CLASSICAL SPINOR STRUCTURES ON QUANTUM SPACES

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Abstract. A noncommutative-geometric generalization of the classical concept of spinor structure is presented. This is done in the framework of the formalism of quantum principal bundles. In particular, analogs of the Dirac operator and the Laplacian are introduced and analyzed. A general construction of examples of quantum spaces with a spinor structure is presented.

1. Introduction

Classical geometry of spinor structures is based on two extremely reach and effective mathematical concepts. As first, there is a “local”, purely algebraic, concept of spinors, which includes Clifford algebras and spin groups. The second one is a “global” concept of principal bundle, specified as a covering spin-bundle of the bundle of oriented orthonormal frames (of the base space).

In this paper basic concepts of classical spinor geometry will be translated into an appropriate quantum context, in accordance with general principles of noncommutative differential geometry [1, 2].

The starting idea is that in formulating the concept of a spinor structure on a quantum space, only “global” aspects of the classical theory should be quantized. In formalizing this we shall use a general non-commutative-geometric theory of principal bundles, presented in [3, 4, 5]. Spinor structures on quantum spaces will be represented by certain quantum principal bundles, however the structure group will be a classical spin group.

In this sense, “local aspects” of the classical theory will be left unchanged. The reason for this is that introducing in the theory a quantum deformation of the spin group would cause the lack of internal symmetry, because of the inherent geometrical inhomogeneity of (truly) quantum groups [3]. Such a philosophy opens the possibility to generalize, in a more or less straightforward way, fundamental concepts and constructions of the classical theory (presented in [11], for example).

The paper is organized as follows. In the next section the concept of a quantum spin manifold will be introduced, in the framework of the formalism of framed quantum principal bundles [6]. Informally speaking, a spinor structure on a quantum space $M$ will be represented by a principal spin-bundle $P$ over $M$, endowed with an additional structure, which expresses the idea that $M$ is an “oriented Riemannian manifold” such that $P$ “covers” the bundle of oriented orthonormal frames for $M$ (however, this bundle will figure only implicitly in the theory).

Further, we shall introduce the analog of the covariant derivative of the Levi-Civita connection (operating on $P$), and sketch a construction of the canonical
differential calculus on $P$. Then counterparts of the Hodge $*$-operator, Laplace operator and invariant integration on $P$ will be constructed, in a direct analogy with classical differential geometry.

Section 3 deals with “spinor fields”. Starting from $P$ we shall introduce a space $E$, interpretable as consisting of “smooth sections” of the corresponding “associated” spinor bundle $E$. A quantum counterpart of the Dirac operator will be then introduced and analyzed. In particular, it turns out that the difference between the square of the Dirac operator and the Laplacian (acting in $E$) is proportional to the (scalar) curvature of the Levi-Civita connection, as in classical geometry [9]. We shall also define the space of “$E$-valued” differential forms on $M$, and the quantum counterpart of the Clifford bundle.

In Section 4 the theory is illustrated in a simple, but sufficiently reach class of examples of inherently “curved” quantum spin manifolds.

Finally, in Section 5 some concluding remarks are made.

2. General Considerations

In this section we shall introduce and analyze spinor structures on quantum spaces. As first, let us fix the notation. The Clifford algebra associated with the complex Euclidean space $\mathbb{C}^n$ will be denoted by $\mathcal{C}l_n$.

Since in all formal considerations quantum spaces are represented “dual” by the corresponding functional algebras, we shall also use this description for the classical groups $S(n)$ (the spin groups) and $SO(n)$, in parallel with the standard one. A (commutative) $*$-algebra of polynomial functions on $S(n)$ will be denoted by $S_n$. The group structure on $S(n)$ induces a Hopf algebra structure on $S_n$, which is determined by the coproduct $\phi: S_n \rightarrow S_n \otimes S_n$, counit $\epsilon: S_n \rightarrow \mathbb{C}$ and the antipode $\kappa: S_n \rightarrow S_n$ (we follow the notation of [10], although in the classical context). In the “dual” picture, the elements of $S(n)$ are naturally interpretable as $*$-characters $g: S_n \rightarrow \mathbb{C}$ (non-trivial hermitian multiplicative linear functionals).

Furthermore $SO(n)$ is represented by a $*$-Hopf subalgebra $A_n$ of $S_n$, consisting of polynomial functions on $S(n)$ invariant under the map $g \mapsto -g$. At the level of spaces, the inclusion $A_n \hookrightarrow S_n$ becomes the canonical (universal covering) epimorphism $\Pi: S(n) \rightarrow SO(n)$. We shall denote by $u$ the canonical representation of $S(n)$ in $\mathbb{C}^n$ (obtained by composing $\Pi$ and the standard representation of $SO(n)$ in $\mathbb{C}^n$). Explicitly,

$$u_g(x) = gxg^{-1}$$

for each $x \in \mathbb{C}^n$ and $g \in S(n)$ (and $S(n)$ and $\mathbb{C}^n$ are embedded in $\mathcal{C}l_n$). Equivalently, $u$ is understandable as a map (a right $\mathcal{S}_n$-comodule structure on $\mathbb{C}^n$) of the form $u: \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathcal{S}_n$ so that $u_g = (id \otimes g)u$. Explicitly,

$$u(e_i) = \sum_{j=1}^n e_j \otimes u_{ji}$$

where $u_{ji} \in \mathcal{S}_n$ are matrix elements of $u$ (actually $u_{ij}$ generate the $*$-algebra $A_n$).

