ON 4-DIMENSIONAL LORENTZIAN AFFINE HYPERSURFACES WITH AN ALMOST SYMPLECTIC FORM

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Abstract. In this paper we study 4-dimensional affine hypersurfaces with a Lorentzian second fundamental form additionally equipped with an almost symplectic structure $\omega$. We prove that the rank of the shape operator is at most one if $R^k \cdot \omega = 0$ or $\nabla^k \omega = 0$ for some positive integer $k$. This result is the final step in a classification of Lorentzian affine hypersurfaces with higher order parallel almost symplectic forms.

1. Introduction

Parallel structures are of great interest in classical Riemannian geometry (see [7, 17, 11]) as well as in affine differential geometry ([2, 9, 13, 15, 12, 14]). Higher order parallel structures are natural generalization of parallel structures and are widely studied as well ([7, 8, 25, 26, 24]). There exist also some classification results in context of induced almost contact and almost paracontact structures ([22, 23]).

On the other hand O. Baues and V. Cortés studied affine hypersurfaces equipped with an almost complex structure ([3]). They proved that every simply connected special Kähler manifold ([11]) can be realized in a canonical way as an improper affine hypersphere. In 2006 V. Cortés together with M.-A. Lawn and L. Schäfer ([5]) proved a similar result for special para-Kähler manifolds ([6]). Such hyperspheres were called by the authors special affine hyperspheres. In both cases an important role was played by the Kählerian (resp. para-Kählerian) symplectic form $\omega$. Later special affine hyperspheres were generalized by the first author in [21]. These results show that there are interesting relations between symplectic (in particular Kähler and para-Kähler) geometry and affine differential geometry.

Motivated by the above results as well as M. Kon results ([16]) the first author studied affine hypersurfaces $f: M \to \mathbb{R}^{2n+1}$ with a transversal vector field $\xi$ additionally equipped with an almost symplectic structure $\omega$. In [19] the following result was obtained:

Theorem 1.1 ([19]). Let $f: M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface with a transversal vector field $\xi$ and an almost symplectic form $\omega$. Equality $R(X,Y)\omega = 0$ for every $X,Y \in \mathcal{X}(M)$ holds if and only if $\dim M = 2$ and $\xi$ is locally equiaffine or $\dim M \geq 4$ and $\nabla$ is flat.

In the case when the second fundamental form is positive definite and the transversal vector field $\xi$ is locally equiaffine the above theorem generalizes to an arbitrary power of $R$. Namely, we have

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Theorem 1.2 ([19]). Let $f: M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface ($\dim M \geq 4$) with a locally equiaffine transversal vector field $\xi$ and an almost symplectic form $\omega$. Additionally assume that the second fundamental form is positive definite on $M$. If $R^l \omega = 0$ for some positive integer $l$ then $\nabla$ is flat.

As a consequence of the above theorem we obtain

Theorem 1.3 ([19]). Let $f: M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface ($\dim M \geq 4$) with a locally equiaffine transversal vector field $\xi$ and an almost symplectic form $\omega$. Additionally assume that the second fundamental form is positive definite on $M$. If $\nabla^k \omega = 0$ for some positive integer $k$ then $\nabla$ is flat.

Later in [20] it was shown that although the above theorems are not true in general when the second fundamental form is Lorentzian, we still have strong constrains on the shape operator if only $\dim M \geq 6$. Namely we have the following theorems:

Theorem 1.4 ([20]). Let $f: M \to \mathbb{R}^{2n+1}$ ($\dim M \geq 6$) be a non-degenerate affine hypersurface with a locally equiaffine transversal vector field $\xi$ and an almost symplectic form $\omega$. If $R^k \omega = 0$ for some $k \geq 1$ and the second fundamental form is Lorentzian on $M$ (that is has signature $(2n-1,1)$) then the shape operator $S$ has the rank $\leq 1$.

Theorem 1.5 ([20]). Let $f: M \to \mathbb{R}^{2n+1}$ ($\dim M \geq 6$) be a non-degenerate affine hypersurface with a locally equiaffine transversal vector field $\xi$ and an almost symplectic form $\omega$. If $\nabla^k \omega = 0$ for some $k \geq 1$ and the second fundamental form is Lorentzian on $M$ (that is has signature $(2n-1,1)$) then the shape operator $S$ has the rank $\leq 1$.

The main purpose of this paper is to prove that Theorem 1.4 and Theorem 1.5 hold also for 4-dimensional affine hypersurfaces. Although some results obtained in [20] stay true in 4-dimensional case, the key step of proof cannot be easily repeated. Simply there is not enough “room” in 4-dimensional space and results from [20] do not provide enough information about structure of eigen values of the shape operator. For this reason in this paper we need to develop a bit different methods. In particular, we consider two separate cases and find several new properties of $R^k \omega$ tensor.

In Section 2 we briefly recall the basic formulas of affine differential geometry. We also recall some basic definitions from symplectic geometry that will be used later in this paper.

The Section 3 contains the main results of this paper. We show that if there exists an almost symplectic structure $\omega$ satisfying condition $R^k \cdot \omega = 0$ or $\nabla^k \omega = 0$ for some positive integer $k$ then the shape operator must have a very special form. More precisely, we obtain that the rank of the shape operator $S$ must be $\leq 1$ if only the transversal vector field is locally equiaffine.

2. Preliminaries

We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [18]. Let $f: M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable $n$-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection $D$. Then for any transversal vector field $\xi$ we have

\begin{equation}
D_X f_* Y = f_*(\nabla_X Y) + h(X,Y)\xi
\end{equation}
and
\[ D_X \xi = -f_*(SX) + \tau(X)\xi, \]
where \(X, Y\) are vector fields tangent to \(M\). It is known that \(\nabla\) is a torsion-free connection, \(h\) is a symmetric bilinear form on \(M\), called the second fundamental form, \(S\) is a tensor of type \((1,1)\), called the shape operator, and \(\tau\) is a 1-form, called the transversal connection form. The vector field \(\xi\) is called equiaffine if \(\tau = 0\). When \(d\tau = 0\) the vector field \(\xi\) is called locally equiaffine.

