Game theory to characterize solutions of a discrete-time Hamilton-Jacobi equation

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Abstract. We study the behavior of solutions of a discrete-time Hamilton-Jacobi equation in a minimax framework of game theory. The solutions of this problem represent the optimal payoff of a zero-sum game of two players, where the number of moves between the players converges to infinity. A real number, called the critical value, plays a central role in this work; this number is the asymptotic average action of optimal trajectories. The aim of this paper is to show the existence and characterization of solutions of a Hamilton-Jacobi equation for this kind of games.

1. Introduction
The method of dynamic programming is an important tool in the study of control theory, with applications to diverse areas of engineering, economics and management science. In dynamic programming the goal is to optimize some payoff function $P$ of a dynamical system, where a value function $L$ is introduced which is the optimum value of the payoff considered as a function of initial data. In this paper the behavior of the dynamical system is modeled by a discrete-time zero-sum game of two players. The value function satisfies a discrete-time Hamilton-Jacobi equation.

Consider a zero-sum static game. Let $M$ and $N$ be two compact metric spaces, and $P : M \times N \to \mathbb{R}$ a bounded function called the game payoff. There are two players $J_M$ and $J_N$ with opposite goals. The function $P$ is the amount that player $J_M$ obtains from $J_N$, so player $J_M$ wishes to maximize $P$, while player $J_N$ wishes to minimize it. The lower value of the payoff of the static game is defined by

$$L := \sup_{w \in M} \inf_{z \in N} P(w, z),$$

where $J_M$ chooses $w \in M$ first, and $w$ is know when $J_N$ chooses $z \in N$. It is observed in [1] that it is possible to express the lower value (1) as follows

$$L = \inf_{\theta(w)} \sup_{w \in M} P(w, \theta(w)),$$

where $\theta$ is a function from $M$ into $N$. Such a $\theta$ is called a strategy for player $J_N$. The strategies will play a very important role in this work, these functions will be defined more generally in Section 2, for a zero-sum game with finite horizon.

In Section 3 we will see that the lower value function is a solution of the Hamilton-Jacobi equation (4),

$$H(u, x, y) = c.$$
And finally, in Section 4, we will consider an infinite horizon problem and find a characterization of the constant $c$.

2. Zero sum game

For $n \in \mathbb{Z}^+$ we will consider a finite horizon problem to $n$–time. From now on we make the following assumptions:

- $M$ and $N$ are compact metric spaces,
- $V : M \times N \times M \times N \to \mathbb{R}$ is a Lipschitz function.

The state of the system, for a zero sum discrete-time game of two players $J_M$ and $J_N$, depends on the moves made by players to go from $(x_i, y_i) \in M \times N$ to $(x_{i+1}, y_{i+1}) \in M \times N$ and initial data $(x_0, y_0) \in M \times N$. These paths are described by a couple of finite sequences $\pi(n) = \{x_0, x_1, \ldots, x_n\}$ in $M$ and $\gamma(n) = \{y_0, y_1, \ldots, y_n\}$ in $N$, respectively for players $J_M$ and $J_N$. If $(x, y) \in M \times N$, we denote by

$$S_M(x, n) := \{\pi(n) = \{x_i\}_{i=0}^n : x_0 = x\},$$

$$S_N(y, n) := \{\gamma(n) = \{y_i\}_{i=0}^n : y_0 = y\},$$

to the sets of sequences in $M$ and $N$, respectively.

For $g \in C(M \times N, \mathbb{R})$ and $(x, y) \in M \times N$, the game payoff $P_g : S_M(x, n) \times S_N(y, n) \to \mathbb{R}$ is defined by

$$P_g(\pi(n), \gamma(n)) := g(x_n, y_n) - A^n(\pi(n), \gamma(n)),$$

where

$$A^n(\pi(n), \gamma(n)) := \sum_{i=0}^{n-1} V(x_i, y_i, x_{i+1}, y_{i+1})$$

is the action of the game. In this case the lower value of the game can get step by step. Given a point $x_i \in M$, $J_N$ chooses $y_i$ to minimize the payoff, with $y_i = y_i(x_i)$ depending on $x_i$ and the above points, then the player $J_M$ chooses $x_i$ in order to maximize the payoff, with initial data $(x, y) \in M \times N$. More precisely, the lower value of the game with finite horizon $n$ for initial data $(x, y) \in M \times N$, is defined by

