Mode intermittence in a wake from two cylinders

G V Gembarzhevskii ¹, ², A K Lednev ¹ and K Yu Osipenko ¹, ³
¹ A. Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia
² Moscow Aviation Institute (National Research University), Moscow, Russia
³ National Research University Moscow State University of Civil Engineering (NRU MGSKU), Moscow, Russia

E-mail: gvgemb@ipmnet.ru

Abstract. Near turbulent intermittent wake from a set contained of two cylinders is investigated in the case of moderate Reynolds number when distinguished wake global modes are competing. This study is performed within the framework of the reduced-order oscillator model where each von Karman vortex street behind a cylinder is modeled by an oscillator of van der Pole type. First distinguished feature of the model is using a nonlinear form in model equations for treating of these streets interaction. The second feature is that the oscillator equations are generalized for explicit dependence of the oscillation frequency on oscillation amplitude. Dependence of the global modes frequencies of oscillation on the model parameters is adduced. An example of attraction domain for an asymmetric wake global mode is studied.

1. Justification
The problem of our interest is in the field of turbulent flows in the electrical constant power industrial lasers, some relevant references to this topic can be found in papers [1 - 4]. We try to adopt the technique of flow control by the electrical discharge in the fast-flow CO2-laser where the glow discharge exists ab initio. A promising result has been achieved for the plasma flow of near wake from two parallel cylinders as is in figure 1 [5]. In that study the wake from cylinders in the case of side-by-side arrangement was studied for the mixture composed of nitrogen and helium (at pressure of tens mbar). The cross section of the discharge chamber was of 55x940 mm, the diameter of the quartz cylinders was of 15 mm and the diameter of the copper water-cooled cathode was of 4 mm. The cylinders touched the cathode in cross arrangement as one can see from figure 1.

Figure 1. Wake flow in the discharge chamber.

This cylinders wake may be treated as a incompressible turbulent flow since the value of the flow Mach number wasn’t higher than 0.15 and the value of Reynolds number defined in terms of cylinder
diameter and non-perturb velocity was of Re ~ 1000. Mean velocity was measured by means of Prandtl tube. One-point spectra of the wake velocity were measured in 120 mm beneath the axis of one cylinder by means of a transducer of Siddon-Ribner type. Its signal was processed by a “PALSE 3560 C” complex of Brue & Kjar. Experimental velocity spectra for the wake at non-self-sustained glow discharge and relevant spectra for the neutral gas flow (that is a reference data) were compared. When the distance between cylinder axes was about of \( L / D \approx 2.2 \), the velocity spectrum with one peak was observed in the case of gas wake, as is seen from figure 2a [5].

![Figure 2](image)

**Figure 2.** The wake velocity spectra (in arbitrary units): a - for neutral gas flow, b - for discharge plasma wake.

This result was treated as a manifestation of one global hydrodynamic mode existing in the case of gas wake with above mentioned parameters. On the contrary, for a relevant plasma wake flow the two-peak velocity spectrum was observed and interpreted as a property of intermittent flow of two concurrent global modes. One of global modes of the plasma wake was the same as for neutral gas flow (since its oscillation frequency was the same), whereas another new mode had lower frequency as compared to previous one. When current density at cathode tube was of about \( 2A/m \) (the case in figure 2b), the dominant part of the energy of coherent oscillations was concentrated in the lower frequency peak. For treatment of such effect one has to use some model of the wake. The previous variant of the model used by us see in [6].

2. Model of wake from two cylinders

Brief sketch of von Karman street formation behind the bluff body is as follows. When fluid pusses cylinder a stagnant zone forms on leeward side of the cylinder where fluid dragged in boundary layer (on one side of the cylinder, principally) accumulates. After half a period this stagnant fluid leaves the cylinder in leeward direction what forms favorable conditions for accumulation of stagnant fluid flowing out boundary layer on opposite side of the cylinder (and with opposite sine of circulation). After second half a period this portion of stagnant fluid takes off the cylinder also and so on. Sequence of rotating in opposite direction moles forms von Karman street behind the cylinder. Accordingly, using some oscillator model for Karman street modeling is natural [7]. Van-der-Pole model is a generic one among oscillator models then we used it as a basis for modeling two cylinders wake. Two von Karman streets constituting the wake of two cylinders are modeled, correspondingly, by two nonlinearly interacting van der Pol oscillators according to the set of equations (1) and (2):

\[
\left(1 + \Delta \left( \rho^2 - 4 \right) \right) \frac{d^2 X_1}{dt^2} + X_1 - \varepsilon \left[ 1 - X_1^2 - \lambda X_1^2 - lX_1X_2 \right] \frac{d X_1}{dt} = h_1 
\]

\[
\left(1 + \Delta \left( \rho^2 - 4 \right) \right) \frac{d^2 X_2}{dt^2} + X_2 - \varepsilon \left[ 1 - X_2^2 - \lambda X_2^2 - lX_1X_2 \right] \frac{d X_2}{dt} = h_2 
\]

