NOTES ON RICCI FLOWS WITH COLLAPSING TIME-SLICES (I):
DISTANCE DISTORTION

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ABSTRACT. In this note, we prove a uniform distance distortion estimate for Ricci flows with uniformly bounded scalar curvature, independent of the lower bound of the initial $\mu$-entropy. Our basic principle tells that once correctly renormalized, the metric-measure quantities obey similar estimates as in the non-collapsing case; especially, the lower bound of the renormalized heat kernel, observed on a scale comparable to the initial diameter, matches with the lower bound of the renormalized volume ratio, giving the desired distance distortion estimate.

1. Introduction

For a fixed Ricci flow, a fundamental question of Richard Hamilton (see Section 17 of [18]) is to obtain a uniform distance distortion estimate depending on a minimal requirement of the space-time curvature bound. A natural and non-trivial condition is to assume a uniform bound of the scalar curvature in space-time, as evidenced by Kähler-Ricci flows on Fano manifolds. The distance distortion problem in this case is completely settled by Chen-Wang in [18], and again in [10] as an important intermediate step towards their main result. The Kähler condition was then dropped by Bamler-Zhang in [2]. See also the previous works of Richard Hamilton [18], Miles Simon [22] and Tian-Wang [23] for several important partial results. However, all these estimates, including the ones of Chen-Wang and Bamler-Zhang, rely on the uniform lower bound of the initial $\mu$-entropy, a crucial condition that we will relax in this note.

As a second motivation, in studying the uniform behavior of all Ricci flows, one may have to encounter a family of Ricci flows without a uniform lower bound for the initial $\mu$-entropy. A very common situation is when the family of initial data have their diameter uniformly bounded, but volume degenerating to 0, causing the initial $\mu$-entropy to approach negative infinity. A natural question would then be whether there is a limiting metric space whose metric evolves in a way determined by the Ricci flows (see Proposition 4.5). In this note, we make efforts towards this direction via the following uniform distance distortion estimate along the Ricci flows:

**Theorem 1.1.** Let $(M, g(t))$ be a complete Ricci flow solution on $[0, T]$ with initial diameter $D_0$ and initial volume $V$, and assume the following conditions:

1. $(M, g(0))$, as a closed Riemannian manifold, has its doubling constant uniformly bounded above by $C_D$, and its $L^2$-Poincaré constant by $C_P$, and
2. the scalar curvature is uniformly bounded in space-time: $\sup_{M \times [0, T]} |R_{g(t)}| \leq C_R$.

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There exist two positive constants $\alpha = \alpha(\theta \mid C_D, C_P, C_R, D_0, n, T) < 1$ with
$$
\lim_{\theta \to 0} \alpha(\theta \mid C_D, C_P, C_R, D_0, n, T) = 0,
$$
and $\nu = \nu(C_D, C_P, C_R, n) < 1$, such that whenever $VD_0^{-n} \leq \nu \omega_n$, for fixed $t \in [0, T]$ and $r \in (0, \sqrt{t})$, if we set $\theta := \min(1, r/D_0)$, then
$$
\forall x, y \in M \text{ with } d_{g(t)}(x, y) \geq r, \text{ and } \forall s \in (t - \alpha r^2, \min\{T, t + \alpha r^2\}),
$$
we have
$$
(1.1) \quad \alpha(\theta)d_{g(t)}(x, y) \leq d_{g(s)}(x, y) \leq \alpha(\theta)^{-1}d_{g(t)}(x, y).
$$

**Remark 1.** The requirement that $VD_0^{-n} < \nu \omega_n$ indicates that the initial data is volume collapsing with bounded diameter. Notice that with $\omega_n$ being the volume of the $n$-dimensional Euclidean unit ball, $\nu \omega_n$ is a dimensional constant only depending on $C_D, C_P$ and $C_R$.

In the statement of the theorem, $\theta$ refers to the relative size of the scale on which we consider distance distortion compared to the initial diameter, and $\alpha \approx \theta^{n\nu}e^{-\theta^{-1}}$. The bound $\alpha$ becomes worse as the scale on which we observe becomes smaller compared to the initial diameter.

This is reasonable, as demonstrated in the case of collapsing initial data with bounded curvature and diameter: there will be no uniform estimate of the distance distortion in the fiber directions. However, we notice that such estimate is not needed for providing a rough metric structure on the collapsing limit, since eventually it is the estimates in the base directions that we will need. Therefore, regardless of how small the relative scale we are considering, a *uniform* estimate, even though depending on such scale, is indeed what we need.

The previous distance distortion estimates are based on the estimates of the volume ratio change along the Ricci flow: with uniformly bounded scalar curvature and initial entropy, the volume ratio at a point can neither suddenly decrease (no local collapsing theorem of Perelman [20]), nor suddenly increase (non-inflation property due to Chen-Wang [9] and Qi S. Zhang [33]). Discretizing the geodesic distance by the number of fix-sized geodesic balls that suitably cover the minimal geodesic, these non-collapsing and non-inflation properties together provide the desired control of the distance distortion. This type of “ball containment” argument is succinctly discribed in the third section of Chen-Wang [11].

In order to obtain uniform estimates of the change of volume ratio along the Ricci flow, in the Kähler case Chen-Wang [10] studied the Bergman kernel, while in the Riemannian case, Bamler-Zhang [2] relies on Qi S. Zhang’s heat kernel estimates in [33].

Our theorem is proven along the same paths that lead to such estimates. However, we need to start from scratch: underlying the estimates of the heat kernel, a corner stone is the expression of the log-Sobolev constant in terms of the initial $\mu$-entropy (see [27] and [29]), which, in the current note, will be replaced by a renormalized version involving the *initial global volume ratio* $VD_0^{-n}$.

Heuristically speaking, collapsing is a geometric phenomenon, while the behavior of the heat kernel (which reflects the volume ratio) is analytic in nature. The monotonicity of Perelman’s functionals along the Ricci flow is another instance where a geometric deformation bears an analytic meaning. A basic principle in dealing with the analytic information associated with collapsing, especially the Dirichlet energy and related objects, is making a correct renormalization. This was
first noticed by Kenji Fukaya [15] in the setting of collapsing with bounded curvature and diameter, and then strengthened through a series of work by Cheeger-Colding (see [5], [11], [6] and [7]) to the case with only Ricci curvature lower bound.

Our third and major motivation of this note is therefore to demonstrate the necessity of the above renormalization principle in the setting of Ricci flows with collapsing initial data: the initial collapsing is a geometric phenomenon, yet in order to obtain the distance distortion estimate, we need to control the analytic quantities — the heat kernel bounds — which could only be made possible through a correct renormalization.

We now outline the series of estimates of the renormalized quantities that lead to the uniform distance distortion estimate. We emphasize that these inequalities are invariant under the parabolic rescaling of the Ricci flow, a crucial point for them to work in a geometric setting. Also notice that the constants involved are determined by $C_D, C_P, C_R, D_0, n$, but we only write explicitly their dependence on $T$. Our starting point is a renormalized $L^2$-Sobolev inequality (see [1] and [21]):

$$\forall u \in H^1(M, g(0)), \quad \left( \int_M u^2 \, dV_{g(0)} \right)^{\frac{1}{2}} \leq C_S (V D_0^{-n})^{-\frac{1}{2}} \int_M |\nabla u|^2 + D_0^2 u^2 \, dV_{g(0)},$$

where $D_0$ is the initial diameter and $V := \int_M 1 \, dV_{g(0)}$ is the initial volume.

Following classical arguments and the definition of the $W$-functional, this gives a lower bound of the initial entropy $\mu(g(0), \tau) \geq \log V D_0^{-n} - (C_R D_0^2 + D_0^2) \tau - \frac{n}{2} \log(8n \pi e C_S)$. Here we would like to raise the readers’ attention that it is not just the initial total volume $V$, but the initial global volume ratio $V D_0^{-n}$, that controls the lower bound of the entropy. This quantity not only technically makes the inequality scaling-correct, but also conceptually reveals the meaning of collapsing initial data — volume collapsing with bounded diameter.

Following Perelman’s classical argument [20], we could deduce the lower bound of the renormalized volume ratio (see Proposition 4.2): there is a uniform $C_{VR}(T) > 0$, such that

$$\forall t \in (0, T], \forall r \in (0, \sqrt{t}], \quad (V D_0^{-n})^{-1} |B_t(x, r)| \geq C_{VR}(T) r^n.$$

Here we start seeing the effect of the correct renormalization: even if the volume ratio fails to have a uniform lower bound, once renormalized by $(V D_0^{-n})^{-1}$, it is indeed bounded below by $C_{VR}(T)$.

Further exploring the definition and monotonicity of the $W$-functional, and following Qi S. Zhang’s application [33] of the method of Edward Davies [14], we obtain the following rough upper bound of the renormalized heat kernel (see Proposition 5.1): there is a uniform $C_H(T) > 0$ such that

$$\forall t \in (0, T], \forall s \in (0, t), \forall x, y \in M, \quad V D_0^{-n} G(x, s; y, t) \leq C_H(T) (t - s)^{-\frac{n}{2}}.$$

For the definition of $G(x, s; y, t)$ see Subsection 2.3. Here we see the duality between the heat and the volume of a Riemannian manifold. Intuitively, the collapsing is an intrinsic geometric procedure, and it should not cause the addition or loss of the total heat. Therefore, if the global volume ratio behaves like $V D_0^{-n} \to 0$, then the heat density should in general behave like $(V D_0^{-n})^{-1} \to \infty$.

Up to this stage it is basically just the interplay between the Sobolev inequality and the $W$-functional: purely analytic in nature. In order to estimate the distance distortion, we still need a lower bound of the renormalized heat kernel. The original argument of Chen-Wang [9] and Qi S.
Zhang [33], however, will not give us the desired bound: their argument, based on the estimate of the reduced length of a space-constant curve at the base point of the heat kernel, is valid regardless of scales; but in our setting there is a drastic difference between the very small scales, which resemble the locally $n$-dimensional Euclidean property of the manifold, and the large scales, on which the collapsing to a lower dimensional space is observed.

We will overcome this difficulty by obtaining a positive-time diameter bound in terms of the initial diameter, and stick to our principle of keeping the heat-volume duality. The following diameter bound is deduced following an argument of Peter Topping in [25] (see Proposition 4.4): there exists a uniform constant $C_{\text{diam}} > 0$ such that if the initial global volume ratio is sufficiently small, i.e. $VD_0^{-n} < \nu \omega_n$ for some uniform $\nu \in (0, 1]$, then

$$\forall t \in (0, T], \quad \text{diam}(M, g(t)) \leq C_{\text{diam}} e^{2C\epsilon t} D_0.$$  

This diameter bound is of great technical importance for us, since we will soon use it to deduce an on-diagonal lower bound of the renormalized heat kernel. Conceptually, this bound tells that scales that are comparable to the initial diameter, remain comparable to the diameter at a positive time, up to a uniform factor depending on the time elapsed.

At this stage, we could already prove some weak compactness result, Proposition 4.5, asserting the existence of a Gromov-Hausdorff limit for positive time-slices — recall that a priorily, we only assume a uniform scalar curvature bound on these time-slices.