When \(h\) is non-degenerate then \(h\) defines a pseudo-Riemannian metric on \(M\). In this case we say that the hypersurface or the hypersurface immersion is non-degenerate. In this paper we always assume that \(f\) is non-degenerate. We have the following

**Theorem 2.1 ([18], Fundamental equations).** For an arbitrary transversal vector field \(\xi\) the induced connection \(\nabla\), the second fundamental form \(h\), the shape operator \(S\), and the 1-form \(\tau\) satisfy the following equations:

\[ R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \]
\[ (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \]
\[ (\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \]
\[ h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \]

The equations (2.3), (2.4), (2.5), and (2.6) are called the equations of Gauss, Codazzi for \(h\), Codazzi for \(S\) and Ricci, respectively.

Let \(\omega\) be a non-degenerate 2-form on manifold \(M\). The form \(\omega\) we call an almost symplectic structure. It is easy to see that if a manifold \(M\) admits some almost symplectic structure then \(M\) is orientable manifold of even dimension. Structure \(\omega\) is called a symplectic structure, if it is almost symplectic and additionally satisfies \(d\omega = 0\). Pair \((M, \omega)\) we call (almost) symplectic manifold, if \(\omega\) is (almost) symplectic structure on \(M\).

Recall ([1]) that affine connection \(\nabla\) on an almost symplectic manifold \((M, \omega)\) we call an almost symplectic connection if \(\nabla \omega = 0\). An affine connection \(\nabla\) on an almost symplectic manifold \((M, \omega)\) we call a symplectic connection if it is almost symplectic and torsion-free.

For a tensor field \(T\) of type \((0,p)\) its covariant derivation \(\nabla T\) is a tensor field of type \((0,p+1)\) given by the formula:

\[ (\nabla T)(X_1, X_2, \ldots, X_{p+1}) := X_1(T(X_2, \ldots, X_{p+1})) - \sum_{i=2}^{p+1} T(X_2, \ldots, \nabla_{X_1} X_i, \ldots, X_{p+1}). \]

Higher order covariant derivatives of \(T\) can be defined by recursion:

\[ (\nabla^{k+1} T) = \nabla(\nabla^k T). \]

To simplify computation it is often convenient to define \(\nabla^0 T := T\).

If \(R\) is a curvature tensor for an affine connection \(\nabla\), one can define a new tensor \(R \cdot T\) of type \((0,p+2)\) by the formula

\[ (R \cdot T)(X_1, X_2, \ldots, X_{p+2}) := -\sum_{i=3}^{p+2} T(X_3, \ldots, R(X_1, X_2)X_i, \ldots, X_{p+2}). \]
Analogously to the previous case, we may define a tensor $R^k \cdot T$ of type $(0,2k+p)$ using the following recursive formula:

$$R^k \cdot T = R \cdot (R^{k-1} \cdot T)$$

and additionally $R^0 \cdot T := T$.

3. Hypersurfaces with "higher order" parallel symplectic structure

In this section we study properties of 4-dimensional affine hypersurfaces $f: M \to \mathbb{R}^5$ with a Lorentzian second fundamental form. We assume that our hypersurfaces are equipped with an almost symplectic structure $\omega$ satisfying condition $R^k \omega = 0$ for some positive integer $k$. In particular we obtain constrains on hypersurfaces with the property $\nabla^k \omega = 0$.

First we recall the following lemma from [19].

**Lemma 3.1** ([19]). Let $T$ be a tensor of type $(0,p)$ and let $\nabla$ be an affine torsion-free connection. Then for every $k \geq 1$ and for any $2k+p$ vector fields $X_{11}^1, \ldots, X_{11}^k, Y_1, \ldots, Y_p$ the following identity holds:

$$\nabla^k \cdot T (X_{11}^1, X_{11}^2, \ldots, X_{11}^k, Y_1, \ldots, Y_p) = \sum_{a \in \mathcal{J}} sgn a (\nabla^k \cdot T)(X_{a(1)}^1, X_{a(1)}^2, \ldots, X_{a(1)}^k, X_{a(k)}^1, X_{a(k)}^2, \ldots, X_{a(k)}^k, Y_1, \ldots, Y_p),$$

where $\mathcal{J} = \{ a: I_k \to \{-1,1\} \}$ and $sgn a := a(1) \cdots a(k)$.

In order to simplify the notation, we will be often omitting "\" in $R^k \cdot T$ when no confusion arises. Thus we will be writing often $R^k T$ instead of $R^k \cdot T$.

In all the below lemmas we assume that $f: M \to \mathbb{R}^5$ is a non-degenerate affine hypersurface with a locally equiaffine transversal vector field $\xi$ and an almost symplectic form $\omega$. About objects $\nabla$, $h$, $S$ and $\tau$ we assume that they are induced by $\xi$.

First note that combining Lemma 3.6 and Lemma 3.11 from [20] and adapting it to 4-dimensional case we have the following:

**Lemma 3.2** ([20]). Let $f: M \to \mathbb{R}^5$ be a non-degenerate Lorentzian affine hypersurface with a locally equiaffine transversal vector field $\xi$ and an almost symplectic form $\omega$. If $R^k \omega = 0$ for some $k \geq 1$ then for every point $x \in M$ there exists a basis $e_1, \ldots, e_4$ of $T_x M$ such that the shape operator $S$ and the second fundamental form $h$ can be expressed in this basis either in the form

$$S = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

where $\lambda_1, \ldots, \lambda_4 \in \mathbb{R}$, or in the form

$$S = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha & \gamma \\ 0 & 0 & -\gamma & \beta \end{bmatrix}, h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

where $\lambda_1, \lambda_2, \alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 0$.

Let us recall yet another lemma from [20] (again adapted to 4-dimensional case).
Lemma 3.3 ([20]). If $S$ and $h$ are of the form (3.3) then for every $k \geq 1$ we have

\[ R^{2k} \omega(e_3, e_4, \ldots, e_3, e_4) \]

\[ = \det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_i, e_4) \]

if $i < 3$,

\[ R^{2k+1} \omega(e_3, e_4, \ldots, e_3, e_4, e_1, e_1, X) \]

\[ = 4^k \gamma \det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_3, e_4) \]

for $X = e_3$ or $X = e_4$,

\[ R^{2k+1} \omega(e_3, e_4, \ldots, e_3, e_4, e_1, e_1, X, Y) \]

\[ = 2 \cdot 4^{k-1} (\alpha - \beta) \det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix}^k \omega(e_3, e_4), \]

for $X = e_3$ and $Y = e_4$ or $X = e_4$ and $Y = e_3$.

