This paper gives an overview of recently developed model for the QCD analytic invariant charge. Its underlying idea is to bring the analyticity condition, which follows from the general principles of local Quantum Field Theory, in perturbative approach to renormalization group (RG) method. The concrete realization of the latter consists in explicit imposition of analyticity requirement on the perturbative expansion of \( \beta \) function for the strong running coupling, with subsequent solution of the corresponding RG equation. In turn, this allows one to avoid the known difficulties originated in perturbative approximation of the RG functions. Ultimately, the proposed approach results in qualitatively new properties of the QCD invariant charge. The latter enables one to describe a wide range of the strong interaction processes both of perturbative and intrinsically nonperturbative nature.

**Keywords**: Nonperturbative QCD; analytic approach; renormalization group; strong running coupling.

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### 1. Introduction

Theoretical analysis of strong interaction processes rests on the quantum non-Abelian gauge field theory, namely, Quantum Chromodynamics (QCD). This theory is dealing with quarks and gluons. The former are the constituents of hadrons, while the latter are quanta of a massless gauge vector field, which provides interaction between quarks. Gluons and quarks carry a specific quantum number, which plays a key role in hadron physics. It was first introduced in the mid 1960s in Ref. [1] and later it was named[2] “color”. The comprehensive theoretical and experimental investigations revealed that the strong interaction possesses two distinctive features.

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First, the interaction between quarks inside the hadrons weakens with increasing the characteristic energy of a process. In other words, the invariant charge$^a$ vanishes at large momenta transferred, representing the so-called asymptotic freedom of the theory. Second, free quarks and gluons have not been observed, at least until the present time. Here we are dealing with the color confinement.

These two phenomena are related to hadron dynamics at different energy scales. The asymptotic freedom takes place at large values of energies, or in the so-called ultraviolet (UV) region. It corresponds to small ($r \lesssim 0.1$ fm) quark separations. On the contrary, the color confinement is related to the low-energy, or infrared (IR) domain, that corresponds to large quark separations, namely, $r \gtrsim 1.0$ fm.

Theoretical description of the asymptotic freedom was first performed in the mid 1970s in Refs. 3. These papers gave rise to wide employment of perturbative calculations in Quantum Chromodynamics. Indeed, if the value of invariant charge is small, then one can parameterize unknown quantities by perturbative power series in the running coupling. On the one hand, this drastically simplifies analysis of hadron interactions in the UV region. On the other hand, the hadron dynamics at low energies, in particular, the quark confinement, entirely remains beyond the scope of perturbation theory. As a rule, description of the QCD experimental data in the IR region requires invoking other nonperturbative approaches, for instance, phenomenological potential models (Refs. 4–12), sum rules method (Refs. 13–16), string models, bag models, variational perturbation theory (Refs. 20–23), and lattice simulations (Refs. 24–29).

The renormalization group (RG) method plays a fundamental role in the framework of Quantum Field Theory (QFT) and its applications. It was developed almost half a century ago in Refs. 30 and 31 (see also Ref. 32). Nowadays, theoretical analysis of hadron dynamics relies, in main part, upon the RG method. Meanwhile, the construction of exact solutions to the renormalization group equation, which is a rather nontrivial mathematical problem itself, is still far from being feasible. Usually, in order to describe the strong interaction processes in asymptotical UV region, one applies the RG method together with perturbative calculations. In this case, the a priori unknown renormalization group functions are parameterized by the power series in the QCD running coupling. Ultimately, this leads to approximate solutions of RG equation, which are commonly used for quantitative analysis of experimental data. However, employment of perturbation theory in the framework of renormalization group method generates unphysical singularities of such solutions. The latter are in contradiction with the basic principles of local Quantum Field Theory. For example, solution to RG equation for invariant charge gains unphysical singularities due to perturbative approximation of the $\beta$ function. At the one-loop level there is the so-called Landau pole, and account of the higher-loop corrections just introduces additional singularities of the cut type into expression

$^a$Sometimes invariant charge $\alpha(q^2) = \tilde{g}^2(q^2)/(4\pi)$ is also called the (strong) running coupling of QCD.
for the running coupling (see Appendix A).

An effective way to overcome such difficulties was proposed in the late 1950s by P.J. Redmond. It consists in invoking into consideration the analyticity requirement, which follows from the general principles of local QFT. This prescription became the underlying idea of the so-called analytic approach to QFT, which was formulated in the framework of Quantum Electrodynamics (QED) by N.N. Bogoliubov, A.A. Logunov, and D.V. Shirkov. Really, the QED running coupling is proportional to the transverse part of the dressed photon propagator. Therefore, in accordance with the basic principles of local QFT the spectral representation of the Källén–Lehmann type holds for it. The latter implies the definite analytic properties in $q^2$ variable for the invariant charge. Namely, it should be the analytic function in the complex $q^2$–plane with the only cut along the negative semiaxis of real $q^2$.

In the general case the QCD invariant charge is defined as a product of propagators and a vertex function (see, e.g., Ref. 35). Therefore, one might pose a question concerning the analytic properties of this quantity. This matter has been examined in Ref. 36 and it was shown therein that in this case the integral representation of the Källén–Lehmann type holds for the running coupling, too. Proceeding from these motivations, the analytic approach was lately extended to Quantum Chromodynamics by D.V. Shirkov and I.I. Solovtsov, and applied to the “analytization” of perturbative series for the QCD observables (see reviews and Refs. 39–42 for details). The term “analytization” means here the restoring of the correct analytic properties in the $q^2$ variable of a quantity under consideration by making use of the Källén–Lehmann integral representation

$$\left\{A(q^2)\right\}_{\text{an}} = \int_0^\infty \frac{\varrho(\sigma)}{\sigma + q^2} d\sigma.$$  \hspace{1cm} (1)

In this equation the spectral function $\varrho(\sigma)$ is determined by the initial (perturbative) expression for a quantity in hand

$$\varrho(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ A(-\sigma - i\varepsilon) - A(-\sigma + i\varepsilon) \right], \quad \sigma > 0.$$  \hspace{1cm} (2)

It is worth mentioning also the appealing features of the analytic approach to Quantum Field Theory, namely, the absence of unphysical singularities and fairly well higher loop and scheme stability of outcoming results.

The description of hadron interaction at low energies remains a long–standing challenge of Quantum Chromodynamics. Some insight into many basic problems of elementary particle physics can be gained form constructing the models, which adequately account both the perturbative and intrinsically nonperturbative traits

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A metric with signature $(-1,1,1,1)$ is used, so that positive $q^2$ corresponds to a spacelike momentum transfer.

This method has been named the analytic perturbation theory (APT) thereafter.
of hadron dynamics. Certainly, such studies are useful for further development of the strong interaction theory. Besides, the description and interpretation of many experimental data can be performed only by invoking such models. Thus, the results obtained in this field could expand our scope of knowledge concerning the nature of fundamental interactions.

This paper overviews a new model for the QCD analytic invariant charge. The model is formulated in detail with a specific attention to the properties of the analytic running coupling and to verification of the model’s self-consistency. The principally new way of involving the analyticity condition into the RG formalism is a distinguishing feature of this model. Namely, the analyticity requirement is imposed on the perturbative expansion of the renormalization group $\beta$ function. Eventually it leads to a number of profitable traits of this running coupling, that enables one to describe a wide range of the strong interaction processes both of perturbative and intrinsically nonperturbative nature. It is worth noting also that this model for the strong running coupling has no adjustable parameters. Therefore, similarly to perturbative approach, $\Lambda_{\text{QCD}}$ remains the basic characterizing parameter of the theory.

The layout of the review is as follows.

In Section 2 the model for QCD analytic invariant charge is discussed. The key point here is the restoring of the correct analytic properties of perturbative expansion for the $\beta$ function, and the subsequent solution of the corresponding RG equation. The explicit expression for the one-loop analytic invariant charge (AIC) is obtained. It is proved to have qualitatively new properties. In particular, the analytic invariant charge has no unphysical singularities, and it incorporates UV asymptotic freedom with IR enhancement in a single expression. The AIC is found to be in agreement with the results of various nonperturbative studies of hadron dynamics in the infrared domain. For example, this strong running coupling explicitly reproduces the recently discovered symmetry related to the size distribution of instantons. The consistency of the considered model with general definition of the QCD invariant charge is proved. The analytic running coupling is also represented in explicitly renorminvariant form, and the properties of the relevant $\beta$ function are studied. A number of symmetry relations for the analytic invariant charge and the corresponding $\beta$ function, which establish a link between the ultraviolet and infrared domains, are derived. The different ways of incorporating the analyticity requirement into the RG formalism are discussed.

Section 3 is devoted to investigation of the analytic invariant charge at the higher-loop levels. The integral representation for this running coupling is derived herein. Similarly to the one-loop level, analytic invariant charge is shown to have no unphysical singularities. The developed model possesses a good higher loop and scheme stability. The investigation of the corresponding $\beta$ function revealed that AIC has the universal asymptotics in both the ultraviolet and infrared regions, irrespective of the loop level. Further, the model in hand is extended to the timelike domain. The difference between values of the analytic running coupling in respec-
tive spacelike and timelike regions is found to be considerable for intermediate and low energies. Apparently, this circumstance must be taken into account when one handles the experimental data. The result obtained in this section confirms the hypothesis due to Schwinger concerning the $\beta$ function proportionality to the relevant spectral density.

The applications of the analytic invariant charge to study of a number of strong interaction processes are gathered in Section 4. In particular, the static quark–antiquark potential constructed by making use of the AIC is proved to be confining at large distances. At the same time, at small distances it has the standard behavior originated in asymptotic freedom. Further, the developed approach is applied to description of gluon condensate, inclusive $\tau$ lepton decay, and electron–positron annihilation into hadrons. The congruity of estimated values of the scale parameter $\Lambda_{\text{QCD}}$ testifies that the analytic invariant charge substantially incorporates, in a consistent way, both perturbative and intrinsically nonperturbative aspects of Quantum Chromodynamics.

In the Conclusions (Section 5), the properties of the analytic invariant charge are summarized and the basic results are formulated in a compact way.

The auxiliary materials are collected in the appendixes. In Appendix A, the perturbative solutions to the RG equation for QCD running coupling are derived at different loop levels. In Appendix B, the Lambert $W$ function is briefly described. The function $N(a)$, which essentially simplifies investigation of the $\beta$ function corresponding to the one-loop AIC, is defined and studied here. Appendix C contains explicit expressions for the spectral functions used in construction of the analytic invariant charge. In Appendix D, the coefficients specifying the expansion of the quark–antiquark potential are presented.

2. The QCD Analytic Invariant Charge

2.1. Formulation of the model

First of all, let us consider the renormalization group equation for the strong running coupling (see, e.g., Refs. 32 and 44):

$$
\frac{d \ln [g^2(\mu)]}{d \ln \mu^2} = \beta(g(\mu)).
$$

(3)

In the framework of perturbative approach, the $\beta$ function on the right-hand side of this equation can be represented as a power series

$$
\beta(g(\mu)) = -\left\{ \beta_0 \left[ \frac{g^2(\mu)}{16\pi^2} \right] + \beta_1 \left[ \frac{g^2(\mu)}{16\pi^2} \right]^2 + \ldots \right\},
$$

(4)

where $\beta_0 = 11 - 2n_f/3$, $\beta_1 = 102 - 38n_f/3$, and $n_f$ is the number of active quarks. Introducing the standard notations $\alpha_s(\mu^2) = g^2(\mu)/(4\pi)$ and $\beta_0(\mu^2) = \alpha(\mu^2) \beta_0/(4\pi)$,
one can reduce the RG equation \(^{34}\) at the \(\ell\)-loop level to the form

\[
\frac{d \ln \left[ \tilde{\alpha}_s(\mu^2) \right]}{d \ln \mu^2} = - \sum_{j=0}^{\ell-1} B_j \left[ \left. \left( \frac{\tilde{\alpha}_s(\mu^2)}{\beta_0} \right)^{j+1} \right|_0 \right].
\]  

(5)

It is well-known that the solution to this equation has unphysical singularities at any loop level. So, there is the Landau pole at the one-loop level, and account of the higher loop corrections just introduces the additional singularities of the cut type into expression for the invariant charge.\(^3\) However, the fundamental principles of local Quantum Field Theory require the invariant charge \(\alpha(q^2)\) to have a definite analytic properties in the \(q^2\) variable. Namely, there must be the only cut along the negative semiaxis of real \(q^2\) (see Ref. \(^{37}\)).

The solution to the renormalization group equation \(^3\) gains unphysical singularities due to perturbative approximation of the \(\beta\) function \(^4\). Such a representation of the right-hand side of the RG equation proves to be justified in the ultraviolet region, where the asymptotic freedom takes place. However, the approximation \(^4\) does not ultimately lead to the correct analytic properties of the QCD invariant charge. In fact, since the strong running coupling \(\alpha(q^2)\) is of a fixed sign, the condition that invariant charge has the only left cut in \(q^2\) means that the right-hand side of the RG equation \(^3\), as a function of \(q^2\), has no singularities in the region of positive \(q^2\). Apparently, the perturbative expansion \(^4\) violates this condition.

In the framework of developed model \(^{45,46}\) the analyticity requirement is imposed on the \(\beta\) function perturbative expansion for restoring its correct analytic properties. This leads to the following RG equation for the QCD analytic invariant charge:

\[
\frac{d \ln \left[ \tilde{\alpha}_{\text{an}}(\mu^2) \right]}{d \ln \mu^2} = - \left\{ \sum_{j=0}^{\ell-1} B_j \left[ \left. \left( \frac{\tilde{\alpha}_s(\mu^2)}{\beta_0} \right)^{j+1} \right|_0 \right] \right\}_{\text{an}}, \quad B_j = \frac{\beta_j}{\beta_0^{j+1}}.
\]  

(6)

In this equation \(\tilde{\alpha}_{\text{an}}(\mu^2)\) denotes the \(\ell\)-loop analytic invariant charge, \(\tilde{\alpha}_s(\mu^2)\) is the perturbative running coupling at the \(\ell\)-loop level, and the braces \(\{ \ldots \}_{\text{an}}\) mean the analytization of the expression within them by making use of the Källén–Lehmann spectral integral \(^{11}\) (see also Refs. \(^{47,48}\)).

**2.2. The one-loop analytic invariant charge**

Since the proposed model rests on invoking the analyticity requirement into the renormalization group formalism, one may anticipate that the strong running coupling acquires qualitatively new properties. Let us proceed to construction of solution to the RG equation \(^3\), restricting ourselves to the leading loop approximation at this stage.

