Invariant measures for Glauber dynamics of continuous systems

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Abstract
We consider Glauber-type stochastic dynamics of continuous systems [BCC02], [KL03], a particular case of spatial birth-and-death processes. The dynamics is defined by a Markov generator in such a way that Gibbs measures of Ruelle type are symmetrizing, and hence invariant for the stochastic dynamics. In this work we show that the converse statement is also true. Namely, all invariant measures satisfying Ruelle bound condition are grand canonical Gibbsian for the potential defining the dynamics. The proof is based on the observation that the well-known Kirkwood-Salsburg equation for correlation functions is indeed an equilibrium equation for the stochastic dynamics.

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1 Introduction

For Gibbs states \( \mu \) on the space \( \Gamma \) of all locally finite subsets (configurations) of \( \mathbb{R}^d \) and being either of the Ruelle type or corresponding to a positive potential, it has been constructed in the recent work \cite{KL03} an equilibrium Glauber-type dynamics on \( \Gamma \) having \( \mu \) as an invariant measure. That is, \( H^* \mu = 0 \) where \( H^* \) is the dual operator of the generator \( H \) of the dynamics. The dual relation between observables and states yields a further interpretation for this invariance result, namely, the Gibbs states as above are stationary distributions of the Glauber dynamics generated by \( H \). In this work we study the converse problem related to the question of whether all invariant measures are Gibbsian. For this purpose, we begin by enlarging the class of invariant measures to a new class of measures outside of the semigroup setting. These new elements, called infinitesimally invariant measures (corresponding to \( H \)), are probability measures \( \mu \) on \( \Gamma \) with finite moments of all orders which satisfy Ruelle bound condition, and

\[
\int_{\Gamma} (HF)(\gamma) \, d\mu(\gamma) = 0
\]

for all functions \( F \) in a proper dense set in the space \( L^1(\Gamma, \mu) \). We effectively formulate the notion of infinitesimally invariant measures by using the combinatorial harmonic analysis on configuration spaces introduced and developed in \cite{KK02}, \cite{KK03b}, \cite{Kun99} (Section 2). The special nature of this analysis yields, in particular, natural relations between states, observables, and correlation measures. We exploit these relations to show that for potentials fulfilling the usual integrability, stability, and lower regularity conditions, any infinitesimally invariant measure is Gibbsian (Theorem 5). In particular, this result applies to any invariant measure of the stochastic dynamics generated by \( H \). This answers an old open question usually known as the Gibbs conjecture for stochastic dynamics. Originally this question was formulated for the Hamiltonian case, and then generalized to other dynamics. For the Hamiltonian case the problem is partially solved, and the most meaningful contributions obtained on this direction are essentially due to B. M. Gurevich, Ya. G. Sinai, Yu. M. Suhov (see e.g. the review work \cite{Dob94} and the references therein). Considerable progress in the stochastic dynamics direction have been achieved in \cite{HS81}, \cite{Fri82}, \cite{Fri86}, \cite{FLO97}, \cite{FRZ98} for the diffusion dynamics of infinite lattice systems over \( \mathbb{Z}^d \), for \( d \leq 2 \). Recently, these results have been generalized in \cite{BRW02} to any dimension and,
moreover, to spin spaces not necessarily compact. In our case, we obtain an analogous result for Glauber-type stochastic dynamics of continuous systems. Besides the statement formulated in Theorem 5, the proof itself encloses an additional interpretation for the well-known Kirkwood-Salsburg equation for correlation functions. More precisely, it shows that the Kirkwood-Salsburg equation is indeed an equilibrium equation of the stochastic dynamics generated by $H$. As an aside, the existence of solutions for this equation, in the case of positive potentials in the high temperature-low activity regime, can easily be demonstrated. We postpone this subject to a forthcoming publication devoted solely to the problem of existence of non-equilibrium Glauber dynamics corresponding to more general potentials.

2 Harmonic analysis on configuration spaces

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all locally finite subsets of $\mathbb{R}^d$,

$$\Gamma := \{ \gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d)$, where $\varepsilon_x$ is the Dirac measure with mass at $x$, $\sum_{x \in \emptyset} \varepsilon_x$ is, by definition, the zero measure, and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d)$. This procedure allows to endow $\Gamma$ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$. We denote the Borel $\sigma$-algebra on $\Gamma$ by $\mathcal{B}(\Gamma)$.

Let us now consider the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)},$$

where $\Gamma^{(n)} := \Gamma^{(n)}_{\mathbb{R}^d} := \{ \gamma \in \Gamma : |\gamma| = n \}$ for $n \in \mathbb{N}$ and $\Gamma^{(0)} := \{ \emptyset \}$. For $n \in \mathbb{N}$, there is a natural bijection between the space $\Gamma^{(n)}$ and the symmetrization $\overline{(\mathbb{R}^d)^n} / S_n$ of the set $(\mathbb{R}^d)^n := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j \}$ under the permutation group $S_n$ over $\{1, ..., n\}$ acting on $(\mathbb{R}^d)^n$ by permuting the coordinate index. This bijection induces a metrizable topology on $\Gamma^{(n)}$ and we will endow $\Gamma_0$ with the topology of disjoint union of topological spaces.
By $\mathcal{B}(\Gamma^{(n)})$ and $\mathcal{B}(\Gamma_0)$ we denote the corresponding Borel $\sigma$-algebras on $\Gamma^{(n)}$ and $\Gamma_0$, respectively.

