The $F^A$ Quantifier Fuzzification Mechanism: analysis of convergence and efficient implementations.

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Abstract

The fuzzy quantification model $F^A$ has been identified as one of the best behaved quantification models in several revisions of the field of fuzzy quantification. This model is, to our knowledge, the unique one fulfilling the strict Determiner Fuzzification Scheme axiomatic framework that does not induce the standard min and max operators. The main contribution of this paper is the proof of a convergence result that links this quantification model with the Zadeh’s model when the size of the input sets tends to infinite. The convergence proof is, in any case, more general than the convergence to the Zadeh’s model, being applicable to any quantitative quantifier. In addition, recent revisions papers have presented some doubts about the existence of suitable computational implementations to evaluate the $F^A$ model in practical applications. In order to prove that this model is not only a theoretical approach, we show exact algorithmic solutions for the most common linguistic quantifiers as well as an approximate implementation by means of Monte Carlo. Additionally, we will also give a general overview of the main properties fulfilled by the $F^A$ model, as a single compendium integrating the whole set of properties fulfilled by it has not been previously published.

Index Terms

Fuzzy quantification, theory of generalized quantifiers, quantifier fuzzification mechanisms, Zadeh’s quantification model.

I. INTRODUCTION

A great range of models have been proposed for the evaluation of fuzzy quantified sentences, being [2], [7], [8], [26], [10], [11], [14], [12], [13], [16], [18], [20], [21], [23], [30], [25], [28], [29], [5], [31] only an example. Several revision papers have also been published, being [8] possibly the one that makes a more exhaustive comparison. Other revision works is worth to mention are [2], [7], [21], [9], [5]. There also exists an specific paper [15], comparing the models following the quantification framework presented in [21].

Moreover, fuzzy quantifiers have been used in a wide range of applications like fuzzy control, temporal reasoning, fuzzy databases, information retrieval, multi-criteria decision making, data fusion, natural language generation, etc. In [8] a list of the main applications of fuzzy quantifiers is presented.

This paper is devoted to present some relevant new results of the $F^A$ quantification model proposed in [13], [14]. The main result we will present is a convergence proof that, in the particular case of proportional quantifiers, assures the convergence of the $F^A$ model to the Zadeh’s quantification model [31] when the intersection of fuzzy sets is modelled by means of the probabilistic $tnorm$ operator. Moreover, although several revisions of the fuzzy quantification field [8], [5], [15] have presented this model as one of the best fuzzy quantification models available, some doubts persist about the possibility of efficiently implementing it [8], [5]. For this reason, we will also provide efficient computational implementations of the $F^A$ quantification model for the most common linguistic quantifiers as well as the explanation of how to extend these implementations to other types of linguistic quantifiers.

The definition of the $F^A$ quantification model follows the Glöckner’s approximation to fuzzy quantification [21] instead of the common one based on type I, type II quantified expressions proposed by Zadeh [31]. Glöckner’s approximation generalizes the concept of generalized classic quantifier [3] (second order predicates or set relationships) to the fuzzy case as fuzzy relationships between fuzzy sets. Following this idea he recasts the problem of evaluating fuzzy quantified expressions as a problem of searching for adequate mechanisms to transform semi-fuzzy quantifiers (specification means) into fuzzy quantifiers (operational means). The author denominates these transformation mechanisms Quantifier Fuzzification Mechanism (QFMs). The followed approach also generalizes the Theory of Generalized Quantifiers (TGQ), that deals with the analysis and modelling of the phenomena of quantification in natural language, [3] to the fuzzy case.

In his proposal the author also defined a rigorous axiomatic framework to ensure the good behavior of QFMs. Models fulfilling this strict framework are denominated Determiner Fuzzification Schemes (DFSs) and they comply with a broad set of properties that guarantee a good behavior from a linguistic and fuzzy point of view. In [26] or [21] can be consulted a comparison between Zadeh’s and Glöckner’s approaches.

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The \( \mathcal{F}^A \) model is a QFM which fulfills the strict axiomatic framework proposed by Glöckner, which makes it a DFS. It is to our knowledge the unique non-standard DFS (a DFS non inducing the standard max and min operators).

The structure of this paper is the following. First, we will introduce the fuzzy quantification framework proposed by Glöckner. Second, we will present two alternative definitions of the \( \mathcal{F}^A \) QFM, one based on the use of fuzzy connectives and the other on a probabilistic interpretation of fuzzy sets. After that, we will give a brief review of the main properties fulfilled by the \( \mathcal{F}^A \) QFM. The following section is dedicated to introduce the convergence results of the \( \mathcal{F}^A \) QFM, that as a particular case includes the convergence to the Zadeh’s model. However, the rate of convergence is too slow to be of utility in most applications, but thanks to this result is argued that we can expect very good approximations of the \( \mathcal{F}^A \) QFM by means of Monte Carlo. The final section is devoted to present exact and approximate algorithmic implementations.

II. THE FUZZY QUANTIFICATION FRAMEWORK

In the specification of the fuzzy quantification framework \([21]\), the author rewrote the problem of defining fuzzy quantification models as the problem of looking for adequate means to convert semi-fuzzy quantifiers (i.e., mechanisms adequate to specify the meaning of linguistic quantifiers) into fuzzy quantifiers (i.e., operational means adequate to apply semi-fuzzy quantifiers to fuzzy inputs).

In this framework, fuzzy quantifiers are just a generalization of crisp quantifiers to fuzzy sets. We will show below the definition of classic quantifiers conforming to TGQ.

**Definition 1:** A two valued (generalized) quantifier on a base set \( E \neq \emptyset \) is a mapping \( \mathcal{Q} : \mathcal{P}(E)^n \rightarrow 2 \), where \( n \in \mathbb{N} \) is the arity (number of arguments) of \( \mathcal{Q} \). \( 2 = \{0,1\} \) denotes the set of crisp truth values, and \( \mathcal{P}(E) \) is the powerset of \( E \).

Below we show two examples of classic quantifiers:

\[
\text{all} (Y_1, Y_2) = Y_1 \subseteq Y_2
\]

\[
\text{at least 60\%} (Y_1, Y_2) = \begin{cases} \frac{|Y_1 \cap Y_2|}{|Y_2|} \geq 0.60 & Y_1 \neq \emptyset \\ 1 & Y_1 = \emptyset \end{cases}
\]

From here on, we will denote \( |E| = m \).

A fuzzy quantifier assigns a fuzzy value to each possible choice of \( X_1, \ldots, X_n \in \mathcal{P}(E) \), where by \( \mathcal{P}(E) \) we are denoting the fuzzy powerset of \( E \).

**Definition 2:** \([21]\) definition 2.6 An n-ary fuzzy quantifier \( \mathcal{Q} \) on a base set \( E \neq \emptyset \) is a mapping \( \mathcal{Q} : \mathcal{P}(E)^n \rightarrow \mathbb{I} = [0,1] \).

The next example shows a possible definition of the fuzzy quantifier \( \text{all} : \mathcal{P}(E)^2 \rightarrow \mathbb{I} \):

\[
\text{all} (X_1, X_2) = \inf \{ \max (1 - \mu_{X_1}(e), \mu_{X_2}(e)) : e \in E \}
\]

where by \( \mu_X(e) \) we are denoting the membership function of \( X \in \mathcal{P}(E) \).

Previous definition of the fuzzy quantifier \( \text{all} \) seems plausible. However, the reader could think of in other possible plausible expressions to model \( \text{all} \) by simply changing the tconorm operator max by other tconorm operator. For other quantifiers, like ‘at least sixty percent’, the problem of establishing adequate models is far from obvious.

In the search of possible solutions for defining fuzzy quantifiers, in \([21]\) the concept of semi-fuzzy quantifier was introduced to work as a ‘middle point’ between classic and fuzzy quantifiers. Semi-fuzzy quantifiers are close but more powerful than the Zadeh’s concept of linguistic quantifiers \([31]\). Semi-fuzzy quantifiers only accept crisp arguments, as classic quantifiers, but they have a fuzzy output, as in the case of fuzzy quantifiers. Semi-fuzzy quantifiers are adequate to capture the semantics of linguistic quantified expressions.

**Definition 3:** \([21]\) definition 2.8 An n-ary semi-fuzzy quantifier \( Q \) on a base set \( E \neq \emptyset \) is a mapping \( Q : \mathcal{P}(E)^n \rightarrow \mathbb{I} \).

\( Q \) assigns a gradual result to each pair of crisp sets \( (Y_1, \ldots, Y_n) \). Some examples of semi-fuzzy quantifiers are:

\[
\text{about 10} (Y_1, Y_2) = T_{0.6,12,14} (|Y_1 \cap Y_2|)
\]

\[
\text{about 60\% or more} (Y_1, Y_2) = \begin{cases} S_{0.4,0.6} \left( \frac{|Y_1 \cap Y_2|}{|Y_1|} \right) & Y_1 \neq \emptyset \\ 1 & Y_1 = \emptyset \end{cases}
\]
where \( T_{a,b,c,d}(x) \) and \( S_{\alpha,\gamma}(x) \) represent the ordinary trapezoida and \( S \) fuzzy numberb.

We generally will denote the fuzzy numbers used in the definition of the semi-fuzzy quantifiers as ‘support functions of the semi-fuzzy quantifiers’.

Although the semantics of semi-fuzzy quantifiers is intuitive, they do not permit to evaluate fuzzy quantified expressions. In [21], the author proposes to use an additional mechanism to transform semi-fuzzy quantifiers into fuzzy quantifiers. This mechanism allows to map semi-fuzzy quantifiers into fuzzy quantifiers:

**Definition 4:** [21] definition 2.10] A quantifier fuzzification mechanism (QFM) \( \mathcal{F} \) assigns to each semi-fuzzy quantifier \( Q : \mathcal{P}(E)^n \rightarrow I \) a corresponding fuzzy quantifier \( \mathcal{F}(Q : \mathcal{P}(E)^n \rightarrow I \) of the same arity \( n \in \mathbb{N} \) and on the same base set \( E \).

### III. The QFM \( \mathcal{F}^A \)

In this section we present the finite QFM \( \mathcal{F}^A \) [14, 13, 9, 12, 15]. The \( \mathcal{F}^A \) QFM can be defined using two different strategies. The first definition uses the equipotence concept and remains purely on the use of fuzzy operators. The second is based on a probabilistic interpretation of fuzzy sets. Both definitions are equivalent.

Following [11] the equipotence between a crisp set \( Y \) and a fuzzy set \( X \) can be defined as:

\[
Eq(Y, X) = \wedge_{e \in E} (\mu_X(e) \rightarrow \mu_Y(e)) \land (\mu_Y(e) \rightarrow \mu_X(e)).
\]

The concept of equipotence is basically a measure of equality between fuzzy sets.

Let us consider the product tnorm \( \land (x_1, x_2) = x_1 \cdot x_2 \) and the Lukasiewicz implication \( \rightarrow (x_1, x_2) = \min (1, 1 - x_1 + x_2) \). In previous expression, if \( e \in Y \) then \( \mu_Y(e) = 1 \) and if \( e \notin Y \) then \( \mu_Y(e) = 0 \). Then

\[
(\mu_X(e) \rightarrow \mu_Y(e)) \land (\mu_Y(e) \rightarrow \mu_X(e)) = \begin{cases} 
\mu_X(e) & : e \in Y \\
1 - \mu_X(e) & : e \notin Y
\end{cases}
\]

(2)

Then from (1) and (2)

\[
Eq(Y, X) = \prod_{e \in Y} \mu_X(e) \prod_{e \notin Y} (1 - \mu_X(e)).
\]

Using the equipotence concept, the \( \mathcal{F}^A \) model can be defined as:

**Definition 5:** Let \( Q : \mathcal{P}(E)^n \rightarrow I \) be a semi-fuzzy quantifier, \( E \) finite. The QFM \( \mathcal{F}^A \) is defined as:

\[
\mathcal{F}^A(X_1, \ldots, X_n) = \bigvee_{Y_1 \in \mathcal{P}(E)} \ldots \bigvee_{Y_n \in \mathcal{P}(E)} Eq(Y_1, X_1) \land \ldots \land Eq(Y_n, X_n) \land Q(Y_1, \ldots, Y_n)
\]

where \( \bigvee \) is the Lukasiewicz tconorm \( \bigvee (x_1, x_2) = \min (x_1 + x_2, 1) \), and \( \land \) is the product tnorm \( \land (x_1, x_2) = x_1 \cdot x_2 \).

Now, we will present an alternative definition based on a probabilistic interpretation of fuzzy sets. The semantic interpretation of fuzzy sets based on likelihood functions [24, 27, 4, 17] simply interprets vagueness in the data as a consequence of making a random experiment in which a set of individuals are asked about the fulfillment of a certain property.

For example, let us consider \( h \in \mathbb{R} \). We can define the degree of fulfillment of the statement “the value of height \( h \) is tall” as:

\[
\mu (\text{“}h \text{ is tall”) = Pr (\text{“}h \text{ is considered tall”) = } |v \in V : C(v, “h \text{ is considered tall”)} = 1|}
\]

where \( V \) is a set of voters and \( C(v, “h \text{ is considered tall”) denotes the answer of the voter \( v \) to the question “\( h \text{ is considered tall}”.

