Closed stable orbits in a strongly coupled resonant Wilberforce pendulum

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Abstract

We prove the existence of closed stable orbits in a strongly coupled Wilberforce pendulum, for the case of a $1 : 2$ resonance, by using techniques of geometric singular symplectic reduction combined with the more classical averaging method of Moser.

1 Introduction

The Wilberforce pendulum is a physical device composed of a long suspended spring with a mass attached at the lower end, which is free to rotate around the vertical axis, twisting the spring through a non-linear coupling.
It can be found in any Physics laboratory, where it is used to demonstrate the periodic motion arising from transference of energy between the two main modes of oscillation. If the spring is initially stretched, with a certain initial torsion, and then released from rest, the motion will start being dominated by an ‘up and down’ swinging, which gradually converts itself into a purely rotational oscillation of the hanging mass.

This striking motion is crucially related to the non-linear coupling between both oscillating modes. In the usual setting, that coupling is as weakest as possible, being given by a quadratic term in the generalized coordinates. To be more specific, consider the spring to be massless, with elastic and torsional constants being $\kappa$ and $\rho$, respectively. Let the moment of inertia of the hanging mass $m$ be $I$. Finally, denote by $x$ the elongation of the spring and by $y$ the torsion angle. The complete Lagrangian in the case of a quadratic coupling is then

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{y}^2 - \left( \frac{1}{2} \kappa x^2 + \frac{1}{2} \rho y^2 + \varepsilon xy \right).$$

This case is well known, as general references we can cite [Köpf(1990), Berg and Marshall(1991), Plavčić et al(2009)]. In this work, we will be interested in the case of a stronger coupling, namely, one given by a quartic non-linear term. The Lagrangian (1) has then to be modified to read

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{y}^2 - \left( \frac{1}{2} \kappa x^2 + \frac{1}{2} \rho y^2 + \varepsilon x^2 y^2 \right),$$

with its corresponding Hamiltonian

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2I} p_y^2 + \frac{1}{2} \kappa x^2 + \frac{1}{2} \rho y^2 + \varepsilon x^2 y^2,$$

and our goal is to study the existence of closed stable motions.

As stated above, the problem falls within the reach of perturbation theory. On physical grounds, it was to be expected that a mild non-linear coupling would lead to periodic motions, ‘inherited’ from the two independent oscillation modes that would exist in the absence of coupling, but the situation is not so clear in the presence of a strong non-linear coupling, as these interactions typically lead to a chaotic evolution, as shown in many textbooks such as [Ott(2002), Strogatz(2015)].

The standard tool for determining the onset of chaos in any dynamical system (in particular, perturbed ones) is the construction of a suitable
Poincaré section in the phase space; we do this in the next section, showing the progressive destruction of the integrable tori associated with increasing values of the perturbation parameter. In the case in which the non-perturbed system is integrable, the Kolgomorov-Arnold-Moser (KAM) theorem guarantees the persistence of some of these tori, thus proving the existence of stable periodic orbits winding around some of the corresponding orbits of the unperturbed system in the absence of resonances. In this work, however, we will be interested in the 1 : 2 resonance (although the ideas presented will work for an arbitrary \( m : n \) resonance), so the KAM theorem will not be applicable.

Aside from the KAM theorem, there are other techniques in perturbation theory that are well suited to the kind of problem at hand. Our path here will be to put first the system into normal form by using the Lie-Deprit method (see [Deprit(1969)]), and then to apply a singular geometric reduction (as in [Cushman(1994), Cushman and Bates(1997)]) to pass to a reduced phase space where closed stable orbits can be detected either by applying Moser’s theorem on reduction (see [Moser(1970), Churchill et al(1983)]) or by determining the fixed points on a suitable Poincaré surface. This approach has been applied successfully to a qualitatively different system, the Pais-Uhlenbeck oscillator, in [Avendaño-Camacho et al(2017)] so it can be considered to be of a general nature.

