THE RATIONALITY OF THE MODULI SPACES OF TRIGONAL CURVES

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ABSTRACT. The moduli spaces of trigonal curves are proven to be rational when the genus is divisible by 4.

1. Introduction

A smooth projective curve is called trigonal if it carries a free $g_1^3$. When the curve has genus $\geq 5$, such a pencil is unique if it exists. The object of our study is the moduli space $T_g$ of trigonal curves of genus $g \geq 5$. This space has been proven to be rational when $g \equiv 2 \pmod{4}$ by Shepherd-Barron [10], and when $g$ is odd in [8]. In the present article we prove that $T_g$ is rational in the remaining case $g \equiv 0 \pmod{4}$, completing the following.

Theorem. The moduli space $T_g$ of trigonal curves of genus $g$ is rational for every $g \geq 5$.

$T_g$ is naturally regarded as a sublocus of the moduli space $M_g$ of genus $g$ curves. The rationality of $T_g$ can be seen as an extension of that of the hyperelliptic locus due to Katsylo and Bogomolov [6], [2]. It would be interesting whether the tetragonal and pentagonal loci are rational as well. They are unirational (see, e.g., [1], [12]), but at present known to be rational only for tetragonal of genus 7 (3). A related question is whether one can find a rational locus in $M_g$ of larger dimension. When $g \geq 23$, Castorena and Ciliberto [4] show that $T_g$ has larger dimension than any other locus that is (generically) the natural image of a linear system on a surface. Thus, for the above question, one would next look at curves in a variety of dimension $\geq 3$ whose ideals have simple description. Note that tetragonal and pentagonal curves can be constructed in such ways ([12]).

We approach our problem from invariant theory for $\text{SL}_2 \times \text{SL}_2$. Let $V_{a,b} = H^0(O_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b))$ be the space of bi-forms of bidegree $(a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which is an irreducible representation of $\text{SL}_2 \times \text{SL}_2$. It is classically known that a general trigonal curve $C$ of genus $g = 4N$ is canonically embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth curve of bidegree $(3, 2N + 1)$. This is based on the fact that the canonical model of $C$ lies on a unique rational normal scroll which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. As a
consequence, we have a natural birational equivalence

$$\mathcal{T}_{4N} \sim \mathbb{P}V_{3,2N+1}/\text{SL}_2 \times \text{SL}_2.$$ 

Hence the problem is restated as follows.

**Theorem 1.1.** The quotient $\mathbb{P}V_{3,b}/\text{SL}_2 \times \text{SL}_2$ is rational for every odd $b \geq 5$.

To prove this, we adopt the traditional and computational method of double bundle ([2], [11]) as follows. By examining the Clebsch-Gordan formula for $\text{SL}_2 \times \text{SL}_2$, we take a suitable $\text{SL}_2 \times \text{SL}_2$-bilinear mapping (bi-transvectant)

$$T : V_{3,b} \times V_{a',b'} \to V_{a'',b''}$$

such that $\dim V_{a',b'} > \dim V_{a'',b''}$. Put $c = \dim V_{a',b'} - \dim V_{a'',b''}$ and let $G(c, V_{a',b'})$ be the Grassmannian of $c$-dimensional subspaces of $V_{a',b'}$. Then $T$ induces the rational map

$$V_{3,b} \dashrightarrow G(c, V_{a',b'}), \quad v \mapsto \ker(T(v, \cdot)).$$

We shall find a bi-transvectant for which (1.1) is well-defined as a rational map and is dominant. In that case, (1.1) makes $V_{3,b}$ birationally an $\text{SL}_2 \times \text{SL}_2$-linearized vector bundle over $G(c, V_{a',b'})$. Utilizing this bundle structure and taking care of $-1$ scalar action, we reduce the rationality of $\mathbb{P}V_{3,b}/\text{SL}_2 \times \text{SL}_2$ to a stable rationality of $G(c, V_{a',b'})/\text{SL}_2 \times \text{SL}_2$, which in turn can be shown in a more or less standard way.

The point for this proof is to choose the bi-transvectant $T$ carefully so that (i) $a', b', c$ are odd (to care $-1$ scalar action) and that (ii) $c$ is small (for $V_{3,b}$ to have larger dimension than $G(c, V_{a',b'})$). For that, we will provide $T$ according to the remainder of $b$ modulo 5, based on some easy calculation. Then the bulk of proof is devoted to verifying non-degeneracy of (1.1), which is facilitated by keeping $c$ small but is still somewhat laborious.

The rest of the article is as follows. In §2.1 we recall bi-transvectants. We explain the method of double bundle in §2.2. In §3 we prepare some stable rationality results in advance, to which the rationality of $\mathbb{P}V_{3,b}/\text{SL}_2 \times \text{SL}_2$ will be eventually reduced. Then we prove Theorem 1.1 in §4.

We work over the complex numbers. The Grassmannian $G(a, V)$ parametrizes $a$-dimensional linear subspaces of the vector space $V$.

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## 2. Bi-transvectant

### 2.1. Bi-transvectant

We write $V_d$ for the $\text{SL}_2$-representation $H^0(O_{\mathbb{P}^1}(d))$, the space of binary forms of degree $d$. Let $e \leq d$. According to the Clebsch-Gordan decomposition

$$V_d \otimes V_e = \bigoplus_{r=0}^{e} V_{d+e-2r},$$

$$V_{3,5} \times V_{1,5} \to V_{2,5}.$$
there exists a unique (up to constant) \( SL_2 \)-bilinear mapping
\[
T^{(e)} : V_d \times V_e \to V_{d+e-2r},
\]
which is called the \( r \)-th transvectant. For two binary forms \( F(X, Y) \in V_d \) and \( G(X, Y) \in V_e \), we have the well-known explicit formula (cf. [9])
\[
(2.2) \quad T^{(e)}(F, G) = \frac{(d-r)!(e-r)!}{d!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r F}{\partial X^r i \partial Y^i} \frac{\partial^r G}{\partial X^r i \partial Y^i}.
\]
We will need this formula when \( r = e \) and \( r = e - 1 \).

