Extended period domains and algebraic groups

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Abstract

For a linear algebraic group $G$ over $\mathbb{Q}$, we consider the period domains $D$ classifying $G$-mixed Hodge structures, and construct the extended period domains $D_{\Sigma}$. In particular, we give toroidal partial compactifications of mixed Mumford–Tate domains, mixed Shimura varieties over $\mathbb{C}$, and higher Albanese manifolds.

0 Introduction

For a linear algebraic group $G$ over $\mathbb{Q}$, we consider the period domains $D$ for $G$-mixed Hodge structures. We construct the extended period domains $D_{\Sigma}$, the space of nilpotent orbits.

In the case where $D$ is the pure Mumford–Tate domain (cf. Green–Griffiths–Kerr’s book [4]), $D_{\Sigma}$ essentially coincides with the one by Kerr–Pearlstein (9).

In the general case, we define $D$ in Section 1 by modifying the definition of Shimura variety over $\mathbb{C}$ by Deligne (2). In Section 2 we introduce the extended period domain $D_{\Sigma}$ and state the main results 2.3.1, 2.3.3. In Section 3 we explain the relation with the theory for the usual period domains (8), and as examples, the Mumford–Tate domains, mixed Shimura varieties, and higher Albanese manifolds.

For the $p$-adic variant of this paper, see 7.

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We omit the details of constructions and proofs in this paper, which are to be published elsewhere.

1 The period domain $D$

Let $G$ be a linear algebraic group over $\mathbb{Q}$. Let $G_u$ be the unipotent radical of $G$. Let $\text{Rep}(G)$ be the category of finite-dimensional linear representations of $G$ over $\mathbb{Q}$.

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1.1  G-mixed Hodge structures

1.1.1. Let $S_{C/R}$ be the Weil restriction ([2]) of $G_m$ from $C$ to $R$. Let $w : G_{m,R} \to S_{C/R}$ be the canonical homomorphism. A linear representation of $S_{C/R}$ over $R$ is equivalent to a finite-dimensional $R$-vector space $V$ endowed with a decomposition $V_C := C \otimes_R V = \bigoplus_{p,q \in \mathbb{Z}} V_{C}^{p,q}$ such that $V_{C}^{p,q} = V_{C}^{q,p}$ for any $p, q$. The corresponding decomposition is given by $V_{C}^{p,q} = \{v \in V_C \mid [z]v = z^p \bar{z}^q v \text{ for } z \in C^\times\}$. Here $[z]$ denotes $z$ regarded as an element of $S_{C/R}(R)$.

1.1.2. Let $h_0 : S_{C/R} \to (G/G_u)_R$ be a homomorphism. Assume that the composite $G_{m,R} \xrightarrow{w} S_{C/R} \to (G/G_u)_R$ is $\mathbb{Q}$-rational and central. Assume also that for one (and hence any) lifting $G_{m,R} \to G_R$ of this composite, the adjoint action of $G_{m,R}$ on $\text{Lie}(G_u)_R$ is of weight $\leq -1$.

Then, for any $V \in \text{Rep}(G)$, the action of $G_m$ on $V$ via a lifting $G_m \to G$ of the above $G_m \to G/G_u$ defines a rational increasing filtration $W$ on $V$ called the weight filtration, which is independent of the lifting.

A $G$-mixed Hodge structure ($G$-MHS, for short) is an exact $\otimes$-functor from $\text{Rep}(G)$ to the category of $\mathbb{Q}$-MHS keeping the underlying vector spaces with the weight filtrations.

We define the period domain $D$ associated to $G$ and $h_0$ as the set of all isomorphism classes of $G$-MHS whose associated homomorphism $S_{C/R} \to (G/G_u)_R$ is $(G/G_u)(R)$-conjugate to $h_0$.

1.1.3. Let $Y$ be the set of all isomorphism classes of exact $\otimes$-functors from $\text{Rep}(G)$ to the category of triples $(V, W, F)$, where $V$ is a finite-dimensional $\mathbb{Q}$-vector space, $W$ is an increasing filtration on $V$ (called the weight filtration), and $F$ is a decreasing filtration on $V_C$, preserving $V$ and $W$.