---

a Function \( T_{a,b,c,d}(x) \) is defined as

\[
T_{a,b,c,d}(x) = \begin{cases} 
0 & : x \leq a \\
\frac{x - a}{b - a} & : a < x \leq b \\
1 & : b < x \leq c \\
\frac{d - x}{d - c} & : c < x \leq d \\
0 & : d < x
\end{cases}
\]

b Function \( S_{\alpha,\gamma}(x) \) is defined as

\[
S_{\alpha,\gamma}(x) = \begin{cases} 
0 & : x \leq \alpha \\
2 \left( \frac{(x - \alpha)}{\gamma - \alpha} \right)^2 & : \alpha < x \leq \frac{\alpha + \gamma}{2} \\
1 - 2 \left( \frac{(x - \gamma)}{\gamma - \alpha} \right)^2 & : \frac{\alpha + \gamma}{2} < x \leq \gamma \\
1 & : \gamma < x
\end{cases}
\]
We can apply the same idea to compute the probability that a crisp set \( Y \in \mathcal{P}(E) \) is a representative of a fuzzy set \( X \in \tilde{\mathcal{P}}(E) \) when we suppose the base set \( E \) finite and that the probabilities of the different elements are independent. The intuition is to measure the probability that only the elements in \( Y \) belongs to \( X \):

**Definition 6:** Let \( X \in \tilde{\mathcal{P}}(E) \) be a fuzzy set, \( E \) finite. The probability of the crisp set \( Y \in \mathcal{P}(E) \) being a representative of the fuzzy set \( X \in \tilde{\mathcal{P}}(E) \) is defined as

\[
\Pr(\text{representative}_X = Y) = m_X(Y) = \prod_{e \in Y} \mu_X(e) \prod_{e \notin E \setminus Y} (1 - \mu_X(e))
\]

We would like to point out that in the previous definition the probability points are the subsets of \( E \). In this way the \( \sigma \)-algebra on which the probability is defined is \( \mathcal{P}(E) \).

Using expression 4 the definition of the QFM \( \mathcal{F}^A \) is easily made:

**Definition 7:**[14] page. 1359]. Let \( Q : \mathcal{P}(E)^n \to \mathbb{I} \) be a semi-fuzzy quantifier, \( E \) finite. The QFM \( \mathcal{F}^A \) is defined as

\[
\mathcal{F}^A(Q)(X_1, \ldots, X_n) = \sum_{Y_1 \in \mathcal{P}(E)} \ldots \sum_{Y_n \in \mathcal{P}(E)} m_{X_1}(Y_1) \ldots m_{X_n}(Y_n) Q(Y_1, \ldots, Y_n)
\] (3)

for all \( X_1, \ldots, X_n \in \tilde{\mathcal{P}}(E) \).

In expression 3 we are assuming that the probability of being \( Y_i \) a representative of the fuzzy set \( X_i \) is independent of the probability of being \( Y_j \) a representative of the fuzzy set \( X_j \) for \( i \neq j \). \( \mathcal{F}^A(Q)(X_1, \ldots, X_n) \) can then be interpreted as the average opinion of voters 3.

The following example shows the application of the QFM \( \mathcal{F}^A \):

**Example 8:** Let us consider the sentence:

"Nearly all big houses are expensive"

where the semi-fuzzy quantifier \( Q = \text{‘nearly all’} \), and the fuzzy sets ‘big houses’ and ‘expensive’ take the following values:

- \text{big houses} = \{0.8/e_1, 0.9/e_2, 1/e_3, 0.2/e_4\}
- \text{expensive} = \{1/e_1, 0.8/e_2, 0.3/e_3, 0.1/e_4\}

\( Q(X_1, X_2) = \max \left\{ \frac{2(X_1 \cap X_2)}{|X_1|} - 1, 0 \right\} \quad X_1 \neq \emptyset \quad X_1 = \emptyset .

We compute the probabilities of the representatives of the fuzzy sets ‘big houses’ and ‘expensive’:

\[
m_{\text{big houses}}(\emptyset) = (1 - 0.8)(1 - 0.9)(1 - 1)(1 - 0.2) = 0,
\]

\[
\ldots
\]

\[
m_{\text{expensive}}(\{e_1, e_2, e_3, e_4\}) = 0.8 \cdot 0.9 \cdot 1 \cdot 0.2 = 0.144,
\]

\[
m_{\text{expensive}}(\{e_2\}) = (1 - 1)(1 - 0.8)(1 - 0.3)(1 - 0.1) = 0,
\]

\[
m_{\text{expensive}}(\{e_1\}) = 1 \cdot (1 - 0.8)(1 - 0.3)(1 - 0.1) = 0.126,
\]

\[
\ldots
\]

\[
m_{\text{expensive}}(\{e_1, e_2, e_3, e_4\}) = 0.8 \cdot 0.9 \cdot 1 \cdot 0.2 = 0.144.
\]

And using expression 3:

\[
\mathcal{F}^A(Q)(\text{big houses, expensive}) = \sum_{Y_1 \in \mathcal{P}(E)} \sum_{Y_2 \in \mathcal{P}(E)} m_{X_1}(Y_1) m_{X_2}(Y_2) Q(Y_1, Y_2) = 0.346.
\]

3We would like to point out that the probabilistic interpretation of the QFM \( \mathcal{F}^A \) holds some relationships with a similar probabilistic interpretation of the Zadeh’s model. Let \( X \in \tilde{\mathcal{P}}(E) \) be a fuzzy set representing the linguistic concept ‘big houses’, and let us suppose we want to select an element \( e \in E \). Let us assume we have the same probability of selecting each element, and that \( \mu_e(X) \) represents the probability that the element \( e \) fulfills the property of being a big house. Let \( f_q : [0, 1] \to \mathbb{I} \) be a function representing a proportional unary linguistic quantifier (e.g. ‘most’). Then, the Zadeh’s model just applies the linguistic quantifier to the average probability of selecting an element fulfilling the property of being a ‘big house’:

\[
f_q(\text{Avg}(\Pr(e \text{ is big} | e \text{ is selected}))) = f_q \left( \frac{1}{m} \sum_{e \in E} \mu_e(X) \right)
\]

. In contrast, the \( \mathcal{F}^A \) QFM computes the probability of every possible combination in which the elements of \( E \) can fulfill the property of ‘being a big house’. After computing the probability of each combination, we compute the average of applying the support function of the quantifier \( f_q \) to the possible combinations.
TABLE I
CONDITIONS OF A DFS FOR ALL SEMI-FUZZY QUANTIFIERS \( Q : \mathcal{P} (E)^n \rightarrow I \)

| Name                     | Condition                                                                 | Reference |
|--------------------------|---------------------------------------------------------------------------|-----------|
| Correct generalization   | \( U (F(Q)) = Q \) if \( n \leq 1 \)                                      | (Z-1)     |
| Projection quantifiers    | \( F(Q) = \pi_e \) if \( Q = \pi_e \) for some \( e \in E \)             | (Z-2)     |
| Dualisation              | \( F(Q_{\cup}) = F(Q)_{\cup} \) if \( n > 0 \)                           | (Z-3)     |
| Internal joins           | \( F(Q_{\cap}) = F(Q)_{\cap} \) if \( n > 0 \)                           | (Z-4)     |
| Preservation of monotonicity | If \( Q \) is nonincreasing in the \( n \)-th arg, then \( F(Q) \) is   | (Z-5)     |
|                          | nonincreasing in \( n \)-th arg, \( n > 0 \)                             |           |
| Functional application   | \( F \left( Q \circ \bigtimes_{i=1}^{n} f_i \right) = F(Q) \circ \bigtimes_{i=1}^{n} \tilde{F}(f_i) \) where \( f_1, \ldots, f_n : E' \rightarrow E, E' \neq \emptyset \) | (Z-6)     |

IV. THE DFS AXIOMATIC FRAMEWORK

We will present now the definition of the Determiner fuzzification scheme (DFS) axiomatic framework [21]. It is impossible in this paper to explain in full detail the DFS axiomatic framework, as in [21] the author needed chapters three and four to present it in adequate detail. We will limit us to introduce the framework, referring the reader to the previous reference for further study.

**Definition 9**: A QFM \( F \) is called a determiner fuzzification scheme (DFS) if the conditions listed in TABLE I are satisfied for all semi-fuzzy quantifiers \( Q : \mathcal{P} (E)^n \rightarrow I \).

In the following section, we will present the main properties of the models fulfilling the framework in relation to the \( \mathcal{F}^{A} \) QFM, and some others that are not consequence of the DFS axiomatic framework but which are important in order to adequately characterize the behavior of QFMs.

V. ANALYSIS OF THE BEHAVIOR OF THE \( \mathcal{F}^{A} \) QFM

In this section we will give a general overview of the main properties of the \( \mathcal{F}^{A} \) QFM referring the different publications where the proofs and extended explanations can be found.

A. Main properties of the \( \mathcal{F}^{A} \) QFM derived from the DFS framework

As we have advanced, the \( \mathcal{F}^{A} \) QFM fulfills the DFS axiomatic framework. This fact guarantees that it also fulfills all the adequacy properties the framework guarantees. We will present now the main properties derived from it in relation with the behavior of the \( \mathcal{F}^{A} \) QFM.

1) Correct generalization (P1): This property is possibly the most important property derived from the DFS framework. Correct generalization requires that the behavior of a fuzzy quantifier \( F(Q) \) when we apply it to crisp arguments would be equal to the application of the semi-fuzzy quantifier \( Q \) over the same crisp arguments. That is, for all the crisp subsets \( Y_1, \ldots, Y_n \in \mathcal{P}(E) \), then it holds that \( F(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n) \). For example, given the crisp sets big houses, expensive \( \in \mathcal{P}(E) \), this property guarantees that:

\[
F(\text{some})(\text{big houses, expensive}) = \text{some}(\text{big houses, expensive})
\]

The proof of this property for the \( \mathcal{F}^{A} \) QFM can be found in [9, page 291], [12, page 31] for the unary case, as in conjunction with the other axioms of the DFSs is enough to assure the fulfillment of the property in the general case.

2) Quantitativity (P2): In TGQ, a quantifier is quantitative if it does not depend on any particular property fulfilled by the elements. Most common examples of quantifiers we can find in the literature are quantitative (e.g., ‘many’, ‘about 10’). Non-quantitative quantifiers involve the reference to particular elements of the base set (e.g., ‘Spain’ in a set of countries). A QFM \( F \) retains the quantitativity property if quantitative semi-fuzzy quantifiers are converted into quantitative fuzzy quantifiers by the application of \( F \). The fulfillment of the quantitativity property for the \( \mathcal{F}^{A} \) QFM is a consequence of the fulfillment of the DFS framework.

3) Projection quantifier (P3): The Axiom Z-2 of the DFS framework establishes that the projection crisp quantifier \( \pi_e(Y) \) (which returns 1 if \( e \in Y \) and 0 in other case) is transformed into the fuzzy projection quantifier \( \tilde{\pi}_e(X) \) (which returns \( \mu_X(e) \)). The proof of this property for the \( \mathcal{F}^{A} \) QFM can be found in [9, page 272], [12, page 31].
4) **Induced propositional logic (P4):** In [21] a mechanism was proposed to embed crisp logical functions (¬(x), ∧(x₁,x₂), ∨(x₁,x₂), →(x₁,x₂)) into semi-fuzzy quantifiers. For example, the ‘and’ function can be embedded into a semi-fuzzy quantifier \( Q_\land : \mathcal{P} \{\{e_1,e_2\}\} \rightarrow \{0,1\} \) such that \( Q_\land (\emptyset) = Q_\land (\{e_1\}) = Q_\land (\{e_2\}) = 0 \) and \( Q_\land (\{e_1,e_2\}) = 1 \). The property of induced propositional logic assures that crisp logical functions are transformed into acceptable fuzzy logical functions. In the case of the \( F^A \) QFM the induced propositional functions are respectively the strong negation, the probabilistic t-norm and the Rechenbach fuzzy implication. The proof of this property for the \( F^A \) QFM can be found in [9] page 265, [12] page 28.

5) **External negation (P5):** We will say that a QFM fulfills the external negation property if \( F(\neg Q) \) is equivalent to \( \neg F(Q) \). In words, equivalence of expressions “it is false that at least 60% of the good students are good athletes” and “less than 40% of the good students are good athletes” is assured. Here, \( \neg \) is assumed to be the induced negation of the QFM (the strong negation for the \( F^A \) QFM). The proof of this property for the \( F^A \) QFM can be found in [9] page 273, [12] page 32.

6) **Internal negation (P6):** The internal negation or antonym of a semi-fuzzy quantifier \( Q : \mathcal{P} (E)^n \rightarrow \mathcal{I} \) is defined as \( Q^\neg (Y_1, \ldots, Y_n) = Q (\neg Y_1, \ldots, \neg Y_n) \). For example, ‘all’ is the antonym of ‘no’ as \( \text{no} (Y_1, Y_2) = \text{all} (\neg Y_1, Y_2) = \text{all} (Y_1, \neg Y_2) \). Fulfillment of the internal negation property assures that internal negation transformations are translated to the fuzzy case. The proof of this property for the \( F^A \) QFM can be found in [9] page 273, [12] page 32.

