Strong convergence rate of a full discretization for stochastic Cahn–Hilliard equation driven by space-time white noise

Jianbo Cui, Jialin Hong, Liying Sun

Abstract

In this article, we consider the stochastic Cahn–Hilliard equation driven by space-time white noise. We discretize this equation by using a spatial spectral Galerkin method and a temporal accelerated implicit Euler method. The optimal regularity properties and uniform moment bounds of the exact and numerical solutions are shown. Then we prove that the proposed numerical method is strongly convergent with the sharp convergence rate in a negative Sobolev space. By using an interpolation approach, we deduce the spatial optimal convergence rate and the temporal super-convergence rate of the proposed numerical method in strong convergence sense. To the best of our knowledge, this is the first result on the strong convergence rates of numerical methods for the stochastic Cahn–Hilliard equation driven by space-time white noise. This interpolation approach is also applied to the general noise and high dimension cases, and strong convergence rate results of the proposed scheme are given.

Keywords stochastic Cahn–Hilliard equation, spectral Galerkin method, implicit Euler method, strong convergence rate, interpolation approach.

AMS 60H35, 60H15, 65C30.

1 Introduction

Let $\mathcal{O} = [0, L]^d$, $d \leq 3$. Let $H = L^2(\mathcal{O})$ be the real separable Hilbert space endowed with usual inner product. In this article, we mainly focus on the following stochastic Cahn–Hilliard equation

$$dX(t) + A(AX(t) + f(X(t)))dt = dW(t), \quad t \in (0, T]$$

$$X(0) = X_0,$$

$$t \in (0, T].$$
where \(0 < T < \infty\), \(A : D(A) \subset H \rightarrow H\) is the Neumann Laplacian operator and \(\{W(t)\}_{t \geq 0}\) is a generalized \(Q\)-Wiener process on a filtered probability space \((\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})\). The nonlinearity \(f\) is assumed to be the Nemyskii operator of \(F'\), where \(F\) is a polynomial of degree 4, i.e., \(c_4 \xi^4 + c_3 \xi^3 + c_2 \xi^2 + c_1 \xi + c_0\) with \(c_i \in \mathbb{R}, i = 0, \cdots, 4, c_4 > 0\). A typical example is \(F = \frac{1}{4}(\xi^2 - 1)^2\), which is double well potential and corresponds to the stochastic Cahn–Hilliard–Cook equation.

Numerical approximations for SPDEs with globally Lipschitz continuous and regular nonlinearities have been studied over past decades. In contrast to the Lipschitz case, convergence and convergence rate of numerical approximation for SPDEs with non-globally Lipschitz continuous nonlinearity, become more involved recently (see e.g. \([1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 20, 22, 23, 25, 28, 29, 30, 31]\)) and are far from well-understood. Up to now, there have been some works focusing on the strong convergence and strong convergence rates of numerical schemes for the stochastic Cahn–Hilliard equation (see e.g. \([12, 20, 21, 24, 26, 27]\)). For instance, the authors in \([20, 26]\) show the strong convergence and obtain the convergence rate in a large subsample space of the finite element method and its implicitly full discretization for Eq. (1) with \(d \leq 3\) driven by spatial regular noise. The authors in \([21]\) study the exponential integrability of the spectral Galerkin method and deduce the strong convergence rate of the spectral Galerkin method for one dimensional Eq. (1) driven by trace class noise. For the strong convergence rate of numerical schemes for the linearized stochastic Cahn–Hilliard equation, we refer to \([12, 27, 24]\) and the reference therein.

To the best of our knowledge, there exists no result about the strong convergence of numerical schemes for stochastic Cahn–Hilliard equation driven by space-time white noise, as well as about the strong convergence rates of the temporal discretization and full discretization for stochastic Cahn–Hilliard equation. The present work makes further contributions on the strong convergence rates of numerical schemes for SPDEs with non-globally monotone continuous nonlinearity, especially for stochastic Cahn–Hilliard equation driven by additive noise. Instead of studying the strong convergence problem in \(H\) directly, we use an interpolation approach to solve this problem. The main idea of the interpolation approach is firstly deducing the optimal spatial regularity estimates of the exact and numerical solutions in strong sense. Then we investigate the strong convergence in a large Sobolev space, such as a negative Sobolev space in the case of stochastic Cahn–Hilliard equation, and obtain the optimal strong convergence error estimate in this large space. Combining the established optimal regularity estimate and error estimate in the large space, with the interpolation inequality for the Sobolev space, we are able to obtain the optimal strong convergence rate of numerical scheme.

One main contribution of this article is applying the interpolation approach, and deducing the optimal strong convergence rate of the spectral Galerkin method for Eq. (1) driven by space-time white noise. We consider the spa-
tial spectral Galerkin method, whose solution satisfies
\[ dX^N(t) + A(AX^N(t) + P^N f(X^N(t)))dt = P^N dW(t), \]
\[ X^N(0) = P^N X_0, \]
where \( P^N \) is the spectral Galerkin projection. To overcome the difficulty brought by both the non-globally monotone nonlinearity and the driven noise which is rougher than the trace class noise, we first show the optimal spatial and temporal regularity estimates of the exact and numerical solutions in Section 3 and Section 4. Inspired by the fact that the deterministic Cahn–Hilliard equation defines a gradient flow in \( \mathbb{H}^{-1} \) for the energy functional, we focus on the strong convergence problem in \( \mathbb{H}^{-1} \). By using the equivalence between Eq. (1) and a random PDE, and the monotonicity of \( f \) in \( \mathbb{H}^{-1} \), we deduce the sharp strong convergence error estimate in \( \mathbb{H}^{-1} \). Then based on the Sobolev interpolation inequality, we recover the optimal strong convergence rate of the spectral Galerkin method for Eq. (1) driven by space-time white noise. Let \( (I - \mathbb{L})X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2}) \), \( N \in \mathbb{N}^+ \) and \( \rho \geq 1 \). The numerical solution \( X^N \) is shown to strongly converge to \( X \) and satisfies
\[ \|X^N(t) - X(t)\|_{L^p(\Omega; H)} \leq C(X_0, T, p)\lambda_N^{-\frac{\gamma}{2}} \]
for a positive constant \( C(X_0, T, p) \).

Another main contribution is about the strong convergence rate of an implicit full discretization for Eq. (1) based on the interpolation approach and an appropriate error decomposition. Let \( \delta t \) be the time stepsize such that \( T = K\delta t \), \( K \in \mathbb{N}^+ \). The accelerated full discretization is defined as
\[
Y^N_{k+1} = Y^N_k - A^2 Y^N_{k+1} \delta t - A P^N f(Y^N_{k+1} + Z^N(t_{k+1})) \delta t, \\
X^N_{k+1} = Y^N_{k+1} + Z^N(t_{k+1}), \quad k \leq K - 1
\]
with the initial data \( Y^N(0) = X^N(0), Z^N(0) = 0 \), and \( Z^N \) being the spectral projection of the stochastic convolution. Following the interpolation approach, we first prove the optimal regularity of the full discretization. Then by introducing an auxiliary process \( \tilde{Y}^N_k, k \leq K \), we divide the temporal error in \( \mathbb{H}^{-1} \) into two parts, \( \|Y^N(t_k) - \tilde{Y}^N_k\|_{\mathbb{H}^{-1}} \) and \( \|\tilde{Y}^N_k - Y^N_k\|_{\mathbb{H}^{-1}} \). Based on the monotonicity of \( f \) in \( \mathbb{H}^{-1} \) and the interpolation arguments, we obtain the strong convergence of the proposed full discretization, i.e., let \( (I - \mathbb{L})X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2}) \), \( N \in \mathbb{N}^+ \) and \( \rho \geq 1 \), then the numerical solution \( X^N_k \) is strongly convergent to \( X \) and satisfies
\[ \|X^N_k - X(t_k)\|_{L^p(\Omega; H)} \leq C(X_0, T, p)(\delta t^{\frac{\gamma}{2}} + \lambda_N^{-\frac{\gamma}{2}}) \]
for a positive constant \( C(X_0, T, p) \). We also remark that this approach is also available for deducing the strong convergence rates of numerical schemes for Eq. (1) driven by general noise for Eq. (1) with \( d \leq 3 \) (see Section 5).

The outline of this paper is as follows. In the next section, some preliminaries are listed. Section 3 is devoted to giving the useful regularity and a priori
estimates of Eq. (1). In Section 4, we prove some a priori estimates of the numerical solutions, and use the interpolation approach to study the strong convergence rates of both the spectral Galerkin method and its accelerated implicit full discretization. Some applications of the interpolation approach to the cases \( d \leq 3 \) and general noises are also discussed in Section 5.

2 Preliminaries

Given two separable Hilbert spaces \((\mathcal{H}, \| \cdot \|_\mathcal{H})\) and \((\bar{\mathcal{H}}, \| \cdot \|_{\bar{\mathcal{H}}})\), \(\mathcal{L}(\mathcal{H}, \bar{\mathcal{H}})\) and \(\mathcal{L}_1(\mathcal{H}, \bar{\mathcal{H}})\) are the Banach spaces of all linear bounded operators and the nuclear operators from \(\mathcal{H}\) to \(\bar{\mathcal{H}}\), respectively. The trace of an operator \(T \in \mathcal{L}_1(\mathcal{H})\) is

\[
\text{tr}[T] = \sum_{k \in \mathbb{N}^+} \langle T f_k, f_k \rangle_{\mathcal{H}},
\]

where \(\{f_k\}_{k \in \mathbb{N}^+} = \{1, 2, \cdots\}\) is any orthonormal basis of \(\mathcal{H}\). In particular, if \(T \geq 0\), \(\text{tr}[T] = \|T\|_{\mathcal{L}_1}\). Denote by \(\mathcal{L}_2(\mathcal{H}, \bar{\mathcal{H}})\) the space of Hilbert–Schmidt operators from \(\mathcal{H}\) into \(\bar{\mathcal{H}}\), equipped with the usual norm given by \(\|T\|_{\mathcal{L}_2(\mathcal{H}, \bar{\mathcal{H}})} = \left(\sum_{k \in \mathbb{N}^+} \|f_k\|^2_{\bar{\mathcal{H}}}^2\right)^{\frac{1}{2}}\). The following useful property and inequality hold

\[
\|S T\|_{\mathcal{L}_2(\mathcal{H}, \bar{\mathcal{H}})} \leq \|S\|_{\mathcal{L}_2(\mathcal{H}, \bar{\mathcal{H}})} \|T\|_{\mathcal{L}_1(\mathcal{H})}, \quad T \in \mathcal{L}_1(\mathcal{H}), \quad S \in \mathcal{L}_2(\mathcal{H}, \bar{\mathcal{H}}),
\]

\[
\text{tr}[Q] = \|Q^\frac{1}{2}\|_{\mathcal{L}_2(\mathcal{H}, \bar{\mathcal{H}})}^2 = \|T\|_{\mathcal{L}_2(\mathcal{H}, \bar{\mathcal{H}})}^2, \quad Q = T T^*, \quad T \in \mathcal{L}_2(\bar{\mathcal{H}}, \mathcal{H}),
\]

where \(T^*\) is the adjoint operator of \(T\).

Given a Banach space \((E, \| \cdot \|_E)\), we denote by \(\gamma(\mathcal{H}, E)\) the space of \(\gamma\)-radonifying operators endowed with the norm \(\|T\|_{\gamma(\mathcal{H}, E)} = \left(\mathbb{E}\left\| \sum_{k \in \mathbb{N}^+} \gamma_k T f_k \right\|^2_E \right)^{\frac{1}{2}}\), where \(\{\gamma_k\}_{k \in \mathbb{N}^+}\) is a sequence of independent \(\mathcal{N}(0, 1)\)-random variables on a probability space \((\Omega, \mathcal{F}, \mathbf{P})\). For convenience, let \(H = L^2(O)\) equipped with \(\| \cdot \| = \| \cdot \|_H\) and \(\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H\), \(L^q = L^q(O), 1 \leq q < \infty\) and \(E = C(O)\) equipped with the usual inner product and norm. We also need the following Burkholder inequality in \(E\),

\[
\sup_{t \in [0, T]} \left\| \int_0^t \phi(r) d\bar{W}(r) \right\|_E \leq C_p \|\phi\|_{L^p(\Omega; L^2([0, T]; \gamma(H; E)))} \leq C_p \left(\mathbb{E}\left(\int_0^T \left\| \sum_{k \in \mathbb{N}^+} (\phi(t) e_k)^2 \right\|_E^2 dt \right)^{\frac{1}{2}}\right)^{\frac{1}{2}},
\]

where \(\bar{W}\) is the \(H\)-valued cylindrical Wiener process and \(\{e_k\}_{k \in \mathbb{N}}\) is an orthonormal basis of \(H\).