\(^d\)Solutions to equation \(^3\) at different loop levels are considered in detail in Appendix A.
At the one-loop level the renormalization group equation (6) takes the form

\[
\frac{d \ln [\tilde{\alpha}_{\text{an}}(\mu^2)]}{d \ln \mu^2} = -\left\{\tilde{\alpha}^{(1)}(\mu^2)\right\}_{\text{an}}.
\]

(7)

Here \(\alpha^{(1)}_{\text{an}}(\mu^2)\) is the one-loop analytic invariant charge and \(\alpha^{(1)}_{s}(\mu^2)\) denotes the perturbative running coupling at the one-loop level. Taking into account Eq. (1), one can represent the right-hand side of the RG equation (7) as follows

\[
\frac{d \ln [\tilde{\alpha}^{(1)}_{\text{an}}(\mu^2)]}{d \ln \mu^2} = -\int_0^\infty \frac{\mathcal{R}^{(1)}(\sigma)}{\sigma + \mu^2} d\sigma,
\]

(8)

where \(\mathcal{R}(\sigma) = [\ln^2(\sigma/\Lambda^2) + \pi^2]^{-1}\). Upon the integration with respect to \(\sigma\) and the introduction of the dimensionless variable \(\nu^2 = \mu^2/\Lambda^2\), Eq. (8) acquires the form

\[
\frac{d \ln [\tilde{\alpha}^{(1)}_{\text{an}}(\nu^2\Lambda^2)]}{d \ln \nu^2} = -\left[\frac{1}{\ln \nu^2} + \frac{1}{1 - \nu^2}\right].
\]

(9)

Further, integrating this equation with respect to \(\nu^2\) in finite terms, one gets

\[
\frac{\tilde{\alpha}^{(1)}_{\text{an}}(q^2)}{\tilde{\alpha}^{(1)}_{\text{an}}(q^2_0)} = \frac{z - 1}{z \ln z}, \quad z = q^2/\Lambda^2.
\]

(10)

where \(z = q^2/\Lambda^2\) and \(z_0 = q_0^2/\Lambda^2\). Thus, the QCD analytic invariant charge at the one-loop level has the form

\[
\alpha^{(1)}_{\text{an}}(q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z}, \quad z = \frac{q^2}{\Lambda^2}.
\]

(11)

It is worth mentioning also that for this strong running coupling the Källén–Lehmann representation

\[
\alpha^{(1)}_{\text{an}}(q^2) = \frac{4\pi}{\beta_0} \int_0^\infty \frac{\rho^{(1)}(\sigma)}{\sigma + z} d\sigma
\]

(12)

holds, where \(\rho^{(1)}(\sigma)\) is the one-loop spectral density

\[
\rho^{(1)}(\sigma) = \left(1 + \frac{1}{\sigma}\right) \frac{1}{\ln^2 \sigma + \pi^2}.
\]

(13)

Figure 1 depicts the analytic invariant charge \(\tilde{\alpha}^{(1)}_{\text{an}}(q^2) = \alpha^{(1)}_{\text{an}}(q^2)\beta_0/(4\pi)\) together with the one-loop perturbative running coupling \(\tilde{\alpha}^{(1)}_{s}(q^2) = 1/\ln(q^2/\Lambda^2)\). The obtained solution to the RG equation (11) possesses a number of profitable features. First of all, the integral representation of the Källén–Lehmann type (12)

\*Equation (11) has been derived by employing the condition \(\alpha^{(1)}_{\text{an}}(q^2) \rightarrow \alpha^{(1)}_{s}(q^2)\), when \(q^2 \rightarrow \infty\) (see also discussion in Subsection 3.1).
Fig. 1. The one-loop analytic invariant charge $\tilde{\alpha}^{(1)}_{an}(q^2)$ defined by Eq. (11) (solid curve) and its perturbative analog $\tilde{\alpha}^{(1)}_{s}(q^2)$ (dashed curve), $z = q^2/\Lambda^2$.

implies that the analytic invariant charge $\alpha^{(1)}_{an}(q^2)$ has the only cut $^f_i$ along the negative semiaxis of real $q^2$. Thus, the prescription proposed ultimately leads to elimination of the Landau pole in a multiplicative way. One has to note that the analytic invariant charge $^{[11]}_i$ incorporates the ultraviolet asymptotic freedom with the infrared enhancement in a single expression. In particular, this trait will play an essential role in applications of the elaborated model, see Section 4 further. It is worth emphasizing also that the developed model contains no adjustable parameters, i.e., similarly to perturbative approach, $\Lambda_{QCD}$ remains the basic characterizing parameter of the theory.

A crucial insight into the nonperturbative aspects of the low-energy hadron dynamics can be provided by the relevant lattice simulations. One should mention here a recent lattice investigation of the topological structure of the SU(3) vacuum by the UKQCD Collaboration.$^{[49]}$ It was revealed therein that the size distribution of instantons $D(\rho)$ has a conspicuous peak shape, and it is severely suppressed at large scales $\rho$. In turn, this is compatible neither with perturbative results, nor with the freezing of the strong running coupling to some constant value at large distances.$^{[49,50]}$ Remarkably, the analytic invariant charge $^{[11]}_i$ is proved to reproduce explicitly a special kind of symmetry related to this issue$^{[13]}$ (see Subsection 2.3 for details). Meantime, the strong running coupling itself can also be studied on the lattice. In particular, a recent investigation of its low-energy behavior by ALPHA Collaboration$^{[51]}$ testify to the infrared enhancement of the QCD invariant charge. The study of the Schwinger–Dyson equations is another source of the nonperturbative information concerning the strong interaction issues. It is worth noting here

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$f_i$The function $a(z) = (z - 1)/(z \ln z)$ has a smooth behavior in the vicinity of the point $z = 1$, namely $a(1 + \varepsilon) \simeq 1 - \varepsilon/2 + \mathcal{O}(\varepsilon^2)$, $\varepsilon \to 0$. 

$^i$
that the behavior of the QCD invariant charge, similar to Eq. (11), is found to be in agreement with the solution of the Schwinger–Dyson equations (see papers 52, 53 and references therein for details).

In general, the QCD invariant charge is defined as the product of the corresponding Green functions and vertexes. For example, in the transverse gauge the following definition

\[ \alpha(q^2) = \alpha(\mu^2) G(q^2) g(q^2) \]  

(14)
takes place (see, e.g., Ref. 35). In this equation \( \alpha(q^2) \) is the QCD running coupling, \( \alpha(\mu^2) \) denotes its value at a normalization point \( \mu^2 \), \( G(q^2) \) and \( g(q^2) \) are dimensionless gluon and ghost propagators, respectively. It is essential to note here that the Green functions possess the correct analytic properties in the \( q^2 \) variable before the RG summation. Indeed, in massless theory the Green function can be written as the perturbative power series in \( \alpha(\mu^2) \) with the coefficients being a polynomial in \( \ln(q^2/\mu^2) \). Obviously, in this case the integral representation of the Källén–Lehmann type must hold for any finite order of perturbative expansion for the Green function. However, the RG summation leads to violation of such a property of the Green functions (e.g., in the simplest one-loop case the Landau pole appears). As it has already been mentioned, the analytic approach to QCD seeks to restore the correct analytic properties of the relevant quantities. It turns out that for the consistency of involving the analyticity requirement with the definition of the invariant charge (14), one has to apply the analytization procedure (1) to the logarithmic derivative of the Green function \( d \ln G(q^2)/d \ln q^2 \). Really, if the spectral function of the Källén–Lehmann representation for the Green function \( G(q^2) \) is of a fixed sign (that is true for the leading orders of perturbation theory), then \( G(q^2) \) has no zeros in the complex \( q^2 \)-plane. Hence, the derivative \( d \ln G(q^2)/d \ln q^2 \) can also be represented as the spectral integral of the Källén–Lehmann type (cf. equations (8) and (12) for the one-loop level, and Eqs. (37) and (42) for the higher loop levels).

In the framework of perturbative approach the Green functions considered above have the following form: \( g(q^2) = (1/\ln z)^{d_g} \) and \( G(q^2) = (1/\ln z)^{d_G} \), where \( z = q^2/\Lambda^2 \), \( d_g \) and \( d_G \) denote the corresponding anomalous dimensions. For the latter the relationship \( 2d_g + d_G = 1 \) holds, that plays the key role in definition (14).

Applying the analytization procedure to these functions in the way specified above, one arrives at the following result: \( g_{\text{an}}(q^2) = [(z-1)/(z \ln z)]^{d_g} \) and \( G_{\text{an}}(q^2) = [(z-1)/(z \ln z)]^{d_G} \). Therefore, in this case invoking the analyticity condition does not affect the definition of the invariant charge (14). Thus, we infer that the analytic running coupling (11) is consistent with the general definition of the QCD invariant charge (see also discussion of this matter in Ref. 54).

\( ^8 \)Namely, there is the only cut along the negative semiaxis of real \( q^2 \).

\( ^6 \)One should note that the relevant spectral function here can not be directly identified with the spectral density.

\( ^{1} \)It is interesting to note that the similar situation takes place in the nonperturbative \( a \)-expansion method also. 20, 22
2.3. Properties of the one-loop analytic invariant charge

Let us address now the renorminvariance of the analytic running coupling \( \alpha \). In general, any expression for the invariant charge makes sense only if the relevant definition of the parameter \( \Lambda_{\text{QCD}} \) is provided. Otherwise, the running coupling may not be a renorminvariant quantity at all. In fact, the renormalization invariance of a solution to the RG equation (5) is ensured by a certain relation between the parameter \( \Lambda_{\text{QCD}} \) and the value of the running coupling \( \alpha(\mu^2) \) at a reference point \( \mu^2 \) (see Eqs. (A.6) and (A.10) also). However, the way of introduction of the parameter \( \Lambda_{\text{QCD}} \), considered in the previous section, blurs such a relation. Nevertheless, in the framework of the model in hand, the latter can be recovered by solving the equation

\[
\left. \alpha_{\text{an}}^{(1)}(q^2) \right|_{q^2=\mu^2} = \alpha(\mu^2) \tag{15}
\]

with respect to parameter \( \Lambda \). In Appendix B, equations of this type are studied in detail. Thus, the solution to Eq. (15) can be explicitly written down in terms of the Lambert \( W \) function which is defined by the relation \( k \)

\[
W_k(x) \exp[W_k(x)] = x. \tag{16}
\]

For our purposes it is convenient to introduce the function \( N(a) \)

\[
N(a) = \begin{cases} 
N_0(a), & 0 < a \leq 1, \\
N_{-1}(a), & 1 < a,
\end{cases} 
N_k(a) = -a W_k \left[ -\frac{1}{a} \exp \left( -\frac{1}{a} \right) \right]. \tag{17}
\]

In Eqs. (16) and (17), \( k \) denotes the branch index of the Lambert \( W \) function. It is straightforward to verify that the solution to Eq. (15) has the form

\[
\Lambda^2 = \mu^2 N \left[ \frac{\beta_0}{4\pi} \alpha(\mu^2) \right]. \tag{18}
\]

Let us note here that the right-hand side of this equation tends to the one-loop perturbative form when \( \mu^2 \to \infty \) (see Eqs. (A.6) and (B.16) also):

\[
\Lambda_n^2 = \mu^2 \exp \left[ -\frac{4\pi}{\beta_0} \frac{1}{\alpha(\mu^2)} \right]. \tag{19}
\]

Thus, the renorminvariant expression for the analytic running coupling (11) is the following:

\[
\alpha_{\text{an}}^{(1)}(q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z}, \quad z = \frac{q^2}{\Lambda^2}, \quad \Lambda^2 = \mu^2 N \left[ \frac{\beta_0}{4\pi} \alpha(\mu^2) \right], \tag{20}
\]

where the function \( N(a) \) is defined in Eq. (17) (see also Ref. [56] for details).

\(^1\)Equation (18) is identical to the normalization condition \( \bar{g}^2(1, g) = g^2 \) for the QCD invariant charge \( \bar{g}^2(q^2/\mu^2, g)/(4\pi) = \alpha(q^2) \) (see, e.g., Ref. [55].

\(^k\)The properties of the Lambert \( W \) function (16) and the function \( N(a) \) (17) are described in detail in Appendix B.
In the framework of the elaborated model the analyticity requirement is imposed on the \( \beta \) function perturbative expansion for restoring its correct analytic properties. Apparently, the form of the \( \beta \) function in this case is affected by the analytization procedure. Thus, it is of a particular interest to study both the \( \beta \) function itself and its properties within the approach developed.

In general, the \( \beta \) function corresponding to the one-loop analytic invariant charge \( (11) \) is defined by

\[
\beta_{\text{an}}^{(1)}(a) = \frac{d \ln a(\mu^2)}{d \ln \mu^2}, \quad a(\mu^2) = \tilde{\alpha}_{\text{an}}^{(1)}(\mu^2).
\]

(21)

The function \( N(a) \) (see Eq. (17)) enables one to obtain the explicit expression for the right-hand side of Eq. (21):

\[
\beta_{\text{an}}^{(1)}(a) = \frac{1 - N(a)/a}{\ln |N(a)|}.
\]

(22)

In order to examine the asymptotics of the \( \beta \) function \( (22) \), it is worth employing the Eqs. (B.16)–(B.18). It turns out that for small values of the running coupling \( a(\mu^2) \) equation \( (22) \) coincides with the well-known perturbative result

\[
\beta_{\text{an}}^{(1)}(a) = -a + O\left(\frac{1}{a}\exp\left(-\frac{1}{a}\right)\right), \quad a \to 0^+.
\]

(23)

The second term in this equation implies the intrinsically nonperturbative nature of the \( \beta \) function \( (22) \). Then, for large values of \( a(\mu^2) \) one has

\[
\beta_{\text{an}}^{(1)}(a) = -1 + O\left[\frac{\ln(\ln a)}{(\ln a)^2}\right], \quad a \to \infty.
\]

(24)

Such a behavior of the \( \beta \) function leads to the infrared enhancement of the QCD invariant charge, namely (up to a logarithmic factor) \( \tilde{\alpha}(q^2) \simeq \Lambda^2/q^2 \), when \( q^2 \to 0 \).

It is worth mentioning also that

\[
\beta_{\text{an}}^{(1)}(1 + \varepsilon) = -\frac{1}{2} - \frac{\varepsilon}{6} + O(\varepsilon^2), \quad \varepsilon \to 0,
\]

(25)

i.e., the function \( (22) \) has a smooth behavior at the point of interchange between the real branches of the Lambert \( W \) function. Moreover, \( \beta_{\text{an}}^{(1)}(a) \leq 0 \) for any non-negative value of \( a(\mu^2) \), that ensures the asymptotic freedom of the theory.

Figure 2 presents the \( \beta \) function \( (22) \) together with its perturbative analog \( \beta_s^{(1)}(a) = -a \), which corresponds to the one-loop running coupling \( \tilde{\alpha}_s^{(1)}(q^2) = 1/\ln(q^2/\Lambda^2) \). In particular, this plot visualizes the reproduction of perturbative limit \( (23) \) within the model elaborated.

Thus, the proposed way of involving the analyticity requirement into RG formalism eventually leads to the qualitatively new features of the \( \beta \) function \( (22) \). Namely, it incorporates the asymptotics related to both the UV asymptotic freedom \( (23) \)

1In Ref. 56 the definition \( \beta(a) = da(\mu^2)/d \ln \mu^2 \) was used.
Fig. 2. The $\beta$ function (solid curve) corresponding to the one-loop analytic invariant charge (11). The relevant perturbative result $\beta_s^{(1)}(a) = -a$ is shown by the dot-dashed line.

and the IR enhancement of the QCD invariant charge. In particular, such behavior of the $\beta$ function corresponds to the so-called “V–scheme” (see Refs. 9–11 and 57), which is widely used in studies of the quark confinement.

Let us address now the symmetries of the analytic invariant charge $\alpha_{\text{an}}^{(1)}(q^2)$ and the corresponding $\beta$ function $\beta_{\text{an}}^{(1)}(a)$, which relate these quantities in the ultraviolet and infrared domains. Obviously, such relations are decisive, since they provide us with insight into the intrinsically nonperturbative features of the strong interaction. The symmetry of this kind for the analytic invariant charge (11) was found in Refs. 43 and 54:

$$\alpha_{\text{an}}^{(1)}(q^2) = \frac{\Lambda^2}{q^2} \alpha_{\text{an}}^{(1)} \left( \frac{\Lambda^4}{q^2} \right).$$  \hspace{1cm} (26)

It is interesting that there are similar symmetry relations for the $\beta$ function (22) also. In particular,\(^m\)

$$\beta \left( \tilde{\alpha}_{\text{an}}^{(1)}(q^2) \right) + \beta \left( \tilde{\alpha}_{\text{an}}^{(1)}(\Lambda^4/q^2) \right) = -1$$ \hspace{1cm} (27)

and

$$\beta \left( \tilde{\alpha}_{\text{an}}^{(1)}(q^2) \right) + \beta \left( \tilde{\alpha}_{\text{an}}^{(1)}(\Lambda^4/q^2) \right) = -1,$$ \hspace{1cm} (28)

where $\beta(a(\mu^2)) = d\ln a(\mu^2)/d\ln \mu^2$. In fact, there is a perturbative analog of Eq. (28), namely $\beta(\alpha_{\text{an}}^{(1)}(q^2)) = -1$, which holds for all non-negative $q^2$.

