A derivation of Weyl gravity

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Abstract

In this paper, two things are done. (i) Using cohomological techniques, we explore the consistent deformations of linearized conformal gravity in 4 dimensions. We show that the only possibility involving no more than 4 derivatives of the metric (i.e., terms of the form $\partial^4 g_{\mu\nu}$, $\partial^3 g_{\mu\nu} \partial g_{\alpha\beta}$, $\partial^2 g_{\mu\nu} \partial^2 g_{\alpha\beta}$, $\partial^2 g_{\mu\nu} \partial g_{\alpha\beta} \partial g_{\rho\sigma}$ or $\partial g_{\mu\nu} \partial g_{\alpha\beta} \partial g_{\rho\sigma} \partial g_{\gamma\delta}$ with coefficients that involve undifferentiated metric components - or terms with less derivatives) is given by the Weyl action

$$\int d^4 x \sqrt{-g} W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta},$$

in much the same way as the Einstein-Hilbert action describes the only consistent manner to make a Pauli-Fierz massless spin-2 field self-interact with no more than 2 derivatives. No a priori requirement of invariance under diffeomorphisms is imposed: this follows automatically from consistency. (ii) We then turn to “multi-Weyl graviton” theories. We show the impossibility to introduce cross-interactions between the different types of Weyl gravitons if one requests that the action reduces, in the free limit, to a sum of linearized Weyl actions. However, if different free limits are authorized, cross-couplings become possible. An explicit example is given. We discuss also how the results extend to other spacetime dimensions.

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1 Introduction

1.1 From the linearized Weyl action to the full Weyl action

The discovery by Becchi, Rouet and Stora [1] and Tyutin [2] of the symmetry that now bears their names ("BRST symmetry") is a landmark in the development of local gauge field theories since it has brought in powerful algebraic methods which have shed a new and deeper light on the structures underlying renormalization and anomalies. The application of the BRST approach is, however, not confined to the quantum domain, since it is quite useful already classically. For instance, it can be used to analyse higher-order conservation laws for Yang-Mills gauge models and Einstein gravity [3, 4, 5]. It is also relevant to the problem of consistent interactions, where it provides a cohomological reformulation of the Noether method [6]. In this paper, we analyse the problem of consistent deformations of linearized Weyl gravity from the BRST standpoint.

There exist many ways to arrive at the Einstein equations [7]. One of them starts with the free action for a massless spin-2 field $h_{\mu\nu}^{PF}$ (Pauli-Fierz action [8])

$$S_{PF}[h_{\mu\nu}^{PF}] = \int d^{4}x \left[ -\frac{1}{2} \left( \partial_{\mu} h_{\nu\rho}^{PF} \right) \left( \partial^{\mu} h^{PF}_{\nu\rho} \right) + \left( \partial_{\mu} h_{\nu\rho}^{PF} \right) \left( \partial^{\mu} h^{PF}_{\nu\rho} \right) \right] .$$

(1.1)

and investigates the consistent manners to self-couple $h_{\mu\nu}^{PF}$. Under reasonable additional conditions, one can show that the only possibility is described by the Einstein-Hilbert action

$$S_{EH}^{PF}[g_{\mu\nu}] = \frac{2}{\kappa^{2}} \int d^{4}x \sqrt{-g} \left( R - 2\Lambda \right) , \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}^{PF}$$

(1.2)

where $R$ is the scalar curvature of the metric $g_{\mu\nu}$ and $g$ its determinant. In (1.2), the coupling constant $\kappa$ is of mass dimension $-1$. At the same time, the abelian gauge invariance of (1.2), namely,

$$\delta_{\xi} h_{\mu\nu}^{PF} = \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}$$

(1.3)

becomes elevated to diffeomorphism invariance,

$$\frac{1}{\kappa} \delta_{\xi} g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}.$$

(1.4)
This approach to the Einstein theory, which has a long history \[9, 10\], exhibits clearly the deep connection between massless spin-2 fields and diffeomorphism invariance.

In connection with the AdS/CFT correspondance, there has been renewed interest recently in Weyl gravity \[11\], described by the conformally invariant action

\[
S = \frac{1}{2\alpha^2} \int d^4x \sqrt{-g} W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta},
\]

\[\text{(1.5)}\]

where \(W_{\alpha\beta\gamma\delta}\) is the conformally invariant Weyl tensor,

\[
W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - 2 \left( \delta_{[\alpha} K_{\delta]\beta - g_{\beta[\gamma} K_{\delta]\alpha} \right)
\]

\[\text{(1.6)}\]

(for an other recent work on Weyl gravity, see also \[12\]). We assign weight 1 to symmetrized and antisymmetrized expressions. Here, \(K_{\alpha\beta}\) is defined through

\[
K_{\alpha\beta} = \frac{1}{n-2} \left( R_{\alpha\beta} - \frac{1}{2(n-2)} g_{\alpha\beta} R \right),
\]

where \(R_{\alpha\beta}\) and \(R = R_{\alpha\beta} g^{\alpha\beta}\) are the Riemann tensor, the Ricci tensor and the scalar curvature, respectively. This action leads to fourth order differential equations and involves no dimensional coupling constant. It possesses, together with its supersymmetric extensions, remarkable properties (see \[13\] for a review of conformal gravity and conformal supergravity).

In the linearized limit, \(\text{(1.5)}\) reduces to

\[
S_{0}[h_{\mu\nu}] = \frac{1}{2} \int d^4x W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta},
\]

\[\text{(1.7)}\]

for \(g_{\mu\nu} = \eta_{\mu\nu} + \alpha h_{\mu\nu}\), \(h_{\mu\nu}\) and \(\alpha\) being dimensionless. Here, \(W_{\alpha\beta\gamma\delta}\) is the linearized Weyl tensor constructed out of the \(h_{\mu\nu}\) according to formula \(\text{(3.5)}\) below.

The free action \(\text{(1.7)}\) is invariant under both linearized diffeomorphisms and the linearized version of the Weyl rescalings,

\[
\delta_{\eta,\phi} h_{\mu\nu} = \partial_{\mu} \eta_{\nu} + \partial_{\nu} \eta_{\mu} + 2 \phi \eta_{\mu\nu}.
\]

\[\text{(1.8)}\]

This paper investigates the extent to which the complete Weyl action \(\text{(1.5)}\) follows from the free action \(\text{(1.7)}\) and the requirement of consistent self-interactions. We show that \(\text{(1.5)}\) is in fact the only way \(\text{-under precise}

\[\text{Our conventions are as follows : the metric has signature } (-, +, \ldots, +), \text{ the Riemann tensor is defined by } R_{\alpha\beta\gamma\delta} = \partial_{\gamma} \Gamma_{\beta\delta}^{\alpha} + \Gamma_{\gamma\lambda}^{\alpha} \Gamma_{\beta\delta}^{\lambda} - (\gamma \leftrightarrow \delta) \text{ and the Ricci tensor is given by } R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}.\]
assumptions which will be spelled out in detail below- to make the fields $h_{\mu\nu}$ self-interact given the starting point (1.7). In the same manner, the gauge transformations

$$\frac{1}{\alpha} \delta_{\eta,\phi} g_{\mu\nu} = \eta_{\mu,\nu} + \eta_{\nu,\mu} + 2 \phi g_{\mu\nu}$$  \hfill (1.9)

constitute the only possibility to deform the abelian gauge symmetry (1.8) (in a way compatible with the existence of a variational principle that reduces to (1.7) in the free limit).

The conditions under which these conditions hold are

- the deformation is smooth in a formal deformation parameter $\alpha$ and reduces, in the free limit where $\alpha = 0$, to the original theory;
- the deformation preserves Lorentz invariance;
- the deformed action involves at most four derivatives of $h_{\mu\nu}$.

This latter condition is equivalent to the requirement that the coupling constants be dimensionless or of positive mass dimension. Note that since the field $h_{\mu\nu}$ is dimensionless, the space of dimension-4 polynomials in $h_{\mu\nu}$ and its derivatives is infinite-dimensional.

As announced, our approach is based on the BRST-antifield formalism [14, 15, 16, 17, 18, 19] where consistent deformations of the action appear as deformations of the solution of the master equation that preserve the master equation [6, 20, 21]. In that view, the first-order consistent deformations define cohomological class of the BRST differential at ghost number zero. We shall follow closely the procedure of [10], where BRST techniques were used in the context of the linearized Einstein theory. A crucial role is played in the calculation by the so-called invariant characteristic cohomology, which, just as in the Yang-Mills case [3, 4, 22] or in the Pauli-Fierz theory [10], controls the antifield-dependence of the BRST cocycles. We refer to [6, 20, 21] for background material on the BRST approach to the problem of consistent deformations. The application of BRST techniques to the conformal gravity context should also prove useful in the investigation of the cohomological questions raised in [23].

It should be stressed that the requirement of (background) Lorentz invariance is the only invariance requirement imposed on the interactions. There is no a priori condition of diffeomorphism invariance or Weyl invariance. These automatically follow from the consistency conditions. No condition
on the polynomial degree of the first order vertices or of the gauge transformation is imposed either. Finally, it is not required that the deformed gauge symmetries should close off-shell. This also follows automatically from the consistency requirement. In fact, the structure of the deformed gauge algebra is severely constrained even if one does not impose conditions on the number of derivatives, see section 6 below. Off-shell closure is automatic (to first-order in the deformation parameter) no matter how many derivatives one allows in the interactions. Furthermore, there are only three possible deformations of the gauge algebra, two of them involving more derivatives than (1.9) and being excluded if one imposes the third condition above.

1.2 Interactions for a collection of Weyl gravitons

Another remarkable feature of Einstein theory, besides the uniqueness of the graviton vertex, is that it involves a single type of gravitons (in contrast to Yang-Mills theory, which involves a collection of spin-1 carriers of the YM interactions). As discussed recently in [10], this is not an accident. Namely, the only consistent deformations of the free action

\[ S^P_F[h^a_{\mu\nu}] = \sum_{a=1}^{N} S^P_F[h^a_{\mu\nu}] \]  

(1.10)

for a collection of massless spin-2 fields, involving no more derivatives than the free action, cannot introduce true cross-couplings between the different types of gravitons: by a change of basis in the internal space of the gravitons, one can always remove any such cross-interactions.

One may wonder whether a similar property holds in the conformal case. We have thus also investigated the deformations of the free action

\[ S_0[h^a_{\mu\nu}] = \frac{1}{2} \sum_{a=1}^{N} \int d^4 x \mathcal{W}^{a}_{\alpha\beta\gamma\delta} \mathcal{W}^{a\alpha\beta\gamma\delta} \]  

(1.11)

describing a collection of free Weyl gravitons. We have found that there is again no unremovable cross-interactions that can be introduced among the Weyl gravitons. A key role is played in the derivation of this result by the requirement that the action should reduce to (1.11) in the free limit. If, instead of (1.11), one allows a free action in which the positive definite metric \( \delta_{ab} \) in the internal space of the Weyl gravitons is replaced by a metric \( k_{ab} \) with indefinite signature, then cross-interactions become possible. We give
an explicit example in the paper. In the case of ordinary gravitons, it is meaningful to request that all the gravitons should come with the same sign in the action (1.10) since otherwise, the energy would be unbounded from below. However, in the conformal case, the energy is in any case unbounded already for a single Weyl graviton so the legitimacy of this requirement does not appear to be as clear as in the standard case.

1.3 Outline of paper

The paper is organized as follows. In the next section, we set up our notations and formulate precisely the cohomological question. We then compute the cohomology of the differential $\gamma$ related to the gauge transformations (1.8) (section 3). In section 4 we apply standard cohomological results to compute various cohomologies in the particular case of linearized Weyl gravity. Section 5 is the core of our paper. We calculate the invariant characteristic cohomology $H^{inv}(\delta|d)$ in form degree $n$ and antifield number $\geq 2$. Our computation is quite general in that no restriction on the dimensionality of the cycles is imposed at this stage. The calculation of $H^{inv}(\delta|d)$ follows the general pattern developed previously for the Yang-Mills field [3, 4] or the massless spin-2 field [10], but present additional novel features related to the algebraic nature of the Weyl symmetry (no derivative of gauge parameters). The relevance of $H^{inv}(\delta|d)$ for the problem of consistent interactions appears clearly in section 6, where we derive all the consistent deformations fulfilling the above conditions of Lorentz invariance and dimensionality. We show that there is in fact a unique deformation, given by the $\mathcal{O}(\alpha)$-term in the action (1.11). Uniqueness to all orders is then easily established. The analysis is carried out up to this stage with a single symmetric field $h_{\mu\nu}$. In section 7, we discuss deformations for the action (1.11) involving many symmetric fields $h^a_{\mu\nu}$ and show that cross-interactions are impossible (in four dimensions) if one insists that the free action be indeed (1.11), but that cross-interactions can be introduced if one allows a free limit in which some of the Weyl gravitons are described by an action with the opposite sign. Section 7 is devoted to the conclusions and a brief discussion of the extension of our results to other spacetime dimensions $\geq 4$ as well as $n = 3$. The case $n = 2$ has been studied in [24, 25].
2 BRST-symmetry and conventions

2.1 Differentials $\delta$, $\gamma$ and $s$

We start the calculation in all spacetime dimensions $\geq 3$. The restriction $n = 4$ will be imposed only at the very end. Similarly, we do not impose Lorentz invariance or any condition on the derivative order. These will not be needed before section 6. By following the general prescriptions of the antifield formalism \[16, 19\], one finds that the spectrum of fields, ghosts and antifields, with their respective gradings, is given by

\[
\begin{align*}
Z & | \text{puregh}(Z) | \text{antigh}(Z) | \text{gh}(Z) \\
h_{\mu\nu} & | 0 | 0 | 0 \\
C_\mu & | 1 | 0 | 1 \\
\xi & | 1 | 0 | 1 \\
h^{*\mu\nu} & | 0 | 1 | -1 \\
C^{*\mu} & | 0 | 2 | -2 \\
\xi^* & | 0 | 2 | -2 \\
\end{align*}
\]

where $C_\mu$ are the ghosts for the diffeomorphisms and $\xi$ the ghost corresponding to the Weyl transformation. The variables $h^{*\mu\nu}$, $C^{*\mu}$ and $\xi^*$ are the antifields. The antighost number is also called the antifield number and we will use both terminologies here. The BRST-differential for the linear theory (1.7), (1.8) is given by

\[
s = \delta + \gamma
\]

where $\delta$ is the Koszul-Tate differential and $\gamma$ is the exterior derivative along the gauge orbits. Explicitly,

\[
\begin{align*}
\gamma h_{\mu\nu} &= 2 \partial_\mu (C_\nu) + 2 \eta_{\mu\nu} \xi; & \gamma \xi &= \gamma C_\nu = 0; \\
\gamma h^{*\mu\nu} &= 0 = \gamma C^{*\mu} = \gamma \xi^*; \\
\delta h_{\mu\nu} &= 0, & \delta C^{*\mu} &= -2 \partial_\nu h^{*\mu\nu}; & \delta \xi^* &= 2 \eta_{\mu\nu} h^{*\mu\nu}; \\
\delta h^{*\mu\nu} &= \frac{\delta \mathcal{L}_0}{\delta h_{\mu\nu}}. \\
\end{align*}
\]

where $\mathcal{L}_0$ is the free Lagrangian. In $n$ dimensions, the free Lagrangian contains $n$ derivatives of $h_{\mu\nu}$ (more on its structure in the conclusions). The
BRST-differential $s$ raises the ghost number and $\gamma$ raises the pureghost number by one unit, while $\delta$ decreases the antighost number by one unit.