Let $M$ be a quantum space represented by a non-commutative $*$-algebra $\mathcal{V}$. The elements of $\mathcal{V}$ are interpretable as “smooth functions” on $M$. Let $P = (\mathcal{B}, i, F)$ be a (quantum) principal $S(n)$-bundle over $M$. Here, $\mathcal{B}$ is a $*$-algebra consisting of appropriate “functions” on the quantum space $P$, while $i: \mathcal{V} \rightarrow \mathcal{B}$ is a $*$-monomorphism.
playing the role of “the dualized projection” of $P$ on $M$. Finally, $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{S}_n$ is a $\ast$-homomorphism playing the role of the dualized right action of $\mathcal{S}(n)$ on $P$. This interpretation is formalized in the following conditions

\begin{align}
(id \otimes \phi)F &= (F \otimes id)F \\
(id \otimes \epsilon)F &= id.
\end{align}

Speaking geometrically, the space $M$ can be identified, via the “projection map” with the “orbit space” corresponding to the action $F$. Formally, it means that $i(\mathcal{V})$ is consisting precisely of those elements $b \in \mathcal{B}$ which are $F$-invariant, in the sense that $F(b) = b \otimes 1$. In what follows, we shall identify the elements of $\mathcal{V}$ with their images in $i(\mathcal{V})$.

The action $F$ is “free” in the sense that for each $a \in \mathcal{S}_n$ there exist elements $q_i, b_i \in \mathcal{B}$ satisfying

\begin{equation}
\sum_i q_i F(b_i) = 1 \otimes a.
\end{equation}

The actual action of elements of $\mathcal{S}(n)$ on $\mathcal{B}$ is described by $\ast$-automorphisms

\[ F_g = (id \otimes g)F. \]

By definition [6] a frame structure on the bundle $P$ (relative to $u$) is every $n$-tuple $\tau = (\partial_1, \ldots, \partial_n)$ of hermitian derivations $\partial_i: \mathcal{V} \rightarrow \mathcal{B}$ satisfying

\[ F\partial_i(f) = \sum_{j=1}^n \partial_j(f) \otimes u_{ji} \]

for each $f \in \mathcal{V}$ and $i \in \{1, \ldots, n\}$ and such that there exist elements $b_{i\alpha} \in \mathcal{B}$ and $v_{i\alpha} \in \mathcal{V}$ with the property

\[ \sum_\alpha b_{i\alpha} \partial_j(v_{i\alpha}) = \delta_{ij}1 \]

for each $i, j \in \{1, \ldots, n\}$.

A frame structure $\tau$ is called integrable iff there exists a system $\tilde{\tau} = (X_1, \ldots, X_n)$ of hermitian derivations $X_j: \mathcal{B} \rightarrow \mathcal{B}$ which extend derivations $\partial_i$ and such that the following identities hold

\[ 0 = X_i \partial_j - X_j \partial_i \]

\[ FX_i(b) = \sum_{j=1}^n \sum_k X_j(b_k) \otimes u_{ji}c_k, \]

where $\sum_k b_k \otimes c_k = F(b)$.

Speaking geometrically, frame structures formalize the idea that $M$ is an “oriented Riemannian manifold”. Furthermore, endowed with a frame structure $\tau$, the bundle $P$ becomes a “covering bundle” of the bundle $P^*$ of oriented orthonormal frames for $M$. 

Starting from an integrable frame structure $\tau$ it is possible to construct [6] the whole differential calculus on the bundle (including the calculus on $M$). The construction has several steps. As first, a graded *-algebra $\mathfrak{hor}_P$, representing horizontal forms on $P$ can be defined as

$$\mathfrak{hor}_P = B \otimes \mathbb{C}_n^\wedge$$

where $(\,)^\wedge$ denotes the corresponding external algebra. The *-structure on $\mathbb{C}_n^\wedge$ is specified by $e_i^* = e_i$ and extended by antilinearity and multiplicativity on the whole $\mathbb{C}_n^\wedge$. Then the formula

$$\nabla (b \otimes \vartheta) = \sum_{k=1}^n X_k(b) \otimes (e_k \wedge \vartheta)$$

consistently defines a hermitian first-order antiderivation $\nabla$ on $\mathfrak{hor}_P$. This map plays the role of the covariant derivative (induced by the Levi-Civita connection).

