The existence and stability of onsite solitons in a driven discrete nonlinear nonlocal Schrödinger equation

G Putra¹, M Syafwan¹,⋆ and H Haripamyu¹

¹Department of Mathematics, Universitas Andalas, Padang, 25163 Indonesia

⋆Corresponding e-mail: mahdhivan@sci.unand.ac.id

Abstract. In this paper, we examine numerically the existence and stability of onsite solitons in a driven discrete nonlinear nonlocal Schrödinger equation. The equation interpolates cubic and Ablowitz-Ladik nonlocal equations. We obtain that the solitons are always stable for small interpolation parameter. We also obtain that the driving parametric can destabilize the soliton solution.

1. Introduction

The discrete nonlinear Schrödinger (DNLS) is one of the most important model that describes many phenomena in physics and biology [1]. As a lattice systems, DNLS has various applications mainly in the fields of physics [2]. In optical fields, DNLS is a nonlinear arrays model of coupled waveguides [3] and describing the nonlinear waves in Bose-Einstein condensates [4]. The DNLS equation is also an effective model of certain biophysical systems, molecular crystals, and atomic chains [5].

One of the most interesting features of DNLS is the existence of its soliton solution. Soliton, as coined by Zabusky and Kruskal in 1965, is a particular wave of permanent form which is localized, decaying or becoming constant at infinity and may interact strongly with other solitons so that after the interaction it retains its form [6]. In physical optics, solitons have been proposed and observed in dielectric waveguide arrays under self-focussing [7].

There are numerous works on the existence and stability of soliton in DNLS equations. We note that two most commonly studied solitons are the onsite soliton (i.e., bond-centered solution) and the intersite soliton (i.e., site-centered solution) [1]. In the focussing case, Susanto et al. has investigated the parametrically driven DNLS equation in [8] where it was confirmed that the parametric driving can destabilize the onsite soliton. The existence and stability of discrete soliton in DNLS equation with inclusion of parametric driving also have been investigated in detail in [9].

The paper is organized as follows. The considered model equation and its preliminary analyses are presented in Section 2. In Section 3 we perform the numerical calculations for the time independent solutions of the system. The numerical scheme is set up in order to obtain the onsite soliton solutions. To check the stability of the obtained results, we solve the generalized eigenvalue problem (EVP) in corresponds with our considered model. Finally, we summarize our findings and point out some interesting subjects for future study in Section 4.

2. The model and preliminary analysis

The DNLS model we consider in this paper is the generalized parametrically driven discrete nonlinear nonlocal Schrödinger (PDNNLS) equation
\[ i\phi_n = \varepsilon \Delta_2 \phi_n - \Delta \phi_n - \sigma \bar{\phi}_n + \frac{\gamma}{2} \phi_n \bar{\phi}_{-n}(\phi_{n+1} + \phi_{n-1}) + (1 - \gamma)|\phi_n|^2 \bar{\phi}_{-n}. \quad (1) \]

In the model, \( \phi_n \equiv \phi_n(t) \) is a complex-valued function at time \( t \) and site \( n \), \( i = \sqrt{-1} \), the time derivative and complex conjugation are denoted by the overdot and the overline, respectively, \( \varepsilon > 0 \) is the coupling coefficient between two adjacent sites, \( \Delta_2 \phi_n = \phi_{n+1} - 2\phi_n + \phi_{n-1} \) is the discrete Laplacian in 1D, \( \gamma \) is the interpolation parameter, \( \sigma \) and \( \Lambda \) are the parametric driving and the frequency coefficient, respectively.

Model (1) can be interpreted as an equation that interpolates cubic discrete nonlinear nonlocal Schrödinger equation when \( \gamma = 0 \) and Ablowitz-Ladik discrete nonlinear nonlocal Schrödinger equation when \( \gamma = 1 \). As far as we know, the cubic DNLS equation, often referred to as "standard DNLS" or simply "DNLS", is perhaps the most studied version of DNLS equations. On the other hand, the Ablowitz-Ladik equation is a model which formulated by Ablowitz and Ladik in 1976.

In order to obtain the steady-state localized solutions, we set \( \phi_n \) as real-valued and time independent wave function. For this purpose, \( \phi_n \) satisfies

\[ \varepsilon \Delta_2 \phi_n - (\Lambda + \sigma) \phi_n + \frac{\gamma}{2} \phi_n \phi_{-n}(\phi_{n+1} + \phi_{n-1}) + (1 - \gamma)\phi_n^2 \phi_{-n} = 0 \quad (2) \]

In correspond to onsite soliton solutions, we let

\[ \phi_n \to 0 \quad \text{as} \quad n \to \pm \infty. \quad (3) \]

Under transformation \( \phi_{-n} \to \phi_n \), when \( \sigma = 0 \) (i.e., in the absence of parametric driving), it was shown in [10, 11] that any localized solutions of model (1) satisfying condition (3) are real-valued. In the other hand, when \( \sigma > 0 \), the solution of the stationary PDNNLS (1) takes one of the two possibilities, i.e., either real or imaginary where the solution for the imaginary case is always unstable. We only interested to investigate the real stationary solution in this paper.

3. Soliton and numerical stability

We seek for steady-state localized solution in the form of onsite soliton. For this purpose, we employ the Newton's method. In the following, we set \( \Lambda = 1 \). First, we begin the numerical calculation for \( \varepsilon = \gamma = 0 \) by choosing the initial guess

\[ \phi_n = \begin{cases} 0, & n \neq 0, \\ 1, & n = 1. \end{cases} \]

Next, we use the obtained result as the seed for next computation with different values of \( \varepsilon \) and \( \gamma \). We present the results in Fig. 1 and Fig. 2.

