A PROFINITE GROUP INVARIANT FOR HYPERBOLIC TORAL AUTOMORPHISMS

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Abstract. For a hyperbolic toral automorphism, we construct a profinite completion of an isomorphic copy of the homoclinic group of its right action using isomorphic copies of the periodic data of its left action. The resulting profinite group has a natural module structure over a ring determined by the right action of the hyperbolic toral automorphism. This module is an invariant of conjugacy that provides means in which to characterize when two similar hyperbolic toral automorphisms are conjugate or not. In particular, this shows for two similar hyperbolic toral automorphisms with module isomorphic left action periodic data, that the homoclinic groups of their right actions play the key role in determining whether or not they are conjugate. This gives a complete set of dynamically significant invariants for the topological classification of hyperbolic toral automorphisms.

1. Introduction. The topological classification of hyperbolic, irreducible, toral automorphisms brings together the subjects of dynamical systems, algebra, and algebraic number theory. It is well-known [1] that two $\mathbb{T}^n$ automorphisms $T_A$ and $T_B$ induced by $A, B \in GL(n, \mathbb{Z})$ are topologically conjugate if and only if $A$ and $B$ are conjugate in the group $GL(n, \mathbb{Z})$, i.e., there is $C \in GL(n, \mathbb{Z})$ such that $AC = CB$. Furthermore, by a well-known result of Latimer, MacDuffee and Taussky (see [14, 17, 18]), this happens if and only if $A$ and $B$ are associated to the same ideal class in the number ring determined by their common characteristic polynomial. In both these settings, algorithms have been developed that, in principle, determine when two automorphisms are conjugate [4, 6]. For instance, the case of $n = 2$ uses classical results about continued fractions [2, 11], while the case $n \geq 3$ uses recently developed geometric continued fractions [9]. Yet there is not yet a clear understanding of the topological classification as would be given by a complete set of dynamically significant invariants.

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The periodic data for a hyperbolic, irreducible, toral automorphism is insufficient to characterize its conjugacy class, and yet provides a dynamically significant invariant of conjugacy (see [13]). For \( k \in \mathbb{N} \), let \( \text{Per}_k(T_A) \) denote the finite group of \( k \)-periodic points for the left action of \( T_A \) on \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \), where elements of \( \mathbb{T}^n \) are represented by columns vectors. It is possible for the hyperbolic, irreducible, toral automorphisms \( T_A \) and \( T_B \) to have conjugate actions on the \( \text{Per}_k \) level, for every \( k \), without \( A \) and \( B \) being conjugate while being similar. (By similar we mean there is a \( C \in \text{GL}(n, \mathbb{Q}) \) such that \( AC = CB \).) More generally, suppose that \( A \) and \( B \) have the same irreducible characteristic polynomial \( p(x) \), let \( R = \mathbb{Z}[x]/(p(x)) \), and set \( \text{Per}_g(T_A) = \ker(g(A)) \subset \mathbb{T}^n \), where \( g \in \mathbb{Z}[x] \) with \( g(A) \) invertible. It is still possible that \( T_A \) and \( T_B \) are not conjugate on \( \mathbb{T}^n \), and yet are strongly Bowen-Franks equivalent (also written strongly BF-equivalent), i.e., associated to every \( g \in \mathbb{Z}[x] \) with \( g(A) \) invertible, is an \( R \)-module isomorphism

\[
\phi_g : \text{Per}_g(T_A) \rightarrow \text{Per}_g(T_B)
\]

that conjugates \( T_A \) and \( T_B \) on the level of periodic points determined by \( g \). It is then natural to consider what additional dynamically significant invariants are needed to determine when two strongly BF-equivalent similar hyperbolic, irreducible, toral automorphisms are conjugate or not, i.e., what else is needed with the level conjugacies to imply the existence of a global conjugacy.

This paper explores this approach, from a dual point of view. Instead of the direct limit of the \( R \)-modules \( \text{Per}_k(T_A) \), \( k \in \mathbb{N} \), the main object is a profinite limit \( G_A \) of the Pontryagin dual \( R \)-modules, or Bowen-Franks groups,

\[
\text{BF}_k(A) = \mathbb{Z}^n / \mathbb{Z}^n(A^k - I), \quad k \in \mathbb{N},
\]

given by the right action of \( A^k - I \) on \( \mathbb{Z}^n \), where the elements of \( \mathbb{Z}^n \) are represented by row vectors. (Throughout this paper, we use the basic definitions, notations, and properties of profinite groups as found in [15] and their appropriate adaptations to the \( R \)-module setting without further reference.) In Section 2, we present the details of the construction of \( G_A \) associated to a hyperbolic \( A \in \text{GL}(n, \mathbb{Z}) \). This \( G_A \) is an \( R \)-module where \( A \) induces a topological automorphism \( \Gamma_A \) on \( G_A \). The profinite group \( G_A \) is a profinite completion of the right \( \mathbb{Z}[A] \)-module \( \mathbb{Z}^n \), where the right \( \mathbb{Z}[A] \)-module \( \mathbb{Z}^n \) is \( R \)-module isomorphic to the homoclinic group \( H_A \) of the right action of hyperbolic toral automorphism \( T_A \). In a sense, \( G_A \) is akin to the profinite completion of \( \mathbb{Z} \). As an invariant of conjugacy, the \( R \)-module \( H_A \) naturally embeds into \( G_A \) and plays the key role in characterizing conjugacy.

Strong BF-equivalence is obviously a necessary condition for conjugacy of similar hyperbolic toral automorphisms. In Section 3, we describe conditions by which two similar hyperbolic toral automorphisms are strongly BF-equivalent in the dual sense, i.e., \( \mathbb{Z}^n / \mathbb{Z}^n g(A) \) is \( R \)-module isomorphic to \( \mathbb{Z}^n / \mathbb{Z}^n g(B) \) for every \( g \in \mathbb{Z}[x] \) with \( g(A) \) invertible. Then in Section 4 we show that strong BF-equivalence of similar hyperbolic \( A, B \in \text{GL}(n, \mathbb{Z}) \) implies the existence of a topological \( R \)-module isomorphism \( \Psi : G_A \rightarrow G_B \). In Section 5, we detail the exact manner by which the embedded copies of \( H_A \) and \( H_B \) in \( G_A \) and \( G_B \) respectively, characterize when \( A \) and \( B \) are conjugate or not.

