An Asymptotic Analysis of the Gradient Remediability Problem for Disturbed Distributed Linear Systems

Soraya Rekkab1*, Samir Benhadid1*, Raheam Al-Saphory2*

1Department of Mathematics, Faculty of Exact Science, University of Mentouri, Constantine, Algeria.
2Department of Mathematics, College of Education for Pure Sciences, University of Tikrit, Tikrit, Iraq.
*Corresponding author: rekkabsoraya@gmail.com
E-mail addresses: ihebmaths@yahoo.fr, sahory@tu.edu.iq

Abstract:
The goal of this work is demonstrating, through the gradient observation of a disturbed distributed parameter systems of type linear (DDPL-systems), the possibility for reducing the effect of any disturbances (pollution, radiation, infection, etc.) asymptotically, by a suitable choice of related actuators of these systems. Thus, a class of asymptotically gradient remediable system (AGR-system) was developed based on finite time gradient remediable system (GR-system). Furthermore, definitions and some properties of this concept AGR-system and asymptotically gradient controllable system (AGC-controllable) were stated and studied. More precisely, asymptotically gradient efficient actuators ensuring the weak asymptotically gradient compensation system (WAGC-system) of known or unknown disturbances are examined. Consequently, under convenient hypothesis, the existence and the uniqueness of the control of type optimal, guaranteeing the asymptotically gradient compensation system (AGC-system), are shown and proven. Finally, an approach that leads to a Mathematical approximation algorithm is explored.

Keywords: Asymptotic analysis, Controllability, Disturbance, Optimal control, Remediability.

Introduction:
Driven by environmental, pollution1, radiation and infection problems 2–3, the authors have studied the problem with regard to the gradient observation of a class of DDPL-systems considering the possibility of lessening or compensating asymptotically the effect of any disturbances. Thus, the study constitutes a development to the case of asymptotic type for the previous investigates to the remediability linear parabolic problem of different systems, introduced in the finite time case 4–7 and asymptotic case 4,8,9.

One can note that studying compensation problem with respect to the gradient observation and the so-called gradient remediability, is of considerable interest 10. Thus, it was shown that there exists a system that is not remediable, however may be gradient remediable.

Gradient remediability concept in usual and regional case is considered and studied for DPL-systems 10–12. Regarding the asymptotic case aspect 13, the great importance of the asymptotic analysis in systems theory 14–15, takes into consideration the problem of AGC-systems and studies a prospective extension of the development methods, in addition to analyzing the results in finite time. Hereafter, through likeness the relationship among the remediability and controllability of the gradient case has been inspected and studied in a considerable time.

Also, the link among remediability and controllability in asymptotic gradient case has been studied and analyzed.

This paper is structured as follows: Section 2, is devoted to the introduction of the gradient remediability concepts of type exact and weak under convenient hypothesis.

Section 3 relates to the asymptotic form in various cases in connection with suitable actuators and sensors. Also, an asymptotically gradient efficient actuators enable the guaranteeing an asymptotic gradient compensation of weak type is presented.

In section 4, weakly and exactly a asymptotically gradient controllable system
(WEAGC-system) are defined and characterized. Then, the link between WEAGC-system and weakly and exactly asymptotically gradient remediable system (WEAGR-system) are studied and analyzed, and it is shown that AGR-system is dependent on the appropriate sensors with corresponding actuators. While, in section 5, the AGR-problem through the energy of type minimum is examined.

In the last section, the control of optimal type, is used to obtain a mathematical algorithm approach.

**Formulation of the Considered Problem:**

Assume that $\Omega$ stands as an open and bounded set in $IR^n$, with a boundary of smooth type $\partial \Omega$. Considering a class of DDPL-system defined by the form:

$$\begin{aligned}
(S) \{ \dot{y}(t) &= A y(t) + B u(t) + f(t) ; 0 < t < T \\
y(0) &= y_0
\end{aligned}$$

where $A$ generates a strongly continuous semi-group $S(t)_{t \geq 0}, B \in L(U, Y), u \in L^2(0, T; U), \ Y$ is a space of Hilbert type denoted the input space and $X = H^1_0(\Omega)$, the space of state.

The system $(S)$ admits a unique solution $y \in C \left(0, T; H^1_0(\Omega) \right) \cap C^1 \left(0, T; L^2(\Omega) \right)$ given by 13:

$$y(t) = S(t) y_0 + \int_0^t S(t-s) B u(s) ds + \int_0^t f(s) ds$$

The system $(S)$ is augmented by the following output (gradient observation) equation:

$$(O) z_{u,f}(t) = C \nabla y(t) ; 0 < t < T$$

where $C \in L \left( \left( L^2(\Omega) \right)^n, \ Y \right),$ $Y$ is a Hilbert space (gradient observation space) and $\nabla$ is the operator defined by:

$$\nabla : H^1_0(\Omega) \rightarrow \left( L^2(\Omega) \right)^n$$

$$y \rightarrow \nabla y = \left( \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n} \right)$$

while $\nabla^*$ its adjoint operator. Then, the gradient observation at the final time $T$ is given by:

$$z_{u,f}(T) = C \nabla S(T) y_0 + C \nabla H_T u + C \nabla F_T f$$

where $H_T$ and $F_T$ are operators formulated by

$$H_T : L^2 \left( 0, T; U \right) \rightarrow X$$

$$u \rightarrow H_T u = \int_0^T S(T-s) B u(s) ds$$

and

$$F_T : L^2 \left( 0, T; \rightarrow X \right)$$

$$f \rightarrow F_T f = \int_0^T S(T-s) f(s) ds$$

In the autonomous case, that is to say, deprived of disturbance ($f = 0$) and control ($u = 0$) the observation of gradient, $z_{0,0}(\cdot) = C \nabla S(\cdot) y_0$, is then normal. But if the system is disturbed by a term $f$, the gradient observation becomes

$$z_{0,f}(T) = C \nabla S(T) y_0 + C \nabla F_T f$$

Generally $z_{0,f}(\cdot) \neq C \nabla S(\cdot) y_0$. Then a control term $Bu$ is introduced in order to reduce, in finite time, the effect of this disturbance according to the gradient observation, such that: For any $f \in L^2(0, T; X)$, there exists $u \in L^2(0, T; U)$ satisfying

$$C \nabla H_T u + C \nabla F_T f = 0$$

The next definition 1 characterizes the gradient remediable notion of type exactly and weakly in finite time as follows:

**Definition 1**

1. System $(S)$ augmented by $(O)$, (or $(S) + (O)$) is called exactly gradient remediable (EGR-system) on $[0, T]$, if for every $f \in L^2(0, T; X)$, there exists a control $u \in L^2(0, T; U)$ such that $C \nabla H_T u + C \nabla F_T f = 0$.

2. $(S) + (O)$ is called weakly gradient remediable (WGR-system) on $[0, T]$, if for every $f \in L^2(0, T; X)$ and for every $\epsilon > 0$, there exists a control $u \in L^2(0, T; U)$ such that $\|C \nabla H_T u + C \nabla F_T f\|_Y < \epsilon$.

**Remark 1**

The finite time gradient compensation problem is equivalent to:

For any $f \in L^2(0, T; X)$, does there exists a control $u \in L^2(0, T; U)$ such that

$$\int_0^T C \nabla S(T-s) B u(s) ds + \int_0^T C \nabla S(T-s) f(s) ds = 0$$

or equivalently

$$\int_0^T C \nabla S(t) B v(t) dt + \int_0^T C \nabla S(t) g(t) dt = 0$$

where $g(t) = f(t) - t$ and $v(t) = u(t) - t$.