With the help of the diameter bound above, we have the following lower bound of the renormalized heat kernel (see Lemma 5.2): there exists a uniform constant $C_{H}^{-}(T) > 0$ and a positive function $\Psi(\theta \mid T)$ with $\lim_{\theta \to 0} \Psi(\theta \mid T) = 0$, such that if $VD_0^{-n} \leq \nu \omega_n$,

$$\forall t \in (0, T], \forall s \in (0, t), \forall x \in M, \quad VD_0^{-n} G(x, s; x, t) \geq C_{H}^{-}(T) \Psi(\theta(s) \mid T) (t - s)^{-\frac{2}{n}}.$$

Here we see that the effect of scales enters into the picture via the factor $\Psi(\theta \mid T)$: for any $t \in (0, T]$ and any $s \in (0, t)$, $\theta(s) := \sqrt{t - s} / D_0$ is the ratio of the (parabolic) scale under consideration compared to the initial diameter; when the scale that we observe approaches 0, relative to the initial diameter, then the lower bound of the renormalized heat kernel will also approach 0. (Rigorously speaking, we actually have $\theta = \sqrt{t - s} / \text{diam}(M, g(t))$ in our mind, but the diameter bound above allows us to compare $r$ directly with $D_0$, making the definition more canonical.) This estimate naturally leads to a Gaussian type lower bound of the renormalized heat kernel, as well as the non-inflation property of the renormalized volume ratio.

The bounds of the renormalized heat kernel, together with the previous lower bound of the renormalized volume ratio, are enough to prove the desired distance distortion estimate, in view of the arguments in proving Theorem 1.1 of [2].

The current note consists of seven sections: We will start with recalling the necessary background in Section 2. In section 3, we apply the renormalized Sobolev inequality to obtain an initial entropy lower bound, explicitly involving the initial global volume ratio. This will be used in the following section to deduce a lower bound of the renormalized volume ratio, as well as an upper bound of the positive-time diameter. In section 5, we obtain the bounds of the renormalized heat kernel, and the proof of our main result is contained in section 6. We will also discuss future work to be done in the final section.

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2. Background

Our consideration will be on a closed Riemannian manifold \((M, g(0))\) whose volume is \(V\) and diameter is \(D_0\). We assume that there exists a Ricci flow up to time \(T\), i.e. there is a family of smooth Riemannian metrics \(g(t)\) on \(M\) satisfying the differential equation of symmetric two tensors:

\[
\forall t \in [0, T], \quad \partial_t g = -2\mathcal{R}_{g(t)}.
\]

We will also assume that the doubling constant of \((M, g(0))\) is given by \(C_D\), and its \(L^2\)-Poincaré constant by \(C_P\). In this section, we will recall the renormalized Sobolev inequality determined by \(C_D\) and \(C_P\), then Perelman’s \(W\)-functional and \(\mu\)-entropy, and finally the gradient estimates due to Bamler-Zhang. Instead of quoting directly the original statements in the most general form, we will adapt these results in a form that we could later make a direct use.

2.1. The renormalized Sobolev inequality. We are inspired by the Sobolev constant estimate due to Michael Anderson [11] (see also [16]), in the situation where a uniform Ricci curvature lower bound is assumed: for a fixed geodesic ball \(B(x, r)\), its Sobolev constant is comparable to \((|B(x, r)|r^{-n})^{-\frac{2}{n}}\). The lesson is to consider explicitly the effect of a correct renormalization, when applying the Sobolev inequality to the study of Ricci flows.

In another direction, using methods in stochastic analysis and the Moser iteration technique, Laurent Saloff-Coste has shown an even more general Sobolev inequality in [21], where the Sobolev constant only depends on the doubling constant and the \(L^2\)-Poincaré constant. This is the inequality that we will employ in this note:

**Proposition 2.1** (Renormalized \(L^2\)-Sobolev inequality). Let \((M^n, g)\) be a Riemannian manifold such that the doubling constant and the \(L^2\)-Poincaré constant are bounded from above by \(C_D\) and \(C_P\) respectively. Then there is a constant \(C_S = C_S(n, C_D, C_P)\) such that for any \(B(x, r) \subset M\) and any \(u \in H^1_0(B(x, r))\), the following renormalized Sobolev inequality holds:

\[
(\int_{B(x,r)} u^{2^n} \, dV_g)^{\frac{2^n}{2^n - 2}} \leq C_S (|B(x, r)|r^{-n})^{-\frac{2}{n}} \int_{B(x,r)} |\nabla u|^2 + r^{-2}u^2 \, dV_g.
\]

**Remark 2.** This is of course just one version of the Sobolev inequality. We call it renormalized just to emphasize the independence of the Sobolev constant from the volume, since eventually the volume will be sent to zero.

The main point of this note is then to explore the geometric consequences of the renormalization \((|B(x, r)|r^{-n})^{-\frac{2}{n}}\) in the setting of Ricci flows. We notice that inequality [12] is just a (weaker) global version of this inequality.

2.2. Perelman’s \(W\)-entropy. As mentioned in the introduction, the monotonicity of Perelman’s \(W\)-functional along the Ricci flow is an instance where a geometric deformation bears an analytic meaning. This connection is the foundation of the current note. We now recall Perelman’s \(W\)-functional [20]: for any \(\bar{t} \in (0, T]\), any \(v^2 \in C^1(M, g(\bar{t}))\) and any \(\tau > 0\),

\[
W(g(\bar{t}), v^2, \tau) := \int_M \tau \left(4|\nabla v|^2 + \mathcal{R}(g(\bar{t}))v^2\right) - v^2 \log v^2 - n \left(1 + \frac{1}{2} \log(4\pi\tau)\right)v^2 dV_{g(\bar{t})}.
\]
If we require \( \int_M v^2 \, dV_{g(t)} = 1 \), let \( \tau \) solve \( \partial_t \tau(t) = -1 \), and let \( u(t) \) solve the conjugate heat equation along the Ricci flow: \((\partial_t - \Delta - R)u = 0\) with the prescribed final data \( u(t) := v^2 \), then we have the monotone increasing property of the \( \mathcal{W} \)-functional:

\[
\frac{d}{dt} \mathcal{W}(g(t), u(t), \tau) \geq 0.
\]

The \( \mu \)-entropy is defined as

\[
\mu(g(t), \tau) := \inf_{\int_M v^2 \, dV_{g(t)} = 1} \mathcal{W}(g(t), v^2, \tau),
\]

and letting the data varying similarly as in the \( \mathcal{W} \)-functional, we also obtain the monotone increasing property of the \( \mu \)-entropy.

### 2.3. Heat equation solutions coupled with the Ricci flow.

In this subsection we collect some point-wise estimates of heat equation solutions coupled with the Ricci flow. For any \( x, y \in M \) and \( 0 \leq s < t < T \), we will let \( G(x, s; y, t) \) denote the heat kernel coupled with the Ricci flow based at \( (x, s) \), i.e. fixing \( (x, s) \in M \times [0, T) \), we have

\[
(\partial_t - \Delta_{g(t)})G(x, s; -, -) = 0, \quad \text{and} \quad \lim_{t \searrow s} G(x, s; -, -) = \delta_{(x, s)},
\]

where \( \delta_{(x, s)} \) is the space-time Dirac delta function at \( (x, s) \in M \times [0, T) \). On the other hand, fixing \( (y, t) \in M \times (0, T) \) and setting \( (x, s) \) free, this same function satisfies

\[
(\partial_s + \Delta_{g(s)} + R_{g(s)})G(\cdot, -, s; y, t) = 0, \quad \text{and} \quad \lim_{s \uparrow t} G(\cdot, -, s; y, t) = \delta_{(y, t)},
\]

i.e. \( G(\cdot, -, s; y, t) \) is the conjugate heat kernel coupled with the Ricci flow based at \( (y, t) \).

Our heat kernel lower bound of Gaussian type will be base on the following key gradient estimate due to Qi S. Zhang, see Theorem 3.3 in [28]:

**Proposition 2.2** (Gradient estimate). Let \((M, g(t))\) be a Ricci flow on a complete n-manifold \( M \) over time \([0, T)\) and let \( u \in C^\infty(M \times [0, T))\) be a positive solution to the heat equation \((\partial_t - \Delta)u = 0\), \( u(\cdot, 0) = u_0 \) coupled with the Ricci flow. Then there is a constant \( B < \infty \) depending only on \( n \), such that if \( u \leq a \) on \( M \times [0, T] \) for some constant \( a > 0 \), then \( \forall (x, t) \in M \times (0, T) \),

\[
\frac{|\nabla u(x, t)|}{u(x, t)} \leq \sqrt{\frac{1}{t} \log \frac{a}{u(x, t)}}.
\]

Note that this inequality also reads

\[
|\nabla \sqrt{\log \frac{a}{u}}(x, t)| \leq \frac{1}{\sqrt{t}}
\]

for any \((x, t) \in M \times (0, T)\).

Now for any fixed \((x, t_0) \in M \times [0, T]\), let \( G(x, t_0; -, -) \) be the coupled heat kernel described above. Viewing \( u(y, s) = G(x, t_0; y, s) \) as a coupled heat equation solution on \( M \times [\frac{t_0 + t}{2}, t] \), and integrating the above inequality along minimal geodesics, we could get a Harnack inequality for heat equation solutions coupled with the Ricci flow, also see inequality (3.44) of [28]:

...
Corollary 2.3. We have \( \forall y(t), (y', t) \in M \times (t_0, T) \),

\[
G(x, t_0; y, t) \leq H(n) \left( \sup_{M \times [(t_0+\varepsilon)\cdot 2, t]} G(x, t_0; y', -) \right)^\frac{1}{2} G(x, t_0; y', t)^\frac{1}{2} e^{H(n)d(y, y')^2 / (t-t_0)},
\]

where \( H(n) \) and \( H'(n) \) are dimensional constants.

In order to estimate the distance distortion we also need a time derivative bound of the coupled heat kernel. This is achieved by the following estimate, which is Lemma 3.1(a) in [2]:

**Proposition 2.4.** Let \((M, g(t))\) be a Ricci flow on a closed \( n \)-manifold \( M \) over time \([0, T] \) and let \( u \in C^\infty(M \times [0, T]) \) be a positive solution to the heat equation \( (\partial_t - \Delta) u = 0, u(\cdot, 0) = u_0 \) coupled with the Ricci flow. Then there is a constant \( B < \infty \) depending only on \( n \), such that if \( u \leq a \) on \( M \times [0, T] \) for some constant \( a > 0 \), then \( \forall (x, t) \in M \times (0, T) \),

\[
|\Delta u| + \left| \frac{\|\nabla u\|^2}{u} - dR \right| (x, t) \leq \frac{aB(n)}{t}.
\]

Again, setting \( u(y, t) = G(x, t_0; y, t) \) for \( t > t_0 \), and considering it as a coupled heat equation solution on \( M \times (t_0 + \varepsilon / 2, t) \), we immediately obtain

\[
|\partial_t G(x, t_0; y, t)| + \frac{|\nabla_y G(x, t_0; y, t)|^2}{G(x, t_0; y, t)} \leq \sup_{M \times [(t_0+\varepsilon)/2, t]} G(x, t_0; y, -) \left( R_{g(t)}(x, t_0; y, t) + \frac{B(n)}{t - t_0} \right).
\]

3. **A uniform renormalized Sobolev inequality along the Ricci flow**

A uniform Sobolev inequality along Ricci flows will enable us to do analysis on positive time slices. Notice that the lower bound of the \( \mu \)-entropy reflects the upper bound of the log-Sobolev constant, and the monotone increasing property of the \( \mu \)-entropy will further preserve, rather than destroying, the log-Sobolev constant. In this section, we will see that the information of initial global volume ratio is encoded in the initial \( \mu \)-entropy via a log-Sobolev inequality, deduced following a classical argument, but with the renormalized Sobolev inequality (1.2) as our starting point. We will also deduce a uniform renormalized Sobolev inequality along the Ricci flow, which clearly shows how the initial global volume ratio affects the Sobolev constants on positive time slices. For previous results we refer the readers to the works of Rugang Ye [27] and Qi S. Zhang [29], [30], [31].

