Capturing strong correlations in spin, electron and local moment systems

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Abstract

We address the question of identifying degrees of freedom for quantum systems. Typically, quasi-particle descriptions of correlated matter are based upon the canonical algebras of bosons or fermions. Here we highlight that a special class of non-canonical algebras also offer useful quantum degrees of freedom, allowing for the development of quasi-particle descriptions which go beyond the weakly correlated paradigm. We give a broad overview of such algebras for spin, electron and local moment systems, and outline important test problems upon which to develop the framework.

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1 Introduction

The task of understanding interacting quantum systems is an inherently challenging one, as the complexity of a quantum system increases exponentially with size. Nevertheless our microscopic understanding of the world is built upon quantum theory. Central to the success of the quantum framework is the semi-classical notion of a quasi-particle, which reflects the organisation of correlations around underlying quantum degrees of freedom \[1,2\]. As with all modelling, the identification of good degrees of freedom permits the most relevant correlations to be isolated, allowing for an accurate and efficient description of a system.

The best understood quantum degrees of freedom are the canonical ones, bosons and fermions. The semi-classical ideology is encapsulated by the Schwinger–Dyson equation

\[
G^{-1} = G_0^{-1} - \Sigma,
\]

which relates a system’s Green’s function to that of a representative non-interacting system, together with a self-energy functional which encodes the correlations. In principle the self-energy obeys an exact closed equation, while in practice it is computed within a perturbative framework where the lowest order contributions are hoped to capture the most relevant behaviour. Systems which are well described in this way are commonly referred to as ‘weakly correlated’. In particular, this formalism provides the microscopic foundation of Fermi liquid theory, the scheme by which the electronic properties of a wide variety of systems are understood \[2\]. The quantum nature of correlations is reflected in the decay of the electronic quasi-particles and the emergence of collective excitations such as plasmons.

It is well known however that the canonical paradigm is insufficient to capture the great wealth of behaviour of interacting quantum systems. There exists a multitude of materials which exhibit behaviour defying a weakly correlated description, e.g. transition metal oxides \[3\], iron pnictides and chalcogenides \[4\], intermetallic alloys \[5,6\], and organic crystals \[7\]. A prominent example are the cuprate oxides, exhibiting strongly correlated phenomena such as the Mott metal–insulator transition, a pseudogap and strange metallic behaviour, as well as hosting a myriad of exotic orders and exhibiting superconductivity at some of the highest recorded temperatures \[8\]. The challenge to coherently characterise such phenomena provides an underlying motivation for this article.

At the same time, bosons and fermions do not exhaust quantum degrees of freedom. Notable examples are to be found in the distinguished setting of one-dimension where, due to the constrained geometry, there exist truly interacting quantum degrees of freedom. In particular there are models possessing high symmetry where the spectrum can be completely characterised through the dispersion of stable particles which scatter elastically off one another, neatly captured by the quantisation conditions on a ring of length \(L\),

\[
e^{i\sum_j p_j L} \prod_{k \neq j} S(p_j, p_k) = 1,
\]

as first described by Bethe \[9\]. Generically, behaviour in the low-temperature low-frequency regime is determined by collective excitations, described through the Luttinger liquid framework \[10,11\].

A question central to this work is whether there exist alternatives to the canonical paradigm in dimensions greater than one. The simplest setting to consider are spin systems. Here the
local operators are cast through the $su(2)$ algebra

$$[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma} S^\gamma,$$

(3)

with $\alpha, \beta, \gamma \in \{x, y, z\}$, whose relations are non-canonical, i.e. they yield non-trivial operators. This gives rise to rich behaviour in spin systems. On the one hand spin systems may order, which allows for a quasi-particle description based on magnon spin-wave excitations defined with respect to the orientation of polarisation \cite{12,13}. Traditionally this is achieved by mapping the spin system to a representative bosonic system through the Holstein–Primakoff/Dyson–Maleev transformation, thereby organising correlations in the weakly correlated sense. Alternatively spin systems may remain disordered, even down to the lowest temperatures. Then all $\langle S^\alpha \rangle = 0$, which obstructs the development of a quasi-particle description based on the $su(2)$ algebra. An alternative approach is to invoke fractionalisation of the spin operators into canonical bosons or fermions \cite{17,18}. A key result in this direction is Kitaev’s exact solution of a frustrated spin model on the honeycomb lattice by fractionalising the spin operators through Majorana fermions, revealing a $Z_2$ quantum spin liquid ground state \cite{21}. In the context of local moment systems, fractionalisation of the spin moments through canonical fermions is inherent to the standard theoretical treatment of heavy fermion formation \cite{22,23}.

Efforts to invoke non-canonical algebras for treating electronic correlations were initiated by Hubbard in the 1960’s \cite{24,25}. He considered the graded algebra of local projection operators, and demonstrated how their correlation functions give directly the electronic Green’s function. A difficulty however lay in the development of a systematic framework, due in part to the inapplicability of Wick’s theorem to non-canonical operators. Efforts to overcome this have primarily focused on obtaining an expansion about the atomic limit. In particular a large body of work has focused on the development and use of a diagrammatic framework where all off-site terms in a Hamiltonian are treated as perturbations \cite{26,27}. Another development is the composite operator method which employs non-canonical operators to identify a local orthogonal basis for computing correlation functions \cite{30}. Neither approach however offers a quasi-particle description on a par with the weakly correlated paradigm, for which the representative solvable model is one of dispersing particles as opposed to an isolated system with a discrete spectrum. Alternatively, fractionalisation of non-canonical operators has also been extensively studied in the electronic context through slave-particle theories \cite{31,32}. These again offer a reformulation in terms of canonical particles, but with one drawback that it is necessary to impose non-trivial local constraints on permissible states, and another that the electronic Green’s function becomes a higher-point correlation function.

Now we come to the focus of the present article, which is to argue that there exist quantum degrees of freedom which are truly non-canonical. Specifically, we highlight a special class of non-canonical algebras which allow for the development of a quasi-particle description. This is motivated in part by the work of Shastry \cite{37,38}, which demonstrates how the Schwinger–Dyson formalism can be adapted to treat non-canonical operators by introducing a second exact functional alongside the self-energy. Recently this has been applied in both the electronic and local moment settings \cite{39,40}, where it is seen to give access to regimes of behaviour inaccessible to the canonical degrees of freedom. In the present work we wish to clarify the origins of the algebraic structures employed therein, while also emphasising both the generality and inherent simplicity of such degrees of freedom.

The article is organised as follows. Section 2 contrasts the usefulness of canonical and
non-canonical algebras as quantum degrees of freedom, highlighting a special class of non-canonical algebras. In Sec. 3 we then summarise how a quasi-particle description is obtained through the introduction of a second functional. Section 4 provides a technical overview of the algebraic structures relevant for the three settings which we then consider: spin systems in Sec. 5, electronic systems in Sec. 6 and local moment systems in Sec. 7. We include a Mathematica notebook along with the arXiv submission providing checks of the formulae in these three sections. The discussion in Sec. 8 outlines some future directions, and we conclude in Sec. 9.

2 Quantum degrees of freedom

A quantum degree of freedom is specified by the algebra it obeys. Important examples are bosons and fermions, which correspond to the canonical algebras \([b, b^\dagger] = 1 \) and \(\{c, c^\dagger\} = 1\). Interactions are induced by a system’s Hamiltonian through the equation of motion, \(\dot{O} = i\hbar[H, O]\), and it is the degree of freedom’s algebra which determines how correlations develop.

In this work we focus on a special class of non-canonical degrees of freedom. To motivate this we provide a brief discussion where we contrast the following three ‘toy’ algebras,

\begin{align}
(i) \quad [a, a^\dagger] &= 1, & [n, a^\dagger] &= a^\dagger, & [n, a] &= -a, \\
(ii) \quad [a, a^\dagger] &= n, & [n, a^\dagger] &= a^\dagger, & [n, a] &= -a, \\
(iii) \quad [a, a^\dagger] &= 1 - \lambda n, & [n, a^\dagger] &= a^\dagger, & [n, a] &= -a,
\end{align}

and consider how correlations develop under a simple Hamiltonian of the form \(H \sim a^\dagger a + \lambda nn\). We keep the discussion schematic, suppressing spatial indices and coupling constants, as well structure constants and the grading of the algebra.

(i) **canonical algebra**: in a canonical algebra the bracket \([a, a^\dagger] = 1\) yields a scalar, and so time evolution has the form \(\dot{a} \sim [H, a] \sim a + \lambda a\). For \(\lambda = 0\) the action under \(H\) is linear, which allows for the identification of single-particle modes. A systematic quasi-particle description is then obtained by incorporating the correlations induced by \(\lambda \neq 0\) according to an appropriate perturbative scheme.

(ii) **generic non-canonical algebra**: in contrast, for a generic non-canonical algebra the bracket \([a, a^\dagger] = n\) yields another operator. Time evolution then results in \(\dot{a} \sim [H, a] \sim na + \lambda na\). The absence of a linear term here has obstructed efforts to develop a quasi-particle description directly from such algebras, leaving their usefulness as quantum degrees of freedom unclear.

(iii) **special non-canonical algebra**: in this work we focus on non-canonical algebras of the form \([a, a^\dagger] = 1 - \lambda n\), where \(\lambda\) is a parameter inherent to the algebra. Here time evolution again gives \(\dot{a} \sim [H, a] \sim a + \lambda na\). As for a canonical algebra, the action is linear when \(\lambda = 0\). This again allows for the development of a systematic quasi-particle description, as elaborated upon in the following section. We can thus regard these as legitimate quantum degrees of freedom.