7) **Dualisation (P7):** The dualisation property is a consequence of the fulfillment of the external negation and internal negation properties. In conjunction, these negation properties assure the maintenance of the equivalences in the ‘Aristotelian square’ [19]. It forms part of the DFS framework (Z-3 axiom), being the dual of a semi-fuzzy quantifier \( Q : \mathcal{P} (E)^n \rightarrow \mathcal{I} \) defined as \( Q^\neg (Y_1, \ldots, Y_n) = \neg Q (\neg Y_1, \ldots, \neg Y_n) \) equivalently in the fuzzy case. As an example, the equivalence of \( F(\text{no}) \) (big houses, expensive) and \( F(\text{all}) \) (big houses, expensive) is assured; or in words, “no big house is not expensive” and “all big houses are expensive” are equivalent. The proof of this property for the \( F^A \) QFM can be found in [9] page 275, [12] page 33.

8) **Union/intersection of arguments (P8):** The properties of union and intersection of arguments guarantee the compliance with some transformations to construct new quantifiers using unions and intersections of arguments. Being \( Q : \mathcal{P} (E)^{n+1} \rightarrow \mathcal{I} \) an \( n + 1 \)-ary semi-fuzzy quantifier, \( Q \cup \) is defined as \( Q \cup (Y_1, \ldots, Y_{n+1}) = Q (Y_1, \ldots, Y_n \cup Y_{n+1}) \) equivalently in the fuzzy case. Z-4 axiom specifies this property for the union of quantifiers, as the property is also fulfilled for the intersection of arguments as a consequence of the DFS axiomatic framework.

One particular example of the consequences of fulfilling these properties is that the equivalence between absolute unary and binary quantifiers is assured, guaranteeing that we obtain the same result when we evaluate “around 5 big houses are expensive” and “there are around 5 houses that are big and expensive”, where the evaluation of the first quantified expression is computed by means of an absolute binary quantifier and the evaluation of the second expression is computed by applying the corresponding absolute unary quantifier to the intersection of ‘big houses’ and ‘expensive houses’ computed by means of the induced t-norm. In combination with the internal and external negation properties, they allow the preservation of the boolean argument structure that can be expressed in natural language when none of the boolean variables \( X_i \) occurs more than once [21] section 3.6]. The proof of these properties for the \( F^A \) QFM can be found in [9] page 275, [12] page 33.

9) **Coherence with standard quantifiers (P9):** By standard quantifiers we mean the classical quantifiers \( \exists, \forall \) and their binary versions some and all. Every QFM fulfilling the DFS axiomatic framework guarantees that the fuzzy version of these classical quantifiers is the expected. For example, the \( F^A \) QFM fulfills (where \( \forall, \exists, \neg \) are the logical operators induced by the \( F^A \) model):

\[
F ( \exists ) (X) = \sup \left\{ \bigvee_{i=1}^{m} \mu_X (a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P} (E), a_i \neq a_j \text{ if } i \neq j \right\}
\]

\[
F ( \text{all} ) (X_1, X_2) = \inf \left\{ \bigwedge_{i=1}^{m} \mu_X (a_i) \rightarrow \mu_{X_2} (a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P} (E), a_i \neq a_j \text{ if } i \neq j \right\}
\]

This property is a consequence of being the \( F^A \) QFM a DFS.

10) **Monotonicity in arguments (P10):** In [21] different definitions to assure the preservation of monotonicity relationships were included. The property of monotonicity in arguments, which forms part of the DFS axiomatic framework (axiom Z5) assures that monotonic behaviors in arguments are translated from the semi-fuzzy to the fuzzy case. For example, for the binary semi-fuzzy quantifier ‘most’, that is increasing in its second argument (e.g. “most politics are rich”), the fulfillment of this property guarantees that its fuzzy version will also be increasing in its second argument. The DFS framework also guarantees the maintenance of ‘local’ monotonic properties [21] section 4.11]. The proof of these properties for the \( F^A \) QFM can be found in [9] page 282, [12] page 39.

11) **Monotonicity between quantifiers (P11):** The DFS axiomatic framework also guarantees the preservation of monotonicity relationships between quantifiers. For example, ‘between 4 and 6’ is more specific than ‘between 2 and 8’. Thanks to this property, monotonicity relationships between semi-fuzzy quantifiers are preserved between fuzzy quantifiers. The fulfillment of the property of monotonicity in quantifiers is a consequence of the DFS axiomatic framework.
12) Crisp argument insertion P12: The operator of crisp argument insertion, applied to a semi-fuzzy quantifier \( Q : \mathcal{P}(E)^n \rightarrow I \), allows to construct a new quantifier \( Q : \mathcal{P}(E)^{n-1} \rightarrow I \) by means of the restriction of \( Q \) by a crisp set \( A \). More explicitly, the crisp argument insertion \( Q \preceq A \) is defined as \( Q \preceq A(Y_1, \ldots, Y_{n-1}) = Q(Y_1, \ldots, Y_{n-1}, A) \). A QFM preserves this property if \( F(Q \preceq A) = F(Q) \preceq A \); that is, the crisp argument insertion commutes for semi-fuzzy and fuzzy quantifiers. Crisp argument insertion allows to model the ‘adjectival restriction’ of natural language in the crisp case. The fulfillment of this property by the \( F^A \) QFM is also a consequence of the DFS axiomatic framework.

B. Some relevant properties considered in the QFM framework but not derived from the DFS axioms

In [21, chapter six] it can be found the definition of some additional adequacy properties for characterizing QFMs. These properties were not included in the DFS framework in some cases, for not being compatible with it, and in other cases, in order to not excessively constrain the set of theoretical models fulfilling the DFS framework. We will present now the most relevant ones:

1) Continuity in arguments P13: The property of continuity in arguments assures the continuity of the models with respect to the input sets. It is fundamental to guarantee that small variations in the inputs do not cause jumps in the outputs.

The \( F^A \) QFM is a finite DFS and it is continuous. The proof of this property for the \( F^A \) QFM can be found in [9, page 293], [12, page 48].

2) Continuity in quantifiers P14: The property of continuity in quantifiers assures the continuity of the QFMs with respect to small variations in the quantifiers. The proof of this property for the \( F^A \) QFM can be found in [9, page 297], [12, page 48].

3) Propagation of fuzziness P15: Propagation of fuzziness properties assure that fuzzier inputs (understood as fuzzier input sets) and fuzzier quantifiers produce fuzzier outputs. This property is not fulfilled by the \( F^A \) QFM because it is not fulfilled by the induced product \( t\text{norm} \) and the induced probabilistic sum \( t\text{conorm} \) of the model. An extensive analysis of the fulfillment of this property by the main QFMs that can be found in the literature is presented in [13].

4) Fuzzy argument insertion P16: The property of fuzzy argument insertion is the fuzzy counterpart of the crisp argument insertion. To our knowledge, this property has only been proved for the DFSs \( \mathcal{M}_{CX} \) [21, definition 7.56] and \( F^A \) [9, page 292], [12, page 48].

C. Additional properties fulfilled by the \( F^A \) QFM not included in the QFM framework

In this section we summarize three other properties fulfilled by the \( F^A \) QFM that do not form part of the ones considered in the QFM framework by Glückner [21]. We will explain these properties in some more detail as they are not commonly considered in the bibliography about fuzzy quantification. In [13] these properties were used, in combination with other criteria, to present a comparison of the behavior of different QFMs thinking in their convenience for practical applications.

1) Property of averaging for the identity quantifier: The fulfillment of this property by a QFM \( F \) assures that when we apply the model to the unary semi-fuzzy quantifier identity \( Y = \frac{|Y|}{|E|}, Y \in \mathcal{P}(E) \) we obtain the average of the membership grades. For the ‘identity’ semi-fuzzy quantifier the addition of one element increases the result in \( \frac{1}{m} \). We could expect that a QFM \( F \) would translate this linearity relationship into the fuzzy case.

The QFM \( F^A \) fulfills the property of averaging for the identity quantifier that assures:

\[
F^A(\text{identity})(X) = \frac{1}{m} \sum_{j=1}^{m} \mu_X(e_j)
\]

The proofs can be found in [9, page 298] or in [12, page 50].

2) Property of the probabilistic interpretation of quantifiers: Let us suppose we use a set of semi-fuzzy quantifiers (“at most about 20%”, “between 20% and 80%”, “at least about 80%”) to split the quantification universe. We will say that a set of semi-fuzzy quantifiers \( Q_1, \ldots, Q_r : \mathcal{P}(E) \rightarrow I \) forms a quantified Ruspini partition of the quantification universe if for all \( Y_1, \ldots, Y_n \in \mathcal{P}(E) \) it holds that

\[
Q_1(Y_1, \ldots, Y_n) + \ldots + Q_r(Y_1, \ldots, Y_n) = 1
\]

The QFM \( F^A \) translates this relationship to the fuzzy case. Forming \( Q_1, \ldots, Q_r : \mathcal{P}(E)^n \rightarrow I \) a quantified Ruspini partition it is fulfilled:

\[
F^A(Q_1)(X_1, \ldots, X_n) + \ldots + F^A(Q_r)(X_1, \ldots, X_n) = 1
\]

This property is very interesting because it will permit to interpret the result of evaluating a fuzzy quantified expression as a probability distributed over the labels related to the quantifiers. Proofs can be found in [9, page 298] or in [12, page 52].

*Let be \( \leq_e \) a partial order in \( I \times I \) defined as [21, section 5.2 and 6.3]:

\[
x \leq_e y \iff y \leq x \leq \frac{1}{2} \text{ or } \frac{1}{2} \leq x \leq y
\]

for \( x, y \in I \). A fuzzy set \( X_1 \) is at least as fuzzy as a fuzzy set \( X_2 \) if for each \( e \in E, \mu_{X_1}(e) \leq_e \mu_{X_2} \); that is, membership degrees of \( X_1 \) are closer to 0.5 than membership degrees of \( X_2 \). In the case of fuzzy quantifiers a similar definition is applied.
3) Fine distinction between objects: This property is particularly useful for the application of fuzzy quantifiers in ranking problems. Let us consider a set of objects \(o_1, \ldots, o_N\) for which the fulfillment of a set of criteria \(p_1, \ldots, p_m\) is represented by means of a fuzzy set \(X^{o_i} = \{\mu_{X^i}(p_1)/p_1, \ldots, \mu_{X^i}(p_m)/p_m\}\), where \(\mu_{X^i}(p_j)/p_j\) indicates the fulfillment of the criteria \(p_j\) by the object \(o_i\). Generally, we also have a set of weights \(W = \{\mu_W(p_1)/p_1, \ldots, \mu_W(p_m)/p_m\}\) to indicate the relative relevance of the different criteria \(p_1, \ldots, p_m\).

Using fuzzy quantification, a ranking can be constructed assigning to each object a weight computed by means of an unary proportional quantified expression \(r^{o_i} = Q(X^{o_i})\) (in the case that a vector of weights is not involved) or a binary proportional quantified expression \(r^{o_i} = Q(W, X^{o_i})\) (in the case that a vector of weights \(W\) is used to indicate the relative importance of each criteria). In this way, computing \(r^{o_i}\) for each \(i = 1, \ldots, N\), we can sort the objects of the collection with respect to the linguistic expression ‘how \(Q\)’ criteria are fulfilled (e.g., for \(Q = \text{many}, \text{‘how many’}\)).

In order to guarantee a sufficient discriminative power, even small variations in the inputs should produce some effect in the outputs. In [15, section 5.6] it was proposed to analyze the behavior of QFMs with respect to the following semi-fuzzy quantifiers defined by means of increasing fuzzy numbers:

**Definition 10:** Let \(h(x) : [0, 1] \to I\) be an strictly increasing continuous mapping; i.e., \(h(x) > h(y)\) for every \(x > y\). We define the unary and binary semi-fuzzy quantifiers \(Q_h : \mathcal{P}(E) \to I\) and \(Q_h : \mathcal{P}(E)^2 \to I\) as

\[
Q_h(Y) = h(|Y|), Y \in \mathcal{P}(E)
\]

\[
Q_h(Y_1, Y_2) = \begin{cases} 
   h\left(\frac{|Y_1 \cap Y_2|}{|Y_1|}\right) & Y_1 \neq \emptyset \\
   1 & Y_1 = \emptyset
\end{cases}
\]

And then, to require to a QFM \(F\) the maintainance of the strictly increasing relationships in the fuzzy case. That is, that any increase in the fulfillment of a criteria will increase \(F(Q_h)\) in the unary case, and that any increase in the fulfillment of a criteria associated with a strictly positive weight will increase \(F(Q_h)\) in the binary case.

The \(F^A\) DFS fulfills this property as can be found in [15, section 5.6].

