SMART CRITICALITY

ALEXANDER BLOKH, LEX OVERSTEEGEN, ROSS PTACEK,
AND VLADLEN TIMORIN

ABSTRACT. Invariant laminations were introduced by W. Thurston as models for the topological dynamics of polynomials on their Julia sets (provided that the latter are locally connected). A quadratic invariant lamination has one or two longest leaves, called major(s). The image of a longest leaf is called a minor. A crucial fact established by Thurston is that distinct minors do not cross inside the unit disk; this led to his construction of a combinatorial model of the Mandelbrot set. Thurston’s argument is based upon the fact that majors of a quadratic lamination never enter the region between them, a result that fails in the cubic case.

In this paper, devoted to laminations of any degree, we use an alternative approach in which the fate of sets of intersecting leaves of two distinct laminations is studied. We show that, under some natural assumptions, these sets of intersecting leaves behave like gaps of a lamination. Relying upon this, we rule out certain types of mutual location of critical sets of distinct laminations (this can be viewed as a partial generalization of Thurston’s theorem that quadratic minors do not cross inside the unit disk). The main application is to the cubic case.

1. INTRODUCTION

We assume basic knowledge of complex dynamics and use standard notation (for precise definitions see Section 2). Let $P$ be a polynomial of degree $d$ with locally connected (and hence connected) Julia set $J(P)$. By a theorem of Carathéodory, the conformal isomorphism $\Phi$ between $\mathbb{C} \setminus \mathbb{D}$ and the complement $U$ of the filled Julia set $K(P)$ that is asymptotic to the identity at infinity can be extended to a continuous map $\Phi_P : \mathbb{C} \setminus \mathbb{D} \to U$.

In this paper, we study combinatorial objects that can be associated with $P$ and the relationships between these objects.

Date: January 20, 2014.
2010 Mathematics Subject Classification. Primary 37F20; Secondary 37F10, 37F50.
Key words and phrases. Complex dynamics; laminations; Mandelbrot set; Julia set.
The first and the third named authors were partially supported by NSF grant DMS–1201450.
The fourth named author was partially supported by the Dynasty foundation grant, the Simons-IUM fellowship, RFBR grants 11-01-00654-a, 12-01-33020, and AG Laboratory NRU-HSE, MESRF grant ag. 11.G34.31.0023.
Call polynomials with only repelling cycles dentritic. Let us start by considering dendritic quadratic polynomials $P_c(z) = z^2 + c$ with locally connected Julia sets and all cycles repelling (the parameter values $c$, for which all these assumptions are fulfilled, are also called dentritic). For a dendritic parameter $c$, let $G_c$ denote the convex hull of all points in the unit circle that map to $c$ under $\Phi_{P_c}$. The set $G_c$ is a convex polygon (or a chord, or a point) in the closed unit disk. An important result, going back to Thurston, implies that $G_c$ and $G_{c'}$ are either equal or disjoint for any two dendritic parameters $c$ and $c'$. Moreover, the mapping $c \mapsto G_c$ is upper semicontinuous in a natural sense (if a sequence of dendritic parameters $c_n$ converges to a dendritic parameter $c$, then the limit set of the corresponding convex sets $G_{c_n}$ is a subset of $G_c$). The set of dendritic parameters projects continuously to the set of combinatorial objects $G_c$.

We generalize these results to the case of cubic polynomials as an application of the new combinatorial machinery introduced in this paper. In fact, our tools are developed for polynomials of any degree and weaker results are obtained for them. However, even though some of the facts are established in the more general setting, the current paper is designed to make full use of the restriction to the cubic dentritic case. This leads to a simpler and cleaner generalization of Thurston’s results.

We now give a precise (although not the strongest) statement concerning cubic dendritic polynomials. Let $P$ be a cubic polynomial with all cycles repelling and with locally connected Julia set. Suppose that $P$ has two distinct critical points. Such cubic polynomials $P$ will be referred to as dentritic bicritical cubic polynomials. A co-critical point of $P$ associated to a critical point $\omega_1$ of $P$ is the only point $\eta_1 \neq \omega_1$ with the property $P(\eta_1) = P(\omega_1)$. We write $\omega_1, \omega_2$ for the two critical points of $P$ and $\eta_1, \eta_2$ for the corresponding co-critical points. Let $G_i, i = 1, 2$, be the convex hull of all points in the unit circle that are mapped to $\eta_i$ under $\Phi_P$. With every dendritic bicritical polynomial, we associate the corresponding co-critical tag $G_1 \times G_2$. Below is a special case of what we prove in this paper.

**Theorem.** Co-critical tags of dendritic bicritical cubic polynomials $P$ are disjoint or coincide. The dependence of the co-critical tag on $P$ is upper semicontinuous: if $P_n$ is a sequence of dendritic bicritical polynomials converging to a dendritic bicritical polynomial $P$, then the co-critical tags $T_n$ of $P_n$ accumulate on a subset of the bicritical tag of $P$.

By a chord we always mean a chord in $\overline{D}$. A chord with endpoints $e^{2\pi i a}$ and $e^{2\pi i b}$ is denoted $ab$ (we often identify points $a$ of the circle $S = \mathbb{R}/\mathbb{Z}$ with the corresponding complex numbers $e^{2\pi i a}$ on the boundary of the unit disk). We extend $\sigma_d$ linearly over all chords and write $\sigma_d(ab) = \sigma_d(a)\sigma_d(b)$. 
Similarly, given a closed subset $A \subset S$ with convex hull $CH(A) = G \subset \mathbb{D}$, we denote by $\sigma_d(G)$ the convex hull $CH(\sigma_d(A))$ of $\sigma_d(A)$.

Given a polynomial $P$ with a locally connected Julia set $J(P)$, define an equivalence relation $\sim_P$ on $S$ by saying that points $x, y$ are equivalent if and only if $\overline{\Phi}(x) = \overline{\Phi}(y)$. Such an equivalence relation $\sim_P$ is called the $(\sigma_d)$-invariant lamination (generated by $P$) (clearly, $\sim_P$ is invariant under the action of $\sigma_d$ on $S$). The quotient space $S/\sim_P$ of $S$, with map induced by $\sigma_d$, can be viewed as a model of $P|_{J(P)}$. Convex hulls of $\sim_P$-classes are always pairwise disjoint. If $P$ is dendritic then $J(P)$ is a dendrite (which explains the terminology). One can define equivalence relations similar to $\sim_P$ in the absence of any polynomial; such equivalence relations $\sim$ are called laminations (we always assume that $\sim$-classes are finite). A $\sim$-class is called critical if it maps $k$-to-1 onto its image for some $k > 1$. A lamination $\sim$ is dendritic if $S/\sim$ is a dendrite.

One can associate to a lamination $\sim$ (e.g., to $\sim_P$) the family of edges of convex hulls of $\sim$-classes; this is a closed family $L_{\sim}$ (in the polynomial case, $L_{\sim_P} = L_P$) of chords with specific properties. A central idea in Thurston’s work [Thu85] was to study dynamics of families of chords with these properties without referring to polynomials. Such families of chords are called $\sigma_d$-invariant geometric laminations, and the chords in these families are called leaves. We will abbreviate geometric (a.k.a. geodesic) laminations as geolaminations. The main property of a $\sigma_d$-invariant geolamination $L$ is that $\sigma_d(\ell)$ is a leaf of $L$ (possibly degenerating to a point in $S$) for every leaf $\ell$ of $L$. A gap of $L$ is defined as the closure of a complementary component of $L^+ = \bigcup_{t \in L} \ell$; a gap is said to be finite or infinite according to whether it has finitely many or infinitely many points on the unit circle.

Using these notions we can state Theorem 1.1 (a weaker version of one of our results) which helps in the proof of the above Theorem. Call a dendritic lamination full if it has $d - 1$ critical classes. Also, two quadrilaterals are called strongly linked if their vertices alternate (see Figure 1).

**Theorem 1.1.** Suppose that $L_1, L_2$ are full dendritic laminations with critical sets $C_1, \ldots, C_{d-1}$ and $D_1, \ldots, D_{d-1}$, respectively. Suppose also that in each $C_i$ we can choose a critical quadrilateral or leaf $Q_i$ and in each $D_i$ we can choose a critical quadrilateral or leaf $T_i$ so that for each $i$, the sets $Q_i$ and $T_i$ either coincide or are strongly linked. Then $L_1 = L_2$.

In the proofs we use several new tools, including so-called accordions; to motivate them we now briefly go over Thurston’s main results [Thu85] (a more detailed description can be found in Subsection 3.1).

A major idea of Thurston’s was that the space of $\sigma_2$-invariant laminations can be modeled by a lamination. Here each $\sigma_2$-invariant lamination $\sim$ (and the geolamination $L = L_\sim$) is associated to its minor $m_\sim$ (which is the
image of a longest leaf $M_L$ of $\mathcal{L}$ called a major). Thurston showed that the family of minors is disjoint inside the unit disk and generates what he called the \textit{quadratic minor lamination} and denoted $\text{QML}$ (see Section 2 for formal definitions). Denote the corresponding equivalence relation by $\sim_{\text{QML}}$; by Theorem A.6 of [Sch09] (see also [Dou93, Thu85]) the quotient $\mathbb{S}/\sim_{\text{QML}}=\mathcal{M}^2_{\text{Comb}}$ is the monotone (i.e. such that all fibers are continua) image of the boundary of the Mandelbrot set $\mathcal{M}^2$.

A crucial ingredient of Thurston’s argument is Lemma 1.2. Given a quadratic invariant geolamination $\mathcal{L}$ and a leaf $\ell \in \mathcal{L}$ which is not a diameter, let $\ell'$ be the unique leaf of $\mathcal{L}$ disjoint from $\ell$ with $\sigma_d(\ell) = \sigma_d(\ell')$. Let $C(\ell)$ be the open strip of $\overline{\mathbb{D}}$ between $\ell$ and $\ell'$ and $L(\ell)$ be the length of the shorter component of $\mathbb{S} \setminus \ell$ (we normalize length on $\mathbb{S}$ so that its length is 1).

\textbf{Lemma 1.2} (Central Strip Lemma, Lemma II.5.1 [Thu85]). Let $\frac{1}{3} \leq L(\ell) < \frac{1}{2}$ for a leaf $\ell \in \mathcal{L}$. Suppose that $k$ is the least number such that $\sigma_d^k(\ell) \subset C(\ell)$ (except, possibly, for the endpoints). Then $\sigma_d^k(\ell)$ separates $\ell$ and $\ell'$. In particular, if $\ell = M$ is a major, then $M$ cannot enter $C(M)$.

Thurston used the Central Strip Lemma to obtain beautiful and surprising results for quadratic invariant laminations. One of them is the absence of so-called \textit{wandering triangles} in the quadratic case. Another one is the fact that $\sigma_d$ is transitive on vertices of a periodic cycle of finite gaps. Finally, Thurston used both these results to show that minors of distinct quadratic laminations are disjoint in $\mathbb{D}$. However, both the Central Strip Lemma and its consequences fail in the cubic case (see Subsection 3.1 for more detail).

This justifies our introducing in this paper a new set of tools meant to replace the Central Strip Lemma and the No Wandering Triangle Theorem in the context of higher-degree laminations. The central concepts here are those of \textit{smart criticality} and of an \textit{accordion}, which are intimately related to each other. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two geolaminations. Let $\ell_1 \in \mathcal{L}_1$ be a
leaf. The *accordion* \( A_{\mathcal{L}_2}(\ell_1) \) is by definition the union of \( \ell_1 \) and the set of all leaves of \( \mathcal{L}_2 \) crossing \( \ell_1 \) inside \( \mathbb{D} \). It turns out that under certain natural conditions (similar to those stated in Theorem 1.1 and in the quadratic case corresponding to the non-disjointness of minors) we can observe that, given \( i \), there exists a collection of critical chords and leaves of \( \mathcal{L}_2 \) disjoint from \( \sigma_d^i(\ell_1) \) inside \( \mathbb{D} \). This implies that the accordions \( A_{\mathcal{L}_2}(\ell_1) \) map forward in an orientation-preserving fashion; in particular, two non-disjoint leaves of different linked laminations never map to disjoint leaves under the iterates of \( \sigma_d \). This statement replaces the Central Strip Lemma for the higher-degree applications.

There are several papers which study geolaminations of higher degrees and their “mutual location”. One of them is due to Dierk Schleicher [Sch04] who extended Thurston’s results to geolaminations of any degree with one critical set. Otherwise we have heard of an old preprint of D. Ahmadi and M. Rees in which they study cubic laminations.

**Organization of the paper.** In Section 2, we introduce basic notation and terminology; then we establish a few initial facts about all geolaminations. Finally in Section 2 we consider several special types of laminations. Section 3 is devoted to the study of accordions. In Section 4 we prove our main general results concerning the mutual location of critical sets of geolaminations. Finally, in Section 5 the main results in the cubic case are obtained.

## 2. GEOLAMINATIONS AND THEIR PROPERTIES

In this section we give basic definitions, list some known results on geolaminations, and establish some new facts about them.

### 2.1. Basic definitions. A geolamination is a collection \( \mathcal{L} \) of chords (possibly degenerate) of \( \mathbb{D} \) which are pairwise disjoint in \( \mathbb{D} \) such that \( \mathcal{L}^+ = \bigcup_{\ell \in \mathcal{L}} \ell \) is closed, and all points of \( S = \text{Bd}(\mathbb{D}) \) are elements of \( \mathcal{L} \).

#### 2.1.1. Sibling invariant geolaminations. Let us introduce the notion of a (sibling) \( \sigma_d \)-invariant geolamination which is a slight modification of an invariant geolamination introduced by Thurston.

**Definition 2.1** (Invariant geolaminations [BMOV12]). A geolamination \( \mathcal{L} \) is (sibling) \( \sigma_d \)-invariant provided:

1. for each \( \ell \in \mathcal{L} \), we have \( \sigma_d(\ell) \in \mathcal{L} \),
2. for each \( \ell \in \mathcal{L} \) there exists \( \ell' \in \mathcal{L} \) so that \( \sigma_d(\ell') = \ell \).
3. for each \( \ell \in \mathcal{L} \) so that \( \sigma_d(\ell) = \ell' \) is a leaf, there exist \( d \) disjoint leaves \( \ell_1, \ldots, \ell_d \) in \( \mathcal{L} \) so that \( \ell = \ell_1 \) and \( \sigma_d(\ell_i) = \ell' \) for all \( i = 1, \ldots, d \).
We call the leaf \( \ell' \) in (2) a pullback of \( \ell \) and the leaves \( \ell_1, \ldots, \ell_d \) in (3) siblings of \( \ell \). In a broad sense a sibling (leaf) of \( \ell \) is a leaf with the same image but distinct from \( \ell \). Definition 2.1 is slightly more restrictive than Thurston’s definition of an invariant geolamination. By [BMOV12], all \( \sigma_d \)-invariant geolaminations are invariant in the sense of Thurston [Thu85] and are, in particular, gap invariant (see Theorem 2.2). From now on by \((\sigma_d)\)-invariant geolaminations we mean sibling \(\sigma_d\)-invariant geolaminations.

**Theorem 2.2 ([BMOV12])**. Suppose that \( L \) is a \( \sigma_d \)-invariant geolamination and \( G \) is a gap of \( L \). Let \( H \) be the convex hull of \( \sigma_d(G \cap S) \). Then \( H \) is a point, a leaf of \( L \), or a gap of \( L \). Moreover, in the latter case, the map \( \sigma_d|_{\partial G} : \partial G \to \partial H \) is the positively oriented composition of a monotone map and a covering map.

Two distinct chords of \( \mathbb{D} \) are said to be linked if they intersect in \( \mathbb{D} \); we will also say that two chords cross each other if they are linked. Geolaminations considered in the paper are \( \sigma_d \)-invariant for some \( d \). A gap is called infinite (finite, uncountable) if \( G \cap S \) is infinite (finite, uncountable). Uncountable gaps are also called Fatou gaps. By the degree of a gap \( G \) we mean the number of components in the full preimage in \( G \) of a point of \( \sigma_d(G \cap S) \) except when a gap (leaf) \( G \) collapses to a point in which case the degree of \( G \) is the number of point inverses in \( G \cap S \).