Using the relations (2.3), (2.4) and the Noether identities for the Euler-Lagrange derivatives of $L_0$ (i.e., $\partial_\mu (\delta L_0 / \delta h_{\mu\nu}) = 0$, $\delta L_0 / \delta h_{\mu\nu} \eta_{\mu\nu} = 0$), together with the invariance of these Euler-Lagrange derivatives under (1.8), it is easy to check that
\[
\delta^2 = 0, \quad \gamma^2 = 0, \quad \delta \gamma + \gamma \delta = 0. \tag{2.5}
\]
Hence,
\[
s^2 = 0. \tag{2.6}
\]

### 2.2 Consistent deformations and cohomology

The solution $(0) W$ of the master equation for the free theory
\[
(W, W) = 0, \tag{2.7}
\]
is
\[
(0) W = S_0 + 2 \int d^nx h^{\mu\nu} (\partial_\nu C_\mu + \eta_{\mu\nu} \xi). \tag{2.8}
\]
The BRST differential $s$ is related to $(0) W$ through
\[
sA \equiv (0) (W, A). \tag{2.9}
\]
As it is well known, the solution of the master equation captures all the information about the gauge invariant action, the gauge symmetries and their algebra. The BRST approach to the problem of consistent interactions consists in deforming $(0) W$,
\[
W \rightarrow W = (0) W + \alpha (1) W + O(\alpha^2) \tag{2.10}
\]
in such a way that the master equation is fulfilled to each order in $\alpha$ \cite{3, 20, 21}
\[
(W, W) = 0. \tag{2.11}
\]
This guarantees gauge invariance by construction (this is what we mean by “consistency” in this paper: the deformed action is “consistent” if and only
if it is invariant under a set of gauge transformations which reduce to (1.9) in the free limit).

One then proceeds order by order in $\alpha$. The first-order deformations $(1) W$ must fulfill

$$\langle (0), (1) W, (1) W \rangle \equiv s (1) W = 0. \quad (2.12)$$

Trivial solutions of this BRST-cocycle condition, of the form $sK$ for some $K$, correspond to trivial deformations that can be undone by a change of variables. Thus, the non-trivial first-order deformations are described by the cohomology group $H^0(s, F)$ of the BRST differential in the space $F$ of local functionals, or, what is the same, the group $H^{0,n}(s | d)$ of the BRST differential acting in the space of local $n$-forms ($0 = \text{ghost number}, n = \text{form-degree}$). The local $n$-forms are by definition given by

$$\omega = f([h_{\mu\nu}], [C_{\mu}], [\xi], [h^{*\mu\nu}], [C^{*\mu}], [\xi^*]) \ dx^0 \wedge \cdots \wedge dx^{n-1} \quad (2.13)$$

where $f$ is a function of $h_{\mu\nu}, C_{\mu}, \xi, h^{*\mu\nu}, C^{*\mu}, \xi^*$ and their derivatives up to some finite (but unspecified) order (“local function”). This is what is meant by the notation $f([h_{\mu\nu}], [C_{\mu}], [\xi], [h^{*\mu\nu}], [C^{*\mu}], [\xi^*])$. Thus, a “local function” $f$ of $\phi$, $f = f([\phi])$, is a function of $\phi$ and a finite number of its derivatives. In fact, we shall assume that $f$ is polynomial in all the variables except, possibly, $h_{\mu\nu}$.

The knowledge of $H^0(s | d)$ requires the computation of the following cohomological groups: $H(\gamma), H(\gamma | d), H(\delta), H(\delta | d)$ and $H^{0,n}(\delta | d)$ (see [3, 4]). Our first task is therefore to compute these groups.

## 3 Cohomology of $\gamma$

In this section we calculate explicitly

$$H(\gamma) \equiv \frac{\text{Ker}(\gamma)}{\text{Im}(\gamma)}. \quad (3.1)$$

For that purpose, it is convenient to split $\gamma$ as the sum of the operator $\gamma_0$ associated with linearized diffeomorphisms plus the operator $\gamma_1$ associated with linearized Weyl transformations:

$$\gamma = \gamma_0 + \gamma_1. \quad (3.2)$$
The grading associated to this splitting is the number of ghosts $\xi$ and their derivatives ($\gamma_1$ increases this number by one unit, while $\gamma_0$ does not affect it).

### 3.1 Linearized conformal invariants

First, we recall a few known facts. Let $f([h_{\mu\nu}])$ be a local function of $h_{\mu\nu}$. This function is invariant under linearized diffeomorphisms if and only if it involves only the linearized Riemann tensor $R_{\alpha\beta\mu\nu}$ and its derivatives,

$$\gamma_0 f = 0 \Leftrightarrow f = f([R_{\alpha\beta\mu\nu}])$$  \hspace{1cm} (3.3)

with

$$R_{\alpha\beta\mu\nu} = -\frac{1}{2}(\partial_{\alpha\mu} h_{\beta\nu} + \partial_{\beta\nu} h_{\alpha\mu} - \partial_{\alpha\nu} h_{\beta\mu} - \partial_{\beta\mu} h_{\alpha\nu}).$$  \hspace{1cm} (3.4)

To discuss Weyl invariance, one introduces the linearized Weyl tensor $W_{\alpha\beta\mu\nu}$, given by

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{2}{n-2} (\eta_{\alpha}[\gamma R_{\delta]\beta - \eta_{\beta}[\gamma R_{\delta}\alpha]) + \frac{2}{(n-2)(n-1)} R \eta_{\alpha}[\gamma \eta_{\delta} \beta],$$  \hspace{1cm} (3.5)

where

$$R_{\mu\nu} = \frac{1}{2} \eta^{\alpha\lambda}(h_{\lambda\nu,\mu\alpha} - h_{\mu\nu,\lambda\alpha} - h_{\lambda\alpha,\mu\nu} + h_{\mu\alpha,\lambda\nu}),$$  \hspace{1cm} (3.6)

and

$$R = \eta^{\alpha\lambda} h_{\mu\alpha,\lambda\nu} - h_{\mu\nu,\alpha\lambda}. $$  \hspace{1cm} (3.7)

The symmetries of the Weyl tensor are $W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\delta\gamma} = W_{\gamma\delta\alpha\beta}$ as well as the cyclic identity $W_{\alpha[\beta\gamma\delta]} = 0$. All the traces of $W_{\alpha\beta\gamma\delta}$ vanish. One also introduces the linearization $K_{\alpha\beta}$ of the tensor $K_{\alpha\beta}$ defined in the introduction,

$$K_{\mu\nu} = \frac{1}{n-2} [R_{\mu\nu} - \frac{1}{2(n-1)} \eta_{\mu\nu} R].$$  \hspace{1cm} (3.8)

One has

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - 2(\eta_{\alpha}[\gamma K_{\delta]\beta - \eta_{\beta}[\gamma K_{\delta}\alpha].$$  \hspace{1cm} (3.9)

\(3\) The components of the linearized Riemann tensor and their derivatives are not independent because of the linearized Bianchi identities but this does not affect the argument - it simply implies that there are many ways to write the same function. One could choose a set of independent components and work exclusively with this set, but this will not be necessary for our purposes.
The linearized Weyl tensor $\mathcal{W}_{\alpha\beta\mu\nu}$ is annihilated by $\gamma_1$,
\[ \gamma_1 \mathcal{W}_{\alpha\beta\mu\nu} = 0, \quad (3.10) \]
while one has
\[ \gamma_1 \mathcal{K}_{\alpha\beta} = -\partial^2_{\alpha\beta} \xi. \quad (3.11) \]
The (linearized) Cotton tensor $\mathcal{C}_{\alpha\beta\mu}$ is obtained by taking the antisymmetrized derivatives of $\mathcal{K}_{\alpha\beta}$,
\[ \mathcal{C}_{\alpha\beta\mu} = 2\partial_{[\mu} \mathcal{K}_{\beta]\alpha} \quad (3.12) \]
and is clearly Weyl-invariant
\[ \gamma_1 \mathcal{C}_{\alpha\beta\mu} = 0. \quad (3.13) \]
The linearized Bianchi identities imply
\[ \partial_{\alpha} \mathcal{W}^{\alpha\beta\mu\nu} = (3 - n) \mathcal{C}_{\beta\mu\nu}. \quad (3.14) \]
In $n \geq 4$ spacetime dimensions, the Cotton tensor is therefore not independent from the derivatives of the Weyl tensor. Any local function of the Riemann tensor can be expressed as a local function of the Weyl tensor and the symmetrized derivatives of $\mathcal{K}_{\alpha\beta}$,$$
\begin{align*}
f ([R_{\alpha\beta\mu\nu}]) & \sim f ([\mathcal{W}_{\alpha\beta\mu\nu}], \mathcal{K}_{\alpha\beta}, \partial_{(\mu} \mathcal{K}_{\alpha\beta)}, \partial^2_{(\mu\nu} \mathcal{K}_{\alpha\beta)}, \cdots).}
(3.15)
\end{align*}
\]
Because of (3.11), it is invariant under (linearized) Weyl transformations if and only if it depends only on the Weyl tensor and its derivatives,
\[ \gamma_1 f ([R_{\alpha\beta\mu\nu}]) = 0 \iff f = f ([\mathcal{W}_{\alpha\beta\mu\nu}]) \quad (n \geq 4). \quad (3.16) \]
In three spacetime dimensions, the Weyl tensor identically vanishes. Any local function of the Riemann tensor can be expressed as a local function of the tensor $\mathcal{K}_{\alpha\beta}$, or, what is the same, as a local function of the Cotton tensor and the symmetrized derivatives of $\mathcal{K}_{\alpha\beta}$,$$
\begin{align*}
f ([R_{\alpha\beta\mu\nu}]) & \sim f ([\mathcal{C}_{\alpha\beta\mu}], \mathcal{K}_{\alpha\beta}, \partial_{(\mu} \mathcal{K}_{\alpha\beta)}, \partial^2_{(\mu\nu} \mathcal{K}_{\alpha\beta)}, \cdots).}
(3.17)
\end{align*}
\]
It is invariant under (linearized) Weyl transformations if and only if it depends only on the Cotton tensor and its derivatives,
\[ \gamma_1 f ([R_{\alpha\beta\mu\nu}]) = 0 \iff f = f ([\mathcal{C}_{\alpha\beta\mu}]) \quad (n = 3). \quad (3.18) \]
Note that the Cotton tensor is equivalent to its “dual” defined by $\mathcal{C}_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\rho} \varepsilon^{\rho\lambda\nu} \mathcal{C}_{\beta\mu\lambda}$ (in 3 dimensions). This invariant tensor $\mathcal{C}_{\alpha\beta} = \partial^{\mu}[(-)\varepsilon_{\mu\alpha\nu} \mathcal{K}^{\nu}]$ is symmetric by virtue of the Bianchi identity and clearly fulfills $\partial_{\alpha} \mathcal{C}^{\alpha\beta} = 0$ as well as $\mathcal{C}^{\alpha\beta} \eta_{\alpha\beta} = 0$. 

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3.2 The Bach tensor

Of particular importance in 4 spacetime dimensions is the (linearized) Bach tensor \[26\],

\[ B^{\alpha\beta} = (-2)\partial_\rho C^{\alpha\beta\rho} \]  

(3.19)

which is such that

\[ \frac{\delta L_0}{\delta h_{\alpha\beta}} = B^{\alpha\beta} \quad (n = 4). \]  

(3.20)

The linearized Bach tensor is symmetric (by virtue of the Bianchi identities),
gauge-invariant, Lorentz-covariant, traceless and divergence-free,

\[ B^{\alpha\beta} = B^{\beta\alpha}, \quad \gamma B^{\alpha\beta} = 0, \quad \partial_\alpha B^{\alpha\beta} = 0, \quad B^{\alpha}_{\alpha} = 0. \]  

(3.21)

These last two properties are direct consequence of the definition (3.19) where
we stress that \[C^{\alpha\beta\rho} = -C^{\alpha\rho\beta}, \quad C^{\alpha\beta\rho} \eta_{\alpha\beta} = 0.\] Actually, the Bach tensor is the only tensor containing four derivatives (or less) of \(h_{\alpha\beta}\) with these properties. Indeed, such a tensor must be obtained by contracting indices in \(\partial_\rho \partial_\sigma W^{\alpha\beta\gamma\delta}\), \(W^{\alpha\beta\gamma\delta} W^{\lambda\mu\nu\rho}\) (four derivatives), \(\partial_\rho W^{\alpha\beta\gamma\delta}\) (three derivatives) or \(W^{\alpha\beta\gamma\delta}\) (two derivatives). The last two possibilities are clearly excluded. There is only one independent way to contract indices in \(\partial_\rho \partial_\sigma W^{\alpha\beta\gamma\delta}\) and this produces a tensor proportional to \(B_{\alpha\gamma}\). Finally, the only possible contraction \(W^{\alpha\beta\gamma\delta} W^{\lambda\beta\gamma\delta}\) is ruled out because it fails to be traceless.