Specifically, for strong BF-equivalent similar hyperbolic \( A \) and \( B \), it is how the image of the embedded copy of \( H_A \) under any topological \( R \)-module isomorphism \( \Psi : G_A \rightarrow G_B \) intersects the embedded copy of \( H_B \) in \( G_B \), that determines conjugacy or the lack thereof. This characterization of conjugacy applies when the characteristic polynomial of the similar \( A \) and \( B \) is irreducible or reducible. When the similar hyperbolic \( A \) and \( B \) have an irreducible characteristic polynomial...
polynomial, we show in Section 6 that the intersection of the image of the embedded copy of \( H_A \) in \( G_A \) under any topological \( R \)-module isomorphism \( \Psi : G_A \to G_B \) with the embedded copy of \( H_B \) in \( G_B \) is either trivial or has finite index in the embedded copy of \( H_B \) in \( G_B \).

The characterization of conjugacy for strongly BF-equivalent similar hyperbolic \( A, B \in \text{GL}(n, \mathbb{Z}) \) shows that the embedded copies of \( H_A \) in \( G_A \) and \( H_B \) in \( G_B \) are the additional dynamically significant invariants that are needed. It is known [12] that the homoclinic groups \( H_A \) and \( H_B \) are complete invariants of conjugacy when \( A \) and \( B \) are Pisot, i.e., have one real eigenvalue larger than 1 in modulus with all the remaining eigenvalues smaller than 1 in modulus. Our characterization of conjugacy extends the role of the homoclinic groups as classifying invariants from the Pisot case to the general case. This characterization of conjugacy also resolves the problem of the classification of quasiperiodic flows of Koch type according to the equivalence relation of projective conjugacy (see [3]), and gives a dynamical systems resolution to the ideal class problem in algebraic number theory.

From a computational point of view, our dynamical characterization of conjugacy for strongly BF-equivalent similar hyperbolic \( A, B \in \text{GL}(n, \mathbb{Z}) \) is not completely satisfactory. We would like to better understand how to computationally detect when our characterizing condition for conjugacy holds or is violated for the \( R \)-submodules \( H_A \) in \( G_A \) and \( H_B \) in \( G_B \). In this direction, one avenue is the nature of the topological dynamical system determined by the action of \( \Gamma_A \) on \( G_A \). Another avenue is the nature of the pairing of \( G_A \) with its Pontryagin dual, the direct limit of \( \text{Per}_k(T_A), k \in \mathbb{N} \). We are currently investigating these avenues for the desired detection that we hope to include in a subsequent paper.

2. A \( \mathbb{Z}[A] \)-module for hyperbolic \( A \). We use the Pontryagin dual of periodic data for the left action of a hyperbolic toral automorphism to construct a profinite group associated to it. Let \( A \in \text{GL}(n, \mathbb{Z}) \) be hyperbolic, and let \( T_A \) be the hyperbolic toral automorphism that the left action of \( A \) induces on \( \mathbb{T}^n \), i.e., \( T_A\pi = \pi A \) where \( \pi : \mathbb{R}^n \to \mathbb{T}^n \) is the canonical covering epimorphism, with \( \mathbb{R}^n \) and \( \mathbb{T}^n \) represented by column vectors. Hyperbolicity of \( A \) implies that \( \det(A^r - I) \neq 0 \) for all \( r \in \mathbb{Z} \). The right action of \( A \) on \( \mathbb{Z}^n \) is \( m \to mA \), where \( \mathbb{Z}^n \) here is represented by row vectors. This makes \( \mathbb{Z}^n \) a right \( \mathbb{Z}[A] \)-module. For each \( k \in \mathbb{N} \), we define the abelian groups

\[
N_{k,A} = \mathbb{Z}^n(A^k - I) \quad \text{and} \quad G_{k,A} = \mathbb{Z}^n/N_{k,A}.
\]

The finite abelian groups \( G_{k,A} \) are the Bowen-Franks groups which are isomorphic to the Pontryagin duals of \( \text{Per}_k(T_A) \), the group of periodic points of period \( k \) for the left action of \( T_A \) on \( \mathbb{T}^n \). Obviously, the collection

\[
\mathcal{N}_A = \{N_{k,A} : k \in \mathbb{N}\}
\]

consists of right \( \mathbb{Z}[A] \)-submodules of \( \mathbb{Z}^n \) of finite index. For \( l, k \in \mathbb{Z} \), we make \( \mathcal{N}_A \) a directed poset by the binary relation \( \preceq \) defined by \( N_{l,A} \preceq N_{k,A} \) if and only if \( N_{l,A} \supseteq N_{k,A} \). To simplify notation, we write \( l \preceq k \) in place of \( N_{l,A} \preceq N_{k,A} \). Clearly \( \preceq \) is a partial ordering, i.e., reflexive, transitive, and antisymmetric. The “directed” part of \( \preceq \) is the property that for all \( k_1, k_2 \in \mathbb{N} \) there exists \( k_3 \in \mathbb{Z} \) such that \( k_1 \leq k_3 \) and \( k_2 \leq k_3 \), which is a consequence of the following result.

