}$ algebra structure on $A$. This map factors to give a morphism of $\text{E}_{d-1}$–$A$–modules $p_A : A^{sc} \to A$.

3.2.5 Remark By [21, Section 5] we can conclude that $\text{Op}(\text{Top}^\Sigma_{\leq 1})$ is tensored over $\text{Top}$. If $O = (O^0, O^1)$ is an operad in degree $0$–1 symmetric sequences of topological spaces and $K \in \text{Top}$, then $K \otimes O$ is defined to be the coequalizer

$$F(F(K \otimes O)) \xrightarrow{\text{coeq}} F(K \otimes O),$$

where $K \otimes O$ is the symmetric sequence of degree $0$–1 symmetric sequences whose arity-$m$, degree-$i$ component is $K \times O^i(m)$, and $F$ is the free operad functor.

If $O$ is a cofibrant operad in $\text{Top}^\Sigma_{\leq 1}$, then $[0, 1] \otimes O$ is a cylinder object and a homotopy $h : [0, 1] \otimes O \to P$ gives the data of maps $h^i(m) : [0, 1] \times O^i(m) \to P^i(m)$, which assemble into $h^i(m') : [0, 1] \times O^i(m'; m) \to P(m'; m)$ for $i = 0, 1$ and $m', m \geq 0$. The homotopy $h$ is compatible with operad composition in the sense that, if $\alpha \in O^i(m)$ and $\beta \in O^j(m'; m)$, $i + j \leq 1$ and $t \in [0, 1]$, then

$$h^i(m)(t, \alpha) \circ h^j(m'; m)(t, \beta) = h^{i+j}(m'; m)(t, \alpha \circ \beta).$$

3.2.6 Lemma For any $\text{E}_{d-1}$ algebra $A \in C$, the map $A^{sc} \to A$ is a weak equivalence of $\text{E}_{d-1}$–$A$–modules.

Proof Abusing notation, we write $p := (\text{id}, p)$ for the morphism of operads from Definition 3.2.4. One can show that $p$ is a weak equivalence of fibrant and cofibrant operads in degree $0$–1 symmetric sequences of topological spaces. Therefore, there is a map of operads $i : (\text{E}_{d-1})_{\leq 1}^\Sigma \to \text{SC}_d^h$ and there are homotopies $h : \text{id}_{\text{SC}_d^h} \simeq ip$ and $g : \text{id}_{(\text{E}_{d-1})_{\leq 1}^\Sigma} \simeq pt$. Since $[0, 1] \otimes -$ distributes over coequalizers, we can use $h$ to
define a homotopy $h_A: [0, 1] \otimes A^{sc} \to A^{sc}$:

$$
\begin{array}{ccc}
\bigcup_m [0, 1] \otimes SC^h_d(1, m) \otimes A^\otimes m & \xrightarrow{\cup h^1(m) \otimes id_A^\otimes m} & \bigcup_m SC^h_d(1, m) \otimes A^\otimes m \\
\downarrow h_A & & \downarrow A^{sc} \\
\bigcup_m SC^h_d(1, m) \otimes A^\otimes m & \xrightarrow{s} & A^{sc},
\end{array}
$$

where $h^1(m)$ is defined from $h$ as in Remark 3.2.5. When $t = 1$, $h_A(1, -)$ factors as

$$
\begin{array}{ccc}
\bigcup_m SC^h_d(1, m) \otimes A^\otimes m & \xrightarrow{s} & A \\
\downarrow p_A & & \downarrow \iota_A \\
\bigcup_m SC^h_d(1, m) \otimes A^\otimes m & \xrightarrow{\iota_A} & A^{sc},
\end{array}
$$

where $p_A$ is the map from Definition 3.2.4 and $\iota_A$ is the evident composite in the diagram using $s$. The map $s$ is a section of the middle horizontal arrow, defined using the identity of the operad $E_{d-1}$,

$$
A \simeq \{1_{E_{d-1}}\} \otimes A \to E_{d-1}(1) \otimes A \to \bigcup_m E_{d-1}(m) \otimes A^\otimes m.
$$

The map $\iota_A$ is a map of $E_{d-1} - A$–modules and $h_A$ is a homotopy $id_{A^{sc}} \simeq \iota_A p_A$. Similarly, $g$ defines a homotopy $g_A: id_A \to p_A \iota_A$.

The precise sense in which Hochschild cohomology can be obtained from the degree 0–1 pieces of the Swiss cheese operad is contained in the following proposition:

**3.2.7 Proposition** Let $A$ be a fibrant and cofibrant $E_{d-1}$ algebra. Then the Hochschild cochain object of $A$ can be computed as

$$
\text{Hoch}(A) \simeq \text{hom}^A_{E_{d-1}}(A^{sc}, A).
$$

**Proof** We are using the projective model structure from Definition 3.1.8, so $A$ is fibrant as an object of $C$ and thus as an $E_{d-1} - A$–module. By Corollary 3.2.3 and Lemma 3.2.6, $A^{sc}$ is a cofibrant replacement for $A$ as an $E_{d-1} - A$–module. By Definition 3.1.11, this proves the proposition. 

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4 The universal cheese theorem

We now shift our attention to $K$–colored algebras over $\text{SC}_d^{h1}$. Take the $K$–colored endomorphism $\text{End}(B, A)$ defined in (4) and (5). Apply the functor in Definition 2.3.3 to this operad to get $\text{End}^{h1}(B, A) \in \text{Op}(\text{Top}^{\Sigma_{\leq 1}})$. With this endomorphism operad we have a separate notion of algebra over $\text{SC}_d^{h1}$. We call this notion of algebra a $K$–colored algebra over $\text{SC}_d^{h1}$. As a point of contrast, the former notion, detailed in Section 3.2, might be called a 1–colored algebra. Recall that a 1–colored $\text{SC}_d^{h1}$ algebra on $(X, Y)$ is equivalent to an $E_{d-1}$ structure on $X$, an $E_{d-1}$ $X$–structure on $Y$ and an $E_{d-1}$ $X$–module map $X^{sc} \to Y$. We will see in Proposition 4.1.2 that a $K$–colored algebra on $(B, A)$ is like a family of 1–colored algebra structures on $(A, A)$. More precisely, it is an $E_{d-1}$ algebra structure on $A$, together with any map $B \to \text{hom}_{E_{d-1}}(A^{sc}, A)$. In other words, $\text{Hoch}(A)$ is the universal object of $C$ acting on the $E_{d-1}$ algebra $A$ through $\text{SC}_d^{h1}$. This result is generalized twice, first in Corollary 4.1.4, then in Proposition 4.1.9. We refer to Proposition 4.1.9 as the universal cheese theorem, since it replaces $E_{d}$ in the Swiss cheese theorem with an arbitrary operad.

4.1.1 Definition Let $A$ be an $E_{d-1}$ algebra. Let $C_{/\text{Hoch}(A)}$ denote the over category of $\text{Hoch}(A) \in C$. More precisely, the objects are $C$–morphisms $B \to \text{Hoch}(A)$ and the morphisms are $C$–morphisms $B \to B'$ commuting with the maps to $\text{Hoch}(A)$. In addition, let $\text{Alg}_{\text{SC}_d^{h1}}^A(C)$ denote the category of $\text{SC}_d^{h1}$ algebras of the form $(B, A)$, where the induced $E_{d-1}$ structure on $A$ is the one given. Morphisms are maps of $\text{SC}_d^{h1}$ algebras which are the identity on $A$.

Given a pair of objects $(B, A)$ of $C$, we let $\text{End}^{h1}(B, A)$ denote the operad obtained by applying the forgetful functor $\text{Op}_K \to \text{Op}(\text{Top}^{\Sigma_{\leq 1}})$ to $\text{End}(B, A)$ from (4) and (5).

Let $\text{Alg}_{\text{SC}_d^{h1}}^A(C)$ denote the category of objects $B \in C$ together with a morphism $\text{SC}_d^{h1} \to \text{End}^{h1}(B, A)$ in $\text{Op}(\text{Top}^{\Sigma_{\leq 1}})$, extending the $E_{d-1}$ structure on $A$. A morphism $B \to B'$ in $C$ induces a map of operads $\text{End}^{h1}(B', A) \to \text{End}^{h1}(B, A)$. Such a morphism gives a map in $\text{Alg}_{\text{SC}_d^{h1}}^A(C)$ if this induced map respects maps from $\text{SC}_d^{h1}$.

4.1.2 Proposition There is an equivalence of relative categories

$$\text{Alg}_{\text{SC}_d^{h1}}^A(C) \cong C_{/\text{Hoch}(A)}.$$

Proof The data of an algebra on the left-hand side is an object $B \in C$ together with maps

$$\text{SC}_d^{h1}(1, m) \otimes B \otimes A^\otimes m \to A.$$
for each $m \geq 0$. The conditions on (22) are that the following diagrams commute:

\[
\begin{align*}
E_{d-1}(m') \otimes SC_d^h(1, m; 0, m') & \otimes B \otimes A^{\otimes m} \to E_{d-1}(m') \otimes A^{\otimes m'} \\
SC_d^h(1, m) \otimes B \otimes A^{\otimes m} & \to A
\end{align*}
\]

(23)

\[
\begin{align*}
SC_d^h(1, m') \otimes E_{d-1}(m; m') \otimes B \otimes A^{\otimes m} & \to SC_d^h(1, m) \otimes B \otimes A^{\otimes m} \\
SC_d^h(1, m') \otimes B \otimes A^{\otimes m'} & \to A
\end{align*}
\]

(24)

Equivalently, we can use the hom-tensor adjunction and assemble the maps in (22) to a single map $B \to \hom_C(\tilde{A}^{sc}, A)$ (see (19)). The commutativity of diagram (24) is equivalent to this map lifting to

\[
B \to \hom_C(A^{sc}, A).
\]

Note that, dual to (20), $\hom_C(A^{sc}, A)$ is given by the equalizer

\[
\hom_C(A^{sc}, A) \overset{\text{eq}}{\to} \hom_C(\tilde{A}^{sc}, A) \to \hom_C\left(\prod_{m, m'} SC_d^h(1, m') \otimes E_{d-1}(m; m') \otimes A^{\otimes m}, A\right).
\]

With this observation, we can now rewrite diagram (23) as:

\[
\begin{align*}
B \otimes F_{E_{d-1}}^A(A^{sc}) & \to F_{E_{d-1}}^A(A) \\
B \otimes A^{sc} & \to A
\end{align*}
\]

(26)

Recall that $F_{E_{d-1}}^A : C \to C$ is the free $E_{d-1}-A$ module functor from Definition 3.1.6. Clearly, diagram (23) commutes if and only if diagram (26) commutes and the map (25) factors through $\text{Hoch}(A) = \hom_{E_{d-1}}^A(A^{sc}, A)$. We conclude that the data of an $SC_d^{h1}$ algebra structure on $(B, A)$ is the data of an $E_{d-1}$ algebra structure on $A$ together with a $C$–morphism $B \to \text{Hoch}(A)$.

It is clear that a map $(B, A) \to (B', A)$ which is identity on $A$ gives an $SC_d^{h1}$ algebra morphism if and only if the map $B \to B'$ commutes with the corresponding morphisms to $\text{Hoch}(A)$.  

\[\Box\]

4.1.3 Definition Define $SC_d^{h\infty} \in \text{Op}(\text{Top}^\Sigma)$ as the image of the left adjoint of $\text{Op}(\text{Top}^\Sigma) \to \text{Op}(\text{Top}^\Sigma^{\leq 1})$ applied to $SC_d^{h1}$. As in Definition 4.1.1, define $\text{Alg}_{SC_d^{h\infty}}^A(C)$ as the category of objects $B \in C$ together with a morphism $SC_d^{h\infty} \to \text{End}^h(B, A)$ in $\text{Op}(\text{Top}^\Sigma)$ extending the $E_{d-1}$ structure on $A$. 

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4.1.4 Corollary  There is an equivalence of relative categories
\[ \text{Alg}^A_{\text{Sc}^\infty_d}(C) \cong C/\text{Hoch}(A). \]

**Proof**  The adjunction isomorphism puts operad maps \( \text{Sc}^1_d \to \text{End}^h(B, A) \) in one-to-one correspondence with operad maps \( \text{Sc}^\infty_d \to \text{End}^h(B, A) \). This gives an isomorphism between \( \text{Alg}^A_{\text{Sc}^\infty_d}(C) \) and \( \text{Alg}^A_{\text{Sc}^1_d}(C) \). Now apply Proposition 4.1.2.  \( \square \)

4.1.5 Definition  For \( n \geq 0 \), let \( \text{Sc}^\infty_d(n, \cdot) \) denote the operad in \( \text{Op}(\text{Top}^\Sigma) \) whose \( m \)-ary component is the degree \( 0\)–\( 1 \) symmetric sequence \( (\text{Ed}_{d-1}(m), \text{Sc}^\infty_d(n, m)) \). One may think of \( \text{Sc}^\infty_d(n, \cdot) \) as a bimodule over \( \text{Ed}_{d-1} \). Let \( \text{End}(\text{Sc}^\infty_d) \) be the symmetric sequence whose \( n \)-th space is the \( \text{Ed}_{d-1} \) bimodule maps \( \text{End}(\text{Sc}^\infty_d)(n) := \text{map}_{\text{Ed}_{d-1}}(\text{Sc}^\infty_d(1, \cdot), \text{Sc}^\infty_d(n, \cdot)) \).

