Orientation Waves in a Director Field with Rotational Inertia

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Abstract

We study the propagation of orientation waves in a director field with rotational inertia and potential energy given by the Oseen-Frank energy functional from the continuum theory of nematic liquid crystals. There are two types of waves, which we call splay and twist waves. Weakly nonlinear splay waves are described by the quadratically nonlinear Hunter-Saxton equation. Here, we show that weakly nonlinear twist waves are described by a new cubically nonlinear, completely integrable asymptotic equation. This equation provides a surprising representation of the Hunter-Saxton equation for $u$ as an advection equation for $v$, where $u_{xx} = v_x^2$. There is an analogous representation of the Camassa-Holm equation. We use the asymptotic equation to analyze a one-dimensional initial value problem for the director-field equations with twist-wave initial data.

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1 Introduction

In this paper, we analyze a system of partial differential equations that models the propagation of orientation waves in a massive director field. The restoring

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force for the waves is provided by the Oseen-Frank energy (2.1), used in the continuum theory of nematic liquid crystals, and the inertia is provided by the rotational inertia of the director field.

The motion of the director field is then described by the variational principle (2.4). The corresponding Euler-Lagrange equation (2.5) is scale-invariant, and forms a non-dispersive hyperbolic system of wave equations. The system supports two types of waves, which we call ‘splay’ and ‘twist’ waves, respectively. The principal nonlinear effects are that the wave speeds depend on the angle between the propagation direction and the director field, and that the twist waves generate splay waves.

Splay waves were investigated by Saxton [1] and Hunter and Saxton [2]. In that case, the one-dimensional system of equations for the director field (see Section 4) reduces to a scalar wave equation for an angle \( \varphi(x, t) \),

\[
\varphi_{tt} - \left[ a^2(\varphi)\varphi_t \right]_x + a(\varphi)a'(\varphi)\varphi_x^2 = 0. \tag{1.1}
\]

Here, the wave-speed \( a \) is a smooth, nonzero function of \( \varphi \), given in (4.2), and the prime denotes a derivative with respect to \( \varphi \).

The wave equation (1.1) is obtained from the variational principle

\[
\delta \int \frac{1}{2} \left\{ \varphi_t^2 - a^2(\varphi)\varphi_x^2 \right\} \, dx \, dt = 0,
\]

which is one of the simplest nonlinear generalizations of the variational principle for the linear wave equation one can imagine. The effects of this nonlinearity include the formation of cusp-type singularities [3]. Smooth solutions can be extended by global weak solutions, but in sharp contrast with the more familiar case of entropy solutions of hyperbolic conservation laws [4], equation (1.1) possesses conservative weak solutions that are compatible with its variational and Hamiltonian structure. The global well-posedness of the initial value problem for (1.1) for conservative weak solutions is proved in [5].

The use of weakly nonlinear asymptotics to study the behavior of solutions of (1.1) that consist of small, localized perturbations \( u(x, t) \) of a constant state \( \varphi_0 \) leads (after an appropriate normalization, and provided that \( a'(\varphi_0) \neq 0 \)) to the Hunter-Saxton (HS) equation [2],

\[
(u_t + uu_x)_x - \frac{1}{2} u_x^2 = 0. \tag{1.2}
\]
This equation is derived from the variational principle

\[ \delta \int \frac{1}{2} \left\{ u_x u_t + uu_x^2 \right\} \, dx dt = 0. \tag{1.3} \]

It is completely integrable [6,7,?], and possesses global dissipative and conservative weak solutions [9,10,11].

In this paper, we study the full system of director-field equations and show that qualitatively new phenomena arise which are not present for the scalar wave equation (1.1). The structure of the system is most easily seen in the case of one space-dimension; the director-field equations then reduce to two coupled wave equations (4.3)–(4.4) for angles \( \varphi(x,t), \psi(x,t) \), given by the variational principle (4.1). The angle \( \varphi \), with wave-speed \( a \), corresponds to splay waves; the angle \( \psi \), with wave speed \( b \), corresponds to twist waves. Both wave speeds \( a, b \) depend only on \( \varphi \), and the wave equation for \( \varphi \) is forced by source terms that are proportional to quadratic functions of the derivatives of \( \psi \). If \( \psi \) is constant, then the system (4.3)–(4.4) reduces to the scalar wave equation (1.1).

Our main result is the following asymptotic PDE for a localized, weakly-nonlinear twist wave with amplitude \( v(x,t) \):

\[
\begin{align*}
(v_t + uv_x)_x &= 0, \\
u_{xx} &= v_x^2.
\end{align*}
\tag{1.4}
\]

This cubically-nonlinear, scale-invariant system consists of an advection equation for \( v \) in which the advection velocity \( u \) is reconstructed nonlocally and quadratically from \( v \). It is obtained from the variational principle

\[ \delta \int \frac{1}{2} \left\{ v_x v_t + uv_x^2 + \frac{1}{2}u_x^2 \right\} \, dx dt = 0. \tag{1.5} \]

The second dependent variable \( u(x,t) \) in (1.4) may be interpreted as the amplitude of a splay wave that is generated nonlinearly by the twist wave, which then affects the velocity of the twist wave.

Remarkably, the elimination of \( v \) from (1.4) implies that \( u \) satisfies the HS-equation

\[ \left[ (u_t + uu_x)_x - \frac{1}{2}u_x^2 \right]_x = 0. \tag{1.6} \]

Thus, a solution \( v \) of (1.4) is related to a solution \( u \) of (1.6) by \( v_x^2 = u_{xx} \). Under this change of variables, the variational principle (1.5) transforms into
a second variational principle for (1.6), distinct from (1.3) but compatible with it, that is associated with a Lie-Poisson structure (see Proposition 4).

The HS-equation therefore arises from the system of director-field equations in two different asymptotic limits for two different types of waves, with two separate variational structures. The resulting bi-Hamiltonian structure explains why (1.6) is completely integrable [6], and it follows that (1.4) is also completely integrable (see Section 6). Furthermore, as we show in Section 5, equation (1.4) may be solved explicitly by the method of characteristics; the key point is that the Jacobian of the transformation between spatial and characteristic coordinates satisfies an integrable Liouville equation.

The correspondence between solutions of (1.4) and (1.6) is neither one-to-one or onto. Only convex solutions of (1.6) with \( u_{xx} \geq 0 \) can be obtained from solutions of (1.4), and solutions \( v \) of (1.4) for which \( v_x \) has the same magnitude but different signs (which may depend on \( x \)) correspond to the same solution of (1.6). Moreover, because of the nonlinear nature of the transformation, distributional solutions of (1.6) do not necessarily transform into distributional solutions of (1.4). Nevertheless, equation (1.4) provides an interesting representation of the HS-equation (1.6) for \( u \) as an advection equation for a new variable \( v \). There is a similar representation of the closely related Camassa-Holm (CH) equation [12] (see (6.1)–(6.2) below, with \( Mu = u_{xx} - u \)).

The twist and splay waves exhibit a basic difference in their nonlinearity. Twist waves are linearly degenerate, meaning that the derivative of their wave speed with respect to the wave amplitude is identically zero, as stated in (2.12). This linear degeneracy may be seen in the asymptotic system (1.4), where the advection velocity \( u \) of \( v \) is independent of \( v \), and in the one-dimensional system (4.3)–(4.4), where the wave speed \( b(\varphi) \) of \( \psi \) is independent of \( \psi \). By contrast, excluding exceptional values of \( \varphi \) where \( a'(\varphi) = 0 \), splay waves are genuinely nonlinear, meaning that the derivative of their wave speed with respect to the wave amplitude is non-zero, as stated in (2.11). Despite the linear degeneracy of the twist waves, their interaction with splay waves leads to nontrivial, cubically nonlinear dynamics.

Equation (1.4) differs from another, more obvious, cubically-nonlinear, scale-invariant modification of the HS-equation [2,13],

\[
(u_t + u^2 u_x)_x = uu_x^2, \tag{1.7}
\]

given by the variational principle

\[
\delta \int \frac{1}{2} \left\{ u_x u_t + u^2 u_x^2 \right\} \, dx \, dt = 0.
\]
Unlike (1.4), equation (1.7) is an asymptotic limit of the scalar wave equation (1.1), and arises when there is a loss of genuine nonlinearity at the unperturbed state $\varphi_0$ such that $a'(\varphi_0) = 0$ but $a''(\varphi_0) \neq 0$. In Section 3.3, we derive from the full director-field system a vector generalization of (1.7), given in (3.14) or (3.16)–(3.17), that describes the propagation of polarized orientation waves in the same direction as the unperturbed director field.

We now outline the contents of this paper. In Section 2, we describe the system of PDEs for the director-field, and briefly compare it with some related variational field theories. We show that the system is hyperbolic, and study the linearized orientation waves. In Section 3, we summarize the asymptotic equations for weakly nonlinear splay and twist waves.

In Section 4, we write out the one-dimensional director-field equations in terms of spherical polar angles, and construct an asymptotic solution of a one-dimensional initial value problem with initial data corresponding to a small-amplitude, compactly supported twist wave. This solution illustrates, in particular, the generation of splay waves by twist waves, and our main goal is to formulate equations for the resulting splay waves.

There are two cases, depending on whether the twist waves are faster or slower than the splay waves (see Figure 2). When the twist waves are faster, they move at a constant velocity into an unperturbed director field, and we obtain a Cauchy problem for the splay-wave equation (1.1) with data for $\varphi, \varphi_x$ given on the space-like twist-wave trajectories. When the twist waves are slower, they are embedded inside the splay wave, and we obtain a free-boundary problem for (1.1). The twist-wave trajectories are not known \textit{a priori}, and $\varphi_x$ satisfies a jump condition across the trajectories that follows from an integration of the asymptotic equation (1.4).

In Section 5, we solve (1.4) by the method of characteristics, and use the result to solve the initial-boundary value problem for (1.4) that arises in Section 4. In Section 6, we show that (1.4) leads to the HS-equation, and describe its bi-Hamiltonian structure and integrability. We also consider the CH-equation. In Section 7, we derive an apparently non-integrable generalization of (1.4), given in (7.1)–(7.2), that applies to periodic twist waves with mean-field interactions.

## 2 Director fields

The system of wave equations we study here is motivated by the theory of nematic liquid crystals. We consider an unbounded anisotropic medium whose orientation (in three space dimensions) is described by a time-dependent director field $n(x,t)$ of unit vectors, $n : \mathbb{R}^3 \times \mathbb{R} \to S^2$. 
For uniaxial nematic liquid crystals, the directions $n$ and $-n$ are identified, and then $n: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{RP}^2$. In most of the problems we consider here, the director field is a small perturbation of a given smooth field, and the global topological distinctions between $S^2$ and $\mathbb{RP}^2$ are not relevant.

We suppose that the potential energy $W$ of a director field $n$ is given by the Oseen-Frank energy functional [14]

$$W(n, \nabla n) = \frac{1}{2} \alpha (\text{div} n)^2 + \frac{1}{2} \beta (n \cdot \text{curl} n)^2 + \frac{1}{2} \gamma |n \times \text{curl} n|^2. \quad (2.1)$$

Up to a null-Lagrangian, $W$ is the most general quadratic function of $\nabla n$ with coefficients depending on $n$ that is invariant under the transformations $n \mapsto -n$ and

$$x \mapsto R x, \quad n \mapsto R n \quad \text{for all orthogonal maps } R. \quad (2.2)$$

The positive coefficients $\alpha, \beta, \gamma$ are elastic constants of splay, twist, and bend, respectively. For liquid crystals, we typically have $0 < \beta < \alpha < \gamma$.

In the one-constant approximation, $\alpha = \beta = \gamma$, the Oseen-Frank energy reduces (up to a null-Lagrangian) to the harmonic map energy,

$$W(\nabla n) = \frac{1}{2} \alpha |\nabla n|^2.$$

This energy does not depend explicitly on $n$, and is invariant under a larger group of transformations, namely

$$x \mapsto R x, \quad n \mapsto S n \quad \text{for all orthogonal maps } R, S. \quad (2.3)$$

The nonlinear effects we study here vanish in this case, and we assume that the elastic constant are distinct.

We further suppose that the director field has rotational kinetic energy and is not subject to damping or dissipation. Its motion is then governed by the variational principle

$$\delta \int_{\mathbb{R}^3 \times \mathbb{R}} \left\{ \frac{1}{2} n^2 - W(n, \nabla n) \right\} \, dx dt = 0, \quad n \cdot n = 1, \quad (2.4)$$

where we have normalized the moment of inertia per unit volume of the director field to one.
The Euler-Lagrange equation associated with (2.4) is
\[
\frac{\partial n}{\partial t} = \alpha \nabla (\text{div} n) - \beta \{A \text{curl} n + \text{curl} (An)\} \\
+ \gamma \{B \times \text{curl} n - \text{curl} (B \times n)\} + \lambda n,
\] (2.5)
where
\[A = n \cdot \text{curl} n, \quad B = n \times \text{curl} n.\]

The Lagrange multiplier \(\lambda(x, t)\) is chosen so that \(n \cdot n = 1\), which implies that
\[\lambda = -n_i^2 + \alpha \left[|\nabla n|^2 - |\text{curl} n|^2\right] + 2 \left[\beta A^2 + \gamma B^2\right] + (\alpha - \gamma) \text{div} B.\]

Equation (2.5) provides a natural, geometrical model for the propagation of orientation waves in an anisotropic medium, and is representative of a large class of variational systems of wave equations in which the wave speeds are functions of the dependent variables \([15]\). It is not applicable to standard liquid crystal hydrodynamics where the motion of the director field is dominated by viscosity and the effects of rotational inertia are negligible. Nevertheless, it is conceivable that the nonlinear phenomena analyzed here could be observed in high-frequency excitations of liquid crystals whose molecules possess a large moment of inertia (such as, perhaps, liquid crystals obtained from carbon nanotubes).