The \( e \)-th transvectant \( T^{(e)} : V_d \times V_e \to V_{d-e} \) is especially called the apolar covariant. By (2.2), \( T^{(e)}(F, G) \) is calculated by applying the differential polynomial \((d!)^{-1}(d-e)!G(\partial Y, \partial X)\) to \( F(X, Y) \). In particular, we have
\[
T^{(e)}(X^i Y^{d-i}, X^e Y^j) = \begin{cases} (-1)^{e-j}(d-i)! \binom{d-e}{i} X^{i-j}Y^{(d-e)-(i-j)}, & j \leq i, \ e-j \leq d-i, \\ 0, & \text{otherwise}. \end{cases}
\]
For the \((e-1)\)-th transvectant \( T^{(e-1)} : V_d \times V_e \to V_{d+e+2} \), we have
\[
T^{(e-1)}(X^i Y^{d-i}, X^{e-j} Y^j) = (-1)^{e-j-1} \frac{(d-e+1)!}{d!} \left\{ jY \partial_{X}^{i-j} \partial_{Y}^{i-j} - (e-j)X \partial_{X}^{i-j} \partial_{Y}^{i-j} \right\},
\]
where \( \partial_{X}^{i-j} = \partial_{Y}^{i-j} = 0 \) by convention. Therefore
\[
T^{(e-1)}(X^i Y^{d-i}, X^{e-j} Y^j) = \begin{cases} AX^{i+1}Y^{d-i-(e-j)+1}, & j \leq i + 1, \ e-j \leq d-i+1, \\ 0, & \text{otherwise}, \end{cases}
\]
where
\[
A = (-1)^{e-j}(d-i)! \binom{d-e+2}{i-j+1} \frac{j(d+2)-(i+1)e}{e(d-e+2)}.
\]
We stress in particular that

**Lemma 2.1.** Let \( 0 \leq j \leq i + 1 \) and \( 0 \leq e-j \leq d-i+1 \). The bilinear map
\[
T^{(e-1)} : \mathbb{C}X^i Y^{d-i} \times \mathbb{C}X^{e-j} Y^j \to \mathbb{C}X^{i+1}Y^{d-i-(e-j)+1}
\]
is non-degenerate if and only if \( j(d+2) \neq (i+1)e \). This is always the case when \( d+2 \) is coprime to \( e \).

Now we consider \( SL_2 \times SL_2 \)-representations. The space \( V_{a,b} = H^0(O_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)) \) is the tensor representation \( V_d \otimes V_b \). Substituting (2.1) into
\[
V_{a,b} \otimes V_{a',b'} = (V_d \otimes V_{a'}) \otimes (V_b \otimes V_{b'}),
\]
we obtain the Clebsch-Gordan decomposition for \( SL_2 \times SL_2 \),
\[
V_{a,b} \otimes V_{a',b'} = \bigoplus_{r,s} V_{a+a'-2r, b+b'-2s},
\]
where \( 0 \leq r \leq \min\{a, a'\} \) and \( 0 \leq s \leq \min\{b, b'\} \). To each irreducible summand \( V_{a+a'-2r, b+b'-2s} \) is associated the \((r, s)\)-th bi-transvectant
\[
T^{(r,s)} : V_{a,b} \times V_{a',b'} \to V_{a+a'-2r, b+b'-2s}.
\]
This $\SL_2 \times \SL_2$-bilinear mapping is calculated from the transvectants by
\[ T^{(r,s)}(F \boxtimes G, F' \boxtimes G') = T^{(r)}(F, F') \boxtimes T^{(s)}(G, G'), \]
where $F \in V_a, G \in V_b, F' \in V_{a'}$, and $G' \in V_{b'}$.

2.2. The method of double bundle. In §4 we will use the method of double bundle (21) and its generalization (11). We here give some account in the present situation. The strategy is to find a certain bi-transvectant which introduces on the target $\PP V_{a,b}$ a fibration structure over a Grassmannian, and then reduce the rationality of $\PP V_{a,b}/\SL_2 \times \SL_2$ to a stable rationality of the quotient of the latter.

Suppose we have a bi-transvectant
\[ T = T^{(r,s)} : V_{a,b} \times V_{a',b'} \to V_{a'',b''} \]
such that $c := \dim V_{a',b'} - \dim V_{a'',b''}$ is positive and that $\dim V_{a,b} > c \cdot \dim V_{a',b''}$. This bilinear map induces an $\SL_2 \times \SL_2$-linear embedding
\[ V_{a,b} \subset \text{Hom}(V_{a',b'}, V_{a'',b''}). \]
The space $\text{Hom}(V_{a',b'}, V_{a'',b''})$ is birationally fibered over $G(c, V_{a',b'})$, by sending a surjective linear map to its kernel. We can thus consider an $\SL_2 \times \SL_2$-equivariant rational map
\[ \varphi : V_{a,b} \dashrightarrow G(c, V_{a',b'}), \quad v \mapsto \text{Ker}(T(v, \cdot)). \]
We assume (hope) that
\[ (\dagger) \quad \varphi \text{ is defined on a non-empty locus, and is dominant.} \]
This means that the position of $V_{a,b}$ inside $\text{Hom}(V_{a',b'}, V_{a'',b''})$ is "non-degenerate" with regards to the fibration over $G(c, V_{a',b'})$. The inequality $\dim V_{a,b} > c \cdot \dim V_{a',b''}$ above is the dimension condition necessary for the dominance of $\varphi$ to be possible. If $\text{(\dagger)}$ holds, then $V_{a,b}$ becomes birational to the unique component $E$ of the incidence
\[ X = \{(v, P) \in V_{a,b} \times G(c, V_{a',b'}), \quad T(v, P) \equiv 0\} \]
that dominates $G(c, V_{a',b'})$. Indeed, the first projection $\pi : X \to V_{a,b}$ is isomorphic over the domain $U$ of regularity of $\varphi$, and then the dominance of $\varphi$ implies that $\pi^{-1}(U)$ is contained in $E$. Since $E$ is (generically) a sub vector bundle of $V_{a,b} \times G(c, V_{a',b'})$ preserved under the $\SL_2 \times \SL_2$-action, it is an $\SL_2 \times \SL_2$-linearized vector bundle over $G(c, V_{a',b'})$. We shall then try to apply the following no-name lemma (cf. [5]).

