TAUTOLOGICAL ALGEBRA OF THE MODULI STACK OF SEMISTABLE BUNDLES OF RANK 2 ON A GENERAL CURVE

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Abstract. Our aim in this paper is to determine the tautological algebra generated by the cohomology classes of the Brill-Noether loci in the rational cohomology of the moduli stack $\mathcal{U}_C(n, d)$ of semistable bundles of rank $n$ and degree $d$. When $C$ is a general smooth projective curve of genus $g \geq 2$, $d = 2g - 2$, the tautological algebra of $\mathcal{U}_C(2, 2g - 2)$ (resp. the moduli stack $\mathcal{SU}_C(2, \mathcal{L})$ of semistable bundles of rank 2 and determinant $\mathcal{L}$ with deg($\mathcal{L}$) = $2g - 2$) is generated by the divisor classes (resp. the class of the Theta divisor $\Theta$).

1. Introduction

Suppose $C$ is a complex smooth projective curve of genus $g$. The Jacobian variety $J_d(C)$ parametrises isomorphism classes of degree $d$ line bundles on $C$. The classical Brill-Noether subvarieties $W^r_d$ of $J_d(C)$ parametrise line bundles with at least $r + 1$ linearly independent sections. The questions on non-emptiness, dimension, and irreducibility of the loci have been classically studied (cf. [ACGH, Chapter IV], [Ge-Hr 1] and [Su]). In other direction, the classical Poincaré formula expresses the cohomological classes of $W^0_i$, in terms of the Theta divisor on $J(C)$:

$$[W^0_i] = \frac{1}{(g - i)!}[\Theta]^{g - i} \in H^*(J(C), \mathbb{Q}).$$

Here we identify $J_d(C) \cong J_0(C) = J(C)$ (cf. [ACGH, Chapter 1, §5; p. 25]).

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A similar formula holds amongst the cohomology classes of $W^r_d$, for varying $r$ in $J_d(C)$, for a general smooth curve $C$ (see Theorem 6.5):

$$[W^r_d] = \prod_{a=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} [\Theta_d]^{g-\rho}.$$

Here $\rho = \dim(W^r_d)$. When $\rho = 0$, the above formula is known as the Castelnuovo formula or the Porteous formula, in general.

For $n, d \geq 1$ we denote by $U_C(n, d)$ the moduli stack of semistable vector bundles of rank $n$ and degree $d$. For $L \in \text{Pic}(C)$ we denote by $SU_C(n, L)$ the moduli stack of semistable vector bundles of rank $n$ and determinant $L$. Their coarse moduli spaces $U_C(n, d)$ and $SU_C(n, L)$ have been widely studied. The Brill-Noether loci have been similarly defined and investigated, and significant results have been obtained (cf. [B 1], [B 2], [Br-Gz-Ne], [Me 1], [Me 2] and [Su]). More recent developments on non-emptiness of the Brill-Noether loci can be found in [La-Ne-St], [La-Ne-Pr], [La-Ne 1], [La-Ne 2] and [La-Ne 3].

Some questions on the cohomology classes have been raised by C. S. Seshadri and N. Sundaram in [Su, p. 176]. As pointed out by P. Newstead, the virtual cohomology classes can be worked out in terms of the known generators of the cohomology of moduli space when $n$ and $d$ are coprime, using the determinantal structure of the Brill-Noether loci. But it is difficult in general to determine whether a polynomial in the generators gives a non-zero cohomology class. Even in rank two, where the generators for the relations are well known, this is difficult. In general the questions are open and in this paper, we illustrate a method to address these questions. We consider rank two situation, and when the degree $d = 2g - 2$.

Our aim in this paper is to obtain a Poincaré type relation amongst the cohomological classes of the geometric Brill-Noether loci, when the curve $C$ is general. As a first step, in [Mk], when $g = 1$ the relations were found to be similar to the Poincaré relations. The moduli spaces $U_C(2, 2g - 2)$ and $SU_C(2, L)$ are singular varieties, and the cohomology class of a Brill-Noether locus is well-defined in the cohomology of the moduli stack $H^*(U_C(2, 2g - 2), \mathbb{Q})$ (resp. $H^*(SU_C(2, L), \mathbb{Q})$) (see §).

We show the following:

**Theorem 1.1.** Suppose $C$ is a general smooth projective curve of genus $g$, and $g \geq 2$. The cohomology class of a Brill-Noether locus on the moduli stack $U_C(2, 2(g-1))$ can be expressed as a polynomial on the divisor classes, with rational coefficients.

See Theorem 9.6.

Similarly, consider the moduli stack $SU_C(2, L)$ of semistable bundles of rank $r$ and fixed determinant $L$ of degree $2g - 2$ on $C$. Then we obtain the following:

**Corollary 1.2.** The cohomology class of a Brill-Noether locus $\tilde{W}^r_{2, 2(g-1)}$ in the moduli stack $SU_C(2, L)$ is a polynomial expression on the class of the Theta divisor, with rational coefficients. In other words, the tautological algebra generated by the Brill-Noether loci is generated by the class of the theta divisor.

See Corollary 9.7.
The nontriviality of the above algebra follows from [La-Ne-Pr].

The key idea is to relate the Brill-Noether loci on the moduli space with the Brill-Noether loci on the Jacobian variety of a general spectral curve. We utilise the rational map obtained in [Be-Na-Ra] from the Jacobian of a general spectral curve to the moduli space $U_C(2, 2(g - 1))$. Another ingredient is to note that the Hodge conjecture holds for the Jacobian of a general spectral curve, via a computation of the Mumford-Tate group [Bi]. We also use the fact that the moduli stacks are quotients stacks [Go]. Hence their cohomology is the equivariant cohomology of (an open subset of) Quot scheme. This enables us to define the Brill-Noether classes in the cohomology of the moduli stack. See §8.

For moduli stacks $U_C(n, d)$, when $n > 2$ and $d \neq 2g - 2$, the same proofs and techniques hold. However to obtain the final conclusion, we need to have that the Hodge conjecture holds for Jacobian of a higher degree general spectral curve (cf. [Ar], for unramified coverings). The proofs employed in Theorem 1.1 will then also hold for higher rank moduli spaces. For degree $d \neq 2g - 2$, we need to know that the divisor classes on the Jacobian of spectral curve descend on the moduli space (see Lemma 9.5). P. Newstead informed us that the descent of divisor classes is known in several other cases too.

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2. Notations

All the varieties are defined over complex numbers.

1. $U_C(n, d)$ denotes the moduli stack of $S$-equivalence classes of semistable bundles of rank $r$ and degree $d$ over $C$. $U_C(n, d)$ denotes its coarse moduli scheme.

2. $SU_C(n, \mathcal{L})$ denotes the moduli stack of $S$-equivalence classes of semistable bundles of rank $r$ and fixed determinant $\mathcal{L}$ of degree $d$ over $C$. $SU_C(n, \mathcal{L})$ denotes its coarse moduli scheme.

3. $J(C)$ is the Jacobian variety of isomorphism classes of line bundles of degree 0 over $C$.

4. $J_d(C)$ is the isomorphism classes of line bundles of degree $d$ over $C$.

5. $\mathcal{O}(D)$ denotes the line bundle corresponding to a divisor $D$ on $C$.

6. Given a closed subvariety $W \subset X$, $[W]$ denotes the cohomology class in the integral or rational cohomology group of $X$.

3. Spectral curves and Moduli Spaces

In this section we recall the construction of spectral curve from [Be-Na-Ra] which will be needed in this paper.
3.1. **Spectral curve.** Let $C$ be a smooth projective curve of genus $g \geq 2$ defined over complex numbers. Fix $n \geq 1$. Let $L$ be a line bundle on $C$ and $s = (s_k)$ be sections of $L^k$ for $k = 1, 2, \cdots, n$. Let $\pi : \mathbb{P}(\mathcal{O}_C \oplus L^*) \to C$ be the natural projection map and $\mathcal{O}(1)$ be the relatively ample bundle. Then $\pi_*(\mathcal{O}(1))$ is naturally isomorphic to $\mathcal{O}_C \oplus L^*$ and therefore has a canonical section. This provides a section of $\mathcal{O}(1)$ denoted by $y$. By projection formula we have:

$$\pi_*(\pi^*L \otimes \mathcal{O}(1)) \cong L \otimes \pi_*(\mathcal{O}(1)) \cong L \otimes (\mathcal{O}_C \oplus L^*) = L \oplus \mathcal{O}_C.$$  

Therefore $\pi_*(\pi^*L \otimes \mathcal{O}(1))$ also has a canonical section and we denote the corresponding section of $\pi^*L \otimes \mathcal{O}(1)$ by $x$. Consider the section

$$x^n + (\pi^*s_1)y x^{n-1} + \cdots + (\pi^*s_n)y^n \quad (1)$$

of $\pi^*L^n \otimes \mathcal{O}(n)$. Zero scheme of this section is a subscheme of $\mathbb{P}(\mathcal{O}_C \oplus L^*)$ and is called *spectral curve* of the given curve $C$ and is denoted by $\widetilde{C}_s$ or $\widetilde{C}$ in short. Let $\pi : \widetilde{C} \to C$ be the restriction of the natural projection $\pi : \mathbb{P}(\mathcal{O}_C \oplus L^*) \to C$. It can be checked that $\pi : \widetilde{C} \to C$ is finite and its fiber over any point $c \in C$ is a subscheme of $\mathbb{P}^1$ given by

$$x^n + a_1 y x^{n-1} + \cdots + a_n y^n = 0,$$

where $(x, y)$ is a homogeneous co-ordinate system and $a_i$ is the value of $s_i$ at $c$.

