A Tree Search Method for Iterative Decoding of Underdetermined Multiuser Systems

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Abstract—Application of the turbo principle to multiuser decoding results in an exchange of probability distributions between two sets of constraints. Firstly, constraints imposed by the multiple-access channel, and secondly, individual constraints imposed by each users’ error control code. A-posteriori probability computation for the first set of constraints is prohibitively complex for all but a small number of users. Several lower complexity approaches have been proposed in the literature. One class of methods is based on linear filtering (e.g. LMMSE). A more recent approach is to compute approximations to the posterior probabilities by marginalising over a subset of sequences (list detection). Most of the list detection methods are restricted to non-singular systems. In this paper, we introduce a transformation that permits application of standard tree-search methods to underdetermined systems. We find that the resulting tree-search based receiver outperforms existing methods.

I. INTRODUCTION

It is well known that joint decoding can improve performance in multiple-access systems. Joint maximum likelihood (ML) decoding, which minimizes the overall probability of error is however prohibitively complex [1]. Brute force computation of the jointly ML codeword sequences for $K$ users is $O(Q^{K^2})$ for $Q$-ary modulation and constraint length $K$ codes.

The good performance and low complexity of the turbo principle to joint multiuser decoding. Figure 1 shows a schematic representation of the “canonical” iterative multiuser decoder [3–6]. This decoder treats the users’ forward error correction codes as an “outer code” and the interdependency introduced by the multiple access channel as an “inner code”. The decoder iterates between a-posteriori probability (APP) computation for the inner code and individual APP decoding of each user’s FEC code. The multiuser APP computation is $O(Q^K)$, an improvement over joint ML decoding, but still prohibitive.

One low complexity alternative is to replace the inner APP decoder with a linear filter. Examples include soft interference cancellation [7, 8] and linear minimum mean-squared error filtering [9]. These approaches can work quite well, but there is still room for improvement compared to the exact computation of the multiuser APP.

A more powerful approach is to compute an approximation of the multiuser APP by marginalizing over a subset of sequences (in many cases only a small subset is required), found using various different list detectors [12–19]. Most of these list-detection based methods rely on Cholesky decomposition of the correlation matrix as a first step. In underdetermined systems (more users than signaling dimensions), this decomposition cannot be performed. First steps towards avoiding this problem have been made in [20, 21], based on filtering, followed by lattice reduction.

The main contribution of this paper is a simple transformation which creates a virtual full rank system, which permits Cholesky decomposition and the straightforward application of tree-search methods in underdetermined multiuser systems. The method requires less computational complexity than the one described in [20, 21]. Numerical results demonstrate the superior performance of the approach, which is compared to other techniques from the literature.

II. SYSTEM MODEL AND CANONICAL DECODER

Consider a multiple-access system with $K$ radio terminals, or users, simultaneously transmitting forward error correction coded digital data across an additive white Gaussian noise (AWGN) channel. The encoder for user $k = 1, 2, \ldots, K$ operates as follows. A length $I$ frame of independent equiprobable information bits $u_k$ is encoded by a rate $R_c$ code $C_k$. The $I/R_c$ coded bits $c_k$ are then permuted with the interleaver $\Pi_k$, and parsed into $\log Q$ segments. These segments are mapped onto a $Q$-ary modulation vector of length $I/R_c \log Q$ constellation symbols according to some memoryless mapping, and then multiplexed onto the symbol sequences of length $I/R_c \log Q$. Each user transmits at a rate of $R_c \log Q$ bits per channel use.

A data vector $d = (d_1, \ldots, d_K)^T \in D^K$ represents all users’ symbols in a given symbol interval (assuming synchronous transmission for simplicity of explanation). The complex constellation $D \subset C$ has $|D| = Q$ unique elements, with moment constraints $E[d] = 0$, and $E[|d|^2] = PI_K$, and symbols are equiprobable. The average transmit power per user is $P$.

Each symbol is multiplied by a length $L$ modulation vector $s_k$, which has real random elements chosen uniformly from $\pm1/\sqrt{L}$. A vector $z \in C^L$ with independent white zero-mean Gaussian element represents thermal noise with variance $\sigma^2$ per real dimension. In a coded system where each user employs a rate $R$ code and transmits with power $P$, the appropriate signal-to-noise measure we will use is $E_k/N_0 = P/2\sigma^2R\log Q$. 