Lemma 2.2 (no-name lemma). Let $G$ be an algebraic group and $E \to X$ a $G$-linearized vector bundle of rank $N + 1$. Suppose that $G$ acts on $X$ almost freely. Then
\[ \PP E/G \sim \PP^N \times (X/G). \]

In the present situation, however, $\SL_2 \times \SL_2$ never acts on $G(c, V_{a',b'})$ almost freely because of the presence of $(\pm 1, \pm 1) \in \SL_2 \times \SL_2$. So we should take $G = \PGL_2 \times \PGL_2$, whose action on $G(c, V_{a',b'})$ is now almost free in most cases, but then the $\SL_2 \times \SL_2$-linearization on $E$ may not descend to that of $G$. To deal with
this problem, we want to tensor $\mathcal{E}$ with an $\SL_2 \times \SL_2$-linearized line bundle $\mathcal{L}$ that kills the action of $(\pm 1, \mp 1)$ on $\mathcal{E}$. If this was successful, we would have
\[(2.3) \quad \mathbb{P}\mathcal{E}/G = \mathbb{P}(\mathcal{E} \otimes \mathcal{L})/G \sim \mathbb{P}^N \times (G(c, V_{a', b'})/G)\]
where $N = \dim \mathbb{P}V_{a,b} - \dim G(c, V_{a', b'})$. Thus the rationality of $\mathbb{P}V_{a,b}/G$ could be reduced to a stable rationality of $G(c, V_{a', b'})/G$, which is much easier to prove: we prepare results of this sort in the next §3.

In practice, we will check the non-degeneracy requirement (4) as follows.

**Lemma 2.3 (cf. [2]).** The condition (4) is satisfied if and only if there exist vectors $v \in V_{a,b}$ and $w_1, \cdots, w_c \in V_{a', b'}$ such that
(i) $w_1, \cdots, w_c$ are linearly independent,
(ii) $T(v, w_i) = 0$ for every $w_i$,
(iii) the map $T(v, \cdot) : V_{a', b'} \to V_{a', b''}$ is surjective, and
(iv) the map $T(v, w_1) \cdots, T(v, w_c)) : V_{a,b} \to V_{a', b''}$ is surjective.

**Proof.** Let $P \in G(c, V_{a', b'})$ be the span of $w_1, \cdots, w_c$. The conditions (ii) and (iii) mean that $v$ is contained in the domain $U$ of regularity of $\varphi$ with $\varphi(v) = P$, whence $U \neq \emptyset$. Then (iv) implies that the fiber of the morphism $\varphi : U \to G(c, V_{a', b'})$ over $P$ has the expected dimension $\dim V_{a,b} - \dim G(c, V_{a', b'})$. Hence $\varphi(U)$ has dimension $\geq \dim G(c, V_{a', b'})$, and so $\varphi$ is dominant. \hfill $\square$

3. SOME STABLE RATIONALITY

A variety $X$ is said to be **stably rational of level $N$** if $X \times \mathbb{P}^N$ is rational. In this section we prepare stable rationality results for some quotients of Grassmannians, to which the proof of Theorem 1.1 will be finally reduced. We set $\overline{G} = \SL_2 \times \SL_2/(-1, -1)$. When $a, b \geq 0$ are odd, the element $(-1, -1)$ of $\SL_2 \times \SL_2$ acts on $V_{a,b}$ trivially so that $\overline{G}$ acts on $V_{a,b}$. This linear $\overline{G}$-action is almost free if $\PGL_2 \times \PGL_2$ acts on $\mathbb{P}V_{a,b}$ almost freely, that is, general bidegree $(a, b)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ have no non-trivial stabilizer.

**Lemma 3.1.** The group $\overline{G}$ acts on $V_{1,1}^{\mathbb{R}^3}/\SL_2 \times \SL_2$ almost freely with the quotient $V_{1,1}^{\mathbb{R}^3}/\SL_2 \times \SL_2$ rational.

**Proof.** The first assertion follows from the almost freeness of the $\PGL_2 \times \PGL_2$-action on $(\mathbb{P}V_{1,1})^3$. For the second assertion, we first note that
$$V_{1,1}^{\mathbb{R}^3}/\SL_2 \times \SL_2 \sim (V_{1,1}^{\mathbb{R}^3}/\PGL_2 \times \PGL_2) \times \mathbb{C}^\infty.$$ The group $\PGL_2 \times \PGL_2$ acts on $V_{1,1}$ almost transitively with the stabilizer of a general point isomorphic to $\PGL_2$ (identify $V_{1,1}$ with $\Hom(V_1, V_1)$). Hence, applying the slice method (cf. [5]) to the first projection $V_{1,1}^{\mathbb{R}^3} \to V_{1,1}$, we obtain
$$V_{1,1}^{\mathbb{R}^3}/\PGL_2 \times \PGL_2 \sim V_{1,1}^{\mathbb{R}^2}/\PGL_2,$$
where $\PGL_2$ acts on $V_{1,1}^{\mathbb{R}^2}$ linearly in the right hand side. Then the quotient $V_{1,1}^{\mathbb{R}^2}/\PGL_2$ is rational by the result of Katsylo [7]. \hfill $\square$
Corollary 3.2. Let \( n > 0 \) be an odd number. Then \( \mathbb{P}V_{1,n}/\mathbb{SL}_2 \times \mathbb{SL}_2 \) and \( \mathbb{P}V_{3,n}/\mathbb{SL}_2 \times \mathbb{SL}_2 \) are stably rational of level 13.