**VI. LIMIT CASE APPROXIMATION OF THE \(F^A\) QFM**

In this section we will prove that in the general case of semi-fuzzy quantifiers defined by means of continuous proportional fuzzy numbers (i.e., ‘unary proportional’, ‘binary proportional’, ‘comparative proportional’, etc.) the \(F^A\) QFM can be approximated by simply evaluating the fuzzy number that supports the quantifier over a function which depends on the average of the different boolean combinations of the input sets (more details below). As an additional result, in the specific case of unary and binary proportional linguistic quantifiers, the \(F^A\) QFM converges to the Zadeh’s model when the intersection of the inputs sets is computed with the probabilistic tconorm for binary proportional quantifiers.

Before proceeding, we will make a brief summary of the ideas of the proof in order to facilitate its understanding. In the proof, we will start introducing some previous results which guarantee that quantitative quantifiers can be expressed by means of a function of the cardinalities of the boolean combinations of the input sets. This will allow us to develop a general proof, valid for each quantitative quantifier defined by means of a proportional fuzzy number.

After that, we will use the fact that in the definition of the \(F^A\) QFM we are interpreting membership degrees \(\mu_X(e_i)\) as probabilities, and that independence is fulfilled for \(\mu_X(e_i)\mu_X(e_j), i \neq j\). In this case, a fuzzy set \(X = \{a_1/e_1, \ldots, a_m/e_m\}\) will induce a specific probability distribution over the function of the possible cardinalities \(0, \ldots, m\) of the set. In other words, as we are interested in the number of elements of \(X\) fulfilling the property, each possible cardinality \(i\) will have a probability value measuring the probability that exactly ‘\(i\)’ elements fulfill the property. We will see that this probability follows a poisson binomial distribution. Moreover, we will also prove that the projections of the probability function \(f(i_1, \ldots, i_K)\) induced by the \(F^A\) QFM for \(n\)-ary quantifiers follow poisson binomial distributions. In that case, the probability parameters of the \(j\) projection will be determined by the \(j\)-th boolean combination used in the specification of the semi-fuzzy quantifier.

When \(m\) tends to infinite, the fuzzy set \(X = \{a_1/e_1, \ldots, a_m/e_m\}\) will induce a sequence \(B_1, B_2, \ldots\) of poisson binomial distributions on \(0, \ldots, m\). But we will see that the variance of \(Z_i = B_i/m\) will tend to 0. As in the definition of proportional quantifiers we use fuzzy numbers defined over \([0, 1]\) instead of \([0, m]\), when we normalize the probability distribution \(f(i_1, \ldots, i_K)\) to \([0, 1]^m\) we will obtain a probability distribution whose projections are poisson binomial distributions such that their average converge in probability to the average of the membership degrees of the fuzzy set ‘induced’ by the boolean combination, and their variance tend to 0. Then, as each marginal distribution converges in probability to a constant, by the theorem of the continuous mapping the joint distribution converges in probability to a constant. In practice, this implies that the probability distribution will be more and more concentrated around the average of the boolean combinations as the size of the input sets tends to infinite. As a consequence, we could simply evaluate \(F^A\) computing the value of the proportional fuzzy number used in the definition of \(Q\) over the average of the boolean combinations.

After this summary we will present the proof in full detail.

The next theorem establishes that, in the finite case, quantitative semi-fuzzy quantifiers can be expressed by means of a function of the cardinalities of the boolean combinations of the input sets.
Theorem 11: [21] Theorem 11.32, chapter 11] A semi-fuzzy quantifier \( Q : \mathcal{P}(E)^n \rightarrow \mathbf{I} \) on a finite base set \( E \neq \emptyset \) is quantitative if and only if \( Q \) can be computed from the cardinalities of its arguments and their Boolean combinations, i.e. there exist Boolean expressions \( \Phi_1(Y_1, \ldots , Y_n), \ldots , \Phi_K(Y_1, \ldots , Y_n) \) for some \( K \in \mathbb{N} \), and a mapping \( q : \{0, \ldots , m\}^K \rightarrow \mathbf{I} \) such that
\[
Q(Y_1, \ldots , Y_n) = q(\{\Phi_1(Y_1, \ldots , Y_n)\}, \ldots , \{\Phi_K(Y_1, \ldots , Y_n)\})
\]
for all \( Y_1, \ldots , Y_n \in \mathcal{P}(E) \).

We will also introduce the following notation for denoting the boolean combinations:

Let be \( l_1, \ldots , l_n \in \{0, 1\} \), we define \( \Phi_{l_1, \ldots , l_n}(Y_1, \ldots , Y_n) \) as:
\[
\Phi_{l_1, \ldots , l_n}(Y_1, \ldots , Y_n) = Y_1^{(l_1)} \cap \cdots \cap Y_n^{(l_n)}
\]
where
\[
Y^{(l)} = \begin{cases} Y : & l = 1 \\ \neg Y : & l = 0 \end{cases}
\]

Let us remember we are denoting \(|E| = m\). Then, in the finite case, quantitative semi-fuzzy quantifiers can be expressed by means of a function \( q : \{0, \ldots , m\}^K \rightarrow \mathbf{I} \) depending only of the cardinalities of the boolean combinations of \( Y_1, \ldots , Y_n \). For example, proportional binary semi-fuzzy quantifiers can be defined by means of the boolean combinations \( \Phi_1(Y_1, Y_2) = Y_1 \cap Y_2 \) and \( \Phi_2(Y_1, Y_2) = Y_1 \cap \neg Y_2 \).

Let \( Q : \mathcal{P}(E)^n \rightarrow \mathbf{I} \) be a quantitative semi-fuzzy quantifier \( Q : \mathcal{P}(E)^n \rightarrow \mathbf{I} \) on a finite base set \( E \neq \emptyset \), and let us suppose it can be expressed following expression [4] for some set \( \Phi(Y_1, \ldots , Y_n), \ldots , \Phi_K(Y_1, \ldots , Y_n) \) of boolean combinations and some \( q : \{0, \ldots , m\}^K \rightarrow \mathbf{I} \). For convenience, we will define \( q' : \{0, 1\}^K \rightarrow \mathbf{I} \) such that:
\[
q'(\{\Phi_1(Y_1, \ldots , Y_n)\}, \ldots , \{\Phi_K(Y_1, \ldots , Y_n)\}) = q(\{\Phi_1(Y_1, \ldots , Y_n)\}, \ldots , \{\Phi_K(Y_1, \ldots , Y_n)\})
\]

\( q' : \{0, 1\}^K \rightarrow \mathbf{I} \) simply normalizes \( q \) in the interval of proportions \([0, 1]^A\).

We introduce now the definition of the **poisson binomial distribution**. Let us consider a sequence of \( m \) independent bernoulli trials \( B = P_1, \ldots , P_m \) that are not necessarily identically distributed. Let be \( p_1, \ldots , p_m \) the corresponding probabilities of the independent bernoulli trials. The probability function of the poisson binomial distribution is:
\[
\Pr(K = k) = \sum_{A \in F_k} \prod_{i \in A} p_i \prod_{j \in A^c} (1 - p_j)
\]
where \( F_k \) is the set of all subsets of \( k \) integers that can be selected from \( \{1, 2, 3, \ldots , m\} \).

We now introduce a notation for representing the poisson bernoulli succession \( B = P_1, \ldots , P_m \) with probabilities \( p_1, \ldots , p_m \) by means of a fuzzy set:

**Notation 12:** Let be \( B = P_1, \ldots , P_m \) a poisson bernoulli succession with probabilities \( p_1, \ldots , p_m \). We will denote by \( X^B \in \overline{\mathcal{P}}(E) \) the fuzzy set defined in the following way:
\[
\mu_{X^B}(e_i) = p_i
\]

Under the probabilistic interpretation of the \( \mathcal{F}^A \) QFM, a crisp set \( Y \in \mathcal{P}(E) \) can be interpreted as a realization of a poisson bernoulli succession \( B = P_1, \ldots , P_m \) with probabilities \( p_1, \ldots , p_m \) such that \( \chi_Y(e_i) = P_i \). In this sense,
\[
\Pr(Y) = \prod_{i|e_i \in Y} p_i \prod_{j|e_j \notin Y} (1 - p_j) = \prod_{i|e_i \in Y} (P_i = 1) \prod_{j|e_j \notin Y} (P_j = 0) = m_{X^B}(Y).
\]

Now, we will compute the projection of the probability function used in the definition of the \( \mathcal{F}^A \) QFM for the cardinalities of each possible boolean combination associated to a quantitative semi-fuzzy quantifier \( Q : \mathcal{P}(E)^n \rightarrow \mathbf{I} \). As \( Q \) is quantitative,\(^5\)

---

\(^5\)For simplicity of the notation, we will use \([0, 1]^K\) instead of \([0, \frac{1}{m}, \ldots , \frac{m-1}{m}, 1]^K\).

\(^6\)By \( \chi_Y(e_i) \) we are representing the characteristic function of \( Y \); that is: \( \chi_Y(e_i) = 1 \) if \( e_i \in Y \) and 0 otherwise.
by theorem \( \| I \| \) it can be defined by means of a function \( q : \{0, \ldots, |E|\}^K \rightarrow I \) depending on the cardinalities of the boolean combinations of the input sets \( \{\Phi_1(Y_1, \ldots, Y_n)\}, \ldots, \{\Phi_K(Y_1, \ldots, Y_n)\} \). Then:

\[
F^A(Q)(X_1, \ldots, X_n) = \sum_{Y_1 \in P(E)} \cdots \sum_{Y_n \in P(E)} m_{X_1}(Y_1) \cdots m_{X_n}(Y_n) Q(Y_1, \ldots, Y_n)
\]

\[
= \sum_{(i_1, \ldots, i_K) \in \{0, \ldots, m\}^K} m_{X_1}(Y_1) \cdots m_{X_n}(Y_n) \times
q(\{\Phi_1(Y_1, \ldots, Y_n)\}, \ldots, \{\Phi_K(Y_1, \ldots, Y_n)\})
\]

\[
= \sum_{(i_1, \ldots, i_K) \in \{0, \ldots, m\}^K} q(i_1, \ldots, i_K) \sum_{Y_1, \ldots, Y_n \in P(E)} m_{X_1}(Y_1) \cdots m_{X_n}(Y_n).
\]

Let us denote by

\[
f(i_1, \ldots, i_K) = \sum_{Y_1, \ldots, Y_n \in P(E)} m_{X_1}(Y_1) \cdots m_{X_n}(Y_n)
\]

where \( f(i_1, \ldots, i_K) \) is a probability function. Take into account that \( m_{X_1}(Y_1) \cdots m_{X_n}(Y_n) \) define a probability over \( (Y_1, \ldots, Y_n) \in P(E)^n \), and \( f(i_1, \ldots, i_K) \) simply distributes the probabilities of \( (Y_1, \ldots, Y_n) \in P(E)^n \) over the cardinalities of the \( K \) boolean combinations.

**Theorem 13:** Let \( f(i_1, \ldots, i_K) \) be the probability distribution that is obtained when we compute the probability induced by the \( X_1, \ldots, X_n \in P(E)^n \) fuzzy sets over the cardinalities of the boolean combinations \( \{\Phi_1(Y_1, \ldots, Y_n)\}, \ldots, \{\Phi_K(Y_1, \ldots, Y_n)\} \) following equation 5. The probability projection \( j \) of \( f(i_1, \ldots, i_K) \) will follow a poisson binomial distribution of parameters:

\[
p_j = \mu_{X_1(i_1, j)X_2(i_2, j)\ldots X_n(i_n, j)}(e_1)
\]

\[
\ldots
\]

\[
p_m = \mu_{X_1(i_1, j)X_2(i_2, j)\ldots X_n(i_n, j)}(e_m)
\]

where \( X_1(i_1, j)X_2(i_2, j)\ldots X_n(i_n, j) = \Phi_j(X_1, \ldots, X_n) \) is the \( j \)-th boolean combination.

**Proof.** We will only give an intuitive idea of this result. In appendix A an analytical proof can be consulted. By assumption, the \( F^A \) QFM is interpreting membership grades of the input sets as probabilities, and considering that the independence assumption is always fulfilled between different elements and sets. The probability projection \( j \) of \( f(i_1, \ldots, i_K) \) simply denotes the probability of the different cardinalities of one of these boolean combinations. But the probability of an element \( e_s \) of pertaining to the boolean combination \( \Phi_j(Y_1, \ldots, Y_n) \) is just the probability of \( e_s \) pertaining to every fuzzy set \( X_r(e_s) \) such that \( l_r = 1 \) and non pertaining to every fuzzy set \( X_r(e_s') \) set such that \( l_r = 0 \). As this is fulfilled for every \( e \in E \), the cardinality of the boolean combination follows a poisson binomial distribution with the indicated parameters.

