Interacting spin 0 fields with torsion via Duffin-Kemmer-Petiau theory

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Here we study the behaviour of spin 0 sector of the DKP field in spaces with torsion. First we show that in a Riemann-Cartan manifold the DKP field presents an interaction with torsion when minimal coupling is performed, contrary to the behaviour of the KG field, a result that breaks the usual equivalence between the DKP and the KG fields.

Next we analyse the case of Teleparallel Equivalent of General Relativity (Weitzenböck manifold), showing that in this case there is a perfect agreement between KG and DKP fields. The origins of both results are also discussed.

I. INTRODUCTION

The Duffin-Kemmer-Petiau (DKP) equation is a first order relativistic wave equation for spin 0 and 1 bosons given by

\[(i\beta^a \partial_a - m) \psi = 0,\]  

where the matrices \(\beta^a\) obey the algebraic relations

\[\beta^a \beta^b \beta^c + \beta^c \beta^b \beta^a = \beta^a \eta^{bc} + \beta^c \eta^{ba};\]  

where \(a, b = 0, 1, 2, 3,\) with \(\eta^{ab}\) being the metric tensor of Minkowski space-time with signature \((+ - - -)\). The latin alphabet will be used all through this paper to indicate Minkowski indexes, while Riemann-Cartan indexes will be indicated by greek letters.

The DKP equation is very similar to Dirac’s equation but the algebraic properties of \(\beta^a\) matrices, which have no inverses, make it more difficult to deal with.

This equation can also be obtained from the Lagrangian density

\[L = \frac{i}{2} \overline{\psi} \beta^a \gamma^a \partial_a \psi - m \overline{\psi} \psi,\]  

where \(\overline{\psi}\) is defined as

\[\overline{\psi} = \psi^\dagger \eta^0,\]  

and \(\eta^0\) is a matrix defined as

\[\eta^a = 2 (\beta^a)^2 - \eta^{aa}.\]  

An equation for \(\overline{\psi}\) can also be obtained from variational principles from the above Lagrangian density or by hermitian conjugation if we choose \(\beta^0\) to be hermitian and \(\beta^i\) \((i = 1, 2, 3)\) anti-hermitian. If one uses two sets of Dirac matrices \(\gamma^a\) and \(\gamma'^a\) acting on different indexes of a 16 component \(\psi\) wave function it can be verified that the matrices

\[\beta^a = \frac{1}{2} (\gamma^a I' + I \gamma'^a)\]  

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The Latin indexes being raised and lowered by the Minkowski metric $\eta$ to a 10 component $\psi$ that describes a spin 0 boson; and a 10 degree one, corresponding to a 5 component $\psi$ describing a spin 1 boson.

Moreover, for any representation, one can define a set of “projectors” which select the scalar and vector sectors of the DKP field [9]. A detailed analysis of the DKP equation and of the properties of $\beta$ and $\eta$ matrices can be found in the original works [1][2], while a similar analysis, complemented by the definitions of the projectors for the scalar and vector sectors of the theory and their properties, can be found in reference [4]. A rather complete introduction to DKP theory, that covers all properties necessary to the reading of this work and uses the same metric signature used here, can be found in references [3] or [5]. For a historical development of this theory, among others, until the 70’s reference [7] is suggested, while the application of DKP field to scalar QED can be found in [8].

More recently there arose some new interest in DKP theory, specifically it has been applied to QCD (large and short distances) by Gribov [9], to covariant Hamiltonian dynamics [10], have been studied in curved space-time [11] in Causal Approach [12] and with five-dimensional Galilean covariance [13]. There also have been given detailed proofs of the equivalence between DKP and KG fields in various situations [14][15] and some points regarding DKP interaction with electromagnetic field have been clarified [16]. In this work we will analyse the interaction of the spin 0 sector of the DKP field with torsion in the cases of Einstein-Cartan (EC) theory, which uses a Riemann-Cartan (RC) manifold to describe gravitational interaction, and of the Teleparallel Equivalent of General Relativity (TEGR).

### II. DKP FIELD IN RIEMANN-CARTAN SPACE-TIME

To interact the DKP field with Riemann-Cartan space-time $\mathcal{U}^4$ we introduce a tetrad set and follow the usual procedure used in the case of Dirac’s spinor. Details of this procedure can be found in literature and we suggest specifically the reference [18], which notations and definitions we follow. Below we just mention some basic identities of torsion, such that they will be used here whenever it is possible.

