ON THE DIAMETER OF THE COMMUTING GRAPH OF THE FULL MATRIX RING OVER THE REAL NUMBERS

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ABSTRACT. In a recent paper C. Miguel proved that the diameter of the commuting graph of the matrix ring \( M_n(\mathbb{R}) \) is equal to 4 if either \( n = 3 \) or \( n \geq 5 \). But the case \( n = 4 \) remained open, since the diameter could be 4 or 5. In this work we close the problem showing that also in this case the diameter is 4.

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1. Introduction

For a ring \( R \), the commuting graph of \( R \), denoted by \( \Gamma(R) \), is a simple undirected graph whose vertices are all non-central elements of \( R \), and two distinct vertices \( a \) and \( b \) are adjacent if and only if \( ab = ba \). The commuting graph was introduced in [1] and has been extensively studied in recent years by several authors [2–7,12,13].

In a graph \( G \), a path \( P \) is a sequence of distinct vertices \( (v_1, \ldots, v_k) \) such that every two consecutive vertices are adjacent. The number \( k - 1 \) is called the length of \( P \). For two vertices \( u \) and \( v \) in a graph \( G \), the distance between \( u \) and \( v \), denoted by \( d(u,v) \), is the length of the shortest path between \( u \) and \( v \), if such a path exists. Otherwise, we define \( d(u,v) = \infty \). The diameter of a graph \( G \) is defined

\[
\text{diam}(G) = \sup\{d(u,v) : u \text{ and } v \text{ are vertices of } G\}.
\]

A graph \( G \) is called connected if there exists a path between every two distinct vertices of \( G \), equivalently, \( \text{diam}(G) < \infty \).

Most research has been conducted regarding the diameter of commuting graphs of certain classes of rings [3,7–10]. Here, we deal with the full matrix rings over fields. Let \( \mathbb{F} \) be an arbitrary field. We know that \( \Gamma(M_2(\mathbb{F})) \) is never
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connected. It was proved in [4] that $\Gamma(M_n(\mathbb{F}))$ is connected if and only if every field extension of $\mathbb{F}$ of degree $n$ contains a proper intermediate field. Moreover, it was shown in [3] that if $\Gamma(M_n(\mathbb{F}))$ is connected, then $4 \leq \text{diam}(\Gamma(M_n(\mathbb{F}))) \leq 6$ and it is conjectured that $\text{diam}(\Gamma(M_n(\mathbb{F}))) \leq 5$. Let $\mathbb{Q}$ and $\mathbb{R}$ be the fields of rational and real numbers, respectively. We know from [3, 4] that $\Gamma(M_n(\mathbb{Q}))$ is disconnected for any $n \geq 2$ and $\text{diam}(\Gamma(M_n(\mathbb{F}))) = 4$ for every algebraically closed field $\mathbb{F}$ and $n \geq 3$. Quite recently, C. Miguel [11] has verified this conjecture for $\mathbb{R}$, proving the following result.

**Theorem 1.1.** Let $n \geq 3$ be any integer. Then, $\text{diam}(\Gamma(M_n(\mathbb{R}))) = 4$ for $n \neq 4$ and $4 \leq \text{diam}(\Gamma(M_4(\mathbb{R}))) \leq 5$.

Unfortunately, this result left open the question whether $\text{diam}(\Gamma(M_4(\mathbb{R})))$ is 4 or 5. In this paper we solve this open problem. Namely we will prove the following result.

**Theorem 1.2.** The diameter of $\Gamma(M_4(\mathbb{R}))$ is equal to 4.

2. On the diameter of $\Gamma(M_n(\mathbb{R}))$

Before we proceed, let us introduce some notation. If $a, b \in \mathbb{R}$, we define the matrix $A_{a,b}$ as

$$A_{a,b} := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$  

Now, given two matrices $X, Y \in M_n(\mathbb{R})$, we define

$$X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in M_{2n}(\mathbb{R}).$$

Finally, two matrices $A, B \in M_n(\mathbb{R})$ are called similar and are written as $A \sim B$ if there exists an invertible matrix $P$ such that $P^{-1}AP = B$.

The proof of Theorem 1.1 in [11] relies on the possible forms of the Jordan canonical form of a real matrix. In particular, it is proved that the distance between two matrices $A, B \in M_4(\mathbb{R})$ is at most 4 unless we are in the situation where $A$ and $B$ have no real eigenvalues and only one of them is diagonalizable over $\mathbb{C}$. In other words, the case when

$$A = \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix}, \quad B = \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix},$$

The following result will provide us the main tool to prove that the distance between $A$ and $B$ is at most 4 also in the previous setting. It is true for any division ring $D$. In what follows, given a matrix $A$, $L_A$ and $R_A$ will denote the left and right multiplication by $A$, respectively.

**Proposition 2.1.** Let $A, B \in M_n(D)$ matrices such that $A^2 = A$ and $B^2 = 0$. Then, there exists a non-scalar matrix commuting with both $A$ and $B$. 