Thanks to the above lemma we have the following:

**Corollary 3.4.** If $S$ and $h$ are of the form (3.3) and $R^k \omega = 0$ for some $k \geq 1$ then

\[ \det \begin{bmatrix} \alpha & \gamma \\ -\gamma & \beta \end{bmatrix} = \alpha \beta + \gamma^2 = 0. \]

**Proof.** If $R^k \omega = 0$ then $R^{2k} \omega = 0$ and $R^{2k+1} \omega = 0$. Since $\omega$ is non-degenerate we can find $i < 4$ such that $\omega(e_i, e_4) \neq 0$. If $i = 1$ or $i = 2$ then by formula (3.4) we get $\alpha \beta + \gamma^2 = 0$. If $i = 3$ then by formula (3.5) we again obtain $\alpha \beta + \gamma^2 = 0$ (since $\gamma \neq 0$).

Now, we shall consider two separate cases: when $\beta^2 - \gamma^2 \neq 0$ and when $\beta^2 - \gamma^2 = 0$. In the first case, using suitable change of the basis one may show that $S$ is diagonalisable. Namely, we have

**Lemma 3.5.** If $S$ and $h$ are of the form (3.3) and $\beta^2 - \gamma^2 \neq 0$ and $R^k \omega = 0$ for some $k \geq 1$ then there exists a basis $e'_1, \ldots, e'_4$ of $T_xM$ such that the shape operator $S$ and the second fundamental form $h$ can be expressed in this new basis in the following form:

\[ S = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + \beta & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{bmatrix} \]

where $\epsilon = \text{sgn}(\beta^2 - \gamma^2) \in \{1, -1\}$. 
Proof. Let us define a matrix

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \gamma \sqrt{|\beta^2 - \gamma^2|} & \beta \sqrt{|\beta^2 - \gamma^2|} \\
0 & 0 & \beta \sqrt{|\beta^2 - \gamma^2|} & \gamma \sqrt{|\beta^2 - \gamma^2|}
\end{bmatrix}.
\]

Since \(\det P = \pm 1\) the matrix \(P\) is non-singular and we can define a new basis of \(T_x M\) by the formula \(e'_i := Pe_i\) for \(i = 1, \ldots, 4\). By straightforward computations we check that \(S\) and \(h\) in this new basis take the form:

\[
S = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \frac{\beta^2 - 3\gamma^2}{\sqrt{|\beta^2 - \gamma^2|}} & -\frac{(\alpha + \gamma^2)\gamma}{\beta^2 - \gamma^2} \\
0 & 0 & -\frac{(\alpha + \gamma^2)\gamma}{\beta^2 - \gamma^2} & \frac{\beta^2 - 3\gamma^2}{\sqrt{|\beta^2 - \gamma^2|}}
\end{bmatrix},
\]

\[
h = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -\epsilon & 0 \\
0 & 0 & 0 & \epsilon
\end{bmatrix}.
\]

Eventually, using Corollary 3.4 we see that \(S\) simplify to (3.7). \(\square\)

When \(\beta^2 - \gamma^2 = 0\) the situation is much more complicated. In this case we have \(\alpha = \pm \gamma\) and \(\beta = \mp \gamma\). Most part of this section is devoted to this case.

In order to simplify further computations, let us introduce the following notation:

\[
A_k := R^k \omega(e_1, e_4, e_3, e_4, \ldots, e_3, e_4); \\
B_k := R^k \omega(e_3, e_4, e_1, e_4, e_3, e_4, \ldots, e_3, e_4); \\
C_k := R^k \omega(e_1, e_3, e_4, e_3, e_4, \ldots, e_3, e_4); \\
D_k := R^k \omega(e_3, e_4, e_1, e_3, e_4, \ldots, e_3, e_4)
\]

for \(k \geq 1\).

Lemma 3.6. If \(S\) and \(h\) are of the form (3.3) and \(\alpha = \pm \gamma\) and \(\beta = \mp \gamma\) then for every \(k \geq 1\) we have

\[
B_{k+1} = \pm \gamma C_k - \gamma A_k, \\
D_{k+1} = \gamma C_k \mp \gamma A_k.
\]

Proof. We shall prove only (3.9). The proof of (3.10) goes in a similar way. First note, that by the Gauss equation we have

\[
R(e_3, e_4)e_1 = R(e_3, e_4)e_2 = 0, \\
R(e_3, e_4)e_3 = -Se_4 = -\gamma e_3 \pm \gamma e_4, \\
R(e_3, e_4)e_4 = -Se_3 = \mp \gamma e_3 + \gamma e_4.
\]
Now we compute

\[ B_{k+1} = R^{k+1} \omega(e_3, e_4, e_1, e_2, e_3, e_4, \ldots, e_3, e_4) \]

\[ = R(e_3, e_4) \cdot R^k \omega(e_1, e_4, e_3, e_4, \ldots, e_3, e_4) \]

\[ = -R^k \omega(R(e_3, e_4)e_1, e_4, e_3, e_4, \ldots, e_3, e_4) \]

\[ - R^k \omega(e_1, R(e_3, e_4)e_4, e_3, e_4, \ldots, e_3, e_4) \]

\[ - R^k \omega(e_1, e_4, R(e_3, e_4)e_3, e_4, \ldots, e_3, e_4) \]

\[ \cdots \]

\[ - R^k \omega(e_1, e_4, e_3, e_4, \ldots, e_3, R(e_3, e_4)e_4). \]

