Pair Density Wave in the Doped $t$-$J$ Model with Ring Exchange on a Triangular Lattice

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In our previous work (PRL 121 046401 (2018)), we found a quantum spin liquid phase with spinon Fermi surface in the two dimensional spin-1/2 Heisenberg model with four-spin ring exchange on a triangular lattice. In this work we dope the spinon Fermi surface phase by studying the $t$-$J$ model with four-spin ring exchange. We perform density matrix renormalization group calculations on four-leg cylinders of a triangular lattice and find that the dominant pair correlation function is that of a pair density wave, i.e. it is oscillatory while decaying with distance with a power law. The doping dependence of the period is studied. This is the first example where pair density wave is the dominant pairing in a generic strongly interacting system where the pair density wave cannot be explained as a composite order and no special symmetry is required.

Pair density wave (PDW) is a superconducting state in which Cooper pairs have finite momentum. The first example of PDW is the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [1, 2] which can arise in superconductors in strong magnetic fields when the Fermi surface is split by Zeeman effect. Recently PDW has come into prominence in the context of underdoped when the Fermi surface is split by Zeeman effect. Recently [1, 2], this is the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [1, 2] that is responsible for many of the anomalous properties of the pseudo-gap regime [7]. Experimentally a direct observation of PDW has been made via local Cooper pair tunnelling in Bi$_2$Sr$_2$CaCu$_2$O$_{8+x}$ [8]. They found Cooper pair density modulation with period $4a_0$ where $a_0$ is the length of unit cell, and the magnitude of the modulation is five percent of the uniform pairing background. This may be interpreted as a subsidiary PDW being generated by the period 4$a_0$ charge order together with the uniform d-wave pairing. In this sense the recent report [9] of short range period 8$a_0$ charge order in the vicinity of the vortex core is even more exciting, because it may be the signature of a hidden period 8$a_0$ PDW [10, 11].

Theoretically, there are very few microscopic models which are shown to have PDW ground states. Berg et al. [12] studied a Kondo-Heisenberg model with 1D electron gas coupled to a spin chain. They found a spin gapped phase with PDW correlations oscillating with period $2a_0$, which matches the period of the ordering tendency of the spin chain. An extended two-leg Hubbard-Heisenberg model is also found to have a spin gapped phase with a PDW [13]. In all these examples, the PDW is commensurate and can either be interpreted as a composite order between short range spin order with the same commensurate period and another short range triplet pairing order, or requires the specially tuned symmetry of $\pi$ flux through the ladder plaquette. Dodaro et al. [14] searched for PDW in a more standard $t$-$J$ model for doped cuprates, but they did not find any evidence of PDW ordering even when they include next nearest neighbor (NNN) hopping and NNN exchange coupling. On a more speculative level, another mechanism to generate PDW is the Amperian pairing [7, 15], which was first proposed for quantum spin liquids with spinon Fermi surface. The Ampere effect of the gauge magnetic field produces attractive interactions between spinons moving in the same directions, which creates PDW with momentum $2k_F$ at a given point on the Fermi surface.

Recently, we found the spinon Fermi surface phase in a two dimensional spin-1/2 Heisenberg model with four-spin ring exchange on a triangular lattice [16] using density matrix renormalization group (DMRG) method. The model is introduced as a microscopic model for Mott insulator phase of 1T-TaS$_2$. The spinon Fermi surface phase was also proposed in early works for organic compounds [17, 18] and confirmed in the two-leg and four-leg ladder DMRG simulations [19, 20]. We noticed a curious absence of spin structure factor peak along the $\Gamma$ to $M$ direction and speculated on the possibility of Amperian pairing between the spinons [16]. It is then natural to extend this work to the doped case to see if any evidence of superconductivity emerges. Based on DMRG calculations on four-leg ladders, we find that upon doping the spinon Fermi surface state, the dominant pairing channel has oscillatory correlations, with a period which depends smoothly on doping and therefore appears to be incommensurate. Also generically the period does not match that of any other charge or spin order, implying that there is no simple interpretation of the PDW as a composite order. The PDW phase we found is very unique and to the best of our knowledge it is the first example of PDW found in a generic interaction driven one band model. This is the key finding in this work.