**Proposition 14:** Let \( Q : P(E)^n \rightarrow I \) be a semi-fuzzy quantitative quantifier on a finite base set \( E \neq \emptyset \), \( \Phi_1(Y_1, \ldots, Y_n), \ldots, \Phi_K(Y_1, \ldots, Y_n) \in \mathbb{N} \) boolean combinations, and \( q : \{0, \ldots, m\}^K \rightarrow I \) the corresponding function for which:

\[
Q(Y_1, \ldots, Y_n) = q(\{\Phi_1(Y_1, \ldots, Y_n)\}, \ldots, \{\Phi_K(Y_1, \ldots, Y_n)\})
\]

\[
= \left( \frac{\sum_{i=1}^m \mu_{\Phi_1(X_1, \ldots, X_n)}(e_1)}{m}, \ldots, \frac{\sum_{i=1}^m \mu_{\Phi_K(X_1, \ldots, X_n)}(e_1)}{m} \right)
\]

If \( q' : [0,1]^K \rightarrow I \) is continuous around

\[
\left( \frac{\sum_{i=1}^m \mu_{\Phi_1(X_1, \ldots, X_n)}(e_1)}{m}, \ldots, \frac{\sum_{i=1}^m \mu_{\Phi_K(X_1, \ldots, X_n)}(e_1)}{m} \right)
\]

then the following result will be fulfilled when the size of \( E \) tend to infinite:

\[
\lim_{|E| \rightarrow \infty} F^A(Q)(X_1, \ldots, X_n) = q' \left( \frac{\sum_{i=1}^m \mu_{\Phi_1(X_1, \ldots, X_n)}(e_1)}{m}, \ldots, \frac{\sum_{i=1}^m \mu_{\Phi_K(X_1, \ldots, X_n)}(e_1)}{m} \right)
\]

Before proving proposition 14 we would like to make some appointments about the applicability of the result. In general, we always could find a \( q' \) continuous around \( \left( \frac{\sum_{i=1}^m \mu_{\Phi_1(X_1, \ldots, X_n)}(e_1)}{m}, \ldots, \frac{\sum_{i=1}^m \mu_{\Phi_K(X_1, \ldots, X_n)}(e_1)}{m} \right) \) such that previous result would be applicable. But in choosing a 'proportional expression' for \( q' \), we are indicating that the types of fuzzy quantifiers in which
we are mainly interested are ‘proportional quantifiers’. In practical applications, support functions associated to proportional quantifiers are generally defined by means of ‘smooth’ fuzzy numbers over \([0, 1]\), which guarantees a good approximation when the size of the referential set is sufficiently large.

**Proof.** Let \(f(\mathbf{i}_1, \ldots, \mathbf{i}_K)\) be the probability distribution that is obtained when we compute the probability induced by the \(X_1, \ldots, X_n \in \mathcal{P}(E)^n\) fuzzy sets over the cardinalities of the boolean combinations \(|\Phi_1(Y_1, \ldots, Y_n)|, \ldots, |\Phi_K(Y_1, \ldots, Y_n)|\).

We know that the probability projection \(F^\mathcal{A}(\mathbf{i}_s)\) follows a poisson binomial distribution of parameters

\[
p_i^j = \mu_{X_1(i_1), \ldots, X_n(i_n)}(e_1)
\]

\[
\vdots
\]

\[
p_m^j = \mu_{X_1(i_1), \ldots, X_n(i_n)}(e_m)
\]

Moreover,

\[
F^\mathcal{A}(Q)(X_1, \ldots, X_n) = \sum_{(i_1, \ldots, i_K) \in \{0, \ldots, m\}^K} q(i_1, \ldots, i_K) \sum_{Y_1, \ldots, Y_n \in \mathcal{P}(E)} m_{X_1}(Y_1) \ldots m_{X_n}(Y_n)
\]

\[
= \sum_{(i_1, \ldots, i_K) \in \{0, \ldots, m\}^K} q'(\frac{i_1}{m}, \ldots, \frac{i_K}{m}) f(i_1, \ldots, i_K)
\]

Let \(f^\prime : [0, 1]^K \rightarrow \mathbf{I}\) be probability distribution defined by:

\[
f^\prime(s_1, \ldots, s_K) = f(m \times i_1, \ldots, m \times i_K)
\]

that normalizes \(f\) in the interval \([0, 1]^K\). Then,

\[
F^\mathcal{A}(Q)(X_1, \ldots, X_n) = \sum_{(i_1, \ldots, i_K) \in \mathcal{P}(E)} q'(\frac{i_1}{m}, \ldots, \frac{i_K}{m}) f^\prime(\frac{i_1}{m}, \ldots, \frac{i_K}{m})
\]

As we are normalizing \(f\) by \(m\), the corresponding \(F^\mathcal{A}(\mathbf{i}_s)\) projection of \(f^\prime\) will follow a probability distribution such that:

\[
\text{average } (f^\prime) = \frac{\text{average } (f^\prime)}{m} = \frac{\sum_{i=1}^m p_i^j}{m}
\]

\[
\text{var } (f^\prime) = \frac{1}{m^2} \text{var } (f^\prime) = \frac{1}{m^2} \sum_{i=1}^m p_i^j (1 - p_i^j)
\]

but when \(m \rightarrow \infty\) the variance tends to 0.

And as the variance tends to 0, \(f^\prime \rightarrow \sum_{i=1}^m p_i^j\), and as \(q(s_1, \ldots, s_K)\) is continuous around \((\sum_{i=1}^m p_i^1, \ldots, \sum_{i=1}^m p_i^K)\), by continuous mapping theorem\(\textsuperscript{7}\):

\[
\lim_{m \rightarrow \infty} \sum_{(i_1, \ldots, i_K) \in \mathcal{P}(E)} q'(\frac{i_1}{m}, \ldots, \frac{i_K}{m}) f(\frac{i_1}{m}, \ldots, \frac{i_K}{m}) \rightarrow q\left(\sum_{i=1}^m p_i^1, \ldots, \sum_{i=1}^m p_i^K\right)
\]

\[
= q'\left(\sum_{i=1}^m \mu_{\Phi_1(X_1, \ldots, X_n)}, \ldots, \sum_{i=1}^m \mu_{\Phi_K(X_1, \ldots, X_n)}\right).
\]

\(\square\)

This result guarantees that the \(\mathcal{F}^\mathcal{A}\) QFM converges to the Zadeh’s model for unary proportional and binary proportional quantifiers when the size of the referential set tends to infinite and the intersection is modelled by means of the \textit{product tnorm} in the proportional case, as these quantifiers basically depend on\(\textsuperscript{8}\):

\[
q : \frac{|Y_1 \cap Y_2|}{m} \rightarrow \mathbf{I} \quad : \text{unary quantifiers}
\]

\[
q : \left(\frac{|Y_1 \cap Y_2|}{m}, \frac{|Y_1 \cap Y_3|}{m}\right) \rightarrow \mathbf{I} \quad : \text{binary quantifiers}
\]

\(\textsuperscript{7}\)Take into account that, as the variance tends to 0, by the \textit{Chebyshev inequality} we always could find an interval around \(\text{average } (f^\prime)\) as small and containing a probability mass as high as desired for any \(j\). This will allow to put as much probability around \((\sum_{i=1}^m p_i^1, \ldots, \sum_{i=1}^m p_i^K)\) as we wanted, where \(q'\) is continuous by hypothesis.

\(\textsuperscript{8}\)Take into account that for proportional quantifiers \(\frac{|Y_1 \cap Y_2|}{|Y_1|} = \frac{|Y_1 \cap Y_2|}{m} / \left(\frac{|Y_1 \cap Y_2|}{m} + \frac{|Y_1 \cap Y_3|}{m}\right)\). In this case, the \(\mathcal{F}^\mathcal{A}\) QFM will converge to

\[
f_Q \left(\sum_{e \in \mathcal{P}(E) \setminus \mu_{X_1}(e) \mu_{X_2}(e)} \frac{|Y_1 \cap Y_2|}{\sum_{e \in \mathcal{P}(E)} \mu_{X_1}(e) \mu_{X_2}(e)}\right).
\]
As we introduced below, the normalization by \( m \) is coherent with proportional linguistic quantifiers, that are generally defined by means of ‘smooth’ fuzzy numbers in \([0, 1]\). In these situations, the result guarantees that the probability of the projections of \( f'(s_1, \ldots, s_s) \) will concentrate around the average of the projections as we increase the size of the referential set. As a consequence, if the variation of the fuzzy number that supports the linguistic quantifier is small around this average, we could expect a good approximation of the \( F^A \) QFM using ?? when the size of the referential set tends to infinite.

VII. QUALITY OF THE CONVERGENCE AND MONTE CARLO APPROXIMATION OF THE \( F^A \) QFM

In section VIII we will present some computational exact implementations of the \( F^A \) QFM for evaluating the most common linguistic quantifiers. We advance that the complexity of the exact implementation of the quantifiers, \( O(m^2) \) for unary quantifiers, \( O(m^3) \) for binary proportional quantifiers and \( O(m^{r+1}) \) in the general case, being \( r \) the number of boolean combinations that are necessary for the definition of the semi-fuzzy quantifier. For some applications, and specifically for quantifiers depending on a high value of \( r \), this complexity could be too high for applying the model to big fuzzy sets.

One consequence of the result of the previous section is that the \( F^A \) QFM can be approximated in linear time for fuzzy sets containing a sufficiently large number of elements. But we do not know if the proposed approximation is sufficiently accurate for problems where the exact implementation could not be applied due to its computational demands. We will make now a deeper analysis about the applicability of the results of previous section for approximating the \( F^A \) QFM, connecting them with a proposal to use a Monte Carlo simulation. Let us consider the following example:

**Example 15:** Let us consider a fuzzy set \( X = \{0.5/e_1, \ldots, 0.5/e_m\} \). In this situation, the probability distribution subjacent to the \( F^A \) QFM is a binomial distribution with parameters \((m, 0.5)\). Let us consider a trapezoidal function \( T_{0.5,0.6,\infty,\infty}(x) \) and the unary semi-fuzzy quantifier defined as \( Q(Y) = T_{0.5,0.6,\infty,\infty}(|Y|) \). The following table compares the result of the application of the \( F^A \) QFM with its approximation by means of the Zadeh’s model:

| \( m, X \) | \( F^A(X) \) | \( f_Q(X) \) |
|-----------|------------|-------------|
| 50, \( X = \{0.5, \ldots, 0.5\}_{50} \) | 0.260 | 0 |
| 100, \( X = \{0.5, \ldots, 0.5\}_{100} \) | 0.195 | 0 |
| 500, \( X = \{0.5, \ldots, 0.5\}_{500} \) | 0.089 | 0 |

Previous example proves that, even for a large fuzzy set containing 500 elements, the error of the approximation is not negligible for a semi-fuzzy quantifier defined by means of a fuzzy number that seems very plausible from a practical viewpoint. Moreover, the error will be greater for a semi-fuzzy quantifier defined by means of a fuzzy number with a higher slope.

We will now introduce a theorem applicable to the poisson binomial distribution [6, page 263].

**Theorem 16:** Central limit theorem applied to Bernoulli variables. Let \( X_1, \ldots, X_m \) be independent random variables, each \( X_i \) following a Bernoulli distribution with parameter \( p_i \). Moreover, let us suppose that the infinite sum \( \sum_{i=1}^{\infty} p_i (1 - p_i) \) is divergent and let \( Y_m \) be

\[
Y_m = \frac{\sum_{i=1}^{m} X_i - \sum_{i=1}^{m} p_i}{(\sum_{i=1}^{m} p_i q_i)^{1/2}}.
\]

Then

\[
\lim_{n \to \infty} \Pr(Y_m \leq x) = \Phi(x)
\]

where \( \Phi(x) \) is the standard normal distribution function.

In practical situations, this result allow us to approximate a poisson binomial distribution by a normal distribution when the variance of the distribution is high (take into account that we are interpreting the cardinality of a fuzzy set as a poisson binomial distribution). Cases of a low variance for poisson binomial distributions with a high number of parameters will be associated to situations in which most parameters are really close to 0 or 1. In these cases, the approximation by means of the normal distribution will be poor, but the probability distribution will be extremely concentrated around the average, which will guarantee an even better approximation by means of Montecarlo.

\(^9\)For many quantifiers, results of the \( F^A \) QFM and of the Zadeh’s model will be extremely close even for small fuzzy sets. There are two main reasons for that. Once is that the variance of the probability projections associated to the different boolean combinations was very low and as a consequence, that the probability distributions would be very concentrated around the average. The other situation is that the fuzzy number used in the definition of the semi-fuzzy quantifier was approximately linear in the area in which much of the probability is concentrated. In this situation, the symmetry of the normal distribution (to which the poisson binomial distribution converges) will cause that the result of the evaluation will be really close to the result of the Zadeh’s model.
Let us consider again a fuzzy set $X = \{0.5/e_1, \ldots, 0.5/e_m\}$ whose underlying probability distribution following the $F^A$ QFM interpretation is a binomial distribution with parameters $(m, 0.5)$. We will compute the confidence intervals for the 0.95 and 0.99 probability mass approximating the underlying probability of the $F^A$ QFM by means of a normal distribution.