Using (3.11)–(3.13) we obtain

\[ B_{k+1} = 0 - R^k \omega(e_1, \mp \gamma e_3 + \gamma e_4, e_3, e_4, \ldots, e_3, e_4) \]

\[ - R^k \omega(e_1, e_4, -\gamma e_3 \pm \gamma e_4, e_4, \ldots, e_3, e_4) \]

\[ \cdots \]

\[ - R^k \omega(e_1, e_4, e_3, e_4, \ldots, e_3, \mp \gamma e_3 + \gamma e_4) \]

\[ = \pm \gamma C_k - \gamma A_k \]

\[ + (\gamma A_k - \gamma A_k) + \cdots + (\gamma A_k - \gamma A_k) \]

\[ = \pm \gamma C_k - \gamma A_k. \]

□

Now, let us define a family of 2-forms on \( T_x M \) as follows:

\( (3.14) \quad E^i_k(X, Y) := R^k \omega(e_3, e_4, e_3, e_4, \ldots, e_3, e_4) \)

for \( k \geq 1, i \in \{1, \ldots, k + 1\} \). We have the following lemma:

**Lemma 3.7.** If \( S \) and \( h \) are of the form (3.3) and \( \alpha = \pm \gamma \) and \( \beta = \mp \gamma \) then for every \( k \geq 2, i \in \{3, \ldots, k + 1\} \) and \( X, Y \in \{e_1, e_3, e_4\} \) we have \( E^i_k(X, Y) = 0 \).

**Proof.** For \( k = 2 \) and \( i = 3 \) by straightforward computation we check that \( E^3_2(X, Y) = 0 \). Assume now that \( E^i_k(X, Y) = 0 \) for some \( k \geq 2 \) and for every \( i \in \{3, \ldots, k + 1\} \).
Let us fix \( i \in \{3, \ldots, k + 2\} \). Then we have

\[
E_{k+1}^i(X, Y) = R^{k+1}\omega(e_3, e_4, e_3, e_4, \ldots, e_3, e_4, X, Y, e_3, e_4, e_3, e_4) \\
= -R^k\omega(R(e_3, e_4)e_3, e_4, \ldots, e_3, e_4, X, Y, e_3, e_4, e_3, e_4) \\
- R^k\omega(e_3, R(e_3, e_4)e_4, \ldots, e_3, e_4, X, Y, e_3, e_4, e_3, e_4) \\
\vdots \\
- R^k\omega(e_3, e_4, \ldots, e_3, e_4, X, R(e_3, e_4)e_4, e_3, e_4, e_3, e_4) \\
\vdots \\
- R^k\omega(e_3, e_4, \ldots, e_3, e_4, X, R(e_3, e_4)e_4, e_3, e_4, e_3, e_4) \\
- R^k\omega(e_3, e_4, \ldots, e_3, e_4, X, R(e_3, e_4)e_4, e_3, e_4, e_3, e_4)
\]

where the last equality follows from (3.12)–(3.13). The above formula can be rewritten as follows:

\[
E_{k+1}^i(X, Y) = -E_{k}^{i-1}(R(e_3, e_4)X, Y) - E_{k}^{i-1}(X, R(e_3, e_4)Y).
\]

Let \( i > 3 \). Taking into account that \( X, Y \in \{e_1, e_3, e_4\} \) and using (3.11)–(3.13) we obtain that \( E_{k+1}^i(X, Y) \) can be expressed as a linear combination of \( E_{k}^{i-1}(Z, W) \), where \( Z, W \in \{e_1, e_3, e_4\} \). Since \( i > 3 \) we have \( i - 1 \geq 3 \) and by assumption \( E_{k}^{i-1}(Z, W) = 0 \). Now it follows that \( E_{k+1}^i(X, Y) = 0 \).

Assume now that \( i = 3 \), then we have

\[
E_{k+1}^3(X, Y) = -R^k\omega(e_3, e_4, R(e_3, e_4)X, Y, e_3, e_4, \ldots, e_3, e_4) \\
- R^k\omega(e_3, e_4, X, R(e_3, e_4)Y, e_3, e_4, e_3, e_4).
\]

For any pair \((X, Y)\) there exists \( i, j \in \{1, 3, 4\} \) such that \( X = e_i \) and \( Y = e_j \). Since \( E_{k+1}^3(X, Y) \) is antisymmetric relative to \( X, Y \) it is enough to show that \( E_{k+1}^3(e_i, e_j) = 0 \) for \( i, j \in \{1, 3, 4\}, \ i < j \). We have the following possibilities:

(i) \( (X, Y) = (e_1, e_3) \). In this case we have

\[
E_{k+1}^3(X, Y) = -R^k\omega(e_3, e_4, e_1, R(e_3, e_4)e_3, e_3, e_4, \ldots, e_3, e_4) \\
= -R^k\omega(e_3, e_4, e_1, -\gamma e_3 \pm \gamma e_4, e_3, e_4, \ldots, e_3, e_4) \\
= \gamma D_k \mp \gamma B_k = 0,
\]
where the last equality follows from Lemma 3.6.

(ii) $(X, Y) = (e_1, e_4)$ In this case we have

$$E_{k+1}^3(X, Y) = -R^k\omega(e_3, e_4, e_1, R(e_3, e_4)e_4, e_3, e_4, \ldots, e_3, e_4)$$

$$= -R^k\omega(e_3, e_4, e_1, \mp\gamma e_3 + \gamma e_4, e_3, e_4, \ldots, e_3, e_4)$$

$$= \pm\gamma D_k - \gamma B_k = 0,$$

where the last equality is also consequence of Lemma 3.6.

(iii) $(X, Y) = (e_3, e_4)$ In this case $E_{k+1}^3(X, Y) = 0$ thanks to (3.12) and (3.13).

Summarising we have shown that $E_{i+1}^k(X, Y) = 0$ for all $i \in \{3, \ldots, k+2\}$.

Now by induction principle $E_{i+1}^k(X, Y) = 0$ for all $k \geq 1$ and $X, Y \in \{e_1, e_3, e_4\}$.

As a consequence of Lemma 3.7 one may prove the following

**Lemma 3.8.** If $S$ and $h$ are of the form (3.3) and $\alpha = \pm\gamma$ and $\beta = \mp\gamma$ then for every $k \geq 1$, $i \in \{1, \ldots, k+1\}$ and $X, Y \in \{e_3, e_4\}$ we have $E_i^k(SX, Y) = 0$.