Transforming the original perturbed Hamiltonian (2) \( H = H_0 + \varepsilon H_1 \), where
\[
H_0 = \frac{1}{2m} p_x^2 + \frac{1}{2I} p_y^2 + \frac{1}{2} \kappa x^2 + \frac{1}{2} \rho y^2
\]
is the sum of two independent oscillators, into another one in normal form \( N = H_0 + \sum_{j=1}^{\infty} \varepsilon^j N_j \), where \( \{H_0, N_j\} = 0 \) for each \( j \in \mathbb{N} \) (with \( \{,\} \) the canonical Poisson bracket on the algebra of smooth functions on the phase space, \( C^\infty(\mathbb{R}^4) \)), requires solving a set of equations known (following V. I. Arnold) as the homological equations. To this end, it is convenient to make use of the averaging method, based on two averaging operators, \( \langle \cdot \rangle \) and \( \mathcal{S} \), acting on tensor fields defined on the phase space manifold (in particular, on smooth functions on phase space). This allows us to obtain recursive formulas for computing the sub-Hamiltonians \( N_j, j \in \mathbb{N} \). Moser’s theorem extracts information about the existence of closed stable orbits from the critical points of the first-order normal perturbation \( N_1 \) on the reduced phase space, but when there are degeneracies (as it turns out to be our case) one has to go over \( N_2 \) (by considering \( N = H_0 + \varepsilon (N_1 + \varepsilon N_2) \) as the new perturbed
Hamiltonian). We will use the explicit expressions for $N_1$ and $N_2$ deduced in [Avendaño-Camacho et al(2013)], which have the advantage of not relying on the introduction on action-angle variables; on the contrary, they only depend on the averaging operators mentioned above, and the canonical Poisson bracket. Finally, from the study of the critical points of $N_1$ and $N_2$ on the reduced phase space, we will be able to conclude the existence of closed stable orbits in the Wilberforce pendulum for any fixed value of the perturbation parameter $\varepsilon > 0$.

Let us remark that averaging techniques have been used to study a different problem, the existence of periodic orbits in a perturbed weakly coupled Wilberforce pendulum, see [de Bustos et al(2016)]. There, the authors parametrize the periodic solutions of that perturbed system by the simple zeros of an associated system of nonlinear equations.

2 Numerical analysis

To make explicit the characteristic frequencies of the system $w_1, w_2$, let us introduce them through $\kappa = mw_1^2$ and $\rho = Iw_2^2$. The Hamiltonian is

$$H = H_0 + \varepsilon H_1 = \frac{1}{2} (p_x^2 + mw_1^2 x^2 + p_y^2 + Iw_2^2 y^2) + \varepsilon x^2 y^2. \quad (3)$$

The corresponding Hamilton equations of motion are given by the first-order system

$$\dot{x} = p_x, \quad \dot{p}_x = - \left(mw_1^2 x + 2\varepsilon xy^2\right), \quad \dot{y} = p_y, \quad \dot{p}_y = - \left(Iw_2^2 y + 2\varepsilon yx^2\right). \quad (4)$$

We solve this system numerically with the symplectic velocity Verlet
method, see [Holmes(2007)]. The numerical scheme in this case is

\[ x_{i+1} = x_i + k(p_x)_i + \frac{k^2}{2} F_1(x_i, y_i; \varepsilon), \]

\[ (p_x)_{i+1} = (p_x)_i + \frac{k}{2} (F_1(x_{i+1}, y_{i+1}; \varepsilon) + F_1(x_i, y_i; \varepsilon)), \]

\[ y_{i+1} = y_i + k(p_y)_i + \frac{k^2}{2} F_2(x_i, y_i; \varepsilon), \]

\[ (p_y)_{i+1} = (p_y)_i + \frac{k}{2} (F_2(x_{i+1}, y_{i+1}; \varepsilon) + F_2(x_i, y_i; \varepsilon)), \]

where \( k > 0 \) is the step size and we have written \( F_1(x, y; \varepsilon) = -m\omega_1^2 x - 2\varepsilon xy^2, \)

\( F_2(x, y; \varepsilon) = -I\omega_2^2 y - 2\varepsilon yx^2. \)

The resulting dynamics in the \( xy \) plane is displayed in Figure 1, where we have taken as initial condition \((1, 1, 1, 1)\), along with values \( m = I = w_1 = 1, w_2 = 2\), which will be assumed in what follows. Notice that the expected Lissajous figure appears when the coupling is switched off, and the pattern becomes fuzzier around \( \varepsilon = 0.5 \).

The chaotic behavior is even more apparent when considering a Poincaré section. We will take a surface \( \sigma \) transversal to the flow of (4) constructed in the following way\(^1\): First, we fix an energy value \( H = h \) in (3), and write \( p_2 \) in terms of \((p_1, q_1, q_2, h)\):

\[
p_2 = \pm \sqrt{2 \left( h - \varepsilon (q_1q_2)^2 \right) - (p_1^2 + q_1^2 + 4q_2^2)}.
\] (5)

Next, we restrict the solution to the level set \( \Sigma_h := \{ (p_1, q_1, q_2) : H = h \} \).

The Poincaré section is then \( \sigma = \{ (p_1, q_1) \in \Sigma_h : q_2 = 0 \} \).

