The one-dimensional nonlocal Dominative $p$-Laplace equation

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Abstract

The explicit solution to the Dirichlet problem for a class of mean value equations on the real line is derived. It shed some light on the behavior of solutions to general nonlocal elliptic equations.

1 Introduction

Imagine you are standing at a point $x$ on a ledge $[-1, 1]$. A number $-1 \leq r \leq 1$ is picked at random. If it is positive you get to walk $r$ metres to the right, but if it is negative you have to take a step $|r|$ metres towards, and perhaps over, the edge at $x = -1$. The process is repeated until you either have reached solid ground at $x > 1$, or until you fall into the abyss $x < -1$. What are the chances of surviving?

![Figure 1: The probability $y$ of surviving when starting from $x \in [-1, 1]$.](image_url)
We shall see that the answer is

\[ \frac{1}{2} \left( \frac{\cos(1/2)}{1 - \sin(1/2)} \sin(x/2) + 1 - \text{sgn}(x) \left[ 1 - \cos(x/2) \right] \right). \]

It is the solution to the mean value equation

\[ u(x) = \frac{1}{2} \int_{x-1}^{x+1} u(y) \, dy, \quad x \in [-1, 1], \]

with boundary conditions

\[ u(x) = \begin{cases} 0, & -2 \leq x < -1, \\ 1, & 1 < x \leq 2. \end{cases} \]

The equation above is a special case of the nonlinear problem

\[ u(x) = \frac{N+2}{N+p} \int_{B_r(x)} u(y) \, dy + \frac{p-2}{N+p} \sup_{|\xi|=1} \left( \frac{u(x-\epsilon \xi) + u(x + \epsilon \xi)}{2} \right), \quad x \in \overline{\Omega}, \quad (1.1) \]

\[ u(x) = f(x), \quad x \in \Gamma^\epsilon, \quad (1.2) \]

which is investigated in the paper [BLM20]. Here, \( p \in [2, \infty) \) and \( \epsilon > 0 \) are fixed parameters, \( \Omega \subseteq \mathbb{R}^N \) is open and bounded, and \( \Gamma^\epsilon := \{ x \notin \Omega \mid \text{dist}(x, \Omega) \leq \epsilon \} \) is the outer strip of width \( \epsilon \). Furthermore, \( f: \Gamma^\epsilon \cup \partial \Omega \to \mathbb{R} \) is a given bounded and integrable function. (1.1) approximates the Domimative \( p \)-Laplacian equation

\[ 0 = D_p u := \Delta u + (p - 2) \lambda_{\max}(D^2 u) \]

in the sense that, if we denote the mean value operator by \( M^\epsilon_p \), then

\[ M^\epsilon_p \phi(x) - \phi(x) = \frac{\epsilon^2}{2(N+p)} D_p \phi(x) + o(\epsilon^2) \]

as \( \epsilon \to 0 \) for \( C^2 \) functions \( \phi \). The uniformly elliptic operator \( D_p \) was introduced in [Bru20] in order to explain a superposition principle in the \( p \)-Laplace equation.

By Lemma 2.2 in [BLM20] we know that there is a unique solution to (1.1) - (1.2). We are going to derive the solution in the one-dimensional case,

\[ u(x) = \frac{3}{p+1} \cdot \frac{1}{\epsilon^2} \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy + \frac{p-2}{p+1} \cdot \frac{1}{2} \left( u(x - \epsilon) + u(x + \epsilon) \right), \quad x \in [a, b], \]

\[ u(x) = f(x), \quad x \in [a - \epsilon, a) \cup (b, b + \epsilon]. \quad (1.3) \]
That is, when \( N = 1 \) and when \( \Omega = (a, b) \) is an interval in \( \mathbb{R} \). We shall assume that \( \epsilon = (b - a)/n \) for some even number \( n = 2m \), and that \( f \) is continuous. The main result of [BLM20] states that the solutions of (1.1) - (1.2), as \( \epsilon \to 0 \), converge uniformly to the solution of the corresponding local Dirichlet problem. At least for well-behaved domains and boundary values.

In our case, this amounts to the simple equation \( u''(x) = 0 \) and the nonlocal solutions will therefore converge to the affine function with endpoint values \( f(a) \) and \( f(b) \). Our explicit formulas give some insight to the nature of this convergence. In Section 2 we solve the problem for \( p = 2 \). The case \( p > 2 \), including the infinity-equation

\[
\begin{align*}
\hat{u}(x, \tau) &= \left( \frac{1}{2} \right) \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy, \\
\hat{u}(x, \tau) &= \left( \frac{1}{2} \right) \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy, \\
\hat{u}(x, \tau) &= \left( \frac{1}{2} \right) \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy,
\end{align*}
\]

which is obtained by sending \( p \to \infty \), is considered in Section 3.

The stochastic interpretation of the Dirichlet problem (1.3) is as follows. Suppose you start a random walk from \( x_0 \in [a, b] \) where each step is chosen from the \( \epsilon \)-neighbourhood of the previous one according to the rule

\[
\begin{align*}
\text{with probability } \frac{3}{p+1}, \text{ the point } x_{k+1} \in [x_k - \epsilon, x_k + \epsilon] \text{ is picked at random.} \\
\text{with probability } \frac{1}{2} \cdot \frac{p-2}{p+1}, \text{ we set } x_{k+1} = x_k - \epsilon. \\
\text{with probability } \frac{1}{2} \cdot \frac{p-2}{p+1}, \text{ we set } x_{k+1} = x_k + \epsilon.
\end{align*}
\]

You stop the walk once you have left \([a, b]\) at, say, step \( k = \tau \). Then \( u(x_0) \) is the expected value of the random variable \( f(x_\tau) \). In particular, if \( f = 0 \) on \([a - \epsilon, a]\) and \( f = 1 \) on \([b, b + \epsilon]\), then \( u(x) \) is the probability of exiting at the right when starting the walk from \( x \).