Model and method — We consider a $t$-$J$ model with four-spin ring exchange terms on a triangular lattice, $H = \hat{\mathcal{P}} (H_{t-J} + H_K) \hat{\mathcal{P}}$, where $\hat{\mathcal{P}}$ excludes doubly occupied states. The hopping and two-spin exchange term $H_{t-J}$ and four-spin ring exchange term $H_K$ are written as

$$H_{t-J} = -t \sum_{\langle i,j \rangle \sigma} \left( \hat{c}_{i \sigma}^\dagger \hat{c}_{j \sigma} + \text{h.c.} \right) + J \sum_{\langle i,j \rangle} \hat{S}_i \cdot \hat{S}_j , \quad (1)$$

$$H_K = K \sum_{\langle i,j,k,l \rangle} \left[ (\hat{S}_i \cdot \hat{S}_j) (\hat{S}_k \cdot \hat{S}_l) + (\hat{S}_j \cdot \hat{S}_k) (\hat{S}_i \cdot \hat{S}_l) - (\hat{S}_i \cdot \hat{S}_k) (\hat{S}_j \cdot \hat{S}_l) \right] , \quad (2)$$

where $\langle i,j \rangle$ denotes nearest neighbor bond, and $\langle i,j,k,l \rangle$ runs over all compact rhombuses. The ring exchange terms simulate...
deep into the spinon Fermi surface phase and $t/J = 2$ which is a conventional value in Mott insulator materials. We perform large-scale DMRG calculations on four-leg ladders with long direction length $L_x$ up to 72. We take periodic boundary conditions in short direction of the ladder, and open boundary conditions in the long direction. The good quantum numbers of total spin $S_{\text{tot}}^z$ and total number of fermions $N_{\text{tot}}$ are used. All the calculations are performed in the $S_{\text{tot}}^z = 0$ and $N_{\text{tot}} = N(1 - p)$ sector, where $N$ is the total number of sites and $p$ is the doping level. The calculations are performed with bond dimensions up to $m = 5120$ and corresponding truncation error is less than $10^{-5}$. All the results showed in the following are after extrapolating to infinite bond dimension if it is not specified. More details are presented in the Supplemental Material [27].

**Results** — To study the pairing properties, we measured the pairing correlation in real space. We considered both spin singlet and triplet pairing, and find the magnitude of singlet pairing is always larger than the triplet one, so in the following we will focus on singlet pairing data. The singlet pairing order parameter is defined as

$$\Delta_0(i) = c_{i\uparrow}c_{i+\delta_{\alpha\downarrow}} - c_{i\downarrow}c_{i+\delta_{\alpha\uparrow}},$$

where $\delta_{\alpha}$ with $\alpha = 1, 2$ takes the value of primitive vectors $a_1$ and $a_2$ respectively, denoting different orientation of pairs as showed in the inset of

FIG. 1. Pairing correlation along the long direction of the ladder for 1/16 hole doping with $K/J = 0.8$, $t/J = 2$ and four-leg ladder length $L_x = 72$. (a) Log-log plot of the pairing correlation $P_{22}$ and $P_{12}$. Here $P_{22}$ is the correlation between pairing order parameters defined both in $a_2$ orientation. $P_{12}$ is the correlation between pairing order parameters defined in $a_1$ and $a_2$ orientation, as showed in the inset. The red thin diamond is for $P_{22}$, olive diamond is for $P_{12}$, and the data for $P_{12}$ have been shifted vertically for clarity. The solid symbol is for positive value, while the open symbol is for negative value and we only plot the magnitude. The power law function $f(x)$ in the plot is a fit through the magnitude of the data points. (b) Pairing correlation $P_{22}$ and $P_{12}$ normalized by the power law function $f(x)$, which directly reflects the oscillation of the pairing correlation. The out of phase oscillation of $P_{22}$ and $P_{12}$ indicates a $d$-wave type pairing. (c) Fourier transformation of the oscillation part of $P_{22}$ and $P_{12}$ for doping 1/16 and 1/8, with parameters $K/J = 0.8$, $t/J = 2$ and four-leg ladder length $L_x = 72$. (a) Power law fitting of the magnitude of pairing correlation $P_{22}$ for each doping. The data for 1/12 and 1/8 hole doping have been shifted vertically for clarity. (b) Normalized pairing correlation which shows the oscillation part of $P_{22}$ for doping 1/12 and 1/8, while 1/16 doping is already showed in Fig. 1(b). (c) Fourier transformation of the oscillation part of $P_{22}$ for each doping, which gives pairing momentum about 0.15(2$\pi/a_0$), 0.17(2$\pi/a_0$) and 0.25(2$\pi/a_0$) for 1/16, 1/12 and 1/8 hole doping, respectively.

the proximity of the undoped insulator to the Mott transition. We already know from our earlier work on six-leg and eight-leg ladders DMRG simulations that the undoped system enters the spinon Fermi surface phase for $K/J > 0.3$. Here we investigate the effect of hole doping.