Proof: We treat the case of \( V_{1,n} \). For dimensional reason we may assume \( n > 3 \). Then the group \( G \) acts on \( V_{1,n} \) almost freely. Hence we may apply the no-name lemma to both projections \( V_{1,1}^{3} \oplus V_{1,n} \rightarrow V_{1,n} \) and \( V_{1,1}^{3} \oplus V_{1,n} \rightarrow V_{1,1}^{3} \) to see that

\[
(V_{1,n}/\mathbb{SL}_2 \times \mathbb{SL}_2) \times \mathbb{C}^{12} \sim (V_{1,1}^{3}/\mathbb{SL}_2 \times \mathbb{SL}_2) \times \mathbb{C}^{2n+2}.
\]

By Lemma 3.1 \( V_{1,n}/\mathbb{SL}_2 \times \mathbb{SL}_2 \) is stably rational of level 12. Since \( V_{1,n}/\mathbb{SL}_2 \times \mathbb{SL}_2 \) is birational to \( \mathbb{C}^{x} \times (\mathbb{P}V_{1,n}/\mathbb{SL}_2 \times \mathbb{SL}_2) \), our assertion is proved. The case of \( V_{3,n} \) is similar: just replace \( V_{1,n} \) by \( V_{3,n} \) in this argument, now with \( n > 1 \). The only change is that the factor \( \mathbb{C}^{2n+2} \) in (3.1) is replaced by \( \mathbb{C}^{4n+4} \).

Proposition 3.3. When \( n > 1 \) is odd, \( G(3,V_{3,n})/\mathbb{SL}_2 \times \mathbb{SL}_2 \) is stably rational of level 2.

Proof: Let \( \mathcal{F} \rightarrow G(3,V_{3,n}) \) be the universal sub vector bundle of rank 3, on which \( \mathbb{SL}_2 \times \mathbb{SL}_2 \) acts equivariantly. The elements \( (\pm 1, \mp 1) \in \mathbb{SL}_2 \times \mathbb{SL}_2 \) act on \( \mathcal{F} \) by multiplication by \( -1 \). Since \( \mathcal{F} \) has odd rank, they act on the line bundle \( \text{det}\mathcal{F} \) also by \( -1 \). Hence the bundle \( \mathcal{F}' = \mathcal{F} \otimes \text{det}\mathcal{F} \) is \( \mathbb{P}G(2) \times \mathbb{P}G(2) \)-linearized. Note that \( \mathbb{P}\mathcal{F} \) is canonically identified with \( \mathbb{P}\mathcal{F}' \). Since \( \mathbb{P}G(2) \times \mathbb{P}G(2) \) acts on \( G(3,V_{3,n}) \) almost freely, we can apply the no-name lemma to \( \mathcal{F}' \) to see that

\[
\mathbb{P}\mathcal{F}/\mathbb{SL}_2 \times \mathbb{SL}_2 \sim \mathbb{P}\mathcal{F}'/\mathbb{SL}_2 \times \mathbb{SL}_2 \sim \mathbb{P}^2 \times (G(3,V_{3,n})/\mathbb{SL}_2 \times \mathbb{SL}_2).
\]

Thus it suffices to show that \( \mathbb{P}\mathcal{F}/\mathbb{SL}_2 \times \mathbb{SL}_2 \) is rational.

Regarding \( \mathbb{P}\mathcal{F} \) as an incidence in \( G(3,V_{3,n}) \times \mathbb{P}V_{3,n} \), we have second projection \( \mathbb{P}\mathcal{F} \rightarrow \mathbb{P}V_{3,n} \). Its fiber over \( Cl \in \mathbb{P}V_{3,n} \) is the sub Grassmannian in \( G(3,V_{3,n}) \) of 3-planes containing \( Cl \), and hence identified with \( G(2,V_{3,n}/Cl) \). Therefore, if \( \mathcal{G} \rightarrow \mathbb{P}V_{3,n} \) is the universal quotient bundle of rank \( \dim V_{3,n} - 1 \), then \( \mathbb{P}\mathcal{G} \) is identified with the relative Grassmannian \( G(2,\mathcal{G}) \). The elements \( (\pm 1, \mp 1) \in \mathbb{SL}_2 \times \mathbb{SL}_2 \) act on \( \mathcal{G} \) by multiplication by \( -1 \), and also on \( O_{\mathbb{P}V_{3,n}}(1) \) by \( -1 \). Thus the bundle \( \mathcal{G}' = \mathcal{G} \otimes O_{\mathbb{P}V_{3,n}}(1) \) is \( \mathbb{P}G(2) \times \mathbb{P}G(2) \)-linearized, and \( G(2,\mathcal{G}) \) is canonically isomorphic to \( G(2,\mathcal{G}') \). Since \( \mathbb{P}G(2) \times \mathbb{P}G(2) \) acts on \( \mathbb{P}V_{3,n} \) almost freely, we can use the no-name lemma to trivialize the \( \mathbb{P}G(2) \times \mathbb{P}G(2) \)-bundle \( \mathcal{G}' \) locally in the Zariski topology. Hence we have

\[
G(2,\mathcal{G}')/\mathbb{SL}_2 \times \mathbb{SL}_2 \sim G(2,\mathbb{C}^{4n+3}) \times (\mathbb{P}V_{3,n}/\mathbb{SL}_2 \times \mathbb{SL}_2).
\]

Since \( \dim G(2,\mathbb{C}^{4n+3}) > 13 \) for \( n > 1 \), our assertion follows from Corollary 3.2.

We also treat \( G(3,V_{3,1}) \) which is excluded above.

Proposition 3.4. The quotient \( G(3,V_{3,1})/\mathbb{SL}_2 \times \mathbb{SL}_2 \) is stably rational of level 5.

