A Systematic Framework and Characterization of Influence-Based Network Centrality

Wei Chen  
Microsoft Research  
weic@microsoft.com

Shang-Hua Teng  
University of Southern California  
shanghua@usc.edu

Hanrui Zhang  
Duke University  
hrzhang@cs.duke.edu

October 24, 2018

Abstract

In this paper, we present a framework for studying the following fundamental question in network analysis:

*How should one assess the centralities of nodes in an information/influence propagation process over a social network?*

Our framework systematically extends a family of classical graph-theoretical centrality formulations, including degree centrality, harmonic centrality, and their “sphere-of-influence” generalizations, to influence-based network centralities. We further extend natural group centralities from graph models to influence models, since group cooperation is essential in social influences. This in turn enables us to assess individuals’ centralities in group influence settings by applying the concept of Shapley value from cooperative game theory.

Mathematically, using the property that these centrality formulations are Bayesian\(^1\), we prove the following characterization theorem: Every influence-based centrality formulation in this family is the unique Bayesian centrality that conforms with its corresponding graph-theoretical centrality formulation. Moreover, the uniqueness is fully determined by the centrality formulation on the class of layered graphs, which is derived from a beautiful algebraic structure of influence instances modeled by cascading sequences. Our main mathematical result that layered graphs in fact form a basis for the space of influence-cascading-sequence profiles could also be useful in other studies of network influences. We further provide an algorithmic framework for efficient approximation of these influence-based centrality measures.

Our study provides a systematic road map for comparative analyses of different influence-based centrality formulations, as well as for transferring graph-theoretical concepts to influence models.

1 Introduction

Network influence is a fundamental subject in network sciences \([28, 15, 21, 7]\). It arises from vast real-world backgrounds, ranging from epidemic spreading/control, to viral marketing, to innovation, and to political campaign. It also provides a family of concrete and illustrative examples for studying network phenomena — particularly regarding the interplay between network dynamics and graph structures — which require solution concepts beyond traditional graph theory \([11]\). As a result,

\(^1\)Meaning that they are linear to the convex combination of influence instances.
network influence is a captivating subject for theoretical modeling, mathematical characterization, and algorithmic analysis [28, 15, 21, 10].

In contrast to some network processes such as random walks, network influence is defined not solely by the static graph structure. It is fundamentally defined by the interaction between the *dynamic* influence models and the *static* network structures. Even on the same static network, different influence propagation models — such as the popular *independent cascade* and *linear-threshold* models — induce different underlying relationships among nodes in the network. The characterization of this interplay thus requires us to reformulate various fundamental graph-theoretical concepts such as centrality, closeness, distance, neighborhood (sphere-of-influence), and clusterability, as well as to identify new concepts fundamental to emerging network phenomena.

In this paper, we will study the following basic question in network science with focusing on *influence-based network centrality*.

*Is there a systematic framework to expand graph-theoretical concepts in network sciences?*

### 1.1 Motivations

Network centrality — a basic concept in network analysis — measures the importance and the criticality of nodes or edges within a given network. Naturally, as network applications vary — being Web search, internet routing, or social interactions — centrality formulations should adapt as well. Thus, numerous centrality measures have been proposed, based on degree, closeness, betweenness, and random-walks (e.g., PageRank) (cf. [25]) to capture the significance of nodes on the Web, in the Internet, and within social networks. Most of these centrality measures depend only on the static graph structures of the networks. Thus, these traditional centrality formulations could be inadequate for many real-world applications — including social influence, viral marketing, and epidemics control — in which static structures are only part of the network data that define the dynamic processes. Our research will focus on the following basic questions:

*How should we summarize influence data to capture the significance of nodes in the dynamic propagation process defined by an influence model? How should we extend graph-theoretic centralities to the influence-based centralities? What does each centrality formulation capture? How should we comparatively evaluate different centrality formulations?*

At WWW’17, Chen and Teng [11] presented an axiomatic framework for characterizing influence-based network centralities. Their work is motivated by studies in multiple disciplines, including social-choice theory [2], cooperative game theory [32], data mining [18], and particularly by [27] on measures of intellectual influence and [1] on PageRank. They present axiomatic characterizations for two basic centrality measures: (a) *Single Node Influence* (SNI) centrality, which measures each node’s significance by its influence spread;⁡ (b) *Shapley Centrality*, which uses the Shapley value of the influence spread function — formulated based on a fundamental cooperative-game-theoretical concept. Mathematically, the axioms are structured into two categories.

---

⁡The influence spread — as defined in [21] — of a group is the expected number of nodes this group can activate as the initial active set, called *seed set*. 

---

2
• **Principle Axioms**: The set of axioms that all desirable influenced-based centrality formulations should satisfy. In [11], two principle axioms, **Anonymity** and **Bayesian**, are identified. **Anonymity** is an ubiquitous and exemplary principle axiom, which states that centrality measures should be preserved under isomorphisms among influence instances. **Bayesian** states that influence-based centrality is a linear measure for mixtures of influence instances.

• **Choice axioms**: A (minimal) set of axioms that together with the principle axioms uniquely determine a given centrality formulation.

Such characterizations and the taxonomy of axioms precisely capture the essence of centrality formulations as well as their fundamental differences. In particular, the choice axioms succinctly distill the comparative differences between different centrality formulations.

However, the axiomatic characterization in [11] has two limitations preventing it to be generalized to study more influence-based centralities. First, it makes a significant simplification of the influence process: each influence instance \( I \) only captures the probability distributions of the final influenced nodes given any initial seed set, which we refer as the seeds-targets (probabilistic) profile. Essentially, it compressed out all intermediate steps in a cascading process and only takes the initial seed nodes and the final target nodes into account. This simplification is enough to study centrality measures concerning the final influence spread of the diffusion model, but is inadequate for characterizing influenced-based centrality measures that can capture the propagation details of network influences, such as neighborhood, closeness, sphere-of-influence centralities. Second, its choice axioms are based on a family of critical set instances, which do not have a graph-theoretical interpretation. This makes it less powerful in explaining the connection between graph-theoretical centralities and influence-based centralities.

### 1.2 Our Contributions

In this paper, we address both of the above issues in [11] and significantly expand the characterization of influence-based network centrality. First, influence instance \( I \) is now defined as the probabilistic profile on the more detailed cascading sequences, as formally defined below.

**Definition 1.1 (Cascading Sequence)**. For a directed network \( G = (V, E) \) with \( n = |V| \), a set sequence \((S_0, S_1, \ldots, S_{n-1})\) is a cascading sequence if it is both (1) **Monotonic**: \( \emptyset \subset S_0 \subset S_1 \subset \cdots \subset S_{t-1} \subset S_t = S_{t+1} = \cdots S_{n-1} \subseteq V \) for some \( t = 0, 1, \ldots, n - 1 \), and (2) **G-continuous**: for all \( t \in [T] \), every node in \( \Delta_t = S_t - S_{t-1} \) can be reached directly from some nodes in \( \Delta_{t-1} \) (where \( \Delta_0 = S_0 \)).

In the above definition, \( S_t \) represents the set of network nodes that become active by step \( t \) during the propagation, and \( \Delta_t = S_t - S_{t-1} \) denotes the set of nodes newly activated at step \( t \). Thus, the cascading sequence \((S_0, S_1, \ldots, S_{n-1})\) provides a layered structure starting from seed set \( S_0 \), similar to network broadcasting.

However, unlike broadcasting, the layered cascading sequences in network influence are formed stochastically. In each time step, already activated nodes stochastically activate more nodes in the next step, and this stochastic propagation ends when no new nodes are activated in a step. Therefore, when describing an influence instance, we need to specify the probabilistic distribution of the possible cascading sequences. This is formally defined as influence profile below.
Figure 1: an example of a layered-graph instance with 4 layers \((R_0, R_1, R_2, R_3)\) where \(R_0 = \{v_1, v_2\}\), \(R_1 = \{v_3, v_4, v_5\}\), \(R_2 = \{v_6, v_7\}\), \(R_3 = \{v_8, v_9, v_{10}\}\).

**Definition 1.2** (Influence Profile). An influence profile (also called an influence instance) is a tuple \(\mathcal{I} = (V, E, P)\), where \(G = (V, E)\) is a directed graph, and \(P : (2^V)^n \rightarrow \mathbb{R}\) is a probability profile defining a probability distribution of cascading sequences for every seed set \(S_0 \subseteq V\). That is, \(P(S_0, S_1, \ldots, S_{n-1}) \in [0, 1]\) specifies the probability that the influence instance generates cascading sequence \((S_0, S_1, \ldots, S_{n-1})\) when given seed set \(S_0\).

Then an influence-based centrality measure is defined as a mapping from the above influence profiles to real-valued vectors assigning centrality values to each node. We further transfer the concept of graph distance to cascading distance for cascading sequences. By using the cascading-sequence based influence profile and cascading distance, we are able to extend a large family of graph theoretical centralities based on graph distances from individual nodes, such as degree, closeness, harmonic, reachability centralities to influence based centralities. We refer to them as the stochastic sphere-of-influence centralities, and use a generic distance function \(f\) to summarize all of them.

Second, we provide a key technical contribution of the paper, which characterizes all influence instances by layered graphs. A layered graph is a directed graph with multiple layers and all nodes in one layer connect to all nodes in the next layer (Fig. 1). A layered graph instance is simply treating the graph as an influence instance following the breadth-first-search (BFS) propagation pattern. Surprisingly, we show that the set of all layered-graph instances form the basis in the vector space of all influence profiles, meaning that every influence profile is a linear combination of layered-graph instances. The result is instrumental to our centrality characterization, and is a powerful result by its own right.

The above layered-graph characterization allows us to connect influence propagation with static graphs. Therefore, by combining the principle axioms (Axioms **Anonymity** and **Bayesian**), we are able to show that our extended stochastic sphere-of-influence centrality with a distance function \(f\) is the unique centrality satisfying Axioms **Anonymity** and **Bayesian** that conforms with the corresponding graph-theoretical centrality with the same distance function \(f\), and the centrality is uniquely determined by their values on layered-graph instances. This characterization illustrates that (a) our extension of graph-theoretical centralities to influence-based centralities is not only reasonable but the only feasible mathematical choice, and (b) layered graphs are the key family of graphs comparing different influence measures, since a graph-theoretical centrality measure on layered graphs fully determines the conforming influence-based centrality measure.

The above centrality extension focuses on individual nodes. As group cooperation plays an important role in influence propagation, we further extend individual centralities to group centralities, which provide a value for every subset of nodes in the network. Similar to individual centrality, we
provide a characterization theorem showing that influence-based group centrality is also uniquely characterized by their values on layered-graph instances, as long as they satisfy the group version of Axioms Anonymity and Bayesian.

A group centrality measure has $2^n$ dimensions, so we further use the Shapley value [32] in cooperative game theory to reduce it to $n$ dimensions, and refer to it as the influence-based Shapley centrality. The Shapley centrality of a node measures its importance when the node collaborate with other nodes in groups. Due to the linearity of the Shapley value, we obtain the same characterization for the Shapley centrality: it is the unique one conforming with the graph-theoretical Shapley centrality and satisfying Axioms Anonymity and Bayesian, and the uniqueness is fully determined by its values on layered-graph instances.