Let $\tilde{g}$ be the genus of $\widetilde{C}$. As $\pi_*(\mathcal{O}_{\widetilde{C}}) \cong \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)}$, we have the following relation between genus $\tilde{g}$ of the spectral curve $\widetilde{C}$ and genus $g$ of $C$ using Riemann-Roch theorem

$$1 - \tilde{g} = \chi(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) = \chi(C, \pi_*(\mathcal{O}_{\widetilde{C}})) = \sum_{i=0}^{n-1} \chi(C, L^{-i}) = -(\deg L) \cdot \frac{n(n-1)}{2} + n(1 - g).$$

Hence we have:

$$\tilde{g} = (\deg L) \cdot \frac{n(n-1)}{2} + n(g - 1) + 1. \quad (2)$$

Moreover if we take the line bundle $L$ to be of degree $2g - 2$, say the canonical line bundle $K_C$ for example, then from (2) the genus $\tilde{g}$ of the corresponding spectral curve $\widetilde{C}$ is given by:

$$\tilde{g} = n^2(g - 1) + 1 = \dim U_C(n, d). \quad (3)$$

3.2. **Spectral curve and moduli space of semistable bundles.** Here we relate the spectral curve $\widetilde{C}$ with the moduli space of semistable bundles of fixed rank and degree over $C$. Consider the following theorem.

**Theorem 3.1.** Let $C$ be any curve and $L$ any line bundle on $C$. Let $s = ((s_i)) \in \Gamma(L) \oplus \Gamma(L^2) \oplus \cdots \oplus \Gamma(L^n)$ be so chosen such that the corresponding spectral curve $\widetilde{C}_s$ is integral, smooth and non-empty. Then there is a bijective correspondence between isomorphism classes of line bundles on $\widetilde{C}_s$ and isomorphism classes of pairs $(E, \phi)$ where $E$ is a vector bundle of rank $n$ and $\phi : E \to L \otimes E$ a homomorphism with characteristic coefficients $s_i$.

**Proof.** See [Be-Na-Ra, Proposition 3.6, Remark 3.1,3.5 and 3.8; p. 172-174].
Let \( n \) be any positive integer. Then following the construction of spectral curve, by Theorem 3.1 we get a smooth, irreducible curve \( \tilde{C} \) and an \( n \)-sheeted branched covering \( \pi : \tilde{C} \to C \) such that a general \( E \in U_C(n, d) \) is the direct image \( \pi_*l \) of a \( l \in J_\delta(\tilde{C}) \). The relation between \( \delta \) and \( d \) can be calculated as follows (cf. \[Be-Tu\], p. 332). By the Leray spectral sequence we have:

\[
H^i(\tilde{C}, l) = H^i(C, \pi_*l) \tag{4}
\]

for all \( i \). Hence we have:

\[
\chi(\tilde{C}, l) = \chi(C, \pi_*l) = \chi(C, E).
\]

So by Riemann-Roch theorem we get,

\[
\chi(\tilde{C}, l) = \chi(C, E)
\Rightarrow \delta - (\bar{g} - 1) = d - n(g - 1)
\Rightarrow \delta = d + (\bar{g} - 1) - n(g - 1).
\]

Therefore by (3) we get the following relation between \( \delta(= \deg l) \) and \( d(= \deg E) \):

\[
\delta = d + (n^2 - n)(g - 1). \tag{5}
\]

As direct image of a line bundle is not necessarily semistable, the map

\[
\pi_* : J_\delta(\tilde{C}) \to U_C(n, d)
\]

is only a rational map. Let us denote by \( J^{ss} \) the semistable locus of \( J_\delta(\tilde{C}) \) defined as:

\[
J^{ss} := \{ l \in J_\delta(\tilde{C}) \mid \pi_*l \in U_C(n, d) \}.
\]

Then \( J^{ss} \) is a Zariski open subset of \( J_\delta(\tilde{C}) \) and the map

\[
\pi_* : J^{ss} \to U_C(n, d) \tag{6}
\]

is a regular dominant map (cf. \[Be-Na-Ra\] Theorem 1; p. 169). Moreover, the following theorem shows that the map \( \pi_* \) is a generically finite map.

**Theorem 3.2.** The map \( \pi_* : J^{ss} \to U_C(n, d) \) is of degree \( 2^{3g-3} \cdot 3^{5g-5} \cdots n^{(2n-1)(g-1)} \).

**Proof.** See \[Be-Na-Ra\] Remark 5.4; p. 177. \( \square \)

4. **Jacobian of a spectral curve and cycle class maps**

In this section we consider the moduli space \( U_C(2, d) \). For a general \( E \in U_C(2, d) \) we get a spectral curve \( \pi : \tilde{C} \to C \) where the map \( \pi \) is a 2-sheeted branched covering. Let \( n \) be the number of branch points. Then by Riemann-Hurwitz formula we get:

\[
\bar{g} = \frac{n}{2} + 2g - 1. \tag{7}
\]

Also we have from (3):

\[
\tilde{g} = 4g - 3. \tag{8}
\]

Therefore from (7) and (8) we get,

\[
n = 4g - 4 \neq 0 \text{ as } g \geq 2
\]

that is, \( \pi : \tilde{C} \to C \) is ramified with \( 4g - 4 \) branch points. Now we have the following lemma.
Lemma 4.1. The map \( \pi^* : J(C) \to J(\tilde{C}) \) is injective.

Proof. See [Mu, Lemma; p. 332].

Consider the following Norm map, denoted by \( \text{Nm}(\pi) \), associated to the map \( \pi : \tilde{C} \to C \).

\[
\text{Nm}(\pi) : J(\tilde{C}) \to J(C)
\]

\[
\sum_{i=1}^{m} n_i \tilde{x}_i \mapsto \sum_{i=1}^{m} n_i \pi(\tilde{x}_i).
\]

Identity component of \( \text{Ker}(\text{Nm}(\pi)) \) is defined to be the Prym variety associated to the covering \( \pi : \tilde{C} \to C \). But in our context the definition of Prym variety can be further improved. For that consider the following Lemma.

Lemma 4.2. The following conditions are equivalent.

1. The map \( \pi^* : J(C) \to J(\tilde{C}) \) is injective.
2. \( \text{Ker}(\text{Nm}(\pi)) \) is connected.

Proof. See [Ka, Lemma 1.1; p. 337].

So by Lemma 4.1 and 4.2 Prym variety associated to the covering \( \pi : \tilde{C} \to C \) is nothing but \( \text{Ker}(\text{Nm}(\pi)) \). Moreover we have, \( J(\tilde{C}) \cong J(C) + P \), where \( P \) is the Prym variety associated to the covering \( \pi : \tilde{C} \to C \). But this sum is not a direct sum as cardinality of \( J(C) \cap P \) is a non-zero finite number. Let \( H \) be the kernel of the map

\[
J(C) \times P \to J(\tilde{C}) \cong J(C) + P
\]

\[
(x, y) \mapsto x + y.
\]

Then we have:

Theorem 4.3.

\[
J(\tilde{C}) \cong \frac{J(C) \oplus P}{H},
\]

where \( P \) is a Prym variety and \( H \) is a finite group. In other words, there exists an isogeny from \( J(C) \times P \) to \( J(\tilde{C}) \).

Proof. See [Mu, Corollary 1; p. 332].

4.1. The Hodge \((p,p)\)-conjecture. Let \( X \) be a smooth projective variety over complex numbers. Let \( Z^p(X) \) be the free abelian group generated by the irreducible subvarieties of codimension \( p \) in \( X \). The elements of \( Z^p(X) \) are called algebraic cycles of codimension \( p \).

Let \( CH^p(X) \) be the \( p \)-th graded piece of the Chow ring \( CH^*(X) \). Here \( CH^p(X) \) is defined as

\[
CH^p(X) := Z^p(X)/\sim,
\]

“\( \sim \)” being the rational equivalence in the sense of [Fu, Chapter 1; p. 6]. Consider the cycle class map defined as follows.

\[
cl : CH^p(X) \to H^{2p}(X, \mathbb{Z})
\]

\[
Z \mapsto [Z],
\]
where $Z$ is an irreducible subvariety of codimension $p$ in $X$ and extend it linearly to the whole of $CH^p(X)$. Let $CH^*(X) \otimes \mathbb{Q}$ be denoted by $CH^*(X)_{\mathbb{Q}}$ and $j_{\mathbb{Q}} : H^{2p}(X, \mathbb{Q}) \to H^{2p}(X, \mathbb{C})$ be the natural map. Then the subspace of Hodge classes of $H^{2p}(X, \mathbb{Q})$, denoted by $H^{2p}_{\text{Hodge}}(X)$, is defined as:

$$H^{2p}_{\text{Hodge}}(X) := H^{2p}(X, \mathbb{Q}) \cap j_{\mathbb{Q}}^{-1}(H^{p,p}(X)).$$

(10)

Consider the cycle class map $cl : CH^p(X)_{\mathbb{Q}} \to H^{2p}(X, \mathbb{Q})$ defined similarly as in (9). The image of this map is denoted by $H^{2p}(X, \mathbb{Q})_{\text{alg}}$ and the elements in $H^{2p}(X, \mathbb{Q})_{\text{alg}}$ are called rational algebraic classes. The Hodge $(p,p)$-conjecture asserts the following:

**Hodge conjecture:**

$$H^{2p}(X, \mathbb{Q})_{\text{alg}} = H^{2p}_{\text{Hodge}}(X)$$

that is, any rational algebraic class is a Hodge class and vice versa.