3.3 Computation of \(H(\gamma)\)

We want to find the most general solution of the cocycle condition

\[ \gamma a = 0 \]  

(3.22)

where \(a\) has a definite pure ghost number, say \(k\). Let us expand \(a\) with respect to the powers of \(\xi\) and its derivatives,

\[ a = a_0 + a_1 + a_2 + \ldots \]  

(3.23)

where \(a_0\) contains neither \(\xi\) nor its derivatives, \(a_1\) is linear in \(\xi\) or one of its derivatives, etc. The expansion stops at order \(k\) equal to the pure ghost number of \(a\) (or earlier). If one plugs this expansion into \(\gamma a = 0\) with \(\gamma = \gamma_0 + \gamma_1\), one gets at zeroth order \(\gamma_0 a_0 = 0\). This implies
$a_0 = P_\Delta([\mathcal{R}_{\mu\nu\alpha\beta}, [\Phi^*]] w^\Delta(C_\mu, \partial_\mu C_\nu) + \gamma_0 b_0$, where $\Phi^*$ denotes collectively all the antifields and where the $\{w^\Delta\}$ form a basis of polynomials in $C_\mu$, $\partial_\mu C_\nu$. By the trivial redefinition $a \rightarrow a - \gamma b_0$ we can set $a_0 = P_\Delta([\mathcal{R}_{\mu\nu\alpha\beta}, [\Phi^*]] w^\Delta(C_\mu, \partial_\mu C_\nu)$.

At first order, the cocycle condition reads $\gamma_1 a_0 + \gamma_0 a_1 = 0$, that is, $\gamma_1 P_\Delta([\mathcal{R}_{\mu\nu\alpha\beta}, [\Phi^*]] w^\Delta(C_\mu, \partial_\mu C_\nu) + \gamma_0 a_1 = 0$. But $\gamma_1 P_\Delta$ is a polynomial in $\mathcal{R}_{\mu\nu\alpha\beta}$, $\Phi^*$, $\xi$ and their derivatives, which cannot be $\gamma_0$-exact unless it vanishes $[10]$ ($\xi$ and its derivatives are $\gamma_0$-closed but not $\gamma_0$-exact). Thus, one gets $\gamma_1 P_\Delta = 0$, which implies $P_\Delta = P_\Delta([\mathcal{W}_{\mu\nu\alpha\beta}, [\Phi^*]) (n \geq 4)$ or $P_\Delta = P_\Delta([\mathcal{C}_{\mu\nu\alpha\beta}, [\Phi^*]) (n = 3)$. In the sequel, we shall assume for definiteness that $n \geq 4$. Thus $a_0 = P_\Delta([\mathcal{W}_{\mu\nu\alpha\beta}, [\Phi^*]) w^\Delta(C_\mu, \partial_\mu C_\nu)$. There remains $\gamma_0 a_1 = 0$, which implies $a_1 = Q_\Delta([\mathcal{R}_{\mu\nu\alpha\beta}, [\xi], [\Phi^*]) w^\Delta(C, \partial_\mu C_\nu) + \gamma_0 b_1$. We redefine $a \rightarrow a - \gamma b_0$ to eliminate the $\gamma_0 b_1$ term from $a_1$.

At second order $\gamma_1 a_1 + \gamma_0 a_2 = 0$ reads $\gamma_1 Q_\Delta([\mathcal{R}_{\mu\nu\alpha\beta}, [\xi], [\Phi^*]) w^\Delta(C, \partial_\mu C_\nu) + \gamma_0 a_2 = 0$. As before, $\gamma_1 Q_\Delta$ cannot be $\gamma_0$-exact unless it vanishes, thus $\gamma_1 Q_\Delta = 0$. The solve this, we note that the cohomology of $\gamma$ in the space of functions $f([\mathcal{R}_{\mu\nu\alpha\beta}, [\xi], [\Phi^*])$ is easily evaluated since the tensor $\mathcal{K}_{\alpha\beta}$ and its successive symmetrized derivatives form trivial pairs with the derivatives $\partial_\mu \cdots \partial_\mu \xi$, $\ell \geq 2$. Thus, only $\mathcal{W}_{\mu\nu\alpha\beta}$ and its derivatives, $\xi$ and $\partial_\mu \xi$ remain in cohomology. This implies that $Q_\Delta = Q_\Delta([\mathcal{W}_{\mu\nu\alpha\beta}, [\Phi^*], \xi, \partial_\mu \xi] + \gamma_1 K$ with some $K$ that involves the $h_{\mu\nu}$ only through the curvatures, i.e. is such that $\gamma_0 K = 0$. Therefore, $a_1$ reads $a_1 = Q_\Delta([\mathcal{W}_{\mu\nu\alpha\beta}, [\Phi^*]) w^\Delta(C_\mu, \partial_\mu C_\nu; \xi, \partial_\mu \xi) + \gamma_1 K$ and one can remove $\gamma_1 K$ by a trivial redefinition. Here, the $\{w^\Delta(C_\mu, \partial_\mu C_\nu; \xi, \partial_\mu \xi)\}$ form a basis of polynomials in $C_\mu$, $\partial_\mu C_\nu$, $\xi$ and $\partial_\mu \xi$. The condition for $a_2$ becomes then $\gamma_0 a_2 = 0$ and the procedure continues to all orders in powers of the Weyl ghost and its derivatives.

**Conclusion**: the cohomology of $\gamma$ is isomorphic to the space of functions of $\mathcal{W}_{\mu\nu\alpha\beta}$, $\Phi^*$, their derivatives, $C_\mu$, $\partial_\mu C_\nu$, $\xi$ and $\partial_\mu \xi$,

$$H(\gamma) \simeq \left\{ f([\mathcal{W}_{\mu\nu\alpha\beta}], [\Phi^*], \xi, \partial_\mu \xi, C_\mu, \partial_\mu C_\nu) \right\} \quad (n \geq 4). \quad (3.24)$$

\(^4\)The differential $\gamma_0$ is just the differential $\gamma$ of $[10]$, with the addition of the “$\gamma_0$-singlet” variables $\xi$, $\xi^*$ and their derivatives, which are annihilated by $\gamma_0$ and define new independent generators in the $\gamma_0$-cohomology. As shown in $[10]$, the $\gamma_0$-cocycle condition forces the cocycles to depend on $h_{\mu\nu}$ only through the linearized Riemann tensor and its derivatives, and eliminates all second and higher derivatives of $C_\mu$ as well as the symmetrized derivatives $\partial_\mu C_\nu + \partial_\nu C_\mu$ (up to trivial terms).
In three dimensions,

\[ H(\gamma) \simeq \left\{ f([C_{\alpha\beta\gamma}], [\Phi^*], \xi, \partial_\mu \xi, C_\mu, \partial_{[\mu} C_{\nu]} \right\} \quad (n = 3). \quad (3.25) \]

Note that the ghost derivatives \( \xi, \partial_\mu \xi, C_\mu \) and \( \partial_{[\mu} C_{\nu]} \) that survive in cohomology are in number equal to the number of infinitesimal transformations of the conformal group. In pure ghost number zero, the polynomials \( \alpha([W_{\mu\nu\alpha\beta}], [\Phi^*]) \) (or \( \alpha([C_{\mu\nu\alpha}], [\Phi^*]) \)) are called "invariant polynomials".

4 Standard material : \( H(\gamma|d), H(\delta), H(\delta|d) \)

4.1 General properties of \( H(\gamma|d) \)

Now that we know \( H(\gamma) \), we can consider \( H(\gamma|d) \), the space of equivalence classes of forms \( a \) such that \( \gamma a + db = 0 \), identified by the relation \( a \sim a' \Leftrightarrow a' = a + \gamma c + df \). We shall need properties of \( H(\gamma|d) \) in strictly positive antighost (= antifield) number. To that end, we first recall the following theorem on invariant polynomials (pure ghost number = 0):

**Theorem 4.1** In form degree less than \( n \) and in antifield number strictly greater than 0, the cohomology of \( d \) is trivial in the space of invariant polynomials.

**Proof**: This just follows from the standard algebraic Poincaré lemma in the sector of the antifields [27].

Theorem 4.1, which deals with \( d \)-closed invariant polynomials that involve no ghosts, has the following useful consequence on general mod-\( d \) \( \gamma \)-cocycles with \( \text{antigh} > 0 \).

**Consequence of Theorem 4.1**

If \( a \) has strictly positive antifield number (and involves possibly the ghosts), the equation

\[ \gamma a + db = 0 \quad (4.1) \]

is equivalent, up to trivial redefinitions, to

\[ \gamma a = 0. \quad (4.2) \]

That is, one can add \( d \)-exact terms to \( a \), \( a \rightarrow a' = a + dv \) so that \( \gamma a' = 0 \). Thus, in antighost number > 0, one can always choose representatives of \( H(\gamma|d) \) that are strictly annihilated by \( \gamma \).
This does not imply that $H(γ|d)$ and $H(γ)$ are isomorphic: although the cocycle conditions are equivalent, the coboundary conditions are different. However, this is all that we shall need about the equation $γa + db = 0$.

In order to prove that $γa + db = 0$ can be replaced by $γa = 0$ in strictly positive antifield number, we consider the descent associated with $γa + db = 0$: by using the properties $γ^2 = 0$, $γd + dγ = 0$ and the triviality of the cohomology of $d$ (algebraic Poincaré lemma), one infers that $γb + dc = 0$ for some $c$. Going on in the same way, we introduce the chain of equations $γc + de = 0$, $γe + df = 0$, etc, in which each successive equation has one less unit in form-degree. The descent ends with the last two equations $γm + dn = 0$, $γn = 0$ (the last equation is $γn = 0$ either because $n$ is a zero-form, or because one stops earlier with a $γ$-closed term).

Now, because $n$ is $γ$-closed, one has, up to trivial, irrelevant terms, $n = α_J ω^J$ where (i) the $α_J = a_J([W_αβγδ], [Φ^*])$ are invariant polynomials; and (ii) the $ω^J = ω^J(ξ, ∂_μξ, C_μ, ∂_μC_ν)$ form, as before, a basis of the space of polynomials in $ξ$, $∂_μξ$, $C_μ$, $∂_μC_ν$. The $dx^μ$’s are included in the $α_J$’s, i.e., the $ω^J$’s are 0-forms. Inserting this expression into the previous equation in the descent yields

$$d(α_J)ω^J ± α_J dω^J + γm = 0. \quad (4.3)$$

In order to analyse (4.3), we introduce a new differential $D$, whose action on $h_μ$, $h_μ^*$, $C_α^*$ and all their derivatives is the same as the action of $d$, but whose action on the ghosts occurring in $ω^J$ is given by:

$$DC_μ = dx^ρ C_{μ, ρ} - dx^ρ η_μ ξ$$
$$D(∂_ρ C_μ) = 2dx^ρ η_μ C_ρ \xi$$
$$Dξ = dx^ρ ∂_ρ \xi$$
$$D(∂_ρ ξ) = 0. \quad (4.4)$$

The operator $D$ coincides with $d$ up to $γ$-exact terms. It follows from the definitions that $Dω^J = A_I^J ω^J$ for some constant matrix $A_I^J$ linear in $dx^μ$.

One can rewrite (4.3) as

$$d(α_J)ω^J ± α_J Dω^J + γm' = 0 \quad (4.5)$$

which implies,

$$d(α_J)ω^J ± α_J Dω^J = 0 \quad (4.6)$$

since a term of the form $β_J ω^J$ (with $β_J$ invariant) is $γ$-exact if and only if it is zero. It is convenient to further split $D$ as the sum of an operator $D_0$ and
an operator $D_1$, by assigning a $D$-degree to the variables occurring in the \( \gamma \)-cocycles \( \alpha_I \omega^I \) as follows. Everything has $D$-degree zero, except the ghosts and their derivatives to which we assign:

\[
\begin{align*}
C_\mu &\rightarrow D\text{-degree } 0, \\
\partial_\mu C_\mu &\rightarrow D\text{-degree } 1, \\
\xi &\rightarrow D\text{-degree } 1, \\
\partial_\mu \xi &\rightarrow D\text{-degree } 2.
\end{align*}
\]

(4.7)

The $D$-degree is bounded because there is a finite number of $C_{[\mu,\nu]}$, $\xi$, $\partial_\mu \xi$, which are anticommuting. One has $D = D_0 + D_1$, where $D_0$ has the same action as $D$ on $h_{\mu\nu}$, $h^*_{\mu\nu}$, $C^*_\alpha$ and all their derivatives, and gives 0 when acting on the ghosts. $D_1$ gives 0 when acting on all the variables but the ghosts, on which it reproduces the action of $D$. $D_1$ raises the $D$-degree by one unit, while $D_0$ leaves it unchanged.