As usual the tetrad vector fields $e_{\mu}^{\ a}(x)$ satisfy, at each point $x$ of $\mathcal{U}^4$, the relations

$$\eta^{ab} = e_{\mu}^{\ a}(x) e_{\nu}^{\ b}(x) g^{\mu\nu}(x), \quad \eta_{ab} = e^{\mu}_{\ a}(x) e^{\nu}_{\ b}(x) g_{\mu\nu}(x), \quad (7)$$

$$g_{\mu\nu}(x) = \epsilon^{\mu}_{\ a}(x) e_{\nu}^{\ a}(x) \eta_{ab}, \quad g^{\mu\nu}(x) = e^{\mu}_{\ a}(x) e^{\nu}_{\ b}(x) \eta^{ab}, \quad (8)$$

the Latin indexes being raised and lowered by the Minkowski metric $\eta^{ab}$ and the Greek ones by the metric $g^{\mu\nu}$ of the $\mathcal{U}^4$ manifold. The components $B^{ab}$ in the tangent Minkowski space-time $\mathcal{M}^4$ of a tensor $B^{\mu\nu}$ defined on $\mathcal{U}^4$ are given by

$$B^{ab} = e^{\mu}_{\ a} e_{\nu}^{\ b} B^{\mu\nu}, \quad B_{ab} = e^{\mu}_{\ a} e_{\nu}^{\ b} B_{\mu\nu}. \quad (9)$$

The Lorentz covariant derivative $D_{\mu}$ is defined as

$$D_{\mu} B^{\alpha} = \partial_{\mu} B^{\alpha} + \omega^{\alpha}_{\ \ {\mu} \beta} (x) B^{\beta}, \quad (10)$$

$$D_{\mu} B_{\alpha} = \partial_{\mu} B_{\alpha} - \omega^{\beta}_{\ \ {\mu} \alpha} B_{\beta}, \quad (11)$$

where $\omega^{\alpha}_{\ \ {\mu} \beta}$ is the connection on the tangent Minkowski space-time, i.e. the spin connection.

The total covariant derivative $\nabla_{\mu}$ of a quantity $B^{\nu}_{\ \alpha}$ will then be

$$\nabla_{\mu} B^{\nu}_{\ \alpha} = D_{\mu} B^{\nu}_{\ \alpha} - \Gamma^{\nu}_{\ {\mu} \alpha} B^{\alpha}, \quad (12)$$

or

$$\nabla_{\mu} B^{\nu\alpha} = D_{\mu} B^{\nu\alpha} + \Gamma^{\alpha}_{\ {\mu} \nu} B^{\nu\alpha}, \quad (13)$$

where $\Gamma^{\alpha}_{\ {\mu} \nu}$ is the nonsymmetric connection on the $\mathcal{U}^4$ manifold given as

$$\Gamma^{\alpha}_{\ {\mu} \nu} = \Gamma^{\alpha}_{\ {\mu} \nu} - K^{\alpha}_{\ {\mu} \nu} = \left\{ \alpha_{\ {\mu} \nu} \right\} - K^{\alpha}_{\ {\mu} \nu}, \quad (14)$$

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where $\Gamma_{\mu\nu}^\alpha = \{^\alpha_{\mu\nu}\}$ is the connection of General Relativity given by the Christoffel symbols

\[
\begin{bmatrix}
\frac{\partial}{\partial \sigma} g_{\beta\mu} + \frac{\partial}{\partial \tau} g_{\beta\mu} - \frac{\partial}{\partial \tau} g_{\mu\beta}
\end{bmatrix} = \frac{1}{2} g^{\alpha\beta} \left( \frac{\partial}{\partial \sigma} g_{\beta\mu} + \frac{\partial}{\partial \tau} g_{\beta\mu} - \frac{\partial}{\partial \tau} g_{\mu\beta} \right),
\]

(15)

and $K_{\mu\nu}^\alpha$ is the contorsion tensor, antisymmetric in the two last indexes, defined in terms of the torsion tensor $Q_{\alpha|\beta\gamma} = \Gamma_{[\alpha|\beta\gamma]}$, as

\[
K_{\mu\alpha\beta} = -Q_{\mu\alpha\beta} - Q_{\beta\mu\alpha} + Q_{\alpha\beta\mu}.
\]

(16)

The derivatives above reduce to the Lorentz derivative $D_\mu$ given in equation (10) if the quantity $B$ has no $U^4$ indexes and to the usual Riemann-Cartan covariant derivative, i.e.

\[
\nabla_\mu B_\nu = \partial_\mu B_\nu + \Gamma_\mu_\nu^\alpha B_\alpha,
\]

(17)

if $B$ has no $M^4$ indexes. From the requirement that $\nabla_\mu e_\nu^a = 0$ we get a relation between the connections $\omega_\mu^{ab}$ and $\Gamma_{\mu\nu}^\alpha$, i.e.