From the general viewpoint, one might anticipate that these symmetries, interrelating the hadron dynamics at short and long distances, could be revealed also\(^m\)

\(^m\)Equation (27) can be deduced from Eq. (26).
in some intrinsically nonperturbative strong interaction processes. It is worth mentioning here the recently discovered conformal inversion symmetry related to the size distribution of instantons. At the classical level, the whole instanton sector is proved to be invariant under the four-dimensional conformal transformations. In particular, the coordinate inversion

$$x_\mu \rightarrow x'_\mu = \frac{\rho_0^2}{x^2} x_\mu \quad (29)$$

transforms the SU(2) Yang–Mills instanton solution of size $\rho$ in singular gauge to the anti-instanton solution of size $\rho_0^2/\rho$ in regular gauge. However, at the quantum level it is rather difficult to expect that the instanton sector is still invariant under the conformal group transformations since the scale invariance is broken via the regularization and renormalization. Nevertheless, there are evidences that the invariance under the coordinate inversion (29), which has a number of important physical implications, exists, in a certain form, at the quantum level, too. Thus, the essential result in this field was obtained in Ref. where it was observed that the high-quality lattice simulation data for the quantity

$$D_{CI}^{Cl}(\rho) = D(\rho) \left( \frac{\rho}{\rho_0} \right)^{N_c/6}, \quad D(\rho) = b \left[ \frac{2\pi}{\alpha(\rho)} \right]^{2N_c} \exp \left[ -\frac{2\pi}{\alpha(\rho)} \right] \quad (30)$$

appear to satisfy with high precision the conformal inversion symmetry $D_{CI}^{Cl}(\rho) = D_{CI}^{Cl}(\rho_0^2/\rho)$. Here $D(\rho)$ is the size distribution of instantons, $\rho$ is the radius of an instanton, $\alpha(\rho)$ denotes the strong running coupling, $\rho_0$ is the peak position of $D_{CI}^{Cl}(\rho)$, $b$ is a constant factor depending on the gauge group, and $N_c$ is the number of colors. This symmetry of $D_{CI}^{Cl}(\rho)$ is evidently originated in the mentioned above conformal invariance of the instanton sector at the classical level, and it turns out to be at most weakly broken at the quantum level.

Apparently, the conformal inversion symmetry $D_{CI}^{Cl}(\rho) = D_{CI}^{Cl}(\rho_0^2/\rho)$ imposes a certain constraint onto the QCD invariant charge, leading to an important relation between the properties of the $\alpha(\rho)$ at small and large distances. In fact, one can derive the expression for the nonperturbative strong running coupling by employing the symmetry of this kind. Following this way, the analytic invariant charge has recently been rediscovered and proved to reproduce explicitly the conformal inversion symmetry of $D_{CI}^{Cl}(\rho)$. This remarkable result opens up an alluring opportunity to build up a new general method of consistent description of the both perturbative and intrinsically nonperturbative aspects of hadron dynamics towards the infrared regime.

2.4. Other models

Several decades ago, when the analytic approach to Quantum Electrodynamics was developing, it was noted that there is no unique way of incorporating the analyticity condition into the RG formalism. There was also an attempt to resolve this ambiguity by involving an additional condition, in particular, by making use
of the equations of motion. Following this way the photon propagator has been derived \(^{63}\) the unphysical singularity being removed in a multiplicative way (i.e., similarly to Eq. (11)). Furthermore, the presence of nonperturbative terms both in the running coupling itself \(^{11}\) and in the corresponding spectral density \(^{13}\) is another common feature of the model in hand and the approach considered in Ref. \(^{63}\) (see also discussion of this issue in Refs. \(^{46}\), \(^{54}\) and \(^{56}\)).

The identical situation arises in Quantum Chromodynamics, namely, different models for the strong running coupling can be proposed in the framework of the analytic approach to QCD. As it was mentioned above, our model \(^{11}\) is based on recovering the correct analytic properties of the perturbative expansion of the RG \(\beta\) function \(^{5}\). Meanwhile, in the original model due to Shirkov and Solovtsov\(^{37}\), the analytization procedure \(^{11}\) is directly applied to the perturbative invariant charge (see also reviews \(^{38}\)). At the one-loop level it gives

\[
\alpha^{(1)}_{\text{SS}} (q^2) = \frac{4\pi}{\beta_0} \left[ \frac{1}{\ln z} + \frac{1}{1 - z} \right], \quad z = \frac{q^2}{\Lambda^2}. \tag{31}
\]

From the general point of view, one can anticipate the different properties of these two running couplings. On the one hand, both these models have no unphysical singularities and reproduce perturbative limit in the ultraviolet region. On the other hand, there is a difference between them in the infrared domain. Namely, the analytic invariant charge \(^{11}\) possesses the IR enhancement, while the running coupling \(^{31}\) freezes to the universal constant value \(\alpha_{\text{SS}}(0) = \frac{4\pi}{\beta_0}\). One has to emphasize here that both these models have no adjustable parameters, so they are the “minimal” ones in this sense. A brief discussion of this issue can also be found in Refs. \(^{46}\), \(^{54}\) and \(^{64}\).

It is worth mentioning some other models for the strong running coupling within the analytic approach to QCD. Let us start with the model developed by Alekseev and Arbuzov\(^{52,53}\). By making use of a special solution to the Schwinger–Dyson equations for the gluon propagator, these authors proposed the following expression for the QCD running coupling:

\[
\alpha^{(1)}_{\text{AA}} (q^2) = \frac{4\pi}{\beta_0} \left[ \frac{1}{\ln z} + \frac{1}{1 - z} + \frac{c}{z + m_g^2/\Lambda^2} \right], \quad z = \frac{q^2}{\Lambda^2}, \tag{32}
\]

where \(m_g\) is the gluon mass and \(c\) denotes a dimensionless parameter fixed by the phenomenological value of the gluon condensate. Likewise the analytic invariant charge \(^{11}\), the running coupling \(^{32}\) has enhancement in the infrared domain.

It is interesting to note here that another similar model for the QCD invariant charge

\[
\alpha^{(1)}_{\text{lat}} (q^2) = \frac{4\pi}{\beta_0} \left[ \frac{1}{\ln z} + \frac{1}{1 - z} + \frac{\eta}{z} \right], \quad z = \frac{q^2}{\Lambda^2} \tag{33}
\]

has been put forward in Ref. \(^{65}\) for the analysis of the lattice simulation data on the low–energy behavior of the QCD running coupling\(^{65,66}\). The model \(^{65}\) also possesses the infrared enhancement.
By making use of a certain phenomenological reasoning, Webber suggested the
strong running coupling of the following form \(67\)

\[
\alpha_{W}^{(1)}(q^2) = \frac{4\pi}{\beta_0} \left[ \frac{1}{\ln z} + \frac{1}{1 - z} \frac{z + b}{1 + b} \left( 1 + c \right)^p \right], \quad z = \frac{q^2}{\Lambda^2}, \quad (34)
\]

with a specific choice of the parameters \(b = 1/4, c = 4, p = 4\). On the contrary, this model has infrared finite value \(\alpha_{W}^{(1)}(0) \approx 2\pi/\beta_0\).

There is also a generalization of the analytic invariant charge \(11\). Thus, the authors of Ref. \(68\) proposed the IR enhanced model for the QCD running coupling

\[
\alpha_{\text{IR}}^{(1)}(q^2) = \left[ \frac{1}{\alpha_{\text{IR}}^{(1)}(\Lambda^2)} + \frac{\beta_0}{4\pi} \int_0^\infty \frac{(z - 1) z^p}{(z + 1 + i\varepsilon)(1 + z^p)} \, d\sigma \right]^{-1}, \quad (35)
\]

where \(z = q^2/\Lambda^2\) and \(0 < p \leq 1\). This formula coincides with Eq. \(11\) when \(p = 1\).

Several phenomenological models for the QCD running coupling have been proposed in Ref. \(69\). It should be mentioned that the ideas, similar to that of the analytic perturbation theory \(37,38\), were also used in analysis of the electron–positron annihilation into hadrons \(70\) and investigation of the inclusive \(\tau\) lepton decay \(71\).

The thorough study of the power corrections to the strong running coupling was performed in Refs. \(72,73\). There is also a number of methods of the RG improvement of perturbative series for the QCD observables (see, e.g., Ref. \(74\)).

3. Higher Loop Levels

3.1. Analytic invariant charge at the higher loop levels

As it has been shown in Section \(2\), the one-loop analytic invariant charge possesses a number of appealing features. The absence of unphysical singularities and incorporation of the ultraviolet asymptotic freedom with the infrared enhancement in a single expression are the most remarkable ones. Apparently, the question of a primary interest here is whether these results are affected by the higher loop corrections or not. So, let us proceed to the study of the analytic running coupling at the higher loop levels.

In accordance with the model proposed (see Subsection \(2.1\)), the QCD analytic invariant charge \(\alpha_{\text{an}}^{(\ell)}(q^2)\) at the \(\ell\)-loop level is the solution to the renormalization group equation \(46\)

\[
\frac{d}{d \ln \mu^2} \left[ \alpha_{\text{an}}^{(\ell)}(\mu^2) \right] = - \left\{ \sum_{j=0}^{\ell-1} B_j \left[ \alpha_{\text{an}}^{(\ell)}(\mu^2) \right]^{j+1} \right\}_{\text{an}}, \quad B_j = \frac{\beta_j}{\beta_0^{j+1}}, \quad (36)
\]

where \(\alpha_{\text{an}}^{(\ell)}(\mu^2)\) is the \(\ell\)-loop perturbative running coupling, \(\tilde{\alpha}(\mu^2) = \alpha(\mu^2)\beta_0/(4\pi)\), and \(\beta_j\) are the \(\beta\) function expansion coefficients (see Appendix \(A\)). By employing
the relation \[ \frac{d \ln \left[ \tilde{\alpha}_\text{an}^{(\ell)}(\mu^2) \right]}{d \ln \mu^2} = - \int_0^\infty \frac{\mathcal{R}^{(\ell)}(\sigma)}{\sigma + \mu^2} \, d\sigma, \] (37)

where

\[ \mathcal{R}^{(\ell)}(\sigma) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \sum_{j=0}^{\ell-1} B_j \left\{ \left[ \tilde{\alpha}_s^{(\ell)}(-\sigma - i\varepsilon) \right]^{j+1} - \left[ \tilde{\alpha}_s^{(\ell)}(-\sigma + i\varepsilon) \right]^{j+1} \right\}. \] (38)

The explicit forms of \( \mathcal{R}^{(\ell)}(\sigma) \) at different loop levels are given in Appendix C. Integrating Eq. (37) with respect to \( \mu^2 \) in finite terms, we obtain

\[ \alpha^{(\ell)}_{\text{an}}(\sigma_2) = \alpha^{(\ell)}_{\text{an}}(\sigma_2^0) \exp \left( \int_0^\infty \mathcal{R}^{(\ell)}(\sigma) \ln \left( \frac{1 + \sigma/z}{1 + \sigma/z_0} \right) \frac{d\sigma}{\sigma} \right), \] (39)

where \( z = q^2/\Lambda^2 \) and \( z_0 = q_0^2/\Lambda^2 \).

Expression (39) contains the explicit dependence on the normalization point \( q_0^2 \) and on the value of the strong running coupling at this point. It is worth mentioning here that this equation possesses the obvious renormalization invariance. In fact, having the value of the QCD running coupling at some momenta transferred, one might exploit Eq. (39). However, usually it turns out to be more convenient to deal with the explicit expression for the strong running coupling independent of the normalization point.

For the case of the model in hand this objective can be achieved by invoking the physical requirement\(^n\) \( \alpha^{(\ell)}_{\text{an}}(q^2) \to \alpha^{(1)}_{\text{an}}(q^2) \), when \( q^2 \to \infty \) (see Refs. 46 and 54 for details). In fact, this condition has already been employed at the one-loop level, see Eqs. (10) and (11). It is worth emphasizing here that the imposition of such requirement does not affect the renormalization invariance of Eq. (39). In general, one is able to derive the integral representation for the QCD analytic invariant charge, independent of the normalization point, by invoking the similar requirement \( \alpha^{(\ell)}_{\text{an}}(q^2) \to \alpha^{(1)}_{\text{an}}(q^2) \), when \( q^2 \to \infty \). For this purpose let us consider the ratio of the \( \ell \)-loop analytic running coupling \( \alpha^{(\ell)}_{\text{an}}(q^2) \) normalized at a point \( \sigma_2^0 \) to the one-loop AIC written in the form \( \alpha^{(1)}_{\text{an}}(q^0^2) \):

\[ \frac{\alpha^{(\ell)}_{\text{an}}(q^2)}{\alpha^{(1)}_{\text{an}}(q^0^2)} = \frac{\alpha^{(\ell)}_{\text{an}}(q_0^2)}{\alpha^{(1)}_{\text{an}}(q_0^2)} \exp \left[ \int_0^\infty \Delta \mathcal{R}^{(\ell)}(\sigma) \ln \left( \frac{1 + \sigma/z}{1 + \sigma/z_0} \right) \frac{d\sigma}{\sigma} \right], \] (40)

where \( \Delta \mathcal{R}^{(\ell)}(\sigma) = \mathcal{R}^{(\ell)}(\sigma) - \mathcal{R}^{(1)}(\sigma) \). Proceeding to the limit \( \sigma_0^2 \to \infty \), one arrives at the expression for the \( \ell \)-loop analytic invariant charge\(^5\),

\[ \alpha^{(\ell)}_{\text{an}}(q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z} \exp \left[ \int_0^\infty \Delta \mathcal{R}^{(\ell)}(\sigma) \ln \left( 1 + \frac{\sigma}{z} \right) \frac{d\sigma}{\sigma} \right], \quad z = q^2/\Lambda^2. \] (41)

\(^n\)This condition has also been used in Ref. 47 for evaluating the normalization coefficients.
It is worthwhile to note that the integral representation of the Källén–Lehmann type also holds for the AIC
\[
\alpha^{(\ell)}_{\text{an}}(q^2) = \frac{4\pi}{\beta_0} \int_0^\infty \frac{\rho^{(\ell)}(\sigma)}{\sigma + z} d\sigma, 
\]
where \(\rho^{(\ell)}(\sigma)\) is the \(\ell\)-loop spectral density
\[
\rho^{(\ell)}(\sigma) = \rho^{(1)}(\sigma) \exp \left[ \int_0^{\infty} \Delta R^{(\ell)}(\zeta) \ln \left| 1 - \frac{\zeta}{\sigma} \right| \frac{d\zeta}{\zeta} \right] \cos \psi^{(\ell)}(\sigma) + \ln \frac{\sigma}{\pi} \sin \psi^{(\ell)}(\sigma),
\]
the one-loop spectral density \(\rho^{(1)}(\sigma)\) has been specified in Eq. (13), and
\[
\psi^{(\ell)}(\sigma) = \pi \int_{\sigma}^{\infty} \Delta R^{(\ell)}(\zeta) \frac{d\zeta}{\zeta}.
\]
In the exponent of Eq. (43) the principle value of the integral is assumed. However, in practical use the expression turns out to be more convenient. One should emphasize here that, similarly to the one-loop case considered above, the analytic invariant charge has no unphysical singularities, it contains no free parameters, and it incorporates the ultraviolet asymptotic freedom with the infrared enhancement in an unified manner.

Figure 3 shows the analytic invariant charge \(\bar{\alpha}^{(\ell)}_{\text{an}}(q^2) = \alpha^{(\ell)}_{\text{an}}(q^2) \beta_0/(4\pi)\) at the one-, two-, and three-loop levels. It is obvious that AIC possesses the higher loop stability. Thus, the curves corresponding to the two- and three-loop levels are practically indistinguishable. Proceeding from this one can also draw the conclusion...
concerning the scheme stability of the current approach. In particular, in Fig. 3 the curve corresponding to the three-loop analytic running coupling $\tilde{\alpha}_\text{an}^{(3)}(q^2)$ is plotted by making use of the coefficient $\beta_2 = \frac{2857}{2} - 5033 n_f/18 + 325 n_f^2/54$ computed in the $\overline{\text{MS}}$ scheme. The account of the third term on the right-hand side of Eq. (6) does not lead to a valuable quantitative variation of its solution in comparison with the two-loop approximation. Therefore it is evident that using the coefficient $\beta_2$ computed in another subtraction scheme does not lead to significant variation of the solution to Eq. (6) in comparison with the considered case of the $\overline{\text{MS}}$ scheme. This statement follows also from the fact that the contribution of every subsequent term on the right-hand side of Eq. (6) is substantially suppressed by the contributions of the preceding ones (see also Ref. 54 and Appendix C).

