Three-nucleon force study in $A = 3, 4$ systems

A Kievsky$^1$, S Rosati$^{2,1}$, M Viviani$^1$, L E Marcucci$^{2,1}$ and L Girlanda$^1$

1 Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, Largo B. Pontecorvo 3, I-56127, Pisa, Italy
2 Department of Physics “E. Fermi”, University of Pisa, Largo B. Pontecorvo 3, I-56127, Pisa, Italy
E-mail: kievsky@pi.infn.it

Abstract. Using modern nucleon-nucleon interactions in the description on $A = 3, 4$ nuclei, it is not possible to reproduce both the three- and four-nucleon binding energies. This is one manifestation of the need to include a three-nucleon force in the nuclear Hamiltonian. In this paper we will analyze the parametrization of some three-nucleon force models and their capability to describe simultaneously the aforementioned binding energies and selected $N - d$ low energy scattering observables.

1. Introduction

Realistic nucleon-nucleon (NN) potentials reproduce the experimental NN scattering data up to energies of 350 MeV with a $\chi^2$ per datum close to 1. However, the use of these potentials in the description of the three- and four-nucleon bound and scattering states gives a $\chi^2$ per datum much larger than 1 (see for example Ref.[1]). In order to improve this situation, different three-nucleon force (TNF) models have been introduced. Widely used in the literature are the Tucson-Melbourne (TM) and the Urbana IX (URIX) models [2, 3]. These models are based on the exchange mechanism of two pions between three nucleons with the intermediate excitation of a $\Delta$ resonance. The TM model has been revisited within a chiral symmetry approach [4], and it has been demonstrated that the contact term present in it should be dropped. This new TM potential, known as TM', has been subsequently readjusted [5]. The final operatorial structure coincides with that one given in the TNF of Brazil already derived many years ago [6].

More recently, TNF models have been derived [7] based on chiral effective field theory at next-to-next-to-leading order. A local version of this interaction (hereafter referred as N2LO) can be found in Ref. [8]. All these models contain a certain number of parameters that fix the strength of the different terms. It is a common practice to determine these parameters from the three- and four-nucleon binding energies. In fact, a TNF is in general associated to a NN potential and the sum of the two interactions forms the nuclear potential energy. Therefore, the parametrization of a particular TNF can change when used with different NN potentials since different NN potentials predict different $A = 3, 4$ binding energies. In this paper we will make a critical analysis of the agreement obtained in some observables of the $A = 3, 4$ systems, when the potential energy consists of a NN potential plus a TNF. We will study the corresponding binding energies as well as the predictions for the doublet $n - d$ scattering length $a_{nd}$ and some polarization observables in $p - d$ scattering. For this purpose we use the hyperspherical harmonic
(HH) method as developed by some of the authors to describe bound and scattering states in $A = 3, 4$ systems [9, 10, 11, 12] (for a recent review see Ref. [13]).

2. Binding energies and scattering lengths for $A = 3, 4$

From the results obtained in Ref. [13], we report in table 1 the triton and $^4$He binding energies, and the doublet $n - d$ scattering length $^2a_{nd}$. These results were obtained using the AV18 or the N3LO-Idaho two-nucleon potentials together with the AV18+URIX, AV18+TM' and N3LO-Idaho+N2LO TNF models. The results are compared to the experimental values also reported in the table. Worthy of notice is the recent very accurate datum for $^2a_{nd}$ [14].

| Potential                   | $B(^3\text{H})$ (MeV) | $B(^4\text{He})$ (MeV) | $^2a_{nd}$ (fm) |
|-----------------------------|------------------------|------------------------|-----------------|
| AV18                        | 7.624                  | 24.22                  | 1.258           |
| N3LO-Idaho                  | 7.854                  | 25.38                  | 1.100           |
| AV18+TM'                    | 8.440                  | 28.31                  | 0.623           |
| AV18+URIX                   | 8.479                  | 28.48                  | 0.578           |
| N3LO-Idaho+N2LO             | 8.474                  | 28.37                  | 0.675           |
| Exp.                        | 8.482                  | 28.30                  | 0.645±0.003±0.007 |

From the table we may observe that only the results obtained using an interaction model that includes a TNF are close to the corresponding experimental values. In the case of the AV18+TM', the strength of the TM' potential has been fixed to reproduce the $^4$He binding energy and, as can be seen from the table, the triton binding energy is underpredicted. Conversely, the strength of the URIX potential has been fixed to reproduce the triton binding energy giving too much binding for $^4$He. The parametrization of the N2LO potential has been fixed to reproduce simultaneously the triton and the $^4$He binding energies. In the three cases the predictions for the doublet scattering length are not in agreement with the experimental value, in particular for the AV18+URIX model.