Note that the sup disappears in (1.3). Thus, in contrast to the higher dimensional situation, there is no control over the stochastic process in one dimension and the equation remains linear for \( p > 2 \).

\[2\] The nonlocal Laplace equation with uniform distribution

With \( N = 1 \) and \( p = 2 \) the Dirichlet problem reads

\[
\begin{align*}
\begin{cases}
\hat{u}(x) = \frac{1}{2} \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy, & \text{for } x \in [a, b], \\
\hat{u}(x) = f(x), & \text{for } x \in [a - \epsilon, a) \cup (b, b + \epsilon].
\end{cases}
\end{align*}
\]
If \( f \) is constant on each of the two boundary parts, we shall show that \( u \) is a piecewise trigonometric function. Specifically, on each of the \( n \) intervals \([a + (k - 1)\epsilon, a + k\epsilon]\), \( k = 1, \ldots, n \), of length \( \epsilon \), \( u \) is on the form

\[
 u(x) = a_k + \sum_{j=1}^{m} b_{k,j} \sin \left( \frac{\lambda_j}{\epsilon} x \right) + c_{k,j} \cos \left( \frac{\lambda_j}{\epsilon} x \right) \tag{2.2}
\]

for computable coefficients \( a_k, b_{k,j}, c_{k,j} \), and where

\[
 \lambda_j = \cos \left( \frac{j\pi}{n+1} \right). 
\]

When \( f \) is not constant, the above formula is supplemented with some additional terms involving trigonometric convolutions of the data. These terms are written out in (2.10) for the case \([a, b] = [-1, 1]\).

![Figure 2: The solution of (2.1) with constant boundary values and \( n = 4 \).](image)

Although the solution has oscillations for all \( \epsilon = (b-a)/n \), our plots show that the graph is almost indistinguishable from a straight line on the inner part of the interval already from \( n \geq 4 \). However, near the endpoints the graph is unmistakably curved and the numerics indicate that the convergence \( u(a) \to f(a) \) and \( u(b) \to f(b) \) is no better than linear in \( \epsilon \). See Figure 3. On the other hand, the comparison principle (Lemma 2.5 [BLM20]) ensures that the convergence is linear, since the solution has to lie between the two lines passing through the points \((a - \epsilon, f(a)), (b, f(b))\) and \((a, f(a)), (b + \epsilon, f(b))\), respectively. Unfortunately, this argument is purely one-dimensional and does not work for \( N \geq 2 \).
2.1 The functional differential equation

One may easily show that the solution \( u \) of (2.1) is \( C/(2\epsilon) \)-Lipschitz on \( [a, b] \), where \( C = \max f - \min f \). It follows that \( u \) is differentiable in \( (a, b) \setminus \{a + \epsilon, b - \epsilon\} \) with

\[
u'(x) = \frac{1}{2\epsilon} \left( u(x + \epsilon) - u(x - \epsilon) \right).
\]  

(2.3)

\( u \) is generally not differentiable at \( a + \epsilon \) or \( b - \epsilon \) since it is no reason \( u \) should be continuous at \( a \) or \( b \). A close inspection of Figure 2 reveals kinks in the graph of \( u \) at these points. The formula (2.3) provides, however, a weak derivative for \( u \) on \( [a, b] \).

2.2 Constant boundary values

Let \( u \) be the solution of

\[
\begin{aligned}
u(x) &= \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy, & \text{for } x \in [a, b], \\
u(x) &= c_l, & \text{for } x \in [a - \epsilon, a), \\
u(x) &= c_r, & \text{for } x \in (b, b + \epsilon].
\end{aligned}
\]  

(2.4)

where \( a < b \), \( \epsilon = (b-a)/n \), for some even number \( n = 2m \), and where \( c_l \neq c_r \) are two constants. If \( c_l = c_r \) the solution is identically equal to this common constant.

To exploit the symmetry in the problem, we assume that \( u \) is scaled, shifted, and translated so that \( [a, b] = [-1, 1] \), and \( c_l = -1 \) and \( c_r = 1 \). The solution will then be odd. This is possible because the equation is linear and translation invariant. Also, constants are solutions.

Divide the domain into \( n \) parts of length \( \epsilon \). For \( k = 1, \ldots, n \) define the
functions $v_k: [0, 1] \to \mathbb{R}$ as

$$v_k(t) := \frac{2}{c_r - c_l} \left( u(et + a + (k - 1)e) - \frac{c_l + c_r}{2} \right)$$  \hspace{1cm} (2.5)$$

$$= u(et - 1 + (k - 1)e).$$

Now, each $v_k$ is differentiable in $(0, 1)$. For $k = 1$ we have by (2.3)

$$v'_1(t) = \frac{1}{2} \left( u'(et - 1) \right)$$

Similarly, $v'_n(t) = \frac{1}{2} \left( 1 - v_{n-1}(t) \right)$, and for $k = 2, \ldots, n - 1$ we simply have

$$v'_k(t) = \frac{1}{2} \left( v_{k+1}(t) - v_{k-1}(t) \right).$$

This defines a non-homogeneous linear system of ODEs,

$$
\begin{bmatrix}
  v'_1 \\
  v'_2 \\
  v'_3 \\
  \vdots \\
  v'_{n-1} \\
  v'_n
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
  0 & 1 & 0 & 0 & \cdots & 0 \\
  -1 & 0 & 1 & 0 & \cdots & 0 \\
  0 & -1 & 0 & 1 & \cdots & 0 \\
  \vdots \\
  0 & \cdots & 0 & -1 & 0 & 1 \\
  0 & \cdots & 0 & -1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  \vdots \\
  v_{n-1} \\
  v_n
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
  1 \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  1
\end{bmatrix}
$$

or

$$\mathbf{v}'(t) = A\mathbf{v}(t) + \mathbf{c}$$  \hspace{1cm} (2.6)$$

in vector notation. The general solution is \( \mathbf{v}(t) = e^{tA}(\mathbf{v}_0 + A^{-1}\mathbf{c}) - A^{-1}\mathbf{c} \).

where

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

is the matrix exponential.

