Subtleties of the clock retardation

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Abstract

For a simple electromagnetic model of a clock introduced by Jeffimenko (clock # 1 in 1996 Am. J. Phys. 64 812), a change of the rate of the clock when it is set in uniform motion is calculated exactly, employing the correct equation of motion of a charged particle in the electromagnetic field and the universal boostability assumption. Thus, for the clock under consideration, dynamical contents of the clock retardation are demonstrated. Somewhat surprisingly, the analysis presented discloses that some familiar relativistic generalizations concerning the retardation of clocks have to be amended.

Keywords: special relativity, clock retardation, Lorentz transformation, Maxwell’s equations

(Some figures may appear in colour only in the online journal)

1. Introduction

Consider a thin massive ring of radius $a$ at rest in the laboratory frame $S$, carrying a uniformly distributed charge $q_1$. Let the axis of the ring be the $x$-axis, with the origin at the centre of the ring. The electrostatic field of the ring on its axis is

$$E(x, 0, 0) = \frac{\kappa q_1 x \hat{x}}{(a^2 + x^2)^{3/2}},$$

where $\kappa \equiv 1/4\pi\varepsilon_0$. A point charge $q_2$ of opposite sign ($q_1 q_2 < 0$), whose mass is $m$, is set at the point on the positive $x$-axis with $x = A$. If the charge $q_2$ is released with zero initial velocity to move under the action of the electrostatic field of the ring, it will oscillate along the axis between the points $x = \pm A$. The system ‘charged ring and $q_2$’ thus constitutes a primitive electromagnetic clock (clock # 1 in [1]); denote the period of oscillations of $q_2$ for the clock at rest by $T_0$.

Assume now that the ring and the point charge are set in motion with constant velocity $v_0 = v_0 \hat{x}$ along the positive $x$-axis. What is the period of the clock in uniform motion?
The problem was posed by Jefimenko, who solved it by a direct calculation of the period in the laboratory frame, employing Heaviside’s formulae for the electric and magnetic fields of a uniformly moving point charge and the longitudinal mass of $q_2$ [1]. The author found that the period of oscillations of $q_2$ for the moving clock is

$$T_M = \frac{T_0}{\sqrt{1 - \frac{v_0^2}{c^2}}}$$

where $c^2 \equiv 1/\epsilon_0\mu_0$; $c$ is the speed of electromagnetic waves in vacuo and at the same time the speed of light in vacuo. Thus the clock ‘consisting of the charged ring and the point charge runs slower when the clock is moving, and the rate of the moving clock is $(1 - \frac{v_0^2}{c^2})^{-1/2}$ times the rate of the same stationary clock’, as predicted by Special Relativity. Since the conclusion was reached via a dynamical argument only, Jefimenko inferred that, for the clock under consideration, his calculations ‘provide a dynamic cause-and-effect type explanation of time dilation’ [1]. Moreover, pursuing this line of reasoning, he noted that ‘if the slow rate of moving clocks can be indeed explained as a dynamic cause-and-effect phenomenon rather than as the kinematic effect enunciated by Einstein, then the slow rate of moving clocks cannot be interpreted as a proof of time dilation’ [2].

Unfortunately, Jefimenko’s argument involves a confusing assumption that the velocity $v_0$ of the moving clock is much larger than the maximum velocity of $q_2$ relative to the ring; moreover, the desired conclusion (2) is reached due to the additional approximation of small oscillations, i.e., $A \ll a$. Thus, while Jefimenko correctly pointed out dynamical contents of the clock retardation, his approach masks some important aspects of the phenomenon, making some of his inferences fallacious.

In this paper, an exact dynamical analysis of Jefimenko’s clock $\neq 1$ is given. It is demonstrated that if the clock in motion is to be relativistically valid, i.e. to serve as an identical standard of time also for a co-moving inertial observer, analysis of its clockwork requires two frames, the lab frame $S$ and the rest frame of the clock $S'$. Thus, contrary to Jefimenko’s claim, one cannot avoid Einstein’s principle approach to Special Relativity; for understanding of the details of the clockwork, a dynamical (implicitly Lorentz-covariant) analysis has to be complemented by the principle approach. Moreover, the analysis presented reveals that for this ‘longitudinal’ relativistic clock, while equation (2) still applies, the clock retardation exhibits non-uniformity, which seems to be overlooked in the literature.

