Efficient quantum algorithms to construct arbitrary Dicke states

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Abstract In this paper, we study several quantum algorithms toward the efficient construction of arbitrary arbitrary Dicke state. The proposed algorithms use proper symmetric Boolean functions that involve manipulation with Krawtchouk polynomials. Deutsch–Jozsa algorithm, Grover algorithm, and the parity measurement technique are stitched together to devise the complete algorithm. In addition to that we explore how the biased Hadamard transformation can be utilized into our strategy, motivated by the work of Childs et al. (Quantum Inf Comput 2(3):181–191, 2002).

Keywords Biased Hadamard transform · Deutsch–Jozsa algorithm · Dicke state · Grover algorithm · Krawtchouk polynomial · Parity measurement · Symmetric boolean functions

1 Introduction

Multipartite entanglement has been investigated to understand the nature of quantum physics, the nature of computational power, and to use them for many quantum infor-
information applications. Although lots of studies have been done, we still do not fully understand the multipartite entanglement. Among many methods for this purpose, there is an approach such as the study of the preparation complexity of a target multipartite state. Actually, the preparation of a target multipartite state is very important from the practical and fundamental point of views because we should prepare the target state correctly and the complexity of preparation is related to the efficiency of the application.

Under this purpose, we study Dicke state [7] in this work. The n-qubit weight \( w \) Dicke state, \(|D^n_w\rangle\), is an equal superposition of all \( n \)-qubit states of weight \( w \). Note that the simplest Dicke state form is W state where \( w = 1 \). Meanwhile, GHZ state is the superposition of \(|D^n_0\rangle\) and \(|D^n_n\rangle\). The transformation of Dicke state into W or GHZ states has been investigated [17].

Likewise many multipartite entanglements, Dicke state has gained lots of interests. From the physical viewpoint, its entanglement properties have been studied [8,22]. Specially, the decoherence property has been analyzed as well [4,13]. Also its physical experiments have been done and implemented with the neutral atom [31,32], ion trap [22,23,29,33], quantum dot [38], linear optics [9,17,28,32], and cavity QEC [14,31,36]. Also some specific implementations have been shown for \(|D^2_1\rangle\) [18], for \(|D^4_2\rangle\) [17], and for \(|D^6_3\rangle\) [34]. From the quantum information application viewpoint, many practical applications of Dicke state have been proposed such as secrete sharing [35], telecloning [3,26], open destination teleportation [16,37], quantum networking [3,19], and precise measurement [1].

Because of the above reasons, the preparation of Dicke state is very important, which was also mentioned in [20]. Unfortunately, Only [2] investigated this issue very briefly. In this work, to extend the previous work, we show how one can efficiently construct Dicke states by using the combinatorial properties of symmetric Boolean functions, two well-known quantum algorithms, and the generalized parity measurement. By efficient, we mean that the resource requirements in terms of quantum circuits and the number of execution steps are \( \text{poly}(n) \) to obtain \(|D^n_w\rangle\). For precise explanation, we define a few states. Let us consider \( n \)-qubit states in the computational basis \( \{0,1\}^n \) that can be written in the form \( \sum_{x \in \{0,1\}^n} a_x |x\rangle \), where \( \sum_{x \in \{0,1\}^n} |a_x|^2 = 1 \). Thus, \( x \) can also be interpreted as a binary string and the number of 1’s in the string is called the (Hamming) weight of \( x \) and denoted as \( w_I(x) \). Based on this, an arbitrary Dicke state can be expressed as follows:

\[
|D^n_w\rangle = \sum_{x \in \{0,1\}^n, w_I(x) = w} \frac{1}{\sqrt{n_w}} |x\rangle.
\]

Let us also define a symmetric \( n \)-qubit state as follows

\[
|S^n\rangle = \sum_{x \in \{0,1\}^n} a_{w_I(x)} |x\rangle,
\]

where \( \sum_{i=0}^{n} \binom{n}{i} |a_i|^2 = 1 \). Based on these definitions, our contributions are summarized as follows.

- First, we show how one can prepare a symmetric \( n \)-qubit state with the property that \( \binom{n}{w} |a_w|^2 \) is \( \Omega \left( \frac{1}{\sqrt{n}} \right) \) by using Deutsch–Jozsa algorithm [6]. This requires certain
novel combinatorial observation related to symmetric Boolean functions. Then, the quantum state out of Deutsch–Jozsa algorithm is measured using the parity measurement technique [15] to obtain $|D^n_w\rangle$ with a probability $\Omega(1/\sqrt{n})$. Thus, $O(\sqrt{n})$ runs are sufficient to obtain the required Dicke state. Note that a direct approach to construct a symmetric state has been presented in [2] using a biased Hadamard transform. While the order of probability to obtain Dicke state by ours and that of [2] are the same, some enumeration result shows that the exact probability value is better in our case than that of [2].

– Further, motivated by the idea in [2], we improve our algorithm further with a modified Deutsch–Jozsa operator that involves a biased Hadamard transform. Since a biased Hadamard transform also helps to generate the target symmetric state, the overall probability to obtain the Dicke state increases.

– Finally, we can also apply the Grover operator [12] before the measurement. Since Grover algorithm amplifies the amplitude of target symmetric state, this helps to reduce the necessary number of steps into $O(4/\sqrt{n})$.

The organization of this paper is as follows. Section 2 shows the properties of symmetric Boolean functions in terms of Walsh spectrum, its relation with Krawtchouk polynomials, and its implementation. Section 3 discusses a method to find an appropriate Boolean function maximizing Walsh spectrum, the relation between the Deutsch–Jozsa algorithm and the Walsh spectrum of the chosen symmetric Boolean function, an quantum algorithm for preparing Dicke state based on these relations, and its performance comparison over the previous approach. Section 4 shows how to improve the first approach by exploiting a biased Hadamard operator. Section 5 compares the performance of the previous and the proposed two methods. Section 6 illustrates how to use the Grover operator to further improve the proposed approaches. Finally, Sect. 7 concludes this work and discuss some open problems.