**Lemma 2.1.** For a hyperbolic \( A \in \text{GL}(n, \mathbb{Z}) \), the collection \( \mathcal{N}_A \) is filtered from below, i.e., for \( N_{k_1,A} \) and \( N_{k_2,A} \in \mathcal{N}_A \) there exists \( N_{k_3,A} \in \mathcal{N}_A \) such that \( N_{k_3,A} \subset N_{k_1,A} \cap N_{k_2,A} \).
Proof. For \( k_3 = k_1 k_2 \), there is the factorization

\[
A^{k_3} - I = (A^{k_3-k_1} + A^{k_3-2k_1} + \cdots + A^{k_1} + I)(A^{k_1} - I).
\]

Since the term in the first set of braces on the right side of the factorization is an endomorphism of \( \mathbb{Z}^n \), it follows that \( N_{k_1,A} \geq N_{k_3,A} \). Similarly, \( N_{k_2,A} \geq N_{k_3,A} \), and so \( N_{k_3,A} \leq N_{k_1,A} \cap N_{k_2,A} \). Hence \( \mathcal{N}_A \) is filtered from below. \(\square\)

We associate to a hyperbolic \( A \in \text{GL}(n, \mathbb{Z}) \) a right \( \mathbb{Z}[A] \)-module \( G_A \) as follows. We denote the elements of \( G_{k,A} \) by \( [m]_{k,A} = m + N_{k,A} \) for \( m \in \mathbb{Z}^n \) and endow the finite \( G_{k,A} \) with the discrete topology. The \( \mathbb{Z}[A] \)-module structure on \( G_{k,A} \) is induced by the topological automorphism \( A_k \) of \( G_{k,A} \) defined by

\[
A_k([m]_{k,A}) = [mA]_{k,A}.
\]

For \( l \leq k \), there are the continuous canonical \( \mathbb{Z}[A] \)-module epimorphisms \( \varphi_{k,l,A} : G_{k,A} \to G_{l,A} \) defined by

\[
\varphi_{k,l,A}([m]_{k,A}) = [m]_{l,A}.
\]

The inverse limit of the surjective inverse system \( \{ G_{k,A}, \varphi_{k,l,A}, \mathbb{N} \} \) is the compact, Hausdorff, totally disconnected, right \( \mathbb{Z}[A] \)-module

\[
G_A = \lim_{\longrightarrow} k \in \mathbb{N} \frac{G_{k,A}}{G_{k,A} \leq \prod_{k \in \mathbb{N}} G_{k,A}}
\]

consisting of the tuples \( \{ [m]_{k,A} \}_{k \in \mathbb{N}} \) for \( m_k \in \mathbb{Z}^n \) satisfying

\[
[m]_{l,A} = \varphi_{k,l,A}([m]_{k,A}) = [m]_{l,A}
\]

when \( l \leq k \), together with the continuous canonical \( \mathbb{Z}[A] \)-module epimorphisms \( \varphi_{k,A} : G_A \to G_{k,A} \) defined by

\[
\varphi_{k,A}(\{ [m]_{k,A} \}_{k \in \mathbb{N}}) = [m]_{k,A}.
\]

The topology on \( G_A \) is the subspace topology coming from the product topology on \( \prod_{k \in \mathbb{N}} G_{k,A} \). The \( \mathbb{Z}[A] \)-module structure on \( G_A \) is induced by the topological automorphism \( \Gamma_A \) defined by

\[
\Gamma_A(\{ [m]_{k,A} \}_{k \in \mathbb{N}}) = \{ A_k[m]_{k,A} \}_{k \in \mathbb{N}}.
\]

The collection \( \{ \ker(\varphi_{k,A}) : k \in \mathbb{N} \} \) forms a fundamental system of open \( \mathbb{Z}[A] \)-submodules of 0 in \( G_A \).

For a hyperbolic \( A \), an isomorphic copy of the homoclinic group for the right action of \( T_A \) sits inside \( G_A \). With elements of \( \mathbb{T}^n \) represented by row vectors, the homoclinic group of the right action of \( T_A \) is the right \( \mathbb{Z}[A] \)-module

\[
H_A = \{ x \in \mathbb{T}^n : x(T_A)^k \to 0 \text{ as } |k| \to \infty \}.
\]

There is a right \( \mathbb{Z}[A] \)-module isomorphism \( \theta_A : \mathbb{Z}^n \to H_A \) (see [10]), and there is a canonical right \( \mathbb{Z}[A] \)-module homomorphism \( \iota_A : \mathbb{Z}^n \to G_A \) defined by

\[
\iota_A(m) = \{ [m]_{k,A} \}_{k \in \mathbb{N}}.
\]

Thus \( \iota_A(\theta_A^{-1}(H_A)) \subset G_A \). The homomorphism \( \iota_A \) is a monomorphism if and only if \( G_A \) is residually finite, i.e.,

\[
\bigcap_{k \in \mathbb{N}} N_{k,A} = \{ 0 \}.
\]

Fortunately, that \( \iota_A \) is a monomorphism, is the case in our situation.

**Lemma 2.2.** If \( A \in \text{GL}(n, \mathbb{Z}) \) is hyperbolic, then \( G_A \) is residually finite.
Proof. We show that the intersection of all the $N_{k,A}$ is $\{0\}$ by contradiction. Suppose there is $m \in \mathbb{Z}^n \setminus \{0\}$ such that

$$m \in \bigcap_{k \in \mathbb{N}} N_{k,A}.$$ 

Then for each $k \in \mathbb{N}$, there is $\xi_k \in \mathbb{Z}^n \setminus \{0\}$ such that $m = \xi_k(A^k - I)$. This means for each $k \in \mathbb{N}$ that $A^k$ sends the element $\xi_k$ of $\mathbb{Z}^n$ to the element $\xi_k A^k$ of $\mathbb{Z}^n$ that is a distance $\|m\|$, independent of $k$, away from $\xi_k$. (Here $\|\cdot\|$ is the usually Euclidean norm on $\mathbb{R}^n$.) Let $E^+(A)$ and $E^-(A)$ be the stable and unstable spaces of the right action of $A$ on $\mathbb{R}^n$. The only element of $\mathbb{Z}^n$ that belongs to $E^+(A) \cup E^-(A)$ is $0$ because no lattice point can iterate under $A$ or $A^{-1}$ to $0$. The hyperbolicity of $A$ thus implies that $\|\xi_k A^k\|$, for $\xi \in \mathbb{Z}^n \setminus \{0\}$, goes to $\infty$ with exponential speed as $k \to \infty$ (see Proposition 1.2.8 on p. 28 in [7]). This means for all sufficiently large $k$, that each $A^k$ sends $\xi_k$ to the element $\xi_k A^k$ whose distance from $\xi_k$ is much larger than $\|m\|$. This is a contradiction. □

