On Free Boson Representations of the Quantum Affine Algebra $U_q(\widehat{\mathfrak{sl}}_2)$

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Dedicated to Professor N. Nakanishi on the occasion of his sixtieth birthday

ABSTRACT

A boson representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is realized based on the Wakimoto construction. We discuss relations with the other boson representations.
1. Introduction

It is established that conformal field theories play an important role in studies on two dimensional models. In particular the Wess-Zumino-Witten (WZW) model provides us with powerful tools to investigate models in the language of affine Kac-Moody algebras. It is also recognized that free field realization is indeed useful to investigate representation of Virasoro and affine Kac-Moody algebras. Wakimoto [1] first introduced free realization of the affine Kac-Moody algebra \( \hat{\mathfrak{sl}}_2 \) and there exist many works on this realization.

Recently Frenkel and Reshetikhin [2] have constructed certain \( q \)-deformed chiral vertex operators (qVOs) of the WZW model based on the representation theory of the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \). They showed that the correlation functions satisfy a \( q \)-difference equation called the \( q \)-deformed Knizhnik-Zamolodchikov (qKZ) equation, and the connection matrix of the qKZ equation indeed corresponds with the elliptic solution of the Yang-Baxter equation. As an application XXZ models are analyzed by the technique of \( q \)-vertex operators [3][4][5].

In these situations it is desirable to construct a concrete realization of quantum affine algebras and \( q \)-deformed chiral vertex operators. Frenkel and Jing first found a \( q \)-deformation of the Frenkel-Kac construction which corresponds to boson representation of \( U_q(\hat{\mathfrak{sl}}_2) \) of level one [6]. They show the Drinfeld realization [7] can be treated in terms of currents in which the technique of operator product expansions (OPEs) is powerful. Following this work Jimbo et al. introduced explicit forms of \( q \)-deformed chiral vertex operators and calculated the trace of the product of the vertex operators. Recently some papers appear to attempt extending to the case of arbitrary level. There are two kinds of boson realization in these papers. One is based on the Wakimoto construction with a bosonization of bosonic ghost system [8] and the other on the realization in terms of bosonized parafermions [9][10]. In this letter we construct another boson representation à la Wakimoto following the line of the Matsuo’s realization [9]. In his formulation currents of \( U_q(\hat{\mathfrak{sl}}_2) \) split into two parts of creation and annihilation, which is useful to investigate
the structure of the algebra. We also show explicit relations between his realization and ours.

2. Wakimoto Construction of the algebra \( \hat{\mathfrak{sl}}_2 \)

We begin with a brief review of the Wakimoto description of the algebra \( \hat{\mathfrak{sl}}_2 \) and its bosonized representation. In this construction the \( \hat{\mathfrak{sl}}_2 \) WZW model is described in terms of one free boson \( \varphi \) and a bosonic \((\beta, \gamma)\) system of weight \((1,0)\). We can express the algebraic relations in the language of OPEs:

\[
\beta(z) \gamma(w) = \frac{1}{z-w} + :\beta(z)\gamma(w):, \\
\varphi(z)\varphi(w) = \ln(z-w) + :\varphi(z)\varphi(w):, \\
\varphi(z)\varphi(w) = \ln(z-w) + :\varphi(z)\varphi(w):,
\]

where \( : \) denotes normal ordering. With the mode expansion:

\[
\beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-(n+1)}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n}, \\
\varphi(z) = \alpha - \alpha_0 \ln z + \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n},
\]

we have

\[
[\beta_m, \gamma_n] = \delta_{m+n,0}, \\
[\alpha_0, \alpha] = 1, \quad [\alpha_m, \alpha_n] = m\delta_{m+n,0}.
\]