2 Properties of symmetric boolean functions

2.1 Walsh spectrum of symmetric boolean functions

A Boolean function on $n$ variables may be viewed as a mapping from $\{0, 1\}^n$ into $\{0, 1\}$. Let us denote the addition operator over $GF(2)$ by $\oplus$. Let $x = (x_1, \ldots, x_n)$ and $\omega = (\omega_1, \ldots, \omega_n)$ both belong to $\{0, 1\}^n$ and the inner product

$$x \cdot \omega = x_1\omega_1 \oplus \cdots \oplus x_n\omega_n.$$ 

Let $f(x)$ be a Boolean function on $n$ variables. Then, the Walsh transform of $f(x)$ is a real valued function over $\{0, 1\}^n$ which is defined as follows
\[ W_f(\omega) = \sum_{x \in \{0, 1\}^n} (-1)^{f(x) \oplus x \cdot \omega}. \]

An \( n \)-variable Boolean function \( f \) is called symmetric if \( f(x) = f(y) \) for all \( x, y \in \{0, 1\}^n \) such that \( wt(x) = wt(y) \). Henceforth, we will denote the set of \( n \)-variable symmetric Boolean functions as \( SB_n \).

In the truth table of \( f \in SB_n \), it is enough to provide outputs corresponding to different weights of elements of \( \{0, 1\}^n \) only. So an \( n \)-variable symmetric function can be expressed by an \((n + 1)\) length bit string as follows

\[ re_f = [f_0, f_1, \ldots, f_n], \]

where \( f_i \) is the output at the inputs of weight \( i \) and \( re_f \) is referred to as a simplified value vector. When \( f \in SB_n \), one may note that \( W_f(x) = W_f(y) \) for all \( x, y \in \{0, 1\}^n \) such that \( wt(x) = wt(y) \). Therefore, the Walsh spectrum of \( f \) can be represented by an \((n + 1)\) length integer string

\[ rw_f = [rw_f(0), rw_f(1), \ldots, rw_f(n)], \]

where \( rw_f(i) \) represents the Walsh spectrum value at the inputs of weight \( i \).

### 2.2 Relation between Walsh spectrum and Krawtchouk polynomials

We now relate the Walsh spectrum of the symmetric functions [30] with Krawtchouk polynomials [21,24]. Krawtchouk polynomial of degree \( i \) is given by

\[ K_i(\eta, n) = \sum_{j=0}^{i} (-1)^j \binom{n}{{i}} \binom{n - \eta}{i - j}. \]

From [30], we get that if \( wt(\omega) = k \), then

\[ W_f(\omega) = \sum_{i=0}^{n} (-1)^{i} K_i(k, n). \]

The \((n + 1) \times (n + 1)\) matrix, which has \( K_i(k, n) \) as the \((i, k)\)th element, is known as the Krawtchouk matrix [10,11].

For example, let us present Krawtchouk matrices for \( n = 5 \) and \( n = 6 \) as follows:

For \( n = 5 \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
5 & 3 & 1 & -1 & -3 \\
10 & 2 & -2 & 2 & 10 \\
10 & -2 & -2 & 2 & -10 \\
5 & -3 & 1 & 1 & -3 \\
1 & -1 & 1 & -1 & -1
\end{bmatrix}
\]

for \( n = 6 \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
6 & 4 & 2 & 0 & -2 & -4 \\
15 & 5 & -1 & -3 & -1 & 5 \\
20 & 0 & -4 & 0 & 4 & 0 \\
15 & -5 & -1 & 3 & -1 & -5 \\
6 & -4 & 2 & 0 & -2 & 4 \\
1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}
\]
In these two matrices, one can verify the properties related to the Krawtchouk matrix given in Proposition 1.

To determine all the Walsh spectrum values of \( f \in SB_n \), it is enough to multiply \((-1)^{f_0}, \ldots, (-1)^{f_n}\) with the \((n + 1) \times (n + 1)\) Krawtchouk matrix. Applying Krawtchouk matrix, the analysis of the Walsh spectra of symmetric functions becomes combinatorially interesting. Elements of a Krawtchouk matrix have nice combinatorial properties, and they follow nice symmetry [21] too. We list some of them in the following proposition.

**Proposition 1**

1. \( K_0(k, n) = 1 \), \( K_1(k, n) = n - 2k \),
2. \((i + 1)K_{i+1}(k, n) = (n - 2k)K_i(k, n) - (n - i + 1)K_{i-1,n}(k, n)\),
3. \( K_i(k, n) = (-1)^kK_{n-i}(k, n)\),
4. \( \binom{n}{k}K_i(k, n) = \binom{n}{k}K_k(i, n)\),
5. \( K_i(k, n) = (-1)^iK_i(n - k, n)\),
6. \((n - k)K_i(k + 1, n) = (n - 2i)K_i(k, n) - kK_i(k - 1, n)\),
7. \((n - i + 1)K_i(k, n + 1) = (3n - 2i - 2k + 1)K_i(k, n) - 2(n - k)K_i(k, n - 1)\).

2.3 Implementation of symmetric boolean functions

The symmetric Boolean functions can be efficiently implemented. As described in [5], the circuit complexity of \( n \)-variable symmetric Boolean functions is \( 4.5n + o(n) \). It is known that given a classical circuit \( f \), there is a quantum circuit of comparable efficiency which computes the transformation \( U_f \) that takes an input like \( |x, y\rangle \) and produces an output like \( |x, y \oplus f(x)\rangle \). Thus, we will consider that for \( f \in SB_n \), the quantum circuit \( U_f \) can be efficiently implemented using \( O(n) \) circuit complexity.