**Proof.** For $X \in \{e_3, e_4\}$ we have that $SX$ is a linear combination of $e_3$ and $e_4$. Since $E_i^k$ is antisymmetric 2-form we conclude that there exists a constant $c_0 \in \mathbb{R}$ such that

$$E_i^k(SX, Y) = c_0 \cdot E_i^k(e_3, e_4).$$

Now, if $k \geq 2$ the thesis follows from Lemma 3.7. If $k = 1$ we check by direct computation that $E_1^1(e_3, e_4) = 0$ for $i = 1, 2$. \hfill $\square$

Now we are at the position to prove the following lemma:

**Lemma 3.9.** If $S$ and $h$ are of the form (3.3) and $\alpha = \pm\gamma$ and $\beta = \mp\gamma$ then for every $k \geq 1$ we have

\[(3.15)\quad A_{k+1} = -\lambda_1(C_k + D_k),\]

\[(3.16)\quad C_{k+1} = -\lambda_1(A_k + B_k).\]

**Proof.** We compute

$$A_{k+1} = R^{k+1}\omega(e_1, e_4, e_3, e_4, \ldots, e_3, e_4)$$

$$= R(e_1, e_4) \cdot R^k\omega(e_3, e_4, \ldots, e_3, e_4)$$

$$= -R^k\omega(R(e_1, e_4)e_3, e_4, \ldots, e_3, e_4)$$

$$- R^k\omega(e_3, R(e_1, e_4)e_4, \ldots, e_3, e_4)$$

$$\ldots$$

$$- R^k\omega(e_3, e_4, \ldots, e_3, R(e_1, e_4)e_4).$$
Since \( R(e_1, e_4)e_3 = 0 \) and \( R(e_1, e_4)e_4 = -\lambda_1 \) the above formula can be simplified as follows:

\[
A_{k+1} = \lambda_1 R^k \omega(e_3, e_1, e_3, \ldots, e_3, e_4) \\
+ \lambda_1 R^k \omega(e_3, e_4, e_3, e_1, \ldots, e_3, e_4) \\
\vdots \\
+ \lambda_1 R^k \omega(e_3, e_4, e_3, e_4, \ldots, e_3, e_1) \\
= \lambda_1 \cdot \sum_{i=1}^{k+1} E^i_k(e_3, e_1).
\]

If \( k = 1 \) we have

\[
A_2 = \lambda_1 (E^1_1(e_3, e_1) + E^2_1(e_3, e_1)) = -\lambda_1 (C_1 + D_1).
\]

If \( k \geq 2 \), by Lemma 3.7, \( E^i_k(e_3, e_1) = 0 \) for \( i = 3, \ldots, k + 1 \). That is we obtain

\[
A_{k+1} = \lambda_1 (E^1_k(e_3, e_1) + E^2_k(e_3, e_1)) = -\lambda_1 (C_k + D_k).
\]

Eventually we have shown (3.15). The proof of (3.16) is similar. □

**Lemma 3.10.** If \( S \) and \( h \) are of the form (3.3) and \( \alpha = \pm \gamma \) and \( \beta = \mp \gamma \) then for every \( k \geq 0 \) we have

\[
A_{2k+1} = -\lambda_1^{2k+1} \omega(e_1, e_3),
\]

\[
C_{2k+1} = -\lambda_1^{2k+1} \omega(e_1, e_4).
\]

**Proof.** By straightforward computations we get

\[
A_1 = R \omega(e_1, e_4, e_3, e_4) = -\lambda_1 \omega(e_1, e_3),
\]

\[
B_1 = \pm \gamma (\omega(e_1, e_3) \mp \omega(e_1, e_4)),
\]

\[
C_1 = R \omega(e_1, e_3, e_3, e_4) = -\lambda_1 \omega(e_1, e_4),
\]

\[
D_1 = \gamma (\omega(e_1, e_3) \mp \omega(e_1, e_4)).
\]

By Lemma 3.6 we also have

\[
B_{k+1} = \pm \gamma (C_k \mp A_k),
\]

\[
D_{k+1} = \gamma (C_k \mp A_k)
\]

for all \( k \geq 1 \). Summarising we have

\[
D_k = \pm B_k
\]

for \( k \geq 1 \). Now, using Lemma 3.9 we obtain

\[
C_{k+1} \mp A_{k+1} = -\lambda_1 (A_k + B_k) \pm \lambda_1 (C_k + D_k)
\]

\[
= \pm \lambda_1 (C_k \mp A_k + D_k \mp B_k)
\]

\[
= \pm \lambda_1 (C_k \mp A_k)
\]

where the last equality is a consequence of (3.19). The above implies, that \( C_k \mp A_k \) is a geometric sequence, that is for \( k \geq 1 \) we have

\[
C_k \mp A_k = (\pm \lambda_1)^{k-1} (C_1 \mp A_1).
\]
In particular we obtain explicit formulas for $B_{k+1}$ and $D_{k+1}$:

\[ B_{k+1} = \pm \gamma (\pm \lambda_1)^{k-1} (C_1 \mp A_1), \]

\[ D_{k+1} = \gamma (\pm \lambda_1)^{k-1} (C_1 \mp A_1). \]

Using (3.15)–(3.16) and (3.21) for all $k \geq 1$ we have

\[ A_{2k+1} = -\lambda_1 (C_{2k} + D_{2k}) \]

\[ = \lambda_1^2 (A_{2k-1} + B_{2k-1}) - \lambda_1 D_{2k} \]

\[ = \lambda_1^2 A_{2k-1} + \lambda_1^2 B_{2k-1} - \lambda_1 \gamma (\pm \lambda_1)^{2k-2} (C_1 \mp A_1) \]

\[ = \lambda_1^2 A_{2k-1} + \lambda_1^2 B_{2k-1} - \gamma \lambda_1^{2k-1} (C_1 \mp A_1). \]

If $k = 1$ we directly check that

\[ \lambda_1^2 B_1 - \gamma \lambda_1 (C_1 \mp A_1) = 0. \]

If $k > 1$, using (3.20) we obtain

\[ \lambda_1^2 B_{2k-1} - \gamma \lambda_1^{2k-1} (C_1 \mp A_1) = \lambda_1^2 (\pm \gamma (\pm \lambda_1)^{2k-3} (C_1 \mp A_1)) - \gamma \lambda_1^{2k-1} (C_1 \mp A_1) \]

\[ = \lambda_1^2 \gamma \lambda_1^{2k-3} (C_1 \mp A_1) - \gamma \lambda_1^{2k-1} (C_1 \mp A_1) \]

\[ = 0. \]

Finally, for any $k \geq 1$ we have

\[ A_{2k+1} = \lambda_1^2 A_{2k-1}. \]

Since $A_1 = -\lambda_1 \omega (e_1, e_3)$ we immediately get (3.17).