The currents[1] are given by these fields:

\[
J^+(z) = \beta(z), \\
J^-(z) = -:\beta(z)\gamma^2(z): + \sqrt{2(k+2)} \partial \varphi(z) \gamma(z) + k \partial \gamma(z), \\
J^3(z) = -:\beta(z)\gamma(z): + \sqrt{\frac{k+2}{2}} \partial \varphi(z),
\]

where \( k \) is a level of the Kac-Moody algebra. OPEs of these currents are derived from ones of the fields \( \varphi, (\beta, \gamma) \):

\[
J^3(z)J^3(w) \sim \frac{k}{(z-w)^2}, \\
J^3(z)J^\pm(w) \sim \pm \frac{1}{z-w} J^\pm(w), \quad |z| > |w|, \\
J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} + \frac{1}{z-w} J^3(w),
\]

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which is equivalent to the commutation relations:

\[
[J^3_m, J^3_n] = kn\delta_{m+n,0},
\]

\[
[J^3_m, J^3_n^\pm] = J^{\pm}_{m+n},
\]

\[
[J^3_m, J^-_{n}] = J^3_{m+n} + km\delta_{m+n,0},
\]

where

\[
H(z) = \sum_{n\in\mathbb{Z}} J^3_n z^{-(n+1)}, \quad J^\pm(z) = \sum_{n\in\mathbb{Z}} J^\pm_n z^{-(n+1)}.
\]

The primary fields of the spin \(j\) representation and screening charge are written as

\[
\phi_{j,m}(z) = \{\gamma(z)\}^{j-m} : \exp\left\{\sqrt{\frac{2}{k+2}} j\varphi(z)\right\} : , \quad 0 \leq j \leq k, \quad -j \leq m \leq j,
\]

\[
J^3(z)\phi_{j,m}(w) \sim \frac{m}{z-w} \phi_{j,m}(w).
\]

\[
J^S(z) = -\beta(z) : \exp\left\{-\sqrt{\frac{2}{k+2}} \varphi(z)\right\} : .
\]

In order to construct \(q\)-deformation of the Wakimoto currents, it is convenient to use the bosonized representation of the \((\beta, \gamma)\) system [11]:

\[
\beta = - \partial \chi(z) \exp\{-\chi(z) + i\sigma(z)\} ; , \quad \gamma(z) = : \exp\{\chi(z) - i\sigma(z)\} : ,
\]

where \(\chi(z)\) and \(\sigma(z)\) are bosonic fields and OPEs of them are same as those of \(\varphi\). With these fields the bosonized Wakimoto currents[12] are rewritten as

\[
J^+(z) = - \partial \chi(z) \exp\{-\chi(z) + i\sigma(z)\} ; ,
\]

\[
J^-(z) =: [(k+2)\partial \{\chi(z) - i\sigma(z)\} - \partial \chi(z) + \sqrt{2(k+2)}\partial \varphi(z)] \exp\{\chi(z) - i\sigma(z)\} :,
\]

\[
J^3(z) = -i\partial \sigma(z) + \frac{\sqrt{2}}{k+2} \partial \varphi(z).
\]

The primary fields and screening charge now become

\[
\phi_{j,m}(z) =: \exp\left[(j-m)\{\chi(z) - i\sigma(z)\}\right] \exp\left\{\sqrt{\frac{2}{k+2}} j\varphi(z)\right\} :,
\]

\[
J^S(z) = - \partial \chi(z) \exp\{-\chi(z) + i\sigma(z)\} \exp\{-\sqrt{\frac{2}{k+2}} \varphi(z)\} : .
\]
3. Quantum Affine Algebra $U_q(\hat{\mathfrak{sl}}_2)$

It is known that the algebra $U_q(\hat{\mathfrak{sl}}_2)$ can be realized in terms of a $q$-deformed harmonic oscillator. First we introduce $q$-deformed oscillators of annihilation and creation, $a$ and $a^\dagger$, which satisfy following relations:

\[
\begin{align*}
    aa^\dagger - qa^\dagger a &= q^{-N}, \\
    aa^\dagger - q^{-1}a^\dagger a &= q^N, \\
    [N, a^\dagger] &= a^\dagger, \\
    [N, a] &= -a,
\end{align*}
\]

where $q$ is a deformation parameter. Hamiltonian of a $q$-deformed harmonic oscillator is given by $\mathcal{H} = [N] + \frac{1}{2}$, where we use the notation $[N] = \frac{q^N - q^{-N}}{q - q^{-1}}$. One can obtain the $q$-deformed $\mathfrak{sl}(2)$ algebra from these oscillators:

\[
\begin{align*}
    J^+ &= a^\dagger, \\
    J^- &= a[\lambda + 1 - N], \\
    J^3 &= N - \frac{\lambda}{2},
\end{align*}
\]

where $\lambda$ is constant corresponding to a value of spin. In the case of the irreducible representation $\lambda = j (j \text{ integer})$, weight vectors are given by

\[
| l > = \frac{(a^\dagger)^l | 0 >}{\sqrt{[l]!}}, \quad [N] | 0 > = 0, \quad l = 0, 1, 2, \cdots, j.
\]

These are eigenvectors of the $q$-deformed Number operator $[N]$ belonging to eigenvalues $[l]$. It is easy to check these operators satisfy the commutation relations of the quantum $\mathfrak{sl}(2)$ algebra:

\[
\begin{align*}
    [J^3, J^\pm] &= \pm J^\pm, \\
    [J^+, J^-] &= [2J^3] = \frac{q^{2J^3} - q^{-2J^3}}{q - q^{-1}}.
\end{align*}
\]

Now we are in a position to construct the $q$-deformation of the Wakimoto current in the same spirit as the $\hat{\mathfrak{sl}}_2$ case. First we introduce $q$-deformation of the boson $\varphi(z)$:

\[
\varphi(z) = \alpha - \alpha_0 \ln z + \sum_{n \neq 0} \frac{\alpha_n}{[n]} z^{-n},
\]
In the case of a system with finite freedom, one has to use $q$-deformed oscillators. However, in field theories one can use ordinary oscillators with adequate normalization. As the Wakimoto currents contain three bosons $\{\sigma(z), \chi(z), \varphi(z)\}$, we prepare three kinds of oscillators $\{a_n, \bar{a}_n, b_n; n \in \mathbb{Z}; a, \bar{a}, b\}$ satisfying the following commutation relations:

$$[a_m, a_n] = -\delta_{m+n,0} \frac{[2m][2m]}{m}, \quad [a_0, a] = -4,$$

$$[\bar{a}_m, \bar{a}_n] = \delta_{m+n,0} \frac{[2m][2m]}{m}, \quad [\bar{a}_0, \bar{a}] = 4,$$

$$[b_m, b_n] = \delta_{m+n,0} \frac{[2m][(k+2)m]}{m}, \quad [b_0, b] = 2(k+2).$$

The other commutation relations of oscillators are equal to zero. We define currents of the algebra $U_q(\mathfrak{sl}_2)$ by using the oscillators (16) as follows:

$$K_+(z) = \exp\left\{ (q - q^{-1}) \sum_{m=1}^{\infty} z^m (a_m + b_m) \right\} q^{(a_0 + b_0)},$$

$$K_-(z) = \exp\left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} z^m (q^{-k+2}a_{-m} + q^{-2m}b_{-m}) \right\} q^{-(a_0 + b_0)},$$

$$X^+(z) = \frac{1}{q - q^{-1}} : \{ Y^+(z) Z_+ (q^{-\frac{k+2}{2}} z) - Z_-(q^{\frac{k+2}{2}} z) Y^+(z) \} :,$$

$$X^-(z) = -\frac{1}{q - q^{-1}} : \{ Y^-(z) Z_+ (q^{\frac{k+2}{2}} z) U_+ (q^{\frac{k}{2}} z) W_+ (q^{\frac{k}{2}} z)$$

$$-Z_- (q^{-\frac{k+2}{2}} z) U_- (q^{-\frac{k}{2}} z) W_- (q^{-\frac{k}{2}} z) Y^-(z) \} :,$$

where

$$Y^+(z) = \exp\left\{ -\sum_{m=1}^{\infty} q^{-\frac{1}{2} m} \frac{z^m}{[2m]} (a_{-m} + \bar{a}_{-m}) \right\} e^{-\frac{a_0 + \bar{a}_0}{2} z - \frac{a_0 + \bar{a}_0}{2}}$$