Soon after the discovery of high $T_c$ Cuprates, Anderson [21] proposed that doping a Mott insulator may lead to a correlation driven superconductor. Since that time, many methods including mean field theory, Gutzwiller variational methods and DMRG simulations and exact diagonalization have found $d$-wave type superconductivity both on square and triangular lattice in some parameter region and doping level [22–26]. We confirm that for the standard $t$-$J$ model ($K = 0$), doping of the Néel ordered state produces uniform $d$-wave pair correlations (see Supplemental Material [27]). However, the situation changes completely when we dope into the spinon Fermi surface state. We choose $K/J = 0.8$ which put us quite
Fig. 1(a). To reduce finite size effects, we measure the correlation functions with a summation over the short direction \( P_{\Delta r}(i_x - i_{x_0}) = \sum_i \langle \Delta_x(i_{x_0}, i_y) \Delta_x'(i_x, i_y) \rangle \), where \((i_x, i_y)\) is the coordinate of site \(i\) in the unit of primitive vectors, \(i_{x_0}\) is a reference coordinate in the long direction, we take \(i_{x_0} > \frac{L_x}{2}\) to reduce boundary effect, and the final results are averaged over several \(i_{x_0}\)'s. We denote the relative distance in the correlator by \(x = i_x - i_{x_0}\). Fig. 1 shows the pairing correlation at doping 1/16 for both \(P_{22}\) and \(P_{12}\). The pairing correlation shows a clear oscillation with an amplitude which is consistent with a power law decay over a large range of \(x\). (We attribute the deviation from a power law for large \(x\) to finite size effect and a lack of convergence.) In the figure, the fit is made to the amplitude of the individual data points which tend to over-estimate the exponent of the power law decay. For an oscillatory function, the proper way to fit the exponent requires first extracting the envelop function which we have not attempted here. The purpose of the fit we did is to allow us to display the oscillations on a linear scale, as is done in Fig. 1(b). While the value of the exponent from the fit should not be taken seriously, it is apparently larger than 2, which means that the Fourier transform of the response function will not show divergence for small \(q\) and \(\omega\). For larger doping of 1/12 and 1/8, we also see similar oscillation behavior, but with faster decay and shorter period with increasing doping (see Fig. 2). The periods are about 7\(a_0\), 6\(a_0\) and 4\(a_0\) for 1/16, 1/12, 1/8 doping respectively.

In order to identify the pairing symmetry, we analyze the relative phase of \(P_{22}\) and \(P_{12}\). It is clear from Fig. 1(b), that \(P_{22}\) and \(P_{12}\) have out of phase oscillation. Thus we conclude that we have found a \(d\)-wave type PDW.

In addition to the power law decay of the pairing correlation, we also found power law decay of both spin and dimer correlation, corresponding to gapless spin and charge degrees of freedom. In Fig. 3, we show the log-log plot of those correlations. The spin correlations are defined as \(S(i_x - i_{x_0}) = \sum_i \langle S_{i_{x_0}, i_y} \cdot S_{i_x, i_y} \rangle \) and dimer correlations are defined as \(D(i_x - i_{x_0}) = \sum_i \langle (d(i_{x_0}, i_y) d(i_x, i_y)) - \langle d(i_{x_0}, i_y) \rangle \langle d(i_x, i_y) \rangle \rangle \) with \(d(i_{x_0}, i_y) = S_{i_{x_0}, i_y} - S_{i_x, i_y} \) the long direction dimer operator. The spin and dimer correlations show slower power law decay compared with the PDW and also show oscillations, and we also analyze their period by Fourier transformation. We find period 2\(a_0\) for both spin and dimer at hole doping 1/12 and 1/16, while at doping 1/8, we have period 3\(a_0\) for dimer but period 4\(a_0\) for spin.