Consequently, the finite time gradient remediability of $(S) + (E)$ can be also formulated as follows:

For any $g \in L^2(0, T; X)$, there exists a control $v \in L^2(0, T; U)$ satisfying Eq.1.

The characterization consequences on the WTEGR-systems and in limited time have been

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established by Rekkab and Benhadid, and they have shown that the remediability concept of type gradient is a weaker than controllability of type gradient 10.

Asymptotic Gradient Compensation Problem: Formalism statement:

An asymptotic analysis of the problem is given by considering the system:

\[
(S_\infty) \begin{cases} 
\dot{y}(t) = Ay(t) + Bu(t) + f(t) \; ; t > 0 \\
y(0) = y_0 
\end{cases}
\]

augmented by the output (gradient observation) equation:

\[
(O_\infty) \quad z_{uf}(t) = C \nabla y(t) ; t > 0
\]

with \( f \in L^2(0, +\infty; \mathcal{X}) \) and \( u \in L^2(0, +\infty; \mathcal{U}) \). Let us consider the following operators

\[
H_\infty: L^2(0, +\infty; \mathcal{U}) \rightarrow \mathcal{X}
\]

\[
u \rightarrow H_\infty \nu = \int_0^{+\infty} S(s)B\nu ds
\]

and

\[
F_\infty: L^2(0, +\infty; \mathcal{X}) \rightarrow \mathcal{X}
\]

\[
f \rightarrow F_\infty f = \int_0^{+\infty} S(s)f(s) ds
\]

The asymptotic gradient remediability problem was studied to consist an investigation regarding the output operator \( \mathcal{C} \), the existence of an input one \( \mathcal{B} \) confirming the gradient compensation asymptotically of any disturbance, that is: For any \( f \in L^2(0, +\infty; \mathcal{X}) \), there exists \( u \in L^2(0, +\infty; \mathcal{U}) \) such that

\[
\nabla C H_\infty \nu + \nabla F_\infty f = 0
\]

Note that the operators \( H_\infty \) and \( F_\infty \) are not generally well defined. They are, if and only if the following condition is verified 14:

\[
\exists k \in L^2(0, +\infty; \mathbb{R}^+) \text{ such that } \|S(t)\| \leq k(t) ; \forall t \geq 0
\]

Remark 2

- If \( (S(t))_{t \geq 0} \) is exponentially stable, that is to say, if \( \exists \beta > 0 \text{ and } \exists \alpha > 0 \) such that

\[
\|S(t)\| \leq \beta e^{-\alpha t} ; \forall t \geq 0
\]

then Eq.3 is satisfied with \( k(t) = \beta e^{-\alpha t} \in L^2(0, +\infty; \mathbb{R}^+) \), consequently \( H_\infty \) and \( F_\infty \) are well defined. This hypothesis concern the choice of the dynamics \( \mathcal{A} \) of the system through the semi-group \( (S(t))_{t \geq 0} \) and also the input operator \( \mathcal{B} \).

- Actually, this work is concerned with the operators \( K_\infty^C \) and \( R_\infty^C \) which are defined by

\[
K_\infty^C : L^2(0, +\infty; \mathcal{U}) \rightarrow \mathcal{Y}
\]

\[
u \rightarrow K_\infty^C \nu = \int_0^{+\infty} C \nabla S(t)B\nu (t)dt
\]

and

\[
R_\infty^C : L^2(0, +\infty; \mathcal{X}) \rightarrow \mathcal{Y}
\]

\[
f \rightarrow R_\infty^C f = \int_0^{+\infty} C \nabla S(t)f(t) dt
\]

Then some weaker hypotheses are needed than Eq.3. Certainly, it is supposed that \( \exists k \in L^2(0, +\infty; \mathbb{R}^+) \) satisfied

\[
\|C \nabla S(t)\| \leq k(t) ; \forall t \geq 0
\]

In this case, \( K_\infty^C \) and \( R_\infty^C \) are well defined and Eq.2 becomes:

\[
K_\infty^C u + R_\infty^C f = 0
\]

Under hypothesis Eq.4, therefore, the \textit{WEAGR}-system can be expressed in the next manner:

**Definition 2**

(i) \( (S_\infty) + (O_\infty) \) is called \textit{EAGR}-system, if \( \forall f \in L^2(0, +\infty; \mathcal{X}) \), there exists a control \( u \in L^2(0, +\infty; \mathcal{U}) \) such that \( K_\infty^C u + R_\infty^C f = 0 \).

(ii) \( (S_\infty) + (O_\infty) \) is called \textit{WAGR}-system, if \( \forall f \in L^2(0, +\infty; \mathcal{X}) \) and every \( \varepsilon > 0 \) there exists a control \( u \in L^2(0, +\infty; \mathcal{U}) \) such that \( \|K_\infty^C u + R_\infty^C f\|_{\mathcal{IR}^q} < \varepsilon \).

Let us note that for \( T > 0 ; \; f \in L^2(0, +\infty; \mathcal{X}) \) and \( u \in L^2(0, +\infty; \mathcal{U}) \) and under hypothesis Eq.4, it follows that:

\[
K_\infty^C u + R_\infty^C f = \int_0^T C \nabla S(t)B u(t)dt + \int_0^{+\infty} C \nabla S(t)f(t)dt
\]

\[
+ \int_0^T C \nabla S(t)B u(t)dt + \int_0^{+\infty} C \nabla S(t)f(t)dt
\]

\[
= \int_0^T C \nabla S(t)B u(t)dt + \int_0^T C \nabla S(t)f(t)dt + \left[ \varepsilon_1(T) + \varepsilon_2(T) \right]
\]

where \( \varepsilon_1(T) = \int_T^{+\infty} C \nabla S(t)B u(t)dt \) and
\[ \varepsilon_2(T) = \int_T^{+\infty} \nabla V_S(t)f(t)dt, \]
\[ \varepsilon_1(T) + \varepsilon_2(T) \to 0 \text{ when } T \to +\infty, \text{ then for any } f \in L^2(0, +\infty; X) \text{ and } u \in L^2(0, +\infty; U), \text{ it follows that} \]
\[ \lim_{T \to +\infty} \left( \int_0^T \nabla V_S(t)Bu(t)dt + \int_0^T \nabla V_S(t)f(t)dt \right) = K_\theta^\infty u + R_\theta^\infty f \]

**Characterization:**

For the following results, let \( B^* \) and \( C^* \) be the adjoint operators of \( B \) and \( C \) respectively and \((S^*(t))_{t \geq 0}\) is considered for the semigroup of \((S(t))_{t \geq 0}\) of type adjoint. Let also \( X', U' \) and \( Y' \) be the dual space of \( X, U \) and \( Y \). Under hypothesis Eq.4, the following general characterization results are obtained:

**Proposition 1**

The following properties are equivalent

(i) \( (S_\infty) + (E_\infty) \) is EAGR-system.
(ii) \( \text{Im}(R_\infty^C) = \text{Im}(K_\theta^\infty) \).
(iii) \( \exists \gamma > 0 \text{ such that } \forall \theta \in Y' \), it follows that
\[ \|S^*(.)V^*C^*\theta\|_{L^2(0, +\infty; X')} \leq \gamma \|B^*S^*(.)V^*C^*\theta\|_{L^2(0, +\infty; U')} \]

**Proof**

(i) \( \iff \) (ii) Derives from Definition 1. Indeed, it is assumed that \( (S_\infty) + (E_\infty) \) is EAGR-system.