3 Algorithm 1: Deutsch–Jozsa algorithm with special symmetric function

3.1 Find a special symmetric boolean function which maximizes the Walsh spectrum

Consider that we want to maximize the Walsh spectrum value corresponding to weight \( w \) points and naturally, from the property of symmetric functions, where all of them will be equal. Now, we present an important combinatorial result to show how to find such symmetric Boolean functions.

**Theorem 1** Consider \( f \in SB_n \). The function \( f \), represented as \( re_f \), for which the Walsh spectrum corresponding to the \( w \) weight points will be maximized, can be written as follows

\[
f_i = \begin{cases} 
0, & \text{if } K_i(w, n) > 01, \\
1 & \text{if } K_i(w, n) < 00 \text{ or } 1, \\
0 & \text{if } K_i(w, n) = 01
\end{cases}
\] (3)

**Proof** We have \( W_f(\omega) = \sum_{i=0}^{n}(-1)^{f_i}K_i(k, n) \). One may note that the maximum value of \( \sum_{i=0}^{n}(-1)^{f_i}K_i(k, n) \) is \( \sum_{i=0}^{n}|K_i(k, n)| \). This is attained when we take the function of the form as described in the theorem.
Example 1 As example, consider \( n = 6 \) and \( w = k = 2 \). In the corresponding column of the \((6 + 1) \times (6 + 1)\) matrix, we get the values as \(1, 2, -1, -4, -1, 2\). Thus, we will consider the function with \( r_{eff} \) as \([0, 0, 1, 1, 0, 0]\). For such a \( f \in SB_6 \), the Walsh spectrum values at the points \( \omega \), such that \( w = wt(\omega) = 2 \), will be maximized, which is \(1 + 2 + 1 + 4 + 1 + 2 + 1 = 12\).

3.2 Walsh spectrum of the special symmetric boolean function by combinatorial property of Krawtchouk matrix

Next, we present certain results related to column sum of a Krawtchouk matrix.

Lemma 1 \( \sum_{i=0}^{n} |K_i \left( \left\lceil \frac{n}{2} \right\rceil, n \right) | = \sum_{i=0}^{n} |K_i \left( \left\lfloor \frac{n}{2} \right\rfloor, n \right) | = 2^\left\lceil \frac{n}{2} \right\rceil \).

Proof Let us first prove this for even \( n \).

Following Proposition 1(2), we have

\[
(i + 1)K_{i+1}(k, n) = (n - 2k)K_{i}(k, n) - (n - i + 1)K_{i-1, n}(k, n).
\]

For \( n \) even, and \( k = \frac{n}{2} \), we get,

\[
K_{i+1} \left( \frac{n}{2}, n \right) = -\frac{n - i + 1}{i + 1} K_{i-1, n} \left( \frac{n}{2}, n \right).
\]

That is, the recurrence relation follows:

\[
K_{i} \left( \frac{n}{2}, n \right) = -\frac{n - i + 2}{i} K_{i-2, n} \left( \frac{n}{2}, n \right),
\]

with the initial conditions \( K_0 \left( \frac{n}{2}, n \right) = 1 \) and \( K_1 \left( \frac{n}{2}, n \right) = 0 \) as available from Proposition 1(1). Thus, one may note that for odd \( i \), \( K_i \left( \frac{n}{2}, n \right) = 0 \). Further, using induction, for even \( i \), we get

\[
|K_i \left( \frac{n}{2}, n \right) | = \left( \frac{n}{2} \right). \]

Thus,

\[
\sum_{i=0}^{n} |K_i \left( \frac{n}{2}, n \right) | = \sum_{i=0}^{\frac{n}{2}} \left( \frac{n}{2} \right),
\]

putting \( i = 2l \). Hence,

\[
\sum_{i=0}^{n} |K_i \left( \frac{n}{2}, n \right) | = 2^\left\lceil \frac{n}{2} \right\rceil.
\]

Now let us prove this for odd \( n \).

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For $n$ odd and $k = \frac{n-1}{2}$, from Proposition 1(2) we get

$$K_{i+1}\left(\frac{n-1}{2}, n\right) = \frac{1}{i+1} K_i\left(\frac{n-1}{2}, n\right) - \frac{n-i+1}{i+1} K_{i-1}\left(\frac{n-1}{2}, n\right).$$

That is, the recurrence relation is as follows:

$$K_i\left(\frac{n-1}{2}, n\right) = \frac{1}{i} K_{i-1}\left(\frac{n-1}{2}, n\right) - \frac{n-i+2}{i} K_{i-2}\left(\frac{n-1}{2}, n\right).$$

One can now show by induction that

$$K_{2i}\left(\frac{n-1}{2}, n\right) = K_{2i+1}\left(\frac{n-1}{2}, n\right), \forall i, \ 1 \leq i \leq \frac{n-1}{2}.$$

Using the above two identities and induction, one can verify that $|K_{2i}\left(\frac{n-1}{2}, n\right)| = \left(\frac{n-1}{i}\right)$. Thus,

$$\sum_{i=0}^{n} |K_i\left(\left\lfloor \frac{n}{2} \right\rfloor, n\right)| = 2 \sum_{l=0}^{\frac{n-1}{2}} \left(\frac{n-1}{l}\right).$$

where $i = 2l$. Hence, we get,

$$\sum_{i=0}^{n} |K_i\left(\left\lfloor \frac{n}{2} \right\rfloor, n\right)| = 2 \cdot 2^{n-1} = 2^{n+1}.$$

Using Proposition 1(5), we get that

$$\sum_{i=0}^{n} |K_i\left(\left\lfloor \frac{n}{2} \right\rfloor, n\right)| = \sum_{i=0}^{n} |K_i\left(\left\lfloor \frac{n}{2} \right\rfloor, n\right)|.$$

That completes the proof.