The TM' potential has three parameters fixing the strengths of the three terms arising from the exchange of two pions between three nucleons. One term is coming from $\pi N$ S-wave scattering and the corresponding constant is called $a$. The other two terms, which are the most important, come from $\pi N$ P-wave scattering and the corresponding constants are called $b$ and $d$. Following Ref. [15], we have used the values $a = -0.87 \frac{m_\pi}{\Lambda}$, $b = -2.58 \frac{m_\pi^3}{\Lambda}$, and $d = -0.753 \frac{m_\pi^2}{\Lambda}$, with $m_\pi$ the pion mass. In addition, this model includes a monopole form factor at the diagram vertices

$$f(q, \Lambda) = \left[ \frac{\Lambda^2 - m_\pi^2}{\Lambda^2 + q^2} \right]^2,$$

with $\Lambda$ a cutoff in the momenta, that regularizes the behaviour of the potential at short distances. In table 1 the calculations have been done using the value $\Lambda = 4.756 \frac{m_\pi}{\Lambda}$.

We would like to see if it is possible to reproduce simultaneously the triton binding energy and the doublet $n - d$ scattering length for some values of the parameters. The $a$-term gives a very small contribution to these quantities, therefore we maintain it fixed at the value given above. The other parameters have been varied as shown in figure 1. In figure 1(a) the parameter $d$ and the cutoff $\Lambda$ have been varied maintaining the value $b = -2.58 \frac{m_\pi^3}{\Lambda}$ fixed. In figure 1(b)
Figure 1. (a) The parameter $d$ as a function of $\Lambda$ for $b = -2.58 \, m_{\pi}^2$, and (b) the parameter $d$ as a function of $b$ for $\Lambda = 4.8 \, m_{\pi}$. At each point of the two lines the set of parameters describe correctly the triton binding energy.

The parameters $d$ and $b$ have been varied maintaining the value of the cutoff $\Lambda = 4.8 \, m_{\pi}$ fixed. Each point on the two lines of the figure corresponds to a set of parameters that, in connection with the AV18 potential, reproduces the triton binding energy.

The predictions for the doublet scattering length $^2a_{nd}$ are given in table 2 for selected values of the parameters extracted from figure 1 and indicated as points in the figure. The last point of panel (a) and the first point of panel (b) correspond to the same set of parameters. In the first four rows the values are extracted from panel (a) whereas the last four rows are from panel (b) (the fourth row belongs to both panels). We can observe that the doublet scattering length remains almost constant around the value $^2a_{nd} \approx 0.595 \, \text{fm}$. The same occurs using set of parameters corresponding to other points on the two lines of figure 1. We can compare these results to that one given in table 1 for the AV18+TM’ in which the triton binding energy is not well reproduced. We observe that when the model is forced to reproduce the triton binding energy, the doublet scattering length goes to a value around 0.595 fm. So, we can extract the first, very important conclusion: with reasonable ranges for the parameters of the AV18+TM’ potential it is not possible to describe simultaneously the triton binding energy and the $n - d$ doublet scattering length.

The motivation for this fact is the following. The main contribution of the TM’ potential comes from the $b$- and $d$- terms which are attractive. Therefore the TM’ potential is an attractive potential; it does not include explicitly a repulsive term. Fixing appropriately its strength, it can supply the extra binding needed to correctly describe the three-nucleon binding energy. On the other hand, the scattering length is sensitive to the balance between the attractive part and the repulsive part of the complete interaction. Therefore, it seems that supplying an attraction fixed to reproduce the triton binding energy, this balance results to be incorrect. From this analysis we conclude that the TNF has to include, not only an attractive part, but also a repulsive term, and their strengths could be fixed to reproduce simultaneously the triton binding energy and the $n - d$ doublet scattering length $^2a_{nd}$.