4.1.6 Lemma  Operadic composition in \( \text{Sc}^\infty_d \) induces the structure of an operad on the symmetric sequence \( \text{End}(\text{Sc}^\infty_d) \).

**Proof** Define an operad \( \mathcal{E} \in \text{Op}(\text{Top}^\Sigma) \) by setting \( \mathcal{E}(n, m) = \text{map}_{\text{Top}^\Sigma}(\text{End}(\text{Sc}^\infty_d)\cdot n), \text{Sc}^\infty_d\cdot m). \) Let \( n_0 + \cdots + n_m = n \) and \( \ell_1 + \cdots + \ell_m = \ell \). The monoidal structure \( \otimes \) on symmetric sequences gives a map from \( \mathcal{E}(n_0, m) \times \prod_{i=1}^m \mathcal{E}(n_i, \ell_i) \) to
\[
\text{map}_{\text{Top}^\Sigma} \left( \bigotimes_{i=0}^m \text{End}(\text{Sc}^\infty_d)\cdot n_i, \text{Sc}^\infty_d\cdot m \otimes \left( \bigotimes_{i=1}^m \text{Sc}^\infty_d\cdot \ell_i \right) \right).
\]

Now push forward from (27) via the operad structure on \( \text{Sc}^\infty_d \),
\[
\text{Sc}^\infty_d\cdot m \otimes \left( \bigotimes_{i=1}^m \text{Sc}^\infty_d\cdot \ell_i \right) \to \text{Sc}^\infty_d\cdot \ell,
\]
and pull back from (27) by
\[
\text{End}(\text{Sc}^\infty_d)\cdot n \to \bigotimes_{i=0}^m \text{End}(\text{Sc}^\infty_d)\cdot n_i.
\]

This defines
\[
\mathcal{E}(n_0, m) \times \prod_{i=1}^m \mathcal{E}(n_i, \ell_i) \to \mathcal{E}(n, \ell).
\]
The morphism in (28) comes from the sequence of maps
\[
\text{End}(\mathcal{S}_d^{\infty})(k; n) = \bigoplus_{f: k \to n} \prod_{i=1}^n \text{End}(\mathcal{S}_d^{h\infty})(f^{-1}(i))
\]
\[
= \bigoplus_{f: k \to n} \prod_{j \in 0 \sqcup m} \prod_{i \in g^{-1}(j)} \text{End}(\mathcal{S}_d^{h\infty})(f^{-1}(i))
\]
\[
\to \bigoplus_{\tilde{f}: k \to 0 \sqcup m} \prod_{j \in 0 \sqcup m} \text{End}(\mathcal{S}_d^{h\infty})(\tilde{f}^{-1}(j); n_j).
\]
The first equality holds by definition; the second is a regrouping. The decomposition \(\sum_{i=0}^m n_i = n\) defines a map \(g: n \to 0 \sqcup m\), where \(|g^{-1}(i)| = n_i\). The third map sends the component corresponding to \(f: k \to n\) to the component corresponding to \(fg: k \to 0 \sqcup m\). For each \(n\) and \(m\) there is a map
\[
\mathcal{S}_d^{h\infty}(n, m) \to \mathcal{E}(n, m). \tag{29}
\]
When \(n = 0, 1\), the map (29) is canonical. Restricting to degrees 0 and 1 gives a map of \(\text{Op}(\text{Top}^{\Sigma_+})\) operads \(\mathcal{S}_d^{h1} \to \mathcal{E}^1\), where \(\mathcal{E}^1\) is the degree 0–1 part of \(\mathcal{E}\). Since \(\mathcal{S}_d^{h\infty}\) is freely generated by its degree 0 and 1 pieces, we get (29) for all \(n\), assembling into a map of operads in \(\text{Op}(\text{Top}^{\Sigma})\). This guarantees that (30) can be used to define an operadic composition law on \(\text{End}(\mathcal{S}_d^{h\infty})\).

\[
\mathcal{S}_d^{h\infty}(n, m) \times \text{End}(\mathcal{S}_d^{h\infty})(k_1) \times \cdots \times \text{End}(\mathcal{S}_d^{h\infty})(k_n) \to \mathcal{S}_d^{h\infty}(k, m). \tag{30}
\]

\[\square\]

4.1.7 Definition Let \(\mathcal{O}\) be a 1–colored topological operad and let \(\rho: \mathcal{O} \to \text{End}(\mathcal{S}_d^{h\infty})\) be a map of operads. Define the \(K\)–colored operad \(\mathcal{S}_d^{h\infty} \rtimes_{\rho} \mathcal{O}\) by setting
\[
(\mathcal{S}_d^{h\infty} \rtimes_{\rho} \mathcal{O})^h(n, m) = \mathcal{S}_d^{h\infty}(n, m), \quad (\mathcal{S}_d^{h\infty} \rtimes_{\rho} \mathcal{O})^f(n, m) = \begin{cases} \mathcal{O}(n) & m = 0, \\ \emptyset & m > 0. \end{cases}
\]
Composition in \(\mathcal{S}_d^{h\infty} \rtimes_{\rho} \mathcal{O}\) uses composition in \(\mathcal{O}\), composition in \(\mathcal{S}_d^{h\infty}\) and the action of \(\mathcal{O}\) on \(\mathcal{S}_d^{h\infty}\) defined by \(\rho\).

\[
\mathcal{S}_d^{h\infty}(n, m) \times \mathcal{O}(k; n) \to \mathcal{S}_d^{h\infty}(n, m) \times \text{End}(\mathcal{S}_d^{h\infty})(k; n) \to \mathcal{S}_d^{h\infty}(k, m),
\]
where the right arrow above is the one in (30).

4.1.8 Lemma Let \(\mathcal{O}\) be a topological operad and let \(\rho: \mathcal{O} \to \text{End}(\mathcal{S}_d^{h\infty})\) be a map of operads; then the \(\mathcal{S}_d^{h\infty}\) structure on \((H, A)\) naturally extends to a \(\mathcal{S}_d^{h\infty} \rtimes_{\rho} \mathcal{O}\) structure on \((H, A)\). In particular, \(H = \text{Hoch}(A)\) inherits an \(\mathcal{O}\) algebra structure.
Proof We only need to show there is a map of operads \( \text{End}(\mathcal{S}_d^{h\infty}) \to \text{End}(H) \) compatible with the action of \( \text{End}(\mathcal{S}_d^{h\infty}) \) on \( \mathcal{S}_d^{h\infty} \) and the action of \( \mathcal{S}_d^{h\infty} \) on \( H \). Indeed, the map

\[
\text{End}(\mathcal{S}_d^{h\infty})(n) \otimes H^\otimes n \to H
\]

is adjoint to the maps, for all \( m \geq 0 \),

\[
\mathcal{S}_d^{h\infty}(1, m) \otimes \text{End}(\mathcal{S}_d^{h\infty})(n) \otimes H^\otimes n \otimes A^\otimes m \to \mathcal{S}_d^{h\infty}(n, m) \otimes H^\otimes n \otimes A^\otimes m \to A,
\]

where the first arrow is (30) and the second arrow is the \( \mathcal{S}_d^{h\infty} \) structure on \( (H, A) \). To check that (31) is compatible with composition in \( \text{End}(\mathcal{S}_d^{h\infty}) \) observe that there are two morphisms of operads in \( \text{Op}(\text{Top}^\Sigma) \),

\[
\mathcal{S}_d^{h\infty} \to \text{End}^h(\text{End}(\mathcal{S}_d^{h\infty}))(H), A),
\]

where \( \text{End}(\mathcal{S}_d^{h\infty}))(H) \) is the free \( \text{End}(\mathcal{S}_d^{h\infty}) \) algebra generated by \( H \). One of the arrows uses the action of \( \text{End}(\mathcal{S}_d^{h\infty}) \) on \( \mathcal{S}_d^{h\infty} \), while the other uses the map \( \text{End}(\mathcal{S}_d^{h\infty}))(H) \to H \) defined by (31). To check that these arrows agree, we only need to check that they agree out of \( \mathcal{S}_d^{h\infty}(n, m) \) when \( n = 0, 1 \). This is because \( \mathcal{S}_d^{h\infty} \) is freely generated in degrees 0 and 1. When \( n = 0 \), the maps are obviously the same. When \( n = 1 \), the maps are the same by definition of the \( \mathcal{S}_d^{h1} \) structure on \( (H, A) \). 

4.1.9 Proposition (The universal cheese theorem) Let \( \mathcal{O} \) be a topological operad and let \( \rho: \mathcal{O} \to \text{End}(\mathcal{S}_d^{h\infty}) \) be a map of operads. Then using the induced \( \mathcal{S}_d^{h\infty} \times_\rho \mathcal{O} \) structure on \( (\text{Hoch}(A), A) \) from Lemma 4.1.8 gives an equivalence of relative categories

\[
\text{Alg}^A_{(\mathcal{S}_d^{h\infty} \times_\rho \mathcal{O})}(C) \cong \text{Alg}_{\mathcal{O}}(C)_{/\text{Hoch}(A)}.
\]

Proof Given any \( C \)-morphism \( B \to H \) we can form the following diagram (for brevity, we have deleted appearances of \( \otimes \)):

\[
\begin{array}{cccccccc}
\mathcal{S}_d^{h\infty}(1, m)\mathcal{O}(n)B^nA^m & \to & \mathcal{S}_d^{h\infty}(n, m)B^nA^m \\
\downarrow & & \downarrow \\
\mathcal{S}_d^{h\infty}(1, m)\mathcal{O}(n)H^nA^m & \to & \mathcal{S}_d^{h\infty}(n, m)H^nA^m \\
\downarrow & & \downarrow \\
\mathcal{S}_d^{h\infty}(1, m)BA^m & \to & A
\end{array}
\]

(32)

Let \( (B, A) \) be a \( \mathcal{S}_d^{h\infty} \times_\rho \mathcal{O} \) algebra extending the given \( E_{d-1} \) structure on \( A \), then by Corollary 4.1.4 we get a \( C \) morphism \( B \to H = \text{Hoch}(A) \) making the right face of the cube (32) commute. The front face commutes by Lemma 4.1.8. The back face...
commutes by assumption. The bottom face commutes by definition, and the top face commutes trivially.

This implies that, after composition with the maps whose codomain is \( A \), the left face of the cube commutes. By adjointness, the two maps \( \mathcal{O}(B) \Rightarrow H \) agree, implying that \( B \rightarrow H \) is indeed an \( \mathcal{O} \) algebra morphism.

On the other hand, given an \( \mathcal{O} \) algebra \( B \) together with an \( \mathcal{O} \) algebra map \( B \rightarrow H \), we get an \( \mathcal{S} \mathcal{C}_d^{h\infty} \) structure on \( (B, A) \) from the underlying \( C \)–morphism. We only need to check that the \( \mathcal{O} \) structure on \( B \) and the \( \mathcal{S} \mathcal{C}_d^{h\infty} \) structure on \( (B, A) \) are compatible via \( \rho \). Indeed, since \( \mathcal{S} \mathcal{C}_d^{h\infty} \) is freely generated in degrees 0 and 1, it is enough to check that the back face of the cube commutes. But this holds because all other faces commute. Most importantly, the left face commutes because \( B \rightarrow H \) is an \( \mathcal{O} \) algebra map.

It is easy to see that each of these constructions are natural in \( B \) and are inverse to one another. \( \square \)

5 The homotopy \( E_d \) structure on Hoch\((A)\)

In light of Proposition 4.1.9, to prove the Swiss cheese theorem, Theorem 2.2.1, we need to construct \( E \simeq E_d \) and an operad morphism \( E \rightarrow \text{End}(\mathcal{S} \mathcal{C}_d^{h\infty}) \) in such a way that the corresponding \( K \)–colored operad \( \mathcal{S} \mathcal{C}_d^{h\infty} \times E \) is equivalent to \( \mathcal{S} \mathcal{C}_d \). While \( \mathcal{S} \mathcal{C}_d^{h\infty} \) has no obvious action of \( E_d \), it is equivalent to something that does have an \( E_d \) action. The following theorem is proven in Section 6.

5.1.1 Theorem The natural map \( \mathcal{S} \mathcal{C}_d^{h\infty} \rightarrow \mathcal{S} \mathcal{C}_d^h \) is an acyclic cofibration of operads in \( \text{Op}(\text{Top}^\Sigma) \).