**Remark 4.4.** Hodge $(p,p)$-conjecture is trivially true for $p = 0$. For $p = 1$, see [Le].

4.2. **The Hodge conjecture for a general Jacobian.** Let $X$ be an abelian variety. The subring of $H^{2p}_{\text{Hodge}}(X)$ generated by $H^{0}_{\text{Hodge}}(X)$ and $H^{2p}_{\text{Hodge}}(X)$ is denoted by $D^*(X)$. The cycle classes in $D^*(X)$ are all algebraic by Remark 4.4. Let $D^p(X)$ be the $p$th graded piece of $D^*(X)$. In particular, the Hodge $(p,p)$-conjecture is true if

$$D^p(X) = H^{2p}_{\text{Hodge}}(X).$$

Mattuck proved that Hodge conjecture is true for a general polarised abelian variety (cf. [Ma]). Tate proved the Hodge conjecture for self product of an elliptic curve (cf. [Ta] and [Gr §3]) and Murasaki did some explicit computations for the same (cf. [Mr]). Then using degeneration technique one can prove that Hodge conjecture holds for a general polarised Jacobian variety with Theta divisor $\Theta$ as a polarisation.

**Theorem 4.5.** For a general polarised Jacobian $(J(C), \Theta)$ of dimension $g$

$$H^{2p}_{\text{Hodge}}(J(C)) = D^p(J(C)) \cong \mathbb{Q}$$

for all $p = 0, \ldots, g$.

**Proof.** See [La-Bk] Theorem 17.5.1; p. 561. \qed

5. **Equivariant Cohomology and Equivariant Chow groups**

In this section, we recall some preliminary facts on the equivariant groups for a smooth variety $X$ of dimension $d$, which is equipped with an action by a linear reductive algebraic group $G$. The equivariant groups and their properties that we recall below were defined by Borel, Totaro, Edidin-Graham, Fulton [Bo], [To], [Ed-Gr], [Fu], [Ed].
5.1. **Equivariant cohomology** \( H^*_G(X, \mathbb{Z}) \) of \( X \). Suppose \( X \) is a variety with an action on the left by an algebraic group \( G \). Borel defined the equivariant cohomology \( H^*_G(X) \) as follows. There is a contractible space \( EG \) on which \( G \) acts freely (on the right) with quotient \( BG := EG/G \). Then form the space

\[
EG \times_G X := EG \times X/(e.g, x) \sim (e, g.x).
\]

In other words, \( EG \times_G X \) represents the (topological) quotient stack \([X/G]\).

**Definition 5.1.** The equivariant cohomology of \( X \) with respect to \( G \) is the ordinary singular cohomology of \( EG \times_G X \):

\[
H^i_G(X) = H^i(EG \times_G X).
\]

For the special case when \( X \) is a point, we have

\[
H^i_G(\text{point}) = H^i(BG).
\]

For any \( X \), the map \( X \to \text{point} \) induces a pullback map \( H^i(BG) \to H^i_G(X) \). Hence the equivariant cohomology of \( X \) has the structure of a \( H^i(BG) \)-algebra, at least when \( H^i(BG) = 0 \) for odd \( i \).

5.2. **Equivariant Chow groups** \( CH^i_G(X) \) of \( X \). [Ed-Gr]

As in the previous subsection, let \( X \) be a smooth variety of dimension \( n \), equipped with a left \( G \)-action. Here \( G \) is an affine algebraic group of dimension \( g \). Choose an \( l \)-dimensional representation \( V \) of \( G \) such that \( V \) has an open subset \( U \) on which \( G \) acts freely and whose complement has codimension more than \( n - i \). The diagonal action on \( X \times U \) is also free, so there is a quotient in the category of algebraic spaces. Denote this quotient by \( X_G := (X \times U)/G \).

**Definition 5.2.** The \( i \)-th equivariant Chow group \( CH^i_G(X) \) is the usual Chow group \( CH^i(X_G) \).

The codimension \( i \) equivariant Chow group \( CH^i_G(X) \) is the usual codimension \( i \) Chow group \( CH^i(X_G) \).

Note that if \( X \) has pure dimension \( n \) then

\[
CH^i_G(X) = CH^i(X_G) = CH^i_{n+l-g-i}(X_G) = CH^G_{n-i}(X).
\]

**Proposition 5.3.** The equivariant Chow group \( CH^i_G(X) \) is independent of the representation \( V \), as long as \( V - U \) has codimension more than \( n - i \).

**Proof.** See [Ed-Gr, Definition-Proposition 1]. \( \square \)

**Lemma 5.4.** If \( Y \subset X \) is an \( m \)-dimensional subvariety which is invariant under the \( G \)-action, and compatible with the \( G \)-action on \( X \), then it has a \( G \)-equivariant fundamental class \([Y]_G \in CH^m_G(X)\).
Proof. Indeed, we can consider the product \((Y \times U) \subset X \times U\), where \(U\) is as above and the corresponding quotient \((Y \times U)/G\) canonically embeds into \(X_G\). The fundamental class of \((Y \times U)/G\) defines the class \([Y]_G \in CH^G_m(X)\). More generally, if \(V\) is an \(l\)-dimensional representation of \(G\) and \(S \subset X \times V\) is an \(m+l\)-dimensional subvariety which is invariant under the \(G\)-action, then the quotient \((S \cap (X \times U))/G \subset (X \times U)/G\) defines the \(G\)-equivariant fundamental class \([S]_G \in CH^G_m(X)\) of \(S\). □

Proposition 5.5. If \(\alpha \in CH^G_m(X)\) then there exists a representation \(V\) such that \(\alpha = \sum a_i[S_i]_G\), for some \(G\)-invariant subvarieties \(S_i\) of \(X \times V\).

Proof. See [Ed-Gr, Proposition 1]. □

5.3. Functoriality properties. Suppose \(f : X \to Y\) is a \(G\)-equivariant morphism. Let \(S\) be one of the following properties of schemes or algebraic spaces: proper, flat, smooth, regular embedding or l.c.i.

Proposition 5.6. If \(f : X \to Y\) has property \(S\), then the induced map \(f_G : X_G \to Y_G\) also has property \(S\).

Proof. See [Ed-Gr, Proposition 2]. □

Proposition 5.7. Equivariant Chow groups (resp. equivariant cohomology) have the same functoriality as ordinary Chow groups (resp. cohomology) for equivariant morphisms with property \(S\).

Proof. See [Ed-Gr, Proposition 3]. □

If \(X\) and \(Y\) have \(G\)-actions then there are exterior products

\[
CH^G_i(X) \otimes CH^G_j(Y) \to CH^G_{i+j}(X \times Y).
\]

In particular, if \(X\) is smooth then there is an intersection product on the equivariant Chow groups which makes \(\oplus_j CH^G_j(X)\) into a graded ring.

5.4. Cycle class maps. [Ed-Gr] §2.8

Suppose \(X\) is a complex algebraic variety and \(G\) is a complex algebraic group. The equivariant Borel-Moore homology \(H^{BM,i}_G(X)\) is the Borel-Moore homology \(H^{BM,i}(X_G)\), for \(X_G = X \times_G U\). This is independent of the representation as long as \(V - U\) has sufficiently large codimension. This gives a cycle class map,

\[
cl_i : CH^G_i(X) \to H^{BM,2i}_G(X, Z),
\]

compatible with usual operations on equivariant Chow groups. Suppose \(X\) is smooth of dimension \(d\) then \(X_G\) is also smooth. In this case the Borel-Moore cohomology \(H^{BM,2i}_G(X, Z)\) is dual to \(H^{2d-i}_G(X_G) = H^{2d-i}(X \times G U)\).

This gives cycle class maps

\[
cl^i : CH^G_i(X) \to H^{2i}_G(X, Z).
\]
There are also maps from the equivariant groups to the usual groups:

\[ H^i_G(X, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \tag{12} \]

and

\[ CH^i_G(X) \to CH^i(X). \tag{13} \]

**Proposition 5.8.** If \( X \) is a smooth variety with a \( G \)-action, then the map

\[ \text{Pic}^G(X) \to CH^1_G(X), \ L \mapsto c_1(L) \]

is an isomorphism.

**Proof.** See [Ed, Corollary 1]. \( \square \)

5.5. **Equivariant Chow groups and moduli stacks:** [Ed]. Suppose \( X \) is a complex variety and \( G \) is an algebraic group acting on \( X \). Let \( \mathcal{X} \) denote the quotient stack \( [X/G] \). We refer to [Ed], for a discussion on quotient stacks and the cohomology of the quotient stacks, with integral coefficients. We have:

**Theorem 5.9.** 1) There is an equality of cohomology rings (resp. Chow rings):

\[ H^*(\mathcal{X}) = H^*_G(X). \]

2) \n
\[ CH^*(\mathcal{X}) = CH^*_G(X) \]

**Proof.** See [Ed, Theorem 3.16] and [Ed-Gr, Proposition 19] or [Ed, Proposition 3.26]. \( \square \)

**Proposition 5.10.** If \( X \) is smooth of dimension \( n \) then there is an isomorphism

\[ CH^k(\mathcal{X}) \to CH^G_{n-k}(X) \]

where \( \mathcal{X} = [X/G] \). Moreover the ring structure on \( CH^*(\mathcal{X}) \) is given by composition of operations, and is compatible with the ring structure on \( CH^*_G(X) \), given by equivariant intersection product.

**Proof.** See [Ed, Proposition 3.28]. \( \square \)

Furthermore, if \( X \) is a smooth complex variety then there is a degree doubling cycle class map:

\[ cl : CH^*(\mathcal{X}) \to H^*(\mathcal{X}) \tag{14} \]

having the same formal properties as the cycle class map on smooth complex varieties.

We will utilise the isomorphism in above proposition, to define the cohomology classes and relations amongst the Brill-Noether loci on the moduli stack \( \mathcal{U}_C(2, 2g - 2) \).
5.6. **Moduli stacks and coarse moduli spaces.** In this subsection, we recall the relation between the cohomology of the moduli stack with that of its coarse moduli space. However we will not be using this in the later sections.

**Proposition 5.11.** Let $G$ be an algebraic group acting on a scheme $X$. If $X \to M$ is a geometric quotient in the sense of [MFK, Definition 0.6] then $M$ is a coarse moduli scheme for $\mathcal{X} = [X/G]$.