**Theorem 2** Let $f \in SB_n$ be as explained in Theorem 1 toward maximizing the Walsh spectrum values at weight $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. Then,

$$\left(\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(r_w f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right)^2 = \left(\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(r_w f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right)^2 \right) \right)^2 \text{ is } \Omega\left(\frac{2^{2n}}{\sqrt{n}}\right).$$

**Proof** The Walsh spectrum in this case is as follows
Thus, the total sum of the squares of the Walsh spectrum values at weight $\lceil \frac{n^2}{2} \rceil$ or $\lfloor \frac{n^2}{2} \rfloor$ is

$$\binom{n}{\lceil \frac{n^2}{2} \rceil} \left( \sum_{i=0}^{n} |K_i(\lceil \frac{n^2}{2} \rceil, n)| \right)^2$$

which is $\Omega \left( \frac{2^{2n}}{\sqrt{n}} \right)$, by Stirling’s approximation. \(\square\)

One may similarly note that for the trivial cases of $w = 0$ or $n$, if one chooses $f \in SB_n$ following Theorem 1, then $\binom{n}{w} (rw_f(w))^2 = 2^{2n}$. However, proving the result similar to Theorem 2 for any $n$ and any weight $w$, in general, seems to be quite tedious. Thus, we make detailed enumerations to obtain $c(n) = \min_{w=0}^{n} \binom{n}{w} \frac{(rw_f(w))^2}{2^{2n} \sqrt{n}}$ that has been verified for $n \leq 1000$ and we note that the values stabilize as $c(999) = 1.24793$ and $c(1000) = 0.797685$. The graph of this is plotted in Fig. 1 for $1 \leq n \leq 100$, where the points for odd $n$ are coming above and those for even $n$ are coming below. Since we are not providing a proof of this, we refer this as follows.

**Conjecture 1** Let $f \in SB_n$ be as explained in Theorem 1 toward maximizing the Walsh spectrum values at weight $w$. Then, the total sum of the squares of the Walsh spectrum values at weight $w$, $\binom{n}{w} (rw_f(w))^2$, is $\Omega \left( \frac{2^{2n}}{\sqrt{n}} \right)$. 

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The proof of the conjecture seems to be quite tedious and elusive, and we leave it as an open problem.

3.3 Relation between Deutsch–Jozsa algorithm and the Walsh spectrum of symmetric boolean function

Given \( f \) is either constant or balanced, if the corresponding quantum implementation \( U_f \) is available, Deutsch and Jozsa [6] provided a quantum algorithm that decide in constant time which one it is. Let us now describe our interpretation of Deutsch–Jozsa algorithm in terms of Walsh spectrum values. We denote the operator for Deutsch–Jozsa algorithm as follows

\[
D_f = H^\otimes n U_f H^\otimes n, \tag{4}
\]

where the Boolean function \( f \) is available as an oracle \( U_f \). For brevity, we abuse the notation and do not write the auxiliary qubit, i.e., \( |0\rangle - |1\rangle \sqrt{2} \) and the corresponding output in this case.

Now one can observe that [25]

\[
D_f |0\rangle^\otimes n = \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} \frac{(-1)^{x \cdot z \oplus f(x)}}{2^n} |z\rangle,
\]

\[
= \sum_{z \in \{0,1\}^n} \frac{W_f(z)}{2^n} |z\rangle.
\]

Note that the associated probability with a state \( |z\rangle \) is \( \frac{W_f^2(z)}{2^{2n}} \). Hence, we have the following technical result as pointed out in [25] with our interpretation for symmetric functions.

**Proposition 2** Given an \( n \)-variable Boolean function \( f \), \( D_f |0\rangle^\otimes n \) produces a superposition of all states \( z \in \{0,1\}^n \) with the amplitude \( \frac{W_f(z)}{2^n} \) corresponding to each state \( |z\rangle \). Specially, if \( f \in SB_n \), then the amplitude corresponding to \( |z\rangle \) is \( \sum_{i=0}^{n} (-1)^i K_i(wt(z), n) \).

3.4 Algorithm

Based on the overall properties, we propose a quantum algorithm as shown in Algorithm 1. The following result provides the estimate of the complexity of our algorithm.

**Theorem 3** Let \( f \in SB_n \) be as explained in Theorem 1 toward maximizing the Walsh spectrum values at weight \( w \). Given that Conjecture 1 is true, the Deutsch–Jozsa algorithm produces a symmetric \( n \)-qubit state (before the measurement) \( |S^n\rangle = \sum_{x \in \{0,1\}^n} a_{wt(x)} |x\rangle \), such that \( \binom{n}{w} |a_w|^2 \) is \( \Omega(\frac{1}{\sqrt{n}}) \).
Algorithm 1 Deutsch–Jozsa Algorithm with Special Symmetric Function

1. Choose $f \in SB_n$ as explained in Theorem 1 to maximize the Walsh spectrum values at weight $w$.
2. Use the Deutsch–Jozsa algorithm to obtain a symmetric $n$-qubit state $|S^n\rangle = \sum_{x \in \{0,1\}^n} a_{wt(x)} |x\rangle$, such that $\left(\binom{n}{w}|a_w|\right)^2 \geq \frac{1}{\sqrt{n}}$.
3. Apply the parity measurement strategy. If the ancilla state is measured at the basis $U_w |\zeta\rangle$, then $|D_w^n\rangle$ is successfully obtained. Else go to Step 2 and iterate.