At a more general level, liquid crystalline phases are the result of a continuous breaking of rotational symmetry, and the orientation waves we analyze here may be regarded as associated Goldstone modes \([16]\). Similar phenomena should occur for non-dispersive Goldstone modes in other systems with continuously broken symmetry in which the energy density associated with a set of order parameters \(\Psi^a\) has the anisotropic form
\[A_{ab}^{ij}(\Psi) \frac{\partial \Psi^a}{\partial x^i} \frac{\partial \Psi^b}{\partial x^j},\]
rather than the more commonly assumed isotropic Ginzburg-Landau form proportional to \(|\nabla \Psi|^2\). For liquid crystals, the isotropic form arises only in the one-constant approximation.

As liquid crystals illustrate, the Ginzburg-Landau energy density is not dictated by general symmetry arguments in anisotropic media. For long-wave variations, it is natural to retain the leading-order terms in the energy that are quadratic in the spatial derivatives of the order parameters, but the order parameters themselves may vary by a large amount, in which case one should
retain any dependence of the coefficients of the spatial derivatives on the order parameters.

A similar situation occurs in classical field theories, where nonlinear sigma-models lead to wave equations with wave speeds that are independent of the dependent variables. This class of field theories includes the wave-map equations from Minkowski space into $\mathbb{S}^2$ to which (2.5) reduces in the one-constant approximation. On the other hand, general relativity leads to a form of nonlinearity that is analogous to that of the general director-field equations: the wave operator in the Einstein equations acts on the metric and has coefficients that are functions of the metric. The effects of this nonlinearity in the Einstein equations are, however, much more degenerate than in the director-field equations [17].

We remark that the invariance properties of these equations also differ in an analogous way. Nonlinear sigma-models are invariant under separate transformations of the independent and dependent variables, as in (2.3), whereas the gauge-invariance of the Einstein equations involves a simultaneous transformation of the independent and dependent variables, as in (2.2).

2.1 Orientation waves

In this section, we show that (2.5) forms a hyperbolic system of PDEs, and describe the corresponding waves. We consider solutions of (2.5) of the form

$$n(x, t) = n_0 + n'(x, t),$$

where $n'$ is a small perturbation of a constant director field $n_0$, linearize the resulting equations for $n'$, and look for Fourier solutions of the linearized equations of the form

$$n'(x, t) = Ne^{i(k \cdot x - i\omega t)\hat{n}}.$$

Here, $N \in \mathbb{R}$ is an arbitrary amplitude, $k \in \mathbb{R}^3$ is the wavenumber vector, $\omega \in \mathbb{R}$ is the frequency, and $\hat{n} \in \mathbb{R}^3$ is a normalized constant vector.

We find that (see Appendix A.1) $\omega$ satisfies the linearized dispersion relation

$$\omega^2 - a^2(k; n_0)\omega^2 - b^2(k; n_0) = 0, \quad (2.6)$$

where, with $k = |k|$, ....
Thus, if $\alpha \neq \beta$, the characteristic variety of (2.5) consists of two nested elliptical cones (see Figure 1). There is a loss of strict hyperbolicity when $\mathbf{k}$ is parallel to $\mathbf{n}_0$, corresponding to a wave that propagates in the same direction as the unperturbed director field.

The waves associated with the branch $\omega^2 = a^2(\mathbf{k}; \mathbf{n}_0)$ carry perturbations of the director field in the same plane as $\mathbf{n}_0$ and $\mathbf{k}$, in which $\mathbf{n} = \mathbf{R}$ where

$$
\mathbf{R} (\mathbf{k}; \mathbf{n}_0) = \mathbf{k} - (\mathbf{k} \cdot \mathbf{n}_0) \mathbf{n}_0. 
$$

We call these waves splay waves. We note that $\mathbf{R}$ is orthogonal to $\mathbf{n}_0$, as required by the linearization of the constraint that $\mathbf{n}$ is a unit vector.

The waves associated with the branch $\omega^2 = b^2(\mathbf{k}; \mathbf{n}_0)$ carry transverse perturbations of the director field orthogonal to $\mathbf{n}_0$ and $\mathbf{k}$, in which $\mathbf{n} = \mathbf{S}$ where

$$
\mathbf{S} (\mathbf{k}; \mathbf{n}_0) = \mathbf{k} \times \mathbf{n}_0.
$$

We call these waves twist waves.

There is a fundamental difference in the dependence of the splay and twist waves on $\mathbf{n}_0$. From (2.7), we compute that the derivative of the splay-wave
speed in the direction of the perturbation carried by the wave is given by

$$
\nabla_{n_0} a(k; n_0) \cdot R(k; n_0) = -\frac{(\alpha - \gamma)}{a(k; n_0)} (k \cdot n_0) \left[ k^2 - (k \cdot n_0)^2 \right].
$$

(2.11)

If $\alpha \neq \gamma$, this quantity is nonzero provided that $k$ is not parallel or orthogonal to $n_0$. On the other hand, from (2.8), we see that the derivative of the twist-wave speed in the direction of the wave is identically zero,

$$
\nabla_{n_0} b(k; n_0) \cdot S(k; n_0) = 0.
$$

(2.12)

By analogy with the terms introduced by Lax in the context of first-order hyperbolic systems of conservation laws [4], we say that a wave is genuinely nonlinear if the derivative of the wave speed in the direction of the wave is nonzero, and linearly degenerate if the derivative is identically zero. Thus, the splay waves are genuinely nonlinear when they do not propagate in directions parallel or orthogonal to the director field, and the twist waves are linearly degenerate.

3 Weakly nonlinear waves

In this section, we describe the asymptotic equations for weakly nonlinear splay and twist waves. We also consider the case of waves that propagate in the same direction as the unperturbed director field, when there is a loss of strict hyperbolicity and the orientation waves are polarized. The algebraic details of the derivations are summarized in Appendix A.

3.1 Splay waves

Weakly nonlinear asymptotics for genuinely nonlinear splay waves leads to the quadratically nonlinear Hunter-Saxton (HS) equation [2]. We summarize here the expansion for weakly nonlinear, non-planar splay waves.

We look for a high-frequency asymptotic solution $n^\varepsilon$ of (2.5) with phase $\Phi(x, t)$, depending on a small parameter $\varepsilon$, of the form

$$
n^\varepsilon(x, t) = n \left( \frac{\Phi(x, t)}{\varepsilon}, x, t; \varepsilon \right),
$$

(3.1)

$$
n(\theta, x, t; \varepsilon) = n_0(x, t) + \varepsilon n_1(\theta, x, t) + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+.
$$

(3.2)
In (3.2), \( n_0(x, t) \) is a smooth solution of (2.5) that is independent of \( \varepsilon \).

We consider localized waves, such as pulses or fronts, rather than periodic waves. The resulting asymptotic solutions are valid near the wavefront \( \Phi(x, t) = 0 \), where \( \theta = O(1) \), but they need not be uniformly valid in \( \theta \). Global solutions may be obtained by matching these ‘inner’ solutions for localized waves with suitable ‘outer’ solutions (see Section 4). For periodic waves, there are additional mean-field interactions (see Section 7).

We define the local frequency \( \omega(x, t) \) and wavenumber \( k(x, t) \) by

\[
\omega = -\Phi_t, \quad k = \nabla \Phi. \tag{3.3}
\]

We assume that \( k \neq 0 \). It follows from the expansion that \( \omega, k \) satisfy the linearized dispersion relation (2.6), and we suppose that they satisfy the splay-wave dispersion relation, \( \omega^2 = a^2 (k; n_0) \). The phase \( \Phi \) then satisfies the linearized eikonal equation

\[
\Phi_t^2 - a^2 (\nabla \Phi; n_0) = 0.
\]

We assume that \( \Phi(x, t) \) is single-valued and caustics do not arise.

We find that (see Appendix A.2)

\[
n_1(\theta, x, t) = u(\theta, x, t) R(x, t),
\]

where \( R(x, t) \) is defined by (2.9). The scalar wave-amplitude function \( u(\theta, x, t) \) satisfies the HS-equation

\[
(u_t + a \cdot \nabla u + \Gamma uu_\theta + Pu)_\theta = \frac{1}{2} \Gamma u^2_\theta. \tag{3.4}
\]

In this equation, \( a(x, t) \) is the linearized group velocity vector \( (a = \nabla_k \omega) \),

\[
a = \frac{1}{\omega} [\alpha k - (\alpha - \gamma) (k \cdot n_0) n_0],
\]

\( \Gamma(x, t) \) is the genuine-nonlinearity coefficient \( (\Gamma = \nabla_{n_0} \omega \cdot R) \),

\[
\Gamma = -\left( \frac{\alpha - \gamma}{\omega} \right) (k \cdot n_0) \left[ k^2 - (k \cdot n_0)^2 \right],
\]

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and \( P(x, t) \) is given by
\[
P = \frac{\left(\omega R^2 \right)_t + \text{div} \left(\omega R^2 a\right)}{2\omega R^2},
\]
where \( R^2 = k^2 - (k \cdot n_0)^2 \). Equation (3.4) follows from the variational principle
\[
\delta \int 1 \frac{1}{2} \omega R^2 \left[u_\theta (u_t + a \cdot \nabla u) + \Gamma uu_\theta^2\right] d\theta dx dt = 0.
\]
Introducing a derivative along the rays associated with \( \Phi \),
\[
\partial_s = \partial_t + a \cdot \nabla,
\]
we may write (3.4) as an evolution equation for \( u \) along a ray,
\[
(u_s + \Gamma uu_\theta + Pu_\theta) u_\theta = \frac{1}{2} \Gamma u_\theta^2. \tag{3.5}
\]
If \( J(x, t) \) is a non-zero ray-density function such that \( J_t + \text{div} \left( Ja \right) = 0 \), then we have
\[
P = \frac{\left(\omega R^2 \right)_s}{2\omega R^2} - \frac{J_s}{2J},
\]
and the change of variables
\[
u \mapsto \sqrt{\frac{\omega R^2}{J}} u, \quad \partial_\theta \mapsto \partial_x, \quad \partial_s \mapsto \sqrt{\frac{J}{\omega R^2}} \Gamma \partial_t
\]
reduces (3.5) to (1.2).

3.2 Twist waves

We consider the propagation of weakly nonlinear twist waves with non-zero wavenumber vector \( k \) through an unperturbed director field \( n_0 \). We assume that \( \alpha \neq \beta \) and \( k \) is not parallel to \( n_0 \). These conditions ensure the strict hyperbolicity of the system.

As a result of their linear degeneracy, the effect of nonlinearity on the twist waves is cubic. We therefore look for an asymptotic solution of (2.5) of the form
\[ n^\varepsilon(x, t) = n \left( \frac{\Phi(x, t)}{\varepsilon}, t; \varepsilon \right), \tag{3.6} \]

\[ n(\theta, x, t; \varepsilon) = n_0(x, t) + \varepsilon^{1/2} n_1(\theta, x, t) + \varepsilon n_2(\theta, x, t) + O(\varepsilon^{3/2}) \tag{3.7} \]

as \( \varepsilon \to 0^+ \), where \( n_0 \) is a solution of (2.5). We assume that the local frequency and wavenumber (3.3) satisfy the linearized dispersion relation for twist waves, \( \omega^2 = b^2 (k; n_0) \), so that the phase \( \Phi \) satisfies the eikonal equation

\[ \Phi_t^2 - b^2 (\nabla \Phi; n_0) = 0. \]

We find that (see Appendix A.3)

\[ n_1 = v S, \tag{3.8} \]

\[ n_2 = - \left( \frac{\beta - \gamma}{\alpha - \beta} \right) (k \cdot n_0) u R + \frac{1}{2} v^2 (k \times S), \tag{3.9} \]

where \( R(x, t), S(x, t) \) are defined in (2.9), (2.10) and the scalar amplitude-functions \( u(\theta, x, t), v(\theta, x, t) \) satisfy

\[ (v_t + b \cdot \nabla v + \Lambda u v_\theta + Qv)_{\theta} = 0, \tag{3.10} \]

\[ u_{\theta\theta} = v_\theta^2. \tag{3.11} \]

Here, \( b(x, t) \) is the group velocity vector,

\[ b = \frac{1}{\omega} \left[ \beta k - (\beta - \gamma) (k \cdot n_0) n_0 \right], \]

\( \Lambda(x, t) \) is given by

\[ \Lambda = \frac{(\beta - \gamma)^2}{\omega(\alpha - \beta)} (k \cdot n_0)^2 \left[ k^2 - (k \cdot n_0)^2 \right], \]

and \( Q(x, t) \) is given by

\[ Q = \frac{(\omega S^2)_t + \text{div} (\omega S^2 b)}{2\omega S^2}, \]

where \( S^2 = k^2 - (k \cdot n_0)^2 \). (With the normalization we adopt for \( R \) and \( S \), we have \( R^2 = S^2 \).) Equations (3.10)–(3.11) follow from the variational principle

\[ \delta \int \frac{1}{2} \omega S^2 \left[ v_\theta (v_t + b \cdot \nabla v) + \Lambda \left( u v_\theta^2 + \frac{1}{2} u_\theta^2 \right) \right] d\theta dx dt = 0. \]
The solution (3.6)–(3.9) consists of a leading-order twist wave with amplitude \( v \) and a higher-order forced splay wave with amplitude \( u \) that propagates at the twist-wave velocity. The amplitude and frequency of the forced splay wave are \( O(\varepsilon) \) and \( O(1/\varepsilon) \), respectively, which is the same scaling as in the weakly nonlinear solution for a free splay wave given in Section 3.1. The term proportional to \( v^2 \) in \( n_2 \) ensures that \( n \) is a unit vector up to the first order in \( \varepsilon \).

The coefficient \( \Lambda \) of the nonlinear term in (3.10) is non-zero if \( \beta \neq \gamma \) and \( k \) is not parallel or orthogonal to \( n_0 \). If \( k, n_0 \) are constant and \( k \) is orthogonal to \( n_0 \), then there are exact large-amplitude traveling twist-wave solutions \([18,19,20]\) and no weakly nonlinear effects arise. If \( k \) is parallel to \( n_0 \), then there is a loss of strict hyperbolicity and one obtains a system of asymptotic equations instead of a scalar equation. We consider this case in Section 3.3.