$$\exp\left\{ \sum_{m=1}^{\infty} q^{-\frac{1}{2} m} q^{(k+2)m} \frac{z^m}{[2m]} (a_m + \bar{a}_m) \right\},$$

$$Y^-(z) = \exp\left\{ \sum_{m=1}^{\infty} q^{\frac{k}{2} m} \frac{z^m}{[2m]} (a_{-m} + \bar{a}_{-m}) \right\} e^{\frac{a_0 + \bar{a}_0}{2} z - \frac{a_0 + \bar{a}_0}{2}}$$

$$\exp\left\{ -\sum_{m=1}^{\infty} q^{\frac{k}{2} m} q^{(k+2)m} \frac{z^m}{[2m]} (a_m + \bar{a}_m) \right\},$$

$$[\alpha_0, \alpha] = 2,$$

$$[\alpha_m, \alpha_n] = \delta_{m+n,0} \frac{[2m][m]}{m}. \quad (15)$$
\[ W_+(z) = \exp \left\{ (q - q^{-1}) \sum_{m=1}^{\infty} z^{-m} b_m \right\} q^{b_0}, \]
\[ W_-(z) = \exp \left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} q^{-2m} z^m b_m \right\} q^{-b_0}, \]
\[ Z_+(z) = \exp \left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} z^{-m} \left[ \frac{m}{2m} a_m \right] \right\} q^{-\frac{1}{2} a_0}, \]
\[ Z_-(z) = \exp \left\{ (q - q^{-1}) \sum_{m=1}^{\infty} q^{-k} z^m \left[ \frac{m}{2m} \bar{a}_{-m} \right] \right\} q^{\frac{1}{2} \bar{a}_0}, \]
\[ U_+(z) = \exp \left\{ (q - q^{-1}) \sum_{m=1}^{\infty} q^{km} z^{-m} \left[ \frac{(k+2)m}{2m} (a_m + \bar{a}_m) \right] \right\} q^{k+2} (a_0 + \bar{a}_0), \]
\[ U_-(z) = \exp \left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} q^{-2k} z^m \left[ \frac{(k+2)m}{2m} (a_{-m} + \bar{a}_{-m}) \right] \right\} q^{-\frac{k+2}{2} (a_0 + \bar{a}_0)}. \]

As we arrange operators in the way of normal ordering except for the zero modes, \( \cdot \cdot \) denotes normal ordering with respect to the zero modes, \( \alpha < \alpha_0, \bar{\alpha} < \bar{\alpha}_0 \) and \( \beta < \beta_0 \). It is the essential idea of this construction that we define \( q \)-deformations of derivative, namely, the field, \( \partial \chi(z), \partial (\chi(z) - \sqrt{-1} \sigma(z)) \) and \( \partial \varphi(z) \) as

\[ \frac{-2}{q - q^{-1}} [Z_+(z) - Z_-(z)], \]
\[ \frac{1}{q - q^{-1}} [U_+(z) - U_-(z)], \]
\[ \frac{1}{q - q^{-1}} [W_+(z) - W_-(z)], \]

respectively *. By taking a limit \( q \to 1 \), they become

\[ \sum_{m \in \mathbb{Z}} \bar{a}_m z^{-m}, \quad \frac{k + 2}{2} \sum_{m \in \mathbb{Z}} (a_m + \bar{a}_m) z^{-m}, \quad \sum_{m \in \mathbb{Z}} b_m z^{-m}, \]

which correspond to the fields \( z \partial \chi(z), z \partial (\chi(z) - \sqrt{-1} \sigma(z)) \) and \( z \partial \varphi(z) \), respectively. The