**Definition 2.3**. A leaf \( \ell \) of a geolamination \( L \) is called critical if \( \sigma_d(\ell) \) is a single point. A gap is all-critical if its image is a point. A gap \( G \) is said to be critical if the degree of \( G \) is greater than one. We say that \( \overline{a} \) is a chord of a lamination \( L \) when \( \overline{a} \) is unlinked with all leaves of \( L \). A critical set is either a critical leaf or a critical gap. A critical chord (of \( L \)) \( \overline{ab} \) is a chord (of \( L \)) such that \( \sigma_d(a) = \sigma_d(b) \).

Given a geolamination \( L \) we linearly extend \( \sigma_d \) onto its leaves. Clearly, this extension is well-defined.

2.1.2. Laminations as equivalence relations. A lot of geolaminations naturally appear in the context of invariant equivalence relations on \( S \) (laminations) satisfying special conditions.

**Definition 2.4** (Laminations). An equivalence relation \( \sim \) on the unit circle \( S \) is called a lamination if either \( S \) is one \( \sim \)-class (such laminations are called degenerate), or the following holds:

1. (E1) the graph of \( \sim \) is a closed subset in \( S \times S \);
2. (E2) the convex hulls of distinct equivalence classes are disjoint;
3. (E3) each equivalence class of \( \sim \) is finite.

**Definition 2.5** (Laminations and dynamics). An equivalence relation \( \sim \) is called \((\sigma_d)\)-invariant if:
(D1) ∼ is forward invariant: for a ∼-class g, the set \( \sigma_d(g) \) is a ∼-class;
(D2) for any ∼-class g, the map \( \sigma_d : g \to \sigma_d(g) \) extends to \( \mathbb{S} \) as an orientation preserving covering map such that g is the full preimage of \( \sigma_d(g) \) under this covering map.

A lamination ∼ admits a canonical extension over \( \mathbb{C} \): classes of this extension are either convex hulls of classes of ∼, or points which do not belong to such convex hulls. By Moore’s Theorem the quotient space \( \mathbb{C}/\sim \) is homeomorphic to \( \mathbb{C} \). The quotient map \( p_\sim : \mathbb{S} \to \mathbb{S}/\sim \) extends to the plane with the only non-trivial point-preimages (fibers) being the convex hulls of non-degenerate ∼-classes. From now on we will always consider such extensions of the quotient map.

For a lamination ∼ consider the topological Julia set \( \mathbb{S}/\sim = J_\sim \) and the topological polynomial \( f_\sim : J_\sim \to J_\sim \) induced by \( \sigma_d \). One can extend \( f_\sim \) to a branched-covering map \( f_\sim : \mathbb{C} \to \mathbb{C} \) of degree \( d \) called a topological polynomial too. The map \( p_\sim \) restricted to \( \mathbb{S} \) semi-conjugates \( \sigma_d \) with \( f_\sim|_{J_\sim} \).

The complement \( K_\sim \) of the unique unbounded component \( U_\sim \) of \( \mathbb{C} \setminus J_\sim \) is called the filled topological Julia set. For a closed convex set \( H \), straight segments from \( \text{Bd}(H) \) are called edges of \( H \). Define the canonical geolamination \( L_\sim \) generated by ∼ as the collection of edges of convex hulls of all ∼-classes and all points of \( \mathbb{S} \).

**Theorem 2.6** ([BMOV12]). The geolamination \( L_\sim \) is \( \sigma_d \)-invariant.

The family of all canonical geolaminations of laminations is denoted by \( \mathcal{EQ} \) (or \( \mathcal{EQ}_d \) if the degree is \( d \)).

**2.1.3. Other useful notions.** We now consider objects that may appear in laminations. However, we do not fix the laminations containing these objects.

**Definition 2.7.** By a periodic gap or leaf we mean a gap or leaf \( G \) for which there exists the least number \( n \) (called the period of \( G \)) such that \( \sigma_d^n(G) = G \). Then we call the map \( \sigma_d^n : G \to G \) the remap. By refixed, reperiodic points (of a certain reperiod), and reorbits of points in \( G \) we mean the corresponding notions as applies to the remap.

Given two points \( a, b \in \mathbb{S} \) we denote by \( (a, b) \) the positively oriented arc from \( a \) to \( b \) (i.e., moving from \( a \) to be \( b \) within \( (a, b) \) takes place in the counterclockwise direction). For a closed set \( G' \subset \mathbb{S} \) of cardinality at least three we call components of \( \mathbb{S} \setminus G' \) holes. If \( \ell = \overline{ab} \) is an edge of \( G = \text{CH}(G') \), then we denote by \( H_G(\ell) \) the component of \( \mathbb{S} \setminus \{a, b\} \) disjoint from \( G' \).

**Definition 2.8.** If \( A \subset \mathbb{S} \) is a closed set such that all the sets \( \text{CH}(\sigma_d^n(A)) \) are pairwise disjoint, then \( A \) is called wandering. If there exists \( n \geq 1 \)
such that all the sets $\text{CH}(\sigma^i_d(A)), i = 0, \ldots, n - 1$ have pairwise disjoint relative interiors while $\sigma^n_d(A) = A$, then $A$ is called *periodic* of period $n$. If there exists $m > 0$ such that all $\text{CH}(\sigma^i_d(A)), 0 \leq i \leq m + n - 1$ have pairwise disjoint relative interiors and $\sigma^m_d(A)$ is periodic of period $n$, then we call $A$ *preperiodic*. If $A$ is wandering, periodic or preperiodic, and for every $i \geq 0$ and every hole $(a, b)$ of $\sigma^i_d(A)$ either $\sigma^i_d(a) = \sigma^i_d(b)$, or the positively oriented arc $(\sigma^i_d(a), \sigma^i_d(b))$ is a hole of $\sigma^{i+1}_d(A)$, then we call $A$ (and $\text{CH}(A)$) a $(\sigma_d)$-*laminational set*; we call both $A$ and $\text{CH}(A)$ *finite* if $A$ is finite. A $(\sigma_d)$-*stand alone gap* is defined as a laminational set with non-empty interior.

Observe that all the results mentioned above for gaps of laminations (such as No Wandering Triangle Theorem or Theorem 3.2) hold for stand alone gaps too. Denote by $<$ the *positive* (counterclockwise) circular order on $S = \mathbb{R}/\mathbb{Z}$ induced by the usual order of $\mathbb{R}$. Note that this order is only meaningful for sets of cardinality at least three. For example, we say that $x < y < z$ provided that moving from $x$ in the positive direction along $S$ we meet $y$ before meeting $z$.

**Definition 2.9** (Order preserving). Let $X \subset S$ be a set with at least three points. We say that $\sigma_d$ is *order preserving* on $X$ if $\sigma_d|_X$ is one-to-one and, for every triple $x, y, z \in X$ with $x < y < z$, we have $\sigma_d(x) < \sigma_d(y) < \sigma_d(z)$.

**Definition 2.10.** A subset $A \subset S$ is said to map in a *monotone* fashion provided $\sigma_d|_A$ extends to a monotone map of $S$.

### 2.2. General properties of geolaminations.

**Lemma 2.11** (Lemma 3.7 [BMOV12]). If $ab$ and $ac$ are two leaves of a geolamination $L$ such that $\sigma_d(a), \sigma_d(b)$ and $\sigma_d(c)$ are all distinct points then the order among points $a, b, c$ is preserved under $\sigma_d$.

We prove a few corollaries of Lemma 2.11.

**Lemma 2.12.** If $L$ is a geolamination, $\ell = ab$ is a leaf of $L$ and points $a, b$ are periodic, then the period of $a$ equals the period of $b$.

**Proof.** Suppose that the period of $a$ is $k$, the period of $b$ is $m$, and $k < m$. Then the $\sigma^k_d$-orbit $L$ of $\ell$ consists of several leaves with the common endpoint $a$. By Lemma 2.11 the map $\sigma^k_d$ preserves the circular order among vertices of each arc in $L$ formed by two images of $\ell$ concatenated at $a$. Hence the circular order among all vertices of $L$ must be preserved by $\sigma^k_d$ which is impossible (recall that $\sigma^k_d(a) = a$).

We will need the following elementary lemma.
Lemma 2.13. Suppose that $x \in S$ is such that the chords $\sigma_d^i(x)\sigma_d^{i+1}(x)$, $i = 0, 1, \ldots$ are pairwise unlinked. Then $x$ (and hence the leaf $x \sigma_d(x) = \ell$) is (pre)periodic.

Proof. The sequence of leaves from the lemma is the $\sigma_d$-orbit of $\ell$ in which consecutive images are concatenated and no two leaves are linked. If for some $i$ the leaf $\sigma_d^i(x)\sigma_d^{i+1}(x) = \sigma_d^i(\ell)$ is critical then $\sigma_d^{i+1}(\ell) = \{\sigma_d^{i+1}(x)\}$ is a $\sigma_d$-fixed point, which proves the claim in this case. Assume now that $\ell$ is not (pre)critical. If $x$ is not (pre)periodic, then, by topological considerations, leaves $\sigma_d^n(\ell)$ must converge to a limit leaf (or point) $\bar{\pi}$. Clearly, $\bar{\pi}$ is $\sigma_d$-invariant. However $\sigma_d$ is expanding, which makes such convergence impossible, a contradiction. □

Lemma 2.13 easily implies Lemma 2.14.

Lemma 2.14. Let $\mathcal{L}$ be a geolamination. Then the following holds.

1. If $\ell$ is a leaf of $\mathcal{L}$ and for some $n > 0$ the leaf $\sigma_d^n(\ell)$ is concatenated to $\ell$ then $\ell$ is (pre)periodic.
2. If $\ell$ has a (pre)periodic endpoint then $\ell$ is (pre)periodic.
3. If two leaves $\ell_1, \ell_2$ from geolaminations $\mathcal{L}_1, \mathcal{L}_2$ share the same (pre)periodic endpoint then they are (pre)periodic with the same eventual period of their endpoints.

Proof. (1) If $\ell = uw$ is (pre)critical, the claim easily follows. Suppose that $\ell$ is not (pre)critical, that $n$ is the least number with $\ell$ and $\sigma_d^n(\ell)$ concatenated, and that $\sigma_d^n(\ell) \neq \ell$. There are two cases. First, assume that $\sigma_d^n(u) = u$, and, for the sake of definiteness, that $u < v < \sigma_d^n(v)$. Then by Lemma 2.11 for every $i$ we have that $u < \sigma_d^{in}(v) < \sigma_d^{(i+1)n}(v) < u$. Expansion properties of $\sigma_d$ imply now that for some $N$ the leaf $u \sigma_d^{Nn}(v)$ is $\sigma_d^n$-fixed. Second, assume that $\sigma_d^n(u) = v$. Then the claim follows from Lemma 2.13.

(2) Follows from (1).

(3) Follows from (2) and Lemma 2.12. □

A similar conclusion can be made for edges of periodic gaps.

Lemma 2.15. Any edge of a periodic gap is (pre)periodic or (pre)critical.

Proof. Let $G$ be a fixed gap and $\ell$ be its edge which is not (pre)critical. The length $s_n$ of the “hole” $H_G(\sigma_d^n(\ell))$ of $G$ behind the leaf $\sigma_d^n(\ell)$ grows with $n$ as long as $s_n$ stays sufficiently small (it is easy to see that the correct bound on $s_n$ is that $s_n < \frac{1}{d+1}$). This implies that the sequence $\{s_i\}$ will contain infinitely many numbers greater than or equal to $\frac{1}{d+1}$. Since there are only finitely many distinct holes of $G$ of length greater $\frac{1}{d+1}$ or bigger, this implies that $\ell$ is (pre)periodic. □
Lemma 2.16 studies a polygon with disjoint interiors of images.

Lemma 2.16 (Lemma 4.5 [BMOV12]). If $B$ is a non-(pre)critical polygon for $\sigma_d$ and all sets $\sigma^j_d(B)$ have pairwise disjoint interiors then in fact all sets $\sigma^i_d(B)$ are pairwise disjoint.

Given $v \in S$, let $E(v)$ be the set of all endpoints $u$ of leaves $uv$ of $L$ (if $E(v)$ accumulates on $v$, then we include $v$ into $E(v)$).

Lemma 2.17. If $v$ is not (pre)periodic then $E(v)$ is at most finite. If $v$ is (pre)periodic then $E(v)$ is at most countable.

Proof. The first claim is proven in [BMOV12 Lemma 4.7]. The second claim follows from Lemma 2.14.

Properties of individual wandering polygons were studied, e.g., in [Kiw02]; properties of collections of wandering polygons were studied in [BL02]; their existence was established in [BO08]. The most detailed results on wandering polygons and their collections are due to Childers [Chi07]. Theorem 2.18 encompasses a small portion of Childers’ results.

Theorem 2.18 ([Chi07]). If $A$ is a wandering non-(pre)critical $k$-gon then there exist $k - 1$ critical chords with distinct grand orbits onto which the orbit of $A$ accumulates.

If $\ell$ is a periodic leaf of a geolamination, then by Lemma 2.12 the periods of its endpoints coincide. It is easy to describe the entire $\sigma_d$-orbit of $\ell$.

Proposition 2.19. Let $\ell = xy$ be a chord with $\sigma_d$-periodic endpoints of the same period $N$ such that the iterated $\sigma_d$-images of $\ell$ are pairwise unlinked. Then the components of the union of the $\sigma_d$-orbit of $\ell$ are leaves or polygons rotating transitively under the remap.

Proof. If there are non-disjoint but distinct images of $\ell$ then there exists a unique $s < N$ with $\sigma^s_d(x) = y$. The leaves $\ell, \sigma_d(\ell), \ldots$ form a concatenation $B_1$ which eventually “dead ends” into the point $x$ (because $x$ is periodic). By the assumptions, $B_1$ is the boundary of the convex hull $P_1$ of $B_1 \cap S$. Then $P_1$ is one of the polygons mentioned in the claim. Clearly, the other images of $x$ are disjoint from $P_1$ and $\sigma_d(P_1), \sigma^2_d(P_1), \ldots, \sigma^{s-1}_d(P_1)$ are other polygons from the lemma.

The last results of this subsection deal with infinite periodic gaps of geolaminations of degree one. It is well-known [Kiw02] that any infinite gap $G$ of a geolamination $L$ is (pre)periodic. By a vertex of $G$ we mean any point of $G \cap S$. The next lemma is easy to prove.

Lemma 2.20. Let $G$ be a periodic gap of period $n$ and set $K = \text{Bd}(G)$. Then $\sigma^i_d|_K$ is either a monotone map of the Jordan curve $K$ onto itself, or
the composition of a covering map and a monotone map of $K$. If $\sigma^n_{d}|_K$ is of degree one then one of the following holds.

1. The gap $G$ has finitely many $\sigma^n_{d}$-periodic edges or vertices. Between any pair of adjacent $\sigma^n_{d}$-periodic points $a$ and $b$ of $K$, there are one or more concatenated $\sigma^n_{d}$-critical edges at $a$ or $b$ (say, at $a$) which collapse to $a$. The gap $G$ has countably many vertices.

2. The map $\sigma^n_{d}|_K$ is monotonically semiconjugate to an irrational circle rotation so that each fiber of this semiconjugacy is a finite concatenation of (pre)critical edges of $G$.

Proof. We will prove only the very last claim. Denote by $\varphi$ the semiconjugacy from (2). Let $T \subset K$ be a fiber of $\varphi$. By Lemma 2.15 all edges of $G$ are (pre)critical. Hence if $T$ contains infinitely many edges, then the forward images of $T$ will hit critical leaves of $\sigma^n_{d}$ infinitely many times as $T$ cannot collapse under a finite power of $\sigma^n_{d}$. This would imply that an irrational circle rotation has periodic points, a contradiction. □

Lemma 2.20 implies Corollary 2.21.

Corollary 2.21. Suppose that $G$ is a periodic gap of a geolamination $L$, whose remap has degree one. Then at most countably many pairwise unlinked leaves of other geolaminations can be located inside $G$.

Proof. Clearly, any chord located inside $G$ has its endpoints at vertices of $G$. Since in case (1) of Lemma 2.20 there are countably many vertices of $G$, we may assume that case (2) of Lemma 2.20 holds. Applying the semiconjugacy $\varphi$ from this lemma we see that if a leaf $\ell$ is located inside $G$ and its endpoints are not mapped to the same point by $\varphi$, then $\ell$ will eventually cross itself. If there are uncountably many leaves of geolaminations inside $G$ then among them there must exist a leaf $\ell$ whose endpoints belong to distinct fibers of $\varphi$. By the above some forward images of $\ell$ cross each other, a contradiction. □

2.3. Some special types of geolaminations. Below, we discuss proper geolaminations, perfect geolaminations, and full dendritic geolaminations.

