Multifractal dimensions for critical random matrix ensembles

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Abstract – Based on heuristic arguments we conjecture that an intimate relation exists between the eigenfunction multifractal dimensions $D_q$ of the eigenstates of critical random matrix ensembles $D_q \approx q D_q'(q' + (q - q') D_q')^{-1}, 1 \leq q \leq 2$. We verify this relation by extensive numerical calculations.
We also demonstrate that the level compressibility $\chi$ describing level correlations can be related to $D_q$ in a unified way as $D_q = (1 - \chi)[1 + (q - 1)\chi]^{-1}$, thus generalizing existing relations with relevance to the disorder-driven Anderson transition.

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Introduction. – It is well-known that the spatial fluctuations of the eigenstates in a disordered system at the Anderson transition show multifractal characteristics [1,2] which has been demonstrated recently in a series of experiments [3]. Therefore the modeling and analysis of multifractal states has become of central importance producing many interesting results. For this purpose random matrix models have been invoked and studied recently [4–6].

Since the exact, analytical prediction of the multifractal dimensions of the states for the experimentally relevant Anderson transition in $d = 3$ or the integer quantum-Hall transition in $d = 2$ seems to be out of reach, it is desirable to search for heuristic relations in order to understand the complexity of the states at criticality. In the present paper we propose such heuristic relations that are numerically verified using various ensembles of random matrices.

The spatial fluctuations of the eigenstates can be described by a set of multifractal dimensions $D_q$ defined by the scaling of the inverse mean eigenfunction participation numbers with the system size $N$:

$$
\left\langle \sum_{i=1}^{N} |\Psi_i|^2 q \right\rangle \sim N^{-(q-1)D_q},
$$

where $\langle \cdots \rangle$ is the average over some eigenvalue window and over random realizations of the matrix. For strongly localized eigenstates these quantities do not scale with system size, i.e. $D_q \rightarrow 0$ for all $q$, while extended states always feel the entire system, i.e. $D_q \rightarrow d$, for all $q$.

Multifractal states, on the other hand, should be described by the series of the $D_q$, which are a nonlinear function of the parameter $q$.

Spectral fluctuations can be characterized in many ways. A usual, often employed quantity is the level compressibility $\chi$, which is extracted from the limiting behavior of the spectral number variance as $\Sigma^{(2)}(E) = \langle (n(E))^2 \rangle - \langle n(E) \rangle^2 \sim \chi E$, where $n(E)$ is the number of eigenstates in an interval of length $E$. The spectral fluctuations in a metallic system with extended states yield a vanishing compressibility, $\chi \rightarrow 0$, while in a strongly disordered insulating system the levels are uncorrelated, so they are easily compressible, $\chi = 1$. However, for the multifractal states an intermediate statistics exists, $0 < \chi < 1$, furthermore the spectral and eigenstate statistics are supposed to be coupled, which has been pointed out first in ref. [7].

One of the most important generalized dimensions often used in this context is the information dimension $D_I$. It is defined through the scaling of the mean eigenfunction entropy with the logarithm of the system size:

$$
\left\langle - \sum_{i=1}^{N} |\Psi_i|^2 \ln |\Psi_i|^2 \right\rangle \sim D_I \ln N.
$$

A further, well-known and widely used dimension is called the correlation dimension $D_2$, which is extracted from the inverse participation number from eq. (1) using $q = 2$. 

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In a recent work [4] Bogomolny and Giraud have shown that in a d-dimensional critical system the information dimension \( D_1 \) and the level compressibility \( \chi \) are simply related as
\[
\chi + D_1/d = 1, \tag{3}
\]
furthermore the generalized dimensions \( D_q \) can be expressed as
\[
\frac{D_q}{d} = \begin{cases} 
\frac{\Gamma(q - 1/2)}{\sqrt{\pi}(q)}(1 - \chi), & 1 - \chi \ll 1, \\
1 - q\chi, & \chi \ll 1.
\end{cases} \tag{4}
\]
These expressions have been shown to be valid for various critical random matrix ensembles in ref. [4].