Namely, Einstein in 1905 inferred from the Lorentz transformation, for a (practically) point clock, that a clock travelling with velocity $v$, ‘when viewed from the stationary system’, runs slower by the factor $(1 - \frac{v^2}{c^2})^{-1/2}$ than the same clock at rest in the stationary system ([3], see also [4]). Later, he generalized this statement to a clock with a second hand and even declared that ‘every happening in a physical system slows down when the system is set into translational motion’ [5]. (Møller paraphrased this by stating that ‘any physical system which is moving relative to a system of inertia must have a slower course of development than the same system at rest’ [6].) Thus, according to Einstein, not only moving clocks run slow but time itself is ‘dilated’ in moving systems, ‘but this slowing occurs only from the standpoint of a non-comoving coordinate system (observer)’ [5]. However, our analysis of Jefimenko’s ‘longitudinal’ clock demonstrates that Einstein’s familiar generalization about slowing down of processes in a moving physical system needs to be amended. Generally, some happenings in a physical system speed up when the system is set into translational motion. Also, for a periodic process, in a physical system travelling with a velocity $v_0$ relative to a system of

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1 The somewhat misleading term ‘time dilation’ was probably introduced by Tolman [7]; incidentally, it appears that ‘dilatation’ would be the grammatically correct variant [8].
inertia, while equation (2) applies, generally it is not true that development of the process has a slower course by the factor \((1 - v_0^2/c^2)^{-1/2}\) than in the same system at rest: the clock retardation may be non-uniform. In this paper, dynamical contents of the phenomenon are illustrated by a detailed examination of Jefimenko’s clock \# 1 [1, 2].

2. Direct calculation of the retardation of Jefimenko’s clock

Now we shall calculate exactly the period of Jefimenko’s clock described in Introduction, when it is at rest in the lab frame \(S\), and when it is in uniform motion with velocity \(v_0 = v_0\hat{x}\) along the \(x\)-axis (the axis of the ring) with respect to the lab. As is well known, in order to be relativistically valid, a clock must operate according to some Lorentz-covariant laws. This can be fulfilled for Jefimenko’s clock, since its operation is based on Maxwell’s equations (which can be made to be Lorentz-covariant (see, e.g., [9, 10])) and the equation of motion of a charge \(q^*\) in the electromagnetic field

\[
\frac{d}{dt} \left( \frac{m^* v}{\sqrt{1 - v^2/c^2}} \right) = q^* E + q^* v \times B, \tag{3}
\]

where \(m^*\) is the mass of the charge, \(v\) is its instantaneous velocity, \(E\) is the electric field and \(B\) is the magnetic flux density; the time parameter \(t\) in the \(S\) frame is interpreted in the standard way, employing propagation of light signals in vacuo as the absolute time keeper, assuming that the Einstein–Poincaré clock synchronization is a valid procedure [6]. Equation (3) fits the experimental facts if the additional independent assumption that \(m^*\) is time-independent is introduced; with that assumption, equation (3) can be Lorentz-covariant too [11].

2.1. Clock at rest

In this subsection we discuss the clockwork of Jefimenko’s clock at rest.

The correct equation of motion of the charge \(q_2\) in an electric field is

\[
m \frac{d}{dt} \left( \frac{v}{\sqrt{1 - v^2/c^2}} \right) = q_2 E, \tag{4}
\]

where the mass \(m\) of \(q_2\) is assumed to be time-independent. Equation (4) and identity

\[
v \cdot \frac{d}{dt} \left( \frac{v}{\sqrt{1 - v^2/c^2}} \right) \equiv c^2 \frac{d}{dt} \left( \frac{1}{\sqrt{1 - v^2/c^2}} \right), \tag{5}
\]

imply that

\[
m c^2 \frac{d}{dt} \left( \frac{1}{\sqrt{1 - v^2/c^2}} \right) = q_2 E \cdot v. \tag{6}
\]

Specifying to our problem, \(E\) is the electrostatic field of the ring on its axis given by equation (1), and \(v = v_x \hat{x}\), since the motion is along the \(x\)-axis. Using equations (4), (6) and (1) one obtains