Let \( y \in \text{Im}(R_\infty^C) \), then there exists \( f \in L^2(0, +\infty; X) \) such that \( y = R_\infty^C f \).

From the property of exact asymptotic gradient remediability for the considered system, there exists \( u \in L^2(0, +\infty; U) \) such that \( K_\theta^\infty u + R_\theta^\infty f = 0 \implies R_\infty^C f = -K_\theta^\infty u \).

By the linearity of the operator \( K_\theta^\infty \), it follows that \( y = R_\infty^C f = K_\theta^\infty (-u) \), then \( y \in \text{Im}(K_\theta^\infty) \).

The other inclusion is obtained as the previous one. Then, it follows that \( \text{Im}(R_\infty^C) = \text{Im}(K_\theta^\infty) \).

- **Conversely**, it is assumed that \( \text{Im}(R_\infty^C) = \text{Im}(K_\theta^\infty) \) and one can show that \( (S_\infty) + (E_\infty) \) is EAGR-system.

Let \( f \in L^2(0, +\infty; X) \), then \( R_\infty^C f \in \text{Im}(R_\infty^C) \). Since \( \text{Im}(R_\infty^C) \subset \text{Im}(K_\theta^\infty) \), it follows that \( R_\infty^C f \in \text{Im}(K_\theta^\infty) \) then there exists \( u \in L^2(0, +\infty; U) \) such that \( R_\infty^C f = K_\theta^\infty u \), this gives \( R_\infty^C f - K_\theta^\infty u = 0 \) and by putting \( u_1 = -u \in L^2(0, +\infty; U) \). Thus \( R_\infty^C f + K_\theta^\infty u_1 = 0 \) where \((S) + (E)\) is EAGR-system.

(ii) \( \iff \) (iii) Derives from the fact that the adjoint operators \((R_\infty^C)^*\) and \((K_\theta^\infty)^*\) of \((R_\infty^C)\) and \((K_\theta^\infty)\) respectively, are defined by
\[ (K_\theta^\infty)^*: Y' \to L^2(0, +\infty; X') \]
\[ \theta \to (R_\infty^C)^*\theta = S^*(.)V^*C^*\theta \]

and
\[ (K_\theta^\infty)^*: Y' \to L^2(0, +\infty; U') \]
\[ \theta \to (K_\theta^\infty)^*\theta = B^*(R_\infty^C)^*\theta = B^*S^*(.)V^*C^*\theta \]
Set \( P = (R_\infty^C)^* \), \( Q = (K_\theta^\infty)^* \) and use the following lemma.

**Lemma 1**

Let \( X, Y, Z \) be spaces of Banach reflexive type, \( P \in \Psi(X, Z) \) and \( Q \in \Psi(Y, Z) \). There is an equivalence between:

\[ \text{Im}(P) \subset \text{Im}(Q) \]

and
\[ \exists \gamma > 0 \text{ such that } \|Pz^*\|_{X'} \leq \gamma \|Qz^*\|_{Y'}, \forall z^* \in Z'. \]

The following **Proposition 2** is proved with regard of the weak asymptotic gradient remediability characterization.

**Proposition 2**

There is equivalence between

(i) \( (S_\infty) + (O_\infty) \) is WAGR-system.
(ii) \( \text{Im}(R_\infty^C) \subset \text{Im}(K_\infty^\theta) \).
(iii) \( \ker(B^*(R_\infty^C)^*) = \ker((R_\infty^C)^*) \).

**Proof**

(i) \( \iff \) (ii) Derives from Definition 1. Indeed, it is assumed that \( (S_\infty) + (O_\infty) \) is WAGR-system.

Let \( f \in L^2(0, +\infty; X) \), then \( \forall \varepsilon > 0, \exists u \in L^2(0, +\infty; U) \) such that
\[ \|K_\theta^\infty u + R_\theta^\infty f\|_Y < \varepsilon, \text{ that is to say } \|K_\theta^\infty u\|_Y < \varepsilon. \]

Set \( u = -u \in L^2(0, +\infty; U) \), then \( \forall \varepsilon > 0, \exists u_1 \in L^2(0, +\infty; U) \) such that \( \|R_\infty^C f - K_\theta^\infty u_1\|_Y < \varepsilon \), this gives \( R_\infty^C f \in \text{Im}(K_\theta^\infty) \), where \( \text{Im}(R_\infty^C) \subset \text{Im}(K_\theta^\infty) \).

Conversely, assume that \( \text{Im}(R_\infty^C) \subset \text{Im}(K_\theta^\infty) \) and let \( f \in L^2(0, +\infty; X) \), then \( R_\infty^C f \in \text{Im}(K_\theta^\infty) \), then \( \forall \varepsilon > 0, \exists u_1 \in L^2(0, +\infty; U) \) such that \( \|R_\infty^C f - K_\theta^\infty u_1\|_Y < \varepsilon \).

Put \( u_1 = -u \in L^2(0, +\infty; U) \), then \( \forall \varepsilon > 0, \exists u \in L^2(0, +\infty; U) \) such that \( \|R_\infty^C f + K_\theta^\infty u\|_Y < \varepsilon \) where \( (S_\infty) + (E_\infty) \) is WAGR-system.

(ii) \( \iff \) (iii) by considering orthogonal. Indeed, it is assumed that \( (S_\infty) + (O_\infty) \) is WAGR-system.

**One can show that** \( \ker(B^*(R_\infty^C)^*) = \ker((R_\infty^C)^*) \).

Let \( \theta \in IR^q \text{ such that } B^*(R_\infty^C)^* \theta = 0 \).

In addition, \( (K_\theta^\infty)^* = B^*(R_\infty^C)^* \), this gives \( \theta \in \ker((K_\theta^\infty)^*) \). Thus \( \text{Im}(K_\theta^\infty) = \ker((K_\theta^\infty)^*) \).

By Proposition 3.5, if follows that \( \text{Im}(R_\infty^C) \subset \text{Im}(K_\theta^\infty) \). Then, \( \text{Im}(R_\infty^C) \subset \text{Im}(K_\theta^\infty) \).

\[ \forall f \in L^2(0, +\infty; X), R_\infty^C f (\in \ker((K_\theta^\infty)^*)) \]

\[ \implies (R_\infty^C f, \theta) = 0, \text{ because } \theta \in \ker((K_\theta^\infty)^*) \]

\[ \implies (\ker(B^*(R_\infty^C)^*)) \]

and \( \ker(B^*(R_\infty^C)^*) \subset \ker((R_\infty^C)^*) \) where \( \ker(B^*(R_\infty^C)^*) = \ker((R_\infty^C)^*) \).
Conversely, assume that $\text{Ker}(B^*(R_0^C)') = \text{Ker}(R_0^C)$ and one can show that $\text{Im}(R_0^C) \subset \text{Im}(K_{C_0}^C)$. Let $f \in L^2(0, +\infty; X)$ such that $f \in \text{Im}(R_0^C)$, it follows that $\text{Im}(K_{C_0}^C) = (\text{Ker}(K_{C_0}^C))^\perp$.

For every $\theta \in IR^q$ such that $(R_0^C)\theta = 0$ that is, $B^*(R_0^C)\theta = 0$, it follows that $(R_0^C)\theta = 0$ because $\text{Ker}(B^*(R_0^C)') = \text{Ker}((R_0^C)')$, then $(R_0^C f, \theta) = 0$.