2.3.1. Proper geolaminations. The construction of $L_\sim$ is based on choosing all edges of convex hulls of all $\sim$-classes. Although this is a natural choice of leaves associated with a $\sim$-class, it is not unique; other choices will result into different geolaminations.

Example 2.22. Assume that there is a bounded domain $U$ complementary to $J_\sim$. Then $p^{-1}_\sim(U) = H$ is the Fatou domain of $L_\sim$. Suppose that there is a periodic point $x \in \text{Bd}(U)$ of period $m$ which cuts $J_\sim$ into $k > 2$ components. Then the $\sim$-class $p^{-1}_\sim(x)$ consists of $k$ points of $S$. The corresponding
gap $G$ of $\mathcal{L}_\sim$ is the convex hull of $p_\sim^{-1}(x)$. Assume that each edge of $G$ maps to itself under $\sigma_d^n$ and erase the edge shared by $H$ and $G$ with all its preimages. It is easy to see that the resulting geolamination $\hat{\mathcal{L}}_\sim \subsetneq \mathcal{L}_\sim$ is a closed $\sigma_d$-invariant geolamination.

To account for various geolaminations related to a lamination, we use an approach which dates back to Kiwi [Kiw04]. This approach allows one to associate a natural equivalence relation to a given geolamination.

**Definition 2.23.** Given a geolamination $\mathcal{L}$, define an equivalence relation $\approx_{\mathcal{L}}$ as follows: for points $x, y \in \mathcal{S}$, we set $x \approx_{\mathcal{L}} y$ if and only if there is a finite concatenation of leaves of $\mathcal{L}$ connecting $x$ and $y$.

Clearly, $\approx_{\mathcal{L}}$ is an equivalence relation. Moreover, we prove in Lemma 2.17 that any $\approx_{\mathcal{L}}$-class is at most countable. Note that $\approx_{\mathcal{L}}$ is not necessarily a closed equivalence relation.

**Definition 2.24.** Assume that a lamination $\sim$ is given. Suppose that $\mathcal{L}$ is a geolamination such that equivalence relations $\approx_{\mathcal{L}}$ and $\sim$ coincide. Then $\mathcal{L}$ and $\sim$ are said to be compatible.

Clearly, $\mathcal{L}_\sim$ and $\sim$ are compatible (i.e., $\approx_{\mathcal{L}_\sim}=\sim$). In Example 2.22 we consider, for a lamination $\sim$ with certain properties, a specific geolamination $\hat{\mathcal{L}}_\sim \subsetneq \mathcal{L}_\sim$ compatible with $\sim$. A natural class of geolaminations $\mathcal{L}$ such that $\approx_{\mathcal{L}}$ is a lamination are the so-called proper geolaminations forming the family $\mathcal{PR}$ (we write $\mathcal{PR}_d$ if the degree is $d$). Proper geolaminations $\mathcal{L}$ can be defined as such that any non-periodic leaf $\ell$ of $\mathcal{L}$ has non-periodic endpoints (see [BMOV12] and Lemma 2.27). This is equivalent to the following.

**Definition 2.25.** Let $\mathcal{L}$ be a geolamination. A wedge (of $\mathcal{L}$) is a pair of leaves $L = \{\ell_1, \ell_2\}$ which share a common endpoint $v$ called the vertex of the wedge. A wedge $\{\ell_1, \ell_2\}$ is critical if $\sigma_d(\ell_1) = \sigma_d(\ell_2)$. A geolamination $\mathcal{L}$ is proper if it has no critical leaves with periodic endpoint and no critical wedges with periodic vertex.

The following is the main result of Section 4 of [BMOV12].

**Theorem 2.26 ([BMOV12], Theorem 4.9).** Let $\mathcal{L}$ be a proper invariant geolamination. Then $\approx_{\mathcal{L}}$ is a lamination.

A useful equivalence concerning proper geolaminations is stated below.

**Lemma 2.27.** The following properties are equivalent.

(1) A geolamination $\mathcal{L}$ is proper.

(2) The relation $\approx_{\mathcal{L}}$ is a lamination.

(3) Any non-periodic leaf $\ell$ of $\mathcal{L}$ has non-periodic endpoints.
Proof. If $\mathcal{L}$ is proper, then $\approx_{\mathcal{L}}$ is a lamination by Theorem 2.26. Now, suppose that $\approx_{\mathcal{L}}$ is a lamination. Let us show that its classes are all finite. Indeed, by Lemma 2.17 $\approx_{\mathcal{L}}$-classes are at most countable. On the other hand, it is well-known (see, e.g., [BL02]) that classes of any lamination are either finite or uncountable. Hence $\approx_{\mathcal{L}}$-classes are finite.

Let us show that this implies that $\mathcal{L}$ has no non-periodic leaves with a periodic endpoint. Indeed, suppose otherwise. We may assume that $\ell = ab$ is such that $a$ is fixed while $b$ is not fixed. Then we can pull $\ell$ back to a leaf $\ell_{-1} \neq \ell$ with an endpoint $a$ which is also non-periodic. Repeating this infinitely many times we see that $\approx_{\mathcal{L}}$-class of $a$ is infinite, a contradiction.

We claim that if any non-periodic leaf $\ell$ of $\mathcal{L}$ has non-periodic endpoints then $\mathcal{L}$ is proper. Suppose that $\mathcal{L}$ is not proper. Then it may happen so that there exists a leaf $\ell = ab$ of $\mathcal{L}$ such that $a$ is periodic and $\ell$ is critical. Clearly, this contradicts the assumption. Now, suppose that there are two leaves $\overline{ab}$ and $\overline{ax}$ of $\mathcal{L}$ such that $a$ is periodic and $\sigma_d(b) = \sigma_d(x)$. Then either $b$ or $x$ is non-periodic, a contradiction. Hence $\mathcal{L}$ is proper as desired. \hfill $\Box$

While a proper geolamination $\mathcal{L}$ does not have to coincide with the canonical geolamination $\mathcal{L}_{\approx_{\mathcal{L}}}$ generated by $\approx_{\mathcal{L}}$, it is easy to see that any proper geolamination $\mathcal{L}$ is compatible with the lamination $\approx_{\mathcal{L}}$ (i.e., leaves of $\mathcal{L}$ and leaves of $\mathcal{L}_{\approx_{\mathcal{L}}}$ are pairwise unlinked). It follows that proper invariant geolaminations are precisely invariant geolaminations compatible with canonical geolaminations of laminations.

2.3.2. Perfect geolaminations. A geolamination $\mathcal{L}$ is said to be perfect if no leaf of $\mathcal{L}$ is isolated. We claim that every geolamination contains a maximal perfect sublamination, and this sublamination contains all degenerate leaves. Indeed, consider $\mathcal{L}$ as a metric space with the Hausdorff metric and denote it by $\mathcal{L}^*$. Then $\mathcal{L}^*$ is a compact metric space which has a maximal perfect subset $\mathcal{L}^c$ called the perfect sublamination of $\mathcal{L}$.

Lemma 2.28 is left to the reader (its first part is easy to verify, and its second part is a well-known topological fact).

Lemma 2.28. The lamination $\mathcal{L}^c$ is an invariant perfect geolamination. For every $\ell \in \mathcal{L}^c$ and every neighborhood $U$ of $\ell$ there exist uncountably leaves of $\mathcal{L}^c$ in $U$.

The process of finding $\mathcal{L}^c$ was described in much more detail in [BOPT10]. Observe that if $\ell$ is a critical leaf of $\mathcal{L}^c$ then, since by Lemma 2.17 there are at most countably many leaves of $\mathcal{L}^c$ emanating from the endpoints of $\ell$, we see that $\ell$ is the limit of leaves of $\mathcal{L}$ disjoint from $\ell$. Hence $\sigma_d(\ell)$ is a point separated from the rest of the circle by images of those leaves. It follows that $\ell$ is either disjoint from all other leaves or gaps of $\mathcal{L}^c$ or is an edge of an all-critical gap of $\mathcal{L}^c$. 
Definition 2.23 helps relate perfect geolaminations and laminations.

**Theorem 2.29.** If $\mathcal{L}$ is perfect then $\mathcal{L}$ is proper and $\approx_{\mathcal{L}}$ is a lamination.

**Proof.** If $\mathcal{L}$ contains three leaves with a common endpoint then the leaf in the middle would have a neighborhood $U$ that, by Lemma 2.17, contains at most countably many leaves of $\mathcal{L}$ which is impossible for perfect geolaminations. Now, if $\pi x$ is a leaf of $\mathcal{L}$ such that $x$ is periodic and $y$ is not then there exist two distinct pullbacks of $\pi y$ with endpoint $x$ each of which is distinct from $\pi y$, a contradiction. By Lemma 2.27 this implies that $\mathcal{L}$ is proper and so, again by Lemma 2.27, $\approx_{\mathcal{L}}$ is a lamination. □

2.3.3. Full dendritic geolaminations. A lamination $\sim$ and the associated geolamination $\mathcal{L}_{\sim}$ are called dendritic if all $\sim$-classes are finite and the topological Julia set $J_{\sim}$ is a dendrite. The family of all dendritic geolaminations is denoted by $\mathcal{D}_{d}$.

The following claim is well-known and is given without a proof.

**Lemma 2.30.** Dendritic geolaminations $\mathcal{L}$ are perfect.

Dendritic geolaminations are closely related to polynomials. Indeed, by Jan Kiwi’s results [Kiw04] if a polynomial $P$ with connected Julia set $J(P)$ has no Siegel or Cremer periodic points (i.e., irrationally indifferent periodic points whose multiplier is of the form $e^{2\pi i \theta}$ for some irrational $\theta$) then there exists a special lamination $\sim_P$, determined by $P$, with the following property: $P|_{J(P)}$ is monotonically semiconjugate to $f_{\sim_P}|_{J_{\sim_P}}$. Moreover, all $\sim_P$-classes are finite and the semiconjugacy is one-to-one on all (pre)periodic points of $P$ (analogous results for all polynomials with connected Julia sets but without the conclusion about finiteness of all $\sim_P$-classes can be found in [BCO11]).

Denote by $\mathcal{R}$ the space of all polynomials with only repelling periodic points and by $\mathcal{R}_d$ the space of all such polynomials of degree $d$. By [Kiw04] $P \in \mathcal{R}$ if and only if the lamination $\sim_P$ is dendritic. Strong conclusions about the topology of the Julia sets of non-renormalizable polynomials $P \in \mathcal{R}$ follow from [KvS06]. Building upon earlier results by Kahn and Lyubich [KL09a, KL09b] and by Kozlovskii, Shen and van Strien [KSvS07a, KSvS07b], Kozlovskii and van Strien generalized results of Avila, Kahn, Lyubich and Shen [AKLS09] and proved in [KvS06] that if all periodic points of $P$ are repelling and $P$ is non-renormalizable then $J(P)$ is locally connected; moreover, by [KvS06] two such polynomials that are topologically conjugate are in fact quasi-conformally conjugate. Thus, in this case $f_{\sim_P}|_{J_{\sim_P}}$ is a precise model of $P|_{J(P)}$. Finally, for a given dendritic laminations $\sim$ it follows from another result of Jan Kiwi [Kiw05] that there exists a polynomial $P$ with $\sim = \sim_P$. 
Thus, we can associate polynomials from $\mathcal{R}$ with their laminations $\sim_P$ and therefore with their geolaminations $\mathcal{L}_P = \mathcal{L}_{\sim_P}$. By [Kiw05], this maps polynomials from $\mathcal{R}_d$ onto $\mathcal{D}_d$. The relation between polynomials and dendritic geolaminations is simpler if we further specify the class of geolaminations (and polynomials) we are studying.

**Definition 2.31.** A geolamination $\mathcal{L}$ is said to be full if it has a collection $\mathcal{C}_2(\mathcal{L})$ of $d - 1$ distinct critical sets such that (1) each two of these sets can intersect inside $\mathbb{D}$ only if both are gaps sharing a common edge, and (2) there are no loops of critical leaves in $\mathcal{C}_2(\mathcal{L})$. The collection $\mathcal{C}_2(\mathcal{L})$ is called the degree two critical portrait of $\mathcal{L}$. The space of all degree two critical portraits of full geolaminations of degree $d$ is denoted by $\mathcal{C}_d^2$.

Observe that if a geolamination is full then all critical sets have degree two with respect to $\sigma_d$ (whence the name “degree two critical portrait”). The choice of critical sets from Definition 2.31 for $\mathcal{L}$ is not always unique because if $\mathcal{L}$ has an all-critical set then the choice of some of its edges which should be included in the collection is not unique. Moreover, even if the choice of critical sets is made, in degree two critical portraits sets are considered with order so that the same collection of sets appears $(d - 1)!$ times as a degree two critical portrait. When talking about a full geolamination we always assume that its degree two critical portrait is chosen; if a different degree two critical portrait is chosen, we view $\mathcal{L}$ with it as a different geolamination.

The family of all $\sigma_d$-invariant full dendritic geolaminations with chosen degree two critical portraits is denoted by $\mathcal{FD}_d$. Let us denote by $\mathcal{RF}_d$ the set of polynomials $P \in \mathcal{R}_d$ with $\mathcal{L}_P \in \mathcal{FD}_d$. According to the above, in $\mathcal{FD}_d$ the same full dendritic geolamination $\mathcal{L}$ is considered $(d - 1)!$ (or more) times depending on the chosen order among its critical points; respectively, in $\mathcal{RF}_d$ each polynomial $P$ is considered many times depending on the chosen order among its critical points and on the chosen critical portrait.

To study the association of polynomials from $\mathcal{RF}_d$ with their geolaminations we need Lemma 2.32 which is due to Goldberg and Milnor.

**Lemma 2.32 (GM93).** Let $P$ be a polynomial, and $I$ be a finite collection of its (pre)periodic external rays landing at iterated preimages of repelling periodic points. Then, for any polynomial $P_1$ sufficiently close to $P$, there is a family $I_1$ of (pre)periodic rays uniformly close to the corresponding rays of $I$ which have the same external arguments.

We also need Lemma 2.33 from [BCMO11].

**Lemma 2.33 (BCMO11).** Suppose that $\sim$ is a lamination such that $\mathcal{L}_{\sim} \in \mathcal{D}_d$. Then each leaf of $\mathcal{L}_{\sim}$ can be approximated by (pre)periodic leaves.
Lemmas 2.32 and 2.33 easily imply Lemma 2.34.

**Lemma 2.34.** Let \( P \in \mathcal{RF}_d \) and let \( C_1, \ldots, C_{d-1} \) be critical sets of \( \mathcal{L}_P \). Let \( U_1, \ldots, U_{d-1} \) be neighborhoods of \( C_1, \ldots, C_{d-1} \). Then there exists a neighborhood \( W \) of \( P \) in \( \mathbb{R}^d \) such that any polynomial \( \hat{P} \in W \) has critical sets \( \hat{C}_1 \subset U_1, \ldots, \hat{C}_{d-1} \subset U_{d-1} \).

By Lemma 2.34, critical sets of geolaminations \( \mathcal{L}_P, P \in \mathcal{RF}_d \) cannot explode under perturbation of \( P \) (they may implode though). Thus, we obtain the following corollary.

**Corollary 2.35.** The set \( \mathcal{RF}_d \) is an open subset of \( \mathbb{R}^d \). Critical sets of \( \mathcal{L}_P \) depend on \( P \in \mathcal{RF}_d \) upper semicontinuously.

In the end of the paper we propose a geometric (visual) way to parameterize \( \mathcal{F}_d \); through the association of \( P \) with its geolamination \( \mathcal{L}_P \) this implies the corresponding parameterization of \( \mathcal{RF}_d \) and gives an important application of our tools.

## 3. Accordions of Laminations

In Introduction we briefly explained that some of Thurston’s tools from [Thu85] fail in the cubic case. This motivates us to develop new tools (so-called *accordions*) which basically are tracking linked leaves from different geolaminations. In this section we first elaborate upon a discussion in the Introduction, and then study accordions in detail. In Sections 3 – 4 we assume that \( \mathcal{L}_1, \mathcal{L}_2 \) are \( \sigma_d \)-invariant geolaminations and \( \ell_1, \ell_2 \) are leaves of \( \mathcal{L}_1, \mathcal{L}_2 \), respectively.

