MORITA CONTEXTS AS LAX FUNCTORS

STEPHEN LACK

ABSTRACT. Monads are well known to be equivalent to lax functors out of the terminal category. Morita contexts are here shown to be lax functors out of the chaotic category with two objects. This allows various aspects in the theory of Morita contexts to be seen as special cases of general results about lax functors. The account we give of this could serve as an introduction to lax functors for those familiar with the theory of monads. We also prove some very general results along these lines relative to a given 2-comonad, with the classical case of ordinary monad theory amounting to the case of the identity comonad on Cat.

1. Introduction

Some of the fundamental aspects of the theory of monads are:
(1) the fact that every adjunction generates a monad,
(2) the construction of the Eilenberg-Moore category of a monad, and the fact that this is part of an adjunction generating the given monad,
(3) the universal property of this construction,
(4) Beck’s theorem characterizing which adjunctions arise in this way.

All of these can be formulated and proved in a suitable 2-category; the resulting theory is known as the formal theory of monads [10].

The recent paper [4] formulated and proved analogous results involving Morita contexts rather than monads. A Morita context, in the sense of [4], involves a monad \( t \) on a category \( A_0 \), a monad \( s \) on a category \( A_1 \), functors \( f : A_0 \to A_1 \) and \( g : A_1 \to A_0 \), and various further natural transformations subject to compatibility conditions. The corresponding notion of adjunction consists of functors \( u_0 : B \to A_0 \) and \( u_1 : B \to A_1 \) with the same domain, each possessing a left adjoint. These were studied in [3] under the name double adjunction. The word “double” refers to the fact that there are two adjunctions, not to any connection with double categories.

The main purpose of this note is to point out a common generalization of monads and Morita contexts, in which the various results listed above are all known.

Bénabou observed in [1] that to give a monad is equivalently to give a lax functor from the terminal 2-category \( 1 \) to \( \text{Cat} \). Building on this, Street [11] studied lax functors from \( X \) to \( \text{Cat} \) for an arbitrary small category \( X \), and managed to formulate and prove results which, when specialized to the case \( X = 1 \), gave each of the results about monads mentioned above.

After recalling the notion of lax functor, we shall quickly focus on the case where \( X \) is the category \( \text{Iso} \) with two objects 0 and 1, and exactly one arrow in each of the four hom-sets. This is often called the “chaotic” or “indiscrete” category on two objects;
it is also called the “free-living isomorphism”, since a functor from $\text{Iso}$ to a category $D$ is precisely an isomorphism in $D$. The key observation of this paper is that a lax functor from $\text{Iso}$ to $\text{Cat}$ is precisely a Morita context. After explaining this, we then recall aspects of [11] and explain how they can be used to obtain the main theoretical results in [4].

There is also another bicategorical treatment of Morita contexts, introduced in [5], and going under the name of wide Morita context. These wide Morita contexts are the special case of the Morita contexts studied here and in [4] in which the two monads are both trivial (identity monads). Corresponding to our observation that Morita contexts in $\mathcal{M}$ are the same as lax functors from $\text{Iso}$ to $\mathcal{M}$ is the observation made in [9] that wide Morita contexts in $\mathcal{M}$ are the same as normal lax functors from $\text{Iso}$ to $\mathcal{M}$, where a lax functor is said to be normal when it strictly preserves the identities.

Thus wide Morita contexts in a bicategory $\mathcal{M}$ are a special case of Morita contexts in $\mathcal{M}$; on the other hand, there is also a way to see Morita contexts as a special case of wide Morita contexts, as we now explain. For a bicategory $\mathcal{M}$ whose hom-categories have reflexive coequalizers, preserved by composition on either side, there is a bicategory $\text{Mod}(\mathcal{M})$ having monads in $\mathcal{M}$ as its objects, and “2-sided actions” as 1-cells: when $\mathcal{M}$ is the one-object bicategory $\mathcal{M}$ corresponding to the monoidal category $\text{Ab}$ of abelian groups, the corresponding $\text{Mod}(\mathcal{M})$ is the bicategory $\text{Mod}$ of rings, bimodules, and homomorphisms of bimodules. There is a natural bijection between lax functors with codomain $\mathcal{M}$ and normal lax functors with codomain $\text{Mod}(\mathcal{M})$; taking the case where the domain is $\text{Iso}$, we obtain a bijection between Morita contexts in $\mathcal{M}$ and wide Morita contexts in $\text{Mod}(\mathcal{M})$. A classical Morita context between rings is a Morita context in the one-object bicategory corresponding to $\text{Ab}$; or, equivalently, a wide Morita context in $\text{Mod}$.

There is a “formal theory of lax functors with domain $X$”, dealing with lax functors from $X$ to a general 2-category $\mathcal{M}$, and containing the formal theory of monads as the special case $X = 1$, but we shall restrict ourselves to the case $\text{Cat}$. On the other hand in the final two sections we describe a still more general setting using the language of 2-dimensional monad theory [2], which includes the case of lax functors from an arbitrary small 2-category $X$ to a complete 2-category $\mathcal{M}$. Readers who enjoy the adrenaline rush of Extreme 2-Category Theory are welcome to skip straight to Section 8.

For an object $x$ of a category or 2-category we shall often write $x$ for the corresponding identity 1-cell; similarly identity 2-cells in 2-categories will sometimes be denoted by the name of the corresponding 1-cell. Alternatively, we may just write 1 for an identity if the domain/codomain is clear from the context.

We shall often consider a category, such as $X$, as a 2-category with no non-identity 2-cells.

2. Monads and lax functors

A lax functor from a category $X$ to a 2-category $\mathcal{K}$ consists of the following assignments. For each object $x \in X$ there is a specified object $A_x \in \mathcal{K}$, for each morphism $\xi: x \to y$ in $X$ there is a morphism (1-cell) $t_{\xi}: A_x \to A_y$ in $\mathcal{K}$, for each composable pair of morphisms $\xi: x \to y$ and $\zeta: y \to z$ there is a 2-cell $\mu_{\xi, \zeta}: t_{\zeta \xi} \to t_{\xi} t_{\zeta}$ in $\mathcal{K}$, and for each object $x \in X$ there is a 2-cell $\eta_x: 1_{A_x} \to t_x = t_{1_x}$ in $\mathcal{K}$. We shall sometimes omit
the subscripts on $\mu$ and $\eta$. These are required to satisfy the associativity condition stating that the diagram

\[
\begin{array}{c}
t_\tau t_\zeta t_\xi \xrightarrow{\mu t_\xi} t_\tau t_\zeta t_\xi \\
t_\tau \downarrow \quad \mu \downarrow \\
t_\tau t_\zeta \xrightarrow{\mu} t_\tau t_\zeta t_\xi
\end{array}
\]

commutes for all composable triples $\xi, \zeta, \tau$. They are also required to satisfy the unit conditions asserting the commutativity of the triangles

\[
\begin{array}{c}
t_\xi 1 \xrightarrow{t_\xi \eta} t_\xi t_1 \\
\mu \downarrow \quad \mu \downarrow \\
t_\xi \xrightarrow{t_\xi}
\end{array} \quad \begin{array}{c}
1 t_\xi \xrightarrow{\eta t_\xi} t_1 t_\xi \\
\mu \downarrow \quad \mu \downarrow \\
1 \xrightarrow{t_\xi}
\end{array}
\]

for all arrows $\xi$.