**Asymptotic Gradient Remediability via Actuators and Sensors:**

In connection with the system $(S_{o_0})$ is motivated by $(\Omega_k, g_k)_{k \in \mathbb{N}}$, actuators suite of type zone with $g_i \in L^2(\Omega_k)$ and $\Omega_k = \text{Supp}(g_k) \subset \Omega, \forall k = 1, \ldots, p$, with control space $U = \mathbb{R}^p$ and $B$ is specified by

$$B : \mathbb{R}^p \rightarrow X$$

$u(t) \rightarrow Bu(t) = \sum_{k=1}^{p} X_{\Omega_k}g_ku_k(t)$

and where $u = (u_1, \ldots, u_p) \in L^2(0, +\infty; \mathbb{R}^p)$. Its adjoint is given by

$$B^* z = \left( \langle g_1, z_1 \rangle_{L^2(\Omega_1)}, \ldots, \langle g_p, z_p \rangle_{L^2(\Omega_p)} \right) \in \mathbb{R}^p$$

then, the following result is obtained:

**Corollary 1**

$(S_{o_0}) + (O_{o_0})$ is EAGR-systems $\iff \exists \gamma > 0$ satisfied the next inequality

$$\int_0^{+\infty} \|S^*(t)P^*C^*\theta\|_{L^2}^2 dt \leq \gamma \int_0^{+\infty} \sum_{k=1}^{p} \langle (g_k, S^*(t)P^*C^*\theta) \rangle^2 dt$$

for every $\theta \in Y$.

**Proof**

By supposing the output function $(S_{o_0})$ is specified via suite of sensor of type zones $(D_p, h_i)_{1 \leq i \leq q}$, $h_i \in L^2(D_i)$, represent the distribution zone sensor, $D_i = \text{Supp} h_i \subset \Omega$, intended for $l = 1, \ldots, q$ as well as $D_l \cap D_j = \emptyset$ for $l \neq j, Y = \mathbb{R}^q$ and the operator $C$ is formed by

$$C : (L^2(\Omega))^n \rightarrow \mathbb{R}^q$$

$$y(t) \rightarrow C y(t) = \left( \sum_{i=1}^{n} (h_i, y_i(t))_{D_i}, \ldots, \sum_{i=1}^{n} (h_q, y_q(t))_{D_q} \right)$$

its adjoint is given by $C^* \text{ with for } \theta = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$

$$C^* \theta = \left( \sum_{i=1}^{q} \chi_{D_l} \theta_i, \ldots, \sum_{i=1}^{q} \chi_{D_l} \theta_i \right) \in (L^2(\Omega))^n$$

Without loss of generality, consider the system $(S_{o_0})$ with a dynamics $A$ having the form

$$A y = \sum_{m=1}^{+\infty} \lambda_m \sum_{j=1}^{r_m} \langle y, \varphi_m \rangle_{L^2(\Omega)} \varphi_m, \forall y \in D(A)$$

where $\lambda_1, \lambda_2, \ldots$ are real parameters such that $\lambda_1 > \lambda_2 > \lambda_3 > \ldots, (\varphi_m)_{1 \leq m \leq r_m}$ is an orthogonal basis in $H_0^1(\Omega)$ of eigenvectors for $A$ which is orthonormal in $L^2(\Omega)$, related to eigenvalues $\lambda_n$ with a multiplicity $\gamma_n$. It is well known that $A$ produces a semi – group $(S(t))_{t \geq 0}$ of type strongly continuous given by 12, 13:

$$S(t) y = \sum_{m=1}^{+\infty} e^{\lambda_m t} \sum_{j=1}^{r_m} \langle y, \varphi_m \rangle_{L^2(\Omega)} \varphi_m$$

Obviously, if $\sup \lambda_n = \lambda_1 < 0, (S(t))_{t \geq 0}$ is exponentially stable.

**The following characterization results have obtained**

**Corollary 2**

$(S_{o_0}) + (E_{o_0})$ is EAGR-systems $\iff \exists \gamma > 0$ satisfied the next inequality

$$\sum_{m=1}^{+\infty} \frac{1}{2\lambda_m} \sum_{j=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega)}^2 \leq \gamma \sum_{k=1}^{p} \sum_{m=1}^{+\infty} \frac{1}{2\lambda_m} \sum_{j=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega)}^2 \leq \gamma \sum_{k=1}^{p} \sum_{m=1}^{+\infty} \sum_{j=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega)}^2 \leq \gamma \sum_{k=1}^{p} \sum_{m=1}^{+\infty} \langle (g_k, S^* C^* \theta)_{L^2(\Omega)} \rangle^2$$

for every $\theta = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$.

**Proof**

Since Corollary 1, $(S_{o_0}) + (E_{o_0})$ is EAGR-systems $\iff \exists \gamma > 0$ satisfied that $\forall \theta = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$, yields that

$$\int_0^{+\infty} \|S^*(t)\nabla \varphi \theta\|_{L^2(\Omega)}^2$$

and the operator $C$ is formed by

$$C : (L^2(\Omega))^n \rightarrow \mathbb{R}^q$$

with $y(t) \rightarrow C y(t) = \left( \sum_{i=1}^{n} (h_i, y_i(t))_{D_i}, \ldots, \sum_{i=1}^{n} (h_q, y_q(t))_{D_q} \right)$

and

$$\int_0^{+\infty} (g_k, S^*(t) \nabla C^* \theta)_{L^2(\Omega)}^2 \leq \gamma \int_0^{+\infty} \sum_{k=1}^{p} \langle (g_k, S^* C^* \theta)_{L^2(\Omega)} \rangle^2$$

for every $\theta = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$.
Since
\[ S(t) y = \sum_{m \geq 1} e^{\lambda_m t} \sum_{r=1}^{r_m} \langle y, \varphi_m \rangle_{L^2(\Omega)} \varphi_m \]
it follows that
\[ \int_0^{+\infty} \| S^* (t) \mathcal{P}^* C^* \theta \|^2_{L^2(\Omega)} dt \leq \int_0^{+\infty} \| S^* (t) \mathcal{P}^* C^* \theta \|^2_{L^2(\Omega)} dt \]
\[ = \sum_{m \geq 1} e^{2 \lambda_m t} \sum_{r=1}^{r_m} \langle (\mathcal{P}^* C^* \theta, \varphi_m) \rangle_{L^2(\Omega)}^2 dt \]
\[ = \sum_{m \geq 1} \frac{-1}{2 \lambda_m} \sum_{r=1}^{r_m} \langle (C^* \theta, \nabla \varphi_m) \rangle_{L^2(\Omega)}^2 \]
and
\[ \int_0^{+\infty} \sum_{k=1}^{p} ((g_k, S^* (t) \mathcal{P}^* C^* \theta))^2 dt = \sum_{k=1}^{p} \int_0^{+\infty} e^{2 \lambda_m t} \sum_{r=1}^{r_m} \langle (\mathcal{P}^* C^* \theta, \varphi_m) \rangle_{L^2(\Omega)}^2 dt \]
\[ = \sum_{k=1}^{p} \sum_{m \geq 1} \frac{-1}{2 \lambda_m} \sum_{r=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega)}^2 \]
By using Eq.6, the formula of the operator \( C^* \), the following Corollary is obtained:

**Corollary 3**

\( (S_{\infty}) + (E_{\infty}) \) is \( \text{EAGR}-\text{systems} \) \( \iff \exists \theta > 0 \) satisfied the next inequality

\[ \sum_{m \geq 1} \frac{-1}{2 \lambda_m} \sum_{r=1}^{r_m} \langle g_k, \varphi_m \rangle_{L^2(\Omega_k)}^2 \sum_{l=1}^{n} \langle \theta_l h_l, \frac{\partial q_m}{\partial x_l} \rangle_{L^2(D_l)}^2 \]
\[ \leq \gamma \sum_{k=1}^{p} \sum_{m \geq 1} \frac{-1}{2 \lambda_m} \sum_{r=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega_k)}^2 \]
\[ \sum_{m \geq 1} \frac{-1}{2 \lambda_m} \sum_{r=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega_k)}^2 \langle g_k, \varphi_m \rangle_{L^2(\Omega_k)}^2 \]
\[ \sum_{k=1}^{p} \sum_{m \geq 1} \frac{-1}{2 \lambda_m} \sum_{r=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega_k)}^2 \langle g_k, \varphi_m \rangle_{L^2(\Omega_k)}^2 \]

\[ \gamma \sum_{k=1}^{p} \sum_{m \geq 1} \frac{-1}{2 \lambda_m} \sum_{r=1}^{r_m} \langle C^* \theta, \nabla \varphi_m \rangle_{L^2(\Omega_k)}^2 \langle g_k, \varphi_m \rangle_{L^2(\Omega_k)}^2 \]

**Asymptotic Gradient Efficient Actuators and Sensors:**

The notion of asymptotic gradient efficient actuator have been presented analogy to the concept of gradient efficient actuator in finite time given as follows:

**Definition 3**

The suite \( \{ \Omega_k, g_k \}_{1 \leq k \leq p} \) is called asymptotic gradient efficient actuators (\( \text{AGE}-\text{actuators} \)) if, \( (S_{\infty}) + (E_{\infty}) \) is \( \text{WAGR}-\text{systems} \).

**Proposition 3**

The suite \( \{ \Omega_k, g_k \}_{1 \leq k \leq p} \), \( \text{AGE}-\text{actuators} \) if and only if

\[ \bigcap_{m \geq 1} \text{Ker} M_m f_m = \text{Ker} B^* (R_{\infty}^C)^* \]

anywhere, for \( m \geq 1, M_m \) is the matrix of order \( (p \times r_m) \) defined by

\[ M_m = \left( (g_k, \varphi_m)_{L^2(\Omega_k)} \right)_{k \leq p \leq r_m} \]
and

1. \( 1 \leq j \leq r_m \)
\[ f_m : \theta \in \mathbb{R}^d \rightarrow f_m(\theta) \]
\[ \in \mathbb{R}^{m} \]

**Proof**

Since Proposition 3, \((\mathcal{S}_{\infty}) + (\mathcal{O}_{\infty})\) is WAGR-systems if and only if

\[ \text{Ker} \ (B^*(R^\infty_C)^*) = \text{Ker} \ ((R^\infty_C)^*) \]

Let \( \theta \in \mathbb{R}^d \), then

\[ B^*(R^\infty_C)^* \theta = B^*S^*(\cdot)V_C^* \theta = \]

\[ (g_1, S^*(\cdot)V_C^* \theta)_{L^2(\Omega)}(g_1, \varphi_{m_1})_{L^2(\Omega)} \]

\[ (g_2, S^*(\cdot)V_C^* \theta)_{L^2(\Omega)}(g_2, \varphi_{m_2})_{L^2(\Omega)} \]

\[ \vdots \]

\[ (g_p, S^*(\cdot)V_C^* \theta)_{L^2(\Omega)}(g_p, \varphi_{m_p})_{L^2(\Omega_p)} \]

\[ \sum_{m=1}^{\infty} e^{\lambda_m} \sum_{j=1}^{r_m} \langle V_C^* \theta, \varphi_{m_j} \rangle_{L^2(\Omega)} \langle g_j, \varphi_{m_j} \rangle_{L^2(\Omega)} \]

\[ \sum_{m=1}^{\infty} e^{\lambda_m} \sum_{j=1}^{r_m} \langle V_C^* \theta, \varphi_{m_j} \rangle_{L^2(\Omega)} \langle g_2, \varphi_{m_j} \rangle_{L^2(\Omega_2)} \]

\[ \vdots \]

\[ \sum_{m=1}^{\infty} e^{\lambda_m} \sum_{j=1}^{r_m} \langle V_C^* \theta, \varphi_{m_j} \rangle_{L^2(\Omega)} \langle g_p, \varphi_{m_j} \rangle_{L^2(\Omega_p)} \]

and then, for \( m \geq 1 \),

\[ M_m f_m(\theta) = \]

\[ \begin{pmatrix} \sum_{j=1}^{r_m} \langle V_C^* \theta, \varphi_{m_j} \rangle_{L^2(\Omega)} \langle g_1, \varphi_{m_j} \rangle_{L^2(\Omega_1)} \\ \vdots \\ \sum_{j=1}^{r_m} \langle V_C^* \theta, \varphi_{m_j} \rangle_{L^2(\Omega)} \langle g_p, \varphi_{m_j} \rangle_{L^2(\Omega_p)} \end{pmatrix} \]

Assume that \( \theta \in \cap_{m \geq 1} \text{Ker} \ (M_m f_m) \), this gives

\[ \theta \in \text{Ker} \ (M_m f_m), \forall m \geq 1 \implies \]

\[ \sum_{j=1}^{r_m} \langle V_C^* \theta, \varphi_{m_j} \rangle_{L^2(\Omega)} \langle g_k, \varphi_{m_j} \rangle_{L^2(\Omega_k)} = 0, \]

\[ \forall k \in \{1, 2, \ldots, p\}, \forall m \geq 1 \implies \]

\[ \sum_{m=1}^{\infty} e^{\lambda_m} \sum_{j=1}^{r_m} \langle V_C^* \theta, \varphi_{m_j} \rangle_{L^2(\Omega)} \langle g_k, \varphi_{m_j} \rangle_{L^2(\Omega_k)} = 0, \forall k \in \{1, 2, \ldots, p\}, \forall m \geq 1 \implies \]

\[ B^*(R^\infty_C)^* \theta = 0 \implies \theta \in \text{Ker} \ (B^*(R^\infty_C)^*) \]

Where

\[ \bigcap_{m \geq 1} \text{Ker} \ (M_m f_m) \subset \text{Ker} \ (B^*(R^\infty_C)^*) \]

that is

\[ \bigcap_{m \geq 1} \text{Ker} \ (M_m f_m) = \text{Ker} \ (B^*(R^\infty_C)^*) \]

\[ \square \]
Corollary 4
The suite \((\Omega_k, g_k)_{k \leq k_p}\) is AGE-actuators if and only if
\[
\bigcap_{m \geq 1} \text{Ker} \left( M_m G_m^\text{tr} \right) = \{0\}
\]

Corollary 5
If
\[
\text{rank} \left( M_{m_0} G_{m_0}^\text{tr} \right) = q \quad \text{or} \quad \text{rank} M_{m_0} = r_{m_0}
\]
Then, the suite \((\Omega_k, g_k)_{k \leq k_p}\) is AGE-actuators

Asymptotic Gradient Remediability and Asymptotic Gradient Controllability:
The case of asymptotic relation is difficult and requires more conditions.

Asymptotic Gradient Controllability:
Assuming the system that is described by the following equation:
\[
(S_0) \quad \begin{cases}
\dot{y}(t) = Ay(t) + Bu(t) ; t > 0 \\
y(0) = y
\end{cases}
\]
and A is supposed generates a strongly continuous semi-group \((S(t))_{t \geq 0}\) such that
\[
\|\nabla S(t)\| \leq k(t) ; \forall t \geq 0
\]
Next, some sufficient conditions to characterize the AGC-system are given in the following results.