**Example 17:** The following table shows the confidence intervals for the underlying probability distribution of a fuzzy set $X = \{0.5/e_1, \ldots, 0.5/e_m\}$, following the $F^A$ QFM interpretation:

| $m$  | $X$ | $\frac{X}{m}$ | $\frac{X}{m}$ |
|------|-----|---------------|---------------|
| 50   | 25  | (0.36, 0.64)  | (0.32, 0.68)  |
| 100  | 50  | (0.40, 0.60)  | (0.37, 0.63)  |
| 1000 | 500 | (0.47, 0.53)  | (0.45, 0.54)  |
| 10000| 5000| (0.49, 0.51)  | (0.49, 0.51)  |

Previous example shows that the probability distribution is really concentrated around the average for medium size fuzzy sets. In previous example, we have chosen the binomial distribution with parameter 0.5 as it is the highest variance distribution in the family of poisson binomial distributions. Take into account that for a poisson binomial distribution $B$, $\text{var}(B) = \sum_{i=1}^{m} p_i (1 - p_i)$, and that the maximum of $p_i (1 - p_i)$ is obtained for $p_i = 0.5$.

The idea of the Monte Carlo simulation is simply to generate, for each $X_i$, a random binary vector using a Bernoulli trial of probability $\mu_{X_i}(j)$ for each $e_j$. Previous example indicates that the $f^p(i_s)$ projections of $f^p$ would be very concentrated around the average when the size of the referential set contains a large number of elements, which will allow to expect a really good approximation of the $F^A$ QFM by means of a Monte Carlo simulation. Moreover, a Monte Carlo simulation can be easily parallelized. In section VIII-C, the algorithm for unary quantifiers is presented. The extension to higher arity quantifiers is trivial.

VIII. EFFICIENT IMPLEMENTATION OF THE $F^A$ MODEL

For quantitative quantifiers is possible to develop polynomial algorithms for the $F^A$ DFS. Let us remember that the class of quantitative quantifiers is composed of the semi-fuzzy quantifiers that are invariant under automorphisms [21, section 4.13], and that they can be expressed as a function of the cardinalities of their arguments and their boolean combinations. The class of quantitative quantifiers include the most interesting ones for applications, and in particular the common absolute, proportional and comparative quantifiers.

A. Quantitative unary quantifiers

Let $Q : \mathcal{P}(E) \to I$ be an unary semi-fuzzy quantifier defined over a referential set $E^m = \{e_1, \ldots, e_m\}$. Quantitative unary semi-fuzzy quantifiers can always be expressed by means of a function $q : \{0, \ldots, |E|\} \to I$ (theorem [11]); that is, a function that goes from cardinality values in $I$. In this way, there exists $q$ such that $q(j) = Q(Y_j)$ where $Y_j \in \mathcal{P}(E)$ is an arbitrary set of cardinality $j (|Y_j| = j)$.

Let $X \in \mathcal{P}(E)$ be a fuzzy set. Then,

$$F^A(Q)(X) = \sum_{Y \in \mathcal{P}(E)} m_X(Y) Q(Y)$$

$$= \sum_{Y \in \mathcal{P}(E), \ |Y| = 0} m_X(Y) Q(Y) + \ldots + \sum_{Y \in \mathcal{P}(E), \ |Y| = m} m_X(Y) Q(Y)$$

$$= \sum_{Y \in \mathcal{P}(E), \ |Y| = 0} m_X(Y) q(0) + \ldots + \sum_{Y \in \mathcal{P}(E), \ |Y| = m} m_X(Y) q(m)$$

$$= \sum_{j=0}^{m} \text{Pr}(\text{card}_X = j) q(j)$$

The algorithm we will present uses the fact that it is possible to compute the probability $\text{Pr}_{E^m}(\text{card}_X = j), j = 0, \ldots, m$ for a referential set $E^m$ of $m$ elements using the probabilities $\text{Pr}_{E^{m-1}}(\text{card}_{X^{m-1}} = j), j = 0, \ldots, m - 1$ where $E^{m-1} = \{e_1, \ldots, e_{m-1}\}$ and $X^{m-1}$ is the projection of $X$ over $E^{m-1}$ (that is, the fuzzy set $X$ without the element $e_m$). In this way, it is easy to develop a recursive function for computing the probabilities of the cardinalities in $E^m$. 

In the case of a referential set of one element \((E^1 = \{e_1\})\) the probabilities of the cardinalities of a fuzzy set \(X \in \mathcal{P}(E^1)\) are simply:

\[
\begin{align*}
\Pr(\text{card}_X = 0) &= m_X(\emptyset) = 1 - \mu_X(e_1) \\
\Pr(\text{card}_X = 1) &= m_X(\{e_1\}) = \mu_X(e_1)
\end{align*}
\]

Let us suppose now a referential set of \(m + 1\) elements \((E^{m+1} = \{e_1, \ldots, e_m, e_{m+1}\})\), let \(X \in \mathcal{P}(E^{m+1})\) be a fuzzy set on \(E^{m+1}\), \(E^m = \{e_1, \ldots, e_m\}\) and \(X^{E^m} \in \mathcal{P}(E^m)\) the projection of \(X\) in \(E^m\); that is, \(\mu_{X^{E^m}}(e_j) = \mu_X(e_j), 1 \leq j \leq m\). Moreover, let us suppose we know the probabilities of the cardinalities associated to \(X^{E^m}\) \((\Pr(\text{card}_{X^{E^m}} = 0), \ldots, \Pr(\text{card}_{X^{E^m}} = m))\).

Now, we will compute the probabilities of \(X\) using the probabilities of the cardinalities on \(X^{E^m}\):  

**Case 1:** \(\Pr(\text{card}_X = 0)\)

\[
\Pr(\text{card}_X = 0) = \sum_{Y \in \mathcal{P}(E^{m+1}) | |Y| = 0} m_X(Y) = m_X(\emptyset) = (1 - \mu_X(e_1)) \cdots (1 - \mu_X(e_m)) (1 - \mu_X(e_{m+1})) = \mu_X(e_1) \cdots \mu_X(e_m) \mu_X(e_{m+1}) = \Pr(\text{card}_{X^{E^m}} = 0) (1 - \mu_X(e_{m+1}))
\]

**Case 2:** \(\Pr(\text{card}_X = m + 1)\)

\[
\Pr(\text{card}_X = m + 1) = \sum_{Y \in \mathcal{P}(E^{m+1}) | |Y| = m+1} m_X(Y) = m_X(E^{m+1}) = \mu_X(e_1) \cdots \mu_X(e_m) \mu_X(e_{m+1}) = \Pr(\text{card}_{X^{E^m}} = m) \mu_X(e_{m+1})
\]

**Case 3:** \(\Pr(\text{card}_X = j), 0 < j < m + 1\)

\[
\Pr(\text{card}_X = j) = \sum_{Y \in \mathcal{P}(E^{m+1}) | |Y| = j} m_X(Y) = \sum_{Y \in \mathcal{P}(E^{m+1}) | |Y| = j \wedge e_{m+1} \notin Y} m_X(Y) + \sum_{Y \in \mathcal{P}(E^{m+1}) | |Y| = j \wedge e_{m+1} \in Y} m_X(Y)
\]

\[
= \sum_{Y \in \mathcal{P}(E^{m}) | |Y| = j} m_{X^{E^m}}(Y) (1 - \mu_X(e_{m+1})) + \cdots + \sum_{Y \in \mathcal{P}(E^{m}) | |Y| = j} m_X(Y) \mu_X(e_{m+1})
\]

\[
= \Pr(\text{card}_{X^{E^m}} = j) (1 - \mu_X(e_{m+1})) + \Pr(\text{card}_{X^{E^m}} = j - 1) \mu_X(e_{m+1})
\]

Previous computations are summarized in expression ?? . In algorithm ?? the code for evaluating \(\mathcal{F}^A(Q)(X)\) is presented. Complexity of the algorithm is \(O(n^2)\).

\[
\Pr(\text{card}_X = j) = \begin{cases} 
\Pr(\text{card}_{X^{E^m}} = 0) (1 - \mu_X(e_{m+1})) & : j = 0 \\
\Pr(\text{card}_{X^{E^m}} = j) (1 - \mu_X(e_{m+1})) + \Pr(\text{card}_{X^{E^m}} = j - 1) \mu_X(e_{m+1}) & : 1 \leq j \leq m \\
\Pr(\text{card}_{X^{E^m}} = m) \mu_X(e_{m+1}) & : j = m + 1
\end{cases}
\]

**B. Conservative binary quantifiers**

In this section we will present the algorithm for evaluating conservative binary quantifiers ?? , which includes proportional quantitative quantifiers as a particular case. The strategy we are going to detail can be easily generalized for implementing other kinds of quantitative quantifiers.

A semi-fuzzy conservative quantitative quantifier \(Q(Y_1, Y_2)\) depends on the cardinalities of \(|Y_1|\) and \(|Y_1 \cap Y_2|\); that is, there exists a function \(q : \{0, \ldots, m\}^2 \to I\) such that:

\[
Q(Y_1, Y_2) = q(|Y_1|, |Y_1 \cap Y_2|)
\]
Algorithm 1: Algorithm for computing unary quantitative quantifiers $F^\mathcal{A}(\mathcal{Q})(X)$

Input: The fuzzy set $X[0, \ldots, m-1]$, $m \geq 1$, and a quantitative unary semi-fuzzy quantifier $q: \{0, \ldots, m\} \to I$.

Output: The result of the quantifier.

1: // Assume all vector elements are initialized to zero */
2: pr\_aux\_i = 0
3: pr\_aux\_i\_minus\_1 = 0
4: pr[i] ← [0, ..., m]
5: result ← 0
6: pr[0] ← 1
7: for $j ← 0; j < m; j++$ do
8:   pr\_aux\_i = pr[0]
9:   for $i = 1; i ≤ j; i++$ do
10:     pr\_aux\_i = pr[0] × pr\_aux\_i
11:   result ← result + pr[j] × q(j)
12: return result

for all $Y_1, Y_2 \in \mathcal{P}(E)$, $X_1, X_2 \in \mathcal{P}(E)$ be two fuzzy sets. By

$$
\Pr(\text{card}_{X_1,X_2} = (j,k)) = \sum_{Y_1 \in \mathcal{P}(E)} \sum_{Y_2 \in \mathcal{P}(E)} m_{X_1}(Y_1) m_{X_2}(Y_2), 0 ≤ j, k ≤ m
$$

we will denote the probability of choosing a pair of representatives $Y_1, Y_2 \in \mathcal{P}(E)$ of $X_1, X_2 \in \mathcal{P}(E)$ such that $|Y_1| = j$ and $|Y_2| = k$. It should be noted that for $k > j$ $\Pr(\text{card}_{X_1,X_2} = (j,k)) = 0$.

Let $X_1, X_2 \in \mathcal{P}(E^m)$ be two fuzzy sets over $E^m = \{e_1, \ldots, e_m\}$. And let us suppose we know the probabilities:

$$
\Pr(\text{card}_{X_1,X_2} = (j,k))
$$

for all $j, k$ such that $0 ≤ j, k ≤ m$. Let us suppose now we add an element $e_{m+1}$ to the referential. That is, the new referential set is $E^{m+1} = \{e_1, \ldots, e_{m+1}\}$. And let $X_1', X_2' \in \mathcal{P}(E^{m+1})$ be two fuzzy sets in $E^{m+1}$ resulting of adding $e_{m+1}$. That is, $(X_1')^{E^m} = X_1, (X_2')^{E^m} = X_2$, where by $(Y)^E$ we are denoting the projections of $Y$ over $E$.

By definition of the $F^\mathcal{A}$ DFS, belongniness of $e_{m+1}$ to the set $X_1', i = 1, 2$ is an event of probability $\mu_{X_1'}(e_{m+1})$ and this probability is independent of the belongniness of other elements. Then, if the cardinality of $X_1, X_2$ were $(j,k)$ and it would happen that $e_{m+1} \in X_1'$ and $e_{m+1} \in X_2'$ then the cardinality of $X_1' \cap X_2'$ would be $(j + 1, k + 1)$.

Let $0 ≤ j, k ≤ m$ be arbitrary indexes and let us consider the probability $\Pr(\text{card}_{X_1,X_2} = (j,k))$. When we include the element $e_{m+1}$ the probability of $e_{m+1}$ contributes to the probability $\Pr(\text{card}_{X_1',X_2'} = (j,k))$ with:

$$
(1 - \mu_{X_1'}(e_{m+1})) (1 - \mu_{X_2'}(e_{m+1})) \Pr(\text{card}_{X_1,X_2} = (j,k)) + (1 - \mu_{X_1'}(e_{m+1})) \mu_{X_2'}(e_{m+1}) \Pr(\text{card}_{X_1,X_2} = (j,k))
$$

That is, if we know that the cardinality $\text{card}_{X_1,X_2}$ is $(j,k)$ then the cardinality $\text{card}_{X_1',X_2'}$ would be $(j, k)$ with probability $(1 - \mu_{X_1'}(e_{m+1}))$. It should be noted that if $e_{m+1} \notin X_1'$ and $e_{m+1} \notin X_2'$ then $e_{m+1} \notin X_1' \cap X_2'$.

Similarly, the contribution to $\Pr(\text{card}_{X_1',X_2'} = (j+1,k))$ will be:

$$
\mu_{X_1'}(e_{m+1}) (1 - \mu_{X_2'}(e_{m+1})) \Pr(\text{card}_{X_1,X_2} = (j,k))
$$

That is, as the cardinality $\text{card}_{X_1,X_2}$ is $(j,k)$ then the cardinality $\text{card}_{X_1',X_2'}$ will be $(j + 1, k)$ with probability $\mu_{X_1'}(e_{m+1}) (1 - \mu_{X_2'}(e_{m+1}))$.