Let us expand (4.6) according to the $D$-degree. At lowest order, we get

\[
d\alpha_{J_0} = 0
\]

(4.8)

where $J_0$ labels the $\omega^J$ that contain zero derivatives of $C_\mu$, and no $\xi$ ($D\omega^J = D_1\omega^J$ contains at least one derivative of $C_\mu$ or one $\xi$). This equation implies, according to Theorem 4.1, that $\alpha_{J_0} = d\beta_{J_0} + \gamma$-exact terms, where $\beta_{J_0}$ is an invariant polynomial. Accordingly, one can write

\[
\begin{align*}
\alpha_{J_0} \omega^{J_0} &= (d\beta_{J_0}) \omega^{J_0} + \gamma\text{-exact terms} \\
&= d(\beta_{J_0} \omega^{J_0}) + \beta_{J_0} d\omega^{J_0} + \gamma\text{-exact terms} \\
&= d(\beta_{J_0} \omega^{J_0}) + \beta_{J_0} D_1 \omega^{J_0} + \gamma\text{-exact terms} \\
&= d(\beta_{J_0} \omega^{J_0}) + \beta_{J_0} A_{J_1} \omega^{J_1} + \gamma\text{-exact terms}.
\end{align*}
\]

(4.9)

As explicitly written, the term $\beta_{J_0} A_{J_1} \omega^{J_1}$ has $D$-degree equal to 1. Thus, by adding trivial terms to the last term $n$ in the descent, we can assume that $n$ contains no term of $D$-degree 0. One can then successively removes the terms of $D$-degree 1, $D$-degree 2, etc, until one gets that the last term $n$ in the descent vanishes. One then repeats the argument for $m$ and the previous terms in the descent until one reaches the conclusion $b = 0$, i.e., $\gamma a = 0$, as announced.
4.2 Characteristic cohomology $H(\delta|d)$

We now turn to the cohomological groups involving the Koszul-Tate differential $\delta$. A crucial aspect of the differential $\delta$ defined through

$$\delta h^*\alpha\beta = \frac{\delta L_0}{\delta h_{\alpha\beta}},$$

$$\delta \xi^* = 2\eta_{\mu\nu}h^{*\mu\nu}$$

and $\delta C^* = -2\partial_\beta h^{*\beta\alpha}$ is that it is related to the dynamics of the theory. This is obvious since $h^*_{\alpha\mu\nu}$ reproduces the Euler-Lagrange derivatives of the Lagrangian. In fact, one has the following important (and rather direct) results about the cohomology of $\delta$ [28, 19]

1. Any form of zero antifield number which is zero on-shell is $\delta$-exact;
2. $H^p_i(\delta) = 0$ for $i > 0$, where $i$ is the antifield number, in any form-degree $p$. [The antifield number is written as a lower index; the ghost number is not written because it is irrelevant here.]

We now consider $H(\delta|d)$ (known as the “characteristic cohomology” because of an isomorphism theorem established in [3] and not needed here). It has been shown in [29] that $H(\delta|d)$ is trivial in the space of forms with positive pure ghost number. Thus we need only $H(\delta|d)$ in the space of local forms that do not involve the ghosts, i.e., having $\text{puregh} = 0$. The following vanishing theorem on $H^n_p(\delta|d)$ can be proven:

**Theorem 4.2** The cohomology groups $H^n_p(\delta|d)$ vanish in antifield number strictly greater than 2,

$$H^n_p(\delta|d) = 0 \text{ for } p > 2.$$  \hspace{1cm} (4.10)

The proof of this theorem is given in [3] and follows from the fact that linearized conformal gravity is a linear, irreducible, gauge theory.

In antifield number two, the cohomology is given by the following theorem:

**Theorem 4.3** A complete set of representatives of $H^2_2(\delta|d)$ is given by the antifields $C^\mu$ conjugate to the diffeomorphism ghosts, i.e.,

$$\delta a^\mu_2 + da^{\mu-1}_1 = 0 \Rightarrow a^\mu_2 = \lambda_\mu C^\mu dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{n-1} + \delta b^\mu_3 + db^{\mu-1}_2$$  \hspace{1cm} (4.11)

where the $\lambda_\mu$ are constants.

**Proof of Theorem 4.3** Let $a$ be a solution of the cocycle condition for $H^2_2(\delta|d)$, written in dual notations,

$$\delta a + \partial_\mu V^\mu = 0.$$  \hspace{1cm} (4.12)
Without loss of generality, one can assume that $a$ is linear in the undifferentiated antifields, since the derivatives of $C^*\mu$ or $\xi^*$ can be removed by integrations by parts (which leaves one in the same cohomological class of $H^n_2(\delta|d)$). Thus,

$$a = \lambda_\mu C^*\mu + \lambda\xi^* + \mu$$

(4.13)

where $\mu$ is quadratic in the antifields $h^*\mu\nu$ and their derivatives, and where the $\lambda_\mu$ and $\lambda$ can be functions of $h_{\mu\nu}$ and their derivatives. Because $\delta\mu \approx 0$, the equation (4.12) implies the linearized conformal Killing equations for $\lambda_\mu$ and $\lambda$,

$$\partial_\nu\lambda_\mu + \partial_\mu\lambda_\nu - 2\eta_{\mu\nu}\lambda \approx 0.$$  

(4.14)

To solve this equation, we observe that it implies

$$\partial_\alpha\partial_\nu\lambda_\mu \approx \eta_{\mu\alpha}\partial_\nu\lambda + \eta_{\mu\nu}\partial_\alpha\lambda - \eta_{\alpha\nu}\partial_\mu\lambda,$$

(4.15)

and

$$\partial_\alpha\partial_\beta\lambda \approx 0.$$  

(4.16)

The second of these equations yields $\partial_\beta\lambda \approx b_\beta$ for some constant $b_\beta$ (since $\partial_\mu f \approx 0$ implies $f \approx C$, see [3]), from which one gets

$$\partial_\alpha\partial_\nu\lambda_\mu \approx \eta_{\mu\alpha}b_\nu + \eta_{\mu\nu}b_\alpha - \eta_{\alpha\nu}b_\mu.$$  

(4.17)

Defining

$$\lambda_\mu = \lambda'_\mu + b_\mu x^2 - 2x_\mu b_\nu x_\nu, \quad \lambda = \lambda' + b_\mu x^\mu,$$

(4.18)

we get for $\lambda'_m$ and $\lambda'$ the same conformal Killing equation (4.14) but now $\partial_\alpha\partial_\nu\lambda'_\mu \approx 0$ and $\partial_\alpha\lambda' \approx 0$. This last equation gives $\lambda' \approx f$ where $f$ is a constant. Redefining

$$\lambda'_\mu = \lambda''_\mu + fx_\mu, \quad \lambda' = \lambda'' + f,$$

(4.19)

yields the weak Killing equation for $\lambda''_\mu$

$$\partial_\nu\lambda''_\mu + \partial_\mu\lambda''_\nu \approx 0.$$  

(4.20)

together with $\lambda'' \approx 0$. The general solution of (4.20) is $\lambda''_\mu \approx a_\mu + \omega_\mu^\nu x_\nu$ with $\omega_\mu\nu = -\omega_\nu\mu$ (see [11]). Putting everything together, one gets

$$\lambda_\mu(x) \approx a_\mu + \omega_\mu^\nu x_\nu + fx_\mu + b_\mu x^2 - 2x_\mu b_\nu x_\nu, \quad \lambda \approx f + b_\mu x^\mu.$$  

(4.21)
i.e., $\lambda_\mu$ is on-shell equal to a conformal Killing vector (as expected). The parameters $\omega_{\mu\nu}$, $a_\mu$, $f$ and $b_\mu$ describe respectively infinitesimal Lorentz transformations, translations, dilations and so-called “special conformal transformations”.

We are interested in solutions of (4.14) that do not depend explicitly on $x^\mu$ (since we do not want the Lagrangian to have an explicit $x$-dependence). This forces $\omega_{\mu}^\nu = f = b_\mu = 0$ in (4.21) and gives

$$\lambda_\mu \approx a_\mu \quad \lambda \approx 0.$$  

Substituting this expression into (4.13), and noting that the term proportional to the equations of motion can be absorbed through a redefinition of $\mu$, one gets

$$a = \lambda_\mu C^{*\mu} + \mu'$$  

(up to trivial terms). Now, the first term in the right-hand side of (4.23) is a solution of $\delta a + \partial_\nu V^\nu = 0$ by itself. This means that $\mu'$, which is quadratic in the $h_\mu^{*\nu}$ and their derivatives, must be also a $\delta$-cocyle modulo $d$. But it is well known that all such cocycles are trivial [3]. Thus, $a$ is given by

$$a = \lambda_\mu C^{*\mu} + \text{trivial terms}$$  

where $\lambda_\mu$ are constants, as we claimed. This proves the theorem.

**Comments**

(i) The above theorems provide a complete description of $H^1_{\delta}(\delta|n)$ for $k > 1$. These groups are zero ($k > 2$) or finite-dimensional ($k = 2$). In contrast, the group $H^1_{\delta}(\delta|d)$, which is related to ordinary conserved currents, is infinite-dimensional since the theory is free. To our knowledge, it has not been completely computed. Fortunately, we shall not need it below. (ii) In a recent interesting paper, deformations involving all conformal Killing vectors have been investigated [30].

5 **Invariant cohomology of $\delta$ modulo $d$.**

5.1 **Central theorem**

We now establish the crucial result that underlies all our discussion of consistent interactions for linear Weyl gravity. This result concerns the invariant cohomology of $\delta$ modulo $d$. The group $H^{inv}_{\delta}(\delta|d)$ is important because it
controls the obstructions to removing the antifields from a $s$-cocycle modulo $d$, as we shall see below.

Throughout this section, there will be no ghost; i.e., the objects that appear involve only the fields, the antifields and their derivatives. The central result that gives $H^{\text{inv}}(\delta|d)$ in antighost number $\geq 2$ is

**Theorem 5.1** Assume that the invariant polynomial $a_k^p$ ($p = \text{form-degree, } k = \text{antifield number}$) is $\delta$-trivial modulo $d$,

$$a_k^p = \delta \mu_{k+1}^p + d \mu_{k-1}^p \quad (k \geq 2). \quad (5.1)$$

Then, one can always choose $\mu_{k+1}^p$ and $\mu_{k-1}^p$ to be invariant.

Hence, we have $H_k^{n,\text{inv}}(\delta|d) = 0$ for $k > 2$ while $H_k^{2,\text{inv}}(\delta|d)$ is given by Theorem 4.3.

### 5.2 Useful lemmas

To prove the theorem, we need the following three lemmas:

**Lemma 5.1** If $a$ is an invariant polynomial that is $\delta$-exact in the space of all polynomials, $a = \delta b$, then, $a$ is also $\delta$-exact in the space of invariant polynomials. That is, one can take $b$ to be invariant.

**Demonstration of the lemma:** Any function $f([h], [h^*], [C^*], [\xi^*])$ can be viewed as a function $f(\tilde{h}, [W], [h^*], [C^*], [\xi^*])$ ($[C]$ instead of $W$ for $n = 3$). Here, the $\tilde{h}$ denote a complete set of (non-invariant) derivatives of $h_{\mu\nu}$ complementary to $[W]$, i.e., $\{\tilde{h}\} = \{h_{\mu\nu}, \partial_\rho h_{\mu\nu}, \ldots\}$, where the $\ldots$ denotes combinations of derivatives independent from $[W]$ (or $[C]$ for $n = 3$). The $[W_{\alpha\beta\gamma\delta}]$’s are not independent because of the linearized Bianchi identities, but this does not affect the argument. An invariant function is just a function that does not involve $\tilde{h}$, so one has (if $f$ is invariant), $f = f_{|\tilde{h}=0}$. Now, the differential $\delta$ commutes with the operation of setting $\tilde{h}$ to zero. So, if $a = \delta b$ and $a$ is invariant, one has $a = a_{|\tilde{h}=0} = (\delta b)_{|\tilde{h}=0} = \delta(b_{|\tilde{h}=0})$, which proves the lemma since $b_{|\tilde{h}=0}$ is invariant. \(\diamondsuit\)

To explain and establish the second lemma, we first derive a chain of equations with the same structure as (5.1) [4]. Acting with $d$ on (5.1), we get

$$da_k^p = -\delta \mu_{k+1}^p.$$

Using the lemma and the fact that $da_k^p$ is invariant, we can also write $da_k^p = -\delta a_{k+1}^{p+1}$ with $a_{k+1}^{p+1}$ invariant. Substituting this in
\[ da_k^p = -\delta d\mu_{k+1}^p, \] we get \[ \delta \left[ a_{k+1}^{p+1} - d\mu_{k+1}^p \right] = 0. \] As \( H(\delta) \) is trivial in antifield number \( > 0 \), this yields

\[ a_{k+1}^{p+1} = \delta \mu_{k+2}^{p+1} + d\mu_{k+1}^p \]  

(5.2)

which has the same structure as (5.1). We can then repeat the same operations, until we reach form-degree \( n \),

\[ a_{k+n-p}^n = \delta \mu_{k+n-p+1}^n + d\mu_{k+n-p-1}^{n-1}. \]  

(5.3)

Similarly, one can go down in form-degree. Acting with \( \delta \) on (5.1), one gets \( \delta a_k^p = -d(\delta \mu_k^{p-1}) \). If the antifield number \( k - 1 \) of \( \delta a_k^p \) is greater than or equal to one (i.e., \( k > 1 \)), one can rewrite, thanks to Theorem 4.1, \( \delta a_k^p = -da_{k-1}^{p-1} \) where \( a_{k-1}^{p-1} \) is invariant. (If \( k = 1 \) we cannot go down and the bottom of the chain is (5.1) with \( a_1^0 = \delta \mu_2^1 + d\mu_1^{p-1} \).) Consequently

\[ d \left[ a_{k-1}^{p-1} - \delta \mu_k^{p-1} \right] = 0 \]

and, as before, we deduce another equation similar to (5.1):

\[ a_{k-1}^{p-1} = \delta \mu_k^{p-1} + d\mu_{k-1}^{p-1}. \]  

(5.4)

Applying \( \delta \) on this equation the descent continues. This descent stops at form degree zero or antifield number one, whichever is reached first, i.e.,

\[ a_{k-p}^0 = \delta \mu_{k-p+1}^0 \]

\[ a_{1-k+1}^1 = \delta \mu_{2-k+1}^1 + d\mu_1^{p-k}. \]  

(5.5)

Putting all these observations together we can write the entire descent as

\[ a_{k+n-p}^n = \delta \mu_{k+n-p+1}^n + d\mu_{k+n-p}^{n-1} \]

\[ a_{k+1}^{p+1} = \delta \mu_{k+2}^{p+1} + d\mu_{k+1}^p \]

\[ a_k^p = \delta \mu_{k+1}^p + d\mu_{k-1}^{p-1} \]

\[ a_{k-1}^{p-1} = \delta \mu_k^{p-1} + d\mu_{k-2}^{p-2} \]

\[ \vdots \]

\[ a_{k-p}^0 = \delta \mu_{k-p+1}^0 \]

\[ a_{1-k+1}^1 = \delta \mu_{2-k+1}^1 + d\mu_1^{p-k} \]  

(5.6)

where all the \( a_{k \pm i}^{p \pm i} \) are invariants.
Lemma 5.2 If one of the $\mu$’s in the chain (5.6) is invariant, one can choose all the other $\mu$’s in such a way that they share this property.