Definition 4
System \((S_0)\) is called
• EAGC-system if for every \(y \in E = (L^2(\Omega))^n\), there exists \(u \in L^2(0, +\infty; U)\) such that
\[
\nabla y_0 + \nabla H_{\omega_0} u = y,
\]
and equivalently
\[
\text{Im} \nabla_H \omega_0 = (L^2(\Omega))^n.
\]

\(WAGC\)-system if for every \(y \in E = (L^2(\Omega))^n\), and every \(\epsilon > 0\), there exists \(u \in L^2(0, +\infty; U)\) such that
\[
\|\nabla y_0 + \nabla H_{\omega_0} u - y \| < \epsilon,
\]
and equivalently
\[
\text{Im} \nabla_H \omega_0 = (L^2(\Omega))^n.
\]

Let \(E', U'\) be the dual spaces of \(E\) and \(U\) respectively, then using Lemma 1, it is easy to show the following results the following proposition 5 characterizes the EAGC-systems, and WAGC-systems.

Proposition 5
The system \((S_0)\) is
(i) EAGC-systems if and only if
\[
\exists \gamma > 0 \text{ such that } \forall z^* \in E', \quad \|z^*\|_{E'} \leq \gamma \|\nabla H_{\omega_0}\| z^* \|_{l^2(0, +\infty; U)}
\]
Or equivalently
\[
\exists \gamma > 0 \text{ such that } \forall z^* \in E', \quad \|z^*\|_{E'} \leq \gamma \|B^* S^* (\cdot) \nabla z^*\|_{l^2(0, +\infty; U)}
\]
(ii) WAGC-systems if and only if
\[
\text{Ker} \left( (\nabla H_{\omega_0})^* \right) = \{0\}
\]

The following results in proposition 6 demonstrate that the asymptotic controllability concept of type gradient is strongest than the asymptotic remediability of type gradient in various situations.

Proposition 6
If \((S_0)\) is EAGC-system (resp. WAGC-system), then, \((S'_0) + (E'_0), \text{ it is EAGR-system (resp. WAGR-system).}

Proof
• By hypothesis Eq.8, if follows that, for \(\theta \in Y'\),
\[
\|S(\cdot)\nabla C^* \theta\|_{l^2(0, +\infty; X')}
\]
\[
= \left( \int_0^{+\infty} \|S(\cdot) \nabla C^* \theta\|_{X'}^2 dt \right)^{1/2}
\]
\[
\leq \left( \int_0^{+\infty} \|S(\cdot) \nabla\|^2 \|C^* \theta\|^2_{E'} dt \right)^{1/2}
\]
\[
= \left( \int_0^{+\infty} \|(\nabla S(\cdot))\|^2 \|C^* \theta\|^2_{E'} dt \right)^{1/2} \leq k \|C^* \theta\|_{E'} ; \text{ with } k > 0.
\]
from Proposition 5, and since \((S_0)\) is EAGC-system, \(\exists \gamma > 0\), with
\[
\|C^* \theta\|_{E'} \leq \gamma \|B^* S^* (\cdot) \nabla C^* \theta\|_{l^2(0, +\infty; U)}
\]
then,
\[
\|S(\cdot) \nabla C^* \theta\|_{l^2(0, +\infty; X')} \leq M \|B^* S^* (\cdot) \nabla C^* \theta\|_{l^2(0, +\infty; U)}
\]
with \(M = k\gamma > 0\). By using the equivalence of part (i) and part (ii) in proposition 1, \((S'_0) + (E'_0)\) is EAGR-systems.

• From Proposition 5, \((S'_0) + (E'_0)\) is WAGC-system and remains equivalent to,
\[
\text{Ker} \left( B^* (R_{E_0}^c) \right) = \text{Ker} \left( (R_{E_0}^c) \right), \text{ that is to say}
\]
\[
\text{Ker} \left( B^* (R_{E_0}^c) \right) \subseteq \text{Ker} \left( (R_{E_0}^c) \right).
\]
This is equivalent to \(\text{Ker} (K_{E_0}^c) \subseteq \text{Ker} (K_{E_0}^c)\), because \((K_{E_0}^c) = B^* (R_{E_0}^c)\). For \(\theta \in \text{Ker} \left( (K_{E_0}^c) \right)\), it follows that
\[
(K_{E_0}^c) \theta = B^* S^* (\cdot) \nabla C^* \theta = (\nabla H_{\omega_0})^* C^* \theta = 0,
\]
then \(C^* \theta = 0\) because \(\text{Ker} \left( (\nabla H_{\omega_0})^* \right) = \{0\}\), and then \(\theta \in \text{Ker} (C^*) \subseteq \text{Ker} (K_{E_0}^c)\).
boosted via observation function allows by \( q \) sensors of type zone

\[
\begin{aligned}
(\Omega_1) \quad z_{af}(t) &= CV y(t) \\
= & \left( \sum_{i=1}^{n} (h_i, \frac{\partial y(t)}{\partial x_i})_{\partial t}, ..., \sum_{i=1}^{n} (h_q, \frac{\partial y(t)}{\partial x_i})_{\partial t} \right)
\end{aligned}
\]

So, \( \Omega = [0,1] \) gives the corresponding operator \( \Delta \) of type Laplace that confesses an appropriate basis of eigenfunctions via next form

\[
\phi_m(x) = \sqrt{2} \sin (m \pi x); m \geq 1
\]

The correspondent eigenvalues are specified through \( \lambda_m = -m^2 \pi^2; m \geq 1 \). The operator \( \Delta \) generates a self adjoint strongly continuous semi group \( S(t) \) defined by

\[
S(t) y = \sum_{m \geq 1} e^{-m^2 \pi^2 t} (y, \phi_m) \phi_m
\]

is exponentially stable \(^{14}\) with the transformations

\[
H_{\omega f} = \sum_{k=m=1}^{+\infty} \int_{0}^{+\infty} e^{-m^2 \pi^2 t} u_k(t) dt \langle g_k, \phi_m \rangle \phi_m
\]

and

\[
F_{\omega f} = \sum_{m=1}^{+\infty} \int_{0}^{+\infty} e^{-m^2 \pi^2 t} \langle f(., .), \phi_m \rangle \phi_m dt
\]

are well defined and since Corollary 3, \( (S_1) \) is \( EAGR \)-systems if and only if \( \exists \gamma > 0 \) such that

\[
\sum_{m=1}^{+\infty} \sum_{q=1}^{q} (\theta, h_i, \frac{\partial \phi_m}{\partial x_i}) L^2(\rho_i) \leq \gamma \sum_{m=1}^{+\infty} \sum_{q=1}^{q} (g, \phi_m) L^2(\rho_i) \langle \theta, h_i, \frac{\partial \phi_m}{\partial x_i} \rangle L^2(\rho_i)
\]

for every \( \theta = (\theta_1, ..., \theta_q) \in \mathbb{R}^q \)