Here, variables \( X_i \) are identified with the transverse velocity component at the characteristic points of wake from each cylinder (behind the region of the wake formation). Variables \( h_i \) in the right-hand
sides of these equations represent the stochastic processes (a certain analog of the Langevin forces), which describe the emergence of stochasticity of the trace. In contrast to analogous models [8–10], nonlinear interaction between the streets (represented by the oscillators) is taken into account by means of a quadratic form of general view (within the square brackets in the set (1)–(2)). Moreover, the explicit dependence of an oscillation frequency on the oscillator amplitude is taken into account and is described by the expression in the parentheses. Accordingly, the set of equations (1) and (2) contains four parameters $\varepsilon, \Delta, \lambda, l$ while the inheritable parameter of the van der Pole oscillator is assumed to be small $\varepsilon \ll 1$ according to the experimental data.

The Krylov–Bogolyubov method known from the theory of nonlinear oscillations [11, 12] completed by the above mentioned smallness of the parameter $\varepsilon$ appears to be suitable for the problem under consideration. The solution to equations (1) and (2) is searched for in the form of oscillations at fundamental frequency with slowly varying amplitude and phase.

\[
\begin{align*}
X_1 &= \rho \cos \left( \int_0^\tau \frac{d\tau}{\sqrt{1+\Delta\left(\rho^2-4\right)}} + \theta \right) \\
X_2 &= r \cos \left( \int_0^\tau \frac{d\tau}{\sqrt{1+\Delta\left(r^2-4\right)}} + \psi \right)
\end{align*}
\]

(3)

By performing averaging over fast oscillations, one obtains the "reduced" system of "truncated" equations (4)–(6) for slowly varying amplitudes a phase.

\[
\begin{align*}
\Phi_\rho \frac{d\rho^2}{dt} &= \varepsilon \rho^2 \left[1 - \frac{\rho^2 + \lambda r^2 \left(2 - \cos(2P)\right) + l \rho r \cos(P)}{4}\right] + h_\rho, \\
\Phi_r \frac{d\rho^2}{dt} &= \varepsilon \rho^2 \left[1 - \frac{r^2 + \lambda \rho^2 \left(2 - \cos(2P)\right) + l \rho r \cos(P)}{4}\right] + h_r, \\
\frac{dP}{dt} &= \frac{1}{\sqrt{1+\Delta\left(P^2-4\right)}} - \frac{1}{\sqrt{1+\Delta\left(r^2-4\right)}} \\
&- \frac{\varepsilon}{8} \left[ \lambda \left( \frac{r^2}{\langle \Phi_\rho \rangle} + \frac{\rho^2}{\langle \Phi_r \rangle} \right) \sin(2P) + l \rho r \left( \frac{1}{\langle \Phi_\rho \rangle} + \frac{1}{\langle \Phi_r \rangle} \right) \sin(P) \right] + h_r
\end{align*}
\]

(4)

(5)

(6)

(7)

Here, we use the definitions

\[
\begin{align*}
P &= \int_0^\tau \left( \frac{1}{\sqrt{1+\Delta\left(\rho^2-4\right)}} - \frac{1}{\sqrt{1+\Delta\left(r^2-4\right)}} \right) d\tau + \theta - \psi \\
\Phi_\rho &= 1 + \Delta \left(0,5 \rho^2 - 4\right), \quad \Phi_r = 1 + \Delta \left(0,5 r^2 - 4\right)
\end{align*}
\]

The dynamic model (4)–(7) makes it possible to obtain global harmonic modes of a wake as being the stable stationary points of system (4)–(6) with zero Langevin forces $h_j$ in the equations. The result of the analysis is that the wake from two cylinders may exist in the form of five different global modes [13]. They are: an asymmetric mode consisted of two different Karman streets, another asymmetric mode with one street fully damped, and three symmetric modes of equal Karman streets.
which are in-phase, anti-phase, and variable angle synchronized (in accord to general consideration [14]). In some domains of the model parameters some wake modes intermit. In all there are six different intermittence regimes within the framework of our model [13]. Configuration of all modes and its stability analysis may be done straightforward analytically except for the case of asymmetric mode \( A_{\rho, r} \) with unequal oscillation amplitudes of oscillators-streets \( \rho \neq r \).