3.2. Properties of the AIC at higher loop levels

At the one-loop level there is an explicit expression for the analytic invariant charge. This allows one to perform the investigation of its properties manifestly. However, at higher loop levels only the integral representation has been obtained for $\alpha_\text{an}^{(\ell)}(q^2)$ so far. This fact significantly complicates the issue in hand, and eventually leads to the necessity of applying the numerical computation. Nevertheless, the study of the relevant $\beta$ function enables one to elucidate some important questions here, in particular, the asymptotic behavior of the analytic invariant charge (see Ref. 54 for details).

The $\beta$ functions corresponding to the analytic invariant charge at the higher

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At least, in schemes that do not have unnaturally large expansion coefficients (see paper 76 and references therein for the detailed discussion of this issue).
are presented in Figures 4 and 5. Figure 4 shows the two-loop β function \( \beta^{(2)}_{an} (a) \) together with respective perturbative result \( \beta^{(2)}_{s} (a) = -a - B_1 a^2 \). Figure 5 depicts the analogous functions at the three-loop level, namely \( \beta^{(3)}_{an} (a) \) and \( \beta^{(3)}_{s} (a) = -a - B_1 a^2 - B_2 a^3 \). It is clear from Figures 2, 4, and 5 that the β function corresponding to the analytic running coupling (41) coincides with its perturbative analog in the region of small values of the invariant charge at any loop level. In other words, in the framework of the model under consideration the complete recovering of the perturbative limit in the ultraviolet domain takes place.

Let us turn now to the asymptotics of the β function (45) at large values of the running coupling. Figure 6 presents the β functions \( \beta^{(\ell)}_{an} (a) \), \( \ell = 1, 2, 3 \) and the one-loop perturbative result \( \beta^{(1)}_{s} (a) = -a \). This figure clearly shows the perturbative limit at small \( a \), as well as the universal asymptotic behavior of the β function in hand: \( \beta^{(\ell)}_{an} (a) \to -1 \), when \( a \to \infty \). The latter statement can also be proved in an independent way. Indeed, due to the infrared enhancement of the analytic running coupling, the value of the right-hand side of Eq. (36) (it is nothing but the β function at the relevant loop level) when \( \mu^2 \to 0 \) corresponds to the limit \( \tilde{\alpha}^{(\ell)}_{an} (\mu^2) \to \infty \). One can show that irrespective of the loop level

\[
\lim_{\mu^2 \to 0} \left\{ \tilde{\alpha}^{(\ell)}_{an} (\mu^2) \right\} = 1, \quad \lim_{\mu^2 \to 0} \left\{ \left[ \tilde{\alpha}^{(\ell)}_{s} (\mu^2) \right]^{j+1} \right\} = 0, \tag{46}
\]

where \( j \) is a natural number (\( j = 1, 2, 3, \ldots \)). Therefore, we infer that at any loop level the β function (45) corresponding to the analytic invariant charge (41) tends
Fig. 6. The $\beta$ function corresponding to the analytic invariant charge (41) at different loop levels (solid curves). The one-loop perturbative result is shown as the dot-dashed curve.

to the universal limit

$$\lim_{a \to \infty} \beta^{(\ell)}_{an}(a) = -1.$$ (47)

As it has been mentioned above, such a behavior of the $\beta$ function leads to the IR enhancement of the running coupling, namely $\tilde{\alpha}(q^2) \simeq \Lambda^2/q^2$, when $q^2 \to 0$.

Thus, the $\beta$ function corresponding to the analytic invariant charge (41) has the universal asymptotic behaviors at both small ($\beta^{(\ell)}_{an}(a) \simeq -a$, when $a \to 0$) and large ($\beta^{(\ell)}_{an}(a) \simeq -1$, when $a \to \infty$) values of the running coupling irrespective of the loop level. Therefore, the analytic invariant charge itself possesses the universal asymptotics both in the ultraviolet ($\tilde{\alpha}^{(\ell)}_{an}(q^2) \simeq 1/\ln(q^2/\Lambda^2)$, when $q^2 \to \infty$) and infrared ($\tilde{\alpha}^{(\ell)}_{an}(q^2) \simeq \Lambda^2/q^2$, when $q^2 \to 0$) regions at any loop level. In particular, this implies that using the AIC (41) at the higher loop levels will also result in the confining quark–antiquark potential (see Subsection 4.1).

3.3. Extension to the timelike region

In the previous sections the model for the QCD analytic invariant charge has been studied in the spacelike (Euclidean) region. However, for congruous description of a number of strong interaction processes (for example, inclusive $\tau$ lepton decay or electron-positron annihilation into hadrons) one has to use the continuation of the invariant charge to the timelike (Minkowskian) region. In general, one might expect such a continuation to affect the properties of the strong running coupling. Therefore, it is of a certain importance to extend the analytic invariant charge to the timelike domain, and to compare the functions obtained.

The consistent description of hadron dynamics in spacelike and timelike domains remains the goal of many studies for quite a long time. Thus, the first attempts to build up the expression for the strong running coupling valid for the timelike momenta transferred were made in the early 1980s (see Refs. 77, 79). Recently, in
Ref. 39 the procedure of continuation of the invariant charge from the spacelike to the timelike region (and vice versa) was elaborated by making use of the dispersion relation for the Adler $D$ function. In the framework of this method the integral relationships between the QCD running coupling in Euclidean and Minkowskian domains were established. In particular, the continuation of the strong running coupling to the timelike region is connected with the “spacelike” invariant charge through the relation (see paper 39 and references therein)

$$\hat{\alpha}(s) = \frac{1}{2\pi i} \int_{s+i\varepsilon}^{s-i\varepsilon} \alpha(-\zeta) \frac{d\zeta}{\zeta}, \quad s = -q^2 > 0. \quad (48)$$

In this equation the integration contour goes from the point $s + i\varepsilon$ to the point $s - i\varepsilon$ and lies in the region of analyticity of the function $\alpha(-\zeta)$. Here and further the running coupling in the spacelike region is denoted by $\alpha(q^2)$, and in the timelike region by $\hat{\alpha}(s)$. It is worth mentioning also that the relation inverse to (48) is

$$\alpha(q^2) = q^2 \int_0^\infty \frac{\hat{\alpha}(s)}{(s+q^2)^2} ds. \quad (49)$$

It is convenient to choose the integration contour in Eq. (48) in the form presented in Figure 7. Namely, the integration path starts from the point $s + i\varepsilon$, proceeds in a parallel way with the real axis to infinity, then along the circle of infinitely large radius it goes counter-clockwise to the point $(\infty - i\varepsilon)$, and then it goes in a parallel way with the real axis to the point $s - i\varepsilon$. Since the QCD running coupling possesses the property of asymptotic freedom (namely, $\hat{\alpha}(q^2) \simeq 1/\ln(q^2/\Lambda^2)$, when $q^2 \to \infty$), the integral along the circle of an infinitely large radius gives vanishing
Fig. 8. The one-loop analytic running coupling in the spacelike ($q^2 > 0$) and timelike ($s = -q^2 > 0$) regions, $z = q^2/\Lambda^2$.

contribution to the right-hand side of Eq. (48). Then, the remaining expression can be reduced to

$$\tilde{\alpha}(s) = \int_{s}^{\infty} \varphi(\zeta) \frac{d\zeta}{\zeta},$$

(50)

where $\varphi(\zeta)$ denotes the spectral density of the Källén–Lehmann representation for the running coupling $\alpha(q^2)$:

$$\varphi(\zeta) = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left[ \alpha(-\zeta - i\epsilon) - \alpha(-\zeta + i\epsilon) \right].$$

(51)

Thus, the continuation of the analytic invariant charge to the timelike region is given by

$$\tilde{\alpha}_{an}^{(l)}(s) = \frac{4\pi}{\beta_0} \int_{s/\Lambda^2}^{\infty} \rho^{(l)}(\zeta) \frac{d\zeta}{\zeta}$$

(52)

with the spectral density $\rho^{(l)}(\zeta)$ denoted in Eqs. (48) and (49) (see Ref. 53 for details).

The one-loop running coupling has the following asymptotic in the ultraviolet limit:

$$\tilde{\alpha}_{an}^{(1)}(s) \approx \frac{4\pi}{\beta_0} \frac{1}{\ln z} \left\{ 1 - \frac{\pi^2}{3} \frac{1}{(\ln z)^2} + O\left[ \frac{1}{(\ln z)^4}, \frac{1}{z} \right] \right\}, \quad s \to \infty,$$

(53)

where $z = s/\Lambda^2$. On the one hand, this running coupling has correct UV behavior determined by the asymptotic freedom. On the other hand, the so-called $\pi^2$-terms have also appeared in expansion. These terms play a key role in the description of the strong interaction processes in the timelike domain (see discussion of this issue
Fig. 9. The relative difference \( \Delta(z) = \left[ \frac{\alpha_{\text{an}}^{(1)}(q^2)}{\hat{\alpha}_{\text{an}}^{(1)}(-q^2)} - 1 \right] \times 100\% \) between the values of the one-loop analytic running coupling in the spacelike and timelike regions, \( z = q^2/\Lambda^2 \).

in Refs. [38, 39, 80, and 81]. It is worthwhile to mention also that, similarly to the “spacelike” case, there is the IR enhancement of the one-loop running coupling (52):

\[
\hat{\alpha}_{\text{an}}^{(1)}(s) \simeq \frac{4\pi}{\beta_0} \frac{1}{z (\ln z)^2}, \quad s \to 0.
\] (54)

However, here the type of the infrared singularity differs from that of the invariant charge (11). Nevertheless, it is this behavior of the running coupling (54) that enables one to handle the integrals over the infrared domain of the form (see, e.g., Ref. [82])

\[
\bar{A}(s) = \frac{1}{s} \int_0^s \hat{\alpha}_{\text{an}}(s') \, ds'.
\] (55)

It is interesting to note also that within the current approach a simple relation holds between the \( \beta \) function corresponding to the running coupling in the timelike region (54) and the relevant spectral density:

\[
\beta(\hat{\alpha}(s)) = \frac{d}{d \ln s} \left[ \hat{\alpha}_{\text{an}}^{(1)}(s) \right] = -\frac{4\pi}{\beta_0} \rho^{(t)}(s).
\] (56)

Thus, the \( \beta \) function in hand proves to be proportional to the spectral density \( \rho^{(t)}(s) \). Obviously, this result justifies the attempts, originated by Schwinger [83], to find a direct physical interpretation of the \( \beta \) function (the so-called Schwinger’s hypothesis, see also Refs. [39 and 84]).

The plots of the functions \( \alpha_{\text{an}}^{(1)}(q^2) \) and \( \hat{\alpha}_{\text{an}}^{(1)}(s) \) are shown in Figure 8. In the ultraviolet limit these expressions have identical behavior determined by the asymptotic freedom. However, there is asymmetry between them in the intermediate- and low-energy regions. The ratio of the one-loop analytic running coupling in the spacelike region \( \alpha_{\text{an}}^{(1)}(q^2) \) to its continuation to the timelike region \( \hat{\alpha}_{\text{an}}^{(1)}(-q^2) \) is presented
in Figure 9. The relative difference between these functions is about several percent at the scale of the Z boson mass, and increases when approaching the infrared domain. Apparently, this circumstance must be taken into account when one handles the experimental data. In particular, this will play a crucial role in description of the inclusive \( \tau \) lepton decay and the \( e^+e^- \) annihilation into hadrons in the framework of the approach developed (see Subsections 4.3 and 4.4 respectively).

4. Phenomenological Applications

As has been mentioned above, there are several ways of incorporating the analyticity requirement into the renormalization group method. Eventually, this leads to different models for the running coupling in the framework of the analytic approach to QCD. Certainly, every such model, as well as any new model for the strong interaction, must be verified thoroughly. A decisive test of a model’s self-consistency is its applicability to description of diverse QCD processes. Since we are working within a nonperturbative approach, the study of intrinsically nonperturbative phenomena is of a primary importance here. Furthermore, the elaborated model for the analytic invariant charge contains no free parameters, i.e., similarly to perturbative approach, \( \Lambda_{\text{QCD}} \) remains the basic characterizing parameter of the theory. Hence, the congruity of the estimations of this parameter, obtained for different strong interaction processes, would imply the quantitative consistency of the developed model (see also papers 47, 48, 85 and references therein).

4.1. Static quark–antiquark potential

Let us proceed to construction of the static quark–antiquark potential in the framework of the one-gluon exchange model. This potential is related to the strong running coupling \( \alpha(q^2) \) by the three-dimensional Fourier transformation

\[
V(r) = -\frac{16\pi}{3} \int_0^\infty \frac{\alpha(q^2) \exp(iqr)}{q^2} \frac{dq}{(2\pi)^3}
\]

(57)

(see, e.g., reviews 8, 11 and references therein for details). In fact, the lowest–lying bound states of heavy quark systems may well be described by the potential (57) even with the perturbative expression for the QCD running coupling. However, at large distances, which play the crucial role in the hadron spectroscopy, the perturbative approach blows up due to unphysical singularities (such as the Landau pole) of the strong running coupling.

For the construction of the quark–antiquark \( (Q\overline{Q}) \) potential we shall use here the analytic invariant charge (11) (see also Refs. 87, 45, 88, and 89). After integration over the angular variables, Eq. (57) acquires the form

\[
V(r) = -\frac{32}{3\beta_0} \int_0^\infty \frac{p^2 - 1}{p^4} \ln \frac{p R}{p R} dp,
\]

(58)
where \( p = qr_0, \ R = r/r_0, \) and \( r_0 \) is a reference scale of the dimension of length, which will be specified below. This equation is rather complicated to be integrated explicitly. Nevertheless, one is able to study the asymptotics of the quark–antiquark potential \((58)\) by making use of the following prescription. First of all, let us introduce a dimensionless variable \( Q = pR \) and rewrite Eq. \((58)\) as follows:

\[
V(r) = \frac{16}{3\beta_0} \frac{1}{r_0} \frac{1}{R} \int_0^\infty \left( 1 - \frac{R^2}{Q^2} \right) \frac{\sin Q}{Q} \ln R - \ln Q \, dQ. \tag{59}
\]

Then, formally expanding the denominator of the integrand,\(^p\) one can find the asymptotic behavior of \( V(r) \) at both small and large distances:

\[
V(r) \simeq \frac{16}{3\beta_0} \frac{1}{r_0} \frac{1}{R} \left[ \frac{1}{R} \sum_{n=0}^{n_0} \frac{1}{(\ln R)^{n+1}} \int_0^\infty \frac{\sin Q}{Q} (\ln Q)^n \, dQ \right] \tag{60}
\]

The values of \( n_0 \) and \( m_0 \) will be specified below.