Next, we introduce some spaces associated with \(A\) and give some assumptions on \(W\). Let \(O = [0, L]^d\), \(L > 0\) and \(d \leq 3\). We denote by \(H^k := H^k(\mathcal{O})\) the standard Sobolev space. Define \(A = -\Delta\), the Neumann Laplacian operator with

\[
D(A) = \left\{ v \in H^2(O) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial \mathcal{O} \right\}.
\]
Denote $L$ the projection $Lv = |\mathcal{O}|^{-1} \int_{\mathcal{O}} v dx$, $v \in H$. Let $\mathbb{H} = (I - L)H$. It is known that $A$ is a positive definite, self-adjoint and unbounded linear operator on $\mathbb{H}$. By extending $A$ on $H$, $A$ has an orthonormal eigensystem $\{ (\lambda_j, e_j) \}_{j \in \mathbb{N}}$ such that $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$ with $\lambda_j \sim j^{\frac{2}{\gamma}}$, $\epsilon_0 = |\mathcal{O}|^{-\frac{2}{\gamma}}$. We also define $\mathbb{H}^\alpha$, $\alpha \in \mathbb{R}$ as the space of the series $v := \sum_{j=1}^{\infty} v_j e_j$, $v_j \in \mathbb{R}$, such that $\|v\|_{\mathbb{H}^\alpha} := (\sum_{j=1}^{\infty} \lambda_j^{\alpha} v_j^2)^{\frac{1}{2}} < \infty$. Equipped with the norm $\| \cdot \|_{\mathbb{H}^\alpha}$ and corresponding inner product, the Hilbert space $\mathbb{H}^\alpha$ of $\mathbb{H}$ is equivalent to the Sobolev norm $\| \cdot \|_{H^\alpha}$, i.e., for $v \in \mathbb{L}H \oplus \mathbb{H}^\alpha$,

$$\|v\|_{\mathbb{H}^\alpha}^2 := \|v\|_{H^\alpha}^2 + |(v, e_0)|^2 = \|\nabla^\alpha v\|^2 + |(v, e_0)|^2 \sim \|v\|_{H^\alpha}.$$ 

Throughout this article, we suppose that there exists some $\gamma > 0$ such that the covariance operator $Q$ satisfies $\|A^{\frac{\gamma-2}{2}} Q^{\frac{1}{2}}\|_{L_2} < \infty$. We denote $L_f$ is the one-side Lipschitz coefficient of $f$. In Sections 3 and 4, we assume that $Q$ equals $I$ and $d = 1$. In this case, it is known (see e.g. [19]) that for any $p \geq 1$ and $\gamma < \frac{2}{p}$,

$$E \left[ \sup_{t \in [0,T]} \| (I - L)Z(t) \|_{E}^p \right] + E \left[ \| (I - L)Z(t) \|_{H_1}^p \right] \leq C(p, T). \tag{5}$$

and for $s \leq t$,

$$E \left[ \| Z(t) - Z(s) \|_{E}^p \right] \leq C(T, p)(t - s)^{\frac{2\gamma}{\gamma}}. \tag{6}$$

where $Z(t)$ is the stochastic convolution. In Section 5, we consider the general case of $Q$ and $d \leq 3$. We also remark that $X = L X + (I - L)X$, where $LX$ satisfies

$$dLX + A^2 L X dt + A L (f(X)) dt = L dW(t)$$

and $(I - L)X$ satisfies

$$d(I - L)X + A^2 (I - L)X dt + A(I - L)(f(X)) dt = (I - L) dW(t).$$

Meanwhile, $LX = LY + LZ$ and $(I - L)X = (I - L)Y + (I - L)Z$, where $Y$ and $Z$ satisfy

$$dY + (A^2 Y + A f(Y + Z)) dt = 0, \; Y(0) = X_0, \; dZ + A^2 Z dt = dW(t), \; Z(0) = 0.$$

### 3 A priori and regularity estimates of stochastic Cahn–Hilliard equation

In this section, based on the a priori estimates of both $Z$ and $Y$, we present both the optimal spatial and temporal regularity estimates of the exact solution $X$ for the one dimensional Eq. (1) driven by space-time white noise.

For the well-posedness of Eq. (1), we refer to [2, 19] and the references therein. Now, we consider the $p$-th moment of $\|X\|$ in the following lemma.
Lemma 3.1. Let \( X_0 \in H \) and \( p \geq 1 \). There exists a unique mild solution \( X \) of Eq. (1) satisfying

\[
E \left[ \sup_{t \in [0,T]} \|X(t)\|^p \right] \leq C(X_0,T,p)
\]

for a positive constant \( C(X_0,T,p) \).

Proof. Since \((I - \mathbb{L})X = (I - \mathbb{L})Y + (I - \mathbb{L})Z\), where \((I - \mathbb{L})Y\) satisfies the following random PDE

\[
d(I - \mathbb{L})Y + A^2(I - \mathbb{L})Ydt + A(I - \mathbb{L})(f(Y + Z))dt = 0,
\]

\((I - \mathbb{L})Y(0) = (I - \mathbb{L})X(0),\)

and \((I - \mathbb{L})Z\) satisfies

\[
d(I - \mathbb{L})Z + A^2(I - \mathbb{L})Zdt = (I - \mathbb{L})dW(t), \ (I - \mathbb{L})Z(0) = 0,
\]

it suffices to estimate the term \(\|(I - \mathbb{L})Y\|\) by the a priori estimate (5) of \(Z\). Before that, we give the estimate of \(\|(I - \mathbb{L})Y\|_{\mathbb{H}^{p-1}}\). For small \(\epsilon > 0\), it follows from the chain rule, Hölder and Young inequalities that

\[
\|(I - \mathbb{L})Y(t)\|_{\mathbb{H}^{p-1}}^2 \leq \|(I - \mathbb{L})X_0\|_{\mathbb{H}^{p-1}}^2 - 2 \int_0^t \|(I - \mathbb{L})Y(s)\|^2_{\mathbb{H}^{p}} ds

- 2 \int_0^t (f(Y(s) + Z(s)), (I - \mathbb{L})Y(s)) ds

\leq \|(I - \mathbb{L})X_0\|_{\mathbb{H}^{p-1}}^2 - (1 - \epsilon) \int_0^t \|(I - \mathbb{L})Y(s)\|^2_{\mathbb{H}^{p}} ds

+ \int_0^t \|(I - \mathbb{L})Y(s)\|^2_{\mathbb{H}^{p-1}} ds - 8(c_4 - \epsilon) \int_0^t \|(I - \mathbb{L})Y(s)\|^4_{L^4} ds

+ C(\epsilon) \int_0^t \left( \|LY(s)\|^4_{L^4} + \|Z(s)\|^4_{L^4} + \|A^{-\frac{4}{5}}(I - \mathbb{L})Z(s)\|^2_{L^4} \right) ds.
\]

Then the Gronwall inequality leads that for \(t \leq T\),

\[
\|(I - \mathbb{L})Y(t)\|_{\mathbb{H}^{p-1}}^2 \leq e^{T\epsilon} \|(I - \mathbb{L})X_0\|_{\mathbb{H}^{p-1}}^2 + C(\epsilon,T) \int_0^T \left( \|LY(s)\|^4_{L^4} ds

+ \|Z(s)\|^4_{L^4} + \|A^{-\frac{4}{5}}(I - \mathbb{L})Z(s)\|^2 \right) ds,
\]

which also implies that

\[
(1 - \epsilon) \int_0^t \|(I - \mathbb{L})Y(s)\|^2_{\mathbb{H}^{p-1}} ds + 8(c_4 - \epsilon) \int_0^t \|(I - \mathbb{L})Y(s)\|^4_{L^4} ds

\leq C(\epsilon,T) \left( \|(I - \mathbb{L})X_0\|_{\mathbb{H}^{p-1}}^2 + \int_0^T \left( \|LY(s)\|^4_{L^4} + \|Z(s)\|^4_{L^4} + \|Z(s)\|^2 \right) ds \right).
\]
Due to the definition of $\mathbb{L}$ and
\[
d\mathbb{L}Y + A^2 \mathbb{L}Y dt + A\mathbb{L}(f(Y + Z))dt = 0, \quad \mathbb{L}Y(0) = \mathbb{L}X_0,
\]
we have $\mathbb{L}Y(t) = \mathbb{L}X_0$, which immediately implies that $\|\mathbb{L}Y(t)\|_{L^4} \leq \|\mathbb{L}X_0\|_{L^4}$. The a priori estimates of $Z$ and $\mathbb{L}Y$ yield the uniformly boundedness of both the $p$-th moment of $\int_0^T \|(I - \mathbb{L})Y(s)\|_{L^4}^2 ds$ and that of $\int_0^T \|(I - \mathbb{L})Y(s)\|_{L^2}^4 ds$.

Now we are in the position to give the a priori estimate of $\|(I - \mathbb{L})Y\|$. By applying the chain rule, integration by parts and the dissipative of $-f$, we obtain
\[
\|(I - \mathbb{L})Y(t)\|^2 = \|(I - \mathbb{L})X_0\|^2 - 2 \int_0^t \|(-A)(I - \mathbb{L})Y(s)\|^2 ds
\]
\[
- 2 \int_0^t \langle \nabla f(Y(s) + Z(s)), \nabla (I - \mathbb{L})Y(s) \rangle ds
\]
\[
\leq \|(I - \mathbb{L})X_0\|^2 - (2 - \epsilon) \int_0^t \|(-A)(I - \mathbb{L})Y(s)\|^2 ds
\]
\[
- (24c_4 - \epsilon) \int_0^t \|Y(s)\|_{H^3}^2 ds + C(\epsilon) \int_0^t \|Z(s)\|_{L^2}^2 \|Y(s)\|_{L^4}^4 ds
\]
\[
+ C(\epsilon) \int_0^t (1 + \|\nabla (I - \mathbb{L})Y(s)\|^2 + \|Z(s)\|_{L^6}^2) ds.
\]
By the equivalence of norms in $\mathbb{H}^1$ and $H^1$ for the functions in $\mathbb{H}^1$, we have
\[
\|(I - \mathbb{L})Y(t)\|^2 + (2 - \epsilon) \int_0^t \|(-A)(I - \mathbb{L})Y(s)\|^2 ds
\]
\[
\leq \|(I - \mathbb{L})X_0\|^2 + C(\epsilon) \int_0^t (1 + \|\mathbb{L}Y(s)\|_{\mathbb{H}^3}^2 + \|Y(s)\|_{L^4}^2 + \|Z(s)\|_{L^6}^2) ds.
\]
From the uniformly boundedness of $\int_0^T \|\mathbb{L}Y(s)\|_{\mathbb{H}^3}^2 ds$, $\int_0^T \|(I - \mathbb{L})Y(s)\|_{L^4}^4 ds$ and the estimate $\mathbb{E}\left[\sup_{s \in [0,T]} \|Z(s)\|_{L^2}^p\right] \leq C(p, T)$, it follows that for $p \geq 1$,
\[
\mathbb{E}\left[\sup_{t \in [0,T]} \|(I - \mathbb{L})Y(t)\|_{\mathbb{H}^3}^{2p} + \left(\int_0^T \|(-A)(I - \mathbb{L})Y(s)\|^2 ds\right)^p\right] \leq C(X_0, T, p),
\]
which, together with the a priori estimates of both $\|\mathbb{L}Y\|$ and $\|Z\|$ and H"older inequality, completes the proof.

Based on the a priori estimate of $\|X\|$, we give the regularity estimate of $X$.

**Proposition 3.1.** Let $(I - \mathbb{L})X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0, \frac{3}{2})$, $p \geq 1$. The unique mild solution $X$ of Eq. (1) satisfies
\[
\mathbb{E}\left[\sup_{t \in [0,T]} \|(I - \mathbb{L})X(t)\|_{\mathbb{H}^\gamma}^p\right] \leq C(X_0, T, p) \tag{10}
\]
for a positive constant \(C(X_0, T, p)\).