To analyze configuration and stability of the modes it is useful to introduce the collective variables \([6] R^2, D, \text{ and } P\) by using the transform

\[
R^2 = \rho^2 + r^2, \quad D = 0.5\left[1 - 2\rho r / \left(\rho^2 + r^2\right)\right]
\]

Than coordinates of asymmetric mode \( A_{\rho, r} \) in new variables are solutions to the set of equations (9), (10), and (11) which give stable stationary points to system (4)–(6) with zero Langevin forces

\[
\cos^2 P = \frac{3\lambda - 1}{2\lambda}
\]

\[
R^2 = \frac{4}{1 + l(0.5 - D)\cos P}
\]

\[
-\frac{\varepsilon R^2}{8} \left[ \frac{0.5 - N}{\Phi_{\rho}} + \frac{0.5 + N}{\Phi_{r}} \right] \sin(2P) + l(0.5 - D) \left( \frac{1}{\Phi_{\rho}} + \frac{1}{\Phi_{r}} \right) \sin(P) = 0
\]

Here

\[
N = \sqrt{D(1 - D)}
\]

Accordingly to equation (11) only low values of variable \( D << 0.5 \) may serve as solution in the case of \( \varepsilon \to 0 \), than configuration and stability analysis for the mode \( A_{\rho, r} \) may be performed by semi-analytical way [13]. The set of equations (9–12) may have in principle a four-branch solution as angle \( P \) may settle in four quadrants accordingly to equation (9). Really, the set (9–12) has solutions in three quadrants only (the second, the third and the fours for the case \( l > 0 \)). Among them we found one branch stable to infinitesimal perturbations in some domain of model parameters. It is branch (13).

\[
P = \frac{-l}{|l|} \arccos \left( -\frac{1}{|l|} \sqrt{\frac{3 - \frac{1}{\lambda}}{2}} \right)
\]

The stability domain for branch (13) is presented in figure 3 for intermediate value of parameter \( \Delta = 0.125 \) (half-analytical evaluation for the case of \( \varepsilon \to 0 \), [13]). Numerical calculations of model system of “exact” equations (4)–(6) with sufficiently low values of parameter \( \varepsilon \) and with initial conditions belonging to this domain on figure 3 demonstrate stable focuses or stable cycles. Moreover, stable cycles exist even in essentially wider domain of parameters \( \lambda \) and \( l \), than it is on figure 3.
Figure 3. Domain (in red) of existence of stable focuses of wake mode $\mathcal{A}_0$, on the plane of model parameters $\lambda$ and $l > 0$ in the case $\Delta = 0.125$; $\varepsilon \to 0$. (The domain right boundary corresponds to criterion of stability $a_i a_z - a_i > 0$ in notations [15]).

3. Distribution of oscillation frequencies of the wake modes
Principle feature of the wake is its oscillation frequency. It is of special interest for comparison with the experimental data. Distribution of the wake modes oscillation frequencies is adduced beneath. The frequencies are in non-dimensional form such that solitary cylinder demonstrates the oscillation frequency being equal to unity.

I. For symmetric mode $S_0$ of wake consisted of identical Karman streets synchronized sin-phase

$$\rho^2 = r^2 = 4 / (1 + \lambda + l), \quad P = 0$$

and the oscillation frequency is as \( \omega^2 = \frac{1 + \lambda + l}{1 + (1 - 4\Delta)(\lambda + l)} \). The domain of existence and stability of this mode is \( \lambda < 1, l > -2\lambda \).

II. Symmetric mode $S_\pi$ of identical streets synchronized in anti-phase

$$\rho^2 = r^2 = 4 / (1 + \lambda - l), \quad P = \pi$$

possesses oscillation frequency \( \omega^2 = \frac{1 + \lambda - l}{1 + (1 - 4\Delta)(\lambda - l)} \). The domain of existence and stability of this mode is \( \lambda < 1, l < 2\lambda \).

III. Symmetric mode $S_\varphi$ of streets synchronized at a variable angle differed from 0 or $\pi$: $P = \arccos(-l / 2\lambda)$;

$$\rho^2 = r^2 = 4 / (1 + 3\lambda - l^2 / \lambda)$$

Its oscillation frequency is \( \omega^2 = \frac{1 + 3\lambda - \left(l^2 / \lambda\right)}{1 + (1 - 4\Delta)(3\lambda - \left(l^2 / \lambda\right))} \). This mode exists and is stable in the domain $-(1/3) < \lambda < 0, |l| < -2\lambda$ and $\lambda < -1(1/3), \sqrt{\lambda(1 + 3\lambda)} < |l| < -2\lambda$.