While the discussion here is greatly simplified, we hope it serves to illustrate the point. That is, there exist a special form of non-canonical degrees of freedom (iii) which offer an alternative to the canonical paradigm. The purpose of this article is to advocate that these offer a powerful route towards characterising strongly correlated behaviour in a broad class of settings.
3 Organising correlations

In the previous section we argued that a special class of non-canonical algebras have potential as valid quantum degrees of freedom, those of the form Eq. (4.iii). In this section we justify this by summarising how the correlations induced by \( \lambda \neq 0 \) can be organised.

For concreteness consider a Hamiltonian built from such operators as follows

\[
H = -\sum_{\langle i,j \rangle} t_{ij} a_i^\dagger a_j + \lambda \sum_{\langle i,j \rangle} V_{ij} n_i n_j,
\]

where \( \langle \cdot, \cdot \rangle \) denotes summation over pairs of sites, and \( t \) and \( V \) denote hopping and interaction parameters respectively. The parameter \( \lambda \) is tied to that appearing in the non-canonical algebraic relations

\[
[a, a^\dagger] = 1 - \lambda n, \quad [n, a^\dagger] = a^\dagger, \quad [n, a] = -a,
\]

so that the limit \( \lambda \to 0 \) admits an interpretation as dispersing single-particle modes.

We focus on obtaining the thermal Green’s function \( \mathcal{G}_{ij}(\tau, \tau') = \langle a_i(\tau) a_j^\dagger(\tau') \rangle \), with \( \mathcal{O}(\tau) = e^{\tau H} O e^{-\tau H} \), which gives access to both the retarded and advanced Green’s functions through the Matsubara formalism \[41\]. This is fixed through its equation of motion

\[
\partial_{\tau} \mathcal{G}_{ij}(\tau, \tau') - \langle [H, a_i(\tau)] a_j^\dagger(\tau') \rangle = \delta_{ij} \delta(\tau - \tau') \langle [a_i(\tau), a_j^\dagger(\tau)] \rangle,
\]

together with the Kubo–Martin–Schwinger condition of periodicity in \( \tau \). Here both brackets yield terms encoding correlations. To treat these a source term \( S(\tau) = \sum_i \zeta_i(\tau) n_i(\tau) \) is incorporated into the thermal expectation value

\[
\langle \mathcal{O} \rangle = \frac{\text{Tr} \left( e^{-\beta H} \left[ e^{i\int_{\tau}^{\tau'} d\tau S(\tau)\mathcal{O}} \right] \right)}{\text{Tr} \left( e^{-\beta H} \right)},
\]

with \( \beta \) the inverse temperature and \( \mathcal{T} \) the \( \tau \)-ordering operator, so that higher-point correlation functions can be obtained through functional derivatives

\[
\langle n_i(\tau) \mathcal{O} \rangle = \langle n_i(\tau) \rangle \langle \mathcal{O} \rangle + \frac{\delta}{\delta \zeta_i(\tau)} \langle \mathcal{O} \rangle.
\]

The equation of motion then comes to the schematic form

\[
\left[ h_0 - \zeta - \lambda \left( \langle n \rangle + \frac{\delta}{\delta \zeta} \right) \right] \mathcal{G} = 1 - \lambda \langle n \rangle,
\]

where \( h_0 \) collects all the uncorrelated contributions, aside from the source term \( \zeta \), and we suppress model parameters, spatial indices and \( \tau \)-dependence for simplicity.

Equation (10) provides a functional equation for \( \mathcal{G} \), and we are interested in obtaining the solution in the zero source limit \( \zeta \to 0 \). The primary difficulty induced by the non-canonical nature of the algebra is the non-trivial expectation value on the right-hand side. This obstructs a straight inversion of \( \mathcal{G} \) through the Schwinger–Dyson equation Eq. (11), as is done for a canonical degree of freedom.

The difficulty is overcome by employing Shastry’s factorisation technique \[37, 38\]. This proceeds by factorising the Green’s function in two \( \mathcal{G} = g \Omega \), with the relative magnitude of \( g \) and \( \Omega \) to be fixed momentarily. Employing the product rule, \( \frac{\delta \mathcal{G}}{\delta \zeta} = \frac{\delta g}{\delta \zeta} \Omega + g \frac{\delta \Omega}{\delta \zeta} \), brings Eq. (10) to the form

\[
\left( \left[ h_0 - \zeta - \lambda \left( \langle n \rangle + \frac{\delta}{\delta \zeta} \right) \right] g \right) \Omega = 1 - \lambda \langle n \rangle + \lambda g \frac{\delta \Omega}{\delta \zeta},
\]

5
where the functional derivative on the left-hand side does not act on $\Omega$, and the contribution of $\frac{\delta}{\delta \xi}$ is brought to the right-hand side. The freedom in the definition of $g$ and $\Omega$ is then exploited to simply set

$$\Omega = 1 - \lambda \langle n \rangle + \lambda_0 \frac{\delta \Omega}{\delta \xi},$$

which provides an exact functional equation for $\Omega$, while also resulting in a reduction of the equation of motion to the canonical form

$$\left[h_0 - \zeta - \lambda (\langle n \rangle + \frac{\delta \zeta}{\delta \xi})\right]g = 1.$$  \hspace{1cm} (13)

Then $g$ can be obtained through a Schwinger–Dyson inversion

$$g^{-1} = g^{-1}_0 - \Sigma,$$

where $g^{-1}_0 = h_0 - \zeta$ is known exactly, and $\Sigma$ obeys an exact functional equation of the form

$$\Sigma = \lambda \langle n \rangle + \lambda_0 g + \lambda_0 g \frac{\delta \zeta}{\delta \xi}.$$  \hspace{1cm} (14)

Thus the non-canonical Green’s function is expressed as

$$G = \frac{\Omega}{g^{-1}_0 - \Sigma},$$

where $g_0$ is known exactly, and $\Sigma$ and $\Omega$ obey exact functional equations. This is the analogue of the Schwinger–Dyson Eq. (1) for a non-canonical degree of freedom, with the new functional $\Omega$ arising directly from the non-canonical nature of the algebra. Like the Schwinger–Dyson equation this result is implicit, in general it is not possible to obtain explicit solutions. Instead approximate solutions are sought, such as by expanding in $\lambda$, which thereby organises the correlations in the system. In practice there exist multiple perturbative schemes which may be employed \[42\]. Examples include Hedin’s GW approach which employs a dressed source $\zeta' = \zeta + \lambda \langle n \rangle$ \[43,44\], and approaches which directly make a resummation of the interaction vertex $\Lambda = -\frac{g^{-1}}{\delta \xi}$ such as the T-matrix/ladder approximation $[45,46]$. We regard the development of such methods for non-canonical degrees of freedom to be an important topic for future research, and we outline some key directions in the Discussion.

4 Algebraic structures

In the following three sections we will identify non-canonical degrees of freedom for spin, electron, and local moment systems. Before proceeding we first provide a relatively broad overview of the formal mathematical structures which we will employ, specifically the graded (super) Lie algebras $su(M|N)$. We hope this offers useful orientation for the reader, as well as providing a robust foundation for what follows. The section may however be safely skipped by readers not interested in this level of formality. We adapt the notations of $[47–49]$.  

The algebra $su(M|N)$ is generated by the set of generators $P^a_\alpha$, $Q^a_\alpha$, $L^a_\beta$, $R^a_\beta$ and $C$, with $\alpha = 1, 2, \ldots, M$ and $a = 1, 2, \ldots, N$. The pair $P^a_\alpha$ and $Q^a_\beta$ are fermionic, obeying the anti-commutation relations

$$\{P^a_\alpha, Q^b_\beta\} = \delta^a_\beta \delta^b_a C + \delta^b_a L^a_\beta + \delta^b_\beta R^a_\beta.$$

(16)

The $L^a_\beta$ and $R^a_\beta$ generate two bosonic sub-algebras, $su(M)$ and $su(N)$ respectively,

$$[L^a_\beta, J^\gamma] = \delta^a_\gamma J^\alpha - \frac{1}{M} \delta^a_\beta J^\gamma, \hspace{1cm} [R^a_\beta, J^c] = \delta^a_c J^a - \frac{1}{N} \delta^a_\beta J^c,$$

$$[L^a_\beta, J^\gamma] = -\delta^a_\gamma J^\beta + \frac{1}{M} \delta^a_\beta J^\gamma, \hspace{1cm} [R^a_\beta, J^c] = -\delta^a_c J^b + \frac{1}{N} \delta^a_\beta J^c.$$  \hspace{1cm} (17)
where $J$ denotes any generator with appropriate index. The generator $C$ obeys
\begin{equation}
[C, P^{\alpha}_a] = \frac{N-M}{MN} P^{\alpha}_a, \quad [C, Q^{\alpha}_a] = \frac{M-N}{MN} Q^{\alpha}_a,
\end{equation}
and is central if $M = N$, i.e. it then commutes with all generators.

It is useful to introduce an oscillator (slave-particle) realisation of $su(M|N)$ as follows
\begin{align}
P^{\alpha}_a &= f^\dagger_\alpha b_a, \quad Q^{\alpha}_a = b^\dagger_\alpha f_\alpha, \\
L^{\alpha}_\beta &= f^\dagger_\alpha f_\beta - \frac{1}{M} \delta^{\alpha}_\beta f^\dagger_\gamma f_\gamma, \quad R^\alpha_\beta = b^\dagger_\alpha b_\beta - \frac{1}{N} \delta^\alpha_\beta b^\dagger_\gamma b_\gamma \\
C &= \frac{1}{M} f^\dagger_\alpha f_\alpha + \frac{1}{N} b^\dagger_\alpha b_\alpha,
\end{align}
where $f_\alpha$ and $b_\alpha$ are canonical fermions and bosons respectively, and summation over repeated indices is implied. We denote the common vacuum of $f_\alpha$ where $f^\dagger_\alpha f_\alpha$ is proportional to the identity.

