Deriving sorting algorithms via abductive logic program transformation

Manuel Hernández
Instituto de Computación
Universidad Tecnológica de la Mixteca
C.P. 69000, Huajuapan de León, Oaxaca, México
manuelhg@mixteco.utm.mx

Abstract. Logic program transformation by the unfold/fold method advocates the writing of correct logic programs via the application of some rules to a naive program. This work focuses on how to overcome subgoal-introduction difficulties in synthesizing efficient sorting algorithms from an naive sorting algorithm, through logic program transformation and abductive reasoning.

1 Introduction

Logic program transformation (LPT) helps us to solve the following problem: Given naive but inefficient logic program, find an efficient version of this program. The sorting problem consists in obtaining an ordered succession of comparable objects from an unordered succession. Because some human-guided transformations can involve the adding of subgoals in the body of clauses, we take some techniques from abductive logic programming (ALP) to justify the selection these subgoals. In this work we apply some transformational techniques and abductive logic programming to a naive sorting algorithm to derive some efficient sorting algorithms.

When we derive by transformational methods some of the sorting algorithms, we note that such algorithms are consequence of specific design decisions implicit in the supporting definitions. However, some other design decisions do not follow a strict deductive analysis. It is required a complementary technique for synthesizing some concrete sorting algorithms: some explanations to be entailed within a theory. These explanations are carried out by adding certain atoms to the body of some clauses, preparing this body for some potential applications of the folding rule. Because there are many possible explanations, we have to justify how to obtain the suitable explanations.

Structure of this work

This work is organized as follows. Section 2 gives some preliminaries. Section 3 presents some permutation and order-check algorithms. We exemplify our transformations first deriving an $O(n^3)$ algorithm in Section 4; next, $O(n^2)$ algorithms in Section 5 and 6, and finally an $O(n \log(n))$ algorithm in Section 7. Section 8 compares with related work, and we finalize with some conclusions in Section 9.
2 Preliminaries

We assume a basic familiarity with logic programming. Now, let $S$ be a finite sequence of objects, where these objects are taken from a set $A$, with $A$ having a complete order relation. The sorting problem consists in finding an ordered version sequence $T$ from an unordered sequence $S$. A naive algorithm for solving the sorting problem relies on considering $T$ is a permutation of $S$.

In the insertion algorithm we take $S$ and $T$ as lists, $T$ is initially the empty list, and then we proceed as follows: We take the first element of $S$, $a$, and we insert $a$ into $T$. Now we take the second element of $S$, $b$, and we insert $b$ in the correct position within $T$, and so on, until list $S$ be empty.

In the selection algorithm we select a minimal element of $S$, $a$, we delete $a$ from $S$, obtaining $S'$ and we place $a$ into a new list $T$. Now we proceed to deal with $S'$ to find another minimal element $b$ of $S'$, deleting from $S'$, and placing it into $T$ after $a$, and so on, until list $S$ be empty.

For the mergesort algorithm we split $S$ into two lists, almost of the same size, $S_1$ and $S_2$. Next, we sort $S_1$ and $S_2$, and finally, we merge $S_1$ and $S_2$ into a new list $T$ by intermixing their elements always placing them in the correct position.

Finally, the quicksort algorithm proceeds as follows: Given the list $S$, we select an arbitrary element $a$ in $S$. Now we partition $S$ into two sublists, $S_1$ and $S_2$, with $S_1$ consisting of those elements of $S$ being less or equal to $a$, and $S_2$ consisting of those elements of $S$ being greater than $a$. We apply recursively the same quicksort algorithm to $S_1$ and $S_2$, obtaining $S'_1$ and $S'_2$, respectively, and the final result is the concatenation of $S'_1$, $a$, and $S'_2$.

It has been observed [Dar78] that the insertion algorithm is a particular case of the mergesort algorithm, and the selection algorithm is a particular case of the quicksort algorithm. Also, the mergesort and the quicksort algorithms are representative elements of algorithms following the general strategy of divide-and-conquer [Smi85].

2.1 Abductive Logic Programming

Abductive logic programming (ALP) can overcome certain limitations of logic programming with respect to higher level knowledge representation and reasoning tasks [KKT93]; in our case, the higher level knowledge is that of algorithms designed by human beings, and the reasoning tasks are those of formally deriving through logic program transformation some of these algorithms.

ALP is a suitable framework for declarative problem solving that complements LPT, because in LPT the introduction of subgoals is not easily justified as significant and useful program development step; at least, not at the same degree as the elimination of clauses via the subsumption rule [PPR97] or the application of the unfolding rule and simplification.

There are some intuitive interpretations to understand this landscape of reasoning. We have: (i) Deduction: If we have axioms $A$ and an inference rule $R$, we want to draw some conclusions $C$. (ii) Induction: If we have causes $A$ and some effects $C$, we want to know an inductive rule $R$. (iii) Abduction: If we have
an inference rule \( R \) and some observations \( C \), we want to know the causes \( A \) of these observations.

3 Abductive Logic Programming

We give now the general framework of abductive logic (AL). Given a theory presentation \( T \) and a sentence or observation \( G \), the AL problem consists in finding a set of sentences \( \Delta \), the abductive explanation of \( G \), such that: (1) \( T \cup \Delta \models G \), and (2) \( T \cup \Delta \) is consistent. In the framework of abductive logic programming (ALP), the theory presentation \( T \) is a logic program \( P \) (augmented with the axioms of the Clark equality theory) and the SLDNF-resolution rule.

This an instance of this framework applied to LPT: Let \( A \) be an algorithm that solves efficiently the sorting problem. Suppose that \( A \) is expressed as a logic program \( P \). Having constructed a finite set of logic programs \( S = \{ C_1, \ldots, C_k \} \) \((k \geq 1)\) by transformation (where \( C_1 \) solves directly the sorting problem), we have to find which significant atoms are necessary to introduce as subgoals in the body of some clauses belonging to the logic program \( C_k \) for deriving the logic program \( P \) or a close variant. Now, \( G \) is identified as the algorithm \( A \), expressed as a logic program \( P \). The theory presentation \( T \) is the set of logic programs \( S \) together with some properties obtained by the context of the problem that the algorithm \( A \) solves. The set \( \Delta \) is a set of atoms to be introduced. The new theory \( T \cup \Delta \) should be consistent (integrity constraint), but in our case the problem is simplified to selecting a significant set of atoms and preserving completeness, because we already have the following important property: goal introduction always preserves correctness although completeness can be altered by thinning. Part of our contribution consists in finding what atoms are significant to preserve completeness. Deleting subgoals within the body of some clauses, in contrast to goal introduction, increases the LHM. Also, correctness is also preserved by deleting subgoals, but we should notice the possible superset so created.