For evaluation of the expansion coefficients in Eq. \((60)\) it is worth considering an auxiliary integral of the form

\[
\int_0^\infty Q^t \sin Q \, dQ = \sqrt{\pi} \frac{2^t \Gamma \left( 1 + \frac{t}{2} \right)}{\Gamma \left( \frac{t}{2} - \frac{1}{2} \right)}. \tag{61}
\]

Differentiating this equation \( n \) times with respect to variable \( t \) we obtain

\[
\int_0^\infty Q^t (\ln Q)^n \sin Q \, dQ = v(n, t), \tag{62}
\]

where

\[
v(n, t) = \sqrt{\pi} \frac{d^n}{dt^n} \left[ 2^t \frac{\Gamma \left( 1 + \frac{t}{2} \right)}{\Gamma \left( \frac{t}{2} - \frac{1}{2} \right)} \right]. \tag{63}
\]

In Eq. \((61)\) the parameter \( t \) takes the values \( 0 \leq |\text{Re} \ (t + 1)| < 1 \) (see, e.g., Ref. \(90\)). However, in order to determine the expansion coefficients of the second sum in Eq. \((60)\), one has to go to the point \( t = -3 \), which is outside of this range. Nevertheless, it can be done by making use of the analytic continuation of the left-hand side of Eq. \((61)\). This continuation is unique and it is defined obviously by the right-hand side of Eq. \((61)\) all over the complex \( t \)–plane except for the points

\(^pA\) similar method has also been used in Ref. \(6\)
\( t = -2N \), with \( N \) being a natural number. Fortunately, we are not dealing with these values of the parameter \( t \), and we can put

\[
\int_{0}^{\infty} \frac{\sin Q}{Q^s} (\ln Q)^n dQ = v(n, t) \bigg|_{t = -s}, \quad s \neq 2N \quad (N = 1, 2, 3, \ldots). \tag{64}
\]

It should be noted here that this analytical continuation plays the role of regularization of the expansion coefficients in Eq. (60). It is also convenient to introduce the notations

\[
u_n = \frac{2}{\pi} v(n, t) \bigg|_{t = -1}, \quad \omega_m = -\frac{2}{\pi} v(m, t) \bigg|_{t = -3}. \tag{65}
\]

The explicit expressions for these coefficients can be found in Appendix D.

Thus, the static quark–antiquark potential (60) can be represented now in the following form:

\[
V(r) \simeq 8\pi \frac{1}{3\beta_0} \frac{1}{r_0} \left[ \frac{1}{R} \sum_{n=0}^{m_0} \frac{u_n}{(\ln R)^{n+1}} + R \sum_{m=0}^{m_0} \frac{\omega_m}{(\ln R)^{m+1}} \right], \quad R = \frac{r}{r_0}. \tag{66}
\]

At small distances the potential (66) possesses the standard behavior, determined by the asymptotic freedom

\[
V(r) = 8\pi \frac{1}{3\beta_0} \frac{1}{r_0} \frac{1}{R \ln R}, \quad r \to 0. \tag{67}
\]

At the same time, it proves to be rising at large distances

\[
V(r) = 8\pi \frac{1}{3\beta_0} \frac{1}{r_0} \frac{R}{2 \ln R}, \quad r \to \infty, \tag{68}
\]

implying the confinement of quarks. It is of a particular interest to mention here that a similar rising behavior of the \( Q\bar{Q} \) potential has been proposed [3] a long time ago proceeding from the phenomenological assumptions.

As it has been shown in Subsection 3.2, the analytic invariant charge \( (41) \) has the universal asymptotics both in the ultraviolet and infrared regions at any loop level. Furthermore, the approach in hand possesses a good scheme stability. Therefore, neither higher loop corrections, nor scheme dependence, can affect qualitatively the results obtained. The use of different subtraction schemes (or different loop approximations) will result in redefinition of the scale parameter, as it also takes place within the perturbative approach.

Equation (66) describes the behavior of the static quark–antiquark potential (58) at small and large distances. However, its straightforward extrapolation to all distances encounters poles of different orders at the point \( R = 1 \), which apparently is an artifact of the expansion (60). For the practical purposes it would be undoubtedly useful to derive an explicit expression for the \( Q\bar{Q} \) potential applicable for \( 0 < r < \infty \).
Fig. 10. The quark–antiquark potential (69) in dimensionless units at different levels of approximation: \( n_0 = m_0 = 0 \) (dashed curve), \( n_0 = m_0 = 4 \) (solid curve), \( n_0 = m_0 = 5 \) (▲), and \( n_0 = m_0 = 10 \) (■). The one-loop perturbative result is shown by dotted curve.

In order to construct an explicit interpolating formula for the quark–antiquark potential (58) we shall employ here the following method. Let us modify the expansion (66) in a “minimal” way, by adding terms which only subtract the singularities at the point \( R = 1 \) and do not contribute to the derived asymptotics. This leads to the following expression for the static quark–antiquark potential

\[
V(r) = \frac{8\pi}{3\beta_0 r_0} \left\{ \sum_{n=0}^{n_0} u_n \left[ \frac{1}{R (\ln R)^{n+1}} \right]_{\text{Reg}} + \sum_{m=0}^{m_0} \omega_m \left[ \frac{R}{(\ln R)^{m+1}} \right]_{\text{Reg}} \right\},
\]

where \( R = r/r_0 \). The functions \( [1/(R (\ln R)^n)]_{\text{Reg}} \) and \( [R/(\ln R)^m]_{\text{Reg}} \) are presented in an explicit form in Appendix D.

The numerical analysis of Eq. (69) revealed that for practical purposes it is enough to retain the first five expansion terms \( (n_0 = m_0 = 4) \) therein. It turns out that the potential (69) itself and the corresponding estimation of the value of parameter \( \Lambda_{QCD} \) are not affected by higher–order contributions. In particular, the curves (69) for \( n_0 = m_0 = 5 \) and for \( n_0 = m_0 = 10 \) are practically indistinguishable of the curve corresponding to \( n_0 = m_0 = 4 \) over the whole region \( 0 < r < \infty \) (see Figure 10). And the higher–order estimations of the parameter \( \Lambda_{QCD} \) vary within 0.5 % of the value obtained at the \( n_0 = m_0 = 4 \).

Thus, we arrive at the following explicit expression for the static quark–antiquark potential

\[
V(r) = V_0 + \frac{8\pi}{3\beta_0 r_0} \left\{ \frac{1}{R} \left[ \frac{1}{\ln R} - \frac{0.577}{(\ln R)^2} + \frac{1.156}{(\ln R)^3} - \frac{4.021}{(\ln R)^4} + \frac{15.018}{(\ln R)^5} \right] \right\}.
\]
Fig. 11. Comparison of the quark–antiquark potential $V(r)$ defined by Eq. (70) (solid curve) with the quenched lattice simulation data (•). The values of the parameters are: $\Lambda = 670$ MeV, $n_f = 0$, $V_0 = -3.164$ GeV. The dashed curve corresponds to the relevant one-loop perturbative result.

\[
+ R \left[ \frac{0.500}{\ln R} + \frac{0.461}{(\ln R)^2} + \frac{1.462}{(\ln R)^3} + \frac{3.185}{(\ln R)^4} + \frac{17.844}{(\ln R)^5} \right] \\
+ \frac{21.489}{1 - R} \left[ 1 - \frac{62.484}{(1 - R)^2} + \frac{98.694}{(1 - R)^3} - \frac{84.143}{(1 - R)^4} + \frac{32.861}{(1 - R)^5} \right],
\]

where $R = r/r_0$. In order to reproduce the correct short-distance behavior of the Q\bar{Q} potential, the dimensional parameter $r_0$ in this equation has to be identified with $\Lambda_{\overline{MS}}$ by the relation $r_0^{-1} = \Lambda \exp(\gamma)$, where $\gamma \approx 0.57721...$ denotes the Euler’s constant (see, e.g., Refs. 57, 9 and 12). It is straightforward to verify that the potential (70) satisfies also the concavity condition

\[
\frac{dV(r)}{dr} > 0, \quad \frac{d^2V(r)}{dr^2} \leq 0,
\]

which is a general property of the gauge theories (see Ref. 91 for details).

Comparison of the quark–antiquark potential (70) with the quenched lattice simulation data (27) has been performed with the use of the least square method (87). The varied parameters in Eq. (70) were $r_0$ and the additive self–energy constant $V_0$. The result of this fit is presented in Figure 11. This figure shows that in the nonperturbative physically-relevant range $0.3 \text{ fm} \lesssim r \lesssim 1.2 \text{ fm}$, in which the average quark separations $\sqrt{\langle r^2 \rangle}$ for quarkonia sits (26) the Q\bar{Q} potential (70) reproduces the lattice data (27) fairly well. At the same time, in the region $r \lesssim 0.05 \text{ fm}$ the derived potential coincides with the perturbative result (dashed curve in Fig. 11). The difference between the lattice data and the expression (70) in the intermediate range $0.05 \text{ fm} \lesssim r \lesssim 0.3 \text{ fm}$ can be explained by the presence of additional nonperturbative contributions at small distances (see, e.g., Ref. 92).
Thus, the estimation of parameter $\Lambda_{QCD}$ in the course of this comparison gives $\Lambda^{(n_f=0)} = (670 \pm 8)$ MeV (this value corresponds to the one-loop level with $n_f = 0$ active quarks). It is worth noting that the fit with the use of the maximum likelihood method also gives a similar value $\Lambda^{(n_f=0)} \approx 650$ MeV. Further, in order to collate the obtained result with forthcoming estimations, one has to continue it to the region of three active quarks. It was performed by employing the matching procedure, and gives the value $\Lambda^{(n_f=3)} = (590 \pm 10)$ MeV.

### 4.2. Gluon condensate

Let us proceed to another nonperturbative aspect of the strong interaction, namely, to the gluon condensate. The latter is determined as the matrix element

$$K = \lim_{x \to y} \left\langle \text{vac} \left| \frac{\alpha}{\pi} : G^{\alpha}_{\mu\nu}(x)G^{\alpha}_{\mu\nu}(y) : \right| \text{vac} \right\rangle, \quad (72)$$

the averaging being performed over the “true physical” vacuum state $|\text{vac}\rangle$ (see Ref. 93 for a recent review of this issue). In the framework of perturbative approach the quantity (72) identically equals to zero. However, lattice simulations and comparison of the QCD sum rules with the experimental data on the low-energy hadron dynamics, testify to the nonzero value of gluon condensate. The quantity (72) is the subject of thorough studies for a long time (see, e.g., reviews 93, 96 and Refs. 52, 53). In particular, the integral relation between the gluon condensate (72) and the nonperturbative part of the strong running coupling was established in Refs. 15, 53 and 96:

$$K = \frac{3}{\pi^3} \int_0^\infty \alpha_{np}(k^2)k^2 \, dk^2, \quad (73)$$

where $\alpha_{np}(k^2) = \alpha(k^2) - \alpha_s(k^2)$. Here $\alpha(k^2)$ is the QCD invariant charge, and $\alpha_s(k^2)$ denotes the perturbative running coupling.

Thus, in the framework of the current approach, the gluon condensate (73) at the one-loop level takes the form (see Eq. (11)):

$$K = -\frac{12}{3\beta_0\pi^2} \Lambda^4 \int_0^\infty \frac{dz}{\ln z}, \quad (74)$$

where $z = k^2/\Lambda^2$. However, the integral on the right-hand side of this equation diverges. In order to regularize it we employ here the method proposed in Ref. 96. Namely, let us introduce the cutoff parameter $k_0$, which has the physical meaning of the characteristic scale of the nonperturbative effects. The phenomenological

---

4. It is interesting to note here that similar values of the parameter $\Lambda_{QCD}$ have also been obtained within different approaches to this issue (see Refs. 5, 7).
estimation of this parameter gives \( k_0 = (700 \pm 100) \text{ MeV} \). Then, assuming the integration in Eq. (74) as the principle value one, we obtain
\[
K = -\frac{12}{\beta_0 \pi^2} \Lambda^4 \text{Li} \left( \frac{k_0^2}{\Lambda^2} \right),
\]
where \( \text{Li}(x) \) denotes the logarithm–integral:
\[
\text{Li}(x) = \int_0^x \frac{dt}{\ln t}, \quad \text{Re } x > 1.
\]

The solution to Eq. (75) with the phenomenological value of the gluon condensate \( K = (0.36 \pm 0.02 \text{ GeV})^4 \) gives \( \Lambda = (631 \pm 79) \text{ MeV} \) (this estimation corresponds to the one-loop level with \( n_f = 3 \) active quarks). This value of the parameter \( \Lambda_{\text{QCD}} \) agrees fairly well with its estimation obtained in Subsection 4.1.

4.3. **Inclusive \( \tau \) lepton decay**

In the previous subsections the analytic invariant charge has been applied to description of some intrinsically nonperturbative issues of QCD, the congruous estimation of the parameter \( \Lambda_{\text{QCD}} \) being obtained. For further verification of the self–consistency of the model proposed, it is worth proceeding to the hadron processes which are commonly described within the perturbative approach. Investigation of such processes at relatively small energies is of a primary interest here. Among them, the inclusive \( \tau \) lepton decay is a most sensitive to the infrared behavior of the strong running coupling. Thus, we turn now to this hadron process, restricting ourselves at this stage to the one-loop level (see Ref. [46] for details).

The experimentally measurable quantity here is the inclusive semileptonic branching ratio
\[
R_\tau = \frac{\Gamma(\tau^\rightarrow \rightarrow \text{hadrons}^\rightarrow \rightarrow \nu_\tau)}{\Gamma(\tau^\rightarrow \rightarrow e^\rightarrow \nu_e \bar{\nu}_\tau)}.
\]

One can split this ratio into three parts, namely \( R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S} \). The terms \( R_{\tau,V} \) and \( R_{\tau,A} \) account the contributions to Eq. (77) of the decay modes with the light quarks only, and they correspond to the vector (V) and axial–vector (A) quark currents, respectively. The accuracy of the experimental measurement of these terms is several times higher than the accuracy of the strange width ratio \( R_{\tau,S} \), which accounts the s quark contribution to Eq. (77). Thus, let us proceed with the nonstrange part of the \( R_\tau \) ratio associated with the vector quark currents (see Refs. [98, 99, 97] and [10] for detailed description of this issue):
\[
R_{\tau,V} = \frac{N_c}{2} |V_{ud}|^2 S_{\text{BW}} (1 + \delta_{\text{QCD}}).
\]

In this equation \( N_c = 3 \) is the number of quark colors, \( |V_{ud}| = 0.9734 \pm 0.0008 \) denotes the Cabibbo–Kobayashi–Maskawa matrix element, and \( S_{\text{BW}} = 1.0194 \pm 0.0040 \)
is the electroweak factor, \( \delta_{\text{QCD}} \) is the strong correction. A recent experimental measurement of the ratio \( R_{\tau, V} \) by ALEPH Collaboration gave \( R_{\tau, V} = 1.775 \pm 0.017 \).

In the framework of the approach in hand the one-loop QCD correction is determined by the integral:

\[
\delta_{\text{QCD}} = \frac{2}{\pi} \int \frac{ds}{M_{\tau}^2} \left( 1 - \frac{s}{M_{\tau}^2} \right)^2 \left( 1 + 2 \frac{s}{M_{\tau}^2} \right) \hat{\alpha}_{\text{an}}^{(1)}(s),
\]

(79)

where \( \hat{\alpha}_{\text{an}}^{(1)}(s) \) is the analytic running coupling in the timelike region \( \alpha_1 \), \( M_{\tau} = (1776.99^{+0.26}_{-0.25}) \) MeV denotes the \( \tau \) lepton mass, and \( m = (4.0 \pm 1.5) \) MeV stands for the light quark mass \( \mu_\text{light} = (4.0 \pm 1.5) \) MeV at a normalization scale of 2 GeV. It is worth noting here that there is no need to involve the contour integration in Eq. (79), since analytic running coupling \( \alpha_1 \) contains no unphysical singularities in the region \( s > 0 \). In other words, the integration in Eq. (79) can be performed in a straightforward way. Introducing the notations \( x = s/M_{\tau}^2 \), \( x_0 = 4m^2/M_{\tau}^2 \), and \( c_0 = x_0(2 - 2x_0 + x_0^2) \), one can rewrite Eq. (79) in a more convenient form:

\[
\delta_{\text{QCD}} = \frac{1}{\pi} \left[ \hat{\alpha}_{\text{an}}^{(1)}(M_{\tau}^2) - c_0 \hat{\alpha}_{\text{an}}^{(1)}(4m^2) \right] + \frac{4}{\beta_0} \int_{x_0}^{1} (2 - 2x^2 + x^3) \rho^{(1)} \left( x \frac{M_{\tau}^2}{\Lambda^2} \right) dx,
\]

(80)

where the spectral density \( \rho^{(1)}(\sigma) \) is defined in Eq. (13).