Introducing a derivative along the rays associated with \( \Phi \),

\[
\partial_s = \partial_t + b \cdot \nabla,
\]

we may write (3.10)–(3.11) as

\[
(v_s + \Lambda uv_\theta + Qv)_{\theta} = 0, \quad u_{\theta\theta} = v_\theta^2.
\] (3.12)

If \( K(x,t) \) is a non-zero ray-density function such that \( K_t + \text{div}(Kb) = 0 \), then we have

\[
Q = \frac{(\omega S^2)_s}{2\omega S^2} - \frac{K_s}{2K},
\]

and the change of variables

\[
u \mapsto \sqrt{\frac{\omega S^2}{K}} u, \quad v \mapsto \sqrt{\frac{\omega S^2}{K}} v, \quad \partial_{\theta} \mapsto \partial_x, \quad \partial_s \mapsto \frac{K\Lambda}{\omega S^2} \partial_t
\]

reduces (3.12) to (1.4).

### 3.3 Polarized waves

The equations of motion (2.5) are invariant under spatial rotations and reflections that leave \( n \) fixed. As a consequence of this invariance, there is a loss of strict hyperbolicity and genuine nonlinearity for waves that propagate in the same direction as \( n \). The resulting polarized orientation waves are described by a cubically nonlinear, rotationally invariant asymptotic equation.
An analogous phenomenon occurs for rotationally invariant waves in first-order hyperbolic systems of conservation laws [21].

We suppose that the unperturbed director field \( \mathbf{n}_0 \) is constant, and consider waves with constant wavenumber vector \( \mathbf{k} = k \mathbf{n}_0 \) parallel to \( \mathbf{n}_0 \). We look for an asymptotic solution \( \mathbf{n}^\varepsilon \) of (2.5) of the form

\[
\mathbf{n}^\varepsilon = \mathbf{n} \left( \frac{\mathbf{k} \cdot \mathbf{x} - \omega t}{\varepsilon}, \mathbf{x}, t; \varepsilon \right),
\]

\[
\mathbf{n}(\theta, \mathbf{x}, t; \varepsilon) = \mathbf{n}_0 + \varepsilon^{1/2} \mathbf{n}_1(\theta, \mathbf{x}, t) + \varepsilon \mathbf{n}_2(\theta, \mathbf{x}, t) + O(\varepsilon^{3/2})
\]
as \( \varepsilon \to 0^+ \).

We find that \( \omega^2 = \gamma k^2 \) (see Appendix A.4), and

\[
\mathbf{n}_1 = \mathbf{u} \quad \text{where} \quad \mathbf{n}_0 \cdot \mathbf{u} = 0,
\]

\[
\mathbf{n}_2 = -\frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) \mathbf{n}_0.
\]

The leading-order perturbation \( \mathbf{u}(\theta, \mathbf{x}, t) \) satisfies the equation

\[
\mathbf{u}_{\theta t} + \frac{\omega}{k} \mathbf{n}_0 \cdot \nabla \mathbf{u}_\theta + \frac{(\alpha - \beta) k^2}{2\omega} (\mathbf{u} \cdot \mathbf{u})_{\theta} \mathbf{u} \\
+ \frac{(\beta - \gamma) k^2}{2\omega} \left\{ (\mathbf{u} \cdot \mathbf{u}) \mathbf{u}_\theta - (\mathbf{u}_\theta \cdot \mathbf{u}) \mathbf{u} \right\} = 0,
\]

which is derived from the variational principle

\[
\delta \int \frac{1}{2} \left\{ \mathbf{u}_{\theta t} \cdot \left( \mathbf{u}_t + \frac{\omega}{k} \mathbf{n}_0 \cdot \nabla \mathbf{u} \right) + \frac{(\alpha - \beta) k^2}{2\omega} (\mathbf{u} \cdot \mathbf{u})^2 \right. \\
+ \frac{(\beta - \gamma) k^2}{2\omega} (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u}_\theta \cdot \mathbf{u}_\theta) \left\} \, d\theta dx dt = 0.
\]

Making the change of variables

\[
\partial_t + \frac{\omega}{k} \mathbf{n}_0 \cdot \nabla \mapsto \partial_t, \quad \partial_\theta \mapsto \partial_x,
\]

and rescaling \( \mathbf{u} \), we can write (3.13) in a normalized form for \( \mathbf{u}(x, t) \in \mathbb{R}^2 \) as

\[
\mathbf{u}_{xt} + (\mu - \nu) (\mathbf{u} \cdot \mathbf{u}_x) \mathbf{u} + \nu [(\mathbf{u} \cdot \mathbf{u}) \mathbf{u}_x]_x - \nu (\mathbf{u}_x \cdot \mathbf{u}) \mathbf{u} = 0,
\]

(3.14)
where
\[ \mu = \frac{\alpha - \gamma}{\gamma}, \quad \nu = \frac{\beta - \gamma}{\gamma}. \]

The corresponding variational principle is
\[ \delta \int \frac{1}{2} \left\{ u_x \cdot u_t + (\mu - \nu) (u \cdot u_x)^2 + \nu (u \cdot u) (u_x \cdot u_x) \right\} \, dx \, dt = 0. \]

Writing \( u = (u \cos v, u \sin v) \), we find that this variational principle becomes
\[ \delta \int \frac{1}{2} \left\{ u_x u_t + \mu u^2 u_x^2 + u^2 (v_x v_t + \nu u^2 v_x^2) \right\} \, dx \, dt = 0. \] (3.15)

This result is consistent with what we obtain by expanding the one-dimensional variational principle (4.1)–(4.2) as \( \varphi \to 0 \), when
\[ a^2 \sim a_0^2 + (\alpha - \gamma) \varphi^2, \quad b^2 \sim a_0^2 + (\beta - \gamma) \varphi^2, \quad q^2 \sim \varphi^2, \]
and making a unidirectional approximation \( \partial_t \sim -a_0 \partial_x \) in the resulting Lagrangian.

The Euler-Lagrange equations for (3.15) are
\begin{align*}
(u_t + \mu u^2 u_x)_x - \mu uu_x^2 - uu_x (v_t + 2\nu u^2 v_x) &= 0, \quad (3.16) \\
(v_t + \nu u^2 v_x)_x + 2\nu uu_x v_x + \frac{1}{u} (u_x v_t + u_t v_x) &= 0. \quad (3.17)
\end{align*}

This system is a coupled pair of wave equations for \( u \) and \( v \). The radial mode \( u \) has velocity \( \mu u^2 \), so it is genuinely nonlinear when \( u \neq 0 \), while the angular mode \( v \) has velocity \( \nu u^2 \), so it is linearly degenerate. If \( v \) is constant, corresponding to a plane-polarized wave, we recover the scalar cubic equation (1.7) for \( u \). Nonlinear circularly polarized waves do not exist, however, since variations in the angular variable \( v \) force variations in the radial variable \( u \).

4 One-dimensional equations

We consider a director field
\[ \mathbf{n}(x, t) = (n_1(x, t), n_2(x, t), n_3(x, t)) \]
that depends upon a single space variable \( x = x_1 \). Writing
\[
\mathbf{n} = (\cos \varphi, \sin \varphi \cos \psi, \sin \varphi \sin \psi),
\]
where \( \varphi(x, t), \psi(x, t) \) are spherical polar angles, we find that the variational principle (2.4) becomes
\[
\delta \int_{\mathbb{R}} \left\{ \frac{1}{2} \left( \varphi_t^2 - a^2(\varphi) \varphi_x^2 + q^2(\varphi) \left[ \psi_t^2 - b^2(\varphi) \psi_x^2 \right] \right) \right\} \, dx \, dt = 0,
\]
(4.1)
with
\[
\begin{align*}
a^2(\varphi) &= \alpha \sin^2 \varphi + \gamma \cos^2 \varphi, \\
b^2(\varphi) &= \beta \sin^2 \varphi + \gamma \cos^2 \varphi, \\
q^2(\varphi) &= \sin^2 \varphi.
\end{align*}
\]
(4.2)

The Euler-Lagrange equation associated with (4.1) is a system of wave equations,
\[
\begin{align*}
\varphi_{tt} - \left( \frac{a^2 \varphi_x}{x} \right)_x + a a' \varphi_x^2 + q^2 b b' \psi_x^2 - q q' \left( \psi_t^2 - b^2 \psi_x^2 \right) &= 0, \\
\psi_{tt} - \left( \frac{b^2 \psi_x}{x} \right)_x + \frac{2q'}{q} \left( \varphi_t \psi_t - b^2 \varphi_x \psi_x \right) &= 0,
\end{align*}
\]
(4.3)
(4.4)
where the prime denotes a derivative with respect to \( \varphi \). The angle \( \varphi \) corresponds to splay waves and the angle \( \psi \) to twist waves. Both wave-speeds \( (a, b) \) are functions only of \( \varphi \), and the wave equation for \( \varphi \) is forced by terms that are proportional to quadratic functions of derivatives of \( \psi \).

\section{The initial value problem}

Hunter and Saxton [2] construct an asymptotic solution \( \varphi(x, t; \varepsilon) \) of the initial value problem for the scalar wave equation (1.1) in \(-\infty < x < \infty \) and \( t > 0 \) with weakly nonlinear splay-wave initial data,
\[
\varphi_0(x, 0; \varepsilon) = \varphi_0 + \varepsilon f \left( \frac{x}{\varepsilon} \right), \quad \varphi_t(x, 0; \varepsilon) = g \left( \frac{x}{\varepsilon} \right),
\]
where \( f, g \) have compact support, and \( \varepsilon \) is a small parameter. The solution consists of a superposition of right and left moving weakly nonlinear splay waves that originate from \( x = 0 \), whose width in \( x \) is of the order \( \varepsilon \). The splay
waves are separated by a slowly-varying, small-amplitude perturbation of the constant state $\varphi_0$ that satisfies a linearized wave equation.

Here, we construct an asymptotic solution for $\varphi(x,t;\varepsilon)$, $\psi(x,t;\varepsilon)$ of the system (4.3)–(4.4) with weakly nonlinear twist-wave initial data,

$$\begin{align*}
\varphi(x,0;\varepsilon) &= \varphi_0, & \varphi_t(x,0;\varepsilon) &= 0, \\
\psi(x,0;\varepsilon) &= \varepsilon^{1/2} f\left(\frac{x}{\varepsilon}\right), & \psi_t(x,0;\varepsilon) &= \varepsilon^{-1/2} g\left(\frac{x}{\varepsilon}\right),
\end{align*}$$

(4.5)

where $f$ and $g$ are smooth, compactly supported functions. We suppose, without loss of generality, that the unperturbed constant value of $\psi$ is equal to zero.

We use a 0-subscript on a function of $\varphi$ to denote evaluation at $\varphi = \varphi_0$. We assume that $a_0 \neq b_0$, so that (4.3)–(4.4) is strictly hyperbolic at $\varphi_0$. The system will then remain strictly hyperbolic, at least in some time-interval of the order one. We further assume that $b'_0 \neq 0$, otherwise the leading-order nonlinear effects studied below vanish. In the case of the director-field wave speeds (4.2), these assumptions mean that $\varphi_0 \neq n\pi/2$ for $n \in \mathbb{Z}$.

Although $\varphi$ is initially constant, it does not remain so. As we will show, the weakly nonlinear twist wave, whose initial energy

$$\frac{1}{2} \int_{-\infty}^{\infty} \left\{ \psi_t^2 + b^2(\varphi) \psi_x^2 \right\} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \left\{ g^2(\theta) + b_0'^2 f_\theta^2(\theta) \right\} \, d\theta$$

is of the order one, generates a slowly-varying ‘outer’ splay-wave whose amplitude is of the order one.

We will construct an asymptotic solution of this initial value problem as follows.

(1) In a short initial layer, when $t = O(\varepsilon)$, we use linearized theory. The initial data splits up into right and left moving twist waves.

(2) For $t = O(1)$, nonlinear effects become important, and we use the method of matched asymptotic expansions, with different expansions for the twist and splay waves.

(a) The twist waves are small-amplitude, localized waves, which vary on a spatial scale of the order $\varepsilon$. We describe them by means of the weakly nonlinear asymptotic equations for twist waves derived above. We call these the ‘inner’ solutions.

(b) Away from the twist waves, the leading-order solution is a large-amplitude splay wave, which varies on a spatial scale of the order 1. We call this the ‘outer’ solution.
Fig. 2. Characteristic structure for the solution of the initial value problem (4.3)–(4.5): (a) Fast twist waves; (b) Slow twist waves. The dashed lines are the trajectories of the twist waves, and the solid lines are the characteristic curves associated with the splay wave.

(c) We obtain jump conditions for the ‘outer’ splay-wave solution across the trajectories of the right and left moving twist waves by matching it with the ‘inner’ twist-wave solutions.
(d) We obtain initial data for the nonlinear solution by matching it with the linearized solution as $t \to 0^+$.

The matching between the inner twist waves and the outer splay wave, and the structure of the nonlinear solution, depend on whether the twist waves are faster or slower than the splay waves (see Figure 2). In deriving the jump conditions for the splay wave below, we will consider these cases separately. For liquid crystals, we have $\beta < \alpha$, meaning that twist deformations are not as ‘stiff’ as splay deformations, and the twist waves are slower.

4.2 Initial layer

First, we consider the solution of (4.3)–(4.4), (4.5) in a short initial layer when $t = O(\varepsilon)$. Because of the finite propagation speed of the system, the solution is constant, with $\varphi = \varphi_0$, $\psi = 0$, outside an interval of width of the order $\varepsilon$ containing $x = 0$.