Initial conditions are taken of the form \((j/100, 1.5, p_2, 0.01)\), where \( j \) is a random number in \([-100, 100] \) and \( p_2 \) is given in (5). In all cases, the value \( h = 3 \) has been chosen. Again, the evolution is computed with the velocity Verlet method, recording the points that cross \( \sigma \) by looking at sign changes. Notice that two set of solutions are obtained, one for each sign of (5), which are superimposed to get the final Poincaré map for each value of the parameter \( \varepsilon \). The results are shown in Figure 2; as stated above, the destruction of the integrable tori is quite visible here. However, certain ‘islands of stability’

\(^1\)Of course, this is just a choice. There are many possibilities for constructing a Poincaré surface, but the idea of the numerical procedure is the same in all of them. Our choice is determined by reasons of graphic cleanliness.
Figure 1: Strongly coupled Wilberforce pendulum dynamics for different values of the parameter $\varepsilon$. The resonance 1:2 is shown.
survive (as in the non-resonant case describes by KAM theorem), and in the next sections we prove their existence analytically.

3 Normal forms in perturbation theory

Given a Poisson manifold \((M, \{\cdot, \cdot\})\), consider a perturbed Hamiltonian of the form \(H = H_0 + \varepsilon H_1\), where \(H_0\) is supposed to be integrable. Hamilton’s equations for \(H\) are a coupled non-linear system of differential equations whose solutions, in general, do not have a closed form. The Lie-Deprit approach to this problem substitutes the system of Hamiltonian equations by a simpler one, suitable to be studied by analytic tools, while providing some criterion to determine the degree of accuracy of the approximation. The perturbed Hamiltonian \(H\) is said to admit a normal form of order \(n\) if there exist a near-identity canonical transformation on phase space such that \(H\) is transformed into

\[
H = \sum_{i=0}^{n} \varepsilon^i N_i + R_H ,
\]

where \(N_0 = H_0\) and

\[
\{N_i, H_0\} = 0, \text{ for all } 1 \leq i \leq n .
\]

The truncated function \(N = \sum_{i=0}^{n} \varepsilon^i N_i\) is the normal form (of order \(n\)) of \(H\). This approach is based on the fact that whenever \(\|H - N\| = \|R_H\|\) is small in a suitable norm, the trajectories of \(N\) provide us with good approximations to the true trajectories of \(H\). In particular, closed orbits for \(H\) can be detected through the existence of closed orbits for \(N\).

The normal form is obtained from a family of canonical transformations depending on the parameter \(\varepsilon\), \(x \mapsto y(x; \varepsilon)\) (where \(x\) denotes collectively the coordinates on \(M\)), such that \(y(x; 0) = x\). To assure that these transformations are canonical, they are derived from a generating function \(S = S(\varepsilon)\):

\[
\frac{\partial y_j}{\partial \varepsilon} = \{S, y_j\} = \mathcal{L}_{X_S} y_j, \text{ for } j \in \{1, \ldots, \dim M\} .
\]

with \(X_S = \{S, \cdot\}\) the Hamiltonian vector field determined by \(S\). Geometrically, \(X_S\) is the ‘\(\varepsilon\)-flow generator’, much in the same way as \(H\) is the time-flow generator.
Figure 2: Poincar maps of the strongly coupled Wilberforce pendulum for different values of $\varepsilon$. The resonance $1:2$ is shown.
The Lie-Deprit method proceeds by developing $S$ in a formal series $S = \sum_{j=0}^{n} \varepsilon^j S_j$ and translating the condition of being a generating function for canonical transformations into a set of equations, one for each term $S_j$, having the structure

$$\mathcal{L}_{X_{H_0}} S_j = F_j - (j + 1) N_{j+1} \quad j \geq 0,$$

(9)

where the $F_j$ functions are determined by quantities already calculated in previous steps. What is remarkable (see [Deprit(1969)]) is that these equations (called the homological equations) have a recursive structure (the Deprit’s triangle) and they can be solved in terms of $H_0$ and the sub-Hamiltonians $N_j$. The usual method of solution is based on the introduction of action-angle coordinates, thus having a local character and requiring a symplectic phase space. To avoid these issues here we follow [Avendaño-Camacho et al(2013)], where a global method of solution is presented in the case of a system admitting a $U(1)$−action such that the Hamiltonian vector field $X_{H_0}$ has periodic flow, as is the case with the Wilberforce pendulum.