**Proof.** See [Ed, Proposition 4.26]. □

Ideally, we would like to identify the rational Chow ring and cohomology of the quotient stack with that of its coarse moduli space. This is true when the quotient is a Deligne-Mumford stack.

**Proposition 5.12.** Let $\mathcal{X} = [X/G]$ be a DM quotient stack and let $p : \mathcal{X} \to M$ be its coarse moduli scheme.

1) If $X$ is defined over $\mathbb{C}$, then there are isomorphisms:

$$H^*(\mathcal{X}) \otimes \mathbb{Q} \to H^*(M, \mathbb{Q}).$$

2) In the algebraic category, there are analogous isomorphisms:

$$CH^*(\mathcal{X}) \otimes \mathbb{Q} \to CH^*(M) \otimes \mathbb{Q}.$$

**Proof.** See [Ed, Theorem 4.40]. □

**Remark 5.13.** Unfortunately, in our context, the moduli stack of semi stable bundles is an Artin stack, and we are not able to make use of above identifications. Hence, we will consider cycle classes on the moduli stack $U_C(r, d)$ and not on the coarse moduli scheme $U_C(r, d)$.

6. **Tautological algebra generated by the Brill-Noether loci on $J_d(C)$**

In this section, we investigate the cohomology algebra generated by the Brill-Noether subvarieties of $J(C)$ and $J_d(C)$. This problem is motivated by the classical Poincaré formula on $J(C)$.

6.1. **Brill-Noether loci on $J(C)$**. Let us fix a point $P \in C$. Consider the classical Abel-Jacobi map $u : S^d(C) \to J(C)$, where $u = \otimes \mathcal{O}(-dP) \circ \phi_d$ and $\phi_d : S^d(C) \to J_d(C)$, $\otimes \mathcal{O}(-dP) : J_d(C) \to J(C)$ defined as follows.

$$S^d(C) \xrightarrow{\phi_d} J_d(C) \xrightarrow{\otimes \mathcal{O}(-dP)} J(C)$$

$$x_1 + x_2 + \cdots + x_d \xrightarrow{\mathcal{O}(x_1 + x_2 + \cdots + x_d)} \mathcal{O}(x_1 + x_2 + \cdots + x_d - dP).$$

Now define $W_d^0$, for all $d$, $1 \leq d \leq g$, called the Brill-Noether subvarieties of $J(C)$, as follows:

$$W_d^0 := u(S^d(C)).$$
Let $\Theta := u(S^{g-1}(C))$. The classical Poincaré relations determine the relations between the cohomological classes of $W^0_i$ on $J(C)$: 

$$[W^0_i] = \frac{1}{(g - i)!}[\Theta]^{g-i} \in H^*(J(C), \mathbb{Q}).$$

See [ACGH, Chapter 1, § 5, p. 25].

6.2. Brill-Noether loci in $J_d(C)$. For a fixed $d$, we recall the Brill-Noether locus $W^r_d$, which are defined to be certain natural closed subschemes of $J_d(C)$ and discuss some of its properties relevant to us.

**Definition 6.1.** As a set, for $r \geq 0$, we define

$$W^r_d := \{L \in J_d(C) : h^0(L) \geq r + 1\} \subseteq J_d(C).$$

It is clear from semicontinuity theorem (cf. [Ha, Theorem 12.8; p. 288]) that $W^r_d$ is closed. In fact, $W^r_d$ has a natural scheme structure as determinantal locus (cf. [ACGH, § 4, Chapter II; p. 83]) of certain morphisms of vector bundles over $J_d(C)$. We define these morphisms as follows:

Let us fix a Poincaré bundle $\mathcal{L}$ over $C \times J_d(C)$. Let $E$ be an effective divisor on $C$ with $\deg E = m \geq 2g - d - 1$.

Let $\Gamma := E \times J_d(C)$. Then, over $C \times J_d(C)$ we have the exact sequence:

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{L}(\Gamma) \rightarrow \mathcal{L}(\Gamma)|_{\Gamma} \rightarrow 0. \tag{15}$$

Let $v$ be the projection from $C \times J_d(C) \rightarrow J_d(C)$. Now, applying the functor $v_*$ to the morphism $\mathcal{L}(\Gamma) \rightarrow \mathcal{L}(\Gamma)|_{\Gamma}$ as in (15), we get a morphism

$$\gamma := v_*(\mathcal{L}(\Gamma)) \rightarrow v_*(\mathcal{L}(\Gamma)|_{\Gamma}).$$

Note that, by the choice of the degree of $E$ and Grauert’s theorem (cf. [Ha, Corollary. 12.9, p. 288]), we get that both $v_*(\mathcal{L}(\Gamma))$ and $v_*(\mathcal{L}(\Gamma)|_{\Gamma})$ are vector bundles of rank $d + m - g + 1$ and $m$ respectively.

**Definition 6.2.** The Brill-Noether locus $W^r_d$ is defined to be the $(m + d - g - r)$-th determinantal locus associated to the morphism $\gamma$.

To see that Definition 6.2 indeed agrees with Definition 6.1 in the sense that the set theoretic support of $6.2$ is exactly $6.1$, we refer to [ACGH, Lemma 3.1; p. 178].

From general properties of determinantal loci, we have the following lemma:

**Lemma 6.3.** [ACGH, Lemma 3.3; p. 181] Suppose $r \geq d - g$. Then every component of $W^r_d$ has dimension greater or equal to the Brill-Noether number

$$\rho := g - (r + 1)(g - d + r).$$

**Remark 6.4.** Note that if $r \leq d - g - 1$, then by Riemann-Roch theorem $W^r_d = J_d(C)$. So, from here onwards, we will assume that $r \geq d - g$. 

In general, the above inequality can be strict (cf. [ACGH, Theorem 5.1; p. 191]). Even, in the case when equality holds, $W^r_d$ can have more than one components (cf. [ACGH, Chapter V; p. 208]).

We recall the following theorem due to Griffith and Harris.

**Theorem 6.5.**  
1. For any smooth projective curve $C$ of genus $g$
   \[ \dim(W^r_d) \geq \rho. \]
2. For a general curve $C$
   \[ \dim(W^r_d) = \rho. \]

Furthermore,
\[ [W^r_d] = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} [\Theta_d]^{g-\rho}. \]

**Proof.** See [Gf-Hr I, Main Theorem; p. 235]. \( \square \)

When $\rho = 0$ the above formula is called the Castelnuovo formula. Regarding the irreducibility, we have:

**Theorem 6.6.** If $C$ is general and $\rho > 0$, then $W^r_d$ is irreducible.

**Proof.** See [Fu-La, Corollary 2.4; p. 280]. \( \square \)

### 6.3. Tautological algebra generated by the Brill-Noether loci in $J(\tilde{C})$.

Let $\pi : \tilde{C} \to C$ be a spectral curve which was defined in Section 3. In this section we investigate the subalgebra of $H^*(J(\tilde{C}), \mathbb{Q})$ generated by the Brill-Noether loci on $J(\tilde{C})$. Towards this, we consider the case when we have a ramified double cover $\pi : \tilde{C} \to C$.

Let $\mathcal{R}_g^r$ denote the moduli space of ramified two sheeted covering of a connected smooth projective curves of genus $g$ with fixed ramification $r$. Then we have the following theorem.

**Theorem 6.7.** The Néron-Severi group of the Jacobian of a general element of $\mathcal{R}_g^r$ is generated by two elements; the two elements are obtained from the decomposition (up to isogeny) of the Jacobian of a covering curve (see Theorem 4.3). Furthermore, the Néron-Severi group generates the algebra of Hodge cycles (of positive degree) on the Jacobian of the general double cover.

**Proof.** See [Bi, Corollary 5.3; p. 634]. \( \square \)

Note that even if $C$ is general, $\tilde{C}$ may not be general. However, in our situation, we will check that the above theorem still holds.

**Theorem 6.8.** The cohomology class of a Brill-Noether locus on the Jacobian $J(\tilde{C})$ of a general 2-sheeted spectral curve $\pi : \tilde{C} \to C$ can be expressed as a sum of powers of divisor classes. In particular the tautological algebra is generated by the divisor classes.
Proof. We only need to check that Theorem 6.7 can be applied to the Jacobian of a general spectral curve. Fix a degree $d > 0$. Denote $S_{g,s}$ the moduli space of tuples
\[ \{(C, L, s) = (s_0, s_1)\}, \]
where $C$ is a curve of genus $g$, $L$ is a line bundle on $C$ of degree $d$, and $s_0 \in H^0(C, L)$, $s_1 \in H^0(C, L^2)$. This moduli space can be interpreted as the moduli space of spectral curves, as in §3. There is a dominant rational map (on the component where $(s_0 = 0)$)
\[ S_{g,s}^0 \to \mathcal{R}_g^r \to \mathcal{U}_g. \]
Here $r$ is the ramification type corresponding to a general section $s$ equivalently the zeroes of the equation (1) (cf. [Ba-Ci-Ve] for a similar moduli space). The maps are given by
\[ (C, L, s) \mapsto (C, L, B(s)) \mapsto C, \]
where $B$ is the branch divisor of the spectral curve $\tilde{C}_s \to C$, such that $L^2 = \mathcal{O}(B)$. Since $J(\tilde{C}_s)$ depends only the ramification type $B$ and $L$, Theorem 6.7 can be applied to the Jacobian of a general spectral curve.

\[ \square \]

7. Brill-Noether loci on $U_C(n, d)$

To define the Brill-Noether loci for $U_C(n, d)$, we start with a more general set up. Let $S$ be an algebraic scheme over $\mathbb{C}$. Let $\mathcal{E}$ be a vector bundle over $C \times S$ such that for all $s \in S$, $\mathcal{E}_s := \mathcal{E}|_{C \times s}$ is a vector bundle of rank $n$ and degree $d$ over $C$.