In a similar way one may show that

\[ C_{2k+1} = \lambda_1^2 C_{2k-1}. \]

and in consequence (3.18). \qed

To simplify further computations we need to introduce the following notation:

\[ T^k_{p,q,r}(X,Y) := R^k \omega (e_3, e_4, \ldots, e_3, e_4, e_1, X, e_3, e_4, \ldots, e_3, e_4, Y, e_3, e_4, \ldots, e_3, e_4), \]

\[ U^k_{p,q,r}(X,Y) := -\lambda_1 h(X, e_3) T^k_{p,q,r}(e_4, Y) + \lambda_1 h(X, e_4) T^k_{p,q,r}(e_3, Y), \]

\[ \hat{U}^k_{p,q,r}(X,Y) := -\lambda_1 h(X, e_3) T^k_{p,q,r}(Y, e_4) + \lambda_1 h(X, e_4) T^k_{p,q,r}(Y, e_3). \]

where $k \geq 1$, $p, q, r \geq 0$, $p + q + r = k - 1$.

We have the following lemma:

**Lemma 3.11.** If $S$ and $h$ are of the form (3.3) and $\alpha = \pm \gamma$ and $\beta = \mp \gamma$ then for every $k \geq 1$, $q, r \geq 0$, $q + r = k$ and $X, Y \in \{ e_3, e_4 \}$ we have

\[ T^{k+1}_{q,r}(X,Y) = \sum_{i=0}^{q-1} t^k_{i,q-1-i,r}(X,Y) + \sum_{i=0}^{r-1} \hat{t}^k_{q,i,r-1-i}(X,Y). \]

Note that it may happen that $q = 0$ (respectively $r = 0$) in such case the sum $\sum_{i=0}^{q-1}$ (respectively the sum $\sum_{i=0}^{r-1}$) is not present in the above formula.
Proof. We compute
\[
T_{0,q,r}^{k+1}(X, Y) = R(e_1, X) \cdot R^k \omega(e_1, e_4, \ldots, e_3, e_4, e_1, Y, e_3, e_4, \ldots, e_3, e_4)
\]
\[
= -R^k \omega(R(e_1, X)e_3, e_4, \ldots, e_3, e_4, e_1, Y, e_3, e_4, \ldots, e_3, e_4)
\]
\[
\cdots
\]
\[
= R^k \omega(e_3, e_4, \ldots, e_3, e_4, R(e_1, X)e_4, e_1, Y, e_3, e_4, \ldots, e_3, e_4)
\]
\[
\cdots
\]
\[
= R^k \omega(e_3, e_4, \ldots, e_3, e_4, e_1, R(e_1, X)e_3, e_4, \ldots, e_3, e_4)
\]
\[
\cdots
\]
\[
= R^k \omega(e_3, e_4, \ldots, e_3, e_4, e_1, Y, R(e_1, X)e_3, e_4, \ldots, e_3, e_4)
\]
\[
\cdots
\]
\[
= R^k \omega(e_3, e_4, \ldots, e_3, e_4, e_1, Y, e_3, e_4, \ldots, e_3, R(e_1, X)e_4).
\]

Using the Gauss equation we obtain
\[
T_{0,q,r}^{k+1}(X, Y) = \sum_{i=0}^{q-1} U_{i,q-r-i}(X, Y)
\]
\[
- R^k \omega(e_3, e_4, \ldots, e_3, e_4, -SX, Y, e_3, e_4, \ldots, e_3, e_4)
\]
\[
- R^k \omega(e_3, e_4, \ldots, e_3, e_4, e_1, h(X, Y)\lambda_1 e_1, e_3, e_4, \ldots, e_3, e_4)
\]
\[
+ \sum_{i=0}^{r-1} U_{i,q-r-i}(X, Y)
\]
\[
= \sum_{i=0}^{q-1} U_{i,q-r-i}(X, Y) + E^{q+1}_k(SX, Y) + \sum_{i=0}^{r-1} \hat{U}_{i,q-r-i}(X, Y).
\]

Since \(X, Y \in \{e_3, e_4\}\) Lemma \(3.8\) implies \(3.22\).

Now we can prove

Lemma 3.12. If \(S\) and \(h\) are of the form \(3.3\) and \(\alpha = \pm \gamma\) and \(\beta = \mp \gamma\) then for every \(k \geq 1\) we have
\[
T_{p,q,r}^k(e_3, e_3) = T_{p,q,r}^k(e_4, e_4) = \pm T_{p,q,r}^k(e_3, e_4) = \pm T_{p,q,r}^k(e_4, e_3)
\]
where \(p, q, r \geq 0\) and \(p + q + r = k - 1\)