$$L^n g(x, y) := \sup_{x_i \in M} \inf_{y_i \in N} \sum_{x_j \in M} \inf_{y_j \in N} \sup_{x_k \in M} \inf_{y_k \in N} P_g(\pi(n), \gamma(n)),$$

where $\pi(n) = \{x, x_1, \ldots, x_n\} \in S_M(x, n)$ and $\gamma(n) = \{y, y_1, \ldots, y_n\} \in S_N(y, n)$. Instead of this, we use the concept of strategy to calculate the game value, like in the static game adapted to this case (see [1, 2, 3]). A progressive strategy $\Theta(n)$ for player $J_N$ is a function $\Theta(n) : S_M(x, n) \to S_N(y, n)$ with following property: for each $m \in \mathbb{Z}^+$, such that $0 \leq m \leq n$, and $\pi(n), \gamma(n) \in S_M(x, n)$, we have

$$x_i = z_i, 0 \leq i \leq m \Rightarrow \Theta(n)[\pi(n)]_i = \Theta(n)[\gamma(n)]_i, 0 \leq i \leq m,$$

where $\Theta(n)[\pi(n)]_i$ is the $i$–th element in the sequence $\Theta(n)[\pi(n)]$. We denote the progressive strategies set of player $J_N$ by

$$E_N(x, y, n) := \{\Theta(n) : S_M(x, n) \to S_N(y, n), \text{progressive strategy}\}.$$  

According to [1] and [3], the lower value of the game payoff is defined by

$$L^n g(x, y) := \sup_{\Theta(n)[\pi(n)]} \inf_{\Theta(n)[\gamma(n)]} P_g(\pi(n), \Theta(n)[\pi(n)]),$$  

where $\pi(n) \in S_M(x, n)$ and $\Theta(n) \in E_N(x, y, n)$. We call $L$ the Lax operator.

Considering the one step case, the Lax operator can be expressed by

$$L^1 g(x, y) = \sup_{x_i \in M} \inf_{y_i \in N} \{g(x_1, y_1) - V(x, y, x_1, y_1)\}.$$  

(3)
3. Hamilton-Jacobi equation
We study a discrete-time analog of Hamilton-Jacobi equation associated with $V$,

$$H(u, x, y) = c,$$

(4)

where

$$H(w, x, y) := \sup_{x_1 \in M} \inf_{y_1 \in N} \{ w(x_1, y_1) - w(x, y) - V(x, y, x_1, y_1) \}. $$

See [4] for a minimal discrete-time case. We are interested in finding solutions $u$ of the equation (4) and characterize the constant $c$. To this end we follow Fathi [5] to obtain a “fixed point” of the Lax operator (2).

Is possible to show that the Lax operator satisfies a regularity property. More precisely, there exist a real number $K$, such that $L^ng$ is $K$–Lipschitz, for any $g \in C(M \times N, \mathbb{R})$ and $n \in \mathbb{Z}^+$. In addition $L^n$ satisfies the semigroup property also call dynamic programming principle (see for example [1, 3]).

**Theorem 3.1** If $g \in C(M \times N, \mathbb{R})$ and $m, n \in \mathbb{Z}^+$, then

$$L^{m+n} g(x, y) = L^m \circ L^n g(x, y) = \inf_{\Theta(m) \in \mathcal{E}_N} \sup_{\Psi(m) \in \mathcal{S}_M} \{ L^m g(x_m, \Theta(m) [\Psi(m)]_m) - A^n (\Psi(m), \Theta(m) [\Psi(m)]) \},$$

where $\Theta(m) \in \mathcal{E}_N (x, y, m)$ and $\Psi(m) \in S_M (x, m)$.

Moreover, it is easy to verify that the Lax operator satisfies the following properties, which are needed to determine the existence of solutions of (4).