We proceed to consider the $K$-transform. Let $\mathcal{O}_c(\mathbb{R}^d)$ denote the set of all compact sets in $\mathbb{R}^d$, and for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ let $\Gamma_\Lambda := \{ \eta \in \Gamma : \eta \subset \Lambda \}$. Evidently $\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}_\Lambda$, where $\Gamma^{(n)}_\Lambda := \Gamma_\Lambda \cap \Gamma^{(n)}$ for all $n \in \mathbb{N}_0$, leading to a situation similar to the one for $\Gamma_0$, described above. We endow $\Gamma_\Lambda$ with the topology of the disjoint union of topological spaces and with the corresponding Borel $\sigma$-algebra $\mathcal{B}(\Gamma_\Lambda)$. To define the $K$-transform let us consider the space $B_{bs}(\Gamma_0)$ of all complex-valued bounded $\mathcal{B}(\Gamma_0)$-measurable functions $G$ with bounded support, i.e., $G|_{\Gamma_0 \setminus \bigsqcup_{n=0}^{N} \Gamma^{(n)}_\Lambda} \equiv 0$ for some $N \in \mathbb{N}_0, \Lambda \in \mathcal{O}_c(\mathbb{R}^d)$. The $K$-transform of any $G \in B_{bs}(\Gamma_0)$ is the mapping $KG : \Gamma \to \mathbb{C}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\eta \subset \gamma, |\eta| < \infty} G(\eta).$$

Note that for every $G \in B_{bs}(\Gamma_0)$ the sum in (1) has only a finite number of summands different from zero and thus $KG$ is a well-defined function on $\Gamma$. Moreover, if $G$ has support described as before, then the restriction $(KG)|_{\Gamma_\Lambda}$ is a $\mathcal{B}(\Gamma_\Lambda)$-measurable function and $(KG)(\gamma) = (KG)|_{\Gamma_\Lambda}(\gamma\Lambda)$ for all $\gamma \in \Gamma$, i.e., $KG$ is a cylinder function. In addition, for any $L \geq 0$ such that $|G| \leq L$, one finds $|(KG)(\gamma)| \leq L(1 + |\gamma\Lambda|)^N$ for all $\gamma \in \Gamma$, i.e., $KG$ is polynomially bounded. It has been shown in [KK02] that the $K$-transform is indeed a linear isomorphism between the spaces $B_{bs}(\Gamma_0)$ and $\mathcal{FP}_{bc}(\Gamma) := K(B_{bs}(\Gamma_0))$. This leads, in particular, to an explicit description of all functions in $\mathcal{FP}_{bc}(\Gamma)$ which may be found in [KK02] and [KK002]. However, throughout this work we shall only make use of the above described cylindricity and polynomial boundedness properties fulfilled by the elements in $\mathcal{FP}_{bc}(\Gamma)$. The inverse mapping of the $K$-transform is defined on $\mathcal{FP}_{bc}(\Gamma)$ by

$$(K^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta|\setminus|\xi|} F(\xi), \quad \eta \in \Gamma_0.$$ 

Besides the functions in $B_{bs}(\Gamma_0)$ we also consider the so-called coherent states $e_\lambda(f)$ of $\mathcal{B}(\mathbb{R}^d)$-measurable functions $f$, defined by

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1.$$
For a $B(\mathbb{R}^d)$-measurable function $f$ with compact support, we observe that the image of $e_\lambda(f)$ under the $K$-transform is still a well-defined function on $\Gamma$ and has an especially simple form given by

$$(Ke_\lambda(f))(\gamma) = \prod_{x \in \gamma} (1 + f(x)), \quad \gamma \in \Gamma.$$  

From the algebraic point of view let us consider the $\ast$-convolution defined on $B(\Gamma_0)$-measurable functions $G_1$ and $G_2$ by

$$(G_1 \ast G_2)(\eta) := \sum_{(\eta_1, \eta_2, \eta_3) \in P_3(\eta)} G_1(\eta_1 \cup \eta_2)G_2(\eta_2 \cup \eta_3), \quad \eta \in \Gamma_0,$$

where $P_3(\eta)$ denotes the set of all partitions of $\eta$ in three parts which may be empty [KK02]. It is straightforward to verify that the space of all $B(\Gamma_0)$-measurable functions endowed with this product has the structure of a commutative algebra with unit element $e_\lambda(0)$. Furthermore, for every $G_1, G_2 \in B_{bs}(\Gamma_0)$ we have $G_1 \ast G_2 \in B_{bs}(\Gamma_0)$, and

$$K (G_1 \ast G_2) = (KG_1) \cdot (KG_2)$$  

(cf. [KK02]. The $\ast$-convolution applied, in particular, to coherent states yields

$$e_\lambda(f) \ast e_\lambda(g) = e_\lambda(f + g + fg).$$  

As well as the $K$-transform, its dual operator $K^*$ will also play an essential role in our setting. In the sequel we denote by $\mathcal{M}^1_{im}(\Gamma)$ the set of all probability measures $\mu$ on $(\Gamma, B(\Gamma))$ with finite moments of all orders, i.e.,

$$\int_{\Gamma} |\gamma_\Lambda|^n \, d\mu(\gamma) < \infty \quad \text{for all } n \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{O}_c(\mathbb{R}^d).$$