\[
\omega_\mu^{ab} = \gamma_\mu^{ab} - K_\mu^{ba},
\]

(19)

where

\[
K_\mu^{ba} = -K_\mu^{ab} = e^a_{\mu i} \left(C_{iab} - C_{iab} - C_{iab}\right),
\]

(20)

is the contorsion tensor; while

\[
\gamma_\mu^{ab} = -\gamma_\mu^{ba} = e_{\mu i} \left(C_{iab} - C_{iab} - C_{iab}\right),
\]

(21)

is the Riemannian part of the connection $\omega_\mu^{ab}$ obtained from the Ricci rotation coefficients $C_{iab}$

\[
C_{iab} = e_{\mu a} (x) e_{\nu b} (x) \partial_{[\mu} e_{\nu i]}.
\]

(22)

The covariant derivative of the DKP field is then given by

\[
\nabla_\mu \psi = D_\mu \psi = \left(\partial_\mu + \frac{1}{2} \omega_{\mu ab} S_{ab}\right) \psi,
\]

(23)

and

\[
\nabla_\mu \overline{\psi} = D_\mu \overline{\psi} = \partial_\mu \overline{\psi} - \frac{1}{2} \omega_{\mu ab} \overline{\psi} S_{ab},
\]

(24)

where

\[
S_{ab} = [\beta_a, \beta_b] = \beta_a \beta_b - \beta_b \beta_a.
\]

(25)

Finally we will use semicolons to indicate the Riemannian part of the covariant derivative of an object $A^{\nu a}$; i.e. its covariant derivative in a Riemann manifold

\[
A^{\nu a}_{\mu} = \partial_\mu A^{\nu a} + \gamma_\mu^{a b} A^{\nu b} + \left\{^\nu_{\mu \alpha}\right\} A^{\alpha a}.
\]

(26)

Specifically for DKP field we have

\[
\psi_{;\mu} = \left(\partial_\mu + \frac{1}{2} \gamma_{\mu ab} S_{ab}\right) \psi.
\]

(27)

1 We use square brackets to designate antisymmetrization, i.e. $A_{[\nu \sigma]} = \frac{1}{2} (A_{\nu \sigma} - A_{\sigma \nu})$, and round brackets to designate symmetrization, i.e. $A_{(\nu \sigma)} = \frac{1}{2} (A_{\nu \sigma} + A_{\sigma \nu})$. 3
A. Minimal coupling of DKP field

Now we can consider the Lagrangian density of DKP field minimally coupled to the $U^4$ manifold

$$\mathcal{L} = \sqrt{-g} \left[ \frac{1}{2} \left( \bar{\psi} \beta^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \beta^\mu \psi \right) - m \bar{\psi} \psi \right],$$  \hspace{1cm} (28)

where

$$\beta^\mu = e^\mu_a \beta^a,$$  \hspace{1cm} (29)

and satisfy

$$\beta^\mu \beta_\nu \beta^\alpha + \beta^\alpha \beta_\nu \beta^\mu = \beta^\mu g^{\nu\alpha} + \beta^\alpha g^{\nu\mu}. $$  \hspace{1cm} (30)

From the Lagrangian above we get the generalized DKP equation of motion

$$i \beta^\mu \nabla_\mu \psi + \frac{i}{2} K_{\mu b} \beta^b \psi - m \psi = 0,$$  \hspace{1cm} (31)

where we made use of

$$\partial_\nu \left( \sqrt{-g} e_\nu^a \right) = \sqrt{-g} g^b_{\nu a} = \sqrt{-g} g^\mu_{\nu a},$$

valid for a general Riemann-Cartan manifold. Some considerations about this result are necessary. First of all, we can see from equation (31) that the minimal coupling procedure performed in the Lagrangian density leads to an equation of motion which is not minimally coupled. This result is similar to the one obtained in the case of Dirac’s field, as can be seen in reference [18].

Moreover, the generalized DKP equation (31) can be cast in the form

$$i \beta^a \psi_{, a} + \frac{3}{2} K_{abc} \beta^a \beta^b \beta^c \psi - \frac{i}{2} K_{abc} \beta^a \beta^b \beta^c \psi - m \psi = 0,$$  \hspace{1cm} (32)

showing that, differently from the Dirac’s field case [18], the coupling to the contorsion tensor is not only to its totally antisymmetric part due to the presence of the term in $K_{abc} \beta^a \beta^b \beta^c$. Finally we must notice that there is an interaction with torsion. One could think that this interaction would disappear when we select the spin 0 sector of the theory but, as we will see next, this does not happen.