The slightly idiosyncratic choice of notation is of course designed to emphasize that if $X = 1$ then this is just a monad.

Now consider the case $X = \text{Iso}$. We write $\alpha: x \to y$ and $\beta: y \to x$ for the two non-identity arrows of $X$. There are exactly 8 composable pairs and exactly 16 composable triples. A lax functor $\text{Iso} \to \text{Cat}$ involves categories $A_x$ and $A_y$, a monad $(t_x, \mu, \eta)$ on $A_x$ and a monad $(t_y, \mu, \eta)$ on $A_y$. There are also functors $t_\alpha: A_x \to A_y$ and $t_\beta: A_y \to A_x$ which we shall call $f$ and $g$ respectively. There are 6 further 2-cells, corresponding to the 6 remaining composable pairs in $\text{Iso}$; explicitly, they are $\rho_f = \mu_{\alpha,y}: t_y f \to f$, $\lambda_f = \mu_{x,\alpha}: ft_x \to f$, $\rho_g = \mu_{x,\beta}: gt_y \to g$, $\lambda_g = \mu_{\beta,y}: t_x g \to g$, $\varepsilon_x = \mu_{\alpha,\beta}: gf \to t_x$, and $\varepsilon_y = \mu_{\beta,\alpha}: fg \to t_y$. There are 14 further associativity conditions and 4 further unit conditions, corresponding precisely to the conditions in the definition of Morita context: see [4].

### 3. Algebras and lax natural transformations

Let $A: X \to \text{Cat}$ be a lax functor, involving objects $A_x$, morphisms $t_\xi$, and 2-cells $\mu_{\xi,\xi}$ and $\eta_\xi$ as above; and let $B: X \to \text{Cat}$ be another lax functor, involving $B_x$, $s_\xi$, $\mu_{\xi,\xi}$, and $\eta_\xi$.

A lax natural transformation $u$ from $B$ to $A$ consists of a morphism $a_x: B_x \to A_x$ for each $x \in X$ and a 2-cell $\alpha_\xi$ as in

\[
\begin{array}{ccc}
B_x & \xrightarrow{a_x} & A_x \\
s_\xi \downarrow & \Downarrow \alpha_\xi & \downarrow t_\xi \\
B_x & \xrightarrow{a_y} & A_y
\end{array}
\]

for each morphism $\xi: x \to y$ in $X$. These are required to be compatible with the $\mu$ and with the $\eta$. The precise conditions can be found for example in [11, Section 1], where the name left lax transformation is used.

In the case where $X = 1$, so that a lax functor $X \to \text{Cat}$ is just a monad, this reduces to a functor $a: B \to A$ equipped with a 2-cell $\alpha: ta \to as$ satisfying the two conditions for $(a, \alpha)$ to be a monad morphism from $(B, s)$ to $(A, t)$, in the sense of [10].
Specializing further, if $B$ is strict, so that $s = 1$, this reduces to a functor $a: B \to A$ equipped with an action $\alpha: ta \to a$; and if finally $B$ is the terminal category, this is just an object $a \in A$ equipped with a $t$-algebra structure $\alpha: ta \to a$.

Reverting to the general case, there is also a notion of oplax natural transformation, called right lax transformation in [11], in which the direction of the $\alpha_\xi$ are reversed.

There are also modifications between lax natural transformations, which capture the monad 2-cells of [10]. A modification between lax natural transformations $(a, \alpha)$ and $(b, \beta)$ as above consists of a natural transformation $\phi_\xi: a_x \to b_x$ for each $x \in X$ suitably compatible with $\alpha$ and $\beta$: see [11]. If $X = 1$, and $B$ is the strict functor with value the terminal category, so that $(a, \alpha)$ and $(b, \beta)$ are just $t$-algebras, then such a $\varphi$ is precisely a morphism of $t$-algebras.

There is a 2-category $\text{Lax}(X, \text{Cat})_\ell$ of lax functors, lax natural transmorphisms, and modifications which reduces, in the case $X = 1$, to Street’s 2-category $\text{Mnd}(\text{Cat})$ of monads in $\text{Cat}$.

### 4. Adjunctions

Let $B: X \to \text{Cat}$ be a lax functor, and suppose that for each object $x \in X$ we are given a category $A_x$, a functor $u_x: B_x \to A_x$ and a left adjoint $f_x \dashv u_x$ with unit $\sigma_x$ and counit $\varepsilon_x$.

For each $\xi: x \to y$ in $X$, let $s_\xi: A_x \to A_y$ be given by the composite

\[
A_x \xrightarrow{f_x} B_x \xrightarrow{t_\xi} B_y \xrightarrow{u_y} A_y.
\]

For $\xi$ as above and $\zeta: y \to z$, define $\mu_{\xi, \zeta}$ for $A$ to be the composite

\[
\begin{array}{ccc}
A_y & \xrightarrow{s_\xi} & B_y \\
\downarrow{\varepsilon_y} & & \downarrow{1} \\
B_x & \xrightarrow{t_\xi} & B_y \\
\downarrow{1} & & \downarrow{s_\xi} \\
A_x & \xrightarrow{f_x} & B_x
\end{array}
\]

and define the $\eta$ for $A$ to be the composite

\[
\begin{array}{ccc}
A_x & \xrightarrow{f_x} & B_x \\
\downarrow{1} & \Downarrow{\eta_x} & \downarrow{u_x} \\
A_x & \xrightarrow{f_x} & A_x
\end{array}
\]

Proposition 4.1 (Street). This defines a lax functor $A$; the $f_x$ become the components of an oplax natural transformation and the $u_x$ become the components of a lax natural transformation.

The most important case is the following.
Definition 4.2. An $X$-adjunction consists of a strict functor $B : X \to \text{Cat}$ equipped with an adjunction $f_x \dashv u_x : B_x \to A_x$ for each object $x \in X$.

If $X = 1$, then $B$ reduces to a single category, and we are given a functor $u : B \to A$ with a left adjoint $f \dashv u$, so that a 1-adjunction is just an adjunction. The induced lax functor $A : 1 \to \text{Cat}$ is the monad induced by the adjunction.