Just as in §6.1, we have the following definition of the Brill-Noether locus as a closed set.

Definition 7.1. We define the Brill-Noether locus $W_{S, \mathcal{E}}^r$ associated to pair $(S, \mathcal{E})$ to be the set
\[ W_{S, \mathcal{E}}^r := \{s \in S|h^0(C, \mathcal{E}_s) \geq r + 1\}. \]

By [Hu-Ln] Lemma 1.7.6; p. 28], since the family $\mathcal{E}$ is a bounded family, we can choose a divisor $D$ in $C$ of sufficiently high degree such that $H^1(C, \mathcal{E}_s(D)) = 0$ for all $s \in S$. For notational convenience, we continue to denote the pullback of $D$ to $C \times S$ by $D$. Then, over $C \times S$ we have the exact sequence:
\[ 0 \to \mathcal{E} \to \mathcal{E}(D) \to \mathcal{E}(D)|_D \to 0. \]

Let $v : C \times S \to S$ be the projection.
Then, we have the morphism
\[ f : v_*(\mathcal{E}(D)) \to v_*(\mathcal{E}(D)|_D). \]

Now for any $s \in S$ we have $h^1(C, \mathcal{E}(D)_s) = h^1(C, \mathcal{E}_s(D)) = 0$. By Riemann-Roch theorem we get
\[ h^0(C, \mathcal{E}(D)_s) = d + n \deg D + n(1 - g), \]
\[ h^0(C, (\mathcal{E}(D)|_D)_s) = n \deg D. \]
Hence, by [Ha] Theorem 12.11; p. 290], we get that both \( v_*(E(D)) \) and \( v_*(E(D)|D) \) are vector bundles and for any \( s \in S \), we have isomorphisms:

\[
\begin{align*}
v_*(E(D))|_s & \cong H^0(C, E|_{C \times S}(D)), \\
v_*(E(D)|D)|_s & \cong H^0(C, E|_{C \times S}(D)|_D).
\end{align*}
\]

Using Riemann-Roch theorem, we get that

\[
\begin{align*}
\text{rank } v_*(E(D)) & = d + n \deg D + n(1 - g), \\
\text{rank } v_*(E(D)|D) & = n \deg D.
\end{align*}
\]

**Definition 7.2.** We define \( W_{S, E}^r \) to be the \((d + n \deg D + n(1 - g) - (r + 1))\)-th determinantal locus associated to the morphism \( f \).

**Remark 7.3.** To see that the set-theoretic support of \( W_{S, E}^r \) is indeed \( W_{S, E}^r \), note that we have the following commutative diagram:

\[
\begin{array}{ccc}
v_*(E(D))|_s & \xrightarrow{f|_s} & v_*(E(D)|D)|_s \\
\downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & H^0(C, E_s) \longrightarrow H^0(C, E_s(D)) \longrightarrow H^0(C, E_s(D)|_D)
\end{array}
\]

Hence,

\[
\text{rank } f|_s \leq d + n \deg D + n(1 - g) - (r + 1) \iff H^0(C, E_s) \geq r + 1.
\]

From this, it follows that definition \( W_{S, E}^r \) agrees with definition \( W_{S, E}^r \).

**Lemma 7.4.** If \( W_{S, E}^r \neq \emptyset \), then, codimension of each component of \( W_{S, E}^r \leq (r + 1)(r + 1 - d + n(g - 1)) \).

**Proof.** This follows from [ACGH] §4, Chapter II; p. 83.

**Lemma 7.5.** Let \( S_1, S_2 \) be two algebraic schemes over \( \mathbb{C} \) and let \( E \) be a bundle on \( C \times S_2 \) such that for all \( s \in S_2 \), \( E_s \) is a vector bundle of rank \( n \) and degree \( d \). If \( g : S_1 \to S_2 \) be a morphism, then

\[
g^{-1}W_{S_2, E}^r = W_{S_1, (\text{id}_C \times g)^*E}^r.
\]

**Proof.** Let \( v_1 : C \times S_1 \to S_1 \) and \( v_2 : C \times S_2 \to S_2 \) be the projections. Let \( G := \text{id}_C \times g : C \times S_1 \to C \times S_2 \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
C \times S_1 & \xrightarrow{G} & C \times S_2 \\
\downarrow v_1 & & \downarrow v_2 \\
S_1 & \xrightarrow{g} & S_2
\end{array}
\]

This induces the following commutative diagram:

\[
\begin{array}{ccc}
g^*v_2)_*E(D)) & \longrightarrow & g^*(v_2)_*(E(D)|_D) \\
\downarrow & & \downarrow \\
(v_1)_*G^*(E(D)) & \longrightarrow & (v_1)_*G^*(E(D)|_D)
\end{array}
\]
By (16), we get that the vertical arrows in the above diagram are isomorphisms. Now, the lemma follows from general properties of determinantal loci.

Now suppose \( \tilde{C} \) be a smooth projective curve of genus \( \tilde{g} \) and \( \pi : \tilde{C} \to C \) be a finite morphism. Let \( \mathcal{E} \) be a vector bundle over \( \tilde{C} \times S \) such that \( \mathcal{E}_s \) is of rank \( n \) and degree \( d \) for all \( s \in S \). Since the map \( \pi \times id : \tilde{C} \times S \to C \times S \) is a finite flat morphism, we get that \((\pi \times id)_s \mathcal{E} \) is a vector bundle over \( C \times S \) and in fact,

\[
((\pi \times id)_s \mathcal{E})_s = \pi_*(\mathcal{E}_s) \text{ for all } s \in S.
\]

We will denote this bundle \((\pi \times id)_s \mathcal{E}\) by \( \mathcal{E}' \). Note that rank of \( \mathcal{E}' \) is \( n' := n (\deg \pi ) \), for all \( s \in S \).

Let \( d' := \deg \mathcal{E}'_s \). Since \( \mathcal{E}'_s = \pi_*(\mathcal{E}_s) \), by Riemann-Roch we have

\[
d + n (1 - g) = \chi(\tilde{C}, \mathcal{E}_s) = \chi(C, \mathcal{E}'_s) = d' + n'(1 - g).
\]

Hence we have

\[
d' = d + n(1 - \tilde{g}) - (n \deg \pi)(1 - g).
\]

Then we have the following lemma:

**Lemma 7.6.** \( W^r_{S, \mathcal{E}} = W^r_{S, \mathcal{E}'} \).

**Proof.** We have the commutative diagram:

\[
\begin{array}{ccc}
\tilde{C} \times S & \xrightarrow{\pi \times id} & C \times S \\
\downarrow{\tilde{v}} & & \downarrow{v} \\
S & & S
\end{array}
\]

Fix \( D \) a divisor on \( C \) such that \( h^1(\mathcal{E}'_s(D)) = 0 \) for all \( s \in S \). Then

\[
h^1(\tilde{C}, \mathcal{E}_s(\pi^*D)) = h^1(C, \pi_*(\mathcal{E}_s(\pi^*D))) = h^1(C, \mathcal{E}'_s(D)) = 0
\]

Therefore we can use the divisor \( \pi^*D \) for the construction of \( W^r_{S, \mathcal{E}} \).

Let us denote the morphism

\[
f : \mathcal{E}(\pi^*D) \to (\mathcal{E}(\pi^*D))|_{\pi^*D}.
\]

Then \( W^r_{S, \mathcal{E}} \) is defined to be the \((d + n \deg \pi^*D + n(1 - \tilde{g}) - (r + 1))\)-th determinantal locus of the morphism \( \tilde{v}_*f \). Now \( \tilde{v} = v \circ (\pi \times id) \). It follows from projection formula that \((\pi \times id)_s f \) is nothing but the morphism

\[
\mathcal{E}'(D) \to \mathcal{E}'(D)|_D
\]

and therefore, \( W^r_{S, \mathcal{E}'} \) is the \( d' + n'(1 - g) - (r + 1) \)-th determinantal locus of \( v_s(\pi \times id)_* f = \tilde{v}_* f \). It can be checked easily that

\[
d' + n'(1 - g) - (r + 1) = d + n \deg \pi^*D + n(1 - \tilde{g}) - (r + 1).
\]
Next, we will define the Brill-Noether Loci for $U_C(n,d)$. Note that if $(n,d) = 1$, we have a universal bundle over $C \times U_C(n,d)$ (cf. [Hu-Ln, Corollary 4.6.6; p. 119]) and hence, we can apply the previous construction to get the notion of the Brill-Noether loci in this case. However, in general we don’t have a universal bundle.

Recall that $U_C(n,d)$ was constructed as a good quotient of certain Quot Schemes (cf. [Hu-Ln, §4.3; p. 88]). We recall the definition of this Quot Scheme. Fix a line bundle $O(1)$ of degree 1 over $C$. Choose an $m \gg 0$ such that any semistable vector bundle $E$ over $C$ of rank $n$ and degree $d$ is $m$-regular.

In particular, we have that

1. $h^1(C, E(m)) = 0$
2. $h^0(C, E(m)) = d + mn + n(1 - g) =: N$
3. The natural map $H^0(C, E(m)) \otimes O \to E(m)$ is surjective.

Now, define $Q$ to be the Quot Scheme parametrizing quotients of $O^N$ of rank $n$ and degree $d + mn$. Let

$$O^N_{C \times Q} \to F$$

be the universal quotient.

Note that the group scheme $GL(N)$ acts on $Q$ in the following manner:

Let $T$ be an algebraic scheme over $C$.

Let $g \in GL(N)(T)$ be an automorphism $O^N_{C \times T} \xrightarrow{g} O^N_{C \times T}$. Let $[O^N_{C \times T} \to F_T] \in Q(T)$.

Then, define $g.[O^N_{C \times T} \to F_T] := [O^N_{C \times T} \xrightarrow{g} O^N_{C \times T} \to F_T]$.