Another way to view $G_A$ is as a profinite completion of $\mathbb{Z}^n$. The collection of subgroups $N_A$ of finite index of $\mathbb{Z}^n$, being filtered from below by Lemma 2.1, determines a topology on $\mathbb{Z}^n$ by considering $N_A$ as a fundamental system of neighborhoods of the identity 0 of $\mathbb{Z}^n$. The profinite completion of $\mathbb{Z}^n$ with respect to this topology is the finitely generated, uncountable, profinite group $G_A$, where $\iota_A(\mathbb{Z}^n)$ is a dense $\mathbb{Z}[A]$-submodule of $G_A$. By Lemma 2.2, we think of $G_A$ as the profinite completion of a $\mathbb{Z}[A]$-module isomorphic copy of homoclinic group $H_A$ of the right action of $T_A$, determined by the $\mathbb{Z}[A]$-module Pontryagin duals $G_{k,A}$ of the periodic groups $\text{Per}_k(T_A)$ for the left action of $T_A$. For a hyperbolic $A$, this profinite completion view of $G_A$ is really useful and sound only because $G_A$ is residually finite.

For another hyperbolic $B \in \text{GL}(n,\mathbb{Z})$ that is similar to $A$, there is a ring $R$ such that $G_{k,A}, G_{k,B}, G_A, G_B$, etc., are all $R$-modules. Similarity of $A$ and $B$ implies that $\mathbb{Z}[A]$ and $\mathbb{Z}[B]$ are isomorphic as rings. We denote by $R$ a ring that is isomorphic to $\mathbb{Z}[A]$ and $\mathbb{Z}[B]$, and understand now that the $R$-module structure on $G_A$ and $G_B$ is given by $\mathbb{Z}[A]$ and $\mathbb{Z}[B]$ respectively. When $A$ and $B$ have the same irreducible characteristic polynomial $p(x)$, we take

$$R = \mathbb{Z}[x]/(p(x)),$$

where $(p(x)) = p(x)\mathbb{Z}[x]$ is the principal ideal in $\mathbb{Z}[x]$ generated by $p(x)$. When the characteristic polynomial $p(x)$ of $A$ and $B$ is not irreducible, we take

$$R = (\mathbb{Z}[x]/(p(x)))/{\ker(\vartheta_D)}$$

where, for $D = A$ or $D = B$, the map $\vartheta_D : \mathbb{Z}[x]/(p(x)) \to \mathbb{Z}[D]$ is the ring endomorphism defined by $\vartheta_D(q(x) + (p(x))) = q(D)$; it does not matter whether $D = A$ or $D = B$ because the similarity of $A$ and $B$ implies that $\ker(\vartheta_A) = \ker(\vartheta_B)$.

3. Sufficient conditions for strong BF-equivalence. For similar hyperbolic $A, B \in \text{GL}(n,\mathbb{Z})$, the finite $R$-modules $G_{k,A}, G_{k,B}$ for $k \in \mathbb{N}$ are invariants of the conjugacy classes of $A$ and $B$ respectively in terms of their $R$-module structure. More generally, for $g \in \mathbb{Z}[x]$ with $g(A)$ (and hence $g(B)$) invertible, the finite $R$-modules $\mathbb{Z}^n/\mathbb{Z}^n g(A)$ and $\mathbb{Z}^n/\mathbb{Z}^n g(B)$ are invariants of the conjugacy classes of $A$ and $B$ in terms of their $R$-module structures [13]. The generalized Bowen-Franks groups are $\text{BF}_g(A) = \mathbb{Z}^n/\mathbb{Z}^n g(A)$ for $g \in \mathbb{Z}[x]$ with $g(A)$ invertible. We note that $G_{k,A} = \text{BF}_{g_k}(A)$ where $g_k(x) = x^k - 1$, $k \in \mathbb{N}$. For two $R$-modules $G_1$ and $G_2$, we
we write \( G_1 \cong_R G_2 \) when \( G_1 \) and \( G_2 \) are \( R \)-module isomorphic. Similar hyperbolic \( A, B \in \text{GL}(n, \mathbb{Z}) \) are said to be strongly BF-equivalent when

\[
\text{BF}_g(A) \cong_R \text{BF}_g(B) \text{ for all } g \in \mathbb{Z}[x] \text{ with } g(A) \text{ invertible.}
\]

When \( \text{BF}_g(A) \) and \( \text{BF}_g(B) \) are not \( R \)-module isomorphic for some \( g \in \mathbb{Z}[x] \) with \( g(A) \) invertible, then the similar \( A \) and \( B \) fail to be conjugate. So a necessary condition for similar hyperbolic \( A \) and \( B \) to be conjugate is that \( A \) and \( B \) are strongly BF-equivalent.

**Example 3.1.** The hyperbolic matrices

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 6 & -2 & 23 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 12 \\ 1 & 0 & -4 \\ 0 & 2 & 23 \end{bmatrix}
\]

have the same irreducible characteristic polynomial of \( p(x) = x^3 - 23x^2 + 7x - 1 \). Using the Smith Normal Form, we compute that \( \text{BF}_{x+1}(A) = \mathbb{Z}_4 \oplus \mathbb{Z}_8 \) while \( \text{BF}_{x+1}(B) = \mathbb{Z}_2 \oplus \mathbb{Z}_{16} \). Thus \( A \) and \( B \) are not strongly BF-equivalent, and hence are not conjugate.