Proof The proof follows from Theorem 1, Conjecture 1 and Proposition 2. \qed

Now, the final step is to measure the symmetric state until we get the target Dicke state by using parity measurement method [15, Section IIIA]. Note [15, Section IIIA] assumes to use $n$-dimensional qudit ancilla, but we consider a qudit $|\zeta\rangle$ of dimension $n + 1$ here. A certain unitary operator $U$ is designed such that

$$|\zeta\rangle, U|\zeta\rangle, U^2|\zeta\rangle, \ldots, U^{n-1}|\zeta\rangle, U^n|\zeta\rangle$$

are all orthogonal to each other and $U^{n+1}|\zeta\rangle = |\zeta\rangle$. Since $|\zeta\rangle$ is an $(n+1)$-dimensional state, one can indeed obtain a set of such $n + 1$ orthogonal states. The parity measurement is done on the basis $\{ |\zeta\rangle, U|\zeta\rangle, U^2|\zeta\rangle, \ldots, U^n|\zeta\rangle\}$.

Fig. 2 Generalized parity measurement module as shown in [15, Fig. 1]. If $wt(x) = w$, then the ancilla will become $U_w^w |\zeta\rangle$. 

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Since the probability of target Dicke state is \( \Omega(\frac{1}{\sqrt{n}}) \), we should repeat the whole procedure at most \( O(\sqrt{n}) \).

**Example 2** Let us have an example taking \( n = 6, w = 2 \) to outline our method. In this case, the Dicke state will be

\[
|D_{6}^{2}\rangle = \sum_{x \in \{0,1\}^6, w(x)=2} \frac{1}{\sqrt{15}} |x\rangle.
\]

We start with an \( n = 6 \) variable symmetric Boolean function having Walsh spectrum value at each of the weight \( w = 2 \) point as 12 (following Theorem 1, one may refer to Example 1 also). There are \( \binom{n}{w} = \binom{6}{2} = 15 \) such points. The amplitude associated with each point, after the Deutsch–Jozsa algorithm, is \( \frac{12}{\sqrt{16}} = \frac{3}{4} \). Thus, we get

\[
\sum_{x \in \{0,1\}^6, w(x)=2} \frac{3}{16} |x\rangle + \sum_{x \in \{0,1\}^6, w(x)\neq2} d_x |x\rangle
\]

initially. Thus, the probability associated with \( |D_{6}^{2}\rangle \) will be \( \binom{6}{2} (\frac{12}{\sqrt{16}})^2 = \frac{144}{256} = 0.552734375 \), and hence, one may note that the probability, where it will land into \( |D_{6}^{2}\rangle \) after parity measurement, is quite high.

### 3.5 Comparison with a previous method

It was explained in [15] how one can obtain \( |D_{w}^{n}\rangle \) from \( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \). However, the idea explained in [15, Section IIIA] works efficiently only for \( w = \lceil \frac{n}{2} \rceil \) or \( \lfloor \frac{n}{2} \rfloor \). The most general work in this direction has appeared in [2] where a biased Hadamard transformation was exploited. The strategy of [2] uses a biased Hadamard transformation as follows

\[
\left[ \sqrt{1 - \frac{w}{n}} \ - \sqrt{\frac{w}{n}} \right] \otimes \cdots \left[ \sqrt{1 - \frac{w}{n}} \ - \sqrt{\frac{w}{n}} \right]
\]

on \( |0\rangle \otimes^n \) such that the probability associated with \( |D_{w}^{n}\rangle \) will be \( \binom{n}{w} (\frac{w}{n})^w (1 - \frac{w}{n})^{n-w} \geq \frac{\sqrt{2}}{n\pi} \), i.e., \( \Omega(\frac{1}{\sqrt{n}}) \). Thus, the probability of our case and also in [2] is of the same order. While the theoretical comparison of the exact probability values seems elusive, we have made detailed enumerations to observe that the exact probability values in our case are better than that of [2]. First, we present two graphs to show the probability values associated with \( |D_{w}^{n}\rangle \) for all \( w \), when \( n = 999 \) (to represent odd case) or \( 1000 \) (to represent even case). For our case, it is \( \binom{n}{w} (\frac{w}{n})^w (1 - \frac{w}{n})^{n-w} \) (after application of Deutsch–Jozsa algorithm without measurement), and for the case of [2], it is \( \binom{n}{w}w (1 - \frac{w}{n})^{n-w} \) (after application of biased Hadamard transform). From Figs. 3 and 4, it is clear to note that our method provides higher probability (the upper curve) in all the cases except \( w = 0, n \) (which are trivial ones) and \( w = \frac{n}{2} \) for \( n = 1000 \).
In both figures, the present algorithm shows some variation of probability when the weight is around $\lfloor \frac{n}{4} \rfloor$. To check whether or not these cases still show the higher probability than the previous method, we look into the $w = \lfloor \frac{n}{4} \rfloor$ case a little bit more. From Fig. 5, one may note that our probability values (the upper curve) are higher than the case of [2]. These results explain the advantage of the use of a suitable symmetric Boolean function which shows higher Walsh spectrum values for the given weight.

4 Algorithm 2: Additional improvement by exploiting a biased Hadamard operator

We have provided numerical evidences that using proper symmetric Boolean functions in Deutsch–Jozsa algorithm provides better probability than the use of a biased Hadamard transform as described in [2]. However, motivated by [2], a natural extension
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should be to couple a biased Hadamard transform into Deutsch–Jozsa algorithm instead of (unbiased) Hadamard transform. Thus, let us refer to the general description of a Hadamard type transformation (biased or unbiased) that can be written as follows

\[ B_{r,n} = \begin{bmatrix} \sqrt{1 - \frac{r}{n}} & \sqrt{\frac{r}{n}} \\ \sqrt{\frac{r}{n}} & -\sqrt{1 - \frac{r}{n}} \end{bmatrix}. \]  

(5)

We will replace the standard notation of \( w \) here by \( r \) as we will not restrict ourselves to integer values \( w \in [0, \ldots, n] \), but use any real number \( r \in [0, n] \) to obtain the optimum probability of success to get a Dicke state.

Instead of using the operator \( D_f = H^\otimes n U_f H^\otimes n \), let us first describe the most general operator of the form

\[ D'_f = B_{r_1,n}^\otimes n U_f B_{r_2,n}^\otimes n, \]

where \( r_1, r_2 \) are real numbers in \([0, n]\).