Representations obtained in this way are commonly referred to as ‘atypical’ or ‘short’. In particular if $M = N$ then $C$ is central and its eigenvalue is fixed through $NC = N$, referred to as the shortening condition.

The fundamental representation of $su(M|N)$ is $(M + N)$-dimensional. In the oscillator realisation this corresponds to taking $N = 1$, i.e. the basis is given by the one-particle states. It is instructive also to consider the matrix realisation. The generators of $su(M|N)$ are then regarded as the $(M + N) \times (M + N)$ matrices with zero supertrace, where for a general matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the condition of zero supertrace is $\text{str } M = \text{tr } A - \text{tr } D = 0$. Schematically the generators have the form
\begin{equation}
P = \begin{pmatrix} 0 & * \\ * & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ * & 0 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ * & * \end{pmatrix},
\end{equation}
where $*$ denotes the existence of non-zero entries. The generator $C$ is diagonal, and for $M = N$ it is proportional to the identity.

The algebra $su(M|N)$ is extended to $u(M|N)$ by incorporating a generator with non-zero supertrace. The additional relations can be written as
\begin{equation}
[D, P^{\alpha}_a] = 2P^{\alpha}_a, \quad [D, Q^{\alpha}_a] = -2Q^{\alpha}_a,
\end{equation}
which corresponds to $D = f^\dagger_\alpha f_\alpha - b^\dagger_\alpha b_\alpha$ in the the oscillator realisation, and $D = \begin{pmatrix} \mathbb{I}_{M \times M} & 0 \\ 0 & -\mathbb{I}_{N \times N} \end{pmatrix}$ for the matrix realisation of the fundamental representation.

This completes our general discussion of $su(M|N)$. Next we focus on several special cases relevant for this article.

### 4.1 $su(2)$

First we consider the $su(2)$ algebra, given above in Eq. (3). It is convenient to re-express the relations through $S^\pm = S^x \pm i S^y$ as follows
\begin{equation}
[S^+, S^-] = 2S^z, \quad [S^z, S^\pm] = \pm S^\pm.
\end{equation}
We can regard $su(2)$ in terms of $su(M|N)$ as either $M = 2$, $N = 0$ or $M = 0$, $N = 2$. In either case there are no fermionic generators and the algebra reduces to one or other of the bosonic sub-algebras.

In the first case $M = 2$, $N = 0$, and the algebra reduces that of the $L$. The generators are related to the $S$ through

$$S^z = L_2^2 = -L_1^1, \quad S^+ = L_2^1, \quad S^- = L_1^2.$$  \hfill (24)

The oscillator realisation of $L$ gives the Abrikosov fermion formulation for the spin-1/2 representation of $su(2)$ \cite{50}. Indeed due to the Pauli exclusion principle for fermions, the oscillator realisation provides a non-trivial representation only for $N = 1$, with the doublet basis

$$|\downarrow\rangle = f_1^\dagger |\Omega\rangle, \quad |\uparrow\rangle = f_2^\dagger |\Omega\rangle.$$  \hfill (25)

The alternative case $M = 0$, $N = 2$ corresponds to $R$ remaining non-trivial. Again the generators are related to $S$ as above

$$S^z = R_2^2 = -R_1^1, \quad S^+ = R_2^1, \quad S^- = R_1^2.$$  \hfill (26)

Here the oscillator realisation gives the Schwinger boson formulation of $su(2)$. The representations determined through the constraint

$$b_1^\dagger b_1 + b_2^\dagger b_2 = N,$$  \hfill (27)

then provide all $(2S + 1)$-dimensional multiplets $|S, m\rangle$ of $su(2)$, where $N = 2S$ and $S$ is the magnitude of the spin fixed via the Casimir identity $\vec{S} \cdot \vec{S} = S(S + 1)$. Specifically, the basis can be identified through

$$|S, m\rangle = \frac{(b_1^\dagger)^{S+m}(b_1^\dagger)^{S-m}}{\sqrt{(S+m)!(S-m)!}} |\Omega\rangle, \quad m \in \{-S, -S + 1, \ldots, S\}.$$  \hfill (28)

The matrix realisation of the generators for the fundamental representation is given by the traceless matrices

$$S^z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  \hfill (29)

The extension to $u(2)$ is given by adding a matrix with non-zero trace, which here corresponds to incorporating the identity.

### 4.2 $su(1|1)$

The simplest graded Lie algebra is $su(1|1)$, which corresponds to the familiar canonical fermion algebra $\{c, c^\dagger\} = 1$. To see this we note that for $M = N = 1$ the two bosonic subalgebras trivialise, i.e. $L = R = 0$, and the only non-trivial relation is $\{P_1^1, Q_1^1\} = C$, where $C = CI$ is proportional to the identity $I$. One can thus interpret $c = \frac{1}{\sqrt{C}} P_1^1$ and $c^\dagger = \frac{1}{\sqrt{C}} Q_1^1$, or vice-versa. We remark that the representations obtained through the oscillator realisation are two dimensional for any $N$, with basis

$$\frac{(b_1^\dagger)^{N-1}}{\sqrt{(N-1)!}} f_1^\dagger |\Omega\rangle, \quad \frac{(b_1^\dagger)^N}{\sqrt{N!}} |\Omega\rangle,$$  \hfill (30)
and $C = \mathcal{N}$.

Focusing on the fundamental $\mathcal{N} = 1$ representation we can identify $|0\rangle = f_1^\dagger |\Omega\rangle$ and $c^\dagger |0\rangle = b_1^\dagger |\Omega\rangle$, and the corresponding matrix realisation is given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (31)$$

The extension to $u(1|1)$ is given by a matrix with non-zero supertrace, for example $n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Contrasting $su(1|1)$ with $su(2)$ allows us to emphasise an important point. While their matrix realisations are very similar, the algebras are distinguished by their grading. This feature is not very significant when dealing with operators at the same site, but it becomes crucial when operators at different sites are involved. In particular, that $c$ and $c^\dagger$ at different sites anti-commute to zero rather than commute to zero has the consequence that $su(1|1)$ degrees of freedom inherently encode long-range fermionic correlations. The same is true for $su(2|2)$, the algebra we consider next.

### 4.3 $su(2|2)$

The algebra $su(2|2)$ underlies our treatment of the electronic and local moment degrees of freedom in Secs. 6 and 7. Here the generators consist of the two $su(2)$ bosonic sub-algebras $L$ and $R$, the eight fermionic generators $P^a$, $Q^{\alpha}_a$, and $C = CI$ which is proportional to the identity.

The oscillator realisation provides a basis of the fundamental 4-dimensional representation

$$f_1^\dagger |\Omega\rangle, \quad f_2^\dagger |\Omega\rangle, \quad b_1^\dagger |\Omega\rangle, \quad b_2^\dagger |\Omega\rangle. \quad (32)$$

The fermionic and bosonic states are respectively $su(2)$ doublets of $L$ and $R$. This basis is naturally identified with the four states of an electron

$$|\circ\rangle, \quad |\bullet\rangle, \quad |\downarrow\rangle, \quad |\uparrow\rangle, \quad (33)$$

where the empty $|\circ\rangle$ and doubly occupied $|\bullet\rangle$ states form a charge doublet, and the spin down $|\downarrow\rangle$ and spin-up $|\uparrow\rangle$ states form a spin doublet. We postpone the full connection between $su(2|2)$ and the electronic degree of freedom to Sec. 6. Let us just emphasise here that unlike slave particle descriptions of correlated matter we will not regard the oscillators $b$ and $f$ as degrees of freedom, and our treatment will not depend on whether charge/spin is assigned to the fermionic/bosonic sector.

The choice of assigning charge/spin to the fermionic/bosonic sector is however important for constructing higher dimensional representations. As the Pauli exclusion principle limits growth of the fermionic sector, increasing $\mathcal{N}$ primarily corresponds to an increasing number of bosons. To proceed we focus on the case of assigning spin to the bosonic sector, though the alternative possibility is also of interest as discussed briefly in Sec. 7. For the fundamental $\mathcal{N} = 1$ representation we thus adopt the identification of states

$$|\circ\rangle = f_1^\dagger |\Omega\rangle, \quad |\bullet\rangle = f_2^\dagger |\Omega\rangle, \quad |\downarrow\rangle = b_1^\dagger |\Omega\rangle, \quad |\uparrow\rangle = b_2^\dagger |\Omega\rangle. \quad (34)$$
Increasing $\mathcal{N}$ then yields $4\mathcal{N}$ states which are comprised of four $su(2)$ multiplets as follows: two spin-$S$ multiplets,

$$
\frac{(b_2^\dagger)^{S+m}(b_1^\dagger)^{S-m}}{\sqrt{(S+m)!(S-m)!}} f_1^\dagger |\Omega\rangle, \quad m \in \{-S, -S + 1, \ldots, S\}, \tag{35}
$$

and a spin-$(S + \frac{1}{2})$ multiplet,

$$
\frac{(b_2^\dagger)^{S+1/2+m}(b_1^\dagger)^{S+1/2-m}}{\sqrt{(S + 1/2 + m)!(S + 1/2 - m)!}} f_1^\dagger f_2^\dagger |\Omega\rangle, \quad m \in \{-S \frac{1}{2}, -S + 1/2, \ldots, S + 1/2\}, \tag{36}
$$