Proof: In this case the \( \mathbb{P}G(2) \times \mathbb{P}G(2) \)-action on \( \mathbb{P}V_{3,1} \) is not almost free, having the Klein 4-group as a general stabilizer, so that we cannot apply the above proof. But the following modification will work: replace \( \mathcal{F} \) with \( \mathcal{F} \odot 2 \), and the projection \( \mathbb{P}\mathcal{F} \rightarrow \mathbb{P}V_{3,1} \) with

\[
\mathbb{P}(\mathcal{F} \odot 2) \rightarrow \mathbb{P}(V_{3,1}^{\odot 2}), \quad (P, (v_1, v_2)) \mapsto (v_1, v_2),
\]
Thus it suffices to prove that \( \mathbb{P}(V_{3,1}^\oplus)/\text{SL}_2 \times \text{SL}_2 \) is stably rational of level 5.

Consider the representation \( W = V_{1,1} \oplus V_{3,1}^\oplus \). We apply the no-name lemma to both projections \( \mathbb{P}W \to \mathbb{P}(V_{3,1}^\oplus) \) and \( \mathbb{P}W \to \mathbb{P}(V_{1,1} \oplus V_{3,1}) \) to see that

\[
\mathbb{C}^4 \times (\mathbb{P}(V_{3,1}^\oplus)/\text{SL}_2 \times \text{SL}_2) \sim \mathbb{C}^8 \times (\mathbb{P}(V_{1,1} \oplus V_{3,1})/\text{SL}_2 \times \text{SL}_2).
\]

Using the slice method for the projection \( V_{1,1} \oplus V_{3,1} \to V_{1,1} \), we then have

\[
(V_{1,1} \oplus V_{3,1})/\text{GL}_2 \times \text{GL}_2 \sim V_{3,1}/\text{GL}_2.
\]

Finally, \( V_{3,1}/\text{GL}_2 \) is rational by Katsylo \([7]\).

\[\square\]

4. Proof of Theorem 1.1

Let \( b \geq 5 \) be an odd number. In this section we prove that \( \mathbb{P}V_{3,b}/\text{SL}_2 \times \text{SL}_2 \) is rational (Theorem 1.1) by executing the method of double bundle explained in \S 2.2. In logical order, the proof proceeds in the following line.

(1) We choose a bi-transvectant

\[
T = T^{(r,s)} : V_{3,b} \times V_{a',b'} \to V_{a'',b''}
\]

according to Table 1 below. This satisfies that \( c := \dim V_{a',b'} - \dim V_{a'',b''} \) is either 1 or 3, \( \dim V_{3,b} > c \cdot \dim V_{a'',b''} \), and that both \( a' \) and \( b' \) are odd.

(2) We check that \( T \) satisfies the non-degeneracy condition \((4)\) by finding vectors \( v \in V_{3,b}, w_1, \ldots, w_c \in V_{a',b'} \) as in Lemma 2.3.

(3) Then, as shown in \S 2.2, \( V_{3,b} \) gets birationally realized as an \( \text{SL}_2 \times \text{SL}_2 \)-linearized vector bundle \( E \) over \( G(c,V_{a',b'}) \) which is a sub bundle of \( V_{3,b} \times G(c,V_{a',b'}) \). (In case \( c = 1 \), \( G(c,V_{a',b'}) \) is just \( \mathbb{P}V_{a',b'} \).)

(4) Since 3 and \( b \) are odd, the elements \((\pm 1, \mp 1)\) in \( \text{SL}_2 \times \text{SL}_2 \) act on \( E \) by multiplication by \(-1\).

(5) Since \( a' \) and \( b' \) are odd, \((\pm 1, \mp 1)\) act on the universal sub bundle \( F \) over \( G(c,V_{a',b'}) \) also by \(-1\). Since \( F \) has odd rank \((= c)\), \((\pm 1, \mp 1)\) act on \( \det F \) by \(-1\). Hence \( E \otimes \det F \) is \( \text{PGL}_2 \times \text{PGL}_2 \)-linearized.

(6) It is not difficult to see that \( \text{PGL}_2 \times \text{PGL}_2 \) acts on \( G(c,V_{a',b'}) \) almost freely. Then by the no-name lemma we have

\[
\mathbb{P}E/\text{PGL}_2 \times \text{PGL}_2 \sim \mathbb{P}^N \times (G(c,V_{a',b'})/\text{PGL}_2 \times \text{PGL}_2)
\]

as explained in \((2,3)\), where \( N = \dim V_{3,b} - \dim G(c,V_{a',b'}) \).

(7) The quotient \( G(c,V_{a',b'})/\text{SL}_2 \times \text{SL}_2 \) is stably rational of level \( \leq N \) by Corollary 3.2 and Propositions 3.3 and 3.4 (see the values of \( c, (a', b') \), \( N \) below.) This concludes that \( \mathbb{P}V_{3,b}/\text{SL}_2 \times \text{SL}_2 \) is rational.

The bi-transvectant \( T^{(r,s)} \) is provided systematically according to the remainder \([b] \in \mathbb{Z}/5\mathbb{Z}\), except the case \( b = 7 \).
Proposition 4.1. For odd \( b \geq 5 \) we set the values of \( (r, s) \) and \((a', b') \) (and hence \((a'', b'') \), \( c \) and \( N \)) by the following Table [7] Here \( n \) is even when \( b \equiv 1,3 \) (5), odd

| \( b \) | \((r, s)\) | \((a', b')\) | \((a'', b'')\) | \( c \) | \( N \) |
|-------|---------|---------|---------|--------|--------|
| \( 5n \) | \((3, n)\) | \((3, n)\) | \((0, 4n)\) | 3      | 8n     |
| \( 5n + 1 \) | \((1, 3n + 1)\) | \((1, 3n + 1)\) | \((2, 2n)\) | 1      | 14n + 4|
| \( 5n + 2 \) | \((3, n)\) | \((3, n)\) | \((0, 4n + 2)\) | 1      | 16n + 8|
| \( 5n + 3 \) | \((3, n)\) | \((3, n + 1)\) | \((0, 4n + 4)\) | 3      | 8n     |
| \( 5n + 4 \) | \((1, 3n + 3)\) | \((1, 3n + 4)\) | \((2, 2n + 2)\) | 1      | 14n + 10|
| \( 7 \) | \((2, 3)\) | \((3, 3)\) | \((2, 4)\) | 1      | 16     |

when \( b \equiv 0, 2, 4 \) (5), and \( n > 1 \) when \( b \equiv 2 \) (5). Then the above argument (1), ..., (7) works.