Let \( P \) be a logic program. Let \( C \) be a clause belonging to \( P \), \( C : p \leftarrow q_1 \land q_2 \land \ldots \land q_m \). If we add a new subgoal \( q \) to the body of \( C \), we have the new clause \( C' : p \leftarrow q \land q_1 \land q_2 \land \ldots \land q_m \), and a new logic program \( P' \) that differs from \( P \) only by clause \( C \) and \( C' \). By taking the fix-point operator, we can obtain the least Herbrand model of \( P \) (LHM(\( P \))). Now, through the application of the subgoal introduction already made, we can conclude that LHM(\( P' \)) \( \subset \) LHM(\( P \)); if LHM(\( P' \)) \( \subset \) LHM(\( P \)), we have thinned the set LHM(\( P \)) and we lose completeness but not correctness. Integrity constraints, for us, is to try maintaining the original LHM (completeness) because correctness is for sure. Our integrity constraints are intended to maintain invariant the original certainty of the body of clauses. In the following sections we will look at specific techniques and examples of the application of these general steps.
4 Goal introduction keys

In this section we want to characterize the abductive atoms to be added to the body of clauses. This characterization should give us criteria to define a suitable search space \( S \) to identify some atoms as good candidates to continue with a transformation process.

Goal introduction is, at first sight, a pessimistic rule: instead of decreasing the number of resolution steps, we increase it. However, we will use the goal introduction rule as an intermediate step for the application of the folding rule. We call both steps in sequence, an application of the goal introduction followed by an application of the folding rule that takes advantage of the atom added, abductive folding. We want to look for the best explanation (optimality and utility in a certain sense) of the subgoal to be introduced. Let us consider the following logic program:

\[
\begin{align*}
(a) & \quad p \leftarrow q \land t \\
(b) & \quad s \leftarrow q \land r \\
(c) & \quad u \leftarrow m \land t \\
(d) & \quad r \leftarrow \\
(e) & \quad q \leftarrow \\
(f) & \quad t \leftarrow
\end{align*}
\]

We want to add an atom \( Q \) to the body of clause (a) for a posterior folding. This example shows that we can fold in (a) with respect to \( q \) and \( r \), if \( Q = r \), or with respect to \( m \) and \( t \), if \( Q = m \). In the first case, we have \( p \leftarrow s \land t \); in the second case, we have \( p \leftarrow q \lor u \). Later we will see why we prefer to fold with respect to \( s \). In the next, we identify some desirable properties of abductive explanations. These properties facilitate the systematic application of the unfold/fold method, and more specifically, the application of the folding rule.

*Subgoals missing for applying the folding rule.* Our first desirable property of abductive explanations via subgoal introduction is intended to satisfy the application of a folding rule. At this point, we see goal introduction as an auxiliary rule for the folding rule.

*Subgoals preserving successful paths.* Even with a good characterization of candidate atoms to be introduced, subgoal introduction is by no means deterministic. In the previous example we explore two possibilities for adding atoms: If \( Q = r \), we have: \( p \leftarrow s \land t \). If \( Q = m \) then we have \( p \leftarrow q \land u \). We prefer to use the definition of \( s \) because the query \( \leftarrow s \) is successful, in contrast to \( \leftarrow u \). Our second desirable property for this choice here is to preserve completeness. Similarly, we prefer explanations participating on the major number of conclusions (relevancy).

*Variable’s coordination.* Our third desirable property is enunciated as a request to coordinate the occurrence of variables. This is also better illustrated through an example, this time involving predicates having variables as arguments:

\[
\begin{align*}
p(X,Y) & \leftarrow q(X) \land r(Y) \\
q(X) & \leftarrow s(X) \\
r(Y) & \leftarrow t(Y) \\
m(X,Y) & \leftarrow q(X) \land l(Y) \\
n(X,Y) & \leftarrow o(X) \land r(Y)
\end{align*}
\]
We introduce \( l(Y) \) to obtain \( p(X, Y) \leftarrow q(X) \land l(Y) \land r(Y) \) so that \( Y \) is now to satisfy \( l/1 \) and \( r/1 \) instead of only \( r/1 \). This is like a filter for discarding some possible terms \( Y \). Folding, we have \( p(X, Y) \leftarrow m(X, Y) \land r(Y) \). The point is: We do not constraint original variables unless they are linked to others.

**General constraints of ALP.** Other constraints are valid here as in abductive logic in general: minimality (\( o(X) \) is preferred to \( o(X) \land o(X) \)) and most specific terms to link subgoals through variables: we prefer \( o(X) \) instead of \( o(g(X)) \) or we prefer \( o(X) \) to \( o(a) \) (\( a \) is a constant).

**Occam’s razor.** Finally, we look for properties obtained from the existent subgoals. Because goal introduction is so demanding (because the big search space), we want to exploit the information already provided by the existent atoms instead of introducing new ones or, at least, to prepare the ground for introducing new ones.

If we add atoms to the body of clauses, these atoms should be characterized in some of the following types (see page 10, mod-survey.dvi):

1. The new information is already deducible from the current atoms; we can explicitly extract information from the existent atoms through semantic domain properties.
2. Some parts are subsumed: we can delete them.
3. The new subgoal leads to contradiction: clause would be erased. We have to avoid having an implosion (\( \text{LHM}(P) = \emptyset \)). So that we should be careful about collapsing the LHM.
4. The new information cannot derived from the current atoms.

Other cases are possible. For example, we can need new syntactical versions of terms.

## 5 Further Details about Abductive Folding

In this section we elaborate on some details about the description and the application of abductive folding.