As discussed above the TM’ potential is a modification of the original TM potential compatible with chiral symmetry. At the same order (next-to-next-to-leading order) in the
Table 2. The doublet scattering length $^2a_{nd}$ for different choices of the TM’ parameters. In all cases the parameters give a correct description of the triton binding energy in connection with the AV18 potential.

| $a$ [$m^{-1}$] | $b$ [$m^{-3}$] | $d$ [$m^{-3}$] | $\Lambda$ [$m_\pi$] | $^2a_{nd}$ [fm] |
|----------------|----------------|----------------|----------------------|----------------|
| -0.87          | -2.58          | -1.500         | 4.590                | 0.596          |
| -0.87          | -2.58          | -1.200         | 4.670                | 0.596          |
| -0.87          | -2.58          | -1.000         | 4.725                | 0.597          |
| -0.87          | -2.58          | -0.753         | 4.800                | 0.595          |
| -0.87          | -2.00          | -1.380         | 4.800                | 0.591          |
| -0.87          | -2.29          | -1.080         | 4.800                | 0.593          |
| -0.87          | -3.00          | -0.230         | 4.800                | 0.594          |

chiral effective field theory two extra terms appear (see Ref. [7] and references therein). The configuration space form of these terms is not unique, however a local form is possible with a particular choice of the cutoff [8]. Here we introduce the following additional term to the TM’ potential based on a contact term of three nucleons

$$W_{3R} = W_0 \sum_{cyc} Z_0(r_{ij})Z_0(r_{jk})$$

with

$$Z_0(r) = \frac{1}{2\pi^2 m_\pi^3} \int_0^\infty dq q^2 j_0(qr) \left[ \frac{\Lambda^2 - m_\pi^2}{\Lambda^2 + q^2} \right]^2 = \frac{1}{8\pi m_\pi^3} \left(1 - \frac{m_\pi^2}{\Lambda^2}\right)^2 e^{-\Lambda r} .$$

A similar term exists in the URIX potential, though with a different form of the function $Z_0(r)$. For positive values of $W_0$ the new term is repulsive, and we include it in the following analysis of the TM’ potential. Fixing the value of the parameters $a$ and $b$ at the value normally used in the literature, $a = -0.87 m_\pi$ and $b = -2.58 m_\pi^3$, we vary the parameter $d$ and the new parameter $W_0$ in order to reproduce simultaneously the triton binding energy and $^2a_{nd}$. The calculations have been done for several values of the cutoff $\Lambda$. The results are shown in figure 2 for $\Lambda/m_\pi$ as a function of $d$. To be noticed that in each point of the curve the value of $W_0$ is chosen to reproduce the aforementioned quantities. It ranges from a value of 1.228 MeV at $\Lambda/m_\pi = 7.2$ to 0.308 MeV at $\Lambda/m_\pi = 5.0$.

It is interesting to analyze the different contributions of the binding energy produced by the inclusion of the new term in the AV18+TM’ model from sets of parameters taken at different points of the curve. In table 3, the mean values of the kinetic energy and potential energy of the two-body AV18 interaction, as well as those ones of the attractive part of the TNF $V_A(3N)$, corresponding to the sum of the $a$, $b$ and $d$ terms, and the repulsive part $V_R(3N)$ corresponding to the $W_0$ term, are given for selected values of the parameters (indicated as points in the curve). The last two columns show the triton binding energy and $^2a_{nd}$. As can be seen, the sets of parameters on the curve reproduce these quantities.

In the following we analyze the URIX potential which has two free parameters that can be conveniently fixed. The first one, called $A_{PW}^{2\pi}$, is related to the strength of the term produced by a $2\pi$-exchange with an intermediate $\Delta$ excitation. This term is constructed from the sum of two contributions, corresponding to the $b$-term and $d$-term of the TM potential. In the URIX potential they are not independent, their relative strength $D_{2\pi}^{PW}$ is taken to be 1/4. The second constant, called $A_R$, fixes the strength of a purely central repulsive term introduced.
Figure 2. The cutoff $\Lambda$ as a function of $d$. In each point of the curve the parameter $W_0$ is chosen to reproduce simultaneously the triton binding energy and the doublet scattering length $^{2}a_{nd}$.