Near $x = 0$, we look for an asymptotic solution of the form

$$\varphi = \Phi_0(X, T) + O(\varepsilon), \quad \psi = \varepsilon^{1/2} \Phi_1(X, T) + O(\varepsilon), \quad X = \frac{x}{\varepsilon}, \quad T = \frac{t}{\varepsilon}.$$ 

Using this expansion in (4.3)–(4.5), we find that $\Phi_0, \Psi_1$ satisfy
\[
\Phi_{0TT} - \left( a^2(\Phi_0)\Phi_{0XX} \right)_X + a(\Phi_0)a'(\Phi_0)\Phi_{0X}^2 = 0,
\]
\[
\Psi_{1TT} - \left( b^2(\Phi_0)\Psi_{1XX} \right)_X + \frac{2q'(\Phi_0)}{q(\Phi_0)} \left( \Phi_{0T}\Psi_{1T} - b^2(\Phi_0)\Phi_{0X}\Psi_{1X} \right) = 0,
\]
\[
\Phi_0(X, 0) = \varphi_0, \quad \Phi_{0T}(X, 0) = 0,
\]
\[
\Psi_1(X, 0) = f(X), \quad \Psi_{1T}(X, 0) = g(X).
\]

The initial value problem for \( \Phi_0 \) has the constant solution \( \Phi_0 = \varphi_0 \), and therefore \( \Psi_1 \) satisfies the linear wave equation

\[
\Psi_{1TT} - b_0^2\Psi_{1XX} = 0,
\]
\[
\Psi_1(X, 0) = f(X), \quad \Psi_{1T}(X, 0) = g(X).
\]

The solution is

\[
\Psi_1(X, T) = F_R(X - b_0T) + F_L(X + b_0T), \quad (4.6)
\]
\[
F_R(\theta) = \frac{1}{2} f(\theta) + \frac{1}{2b_0} \int_{-\infty}^{\infty} g(\xi) \, d\xi, \quad (4.7)
\]
\[
F_L(\theta) = \frac{1}{2} f(\theta) - \frac{1}{2b_0} \int_{-\infty}^{\infty} g(\xi) \, d\xi. \quad (4.8)
\]

Since \( f, g \) have compact support, the functions \( F_{R\theta}, F_{L\theta} \) have compact support.

4.3 Twist waves

Next, we consider the propagation of the twist-waves for \( t = O(1) \) through a possibly non-uniform splay-wave field \( \varphi(x, t) \). For definiteness, we consider the right-moving twist wave, which moves with velocity \( b(\varphi) \). The trajectory \( x = s_R(t) \) of the wave satisfies

\[
\frac{ds_R}{dt} = b_R,
\]

where an \( R \)-subscript on a function of \( \varphi \) denotes evaluation at \( \varphi = \varphi_R(t) \), with \( \varphi_R(t) = \varphi(s_R(t), t) \). For the initial value problem, we have \( s_R(0) = 0 \) and \( \varphi_R(0) = \varphi_0 \).

We introduce a stretched inner variable near this trajectory,

\[
\theta = \frac{x - s_R(t)}{\varepsilon}.
\]
We find that the weakly nonlinear twist-wave solution of (4.3)–(4.4) is
\[
\varphi = \varphi_R(t) + \varepsilon \varphi_2(\theta, t) + O(\varepsilon^{3/2}),
\]
\[
\psi = \varepsilon^{1/2} \psi_1(\theta, t) + O(\varepsilon),
\]
where \( \varphi_2, \psi_1 \) satisfy
\[
\left[ \psi_{1t} + b'_R \varphi_2 \psi_{1\theta} + \left( \frac{b_R t}{2b_R} + \frac{q_R t}{q_R} \right) \psi_1 \right] = 0,
\]
\[
\varphi_{2\theta} = \left( \frac{q_R^2 b_R b'_R}{a_R^2 - b_R^2} \right) \psi_{1\theta}^2.
\]

We may transform (4.11)–(4.12) into (1.4) by a suitable change of variables, in which \( \theta \) corresponds to \( x \), \( \psi_1 \) to \( v \), and \( \varphi_2 \) to \( u \).

Matching the twist-wave solution (4.10) as \( t \to 0^+ \) with the linearized solution (4.6) as \( T \to \infty \), we get the initial condition
\[
\psi_1(\theta, 0) = F_R(\theta),
\]
where \( F_R \) is given in (4.7). Equations (4.11)–(4.12) are supplemented with suitable boundary conditions for \( \varphi_2 \) and \( \psi_1 \) at \( \theta = \pm \infty \), which we consider further below.

The main result we need in order to obtain equations for the ‘outer’ splay wave solution is the following jump condition for \( \varphi_2 \) across the twist wave.

**Proposition 1** Suppose that \( \varphi_2, \psi_1 \) are smooth solutions of (4.11)–(4.12) such that \( \psi_{1\theta}(\cdot, t) \) has compact support. Then
\[
\frac{d}{dt} \left\{ \left( \frac{a_R^2 - b_R^2}{b'_R} \right) [\varphi_{2\theta}] \right\} + \left( \frac{a_R^2 - b_R^2}{2} \right) [\varphi_{2\theta}^2] = 0,
\]
where \([\cdot]\) denotes the jump from \( \theta = -\infty \) to \( \theta = \infty \).

**Proof.** Since \( \psi_{1\theta} \) has compact support, it follows from (4.12) that \( \varphi_2 \) is a linear function of \( \theta \) for large negative and positive values of \( \theta \). Moreover,
\[
[\varphi_{2\theta}] = \left( \frac{q_R^2 b_R b'_R}{a_R^2 - b_R^2} \right) \int_{-\infty}^{\infty} \psi_{1\theta}^2 d\theta.
\]
Multiplying (4.11) by \( \psi_{1\theta} \) and using (4.12) to rewrite the result, we get

\[
\left( \psi_{1\theta}^2 \right)_t + \left[ b_R \varphi_2 \psi_{1\theta}^2 + \frac{1}{2} \left( \frac{a_R^2 - b_R^2}{q_R^2 b_R} \right) \varphi_{2\theta}^2 \right]_\theta + \left( \frac{b_{Rt}}{b_R} + \frac{2q_{Rt}}{q_R} \right) \psi_{1\theta}^2 = 0.
\]

Integrating this equation with respect to \( \theta \), and using the fact that \( \psi_{1\theta} \) has compact support, we find that

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \psi_{1\theta}^2 d\theta + \frac{1}{2} \left( \frac{a_R^2 - b_R^2}{q_R^2 b_R} \right) \int_{-\infty}^{\infty} \varphi_{2\theta}^2 d\theta + \left( \frac{b_{Rt}}{b_R} + \frac{2q_{Rt}}{q_R} \right) \int_{-\infty}^{\infty} \psi_{1\theta}^2 d\theta = 0.
\]

Using (4.15) to eliminate the integral of \( \psi_{1\theta}^2 \) from this equation, and rewriting the result, we get (4.14).

4.4 Matching: fast twist waves

This case corresponds to \( 0 < \alpha < \beta \), when \( 0 < a < b \). Since the twist waves are faster than the splay waves, they propagate into a constant state \( \varphi = \varphi_0 \), \( \psi = 0 \) ahead of them, and generate splay waves behind them. (See Figure 2(a).) It follows that the right and left moving twist waves move at a constant velocity along the trajectories \( x = b_0 t \) and \( x = -b_0 t \), respectively.

We consider the right-moving twist wave for definiteness. The appropriate inner variable is then

\[
\theta = \frac{x - b_0 t}{\varepsilon}.
\]

The weakly nonlinear solution inside the twist wave must match for large positive \( \theta \) with the constant initial state ahead of the wave. This condition implies that \( \varphi_2(\theta, t) \) and \( \psi_1(\theta, t) \) in (4.9)--(4.10), with \( \varphi_R = \varphi_0 \), satisfy the boundary conditions

\[
\varphi_2(\infty, t) = 0, \quad \varphi_{2\theta}(\infty, t) = 0, \quad \psi_1(\infty, t) = 0.
\]

The derivative \( \varphi_{2\theta} \) jumps from zero at \( \theta = \infty \) to a value

\[
\varphi_{2\theta}(-\infty, t) = \sigma_R(t)
\]
at $\theta = -\infty$. It follows from the jump condition (4.14), with $a_R = a_0$, $b_R = b_0$, $b'_R = b'_0$ constants, that $\sigma_R$ satisfies

$$
\frac{d\sigma_R}{dt} + \frac{1}{2} b'_0 \sigma_R^2 = 0.
$$

From (4.12) and (4.13), we have $\sigma_R(0) = \sigma_{R0}$ where

$$
\sigma_{R0} = -\left( \frac{q_0^2 b_0 b'_0}{a_0^2 - b_0^2} \right) \int_{-\infty}^{\infty} F_{R0}^2 \, d\theta.
$$

The solution of this Riccati equation,

$$
\sigma_R(t) = \frac{\sigma_{R0}}{1 + \sigma_{R0} b'_0 t/2},
$$

is defined for all $t \geq 0$, since $\sigma_{R0} b'_0 \geq 0$ when $0 < a_0 < b_0$.

In Section 5, we prove that when $a_0 < b_0$, equations (4.11)–(4.13), (4.17), with $a_R = a_0$ and so on, have a smooth solution defined for all $t \geq 0$. We note that the derivative $\varphi_{2\theta}(t)$ decays as $t \to \infty$. This is a result of the fact that the twist wave radiates energy away from it in the form of splay waves. It also follows from the solution that the twist wave is a rarefaction, in the sense that its characteristics spread out with increasing time.

A similar analysis applies to the left-moving twist wave, in which

$$
\theta = \frac{x + b_0 t}{\varepsilon},
$$

and $\varphi_{2\theta}(\theta, t) = 0$ for $\theta$ sufficiently large and negative. We find that $\varphi_{2\theta} = \sigma_L$ for $\theta$ sufficiently large and positive, where

$$
\sigma_L(t) = \frac{\sigma_{L0}}{1 - \sigma_{L0} b'_0 t/2},
$$

with

$$
\sigma_{L0} = \left( \frac{q_0^2 b_0 b'_0}{a_0^2 - b_0^2} \right) \int_{-\infty}^{\infty} F_{L0}^2 \, d\theta.
$$

These 'inner' twist-wave solutions provide matching conditions for an 'outer' splay-wave solution $\varphi(x, t)$. Using (4.9) with $\varphi_R = \varphi_0$ to rewrite the condition
for the right-moving twist wave,
\[ \varphi_{2\theta}(\theta, t) \sim \sigma_R(t) \quad \text{as } \theta \to -\infty \]

where \( \theta \) is given by (4.16), in terms of the outer solution \( \varphi(x, t) \), and equating the outer limit of the inner solution with the inner limit of the outer solution, we find that
\[ \varphi_x(x, t) \sim \sigma_R(t) \quad \text{as } x \to b_0 t^- . \]

Furthermore, the leading-order outer solution for \( \varphi \) is continuous across the twist-wave, and \( \psi \) is higher-order in \( \varepsilon \). We obtain a condition for \( \varphi \) as \( x \to -b_0 t^+ \) in an analogous way.

Summarizing these results for the leading-order outer splay-wave solution \( \varphi(x, t) \), we find that \( \varphi = \varphi_0 \) is constant if \( x > b_0 t \) or \( x < -b_0 t \). Inside the region \( -b_0 t < x < b_0 t \), we find that \( \varphi \) satisfies (1.1), with data on the space-like lines \( x = \pm b_0 t \) given by
\[
\begin{align*}
\varphi(b_0 t, t) &= \varphi_0, & \varphi_x(b_0 t, t) &= \sigma_R(t), \\
\varphi(-b_0 t, t) &= \varphi_0, & \varphi_x(-b_0 t, t) &= \sigma_L(t),
\end{align*}
\]

where \( \sigma_R, \sigma_L \) are given by (4.18), (4.19), respectively.

Since the initial value problem for (1.1) is well-posed, this Cauchy problem is presumably solvable. The solution \( \varphi \) may form singularities, in which case it would have to be continued by a weak solution.

4.5 Matching: slow twist waves

This case corresponds to \( \alpha > \beta > 0 \), when \( a > b > 0 \). Since the twist waves are slower than the splay waves, they generate splay waves both in front and behind them. As a result, the twist waves are embedded inside a splay-wave field. (See Figure 2(b).) The speeds of the twist waves depend on the splay-wave field, leading to a free-boundary problem for the trajectories of the twist waves, coupled with a wave equation for the splay wave that is subject to jump conditions across the twist-wave trajectories.

We will not write out detailed asymptotic equations for the weakly nonlinear twist waves in this case, but we summarize the equations satisfied by the leading-order outer splay-wave solution \( \varphi(x, t) \). The main point is the derivation of jump conditions for \( \varphi_x \) across the twist-wave trajectories.
The right and left moving twist waves are located at $x = s_R(t)$ and $x = s_L(t)$, respectively, where

$$\frac{ds_R}{dt}(t) = b(\varphi(s_R(t), t)),$$  \hspace{1cm}  (4.20)
$$\frac{ds_L}{dt}(t) = -b(\varphi(s_L(t), t)),$$  \hspace{1cm}  (4.20)

$s_R(0) = 0, \quad s_L(0) = 0.$  \hspace{1cm}  (4.21)

The solution $\varphi$ is continuous across $x = s_R(t)$ and $x = s_L(t)$, so that

$$\left[ \varphi \right]_R = 0, \quad \left[ \varphi \right]_L = 0.$$  \hspace{1cm}  (4.22)

Here, and below, we use $[\cdot]_R$, $[\cdot]_L$ to denote the jumps across $x = s_R(t)$, $x = s_L(t)$, respectively, meaning that

$$\left[ \varphi \right]_R(t) = \lim_{x \to s_R(t)^+} \varphi(x, t) - \lim_{x \to s_R(t)^-} \varphi(x, t),$$  \hspace{1cm}  (4.22)
$$\left[ \varphi \right]_L(t) = \lim_{x \to s_L(t)^+} \varphi(x, t) - \lim_{x \to s_L(t)^-} \varphi(x, t).$$  \hspace{1cm}  (4.22)

Considering the right-moving twist wave for definiteness, we have

$$\varphi_{2\theta}(\theta, t) \to \sigma_+(t) \quad \text{as} \ \theta \to \infty,$$  \hspace{1cm}  (4.23)
$$\varphi_{2\theta}(\theta, t) \to \sigma_-(t) \quad \text{as} \ \theta \to -\infty,$$  \hspace{1cm}  (4.24)

for some functions $\sigma_+(t), \sigma_-(t)$. The corresponding matching conditions for $\varphi(x, t)$ are

$$\lim_{x \to s_R^+(t)} \varphi_x(x, t) = \sigma_+(t), \quad \lim_{x \to s_R^-(t)} \varphi_x(x, t) = \sigma_-(t).$$  \hspace{1cm}  (4.25)

From (4.14), (4.23)–(4.24), (4.25), and the analogous equations for the left-moving twist wave, we find that $\varphi_x$ satisfies the following jump conditions across the twist waves:

$$\frac{d}{dt} \left\{ \left( \frac{a^2_R - b^2_R}{b'_R} \right) [\varphi_x]_R \right\} + \left( \frac{a^2_R - b^2_R}{2} \right) [\varphi^2_x]_R = 0,$$  \hspace{1cm}  (4.26)
$$\frac{d}{dt} \left\{ \left( \frac{a^2_L - b^2_L}{b'_L} \right) [\varphi_x]_L \right\} - \left( \frac{a^2_L - b^2_L}{2} \right) [\varphi^2_x]_L = 0.$$  \hspace{1cm}  (4.27)

Furthermore, from (4.13) and (4.15), and their analogs for left-moving waves, we get the initial conditions
Finally, matching the outer solution with the initial, linearized solution, we find that \( \varphi(x, t) \) satisfies the initial conditions

\[
\varphi(x, 0) = \varphi_0, \quad \varphi_t(x, 0) = 0, \quad \text{for } x \neq 0.
\]  

(4.30)

Summarizing, we find that the free-boundary problem for \( \varphi, s_R, s_L \) consists of (1.1) for \( \varphi(x, t) \) in \( -\infty < x < \infty, \ t > 0 \) with the initial condition (4.30). The functions \( s_R(t), s_L(t) \) satisfy (4.20)–(4.21), and \( \varphi(x, t) \) satisfies the jump conditions (4.22), (4.26)–(4.29) across the curves \( x = s_R(t), x = s_L(t) \).