Then $G(C)$ acts on $Y$ by changing the Hodge filtration $F$. Let $\tilde{D} := G(C)D \subset Y$. Then $\tilde{D}$ is a $G(C)$-orbit in $Y$, $D$ is a $G(R)/G_u(C)$-orbit in $Y$, and $D$ is open in $\tilde{D}$. $\tilde{D}$ has a structure of a complex analytic manifold as a $G(C)$-homogeneous space, and $D$ is an open submanifold of $\tilde{D}$.

1.2  Polarizability

For a linear algebraic group $G$, let $G'$ be the commutator algebraic subgroup.

1.2.1. Let $h_0 : S_{C/R} \to (G/G_u)_R$ be as in [1.1.2]. We say that it is $R$-polarizable if \{a \in (G/G_u)'(R) \mid Ca = aC\} is a maximal compact subgroup of $(G/G_u)'(R)$, where $C$ is the image of $i \in C^\times = S_{C/R}(R)$ by $h_0$ in $(G/G_u)(R)$.

1.2.2. A relationship with the usual $R$-polarizability is as follows ([3] 2.11). Let $h_0$ be as in [1.1.2]. Let $H$ be a $G$-MHS such that the associated $S_{C/R} \to (G/G_u)_R$ is $R$-polarizable. Let $V \in \text{Rep}(G)$. Then for each $w \in \mathbb{Z}$, there is an $R$-bilinear form on $\text{gr}_w^W(V)_R$ which is stable under $(G/G_u)'$ and which polarizes $\text{gr}_w^WH(V)$.

1.2.3. We will often consider a subgroup $\Gamma$ of $G(\mathbb{Q})$ satisfying the following condition.

There is a faithful $V \in \text{Rep}(G)$ and a $\mathbb{Z}$-lattice $L$ in $V$ such that $L$ is stable under the action of $\Gamma$. 

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1.2.4. Assume that $h_0 : \mathbb{S}/\mathbb{R} \to (G/G_u)\mathbb{R}$ is $\mathbb{R}$-polarizable. Let $\Gamma$ be a subgroup of $G(\mathbb{Q})$ satisfying the condition in 1.2.3. Then the quotient space $\Gamma \setminus D$ is Hausdorff.

2 Space of nilpotent orbits $D_\Sigma$

We define the extended period domain $D_\Sigma \supset D$ as the space of nilpotent orbits, and state the main results. We fix $G$ and $h_0$ as in 1.1.2. Assume that $h_0$ is $\mathbb{R}$-polarizable.

2.1 Definition of $D_\Sigma$

2.1.1. A nilpotent cone is a subset $\sigma$ of $\text{Lie}(G)\mathbb{R}$ satisfying the following (i)–(iii).

(i) $\sigma = \mathbb{R}_{\geq 0}N_1 + \cdots + \mathbb{R}_{\geq 0}N_n$ for some $N_1, \ldots, N_n \in \text{Lie}(G)\mathbb{R}$.

(ii) For any $V \in \text{Rep}(G)$, the image of $\sigma$ under the induced map $\text{Lie}(G)\mathbb{R} \to \text{End}_\mathbb{R}(V)$ consists of nilpotent operators.

(iii) $[N, N'] = 0$ for any $N, N' \in \sigma$.

2.1.2. Let $F \in \check{D}$ and let $\sigma$ be a nilpotent cone. We say that the pair $(\sigma, F)$ generates a nilpotent orbit if the following (i)–(iii) are satisfied.

(i) There is a faithful $V \in \text{Rep}(G)$ such that the action of $\sigma$ on $V_\mathbb{R}$ is admissible with respect to $W$.

(ii) $N^pF^p \subset F^{p-1}$ for any $N \in \sigma$ and $p \in \mathbb{Z}$.

(iii) Let $N_1, \ldots, N_n$ be as in (i) in 2.1.1. Then $\exp(\sum_{j=1}^n z_j N_j)F \in D$ if $z_j \in \mathbb{C}$ and $\text{Im}(z_j) \gg 0$ (1 $\leq j \leq n$).