* This idea is due to the paper[9]
currents of $U_q(\hat{sI}_2)$ satisfy the following relations\cite{6}:

\begin{align}
K_+(z)X^\pm(w) &= \left(\frac{q^2z - q^{\frac{k}{2}}w}{z - q^{2+\frac{k}{2}}w}\right)^{\pm1}X^\pm(w)K_+(z), \\
K_-(z)X^\pm(w) &= \left(\frac{q^2w - q^{\frac{k}{2}}z}{w - q^{2+\frac{k}{2}}z}\right)^{\pm1}X^\pm(w)K_-(z), \\
\frac{q^{2+k}z - w}{q^kz - q^2w}K_-(z)K_+(w) &= \frac{q^{2-k}z - w}{q^kz - q^2w}K_+(w)K_-(z), \\
X^\pm(z)X^\pm(w) &= \frac{q^{2+2}z - w}{z - q^{2+2}w}X^\pm(w)X^\pm(z), \\
X^+(z)X^-(w) &\sim \frac{1}{q - q^{-1}}\left(\frac{z}{z - q^kw}K_+(q^kz) - \frac{z}{z - q^{-k}w}K_-(q^{-\frac{k}{2}}w)\right).
\end{align}

Here we define the mode expansions of these currents as

\begin{align}
K_+(z) &= \sum_{m \in Z_{\geq 0}} \psi_m z^{-m} = q^{a_0 + b_0} \exp\left\{ (q + q^{-1}) \sum_{m=1}^{\infty} H_m z^{-m} \right\}, \\
K_-(z) &= \sum_{m \in Z_{\geq 0}} \psi_m z^{-m} = q^{-(a_0 + b_0)} \exp\left\{ -(q + q^{-1}) \sum_{m=1}^{\infty} H_m z^{-m} \right\}, \\
H(z) &= \sum_{m \in Z} H_m z^{-m}, \quad X^\pm(z) = \sum_{m \in Z} X^\pm_m z^{-m}.
\end{align}

Putting $K = q^{a_0 + b_0}$ we obtain the relations of the Drinfeld realization of $U_q(\hat{sI}_2)$ for level $k\cite{7}$:

\begin{align}
[H_m, H_n] &= \delta_{m+n,0} \frac{1}{m}[2m][km], \quad m \neq 0, \\
[H_m, K] &= 0, \\
K X^\pm_m K^{-1} &= q^{\pm2} X^\pm_m, \\
[H_m, X^\pm_n] &= \pm \frac{1}{m}[2m]q^{\frac{k(m+n)}{2}} X^\pm_{m+n}, \\
X^\pm_{m+1}X^\pm_n - q^{\pm2}X^\pm_nX^\pm_{m+1} &= q^{\pm2} X^\pm_n X^\pm_{m+1} - X^\pm_{n+1} X^\pm_m, \\
[X^+_m, X^-_n] &= \frac{1}{q - q^{-1}} \left( q^{\frac{k(m-n)}{2}} \psi_{m+n} - q^{\frac{k(n-m)}{2}} \varphi_{m+n} \right).
\end{align}

The Drinfeld realization of $U_q(\hat{sI}_2)$ corresponds to one of the $q$-deformation of the algebra $\hat{sI}_2(6)$.

Next we introduce the Fock module $F_{l,m_1,m_2}(l \in \frac{1}{2}Z; m_1, m_2 \in Z)$ freely generated by
\{a_n, \bar{a}_n, b_n; n \in \mathbb{Z}_{>0}\}$ from a vector

\[
|l, m_1, m_2> = \exp\left\{ \frac{l}{k+2} + m_1 \frac{a}{2} - m_2 \frac{\bar{a}}{2} \right\} |0>.
\]

Here a vector $|0>$ has the following properties:

\[
b_n |0> = 0, \quad a_n |0> = 0, \quad \bar{a}_n |0> = 0, \quad n \geq 0.
\]

The vector $|l, m_1, m_2>$ is an eigenvector of $b_0, a_0$ and $\bar{a}_0$ belonging to eigenvalues $2l, 2m_1$ and $2m_2$, respectively. From the following relations:

\[
[H(\phi), a_0 + \bar{a}_0] = 0, \quad [X^\pm(\phi), a_0 + \bar{a}_0] = 0,
\]

we can restrict the Fock module $F_{l,m_1,m_2}$ to the sector in which the eigenvalue of $a_0 + \bar{a}_0$ is equal to zero[11]. It is easy to check the vector $|\frac{i}{2}, \frac{i}{2}, \frac{i}{2}>$ satisfies the highest weight conditions.