2 The force \(q_2 E\) is always parallel to the instantaneous velocity \(v\) of the charge \(q_2\) so that equation (7) can be derived using the concept of ‘longitudinal’ mass, taking into account that the Lorentz force expression is a pure force (see [11, 12]); I preferred not to employ here the potentially misleading concept of ‘longitudinal’ mass.
\[
\frac{dv_x}{dt} = -\frac{\kappa|q_1q_2|}{m} \left(1 - \frac{v_x^2}{c^2}\right)^{3/2} \frac{x}{(a^2 + x^2)^{3/2}},
\]
(7)
which can obviously be recast into
\[
\frac{dv_x}{dx}v_x = -\frac{\kappa|q_1q_2|}{m} \left(1 - \frac{v_x^2}{c^2}\right)^{3/2} \frac{x}{(a^2 + x^2)^{3/2}}.
\]
(8)
Separating variables and integrating, setting \(v_x = 0\) when \(x = A\) and solving for \(v_x\) yields
\[
v_x = \frac{dx}{dt} = \mp c \{\ldots\}^{1/2},
\]
(9)
where \(\mp\) and \(+\) sign corresponds to the motion of \(q_2\) in the direction of decreasing \(x\) and increasing \(x\), respectively, and
\[
\{\ldots\} \equiv \left\{1 - \left[1 + \left(\kappa|q_1q_2|/mc^2\right)\left(1/\sqrt{a^2 + x^2} - 1/\sqrt{a^2 + A^2}\right)\right]^{-2}\right\}.
\]
(10)
Equation (9) implies that, for the oscillator at rest, passage of \(q_2\) from \(x\) to \(x + \delta x\) lasts time interval
\[
dt = \mp (1/c) \{\ldots\}^{-1/2} \delta x.
\]
(11)
Thus, the period \(T_0\) of the oscillator at rest is given by
\[
T_0 = \frac{2}{c} \int_{-A}^{A} \{\ldots\}^{-1/2} \delta x.
\]
(12)
In the case of small oscillations, i.e. when \(A < a\), from equation (12) one obtains that
\[
T_0 \approx \frac{2}{c} \int_{-A}^{A} \left\{1 - \left[1 + \left(\kappa/2mc^2\right)(a^2 - x^2)\right]^{-2}\right\}^{-1/2} \delta x.
\]
(13)
where \(\kappa' \equiv \kappa|q_1q_2|/a^3\). The same result is obtained if, instead of the exact expression (1) for the electrostatic field on the axis of the ring, the approximate formula
\[
E(x, 0, 0) \approx \frac{\kappa q_1 \hat{\mathbf{k}}}{a^3},
\]
(14)
is employed in equation (4). As can be seen, equation (13) coincides with the corresponding Møller’s result for the period of his gravity-free “relativistic” oscillator, equation (60) in [13], as it should. Note that when \(\kappa'A^2 \ll mc^2\), from equation (13) one obtains the familiar amplitude–independent expression for the period of the simple harmonic oscillator,
\[
T_0 = 2\pi \sqrt{\frac{m}{\kappa'}}
\]
(15)
(see also [13, 14]). The last expression coincides with Jefimenko’s result for the period of his clock at rest [1, 2]. However, Jefimenko’s starting equation of motion of \(q_2\) was basically
\[
\frac{dv_x}{dt} = -\frac{\kappa|q_1q_2|}{m} \frac{x}{a^3},
\]
(16)
instead of exact equation (7).
2.2. Clock in uniform motion

Assume now that the same clock is set in uniform motion along the $x$-axis (the axis of the ring) with constant velocity $v_0 = v_0 \hat{x}$, so as to be relativistically valid, i.e. to serve as an identical standard of time also for a co-moving observer. The $x$ coordinate of the charge $q_2$, $x$, can be expressed as

$$x = x_c + x_r,$$

(17)

where $x_c$ is the $x$ coordinate of the centre of the ring, and $x_r \equiv x - x_c$ is the relative coordinate of $q_2$ with respect to the instantaneous position of the centre. Since $\dot{x}_c = dx/dr$ and $v_0 = dx_0/dr$, one has

$$\dot{x}_r = v_0 + \dot{v}_x,$$

(18)

where $\dot{v}_x = dx_x/dr$ is the relative velocity of $q_2$ with respect to the centre of the ring, measured in the lab frame $S$.

The charge $q_2$ moves under the action of the electromagnetic field on the axis of the uniformly moving charged ring (the ring constitutes the framework of Jefermenko’s clock). The electromagnetic field is calculated using the formulae for the $E$ and $B$ fields of a point charge $q$ moving with constant velocity $v_0$, that were first obtained by Heaviside [15, 16] (the $B$ field was rederived by Thomson [17], see Jefermenko [18] and references therein), long before the advent of Special Relativity. The electric field of $q$ (radial but not spherically symmetrical) is given by

$$E(r, t) = \frac{\kappa qr}{r^3} \frac{1 - v_0^2/c^2}{(1 - v_0^2 \sin^2 \theta/c^2)^{3/2}},$$