In a general setting, if we have a phase space which is a Poisson manifold $(M, \{\cdot, \cdot\})$, given the Hamiltonian $H = H_0 + \varepsilon H_1$ we can set up the homological equations (9). Now, suppose that the vector field $X_{H_0} = \{H_0, \cdot\}$ is complete and has periodic flow $\text{Fl}_{X_{H_0}}^t$. The periodicity condition means that there exists a period function $T : M \to \mathbb{R}$ such that $\text{Fl}_{X_{H_0}}^t(p) = \text{Fl}_{X_{H_0}}^{t+T(p)}(p)$. This flow induces a $U(1)$−action by putting $(t,p) \mapsto \text{Fl}_{X_{H_0}}^{t/w}(p)$, where $w = 2\pi/T > 0$ is the frequency function. A straightforward computation shows that the generator of this $U(1)$−action is given by the vector field

$$\Upsilon = \frac{1}{w} X_{H_0} \in \mathcal{X}(M).$$

Now, for any function $f \in C^\infty(M)$, its $U(1)$−averaging is defined in terms of the pullback by the flow:

$$\langle f \rangle = \frac{1}{T} \int_0^T (\text{Fl}_T^t)^* f \, dt.$$

Also, an $S$ operator, mapping $C^\infty(M)$ into itself, is defined as

$$S(f) = \frac{1}{T} \int_0^T (t - \pi)(\text{Fl}_T^t)^* f \, dt.$$
The solution to the homological equations can be expressed in terms of these operators (see [Avendaño-Camacho et al. (2013)]). In particular, the lowest order expressions for the normal forms of the perturbed Hamiltonian are

\[ N_1 = \langle H_1 \rangle = \frac{1}{T} \int_0^T (\text{Fl}_T^t)^* H_1 \, dt, \tag{10} \]

and

\[ N_2 = \frac{1}{2} \left\langle \left\{ S \left( \frac{H_1}{w} \right), H_1 \right\} \right\rangle. \tag{11} \]

4 Invariants of the Hamiltonian flow of the harmonic oscillator with two degrees of freedom.

Consider the harmonic oscillator with two degree of freedom on \( T^* \mathbb{R}^2 \) with coordinates \((q_1, p_1, q_2, p_2)\) and the Poisson bracket induced by the usual canonical symplectic structure, whose Hamiltonian is

\[ H_0(q_1, p_1, q_2, p_2) = \frac{1}{2} \left( p_1^2 + \omega_1^2 q_1^2 + p_2^2 + \omega_2^2 q_2^2 \right). \tag{12} \]

The associated Hamiltonian vector field is readily found to be

\[ X_{H_0} = p_1 \frac{\partial}{\partial q_1} - \omega_1^2 q_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_2} - \omega_2^2 q_2 \frac{\partial}{\partial p_2}. \]

The integral curves of \( X_{H_0}, c : I \subset \mathbb{R} \to T^* \mathbb{R}^2 \), can be parametrized as \( c(t) = (q_1(t), p_1(t), q_2(t), p_2(t)) \), and satisfy the decoupled system (where the dots denote time derivatives)

\[ \begin{cases} \dot{q}_1 + \omega_1^2 q_1 = 0 \\ \dot{q}_2 + \omega_2^2 q_2 = 0. \end{cases} \]

Hence, we have an action on \( T^* \mathbb{R}^2 \simeq \mathbb{R}^4 \) given by the (linear) flow of \( X_{H_0} \):

\[ \text{Fl}_{X_{H_0}}^t \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \cos \omega_1 t + \frac{p_1}{\omega_1} \sin \omega_1 t \\ -\omega_1 q_1 \sin \omega_1 t + p_1 \cos \omega_1 t \\ q_2 \cos \omega_2 t + \frac{p_2}{\omega_2} \sin \omega_2 t \\ -\omega_2 q_2 \sin \omega_2 t + p_2 \cos \omega_2 t \end{pmatrix}. \]
This flow is periodic whenever $\omega_1$ and $\omega_2$ are commensurable so, by a suitable rescaling in time, we actually have a $U(1)$ action. In particular, if $\omega_1, \omega_2 \in \mathbb{Z}$ are coprime, as in the case of the $1:2$ resonance that we will consider (that is, $\omega_1 = 1, \omega_2 = 2$), then $\text{Fl}_{X_{H_0}}$ is already $2\pi$-periodic. Notice that periodic orbits will be invariant sets under the action of this flow, so we expect to be able of finding them by studying the invariant functions. In fact, we will restrict our attention to the set of invariant polynomials under the action of this flow; the reason is that any other invariant will be a smooth function of these, as we will see below.