Sometimes we need to discover the most direct way to explain a conclusion. For example, in the following program (a) \( p \leftarrow q \), (b) \( q \leftarrow r \) and (c) \( q \leftarrow s \), on the one hand, there is an explanation for \( p \), namely, \( q \). The conclusion \( q \), on the other hand, has two possibilities: either \( r \) or \( s \).

Now consider the following program: (a) \( p \leftarrow q \) and (b) \( q \leftarrow G \). Here we have an explanation for \( p \), but we do not have any explanation for \( q \) (the meta-variable \( G \) represents a possible explanation). By transitivity, really we do not have any explanation for \( p \) either. This transitivity can be made explicit as follows: From \( p \leftarrow q \), and \( q \leftarrow G \), by unfolding \( q \) in \( p \leftarrow q \) we get: \( p \leftarrow G \). Now it is evident that we did not know how to conclude \( p \). We call \( q \) a weak predicate. Weakness of predicates is made explicit through the unfolding rule.
Now suppose $q$ has two possible explanations: $r$ and $s$. If we know that $r$ is false, we discard $r$. If we know that $s$ is true, we prefer $s$ to $r$. To preserve the major possible completeness, we prefer true explanations because false (or nonexistent, in negation as failure) explanations lead us to failed paths.

After finding some points where goal introduction would be possible, we have to corroborate a sensible use of the subgoal within the body of the clause. As we already seen, a first requirement is that the subgoal contributes to apply the folding rule. In contrast to some proposal of abductive reasoning, here the predicate of the atom used as subgoal can be the same than the head of the clause. This is often discarded because we would incur in petitio principii (to beg the question, to call the question), trying to explain an effect through the same effect. However, when arguments of predicates are given, we can use the concept of well-founded recursion: We can explain an effect of an object by the effects of the smaller constituent objects.

Now we consider the calculus of the subgoals missing for applying the folding rule. We show on Fig. 1 the usual folding (by using in this example only one clause).

![Fig. 1. Usual and abductive folding.](image)

Abductive folding requires at least two steps, see (b) on Fig. 1. Other steps such as calculating plain complements or coordinating variables are also important. We explain these concepts.

If we want to fold $p$ by using $C$, we need to calculate the plain complement of folding some part of the body of $p$ by considering the body of $q$. The plain complement in the previous example is: $\{q_1, \ldots, q_n\} \setminus \{q_2, \ldots, q_n\} = \{q_1\}$ so that $\{q_1\}$ is the set of candidates to add to the body of $B$. The plain complement of folding is given by trying to preserve the original terms occurring in $\{q_2, \ldots, q_n\}$ in clause $B$ and instantiating (by specialization).

It is also necessary to speak about a strategy of goal introduction. In our case, we use a greedy strategy to take benefit from the occurrence of well-founded recursion: If we find a component of a possible folding instantiated more closely to the base case, we introduce the plain complement with a substitution $\theta$ for matching the folding clause.
Plain complements indicate us what atoms are missing to fold. Well-founded search indicates us when is appropriate to fold to introduce well-founded recursivity. Both techniques help us to implement an algorithmic strategy for subgoal introduction, although this strategy is greedy and, in any case, the strategy would require at least an approval by a human being. The part of well-foundedness is taken from the partial order $\leq_\tau$, when possible, defined over the Herbrand universe.

To resume, we have described the following meta-algorithm:

1. Use the “need-for-folding” heuristics;
2. identify complements of atoms;
3. when atoms have arguments smaller than head-arguments, choose as candidate for folding;
4. fold.

To summarize, our abductive proposal is: First, considering a good algorithm ($O(n \cdot \log(n))$, $O(n^2)$, $O(n^3)$) as is already known in literature in a procedural way. Second, we transform some clauses to obtain a structure already seen or known in this algorithm. Third, we add goals to the body of these clauses for allowing to fold with respect to previous definitions. We apply the folding rule and, finally, we check whether the selected sorting algorithm has been obtained (when not, we can assess whether the current version so obtained is good enough).

6 Permutation and order-check algorithms

To explain sorting algorithms from sorting by permutation adopt the following general guidelines: By using a permutation algorithm based on merging, and goal introduction, we can explain mergesort and quicksort. Similarly, by using a permutation algorithm based on insertion, and goal introduction, we can explain insertion sort. Finally, by using a permutation algorithm based on selection, and goal introduction we can obtain the selection sort algorithm. Permutation algorithms give us the skeleton of generation of some sorting algorithms, and goal introduction allows to filter some answers before using them, obtaining the concrete recursive calls.

We choose the least Herbrand model augmented with equations and inequations $a < b$, $a \leq b$ as constraints between terms $a$ and $b$ as the meaning of logic programs. Let us consider the sorting problem over the integers and the order relation $a$ less than $b$ (written as $<$, from where we derive by definition the notation of $\leq$, $>$, and $\geq$). The following logic program solves the sorting problem correctly:

$$\text{sort}(Ls_1, Ls_2) \leftarrow \text{perm}(Ls_1, Ls_2) \land \text{ord}(Ls_2)$$

(5)

where $\text{perm}(Ls_1, Ls_2)$ holds if the list $Ls_2$ is a permutation of the list $Ls_1$, and $\text{ord}(Ls_2)$ holds if the list $Ls_2$ is (non-decreasing) ordered by the relation $\leq$.

Under the SLD resolution rule, and given the definitions of $\text{perm}$ and $\text{ord}$, the naive algorithm solves the sorting problem correctly, but if we suppose that the
list $Ls$ has length $n$, the order of execution of this algorithm is $O(n!)$. Our objective is to formulate algorithms more efficient than the naive algorithm through LPT and ALP.

Let us consider three permutation algorithms written as logic programs. Each permutation algorithm will determine the main structure of a distinct sorting algorithm (cf. [Dar78]). Having this structure, the next step will be to identify which atoms are necessary to add to the body of some clauses (via suitable properties where these atoms appear) for a possible application of the folding rule. Our idea is to apply the techniques previously described about plain complements and well-founded terms.

