Characterization of Diskcyclic Criterion

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Abstract. It is known that there are diskcyclic operators that are not satisfy the diskcyclic criterion. This work introduced some kinds of diskcyclic operators, and studied its relation with diskcyclic criterion, as: power of operators, infinite direct summand of operators and introduced the concept of hereditarily diskcyclic operators, which are kind of diskcyclic operators.

1. Introduction.

Let $H$ be a separable infinite dimensional Hilbert space, and $T \in B(H)$ is said to be a hypercyclic operator if the orbit of $T$, $\text{orbit}(T, x) := \{T^n x : n \geq 0\}$, is dense in $H$ [3]. The first sufficient condition for hypercyclic (the Hypercyclic Criterion) discovered independently by Kitai [6] and Godefroy and Shapiro [4]. In 1974 Hilden and Wallen [5], generalized the definition of hypercyclic operators to supercyclic by cone orbit, $\text{Corb}(T, x) := \{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\}$, is dense in $H$. Jamil in her Ph. D. thesis [8], partition the cone orbit into three parts, according to unit circle as: diskcyclic operators when $|\alpha| \leq 1$, circle cyclic operators when $|\alpha| = 1$ and codiskcyclic operators when $|\alpha| \geq 1$. But Saavedra and Müller [9], proved that every circle cyclic operators are hypercyclic.

There are many authors studied diskcyclic operators from multiple aspects like: Bamerni defined subspace disk-cyclic and multidiskcyclic operators on Banach Spaces [2], and Yu-Xia Iiang and Ze-Hua Zhou introduced Disk-cyclic and Codisk-cyclic tuples of the adjoint weighted composition operators on Hilbert spaces [7].

A sufficient conditions for diskcyclic operators were found by Jamil in 2002 [8], which is called diskcyclic criterion. In 2016 Bamerni [2], provided another version to the diskcyclic criterion, which is simpler than the main diskcyclic criterion. Moreover, proved that ($T$ satisfies diskcyclic criterion if and only if $\bigoplus_{k=1}^{\infty} T$ is diskcyclic for all $k \geq 2$) [2].

Abdulkareem and Jamil in 2019 gave new versions for diskcyclic criterion, also provided equivalent between $T \otimes T$ is diskcyclic operator and $T$ satisfies any one of the new versions of diskcyclic criterion [1].

In general every operator satisfies diskcyclic criterion is diskcyclic operator but the inverse is incorrect, e.g. $T(x) = 2x$ is diskcyclic operator and $T$ satisfies any one of the new versions of diskcyclic criterion [1].

That is necessary to ask which kinds of diskcyclic operators satisfy diskcyclic criterion?

In this paper, we tried to answer this question in the following cases: first, by power of operator (2.3), second, infinite direct summand of operators (2.4), finally by defined hereditarily diskcyclic operators and clarify its relationship with diskcyclic criterion (3.5).

We remark that $\text{Dc}(T)$ stands for a diskcyclic vectors of $T$, $\mathcal{R}(T)$ is the range of operator of $T$.

\[ \mathbb{N} = \{1, 2, \ldots\}. \]

2. Power and Infinite Direct Summand of Operators.

In this section, we have presented the relation between operators satisfy diskcyclic criterion and their power or direct summand of them. And give application about them.

The following theorem gives specific certain of diskcyclic criterion use in this paper.
2.1 Definition (Diskcyclic Criterion).
Suppose that \( T \in B(H) \) is called satisfies diskcyclic criterion. If there exists an increasing sequence of positive integers \( \{n_k\} \in \mathbb{N} \) and \( \{\alpha_{n_k}\} \in \mathbb{C}; \ 0 < |\alpha_{n_k}| \leq 1 \). For which there are a dense subsets \( Y, X \) in \( H \) and a sequence of mappings,
\[
S_{n_k}: Y \rightarrow H, \text{as } k \rightarrow \infty \text{ such that:}
\]
\[
\alpha_{n_k} T^{n_k} x \rightarrow 0 \text{ for all } x \in X
\]
\[
a) \ \frac{1}{\alpha_{n_k}} S_{n_k} y \rightarrow 0 \text{ for all } y \in Y
\]
\[
b) \ T^{n_k} S_{n_k} y \rightarrow y \text{ for all } y \in Y.
\]

2.2 Theorem [1]:
Every operator satisfies diskcyclic criterion is diskcyclic operator.

2.3 Proposition [2]:
\( T \in B(H) \) satisfies the diskcyclic criterion if and only if \( \bigoplus_{i=1}^{\infty} T \) is diskcyclic operator for all \( r \geq 1 \).