### 3.1. Motivation

We start by recalling some facts from the Introduction. For a quadratic invariant geolamination \( \mathcal{L} \) and a leaf \( \ell \) of \( \mathcal{L} \) which is not a diameter, let \( \ell' \) be the sibling of \( \ell \) (disjoint from \( \ell \)). Denote by \( C(\ell) \) the open strip of \( \mathbb{D} \) between \( \ell \) and \( \ell' \) and by \( L(\ell) \) the length of the shorter component of \( S \setminus \ell \). Suppose that \( \frac{1}{3} \leq L(\ell) < \frac{1}{2} \), and that \( k \) is the smallest number such that \( \sigma^k_2(\ell) \subset C(\ell) \). The Central Strip Lemma [1.2] claims that \( \sigma^k_2(\ell) \) separates \( \ell \) and \( \ell' \). In particular, if \( \ell = M \) is a major then \( M \) cannot enter \( C(M) \).

Let us list Thurston’s results for which the Central Strip Lemma is crucial. A *wandering triangle* for \( \sigma_2 \) is a triangle with vertices \( a, b, c \) on the unit circle such that the convex hull \( T_n \) of \( \sigma^n_2(a), \sigma^n_2(b), \sigma^n_2(c) \) is a non-degenerate triangle for every \( n = 0, 1, \ldots \), and all these triangles are pairwise disjoint.

**Theorem 3.1** (No Wandering Triangle Theorem [Thu85]). There are no wandering triangles for \( \sigma_2 \).
The next Theorem 3.2 follows from the Central Strip Lemma and is due to Thurston for $d = 2$; for arbitrary $d$ it is was proven by J. Kiwi who used different tools.

**Theorem 3.2** ([Thu85, Kiw02]). If $A$ is a finite $\sigma_d$-periodic gap of period $k$ then either $A$ is a $d$-gon and $\sigma_d^k$ fixes all vertices of $A$, or there are at most $d - 1$ periodic orbits of vertices of $A$. Thus, for $d = 2$ the remap is transitive on the vertices of a finite periodic gap.

Another crucial result by Thurston is that minors of distinct quadratic invariant geolaminations are disjoint in $\mathbb{D}$.

Let $m_1$ and $m_2$ be the minors of two invariant geolaminations $L_1 \neq L_2$ which are non-disjoint in $\mathbb{D}$. Let $M_1$, $M'_1$, $M_2$, $M'_2$ be the two pairs of corresponding majors. Then we may assume that $M_1$, $M_2$ are non-disjoint in $\mathbb{D}$, that $M'_1$, $M'_2$ are non-disjoint in $\mathbb{D}$, but $(M_1 \cup M_2) \cap (M'_1 \cup M'_2) = \emptyset$ (see Figure 2) so that there is a diameter $\tau$ with strictly preperiodic endpoints separating $M_1 \cup M_2$ from $M'_1 \cup M'_2$. Thurston shows that there exists a unique invariant geolamination $\mathcal{L}$, with only finite gaps, whose major is $\tau$. By the Central Strip Lemma, forward images of $m_1$, $m_2$ do not intersect $\tau$. This implies that $m_1 \cup m_2$ is contained in a finite gap $G$ of $\mathcal{L}$. By the No Wandering Triangle Theorem $G$ is eventually periodic; now by Theorem 3.2 images of $m_1$ intersect themselves inside $\mathbb{D}$, a contradiction.

Simple examples show that statements analogous to the Central Strip Lemma already fail in the cubic case. Indeed, Figure 3 shows a leaf $M = \frac{342}{728} \frac{576}{728}$ of period 6 under $\sigma_3$ and its $\sigma_3$-orbit together with the leaf $M'$ (which has the same image as $M$ forming together with $M$ a narrower “critical strip” $S_n$) and the leaf $N'$ (which has the same image as $N = (\sigma_3)^4(M)$ forming together with $N$ a wider “critical strip” $S_w$). Observe that $\sigma_3(M) \subset S_w$ which shows that Lemma 1.2 does not hold in the cubic case (orbits of
periodic leaves may contain “critical strips” containing some elements of these orbits of leaves). This apparently makes a direct extension of the arguments from the previous paragraph impossible leaving the issue of whether and how minors of cubic geolaminations can be linked unresolved.

Another consequence of the failure of the Central Strip Lemma in the cubic case is the failure of the No Wandering Triangle Theorem (a counterexample was given in [BO08]). Properties of wandering polygons were studied in [Kiw02, BL02, Chi07].

3.2. Properties of accordions. Recall the definition of accordions.

Definition 3.3. Let $A_{\mathcal{L}_2}(\ell_1)$ be the collection of leaves of $\mathcal{L}_2$ linked with $\ell_1$ together with $\ell_1$. Let $A_{\sigma d}(\ell_1)$ be the collection of leaves from the forward orbit of $\ell_2$ which are linked with $\ell_1$ together with $\ell_1$. The sets defined above are called accordions (of $\ell_1$) while $\ell_1$ is called the axis of the accordion. Sometimes we will also use $A_{\mathcal{L}_2}(\ell_1)$ and $A_{\sigma d}(\ell_1)$ to mean the union of the leaves constituting these accordions.

In general, accordions do not behave nicely with respect to $\sigma_d$ as leaves which are linked may have unlinked images. To avoid these problems, for the rest of this section we will impose the following conditions on accordions. While fairly strong, they are satisfied in a number of situations and force a rigid structure on the orbits of accordions.

Definition 3.4. A leaf $\ell_1$ is said to have order preserving accordions with respect to $\mathcal{L}_2$ (respectively, to a leaf $\ell_2$) if $A_{\mathcal{L}_2}(\ell_1) \neq \{\ell_1\}$ (respectively, $A_{\sigma d}(\ell_1) \neq \{\ell_1\}$), and for each $k \geq 0$, the map $\sigma_d$ restricted to $A_{\mathcal{L}_2}(\sigma_d^k(\ell_1)) \cap S$ (respectively, to $A_{\sigma d}(\sigma_d^k(\ell_1)) \cap S$) is order preserving. Say that $\ell_1$ and
have mutually order preserving accordions if \( \ell_1 \) has order preserving accordions with respect to \( \ell_2 \), and vice versa. In particular, all powers of \( \sigma_d \) act one-to-one on the endpoints of leaves included into the accordions.

The proof of Proposition 3.5 is left to the reader.

**Proposition 3.5.** If \( \sigma_d \) is order preserving on an accordion \( A \) with axis \( \ell_1 \) and \( \ell \in A, \ell \neq \ell_1 \) then \( \sigma_d(\ell) \) and \( \sigma_d(\ell_1) \) are linked. In particular, if \( \ell_1 \) has order preserving accordions with respect to \( \ell_2 \) then \( \sigma_d^{k}(\ell) \in A_{\ell_2}(\sigma_d^{k}(\ell_1)) \) for every \( \ell \in A_{\ell_2}(\ell_1), \ell \neq \ell_1 \) and every \( k \geq 0 \).

We now explore more closely the orbits of leaves from Definition 3.4.

**Proposition 3.6.** Suppose that \( \ell_1 \) and \( \ell_2 \) are linked, \( \ell_1 \) has order preserving accordions with respect to \( \ell_2 \), and \( \sigma_d^{k}(\ell_2) \in A_{\ell_2}(\ell_1) \) for some \( k > 0 \). In this case, if \( \ell_2 = \overline{xy} \), then either \( \ell_1 \) separates \( x \) from \( \sigma_d^{k}(x) \) and \( y \) from \( \sigma_d^{k}(y) \), or \( \ell_2 \) has \( \sigma_d^{k} \)-fixed endpoints.

**Proof.** Suppose that \( \ell_2 \) is not \( \sigma_d^{k} \)-fixed. Denote by \( x_0 = x, y_0 = y \) the endpoints of \( \ell_2 \); set \( x_i = \sigma_d^{k}(x_0), y_i = \sigma_d^{k}(y_0) \) and \( A_t = A_{\ell_2}(\sigma_d^{k}(\ell_1)), t = 0, 1, \ldots \). If \( \ell_1 \) does not separate \( x_0 \) and \( x_1 \), then either \( x_0 < x_1 < y_1 < y_0 \) or \( x_0 < y_0 < y_1 < x_1 \). We may assume the latter (cf. Figure 4).

![Figure 4](image)

**Figure 4.** This figure illustrates Proposition 3.6. Although in the figure \( \overline{x_2y_2} \) is linked with \( \ell_1 \), the argument does not assume this.

Since \( \sigma_d^{k} \) is order preserving on \( A_0 \cap \mathbb{S} \), then \( x_0 < y_0 < y_1 < y_2 < x_2 < x_1 < x_0 \) while the leaves \( \overline{x_1y_1} \) and \( \overline{x_2y_2} \) belong to the accordion \( A_k \) so that the above inequalities can be iterated. Inductively we see that

\[
x_0 < y_0 < \ldots < y_{m-1} < y_m < x_m < x_{m-1} < \ldots < x_0.
\]

All leaves \( \overline{x_iy_i} \) are pairwise distinct as otherwise there exists \( n \) such that \( \overline{x_{n-1}y_{n-1}} \neq \overline{x_ny_n} = \overline{x_{n+1}y_{n+1}} \) contradicting \( \sigma_d^{k} \) being order preserving.
on $A_{k(n-1)}$. Hence the leaves $x_i y_i$ converge to a $\sigma_d^k$-fixed point or leaf, contradicting the expansion property of $\sigma_d^k$.

In what follows denoting leaves we often use one of their endpoints as the subscript in our notation for them.

**Lemma 2.12.** Points $x, y$ assume that by Proposition 3.6 and the order preservation, case (4) holds. Thus we can have case (1).

Let us show that then

$$\text{Proof.}$$

the pairs of images of $x, y$ are also of period $k$. Suppose that $x = \sigma_d^k(x), y = \sigma_d^k(y)$. Since by Lemma 2.12 the points $x$ and $y$ have the same period (say, $m$), then $m$ divides $k$. Similarly, $k$ divides $m$. Hence $k = m$. □

Mostly we will use the following corollary of the above results.

**Corollary 3.8.** Suppose that $\ell_a = \overline{ab}$ and $\ell_x = \overline{xy}$ with $a < x < b < y$ are linked leaves. If $\ell_a$ has order preserving accordions with respect to $\ell_x$ then there are the following possibilities for $A = A_{\ell_x}(\ell_a)$.

1. $A = \{\ell_a, \ell_x\}$ and no forward image of $\ell_x$ crosses $\ell_a$.
2. $A = \{\ell_a, \ell_x\}$ and there exists $j$ such that $\sigma_d^j(x) = y, \sigma_d^j(y) = x$ and either $\sigma_d^j(a) = b, \sigma_d^j(b) = a$, or $\sigma_d^j(a) \neq b, \sigma_d^j(b) \neq a$, and $\ell_x$ separates points $a, \sigma_d^j(b)$ (located on one side of $\ell_x$) from points $b, \sigma_d^j(a)$ (located on the other side of $\ell_x$). In any case, $\sigma_d^{2j}(a) = a$ and $\sigma_d^{2j}(b) = b$.
3. $A = \{\ell_a, \ell_x\}$, the points $a, b, x, y$ are of the same period, $x, y$ have distinct orbits, and $a, b$ have distinct orbits.
4. There exists $i > 0$ such that $A = \{\ell_a, \ell_x, \sigma_d^i(\ell_a)\}$ and either $a < x < y \leq \sigma_d^i(x) < b < \sigma_d^i(y) \leq x$ or $x \leq \sigma_d^i(y) < a < \sigma_d^i(x) \leq y < b$, as shown in Figure 6.

**Proof.** Three distinct images of $\ell_x$ cannot cross $\ell_a$ as if they do then it is impossible for the separation required in Proposition 3.6 to occur for all of the pairs of images of $\ell_x$. Hence at most two images of $\ell_x$ cross $\ell_a$.

If there are two distinct leaves from the orbit of $\ell_x$ linked with $\ell_a$, then, by Proposition 3.6 and the order preservation, case (4) holds. Thus we can assume that $A = \{\ell_a, \ell_x\}$. If no forward image of $\ell_x$ is linked with $\ell_a$, then we have case (1).

In all remaining cases we have $\sigma_d^k(\ell_x) = \ell_x$ for some $k > 0$. By Lemma 2.12 points $x$ and $y$ are of the same period. Suppose that $x, y$ belong to the same periodic orbit. Choose the least $j$ such that $\sigma_d^j(x) = y$. Let us show that then $\sigma_d^j(y) = x$. Indeed, assume that $\sigma_d^j(y) \neq x$. Since by the assumption the only leaf from the forward orbit of $\ell_x$, linked with
\( \ell_a \), is \( \ell_x \), we may assume (for the sake of definiteness) that \( y < \sigma_d^j(y) \leq b \). Then a finite concatenation of further \( \sigma_d^j \)-images of \( \ell_x \) will connect \( y \) with \( x \). Again since \( A = \{ \ell_a, \ell_x \} \) then one of their endpoints will coincide with \( b \). Thus, \( y < \sigma_d^j(y) \leq b < \sigma_d^j(b) \leq x \). Let us now apply \( \sigma_d^j \) to \( A \); by order preservation \( y < \sigma_d^j(a) < \sigma_d^j(y) \leq b < \sigma_d^j(b) \leq x < a \). Hence, \( \sigma_d^j(\ell_a) \) is linked with \( \ell_a \), a contradiction.

Thus, \( \sigma_d^j(y) = x \) (i.e., \( \sigma_d^j \) flips \( \ell_x \) onto itself), \( k = j \), the points \( x \) and \( y \) are of period \( 2j \) and by Lemma 3.7 \( a \) and \( b \) are also of period \( 2j \). If \( \sigma_d^j(a) = b \) then \( \sigma_d^j(b) = a \) (by Lemma 3.7, \( a \) and \( b \) are of period \( 2j \)), and if \( \sigma_d^j(b) = a \) then \( \sigma_d^j(a) = b \) (for the same reason). Now, if \( \sigma_d^j(a) \neq b \) and \( \sigma_d^j(b) \neq a \) then by order preservation \( \ell_x \) separates points \( a, \sigma_d^j(b) \) (located on one side of \( \ell_x \)) from points \( b, \sigma_d^j(a) \) (located on the other side of \( \ell_x \)). So, case (2) holds.

Assume that \( x \) and \( y \) belong to distinct periodic orbits of period \( k \). By Lemma 3.7 points \( a, b \) are of period \( k \). Let points \( a \) and \( b \) have the same orbit. Then if \( k = 2j \) and \( \sigma_d^j \) flips \( \ell_x \) onto itself it would follow by order preservation that \( \sigma_d^j(\ell_x) \) is linked with \( \ell_a \). Since \( \ell_x \) is the unique leaf from the orbit of \( \ell_x \) linked with \( \ell_a \) this would imply that \( \sigma_d^j \) flips \( \ell_x \) onto itself, a contradiction with \( x, y \) having disjoint orbits. Hence we may assume that for some \( j \) and \( m > 2 \) we have that \( \sigma_d^j(a) = b, jm = k \), and a concatenation of leaves \( \ell_a, \sigma_d^j(\ell_a), \ldots, \sigma_d^{j(m-1)}(\ell_a) \) forms a polygon \( P \).

If one of these leaves distinct from \( \ell_a \) (say, \( \sigma_d^{js}(\ell_a) \)) is linked with \( \ell_x \), we can apply the map \( \sigma_d^{js(m-s)} \) to \( \sigma_d^{js}(\ell_a) \) and \( \ell_x \); by order preservation we will see then that \( \ell_a \) and \( \sigma_d^{js(m-s)}(\ell_x) \neq \ell_x \) are linked, a contradiction with the assumption that \( A = \{ \ell_a, \ell_x \} \). If none of the leaves \( \sigma_d^j(\ell_a), \ldots, \sigma_d^{j(m-1)}(\ell_a) \) is linked with \( \ell_x \) then \( P \) has an endpoint of \( \ell_x \) as one of its vertices. Similarly to the argument given above, we can then apply \( \sigma_d^j \) to \( A \) and observe that
by order preservation the $\sigma_d^{-j}$-image of $\ell_x$ is forced to be linked with $\ell_x$, a contradiction. Hence $a$ and $b$ have disjoint orbits and case (3) holds.