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We assume that each user’s signals are received with identical power, phase and delay, although these are not fundamental restrictions imposed by the proposed receiver. After standard manipulations the multiple-access channel may be represented by

\[ r = Sd + z \]  

where \( S = (s_1, \ldots, s_K) \in \{ \pm 1/\sqrt{L} \}_{L \times K} \). We only consider the case that \( K > L \), i.e. the number of users exceeds the number of independent observations.

The canonical iterative decoder is shown in Fig. 1. The goal is to infer the value of \( u_k \), \( k = 1, \ldots, K \), based on \( r \), \( S \) and the constraints \( c_k \). The module labelled Multiuser APP computes the marginal posterior probability matrix \( \omega(d) \in \mathbb{P}^{Q \times K} \), which has as columns the probability mass functions for the corresponding symbols, based on all the available information using the constraint (1), as well as the prior probability matrices \( \omega_\nu(d) \). This inner decoder is the focus of this work.

The other constraint, separated from the first by an interleaver, is the single-user decoders, which calculate extrinsic probabilities \( \omega_{0}(c_k) \) based on the codes and \( \omega_{\nu}(c_k) \) for all \( k \). The process repeats by iteratively exchanging information in the form of extrinsic probability matrices between the two modules. The individual APP decoders also compute, on the final iteration, the data sequence probabilities \( \omega(u_k) \).

III. APPROXIMATION OF THE MULTIUSER APP

The posterior log joint-probability of a particular hypothesis sequence \( d' \) is equal to

\[ \log p(r, d'|S, N_0) = c - \frac{1}{N_0} \| r - Sd' \|^2 + \log p(d') \]  

where \( c \) is a constant and \( p(d') \) is the prior probability of the sequence. Expand the squared-distance term as

\[ \| r - Sd' \|^2 = r^*r - 2\Re\{r^*Sd'\} + d'^* \| Sd' \|^2 \]

\[ = c + \Re\{yd'\} + \| Sd' \|^2 \]  

where \( y = -2r^*S \). In order to simplify the search for sequences \( d' \) that minimise (3), a recursive expression in \( d'_1, \ldots, d'_K \) may be obtained if \( G = S^*S \) is positive-definite. This cannot be the case when \( K > L \). In our model \( S \) is not even guaranteed to have rank \( L \). In order to obtain an equivalent full-rank system, we exploit the following representation.

\[ \| Sd' \|^2 = d'^*Gd' = \sum_{k=1}^{K} g_{kk}d_k^2 + \sum_{j,k=1 \atop j \neq k}^{K} g_{jk}d_jd_k \]

where \( \rho_k \in \mathbb{R} \) is a free parameter. The terms in brackets define the columns of a new matrix, and a sufficient condition for that matrix to be positive-definite is that each term is positive irrespective of \( d' \). The following procedure is used to transform the log-likelihood into an additive recursive metric with \( K \) terms. This technique, which we believe to be new, is the main contribution of this paper.

1) Choose a positive constant \( \rho \in \mathbb{R}^+ \) satisfying

\[ \rho > (K-1) \max_{i,j} \frac{-\Re\{D_i^*D_j\}}{|D_i|^2} \]  

where \( D_1, \ldots, D_Q \) are the elements of \( D \).

2) Construct the vector \( u = \text{diag}(G) - \rho 1 \in \mathbb{R}^K \), where \( 1 \) is the all-ones vector.

3) Construct a new matrix \( \tilde{G} \) by setting all diagonal elements of \( G \) equal to \( \rho \). The matrix \( \tilde{G} \) is guaranteed to be positive-definite.

4) Compute the factorisation \( T^*T = \tilde{G} \), where \( T \in \mathbb{R}^{K \times K} \) is lower-triangular.

By setting the prior term \( \mathcal{L}(d') = -N_0 \log p(d') \), and assuming statistical independence of the prior symbol probabilities due to the interleaver, (2) may be written as

\[ -N_0 \log p(r, d'|S, N_0) = c + \sum_{k=1}^{K} \Re\{yd_kd_k^*\} + \sum_{j=1}^{k} t_{kj}d_j^2 + \mathcal{L}(d_k') + u_k|d_k'|^2 \]

\[ \sum_{j=1}^{k} t_{kj}d_j^2 + \mathcal{L}(d_k') + u_k|d_k'|^2 \]  

FIG. 1. Iterative joint-APP multi-user receiver.
For constant energy symbol constellations such as $Q$-ary PSK, the last term in (5) is absorbed into $c$, and we only require $\rho > (K - 1)$.