3.1. **Lower bound of initial entropy via the renormalized Sobolev inequality.** From (1.2), we see that if \( \int_M v^2 dV_{g(0)} = 1 \), then

\[
\left( \int_M v^{\frac{2n}{n-2}} dV_{g(0)} \right)^{\frac{n-2}{n}} \leq 4C_S V^{\frac{n}{2}} \left( D_0^2 \int_M |\nabla v|^2 dV_{g(0)} + 1 \right).
\]

Due to the uniform bound of the scalar curvature, we could further obtain

\[
\left( \int_M v^{\frac{2n}{n-2}} dV_{g(0)} \right)^{\frac{n-2}{n}} \leq 4C_S V^{\frac{n}{2}} \left( D_0^2 \int_M (4|\nabla v|^2 + R_{g(0)} v^2) dV_{g(0)} + C_R D_0^2 + 1 \right).
\]
Since the logarithm function is concave, and since \( v^2 \text{d}V_{g(0)} \) defines a probability measure on \( M \), by Jensen’s inequality, we have

\[
\forall u \in L^1(M, v^2 \text{d}V_{g(0)}), \quad \int_M (\log |u|) \ v^2 \text{d}V_{g(0)} \leq \log \int_M |u| \ v^2 \text{d}V_{g(0)}.
\]

With \( u = v^{q-2} \) with \( q = \frac{2n}{n-2} \) (notice that \( \frac{q}{q-2} = \frac{4}{n-2} \)), the above inequality gives

\[
\int_M v^2 \log v^2 \text{d}V_{g(0)} = \int_M \frac{2}{q-2} (\log v^{q-2}) \ v^2 \text{d}V_{g(0)} \leq \frac{2}{q-2} \log \int_M v^q \text{d}V_{g(0)},
\]

which is exactly \( \frac{n}{2} \log \|v\|_{L^q(M)}^2 \); furthermore,

\[
\frac{n}{2} \log \|v\|_{L^q(M)}^2 \leq \frac{n}{2} \log \left( D_0^2 \int_M (4|\nabla v|^2 + R_{g(0)} v^2) \text{d}V_{g(0)} + C_R D_0^2 + 1 \right) - \log V + \frac{n}{2} \log 4C_S.
\]

Now applying the elementary inequality \( \log u \leq au - 1 - \log a \) for all \( a > 0 \) to the first term in the right-hand side of this last inequality, we obtain

\[
\int_M v^2 \log v^2 \text{d}V_{g(0)} \leq \frac{n}{2} \log \left( D_0^2 \int_M (4|\nabla v|^2 + R_{g(0)} v^2) \text{d}V_{g(0)} + C_R D_0^2 + 1 \right) - \log V + \frac{n}{2} \log 4C_S
\]

\[
\leq \frac{an}{2} \left( D_0^2 \int_M (4|\nabla v|^2 + R_{g(0)} v^2) \text{d}V_{g(0)} + C_R D_0^2 + 1 \right) - \log V + \frac{n}{2} \left( \log 4C_S - 1 - \log \alpha \right).
\]

Recalling the definition of the \( \mathcal{W} \)-functional (2.2), and taking \( \alpha = \frac{2\pi}{\mu D_0} \) in (3.1), we immediately see

\[
\mathcal{W}(g(0), v^2, \tau) \geq \log VD_0^{-n} - (C_R D_0^2 + D_0^{-2}) \tau - \frac{n}{2} \log(8\pi e C_S).
\]

Here \( \tau \), as a multiple of \( \alpha \), could be any positive number. Since this is valid for any function \( v \) on \( M \) with unit \( L^2 \)-norm, we have, for any \( \tau \in [T, 2T] \),

\[
\mu(g(0), \tau) \geq \log VD_0^{-n} - (C_R D_0^2 + D_0^{-2}) \tau - \frac{n}{2} \log(8\pi e C_S).
\]

Here we notice that both sides of the inequalities above are invariant under a parabolic rescaling.

Remark 3. It is well-know that collapsing initial data implies that there is no uniform lower bound of the \( \mathcal{W} \)-entropy, and here we give an explicit lower bound in terms of the initial global volume ratio.

Now suppose we evolve \( v^2 \) at some \( \bar{t} \)-slice backward by the conjugate heat equation, i.e. we consider a function \( u \) such that

\[
u(\bar{t}) = v^2; \quad (\partial_t + \Delta_{g(\bar{t})} - \mathcal{R}_{g(\bar{t})})u = 0; \quad \partial_\bar{t}g = -2\mathcal{R}_{g(\bar{t})},
\]
then \( \mathcal{W}(g(t), u(t), \tau(t)) \) is increasing in \( t \) where \( \tau' = -1 \). Therefore, for any \( v^2 \) with unit \( L^1(g(\bar{t})) \)-norm, we have, by the monotone increasing property of the \( \mathcal{W} \)-functional, that
\[
\mathcal{W}(g(\bar{t}), v^2, \tau(\bar{t})) \geq \mathcal{W}(g(0), u(0), \tau(\bar{t}) + \bar{t})
\geq \log VD_0^n - (C_R + D_0^{-2})(\tau(\bar{t}) + \bar{t}) - \frac{n}{2} \log(8\pi e C_S),
\]
or in the form of the log-Sobolev inequality,
\[
\int_M \tau(t) |\nabla v|^2 + \mathcal{R}(\bar{t}) v^2 - v^2 \log v^2 \, dV_{g(\bar{t})} \geq \log V \left( \frac{\tau}{D_0^2} \right)^n - (C_R + D_0^{-2})(\tau + \bar{t}) - C_{IS},
\]
where \( C_{IS} := \frac{n}{2} \log(2\pi e C_S) \) and \( \tau \) is any positive number.

3.2. Uniform renormalized Sobolev inequality along the Ricci flow. In this subsection, we establish a uniform renormalized Sobolev inequality along the Ricci flow. We will follow the exposition of [32], which is based on the argument of Edward Davies [14] in the case of a fixed Riemannian manifold. The result of this subsection will not be needed in our estimate of the distance distortion, yet we still include it here because we will later use a similar argument to prove a rough upper bound of the renormalized heat kernel in Section 5.1.

The first step would be using the uniform log-Sobolev inequality (3.4) to obtain an upper bound of the heat kernel on a fixed future time slice \((M, g(\bar{t}))\). Now let \( u \) be any solution to the equation
\[
(\partial_t - \Delta_{g(\bar{t})} + \mathcal{R}(\bar{t})) u = 0
\]
on the fixed Riemannian manifold \((M, g(\bar{t}))\). Consider for any fixed \( t > 0 \), the exponent \( p(s) := \frac{t}{t-s} \) for \( s \in [0, t] \). We immediately see that \( p'(s) = (t-s)^{-2} > 0 \), moreover,
\[
0 \leq \frac{p(s) - 1}{p'(s)} = \frac{s(t-s)}{t} \leq \frac{t}{4},
\]
and
\[
\frac{p'(s)}{p^2(s)} = \frac{1}{t}.
\]
We also let
\[
v(x, s) := u(x, s) \frac{p(s)}{p'(s)} ||u||_{L^p(M, g(\bar{t}))}^{-1} \in L^p(M, g(\bar{t})),
\]
so that \( ||v||_{L^2(M, g(\bar{t}))} = 1 \). Routine computations give
\[
p^2(s) \partial_s \log ||u||_{L^p(M, g(\bar{t}))} = p'(s) \int_M v^2 \log v^2 \, dV_{g(\bar{t})} - 4(p(s) - 1) \int_M |\nabla v|^2 \, dV_{g(\bar{t})} - p^2(s) \int_M \mathcal{R}(\bar{t}) v^2 \, dV_{g(\bar{t})} \leq p'(s) \left( \int_M v^2 \log v^2 \, dV_{g(\bar{t})} - \frac{p(s) - 1}{p'(s)} \int_M 4|\nabla v|^2 + \mathcal{R}(\bar{t}) v^2 \, dV_{g(\bar{t})} + \frac{3t}{4} C_R \right).
\]
Thus if we plug \( \tau = \frac{p(s) - 1}{p'(s)} \) into (3.4), then the above computation, together with (3.5) give
\[
\partial_t \log ||u||_{L^p(M, g(\bar{t}))} \leq \frac{1}{t} \left( -\frac{n}{2} \log \frac{s(t-s)}{t} - \log VD_0^n - (C_R + D_0^{-2}) \left( \bar{t} + \frac{s(t-s)}{t} \right) + C_{IS} + \frac{3t}{4} C_R \right).
\]
Notice that \( p(0) = 1 \) and \( p(t) = \infty \), we integrate the above inequality (with respect to \( x \)) from 0 to \( t \) to obtain for any \( t > 0 \),

\[
\log \frac{\|u(\cdot, t)\|_{L^\infty(M, g(t))}}{\|u(\cdot, t)\|_{L^1(M, g(t))}} \leq -\frac{n}{2} \log t - \log VD_0^{-n} + (C_R - D_0^{-2})t + C_H(t),
\]

where \( C_H(t) = 2(\bar{t} + 1)(C_R + D_0^{-2}) + C_{IS} + n \). Now let \( G_{\bar{t}}(x, t, y) \) be the heat kernel of \((M, g(\bar{t}))\) centered at \( x \in M \), then

\[
u(x, t) = \int_M G_{\bar{t}}(x, t, y)u(y, 0) \, dV_{g(\bar{t})}(y),
\]

and \( \|u(\cdot, t)\|_{L^1(M, g(t))} = \int_M u(y, t) \, dV_{g(\bar{t})}(y) \),

we conclude that

\[
(VD_0^{-n})G_{\bar{t}}(x, t, y) \leq e^{C_H(\bar{t})+(2C_R+D_0^{-2})\bar{t}}t^{-\frac{n}{2}}.
\]

Now consider \( \tilde{G}_{\bar{t}}(-t, -) := e^{-(2C_R+D_0^{-2})\bar{t}}G_{\bar{t}}(-t, -) \), then \( \tilde{G}_{\bar{t}} \) is the fundamental solution to the equation

\[
\left( \partial_t - \Delta_t + \mathcal{R}_{g(\bar{t})} \right) + (2C_R + D_0^{-2})u = 0,
\]

and by (3.8) we have the control

\[
\forall t > 0, \quad \tilde{G}_{\bar{t}}(-t, -) \leq C_H(\bar{t})(VD_0^{-n})^{-1}t^{-\frac{n}{2}}.
\]

Notice that \( \tilde{C}_H(\bar{t}) = 2\bar{t}(C_R + D_0^{-2}) + C_{IS} + n \) is independent of time and space variables on the fixed manifold \((M, g(\bar{t}))\); also notice that it is invariant under the parabolic rescaling.