First, we consider the case when \( r_1 = \frac{n}{2} \), i.e., \( B_{r_1,n} = H \), but \( r_2 = r \) varies toward optimization. That is,

\[ D'_f = H^\otimes n U_f B_{r,n}^\otimes n. \]

One may note that the application of \( D'_f \) on \( |0\rangle^\otimes n \) will produce

\[ D'_f |0\rangle^\otimes n = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} \left( \sum_{x \in \{0,1\}^n} (-1)^{x \cdot z} f(x) (1 - \frac{r}{n})^\frac{n-d(x,z)}{2} \left( \frac{r}{n} \right)^\frac{d(x,z)}{2} \right) |z\rangle, \]

where \( d(x, z) \) is the (Hamming) distance between two same length binary strings \( x \) and \( z \). Before proceeding further, we also have the following technical result.

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**Fig. 5** Plot of probabilities associated with \( |D'_w\rangle \) against \( n \) in our case (above) and in [2] (below) for \( n = 4–1,000 \) and \( w = \lfloor \frac{n}{4} \rfloor \).
Proposition 3 Let $\mathcal{D}'_f = H^\otimes n U_f B^\otimes n_r$. If $f \in SB_n$ then $\mathcal{D}'_f|0\rangle^\otimes n$ is a symmetric state.

Proof We need to prove that $\sum_{x \in \{0,1\}^n} (-1)^{x \cdot u \oplus f(x)} (1 - \frac{r}{n}) \frac{n - d(x,u)}{2} (\frac{r}{n}) \frac{d(x,u)}{2}$ is the same for all the $z \in \{0,1\}^n$ having the same Hamming weight. Let us consider $u, v \in \{0,1\}^n$ such that $u \neq v$, but $wt(u) = wt(v)$. Then, we need to prove that

$$\sum_{x \in \{0,1\}^n} (-1)^{x \cdot u \oplus f(x)} (1 - \frac{r}{n}) \frac{n - d(x,u)}{2} (\frac{r}{n}) \frac{d(x,u)}{2} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot v \oplus f(x)} (1 - \frac{r}{n}) \frac{n - d(x,v)}{2} (\frac{r}{n}) \frac{d(x,v)}{2}.$$ 

The proof follows from the fact that

$$\sum_{x \in \{0,1\}^n, wt(x) = w} (-1)^{x \cdot u \oplus f(x)} (1 - \frac{r}{n}) \frac{n - d(x,u)}{2} (\frac{r}{n}) \frac{d(x,u)}{2} = \sum_{x \in \{0,1\}^n, wt(x) = w} (-1)^{x \cdot v \oplus f(x)} (1 - \frac{r}{n}) \frac{n - d(x,v)}{2} (\frac{r}{n}) \frac{d(x,v)}{2},$$

given that $f$ is symmetric. $\square$

The main problem in this case is that we need to go for trial and error by modifying the symmetric Boolean functions and trying out different values of $\frac{r}{n}$. So far, we could not obtain the exact characterization of symmetric functions, while a biased Hadamard transform is used. Based on this, we propose an improved algorithm as shown in Algorithm 2.

As we could not characterize this, to provide some experimental results in this direction (see Table 1), we used the following method for some small values of $n$ ($n = 4$ to $9$). We select each of the Boolean functions $f$ from $SB_n$. Given $f$ and a specific weight $w$, $1 \leq w \leq n$, we write the expression of success probability as a function of $r$. Then, we apply the Maximize function available in Mathematica 8.0 to compute the optimum value of $r$ given $f, w$, so that the success probability becomes maximum. Note that, as we could not characterize the process yet, this is an exhaustive task and for each $n$, it requires checking of $2^{n+1}$ symmetric Boolean functions. That is the reason, we can study it for only a few small values of $n$. However, this is a classical computation that can be done as an off-line work. Once such programs are executed, we can have a database of proper $f \in SB_n$ and the corresponding $r$ to have

**Algorithm 2** Improved Algorithm by Exploiting a Biased Hadamard Operator

1. Apply $\mathcal{D}'_f$ to $|0\rangle^\otimes n$ to obtain a symmetric $n$-qubit state. The value of $\frac{r}{n}$ and the choice of the symmetric Boolean functions are achieved heuristically.
2. Use parity measurement strategy. If the ancilla state is measured at the basis $U^w|\xi\rangle$, then $|D^w_r\rangle$ is successfully obtained. Else take the parameters as in Step 1 once again and iterate Step 2.
Table 1  Probability values using a biased Hadamard transform (in this case, we provide the corresponding symmetric function $f$ represented as a hexadecimal number of the $(n + 1)$ length bit string $f_n, f_{n-1}, \ldots, f_1, f_0$ and the value of $r$), using (standard) Hadamard transform and the method of [2]