Notice that we have to separate the case \( b = 7 \) because the \( \text{PGL}_2 \times \text{PGL}_2 \) action on \( G(c, V_{a', b'}) = \mathbb{P}V_{3, 1} \) is not almost free, so that the step (6) would not work with \( (r, s) = (a', b') = (3, 1) \).

For the proof of Proposition 4.1 we are now only left with the step (2) to fill out. In the remainder of the article we choose vectors \( v \in V_{3, b} \) and \( w_1, \cdots, w_c \in V_{a', b'} \) that should satisfy the conditions (i), ..., (iv) of Lemma 2.3. In any case the equality \( T(v, w_i) = 0 \) (the condition (ii)) can be checked with a direct calculation using the formula of \( T = T^{(c, s)} \) given in [2, 3]. We leave this to the reader. The linear independence of \( w_1, \cdots, w_c \) (the condition (i)) can be seen at a glance, and we also omit it. Note that this is even trivial when \( c = 1 \). Thus what we are going to verify below is the surjectivity conditions (iii) and (iv).

We shall use the notation \( ([x, y], [X, Y]) \) for the bi-homogeneous coordinate of \( \mathbb{P}^1 \times \mathbb{P}^1 \). Thus elements of \( V_{a, b} \) will be expressed as

\[
\sum_i F_i(x, y)G_i(X, Y),
\]

where \( F_i, G_i \) are binary forms of degree \( a, b \) respectively.

4.1. The case \( b \equiv 0 \) (5). We take vectors \( v \in V_{3, 5n}, w = (w_1, w_2, w_3) \in (V_{3, n})^3 \) by

\[
v = \binom{5n}{n}X^ny^{4n}x^3 + 3\binom{5n}{2n}X^{2n}y^{3n}x^2y + 3\binom{5n}{2n}X^{3n}y^{2n}xy^2 + \binom{5n}{n}X^{4n}y^{n}x^3,
\]

\[
w_1 = Y^n x^3 - X^n x^2 y,
\]

\[
w_2 = Y^n x^2 y - X^n xy^2,
\]

\[
w_3 = Y^n xy^2 - X^n y^3.
\]

The map \( T(v, \cdot) \) : \( V_{3, n} \rightarrow V_{0, 4n} \) is surjective because

\[
T(v, V_n x^3) = \mathbb{C}(X^{4n}, \cdots, X^{3n} Y^n), \quad T(v, V_n x^2 y) = \mathbb{C}(X^{3n} Y^n, \cdots, X^{2n} Y^{2n}),
\]

\[
T(v, V_n xy^2) = \mathbb{C}(X^{2n} Y^{2n}, \cdots, X^n Y^{3n}), \quad T(v, V_n y^3) = \mathbb{C}(X^n Y^{3n}, \cdots, Y^{4n}).
\]
To see the surjectivity of \( T(\cdot, \vec{w}) : V_{3,5n} \to V_{0,4n}^{\oplus 3} \) we note that
\[
T(V_{5n}x^3 \oplus V_{5n}y^3, \vec{w}) = (V_{0,4n}, 0, V_{0,4n}) \subseteq V_{0,4n}^{\oplus 3}.
\]
Since \( T(V_{5n}x^2y, w_2) = V_{0,4n} \), then \( (0, V_{0,4n}, 0) \subseteq V_{0,4n}^{\oplus 3} \) is also contained in the image of \( T(\cdot, \vec{w}) \).

4.2. The case \( b = 1 \) (5). We take the following vectors of \( V_{3,5n+1} \) and \( V_{1,3n+1} \):

\[
v = \left( \frac{5n+1}{2n} \right) x^{3n+1} y^{2n} x^3 + 3 \left( \frac{5n+1}{n} \right) x^{4n+1} y^n x^2 y
\]
\[
+w = (x^{3n+1} - y^{3n+1}) x - (x^n y^{2n+1} - x^{2n+1} y^n)y.
\]

We shall prove the surjectivity of \( T(v, \cdot) : V_{1,3n+1} \to V_{2,2n} \) by showing that its kernel is 1-dimensional. Suppose we have a vector \( w' = G_+(X, Y)x + G_-(X, Y)y \) in \( V_{1,3n+1} \) with \( T(v, w') = 0 \). Then we have

\[
T(3n+1)(X^n Y^{4n+1}, G_+) = b_0 T(3n+1)(X^{2n} Y^{3n+1}, G_-),
\]
\[
T(3n+1)(X^{2n} Y^{3n+1}, G_+), = b_1 T(3n+1)(X^{4n+1} Y^n, G_-),
\]
\[
T(3n+1)(X^{4n+1} Y^n, G_+), = b_2 T(3n+1)(X^{3n+1} Y^{2n}, G_-).
\]

for suitable constants \( b_i \). Expanding \( G_{\pm} = \sum_i \alpha_i^{\pm} X^{3n+1-i} Y^{i} \), we obtain

\[
\alpha_i^+ = c_1 \alpha_i^- (0 \leq i \leq n), \quad \alpha_i^- = 0 \quad (0 \leq i \leq n - 1),
\]
\[
\alpha_i^+ = c_2 \alpha_i^- (0 \leq i \leq n), \quad \alpha_i^- = 0 \quad (n + 1 \leq i \leq 2n),
\]
\[
\alpha_i^+ = c_3 \alpha_i^- (n + 1 \leq i \leq 2n + 1), \quad \alpha_i^- = 0 \quad (2n + 2 \leq i \leq 3n + 1),
\]

for some fixed constants \( c_i \). This reduces to the relations

\[
\alpha_0^+ = d_1 \alpha_{3n+1}^+ = d_2 \alpha_n^+ = d_3 \alpha_{2n+1}^-
\]

where \( d_i \) are appropriate constants, and \( \alpha_i^+ = 0 \) for other \( i \). This implies our assertion.