As for the critical, three-dimensional Anderson transition and the two-dimensional quantum-Hall transition it has been shown earlier that another relation holds between the level compressibility \( \chi \) and the correlation dimension \( D_2 \) [7]:
\[
2\chi + D_2/d = 1. \tag{5}
\]
This relation should obviously hold approximately only since \( 0 < D_2/d < 1 \) but \( 0 < \chi < 1 \), leaving the range of validity for the limit of weak multifractality.

In the present work we show a series of relations between various generalized dimensions, \( D_q \) and \( D_q' \), and the level compressibility \( \chi \) allowing for a generalization that for particular cases yields eq. (3) exactly and eq. (5) in the appropriate limit. In order to prove that, numerical simulations of various critical random matrix ensembles will be used. Further implications and more details will be presented elsewhere [8].

**Model and heuristic relations.** — In ref. [4] eqs. (3) and (4) were shown to be correct numerically for the power-law banded random matrix (PBRM) model [2,9,10] at criticality. Below we will make use of this model to derive our main results.

The PBRM model describes one-dimensional (1d) samples of length \( N \) with random long-range hoppings. This model is represented by \( N \times N \) real symmetric \((\beta=1)\) or complex Hermitian \((\beta=2)\) matrices whose elements are statistically independent random variables drawn from a normal distribution with zero mean and a variance given by
\[
(\langle |H_{mm}|^2 \rangle = \beta^{-1} \tag{6}
\]
where \( b \) and \( \mu \) are parameters. In eq. (6) the PBRM model is in its periodic version; i.e. the 1d sample is in a ring geometry. Theoretical considerations [9,11] and detailed numerical investigations [2,12,13] have verified that the PBRM model undergoes a transition at \( \mu = 1 \) from localized states for \( \mu > 1 \) to delocalized states for \( \mu < 1 \). This transition shows key features of the disorder-driven Anderson metal-insulator transition [2], including multifractality of eigenfunctions and nontrivial spectral statistics. Thus the PBRM model possesses a line of critical points \( b \in (0, \infty) \) in the case of \( \mu = 1 \). In the following we will focus on the PBRM model at criticality, \( \mu = 1 \). By tuning the parameter \( b \), from \( b < 1 \) to \( b > 1 \), the states cross over from the nature of strong multifractality \((D_q \to 0)\) which corresponds to localized-like or insulator-like states to weak multifractality \((D_q \to 1)\) showing rather extended, i.e. metallic-like states. Meanwhile at the true Anderson transition in \( d = 3 \) or at the integer quantum-Hall transition in \( d = 2 \), the states belong to the weakly multifractal regime, i.e. \( d - D_2 < d \), the PBRM model allows for an investigation without such a limitation. The evolution of the generalized dimensions as a function of the parameter \( b \) therefore represents this behavior, i.e. \( D_q \to 1 \) for \( b \gg 1 \) and in the other limit of \( b \ll 1 \) the multifractal dimensions vanish as \( D_q \sim b \) [2,10].

Previously, for the PBRM model at criticality with \( \beta = 1 \), we have observed that both \( D_1 \) and \( D_2 \) can be approximated simply as \([14] \ D_1 \approx [1 + (\alpha_1 b)^{-1}]^{-1} \) and \( D_2 \approx [1 + (\alpha_2 b)^{-1}]^{-1} \), where \( \alpha_1, \alpha_2 \) are fitting constants. This continuous function is a trivial interpolation between the limiting cases of low \( b \) and large \( b \) taking the half of the harmonic mean of the two. Here we generalize and propose the following heuristic expression for a wider range of the parameter \( q \):
\[
D_q \approx [1 + (\alpha b)^{-1}]^{-1} \tag{7}
\]
as a global fit for the multifractal dimensions \( D_q \) of the PBRM model in both symmetries, \( \beta = 1 \) and \( \beta = 2 \). In fig. 1 we show fits of eq. (7) to numerically obtained \( D_q \) as a function of \( b \) for some values of \( q \) and in fig. 2 we plot the values of \( \alpha_q \) extracted from the fittings\(^1\). We observe that eq. (7) fits reasonably well the numerical \( D_q \) for \( q > 1/2 \). It is important to stress that eq. (7) reproduces well the \( b \)-dependencies predicted analytically [2] for the limits \( b \ll 1 \) and \( b \gg 1 \). However, for \( q < 1/2 \), eq. (7) can not be directly applied. The regime \( q < 1/2 \) could also be explored within our approach by the combination of eq. (7) and relation [15] \( \Delta_q = \Delta_{1-q} \), with \( \Delta_q = D_q(q - 1) - d(q - 1) \), implying a symmetry of the multifractal spectrum that links the multifractal dimensions with indexes \( q < 1/2 \) to those with \( q > 1/2 \). We will report results in this direction elsewhere [8].