We also measured the Fermi vectors \(k_F\)'s, which can be estimated by the singular positions of the density in momentum space of the fit we did is to allow us to display the oscillations in the \(k_x\) direction \(\partial n(k) / \partial k_x\), with parameters \(K/J = 0.8, t/J = 2\) and four-leg ladder length \(L_x = 72\). In the plot, \(k_1\) and \(k_2\) are the coordinates of \(k\) points in the unit of primitive vectors in reciprocal space \(\mathbf{k} = k_1 \mathbf{b}_1 + k_2 \mathbf{b}_2\). As there is inversion symmetry, only two cuts \(k_2 = -0.25\) and \(k_1 = 0\) are shown. The peak and dip positions (denoted by black arrows) of \(\partial n(k) / \partial k_x\) give the \(k_F\)'s. We have \(k_F = \pm \frac{\pi}{2 a_1}, (\pm \frac{1}{2} \mathbf{b}_1 - \frac{1}{2} \mathbf{b}_2)\) for 1/8 hole doping (figure (a) and (b)), \(k_F = \pm \frac{\pi}{2 a_1}, (\pm \frac{1}{2} \mathbf{b}_1 - \frac{1}{2} \mathbf{b}_2)\) and \((- \frac{1}{2} \mathbf{b}_1 - \frac{1}{2} \mathbf{b}_2)\) for 1/12 (figure (c) and (d)) and 1/16 hole doping (figure (e) and (f)). The precision of these values are limited by the finite system size.
space \( n(\mathbf{k}) \). The momentum space Fermi density is calculated as
\[ n(\mathbf{k}) = \frac{1}{\pi} \sum_{ijr} e^{\eta (r-r')} (c_{\mathbf{k}r}^\dagger c_{\mathbf{k}r'}). \]
Fig. 4 shows \( n(\mathbf{k}) \) along different cuts of the Brillouin zone (BZ). We collect all the singular points in \( n(\mathbf{k}) \) where the first derivative \( \partial n(\mathbf{k})/\partial \mathbf{k}_1 \) has a dip or peak and get the \( \mathbf{k}_F \)'s shown in the caption of Fig. 4. Although those estimations of \( \mathbf{k}_F \) are rather crude, we can make a consistent check of the Fermi surface area based on these \( \mathbf{k}_F \)'s. For example for 1/8 hole doping, we add up the distances between the Fermi crossings along the 3 lines given by \((0, \pm \frac{1}{4})b_2\) and multiply by the width of each line which is 1/4 (in the unit of width of the first BZ) to get a total area of 11/24 = 0.4583 (in the unit of area of the first BZ). This is close to the free fermion value of \((1-p)/2 = 0.4375\) for the 2D Fermi surface. If we focus on the change of Fermi surface area from 1/8 to 1/12 hole doping, we find 1/48 from this estimate, in precise agreement with what is expected for free Fermions. Thus we conclude that our interacting quasi-1D system retains the Fermi surface structure expected for Luttinger liquids.

The three crossings of Fermi surface from the measurements of \( n(\mathbf{k}) \) correspond to six gapless modes. To verify it, we measured the sub-system entanglement entropy and fit the central charge with formula \( S(l, N) = \frac{\pi}{2} \log \left( \frac{\sin \frac{l \pi}{2N}}{\sin \frac{\pi}{2N}} \right) + A \), where \( l \) is the length of subsystem, \( N \) is the total number of sites, \( c \) is the central charge, \( A \) is a constant and the \( l = 1 \) entropy gives a very good estimation of it when \( N \) is sufficiently large. While we have not reached convergence, we find that central charge is at least 4 and is consistent with central charge of \( c = 6 \) which supports six gapless modes due to three crossings. The details of the estimations are presented in Supplemental Material [27].