Demonstration: The proof was given in [4] but we repeat it here for completeness. Let us thus assume that $\mu^{c-1}_{b}$ is invariant. This $\mu^{c-1}_{b}$ appears in two equations of the descent:

\begin{align*}
a^c_{b} &= \delta \mu^{c+1}_{b+1} + d \mu^{c-1}_{b}, \\
a^{c-1}_{b-1} &= \delta \mu^{c-2}_{b-1} + d \mu^{c-1}_{b-1}
\end{align*} (5.7)

(if we are at the bottom or at the top, $\mu^{c-1}_{b}$ occurs in only one equation, and one should just proceed from that one). The first equation tells us that $\delta \mu^{c+1}_{b+1}$ is invariant. Thanks to Lemma 5.1 we can choose $\mu^{c+1}_{b+1}$ to be invariant. Looking at the second equation, we see that $d \mu^{c-2}_{b-1}$ is invariant and by virtue of theorem 4.1, $\mu^{c-2}_{b-1}$ can be chosen to be invariant since $b-1 \geq 1$. These two $\mu$’s appear each one in two different equations of the chain, where we can apply the same reasoning. The invariance property propagates then to all the $\mu$’s. ♦

The third lemma is:

Lemma 5.3 If $a^a_k$ is of antifield number $k > n$, then the $\mu$’s in (5.1) can be taken to be invariant.

Demonstration: Indeed, if $k > n$, the last equation of the descent is $a^k_{k-n} = \delta \mu^0_{k-n+1}$. We can, using Lemma 5.1, choose $\mu^0_{k-n+1}$ invariant, and so, all the $\mu$’s can be chosen to have the same property. ♦

5.3 Demonstration of Theorem 5.1

From we have just shown, it is sufficient to demonstrate Theorem 5.1 in form degree $n$ and in the case where the antifield number satisfies $k \leq n$. Rewriting the top equation (i.e. (5.1) with $p = n$) in dual notation, we have

\[ a_k = \delta b_{k+1} + \partial \rho j^0_k, \quad (k \geq 2). \] (5.8)

We will work by induction on the antifield number, showing that if the property is true for antighost numbers $\geq k + 1$ (with $k > 0$), then it is true for $k$. As we already know that it is true in the case $k > n$, the theorem will be demonstrated.
The idea of the proof is to reconstruct $a_k$ from its Euler-Lagrange (E.L.) derivatives using the homotopy formula

$$a_k = \int_0^1 dt [\delta^L a_k(t) C^* + \delta^L a_k(t) \xi^* + \delta^L a_k(t) h^{*\mu\nu} + \delta^L a_k(t) h_{\mu\nu}] + \partial_\mu V^\mu,$$

(5.9)

and to control these E.L. derivatives from (5.8).

### 5.3.1 Euler-Lagrange derivatives of $a_k$

Let us take the E.L. derivatives of (5.8). A direct calculation yields

$$\frac{\delta^L a_k}{\delta C^*} = \delta Z_{k-1|\alpha},$$

(5.10)

with $Z_{k-1|\alpha} = \frac{\delta^L b_{k+1}}{\delta C^*}$. Similarly,

$$\frac{\delta^L a_k}{\delta \xi^*} = \delta Z_{k-1},$$

(5.11)

with $Z_{k-1} = \frac{\delta^L b_{k+1}}{\delta \xi^*}$.

For the E.L. derivatives with respect to $h^{*\mu\nu}$ we obtain

$$\frac{\delta^L a_k}{\delta h^{*\mu\nu}} = -\delta X_{k|\mu\nu} + 2\partial_\mu (Z_k)_|\nu + 2\eta_{\mu\nu} Z_{k-1}.$$

(5.12)

with $X_{k|\mu\nu} = \frac{\delta b_{k+1}}{\delta h^{*\mu\nu}}$. Finally, let us compute the E.L. derivatives of $a_k$ with respect to the fields. We get :

$$\frac{\delta^L a_k}{\delta h_{\mu\nu}} = \delta Y_{k+1}^{\mu\nu} + D^{\mu\nu\rho\sigma} X_{k|\rho\sigma},$$

(5.13)

where $Y_{k+1}^{\mu\nu} = \frac{\delta b_{k+1}}{\delta h_{\mu\nu}}$ and where $D^{\alpha\beta\rho\sigma}$ is the differential operator appearing in the equations of motion,

$$\frac{\delta L_0}{\delta h_{\rho\sigma}} = D^{\rho\sigma\alpha\beta} h_{\alpha\beta}.$$ 

(5.14)

One has

$$D^{\mu\nu\rho\sigma} X_{k|\rho\sigma} = \partial_\alpha B_{k}^{\alpha\mu
\nu}$$

(5.15)
for some $B^{\alpha\mu\nu}_k$ such that $B^{\alpha\mu\nu}_k = -B^{\mu\alpha\nu}_k$ and $\eta_{\mu\nu}B^{\alpha\mu\nu}_k = 0$.

On the other hand, since $a_k$ is an invariant object, it depends only on the linearized Weyl tensor (Cotton tensor for $n = 3$) and its derivatives. Using the chain rule, one can thus also rewrite the E.L. derivatives of $a_k$ with respect to the fields as:

$$\frac{\delta L}{\delta h_{\mu\nu}} = \partial_{\alpha} U^{\alpha\mu\nu}_k,$$

for some invariant $U^{\alpha\mu\nu}_k$ such that $U^{\alpha\mu\nu}_k = -U^{\mu\alpha\nu}_k$ and $\eta_{\mu\nu}U^{\alpha\mu\nu}_k = 0$.

Because the E.L. derivatives of $a_k$ are invariant, one can replace, thanks to lemma 5.1, the quantities $Z_{k-1|\alpha}$, $Z_{k-1}$, $X_{k|\mu\nu}$ and $Y_{k+1}^{\mu\nu}$ appearing in Eqs. (5.10), (5.11), (5.12) and (5.13) by invariant quantities, i.e., rewrite Eqs. (5.10), (5.11), (5.12) and (5.13) as:

$$\frac{\delta L}{\delta C^{\kappa\alpha}} = \delta Z_{k-1|\alpha}$$

$$\frac{\delta L}{\delta \xi} = \delta Z_{k-1}$$

$$\frac{\delta L}{\delta h_{\mu\nu}} = -\delta X_{k|\mu\nu} + 2\partial_{(\mu}Z_{\nu)|k-1} + 2\eta_{\mu\nu}Z_{k-1}$$

$$\frac{\delta L}{\delta h_{\mu\nu}} = \delta Y_{k+1}^{\mu\nu} + \mathcal{D}^{\mu\nu\rho\sigma}X_{k|\rho\sigma}$$

for some $Z_{k-1|\alpha}$, $Z_{k-1}$, $X_{k|\mu\nu}$ and $Y_{k+1}^{\mu\nu}$ that are invariant (and not necessarily equal to the E.L. derivatives of $b_{k+1}$ any more). Without loss of generality, one can assume that $Y_{k+1}^{\mu\nu}$ is traceless because one has $\eta_{\mu\nu}\delta Y_{k+1}^{\mu\nu} = \delta(\eta_{\mu\nu}Y_{k+1}^{\mu\nu}) = 0$ from (5.15), (5.16) and (5.20) so, by the triviality of $H(\delta)$, we can write $\eta_{\mu\nu}Y_{k+1}^{\mu\nu} = \delta \psi_{k+1}$ where $\psi_{k+1}$ is invariant by lemma 5.1. Thus one can redefine in (5.20) $Y_{k+1}^{\mu\nu} \rightarrow Y_{k+1}^{\mu\nu} - (1/n)\eta_{\mu\nu}\delta \psi_{k+1}$ if necessary.

### 5.3.2 Analysis of $Y_{k+1}^{\mu\nu}$

Now, the tensor $Y_{k+1}^{\mu\nu}$ is not only invariant, but, in view of (5.20), (5.15), (5.16) and the structure of $\mathcal{D}^{\mu\nu\rho\sigma}$, it fulfills also

$$\delta Y_{k+1}^{\mu\nu} = \partial_{\alpha} M_k^{\alpha\mu\nu}$$
for some invariant \( M_k^{\alpha \mu} \) such that \( M_k^{\alpha \mu} \) is antisymmetric in \( \alpha, \mu \) and \( \eta_{\mu \nu} M_k^{\alpha \mu} = 0 \). This severely constrains its form.

To see this, we first note that the equation (5.21) tells us that for fixed \( \nu \), \( Y_{k+1}^{\nu} \) is a \( \delta \)-cocycle modulo \( d \), in form degree \( n-1 \) and antifield number \( k+1 \). \( Y_{k+1}^{\nu} \) is thus \( \delta \)-exact modulo \( d \) \( (H_{k+1}^{1,1}(\delta[d]) \simeq H_{k+1}^{n}(\delta[d]) \simeq 0 \) because \( k > 0 \)), \( Y_{k+1}^{\nu} = \delta A_{k+2}^{\nu} + \partial_{\mu} T_{k+1}^{\mu \nu \nu} \) where \( T_{k+1}^{\mu \nu \nu} \) is antisymmetric in \( \rho \) and \( \mu \). By the induction hypothesis, which reads \( H_{k+2}^{1}(\delta[d]) \simeq 0 \) for all \( k \) and thus in particular, \( H_{k+2}^{1}(\delta[d]) \simeq 0 \), \( A_{k+2}^{\nu} \) and \( T_{k+1}^{\mu \nu \nu} \) can be assumed to be invariant. Since \( Y_{k+1}^{\nu} \) is symmetric in \( \mu \) and \( \nu \), we have also \( \delta A_{k+2}^{\nu} + \partial_{\mu} T_{k+1}^{\mu \nu \nu} = 0 \). The triviality of \( H_{k+2}^{1}(\delta[d]) \) and the theorem 8.2 of [4] implies again that \( A_{k+2}^{\nu} \) and \( T_{k+1}^{\mu \nu \nu} \) are trivial, in particular, \( T_{k+1}^{\mu \nu \nu} = \delta Q_{k+2}^{\mu \nu} + \partial_{\alpha} S^{\alpha \mu \nu \nu} \), where \( S^{\alpha \mu \nu \nu} \) is antisymmetric in \( (\alpha, \rho) \) and in \( (\mu, \nu) \), respectively. The induction assumption allows us to choose \( Q^{\mu \nu} \) and \( S^{\alpha \mu \nu \nu} \) to be invariant. Writing \( E_{k+1}^{\alpha \beta \nu \nu} = -[S_{k+1}^{\alpha \mu \nu \nu} + S^{\nu \beta \alpha \nu}] \) and computing \( \partial_{\alpha \beta} E_{k+1}^{\alpha \beta \nu \nu} \), we get finally that

\[
Y_{k+1}^{\nu} = \delta F_{k+2}^{\nu} + \partial_{\alpha} E_{k+1}^{\alpha \beta \nu \nu}
\]

for some invariant \( F_{k+2}^{\nu} \). By construction, \( E_{k+1}^{\alpha \beta \nu \nu} \) is invariant and has the symmetries \( E_{k+1}^{\alpha \beta \nu \nu} = E_{k+1}^{\nu \beta \alpha \nu} \), \( E_{k+1}^{\alpha \beta \nu \nu} = E_{k+1}^{\beta \alpha \nu \nu} \) and \( E_{k+1}^{\alpha \beta \nu \nu} = E_{k+1}^{\nu \alpha \beta \nu} \).