In this section, we focus our attention to the stability of the obtained solutions of the PDNNLS (1). Once such discrete solitons have been found, let say it \( \varphi_n \), we can proceed to determine their linear stability by solving a corresponding EVP. First, we make use of a linearization function

\[ \varphi_n(t) = \varphi_n + \delta \varphi_n(t), \quad \delta \ll 1, \]

and substitute the function into PDNNLS (1). Let \( \psi_n(t) = (\eta_n + i\xi_n)e^{\lambda t} \), we obtain an eigenvalue problem

\[ \lambda \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix} = \mathcal{M} \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix}, \]

where

\[ \mathcal{M}_{1,1} = 0, \quad \mathcal{M}_{1,2} = \begin{pmatrix} c + \frac{\gamma}{2} \varphi_n \varphi_{-n} \Delta_2 + 2 \left( 1 - \frac{\gamma}{2} \right) \varphi_n \varphi_{-n} + \frac{\gamma}{2} \varphi_{-n}(\varphi_{n+1} + \varphi_{n-1}) + \sigma - 1 \\ (\gamma - 1)\varphi_n^2 - \frac{\gamma}{2} \varphi_{-n}(\varphi_{n+1} + \varphi_{n-1}) \end{pmatrix}, \]
Figure 1. Profile of onsite solitons of model (1) and their numerical stability for several values of $\varepsilon$ in the absence of interpolation parameter ($\gamma = 0$).

$$
M_{2,1} = -\left(\varepsilon + \frac{\gamma}{2} \varphi_{n-1} \varphi_{n-1}\right) d_2 - 2 \left(1 - \frac{\gamma}{2}\right) \varphi_n \varphi_{n-1} - \frac{\gamma}{2} \varphi_{n-1} (\varphi_{n+1} + \varphi_{n-1}) + \sigma + 1
- (\gamma - 1) \varphi_{n-1}^2 + \frac{\gamma}{2} \varphi_{n-1} (\varphi_{n+1} + \varphi_{n-1}),
$$

$$
M_{2,2} = 0,
$$

where the eigenvalues $\lambda$ and the corresponding eigenvectors $[\eta_n, \xi_n]^T$ can be obtained numerically. We note that a soliton is stable when $\text{Re}(\lambda) < 0$.

In Fig. 1, we plot the onsite solitons and their corresponding eigenvalue structures for a fixed parametric driving $\sigma = 0.02$ and interpolation parameters $\gamma = 0$. We observed that the solitons are stable for small coupling strength $\varepsilon$. Note that we choose $\varepsilon = 1, 1.4$ in order to confirm our results with the one in [8]. In the presence of interpolation parameter, i.e., $\gamma = 0.02$, Fig. 2 shown that the solitons for small $\varepsilon$ are also stable.

Regarding the intervals of stability/instability of the onsite solitons, Fig. 3 shown the a description of the dynamics of PDNNLS model in the presence of interpolation parameter $\gamma$. For small $\gamma = 0.02$, the obtained result in the top panel of Fig. 3 is similar with the one in [8]. It means that the driving parametric $\sigma$ has a contribution in stabilizing/destabilizing the solitons. For larger value of the interpolation parameter, i.e., $\gamma = 0.5$, the driving parametric $\sigma$ also can stabilize/destabilize the
solitons. We note that the zero solution appears for larger value of driving parametric (i.e., $\gamma > 0.5$), hence the analysis for its stability regions is omitted.

**Figure 2.** Profile of onsite solitons of model (1) and their numerical stability for several values of $\epsilon$ in the presence of interpolation parameter ($\gamma = 0.02$).
4. Conclusions
We have studied the soliton solution of PDNNLS equation and its stability by means of numerical methods. We obtain that the soliton solution of PDNNLS equation is stable for small interpolation parameter. The driving parametric also play an important role in stabilizing/destabilizing the solitons. For future work, analytical approach, e.g. the variational approximation, can be employed to study the soliton solution of PDNNLS equation. The validity of the analytical methods is also an interesting subject to be considered for the future study.

References
[1] Kevrekidis P G 2009 Discrete Nonlinear Schrödinger Equation: Mathematical Analysis, Numerical Computations and Physical Perspectives (Berlin: Springer)
[2] Rusin R, Kusdiantara R and Susanto H 2018 J. Phys. A: Math. Theor. 51 475202
[3] Christodoulides D N and Joseph R I 1988 Opt. Lett. 13 794
[4] Carretero-Gonzales R, Frantzeskakis D J and Kevrekidis P G 2008 Nonlinear Waves in Bose-Einstein Condensates: Physical Relevance and Mathematical Techniques Nonlinearity 21 R139
[5] Ablowitz M J and Musslimani Z H 2014 J. Phys. Rev. E 87 032912
[6] Drazin P G 1989 *Solitons* (Cambridge: Cambridge University Press)

[7] Liu Y, Bartal G, Genov D A and Zhang X 2007 *J. Phys. Rev. Lett.* **99** 153901

[8] Susanto H, Hoq Q E and Kevrekidis P G 2006 *Phys. Rev. E* **74** 067601

[9] Syafwan M, Susanto H and Cox S M 2010 *Phys Rev E* **81** 026207

[10] Kevrekidis P G, Rasmussen Ø and Bishop A R 2001 *Internal. J. Modern. Phys. B* **15** 2833

[11] Hennig D and Tsironis G 1999 *Phys. Rep.* **307** 333

[12] Syafwan M 2012 The Existence and Stability of Solitons in Discrete Nonlinear Schrödinger Equations (PhD Thesis, The University of Nottingham).