where $S$ is related to $\mathcal{N}$ through $\mathcal{N} = 2S + 1$. This basis of states admits a natural interpretation as combining a spin-$S$ local moment with the electron. Firstly, the two spin-$S$ multiplets of Eq. (35) can be written as $|\circ; S, m\rangle$ and $|\bullet; S, m\rangle$, which are the states of a spin moment with the electronic state respectively unoccupied and doubly occupied. The remaining two multiplets can be associated with the decomposition $S \otimes \frac{1}{2} = (S + \frac{1}{2}) \oplus (S - \frac{1}{2})$ arising from combining the spin moment with the spinful singly occupied electronic states. Specifically, the counterpart of the spin-$(S + \frac{1}{2})$ multiplet is

$$
\gamma^+_{m+\frac{1}{2}} |\uparrow; S, m - \frac{1}{2}\rangle + \gamma^-_{m-\frac{1}{2}} |\downarrow; S, m + \frac{1}{2}\rangle, \tag{38}
$$

and that of the spin-$(S - \frac{1}{2})$ multiplet is

$$
\gamma^-_{m-\frac{1}{2}} |\uparrow; S, m - \frac{1}{2}\rangle - \gamma^+_{m+\frac{1}{2}} |\downarrow; S, m + \frac{1}{2}\rangle, \tag{39}
$$

with $\gamma^\pm_m = \sqrt{\frac{S+m}{2S+1}}$. This identification between these higher dimensional representations and an electron combined with a spin moment lays the foundation for Sec. \ref{sec:spinorbitals}.

\subsection{4.4 Extended $su(2|2)$}

Finally we highlight a special feature of $su(2|2)$ which plays a key role in our subsequent analysis: the algebra admits an exceptional central extension. While for general $su(M|N)$ the fermionic generators obey

$$
\{P_\alpha^a, P_\beta^b\} = 0, \quad \{Q_\alpha^a, Q_\beta^b\} = 0, \tag{40}
$$

for $su(2|2)$ these relations can be made non-trivial

$$
\{P_\alpha^a, P_\beta^b\} = \epsilon^{\alpha\beta}_{a} e_{ab} A, \quad \{Q_\alpha^a, Q_\beta^b\} = \epsilon^{ab} e_{\alpha\beta} B, \tag{41}
$$
where the generators $A, B$ are central, and $\epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$. This is special to $M = N = 2$, as only then are the antisymmetric tensors $\epsilon_{ab}$ and $\epsilon_{a\beta}$ defined. So as not to labour notations we refer to this extended algebra also as $su(2|2)$. The extension deforms the oscillator realisation of the generators to

$$\begin{align*}
P^\alpha_a &= u_+ f^\alpha_a b^\dagger_a + u_- \epsilon^{a\beta} \epsilon_{ab} b^\dagger_b f^\beta_b, & Q^\alpha_a &= v_+ b^\dagger_a f^\alpha_a + v_- \epsilon^{a\beta} \epsilon_{ab} f^\beta_b, \\
L^\alpha_3 &= f^\alpha_a f^\beta - \frac{1}{2} \delta^\alpha_3 f^\gamma, & R^a_0 &= b^\dagger_a b^a - \frac{1}{2} \delta^a_0 b^\dagger_c b^c, \\
A &= u_+ u_- (f^\alpha_a f^\alpha_a + b^\dagger_a b^a), & B &= v_+ v_- (f^\alpha_a f^\alpha_a + b^\dagger_a b^a), \\
C &= \frac{1}{2} (u_+ v_+ + u_- v_-)(f^\alpha_a f^\alpha_a + b^\dagger_a b^a),
\end{align*}$$

(42)

where the deformation parameters are constrained to obey $u_+ v_+ - u_- v_- = 1$. The shortening condition becomes $2\sqrt{C^2 - AB} = N$, where $A, B, C$ are the eigenvalues of $A, B, C$. The discussion of representations above in Sec. 4.3 carries over unchanged.

## 5 Spin degrees of freedom

The first setting we focus on are spin systems. Indeed these provide the simplest setting where the non-canonical framework pursued here can be employed and investigated. This example provides a useful testing ground for benchmarking and developing the formalism, while also serving to illustrate the generality of the approach.

Spin systems are inherently bosonic, being governed by the $su(2)$ relations

$$[S^z, S^±] = 2S^±, \quad [S^z, S^z] = \pm S^z,$$

(43)

with $S^± = S^z \pm iS^y$. These are of the generic non-canonical form of Eq. (4ii), and thus do not offer a clear way to organise correlations in general. As discussed in the Introduction however, a quasi-particle description can nevertheless be obtained when the system is magnetically ordered. In this section we wish to highlight that this can be achieved through the formalism of Sec. 3 by recasting Eq. (4.iii) in the special non-canonical form of Eq. (4.iii). Indeed this is achieved through the well-known identification

$$a = \frac{1}{\sqrt{2S}} S^+, \quad a^\dagger = \frac{1}{\sqrt{2S}} S^-, \quad n = S - S^z,$$  

(44)

which brings the $su(2)$ relations to the form

$$[a, a^\dagger] = 1 - \frac{1}{2S} n, \quad [n, a^\dagger] = a^\dagger, \quad [n, a] = -a,$$

(45)

where we assume for simplicity that the local spin polarisation is orientated in the $z$-direction. The parameter $S$ is naturally identified as the magnitude of the spin, so that $n$ takes integer values between 0 and $2S$, and $\langle n \rangle = 0$ if the spin is maximally polarised. The Casimir identity here translates to the following useful relation

$$n = a^\dagger a + \frac{1}{2S} n(n - 1),$$

(46)

(which simplifies to $n = a^\dagger a$ for $S = 1/2$). We highlight that in the large-$S$ limit the number of states diverges, and the algebra simplifies to that of canonical bosons.
It is worthwhile to compare this with the standard theoretical treatments of magnetically ordered phases. In its conventional formulation, spin-wave theory organises correlations through the weakly correlated paradigm \[12\,16\]. This is achieved by employing a representation of the spin generators in terms of canonical bosons as follows

\[
S^+ = \sqrt{2S} \left(1 - \frac{1}{2S} b^\dagger b\right)^{\frac{1-\gamma}{2}} b, \quad S^- = \sqrt{2S} b^\dagger \left(1 - \frac{1}{2S} b^\dagger b\right)^\gamma, \quad S^z = S - b^\dagger b,
\]

e.g. the Holstein–Primakoff representation at \(\gamma = 1/2\), or the Dyson–Maleev representation at \(\gamma = 0\). While this approach has proved useful in practice, it lacks the robustness of a true weakly correlated theory. An issue is that the Hilbert space of the representative bosonic system includes multitudes of spurious states, as a boson has infinitely many states per site while a spin has only a finite number. Although the relations Eq. (47) incorporate a Hilbert space restriction, it is not possible to take this into account at finite temperature. By contrast a non-canonical description accounts for the correct number of states to begin with, i.e. the thermal traces appearing for example in Eq. (8) are defined over the states of the local spins. Indeed this feature is taken advantage of in an alternative approach to describing magnetism which develops a diagrammatic framework organised about the atomic limit \[27\,51\]. As mentioned in the Introduction however, a drawback of such an approach is that the representable solvable system is not composed of dispersing modes. We thus highlight that the merit of the formalism of Sec. 3 is that it offers a quasi-particle description which is formulated on the correct Hilbert space.

6 Electronic degrees of freedom

The next setting we consider are electronic systems, which provide the core setting for the strongly correlated behaviour highlighted in the Introduction. Specifically, we consider lattice models with four states per site

\[
|\circ\rangle = |0\rangle, \quad |\downarrow\rangle = c^\dagger_\downarrow |0\rangle, \quad |\uparrow\rangle = c^\dagger_\uparrow |0\rangle, \quad |\bullet\rangle = c^\dagger_\downarrow c^\dagger_\uparrow |0\rangle.
\]

(48)

A challenge often associated with electrons are the non-local correlations arising from their inherent fermionic character, corresponding to a relative grading between the pairs of states \(|\circ\rangle, |\bullet\rangle\) and \(|\downarrow\rangle, |\uparrow\rangle\). Such correlations are however automatically taken into account by casting the electronic degree of freedom through graded Lie algebras. Naively there are two possibilities, corresponding to the two factorisations \(4 = 2 \times 2\) or \(1 \times 4\) of the four electronic states.