**Proof.** Due to (5), it suffices to give a regularity estimate for \((I - L)Y\). Before that, we give the following estimate of \(\|(I - L)Y(t)\|_{L^6}\). The Sobolev embedding theorem, the smooth effect of \(e^{-\Delta t}\) and the Gagliardo–Nirenberg inequality yield that

\[
\|(I - L)Y(t)\|_{L^6} \leq \|e^{-\Delta t}(I - L)X_0\|_{L^6} + \int_0^t \|e^{-\Delta(t-s)}A(-L)f(Y(s) + Z(s))\|_{L^6} ds
\]

\[
\leq C\|X_0\|_{H^1} + C \int_0^t (t-s)^{-\frac{5}{6}} (1 + \|(I - L)Y(s)\|_{L^6}^3 + \|Z(s)\|_{L^6}^3 + \|LY(s)\|_{L^6}^3) ds
\]

From the Hölder inequality, the a priori estimates of both \(Z\) and \(LY\) and the estimate (9), it follows that for any \(p \geq 1\),

\[
E \left( \sup_{t \in [0,T]} \| (I - L)Y(t) \|_{L^6}^{p} \right) \leq C(p)\|X_0\|_{H^1}^{p} + C(p)E \left[ \int_0^T \|(-A)(I - L)Y(s)\|_{L^6}^{2p} ds \right]^{\frac{p}{p-1}}
\]

\[
+ C(p) \int_0^T (t-s)^{-\frac{5}{6}} ds \|Z(s)\|_{L^6}^{3p} E \left[ \sup_{s \in [0,T]} \| (I - L)Y(s) \|_{L^6}^{10p} \right]^{\frac{p}{p-1}}
\]

\[
+ C(p) \int_0^T (t-s)^{-\frac{5}{6}} ds \|LY(s)\|_{L^6}^{3p} E \left[ \sup_{s \in [0,T]} \| (I - L)Y(s) \|_{L^6}^{3p} \right]
\]

\[
\leq C(T, X_0, p).
\]

From the mild form of \((I - L)Y(t)\) for Eq. (8), it follows that

\[
\|(I - L)Y(t)\|_{H^\gamma} \leq \|e^{-\Delta t}(I - L)X_0\|_{H^\gamma} + \int_0^t \|e^{-\frac{t-s}{2}}(t-s)^{-\frac{\gamma}{2}} A(I - L)f(Y(s) + Z(s))\|_{H^\gamma} ds
\]

\[
\leq C\|X_0\|_{H^\gamma} + C \int_0^t (t-s)^{-\frac{\delta}{2}} \|e^{-\frac{t-s}{2}}A(t-s)^{\frac{\gamma}{2}} f(Y(s) + Z(s))\|_{H^\gamma} ds
\]

\[
\leq C\|X_0\|_{H^\gamma} + C \int_0^t (t-s)^{-\frac{\delta}{2}} \|1 + \|(I - L)Y(s)\|_{L^6}^3 + \|LY(s)\|_{L^6}^3 \|Z(s)\|_{L^6}^3 ds.
\]

By taking \(p\)-th moment and making use of the a priori estimates of \(\|LY(s)\|_{L^6}\), \(\|(I - L)Y(s)\|_{L^6}\) and \(\|Z(s)\|_{H^\gamma}\), we finish the proof. \(\square\)

**Remark 3.1.** Let the conditions of Proposition 3.1 hold. If in addition assume that \((I - L)X_0 \in H^{\beta}\) for \(\beta < 2\), we can obtain the higher regularity of \(Y\). Indeed,
we have

$$\mathbb{E}
\left[
\sup_{t \in [0,T]}
\|(I - L)Y(t)\|_E^p
\right] \leq C(T, X_0, p)$$

for \( \beta < 2 \).

**Corollary 3.1.** Let \((I - L)X_0 \in E\) and \(p \geq 1\). The unique mild solution \(X\) of Eq. (1) satisfies

$$\mathbb{E}
\left[
\sup_{t \in [0,T]}
\|(I - L)X(t)\|_E^p
\right] \leq C(X_0, T, p) \quad (11)$$

for a positive constant \(C(X_0, T, p)\).

**Proposition 3.2.** Let \((I - L)X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{2}{\beta})\), \(p \geq 1\). The unique mild solution \(X\) of Eq. (1) satisfies

$$\mathbb{E}
\left[
\|X(t) - X(s)\|^p
\right] \leq C(X_0, T, p)(t - s)^{\frac{2p}{\beta}} \quad (12)$$

for a positive constant \(C(X_0, T, p)\) and \(0 \leq s \leq t \leq T\).

**Proof.** Due to the continuity of \(Z\) (6) and the fact that \(LZ(t) = XZ_0\), we focus on the estimate of \(\|(I - L)(Y(t) - Y(s))\|\). For the term \(\|(I - L)(Y(t) - Y(s))\|\), we obtain for some \(\beta < 2\),

\[
\|(I - L)Y(t) - Y(s)\| \\
\leq \|(I - L)e^{-A^2s}(e^{-A^2(t-s)} - I)X_0\| \\
+ \int_0^s \left\|\left(e^{-A^2(t-r)} - e^{-A^2(s-r)}\right)(I - L)f(Y(r) + Z(r))\right\|dr \\
+ \int_s^t \left\|e^{-A^2(t-r)}A(f(Y(r) + Z(r))\right\|dr \\
\leq C\|(I - L)X_0\|_{\mathbb{H}^\gamma} (t - s)^{\frac{2}{\beta}} + \int_0^s (t - s)^{-\frac{1}{2}} \|f(Y(r) + Z(r))\|dr \\
+ \int_0^s (s - r)^{-\frac{1}{2} - \frac{\beta}{2}} \|A^{-\beta/2}(e^{-A^2(t-s)} - I)\|f(Y(r) + Z(r))\|dr \\
\leq C\left(\|(I - L)X_0\|_{\mathbb{H}^\gamma} (t - s)^{\frac{2}{\beta}} + (t - s)^{\frac{1}{2}} (1 + \sup_{r \in [0,T]} \|Y(r)\|_{L^2}^3 + \sup_{r \in [0,T]} \|Z(r)\|_{L^2}^3)\right) \\
+ C(t - s)^{\frac{1}{2}} \left(1 + \sup_{r \in [0,T]} \|Y(r)\|_{L^2}^3 + \sup_{r \in [0,T]} \|Z(r)\|_{L^2}^3\right) \int_0^s (s - r)^{-\frac{1}{2} - \frac{\beta}{2}} dr.
\]

By taking \(p\)-th moment and using the a priori estimates of \(Y\) and \(Z\), we get

$$\mathbb{E}
\left[
\|(I - L)(Y(t) - Y(s))\|^p
\right] \leq C(T, X_0, p)(t - s)^{\frac{2p}{\beta}}.$$

Combining all the above estimates, we complete the proof.  \(\square\)
Remark 3.2. Let the conditions of Proposition 3.2 hold. If in addition $(I - L)X_0 \in H^\beta$, $\beta < 2$, we can obtain the higher temporal regularity of $Y$. Indeed, we have

$$\mathbb{E} \left[ \|Y(t) - Y(s)\|^p \right] \leq C(T, X_0, p)(t - s)^{\frac{\beta p}{4}}$$

for $\beta < 2$.

4 Strong convergence rate of full discretization

After giving the optimal spatial and temporal regularity of $X$, we first show the semi-discretization and full discretization of Eq. (1) in this section. Then we focus on the strong convergence rate of the proposed full discretization.

Denote $P^N$ the spectral Galerkin projection into the linear space spanned by the first $N+1$ eigenvectors $\{e_0, e_1, \ldots, e_N\}$. Then the spatial semi-discretization is

$$dX^N + A(AX^N + P^N(f(X^N))) dt = P^N dW, \quad t \in (0, T]; \quad X^N(0) = P^N X_0.$$  (13)

Let $\delta t$ be the time stepsize such that $T = K\delta t$, $K \in \mathbb{N}^+$. By using the accelerated idea, we propose the full discrete numerical scheme

$$
\begin{align*}
Y^N_{k+1} &= Y^N_{k} - A^2 Y^N_{k+1}\delta t - AP^N f(Y^N_{k+1} + Z^N(t_{k+1}))\delta t, \\
X^N_{k+1} &= Y^N_{k+1} + Z^N(t_{k+1}), \quad k \leq K - 1
\end{align*}
$$

(14)

with the initial data $Y^N_0 = X^N(0), Z^N(0) = 0$. Then we have the mild form of $Y^N_{k}$,

$$
\begin{align*}
Y^N_{k+1} &= T_{\delta t} Y^N_k - \delta t T_{\delta t} AP^N (f(Y^N_{k+1} + Z^N(t_{k+1}))), \quad k \leq K - 1,
\end{align*}
$$

(15)

where $T_{\delta t} = (I + A^2 \delta t)^{-1}$.

To analyze the strong convergence rate of the proposed spectral Galerkin method, we introduce an interpolation approach, which is also used to study the strong convergence rate of the full discretization.

4.1 Strong convergence rate of spectral Galerkin method

The main idea of deducing the optimal convergence rates of numerical schemes lies on an interpolation arguments. Let us briefly describe it. First, we need to obtain the optimal regularity of the exact and numerical solutions in an interpolation space $E_{\theta_1}, \theta_1 \in \mathbb{R}$, which is similar to the idea in Section 3. Then we need the optimal convergence rate of the proposed numerical scheme in another interpolation space $E_{\theta_0}, \theta_0 \in \mathbb{R}, \theta_0 < \theta_1$. By the interpolation inequality, we deduce the optimal convergence rate of the numerical scheme in any interpolation space between $E_{\theta_0}$ and $E_{\theta_1}$.
In our case, the interpolation space $E_{\theta_1}$ is the Sobolev space $H^{\gamma}$, $\gamma < \frac{3}{2}$ and $E_{\theta_0}$ is the space $H^{-1}$. Notice that $X^N = L X^N + (I - L)X^N$, $X^N = Y^N + Z^N$, where $Y^N$ and $Z^N$ satisfy
\[
d Y^N + P^N(A^2 Y^N + A f(Y^N + Z^N))dt = 0, \quad Y^N(0) = X^N(0),
\]
\[
d Z^N + A^2 Z^N dt = P^N dW(t), \quad Z^N(0) = 0.
\]
It is obvious that the strong error estimate of $\|X^N - X\|$ can be split as
\[
\|X^N - X\| \leq \|L(Y^N - Y)\| + \|(I - L)(Y^N - Y)\| + \|Z^N - Z\|.
\]
The first term is 0 due to the definition of $L$ and $P^N$, and the last term is controlled directly by
\[
E\left[\|Z^N - Z\|^p\right] \leq C(X_0, T, p) \lambda_N^{-\frac{2p}{2}}
\]
for $\gamma < \frac{3}{2}$. Thus it suffices to estimate the term $\|(I - L)(Y^N - Y)\|$. It meets a lot of troubles to give the strong convergence rate of numerical schemes for Eq. (1) due to the nonlinear term $A f(X)$. One of the main difficulties lies on the loss of the Gronwall inequality to deduce the convergence rate. Another difficulty is the lack of the regularity property of the exact solution due to the space-time white noise. If the driven noise has more regularity in space, such as the trace class noise, one can use the exponential integrability of numerical and exact solutions to deduce the strong convergence rate. To overcome these difficulties, we use the interpolation approach to deal with the term $\|(I - L)(Y^N - Y)\|$. The first step is the following optimal regularity of $X^N$, which is obtained by the similar arguments in the proofs of Propositions 3.1 and 3.2.

**Lemma 4.1.** Let $(I - L)X_0 \in H^{\gamma}$, $\gamma \in (0, \frac{3}{2})$, $p \geq 1$. Then $Y^N$ satisfies
\[
E\left[\sup_{t \in [0, T]} \|(I - L)Y^N(t)\|^p_{H^\gamma}\right] \leq C(X_0, T, p),
\]
and
\[
E\left[\|Y^N(t) - Y^N(s)\|^p\right] \leq C(X_0, T, p)(t - s)^{\frac{2p}{2}}
\]
for a positive constant $C(X_0, T, p)$.

**Corollary 4.1.** Let $(I - L)X_0 \in H^{\gamma}$, $\gamma \in (0, \frac{3}{2})$, $p \geq 1$. The unique mild solution $X^N$ of Eq. (13) satisfies
\[
E\left[\sup_{t \in [0, T]} \|(I - L)X^N(t)\|^p_{H^\gamma}\right] \leq C(X_0, T, p),
\]
and
\[
E\left[\|X^N(t) - X^N(s)\|^p\right] \leq C(X_0, T, p)(t - s)^{\frac{2p}{2}}
\]
for a positive constant $C(X_0, T, p)$. 