Now we can conclude that the operator of integrating against the kernel \( \tilde{G}_{\bar{t}} \) is a contraction, and standard argument gives the \( L^2 \)-Sobolev inequality on \((M, g(\bar{t}))\):

\[
\left\| f \right\|^2_{L^2(M, g(\bar{t}))} \leq C_{Sob}(\bar{t})V^{-\frac{n}{2}}D_0^2 \left\| \nabla f \right\|^2_{L^2(M, g(\bar{t}))} + (2C_R + D_0^{-2})\left\| f \right\|^2_{L^2(M, g(\bar{t}))},
\]

where \( C_{Sob}(\bar{t}) = (2(\bar{t} + 1)(C_R + D_0^{-2}) + C_{IS} + n)^{\frac{n}{2}} \) is uniformly bounded for bounded \( \bar{t} \), independent of \( V \) and the flow.

### 4. Estimating the geometric quantities along the Ricci flow

In this section we give a lower bound of the renormalized volume ratio on any scale, and a scaling invariant upper bound of the diameter along the Ricci flow. The estimates only depend on the initial doubling constant \( C_D \), the initial \( L^2 \)-Poincaré constant \( C_P \), the initial diameter \( D_0 \), the space-time scalar curvature bound \( C_R \), and the time elapsed from the beginning.

Both estimates are based on the idea that the \( \mathcal{W} \)-functional, when tested against a suitable spacial cut-off function, bounds from below the volume ratio at the given time slice, and then the monotone increasing property of the \( \mathcal{W} \)-functional further provides the desired renormalization by the initial total volume, as shown in (3.2).

More specifically, throughout this section, we fix a time slice \( \bar{t} \in (0, T] \) and a scale \( r \) such that \( r^2 \in (0, \bar{t}] \). For any fixed \( x \in M \), we could define a spacial cut-off function as

\[
h^2(y) = e^{-A(4\pi r^2)^{-\frac{n}{2}}} \left( r^{-1}d_\bar{t}(x, y) \right),
\]
with $\eta$ being a smooth cut-off function supported on $[0, 1)$, constantly equal to 1 on $[0, \frac{1}{2})$ and $-2 \leq \eta' \leq 0$ on $(\frac{1}{2}, 1)$. Moreover, $A$ is chosen so that $\int_M h^2 \, dV_{g(\tilde{t})} = 1$, and we immediately see

\[
\frac{|B_{\tilde{t}}(x, \frac{r}{2})|}{(4\pi r^2)^{\frac{3}{2}}} \leq e^A = \int_M \frac{\eta(y) (1 - d_h(x, y))}{(4\pi r^2)^{\frac{3}{2}}} \, dV_{g(\tilde{t})}(y) \leq \frac{|B_{\tilde{t}}(x, r)|}{(4\pi r^2)^{\frac{3}{2}}}.
\]

(4.1)

Recall that the $\mathcal{W}$-functional for $(M, g(\tilde{t}), h^2)$ is defined as

\[
\mathcal{W}(g(\tilde{t}), h^2, r^2) = \int_M 4r^2|\nabla h|^2 + r^2 R_{g(\tilde{t})} h^2 - h^2 \log h^2 \, dV_{g(\tilde{t})} - \frac{n}{2} \log(4\pi r^2) - n.
\]

We now roughly estimate some terms of the right-hand side of this inequality:

Since $|\nabla d_t| \leq 1$, we have

\[
\int_M 4r^2|\nabla h|^2 \, dV_{g(\tilde{t})} \leq \int_{B_{\tilde{t}}(x, r)} \frac{16r^2 e^{-8 |\nabla d_t(x, y)|^2}}{(4\pi r^2)^{\frac{3}{2}}} \, dV_{g(\tilde{t})}(y)
\]

\[
\leq \frac{64|B_{\tilde{t}}(x, r)|}{e^A(4\pi r^2)^{\frac{3}{2}}} 
\leq \frac{64|B_{\tilde{t}}(x, r)|}{|B_{\tilde{t}}(x, \frac{r}{2})|}
\]

(4.2)

where we have used (4.1); moreover, since $h^2$ is supported in $B_t(x, r)$, and since the mapping $\sigma \mapsto -\sigma \log \sigma$ is concave, we apply this to $\sigma = h^2$ and use Jensen’s inequality see

\[
\int_M -h^2 \log h^2 \, dV_{g(\tilde{t})} - \frac{n}{2} \log 4\pi r^2 - n
\]

\[
\leq -\int_{B_{\tilde{t}}(x, r)} h^2 \, dV_{g(\tilde{t})} \left( \log \int_{B_{\tilde{t}}(x, r)} h^2 \, dV_{g(\tilde{t})} \right) - \frac{n}{2} \log 4\pi r^2 - n
\]

\[
= \log |B_{\tilde{t}}(x, r)| r^{-\eta} - \frac{n}{2} \log 4\pi e^2.
\]

4.1. **Lower bound of the renormalized volume ratio.** It is well known, as Perelman’s no local collapsing theorem tells, that the lower bound of the initial $\mu$-entropy and the upper bound of the scalar curvature together give a lower bound of the volume ratio, see [20] and [19]. Following this classical argument, but with the more explicit lower bound of the initial $\mu$-entropy, we obtain a generalized lower bound of the renormalized volume ratio. We begin with the following lemma:

**Lemma 4.1.** For the fixed time slice $\tilde{t}$ and any positive $r \leq \sqrt{R}$, suppose the doubling property

\[
|B_{\tilde{t}}(x, \frac{r}{2})| \geq 3^{-\eta} |B_{\tilde{t}}(x, r)|
\]

(4.4)

holds, then there is a constant $C_{VR}^{-}(T) = C_{\mathcal{R}}^{-}(C_R, C_S, D_0, T)$ such that

\[
\frac{|B_{\tilde{t}}(x, r)| r^\eta}{r^n} \geq C_{VR}^{-}(T) V D_0^{-n}.
\]

(4.5)

**Proof.** We examine the upper bound of $\mathcal{W}(g(\tilde{t}), h^2, r^2)$ with the help of (4.4).

Since $\sup_{M \times [0, 2T]} |\mathcal{R}_{g(\tilde{t})}| \leq C_R$, we have

\[
\int_M r^2 \mathcal{R}_{g(\tilde{t})} h^2 \, dV_{g(\tilde{t})} \leq 2C_RT;
\]

(4.6)
moreover, from (4.2) and (4.4) we have
\[ \int_M 4r^2 |\nabla h|^2 \, dV_{g(\tilde{t})} \leq 3^{n+4}. \]

These estimates, together with (4.3) give
\[ (4.7) \quad \mathcal{W}(g(\tilde{t}), h^2, r^2) \leq \log \frac{|B_r(x, r)|}{r^n} + 2C_R T + 3^{n+5} - \frac{n}{2} \log 4\pi e^2. \]

On the other hand, since $||h||_{L^2(M,g(\tilde{t}))} = 1$, we could evolve $h^2$ by the conjugate heat equation along the Ricci flow $\partial_t g(t) = -2\mathcal{R}c_{g(t)}$, i.e. we solve $(\partial_t + \Delta - \mathcal{R})u = 0$ with final value $u(\tilde{t}) = h^2$.

By the monotone increasing property of $\mathcal{W}(g(t), u(t), \tau)$ in $t$ (with $\tau'(t) = -1$), we may apply the initial lower bound (4.2) to see
\[ \mathcal{W}(g(\tilde{t}), h^2, r^2) \geq \mathcal{W}(g(0), u(0), \tilde{t} + r^2) \]
\[ \geq \log VD_0^{-n} - (CRD_0^2 + 1) \tilde{t} - \frac{n}{2} \log 8n\pi eC_5 D_0^3, \]

therefore by (4.7), we have the following lower bound of the log volume ratio:
\[ \frac{2T(2CR_D^2 + 1)}{D_0^2} - 3^{n+5} - \frac{n}{2} \log 2ne^{-1}C_5 D_0^2, \]

which is
\[ (4.8) \quad \frac{|B_r(x, r)|}{r^n} \geq C_{VR}^{-}(T)(C_R, D_0, T)VD_0^{-n}, \]

where $C_{VR}^{-}(T) := (2ne^{-1}C_5)^{-\frac{1}{4}} \exp(2T(2CR_D + D_0^2) - 3^{n+5})$, which ultimately also depends on the $L^2$-Poincaré constant $C_P$ and the doubling constant $C_D$ of the initial metric, as encoded in $C_5$. Again, $C_{VR}^{-}(T)$ is invariant under the parabolic rescaling of the Ricci flow. \(\square\)

We now prove the local volume doubling property (4.4), which follows directly from the original contradiction argument due to Grisha Perelman, if we notice that the constant $C_{VR}^{-}(T)$ is independent of $\tilde{t}$ and $r^2$ as long as $\tilde{t} \leq T$ and $r^2 \leq \tilde{t}$.

Now suppose (4.5) fails for some scale $r \in (0, \sqrt{\tilde{t}})$ at time $\tilde{t} \leq T$ and a point $x \in M$, then (4.4) must fail for this $r$, and it will also fail at scale $\frac{\tilde{t}}{2}$, otherwise, the above argument applied to the $\frac{\tilde{t}}{2}$-ball around $x \in M$ will produce
\[ |B_r(x, \frac{r}{2})| \geq 2^{-n}C_{VR}^{-}(T)(C_R, D_0, T)Vr^n, \]
\[ \geq 2^{-n}|B_{\frac{r}{2}}(x, r)|, \]

where we have used the converse of (4.5), but contradicts the failure of (4.4). Therefore, if the converse of (4.4) is observed at any point and scale, then it will pass down to all smaller scales at that point, i.e. the converse of (4.4) implies for any $k \geq 1$, \[ |B_r(x, 2^{-k} r)| \leq 3^{-nk}|B_r(x, r)|, \]

which is impossible for $k$ sufficiently large, since $(M, g(\tilde{t}))$ is locally Euclidean. Therefore, we have the following
Proposition 4.2 (Lower bound of renormalized volume ratio). Let \((M, g(t))\) be a Ricci flow solution on \([0, T]\) with initial diameter \(D_0\) and initial volume \(V\). Assume that the scalar curvature is uniformly bounded by \(C_R\) in space-time, then there is a constant \(C_{VR}(T)\) depending on the initial doubling constant \(C_D\), the initial \(L^2\)-Poincaré constant \(C_P\), the initial diameter \(D_0\), the scalar curvature bound \(C_R\) and \(T\), such that for any time \(t \in [0, T]\) and any scale \(r\) such that \(r^2 \in (0, t]\

\[
\frac{|B_t(x, r)|}{r^n} \geq C_{VR}(T) V D_0^{-n}.
\]

Moreover, \(C_{VR}(T)\) is invariant under the parabolic rescaling of the Ricci flow.

4.2. Diameter upper bound. In the same vein, but with the straightforward estimate (4.6) replaced by a more delicate maximal function argument, Peter Topping [25] proved a diameter upper bound in terms of the integral of the scalar curvature (see also [34]). When the scalar curvature is uniformly bounded in space-time, we notice that Topping’s estimates depend on the initial volume, a factor that we hope to avoid in our estimates. However, once the quantities involved are correctly renormalized and the initial entropy lower bound (3.2) is used, Topping’s argument still leads to a diameter upper bound which is independent of the initial volume. In the current subsection we discuss this in detail.