We noticed that by the use of eq. (7), eq. (3) leads to
\[
\chi \approx (1 + \alpha b)^{-1}, \tag{8}
\]
which also well reproduces the \( b \)-dependencies predicted analytically [2,4] in the small- and large-\( b \) limits:
\[
\chi = \begin{cases} 
1 - 4b, & b \ll 1, \\
(2\pi b)^{-1}, & b \gg 1.
\end{cases} \tag{9}
\]
\(^1\)The multifractal dimensions \( D_q \) were extracted from the linear fit of the logarithm of the inverse mean eigenfunction participation numbers vs. the logarithm of \( N \), see eq. (1). \( D_1 \) was extracted from the linear fit of the mean eigenfunction entropy vs. the logarithm of \( N \), see eq. (2). We used \( N = 2^n \), \( 8 \leq n \leq 13 \). The average was performed over \( 2^{16-n} \) realizations of the random matrices.
Then, by equating $b$ in eqs. (7) and (8) we get

$$\chi \approx (1-D_q)[1+(\gamma_q-1)D_q]^{-1},$$

with $\gamma_q = \alpha_1/\alpha_q$. We observed that $\gamma_q \approx q$ in the range $0.8 < q < 2.5$, see fig. 2, so in this range of $q$ values we can write simplified relations between $\chi$ and $D_q$:

$$\chi \approx \frac{1-D_q}{1+(q-1)D_q} \quad \text{and} \quad D_q \approx \frac{1-\chi}{1+(q-1)\chi}. \quad (11)$$

The expression for $D_q$ in eq. (11) reproduces eq. (4) exactly for $q = 1$ and $q = 2$ and approximately for $1 < q < 2.5$. Moreover, eq. (11) combined with eq. (3) allows us to express any $D_q$ in terms of $D_1$:

$$D_q \approx D_1 [q + (1-q)D_1]^{-1}. \quad (12)$$

We also noticed that by equating $\chi$ for different $D_q$’s form eq. (11) we could get recursive relations for them:

$$\frac{q' D_{q'}}{1-D_{q'}} = \frac{q D_q}{1-D_q} \quad \text{and} \quad D_{q'} = \frac{q D_q}{1+(q-q')D_q}, \quad (13)$$

which lead to $D_{q+1} = qD_q(1+q-D_q)^{-1}$, when $q' = q+1$. Notice that the ratio $qD_q(1-D_q)^{-1}$ is independent of $q$.

These expressions also provide a relation between the correlation dimension and the information dimension or between the correlation dimension and the compressibility of the spectrum:

$$D_2 = D_1 (2-D_1)^{-1} = (1-\chi)(1+\chi)^{-1}. \quad (14)$$

It is relevant to add that in the weak multifractal regime, i.e. when $\chi \to 1$, eq. (14) reproduces the relation given in eq. (5) with $d = 1$, reported in [7].

We want to stress that we expect eqs. (11)–(14) to be valid for $0.8 < q < 2.5$, when $\gamma_q \approx q$, see fig. 2.

**Numerical results for the PBRM model.** Here we verify the expressions (10)–(14) for the PBRM model at criticality. Below we concentrate on the case $\beta = 1$ but we have already validated our results for $\beta = 2$ [8].