The generalized Bowen-Franks groups are related to quotients of fractional ideals in algebraic number theory. Suppose that a hyperbolic \( A \in \text{GL}(n, \mathbb{Z}) \) has an irreducible characteristic polynomial \( p(x) \in \mathbb{Z}[x] \). The field of fractions of \( R = \mathbb{Z}[x]/(p(x)) \) is the algebraic number field \( K = \mathbb{Q}(x)/(p(x)) \). For a fixed root \( \beta \) of \( p(x) \), we make the identifications \( R = \mathbb{Z}[\beta] \) and \( K = \mathbb{Q}(\beta) \). Choose a row eigenvector \( v \) of \( A \) for \( \beta \), i.e., \( vA = \beta v \), such that the components of \( v \) form a \( \mathbb{Z} \)-basis for a fractional ideal \( I \) of \( \mathbb{Z}[\beta] \), i.e., a finitely generated \( \mathbb{Z}[\beta] \)-module contained in \( K \). Another fractional ideal \( J \) of \( \mathbb{Z}[\beta] \) is said to be equivalent to \( I \) if there is a nonzero \( z \in K \) such \( J = zI \). The Latimer-MacDuffee-Taussky Theorem asserts that there is a one-to-one correspondence between the conjugacy classes of matrices in \( \text{GL}(n, \mathbb{Z}) \) with characteristic polynomial \( p(x) \) and the equivalence classes of the fractional ideals of \( \mathbb{Z}[\beta] \). This one-to-one correspondence implies [13] that

\[
\text{BF}_g(A) \cong_R I/g(\beta)I,
\]

for every \( g \in \mathbb{Z}[x] \) with \( g(A) \) invertible.

A condition by which \( A, B \in \text{GL}(n, \mathbb{Z}) \) with a common irreducible characteristic polynomial \( p(x) \in \mathbb{Z}[x] \) are strongly BF-equivalent is based on \( \text{BF}_g(A) \cong_R I/g(\beta)I \) and [5]. The ring of coefficients of a fractional ideal \( V \) of \( \mathbb{Z}[\beta] \) is

\[
\mathcal{O}(V) = \{ z \in K : zV \subset V \};
\]

it is contained by the ring of integers \( \mathfrak{o}_K \) of \( K \), and it contains \( \mathbb{Z}[\beta] \). A nonzero fractional ideal \( V \) of \( \mathbb{Z}[\beta] \) is said to be invertible if the fractional ideal

\[
V^{-1} = \{ z \in K : zV \subset \mathcal{O}(V) \}
\]

of \( \mathbb{Z}[\beta] \) satisfies \( VV^{-1} = \mathcal{O}(V) \). Choose a row eigenvector \( w \) for \( B \) for \( \beta \), i.e., \( wB = \beta w \), and associate to \( B \) the fractional ideal \( J \) of \( \mathbb{Z}[\beta] \) generated by \( w \). The ring \( \mathcal{O}(J) \) is an invariant of the conjugacy class of \( B \) in that if \( A \) and \( B \) are conjugate, then \( \mathcal{O}(I) = \mathcal{O}(J) \). If \( A \) and \( B \) are strongly BF-equivalent, then \( \mathcal{O}(I) = \mathcal{O}(J) \) (see [13]). The fractional ideals \( I \) and \( J \) are said to be weakly equivalent if there are fractional ideals \( X \) and \( Y \) of \( \mathbb{Z}[\beta] \) such that \( IX = J \) and \( JY = I \); in this case, \( X = \{ z \in K : zI \subset J \} \), \( Y = \{ z \in K : zJ \subset I \} \), and \( XY = \mathcal{O}(I) = \mathcal{O}(J) \). The latter condition implies that \( X \) and \( Y \) are invertible and inverses of each other. Equivalent fractional ideals of \( \mathbb{Z}[\beta] \) are always weakly equivalent.
Theorem 3.2. For hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$ with the same irreducible characteristic polynomial $p(x)$, if the associated fractional ideals $I$ and $J$ of $\mathbb{Z}[\beta]$ are weakly equivalent, then $A$ and $B$ are strongly BF-equivalent.

Proof. By passing to equivalent fractional ideals, we may choose the associated fractional ideals $I$ and $J$ so that $I \subset J \subset \mathbb{Z}[\beta]$. By hypothesis, there are fractional ideals $X$ and $Y$ of $\mathbb{Z}[\beta]$ such that $IX = J$, $JY = I$, and $XY = D(I) = D(J) = D$. Set $\alpha = g(\beta) \in \mathbb{Z}[\beta]$ for $g \in \mathbb{Z}[x]$ with $g(A)$ invertible. Note that $\mathbb{Z}[\beta] \subset X$ and so $\alpha \in X$. Because $Y$ is an invertible fractional ideal of $\mathbb{Z}[\beta]$ and $X = Y^{-1}$, there are constants $a, b \in Y$ and $\gamma \in X$ such that

$$X = \alpha D + \gamma D, \quad Y = aD + bD, \quad a\alpha + b\gamma = 1$$

(see [13]). Since $IX = J$, multiplication by $\gamma$ produces a map $x \rightarrow \gamma x$ from $I$ to $J$. It obviously maps $\alpha I$ into $\alpha J$. On the other hand, if $\gamma x \in \alpha J$, say $\gamma x = \alpha z$ for $z \in J$, then

$$x = a\alpha x + b\gamma x = \alpha (ax + bz).$$

Here $ax \in YI \subset YJ = I$ since $I \subset J$, and $bz \in YJ = I$. Hence it follows that $x \in \alpha I$. So $\gamma$ induces a $\mathbb{Z}[\beta]$-module isomorphism

$$\gamma_* : I/\alpha I \rightarrow J/\alpha J.$$ 

This implies that $\text{BF}_g(A) \cong_R \text{BF}_g(B)$ where $R = \mathbb{Z}[\beta]$. Since $g \in \mathbb{Z}[x]$ with $g(A)$ invertible, is arbitrary, we have strong BF-equivalence of $A$ with $B$. □

Theorem 3.2 extends a result from [13], which states that for hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$ with the same irreducible characteristic polynomial $p(x)$, whose associated fractional ideals $I$ and $J$ of $\mathbb{Z}[\beta]$ satisfy $D(I) = D(J)$ with $I$ and $J$ invertible, the matrices $A$ and $B$ are strongly BF-equivalent. Invertible fractional ideals $I$ and $J$ of $\mathbb{Z}[\beta]$ are weakly equivalent when $D(I) = D(J)$ (see [5]). One way to guarantee the invertibility of nonzero fractional ideals of $\mathbb{Z}[\beta]$ is when the discriminant of $p(x)$ is square-free; then $\mathbb{Z}[\beta] = \mathfrak{o}_K$ (see [4]), a Dedekind domain in which every nonzero fractional ideal is invertible (see [16]).