The tracelessness of \( Y_{k+1}^{\nu} \) implies \( 0 = \delta F_{k+2} + \partial_{\alpha} E_{k+1}^{\alpha \beta \nu \nu} \) where we have set \( E_{k+1}^{\alpha \beta} = \eta_{\mu \nu} E_{k+1}^{\alpha \beta \mu \nu} \) and \( F_{k+2} = F_{k+2}^{\nu} \). This implies in turn \( F_{k+2} = \delta K_{k+3} + \partial_{\alpha} V_{k+2}^{\alpha} \) and \( \partial_{\beta} E_{k+1}^{\alpha \beta \nu \nu} = -\delta V_{k+2}^{\alpha} + \partial_{\beta} P_{k+1}^{\alpha \beta} \) for some \( K_{k+3}, V_{k+2}^{\alpha}, P_{k+1}^{\alpha \beta} = -P_{k+1}^{\beta \alpha} \) that are invariant. This follows again from our induction hypothesis. The last condition on \( E_{k+1}^{\alpha \beta} \) can be rewritten \( \partial_{\beta}(E_{k+1}^{\alpha \beta} - P_{k+1}^{\alpha \beta}) + \delta U_{k+2}^{\alpha} = 0 \) from which one infers, using again \( H_{k+2}^{1}(\delta[d]) \simeq 0 \), \( E_{k+1}^{\alpha \beta} - P_{k+1}^{\alpha \beta} = \partial_{\rho} Q_{k+1}^{\rho \alpha \beta} + \delta \Phi^{\alpha \beta}_{k+2} \) for some invariants with \( Q_{k+1}^{\rho \alpha \beta} = -Q_{k+1}^{\rho \beta \alpha} \). Taking the symmetric part of this equation in \( \alpha, \beta \) \( (E_{k+1}^{\alpha \beta} \) is symmetric, \( P_{k+1}^{\alpha \beta} \) is antisymmetric) gives

\[
E_{k+1}^{\alpha \beta} = \frac{1}{2} \partial_{\rho}(Q_{k+1}^{\rho \alpha \beta} + Q_{k+1}^{\rho \beta \alpha}) + \delta \Phi^{\alpha \beta}_{k+2}
\]

with \( \Phi^{\alpha \beta}_{k+2} = \Psi^{\beta \alpha}_{k+2} \).

5.3.3 Finishing the proof

We can now complete the argument. Using the homotopy formula (5.9) as well as the expressions (5.17), (5.18), (5.19), (5.20) for these E.L. derivatives,
we get
\[ a_k = \delta\left( \int_0^1 [Z'_{k-1}|_\alpha C^{*\alpha} + Z'_{k-1}|_\xi + X'_k h^{*\alpha\beta} + Y''_{k+1}|_\mu\nu] dt \right) + \partial_p \kappa^p. \] (5.24)

The first three terms in the argument of \( \delta \) are manifestly invariant. To handle the fourth term, we use (5.22). The \( \delta \)-exact term disappears (\( \delta^2 = 0 \)). The other one yields, after integrating by parts twice, a term of the form
\[ \delta\left[ \int_0^1 dt E^\alpha_{\mu\nu} \mathcal{R}_{\mu\nu\beta} \right] = \delta\left[ \int_0^1 dt (E^\alpha_{\mu\nu} \mathcal{W}_{\mu\nu\beta} + 4E^\alpha_{k+1} \mathcal{K}_{\alpha\beta}) \right] \] (see (3.9)). The first term under the integral is invariant. The last step consists in using (5.23) to transform \( E_{k+1}^\alpha \mathcal{K}_{\alpha\beta} \) into an acceptable form
\[ E_{k+1}^\alpha \mathcal{K}_{\alpha\beta} \sim -Q^\alpha_{k+1} \partial_\rho \mathcal{K}_{\alpha\beta} \sim -Q^\alpha_{k+1} \partial_\rho \mathcal{K}_{\alpha\beta}, \] (5.25)
which shows that this term is also invariant, because the antisymmetrized derivatives of \( \mathcal{K}_{\alpha\beta} \) are proportional to the linearized Cotton tensor.

This proves the theorem.

6 First-order consistent interactions : \( H(s|d) \)

We have now developed all the necessary tools for the study of the cohomology of \( s \) modulo \( d \) in form degree \( n \). A cocycle of \( H^{0,n}(s|d) \) must obey
\[ sa + db = 0. \] (6.1)
Furthermore, \( a \) must be of form degree \( n \) and of ghost number 0. To analyse (6.1), we expand \( a \) and \( b \) according to the antifield number, \( a = a_0 + a_1 + ... + a_k \), \( b = b_0 + b_1 + ... + b_k \), where, as shown in [4] and explicitly proved in appendix A, the expansion stops at some finite antifield number. We recall [20] (i) that the antifield-independent piece \( a_0 \) is the deformation of the Lagrangian; (ii) that \( a_1 \), which is linear in the antifields \( h^{*\mu\nu} \), contains the information about the deformation of the gauge symmetries, given by the coefficients of \( h^{*\mu\nu} \); (iii) that \( a_2 \) contains the information about the deformation of the gauge algebra (the term \( C^* A^B \) with \( C^* \equiv C^{*\mu}, \xi^* \) and \( C^A \equiv C^\mu, \xi \)) gives the deformation of the structure functions appearing in the commutator of two gauge transformations, while the term \( h^* h^* CC \) gives the on-shell terms); and (iv) that the \( a_k \) (\( k > 2 \)) give the information about the deformation of the higher order structure functions, which appear only when the algebra does not close off-shell.
Writing \( s \) as the sum of \( \gamma \) and \( \delta \), the equation \( sa + db = 0 \) is equivalent to the system of equations \( \delta a_i + \gamma a_{i-1} + db_{i-1} = 0 \) for \( i = 1, \cdots, k \), and \( \gamma a_k + db_k = 0 \).

6.1 Deformation of gauge algebra

Let us assume \( k \geq 2 \) (the cases \( k < 2 \) will be discussed below). Then, using the consequence of Theorem 4.1, one may redefine \( a_k \) and \( b_k \) so that \( b_k = 0 \), i.e., \( \gamma a_k = 0 \). Then, \( a_k = \alpha_J \omega^J \) (up to trivial terms), where the \( \alpha_J \) are invariant polynomials and where the \( \{ \omega^J \} \) form a basis of polynomials in \( \xi, \partial \mu \xi, C_\mu, \partial \mu C_\nu \). Acting with \( \gamma \) on the second to last equation and using \( \gamma^2 = 0 \), \( \gamma a_k = 0 \), we get \( d\gamma b_{k-1} = 0 \) i.e. \( \gamma b_{k-1} + db_k = 0 \); and then, thanks again to the consequence of theorem 4.1, \( b_{k-1} = \beta_J \omega^J \). Substituting these expressions for \( a_k \) and \( b_{k-1} \) in the second to last equation, we get:

\[
\delta[\alpha_J \omega^J] + D[\beta_J \omega^J] = \gamma(\ldots). \tag{6.2}
\]

As above, this equation implies

\[
\delta[\alpha_J] \omega^J + D[\beta_J] \omega^J \pm \beta_J A^J_I \omega^I = 0 \tag{6.3}
\]

since the only combination \( \lambda_J \omega^J \) (with \( \lambda_J \) invariant) that is \( \delta \)-exact vanishes. We now expand this equation according to the \( D \)-degree. The term of degree zero reads

\[
[\delta \alpha_J + D_0 \beta_J] \omega^J_0 = 0. \tag{6.4}
\]

This equation implies that the coefficient of \( \omega^J_0 \) must be zero, and as \( D_0 \) acts on the objects upon which \( \beta_J \) depends in the same way as \( d \), we get:

\[
\delta \alpha_J + d\beta_J = 0. \tag{6.5}
\]

If the antifield number of \( \alpha_J \) is strictly greater than 2, the solution is trivial, thanks to our results on the cohomology of \( \delta \) modulo \( d \):

\[
\alpha_J = \delta \mu_J + d\nu_J. \tag{6.6}
\]

Furthermore, theorem 5.1 tells us that \( \mu_J \) and \( \nu_J \) can be chosen invariants. We thus get:

\[
a_k = (\delta \mu_J + D_0 \nu_J) \omega^J_0 = s(\mu_J \omega^J_0) + d(\nu_J \omega^J_0) + "\text{more}"
\]

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where ”more” arises from $d\omega^J_0$, which can be written as $d\omega^J_0 = A^J_{0I} \omega^{J^I} + su^J_0$.

The term $\nu_{J_0} A^J_{0I} \omega^{J^I}$ has $D$-degree one, while the term $\nu_{J_0} su^J_0$ differs from the $s$-exact term $s(\pm \nu_{J_0} u^J_0)$ by the term $\pm \delta(\nu_{J_0}) u^J_0$, which is of lowest antifield number. Thus, trivial redefinitions enable one to assume that $a^0_k$ vanishes. Once this is done, $\beta_{J_0}$ must fulfill $d\beta_{J_0} = 0$ and thus be $d$-exact in the space of invariant polynomials by theorem [1.1], which allows one to set it to zero through appropriate redefinitions.

We can then successively remove the terms of higher $D$-degree by a similar procedure, until one has completely redefined away $a_k$ and $b_{k-1}$. One can next repeat the argument for antifield number $k - 1$, etc, until one reaches antifield number 2. This case deserves more attention, but what we can stress already now is the following: we can assume that the expansion of $a$ in $sa + db = 0$ stops at antifield number 2 and takes the form $a = a_0 + a_1 + a_2$ with $b = b_0 + b_1$. Note that this results is independent of any condition on the number of derivatives or of Lorentz invariance. These requirements have not been used so far. The crucial ingredient of the proof is that the cohomological groups $H^\text{inv}_k (\delta | d)$, which controls the obstructions to remove $a_k$ from $a$, vanish for $k > 2$ as shown in the previous section.

Now let us come to the case $k = 2$ and expand $a_2$ in power of the $D$-degree:

$$a_2 = a^m_2 + a^{m+1}_2 + \ldots + a^M_2 = \sum_{i=m}^{i=M} a^i_2 = \sum_{i=m}^{i=M} P_i \omega^{J_i}, \quad (6.8)$$

and

$$b_1 = \sum_{i=m}^{i=M} \beta_i \omega^{J_i}. \quad (6.9)$$

The equation $\delta[P_i \omega^{J_i}] + D[\beta_i \omega^{J_i}] = -\gamma a'_1$ implies as before $\delta[P_i \omega^{J_i}] + D[\beta_i \omega^{J_i}] = 0$. In $D$-degree minimum we thus get $[\delta P_m + D_0 \beta^1 \omega^J m = 0$ or, equivalently, $\delta P_m + d\beta^1 m = 0$. The antighost number is 2 so this equation admits a non-trivial solution in $H_2(\delta | d)$. In $D$-degree zero we have $\omega^J_0 = C_\mu C^\nu$ which cannot be completed into a Lorentz invariant object by multiplication with a single $C^{\ast \alpha}$, so $m > 0$. We must begin with $a^1_2$. This time, Lorentz invariance allows two non-trivial terms, namely $C^{\ast \mu} C^{\nu} C_{[\nu, \mu]}$ and $C^{\ast \mu} C_{\mu} \xi$. The most general $a^2_2$ is therefore $P_1 \omega^{J_1} = C^{\ast \mu} (x_1 C_\mu \xi + x_2 C^{\nu} C_{[\nu, \mu]})$. The equation for $a^2_2 = P_1 \omega^{J_2}$ reads then

$$\delta P_1 + D_0 \beta_2 + \beta_1 A^1_{I_2} = 0 \quad (6.10)$$
where $A_{I_2}^1 \omega^{j_2} = D \omega^{j_2}$. Note that this equation (6.10) does not imply that $P_{I_2}$ belongs to $H_2(\delta|d)$ anymore. Now, we have

$$\beta_{I_1} A_{I_2}^1 \omega^{j_2} = 2(x_1 - x_2) h^{*\mu\nu} C_\mu \partial_\nu \xi + 2 x_2 h^{*\rho} \partial_\rho \xi.$$  

(6.11)

The coefficient of $C_\rho \partial_\rho \xi$ is $\delta$-exact modulo $D_0$ but the coefficient of $C_\mu \partial_\nu \xi$ is not \[\square\] Thus we must impose $x_1 = x_2$. With this condition, $P_{I_2}$ reads $P_{I_2}^2 = -x_1 \xi^*$ up to a solution of the homogeneous equation $\delta P_{I_2} + D_0 \beta_{I_2} = 0$. Lorentz invariance forces one to take this solution to be zero, since it should contain either $C_\alpha \partial_\beta \xi$ or $\partial_\beta C_\alpha \xi$ contracted with one $C^{*\mu}$. Thus, $a_2^3$ is equal to $a_2^3 = -x_1 \xi^* C_\rho \partial_\rho \xi$ and $\beta_{I_2} = 0$.

The equation for $a_2^3$ is then $\delta P_{I_3} + D_0 \beta_{I_3} = 0$, which implies as above $a_2^3 = x_3 C^{*\mu} \partial_\nu \xi + x_4 C^{*\mu} \partial_\nu \xi C_{[\nu, \mu]}$, where $x_3$ and $x_4$ are arbitrary constants. These constants are not constrained by the equation in the next $D$-degree ($n = 4$) because they yield automatically $\delta$-exact (modulo $d$) terms. Furthermore, Lorentz invariance prevents one from adding homogeneous solution of the form $C^{*\mu} \partial_\alpha \xi \partial_\beta \xi$ at $D$-degree 4, so the most general possibility for $a_2$ is

$$a_2 = x_1 (C^{*\mu} C_\mu \xi + C^{*\mu} C_{[\nu, \mu]} - \xi^* C_\rho \partial_\rho \xi) + x_3 C^{*\mu} \partial_\nu \xi + x_4 C^{*\mu} \partial_\nu \xi C_{[\nu, \mu]}.$$  

(6.12)

The term $a_2$ in the deformation $a$ contains the information about the deformation of the algebra of the gauge transformations. The absence of terms quadratic in the antifields $h^{*\mu\nu}$ indicates that the algebra remains closed off-shell. This is not an assumption, it is a consequence of consistency.