(19)

where $r$ is the position vector of a field point with respect to the instantaneous (at the same instant $t$) position of $q$, $\theta$ is the angle between $r$ and the velocity $v_0$, and $c^2 \equiv 1/\epsilon_0 \mu_0$. (Recall that throughout the relativity paper [3], Einstein used the same symbol ($V$) for the speed of light in vacuo and the speed of electromagnetic waves in vacuo ($V \equiv 1/\sqrt{\epsilon_0 \mu_0}$ in SI system, not used by Einstein, he employed the Heaviside–Lorentz units), linking thus Special Relativity with Maxwell’s theory (see [19, p 197])). The magnetic flux density is

$$B(r, t) = \epsilon_0 \mu_0 v_0 \times E(r, t).$$

(20)

A simple analysis employing formula (19) gives the following expression for the electric field on the axis of the moving charged ring

$$E_M = \frac{\kappa q_0}{m} \frac{1 - v_0^2/c^2}{(1 - v_0^2/c^2)^{3/2}} x_c \hat{x},$$

(21)

as Jefermenko pointed out; the subscript $M$ serves as a reminder that the field is due to the moving ring-charge. The $B$ field on the axis of the ring obviously vanishes. Using equations (4), (6) and (21) one obtains

$$\frac{d\dot{x}_c}{dt} = -\frac{\kappa q_0}{m} \left(1 - \frac{v_0^2}{c^2}\right)^{3/2} \frac{x_r}{(a^2 + x_r^2 - a^2 v_0^2/c^2)^{3/2}}.$$

(22)
Using equation (18), the last equation can obviously be recast into

$$\frac{dv_r}{v_r - v_0} = -\frac{\kappa |q_2|}{m} \frac{x_r dx_r \left(1 - \frac{v_0^2}{c^2}\right)}{\left(1 - \frac{v_r^2}{c^2}\right)^{3/2}}. \quad (23)$$

Now we have to specify our clock so as to be {	extit{relativistically valid}}. As can be seen, this cannot be done without recourse to Einstein’s principle approach to Special Relativity, i.e. without employing the Lorentz transformation and the universal boostability assumption (see [14, 19]). A little reflection reveals that one has to choose initial condition $v_x = v_0$ for $x_r = A\sqrt{1 - \frac{v_0^2}{c^2}}$. A glance at equation (22) shows that the charge $q_2$ will oscillate between the points $x_r = \pm A\sqrt{1 - \frac{v_0^2}{c^2}}$, where $v_x = v_0$.

Integration of equation (23), taking into account the initial condition, gives

$$\frac{1 - v_0 v_r/c^2}{\sqrt{1 - v_r^2/c^2}} \frac{1}{\sqrt{1 - v_0^2/c^2}} = 1 + \gamma, \quad (24)$$

where

$$\gamma \equiv \frac{\kappa |q_2|}{mc^2} \left(\frac{1}{\sqrt{a^2 + x_r^2/(1 - v_0^2/c^2)}} - \frac{1}{\sqrt{a^2 + A^2}}\right). \quad (25)$$

Solving for $v_x$ and using equation (18), after a somewhat lengthy but in every step simple calculation, one obtains for $v_x$

$$v_x = \frac{dx_r}{dr} = \pm \frac{c \left(1 - \frac{v_0^2}{c^2}\right) \sqrt{1 - 1/(1 + \gamma)^2}}{1 \mp (v_0/c) \sqrt{1 - 1/(1 + \gamma)^2}}, \quad (26)$$

where $-$ and $+$ sign corresponds to the motion of $q_2$ in the direction of decreasing $x_r$ and increasing $x_r$, respectively. Equation (26) obviously implies that passage of $q_2$ from $x_r$ to $x_r + dx_r$ lasts time interval

$$dt = \pm \frac{1 \mp (v_0/c) \sqrt{1 - 1/(1 + \gamma)^2}}{c \left(1 - \frac{v_0^2}{c^2}\right) \sqrt{1 - 1/(1 + \gamma)^2}} dx_r. \quad (27)$$