In fig. 3 we plot $(1-D_q)[1+(\gamma_q-1)D_q]^{-1}$ and $(1-D_q)[1+(q-1)D_q]^{-1}$ as a function of $b$ for several values of $q$ and observe good correspondence with the analytical prediction for $\chi$; that is, we verify the validity of eqs. (10) and (11), respectively. In the inset of fig. 3(b) we plot $qD_q(1-D_q)^{-1}$ as a function of $b$, see eq. (13), which for the PBRM model acquires the simple form

$$qD_q(1-D_q)^{-1} \approx \alpha_1 b, \quad (15)$$

making evident its independence from $q$.

Then, in fig. 4 we compare $D_q$ and $D_{q'}$ with $D_1[q + (1-q)D_1]^{-1}$ and $D_q[q' + (q-q')D_q]^{-1}$, respectively, for
several values of $q$; that is, we verify the validity of eqs. (12) and (13). Equation (14) is also validated in fig. 4(b).

Additionally, in [16] the duality relation

$$D_2(B) + D_2(B^{-1}) = 1, \quad B \equiv 2^{1/4} \pi b,$$

(16)

was shown to be valid (with maximum deviations of 1%) for the PBRM model at criticality. We also want to comment that by the use of eq. (7) we could write $D_2(B) \approx [1 + (\delta B)^{-1}]^{-1}$, with $\delta \equiv \alpha_2/(2^{1/4} \pi)$, so relation (16) gets the form

$$D_2(B) + D_2(B^{-1}) \approx 1 - \frac{B(\delta - 1)^2}{B + \delta (B^2 + \delta B + 1)}.$$  

(17)

We notice that the quantity $D_2(B) + D_2(B^{-1})$ is very sensitive to the value of $\alpha_2$. So, the error in $\alpha_2$ is magnified in the r.h.s. of eq. (17). The maximal deviation from 1 (of 7.3% and 2% for $\beta = 1$ and $\beta = 2$, respectively) occurs at $B = 1$ where the r.h.s. of eq. (17) acquires the form $1 - [(\delta - 1)/(\delta + 1)]^2$.

**Other critical ensembles.** Remember that relations (11)–(14) were obtained form the combination of eqs. (7) and (8). That is, relations (11)–(14) are expected to work in particular for the PBRM model at criticality. However, eqs. (11) reproduce eqs. (3) and (4), which were shown to be valid for the PBRM model but also for other critical ensembles [4]. Then the question is to which extent relations (11)–(14) are valid for critical ensembles different to the PBRM model. So, in the following we verify the validity of eqs. (11)–(14) for other critical ensembles.

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\footnote{The multifractal dimensions $D_1$ and $D_q$ for those ensembles were extracted numerically by the use of the same matrix sizes and ensemble realizations as for the PBRM model, if not indicated otherwise.}

Fig. 4: (Colour on-line) (a), (b): $D_q$ (open symbols) and $D_1[q + (1 - q)D_1]^{-1}$ (full red symbols) (see eq. (12)); and (c), (d): $D'_q$ (open circles) and $qD_0[q' + (q - q')D_3]^{-1}$ (full red symbols) (see eq. (13)) as a function of $b$ for the PBRM model.

Fig. 5: (Colour on-line) $D_1$, $D_2$, and $D_1(2 - D_1)^{-1}$ as a function of $a$ for the RSE (black open circles, red open diamonds, and blue full circles). Black and red dashed lines are the theoretical predictions for $D_1$ and $D_2$, respectively, given in eqs. (19) and (20). The blue dot-dashed line is the prediction for $D_2$ given by eq. (21).

**The Ruijsemaars-Schneider ensemble (RSE).** The RSE proposed in [17] is defined as matrices of the form

$$H_{mn} = \exp(i\Phi_m) \frac{1 - \exp(2\pi i a)}{N[1 - \exp(2\pi i (m - n + a)/N)]},$$

(18)

where $1 \leq m \leq n$, $\Phi_m$ are independent random phases distributed between 0 and $2\pi$, and $a$ is a free parameter independent of $N$. When $0 < a < 1$, the compressibility and the multifractal dimensions take the form [4]

$$\chi \sim (a - 1)^2$$

and $D_q = 1 - q(a - 1)^2$; \hspace{1cm} (19)

while in the vicinity of an integer $k \geq 2$, when $|a - k| \ll 1$, $\chi \sim (a - k)^2/k^2$ and $D_q = 1 - q(a - k)^2/k^2$. \hspace{1cm} (20)