It is well known that the algebra of invariant polynomials (under the action of the Hamiltonian flow of $X_{H_0}$) is finitely generated, see for example [Churchill et al.(1983), Cushman and Bates(1997)]. Moreover, the generators can be chosen as the so-called the Hopf variables:

$$
\begin{align*}
\rho_1 &= z_1 \bar{z}_1 = \omega_1^2 q_1^2 + p_1^2 \\
\rho_2 &= z_2 \bar{z}_2 = \omega_2^2 q_2^2 + p_2^2 \\
\rho_3 &= \text{Re} \left( z_1^{\omega_1} z_2^{\omega_2} \right) = \text{Re} \left( (p_1 + i\omega_1 q_1)^{\omega_2} (p_2 - i\omega_2 q_2)^{\omega_1} \right) \\
\rho_4 &= \text{Im} \left( z_1^{\omega_1} z_2^{\omega_2} \right) = \text{Im} \left( (p_1 + i\omega_1 q_1)^{\omega_2} (p_2 - i\omega_2 q_2)^{\omega_1} \right).
\end{align*}
$$

For instance, in the case of the $1:2$ resonance we get

$$
\begin{align*}
\rho_1 &= q_1^2 + p_1^2 \\
\rho_2 &= 4q_2^2 + p_2^2 \\
\rho_3 &= p_2(p_1^2 - q_1^2) + 4p_1q_1q_2 \\
\rho_4 &= 2q_2(p_1^2 - q_1^2) - 2q_1p_1p_2.
\end{align*}
$$

There exists a certain algebraic relation satisfied by the $\rho$ variables, namely:

$$\rho_3^2 + \rho_4^2 = \rho_1^2 \rho_2^2 \omega_1^2, \quad \rho_1, \rho_2 \geq 0,$$

which is the equation of a singular algebraic surface in $\mathbb{R}^4$. For the particular case of the $1:2$ resonance, this is

$$\rho_3^2 + \rho_4^2 = \rho_1^2 \rho_2^2, \quad \rho_1, \rho_2 \geq 0.$$
the result in [Schwarz(1975)], which tells us that the smooth observables invariant under the action of $U(1)$ are smooth functions of the polynomial generators $(\rho_1, \rho_2, \rho_3, \rho_4)$.

5 Second-order normal form of the Hamiltonian

In order to prove analytically the existence of periodic orbits for the Wilberforce pendulum and determine their stability, we compute the second order normal form in the case of a quartic interaction and $1:2$ resonance:

$$H(q_1, p_1, q_2, p_2) = H_0 + \varepsilon H_1$$

$$= \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + 4q_2^2) + \varepsilon q_1^2 q_2^2.$$  

Since $\{H_0, N_i\} = \mathcal{L}_{X_{H_0}} N_i = 0$, the first and second order normal forms are invariant under the $U(1)$-action induced by the flow of $H_0$; we will take the quotient of the phase space by this action and get the corresponding Hamiltonian on the reduced phase space in the next section. An important feature of this reduction process is that this reduced Hamiltonian will be a function of only three among the invariant generators $(\rho_1, \rho_2, \rho_3, \rho_4)$. Previous to reduction, we compute in this section the expressions of $N_1$ and $N_2$.

Notice that the Hamiltonian flow $\text{Fl}^{t}_{X_{H_0}}$ in this case is given by

$$\text{Fl}^{t}_{X_{H_0}} \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \cos t + p_1 \sin t \\ -q_1 \sin t + p_1 \cos t \\ q_2 \cos 2t + \frac{p_2}{2} \sin 2t \\ -2q_2 \sin 2t + p_2 \cos 2t \end{pmatrix}$$

and it is $2\pi$-periodic. The second-order normal form of the Wilberforce oscillator is $H_0 + \varepsilon N_1 + \frac{\varepsilon^2}{2} N_2$, where $N_1$ and $N_2$ are given by (10), (11). The computations are straightforward but tedious, and are best done using a computer algebra system (CAS). We have found the CAS Maxima very useful in this regard, and we have written a small Maxima package for this kind of computations, called pdynamics, which is available at https://github.com/josanvallejo/pdynamics.
The resulting normal form sub-Hamiltonians, already written in the Hopf variables, are as follows:

\[ H_0 = \frac{1}{2}(\rho_1 + \rho_2), \]

for the unperturbed part, and

\[ N_1 = \langle H_1 \rangle = \frac{1}{16}\rho_1\rho_2 \quad (17) \]

and

\[ N_2 = \langle \{ S(H_1), H_1 \} \rangle = -\frac{1}{68}(5\rho_1\rho_2^2 + 4\rho_2^3 + 16\rho_1^2), \quad (18) \]

for the first and second-order perturbations, respectively.

We will make use of these explicit expressions in the following sections, to determine the existence of stable periodic orbits in the dynamics of the Wilberforce oscillator.