Quadratic forms such as (5) admit a tree representation [11]. The equation represents a $Q$-ary tree of depth $K$, where each of the $Q^k$ nodes at depth $k$ represent a partial sequence with an associated positive path weight, and the leaf nodes represent sequences $\mathbf{d}'$ with total path weight equal to $c - N_0 \log p(\mathbf{d}', \mathbf{r}; \mathbf{S}, N_0)$. Hence, the problem reduces to finding the $P$ leaf nodes in the tree with minimum weight, which may be approximated using tree search techniques.

The simple manipulations applied to $\mathbf{G}$ described above artificially create a virtual full rank channel from a rank-deficient one, assigning a positive path weight to every node in the tree and allowing sequential search to be applied to (5). This is not to say that extra information is obtained about the symbols via the transformations described; only that the information about the interfering signals is spread out onto a greater number of effective observations, so that any sequential search techniques developed for a full rank channel may also be applied in the overloaded or singular case.

A transformation for overloaded linear systems was presented in [20, 21], which similarly creates a virtual full-rank system to which the full $Q$-ary tree may be assigned. The approach is based on a minimum mean square error generalised decision feedback equaliser filter, followed by lattice reduction, column re-ordering, and then triangular factorisation (if tree/sphere decoding is used). These transformations are significantly more complex than our procedure, and may be unsuitable for time-varying channels. The approach also colours the noise, so that the system no longer lends itself naturally to the iterative APP framework.

A depth-first tree-search was used in [14, 15, 18], which necessitated special treatment of the prior probability on those paths that did not reach full depth. We propose a breadth-first search using the $T$-algorithm. The $T$-algorithm was used in [22, 23] for near-optimal hard-decision decoding of channel codes up to a pre-determined minimum-distance with significant complexity savings over the Viterbi algorithm. The related $M$-algorithm retains exactly $M$ paths at each depth, regardless of the actual weights of each partial path (this approach was used in [12]). In practice the statistical nature of the noise and the spreading sequences for each transmission may require a different number of sequences to approximate the APP. It should also be noted that any other tree search algorithm could be used, with slightly varying levels of performance and complexity. The key step is (5), which admits such tree representations for overloaded systems.

We exploit the heuristic observation that paths with very large partial path weight are unlikely to be components of low weight paths. Rather than retaining a fixed number of paths at each depth, the $T$-algorithm attempts to adapt to the channel conditions by only retaining paths at each depth with weight not exceeding the best weight by more than $T$, where $T$ is a parameter of the algorithm. When the algorithm terminates at the leaves, the best $P$ sequences are used in the marginalisation.

At low SNR in the early iterations, or with few receiver observations compared to the number of transmitters, large numbers of paths will exist with similar path weight. Due to complexity constraints in this scenario the number of retained paths at each depth must be limited to $\mathcal{P}_{\max}$, and the algorithm essentially becomes the $M$-algorithm with $M = \mathcal{P}_{\max}$. In more favourable circumstances however, very few paths are required and the $T$-algorithm adapts automatically to take advantage of the conditions with greatly reduced complexity.

When no prior information is available the $T$-algorithm adapts very well to the channel, automatically finding a good performance/complexity trade-off through the parameter $T$. As a general rule, the $T$-algorithm only tends to approach the $\mathcal{P}_{\max}$ bound in the early iterations, since the search is greatly facilitated by the prior probabilities once they become available. When very strong prior information is available however, the only sequences retained will be those dictated by the priors, since other paths will be discarded in the early depths. In this case the detector will glean little new information, and the information about the symbols will quickly become correlated over iterations. Hence, another parameter $\mathcal{P}_{\min}$ must also be set, forcing the algorithm to consider a certain minimum number of sequences at each depth. The effect is only significant in highly loaded systems where many iterations are required for convergence.

The $T$-algorithm finds full-depth paths through the tree, and the prior probabilities are incorporated in a natural fashion. In contrast, the depth-first strategy of [14, 15] required special handling of the prior probabilities, while the method of [18] required an initial breadth-first search in order to exploit the priors during the main search. The $M$-algorithm based approach used in [12, 13] did not directly incorporate priors (this was done in a separate combining step). Other approaches incorporate the prior probability into spherical or branch-and-bound decoders in various ways [16, 17, 19], but tend to be quite complex and unsuitable for large-dimensional systems.

The receiver complexity is dominated by Cholesky factorisation of the $K \times K$ matrix $\mathbf{G}$, and then by the tree search during the iterations. The complexity of the $T$-algorithm is upper bounded by $KP_{\max}$ node computations per iteration, but this bound only ever tends to be reached in the early iterations, as discussed above. Contrast this for example with the LMMSE filter [9] as the inner detector, which requires an initial matrix inverse, and then a matrix inverse per user on each subsequent iteration.