We will not investigate this problem here. We remark, however, that Proposition 3 in Section 5 implies that there is a smooth solution of the ‘inner’ asymptotic equations for the weakly nonlinear twist wave (4.11)–(4.13), (4.23)–(4.24) whenever the derivatives \( \sigma_\pm(t) = \varphi_x(s_\pm_R(t), t) \) of the ‘outer’ splay-wave solution of the free-boundary problem on either side of the twist wave are smooth functions of time.

5 Method of characteristics

In this section, we solve (1.4) by the method of characteristics. The explicit nature of this solution is related to the complete integrability of the equation, which is discussed in the next section.

**Proposition 2** Let \((\xi, \tau)\) be characteristic coordinates for the PDE (1.4), where \( x = X(\xi, \tau), \ t = \tau, \) and write \( U(\xi, \tau) = u(X(\xi, \tau), \tau), \ V(\xi, \tau) = v(X(\xi, \tau), \tau). \) Then a formal solution of (1.4) is given by

\[
U = X_\tau, \quad V = F + G, \quad X = -\int_0^\xi \frac{F^2(A + B)^2}{2A_\xi B_\tau} d\xi + H,
\]  

(5.1)

where \( A(\xi), B(\tau), F(\xi), G(\tau), H(\tau) \) are arbitrary functions.
**Proof.** Writing (1.4) in terms of characteristic coordinates \((\xi, \tau)\) in which \(\tau = t\) and \(x_\tau = u\), we find that the PDE becomes
\[
X_\tau = U, \quad V_{\xi\tau} = 0, \quad U_{\xi\xi} - \frac{J_\xi}{J} U_\xi = V_\xi^2,
\]
where \(J(\xi, \tau)\) is the Jacobian \(J = X_\xi\). It follows that \(V(\xi, \tau) = F(\xi) + G(\tau)\), where \(F, G\) are functions of integration, and
\[
J_\tau = U_\xi, \quad U_{\xi\xi} - \frac{J_\xi}{J} U_\xi = F_\xi^2.
\]
The elimination of \(U\) from these equations yields a PDE for \(J\),
\[
J_{\xi\tau} - \frac{J_\xi J_\tau}{J} = F_\xi^2.
\]
Making the change of variables \(\eta = \eta(\xi)\) where \(\eta_\xi = F_\xi^2\), and \(J = -e^{-K}\), we find that this PDE transforms into an integrable Liouville equation,
\[
K_{\eta\tau} = e^K.
\]
The general solution is
\[
e^K = \frac{2A_\eta B_\tau}{(A + B)^2},
\]
where \(A(\eta)\) and \(B(\tau)\) are arbitrary functions. Integrating the equation \(X_\xi = -e^{-K}\) with respect to \(\xi\), we find that \(X\) is given by (5.1), which proves the result. \(\square\)

Next, we consider (1.4) in \(-\infty < x < \infty\),
\[
(v_t + uv_x)_x = 0, \quad (5.2)
\]
\[
u_{xx} = v_x^2, \quad (5.3)
\]
supplemented with the initial condition
\[
v(x, 0) = F(x), \quad (5.4)
\]
and the boundary conditions
\[
u_x(\infty, t) = \sigma_+(t), \quad u_x(-\infty, t) = \sigma_-(t). \quad (5.5)
\]
We assume that \( F \) is a smooth function and that \( F_x \) has compact support. Equation (5.2) then implies that \( v_x(x, t) \) has compact support in \( x \) whenever a smooth solution exists, and (5.3) implies that \( u(x, t) \) is a linear function of \( x \) for sufficiently large positive and negative values of \( x \). The boundary condition (5.5) specifies the corresponding values of \( u_x \). We illustrate the structure of the solution schematically in Figure 3.

Computing the jump condition (4.14) for (5.2)–(5.3), we find that

\[
\frac{d}{dt} [u_x] + \frac{1}{2} [u_x^2] = 0,
\]

where \([\cdot]\) denotes the jump from \( x = -\infty \) to \( x = \infty \). It follows that the data \( \sigma_\pm \) must satisfy

\[
\frac{d\sigma_+}{dt} + \frac{1}{2}\sigma_+^2 = \frac{d\sigma_-}{dt} + \frac{1}{2}\sigma_-^2.
\] (5.6)

Moreover, integrating (5.3) with respect to \( x \) at \( t = 0 \) and using (5.4)–(5.5), we find that

\[
\sigma_+(0) - \sigma_-(0) = \int_{-\infty}^{\infty} F_x^2 \, dx.
\] (5.7)

The next proposition establishes the existence of smooth solutions of the IBVP (5.2)–(5.5) for compatible data \( F \) and \( \sigma_\pm \). The solutions are not unique, since we may add an arbitrary function of time to \( v \), and an arbitrary function of time to \( u \) (together with an appropriate time-dependent translation of the spatial coordinate \( x \)). We can remove this non-uniqueness by specifying, for example, \( u, v \) as functions of time at some value of \( x \).
Proposition 3. Suppose that $F: \mathbb{R} \to \mathbb{R}$ is a smooth function and that $F_x$ has compact support. Also suppose that $\sigma_+, \sigma_- : [0, t_*) \to \mathbb{R}$ are smooth functions defined in some time interval $0 \leq t < t_*$, where $0 < t_* \leq \infty$, that satisfy (5.6), (5.7). Then there is a smooth solution of (5.2)–(5.5), defined in $-\infty < x < \infty$, $0 \leq t < t_*$. 

Proof. If $F(x) = F_0$ is constant, then $\sigma_+(t) = \sigma_-(t)$, and a solution is $u = x\sigma_+(t)$, $v = F_0$. We therefore assume that $F$ is not constant.

We then have from (5.7) that $\sigma_-(0) < \sigma_+(0)$. Equation (5.6) implies that 

$$\frac{d}{dt} (\sigma_+ - \sigma_-) + \frac{1}{2} (\sigma_+ + \sigma_-) (\sigma_+ - \sigma_-) = 0,$$

so $\sigma_-(t) < \sigma_+(t)$ for all $0 \leq t < t_*$. We choose constants $\eta_- < \eta_+ < 0$ such that 

$$\eta_+ - \eta_- = \int_{-\infty}^{\infty} F_x^2(\xi) \, d\xi,$$

and define the function $\eta(\xi)$ by 

$$\eta(\xi) = \eta_- + \int_{-\infty}^{\xi} F_x^2(\xi') \, d\xi'$$

$$= \eta_+ - \int_{\xi}^{\infty} F_x^2(\xi') \, d\xi'.$$

We also define a Jacobian $J(\xi, \tau)$ by 

$$J = \frac{[(\eta_+ - \eta) E_+ + (\eta - \eta_-) E_-]^2}{(\eta_+ - \eta_-) (\sigma_+ - \sigma_-)}.$$

(5.8)

where 

$$E_+(\tau) = \exp \left\{ -\frac{1}{4} \int_0^\tau [\sigma_+(\tau') - \sigma_-(\tau')] \, d\tau' \right\},$$

(5.9)

$$E_-(\tau) = \exp \left\{ +\frac{1}{4} \int_0^\tau [\sigma_+(\tau') - \sigma_-(\tau')] \, d\tau' \right\}.$$

(5.10)
One can verify that $J = X_\xi$ is obtained from (5.1) with

$$A(\xi) = \frac{1}{\eta(\xi)}, \quad B(\tau) = -\left[\frac{E_+(\tau) - E_-(\tau)}{\eta_+ E_+(\tau) - \eta_- E_-(\tau)}\right].$$

We then let

$$X(\xi, \tau) = \int_0^\xi J(\xi', \tau) \, d\xi',$$

$$U(\xi, \tau) = X_\tau(\xi, \tau), \quad V(\xi) = F(\xi).$$

Since $E_-, E_+ > 0$, $\eta_- < \eta_+$, and $\eta_- \leq \eta \leq \eta_+$, we see from (5.8) that $J > 0$. It follows that the transformation $x = X(\xi, \tau)$, $t = \tau$ between spatial and characteristic coordinates is smoothly invertible, and, according to Proposition 2, these expressions define a smooth solution of (5.2)–(5.3), as may be verified directly. We show that this solution satisfies the required initial and boundary conditions.

First, at $\tau = 0$, we have $E_+ = E_- = 1$ and $\sigma_+ - \sigma_- = \eta_+ - \eta_-$. It follows from (5.8) that $J = 1$ at $\tau = 0$, so $x = \xi$, and $v(x, 0) = F(x)$.

Second, using the equation

$$u_x = \frac{U_\xi}{X_\xi} = \frac{J_x}{J},$$

we compute from (5.8) that

$$u_x = 2\frac{(\eta_+ - \eta) \, dE_+}{(\eta_+ - \eta) E_+} + \frac{(\eta - \eta_-) \, dE_-}{(\eta - \eta_-) E_-} - \frac{d}{d\tau} (\sigma_+ - \sigma_-) \frac{\sigma_+ - \sigma_-}{\sigma_+ - \sigma_-}.$$ 

From (5.9)–(5.10), we have

$$\frac{dE_+}{d\tau} = -\frac{1}{4} (\sigma_+ - \sigma_-) E_+, \quad \frac{dE_-}{d\tau} = \frac{1}{4} (\sigma_+ - \sigma_-) E_-,$$

and from the jump condition (5.6), we have

$$\frac{d}{d\tau} (\sigma_+ - \sigma_-) = -\frac{1}{2} \left(\sigma_+^2 - \sigma_-^2\right).$$
Using these equations to eliminate $\tau$-derivatives from the expression for $u_x$ and simplifying the result, we get

$$u_x = \frac{(\eta_+ - \eta) \sigma_- E_+ + (\eta - \eta_-) \sigma_+ E_-}{(\eta_+ - \eta) E_+ + (\eta - \eta_-) E_-}.$$ 

It follows that $u_x = \sigma_-$ at $x = -\infty$, when $\eta = \eta_-$, and $u_x = \sigma_+$ at $x = \infty$, when $\eta = \eta_+$. $\square$

For example, let us consider what happens when the derivative $u_x$ vanishes at $x = -\infty$ or $x = \infty$. If $\sigma_- = 0$, then $\sigma_+ > 0$ satisfies the equation

$$\frac{d\sigma_+}{dt} + \frac{1}{2} \sigma_+^2 = 0,$$

which has a global smooth solution forward in time,

$$\sigma_+(t) = \frac{\sigma_+(0)}{1 + \sigma_+(0)t/2}.$$

It follows that (5.2)--(5.5) has a global smooth solution forward in time, which may be specified uniquely by the requirement that $u = v = 0$ at $x = -\infty$. This case corresponds to the one that arises for the fast twist waves analyzed in Section 4.4.

On the other hand, if $\sigma_+ = 0$, then $\sigma_- < 0$ satisfies the equation

$$\frac{d\sigma_-}{dt} + \frac{1}{2} \sigma_-^2 = 0,$$

whose solution

$$\sigma_-(t) = \frac{\sigma_-(0)}{1 + \sigma_-(0)t/2}$$

blows up as $t \uparrow t_*$, where $t_* = -2/\sigma_-(0) > 0$. Thus, a smooth solution of (5.2)--(5.5) exists only in the finite time-interval $0 \leq t < t_*$. The derivative $u_x$ blows up simultaneously in the entire semi-infinite spatial interval to the left of the support of $u_x$, so it does not appear possible to continue the smooth solution by any kind of distributional solution after the singularity forms.
6 Integrability and Hamiltonian structure

In this section, we show that the twist-wave equation (1.4) is a completely integrable, bi-Hamiltonian PDE, and that if \((u, v)\) satisfies (1.4) then \(u\) satisfies the HS-equation (1.6).

We begin by describing the relation between (1.4) and the HS-equation (1.6). In order to describe the corresponding relation for the Camassa-Holm (CH) equation at the same time, we consider the following generalization of (1.4):

\[
(v_t + uv_x)_x = 0, \quad (6.1)
\]

\[
Mu = v_x^2, \quad (6.2)
\]

where \(M\) is a self-adjoint linear operator acting on functions of \(x\) that commutes with \(\partial_x\. If \(M = \partial_x^2\), then (6.1)–(6.2) is (1.4).

We suppose that \(u, v\) are smooth solutions of (6.1)–(6.2). Differentiating (6.2) with respect to \(t\), using (6.1) to write \(v_{xt}\) in terms of \(u, v\) and their spatial derivatives, then using (6.2) to eliminate \(v\) from the result, we find that \(u\) satisfies

\[
m_t + mu_x + (mu)_x = 0 \quad \text{with } m = Mu. \quad (6.3)
\]

If \(M = \partial_x^2\), then (6.3) is the HS-equation (1.6); if \(M = \partial_x^2 - 1\), then (6.3) is the CH-equation [12],

\[
\left[(u_t + uu_x)_x - \frac{1}{2} u_x^2\right]_x = u_t + 3uu_x.
\]

Conversely, if \(m > 0\) is a smooth solution of (6.3) and \(v_x = \sqrt{m}\), then \(v\) satisfies (6.1)–(6.2).