There exists a natural action $F^\otimes: \mathfrak{hor}_P \to \mathfrak{hor}_P \otimes S_n$ of $S(n)$ on $\mathfrak{hor}_P$. It is defined by

$$F^\otimes (b \otimes \vartheta) = F(b) u^\wedge (\vartheta),$$

where $u^\wedge: \mathbb{C}_n^\wedge \to \mathbb{C}_n^\wedge \otimes S_n$ is the representation induced by $u$ (and $B$, $\mathbb{C}_n^\wedge$ are understood as subalgebras of $\mathfrak{hor}_P$). The map $F^\otimes$ is a *-homomorphism and

$$F^*(\varphi) = -\varphi_k \mathcal{C}_k$$

Moreover, there exists the unique map $\varrho: S_n \to \mathfrak{hor}_P$ such that

$$\nabla^2(\varphi) = -\sum_k \varphi_k \mathcal{C}_k$$

where $F^*(\varphi) = \sum_k \varphi_k \mathcal{C}_k$. The map $\varrho$ is interpretable as the corresponding "curvature tensor". We have

$$\varrho(1) = 0$$

where $\varrho(1) = 0$ and $\varrho(ab) = \varrho(a) \varrho(b) + \varrho(b) \varrho(a)$ for each $a, b \in A$. Then the following properties hold

$$\varphi \varrho(a) = \varrho(a) \varphi$$

$$\varrho(a^\ast) = \varrho(a)^\ast$$

$$\varrho(\kappa(a)) = -\varrho(a).$$
Let $\Omega_M$ be the *-subalgebra of $\mathfrak{hor}_P$ consisting of $F^*$-invariant elements. According to (2.8) we have $\nabla(\Omega_M) \subseteq \Omega_M$. Let $d_M: \Omega_M \to \Omega_M$ be the restriction of $\nabla$ on $\Omega_M$. This map is a hermitian first-order antiderivation. Furthermore,

$$d_M^2 = 0$$

(according to (2.9) and (2.10)). The elements of $\Omega_M$ are interpretable as “differential forms” on $M$. The map $d_M$ plays the role of the exterior derivative.

A graded-differential *-algebra $\Omega_P$ describing the whole differential calculus on the bundle can be now constructed as follows. At the level of graded *-algebras,

$$\Omega_P = \mathfrak{hor}_P \hat{\otimes} L^\wedge$$

where $\hat{\otimes}$ denotes the graded tensor product. In the above formula $L = \mathfrak{so}(n)^*$ (where $\mathfrak{so}(n)$ is the Lie algebra of complex antisymmetric $n \times n$ matrices, naturally understood as the Lie algebra of $\text{SO}(n)$) and $L^\wedge$ is endowed with its natural graded-differential *-algebra structure. The differential $d: L^\wedge \to L^\wedge$ is explicitly given by the Mauer-Cartan formula (on the first-order elements and extended on the whole $L^\wedge$ by the graded Leibniz rule). Also, $\mathfrak{hor}_P$ and $L^\wedge$ are naturally understandable as subalgebras of $\Omega_P$.

The differential $d^\wedge: \Omega_P \to \Omega_P$ is specified by

$$d^\wedge(\varphi) = \nabla(\varphi) + (-)^{\varphi} \sum_k \varphi_k \pi(c_k)$$

(2.16)

$$d^\wedge(\vartheta) = R(\vartheta) + d(\vartheta)$$

(2.17)

where $\vartheta \in L$ and $\varphi \in \mathfrak{hor}_P$, while $F^*(\varphi) = \sum_k \varphi_k \otimes c_k$. Next, $\pi: S_n \to L$ is a natural projection, given by $\pi(a)(X) = (Xa)_e$ (here elements $X \in \mathfrak{so}(n)$ are understood as left-invariant vector fields on $S(n)$). Finally $R: A \to \mathfrak{hor}_P$ is a map given by $R\pi = \varrho$ (consistency of this definition is a consequence of (2.12)). Using the above formulas and the graded Leibniz rule the map $d^\wedge$ can be consistently extended to the whole $\Omega_P$.

Let $\varpi^\wedge: L^\wedge \to L^\wedge \otimes S_n^*$ be the *-homomorphism extending the dualized coadjoint action $\varpi: L \to L \otimes S_n^*$ (explicitly given by $\varpi\pi = (\pi \otimes \text{id})\text{ad}$). The formula

$$F^\wedge(\varphi \otimes \vartheta) = F^*(\varphi)\varpi^\wedge(\vartheta)$$

(2.18)

defines a *-homomorphism $F^\wedge: \Omega_P \to \Omega_P \otimes S_n$ extending the action $F^*$. We have

$$F^\wedge d^\wedge = (d^\wedge \otimes \text{id}) F^\wedge$$

(2.19)

$$(F^\wedge \otimes \text{id}) F^\wedge = (\text{id} \otimes \phi) F^\wedge$$

(2.20)

$$\text{id} = (\text{id} \otimes \epsilon) F^\wedge$$

(2.21)

The construction of $\Omega_P$ can be viewed as a special case of a general construction of differential calculus presented in [4]–Section 6.

A map $\omega: L \to \Omega_P$, defined by

$$\omega(\vartheta) = 1 \otimes \vartheta$$

is a regular and multiplicative connection on $P$. It is a counterpart of the Levi-Civita connection (viewed as a connection on the spin-bundle) in classical geometry.
Moreover the map $R$ corresponds to the curvature of $\omega$, and $\nabla$ can be viewed as the covariant derivative associated to $\omega$.