Discussions — As the period of spin correlation in 1/8 hole doping is two times the period of charge correlation and equals the period of PDW, this is reminiscent of the "anti-phase" stripe found in La-based cuprates near 1/8 hole doping, where the onset of pairing correlations coincides with the onset of static spin-stripe order, and they share same periodicity [6]. On the other hand, for doping of 1/12 and 1/16, there is no such relation between the various periodicity and the PDW cannot be interpreted as stripes. While we do not have a clear picture of what controls the PDW period, we find that the empirical relation for the wave-vector, \( 4\pi p/a_0 \), works perfectly for \( p=1/8 \) and 1/12 and within errors for \( p=1/16 \). This reminds us of the discussion of pairing of electrons on the same side of the Fermi surface to form a PDW with wave vector \( 2k_F \) in a one dimensional Luttinger liquid, which was referred to as \( \eta \)-pairing [28]. The power law decay is governed by the exponent \( \kappa_p = 1/\kappa_\eta \), which is always greater than 2 for any value of the Luttinger parameter \( \kappa_\eta \) [28]. If we regard our quasi-1D system as a set of 4 interacting 1D Luttinger liquids, the quantity that is fixed is the sum of the \( 2k_F \) from the Fermi crossing of each band and it is given by \( 4((1-p)/2)/(2\pi/a_0) \) where the factor 4 accounts for the fractional BZ area taken up by each 1D band. It is interesting to note that up to umklapp this is just our empirical formula \( 4\pi p/a_0 \).

We believe that the key reason why we find a PDW upon doping the \( J-K \) model of the triangular lattice is that we are doping into a spin liquid. While the spin liquid in the undoped system has Fermi surfaces, we do not know whether it is a \( U(1) \) spin liquid which has a full Fermi surface, or a Z2 spin liquid with a partially gapped Fermi surface. The latter has spinon pairing and it is natural to expect that doping will immediately lead to a pairing state which may be exotic. We indeed find the emergence of an exotic PDW. We do not believe the introduction of the ring exchange term alone is sufficient. We have added the ring exchange term to the \( t-J \) model on a square lattice and the leading pairing correlator remains uniform d-wave.

In conclusion, we find it encouraging that a dominant PDW correlation emerges upon doping of a model that supports a spinon Fermi surface state. Since this model may be applicable to 1T-TaS2, we continue to encourage experimentalists to dope this material by gating in order not to introduce too much disorder [16]. The existence of fluctuating PDW in a doped Mott insulator model is also encouraging news for the search of fluctuating PDW in underdoped Cuprates [7].

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Cuprate Superconductors,” Phys. Rev. X 4, 031017 (2014).

[8] M. H. Hamidian, S. D. Edkins, Sang Hyun Joo, A. Kostin, H. Eisaki, S. Uchida, M. J. Lawler, E. A. Kim, A. P. Mackenzie, K. Fujita, Jinho Lee, and J. C. Séamus Davis, “Detection of a Cooper-pair density wave in Bi2Sr2CaCu2O8+x,” Nature 532, 343 (2016).

[9] Stephen D Edkins, Andrey Kostin, Kazuhiro Fujita, Andrew P Mackenzie, Hiroshi Eisaki, Shin-Ichi Uchida, Subir Sachdev, Michael J Lawler, Eun-Ah Kim, JC Davis, et al., “Magnetic-field induced pair density wave state in the cuprate vortex halo,” Physical Review B 97, 174510 (2018).

[10] Yuxuan Wang, Stephen D Edkins, Mohammad H Hamidian, JC Séamus Davis, Eduardo Fradkin, and Steven A Kivelson, “Pair density waves in superconducting vortex halos,” Physical Review B 97, 174510 (2018).

[11] Zhehao Dai, Ya-Hui Zhang, T. Senthil, and Patrick A. Lee, “Pair-densitywaves,charge-densitywaves,andvorticesinhight-Tc cuprates,” Phys. Rev. B 97, 174511 (2018).

[12] Erez Berg, Eduardo Fradkin, and Steven A. Kivelson, “Pair-Density-Wave Correlations in the Kondo-Heisenberg Model,” Phys. Rev. Lett. 105, 146403 (2010).

[13] Akbar Jaefari and Eduardo Fradkin, “Pair-density-wave superconducting order in two-leg ladders,” Phys. Rev. B 85, 035104 (2012).

[14] John F. Dodaro, Hong-Chen Jiang, and Steven A. Kivelson, “Intertwined order in a frustrated four-leg t – J cylinder,” Phys. Rev. B 95, 155116 (2017).