The conventional way to interpret the electronic degree of freedom is through the first possibility \(4 = 2 \times 2\), for which it is useful to view the four electronic states grouped as

\[
\{ |0\rangle, |\downarrow\rangle \} \otimes \{ |0\rangle, |\uparrow\rangle \}.
\]

(49)

This corresponds to the spinful canonical fermion algebra

\[
\{c_\sigma, c^\dagger_{\sigma'}\} = \delta_{\sigma\sigma'},
\]

(50)

which underlies the weakly correlated paradigm. In the language of Sec. 4 this algebra corresponds to two copies of the \(su(1\,1)\), one for each spin species. That is, the electronic degree
of freedom is here cast as \( su(1|1) \otimes su(1|1) \). This is extended to \( u(1|1) \otimes u(1|1) \) by including the operators \( n_\sigma = c_\sigma^\dagger c_\sigma \) which obey

\[
[n_\sigma, c_\sigma'] = \delta_{\sigma \sigma'} c_\sigma^\dagger, \quad [n_\sigma, c_\sigma'] = -\delta_{\sigma \sigma'} c_\sigma.
\] (51)

The alternative \( 4 = 1 \times 4 \) possibility corresponds to the algebra \( su(2|2) \). Here it is useful to view the four electronic states grouped as

\[
\{ |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle \}.
\] (52)

The \( su(2|2) \) algebra contains two bosonic sub-algebras: there is a spin \( su(2) \) with generators \( s \) acting on the pair of singly occupied states \( |\downarrow\rangle, |\uparrow\rangle \),

\[
 s^z = \frac{1}{2}(n_+ - n_-), \quad s^+ = c_1^\dagger c_1, \quad s^- = c_{\downarrow}^\dagger c_{\downarrow},
\] (53)

obeying \( [s^+, s^-] = 2s^z \) and \( [s^z, s^\pm] = \pm s^\pm \), as well as a charge \( su(2) \) with generators \( q \) acting on the charge doublet \( |\uparrow\rangle, |\downarrow\rangle \),

\[
 \eta^z = \frac{1}{2}(n_+ + n_- - 1), \quad \eta^+ = c_{\uparrow}^\dagger c_{\uparrow}, \quad \eta^- = c_{\downarrow}^\dagger c_{\downarrow},
\] (54)

obeying \( [\eta^+, \eta^-] = 2\eta^z \) and \( [\eta^z, \eta^\pm] = \pm \eta^\pm \). In addition there are fermionic generators \( q \) which act between the two pairs,

\[
 q_{\sigma\circ}^\dagger = c_{\sigma} - n_{\sigma} c_{\sigma}, \quad q_{\sigma\circ} = \bar{\sigma} n_{\sigma} c_{\sigma},
\] (55)

obeying the anti-commutation relations

\[
\{ q_{\sigma\nu}, q_{\nu\nu'}^\dagger \} = \frac{1}{2} + \nu \eta^z - \sigma s^z,
\]

\[
\{ q_{\nu\circ}, q_{\circ\nu'}^\dagger \} = s^+, \quad \{ q_{\sigma\circ}, q_{\sigma\circ'}^\dagger \} = \eta^+,
\]

\[
\{ q_{\nu\circ}, q_{\nu\circ'}^\dagger \} = s^−, \quad \{ q_{\sigma\circ}, q_{\sigma\circ'}^\dagger \} = \eta^−,
\]

\[
\{ q_{\nu\nu'}, q_{\nu'\nu'} \} = \{ q_{\nu\nu'}, q_{\nu'\nu'} \} = 0,
\] (56)

where \( \sigma \) takes values \(-1, 1\) for \( \sigma = \downarrow, \uparrow \) and \( \bar{\sigma} = -\sigma \), and \( \nu \) takes values \(-1, 1\) for \( \nu = \circ, \bullet \).

Unlike for the canonical \( c \), these relations are non-canonical, yielding the generators of both the spin and charge \( su(2) \) algebras. The commutation relations between the \( q \) and the \( s \) are

\[
[s^z, q_{\sigma\nu}^\dagger] = \frac{\sigma}{2} q_{\nu\nu'}, \quad [s^+, q_{\nu\nu'}^\dagger] = -q_{\nu\nu'}, \quad [s^−, q_{\nu\nu'}^\dagger] = -q_{\nu\nu'},
\] (57)

and between the \( q \) and the \( \eta \) are

\[
[\eta^z, q_{\nu\nu'}^\dagger] = \frac{\nu}{2} q_{\nu\nu'}, \quad [\eta^+, q_{\nu\nu'}^\dagger] = q_{\nu\nu'}, \quad [\eta^−, q_{\nu\nu'}^\dagger] = q_{\nu\nu'},
\]

\[
[\eta^z, q_{\nu\nu'}] = -\frac{\nu}{2} q_{\nu\nu'}, \quad [\eta^+, q_{\nu\nu'}] = -q_{\nu\nu'}, \quad [\eta^−, q_{\nu\nu'}] = -q_{\nu\nu'}.
\] (58)
In total the fifteen operators, \(8 \times q, 3 \times s, 3 \times \eta\) and the identity, generate \(su(2|2)\). The algebra is further extended to \(u(2|2)\) by adding a sixteenth with zero supertrace. Here we choose

\[
\theta = (n_+ - \frac{1}{2})(n_- - \frac{1}{2}),
\]

for which the additional relations are

\[
[\theta, q_{\sigma \nu}^+] = \frac{1}{2} q_{\sigma \nu}^+,
\]

\[
[\theta, q_{\sigma \nu}] = -\frac{1}{2} q_{\sigma \nu}.
\]

In the language of Sec. 4, the \(s\) and \(\eta\) furnish the generators \(L\) and \(R\) of the bosonic sub-algebras of \(su(2|2)\) as follows

\[
L_2^2 = -L_1^2 = \eta^+, \quad L_2^1 = \eta^-, \quad L_2^1 = s^-
\]

\[
R_2^2 = -R_1^1 = s^+, \quad R_1^2 = s^+, \quad R_1^2 = s^-.
\]

the \(q\) furnish the fermionic generators

\[
P_1^1 = q_{\sigma \nu}^+, \quad P_2^1 = q_{\sigma \nu}, \quad Q_1^1 = q_{\sigma \nu}, \quad Q_2^1 = q_{\sigma \nu},
\]

\[
P_2^1 = q_{\sigma \nu}, \quad P_2^1 = q_{\sigma \nu}, \quad Q_2^1 = q_{\sigma \nu}, \quad Q_2^1 = q_{\sigma \nu}.
\]

and \(D = 4\theta\) is the extension to \(u(2|2)\). They are related to the Hubbard projection operators through

\[
X_\sigma = q_{\sigma \nu}^+, \quad X_\nu = \nu \eta^+ + \theta + 1/4, \quad X_\rho = \eta^+, \quad X_\rho = \eta^-,
\]

\[
X_\sigma = q_{\sigma \nu}, \quad X_\nu = \sigma s^+ - \theta + 1/4, \quad X_\nu = s^+, \quad X_\nu = s^-.
\]

for which the graded algebra takes the simple form

\[
X_a^bX_b^d \pm X_b^aX_a^d = \delta_a^bX_a^d \pm \delta_b^dX_b^a,
\]

with \(a, b, c, d \in \{\circ, \bullet, \downarrow, \uparrow\}\), and a plus sign if both \(X_b^a\) and \(X_a^c\) are fermionic and a minus sign otherwise.

In order to consider the \(su(2|2)\) algebra as offering a quantum degree of freedom we wish to cast it in the special non-canonical form of Eq. (4.iii). This we achieve by exploiting the exceptional deformation of \(su(2|2)\) described in Sec. 4.3. In fact, we find that it is possible to obtain two distinct embeddings, which we call \(q\)-form and \(p\)-form, which impose distinct reality conditions on the generators

\[
q\text{-form}: \quad (P_a^\alpha)^\dagger = Q_a^\alpha, \quad (Q_a^\alpha)^\dagger = P_a^\alpha
\]

\[
p\text{-form}: \quad (P_a^\alpha)^\dagger = i \epsilon^{a b \alpha \beta} P_b^\beta, \quad (Q_a^\alpha)^\dagger = i \epsilon^{a \beta \alpha \beta} Q_b^\beta.
\]

We proceed to describe each in turn.

\textbf{q\text{-form}}

A quantum degree of freedom is obtained from \(su(2|2)\) by deforming the generators as follows

\[
q_{\sigma \nu}^+ = \begin{pmatrix} \frac{1 + \lambda}{2} & c_{\sigma} - \lambda n_{\sigma}c_{\sigma}\end{pmatrix}, \quad q_{\sigma \nu}^+ = \bar{\sigma}\left(\frac{1 + \lambda}{2} c_{\sigma} + \lambda n_{\sigma}c_{\sigma}\right).
\]

\[
q_{\sigma \nu} = \begin{pmatrix} \frac{1 + \lambda}{2} & c_{\sigma} - \lambda n_{\sigma}c_{\sigma}\end{pmatrix}, \quad q_{\sigma \nu} = \bar{\sigma}\left(\frac{1 + \lambda}{2} c_{\sigma} + \lambda n_{\sigma}c_{\sigma}\right).
\]

\[
q_{\sigma \nu} = \begin{pmatrix} \frac{1 + \lambda}{2} & c_{\sigma} - \lambda n_{\sigma}c_{\sigma}\end{pmatrix}, \quad q_{\sigma \nu} = \bar{\sigma}\left(\frac{1 + \lambda}{2} c_{\sigma} + \lambda n_{\sigma}c_{\sigma}\right).
\]
with \( \lambda \in \mathbb{R} \). The undeformed case above corresponds to \( \lambda = 1 \), and as \( \lambda \to 0 \) the generators collapse pairwise onto the canonical fermion operators. For general \( \lambda \) the anti-commutation relations of the \( q \) get modified to

\[
\begin{align*}
\{ q_{\sigma \nu}, q_{\sigma \nu}^\dagger \} &= \frac{1+\lambda^2}{4} + \lambda (\nu \eta^z - \sigma s^z), \\
\{ q_{\nu \sigma}, q_{\nu \sigma}^\dagger \} &= \lambda s^+, \quad \{ q_{\sigma 0}, q_{\sigma 0}^\dagger \} = \lambda \eta^+, \\
\{ q_{\nu \sigma}, q_{\nu \sigma}^\dagger \} &= \lambda s^-, \quad \{ q_{\sigma 0}, q_{\sigma 0}^\dagger \} = \lambda \eta^-, \\
\{ q_{\sigma \nu}, q_{\sigma \nu'}^\dagger \} &= \{ q_{\sigma \nu}, q_{\sigma' \nu'}^\dagger \} = \frac{1-\lambda^2}{4} \epsilon_{\sigma' \sigma} \epsilon_{\nu \nu'}, 
\end{align*}
\]

with \( \epsilon_{+} = -\epsilon_{\downarrow} = \epsilon_{\uparrow} = -\epsilon_{\downarrow} = 1 \). The commutation relations between the \( q \) and \( s \) and \( \eta \) are again Eqs. (57), (58) above. The linear action of \( \theta \) on \( q \) gets deformed to

\[
\begin{align*}
[\theta, q_{\sigma \nu}^\dagger] &= \frac{1+\lambda^2}{4\lambda} q_{\sigma \nu}^\dagger + \frac{1-\lambda^2}{4\lambda} \epsilon_{\sigma', \nu} \epsilon_{\nu', \nu'} q_{\sigma', \nu'}, \\
[\theta, q_{\sigma \nu}] &= -\frac{1+\lambda^2}{4\lambda} q_{\sigma \nu} - \frac{1-\lambda^2}{4\lambda} \epsilon_{\sigma', \nu} \epsilon_{\nu', \nu'} q_{\sigma', \nu'}^\dagger.
\end{align*}
\]