4. Relations to other realizations

Now we will give relations between our realization of $U_q(\hat{sl}_2)$ and another one studied by Matsuo. His realization is based on the following currents of $\hat{sl}_2[13]$:

\[
J^\pm(\phi) = \frac{1}{\sqrt{2}} \left[ \sqrt{k+2} \phi_1(\phi) \pm \sqrt{-1} k \phi_2(\phi) \right] \exp\left\{ \pm \sqrt{\frac{2}{k}} [\sqrt{-1} \phi_2(\phi) - \phi_0(\phi)] \right\} ;\]

\[
J^3(\phi) = -\sqrt{\frac{k}{2}} \partial \phi_0(\phi).
\]

We define the following linear transformation of oscillators $\{a_n, \bar{a}_n, b_n; n \in \mathbb{Z}; a, \bar{a}, b\}$:

\[
\begin{pmatrix}
\alpha_m \\
\bar{\alpha}_m \\
\beta_m
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{[(k+2)m] q^{km}} & 0 & \frac{1}{q^{(k+2)m}} \\
\frac{[(k+2)m]}{[2m]} q^{(k+1)m} & \frac{1}{[2m]} & -1 \\
\frac{[(k+2)m]}{[2m]} q^{m} & q^{m} & \frac{1}{q^{m}}
\end{pmatrix}
\begin{pmatrix}
a_m \\
\bar{a}_m \\
b_m
\end{pmatrix}, \quad m \geq 1,
\]

\[
\begin{pmatrix}
\alpha_{-m} \\
\bar{\alpha}_{-m} \\
\beta_{-m}
\end{pmatrix}
= \begin{pmatrix}
\frac{q^{-(k+2)m}}{[(k+2)m] q^{-2m}} & 0 & \frac{q^{-2m}}{q^{-2m}} \\
\frac{[(k+2)m]}{[2m]} q^{-m} & \frac{1}{[2m]} & -q^{-2m} \\
\frac{[(k+2)m]}{[2m]} q^{-m} & q^{-m} & \frac{1}{q^{-m}}
\end{pmatrix}
\begin{pmatrix}
a_{-m} \\
\bar{a}_{-m} \\
b_{-m}
\end{pmatrix}, \quad m \geq 1,
\]

\[
\begin{pmatrix}
\alpha_0 \\
\bar{\alpha}_0 \\
\beta_0
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{k+2} & 0 & 1 \\
-k+2 & -\frac{k}{2} & -1 \\
k+2 & \frac{k+2}{2} & 1
\end{pmatrix}
\begin{pmatrix}
a_0 \\
\bar{a}_0 \\
b_0
\end{pmatrix}.
\]
\[
\begin{pmatrix}
\alpha \\
\bar{\alpha} \\
\beta
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 1 \\
-\frac{k+2}{2} & -\frac{k}{2} & 1 \\
\frac{k+2}{2} & \frac{k}{2} & 1
\end{pmatrix}
\begin{pmatrix}
a \\
\bar{a} \\
b
\end{pmatrix}.
\] (29)

From the commutation relations (16) one can obtain
\[
[a_m, a_n] = \delta_{m+n,0} \frac{[2m][km]}{m}, \quad [a_0, \alpha] = 2k,
\]
\[
[\bar{a}_m, \bar{a}_n] = -\delta_{m+n,0} \frac{[2m][km]}{m}, \quad [\bar{a}_0, \bar{a}] = -2k, \quad m, n \neq 0
\] (30

