Symmetry problems 2

N. S. Hoang†* and A. G. Ramm†‡

†Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA

Abstract

Some symmetry problems are formulated and solved. New simple proofs are given for the earlier studied symmetry problems.

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1 Introduction

Symmetry problems are of interest both theoretically and in applications.

A well-known, and still unsolved, symmetry problem is the Pompeiu problem (see [3], [4]). It consists of proving the following:

If \( D \subset \mathbb{R}^n, n \geq 2 \) is homeomorphic to a ball, and the boundary \( S \) of \( D \) is sufficiently smooth, \( (S \in C^{1,\lambda}, \lambda > 0 \) is sufficient) and if the problem

\[
(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D, \quad u|_S = c, \quad u_N|_S = 0, \quad k^2 = \text{const} > 0,
\]

has a solution, then \( S \) is a sphere.

A similar problem (Schiffer’s conjecture) is also unsolved:

If the problem

\[
(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D, \quad u|_S = 0, \quad u_N|_S = c \neq 0, \quad k^2 = \text{const} > 0,
\]

has a solution, then \( S \) is a sphere.

In [5] it is proved that if

\[
\int_D \frac{dy}{4\pi|x-y|} = \frac{c}{|x|}, \quad \forall x \in B_R', \quad c = \text{const} > 0,
\]

then \( D \) is a ball. Here and below we assume that \( D \subset \mathbb{R}^3 \) is a bounded domain homeomorphic to a ball, with a sufficiently smooth boundary \( S \) (\( S \) is Lipschitz suffices),

*Email: nguyenhs@math.ksu.edu
‡Corresponding author. Email: ramm@math.ksu.edu
$B_R = \{x : |x| \leq R\}$, $B_R \supset D$. By $\mathcal{H}$ we denote the set of all harmonic functions in a domain which contains $D$. By $|D|$ and $|S|$ we denote the volume of $D$ and the surface area of $S$, respectively.

Our goal is to give a simple proof of the three symmetry-type results, formulated in Theorem 1 in Section 2.

In [7] the following result is obtained:

\begin{equation}
\Delta u = 1 \quad \text{in} \quad D, \quad u|_S = 0, \quad u_N = \mu = \text{const} > 0,
\end{equation}

then $S$ is a sphere.

This result is obtained by A. D. Alexandrov’s ”moving plane” argument, and is equivalent to the following result:

\begin{equation}
\frac{1}{|D|} \int_D h(x)dx = \frac{1}{|S|} \int_S h(s)ds, \quad \forall h \in \mathcal{H},
\end{equation}

then $S$ is a sphere.

The equivalence of (4) and (5) can be proved as follows.

Suppose (4) holds. Multiply (4) by an arbitrary $h \in \mathcal{H}$, integrate by parts and get

\begin{equation}
\int_D h(x)dx = \mu \int_S h(s)ds.
\end{equation}

If $h = 1$ in (6), then one gets $\mu = \frac{|D|}{|S|}$, so (6) is identical to (5).

Suppose (5) holds. Then (6) holds. Let $v$ solve the problem $\Delta v = 1$ in $D$, $v|_S = 0$. This $v$ exists and is unique. Using (5), the equation $\Delta h = 0$ in $D$, and the Green’s formula, one gets

\begin{equation}
\mu \int_S h(s)ds = \int_D h(x)dx = \int_D h(x)\Delta vdx = \int_S h(s)v_N ds.
\end{equation}

Thus,

\begin{equation}
\int_S h(s)[v_N - \mu]ds = 0, \quad \forall h \in \mathcal{H}.
\end{equation}

The set of restrictions on $S$ of all harmonic functions in $D$ is dense in $L^2(S)$ (see, e.g., [5]). Thus, (8) implies $v_N|_S = \mu$. Thus, (4) holds.