2.4 Proposition [8]:
Let \( x \in \mathbb{D} \mathcal{C}(T) \), and \( S \in B(H) \) such that, \( ST = TS \) and \( \mathcal{R}(S) \) is dense in \( H \). Then \( Sx \) is a diskcyclic vector for \( T \).
In particular, if \( x \) is a diskcyclic vector for \( T \), then for all \( m \geq 0 \), \( T^m x \) is a diskcyclic vector for \( T \).

In the next proposition show, if \( T \) is a diskcyclic operator so \( T^n, (n \geq 1) \).

2.5 Proposition:
Let \( H \) be a separable infinite dimensional Hilbert space, and \( T \in B(H) \) satisfies diskcyclic criterion. Then \( T^n; n \geq 1 \) satisfies diskcyclic criterion.

Proof:
Since \( T \) satisfies diskcyclic criterion, then \( T \oplus T \) is diskcyclic operator by proposition (2.3), and by proposition (2.4), we get \( T^n \oplus T^n = (T \oplus T) \) is diskcyclic, so \( T^n \) satisfies diskcyclic criterion by proposition (2.3).

Abdulkareem and Jamil in [1], proved the equivalent between \( T \in B(H) \) satisfies the diskcyclic criterion and \( T \oplus T \) is diskcyclic operator. The following proposition generalize this fact into infinite case, we will prove the necessary condition, and we note that if \( \bigoplus_{i=1}^{\infty} T \) satisfies diskcyclic criterion, then \( \bigoplus_{i=1}^{\infty} T \) satisfies diskcyclic criterion for all \( r \geq 1 \). Thus sufficient condition satisfies by proposition (2.3).

2.6 Proposition:
Let \( H \) be a separable infinite dimensional Hilbert space, and \( T \in B(H) \), then \( T \) satisfies the diskcyclic criterion if and only if \( \bigoplus_{i=1}^{\infty} T \) satisfies diskcyclic criterion on \( \prod_{i=1}^{\infty} H \).

Proof:
Let \( T \) satisfies diskcyclic criterion, and \( X, Y \) be dense subsets in \( H \), \( \{n_k\} \in \mathbb{N} \), \( \{\alpha_{n_k}\} \in \mathbb{C}; \ 0 < |\alpha_{n_k}| \leq 1 \) and \( S_{n_k} \) are satisfies (1) and (2) of definition (2.1), then
\[
\prod_{i=1}^{\infty} X \text{ and } \prod_{i=1}^{\infty} Y \text{ are dense in } \prod_{i=1}^{\infty} H.
\]
Hence, for all \( (x_1, x_2, ...) \in \prod_{i=1}^{\infty} X \) and \( (y_1, y_2, ...) \in \prod_{i=1}^{\infty} Y \),
\[
\alpha_{n_k} (\bigoplus_{i=1}^{\infty} T) ^{n_k} (x_1, x_2, ...
\]
\[
= (\alpha_{n_k} T^{n_k} x_1, \alpha_{n_k} T^{n_k} x_2, ... \rightarrow (0,0,...).
\]
\[
a) \ \frac{1}{\alpha_{n_k}} S_{n_k} y_1, \frac{1}{\alpha_{n_k}} S_{n_k} y_2, ...ightarrow (0,0,...).
\]
\[
= (\frac{1}{\alpha_{n_k}} S_{n_k} y_1, \frac{1}{\alpha_{n_k}} S_{n_k} y_2, ... \rightarrow (0,0,...).
\]
b) \((\bigoplus_{i=1}^{n} T)^{n_{k}}(\bigoplus_{i=1}^{n_{k}} S_{n_{k}})(y_{1}, y_{2}, ...)
\quad = (T^{n_{k}} S_{n_{k}} y_{1}, T^{n_{k}} S_{n_{k}} y_{2}, ... ) \to (y_{1}, y_{2}, ...).
\]
So, \(\bigoplus_{i=1}^{n} T\) satisfies the diskcyclic criterion. ■

Given \(H\) be a separable infinite dimensional real Hilbert space, we will denote by \(\bar{H}\) its complexification. That is, \(\bar{H}\) will denote the product space \(H \times H\) endowed with the complex scalar product given by
\[
(a + ib)(x, y) := (ax - by, ay + bx), \quad x, y \in H, \quad a, b \in \mathbb{R}
\]
Also, given \(T \in B(H)\), we will denote its complexification by \(\tilde{T} \in B(\bar{H})\). That is,
\[
\tilde{T}(x, y) := (Tx, Ty), \quad \text{for all } x, y \in H.
\]

2.7 Corollary:
Let \(H\) be a separable infinite dimensional real Hilbert space, and \(T \in B(H)\). If \(\tilde{T} \in B(\bar{H})\) denotes its complexification, then \(T\) satisfies the diskcyclic criterion if and only if \(\tilde{T}\) does. Moreover, \(\tilde{T}\) satisfies the diskcyclic criterion whenever it is diskcyclic.
Proof:
Let \(T\) satisfies diskcyclic criterion then by proposition (2.6) \(\bigoplus_{i=1}^{n} T\) satisfies diskcyclic criterion, and since \(T \oplus T\) commute with \(\tilde{T}\) by proposition (2.4). Hence \(\tilde{T}\) satisfies diskcyclic criterion. By the same arguments the conversely it is true.