[15] Wen-Yu He, XiaoYan Xu, Gang Chen, K. T. Law, and Patrick A. Lee, “Spinon Fermi Surface in a Cluster Mott Insulator Model on a Triangular Lattice and Possible Application to $\kappa$–(BEDT–TTF)$_2$Cu$_2$(CN)$_3$,” Phys. Rev. Lett. 121, 046401 (2018).

[16] Olexei I. Motrunich, “Variational study of triangular lattice spin-1/2 model with ring exchanges and spin liquid state in $\kappa$–(ET)$_2$Cu$_2$(CN)$_3$,” Phys. Rev. B 72, 045105 (2005).

[17] Sung-Sik Lee and Patrick A. Lee, “U(1) gauge theory of the hubbard model: Spin liquid states and possible application to $\kappa$–(BEDT–TTF)$_2$Cu$_2$(CN)$_3$,” Phys. Rev. Lett. 95, 036403 (2005).

[18] D. N. Sheng, Olexei I. Motrunich, and Matthew P. A. Fisher, “Spin bose-metal phase in a spin-$1/2$ model with ring exchange on a two-leg triangular strip,” Phys. Rev. B 79, 205112 (2009).

[19] Matthew S. Block, D. N. Sheng, Olexei I. Motrunich, and Matthew P. A. Fisher, “Spin bose-metal and valence bond solid phases in a spin-1/2 model with ring exchanges on a four-leg triangular ladder,” Phys. Rev. Lett. 106, 157202 (2011).

[20] P. W. Anderson, “The Resonating Valence Bond State in La2CuO4 and Superconductivity,” Science 235, 1196–1198 (1987).

[21] Gabriel Kotliar and Jialin Liu, “Superexchange mechanism and d-wave superconductivity,” Phys. Rev. B 38, 5142–5145 (1988).

[22] Steven R. White and D. J. Scalapino, “Ground states of the doped four-leg t-J ladder,” Phys. Rev. B 55, R14701–R14704 (1997).

[23] Qiang-Hua Wang, Dung-Hai Lee, and Patrick A. Lee, “Doped t – J model on a triangular lattice: Possible application to Na$_x$CoO$_2$ · yH$_2$O and Na$_{1-x}$TiO$_2$,” Phys. Rev. B 69, 092504 (2004).

[24] Hong-Chen Jiang, Zheng-Yu Weng, and Steven A. Kivelson, “Superconductivity in the doped t–J model: Results for four-leg cylinders,” Phys. Rev. B 98, 140505 (2018).

[25] Bo-Xiao Zheng, Chia-Min Chung, Philippe Corboz, Georg Ehlers, Ming-Pu Qin, Reinhard M Noack, Hao Shi, Steven R White, Shiwei Zhang, and Garnet Kin-Lic Chan, “Stripe order in the underdoped region of the two-dimensional hubbard model,” Science 358, 1156–1160 (2017).

[26] See Supplemental Material for details on the DMRG simulation and the results.

[27] V. J. Emery, S. A. Kivelson, and O. Zachar, “Classification and stability of phases of the multicomponent one-dimensional electron gas,” Phys. Rev. B 59, 15641–15653 (1999).
I. DETAILS OF DMRG SIMULATION

We use DMRG to simulate the $t$-$J$ model with four-spin ring exchange on a triangular lattice as defined in the main text, and we repeat it here for the convenience of discussion. The total Hamiltonian is $H = \hat{P} (H_{t-J} + H_K) \hat{P}$, where $\hat{P}$ excludes doubly occupied states. The $t$-$J$ term $H_{t-J}$ and four-spin ring exchange term $H_K$ are written as

$$H_{t-J} = -t \sum_{\langle i,j \rangle \sigma} \left( c_i^{\dagger} c_{j\sigma} + h.c. \right) + J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j,$$

$$H_K = K \sum_{\langle i,j,k,l \rangle} \left[ \left( \mathbf{S}_i \cdot \mathbf{S}_j \right) \left( \mathbf{S}_k \cdot \mathbf{S}_l \right) + \left( \mathbf{S}_j \cdot \mathbf{S}_k \right) \left( \mathbf{S}_l \cdot \mathbf{S}_i \right) - \left( \mathbf{S}_i \cdot \mathbf{S}_k \right) \left( \mathbf{S}_j \cdot \mathbf{S}_l \right) \right],$$