B. Comparison with the KG field

Now we can use the projectors $P$ and $P^a$ that select the spin 0 sector of the theory (see [4–6]). From these projectors defined in $M^4$ we can construct the projectors in $U^4$ as

$$P^\mu = e^\mu_a P^a = e^\mu_a P \beta^a = P \beta^\mu.$$  \hspace{1cm} (33)

From the definitions above and the properties of $P$ and $P^a$ it is easy to verify that

$$P^\mu P^\nu = P g^{\mu \nu}, \quad P \sigma^{\mu \nu} = 0,$$  \hspace{1cm} (34)

and it can also be seen that $P \nabla_\mu \psi = \nabla_\mu (P \psi)$ and $P^\nu \nabla_\mu \psi = \nabla_\mu (P^\nu \psi)$, where $\nabla_\mu (P \psi) = \partial_\mu (P \psi)$ and $\nabla_\mu (P^\nu \psi) = \partial_\mu (P^\nu \psi) + \Gamma_{\mu \rho}^{\nu} P^\rho \psi$, i.e. $P \psi$ is derivated as a scalar and $P^\nu \psi$ as a 4-vector.

So, applying the operators $P^a$ and $P$ to the generalized DKP equation (31) we get, respectively,

\footnote{At this point we depart from the case presented in reference [5] since there we have $\partial_\nu \left( \sqrt{-g} e_\nu^a \right) = 1 - g^a_{\nu b} = 1 - g^{\nu a}$ because the manifold is riemannian.}
\footnote{Here we have $\beta^\alpha \beta^b \beta^c = \frac{1}{6} \left( \beta^a \beta^b \beta^c + \beta^b \beta^c \beta^a + \beta^c \beta^a \beta^b - \beta^a \beta^b \beta^c - \beta^a \beta^c \beta^b - \beta^b \beta^c \beta^a \right)$, the total antisymmetrization of the indexes.}
\[ P^\alpha \psi = \frac{i}{m} \nabla^\alpha (P\psi) + \frac{i}{2m} K_{\mu\nu}^\alpha g^{\alpha\nu} (P\psi), \]  

(35)

and

\[ P\psi = \frac{i}{m} \nabla_\alpha (P^\alpha \psi) + \frac{i}{2m} K_{\mu\alpha}^\mu (P^\alpha \psi). \]  

(36)

We must remember that \( P\psi \) is a scalar and \( P^\alpha \psi \) is a vector\(^4\) so that we can calculate their covariant derivatives \( \nabla^\alpha \) by considering only the connection \( \Gamma^\alpha_{\mu\nu} \) acting on the Riemannian indexes \( \Gamma^\alpha_{\mu\nu} \); specifically

\[
\nabla^\alpha (P\psi) = \partial^\alpha (P\psi),
\]

and

\[
\nabla_\alpha (P^\alpha \psi) = \partial_\alpha (P^\alpha \psi) + \Gamma_\alpha^\mu \nu K^\nu_{\nu\mu \alpha}. 
\]

Combining equations (35) and (36) we get the equation of motion for the scalar field \( P\psi \) as

\[
\nabla^\alpha \nabla_\alpha (P\psi) + m^2 (P\psi) + \frac{1}{2} K_{\mu\alpha}^\alpha (P\psi) \nabla^\mu (P\psi) + \frac{1}{4} K_{\mu\beta}^\mu K_{\beta\alpha}^\alpha (P\psi) = 0,
\]

or in a more compact form

\[
\left( \nabla^\alpha + \frac{1}{2} K_{\mu\alpha}^\mu \right) \left( \nabla_\beta + \frac{1}{2} K_{\mu\beta}^\mu \right) g^{\alpha\beta} (P\psi) + m^2 (P\psi) = 0.
\]

(37)

As we just mentioned above, the interaction with torsion does not disappear, even after we selected the spin 0 sector of the DKP field. This interaction is present both in the connection \( \Gamma^\mu_{\nu\alpha} \) used in the calculation of the covariant derivative \( \nabla^\alpha \) and in the explicit presence of terms containing the contorsion tensor \( K_{\mu\nu\alpha} \) in the equation above. This is in contrast with the result obtained by performing the minimal coupling in the KG field Lagrangian density, i.e.

\[
\mathcal{L} = \sqrt{-g} \left[ \partial^\mu \varphi^* \partial_\mu \varphi - m \varphi^* \varphi \right],
\]

(39)

that results in the equation of motion

\[
\left( \partial^\mu \varphi \right)_{;\mu} + m^2 \varphi = 0,
\]