In the case $X = \text{Iso}$, then $B$ amounts to a pair of categories with an isomorphism between them, but up to isomorphism this is again just a single category. But our notion of adjunction is now a pair of categories between them, but up to isomorphism this is again just a single category. But our notion of adjunction is now a pair of categories $A_x$ and $A_y$, and a pair of functors $u_x : B \to A_x$ and $u_y : B \to A_y$ with adjunctions $f_x \dashv u_x$ and $f_y \dashv u_y$; thus an Iso-adjunction amounts to a double adjunction in the sense of [3]. The induced lax functor $\text{Iso} \to \text{Cat}$ is precisely the induced Morita context described in [4, Section 3.1].

5. Eilenberg-Moore construction

We have seen that a monad is the same thing as a lax functor $1 \to \text{Cat}$, and that an algebra for the monad is then the same thing as a lax natural transformation from the strict functor $1 \to \text{Cat}$ whose image is the terminal category. This strict functor is of course the unique representable functor $1 \to \text{Cat}$. This suggests the importance of lax natural transformations from strict functors, especially representable ones, to lax functors.

Let $[X, \text{Cat}]$ denote the 2-category of strict functors from $X$ to $\text{Cat}$, strict natural transformations between them, and modifications. Recall that we write $\text{Lax}(X, \text{Cat})_\ell$ for the 2-category of lax functors from $X$ to $\text{Cat}$, lax natural transformations, and modifications.

Proposition 5.1 (Street). The inclusion $[X, \text{Cat}] \to \text{Lax}(X, \text{Cat})_\ell$ has a right adjoint.

We shall write $\text{alg}(A) : X \to \text{Cat}$ for the value of the right adjoint at a lax functor $A : X \to \text{Cat}$. The universal property of the adjunction says in part that for any strict functor $B : X \to \text{Cat}$, there is a bijection between lax natural transformations $B \to A$ and strict natural transformations $B \to \text{alg}(A)$. In particular, this should be the case if $B$ is a representable functor $X(x, -) : X \to \text{Cat}$. By the Yoneda lemma, strict natural transformations $X(x, -) \to \text{alg}(A)$ are in bijection with objects of $\text{alg}(A)_x$, and so we deduce that the objects of $\text{alg}(A)_x$ should be the lax natural transformations from $X(x, -)$ to $A$. Similarly the morphisms of $\text{alg}(A)_x$ are the modifications between lax natural transformations from $X(x, -)$ to $A$.

This reasoning tells us what $\text{alg}(A)$ must be; the fact that this actually works is proved in [11, Theorem 4].

If $X = 1$, so that a lax functor $A : 1 \to \text{Cat}$ is just a monad, then $\text{alg}(A) : 1 \to \text{Cat}$ picks out the Eilenberg-Moore category of the monad. This $\text{alg}(A)$ is Street’s “second construction” on lax functors; the first corresponds to the Kleisli construction for monads, and provides a left adjoint to the inclusion of $[X, \text{Cat}]$ in the 2-category $\text{Lax}(X, \text{Cat})_c$ of lax functors, oplax natural transformations, and modifications.

Proposition 5.2 (Street). Let $A : X \to \text{Cat}$ be a lax functor, and $\text{alg}(A) : X \to \text{Cat}$ the strict functor defined above. For each $x \in X$ there is a functor $u_x : \text{alg}(A)_x \to A_x$ sending a lax natural transformation $X(x, -) \to A$ to the value at $1 : x \to x$ of its
x-component $X(x, x) \to A_x$, and this functor $u_x$ has a left adjoint $f_x$. The lax functor $X \to \text{Cat}$ induced by this adjunction is $A$.

For a Morita context, seen as a lax functor $A: \text{Iso} \to \text{Cat}$, the corresponding strict functor $\text{alg}(A): \text{Iso} \to \text{Cat}$ will give a pair of isomorphic categories, but up to isomorphism we may take this just to be a single category. This is exactly the Eilenberg-Moore category associated to the Morita context, as described in [4, Section 3.2].

6. Interlude: parametrized internal coequalizers

The Beck condition characterizing monadic functors involves existence and preservation of certain coequalizers. We shall describe a generalization of Beck’s theorem in the next section, but first we need to develop a notion of coequalizer suitable for our purposes.

Let $\mathcal{K}$ be a 2-category, and $K$ an object of $\mathcal{K}$. We write $\mathcal{K}//K$ for the evident 2-category whose objects are arrows in $\mathcal{K}$ with codomain $K$, and arrows are triangles in $\mathcal{K}$ which commute up to a specified 2-cell. We use the following notation. An object of $\mathcal{K}//K$ is a morphism $A: \partial_0 A \to K$ in $\mathcal{K}$. A morphism $A \to B$ consists of a morphism $\partial_0 f: \partial_0 B \to \partial_0 A$ in $\mathcal{K}$ equipped with a 2-cell $f$ from $A$ to the composite of $B$ and $\partial_0 f$. A 2-cell from $f$ to $g$ is a 2-cell $\partial_0 f \to \partial_0 g$ which when pasted with $g$ gives $f$.

There is an evident forgetful 2-functor $\partial_0: \mathcal{K}//K \to \mathcal{K}$. The image under $\partial_0$ of an object or morphism will sometimes be called its head.

**Example 6.1.** If $\mathcal{K} = \text{Cat}$, and $K \in \text{Cat}$ is a category, then we may identify $K$ with the full subcategory of $\mathcal{K}//K$ consisting of all objects $(A, a)$ with $A$ the terminal category $1$. In other words, $K$ is the fibre of $\partial_0: \text{Cat}//\text{Cat} \to \text{Cat}$ over $1$.

**Definition 6.2.** We say that the object $K \in \mathcal{K}$ has parametrized internal coequalizers if the category $\mathcal{K}//K$ has coequalizers preserved by $\partial_0$.

We shall often want to restrict to the case of parametrized internal coequalizers with specified head; in particular, the example above shows that a category $K$ has ordinary coequalizers if and only if it has parametrized internal coequalizers in $\text{Cat}$ with head the terminal coequalizer diagram

$$1 \longrightarrow 1 \longrightarrow 1.$$ 

Since our parametrized internal coequalizers in $K$ are just ordinary coequalizers in $\mathcal{K}//K$, we can define a parametrized internal coequalizer to be split when it is split as a coequalizer in $\mathcal{K}//K$. Once again, we may choose to specify the splitting in the head.