where $\langle i,j \rangle$ denotes nearest neighbor bond, and $\langle i,j,k,l \rangle$ runs over all compact rhombuses. The simulation is performed on four-leg ladders on a triangular lattice. As showed in Fig. S1, periodic boundary condition is used in the short direction, and open boundary condition is used in the long direction. To accelerate the simulation, good quantum numbers are used, and the simulations are performed in the $S^z_{\text{tot}} = 0$ and $N_{\text{tot}} = N(1 - p)$ subspace. We also exclude the double occupancy states in this subspace. The largest bond dimension we used is 10, and the largest system size we simulated is with length $L_x = 72$, and the corresponding truncation error is less than $10^{-5}$.

II. PAIRING CORRELATIONS

We considered both spin singlet and triplet pairing, and found that the spin singlet pairing is always dominant, so we will only focus on the singlet pairing correlation in the following. We define the singlet pairing correlation as

$$P_{aa'}(i_x - i_{x_0}) = \sum_{i_y} \langle \Delta^1_a(i_{x_0}, i_y) \Delta_{a'}(i_x, i_y) \rangle,$$

where the singlet pairing order parameter $\Delta_a(i)$ is defined on nearest neighbor bonds $\Delta_a(i) = c_i \dagger c_{i+\delta_a,\downarrow} - c_i c_{i+\delta_a,\uparrow}$, with $a = 1, 2$ denoting the long and short direction respectively. To reduce finite size effects, a summation over the short direction is performed. In the above formula, $(i_x, i_y)$ is the coordinate of site $i$ in the unit of primitive vectors, $i_{x_0}$ is a reference coordinate in the long direction. We take $i_{x_0} > \frac{L_x}{2}$ to reduce boundary effect, and the final results are averaged over several $i_{x_0}$s. We denote the relative distance in the correlator by $x = i_x - i_{x_0}$.
A. Pairing correlations of the $t$-$J$ model on a triangular lattice

The pairing correlation without four-spin ring exchange is showed here for a comparison. Fig. S2(a) and Fig. S2(b) show the pairing correlation $P_{22}$ and $P_{12}$ of a $L_x = 48$ system with $K/J = 0, t/J = 2$ and 1/16 hole doping. The data for different bond dimensions are showed here, and the magnitude of the pairing correlations show power law behavior, especially when it is extrapolated to infinite bond dimension. As $P_{22}$ and $P_{12}$ have different signs, they show a $d$-wave type pairing.

B. Pairing correlations with four-spin ring exchange

Fig. S3(a) and Fig. S3(b) show the pairing correlation $P_{22}$ and $P_{12}$ of a $L_x = 72$ system with $K/J = 0.8, t/J = 2$ and 1/16 hole doping. The data for bond dimension $m = 1280, 2560$ and $5120$ are showed. As we see, the pairing correlations with four-spin ring exchange have a significant change. First, there is sign oscillation in the pairing correlation, and the oscillation of the pairing correlation shows up even at $m = 1280$, which indicates the robustness of the PDW. Second, the convergence of the magnitude of the pairing correlation with bond dimension is slow, especially for the long distance part. As we increase the bond dimension, the magnitude of the long distance correlation increases significantly, and it becomes increasingly like power law behavior. Thus we interpret the deviation from power law decay at large distances as being due to a lack of convergence.

III. ENTANGLEMENT ENTROPY

As we showed in Fig.4 of the main text, there are three crossings of the Fermi surface, and we expect six gapless modes in the system. To analyze the number of gapless modes in the system, we measured the subsystem entanglement entropy, and estimated the central charge by the formula

$$S(l, N) = \frac{c}{6} \log \left( \frac{N}{\pi \sin \left( \frac{\pi l}{N} \right) } \right) + A$$ (S4)

where $l$ is the length of subsystem, $N$ is the total number of sites, $c$ is the central charge, $A$ is a constant and the $l = 1$ entropy gives a very good estimation of its value when $N$ is sufficiently large. Fig. S4 shows the subsystem entanglement entropy for a $L_x = 48$ system with $K/J = 0.8, t/J = 2$ and 1/8 hole doping. 1/12 and 1/16 hole doping exhibit similar behavior (not shown here). We find that $c$ is clearly larger than 4 and it is consistent with $c = 6.0$, especially near the boundary part where the entanglement entropy is well converged.