2 Results and proofs

Our main results are formulated in the following theorem

**Theorem 1** Let $D \subset \mathbb{R}^3$ be a bounded domain homeomorphic to a ball, $S$ be its Lipschitz boundary, $D' := \mathbb{R}^3 \setminus D$. If any one of the following assumptions holds, then $S$ is a sphere:

1. \[ u(x) := \frac{1}{4\pi |x-s|} \int_S \frac{ds}{|x-s|} = \frac{c}{|x|}, \quad c = \text{const}, \quad \forall x \in B'_R, \]

where $B'_R := \{x : |x| > R\}$, $D \subset B_R$, $B_R := \mathbb{R}^3 \setminus B'_R$;
2. \[
\frac{1}{|S|} \int_S h(s) ds = h(0), \quad \forall h \in \mathcal{H}; \tag{10}
\]

3. There exists a solution to the problem
\[
\triangle u = \delta(y) \quad \text{in} \quad D, \quad u|_S = 0, \quad u_N|_S = c_1 = \text{const}, \tag{11}
\]
where \(\delta(y)\) is the delta-function.

In \((10)\) 0 is the origin, \(0 \in D\), \(|S|\) is the surface area of \(S\), \(\mathcal{H}\) is the set of all harmonic functions in a domain containing \(D\).

**Proof.**

1. Assume \((9)\). Then \(c = \frac{|S|}{4\pi}\) as one can see by taking \(|x| \to \infty\). If \((9)\) holds for \(\forall x \in B'_R\) then, by the unique continuation property for harmonic functions, \((9)\) holds \(\forall x \in D'\).

Let \(N_s\) be a unit normal to \(S\) at the point \(s \in S\), pointing into \(D'\). The known jump formula for the normal derivative of a single-layer potential (\([2, p.14]\)) yields
\[
u^+_N s = u_N s + 1, \quad u^-_N s = -\frac{|S|}{4\pi} \frac{N_s \cdot s_0}{|s_0|^3}, \quad s_0 \in S, \tag{12}
\]

If \(S\) is not a sphere, then there exists an \(s_0 \in S\), \(|s_0| \leq |s|, \forall s \in S\). The ball \(B_{|s_0|}\) of radius \(|s_0|\), centered at the origin, belongs to \(D\). At the point \(s_0\) the normal \(N_{s_0}\) to \(S\) is directed along the vector \(s_0\), so
\[
u^-_{N_{s_0}} = -\frac{|S|}{4\pi|s_0|^2} < -1, \tag{13}
\]
because \(|S| > 4\pi|s_0|^2\) by the isoperimetric inequality \((11)\). This and formula \((12)\) imply
\[
u^+_{N_{s_0}} < 0. \tag{14}
\]
On the other hand,
\[
u(s) = \frac{1}{4\pi|s|} \leq \frac{1}{4\pi|s_0|}. \tag{15}
\]
So the harmonic and continuous in \(D\) function \(\nu(x)\) attains its maximum on \(S\) at the point \(s_0\), because \(\nu|_S = \frac{1}{4\pi|s_0|^2}|s_0|\). Therefore, by the maximum principle,
\[
u(x) \leq \nu(s_0), \quad \forall x \in D.
\]
In particular, \(\nu(s_0) - \nu(s_0 - \epsilon N_{s_0}) \geq 0\) for all sufficiently small \(\epsilon > 0\). Consequently, \(\nu_{N_{s_0}} \geq 0\). This contradicts \((14)\), and the contradiction proves that \(S\) is a sphere.

2. Assume \((10)\). Let \(h(y) = \frac{1}{4\pi|x-y|^3}, x \in D', y \in D\). This function is a harmonic function in \(D\). Thus, \((10)\) yields \((9)\):
\[
\int_S \frac{ds}{4\pi|x-s|} = \frac{|S|}{4\pi|x|}, \quad c := \frac{|S|}{4\pi}, \quad \forall x \in D'. \tag{16}
\]
We have already proved that \((16)\) implies that \(S\) is a sphere. Therefore, the Assertion 2 of Theorem 1 is established.
3. Assume (11). Multiply (11) by $\frac{1}{4\pi|x-y|}$, $x \in D'$, integrate over $D$, and then integrate by parts to get

$$c_1 \int_S \frac{ds}{4\pi|x-s|} = \frac{1}{4\pi|x|}, \quad \forall x \in D'. \quad (17)$$

By the result, proved in Assertion 1, this implies that $S$ is a sphere. Therefore, Assertion 3 of Theorem 1 is proved.

\[ \square \]

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