By the definition of a dual operator, given a $\mu \in \mathcal{M}^1_{im}(\Gamma)$, $K^*\mu =: \rho_\mu$ is a measure on $(\Gamma, B(\Gamma_0))$ defined by

$$\int_{\Gamma_0} G(\eta) \, d\rho_\mu(\eta) = \int_{\Gamma} (KG)(\gamma) \, d\mu(\gamma),$$  

for all $G \in B_{bs}(\Gamma_0)$. Following the terminology used in the Gibbsian case, we call $\rho_\mu$ the correlation measure corresponding to $\mu$. This definition shows, in particular, that $B_{bs}(\Gamma_0) \subset L^1(\Gamma_0, \rho_\mu)$. Moreover, on $B_{bs}(\Gamma_0)$ the inequality
\[ \|KG\|_{L^1(\mu)} \leq \|G\|_{L^1(\rho_\mu)} \] holds, allowing an extension of the \( K \)-transform to a bounded operator \( K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu) \) in such a way that equality (1) still holds for any \( G \in L^1(\Gamma_0, \rho_\mu) \). For the extended operator the explicit form (1) still holds, now \( \mu \)-a.e. This means, in particular,

\[
(Ke_\lambda(f))(\gamma) = \prod_{x \in \gamma} (1 + f(x)), \quad \mu \text{-a.a. } \gamma \in \Gamma,
\]

for all \( B(\mathbb{R}^d) \)-measurable functions \( f \) such that \( e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu) \).

All the notions described above as well as their relations are graphically summarized in the figure below. In the context of an infinite particle system this figure has a natural meaning. The state of such a system is described by a probability measure \( \mu \) on \( \Gamma \) and the functions \( F \) on \( \Gamma \) are considered as observables of the system and they represent physical quantities which can be measured. The measured values correspond to the expectation values \( \int_\Gamma F(\gamma) \, d\mu(\gamma) \). In this context we call the functions \( G \) on \( \Gamma_0 \) quasi-observables.

**Example 1** On \( \mathbb{R}^d \) consider the intensity measure \( z \, dx, \, z > 0 \), and the Poisson measure \( \pi_z \) defined on \( (\Gamma, \mathcal{B}(\Gamma)) \) by

\[
\int_{\Gamma} \exp \left( \sum_{x \in \gamma} \varphi(x) \right) \, d\pi_z(\gamma) = \exp \left( z \int_{\mathbb{R}^d} (e^{\varphi(x)} - 1) \, dx \right), \quad \varphi \in \mathcal{D}.
\]
Here $\mathcal{D} := C^\infty_0(\mathbb{R}^d)$ denotes the Schwartz space of all infinitely differentiable real-valued functions with compact support. The correlation measure corresponding to the Poisson measure $\pi_z$ is the so-called Lebesgue-Poisson measure

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} m^{(n)},$$

where each $m^{(n)}$, $n \in \mathbb{N}$, is the image measure on $\Gamma^{(n)}$ of the product measure $dx_1...dx_n$ under the mapping $(\mathbb{R}^d)^n \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\} \in \Gamma^{(n)}$. For $n = 0$ we set $m^{(0)}(\{\emptyset\}) := 1$. This special case increases the importance of the coherent states and the space $B_{bs}(\Gamma_0)$ in our setting, mainly, due to the following two technical reasons, used throughout this work. First, $e_\lambda(f) \in L^p(\Gamma_0, \lambda_z)$ whenever $f \in L^p(\mathbb{R}^d, dx)$ for some $p \geq 1$, and, moreover,

$$\int_{\Gamma_0} |e_\lambda(f, \eta)|^p d\lambda_z(\eta) = \exp \left( z \int_{\mathbb{R}^d} |f(x)|^p dx \right). \quad (5)$$

Secondly, the space $B_{bs}(\Gamma_0)$ is dense in $L^2(\Gamma_0, \lambda_z)$.

### 3 Gibbs measures on configuration spaces

Let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a pair potential, that is, a $\mathcal{B}(\mathbb{R}^d)$-measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$ we define a relative energy of interaction between a particle located at $x$ and the configuration $\gamma$ by

$$E(x, \gamma) := \begin{cases} \sum_{y \in \gamma} \phi(x - y), & \text{if } \sum_{y \in \gamma} |\phi(x - y)| < \infty \\ +\infty, & \text{otherwise} \end{cases}.$$ 

For $\gamma = \emptyset$ we set $E(x, \emptyset) := 0$. A grand canonical Gibbs measure (Gibbs measure for short) corresponding to a pair potential $\phi$ and an activity parameter $z > 0$ is usually defined through the Dobrushin-Lanford-Ruelle equation. For convenience, we present here an equivalent definition through the Georgii-Nguyen-Zessin equation ([NZ79, Theorem 2], see also [KK03a, Theorem 3.12], [Kun99, Appendix A.1]). More precisely, a probability measure
\( \mu \) on \((\Gamma, \mathcal{B}(\Gamma))\) is called a Gibbs measure if it fulfills the integral equation

\[
\int_{\Gamma} \sum_{x \in \gamma} H(x, \gamma) \, d\mu(\gamma) = z \int_{\Gamma} \int_{\mathbb{R}^d} H(x, \gamma \cup \{x\}) e^{-E(x, \gamma)} \, dx \, d\mu(\gamma) \quad (6)
\]

for all positive measurable functions \(H : \mathbb{R}^d \times \Gamma \to \mathbb{R}\). In particular, for \(\phi \equiv 0\), (6) reduces to the Mecke identity, which yields an equivalent definition of the Poisson measure \(\pi_z\) \cite[Theorem 3.1]{Mec67}. For Gibbs measures, the corresponding correlation measures are always absolutely continuous with respect to the Lebesgue-Poisson measure \(\lambda_z\). A Radon-Nikodym derivative \(k_\mu := \frac{d\rho}{d\lambda_z}\) is called the correlation function of the measure \(\mu\).