IV. Asymmetric mode $\mathcal{A}_0$ with total quenching of Karman street behind one of the cylinders:

$$\rho^2 = 4, \quad r^2 = 0.$$ Its oscillation frequency is as \( \omega^2 = 1 \). The domain of existence and stability of this mode is \( \lambda > 0.5 \).
V. Asymmetric mode $A\mathcal{S}_{\rho r}$ of two streets with different amplitudes. Its form and the domain of existence and stability discussed earlier, in section 2. Expression for the mode oscillation frequency in the limit $\varepsilon \to 0$ is 
$$\omega^2 = \frac{2 - |l| \sqrt{1.5 - (0.5/\lambda)}}{1 + (1 - 4\Delta)(1 - |l| \sqrt{1.5 - (0.5/\lambda)})}.$$ 

Information concerning the oscillation frequencies of the modes that may concurrent is summarized in figure 4. These data are represented by level lines of square of oscillation frequency on the model parameters plane $\lambda, l$ while the value of the third parameter is intermediate: $\Delta = 0.125$. The value of parameter $\varepsilon$ is essential only for asymmetric mode $A\mathcal{S}_{\rho r}$ data. In this case the parameter $\varepsilon$ assumed to be small enough. Aria $0 < l < 8$ is adduced at this figure only as for area $l > 0$ is a mirror image of the opposite one for $l > 0$.

![Figure 4](image)

Figure 4. Dependence of square of oscillation frequencies for three wake global modes $S_0$, $S_\pi$, and $A\mathcal{S}_{\rho r}$ as functions of model parameters $\lambda, l$. (Area $l < 0$ is a mirror image of the opposite one: $l > 0$). The frequency dependence is represented by level lines and the values of these constant levels of frequency squares are depicted at corresponding lines. Information for $S_0$ mode is given in green, for $S_\pi$ - in blue, and for $A\mathcal{S}_{\rho r}$ - in red. Heavy blue line is a boundary of the mode $S_\pi$ stability domain.

According to preliminary discussion the principle question is as follows. Whether or not the advanced model accommodates the distribution of oscillation frequencies, which has been observed in our experiment? These oscillation frequencies are as in figure 2 whereas the wake frequency for solitary cylinder is of 0.9 kHz. The desired agreement may be achieved if we assume the intermittence of $S_0$ and $S_\pi$ wake modes, for example, in the case of following model parameters: $\lambda = 0.65$; $l = 0.99$ and $\Delta = 0.14$. Another opportunity is $S_0$ and $A\mathcal{S}_{\rho r}$ intermittence in the case of the model parameters: $\lambda = 0.344$; $l = 6.41$; $\Delta = 0.1$ and $\varepsilon = 0.1$, for example. Here it should be mentioned that for the last set of parameters, the $A\mathcal{S}_{\rho r}$ stationary point is unstable and the trajectory localizes in some area around this point (as it was mentioned above and in [13]).
4. Domain of attraction for asymmetric mode $A_{S_{pr}}$

Among properties of the wake mode its attraction domain is of principle one. Information concerning the wake modes attraction domains in phase space may be useful in analysis of velocity spectra and other topics. Accordingly, numerical calculation of the attraction area for asymmetric mode $A_{S_{pr}}$ was performed. The results are illustrated by figure 5 for the case $\lambda = 0.34$, $l = 2.5$, $\Delta = 0.125$, and $\epsilon = 0.1$. One can see that the mode attraction domain has characteristic shape of a roll with central core, and the attraction domain volume is not large as compared to total volume of the phase space. Accordingly, the existence probability of asymmetric mode $A_{S_{pr}}$ have to be low in comparison with existence probability of competing symmetric mode. Precisely this situation (of practically zero intermittence coefficient) one can see in our experimental data for neutral gas wake (figure 2a).

![Figure 5](image)

**Figure 5.** Attraction domain in the phase space $R^2, D, P$ of our model for wake asymmetric mode $A_{S_{pr}}$. The domain is illustrated by level cuts $R^2 = \text{const}$ which colored in green for $R^2 = 8$, blue for $R^2 = 4$, and brown for $R^2 = 2$ (three turns of infinite spiral are illustrated only).

5. Brief resume

If one exclude from consideration mode $A_{S_{01}}$ with total quenching of one Karman street (that is the case of strongly interacting streets [14]), the intermittent wake of two cylinders within the framework of our model may exist in two regimes. These regimes are: the first - intermittence of sin- and anti-phase synchronized von Karman streets ($S_0$ and $S_\pi$ modes), and the second - intermittence between streets synchronized at the angle $\sim \pi / 2$ (that is $A_{S_{pr}}$ mode) and one symmetric mode ($S_0$ or $S_\pi$).

Acknowledgements

This study is an extension of the work supported by Russian Fund for Basic Researches (grant No 13–01–00742) and may ground the appropriate claim No 17–01–00394.

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