Figure 2 summarizes our systematic extension of graph-theoretical centralities (the lower three boxes) to influence-based centralities (the upper three boxes). Starting from the classical graph-theoretical distance-based individual centrality (e.g. harmonic centrality), by transferring the concept of graph distance to cascading distance, we could lift it to the stochastic sphere-of-influence individual centrality (e.g. influence-based harmonic centrality). From individual centralities (either graph-theoretical or influence-based), we could use group distance to extend them to group centralities. From group centralities, we could apply Shapley value to obtain Shapley centralities. Therefore, Figure 2 provides a road map on how to extend many classical graph-theoretical centralities to influence-based centralities.

In addition to studying the systematic framework and characterization of influence-based centralities, we also address the algorithmic aspects of these centralities. We extend the approximation algorithm of [11] to cover all stochastic sphere-of-influence centralities introduced in this paper. The algorithm efficiently provides estimates for the centrality values of all nodes with theoretical guarantees.

To summarize, our contributions include: (a) a systematic extension of graph-theoretical distance-
based centralities to corresponding influence-based centralities, and further extending them to group centralities and Shapley centralities; (b) a key algebraic characterization of influence profiles by layered-graph instances; (c) unique characterization of stochastic sphere-of-influence centralities by their layered graph centralities and the principle axioms; and (d) an efficient algorithmic framework that approximates influence-based centralities for all centralities measures proposed in the paper.

1.3 Related Work

Influence propagation is an important topic in network science and network mining. One well studied problem on influence propagation is the influence maximization problem [16, 29, 21], which is to find $k$ seed nodes that generate the largest influence in the network. Influence maximization has been extensively studied for improving its efficiency or extending it to various other settings (e.g. [22, 23, 3, 12, 13]). However, influence maximization is different from the study of individual centrality as pointed out by [7]. Putting into our group centrality context, influence maximization can be viewed as the task of finding the group with the largest reachability group centrality. Our efficient centrality approximation algorithm is inspired from the scalable influence maximization based on the reverse-reachable set approach [8, 34, 33].

Network centrality has been extensively studied and numerous centrality measures are proposed (e.g. [25, 5, 20, 6, 9, 17]). As discussed in the introduction, most centrality measures (including group centrality [17] and Shapley centrality [24, 19]) are graph theoretical, and only focus on the static graph structure. A recent study by Chen and Teng [11] is the first systematic study combining network centrality with influence propagation. As already detailed in the introduction, our study is motivated by [11], and we aim at overcoming the limitations in [11] and extending the study of influence-based centrality to much wider settings. In particular, the SNI centrality and Shapley centrality studied in [11] are two instances related to reachability in the family of sphere-of-influence centrality measures we cover in this paper.

Axiomatic approach has been used to characterize network centrality [30, 26, 4, 31, 1]. These characterizations are mainly for graph-theoretical centralities. Again, the study in [11] provides the first axiomatic characterization for a couple of influence-based centralities. The characterization we provide in this paper is novel in the sense that it connects general influence profiles with a family of classical layered graphs so that our characterization can be based on graph-theoretical centralities.

2 Algebraic Characterization of Network Influences

In this section, we present our main technical result — a surprising discovery during our research — which provides a graph-theoretical characterization of the space of influence profiles. Specifically, we identify a simple set of classical graphs, and prove that when treated as BFS propagation instances, they form a linear basis in the space of all stochastic cascading-sequence profiles. This graph-theoretical characterization of influence models is instrumental to our systematic characterization of a family of influence-based network centralities. Moreover, we believe that this result is also important on its own right, and is potentially useful in other settings studying general influence propagation models.
2.1 Stochastic Diffusion Model

Stochastic diffusion models describe how information or influence are propagated through a network. A number of models have been well-studied (cf. [21, 10]), and among them independent cascade (IC) and linear threshold (LT) models are most popular ones. Here, we illustrate the stochastic profile of cascading sequences with the triggering model of Kempe-Kleinberg-Tardos [21], which includes IC and LT models as special cases. Triggering model will also be the subject of our algorithmic study.

In a triggering model, the static network structure is modeled as a directed graph $G = (V, E)$ with $n = |V|$. Each node $v \in V$ has a random triggering set $T(v)$ drawn from distribution $D(v)$ over subsets of $v$’s in-neighbors $N^-(v)$. At time $t = 0$, triggering sets of all nodes are sampled from their distributions, and nodes in a given seed set $S \subseteq V$ are activated. At any time $t \geq 1$, a node $v$ is activated if some nodes in its triggering set $T(v)$ was activated at time $t - 1$. The propagation continues until no new nodes are activated in a step.

In general, we can describe propagation in networks as a sequence of node activations, as described in the introduction. For convenience, we restate Definition 1.1 here.

**Definition 1.1 (Cascading Sequence).** For a directed network $G = (V, E)$ with $n = |V|$, a set sequence $(S_0, S_1, \ldots, S_{n-1})$ is a cascading sequence if it is both (1) **Monotonic:** $\emptyset \subset S_0 \subset S_1 \subset \cdots \subset S_{t-1} \subset S_t = S_{t+1} = \cdots S_{n-1} \subseteq V$ for some $t = 0, 1, \ldots, n - 1$, and (2) **G-continuous:** for all $t \in [T]$, every node in $\Delta_t = S_t - S_{t-1}$ can be reached directly from some nodes in $\Delta_{t-1}$ (where $\Delta_0 = S_0$).

Note that $S_t$ in a cascading sequence represents the nodes that are active by time $t$. The monotonicity requirement in the above definition means that (a) active nodes will not be deactivated, and (b) each step should have at least one new active node unless the cascade stops. This corresponds to the progress diffusion model in the literature. Since $S_0$ cannot be empty, and the set must grow by at least one node in each cascading step we have at most $n - 1$ cascading steps in the sequence. The G-continuous condition means that the activation of any new node in $\Delta_t$ at time $t$ must be partially due to the activation of some nodes in $\Delta_{t-1}$ at the previous time step. At this level of abstraction, a stochastic diffusion model can be viewed as a probabilistic mechanism to generate cascading sequences. Similar to [10], in this paper we use the distribution of the cascading sequences as the general specification of the diffusion model, as defined in Definition 1.2 and restated here.

**Definition 1.2 (Influence Profile).** An influence profile (also called an influence instance) is a tuple $I = (V, E, P_I)$, where $G = (V, E)$ is a directed graph, and $P_I : (2^V)^n \to \mathbb{R}$ is a probability profile defining a probability distribution of cascading sequences for every seed set $S_0 \subseteq V$. That is, $P_I(S_0, S_1, \ldots, S_{n-1}) \in [0, 1]$ specifies the probability that the influence instance generates cascading sequence $(S_0, S_1, \ldots, S_{n-1})$ when given seed set $S_0$.

Note that (1) if $(S_0, S_1, \ldots, S_{n-1})$ is not a valid cascading sequence, then $P_I(S_0, S_1, \ldots, S_{n-1}) = 0$. (2) For every $S_0$, $\sum_{S_1, \ldots, S_{n-1}} P_I(S_0, S_1, \ldots, S_{n-1}) = 1$. We also use the notation $P_I(S_0)$ to denote the distribution of sequence $(S_1, \ldots, S_{n-1})$ starting from $S_0$.

In influence propagation models, one key metric is the influence spread of a seed set $S$, which is the expected number of nodes activated when $S$ is the seed set. For an influence profile $I = (V, E, P_I)$, we can define influence spread $\sigma_I(S) = \sum_{S_1, \ldots, S_{n-1}} P_I(S, S_1, \ldots, S_{n-1}) \cdot |S_{n-1}|$. 

7
We remark that in [11], a more coarse-grained seed-target profile is used to study two influence-based centrality measures. The seed-target profile is only suitable for centrality measures addressing the final influence spread, but is not detailed enough to study other centrality extensions including extensions to degree, closeness, harmonic centralities, etc. Therefore, in this paper, we focus on the cascading sequence profile.

Note that for any directed graph \( G = (V, E) \), we can equivalently interpret it as a diffusion model, where the diffusion is carried out by the breadth-first search (BFS). In particular, for any seed set \( S_0 \), we have a deterministic cascading sequence \((S_0, S_1, \ldots, S_{n-1})\), where \( S_t \) is all the nodes that can be reached within \( t \) steps of BFS. Thus, the probability profile \( P_T \) is such that only this BFS sequence has probability 1, and all other sequences starting from \( S_0 \) have probability 0. We call this instance the BFS influence instance corresponding to graph \( G \), and denote it as \( I_G^{(BFS)} \).

### 2.2 A Graph-Theoretical Basis of Influence Profiles

Mathematically, each influence profile \( I \) of stochastic cascading sequences as defined in Definition 1.2, can be represented as a vector of probabilities \([P_T(S_0, S_1, \ldots, S_{n-1})]\) over monotonic cascading sequences. In other words, the vector contains entries for each monotonic cascading sequences. Because for each set \( S_0 \), all valid cascading sequences add up to 1, so one entry is redundant. We remove the entry \( P_T(S_0, S_0, \ldots, S_0) \) from the vector, and express it implicitly. The resulting vector certainly has an exponential number of dimensions, as there are exponential number of monotonic set sequences. We use \( M \) to denote its dimension.

The set of “basis” graphs are the layered graphs, as depicted in Figure 1. Formally, for a vertex set \( V \), for an integer \( t \geq 0 \), and \( t + 1 \) disjoint nonempty subsets \( R_0, R_1, \ldots, R_t \subseteq V \), a layered graph \( L_V(R_0, \ldots, R_t) \) is a directed graph in which every node in \( R_{i-1} \) has a directed edge pointing to every node in \( R_i \), for \( i \in [t] \), and the rest nodes in \( V \setminus \bigcup_{i=1}^t R_i \) are isolated nodes with no connections to and from any other nodes. We say that the BFS influence instance of the layered graph \( L_V(R_0, \ldots, R_t) \), namely \( I_{L_V(R_0, \ldots, R_t)}^{(BFS)} \), is a layered-graph instance, and for convenience we also use \( I_V(R_0, \ldots, R_t) \) to denote this instance. When the context is clear, we ignore \( V \) in the subscript.

A trivial layered graph instance is when \( t = 0 \), in which case all nodes are isolated and there is no edge in the graph. We call this the null influence instance, and denote it as \( I^N \) (or \( I_V(V) \) to make it consistent with the layered-graph notation). Technically, in \( I^N \), only \( P_T(S_0, S_0, \ldots, S_0) = 1 \), and all other probability values are 0, which means its corresponding vector form is the all-zero vector.

Let \( \mathcal{L} = \{I_V(R_0, \ldots, R_t) \mid t = 1, \ldots, n - 1, \emptyset \neq R_i \subseteq V, \text{all } R_i \text{'s are disjoint} \} \), i.e. \( \mathcal{L} \) is the set of all nontrivial layered-graph instances under node set \( V \).

As a fundamental characterization of the mathematical space of influence profiles, we prove the following theorem, which states that all nontrivial layered-graph instances form a linear basis in the space of all cascading-sequence based influence instances:

**Theorem 2.1** (Graph-Theoretical Basis). The set of vectors corresponding to the nontrivial layered-graph instances in \( \mathcal{L} \) forms a basis in \( \mathbb{R}^M \).