Example 3.3. Consider the irreducible polynomial $p(x) = x^3 - 2x^2 - 8x - 1$ whose discriminant of 1957 is square-free (see Table B4 in [4]). The two matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 8 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -5 & 9 & 2 \end{bmatrix}$$

have $p(x)$ as their characteristic polynomials, where $A$ is the the companion matrix for $p(x)$. Let $\beta$ be a root of $p(x)$, set $K = \mathbb{Q}(\beta)$, and let $I$ and $J$ be fractional ideals in $\mathbb{Z}[\beta]$ associated to $A$ and $B$. Then $\mathbb{Z}[\beta] = \mathfrak{o}_K$ because the discriminant of $p(x)$ is square-free, and so $D(I) = D(J) = \mathfrak{o}_K$ and $I$ and $J$ are invertible. Hence $I$ and $J$ are weakly equivalent. By Theorem 3.2, the matrices $A$ and $B$ are strongly BF-equivalent. But, $I$ and $J$ are inequivalent fractional ideals (see [8]), and so $A$ and $B$ are not conjugate.

4. Strongly BF-equivalent hyperbolic Toral automorphisms. Unfortunately, for similar hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$, strong BF-equivalence of $A$ and $B$ does not imply the conjugacy of $A$ and $B$ (see [13]). As illustrated in Example 3.3, an algebraic obstruction for this is the existence of inequivalent but weakly equivalent fractional ideals associated to $A$ and $B$. However, strong BF-equivalence does relate the $R$-modules $G_A$ and $G_B$. 
Theorem 4.1. Suppose hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$ are similar. If $A$ and $B$ are strongly BF-equivalent, then there exists a topological $R$-module isomorphism $\Psi : G_A \to G_B$.

Proof. Under the hypotheses on $A$ and $B$, we prove $G_A \cong_R G_B$ through the open $R$-submodules of $G_A$ and $G_B$. Let $\mathcal{U}_A$ be the collection of all open $R$-submodules of $G_A$, and $\mathcal{U}_B$ the collection of all open $R$-submodules of $G_B$.

We claim that $G_A$ and $G_B$ have the same finite quotients through their respective open $R$-submodules:

$$\{G_A/U : U \in \mathcal{U}_A\} = \{G_B/V : V \in \mathcal{U}_B\},$$

where we understand equality in terms of $R$-module isomorphisms. Let $H = G_A/U$ for $U \in \mathcal{U}_A$. Since $\{\ker(\varphi_{k,A}) : k \in \mathbb{N}\}$ is a fundamental set of open $R$-submodule neighbourhoods of 0, there is a $k \in \mathbb{N}$ such that $\ker(\varphi_{k,A})$ is an $R$-submodule of $U$. Thus $U/\ker(\varphi_{k,A})$ is an $R$-submodule of $G_A/\ker(\varphi_{k,A})$ and

$$(G_A/\ker(\varphi_{k,A}))/\ker(\varphi_{k,A})) \cong_R G_A/U = H,$$

where $G_A/\ker(\varphi_{k,A}) \cong_R G_{k,A}$. By the assumption of strong BF-equivalence of $A$ and $B$, we have that $G_{k,A} \cong_R G_{k,B} \cong_R G_B/\ker(\varphi_{k,B})$. Then there is an open $R$-submodule $V$ of $G_B$ containing $\ker(\varphi_{k,B})$ such that $U/\ker(\varphi_{k,A}) \cong_R V/\ker(\varphi_{k,B})$. This implies that

$$H = G_A/U \cong_R G_B/V$$

for $V \in \mathcal{U}_B$. By reversing the roles of $A$ and $B$ in this argument, we obtain the other inclusion.

The rest of the of proof now follows from well-known results (see Theorem 3.2.7 and/or Corollary 3.28 on pp. 88-89 in [15] for example). \hfill \square

5. A characterization of conjugacy. For similar $A, B \in \text{GL}(n, \mathbb{Z})$, the $R$-modules $G_A$ and $G_B$ are invariants of conjugacy. Furthermore, they provide means to characterize when $A$ and $B$ are conjugate or not in terms of the embedded copies of the $R$-modules $H_A$ and $H_B$ in $G_A$ and $G_B$ respectively. Specifically, for a topological $R$-module isomorphism $\Psi : G_A \to G_B$, it is how the image $\Psi(\iota_A(\mathbb{Z}^n))$ intersects $\iota_B(\mathbb{Z}^n)$.

Theorem 5.1. For similar hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$, the following are equivalent:

(a) $A$ is conjugate to $B$,

(b) there exists a topological $R$-module isomorphism $\Psi : G_A \to G_B$ such that $\Psi(\iota_A(\mathbb{Z}^n)) = \iota_B(\mathbb{Z}^n)$, and

(c) there exists a topological $R$-module isomorphism $\Psi : G_A \to G_B$ such that $\Psi(\iota_A(\mathbb{Z}^n)) \cap \iota_B(\mathbb{Z}^n)$ is $R$-module isomorphic to the right $\mathbb{Z}[B]$-module $\mathbb{Z}^n$.

Proof. (a)$\Rightarrow$(b) Suppose $A$ and $B$ are conjugate, i.e., there is $C \in \text{GL}(n, \mathbb{Z})$ such that $AC = CB$. By a standard argument, the map $\Psi : G_A \to G_B$ defined by

$$\Psi([m_k]_{k \in \mathbb{N}}) = ([m_k C]_{k \in \mathbb{N}})$$

is a topological $R$-module isomorphism. For $m \in \mathbb{Z}^n$, we have

$$\Psi(\iota_A(m)) = \Psi([m]_{k \in \mathbb{N}}) = ([m C]_{k \in \mathbb{N}}),$$

which implies that $\Psi(\iota_A(\mathbb{Z}^n)) \subset \iota_B(\mathbb{Z}^n)$. The invertibility of $C$ implies the opposite inclusion, so that $\Psi(\iota_A(\mathbb{Z}^n)) = \iota_B(\mathbb{Z}^n)$.