As shown in [4], eqs. (19) and (20) satisfy relation (3). Moreover, by direct substitution of eqs. (19) (or eqs. (20)) we verified that eqs. (11)–(14) are also satisfied at leading order in $(a - 1)^2$ $[(a - k)^2]$. In fig. 5 we plot $D_1$ and $D_2$ as a function of $a$ for the RSE. Black and red dashed lines are the theoretical predictions for $D_1$ and $D_2$, respectively, given in eqs. (19) and (20). As it was earlier shown in ref. [4], the analytical form of $D_q$ given in eqs. (19) and (20) reproduces very well the numerically obtained $D_1$. However, we notice that eq. (19) does not describe the numerical $D_2$ well, mainly when $a \to 0$. Now, note that by plotting the numerically obtained $D_1/(2 - D_1)$ we get good agreement with the numerical data for $D_2$, that is eq. (14) works well for this model. Then, if we take $D_1 \approx 1 - (a - 1)^2$ and $D_1 \approx 1 - (a - k)^2/k^2$ as theoretical predictions for $D_1$ and plug them into eq. (14) we get

$$D_2 \approx \frac{1 - (a - 1)^2}{1 + (a - 1)^2}$$

and $D_2 \approx \frac{k^2 - (a - k)^2}{k^2 + (a - k)^2}$. \hspace{1cm} (21)

for $0 < a < 1$ and $|a - k| \ll 1$ with $k \geq 2$, respectively; which in fact work much better than $D_2 \approx 1 - 2(a - 1)^2$ and $D_2 \approx 1 - 2(a - k)^2/k^2$, correspondingly; see fig. 5.
Fig. 6: (Colour on-line) $D_q$ as a function of $q$ for the RSE (black open symbols) for several values of $a$. Blue dot-dashed lines are eq. (22). The green line is eq. (23) with $k = 2$. The red dashed line is eq. (12) with $D_1(a = 1.5) = 0.96$.

To get expressions for $D_q$ we substituted $\chi \sim (a - 1)^2$ and $\chi \sim (a - k)^2/k^2$ [or $D_1 \approx 1 - (a - 1)^2$ and $D_1 \approx 1 - (a - k)^2/k^2$] into eq. (11) (or eq. (12)), to get

$$D_q \approx \left[1 - (a - 1)^2\right] \left[1 + (q - 1)(a - 1)^2\right]^{-1}$$  \hspace{1cm} (22)

and

$$D_q \approx \left[k^2 - (a - k)^2\right] \left[k^2 + (q - 1)(a - k)^2\right]^{-1}.$$  \hspace{1cm} (23)

In fig. 6 we plot $D_q$ as a function of $q$ for the RSE for several values of $a$. We also plot eqs. (22) and (23) and observe rather good correspondence with the numerical data mainly in the range $1 < q < 2$. Notice that neither eq. (22) nor eq. (23) can be used for $a = 1.5$. For that case we substituted the numerically obtained value of $D_1$ into eq. (12) and again we observe good correspondence for $0 < q < 2$, see the red dashed line in fig. 6.

**Intermediate quantum maps.** A variant of the RSE was studied in [18] with the name of intermediate quantum maps (IQM) model. In this model the parameter $a$ of the RSE equals $cn/g$ with $cn = \pm 1$ mod $g$, being $g$ the parameter of the IQM model. For the IQM model the compressibility and the multifractal dimensions take the form [18]

$$\chi \approx 1/g \quad \text{and} \quad D_q \approx 1 - q/g.$$  \hspace{1cm} (24)

As for the RSE, here eqs. (24) satisfy relation (3). Again, by direct substitution of eqs. (24) we verified that eqs. (11)–(14) are satisfied at leading order in $1/g, g \gg 1$.