When it comes to preservation, both the 2-category $\mathcal{K}$ and the object $K$ might be varied. The key observation is that for a 2-functor $F: \mathcal{K} \to \mathcal{L}$ and a morphism $f: FK \to L$ there is an induced 2-functor $F//f: \mathcal{K}//K \to \mathcal{L}//L$ sending an object $A: \partial_0 A \to K$ to the object

$$F(\partial_0 A) \xrightarrow{FA} FK \xrightarrow{f} L$$
of $\mathcal{L}//\mathcal{L}$. We shall say that $F//f$ preserves a parametrized internal coequalizer in $\mathcal{K}//\mathcal{K}$ if $F//f$ sends it to a parametrized internal coequalizer in $\mathcal{L}//\mathcal{L}$. This includes in particular the requirement that $F$ preserve the coequalizer appearing in the head.

7. **Beck theorem**

Consider an $X$-adjunction consisting of a strict functor $B: X \to \text{Cat}$ equipped with functors $u_x: B_x \to A_x$ for each $x \in X$, with left adjoints $f_x \dashv u_x$. We have seen that these adjunctions generate a lax functor $A: X \to \text{Cat}$, and that the $u_x$ become the components of a lax natural transformation $u: B \to A$. Thus by the universal property of the Eilenberg-Moore object (second construction) they correspond to a unique strict natural transformation $k: B \to \text{alg}(A)$.

In the case $X = 1$, this is the canonical comparison functor into the Eilenberg-Moore category; in the case $X = \text{Iso}$ it is the comparison functor of [4, Section 3.5].

We shall say that the $X$-adjunction is monadic if the components $k_c: B_c \to \text{alg}(A)_c$ of $k$ are equivalences of categories. This is not enough to make $k$ into an equivalence in the 2-category $[X, \text{Cat}]$, since the equivalence inverses $\text{alg}(A)_c \to B_c$ need not be strictly natural, but they will be pseudonatural, and in fact the condition is equivalent to saying that $k$ is an equivalence in the 2-category of 2-functors from $X$ to $\text{Cat}$, pseudonatural transformations, and modifications.

This corresponds to the ordinary (up-to-equivalence) notion of monadity in the case $X = 1$, and to what was called “moritability” in the case $X = \text{Iso}$. [4]

Similarly, we may define the $X$-adjunction to be strictly monadic if the $k_c$ are isomorphisms of categories. This is of course equivalent to $k$ being an isomorphism in $[X, \text{Cat}]$.

It is this strict notion of monadity that was used in [11]. The Beck-style theorem of [11, Theorems 10 and 11] involved a colimit-notion called a “universal reflection of a centipede”, which we shall now describe using the notion of parametrized internal coequalizer given in the previous section. The presentation given here looks quite different to that of [11] but the equivalence of the two presentations is an extended but straightforward exercise using the Yoneda lemma.

Write $\text{ob}X$ for the category with the same objects as $X$, but with only identity morphisms. The evident forgetful 2-functor $U: [X, \text{Cat}] \to [\text{ob}X, \text{Cat}]$ given by composition with the inclusion $\text{ob}X \to X$ has both left and right adjoints, given by left and right Kan extension, and in fact $U$ is both monadic and comonadic. In particular, we shall write $F$ for the left adjoint of $U$, and $\varepsilon: FU \to 1$ for the counit and $\eta: 1 \to UF$ for the unit. A straightforward calculation shows that, for an arbitrary strict functor $C: X \to \text{Cat}$, the functor $FUC$ is given by

$$(FUC)_x = \sum_{y \to x} C_y$$

with $\varepsilon: FUC \to C$ induced by the maps $C_{\xi}: C_y \to C_x$ for $\xi: y \to x$. One aspect of the monadcity of $U$ is that the diagram

$$\begin{array}{ccc}
FUFUC & \xrightarrow{\varepsilon FU} & FUC \\
\xrightarrow{FU\varepsilon} & & \xrightarrow{\varepsilon} C
\end{array}$$
is always a coequalizer, and this coequalizer is $U$-split, in the sense that it is sent by $U$ to a split coequalizer, given in the following diagram.

\[
\begin{array}{ccc}
UFUFUC & \xrightarrow{U\varepsilon} & U\varepsilon \circ UC \\
& \downarrow \varepsilon & \downarrow \varepsilon \\
UFU\varepsilon & \xrightarrow{U\varepsilon} & \varepsilon \circ UC
\end{array}
\] (7.1)

For a functor $B: X \to \textbf{Cat}$ and object $x \in X$, an $x$-centred centipede in $B$, in the sense of [11], is now a pair of parallel arrows in $[X, \textbf{Cat}]/B$ with head given by the parallel arrows in

\[
FUFUX(x, -) \xrightarrow{\varepsilon \circ FU} FUX(x, -) \xrightarrow{\varepsilon} X(x, -)
\]

while a universal reflection for the centipede is precisely an internal parametrized coequalizer; of course its head must be given as in the diagram, since we require the coequalizer to be preserved by $\partial_0: [X, \textbf{Cat}]/B \to [X, \textbf{Cat}]$. An $x$-centred centipede as above is split by the family $u_x$, in the sense of [11], just when it is sent by $U//u$ to a split coequalizer in $[\text{ob}X, \textbf{Cat}]/A$ with head given by the canonical splitting appearing in (7.1). We shall say then that the centipede is $U//u$-split.

We may now say that $U//u$ creates ($U//u$)-split parametrized internal coequalizers with canonical head, if for each parallel pair of morphisms in $[X, \textbf{Cat}]/B$ with head as in (7.1) and which is $U//u$-split, there is a unique parametrized internal coequalizer in $[\text{ob}X, \textbf{Cat}]/A$ which is sent by $U//u$ to the specified (split) parametrized internal coequalizer in $[\text{ob}X, \textbf{Cat}]/A$.

**Theorem 7.1** (Street). For an $X$-adjunction $f \dashv u: UB \to A$, the induced comparison map $k: B \to \text{alg}(A)$ is invertible if and only if $U//u$ creates ($U//u$)-split parametrized internal coequalizers with canonical head.

It is straightforward to modify this condition to deal with the “up-to-equivalence” notion of monadicity. One simply asks that $[X, \textbf{Cat}]/B$ have, and that $U//u$ preserve and reflect, parametrized internal coequalizers with canonical head for all parallel pairs whose image under $U//u$ has a split parametrized internal coequalizer with canonical head.

We now investigate what this condition says in the case $X = \textbf{Iso}$ relevant to Morita contexts; as usual, we identify strict functors $\textbf{Iso} \to \textbf{Cat}$ with single categories.

An $X$-adjunction then consists of a pair of right adjoints $u_0: B \to A_0$ and $u_1: B \to A_1$.