To begin with, we recall that the total volume changes as following:

\[
V(t) := |M|_{g(t)} = V e^{-\int_0^t \int_{B_t(x, r)} R_{g(t)} dV_{g(t)} dt} \leq V e^{C_R t}.
\]

Moreover, as discussed above, we evolve the cut-off function \(h^2\) backward by the conjugate heat equation along the Ricci flow. From (3.2), (4.1), (4.2) and (4.3) we get

\[
\log V D_0^{-n} + C_1(T) \leq \frac{64|B_t(x, r)|}{|B_t(x, \frac{r}{2})|} + \frac{r^2}{|B_t(x, \frac{r}{2})|} \int_{B_t(x, r)} |R_{g(t)}| dV_{g(t)} + \log \left( \frac{|B_t(x, r)|}{r^n} \right),
\]

where \(C_1(T) := -2T(C_R + D_0^{-2}) - \frac{d}{2} \log(2ne^{-1}C_S)\), a constant only depending on the initial doubling and \(L^2\)-Poincaré constants, the initial diameter and the space-time scalar curvature bound; especially it is independent of the initial volume.

Now we define the maximal function of the scalar curvature following [25]:

\[
M R(x, r, \bar{t}) := \sup_{\bar{s} \in (0, r]} \frac{|B_t(x, \bar{s})|}{s} \left( \int_{B_t(x, \bar{s})} \left| R_{g(\bar{t})} \right| dV_{g(\bar{t})} \right)^{\frac{\bar{t}}{\bar{s}}}. 
\]

We also define \(C_2 = \min \left\{ \frac{\omega_n D_0^n}{2^d}, e^{C_1(T) - 2^{d+1}} \right\}\), where \(\omega_n\) is the volume of \(n\)-dimensional Euclidean unit ball. Notice that since we are dealing with the case as \(V \to 0\), the constant \(C\) is in fact independent of \(V\), and we put it here just for the convenience of statement.

The key property of \(M R(x, r, \bar{t})\), as described in [25], is that “we cannot simultaneously have small curvature and small volume ratio”, but in our context we should consider the renormalized volume ratio instead, and this is described in the following proposition:

Lemma 4.3. Let \((M, g(t))\) be a Ricci flow solution on \([0, T]\) with initial diameter \(D_0\) and initial volume \(V\). Assume that the scalar curvature is uniformly bounded by \(C_R\) in space-time, then for
any \( t \in (0, T) \) and \( r > 0 \) such that \( r^2 \leq t \), we have

\[
|B_r(x, r)| \leq C_2 V D_0^{-n} r^n \quad \Rightarrow \quad M R(x, r, \bar{t}) \geq C_2 V D_0^{-n},
\]

where the constant \( C_2 \) is defined as above.

**Proof (following [25])**. We first claim that if \( M R(x, r, \bar{t}) \leq C_2 V D_0^{-n} \) then for any \( s \in (0, r) \),

\[
|B_r(x, s)| \leq C_2 V D_0^{-n} s^n \quad \Rightarrow \quad |B_r(x, \frac{s}{2})| \leq 2^{-n} C_2 V D_0^{-n} s^n.
\]

Suppose otherwise, then we could fix some \( s \in (0, r] \) that contradicts the claim, i.e.

\[
|B_r(x, \frac{s}{2})| > (C_2 V D_0^{-n})^{\frac{2}{n+2}} 2^{-n} s^{\frac{n+2}{n+1}} |B_r(x, s)|^{\frac{n+3}{n+1}},
\]

so that we have

\[
\int_{B_r(x,s)} |\mathcal{R}_{g(\bar{t})}| \, dV_{g(\bar{t})} \leq (M R(x, r, \bar{t}))^{rac{2}{n+1}} s^{\frac{n+2}{n+1}} |B_r(x, s)|^{rac{n+3}{n+1}} \\
\leq (C_2 V D_0^{-n})^{\frac{2}{n+1}} s^{\frac{n}{n+1}} |B_r(x, s)|^{rac{n+3}{n+1}} \leq 2^n s^{-2} |B_r(x, \frac{s}{2})|.
\]

By (4.11), we could further deduce

\[
\log V D_0^{-n} + C_1(T) \leq \frac{64|B_r(x, s)|}{|B_r(x, \frac{s}{2})|} + \frac{s^2}{|B_r(x, \frac{s}{2})|} \int_{B_r(x,s)} |\mathcal{R}_{g(\bar{t})}| \, dV_{g(\bar{t})} + \log \left( \frac{|B_r(x, s)|}{s^n} \right) \\
\leq \frac{64|B_r(x, s)|}{|B_r(x, \frac{s}{2})|} + 2^n + \log V D_0^{-n} + \log C_2,
\]

so that \( |B_r(x, s)| \geq 2^n |B_r(x, \frac{s}{2})| \) by the choice of \( C_2 \), whence the claim.

Now by the claim, if there were any \( x \in M \) that has some scale \( s \in (0, r] \) contradicting the statement of the proposition, i.e. \( |B_r(x, s)| \leq C_2 V D_0^{-n} s^n \) and simultaneously \( M R(x, r, \bar{t}) \leq C_2 V D_0^{-n} \), then for any \( m \in \mathbb{N} \), we have, by the choice of \( C_2 \), that

\[
|B_r(x, 2^{-m} s)| \leq 2^{-mn} s^n C_2 V D_0^{-n} \leq \frac{\omega_n (2^{-m} s)^n}{2},
\]

which is impossible for all \( m \) sufficiently large, since as a smooth Riemannian manifold, \((M, g(\bar{t}))\) is locally Euclidean of dimension \( n \).

Now we define \( \nu := \min\left(n e^{-1} C_3 \right)^{\frac{2}{n}} \). Notice that \( \nu \) only depends on the initial data: the initial doubling and \( L^2\)-Poincaré constants. We prove the following diameter bound:

**Proposition 4.4.** Let \((M, g(t))\) be a Ricci flow solution on \([0, T]\) with initial diameter \( D_0 \) and initial volume \( V \), and assume that the scalar curvature is uniformly bounded by \( C_R \) in space-time.

Then there is a constant \( C_{diam} > 0 \) such that if \( V D_0^{-n} < \nu \omega_n \), then

\[
\forall t \in [0, T], \quad \text{diam}(M, g(t)) \leq C_{diam} e^{2C_R t} D_0,
\]

where the constants only depend on \( C_D, C_P, C_R, D_0 \) and are invariant under the parabolic rescaling of the Ricci flow.
Let $N := |\{x_i\}|$, then clearly $N \leq V(\bar{t})/(10^n C_2 V) \leq e^{C_\varepsilon\bar{t}}/C_2$.

Now the set $\gamma \setminus \bigcup_{i=1}^{N} B_i(x_i, D_0/5)$ has at most $N + 1$ connected components, and let $\sigma$ be one of these components with largest length. We either have $\text{diam}(M, g(\bar{t})) = |\gamma|_{g(\bar{t})} = |\sigma|_{g(\bar{t})}$ if $N = 0$; or else, if $N \geq 1$, we then have

\[
\text{diam}(M, g(\bar{t})) \leq (N + 1)|\sigma|_{g(\bar{t})} + 2ND_0/5
\leq 2N(|\sigma|_{g(\bar{t})} + D_0/5)
\leq 2C^{-1}_2 e^{C_\varepsilon\bar{t}}(|\sigma|_{g(\bar{t})} + D_0/5).
\]

In any case, we will need to estimate $|\sigma|_{g(\bar{t})}$ in terms of the initial diameter $D_0$:

For any $x \in \text{Im}(\sigma)$, the maximality of $\{x_i\}$ guarantees that $|B_i(x, D_0/10)| \leq 10^n C_2 V$. Now by Lemma 4.3, we know that

\[
\forall x \in \text{Im}(\sigma), \quad \forall R(x, D_0/10, \bar{t}) \geq C_2 V D_0^{-n}.
\]

Therefore we could find some $s(x) \in (0, D_0/10]$ such that

\[
C_2 V D_0^{-n} \leq \frac{|B_i(x, s)|}{s} \left( \frac{1}{s} \int_{B_i(x, s)} |\mathcal{R}_{g(\bar{t})}| \, dV_{g(\bar{t})} \right)^{\frac{n+1}{n}}
\leq 1 \frac{1}{s} \int_{B_i(x, s)} |\mathcal{R}_{g(\bar{t})}|^{\frac{n+1}{n}} \, dV_{g(\bar{t})}.
\]

Now we could apply Lemma 5.2 of [25] to pick a set of points $\{y_j\} \subset \text{Im}(\sigma)$ such that $\{B_j(y_j, s(y_j))\}$ are mutually disjoint, and that $|\sigma| \leq 6 \sum_j s(y_j)$. We could now estimate

\[
|\sigma|_{g(\bar{t})} \leq 6 \sum_j \frac{D_0^n}{C_2 V} \int_{B_j(y_j, s(y_j))} |\mathcal{R}_{g(\bar{t})}|^{\frac{n+1}{n}} \, dV_{g(\bar{t})}
\leq 6 \frac{D_0^n}{C_2 V} \int_M |\mathcal{R}_{g(\bar{t})}|^{\frac{n+1}{n}} \, dV_{g(\bar{t})}
\leq 6C_2^{-1} e^{C_\varepsilon\bar{t}} D_0^n C_R^{\frac{n+1}{n}}.
\]

Putting the estimates (4.15) and (4.16) together we obtain

\[
\text{diam}(M, g(\bar{t})) \leq C_{\text{diam}} e^{2C_\varepsilon\bar{t}} D_0,
\]

where, recalling the definition of $C_2$, we have $C_{\text{diam}} = 4^{n+4} D_0^{n-1} C_R^{\frac{n+1}{n}}$.

We notice that $C_{\text{diam}}$ is invariant under the parabolic rescaling of the Ricci flow, and is independent of time. \hfill \Box

**Remark 4.** When the scalar curvature is uniformly bounded, the previous renormalized volume ratio lower bound, and the upper bound of the total volume, actually provide a diameter upper bound. This naive estimate, however, fails to provide constants that are invariant under the parabolic rescaling of the Ricci flow.
4.3. A weak compactness result. We now state a proposition that corresponds to our second motivation of the paper: to construct, from a sequence of Ricci flows with collapsing initial data, Gromov-Hausdorff limits of the positive time-slices. Compare a result of Chen-Yuan [12, Theorem 1] in the case where lower bounds, uniform in space-time, of the Ricci curvature and the unit ball volume are assumed.

Proposition 4.5 (Weak compactness for positive time slices). Let \( \{(M_i, g_i(t))\} \) be a sequence of Ricci flows defined for \( t \in [0, T] \), such that they satisfy the same assumptions as in Theorem 1.7.

Then for each \( t \in (0, T) \), there is a subsequence of \( \{(M_i, g_i(t))\} \), a compact metric space \( (X, d_t) \), to which the sequence converges in the Gromov-Hausdorff topology.

Proof. This is a simple consequences of the estimates we proved previously in this section. Recall that in [17, Chapter 5, A] a quantity \( N(\varepsilon, R, X) \) is defined for each complete metric space \( X \), to denote the maximal number of disjoint \( \varepsilon \)-balls that could be possibly fitted into an \( R \)-ball in the metric space \( X \). As shown in [17, Proposition 5.2], as long as \( N(\varepsilon, R, X) \) is uniformly bound for all \( \varepsilon \in (0, R) \), \( R \in (0, \text{diam}(X)) \) and \( X \), the sequence \( \{X\} \) is precompact in the pointed-Gromov-Hausdorff topology.