(b)$\Rightarrow$(c) This follows because $\iota_B$ is an $R$-module monomorphism by Lemma 2.2.
(c)⇒(a) Set
\[ \Delta_\Psi = \Psi(\iota_A(Z^n)) \cap \iota_B(Z^n), \]
and suppose there is an R-module isomorphism \( h : Z^n \to \Delta_\Psi, \) i.e., \( hB = \Gamma_Bh. \)
Then \( h^{-1} : \Delta_\Psi \to Z^n \) is also an R-module isomorphism, i.e., \( Bh^{-1} = h^{-1}\Gamma_B, \) and so the map \( h^{-1}\Psi_\iota_A \) is an R-module isomorphism from the right \( \mathbb{Z}[A]\)-module \( Z^n \) to the right \( \mathbb{Z}[B]\)-module \( Z^n. \) This implies that \( h^{-1}\Psi_\iota_A \) is an automorphism of \( Z^n, \)
and so there is \( C \in \text{GL}(n, \mathbb{Z}) \) such that \( C = h^{-1}\Psi_\iota_A. \) Thus for all \( m \in Z^n, \) we have
\[ mAC = C(mA) = (h^{-1}\Psi_\iota_A A)(m) = (h^{-1}\Psi\Gamma_A)(m) = (h^{-1}\Gamma_B\Psi_\iota_A)(m) = (Bh^{-1}\Psi_\iota_A)(m) = (BC)(m) = mCB. \]
Since \( mAC = mCB \) holds for all \( m \in Z^n, \) we have that \( AC = CB, \) i.e., that \( A \) and \( B \) are conjugate. \( \square \)

We detail another proof of part (b) implying part (a) of Theorem 5.1 that highlights the role that hyperbolicity and the profinite topology play. Suppose that there is a topological R-module isomorphism \( \Psi : G_A \to G_B \) such that \( \Psi(\iota_A(Z^n)) = \iota_B(Z^n). \) Since \( \iota_B : Z^n \to G_B \) is a monomorphism, there is an isomorphism \( \iota_B^{-1} : \iota_B(Z^n) \to Z^n, \) and so \( \Psi(\iota_A(Z^n)) = \iota_B(Z^n) \) implies that
\[ \iota_B^{-1}\Psi_\iota_A : Z^n \to Z^n \]
is an automorphism. Since the automorphism group of \( Z^n \) is \( \text{GL}(n, \mathbb{Z}), \) there is \( C \in \text{GL}(n, \mathbb{Z}) \) such that
\[ \Psi_\iota_A = \iota_B C. \]
For all \( m \in Z^n, \) it follows that
\[ \Psi([m]_{k,A}) = (\Psi_\iota_A)(m) = (\iota_B C)(m) = ([m]_{k,B})_{k \in \mathbb{N}}. \]
Since \( \Psi \) is an R-module isomorphism, we have \( \Gamma_B\Psi = \Psi\Gamma_A, \) and so for all \( m \in Z^n, \)
\[ [mCB]_{k,B} = \Gamma_B([mC]_{k,B})_{k \in \mathbb{N}} = \Gamma_B\Psi([m]_{k,A})_{k \in \mathbb{N}} = \Psi\Gamma_A([m]_{k,A})_{k \in \mathbb{N}} = \Psi([mA]_{k,A})_{k \in \mathbb{N}} = ([mAC]_{k,B})_{k \in \mathbb{N}}. \]
This means for all \( k \in \mathbb{N} \) and for all \( m \in Z^n \) that
\[ m(CB - AC) \in N_{k,B} = Z^n(B^k - I). \]
For the standard basis of row vectors \( e_1, \ldots, e_n \) of \( Z^n, \) we have \( e_j(CB - AC) \in N_{k,B} \) for all \( k \in \mathbb{N} \) and for all \( j = 1, \ldots, n. \) By Lemma 2.2, the hyperbolicity of \( B \) implies that
\[ \bigcap_{k \in \mathbb{N}} N_{k,B} = \{0\}. \]
Thus \( e_j(CB - AC) = 0 \) for \( j = 1, \ldots, n. \) Therefore \( AC = CB, \) and so \( A \) and \( B \) are conjugate. This completes this alternative proof of (b) implies (a) in Theorem 5.1.
When similar hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$ are nonconjugate but strongly BF-equivalent, it is not the lack of a topological $R$-module isomorphism between $G_A$ and $G_B$ that is responsible for lack of conjugacy between $A$ and $B$. By Theorem 4.1 there exists at least one topological $R$-module isomorphism $\Psi : G_A \rightarrow G_B$. By part (b) of Theorem 5.1, the failure of $\Psi(i_A(\mathbb{Z}^n)) = i_B(\mathbb{Z}^n)$ for every topological $R$-module isomorphism $\Psi : G_A \rightarrow G_B$ is responsible for the lack of conjugacy between $A$ and $B$. For instance, there is no way to take any topological $R$-module isomorphism from $G_A$ to $G_B$ and, by adjusting $G_A$ and $G_B$ by topological $R$-module automorphisms, achieve a topological $R$-module isomorphism $\Psi : G_A \rightarrow G_B$ such that $\Psi(i_A(\mathbb{Z}^n)) = i_B(\mathbb{Z}^n)$. By part (c) of Theorem 5.1, similar statements hold regarding the failure of $\Psi(i_A(\mathbb{Z}^n)) \cap i_B(\mathbb{Z}^n)$ to be $R$-module isomorphic to the right $\mathbb{Z}[B]$-module $\mathbb{Z}^n$.