A centipede consists of the following data: objects $b_x \in B$ for each $x \in X$, objects $b_{xy} \in B$ for each $x, y \in X$, morphisms $\beta_{xy}: b_{xy} \to b_x$ and $\beta'_{xy}: b_{xy} \to b_y$ for each $x, y \in X$. A universal reflection of the centipede is a colimit of the diagram
(compare diagram (16) in [4]). The general theorem stated above asks for such universal reflections whenever the centipede is split by \(U/u\), but on inspecting the proof one finds, just as in the case of the classical Beck theorem, that only certain specific colimits involving free algebras are required. It is these specific colimits which are used in [4, Theorem 3.12].

8. Generalizations

As was mentioned in the previous section, the forgetful 2-functor \(U: \mathcal{X}, \text{Cat} \to \text{ob}\mathcal{X}, \text{Cat}\) is both monadic and comonadic. In this section we focus on the comonadicity of \(U\) along with the fact that the comonad in question preserves certain limits (in the case of the previous section, it has an adjoint).

We therefore consider a 2-category \(\mathcal{K}\) equipped with a 2-comonad \(G = (G,d,e)\), and we write \(\mathcal{K}^G\) for the Eilenberg-Moore 2-category of \(G\): the 2-category of strict \(G\)-algebras, strict morphisms of \(G\)-algebras, and algebra 2-cells. Various weakenings of these notions have been studied in detail in the monad case, but much less so for comonads, so we shall recall in full the required definitions; this also serves to fix our conventions for the directions of 2-cells. We discuss in the next section how to dualize these results, and so obtain results about 2-monads.

Example 8.1. The setting of [11] corresponds to the case \(\mathcal{K} = \text{ob}\mathcal{X}, \text{Cat}\), with \(G\) the comonad induced by right Kan extension. The classical case of ordinary monads corresponds to the case where \(\mathcal{K} = \text{Cat}\) and \(G\) is the identity comonad. The setting of [4] corresponds to the comonad \(G\) on \(\text{Cat} \times \text{Cat}\) induced by the diagonal \(\Delta: \text{Cat} \to \text{Cat} \times \text{Cat}\) and its right adjoint; explicitly \(G(A,B) = (A \times B, A \times B)\).

A \text{ lax } G\text{-coalgebra} consists of an object \(A\) equipped with a morphism \(a: A \to GA\) and 2-cells

\[
\begin{array}{ccc}
A & \xrightarrow{a} & GA \\
\downarrow a & & \downarrow \alpha \\
GA & \xrightarrow{dA} & G^2A
\end{array}
\quad \quad \begin{array}{ccc}
A & \xrightarrow{1} & \xrightarrow{eA} A \\
\downarrow a & & \downarrow \alpha_0 \\
GA & \xrightarrow{eA} & A
\end{array}
\]

subject to the following three coherence conditions.
A *lax $G$-morphism* from $(A, a, \alpha, \alpha_0)$ to another lax $G$-algebra $(B, b, \beta, \beta_0)$ consists of a morphism $f: A \to B$ equipped with a 2-cell

$$A \xrightarrow{f} B$$

subject to two coherence conditions

There are also *colax $G$-morphisms* in which the sense of $f$ is reversed. In the case of strict $G$-coalgebras, this could be obtained from lax $G$-morphisms via a formal dualization process, but in our more general setting this is no longer the case. We shall write $\tilde{f}$ for the 2-cell part of a colax $G$-morphism; the coherence conditions are
A $G$-transformation between lax $G$-morphisms $(f, \overline{f})$ and $(g, \overline{g})$ from $(A, a, \alpha, \alpha_0)$ to $(B, b, \beta, \beta_0)$ is a 2-cell $\rho: f \to g$ satisfying the single condition

There is a 2-category Lax-$G$-Coalg $\ell$ of lax $G$-coalgebras, lax $G$-morphisms, and $G$-transformations.

There are also $G$-transformations between colax $G$-morphisms, and a corresponding 2-category, but we shall not need these.

**Example 8.2.** If $G$ is the 2-comonad on $[\text{ob} X, \text{Cat}]$ whose Eilenberg-Moore 2-category is $[X, \text{Cat}]$, then a lax $G$-coalgebra is a lax functor from $X$ to $\text{Cat}$, a lax $G$-morphism is a lax natural transformation, and a colax $G$-morphism is an oplax natural transformation. A $G$-transformation is a modification between oplax natural transformations. In particular, if $G$ is the identity 2-comonad on $\text{Cat}$, a lax $G$-coalgebra is a monad, a lax $G$-morphism is a monad functor, a colax $G$-morphism is a monad opfunctor, and a $G$-transformation is a monad transformation.

Generalizing the fact that every adjunction induces a monad we have:

**Theorem 8.3.** Let $(B, b)$ be a strict $G$-coalgebra, and let $u: B \to A$ be a morphism in $K$ with left adjoint $f \dashv u$, and write $\eta$ and $\varepsilon$ for the unit and counit. Then $A$ becomes a lax $G$-coalgebra $(A, a, \alpha, \alpha_0)$ where $a$, $\alpha$, and $\alpha_0$ are given by
and then $u$ becomes a lax $G$-morphism $(u, \overline{u}) : (B, b, \beta, \beta_0) \to (A, a, \alpha, \alpha_0)$ when we define $\overline{u}$ to be the 2-cell

\[
\begin{array}{ccc}
B & \overset{b}{\rightarrow} & GB \\
\downarrow_{u} & & \downarrow_{Gu} \\
A & \overset{f}{\rightarrow} & GB
\end{array}
\]

Proof. All of the assertions can be verified by direct calculation. Alternatively, one can deduce the results from [7], specifically Theorem 3.5 and the discussion preceding Theorem 4.1. As stated these results require the existence of comma objects in $\mathcal{K}$, but this can be avoided by embedding $\mathcal{K}$ in a larger 2-category if necessary. \qed

Remark 8.4. This generalizes easily to the case where $B$ is only a lax $G$-coalgebra; see [7] again.

By doctrinal adjunction (see [6, Theorem 1.2]) the lax $G$-morphism structure on $u$ determines colax $G$-morphism structure on $f$ in the form of the 2-cell $\tilde{f}$ displayed below.

\[
\begin{array}{ccc}
A & \overset{f}{\rightarrow} & B \\
\downarrow_{f} & & \downarrow_{b} \\
B & \overset{b}{\rightarrow} & GB
\end{array}
\]

The Eilenberg-Moore construction in this context becomes the following theorem. The lax descent objects mentioned in the theorem were defined in [8] via a minor modification of a definition in [12]. They can be constructed out of inserters and equifiers.

Theorem 8.5. If $\mathcal{K}^G$ has lax descent objects for lax coherence data, and so in particular if it has inserters and equifiers, then the inclusion $\mathcal{K}^G \to \text{Lax-}G\text{-Coalg}_\ell$ has a right adjoint.