So far, we have used only Lorentz-invariance. If one imposes in addition the requirement that the deformed Lagrangian should contain no more derivatives than the original Lagrangian, then one must set $x_3 = x_4 = 0$, since these would lead to terms with $n + 2$ derivatives: in $n$ dimensions, the free Lagrangian contains $n$ derivatives. The count of the derivatives proceed as follows: the antifield $C^{*\mu}$ counts for $n - 1$ derivatives, while the ghost $\xi$ counts for one derivative and $C_\mu$ counts for none (see appendix). We shall thus set $x_3 = x_4 = 0$ from now on and redefine $a_2$ by adding a $\gamma$-exact, to get

$$a_2 = -x_1 (C^{*\mu} C_\mu \xi + \xi^* C_\rho \partial_\rho \xi).$$  

(6.13)

The term $a_2$, one can read the deformation of the gauge algebra: in the deformed theory, the commutator of two transformations parametrized by the vectors

\[\square\] Indeed, one has $h^{*\mu\nu} = h^{*\mu\nu} + \frac{\mu}{n} h^*$ and $h^{*\mu\nu} C_\mu \partial_\nu \xi = \hat{h}^{*\mu\nu} C_\mu \partial_\nu \xi + \frac{1}{n} h^{*\nu\sigma} C^\nu \partial_\rho \xi$. The term linear in the trace of $h^{*\mu\nu}$ can be written as the $\delta$ of something, but the term linear in $h^{*\mu\nu}$ cannot be equal to $(\delta P_{I_2}) \omega^{j_2} + (D_0 \beta_{I_2}) \omega^{j_2}$. 

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\( \xi^a_1, \xi^b_2 \) is no longer zero but closes according to the diffeomorphism algebra (as it follows from the term \( C^a \partial^\mu C^b \partial_\mu \) in \( a_2 \)), while the commutator of a diffeomorphism with a Weyl transformation is a Weyl transformation of parameter \( C^\rho \partial_\rho \xi \) (as it follows from the second term in \( a_2 \)).

### 6.2 Deformation of gauge transformations

Having \( a_2 \), one gets \( a_1 \) from \( \delta a_2 + \gamma a_1 + db_1 = 0 \). This gives

\[
a_1 = -x_1 h^{\mu\nu} [2h_{\mu\nu}\xi + C^\rho \partial_\rho h_{\mu\nu} + \partial_\mu C^\rho h_{\rho\nu} + \partial_\nu C^\rho h_{\rho\mu}] \quad (6.14)
\]

up to a solution of the homogeneous equation \( \gamma a'_1 + db'_1 = 0 \). The analysis of this homogeneous equation is precisely the case \( k = 1 \) mentioned at the beginning of the previous section. One can assume \( \gamma a'_1 = 0 \) by using the consequence of Theorem 4.4. Thus, \( a'_1 = \alpha_j \omega^j \), where the \( \alpha_j \) are invariant polynomials, \( \alpha_j = \alpha_j([W_\alpha \beta \mu \nu], [h^{\mu\nu}]) \) and the \( \omega^j \) are linear in the ghosts and their (cohomologically non-trivial) derivatives. The \( \alpha_j \) must also be linear in the antifields \( h^{\mu\nu} \) and their derivatives. Counting derivatives, one sees that Lorentz invariance prevents such a term (\( h^{\mu\nu} \) counts for \( n - 1 \) derivatives and the Weyl tensor for 4; the only term with \( n \) derivatives that matches the requirements is \( h^{\mu\nu} \partial_\mu C_\nu \)), which identically vanishes).

The term \( a_1 \) yields the \( \mathcal{O}(\alpha) \) term of the full non-linear gauge transformations (1.9).

### 6.3 Deformation of Lagrangian

We now restrict the analysis to four spacetime dimensions. Lifting (3.14), we get for \( a_0 \) the cubic vertex of the Weyl action (3.3),

\[
2a_0 = -x_1 \left[ -4W_{\alpha\beta\gamma\delta}W_{\alpha'\beta'\gamma'\delta'} h^{\alpha\alpha'} h^{\beta\beta'} h^{\gamma\gamma'} h^{\delta\delta'} + (2W_{\alpha\beta\gamma\delta} + \frac{h}{2}W_{\alpha\beta\gamma\delta})W_{\alpha'\beta'\gamma'\delta'} h^{\alpha\alpha'} h^{\beta\beta'} h^{\gamma\gamma'} h^{\delta\delta'} \right]
\quad (6.15)
\]

modulo \( \bar{a}_0 \) solution of the equation \( \gamma \bar{a}_0 + db_0 = 0 \). \( W_{\alpha\beta\gamma\delta} \) is given in (3.3) and the conformal Weyl tensor at second order is

\[
W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{2}{n-2} \left( \eta_{\alpha[\gamma} R_{\delta]\beta} + h_{\alpha[\gamma} R_{\delta]\beta} - \eta_{\beta[\gamma} R_{\delta]\alpha} - h_{\beta[\gamma} R_{\delta]\alpha} \right)
+ \frac{2}{(n-1)(n-2)} \left( \eta_{\alpha[\gamma} \eta_{\delta]\beta} R_{\gamma\delta\alpha} + \eta_{\alpha[\gamma} h_{\delta]\beta} R_{\gamma\delta\alpha} + \eta_{\alpha[\gamma} h_{\delta]\beta} R_{\gamma\delta\alpha} \right),
\quad (6.16)
\]

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where \( R_{\alpha\beta\gamma\delta} \), \( R_{\mu\nu} \) and \( R \) are respectively the Riemann tensor, the Ricci tensor and the scalar curvature to second order (with all indices down).

Now, any \( \bar{a}_0 \) solution of \( \gamma \bar{a}_0 + d\bar{b}_0 = 0 \) has Euler-Lagrange derivatives \( \frac{\delta\bar{b}_0}{\delta a_{\alpha\beta}} \) which are (i) invariant; (ii) trace-free; and (iii) divergence-free. In arbitrary spacetime dimensions, there are many candidates. However, in 4 dimensions, there is only one candidate with 4 derivatives, namely the Bach tensor. This corresponds to a \( \bar{a}_0 \) proportional to the original Lagrangian, \( \bar{a}_0 \sim L_0 \), which can be removed by redefinitions. Thus, in 4 dimensions, we can assume \( \bar{a}_0 = 0 \), and there is only one first-order consistent deformation that matches all the requirements (Lorentz invariance, number of derivatives), namely (6.15).

Once the first-order vertex has been shown to be unique and has been identified with the first order Weyl deformation, it is easy to show that the action can be completed to all orders in the deformation parameter \( \alpha \) (we absorb \( x_1 \) in \( \alpha \) through the redefinition \( -\alpha x_1 \rightarrow \alpha \)). The argument proceeds as in the case of Einstein gravity [10] and leads uniquely to the complete Weyl action.

### 7 Consistent couplings for different types of Weyl gravitons

We now turn to the problem of deforming the action (1.11) describing a collection of free “Weyl fields” \( h^a_{\mu\nu} \). The computation of the intermediate cohomologies \( H(\gamma) \), \( \hat{H}^{\text{inv}}(\delta|d) \) etc proceeds as above, so we immediately go to the calculation of \( H(s|d) \) in form degree \( n \) and ghost number zero. Again, one gets that the most general cocycle can be brought to the form \( a = a_0 + a_1 + a_2 \) up to trivial terms. This is because the obstructions to removing \( a_k \) (\( k > 2 \)) are absent since \( H_k^{\text{inv}}(\delta|d) = 0 \) for \( k > 2 \).

The discussion of the cross-couplings of Weyl gravitons follows very much the same pattern as the analysis of multi Einstein gravitons [10].

First we expand \( a_2 \) as before in power of \( D \)-degree. The equation \( \delta[P_{I_1}\omega^{I_1}] + D[\beta_{I_1}\omega^{I_1}] = -\gamma a_1' \) implies \( \delta[P_{I_1}\omega^{I_1}] + D[\beta_{I_1}\omega^{I_1}] = 0 \). The most general \( a_2 \) is

\[
P_{I_1}\omega^{I_1} = C^s_{a}(a^a_{bc}C^b_{\mu}\xi^c + b^b_{ec}C^{\mu}_{[\mu}C^c_{\nu]}).
\]

(7.17)

With this \( a_2 \) we get the following expression for \( \beta_{I_1}A^{I_1}_{I_2}\omega^{I_2} \):

\[
\beta_{I_1}A^{I_1}_{I_2}\omega^{I_2} = -2h^a_{\mu\nu}\xi^c a^a_{bc} + 2(a^a_{bc} - b^a_{bc})h^a_{\mu\nu}\xi^c +
\]

30
\[ +2h_a^{\mu\rho} \partial^{[\rho} C^{\nu]}b_c^a + 2h_a^b C^{\mu\nu} \partial \xi^c b_c^a. \]  
(7.18)

The equation for \( a_2^2 = P_{I_2} \omega^{I_2} \) reads (see (6.10))
\[ \delta P_{I_2} + D_0 \beta_{I_2} + \beta_{I_1} A_{I_2} = 0. \]

For all \( I_2 \), we must be able to write \( \beta_{I_1} A_{I_2} \) as a \( \delta \)-exact term modulo \( D_0 \). Before, we had to impose \( x_1 = x_2 \) for this equation to have a solution. In this new case with \( \beta_{I_1} A_{I_2} \omega^{I_1} \) given above, we have to impose the following two conditions

\[ \bullet \quad a_{bc}^a = b_{bc}^a; \]
\[ \bullet \quad a_{ac}^a = a_{ca}^a. \]

The last condition means that if we view the constants \( a_{bc}^a \) as defining a product in internal space, then we have a commutative algebra, as in [10].

When these conditions are fulfilled, we obtain
\[ a_2^2 = -\xi_a^b C^{b\nu} \partial \xi^c a_{bc}^a, \]  
(7.19)

yielding
\[ a_2 = (\xi_a^b C^{b\nu} \partial \xi^c - C^{b\mu} C^{c\nu} \partial \partial_{\mu} \partial_{\nu} a_{bc}) a_{bc}^a. \]  
(7.20)

where we added a \( \gamma \)-exact term to simplify the expression and, as before, assumed that the deformed Lagrangian does not possess more derivatives than the original one.

Once \( a_2 \) is known, one obtains for \( a_1 \)
\[ a_1 = -a_{abc} h^{a\mu} \epsilon^{\nu \rho} \partial_{\mu} \partial_{\nu} + h^{b\mu} \partial_{\mu} \partial_{\rho} \epsilon^{c \rho} + h^{b\mu} \partial_{\rho} \partial_{\mu} \epsilon^{c \rho}. \]  
(7.21)

It remains then to determine \( a_0 \). It turns out that just as in the Einstein case [10], there is an obstruction to the existence of \( a_0 \), which disappears only if the constants \( a_{abc} \equiv \delta_{ad} \epsilon^{d \rho} \) are completely symmetric. This means that the commutative algebra defined by the \( a_{bc}^a \) should be “symmetric” [10]. The metric \( \delta_{ad} \) that appears here is the metric in internal space defined by the free (quadratic) Lagrangian (1.11).

To see the appearance of the obstructions, it is enough to focus on the following two types of terms: (i) terms involving quadratically the variables of the sector number 1 and linearly the variables of the sector 2; and (ii) terms involving linearly the variables of the sectors 1, 2 and 3. Indeed, the other sectors are treated in the same way (and do not interfere with each others since the numbers \( N_i \) counting the variables of the various sectors commute with all the differentials in the problem).
1. Terms of the form \( \left( h_{\alpha\beta}^{1} \right)^{2} h_{\gamma\delta}^{2} \).

These terms are determined by two constants, namely \( a_{12}^{1} = a_{21}^{1} \) and \( a_{11}^{2} \). It is easy to verify that the construction is unobstructed if these constants are equal, \( a_{12}^{1} = a_{11}^{2} \). If there were another choice of these constants that is unobstructed, then, since the problem is linear, any choice would be unobstructed. In particular, \( a_{11}^{2} = 0 \) would be acceptable. However, it is easy to see that the choice \( a_{11}^{2} = 0 \) is obstructed. Thus, the only acceptable choice is the completely symmetric one, \( a_{12}^{1} = a_{11}^{2} \). That \( a_{11}^{2} = 0 \) leads to an obstruction follows from a direct calculation: with \( a_{11}^{2} = 0 \), \( a_{1} \) reads (in the (1)-sector)

\[
a_{1} = h_{1}^{\mu\nu}[2h_{1\mu\nu}^{1}\xi^{2} + 2h_{1\mu\nu}^{2}\xi^{1} - C^{1\nu}(\partial_{\nu}h_{1\mu\nu}^{2} + \partial_{\mu}h_{1\mu\nu}^{2} - \partial_{\nu}h_{1\mu\nu}^{1})] - C^{2\rho}(\partial_{\rho}h_{1\mu\nu}^{1} + \partial_{\mu}h_{1\mu\nu}^{1} - \partial_{\rho}h_{1\mu\nu}^{2})
\]

and yields

\[
\delta a_{1} = B_{1}^{\nu\lambda}[2h_{1\mu\nu}^{1}\xi^{2} + 2h_{1\mu\nu}^{2}\xi^{1} - C^{1\nu}(\partial_{\nu}h_{1\mu\nu}^{2} + \partial_{\mu}h_{1\mu\nu}^{2} - \partial_{\nu}h_{1\mu\nu}^{1})] - C^{2\rho}(\partial_{\rho}h_{1\mu\nu}^{1} + \partial_{\mu}h_{1\mu\nu}^{1} - \partial_{\rho}h_{1\mu\nu}^{2})
\]

Up to a total derivative, this term must be equal to \( \gamma a_{0} \). Without loss of generality, one can assume \( a_{0} = A^{\nu\mu}(\left[ h_{1\mu\nu}^{1} \right])h_{1\mu\nu}^{2} \) and thus the coefficient of \( h_{1\mu\nu}^{2} \) in \( \gamma a_{0} \) is equal to \( \gamma A^{\nu\mu} \). But the coefficient of \( h_{1\mu\nu}^{2} \) in \( \delta a_{1} \) is (after integration by parts to remove its derivatives and up to manifestly \( g \)-exact terms) equal to \( DB_{1}^{\mu\nu}\xi^{1} + B_{1}^{\nu\mu}C_{1}^{1\nu\rho} + B_{1}^{\nu\mu}C_{1}^{1\rho\nu} - \partial_{\rho}B_{1}^{\mu\nu}C_{1}^{\rho} \), which is not \( \gamma \)-exact. Hence, the construction of \( a_{0} \) is obstructed, as announced.