The $R$-submodule $\Delta_\Psi$ of $i_B(\mathbb{Z}^n)$ defined in the proof of part (c) implies part (a) of Theorem 5.1 plays the key role in the characterization of conjugacy for strongly BF-equivalent similar hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$. Since $\Psi(i_A(\mathbb{Z}^n)) = \Psi(i_A(\theta_A^{-1}(H_A)))$ and $i_B(\mathbb{Z}^n) = i_B(\theta_B^{-1}(H_B))$ as $R$-modules, then

$$\Delta_\Psi = \Psi(i_A(\theta_A^{-1}(H_A))) \cap i_B(\theta_B^{-1}(H_B)).$$

The characterization of conjugacy given in Theorem 5.1 implies that strongly BF-equivalent similar hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$ are conjugate if and only if there exists a topological $R$-module isomorphism $\Psi : G_A \rightarrow G_B$ such that either $\Psi(i_A(\theta_A^{-1}(H_A))) = i_B(\theta_B^{-1}(H_B))$, or $\Psi(i_A(\theta_A^{-1}(H_A))) \cap i_B(\theta_B^{-1}(H_B)) \equiv_R H_B$. Thus it is the embedded copies of $H_A$ and $H_B$ in $G_A$ and $G_B$ respectively, that determine the conjugacy of $A$ with $B$ or the lack thereof.

6. Irreducible hyperbolic Toral automorphisms. No assumption is made about the irreducibility of the characteristic polynomial of the similar hyperbolic $\text{GL}(n, \mathbb{Z})$ matrices in Theorem 4.1 or Theorem 5.1. A hyperbolic $A \in \text{GL}(n, \mathbb{Z})$ is irreducible if its characteristic polynomial is irreducible. For a topological $R$-module isomorphism $\Psi : G_A \rightarrow G_B$, more can be deduced about the nature of the intersection of $\Psi(i_A(\mathbb{Z}^n))$ with $i_B(\mathbb{Z}^n)$ when $A$ and $B$ have the same irreducible characteristic polynomial.

**Theorem 6.1.** Suppose similar hyperbolic $A, B \in \text{GL}(n, \mathbb{Z})$ are irreducible. Then for any topological $R$-module isomorphism $\Psi : G_A \rightarrow G_B$, either $\Psi(i_A(\mathbb{Z}^n)) \cap i_B(\mathbb{Z}^n) = \{0\}$ or $\Psi(i_A(\mathbb{Z}^n)) \cap i_B(\mathbb{Z}^n)$ has finite index in $i_B(\mathbb{Z}^n)$.

**Proof.** Suppose $\Psi : G_A \rightarrow G_B$ is a topological $R$-module isomorphism such that

$$\Delta_\Psi = \Psi(i_A(\mathbb{Z}^n)) \cap i_B(\mathbb{Z}^n) \neq \{0\}.$$

Since $i_A(\mathbb{Z}^n)$ is an $R$-submodule of $G_A$ and $\Psi$ is an $R$-module isomorphism, it follows that

$$\Psi(i_A(\mathbb{Z}^n)) = \Psi(\Gamma_A(i_A(\mathbb{Z}^n))) = \Gamma_B(\Psi(i_A(\mathbb{Z}^n))),$$

and so $\Psi(i_A(\mathbb{Z}^n))$ is an $R$-submodule of $G_B$. Since $i_B(\mathbb{Z}^n)$ is an $R$-submodule of $G_B$, it follows that $\Delta_\Psi$ is an $R$-submodule of $G_B$.

The inclusion of $R$-submodules $\Delta_\Psi < i_B(\mathbb{Z}^n)$ and the conjugacy $\Gamma_{B \cap B} = i_B B$ imply that $i_B^{-1}(\Delta_\Psi)$ is a $B$-invariant subgroup of $\mathbb{Z}^n$:

$$B(i_B^{-1}(\Delta_\Psi)) = i_B^{-1}(\Gamma_B(\Delta_\Psi)) = i_B^{-1}(\Delta_\Psi).$$
Let $S$ be the nontrivial subspace $\mathbb{R}^n$ spanned by a set of generators of $\iota_B^{-1}(\Delta_\Psi)$. This subspace is $B$-invariant, i.e., $B(S) = S$, and it projects to a right action $T_B$-invariant subtorus of $\mathbb{T}^n$. The irreducibility of the characteristic polynomial of $B$ implies that the only $T_B$-invariant subtori of $\mathbb{T}^n$ are $\{0\}$ and $\mathbb{T}^n$ (see Proposition 3.1 on p. 726 in [8]). Since $S$ is nontrivial, it follows that $\iota_B^{-1}(\Delta_\Psi)$ contains a basis for $\mathbb{R}^n$. Thus $\iota_B^{-1}(\Delta_\Psi)$ has finite index in $\mathbb{Z}^n$, and hence $\Delta_\Psi$ has finite index in $\iota_B(\mathbb{Z}^n)$.

By Theorem 6.1, $\Delta_\Psi$ has finite index within $\iota_B(\mathbb{Z}^n)$, when $\Delta_\Psi \neq \{0\}$, and so $\Delta_\Psi$ is group-theoretically isomorphic to $\mathbb{Z}^n$. By part (c) of Theorem 5.1, only when $\Delta_\Psi$ is $R$-module isomorphic to the right $\mathbb{Z}[B]$-module $\mathbb{Z}^n$ can we conclude that $A$ and $B$ are conjugate. For any irreducible similar hyperbolic strongly BF-equivalent $A$ and $B$ that are not conjugate, such as in Example 3.3, any topological $R$-module isomorphism $\Psi : G_A \to G_B$ has either $\Delta_\Psi = \{0\}$ or it has finite index in $\iota_B(\mathbb{Z}^n)$, but there is no topological $R$-module isomorphism $\Psi : G_A \to G_B$ with $\Delta_\Psi = \Psi(\iota_A(\theta_A^{-1}(H_A))) \cap \iota_B(\theta_B^{-1}(H_B))$ being $R$-module isomorphic to $H_B$. Thus, the finite intersection of $\Psi(\iota_A(\theta_A^{-1}(H_A)))$ with $\iota_B(\theta_B^{-1}(H_B))$ by itself is insufficient to distinguish $A$ and $B$ dynamically.

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