Because of the nonlinearity of this transformation, difficulties may arise in its application to distributional solutions. For example, the function

\[
u(x, t) = \begin{cases} 
0 & \text{if } x \leq 0 \\
\frac{2}{t} x/t & \text{if } x > 0 
\end{cases}
\]

is a weak solution of the HS-equation (1.6) in \(t > 0\), and

\[
u_{xx}(x, t) = \frac{2}{t^2} \delta(x)
\]
is non-negative in the sense of distributions. There is, however, no standard way to define a distribution \( v \) such that \( u_{xx} = v_x^2 \).

The HS-equation (1.6) is bi-Hamiltonian and completely integrable [6], so (1.4) is also. Next, we consider the effect of the transformation \( v \mapsto u \) on the Hamiltonian structures of these equations.

The system (6.1)–(6.2) is obtained from the variational principle

\[
\delta \int \frac{1}{2} \left\{ -v_t v_x - u v_x^2 + \frac{1}{2} u M u \right\} \, dx \, dt = 0.
\]

Variations with respect to \( u \) yield (6.2), and variations with respect to \( v \) yield (6.1). We may eliminate \( u \) by means of the constraint equation \( u = M^{-1} (v_x^2) \) to obtain a variational principle for \( v \) alone,

\[
\delta \int \frac{1}{2} \left\{ -v_t v_x - \frac{1}{2} v_x^2 M^{-1} (v_x^2) \right\} \, dx \, dt = 0. \tag{6.4}
\]

The Euler-Lagrange equation for (6.4),

\[
\left[ v_t + M^{-1} (v_x^2) v_x \right]_x = 0, \tag{6.5}
\]

is equivalent to (6.1)–(6.2). Here, and below, we assume that operators such as \( M \) and \( \partial_x \) are invertible; in the case of differential operators, this requires the addition of suitable boundary conditions which we do not specify explicitly.

Making a Legendre transform of the Lagrangian in (6.4), we get the corresponding Hamiltonian form of (6.5),

\[
v_t = \partial_x^{-1} \left( \frac{\delta \mathcal{H}}{\delta v} \right), \quad \mathcal{H} = \frac{1}{4} \int v_x^2 M^{-1} (v_x^2) \, dx, \tag{6.6}
\]

where \( \partial_x^{-1} \) is the constant Hamiltonian operator associated canonically with the variational principle (6.4).

**Proposition 4** Let \( \{ \mathcal{F}, \mathcal{G} \} \) denote the Poisson bracket of functionals \( \mathcal{F}, \mathcal{G} \) of \( v \) associated with the constant Hamiltonian operator \( \partial_x^{-1} \),

\[
\{ \mathcal{F}, \mathcal{G} \} = \int \frac{\delta \mathcal{F}}{\delta v} \partial_x^{-1} \left( \frac{\delta \mathcal{G}}{\delta v} \right) \, dx. \tag{6.7}
\]

Under the change of variables \( u = M^{-1} (v_x^2) \), where \( M \) is a self-adjoint linear operator, the bracket (6.7) transforms formally into a Lie-Poisson bracket.
\{F, G\} = \int \frac{\delta F}{\delta u} J(u) \left( \frac{\delta G}{\delta u} \right) dx,  \tag{6.8} \\
J = -2M^{-1} (m \partial_x + \partial_x m) M^{-1}, \tag{6.9}

where \(m = Mu\).

**Proof.** First, we consider the nonlinear change of variables \(m = v^2_x\). Variations of \(v\) of the form \(v^\varepsilon = v + \varepsilon k + \ldots\) lead to variations \(m^\varepsilon = m + \varepsilon h + \ldots\) of \(m\) where \(h = 2v_x k_x\). For any functional \(F\) of \(m\), with \(F^\varepsilon = F(m^\varepsilon)\), we have

\[
\frac{d}{d\varepsilon} F^\varepsilon \bigg|_{\varepsilon=0} = \int \frac{\delta F}{\delta m} h dx.
\]

Writing \(h\) in terms of \(k\) and using the skew-adjointness of \(\partial_x\), we compute that

\[
\frac{d}{d\varepsilon} F^\varepsilon \bigg|_{\varepsilon=0} = -2 \int \left( \frac{\delta F}{\delta m} v_x \right) k dx.
\]

Since

\[
\frac{d}{d\varepsilon} F^\varepsilon \bigg|_{\varepsilon=0} = \int \frac{\delta F}{\delta v} k dx,
\]

we conclude that

\[
\frac{\delta F}{\delta v} = -2 \left( \frac{\delta F}{\delta m} \sqrt{m} \right)_x.
\]

Using this equation in (6.7) and integrating by parts, we get

\[
\{F, G\} = 4 \int \left( \frac{\delta F}{\delta m} \sqrt{m} \right)_x \frac{\delta G}{\delta m} \sqrt{m} dx
\]

\[
= -2 \int \frac{\delta F}{\delta m} (m \partial_x + \partial_x m) \frac{\delta G}{\delta m} dx.
\]

Making the linear change of variables \(u = M^{-1} m\) in this expression, and using the self-adjointness of \(M\), we get (6.8)–(6.9). \(\square\)

The Hamiltonian form of (6.3) for \(u\) corresponding to (6.6) for \(v\) is therefore

\[
u_t = J \left( \frac{\partial \mathcal{H}}{\partial u} \right), \quad \mathcal{H} = \frac{1}{4} \int uMu dx,
\]
where \(J\) is given in (6.9).

To give a second Hamiltonian structure for (6.1)–(6.2), we define a skew-adjoint operator \(K\), depending on \(v\), by

\[
K = v_x \partial_x^{-1} M^{-1} v_x, \tag{6.10}
\]

where \(M\) is a self-adjoint linear operator commuting with \(\partial_x\), as before.

We find that the operator (6.10) satisfies the Jacobi identity if the quantity

\[
(g h_x - h g_x) M f_x + (h f_x - f h_x) M g_x + (f g_x - g f_x) M h_x
\]

is an exact \(x\)-derivative for arbitrary functions \(f, g, h\). This condition holds for \(M = \partial_x^2\), since

\[
(g h_x - h g_x) f_{xxx} = [(g h_x - h g_x) f_{xx}]_x + h f_{xxx} g_{xx} - g h_{xxx} f_{xx},
\]

and the terms that are not exact derivatives cancel under a cyclic summation. The condition also holds for \(M = \partial_x^2 - 1\). Moreover, in those cases, \((c_1 \partial_x^{-1} + c_2 K)\) satisfies the Jacobi identity for arbitrary real constants \(c_1, c_2\), so that \(\partial_x^{-1}\) and \(K\) define compatible Hamiltonian structures.

The Hamiltonian form of (6.5) with respect to \(K\) is

\[
v_t = K \left( \frac{\delta \mathcal{P}}{\delta v} \right), \quad \mathcal{P} = \int v_x^2 \, dx. \tag{6.11}
\]

If \(M = \partial_x^2\), then \(K = v_x \partial_x^{-3} v_x\), and (6.11) is equivalent to (1.4).

Under the transformation \(m = v_x^2\), equation (6.11) becomes

\[
m_t = \tilde{K} \left( \frac{\delta \mathcal{P}}{\delta m} \right), \quad \mathcal{P} = \int m \, dx,
\]

\[
\tilde{K} = - (m \partial_x + \partial_x m) \left( \partial_x^{-1} M^{-1} \right) (m \partial_x + \partial_x m),
\]

which gives (6.3).

When \(M = \partial_x^2\), equations (6.6) and (6.11) provide a bi-Hamiltonian structure for (1.4). One can then obtain an infinite sequence of commuting Hamiltonian flows by recursion. We will not write them out explicitly here, but we remark
that among them is a Hamiltonian structure for \( v \) which maps to the Hamiltonian structure for \( u \) canonically associated with the variational principle in (1.3).

Lax pairs for (1.4) follow directly by transformation of the Lax pairs for the HS-equation [6]. We define

\[
L = \partial_x^{-1} v_x^2 \partial_x^{-1}, \quad A = \frac{1}{2} (u \partial_x + \partial_x u).
\]

Then, using the identity \( f \partial_x^{-1} - \partial_x^{-1} f = \partial_x^{-1} f x_x \partial_x^{-1} \), we compute that the Lax equation \( L_t = [L, A] \) is equivalent to

\[
\left( v_x^2 \right)_t + \left( uv_x^2 + \frac{1}{2} u_x^2 \right)_x = 0, \quad u_{xx} = v_x^2,
\]

which may be rewritten as (1.4). Alternatively, we can set \( u = \partial_x^{-2} (v_x^2) \) in the original Lax pair.

7 Periodic twist waves

Spatially periodic splay waves are described by the following version of the HS-equation [22,23]

\[
(u_t + uu_x)_x = \frac{1}{2} \left( u_x^2 - \langle u_x^2 \rangle \right).
\]

Here, \( u(x, t) \) is a periodic function of \( x \), and angular brackets denote an average over a period. The wave also drives a mean-field, which evolves on the same time-scale, \( t = O(1) \), as the wave.

The interaction between a weakly nonlinear twist wave and a mean-field is more complicated because the mean-field evolves on a faster time-scale than the wave. This is a consequence of the fact that the nonlinear self-interaction of a weakly nonlinear twist wave is cubic, but the mean-field is driven by quadratic nonlinearities.

In this section, we derive the following generalization of the twist-wave asymptotic equation (1.4) that applies to periodic waves:

\[
\begin{align*}
(v_t + uv_x)_x + \mu \langle v_x^2 \rangle v &= 0, \quad (7.1) \\
u_{xx} &= v_x^2 - \langle v_x^2 \rangle. \quad (7.2)
\end{align*}
\]
Here, $u(x,t), v(x,t)$ are periodic functions of $x$, which we assume to have zero mean without loss generality, and $\mu$ is a constant that cannot be removed by rescaling. We will derive (7.1)–(7.2) from the one-dimensional wave equations (4.3)–(4.4), but a similar derivation would apply to more general systems.

It is interesting to note that the mean-field interaction introduces a dispersive term of Klein-Gordon type into the evolution equation (7.1) for $v$, despite the fact that the original system is scale-invariant and non-dispersive. The scale-invariance is preserved by the fact that the coefficient of the dispersive term is proportional to the mean wave-energy (or momentum) $\langle v_x^2 \rangle$, which is a constant conserved quantity.

The mean-terms prevent the elimination of $v$ from the system (7.1)–(7.2) by cross-differentiation, as is possible in the case of (6.1)–(6.2). Instead, one finds that

$$
\left[ (u_t + uu_x)_x - \frac{1}{2} u_x^2 - \langle v_x^2 \rangle (2u + \mu v^2) \right]_x = 0,
$$

which suggests that (7.1)–(7.2) is not completely integrable when $\mu \neq 0$.

### 7.1 Derivation of the periodic equation

We consider spatially-periodic solutions of (4.3)–(4.4) with period of the order $\varepsilon$ in which $\psi$ has amplitude of the order $\varepsilon^{1/2}$, where $\varepsilon$ is a small parameter. The corresponding time-scale for the nonlinear evolution of the $\psi$-wave is of the order 1. As we will see, the $\psi$-wave generates a mean $\varphi$-field which evolves on a time-scale of the order $\varepsilon^{1/2}$. This mean field modulates the speed of the $\psi$-wave on the same time-scale.

We therefore introduce multiple-scale variables

$$
\begin{align*}
\theta &= \frac{x - \varepsilon^{1/2}s(t/\varepsilon^{1/2})}{\varepsilon}, \\
\tau &= \frac{t}{\varepsilon^{1/2}}, \\
t &= t,
\end{align*}
$$

(7.3)

where $s(\tau)$ is a suitable phase function, and look for an asymptotic solution of (4.3)–(4.4) of the form

$$
\begin{align*}
\varphi &= \varphi_0 (\tau) + \varepsilon \varphi_2 (\theta, \tau, t) + O(\varepsilon^{3/2}), \\
\psi &= \varepsilon^{1/2} \psi_1 (\theta, \tau, t) + \varepsilon \psi_2 (\theta, \tau, t) + O(\varepsilon^{3/2}),
\end{align*}
$$
where all the terms are periodic functions of $\theta$. We will also require below that the terms are periodic functions of $\tau$.

We use this ansatz in (4.3)--(4.4), expand derivatives as

$$
\partial_t \rightarrow -\frac{1}{\varepsilon} s_\tau \partial_\theta + \frac{1}{\varepsilon^{1/2}} \partial_\tau + \partial_t, \quad \partial_x \rightarrow \frac{1}{\varepsilon} \partial_\theta,
$$

Taylor expand the result with respect to $\varepsilon$, and equate coefficients of powers of $\varepsilon^{1/2}$ to zero.

We consider first the $\psi$-equation (4.4). At the order $\varepsilon^{-3/2}$, we obtain that

$$
(s_\tau^2 - b_0^2) \psi_{1\theta\theta} = 0,
$$

where the 0-subscript on a function of $\varphi$ denotes evaluation at $\varphi = \varphi_0$. It follows from this equation that $s_\tau^2 = b_0^2$. For definiteness, we consider a right-moving wave and assume that

$$
s_\tau = b_0, \quad (7.4)
$$

where $b_0 > 0$.

At the order $\varepsilon^{-1}$, we obtain that

$$
2s_\tau \psi_{1\theta\tau} + \left( s_{\tau\tau} + \frac{2q_0}{q_0} s_\tau \right) \psi_{1\theta} = 0. \quad (7.5)
$$

We may choose $\psi_1$ so that it is a zero-mean periodic function of $\theta$. It follows from (7.4) and (7.5) that

$$
\psi_1(\theta, \tau, t) = \frac{v(\theta, t)}{q_0(\tau)b_0^{1/2}(\tau)} \quad (7.6)
$$

where $v(\theta, t)$ is a zero-mean periodic function of $\theta$ which is independent of $\tau$.