It is clear that this action in fact factors through an action of the group scheme $PGL(N)$.

Let $R \subseteq Q$ be the open subset such that for all $x \in R$, $F|_{C \times x}$ is a semistable bundle and $H^0(C, O^N) \to H^0(C, F|_{C \times x})$ is an isomorphism. It is immediate that $R$ is $PGL(N)$-equivariant. Then, we define

$$U_C(n,d) := R \sslash PGL(N).$$

and we have the quotient map

$$\mu : R \to U_C(n,d).$$

(17)

Let us denote $F|_{C \times R}$ by $F'$. By Definition 7.2 we have the closed subscheme $W^r_{R,F'}(-m) \subseteq R$.

**Lemma 7.7.** $W^r_{R,F'}(-m)$ is $GL(N)$-equivariant (consequently $PGL(N)$-equivariant as well), and compatible with the $GL(N)$-action on $R$.

**Proof.** Let $q : T \to W^r_{R,F'}(-m)$ be a $T$-valued point of $W^r_{R,F'}(-m)$. Let $O^N_{C \times T} \to F_T$ be the pullback of the universal quotient under $q$. By Lemma 7.3 we get that $W^r_{T,F_T(-m)} = q^{-1}W^r_{R,F'}(-m) = T$. Let $g \in GL(N)(T)$. By definition, the quotient corresponding to $g.q : T \to R$ is given by

$$O^N_{C \times T} \xrightarrow{g} O^N_{C \times T} \to F_T.$$
Again, by Lemma [7.5] we have

\[(g,q)^{-1}W_{\mathcal{R},\mathcal{F}'(-m)}^r = W_{T,\mathcal{F}'(-m)}^r = T.\]

In other words, we get that \(g.q : T \to \mathcal{R}\) factors through \(W_{\mathcal{R},\mathcal{F}'(-m)}^r\). Hence the closed subscheme \(W_{\mathcal{R},\mathcal{F}'(-m)}^r\) is \(GL(N)\)-equivariant.

**Definition 7.8.** We define the Brill-Noether locus \(\tilde{W}_{n,d}^r(C)\) to be the scheme theoretic image of \(W_{n,d}^r(C)\) under the morphism \(\mu\). In other words,

\[\tilde{W}_{n,d}^r(C) = \mu(W_{n,d}^r(C)).\]

**Notation 7.9.** We will denote \(\tilde{W}_{n,d}^r(C)\) by \(\tilde{W}_{n,d}^r\) when there is no chance of confusion.

**Remark 7.10.** Note that since the morphism

\[\mu : \mathcal{R} \to U_C(n,d)\]

is a good quotient and \(W_{\mathcal{R},\mathcal{F}'(-m)}^r\) is \(PGL(N)\)-equivariant, we get that \(\mu(W_{\mathcal{R},\mathcal{F}'(-m)}^r)\) is a closed subset of \(U_C(n,d)\). Hence, as sets \(\tilde{W}_{n,d}^r = \mu(W_{n,d}^r)\). That is to say, denoting the \(S\)- equivalence class of a semistable bundle \(E\) over \(C\) by \(e\), we get

\[\tilde{W}_{n,d}^r = \{e \in U_C(n,d) \mid \exists E \in e \text{ such that } h^0(C, E) \geq r + 1\}.\]  

(18)

Let us denote by \(U_C^s(n,d)\) the moduli space of stable bundles on \(C\) of rank \(n\) and degree \(d\). Recall that \(U_C^s(n,d)\) is an open subset of \(U_C(n,d)\).

**Definition 7.11.** We define the Brill-Noether locus \(W_{n,d}^r\) of \(U_C^s(n,d)\) to be the closed subscheme

\[W_{n,d}^r := \tilde{W}_{n,d}^r \cap U_C^s(n,d) \subset U_C^s(n,d).\]

**Remark 7.12.** Let \(\mathcal{R}^s \subseteq \mathcal{R}\) be the set of all \(x \in \mathcal{R}\) such that \(\mathcal{F}'|_{C \times x}\) is stable. Let \(\mathcal{F}'' := \mathcal{F}'|_{C \times \mathcal{R}^s}\). Let \(\mu_s : \mathcal{R}^s \to U_C^s(n,d)\) be the restriction of \(\mu\) to \(\mathcal{R}^s\). Then, \(W_{n,d}^r\) is the scheme-theoretic image of \(W_{\mathcal{R}^s,\mathcal{F}''(-m)}^r\) under the map \(\mu_s\).

Now, using the fact that \(\mu_s : \mathcal{R}^s \to U_C^s(n,d)\) is a principal \(PGL(N)\)-bundle (cf. \[Hu-Ln\] Corollary 4.3.5; p. 91), and Lemma [7.4] we have the following lemma:

**Lemma 7.13.** If \(W_{n,d}^r \neq \emptyset\), then dimension of each component of \(W_{n,d}^r\) is at least

\[n^2(g-1) + 1 - (r+1)(r+1-d+n(g-1)).\]

**Definition 7.14.** We define

\[\rho_{n,d}^r := n^2(g-1) + 1 - (r+1)(r+1-d+n(g-1))\]

to be the expected dimension of \(W_{n,d}^r\).

**Remark 7.15.** The above lemma is not true in the case of \(\tilde{W}_{n,d}^r\). It may have components whose dimensions are less than \(\rho_{n,d}^r\). See \[Br-Gz-Ne\] §7 for example.
Lemma 7.16. Let $S$ be an algebraic scheme and $E$ be a vector bundle over $C \times S$ such that for all $s \in S, E_s$ is stable of rank $n$ and degree $d$. If $f : S \to U_C^r(n, d)$ is the induced map, then

$$f^{-1}W_{n,d}^r = W_{S,E}^r.$$  

Proof. First we show that the statement is true in the case when $S = R^*$ and $E = F''(-m)$. As we saw earlier, $W_{R^*,F''(-m)}^r$ is a $PGL(N)$-equivariant subscheme and since $R^* \to U_C^r(n, d)$ is a principal $PGL(N)$-bundle, $W_{R^*,F''(-m)}^r$ descends to a closed subscheme $Z$ in $U_C^r(n, d)$, i.e.

$$\mu_s^{-1}Z = W_{R^*,F''(-m)}^r.$$  

Since $W_{n,d}^r = \mu_s(W_{R^*,F''(-m)}^r)$, it is clear that $Z = W_{n,d}^r$. Hence

$$\mu_s^{-1}W_{n,d}^r = W_{R^*,F''(-m)}^r.$$  

Now let $(S, E)$ be as in the hypothesis. Since $F''(-m)$ is a locally universal family, for any $x \in S$, there exists $U_x \subset R$ which is open and a map $g : U_x \to R^*$ such that

$$(id \times g)^*F''(-m) = E|_{C \times U_x}.$$  

By Lemma 7.5 we have

$$g^{-1}W_{R^*,F''(-m)}^r = W_{S,E}^r \cap U_x.$$  

Since $\mu_s \circ g = f|_{U_x}$, we have

$$(f|_{U_x})^{-1}W_{n,d}^r = W_{S,E}^r \cap U_x.$$  

The lemma now follows from this. \qed

Our aim in this paper is to find a Poincaré type expression for the cohomology class, in the cohomology ring of the moduli stack $U_C^r(2, d)$ (resp. $SU_C(2, \mathcal{L})$ for a line bundle $\mathcal{L}$ on $C$).

Let us now fix different notations of the Brill-Noether subvarieties in different spaces to avoid confusion.

For a given scheme and for a given sheaf $E$ over $C \times S$, we denote the Brill-Noether locus by $W_{S,E}^r$ as in Definition 7.1 or in Definition 7.2. We also denote this by $W_S^r$ when the sheaf involved is clear from the context. In $U_C^r(n, d)$ the Brill-Noether locus is denoted by $W_{n,d}^r$ as in (18). The same is denoted by $W_{n,d}^r$ in $U_C^r(n, d)$ as in Definition 7.11. Inside $J_d(C)$, that is inside $U_C(n, d)$, the Brill-Noether locus $W_{1,d}^r$ is denoted by $W_d^r$ as in Definition 6.1 or in Definition 6.2. Inside $J_d(\tilde{C})$ the same is denoted by $W_d^r(\tilde{C})$.

8. Chow-Cohomology class of the Brill-Noether locus

For generalities on algebraic stacks, and in particular on moduli stacks of vector bundles, we refer to [Go]. Denote $G := GL(N)$.

Consider the semistable locus $R$ of the Quot scheme, together with the quotient morphism (17) to the GIT-quotient:

$$\mu : R \to U_C(n, d) = R \sslash PGL_n.$$
By [Dr-Na], \( \mathcal{R} \) is a smooth variety. Recall that the Brill-Noether locus is defined as (cf. Definition 7.8)

\[
\tilde{W}_{r,n,d} = \mu(W_{\mathcal{R}, F, (-m)}^r).
\]

Also consider the map to the quotient stack (see [Go, Proposition 3.3, p.23]):

\[\mu_{st}: \mathcal{R} \rightarrow \mathcal{U}_C(n,d) = [\mathcal{R}/\text{GL}_n].\]

We use the same notation for the Brill-Noether loci in the moduli stack and its coarse moduli scheme.

We first notice the following:

**Lemma 8.1.** The Brill-Noether loci give well-defined Chow-cohomology class in the equivariant Chow-cohomology of \( \mathcal{R} \), and is compatible with the cycle class map on equivariant groups.