The quantum analog of the Hodge $*$-operator can be introduced in a direct analogy with classical geometry. The formula

$$\star(b \otimes (e_{i_1} \wedge \ldots \wedge e_{i_k})) = b \otimes (e_{j_1} \wedge \ldots \wedge e_{j_l})$$

where $1 \leq i_1 < \ldots < i_k \leq n$ and $1 \leq j_1 < \ldots < j_l \leq n$, while $k + l = n$ and $e_{i_1} \wedge \ldots \wedge e_{i_k} \wedge e_{j_1} \wedge \ldots \wedge e_{j_l} = e_1 \wedge \ldots \wedge e_n$, consistently determines a linear map $\star : \mathfrak{hor}_P \to \mathfrak{hor}_P$.

Let $\nabla^\dagger : \mathfrak{hor}_P \to \mathfrak{hor}_P$ be a linear map defined by

$$\nabla^\dagger(\varphi) = -(-)^{n-k} \star \nabla(\varphi)$$

where $\varphi \in \mathfrak{hor}^k$.

By construction maps $\star$ and $\nabla^\dagger$ are right-covariant, in the sense that

$$F^\star \star = (\star \otimes \text{id}) F^\star$$
$$F^\star \nabla^\dagger = (\nabla^\dagger \otimes \text{id}) F^\star.$$  

In particular $\star$ and $\nabla^\dagger$ are reduced in $\Omega_M$.

Let us denote by $\star_M, d_M^\dagger : \Omega_M \to \Omega_M$ the corresponding restriction maps. Evidently, $(d_M^\dagger)^2 = 0$ and $\star^2 = (-)^{(n-\partial)\partial}$.

The analog of the Laplace operator $\Delta_M : \Omega_M \to \Omega_M$ can be defined by the classical formula

$$\Delta_M = d_M d_M^\dagger + d_M^\dagger d_M = (d_M + d_M^\dagger)^2$$

Evidently

$$\Delta_M d_M = d_M \Delta_M$$
$$\Delta_M d_M^\dagger = d_M^\dagger \Delta_M$$
$$\Delta_M \star_M = \star_M \Delta_M.$$  

The Laplacian $\Delta_M$ is grade-preserving. If $f \in \mathcal{V}$ then

$$\Delta_M(f) = d_M d_M^\dagger(f) = -\sum_{i=1}^n X_i^2(f)$$

Now we shall prove that, under certain additional assumptions, it is possible to define the integration and scalar product in $\Omega_M$, such that $d_M$ and $d_M^\dagger$ become mutually formally adjoint.

As first let us suppose that $\mathcal{B}$ can be realized as a dense $*$-subalgebra of a $C^*$-algebra $\widehat{\mathcal{B}}$.

Secondly, let us assume that there exists a faithful state $\nu$ on $\widehat{\mathcal{B}}$ such that

$$\nu X_i(b) = 0$$
$$\sum_k \nu(b_k) \otimes c_k = \nu(b) \otimes 1$$

for each $b \in \mathcal{B}$ and $i \in \{1, \ldots, n\}$, where $\sum_k b_k \otimes c_k = F(b)$. 


Let $\int_P: \Omega_P \to \mathbb{C}$ be a linear functional given by

$$\int_P \left( b \otimes (e_1 \wedge \ldots \wedge e_n) \otimes W \right) = \nu(b) \quad \int_P \left( \Omega_P^k \right) = \{0\},$$

for $k < n + m$. Here $W \in \Lambda^{m}$ is a volume element ($m = \dim L = n(n-1)/2$).

**Lemma 1.** We have

\begin{align}(2.33)& \quad \sum_k \left( \int_P w_k \right) \otimes c_k = \left( \int_P w \right) \otimes 1 \\
(2.34)& \quad \int_P d^{\wedge}(w) = 0\end{align}

for each $w \in \Omega_P$, where $\sum_k w_k \otimes c_k = F^{\wedge}(w)$.

**Proof.** Formula (2.33) directly follows from (2.32) and from the definition of $\int_P$.
Formula (2.34) is non-trivial only if $\partial w = n + m - 1$. In this case we can write

$$w = \sum_{i=1}^n b_i \otimes (e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_n) \otimes W + \sum_j q_j \otimes (e_1 \wedge \ldots \wedge e_n) \otimes \eta_j$$

where $\eta_j \in (L^{\wedge})^{m-1}$ (and $b_i, q_j \in \mathcal{B}$). Applying (2.31)–(2.32), the definition of $d^{\wedge}$ and the fact that $d((\Lambda^{m-1}) = \{0\}$ we obtain

$$\int_P d^{\wedge}(w) = -\sum_{i=1}^n (-)^i \int_P \left( X_i(b_i) \otimes (e_1 \wedge \ldots \wedge e_n) \otimes W \right)$$

$$= -\sum_{i=1}^n (-)^i \nu(X_i(b_i)) = 0. \quad \Box$$

Let $\int_M^\triangledown: \mathfrak{hor}_P \to \mathbb{C}$ be a linear functional given by

$$\int_M^\triangledown \varphi = \int_P (\varphi \otimes W).$$

**Lemma 2.** We have

\begin{align}(2.35)& \quad \int_M^\triangledown \nabla(\varphi) = 0 \end{align}

for each $\varphi \in \mathfrak{hor}_P$.