In our situation, \( \forall t \in (0, T) \), since we have a uniform diameter upper bound \( (4.14) \), we only need to control \( N\left(\varepsilon, C_{\text{diam}}e^{2C_\varepsilon t}D_0, (M_i, g_i(t))\right) \). In fact, we could easily see that \( \forall \varepsilon \in (0, C_{\text{diam}}e^{2C_\varepsilon t}D_0) \), the total volume upper bound \( (4.10) \) together with the lower bound of renormalized volume ratio \( (4.9) \) gives: denoting \( V_i := \text{Vol}(M_i, g_i(0)) \), we have

\[
N\left(\varepsilon, C_{\text{diam}}e^{2C_\varepsilon t}D_0, (M_i, g_i(t))\right) \leq \frac{V_i e^{C_\varepsilon t}}{C_{VR}(T) V_i D_0^{-n} \varepsilon^n} = \frac{e^{C_\varepsilon t}}{C_{VR} \left(\frac{D_0}{\varepsilon}\right)^n}.
\]

This bound is uniform on the sequence \( \{(M_i, g_i(t))\} \) and therefore there is a metric space \( (X, d_t) \) to which the sequence subconverges in the Gromov-Hausdorff sense.

Clearly, \( \text{diam}(X, d_t) \leq C_{\text{diam}}e^{2C_\varepsilon t} \).

\( \square \)

Remark 5. Especially, we may assume \( \lim_{t \to \infty} V_i D_0^{-n} = 0 \), and this justifies us calling “collapsing initial data”.

5. Estimating the analytic quantities along the Ricci flow

In this section we provide a rough upper bound of the renormalized heat kernel, following Davies’ argument as discussed in Qi S. Zhang’s book and paper; we then apply this rough upper bound, the Harnack inequality \( (2.6) \), and the diameter upper bound \( (4.14) \) to obtain an on-diagonal lower bound lower bound of the renormalized heat kernel, when the initial global volume ratio is sufficiently small. As consequences, we also deduce a Gaussian type lower bound of the renormalized heat kernel, as well as the non-inflation property for the renormalized volume ratio.

5.1. Rough upper bound of the renormalized heat kernel in space time. The technique used to show the uniform Sobolev inequality in Subsection 3.2, i.e. the method of Davies, could be further applied to obtain a rough upper bound of the heat kernel coupled with the Ricci flow. This was first noticed by Qi S. Zhang in [33] and we will follow the exposition there. Notice that recently, Meng Zhu also extended Davies’ method to Ricci flows with a uniform Ricci curvature lower bound in space-time, see [35].
We fix \((x_0, t_0) \in M \times [0, T]\), and consider the heat kernel based at \((x_0, t_0)\), coupled with the Ricci flow, as introduced in Subsection 2.3. More specifically, we denote the heat kernel by

\[ K(x, t) = G(x_0, t_0; x, t), \]

i.e. for any \((x, t) \in M \times (t_0, T]\), \((\partial_t - \Delta_{g(t)})K(x, t) = 0\), and \(\lim_{t \to t_0} K(x, t) = \delta_{x_0}(x)\).

Now fix any \(t \in (t_0, T]\), let \(p(s) := (t - t_0)/(t - s)\) for \(s \in (t_0, t]\). Besides \(p(t_0) = 1\) and \(\lim_{s \to t} p(s) = \infty\), we also notice the following relations:

\[
0 \leq \frac{p(s) - 1}{p'(s)} = \frac{(s - t_0)(t - s)}{t - t_0} \leq t - t_0,
\]

\[
0 < \frac{1}{p'(s)} = \frac{(t - t_0)^2}{t} \leq t,
\]

and \(p'(s) p^{-2}(s) = \frac{1}{t - t_0}\).

Defining for any \((x, s) \in M \times (t_0, t]\),

\[ v(x, s) := \int K(x, s) \frac{p'(s)}{p^2(s)} \| K^{-1} \|^{-1}_{L^2(M, g(s))}, \]

we could compute as before to obtain

\[
\partial_s \log \|K\|_{L^{p(s)}(M, g(s))} = \frac{p'(s)}{p^2(s)} \int_M v^2 \log v^2 - 4\frac{(p(s) - 1)}{p'(s)} |\nabla v|^2 - \frac{p(s) - 1}{p'(s)} R_{g(s)} v^2 \, dV_{g(s)}
\]

\[
\leq \frac{p'(s)}{p^2(s)} \int_M v^2 \log v^2 - \frac{p(s) - 1}{p'(s)} \left( 4 |\nabla v|^2 + R_{g(s)} v^2 \right) \, dV_{g(s)} + C_R.
\]

We now plug \(\tau = \frac{p(s) - 1}{p'(s)}\) and \(\bar{t} = s\) into the uniform log-Sobolev inequality (3.4) (a bit abusing of notation), and obtain from the above calculations:

\[
\partial_s \log \|K\|_{L^{p(s)}(M, g(s))} \leq \frac{p'(s)}{p^2(s)} \left( \frac{n}{2} \log \frac{p(s) - 1}{p'(s)} - \log V D_0^{-n} + (C_R + D_0^2) \left( \bar{t} + \frac{p(s) - 1}{p'(s)} \right) + C_{IS} \right) + C_R.
\]

Integrating \(s\) from \(t_0\) to \(t\), we see that

\[
\log \frac{\|K(-, t)\|_{L^{p(t)}(M, g(t))}}{\|K(-, t_0)\|_{L^{p(t_0)}(M, g(t_0))}} \leq - \log V D_0^{-n}(t - t_0)^{\frac{n}{2}} + 2t(C_R + D_0^2) + C_R(t - t_0) + C_{IS} + n.
\]

Since the \(K(-, s)\) acquires the Delta function property as \(s\) descends to \(t_0\), we clearly have

\[ \|K(-, t_0)\|_{L^1(M, g(t_0))} = 1. \]

Therefore, exponentiating both sides of the above estimate we obtain

\[ V D_0^{-n} G(x_0, t_0; x, t) \leq C^+_{H}(T)(t - t_0)^{-\frac{n}{2}}, \]

where \(C^+_{H}(T) = \exp(2T(C_R + D_0^2) + C_R T + C_{IS} + n)\) is a universal constant independent of \(t - t_0\), and is invariant under the parabolic rescaling of the Ricci flow. We collect the result in the following

**Proposition 5.1** (Rough upper bound of the renormalized heat kernel). Let \((M, g(t))\) be a Ricci flow solution on \([0, T]\) with initial diameter \(D_0\) and initial volume \(V\), and assume that the scalar curvature is uniformly bounded by \(C_R\) in space-time.
Then there is a constant \( C_H^+(T) = C_H^+(C_D, C_P, C_R, D_0, n, T) \) such that for any \((y, t) \in M \times (0, T]\), the conjugate heat kernel \( G(-, -; y, t) \) based at \((y, t)\) obeys the following estimate:

\[
\forall x \in M, \forall s \in (0, t), \quad VD_0^{-n}G(x, s; y, t) \leq C_H^+(T)(t - s)^{-\frac{n}{2}}.
\]

5.2. Gaussian type lower bound of the renormalized heat kernel. In [9] and [33], the non-inflation property of volume ratio was proven based on an on-diagonal heat kernel lower bound, which was obtained by estimating the reduced length of a constant curve in space, and Perelman’s estimate. Such on-diagonal heat kernel lower bound, together with the gradient estimate Theorem 2.2, then gives a Gaussian lower bound of the heat kernel. This lower bound is essential in Bamber-Zhang’s estimate of the distance distortion.

We notice that this lower bound, however, could be not applied in the case of collapsing initial data, basically because of the lack of a correct renormalization. In this subsection, our major task is then to obtain a Gaussian type lower bound of the renormalized heat kernel. We will start similarly with an on-diagonal lower bound, and then apply the gradient estimate to obtain the desired Gaussian lower bound of the renormalized heat kernel.

First let us recall the volume upper bound:

\[
V(t) := |M|_{g(t)} = Ve^{-\int_0^t R(s)dv_{g(t)} ds} \leq Ve^{Cr'_t}.
\]

We also recall the bound of the total heat on the whole manifold: for any \( x, y \in M \) and any \( 0 \leq s < t \leq T \), recall that \( G(x, y, t) \) satisfies the heat equation in the variable \((y, t)\), and it satisfies the conjugate heat equation in the variable \((x, s)\); fixing \((x, s)\) and integrating \( y \in M \) we have \( \forall t' < t \leq T \),

\[
\partial_s \int_M G(x, t'; y, t) \, dv_{g(t)} = \left| \int_M \nabla_y G(x, t'; y, t) - R_{g(t)}(y)G(x, t'; y, t) \, dv_{g(t)}(y) \right| 
\leq C_R \int_M G(x, t'; y, t) \, dv_{g(t)}(y),
\]

therefore integrating in time we see

\[
e^{-Cr'_t(t-s)} \leq \int_M G(x, y, t) \, dv_{g(t)}(y) \leq e^{Cr'_t(t-s)}.
\]

As discussed in the introduction, the collapsing procedure is an intrinsic geometric phenomenon, and it should not cause the addition or loss of the total heat. Therefore, the heat-volume duality should be preserved. If the heat kernel at a future time fails to have a pointwise lower bound of order \((VD_0^{-n})^{-1}\), then the duality between the heat and volume will be contradicted, due to the rough bound of the renormalized heat kernel, the Harnack inequality (2.6), and the volume and diameter upper bounds of the whole manifold.

More specifically, the diameter upper bound in Proposition 4.4 the rough pointwise upper bound of \( G(x, s; -,-) \) in Proposttion 5.1 joint with the Harnack inequality (2.6) give, for any \((y, t) \in M \times (s, T]\),

\[
G(x, s; y, t) \leq H(n) \left( \frac{C_H^+(T)G(x, s; x, t)}{(VD_0^{-n})(t-s)^{-\frac{n}{2}}} \right)^{\frac{1}{2}} \exp \left( \frac{H'(n)(C_{\text{diam}}^2 e^{4C_x T} D_0^2)}{(t-s)} \right),
\]

whenever the initial golbal volume ratio is bounded as \( VD_0^{-n} \leq \nu \omega_n \).
Now we let $\theta := \sqrt{t-s}/D_0$ denote the ratio of the parabolic scale to the initial diameter. Integrating $y \in M$ and involving the lower bound of total heat in \((5.3)\), we see
\[
e^{-C_R(t-s)} \leq \int_M G(x, s; y, t) \, dV_{g(t)}(y)
\leq H(n) V e^{C_R t} \left( \frac{C_H^+(t) G(x, s; x, t)}{V^p} \right)^{\frac{1}{2}} \exp \left( \frac{H'(n)C_{\text{diam}}^2 e^{AC_R T}}{\theta^2} \right)
= H(n) e^{C_R t^{\frac{1}{2}}} \left( C_H^+(t) V G(x, s; x, t) \right)^{\frac{1}{2}} \exp \left( \frac{H'(n)C_{\text{diam}}^2 e^{AC_R T}}{\theta^2} \right),
\]
and thus
\[
V G(x, s; x, t) \geq H(n)^{-2} e^{-C_R (3t-s)} (C_H^+(t))^{-1} \theta^n \exp \left( \frac{-2H'(n)C_{\text{diam}}^2 e^{AC_R T}}{\theta^2} \right).
\]
Multiplying $D_0^n$ on both sides of this inequality we get
\[
VD_0^n G(x, s; x, t) \geq C_{\text{HD}}^-(T) \Psi(\theta \mid T)(t-s)^{-\frac{n}{2}},
\]
where
\[
C_{\text{HD}}^-(T) := H(n)^{-2} e^{-3C_R T} C_H^+(T)^{-1}, \quad \text{and} \quad \Psi(\theta \mid T) := \theta^{2n} \exp \left( -2H'(n)C_{\text{diam}}^2 e^{AC_R T} \theta^2 \right).
\]
Especially, we notice that $C_{\text{HD}}^-(T)$ only depends on the initial diameter, the doubling and $L^2$-Poincaré constants, the space-time scalar curvature upper bound, and the time elapsed from the beginning. On the other hand, we notice that $\Psi(\theta \mid T)$ depends, besides $\theta$ and $T$, only on the initial diameter and the space-time scalar curvature bound, especially it is independent of the initial doubling and $L^2$-Poincaré constants. We clearly see that
\[
\lim_{\theta \to 0} \Psi(\theta \mid T) = 0,
\]
indicating that the renormalization is only valid on scales comparable to the initial diameter. Moreover, both constants $C_{\text{HD}}^-(T)$ and $\Psi(\theta \mid T)$ are invariant under the parabolic rescaling of the Ricci flow. Summarizing, we have the following

**Lemma 5.2** (On-diagonal lower bound of the renormalized heat kernel). Let $(M, g(t))$ be a Ricci flow solution on $[0, T]$ with initial diameter $D_0$ and initial volume $V$, and assume that the scalar curvature is uniformly bounded by $C_R$ in space-time.