In this section, we define the precise sense in which Theorem 5.1.1 gives us our \( E_d \) action on \( \mathcal{S} \mathcal{C}_d^{h\infty} \) up to homotopy. First, we have a lift \( p \) in the following diagram:

\[
\begin{array}{ccc}
\mathcal{S} \mathcal{C}_d^{h\infty} & \xrightarrow{id} & \mathcal{S} \mathcal{C}_d^{h\infty} \\
\downarrow & & \downarrow \\
\mathcal{S} \mathcal{C}_d^h & \xrightarrow{p} & \mathcal{S} \mathcal{C}_d^h
\end{array}
\]

We know that \( \mathcal{S} \mathcal{C}_d^h \) is cofibrant since it is obtained as the \( W \) construction applied to a \( \Sigma \)–cofibrant, well-pointed operad \( \mathcal{S} \mathcal{C}_d^h \), so it fits into the context covered by Berger and Moerdijk [4; 5]. In Spitzweck [21] we see that the corner axiom (or Quillen’s SM7) for monoidal model categories tensored over topological spaces applies to categories of operads in topological spaces. Thus we have an acyclic fibration

\[
\text{map}(\mathcal{S} \mathcal{C}_d^h, \mathcal{S} \mathcal{C}_d^h) \xrightarrow{\tau^*} \text{map}(\mathcal{S} \mathcal{C}_d^{h\infty}, \mathcal{S} \mathcal{C}_d^h)
\]
given by pre-composing with \( \iota \). Since both \( \iota p \) and \( \text{id} \) live over \( \iota \), they must be homotopic. Let \( h : [0, \infty] \otimes \mathcal{SC}_d^h \to \mathcal{SC}_d^h \) be a homotopy with \( h_0 = \text{id} \) and \( h_\infty = \iota p \).

We will use this \( h \) to define a homotopy right \( E_d \)-module structure on \( \mathcal{SC}_d^{h\infty} \). For this we will use a homotopy equivalent version of \( E_d \) which sits inside the \( W \) construction. For simplicity we denote it by \( E \). First we define the category \( LE_d \). This category is not monoidal, but will be used to build \( E \). The letter \( L \) stands for level trees. The objects of the topological category \( LE_d \) are finite sets and the morphism space \( LE_d(n, n') \) is defined to be a quotient of

\[
\coprod_{n_1, \ldots, n_k \geq 0} E_d(n_1, n') \times E_d(n_2, n_1) \times \cdots \times E_d(n, n_k) \times [0, \infty]^k.
\]

A point of the space above is given by a sequence \( \alpha_i \in E_d(n_i, n_{i-1}) \) for \( 1 \leq i \leq k + 1 \) and \( t_i \in [0, \infty] \) for \( 1 \leq i \leq k \). For convenience of notation, we set \( n_0 = n' \), \( n_{k+1} = n \), \( t_0 = \infty \) and \( t_{k+1} = \infty \). We impose the following relations:

**5.1.2 Relations** If \( t_i = 0 \), then we can delete \( t_i \) and replace \((\ldots, \alpha_i, \alpha_{i+1}, \ldots)\) by the composition \((\ldots, \alpha_i \circ \alpha_{i+1}, \ldots)\). If \( n_i = n_{i-1} \) and \( \alpha_i \) is the identity, and \( t_{i-1} = \infty = t_i \), then we can delete \( \alpha_i \) from the sequence and delete \( t_i \) from the sequence.

**5.1.3 Remark** In the \( W \) construction, we could always delete the appearance of an identity and sum the lengths of the surrounding edges. We do not allow that here since we do not have \( h_{s+t} = h_s \circ h_t \).

Composition in the category \( LE_d \) is given by concatenating sequences, setting the new coordinate in the factor \( [0, \infty] \) between the two sequences to be \( \infty \).

We can use the action of \( E_d \) on \( \mathcal{SC}_d^h \) as well as the maps \( h_t \), \( p \) and \( \iota \) to define

\[
LE_d(n, n') \to \text{map}(\mathcal{SC}_d^{h\infty}(n', m), \mathcal{SC}_d^{h\infty}(n, m)).
\]

To do this, represent \( \alpha \in LE_d(n, n') \) with a sequence \( n' = n_0, n_1, \ldots, n_k, n_{k+1} = n \) together with \( \alpha_i \in E_d(n_i, n_{i-1}) \) for \( 1 \leq i \leq k + 1 \) and \( t_i \in [0, \infty] \) for \( 1 \leq i \leq k \). This gives a chain of maps

\[
\mathcal{SC}_d^{h\infty}(n', m) \xrightarrow{\iota} \mathcal{SC}_d^h(n_0, m) \xrightarrow{\alpha_1} \mathcal{SC}_d^h(n_1, m) \xrightarrow{h_{t_1}} \mathcal{SC}_d^h(n_1, m) \xrightarrow{\alpha_2} \cdots \xrightarrow{h_{t_k}} \mathcal{SC}_d^h(n_k, m) \xrightarrow{\alpha_{k+1}} \mathcal{SC}_d^h(n_{k+1}, m) \xrightarrow{p} \mathcal{SC}_d^{h\infty}(n, m).
\]

The maps \( \mathcal{SC}_d^h(n_i, m) \xrightarrow{\alpha_i} \mathcal{SC}_d^h(n_{i+1}, m) \) are defined by the action of \( E_d \) on \( \mathcal{SC}_d^h \),

\[
\mathcal{SC}_d^h(n_i, m) \times E_d(n_{i+1}, n_i) \to \mathcal{SC}_d^h(n_{i+1}, m).
\]
Let us check that Relations 5.1.2 in $L E_d$ are satisfied and that composition in $E_d$ corresponds to composition of maps of $S C^h_d$. Suppose $t_i = 0$ for some $i$. Then $h_0 = \text{id}$, so our chain of arrows contains

$$SC^h_d(n_{i-1}, m) \xrightarrow{\alpha_i} SC^h_d(n_i, m) \xrightarrow{\alpha_i+1} SC^h_d(n_i+1, m).$$

The composition of these two is equal to the map given by $\alpha_i \alpha_i+1 \in E_d(n_{i+1}, n_{i-1})$. This is because $SC^h_d(-, m)$ is a right $E_d$–module.

If $n_i = n_{i-1}$, $\alpha_i$ is the identity and $t_{i-1} = t_i = \infty$, then the composition $h_{t_{i-1}} \circ \alpha_i \circ h_t_{i}$ is equal to $h_{t_{i-1}} = h_\infty$, so we are justified in deleting $\alpha_i$ and $t_i$ from the sequence.

Now suppose we have some $t_i = \infty$, so that $\alpha \in LE_d(n, n')$ decomposes as $\beta_1 \beta_2$ for some $\beta_1 \in E(n_i, n')$ and $\beta_2 \in E(n, n_i)$. The chain of compositions defining the action of $\alpha$ from $SC^h_d(1, m)$ to $SC^h_d(n, m)$ contains the segment:

$$\cdots SC^h_d(n_i, m) \xrightarrow{h_\infty} SC^h_d(n_i, m) \rightarrow \cdots$$

The composite of the actions of $\beta_1$ and $\beta_2$ is computed by joining the chains for $\beta_1$ and for $\beta_2$. This joined chain agrees with the chain for $\alpha$ except for the segment above, which is replaced with the segment:

$$\cdots \rightarrow SC^h_d(n_i, m) \xrightarrow{p} SC^h_d(n_i, m) \xrightarrow{t} SC^h_d(n_i, m) \rightarrow \cdots$$

Since $h_\infty = tp$, these chains of maps have the same composition.

The maps (33) define a functor

(35) $LE_d \rightarrow \text{End}(SC_d^{h\infty})$.

There is no obvious operad structure on $LE_d$, so we take the smallest operad containing $LE_d$. More precisely, (35) is a morphism of topological categories whose objects are finite sets. There is a forgetful functor from operads to the category of such topological categories. The operad E is defined to be the result of applying the left adjoint of this forgetful functor to the category $LE_d$.

5.1.4 Definition Let $F(LE_d)$ be the free 1–colored operad generated by the symmetric sequence $n \mapsto LE_d(n, 1)$. For each $n, n' \geq 0$ let $E(n, n')$ be the topological space given by the coequalizer

$$\bigsqcup_{n''} LE_d(n'', n') \times LE_d(n, n'') \xrightarrow{\text{eq}} F(LE_d)(n, n') \xrightarrow{\text{eq}} E(n, n'),$$

where the two maps are given by composition in either $F(LE_d)$ or $LE_d$ and the inclusion of $LE_d$ into $F(LE_d)$.
5.1.5 Lemma  The category $E$ is an operad and is equivalent to $E_d$.

Proof  Given a tree with its internal edges labeled by lengths $[0, \infty]$, call it a level tree if edges equidistant from the root vertex have the same length. Every morphism in $LE_d(n, 1)$ can be represented by a level tree with vertices labeled by $E_d$. We can represent a point of $F(LE_d)$ with a tree whose vertices are labeled by level trees in $LE_d$. The relation defining $F(LE_d) \to E$ allows us to break up a level tree with at least one level of length $\infty$ into several level trees all of whose levels have finite length. We conclude that $E$ consists of trees labeled by $E_d$ on the vertices and $[0, 1]$ on the internal edges, satisfying the condition that every maximal finite subtree is level.

There is an operad morphism $E \to E_d$ which collapses all edge lengths to 0. On the level of symmetric sequences, there is a homotopy inverse $E_d \to E$. The homotopy $g_t: E \to E$ first collapses lengths of the edges furthest from the root to zero. This preserves the condition that every maximal finite subtree is level. Continuing in this way, we collapse all edge lengths to zero.

The adjoint to (35) is an operad morphism $E \to \text{End}(SC_d^{h\infty})$, which by Definition 4.1.7 we can use to define the $K$–colored operad $SC_d^{h\infty} \times E$.

5.1.6 Lemma  The $\{f, h\}$–colored operad $SC_d^{h\infty} \times E$ is weakly equivalent to the Swiss cheese operad.

Proof  First, note that $SC_d$ is equivalent to the semi-direct product of $SC_d^h$ and $E_d$ where the action of $E_d$ factors through the map $E_d \to E_d$ which sends all lengths of internal edges to zero. This is because the map $SC_d^h \to SC_d^h$ which collapses trees is a weak equivalence and respects the action of $E_d$.

The action of $E$ on $SC_d^{h\infty}$ can be extended to an action on all of $SC_d^h$. The sequence $\alpha_1, t_1, \ldots, t_k, \alpha_{k+1}$ acts via the composition

\[(36) \quad SC_d^h(n_0, m) \xrightarrow{\alpha_1} SC_d^h(n_1, m) \xrightarrow{h_{t_1}} SC_d^h(n_1, m) \to \cdots \to SC_d^h(n_{l-1}, m) \xrightarrow{h_{t_l}} SC_d^h(n_{l-1}, m) \xrightarrow{\alpha_l} SC_d^h(n_{l}, m) \xrightarrow{h_{\infty}} SC_d^h(n_{l}, m) \xrightarrow{h_{t_{l+1}}} SC_d^h(n_{l+1}, m).
\]

Define, for each $s \in [0, \infty]$, a homotopy $h^{[0, s]}: [0, \infty] \otimes SC_d^h \to SC_d^h$ by setting $h^{[0, s]}_t = h_{\min(s, t)}$. We have $h^{[0, s]}_0 = \text{id}$ and $h^{[0, s]}_\infty = h_s$; therefore, we can define an action of $E$ on $[0, \infty] \otimes SC_d^h$ by replacing $h_{t_i}$ in (36) with $h^{[0, s]}_{t_i}$. Then, when $s = 0$, each $h^{[0, 0]}_{t_i}$ is the identity, so the action factors through the map $E \to E_d$ collapsing all edges to 0. When $s = \infty$ we have $h^{[0, \infty]}_{t_i} = h_{t_i}$, so the action of $E$ on $SC_d^h$ is (36).

Thus we have a diagram of equivalences

$$SC_d \leftrightarrow SC_d^h \times_{s=0} E \to ([0, 1] \otimes SC_d^h) \times E \leftarrow SC_d^h \times_{s=\infty} E \to SC_d^{h\infty} \times E.$$
where the map on the left collapses all edge lengths to 0.

6 The equivalence $\text{SC}_d^{h\infty} \to \text{SC}_d^h$

This section is dedicated to proving Theorem 5.1.1.

The proof uses a recasting Definition 6.1.4 of Definition 2.1.1 which is equivalent when considering operads in $\text{Top}$; see Getzler [11]. First we set the context for this new definition, then we prove that $\text{SC}_d^{h\infty} \to \text{SC}_d^h$ is a cofibration. Finally, we show that it is a weak equivalence.

6.1 The category of forests

The following definition is an amalgamation of those found in [17; 8; 11].

6.1.1 Definition Fix a set $K$. A $K$–colored young forest is an uncolored map of finite $K$–colored sets $x: I_x \to J_x$. A $K$–colored forest $f: x \to y$ is a color-preserving isomorphism $f: I_y \sqcup J_x \to J_y \sqcup I_x$ satisfying certain conditions. Before mentioning the conditions, it helps to think of $I_y$ as the set of incoming edges of $f$, $J_x$ as the set of internal vertices of $f$, $J_y$ as the set of outgoing vertices of $f$ and $I_x$ as the set of internal edges of $f$.