11
Lemma 4.2. Let \((I - \mathbb{L})X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2})\), \(p \geq 1\). Then \(X^N\) satisfies that for any small \(\epsilon > 0\)
\[
\|\nabla(I - \mathbb{L})X^N(t)\|_{L^p(\Omega; E)} \leq C(X_0, T, p)t^{-\epsilon}.
\]
Proof. Due to the boundedness of \(Z^N\), it only need to estimate \(\nabla(I - \mathbb{L})Y^N(t)\). From the property of \(e^{-tA^2}\), it follows that for any small \(\epsilon > 0\),
\[
\|\nabla(I - \mathbb{L})Y^N(t)\|_E
\leq \|\nabla e^{-A^2t}(I - \mathbb{L})Y^N(0)\|_E + \int_0^t \|\nabla e^{-A^2(t-s)}A(I - \mathbb{L})f(Y^N(s) + Z^N(s))\|_E ds
\leq Ct^{-\frac{3}{4} - \epsilon + \frac{\gamma}{2}}\|((I - \mathbb{L})Y^N(0))\|_{H^\gamma}
+ C \int_0^t (t-s)^{-\frac{3}{4} - \frac{\gamma}{2} - \epsilon}\|f(Y^N(s) + Z^N(s))\| ds,
\]
which, together the a priori estimates of \(Y^N\) and \(Z^N\), completes the proof. \(\square\)

Remark 4.1. Let \((I - \mathbb{L})X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2})\). Then we have the following estimate, for any small \(\epsilon > 0\),
\[
\|\nabla(I - \mathbb{L})Y^N\|_E \leq c(T, X_0)t^{-\frac{3}{4} - \epsilon + \frac{\gamma}{2}},
\]
where \(c(T, X_0)\) has finite \(p\)-th moment for any \(p \geq 1\). If \((I - \mathbb{L})X_0 \in \mathbb{H}^\beta, \beta > \frac{3}{2}\), we have
\[
\|\nabla(I - \mathbb{L})Y^N\|_{L^p(\Omega; E)} \leq C(T, X_0, p) < \infty.
\]

The second step of the interpolation approach is proving the following optimal strong convergence rate of the spectral Galerkin method in \(\mathbb{H}^{-\lambda}\). Before that, a useful lemma is introduced as follows.

Lemma 4.3. Let \(g : L^4 \to H\) be the Nemskii operator of a polynomial of second degree. Then it holds that for any \(\beta \in (0, 1)\) that
\[
\|(I - \mathbb{L})g(x)\|_{H^{-\lambda}} \leq C\left(1 + \|x\|_{E}^2 + \|(I - \mathbb{L})x\|_{H^\beta}^2\right)\|y\|_{H^{-\lambda}},
\]
where \(x \in E, (I - \mathbb{L})x \in \mathbb{H}^\beta\) and \(y \in \mathbb{H}\).

Proof. The equivalence of norms in \(\mathbb{H}^\beta\) and \(H^\beta\) and Hölder inequality yield that for \(z \in \mathbb{H}^\beta \cap E, \beta < 1\),
\[
\|(I - \mathbb{L})g(x)z\|_{H^\beta}^2
\leq C\|(I - \mathbb{L})g(x)\|_{H^\beta}^2 + C\int_0^L \int_0^L \frac{|g(x(\xi_1))z(\xi_1) - g(x(\xi_2))z(\xi_2)|^2}{|\xi_1 - \xi_2|^{2\beta + 1}} d\xi_1 d\xi_2
\leq C(1 + \|x\|_{H^\beta}^2)\|z\|_{H^\beta}^2 + C(1 + \|x\|_{H^\beta}^2)\|z\|_{H^\beta}^2
+ C(1 + \|x\|_{H^\beta}^2)(\|z\|_{E}^2 + \|(I - \mathbb{L})x\|_{H^\beta}^2)\|z\|_{E}^2.
\]
\[ \leq C(1 + \|x\|_E^2 + \|(I - L)x\|_{H^{3/2}}^2)(\|z\|_{H^{3/2}} + \|z\|_{E^2}).\]

According to the self-adjointness of \( g \) and the Sobolev embedding theorem, we have

\[
\|(I - L)g(x)\|_{H^{3/2}} = \sup_{\|w\|_{H^{3/2}} \leq 1} \left| \langle y, (I - L)g(x)A^{-\frac{3}{2}}w \rangle \right|
\]

\[
= \sup_{\|w\|_{H^{3/2}} \leq 1} \left| \langle A^{-\frac{3}{2}}y, A^{\frac{3}{2}}(I - L)g(x)A^{-\frac{3}{2}}w \rangle \right|
\]

\[
\leq \sup_{\|w\|_{H^{3/2}} \leq 1} \|y\|_{H^{3/2}} \|(I - L)g(x)A^{-\frac{3}{2}}w\|_{H^{3/2}}
\]

\[
\leq C \sup_{\|w\|_{H^{3/2}} \leq 1} \|y\|_{H^{3/2}}(1 + \|x\|_E^2 + \|(I - L)x\|_{H^{3/2}}^2)(\|w\|_{H^{3/2}} + \|A^{-\frac{3}{2}}w\|_E)
\]

\[
\leq C(1 + \|x\|_E^2 + \|(I - L)x\|_{H^{3/2}}^2)\|y\|_{H^{3/2}},
\]

which completes the proof. \( \square \)

**Corollary 4.2.** Under the condition of the above lemma, for any \( \beta \in [1, \min(\eta, 2)) \) and \( \eta > \frac{3}{2} \), we have

\[ \|(I - L)g(x)y\|_{H^{\eta}} \leq C(1 + \|x\|_E^2 + \|\nabla x\|_E^2 + \|(I - L)x\|_{H^{3/2}}^2)\|y\|_{H^{\eta}}, \]

where \( x, \nabla x \in E, (I - L)x \in H^{3/2} \) and \( y \in H^\eta \).

**Remark 4.2.** Similar to Corollary 4.2, we have that for any \( \beta \in [2, \min(\eta, 3)] \) and \( \eta > \frac{3}{2} \),

\[ \|(I - L)g(x)y\|_{H^{\eta}} \leq C(1 + \|x\|_{W^{2,\infty}}^2 + \|(I - L)x\|_{H^{3/2}}^2)\|y\|_{H^{\eta}}, \]

where \( x \in W^{2,\infty}, (I - L)x \in H^{3/2} \) and \( y \in H^\eta \). We also have similar estimations in the cases \( d = 2, 3 \), where \( \eta > s + \frac{3}{2}, s = 1 \) or \( 2, \beta \in [s, \min(\eta, 1 + s)] \).

**Proposition 4.1.** Let \( (I - L)X_0 \in H^\gamma, \gamma \in (0, \frac{3}{2}), N \in \mathbb{N}^+ \) and \( p \geq 1 \). There exists a positive constant \( C(X_0, T, p) \) such that

\[ \|Y^N(t) - Y(t)\|_{L^p(\Omega; H^{\gamma-1})} \leq C(X_0, T, p)\lambda_N^{\frac{1}{2} - \frac{3}{2}}, \] (16)

**Proof.** From the property of the Galerkin projection and the triangle inequality, it follows that

\[ \|Y^N - Y\|_{H^{\gamma-1}} \leq \|Y^N - P^NY\|_{H^{\gamma-1}} + \|(I - P^N)Y\|_{H^{\gamma-1}} \]

\[ \leq \|Y^N - P^NY\|_{H^{\gamma-1}} + C\lambda_N^{\frac{3}{2} - \gamma} \|Y\|_{H^\gamma}. \]

Thus it suffices to estimate the term \( \|Y^N - P^NY\|_{H^{\gamma-1}} \). By the chain rule, the integration by parts, the monotonicity of \(-f\) and the Young inequality, we have

\[ \|Y^N - P^NY\|_{H^{\gamma-1}}^2, \]
\[
= -2 \int_0^t \langle \nabla (Y^N - P^N Y), \nabla (Y^N - P^N Y) \rangle ds \\
- 2 \int_0^t (f(Y^N + Z^N) - f(Y + Z), Y^N - P^N Y) ds \\
\leq -2 \int_0^t \| \nabla (Y^N - P^N Y) \|^2 ds - 2 \int_0^t \langle \int_0^t f'(\theta Y^N + \theta Z^N + (1 - \theta)Y + (1 - \theta)Z) d\theta \rangle \\
(Y^N - P^N Y - (I - P^N)Y - (I - P^N)Z), Y^N - P^N Y \rangle ds \\
\leq -2 \int_0^t \| \nabla (Y^N - P^N Y) \|^2 ds + 2L_f \int_0^t \| Y^N - P^N Y \|^2 ds \\
+ 2 \int_0^t \langle A^- (I - L) \left( \int_0^1 f'(\theta (Y^N + Z^N) + (1 - \theta)(Y + Z)) d\theta \right) \\
\left( (I - P^N)Y + (I - P^N)Z \right), A^- (Y^N - P^N Y) \rangle ds \\
\leq -(2 - \epsilon) \lambda_{1} - 2 \epsilon L_f \int_0^1 \| \nabla (Y^N - P^N Y) \|^2 ds + C(\epsilon) L_f \int_0^t \| Y^N - P^N Y \|^2_{H^{-1}} ds \\
+ C(\epsilon) \int_0^t \left\| \left( \int_0^1 f'(\theta (Y^N + Z^N) + (1 - \theta)(Y + Z)) d\theta ((I - P^N)Y + (I - P^N)Z) \right) \right\|^2_{H^{-1}} ds.
\]

It follows from Lemma 4.3 that for \( \beta \in (0, 1) \),
\[
\| Y^N (t) - P^N Y(t) \|^2_{H^{-1}} \\
\leq \int_0^t \left\| \left( \int_0^1 (I - L) \left( f'(\theta (Y^N + Z^N) + (1 - \theta)(Y + Z)) \right) \\
\left( (I - P^N)Y + (I - P^N)Z \right) d\theta \right\|^2_{H^{-1}} ds \\
\leq C \int_0^t \left( 1 + \left\| (I - L) X^N \right\|^4_{\beta, \beta} + \left\| (I - L) X \right\|^4_{\beta, \beta} + \left\| X^N \right\|^4_{E} + \left\| X \right\|^4_{L^2} \right) \left\| (I - P^N)Y + (I - P^N)Z \right\|^2_{H^{-1}} ds.
\]

Then from taking \( L^p(\Omega) \) norm, the a priori estimates in Proposition 3.1, Corollary 3.1 and Corollary 4.1 and the Hölder inequality, it follows that for \( \beta \in (0, 1) \)
\[
\| Y^N (t) - P^N Y(t) \|^2_{L^p(\Omega; H^{-1})} \\
\leq C(T) \left( 1 + \sup_{s \in [0, T]} \left\| (I - L) X^N \right\|^4_{\beta, \beta} + \left\| (I - L) X \right\|^4_{\beta, \beta} + \left\| X^N \right\|^4_{E} + \left\| X \right\|^4_{L^2} \right) \\
\times \sup_{s \in [0, T]} \left\| (I - P^N)Y(s) + (I - P^N)Z(s) \right\|^2_{L^p(\Omega; H^{-1})} \\
\leq C(T, X_0, p) \left( \sup_{s \in [0, T]} \left\| A^{-\frac{\beta}{2}} (I - P^N)Y(s) \right\|^2_{L^{4p}(\Omega; H^\beta)} + \sup_{s \in [0, T]} \left\| A^{-\frac{\beta}{2}} (I - P^N)Z(s) \right\|^2_{L^{4p}(\Omega; H^\beta)} \right) \\
\leq C(T, X_0, p) \lambda_{\gamma}^{-\beta - \gamma}.
\]

Taking square root on both sides, letting \( \beta \) close to 1 and using Hölder inequality, we complete the proof. \( \square \)
Based on the above two steps in Lemma 4.1 and Proposition 4.1, by using the interpolation inequality for interpolation Sobolev space, we deduce the optimal strong convergence rate of the proposed semi-discretization.

**Theorem 4.1.** Let \((I - \mathbb{L})X_0 \in \mathbb{H}^r, \gamma \in (0, \frac{3}{2}), N \in \mathbb{N}^+\) and \(p \geq 1\). The numerical solution \(X^N\) is strongly convergent to \(X\) and satisfies

\[
\|X^N(t) - X(t)\|_{L^p(\Omega; H)} \leq C(X_0, T, p)\lambda_N^{-\frac{2}{\gamma}}
\]

(17)

for a positive constant \(C(X_0, T, p)\).