Introducing

$$x_r^* = \frac{x_r}{\sqrt{1 - v_0^2/c^2}}, \quad (28)$$

and

$$\gamma^* = \gamma = \frac{\kappa |q_2|}{mc^2} \left(\frac{1}{\sqrt{a^2 + x_r^*}} - \frac{1}{\sqrt{a^2 + A^2}}\right), \quad (29)$$

equation (27) can be recast into

$$dt = \pm \frac{1 \mp (v_0/c) \sqrt{1 - 1/(1 + \gamma^*)^2}}{\sqrt{1 - v_0^2/c^2} c} dx_r^*, \quad (30)$$
or, equivalently,

\[ \frac{1}{\sqrt{1 - v_0^2/c^2}} \frac{1}{c} \left[ \ldots^* \right]^{-1/2} \pm v_0/c \int dx^*_r, \tag{31} \]

where

\[ \left[ \ldots^* \right] = \left\{ 1 - \left[ 1 + \left( \kappa |q_1 q_2|/mc^2 \right) \left( \frac{1}{\sqrt{\alpha^2 + x^*_r}} - \frac{1}{\sqrt{\alpha^2 + \alpha^2}} \right) \right]^{-1/2} \right\}. \tag{32} \]

Comparing equations (31) and (11) taking into account that \( x^*_r \) runs from \( \mathcal{A} \) to \(-\mathcal{A} \) and vice versa, it follows that clock retardation is non-uniform in the case of Jefimenko’s clock; simple slowing down by the factor \( 1/\sqrt{1 - v_0^2/c^2} \) is clearly violated in the ‘life’ of this ‘longitudinal’ clock. Particularly, using equation (31) one finds that travelling of \( q_2 \) downwards from \( x_r = \mathcal{A}\sqrt{1 - v_0^2/c^2} \) to \( x_r = -\mathcal{A}\sqrt{1 - v_0^2/c^2} \) lasts time interval

\[ (\Delta t)_{\text{down}} = \frac{1}{\sqrt{1 - v_0^2/c^2}} \frac{1}{c} \int_{-\mathcal{A}}^{\mathcal{A}} \left[ \ldots^* \right]^{-1/2} dx^*_r - \int_{-\mathcal{A}}^{\mathcal{A}} v_0/c dx^*_r, \tag{33} \]

whereas the reverse travel upwards lasts

\[ (\Delta t)_{\text{up}} = \frac{1}{\sqrt{1 - v_0^2/c^2}} \frac{1}{c} \left[ \int_{-\mathcal{A}}^{\mathcal{A}} \left[ \ldots^* \right]^{-1/2} dx^*_r + \int_{-\mathcal{A}}^{\mathcal{A}} v_0/c dx^*_r \right]. \tag{34} \]

Taking into account equation (12), these expressions can be recast into

\[ (\Delta t)_{\text{down}} = \frac{1}{\sqrt{1 - v_0^2/c^2}} \frac{T_0}{2} \frac{v_0}{c^2} \mathcal{A}, \tag{35} \]

\[ (\Delta t)_{\text{up}} = \frac{1}{\sqrt{1 - v_0^2/c^2}} \frac{T_0}{2} + \frac{v_0}{c^2} \mathcal{A}. \tag{36} \]

Consequently, for the period of Jefimenko’s uniformly moving ‘longitudinal’ clock, one obtains

\[ T_M = (\Delta t)_{\text{down}} + (\Delta t)_{\text{up}} = \frac{T_0}{\sqrt{1 - v_0^2/c^2}}, \tag{37} \]

as expected for a clock whose clockwork is based on some Lorentz-covariant (or that can be made to be Lorentz-covariant, see, e.g., [9, 10]) laws; the period of the clock in motion is by the factor \( 1/\sqrt{1 - v_0^2/c^2} \) greater than the period of the same clock at rest, all with respect to the lab frame \( S \). However, the present analysis reveals a dynamical content of the phenomenon and its non-uniform character, which are hidden in the conventional kinematical approach.

3. Jefimenko’s calculation of the retardation of his clock

For the sake of comparison, we discuss briefly Jefimenko’s calculation of the retardation of his clock in uniform motion [1, 2].

\[ \text{This is in contradistinction to the case of a ‘transverse’ clock where the clock retardation is uniform. A dynamical content of the clock retardation in the case of a simple electromagnetic model of a ‘transverse’ relativistically valid clock was recently analysed exactly in [14].} \]
Jefimenko assumed that \( x_r \ll a \), and also that the speed \( v_0 \) of the moving ring is much larger than the maximum speed of \( q_r \) relative to the ring, and thus that \( |v_{r\ell}| \ll v_0 \). Employing the two assumptions and the standard expression for longitudinal mass, neglecting \( x_r^2/a^2 \) terms, he basically deduced that equation of motion of \( q_2 \) for the clock in motion reads

\[
\frac{dv}{dt} = \frac{dv_r}{dt} = -\frac{\kappa |q_2|}{ma^3} \left( 1 - \frac{v_0^2}{c^2} \right) x_r,
\]

instead of exact equation (22), since \( v_0 = \text{const} \) and \( v_r^2 \approx v_0^2 \). From the last equation, Jefimenko concluded that the (amplitude-independent) period of oscillations of \( q_2 \) for the moving clock is

\[
T_M = 2\pi \sqrt{\frac{m}{\kappa \left( 1 - \frac{v_0^2}{c^2} \right)}},
\]

which is equation (2), since his \( T_0 \) is given by equation (15). Jefimenko thus inferred that, for the clock under consideration, his calculations based on purely dynamical argument applied in one frame only (given the standard expression for longitudinal mass), show that when the clock moves it runs ‘as predicted by special relativity theory’.