And the contribution to the probability $\Pr(\text{card}_{X_1',X_2'} = (j+1,k+1))$ will be:

$$
\mu_{X_1'}(e_{m+1}) \mu_{X_2'}(e_{m+1}) \Pr(\text{card}_{X_1,X_2} = (j,k))
$$

That is, as the cardinality $\text{card}_{X_1,X_2}$ is $(j,k)$ then the cardinality $\text{card}_{X_1',X_2'}$ will be $(j + 1, k + 1)$ with probability $\mu_{X_1'}(e_{m+1}) \mu_{X_2'}(e_{m+1})$.

Using previous expressions a polynomial algorithm can be developed to evaluate conservative semi-fuzzy quantifiers (table 2). Complexity of the algorithm is $O(n^3)$.
Algorithm 2: Algorithm for computing binary quantitative conservative quantifiers $F^A(Q)(X_1,X_2)$.

Input: The fuzzy sets $X_1[0,\ldots,m-1]$ and $X_2[0,\ldots,m-1]$, $m \geq 1$, and a binary quantitative conservative semi-fuzzy quantifier $q : \mathbb{N}^2 \to I$.
Output: The result of the quantifier.

```
/* Assume all vector elements are initialized to zero */
card ← [0, ..., m][0, ..., m]
card_aux ← [0, ..., m][0, ..., m]
result ← 0
i ← 0
card[0,0] ← 1
while i < m do
    clear(card_aux)
    v_i,00 ← (1 - X_1[i]) × (1 - X_2[i])
v_i,01 ← (1 - X_1[i]) × X_2[i]
v_i,10 ← X_1[i] × (1 - X_2[i])
v_i,11 ← X_1[i] × X_2[i]
    for j ← 0; j ≤ i; j++ do
        card_aux[j, k] ← card_aux[j, k] + (v_i,00 + v_i,01) × card[j, k]
        card_aux[j + 1, k] ← card_aux[j + 1, k] + v_i,00 × card[j, k]
        card_aux[j + 1, k + 1] ← card_aux[j + 1, k + 1] + v_i,11 × card[j, k]
    for j ← 0; j ≤ m; j++ do
        for k ← 0; k ≤ j; k++ do
            result ← result + card[j,k] ∗ q(j,k)
return result
```

Algorithm 3: Monte Carlo Approximation for absolute unary quantifiers $F^A(Q)(X)$.

Input: The fuzzy set $X[0,\ldots,m-1]$, $m \geq 1$, the number of iterations num_simu, and a quantitative unary semi-fuzzy quantifier $q : \{0, \ldots, m\} \to I$.
Output: The result of the quantifier.

```
/* Assume all vector elements are initialized to zero */
v_simu ← [0, ..., m]
for i ← 0; i < num_simu; i++ do
    for j ← 0; j < m; j++ do
        /* Simulation of a bernoulli trial with probability X[j] */
        x_bino[j] ← BernoulliTrial(X[j])
        v_simu[Sum(x_bino)]++
    /* Normalization to define a probability */
    v_simu ← v_simu/num_simu
result ← Sum(q^T ∗ v_simu)
return result
```

It is not difficult to generalize the strategy we have presented to other quantifiers. For example, let us consider the case of a ternary comparative quantifier (e.g., “the number of brilliant investors that earn high salaries is about twice the number of brilliant investors that earn low salaries”). For this example, the semi-fuzzy quantifier will follow the expression $Q(Y_1,Y_2,Y_3) = q(\{Y_1 \cap Y_2, |Y_1 \cap Y_3\})$ where $q : \{0, \ldots, m\}^2 \to I$ is the fuzzy number we use to model ‘about twice’. As this quantifier only depends on two boolean combinations, a binary probability matrix will be enough to compute and update the probabilities of the cardinalities. In this way, the complexity of the resulting algorithm will be again $O(n^3)$.

In the general case, if a quantitative semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \to I$ depends on $r$ boolean combinations its complexity will be $O(m^{r+1})$, one iteration to go through the input vectors and $r$ to go through the probability matrix of cardinalities.

C. Monte Carlo Approximation

Monte Carlo simulation permits the approximation of the $F^A$ QFM when efficiency restrictions do not permit the use of the exact implementations presented in previous sections. The idea of the Monte Carlo simulation is simply to generate, for each $X_e$, a random binary vector using a Bernoulli trial of probability $\mu_X(e)$ for each $e$. The code for the Monte Carlo implementation of the $F^A$ model for an unary quantifier can be seen in table 3. The extension to higher arity quantifiers is trivial, by simply generating a random binary vector for each $X_e$. The Monte Carlo approximation can be parallelized by simply dividing the number of simulations between different processors.
IX. Conclusions

In this paper we have presented several relevant results about the $F^A$ QFM. First, we summarized some of the most relevant properties fulfilled by this model, in order to give a comprehensive and integrative summary of its behavior. After that, we introduced a convergence result that guarantees that, in the limit case, the model converges to the Zadeh’s model for semi-fuzzy quantifiers defined by means of proportional continuous fuzzy numbers. Moreover, this result is more general than the specific convergence to the Zadeh’s model, being applicable to every proportional quantitative quantifier. For sufficiently big fuzzy sets, this will allow to approximate the $F^A$ QFM in linear time.

However, the rate of convergence could be too slow to make this approximation useful in most applications. For this reason, we also provided the exact computational implementation for some of the most common quantifiers (unary and proportional quantitative quantifiers), introducing a scheme that can be easily extended to other types of quantifiers. Complexity of the exact implementation is $O\left(n^{r+1}\right)$, being $r$ the number of boolean combinations that are involved in the definition of the semi-fuzzy quantifier.

Finally, the convergence result has a strong implication. The underlying probability of the $F^A$ QFM will concentrate around the average of the boolean combinations necessary to define the semi-fuzzy quantifier, as we increase the number of elements of the input fuzzy sets. This property was used to propose a Monte Carlo approximation of the $F^A$ QFM that can be used when the complexity of the exact implementation is too elevate to compute an exact solution.

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APPENDIX A

APPENDIX: COMPUTATION OF THE PROBABILITY OF THE PROJECTIONS OF THE BOOLEAN COMBINATIONS

In this section we will develop an analytic proof to show that the probability projection \( j \) of \( f(i_1, \ldots, i_K) \) follows a binomial Poisson distribution.

For developing the proof, we will need to introduce the complete definition of some of the axioms of the DFS framework and of its derived properties.

The next definition allows the construction of a new semi-fuzzy quantifier that simply permutes the arguments in the input:

**Definition 18 (Argument permutations):** \([21]\) Definition 4.13] Let \( Q : P(E)^n \rightarrow I \) be a semi-fuzzy quantifier and \( \beta : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \) a permutation. By \( Q\beta : P(E)^n \rightarrow I \) we denote the semi-fuzzy quantifier defined by:

\[
Q\beta (Y_1, \ldots, Y_n) = Q(Y_{\beta(1)}, \ldots, Y_{\beta(n)})
\]

for all \( Y_1, \ldots, Y_n \in P(E) \). In the case of fuzzy quantifiers \( \tilde{Q}\beta : \bar{P}(E)^n \rightarrow I \) is defined analogously.

The next definition will allow us to rewrite permutations as a combination of transpositions:

**Definition 19 (Transposition):** \([21]\) Definition 4.14] For all \( n \in \mathbb{N} (n > 0) \) and \( i, j \in \{1, \ldots, n\} \), the transposition \( \tau_{i,j} : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \) is defined as:

\[
\tau_{i,j} (k) = \begin{cases} 
  i & : k = j \\
  j & : k = i \\
  k & : k \neq j \land k \neq i
\end{cases}
\]

for all \( k \in \{1, \ldots, n\} \). Moreover, by \( \tau_i \) we will denote the transposition \( \tau_{i,n} \) (that interchanges positions \( i \) and \( n \)). It should be noted that \( \tau_{i,n} = \tau_i \circ \tau_j \circ \tau_i \).

We also can apply \([19]\) to fuzzy and semi-fuzzy quantifiers:

**Definition 20 (Argument transpositions):** Let \( Q : P(E)^n \rightarrow I \) be a semi-fuzzy quantifier, \( n > 0 \). By \( Q\tau_i : P(E)^n \rightarrow I \) we denote the semi-fuzzy quantifier defined by:

\[
Q\tau_i (Y_1, \ldots, Y_{i-1}, Y_i, Y_{i+1}, \ldots, Y_n) = Q(Y_1, \ldots, Y_{i-1}, Y_n, Y_{i+1}, \ldots, Y_i)
\]

for all \( Y_1, \ldots, Y_n \in P(E) \). In the case of semi-fuzzy quantifiers \( \tilde{Q}\tau_i : \bar{P}(E)^n \rightarrow I \) is defined analogously.

The DFS axiomatic framework guarantees the adequate generalization of argument transpositions:

**Theorem 21:** \([21]\) Theorem 4.16] Every DFS \( F \) is compatible with argument transpositions, i.e. for every semi-fuzzy quantifier \( Q : P(E)^n \rightarrow I \) \( i \in \{1, \ldots, n\} \),

\[
F(Q\tau_i) = F(Q) \tau_i
\]

Note that as permutations can be expressed as compositions of transpositions, every DFS \( F \) also commutes with permutations. We will also need to introduce the full definition of external and internal negation.

**Definition 22 (External negation):** \([21]\) Definition 3.8] The external negation of a semi-fuzzy quantifier \( Q : P(E)^n \rightarrow I \) is defined by

\[
(\overline{\neg}Q)(Y_1, \ldots, Y_n) = \overline{\neg}(Q(Y_1, \ldots, Y_n))
\]

for all \( Y_1, \ldots, Y_n \in P(E) \). The definition of \( \overline{\neg}Q : \bar{P}(E) \rightarrow I \) in the case of fuzzy quantifiers \( \tilde{Q} : \bar{P}(E) \rightarrow I \) is analogous\(^{[4]}\).

The next theorem expresses that every DFS correctly generalizes the external negation property:

**Theorem 23:** \([21]\) Theorem 4.20] Every DFS \( F \) is compatible with the formation of negations. Hence if \( Q : P(E)^n \rightarrow I \) is a semi-fuzzy quantifier then \( F(\overline{\neg}Q) = \overline{\neg}F(Q) \).

The internal negation or antonym of a semi-fuzzy quantifier is defined as:

**Definition 24 (Internal negation/antonym):** \([21]\) Definition 3.9] Let a semi-fuzzy quantifier \( Q : P(E)^n \rightarrow I \) of arity \( n > 0 \) be given. The internal negation \( Q\overline{} : P(E)^n \rightarrow I \) of \( Q \) is defined by

\[
Q\overline{} (Y_1, \ldots, Y_n) = Q(Y_1, \ldots, \neg Y_n)
\]

for all \( Y_1, \ldots, Y_n \in P(E) \). The internal negation \( \overline{Q}\overline{} : \bar{P}(E)^n \rightarrow I \) of a fuzzy quantifier \( \tilde{Q} : \bar{P}(E)^n \rightarrow I \) is defined analogously, based on the given fuzzy complement \( \overline{\neg} \).

The next theorem expresses that every DFS correctly generalizes the internal negation property:

\(^{[4]}\)The reasonable choice of the fuzzy negation \( \overline{\neg} : I \rightarrow I \) is the induced negation of the QFM.
Theorem 25: [21] Theorem 4.19] Every DFS \( \mathcal{F} \) is compatible with the negation of quantifiers. Hence if \( Q : \overline{P}(E)^n \to \mathbb{I} \) is a semi-fuzzy quantifier, then \( \mathcal{F}(Q\neg) = \mathcal{F}(Q)\overline{\neg} \).

We will now show the necessary definitions to establish the compatibility with unions and intersections of quantifiers:

Definition 26 (Union quantifier): [21] Definition 3.12] Let a semi-fuzzy quantifier \( Q : \mathcal{P}(E)^n \to \mathbb{I} \) of arity \( n > 0 \) be given. We define the fuzzy quantifier \( Q_\cup : \mathcal{P}(E)^n+1 \to \mathbb{I} \) as

\[
Q \cup (Y_1, \ldots, Y_n, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})
\]

for all \( Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E) \). In the case of fuzzy quantifiers \( \overline{Q}\cup \) is defined analogously, based on a fuzzy definition of \( \overline{\cup} \).

Analogously, the definition of the intersection of quantifiers is:

Definition 27 (Intersection quantifier): Let \( Q : \mathcal{P}(E)^n \to \mathbb{I} \) a semi-fuzzy quantifier, \( n > 0 \), be given. We define the semi-fuzzy quantifier \( Q \cap : \mathcal{P}(E)^n+1 \to \mathbb{I} \) as

\[
Q \cap (Y_1, \ldots, Y_n, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cap Y_{n+1})
\]

for all \( Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E) \). In the case of fuzzy quantifiers \( \overline{Q}\cap \) is defined analogously, based on a fuzzy definition of \( \overline{\cap} \).