3. Hereditarily Diskcyclic Operators.
In this section, we will introduce the concept of hereditarily diskcyclic operators and study their relation with diskcyclic criterion.

3.1 Definition:
Let \(T \in B(H)\), a sequence \(\{T^{n_{k}}\}\) is called diskcyclic operator if there exist a vector \(x \in H\) such that
\[
\{\alpha T^{n_{k}} x : k \geq 1, \alpha \in \mathbb{C}; 0 < |\alpha| \leq 1\} = H, \quad \text{and } x \text{ is called diskcyclic vector for } \{T^{n_{k}}\}. \]
We will denote the set \(\{\alpha T^{n_{k}} x : k \geq 1, \alpha \in \mathbb{C}; 0 < |\alpha| \leq 1\}\) by \(\text{Dorbt}(T^{n_{k}}, x)\).

In the next lemma shows, if \(\{T^{n_{k}}\}\) is a diskcyclic operator then every power \(T^{m}\) dose \((m \geq 1)\).

3.2 Lemma:
Let \(x\) be a diskcyclic vector for \(\{T^{n_{k}}\}\), where \((n_{k})\) be a subsequence \((n)\), \(S \in B(H)\),such that,
\(ST = TS\) and \(\mathcal{R}(S)\) is dense in \(H\). Then \(Sx\) is a diskcyclic vector for \(\{T^{n_{k}}\}\). In particular, if \(x\) is a diskcyclic vector for \(\{T^{n_{k}}\}\) then for all \(m \geq 1\), \(T^{m} x\) is a diskcyclic vector for \(\{T^{n_{k}}\}\).
Proof:
Let \(x\) be a diskcyclic vector of \(\{T^{n_{k}}\}\), then \(\text{Dorbt}(T, x)\) is dense in \(H\). Therefore
\[
\text{Dorbt}(T^{m}, Sx) = \{[\alpha T^{n_{k}}(Sx) : k \geq 1, \alpha \in \mathbb{C}; 0 < |\alpha| \leq 1]\} = \mathcal{R}(S(T^{m} x)) = S(H) = H. \] ■
Next result discusses the equivalent statements on the diskcyclic operators. Since \((3) \Rightarrow (1)\) is trivial we will omit its proof.

3.3 Proposition:
Let \(H\) be a separable infinite dimensional Hilbert space, and \(\{T^{n_k}\}_{k \geq 1} \in B(H)\)

the following are equivalent:

1- The sequence \(\{T^{n_k}\}\) is diskcyclic.

2- For each non-empty open sets \(U, V\) there are \(\alpha \in \mathbb{C} ; 0 < |\alpha| \leq 1\) and \(r \in \mathbb{N}\) such that \(T^{n_r}(\alpha U) \cap V \neq \emptyset\).

3- The set \(\mathbb{D}(\{T^{n_k}\}_{k \geq 1})\) of diskcyclic vectors for \(\{T^{n_k}\}_{k \geq 1}\) is a dense \(G_\delta\) subset of \(H\).

Proof:

1) \(\Rightarrow\) 2): Let \(U, V\) be non-empty open sets in \(H\), and take \(x\) any diskcyclic vector for \(\{T^{n_k}\}_{k \geq 1}\). By definition (3.1), choose \(k_1 \in \mathbb{N}\) and \(\alpha_1 \in \mathbb{C} ; 0 < |\alpha_1| \leq 1\) such that \(\alpha_1 T^{n_{k_1}} x \in U\).

Put \(x_0 = \alpha_1 T^{n_{k_1}} x\), by lemma (3.2) \(x_0\) is a diskcyclic vector for \(\{T^{n_k}\}_{k \geq 1}\). So there is \(k_2 \in \mathbb{N}\) arbitrarily large enough, and \(\alpha_2 \in \mathbb{C} ; 0 < |\alpha_2| \leq 1\), such that \(\alpha_2 T^{n_{k_2}} x_0 \in V\).

Hence \(\alpha_2 T^{n_{k_2}}(U) \cap V \neq \emptyset\).

2) \(\Rightarrow\) 3): Let \(\{B_x\}_{x \geq 1}\) be a countable basis for the topology on \(H\).