Defining

$$\tilde{e} := \sum_{k=1}^{n} (-1)^k e_k,$$
we note that

\[
A\tilde{e} = \frac{1}{2} \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
-1 \\
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
= \frac{1}{2} \begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
1
\end{bmatrix}
= c
\]

and thus

\[
v(t) = e^{tA}(v_0 + \tilde{e}) - \tilde{e}.
\] (2.7)

The initial value \(v_0 = v(0)\) contains the values of \(u\) at the nodes \(-1 + (k-1)\epsilon\) and is of course unknown. By definition, we have the \(n-1\) identities \(v_k(0) = v_{k-1}(1), \quad k = 2, \ldots, n\). An \(n\)'th equation can be obtained by using the fact that \(u\) is an odd function. For example, \(u(-1) = -u(1)\) which corresponds to \(v_1(0) = -v_n(1)\). Thus,

\[
v(0) = Bv(1)
\] (2.8)

where \(B\) is the orthogonal matrix

\[
B := \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \cdots & 1 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
0^\top & -1 \\
I_{n-1} & 0
\end{bmatrix}.
\]

Inserting (2.7) into (2.8) with \(t = 1\), the continuity of \(u\) in \([-1, 1]\) gives a linear equation for \(v_0\),

\[
(I - Be^A)v_0 = B(e^A - I)\tilde{e},
\]

and the unique solution of (2.6), (2.8) is, after some simplifications,

\[
v(t) = e^{tA}(I - Be^A)^{-1}(I - B)\tilde{e} - \tilde{e}, \quad 0 \leq t \leq 1.
\]

The invertability of \(I - Be^A\) is discussed later. Observe that \(v(t)\) does not depend on anything but \(n\).

The solution \(u\) of the original problem (2.4) can now be assembled by inverting the relations in (2.5):
\[ u(x) = \begin{cases} c_l, & x \in [a - \epsilon, a), \\ \frac{x - c_l}{\epsilon} v_k \left( \frac{x - a}{\epsilon} + 1 - k \right) + \frac{x + c_l}{\epsilon}, & x \in [a + (k - 1)\epsilon, a + k\epsilon], \quad k = 1, \ldots, n, \\ c_r, & x \in (b, b + \epsilon]. \end{cases} \]

\[ = \begin{cases} -1, & x \in [-1 - \epsilon, -1), \\ v_k \left( \frac{x + 1}{\epsilon} + 1 - k \right), & x \in [-1 + (k - 1)\epsilon, -1 + k\epsilon], \quad k = 1, \ldots, n, \\ 1, & x \in (1, 1 + \epsilon]. \end{cases} \]

\section*{2.3 Analysis}

The coefficient matrix of the linear system is the skew-symmetric and tridiagonal \( n \times n \) matrix

\[ A := \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix} \]

where \( n = 2m \) is even.

In order to diagonalize \( A \), we define the vectors \( \xi_j \in \mathbb{C}^n \) with components

\[ e_k^\top \xi_j = c_n i^{k+2j} \sin \left( kj \frac{\pi}{n+1} \right), \quad c_n := \sqrt{\frac{2}{n+1}}, \quad i := \sqrt{-1}. \]

They have unit length since

\[ \sum_{k=1}^n \sin^2 \left( kj \frac{\pi}{n+1} \right) = \frac{n + 1}{2} \]