A set of axioms that defines a permutation of a list $Ls$ is:

**Clauses 1 (Perm1, based on insert)**


clause 1

\[
\begin{align*}
\text{perm1}([], []) & \leftarrow \quad (6) \\
\text{perm1}([A | Ls_1], Ls_3) & \leftarrow \text{perm1}(Ls_1, Ls_2) \land \text{insert}(A, Ls_2, Ls_3) \quad (7) \\
\text{insert}(A, Ls, [A | Ls]) & \leftarrow \quad (8) \\
\text{insert}(A, [B | Ls_1], [B | Ls_2]) & \leftarrow \text{insert}(A, Ls_1, Ls_2) \quad (9)
\end{align*}
\]

A second definition of a permutation algorithm is:

**Clauses 2 (Perm2, based on delete)**


clause 2

\[
\begin{align*}
\text{perm2}([], []) & \leftarrow \quad (10) \\
\text{perm2}(Ls, [A | Ls_1]) & \leftarrow \text{delete}(A, Ls, Ls_2) \land \text{perm2}(Ls_2, Ls_1) \quad (11) \\
\text{delete}(A, [A | Ls], Ls) & \leftarrow \quad (12) \\
\text{delete}(A, [B | Ls_1], [B | Ls_2]) & \leftarrow \text{delete}(A, Ls_1, Ls_2) \quad (13)
\end{align*}
\]

To show our third algorithm, first we present a generic definition of permutation, having a double recursive structure:

**Clauses 3 (PermG, based on shuffle)**


clause 3

\[
\begin{align*}
\text{permG}([], []) & \leftarrow \quad (14) \\
\text{permG}([A], [A]) & \leftarrow \quad (15) \\
\text{permG}(Ls, Ls_5) & \leftarrow \text{union}(Ls_1, Ls_2, Ls) \\
& \land \text{permG}(Ls_1, Ls_3) \\
& \land \text{permG}(Ls_2, Ls_4) \\
& \land \text{shuffle}(Ls_3, Ls_4, Ls_5) \quad (16)
\end{align*}
\]

where $\text{union}(Ls_1, Ls_2, Ls)$ denotes a disjunct, nondeterministic choice of the subsets $Ls_1, Ls_2$ of $Ls$ such that if $\text{append}(Ls_1, Ls_2, Ls_3)$ and $\text{union}(Ls_1, Ls_2, Ls)$ hold, then $\text{sort}(Ls_3, Ms)$ and $\text{sort}(Ls, Ms)$ for some sorting algorithm, and the length of $Ls$ is greater than or equal to two. The $\text{shuffle}$ predicate is defined as follows (Tho86, M93):
Clauses 4 (Shuffle)

\[
\begin{align*}
\text{shuffle}([], Ls, Ls) & \leftarrow \tag{17} \\
\text{shuffle}(Ls, [], Ls) & \leftarrow \tag{18} \\
\text{shuffle}([A|Ls_1], [B|Ls_2], [A|Ls_3]) & \leftarrow \text{shuffle}(Ls_1, [B|Ls_2], Ls_3) \tag{19} \\
\text{shuffle}([A|Ls_1], [B|Ls_2], [B|Ls_3]) & \leftarrow \text{shuffle}([A|Ls_1], Ls_2, Ls_3) \tag{20}
\end{align*}
\]

One way of translating \textit{union}(Ls_2, Ls_3, Ls_1) into a logic program is:

Clauses 5

\[
\begin{align*}
\text{split}([], [], []) & \leftarrow \tag{21} \\
\text{split}([A], [], [A]) & \leftarrow \tag{22} \\
\text{split}([A, B|Ls_1], [A|Ls_2], [B|Ls_3]) & \leftarrow \text{split}(Ls_1, Ls_2, Ls_3) \tag{23}
\end{align*}
\]

The goal \textit{split}(Ls_1, Ls_2, Ls_3) divides the list \(Ls_1\) into two sublists, \(Ls_2\) and \(Ls_3\), and \(|Ls_2| = |Ls_3|\) or \(|Ls_2| + 1 = |Ls_3|\), where \(|Ls|\) denotes the length of a list \(Ls\). Now, we can formulate a concrete logical definition of our third permutation algorithm:

Clauses 6 (Perm3, based on \textit{split} and \textit{shuffle})

\[
\begin{align*}
\text{perm}_3([], []) & \leftarrow \tag{24} \\
\text{perm}_3([A], [A]) & \leftarrow \tag{25} \\
\text{perm}_3([A, B|Ls_1], Ls_6) & \leftarrow \text{split}([A, B|Ls_1], Ls_2, Ls_3) \\
& \quad \land \text{perm}_3(Ls_2, Ls_4), \text{perm}_3(Ls_3, Ls_5) \\
& \quad \land \text{shuffle}(Ls_4, Ls_5, Ls_6) \tag{26}
\end{align*}
\]

We also need some possible definitions of \textit{ord}. We give two definitions of \textit{ord}: \textit{ord}_1 accesses consecutively elements of a list, and \textit{ord}_2 delegates further comparisons to a predicate named \textit{minlist}.

Clauses 7 (Ord1, linear)

\[
\begin{align*}
\text{ord}_1([], []) & \leftarrow \tag{27} \\
\text{ord}_1([A], []) & \leftarrow \tag{28} \\
\text{ord}_1([A, B|Ls]) & \leftarrow A \leq B \land \text{ord}_1([B|Ls]) \tag{29}
\end{align*}
\]

Another definition of \textit{ord} is the following:

Clauses 8 (Ord2, subset)

\[
\begin{align*}
\text{ord}_2([], []) & \leftarrow \tag{30} \\
\text{ord}_2([A|Ls]) & \leftarrow \text{minlist}(A, Ls) \land \text{ord}_2(Ls) \tag{31}
\end{align*}
\]

where \textit{minlist}(A, Ls) holds if \(A\) is a lower bound of all the elements of the list \(Ls\) (\(A\) may or may not belong to the set of elements in \(Ls\)), but it is only a check; \textit{minlist} is not able to find this minimum.
Clauses 9

\[ minlist(A, []) \leftarrow \quad (32) \]
\[ minlist(A, [B|Ls]) \leftarrow A \leq B \wedge minlist(A, Ls) \quad (33) \]

Theoretically, this predicate should be invertible, but the inequality in the body of Clause (33) does not allow it; to adjust invertibility, we define a new predicate to find the minimum of a list:

Clauses 10

\[ findmin(-\infty, []) \leftarrow \quad (34) \]
\[ findmin(A, [A]) \leftarrow \]
\[ findmin(A, [B | Ls]) \leftarrow findmin(C, Ls) \wedge min(B, C, A) \quad (35) \]
\[ min(A, B, C) \leftarrow A < B \wedge C = A \quad (36) \]
\[ min(A, B, C) \leftarrow B \leq A \wedge C = B \quad (37) \]

Now, we need to formulate some properties about relationships between order-check and some other predicates that we use. We symbolize with \( P \implies Q \) an implicative and universally quantified logical formula between \( P \) and \( Q \).