1. (Monotony) If $f, g \in C(M \times N, \mathbb{R})$ and $g \leq f$, then $L^ng \leq L^nf$.
2. If $g \in C(M \times N, \mathbb{R})$ and $k \in \mathbb{R}$, then $L^n (k + g) = k + L^ng$.
3. (Weak contraction) If $f, g \in C(M \times N, \mathbb{R})$ and $\| L^ng - L^nf \|_\infty \leq \| g - f \|_\infty$.

Using Theorem 3.1 and above properties, Fathi proves in [5] the following result.

**Theorem 3.2** There are a constant $c \in \mathbb{R}$ and a Lipschitz function $u \in C(M \times N, \mathbb{R})$, such that

$$L^nu = u + nc,$$

for any $n \in \mathbb{Z}^+$.

Suppose that there are $c_1 > c_2$ and functions $u_1, u_2 \in C(M \times N, \mathbb{R})$ such that $L^nu_i = u_i + nc_i$, for $i = 1, 2$. If $\varepsilon > 0$ we can choose $\Theta^*(n) \in \mathcal{E}_N (x, y, n)$ such that

$$L^n u_2(x, y) > \sup_{\Psi(n)} \{ u_2(x_n, \Theta^*(n) [\Psi(n)]_n) - A^n (\Psi(n), \Theta^*(n) [\Psi(n)]) \} - \varepsilon.$$

In addition, there is $\Psi^*(n) \in S_M (x, n)$ such that

$$L^nu_1(x, y) < u_1(x_n, \Theta^*(n) [\Psi^*(n)]_n) - A^n (\Psi^*(n), \Theta^*(n) [\Psi^*(n)]) + \varepsilon.$$

Therefore, we deduce that

$$u_1(x, y) + nc_1 < u_1(x_n, \Theta^*(n) [\Psi^*(n)]_n) - A^n (\Psi^*(n), \Theta^*(n) [\Psi^*(n)]) + \varepsilon,$$

$$u_2(x, y) + nc_2 > u_2(x_n, \Theta^*(n) [\Psi^*(n)]_n) - A^n (\Psi^*(n), \Theta^*(n) [\Psi^*(n)]) - \varepsilon.$$
Then
\[
n(c_1 - c_2) < \sup_{M \times N} (u_1 - u_2) - \inf_{M \times N} (u_1 - u_2) + 2\varepsilon.
\]

On the other hand, because \( c_1 > c_2 \), then \( n(c_1 - c_2) \to \infty \) as \( n \to \infty \); which is a contradiction. Consequently the number \( c \) in Theorem (3.2) is unique. According to this result, we say that \( u \) is a fixed point of Lax operator (2) with critical value \( c \). Considering \( \mathcal{L}^1 u(x, y) \), (3) shows that
\[
\sup_{x_1 \in M \forall y_1 \in N} \inf_{\Theta(n)} \{ u(x_1, y_1) - u(x, y) - V(x, y, x_1, y_1) \} = c,
\]
we obtain a solution of the Hamilton-Jacobi equation (4).

**Corollary 3.3** A function \( u \in C(M \times N, \mathbb{R}) \) is a fixed point of Lax operator (2) with critical value \( c \) if and only if \( u \) is a solution of the Hamilton-Jacobi equation (4).

### 4. Peierls Barrier

We now consider an infinite horizon problem. Let \( k \in \mathbb{R} \), following Mañé [6] and Mather [7], we define the lower Peierls barrier by

\[
h_k^-(x, y) := \liminf_{n \to \infty} \left\{ \sup_{\Theta(n)} A^n(\pi(n), \Theta(n)[\pi(n)]) + nk \right\},
\]

where \( \pi(n) \in S_M(x, n) \) and \( \Theta(n) \in \mathcal{E}_N(x, y, n) \). This function has several properties, including the following. For any \( k \in \mathbb{R} \)

- If \( h_k^-(x, y) = -\infty \) for \( (x, y) \in M \times N \), then \( h_k^-(z, w) = -\infty \), for all \( (z, w) \in M \times N \).
- If \( h_k^-(x, y) = +\infty \) for \( (x, y) \in M \times N \), then \( h_k^-(z, w) = +\infty \), for all \( (z, w) \in M \times N \).