Table 3. Mean values of the kinetic energy $T$, the two-body potential $V(2N)$ and the attractive $V_A(3N)$ and repulsive $V_R(3N)$ contributions of the modified TM' potential for the indicated values of the parameters. In the last two columns the triton binding energy and the doublet $n-d$ scattering length are shown.

| $d$ [m$_{\pi^{-3}}$] | $W_0$ [MeV] | $\Lambda$ [m$_{\pi}$] | $T$ [MeV] | $V(2N)$ [MeV] | $V_A(3N)$ [MeV] | $V_R(3N)$ [MeV] | $B(^3H)$ [MeV] | $^{2}a_{nd}$ [fm] |
|----------------------|-------------|-----------------------|-----------|--------------|--------------|--------------|----------------|---------------|
| -0.753               | 1.228       | 7.2                   | 51.420    | -57.911      | -1.996       | 2.042        | 8.486          | 0.644         |
| -1.506               | 0.512       | 6.0                   | 51.011    | -57.901      | -3.180       | 1.553        | 8.478          | 0.644         |
| -2.106               | 0.402       | 5.6                   | 50.953    | -57.852      | -3.000       | 1.423        | 8.477          | 0.645         |
| -2.688               | 0.333       | 5.3                   | 50.890    | -57.815      | -2.863       | 1.313        | 8.475          | 0.644         |
| -3.718               | 0.308       | 5.0                   | 50.846    | -57.706      | -2.942       | 1.327        | 8.475          | 0.644         |

Table 3. Mean values of the kinetic energy $T$, the two-body potential $V(2N)$ and the attractive $V_A(3N)$ and repulsive $V_R(3N)$ contributions of the modified TM' potential for the indicated values of the parameters. In the last two columns the triton binding energy and the doublet $n-d$ scattering length are shown.

To compensate the attraction of the previous term, which by itself would produce a large overbinding in infinite nuclear matter. Its form is similar to the one given in eq.(2), with $Z_0(r) = T^2(m_\pi r)$, and

$$T(x) = \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) e^{-x} \xi_T^2(r), \quad (4)$$

where $x = m_\pi r$ and $\xi_T(r) = 1 - e^{-c x^2}$ is a cutoff function with $c = 2.1$ fm$^{-2}$. The original values of the parameters of the URIX model are $A_{2s}^{PW} = -0.0293$ MeV, $A_R = 0.0048$ MeV and $D_{2s}^{PW} = 0.25$. The corresponding results for the quantities of interest are given in table 1.

In order to improve the agreement with $^{2}a_{nd}$ and the $^4$He binding energy, we have varied the constants $A_{2s}^{PW}$, $A_R$ and the relative strength $D_{2s}^{PW}$. For a given value of $A_{2s}^{PW}$, we have varied $A_R$ and $D_{2s}^{PW}$ to reproduce $B(^3H)$ and $^{2}a_{nd}$. Then we have calculated $B(^4He)$. Surprisingly this last result turned out to be quite close to the experimental value. The results of the analysis are
given in table 4 where five sets of values which reproduce the mentioned quantities are reported.

Table 4. Different choices for the parameters of the URIX potential and the corresponding triton and $^4$He binding energies (in MeV) and scattering length $^2a_{nd}$ (in fm), calculated with the AV18+URIX potential.

| $A_{2\pi}^{PW}$ (MeV) | $D_{2\pi}^{PW}$ | $A_R$ (MeV) | $B(^3\text{H})$ | $B(^4\text{He})$ | $^2a_{nd}$ |
|------------------------|----------------|-------------|-----------------|-----------------|-----------|
| -0.0200                | 1.625          | 0.0176      | 8.474           | 28.33           | 0.644     |
| -0.0250                | 1.25           | 0.0182      | 8.474           | 28.34           | 0.644     |
| -0.0293                | 1.00           | 0.0181      | 8.474           | 28.33           | 0.643     |
| -0.0350                | 0.8125         | 0.0191      | 8.474           | 28.33           | 0.645     |
| -0.0400                | 0.6875         | 0.0198      | 8.474           | 28.38           | 0.645     |

From the table we observe that the values considered for $D_{2\pi}^{PW}$ and $A_R$ are quite far from the original ones. In particular, the relative strength $D_{2\pi}^{PW}$ differs from the original value of 1/4. To extend the analysis further, the obtained set of values can be used to study $p - d$ scattering at low energy. In figure 3 the $p - d$ analyzing power $A_y$ at $E_{lab} = 3$ MeV is shown in correspondence to calculations performed using the original AV18+URIX model (solid line) and using the first three sets of values given in table 4 and indicated by (a), (b) and (c). The calculations have been done considering the coulomb potential as well as the magnetic moment interaction as explained in Ref. [16]. As can be seen, the results for the models (a), (b) and (c) are very close to each other. The same is true for the last two choices of table 4; however, corresponding results are not shown for the sake of clarity. Besides the usual underprediction of this observable found within the AV18+URIX model, we observe a substantially worse description when the new sets of constants are used. In figure 4 the tensor analyzing power $T_{21}$ is shown corresponding to the same choice of parameters. The original AV18+URIX model overpredicts the minimum close to 90°. Again the curves listed (a), (b) and (c) nearly overlap and there is a substantially worse description of the observable between 40° and 120°.