To form the conditions $f$ must satisfy to qualify as a $K$–colored forest, we use $f$, $x$ and $y$ to build an endomorphism $(y, f, \text{id}, x)$ of $I_y \sqcup J_x \sqcup J_y \sqcup I_x$ which is $y$ on $I_y$, $f$ on $J_x$, the identity on $J_y$ and $x$ on $I_x$. We require that, for every $p \in I_y \sqcup J_x \sqcup J_y \sqcup I_x$, there is a $k \geq 0$ such that $(y, f, \text{id}, x)^k(p) \in J_y$. Moreover, if $p \in I_y$, we require $(y, f, \text{id}, x)^k(p)$ to be equal to $y(p)$. An example is shown in Figure 1. We will abuse notation and write $(f \circ x)^k(p)$ when $p \in I_x$, $(y, f, \text{id}, x)^k(p) \in J_y$ and $(y, f, \text{id}, x)^{k-1}(p) \notin J_y$.

6.1.2 Definition Given a $K$–colored forest $f: x \to y$, we call $V(f) := J_x$ the set of internal vertices of $f$. We call $\text{in}(f) := I_y$ the set of input vertices of $f$ and $\text{rt}(f) := J_y$ the set of root vertices of $f$. In addition, $\text{Edge}(f) = J_x \sqcup I_y \cong J_y \sqcup I_x$ is called the set of extended edges of $f$ and $E(f) := J_x \times_f I_x$ is the set of internal edges of $f$.

6.1.3 Definition If $g: x \to y$ and $f: y \to z$ are forests, we can define a composite forest $fg: x \to z$. We use concatenation to denote this composition and $\circ$ to denote composition of maps of finite sets. The forest $fg$, as a map $I_z \sqcup J_x \to J_z \sqcup I_x$, is defined by the following rule. If $i \in I_z$, then there is a $k \geq 0$ and an $\varepsilon \in \{0, 1\}$
Figure 1: Here, $f$ is a forest from $x$ to $y$, where $x: I_x \to J_x$ and $y: I_y \to J_y$ are maps of finite $K$–colored sets for $I_x = \{i_{x_1}, i_{x_2}, i_{x_3}, i_{x_4}, i_{x_5}\}$ and $J_x = \{j_{x_1}, j_{x_2}\}$. Similarly, $I_y = \{i_{y_1}, i_{y_2}, i_{y_3}, i_{y_4}\}$ and $J_y = \{j_{y_1}, j_{y_2}\}$. The map $x$ is denoted by dotted edges. The map $y$ is denoted by dashed edges, and the map $f$ is denoted, where applicable, by solid edges. The picture demonstrates the second condition for $f$ to qualify as a forest. Namely, $f(x(f(x(f(i_{y_1}))))) = f(x(f(x(f(i_{y_2})))) = f(x(f(i_{y_3}))) = f(x(f(i_{y_4}))) = y(i_{y_1}) = y(i_{y_2}) = y(i_{y_3}) = y(i_{y_4}) = j_{y_1}$.

such that $g^e(f \circ g)^k(i) \in J_z \sqcup I_x$. Similarly, if $i \in J_x$ there is a $k$ and $e$ such that $f^e(g \circ f)^k(i) \in J_z \sqcup I_x$. In [23], we show that forest composition is associative. This, together with disjoint union, makes young forests the objects and forests the morphisms of a symmetric monoidal category denoted For.

6.1.4 Definition A $K$–colored operad $O$ is a strong symmetric monoidal functor $(\text{For}_K, \sqcup) \to (\text{Top}, \times)$. The category of operads is the category $\text{Fun}^\otimes(\text{For}_K, \text{Top})$ of symmetric monoidal functors and natural transformations. We denote this category by $\text{Op}_K$ just as in Definition 2.3.3.

6.1.5 Remark The category of $K$–colored operads $\text{Op}_K$ as defined in Definition 2.3.3 is naturally isomorphic to the functor category $\text{Fun}^\otimes(\text{For}_K, \text{Top})$. Indeed, given $O$ from Definition 2.1.1, as in Definition 2.3.3 we define $O(x)$ for a young forest $x$ to be $\bigotimes_{j \in J_x} O(x^{-1}(j); j)$.
6.1.6 Remark  When the set of colors $K$ is understood, we often drop it from the notation. Forests and young forests are always $K$-colored for some set $K$. The category $\text{For}_K$ will be abbreviated $	ext{For}$ and $\text{Op}_K$ will be denoted by $\text{Op}$.

6.1.7 Definition  Let $(C, \otimes)$ be a symmetric monoidal category. Call an object $c \in C$ indecomposable if it cannot be written as a tensor product $c \cong c_1 \otimes c_2$ for any $c_1, c_2 \in C$. Let $\text{Fun}^\otimes(C, \text{Top})$ denote the category of strong symmetric monoidal functors $(C, \otimes) \to (\text{Top}, \times)$. We call a morphism $\psi: \mathcal{O} \to \mathcal{P}$ in $\text{Fun}^\otimes(C, \text{Top})$ a fibration (respectively weak equivalence) if $\psi(c): \mathcal{O}(c) \to \mathcal{P}(c)$ is a fibration (respectively weak equivalence) for every indecomposable $c \in C$. Define the class of cofibrations in the usual manner (see Hovey [13]).

For every symmetric monoidal category $C$ we consider in this paper, we will use Definition 6.1.7 to define cofibrations, fibrations and weak equivalences regardless of whether or not these form a model structure.

6.1.8 Remark  A young forest $x$ is indecomposable (Definition 6.1.7) if $J_x \simeq \ast$. In this case we say $x$ is a young tree. We say a forest $f: y \to x$ is a tree if $x$ is a young tree.

6.1.9 Definition  For a category $C$, let $C^\times$ denote the category with the same objects as those of $C$, but only invertible morphisms. Let $C_{\text{indec}}^\times$ denote the full subcategory (not monoidal) of $C^\times$ consisting indecomposable objects. Let $C_{\text{indec}}^{\times\times} = (C_{\text{indec}}^\times)^\times$. The functor category $\text{Top}^{C_{\text{indec}}}^\times$ is called the category of $C$–symmetric sequences, denoted by $C$–$\text{sSeq}$.

6.1.10 Remark  A forest $f: x \to y$ is invertible if and only if $f(J_x) = J_y$ and $f(I_y) = I_x$. Thus an invertible forest gives a pair of isomorphisms of $K$–colored finite sets $J_f: J_x \to J_y$ and $I_f: I_y \to I_x$ which are compatible with the maps $x: I_x \to J_x$ and $y: I_y \to J_y$. We conclude that the category $\text{For}_{\text{indec}}^{\times\times}$ is isomorphic to the opposite of the category of $K$-colored finite sets and bijections. In the case $K \simeq \{\ast\}$, we get $\text{For}_{\text{indec}}^{\times\times} \cong \Sigma$, where $\Sigma$ is as in Definition 2.3.1. Moreover, in this case the category of $\text{For}$–symmetric sequences is isomorphic to the category $\text{Top}^\Sigma$ of symmetric sequences.

6.1.11 Notation  If $C \to \text{For}$ is any symmetric monoidal functor, we will denote the category $\text{Fun}^\otimes(C, \text{Top})$ of strong symmetric monoidal functors by $C$–$\text{Op}$ unless we say otherwise (for example, we do not use this notation in Definition 6.3.4). We call the objects of $C$–$\text{Op}$ $C$–operads. In all cases we consider, the functor $C \to \text{For}$ will be apparent from the category $C$, so we leave the functor out of the notation. If $C \to D \to \text{For}$ is a pair of symmetric monoidal functors, we denote the forgetful functor $D$–$\text{Op} \to C$–$\text{Op}$ by $U^D_C$ and denote its left adjoint by $F^C_D$.
6.2 The $W$ construction

We show how the $W$ construction of [6] can be realized as a coend construction using For. In nice situations the $W$ construction gives a cofibrant replacement for an operad, as shown in [4]. We use $[0, \infty]$ as our edge labels as in [16].

Given any young forest $z$ there is a contravariant functor $W: \text{For}_{/z} \to \text{Top}$ from the over category of $z$ to topological spaces. For any object $g: y \to z$ of this over category, set $W(g) = \text{map}(E(g), [0, \infty])$. If $f: x \to y$ is a forest, the map $W(g) \to W(gf)$ is denoted by $W_{\Sigma}(f)$. This map uses the sum operation on $[0, \infty]$. This is an extension of $+$ on $[0, \infty)$ such that $t + \infty = \infty = \infty + t$ for all values of $t$. Concretely, $g$ and $gf$ define maps of sets

$$E(g) \overset{\tilde{g}}{\rightarrow} J_z \sqcup I_x \overset{\tilde{g}f}{\leftarrow} E(gf)$$

We can turn a function $t \in W(g)$ to a function $W_{\Sigma}(t) \in W(gf)$ by pushing forward along $\tilde{g}$ then pulling back along $\tilde{g}f$.

Pushing forward means summing over fibers, which is well defined since all the sets we are considering are finite.

If $h: z \to w$ is a forest there is a morphism $W_{\infty}(h): W(g) \to W(hg)$ which uses the maps $E(g) \leftrightarrow I_y \leftrightarrow E(hg)$. In this case we do not push forward and pull back functions. Rather, we extend a function $t: E(g) \to [0, \infty]$ to a function on $E(hg)$ by setting $t(e) = \infty$ if $e \notin E(g)$. This defines a natural transformation $W_{\infty}(h): W \to Wh_*^\text{op}$, where $h_*: \text{For}_{/z}^{\text{op}} \to \text{For}_{/w}^{\text{op}}$ is induced by $h: z \to w$. In other words, this diagram commutes for all composable triples $h$, $g$ and $f$:

$$
\begin{array}{ccc}
W(g) & \xrightarrow{W_{\Sigma}(f)} & W(gf) \\
W_{\infty}(h) \downarrow & & \downarrow W_{\infty}(h) \\
W(hg) & \xrightarrow{W_{\Sigma}(f)} & W(hgf)
\end{array}
$$

Consider an operad $\mathcal{O}$ as a collection of functors $\mathcal{O}_{/z}: \text{For}_{/z} \to \text{Top}$ by setting $\mathcal{O}_{/z}(g) := \mathcal{O}(y)$ for $g: y \to z$. For a young forest $z$ the topological space $W\mathcal{O}(z)$ is the coend

$$W\mathcal{O}(z) = W \otimes_{\text{For}_{/z}} \mathcal{O}_z = \left( \coprod_{g: y \to z} W(g) \times \mathcal{O}(y) \right) / \sim,$$

where $(W_{\Sigma}(f)t, \alpha) \sim (t, \mathcal{O}(f)\alpha)$ for every $f: x \to y$, $t \in W(g)$ and $\alpha \in \mathcal{O}(x)$. Now a forest $h: z \to w$ with the natural transformations above gives us a map

$$W\mathcal{O}(z) = W \otimes_{\text{For}_{/z}} \mathcal{O}_z \xrightarrow{W_{\infty}(h) \otimes \text{id}} Wh_* \otimes_{\text{For}_{/z}} \mathcal{O}_z \to W \otimes_{\text{For}_{/w}} \mathcal{O}_w = W\mathcal{O}(w).$$
This defines $W\mathcal{O}$ as a functor $\mathcal{O} \to \text{Top}$. This functor is symmetric monoidal, so $W\mathcal{O}$ is a $K$–colored operad.

6.2.1 Remark The space $W\mathcal{O}(z)$ consists of labeled forests $f : y \to z$. A labeled forest is one whose internal edges each have a length in $[0, \infty]$ and whose vertices $j \in J_x$ have a corresponding label in $\mathcal{O}(x^{-1}(j); j)$.

In the sequel, we will define several variants on the category $\mathcal{O}$. Each of these variants admits a functor to $\mathcal{O}$ and we want a corresponding $W$ construction for each.

6.2.2 Definition Suppose $\mathcal{O}$ is an operad $\mathcal{O} \to \text{Top}$ and $C$ is any symmetric monoidal category equipped with a symmetric monoidal functor $G : C \to \text{For}$. From this data we can construct an operad $W_C \mathcal{O} \in C–\text{Op}$ using formulas analogous to (38) and (39). Specifically, let $W_C$ be the composite $W \circ G_c : (C/c)^{\text{op}} \to (\text{For}/G(c))^{\text{op}} \to \text{Top}$, where $c \in C$. We define

\begin{equation}
W_C \mathcal{O}(c) = W_C \otimes C/c (G^* \mathcal{O})_c = \left( \coprod_{f : b \to c} W(G(f)) \times \mathcal{O}(G(y)) \right) / \sim.
\end{equation}

We define $W_C \mathcal{O}(c) \to W_C \mathcal{O}(d)$ for a $C$–morphism $g : c \to d$ just as in (39).