We can see that the Peierls barrier has a radical change in their values, to this end we should note that there is \( k > 0 \) large enough, such that \( h_{-k}^-(x, y) = -\infty \) and \( h_k^- (x, y) = +\infty \). It is also clear that \( h_k^- \) is monotone for \( k \): considering \( k, k_1, k_2 \in \mathbb{R} \) with \( k_1 < k < k_2 \) and \( (x, y) \in M \times N \), we have

- If \( h_k^-(x, y) = -\infty \), then \( h_{k_1}^-(x, y) = -\infty \).
- If \( h_k^-(x, y) = +\infty \), then \( h_{k_2}^-(x, y) = +\infty \).
- Moreover, if \( h_k^-(x, y) \in \mathbb{R} \), then \( h_{k_1}^-(x, y) = -\infty \) and \( h_{k_2}^-(x, y) = +\infty \).

On account of previous properties, there are inf \( \{ k \in \mathbb{R} : h_k^- = +\infty \} \), sup \( \{ k \in \mathbb{R} : h_k^- = -\infty \} \) \( \in \mathbb{R} \). In addition, given \( (x, y) \in M \times N \), we can choose \( k_1, k_2 \in \mathbb{R} \) such that \( h_{k_1}^- = -\infty \) and \( h_{k_2}^- = +\infty \), then \( k_1 \leq k_2 \). Therefore sup \( \{ k \in \mathbb{R} : h_k^- = -\infty \} \leq k_2 \) and, in consequence,
\[
\sup \{ k \in \mathbb{R} : h_k^- = -\infty \} \leq \inf \{ k \in \mathbb{R} : h_k^- = +\infty \}.
\]

Suppose that sup \( \{ k \in \mathbb{R} : h_k^- = -\infty \} < \inf \{ k \in \mathbb{R} : h_k^- = +\infty \} \), then there are \( k_0 \in \mathbb{R} \) and \( \varepsilon > 0 \) such that \( h_{k_0}^-(x, y) \in \mathbb{R} \) and
\[
\sup \{ k \in \mathbb{R} : h_k^- = -\infty \} < k_0 - \varepsilon < k_0 + \varepsilon < \inf \{ k \in \mathbb{R} : h_k^- = +\infty \}.
\]

Then we have \( h_{k_0 - \varepsilon}^- = -\infty \) and \( h_{k_0 + \varepsilon}^- = +\infty \), this is a contradiction. Hence
\[
\sup \{ k \in \mathbb{R} : h_k^- = -\infty \} = \inf \{ k \in \mathbb{R} : h_k^- = +\infty \}.
\]
Let $u$ be a fixed point for Lax operator with critical value $c$. For $(x, y) \in M \times N$ and $n \in \mathbb{Z}^+$,
\[
u(x, y) + nc = \mathcal{L}^n u(x, y) = \inf_{\Theta(n) \pi(n)} \sup \{u(x_n, \Theta(n) [\pi(n) n]) - A^n (\pi(n), \Theta(n) [\pi(n)])\},
\]
therefore
\[
\inf u + nc \leq \sup \inf_{\Theta(n) \pi(n)} A^n (\pi(n), \Theta(n) [\pi(n)]),
\]
then $h_c^-(x, y) < +\infty$; on that account $c \leq \inf \{k \in \mathbb{R} : h_k^- = +\infty\}$. Analogously, from (6) we deduce that
\[
\sup u + nc \geq \inf u - \sup \inf_{\Theta(n) \pi(n)} A^n (\pi(n), \Theta(n) [\pi(n)]),
\]
consequently $h_c^-(x, y) > -\infty$ and in addition $c \geq \sup \{k \in \mathbb{R} : h_k^- = -\infty\}$. This provides us a characterization of the critical value $c$.