Although the proof of this theorem is quite technical, its underlying principle is quite basic. Here we provide some intuitions. Note first that \( M = |\mathcal{L}| \). Thus, the central argument in the proof is to show that elements in \( \mathcal{L} \) are independent, which we will establish using proof-by-contradiction: Suppose the profiles corresponding to layered graphs are not independent. That is, there are not-all-zero coefficients \( \lambda_T \) such that a linear combination (denoted by \( P = \sum_{I \in \mathcal{L}} \lambda_T P_I \)) of these profiles...
is zero. We consider a carefully-designed inclusion-exclusion form linear combination of the entries of P and show that this combination is exactly some \( \lambda I \neq 0 \), which means \( P \neq 0 \).

## 3 Influence-based Centrality: Single Node Perspective

Recall that a graph-theoretical centrality, such as degree, distance, PageRank, and betweenness, summarizes network data to measure the importance of each node in a network structure. Likewise, the objective of influence-based centrality formulations is to summarize the network-influence data in order to measure the importance of every node in influence diffusion processes. Formally (in the cascading-sequence model):

**Definition 3.1** (Influence-based Centrality Measure). An influence-based centrality measure \( \psi \) is a mapping from an influence profile \( \mathcal{I} = (V, E, P) \), to a real-valued vector \( (\psi_v(\mathcal{I}))_{v \in V} \in \mathbb{R}^n \).

The objective is to formulate network centrality measures that reflect dynamic influence propagation. Theorem 2.1 lays the foundation for a systematic framework to generalize graph-theoretical centrality formulation to network centrality measures. To this end, we first examine a unified centrality family that are natural for layered graphs.

### 3.1 A Unified Family of Sphere-of-Influence Centralities

In this subsection, we discuss a family of graph-theoretical centrality measures that contains various forms of “sphere-of-influence” and closeness centralities. These centrality measures have a common feature: the centrality of node \( v \) is fully determined by the distances from \( v \) to all nodes.

Consider a directed graph \( G = (V, E) \). Let \( N^+_G(v) \) and \( N^-_G(v) \) denote the set of out-neighbors and in-neighbors, respectively, of a node \( v \). Let \( d_G(u, v) \) be the graph distance from \( u \) to \( v \) in \( G \). If \( v \) is not reachable from \( u \) in \( G \) then we set \( d(u, v) = \infty \). Let \( d_G(S, v) = \min_{u \in S} d_G(u, v) \) be the distance from a subset \( S \subseteq V \) to node \( v \). Let \( \Gamma_G(S) \) be the set of nodes reachable from \( S \subseteq V \) in \( G \). When the context is clear, we would remove \( G \) from the subscripts in the above notations.

Recall that a graph-theoretical centrality measure is a mapping \( \mu \) from a graph \( G \) to a real-valued vector \( (\mu_v(G))_{v \in V} \in \mathbb{R}^n \), where \( \mu_v(G) \) denote the centrality of \( v \) in \( G \).

For every \( S \subseteq V \) and \( v \in V \), let \( \bar{d}_G(S) \) be the vector in \( \mathbb{R}^n \) consisting of the distance from \( S \) to every node \( u \), i.e. \( \bar{d}_G(S) = (d(S, u))_{u \in V} \). Let \( \bar{d}_G(v) = \bar{d}_G(\{v\}) \). We use \( \mathbb{R}_\infty \) to denote \( \mathbb{R} \cup \{\infty\} \). For each \( f : \mathbb{R}^n_\infty \rightarrow \mathbb{R} \), we can define:

**Definition 3.2** (Distance-based Centrality). A distance-based centrality \( \mu^{\text{ind}}[f] \) with function \( f : \mathbb{R}^n_\infty \rightarrow \mathbb{R} \) is defined as \( \mu^{\text{ind}}[f](G) = f(\bar{d}_G(v)) \).

Definition 3.2 is a general formulation. It includes several classical graph-theoretical centrality formulations as special cases: (a) The degree centrality (or immediate sphere of influence), \( \mu^{\text{ind-deg}} \), is defined as the out-degree of a node \( v \) in graph \( G \), that is, \( \mu^{\text{ind-deg}}_v(G) = |N^+_G(v)| \). It is defined by \( f^{\text{deg}}(\bar{d}) = |\{u \in V \mid d_u = 1\}| \). (b) The closeness centrality, \( \mu^{\text{ind-cls}} \), is defined as the reciprocal of the average distance to other nodes, \( \mu^{\text{ind-cls}}_v(G) = \frac{1}{\sum_{u \neq v} d_G(v, u)} \). It is defined by \( f^{\text{cls}}(\bar{d}) = \frac{1}{\sum_{u \in V} d_u} \).

If \( G \) is not strongly connected, then \( \mu^{\text{ind-cls}}_v(G) = 0 \) for any \( v \) that cannot reach all other nodes, and thus closeness centrality is not expressive enough for such graphs. (c) harmonic centrality, \( \mu^{\text{ind-har}} \), is defined: \( \mu^{\text{ind-har}}_v(G) = \sum_{u \neq v} \frac{1}{d_G(v, u)} \). It is defined by \( f^{\text{har}}(\bar{d}) = \sum_{u \in V, d_u > 0} \frac{1}{d_u} \). Note that harmonic centrality is closely related with closeness centrality, and is applicable to network with
Proposition 3.2. For any function \( f : \mathbb{R}_\infty^n \to \mathbb{R} \), \( \psi^{\text{ind}}[f] \) conforms with \( \mu^{\text{ind}}[f] \).

3.2 Stochastic Sphere-of-Influence: Lifting from Graph to Influence Models

Thus, Definition 3.2 represents a unified family of sphere-of-influence centralities for graphs. The function \( f \) — which is usually a non-increasing function of distance profiles — captures the scale of the impact, based on the distance of nodes from the source. By unifying these centralities under one general centrality class, we are able to systematically derive and study their generalization in the network-influence models. The key step is to transfer the graph distance in directed graph to cascading distance in cascading sequences. For any cascading sequence \((S_0, S_1, \ldots, S_{n-1})\) starting from seed set \( S_0 \), let \( d_u(S_0, S_1, \ldots, S_{n-1}) = t \) if \( u \in \Delta_t = S_t \setminus S_{t-1} (\Delta_0 = S_0) \), and \( d_u(S_0, S_1, \ldots, S_{n-1}) = \infty \) if \( u \not\in S_{n-1} \). We call \( d_u(S_0, S_1, \ldots, S_{n-1}) \) the cascading distance from seed set \( S_0 \) to node \( u \), since it represents the number of steps needed for \( S_0 \) to activate \( u \) in the cascading sequence. Then, we define the cascading distance vector \( \vec{d}(S_0, S_1, \ldots, S_{n-1}) \) as \( (d_u(S_0, S_1, \ldots, S_{n-1}))_{u \in V} \in \mathbb{R}^n \).

In particular, when we consider a cascading sequence \((\{v\}, S_1, \ldots, S_{n-1})\) starting from a single node \( v \), set \( \Delta_1 = S_1 \setminus \{v\} \) can be viewed as the out-neighbor of \( v \), set \( S_{n-1} \) can be viewed as the all nodes reachable from \( v \), and for every node \( u \in \Delta_t = S_t \setminus S_{t-1} \), the distance from \( v \) to \( u \) is \( t \).

Definition 3.3 (Individual Stochastic Sphere-of-Influence Centrality). For each function \( f : \mathbb{R}_\infty^n \to \mathbb{R} \), the influence-based individual stochastic sphere-of-influence centrality \( \psi^{\text{ind}}[f] \) is defined as:

\[
\psi^{\text{ind}}[f]_v(\mathcal{I}) = \mathbb{E}_{(S_1, \ldots, S_{n-1}) \sim P_Z(\{v\})}[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))].
\]

Definition 3.3 systematically extended the family of graph-theoretical centralities of Definition 3.2 to influence models. Natural influence-based centralities, e.g., the single-node influence (SNI) centrality defined in [11] (using each node \( v \)’s influence spread \( \sigma_Z(\{v\}) \) as the measure of its influence-based centrality), can be expressed by this extension:

Proposition 3.1. \( \forall \) influence profile \( \mathcal{I} = (V, E, P_Z) \):

\[
\text{SNI}(\mathcal{I}) = \psi^{\text{ind}}[f^{\text{rch}}](\mathcal{I}).
\]

The influence-based centrality formulations of Definition 3.3 enjoy the following graph-theoretical conformity property.

Definition 3.4 (Graph-Theoretical Conformity). An influence-based centrality measure \( \psi \) conforms with a graph-theoretical centrality measure \( \mu \) if for every directed graph \( G \), \( \psi(\mathcal{I}_G^{(\text{BFS})}) = \mu(G) \).

Proposition 3.2. For any function \( f : \mathbb{R}_\infty^n \to \mathbb{R} \), \( \psi^{\text{ind}}[f] \) conforms with \( \mu^{\text{ind}}[f] \).
3.3 Characterization of Influence-Based Centrality Measures

Given the multiplicity of the (potential) centrality formulations, “how should we characterize each formulation?” and “how should we compare different formulations?” are fundamental questions in network analysis. Inspired by the pioneering work of Arrow [2] on social choice, Shapley [32] on cooperation games and coalition, Palacios-Huerta & Volij [27] on measures of intellectual influence, and Altman & Tennenholtz [1] on PageRank, Chen and Teng [11] proposed an axiomatic framework for characterizing and analyzing influence-based network centrality. They identify two principle axioms that all desirable influence-based centrality formulations should satisfy.

Principle Axioms for Influenced-Based Centrality

The first axiom — ubiquitous axiom for centrality characterization, e.g. [30] — states that labels on the nodes should have no effect on centrality measures.

**Axiom 3.1** (Anonymity). For any influence instance \( \mathcal{I} = (V,E,P) \), and permutation \( \pi \) of \( V \),

\[
\psi_v(\mathcal{I}) = \psi_{\pi(v)}(\pi(\mathcal{I})), \quad \forall v \in V.
\]  

(1)

In Axiom 3.1, \( \pi(\mathcal{I}) = (\pi(V), \pi(E), \pi(P)) \) denotes the isomorphic instance: (1) \( \forall u, v \in V, (\pi(u), \pi(v)) \in \pi(E) \) if and only if \( (u, v) \in E \), and (2) for any cascading sequence \( (S_0, S_1, \ldots, S_{n-1}) \),

\[
P_{\pi(\mathcal{I})}(\pi(S_0), \pi(S_1), \ldots, \pi(S_{n-1})) = P_{\pi}(S_0, S_1, \ldots, S_{n-1}).
\]

The second axiom concerns Bayesian social influence [11] through a given network: For any three influence profiles \( \mathcal{I}, \mathcal{I}_1, \mathcal{I}_2 \) over the same vertex set \( V \), we say \( \mathcal{I} \) is a Bayesian of \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) if there exists \( \alpha \in [0,1] \) such that \( P_\mathcal{I} = \alpha P_{\mathcal{I}_1} + (1-\alpha) P_{\mathcal{I}_2} \). In other words, \( \mathcal{I} \) can be interpreted as a stochastic diffusion model where we first make a random selection — with probability \( \alpha \) — and then carry out the diffusion process according to the selected model. We also say that \( \mathcal{I} \) is a convex combination of \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \). The axiom reflects the linearity-of-expectation principle. If an influence instance is a convex combination of two other influence instances, the centrality value of a vertex is the same convex combination of the corresponding centrality values in the two other instances.