We thus see that although the anti-commutation relations of the \( q \) do remain non-canonical, now all non-canonical terms come with the scalar factor \( \lambda \), and so the algebra is of the special non-canonical form of Eq. (4.iii), with the \( q \) playing the role of the \( a \) and the \( s, \eta \) and \( \theta \) playing the role of the \( n \).

The framework outlined in Sec. 3 can then be employed to systematically compute two-point functions of the \( q \), as described in explicit detail in Ref. [39]. The \( q \) are inherently correlated from a canonical perspective as they are non-linear in the \( c \). A remarkable fact however is that the relations linking the \( q \) back to the \( c \) are incredibly simple. They are linear

\[
c^\dagger_\gamma = q^\dagger_{\gamma 0} + q^\dagger_{\gamma \bullet}, \quad c^\dagger_\pi = q^\dagger_{\pi 0} - q^\dagger_{\pi \bullet}
\]

and independent of \( \lambda \). This is dubbed a splitting of the electron, in contrast with fractionalisation which takes a product form. A powerful consequence is that once the two-point functions of the \( q \) are computed, the two-point functions of the \( c \) follow immediately upon taking linear combinations. As a corollary, the expectation values of both \( s \) and \( \eta \) follow also.

Another significant feature is the linear action of \( \theta \) on the \( q \). In the language of Sec. 3 it thus enters \( s_0 \) only, and so does not directly contribute to the correlation encoding functionals \( \Sigma \) and \( \Omega \). This is in stark contrast with its action on the \( c \), whose cubic form is responsible for the difficulty in addressing the Hubbard model within the weakly correlated framework. From the perspective of the \( q \), a Hubbard interaction term \( U \theta \) is akin to incorporating an emergent chemical potential which accounts for a distinction of doubly occupied sites from pairs of singly occupied sites. Spin and charge correlations take the form of collective modes.

In the language of Sec. 4.4 the \( q \)-form generators are given by

\[
\begin{align*}
q^\dagger_{\gamma 0} &= \sqrt{\lambda} P^\dagger_{\gamma 1}, \quad q^\dagger_{\gamma \bullet} = \sqrt{\lambda} P^\dagger_{\gamma 2}, \quad q^\dagger_{\pi 0} = \sqrt{\lambda} Q^\dagger_{\pi 1}, \quad q^\dagger_{\pi \bullet} = \sqrt{\lambda} Q^\dagger_{\pi 2},
q^\dagger_{\pi 0} &= \sqrt{\lambda} P^\dagger_{\pi 1}, \quad q^\dagger_{\pi \bullet} = \sqrt{\lambda} P^\dagger_{\pi 2}, \quad q^\dagger_{\gamma 0} = \sqrt{\lambda} Q^\dagger_{\gamma 1}, \quad q^\dagger_{\gamma \bullet} = \sqrt{\lambda} Q^\dagger_{\gamma 2},
\end{align*}
\]

and the deformation parameters are \( u_\pm = v_\pm = \frac{1+\lambda}{2\sqrt{\lambda}} \).

**p-form**

An alternative quantum degree of freedom for the electron can be obtained from \( su(2|2) \) by modifying the hermiticity relationship between the generators as in Eq. (65). Essentially this
amounts to taking \( \lambda = i\gamma \). Whether this is capable of capturing different correlations from the \( q \)-form, or whether it corresponds to a topologically distinct correlated regime are questions we leave to the future. Here we focus on characterising the degree of freedom. Specifically we write the fermionic generators as \( p_{\sigma \tau} \), with \( \tau \in \{-, +\} \), given explicitly by

\[
\begin{align*}
p_{\sigma -} &= \frac{1 + i\gamma}{2} c_\sigma^\dagger + i\gamma n_\sigma c_\sigma^\dagger, \\
p_{\sigma +} &= \frac{1 + i\gamma}{2} c_\sigma^\dagger - i\gamma n_\sigma c_\sigma^\dagger,
\end{align*}
\]

(71)

with \( \gamma \in \mathbb{R} \). Here \( p_{\sigma -} \) and \( p_{\sigma +} \) collapse pairwise onto \( c \) as \( \gamma \to 0 \), but there is no real value of \( \gamma \) for which the undeformed generators are reobtained. The diagonal anti-commutation relations of the \( p \) are canonical

\[
\{p_{\sigma \tau}, p_{\sigma \tau}^\dagger\} = \frac{1 + i\gamma^2}{4}.
\]

The non-canonical nature of the algebra appears through the relations for \( \tau \neq \tau' \),

\[
\begin{align*}
\{p_{\tau \tau}, p_{\sigma \tau}^\dagger\} &= \frac{1 - \gamma^2}{4} + i\gamma \epsilon_{\tau\tau'} (\sigma s^\tau - \eta^\tau), \\
\{p_{\tau \tau}, p_{\sigma \tau}^\dagger\} &= i\gamma \epsilon_{\tau\tau'} s^\tau, \\
\{p_{\tau \tau}, p_{\tau \tau}^\dagger\} &= i\gamma \epsilon_{\tau\tau'} \eta^\tau,
\end{align*}
\]

(73)

with \( \epsilon_{-+} = -\epsilon_{+-} = 1 \), yielding the generators of the two \( su(2) \) algebras. Here the commutation relations between the \( p \) and \( s \) are

\[
\begin{align*}
[s^\tau, p_{\sigma \tau}^\dagger] &= \frac{\sigma}{2} p_{\sigma \tau}^\dagger, \\
[s^\tau, p_{\sigma \tau}] &= -\frac{\sigma}{2} p_{\sigma \tau}, \\
[s^\tau, p_{\tau \tau}^\dagger] &= p_{\tau \tau}^\dagger, \\
[s^\tau, p_{\tau \tau}] &= -p_{\tau \tau},
\end{align*}
\]

(74)

and between the \( p \) and the \( \eta \) are

\[
\begin{align*}
[\eta^\tau, p_{\sigma \tau}] &= \frac{1}{2} p_{\sigma \tau}, \\
[\eta^\tau, p_{\tau \tau}] &= -p_{\tau \tau}^\dagger, \\
[\eta^\tau, p_{\tau \tau}^\dagger] &= p_{\tau \tau}, \\
[\eta^\tau, p_{\tau \tau}] &= p_{\tau \tau}, \\
\end{align*}
\]

(75)

The action of \( \theta \) on \( p \) is again linear

\[
\begin{align*}
[\theta, p_{\sigma \tau}^\dagger] &= \tau \left( \frac{1 + i\gamma^2}{4\gamma} p_{\sigma \tau}^\dagger - \frac{1 - i\gamma^2}{4\gamma} p_{\sigma \tau} \right), \\
[\theta, p_{\sigma \tau}] &= \tau \left( \frac{1 + i\gamma^2}{4\gamma} p_{\sigma \tau} - \frac{1 - i\gamma^2}{4\gamma} p_{\sigma \tau}^\dagger \right).
\end{align*}
\]

(76)

Thus once more the algebra takes the special non-canonical form of Eq. (12), here with \( p \) as the \( a \), the generators \( s \), \( \eta \) and \( \theta \) as the \( n \), and \( \gamma \) as \( \lambda \). The \( p \)-form similarly corresponds to a splitting of the electron, here with

\[
c_\dagger^\tau = p_{\tau -}^\dagger + p_{\tau +}^\dagger, \quad c^\tau = p_{\tau -} + p_{\tau +}^\dagger,
\]

(77)

independent of \( \gamma \). Thus again Sec. 3 offers a quasi-particle framework giving access to the electronic Green’s function.

In the language of Sec. 4, the \( p \)-form generators are given by

\[
\begin{align*}
p_{\tau -} &= \sqrt{\gamma} P_{1\tau}, \\
p_{\tau +} &= \sqrt{\gamma} P_{2\tau}, \\
p_{\tau -}^\dagger &= \sqrt{\gamma} Q_{1\tau}, \\
p_{\tau +}^\dagger &= \sqrt{\gamma} Q_{2\tau},
\end{align*}
\]

(78)

with deformation parameters \( u_\pm = v_\pm = \frac{1 + i\gamma}{2\sqrt{i\gamma}} \).
7 Local moment degrees of freedom

The third and final setting we consider are local moment systems. These correspond to a combination of a spin and an electron at each site. Materials which admit such a description are known for the wide range of strongly correlated phenomena they exhibit, including heavy fermion formation, quantum criticality, magnetism and unconventional superconductivity.