A nilpotent orbit is a pair $(\sigma, Z)$ of a nilpotent cone and an $\exp(\sigma_\mathbb{C})$-orbit in $\check{D}$ satisfying that for any $F \in Z$, $(\sigma, F)$ generates a nilpotent orbit. Here $\sigma_\mathbb{C}$ denotes the $\mathbb{C}$-linear span of $\sigma$ in $\text{Lie}(G)_\mathbb{C}$.

2.1.3. A weak fan $\Sigma$ in $\text{Lie}(G)$ is a nonempty set of sharp rational nilpotent cones satisfying the conditions that it is closed under taking faces and that any $\sigma, \sigma' \in \Sigma$ coincide if they have a common interior point and if there is an $F \in \check{D}$ such that both $(\sigma, F)$ and $(\sigma', F)$ generate nilpotent orbits.

Let $D_\Sigma$ be the set of all nilpotent orbits $(\sigma, Z)$ such that $\sigma \in \Sigma$. Then $D$ is naturally embedded in $D_\Sigma$.

Let $\Gamma$ be a subgroup of $G(\mathbb{Q})$ satisfying the condition in 1.2.3. We say that $\Sigma$ and $\Gamma$ are strongly compatible if $\Sigma$ is stable under the adjoint action of $\Gamma$ and if any $\sigma \in \Sigma$ is generated by elements whose exp in $G(\mathbb{R})$ belong to $\Gamma$. If this is the case, $\Gamma$ naturally acts on $D_\Sigma$.

2.2 Log mixed Hodge structures

2.2.1. Let $S$ be an object of the category $\mathcal{B}(\log)$ (8 III 1.1). We denote by $\text{LMH}(S)$ the category of log $\mathbb{Q}$-mixed Hodge structures over $S$ (8 III 1.3).

A $G$-log mixed Hodge structure ($G$-LMH, for short) over $S$ is an exact $\otimes$-functor from $\text{Rep}(G)$ to $\text{LMH}(S)$. 
Let $\Gamma$ be a subgroup of $G(Q)$ satisfying the condition in 1.2.3. A $G$-LMH over $S$ with a $\Gamma$-level structure is a $G$-LMH $H$ over $S$ endowed with a global section of the quotient sheaf $\Gamma \backslash I$, where $I$ is the following sheaf on $S^{\log}$. For an open set $U$ of $S^{\log}$, $I(U)$ is the set of all isomorphisms $H_S|_U \cong \text{id}$ of $\otimes$-functors from $\text{Rep}(G)$ to the category of local systems of $Q$-modules over $U$.

2.2.2. Let $(G, h_0)$ be as in 1.1.2 let $\Gamma$ be a subgroup of $G(Q)$ satisfying the condition in 1.2.3 and let $\Sigma$ be a weak fan in $\text{Lie}(G)$ which is strongly compatible with $\Gamma$. A $G$-LMH $H$ over $S$ with a $\Gamma$-level structure $\lambda$ is said to be of $\Sigma$-type if the following (i) and (ii) are satisfied for any $s \in S$ and any $t \in s^{\log}$. Take a $\otimes$-isomorphism $\tilde{\lambda}_t : H_{Q,t} \cong \text{id}$ which belongs to $\lambda_t$.

(i) There is a $\sigma \in \Sigma$ such that the logarithm of the action of $\text{Hom}((M_S/O_S^s)_s, N) \subset \pi_1(s^{\log})$ on $H_{Q,t}$ is contained, via $\tilde{\lambda}_t$, in $\sigma \subset \text{Lie}(G)_R$.

(ii) Let $\sigma \in \Sigma$ be the smallest cone satisfying (i). Let $a : O^{\log}_{S,t} \rightarrow C$ be a ring homomorphism which induces the evaluation $O_{S,s} \rightarrow C$ at $s$ and consider the element $F : V \mapsto \tilde{\lambda}_t a(H(V))$ of $Y (1.1.3)$. Then this element belongs to $D$ and $(\sigma, F)$ generates a nilpotent orbit $\Sigma$.