At the order $\varepsilon^{-1/2}$, we obtain that

$$
2s_\tau \psi_{2\theta\tau} + \left( s_{\tau\tau} + \frac{2q_0}{q_0} s_\tau \right) \psi_{2\theta} + 2s_\tau \psi_{1\theta t} + (2b_0 b'_0 \varphi_2 \psi_{1\theta})_\theta - \psi_{1\tau\tau} = 0.
$$

Using (7.4), we may write this equation as

$$
\left( q_0 b_0^{1/2} \psi_{2\theta} \right)_\tau + q_0 b_0^{1/2} \psi_{1\theta t} + \left( q_0 b_0^{1/2} b'_0 \varphi_2 \psi_{1\theta} \right)_\theta - \frac{q_0}{2b_0^{1/2}} \psi_{1\tau\tau} = 0. \quad (7.7)
$$
We will return to (7.7) after we expand the \( \varphi \)-equation.

The leading-order terms in the expansion of the \( \varphi \)-equation (4.3) are of the order \( \varepsilon^{-1} \), and give
\[
\left( s_\tau^2 - a_0^2 \right) \varphi_{2\theta\theta} + q_0^2 b_0 b_1' \psi^2_{1\theta} + \varphi_{0\tau\tau} = 0.
\]

Using (7.4) and (7.6) in this equation, we get
\[
\left( b_0^2 - a_0^2 \right) \varphi_{2\theta\theta} + b_0' v^2_{\theta} + \varphi_{0\tau\tau} = 0. \tag{7.8}
\]

Averaging this equation with respect to \( \theta \), we find that
\[
\varphi_{0\tau\tau} + \langle v^2_{\theta} \rangle b_0' = 0, \tag{7.9}
\]
where the angular brackets denote an average with respect to \( \theta \). As we will see, for smooth solutions, the quantity \( \langle v^2_{\theta} \rangle \) is a constant independent of \( t \), so (7.9) is consistent with the ansatz that \( \varphi_0 \) depends only on \( \tau \).

Equation (7.9) provides an ODE for \( \varphi_0 \), corresponding to motion in a potential proportional to the twist-wave speed \( b_0 = b(\varphi_0) \). For definiteness, we assume that the solution of (7.9) for \( \varphi_0(\tau) \) is a periodic function of \( \tau \). We then require that all other terms in the expansion are periodic functions of \( \tau \).

Subtracting (7.9) from (7.8), we find that
\[
\left( b_0^2 - a_0^2 \right) \varphi_{2\theta\theta} + b_0' \left( v^2_{\theta} - \langle v^2_{\theta} \rangle \right) = 0.
\]

Hence, we may write
\[
\varphi_2(\theta, \tau, t) = \left[ \frac{b_0'(\tau)}{a_0^2(\tau) - b_0^2(\tau)} \right] u(\theta, t), \tag{7.10}
\]
where \( u \) satisfies
\[
u_{0\theta} = v^2_{\theta} - \langle v^2_{\theta} \rangle.
\]

Using (7.6) and (7.10) in (7.7), we get
\[
\left( q_0 b_0^{1/2} \psi_{2\theta} \right)_\tau + v_{\theta t} + \left[ \frac{(b_0')^2}{a_0^2 - b_0^2} \right] (u v_{\theta})_\theta - \frac{q_0}{2 b_0^{1/2}} \left( \frac{1}{q_0 b_0^{1/2}} \right) \tau \tau v = 0.
\]
Averaging this equation with respect to $\tau$, we get
\[
(v_t + \Lambda w_\theta)_\theta + N v = 0,
\]
where
\[
\Lambda = \oint \frac{(b_0')^2}{a_0^2 - b_0^2} \, d\tau,
\]
\[
N = -\frac{1}{2} \oint \frac{q_0}{b_0^{1/2}} \left( \frac{1}{q_0 b_0^{1/2}} \right)_{\tau\tau} \, d\tau.
\]
Here, $\oint$ denotes an average over a period in $\tau$.

To make the dependence of $\varphi_0$ on $v$ explicit, we introduce a new time variable
\[
T = \langle v_\theta^2 \rangle^{1/2} \tau.
\]
We may then rewrite (7.9) as
\[
\varphi_0 TT + b_0' = 0. \tag{7.11}
\]
Given a solution of this equation for $\varphi_0(T)$, we have $N = \langle v_\theta^2 \rangle M$, where
\[
M = -\frac{1}{2} \oint \frac{q_0}{b_0^{1/2}} \left( \frac{1}{q_0 b_0^{1/2}} \right)_{TT} \, dT
\]
is a constant independent of $v$, and $\oint$ denotes an average with respect to $T$ over a period. From (7.11), we have
\[
\frac{1}{2} \varphi_{0TT} + b_0 = E
\]
for some constant $E$. Using an integration by parts, we may also write $M$ as
\[
M = \frac{1}{2} \oint \frac{1}{b_0} \left( \frac{b_0^2}{4b_0^2} - \frac{q_0^2}{q_0^2} \right) dT
\]
\[
= \oint \frac{E - b_0}{b_0} \left[ \frac{(b_0^2)^2}{4b_0^2} - \frac{(q_0^2)^2}{q_0^2} \right] dT.
\]
Thus, the final equations for $u(\theta, t)$, $v(\theta, t)$ are
\[(v_t + \Lambda uv_\theta)_\theta + M\langle v^2_\theta \rangle v = 0, \quad (7.12)\]
\[u_{\theta\theta} = v^2_\theta - \langle v^2_\theta \rangle. \quad (7.13)\]

It follows from these equations that, for smooth solutions,

\[(v^2_\theta)_t + \left[\Lambda \left(uv^2_\theta - \frac{1}{2}u^2_\theta + \langle v^2_\theta \rangle u_\theta\right) + M\langle v^2_\theta \rangle v^2_\theta\right]_\theta = 0.\]

Taking the average of this equation with respect to \(\theta\), we find that \(\langle v^2_\theta \rangle\) is constant in time, as stated earlier.

In summary, the asymptotic solution of (4.3)–(4.4) is given by

\[
\varphi = \varphi_0(\tau) + \frac{\varepsilon b'_0(\tau)}{a^2_0(\tau) - b^2_0(\tau)} u(\theta, t) + O(\varepsilon^{3/2}),
\]
\[
\psi = \frac{\varepsilon^{1/2}}{g_0(\tau)b^{1/2}_0(\tau)} v(\theta, t) + O(\varepsilon),
\]

where the multiple-scale variables \(\theta, \tau\) are evaluated at (7.3), \(s\) satisfies (7.4), \(\varphi_0\) satisfies (7.9), and \(u, v\) satisfy (7.12)–(7.13).

If \(\Lambda \neq 0\) then we may rescale variables in (7.12)–(7.13) to get (7.1)–(7.2) with

\[\mu = \frac{M}{\Lambda}.\]

As an example, let us consider the wave speeds in (4.2) arising from the one-dimensional director field equations, where

\[b(\varphi) = \sqrt{\beta \sin^2 \varphi + \gamma \cos^2 \varphi}.\]

If \(\beta < \gamma\), then \(b\) has a minimum at \(\varphi = \pi/2\), and (7.9) has periodic solutions for the mean field \(\varphi_0\) that oscillate around \(\pi/2\). Our asymptotic solution applies in this case. If \(\beta > \gamma\), then \(b\) has a minimum at \(\varphi = 0\). Although (7.9) also has periodic solutions in this case, there is a loss of strict hyperbolicity at \(\varphi = 0\), where \(a = b\), and the asymptotic solution breaks down.

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A Algebraic details

A.1 Linearized equations

The linearization of (2.5) at \( n_0 \) is

\[
n''_t = \alpha \nabla (\text{div} \ n') - \beta \text{curl} \ (A'n_0) - \gamma \text{curl} \ (B' \times n_0) + \lambda' n_0, \tag{A.1}
\]

where

\[
A' = n_0 \cdot \text{curl} \ n', \quad B' = n_0 \times \text{curl} \ n',
\]

and the Lagrange multiplier \( \lambda' \) is chosen so that \( n_0 \cdot n' = 0 \).

The Fourier mode

\[
n'(x, t) = \hat{n} e^{i k \cdot x - i \omega t}, \quad \lambda'(x, t) = \hat{\lambda} e^{i k \cdot x - i \omega t}
\]

is a solution of the linearized equations (A.1) if

\[
L \hat{n} - \hat{\lambda} n_0 = 0, \quad n_0 \cdot \hat{n} = 0, \tag{A.2}
\]

where the linear map \( L : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by

\[
L \hat{n} = \omega^2 \hat{n} - \alpha (k \cdot \hat{n}) k + \beta \hat{A} (k \times n_0) + \gamma k \times (\hat{B} \times n_0), \tag{A.3}
\]

\[
\hat{A} = n_0 \cdot (k \times \hat{n}), \quad \hat{B} = n_0 \times (k \times \hat{n}).
\]

We solve this eigenvalue problem in the next proposition.

**Proposition 5** Suppose that \( n_0, k \in \mathbb{R}^3 \) where \( n_0 \) is a unit vector and \( k \) is non-zero, and \( \alpha, \beta, \gamma \in \mathbb{R} \) are distinct positive constants. For \( \omega \in \mathbb{R} \) let \( L : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear map defined in (A.3). Then the linear system

\[
L m - \lambda n_0 = F, \quad n_0 \cdot m = G \tag{A.4}
\]

has a unique solution for \( \{m, \lambda\} \in \mathbb{R}^3 \times \mathbb{R} \) for every \( \{F, G\} \in \mathbb{R}^3 \times \mathbb{R} \) unless \( \omega^2 = a^2(k; n_0) \) or \( \omega^2 = b^2(k; n_0) \), where \( a^2, b^2 \) are defined in (2.7)–(2.8).

(a) If \( k \) is not parallel to \( n_0 \) and \( \omega^2 = a^2(k; n_0) \), then the general solution of (A.4) when \( F = 0, G = 0 \) is

\[
m = c R, \quad \lambda = -c (\alpha - \gamma) (k \cdot n_0) (k \cdot R), \tag{A.5}
\]
where \( c \) is an arbitrary constant and \( \mathbf{R} = \mathbf{k} - (\mathbf{k} \cdot \mathbf{n}_0) \mathbf{n}_0 \). Equation (A.4) is solvable for \( \{\mathbf{m}, \lambda\} \) if and only if \( \{\mathbf{F}, \mathbf{G}\} \) satisfy
\[
\mathbf{R} \cdot \mathbf{F} + (\alpha - \gamma) (\mathbf{k} \cdot \mathbf{n}_0) (\mathbf{k} \cdot \mathbf{R}) \mathbf{G} = 0.
\] (A.6)

(b) If \( \mathbf{k} \) is not parallel to \( \mathbf{n}_0 \) and \( \omega^2 = b^2 (\mathbf{k}; \mathbf{n}_0) \), then the general solution of (A.4) when \( \mathbf{F} = 0, \mathbf{G} = 0 \) is
\[
\mathbf{m} = c \mathbf{S}, \quad \lambda = 0,
\] (A.7)
where \( c \) is an arbitrary constant and \( \mathbf{S} = \mathbf{k} \times \mathbf{n}_0 \). Equation (A.4) is solvable for \( \{\mathbf{m}, \lambda\} \) if and only if \( \mathbf{F} \) satisfies
\[
\mathbf{S} \cdot \mathbf{F} = 0.
\] (A.8)

(c) If \( \mathbf{k} \) is parallel to \( \mathbf{n}_0 \) and \( \omega^2 = \gamma (\mathbf{k} \cdot \mathbf{n}_0)^2 \), then the general solution of (A.4) when \( \mathbf{F} = 0, \mathbf{G} = 0 \) is \( \{\mathbf{m}, \lambda\} = \{\mathbf{m}^\perp, 0\} \) where \( \mathbf{m}^\perp \) is any vector orthogonal to \( \mathbf{n}_0 \). Equation (A.4) is solvable for \( \{\mathbf{m}, \lambda\} \) if and only if \( \mathbf{F} \) is parallel to \( \mathbf{n}_0 \).

**Proof.** First, we suppose that \( \mathbf{k} \) is not parallel to \( \mathbf{n}_0 \). Expanding
\[
\mathbf{m} = m_1 \mathbf{k} + m_2 \mathbf{R} + m_3 \mathbf{S},
\]
\[
\mathbf{F} = F_1 \mathbf{k} + F_2 \mathbf{n}_0 + F_3 \mathbf{S},
\]
we find, after some algebra, that (A.4) is equivalent to
\[
(\mathbf{k} \cdot \mathbf{n}_0) m_1 = G,
\]
\[
\lambda + (\mathbf{k} \cdot \mathbf{n}_0) (\omega^2 - \gamma k^2) m_2 = -F_2,
\]
\[
(\omega^2 - \alpha k^2) m_1 + (\omega^2 - \alpha^2) m_2 = F_1,
\]
\[
(\omega^2 - b^2) m_3 = F_3.
\]
The first two equations determine \( m_1 \) and \( \lambda \), and the remaining two equations determine \( m_2, m_3 \) unless \( \omega^2 = \alpha^2 \) or \( \omega^2 = b^2 \).

If \( \omega^2 = \alpha^2 \), then \( \omega^2 \neq b^2 \) (since \( \mathbf{k} \) is not parallel to \( \mathbf{n}_0 \)) and the fourth equation is solvable for \( m_3 \). The third equation is solvable for \( m_2 \) if and only if
\[
(\mathbf{k} \cdot \mathbf{n}_0) F_1 - (\alpha^2 - \alpha k^2) G = 0.
\]

Using the equations
\[
a^2 - \alpha k^2 = - (\alpha - \gamma) (\mathbf{k} \cdot \mathbf{n}_0)^2, \quad \mathbf{R} \cdot \mathbf{F} = (\mathbf{k} \cdot \mathbf{R}) F_1
\]
in this condition, we get (A.6). If \( F = 0, G = 0 \), then we find that \( m_1 = 0, m_3 = 0 \) and \( m_2 = c \), where \( c \) is an arbitrary constant. Computing the corresponding value of \( \lambda \), we get (A.5).