Throughout this work we shall consider potentials \(\phi\) fulfilling the standard integrability (I) and stability (S) conditions:

(I) \[\int_{\mathbb{R}^d} |1 - e^{-\phi(x)}| \, dx < \infty.\]

(S) There is a \(B \geq 0\) such that

\[\forall \eta \in \Gamma_0, \quad E(\eta) := \sum_{\{x,y\} \subset \eta} \phi(x - y) \geq -B|\eta| \quad (E(\emptyset) := E(\{x\}) := 0)\]

Let us note that if \(\phi\) is semi-bounded from below, then condition (I) is equivalent to the integrability of \(\phi\) on the set \(\mathbb{R}^d \setminus \{\phi \geq 1\}\) whenever \(\{\phi \geq 1\}\) has finite Lebesgue measure. Of course, the stability condition (S) implies the semi-boundeness of \(\phi\) from below, namely, \(\phi \geq -2B\) on \(\mathbb{R}^d\). We will also use the superstability condition (SS), stronger than (S), and the lower regularity condition (LR), which may be found in \cite{Rue70}.

For potentials fulfilling (I), (SS), and (LR), D. Ruelle proved \cite{Rue70} the existence of tempered Gibbs measures (Ruelle measures for short). For positive potentials, condition (I) is sufficient to insure the existence of Gibbs measures (see e.g. \cite[Proposition 7.14]{KK03a}, \cite[Proposition 2.7.15]{Kun99}). In either case, the corresponding correlation functions fulfill the so-called Ruelle bound (RB):

\[\exists C > 0 : \quad k_\mu(\eta) \leq e_\lambda(C, \eta) = C^{||\eta||}, \quad \forall \eta \in \Gamma_0,\]

cf. \cite{Rue70}. Condition (RB) implies, in particular, that any Gibbs measure \(\mu\) has all local moments finite, i.e., \(\mu \in \mathcal{M}^1_{\text{fin}}(\Gamma)\).
4 Infinitesimally invariant measures

In the recent work [KL03], the authors have shown that the operator $H$ defined on a proper set of cylinder functions by

$$(HF)(\gamma) := \sum_{x \in \gamma} (F(\gamma \setminus \{x\}) - F(\gamma)) + z \int_{\mathbb{R}^d} e^{-E(x,\gamma)} (F(\gamma \cup \{x\}) - F(\gamma)) \, dx$$

is the generator of an equilibrium Glauber-type dynamics. More precisely, for Gibbs measures $\mu$ corresponding to an activity parameter $z$ and a pair potential $\phi$ fulfilling either conditions (I), (SS), and (LR) or conditions $\phi \geq 0$ on $\mathbb{R}^d$ and (I), it is proved that $H$ is a positive definite symmetric operator on $L^2(\Gamma, \mu)$. This allows the use of standard Dirichlet forms techniques to construct a Markov process on $\Gamma$, called an equilibrium Glauber dynamics, having $\mu$ as an invariant measure. That is, $H^* \mu = 0$ in the sense that

$$\int_{\Gamma} (HF)(\gamma) \, d\mu(\gamma) = 0$$

for all the cylinder functions $F$ as considered in [KL03]. The dual relation between observables and states yields a further interpretation for this invariance result. Since the semigroup $T_t = e^{-tH}$ associated to $H$ on $L^2(\Gamma, \mu)$ is related to the Kolmogorov equation

$$\frac{d}{dt} F_t = -HF_t, \quad t \geq 0$$

on the space of observables, it is seen from the aforementioned dual relation that, for $\mu_t = T_t^* \mu = e^{-tH^*} \mu$, on the space of states one has

$$\begin{cases} 
\frac{d}{dt} \mu_t = -H^* \mu_t, & t \geq 0 \\
\mu_0 = \mu 
\end{cases}$$

This consideration shows that the Gibbs measures studied in [KL03] are stationary distributions of the dynamics generated by $H$ described above. One of our aims is to study the converse problem related to the question of whether all invariant measures are Gibbsian. As a first step for this purpose, we shall enlarged the class of invariant measures to a new class of measures.
outside of the semigroup setting. These new elements, so-called infinitesimally invariant measures, will be measures \( \mu \in M_1^\text{inf}(\Gamma) \) such that, in a proper sense defined below (Definition 4), verify \( H^* \mu = 0 \).

In order to define the notion of infinitesimally invariant measures corresponding to the operator \( H \), first we shall extend the action of \( H \) to the set of cylinder functions \( \mathcal{FP}_{bc}(\Gamma) \). As \( \mathcal{FP}_{bc}(\Gamma) = K(B_{bs}(\Gamma_0)) \), this procedure naturally leads to the operator \( \hat{H} := K^{-1}HK \) on the space of quasi-observables.