**Axiom 3.2** (Bayesian). For any \( \alpha \in [0,1] \), for any influence profiles \( \mathcal{I}, \mathcal{I}_1 \) and \( \mathcal{I}_2 \) over common vertex set \( V \) such that \( P_\mathcal{I} = \alpha P_{\mathcal{I}_1} + (1-\alpha) P_{\mathcal{I}_2} \),

\[
\psi_v(\mathcal{I}) = \alpha \psi_v(\mathcal{I}_1) + (1-\alpha) \psi_v(\mathcal{I}_2), \quad \forall v \in V.
\]  

(2)

Characterization of Influence-Based Centrality

We now use Theorem 2.1 and Axioms Anonymity and Bayesian to establish a complete characterization of the family of stochastic sphere-of-influence centralities formulated in Definition 3.3.

**Proposition 3.3.** If a function \( f : \mathbb{R}_\infty^n \to \mathbb{R} \) is anonymous — i.e., \( f(\vec{d}) \) is permutation-invariant — then \( \psi^{\text{ind}}[f] \) (as defined in Definition 3.3) satisfies Axiom Anonymity and \( \mu^{\text{ind}}[f] \) (as defined in Definition 3.2) satisfies the graph-theoretical counterpart of Axiom Anonymity.

**Proposition 3.4.** For any function \( f : \mathbb{R}_\infty^n \to \mathbb{R} \), \( \psi^{\text{ind}}[f] \) satisfies Axiom Bayesian.

Theorem 2.1 shows that all influence profiles can be represented as a linear combination of non-trivial layered-graph instances. This enables us to study and compare centrality measures by looking at their instantiation in the simple layered graph instances. The Bayesian property together with the linear basis of nontrivial layered-graph instances leads to the following characterization theorem.
**Theorem 3.1.** A Bayesian influence-based centrality measure is uniquely determined by its values on layered-graph instances (including the null instance).

Since layered-graph instances are all BFS instances derived from a special class of directed graphs, a direct implication of Theorem 3.1 is that any Bayesian centrality conforming with a classical graph-theoretical centrality is unique. Therefore, we have:

**Theorem 3.2** (Characterization of Individual Centrality). For any anonymous function $f : \mathbb{R}_\infty^n \to \mathbb{R}$, $\psi^{\text{ind}}[f]$ (defined in Definition 3.3) is the unique influence-based centrality that conforms with $\mu^{\text{ind}}[f]$ (defined in Definition 3.2) that satisfies both Axiom Anonymity and Axiom Bayesian.

The above uniqueness result show that our generalized definitions of influence-based centralities Definition 3.3 are not only reasonable but the only feasible choice (to the extent of Bayesian centralities).

### 4 Influence-based Centrality: Group Perspective and Shapley Centrality

As highlighted in Domingos-Richardson [28, 15] and Kempe-Kleinberg-Tardos [21], social influence propagation and viral marketing are largely group-based phenomena. Besides characterizing individuals’ influential centralities, perhaps the more important task is to characterize the influential centrality of groups, and individuals’ roles in group cooperation. This is the group centrality and Shapley centrality introduced in this section. When distinction is necessary, we refer to the centrality defined in Section 3 as individual centrality.

#### 4.1 Group Centrality

Group centrality measures the importance of each group in a network. Formally,

**Definition 4.1** (Influence-based Group Centrality). An influence-based group centrality measure $\psi^{\text{grp}}$ is a mapping from an influence profile $I = (V, E, P_I)$ to a real-valued vector $(\psi^{\text{grp}}(S))_{S \in 2^V} \in \mathbb{R}^{2^n}$.

For any function $f : \mathbb{R}_\infty^n \to \mathbb{R}$, both Definition 3.2 and Definition 3.3 have a natural extension: For $S \subseteq V$,

$$\mu^{\text{grp}}[f]_S(G) = f(\tilde{d}_G(S)),$$

$$\psi^{\text{grp}}[f]_S(I) = \mathbb{E}_{(S_1, \ldots, S_{n-1}) \sim P_I(\{v\})}[f(\tilde{d}(S, S_1, \ldots, S_{n-1}))].$$

Axioms Anonymity and Bayesian extend naturally as well as the following characterization based on Theorem 2.1.

**Theorem 4.1** (Characterization of Group Centrality). For any anonymous function $f : \mathbb{R}_\infty^n \to \mathbb{R}$, $\psi^{\text{grp}}[f]$ is the unique influence-based group centrality that conforms with $\mu^{\text{grp}}[f]$ and satisfies both Axiom Anonymity and Axiom Bayesian.

Therefore, we can again reduce the analysis of an influence-based group centrality to the analysis of the measure on the particular layered-graph instances.
4.2 Cooperative Games and Shapley Value

A cooperative game [32] is defined by tuple $(V, \tau)$, where $V$ is a set of $n$ players, and $\tau : 2^V \rightarrow \mathbb{R}$ is called characteristic function specifying the cooperative utility of any subset of players. In cooperative game theory, a ranking function $\phi$ is a mapping from a characteristic function $\tau$ to a vector $(\phi_v(\tau))_{v \in V} \in \mathbb{R}^n$, indicating the importance of each individual in the cooperation. One famous ranking function is the Shapley value $\phi^{\text{Shapley}} [32]$, as defined below. Let $\Pi$ be the set of all permutations of $V$, and $\pi \sim \Pi$ denote a random permutation $\pi$ drawn uniformly from set $\Pi$. For any $v \in V$ and $\pi \in \Pi$, let $S_{\pi,v}$ denote the set of nodes in $V$ preceding $v$ in permutation $\pi$. Then, $\forall v \in V$:

$$
\phi_v^{\text{Shapley}}(\tau) = \frac{1}{n!} \sum_{\pi \in \Pi} (\tau(S_{\pi,v} \cup \{v\}) - \tau(S_{\pi,v}))
= \sum_{S \subseteq V \setminus \{v\}} \frac{|S|!(n - |S| - 1)!}{n!} (\tau(S \cup \{v\}) - \tau(S))
= \mathbb{E}_{\pi \sim \Pi}[\tau(S_{\pi,v} \cup \{v\}) - \tau(S_{\pi,v})].
$$

The Shapley value of a player $v$ measures the expected marginal contribution of $v$ on the set of players ordered before $v$ in a random order. Shapley [32] proved a remarkable representation theorem: The Shapley value is the unique ranking function that satisfies all the following four conditions: (1) **Efficiency**: $\sum_{v \in V} \phi_v(\tau) = \tau(V)$. (2) **Symmetry**: For any $u, v \in V$, if $\tau(S \cup \{u\}) = \tau(S \cup \{v\})$, $\forall S \subseteq V \setminus \{u, v\}$, then $\phi_u(\tau) = \phi_v(\tau)$. (3) **Linearity**: For any two characteristic functions $\tau$ and $\omega$, for any $\alpha, \beta > 0$, $\phi(\alpha \tau + \beta \omega) = \alpha \phi(\tau) + \beta \phi(\omega)$. (4) **Null Player**: For any $v \in V$, if $\tau(S \cup \{v\}) - \tau(S) = 0$, $\forall S \subseteq V \setminus \{v\}$, then $\phi_v(\tau) = 0$. **Efficiency** states that the total utility is fully distributed. **Symmetry** states that two players’ ranking values should be the same if they have the identical marginal utility profile. **Linearity** states that the ranking values of the weighted sum of two coalition games is the same as the weighted sum of their ranking values. **Null Player** states that a player’s ranking value should be zero if the player has zero marginal utility to every subset.

4.3 Shapley Centrality

Shapley’s celebrated concept — as highlighted in [11] — offers a formulation for assessing individuals’ performance in group influence settings. It can be used to systematically compress exponential-dimensional group centrality measures into $n$-dimensional individual centrality measures.

**Definition 4.2** (Influence-based Shapley Centrality). An influence-based Shapley centrality $\psi^{\text{Shapley}}$ is an individual centrality measure corresponding to a group centrality $\psi^{\text{grp}}$:

$$
\psi_v^{\text{Shapley}}(\mathcal{I}) = \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I}))
= \mathbb{E}_{\pi \sim \Pi}[\psi^{\text{grp}}_{S_{\pi,v} \cup \{v\}}(\mathcal{I}) - \psi^{\text{grp}}_{S_{\pi,v}}(\mathcal{I})].
$$

We also denote it as $\psi^{\text{Shapley}} = \phi^{\text{Shapley}} \circ \psi^{\text{grp}}$.

In [11], Chen and Teng analyze the Shapley value of the influence-spread function, which is a special case of the following “Shapley extension” of Definition 3.3.

**Definition 4.3** (Shapley Centrality of Stochastic Sphere-of-Influence). For each $f : \mathbb{R}_\infty^n \rightarrow \mathbb{R}$, the Shapley centrality of Stochastic Sphere-of-Influence $\psi^{\text{Shapley}}[f]$ is defined as:

$$
\psi_v^{\text{Shapley}}[f](\mathcal{I}) = \phi_v^{\text{Shapley}}(\psi^{\text{grp}}[f](\mathcal{I})).
$$
Shapley centrality $\mu^{\text{Shapley}}$ can also be defined similarly based on graph-theoretical group centrality (see, for example, [24]). We will refer to the extension of Definition 3.2 as $\mu^{\text{Shapley}}[f]$. Using Theorem 2.1, we can establish the following characterization.

**Theorem 4.2** (Characterization of Shapley Centrality). For any anonymous function $f : \mathbb{R}^n_{\infty} \to \mathbb{R}$, $\psi^{\text{Shapley}}[f]$ is the unique influence-based centrality that conforms with $\mu^{\text{Shapley}}[f]$ and satisfies both Axiom Anonymity and Axiom Bayesian.

Theorem 4.2 systematically extends the work of [11] to all sphere-of-influence formulations. The SNI and Shapley centrality analyzed in [11] are $\psi^{\text{ind}[f^{\text{rch}}]}$ and $\psi^{\text{Shapley}[f^{\text{rch}}]}$, respectively. In our process of generalizing the work of [11], we also resolve an open question central to the axiomatic characterization of [11], which is based on a family of critical set instances that do not correspond to a graph-theoretical interpretation. In fact, the influence-spread functions of these “axiomatic” critical set instances used in [11] are not submodular. In contrast, influence-spread functions of the popular independent cascade (IC) and linear threshold (LT) models, as well as, the trigger models of Kempe-Kleinberg-Tardos, are submodular. The submodularity of these influence-spread functions plays an instrumental role in influence maximization algorithms [21, 10]. Thus, it is a fundamental and mathematical question whether influence profiles can be characterized by “simpler” influence instances. Our layered-graph characterization (Theorem 2.1) resolves this open question by connecting all influence profiles with simple BFS cascading sequence in the layered graphs, which is a special case of the IC model and possess the submodularity property. In summary, our layered-graph characterization is instrumental to the series of characterizations we could provide in this paper for influence-based individual, group, and Shapley centralities (Theorem 3.2, 4.1, 4.2).