**Proof.** Applying (2.34) and (2.16)–(2.17), and the above definition we find

$$\int_M^\triangledown \nabla(\varphi) = \int_P \nabla(\varphi) \otimes W = \int_P d^{\wedge}(\varphi) W = \int_P d^{\wedge}(\varphi W) = 0. \quad \Box$$
The formula

(2.36) \((\psi, \varphi) = \int_M \bar{\psi} \star (\varphi)\)

where the bar denotes the standard \(*\)-operation, determines a (strictly positive) scalar product in \(\mathfrak{hor}_P\) (and in particular in \(\Omega_M\)). We have

(2.37) \((\nabla \psi, \varphi) = (\psi, \nabla \varphi)\)

for each \(\varphi, \psi \in \mathfrak{hor}_P\). In particular, \(d_M\) and \(d_M^\dagger\) are formally adjoint in \(\Omega_M\) and the operator \(\Delta_M\) is symmetric and positive. The map \(\star\) is isometric.

3. Associated Spinor Bundles

Let \(\Sigma\) be the space of algebraic spinors for \(\mathbb{O}_n\). Explicitly [7], \(\Sigma\) can be constructed as follows. For simplicity it will be assumed that \(n\) is even. Then \(\mathbb{C}_n = V_+ \oplus V_-\) where \(V_\pm\) are (naturally mutually dual) isotropic subspaces spanned by vectors \(e_{2j-1} \pm ie_{2j}\) while \(j \in \{1, \ldots, r\}\) and \(2r = n\). By definition

\[\Sigma = V_+^\wedge.\]

The space \(\Sigma\) is an irreducible left \(\mathbb{O}_n\)-module. Elements of the generating space \(\mathbb{C}_n\) of \(\mathbb{O}_n\) act on \(\Sigma\) as follows

\[x \xi = x_+ \wedge \xi + x_- \sqcup \xi\]

where \(x_\pm \in V_\pm\) are such that

\[x = \frac{x_- + x_+}{\sqrt{2}}\]

and \(\sqcup\) is the contraction map.

Let \(\{\sigma_I | I \subseteq \{1, \ldots, r\}\}\) be a basis in \(\Sigma\) given by

\[\sigma_I = \prod_{j \in I} \left(\frac{e_{2j-1} + ie_{2j}}{\sqrt{2}}\right)\]

It is natural to introduce a scalar product \((,\) in \(\Sigma\) by requiring that \(\sigma_I\) are orthonormal spinors. Then the action of \(\mathbb{O}_n\) on \(\Sigma\) is hermitian, in the sense that

\[(Q \xi, \zeta) = (\xi, Q^* \zeta)\]

for each \(\xi, \zeta \in \Sigma\) and \(Q \in \mathbb{O}_n\), where \(*: \mathbb{O}_n \to \mathbb{O}_n\) is the anticommutative antilinear involution extending the complex conjugation in \(\mathbb{C}_n\).

We shall denote by \(\gamma_i: \Sigma \to \Sigma\) operators representing the basis vectors \(e_i\).

The spinorial conjugation operator \(C: \Sigma \to \Sigma\) is given by

\[C(\psi) = (-)^{k(k+1)/2} \star_{\Sigma} (\bar{\psi}),\]

where \(\psi \in \Sigma^k\) and \(\star_{\Sigma}\) is the corresponding Hodge \(*\)-operator. The bar denotes the complex conjugation in \(\Sigma\), defined by requiring reality of basis vectors \(\sigma_I\). We have

\[C \gamma_i + \gamma_i C = 0\]

for each \(i \in \{1, \ldots, n\}\). Further, \(C\) is isometric and

\[C^2 = (-)^{(r+1)/2}.\]
Restricted on $S(n)$ the action of $Cl_n$ becomes a unitary representation $U$. We have

$$\Sigma = \Sigma_+ \oplus \Sigma_-$$

where $\Sigma_+$ and $\Sigma_-$ are subspaces corresponding to even and odd multivectors. These subspaces are invariant under $U$, moreover $U \upharpoonright \Sigma_{\pm}$ are irreducible. The operator $C$ intertwines $U$.

We shall also interpret the representation $U$ as a right comodule structure map $U: \Sigma \rightarrow \Sigma \otimes S_n$.

In the case $n = 2r + 1$ similar algebraic constructions can be performed.

Now, the associated spinor bundle will be defined. Let $E$ be the space of all linear maps $\varsigma: \Sigma^* \rightarrow B$ satisfying

$$F\varsigma = (\varsigma \otimes \text{id})U_c$$

(intertwiners between the contragradient representation $U_c: \Sigma^* \rightarrow \Sigma^* \otimes S_n$ and $F$). The space $E$ is a $\mathcal{V}$-bimodule, in a natural manner. The elements of $E$ are interpretable as “smooth sections” of the “associated spinor bundle” $E$.

Similarly, intertwiners $\varsigma: \Sigma^* \rightarrow \mathfrak{hor}_P$ between $U_c$ and $F^*$ constitute a graded $\Omega_M$-bimodule $E_{\Omega}$ the elements of which are interpretable as $E$-valued differential forms on $M$. We have

$$E_{\Omega} = \sum_{k=1}^{n} \oplus E^k$$

with $E^0 = E$.