by Lagrange’s identity. Using the rule

\[ \sin \theta + \sin \phi = 2 \cos \left( \frac{\theta - \phi}{2} \right) \sin \left( \frac{\theta + \phi}{2} \right), \]
shows that
\[
\mathbf{e}_k^\top A \xi_j = \frac{c_n}{2} (i^{k+2j+1} \sin \left( (k+1)j \frac{\pi}{n+1} \right) - i^{k-1+2j} \sin \left( (k-1)j \frac{\pi}{n+1} \right))
= \frac{i^{k+1+2j} c_n}{2} \left( \sin \left( (k+1)j \frac{\pi}{n+1} \right) + \sin \left( (k-1)j \frac{\pi}{n+1} \right) \right)
= i^{k+1+2j} c_n \cos \left( j \frac{\pi}{n+1} \right) \sin \left( k \frac{\pi}{n+1} \right)
= i \cos \left( j \frac{\pi}{n+1} \right) \mathbf{e}_k^\top \xi_j.
\]
That is, \( \xi_j \) is an eigenvector to \( A \) with eigenvalue \( i \cos \left( j \frac{\pi}{n+1} \right) \). Since the eigenvalues are distinct, this also implies that the eigenvectors are orthogonal. Moreover, for \( j = 1, \ldots, m \) one can show that \( \xi_{n+1-j} = \bar{\xi}_j \), and \( A \) is thus diagonalized by the unitary matrix
\[
U := [\xi_1, \bar{\xi}_1, \ldots, \xi_m, \bar{\xi}_m] \in \mathbb{C}^{n \times n},
\]
producing
\[
\Lambda := \mathbf{U}^\top A U = \text{diag}(i\lambda_1, -i\lambda_1, \ldots, i\lambda_m, -i\lambda_m)
\]
where
\[
0 < \lambda_j := \cos \left( \frac{j}{n+1} \pi \right) < 1, \quad j = 1, \ldots, m.
\]
The real and imaginary parts of the eigenvectors \( \xi_j = a_j + i b_j \) are
\[
a_j = c_n \sum_{k=1}^{m} (-1)^{k+j} \sin \left( 2k \right) \frac{\pi}{n+1} \mathbf{e}_{2k},
\]
\[
b_j = c_n \sum_{k=1}^{m} (-1)^{k+j+1} \sin \left( (2k-1) \right) \frac{\pi}{n+1} \mathbf{e}_{2k-1}.
\]
Here,
\[
a_j^\top b_k = 0 \quad \text{and} \quad a_j^\top a_k = b_j^\top b_k = \delta_{j,k}
\]
since \( \xi_j^\top \xi_k = 0 \) and \( \bar{\xi}_j^\top \xi_k = \delta_{j,k} \). If we define the real skew-symmetric matrices
\[
A_j := -2 \text{Im} \, \xi_j^\top \bar{\xi}_j = 2(a_j b_j^\top - b_j a_j^\top), \quad j = 1, \ldots, m,
\]
then
\[
A = U \Lambda U^T = \sum_{j=1}^{m} i \lambda_j \left( \xi_j \xi_j^\top - \xi_j^\top \xi_j \right) = \sum_{j=1}^{m} \lambda_j A_j.
\]

Furthermore,
\[
A_j^2 = -2(a_j a_j^\top + b_j b_j^\top) = -2 \text{Re} \xi_j \xi_j^\top
\]
is the negative of a two-rank symmetric projection and
\[
e^{tA} = U e^{t \Lambda} U^T = \sum_{j=1}^{m} e^{i \lambda_j t} \xi_j \xi_j^\top + e^{-i \lambda_j t} \xi_j^\top \xi_j = 2 \text{Re} \sum_{j=1}^{m} e^{i \lambda_j t} \xi_j \xi_j^\top
\]
\[
= \sum_{j=1}^{m} \sin(\lambda_j t) A_j - \cos(\lambda_j t) A_j^2.
\]
The solution on the interval \([-1 + (k - 1)\epsilon, -1 + k\epsilon]\) is therefore
\[
u(x) = v_k \left( \frac{x + 1}{\epsilon} + 1 - k \right)
\]
\[
= e_k^\top v \left( \frac{x + 1}{\epsilon} + 1 - k \right)
\]
\[
= e_k^\top e^{(m+1-k)A} (v_0 + \tilde{e}) - e_k^\top \tilde{e}
\]
\[
= (1)^{k+1} + e_k^\top e^{(m+1-k)A} e^{mxA} (v_0 + \tilde{e}), \quad \frac{1}{\epsilon} = m,
\]
\[
= (1)^{k+1} + e_k^\top e^{(m+1-k)A} \left( \sum_{j=1}^{m} \sin(\lambda_j mx) A_j - \cos(\lambda_j mx) A_j^2 \right) (v_0 + \tilde{e}),
\]
which is on the form \((2.2)\).

### 2.4 The case \(n = 2\)

When \(n = 2\),
\[
A = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]
with eigenvalues ±i/2. We have \( \lambda_1 = 1/2 = \cos(1 \cdot \pi / (n + 1)) \) and

\[
A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

so that \( A = \lambda_1 A_1 \). Next, \( A_1^2 = -I \) and it follows that

\[
e^{tA} = \sin(\lambda_1 t) A_1 + \cos(\lambda_1 t) I = \begin{bmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{bmatrix}.
\]

Also, \( B = -A_1 = A_1^\top \) and

\[
I - Be^A = (1 - \sin(1/2)) I + \cos(1/2) A_1 = \begin{bmatrix} 1 - \sin(1/2) & \cos(1/2) \\ -\cos(1/2) & 1 - \sin(1/2) \end{bmatrix}
\]

with inverse

\[
(I - Be^A)^{-1} = \frac{1}{2(1 - s)} \begin{bmatrix} 1 - s & -c \\ c & 1 - s \end{bmatrix} = \frac{(I - Be^A)^\top}{2(1 - s)} = \frac{I - e^{-A}A_1}{2(1 - s)}.
\]

This makes

\[
v_0 = (I - Be^A)^{-1}(Be^A - B)\hat{e}
= \frac{1}{2(1 - s)}(I - A_1e^{-A})(I - e^A)A_1\hat{e}
= \frac{1}{2(1 - s)}(I - sA_1 - cI - A_1(-sA_1 + cI) + A_1)A_1\hat{e}
= \frac{1 - \sin(1/2) - \cos(1/2)}{1 - \sin(1/2)} \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

and

\[
v(t) = e^{tA}(v_0 + \hat{e}) - \hat{e}
= \begin{bmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{bmatrix} \begin{bmatrix} -\cos(1/2) \\ 1 - \sin(1/2) \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\]

For \( k = 1, 2 \), the solution in \([-2+k, -1+k]\) is therefore \( u(x) = v_k(x + 2 - k) \).