We have to show that the set \(\mathbb{D}(\{T^{n_k}\}_{k \geq 1}) = \cap_s(\cup_{\beta \in \mathbb{C}} \cup_k T^{-n_k}(\beta B_\beta))\). In fact,

\[
\text{let } U \text{ be a non-empty open set in } H, \text{ then by (2) there exist } s \geq 1, \alpha \in \mathbb{C} ; 0 < |\alpha| \leq 1 \text{ and } k \in \mathbb{N}, \text{ such that } \frac{1}{\alpha} T^{-n_k} B_\beta \cap U \neq \emptyset. \text{ Then, } \cup_{\beta \in \mathbb{C}} \cup_k T^{-n_k}(\beta B_\beta) \text{ is dense in } H.
\]

the result is done by (Baire theorem), and since \(\mathbb{D}(\{T^{n_k}\}_{k \geq 1}) = \cap_s(\cup_{\beta \in \mathbb{C}} \cup_k T^{-n_k}(\beta B_\beta))\)

Now we will present the definition of hereditarily diskcyclic.

3.4 Definition:
A \(T \in B(H)\) said to be hereditarily diskcyclic if there exists a non negative integer sequence \((n_k)\) such that for all subsequences \((n_{k_j})\) of \((n_k)\) ; \(\{T^{n_{k_j}}\}_{j \geq 1}\) is diskcyclic.

Next theorem shows the relation between diskcyclic criterion and hereditarily diskcyclic.

3.5 Theorem:
Let \(H\) be a separable infinite dimensional Hilbert space, and let \(T \in B(H)\). Then \(T\) satisfies the diskcyclic criterion if and only if \(T\) is hereditarily diskcyclic.

Proof:

\(\Rightarrow\) Let \(T \in B(H)\) satisfies the diskcyclic criterion, then there exist \(X, Y\) are dense sets in \(H\), \((n_k)\) be a sequence of nonnegative integers and \(S_{n_k} : Y \to H\) satisfies (1) and (2) above.

Let \((n_{k_j})\) be a subsequence of \((n_k)\), and \(\alpha_{n_{k_j}} \in \mathbb{C} ; 0 < |\alpha_{n_{k_j}}| \leq 1\) then:

\[
\alpha_{n_{k_j}} T^{n_{k_j}} x \to 0 \text{ for all } x \in X.
\]

\[
a_{n_{k_j}} S_{n_{k_j}} y \to 0 \text{ for all } y \in Y.
\]
b) \( T^{n_k} S_{n_k} y \to y \) for all \( y \in Y \).

Hence we have to show \( \{ T^{n_k} \}_j \) is diskcyclic operator.

Now, let \( U \) and \( V \) be non-empty open sets in \( H \), pick \( x \in X, y \in Y, \varepsilon > 0 \).

Then there exist \( x \in U \cap X \) and \( y \in V \cap Y \). Hence by (1) and (2).

\[
x + \frac{1}{a_{n_r}} S_{n_r} y \to x \text{ and } T^{n_r} S_{n_r}(x + \frac{1}{a_{n_r}} S_{n_r} y) \to y.
\]

Thus there is \( k_j \) arbitrarily large enough, satisfying

\[
x + \frac{1}{a_{n_r}} S_{n_r} y \in U \text{ and } \alpha a_{n_r} T^{n_r}(x + \frac{1}{a_{n_r}} S_{n_r} y) \in V.
\]

Therefore \( \alpha a_{n_r} T^{n_r}(U) \cap V \neq \emptyset \), then by proposition (3.3), \( \{ T^{n_k} \}_j \) is diskcyclic operator.

\( \Rightarrow \) Suppose that \( T \) is hereditarily diskcyclic, let \( U_i, V_i \) be non-empty open subsets of \( H \); \( i = 1, 2 \).

Since \( \{ T^{n_k} \}_j \) is diskcyclic for all subsequence \( (n_{k_j}) \) of \( (n_k) \), then by proposition (3.3), there exists a subsequence \( (n_{k_j}) \) of \( (n_{k_j}) \) and, for all \( r \geq 1, \alpha_r \in \mathbb{C} ; 0 < |\alpha_r| \leq 1 \) with \( T^{n_{k_j}}(\alpha_r U_1) \cap V_1 \neq \emptyset \).

But \( \{ T^{n_{k_j}} \}_j \) is also hereditarily, hence by proposition (3.3), there exists \( m \in (n_{k_{j_m}}) \) an arbitrarily large enough so that \( T^m(\alpha U_2) \cap V_2 \neq \emptyset \). where \( \alpha \in \mathbb{C} ; 0 < |\alpha| \leq 1 \), so \( T^m(\alpha U_1) \cap V_1 \neq \emptyset ; m \in (n_{k_{j_m}}) \).

Thus \( T \oplus T \) is diskcyclic operator and by proposition (2.3), \( T \) satisfies the diskcyclic criterion. ■

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