**Proof.** Using Lemma 7.7 and Lemma 5.4, we know that \( W_{\mathcal{R}, F, (-m)}^r \) is equivariant for the \( \text{GL}_n \)-action. Hence, it corresponds to a \( \text{GL}_n \)-equivariant cohomology class:

\[\left[\tilde{W}_{r,n,d}\right] \in \bigoplus_i H^2_G(\mathcal{R}, \mathbb{Z}).\]

Similarly, we obtain an equivariant Chow class:

\[\left[\tilde{W}_{r,n,d}\right] \in \bigoplus_i CH^i_G(\mathcal{R}).\]

Since we do not know if the Brill-Noether locus is of pure dimension, we will use the cycle class map on the equivariant Chow ring:

\[CH^*_G(\mathcal{R}) \rightarrow \bigoplus_i H^2_G(\mathcal{R}, \mathbb{Z}).\]

Via the cycle class map, the equivariant Chow class

\[\left[\tilde{W}_{r,n,d}\right] \in CH^*_G(\mathcal{R}) = \bigoplus_i CH^i_G(\mathcal{R})\]

maps to the Brill-Noether cohomology class

\[\left[\tilde{W}_{r,n,d}\right] \in \bigoplus_i H^2_G(\mathcal{R}, \mathbb{Z}).\]

\(\square\)

8.1. **Brill Noether Chow- Cohomology class on the moduli stack** \( \mathcal{U}_C(r, 2g - 2) \).

Henceforth, we write \( d = 2g - 2 \). (Note that most of the discussions remain true for other degrees as well, except for certain results in the final section).

Since \( \mathcal{U}_C(2, 2(g - 1)) \) is a singular variety, we will consider Chow groups \( CH^*(\mathcal{U}_C(r, d)) \) of the moduli stack, instead of the Chow groups of the coarse moduli scheme \( \mathcal{U}_C(2, 2g - 2) \):

**Lemma 8.2.** The Brill Noether loci give a well-defined Chow class

\[\left[\tilde{W}_{r,n,d}\right] \in CH^*(\mathcal{U}_C(n,d))\]

and a cohomology class

\[\left[\tilde{W}_{r,n,d}\right] \in H^*(\mathcal{U}_C(n,d), \mathbb{Z}).\]

The classes are compatible under the cycle class map in (14).
Proof. Since $R$ is a smooth variety, the lemma follows from the identification in Proposition 5.10 and above Lemma 8.1. □

In particular, we have the following lemma.

**Lemma 8.3.** The Brill-Noether class $[W^r_R]$ is non-zero if and only if $[\tilde{W}^r_{n,d}]$ is non-zero, in Chow cohomology (resp. in cohomology ring).

Proof. Using (13) and Theorem 5.9 (similarly for cohomology: use (12)), we have:

$$CH^*(U_C(r,d)) = CH^*_G(R) \to CH^*(R)$$

and compatible with the cycle class maps:

$$CH^*(U_C(n,d)) \xrightarrow{\mu^*} \odot_i H^{2i}(U_C(n,d), \mathbb{Z}) \xrightarrow{\mu^*} \odot_i H^{2i}(R, \mathbb{Z}).$$

We only need to note that the $GL(N)$-equivariant subvariety $W^r_R$ is represented by the class $[\tilde{W}^r_{n,d}]$ in the equivariant cohomology of $R$.

Hence the lemma is clear. □

**Lemma 8.4.** The Brill-Noether loci in $R$ is non-empty (consequently in $U_C(n,d)$) if and only if its cohomology class is non-empty, in $H^*(R, \mathbb{Q})$ or in $H^*(U_C(n,d), \mathbb{Q})$.

Proof. Clear. □

9. Main theorems, when the rank is two

We want to give some relations amongst the Brill-Noether loci in $U_C(2, d)$. In our context we fix degree $d$ to be $2g - 1$). Denote $G := GL(N)$.

In this case, Sundaram (cf. [Su]) proved that $W^0_{2,2(g-1)}$ is a divisor in $U_C(2, 2(g-1))$. We give some relations between the cohomology classes of the Brill-Noether loci in terms of cohomology class of $W^0_{2,2(g-1)}$ in the moduli stack $U_C(2, 2g - 2)$. Since the moduli spaces $U_C(2, 2g - 2)$ and $SU_C(2, L)$ are singular varieties, Chow rings and cohomology rings of the moduli stacks seem appropriate to consider.

Consider the map $\pi_s : J^{ss} \subseteq J_{4(g-1)}(\tilde{C}) \to U_C(2, 2g - 1)$ as in (6). Note that as we have taken $d = 2g - 1$, therefore it follows from (5) that $\delta = 4(g - 1)$. Also from (3), we have

$$\delta = 4(g - 1) = \{4(g - 1) + 1\} - 1 = \tilde{g} - 1.$$ 

Hence we have the Theta divisor $\Theta := W^0_{4(g-1)}(\tilde{C})$ in $J_{4(g-1)}(\tilde{C})$. Following theorem says that the Theta divisor of $\tilde{C}$ intersects both $J^{ss}$ and its complement in $J_{4(g-1)}(\tilde{C})$.

**Theorem 9.1.** The Theta divisor of $J_{4(g-1)}(\tilde{C})$, denoted by $\Theta$, does not lie inside the complement of $J^{ss}$ in $J_{4(g-1)}(\tilde{C})$. More precisely,
(1) For any point \( l \in J_{4(g-1)}(\tilde{C}) - \Theta \), \( \pi_{*}(l) \) is semistable.

(2) There is a point \( \xi \in \Theta \) such that \( \pi_{*}(\xi) \) is semistable.

Proof. See [Be-Na-Ra, Proposition 5.1; p. 176]. \( \square \)

Moreover we have that pullback of the divisor \( \widetilde{W}_{2,2(g-1)}^{0} \) of \( U_{C}(2, 2(g-1)) \) is the restriction of \( \Theta \) to \( J^{ss} \).

**Theorem 9.2.** Let us denote the restriction of \( \Theta \) to \( J^{ss} \) by \( \Theta |_{J^{ss}} \). Then
\[
\pi_{*}^{-1}(\widetilde{W}_{2,2(g-1)}^{0}) = \Theta |_{J^{ss}}.
\]

Proof. See [Bg-Tu, Lemma 6; p. 335]. Also follows directly from the fact that \( H^{0}(\tilde{C}, l) = H^{0}(C, \pi_{*}l) \). \( \square \)

Now we want to check whether Theorem 9.1 and 9.2 hold for other Brill-Noether subvarieties of higher codimension. By (4), we have
\[
\pi_{*}^{-1}( \widetilde{W}_{2,2(g-1)}^{r} ) = W_{2,2(g-1)}^{r} |_{J^{ss}}.
\]

For our purpose, we consider a scheme \( S \) to give relations between the cohomology classes Brill-Noether loci, on \( U_{C}(2, 2(g-1)) \).

We can construct a scheme \( S \) with the following properties.

1. \( S \) is a smooth projective variety.
2. There is a morphism \( q : S \to Q \) such that \( \psi = \mu \circ q \), wherever \( \mu \) is defined.
3. There exists a birational morphism \( \phi : S \to J_{4(g-1)}(\tilde{C}) \) and a generically finite morphism \( \psi : S \to U_{C}(2, 2(g-1)) \).
4. \( \phi : \phi^{-1}(J^{ss}) \to J^{ss} \) is an isomorphism.
5. The following diagram is commutative.

\[
\begin{array}{ccc}
S & \xrightarrow{s} & J_{4(g-1)}(\tilde{C}) - - - - - - - - - - - U_{C}(2, 2(g-1)) \\
\phi & & \psi \\
J^{ss} & \xrightarrow{s} & \pi_{*}U_{C}(2, 2(g-1)) \\
\phi^{-1}(J^{ss}) & & \psi \\
\end{array}
\]

Moreover this diagram is commutative whenever the domains of the involved rational maps are chosen properly. In particular, we have the following commutative diagram.

\[
\begin{array}{ccc}
S & \xrightarrow{s} & J_{4(g-1)}(\tilde{C}) - - - - - - - - - - - U_{C}(2, 2(g-1)) \\
\phi & & \psi \\
J^{ss} & \xrightarrow{s} & \pi_{*}U_{C}(2, 2(g-1)) \\
\phi^{-1}(J^{ss}) & & \psi \\
\end{array}
\]

To construct \( S \) as above, we use the Poincaré bundle \( P \) over \( \tilde{C} \times J_{4(g-1)}(\tilde{C}) \), and the morphism on the semistable points \( J^{ss} \subset J_{4(g-1)}(\tilde{C}) \):
\[
J^{ss} \to Q
\]
is induced by the family \( (\pi \times id)_{*}P|_{C \times J^{ss}} \). Resolving this map and its singularities corresponds to a smooth projective variety \( S \), with above properties.
Then we have the following diagram.

\[
\begin{array}{c c c c c}
\tilde{C} \times S & \xrightarrow{\pi \times id} & C \times S & \xrightarrow{\psi} & S \\
\downarrow{id \times \phi} & & \downarrow{id \times \psi} & & \\
\tilde{C} \times J_{4(g-1)}(\tilde{C}) & \to & C \times U_C(2, 2(g-1)) & \to & U_C(2, 2(g-1))
\end{array}
\]

Let us denote by \( J^* \) the following set:

\[
J^* := \{ l \in J_{4(g-1)}(\tilde{C}) \mid \pi \times l \in U_C^*(2, 2(g-1)) \}.
\]

Define \( S_0 := \phi^{-1}(J^*) \). Then we have the following lemma:

**Lemma 9.3.** \( \psi^{-1}(W^r_{2,2(g-1)}) \cap S_0 = \phi^{-1}(W^r_{2,2(g-1)}(\tilde{C})) \cap S_0 \).

**Proof.** If \( P \) is a Poincaré bundle over \( \tilde{C} \times J_{4(g-1)}(\tilde{C}) \), then the morphism \( S_0 \to U_C(2, 2(g-1)) \) is induced by the family \( (\pi \times id)_*(id \times \phi)^*P) \mid_{C \times S_0} \). Now by Lemma \[7,16] we get that

\[
W^r_{S,(\pi \times id)_*((id \times \phi)^*P)} \cap S_0 = W^r_{S,(\pi \times id)_*((id \times \phi)^*P)} \cap S_0.
\]

The Lemma \[7,16] then implies

\[
W^r_{S,(\pi \times id)_*((id \times \phi)^*P)} \cap S_0 = \psi^{-1}W^r_{2,2(g-1)}
\]

and Lemma \[7,5] implies

\[
W^r_{S,(id \times \phi)^*P} \cap S_0 = \phi^{-1}W^r_{2,2(g-1)}(\tilde{C}) \cap S_0.
\]

Hence we get

\[
\psi^{-1}(W^r_{2,2(g-1)}) \cap S_0 = \phi^{-1}(W^r_{2,2(g-1)}(\tilde{C})) \cap S_0.
\]

\[\square\]

Hence we obtain the following:

**Lemma 9.4.** We have the equality of the closures

\[
\psi^{-1}(W^r_{2,2(g-1)}) \cap S_0 = \phi^{-1}(W^r_{2,2(g-1)}(\tilde{C})) \cap S_0
\]

of a component of the Brill-Noether locus on \( S \). In particular, of the corresponding cohomology classes in \( H^*(S, \mathbb{Z}) \).