3.3. Accordions are (pre-)periodic or wandering. Here we prove Theorem 3.12 which is the main result of Section 3.

**Definition 3.9.** A finite sequence of points $x_0, \ldots, x_{k-1} \in S$ is positively ordered if $x_0 < x_1 < \cdots < x_{k-1} < x_0$. If the inequality is reversed then we say that points $x_0, \ldots, x_{k-1} \in S$ are negatively ordered. A sequence $y_0, y_1, \ldots$ is said to be positively circularly ordered if it is either positively ordered or there exists $k$ such that $y_i = y_i \mod k$ and $y_0 < y_1 < \cdots < y_{k-1} < y_0$. Similarly we define points that are negatively circularly ordered.

A positively (negatively) circularly ordered sequence which is not positively (negatively) ordered is a sequence whose points repeat themselves after the initial collection of points that are positively (negatively) ordered.

**Definition 3.10.** Suppose that the chords $\overline{t}_1, \ldots, \overline{t}_n$ are edges of the closure $Q$ of a single component of $D \setminus \bigcup \overline{t}_i$. For each $i$, let $m_i$ be the midpoint of the hole $H_Q(\overline{t}_i)$. We write $\overline{t}_1 < \overline{t}_2 < \cdots < \overline{t}_n$ if the points $m_i$ form a positively ordered set and call the chords $\overline{t}_1, \ldots, \overline{t}_n$ positively ordered. If the points $m_i$ are positively circularly ordered, then we say that $\overline{t}_1, \ldots, \overline{t}_n$ are positively circularly ordered. Negatively ordered and negatively circularly ordered chords are defined similarly.

Lemma 3.11 is used in the main result of this section.

**Lemma 3.11.** If $\ell_a$ and $\ell_x$ are linked, have mutually order preserving accordions, and $\sigma_d^{-k} (\ell_x) \in A_{\ell_x}(\ell_a)$ for some $k > 0$ then for every $j > 0$ the leaves $\sigma_d^{-ki} (\ell_x), i = 0, \ldots, j$ are circularly ordered, and $\ell_a, \ell_x$ are periodic with endpoints of the same period.
Proof. By Lemma \[3.7\] we may assume that case (4) of Corollary \[3.8\] holds (and so \(\sigma^k_a(\ell_x) \neq \ell_x\)). Set \(B = \{\ell_a, \ell_x\}, \ell_a = ab, \ell_x = \pi y\) and let \(a_i, b_i, x_i, y_i\) denote the \(\sigma^k_i\)-images of \(a, b, x, y\), respectively \((i \geq 0)\). We may assume that the first possibility from case (4) holds and \(x_0 < a_0 < y_0 \leq x_1 < b_0 < y_1 \leq x_0\) (see the left part of Figure 6). By the assumption of mutually order preserving accordions applied to \(B\), we have \(x_i < a_i < y_i \leq x_{i+1} < b_i < y_{i+1} \leq x_i\) \((i \geq 0)\), in particular \(x_1 < a_1 < y_1\). Then there are two cases depending on the location of \(a_1\). We consider one of them as the other one can be considered similarly; namely, we assume that \(b_0 < a_1 < y_1\). In this case we proceed by induction for \(m\) steps and observe that

\[x_0 < a_0 < y_0 \leq x_1 < b_0 \leq a_1 < \ldots \leq x_m < b_{m-1} < a_m < y_m \leq x_0.\]

Thus, the first \(m\) iterated \(\sigma^k_d\)-images of \(\ell_x\) are circularly ordered and alternately linked with the first \(m - 1\) \(\sigma^k_d\)-images of \(\ell_a\). In the rest of the proof we exploit the following fact.

**Claim A.** Further images of \(\ell_a\) or \(\ell_x\) distinct from the already existing ones cannot cross leaves \(\ell_a, \sigma^k_a(\ell_x), \ldots, \sigma^{k(m-1)}_a(\ell_a), \sigma^{km}_d(\ell_x)\) because either it would mean that leaves from the same geolamination are linked, or it would contradict Corollary \[3.8\].

By Claim A, we have \(b_m \in (y_m, a_0]\). Consider possible locations of \(b_m\).

(1) If \(x_0 < b_m \leq a_0\) then \(a_m b_m\) is linked with \(x_m y_m, x_{m+1} y_{m+1}\) and \(x_0 y_0\) which by Corollary \[3.8\] implies that \(x_{m+1} y_{m+1} = x_0 y_0\) and we are done (observe that in this case by Lemma \[3.7\] points \(a_0, b_0\) are periodic of the same period as \(x_0, y_0\)).

---

**Figure 7.** This figure illustrates Lemma \[3.11\]. Images of \(\ell_a\) cannot cross other images of \(\ell_a\), neither can they cross images of \(\ell_x\) that are already linked with two images of \(\ell_a\) (by Corollary \[3.8\]). Similar claims hold for \(\ell_x\).
(2) \( x_0 = b_m \) is impossible because if \( x_0 = b_m \) then by order preservation and by Claim A the leaf \( x_{m+1}y_{m+1} = \sigma_d^{k(m+1)}(\ell_x) \) is forced to be linked with \( \ell_x \), a contradiction.

(3) Otherwise we have \( y_m < b_m < x_0 \) and hence by the order preservation \( y_m \leq x_{m+1} < b_m \). Then, by Claim A and because images of \( \ell_x \) do not cross, \( b_m < y_{m+1} \leq x_0 \). If \( y_{m+1} = x_0 \), then by the order preservation and Claim A, \( \sigma_d(x_{m+1}y_{m+1}) = x_0y_0 \) and we are done (as before, we need to rely on Lemma 3.7 here). Otherwise \( b_m < y_{m+1} < x_0 \) and the arguments can be repeated because leaves \( \sigma_d^{k}(\ell_x) \), \( i = 0, \ldots, m + 1 \) are circularly ordered. Therefore, either \( \ell_x \) is periodic, \( x_ny_m = x_0y_0 \) for some \( n \), and all leaves in \( \sigma_d^{k} \)-orbit of \( \ell_x \) are circularly ordered, or the leaves \( x_ny_m \) converge monotonically to a point of \( S \). The latter is impossible since \( \sigma_d^{k} \) is expanding. By Lemma 3.7 \( \ell_y \) is periodic and its endpoints have the same period as the endpoints of \( \ell_x \).

Theorem 3.12 is the main result of this section.

**Theorem 3.12.** If \( \ell_a, \ell_x \) are linked chords with mutually order preserving accordions and \( B = CH(\ell_a, \ell_x) \) then (1) or (2) holds.

1. All forward images of \( B \) are pairwise disjoint.
2. There exists \( r \) such that (a) interiors of polygons \( \sigma_d^i(B), 0 \leq i < r \), are disjoint from all other images of \( B \), (b) if \( X \) is the union of polygons in the forward orbit of \( \sigma_d^r(B) \) and \( Q \) is a component of \( X \), then \( Q' = Q \cap S \) is a finite set, \( \sigma_d|Q' \) is order preserving, the vertices of \( \sigma_d^r(B) \) are periodic of the same period and belong to either two, or three, or four distinct periodic orbits, and the first return map on \( Q' \) can be identity only if \( Q = \sigma_d^r(B) \) is a quadrilateral, (c) \( \ell_a, \ell_x \) are (pre)periodic of the same eventual period of endpoints.

**Proof.** By Lemma 2.16 we may assume that there are two forward images of \( B \) with non-disjoint interiors. We can always choose the least \( r \) such that the interior of \( \sigma_d^r(B) \) intersects some forward images of \( B \); then (a) holds. We may assume that \( r = 0 \) and for some (minimal) \( k > 0 \) the interior of the set \( \sigma_d^k(B) \) intersects the interior of \( B \) so that \( \sigma_d^k(\ell_x) \in A_{\ell_x}(\ell_a) \).

Let \( \ell_x = x_0y_0, \ell_a = a_0b_0, \sigma_d^{ik}(a_0) = a_i, \sigma_d^{ik}(b_0) = b_i, \sigma_d^{ik}(x_0) = x_i, \sigma_d^{ik}(y_0) = y_i \). By Lemma 3.11 applied to both leaves, by the assumption of mutually order preserving accordions, and because leaves in the forward orbit of \( \ell_a, \ell_x \) are pairwise unlinked, we may assume, without loss of generality, that, for some \( m \geq 1 \),

\[
x_0 < a_0 < y_0 \leq x_1 < b_0 \leq a_1 < \ldots \leq x_m < b_{m-1} \leq a_m < y_m < b_m
\]

and \( x_m = x_{m+1}y_m = y_0, a_m = a_0, b_m = b_0 \). Thus, for every \( i = 0, \ldots, k - 1 \) there is a loop \( L_i \) of alternately linked \( \sigma_d^i \)-images of \( \sigma_d^j(\ell_a) \) and \( \sigma_d^j(\ell_x) \). If
the $\sigma_d^k$-images of $\sigma_d^k(\ell_a)$ are concatenated to each other, then their endpoints belong to the same periodic orbit, otherwise they belong to two distinct periodic orbits. A similar claim holds for $\sigma_d^k$-images of $\sigma_d^k(\ell_x)$. Thus, the endpoints of $\sigma_d^k(B)$ belong to two, three or four distinct periodic orbits. Set $\text{CH}(L_i) = T_i$ and consider some cases.

(1) Let $m > 1$ (this includes the “flipping” case from Corollary 3.8(2)). Let us show that the convex hulls of the $L_i$’s either coincide or are disjoint. Every image $\hat{\ell}$ of $\ell_a$ in $L_i$ crosses two images of $\ell_x$ in $L_i$ (if $m = 2$ and $\ell_x$ is “flipped” by $\sigma_d^k$ we still consider $\ell_x$ and $\sigma_d^k(\ell_x)$ as distinct leaves). By Corollary 3.8 no other image of $\ell_x$ crosses $\hat{\ell}$.

Suppose that interiors of $T_i$ and $T_j$ intersect. Let $\mathcal{T}$ be an edge of $T_i$ and $I = H_{T_i}(\mathcal{T})$ be the corresponding hole of $T_i$. Then the union of two or three images of $\ell_a$ or $\ell_x$ from $L_i$ separates $I$ from $\mathbb{S} \setminus I$ in $\mathbb{D}$. Therefore if there are vertices of $T_j$ in $I$ and in $\mathbb{S} \setminus I$ then there is a leaf of $L_j$ crossing leaves of $L_i$, a contradiction with the above and Corollary 3.8. Thus, the only way $T_i \neq T_j$ can intersect is if they share a vertex or an edge. Let us show that this is impossible. Indeed, $T_i \neq T_j$ cannot share a vertex as otherwise it must be $\sigma_d^k$-invariant while all vertices of any $T_r$ map to other vertices (sets $T_r$ “rotate” under $\sigma_d^k$). Finally, suppose that $T_i$ and $T_j$ share an edge $\ell$. The same argument shows that $\sigma_d^k$ cannot fix the endpoints of $\ell$, hence it “flips” under $\sigma_d^k$. However this is impossible as each set $T_r$ has at least four vertices and its edges “rotate” under $\sigma_d^k$.

Hence, the component $Q_i$ of $X = \bigcup_{i=0}^{k-1} T_i$ containing $\sigma_d^k(\ell_a)$ is $T_i$. By Lemma 3.11 $\sigma_d|_{T_i \cap \mathbb{S}}$ is order preserving/reversing. Since $\sigma_d$ preserves order on any single accordion, $\sigma_d|_{T_i \cap \mathbb{S}}$ is order preserving. The result now follows; note that the first return map on $Q$ is not the identity map.

(2) Let $m = 1$. This corresponds to Corollary 3.8(3): both $\ell_a$ and $\ell_x$ have endpoints of minimal period $k$ and the orbit of $\ell_a$ ($\ell_x$) consists of $k$ pairwise disjoint leaves. If all sets $T_i$ are pairwise disjoint then $T_0$ is a quadrilateral
and the first return map on $T_0$ is the identity map. Consider now the case when not all sets $T_i$ are pairwise disjoint.

By the above, the only way a $T_i$ can meet $T_0$ is when $\sigma^i_d(\ell_a)$ shares an endpoint with $\ell_x$ or vice versa. Assume that for some $i$

$$x_0 < a_0 < y_0 < b_0 = \sigma^i_d(x_0) < \sigma^i_d(a_0) < \sigma^i_d(y_0) < \sigma^i_d(b_0) \leq x_0.$$ 

Note that the chord $x_0b_0$ satisfies the assumptions of Proposition 2.19. Then $R_i = \bigcup_{j=0}^{\infty} \sigma^j_d L_0$ is connected (as $\ell_a, \ell_x$ are periodic of period $k$, there are at most $k$ distinct images of $L_0$ in $R_i$). Since $\sigma^i_d$ preserves order on each $A_{\ell_x}(\sigma^j_d(\ell_a)) \cap \mathbb{S}$, the whole $R_i \cap \mathbb{S}$ maps forward order preserving (as in (1) above). Finally, convex hulls of the sets $R_i$ are pairwise disjoint by construction as all possible ways for the orbits of $\ell_a, \ell_x$ to intersect have been accounted for. Clearly, in this case the endpoints of $\sigma^i_d(B)$ belong to three distinct periodic orbits. This completes the proof. \hfill \Box

For a leaf $\ell_1 \in \mathcal{L}_1$ let $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ be the collection of all leaves $\ell_2 \in \mathcal{L}_2$ which are linked with $\ell_1$ and have mutually order preserving accordions with $\ell_1$. Observe that if $\ell_1$ is (pre)critical then $\mathcal{B}_{\mathcal{L}_2}(\ell_1) = \emptyset$ by Definition 3.4. Similarly, no leaf from $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is (pre)critical.

**Corollary 3.13.** If $\ell_1$ is not (pre)periodic then the collection $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is finite; if $\ell_1$ is (pre)periodic then $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is at most countable.

**Proof.** Clearly, $\ell_1$ not being (pre)periodic corresponds to the situation of Theorem 3.12(1) while $\ell_1$ being (pre)periodic corresponds to the situation of Theorem 3.12(2).

Suppose first that $\ell_1$ is not (pre)periodic. Let us show that the convex hull $B$ of $\ell_1$ and leaves $\overline{\ell_1}, \ldots, \overline{\ell_s}$ from $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is wandering. By Theorem 3.12 for each $i$ the set $B_i = \text{CH}(\ell_1, \overline{\ell_i})$ is wandering (because $\ell_1$ is not (pre)periodic). This implies that if $i \neq j$ then $\sigma^i_d(B_i)$ and $\sigma^j_d(B_i)$ are disjoint (otherwise $\sigma^i_d(B_i)$ and $\sigma^j_d(B_i)$ are non-disjoint). Moreover, $\sigma^i_d(\ell_1)$ and
\[ \sigma_d^j(\ell_1) \text{ are disjoint as otherwise by Lemma } 2.14 \text{ } \ell_1 \text{ is (pre)periodic. Therefore } \sigma_d^j(\ell_1) \text{ is disjoint from } \sigma_d^j(B). \]

Now, suppose that \( \sigma_d^j(B) \) and \( \sigma_d^j(B) \) are non-disjoint. By the just proven this means that, say, \( \sigma_d^j(\overline{\tau_1}) \) is non-disjoint from \( \sigma_d^j(B) \). Again by the just proven \( \sigma_d^j(\overline{\tau_1}) \) is disjoint from \( \sigma_d^j(\ell_1) \). Hence the only intersection is possible between \( \sigma_d^j(\overline{\tau_1}) \) and, say, \( \sigma_d^j(\overline{\tau_2}) \). Moreover, since \( \sigma_d^j(\ell_1) \) is disjoint from \( \sigma_d^j(\tau_1) \), then \( \sigma_d^j(\overline{\tau_1}) \neq \sigma_d^j(\overline{\tau_2}) \) and, moreover, as distinct leaves of the same lamination, the leaves \( \sigma_d^j(\overline{\tau_1}), \sigma_d^j(\overline{\tau_2}) \) cannot cross. Hence the only way \( \sigma_d^j(\overline{\tau_1}) \) and \( \sigma_d^j(\overline{\tau_2}) \) are non-disjoint is when \( \sigma_d^j(\overline{\tau_1}) \) and \( \sigma_d^j(\overline{\tau_2}) \) are concatenated.