| $n$ | $w \rightarrow$ | 1       | 2       | 3       | 4       | 5       | 6       | 7       | 8       |
|-----|----------------|---------|---------|---------|---------|---------|---------|---------|---------|
|     |                | 0.833609| 0.981763| 0.833609| –       | –       | –       | –       | –       |
| 4   |                | 0.468136| 0.298698| 0.468136| –       | –       | –       | –       | –       |
|     | $f$            | 01      | 02      | 05      | –       | –       | –       | –       | –       |
|     | $r$            | 1.42458 | 0.313077| 0.313077| 3.57542 | –       | –       | –       | –       |
|     | $D_f\{0\}^{\otimes n}$ | 0.703125| 0.625   | 0.625   | 0.703125| –       | –       | –       | –       |
|     | [2]            | 0.4096  | 0.3456  | 0.3456  | 0.4096  | –       | –       | –       | –       |
|     | $D'\{0\}^{\otimes n}$ | 0.730278| 0.823495| 0.954987| 0.823495| 0.730278| –       | –       | –       |
|     | $f$            | 03      | 02      | 05      | 16      | –       | –       | –       | –       |
| 5   | $r$            | 1.48129 | 0.357282| 0.277975| 0.357282| 4.51871 | –       | –       | –       |
|     | $D_f\{0\}^{\otimes n}$ | 0.585938| 0.527344| 0.3125  | 0.527344| 0.585938| –       | –       | –       |
|     | [2]            | 0.401878| 0.329218| 0.3125  | 0.329218| 0.401878| –       | –       | –       |
|     | $D'\{0\}^{\otimes n}$ | 0.704306| 0.754753| 0.907588| 0.907588| 0.754753| 0.704306| –       | –       |
|     | $f$            | 07      | 60      | 05      | 0A      | 29      | –       | –       | –       |
| 6   | $r$            | 2.44507 | 5.93733 | 0.27984 | 0.27984 | 5.93733 | 2.44507 | –       | –       |
|     | $D_f\{0\}^{\otimes n}$ | 0.683594| 0.512695| 0.546875| 0.546875| 0.512695| 0.683594| –       | –       |
|     | [2]            | 0.396569| 0.318745| 0.293755| 0.293755| 0.318745| 0.396569| –       | –       |
|     | $D'\{0\}^{\otimes n}$ | 0.698181| 0.710643| 0.813922| 0.92625 | 0.813922| 0.710643| 0.698181| –       |
|     | $f$            | 3F      | C0      | BF      | A0      | AF      | AC      | AD      | –       |
| $n$ | $w \rightarrow$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   | $r$             | 5.51859 | 6.91248 | 7.69903 | 7.74472 | 7.69903 | 6.91248 | 5.51859 | –   |
|     | $\mathcal{D}_f | 0 \rangle \otimes_0 $ | 0.598145 | 0.553711 | 0.413574 | 0.273438 | 0.413574 | 0.553711 | 0.598145 | –   |
| [2] |                 | 0.392696 | 0.311462 | 0.281632 | 0.273438 | 0.281632 | 0.311462 | 0.392696 | –   |
|     | $\mathcal{D}'_f | 0 \rangle \otimes_0 $ | 0.684842 | 0.651002 | 0.76886 | 0.884277 | 0.884277 | 0.76886 | 0.651002 | 0.684842 |
|     | $f$             | 0F   | 180  | 0D   | 140  | 15F  | 6A  | 153  | 16A  |
| 9   | $r$             | 3.4566 | 7.86171 | 0.858163 | 8.7469 | 8.7469 | 0.858153 | 7.86171 | 3.4566 |
|     | $\mathcal{D}_f | 0 \rangle \otimes_0 $ | 0.672913 | 0.430664 | 0.415283 | 0.492188 | 0.492188 | 0.415283 | 0.430664 | 0.672913 |
| [2] |                 | 0.389744 | 0.306102 | 0.273129 | 0.260182 | 0.260182 | 0.273129 | 0.306102 | 0.389744 |
the optimal success probability to obtain $|D^n_w\rangle$. Given these data, the actual quantum algorithm to obtain Dicke states can be efficiently implemented.

5 Numerical comparison of three approaches

Now, we compare three approaches:

– Using a biased Hadamard operator as shown in [2],
– Algorithm 1 based on $D_f|0\rangle^\otimes n$, and
– Algorithm 2 based on $D'_f|0\rangle^\otimes n$.

The first two cases need $O(\sqrt{n})$ complexity, and the third one is a heuristic that shows improved results than the first two. Some numerical results of the probability associated with $|D^n_w\rangle$ are shown in Table 1 for $n = 4, \ldots, 9$. As shown in the table, we note that Algorithm 2 provides the highest probability than others. There are a few interesting observations from the enumeration results.

– We note that the improvements using $D'_f$ are highly significant at $w = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ and the significance reduces as $w$ moves away from the middle, i.e., toward $w = 1$ or $n - 1$.
– In the case of using $D'_f$, the success probabilities at $w$ and $n - w$ weights are the same for all the values of $w$, i.e., $w \geq 1$. However, the values of $r$ in those cases are the same at $w$ and $n - w$ weights for $w \geq 2$ only.
– Since the case $w = 1$ corresponds to $W$ state, it is interesting to check their properties. Since the number of basis states for $W$ state is $n$, and hence, the necessary complexity is supposed to be smaller than other cases, the success probability for this state is relatively higher than other states. For example, the success probabilities of [2] and Algorithm 1 for $n = 8$, $w = 1$ state are 0.392696 and 0.598145 which are higher than 0.273438 for $n = 8$, $w = 4$ state, respectively. However, as explained in the first item, Algorithm 2 shows different property which shows the smallest success probability with smaller or larger $w$.

6 Algorithm 3: The complete strategy using grover algorithm

Quadratic improvement by Grover’s algorithm [12] is achieved in several applications. We point out here how that can be exploited in our algorithm. Although we can construct the target Dicke state by measuring the intermediate quantum state, we may increase the efficiency further by using the amplitude amplification method. Based on this, an adiabatic evolution has been used toward amplitude amplification of the desired states in [2], but no complexity analysis was shown. In this work, instead, we apply the conventional Grover algorithm [12] as it provides a quadratic speed-up.

Instead of equal superposition $|\psi\rangle = H^\otimes n|0\rangle^\otimes n = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} |x\rangle$ in Grover algorithm, we will use the symmetric state of the form $|\psi\rangle = D_f(|0\rangle^\otimes n) = \sum_{x \in \{0,1\}^n} \frac{W_f(x)}{2^n} |x\rangle$ exploiting the properly chosen Boolean function $f(x)$, as explained in the previous sections.
Further, toward inverting the phase, we will use another symmetric Boolean function $g(x)$, different from $f(x)$, where $g(x) = 1$, when $wt(x) = w$, and $g(x) = 0$, otherwise. Based on $g(x)$, we implement the inversion operator as $O_g$ that inverts the phase of the states $|x\rangle$ where $x \in \{0, 1\}^n$ $wt(x) = w$. Thus, we consider the operator

$$G_t = \left[ (2|\Psi\rangle\langle\Psi| - I)O_g \right]^t$$

on $|\Psi\rangle$ to get $|\Psi_t\rangle$.