(40)

so that there is no interaction with torsion. Then we find that, in Einstein-Cartan theory with minimal coupling procedure, DKP and KG fields yield qualitatively different results: the first has an interaction with torsion in its spin 0 sector while the second does not. So, differently from others situations\(^5\), DKP and KG theories for spin 0 particles are inequivalent in this context. The origin of this inequivalence can be identified with the fact that the free DKP field Lagrangian reduces, by a suitable choice of a rank 5 representation of the \( \beta \) matrices, to the second order scalar field Lagrangian\(^6\)

\[
\mathcal{L} = -\frac{1}{2} (\varphi^* \Box \varphi + \varphi \Box \varphi^*) - m^2 \varphi^* \varphi,
\]

(41)

\( \Box = \partial^\mu \partial_\mu \) and not the first order KG Lagrangian. This is irrelevant in the free field case since these two Lagrangians differ only by a complete divergence, what is also true in most of the interaction cases. In the case of electromagnetic interaction, for example, the minimally coupled DKP Lagrangian can be reduced to a second order scalar field Lagrangian which is exactly the one obtained by performing the minimal electromagnetic coupling in the second order free Lagrangian\(^7\) above. But this second order Lagrangian differs from the minimally coupled first order KG Lagrangian only by a complete divergence, so that there is a complete equivalence between them\(^8\).

As DKP Lagrangian with interaction can always be reduced to a second order interacting scalar field Lagrangian, the equivalence between interacting DKP Lagrangian and first order KG Lagrangian depends on the equivalence between this second order Lagrangian, obtained from the DKP one, and the first order KG Lagrangian, i.e. whether they differ only by a complete divergence. In General Relativity, as another example, there is such equivalence\(^9\) because DKP Lagrangian can be reduced to a second order scalar field Lagrangian, the same obtained by performing minimal coupling in the Lagrangian\(^10\), which differs from the first order KG Lagrangian only by a complete divergence.

\(^4\)Being rigorous, each component of the columns vectors \( P\psi \) and \( P^\alpha \psi \) is, respectively, a scalar and a vector.
Similarly, in the case of Einstein-Cartan theory interaction, the specific choice of \( \beta \) matrices given in \( [4] \) will give as result

\[
P\psi = \left( \begin{array}{c} 0_{4 \times 1} \\ \psi_4 \end{array} \right), \quad P^\mu \psi = e^{\mu a} P_a \psi = e^{\mu a} \left( \begin{array}{c} 0_{4 \times 1} \\ \psi_a \end{array} \right),
\]

such that we will have, with the use of \( [35] \), that the DKP field can be written as

\[
\psi = \left( \frac{1}{\sqrt{m}} \left( \nabla a \varphi + \frac{1}{2} K_{\mu a} \frac{\partial}{\partial \psi} \varphi \right) \right) = \left( \frac{1}{\sqrt{m}} \left( \partial a \varphi + \frac{1}{2} K_{\mu a} \frac{\varphi}{\sqrt{m} \varphi} \right) \right),
\]

where \( \varphi \) is a scalar field. If we compare this result with the similar expression obtained in the case of electromagnetic interaction (equation (54) of reference \([4]\)), we see that the expression above for \( \psi \) makes explicitly clear the existence of an interaction between the scalar field and torsion. Moreover, one can expect a nonminimal coupling since the derivative present in first four components of \( \psi \) is not the covariant derivative of Einstein-Cartan theory. Indeed, it can be seen that the corresponding second order scalar field Lagrangian will be

\[
\mathcal{L} = \sqrt{-g} \left[ -\frac{1}{2} \left( \varphi^* \nabla \mu \varphi + \varphi \nabla \mu \varphi^* \right) + \frac{1}{4} K_{\mu a} \frac{\varphi}{\sqrt{m} \varphi} - m^2 \varphi^* \varphi \right],
\]

which has equation \( [37] \) as equation of motion but is not obtained by minimal coupling from the free field Lagrangian \( [11] \) due to the presence of the term \( \frac{1}{4} K_{\mu a} \frac{\varphi}{\sqrt{m} \varphi} \).

Now, if one performs an integration by parts in the minimally coupled first order KG Lagrangian \( [34] \), it is not possible to obtain a term containing torsion since the derivative \( \partial \sqrt{-g} \) will only generate terms containing the Christoffel symbol \( [11] \). So, the second order Lagrangian obtained in this way will not have interaction with torsion, differently from the Lagrangian \( [44] \), where there is an explicit interaction with torsion.