6.2.3 Example Define the full subcategory $D \hookrightarrow \mathcal{O}$ to be given by those young forests $x$ where $J_x$ has only color $h$. Restricting the Swiss cheese operad to this full subcategory gives the operad $\mathcal{SC}^h_d$ as $W_D \mathcal{SC}^h_d$. To get $\mathcal{SC}^{h_1}_d$, we use $C \hookrightarrow \mathcal{O}$, the full subcategory of young forests $x$ where $J_x$ has color $h$ and, for each $j \in J_x$, there is at most one element of $x^{-1}(j)$ of color $f$.

It should be clear that $D–\text{Op} \simeq \text{Op}(\text{Top}^\Sigma)$: the forgetful functor $\text{Op}_K \to \text{Op}(\text{Top}^\Sigma)$ from Definition 2.3.3 is given by pulling back along the symmetric monoidal functor $D \to \text{For}_K$. Finally, note that $\mathcal{SC}^{h_\infty}_d$ as defined in Definition 4.1.3 is $F_D^C \mathcal{SC}^{h_1}_d$, where $F_D^C$ is as defined in Notation 6.1.11.

6.3 Weighted forests

We need to define the category of weighted forests to prove the following half of Theorem 5.1.1:

6.3.1 Theorem The natural map $\mathcal{SC}^{h_\infty}_d \to \mathcal{SC}^h_d$ is a cofibration in $\text{Op}(\text{Top}^\Sigma)$. 

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The proof follows the work of Berger and Moerdijk [4] closely. From an operad $O$ they construct an increasing chain of symmetric sequences

\begin{equation}
W_0O \rightarrow W_0^+O \rightarrow W_1O \rightarrow W_1^+O \rightarrow W_2O \rightarrow \cdots.
\end{equation}

The symmetric sequence $W_0O$ is just the underlying symmetric sequence of $O$. If $O$ is cofibrant as a symmetric sequence, $O$ is said to be $\Sigma$–cofibrant. If the operadic unit maps of $O$ are cofibrations, $O$ is said to be well-pointed. In the case $O$ is well-pointed and $\Sigma$–cofibrant, Berger and Moerdijk show that $WO := \operatorname{colim}_k W_kO$ is a cofibrant replacement of $O$ as an operad. In the course of the proof, they show that $W_kO$ is a $k$–operad, which is a partial operad in a certain sense. This partial operad structure will be encoded here in the category $\text{Op}_k$ from Definition 6.3.4. Each $W_k^+O$ is an operad in $\text{Op}_{k+1}$ and, in our context, is given by the left adjoint to a forgetful functor $\text{Op}_{k+1} \rightarrow \text{Op}_k$ applied to $W_kO$. Concretely, the points of $W_kO$ are given by trees with at most $k$ internal edges whose vertices are labeled by $O$ and whose internal edges are labeled by $[0, \infty]$.

**6.3.2 Definition** Let $f : x \rightarrow y$ be a forest. Let $(f \mid x)$ denote the endomorphism of $I_y \sqcup J_y \sqcup I_x \sqcup J_x$ which is $f$ on $I_y \sqcup J_x$, $x$ on $I_x$ and the identity on $J_y$. Since $f$ is a forest, there is a $k \geq 0$ such that, for every $i$, we have $(f \mid x)^k(i) \in J_y$. Let $[f \mid x]$ denote $(f \mid x)^\infty$.

**6.3.3 Definition** For $I \subset I_y \sqcup I_x \sqcup J_x$ and $j \in J_y$, let $I(j)$ denote $I \cap [f \mid x]^{-1}(j)$, the set of elements of $I$ living over $j$. A **weighted young forest** is a pair $(x, \omega_x)$ where $x$ is a young forest and $\omega_x : J_x \rightarrow \mathbb{Z}_{\geq 0}$ is any function, called the **weight** of $x$. A **weighted forest** $f : (x, \omega_x) \rightarrow (y, \omega_y)$ is a forest $f : x \rightarrow y$ such that, for all $j \in J_y$,

\begin{equation}
\omega_y(j) \geq \#E(f) + \sum_{i \in J_x(j)} \omega_x(i).
\end{equation}

If $g : (y, \omega_y) \rightarrow (z, \omega_z)$ is a weighted forest, then one can show that $gf : x \rightarrow z$ defines a weighted forest $gf : (x, \omega_x) \rightarrow (z, \omega_z)$.

**6.3.4 Definition** Disjoint union of forests extends to disjoint union of weighted forests. Let $\text{For}_\omega$ denote the symmetric monoidal category of weighted forests. For each $k \geq 0$, let $\text{For}_k$ denote the full subcategory of $\text{For}_\omega$ generated by objects of the form $(x, \omega_x)$ such that $\omega_x(j) \leq k$ for every $j \in J_x$. Let $\text{Op}_k$ denote the category of $(K$–colored) **weight-$k$ operads**, which are strong symmetric monoidal functors $\text{For}_k \rightarrow \text{Top}$. Note that there is a symmetric monoidal functor $\text{For}_\omega \rightarrow \text{For}$ which forgets the weights.

**6.3.5 Remark** If a forest $f : x \rightarrow y$ has no internal edges then $f$ is a disjoint union of isomorphisms and maps of the form $[\emptyset \rightarrow \emptyset] \rightarrow [\{\kappa\} \rightarrow \{\kappa\}]$, where $\kappa \in K$. The restriction of an operad $O$ to $\text{For}_0$ remembers only
• the spaces $\mathcal{O}(I; \kappa)$ for each $K$–colored set $I$ and each color $\kappa$, together with
the right aut $I$ action on $\mathcal{O}(I; \kappa)$, and
• the operadic unit maps $\ast \to \mathcal{O}(\kappa; \kappa)$ for each color $\kappa \in K$.

Thus $\mathcal{O}$ is cofibrant (Definition 6.1.7) as an object of $\text{Fun}^\Sigma(\text{For}_0, \text{Top})$ if and only if it
is well-pointed and $\Sigma$–cofibrant as in [4, Section 3].

**Proof of Theorem 6.3.1** In Example 6.2.3 we constructed categories $C$ and $D$ such
that $\mathcal{S}_d^{h1} = W_C \mathcal{S}_d$, $\mathcal{S}_d^{h} = W_D \mathcal{S}_d$ and the map $\mathcal{S}_d^{h\infty} \to \mathcal{S}_d^{h}$ is

$$F^C_D W_C \mathcal{S}_d \to W_D \mathcal{S}_d.$$ 

Lemma 6.3.12 applies since $C$ is a full subcategory of $D$, which is a full subcategory of $\text{For}$, and $\mathcal{S}_d: \text{For} \to \text{Top}$ is cofibrant as a functor $\text{For}_0 \to \text{Top}$ (see Remark 6.3.5). □

**6.3.6 Definition** Let $C$ be any full subcategory of $\text{For}$. Define $C_\omega$ to be the full
subcategory of $\text{For}_\omega$ given by young weighted forests $(x, \omega_x)$ such that $x \in C$. The
functor $\text{For}_\omega \to \text{For}$ induces a functor $C_\omega \to C$.

If $C \hookrightarrow \text{For}$ is a full subcategory, there is a map of $C_\omega$ operads $W_{C_\omega} \to U^C_{C_\omega} W_C \mathcal{O}$.

Concretely, for each weighted young forest $(x, \omega_x)$ we have a map

$$(43) \prod_{f: (y, \omega_y) \to (x, \omega_x)} W(f) \times \mathcal{O}(y) \to \prod_{f: y \to x} W(f) \times \mathcal{O}(y),$$

defined in the obvious way. This descends to give a morphism

$$(44) W_{C_\omega} \mathcal{O} \to U^C_{C_\omega} W_C \mathcal{O}$$

**6.3.7 Lemma** For any full subcategory $C \hookrightarrow \text{For}$, the left adjoint of (44),

$$F^C_{C_\omega} W_{C_\omega} \mathcal{O} \to W_C \mathcal{O},$$

is an isomorphism.

**Proof** The left adjoint $F^C_{C_\omega} W_{C_\omega} \mathcal{O}$ can be computed at a young tree $x$ as

$$\text{colim}_{k \to \infty} (W_{C_\omega} \mathcal{O})(x, k).$$

For each $k$, a point of $(W_{C_\omega} \mathcal{O})(x, k)$ is given by a labeled tree (Remark 6.2.1) $f: y \to x$
with at most $k$ internal edges. Taking the colimit as $k$ goes to $\infty$, we get all labeled
trees over $x$, which is $W_C \mathcal{O}(x)$. □
Let $C$ be any symmetric monoidal full subcategory of $\text{For}_\omega$. Suppose $\psi: \mathcal{O} \to \mathcal{P}$ is a morphism of operads in $C-\text{Op}$ which is a cofibration of $C-$symmetric sequences. Further suppose that, for every young forest $x \in C$ such that there is a tree $f: x \to y$ in $C$ with at least one internal edge, we have that $\psi(x): \mathcal{O}(x) \to \mathcal{P}(x)$ is an isomorphism. These conditions imply that $\psi$ is a cofibration of $C-$operads.

**Proof** Suppose $Q \to Q'$ is an acyclic fibration of $C-$operads. By Definition 6.1.7 this means that, for every young tree $x \in C$, $Q(x) \to Q'(x)$ is an acyclic fibration of topological spaces. Suppose we have a commutative diagram of $C-$operads:

$$
\begin{array}{ccc}
\mathcal{O} & \longrightarrow & Q \\
\downarrow & & \downarrow \\
\mathcal{P} & \longrightarrow & Q'
\end{array}
$$

By assumption, there is a lift $\rho: \mathcal{P} \to Q$ of $C-$symmetric sequences. We claim that $\rho$ is automatically a morphism of $C-$operads. To prove this claim, it is enough to show that the square on the right in the following diagram commutes for every tree $f: x \to y$ in $C$:

$$
\begin{array}{ccc}
\mathcal{O}(x) & \xrightarrow{\psi(x)} & \mathcal{P}(x) & \xrightarrow{\rho(x)} & Q(x) \\
\mathcal{O}(y) & \xrightarrow{\psi(y)} & \mathcal{P}(y) & \xrightarrow{\rho(y)} & Q(y)
\end{array}
$$

Suppose $f$ is an isomorphism. Then we know the square commutes because $\rho$ is a map of $C-$symmetric sequences. Suppose $f$ is not an isomorphism and has no internal edges. Since $f$ is a tree it must be of the form $[\emptyset \to \emptyset] \to [{c} \to \{c\}]$ for some $c \in K$. In this case we have $\mathcal{O}(x) = \mathcal{P}(x) = Q(x) = \ast$; in particular, $\psi(x)$ is an isomorphism. Thus our assumption shows that if $f$ is not an isomorphism then $\psi(x)$ is an isomorphism. We can deduce that the square on the right commutes in this case from the fact that the square on the left and the outer square commute. \qed

Given a full subcategory $D \hookrightarrow \text{For}$ and a full subcategory $C \hookrightarrow D$, we can interpolate between $C_\omega$ and $D_\omega$ with a sequence of subcategories of $\text{For}_\omega$. For each $k \geq 0$, let $D_k$ be the full subcategory of $\text{For}_\omega$ given by disjoint unions of young weighted trees $(x, \omega_x)$ where either $x \in C$, or $x \in D$ and $\omega_x \leq k$. Note that $D_{-1} = C_\omega$. We have left the inclusion $C \hookrightarrow D$ implicit in the notation $D_k$. We have a commutative diagram of symmetric monoidal functors:

$$
\begin{array}{ccc}
C & \longrightarrow & D \\
\uparrow & & \uparrow \\
C_\omega = D_{-1} & \longrightarrow & D_0 \longrightarrow \cdots \longrightarrow D_\omega
\end{array}
$$
The bottom row consists of successively larger full subcategories of \( \text{For}_\omega \) and the top row consists of full subcategories of \( \text{For} \).

From the symmetric monoidal functors in (47) we get the categories and forgetful functors in:

\[
\begin{array}{c}
\text{C–Op} \xrightarrow{\text{C}_1} \text{D–Op} \\
\downarrow \quad \downarrow \\
\text{C}_0–\text{Op} \leftarrow \text{D}_0–\text{Op} \leftarrow \cdots \leftarrow \text{D}_\omega–\text{Op}
\end{array}
\]

\[\text{(48)}\]

6.3.9 Remark For \( k, \ell \in \{-1, 0, 1, \ldots \} \cup \{\omega\} \) with \( k \geq \ell \), the forgetful functor \( \text{D}_k–\text{Op} \rightarrow \text{D}_\ell–\text{Op} \) clearly preserves fibrations (Definition 6.1.7); thus, the left adjoint \( F^{D_k}_{D_\ell} \) preserves cofibrations.

6.3.10 Lemma For each \( k \geq 0 \) there is a natural map in \( \text{D}_\omega–\text{Op} \),

\[
F^{D_k}_{D_\omega} \mathcal{O} \rightarrow \mathcal{O}.
\]

The colimit of these maps as \( k \rightarrow \infty \) is an isomorphism.