The surjectivity of \( T(\cdot, \vec{w}) : V_{3,5n+1} \to V_{2,2n} \) can be seen by noticing that

\[
T(V_{5n+1} x^3, w) = V_{2n} y^2, \quad T(V_{5n+1} x^3, w) = V_{2n} x^2,
\]
\[
T(V_{5n+1} x^2 y, (X^{3n+1} - Y^{3n+1}) x) = V_{2n} xy.
\]

4.3. The case \( b = 2 \) (5). We take vectors in \( V_{3,5n+2} \) and \( V_{3,n} \) by

\[
v = x^n y^{4n+2} x^3 + X^{2n+1} y^{3n+1} x^2 y + X^{3n+1} y^{2n+1} x^2 y^2 + X^{4n+2} y^n x^3,
\]
\[
w = Y^n x^2 y - X^n xy^2.
\]

The map \( T(v, \cdot) : V_{3,n} \to V_{0,4n+2} \) is surjective because

\[
T(v, V_n x^3) = \mathbb{C}(X^{4n+2}, \ldots, X^{3n+2} y^n),
\]
\[
T(v, V_n x^2 y) = \mathbb{C}(X^{3n+1} y^{n+1}, \ldots, X^{2n+1} y^{2n+1}),
\]
\[
T(v, V_n y x^2) = \mathbb{C}(X^{2n+1} y^{2n+1}, \ldots, X^{n+1} y^{3n+1}),
\]
\[
T(v, V_n^3) = \mathbb{C}(X^n Y^{3n+2}, \ldots, Y^{4n+2}).
\]
On the other hand, we have \( T(V_{5n+2} x^2, w) = V_{0,4n+2} \) so that the map \( T(\cdot, w) : V_{3,5n+2} \to V_{0,4n+2} \) is also surjective.

4.4. The case \( b \equiv 3 \pmod{5} \). We take the following vectors of \( V_{3,5n+3} \) and \( V_{3,n+1} \) according to the remainder of \( n \) modulo 5:

1. When \( n \not\equiv 4 \pmod{5} \), we set
   \[
   v = \left( \frac{5n+3}{n} \right) x^n y^{4n+3} x^3 + \left( \frac{5n+3}{2n+1} \right) x^{2n+1} y^{3n+2} x^2 y \\
   + \left( \frac{5n+3}{2n+1} \right) x^{3n+2} y^{2n+1} x y^2 + \left( \frac{5n+3}{n} \right) x^{4n+3} y^{n+3},
   \]
   \[
   w_1 = X^{n+1} y^3 + Y^{n+1} x y^2,
   \]
   \[
   w_2 = X^{n+1} x y^2 + Y^{n+1} x^2 y,
   \]
   \[
   w_3 = X^{n+1} x^2 y + Y^{n+1} x^3.
   \]

2. When \( n \equiv 4 \pmod{5} \), we denote \( n = 2m \) (remember \( n \) is even) and set
   \[
   v = \left\{ \begin{array}{c}
   \left( \frac{5n+3}{m+1} \right) x^m y^{9m+3} + X^9 y^{m-2} \\
   \left( \frac{5n+3}{3m+2} \right) X^{3m+1} y^{7m+2} x^2 y + \left( \frac{5n+3}{5m+2} \right) x^{5m+2} y^{5m+1} x y^2 \\
   \left( \frac{5n+3}{3m+1} \right) X^{7m+3} y^{3m+3},
   \end{array} \right.
   \]
and use the same \( w_i \) as above.

When \( n \not\equiv 4 \pmod{5} \), we have no \( 0 \leq j \leq n+1 \) with \( j(5n+5) = (i+1)(n+1) \) for \( i = n, 2n+1, 3n+2, 4n+3 \). Hence by Lemma \([2.1] \) for those \( i \) the bilinear map

\[
T^{(n)} : \mathbb{C} X^i Y^{5n+3-i} \times \mathbb{C} X^{n+1-j} Y^j \to \mathbb{C} X^{i-j+1} Y^{4n+3-i+j}
\]

is non-degenerate for any \( j \), as far as the indices are non-negative. It follows that

\[
T(v, V_{n+1} x^3) = \mathbb{C} (X^{4n+4}, \ldots, X^{3n+3} Y^{n+1}),
\]

\[
T(v, V_{n+1} x^2 y) = \mathbb{C} (X^{3n+3} y^{n+1}, \ldots, x^{2n+2} y^{2n+2}),
\]

\[
T(v, V_{n+1} x y^2) = \mathbb{C} (X^{2n+2} y^{2n+2}, \ldots, x^{n+1} Y^{3n+3}),
\]

\[
T(v, V_{n+1} y^3) = \mathbb{C} (X^{n+1} Y^{3n+3}, \ldots, y^{4n+4}),
\]
whence the map \( T(v, \cdot) : V_{3,n+1} \to V_{0,4n+4} \) is surjective. We leave it to the reader to check similar surjectivity when \( n \equiv 4 \pmod{5} \). In that case, since \( m \equiv 2 \pmod{5} \), we have no \( j \) with \( j(5n+5) = (i+1)(n+1) \) for \( i = m + k(n+1), 0 \leq k \leq 3, \) and \( i = 9m+5 \). Hence for those \( i \) the map \((4.1)\) is non-degenerate for any relevant \( j \), again by Lemma \([2.1] \).

To see that

\[
T(\cdot, w) = (T(\cdot, w_1), T(\cdot, w_2), T(\cdot, w_3)) : V_{3,5n+3} \to V_{0,4n+4} \]
is surjective (regardless of \( [n] \in \mathbb{Z}/5\mathbb{Z} \)), we note that the bilinear maps

\[
T^{(n)}(\cdot, X^{n+1}) : \mathbb{C} X^i Y^{5n+3-i} \to \mathbb{C} X^{i+1} Y^{4n+3-i}
\]
are non-degenerate whenever the indices are non-negative. It follows that

\[ T(V^{n+3}x, w) = (\mathbb{C}(Y^{n+4}, \ldots, XY^{n+3}), 0, 0), \]

so that \( (V_{0,4n+4}, 0, 0) \subset V^{3n}_{0,4n+4} \) is contained in the image of \( T(\cdot, \vec{w}) \). Similarly, we see that \( (0, 0, V_{0,4n+4}) \subset V^{3n}_{0,4n+4} \) is contained in the image too. Finally, since \( T(\cdot, w_2) \) maps the space \( V_{5n+3}Y^3y \oplus V_{n+3}xy^2 \) onto \( V_{0,4n+4} \), we find using the above results that \( (0, V_{0,4n+4}, 0) \) is also contained in the image.