3. Conclusions
Stimulated by the fact that the commonly used TNF models do not reproduce simultaneously the triton and $^4$He binding energy and the $n - d$ doublet scattering length $^2a_{nd}$, we have analyzed possible modifications of the AV18+TM’ and AV18+URIX potentials. In the case of the AV18+TM’ interaction we have shown the impossibility of describing simultaneously the triton binding energy and $^2a_{nd}$. The TM’ potential is attractive, accordingly, it provides more binding. Its strength can be fixed in order to reproduce the triton binding energy. However, the doublet scattering length is sensitive to the balance between the attraction and the repulsion provided by the whole potential. From the analysis given here, it has emerged that the TNF, besides providing more attraction, has to provide a certain amount of repulsion. To this end we have included a local repulsive term in the TM’ potential. This term is based on a three-nucleon contact term which appears at next-to-next-to-leading order in the chiral effective field theory and is similar to the repulsive term already present in the URIX model. Varying the $d$ parameter and the cutoff $\Lambda$, and adjusting the strength of the repulsive term $W_0$, we found a continuity of sets of parameters that reproduce the triton binding energy and $^2a_{nd}$.

In the case of the AV18+URIX interaction, we have varied the original parameter values of the model so as to improve the description of triton binding energy and $^2a_{nd}$. Five choices of the parameters have been considered and we have shown that it is possible to reproduce simultaneously these quantities. The obtained values of the parameters substantially differ from...
Figure 3. The $p - d$ analyzing power $A_y$ for the models discussed in the text. Experimental data are from Ref. [17].

Figure 4. The $d - p$ analyzing power $T_{21}$ for the models discussed in the text. Experimental data are from Ref. [17].

the original ones. Values of the relative strength $D_{2n}^{PW}$ ranging from 0.68 to 1.63 have been considered. Its original value of 1/4 produces unnatural values of the other constants $A_{2n}^{PW}$ and $A_R$. In the range considered, the value of $A_R$ results almost constant. We have extended the analysis of the AV18+URIX interaction to the description of $p - d$ scattering. It has emerged
that the selected set of parameters worsen the description of some $p - d$ polarization observables at low energies. Further work on this problem is in progress.

References
[1] Kievsky A, Viviani M and Rosati S 2001 Phys. Rev. C 64 024002
[2] Coon S A and Glöckle W. 1981 Phys. Rev. C 23 1790
[3] Pudliner B S et al 1995 Phys. Rev. Lett. 51 4396
[4] Friar J L, Hübner D and van Kolck U 1999 Phys. Rev. C 59 33
[5] Coon S A and Han H K 2001 Few-Body Syst. 30 131
[6] Coelho H T, Das T K and Robilotta M R 1983 Phys. Rev. C 28 1812; Robilotta M R and Coelho H T 1986 Nucl. Phys. A 460 645
[7] Epelbaum E et al 2002 Phys. Rev. C 66 064001
[8] Navratil P 2007 Few-Body Syst. 41 117
[9] Kievsky A, Viviani M and Rosati S 1993 Nucl. Phys. A 551 241; Kievsky A, Viviani M and Rosati S 1993 Nucl. Phys. A 577 511
[10] Kievsky A 1997 Nucl. Phys. A 624 125
[11] Viviani M, Kievsky A and Rosati S 1995 Few-Body Syst. 18 25; Viviani M, Kievsky A and Rosati S 2005 Phys. Rev. C 71 024006
[12] Viviani M et al 2001 Phys. Rev. Lett. 86 3739; Fisher B M et al 2006 Phys. Rev. C 74 034001
[13] Kievsky A, Rosati S, Viviani M, Marcucci L E and Girlanda L 2008 J. Phys. G: Nucl. Part. Phys. 35, 063101
[14] Schoen K et al 2003 Phys. Rev. C 67 044005
[15] Nogga A, Kamada H, Glöckle W and Barrett B R 2002 Phys. Rev. C 65 054002
[16] Kievsky A, Viviani M and Marcucci L E 2004 Phys. Rev. C 69 014002
[17] Shimizu S et al 1995 Phys. Rev. C 52 1193