6 Constructing the reduced phase space

We begin by identifying the geometry of the the reduced phase space. Then, we find an explicit expression for the reduced Hamiltonian, that is, the normal form Hamiltonian

\[ N = H_0 + \varepsilon N_1 + \frac{1}{2}\varepsilon^2 N_2 + O(\varepsilon^3) \]

restricted to the reduced phase space. We follow the technique described in [Churchill et al(1983), Cushman(1994)] to prove that (14) and the condition of constant energy \( H_0 = h > 0 \), give the algebraic description of the reduced phase space. We use a result in [Poènaru(1976)], which states that the basic invariant polynomials separate the orbits of the Hamiltonian flow \( \text{Fl}_{XH_0} \). In our case this implies\(^2\) that the equality \( (\rho_1(q,p), \ldots, \rho_4(q,p)) = (\rho_1(q',p'), \ldots, \rho_4(q',p')) \) holds if and only if \( (q,p) \) and \( (q',p') \) belong to the same orbit. Thus, it is enough to prove that for every \( (u_1, u_2, u_3, u_4) \) such that \( u_3^2 + u_4^2 = u_1^2u_2 \), its inverse image under the map \( (q,p) \mapsto (\rho_1(q,p), \ldots, \rho_4(q,p)) \) is precisely a single orbit of the flow \( \text{Fl}_{XH_0}^L \). For instance, if \( u_2 = 0 \) then \( \rho_2(q,p) = 0 \) and necessarily \( q_2 = 0 = p_2 \) (from (13)). This, in turn, implies that \( \rho_3 = 0 = \rho_4 \) so we have the inverse image of \( (u_1, 0, 0, 0) \), where \( u_1 \geq 0 \), which is the set \{ \( (q_1, p_1, 0, 0) \in \mathbb{R}^4 : q_1^2 + p_1^2 = u_1 \} \), and this is clearly an orbit of \( \text{Fl}_{XH_0}^L \). The remaining cases

\(^2\)Here we collectively denote \( (q_1, p_1, q_2, p_2) \) by \( (q, p) \).
can be done along similar lines, and will not be repeated here. The reduced phase space is then given by the set of equations

\[
\begin{align*}
\rho_3^2 + \rho_4^2 &= \rho_2^2 \rho_1, \\
\rho_1 + \rho_2 &= 2h,
\end{align*}
\]

that is,

\[
\rho_3^2 + \rho_4^2 = \rho_1^2 (2h - \rho_1), \quad 0 \leq \rho_1 \leq 2h.
\]

As mentioned above, (19) is the equation of a singular algebraic surface \( S \in \mathbb{R}^3 \). Topologically, this surface is a pinched sphere with a singularity at the point \((\rho_1, \rho_3, \rho_4) = (0, 0, 0)\) (see Figure 3).

Figure 3: Reduced phase space of the 1:2 resonance.

One of the most important results in the theory is a theorem by Moser (see [Moser(1970), Churchill et al(1983)]), which can be stated as follows: Let \( H = H_0 + \varepsilon H_1 \) be a perturbed Hamiltonian, with \( S \) the hypersurface \( H_0 = h \). Suppose that the orbits of the Hamiltonian flow \( \text{Fl}^{t}_{X_{H_0}} \) are all periodic with period \( 2\pi \) and let \( M_h \) be the quotient with respect to the induced \( U(1) \)–action on \( S \). Then, to every non-degenerate critical point \( p \in M_h \) of the restricted averaged perturbation \( N_1|_S = \langle H_1 \rangle|_{M_h} \) corresponds a periodic trajectory of the full Hamiltonian vector field \( X_H \), that branches off from the orbit represented by \( p \) and has period close to \( 2\pi \).

In order to apply this result, we must first characterize the critical points of Hamiltonian vector fields in the reduced space. First, observe that the
commutator relations among generators \((\rho_1, \rho_2, \rho_3, \rho_4)\) are given by
\[
\begin{align*}
\{\rho_1, \rho_2\} &= 0, \quad \{\rho_1, \rho_3\} = -4\rho_4, \quad \{\rho_1, \rho_4\} = 4\rho_3, \\
\{\rho_2, \rho_3\} &= 4\rho_4, \quad \{\rho_2, \rho_4\} = -4\rho_3, \\
\{\rho_3, \rho_4\} &= -4\rho_1(\rho_1 - 2\rho_2).
\end{align*}
\] (20)