Proof. This is a straightforward modification of the argument given in [8] for the case of lax $T$-algebras and lax $T$-morphisms for a 2-monad $T$. The value at a lax $G$-coalgebra $(A, a, \alpha, \alpha_0)$ of the right adjoint is the universal strict $G$-coalgebra $\text{alg}(A)$ equipped with a strict $G$-morphism $v : \text{alg}(A) \to GA$ and a $G$-transformation

\[
\begin{array}{ccc}
\text{alg}(A) & \xrightarrow{v} & GA \\
\downarrow_{v} & \searrow^{\psi} & \downarrow_{Ga} \\
GA & \xrightarrow{d_A} & G^2A
\end{array}
\]
satisfying the two conditions

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,0) {\text{alg}(A)};
  \node (b) at (2,2) {G^2 A};
  \node (c) at (4,2) {G^3 A};
  \node (d) at (0,-2) {G A};
  \node (e) at (2,-2) {G^2 A};
  \node (f) at (4,-2) {G^3 A};
  \node (g) at (6,0) {G A};
  \node (h) at (8,0) {G^2 A};
  \node (i) at (10,0) {G^3 A};
  \draw[->] (a) to node {$v$} (d);
  \draw[->] (d) to node {$G a$} (b);
  \draw[->] (b) to node {$G^d a$} (c);
  \draw[->] (c) to node {$d G a$} (i);
  \draw[->] (d) to node {$d A$} (e);
  \draw[->] (e) to node {$G v$} (f);
  \draw[->] (f) to node {$G a$} (g);
  \draw[->] (g) to node {$G a$} (h);
  \draw[->] (h) to node {$G a$} (i);
  \draw[->] (a) to node [swap] {$\psi$} (d);
  \draw[->] (d) to node [swap] {$\psi$} (g);
  \draw[->] (d) to node [swap] {$\psi$} (h);
\end{tikzpicture}
\end{array}
\]

(8.1)

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,0) {\text{alg}(A)};
  \node (b) at (2,2) {G^2 A};
  \node (c) at (4,2) {G A};
  \node (d) at (0,-2) {G A};
  \node (e) at (2,-2) {G^2 A};
  \node (f) at (4,-2) {G^3 A};
  \node (g) at (6,0) {G A};
  \node (h) at (8,0) {G^2 A};
  \node (i) at (10,0) {G^3 A};
  \draw[->] (a) to node {$v$} (d);
  \draw[->] (d) to node {$G a$} (b);
  \draw[->] (b) to node {$G a$} (c);
  \draw[->] (c) to node {$G a$} (i);
  \draw[->] (d) to node {$d A$} (e);
  \draw[->] (e) to node {$G d A$} (f);
  \draw[->] (f) to node {$G^2 A$} (g);
  \draw[->] (g) to node {$G a$} (h);
  \draw[->] (h) to node {$G a$} (i);
  \draw[->] (a) to node {$v$} (d);
  \draw[->] (d) to node {$G a$} (g);
  \draw[->] (d) to node {$G a$} (h);
\end{tikzpicture}
\end{array}
\]

(8.2)

The component at \((A,a,\alpha,\alpha_0)\) of the counit is the lax morphism \((q,\overline{q}): \text{alg}(A) \to (A,a,\alpha,\alpha_0)\) where \(q\) is the composite \(e A.v\) and \(\overline{q}\) is given by the pasting composite

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,0) {\text{alg}(A)};
  \node (b) at (2,2) {G^2 A};
  \node (c) at (4,2) {G A};
  \node (d) at (0,-2) {G A};
  \node (e) at (2,-2) {G^2 A};
  \node (f) at (4,-2) {G A};
  \node (g) at (6,0) {G A};
  \node (h) at (8,0) {G^2 A};
  \node (i) at (10,0) {G^3 A};
  \draw[->] (a) to node {$v$} (d);
  \draw[->] (d) to node {$G a$} (b);
  \draw[->] (b) to node {$G a$} (c);
  \draw[->] (c) to node {$G a$} (i);
  \draw[->] (d) to node {$d A$} (e);
  \draw[->] (e) to node {$G d A$} (f);
  \draw[->] (f) to node {$G^2 A$} (g);
  \draw[->] (g) to node {$G a$} (h);
  \draw[->] (h) to node {$G a$} (i);
  \draw[->] (a) to node {$\psi$} (d);
  \draw[->] (d) to node {$\psi$} (g);
  \draw[->] (d) to node {$\psi$} (h);
\end{tikzpicture}
\end{array}
\]

in which \(a'\) denotes the coalgebra structure of \(\text{alg}(A)\).

\[\square\]

**Theorem 8.6.** If \(K\) has lax descent objects for lax coherence data, and \(G\) preserves them, then the inclusion \(K^G \to \text{Lax-}G\text{-Coalg}_\ell\) has a right adjoint. Furthermore, the component at a lax coalgebra \((A,a,\alpha,\alpha_0)\) of the counit of the adjunction is a lax morphism \((q,\overline{q}): (\text{alg}(A),a') \to (A,a,\alpha,\alpha_0)\) for which \(q: \text{alg}(A) \to A\) has a left adjoint \(p\), and this \(p\) inherits a colax \(G\)-morphism structure.

**Proof.** The first sentence holds because the forgetful 2-functor \(U: K^G \to K\) creates limits of any type which are preserved by \(G\). Thus we may construct \(\text{alg}(A)\) as the universal object in \(K\) equipped with a morphism \(v: \text{alg}(A) \to G A\) and 2-cell \(\psi\) as in the proof of the previous theorem. There is then a unique induced morphism \(a': \text{alg}(A) \to G \text{alg}(A)\) making \(\text{alg}(A)\) into a strict \(G\)-coalgebra, \(v\) into a strict morphism of \(G\)-coalgebras, and \(\psi\) into a \(G\)-transformation, and the construction of \(q\) and \(\overline{q}\) was described in the previous theorem.