Proof For each young tree \( x \) and weight \( \omega_x = \ell \in \mathbb{Z}_{\geq 0} \), the space \( (F^{D_k}_{D_\omega} \mathcal{O})(x, \ell) \) can be described as the subspace of \( (\mathcal{O})(x, \ell) \) given by labeled trees \( f: y \rightarrow x \) with at most \( \ell \) internal edges such that, after cutting apart \( f \) at the edges labeled \( \infty \), each remaining subtree of \( f \) has at most \( k \) internal edges. Taking the colimit as \( k \rightarrow \infty \), we get all labeled trees \( f: x \rightarrow y \) with at most \( \ell \) internal edges. That is, we get all of \( (\mathcal{O})(x, \ell) \).

6.3.11 Lemma Suppose \( \mathcal{O} \in \text{Op} \) is cofibrant as an object of \( \text{Fun}^\otimes(\text{For}_0, \text{Top}) \) (see Remark 6.3.5). Then, for each \( k \geq 0 \), the natural map \( \iota_k: F^{D_k}_{D_{k-1}} \mathcal{O} \rightarrow \mathcal{O} \) is a cofibration in \( \text{D}_k–\text{Op} \).

Proof Observe that, if \( f: x \rightarrow y \) is a tree in \( \text{D}_k \) with at least one internal edge, then \( x \in \text{D}_{k-1} \), so \( \iota_k(x) \) is an isomorphism. By Lemma 6.3.8 we only need to show that \( \iota_k \) is a cofibration of \( \text{D}_k–\text{symmetric sequences} \).

In the case \( k = 0 \), if \( x \) is a tree in \( \text{C} = \text{D}_{-1} \) then \( \iota_0(x, 0) \) is an isomorphism. If \( x \) is in \( \text{D}–\text{C} \) then \( \iota(x, 0) \) is the map \( \emptyset \rightarrow \mathcal{O}(x) \). The assumption that \( \mathcal{O} \) is cofibrant as a functor \( \text{Fun}^\otimes(\text{For}_0, \text{Top}) \) implies in particular that each \( \mathcal{O}(x) \) is \( \text{aut}(x)–\text{cofibrant} \). That is, \( \mathcal{O} \) is cofibrant as a \( \text{For}–\text{symmetric sequence} \), so \( \iota_0 \) is a cofibration of \( \text{D}_0–\text{symmetric sequences} \). We now consider the case \( k > 0 \). We follow \cite[Lemma 5.4]{4}. For a young tree \( z \in \text{For}_k \) and \( g: y \rightarrow z \) a tree in \( \text{For}_k \), let \( (W \times \mathcal{O})^+_k(g) \) be \( W(g) \times \mathcal{O}(y) \) if \( g \) has at most \( k - 1 \) internal edges. Otherwise, let \( (W \times \mathcal{O})^+_k(g) \subset W(g) \times \mathcal{O}(y) \) be the set of \( (t, \alpha) \)
such that \( t(\epsilon) = 0 \) or \( t(\epsilon) = \infty \) for some \( \epsilon \in E(g) \) or \( \alpha(j) = \text{id} \) for some \( j \in V(g) \). Using the techniques of [4, Section 2] one can also show that if \( \mathcal{O} \) is cofibrant as an object of \( \text{Fun}^{\otimes} \left( \text{For}_0, \text{Top} \right) \) (see Remark 6.3.5) then \( (W \times \mathcal{O})_k^+ \to W(g) \times \mathcal{O}(\text{dom } g) \) is an \( \text{aut}(g) \)–cofibration, where \( \text{aut } g \) is the automorphism group of \( g \) as an object of the category \( \text{For}_{k/z} \). Define the map \( (W \times \mathcal{O})_k^+ \to F_{D_{k-1}}^{D_k} W_{D_{k-1}} \mathcal{O}(z) \) by collapsing any edge labeled 0 and deleting any vertex labeled with the identity.

In the diagram below, \( \pi_0 \text{For}_{k/z} \) is the set of isomorphism classes of forests \( g: y \to z \) in \( \text{For}_k \) and the domain, \( y \), of a forest \( g: y \to z \) is denoted by \( \text{dom } g \):

\[
\begin{array}{cccc}
\bigcup_{[g] \in \pi_0 \text{For}_{k/z}} ((W \times \mathcal{O})_k^+ \to F_{D_{k-1}}^{D_k} W_{D_{k-1}} \mathcal{O}(z)) \\
\bigcup_{[g] \in \pi_0 \text{For}_{k/z}} \left( (W(g) \times \mathcal{O}(\text{dom } g)) \to (W_{D_{k}} \mathcal{O})(z) \right)
\end{array}
\]

The square in diagram (49) is a pushout. By remarks above, we know the map on the left in (49) is an \( \text{aut}(z) \)–cofibration. We conclude \( F_{D_{k-1}}^{D_k} W_{D_{k-1}} \mathcal{O} \to W_{D_{k}} \mathcal{O} \) is a cofibration of \( D_k \)–symmetric sequences.

**6.3.12 Lemma** If \( C \) is a full subcategory of \( D \) and \( D \) is a full subcategory of \( \text{For} \), and \( \mathcal{O} \) is an operad \( \text{For} \to \text{Top} \) which is cofibrant as a functor \( \text{For}_0 \to \text{Top} \), then the natural map \( F_{C_{D}}^{C_{D}} W_C \mathcal{O} \to W_D \mathcal{O} \) is a cofibration of \( d \)–operads.

**Proof** By Lemma 6.3.7 we have

\[
W_C \mathcal{O} \cong F_{C_{D}}^{C_{D}} W_C \mathcal{O} = F_{C_{D}}^{C_{D}} W_{D_{-1}} \mathcal{O}.
\]

The commutative diagram (48) shows that \( F_{C_{D}}^{C_{D}} F_{C_{D}}^{C_{D}} = F_{D_{D}}^{D_{D}} F_{D_{D}}^{D_{D}} \). This gives the first equality in

\[
F_{C_{D}}^{C_{D}} W_{C_{D}} \mathcal{O} = F_{D_{D}}^{D_{D}} W_{D_{-1}} \mathcal{O} \to \text{colim}_k F_{D_{D}}^{D_{D}} W_{D_k} \mathcal{O} = F_{D_{D}}^{D_{D}} W_{D_{-1}} \mathcal{O} \cong W_D \mathcal{O}.
\]

The second equality comes from Lemma 6.3.10 and the isomorphism from Lemma 6.3.7. The arrow is a cofibration: Indeed, by repeated application of Lemma 6.3.11 together with the fact that each \( F_{D_{k}}^{D_{k}} \) preserves cofibrations (Remark 6.3.9) we see that

\[
F_{D_{k}}^{D_{k}} \ldots F_{D_0}^{D_0} W_{D_{-1}} \mathcal{O} \to W_{D_k} \mathcal{O}
\]

is a cofibration in \( D_k \)–Op. Again, by Remark 6.3.9 \( F_{D_{D}}^{D_{D}} \) and \( F_{D_{D}}^{D_{D}} \) preserve cofibrations. Thus the arrow is a cofibration in \( D \)–Op. \( \square \)
6.4 Weak equivalence proof

This section contains a proof of:

6.4.1 Theorem  The natural map \(SC_d h^{\infty} \to SC_d h\) is a weak equivalence of operads in \(\text{Op}(\text{Top}^\Sigma)\).

The idea of the proof is to consider the maps \(p_1: SC_d h^{\infty}(n, m) \to SC_d h^{\infty}(n - 1, m)\) and \(p: SC_d h(n, m) \to SC_d h(n - 1, m)\) given by forgetting the \(n\)th disk. By induction, we can suppose \(SC_d h^{\infty}(n - 1, m) \to SC_d h(n - 1, m)\) is a weak equivalence. We continue the induction by showing that \(p_1^{-1}(\alpha) \to p^{-1}(\alpha)\) is a weak equivalence for every \(\alpha \in SC_d h^{\infty}(n - 1, m)\).

To make the computation of \(p_1^{-1}(\alpha)\) and \(p^{-1}(\alpha)\) accessible, we will collapse the \(n\)th disk of \(\alpha \in SC_d h(n, m)\) to a point. Our goal in the next section is to make this precise.

6.4.1 Defining \(SC_d,\star(k, \ell \mid n, m)\)  When we collapse the \(n\)th disk of \(\alpha \in SC_d h(n, m)\) to its center, we think of the result \(\tilde{\alpha}\) as living in a 4-colored operad, which we denote by \(SC_d,\star\). We add the colors \(f, h\). Let \(K = \{f, h, f, h\}\) be the set of colors for this new operad. The color \(f\) stands for collapsed full-disks. It is convenient to also allow a collapsed half disk, which we color with \(h\). Let \((k, \ell \mid n, m)\) denote the \(K\)–colored finite set with \(k, \ell, n\) and \(m\) elements of color \(f, h, f\) and \(h\), respectively. Let \(D_{\star}\) denote the full subcategory of \(\text{For}_{K_{\star}}\) with objects isomorphic to disjoint unions of the young trees

\[
\begin{align*}
(0, 0 \mid n, m) & \to \{h\}, \\
(1, 0 \mid n, m) & \to \{h\}, \\
(0, 1 \mid n, m) & \to \{h\}, \\
(1, 0 \mid 0, 0) & \to \{h_{\star}\}.
\end{align*}
\]

To define \(SC_d,\star: D_{\star} \to \text{Top}\) we need the notion of the geometric realization of \(\beta\) in \(SC_d h(n, m)\).

6.4.2 Definition  Given \(\beta \in SC_d h(n, m)\), let \(|\beta|\) be its geometric realization. This is the subset of \(\mathbb{R}^d\) given by deleting the open disks and half-disks of \(\beta\) from the closed unit half-disk. More precisely, if \(\overline{D}_+^d\) is the closed unit half-disk in \(\mathbb{R}^d\), \(\{(D_{\star}^d)\}_{j=1}^m\) are the open disks of \(\beta\), and \(\{(D_{h,j}^d)\}_{i=1}^n\) are the open half-disks of \(\beta\) considered as open disks in \(\mathbb{R}^d\) whose center lies in \(\mathbb{R}^{d-1}\), then

\[
|\beta| = \overline{D}_+^d - \left( \left( \bigcup_{i=1}^{n} (D_{\star}^d)_{i} \right) \cup \left( \bigcup_{j=1}^{m} (D_{h,j}^d)_{j} \right) \right).
\]

Let

\[
\partial_h|\beta| := \partial \left( \overline{D}_+^d - \left( \bigcup_i (D_{h,i}^d) \right) \right)
\]
Kontsevich’s Swiss cheese conjecture

Figure 2: The collapsed disks are denoted by dots and the collapsed half-disks by tick marks. Collapsed disks are color $f_*$ input edges and collapsed half-disks are color $h_*$ input edges. To keep the collapsed disks and half-disks from coinciding, we only allow one or the other in any composition. Composition in $SC_{d,*}$ takes place only in the half-disks and collapsed half-disks. The only composition we can do in a collapsed half-disk is given by plugging in a collapsed disk. The result is a collapsed disk replacing the collapsed half-disk.

We think of the point $q \in |\alpha|$ as a collapsed disk and the point $q \in |\alpha| \cap \mathbb{R}^{d-1}$ as a collapsed half-disk. Composition in $SC_{d,*}$ takes place in the half-disks and collapsed half-disks only. The un-collapsed disks play no part in composition. However, the collapsed half-disks and collapsed disks only play a part in composition when we plug a collapsed disk into a collapsed half-disk. The result is a collapsed disk which happens to live on the boundary of the geometric realization. See Figure 2.

6.4.3 Definition Let $C_{\ast}$ denote the full subcategory of $D_{\ast}$ with objects isomorphic to disjoint unions of the young forests from (50) with $n \leq 1$. Recall from Example 6.2.3
that we used $D$ to denote the full subcategory of $\text{For}_K$ whose young trees are isomorphic to the trees in (50) with $k = \ell = 0$. Likewise, $C$ is the full subcategory of $D$ given by trees isomorphic to disjoint unions of forests from (50) with $k = \ell = 0$ and $n \leq 1$. We write $x \leq 1$ if $x \in C$ or $x \in C_\ast$. We have that $\text{SC}^\text{h}_d^1$ is the restriction of $\text{SC}^\text{h}_d$ to $C$ and $\text{SC}^\text{h}_d^\ast$ is the restriction of $\text{SC}^\text{h}_d$ to $D$. Let $\text{SC}^\text{h}_d^1$ denote the restriction of $\text{SC}^\text{h}_d_\ast$ to $C_\ast$.

Let

\[ \text{SC}^\text{h}_d_\ast \quad \text{and} \quad \text{SC}^\text{h}_d^1 \]

denote the $W$ construction applied to the 4–colored operads $\text{SC}^\text{h}_d_\ast$ and $\text{SC}^\text{h}_d^1$ via Definition 6.2.2. Let $F$ denote Kan extension along $C_\ast \to D_\ast$, that is, $F = F_d^C$ using Notation 6.1.11.