If a unique actuator (sensor) represents the input (output) of system \( (S_1) + (O_1) \), then the last inequality becomes as follows:

\[
\sum_{m=1}^{+\infty} \sum_{q=1}^{q} (\theta, h_i, \frac{\partial \phi_m}{\partial x_i}) L^2(\rho_i) \leq \gamma \sum_{m=1}^{+\infty} \sum_{q=1}^{q} (g, \phi_m) L^2(\rho_i) \langle \theta, h_i, \frac{\partial \phi_m}{\partial x_i} \rangle L^2(\rho_i)
\]

for \( g = \phi_m \) with \( m_0 \geq 1 \), it is obtained that

\[
\frac{1}{2m_0^2 \pi^2} (h, \frac{\partial \phi_m}{\partial x}) L^2(\rho) \leq \gamma \frac{1}{2m_0^2 \pi^2} (h, \frac{\partial \phi_m}{\partial x}) L^2(\rho)
\]

this is verified for \( \gamma \geq 1 \). But the considered system \( (S_1) \) is not \( EAGC \)-system because it is not \( WAGC \)-system. Indeed, let \( y \in L^2(\Omega) \)

\[
\langle \nabla H \rangle \ast y = (H) \ast \nabla y = B^* S^*(.) \nabla y
\]

\[
= \sum_{m \geq 1} e^{-m^2 \pi^2} \langle y, \nabla \phi_m \rangle B^* \phi_m
\]

for \( g = \phi_m \) with \( m_0 \geq 1 \), it follows that

\[
\langle \nabla H \rangle \ast y = e^{-m_0^2 \pi^2} \langle y, \nabla \phi_m \rangle
\]

Putting \( y(x) = \sin(m_0 \pi x) \), yields that

\[
\langle \nabla H \rangle \ast y = e^{-m_0^2 \pi^2} \int_{0}^{1} y(x) \cos(m_0 \pi x) dx
\]

Putting \( y(x) = \sin(m_0 \pi x) \), yields that

\[
\langle \nabla H \rangle \ast y = e^{-m_0^2 \pi^2} \int_{0}^{1} y(x) \cos(m_0 \pi x) dx
\]

then \( Ker \{ \langle \nabla H \rangle \ast \} \neq \{0\} \) and by proposition 5, the result is proven.

**Asymptotic Gradient Remediability with Minimum Energy:**

Under the condition Eq.7, and the hypothesis of \( WAGC \)-system, then in the present section the problem of \( WAGC \)-system with Minimal Energy is studied. Thus, through \( f \in L^2(0, +\infty; X) \), there exists a control of type optimal \( u \in L^2(0, +\infty; \mathbb{R}^P) \) ensuring, asymptotically, the gradient remediability of the disturbance \( f \) such that \( K_C^\infty u + R_C^\infty f = 0 \), are studied. That is the set defined by

\[
D = \{ u \in L^2(0, +\infty; \mathbb{R}^P); K_C^\infty u + R_C^\infty f = 0 \}
\]

is non empty. Next, the following function is considered

\[
J(u) = ||K_C^\infty u + R_C^\infty f||_{\mathbb{R}^P} + ||u||_{L^2(0, +\infty; \mathbb{R}^P)}^2
\]

The considered problem becomes \( \min_{u \in D} J(u) \).

For its resolution, one can use a modification of \( (H, U, M) \) \(^{14}\). For \( \theta \in \mathbb{R}^q \), it is noted that

\[
||\theta||_* = \left( \int_{0}^{+\infty} ||B^* S^*(t) \nabla C^* \theta, B^* S^*(t) \nabla C^* \sigma dt \right)^\frac{1}{2}
\]

The conforming inner product is specified by

\[
(\theta, \sigma)_* = \int_{0}^{+\infty} (B^* S^*(t) \nabla C^* \theta, B^* S^*(t) \nabla C^* \sigma) dt
\]
and the operator $\Lambda_c^\infty: \mathbb{R}^q \to \mathbb{R}^q$ defined by

$$\Lambda_c^\infty \theta = K_c^\infty (K_c^\infty)^* \theta$$

Then, the following proposition have obtained.

**Proposition 7**

If the condition Eq.7, is verified, then $\|\cdot\|_a$ is a norm on $\mathbb{R}^q$ if and only if $(S_\infty) + (E_\infty)$ is WAGR-system and the operator $\Lambda_c^\infty$ is invertible.

**Proof**

Since,

$$
\|\theta\|_a = \left( \int_0^{+\infty} \|B^* S(t) \nabla C^* \theta\|^2_{\mathbb{R}^q dt} \right)^{\frac{1}{2}} = 0
$$

$$
\Rightarrow \|B^* S(\cdot) \nabla C^* \theta\|_{L^2(0, +\infty; \mathbb{R}^q)} = 0
$$

$$
\Rightarrow B^* S(\cdot) \nabla C^* = 0
$$

$$
\Rightarrow \theta \in \text{Ker}(B^* (R_c^\infty)^*) = \text{Ker}(B^* (R_c^\infty)^*)
$$

However, from Proposition 3, it follows that

$$\text{Ker}(M_{m,f_0}) \cap \text{Ker}(M_{m,f_0}) = 0, \forall m \geq 1.$$
\[ a_{ij} \equiv \sum_{m=1}^{M} \sum_{m'=1}^{M'} \sum_{n=1}^{N} \sum_{k=1}^{k'} \sum_{r=1}^{r'} \left( \frac{-1}{\lambda_m + \lambda_{m'}} \right) \langle g_r, \varphi_{mj} \rangle_{\Omega_r} \langle g_r, \varphi_{m'h} \rangle_{\Omega_r} \sum_{k'=1}^{n} \left( \frac{\partial \varphi_{m'h}}{\partial x_{k'}} \right)_{D_i} \sum_{k=1}^{n} \left( \frac{\partial \varphi_{mj}}{\partial x_{k}} \right)_{D_j} \]

and \( b_j = -\langle R_C^2 f, e_j \rangle_{\mathbb{R}^n} \).

Because \( N \) represent the number of eigenvectors \( \varphi_j \) \( 1 \leq j \leq r_m \) and really it is infinite.

\[ b_j \equiv -\sum_{m'=1}^{M'} \sum_{n=1}^{n} \sum_{k=1}^{k'} \left( \frac{\partial \varphi_{m*l}}{\partial x_{k}} \right)_{D_j} \int_{0}^{+\infty} e^{\lambda_{m*t}} \langle f(t), \varphi_{m'h} \rangle_{L^2(\Omega)} \, dt \]

**The optimal control:**

In this part, an approximation of the optimal control \( u_{\theta_f}(\cdot) \) is given, which is defined by:

\[ u_{\theta_f}(s) = B^* S^* (t) \nabla C^* \theta_f \]

\[ \equiv \sum_{m'=1}^{M'} \sum_{n=1}^{n} \sum_{k=1}^{k'} \sum_{i=1}^{q} \theta_{i,f} e^{\lambda_{m*t}} \langle g_j, \varphi_{m'h} \rangle_{L^2(\Omega)} \left( \frac{\partial \varphi_{m'h}}{\partial x_{k}} \right)_{D_j} \]

**Cost:**

The minimum energy (cost), for \( N \) sufficiently large, is defined by

\[ \| u_{\theta_f} \|_{L^2(0, +\infty; \mathbb{R}^P)} = \left( \int_{0}^{+\infty} \| B^* S^* (t) \nabla C^* \theta_f \|_{\mathbb{R}^P}^2 \, dt \right)^{\frac{1}{2}} \]

\[ \equiv \left( \sum_{j=1}^{p} \int_{0}^{+\infty} \left( \sum_{m'=1}^{M'} \sum_{n=1}^{n} \sum_{k=1}^{k'} \sum_{i=1}^{q} \theta_{i,f} e^{\lambda_{m*t}} \langle g_j, \varphi_{m'h} \rangle_{L^2(\Omega)} \left( \frac{\partial \varphi_{m'h}}{\partial x_{k}} \right)_{D_j} \right)^2 \, dt \right)^{\frac{1}{2}} \]