For simplicity assume that \( \sigma_d^j(\overline{\tau_2}) \) is concatenated with \( \overline{\tau_1} \) at an endpoint \( x \) of \( \overline{\tau_1} \). Clearly, \( x \) is a common vertex of \( B \) and of \( \sigma_d^j(B) \). Hence \( \sigma_d^j(x) \) is a common vertex of \( \sigma_d(B) \) and \( \sigma_2^j(B) \), etc. Connect points \( x, \sigma^j(x), \sigma^{2j}(x), \ldots \) with consecutive chords \( \overline{m_0, \overline{m_1, \ldots}} \). These chords are pairwise unlinked because, as it follows from above, sets \( \sigma_d^j(B), r = 0, 1, \ldots \) have pairwise disjoint interiors. Hence by Lemma \( 2.13 \) \( x \) is (pre)periodic, a contradiction with the fact that all sets \( B_t = CH(\ell_1, \overline{\tau_1}) \) are wandering. Thus, \( B \) is wandering. Hence by \([Kiw02]\) \( B_{C_2}(\ell_1) \) is a finite collection.

If \( \ell \) is (pre)periodic, Theorem \( 3.12(2) \) implies the lemma. \( \square \)

4. LINKED QUADRATICALLY CRITICAL GEOLAMINATIONS

In this section we first define quadratically critical (qc-)geolaminations. Then we discuss the notion of linked or essentially coinciding qc-geolaminations and study their properties. The main result is that two linked or essentially coinciding qc-geolaminations have the same perfect sublamination.

4.1. Basic definitions. In the quadratic case minors are uniquely determined by the minor. Even though in the cubic case one could define minors similarly to the quadratic case, unlike in the quadratic case, these “minors” do not uniquely determine corresponding majors. The simplest way to see that is to consider distinct pairs of critical leaves with the same images. One can choose two all-critical triangles with so-called aperiodic kneadings as defined by J. Kiwi in \([Kiw04]\) (it is easy to see that this can be done). By \([Kiw04]\), this would imply that any choice of two disjoint critical leaves, one from either triangle, will give rise to the corresponding geolamination; clearly, these two geolaminations are very different even though they have the same images of their critical leaves, i.e. the same minors. For this reason we will mostly be concerned with critical leaves and collapsing quadrilaterals. Definition \( 4.1 \) is inspired by \([Thu85]\) and \([Kiw04]\).

**Definition 4.1** (Quadratic criticality). If \( \mathcal{L} \) is a \( \sigma_d \)-invariant geolamination for which \( d - 1 \) collapsing quadrilaterals or critical leaves of \( \mathcal{L} \) are chosen
so that (1) there are no loops of chosen critical leaves, and (2) for any component of the union of all critical leaves of \( L \) their convex hull has edges that are leaves of \( L \), then \( L \) is said to have \textit{quadratic criticality} or to be a \textit{qc-geolamination}. The family of all \( \sigma_d \)-invariant geolaminations of quadratic criticality is denoted by \( QC \). The chosen quadrilaterals and leaves form a \textit{quadratically critical portrait} (or \textit{qc-portrait}) \( QCP(L) \) of \( L \).

Working with \( \sigma_d \)-invariant geolaminations of quadratic criticality is not restrictive even though critical sets of \( L \) may be gaps which are not collapsing quadrilaterals; indeed, such critical sets may be “tuned” by inserting into them critical leaves and collapsing quadrilaterals. For a geolamination \( L \in QC \) coming with a fixed qc-portrait \( QCP(L) \), we call critical leaves from \( QCP(L) \) \textit{selected}. A \textit{full} collection of critical chords is a collection of \( d - 1 \) critical chords with no loops.

\begin{lemma}
If \( L \in QC \) then, if we choose one diagonal in each collapsing quadrilateral from \( QCP(L) \) and combine them with all selected critical leaves of \( L \), we obtain a full collection of critical chords of \( L \).
\end{lemma}

I.e., the choice made in the lemma does not yield loops of critical chords.

\textit{Proof.} Let \( Q = CH(x, y, x', y') \) be a collapsing quadrilateral of \( L \) and \( \overline{xx'} \) be its diagonal. We claim that no collection of critical chords of \( L \) completes \( \overline{xx'} \) to a Jordan curve \( T \). Indeed, as \( T \) continues from \( x' \), it gets trapped “under” edges \( \overline{yy'} \) or \( \overline{xx'} \); from this location it would not be able to escape back to \( x \) as \( \sigma_d(y) = \sigma_d(y') \neq \sigma(x) = \sigma_d(x') \). Thus, a loop of critical chords of \( L \) does not include diagonals of collapsing quadrilaterals of \( L \). By Definition 4.1 critical leaves from \( QCP(L) \) do not form a loop either. Since by definition \( QCP(L) \) consists of \( d - 1 \) sets, the lemma follows. \( \square \)

Even if the set \( QCP(L) \) is fixed, the choice of a full collection of critical chords from Lemma 4.2 is ambiguous because the choice of diagonals inside quadrilaterals is ambiguous. This is important for the next subsection where we introduce and study the notion of Smart Criticality. More ambiguity there is related to the choice of selected critical leaves of \( L \) that can be changed while still resulting in another \textit{qc-portrait} which can be associated with \( L \).

The part of Definition 4.1 concerning critical leaves narrows the choice of a qc-portrait for a given geolamination. E.g., consider the cubic geolamination \( L \) with a leaf \( 0 \) and Fatou gaps of degree 2 above \( 0 \) and below \( 0 \) (the rest of \( L \) can be recovered by means of pullbacks [Thu85]). Since \( L \) does not have critical leaves or collapsing quadrilaterals, we have to insert them into the Fatou gaps of \( L \) to create a new quadratically critical geolamination. Our definition then restricts our choices here; e.g., \( \{0, 0, 0, 0\} \) is
not a qc-portrait of \(L\) because \(\frac{1}{3}\) cannot be added to \(L\) (it will cross \(\frac{1}{2}\)).

considering quadratically critical geolaminations is not restrictive as one can always insert collapsing quadrilaterals and critical leaves into critical sets of a given geolamination \(L\) to obtain an admissible qc-modification. In what follows we discuss such modifications in more detail.

Recall that two quadrilaterals are strongly linked if their vertices alternate. Our main results on qc-geolaminations show that certain perturbations (e.g., those which keep the collapsing quadrilaterals strongly linked) do not change the perfect sublamination. Thus, these results could be viewed as rigidity results. Also, call two collapsing quadrilaterals or critical leaves \(Q\) and \(T\) compatible if they share a common critical diagonal (leaf). I.e., \(Q\) and \(T\) either coincide, or share a common critical diagonal, or one is a diagonal of the the other one.

**Definition 4.3 (Linked geolaminations).** Let \(L_1\) and \(L_2\) be geolaminations of quadratic criticality with qc-portraits \(\{C_1^i\}_{i=1}^{d-1}\) and \(\{C_2^i\}_{i=1}^{d-1}\). The geolaminations \(L_1\) and \(L_2\) are said to be linked if for \(1 \leq i \leq d - 1\), the sets \(C_1^i\) and \(C_2^i\) are strongly linked quadrilaterals or are compatible, and for at least one \(i\) they are strongly linked. If it is known that for every \(i\), the sets \(C_1^i\) and \(C_2^i\) are compatible, then we say that \(L_1\) and \(L_2\) coincide in essence (or essentially coincide, or are essentially equal). The critical sets \(C_1^i\) and \(C_2^i\), \(1 \leq i \leq d - 1\), are said to be associated (critical sets of the linked \(\sigma_d\)-invariant geolaminations \(L_1\) and \(L_2\)).

In the rest of this section we work with geolaminations \(L_1, L_2 \in QC\) and assume that they either (1) coincide in essence or (2) are linked in which case their critical sets \(\{C_1^i\}_{i=1}^{d-1}, \{C_2^i\}_{i=1}^{d-1}\) are associated for each \(1 \leq i \leq d - 1\). Without loss of generality we may also assume that if \(L_1, L_2\) are linked then \(C_1^i\) and \(C_2^i\) are strongly linked quadrilaterals. Since by definition a selected critical leaf of one geolamination is shared by the other one, we talk of selected critical leaves without mentioning the geolamination. Speaking of shared leaves (chords, polygons etc) we always mean “shared by the two geolaminations which are considered”.

If \(L_1\) and \(L_2\) share a critical chord \(\ell\), then for either geolamination \(\ell\) is either a critical leaf or a diagonal of a collapsing quadrilateral. If, say, \(\ell\) is a diagonal of a collapsing quadrilateral \(Q_1\) of \(L_1\), we can always extend \(L_1\) by adding \(\ell\) with all its preimages inside pullbacks of \(Q_1\) to it. If we do the same for all pairs of compatible quadrilaterals or critical leaves of \(L_1, L_2\), we will in the end construct extensions \(L_1^e\) of \(L_1\) and \(L_2^e\) of \(L_2\). Clearly, these extensions are very closely related to \(L_1\) and \(L_2\).

**4.2. Smart Criticality.** Our aim in this subsection is to introduce Smart Criticality, a principle which allows one to use a flexible choice of critical
Lemma 4.4. If $\ell_1 \in \mathcal{L}_1$ is not a selected critical leaf then each critical set $C^2$ of $\mathcal{L}_2$ contains a critical chord $\ell$ unlinked with $\ell_1$. This gives rise to a full collection of critical chords $\mathcal{A}$ of $\mathcal{L}_2$ consisting of chords unlinked with $\ell_1$. Moreover, if $\ell_1 = \overline{xy}$ is not a side of a shared collapsing quadrilateral and $x$ is not an endpoint of a selected critical leaf then $\mathcal{A}$ can (and always will) be chosen so that $x$ is not an endpoint of a chord from $\mathcal{A}$.

Proof. It suffices to assume that $C^2$ is a collapsing quadrilateral. If $C^2$ is a shared quadrilateral, choose $\ell$ as a diagonal of $C^1$ disjoint from $\ell_1$ (if $\ell_1$ is not a side of $C^2$) or at least unlinked with $\ell_1$ (if $\ell_1$ is a side of $C^2$). Otherwise assume that $C^1$ and $C^2$ are strongly linked associated quadrilaterals. In this case it is easy to see that we can choose a critical diagonal $\ell$ of $C^2$ disjoint from $\ell_1$ (see Figure 10). This proves the first claim of the lemma. To create $\mathcal{A}$, include in it critical chords $\ell$ chosen in each critical set of $\mathcal{L}_2$ (in particular, we include in $\mathcal{A}$ all selected critical leaves). This yields the desired full collection of critical chords $\mathcal{A}$. Now, if $\ell_1 = \overline{xy}$ is not a side of a shared collapsing quadrilateral and $x$ is not an endpoint of a selected critical leaf then by the above construction $\mathcal{A}$ can be chosen so that $x$ is not an endpoint of a chord from $\mathcal{A}$. \hfill \Box

Lemma 4.5 describes critical leaves of our geolaminations.

Lemma 4.5. The set of all critical leaves of $\mathcal{L}_1$ and $\mathcal{L}_2$ consists of finitely many pairwise disjoint polygons $P_1, \ldots, P_k$ whose edges are shared critical leaves and, perhaps, some diagonals of these polygons.

Proof. Let $C$ be a connected component of the union $B$ of all selected critical leaves. We show that $\text{CH}(C)$ is a polygon whose edges are shared critical leaves. Assume that $a$ and $b$ are adjacent (on $\mathbb{S}$) points of $C$ such that...
Corollary 4.6. If $\ell_1 = \overline{ab} \in L_1$ is non-critical and $A = A_{C_1}(\ell_1)$ then $A \cap S$ is contained in the complement of $d - 1$ disjoint open circular arcs of length $\sigma_D$.

\textbf{Corollary 4.6.} If \( \ell_1 = \overline{ab} \in L_1 \) is non-critical and \( A = A_{C_1}(\ell_1) \) then \( A \cap S \) is contained in the complement of \( d - 1 \) disjoint open circular arcs of length \( \sigma_D \).
\(1/2\) and \(\sigma_d|_{A^\text{vs}}\) is (non-strictly) monotone. Let \(\ell_2 = \overline{xy} \in L_2, \ell_1 \cap \ell_2 \neq \emptyset\).

Then:

1. if \(\ell_1\) and \(\ell_2\) are concatenated at a point \(x\) which is not an endpoint of a shared critical chord then \(\sigma_d\) is monotone on \(\ell_1 \cup \ell_2\);
2. if \(\ell_2\) crosses \(\ell_1\) then for each \(i\), we have \(\sigma_d^i(\ell_1) \cap \sigma_d^i(\ell_2) \neq \emptyset\), and one the following holds:
   a. \(\sigma_d^i(\ell_1) = \sigma_d^i(\ell_2)\) is a point or a shared leaf;
   b. \(\sigma_d^i(\ell_1), \sigma_d^i(\ell_2)\) share an endpoint;
   c. \(\sigma_d^i(\ell_1), \sigma_d^i(\ell_2)\) are linked and have the same order of endpoints as \(\ell_1, \ell_2\);
3. points \(a, b, x, y\) are either all (pre)periodic of the same eventual period, or all not (pre)periodic.

Proof. By Lemma 4.4 choose a full collection of critical chords \(A \subset L_2\) for \(\ell_1\). As \(\ell_1 \notin A\), there exists a unique complementary component \(Y\) of the union of all chords in \(A\) that contains \(\ell_1\) (except perhaps for the endpoints). The fact that each leaf of \(L_2\) is unlinked with \(A\) implies that \(A_{L_2}(\ell_1) \subset Y\). This proves the main claim and implies that \(\sigma_d|_{A^\text{vs}}\) is monotone (recall that by monotone we mean non-strictly monotone).

1. By Lemma 4.4, the collection \(A\) can be chosen so that \(x\) is not an endpoint of a chord from \(A\). Let \(Y\) be the complementary component of the union of all leaves of \(A\) containing \(x\); assume that \(x = b\). Then \(\ell_1 \setminus \{a\} \subset Y\), \(\ell_2 \setminus \{y\} \subset Y\), and so \(\ell_1 \cup \ell_2 \subset Y\) which implies the desired.

2. We use induction to prove the claim. If \(\sigma_d^i(\ell_1)\) is critical and linked with \(\sigma_d^i(\ell_2)\), then by Lemma 4.5 both leaves are critical with the same image and we are done. Assume that neither leaf is critical. If they coincide then their images coincide. By induction it remains to consider the case when they share an endpoint and the case when they are linked. If \(\sigma_d^i(\ell_1)\) and \(\sigma_d^i(\ell_2)\) share an endpoint then on the next step they will either coincide, or will share only an endpoint.

Hence the only remaining case is when \(\sigma_d^i(\ell_1)\) and \(\sigma_d^i(\ell_2)\) are linked and the order of their endpoints and the order of endpoints of leaves \(\ell_1\) and \(\ell_2\) is the same. Then by the main claim of the lemma \(\sigma_d\) is monotone on the set of the four endpoints of \(\sigma_d^i(\ell_1)\) and \(\sigma_d^i(\ell_2)\). Hence either the same order is kept on the images of these endpoints, or the leaves \(\sigma_d^{i+1}(\ell_1)\) and \(\sigma_d^{i+1}(\ell_2)\) share only an endpoint, or the leaves \(\sigma_d^{i+1}(\ell_1)\) and \(\sigma_d^{i+1}(\ell_2)\) coincide.

3. By Lemma 2.14 if an endpoint of a leaf of a geolamination is (pre)periodic, then so is the other endpoint of the leaf. Consider two cases. Suppose first that an image of \(\ell_1\) and an image of \(\ell_2\) “collide” (i.e., have a common endpoint \(z\)). By the above if \(z\) is (pre)periodic then all endpoints of our leaves are, and if \(z\) is not (pre)periodic then all endpoints of our leaves are not (pre)periodic. Suppose now that no two images of \(\ell_1, \ell_2\) collide. Then
it follows that \( \ell_1 \) and \( \ell_2 \) have mutually order preserving accordions, and the claim follows from Theorem 3.12.

Lemma 4.4 and Corollary 4.6 implement Smart Criticality. Indeed, given a geolamination \( \mathcal{L} \), bases of its gaps (loosely) consist of points whose orbits avoid critical sets of \( \mathcal{L} \). It follows that any power of the map is order preserving on its basis. It turns out that we can treat sets \( X \) formed by linked leaves of two linked geolaminations similarly by varying our choice of the full collection of critical chords on each step so that the orbit of \( X \) avoids that particular full collection of critical chords on that particular step (thus smart criticality). Therefore, similarly to the case of one geolamination, any power of the map is order preserving on \( X \). This allows one to treat such sets \( X \) almost as sets of one lamination.