That is,

\[
u(x) = \begin{cases} -1, & x \in [-2, -1), \\ -C \cos(x/2 + 1/2) + \sin(x/2 + 1/2) + 1, & x \in [-1, 0], \\ C \sin(x/2) + \cos(x/2) - 1, & x \in [0, 1] \\ 1, & x \in (1, 2], \end{cases}
\]

\[
= \begin{cases} -1, & x \in [-2, -1), \\ C \sin(x/2) - \text{sgn}(x)[1 - \cos(x/2)], & x \in [-1, 1], \\ 1, & x \in (1, 2], \end{cases}
\]
where
\[ C := \frac{\cos(1/2)}{1 - \sin(1/2)}. \]

It is the graph of \((u + 1)/2\) that is shown in Figure 1.

### 2.5 Non-constant \(f\)

Let \(u\) be the solution of

\[
\begin{cases}
  u(x) = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy, & \text{for } x \in [-1, 1], \\
  u(x) = f(x), & \text{for } x \in [-1 - \epsilon, -1) \cup (1, 1 + \epsilon],
\end{cases}
\]

where \(\epsilon = 2/n\). Translate the data to the interval \([0, 1]\) by writing

\[
f_l(t) := f(\epsilon t - 1 - \epsilon), \quad f_r(t) := f(\epsilon t + 1),
\]

and, as usual, define the functions \(v_k : [0, 1] \to \mathbb{R}\) as

\[
v_k(t) = u(x_k + \epsilon t), \quad x_k := -1 + (k - 1)\epsilon.
\]

As before,

\[
v_k'(t) = \frac{1}{2} \left( v_{k+1}(t) - v_{k-1}(t) \right)
\]

for \(k = 2, \ldots, n - 1\), and now

\[
\begin{align*}
v_1'(t) &= \frac{1}{2} \left( v_2(t) - f_l(t) \right), \\
v_n'(t) &= \frac{1}{2} \left( f_r(t) - v_{n-1}(t) \right).
\end{align*}
\]

The system then reads

\[
\begin{bmatrix}
v_1' \\
v_2' \\
v_3' \\
\vdots \\
v_{n-1}' \\
v_n'
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 0 & 1 \\
0 & \cdots & 0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
\vdots \\
v_{n-1} \\
v_n
\end{bmatrix} + \frac{1}{2}
\begin{bmatrix}
f_l \\
f_2 \\
f_3 \\
\vdots \\
f_{n-1} \\
f_n
\end{bmatrix},
\]

or

\[
v'(t) = \mathbf{A}v(t) + \mathbf{f}(t)
\]
in obvious notation. The general solution is
\[ v(t) = e^{tA} \left( v_0 + \int_0^t e^{-sA} f(s) \, ds \right), \quad 0 \leq t \leq 1, \] (2.9)
and the solution \( u \) on the \( k \)'th interval \([x_k, x_{k+1}]\) is
\[ u(x) = e_k^T v \left( \frac{x + 1}{\epsilon} + 1 - k \right) \]
\[ = e_k^T e^{\left(\frac{x + 1}{\epsilon} + 1 - k\right)A} \left( v_0 + \int_0^x e^{-sA} f(s) \, ds \right) \]
\[ = e_k^T e^{\left(\frac{x + 1}{\epsilon} + 1 - k\right)A} v_0 + \frac{1}{\epsilon} \int_{-1+(k-1)\epsilon}^x e_k^T e^{\frac{y+1}{\epsilon} + 1 - k} f \left( y + \frac{1}{\epsilon} + 1 - k \right) \, dy. \] (2.10)

In order to find an expression for the initial condition \( v_0 = v(0) \) we still have the \( n - 1 \) identities
\[ v_k(0) = v_{k-1}(1), \quad k = 2, \ldots, n, \]
but now we cannot assume \( u \) to be odd and \( v_1(0) = -v_n(1) \) is therefore not valid. Thus we are one equation short of determining \( v_0 \).

Set
\[ F_0 := \int_{\Gamma} f \, dx \]
to be the average of the data and let \( U_0 := \int_{-1}^1 u \, dx \) be the total integral of the solution. Then
\[ \sum_{k=1}^n v_k(0) + v_n(1) = \sum_{k=0}^n u(-1 + k\epsilon) \]
\[ = \frac{1}{2\epsilon} \sum_{k=0}^n \int_{-1+(k-1)\epsilon}^{-1+(k+1)\epsilon} u(y) \, dy \]
\[ = \frac{1}{2\epsilon} \left( \int_{-1-\epsilon}^{-1} f(y) \, dy + 2 \int_{-1}^1 u(y) \, dy + \int_1^{1+\epsilon} f(y) \, dy \right) \]
\[ = F_0 + \frac{1}{\epsilon} U_0, \]
while
\[
\sum_{k=1}^{m} v_{2k-1}(0) + v_{n}(1) = \sum_{k=0}^{m} u(-1 + 2k\epsilon)
\]
\[
= \frac{1}{2\epsilon} \sum_{k=0}^{m} \int_{-1+(2k-1)\epsilon}^{-1+(2k+1)\epsilon} u(y) \, dy
\]
\[
= F_0 + \frac{1}{2\epsilon} U_0.
\]

It follows that
\[
F_0 = 2 \left( \sum_{k=1}^{m} v_{2k-1}(0) + v_{n}(1) \right) - \left( \sum_{k=1}^{n} v_k(0) + v_n(1) \right)
\]
\[
= \sum_{k=1}^{n} (-1)^{k-1} v_{k}(0) + v_{n}(1)
\]
\[
= v_1(0) + \sum_{k=1}^{n} (-1)^k v_{k}(1).
\]