5 Efficient Algorithm for Approximating Influence-based Centrality

Besides studying the characterization of influence-based centralities, we also want to compute these centrality measures efficiently. Accurate computation is in general infeasible (e.g. it is $\#P$-hard to compute influence-based reachability centrality $\psi^{\text{ind}[f^{\text{rch}}]}$ in the triggering model [35, 14]). Thus, we are looking into approximating centrality values. Instead of designing one algorithm for each centrality, we borrow the algorithmic framework from [11] and show how to adapt the framework to approximate different centralities. Same as in [11], the algorithmic framework applies to the triggering model of influence propagation. For efficient computation, we further assume that the distance function $f$ is additive, i.e. $f(d) = \sum_{u \in V} g(d_u)$ for some scalar function $g : \mathbb{R}^n_{\infty} \to \mathbb{R}$ satisfying $g(\infty) = 0$. The degree, harmonic, and reachability centralities all satisfy this condition. In particular, we have $f^{\text{deg}}(d) = \sum_{u \in V} g^{\text{deg}}(d_u)$, with $g^{\text{deg}}(d_u) = 1$ if $d_u = 1$ and $g^{\text{deg}}(d_u) = 0$ otherwise; $f^{\text{har}}(d) = \sum_{u \in V} g^{\text{har}}(d_u)$, with $g^{\text{har}}(d_u) = 1/d_u$ if $d_u > 0$ and $g^{\text{har}}(d_u) = 0$ otherwise; and $f^{\text{rch}}(d) = \sum_{u \in V} g^{\text{rch}}(d_u)$, where $g^{\text{rch}}(d_u) = 1$ if $d_u < \infty$ and $g^{\text{rch}}(d_u) = 0$ otherwise.

The algorithmic framework for estimating individual and Shapley forms of sphere-of-influence centrality is given in Algorithm 1, and is denoted ICE-RR (for Influence-based Centrality Estimate via RR set). The algorithm uses the approach of reverse-reachable sets (RR sets) [8, 34, 33]. An RR set $R_v$ is generated by randomly selecting a node $v$ (called the root of $R_v$) with equal probability, and then reverse simulating the influence propagation starting from $v$. In the triggering model, it is simply sampling a random triggering set $T(v)$ for $v$, putting all nodes in $T(v)$ into $R_v$, and then recursively sampling triggering sets for all nodes in $T(v)$, until no new nodes are generated.
**Input:** Network: $G = (V,E)$; Parameters: random triggering set distribution $\{T(v)\}_{v \in V}$, $\varepsilon > 0$, $\ell > 0$, $k \in [n]$, node-wise distance function $g$

**Output:** $\hat{\psi}_v$, $\forall v \in V$: estimated centrality value

1: {Phase 1. Estimate the number of RR sets needed}
2: $LB = 1$; $\varepsilon' = \sqrt{2} \cdot \varepsilon$; $\theta_0 = 0$
3: $est_v = 0$ for every $v \in V$
4: for $i = 1$ to $\lfloor \log_2 n \rfloor - 1$
5: $x = n/2^i$
6: $\theta_i = \left\lceil \frac{n \cdot (n^{(\ell+1)\ln n + \ln \log_2 n + \ln 2) \cdot (2 + 2\varepsilon') \cdot (2 + 2\varepsilon')}{\varepsilon^2 \cdot x} \right\rceil$
7: for $j = 1$ to $\theta_i - \theta_{i-1}$
8: generate a random RR set $R_v$ rooted at $v$, and for each $u \in R_v$, record the distance $d_{R_v}(u,v)$ from $u$ to $v$ in this reverse simulation.
9: if estimating individual centrality then
10: for every $u \in R_v$, $est_u = est_u + g(d_{R_v}(u,v))$
11: else
12: {estimating Shapley centrality}
13: for every $u \in R_v$, $est_u = est_u + \phi_u^{\text{Shapley}}(g(d_{R_v}(.v)))$
14: end if
15: end for
16: $est^{(k)} = \text{the } k\text{-th largest value in } \{est_v\}_{v \in V}$
17: if $n \cdot est^{(k)}/\theta_i \geq (1 + \varepsilon') \cdot x$ then
18: $LB = n \cdot est^{(k)}/((\theta_i \cdot (1 + \varepsilon')$)
19: break
20: end if
21: end for
22: $\theta = \left\lceil \frac{n \cdot (n^{(\ell+1)\ln n + \ln 4) (2 + 2\varepsilon') \cdot (2 + 2\varepsilon')}{\varepsilon^2 \cdot LB} \right\rceil$
23: {Phase 2. Estimate the centrality value}
24: $est_v = 0$ for every $v \in V$
25: for $j = 1$ to $\theta$
26: generate a random RR set $R_v$ rooted at $v$, and for each $u \in R_v$, record the distance $d_{R_v}(u,v)$ from $u$ to $v$ in this reverse simulation.
27: if estimating individual centrality then
28: for every $u \in R_v$, $est_u = est_u + g(d_{R_v}(u,v))$
29: else
30: for every $u \in R_v$, $est_u = est_u + \phi_u^{\text{Shapley}}(g(d_{R_v}(.v)))$
31: end if
32: end for
33: for every $v \in V$, $\hat{\psi}_v = n \cdot est_v/\theta$
34: return $\hat{\psi}_v$, $v \in V$

**Algorithm 1: ICE-RR:** Efficient estimation of sphere-of-influence centralities via RR-sets, for the triggering model and additive distance function $f(\vec{d}) = \sum_{u \in V} g(d_u)$.

The algorithm has two phases. In the first phase (lines 1–22), the number $\theta$ of RR sets needed for the estimation is computed. The mechanism for obtaining $\theta$ follows the IMM algorithm in [33]
and is also the same as in [11]. In the second phase (lines 23–34), \( \theta \) RR sets are generated, and for each RR set \( R_v \), the centrality estimate of \( u \in R_v, \text{est}_u \), is updated properly depending on the centrality type.

Comparing to the algorithm in [11], our change is in lines 8–14 and lines 26–31. First, when generating an RR set \( R_v \), we not only stores the nodes, but for each \( u \in R_v \), we also store the distance from \( u \) to root \( v \) in the reverse simulation paths \( d_{R_v}(u, v) \). Technically, \( d_{R_v}(u, v) \) is the graph distance from \( u \) to \( v \) in the subgraph \( G_{R_v} \), where \( G_{R_v} = (V, E_{R_v}) \) with \( E_{R_v} = \{(w, u) \mid u \in R_v, w \in T(u)\} \) is the subgraph generated by the triggering sets sampled during the reverse simulation. Note that with this definition, for \( u \not\in R_v \), we have \( d_{R_v}(u, v) = \infty \). Next, if we are estimating individual centrality, we simply update the estimate \( \text{est}_u \) by adding \( g(d_{R_v}(u, v)) \). If we are estimating Shapley centrality, we need to update \( \text{est}_u \) by adding \( \phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v))) \), the Shapley value of \( u \) on the set function \( g(d_{R_v}(\cdot, v)) : S \in 2^V \mapsto g(d_{R_v}(S, v)) \subset \mathbb{R} \). We will show below that the computation of \( \phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v))) \) for all \( u \in R_v \) together is linear to \( |R_v| \), so it is in the same order of generating \( R_v \) and does not incur significant extra cost. Note that the algorithm in [11] corresponds to our algorithm with \( g = g^{\text{th}} \). The correctness of the algorithm replies on the following crucial lemma.

**Lemma 5.1.** Let \( R_v \) be a random RR set with root \( v \) generated in a triggering model instance \( \mathcal{I} \). Then, \( \forall u \in V, u \text{'s stochastic sphere-of-influence individual centrality with function } f(\bar{d}) = \sum_{u \in V} g(d_u) \text{ is } \psi[f]_{u}(\mathcal{I}) = n \cdot \mathbb{E}[g(d_{R_v}(u, v))], \text{ where the expectation is taking over the distribution of RR set } R_v. \text{ Similarly, } u \text{'s influence-based Shapley centrality with } f \text{ is } \psi^{\text{Shapley}}[f]_{u}(\mathcal{I}) = n \cdot \mathbb{E}[\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v)))].

From Lemma 5.1, we can understand that lines 28 and 30 are simply accumulating empirical values of \( g(d_{R_v}(u, v)) \) and \( \phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v))) \) for individual centrality and Shapley centrality, respectively, and line 33 averages this cumulative value and then multiply it by \( n \) to obtain the final centrality estimate. With the above lemma, the correctness of the algorithm ICE-RR is shown by the following theorem.

**Theorem 5.1.** Let \( (\psi_v)_{v \in V} \) be the true centrality value for an influence-based individual or Shapley centrality with additive function \( f \), and let \( \psi^{(k)} \) be the \( k \)-th largest value in \( (\psi_v)_{v \in V} \). For any \( \epsilon > 0, \ell > 0, \) and \( k \in [n] \), Algorithm ICE-RR returns the estimated centrality \( (\hat{\psi}_v)_{v \in V} \) that satisfies (a) unbiasedness: \( \mathbb{E}[\hat{\psi}_v] = \psi_v, \forall v \in V; \) and (b) robustness: under the condition that \( \psi^{(k)} \geq 1 \), with probability at least \( 1 - \frac{1}{n^\ell} \):

\[
\begin{align*}
\left\{ \begin{array}{ll}
\hat{\psi}_v - \psi_v & \leq \epsilon \psi_v, & \forall v \in V \text{ with } \psi_v > \psi^{(k)}, \\
|\hat{\psi}_v - \psi_v| & \leq \epsilon \psi^{(k)}, & \forall v \in V \text{ with } \psi_v \leq \psi^{(k)}.
\end{array} \right.
\end{align*}
\]

In terms of time complexity, for individual centrality, lines 10 and 28 take constant time for each \( u \in R_v \), so it has the same complexity as the algorithm in [11]. For Shapley centrality, the following lemma shows that the computation of \( \phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v))) \) for all \( v \) is \( O(|R_v|) \), same as the complexity of generating \( R_v \), so it will not add complexity to the overall running time. Suppose \( R_v \) has \( \Delta \) levels in total (i.e., \( \Delta = \max\{d_{R_v}(u', v) \mid u' \in R_v\} \)), and let \( s_i = |\{u' \mid u' \in R_v, d_{R_v}(u', v) \geq i\}| \).

**Lemma 5.2.** For any function \( g : \mathbb{R}_\infty \rightarrow \mathbb{R} \) with \( g(\infty) = 0 \),

\[
\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v))) = \frac{1}{|R_v|} g(k) + \frac{1}{|R_v|!} \sum_{k < i \leq \Delta} (g(k) - g(i)) \left( \sum_{0 \leq j \leq s_i} \frac{s_i!}{j!} - \sum_{0 \leq j \leq s_{i+1}} \frac{s_{i+1}!}{j!} \right).
\]
In $O(R_v)$ time we can compute this value for all nodes in $R_v$ (assuming infinite precision). For degree centrality, \( \phi_{\text{Shapley}}^{\text{deg}}(g^{\text{deg}}(d_{R_v}(:,v))) = 1/|\{w \mid d_{R_v}(w,v) = 1\}| \) if \( d_{R_v}(u,v) = 1 \), and otherwise it is 0. For reachability centrality, \( \phi_{\text{Shapley}}^{\text{rch}}(g^{\text{rch}}(d_{R_v}(:,v))) = 1/|R_v| \).