Equivalently, $E_{\Omega}$ can be described as the space of elements $\psi \in \mathfrak{hor}_P \otimes \Sigma$ invariant under the product of actions $F^*$ and $U$. The connection between the second and the first definition is given by $\psi \leftrightarrow \varsigma$, with

$$\varsigma(\zeta) = (\text{id} \otimes \varsigma)(\psi)$$
$$\psi = \sum I \varsigma(\sigma^*_I) \otimes \sigma_I$$

where $\sigma^*_I \in \Sigma^*$ are vectors of the corresponding biorthogonal basis.

The decomposition (3.1) induces a bimodule splitting

$$E_{\Omega} = E^+ \oplus E^-.$$ (3.3)

In particular, $E = E^+ \oplus E^-$. Every $\psi \in E_{\Omega}$ can be written in the form

$$\psi = \sum I \psi_I \otimes \sigma_I$$ (3.4)

where $\psi_I \in \mathfrak{hor}_P$ and

$$F^*(\psi_I) = \sum J \psi_J \otimes U^*_J.$$ (3.5)

The space $E$ possesses a natural $^*$-structure, given by a “charge conjugation” $c_E: E \rightarrow E$, where

$$c_E(\psi) = \sum I \tilde{\psi}_I \otimes C(\sigma_I)$$
(the same formula determines a \(*\)-structure on the $\Omega_M$-bimodule $\mathcal{E}_\Omega$).

The corresponding Hodge operator $\ast_E : \mathcal{E}_\Omega \to \mathcal{E}_\Omega$ is given by

$$\ast_E(\psi) = \sum_I \ast(\psi_I) \otimes \sigma_I. \quad (3.6)$$

By construction,

$$c^2_E = (-)^{r(r+1)/2},$$
$$\ast^2_E = (-)^{\partial(n-\partial)}.$$

The formula

$$\langle \psi, \varphi \rangle = \sum_I (\psi_I, \varphi_I) \quad (3.7)$$
defines a (strictly positive) scalar product on $\mathcal{E}_\Omega$. The spaces $\mathcal{E}^k_\Omega$ are mutually orthogonal. The operator $\ast_E$ is isometric. It is worth noticing that $c_E$ is generally not isometric (if the state $\nu$ possesses non-trivial modular properties).

Let $\nabla_E, \nabla^\dagger_E : \mathcal{E}_\Omega \to \mathcal{E}_\Omega$ be operators given by

$$\nabla_E(\psi) = (\nabla \otimes \text{id}) \upharpoonright \mathcal{E}_\Omega \quad (3.8)$$
$$\nabla^\dagger_E(\psi) = (\nabla^\dagger \otimes \text{id}) \upharpoonright \mathcal{E}_\Omega \quad (3.9)$$

In particular,

$$\nabla^\dagger_E(\psi) = (-)^{nk-n} \ast_E \nabla_E \ast_E(\psi) \quad (3.10)$$

for each $\psi \in \mathcal{E}^k_\Omega$.

Operators $\nabla_E$ and $\nabla^\dagger_E$ are mutually formally adjoint. The map $\nabla_E$ plays the role of the spinorial covariant derivative.

The corresponding Laplacian can be introduced by the classical expression

$$\Delta_E = \nabla_E \nabla^\dagger_E + \nabla^\dagger_E \nabla_E. \quad (3.11)$$

We have

$$\langle \Delta_E \psi, \varphi \rangle = \langle \psi, \Delta_E \varphi \rangle$$
$$\Delta_E \ast_E = \ast_E \Delta_E.$$

The operator $\Delta_E$ is grade-preserving and positive. In the “pure spinor” space $\mathcal{E}$ we have $\Delta_E = \nabla^\dagger_E \nabla_E$. Explicitly,

$$\Delta_E(\psi) = -\sum_{i,I} X_i^2(\psi_I) \otimes \sigma_I \quad (3.12)$$

for each $\psi \in \mathcal{E}$.

The map $c_E$ commutes with $\nabla_E, \nabla^\dagger_E, \Delta_E$ and $\ast_E$.

Now we shall introduce an analog of the Dirac operator. Let $D_E : \mathcal{E} \to \mathcal{E}$ be a linear map defined by

$$D_E(\psi) = -i \sum_{i,I} X_i(\psi_I) \otimes \gamma_i \sigma_I. \quad (3.13)$$
By construction, $D_E$ “mixes” $\mathcal{E}^+$ and $\mathcal{E}^-$. Further,

\begin{align}
\langle D_E \psi, \varphi \rangle &= \langle \psi, D_E \varphi \rangle \\
D_E c_E &= c_E D_E.
\end{align}

Let us consider “the curvature operator” $\varrho_E : \mathcal{E} \to \mathcal{E}$ given by

\begin{equation}
\varrho_E(\psi) = -\frac{1}{2} \sum_{ij} \psi_i \varrho_{ij}(U_{ji}) \otimes \gamma_i \gamma_j \sigma_j
\end{equation}

where $\varrho_{ji} = -\varrho_{ij} : \mathcal{S}_n \to \mathcal{B}$ are components of the curvature,

\begin{equation}
\varrho(a) = \frac{1}{2} \sum_{ij} \varrho_{ij}(a) \otimes (e_i \wedge e_j).
\end{equation}

**Lemma 3.** We have

\begin{equation}
D^2_E = \Delta_E + \varrho_E.
\end{equation}

**Proof.** A direct calculation gives

\begin{align*}
D^2_E(\psi) &= -\sum_{ij} X_i X_j (\psi_I) \otimes \gamma_i \gamma_j (\sigma_I) \\
&= -\frac{1}{2} \sum_{ij} [X_i, X_j](\psi_I) \otimes \gamma_i \gamma_j (\sigma_I) - \sum_{ij} X^2_i (\psi_I) \otimes \sigma_I \\
&= \frac{1}{2} \sum_{ij} \psi_j \varrho_{ij}(U_{ji}) \otimes \gamma_i \gamma_j \sigma_I + \Delta_E(\psi) = \varrho_E(\psi) + \Delta_E(\psi). \quad \square
\end{align*}

Actually, the operator $\varrho_E$ can be further simplified so that a complete analogy with the classical formalism [9] holds.