The existence of the adjoint \(p\) is a lax version of \([8, \text{Theorem 3.2}]\), while the colax structure of \(p\) is given by doctrinal adjunction again: see \([6, \text{Theorem 1.2}]\). Explicitly, \(p: A \to \text{alg}(A)\) is the unique morphism in \(K\) whose composite with \(v\) is \(a: A \to G A\) and whose composite with \(\psi\) is \(\alpha\). The unit \(\eta: 1 \to qp\) of the adjunction \(p \dashv q\) is \(\alpha_0\).
The counit is the unique 2-cell \( \varepsilon : qp \to 1 \) whose composite \( v\varepsilon \) with \( v \) is given by

\[
\begin{array}{c}
vpq \arr{a} \quad \arr{a \cdot eA \cdot v} \quad eGA \cdot Ga \cdot v \\
\quad \arr{eGa \cdot \psi} \\
\quad \arr{v \arr{eGa \cdot dA \cdot v}}
\end{array}
\]

The colax structure for \( p \) is given by the unique 2-cell \( \tilde{p} \) for which the pasting composite

\[
\begin{array}{c}
A \arr{p} \quad \arr{\alg(A) \cdot v} \quad GA \\
\quad \\ Ga \arr{Ga \cdot \psi} \\
G2A \arr{Ga \cdot \psi} \quad \arr{G2A}
\end{array}
\]

is equal to \( \alpha \).

Under the hypotheses of the theorem, suppose that \((B, b)\) is a strict \( G \)-coalgebra, and that \( u : B \to A \) is a morphism with a left adjoint \( f \dashv u \). Let \((A, a, \alpha, \alpha_0)\) be the induced lax \( G \)-coalgebra, and \((u, \overline{\alpha}) : (B, b) \to (A, a, \alpha, \alpha_0)\) the induced lax \( G \)-morphism. By the universal property of \( \alg(A) \), this corresponds to a unique strict \( G \)-morphism \( k : (B, b) \to (\alg(A), a') \), with \((q, \overline{q})k = (u, \overline{u})\).

**Definition 8.7.** We call this \( k \) the *canonical comparison*, and say that the original adjunction \( f \dashv u \), or just \( u \), is strictly monadic relative to \( G \) if this \( k \) is an isomorphism, and monadic relative to \( G \) if the underlying morphism \( k : B \to \alg(A) \) in \( K \) is an equivalence.

We now generalize the centipedes of [11] to this setting; of course they have long since lost any similarity to centipedes in the biological sense. The description is a sort of dual of that in Section 6 since our forgetful 2-functor is now comonadic rather than monadic.

For a 2-category \( \mathcal{M} \) and an object \( C \in \mathcal{M} \) we write \( C /// \mathcal{M} \) for the evident 2-category whose objects are arrows \( A : C \to \partial_1 A \) in \( \mathcal{M} \), whose arrows from \((A, \partial_1 A)\) to \((B, \partial_1 B)\) are arrows \( \partial_1 f : \partial_1 B \to \partial_1 A \) equipped with a 2-cell \( f : A \to \partial_1 f \cdot B \), as in

\[
\begin{array}{c}
\partial_1 A \\
\arr{\partial_1 f} \\
\arr{f} \\
C \arr{B} \quad \arr{\partial_1 B}
\end{array}
\]

and whose 2-cells from \((f, \partial_1 f)\) to \((g, \partial_1 g)\) are 2-cells from \( \partial_1 f \to \partial_1 g \) satisfying the evident compatibility condition. There is an evident projection \( \partial_1 : (C /// \mathcal{M}) \to \mathcal{M}^{op} \).

We shall be interested in the case where \( \mathcal{M} = \mathcal{K}^G \).

For \( G \)-coalgebras \((C, c)\) and \((B, b)\), we define a \((C, c)\)-centred centipede in \((B, b)\) to be a parallel pair of arrows in \((C, c) /// \mathcal{K}^G \) which lie over \( Gb : GB \to G^2B \) and \( dB : GB \to G^2B \). We display below such a pair in two ways.
Rather than universal reflection, we shall simply speak of a coequalizer of a centipede, and define this to be a coequalizer in \((C, c)\)\!/\(\mathcal{K}^G\) which is preserved by \(\partial_1\) (and so must be projected to \(b: B \to GB\)).

A morphism \(g: (B, b) \to (B', b')\) in \(\mathcal{K}^G\) sends a centipede in \((B, b)\) to one in \((B', b')\), but it need not of course preserve coequalizers. A coequalizer of a centipede in \((B, b)\) is said to be absolute when it is preserved by all morphisms \(g: (B, b) \to (B', b')\) in \(\mathcal{K}^G\).

Among the absolute coequalizers are the split ones, and we can similarly define split coequalizers of centipedes. But this would require that the diagram

\[
\begin{array}{c}
M \\
\downarrow^m \\
C \\
\downarrow^N \\
GB
\end{array}
\xrightarrow{dB} (M, G^2B) \xrightarrow{(m,Gb)} (N, GB)
\]

is always an equalizer in \(\mathcal{K}^G\), is in fact a split equalizer in \(\mathcal{K}^G\) (not just in \(\mathcal{K}\)). This is not true for a general coalgebra \((B, b)\), but it is true for a cofree one, such as \((GA, dA)\), so we shall only consider split centipedes in cofree coalgebras. We always use the standard splitting for cofree coalgebras, as in the following diagram.

\[
\begin{array}{c}
GA \\
\xrightarrow{dA} G^2A \\
\xrightarrow{GdA} G^3A \\
\xrightarrow{G^2eA} GeA \\
\end{array}
\]

Explicitly, then, a split centipede in \((GA, dA)\) has the form

\[
(M, G^3A) \xrightarrow{(m,GdA)} (N, G^2A) \xrightarrow{(p,dA)} (P, GA)
\]

where \((p, dA)(m, GdA) = (p, dA)(n, dGA)\), \((p, dA)(s, GeA) = 1\), \((s, GeA)(p, dA) = (n, dGA)(t, G^2eA)\), and \((m, GdA)(t, G^2eA) = 1\). Clearly every split coequalizer of a centipede is absolute.

For a morphism \(g: (B, b) \to (B', b')\) in \(\mathcal{K}^G\), we say that a coequalizer of a centipede in \((B, b)\) is \(g\)-split or \(g\)-absolute if the induced centipede in \((B', b')\) is so.

We are now ready to state our version of the Beck monadicity theorem.

**Theorem 8.8.** Suppose that \(\mathcal{K}\) is a 2-category admitting lax descent objects of coherence data and that \(G = (G, d, e)\) is a 2-comonad on \(\mathcal{K}\) for which the 2-functor \(G\) preserves such lax descent objects; for example, \(\mathcal{K}\) could have inserters and equifiers, and \(G\) preserve them. Let \((B, b)\) be a strict \(G\)-coalgebra, and \(u: B \to A\) a right adjoint in \(\mathcal{K}\), and let \(w: B \to GA\) be the unique strict \(G\)-morphism whose composite with \(eA: GA \to A\) is \(u\). The following conditions are equivalent:

(a) \(u\) is strictly monadic relative to \(G\)
(b) \( w \) creates \( w \)-absolute coequalizers of centipedes
(c) \( w \) creates coequalizers of \( w \)-split centipedes in \((B, b)\).