We now have \( n \) equations for \( v_0 \), namely

\[
\begin{bmatrix}
  v_1(0) \\
  v_2(0) \\
  v_3(0) \\
  \vdots \\
  v_n(0)
\end{bmatrix} =
\begin{bmatrix}
  1 & -1 & 1 & -1 & \cdots & -1 \\
  1 & 0 & 0 & 0 & \cdots & 0 \\
  0 & 1 & 0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & 1 & 0 & \cdots
\end{bmatrix}
\begin{bmatrix}
  v_1(1) \\
  v_2(1) \\
  v_3(1) \\
  \vdots \\
  v_n(1)
\end{bmatrix} +
\begin{bmatrix}
  F_0 \\
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix},
\]

or
\[
v_0 = \tilde{B} v(1) + F_0 e_1.
\]

By the general solution (2.9) we may write
\[
v_0 = \tilde{B} e A (v_0 + \int_0^1 e^{-sA} f(s) \, ds) + F_0 e_1,
\]

which means that
\[
v_0 = (I - \tilde{B} e A)^{-1} \left( \tilde{B} e A \int_0^1 e^{-sA} f(s) \, ds + F_0 e_1 \right).
\]

It seems hard to prove that \( I - Be A \) and \( I - \tilde{B} e A \) are invertible or, equivalently, that \( Be A \) and \( \tilde{B} e A \) does not have an eigenvalue equal to 1.
The matrix \( Be^{tA} \) is orthogonal for all \( t \in \mathbb{R} \) and thus have eigenvalues on the unit circle in \( \mathbb{C} \). The numerics indicate that \( \det(I - Be^{tA}) > 0 \) for all \( t \in [0, 1] \) and in all even dimensions \( n \). However, the value \( t = 1 \) seems to play a special role: If we define \( t_n \) to be the smallest positive number such that \( \det(I - Be^{t_n A}) = 0 \), we conjecture that the sequence \((t_n)\) is strictly decreasing with \( \lim_{n \to \infty} t_n = 1 \).

3 The general case \( 2 \leq p \leq \infty \)

Recall that the Dirichlet problem for \( p > 2 \) is

\[
\begin{align*}
  u(x) &= \frac{3}{p+1} \cdot \frac{1}{2x} \int_{x-\epsilon}^{x+\epsilon} u(y) \, dy + \frac{p-2}{p+1} \cdot \frac{1}{2} (u(x-\epsilon) + u(x+\epsilon)) \quad x \in [-1, 1], \\
  u(x) &= f(x), \quad x \in [-1 - \epsilon, -1], \\
  u(x) &= f(x), \quad x \in (1, 1 + \epsilon],
\end{align*}
\]

where \( \epsilon = 2/n \).

As usual, we let \( x_k := -1 + (k - 1)\epsilon \) denote the nodes, and we define the functions \( v_k : [0, 1] \to \mathbb{R} \) as

\[
v_k(t) := u(x_k + \epsilon t), \quad k = 0, \ldots, n + 1.
\]

Also, \( f_l, f_r : [0, 1] \to \mathbb{R} \) are given by

\[
f_l(t) := f(x_0 + \epsilon t), \quad f_r(t) := f(x_{n+1} + \epsilon t).
\]

In order to derive the solution, it is instructive to first look at the case \( p = \infty \). The equation is then

\[
u(x) = \frac{1}{2} \left( u(x-\epsilon) + u(x+\epsilon) \right),
\]
and for $k = 1, \ldots, n$ we have

$$v_k(t) = \frac{1}{2} (v_{k-1}(t) + v_{k+1}(t)).$$

Now, $v_0(t) = f_l(t)$ for $0 \leq t < 1$ and $v_{n+1}(t) = f_r(t)$ for $0 < t \leq 1$. Thus,

$$v(t) = \frac{1}{2} (L + L^\top) v(t) + \frac{1}{2} (f_l(t) e_1 + f_r(t) e_n) \quad \text{for } 0 < t < 1, \quad (3.1)$$

where

$$v(t) := [v_1(t), \ldots, v_n(t)]^\top \in \mathbb{R}^n, \quad L := \begin{bmatrix} 0^\top & 0 \\ I_{n-1} & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

The well-known matrix $2I - L - L^\top$ is invertible and a direct calculation confirms that $(2I - L - L^\top) w_l = e_1$ and $(2I - L - L^\top) w_r = e_n$ where

$$w_l := \sum_{k=1}^{n} \left( 1 - \frac{k}{n+1} \right) e_k, \quad w_r := \sum_{k=1}^{n} \frac{k}{n+1} e_k.$$

The solution of the linear equation (3.1) is therefore

$$v(t) = \sum_{k=1}^{n} \left[ \left( 1 - \frac{k}{n+1} \right) f_l(t) + \frac{k}{n+1} f_r(t) \right] e_k, \quad 0 < t < 1.$$

Figure 5: The solution of (1.3) with $p = \infty$ and $n = 4$. The dotted line indicates how $u(x)$ can be constructed from the boundary values.
For $t = 0$ we have
\[ v_k(0) = \begin{cases} f_0(0) + v_2(0), & \text{for } k = 1, \\ v_{k-1}(0) + v_{k+1}(0), & \text{for } k = 2, \ldots, n, \\ v_n(0) + f_r(1), & \text{for } k = n + 1. \end{cases} \]

If we define
\[ \hat{v}_0 := [v_1(0), \ldots, v_n(0), v_{n+1}(0)]^\top \in \mathbb{R}^{n+1}, \quad \hat{L} := \begin{bmatrix} 0^\top & 0 \\ I_n & 0 \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}, \]
we get the same equation for $\hat{v}_0$ as we did for $v(t)$, but in one dimension higher. Thus,
\[ \hat{v}_0 = \sum_{k=1}^{n+1} \left[ \left( 1 - \frac{k}{n+2} \right) f_i(0) + \frac{k}{n+2} f_r(1) \right] \hat{e}_k. \]