Property 1 (Append) If \( append(Ls_1, Ls_2, Ls_3) \) holds,

\[ ord(Ls_3) \implies ord(Ls_1) \wedge ord(Ls_2) \quad (38) \]

Let \( C \) be a number and \( Ls \) be a list of numbers. We use the following notation: \( C < Ls \) denotes \( C < D \), for every number such that \( D \in \{ Ls \} \), and \( Ls \leq C \) denotes \( D \leq C \), for every \( D \in \{ Ls \} \).

Property 2 (Append and an element) If \( append(Ls_1, [A|Ls_2], Ls) \) holds,

\[ ord(Ls) \vdash ord(Ls_1) \wedge ord(Ls_2) \wedge Ls_1 \triangleleft A \wedge A \leq Ls_2 \quad (39) \]

Property 3 (Insert)

\[ insert(A, Ls_1, Ls_2) \wedge ord(Ls_2) \implies ord(Ls_1) \quad (40) \]

Property 4 (Minlist) If \( append([A], Ls_1, Ls_2) \) holds,

\[ ord(Ls_2) \equiv minlist(A, Ls_1) \wedge ord(Ls_1) \quad (41) \]

Property 5 (Merging)

\[ shuffle(Ls_1, Ls_2, Ls_3) \wedge ord(Ls_3) \equiv \]
\[ ord(Ls_1) \wedge ord(Ls_2) \wedge shuffle(Ls_1, Ls_2, Ls_3) \wedge ord(Ls_3) \]
7 Tamaki and Sato’s sorting algorithm

Let us begin with a reconstruction of a derivation of an $O(n^3)$ sorting algorithm. This derivation was given by Tamaki and Sato in [TS84], p. 135, from a definition of sort given by perm1 and ord1. In the course of the derivation, an atom was added to the body of a clause as a subgoal. We will take this derivation as an instance of the previous general results related to abductive folding: an application of the goal introduction rule followed by an application of the folding rule.

To emphasize local transformational developments, we will apply a renaming of the sort predicate at the beginning of each derivation. In this case we rename sort to sort_TS.

Program 1 (Naive program)

$$sort_{TS}(Ls_1, Ls_2) \leftarrow \text{perm1}(Ls_1, Ls_2) \land \text{ord1}(Ls_2) \quad (42)$$

Unfolding perm1 in the body of Clause (42) we obtain:

Program 2

$$sort_{TS}([], []) \leftarrow \quad (43)$$

$$sort_{TS}([A \mid Ls_1], Ls_3) \leftarrow \text{perm1}(Ls_1, Ls_2) \land F1 \land \text{insert}(A, Ls_2, Ls_3) \land \text{ord1}(Ls_3) \quad (44)$$

By $Ls_1 \leq \tau [A, Ls_1]$, reinforced with Property 3 we must add the subgoal ord1(Ls2) to the body of Clause (44) to proceed to apply abductive folding:

$$sort_{TS}([A \mid Ls_1], Ls_3) \leftarrow \text{perm1}(Ls_1, Ls_2) \land \text{ord1}(Ls_2) \land \text{insert}(A, Ls_2, Ls_3) \land \text{ord1}(Ls_3) \quad (45)$$

Folding the subgoals perm1(Ls1, Ls2) and ord1(Ls2) w.r.t. sort_TS, we get a new program more efficient than the naive program:

Program 3 (Tamaki & Sato)

$$sort_{TS}([], []) \leftarrow \quad (46)$$

$$sort_{TS}([A \mid Ls_1], Ls_3) \leftarrow sort_{TS}(Ls_1, Ls_2) \land \text{insert}(A, Ls_2, Ls_3) \land \text{ord1}(Ls_3) \quad (47)$$

We have obtained an $O(n^3)$ algorithm from an $O(n!)$ algorithm. The moded program [Apt97] described under the left-to-right computation rule is: Given a list $[A \mid Ls]$, we order $Ls$; next, we insert $A$ in $Ls$ at some position; and, finally, we check whether the resultant list is ordered. Next, we will derive another algorithm of order $O(n^2)$ from the naive program, the insertion sort algorithm.
8 The insertion sort algorithm

Beginning now from Prog. (3), let us rename the sort\textunderscore TS predicate to inssort and apply a subgoal introduction intended to take benefit from correctly placing the element \( A \). We divide the list \( Zs \) into two sublists \( Ls_1 \) and \( Ls_2 \) through the atom \( \text{append}(Ls_1, Ls_2, Zs) \). Consider the following definition of insert:

\[
\text{insert}(A, Zs, Ls) \leftarrow \text{append}(Ls_1, Ls_2, Zs) \land \text{append}(Ls_1, [A|Ls_2], Ls) \quad (48)
\]

Now, we unfold insert w.r.t. this definition:

Program 4

\[
\text{inssort}([], []) \leftarrow \quad (49)
\]

\[
\text{inssort}([A | Ls], Ls_3) \leftarrow \text{inssort}(Ls, Zs) \land \text{append}(Ls_1, Ls_2, Zs) \land \text{append}(Ls_1, [A | Ls_2], Ls_3) \land \text{ord}(Ls_3) \quad (50)
\]

Clauses 11

\[
\text{inssort}([A | Ls], Ls_3) \leftarrow \text{inssort}(Ls, Zs) \land \text{append}(Ls_1, Ls_2, Zs) \land \text{append}(Ls_1, [A | Ls_2], Ls_3) \land \text{ord}(Ls_3) \land \text{ord}(Ls_1) \land \text{ord}(Ls_2) \land Ls_1 \triangleleft A \land A \triangleleft Ls_2 \quad (51)
\]

Because \( \text{ord}(Ls_3) \) holds and the property that \( \text{ord}(Ls) \) holds for every sublist of \( Ls_3 \), we create a new definition that correctly places \( A \) in the list \( Ls_3 \).