6.4.4 Definition Define $p: D_\ast \to D$ to send the $K_\ast$–colored young forest $x: I_x \to J_x$ to the $K$–colored forest $px$ with

\[ I_{px} = (I_x)_{t, h} \quad \text{and} \quad J_{px} = (J_x)_{t, h}, \]

where for $K' \subset K_\ast$ we set $I_{K'} = \text{col}^{-1}_{t, h}(K')$, $\text{col}_t: I \to K_\ast$. In (50) we see that $x(I_{px}) \subset J_{px}$, so that we can define $px$ as the restriction of $x$ to $I_{px}$. Observe that $p(1, 0 | n, m) = (n, m)$. If $f: y \to x$ is a forest, then $pf: py \to px$ is defined using $f$. Since $f$ preserves the colorings, $pf$ is indeed a forest from $py$ to $px$. If $f$ is a morphism in $C_\ast$ then $pf$ is a morphism in $C$.

If $\beta \in \text{SC}^\text{h}_d_\ast(z)$ for a $K_\ast$–colored young forest $z$, then we get $p\beta \in \text{SC}^\text{h}_d(pxz)$. To define $p\beta$ write $\beta = (\beta_j)_{j \in J_z}$, where $\beta_j \in \text{SC}^\text{h}_d_\ast(\text{col}^{-1}_{t, h}(j); j)$. Each $\beta_j$ is of the form $(\gamma_j, q_j)$ with $q_j \in |\gamma_j|$ or of the form $\gamma_j \in \text{SC}^\text{h}_d(\text{col}^{-1}_{t, h}(j); j)$. Set $p\beta = (\gamma_j)_{j \in J_{pxz}}$.

If $t \in W(f)$ and $f \in D_\ast$, then $E(pf) \subset E(f)$ and $pt \in W(pf)$ is defined to be the pullback of $t: E(f) \to [0, \infty]$.

If $(f: y \to x, t \in W(f), \alpha \in \text{SC}^\text{h}_d_\ast(x))$ represents a point in $\text{SC}^\text{h}_d_\ast(x)$ then $(pf, pt, p\alpha)$ represents a point in $\text{SC}^\text{h}_d(\alpha)$. This defines the map $p: \text{SC}^\text{h}_d_\ast(1, 0 | n, m) \to \text{SC}^\text{h}_d(n, m)$. The restriction of $p$ to $F(\text{SC}^\text{h}_d_\ast)(1, 0 | n, m)$ factors through the inclusion

\[ \text{SC}^\text{h}_d(1, 0 | n, m) \to \text{SC}^\text{h}_d(n, m). \]

Let $p_1$ be the induced map $F(\text{SC}^\text{h}_d_\ast)(1, 0 | n, m) \to \text{SC}^\text{h}_d(1, 0 | n, m)$.

Consider the commutative diagram of topological spaces, where the horizontal arrows do not assemble to operad maps:

\[
\begin{array}{ccc}
\text{SC}^\text{h}_d(1, 0 | n, m) & \to & F(\text{SC}^\text{h}_d_\ast)(1, 0 | n, m) \\
\downarrow & & \downarrow \\
\text{SC}^\text{h}_d(n, m) & \to & \text{SC}^\text{h}_d(n, m)
\end{array}
\]

\[
\begin{array}{ccc}
\text{SC}^\text{h}_d(1, 0 | n, m) & \to & \text{SC}^\text{h}_d(1, 0 | n, m) \\
\downarrow & & \downarrow \\
\text{SC}^\text{h}_d(n, m) & \to & \text{SC}^\text{h}_d(n, m)
\end{array}
\]
The maps $p_1$ and $p$ delete the collapsed disk and, if necessary, a leftover collapsed half-disk. By induction on $n$ we assume the right vertical arrow is an equivalence. We will show that for each $\alpha \in \text{SC}_d^{h\infty}(n,m)$ the inclusion $p_1^{-1}(\alpha) \rightarrow p^{-1}(\iota \alpha)$ is an equivalence. Then by the long exact sequence of homotopy groups we conclude that the middle vertical arrow is an equivalence. The top left and bottom right maps collapse the $n^{th}$ full-disk. One can show that these are equivalences. We conclude that the left vertical arrow is also an equivalence. This will prove Theorem 6.4.1.

6.4.2 Computing $p^{-1}(\iota \alpha)$ and $p_1^{-1}(\alpha)$ We have shown that the proof rests on Proposition 6.4.5 below. This section is dedicated to the proof of this proposition.

6.4.5 Proposition  Fix $\alpha \in \text{SC}_d^{h\infty}(n,m)$. The inclusion of the fiber $p_1^{-1}(\alpha)$ into the fiber $p_1^{-1}(\iota \alpha)$ is a weak equivalence.

Combining the $W$ construction (38) and the left adjoint $C – \text{Op} \rightarrow D – \text{Op}$ (that is, $\text{Op}(\text{Top}^{\Sigma^{\geq 1}}) \rightarrow \text{Op}(\text{Top}^\Sigma)$) we get

$$\text{SC}_d^{h\infty}(n,m) = \left( \bigsqcup_{f: y \rightarrow (n,m)} W(g) \times \text{SC}_d(z) \right) / \sim,$$

where $y \leq 1$ (Definition 6.4.3). If $\alpha \in \text{SC}_d^{h\infty}(n,m)$ is represented by $(f, g, t, \tilde{\alpha})$ where $f: y \rightarrow (n,m)$, $g: z_{\alpha} \rightarrow y$, $t \in W(g)$ and $\tilde{\alpha} \in \text{SC}_d(z)$, then $\iota \alpha \in \text{SC}_d^h(n,m)$ is represented by $(fg, W_\infty(f)t, \tilde{\alpha})$. Let $T_\alpha = fg: z_{\alpha} \rightarrow (n,m)$ and $t_\alpha = W_\infty(f)t$. Without loss of generality, we may assume $t_\alpha(i) > 0$ for every $i \in E(T_\alpha)$ and that $\tilde{\alpha}(j) \neq \text{id}_{\text{SC}_d}$ for any $j \in J_{z_{\alpha}}$.

6.4.6 Definition  Let $\text{Trees}(1, 0 \mid n, m)$ denote the over category $(D_\ast)/(1,0|n,m)$. Similarly, let $\text{Trees}(n, m) = D/(n,m)$. Let $p: \text{Trees}(1, 0 \mid n, m) \rightarrow \text{Trees}(n, m)$ denote the functor induced by $p$ from Definition 6.4.4. Note that $T_\alpha \in \text{Trees}(n, m)$. Let $(S, v) \in \text{Trees}(1, 0 \mid n, m)/T_\alpha$, where $S: x \rightarrow (1,0 \mid n,m)$ is any $K_\ast$–colored tree and $v: z_{\alpha} \rightarrow px$ is a forest such that $(pS)v = T_\alpha$. Define functors $W_\alpha: \text{Trees}(1, 0 \mid n, m)^{\text{op}}/T_\alpha \rightarrow \text{Top}$ and $\text{SC}_\alpha: \text{Trees}(1, 0 \mid n, m)/T_\alpha \rightarrow \text{Top}$ via the pullbacks:

\begin{equation}
\begin{array}{ccc}
\text{SC}_\alpha(S) & \xrightarrow{p} & W(pS) \\
\downarrow & & \downarrow \\
\* \rightarrow \text{SC}_d(z) & \xrightarrow{t_\alpha} & W(T_\alpha) \\
& \text{SC}_d(v) \rightarrow \text{SC}_d(px) & \\
\end{array}
\end{equation}

We want to replace $\text{Trees}(1, 0 \mid n, m)$ by a much smaller category. First we need the wedge operation on forests.
Figure 3: The edge $i$ and vertex $j$ of $T_\alpha$ give a commutative square in $\text{Trees}_\alpha$. The input vertices are circles. The output vertex ends in an $\times$. The internal vertices are filled dots. The input and internal vertices of $\Gamma_0$ and $\Gamma_1$ are labeled with their colors. In addition, the image of $S_{i,1}$ under the functor $p$ is shown. This makes it clear that the map $T_\alpha \to pS_{i,1}$ is given by inserting a single vertex.

6.4.7 Definition Let $f: x \to y$ be a $K_f$–colored forest and let $g: z \to w$ be a $K_g$–colored forest for some finite sets $K_f$ and $K_g$. Let $\tau: J_w \to J_x$ be any map. Define $x \vee_\tau z$ to be the young $K_f \sqcup K_g$–colored forest $(x, \tau, z): I_x \sqcup J_w \sqcup I_z \to J_x \sqcup J_z$ and define $y \vee_\tau w$ to be the young forest $(y, [f \mid x] \tau w): I_y \sqcup I_w \to J_y$ (see Definition 6.3.2). Finally, set $f \vee_\tau g: x \vee_\tau z \to y \vee_\tau w$ to be the forest

$$(f, g, f, \tau): I_y \sqcup I_w \sqcup J_x \sqcup J_z \to I_x \sqcup J_w \sqcup I_z \sqcup J_y.$$

6.4.8 Definition Let $\Gamma_0$ be the tree with no internal vertices and a single input vertex of color $f_\bullet$. Let $\Gamma_1$ be the tree with a single internal vertex of color $h_\bullet$ and a single input vertex of color $f_\bullet$.

For any edge $i \in \text{Edge}(T_\alpha)_h$ define $\nu(i): T_\alpha \to T_\alpha(i)$ to be the morphism in $\text{Trees}(n, m)$ which inserts a unary vertex along $i$. Call this new vertex $i_v$. Let $S_{i,k} = T_\alpha(i) \vee_{i_v} \Gamma_k$. For any internal vertex $j \in J_{z_\alpha}$ let $S_{j,k} = T_\alpha \vee_j \Gamma_k$. Note that $pS_{i,k} = T_\alpha(i)$ and $pS_{j,k} = T_\alpha$.

Let $\text{Trees}_\alpha$ be the full subcategory of $\text{Trees}(1, 0 \mid n, m)/T_\alpha$ given by the objects $S_{i,k} = (S_{i,k}, \nu(i))$ and $S_{j,k} = (S_{j,k}, \text{id}_{T_\alpha})$, where $i \in (I_{z_\alpha})_h \cup \{rt\}$, $j \in J_{z_\alpha}$ and $k \in \{0, 1\}$.

6.4.9 Remark The advantage of $T_\alpha$ is that it is easy to understand and computes the space $p^{-1} \alpha$ (Lemma 6.4.10). There is a unique morphism $S_{\ell,1}$ to $S_{\ell,0}$ for every $\ell$ and unique morphisms $S_{i,k} \to S_{T_\alpha^{-1}(i), k}$ and $S_{i,k} \to S_{z_\alpha(i), k}$. See Figure 3 for an illustration.
6.4.10 Lemma  The fiber $p^{-1}(\alpha)$ is given by the coend $W_\alpha \otimes \text{Trees}_\alpha \ SC_\alpha$.

Proof  Let $\gamma = [S, s, \gamma] \in SC_d^h(1, 0 | n, m)$, where $S: x \rightarrow (1, 0 | n, m)$ is a forest in $D_\bullet$, $s \in W(S)$ and $\gamma \in SC_d^h(x)$. Let us assume that $\gamma(j) \neq \text{id}$ for all $j \in J_x$ and $s(i) > 0$ for all $i \in E(S)$. Observe that $p\gamma \in SC_d^h(n, m)$ is given by $[pS, ps, p\gamma]$. If $p\gamma = \alpha$ there must be some $v: T_\alpha \rightarrow pS$ in $\text{Trees}(n, m)$ such that $SC_d(v)\alpha = p\gamma$ and $W(v)ps = t_\alpha$. The condition $t_\alpha(i) > 0$ for all $i \in E(T_\alpha)$ implies that $t_\alpha \neq W_\Sigma(v)(t')$ for any $t'$ and any $v$ which collapses any edges. Moreover the condition $p\gamma(j) \neq \text{id}$ for all $j$ implies that $p\gamma(j) \neq \text{id}$ for all $j \in J_x$ such that $x^{-1}(j)_{f_{i, h}}$ is not empty. We conclude that either $v = \text{id}$ or $v$ is the insertion of the unique unary (in $pS$, not in $S$) vertex $j$ such that $x^{-1}(j)_{f_{i, h}}$ is not empty. In the former case we must have $S = S_{j,k}$ for some vertex $j \in J_x$ and some $k \in \{0, 1\}$. In the latter case we have $S = S_{i,k}$ for some edge $i$ of $T_\alpha$ and some $k$. This defines the map $p^{-1}(\alpha) \rightarrow W_\alpha \otimes \text{Trees}_\alpha \ SC_\alpha$. The map in the other direction is clear and the verification that they are inverses is left to the reader. □

In diagram (52) we have $h$–colored edges $i_1$ and $i_2$ of $T_\alpha$ with $z_\alpha(i_1) = j = T_\alpha^{-1}(i_2)$. Thus we get the commutative diagram on the left. The image of this diagram under $SC_\alpha$ is shown on the right:

\begin{equation}
\begin{array}{ccc}
S_{i_1,1} & \rightarrow & S_{j,1} \\
| & \downarrow & | \\
S_{i_2,1} & \leftarrow & S_{j,1} \\
\end{array} \begin{array}{ccc}
S_{i_1,0} & \rightarrow & S_{j,0} \\
| & \downarrow & | \\
S_{i_2,0} & \leftarrow & S_{j,0} \\
\end{array}
\end{equation}

|id_h| \cap \mathbb{R}^{d-1} \rightarrow |\alpha(j)| \cap \mathbb{R}^{d-1} \leftarrow |id_h| \cap \mathbb{R}^{d-1}

The geometric realization of the identity $id_h$ is just $S_+^{d-1}$, the top half of the $(d-1)$–sphere. The input of $\alpha(j)$ corresponding to $i_1$ is a half-disk and the map $|id_h| \rightarrow |\alpha(j)|$ corresponding to $S_{i_1,0} \rightarrow S_{j,0}$ is just $\partial_1|\alpha(j)| \rightarrow |\alpha(j)|$ (see Definition 6.4.2). On the other hand, the image of $S_{i_2,0} \rightarrow S_{j,0}$ is the inclusion of the output boundary $\partial_0|\alpha(j)| \rightarrow |\alpha(j)|$.