The components of the curvature can be written as

\begin{equation}
\varrho_{ij}(a) = \frac{1}{2} \sum_{kl} \varrho_{ijkl} e_{kl}(a),
\end{equation}

where $\varrho_{ijkl}$ belong to the center of $\mathcal{B}$ and $e_{kl} = -e_{lk}$ are canonical generators of $\mathfrak{so}(n) = \mathcal{L}^\ast$.

**Lemma 4.** We have

\begin{equation}
\varrho_E(\psi) = \frac{1}{4} \rho \psi
\end{equation}

for each $\psi \in \mathcal{E}$, where

\begin{equation}
\rho = \sum_{ij} \varrho_{ijij}.
\end{equation}

The element $\rho$ is interpretable as the scalar curvature of the Levi-Civita connection.
Proof. The operator $\varrho_E$ can be transformed in the following way

$$
\varrho_E(\psi) = -\frac{1}{2} \sum_{ij} \psi_I \varrho_{ij} (U_{JI}) \otimes \gamma_i \gamma_j \sigma_I = -\frac{1}{8} \sum_{ijkl} \psi_I \varrho_{ijkl} \otimes \gamma_i \gamma_j \gamma_k \gamma_l \sigma_I
$$

$$
= \frac{1}{4} \sum_{IJ} \sum_{ij} \psi_I \varrho_{ijij} \otimes \sigma_I = \frac{1}{4} \rho \psi.
$$

Here, we have applied the correspondence

$$
e_{kl} \leftrightarrow \frac{1}{4} [\gamma_k, \gamma_l]
$$

(the action of $\mathfrak{so}(n)$ in the spinor space), as well as the following symmetry properties

\begin{align}
-\varrho_{ijkl} &= \varrho_{jikl} = \varrho_{ijlk} \\
\varrho_{ijkl} + \varrho_{iklj} + \varrho_{iljk} &= 0 \\
\varrho_{ijkl} &= \varrho_{klij}.
\end{align}

They can be derived essentially in the same manner as in classical geometry.

4. Example

A large class of “non-commutative spin manifolds” can be constructed starting from appropriate actions of $S(n+1)$ on quantum spaces and then restricting this action on $S(n)$ (viewed as a subgroup of $S(n+1)$, in a natural manner). At the dual level, this inclusion of groups is described by an epimorphism $j_n: S_{n+1} \rightarrow S_n$ of *-Hopf algebras (the restriction map).

Let $\mathcal{B}$ be a *-algebra. Let us assume that $F: \mathcal{B} \rightarrow \mathcal{B} \otimes S_{n+1}$ is a *-homomorphism describing an action of $S(n+1)$ by *-automorphisms of $\mathcal{B}$ and satisfying the “freedom” condition. In other words for each $a \in S_{n+1}$ there exist $q_i, b_i \in \mathcal{B}$ such that (2.3) holds.

The formula

$$
\iota(x) = \begin{pmatrix}
0 & x \\
-x^\top & 0
\end{pmatrix}
$$

defines an embedding $\iota: \mathbb{C}_n \rightarrow \mathfrak{so}(n+1)$.

Let $F^\sharp: \mathcal{B} \rightarrow \mathcal{B} \otimes S_n$ be the restriction of $F$ on $S(n)$. In other words

$$
F^\sharp = (\text{id} \otimes j_n) F.
$$

Let $\mathcal{V} \subseteq \mathcal{B}$ be the fixed-point subalgebra of $\mathcal{B}$ for the action $F^\sharp$, and let $i: \mathcal{V} \hookrightarrow \mathcal{B}$ be the inclusion map. Then $P = (\mathcal{B}, i, F^\sharp)$ is a quantum principal $S(n)$-bundle (over the quantum space $M$ described by $\mathcal{V}$).
The bundle $P$ possesses a canonical (integrable) frame structure. For every $i \in \{1, \ldots, n\}$, let $X_i: \mathcal{B} \to \mathcal{B}$ be a derivation corresponding to $\iota(e_i)$, via the infinitesimalization of $F^\sharp$. Let $\partial_i = (X_i | \mathcal{V}): \mathcal{V} \to \mathcal{B}$ be the corresponding restrictions.

Then $\tau = (\partial_1, \ldots, \partial_n)$ is an integrable frame structure with $\hat{\tau} = (X_1, \ldots, X_n)$. The space $M$ is inherently “curved”. The components $\varrho_{ij}$ of the curvature take values in scalars. Explicitly,

$$\varrho_{ij}(a) = -\{[\iota(e_i), \iota(e_j)](a)\}_e 1$$

for each $a \in S_n$.