Proof. Since \( A^2 = A \); i.e., \( A(I - A) = (I - A)A = 0 \), then one of nullity \( A \) or nullity \((I - A)\) is at least \( n/2 \). Since \( I - A \) is also idempotent and a matrix commutes with \( A \) if and only if it commutes with \( I - A \) we can assume that nullity \( A \geq n/2 \). Moreover, since \( B^2 = 0 \), it follows that nullity \( B \geq n/2 \).

Now, if \( \text{Ker}(A) \cap \text{Ker}(B) \neq \{0\} \) and \( \text{Ker}(A) \cap \text{Ker}(B) \neq \{0\} \) we can apply [3, Lemma 4] and the result follows. Hence, we assume that \( \text{Ker}(A) \cap \text{Ker}(B) = \{0\} \), since in the case \( \text{Ker}(A) \cap \text{Ker}(B) = \{0\} \) we can consider the transposes of \( A \) and \( B \) instead of \( A \) and \( B \), respectively. Note that, in these conditions, \( n = 2r \) and the nullities of \( A \) and \( B \) are equal to \( r \).

Let \( B_1 \) and \( B_2 \) be bases for \( \text{Ker}(A) \) and \( \text{Ker}(B) \), respectively, and consider \( B = B_1 \cup B_2 \) a basis for \( D^n \). Since \( A \) is idempotent, it follows that \( D^n = \text{Ker}(A) \oplus \text{Im}(A) \).

We want to find the representation matrix of \( A \) in the basis \( B \). To do so, if \( v \in B_2 \), we write \( v = a + a' \) with \( a \in \text{Ker}(A) \) and \( a' \in \text{Im}(A) \). If \( a' = Aa'' \) for some \( a'' \in D^n \), then \( Av = Aa + Aa' = 0 + A(Aa'') = Aa'' = a'' = -a + v \). Since \( Av = 0 \) for every \( v \in B_1 \), we get that the representation matrix of \( A \) in the basis \( B \) is of the form

\[
\begin{pmatrix}
0 & A' \\
0 & I_r
\end{pmatrix},
\]

with \( A' \in M_r(D) \).

Now, we want to find the representation matrix of \( B \) in the basis \( B \). Clearly, \( Bw = 0 \) for every \( w \in B_2 \). Let \( w \in B_1 \). Then, \( Bw = w_1 + w_2 \) with \( w_1 \in \text{Ker}(A) \) and so \( w_2 \in \text{Ker}(B) \). Hence, \( 0 = B^2w = Bw_1 \) and \( w_1 \in \text{Ker}(A) \cap \text{Ker}(B) = \{0\} \). Thus, the representation matrix of \( B \) in the basis \( B \) is of the form

\[
\begin{pmatrix}
0 & 0 \\
B' & 0
\end{pmatrix},
\]

with \( B' \in M_r(D) \).

As a consequence of the previous work we can find a regular matrix \( P \) such that:

\[
PAP^{-1} = \begin{pmatrix}
0 & A' \\
0 & I_r
\end{pmatrix}, \quad PBP^{-1} = \begin{pmatrix}
0 & 0 \\
B' & 0
\end{pmatrix}.
\]

Now, if \( A'B' \neq B'A' \), then \( P^{-1}(A'B' \oplus B'A')P \) is a non-scalar matrix commuting with \( A \) and \( B \). If \( A' \) and \( B' \) commute, we can find a non-scalar matrix \( S \in M_r(D) \) commuting with both \( A' \) and \( B' \). Therefore \( P^{-1}(S \oplus S)P \) commutes with both \( A \) and \( B \) and the proof is complete. \( \square \)

We are now in the condition to prove the main result of the paper.

**Theorem 2.2.** The diameter of \( \Gamma(M_4(\mathbb{R})) \) is four.

**Proof.** In [11] it was proved that \( d(A, B) \leq 4 \) for every \( A, B \in M_4(\mathbb{R}) \), unless

\[
A \sim \begin{pmatrix}
A_{a,b} & 0 \\
0 & A_{c,d}
\end{pmatrix} \quad \text{and} \quad B \sim \begin{pmatrix}
A_{s,t} & I_2 \\
0 & A_{s,t}
\end{pmatrix},
\]
for some real numbers \(a, b, c, d, s, t\). Hence, we only focus on this case. Assume that
\[
A = P^{-1} \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix} P \quad \text{and} \quad B = Q^{-1} \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix} Q,
\]
for some invertible matrices \(P\) and \(Q\). Let
\[
M = P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix} P \quad \text{and} \quad N = Q^{-1} \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix} Q.
\]
It is straightforwardly checked that \(M^2 = M, N^2 = 0, AM = MA,\) and \(BN = NB\). Furthermore, Proposition 2.1 implies that there exists a non-scalar matrix \(X\) that commutes both with \(M\) and \(N\).
Thus, we have found a path \((A, M, X, N, B)\) of length 4 connecting \(A\) and \(B\) and the result follows.

\[\square\]

**Corollary 2.3.** For every \(n \geq 3\), \(diam(\Gamma(M_4(\mathbb{R}))) = 4\).

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