**Theorem 4.1** For any $(x, y) \in M \times N$, $h_c^-(x, y) \in \mathbb{R}$ and
\[
c = \inf \{k \in \mathbb{R} : h_k^- = +\infty\} = \sup \{k \in \mathbb{R} : h_k^- = -\infty\}.
\]

As in (5) define the upper Peierls barrier by
\[
h_k^+(x, y) := \limsup_{n \to \infty} \left\{ \sup_{\Theta(n) \pi(n)} \inf A^n (\pi(n), \Theta(n) [\pi(n)]) + nk \right\}.
\]

By similar arguments we made above for $h_k^-$, we find analogous properties for $h_k^+$ and deduce the following result.

**Theorem 4.2** For any $(x, y) \in M \times N$, $h_c^+(x, y) \in \mathbb{R}$ and
\[
c = \inf \{k \in \mathbb{R} : h_k^+ = +\infty\} = \sup \{k \in \mathbb{R} : h_k^+ = -\infty\}.
\]

If $(x, y) \in M \times N$ and $\varepsilon > 0$, because $h_c^-(x, y), h_c^+(x, y) \in \mathbb{R}$, there is $N_1 \in \mathbb{Z}^+$ such that
\[
h_c^-(x, y) - 1 < \sup \inf_{\Theta(n) \pi(n)} A^n (\pi(n), \Theta(n) [\pi(n)]) + nc < h_c^+(x, y) + 1,
\]
for $n \geq N_1$. In addition there is $N_2 \in \mathbb{Z}^+$, where $N_2 \geq N_1$, such that $\frac{1}{n} h_c^+(x, y) + 1 < \varepsilon$ and $\frac{1}{n} h_c^-(x, y) - 1 > -\varepsilon$, for $n \geq N_2$. Therefore
\[
\lim_{n \to \infty} \frac{1}{n} \sup_{\Theta(n) \pi(n)} \inf A^n (\pi(n), \Theta(n) [\pi(n)]) + c = 0.
\]

Consequently we find another characterization of the critical value $c$.

**Corollary 4.3** For any $(x, y) \in M \times N$
\[
c = \lim_{n \to \infty} \frac{1}{n} \sup_{\Theta(n) \pi(n)} \inf A^n (\pi(n), \Theta(n) [\pi(n)]).
\]

Considering $n \in \mathbb{Z}^+$, $(x, y) \in M \times N$ and $u \in C(M \times N, \mathbb{R})$ a solution of the Hamilton-Jacobi equation (4), it is possible to find sequences $\{x^*_i\}_{i=0}^n \subset M$ and $\{y^*_i\}_{i=0}^n \subset N$ such that
\[
c = \lim_{n \to \infty} \frac{1}{n} \sup_{\Theta(n) \pi(n)} A^n (\pi^*(n), \Theta^*(n)),
\]
where $\pi^*(n) := \{x^*_i\}_{i=0}^n$ and $\Theta^*(n) := \{y^*_i\}_{i=0}^n$. 

5. Conclusions

In this paper we showed the existence of a fixed point of the Lax operator $\mathcal{L}^n$ with critical value $c$, which is a solution of the Hamilton-Jacobi equation (4), according to Corollary (3.3). We also showed different characterizations of the critical value $c$ given by (7), (8), (9) and (10). An important aspect of the previous method is that it allows us to study the behavior of a game in an infinite horizon problem without using discount factors or average actions.

References

[1] Fleming W H and Soner H M 2006 Controlled Markov processes and viscosity solutions 2nd ed (New York: Springer) pp 375 – 95
[2] Bardi M and Capuzzo-Dolcetta I 1997 Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations (Boston: Birkhäuser) pp 431 – 70
[3] Evans L C and Souganidis P E 1984 Indiana Univ. Math. J. 33 773–97
[4] Gomes D A 2005 Discrete Contin. Dyn. Syst. 13 103–16
[5] Fathi A 1997 C. R. Acad. Sci. Paris I Math. 324 1043–46
Fathi A 1997 C. R. Acad. Sci. Paris Sr. I Math. 325 649 –52
[6] Mañé R 1997 Bol. Soc. Brasil. Mat. (N.S.) 28 141–53
[7] Mather J N 1993 Ann. Inst. Fourier (Grenoble) 43 1349–86