However, Jefimenko’s inference is fallacious. Namely, according to Special Relativity, equation (2) is an exact result valid for all values of speed \( v_0 < c \), whereas Jefimenko’s analysis deduces equation (39), and thus equation (2), as approximate formulae, which he proved only for comparatively large values of \( v_0 \). Thus, Jefimenko’s one-frame analysis does not provide ‘a dynamic cause-and-effect type’ explanation of the relativistic clock retardation, contrary to his claim in [1] 4.

It is perhaps interesting here to perform exact dynamical analysis of Jefimenko’s clock \# 1, as indicated in section 2, ignoring, however, Special Relativity. Namely, what is the period of the clock in uniform motion if one chooses initial condition \( v_x = v_0 \) for \( x_r = A \), instead of \( x_r = A\sqrt{1 - \frac{v_0^2}{c^2}} \)? (It appears that that would be the natural choice if one ignores Special Relativity.) Performing the same steps, \textit{mutatis mutandis}, as in section 2, one finds that the period of the clock in motion, \( T_M \), is given by

\[
\bar{T}_M = \frac{T_0}{\sqrt{1 - \frac{v_0^2}{c^2}}},
\]

where \( T_0 \) is the period of the same clock at rest but now of amplitude \( A\sqrt{1 - \frac{v_0^2}{c^2}} \); expression for \( T_0 \) is obtained replacing in equation (12) each \( A \) by \( A\sqrt{1 - \frac{v_0^2}{c^2}} \). While result (40) is by no means surprising from the perspective of Special Relativity, it presumably demonstrates that exact one-frame dynamical derivation of the relativistic clock retardation is not possible without introducing special assumptions about acceleration of the clock. Incidentally, relationship between \( \bar{T}_0 \) and \( T_0 \) (the period of the original clock at rest of amplitude \( A \)) does not appear to be simple.

4. Discussion

Our analysis of Jefimenko’s ‘longitudinal’ clock given in sections 2 and 3, involving velocity-dependence of the forces governing the operation of the clock, illustrates dynamical contents
of the clock retardation. Also, it demonstrates that in order for the clock to be relativistically valid (inter alia, to choose adequately the amplitude of oscillations of $q_2$ for the clock in motion), one has to recuse to Einstein’s principle approach to Special Relativity and in particular to length contraction\(^5\). Thus, perhaps somewhat surprisingly for adherents of a constructive dynamical approach to the theory (e.g., [21–23]), it appears that the laws of physics in any one inertial frame cannot ‘account for all physical phenomena, including the observations of moving observers,’ contrary to Bell’s claim in [21]\(^6\).

Another outcome of our analysis of Jeffmenko’s clock is that some familiar relativistic generalizations concerning time need to be amended. Notably, declarations such as ‘any physical system which is moving relative to a system of inertia must have a slower course of development than the same system at rest’ [6], prove to be fallacious in the general case\(^7\). However, Möller’s statement is valid in the special case of a periodic complete process occurring in Jeffmenko’s clock. Indeed ‘moving clocks run slow’ from the standpoint of a non-comoving observer, but with addendum that within one period this slowing is generally non-uniform.

The power and precision of Einstein’s principle approach to Special Relativity manifests itself in the universality of its results, regardless of the type of Lorentz-covariant mechanism responsible for a particular phenomenon. Therefore it is to be expected that some of our basic conclusions concerning the properties of relativistically valid clocks in uniform motion are reachable also via the principle approach. This is indeed so, as the following argument reveals.

Consider the standard Lorentz transformation

$$t = t' + \frac{v_0 x'/c^2}{\sqrt{1 - v_0^2/c^2}}, \quad x = \frac{x' + v_0 t'}{\sqrt{1 - v_0^2/c^2}}, \quad y = y', \quad z = z',$$

(41)

where unprimed coordinates refer to the lab frame $S$ and primed coordinates refer to an inertial frame $S'$ which is in standard configuration with $S$ ($S'$ is uniformly moving at speed $v_0$ along the common positive $x$, $x'$-axes, and the $y$- and $z$-axis of $S$ are parallel to the $y'$- and $z'$-axis of $S'$). Recall that time coordinates are interpreted in the standard way, employing propagation of light signals in vacuo as the absolute time keeper, assuming the validity of the Einstein–Poincaré clock synchronization, in both frames [6].