**Proof.** From the triangle inequality, it follows that

\[
\|X^N(t) - X(t)\|_{L^p(\Omega; H)} \leq \|Y^N(t) - Y(t)\|_{L^p(\Omega; H)} + \|Z^N(t) - Z(t)\|_{L^p(\Omega; H)}
\]

\[
= \|Y^N(t) - Y(t)\|_{L^p(\Omega; H)} + \|Z^N(t) - Z(t)\|_{L^p(\Omega; H)}
\]

\[
\leq \|Y^N(t) - Y(t)\|_{L^p(\Omega; H)} + C(t, p)\lambda_N^{-\frac{2}{\gamma}}
\]

It suffices to estimate \(\|Y^N(t) - Y(t)\|_{L^p(\Omega; H)}\). By using the interpolation inequality, we have

\[
\|Y^N(t) - Y(t)\| \leq C\|Y^N(t) - Y(t)\|_{H^{\gamma-1}} \|Y^N(t) - Y(t)\|_{H^{\gamma}},
\]

which implies that

\[
\|Y^N(t) - Y(t)\|_{L^p(\Omega; H)} \leq C\|Y^N(t) - Y(t)\|_{L^p(\Omega; H^{\gamma-1})} \|Y^N(t) - Y(t)\|_{L^p(\Omega; H^{\gamma})}.
\]

Lemma 4.1 and Remark 3.1 yield that

\[
\|Y^N(t) - Y(t)\|_{L^p(\Omega; H^{\gamma})} \leq C(T, X_0, p).
\]

Together with Proposition 4.1, we have

\[
\|Y^N(t) - Y(t)\|_{L^p(\Omega; H)} \leq C(T, X_0, p)\|Y^N(t) - Y(t)\|_{L^p(\Omega; H^{\gamma-1})}
\]

\[
\leq C(T, X_0, p)\lambda_N^{-\frac{2}{\gamma}},
\]

which completes the proof. \(\square\)

### 4.2 Strong convergence rate of full discretization

In this part, we extend the interpolation approach to the study of strong convergence rate of full discretization. For convenience, we denote \(Z_k^N = Z^N(t_k), k \leq K\). We recall the equivalent form of (14), i.e., \(X_k^N = Y_k^N + Z_k^N, k \leq K\), where

\[
Y_{k+1}^N = Y_k^N - A^2 Y_{k+1}^N \delta t - AP^N f(Y_{k+1}^N + Z_{k+1}^N) \delta t,
\]
\[ dZ^N = -A^2 Z^N \, dt + P^N \, dW(t) \]

with \( k \leq K - 1 \), \( Y^N_0 = X^N(0) \) and \( Z^N_0 = 0 \). To make sure that the implicit method is solvable, we take \( \delta t < \min(1, \frac{1}{(L - \lambda_0) \sqrt{D}}) \). By the estimate for \( t_k = k\delta t \),

\[
\|X^N_k - X(t_k)\| \leq \|X^N_k - X^N(t_k)\| + \|X^N(t_k) - X(t_k)\|
\leq \|Y^N_k - Y^N(t_k)\| + \|X^N(t_k) - X(t_k)\|,
\]

it suffices to estimate the first term \( \|Y^N_k - Y^N(t_k)\| \). To apply the interpolation approach, we need the optimal regularity estimate of \( Y^N_k \) and the sharp strong convergence analysis of \( \|Y^N_k - Y^N(t_k)\|_{H^{-1}} \).

**Lemma 4.4.** Let \( (I - \mathbb{L})X_0 \in \mathbb{H}^\gamma \), \( \gamma \in (0, \frac{3}{2}) \), \( p \geq 1 \). Then \( Y^N_k \) satisfies

\[
\mathbb{E} \left[ \sup_{k \leq K} \| (I - \mathbb{L})Y^N_k \|_{H^\gamma}^p \right] \leq C(X_0, T, p),
\]

and

\[
\mathbb{E} \left[ \| Y^N_k - Y^N_{k+1} \|_{H^\gamma}^p \right] \leq C(X_0, T, p)((k - k_1)\delta t)^\frac{3}{2}
\]

for a positive constant \( C(X_0, T, p) \) and \( k_1, k \leq K \).

**Proof.** The proof is similar to that of Lemma 4.1. Since

\[
(I - \mathbb{L})Y^N_{k+1} = (I - \mathbb{L})Y^N_k - A^2 \delta t (I - \mathbb{L})Y^N_{k+1} - A \delta t (I - \mathbb{L})P^N (f(Y^N_{k+1}) + Z^N_{k+1}),
\]

(18)

taking inner product with \( (I - \mathbb{L})Y^N_{k+1} \) in \( \mathbb{H}^{-1} \) on both sides leads to

\[
\|(I - \mathbb{L})Y^N_{k+1}\|_{H^{-1}}^2 \leq \|(I - \mathbb{L})Y^N_k\|_{H^{-1}}^2 - 2\|\nabla (I - \mathbb{L})Y^N_{k+1}\|^2 \delta t
- 2\langle f(Y^N_{k+1} + Z^N_{k+1}), (I - \mathbb{L})Y^N_{k+1} \rangle \delta t.
\]

From the monotonicity of \(-f\), the equivalence of norms in \( \mathbb{H}^1 \) and \( H^1 \) for functions in \( \mathbb{H}^1 \), and the Young inequality, it follows that for some small \( \epsilon > 0 \),

\[
\|(I - \mathbb{L})Y^N_{k+1}\|_{H^{-1}}^2 + 8(c_4 - \epsilon)\|(I - \mathbb{L})Y^N_{k+1}\|_{L^4}^4 \delta t + (2 - \epsilon)\langle f(I - \mathbb{L})Y^N_{k+1}\rangle_{L^4}^2 \delta t
\leq \|(I - \mathbb{L})Y^N_{k+1}\|_{H^{-1}}^2 + C(\epsilon) \left( 1 + \|\nabla Y^N_{k+1}\|_{L^4}^4 + \|Z^N_{k+1}\|_{L^4}^4 + \|A^{-\frac{1}{2}}(I - \mathbb{L})Z^N_{k+1}\|_{H^{-1}}^2 \right) \delta t.
\]

By taking the \( p \)-th moment, the priori estimate of \( Z^N_k \) and the fact that \( LY^N_k = LX^N(0) \), we have

\[
\mathbb{E} \left[ \sup_{k \leq K} \| (I - \mathbb{L})Y^N_k \|_{H^{-1}}^{2p} \right] + \mathbb{E} \left[ \left( \sum_{k=0}^{K-1} (2\epsilon)\|(I - \mathbb{L})Y^N_{k+1}\|_{L^4}^2 \delta t \right)^p \right]
\]
Next we show the boundedness of \((I - \mathbb{L})Y_k^N\) in \(H\). By taking inner product on both sides of Eq. (18) with \((I - \mathbb{L})Y_k^N\) in \(H\), we obtain

\[
\| (I - \mathbb{L})Y_k^N \|_{L^2}^2 \leq \| (I - \mathbb{L})Y_k^N \|_{L^2}^2 - (2 - \epsilon) \delta t \|A(I - \mathbb{L})Y_k^N\|^2 - (24c_4 - \epsilon) \|Y_k^N\|^2 \|\nabla (I - \mathbb{L})Y_k^N\|^2 \delta t \\
+ C(\epsilon)(\|\nabla (I - \mathbb{L})Y_k^N\|^2 + \|Y_k^N\|^2 \|\nabla Z_k^N\|^2 + \|Z_k^N\|^2) \delta t.
\]

Thus it is concluded that for \(p \geq 1\),

\[
E \left[ \sup_{k \leq K} \| (I - \mathbb{L})Y_k^N \|_{L^2}^p \right] + (2 - \epsilon) E \left[ \left( \sum_{k=0}^{K-1} \|A(I - \mathbb{L})Y_k^N\|_{L^2}^2 \delta t \right)^p \right] \\
\leq C(\epsilon, T) E \left[ \left( \sum_{k=0}^{K-1} \|\nabla (I - \mathbb{L})Y_k^N\|_{L^2}^2 \delta t \right)^p \right] + C(\epsilon, T) E \left[ \left( \sum_{k=0}^{K-1} \|Y_k^N\|_{L^2}^4 \delta t \right)^p \right] \\
+ C(\epsilon, T) \sum_{k=0}^{K-1} E \left[ \|Z_k^N\|_{L^4}^4 + \|\nabla Z_k^N\|_{L^4}^4 + \|\nabla Z_k^N\|^2 \right] \delta t \leq C(X_0, p, T).
\]

Similar to the proof of Proposition 3.1, we need the boundedness of \(||(I - \mathbb{L})Y_k^N\|_{L^6}\). From the Sobolev embedding theorem, the smoothness effect of \(e^{-A^2t}\), the Gagliardo–Nirenberg and Young inequalities, it follows that

\[
\| (I - \mathbb{L})Y_k^N \|_{L^6} \leq \left( T^{k+1}_\delta (I - \mathbb{L})Y_0^N \right) \|_{L^6} + \left( \sum_{j=0}^{k} T^{k+1-j}_\delta A(I - \mathbb{L})f(Y_{j+1}^N + Z_{j+1}^N) \right) \|_{L^6} \delta t \\
\leq C \|Y_0^N\|_{H^3} + C \sum_{j=0}^{k} (k + 1 - j)^{-\bar{\gamma}} \delta t^{-\bar{\gamma}} \left( \|Z_{j+1}^N\|_{L^6}^3 + \|\nabla Y_{j+1}^N\|_{L^6}^3 + \|\nabla Z_{j+1}^N\|_{L^6} \right) \delta t \\
+ C \sum_{j=0}^{k} (k + 1 - j)^{-\bar{\gamma}} \delta t^{-\bar{\gamma}} \sup_{j \leq K} \| (I - \mathbb{L})Y_j^N \|_{L^6}^{\frac{3\gamma}{2\gamma}} \delta t + \sum_{j=0}^{k} \|(-A)(I - \mathbb{L})Y_j^N\|^2 \delta t.
\]

The a priori estimates of \(Y_k^N\) and \(Z_k^N\) yield that for \(p \geq 1\)

\[
E \left[ \sup_{k \leq K} \| (I - \mathbb{L})Y_k^N \|_{L^6}^p \right] \leq C(T, X_0, p).
\]

Now, we are in the position to give the desired regularity estimate. From the mild form of \(Y_k^N\) and the above a priori estimates, we have

\[
E \left[ \sup_{k \leq K} \| (I - \mathbb{L})Y_k^N \|_{L^6}^p \right] \leq C(T, X_0, p).
\]
\[ \leq C\mathbb{E}\left[\|(I - L)Y_0^N\|_{L^p}\right] + C\delta t\mathbb{E}\left[\left(\sum_{j=0}^{K-1}\left\|T_{\delta t}^{K-j}A(I - L)f(Y_{j+1}^N + Z_{j+1}^N)\right\|^p\right)\right] \]

\[ \leq C(p)\|(I - L)X_0^N\|_{L^p}^p + C(p)\sum_{j=0}^{K-1}(T - t_j)^{-\frac{1}{2}}\delta t\mathbb{E}\left[\sup_{j \leq K}\left(1 + \|Y_j^N\|_{L^0}^3 + \|Z_j^N\|_{L^0}^3\right)^p\right] \]

\[ \leq C(p)\|(I - L)X_0^N\|_{L^p}^p + C(T, p)\mathbb{E}\left[\sup_{j \leq K}\left(1 + \|Y_j^N\|_{L^0}^3 + \|Z_j^N\|_{L^0}^3\right)^p\right] \]

\[ \leq C(T, p, X_0). \]

For convenience, we assume that \( k > k_1 \). Similar arguments in the proof of Proposition 3.2 yield that

\[ \mathbb{E}\left[\|Y_k^N - Y_k^N\|_{L^p}^p\right] \leq \|(I - L)T_{\delta t}^{k_1} (T_{\delta t}^{k-k_1} - I)Y_0^N\| \]

\[ + \sum_{j=0}^{k_1-1}\left\|(T_{\delta t}^{k_1-j} (T_{\delta t}^{k-k_1} - I)A(I - L)f(Y_{j+1}^N + Z_{j+1}^N)\right\|\delta t \]

\[ + \sum_{j=k_1}^{k-1}\left\|T_{\delta t}^{k-j}Af(Y_{j+1}^N + Z_{j+1}^N)\right\|\delta t \]

\[ \leq C(X_0, T, p)\|(k - k_1)\delta t\|^p. \]

Combining the above regularity estimates together, we finish the proof. \( \square \)

To deduce the strong convergence rate in time, we introduce an auxiliary process \( \tilde{Y}_k^N, k \leq K \) with \( Y_0^N = X_0^N(0) \), defined by

\[ \tilde{Y}_k^N = Y_k^N - A\delta t\tilde{Y}_k^N - P^N Af(Y_{k+1}^N) + Z_{k+1}^N. \]

Then we split the error of \( Y_k^N - Y^N(t_k) \) as

\[ \|Y_k^N - Y^N(t_k)\| \leq \|Y^N(t_k) - \tilde{Y}_k^N\| + \|\tilde{Y}_k^N - Y_k^N\|. \]

The first error is bounded as the following lemma. The second error will be dealt with the interpolation arguments.