IV. Numerical Results

In this section we consider a benchmark model, with length $L = 8$ random PN spreading sequences and no fading. The model is difficult to work with, since a significant probability exists that the spreading matrix $\mathbf{S}$ will have linearly dependent rows.

The individual users transmit BPSK symbols, which are encoded with a nonsystematic 4-state rate 1/2 convolutional code, described by the feed forward generator polynomials...
(05, 07). A length $2I = 1000$ interleaver between the encoder and the transmitter is generated randomly for each user.

We consider joint iterative decoding of the system using the $T$-algorithm, where $P_{\text{max}}$ is set to 512 and the threshold $T$ is set to $16N_0$. The bound $P_{\text{max}}$ is deliberately set large in order to demonstrate the performance advantage of closely approximating the APP. The $T$-algorithm for the overloaded case is furnished by the matrix manipulations proposed in Section III for computing the log-likelihood.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
5 iterations & 20 iterations \\
\hline
\end{tabular}
\end{center}

Fig. 2. Comparative CDMA system BER performance as a function of number of users $K$. Spreading gain $L = 8$, $E_b/N_0 = 5$ dB.

The performance of the $T$-algorithm in the above model is shown as a function of the number of users in Figure 2, at $E_b/N_0 = 5$ dB. Also shown is the performance of two linear filters commonly used as the multi-user detector in such systems, the PIC [8] and the LMMSE [9] filters. The parameter $P_{\text{min}}$ for the $T$-algorithm is set to 32 for $K \leq 16$, $P_{\text{min}} = 64$ for $K = 17, 18$, and $P_{\text{min}} = 128$ for $K = 19, 20$ users. These values were found by experiment to be sufficiently large for the loads considered.

While very computationally efficient, the PIC can only support 9 users after 20 iterations, and is clearly not suitable for highly loaded systems. The MMSE filter performs better, supporting 14 users after 20 iterations, but requires a matrix inversion per user per iteration.

Estimating the detector APP is a highly non-linear calculation, and the linearised models and assumptions used by the above filters are not necessarily valid in the model under consideration. The performance of the $T$-algorithm in Figure 2 clearly demonstrates the performance advantage of approximating the APP directly using the rules of probability. List-detection using the $T$-algorithm supports 16 users with only 5 iterations and 19 users after 20 iterations at 5 dB. To our knowledge we have not seen loads in such a system approaching those achieved here.

In Figure 3 is shown the performance of list detection as a function of $E_b/N_0$, for various number of users, after 20 receiver iterations. Note that without the log-likelihood transformation of Section III, the tree search with 20 users would require at least one stage with $2^{K-L} = 4096$ node computations, assuming the best case that $S$ has rank $L$. An extrinsic-information transfer chart [24] shows that the $T$-algorithm detector is very well matched in shape to the particular code, which helps to explain the good performance after many iterations, even at very high loads. The charts, which we do not include here, also predict very accurately the convergence characteristics shown in Figures 2 and 3.

![Fig. 3. CDMA system BER performance after 20 iterations as a function of SNR. Spreading gain $L = 8$.](image)

Figure 4 shows the spectral efficiency of the receiver for the system described above, as a function of $E_b/N_0$, measured as the maximum number of users for which single-user performance is reached. Also shown is the maximum spectral efficiency $C$ achievable by using both an optimal joint receiver, and an MMSE detector followed by single-user decoding. These curves were approximated using the large-systems expressions for random spreading given in [25] under the constraint $C = KR/L$ with $R = 1/2$.

The receiver easily approaches optimal joint-processing data rates at low SNR, but cannot maintain this slope with increasing SNR. Nevertheless, the iterative $T$-algorithm receiver outperforms any other practical algorithm we are aware of in terms of system load for the given channel model. Various other results for randomly spread CDMA in AWGN using a rate 1/2 code are available in the literature, utilising the same canonical receiver structure but differing in the multi-user detector implementation. These are shown in Figure 4, along with references to the relevant papers.

V. CONCLUSION

We have shown that near-optimal performance may be achieved with low complexity in a randomly spread CDMA channel by employing the turbo principle in an iterative receiver. This is not a new observation; our contribution is to show that by attempting to calculate the true symbol-APP distributions in the inner detector, the performance is significantly improved over detectors that employ linear filters, or other structures derived using alternative considerations. Close approximation of the desired APP distributions in overloaded or singular channels is practically facilitated by the simple
procedure of Section III, and by the application of simple and well-known sequential algorithms.

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