Then there are positive constants
\[
C_{\text{HD}}^-(T) = C_{\text{HD}}^-(C_D, C_P, C_R, D_0, n, T) \quad \text{and} \quad \Psi(\theta \mid T) = \Psi(\theta \mid C_R, D_0, n, T)
\]
such that for any $(x, t) \in M \times (0, T]$, the conjugate heat kernel $G(-, -; x, t)$ based at $(x, t)$ obeys the following estimate: for any $s \in (0, t)$, setting $\theta := \sqrt{t-s}/D_0$, then
\[
VD_0^n G(x, s; x, t) \geq C_{\text{HD}}^-(T) \Psi(\theta \mid T)(t-s)^{-\frac{n}{2}},
\]
whenever $VD_0^n \leq \nu_{\omega_n}$. Moreover, the constants $C_{\text{HD}}^-(T)$ and $\Psi(\theta \mid T)$ are invariant under the parabolic rescaling of the Ricci flow, and $\lim_{\theta \to 0} \Psi(\theta \mid T) = 0$.

Recall that the positive constant $\nu$ is defined right above Proposition 4.4.

Once this on-diagonal estimate is obtained, we could easily apply the Harnack inequality \((2.6)\) again to obtain a Gaussian lower bound:
Proposition 5.3 (Gaussian type lower bound of the renormalized heat kernel). Let \((M, g(t))\) be a Ricci flow solution on \([0, T]\) with initial diameter \(D_0\) and initial volume \(V\), and assume that the scalar curvature is uniformly bounded by \(C_R\) in space-time.

Then there are positive constants
\[
C_H^{-}(T) = C_H^{-}(C_D, C_P, C_R, D_0, n, T) \quad \text{and} \quad \Psi(\theta | T) = \Psi(\theta | C_R, D_0, n, T)
\]
such that for any \((y, t) \in M \times (0, T]\), the conjugate heat kernel \(G(-, -; y, t)\) based at \((y, t)\) obeys the following estimate: for any \(s \in (0, t)\), setting \(\theta := \sqrt{t - s} / D_0\), then
\[
(5.6) \quad VD_0^{-n}G(x, s; y, t) \geq C_H^{-}(T)\Psi(\theta | T)^2(t - s)^{-\frac{n}{2}} \exp\left(-2H'(n)\frac{d(x, y)^2}{t - s}\right),
\]
whenever \(VD_0^{-n} \leq \nu \omega_n\). Moreover, the constants \(C_H^{-}(T)\) and \(\Psi(\theta | T)\) are invariant under the parabolic rescaling of the Ricci flow, and \(\lim_{\theta \to 0} \Psi(\theta | T) = 0\).

Here the constant is defined as \(C_H^{-}(T) := (C_H^{-}(0))^{-2}(H(n)^2C_H^{-}(T))^{-1}\).

As a direct geometric consequence, we could also deduce the non-inflation property of the volume ratio:

Corollary 5.4 (Non-inflation of the renormalized volume ratio). Let \((M, g(t))\) be a Ricci flow solution on \([0, T]\) with initial diameter \(D_0\) and initial volume \(V\), and assume that the scalar curvature is uniformly bounded by \(C_R\) in space-time.

Then there are positive constants
\[
C_V^{+}(T) = C_V^{+}(C_D, C_P, C_R, D_0, n, T) \quad \text{and} \quad \Psi(\theta | T) = \Psi(\theta | C_R, D_0, n, T)
\]
such that for any \((x, t) \in M \times (0, T]\) and any \(r \in (0, \sqrt{t})\), setting \(\theta = r / D_0\), then
\[
(VD_0^{-n})^{-1}|B_r(x, r)| \leq C_V^{+}(T)\Psi(\theta | T)^2r^n,
\]
whenever \(VD_0^{-n} \leq \nu \omega_n\). Moreover, the constants \(C_V^{+}(T)\) and \(\Psi(\theta | T)\) are invariant under the parabolic rescaling of the Ricci flow, and \(\lim_{\theta \to 0} \Psi(\theta | T) = 0\).

Proof. Fix \((x, t) \in M \times (0, T]\) and \(r \in (0, \sqrt{t})\). Let \(G(x, t - r^2; -, -)\) be the fundamental solution to the conjugate heat equation coupled with the Ricci flow on \(M \times (t - r^2, T]\), based at \((x, t - r^2)\), i.e. \(\lim_{\delta \downarrow 0} G(x, t - r^2; -, s) = \delta\). By the Gaussian type lower bound (5.6) of the renormalized heat kernel, we have
\[
\forall y \in B_r(x, r), \quad VD_0^{-n}G(x, t - r^2; y, t) \geq C_H^{-}(T)\Psi(\theta | T)^2e^{-2H'(n)r^{-n}}.
\]
On the other hand, by (5.3), we have an upper bound of the total heat. Therefore, integrating over \(B_r(x, r)\) we have
\[
e^{C_Hr^2} \geq \int_{B_r(x, r)} G(x, t - r^2; y, t) dV_{g(t)}(y) \geq C_H^{-}(T)\Psi(\theta | T)^2e^{-2H'(n)}|B_r(x, r)|(VD_0^{-n})^{-1},
\]
or equivalently,
\[
(VD_0^{-n})^{-1}|B_r(x, r)| \leq C_V^{+}(T)\Psi(\theta | T)^{-2}r^n,
\]
with \(C_V^{+}(T) := e^{2H'(n) + C_Hr^2} / C_H^{-}(T)\). \(\square\)
Remark 6. Again, we see that the bound becomes worse as \(r/D_0\) becomes smaller. However, for any fixed positive scale, we have a uniform estimate.

6. Estimating the Distance Distortion

In this section we prove the main result of our note: the distance distortion estimate. Once the lower bound of the renormalized volume ratio (4.5) matches with that of the renormalized heat kernel (5.5), the classical argument of counting geodesics suitably covering a minimal geodesic carries over; see Section 5.3 of [10] and Section 3 of [2]. See also Section 3 of Chen-Wang [11] for a thorough exposition. However, we reproduce a detailed proof here, following [2], for the sake of completeness and readers’ convenience.

Before the commencement of the proof, we would like to emphasize the importance of the parabolic-scaling invariance of the constants in our previous estimates: for fixed small scales, we will dilate them to unit size and work with the rescaled quantities.

Proof of Theorem 1.1. Fix \(t_1 \in (0, T]\), and suppose that \(d_1(x_0, y_0) = r\). Let \(\theta = r/D_0\). Then we rescale \(r\) to \(1\) parabolically and denote the rescaled time slice as \(\bar{t}\). Also denote the rescaled metric as \(\bar{g}\).

Let \(\gamma : [0, 1] \rightarrow M\) be a unit speed \(\bar{g}(\bar{t})\)-minimal geodesic that connects \(x_0\) to \(y_0\). Let 
\[
K(x, t) := G(x_0, \bar{t} - \frac{1}{2}; x, t)
\]
be a heat kernel coupled with the Ricci flow, with initial data the Delta function at \((x_0, \bar{t} - \frac{1}{2})\), recall immediately that we have the bound (5.3) of the total heat:

\[
\forall t \in [\bar{t} - \frac{1}{2}, \bar{t} + \frac{1}{2}], \quad \int_M K(\cdot, t) \, dV_{\bar{g}(\bar{t})} \leq e^{C_r r^2}.
\]

By the renormalized heat kernel upper bound (5.1), we have

\[
\forall t \in [\bar{t} - \frac{1}{4}, \bar{t} + \frac{1}{4}], \quad (VD_0^n)K(\cdot, t) \leq C^+_H 2^n;
\]
on the other hand, by the Gaussian type lower bound (5.6), we have

\[
\forall s \in [0, 1], \quad (VD_0^n)K(\gamma(s), \bar{t}) \geq C^-_H e^{-4H(n)} 2^\frac{s}{2} \Psi(\theta | T)^2.
\]

Time derivative bound (2.7) together with (6.2) imply that

\[
\forall (s, t) \in [0, 1] \times [\bar{t} - \frac{1}{4}, \bar{t} + \frac{1}{4}], \quad |\partial_t (VD_0^n)K(\gamma(s), t)| \leq C^+_H 2^n (C_R r^2 + 4B(n));
\]
therefore, setting

\[
\alpha_0(\theta) := \min \left\{ \frac{1}{8} \frac{C^-_H e^{-4H(n)} \Psi(\theta | T)^2}{2^n + 1 \cdot C^+_H (C_R T + 4B(n))} \right\}
\]
and integrating the above time derivative bound we obtain from (6.3) that

\[
\forall (s, t) \in [0, 1] \times [\bar{t} - \alpha_0(\theta), \bar{t} + \alpha_0(\theta)], \quad (VD_0^n)K(\gamma(s), t) \geq \frac{1}{2} C^-_H e^{-4H(n)} 2^\frac{s}{2} \Psi(\theta | T)^2.
\]
Now by the Harnack inequality (2.6), we could estimate

\[(6.4) \quad \forall (s, t) \in [0, 1] \times [\bar{t} - \alpha_0, \bar{t} + \alpha_0], \quad \inf_{B_{\bar{t}}(y(s), 1)} (VD_0^{-\alpha}) K(-, t) \geq C_3(T)\Psi(\theta | T)^{\frac{\alpha}{2}}, \]

where \(C_3(T) = C_3(T)^{-1}\Psi(\theta | T)^{-1}\), a constant only depending on the initial diameter, the initial doubling and \(L^2\)-Poincaré constants, and the space-time scalar curvature bound. Moreover, \(C_3(T)\) is invariant under the parabolic rescaling of the Ricci flow.