Renaming the variables \(\rho_3 = x, \rho_4 = y, \) and \(\rho_1 = z,\) these relations induce a Poisson bracket on the three dimensional Euclidean space \(\mathbb{R}^3 = \{(x, y, z)\}\) given by
\[
\{f, g\} = 2\left\langle \nabla g, \nabla f \times \nabla F \right\rangle,
\] (21)
where \(F\) is the function
\[
F(x, y, z) = x^2 + y^2 - z^2(2h - z),
\] (22)
and the symbols \(\langle \cdot, \cdot \rangle, \times, \nabla\) stand for the usual inner product, cross product and nabla operator in \(\mathbb{R}^3,\) respectively. Hence, for any \(f \in C^\infty(\mathbb{R}^3)\), its Hamiltonian vector field is given by
\[
X_f = 2\nabla f \times \nabla F.
\] (23)

It follows directly from definition (21) that the function \(F(x, y, z)\) (22) is a Casimir of the Poisson structure (21). Thus, the symplectic leaves of the corresponding foliation are precisely the connected components of level sets of \(F.\) If we define the mapping \(P : \mathbb{R}^4 \rightarrow \mathbb{R}^4\) by
\[
P(\rho_1, \rho_2, \rho_3, \rho_4) = (\rho_3, \rho_4, \rho_1),
\]
we get that \(P\) is a Poisson map and \(P(H^{-1}_0(h)) = F^{-1}(0).\) Moreover,
\[
\left( P \circ F|_{\mathcal{L}_{\rho_0}} \right) \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} = P \begin{pmatrix} \rho_1(p_1, q_1, p_2, q_2) \\ \rho_2(p_1, q_1, p_2, q_2) \\ \rho_3(p_1, q_1, p_2, q_2) \\ \rho_4(p_1, q_1, p_2, q_2) \end{pmatrix}.
\]

Therefore, the reduced space is contained in a symplectic leaf of \(F^{-1}(0) \subset \mathbb{R}^3.\) Let us denote by \(M_h\) the reduced space. Then, a realization of it as a smooth manifold\(^3\) is given by
\[
M_h = F^{-1}(0) \text{ and } 0 < z \leq 2h.
\] (24)

\(^3\) Notice that the condition \(z > 0\) removes the singularity at the origin.
Any function $f \in C^\infty(\mathbb{R}^3)$ defines a Hamiltonian vector field $\tilde{X}_f$ on $M_h$ by
\[ \tilde{X}_f := (2\nabla f \times \nabla F) |_{M_h}. \]

It also follows from (23) that the Hamiltonian vector field $\tilde{X}_f$ has a critical point at the point $p \in M_h$ if and only if either $\nabla f(p)$ is orthogonal at $p$ to the reduced space $M_h$, or $\nabla f(p) = 0$.

Next, we describe how to obtain the reduced Hamiltonian vector field corresponding to a function $G \in C^\infty(\mathbb{R}^4)$ such that $\{H_0, G\} = 0$. As discussed above, $G$ can be expressed in terms of the Hopf variables: $G = G(\rho_1, \rho_2, \rho_3, \rho_4)$. Writing $\rho_1 = z, \rho_2 = 2h - z, \rho_3 = x$ and $\rho_4 = y$, we obtain the function $Q(x, y, z) = G(z, 2h - z, x, y)$. Thus, the reduced Hamiltonian vector field associated to $G$ is the vector field
\[ \tilde{X}_G = (2\nabla Q \times \nabla F) |_{M_h}. \]

This expression allows us to compute the critical points of the reduced vector field associated to a function $G \in C^\infty(\mathbb{R}^4)$ such that $\{H_0, G\} = 0$. As discussed above, $G$ can be expressed in terms of the Hopf variables: $G = G(\rho_1, \rho_2, \rho_3, \rho_4)$. Writing $\rho_1 = z, \rho_2 = 2h - z, \rho_3 = x$ and $\rho_4 = y$, we obtain the function $Q(x, y, z) = G(z, 2h - z, x, y)$. Thus, the reduced Hamiltonian vector field associated to $G$ is the vector field
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\[ \tilde{X}_G = (2\nabla Q \times \nabla F) |_{M_h}. \]
Consider the critical point \((0, 0, 2h)\). By a straightforward computation, we get
\[
\frac{\partial F}{\partial z}(0, 0, 2h) = 4h^2 \neq 0.
\]
By the implicit function theorem, \(z = \psi(x, y)\) with \(\psi\) a smooth function at \((0, 0)\) satisfying \(\psi(0, 0) = 2h\) and \(F(x, y, \psi(x, y)) = 0\). Therefore, the function \(K\) in (25) has the form \(\tilde{K} = K(\psi(x, y))\) in a neighborhood of \((0, 0, 2h)\). Another immediate computation shows that
\[
\text{Hess}(\tilde{K}(0, 0)) = \frac{1}{16h^2} > 0.
\]
Thus, the critical point \((0, 0, 2h)\) is non-degenerate and Moser’s theorem (see the version presented as Theorem 6.4 in [Churchill et al(1983)]) implies that, for small enough \(\varepsilon\), the Wilberforce oscillator has a unique stable periodic orbit \(\gamma_\varepsilon\) with energy \(h\) through each point \(p(\varepsilon)\), sufficiently close to \((0, 0, 2h)\), with period \(T(\varepsilon)\), such that \(H_0(p(\varepsilon)) \to h\) and \(T(\varepsilon) \to 2\pi\).