Clauses 12

\[
\text{filter}(A, [], [], []) \leftarrow \quad (52)
\]

\[
\text{filter}(A, [B | Ls_1], [B | Ls_2], Ls_3) \leftarrow B \leq A \land \text{filter}(A, Ls_1, Ls_2, Ls_3) \quad (53)
\]

\[
\text{filter}(A, [B | Ls_1], Ls_2, [B | Ls_3]) \leftarrow A < B \land \text{filter}(A, Ls_1, Ls_2, Ls_3) \quad (54)
\]

\[
\text{filter}(A, Ls_3, Ls_1, Ls_2) \text{ holds if } Ls_3 = Ls_1 \cup Ls_2, Ls_1 \triangleleft A, \text{ and } A \triangleleft Ls_2.
\]

Replacing the subgoal given by (51), we have an insertion algorithm \( O(n^2) \) for correctly solving the sorting problem:

Program 5

\[
\text{inssort}([], []) \leftarrow \quad (55)
\]

\[
\text{inssort}([A | Ls_0], Ls_3) \leftarrow \text{inssort}(Ls_0, Zs) \land \text{filter}(A, Zs, Ls_1, Ls_2) \land \text{append}(Ls_1, [A|Ls_2], Ls_3) \quad (56)
\]

Further minor optimizations are possible (for example, we can use difference lists instead of \( \text{append} \)).
9 Selection algorithm

Let us see again the naive algorithm, this time with the definition of \( \text{perm} \) given by \( \text{perm}^2 \).

\[
selsort(Ls, Ls_1) \leftarrow \text{perm}^2(Ls, Ls_1) \land \text{ord}^2(Ls_1) \tag{57}
\]

Unfolding \( \text{perm}^2 \) in Clause (57), we get

\[
selsort(Ls, [A | Ls_2]) \leftarrow \text{delete}(A, Ls, Ls_1) \land \text{perm}^2(Ls_1, Ls_2) \land \text{ord}^2([A | Ls_2]) \tag{59}
\]

Program 6

where we have made annotations about unfolding and a possible folding point. We have \( \text{ord}^2([A | Ls]) \) if \( \text{minlist}(A, Ls) \) and \( \text{ord}^2(Ls) \). Replacing the subgoals, with \( \text{F1} \) instantiated to \( \text{ord}^2(Ls_2) \), we have:

\[
selsort(Ls, [A | Ls_2]) \leftarrow \text{delete}(A, Ls, Ls_1) \land \text{perm}^2(Ls_1, Ls_2) \land \text{ord}^2(Ls_2) \land \text{minlist}(A, Ls_2) \tag{60}
\]

Now we fold the subgoals \( \text{perm}^2(Ls_1, Ls_2) \) and \( \text{ord}^2(Ls_2) \) w.r.t. the original definition of \( \text{selsort} \):

\[
selsort(Ls, [A | Ls_2]) \leftarrow \text{delete}(A, Ls, Ls_1) \land \text{selsort}(Ls_1, Ls_2) \land \text{minlist}(A, Ls_2) \tag{61}
\]

Now, \( \text{minlist}(A, Ls_2) \equiv \text{minlist}(A, Ls_1) \); hence,

\[
selsort(Ls, [A | Ls_2]) \leftarrow \boxed{\text{delete}(A, Ls, Ls_1) \land \text{minlist}(A, Ls_1)} \land \text{selsort}(Ls_1, Ls_2) \tag{62}
\]

(We have surrounded a conjunction of atoms by a double box anticipating the writing of a new definition and then the application of a folding step w.r.t. this new definition.)

Making \( \text{delete}_{\text{min}}(A, Ls, Ls_1) \leftarrow \text{delete}(A, Ls, Ls_1) \land \text{minlist}(A, Ls_1) \) or, dually, \( \text{delete}_{\text{min}}(A, Ls, Ls_1) \leftarrow \text{findmin}(A, Ls) \land \text{delete}(A, Ls, Ls_1) \) we have:

Program 7 (Selection algorithm)

\[
selsort([], []) \leftarrow \tag{63}
\]

\[
selsort(Ls, [A | Ls_2]) \leftarrow \text{delete}_{\text{min}}(A, Ls, Ls_1) \land \text{selsort}(Ls_1, Ls_2) \tag{64}
\]

plus definitions of \( \text{delete}/3 \) and \( \text{findmin}/2 \)
10 Mergesort algorithm

To begin a new derivation, we rename sort to msort:

\[
\text{msort}(Ls_1, Ls_2) \leftarrow \text{perm3}(Ls_1, Ls_2) \land \text{ord2}(Ls_2)
\]  

(65)

We are now ready to unfold perm3 in the body of Clause (65):

Program 8

\[
\begin{align*}
\text{msort}([], []) & \leftarrow & \quad & (66) \\
\text{msort}([A], [A]) & \leftarrow & \quad & (67) \\
\text{msort}([A, B | Ls_1], B) & \leftarrow \text{split}([A, B | Ls_1], Ls_2, Ls_3) \\
& & & \land \text{perm3}(Ls_2, Ls_4) \land \text{F1} \\
& & & \land \text{perm3}(Ls_3, Ls_5) \land \text{F2} \\
& & & \land \text{shuffle}(Ls_4, Ls_5, Ls_6) \land \text{ord2}(Ls_6)
\end{align*}
\]  

(68)

Now we add two subgoals, using Property 5:

\[
\begin{align*}
\text{msort}([A, B | Ls_1], Ls_6) & \leftarrow \text{split}([A, B | Ls_1], Ls_2, Ls_3) \\
& & \land \text{msort}(Ls_2, Ls_4) \\
& & \land \text{msort}(Ls_3, Ls_5) \\
& & \land \text{shuffle}(Ls_4, Ls_5, Ls_6) \land \text{ord2}(Ls_6)
\end{align*}
\]  

(69)