For the value \( R_{\tau, V} \) given above \( \alpha_1 \), the estimation \( \Lambda_{\text{QCD}}(n_f = 2) = (561 \pm 69) \) MeV has been obtained for two active quarks. The uncertainty here is due to the errors in the values of \( R_{\tau, V}, |V_{ud}|, S_{\text{EW}}, m, \) and \( M_{\tau} \). However, in order to collate this result with the previous estimations of the parameter \( \Lambda_{\text{QCD}} \), one has to continue it to the region of three active quarks (the \( s \) quark mass \( m_s = (117.5 \pm 37.5) \) MeV has been employed here). This gives the value \( \Lambda_{\text{QCD}}(n_f = 3) = (524 \pm 68) \) MeV, which perfectly agrees with all the estimations of \( \Lambda_{\text{QCD}} \) in the framework of the approach in hand.

### 4.4. \( e^+e^- \) annihilation into hadrons

It is worthwhile to consider one more strong interaction process, namely, the electron–positron annihilation into hadrons. The experimentally measurable quantity here is the ratio of two cross–sections

\[
R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}.
\]

(81)

Unlike the inclusive semileptonic branching ratio \( \gamma \), this equation contains the explicit dependence on the center–of–mass energy of the process. In the framework of the approach in hand, at the one-loop level the ratio \( R(s) \) can be represented in
the following form (see, e.g., Refs. 103 and 41):

\[ R(s) = N_c \sum_{f=1}^{n_f} Q_f^2 \left[ 1 + \frac{1}{\pi} \tilde{a}^{(1)}_{\text{an}}(s) \right]. \]  

In this equation \( N_c = 3 \) is the number of colors, \( Q_f \) denotes the charge of the \( f \)-th quark, and \( \tilde{a}^{(1)}_{\text{an}}(s) \) is the analytic running coupling in the timelike region [42].

Recent experimental measurement of the \( R(s) \) by CLEO Collaboration at \( \sqrt{s_0} = 10.52 \) GeV gave \( R(s_0) = 3.56 \pm 0.01 \) (stat.). With this value the estimation of the parameter \( \Lambda_{\text{QCD}} \) for \( n_f = 4 \) active quarks is \( \Lambda^{(n_f=4)} = (393 \pm 65) \) MeV. It is worth noting that this estimation turns out to be rather rough due to a considerable systematic errors of the experimental data at small energies. Nevertheless, the continuation of the obtained result to the three active quarks region gives \( \Lambda^{(n_f=3)} = (490 \pm 76) \) MeV, that agrees fairly well with all the previous estimations of the parameter \( \Lambda_{\text{QCD}} \).

Thus, the applications of the model developed to description of diverse strong interaction processes ultimately lead to congruous estimation of the parameter \( \Lambda_{\text{QCD}} \). Namely, at the one-loop level with \( n_f = 3 \) active quarks \( \Lambda = (557 \pm 36) \) MeV. Apparently, this testifies that the analytic invariant charge substantially incorporates, in a consistent way, both perturbative and intrinsically nonperturbative aspects of Quantum Chromodynamics.

5. Conclusions

Let us summarize the basic properties of the model for the QCD analytic invariant charge, considered in this review, and the main results obtained in this field.

The proposed way of involving the analyticity requirement into the renormalization group method results in qualitatively new features of the strong running coupling. Namely, analytic invariant charge has no unphysical singularities, and it incorporates the ultraviolet asymptotic freedom with infrared enhancement in a single expression. Furthermore, additional parameters are not introduced into the theory. The consistency of the model developed with the general definition of the QCD invariant charge is proved.

The detailed investigation of the one-loop analytic running coupling and the relevant \( \beta \) function is performed. The latter is proved to coincide with its perturbative analog at small values of invariant charge. At the same time, this \( \beta \) function possesses the asymptotic behavior corresponding to the infrared enhancement of the strong running coupling. The renormalization invariance of the analytic running coupling is shown explicitly.

The analytic invariant charge is derived and studied at the higher loop levels also. It is proved that the higher loop corrections do not affect the AIC qualitatively. In particular, the analytic running coupling has the universal behavior both in ultraviolet and infrared regions at any loop level, and it possesses good higher loop and scheme stability.
The analytic invariant charge is extended to the timelike region. It is shown to have asymmetrical behavior in the intermediate- and low-energy domains of the spacelike and timelike regions. This difference turns out to be rather substantial and should be taken into account when one handles the experimental data. The obtained result confirms hypothesis due to Schwinger concerning the relation between the $\beta$ function and the relevant spectral density.

The developed model is applied to description of diverse strong interaction processes both of perturbative and intrinsically nonperturbative nature. The derived static quark–antiquark potential is proved to be confining at large distances. At the same time, it has the standard perturbative behavior at small distances. The analytic invariant charge reproduces explicitly the conformal inversion symmetry related to the size distribution of instantons. The developed model is also applied to description of gluon condensate, inclusive $\tau$ lepton decay and electron–positron annihilation into hadrons. The congruity of the estimated values of the parameter $\Lambda_{QCD}$ testifies to the self-consistency of the approach developed.

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**Appendix A. Perturbative QCD Running Coupling**

The QCD invariant charge $g^2(\mu^2)$ is the solution to the renormalization group equation

$$\frac{d \ln [g^2(\mu^2)]}{d \ln \mu^2} = \beta \left(g(\mu^2)\right).$$

(A.1)

In the framework of perturbative approach, assuming the running coupling being sufficiently small, one can approximate the $\beta$ function on the right-hand side of this equation by the power series

$$\beta \left(g(\mu^2)\right) = -\left\{\beta_0 \left[\frac{g^2(\mu^2)}{16\pi^2}\right] + \beta_1 \left[\frac{g^2(\mu^2)}{16\pi^2}\right]^2 + \beta_2 \left[\frac{g^2(\mu^2)}{16\pi^2}\right]^3 + \cdots\right\},$$

(A.2)
where for the SU(3) gauge group
\[ \beta_0 = 11 - 2n_f/3, \quad \beta_1 = 102 - 38n_f/3, \quad \beta_2 = 2857/2 - 5033n_f/18 + 32n_f^2/54, \]
and \( n_f \) denotes the number of active quarks. The coefficients \( \beta_0 \) and \( \beta_1 \) are scheme independent, while the value given for the three-loop coefficient \( \beta_2 \) is calculated in the \( \overline{\text{MS}} \) scheme. Let us rewrite RG equation (A.1) at the \( \ell \)-loop level as follows:

\[
\frac{d \ln \left[ \tilde{\alpha}_s^{(\ell)}(\mu^2) \right]}{d \ln \mu^2} = - \sum_{j=0}^{\ell-1} B_j \left[ \tilde{\alpha}_s^{(j)}(\mu^2) \right]^{j+1}, \quad B_j = \frac{\beta_j}{\beta_0^{j+1}}, \tag{A.3}
\]

where \( \alpha_s(\mu^2) = g^2(\mu^2)/(4\pi) \) and \( \tilde{\alpha}(\mu^2) = \alpha(\mu^2)\beta_0/(4\pi) \).

At the one-loop level Eq. (A.3) reads as

\[
\frac{d \ln \left[ \tilde{\alpha}_s^{(1)}(\mu^2) \right]}{d \ln \mu^2} = -\tilde{\alpha}_s^{(1)}(\mu^2). \tag{A.4}
\]

Integrating this equation in finite terms, one gets

\[
\frac{1}{\tilde{\alpha}_s^{(1)}(q^2)} - \frac{1}{\tilde{\alpha}_s^{(1)}(q_0^2)} = \ln \left( \frac{q^2}{q_0^2} \right). \tag{A.5}
\]

It is convenient to introduce here the parameter \( \Lambda \) of the dimension of mass, which absorbs all the dependence on the normalization point \( q_0^2 \):

\[
\Lambda_{(1)}^2 = q_0^2 \exp \left[ -\frac{4\pi}{\beta_0} \frac{1}{\tilde{\alpha}_s^{(1)}(q_0^2)} \right]. \tag{A.6}
\]

In this case, the solution to the RG equation (A.3) takes the well-known form\(^1\)

\[
\alpha_s^{(1)}(q^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln z}, \quad z = \frac{q^2}{\Lambda^2}. \tag{A.7}
\]

Let us note here that this expression for the one-loop perturbative running coupling has the unphysical singularity (the so-called Landau pole) at the point \( q^2 = \Lambda^2 \).

At the two-loop level Eq. (A.3) acquires the form

\[
\frac{d \ln \left[ \tilde{\alpha}_s^{(2)}(\mu^2) \right]}{d \ln \mu^2} = -\tilde{\alpha}_s^{(2)}(\mu^2) - B_1 \left[ \tilde{\alpha}_s^{(1)}(\mu^2) \right]^2. \tag{A.8}
\]

Integrating this equation likewise the previous case, one arrives at the transcendental relation,\(^2\) which determines the two-loop perturbative running coupling

\[
\frac{1}{\tilde{\alpha}_s^{(2)}(q^2)} - B_1 \ln \left[ 1 + \frac{1}{B_1 \tilde{\alpha}_s^{(2)}(q^2)} \right] = \ln \left( \frac{q^2}{\Lambda^2} \right). \tag{A.9}
\]

\(^1\)It is worth mentioning here that the relation (A.6) ensures the renormalization invariance of the solution to the RG equation (A.3).

\(^2\)Certainly, Eq. (A.8) has also solutions different from Eq. (A.9). Nevertheless, all the ambiguity here can be eliminated by redefinition of the parameter \( \Lambda \) on the right-hand side of Eq. (A.9).
The parameter $\Lambda$ in this formula differs from the one-loop one \(\text{(A.6)}\)

\[
\Lambda_2^{(2)} = \mu^2 \exp \left\{ -\frac{4\pi}{\beta_0} \frac{1}{\alpha_\text{s}^{(2)}(\mu^2)} + B_1 \ln \left[ 1 + \frac{1}{B_1} \frac{4\pi}{\beta_0} \frac{1}{\alpha_\text{s}^{(2)}(\mu^2)} \right] \right\}. \tag{A.10}
\]

The explicit solution to Eq. \(\text{(A.9)}\) can be written down in terms of the so-called Lambert $W$ function\(^{105}\) (see Appendix B):

\[
\alpha_\text{ex}^{(2)}(q^2) = -\frac{4\pi}{\beta_0} \frac{1}{B_1} \frac{1}{1 + W_k \{ -\exp \left\{ -(1 + B_1^{-1} \ln z) \right\} \}}. \tag{A.11}
\]

where $z = q^2/\Lambda^2$ and $B_1 = \beta_1/\beta_0$. Here the branch index of the Lambert $W$ function may take the values $k = 0$ or $k = -1$. But only the latter one satisfies the asymptotic freedom condition. It is worthwhile to note that two-loop perturbative invariant charge \(\text{(A.11)}\) also has unphysical singularities, namely, the pole at $q^2 = \Lambda^2$ and the cut $0 < q^2 \leq \Lambda^2$. The exact solution \(\text{(A.11)}\) is widely used along with the iterative solution to Eq. \(\text{(A.9)}\):

\[
\alpha_\text{it}^{(2)}(q^2) = \frac{4\pi}{\beta_0} \frac{1}{B_1} \frac{1}{\ln z + B_1 \ln (1 + B_1^{-1} \ln z)}. \tag{A.12}
\]

and its approximate form (see, e.g., Ref. 44):

\[
\alpha_\text{ap}^{(2)}(q^2) = \frac{4\pi}{\beta_0} \frac{1}{B_1} \ln z \left[ 1 - B_1 \frac{\ln (\ln z)}{\ln z} \right]. \tag{A.13}
\]

Since the account of the higher-order corrections leads to substantial technical complications, we present only the explicit expression for the three-loop running coupling here (see, e.g., Ref. 100):

\[
\alpha_3^{(3)}(q^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln z} \left\{ 1 - B_1 \frac{\ln (\ln z)}{\ln z} + \frac{B_2^2}{(\ln z)^2} \left[ (\ln (\ln z))^2 - \ln (\ln z) + \frac{B_2}{B_1^2} - 1 \right] \right\}. \tag{A.14}
\]

The calculation of the four-loop $\beta$ function coefficient $\beta_3$ in the $\text{MS}$ scheme has been performed in Ref. 106. For the relevant four-loop perturbative running coupling see, e.g., Ref. 107.

### Appendix B. The Lambert $W$ Function

For the representation of the analytic running coupling \(\text{(11)}\) in the renorminvariant form and for the investigation of the corresponding $\beta$ function \(\text{(22)}\) it proves to be convenient to use the so-called Lambert $W$ function. As long ago as the middle of 18th century this function is being employed in diverse physical problems\(^{108}\) In contemporary QCD studies the interest to this function arose just a several years ago. Namely, it was revealed\(^{105}\) that the explicit solution to the perturbative RG equation for the invariant charge at the two-loop level can be expressed in terms of the Lambert $W$ function (see also Appendix A).
Fig. 12. The function $xe^x$ (dot-dashed curve) and two real branches of the Lambert $W$ function (B.1): $W_0(x)$ (solid curve) and $W_{-1}(x)$ (dashed curve).

The Lambert $W$ function is defined as a many–valued function $W_k(x)$, which satisfies the equation

$$W_k(x) \exp[W_k(x)] = x,$$

(B.1)

where $k$ denotes the branch index of this function. Only two real branches of the Lambert $W$ function, the principal branch $W_0(x)$ and the branch $W_{-1}(x)$ (see Fig. 12), will be used in our consideration. The other branches of this function take imaginary values. One can show that for the branches $W_0(x)$ and $W_{-1}(x)$ the following expansions hold:

$$W_0(\varepsilon) = \varepsilon - \varepsilon^2 + O(\varepsilon^3), \quad \varepsilon \to 0,$$

(B.2)

$$W_{-1}(-\varepsilon) = \ln \varepsilon + O(\ln \ln \varepsilon), \quad \varepsilon \to 0^+,$$

(B.3)

$$W_0 \left( \frac{1}{e} + \varepsilon \right) = -1 + \sqrt{2\varepsilon e} + O(\varepsilon), \quad \varepsilon \to 0^+,$$

(B.4)

$$W_{-1} \left( \frac{1}{e} + \varepsilon \right) = -1 - \sqrt{2\varepsilon e} + O(\varepsilon), \quad \varepsilon \to 0^+,$$

(B.5)

where $e = 2.71828\ldots$ denotes the base of the natural logarithm. Details concerning the properties of the Lambert $W$ function can be found in Ref. [108].

In order to represent the analytic running coupling (11) in a renorm invariant form and to derive the corresponding $\beta$ function (22) one has to solve the equation (see Subsection 2.3 and Refs. [54, 56])

$$z - 1 = a, \quad z > 0, \quad a > 0$$

(B.6)

for the variable $z$. Let us multiply this equation through by the factor $a^{-1} \ln z$

$$\ln z = \frac{b}{z} - b, \quad z > 0,$$

(B.7)
where \( b = -1/a \). Obviously, the equation obtained has a trivial solution \( z = 1 \), which does not satisfy the initial Eq. (B.6) when \( a \neq 1 \), that is a consequence of the multiplication of the latter by \( \ln z \). Therefore, when solving Eq. (B.7) for the variable \( z \) one has to discard its trivial solution \( z = 1 \). Next, let us represent Eq. (B.7) in the form

\[
\frac{b}{z} \exp \left( \frac{b}{z} \right) = b e^b. \tag{B.8}
\]

Taking into account the definition (B.1), solution to Eq. (B.8) can be expressed in terms of the Lambert \( W \) function

\[
\frac{b}{z} = W_k (b e^b). \tag{B.9}
\]

The branch index \( k \) of the \( W \) function will be specified below.