As a concrete illustration, let us consider bundles based on Cuntz algebras [8]. By definition the Cuntz algebra $\mathcal{C}_d$ (where $d \geq 2$ is an integer) is a *-algebra generated by elements $\psi_1, \ldots, \psi_d$, and relations

$$\psi_i^* \psi_j = \delta_{ij} 1$$
$$\sum_{i=1}^d \psi_i \psi_i^* = 1.$$  

Let us assume that $v: \mathbb{C}_d \to \mathcal{C}_d \otimes S_{n+1}$ is a faithful and irreducible unitary representation of $S(n+1)$. This map is uniquely extendible to a *-homomorphism $F: \mathcal{C}_d \to \mathcal{C}_d \otimes S_{n+1}$ (describing the action of $S(n+1)$ by automorphisms of $\mathcal{C}_d$).

Let us prove that the action $F$ satisfies the “freeness” condition. Matrix elements $v_{ij}$ generate the whole algebra $S_{n+1}$. For this reason it is sufficient to check that equations of the form (2.3) hold for elements $a = v_{ij}$. We have

$$F(\psi_i) = \sum_{k=1}^d \psi_k \otimes v_{ki}.$$  

Multiplying both sides of the above equality by $\psi_j^*$ on the left and using (4.2) we obtain

$$\psi_j^* F(\psi_i) = 1 \otimes v_{ji},$$  

Consequently, the freeness condition holds.

5. CONCLUDING REMARKS

Logically, constructions of the Laplacian $\Delta_M$ and the Hodge operator $\star_M$ (and considerations with the curvature) do not require $S(n)$-bundles. Everything can be performed starting from principal $SO(n)$-bundles (and integrable frame structures on them).

The bundle of “orthonormal oriented frames” can be extracted from the spin-bundle $P$ as follows. Let $\mathcal{O} \subseteq \mathcal{B}$ be a *-subalgebra consisting of elements $b$ satisfying $F_\phi^\sharp(b) = F_{-\phi}^\sharp(b)$. By construction, $i(\mathcal{V}) \subseteq \mathcal{O}$ and $F(\mathcal{O}) \subseteq \mathcal{O} \otimes A_n$. In other words, the action of $S(n)$ can be “projected” to the action of $SO(n)$ on $\mathcal{O}$. In such a way we obtain a quantum principal $SO(n)$-bundle $P^*$ over $M$. Moreover, $X_i(\mathcal{O}) \subseteq \mathcal{O}$ and the frame structure on $P$ induces a frame structure on $P^*$. The bundle $P^*$ is interpretable as consisting of “oriented orthonormal frames” of $M$. At the level of spaces the inclusion $\mathcal{O} \hookrightarrow \mathcal{B}$ becomes the “covering projection” of $P$ onto $P^*$, corresponding to the standard interpretation of spinor structures.
The number $n$ (playing the role of the dimension of $M$) is not fixed. The same quantum space could possess frame structures with different “dimensions”. Moreover, even if $M$ is a classical compact manifold then the class of frame structures described above includes nonstandard geometrical objects based on foliations of $M$.

It is interesting to think about possible physical applications of the presented theory, starting from the idea that space-time is interpretable as a non-commutative (completely pointless) spin manifold. In this conceptual framework it is natural to consider field theories including “spinor” and “metric” fields, as well as the corresponding “gauge fields”. For example, classical Einstein-Dirac type theories [12, 13] include all these concepts in a natural way. Counterparts of such theories (on non-commutative spin manifolds) are in principle constructable following the ideas of classical geometrical formulations.

REFERENCES

1. Connes A Non-commutative differential geometry, IHES: Extrait des Publications Mathématiques 62 (1986)
2. Connes A Geometrie non commutative, InterEditions, Paris (1990)
3. Đurđević M Geometry of Quantum Principal Bundles I, Preprint Qmmp 6/92, Belgrade University
4. Đurđević M Geometry of Quantum Principal Bundles II, Preprint Qmmp 4/93, Belgrade University
5. Đurđević M Quantum Principal Bundles, Proceedings of the XXII-th Conference on Differential Geometric Methods in Theoretical Physics, Ixtapa-Zihuatanejo, México, September 1993
6. Đurđević, M On Framed Quantum Principal Bundles, Preprint, Instituto de Matematicas, UNAM (1994)
7. Cartan E The Theory of Spinors, Dover (1966)
8. Cuntz J Simple $C^*$-algebras generated by isometries, CMP 57 173–185 (1977)
9. Lichnerowicz A Spineurs harmoniques CR Acad Sci Paris 257 S1 7–9 (1963)
10. Woronowicz S L Compact Matrix Pseudogroups, CMP 111 613–665 (1987)
11. Crumeyrolle A Orthogonal and Symplectic Clifford Algebras: Spinor Structures Dordrecht, Netherlands: Kluwer (Mathematics and its Applications 57) (1990)
12. Crumeyrolle A Théorie d’Einstein-Dirac en spin maximal 1 AIHP 22 43–61 (1975)
13. Crumeyrolle A Schémas d’Einstein-Dirac en spin 1/2 AIHP 23 259–274 (1975)