If $(H, \lambda)$ is of $\Sigma$-type, we have a map $S \rightarrow \Gamma \backslash D_{\Sigma}$, called the period map associated to $(H, \lambda)$, which sends $s \in S$ to the class of the nilpotent orbit $(\sigma, Z) \in D_\Sigma$ which is obtained in (ii) in 2.2.2.

2.2.3. Let $(G, h_0, \Gamma, \Sigma)$ be as in 2.2.2. We endow $\Gamma \backslash D_{\Sigma}$ with a topology, a sheaf of rings $O$ over $C$ and a log structure $M$ defined as follows. The topology is the strongest topology for which the period map $S \rightarrow \Gamma \backslash D_{\Sigma}$ is continuous for any $(S, H, \lambda)$, where $S$ is an object of $B(\log)$, $H$ is a $G$-LMH on $S$, and $\lambda$ is a $\Gamma$-level structure which is of $\Sigma$-type. For an open set $U$ of $\Gamma \backslash D_{\Sigma}$, $O(U)$ (resp. $M(U)$) is the set of all $C$-valued functions $f$ on $U$ such that for any $(S, H, \lambda)$ as above with the period map $\phi : S \rightarrow \Gamma \backslash D_{\Sigma}$, the pullback of $f$ on $U' := \phi^{-1}(U)$ belongs to the image of $O_{U'}$ (resp. $M_{U'}$) in the sheaf of $C$-valued functions on $U'$.

These structures of $\Gamma \backslash D_{\Sigma}$ are defined also by defining spaces $E_\sigma$ ($\sigma \in \Sigma$) in a similar way as [8] III. We get the same structures when we use only $S = E_\sigma$ for $\sigma \in \Sigma$ and the universal objects $(H, \lambda)$ over $E_\sigma$ in the above definitions of the structures.

2.3 Main results

Let $(G, h_0, \Gamma, \Sigma)$ be as in 2.2.2. Assume that $h_0$ is $R$-polarizable.

Theorem 2.3.1. (1) $\Gamma \backslash D_{\Sigma}$ is Hausdorff.

(2) When $\Gamma$ is neat, $\Gamma \backslash D_{\Sigma}$ is a log manifold ([8] III 1.1.5).

Here we say that $\Gamma$ is neat if there is a faithful $V \in \text{Rep}(G)$ such that for any $\gamma \in \Gamma$, the subgroup of $C^\times$ generated by all eigenvalues of $\gamma : V_C \rightarrow V_C$ is torsion-free.

2.3.2. The outline of the proof is as follows. As in [8], we can define various spaces $D_{SL(2)}$, $D_{BS}$, $E_\sigma$ etc., and have the theory of CKS map. Then, as in [8], by using the CKS map, good properties of $\Gamma \backslash D_{\Sigma}$ are deduced from those of the space of $SL(2)$-orbits $D_{SL(2)}$, which reduce to the $R$-polarizable version of [8]. We remark that what were shown in [8] by using $Q$-polarizations still hold under $R$-polarizations (1.2.2).
Theorem 2.3.3. When \( \Gamma \) is neat, \( \Gamma \backslash D_\Sigma \) represents the functor \( S \mapsto \{ \text{isomorphism class of } G\text{-LMH over } S \text{ with a } \Gamma\text{-level structure of } \Sigma\text{-type} \} \).

The proof of 2.3.3 is similar to the proof of [8] III, 2.6.6.

3 Examples

We discuss four examples of \( D \) to which our theory can be applied so that we can give \( D_\Sigma \) for these \( D \).

3.1 Usual period domains

We explain that the classical Griffiths domains [5] and their mixed Hodge generalization in [11] are essentially regarded as special cases of the period domains of this paper. In this case, our partial compactifications essentially coincide with those in [8] III.