4.5. The case \( b = 4 \) (5). We take the following vectors of \( V_{1,5n+4} \) and \( V_{1,3n+4} \):

\[
\begin{align*}
v = \frac{3n + 4}{n + 2} & \frac{3n + 4}{n + 1} \frac{(5n + 4)}{2n + 1}(X^{3n+3}Y^{2n+1}x^3 + 3 \frac{3n + 4}{n + 1}X^{n+4}Y^n x^2 y \\
-3(5n + 4) & \left( 2n + 1 \right)^2 X^{2n+1} Y^{3n+3} x^2 y - \frac{n + 2}{3n + 4} \frac{(5n + 4)}{n} X^n Y^{4n+3} y^3,
\end{align*}
\]

\[
\begin{align*}
w = (X^{3n+4} + Y^{3n+4})x + (X^{3n+3}Y^{n+1} + X^{n+1}Y^{2n+3})y.
\end{align*}
\]

We shall show that the kernel of \( T(v, \cdot) : V_{1,3n+4} \rightarrow V_{2,2n+2} \) is 1-dimensional, which then implies its surjectivity. We first note that \( 5n + 6 \) and \( 3n + 4 \) are coprime by the Euclidean algorithm. By Lemma [2.1], the bilinear map

\[
T(3n+3) : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}
\]

is non-degenerate whenever the indices are non-negative. Now suppose a vector \( w' = G_+ (X, Y)x + G_- (X, Y)y \) in \( V_{1,3n+4} \) satisfies \( T(v, w') = 0 \). This is rewritten as

\[
\begin{align*}
T(3n+3)(X^{3n+3}Y^{2n+1}, G_-) &= b_0 T(3n+3)(X^{n+4}Y^n, G_+), \\
T(3n+3)(X^{n+4}Y^n, G_-) &= b_1 T(3n+3)(X^{2n+1}Y^{3n+3}, G_+), \\
T(3n+3)(X^{2n+1}Y^{3n+3}, G_-) &= b_2 T(3n+3)(X^nY^{4n+4}, G_+),
\end{align*}
\]

for some constants \( b_j \). Expanding \( G_\pm (X, Y) = \sum_{j=0}^{3n+4} \alpha^\pm_j X^{3n+4-j}Y^j \), we obtain the relation

\[
\begin{align*}
\alpha^+_j &= c_1 \alpha^-_{n+1} (n + 2 \leq j \leq 2n + 3), & \alpha^-_j &= 0 \ (2n + 4 \leq j \leq 3n + 4), \\
\alpha^+_j &= c_2 \alpha^-_{j+2n+3} \ (0 \leq j \leq n + 1), & \alpha^-_j &= 0 \ (n + 2 \leq j \leq 2n + 2), \\
\alpha^+_j &= c_3 \alpha^-_{j+n+1} \ (0 \leq j \leq n + 1), & \alpha^-_j &= 0 \ (0 \leq j \leq n),
\end{align*}
\]

where \( c_\pm \) are suitable non-zero constants. This is reduced to the relations

\[
\alpha^+_0 = d_1 \alpha^-_1 = d_2 \alpha^-_{2n+3} = d_3 \alpha^+_3
\]

for some constants \( d_j \), and \( \alpha^+_i = 0 \) for other \( i \). This proves our claim.

On the other hand, the surjectivity of \( T(\cdot, w) : V_{3,5n+4} \rightarrow V_{2,2n+2} \) follows by noticing that

\[
\begin{align*}
T(V_{5n+4}x, w) &= V_{2n+2}x^2, & T(V_{5n+4}y, w) &= V_{2n+2}y^2, \\
T(V_{5n+4}x^2, (X^{3n+4} + Y^{3n+4})x) &= V_{2n+2}xy.
\end{align*}
\]
4.6. **The case** $b = 7$. We choose the following vectors of $V_{3, 7}$ and $V_{3, 3}$:

$$
v = \left( \frac{7}{3} \right) x^3 y^4 x^3 - 9 Y^7 x^2 y + \left( \frac{7}{1} \right) x^6 y x^2 + \left( \frac{7}{3} \right) x^4 y^3 x^3,
\quad w = Y^3 x^3 + X^3 y^2 + (XY^2 + Y^3) y^3.
$$

We leave it to the reader to check that $w$ spans the kernel of $T(v, \cdot) : V_{3, 3} \to V_{2, 4}$ (cf. §4.2 and §4.5). We shall show that $T(\cdot, w) : V_{3, 7} \to V_{2, 4}$ is surjective too. First note that the bilinear map

$$T^{(2)} : \mathbb{C} x^3 y^{3-i} \times \mathbb{C} x^3 y^i \to \mathbb{C} x^{j-i+1} y^{j-i+1}$$

is non-degenerate whenever the indices are non-negative, for 3 and 5 are coprime (Lemma 2.1). Then we have

$$T(V_3 y^3, w) = T(V_3 x^3, y^3 x^3) = V_4 x y.$$

Since $T^{(3)}(V_7, x^3) = V_4$, we have $T(V_7 x^3, w) \subset V_4 x^2 \oplus V_4 x y$ with surjective projection $T(V_7 x^3, w) \to V_4 x^2$. Therefore $V_4 x^2$ is also contained in the image of $T(\cdot, w)$. Finally, since $T(V_7 x^2, x^3 y^2) = V_4 y^2$, the space $V_4 y^2$ is contained in the image too.

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