Proof. We shall prove \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)\), following the classical proof as modified by Paré to use absolute coequalizers. Of course the implication \((b) \Rightarrow (c)\) holds trivially since split coequalizers are absolute.

First suppose \((a)\). Suppose that the pair \((m, Gb): (M, G^2B) \rightarrow (N, GB)\) and \((n, dB): (M, G^2B) \rightarrow (N, GB)\) are sent by \( w \) to a centipede in \((GA, dA)\) with an absolute coequalizer

\[
\begin{array}{ccc}
(G^2a.M, G^3A) & \xrightarrow{(G^2.a, G^3.a)} & (Gw.N, G^2A) & \xrightarrow{(q,G^2.a)} & (Q, GA)
\end{array}
\]

Then \(Ga\) preserves the coequalizer, and so the rows of the diagram

\[
\begin{array}{ccc}
(G^3a.G^2w.M, G^4A) & \xrightarrow{(G^3.a.Gw.m.GdA)} & (G^2a.Gw.M, G^3A) & \xrightarrow{(G^2.a,G^3.a)} & (Ga.Q, G^2A)
\end{array}
\]

are coequalizers. It follows by the universal property of the top coequalizer that there is a unique \( \theta \) as in the dotted arrow which makes the whole diagram commute. By considering the coequalizer obtained by applying \( G^2a \) to the top row, one verifies that this \( \theta \) satisfies the coherence condition \((8.1)\), and by using the original coequalizer one checks \((8.2)\); thus there is a unique map \( P: C \rightarrow B \) in \( K^G \) with \( wP = Q \) and \( \psi P = q \).

Commutativity of the square defining \( \theta \) is exactly what is needed to apply the two-dimensional aspect of the universal property of \( G\text{alg}(A) \) (recalling also that \( G \) preserves the limit which defines \( \text{alg}(A) \)), and so there is a unique map \( P: C \rightarrow B \) in \( K^G \) with \( wP = Q \) and \( \psi P = q \).

Thus it remains to show that \((c)\) implies \((a)\). As in the classical proof, we show that the comparison \( k: B \rightarrow \text{alg}(A) \) is invertible by constructing a suitable centipede with a \( w \)-split coequalizer. The centipede is centred at \( \text{alg}(A) \), and is displayed below.

\[
\begin{array}{ccc}
(M, G^2B) & \xrightarrow{(m,Gb)} & (N, GB) & \xrightarrow{(p,B)} & (P, B)
\end{array}
\]
Applying the strict map $w: B \to GA$, and using the fact that $wf = a$ and $Gw.\tilde{f} = \alpha$, we see that the induced centipede is given as in the following diagram

$$
\begin{array}{ccc}
(G^2a.Ga.v, G^3A) & \xrightarrow{(Ga.v, GdA)} & (Ga.v, GdA) \\
(G^2a,\psi, dGA) & \downarrow & (G^2a, Gv, GeA) \\
(G^2a_0.Ga.v, G^2eA) & \xrightarrow{} & (G^2a_0, Gv, GeA)
\end{array}
$$

in which a split coequalizer for the induced centipede is also given. The lifted coequalizer now has the form

$$
\begin{array}{ccc}
(G^2f.Ga.v, G^2B) & \xrightarrow{(Gf.v, GB)} & (Gf.v, GB) \\
(G^2f,\psi, dB) & \downarrow & (G^2f, v, dB) \\
& \xrightarrow{(p, b)} & (P, B)
\end{array}
$$

where $P: \text{alg}(A) \to B$ satisfies $wP = v$ and $G\tilde{w}.bP = \psi$. One now uses the universal property of $\text{alg}(A)$ to check that $kP = 1$, and the uniqueness aspect in the creation of $w$-absolute coequalizers to show that $Pk = 1$. \hfill \Box

We leave to the reader the modifications necessary to prove the following “up-to-equivalence” version of monadicity.

**Theorem 8.9.** In the setting of Theorem 8.8 once again, the following conditions are equivalent:

(a) $u$ is monadic relative to $G$

(b) any centipede in $(B, b)$ which is sent by $w$ to a centipede in $(GA, dA)$ with an absolute coequalizer itself has a coequalizer, and such coequalizers are preserved and reflected by $w$

(c) as in (b) but with split coequalizers rather than absolute ones.

**9. Dualization**

One of the striking things about the paper [10] was the use of duality to obtain both Eilenberg-Moore objects and Kleisli objects for both monads and comonads, all in a single setting.

In this short final section, we very briefly indicate how the results of the previous section can be dualized.

Write $K^{op}$ for the 2-category obtained from $K$ by formally reversing the 1-cells, but not the 2-cells. A 2-comonad $G$ on $K$, as in the previous section, corresponds to a 2-monad $T$ on $K^{op}$. A lax $G$-coalgebra is then the same thing as a lax $T$-algebra. Given lax $G$-coalgebras $A$ and $B$, a lax morphism of $G$-coalgebras from $A$ to $B$ is the same thing as a colax morphism $T$-algebras from $B$ to $A$. Thus Lax-$G$-Coalg$_{\ell} = \text{Lax-}T$-Alg$_c$.

The assumption of certain limits in $K$, preserved by $G$, is equivalent to the existence of certain colimits in $K^{op}$, preserved by $T$.

A right adjoint to the inclusion $K^G \to \text{Lax-}G$-Coalg$_{\ell}$ is equivalent to a left adjoint to the inclusion $K^T \to \text{Lax-}T$-Alg$_c$ of the Eilenberg-Moore 2-category $K^T$ of $T$ into the 2-category Lax-$T$-Alg$_c$ of lax $T$-algebras, colax $T$-morphisms, and $T$-transformations.

Thus our Beck-style theorem of the previous section becomes a recognition theorem for colax morphism classifiers (in the context of lax algebras for a 2-monad).
If the original comonad $G$ is the identity, so that $T$ is also the identity, then these colax morphism classifiers are in fact Kleisli objects for monads.

Write $\mathcal{K}^{\text{co}}$ for the 2-category obtained from $\mathcal{K}$ by formally reversing the 2-cells but not the 1-cells. A 2-comonad $G$ on $\mathcal{K}$ can equally be seen as a 2-comonad on $\mathcal{K}^{\text{co}}$, but now lax $G$-coalgebras in $\mathcal{K}$ are the same as colax $G$-coalgebras in $\mathcal{K}^{\text{co}}$.

If $G$ (and so $T$) are identities, then this reduces to Eilenberg-Moore objects for comonads.

One can also reverse both 1-cells and 2-cells to get a 2-category $\mathcal{K}^{\text{coop}}$, and the theory now generalizes Kleisli objects for comonads.

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Department of Mathematics, Macquarie University

E-mail address: steve.lack@mq.edu.au