**Lemma 4.5.** Let \((I - L)X_0 \in H^\gamma, \gamma \in (0, \frac{3}{2}), N \in \mathbb{N}^+, p \geq 1\). There exists a positive constant \( C(X_0, T, p) \) such that

\[ \|Y^N(t_k) - \tilde{Y}_k^N\|_{L^p(\Omega; H)} \leq C(X_0, T, p)\delta t^\gamma. \quad (19) \]

**Proof.** Denote \([s]_{\delta t} := \max\{0, \delta t, \cdots, k\delta t, \cdots\} \cap [0, s]\) and \([s] = \frac{[s]}{\delta t}\). The mild forms of \( Y^N(t_k) \) and \( \tilde{Y}_k^N \) yield that

\[ \|Y^N(t_k) - \tilde{Y}_k^N\|_{L^p(\Omega; H)} \leq \left\|\int_{0}^{t_k} (e^{-A(t_k-s)} - T_{\delta t}^{k-s})Ap^N f(Y^N(s) + Z^N(s))ds\right\|_{L^p(\Omega; H)} \quad (20) \]

\[ \leq \left\|\int_{0}^{t_k} (e^{-A(t_k-s)} - T_{\delta t}^{k-s})Ap^N f(Y^N(s) + Z^N(s))ds\right\|_{L^p(\Omega; H)} \]

18
\[ + \left\| \int_0^{t_k} T_{\delta t}^{k-s} A P^N \left( f(Y^N(s) + Z^N(s)) - f(Y^N([s]_{\delta t} + \delta t) + Z^N([s]_{\delta t + 1}) \right) ds \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ := I_1 + I_2. \]

By the properties of \(e^{-A^2 t}\) and \(T_{\delta t}^k\), the priori estimates of \(Y^N\) and \(Z^N\) and Lemma 4.2, the first term is estimated as for small \(\epsilon > 0\), \(\gamma < \frac{3}{2}\),
\[ I_1 \leq C \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - [s]_{\delta t})^{-1+\epsilon} \delta t^{\frac{7}{8} - \epsilon} \left\| \left( 1 + \|X^N(s)\|^2_{L^2} + \|\nabla X^N(s)\|^2_{L^2} \right) \right\|_{L^p(\Omega; \mathbb{R})} ds \leq C(T, X_0, p) \delta t^{\frac{7}{8}}. \]

Similarly, by using the Talyor formula and the mild form of \(X^N\), we have
\[ I_2 \leq \left\| \int_0^{t_k} T_{\delta t}^{k-s} A P^N \left( f'(X^N(s))(e^{-A^2([s]_{\delta t} + \delta t - s)} - I)(X^N(s)) \right) ds \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ + \left\| \int_0^{t_k} T_{\delta t}^{k-s} A P^N \left( f'(X^N(s)) \int_s^{[s]_{\delta t} + \delta t} e^{-A^2([s]_{\delta t} + \delta t - r)} A P^N f(X^N(r)) dr \right) ds \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ + \left\| \int_0^{t_k} T_{\delta t}^{k-s} A P^N \left( f'(X^N(s)) \int_s^{[s]_{\delta t} + \delta t} e^{-A^2([s]_{\delta t} + \delta t - r)} P^N dW(r) \right) ds \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ + \left\| \int_0^{t_k} T_{\delta t}^{k-s} A P^N \left( \int_s^{[s]_{\delta t} + \delta t} f''(\lambda X^N(s) + (1 - \lambda)X^N([s]_{\delta t})(1 - \lambda) d\lambda \right) \right\|_{L^p(\Omega; \mathbb{R})} := I_{21} + I_{22} + I_{23} + I_{24}. \]

Then the smooth effect of \(e^{-A^2 t}\), Lemma 4.2 and Corollary 4.2 lead to for \(1 \leq \gamma < \frac{3}{2}\),
\[ I_{21} \leq \left\| \int_0^{t_k} T_{\delta t}^{k-s} A^{1+\frac{7}{8} - \epsilon} A^{-\frac{7}{8} - \epsilon} \left( (I - L) f'(X^N(s))(e^{-A^2([s]_{\delta t} + \delta t - s)} - I)(X^N(s)) \right) ds \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ \leq \int_0^{t_k} (t_k - [s]_{\delta t})^{-\frac{7}{8} - \epsilon} \left\| (I - L) f'(X^N(s))(e^{-A^2([s]_{\delta t} + \delta t - s)} - I)(X^N(s)) \right\|_{L^p(\Omega; \mathbb{R})} ds \]
\[ \leq \int_0^{t_k} (t_k - [s]_{\delta t})^{-\frac{7}{8} - \epsilon} \left\| (1 + \|X^N(s)\|^2_{L^2} + \|\nabla X^N(s)\|^2_{L^2} + \|(I - L) X^N(s)\|^2_{H^\gamma}) \right\|_{L^p(\Omega; \mathbb{R})} ds \]
\[ \leq C(T, X_0, p) \delta t^{\frac{7}{8}}. \]

Similarly, we have
\[ I_{22} \leq \left\| \int_0^{t_k} T_{\delta t}^{k-s} A P^N \left( f'(X^N(s)) \int_s^{[s]_{\delta t} + \delta t} e^{-A^2([s]_{\delta t} + \delta t - r)} A P^N f(X^N(r)) dr \right) ds \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ \leq C \int_0^{t_k} (t_k - [s]_{\delta t})^{-\frac{7}{8}} \left\| f'(X^N(s)) \right\|_{L^p(\Omega; \mathbb{R})} \left\| \int_s^{[s]_{\delta t} + \delta t} e^{-A^2([s]_{\delta t} + \delta t - r)} A^{1-\frac{7}{8}} \right\| \]
\[ \| p^N A^\gamma f(X^N(r)) \|_{L^p(\Omega; \mathbb{R})} \| ds \leq C(T, X_0, p) \delta t^{\frac{1}{2} + \frac{\gamma}{2}}. \]

From the stochastic Fubini theorem, the Burkholder–Davis–Gundy inequality and Hölder inequality, it follows that for \( p \geq 2, \)

\[ I_{23} = \left\| \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} T_{\delta t}^{k-\lfloor \delta t \rfloor} A f'(X^N(s)) \int_{t_j}^{t_{j+1}} e^{-A^2([s]_\delta + \delta t - r)} P^N dW(r) ds \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ = \left\| \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} T_{\delta t}^{k-\lfloor \delta t \rfloor} A f'(X^N(s)) e^{-A^2([s]_\delta + \delta t - r)} P^N ds dW(r) \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ \leq C_p \left\| \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t_{j+1}} T_{\delta t}^{k-\lfloor \delta t \rfloor} A f'(X^N(s)) e^{-A^2(t_{j+1}-r)} P^N ds \right\|_{L^p(\Omega; \mathbb{L}^2)} ds dr \]
\[ \leq C_p \delta t^{\frac{1}{2}} \left\| \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} T_{\delta t}^{k-\lfloor \delta t \rfloor} A f'(X^N(s)) e^{-A^2(t_{j+1}-r)} P^N \right\|_{L^p(\Omega; \mathbb{L}^2)} ds dr \]
\[ \leq C(T, X_0, p) \delta t^{\frac{1}{2} + \frac{\gamma}{2} - \frac{\gamma}{2}}. \]

Due to the continuity of \( Z^N \) and \( Y^N \) and Sobolev embedding theorem, we obtain for small \( \epsilon > 0, \)

\[ I_{24} = \left\| \int_0^{t_k} T_{\delta t}^{k-\lfloor \delta t \rfloor} A^{1+\frac{1}{2}+\epsilon} A^{\frac{3}{2}+\epsilon} P^N \left( \int_0^{1} f''(\lambda X^N(s) + (1-\lambda)X^N([s]_\delta + \delta t)) ds \right) \right\|_{L^p(\Omega; \mathbb{R})} \]
\[ \leq C \int_0^{t_k} \| T_{\delta t}^{k-\lfloor \delta t \rfloor} A^{1+\frac{1}{2}+\epsilon} A^{\frac{3}{2}+\epsilon} P^N \|_{L^p(\Omega; \mathbb{L}^2)} ds \leq C(T, X_0, p) \delta t^{\frac{1}{2}}.\]

Combining (20) and the above regularity estimates, we complete the proof. \( \square \)

Similar arguments yield the following estimate in the negative Sobolev space.

**Corollary 4.3.** Let \((I - L)X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2})\), \( N \in \mathbb{N}^+ \) and \( p \geq 1. \) There exists a positive constant \( C(X_0, T, p) \) such that

\[ \| Y^N(t_k) - \tilde{Y}_k^N \|_{L^p(\Omega; \mathbb{H}^{\gamma-1})} \leq C(X_0, T, p) \delta t^{\frac{1}{2}}. \] (21)

Next, we show the optimal regularity of \( \tilde{Y}_k^N \) and the optimal convergence analysis of \( \tilde{Y}_k^N - Y_k^N \) in \( \mathbb{H}^{-1}. \)

**Lemma 4.6.** Let \((I - L)X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2}), p \geq 1. \) Then \( \tilde{Y}_k^N \) satisfies

\[ \mathbb{E} \left[ \sup_{k \leq K} \| (I - L)\tilde{Y}_k^N \|_{\mathbb{H}^\gamma}^p \right] \leq C(X_0, T, p). \]
Lemma 4.7. Under the condition of Lemma 4.5, there exist $\delta t_0 \leq 1$ and $C(X_0, T, p) > 0$ such that for any $\delta t \leq \delta t_0$, $N \in \mathbb{N}^+$, we have
\[
\|Y^N_k - Y^N_k\|_{L^p(\Omega; H^{-1})} \leq C(X_0, T, p)\delta t^2. \tag{22}
\]
Proof. From the definitions of $Y^N_{k+1}$ and $\tilde{Y}^N_{k+1}$, it follows that
\[
\|Y^N_{k+1} - \tilde{Y}^N_{k+1}\|_{H^{-1}}^2 \leq \|Y^N_k - \tilde{Y}^N_k\|_{H^{-1}}^2, \quad 2\|A^*(Y^N_{k+1} - \tilde{Y}^N_{k+1})\|^{2\delta t} \\
- 2\langle P^N (f(X^N_{k+1}) - f(Y^N(t_{k+1}) + Z^N_{k+1})), (Y^N_{k+1} - \tilde{Y}^N_{k+1}) \rangle \delta t \\
\leq \|Y^N_k - Y^N_k\|_{H^{-1}}^2 - 2\|A^*(Y^N_{k+1} - \tilde{Y}^N_{k+1})\|^{2\delta t} \\
- 2\langle f(Y^N_{k+1}) - f(Y^N(t_{k+1}) + Z^N_{k+1}), (Y^N_{k+1} - \tilde{Y}^N_{k+1}) \rangle \delta t \\
- 2\langle f(\tilde{Y}^N_{k+1} + Z^N_{k+1}) - f(Y^N(t_{k+1}) + Z^N_{k+1}), (Y^N_{k+1} - \tilde{Y}^N_{k+1}) \rangle \delta t \\
\leq \|Y^N_k - Y^N_k\|_{H^{-1}}^2 + 2C\|Y^N_{k+1} - Z^N_{k+1}\|_{H^{-1}}^2 \delta t \\
- 2C\|(I - L)(f(\tilde{Y}^N_{k+1} + Z^N_{k+1}) - f(Y^N(t_{k+1}) + Z^N_{k+1}))\|_{H^{-1}}^2 \delta t.
\]
By using the a priori estimates of $Y^N_{k+1}$, $X^N$ and $Z^N_{k+1}$, and Lemma 4.3, we have
\[
\|Y^N_{k+1} - \tilde{Y}^N_{k+1}\|_{L^2p(\Omega; H^{-1})}^2 \\
\leq C\sum_{j=0}^{k-1} \|(I - L)f(Y^N(t_{j+1}) + Z^N_{j+1}) - f(Y^N(t_{j+1}) + Z^N_{j+1})\|_{L^2p(\Omega; H^{-1})}^2 \delta t \\
\leq C(X_0, p)\sum_{j=0}^{k-1} (1 + \|Y^N(t_{j+1})\|^4_{W^{1,\infty}} + \|\tilde{Y}^N_{j+1}\|^4_{W^{1,\infty}} + \|Z^N_{j+1}\|^4_{W^{1,\infty}}) \\
\quad \times \|\tilde{Y}^N_{j+1} - Y^N(t_{j+1})\|_{L^2p(\Omega; H^{-1})}^2 \delta t \\
\leq C(X_0, T, p)\delta t^2,
\]
which completes the proof by the Hölder inequality. \qed

Based on the interpolation method, Lemmas 4.6 and 4.7, we obtain the following convergence result.