In the physically relevant range \( a > 0 \) the argument of the Lambert \( W \) function in Eq. (B.9) takes the values \(-1/e \leq b e^b < 0\). The only two real branches, the principle branch \( W_0(x) \) and the branch \( W_{-1}(x) \) (see Fig. 12) correspond to this interval. Here one has to handle carefully with the interchange between these branches at the point \( x = -1/e \) (this corresponds to the value \( a = 1 \)). Thus, the nontrivial solution to Eq. (B.7) for the variable \( z \) is

\[
\frac{1}{z} = \begin{cases} 
  b^{-1} W_{-1}(b e^b), & -1 \leq b < 0, \\
  b^{-1} W_0(b e^b), & b < -1
\end{cases} \tag{B.10}
\]

(another choice of branches will be considered below). Therefore, the solution to Eq. (B.6) we are interested in can be written in the form

\[
z = \frac{1}{N(a)}, \tag{B.11}
\]

where the function \( N(a) \) (see Fig. 13) is defined by

\[
N(a) = \begin{cases} 
  N_0(a), & 0 < a \leq 1, \\
  N_{-1}(a), & 1 < a,
\end{cases} \quad N_k(a) = -a W_k \left[ \frac{-1}{a} \exp \left( \frac{-1}{a} \right) \right]. \tag{B.12}
\]

As follows from the definition (B.1), the function \( x e^x \) and the Lambert \( W \) function are mutually inverse. Here one has to distinguish precisely the branches \( W_0 \) and \( W_{-1} \) (see Fig. 12). From this figure it follows directly that for \( x > -1 \) the function \( x e^x \) “corresponds” to the branch \( W_0 \), and for \( x \leq -1 \) the function \( x e^x \) “corresponds” to the branch \( W_{-1} \). Let us introduce for convenience the notations of the inverse functions for these branches

\[
x e^x = \begin{cases} 
  (W_0)^{-1}(x), & x > -1, \\
  (W_{-1})^{-1}(x), & x \leq -1.
\end{cases} \tag{B.13}
\]
Fig. 13. The function $N(a)$ (boldface curve), and the functions $N_0(a)$ (solid curve) and $N_{-1}(a)$ (dashed curve) (see Eq. (B.12)).

Now the function $N(a)$ in Eq. (B.12) can be written in a more compact form

$$N(a) = \begin{cases} -a W_0 \left[ (W_{-1})^{-1} (-a^{-1}) \right], & 0 < a \leq 1, \\ -a W_{-1} \left[ (W_0)^{-1} (-a^{-1}) \right], & 1 < a. \end{cases} \tag{B.14}$$

It is worth noting also that another choice of the branches of the function $N(a)$

$$\tilde{N}(a) = \begin{cases} N_{-1}(a), & 0 < a \leq 1, \\ N_0(a), & 1 < a \end{cases} \tag{B.15}$$

leads to the trivial solution of Eq. (B.7) (in this case $\tilde{N}(a) \equiv 1$). Therefore, the solution $z = 1/\tilde{N}(a)$ does not satisfy Eq. (B.6) when $a \neq 1$.

Let us outline briefly the basic properties of the function $N(a)$ introduced in Eq. (B.12). In the physical range $a > 0$ it is a non-negative, monotonously increasing function (see Fig. 13). With the help of the series (B.2)–(B.5) one can show that for the function $N(a)$ the following expansions hold:

$$N(a \to 0^+) = \exp \left( -\frac{1}{a} \right) \left\{ 1 + \mathcal{O} \left[ \exp \left( -\frac{1}{a} \right) \right] \right\}, \tag{B.16}$$

$$N(1 + \varepsilon) = 1 + 2\varepsilon + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0, \tag{B.17}$$

$$N(a \to \infty) = a \ln a \left\{ 1 + \mathcal{O} \left[ \frac{\ln(\ln a)}{\ln a} \right] \right\}. \tag{B.18}$$
Appendix C. Spectral Functions

The spectral function $R^{(\ell)}(\sigma)$ at the $\ell$-loop level has the following form (see equation (38)):

\[
R^{(\ell)}(\sigma) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \sum_{j=0}^{\ell-1} \frac{\beta_j}{\beta_0^{j+1}} \left\{ \left[ \tilde{\alpha}^{(\ell)}_s(-\sigma - i\epsilon) \right]^{j+1} - \left[ \tilde{\alpha}^{(\ell)}_s(-\sigma + i\epsilon) \right]^{j+1} \right\}. \tag{C.1}
\]

Here $\alpha^{(\ell)}_s(\mu^2)$ is the $\ell$-loop perturbative running coupling and $\beta_j$ denotes the $\beta$ function expansion coefficient (see Appendix A). It turns out to be convenient to rearrange equation (C.1):

\[
R^{(\ell)}(\sigma) = \sum_{j=0}^{\ell-1} B_j \vartheta^{(\ell)}_{j+1}(\sigma), \tag{C.2}
\]

where

\[
\vartheta^{(\ell)}_{j+1}(\sigma) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left\{ \left[ \tilde{\alpha}^{(\ell)}_s(-\sigma - i\epsilon) \right]^{j+1} - \left[ \tilde{\alpha}^{(\ell)}_s(-\sigma + i\epsilon) \right]^{j+1} \right\}, \tag{C.3}
\]

and $B_j = \beta_j / \beta_0^{j+1}$. It is worthwhile to quote also several commonly used relations:

\[
\lim_{\epsilon \to 0^+} \ln(\sigma \pm i\epsilon) = \ln \sigma \pm i\pi, \quad \sigma > 0 \tag{C.4}
\]

and

\[
\ln(a \pm ib) = \frac{1}{2} \ln(a^2 + b^2) \pm i \left[ \frac{\pi}{2} - \arctan\left( \frac{a}{b} \right) \right], \quad b > 0. \tag{C.5}
\]

Here it is assumed that $-\pi/2 \leq \arctan(x) \leq \pi/2$.

The one-loop perturbative running coupling is determined by Eq. (A.7). In this case the spectral function (C.1) takes a simple form

\[
R^{(1)}(\sigma) = \vartheta^{(1)}_1(\sigma) = \frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2}. \tag{C.6}
\]

For the construction of the two-loop spectral function we shall use the expression (A.12), which gives

\[
R^{(2)}(\sigma) = \vartheta^{(2)}_1(\sigma) + B_1 \vartheta^{(2)}_2(\sigma), \tag{C.7}
\]

where

\[
\vartheta^{(2)}_1(\sigma) = \frac{C^{(2)}(y)}{y^2 \left[ A^{(2)}(y) \right]^2 + \pi^2 \left[ C^{(2)}(y) \right]^2}, \tag{C.8}
\]

\[
\vartheta^{(2)}_2(\sigma) = \frac{2yA^{(2)}(y)C^{(2)}(y)}{\left\{ y^2 \left[ A^{(2)}(y) \right]^2 + \pi^2 \left[ C^{(2)}(y) \right]^2 \right\}^2}. \tag{C.9}
\]

\footnote{It has been shown in Ref. [38] that the spectral functions constructed by making use of Eqs. (A.12) and (A.14) practically lead to the same result as the spectral functions corresponding to the exact (numerical) solutions to the RG equation (A.3) at the higher-loop levels.}
with the two-loop functions $A^{(2)}(y)$ and $C^{(2)}(y)$ defined by
\begin{align}
A^{(2)}(y) &= 1 + \frac{B_1}{y} \left\{ \frac{1}{2} \ln \left[ (y + B_1)^2 + \pi^2 \right] - \ln B_1 \right\}, \quad (C.10) \\
C^{(2)}(y) &= 1 + B_1 \left[ \frac{1}{2} - \frac{1}{\pi} \arctan \left( \frac{y + B_1}{\pi} \right) \right]. \quad (C.11)
\end{align}

Here and further we use the notation $y = \ln(\sigma/\Lambda^2)$.

By making use of the perturbative running coupling (A.14), the following result was obtained for the three-loop spectral function (C.1):
\begin{align}
R^{(3)}(\sigma) &= \varrho^{(3)}_1(\sigma) + B_1 \varrho^{(3)}_2(\sigma) + B_2 \varrho^{(3)}_3(\sigma), \quad (C.12)
\end{align}
where
\begin{align}
\varrho^{(3)}_1(\sigma) &= \frac{C^{(3)}(y)}{(y^2 + \pi^2)^3}, \quad (C.13) \\
\varrho^{(3)}_2(\sigma) &= \frac{2A^{(3)}(y)C^{(3)}(y)}{(y^2 + \pi^2)^6}, \quad (C.14) \\
\varrho^{(3)}_3(\sigma) &= \frac{C^{(3)}(y) \left\{ 3 \left[ A^{(3)}(y) \right]^2 - \pi^2 \left[ C^{(3)}(y) \right]^2 \right\}}{(y^2 + \pi^2)^9}, \quad (C.15)
\end{align}
and the three-loop functions $A^{(3)}(y)$ and $C^{(3)}(y)$ are
\begin{align}
A^{(3)}(y) &= y(y^2 - 3\pi^2)R(y) - \pi^2(\pi^2 - 3y^2)J(y), \quad (C.16) \\
C^{(3)}(y) &= y(3\pi^2 - y^2)J(y) - (3y^2 - \pi^2)R(y), \quad (C.17) \\
R(y) &= y^2 - \pi^2 - B_1(ya - \pi^2c) + B_1^2(a^2 - \pi^2c^2 - a - 1) + B_2, \quad (C.18) \\
J(y) &= 2y - B_1(yc + a) + B_1^2c(2a - 1), \quad (C.19) \\
a &= \frac{1}{2} \ln(y^2 + \pi^2), \quad c = \frac{1}{2} - \frac{1}{\pi} \arctan \left( \frac{y}{\pi} \right). \quad (C.20)
\end{align}

The plots of the spectral functions (C.6), (C.7), and (C.12) are presented in Figure 14. It is interesting to note that $R^{(l)}(\sigma) \to R^{(1)}(\sigma)$, when $\sigma \to \infty$, as well as when $\sigma \to 0$, irrespective of the loop level. It is worth mentioning also that the relation $\max |\varrho^{(l)}_j(\sigma)| \gg \max |\varrho^{(l)}_{j+1}(\sigma)|$ holds at any loop level. In particular, this implies that the contribution of the every subsequent term on the right-hand side of Eq. (36) is substantially suppressed by the contributions of the preceding ones. In turn, this results in a good higher loop and scheme stability of the approach in hand.

Appendix D. The Expansion Coefficients of the Quark–Antiquark Potential

In this appendix the explicit expressions for the expansion coefficients of the static quark–antiquark potential generated by the analytic invariant charge (69) are gathered in a concise form. By making use of Eqs. (65) and (68) we find

\begin{align}
A^{(2)}(y) &= 1 + \frac{B_1}{y} \left\{ \frac{1}{2} \ln \left[ (y + B_1)^2 + \pi^2 \right] - \ln B_1 \right\}, \quad (C.10) \\
C^{(2)}(y) &= 1 + B_1 \left[ \frac{1}{2} - \frac{1}{\pi} \arctan \left( \frac{y + B_1}{\pi} \right) \right]. \quad (C.11)
\end{align}
Fig. 14. The spectral functions $\mathcal{R}^{(\ell)}(\sigma)$ defined by Eq. (C.1). The one-, two-, and three-loop levels are shown by the dot-dashed, solid, and dashed curves, respectively.

\[
\begin{align*}
\omega_0 &= 1, \quad \text{(D.1)} \\
\omega_1 &= 3 + \frac{\gamma}{2}, \quad \text{(D.7)} \\
\omega_2 &= \frac{7}{4} + \frac{\pi^2}{24} - \frac{3}{2} \gamma + \frac{1}{2} \gamma^2, \quad \text{(D.8)} \\
\omega_3 &= \frac{45}{8} + \frac{3}{16} \pi^2 - \zeta(3) - \frac{\gamma}{4} \left[ \frac{\pi^2}{2} + 21 \right] + \frac{9}{4} \gamma^2 - \frac{1}{2} \gamma^3, \quad \text{(D.9)} \\
\omega_4 &= \frac{93}{4} + \frac{\pi^2}{8} \left[ \frac{19}{60} \pi^2 + 7 \right] - 6 \zeta(3) + \gamma \left[ 4 \zeta(3) - \frac{45}{2} - \frac{3}{4} \pi^2 \right] + \frac{\gamma^2}{2} \left[ \frac{\pi^2}{2} + 21 \right] - 3 \gamma^3 + \frac{1}{2} \gamma^4, \quad \text{(D.10)}
\end{align*}
\]

where $\gamma \approx 0.57721\ldots$ denotes the Euler’s constant, and $\zeta(x)$ is the Riemann’s Zeta function $\zeta(3) \approx 1.20206\ldots$. Applying the “regularization” procedure specified...
in Subsection 4.1 one obtains (see Eqs. (39) and (70))

\[
\frac{1}{R \ln R} \left|_{\text{Reg}} \right. = \frac{1}{R \ln R} - \frac{1}{R - 1},
\]

(D.11)

\[
\frac{1}{R (\ln R)^2} \left|_{\text{Reg}} \right. = \frac{1}{R (\ln R)^2} - \frac{1}{(R - 1)^2},
\]

(D.12)

\[
\frac{1}{R (\ln R)^3} \left|_{\text{Reg}} \right. = \frac{1}{R (\ln R)^3} - \frac{1}{2 (R - 1)^2} - \frac{1}{(R - 1)^3},
\]

(D.13)

\[
\frac{1}{R (\ln R)^4} \left|_{\text{Reg}} \right. = \frac{1}{R (\ln R)^4} - \frac{1}{6 (R - 1)^2} - \frac{1}{(R - 1)^3} - \frac{1}{(R - 1)^4},
\]

(D.14)

\[
\frac{1}{R (\ln R)^5} \left|_{\text{Reg}} \right. = \frac{1}{R (\ln R)^5} - \frac{1}{24 (R - 1)^2} - \frac{7}{12 (R - 1)^3} - \frac{3}{2 (R - 1)^4} - \frac{1}{(R - 1)^5},
\]

(D.15)

\[
\frac{R}{\ln R} \left|_{\text{Reg}} \right. = \frac{R}{\ln R} - \frac{1}{R - 1},
\]

(D.16)

\[
\frac{R}{(\ln R)^2} \left|_{\text{Reg}} \right. = \frac{R}{(\ln R)^2} - \frac{2}{R - 1} - \frac{1}{(R - 1)^2},
\]

(D.17)

\[
\frac{R}{(\ln R)^3} \left|_{\text{Reg}} \right. = \frac{R}{(\ln R)^3} - \frac{2}{R - 1} - \frac{5}{2 (R - 1)^2} - \frac{1}{(R - 1)^3},
\]

(D.18)

\[
\frac{R}{(\ln R)^4} \left|_{\text{Reg}} \right. = \frac{R}{(\ln R)^4} - \frac{4}{3 R - 1} - \frac{19}{6 (R - 1)^2} - \frac{3}{(R - 1)^3},
\]

(D.19)

\[
\frac{R}{(\ln R)^5} \left|_{\text{Reg}} \right. = \frac{R}{(\ln R)^5} - \frac{2}{3 R - 1} - \frac{65}{24 (R - 1)^2} - \frac{55}{12 (R - 1)^3} - \frac{7}{2 (R - 1)^4} - \frac{1}{(R - 1)^5}.
\]

(D.20)

References

1. O.W. Greenberg, *Phys. Rev. Lett.* 13, 598 (1964); M.Y. Han and Y. Nambu, *Phys. Rev.* 139, 1006 (1965); N.N. Bogoliubov, B.V. Struminsky, and A.N. Tavkhelidze, Joint Institute for Nuclear Research, Report No. D–1968, 1965.

2. H. Fritzsch, M. Gell–Mann, and H. Leutwyler, *Phys. Lett.* B47, 365 (1973).

3. D.J. Gross and F. Wilczek, *Phys. Rev. Lett.* 30, 1343 (1973); H.D. Politzer, *ibid.* 30, 1346 (1973).

4. E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane, and T.M. Yan, *Phys. Rev.* D17, 3090 (1978); C. Quigg and J.L. Rosner, *Phys. Lett.* B71, 153 (1977); J.L. Richardson, *ibid.* B82, 272 (1979); A. Martin, *ibid.* B93, 338 (1980).

5. D.B. Lichtenberg and J.G. Wills, *Nuovo Cimento* A47, 483 (1978); G. Fogleman, D.B. Lichtenberg, and J.G. Wills, *Lett. Nuovo Cimento* 26, 369 (1979).