Proof. For \(k = 1\), by straightforward computations we check that
\[
R \omega(e_1, e_3, e_1, e_3) = R \omega(e_1, e_4, e_1, e_4) = \pm R \omega(e_1, e_3, e_1, e_4) = \pm R \omega(e_1, e_4, e_1, e_3)
\]
so
\[ T_{0,0,0}^1(e_3, e_3) = T_{0,0,0}^1(e_4, e_4) = T_{0,0,0}^1(e_3, e_4) = \pm T_{0,0,0}^1(e_4, e_3). \]
Assume now that (3.23) is true for some \( k \geq 1 \) and all \( p, q, r \geq 0 \) such that \( p+q+r = k-1 \). We compute
\[
U_{p,q,r}^k(e_3, e_3) = -\lambda_1 T_{p,q,r}^k(e_4, e_3) = -\lambda_1 T_{p,q,r}^k(e_3, e_4) = U_{p,q,r}^k(e_4, e_4) = \pm U_{p,q,r}^k(e_3, e_4) = \pm U_{p,q,r}^k(e_4, e_3).
\]
In a similar way we get
\[
\hat{U}_{p,q,r}^k(e_3, e_3) = \hat{U}_{p,q,r}^k(e_4, e_4) = \pm \hat{U}_{p,q,r}^k(e_3, e_4) = \pm \hat{U}_{p,q,r}^k(e_4, e_3).
\]
Now, let us consider \( T_{p,q,r}^{k+1}(X, Y) \) where \( p, q, r \geq 0 \) and \( p+q+r = k \).
If \( p = 0 \) then from Lemma 3.11 we have
\[
T_{0,q,r}^{k+1}(X, Y) = \sum_{i=0}^{q-1} U_{i,q-1-i,r}^k(X, Y) + \sum_{i=0}^{r-1} \hat{U}_{i,r-1-i}^k(X, Y).
\]
Using (3.24) - (3.26) we obtain
\[
T_{0,q,r}^{k+1}(e_3, e_3) = T_{0,q,r}^{k+1}(e_4, e_4) = \pm T_{1,q,r}^{k+1}(e_3, e_4) = \pm T_{0,q,r}^{k+1}(e_4, e_3).
\]
Assume now that \( p > 0 \). First note that
\[
R^k \omega(\ldots, R(e_3, e_4)e_3, e_4, \ldots) = R^k \omega(\ldots, -\gamma e_3 \pm \epsilon e_4, e_4, \ldots)
\]
\[
= -\gamma R^k \omega(\ldots, e_3, e_4, \ldots)
\]
\[
= R^k \omega(\ldots, e_3, \pm \epsilon e_3 - \gamma e_4, \ldots)
\]
\[
= -R^k \omega(\ldots, e_3, R(e_3, e_4)e_4, \ldots).
\]
By (3.27) and using the fact that \( R(e_3, e_4)e_1 = 0 \) we get
\[
T_{p,q,r}^{k+1}(X, Y)
\]
\[
= R(e_3, e_4) \cdot \underbrace{R^k \omega(e_3, e_4, \ldots, e_3, e_4, e_1, X, e_3, e_4, \ldots, e_3, e_4, e_1, Y, e_3, e_4, \ldots, e_3, e_4)}_{2p-2}
\]
\[
= -R^k \omega(e_3, e_4, \ldots, e_3, e_4, e_1, R(e_3, e_4)X, e_3, e_4, \ldots, e_3, e_4, e_1, Y, e_3, e_4, \ldots, e_3, e_4)
\]
\[
- R^k \omega(e_3, e_4, \ldots, e_3, e_4, e_1, X, e_3, e_4, \ldots, e_3, e_4, e_1, R(e_3, e_4)Y, e_3, e_4, \ldots, e_3, e_4)
\]
\[
= -T_{p-1,q,r}^k(R(e_3, e_4)X, Y) - T_{p-1,q,r}^k(X, R(e_3, e_4)Y).
\]
Using (3.24), (3.26), and (3.28), by direct computation, one may check that
\[
T_{p-1,q,r}^k(R(e_3, e_4)X, Y) = T_{p-1,q,r}^k(X, R(e_3, e_4)Y) = 0
\]
for any \( X, Y \in \{e_3, e_4\} \). In consequence we get
\[
T_{p,q,r}^{k+1}(X, Y) = 0
\]
for all \( X, Y \in \{e_3, e_4\} \). Now from (3.28) and (3.26) we obtain
\[
T_{p,q,r}^{k+1}(e_3, e_3) = T_{p,q,r}^{k+1}(e_4, e_4) = \pm T_{p,q,r}^{k+1}(e_3, e_4) = \pm T_{p,q,r}^{k+1}(e_4, e_3)
\]
for every \( p, q, r \geq 0 \), \( p+q+r = k \). By induction principle the formula (3.23) is true for any \( k \geq 1 \).
As an immediate consequence of Lemma 3.12 (see formula (3.28)) we get the following

**Corollary 3.13.** If $S$ and $h$ are of the form (3.33) and $\alpha = \pm \gamma$ and $\beta = \mp \gamma$ then for every $k \geq 2$, $p \geq 1$, $q, r \geq 0$ and $p + q + r = k - 1$ we have

\[ T^k_{p,q,r}(X,Y) = 0 \tag{3.29} \]
for any $X, Y \in \{e_3, e_4\}$.

The above lemmas and corollary allow us to prove the following

**Lemma 3.14.** If $S$ and $h$ are of the form (3.33) and $\alpha = \pm \gamma$ and $\beta = \mp \gamma$ then for every $k \geq 2$ we have

\[ T^k_{0,0,k-1}(e_3, e_3) = 2^{k-2}(\mp \lambda_1)k-\gamma\omega(e_3, e_4). \tag{3.30} \]

**Proof.** Let us consider $T^k_{0,q,r}(e_3, e_3)$, when $q + r = k$, $k \geq 1$. If $q \geq 1$, using Lemma 3.11 and Corollary 3.13 (if $k \geq 2$) we obtain

\[
T^{k+1}_{0,q,r}(e_3, e_3) = \sum_{i=0}^{q-1} U^k_{i,q-1-i,r}(e_3, e_3) + \sum_{i=0}^{r-1} U^k_{q,i,r-1-i}(e_3, e_3)
\]

\[
= -\lambda_1 \sum_{i=0}^{q-1} T^k_{i,q-1-i,r}(e_4, e_3) - \lambda_1 \sum_{i=0}^{r-1} T^k_{q,i,r-1-i}(e_3, e_4)
\]

\[
= -\lambda_1 T^k_{0,q-1,r}(e_4, e_3).
\]

Now, by Lemma 3.12 we have

\[ T^{k+1}_{0,q,r}(e_3, e_3) = -\lambda_1 T^k_{0,q-1,r}(e_4, e_3) = \mp \lambda_1 T^k_{0,q-1,r}(e_3, e_3) \]

that is

\[ T^{k+1}_{0,q,r}(e_3, e_3) = (\mp \lambda_1)^q T^r_{0,0,r}(e_3, e_3). \tag{3.31} \]