If \( \omega^2 = b^2 \), then the first three equations are solvable for \( \lambda, m_1, m_2 \). The last equation is solvable for \( m_3 \) if and only if \( F_3 = 0 \), which gives (A.8). If \( F = 0, G = 0 \) then \( \lambda = 0, m_1 = 0, m_2 = 0, \) and \( m_3 = c \), which gives (A.7).

Finally, we suppose that \( k = k n_0 \) is parallel to \( n_0 \). Then

\[
L m = \left( \omega^2 - \gamma k^2 \right) m - (\alpha - \gamma) k^2 \left( n_0 \cdot m \right) n_0.
\]

Equation (A.4) is therefore uniquely solvable unless \( \omega^2 = \gamma k^2 \), when

\[
L m = - (\alpha - \gamma) k^2 \left( n_0 \cdot m \right) n_0.
\]

In that case, (A.4) is solvable if and only if \( F = F n_0 \) is parallel to \( n_0 \), and the solution is

\[
m = G n_0 + m^\perp, \quad \lambda = - \left[ F + (\alpha - \gamma) k^2 G \right],
\]

where \( m^\perp \) is an arbitrary vector orthogonal to \( n_0 \). \( \square \)

From Proposition 5, the solutions of the eigenvalue problem (A.2) are given by (2.6)–(2.10).

A.2 Weakly nonlinear splay waves

We look for an asymptotic solution of (2.5) of the form (3.1)–(3.2). We expand derivatives as

\[
\partial_t \to - \frac{\omega}{\varepsilon} \partial_\theta + \partial_t, \quad \nabla \to \frac{k}{\varepsilon} \partial_\theta + \nabla,
\]

where \( \omega, k \) are defined in (3.3). The corresponding expansions of \( \lambda, A, B \) are

\[
\lambda = \frac{1}{\varepsilon} \lambda_1 + \lambda_2 + \ldots, \quad A = A_1 + \varepsilon A_2 + \ldots, \quad B = B_1 + \varepsilon B_2 + \ldots.
\]

where
\begin{align*}
A_1 &= n_0 \cdot (k \times n_{1\theta}) + n_0 \cdot \text{curl } n_0, \\
B_1 &= n_0 \times (k \times n_{1\theta}) + n_0 \times \text{curl } n_0, \\
A_2 &= n_0 \cdot (k \times n_{2\theta}) + n_1 \cdot (k \times n_{1\theta}) + n_0 \cdot \text{curl } n_1 + n_1 \cdot \text{curl } n_0, \\
B_2 &= n_0 \times (k \times n_{2\theta}) + n_1 \times (k \times n_{1\theta}) + n_0 \times \text{curl } n_1 + n_1 \times \text{curl } n_0.
\end{align*}

We write these expressions as
\begin{align*}
A_i &= \widehat{A}_i + \tilde{A}_i, \\
B_i &= \widehat{B}_i + \tilde{B}_i, \\
\widehat{A}_i &= n_0 \cdot (k \times n_{i\theta}), \\
\widehat{B}_i &= n_0 \times (k \times n_{i\theta}).
\end{align*}

We use these expansions in (2.5), Taylor expand the result with respect to \(\varepsilon\), and equate coefficients of powers of \(\varepsilon\). At the order \(\varepsilon^{-1}\), we obtain linearized equations for \(n_1\), \(\lambda_1\), which have the form
\begin{align*}
\mathcal{L} n_{1\theta\theta} - \lambda_1 n_0 &= 0, \\
n_0 \cdot n_1 &= 0,
\end{align*}
\(\text{(A.11)}\)
where \(\mathcal{L}\) is the linear map defined in (A.3).

The system (A.11) has the splay-wave eigenvalue \(\omega^2 = a^2 (k; n_0)\), where \(a\) is defined in (2.7). The corresponding solution for \(n_1\), after two integrations with respect to \(\theta\), is
\[n_1(\theta, x, t) = u(\theta, x, t)R(x, t),\]
where \(u\) is an arbitrary scalar-valued function, and \(R\) is defined in (2.9). The solution for \(\lambda_1\) is
\[\lambda_1 = - (\alpha - \gamma) u_{\theta\theta} (k \cdot n_0) (k \cdot R).\]

At the order \(\varepsilon^0\), we obtain equations for \(n_2\), \(\lambda_2\). Using the fact that \(n_0\) is a solution of (2.5), with \(\lambda = \lambda_0\) say, we may write them as
\begin{align*}
\mathcal{L} n_{2\theta\theta} - \tilde{\lambda}_2 n_0 &= F_1, \\
n_0 \cdot n_2 &= G_1,
\end{align*}
\(\text{(A.12)}\)
where \(\tilde{\lambda}_2 = \lambda_2 - \lambda_0\) and
\begin{align*}
F_1 &= 2 \omega n_{1\theta t} + \omega t n_{1\theta} + \alpha \{ (\text{div } n_{1\theta}) k + \nabla (k \cdot n_{1\theta}) \} \\
&\quad - \beta \{ \tilde{A}_{2\theta} (k \times n_0) + A_1 (k \times n_{1\theta}) + k \times (A_1 n_1)_{\theta} \\
&\quad + \tilde{A}_1 \text{curl } n_0 + \text{curl } (\tilde{A}_1 n_0) \} \\
&\quad + \gamma \{ - k \times (\tilde{B}_{2\theta} \times n_0) + B_1 \times (k \times n_{1\theta}) - k \times (B_1 \times n_1)_{\theta} \}.
\end{align*}
\[ G_1 = -\frac{1}{2} n_1 \cdot n_1. \]

From Proposition 5, equation (A.12) is solvable for \( n_{2\theta} \) and \( \tilde{\lambda}_2 \) if and only if

\[ R \cdot F_1 + (\alpha - \gamma) (k \cdot n_0) (k \cdot R) G_{1\theta} = 0. \]

After some algebra, we find that this solvability condition gives (3.4).

### A.3 Weakly nonlinear twist waves

We look for an asymptotic solution of (2.5) of the form (3.6)–(3.7), expanding derivatives as in (A.9). The corresponding expansions of \( \lambda, A, B \) are

\[ \lambda = \frac{1}{\varepsilon^{3/2}} \lambda_1 + \frac{1}{\varepsilon} \lambda_2 + \frac{1}{\varepsilon^{1/2}} \lambda_3 + \ldots, \]
\[ A = \frac{1}{\varepsilon^{1/2}} A_1 + A_2 + \varepsilon^{1/2} A_3 + \ldots, \]
\[ B = \frac{1}{\varepsilon^{1/2}} B_1 + B_2 + \varepsilon^{1/2} B_3 + \ldots, \]

where

\[ A_1 = n_0 \cdot (k \times n_{1\theta}), \]
\[ B_1 = n_0 \times (k \times n_{1\theta}), \]
\[ A_2 = n_0 \cdot (k \times n_{2\theta}) + n_1 \cdot (k \times n_{1\theta}) + n_0 \cdot \text{curl} n_0, \]
\[ B_2 = n_0 \times (k \times n_{2\theta}) + n_1 \times (k \times n_{1\theta}) + n_0 \times \text{curl} n_0, \]
\[ A_3 = n_0 \cdot (k \times n_{3\theta}) + n_1 \cdot (k \times n_{2\theta}) + n_2 \cdot (k \times n_{1\theta}) + n_0 \cdot \text{curl} n_1 + n_1 \cdot \text{curl} n_0, \]
\[ B_3 = n_0 \times (k \times n_{3\theta}) + n_1 \times (k \times n_{2\theta}) + n_2 \times (k \times n_{1\theta}) + n_0 \times \text{curl} n_1 + n_1 \times \text{curl} n_0. \]

We write these expressions as in (A.10).

Using these expansions in (2.5), Taylor expanding the result, and equating coefficients of powers of \( \varepsilon^{1/2} \), we get at the order \( \varepsilon^{-3/2} \) the linearized equations (A.11). From Proposition 5, this system has the twist-wave eigenvalue \( \omega^2 = b^2 (k; n_0) \), where \( b \) is given by (2.8). The corresponding solution for \{ \( n_1, \lambda_1 \) \} is

\[ n_1 (\theta, x, t) = v (\theta, x, t) S (x, t), \quad \lambda_1 = 0, \quad (A.13) \]
where \( v \) is an arbitrary scalar-valued function, and \( S \) is defined in (2.10).

Equating coefficients of the order \( \varepsilon^{-1} \), we obtain that

\[
\mathcal{L} n_{2\theta} - \lambda_2 n_0 = F_1, \quad n_0 \cdot n_2 = G_1, \tag{A.14}
\]

where \( \mathcal{L} \) is defined in (A.3) and

\[
F_1 = -\beta \left[ \tilde{A}_{2\theta} (k \times n_0) + k \times (A_1 n_1)_\theta + A_1 (k \times n_{1\theta}) \right] \\
- \gamma \left[ k \times (B_{2\theta} \times n_0) + k \times (B_1 \times n_1)_\theta - B_1 \times (k \times n_{1\theta}) \right] + \lambda_1 n_1, \\
G_1 = -\frac{1}{2} n_1 \cdot n_1.
\]

Using Proposition 5 to solve (A.14), we find that

\[
\begin{align*}
n_2 &= -ru (k \cdot n_0) R + \frac{1}{2} v^2 (k \times S), \\
\lambda_2 &= v_0^2 \left\{ \beta k^2 + r (\beta - \gamma) (k \cdot n_0)^2 \right\} \left[ k^2 - (k \cdot n_0)^2 \right],
\end{align*}
\]

where \( u(\theta, x, t) \) satisfies (3.11), and

\[
r = \frac{\beta - \gamma}{\alpha - \beta}.
\]

We could add an arbitrary scalar multiple of the null-vector \( S \) to \( n_2 \), but this would not alter our final equations, so we omit it for simplicity.

Equating coefficients of the order \( \varepsilon^{-1/2} \), we obtain that

\[
\mathcal{L} n_{3\theta} - \lambda_3 n_0 = F_2, \quad n_0 \cdot n_3 = G_2, \tag{A.16}
\]

where

\[
F_2 = 2\omega n_{1\theta t} + \omega_t n_{1\theta} + \lambda_2 n_1 + \lambda_1 n_2 + \alpha \left[ \text{div} (n_{1\theta}) k + \nabla (k \cdot n_{1\theta}) \right] \\
- \beta \left[ \tilde{A}_{3\theta} (k \times n_0) + k \times (A_2 n_1)_\theta + k \times (A_1 n_2)_\theta + A_2 (k \times n_{1\theta}) \\
+ A_1 (k \times n_{2\theta}) + A_1 \text{curl} n_0 + \text{curl} (A_1 n_0) \right] \\
- \gamma \left[ k \times (B_{3\theta} \times n_0) + k \times (B_2 \times n_1)_\theta + k \times (B_1 \times n_2)_\theta \\
- B_2 \times (k \times n_{1\theta}) - B_1 \times (k \times n_{2\theta}) \\
- B_1 \times \text{curl} n_0 + \text{curl} (B_1 \times n_0) \right], \\
G_2 = -n_1 \cdot n_2.
\]
From Proposition 5, this system is solvable if $S \cdot F_2 = 0$. After some algebra, we compute that this solvability condition gives (3.10).

### A.4 Weakly nonlinear polarized waves

We assume that $n_0$ and $k$ are constant and $k = kn_0$ is parallel to $n_0$. We look for an expansion of the same form as the one used in the previous section for twist waves, with a plane-wave phase $\Phi(x,t) = k \cdot x - \omega t$.

When $k$ is parallel to $n_0$, the linearized dispersion relation (2.6) reduces to $[\omega^2 - \gamma k^2]^2 = 0$, so $\omega^2 = \gamma k^2$. From Proposition 5, the solution of the leading-order $O(\varepsilon^{-3/2})$-equation (A.11) is then

$$n_1(\theta, x, t) = u(\theta, x, t), \quad \lambda_1 = 0,$$

where $u$ is an arbitrary vector such that $n_0 \cdot u = 0$.

The $O(\varepsilon^{-1})$-equation (A.14) simplifies to

$$\mathcal{L}n_{2\theta} - \lambda_2 n_0 = -\gamma k^2 (u_\theta \cdot u_\theta) n_0,$$

$$n_0 \cdot n_2 = -\frac{1}{2} (u \cdot u),$$

whose solution is

$$n_2 = -\frac{1}{2} (u \cdot u) n_0,$$

$$\lambda_2 = \frac{1}{2} (\alpha - \gamma) k^2 (u \cdot u)_{\theta\theta} + \gamma k^2 (u_\theta \cdot u_\theta).$$

We could add to $n_2$ an arbitrary vector orthogonal to $n_0$, but this would not alter our final equations, so we omit it for simplicity.

Computing $F_2$ in the $O(\varepsilon^{-1/2})$-equation (A.16) and imposing the solvability condition in Proposition 5 that $F_2$ is parallel to $n_0$, we get after some algebra that

$$u_{\theta t} + \frac{\omega}{k} n_0 \cdot \nabla u_\theta + \frac{\alpha k^2}{2\omega} (u \cdot u_\theta)_\theta u$$

$$- \frac{\beta k^2}{2\omega} \left\{ [u \cdot (n_0 \times u_{\theta\theta})] (n_0 \times u) + 2 [u \cdot (n_0 \times u_\theta)] (n_0 \times u_\theta) \right\}$$

$$- \frac{\gamma k^2}{2\omega} \left\{ [(u \cdot u)_\theta]_\theta - (u_\theta \cdot u_\theta) u \right\} = 0.$$

(A.17)
A computation in terms of components show that if $\mathbf{u}$ is orthogonal to the constant unit vector $\mathbf{n}_0$, then

$$[\mathbf{u} \cdot (\mathbf{n}_0 \times \mathbf{u}_\theta)] (\mathbf{n}_0 \times \mathbf{u}) + 2 [\mathbf{u} \cdot (\mathbf{n}_0 \times \mathbf{u}_\theta)] (\mathbf{n}_0 \times \mathbf{u}_\theta) = (\mathbf{u} \cdot \mathbf{u}_\theta)_{\theta} \mathbf{u} - [(\mathbf{u} \cdot \mathbf{u}) \mathbf{u}_\theta]_{\theta} + (\mathbf{u}_\theta \cdot \mathbf{u}_\theta) \mathbf{u}. $$

Using this result in (A.17), we obtain (3.13).

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