Therefore, the time complexity follows [11]:

**Theorem 5.2.** Under the assumption that sampling a triggering set \( T(v) \) takes time at most $O(|N^-(v)|)$ time, and the condition \( \ell \geq (\log_2 k - \log_2 \log_2 n)/\log_2 n \), the expected running time of \( \text{ICE-RR} \) is

\[
O(\ell(m + n) \log n \cdot E[\sigma(\tilde{v})]/(\psi(k)\varepsilon^2)),
\]

where \( E[\sigma(\tilde{v})] \) is the expected influence spread of a random node \( \tilde{v} \) drawn from \( V \) with probability proportional to the in-degree of \( \tilde{v} \).

Theorems 5.1 and 5.2 together show that our algorithm \( \text{ICE-RR} \) provides a framework to efficiently estimate all individual and Shapley centralities in the family of influence-based stochastic sphere-of-influence centralities. We further remark that, although algorithm \( \text{ICE-RR} \) is shown for computing individual centralities and Shapley centralities, it can be easily adapted to computing group centralities as well. Of course, a group centrality has \( 2^n \) values, so it is not feasible to list all of them. But if we consider that the algorithm is to estimate \( n \) group centrality values for \( n \) given sets, then we only need to replace \( \text{est}_u \) with \( \text{est}_S \) for every \( S \) in the input, and change the lines corresponding to individual centrality (lines 10 and 28) to “for each \( S \) in the input, \( \text{est}_S = \text{est}_S + g(d_{R_v}(S,v)) \)”. This change is enough for estimating \( n \) group centrality values.

### 6 Future Work

Many topics concerning the interaction between network centralities and influence dynamics can be further explored. One open question is how to extend other centralities that are not covered by sphere-of-influence to influence-based centralities. For example, betweenness centrality of a node \( v \) is determined not only by the distance from \( v \) to other nodes, but by all-pair distances, while PageRank and other eigenvalue centralities are determined by the entire graph structure. Therefore, one may need to capture further aspects of the influence propagation to provide natural extensions to these graph-theoretical centralities. Another open question is how to characterize centrality for a class of influence profiles, e.g. all submodular influence profiles, all triggering models, etc. Empirical comparisons of different influence-based centralities, as well as studying the applications that could utilize influence-based centralities, are all interesting and important topics worth further investigation.
References

[1] Alon Altman and Moshe Tennenholtz. Ranking systems: The pagerank axioms. In ACM, EC ’05, pages 1–8, 2005.

[2] K. J. Arrow. Social Choice and Individual Values. Wiley, New York, 2nd edition, 1963.

[3] Shishir Bharathi, David Kempe, and Mahyar Salek. Competitive influence maximization in social networks. In WINE, volume 4858, pages 306–311, 2007.

[4] P. Boldi and S. Vigna. Axioms for centrality. Internet Mathematics, 10:222–262, 2014.

[5] Phillip Bonacich. Factoring and weighting approaches to status scores and clique identification,. Journal of Mathematical Sociology, 2:113–120, 1972.

[6] Phillip Bonacich. Power and centrality: A family of measures. American Journal of Sociology, 92(5):1170–1182, 1987.

[7] Stephen P. Borgatti. Identifying sets of key players in a social network. Computational and Mathematical Organizational Theory, 12:21–34, 2006.

[8] Christian Borgs, Michael Brautbar, Jennifer Chayes, and Brendan Lucier. Maximizing social influence in nearly optimal time. In ACM-SIAM, SODA ’14, pages 946–957, 2014.

[9] Sergey Brin and Lawrence Page. The anatomy of a large-scale hypertextual web search engine. Computer Networks, 30(1-7):107–117, 1998.

[10] Wei Chen, Laks V. S. Lakshmanan, and Carlos Castillo. Information and Influence Propagation in Social Networks. Synthesis Lectures on Data Management. Morgan & Claypool Publishers, 2013.

[11] Wei Chen and Shang-Hua Teng. Interplay between social influence and network centrality: A comparative study on shapley centrality and single-node-influence centrality. In WWW, April 2017. The full version appears in arXiv:1602.03780.

[12] Wei Chen, Chi Wang, and Yajun Wang. Scalable influence maximization for prevalent viral marketing in large-scale social networks. In KDD, pages 1029–1038, 2010.

[13] Wei Chen, Yifei Yuan, and Li Zhang. Scalable influence maximization in social networks under the linear threshold model. In ICDM, pages 88–97, 2010.

[14] Wei Chen, Yifei Yuan, and Li Zhang. Scalable influence maximization in social networks under the linear threshold model. In ICDM, pages 88–97, 2010.

[15] Pedro Domingos and Matt Richardson. Mining the network value of customers. In ACM, KDD ’01, pages 57–66, 2001.

[16] Pedro Domingos and Matthew Richardson. Mining the network value of customers. In KDD, pages 57–66, 2001.

[17] M. G. Everett and S. P. Borgatti. The centrality of groups and classes. Journal of Mathematical Sociology, 23(3):181–201, 1999.
[18] Rumi Ghosh, Shang-Hua Teng, Kristina Lerman, and Xiaoran Yan. The interplay between dynamics and networks: centrality, communities, and cheeger inequality. In ACM, KDD ’14, pages 1406–1415, 2014.

[19] Daniel Gómez, Enrique Gonzalez-Arangüena, Conrado Manuel, Guillermo Owen, Monica del Pozo, and Juan Tejada. Centrality and power in social networks: a game theoretic approach. *Mathematical Social Sciences*, 46(1):27–54, 2003.

[20] Leo Katz. A new status index derived from sociometric analysis. *Psychometrika*, 18(1):39–43, March 1953.

[21] David Kempe, Jon M. Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. In *KDD*, pages 137–146, 2003.

[22] David Kempe, Jon M. Kleinberg, and Éva Tardos. Influential nodes in a diffusion model for social networks. In *ICALP*, pages 1127–1138, 2005.

[23] Jure Leskovec, Andreas Krause, Carlos Guestrin, Christos Faloutsos, Jeanne M. VanBriesen, and Natalie S. Glance. Cost-effective outbreak detection in networks. In *KDD*, pages 420–429, 2007.

[24] Tomasz P. Michalak, Karthik V. Aadithya, Piotr L. Szczepanski, Balaraman Ravindran, and Nicholas R. Jennings. Efficient computation of the shapley value for game-theoretic network centrality. *J. Artif. Int. Res.*, 46(1):607–650, January 2013.

[25] Mark Newman. *Networks: An Introduction*. Oxford University Press, Inc., New York, NY, USA, 2010.

[26] U.J. Nieminen. On the centrality in a directed graph. *Social Science Research*, 2(4):371–378, 1973.

[27] Ignacio Palacios-Huerta and Oscar Volij. The measurement of intellectual influence. *Econometrica*, 72:963–977, 2004.

[28] Matthew Richardson and Pedro Domingos. Mining knowledge-sharing sites for viral marketing. In ACM, KDD ’02, pages 61–70, 2002.

[29] Matthew Richardson and Pedro Domingos. Mining knowledge-sharing sites for viral marketing. In *KDD*, pages 61–70, 2002.

[30] G. Sabidussi. The centrality index of a graph. *Psychometrika*, 31(4):581–603, 1966.

[31] D. Schoch and U. Brandes. Re-conceptualizing centrality in social networks. *European Journal of Applied Mathematics*, 27:971–985, 2016.

[32] Lloyd. S. Shapley. A value for n-person games. In H. Kuhn and A. Tucker, editors, *Contributions to the Theory of Games, Volume II*, pages 307–317. Princeton University Press, 1953.

[33] Youze Tang, Yanchen Shi, and Xiaokui Xiao. Influence maximization in near-linear time: a martingale approach. In SIGMOD, pages 1539–1554, 2015.
[34] Youze Tang, Xiaokui Xiao, and Yanchen Shi. Influence maximization: near-optimal time complexity meets practical efficiency. In *SIGMOD*, pages 75–86, 2014.

[35] Chi Wang, Wei Chen, and Yajun Wang. Scalable influence maximization for independent cascade model in large-scale social networks. *DMKD*, 25(3):545–576, 2012.
Appendix

A Omitted Proofs

For convenience, we restate the theorems and lemmas in this section before the proofs.

**Proposition 3.1.** ∀ influence profile $\mathcal{I} = (V, E, P_I)$:

$$\text{SNI} (\mathcal{I}) = \psi^{\text{ind}} [ f^{\text{rch}} ] (\mathcal{I}).$$

**Proof.** For any $v \in V$,

$$\text{SNI}_v (\mathcal{I}) = \sigma_{\mathcal{I}} (v) = \mathbb{E}_{(S_1, \ldots, S_{n-1}) \sim P_I} \sum_{u \in V} \mathbb{I} [d_u \leq \infty] = \mathbb{E}_{(S_1, \ldots, S_{n-1}) \sim P_I} f^{\text{rch}} (\tilde{d} (\{v\}, S_1, \ldots, S_{n-1})) = \psi^{\text{SNSSol}} [ f^{\text{rch}} ] (\mathcal{I}).$$

\[ \Box \]

**Proposition 3.2.** For any function $f : \mathbb{R}^n_{\infty} \to \mathbb{R}$, $\psi^{\text{ind}} [ f ]$ conforms with $\mu^{\text{ind}} [ f ]$.

**Proof.** For any directed graph $G = (V, E)$ and any set $v \in V$, let $(\{v\}, S_1^v, \ldots, S_{n-1}^v)$ be the BFS sequence starting from $v$ in $G$. Then we have

$$\psi [ f ]_v (I_G) = \mathbb{E}_{(S_1, \ldots, S_{n-1}) \sim P_{I_G}} [ f (\tilde{d} (\{v\}, S_1, \ldots, S_{n-1}))] = f (\tilde{d} (\{v\}, S_1^v, \ldots, S_{n-1}^v)) = f (\tilde{d}_G (v)) = \mu [ f ]_v (G).$$

Thus the lemma holds. \[ \Box \]

**Theorem 2.1** (Graph-Theoretical Basis). The set of vectors corresponding to the nontrivial layered-graph instances in $\mathcal{L}$ forms a basis in $\mathbb{R}^M$.

**Proof.** All we have to show is independence since $|\mathcal{L}| = M$. Suppose not, i.e., there is a nontrivial group of $\{\lambda(R_0, \ldots, R_t)\}$ such that

$$\sum_{R_0, \ldots, R_t : I(R_0, \ldots, R_t) \in \mathcal{L}} \lambda(R_0, \ldots, R_t) P_{I(R_0, \ldots, R_t)} = 0.$$

Let $I(R_0, \ldots, R_t)$ be a layered-graph instance

- Such that $\lambda(R_0, \ldots, R_t) \neq 0$;
- Among those satisfying the condition above, with the largest number of layers (i.e. $t^*$);
- Among those satisfying the conditions above, with the largest number of vertices in the first layer (i.e. $|R_0^*|$).
Note that fixing the seed set, the propagation on a layered-graph instance is deterministic. That is, there is exactly one cascading sequence with the fixed seed set, which happens with probability 1. Let \( Seq_{I(R_0, \ldots, R_t)}(S_0) \) be the unique BFS sequence which happens on \( I(R_0, \ldots, R_t) \) with seed set \( S_0 \). We show that

\[
\sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} \sum_{R_0, \ldots, R_t: I(R_0, \ldots, R_t) \in \mathcal{L}} \lambda(R_0, \ldots, R_t) P_{I(R_0, \ldots, R_t)}(Seq_{I(R_0^*, \ldots, R_t^*)}(S_0)) \neq 0,
\]

which contradicts the assumption of non-independence and thereby concludes the proof.