Figure 5 shows the graph of the solution to the infinity-equation
\[ u(x) = \frac{1}{2} (u(x - 1/2) + u(x + 1/2)), \quad x \in [-1, 1], \]
with boundary values
\[ u(x) = \begin{cases} \frac{1}{8} \sin \left( 4\pi (x + 3/2) \right) - 1, & x \in [-3/2, -1), \\ \frac{9}{8} - \frac{1}{12}, & x \in (1, 3/2]. \end{cases} \]

Notice the jumps $\lim_{x \to x_k^-} u(x) < u(x_k) < \lim_{x \to x_k^+} u(x)$.

We now turn to the case $2 \leq p < \infty$. For $k = 0, \ldots, n + 1$ define the integrals
\[ V_k := V_k(1) \quad \text{where} \quad V_k(t) := \int_0^t v_k(s) \, ds. \]
Then
\[ V_0(t) = \int_0^t f_i(s) \, ds =: F_i(t) \quad \text{and} \quad V_{n+1}(t) = \int_0^t f_r(s) \, ds =: F_r(t). \]
The equation can be written as
\[ v_k(t) = u(x_k + \epsilon t) \]
\[ = \frac{3}{p+1} \cdot \frac{1}{2\epsilon} \int_{x_{k-1} + \epsilon t}^{x_{k+1} + \epsilon t} u(y) \, dy + \frac{p-2}{p+1} \cdot \frac{1}{2} \left( u(x_{k-1} + \epsilon t) + u(x_{k+1} + \epsilon t) \right) \]
\[ = \frac{3}{p+1} \cdot \frac{1}{2} \left( V_{k-1} - V_{k-1}(t) + V_k + V_{k+1}(t) \right) + \frac{p-2}{p+1} \cdot \frac{1}{2} \left( v_{k-1}(t) + v_{k+1}(t) \right) \]
for \( k = 1, \ldots, n \). We have \( v_0(t) = f_l(t) \) when \( 0 \leq t < 1 \) and \( v_{n+1}(t) = f_r(t) \) when \( 0 < t \leq 1 \). Thus,

\[
V'(t) = v(t) = \frac{3}{p+1} \cdot \frac{1}{2} ((L^\top - L)V(t) + F_r(t)e_n - F_l(t)e_1 + (L + I)V + F_l e_1) \\
+ \frac{p-2}{p+1} \cdot \frac{1}{2} ((L + L^\top)V'(t) + f_l(t)e_1 + f_r(t)e_n).
\]

That is,

\[
E_p V'(t) = A V(t) + F(t) + \frac{1}{2} (L + I)V, \quad 0 < t < 1,
\]

(3.2)

where

\[
E_p := \frac{1}{3} \left( (p+1)I - \frac{p-2}{2} (L + L^\top) \right),
\]

and where

\[
F(t) := \frac{1}{2} \left( (F_l(1) - F_l(t))e_1 + F_r(t)e_n \right) + \frac{p-2}{6} (f_l(t)e_1 + f_r(t)e_n).
\]

The matrix \( E_p \) is invertible since it is diagonal dominant. Since \( V(t) \) is continuous with \( V(0) = 0 \), the solution of (3.2) is

\[
V(t) = e^{t E_p^{-1} A} \left( 0 + \int_0^t e^{-s E_p^{-1} A} \left[ F(s) + \frac{1}{2} (L + I)V \right] ds \right)
\]

\[
= e^{t E_p^{-1} A} \int_0^t e^{-s E_p^{-1} A} F(s) ds + \frac{1}{2} \left( e^{t E_p^{-1} A} - I \right) A^{-1} (L + I)V.
\]

(3.3)

This gives the linear equation

\[
\left[ I - \frac{1}{2} \left( e^{E_p^{-1} A} - I \right) A^{-1} (L + I) \right] V = e^{E_p^{-1} A} \int_0^1 e^{-s E_p^{-1} A} E_p^{-1} F(s) ds
\]

(3.4)

for \( V = V(1) \), which, numerically, seems to be nondegenerate. In fact, one can show that

\[
2A \left[ I - \frac{1}{2} \left( e^{E_p^{-1} A} - I \right) A^{-1} (L + I) \right] (L^\top + I)^{-1} = I - e^{A E_p^{-1} \tilde{B}}
\]

and the question of solvability of (3.4) is, at least for \( p \) close to 2, equivalent to the solvability for \( v_0 \) in the previous Section.
The formula for $u$ on the interior of the intervals $(x_k, x_{k+1})$ follows now from (3.2):

$$v(t) = E_{p}^{-1} \left( AV(t) + F(t) + \frac{1}{2}(L + I)V \right), \quad 0 < t < 1.$$

When $t = 0$ we have

$$v_k(0) = \frac{3}{p + 1} \cdot \frac{1}{2} (V_{k-1} + V_k) + \frac{p - 2}{p + 1} \cdot \frac{1}{2} (v_{k-1}(0) + v_{k+1}(0))$$

for $k = 1, \ldots, n$. Now,

$$v_1(0) = \frac{3}{p + 1} \cdot \frac{1}{2} (F_l(1) + V_1) + \frac{p - 2}{p + 1} \cdot \frac{1}{2} (f_l(0) + v_2(0)),$$

and

$$v_{n+1}(0) = \frac{3}{p + 1} \cdot \frac{1}{2} (V_n + F_r(1)) + \frac{p - 2}{p + 1} \cdot \frac{1}{2} (v_n(0) + f_r(1)).$$

We set

$$Q := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+1) \times n},$$

and get the solvable equation

$$\hat{E}_{p} \hat{v}_0 = \frac{1}{2} \left( \hat{L} + \hat{I} \right) Q V + \frac{1}{2} (F_l(1) \hat{e}_1 + F_r(1) \hat{e}_{n+1}) + \frac{p - 2}{6} (f_l(0) \hat{e}_1 + f_r(1) \hat{e}_{n+1})$$

for $\hat{v}_0 := [v_1(0), \ldots, v_{n+1}(0)]^\top$ in $\mathbb{R}^{n+1}$.