In the sequel we assume the potential \( \phi \) to fulfil conditions (I) and (S). For functions \( F \in \mathcal{FP}_{bc}(\Gamma) \), these assumptions are sufficient to insure that \( HF \) is a well-defined function on \( \Gamma \). This follows from the fact that for any \( G \in B_{bs}(\Gamma_0) \) there are \( \Lambda \in O_c(\mathbb{R}^d), N \in \mathbb{N}_0 \) and a \( L \geq 0 \) such that \( G|_{\Gamma_0}(\bigcup_{n=0}^N r_n^{(\eta)}) \equiv 0 \) and \( |G| \leq L \), which implies that \( F(\gamma) := (KG)(\gamma) = F|_{\Gamma_\Lambda}(\gamma_\Lambda) \) and \( |F(\gamma)| \leq L(1 + |\gamma_\Lambda|)^N \) for all \( \gamma \in \Gamma \) (cf. Section 2). Therefore,

\[
-(HF)(\gamma) = \sum_{x \in \gamma_\Lambda} (F(\gamma \setminus \{x\}) - F(\gamma)) + z \int_{\Lambda} e^{-E(x,\gamma)} \left( F(\gamma \cup \{x\}) - F(\gamma) \right) dx,
\]

and the semi-boundeness of \( \phi \) from below allows to majorize the integral by the function defined on \( \Gamma \),

\[
e^{2B|\gamma|} \int_{\Lambda} (|F(\gamma_\Lambda \cup \{x\})| + |F(\gamma_\Lambda)|) dx \leq 2Le^{2B|\gamma|}(2 + |\gamma_\Lambda|)^Nm(\Lambda).
\]

Here, and below, \( m(\Lambda) \) denotes the volume of a set \( \Lambda \).

**Proposition 2** The action of \( \hat{H} \) on functions \( G \in B_{bs}(\Gamma_0) \) is given by

\[
-(\hat{H}G)(\eta) = -|\eta|G(\eta) + z \int_{\mathbb{R}^d} \left( e^{\lambda(e^{-\phi(x)} - 1)} \ast G(\cdot \cup \{x\}) \right)(\eta) dx,
\]

for all \( \eta \in \Gamma_0 \).

**Proof.** According to the definitions of the operators \( H \) and \( \hat{H} \), for all \( \eta \in \Gamma_0 \)
we find

\[- (\hat{HG})(\eta) = K^{-1} \left( \sum_{x \in \cdot} ((KG)(\cdot \setminus \{x\}) - (KG)(\cdot)) \right)(\eta) \]

\[+ K^{-1} \left( z \int_{\mathbb{R}^d} e^{-E(x, \cdot)} ((KG)(\cdot \cup \{x\}) - (KG)(\cdot)) \, dx \right)(\eta) \]

\[= \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \sum_{x \in \xi} ((KG)(\xi \setminus \{x\}) - (KG)(\xi)) \]

\[+ z \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \int_{\mathbb{R}^d} e^{-E(x, \xi)} ((KG)(\xi \cup \{x\}) - (KG)(\xi)) \, dx. \]

A direct application of the definitions of the $K$-transform and its inverse mapping yields for the first sum in (7)

\[\sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \sum_{x \in \xi} \sum_{\rho \subset \xi \setminus \{x\}} G(\rho \cup \{x\}) \]

\[= - \sum_{\xi \subset \eta} \sum_{x \in \xi} (-1)^{|\eta \setminus \xi|} (K(G(\cdot \cup \{x\}))(\xi \setminus \{x\})) \]

\[= - \sum_{x \in \eta} \sum_{\xi \subset \eta \setminus \{x\}} (-1)^{|\eta \setminus \xi\}| \sum_{\rho \subset \xi \setminus \{x\}} G(\rho \cup \{x\}) \]

\[= - \sum_{x \in \eta} K^{-1}(KG(\cdot \cup \{x\}))(\eta \setminus \{x\}) \]

\[= - \sum_{x \in \eta} G((\eta \setminus \{x\}) \cup \{x\}) = -|\eta|G(\eta). \]

To compute the second sum in (7), first observe that by the definition of the $K$-transform one has

\[\sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \int_{\mathbb{R}^d} e^{-E(x, \xi)} ((KG)(\xi \cup \{x\}) - (KG)(\xi)) \, dx \]

\[= \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \int_{\{x : x \not\in \xi\}} e^{-E(x, \xi)} ((KG)(\xi \cup \{x\}) - (KG)(\xi)) \, dx \]

\[= \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \int_{\mathbb{R}^d} e^{-E(x, \xi)} \sum_{\rho \subset \xi \setminus \{x\}} G(\rho \cup \{x\}) \, dx \]

\[= \int_{\mathbb{R}^d} \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} (Ke\lambda(e^{-\phi(x, \cdot)} - 1))(\xi)(KG(\cdot \cup \{x\}))(\xi) \, dx. \]
Therefore, by the action of the $K$-transform on the $\star$-convolution \textsuperscript{(2)}, we finally obtain

$$
\int_{\mathbb{R}^d} \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} K(e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup \{x\})) \, d\eta \, dx \\
= \int_{\mathbb{R}^d} K^{-1}(K(e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup \{x\})) \eta) \, d\eta \\
= \int_{\mathbb{R}^d} e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup \{x\})(\eta) \, d\eta.
$$

\[ \square \]

**Proposition 3** Given a $\mu \in \mathcal{M}_{1m}(\Gamma)$ assume that the correlation measure $\rho_\mu$ is absolutely continuous with respect to the Lebesgue-Poisson measure $\lambda_z$, and the correlation function $k_\mu$ fulfills condition (RB) for some $C > 0$. Then, $\hat{H}(B_{bs}(\Gamma_0)) \subset L^1(\Gamma_0, \rho_\mu)$. As a consequence, the operator $H$ maps the space $\mathcal{F}\mathcal{P}_{bc}(\Gamma)$ into $L^1(\Gamma, \mu)$.