We now compute the left hand side of the above formula.

\[
\begin{align*}
\sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} & \sum_{R_0, \ldots, R_t: I(R_0, \ldots, R_t) \in \mathcal{L}} \lambda(R_0, \ldots, R_t) P_{I(R_0, \ldots, R_t)}(Seq_{I(R_0^*, \ldots, R_t^*)}(S_0)) \\
= & \sum_{R_0, \ldots, R_t: I(R_0, \ldots, R_t) \in \mathcal{L}} \lambda(R_0, \ldots, R_t) \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} P_{I(R_0, \ldots, R_t)}(Seq_{I(R_0^*, \ldots, R_t^*)}(S_0)) \\
= & \sum_{t < t^*, R_0, \ldots, R_t: I(R_0, \ldots, R_t) \in \mathcal{L}} \lambda(R_0, \ldots, R_t) \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} P_{I(R_0, \ldots, R_t)}(Seq_{I(R_0^*, \ldots, R_t^*)}(S_0)) \\
+ & \sum_{t > t^*, R_0, \ldots, R_t: I(R_0, \ldots, R_t) \in \mathcal{L}} \lambda(R_0, \ldots, R_t) \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} P_{I(R_0, \ldots, R_t)}(Seq_{I(R_0^*, \ldots, R_t^*)}(S_0)) \\
= & \sum_{R_0, \ldots, R_{t^*}: I(R_0, \ldots, R_{t^*}) \in \mathcal{L}} \lambda(R_0, \ldots, R_{t^*}) \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} P_{I(R_0, \ldots, R_{t^*})}(Seq_{I(R_0^*, \ldots, R_{t^*})}(S_0)) \\
= & \sum_{R_0: R_0 \cap R_0^* \neq \emptyset, I(R_0, R_1^*, \ldots, R_{t^*}) \in \mathcal{L}} \lambda(R_0, R_1^*, \ldots, R_{t^*}) \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} P_{I(R_0, R_1^*, \ldots, R_{t^*})}(Seq_{I(R_0^*, \ldots, R_{t^*})}(S_0)).
\end{align*}
\]

Now consider the summand in the last line of the above equation. For \( R_0 \neq R_0^* \),

\[
\begin{align*}
& \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+\left|S_0\right|} P_{I(R_0, R_1^*, \ldots, R_{t^*})}(Seq_{I(R_0^*, \ldots, R_{t^*})}(S_0)) \\
= & \sum_{\emptyset \neq X \subseteq R_0^* \cap R_0} \sum_{Y \subseteq R_0^* \setminus R_0} (-1)^{1+\left|X\right|+\left|Y\right|} P_{I(R_0, R_1^*, \ldots, R_{t^*})}(Seq_{I(R_0^*, \ldots, R_{t^*})}(X \cup Y)) \\
+ & \sum_{\emptyset \neq Y \subseteq R_0^* \setminus R_0} (-1)^{1+\left|Y\right|} P_{I(R_0, R_1^*, \ldots, R_{t^*})}(Seq_{I(R_0^*, \ldots, R_{t^*})}(Y)) \\
= & \sum_{\emptyset \neq X \subseteq R_0^* \cap R_0} \sum_{Y \subseteq R_0^* \setminus R_0} (-1)^{1+\left|X\right|} (-1)^{\left|Y\right|} \\
= & \sum_{\emptyset \neq X \subseteq R_0^* \cap R_0} (-1)^{1+\left|Y\right|} \times 0 \\
= & 0.
\end{align*}
\]
Thus the lemma holds.

Proposition 3.3. If a function \( f : \mathbb{R}^n_\infty \to \mathbb{R} \) is anonymous — i.e., \( f(\vec{d}) \) is permutation-invariant — then \( \psi^{\text{ind}}[f] \) (as defined in Definition 3.3) satisfies Axiom Anonymity and \( \mu^{\text{ind}}[f] \) (as defined in Definition 3.2) satisfies the graph-theoretical counterpart of Axiom Anonymity.

Proof.

\[
\psi^{\text{ind}}[f]_{\pi(v)}(\pi(I)) = E(S_1, \ldots, S_{n-1} \sim P_{\pi(I)}((\pi(v))))[f(\vec{d}(\{v\}, \pi(S_1), \ldots, \pi(S_{n-1})))]
\]

\[
= E(S_1, \ldots, S_{n-1} \sim P_{\pi(I)}((\pi(v))))[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))]
\]

\[
= \psi^{\text{ind}}[f]_v(I).
\]

Proposition 3.4. For any function \( f : \mathbb{R}^n_\infty \to \mathbb{R} \), \( \psi^{\text{ind}}[f] \) satisfies Axiom Bayesian.

Proof. Suppose \( \psi[f] \) is an influence-based distance-function centrality measure. Let instances \( I, I_1, I_2 \) be that \( P_I = \alpha P_{I_1} + (1 - \alpha) P_{I_2} \). Then for every sequence \((S_1, \ldots, S_{n-1})\) drawn from distribution \( P_I(\{v\}) \), it is equivalently drawn with probability \( \alpha \) from \( P_{I_1}(\{v\}) \), and with probability \( 1 - \alpha \) from \( P_{I_2}(\{v\}) \). Therefore,

\[
\psi[f]_{\pi(I)} = E(S_1, \ldots, S_{n-1} \sim P_{\pi(I)}((\pi(v))))[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))]
\]

\[
= \alpha \cdot E(S_1, \ldots, S_{n-1} \sim P_{\pi(I)}((\pi(v))))[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))]
\]

\[
+ (1 - \alpha) \cdot E(S_1, \ldots, S_{n-1} \sim P_{\pi(I)}((\pi(v))))[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))]
\]

\[
= \alpha \cdot \psi[f]_v(I_1) + (1 - \alpha) \cdot \psi[f]_v(I_2).
\]

Thus the lemma holds.
The proof of Theorem 3.1 relies on a general lemma about linear mapping as given in [11], as restated below.

Lemma A.1 (Lemma 11 of [11]). Let \( \psi \) be a mapping from a convex set \( D \subseteq \mathbb{R}^M \) to \( \mathbb{R}^n \) satisfying that for any vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s \in D \), for any \( \alpha_1, \alpha_2, \ldots, \alpha_s \geq 0 \) and \( \sum_{i=1}^{s} \alpha_i = 1 \), \( \psi(\sum_{i=1}^{s} \alpha_i \cdot \vec{v}_i) = \sum_{i=1}^{s} \alpha_i \cdot \psi(\vec{v}_i) \). Suppose that \( D \) contains a set of linearly independent basis vectors of \( \mathbb{R}^M \), \( \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_M\} \) and also vector \( \vec{0} \). Then for any \( \vec{v} \in D \), which can be represented as \( \vec{v} = \sum_{i=1}^{M} \lambda_i \cdot \vec{b}_i \) for some \( \lambda_1, \lambda_2, \ldots, \lambda_M \in \mathbb{R} \), we have

\[
\psi(\vec{v}) = \psi \left( \sum_{i=1}^{M} \lambda_i \cdot \vec{b}_i \right) = \sum_{i=1}^{M} \lambda_i \cdot \psi(\vec{b}_i) + \left( 1 - \sum_{i=1}^{M} \lambda_i \right) \cdot \psi(\vec{0}).
\]

Theorem 3.1. A Bayesian influence-based centrality measure is uniquely determined by its values on layered-graph instances (including the null instance).

Proof of Theorem 3.1. By the definition of the null influence instance \( I^N \) (same as the trivial layered-graph instance \( I_V(V) \)), we can see that the vector representation of the null influence instance is the all 0 vector, because the entries corresponding to \( P_I^N(S_0, S_0, \ldots, S_0) \) are not included in the vector by definition. Then by Theorem 2.1 and the Lemma A.1, we know that for any Bayesian centrality measure \( \psi \), its value on any influence instance \( I \), \( \psi(I) \), can be represented as a linear combination of the \( I \)'s values on layered-graph instances (including the null instance). Thus, the theorem holds.

Theorem 3.2 (Characterization of Individual Centrality). For any anonymous function \( f : \mathbb{R}_\infty^n \rightarrow \mathbb{R} \), \( \psi^{\text{ind}}[f] \) (defined in Definition 3.3) is the unique influence-based centrality that conforms with \( \mu^{\text{ind}}[f] \) (defined in Definition 3.2) that satisfies both Axiom Anonymity and Axiom Bayesian.

Proof. The theorem follows directly from Propositions 3.4, 3.3 and 3.2.

Theorem 4.1 (Characterization of Group Centrality). For any anonymous function \( f : \mathbb{R}_\infty^n \rightarrow \mathbb{R} \), \( \psi^{\text{grp}}[f] \) is the unique influence-based group centrality that conforms with \( \mu^{\text{grp}}[f] \) and satisfies both Axiom Anonymity and Axiom Bayesian.

Proof. The Bayesian part of the proof is essentially the same as the proof of Proposition 3.4, with subset \( S \) replacing node \( v \). The Anonymity part of the proof is essentially the same as the proof of Proposition 3.3, with subset \( S \) replacing node \( v \). The conformity part of the proof is essentially the same as the proof of Proposition 3.2, with subset \( S \) replacing node \( v \).

Theorem 4.2 (Characterization of Shapley Centrality). For any anonymous function \( f : \mathbb{R}_\infty^n \rightarrow \mathbb{R} \), \( \psi^{\text{Shapley}}[f] \) is the unique influence-based centrality that conforms with \( \mu^{\text{Shapley}}[f] \) and satisfies both Axiom Anonymity and Axiom Bayesian.