Substituting the inverse transformations of (29) into currents (17), they become
\[
K_+(x) = \exp\left\{ (q - q^{-1}) \sum_{m=1}^\infty z^{-m} a_m \right\} q^{a_0},
\]
\[
K_-(x) = \exp\left\{ -(q - q^{-1}) \sum_{m=1}^\infty z^m a_{-m} \right\} q^{-a_0},
\]
\[
X^+(z) = \frac{1}{q - q^{-1}} \left\{ Y^+(z) Z_+(q^{-\frac{k+2}{2}} z) W_+(q^{-\frac{1}{2}} z) - W_-(q^{\frac{1}{2}} z) Z_-(q^{-\frac{k+2}{2}} z) Y^+(z) \right\} ;
\]
\[
X^-(z) = -\frac{1}{q - q^{-1}} \left\{ Y^-(z) Z_+(q^{\frac{k+2}{2}} z) W_+(q^{\frac{1}{2}} z)^{-1} - W_-(q^{-\frac{1}{2}} z)^{-1} Z_-(q^{-\frac{k+2}{2}} z) Y^-(z) \right\} ;
\]

where
\[
Y^+(z) = \exp\left\{ \sum_{m=1}^\infty q^{-\frac{k+2}{2}} z^m \frac{z^m}{km} \right\} \left( \alpha_m + \bar{\alpha}_{-m} \right) e^{\frac{a_0 + \bar{a}_0}{k} z^\frac{a_0 + \bar{a}_0}{k}}
\]
\[
\exp\left\{ -\sum_{m=1}^\infty q^{-\frac{k+2}{2}} z^{-m} \frac{z^{-m}}{km} \right\} \left( \alpha_m + \bar{\alpha}_{m} \right),
\]
\[
Y^-(z) = \exp\left\{ -\sum_{m=1}^\infty q^{\frac{k+2}{2}} z^m \frac{z^m}{km} \right\} \left( \alpha_{-m} + \bar{\alpha}_m \right) e^{-\frac{a_0 + \bar{a}_0}{k} z^{-\frac{a_0 + \bar{a}_0}{k}}}
\]
\[
\exp\left\{ \sum_{m=1}^\infty q^{\frac{k+2}{2}} z^{-m} \frac{z^{-m}}{km} \right\} \left( \alpha_m + \bar{\alpha}_{m} \right),
\]
\[
W_+(z) = \exp\left\{ -(q - q^{-1}) \sum_{m=1}^\infty z^{-m} \frac{m}{2m} \beta_m \right\} q^{-\frac{\beta_0}{2}};
\]
\[
W_-(z) = \exp\left\{ (q - q^{-1}) \sum_{m=1}^\infty z^m \frac{m}{2m} \beta_{-m} \right\} q^{\frac{\beta_0}{2}}.
\] (33)
\[
Z_+(z) = \exp\left\{- (q - q^{-1}) \sum_{m=1}^{\infty} z^{-m} \frac{[m]}{[2m]} \tilde{\alpha}_m \right\} q^{-\frac{1}{2}\tilde{\alpha}_0},
\]
\[
Z_-(z) = \exp\left\{ (q - q^{-1}) \sum_{m=1}^{\infty} z^m \frac{[m]}{[2m]} \tilde{\alpha}_{-m} \right\} q^{\frac{1}{2}\tilde{\alpha}_0}.
\]

(34)

These currents correspond with those in the paper[9] except for a little change because of a normalization of zero mode.

Shiraishi’s representation is also based on the Wakimoto currents. The main difference is a treatment for \(q\)-deformation of derivative. He extends derivative to a \(q\)-difference operator defined as
\[
n\partial_z f(z) \equiv \frac{f(q^n z) - f(q^{-n} z)}{(q - q^{-1})z}.
\]

(35)

It is not easy to show explicitly relations between his representation and ours. However, there is a correspondence between \(H(z), X^\pm(z) = X_+^\pm(z) + X_-^\pm(z)\) in (17)(24) and \(zJ^3(z), zJ^\pm(z) = J^\pm_I(z) + J^\pm_{\bar{I}}\) in the appendix of the paper[14].

Finally the aim of this paper is concentrated on the forms of currents and relations of them. In the line of our construction we can obtain screening currents, qVOs and n-point correlation functions with one screening charge on sphere. In the case of the two-point correlation function, we confirmed correspondence with the results of the paper[14]. Our results will be contained in a forthcoming paper. It is necessary to investigate cohomological structure of the Fock module[15][16] in order to derive irreducible representations and correlation functions on torus. These are under investigation.

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