The conventional way to regard a local moment system is to treat the electron $c_\sigma$ and spin moment $s$ independently. That is, with the electrons governed by the canonical anticommutation relations $\{c_\sigma, c_\sigma^\dagger\} = \delta_{\sigma\sigma'}$, i.e. the $su(1|1) \otimes su(1|1)$ algebra, and the spin moment governed by the non-canonical commutation relations $[S^2, S^\pm] = \pm S^\pm$, $[S^+, S^-] = 2S^z$, i.e. the $su(2)$ algebra. These provide reasonable degrees of freedom for a regime of behaviour where the electrons form a canonical Fermi liquid and the spins are free to order.

We wish to focus on alternatives based on non-canonical degrees of freedom. As recently demonstrated in Ref. [40], the $su(2|2)$ algebra can again be employed in this setting. Here we elaborate upon this, clarifying the algebraic origins of the degree of freedom and placing it in a broader context.

Indeed what we desire for local moment systems is a graded algebra which can account for the combination of the electron and spin moment. One attempt may be to turn to the higher rank $su(M|N)$ algebras. For example, for the case of a spin-1/2 local moment there are eight states locally

$$|\downarrow\rangle, c_{\downarrow}^\dagger |\downarrow\rangle, c_{\uparrow}^\dagger |\downarrow\rangle, c_{\downarrow}^1 c_{\uparrow}^\dagger |\downarrow\rangle, |\uparrow\rangle, c_{\downarrow}^\dagger |\uparrow\rangle, c_{\uparrow}^\dagger |\uparrow\rangle, c_{\downarrow}^1 c_{\uparrow}^\dagger |\uparrow\rangle,$$

and one may consider employing the algebra $su(4|4)$ which has an 8-dimensional fundamental representation. There are two clear difficulties however, one being the absence of an analogue of the deformation of Sec. 4.1, the other that this choice fails to distinguish between the underlying spin and electron. Instead, a natural way to treat local moment systems is through the higher dimensional representations of $su(2|2)$. Specifically, in Sec. 4.3 we constructed a family of $su(2|2)$ representations and showed that they can be regarded precisely as the combination of a spin moment with an electron.

We obtain the generators of $su(2|2)$ in the local moment setting from their explicit form in the oscillator realisation Eq. (19) through the matching of the states described in Eqs. (35)-(39). We obtain the fermionic generators as

$$q_{1,0} = \frac{1}{2} c_{\uparrow} + \frac{1}{2S+1} \left( \frac{1}{2} c_{\downarrow} - n_{\downarrow} c_{\uparrow} + c_{\uparrow} S^- + c_{\downarrow} S^\dagger \right),$$
$$q_{1,0}^\dagger = \frac{1}{2} c_{\downarrow} + \frac{1}{2S+1} \left( \frac{1}{2} c_{\uparrow} - n_{\uparrow} c_{\downarrow} + c_{\downarrow} S^+ + c_{\uparrow} S^- \right),$$
$$q_{1,0}^\dagger = \frac{1}{2} c_{\downarrow} - \frac{1}{2S+1} \left( \frac{1}{2} c_{\uparrow} - n_{\uparrow} c_{\downarrow} + c_{\downarrow} S^- + c_{\uparrow} S^\dagger \right),$$
$$q_{1,0}^\dagger = \frac{1}{2} c_{\uparrow} + \frac{1}{2S+1} \left( \frac{1}{2} c_{\downarrow} - n_{\downarrow} c_{\uparrow} + c_{\uparrow} S^+ + c_{\downarrow} S^- \right),$$

under the identification

$$q_{1,0} = \frac{1}{\sqrt{2S+1}} P_{1,0}, \quad q_{1,0}^\dagger = \frac{1}{\sqrt{2S+1}} P_{1,0}^\dagger, \quad q_{1,0} = \frac{1}{\sqrt{2S+1}} Q_{1,0}, \quad q_{1,0}^\dagger = \frac{1}{\sqrt{2S+1}} Q_{1,0}^\dagger,$$

These again manifest a splitting of the electron

$$c_{\uparrow} = q_{1,0} + q_{1,0}^\dagger, \quad c_{\downarrow} = q_{1,0} - q_{1,0}^\dagger.$$
We identify the bosonic generators through Eq. (61), and find that the charge $su(2)$ is again $\eta$, while the spin $su(2)$ is now given by the total spin operator $\vec{\Sigma} = \vec{s} + \vec{S}$ which combines the electronic and local moment spin. Identifying $\theta$ with $\frac{1}{4} D$ as in the electronic setting we obtain

$$\theta = \frac{1}{2} - \frac{1}{2S+1} \left( \vec{\Sigma} \cdot \vec{\Sigma} + \frac{1}{3} \vec{\eta} \cdot \vec{\eta} \right).$$

(83)

The algebraic relations get modified correspondingly. The anti-commutation relations of the $q$ are as follows

$$\{q_{\sigma\nu}, q_{\sigma\nu}^+\} = \frac{1}{2} + \frac{1}{2S+1} (\nu \eta^\nu - \sigma \Sigma^\sigma),$$

$$\{q_{\lambda\nu}, q_{\lambda\nu}^+\} = \frac{1}{2S+1} \Sigma^\lambda, \quad \{q_{\sigma\sigma}, q_{\sigma\sigma}^+\} = \frac{1}{2S+1} \eta^\sigma,$$

$$\{q_{\nu\nu}, q_{\nu\nu}^+\} = \frac{1}{2S+1} \Sigma^\nu, \quad \{q_{\sigma\sigma}, q_{\sigma\sigma}^+\} = \frac{1}{2S+1} \eta^\sigma,$$

$$\{q_{\sigma\nu}, q_{\sigma'\nu'}\} = \{q_{\sigma\nu}^+, q_{\sigma'\nu'}^+\} = 0,$$

(84)

where $S$ is the magnitude of the spin moment, i.e. $\vec{S} \cdot \vec{S} = S(S+1)$. Here $\Sigma$ obeys Eq. (57) in place of $s$, while $\eta$ again obeys Eq. (58) and $\theta$ of Eq. (83) again obeys Eq. (60).

These relations already take the special non-canonical form, now with $\frac{1}{2S+1}$ as $\lambda$. We may thus obtain a quasi-particle description in the large-$S$ limit. As we expect spin to order as $S \to \infty$, this may be useful for characterising magnetic behaviour in electronic and local moment systems. We highlight here that a similar logic can also be employed for characterising charge order in the electronic setting, for example superconductivity or charge-density waves resulting from electronic correlations. A second family of higher dimensional representations of $su(2|2)$ can be constructed as in Sec. 4.3 by assigning charge to the bosonic sector and spin to the fermionic sector. This can be viewed as combining a fictional charge moment with an electron, which provides an opportunity to organise correlations about the order which appears in the large-charge limit.

We can also cast $su(2|2)$ as a quantum degree of freedom for local moment systems in a way that does not require order. As in the electronic setting, this is achieved by taking advantage of the deformation of the algebra, Sec. 4.3 and likewise there are two embeddings, $q$-form and $p$-form, corresponding to distinct reality conditions on the fermionic generators Eq. (63). Their description largely mirrors that of Sec. 6 and so here we just succinctly highlight the key relations.

$q$-form

Choosing deformation parameters as $u_\pm = v_\pm = \frac{1\pm\lambda}{2\sqrt{\lambda}}$, the fermionic generators are

$$q_{\sigma\sigma}^+ = \frac{1}{2} c_{\sigma} + \frac{\lambda}{2S+1} \left( \frac{1}{2} c_{\sigma} - n_{\sigma} c_{\sigma} + c_{\sigma} S^- + c_{\sigma} S^z \right),$$

$$q_{\lambda\nu}^+ = \frac{1}{2} c_{\nu} + \frac{\lambda}{2S+1} \left( \frac{1}{2} c_{\nu} - n_{\nu} c_{\nu} + c_{\nu} S^+ - c_{\nu} S^z \right),$$

$$q_{\sigma\sigma}^+ = \frac{1}{2} c_{\sigma} - \frac{\lambda}{2S+1} \left( \frac{1}{2} c_{\sigma} - n_{\sigma} c_{\sigma} + c_{\sigma} S^- - c_{\sigma} S^z \right),$$

$$q_{\lambda\nu}^+ = -\frac{1}{2} c_{\nu} + \frac{1}{2S+1} \left( \frac{1}{2} c_{\nu} - n_{\nu} c_{\nu} + c_{\nu} S^+ + c_{\nu} S^z \right),$$

(85)
and obey the anti-commutation relations
\[
\{q_{\sigma\nu}, q_{\sigma\nu}^\dagger\} = \frac{1+i\gamma^2}{4} + \frac{\lambda}{2S+1}(\nu\eta^z - \sigma\Sigma^z),
\]
\[
\{q_{\nu\nu}, q_{\nu\nu}^\dagger\} = \frac{\lambda}{2S+1}\Sigma^+, \quad \{q_{\sigma\nu}, q_{\sigma\nu}^\dagger\} = \frac{\lambda}{2S+1}\eta^+,
\]
\[
\{q_{\nu\nu}, q_{\nu\nu}^\dagger\} = \frac{\lambda}{2S+1}\Sigma^-, \quad \{q_{\sigma\nu}, q_{\sigma\nu}^\dagger\} = \frac{\lambda}{2S+1}\eta^-.
\]
\[
(86)
\]

The linear action of \(\theta\) of Eq. \((83)\) is again given by Eq. \((68)\), and the \(\lambda\)-independent splitting of the electron, Eq. \((69)\), is preserved.