Denote this component \( W^r_{S} \), in \( S \).

9.1. **Poincaré type relations on moduli stacks.** Assume that \( C \) is a general smooth projective curve and \( \tilde{C} \to C \) is a general smooth spectral curve, which is a double ramified covering of \( C \). We recall the cycle class maps between the Chow rings and cohomology rings

\[
CH^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})
\]

of the moduli stacks \( X = U_C(2, 2(g-1)) \) and \( X = SU_C(2, \mathcal{L}) \) (for \( \mathcal{L} \in J_{2(g-1)}(C) \)) (cf. §5.5).

We start with the following lemma.
Lemma 9.5. The divisor classes on \( J_{4(g-1)}(\tilde{C}) \) descend to the moduli stack \( U_C(2, 2(g-1)) \) via the above diagram \([19]\).

Proof. Recall from [Be-Na-Ra, Proposition 5.7] a commutative diagram:

\[
\begin{array}{ccc}
P' \times J_{g-1}(C) & \xrightarrow{\pi_*} & J_{4(g-1)}(\tilde{C}) \\
\downarrow & & \downarrow \\
SU_C(2) \times J_{g-1}(C) & \xrightarrow{\pi} & U_C(2, 2(g-1)).
\end{array}
\]

Here \( P' \) is the Prym variety associate to the covering \( \tilde{C} \to C \) and \( SU_C(2) \) is a fixed determinant (of degree \( 2(g-1) \)) moduli space. Furthermore, it is shown that the indiscrepancy locus of the dominant rational map \( \pi_* \) has codimension at least two and the same is true when restricted to \( P' \). The proof of loc.cit. implies that the polarisations on \( P' \) and \( J_{g-1}(C) \) descend on the moduli space \( U_C(2, 2(g-1)) \). By functoriality, via the diagram \([19]\), the divisor classes descend on \( U_C(2, 2(g-1)) \).

\[\square\]

We now show the following.

Theorem 9.6. The cohomology class of a Brill-Noether subvariety on the moduli stack \( U_C(2, 2(g-1)) \) can be expressed as a polynomial on the divisor classes. In particular, the tautological algebra generated by the Brill-Noether loci is generated by the divisor classes.

Proof. Recall the morphism

\[
\phi : S \to J_{4(g-1)}(\tilde{C}).
\]

Now \( \phi \) is a birational morphism and let \( E \subset S \) be the exceptional locus. Hence, we have the following equalities of cohomology rings:

\[
H^*(S, \mathbb{Q}) = H^*(J_{4(g-1)}(\tilde{C}), \mathbb{Q}) \oplus H^*(E, \mathbb{Q}).
\]

By Theorem [6.8] the cohomology class of the Brill-Noether locus \( W^r_{1,d}(\tilde{C}) \subset J_{4(g-1)}(\tilde{C}) \) is a polynomial expression on the divisor classes in \( H^*(J_{4(g-1)}(\tilde{C})) \). This implies that in \( H^*(S, \mathbb{Q}) \), the pullback of the cohomology class \( [W^r_{1,d}(\tilde{C})] \) is the cohomology class of the Brill-Noether locus \( W^r_S \subset S \) and it is a polynomial expression on the pullback of the divisor classes on \( J_{4(g-1)}(\tilde{C}) \).

Recall that \( S \) was constructed such that \( q : S \to Q \) and \( \psi = \mu \circ q \), wherever \( \mu \) is defined.

Denote \( S' := q^{-1}(\mathcal{R}) \). Since \( \mathcal{R} \) is a smooth variety there are pullback maps on the Chow cohomologies, and using [13], we have:

\[
CH^*(U_C(2, 2(g-1))) = CH^*_G(\mathcal{R}) \xrightarrow{\psi^*} CH^*(\mathcal{R}) \xrightarrow{\phi^*} CH^*(S').
\]

By Lemma [8.3]

\[
q^* \mu^*[W^r_{2,2(g-1)}] = q^*[W^r_{\mathcal{R}}] = [W^r_{S'}].
\]

Since \( \mathcal{R} \subset \mathcal{Q} \) is an open subvariety of \( \mathcal{Q} \), using the localization sequence

\[
CH^*(S) \to CH^*(S') \to 0,
\]
we deduce that \([W^r_S] \mapsto [W^r_{S'}]\).

The above Chow cohomology diagram is compatible, via cycle class maps, with the cohomology rings:

\[
H^*(U_C(2, 2g-2), \mathbb{Q}) \xrightarrow{\nu_2} H^*(\mathcal{R}, \mathbb{Q}) \xrightarrow{q'} H^*(S', \mathbb{Q})
\]

together with a restriction

\[
H^*(S, \mathbb{Q}) \xrightarrow{j} H^*(S', \mathbb{Q}).
\]

In particular, \([W^r_S] \mapsto [W^r_{S'}]\), and it is a polynomial expression on the pullback of the divisor classes on \(J_{4(g-1)}(\widetilde{C})\).

Since the map \(S' \to \mathcal{R}\), is a proper, generically finite morphism, the map on the cohomologies is injective:

\[
\psi^*_{coh} : H^*(\mathcal{R}, \mathbb{Q}) \hookrightarrow H^*(S', \mathbb{Q}).
\]

Furthermore, by (20), we deduce that

\[
\psi^*_{coh}[\widetilde{W}^r_R] = [W^r_{S'}].
\]  

By Lemma 9.4, Lemma 9.5 we know that the divisor classes descend (equivalently they are \(GL(N)-\)invariant classes). In other words, any polynomial expression on the divisor classes lies in the equivariant cohomology \(H_c^*(\mathcal{R}, \mathbb{Q})\). This implies that any polynomial expression on the divisor classes descends on the cohomology of the moduli stack. Now (21) implies that the cohomology class of the Brill-Noether locus in \(H^*(U_C(2, 2(2g-1)), \mathbb{Q})\) is expressible as a polynomial on the divisor classes.

9.2. Relations on the moduli stack \(SU_C(2, \mathcal{L})\). Consider the determinant morphism

\[\text{det} : U_C(2, 2(g-1)) \to J_{2(g-1)}(C)\]

The inverse image \(\text{det}^{-1}([\mathcal{L}]\)) is the sub(moduli) stack \(SU_C(2, \mathcal{L})\), for a line bundle \(\mathcal{L}\) on \(C\) of degree \(2(g-1)\). Denote the Brill-Noether locus

\[
W_{2,2(g-1)}^{r, \mathcal{L}} := W_{2,2(g-1)}^{r, \mathcal{L}} \cap SU_C(2, \mathcal{L}).
\]

**Corollary 9.7.** The cohomology class of a Brill-Noether locus \(W_{2,2(g-1)}^{r, \mathcal{L}}\) in the moduli stack \(SU_C(2, \mathcal{L})\) is a polynomial expression on the class of the Theta divisor, with rational coefficients. In particular the tautological algebra is generated by the class of the Theta divisor \(\Theta\).

**Proof.** Consider the inclusion:

\[j : SU_C(2, \mathcal{L}) \hookrightarrow U_C(2, 2(g-1)).\]

The pullback map on the cohomology ring

\[j^* : H^*(U_C(2, 2(g-1)), \mathbb{Q}) \to H^*(SU_C(2, \mathcal{L}), \mathbb{Q})\]

is a ring homomorphism. By Theorem 9.6, the cohomology class of the Brill-Noether locus is a polynomial expression on the divisor classes on \(SU_C(2, 2(g-1))\). Using Proposition 5.8, we can conclude...
we know that the Picard group of $SU_C(2, 2(g - 1))$ is generated by the Theta divisor $\Theta$. This gives the relation, for any irreducible component:

$$[\overline{W_r, L_{2,2(g-1)}}] = \alpha, [\Theta]^{t(r)} \in H^*(SU_C(2, L), \mathbb{Q})$$

for some $\alpha \in \mathbb{Q}$ and $t(r)$ is the codimension of an irreducible component of the Brill-Noether locus.

$\square$

**Remark 9.8.** Using Theorem 6.5 a), we obtain that the dimension of every component of the Brill-Noether locus on $J_{4(g-1)}(\widetilde{C})$ is at least the expected dimension. This implies that the corresponding cohomology class is non-zero in $H^*(J_{4(g-1)}(\widetilde{C}), \mathbb{Z})$. In turn, the same is true for the pullback class on $S$. Restricting on $S'$, the classes further descend on $U_C(2, 2(g - 1))$ (see proof of Theorem 9.6). The nontriviality will involve a further study of the discrepancy locus of $S \to \mathcal{R}$.

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