6. R. Levine and Y. Tomozawa, *Phys. Rev.* D19, 1572 (1979).
7. W. Buchmuller, G. Grunberg, and S.-H.H. Tye, Phys. Rev. Lett. 45, 103 (1980); 45, 587(E) (1980); W. Buchmuller and S.-H.H. Tye, Phys. Rev. D24, 132 (1981); W. Celmaster and F.S. Henyey, ibid. D18, 1688 (1978).
8. W. Lucha, F.F. Schoberl, and D. Gromes, Phys. Rep. 200, 127 (1991); N. Brambilla and A. Vairo, arXiv: [hep-ph/9904330].
9. V.V. Kiselev, A.E. Kovalsky, and A.I. Onishchenko, Phys. Rev. D64, 054009 (2001).
10. V.V. Kiselev, A.K. Likhoded, O.N. Pakhomova, and V.A. Saleev, Phys. Rev. D65, 034013 (2002); D66, 034030 (2002).
11. V.V. Kiselev and A.K. Likhoded, Usp. Fiz. Nauk 172, 497 (2002) [Phys. Usp. 45, 455 (2002)].
12. M. Melles, Phys. Rev. D62, 074019 (2000).
13. M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. B147, 385 (1979); B147, 448 (1979).
14. S. Narison, QCD Spectral Sum Rules (World Scientific, Singapore, 1990).
15. L.J. Reinders, H. Rubinstein, and S. Yazaki, Phys. Rep. 127, 1 (1985).
16. P. Colangelo and A. Khodjamirian, in At the Frontier of Particle Physics. Handbook of QCD. Edited by M. Shifman (World Scientific, Singapore, 2001), p. 1495.
17. B.M. Barbashov and V.V. Nesterenko, Introduction to the Relativistic String Theory (World Scientific, Singapore, 1990).
18. L.D. Soloviev, Phys. Rev. D58, 035005 (1998); D61, 015009 (1999).
19. P. Hasenfratz and J. Kuti, Phys. Rep. C40, 75 (1978); S.L. Adler and T. Piran, Rev. Mod. Phys. 56, 1 (1984).
20. I.L. Solovtsov, Phys. Lett. B327, 335 (1994); B340, 245 (1994).
21. A.N. Sisakian, I.L. Solovtsov, and O. Shevchenko, Int. J. Mod. Phys. A9, 1929 (1994).
22. A.N. Sisakian and I.L. Solovtsov, Phys. Part. Nucl. 25, 478 (1994); 30, 461 (1999); Phys. Atom. Nucl. 61, 1942 (1998).
23. A.N. Sisakian, I.L. Solovtsov, and O.P. Solovtsova, JETP Lett. 73, 166 (2001).
24. G.S. Bali, Phys. Rep. 343, 1 (2001).
25. G.S. Bali, C. Schlichter, and K. Schilling, Phys. Rev. D51, 5165 (1995).
26. G.S. Bali, K. Schilling, and A. Wachter, Phys. Rev. D56, 2566 (1997); G.S. Bali and P. Boyle, ibid. D59, 114504 (1999).
27. G.S. Bali et al. (SESAM and TyL Collaborations), Phys. Rev. D62, 054503 (2000).
28. C. Bernard et al., Phys. Rev. D62, 034503 (2000); S. Necco and R. Sommer, Nucl. Phys. B622, 328 (2002).
29. T.T. Takahashi, H. Suganuma, Y. Nemoto, and H. Matsufuru, Phys. Rev. D65, 114509 (2002); S. Aoki et al. (JLQCD Collaboration), ibid. D68, 054502 (2003).
30. E.C.G. Stuckelberg and A. Petermann, Helv. Phys. Acta 24, 317 (1951); 26, 499 (1953); M. Gell-Mann and F.E. Low, Phys. Rev. 95, 1300 (1954).
31. N.N. Bogoliubov and D.V. Shirkov, Dokl. Akad. Nauk SSSR 103, 203 (1955); 103, 391 (1955); Nuovo Cimento 3, 845 (1956).
32. N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience, New York, 1980).
33. P.J. Redmond, Phys. Rev. 112, 1404 (1958); P.J. Redmond and J.L. Uretsky, Phys. Rev. Lett. 1, 147 (1958).
34. N.N. Bogoliubov, A.A. Logunov, and D.V. Shirkov, Zh. Eksp. Teor. Fiz. 37, 805 (1959) [Sov. Phys. JETP 37, 574 (1960)].
35. S. Narison, Phys. Rep. 84, 263 (1982).
36. I.F. Ginzburg and D.V. Shirkov, Zh. Eksp. Teor. Fiz. 49, 335 (1965) [Sov. Phys. JETP 22, 234 (1966)].
37. D.V. Shirkov and I.L. Solovtsov, *JINR Rapid Comm.* **2**, 5 (1996); D.V. Shirkov and I.L. Solovtsov, *Phys. Rev. Lett.* **79**, 1209 (1997).

38. D.V. Shirkov, *Teor. Mat. Fiz.* **119**, 55 (1999) [Theor. Math. Phys. **119**, 438 (1999)]; *Eur. Phys. J. C** **22**, 331 (2001); I.L. Solovtsov and D.V. Shirkov, *Teor. Mat. Fiz.* **120**, 482 (1999) [Theor. Math. Phys. **120**, 1220 (1999)].

39. K.A. Milton and I.L. Solovtsov, *Phys. Rev. D** **55**, 5295 (1997); D.59, 107701 (1999); K.A. Milton and O.P. Solovtsova, *ibid.* **D57**, 5402 (1998).

40. K.A. Milton, I.L. Solovtsov, and O.P. Solovtsova, *ibid.* **D57**, 5402 (1998); K.A. Milton, I.L. Solovtsov, and V.I. Yasnov, *Eur. Phys. J. C** **14**, 495 (2000); K.A. Milton and O.P. Solovtsova, *Int. J. Mod. Phys. A** **17**, 3789 (2002).

41. I.L. Solovtsov and D.V. Shirkov, *Phys. Lett.* **B442**, 344 (1998); K.A. Milton, I.L. Solovtsov, and O.P. Solovtsova, *Eur. Phys. J. C** **13**, 497 (2000).

42. K.A. Milton, I.L. Solovtsov, and O.P. Solovtsova, *Phys. Rev. D** **64**, 016005 (2001); **D65**, 076009 (2002); K.A. Milton, I.L. Solovtsov, O.P. Solovtsova, and V.I. Yasnov, *Eur. Phys. J. C** **14**, 495 (2000); K.A. Milton and O.P. Solovtsova, *ibid.* **D65**, 076009 (2002).

43. F. Schrempp, *J. Phys. G** **28**, 915 (2002).

44. F.J. Yndurain, *The Theory of Quark and Gluon Interactions* (Springer–Verlag, Berlin, 1999).

45. A.V. Nesterenko, *Phys. Rev. D** **62**, 094028 (2000).

46. A.V. Nesterenko, *Phys. Rev. D** **64**, 116009 (2001); *Nucl. Phys. (Proc. Suppl.) B** **133**, 59 (2004).

47. A.V. Nesterenko, in *Proceedings of the Sixth Workshop on Non–Perturbative Quantum Chromodynamics*, Paris, France, 2001, edited by H.M. Fried, Y. Gabellini, and B. Muller (World Scientific, Singapore, 2002), p. 340; arXiv: [hep-ph/0106305](https://arxiv.org/abs/hep-ph/0106305).

48. A.V. Nesterenko, in *Proceedings of the Fifth International Conference on Quark Confinement and the Hadron Spectrum*, Gargnano, Italy, 2002, edited by N. Brambilla and G. Prosperi (World Scientific, Singapore, 2003), p. 288; arXiv: [hep-ph/0210122](https://arxiv.org/abs/hep-ph/0210122).

49. D.A. Smith and M.J. Teper (UKQCD Collaboration), *Phys. Rev. D** **58**, 014505 (1998).

50. A. Ringwald and F. Schrempp, *Phys. Lett. B** **459**, 249 (1999); **B503**, 331 (2001).

51. J. Heitger, H. Simma, R. Sommer, and U. Wolff (ALPHA Collaboration), *Nucl. Phys. (Proc. Suppl.) B** **106**, 859 (2002); A. Bode et al. (ALPHA Collaboration), *Phys. Lett. B** **515**, 49 (2001).

52. A.V. Nesterenko, *Phys. Rev. D** **62**, 094028 (2000).

53. A.I. Alekseev and B.A. Arbuzov, *Mod. Phys. Lett. A** **13**, 1747 (1998).

54. A.I. Alekseev, arXiv: [hep-ph/9802372](https://arxiv.org/abs/hep-ph/9802372); arXiv: [hep-ph/9808206](https://arxiv.org/abs/hep-ph/9808206).

55. A.V. Nesterenko and I.L. Solovtsov, *Mod. Phys. Lett. A** **16**, 2517 (2001).

56. G. ’t Hooft, *Phys. Rev. Lett. B** **37**, 8 (1976); *Phys. Rev. D** **14**, 3432 (1976); **D18**, 2199(E) (1978); *Phys. Rep.* **142**, 357 (1986).

57. C.G. Callan, R. Dashen, and D.J. Gross, *Phys. Rev. D** **17**, 2717 (1978); C. Bernard, *ibid.* **D19**, 3013 (1979).

58. D.A. Kirzhnits, Y.A. Fainberg, and E.S. Fradkin, *Zh. Eksp. Teor. Fiz.* **38**, 239 (1960) [Sov. Phys. JETP **11**, 174 (1960)].

59. B.A. Arbuzov, *Dokl. Akad. Nauk SSSR* **128**, 1149 (1959).
64. D.V. Shirkov, Teor. Mat. Fiz. 132, 484 (2002) [Theor. Math. Phys. 132, 1309 (2002)].
65. Ph. Boucaud et al., JHEP 04, 006 (2000); G. Burgio, F.Di Renzo, C. Parrinello, and C. Pittori, arXiv: hep-ph/9808258.
66. G. Burgio, F.Di Renzo, G. Marchesini, and E. Onofri, Phys. Lett. B422, 219 (1998).
67. B.R. Webber, JHEP 10, 012 (1998).
68. Y. Srivastava, S. Pacetti, G. Pancheri, and A. Widom, arXiv: hep-ph/0106005.
69. N.V. Krasnikov and A.A. Pivovarov, Phys. Atom. Nucl. 64, 1500 (2001).
70. D.M. Howe and C.J. Maxwell, Phys. Lett. B541, 219 (1998).
71. B.R. Webber, JHEP 10, 012 (1998).
72. G. Grunberg, Phys. Lett. B93, 429 (1980).
73. J. Fischer, Int. J. Mod. Phys. A12, 3625 (1997); I. Caprini and J. Fischer, Eur. Phys. J. C24, 127 (2002).
74. G. Grunberg, Phys. Rev. D46, 2228 (1992); A. Pineda and J. Soto, Phys. Lett. B495, 323 (2000); C.J. Maxwell and A. Mirjalili, Nucl. Phys. B577, 209 (2000); M.R. Ahmady et al., Phys. Rev. D67, 034017 (2003); V. Elias, arXiv: hep-ph/0305187.
75. O.V. Tarasov, A.A. Vladimirov, and A.Yu. Zharkov, Phys. Lett. B422, 219 (1998).
76. P.A. Raczka, Z. Phys. C65, 481 (1995).
77. B. Schrempp and F. Schrempp, Z. Phys. C6, 7 (1980).
78. A.V. Radyushkin, Joint Institute for Nuclear Research, Report No. 2-82-159, 1982; JINR Rapid Comm. 4, 9 (1996).
79. M.R. Pennington and G.G. Ross, Phys. Lett. B102, 167 (1981); N.V. Krasnikov and A.A. Pivovarov, ibid. B116, 168 (1982).
80. S.G. Gorishny, A.L. Kataev, and S.A. Larin, Phys. Lett. B259, 144 (1991); A.A. Pivovarov, Z. Phys. C53, 461 (1992).
81. L.R. Surguladze and M.A. Samuel, Phys. Rev. Lett. 66, 560 (1991); 66, 2416(E) (1991); D.E. Soper and L.R. Surguladze, Phys. Rev. D54, 4566 (1996); A.A. Pivovarov, Nuovo Cimento A105, 813 (1992).
82. Yu.L. Dokshitzer and B.R. Webber, Phys. Lett. B352, 451 (1995); Yu.L. Dokshitzer, V.A. Khoze, and S.I. Troyan, Phys. Rev. D53, 89 (1996).
83. S. Titard and F.J. Yndurain, Phys. Rev. D49, 6007 (1994).
84. A.V. Nesterenko, Int. J. Mod. Phys. A19, 3471 (2004).
85. A.V. Nesterenko, in Proceedings of the Fourth International Conference on Quark Confinement and the Hadron Spectrum, Vienna, Austria, 2000, edited by W. Lucha and K.M. Maung (World Scientific, Singapore, 2002), p. 255; arXiv: hep-ph/0010257.
86. I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, edited by A. Jeffrey (Academic, London, 1994).
93. V.I. Zakharov, *Int. J. Mod. Phys.* **A14**, 4865 (1999).
94. G. Burgio, F. Di Renzo, G. Marchesini, and E. Onofri, *Phys. Lett.* **B422**, 219 (1998); R. Horsley, P.E.L. Rakow, and G. Schierholz, *Nucl. Phys. (Proc. Suppl.)* **B106**, 870 (2002); T. Doi, N. Ishii, M. Oka, and H. Suganuma, *Phys. Rev.* **D67**, 054504 (2003).
95. V.A. Novikov, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin, and V.I. Zakharov, *Nucl. Phys.* **B237**, 525 (1984).
96. B.A. Arbuzov, *Phys. Part. Nucl.* **19**, 5 (1988).
97. W. Greiner and A. Schafer, *Quantum Chromodynamics* (Springer–Verlag, 1994).
98. E. Braaten, *Phys. Rev. Lett.* **60**, 1606 (1988); *Phys. Rev.* **D39**, 1458 (1989); F. Le Diberder and A. Pich, *Phys. Lett.* **B286**, 147 (1992).
99. B.A. Arbuzov, *Phys. Part. Nucl.* **19**, 5 (1988).
100. W. Greiner and A. Schafer, *Quantum Chromodynamics* (Springer–Verlag, 1994).
101. E. Braaten, *Phys. Rev. Lett.* **60**, 1606 (1988); *Phys. Rev.* **D39**, 1458 (1989); F. Le Diberder and A. Pich, *Phys. Lett.* **B286**, 147 (1992).
102. E. Braaten, S. Narison, and A. Pich, *Nucl. Phys.* **B373**, 581 (1992).
103. K. Hagiwara *et al.* (Particle Data Group), *Phys. Rev.* **D66**, 010001 (2002).
104. E. Braaten and C. S. Li, *Phys. Rev. D42*, 3888 (1990).
105. R. Barate *et al.* (ALEPH Collaboration), *Eur. Phys. J.* **C4**, 409 (1998).
106. T. Applequist and H. Georgi, *Phys. Rev.* **D8**, 4000 (1973); A. Zee, *ibid.* **D8**, 4038 (1973).
107. R. Ammar *et al.* (CLEO Collaboration), *Phys. Rev. D57*, 1350 (1998).
108. E. Gardi, G. Grunberg, and M. Karliner, *JHEP* **07**, 007 (1998); E. Gardi and M. Karliner, *Nucl. Phys. B529*, 383 (1998); B.A. Magradze, arXiv: hep-ph/9808247 arXiv: hep-ph/0305020.
109. T. van Ritbergen, J.A.M. Vermaseren, and S.A. Larin, *Phys. Lett. B400*, 379 (1997).
110. K.G. Chetyrkin, B.A. Kniehl, and M. Steinhauser, *Phys. Rev. Lett.* **79**, 2184 (1997); A.I. Alekseev, *J. Phys.* **G27**, 117 (2001).
111. R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth, *Adv. Comput. Math.* **5**, 329 (1996).