**Proof.** As any $G \in B_{bs}(\Gamma_0)$ fulfills $|G| \leq L$ and $G|_{\Gamma_0 \setminus \bigcup_{n=0}^{N} \Gamma_0^{(n)}} \equiv 0$ for some $L \geq 0$, $N \in \mathbb{N}_0$, and $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, clearly one has

$$
\int_{\Gamma} |\eta||G(\eta)| \, d\rho_\mu(\eta) \leq NL \int_{\Gamma_{\Lambda}} C^{|\eta|} \, d\lambda_z(\eta) < \infty.
$$

Hence to prove the integrability of $\hat{H}G$ for $G \in B_{bs}(\Gamma_0)$ amounts to show the integrability of

$$
\int_{\mathbb{R}^d} (e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup \{x\})) \, d\eta.
$$

In order to do this, first observe that any function $G \in B_{bs}(\Gamma_0)$ described as before verifies $|G| \leq L e_\lambda(\mathbb{I}_\Lambda)$, where $\mathbb{I}_\Lambda$ is the indicator function of $\Lambda$, and thus

$$
\begin{align*}
\int_{\Gamma_0} \left| \int_{\mathbb{R}^d} (e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup \{x\})) \, d\eta \right| \, d\rho_\mu(\eta) \\
\leq L \int_{\mathbb{R}^d} \int_{\Gamma_0} (e_\lambda(|e^{-\phi(x-\cdot)} - 1|) + e_\lambda(\mathbb{I}_\Lambda, \cdot \cup \{x\})) \, d\eta \, d\rho_\mu(\eta) dx.
\end{align*}
$$

(8)
The definition of the $\star$-convolution and its especially simple form (3) for coherent states then allow rewriting the integrals in (8) as
\[
\int_{\mathbb{R}^d} \Pi_\Lambda(x) \int_{\Gamma_0} e^{\lambda \left( \Pi_\Lambda + (\Pi_\Lambda + 1) \right) \left| e^{-\phi(x)} - 1 \right|, \eta} \ d\rho_\mu(\eta) dx,
\]

which, due to the Ruelle boundeness, is bounded by
\[
\int_{\mathbb{R}^d} \Pi_\Lambda(x) \int_{\Gamma_0} e^{\lambda \left( C \Pi_\Lambda + C (\Pi_\Lambda + 1) \right) \left| e^{-\phi(x)} - 1 \right|, \eta} \ d\lambda_z(\eta) dx.
\]
Assumption (I) combined with equality (5) for the $\lambda_z$-expectation of a coherent state completes the proof showing that the latter expression may be bounded by
\[
m(\Lambda) \exp \left( z C \left( m(\Lambda) + 2 \int_{\mathbb{R}^d} \left| e^{-\phi(x)} - 1 \right| \ dx \right) \right) < \infty.
\]
The last assertion arises from $K(\hat{H}G) = H(KG)$ for all $G \in B_{bd}(\Gamma_0)$, and the $K$-transform maps the space $L^1(\Gamma_0, \rho_\mu)$ into $L^1(\Gamma, \mu)$. ■

In this way Proposition 3 yields the following definition.

**Definition 4** A measure $\mu \in \mathcal{M}_{11}(\Gamma)$ as in Proposition 3 is called an infinitesimally invariant measure corresponding to $H$ whenever
\[
\int_{\Gamma} (HF)(\gamma) d\mu(\gamma) = 0
\]
for all $F \in \mathcal{FP}_{bc}(\Gamma)$.

**Theorem 5** Let $\phi$ be a pair potential fulfilling (S), (I), and (LR). Then any infinitesimally invariant measure corresponding to $H$ is Gibbsian.

To prove this result we need the following lemma. We refer e.g. to [Oli02] for its proof.

**Lemma 6** Let $n \in \mathbb{N}$, $n \geq 2$, and $z > 0$ be given. Then
\[
\int_{\Gamma_0} \ldots \int_{\Gamma_0} G(\eta_1 \cup \ldots \cup \eta_n) H(\eta_1, \ldots, \eta_n) d\lambda_z(\eta_1) \ldots d\lambda_z(\eta_n) = \int_{\Gamma_0} G(\eta) \sum_{(\eta_1, \ldots, \eta_n) \in P_n(\eta)} H(\eta_1, \ldots, \eta_n) d\lambda_z(\eta)
\]

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for all positive measurable functions $G : \Gamma_0 \to \mathbb{R}$ and $H : \Gamma_0 \times \cdots \times \Gamma_0 \to \mathbb{R}$ with respect to which both sides of the equality make sense. Here $\mathcal{P}_n(\eta)$ denotes the set of all partitions of $\eta$ in $n$ parts, which may be empty.