Some simplifications can be made when the boundary values are constant.

If $f_l(t) = -1$ and $f_r(t) = 1$, then

$$F(t) = \frac{1}{2} \left( (F_l - F_l(t)) \hat{e}_1 + F_r(t) \hat{e}_n \right) + \frac{p - 2}{6} (f_l(t) \hat{e}_1 + f_r(t) \hat{e}_n)$$

$$= \frac{t}{2} (\hat{e}_1 + \hat{e}_n) + \frac{p - 2}{6} \hat{e}_n - \frac{p + 1}{6} \hat{e}_1$$

$$= \frac{t}{2} (\hat{e}_1 + \hat{e}_n) + \frac{p - 2}{6} (\hat{e}_n - \hat{e}_1) + \frac{1}{2} (L + I) \hat{e},$$

and an integration by parts,

$$\int_0^t se^{-sE_p^{-1}A}E_p^{-1} ds = -\int_0^t se^{-sE_p^{-1}A}(E_p^{-1}A)^{-1}E_p^{-1} + \int_0^t e^{-sE_p^{-1}A}(E_p^{-1}A)^{-1}E_p^{-1} ds$$

$$= -t e^{-tE_p^{-1}A} - (e^{-tE_p^{-1}A} - I) A^{-1}E_pA^{-1},$$

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Figure 6: The solution of (1.3) with \( p = 5, n = 4 \), and boundary data \( \pm 1 \).

yields

\[
\mathbf{V}(t) = e^{tE_p^{-1}A} \int_0^t e^{-sE_p^{-1}A} E_p^{-1} \left[ \frac{2}{p} (e_1 + e_n) + \frac{p-2}{6} (e_n - e_1) + \frac{1}{2} (L + I)(\mathbf{V} + \hat{\mathbf{e}}) \right] \, ds \\
= e^{tE_p^{-1}A} \left( -te^{-tE_p^{-1}A} A^{-1} - \left( e^{-tE_p^{-1}A} - I \right) A^{-1} E_p A^{-1} \right) \frac{1}{2} (e_1 + e_n) \\
+ e^{tE_p^{-1}A} \int_0^t e^{-sE_p^{-1}A} E_p^{-1} \left[ \frac{p-2}{6} (e_n - e_1) + \frac{1}{2} (L + I)(\mathbf{V} + \hat{\mathbf{e}}) \right] \, ds \\
= - \left( tI + \left( I - e^{tE_p^{-1}A} \right) A^{-1} E_p \right) \hat{\mathbf{e}} \\
- \left( I - e^{tE_p^{-1}A} \right) A^{-1} \left( \frac{p-2}{6} (e_n - e_1) + \frac{1}{2} (L + I)(\mathbf{V} + \hat{\mathbf{e}}) \right) \\
= (e^{tE_p^{-1}A} - I) A^{-1} \left( \frac{p-2}{6} (e_n - e_1) + \frac{1}{2} (L + I)(\mathbf{V} + \hat{\mathbf{e}}) + E_p \hat{\mathbf{e}} \right) - t\hat{\mathbf{e}}.
\]

The equation for \( \mathbf{V} + \hat{\mathbf{e}} \) is then

\[
\left[ I - \frac{1}{2} (e^{tE_p^{-1}A} - I) A^{-1} (L + I) \right] (\mathbf{V} + \hat{\mathbf{e}}) = \left( e^{tE_p^{-1}A} - I \right) A^{-1} \left( \frac{p-2}{6} (e_n - e_1) + E_p \hat{\mathbf{e}} \right),
\]

and a differentiation gives

\[
\mathbf{v}(t) = \mathbf{V}'(t) = e^{tE_p^{-1}A} \left( \frac{p-2}{6} E_p^{-1} (e_n - e_1) + \frac{1}{2} E_p^{-1} (L + I) (\mathbf{V} + \hat{\mathbf{e}}) + \hat{\mathbf{e}} \right) - \hat{\mathbf{e}}.
\]

Finally, the equation for the values \( \hat{\mathbf{v}}_0 := [v_1(0), \ldots, v_{n+1}(0)]^T \) at the
The Figures 6 - 8 show the graph of the solution to (1.3) for various \( n \) and \( p \), and when the boundary values are constant. As \( p \to \infty \) the solution converges (slowly) to the solution of the infinity-equation.
When $p > 2$, the matrix $E_p^{-1}A$ is not skew-symmetric and $e^{tE_p^{-1}A}$ is no longer orthogonal. However, since $E_p$ is symmetric and positive definite, the eigenvalues of $E_p^{-1}A$ are still purely imaginary and the eigenvalues of $e^{tE_p^{-1}A}$ are again on the form $\cos(\lambda t) + i\sin(\lambda t)$. The mean value solutions are therefore piecewise trigonometric also in the case $2 < p < \infty$.

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