We want to mention that in [18] it was shown that eq. (24) reproduces well the numerically obtained $D_1$ but underestimates the numerical $D_2$, in particular for small $g$, see fig. 7. Now, notice that by plotting the numerically obtained $D_1/(2 - D_1)$ we nicely reproduce the numerical data for $D_2$, that is eq. (14) works well also for this model. Then, if we take $D_1 \approx 1 - 1/g$ as the theoretical prediction for $D_1$ and plug it into eq. (14) we get

$$D_2 \approx (1 - 1/g)(1 + 1/g)^{-1},$$  \hspace{1cm} (25)

which in fact works much better than $D_2 \approx 1 - 2/g$ in reproducing the numerical $D_2$, see fig. 7.

To get the expression for $D_q$ we substituted $\chi \approx 1/g$ or $D_1 \approx 1 - 1/g$ into eq. (11) or (12), respectively, to get

$$D_q \approx (g - 1)(g + q - 1)^{-1}.$$  \hspace{1cm} (26)

In fig. 8 we plot $D_q$ as a function of $q$ for the IQM model for some values of $g$. We also plot eq. (26) and observe that it falls below the numerical data mainly for small $g$. However, by substituting the numerically obtained values of $D_1$ into eq. (12) we get much better correspondence with the numerical $D_q$, mainly for $1 < q < 2$. In fig. 8 we also include $D_q$ from eq. (24). We may conclude that while eq. (24) well reproduces the numerical $D_q$ for $q < 1$, eq. (26) can serve as the analytical continuation for $q > 1$.

**The critical ultrametric ensemble.** The critical ultrametric ensemble (CUE) proposed in [19] consists of $2^K \times 2^K$ Hermitian matrices whose matrix elements are
Gaussian random variables with zero mean and variance

$$\langle |H_{mn}|^2 \rangle = W^2, \quad \langle |H_{mn}|^2 \rangle = 2^{\frac{d}{2} - d_m} J^2,$$

(27)

where $d_{mn}$ is the ultrametric distance between $m$ and $n$ on the binary tree with $K$ levels and the root of 1. The parameter in this model is the ratio $J/W$. For the CUE, when $J/W \ll 1$, the compressibility and the multifractal dimensions have the form [4,19]

$$\chi = 1 - \frac{J}{W} \frac{\pi}{\sqrt{2} \ln 2} \quad \text{and} \quad D_q = \frac{J}{W} \sqrt{\frac{\pi}{2}} \Gamma(q - 1/2).$$

(28)

Equations (28) satisfy relation (3) at first order in $J/W$ [4]. Again, as for the previous critical ensembles, by direct substitution of eqs. (28) we verified that eqs. (11)–(14) are satisfied at leading order in $J/W$, for $0.8 < q < 2.5$; because in this range of $q$ we have that $\Gamma(q - 0.5)/\sqrt{\pi} \Gamma(q) \approx 1/q$.

In fig. 9 we show $D_q$ as a function of $q$ for the CUE for $J/W = 0.1$ and 3. The data was taken from [19]. The blue dashed line is $D_q$ from eq. (28) for $J/W = 0.1$. Notice that since eq. (28) is only valid when $J/W \ll 1$ and for $q \geq 3/4$ one cannot use it to predict $D_q$ for $J/W = 3$. However, with eq. (12) using as input the numerically obtained $D_1$ we got good predictions for $D_q$ for small and large values of $J/W$ and even for values of $q$ smaller than 3/4. This is shown in fig. 9 where we plot eq. (12) (red dashed lines) using $D_1 = 0.2805$ and 0.8443 for $J/W = 0.1$ and 3, respectively. The values of $D_1$ were obtained by the interpolation of the $D_q$ data. We observe good correspondence between eq. (12) and the numerical $D_q$ for $0 < q < 10$.

Conclusions. – In this paper we propose heuristic relations on one hand between the generalized multifractal dimensions, $D_q$ and $D_q'$, for a relatively wide range of the parameter $q$, and on the other hand between these dimensions and the level compressibility $\chi$. As a result, we find a general framework embracing an earlier [7] and a recent one [4]. Our proposed relations have been backed by numerical simulation on various random matrix ensembles whose eigenstates have multifractal properties. These results call for further theoretical as well as numerical investigations.

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