Now we can fold the subgoals perm3 and ord2 w.r.t. the original definition of msort:

\[
\begin{align*}
\text{msort}([A, B | Ls_1], Ls_6) & \leftarrow \text{split}([A, B | Ls_1], Ls_2, Ls_3) \\
& & \land \text{msort}(Ls_2, Ls_4) \\
& & \land \text{msort}(Ls_3, Ls_5) \\
& & \land \text{shuffle}(Ls_4, Ls_5, Ls_6) \land \text{ord2}(Ls_6)
\end{align*}
\]  

(70)

We write a new definition: \( \text{new}(Ls_1, Ls_2, Ls_3) \leftarrow \text{shuffle}(Ls_1, Ls_2, Ls_3) \land \text{ord2}(Ls_3) \) and fold w.r.t. this new definition:

\[
\begin{align*}
\text{msort}([A, B | Ls_1], Ls_6) & \leftarrow \text{split}([A, B | Ls_1], Ls_2, Ls_3) \\
& & \land \text{msort}(Ls_2, Ls_4) \land \text{msort}(Ls_3, Ls_5) \\
& & \land \text{new}(Ls_4, Ls_5, Ls_6)
\end{align*}
\]  

(71)
From now on, our attention will be centered on new. Unfolding shuffle in the body of new, we get the following clauses:

\[
\begin{align*}
\text{new}([], Ls, Ls) & \leftarrow \text{ord}2(Ls) \\
\text{new}(Ls, [], Ls) & \leftarrow \text{ord}2(Ls) \\
\text{new}([A | Ls_1], [B | Ls_2], [A | Ls_3]) & \leftarrow \\
& \quad \text{shuffle}(Ls_1, [B | Ls_2], Ls_3) \land \text{F1} \land \text{ord}2([A | Ls_3]) \\
\text{new}([A | Ls_1], [B | Ls_2], [B | Ls_3]) & \leftarrow \\
& \quad \text{shuffle}([A | Ls_1], Ls_2, Ls_3) \land \text{F2} \land \text{ord}2([B | Ls_3])
\end{align*}
\]

We should replace the meta-variables F1 and F2 by concrete atoms (or in general, literals) to allow the application of the folding rule. This would give an explanation of having a new predicate self-contained, as is seen in several versions of mergesort algorithms.

\[
\begin{align*}
\text{new}([A | Ls_1], [B | Ls_2], [A | Ls_3]) & \leftarrow \\
& \quad \text{shuffle}(Ls_1, [B | Ls_2], Ls_3) \land \text{ord}2(Ls_3) \land \text{minlist}(A, Ls_3) \\
\text{new}([A | Ls_1], [B | Ls_2], [B | Ls_3]) & \leftarrow \\
& \quad \text{shuffle}([A | Ls_1], Ls_2, Ls_3) \land \text{ord}2(Ls_3) \land \text{minlist}(B, Ls_3)
\end{align*}
\]

After folding, we obtain the following clauses:

\[
\begin{align*}
\text{new}([A | Ls_1], [B | Ls_2], [A | Ls_3]) & \leftarrow \\
& \quad \text{new}(Ls_1, [B | Ls_2], Ls_3) \land \text{minlist}(A, Ls_3) \\
\text{new}([A | Ls_1], [B | Ls_2], [B | Ls_3]) & \leftarrow \\
& \quad \text{new}([A | Ls_1], Ls_2, Ls_3) \land \text{minlist}(B, Ls_3)
\end{align*}
\]

Having folded, the next step is considering how to constrain the general behavior of minlist and its eventual elimination.

From the conjunction \(\text{new}(Ls_1, [B | Ls_2], Ls_3) \land \text{minlist}(A, Ls_3)\) in the body of Clause (78), we can obtain the following consequences: a) \(B \in Ls_3\) because

\[(X \in Ls_1 \lor X \in Ls_2) \land \text{shuffle}(Ls_1, Ls_2, Ls_3) \implies X \in Ls_3;\]

b) \(A \leq X, \forall X. X \in Ls_3\) by the mathematical definition of minlist; c) from a) and b), \(A \leq B\). Thus, we can add the inequality \(A \leq B\) to the body of Clause (78). By a similar argument, we can add the inequality \(B \leq A\) to the body of Clause (79).
new([A | Ls1], [B | Ls2], [A | Ls3]) ← A ≤ B
∧ new(Ls1, [B | Ls2], Ls3) ∧ minlist(A, Ls3) (80)

new([A | Ls1], [B | Ls2], [B | Ls3]) ← B ≤ A
∧ new([A | Ls1], Ls2, Ls3) ∧ minlist(B, Ls3) (81)

The subgoal minlist(A, Ls3) in Clause (80) is unnecessary for the following reasons: First, we observe that the first two arguments of new are already ordered (see Clause (71)). Second, from Clause (78), we have a) A ≤ X, ∀X.X ∈ Ls1; b) B ≤ Y, ∀Y.Y ∈ Ls2; c) A ≤ B ⇒ A ≤ Y, ∀Y.Y ∈ Ls2; d) from a) and c), we have A ≤ Z, ∀Z.Z ∈ Ls3 (because Ls3 is conformed by elements belonging to Ls1 and Ls2). Hence, we can get rid of minlist(A, Ls3) without losing correctness. A similar argument works for Clause (81).

Finally, we arrive at the following program:

**Program 9 (Mergesort algorithm)**

msort([], []) ← (82)

msort([A], [A]) ← (83)

msort([A, B | Ls1], Ls6) ← split([A, B | Ls1], Ls2, Ls3)
∧ msort(Ls2, Ls4) ∧ msort(Ls3, Ls5)
∧ new(Ls4, Ls5, Ls6) (84)

plus definitions of split/3 and new/3 (without minlist) (85)

This program has the essential structure of the mergesort algorithm, and has an \(O(n \log n)\) time complexity order.