Now fix any \(t \in [\bar{t} - \alpha_0(\theta), \bar{t} + \alpha_0(\theta)]\), and cover \(\text{Im}(\gamma) \subset M\) by a minimal number of unit \(\bar{g}(t)\)-geodesic balls \(\{B_t(\gamma(s), 1)\}, i = 1, \cdots, N\). It is easily seen that \(|\gamma|_{\bar{g}(t)} \leq 2N\). Therefore, in order to obtain an upper bound of \(d_t(x_0, y_0)\), it suffices to control \(N\) from above.

By the minimality of the covering, we see that the collection \(\{B_t(\gamma(s), 1/2)\}\) are pairwise disjoint. We could therefore combine the upper bound (6.1) of the total heat, the renormalized lower bound (6.4) of the local heat, together with the lower bound (4.9) of the renormalized volume ratio, to estimate:

\[
e^{C_\theta T} \geq \int_M K(x, t) \, dV_{\bar{g}(t)}
\]

\[
\geq \sum_{i=1}^N \int_{B_t(\gamma(s), \theta^2/2)} K(x, t) \, dV_{\bar{g}(t)}
\]

\[
\geq NC_3(T)^{\frac{\alpha}{2}}\Psi(\theta | T)^{\frac{\alpha}{2}}.
\]

Therefore \(N \leq 2^n e^{C_\theta T} C_3(T)^{-1}\Psi(\theta | T)^{-1}\), a constant independent of specific Ricci flow, especially its initial entropy. On the other hand, recalling that \(d_t(x_0, y_0) = 1\), we get

\[(6.5) \quad \forall t \in [\bar{t} - \alpha_1(\theta), \bar{t} + \alpha_1(\theta)], \quad d_t(x_0, y_0) \leq \alpha_1(\theta)^{-1} d_t(x_0, y_0),\]

where

\[
\alpha_1(\theta) := \min \left\{ \alpha_0(\theta), \frac{C_3(T)\Psi(\theta | T)^{\frac{\alpha}{2}}}{2^n e^{C_\theta T}} \right\}.
\]

This proves one side of the desired distance distortion estimate. To see the other side, we notice that the estimate (6.5) is independent of specific time slice \(\bar{t}\). Therefore, letting \(\alpha(\theta) = \frac{1}{2}\alpha_1(\theta)\), and applying the previous argument at the \(t\)-slice for any \(t \in [\bar{t} - \alpha, \bar{t} + \alpha]\), we see

\[
\forall s \in [t - \alpha_1(\theta), t + \alpha_1(\theta)], \quad d_s(x_0, y_0) \leq \alpha_1(\theta)^{-1} d_s(x_0, y_0).
\]

Especially, since \(\bar{t} \in [t - \alpha_1, t + \alpha_1]\), plugging \(s = \bar{t}\) into the above inequality we get the desired estimate (1) with \(\alpha(\theta)\) in place of \(\alpha_1(\theta)\). \(\square\)

Here we emphasize again that \(\alpha(\theta) \to 0\) as \(\theta \to 0\), reflecting the fact that when we look at smaller scales compared to the initial diameter, the estimate will be less effective.

We could also enhance the above distance distortion estimate in the following

**Corollary 6.1.** Let \((M, g(t))\) be a complete Ricci flow solution on \([0, T]\) with initial diameter \(D_0\) and initial volume \(V\), and assume the following conditions:

1. \((M, g(0))\), as a closed Riemannian manifold, has its doubling constant uniformly bounded above by \(C_D\), and its \(L^2\)-Poincaré constant by \(C_P\), and
2. the scalar curvature is uniformly bounded in space-time: \(\sup_{M \times [0, T]} |R_{g(t)}| \leq C_R\).
There exist two positive constants $\alpha = \alpha(\theta | C_D, C_P, C_R, D_0, n, T) < 1$ with

$$\lim_{\theta \to 0} \alpha(\theta | C_D, C_P, C_R, D_0, n, T) = 0,$$

and $\nu = \nu(C_D, C_P, C_R, n) < 1$, such that whenever $V D_0^{-n} \leq \nu n$, we have,

$$\forall t \in [0, T], \quad \forall x, y \in M \text{ with } d_t(x, y) = r \leq \sqrt{t}$$

and

$$\forall s \in [r^2, T] \text{ with } |s - t| \leq \alpha(\theta) \min\{C^{-1}_R, t\} + r^2,$$

the following estimate:

$$d_s(x, y)^2 \leq \alpha(\theta)^{-1}(d_t(x, y)^2 + |s - t|).$$

The proof is identical to that of Corollary 1.2 of [2], and we will omit it here.

7. Discussion

Although we have achieved the goal of our note — pointing out that meaningful geometric consequences could be proven from a correct renormalization, even if a uniform initial $\mu$-entropy lower bound may be violated — some new research directions are left open, which we now briefly discuss.

7.1. Uniform Hölder continuity of distance function. In a sequential work [3], Bamler-Zhang obtained a 1/2-Hölder continuity of the distance function along the Ricci flows, where the constant is uniformly bounded in terms of the scalar curvature and the initial $\mu$-entropy. This result is parallel to Colding-Naber’s Hölder continuity theorem for manifolds with a uniform Ricci curvature lower bound [13]. See the original work of Perelman [20] for a comparison geometry viewpoint of the Ricci flow, as well as the recent work of Bing Wang [26] for a nice explanation of the similarities between Ricci flows and manifolds with Ricci curvature lower bound.

Notice however, that Colding-Naber’s estimates are independent of the uniform volume non-collapsing assumption, a condition comparable to the uniform initial $\mu$-entropy lower bound in the Ricci flow setting. It is therefore natural to conjecture that Bamler-Zhang’s Hölder continuity could be generalized to Ricci flows with uniformly bounded scalar curvature in space-time, but without an a priori initial $\mu$-entropy lower bound. Again, we may have to impose the control of the initial diameter, as well as a uniform Ricci curvature lower bound for the initial metric.

7.2. Localization! A viable principle in geometric analysis should see a reasonable localization. We expect, in our future work, that locally collapsing initial data with Ricci curvature bounded below should also imply meaningful geometric structures on a positive-time slice. To indicate, we notice that the renormalized Sobolev inequality (1.2) we have used is only the global version of the locally valid estimate (2.1); on the other hand, the recent foundational work on local entropy by Bing Wang [26], provides the necessary technical tools that pass the initial local Sobolev constant estimate to positive-time slices. We would also like to mention the recent work of Gang Tian and Zhenlei Zhang [24], and the book by Qi S. Zhang [32] for related efforts in this direction.
REFERENCES

[1] Michael Anderson, The $L^2$ structure of moduli spaces of Einstein metrics on 4-manifolds. *Geom. Funct. Anal.*, 2 (1992), No. 1, 29-89.

[2] Richard H. Bamler and Qi S. Zhang, Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature. *Adv. Math.* 319 (2015), 396-450.

[3] Richard H. Bamler and Qi S. Zhang, Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature - part II. *Preprint*, arXiv:1506.03154, 2015.

[4] Jeff Cheeger, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.* 9 (1999), No.3, 428-517.

[5] Jeff Cheeger and Tobias Colding, On the structure of spaces with Ricci curvature bounded below. I. *J. Diff. Geom.* 46 (1997), 406-480.

[6] Jeff Cheeger and Tobias Colding, On the structure of spaces with Ricci curvature bounded below. II. *J. Diff. Geom.* 54 (2000), 13-35.

[7] Jeff Cheeger and Tobias Colding, On the structure of spaces with Ricci curvature bounded below. III. *J. Diff. Geom.* 54 (2000), 37-74.

[8] Xiuxiong Chen and Bing Wang, Space of Ricci flows (I). *Comm. Pure Appl. Math.* Vol. LXV (2012), 1399-1457.

[9] Xiuxiong Chen and Bing Wang, On the conditions to extend Ricci flow (III). *Int. Math. Res. Not.* (2013), No.10, 2349-2367.

[10] Xiuxiong Chen and Bing Wang, Space of Ricci flows (II). *Preprint*, arXiv:1405.6797, to appear in *J. Diff. Geom.*

[11] Xiuxiong Chen and Fang Yuan, A note on Ricci flow with Ricci curvature bounded below. *J. Reine Angew. Math.* 726 (2017), 29-44.

[12] Tobias H. Colding and Aaron Naber, Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. *Ann. of Math.* 176 (2012), 1173-1229.

[13] Edward Davies, *Heat kernel and spectral theory*. Cambridge University Press, 1989.

[14] Mikhail Gromov, Paul Levy’s isoperimetric inequality. *Reprint*, www.ihes.fr/~gromov/PDF/11[33].pdf

[15] Mikhail Gromov, Metric structure for Riemannian and non-Riemannian spaces. Modern Birkhäuser Classics, 2006.

[16] Richar Hamilton, The formation of singularities in the Ricci flow. *Surveys in differential geometry*, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA (1995), 7-136.

[17] Bruce Kleiner and John Lott, Notes on Perelman’s papers. *Geom. Topol.* 12 (2008), 2587-2855.

[18] Grisha Perelman, The entropy formula for the Ricci flow and its geometric applications. *Preprint*, arXiv:math/0211159.

[19] Laurent Saloff-Coste, A note on Poincaré, Sobolev and Harnack inequality. *Int. Math. Res. Not.* (1992), No.2, 27-38.

[20] Miles Simon, Ricci flow of almost non-negatively curved three manifolds. *J. Reine Angew. Math.* 630 (2009), 177-217.

[21] Gang Tian and Bing Wang, On the structure of almost Einstein manifolds. *J. Amer. Math. Soc.* 28 (2015), no. 4, 1169-1209.

[22] Gang Tian and Zhenlei Zhang, Relative volume comparison of Ricci flow and its applications. *Preprint*, arXiv:1802.09506.

[23] Peter Topping, Diameter control under Ricci flow. *Comm. Anal. Geom.* 13 (2005), 1039-1055.

[24] Bing Wang, The local entropy along Ricci flow — Part A: the no-local-collapsing theorems. To appear in *Camb. J. Math.*, arXiv:1706.08485.

[25] Rugang Ye, The logarithmic Sobolev and Sobolev inequalities along the Ricci flow. *Commun. Math. Stat.* 3, Issue 1 (2015), 1-36.

[26] Qi S. Zhang, Some gradient estimates for the heat kernel equation on domains and for an equation by Perelman. *Int. Math. Res. Not.* (2006), Art. ID 92314, 39 pp.

[27] Qi S. Zhang, A uniform Sobolev inequality under Ricci flow. *Int. Math. Res. Not.* 17 (2007), Art. ID rnm056, 17 pp.

[28] Qi S. Zhang, Erratum to: A uniform Sobolev inequality under Ricci flow. *Int. Math. Res. Not.* (2007).
[31] Qi S. Zhang, Addendum to: A uniform Sobolev inequality under Ricci flow. Int. Math. Res. Not. (2007).
[32] Qi S. Zhang, Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture. CRC Press, 2011. ISBN 978-1-4398-3459-6.
[33] Qi S. Zhang, Bounds on volume growth of geodesic balls under Ricci flow. Math. Res. Lett. 19 (2012), No. 1, 245-253.
[34] Qi S. Zhang, On the question of diameter bounds in Ricci flow. Illinois J. Math. 58 (2014), No. 1, 113-123.
[35] Meng Zhu, Davies type estimate and the heat kernel bound under the Ricci flow. Trans. Amer. Math. Soc. 368 (2016), No. 3, 1663-1680.

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