2. Terms of the form \( \left( h_{\alpha\beta}^{1} \right)^{2} h_{\gamma\delta}^{3} \).

Again, there is a choice of the three constants \( a_{12}^{3} = a_{23}^{3}, a_{13}^{2} = a_{32}^{2} \) and \( a_{12}^{3} = a_{21}^{3} \) that is unobstructed, namely, the completely symmetric one, \( a_{12}^{3} = a_{23}^{3} = a_{31}^{3} \). If there were a second obstructed (independent) choice, then, using linearity, this would imply that there is also an acceptable choice with, say, \( a_{12}^{3} = a_{21}^{3} = 0 \). However, reasoning as above, one easily checks that this is not the case. Hence, \( a_{12}^{3} = a_{23}^{3} = a_{31}^{3} \) is the only possibility.

With coupling constants \( a_{abc} \) that are completely symmetric, \( a_{0} \) is given by (6.15) with the cubic, 4-derivatives structure \( \left( \partial^{2}\partial h \right)^{a}h^{b}h^{c}a_{abc} \)

\[
2a_{0} = -a_{abc} \left[ -4W_{\alpha\beta\gamma\delta}^{a}W_{\alpha^{'}\beta^{'}}^{b}h^{\alpha\alpha'}\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'} + \right]
\]

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\begin{align}
+ (2 W^{ab}_{\alpha\beta\gamma\delta} + \frac{h^a}{2} W^b_{\alpha\beta\gamma\delta}) W^c_{\alpha'\beta'\gamma'\delta'} \eta^{\alpha\beta'} \eta^{\beta\gamma'} \eta^{\gamma\delta'} \eta^{\delta\delta'}. \tag{7.24}
\end{align}

Proceeding as in [10], one then finds that there is no obstruction at second order in the deformation parameter if and only if the \( a^a_{bc} \) fulfill the identity

\begin{align}
 a^a_{b(c} a^b_{d)f} = 0, \tag{7.25}
\end{align}

which expresses that the algebra that they define is not only commutative and symmetric, but also associative. Since the only such algebras are trivial (direct sums of one-dimensional ideals) when the internal metric is definite positive, one concludes that cross-couplings can be removed, exactly as in [10]. To restate this result: \textit{there is no possibility of consistent cross-couplings (with number of derivatives \( \leq 4 \)) between the various Weyl fields \( h^a_{\mu\nu} \) for the free Lagrangian (1.11).}

Now, in the case of Weyl gravity, there does not appear to be any particularly strong reason for taking the free Lagrangian to be a sum of free Weyl Lagrangians, as in (1.11). Any other choice, corresponding to an internal metric \( k_{ab} \) that need not be definite positive, would seem to be equally good since the energy is in any case not bounded from below (or above). If one allows non positive definite metrics in internal space, then, non trivial algebras of the type studied in [31, 32, 33] exist and lead to non trivial cross interactions among the various types of Weyl gravitons. Rather than developing the general theory, we shall just give an example with two Weyl fields \( h^1_{\mu\nu}, h^2_{\mu\nu} \) and metric \( k_{ab} \) in internal space given by

\begin{equation}
k_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.26}
\end{equation}

The complete, interacting action reads

\begin{equation}
 S[h^a_{\mu\nu}] = \int d^4 x \sqrt{-g} h^2_{\mu\nu} B^{\mu\nu}, \tag{7.27}
\end{equation}

where \( B^{\mu\nu} \) is the (complete) Bach tensor of the metric \( g_{\alpha\beta} = \eta_{\alpha\beta} + \alpha h^1_{\alpha\beta} \) and \( g \) its determinant. The complete gauge transformations are

\begin{align}
 \frac{1}{\alpha} \delta_{\eta^a,\phi^a} g^a_{\mu\nu} &= \eta^1_{\mu\nu} + \eta^1_{\nu\mu} + 2 \phi^1 g_{\mu\nu}, \tag{7.28}
 \delta_{\eta^a,\phi^a} h^2_{\mu\nu} &= \alpha \mathcal{L} \eta^1 h^2_{\mu\nu} + 2 \alpha \phi^1 h^2_{\mu\nu} + \eta^2_{\mu\nu} + \eta^2_{\nu\mu} + 2 \phi^2 g_{\mu\nu}. \tag{7.29}
\end{align}
where covariant derivatives (;) are computed with the metric $g_{\alpha\beta}$. The invariance of the action (7.27) under (7.28) and (7.29) is a direct consequence of the identities fulfilled by the Bach tensor, i.e.

$$B^\alpha{}_{\beta;\delta} = 0, \ B^\alpha{}_{\beta} g_{\alpha\beta} = 0, \ \delta_{\eta,\phi} B^\alpha{}_{\beta} = \mathcal{L}_{\alpha\eta}^1 B^\alpha{}_{\beta} - 6\alpha \phi^1 B^\alpha{}_{\beta}. \quad (7.30)$$

The theory with action (7.27) can probably be given a nice group-theoretical interpretation along the lines of [34], which we plan to investigate. This would be useful for its supersymmetrization. [A complete action and transformation rules for conformal supergravity was given in [35]. Some simplifications of the results were presented in [36]. The simplest treatment is given in [37]. A nice feature of $N=1$ conformal supergravity is that the gauge algebra closes without auxiliary fields.]

8 Conclusions

In this paper, we have found results about the uniqueness of Weyl gravity in four dimensions. Our method relies on the antifield approach and uses cohomological techniques.

8.1 Deformations of the gauge transformations

We have found all the possible deformations of the gauge transformations of linearized Weyl gravity, under the assumptions that the deformed Lagrangian is Lorentz invariant and contains no more derivatives than those appearing in the free Lagrangian. We have shown that the only possibility is given by

$$\frac{1}{\alpha} \delta_{\eta,\phi} g_{\mu\nu} = \eta_{\mu,\nu} + \eta_{\nu,\mu} + 2\phi g_{\mu\nu}, \quad g_{\mu\nu} = \eta_{\mu\nu} + \alpha h_{\mu\nu}. \quad (8.1)$$

(the limit $\alpha = 0$ corresponds to no deformation at all).

If one does not impose the requirement on the number of derivatives, there are two other possibilities for the deformations of the gauge algebra, but we have not investigated them, although we suspect that they will be ultimately inconsistent (if only because they involve coupling constants with negative mass dimensions).
8.2 Deformations of the Lagrangian

We have then derived the most general deformation of the Lagrangian, which was shown to be just the non-linear Weyl gravity Lagrangian (1.5) in four dimensions.

Our results can easily be extended to 3, or higher, dimensions, because it is only in the last step, i.e., in the construction of $a_0$, that we used the assumption $n = 4$.

1. $n = 3$: in 3 spacetime dimensions, the free action is (38)

$$S_0^{CS} = \frac{1}{2} \int d^3x \varepsilon_{\mu\alpha\beta} G^{(1)}_{\alpha\nu} \partial^\mu h^{\beta}_\nu$$

(8.2)

and the free Lagrangian is gauge-invariant only up to a total derivative. It contains three derivatives. Deforming this theory leads uniquely to the theory of [38, 39].

2. $n = 2p \geq 6$: in even spacetime dimensions, the free Lagrangian is given by the quadratic part of the unique conformal invariant that has a non-vanishing quadratic part, which reads, (see for example proposition 3.4. p. 106 of [40])

$$I_n \propto \mathcal{W}_{\alpha\beta\gamma\delta} \tilde{\Delta}_{a_n}^{\alpha\beta\gamma\delta\mu\nu\rho\sigma} \mathcal{W}_{\mu\nu\rho\sigma},$$

(8.3)

where $\tilde{\Delta}_{a_n}^{\alpha\beta\gamma\delta\mu\nu\rho\sigma}$ is a differential operator of order $n - 4$ in dimension $n$. There are, however, other conformal invariants that have no quadratic parts. These can come into the deformation process, through $a_0$. Therefore, deforming the free theory based on (8.3) gives not only the full conformally invariant completion of (8.3), but also all the other possible conformal invariants, the number of which grows with the dimension (see for instance [40, 41]). Thus the deformed Lagrangian is no longer unique. The other results on the deformation of the gauge algebra and the gauge transformations are otherwise unchanged. For instance, in dimension $n = 6$ we have the invariant at the linearized level

$$\mathcal{I} = \partial^\alpha \mathcal{W}^{\beta\gamma\delta\epsilon} \partial_\alpha \mathcal{W}_{\beta\gamma\delta\epsilon} + 16 \mathcal{C}^{\gamma\delta\epsilon} \mathcal{C}_{\gamma\delta\epsilon} - 16 \mathcal{W}^{\alpha\gamma\delta\epsilon} \partial_\alpha \mathcal{C}_{\gamma\delta\epsilon},$$

(8.4)

which, deformed, would give the invariant

$$\sqrt{-g} I_6 = \sqrt{-g} \left[ \nabla^\alpha \mathcal{W}^{\beta\gamma\delta\epsilon} \nabla_\alpha \mathcal{W}_{\beta\gamma\delta\epsilon} + 16 \mathcal{C}^{\gamma\delta\epsilon} \mathcal{C}_{\gamma\delta\epsilon} - 16 \mathcal{W}^{\alpha\gamma\delta\epsilon} \nabla_\alpha \mathcal{C}_{\gamma\delta\epsilon} - 16 \mathcal{K}_{\alpha\beta} \mathcal{W}^{\alpha\gamma\delta\epsilon} \mathcal{W}_{\beta,\gamma\delta\epsilon} \right].$$

(8.5)
The other two known terms \( \Omega_1 = \sqrt{-g} W_{\mu \sigma} W_{\alpha \beta} \) and \( \Omega_2 = \sqrt{-g} W^{\mu \rho} W^{\alpha \beta} W_{\rho \mu} \) can also come in through \( a_0 \).

Furthermore, there exists the possibility of taking \( a_2 \) (and then also \( a_1 \)) equal to zero, i.e., of not deforming the gauge transformations at all, while taking a non-trivial \( a_0 \) polynomial in the linearized Weyl curvatures with the appropriate number of derivatives. In six dimensions, \( W^{\mu \sigma} W_{\alpha \beta} W_{\rho \mu} \) would be possible interactions that do not deform the gauge symmetry. This possibility does not exist in four dimensions because the candidate interactions would contain more than four derivatives.

### 8.3 Interactions for a collection of Weyl fields

We have then investigated interactions for a collection of Weyl fields and have shown that cross interactions were impossible in four dimensions with the prescribed free field limit (1.11), although non trivial possibilities exist with a different free Lagrangian. Many of our considerations hold in higher dimensions. However, one can then also build non trivial cross interactions that do not modify the gauge structure, e.g., by adding \( g_{abc} W_{\alpha \beta \alpha} W_{\gamma \delta} \) to the free Lagrangian (in 6 dimensions).

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### A Counting derivatives

In this appendix we establish the following result: let \( a_0 \) be a consistent first-order deformation of the free Lagrangian with bounded number of derivatives,
i.e., be a solution of $\gamma a_0 + \partial_\mu t^\mu \approx 0$, or, what is the same

$$\gamma a_0 + \delta a_1 + \partial_\mu t^\mu = 0$$  \hspace{1cm} (A.1)$$

for some $a_1$. We assume that the spacetime dimension is such that $n \geq 3$. Then the corresponding BRST cocycle $a = a_1 + a_2 + \ldots$ obtained by completing $a_0$ in such a way that $sa + db = 0$ can be assumed to have a finite expansion.

To prove this, one assigns a new degree to the fields, called the $K$-degree,

$$
\begin{array}{cccccccc}
\text{fields} & h^*_{\mu\nu} & C^*_{\mu} & \xi^* & \partial_\mu & \delta & \gamma & h_{\mu\nu} & C_{\mu} & \xi \\
K\text{-degree} & n-1 & n-1 & n-2 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{array}
$$  \hspace{1cm} (A.2)

The $K$-degree counts in fact the dimension (there is some freedom in the assignments for the ghosts and the antifields; we choose to assign dimension 1 to $\gamma$ and $\delta$ for convenience). Note in particular that the $K$-degree of an expression involving only $h_{\mu\nu}$ and its derivatives is precisely equal to the number of derivatives.

The $K$-degree is increased by one by $\delta, \gamma, \partial_\mu$, so the $K$-degree is the same for $a_0, a_1$, etc. Because $a_0$ has a bounded derivative order, and one may assume it to be homogeneous, so $K(a_0) = N$ for some finite $N$. The claim is : $a = a_0 + \ldots + a_k$ stops at antifield number $k \leq N$.

Proof : From the above, $K(a_k) = N$. One has $k = n_{h^*} + 2n_{C^*} + 2n_{\xi^*}$, where $n_{h^*}$ (resp. $n_{C^*}$ and $n_{\xi^*}$) counts the number of $h^*_{\mu\nu}$ (resp. $C^*_{\mu}$ and $\xi^*$), differentiated or not. On the other hand we know that $K(a_k) \geq (n-1)n_{h^*} + (n-1)n_{C^*} + (n-2)n_{\xi^*}$. The inequality holds because some differentiations may appear. This expression is clearly greater than or equal to the expression $n_{h^*} + 2n_{C^*} + 2n_{\xi^*}$ for $n \geq 4$ so in these cases we proved that

$$k \leq N.$$  \hspace{1cm} (A.3)

In the case $n = 3$ one has $k \leq 2K(a_k)$ and thus $k \leq 2N$.

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