Lemma 4.7 describes the only way the map is non-strictly monotone on \( A \cap S \). A chain \( B \) of critical chords of a geolamination \( \hat{\mathcal{L}} \) such that the endpoints of its chords are monotonically ordered on the circle will be called simply a critical chain (of \( \mathcal{L} \)). We will write \( B^+ \) for the union of all chords in \( B \).

**Lemma 4.7.** Suppose that \( \ell_a = \overline{ab} \in \mathcal{L}_1 \), \( \ell_x = \overline{xy} \in \mathcal{L}_2 \), we have \( a < x < b \leq y < a \) (see Figure 11) and if \( b = y \) then it is not an endpoint of a shared critical chord. Let \( \sigma_d(a) = \sigma_d(x) \). Then there is a critical chain \( B_1 \) of \( \mathcal{L}_1 \) (\( B_2 \) of \( \mathcal{L}_2 \)) connecting \( a \) and \( x \). Also, \( B_1^+ \cap S \subset [a, x], B_2^+ \cap S \subset [a, x] \) and \( B_2 \) has a shared critical leaf \( \overline{ab} \) and \( B_1 \) has a shared critical leaf \( \overline{xy} \).

If none of the arcs \([a, x], [x, b], [b, y] \) and \([y, a] \) contains critical chains of both \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) in its convex hull, then the order of the endpoints of \( \ell_a, \ell_x \) is preserved.

**Proof.** First assume that one of the leaves \( \ell_a, \ell_x \) is critical. If \( \ell_a, \ell_x \) cross then both are critical and Lemma 4.5 implies the claim. If \( \ell_a, \ell_x \) share an endpoint then by Lemma 4.5 this common endpoint is an endpoint of a shared critical chord, a contradiction with the assumptions of the lemma.

Thus, we may assume that neither \( \ell_a \) nor \( \ell_x \) is critical. Hence \( \sigma_d(b) \neq \sigma_d(a), \sigma_d(y) \neq \sigma_d(x) \). By Lemma 4.4 choose a full collection of critical chords \( A_2 \) of \( \mathcal{L}_2 \) unlinked with \( \ell_a \) and a full collection of critical chords \( A_1 \) of \( \mathcal{L}_1 \) unlinked with \( \ell_x \). By the assumptions and Lemma 4.4 if \( b = y \) then \( b = y \notin A_1^+ \cup A_2^+ \). Since \( \sigma_d(a) = \sigma_d(x) \), there is a critical chain \( B_2 \subset A_2 \) of \( \mathcal{L}_2 \) (\( B_1 \subset A_1 \) of \( \mathcal{L}_1 \)) connecting \( a \) and \( x \). Moreover, \( B_2 \) can be chosen so that \( B_2 = B_2^+ \cap S \subset [a, x] \) as \( A_2 \) is unlinked with \( \ell_a \) and \( \sigma_d(y) \neq \sigma_d(x) \) (because \( \ell_x \) is not critical). A similar statement holds for \( B_1 \).

Let us show that \( B_2 \) includes a critical leaf of \( \mathcal{L}_2 \) with an endpoint \( a \); assume otherwise and denote by \( \overline{ab} \) the critical chord of \( B_2 \) which is a critical diagonal of a collapsing quadrilateral \( Q_2 \) of \( \mathcal{L}_2 \). If \( t = x \) then the associated
to $Q_2$ collapsing quadrilateral $Q_1$ of $\mathcal{L}_1$ intersects $\ell_a$, a contradiction; hence $a < t < x$. Clearly, there is a vertex $v$ of $Q_2$ in $(t, a)$, and since edges of $Q_2$ do not cross $\ell_x$, $v \in [y, x)$ ($v \neq x$ as $\sigma_d(v) \neq \sigma_d(a) = \sigma_d(x)$). If $v \in (t, x]$ then $B_2$ is “trapped” in $(t, v)$, a contradiction. Hence $v \in [y, a)$; in particular, $\overline{ax}$ is not a leaf of $\mathcal{L}_2$. It follows that $Q_2 = \text{CH}(a, z, t, v)$ for some point $z \in (a, t)$ and that the leaf $\overline{zx}$ of $\mathcal{L}_2$ is the closest to $x$ from within $(a, x)$. Let $Q_1$ be a collapsing quadrilateral of $\mathcal{L}_1$ associated with $Q_2$. Consider now two cases, $v = b$ and $y < v < a$.

Clearly, the only way $v$ can equal $b$ is when $v = y = b$. In this case $\overline{ab}$ is a shared leaf, hence $Q_1$ must coincide with $Q_2$ which contradicts the assumption of the lemma according to which $y = b$ is not an endpoint of a shared critical leaf. Hence $y < v < a$; in that case $Q_1$ can neither coincide with $Q_2$ nor be linked with $Q_2$ because in either case we get a contradiction with the existence of $\overline{ab}$. This completes the proof.

Definition 4.8 reflects the phenomenon from Lemma 4.7.
Definition 4.8. Non-disjoint leaves $\ell_1 \neq \ell_2$ are said to **collapse around chains of critical chords** if there are two chains of critical chords of our geolaminations connecting two adjacent endpoints of $\ell_1, \ell_2$ as in Lemma 4.7.

Smart Criticality allows one to treat accordions as gaps of one geolamination. The main difficulty here is that linked leaves of linked laminations may have unlinked (but sharing an endpoint) images (see Lemma 4.7).

Lemma 4.9. Let $\overline{m^1}, \overline{m^2}$ be linked leaves from $L_1, L_2$ such that there is no $t$ with $\sigma_d^i(\overline{m^1}), \sigma_d^j(\overline{m^2})$ collapsing around chains of critical chords. Then there exists an $N$ such that the $\sigma_d^N$-images of $\overline{m^1}, \overline{m^2}$ are linked and have mutually order preserving accordions. Conclusions of Theorem 4.12 hold for $\overline{m^1}, \overline{m^2}$ and $B = \text{CH}(\overline{m^1}, \overline{m^2})$ is either wandering or (pre)periodic so that $\overline{m^1}, \overline{m^2}$ are (pre)periodic of the same eventual period of endpoints.

Proof. By the assumptions, by Lemma 4.7 and by Lemma 4.5, $\sigma_d^i(\overline{m^1})$ and $\sigma_d^j(\overline{m^2})$ are linked and not critical for any $i$. Thus, if $u, v$ are endpoints of $\overline{m^1}$ or $\overline{m^2}$ then $\sigma_d^i(u) \neq \sigma_d^j(v)$ for any $i$. By Lemma 2.14 choose $N$ so that each of the leaves $\sigma_d^N(\overline{m^1})$ and $\sigma_d^N(\overline{m^2})$ is periodic or has no (pre)periodic endpoints. Let $\sigma_d^N(\overline{m^1}) = \overline{ab}, \sigma_d^N(\overline{m^2}) = \overline{xy}$. If $\overline{ab}$ and $\overline{xy}$ are periodic, then no collapse around chains of critical leaves on any images of $\overline{ab}, \overline{xy}$ is possible (for set-theoretic reasons). Hence $\sigma_d^N(\overline{m^1}), \sigma_d^N(\overline{m^2})$ are linked and have mutually order preserving accordions as desired.

Let now $\overline{ab}$ have non-(pre)periodic endpoints $a, b$. By increasing $N$ we may assume that for any two endpoints of $\overline{m^1}, \overline{m^2}$ and for any two powers of $\sigma_d$, the images of these endpoints under these powers of $\sigma_d$ are distinct. Indeed, otherwise there exist, say, $i < j$ such that, say, $\sigma_d^i(x) = \sigma_d^j(x)$ (we call this a collision between (the orbits of) $a$ and $x$). If a collision between $a$ and $x$ happens for some other pair of numbers $i, j$, this would imply that both $a$ and $x$ are (pre)periodic, a contradiction with the assumption. Hence a collision between $a$ and $x$ is either non-existing or unique. Increasing $N$ we can make sure that there are no collisions between the orbits of endpoints of $\sigma_d^N(\overline{m^1})$ and $\sigma_d^N(\overline{m^2})$. Then if two leaves from the orbits of $\sigma_d^N(\overline{m^1})$ and $\sigma_d^N(\overline{m^2})$ cross, their images must cross too.

4.3. Linked perfect laminations. Let us study intersections of leaves of $L_1, L_2$.

Lemma 4.10. The set $\mathcal{T}$ of all leaves of $L_2$ non-disjoint from a leaf $\ell_1$ of $L_1$ is at most countable. Thus, if $\ell_1$ is an accumulation point of uncountably many leaves of $L_1$ then $\ell$ is unlinked with any leaf of $L_2$.

Proof. If $\ell_1$ has (pre)periodic endpoints then by Corollary 4.6 any leaf of $L_2$ non-disjoint from $\ell_1$ has (pre)periodic endpoints which implies the first claim of the lemma in this case. Let $\ell_1$ have no (pre)periodic endpoints.
Then, by Corollary 4.6 leaves of $L_2$ non-disjoint from $\ell_1$ have no (pre)periodic endpoints. By Lemma 2.17 there are finitely many leaves with an endpoint which is an eventual image of an endpoint of $\ell_1$. Hence the family of all leaves of $L_2$ which will ever “collide” with a leaf from the orbit of $\ell_1$ is countable. If we remove them from $T$, we will get a new collection $T'$ of leaves. It follows that if $\ell_2' \in T'$ then $\ell_2'$ and $\ell$ have mutually order preserving accordions. By Corollary 3.13 this implies that leaves of $T'$ linked with $\ell_1$ form a finite collection. This completes the proof of the first claim of the lemma. Now, suppose that $\ell_1$ is linked with a leaf $\ell_2 \in L_2$ and is an accumulation point of uncountably many leaves of $L_1$. Then $\ell_2$ is linked with uncountably many leaves of $L_1$, a contradiction. □

Theorem 4.11 is our main result concerning perfect geolaminations.

**Theorem 4.11.** Let $L_1$ and $L_2$ be two linked or essentially equal qc-geolaminations. Then the perfect sublaminations $L_1^c$ and $L_2^c$ coincide.

**Proof.** By way of contradiction we may assume that $L_1^c \not\subset L_2^c$. It follows that then $L_1^c \not\subset L_2$. Hence there exists a leaf $\ell \in L_1^c \setminus L_2$. Then, by Lemma 4.10 the leaf $\ell$ (except for its endpoints) is contained in the interior of a gap $G$ of $L_2$ (otherwise a leaf of $L_2$ linked with $\ell$ would have an uncountable accordion contradicting Lemma 4.10). Since $L_1^c$ is perfect, from at least one side all one-sided neighborhoods of $\ell$ contain uncountably many leaves of $L_1^c$. Hence $G$ is uncountable since if $G$ is finite or countable, then there must exist edges of $G$ which cross leaves of $L_1^c$, a contradiction as above. Thus, there are uncountably many leaves of $L_1^c$ inside $G$; these leaves connect points of $G \cap S$.

Now, interiors of forward images of $G$ are disjoint from the critical sets of $L_2$ since these critical sets are finite. Hence all powers of $\sigma_d$ on $G$ are (non-strictly) monotone. By [BL02, Kiw02] there exists $m$ such that $\sigma_d^m(G)$ is $\sigma_d^N$-periodic. Since $G$ is uncountable, it follows that in fact $\sigma_d^m(G)$ is a Siegel gap and $\sigma_d^N|_{\sigma_d^m(G)}$ is semiconjugate to an irrational circle rotation. However this implies that leaves of $L_1^c$ contained inside $G$ will eventually have linked images, a contradiction. □

**Corollary 4.12.** Let $L_1$ and $L_2$ be two geolaminations. Suppose that $L_1^m \supset \mathcal{L}_1$ and $L_2^m \supset \mathcal{L}_2$ are two qc-geolaminations which are linked or essentially equal. Then the perfect sublamination of $L_1$ is contained in the perfect sublamination of $L_2^m$ and the perfect sublamination of $L_2$ is contained in the perfect sublamination of $L_1^m$.

By Theorem 4.11 the perfect sublamination of a qc-geolamination is determined by the critical data (compare with results in [Kiw04] where...
Jan Kiwi showed that if a geolamination has critical leaves with *aperiodic kneading* then it is completely determined. In fact, Theorem 4.11 takes the issue of how critical data impacts the perfect sublamination of a geolamination further as it considers the dependence of the perfect sublaminations upon critical data while relaxing the conditions on critical sets and studies this dependence under the assumptions of “linked perturbation” of the critical data. From this angle, Theorem 4.11 should be viewed as a rigidity result.

5. Applications

In Section 5 we apply the developed tools. We always assume that our (geo)laminations are $\sigma_d$-invariant. An idea of Thurston’s in [Thu85] was that since geolaminations model polynomials, families of geolaminations may model families of polynomials. We want to parameterize a set of invariant geolaminations and thus model the corresponding set in the parameter space of polynomials. In this section we first introduce the general machinery in Subsection 5.1 and then study the parameter space of cubic bicritical dendritic polynomials in Subsection 5.2.

5.1. Tags of geolaminations. In this Subsection we consider *full* geolaminations $\mathcal{L}$ with *degree two* critical portraits $\text{Cr}^2(\mathcal{L})$, defined in Definition 2.31. Our approach to the parameterization of such geolaminations is based on associating to them special sets called *tags*. Given a positive integer $m$, we use a natural partial order by inclusion among all sets of the form $A_1 \times \cdots \times A_m$: we have $A_1 \times \cdots \times A_m \supset B_1 \times \cdots \times B_m$ if and only if $A_i \supset B_i$ for all $i = 1, \ldots, m$.

**Definition 5.1** (Tags and tagging). A *tagging* is a map $\text{Tag}$ which associates to any degree two critical portrait $\text{Cr}^2(\mathcal{L})$ of a full geolamination $\mathcal{L}$ a product $\text{Tag}(\mathcal{L}) = \text{Tag}(\text{Cr}^2(\mathcal{L})) \subset (\mathbb{D})^k$ of $k$ leaves and gaps of $\mathcal{L}$ such that the following holds:

1. the map $\text{Tag}$ is a homeomorphism between the space $\mathcal{C}_d^2$ of degree two critical portraits with the Hausdorff metric and the space of their images under $\text{Tag}$ with the Hausdorff metric;
2. if $\mathcal{Y} \supset \mathcal{Z}$ are degree two critical portraits then $\text{Tag}(\mathcal{Y}) \supset \text{Tag}(\mathcal{Z})$;
3. if for two geolaminations $\mathcal{L}_1$ and $\mathcal{L}_2$ their tags $\text{Tag}(\mathcal{L}_1)$ and $\text{Tag}(\mathcal{L}_2)$ are non-disjoint then there exist qc-geolaminations $\mathcal{L}_1^q \supset \mathcal{L}_1$ and $\mathcal{L}_2^q \supset \mathcal{L}_2$ which are linked or essentially equal.

In the rest of this subsection we fix a tagging $\text{Tag}$. Tags help one give a geometric interpretation to some families of geolaminations. To explain how they do so we need a few topological results.
Definition 5.2. A collection $D = \{D_\alpha\}$ of compact and disjoint subsets of a metric space $X$ is upper semicontinuous (USC) if for every $D_\alpha$ and every open set $U \supset D_\alpha$ there exists an open set $V$ containing $D_\alpha$ so that for each $D_\beta \in D$, if $D_\beta \cap V \neq \emptyset$, then $D_\beta \subset U$.

We called $\text{Tag}(L)$ the tag of geolamination $L$.

Theorem 5.3 ([Dav86]). If $D$ is an upper semicontinuous decomposition of a separable metric space $X$, then $X/D$ is also a separable metric space.

To apply Theorem 5.3 we need Theorem 5.5. However first we study limits of critical sets of geolaminations from $\mathcal{FD}_d$.

Lemma 5.4. Let $\{C_i\}$ be a sequence of critical sets of geolaminations $L_i \in \mathcal{FD}_d$ converging to a set $C$. Then $C$ is a critical set (in particular, $C$ is not a gap of degree one), and $C$ is not periodic.

Proof. We may assume that all sets $C_i$ have degree $k_i$. Then the degree of $C$ is at least $k$, and hence $C$ is critical. Moreover, $C$ obviously cannot be a periodic leaf. Suppose that $C$ is a periodic gap of period $n$. Then the fact that $\sigma_n^d(C) = C$ implies that any gap $C_i$ sufficiently close to $C$ will have its $\sigma_n^d$-image also close to $C$, and therefore coinciding with $C_i$. Thus, $C_i$ is $\sigma_d$-periodic, which is impossible because $L_i$ is dendritic.