Let \( \Lambda = (H_0, W, (\langle , \rangle_w, (h^{p,q})_{p,q}) \) be as usual as in [8] III. Let \( G \) be the subgroup of \( \text{Aut}(H_0, W) \) consisting of elements which induce similitudes for \( (\langle , \rangle_w \) for each \( w \). That is, \( G := \{ g \in \text{Aut}(H_0, W) \mid \text{ for any } w, \text{ there is a } t_w \in G_m \text{ such that } \langle gx, gy \rangle_w = t_w \langle x, y \rangle_w \text{ for any } x, y \in \text{gr}^W_w \} \). Let \( G_1 := \text{Aut}(H_0, W, (\langle , \rangle_w) \subset G \).

Let \( D(\Lambda) \) be the period domain of [11]. Then \( D(\Lambda) \) is identified with an open and closed part of \( D \) in this paper as follows.

Assume that \( D(\Lambda) \) is not empty and fix an \( r \in D(\Lambda) \). Then the Hodge decomposition of \( \text{gr}^W_r \) induces \( h_0 : S_{C/R} \to (G/G_u)_{R} \). (We have \( \langle [z]x, [z]y \rangle_w = |z|^{2w} \langle x, y \rangle_w \) for \( z \in C^\times \) (see 1.1.1 for \( [z] \)).) Consider the associated period domain \( D \) (11.2). Then \( D \) is a finite disjoint union of \( G_1(R)G_u(C)\)-orbits which are open and closed in \( D \). Let \( \mathcal{D} \) be the \( G_1(R)G_u(C)\)-orbit in \( D \) consisting of points whose associated homomorphisms \( S_{C/R} \to (G/G_u)_{R} \) are \( (G_1/G_u)(R)\)-conjugate to \( h_0 \). Then the map \( H \mapsto H(H_0, Q) \) gives a \( G_1(R)G_u(C)\)-equivariant isomorphism \( \mathcal{D} \cong D(\Lambda) \).

3.2 Mixed Mumford–Tate domains

Let \( H \) be a MHS whose \( \text{gr}^W \) are \( R \)-polarizable.

The Mumford–Tate group \( G \) of \( H \) is the Tannaka group (cf. [10]) of the Tannaka category generated by \( H \) (cf. [11]). Explicitly, it is the smallest \( Q \)-subgroup \( G \) of \( \text{Aut}(H_Q) \) such that \( G_R \) contains the image of the homomorphism \( h : S_{C/R} \to \text{Aut}(H_R) \) and such that \( \text{Lie}(G)_R \) contains \( \delta \). Here \( h \) and \( \delta \) are determined by the canonical splitting of \( H \) ([8] II 1.2). In the case where \( H \) is pure, \( G \) is the smallest \( Q \)-subgroup of \( \text{Aut}(H_Q) \) such that \( G_R \) contains the image of \( S_{C/R} \to \text{Aut}(H_R) \).

The Mumford–Tate domain associated to \( H \) is defined as the period domain \( D \) associated to \( G \) and \( h_0 : S_{C/R} \to (G/G_u)_{R} \) which is defined by \( \text{gr}^W H \).

In the pure case, our \( \Gamma \backslash D_\Sigma \) is essentially the same as the one by Kerr–Pearlstein ([9]).
3.3 Mixed Shimura varieties

See [10] for the generality of mixed Shimura varieties. This is the case where the universal object satisfies Griffiths transversality. $\text{gr}^W_{w} \text{Lie}(G)$ should be 0 unless $w = 0, -1, -2$. The $(p, q)$-Hodge component of $\text{gr}^W_{w} \text{Lie}(G)$ for $w = 0$ (resp. $w = -1$, resp. $w = -2$) should be 0 unless $(p, q)$ is $(1, -1)$, $(0, 0)$, and $(-1, 1)$ (resp. $(0, 1)$ and $(-1, 0)$, resp. $(-1, -1)$). (If this condition is satisfied by one point of $D$, it is satisfied by all points of $D$.)

In the case of PEL (polarizations, endomorphisms, and level structures) type, toroidal compactifications of universal abelian varieties are expressed as $\Gamma \setminus D_\Sigma$.

3.4 Higher Albanese manifolds

The higher Albanese manifold (see [6]) can be explained by using $D$ of this paper so that we can construct a toroidal partial compactification of it by the method of this paper.

