The analysis of the polymer concrete characteristics by fractional calculus

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Abstract. The paper investigates a second-order differential equation with a fractional derivative in the lower term, in which the order of the fractional derivative is in the range from zero to two and is not known in advance. This model is used to describe oscillatory processes in a viscous medium. A fundamentally new method has been developed for the approximate solution of the first boundary-value problem for the equation of string vibration taking into account friction in a medium with fractal geometry. Considering that polymer concrete is a set of granules of a mineral filler located in a viscous medium, the equation of motion of these granules is derived and investigated. A technique is proposed for studying the motion of granules in a medium with fractal geometry.

1. Introduction

Recently, fractional calculus has been the focus of attention of many researchers in the field of science and technology. In this regard, we should mention the work [15], which is a unique comprehensive review of fractional calculus and its application with the authoritative contribution of leading world experts.

First of all, we note that fractional derivatives with respect to space can be used to model anomalous diffusions or dispersions, and fractional derivatives with respect to time can be used to model some processes with a “memory”. Particular attention should be paid to an equation of a kind

$$\frac{\partial^2 u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + C_0 D_{0:}^\alpha u + C_1 D_{0:}^\beta u + F$$

which, in particular, is used to describe the vibration of a string taking into account friction in a medium with fractal geometry. In this work, this equation is used to model changes in the deformation-strength characteristics of polymer concrete under loading.

2. Basic theory

Let’s in $D = \{0 < x < 1, 0 < t < 1\}$ consider the first boundary value problem for equations of vibration of a string with a fractional derivative of order $\alpha$ with respect to patial variable
\[
\frac{\partial^2 u}{\partial t^2} = m \frac{\partial^2 u}{\partial x^2} + C_i D^\alpha_{0x} u, \quad 0 < \alpha < 2, \tag{1}
\]

\[
u(0, t) = u(1, t) = 0, \tag{2}
\]

\[
u(x, 0) = \varphi(x), \tag{3}
\]

\[
u'(x, 0) = \psi(x). \tag{4}
\]

Here, \(0 < \alpha < 2\), \(c\) is an arbitrary constant, \(D^\alpha_{0x} u\) - is fractional derivative in Riemann-Liouville type of order \(\alpha\).

The fractional derivative of order \(\alpha\) of the function \(f(x)\) at the point \(x\) \((0 \leq m - 1 < \alpha < m, \ m \in N)\) defined by the formula [1-5]

\[
D^\alpha f(x) = \frac{d^m}{dx^m} \left( \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - \tau)^{\alpha+1-m} f(\tau) d\tau \right).
\]

For example, the fractional derivative of order \(\alpha = 1/2\) we can calculate by the formula

\[
D^{1/2} f(x) = \frac{d}{dx} \left( \frac{1}{\Gamma(1/2)} \int_0^x (x - \tau)^{1/2} f(\tau) d\tau \right).
\]

For \(1 < \alpha < 2\) using the Riemann-Liouville definition we can write it out

\[
D^\alpha f(x) = \frac{d^2}{dx^2} \left( \frac{1}{\Gamma(2 - \alpha)} \int_0^x (x - \tau)^{2-\alpha} f(\tau) d\tau \right).
\]

These obtained results were used [18, 19] in modeling the change in the deformation-strength characteristics of polymer concrete when subjected to loadings.

In order to solve the problem (1)-(2)-(3)-(4) we use the Fourier method

\[
u(x, t) = X(x)T(t). \tag{5}
\]

Substituting the expression (5) into (1) we obtain the two-point boundary value problem for \(X(x)\)

\[
X''(x) + C_i D^\alpha_{0x} X = \lambda X(x), \tag{6}
\]

\[
X(0) = X(1) = 0. \tag{7}
\]

The solution of the problem (6)-(7) was written out in [2,3]. In particular, there was shown that the value \(\lambda\) is the an eigenvalue of the problem (6)-(7) iff \(\lambda\) is the zero of the function
\[ \omega(\lambda) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda^{n-k} (-C_1)^k}{\Gamma(2n-k\beta + 2)} \]
and corresponding eigen functions \( X_j(x) \) are

\[ X_j(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda^{n-k} (-C_1)^k}{\Gamma(2n-k\beta + 2)} x^{2n+1-k\alpha} , \quad j = 1, 2, 3, \ldots \quad (8) \]

(here \( \lambda_j \) is the \( j \)-th eigenvalue of the problem (6)-(7)). The system of eigenfunctions (8) is complete \([6,8,9-14,16,17]\) but non-biorthogonal, in this way we construct the system

\[ \tilde{X}_j(x) = (1-x) - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k} (-C_1)^k}{\Gamma(2n-k\beta + 2)} (1-x)^{2n+1-k\alpha} , \quad j = 1, 2, 3, \ldots \quad (9) \]

which will be biorthogonal to the system of eigenfunctions

\[ X_j(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k} (-C_1)^k}{\Gamma(2n-k\beta + 2)} x^{2n+1-k\alpha} . \]

In order to construct this biorthogonal system, together with problem (6) - (7), we consider a problem conjugate to problem (6) - (7).