We are now ready to show that Theorem 5.3 applies to taggings.

Theorem 5.5. If $\text{Tag}$ is a tagging then the family $\{\text{Tag}(\mathcal{FD}_d)\}$ of tags of full dendritic geolaminations forms an upper semicontinuous decomposition of the set $\{\text{Tag}(\mathcal{FD}_d)\}^+$.

Proof. We claim that $\text{Tag}(L_1)$ and $\text{Tag}(L_2)$ are disjoint if $L_1 \neq L_2$ are full dendritic geolaminations. Indeed, suppose that $\text{Tag}(L_1) \cap \text{Tag}(L_2) \neq \emptyset$. Then by Definition 5.1 there exist qc-geolaminations $\hat{L}_1 \supset L_1$ and $\hat{L}_2 \supset L_2$ which are linked. By Theorem 4.11 their maximal perfect sublaminations $\hat{L}_1^c$ and $\hat{L}_2^c$ coincide. Since $L_1$ and $L_2$ are themselves perfect and have only finite classes, we conclude that $L_1 = L_2$, a contradiction.

Now, consider a sequence of geolaminations $L_i \in \mathcal{FD}_d$ such that their tags $\text{Tag}(L_i)$ converge (in the Hausdorff metric) to a set $\hat{Z} \subset \mathcal{D}^+$. We claim that if $\hat{Z} \cap \text{Tag}(\hat{L}) \neq \emptyset$ for some $\hat{L} \in \mathcal{FD}_d$ then $\hat{Z} \subset \text{Tag}(\hat{L})$. By compactness we may assume that the sets $L_i^+$ converge to a closed union of chords which, by [BMMOV12], coincides with the set $L^+$ for some geolamination $L$. Then the critical sets of $L_i$ converge to a collection $\mathcal{Y} = \{Y_i\}_{i=1}^{d-1}$ of $d-1$ critical sets which form a degree two critical portrait of $L$. By Definition 5.1, $\hat{Z} = \text{Tag}(\mathcal{Y}) = \text{Tag}(L)$ with $L$ viewed as a full geolamination with selected collection $\mathcal{Y}$ of $d-1$ critical sets.
Thus, again by Definition 5.1, there are $qc$-geolaminations $L^q \supset L$ and $\tilde{L}^q \supset \tilde{L}$ that are linked. By Corollary 4.12, their perfect sublaminations then coincide. Since $\tilde{L} \in F D_d$, the perfect sublamination of $\tilde{L}^q$ equals $\tilde{L}$ itself. Hence $L^q \supset \tilde{L}$, which implies that (1) critical sets of $L^q$ are subsets of critical sets of $\tilde{L}$, and (2) leaves of $L$ do not cross leaves of $\tilde{L}$.

We claim that $L$ does not have infinite gaps. Suppose that $G$ is an infinite gap of $L$. By [Kiw02] we may assume that $G$ is periodic of period $n$. By Corollary 2.21, $\sigma_d \mid \text{Bd}(G)$ is not of degree one (otherwise, since leaves of $L$ do not cross leaves of $\tilde{L}$ and $\tilde{L}$ is perfect, then uncountably many leaves of $\tilde{L}$ intersect $G$, a contradiction with Corollary 2.21). Suppose that $G$ is of degree $k > 1$. Then, by Lemma 5.4, we have $G \neq Y_i$ for any $i = 1, \ldots, d - 1$, which is a contradiction because the degree of $\sigma_d$ is $d$. Thus, $L$ has only finite gaps. Since leaves of $L$ do not cross leaves of $\tilde{L}$, this implies that $L \supset \tilde{L}$. Hence, the critical sets of $L$ are contained in the critical sets of $\tilde{L}$ and, by Definition 5.1, we have $\text{Tag}(L) = \tilde{Z} \subset \text{Tag}(\tilde{L})$ as desired. □

By Theorem 5.3, the space $\{\text{Tag}(F D_d)\}^+ / \{\text{Tag}(F D_d)\}$ is separable and metric. Together with definitions and Corollary 2.35 we obtain Theorem 5.6. Recall that, as was explained in Subsection 2.3.3, to each polynomial $P \in R F_d$ we can associate its geolamination $L_P \in F D_d$, and that polynomials in $R F_d$ come with selected critical portraits.

**Theorem 5.6.** If $\text{Tag}$ is a tagging then the map $\Psi_{\text{Tag}} : P \mapsto \text{Tag}(L_P)$ is a continuous map from $R F_d$ onto $\{\text{Tag}(F D_d)\}^+ / \{\text{Tag}(F D_d)\}$.

**Proof.** Let $P_i \to P$ with $P_i, P \in R F_d$. Then, by Corollary 2.35 we may order critical sets of geolaminations $L_{P_i}, L_P$ to form degree two critical portraits $\text{Cr}^2(L_{P_i}), \text{Cr}^2(L_P)$ so that the sequence of sets $\text{Cr}^2(L_{P_i})$ converges into the set $\text{Cr}^2(P)$. By definition this implies that the sequence of tags $\text{Tag}(\text{Cr}^2(L_{P_i}))$ converges into the tag $\text{Tag}(\text{Cr}^2(L_P))$ as desired. □

### 5.2. The space of cubic bicritical dendritic geolaminations.

In Definition 5.7 we mimic Milnor’s terminology for polynomials.

**Definition 5.7.** A geolamination with one critical set is called unicritical. A geolamination with two disjoint critical sets with a fixed order is called bicritical. The family $F D_3$ of all bicritical dendritic cubic geolaminations is also denoted by $BD^2$.

In other words, if $L \in BD^2$ has critical sets $C_1$ and $C_2$ then $(L, C_1, C_2)$ and $(L, C_2, C_1)$ are two distinct elements of $BD^2$ (cf Definition 2.31 and discussion thereafter).

It is known that if a geolamination $L$ belongs to $BD^2$ then its critical sets $C_1, C_2$ are distinct, finite and contain no periodic points. This is almost a
criterion for $\mathcal{L}$ being dendritic and bicritical, yet there are geolaminations with these properties which have Siegel gaps. An alternative characterization of such geolaminations is due Jan Kiwi [Kiw04] who uses the notion of aperiodic kneading in his description. It is easy to see that a bicritical dendritic geolamination $\mathcal{L}$ is perfect and corresponds to a lamination $\sim$.

We propose a way of parameterizing the space of all cubic bicritical dendritic geolaminations. This approach provides a geometric interpretation of this space analogous to Thurston’s description of the geolamination $\text{QML}$. However our choice for tagging geolaminations is different from Thurston’s: instead of considering minors and postcritical sets we work with siblings of critical sets which we call co-critical sets.

By a sibling of a leaf $\ell$ we mean a leaf $\ell'$ which is disjoint from $\ell$ and has the same image (cf. Definition 2.1). Thurston parameterized the set of all quadratic geolaminations by selecting, for each geolamination $\mathcal{L}$, a special leaf $m$ (the minor of $\mathcal{L}$) which is the image of a leaf $M$ of $\mathcal{L}$ of maximal length. We will tag our cubic laminations not by its “minors” but by something which serves the same purpose as the minor of a quadratic geolamination and determines the choice of the corresponding “major.” For simplicity of this subsection we consider only cubic bicritical dendritic geolaminations.

**Definition 5.8.** A critical set $C$ of a cubic geolamination is called an admissible critical set if

1. it has degree two;
2. the convex hull $S$ of the set of all points of $S \setminus C$ with images in $\sigma_3(C \cap S)$ is unlinked with $C$.

In this case we call $S = \text{co}(C)$ the co-critical set of $C$.

For a (degenerate) leaf or a gap $T$, let $T(1/3)$ be the rotation of $T$ by $2\pi/3$ and $T(2/3)$ be the rotation of $T$ by $4\pi/3$.

**Lemma 5.9.** An admissible critical set $C$ with co-critical set $S$ coincides with the convex hull of the sets $S(1/3)$ and $S(2/3)$. If two admissible critical sets $C$ and $K$ have co-critical sets $S^C$ and $S^K$ then (1) if $S^C$ and $S^K$ have a common vertex then $C$ and $K$ share a critical chord, and (2) if $S^C$ and $S^K$ have linked edges then $C$ and $K$ contain strongly linked quadrilaterals.

**Proof.** The first claim of the lemma is left to the reader. It implies that (1) if $S^C$ and $S^K$ have a common vertex $x$ then the critical chord connecting $x(1/3)$ and $x(2/3)$ is a common critical chord of $C$ and $K$, and (2) if $S^C$ and $S^K$ have linked edges $E^C$ and $E^K$ then the collapsing quadrilateral...
$Q^C = \text{CH}(S^C(1/3), S^C(2/3))$ is contained in $C$, the collapsing quadrilateral $Q^K = \text{CH}(S^K(1/3), S^K(2/3))$ is contained in $K$, and $Q^C$ and $Q^K$ are strongly linked as desired.

**Definition 5.10.** Suppose that $\mathcal{L} \in \mathcal{BD}^2$ is a bicritical dendritic cubic lamination and $C_1, C_2$ are the two critical sets of $\mathcal{L}$. Call the sets $\text{co}(C_1), \text{co}(C_2)$ the co-critical sets of $\mathcal{L}$. Also, we call the set $\text{Tag}(\mathcal{L}, C_1, C_2) = \text{co}(C_1) \times \text{co}(C_2) \subseteq D \times D$ the co-critical tag of $\mathcal{L}$.

It follows from Lemma 5.9 that $\text{Tag}$ is a tagging. The family of co-critical tags of all cubic bicritical dendritic geolaminations is called the cubic co-critical lamination.

Let us remind the reader about Kiwi’s results [Kiw04]. By [Kiw04], if $P$ is a dendritic polynomial $P$ with connected Julia set $J(P)$ then there exists a special lamination $\sim_P$, determined by $P$, with the following property: $P|_{J(P)}$ is semiconjugate by a monotone map $\varphi$ to $f_{\sim_P}|_{J_{\sim_P}}$. Let $\pi : \overline{D} \to J_{\sim_P}$ be the natural extension over $\overline{D}$ of the quotient map from $\mathbb{S}$ to $J_{\sim_P}$. Then to any point $x \in J(P)$ we can associate the set $X = \pi^{-1} \circ \varphi(x)$.

Consider the space $\mathcal{RF}_3$ of cubic polynomials with chosen order of their distinct critical points. In other words, if $f$ is a cubic polynomial with critical points $c_1 \neq c_2$ then $(f, c_1, c_2)$ and $(f, c_2, c_1)$ are two distinct elements of $\mathcal{RF}_3$. To each element $(f, c_1, c_2)$ of $\mathcal{RF}_3$ we associate the element $(\mathcal{L}_P, C_1, C_2)$ of $\mathcal{BD}^2$ where $C_1$ and $C_2$ are sets associated to $c_1$ and $c_2$ in the above sense, respectively (clearly, $C_1$ and $C_2$ are critical sets of $\mathcal{L}_P$ which are distinct by definition). Define the map $\Psi_{\text{Tag}}$ on $\mathcal{RF}_3$ by $\Psi_{\text{Tag}}((P, c_1, c_2)) = \text{Tag}((\mathcal{L}_P, C_1, C_2))$. Theorem 5.6 implies Theorem 5.11.

**Theorem 5.11.** The map $\Psi_{\text{Tag}}$ is a continuous map from $\mathcal{RF}_3$ onto the quotient space $\{\text{Tag}(\mathcal{BD}^2)\}^+ / \{\text{Tag}(\mathcal{BD}^2)\}$.

**References**

[AKLS09] A. Avila, J. Kahn, M. Lyubich, W. Shen, *Combinatorial rigidity for unicritical polynomials*, Ann. of Math. (2) 170 (2009), 783-797

[BCMO11] A. Blokh, D. Childers, J. Mayer, L. Oversteegen, *Non-degenerate quadratic laminations*, Topology Proceedings, 38 (2011), 313–360.

[BCO11] A. Blokh, C. Curry, L. Oversteegen, *Locally connected models for Julia sets*, Advances in Mathematics 226 (2011), 1621–1661.

[BL02] A. Blokh, G. Levin, *Growing trees, laminations and the dynamics on the Julia set*, Ergodic Theory and Dynamical Systems 22 (2002), 63–97.

[BMVV12] A. Blokh, D. Mimbs, L. Oversteegen, K. Valkenburg, *Laminations in the language of leaves*, Trans. Amer. Math. Soc., 365 (2013), 5367–5391.

[BO08] A. Blokh, L. Oversteegen, *Wandering Gaps for Weakly Hyperbolic Polynomials*, in: “Complex dynamics: families and friends”, ed. by D. Schleicher, A K Peters (2008), 139–168.
[BOPT10] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Dynamical cores of topological polynomials*, to appear in Proceedings for the Conference “Frontiers in Complex Dynamics (Celebrating John Milnor’s 80th birthday)”

[BOPT13a] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Laminations from the Main Cuboid*, preprint arXiv:1305.5788 (2013)

[Chi07] D. Childers, *Wandering polygons and recurrent critical leaves*, Ergod. Th. and Dynam. Sys. 27 (2007), 87–107.

[Dav86] R. Daverman, *Decompositions of manifolds*, Academic Press (1986).

[Dou93] A. Douady, *Descriptions of compact sets in $\mathbb{C}$*, Topological methods in modern mathematics (Stony Brook, NY, 1991), ed. by L. Goldberg and A. Phillips, Publish or Perish, Houston, TX (1993), 429-465.

[GM93] L. Goldberg, J. Milnor, *Fixed points of polynomial maps. Part II. Fixed point portraits*, Annales scientifiques de l’ENS 26 (1993), 51–98.

[KL09a] J. Kahn, M. Lyubich, *The quasi-additivity law in conformal geometry*, Ann. of Math. (2) 169 (2009), 561-593.

[KL09b] J. Kahn, M. Lyubich, *Local connectivity of Julia sets for unicritical polynomials*, Ann. of Math. (2) 170 (2009), 413-426.

[Kiw02] J. Kiwi, *Wandering orbit portraits*, Trans. Amer. Math. Soc. 354 (2002), 1473–1485.

[Kiw04] J. Kiwi, *Real laminations and the topological dynamics of complex polynomials*, Advances in Mathematics, 184 (2004), 207–267.

[Kiw05] J. Kiwi, *Combinatorial continuity in complex polynomial dynamics*, Proc. London Math. Soc. (3) 91 (2005), 215–248.

[KvS06] O. Kozlovski, S. van Strien, *Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials*, Proc. of the LMS, 99 (2009), 275–296.

[KSvS07a] O. Kozlovski, W. Shen, W., S. van Strien, *Rigidity for real polynomials*, Ann. of Math. (2) 165 (2007), 749-841.

[KSvS07b] O. Kozlovski, W. Shen, W. van Strien, *Density of hyperbolicity in dimension one*, Ann. of Math. (2) 166 (2007), 145-182.

[Sch04] D. Schleicher, *On fibers and local connectivity of Mandelbrot and Multibrot sets*, Fractal geometry and applications: a jubilee of Benoît Mandelbrot, ed. by M. Lapidus and M. van Frankenhuysen, Proc. Sympos. Pure Math. 72, Part 1, Amer. Math. Soc., Providence, RI (2004), 477-517.

[Sch09] D. Schleicher, *Appendix: Laminations, Julia sets, and the Mandelbrot set*, in: “Complex dynamics: Families and Friends”, ed. by D. Schleicher, A K Peters (2009), 111–130.

[Thu85] W. Thurston. *The combinatorics of iterated rational maps* (1985), published in: “Complex dynamics: Families and Friends”, ed. by D. Schleicher, A K Peters (2009), 1–108.

(Alexander Blokh and Lex Oversteegen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294

(Ross Ptacek and Vladlen Timorin) FACULTY OF MATHEMATICS, LABORATORY OF ALGEBRAIC GEOMETRY AND ITS APPLICATIONS, HIGHER SCHOOL OF ECONOMICS, VAVILOVA ST. 7, 112312 MOSCOW, RUSSIA
(Vladlen Timorin) Independent University of Moscow, Bolshoy Vla-Syevskiy Pereulok 11, 119002 Moscow, Russia

E-mail address, Alexander Blokh: ablokh@math.uab.edu
E-mail address, Lex Oversteegen: overstee@math.uab.edu
E-mail address, Ross Ptacek: rptacek@uab.edu
E-mail address, Vladlen Timorin: vtimorin@hse.ru