Proposition 4.2. Under the condition of Lemma 4.5, there exist $\delta t_0 \leq 1$ and $C(X_0, T, p) > 0$ such that for any $\delta t \leq \delta t_0$, $N \in \mathbb{N}^+$, we have
\[
\|Y^N(t_k) - Y^N_k\|_{L^p(\Omega; H)} \leq C(X_0, T, p)\delta t^2. 
\]
Proof. By the mild form of $\tilde{Y}^N_k$ and $Y^N_k$ and Lemma 4.3, we have for any $\beta < 1$,
\[
\|\tilde{Y}^N_k - Y^N_k\| \leq \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} T^k_{\delta t}[s]P^N A^* A^{-\frac{1}{2}}(f(Y^N([s], t)) + Z^N_{[s]}) - f(Y^N_{[s]} + Z^N_{[s]}))ds \\
\leq C\sum_{j=0}^{k-1} (t_k - t_j)^{-\frac{1}{2}}(1 + \|Y^N(t_j) + Z^N_j\|_E^2 + \|Y^N_j + Z^N_j\|_E^2)\|Y^N_j - Y^N(t_j)\|_{H^{-\beta}} \delta t.
\]

21
Together with Lemmas 4.5, 4.6 and 4.7, we obtain
\[ \|Y^N(t_k) - Y^N_k\|_{L^p(\Omega; H)} \leq \|Y^N(t_k) - \bar{Y}^N_k\|_{L^p(\Omega; H)} + \|\bar{Y}^N_k - Y^N_k\|_{L^p(\Omega; H)} \leq C(T, X_0, p)\delta t^{\frac{1}{2}}. \]

\[ 22 \]

**Theorem 4.2.** Let \((I - L)X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{d}{2})\), \(N \in \mathbb{N}^+\) and \(p \geq 1\). There exist \(\delta t_0 \leq 1\) and \(C(X_0, T, p) > 0\) such that for any \(\delta t \leq \delta t_0, N \in \mathbb{N}^+\), we have
\[ \|X^N_k - X(t_k)\|_{L^p(\Omega; H)} \leq C(X_0, T, p)(\delta t^{\frac{1}{p} + \lambda N^2}). \]

**Proof.** Notice that
\[ \|X^N_k - X(t_k)\| \leq \|X^N_k - X^N(t_k)\| + \|X^N(t_k) - X(t_k)\| \]
\[ \leq \|Y^N_k - Y^N(t_k)\| + \|X^N(t_k) - X(t_k)\|. \]

By using Lemma 4.5, Proposition 4.2 and Theorem 4.1, we complete the proof.

We also remark that the interpolation approach is also available for the numerical analysis on the strong convergence rates of the finite element method and the implicit Euler method for Eq. (1) and will be studied further.

## 5 Application to the cases of general noise and high dimension

Now we extend this approach to study the cases of general noise and high dimension. For convenience, we assume that \(\gamma \in (0, 4]\). In the case that \(d = 1\), \(\gamma > 0, \|A^{\frac{\gamma}{2}}Q^\frac{1}{2}\|_{L^2} < \infty\), we assume that \(Q\) commutes with \(A\) or \(\gamma > \frac{4}{d}\). The result of the case \(\gamma > 4\) is similar, we omit the details.

**Lemma 5.1.** Let \(d = 1, \gamma \in (0, 4], \|A^{\frac{\gamma}{2}}Q^\frac{1}{2}\|_{L^2} < \infty, (I - L)X_0 \in \mathbb{H}^\gamma \cap L^6\) and \(p \geq 1\). In addition, assume that \(Q\) commutes with \(A\) or \(\gamma > \frac{1}{2}\), then the unique mild solution \(X\) of Eq. (1) satisfies
\[ \sup_{t \in [0, T]} \mathbb{E}\left[\| (I - L)X(t)\|_{L^p}^p \right] \leq C(X_0, T, p), \]
and
\[ \mathbb{E}\left[\|X(t) - X(s)\|_{L^p}^p \right] \leq C(X_0, T, p)(t - s)^{\min\left(\frac{d}{2}, \frac{4}{d}\right)p} \]
for a positive constant \(C(X_0, T, p)\) and \(0 \leq s \leq t \leq T\).
Proof. The conditions on $Q$ and $A$ ensure that

$$
E\left[ \sup_{t \in [0,T]} \|Z(t)\|_{L^p}^p \right] + E\left[ \sup_{t \in [0,T]} \|(I - \mathbb{L})Z(t)\|_{H^7}^{2p} \right] \leq C(T, p).
$$

According to (9) and the proof of Proposition 3.1, we have

$$
E\left[ \sup_{t \in [0,T]} \|(I - \mathbb{L})Y(t)\|_{L^p}^{2p} \right] + E\left[ \left( \int_0^T \|(-A)(I - \mathbb{L})Y(s)\|^2 \, ds \right)^p \right] \leq C(T, X_0, p).
$$

Now it suffices to deduce the optimal regularity of $(I - \mathbb{L})Y$. From the mild form of $(I - \mathbb{L})Y(\tau)$ for Eq. (8), it follows that if $\gamma < 2$, then

$$
\|\gamma \leq C\|X_0\|_{H^7} + C \int_0^t \|e^{-A(t-s)}(I - \mathbb{L})f(Y(s) + Z(s))\|_{H^7} \, ds
$$

$$
\leq C\|X_0\|_{H^7} + C \int_0^t \|e^{-A(t-s)}(I - \mathbb{L})f(Y(s) + Z(s))\|_{H^7} \, ds
$$

$$
\leq C\|X_0\|_{H^7} + C \int_0^t \|Y(t)\|_{L^p}^2 + \|Y(t)\|_{L^p}^2 \, ds.
$$

By taking the $p$-th moment and making use of the a priori estimates of $\|\mathbb{L}Y(s)\|_{L^p}$, $\|(I - \mathbb{L})Y(s)\|_{L^p}$ and $\|Z(s)\|_{H^7}$, we finish the proof for the case $\gamma < 2$. For the case $\gamma \geq 2$, repeating the above arguments and using similar arguments in the proof of Lemma 4.3, we obtain for a small $\epsilon > 0$,

$$
\|\gamma \leq C\|X_0\|_{H^7} + C \int_0^t \|Y(t)\|_{L^p}^2 + \|Y(t)\|_{L^p}^2 \, ds.
$$

If $\gamma \in [2, 4]$, from the estimate of $\|(I - \mathbb{L})X(s)\|_{H^\beta}$, $\beta < 2$, it only suffices to bound the moment of $\|X(s)\|_{W^{1,\infty}}$. By the Sobolev embedding theorem, we have for some small $\epsilon_1$ and any $p \geq 1$,

$$
\|X(s)\|_{L^p(\Omega; W^{1,\infty})} \leq C\|X(s)\|_{L^p(\Omega; H^{2+\epsilon_1})} \leq C(T, X_0, p).
$$

If $\gamma = 4$, based on the above estimate for the case $\gamma < 4$, we only need to estimate $\|X(s)\|_{W^{2,\infty}}$. By using the Sobolev embedding theorem, we obtain

$$
\|X(s)\|_{L^p(\Omega; W^{2,\infty})} \leq C\|X(s)\|_{L^p(\Omega; H^{3+\epsilon_1})} \leq C(T, X_0, p).
$$
Combining the estimates in all cases of \( \gamma \), we complete the proof of the optimal spatial regularity estimate (24).

Now we are in the position to show the temporal regularity estimate (25). It is known that for \( s \leq t, \gamma \leq 4 \),
\[
\mathbb{E} \left[ \| Z(t) - Z(s) \|^p \right] \leq C(T, p)(t - s)^{\min \left( \frac{\gamma}{2}, \frac{\gamma}{4} \right)}.
\]

Similar arguments in the proof of Proposition 3.2 yield that if \( \gamma < 2 \),
\[
\mathbb{E} \left[ \| Y(t) - Y(s) \|^p \right] \leq C(T, X_0, p)(t - s)^{\frac{\gamma}{2}p}.
\]

It suffices to prove that for the case \( \gamma \in [2, 4] \), we have
\[
\mathbb{E} \left[ \| (I - \mathbb{L})(Y(t) - Y(s)) \|^p \right] \leq C(T, X_0, p)(t - s)^{\frac{\gamma}{2}p}.
\]

The mild form of \((I - \mathbb{L})Y\) yields that for a small \( \epsilon_1 > 0 \)
\[
\| (I - \mathbb{L})(Y(t) - Y(s)) \|
\leq C\| (I - \mathbb{L})X_0 \|_{\mathcal{H}^\gamma}(t - s)^{\frac{\gamma}{2}} + \int_s^t (t - s)^{-\frac{\gamma}{2}} \| f(Y(r) + Z(r)) \| dr
\]
\[
+ \int_0^s (s - r)^{-\frac{\gamma}{2}} \| A^{-1+\epsilon}(e^{-\lambda_\epsilon(t-s)} - I)A^{-\frac{\gamma}{2}}(I - \mathbb{L})f(Y(r) + Z(r)) \| dr
\]
\[
\leq C \left( \| (I - \mathbb{L})X_0 \|_{\mathcal{H}^\gamma}(t - s)^{\frac{\gamma}{2}} + (t - s)^{\frac{\gamma}{2}} \left( 1 + \sup_{r \in [0, T]} \| Y(r) \|_{H^1}^3 + \sup_{r \in [0, T]} \| Z(r) \|_{H^1}^3 \right) \right)
\]
\[
+ C(t - s)^{\frac{\gamma}{2} - \frac{\gamma}{2}} \left( 1 + \sup_{r \in [0, T]} \| Y(r) \|_{H^1}^3 + \sup_{r \in [0, T]} \| Z(r) \|_{H^1}^3 \right).
\]

By taking the \( p \)-th moment and using the a priori estimates of \( Y \) and \( Z \), we get
\[
\mathbb{E} \left[ \| (I - \mathbb{L})(Y(t) - Y(s)) \|^p \right] \leq C(T, X_0, p)(t - s)^{\frac{\gamma}{2}p}.
\]

Combining all the above estimates, we complete the proof. \( \square \)

**Proposition 5.1.** Let \( d = 1, \gamma \in (0, 4], \| A^{-\frac{\gamma}{2}}Q_{\frac{\gamma}{2}} \|^2_{L^p} < \infty \) and \((I - \mathbb{L})X_0 \in \mathcal{H}^\gamma \cap E\). Suppose that \( Q \) commutes with \( A \) or \( \gamma > \frac{1}{2} \). Then we have
\[
\left\| X_N(t) - X(t) \right\|_{L^p(\Omega; H)} \leq C(T, X_0, p)\lambda_N^{-\frac{\gamma}{2}},
\]
where \( C(T, X_0, p) \) is some positive constant and \( t \in [0, T] \). Furthermore, if \( Q \) commutes with \( A \) and \( \gamma > \frac{1}{2} \), or \( \gamma \in \left( \frac{3}{2}, 4 \right] \), then it holds that
\[
\left\| X_N(t) - X(t) \right\|_{L^p(\Omega; H)} \leq C'(T, X_0, p)\lambda_N^{-\frac{\gamma}{2}},
\]
where \( C'(T, X_0, p) \) is some positive constant and \( t \in [0, T] \).
Proof. If $Q$ commutes with $A$ or $\gamma > \frac{1}{2}$, the procedures in Theorem 4.1 yields the spatial error estimate with an infinitesimal factor. If in addition it holds that $Q$ commutes with $A$ and $\gamma > \frac{1}{4}$, or that $\gamma \in (\frac{3}{4}, 4]$, using the the procedures in Theorem 4.1 and combining with Corollary 4.2, we get the desired result. \hfill \Box

**Proposition 5.2.** Let $d = 1$, $\gamma \in (0, 2]$, $\|A^2 Q\frac{\gamma}{2}\|_{L^2} < \infty$, $(I - L)X_0 \in H^\gamma \cap L^6$ and $p \geq 1$. Suppose that $Q$ commutes with $A$ and $\gamma > \frac{1}{2}$ or that $\gamma \in (\frac{3}{2}, 2)$. Then there exist $\delta t_0 \leq 1$ and $C(X_0, T, p) > 0$ such that for any $\delta t \leq \delta t_0$, $N \in \mathbb{N}^+$, the numerical solution $X^N_k$, $k \leq K$ is strongly convergent to $X$ and satisfies

$$\|X^N_k - X(t_k)\|_{L^p(\Omega; H)} \leq C(X_0, T, p)(\delta t^\frac{\gamma}{2} + \lambda_N^{-\frac{\gamma}{2}}).$$