3.4.1. Let $X$ be a connected smooth algebraic variety over $\mathbb{C}$. Fix $b \in X$. Let $\Gamma$ be a quotient group of $\pi_1(X, b)$ and assume that $\Gamma$ is a torsion-free nilpotent group.

Let $\mathcal{G}$ be the unipotent algebraic group over $\mathbb{Q}$ whose Lie algebra is defined as follows. Let $I$ be the augmentation ideal $\text{Ker}(\mathbb{Q}[\Gamma] \to \mathbb{Q})$. Then $\text{Lie}(\mathcal{G})$ is the $\mathbb{Q}$-subspace of $\varprojlim \mathbb{Q}[\Gamma]/I^n$ generated by all $\log(\gamma)$ ($\gamma \in \Gamma$). The Lie product of $\text{Lie}(\mathcal{G})$ is defined by $[x, y] = xy - yx$. We have $\Gamma \subseteq \mathcal{G}(\mathbb{Q})$.

Then $\text{Lie}(\mathcal{G})$ has a natural MHS of weights $\leq -1$ such that $\text{gr}^W_{w} \text{Lie}(\mathcal{G})$ are polarizable and such that the Lie product $\text{Lie}(\mathcal{G}) \otimes \text{Lie}(\mathcal{G}) \to \text{Lie}(\mathcal{G}) ; \ x \otimes y \mapsto [x, y]$ is a homomorphism of MHS.

3.4.2. Let $\mathcal{C}_{X, \Gamma}$ be the category of variations of $\mathbb{Q}$-MHS $\mathcal{H}$ on $X$ satisfying the following conditions.

(i) For any $w \in \mathbb{Z}$, $\text{gr}^W_{w} \mathcal{H}$ is a constant polarizable Hodge structure.

(ii) $\mathcal{H}$ is good at infinity in the sense of [6] (1.5).

(iii) The monodromy action of $\pi_1(X, b)$ on $\mathcal{H}_{\mathbb{Q}, b}$ (which is unipotent under (i)) factors through $\Gamma$.

For an object $\mathcal{H}$ of $\mathcal{C}_{X, \Gamma}$, we have the fiber $\mathcal{H}(b)$ at $b$ which is a $\mathbb{Q}$-MHS. The $\mathbb{Q}$-vector space $\mathcal{H}(b)_{\mathbb{Q}}$ has a unipotent linear action of $\Gamma$ and hence an action of the Lie algebra $\text{Lie}(\mathcal{G})$.

The theorem of Hain–Zucker says that the functor $\mathcal{H} \mapsto \mathcal{H}(b)$ gives an equivalence of categories

$$
\mathcal{C}_{X, \Gamma} \xrightarrow{\sim} \mathcal{C}'_{X, \Gamma},
$$

where $\mathcal{C}'_{X, \Gamma}$ is the category of $\mathbb{Q}$-MHS $H$ with polarizable $\text{gr}^W$ endowed with an action of the Lie algebra $\text{Lie}(\mathcal{G})$ on $H_{\mathbb{Q}}$ such that $\text{Lie}(\mathcal{G}) \otimes H \to H$ is a homomorphism of MHS.

3.4.3. The higher Albanese manifold $A_{X, \Gamma}$ of $X$ for $\Gamma$ is as follows. Let $\mathcal{F}^0 \mathcal{G}(\mathbb{C})$ be the algebraic subgroup of $\mathcal{G}(\mathbb{C})$ over $\mathbb{C}$ corresponding to the Lie subalgebra $\mathcal{F}^0 \text{Lie}(\mathcal{G})_{\mathbb{C}}$ of $\text{Lie}(\mathcal{G})_{\mathbb{C}}$. Define

$$
A_{X, \Gamma} := \Gamma \setminus \mathcal{G}(\mathbb{C})/\mathcal{F}^0 \mathcal{G}(\mathbb{C}).
$$

In the case where $\Gamma$ is $H_1(X, \mathbb{Z})/(\text{torsion})$ regarded as a quotient group of $\pi_1(X, b)$, $A_{X, \Gamma}$ coincides with the Albanese variety $\Gamma \setminus H_1(X, \mathbb{C})/\mathcal{F}^0 H_1(X, \mathbb{C})$ of $X$. 