Differentiating the first equation (41), assuming that $x' = \text{const}$, one obtains the familiar result

\(^5\) This is analogous to the case of a relativistically valid ‘transverse’ clock, where it seems that an exact one-frame derivation of the clock retardation is impossible too (see [14]).

\(^6\) Even if one could deduce the relativistic length contraction and clock retardation through dynamical analyses performed in one inertial frame only, employing good laws of physics in that frame, that would not be enough for constructing another inertial frame and deriving the Lorentz transformation. Namely, from contraction of moving rods and retardation of moving clock in the lab frame $S$, one can conclude that one clock-two way speed of light is $c$ also in the rods and clock’s rest frame $S'$, but this does not suffice to ‘spread time over space’ in $S'$; Einstein’s principle approach is indispensable. Moreover, only via the principle approach one knows that candidates for good physical laws in one inertial frame must be (or can be made to be) Lorentz-covariant.

\(^7\) Möller’s statement is clearly wrong in the case of the ‘longitudinal’ light-pulse clock (see, e.g., [24, pp 105–9]). Namely, a simple analysis reveals that the ‘downwards’ segment of life of the moving light clock (i.e., duration of the motion of light-pulse in the direction opposite to the direction of motion of the clock itself) lasts shorter than the corresponding segment of life of the same clock at rest, all with respect to the lab frame $S$, since

$$\frac{l_0 \sqrt{1 - v_0^2/c^2}}{c + v_0} < \frac{l_0}{c},$$

where $l_0$ is the rest length of the clock and $v_0$ is its velocity relative to $S$. Incidentally, it appears that the concept of the light clock (without naming it so) is due to Lewis and Tolman [25], see also [26].
\[ dt = \frac{dr'}{\sqrt{1 - \frac{v_0^2}{c^2}}}. \quad (42) \]

Generally speaking, equation (42) relates a time interval \( dt' \) between two neighbouring events which have the same \( x' \), measured in the \( S' \) frame (but need not necessarily occur at the same point in \( S' \)), with time interval \( dt \) between the two events measured in the \( S \) frame. On this standard reading, equation (42) is a universally valid consequence of the Lorentz transformation (41). Examine now whether equation (42) can provide some conclusions as to the rate of some specific types of clocks in motion.

Let \( S' \) be the rest frame of the framework of a relativistically valid clock. Consider first a ‘transverse’ clock, i.e. the one whose clockwork involves oscillations of a mass point with \( x' = \text{const} \), whereas \( y' \) and \( z' \) are variable (see [14]). Clearly, equation (42) applies to this type of clock. Taking into account the principle of (special) relativity, and the universal boostability assumption [19, 27], in this case equation (42) can be given the following interpretation: any segment of ‘life’ of a ‘transverse’ clock that is moving uniformly with the velocity \( v_0 \) lasts longer by the factor \( \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \) than the same segment of ‘life’ of the same clock at rest in \( S \), all with respect to the lab frame \( S \) [14]. The same interpretation applies a fortiori to Einstein’s (practically) point clock as well.

Consider now a ‘longitudinal’ relativistically valid clock, which involves oscillations of a mass point with \( x' = \text{const} \), and \( y' \) and \( z' \) need not be constant. In this case, the first equation (41) implies

\[ dt = \frac{dt' + (v_0/c^2)dx'}{\sqrt{1 - \frac{v_0^2}{c^2}}} = \frac{dt' + (v_0/c^2)v'_y dx'}{\sqrt{1 - \frac{v_0^2}{c^2}}}, \quad (43) \]

where \( v'_y \) is the \( x' \) component of the instantaneous velocity of the mass point in \( S' \). As can be seen, the first equation (43) is tantamount to equation (31), taking into account the universal boostability assumption. Obviously, equation (42) does not apply to this ‘longitudinal’ type of clock; the clock retardation now is non-uniform, what appears to be overlooked in the literature. However, any clock involves by definition a periodic process; thus, integration of equation (43) over a period in \( S' \) yields

\[ T = \frac{T'}{\sqrt{1 - \frac{v_0^2}{c^2}}}, \quad (44) \]

where \( T \) and \( T' \) are the periods of the clock as observed in \( S \) and \( S' \), respectively, since \( x' \) takes its initial value after the period \( T' \). As can be seen, equation (44) can be given the same meaning as equation (37)\(^8\).