Let $d = 1$, $\gamma \in (2, 4]$, $\|A^{2 - \frac{\gamma}{2}} Q\frac{\gamma}{2}\|_{L^2} < \infty$, $(I - L)X_0 \in H^\gamma$ and $p \geq 1$. Then there exist $\delta t_0 \leq 1$ and $C(X_0, T, p) > 0$ such that for any $\delta t \leq \delta t_0$, $N \in \mathbb{N}^+$, we have

$$\|X^N_k - X(t_k)\|_{L^p(\Omega; H)} \leq C(X_0, T, p)(\delta t^\frac{\gamma}{2} + \lambda_N^{-\frac{\gamma}{2}}).$$

Proof. Following the procedures in the proof of Lemmas 4.5 and 4.7, it suffices to deduce the temporal strong convergence rate of $\|Y^N(t_k) - \bar{Y}^N_k\|$. By repeating the procedures in the proof of Lemma 4.5, and using Corollary 4.2 and Remark 4.2, we have that for $\gamma < 2$,

$$\|Y^N(t_k) - \bar{Y}^N_k\|_{L^p(\Omega; H)} \leq C(X_0, T, p)\delta t^\frac{\gamma}{2},$$

and that for $\gamma \in [2, 4]$,

$$\|Y^N(t_k) - \bar{Y}^N_k\|_{L^p(\Omega; H^{-1})} \leq C(X_0, T, p)\delta t.$$

Then combining the proof of the arguments in Theorem 4.2 and Proposition 4.2, and the interpolation equality, we complete the proof. \hfill \Box

**Remark 5.1.** If $d = 1$, $\gamma \in (0, 2]$, $Q$ commutes with $A$, $\|A^{2 - \frac{\gamma}{2}} Q\frac{\gamma}{2}\|_{L^2} < \infty$, $(I - L)X_0 \in H^\gamma \cap L^6$ and $p \geq 1$, then there exist $\delta t_0 \leq 1$ and $C(X_0, T, p) > 0$ such that

$$\|X^N_k - X(t_k)\|_{L^p(\Omega; H)} \leq C(X_0, T, p)(\delta t^\frac{\gamma}{2} + \lambda_N^{-\frac{\gamma}{2}}).$$

In the case that $d = 2, 3$, $\|A^2 Q\frac{\gamma}{2}\|_{L^2} < \infty$, we also obtain the strong convergence rate of the proposed method. Since the deterministic Cahn–Hilliard equation defines a gradient flow in $H^{-1}$ for the energy functional $J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)dx, u \in H^1$, we have the boundedness of $J$ in this case (see e.g. [26, Theorem 3.1]). Then we follow the arguments in the proof of Propositions 3.1 and 3.2 and obtain the following optimal regularity estimates.
Lemma 5.2. Let \( d = 2, 3, \| A^{-1+\frac{1}{2}} Q^{1/2} \|_{L^2} < \infty, \gamma \in [3, 4], (I - L)X_0 \in \mathbb{H}^\gamma \) and \( p \geq 1 \). Then the unique mild solution \( X \) of Eq. (1) satisfies

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \| (I - L)X(t) \|_{L^p}^p \right] \leq C(X_0, T, p),
\]

and

\[
\mathbb{E} \left[ \| X(t) - X(s) \|_p^p \right] \leq C(X_0, T, p)(t - s)^{\frac{\gamma}{2}}
\]

for a positive constant \( C(X_0, T, p) \) and \( 0 \leq s \leq t \leq T \).

Proof. From [26, Theorem 3.1] and Sobolev embedding theorem, it follows that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| (I - L)Y(t) \|_{L^p}^{2p} \right] \leq C(T, X_0, p).
\]

Then the arguments in Lemma 5.1, together with the Sobolev embedding theorem, yields the desired results.

Based on the above regularity estimates, we give the following strong convergence error estimate of the proposed numerical scheme. Since the Sobolev embedding \( L^\infty \rightarrow H^1 \) does not hold, the convergence rate of the numerical scheme by the interpolation approach is not optimal. Thus we need to make use of the regularity properties of exact and numerical solutions in this case.

Proposition 5.3. Under the condition of Lemma 5.2. There exist \( \delta t_0 \leq 1 \) and \( C(X_0, T, p) > 0 \) such that for any \( \delta t \leq \delta t_0, N \in \mathbb{N}^+ \), the numerical solution \( X^N_k, k \leq K \) satisfies

\[
\| X^N_k - X(t_k) \|_{L^p(\Omega; H)} \leq C(X_0, T, p)(\delta t + \frac{1}{N^{\gamma}}).
\]

Proof. Following the proof of Theorem 4.1, we have

\[
\| Y^N(t) - P^N Y(t) \|_{L^2(\Omega; H^{-1})} + \| \int_0^t \| \nabla (Y(s) - Y^N(s)) \|_{L^p(\Omega; \mathbb{R})}^2 ds \|_{L^p(\Omega; H)} \leq C(T, X_0, p)N^{-\gamma}.
\]

By the mild form of \( Y^N(t) \) and \( P^N Y(t) \), we get

\[
\| Y^N(t) - P^N Y(t) \| \leq \int_0^t e^{-A(t-s)}A(f(Y^N + Z^N) - f(Y + Z))ds \leq C \int_0^t (t-s)^{-\frac{1}{2}} \int_0^1 \left\| (f'(\theta(Y^N + Z^N) + (1-\theta)(Y + Z))((I - P^N)Y + (I - P^N)Z)d\theta \right\| ds + C \int_0^t (t-s)^{-\frac{1}{2}} \left\| (f'(\theta(Y^N + Z^N) + (1-\theta)(Y + Z))((Y^N(t) - P^N Y(t)) \right\|_{H^1} d\theta ds.
\]

26
From the regularity estimates in Lemma 5.2, it follows that
\[ \|Y^N_k - Y(t_k)\|_{L^p(\Omega; H)} \leq C\lambda_N^{-\frac{2}{p}}, \]
which, combined with the arguments in Theorem 4.1 yields the desired spatial estimate.

The arguments in the proofs of Lemmas 4.5 and 4.7 lead to
\[ \|\tilde{Y}^N_k - Y^N_k\|_{L^2(\Omega; H^{\frac{1}{p}-1})}^2 + \left\| \int_0^{t_k} \|\nabla(\tilde{Y}^N_{[\sigma]+1} - Y^N_{[\sigma]+1})\|^2 \right\|_{L^p(\Omega; \mathbb{R})} \leq C(T, X_0, p)\delta t^2. \]

Then repeat the above procedures for spatial error estimate and the arguments in Theorem 4.2, we completes the proof.  

References

[1] R. Anton, D. Cohen, and L. Quer-Sardanyons. A fully discrete approximation of the one-dimensional stochastic heat equation. IMA Journal of Numerical Analysis, dry060, https://doi.org/10.1093/imanum/dry060, 2018.

[2] D. C. Antonopoulou, G. Karali, and A. Millet. Existence and regularity of solution for a stochastic Cahn–Hilliard/Allen–Cahn equation with unbounded noise diffusion. J. Differential Equations, 260(3):2383–2417, 2016.

[3] S. Becker, B. Gess, A. Jentzen, and P. E. Kloeden. Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen-Cahn equations. arXiv:1711.02423, 2017.

[4] S. Becker and A. Jentzen. Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg–Landau equations. arXiv:1601.05756, 2016.

[5] H. Bessaih and A. Millet. On strong L2 convergence of time numerical schemes for the stochastic 2d Navier-Stokes equations. arXiv:1801.06455, 2018.

[6] D. Blömker and A. Jentzen. Galerkin approximations for the stochastic Burgers equation. SIAM J. Numer. Anal., 51(1):694–715, 2013.

[7] D. Blömker, M. Kamrani, and S. M. Hosseini. Full discretization of the stochastic Burgers equation with correlated noise. IMA J. Numer. Anal., 33(3):825–848, 2013.

[8] C. E. Bréhier, J. Cui, and J. Hong. Strong convergence rates of semi-discrete splitting approximations for stochastic Allen–Cahn equation. IMA J. Numer. Anal., dry052, https://doi.org/10.1093/imanum/dry052, 2018.
[9] C. E. Bréhier and L. Goudenège. Analysis of some splitting schemes for the stochastic Allen–Cahn equation. arXiv:1801.06455, 2018.

[10] C. E. Bréhier and L. Goudenège. Weak convergence rates of splitting schemes for the stochastic Allen-Cahn equation. arXiv:1804.04061, 2018.

[11] E. Carelli, E. Hausenblas, and A. Prohl. Time-splitting methods to solve the stochastic incompressible Stokes equation. SIAM J. Numer. Anal., 50(6):2917–2939, 2012.

[12] S. Chai, Y. Cao, Y. Zou, and W. Zhao. Conforming finite element methods for the stochastic Cahn-Hilliard-Cook equation. Appl. Numer. Math., 124:44–56, 2018.

[13] C. Chen, J. Hong, and A. Prohl. Convergence of a \(\theta\)-scheme to solve the stochastic nonlinear Schrödinger equation with Stratonovich noise. Stoch. Partial Differ. Equ. Anal. Comput., 4(2):274–318, 2016.

[14] J. Cui and J. Hong. Analysis of a splitting scheme for damped stochastic nonlinear Schrödinger equation with multiplicative noise. SIAM J. Numer. Anal., 56(4):2045–2069, 2018.

[15] J. Cui and J. Hong. Strong and weak convergence rates of finite element method for stochastic partial differential equation with non-globally lipschitz coefficients. arXiv:1806.01564, 2018.

[16] J. Cui, J. Hong, and Z. Liu. Strong convergence rate of finite difference approximations for stochastic cubic Schrödinger equations. J. Differential Equations, 263(7):3687–3713, 2017.

[17] J. Cui, J. Hong, Z. Liu, and W. Zhou. Strong convergence rate of splitting schemes for stochastic nonlinear Schrödinger equations. J. Differential Equations, https://doi.org/10.1016/j.jde.2018.10.034, 2018.

[18] J. Cui, J. Hong, and L. Sun. Weak convergence and invariant measure of a full discretization for non-globally lipschitz parabolic spde. arXiv:1811.04075, 2018.

[19] G. Da Prato and A. Debussche. Stochastic Cahn-Hilliard equation. Nonlinear Anal., 26(2):241–263, 1996.

[20] D. Furihata, M. Kovács, S. Larsson, and F. Lindgren. Strong convergence of a fully discrete finite element approximation of the stochastic Cahn-Hilliard equation. SIAM J. Numer. Anal., 56(2):708–731, 2018.

[21] M. Hutzenthaler and A. Jentzen. On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients. arXiv:1401.0295, 2014.
[22] A. Jentzen, D. Salimova, and T. Welti. Strong convergence for explicit space-time discrete numerical approximation methods for stochastic Burgers equations. J. Math. Anal. Appl., 469(2):661–704, 2019.

[23] C. Kelly and G. J. Lord. Adaptive time-stepping strategies for nonlinear stochastic systems. IMA J. Numer. Anal., 38(3):1523–1549, 2018.

[24] G. T. Kossioris and G. E. Zouraris. Finite element approximations for a linear Cahn-Hilliard-Cook equation driven by the space derivative of a space-time white noise. Discrete Contin. Dyn. Syst. Ser. B, 18(7):1845–1872, 2013.

[25] M. Kovács, S. Larsson, and F. Lindgren. On the discretisation in time of the stochastic Allen-Cahn equation. Math. Nachr., 291(5-6):966–995, 2018.

[26] M. Kovács, S. Larsson, and A. Mesforush. Finite element approximation of the Cahn-Hilliard-Cook equation. SIAM J. Numer. Anal., 49(6):2407–2429, 2011.

[27] S. Larsson and A. Mesforush. Finite-element approximation of the linearized Cahn-Hilliard-Cook equation. IMA J. Numer. Anal., 31(4):1315–1333, 2011.

[28] Z. Liu and Z. Qiao. Strong approximation of stochastic allen-cahn equation with white noise. arXiv:1801.09348, 2018.

[29] A. K. Majee and A. Prohl. Optimal strong rates of convergence for a space-time discretization of the stochastic Allen-Cahn equation with multiplicative noise. Comput. Methods Appl. Math., 18(2):297–311, 2018.

[30] R. Qi and X. Wang. Optimal error estimates of Galerkin finite element methods for stochastic Allen-Cahn equation with additive noise. arXiv:1804.11331, 2018.

[31] X. Wang. An efficient explicit full discrete scheme for strong approximation of stochastic Allen-Cahn equation. arXiv:1802.09413, 2018.