Theorem 28: [21] sections 3.9 and 4.9] Let \( Q : \mathcal{P}(E)^n \to \mathbb{I} \) be a semi-fuzzy quantifier, \( n > 0 \). Every DFS \( \mathcal{F} \) is compatible with the union an intersection of arguments

\[
\mathcal{F}(Q \cup) = \mathcal{F}(Q) \overline{\cup}
\]

\[
\mathcal{F}(Q \cap) = \mathcal{F}(Q) \overline{\cap}
\]

Previous properties guarantee that arbitrary boolean combinations commute between fuzzy and semi-fuzzy quantifiers. This is one to the consequences of the DFS axiomatic framework and it will be fundamental to prove that the projection \( j \) of \( f(i_1, \ldots, i_K) \) follows a binomial poisson distribution.

Example 29: Let \( Q : \mathcal{P}(E) \to \mathbb{I} \) be a semi-fuzzy quantifier. And let \( \Phi(Y_1, Y_2, Y_3) = \neg Y_1 \cap \neg Y_2 \cap Y_3 \) be a boolean combination of the crisp sets \( Y_1, Y_2, Y_3 \in \mathcal{P}(E) \), and \( \Phi_3'(X_1, X_2, X_3) = \neg X_1 \cap \neg X_2 \cap X_3 \) be the analogous boolean combination of fuzzy sets \( X_1, X_2, X_3 \in \mathcal{P}(E) \) where \( \neg, \cap \) are defined by means of the corresponding negation and tnorm induced by a particular DFS \( \mathcal{F} \). Then

\[
\mathcal{F}(Q \circ \Phi)(X_1, X_2, X_3) = \mathcal{F}(Q \cap \neg \tau_1 \tau_2 \tau_3)(X_1, X_2, X_3)
\]

\[
= \mathcal{F}(Q) \neg \tau_1 \neg \tau_2 \neg \tau_3 (X_1, X_2, X_3)
\]

\[
= \mathcal{F}(Q) \circ \Phi(X_1, X_2, X_3)
\]

\[
= \mathcal{F}(Q) (\neg X_1 \neg X_2 \neg X_3)(X_3)
\]

Now, we will introduce some notation to specify that the cardinality of the input sets of a semi-fuzzy quantifiers is exactly of ‘i elements’:

Notation 30: We will denote by \( q_i^{\text{exactly}} : \{0, \ldots, m\} \to \{0, 1\} \) the function defined by

\[
q_i^{\text{exactly}}(x) = \begin{cases}
1 & : \quad x = i \\
0 & : \quad \text{otherwise}
\end{cases}
\]

1 Notation used in the example can result very confusing. For this reason, we will present below the full detail \( Q \cap \neg \tau_1 \neg \tau_2 \neg \tau_2 \), explicitly detailing the application of the different transformations to the semi-fuzzy quantifier:

\[
(Q \cap \neg \tau_1 \neg \tau_2 \neg \tau_2) = (f': (Y'_1, Y'_2) \to Q(Y'_1 \cap Y'_2)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]

\[
= (f''': (Y''_1, Y''_2, Y''_3) \to f'(Y''_1, Y''_2 \cap Y''_3)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]

\[
= (f''': (Y''_1, Y''_2, Y''_3) \to Q(Y''_1 \cap Y''_2 \cap Y''_3)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]

\[
= (f''': (Y''_1, Y''_2, Y''_3) \to f'(Y''_3, Y''_2 \cap Y''_1)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]

\[
= (f''''': (Y'''_1, Y'''_2, Y'''_3, Y'''_4) \to f''''(Y'''_3, Y'''_2 \cap Y'''_1)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]

\[
= (f''''': (Y'''_1, Y'''_2, Y'''_3, Y'''_4) \to Q(\neg Y'''_3 \cap Y'''_2 \cap Y'''_1)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]

\[
= (f''''': (Y'''_1, Y'''_2, Y'''_3, Y'''_4) \to f''''(Y'''_3, Y'''_2 \cap Y'''_1)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]

\[
= (f''''': (Y'''_1, Y'''_2, Y'''_3, Y'''_4) \to Q(\neg Y'''_3 \cap Y'''_2 \cap Y'''_1)) \cap \neg \tau_1 \neg \tau_2 \neg \tau_2
\]
and by \( Q_{i,n}^{\text{exactly}} : \mathcal{P}(E)^n \rightarrow I \) the semi-fuzzy quantifier defined as:

\[
Q_{i,n}^{\text{exactly}}(Y_1, \ldots, Y_n) = q_{i,n}^{\text{exactly}}([Y_1 \cap \cdots \cap Y_n])
\]

where with the superindex \( n \) we are indicating the arity of the semi-fuzzy quantifier.

**Proposition 31:** Let \( X^B \in \mathcal{P}(E) \) a fuzzy set where \( B = P_1, \ldots, P_m \) is its corresponding poisson bernoulli succession with probabilities \( p_1, \ldots, p_m \). It is fulfilled:

\[
\Pr(B_k = k) = F^A \left( Q_{i,n}^{\text{exactly}} \right) \left( X^B \right)
\]

**Proof.** Simply:

\[
F^A \left( Q_{i,n}^{\text{exactly}} \right) \left( X^B \right) = \sum_{Y \in \mathcal{P}(E)} m_{X^B}(Y) Q_{i,n}^{\text{exactly}}(Y)
\]

\[
= \sum_{Y \in \mathcal{P}(E)} m_{X^B}(Y)
\]

\[
= \sum_{Y \in \mathcal{P}(E)} \prod_{e \in Y} \mu_{X^B}(e) \prod_{e \in Y^c} (1 - \mu_{X^B}(e))
\]

\[
= \sum_{A \in \mathcal{F}_k} \prod_{j \in A} (1 - p_j)
\]

\[
= \Pr(B_k = k)
\]

And before proceeding to the main proof of this section, we need to introduce the following lemma:

**Lemma 32:** Let \( X_1, \ldots, X_n \in \mathcal{P}(E) \) and \( \Phi_{l_1, \ldots, l_n}(X_1, \ldots, X_n) = X_1^{(l_1)} \cap \cdots \cap X_n^{(l_n)} \) a boolean combination of \( X_1, \ldots, X_n \), then it is fulfilled:

\[
F^A \left( Q_{i,n}^{\text{exactly}} \right) \left( X_1^{(l_1)} \cap \cdots \cap X_n^{(l_n)} \right) = \sum_{Y \in \mathcal{P}(E) \mid |Y| = j} m_{X_1^{(l_1)} \cap \cdots \cap X_n^{(l_n)}}(Y)
\]

\[
= \sum_{Y \in \mathcal{P}(E) \mid |Y| = j} m_{X_1^{(l_1)}}(Y) \cdots m_{X_n^{(l_n)}}(Y)
\]

\[
= F^A \left( Q_{i,n}^{\text{exactly}} \circ \Phi_{l_1, \ldots, l_n} \right) (X_1, \ldots, X_n)
\]

**Proof.** Being \( F^A \) a DFS, we have seen it commutes with boolean combinations. Then:

\[
F^A \left( Q_{i,n}^{\text{exactly}} \right) \left( X_1^{(l_1)} \cap \cdots \cap X_n^{(l_n)} \right) = \sum_{Y \in \mathcal{P}(E)} m_{X_1^{(l_1)} \cap \cdots \cap X_n^{(l_n)}}(Y) Q_{i,n}^{\text{exactly}}(Y)
\]

\[
= \sum_{Y \in \mathcal{P}(E) \mid |Y| = j} m_{X_1^{(l_1)} \cap \cdots \cap X_n^{(l_n)}}(Y)
\]

\[
= F^A \left( Q_{i,n}^{\text{exactly}} \right) \left( \Phi_{l_1, \ldots, l_n}(X_1, \ldots, X_n) \right)
\]

\[
= F^A \left( Q_{i,n}^{\text{exactly}} \circ \Phi_{l_1, \ldots, l_n} \right) (X_1, \ldots, X_n)
\]

and let \( j_1, \ldots, j_s \in \{1, \ldots, n\} \) the ordered set of indexes in \( l_1, \ldots, l_n \) such that \( l_{j_s} = 0 \) (i.e., the ones that complement the input argument). Example 29 showed that \( \Phi_{l_1, \ldots, l_n} \) is of the form \( \cap_{s=1}^{n} \tau_{j_1} \tau_{j_2} \cdots \tau_{j_s} \). That is:

\[
\Phi_{l_1, \ldots, l_n}(Y_1, \ldots, Y_n) = \cap_{s=1}^{n} \tau_{j_1} \tau_{j_2} \cdots \tau_{j_s} (Y_1, \ldots, Y_n)
\]

Following notation in 11 by \( X_1^{(l_1)} \) we are denoting the set \( X_1 \) in case \( l_1 = 1 \) and \( \tau X_1 \) in case \( l_1 = 0 \).
depending on the cardinalities of the boolean combinations of the input sets (\( F \)).

By theorem 11 every semi-fuzzy quantifier \( Q : \mathcal{P}(E)^n \rightarrow I \) can be defined by means of a function \( q : \{0, \ldots, |E|\}^K \rightarrow I \) depending on the cardinalities of the boolean combinations of the input sets \( \{\Phi_1 (Y_1, \ldots, Y_n), \ldots, \Phi_K (Y_1, \ldots, Y_n)\} \). Then:

\[
F^A (Q) (X_1, \ldots, X_n) = \sum_{Y_1 \in \mathcal{P}(E)} \ldots \sum_{Y_n \in \mathcal{P}(E)} m_{X_1} (Y_1) \ldots m_{X_n} (Y_n) Q (Y_1, \ldots, Y_n)
\]

\[
= \sum_{(i_1, \ldots, i_K) \in m^K} \sum_{Y_1, \ldots, Y_n \in \mathcal{P}(E)} q (\Phi_1 (Y_1, \ldots, Y_n), \ldots, \Phi_K (Y_1, \ldots, Y_n)) m_{X_1} (Y_1) \ldots m_{X_n} (Y_n)
\]

Let us denote by

\[
f (i_1, \ldots, i_K) = \sum_{Y_1, \ldots, Y_n \in \mathcal{P}(E)} q (\Phi_1 (Y_1, \ldots, Y_n), \ldots, \Phi_K (Y_1, \ldots, Y_n)) m_{X_1} (Y_1) \ldots m_{X_n} (Y_n)
\]  

(6)

\( f (i_1, \ldots, i_K) \) is a probability function. Take into account that \( m_{X_1} (Y_1) \ldots m_{X_n} (Y_n) \) define a probability over \( (Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n \), and \( f (i_1, \ldots, i_K) \) simply distributes the probabilities on \( (Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n \) over the cardinalities of the \( K \) boolean combinations.

Theorem 33: Let \( f (i_1, \ldots, i_K) \) be the probability distribution that is obtained when we compute the probability induced by \( X_1, \ldots, X_n \in \mathcal{P}(E)^n \) fuzzy sets over the cardinalities of the boolean combinations \( \{\Phi_1 (Y_1, \ldots, Y_n), \ldots, \Phi_K (Y_1, \ldots, Y_n)\} \). The probability projection \( j \) of \( f (i_1, \ldots, i_K) \) will follow a binomial poisson distribution of parameters

\[
p_1^j = \mu_{X_1^{(i_1,j)} \cap \ldots \cap X_n^{(i_1,n)}} (e_1)
\]

\[
\ldots
\]

\[
p_m^j = \mu_{X_1^{(i_1,j)} \cap \ldots \cap X_n^{(i_1,n)}} (e_m)
\]

where \( X_1^{(i_1,j)} \cap \ldots \cap X_n^{(i_1,n)} = \Phi_j (X_1, \ldots, X_n) \)
Proof. Let us consider the projection $j$ of $f(i_1, \ldots, i_K)$. Using the same ideas than in the proof of lemma 32:

$$f_j(i_j) = \sum_{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_K} f(i_1, \ldots, i_j, \ldots, i_K)$$

$$= \sum_{Y_1, \ldots, Y_n \in \mathcal{P}(E) \mid |\Phi_j(Y_1, \ldots, Y_n)| = i_j} m_{X_1}(Y_1) \ldots m_{X_n}(Y_n) \cdot (Q_{\text{exactly}}^{i_j, 1} \circ \Phi_j)(Y_1, \ldots, Y_n)$$

$$= \mathcal{F}^A \left( Q_{\text{exactly}}^{i_j, 1} \circ \Phi_j \right)(X_1, \ldots, X_n)$$

$$= \sum_{Y \in \mathcal{P}(E) \mid |Y| = j} m_{X_1^{(i_1)}, \ldots, X_n^{(i_n)}}(Y)$$

but this is a binomial poisson bernoulli succession with distribution $\mathbf{B} = P_1, \ldots, P_m$ with probabilities (proposition 31):

$$p_1 = \mu_{X_1^{(i_1)}, \ldots, X_n^{(i_n)}}(e_1)$$

$$\ldots$$

$$p_m = \mu_{X_1^{(i_1)}, \ldots, X_n^{(i_n)}}(e_m)$$

\[\square\]