We will give an understanding of \( A_{X,G} \) by using \( D \) of this paper \(^{3.4.6}\).
We will describe the functor represented by \( A_{X,G} \) (Theorem \(^{3.4.9}\)(1)).

3.4.4. Let \( Q \) be the Mumford–Tate group \(^{3.2}\) of the MHS \( \text{Lie}(\mathcal{G}) \) \(^{3.4.1}\). The action of \( Q \) on \( \text{Lie}(\mathcal{G}) \) induces an action of \( Q \) on \( \mathcal{G} \). By using this action, define the semidirect product \( G \) of \( Q \) and \( \mathcal{G} \) with an exact sequence \( 1 \to \mathcal{G} \to G \to Q \to 1 \). We have \( G \subset G_u \). We have \( h_0 : S_{C/R} \to (Q/Q_u)_{R} = (G/G_u)_{R} \) given by the Hodge decomposition of \( \text{gr}^{W} \text{Lie}(\mathcal{G}) \).

Then \((G, \Gamma)\) satisfies the condition in \(^{1.2.3}\) and \( \Gamma \) is a neat subgroup of \( G(Q) \).

Let \( D_{G} \) (resp. \( D_Q \)) be the period domain \( D \) for \( G \) (resp. \( Q \)) and \( h_0 \). We have a canonical map \( \Gamma \mid D_{G} \to D_{Q} \) induced by the canonical homomorphism \( G \to Q \).

3.4.5. The equivalence in \(^{3.4.2}\) gives an inclusion functor

\[
\text{Rep}(G) \hookrightarrow \mathcal{C}_{X,G}.
\]

3.4.6. Let \( b_{G} \in D_{G} \) be the element induced by \( C_{X,G} \to \text{MHS} ; H \mapsto \mathcal{H}(b) \) and \(^{3.4.5}\). Let \( b_{Q} \in D_{Q} \) be the image of \( b_{G} \).

The map \( G_u(C) \to D_{G} ; g \mapsto g b_{G} \) induces an isomorphism from \( A_{X,G} = \Gamma \mid D_{G}/F^0\mathcal{G}(C) \) to the inverse image of \( b_{Q} \) in \( \Gamma \mid D_{G} \to D_{Q} \).

3.4.7. Let \( \Sigma \) be a weak fan in \( \text{Lie}(G) \) such that \( \sigma \in \text{Lie}(\mathcal{G})_{R} \) for any \( \sigma \in \Sigma \). We have a canonical morphism \( \Gamma \mid D_{G,S} \to D_{Q} \) induced by the homomorphism \( G \to Q \).

Define the toroidal partial compactification \( A_{X,G,S} \) of \( A_{X,G} \) as the subspace of \( \Gamma \mid D_{G,S} \) defined to be the inverse image of \( b_{Q} \). We can endow \( A_{X,G,S} \) with a structure of a log manifold such that for any object \( S \) of \( \mathcal{B}(\text{log}) \), \( \text{Mor}(S, A_{X,G,S}) \) coincides with the set of all morphisms \( S \to \Gamma \mid D_{G,S} \) whose images in \( D_{Q} \) are \( b_{Q} \).

3.4.8. Define contravariant functors

\[
\mathcal{F}_{\Gamma}, \mathcal{F}_{\Gamma,S} : \mathcal{B}(\text{log}) \to (\text{Set})
\]

as follows.

\( \mathcal{F}_{\Gamma,S}(S) \) is the set of isomorphism classes of pairs \((H, \lambda)\), where \( H \) is an exact \( \otimes \)-functor \( C_{X,G} \to \text{LMH}(S) \) and \( \lambda \) is a global section of the sheaf \( \Gamma \mid \mathcal{I} \) on \( S^{\text{log}} \), where \( \mathcal{I} \) is the sheaf of functorial \( \otimes \)-isomorphisms \( H(\mathcal{H})_{Q} \cong \mathcal{H}(b)_{Q} \) of \( Q \)-local systems, satisfying the following conditions (i) and (ii).