6.4.11 Definition  Let $\epsilon_\bullet \in E(S_{\ell,1})$ be the unique internal edge of color $h_\bullet$. If $i \in \text{Edge}(T_\alpha)$ (Definition 6.1.2), let $i_v$ denote the vertex inserted by $v: T_\alpha \rightarrow pS_{i,k}$. Let $i_{in}$ and $i_{out}$ denote the incoming and outgoing edges of $i_v$, respectively, considered as internal edges of $S_{i,k}$. For any object $S_{\ell,k}$ of $\text{Trees}_\alpha$, let $E_\alpha(S_{\ell,k}) = \{\epsilon_\bullet \}^k \sqcup \{(i_{in}, i_{out}) \cap E(S_{\ell,k})\}$. This defines a functor $E_\alpha: \text{Trees}_\alpha^{\text{op}} \rightarrow \text{Set}$.

The image under $W_\alpha$ of the square in diagram (52) is:

\begin{equation}
\begin{array}{ccc}
[0, \infty]^2 & \xrightarrow{(id, 0)} & [0, \infty] \\
(0, id) \uparrow & & \uparrow 0 \\
[0, \infty] & \xleftarrow{\ast} & [0, \infty] \\
\end{array} \begin{array}{ccc}
& \xrightarrow{(id, \infty)} & \\
& \uparrow (0, id) & \\
& \xrightarrow{\infty} & \\
\end{array}
\end{equation}
More precisely,

\[(54) \quad W_\alpha(S) = \{s : E_\alpha(S) \to [0, \infty] \mid s(i_{\text{in}}) + s(i_{\text{out}}) = t_\alpha(i)\},\]

and \(W_\alpha(S) \to W_\alpha(S')\) for a map \(S' \to S\) in \(\text{Trees}_\alpha\) is given by push forward of functions along the map of finite sets \(E_\alpha(S) \hookrightarrow E_\alpha(S')\). There is no condition on \(s(\epsilon_s)\), the length of the edge of color \(h_s\). The isomorphism \(W_\alpha(S_{i,1}) \to [0, \infty]^2\) sends \(s\) to \((s(\epsilon_s), r(s(i_{\text{out}}), s(i_{\text{in}})))\), where

\[r(s_o, s_i) = \frac{1 - e^{-s_o}}{1 - e^{-s_i}},\]

which lands in \([0, \infty]\) because \(s_o + s_i = t_\alpha > 0\). Note that \(s_o = 0\) if and only if \(r(s_o, s_i) = 0\) and \(s_o = t_\alpha\) if and only if \(r(s_o, s_i) = \infty\). Since the morphism \(S_{i_1,1} \to S_{j,1}\) from diagram (52) collapses the edge \((i_1)_{\text{out}}\), we get \(W_\alpha(S_{j,1}) \cong \{(r_*, r) \in W_\alpha(S_{i_1,1}) \mid r = 0\}\). In the same diagram, the morphism \(S_{i_2,1} \to S_{j,1}\) collapses the edge \((i_2)_{\text{in}}\), so we have \(W_\alpha(S_{j,1}) \cong \{(r_*, r) \in W_\alpha(S_{i_2,1}) \mid r = \infty\}\). The unique morphism \(S_{i,1} \to S_{i,0}\) collapses the edge \(i_*,\) so that \(W_\alpha(S_{i,0}) \cong \{(r_*, r) \in W_\alpha(S_{i,1}) \mid r_* = 0\}\). The rest can be deduced from these cases.

**6.4.12 Lemma** For any functor \(F : \text{Trees}_\alpha \to \text{Top}\), the coend \(W_\alpha \otimes_{\text{Trees}_\alpha} F\) is the homotopy colimit of \(F\) over \(\text{Trees}_\alpha\).

**Proof** It is clear from diagrams (53) and (52) that \(W_\alpha(S)\) is the geometric realization of the nerve of the under category of \(S\) for each object \(S \in \text{Trees}_\alpha\). In addition, the maps \(W_\alpha(S) \to W_\alpha(S')\) for \(S' \to S\) agree with the maps obtained from the nerves of under categories. \(\square\)

**6.4.13 Lemma** We can explicitly compute \(p^{-1}(\iota \alpha)\) as

\[p^{-1}(\iota \alpha) \simeq |SC_d(T_\alpha)\tilde{\alpha}| \simeq (S^{d-1})^{\n},\]

where \(SC_d(T_\alpha)\tilde{\alpha}\) is the composition of all vertex labels from \(\iota \alpha\).

**Proof** Let \(\text{Trees}_{\iota \alpha,0}\) denote the full subcategory of \(\text{Trees}_\alpha\) consisting of objects \(S_{j,0}\) and \(S_{i,0}\) for internal vertices \(j\) and internal edges \(i\). This category is homotopy terminal so, by Lemma 6.4.12 and Lemma 6.4.10, we have \(p^{-1}(\iota \alpha) = \text{hocolim}_{\text{Trees}_{\iota \alpha,0}} SC_\alpha\). This is the same as the homotopy colimit of the coequalizer diagram

\[\bigsqcup_{i \in E(T_\alpha)} \text{id}_h \Rightarrow \bigsqcup_{j \in V(T_\alpha)} |\tilde{\alpha}(j)|,\]

where one arrow is given by including into output parts of the boundaries of the \(|\tilde{\alpha}(j)|\) and the other arrow is given by including into input boundaries. These maps are

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cofibrations with disjoint images. Each space in the coequalizer diagram is cofibrant. Thus the coequalizer diagram is already cofibrant as a functor \((\cdot \Rightarrow \cdot) \rightarrow \text{Top}\). Thus we can compute the normal colimit. It is clear that this is the same as composing the \(\tilde{\alpha}(j)\) via \(T_\alpha\) then taking the realization of the result. In addition, \(|\beta|\) is equivalent to a wedge of \(n\) spheres of dimension \(d-1\) if \(\beta \in SC^h_d(n, m)\).

6.4.14 Definition Let \(\text{Trees}_{\alpha,1}\) denote the full subcategory of \(\text{Trees}_\alpha\), where we discard the objects \(S_{j,0}\) and \(S_{i,0}\) for \(j \in J_\alpha\) and \(i \in E(T_\alpha)\). Define a functor \(W_{\alpha,1}: \text{Trees}_{\alpha,1}^{\text{op}} \rightarrow \text{Top}\) by setting

\[
W_{\alpha,1}(S) = \left\{ s: E_\alpha(S) \rightarrow [0, \infty] \mid \sum_{i \in E_\alpha(S)} s(i) = \infty \right\}.
\]

6.4.15 Lemma Suppose \(t_\alpha < \infty\) and \(n = 1\); then \(p^{-1}_1(\alpha)\) is given by the coend

\[
W_\alpha \otimes_{\text{Trees}_{\alpha,1}} SC_\alpha,
\]

where \(SC_\alpha\) is the functor in Definition 6.4.6 restricted to \(\text{Trees}_{\alpha,1}\) and \(W_{\alpha,1}\) is defined in Definition 6.4.14.

Proof Let \(\gamma \in F(SC^h_{d,\bullet})(1, m)\) be such that \(p_1(\gamma) = \alpha\). Pick a representative \((f, g, s, \tilde{\gamma})\), where \(f: y \rightarrow (1, 0 \mid 1, m)\), \(y \leq 1\), \(g: z \rightarrow y\), \(s \in W(g)\) and \(\tilde{\gamma} \in SC^h_{d,\bullet}(z)\). Consider \(\iota \gamma \in SC^h_{d,\bullet}(1, m)\), which is represented by \((fg, W_\infty(f)s, \tilde{\gamma})\). Recall that the condition \(y \leq 1\) means that each connected component of the young forest \(y\) has at most one input whose color lives in \(\{f, f_*\}\). This implies that \(f\) has at least one internal edge \(i \in E(f)\). Thus \(W_\infty(f)s(i) = \infty\) when \(i\) is viewed as an internal edge in \(fg\).

We know \(p\iota \gamma = i_\alpha\), so \(\iota \gamma\) is represented by some triple \((S, s', \tilde{\gamma})\) with \(S \in \text{Trees}_\alpha\), \(s' \in W_\alpha(S)\) and \(\tilde{\gamma} \in SC_\alpha(S)\). The relations in \(SC^h_{d,\bullet}\) preserve edges of length \(\infty\), so we must have \(s'(i) = \infty\) for some \(i \in E(S)\). We are assuming \(t_\alpha(i) < \infty\) for all \(i \in E(T_\alpha)\), so the infinite edge in \(S\) must be in \(E_\alpha(S)\). This implies \(s' \in W_{\alpha,1}(S)\).

Moreover, we cannot have such an infinite edge if \(S = S_{j,0}\) for some vertex \(j\) or \(S = S_{i,0}\) for some internal edge \(i\). Thus \(S \in \text{Trees}_{\alpha,1}\).

This defines the map from \(p^{-1}_1(\alpha)\) to the coend. We leave the rest to the reader. \(\square\)

6.4.16 Lemma For any functor \(F: \text{Trees}_{\alpha,1} \rightarrow \text{Top}\), the coend \(W_{\alpha,1} \otimes_{\text{Trees}_{\alpha,1}} F\) is the homotopy colimit of \(F\) over \(\text{Trees}_{\alpha,1}\).

Proof The argument here is similar to the proof of Lemma 6.4.12. \(\square\)
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6.4.17 Corollary If $t_\alpha < 0$ and $n = 1$, then the fiber $p_1^{-1}(\alpha)$ is equivalent to $\partial_h|SC_d(T_\alpha)\tilde{\alpha}| \simeq S^{d-1}$.

Proof By the same argument as in Lemma 6.4.13, $\hocolim_{\text{Trees}_{\alpha,1}} SC_\alpha$ is equivalent to $\colim_{\text{Trees}_{\alpha,1}} SC_\alpha$. This is easily computed as the $h$–colored boundary of the composite of $\tilde{\alpha}$. \hfill \Box

Proof of Proposition 6.4.5 Recall $\alpha$ is represented by $f: y \to (n, m)$, $y \leq 1$, $g: z_\alpha \to y$, $t \in W(g)$ and $\tilde{\alpha} \in SC_d(z_\alpha)$. By applying relations in $SC_d^{h\infty}$ we may assume $0 < t < \infty$. We may think of $(g, t)$ as representing an element of $SC_d^{h1}(y)$, which we can write as $(\alpha(j))_{j \in J_y}$. If $\alpha(j) \in SC_d^h(n_j, m_j)$ then $n_j \leq 1$. Clearly $p_1^{-1}(\alpha(j)) \simeq p^{-1}(\alpha(j)) \simeq *$ when $n_j = 0$. Since $t_{\alpha(j)} < \infty$ we can use Corollary 6.4.17 to conclude $p_1^{-1}(\alpha(j)) \simeq \partial_h|(SC_d(g)(\alpha))(j)|$. The fiber $p_1^{-1}(\alpha)$ is equal to the colimit of the diagram

$$\bigsqcup_{i \in E(f)} |1_h| \Rightarrow \bigsqcup_{j \in V(f)} p_1^{-1}(\alpha(j)),$$

where one arrow is given by $|1_h| \simeq \partial_i |\alpha(y(i))| \to \partial_h|(SC_d(g)(\alpha))(y(i))|$ and the other by $|1_h| \simeq \partial_\alpha |\alpha(f(i))| \to \partial_h|(SC_d(g)(\alpha))(y(i))|$. This colimit is clearly $(S^{d-1})^\vee n \simeq p^{-1}(t_\alpha)$. \hfill \Box

See Figure 4 for an illustration.

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404 23rd Ave NE, Norman, OK 73071, USA

Department of Mathematics, University of Notre Dame
255 Hurley Hall, Notre Dame, IN 46556-4618, USA

justmas@gmail.com

Proposed: Haynes Miller
Seconded: Peter Teichner, Bill Dwyer

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