Consider the $n$-qubit state $|\Psi\rangle = \sum_{s \in S} u_s |s\rangle + \sum_{s \in \{0, 1\}^n \setminus S} v_s |s\rangle$, where $u_s$, $v_s$ are real and $\sum_{s \in \{0, 1\}^n \setminus S} v_s^2 = 1$. For brevity, let us represent $|\Psi\rangle = \sum_{s \in S} u_s |s\rangle + \sum_{s \in \{0, 1\}^n \setminus S} v_s |s\rangle = u |X\rangle + v |Y\rangle$. That is, $u^2 = \sum_{s \in S} u_s^2$ and $v^2 = \sum_{s \in \{0, 1\}^n \setminus S} v_s^2$.

Let $u = \sin \theta$, $v = \cos \theta$. It is easy to check that the application of $[(2|\Psi\rangle\langle\Psi| - I)O_g]^t$ operator on $|\Psi\rangle$ produces $|\Psi_t\rangle$, in which the probability amplitude of $|X\rangle$ is $\sin(2t + 1)\theta$.

We will now use such states $|\Psi_t\rangle$ in the parity measurement. Consider that after the Deutsch–Jozsa algorithm, we obtain a symmetric $n$-qubit state (before the measurement)

$$|S^n\rangle = \sum_{x \in \{0, 1\}^n} a_{wt(x)} |x\rangle,$$

such that $\frac{n!}{w!} |a_w|^2 = \frac{c}{\sqrt{n}}$, for some constant $c$. Thus, we have the initial amplitude of target states, $\{x \in \{0, 1\}^n$ $wt(x) = w\}$, as $\sin \theta = \frac{n!}{w!} |a_w| = \sqrt{\frac{c}{\sqrt{n}}}$. For large $n$, one can approximate it as $\theta = \frac{\sqrt{c}}{\sqrt{n}}$, and hence, we need $t$ iterations of Grover like strategy such that $(2t + 1)\theta \approx \frac{\pi}{2}$, i.e., $t \approx \frac{\pi \sqrt{n}}{2\sqrt{c}}$.

Here, we have a good (almost exact) estimate of $t$, which is not known priori for application in search algorithms. After the application of Grover like strategy, we will get another symmetric $n$-qubit state $|T^n\rangle = \sum_{x \in \{0, 1\}^n} a'_{wt(x)} |x\rangle$ such that $\frac{n!}{w!} |a'_w|^2$ will be very close to 1, and the parity measurement will produce $|D^n_w\rangle$ mostly in one

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**Algorithm 3** Augmented with Grover Search

1. Let $f \in SB_n$ be as explained in Theorem 1 to maximize the Walsh spectrum values at weight $w$.
2. Use any of the above three strategies (our strategies exploiting Hadamard or biased Hadamard transform or the strategy of [2]) to obtain a symmetric $n$-qubit state $|S^n\rangle = \sum_{x \in \{0, 1\}^n} a_{wt(x)} |x\rangle$, such that $\frac{n!}{w!} |a_w|^2$ is $\Omega(\frac{1}{\sqrt{n}})$.
3. Use $G_t$ on $|S^n\rangle$, $t$ many times, where $t$ is $O(\frac{\sqrt{n}}{c})$ to obtain $|T^n\rangle = \sum_{x \in \{0, 1\}^n} a'_w |x\rangle$ such that $\frac{n!}{w!} |a'_w|^2$ is very close to one.
4. Use parity measurement strategy. If the ancilla state is measured at the basis $U^w |\xi\rangle$, then $|D^n_w\rangle$ is successfully obtained. Else go to Step 2 and iterate.

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step with very high probability. Thus, the exact strategy is similar to Algorithm 1 (Algorithm 2 can be modified with a similar way) in the previous section, where we add one more step as shown in Algorithm 3.

In this algorithm, we need $O(\sqrt{n})$ steps using Grover algorithm in each run and then a parity measurement should provide $|D_n^w\rangle$. Thus, we get a quadratic speed-up (which is quite natural) over just using Deutsch–Jozsa algorithm. The number of parity measurement is $O(\sqrt{n})$ in the earlier case, once in each iteration. Here, it is only a very few (may be 1 in most of the cases).

7 Conclusion and open problems

In this work, we study several quantum algorithms to construct arbitrary Dicke state in a disciplined manner. The key idea is to find a suitable symmetric Boolean function for Deutsch–Jozsa algorithm for the given $n$ and $w$, to use the Grover algorithm and the generalized parity measurement strategy. Further, we show that it is possible to obtain better results using a biased Hadamard transform suitably. Our results improve the probabilities obtained in [2] and thus provide faster method to construct Dicke states. The open problem in this area is to characterize the enumeration results in case of modifying the Deutsch–Jozsa algorithm with a biased Hadamard transform. Obtaining the exact bias ($\frac{r}{n}$) in a biased Hadamard transform with the corresponding symmetric function to optimize the probability corresponding to the Dicke state seems to be an interesting problem.

Though we look at the problem from theoretical angle, the algorithmic blocks used by us have experienced major advancement toward actual implementation. One may refer to [27, Section 7] for literature related to implementation of quantum gates as well as Deutsch–Jozsa algorithm, Grover algorithm and several measurement techniques. As an example, the idea of implementing biased a Hadamard transform is related to the Fabry-Perotcavity [27, p. 299].

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