**The related observation:**

The measurement information related to a given control is described by

\[ z_{u_{\theta_f}}(t) = C \nabla S(t) y^0 + C \nabla \int_{0}^{t} S(t) B u_{\theta_f}(\tau) \, d\tau \]

\[ + C \int_{0}^{t} S(t) f(\tau) \, d\tau \]

Its coordinates \( \left( z_{j,u_{\theta_f}}(\cdot) \right)_{1 \leq j \leq q} \) are achieved for a specific integer \( N \), given by:

\[ z_{j,u_{\theta_f}}(t) \equiv \sum_{m'=1}^{M'} \sum_{n=1}^{n} \sum_{k=1}^{k'} e^{\lambda_{m*t}} \langle y_0, \varphi_{m'h} \rangle_{L^2(\Omega)} \left( \frac{\partial \varphi_{m'h}}{\partial x_{k}} \right)_{D_j} \int_{0}^{t} e^{\lambda_{m't}} u_{j,\theta_f}(\tau) \, d\tau \]

\[ + \sum_{m'=1}^{M'} \sum_{n=1}^{n} \sum_{k=1}^{k'} \langle g_j, \varphi_{m'h} \rangle_{L^2(\Omega)} \left( \frac{\partial \varphi_{m'h}}{\partial x_{k}} \right)_{D_j} \int_{0}^{t} e^{\lambda_{m't}} u_{j,\theta_f}(\tau) \, d\tau \]

\[ + \sum_{m'=1}^{M'} \sum_{n=1}^{n} \sum_{k=1}^{k'} \left( \frac{\partial \varphi_{m'h}}{\partial x_{k}} \right)_{D_j} \int_{0}^{t} e^{\lambda_{m't}} f(\tau, \varphi_{m'h})_{L^2(\Omega)} \, d\tau \]
The Mathematical Approach:
Remember the problem considered above:
\[
\begin{align*}
(P) &\quad \text{Calculate } u^* \in L^2(0, +\infty; U), \text{ with } \\
&\quad \begin{cases}
K^\infty u^* + R^\infty f = 0
\end{cases}
\end{align*}
\]
So, depending on the above result, and employment the preceding consequences in this investigation, one can improve an algorithm which permits to define controls suite which tends to \(u^*\) of (P). The measurement information is specified via Eq.13 and Eq.14.

Algorithm

First Step: Data: domain \(\Omega\), initial state \(\varphi^0\), disturbance function \(f\), sensors \((D, h)\), gradient of efficient actuators \((\sigma, g)\) and precision threshold \(\varepsilon\).

Second Step: Select a truncation low of order \(M = N\).

Third Step: Calculate \(z_{0,0}\): output with \(f = 0\) and \(u = 0\).

Fourth Step: Calculate \(z_{0,f}\): output with \(f \neq 0\) and \(u = 0\).

Fifth Step: Resolve a finite system \(A\theta = b\) such that the parameters are represented by Eq.10 and Eq.11.

Sixth Step: Calculate \(u\) given by Eq.12.

Seventh Step: Compute \(z_{u,f}\): output where \(f \neq 0\) and \(u \neq 0\).

Eighth Step: If \(\|z_{u,f} - z_{0,0}\|_{L^2(\Omega)} \leq \varepsilon\), then stop. Otherwise, Ninth Step: \(M \leftarrow M + 1\) and \(N \leftarrow N + 1\) and return to third step.

Ten Step: Control \(u\) of type optimal links to \(u^*\) the solution of (P).

Conclusion:
In this paper, the problem of AGC analysis has been presented. Certainly, it is based on suitable hypothesis and an appropriate choice of operators and spaces. Furthermore, WEAGR-system and AGER-actuators have been presented firstly. Also the problem of WEAGC-system has been examined under a suitable hypothesis with appropriate choice of spaces and operators. More precisely, the relationship between WEAGC-system and AGR-system has been demonstrated in different important results. Indeed, in the asymptotic case, it has been proved that the controllability concept of gradient type remains stronger than the remediability concept of gradient type, that is to say, AGR-system can be asymptotically gradient remediable but, it is not AGC-system.

Thus, through the choice of sensors and hypothesis of WAGR-system, the problem of EAGR-system with minimum energy has been studied. Moreover, the issue of how to discover an optimal control has been examined in a way compensating for the influence of the disturbances about the observation of gradient via the use of HUM modified.

Regarding the digital processing, some mathematical approximations are proposed, using a multi-step algorithm.

Later, the obtained outcomes have been introduced for class DDPL-systems and may be interesting to expand this work to regional or regional bounded case with other classes under the suitable different select of spaces, for example, the possibility to replace the observability concept in this paper by an asymptotic observer.

Authors' declaration:
- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Mentouri.

Authors' contributions statement:
S. R. and S. B. conceived of the presented idea and developed the theory, R. AL verified the analytical methods and contributed to the analysis of the results and to the writing of the manuscript. All authors discussed the results and contributed to the final manuscript.

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تحليل مقارب لمسائل قابلية معالجة التدرج للأنظمة الخطية التوزيعية المضطربة

سمير بن حديد
صحرا ركاب

قسم الرياضيات، كلية العلوم الدقيقة، جامعة الأخوة منتوري، قسنطينة، الجزائر.
قسم الرياضيات، كلية التربية للعلوم الاقتصادية، جامعة تكريت، العراق.

1. قسم الرياضيات، كلية العلوم الدقيقة، جامعة الأخوة منتوري، قسنطينة، الجزائر.
2. قسم الرياضيات، كلية التربية للعلوم الاقتصادية، جامعة تكريت، العراق.

الخلاصة:
الهدف من هذا العمل، برهان إمكانية التقليل من تأثير أي اضطرابات (تلوث، إشعاع، عدوى، الخ) بشكل تقريبي، من خلال مراقبة تدرج نوع من الأنظمة الخطية المضطربة ذات المعاملات التوزيعية (أنظمة DDPL). بواسطة اختيار مناسب للمحفزات ذات العلاقة ضمن الأنظمة (AFC)، البالغات على منظومة قابلية معالجة التدرج في زمن محدود (منظمة AGC). وعلاوة على ذلك، درست وقدمت تعرِّيف بعض خصائص مفاهيم تتعلق بمنظمات AGC ومنظمة QAGC، التي تتعلق بمنظومة التدرج المقربة. هناك، تم تطوير منظومة QAGC، التي تستخدم في تدفق التدرج المقربة. وعلاوة على ذلك، درست وقدمت تعرِّيف بعض خصائص مفاهيم تتعلق بمنظومة التدرج المقربة. وعلاوة على ذلك، درست وقدمت تعرِّيف بعض خصائص مفاهيم تتعلق بمنظومة التدرج المقربة. (منظمة AGC). وعلاوة على ذلك، درست وقدمت تعرِّيف بعض خصائص مفاهيم تتعلق بمنظومة التدرج المقربة. وعلاوة على ذلك، درست وقدمت تعرِّيف بعض خصائص مفاهيم تتعلق بمنظومة التدرج المقربة. (منظمة AGC). وعلاوة على ذلك، درست وقدمت تعرِّيف بعض خصائص مفاهيم تتعلق بمنظومة التدرج المقربة. وعلاوة على ذلك، درست وقدمت تعرِّيف بعض خصائص مفاهيم تتعلق بمنظومة التدرج المقربة. (منظمة AGC).

الكلمات المفتاحية: تحليل مقارب، قابلية معالجة التدرج، اضطراب، التحكم الأمثل، قابلية المعالجة.