On the other hand, in the case of an aperiodic process involving a point particle, no inference can be deduced from equation (43), except that it is not generally true that ‘every happening in a physical system slows down when the system is set into translational motion [in a rest properties-preserving way]’ [5]. This implies that particular instances of ‘time retardation’ must be carefully considered.

Note that the universality and elegance of the above kinematical deductions concerning clocks in motion may be deceptive. Namely, one should keep in mind that any particular clock retardation involves a complex dynamical process. The dynamical contents are

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\(^8\) But, we derived equation (37) for Jelimenko’s clock, which is strictly ‘longitudinal’ \((y'=z'=0)\), whereas the kinematical approach yields equation (44) for more general ‘longitudinal’ clocks. Incidentally, any real clock involves damping; thus, Jelimenko’s clock represents an ideal clock since the radiation reaction force was neglected in its clockwork.
encrypted in the exact expression for $T_0$ and remain hidden in a purely kinematical derivation; they become manifest only when a dynamical analysis is made, either a simple one, in the clock’s rest frame, or a cumbersome one, in the frame of a non-comoving observer.

For the sake of completeness, discuss in some detail the above conclusion that the non-uniform clock retardation already follows from the Lorentz transformation. Indeed, for any ‘longitudinal’ clock where a mass point oscillates between $x = \pm A$ in the clock’s rest frame, assuming that travel downwards (from $A$ to $-A$) and travel upwards (from $-A$ to $A$) last the same time $T_0/2$ in that frame, one would obtain, using equations (43), expressions of the same form as (35) and (36) for $(\Delta t)_{\text{down}}$ and $(\Delta t)_{\text{up}}$ as derived for the considered model clock. This applies to all possible clock dynamics if only the underlying dynamical theory is compatible with Special Relativity (which is clearly the case for the considered electromagnetic clock).

The particular theory or model just specifies the exact path in a space-time diagram between, say, the two events $(t = t_0, x = A)$ and $(t = t_0 + T_0/2, x = -A)$ and, similarly, $(t_0 + T_0/2, -A)$ and $(t_0 + T_0, A)$, whereas time intervals $(\Delta t)_{\text{down}}$, $(\Delta t)_{\text{up}}$ and the overall time delay is the same irrespective of the particular path. Therefore they are just kinematical...
effects (but note that ‘kinematical’ here means just that no dynamical concepts are needed explicitly in their derivation)\(^9\).

The non-uniform character of the clock retardation may appear striking at first sight, but it is actually by no means surprising. It is obvious that any extended clock whose components are in longitudinal motion relative to the clock’s framework, will be subject to non-uniform retardations. Indeed, in the present model, the velocity of the oscillating charge is symmetric with respect to the clock’s rest frame, but trivially asymmetric as observed in the lab frame. This can be visualized easily with a space-time diagram from the point of view of the lab frame, where the world line of the oscillating charge will be asymmetric, so that clearly not the same time intervals \((\Delta t)_{\text{down}}\) and \((\Delta t)_{\text{up}}\) can be expected, as depicted in figure 1. The exact world line of the charged particle (the thick black curve which connects the red points in figure 1), is not important for that effect.

To summarize, basic inferences concerning slowing down of moving clocks from the standpoint of a non-comoving observer are deduced from Einstein’s two principles of Special Relativity, aided with the universal boostability assumption. The present discussion reveals that for subtler insights, dynamical considerations are needed, as expected. Moreover, the dynamical analysis indicates where to look within Einstein’s principle approach. On the other hand, the principle approach anticipates simplicity and universality behind an intricate dynamical mechanism; for instance, one knows in advance that \(T_M\) depends on dynamical parameters only through dependence of \(T_0\) on those parameters. Thus one can evade cumbersome dynamical calculation for a physical system in uniform motion with respect to the lab frame by combining much simpler dynamical calculation in the rest frame of the system and the transformation laws of relevant quantities known from the principle approach, as Einstein noted long ago for ‘all problems in the optics of moving bodies’ [3].

In his excellent book Special Relativity, French warned to beware of ‘glib statements involving relativity theory’ [24]. This paper, hopefully, represents such a warning, illustrating ever present danger of unjustified generalizations.

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\(^9\) Recall that Bergmann in his well-known book [28] discusses the relativistic length contraction and clock retardation in a section entitled ‘The ‘kinematic’ effects of the Lorentz transformation’.
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