If $q = 0$, by Lemma 3.11 and Lemma 3.12 we have

\[ T^{k+1}_{0,0,k}(e_3, e_3) = -\lambda_1 \sum_{i=0}^{k-1} T^k_{0,i,k-1-i}(e_3, e_4) = \mp \lambda_1 \sum_{i=0}^{k-1} T^k_{0,i,k-1-i}(e_3, e_3) \tag{3.32} \]

for $k \geq 1$. Applying (3.31) to (3.32) we get

\[ T^{k+1}_{0,0,k}(e_3, e_3) = \mp \lambda_1 (\mp \lambda_1)^{k-1} T^k_{0,0,k-1}(e_3, e_3). \tag{3.33} \]

By straightforward computations we check that

\[ T^1_{0,0,0}(e_3, e_3) = \gamma\omega(e_3, e_4) \quad \text{and} \quad T^2_{0,0,1}(e_3, e_3) = \mp \lambda_1 \gamma\omega(e_3, e_4). \]
so in particular (3.30) is true for \( k = 2 \). Let us fix \( k_0 \geq 2 \) and assume that (3.30) is true for any \( k \in \{2, \ldots, k_0\} \). Now we have

\[
T_{0,0,k_0}^{k_0+1}(e_3, e_3) = \mp \lambda_1 \sum_{i=0}^{k_0-2} (\mp \lambda_1)^{i} 2^{k_0-i-2} (\mp \lambda_1)^{k_0-i-1} \gamma \omega(e_3, e_4) \\
+ (\mp \lambda_1)^{k_0} T_{0,0,0}^{1}(e_3, e_3) \\
= (\mp \lambda_1)^{k_0} \gamma \omega(e_3, e_4) \sum_{i=0}^{k_0-2} 2^{k_0-i-2} + (\mp \lambda_1)^{k_0} \gamma \omega(e_3, e_4) \\
= 2^{k_0-1} (\mp \lambda_1)^{k_0} \gamma \omega(e_3, e_4).
\]

That is (3.30) holds also for \( k = k_0 + 1 \). Now, by induction principle (3.30) is true for any \( k \geq 2 \).

Now we are ready to prove main results of this paper. Namely we have

**Theorem 3.15.** Let \( f : M \to \mathbb{R}^5 \) be a non-degenerate affine hypersurface with a locally equiaffine transversal vector field \( \xi \) and an almost symplectic form \( \omega \). If \( R^k \omega = 0 \) for some \( k \geq 1 \) and the second fundamental form is Lorentzian on \( M \) (that is has signature (3, 1)) then the shape operator \( S \) has the rank \( \leq 1 \).

**Proof.** Let \( x \in M \) and let \( \{e_1, \ldots, e_4\} \) be the basis from Lemma 3.2. If \( S \) and \( h \) are of the form (3.2), then in the same way as in the proof of Theorem 1.2 (see [19] for details) we obtain that \( S \) is equal to zero thus rank \( S_x = 0 \).

Let \( S \) and \( h \) have the form (3.3). If \( \beta^2 - \gamma^2 \neq 0 \) then by Lemma 3.3 we can change the basis \( \{e_1, \ldots, e_4\} \) of \( T_x M \) to \( h \)-orthonormal basis \( e_1', \ldots, e_4' \) in such a way that \( S \) and \( h \) are of the form (3.7). In particular, \( S \) is diagonal. Since \( \beta^2 - \gamma^2 \neq 0 \), Corollary 3.4 implies that \( \alpha + \beta \neq 0 \) and \( \mathrm{rank} \ S_x \geq 1 \). However, since \( S \) is diagonal we again can use methods from [19] (proof of Theorem 1.2) and show that rank \( S_x = 0 \), what leads to contradiction. It means that the case \( \beta^2 - \gamma^2 \neq 0 \) is not possible.

Assume now that \( \beta^2 - \gamma^2 = 0 \). By Corollary 3.4 we have \( \alpha \beta + \gamma^2 = 0 \) and in consequence we get that \( \alpha = \pm \gamma \) and \( \beta = \mp \gamma \). Without loss of generality (rearranging \( e_1 \) and \( e_2 \) if needed) we may always assume that \( |\lambda_1| \geq |\lambda_2| \geq 0 \). If \( \lambda_1 = \lambda_2 = 0 \) then rank \( S_x = 1 \) (since \( \gamma \neq 0 \)) and the proof is completed. Let us assume that \( \lambda_1 \neq 0 \). Since \( R^k \omega = 0 \) for some \( k \geq 1 \) then in particular \( R^k \omega = 0 \) and \( R^{2k+1} \omega = 0 \) and hence \( S \) is diagonal. By Corollary 3.4 we obtain \( \omega(e_1, e_3) = \omega(e_1, e_4) = 0 \). Since \( \omega \) is non-degenerate then

\[
\det \omega = (\omega(e_1, e_2) \omega(e_3, e_4) - \omega(e_1, e_3) \omega(e_2, e_4) + \omega(e_1, e_4) \omega(e_2, e_3))^2 \\
= (\omega(e_1, e_2) \omega(e_3, e_4))^2 \neq 0.
\]

In particular \( \omega(e_3, e_4) \neq 0 \). Now Lemma 3.13 implies that \( \lambda_1 \gamma = 0 \) what (since \( \gamma \neq 0 \) and \( \lambda_1 \neq 0 \)) leads us to contradiction. Summarising we must have \( \lambda_1 = 0 \) and in consequence also \( \lambda_2 = 0 \).

From Theorem 3.15 we directly obtain the following

**Theorem 3.16.** Let \( f : M \to \mathbb{R}^5 \) be a non-degenerate affine hypersurface with a locally equiaffine transversal vector field \( \xi \) and an almost symplectic form \( \omega \). If \( \nabla^k \omega = 0 \) for some \( k \geq 1 \) and the second fundamental form is Lorentzian on \( M \) (that is has signature (3, 1)) then the shape operator \( S \) has the rank \( \leq 1 \).
Proof. If $\nabla^k \omega = 0$ for some $k$ then, of course, we have that also $\nabla^{2k} \omega = 0$ and now by Lemma 3.1 we get $R^k \omega = 0$. Now, thesis is an immediate consequence of Theorem 3.15.

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