In particular, for $n = 3$, Lemma 3 yields the following integration result for the $\ast$-convolution.

**Lemma 7** For all positive measurable functions $H, G_1, G_2 : \Gamma_0 \to \mathbb{R}$ and all $z > 0$ one has

$$\int_{\Gamma_0} H(\eta)(G_1 \ast G_2)(\eta) d\lambda_z(\eta)$$

$$= \int_{\Gamma_0} \int_{\Gamma_0} \int_{\Gamma_0} H(\eta_1 \cup \eta_2 \cup \eta_3) G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3) d\lambda_z(\eta_1)d\lambda_z(\eta_2)d\lambda_z(\eta_3).$$

**Proof of Theorem 5** According to Proposition 3, for any infinitesimally invariant measure $\mu$ corresponding to $H$ one has

$$\int_{\Gamma_0} (\hat{H}G)(\eta) k_\mu(\eta) d\lambda_z(\eta) = \int_{\Gamma_0} (\hat{H}G)(\eta) d\rho_\mu(\eta) = \int_{\Gamma} (H(KG))(\gamma) d\mu(\gamma) = 0$$

for all functions $G \in B_{bs}(\Gamma_0)$. Concerning the first expectation, observe that an application of Lemma 7 to the integral expression which appears in the definition of $\hat{H}$ yields

$$z \int_{\Gamma_0} \int_{\Gamma_0} \int_{\Gamma_0} (e^{\lambda(e^{-\phi(z-\cdot)} - 1)} G(\cdot \cup \{x\}))(\eta) k_\mu(\eta) d\lambda_z(\eta)dx$$

$$= z \int_{\Gamma_0} \int_{\Gamma_0} \int_{\Gamma_0} k_\mu(\eta_1 \cup \eta_2 \cup \eta_3) G(\eta_1 \cup \eta_2 \cup \{x\}) \cdot \cdot e^{\lambda(e^{-\phi(z-\cdot)} - 1)}(\eta_2 \cup \eta_3) d\lambda_z(\eta_1)d\lambda_z(\eta_2)d\lambda_z(\eta_3)dx$$

$$= \int_{\Gamma_0} d\lambda_z(\eta_3) e^{\lambda(e^{-\phi(z-\cdot)} - 1)}(\eta_3)$$

$$\int_{\Gamma_0} \int_{\Gamma_0} \int_{\Gamma_0} k_\mu(\eta_1 \cup \eta_2 \cup \eta_3) G(\eta_1 \cup \eta_2 \cup \{x\}) \cdot \cdot e^{\lambda(e^{-\phi(z-\cdot)} - 1)}(\eta_2) dx d\lambda_z(\eta_1)d\lambda_z(\eta_2)$$
with
\[
\int_{\mathbb{R}^d} \int_{\Gamma_0} \int_{\Gamma_0} k_{\mu}(\eta_1 \cup \eta_2 \cup \eta_3) G(\eta_1 \cup \eta_2 \cup \{x\}) 
\cdot e_\lambda(e^{-\phi(x-)} - 1, \eta_2) zdxd\lambda_z(\eta_1) d\lambda_z(\eta_2)
= \int_{\Gamma_0} G(\eta) \sum_{x \in \eta} \sum_{\xi \subset \eta \setminus \{x\}} k_{\mu}((\eta \setminus \{x\}) \cup \eta_3) e_\lambda(e^{-\phi(x-)} - 1, \xi) d\lambda_z(\eta),
\]
by an application of Lemma 6 for \(n = 3\). Moreover, since
\[
\sum_{\xi \subset \eta \setminus \{x\}} e_\lambda(e^{-\phi(x-)} - 1, \xi) = e_\lambda(e^{-\phi(x-)}, \eta \setminus \{x\}) = e^{-E(x, \eta \setminus \{x\})},
\]
we derive
\[
\int_{\Gamma_0} G(\eta) \sum_{x \in \eta} \sum_{\xi \subset \eta \setminus \{x\}} k_{\mu}((\eta \setminus \{x\}) \cup \eta_3) e_\lambda(e^{-\phi(x-)} - 1, \xi) d\lambda_z(\eta)
= \int_{\Gamma_0} G(\eta) \sum_{x \in \eta} e^{-E(x, \eta \setminus \{x\})} k_{\mu}((\eta \setminus \{x\}) \cup \eta_3) d\lambda_z(\eta).
\]
As a result
\[
z \int_{\mathbb{R}^d} \int_{\Gamma_0} (e_\lambda(e^{-\phi(x-)} - 1) \star G(\cdot \cup \{x\}))((\eta)k_{\mu}(\eta)) d\lambda_z(\eta) dx
= \int_{\Gamma_0} G(\eta) \sum_{x \in \eta} e^{-E(x, \eta \setminus \{x\})} .
\]
\[
\cdot \int_{\Gamma_0} e_\lambda(e^{-\phi(x-)} - 1, \rho) k_{\mu}((\eta \setminus \{x\}) \cup \rho) d\lambda_z(\rho) d\lambda_z(\eta).
\]
In this way for all \(G \in B_{bs}(\Gamma_0)\) one finds
\[
0 = \int_{\Gamma_0} (\hat{H}G)(\eta) k_{\mu}(\eta) d\lambda_z(\eta)
= \int_{\Gamma_0} |\eta| G(\eta) k_{\mu}(\eta) d\lambda_z(\eta) - \int_{\Gamma_0} G(\eta) \sum_{x \in \eta} e^{-E(x, \eta \setminus \{x\})}
\]
\[
\cdot \int_{\Gamma_0} e_\lambda(e^{-\phi(x-)} - 1, \rho) k_{\mu}((\eta \setminus \{x\}) \cup \rho) d\lambda_z(\rho) d\lambda_z(\eta).
\]
This implies
\[ |\eta|k_\mu(\eta) = \sum_{x \in \eta} e^{-E(x, \eta \setminus \{x\})} \int_{\Gamma_0} e_\lambda(e^{-\phi(x_{-\cdot})} - 1, \rho)k_\mu((\eta \setminus \{x\}) \cup \rho) d\lambda_z(\rho) \tag{9} \]
for \(\lambda_z\)-a.a. \(\eta \in \Gamma_0\). Note that in terms of the adjoint operator \(\hat{H}^*\) of \(\hat{H}\) on \(L^2(\Gamma_0, \lambda_z)\), equality (9) means \(\hat{H}^*k_\mu = 0\). We proceed to show the equivalence between equation (9) and the so-called Kirkwood-Salsburg equation ((KS)-equation for short), i.e.,
\[ k_\mu(\eta \cup \{x\}) = e^{-E(x, \eta)} \int_{\Gamma_0} e_\lambda(e^{-\phi(x_{-\cdot})} - 1, \rho)k_\mu(\eta \cup \rho) d\lambda_z(\rho), \quad \lambda_z \otimes dx - a.e. \]
Once this is proved, the proof then naturally follows by Proposition 8 below, due to [Rue70]. For non-translation invariant potentials, a similar result has been proved in [Kun99, Section 2.6].