To pose the problem conjugate to problem (6) - (7), in the class \( C^2(0,1) \cap C([0,1]) \) we consider the following Cauchy problem: find a solution to the equation

\[ u'' + \frac{d^{\alpha}}{d(1-x)^{\alpha}} u + \lambda u = 0 \quad (6a) \]

\[ u(1) = 0, \quad u(1) = -1 \quad (7a) \]

where \( \frac{d^{\alpha}}{d(1-x)^{\alpha}} \) is the operator adjoint to the fractional differentiation operator \( D_0^{\alpha} \) of order \( \alpha \) \([7]\). It can be shown that problem (6a), (7a) is equivalent to the equation

\[ u(x) = -\int_1^x \frac{1}{t} K(x,t) u(t) dt + (1-x) , \]

where

\[ K(x,t) = \begin{cases} 0, \quad 0 < t < x < 1 , \\ (t-x)^{1-\alpha} / \Gamma(2-\alpha) + \lambda (t-x), \quad 0 < x < t < 1 . \end{cases} \]

Let’s define the sequence of iterated kernels through recurrence relations
\[ K_{n+1}(x,t) = \int_x^1 K_n(x,\tau)K_j(\tau,t)d\tau. \]

Clearly, that

\[ K_n(x,t,\lambda) = \sum_{m=0}^{\infty} \frac{C_n^m \lambda^{m-n}}{\Gamma(2n-m\alpha)}(t-x)^{2n-1-m\alpha}, \]

then the resolvent will be

\[ R(x,t,\lambda) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \frac{C_n^m \lambda^{n-m}}{\Gamma(2n-m\alpha)}(t-x)^{2n-1-m\alpha}. \]

Therefore, the corresponding solution of the integral equation have the form

\[ u(x) = (1-x) + \int_x^1 \left( \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \frac{C_n^m \lambda^{n-m}}{\Gamma(2n-m\alpha)}(t-x)^{2n-1-m\alpha} \right) (1-t) dt \]

from which it follows that the eigenfunctions are

\[ \tilde{X}_j(x) = (1-x) - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda^{n-k}}{\Gamma(2n-k\beta+2)} (1-x)^{2n+1-k\alpha}. \]

Of course the system of functions \( \{\tilde{X}_j(x)\} \) is the system of eigenfunctions of a problem

\[ X'''(x) + D_{0x}^\alpha X = \lambda X(x), \quad (10) \]

\[ X(0) = X(1) = 0, \quad (11) \]

which conjugate to the problem (6)-(7), and eigenvalues of these two problems, how it was noticed, are the same. Here \( D_{0x}^\alpha X \) - is the operator of fractional differentiation with a start at a point \( x \) and the end at 1, i.e. it is the operator conjugated to the operator of fractional differentiation \( D_{0x}^\alpha \).

Now the solution of the problem (1)-(2)-(3)-(4) is written out by standard way

\[ u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos(\pi nt) + B_n \sin(\pi nt) \right) \chi_n(x) \quad (12) \]

Finally, we will to declare constants \( A_n \) and \( B_n \) so that the initial conditions (3)-(4) are satisfied. To find \( A_n \) let \( t = 0 \) in (12). Then

\[ u(x,0) = \sum_{n=1}^{\infty} A_n \chi_n(x) = \varphi(x). \]

Taking to account that systems \( \{X_j(x)\}_{j=1}^{\infty} \) and \( \{\tilde{X}_j(x)\}_{j=1}^{\infty} \) and are biorthogonal we obtain
\[ A_n = C \int_0^1 \varphi_n(x) \tilde{X}_n(x) \, dx, \quad (n = 1, 2, 3, \ldots) . \]

To find \( B_n \) we differentiate both sides of (12) with respect to \( t \), then obtain

\[ \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (\pi n)(-A_n \sin(\pi nt) + B_n \cos(\pi nt)) \chi_n(x) = \psi(x) . \quad (13) \]