Summing up, the inequivalence between DKP and KG fields in Einstein-Cartan theory is due to the fact that the minimal coupling in DKP Lagrangian corresponds to a nonminimal coupling in a second order scalar Lagrangian, what results in an interaction with torsion, contrary to the minimal coupling in the first order KG Lagrangian. We should also remember the necessity of using the symmetric DKP Lagrangian \( [28] \) and not the nonsymmetric one, i.e.

\[
\mathcal{L} = \sqrt{-g} \left[ i \beta^\mu \nabla \mu \psi - m \psi \right],
\]

because they are not equivalent, contrary to what happens in the cases of General Relativity or electromagnetic interactions. This is due to the fact that the symmetric Lagrangian can not be obtained from the nonsymmetric one by an integration by parts because the derivative \( \partial \sqrt{-g} \) will not generate the contorsion terms necessary to construct the complete Riemann-Cartan connection, present in the covariant derivative of \( \varphi \) contained in the symmetric Lagrangian. Moreover, Lagrangian \( [43] \) would result in a equation of motion for \( \tilde{\psi} \) field which can not be obtained by conjugation from the equation for \( \psi \). But these results are not particularities of DKP field but a general characteristic of fields in Riemann-Cartan manifolds, as can be seen through a similar analysis, for example, of the second order KG or Dirac Lagrangians. We also should mention that there would be no inequivalence if the minimal coupling where performed in the equation of motion instead of in the Lagrangian. In this case the generalized DKP equation in Riemann-Cartan space is given by

\[
i \beta^\mu \nabla \mu \psi - m \psi = 0,
\]

which, by means of the projectors \( P \) and \( P^\alpha \), provides

\[
\nabla^\mu \nabla_\mu (P \psi) - m (P \psi) = \nabla^\mu \partial_\mu (P \psi) - m (P \psi) = 0.
\]

Since \( P \psi \) is a scalar, this result is completely equivalent to the minimal coupling performed in the KG equation of motion, both cases resulting in a coupling with torsion via the contorsion tensor present in the covariant derivative \( \nabla^\mu \). As can be seen from these results, performing minimal coupling in the equation of motion gives different results.

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5 The fact that the coupling is nonminimal is not fundamental. Even if the second order scalar Lagrangian obtained from DKP one were simply that obtained by performing minimal coupling in Lagrangian \( [11] \) there would be an interaction with torsion and an inequivalence with the first order KG Lagrangian.
from the same coupling performed in the Lagrangian, for both DKP and KG fields; what is, once more, a characteristic of fields in Riemann-Cartan manifolds: the same occurs with Dirac field [18].

Finally, it is necessary to mention that this result is qualitatively different from the torsion coupling with the KG field described in reference [19] in the context of Teleparallel Description of Gravity. In the Teleparallel formalism the General Relativity’s description of gravity as a curvature effect is replaced by a description in terms of a flat space with torsion (Weitzenböck spacetime) so that the interaction with torsion is necessary to exist a gravitational interaction at all. The result we have just obtained above, on the contrary, describes the interaction of the spin 0 sector of DKP field with torsion in the context of Einstein-Cartan theory, where both torsion and curvature are present. This point will be made more clear in the next section, where we will analyse the interaction of the DKP field with torsion in the context of Teleparallel Description of Gravity.

III. DKP FIELD IN TELEPARALLEL GRAVITY

Here we will analyse the DKP field in the context of Teleparallel version of General Relativity [20]. In this case we have a Weitzenböck space-time; a particular case of Riemann-Cartan space-time constrained to have zero curvature or, equivalently, to satisfy the teleparallel condition of the tetrad field

\[
\nabla^c_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^a_\mu\nu e^a_\alpha = 0. \tag{48}
\]

In the expression above \(\nabla^c_\mu\) is the covariant derivative with respect only to the Riemann-Cartan connection and differs from the total covariant derivative \(\nabla_\mu\), given in equation (12), by the absence of the \(\omega^a_\mu\) spin connection. This condition implies that the spin connection \(\omega^a_\mu\) vanishes identically and that the connection \(\Gamma\) is the (nonsymmetric) Cartan connection

\[
\Gamma^\alpha_\mu\nu = e^\alpha_i \partial_\mu e^i_\nu. \tag{49}
\]