Let \(k\) be a correlation function solving the (KS)-equation. Then
\[ e^{-E(x, \eta \setminus \{x\})} \int_{\Gamma_0} e_\lambda(e^{-\phi(x_{-\cdot})} - 1, \rho)k((\eta \setminus \{x\}) \cup \rho) d\lambda_z(\rho) = k(\{x\} \cup (\eta \setminus \{x\})) = k(\eta), \]
and summing both sides for all \(x \in \eta\) yields equation (9). To check the converse implication, let us first rewrite equation (9) in the simpler form
\[ \sum_{x \in \eta} I(x, \eta \setminus \{x\}) = 0, \]
where
\[ I(x, \eta \setminus \{x\}) := k(\{x\} \cup (\eta \setminus \{x\})) - e^{-E(x, \eta \setminus \{x\})} \int_{\Gamma_0} e_\lambda(e^{-\phi(x_{-\cdot})} - 1, \rho)k((\eta \setminus \{x\}) \cup \rho) d\lambda_z(\rho). \]
A straightforward application of Lemma 6 for \(n = 2\) then yields
\[ 0 = \int_{\Gamma_0} G(\eta) \sum_{x \in \eta} I(x, \eta \setminus \{x\}) d\lambda_z(\eta) = \int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta \cup \{x\})I(x, \eta) zdxd\lambda_z(\eta), \]
for all functions \(G \in B_{bs}(\Gamma_0)\). This implies that \(dx \otimes \lambda_z\)-a.e. \(I(x, \eta) = 0\), that is, the (KS)-equation.
Proposition 8  Let $\phi$ be a pair potential fulfilling $(S)$, $(I)$, and $(LR)$. Given a $\mu \in \mathcal{M}_{\text{fin}}^{1}(\Gamma)$ assume that the correlation measure $\rho_{\mu}$ is absolutely continuous with respect to a Lebesgue-Poisson measure $\lambda_{z}$ for some $z > 0$, and the correlation function $k_{\mu}$ fulfills $(RB)$. Then, $\mu$ is a Gibbs measure corresponding to $\phi$ and the activity $z$ if and only if $k_{\mu}$ solves the $(KS)$-equation.

Remark 9 Calculations similar to those in the proof of Theorem 8 show that, in terms of Bogoliubov functionals $L_{\mu}$ corresponding to infinitesimally invariant measures $\mu$,

$$L_{\mu}(\varphi) := \int_{\Gamma} \prod_{x \in \gamma} (1 + \varphi(x)) \, d\mu(\gamma), \quad \varphi \in D,$$

one finds the equality

$$\int_{\mathbb{R}^{d}} \varphi(x) \left( L_{\mu}((\varphi + 1)(e^{-\varphi(x-\cdot)} - 1) + \varphi) - \frac{\delta L_{\mu}(\varphi)}{\delta \varphi(x)} \right) \, dx = 0,$$

for all $\varphi \in D$. Here $\frac{\delta L_{\mu}(\varphi)}{\delta \varphi(x)}$ denotes the first variational derivative of $L_{\mu}$ at $\varphi$. This leads to the well-known equilibrium Bogoliubov equation introduced in $[\text{Bog46}]$

$$\frac{\delta L_{\mu}(\varphi)}{\delta \varphi(x)} = L_{\mu} \left( (1 + \varphi) \left( e^{-\varphi(x-\cdot)} - 1 \right) + \varphi \right), \quad dx - a.e.,$$

which yields an equivalent description of Gibbs measures (see $[\text{KK03a}]$, $[\text{KKO03}]$, $[\text{Kun99}]$, and also $[\text{Naz85}]$).

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