Proof. We first show that \( \psi^{\text{Shapley}}[f] \) is Bayesian. Let \( I, I_1, I_2 \) be three influence instances with the same vertex set \( V \), and \( \alpha \in [0, 1] \), where \( P_Z = \alpha P_{I_1} + (1 - \alpha) P_{I_2} \). Then for every node \( v \in V \),
we have

\[
\psi^{\text{Shapley}}(\mathcal{I}) = \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I})) \\
= \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\alpha P_1 + (1 - \alpha) P_2)) \\
= \phi_v^{\text{Shapley}}(\alpha \psi^{\text{grp}}(\mathcal{I}_1) + (1 - \alpha) \psi^{\text{grp}}(\mathcal{I}_2)) \\
= \alpha \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I}_1)) + (1 - \alpha) \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I}_2))
\]

\[(4)\]

where Eq. (4) is due to the linearity of the Shapley value, which is easy to verify by the following derivations:

\[
\phi_v^{\text{Shapley}}(\alpha \tau_1 + \beta \tau_2) = \mathbb{E}_{\pi \sim \Pi}[\{\alpha \tau_1 + \beta \tau_2\}(S_{\pi,v} \cup \{v\}) - (\alpha \tau_1 + \beta \tau_2)(S_{\pi,v})] \\
= \alpha \mathbb{E}_{\pi \sim \Pi}[\tau_1(S_{\pi,v} \cup \{v\}) - \tau_1(S_{\pi,v})] + \beta \mathbb{E}_{\pi \sim \Pi}[\tau_2(S_{\pi,v} \cup \{v\}) - \tau_2(S_{\pi,v})] \\
= \alpha \phi_v^{\text{Shapley}}(\tau_1) + \beta \phi_v^{\text{Shapley}}(\tau_2).
\]

Now we show that \( \psi^{\text{Shapley}}[f] \) conforms with \( \mu^{\text{Shapley}}[f] \). For any directed graph \( G = (V,E) \) and any node \( v \in V \), let \( \{v\}, S_1, \ldots, S_{n-1} \) be the BFS sequence starting from \( v \) in graph \( G \). We have

\[
\psi^{\text{Shapley}}[f]_{\nu}(\mathcal{I}_G) = \phi_v^{\text{Shapley}}(\psi^{\text{grp}}[f](\mathcal{I}_G)) \\
= \mathbb{E}_{\pi \sim \Pi}[\psi^{\text{grp}}[f]|_{S_{\pi,v} \cup \{v\}}(\mathcal{I}_G) - \psi^{\text{grp}}[f]|_{S_{\pi,v}}(\mathcal{I}_G)] \\
= \mathbb{E}_{\pi \sim \Pi}[\mu^{\text{grp}}[f]|_{S_{\pi,v} \cup \{v\}}(G) - \mu^{\text{grp}}[f]|_{S_{\pi,v}}(G)] \\
= \phi_v^{\text{Shapley}}(\mu^{\text{grp}}[f](G)) \\
= \mu^{\text{Shapley}}[f](G),
\]

where Eq. (5) is because influence-based group centrality \( \psi^{\text{grp}}[f] \) conforms with the structure-based group centrality \( \mu^{\text{grp}}[f] \) (Theorem 4.1). Anonymity follows from anonymity of group centralities (Theorem 4.1). Uniqueness then follows from Theorem 3.1. \( \square \)

**Lemma A.2.** For fixed nodes \( u, w \in V \), suppose we generate a random RR set \( R_w \) rooted at \( w \), according to a triggering model instance \( \mathcal{I} \). Then we have

\[
\mathbb{E}[g(d_{R_w}(u, w))] = \mathbb{E}_{(S_1, \ldots, S_{n-1}) \sim P_2(u)}[g(d_{R_w}(\{u\}, S_1, \ldots, S_{n-1}))].
\]

**Proof.** We know that the triggering model is equivalent to the following live-edge graph model [21]: for every node \( v \in V \), sample its triggering set \( T(v) \) and add edges \((u, v)\) to a live-edge graph \( L \) for all \( u \in T(v) \) (these edges are called live edges). Then the diffusion from a seed set \( S \) is the same as the BFS propagation in \( L \) from \( S \). Since reverse simulation for generating RR set \( R_w \) also do the same sampling of the triggering sets, we can couple the reverse simulation process with the forward propagation by fixing a live-edge graph \( L \). For a fixed live-edge graph \( L \), the subgraph \( G_{R_w} \) generated by reverse simulation from the fixed root \( w \) is simply the induced subgraph of \( L \) induced by all nodes that can reach \( w \) in \( L \). Thus \( d_{R_w}(u, w) \) is the fixed distance from \( u \) to \( w \) in \( L \), namely \( d_L(u, w) \). On the other hand, with the fixed \( L \), the cascading sequence starting from \( u \) is the fixed BFS sequence starting from \( u \) in \( L \). Then in this BFS sequence \( d_w(\{u\}, S_1, \ldots, S_{n-1}) \) is the distance from \( u \) to \( w \) in this sequence, which is the same as the distance from \( u \) to \( w \) in
Consider first the influence-based distance-function centrality. We have

\[ E[R(u, w)] = E(S_{\ell - 1}) = E[R(u, w)] \]

Theorem 5.1. Let \( R_v \) be a random RR set with root \( v \) generated in a triggering model instance \( I \). Then, \( \forall u \in V \), u’s stochastic sphere-of-influence individual centrality with function \( f(\vec{d}) = \sum_{v \in V} g(d_u) \) is \( \psi(f)_u(I) = n \cdot E[g(d_{R_v}(u,v))] \), where the expectation is taking over the distribution of RR set \( R_v \). Similarly, u’s influence-based Shapley centrality with f is \( \psi_{\text{Shapley}}[f]_u(I) = n \cdot E[\phi^\text{Shapley}_u(g(d_{R_v}(\cdot,v)))]. \)

Proof. Consider first the influence-based distance-function centrality. We have

\[
n \cdot E[g(d_{R_v}(u,v))] = n \cdot \sum_{w \in V} \Pr\{w = v\} E[g(d_{R_v}(u,v)) | w = v]
\]

\[
= n \cdot \sum_{w \in V} \frac{1}{n} E[g(d_{R_v}(u,w))]
\]

\[
= \sum_{w \in V} E(S_{\ell - 1}) = P_2(\{u\}) [g(d_w(\{u\}, S_{\ell - 1}))]
\]

\[
= E(S_{\ell - 1}) = P_2(\{u\}) [f(\vec{d}(\{u\}, S_{\ell - 1}))]
\]

\[
= \psi(f)_u(I),
\]

where Eq. (6) is by Lemma A.2.

Next consider the influence-based distance-function Shapley centrality.

\[
n \cdot E[\phi^\text{Shapley}_u(g(d_{R_v}(\cdot,v)))] = n \cdot \sum_{w \in V} \Pr\{w = v\} E[\phi^\text{Shapley}_u(g(d_{R_v}(\cdot,v))) | w = v]
\]

\[
= n \cdot \sum_{w \in V} \frac{1}{n} E[\phi^\text{Shapley}_u(g(d_{R_v}(\cdot,w)))]
\]

\[
= \phi^\text{Shapley}_u \left( \sum_{w \in V} E[g(d_{R_v}(\cdot,w))] \right)
\]

\[
= \phi^\text{Shapley}_u \left( \psi^\text{grd}[f](I) \right)
\]

\[
= \psi^\text{Shapley}[f]_u(I),
\]

where Eq. (7) is by the linearity of the Shapley value, as already argued in the proof of Theorem 4.1, and Eq. (8) follows the similar derivation step as in the case of individual centrality above.

Theorem 5.1. Let \( (\psi_v)_{v \in V} \) be the true centrality value for an influence-based individual or Shapley centrality with additive function f, and let \( \psi^{(k)} \) be the k-th largest value in \( (\psi_v)_{v \in V} \). For any \( \epsilon > 0, \ell > 0, \) and \( k \in [n] \), Algorithm ICE-RR returns the estimated centrality \( (\hat{\psi}_v)_{v \in V} \) that satisfies (a)
unbiasedness: $\mathbb{E}[\hat{\psi}_v] = \psi_v, \forall v \in V$; and (b) robustness: under the condition that $\psi^{(k)} \geq 1$, with probability at least $1 - \frac{1}{n^2}$;

$$\left\{ \begin{array}{ll} \left| \hat{\psi}_v - \psi_v \right| \leq \varepsilon \psi_v & \forall v \in V \text{ with } \psi_v > \psi^{(k)}, \\
\hat{\psi}_v - \psi_v \leq \varepsilon \psi^{(k)} & \forall v \in V \text{ with } \psi_v \leq \psi^{(k)}. \end{array} \right.$$

Proof. The proof follows exactly the same proof structure as the proof in [11]. All we need to change is to use our crucial lemma connecting RR sets with centrality measures (Lemma 5.1) to replace the corresponding lemma (Lemma 23) in [11].

Lemma 5.2. For any function $g : \mathbb{R}_\infty \to \mathbb{R}$ with $g(\infty) = 0$,

$$\phi_u^{\text{Shapley}}(g(d_{R_u}(\cdot, v))) = \frac{1}{|R_u|}g(k) + \frac{1}{|R_u|!} \sum_{k \leq i \leq \Delta} (g(k) - g(i)) \left( \sum_{0 \leq j \leq s_i} \frac{s_i^!}{j^!} - \sum_{0 \leq j \leq s_{i+1}} \frac{s_{i+1}^!}{j^!} \right).$$

In $O(|R_u|)$ time we can compute this value for all nodes in $R_u$ (assuming infinite precision). For degree centrality, $\phi_u^{\text{Shapley}}(g^{\text{deg}}(d_{R_u}(\cdot, v))) = 1/|\{ w \mid d_{R_u}(w, v) = 1 \}|$ if $d_{R_u}(u, v) = 1$, and otherwise it is 0. For reachability centrality, $\phi_u^{\text{Shapley}}(g^{\text{rch}}(d_{R_u}(\cdot, v))) = 1/|R_u|.$

Proof. We prove only for general $g$. For $u$ at the $k$-th level,

$$\phi_u^{\text{Shapley}}(g(d_{R_u}(\cdot, v))) = \frac{1}{|R_u|}g(k) + \frac{1}{|R_u|!} \sum_{k \leq i \leq \Delta} (g(k) - g(i)) \sum_{1 \leq j \leq |R_u|} \left[ \binom{s_i}{j} (j - 1)! - \binom{s_{i+1}}{j} (j - 1)! \right]$$

$$= \frac{1}{|R_u|}g(k) + \frac{1}{|R_u|!} \sum_{k \leq i \leq \Delta} (g(k) - g(i)) \left( \sum_{1 \leq j \leq s_i} \frac{s_i^!}{(s_i - j + 1)!} - \sum_{1 \leq j \leq s_{i+1}} \frac{s_{i+1}^!}{(s_{i+1} - j + 1)!} \right)$$

One possible way to compute the value above is:

1. Compute $x!$ and $\sum_{0 \leq i \leq x} \frac{1}{i^!}$ for all $x \in |R_u|$ in time $O(|R_u|)$.

2. Now the term $\sum_{0 \leq j \leq s_i} \frac{s_i^!}{j^!}$ can be computed in additional constant time for any $i$. We can compute

$$\frac{1}{|R_u|!} \sum_{k \leq i \leq \Delta} (g(k) - g(i)) \left( \sum_{0 \leq j \leq s_i} \frac{s_i^!}{j^!} - \sum_{0 \leq j \leq s_{i+1}} \frac{s_{i+1}^!}{j^!} \right).$$
for all $0 \leq k \leq \Delta$ in additional total time $O(\Delta)$ by first computing a suffix sum of

$$g(i) \left( \sum_{0 \leq j \leq s_i} \frac{s_i!}{j!} - \sum_{0 \leq j \leq s_{i+1}} \frac{s_{i+1}!}{j!} \right).$$

That is,

$$\sum_{k<i \leq \Delta} g(i) \left( \sum_{0 \leq j \leq s_i} \frac{s_i!}{j!} - \sum_{0 \leq j \leq s_{i+1}} \frac{s_{i+1}!}{j!} \right),$$

for all $k \in \{0, \ldots, \Delta\}$.

3. Assign the values to vertices in each level in total time $O(|R_v|)$. 

\(\square\)