We also have

\[
\partial_\mu e = ee^\nu_\mu \partial_\mu e^a_\nu = e\Gamma^\nu_\mu , \tag{50}
\]

where \(e = \text{det} e^a_\nu\), and combining equations (48) and (14) we find that

\[
\nabla^c_\mu e^a_\nu = \partial_\mu e^a_\nu - \left\{ \begin{array}{c} \alpha \\ \mu \nu \end{array} \right\} e^a_\alpha = -K^a_\mu\nu e^a_\alpha , \tag{51}
\]

where \(\nabla^c_\mu\) indicates the covariant derivative with respect to the Christoffel connection, i.e. the connection of General Relativity. From the torsionless case of equation (18) we can also write the spin connection \(\tilde{\omega}_\mu^a\) of General Relativity as

\[
\tilde{\omega}_\mu^a = e^{rb} \left\{ \begin{array}{c} \alpha \\ \mu \nu \end{array} \right\} e^a_\alpha - e^{rb} \partial_\mu e^a_\nu = -e^{rb} \nabla^c_\mu e^a_\nu = e^{rb} K^a_\mu\nu e^a_\alpha = -K^a_\mu^{rb}. \tag{52}
\]

Now we must notice that the use of covariant derivative (23), with \(\omega^a_\mu = 0\) as consequence of teleparallel condition, will not provide an equivalence with General Relativity, as can also be seen in the case of Dirac field addressed in reference [21]. But, following the procedure described in this reference, the covariant derivative \(\nabla^c_\mu \psi = \tilde{D}_\mu \psi\) of the DKP field in General Relativity, i.e. in a Riemann manifold, can be written in terms of teleparallel quantities. To do this, first we must notice that such derivative can be obtained from the torsionless case of equation (23) and is given by

\[
\nabla^r_\mu \psi = \tilde{D}_\mu \psi = \left( \partial_\mu + \frac{1}{2} \omega^a_\mu S^a_{\beta\gamma} \right) \psi. \tag{53}
\]

So, with the help of equation (22) above, it can be rewritten in terms of teleparallel quantities as

\[
\nabla^r_\mu \psi = D_\mu \psi = \left( \partial_\mu - \frac{1}{2} K^{\alpha\beta}_{\mu\nu} S^a_{\alpha\beta} \right) \psi. \tag{54}
\]
Analogously, for $\psi$ we have
\[
\nabla_{\mu} \psi = D_{\mu} \psi = \partial_{\mu} \psi + \frac{1}{2} K_{\mu \alpha \beta} S^{\alpha \beta},
\]
so that the minimally coupled Lagrangian is written in terms of teleparallel structure as
\[
\mathcal{L} = e \left[ \frac{i}{2} \left( \bar{\psi} \nabla_{\mu} \psi - \nabla_{\mu} \bar{\psi} \beta^\mu \psi \right) - m \bar{\psi} \psi \right],
\]
or
\[
\mathcal{L} = e \left[ \frac{i}{2} \left( \bar{\psi} \beta^\mu \left( \partial_{\mu} \psi - \frac{1}{2} K_{\mu \alpha \beta} S^{\alpha \beta} \psi \right) - \left( \partial_{\mu} \bar{\psi} + \frac{1}{2} K_{\mu \alpha \beta} \bar{\psi} S^{\alpha \beta} \right) \beta^\mu \psi \right) - m \bar{\psi} \psi \right],
\]
from which we obtain the equation of motion
\[
i \beta^\mu \nabla_{\mu} \psi - m \psi = 0.
\]
Now the resulting coupled equation of motion is exactly the one obtained by performing the minimal coupling in the equation of motion. The above equation of motion is exactly the one obtained in General Relativity [5], but here is written in terms of teleparallel quantities as we can see from equation (54). Moreover, an identical result is obtained by performing the minimal coupling procedure directly in the equation of motion. But this identity is expected since it occurs in General Relativity and we are dealing with its teleparallel equivalent.