Put \( t = 0 \) in (13), then

\[ \psi(x) = \pi \sum_{n=1}^{\infty} nB_n \chi_n(x) . \]

From (14) follows that

\[ \pi n B_n = \int_0^1 \psi(x) \tilde{X}_n(x) \, dx , \]

or, what is the same,

\[ B_n = \frac{1}{\pi n} \int_0^1 \psi(x) \tilde{X}_n(x) \, dx . \]

3. Application of the obtained results in modeling changes in the deformation-strength characteristics of polymer concrete under loading

In [18-24], the equation (6) was used for modeling the deformation-strength characteristics of polymer concrete. Polymer concrete is represented as the set of granules (granules of mineral aggregate) of a mineral filler that are located in a viscoelastic medium. In this case [18-24] the equation of motion of a granule of mass \( m \) under the action of loading \( F \) from a moving vehicle taking into account the viscous resistance represents as equation[18,19,20,21,22, 23, 24]

\[ m \frac{\partial^2 u}{\partial t^2} = -b \frac{\partial X}{\partial t} - kX + F(t) , \]

where \( X \) is the displacement of the granule, \( b \) is the coefficient taking into account the viscous properties of the medium (viscosity modulus), \( k \) is the coefficient characterizing the elastic properties of the medium (stiffness modulus).

The main problem when using models based on fractional derivatives is the problem of identifying the parameters of this model, especially the order of the fractional derivative.

In the few publications of the last ten years, the problem of identifying the parameters of fractional models is mainly solved at the theoretical level, for example, by spectral analysis methods [24]. In [18-24] the model parameters are determined on the basis of several characteristic points obtained in the experiment by substituting the strain values in the analytical solutions of the corresponding problem.

In particular, in papers [18-24] were studied some samples of polymeric concrete based on a polyester resin (dian- and diloangidrid-1,1-dichloro-2,2-diethylene) , where it was shown by a fundamentally new method that the order of the fractional derivative is \( \alpha = 1.47 \).

In order to test the results, we used the experimental data presented in [19]. Comparing the experimental data [19] with the calculated ones (in the case of \( \alpha = 1.47 \)), a conclusion was drawn on the adequacy of the constructed model. To more accurately describe the process under study, we introduce the notation \( u(x; t) \) - the displacement of a granule with an abscissa \( x \) at a time \( t \). In this
paper, we consider transverse vibrations only, and it is assumed that all motions occur in one plane and that the granule moves perpendicular to the axis $0x$. Then, to simulate changes in the deformation-strength characteristics of polymer concrete under loading, we have the following first boundary-value problem

$$\frac{\partial^2 u}{\partial t^2} = m \frac{\partial^2 u}{\partial x^2} + C_iD_{147}^{147} u, \quad 0 < \beta < 2,$$

$$u(0,t) = u(1,t) = 0,$$  \hspace{1cm} (15)

$$u(x,0) = \varphi(x),$$

$$u'(x,0) = \psi(x),$$  \hspace{1cm} (16)

by the formula (12) the solution of this problem has the form

$$u(x,t) = \sum_{n=1}^\infty (A_n \cos(\omega nt) + B_n \sin(\omega nt))\chi_n(x)$$  \hspace{1cm} (17)

where

$$\lambda_j = \frac{\alpha^2}{\beta^2} \lambda_j^{n-k} (-C_i)^k \chi_j^{2n+1-1,47k}, \quad j = 1,2,3,...,$$  \hspace{1cm} (18)

Let's numerically find the eigenvalues $\lambda_j$ using the high-level technical calculation language MATLAB for $\alpha = 1.47, \quad c = 1.8$ (according to [19]). Eigenvalues are presented in table 1:

| eigennumbers | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|--------------|-------------|-------------|-------------|-------------|-------------|
| values       | 16.6        | 59.4        | 125.0       | 213.4       | 323.4       |

Then, an approximate solution to problem (15) - (16) - (17) - (18) will take the form

$$u(x,t) \approx \sum_{n=1}^3 (A_n \cos(\omega nt) + B_n \sin(\omega nt))\chi_n(x)$$  \hspace{1cm} (19)

Thus, formula (22) allows us to write a solution to problem (15) - (16) - (17) - (18) if the functions $\varphi(x)$ and $\psi(x)$ are continuously differentiable.

4. Conclusion

Thus in present paper, assuming that polymer concrete is a set of granules of a mineral filler located in a viscous medium, we obtained and investigated the equation of motion of these granules.
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