(i) For any \( Q \)-MHS \( h \), we have a functorial \( \otimes \)-isomorphism \( H(h_{X}) \cong h \) such that the induced isomorphism of local systems \( H(h_{X})_{Q} \cong h_{Q} = h_{X}(b)_{Q} \) belongs to \( \lambda \). Here \( h_{X} \) denotes the constant variation of \( Q \)-MHS over \( X \) associated to \( h \).

(ii) The following (ii-1) and (ii-2) are satisfied for any \( s \in S \) and any \( t \in s^{\text{log}} \). Let \( \lambda_{t} : H(\mathcal{H})_{Q,t} \cong \mathcal{H}(b)_{Q} \) be a functorial \( \otimes \)-isomorphism which belongs to \( \lambda_{t} \).

(ii-1) There is a \( \sigma \in \Sigma \) such that the logarithm of the action of \( \text{Hom}((M_{s}/\mathcal{O}_{s})_{s}, \mathcal{N}) \subset \pi_{1}(s^{\text{log}}) \) on \( H_{Q,t} \) is contained, via \( \lambda_{t} \), in \( \sigma \subset \text{Lie}(\mathcal{G})_{R} \).

(ii-2) Let \( \sigma \in \Sigma \) be the smallest cone which satisfies (ii-1) and let \( a : \mathcal{O}_{s,t}^{\log} \to \mathcal{C} \) be a ring homomorphism which induces the evaluation \( \mathcal{O}_{s,t} \to \mathcal{C} \) at \( s \). Then for each \( \mathcal{H} \in C_{X,G} \), \((\sigma, \lambda_{t}(a(H(\mathcal{H})))) \) generates a nilpotent orbit in the sense of \(^{8}\) III 2.2.2.
We define a subfunctor $\mathcal{F}_{\Gamma} \subset \mathcal{F}_{\Gamma,\Sigma}$ by replacing $\text{LMH}(S)$ in the above definition of $\mathcal{F}_{\Gamma,\Sigma}$ by the full subcategory $\text{MHS}(S) \subset \text{LMH}(S)$ consisting of objects with no degeneration. That is, $\text{MHS}(S)$ is the category of analytic families of mixed Hodge structures parametrized by $S$. (In this case, the above condition (ii) is empty. Objects of $\text{MHS}(S)$ need not satisfy Griffiths transversality.)

**Theorem 3.4.9.** (1) The functor $\mathcal{F}_{\Gamma}$ is represented by $\mathcal{A}_{X,\Gamma}$.

(2) The functor $\mathcal{F}_{\Gamma,\Sigma}$ is represented by $\mathcal{A}_{X,\Gamma,\Sigma}$.

This theorem is proved in the following way. By 3.4.5, 3.4.6, and Theorem 2.3.3, we have a map from $\mathcal{F}_{\Gamma}(S)$ (resp. $\mathcal{F}_{\Gamma,\Sigma}(S)$) to the fiber $\text{Mor}(S, \mathcal{A}_{X,\Gamma})$ (resp. $\text{Mor}(S, \mathcal{A}_{X,\Gamma,\Sigma})$) of $\text{Mor}(S, \Gamma \setminus D_G) \to \text{Mor}(S, D_Q)$ (resp. $\text{Mor}(S, \Gamma \setminus D_{G,\Sigma}) \to \text{Mor}(S, D_Q)$) over $b_Q$. We can show that this map is a bijection.

**3.4.10.** The higher Albanese map $X \to \mathcal{A}_{X,\Gamma}$ corresponds in 3.4.9 (1) to the evident functor $\mathcal{C}_{X,\Gamma} \to \text{MHS}(X)$. When this extends to a morphism $\overline{X} \to \mathcal{A}_{X,\Gamma,\Sigma}$ for some complex analytic manifold $\overline{X}$ which contains $X$ as a dense open subset such that the complement $\overline{X} \setminus X$ is a divisor with normal crossings, this extended higher Albanese map corresponds in 3.4.9 (2) to the inclusion functor $\mathcal{C}_{X,\Gamma} \to \text{LMH}(\overline{X})$.

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