\(p\)-form

Choosing deformation parameters as \(u_\pm = v_\pm = \frac{1\pm i\gamma}{2\sqrt{1+i^2}}\) the fermionic generators are
\[
\begin{align*}
p^\uparrow_{\sigma\tau} &= \frac{1}{2}c_{\tau}^\dagger + i\tau\frac{\gamma}{2S+1}(c_{\tau}^\dagger - n_{\tau}c_{\tau}^\dagger - c_{\tau}^\dagger S^- - c_{\tau}^\dagger S^+), \\
p^\uparrow_{\nu\nu} &= \frac{1}{2}c_{\nu}^\dagger + i\nu\frac{\gamma}{2S+1}(c_{\nu}^\dagger - n_{\nu}c_{\nu}^\dagger - c_{\nu}^\dagger S^- + c_{\nu}^\dagger S^+), \\
p_{\nu\nu} &= \frac{1}{2}c_{\nu} - i\nu\frac{\gamma}{2S+1}(c_{\nu} - n_{\nu}c_{\nu} + c_{\nu}S^- - c_{\nu}S^+), \\
p_{\sigma\nu} &= \frac{1}{2}c_{\sigma} - i\sigma\frac{\gamma}{2S+1}(c_{\sigma} - n_{\sigma}c_{\sigma} + c_{\sigma}S^- + c_{\sigma}S^+),
\end{align*}
\]
\[
(87)
\]

and here obey
\[
\{p_{\sigma\tau}, p_{\sigma\tau}^\dagger\} = \frac{1+i\gamma^2}{4},
\]
\[
(88)
\]

and
\[
\begin{align*}
\{p_{\sigma\tau}, p_{\sigma\tau'}^\dagger\} &= \frac{1+i\gamma^2}{4} + \frac{i\gamma}{2S+1}\epsilon_{\tau\tau'}(\sigma s^z - \eta^z), \\
\{p_{\nu\nu}, p_{\nu\nu}^\dagger\} &= \frac{i\gamma}{2S+1}\epsilon_{\nu\nu} s^-, \quad \{p_{\nu\nu}, p_{\nu\nu}^\dagger\} = \frac{i\gamma}{2S+1}\epsilon_{\nu\nu} s^+,
\end{align*}
\]
\[
(89)
\]

The linear action of \(\theta\) of Eq. \((83)\) is again given by Eq. \((70)\), and the \(\gamma\)-independent splitting of the electron, Eq. \((77)\), is preserved.

\section{Discussion}

In this work we have argued that a special class of non-canonical algebras provide useful quantum degrees of freedom, specifically for spin, electron and local moment systems. By contrast, the degrees of freedom most commonly employed for modelling condensed matter systems obey canonical algebras, either directly as in the weakly correlated framework, or through fractionalisation. Indeed their use is ubiquitous. On the one hand, magnetic phases are generally treated as weakly correlated in the spirit of Holstein–Primakoff/Dyson–Maleev as described in Sec. 5. On the other, evidence of strongly correlated behaviour in electron and local moment systems is often used to infer underlying fractionalised degrees of freedom, see e.g. \cite{20,35,52,53}. We argue however that magnetism, as well as strongly correlated electronic behaviours such as Mott physics \cite{39} and heavy fermion formation \cite{40} are more appropriately characterised by degrees of freedom which are truly non-canonical.
An objective of this work has been to lay a solid foundation for establishing such degrees of freedom. As much as possible we have adopted a unified language, so as to make clear the connections between different settings, and to highlight the relationships between the various algebraic structures at play.

Going forward, an important step is to develop the formalism for organising the correlations captured by the functionals $\Sigma$ and $\Omega$ entering the exact representation of the Green’s function given by Eq. (15), paralleling that developed within the weakly correlated framework. The two recent studies in the electron and local moment settings focused on static approximations \cite{39, 40}, analogues of the non-interacting and Hartree-Fock approximations. While these revealed essential phenomena such as the opening of a Mott gap and heavy fermion formation, they missed many important correlations. They did not take into account quasi-particle decay or collective excitations, and did not capture effects such as dynamical screening and Kondo coherence. In the framework pursued by Shastry and collaborators, focusing on a $U = \infty$ limit of the Hubbard model while reincorporating $U$ on the projected Hilbert space to enforce the Luttinger sum rule, a variety of improved approximations were considered, up to a self-consistent treatment at second order in $\lambda$, which revealed interesting features such as asymmetry and kinks in spectral lineshapes \cite{37, 54–57}. Moreover it would be interesting to explore schemes which go beyond an expansion in $\lambda$, such as the GW and T-matrix/ladder approximations, which are often necessary in the weakly correlated context in order to adequately describe a system \cite{58, 59}. An interesting future direction is to generalise such schemes to the non-canonical setting, and to establish their relevance for paradigmatic models of strong correlations such as the Heisenberg, Hubbard and Kondo lattice models. In particular, we here identify two problems which provide interesting test cases for developing and benchmarking these methods.

Firstly, spin systems offer a convenient setting for establishing the non-canonical framework. They have been studied in great detail over many decades through a wide range of both analytical and numerical methods. In particular, they are more amenable to numerical simulations than fermionic systems, which makes them well suited for benchmarking analytic approaches. The assertion of Sec. 5 that magnetically ordered phases are best characterised through the non-canonical relations Eq. (45), as opposed to the canonical representation Eq. (47), is not intended to challenge the success of traditional spin-wave theory, but rather to provide a powerful framework for going beyond it where necessary \cite{60, 61}. We illustrate this on a specific problem. Recently the nature of magnon excitations above the Néel ordered ground state of the $S = 1/2$ Heisenberg antiferromagnet on the square lattice has received considerable attention. While traditional spin wave theory captures the overall dispersion very well, it exhibits nearly flat dispersion along the line $(\pi, 0)$–$(\frac{\pi}{2}, \frac{\pi}{2})$ \cite{62, 63}, in contrast with a range of other analytic and numeric studies \cite{64–69}, as well as inelastic neutron scattering experiments \cite{70, 71}. Although numerous works have interpreted the distinction between $(\pi, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$ in terms of spinon deconfinement, there is compelling evidence that the spectral disparity results from correlations inherent to a magnon description which are missed by the bare expansion of spin-wave theory \cite{67, 69}. An improved approximation scheme building upon a magnon quasi-particle description is therefore highly desirable, and a natural option to pursue is the development of a GW approach within the non-canonical formalism. The GW approximation \cite{43, 44} is well suited to capturing screening effects, and these may be particularly important for the 2d antiferromagnetic ground state as it is not completely polarised. The scheme will provide an improved magnon dispersion, and it will be interesting to see if it
can accurately capture the dichotomy between \((\pi, 0)\) and \((\frac{\pi}{2}, \frac{\pi}{2})\). At any rate, the wealth of available data for the transverse and longitudinal dynamical structure factors provide good benchmarks for developing the computation of both quasi-particle and collective excitations. It is worth emphasising that the significance of this problem goes beyond the spin setting. The distinction between \((\pi, 0)\) and \((\frac{\pi}{2}, \frac{\pi}{2})\) is central to the enigmatic behaviour in pseudogap regime of the cuprates [8], which appears upon doping a quasi-2d antiferromagnet. The spin system thus offers an important benchmark for improved approximation schemes which one may wish to employ to address the challenges of strongly correlated electrons.

A second relevant test problem is the Anderson impurity model [72, 73], which is an electronic hopping model hybridised with an impurity site at which there is an energy cost for double occupation. While historically the model has played an important role in our understanding of magnetic impurities and Kondo resonance, it has found a renewed significance within the context of dynamical mean-field theory [74], which underlies modern first principles approaches to strongly correlated materials [75, 76]. At its simplest level, dynamical mean-field theory is a framework which allows for an efficient computation of the electronic Green’s function of the Hubbard model in the limit of infinite dimensions. It works via a self-consistent mapping to the Anderson impurity model, and so a scheme capable of isolating the most relevant correlations in this model is highly desirable. At present numerical methods are most commonly employed [77, 78], which are in turn useful for benchmarking the development of novel approaches. We advocate that the degrees of freedom identified in Sec. 6 may be useful in this context, as when applied to an isolated impurity site they provide an exact solution. To address the Anderson impurity model a particularly natural scheme to adapt is the T-matrix/ladder approximation, which is well-suited to the local nature of the interaction. We highlight that as \(\lambda\) or \(\gamma\) is the parameter which organises correlations, this offers the possibility to capture phenomena which are non-perturbative in either the strength of hybridisation to the impurity or the interaction parameter at the impurity site. If successful, closely related problems upon which the formalism can be further established are the Kondo impurity model, the Hubbard model and the Kondo lattice model. It is to be expected that such a scheme would provide useful insight into Kondo resonance formation and related phenomena.

9 Conclusion

An important objective is a classification of the possible behaviours of quantum systems. This serves both to clarify the origins of observed phenomena, as well as to map out as yet unexplored regimes of behaviour. In this work we have sought to approach this challenge under the guiding principle of classifying the quantum degree of freedom, treating this as a proxy for classifying behaviour.

Specifically we have argued that a special class of non-canonical algebras can be considered as legitimate quantum degrees of freedom. By appealing to a formal classification of algebraic structures, we have collected explicit examples for spin, electron, and local moment systems. These offer a powerful unified framework for characterising strongly correlated behaviour in these important settings. To proceed it will be necessary to adapt the canonical schemes for organising correlations to such non-canonical degrees of freedom, and in the Discussion we identify two key problems which may serve as useful testing grounds.
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