### 11 The quicksort algorithm

To derive the quicksort algorithm, we need to see again the Prog. 9. There, our choice of the split(L, Ls1, Ls2) predicate was motivated by \(L = Ls1 \cup Ls2\), without any major constraint. Now we select another definition, arguably more discriminant that the first one:

\[
\text{partition}([A | Ls], Ls1, [A | Ls2]) ← \\
\quad [A | Ls] = Ls1 ++ [A | Ls2] ∧ \\
\quad Ls1 < A ∧ A < Ls2
\] (86)

If \(X = [A | Ls]\), \(\text{partition}([A | Ls], Ls1, [A | Ls2])\) allows that \(X = \{Ls1\} \cup \{[A | Ls2]\}\). Our new choice allows us to eliminate the following (useless) clauses:

new(Ls, [], Ls) ← (87)

new([A | Ls1], [B | Ls2], [B | Ls3]) ← B < A ∧ new([A | Ls1], Ls2, Ls3) (88)

16
Now, we rename new to append:

\[
\begin{align*}
\text{append}^\prime([],[Ls]) & \leftarrow \\
\text{append}^\prime([A\mid Ls_1],[B\mid Ls_2],[A\mid Ls_3]) & \leftarrow \\
& B < A \land \text{append}^\prime(Ls_1,[B\mid Ls_2],Ls_3) 
\end{align*}
\]  

(89)

(90)

Our next step is to eliminate the (unnecessary) comparison \( B < A \).

To implement our new predicate we define the following clauses: \( L \) is divide into two lists, according to Clause (86):

\[
\begin{align*}
\text{partition}(A,[[],[[],[]]) & \leftarrow \\
\text{partition}(A,[B\mid L],[B\mid Ls_1],Ls_2) & \leftarrow \\
& B < A \land \text{partition}(A,L,Ls_1,Ls_2) \\
\text{partition}(A,[B\mid L],Ls_1,[B\mid Ls_2]) & \leftarrow \\
& B > A \land \text{partition}(A,L,Ls_1,Ls_2) 
\end{align*}
\]  

(91)

(92)

(93)

Therefore, we have derived the quicksort algorithm.

12 Comparison with similar work

The ideas of deriving sorting algorithms have been carried out through mathematical-oriented developments [Dar78], logic program derivation synthesis, and functional program transformation [Par91]. In work [CD78], some justifications were absent and are given within our approach.

In [Lau89] and [LP91] the program synthesis proceeds as follows: For each derivation, the user gives a general scheme in clausal form, following a catalog of possible recursion patterns. However, neither dependency with respect to supporting definitions nor correctness are shown; also, in [LP91] there is not any permutation algorithm, and the base cases are added manually.

In [M93] there is a derivation of the mergesort algorithm by using formal languages, but without following a computing paradigm and without any commitment with a specific implementation. We adapted from [M93] the definition of shuffle to logic programming.

13 Conclusions

In this work we have presented the novel concept of abductive folding as a mechanism to overcome some limitations of the unfold/fold method. Abductive folding is carried out through two consecutive steps: an application of the subgoal introduction rule and an application of the traditional folding rule. We have achieved some characterizations to identify suitable atoms to be added to the body of clauses. Some sorting algorithms were derived: A sorting algorithm devised by Tamaki and Sato, the selection sort algorithm, the insertion sort algorithm, the mergesort algorithm and the quicksort algorithm. Some of the derivations of
these sorting algorithms were carried out through LPT with an occasional and complementary ALP support. ALP has been applied for justifying the selection and the introduction of atoms within the body of clauses, adapting the methods of reasoning belonging to ALP to LPT. This is the part of including non-declarative heuristic and operational control to explain or refine a purely declarative problem description via a logical model \[\text{[KKT93]}\].

Tamaki and Sato in \[\text{[TSS84]}\] gave a short derivation of an $O(n^3)$ sorting algorithm from a naive sorting algorithm by a technique of introducing an atom in the body of a clause. However, Tamaki and Sato’s technique has not been recognized as a general and useful technique within LPT methodologies. Our contribution is to argue that Tamaki and Sato’s technique can be identified as a valuable instance of the ALP approach for complementing LPT techniques.

Further research is required to mechanize (at least partially) the selection and introduction of subgoals. The abductive task, apparently, depends on the presentation theory and some properties explicitly formulated; these properties should involve the predicates occurring within the logic programs at hand. In LPT, the problem of finding abductive explanations for some “dead ends” of derivations seems promising, powerful and mechanically plausible.

References

Apt97. Krzysztof R. Apt. \textit{From Logic Programming to Prolog}. Prentice Hall, 1997.

CD78. Keith L. Clark and John Darlington. Algorithm classification through synthesis. \textit{The Computer Journal}, 23(1):61–65, 1978.

Dar78. John Darlington. A synthesis of several sorting algorithms. \textit{Acta Informatica}, 11:1–30, 1978.

KKT93. Antonis C. Kakas, Robert Kowalski, and Francesca Toni. Abductive logic programming. \textit{Journal of Logic and Computation}, 2(6):719–770, 1993.

Lau89. Kung-Kiu Lau. A Note on Synthesis and Classification of Sorting Algorithms. \textit{Acta Informatica}, 27:73–80, 1989.

LP91. Kung-Kiu Lau and Steven D. Prestwich. Synthesis of a family of recursive sorting procedures. In V. Saraswat and K. Ueda, editors, \textit{Proc. 1991 Int. Logic Programming Symposium}, pages 641–658. MIT Press, 1991.

Möller. Bernhard Möller. Algebraic calculation of graph and sorting algorithms. In B. Bjorner, M. Broy, and I.V. Pottosin, editors, \textit{Formal Methods in Programming and their Applications}, volume 735 of \textit{Lecture Notes on Computer Science}, pages 394–413, 1993.

Par91. Helmut Partsch. \textit{Specification and transformation of programs}. Springer-Verlag, 1991.

PPR97. Alberto Pettorossi, Maurizio Proietti, and Sophie Renault. Enhancing Partial Deduction via Unfold/Fold Rules. In \textit{Logic Program Synthesis and Transformation}, volume 1207 of \textit{Lecture Notes on Computer Science}. LOPSTR’96, Springer-Verlag, August 1997.

Smi85. Douglas R. Smith. Top-dow synthesis of divide-and-conquer algorithms. \textit{Artificial Intelligence}, 27:43–96, 1985.

Tho86. Simon Thompson. Laws in Miranda. In \textit{LFP ’86: Proceedings of the 1986 ACM conference on LISP and functional programming}, pages 1–12, New York, NY, USA, 1986. ACM.
TS84. Hisao Tamaki and Taisuke Sato. Unfold/Fold Transformation of Logic Programs. *International Conference on Logic Programming*, pages 127–138, 1984.