A. Comparison with the KG field

Due to properties of teleparallel structure, the operators $P$ and $P^\nu$ also have the properties $P \nabla_{\mu} \psi = \nabla_{\mu} \bar{P}(P \psi)$ and $P^\nu \nabla_{\mu} \psi = \nabla_{\mu} \bar{P}(P^\nu \psi)$, with $\nabla_{\mu} \bar{P}(P \psi) = \partial_{\mu} \bar{P}(P \psi)$ and $\nabla_{\mu} \bar{P}(P^\nu \psi) = \partial_{\mu} \bar{P}(P^\nu \psi) + \Gamma_{\mu \alpha \beta} P_{\beta} \psi$, as in General Relativity. Consequently, applying the operators $P$ and $P^\nu$ to the equation of motion (58) results in
\[
P \psi = \frac{i}{m} \nabla_{\mu} \bar{P}(P^\mu \psi),
\]
and
\[
P^\lambda \psi = \frac{i}{m} \nabla^\lambda (P \psi) = \frac{i}{m} \partial^\lambda (P \psi),
\]
respectively. Combining these results we get
\[
\nabla_{\mu} \bar{P}^\mu (P \psi) + m^2 P \psi = \nabla_{\mu} \partial^\mu (P \psi) + m^2 P \psi = 0.
\]
This result is exactly what is obtained in the teleparallel description of the KG field presented in reference [19], as it can be seen by writing this last equation, with the help of equation (44), as
\[
\partial_{\mu} \partial^\mu (P \psi) + (\Gamma_{\mu \alpha \beta} K_{\mu \alpha \beta}) \partial^\alpha (P \psi) + m^2 P \psi = 0,
\]
where $D_{\mu} = \partial_{\mu} + \Gamma_{\mu \alpha} + K_{\mu \alpha}$, being $\Gamma$ the Cartan connection, given by equation (49), and $K$ the corresponding contorsion tensor, as given by equation (16). The derivative $D_{\mu}$ is the teleparallel version of General Relativity’s covariant derivative given by equation (26) of reference [19], so equation (63) above is exactly the same as equation (33) of this reference. Furthermore we can also write equation (63) as
\[
\nabla_{\mu} \partial^\mu (P \psi) + m^2 P \psi = -K_{\mu \alpha \beta} \partial^\alpha (P \psi),
\]
which is also identical to equation (34) of reference [19]. So we can see that, in the teleparallel description of General Relativity, the spin 0 sector of the DKP field is exactly equivalent to KG field.

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6We must remember that the definitions here used for the connections indexes and for the contorsion tensor are different from those in the mentioned reference.
IV. CONCLUSIONS AND COMMENTS

We have analysed the minimal coupling of the DKP field with torsion in two contexts. First, we have shown that in a Riemann-Cartan manifold there is an interaction with torsion that persists even when we select the spin 0 sector of the theory. So, in this context, the DKP field provides a coupling of spin 0 particles with torsion when we perform minimal coupling; a result different from that obtained when we describe the spin 0 field using the KG field. This difference comes, as we mentioned before, from the fact that, in a Riemann-Cartan manifold, the DKP Lagrangian corresponds to a second order scalar field Lagrangian, given by equation (44), which is not equivalent to the first order KG Lagrangian given by equation (39); a result that is different from the usual equivalence obtained in the case of other interactions. It should also be noticed that the DKP coupling here is purely classical and has no relation with quantum effects; as those that produce the coupling of quantized electromagnetic field with torsion via second order terms in pertubative series [22].

Next we have shown that in the case of teleparallel description of General Relativity, constructed in a Weitzenböck space-time, there is a perfect agreement between the results of the DKP field and the KG field, even in the presence of an interaction of the spin 0 sector with torsion, as it can be seen in equations (61), (63) or (64) above. This represents no contradiction with the results of Riemann-Cartan case since here we are dealing with a Weitzenböck manifold to construct a teleparallel equivalent of General Relativity. Since such agreement occurs in General Relativity it must also occur in its teleparallel equivalent. The point here is that the origin of the inequivalence does not exist anymore, since the teleparallel condition, equation (48), implies a relation between the contorsion tensor and the Christoffel symbol, since the contributions of each to the curvature must cancel. Such relation is what makes possible to write the Riemannian quantities of General Relativity in terms of the quantities of Weitzenböck space-time and also makes the first order KG Lagrangian in such description again equivalent to the second order scalar Lagrangian obtained from DKP one, since now one can be obtained from the other by an integration by parts. Consequently, the equivalence between both theories will be restored.

We must also stress that the coupling with torsion has completely different physical meanings in each of the cases analysed above. In the teleparallel description we exchanged the description of General Relativity in terms of curvature and metric for a description in terms of torsion and tetrads, without curvature. In this case the interaction with torsion substitutes the interaction with curvature, both in the case of the DKP field described here or in the case of the KG field [3]. Diversely, in the Riemann-Cartan manifold we have both curvature and torsion and the minimally coupled KG field does not interact with torsion in this context while the DKP field does. Finally, we would like to emphasize that the projected equations (37), (38), (47) and (64), which shows explicitly the scalar sector of the DKP field, are used only to provide a comparison with the KG field. It is the linearity of the DKP equation that, despite the complicate algebra of $\beta$ matrices, can be an advantage when studying spin 0 fields in curved space-times and for this it is the DKP equation in the forms obtained here that should be used. Moreover, the natural coupling of the DKP field to torsion in Einstein-Cartan theory makes it an interesting alternative to study interaction of spin 0 fields with torsion without introducing nonminimal couplings.

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*We must stress that this will also occur with the spin 0 sector of any reduible representation of $\beta$ matrices.*
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