7 Stability analysis at the degenerate points

We can not apply Moser’s theorem to the curve of critical points of the preceding section, \(\Gamma_h\), because they are degenerate (non-isolated). In order
to determine if some periodic orbits arise from some points of $\Gamma_h$ we must resort to the second-order normal form of the Hamiltonian \( H_0 + \varepsilon (N_1 + \varepsilon N_2) \) on its own.

Thus, we consider the $\varepsilon$-dependent function $K_\varepsilon(x, y, z) = (N_1 + \varepsilon N_2)|_{M_h}$. By arguments similar to those used in the case of Moser’s theorem, we readily see that a given point of $\Gamma_h$ generates a periodic orbit of the Wilberforce pendulum if it is a non-degenerate critical point of the Hamiltonian vector field $X_{K_\varepsilon} = 2\nabla K_\varepsilon \times \nabla F$ for all $\varepsilon$. Therefore, we need to look for points $p \in \Gamma_h$ such that, either $\nabla K_\varepsilon(p) = 0$, or $\nabla K_\varepsilon(p)$ is parallel to $\nabla F(p)$ for all $\varepsilon$. A straightforward computation gives

$$\nabla K_\varepsilon = \left( \frac{\varepsilon x}{96}, \frac{\varepsilon y}{24} - \frac{h - z}{8} - \frac{5\varepsilon}{768} (4h^2 - 8hz + 3\varepsilon^2) \right).$$

For this vector to vanish, its third component must be zero independently of $\varepsilon$, that is, both the independent term and the coefficient of $\varepsilon$ must vanish separately. These conditions would lead to $z = h$ and $h = 0$, so the only possibility is the point $(0, 0, 0)$ on the particular surface $M_0$, which is the case of the singular point that we consider in the next section. It follows from here that $\nabla K_\varepsilon$ never vanishes for $\varepsilon \neq 0$ and $h \neq 0$, and it is easy to check that it is parallel to $\nabla F$ only at the points $(0, 0, \frac{3}{2}h)$ and $(0, 0, 2h)$, which do not belong to the curve $\Gamma_h$. Consequently, we conclude that no points of $\Gamma_h$ (aside from the singular point $(0, 0, 0)$ in the case $h = 0$) can generate a periodic orbit.

8 Stability analysis at the singular point

Recall that, in order to impose a smooth structure on the reduced space, we left aside the singular point $(0, 0, 0)$. To complete our analysis, in this section we deal with that particular case (which corresponds in the literature to the so-called normal mode, $\gamma$). The existence of closed orbits will be proved by finding fixed points on a suitable Poincaré section.

Let $f_2(p_1, q_1, p_2, q_2) = \frac{1}{2}(p_2^2 + 4q_2^2)$. The Hamiltonian vector field with respect to the canonical symplectic structure on $\mathbb{R}^4$, $X_{f_2}$, has periodic flow with periodic $T = \pi$. This flow generates a free and proper $U(1)$-action on $((\mathbb{R}^2 - (0, 0)) \times \mathbb{R}^2$. For every fixed $h > 0$, the level set $f_1^{-1}(h)$ is foliated by periodic orbits of $X_{f_2}$ and the reduced space is given by $M_h = f_2^{-1}(h)/U(1)$. Let
us make the following change of variables from \((p_1, q_1, p_2, q_2)\) to \((p_1, q_1, L, \theta)\): 

\[
\Psi(p_1, q_1, L, \theta) = (p_1, q_1, -\sqrt{4L} \sin \theta, \sqrt{L} \cos \theta),
\]

with \(L > 0, 0 < \theta < 2\pi/\omega_2\). In these coordinates, the canonical symplectic form on the domain \(\mathbb{R}^2 \times (\mathbb{R}^2 - (0,0))\), given by \(dp_1 \wedge dq_1 + dp_2 \wedge dq_2\), becomes \(dp_1 \wedge dq_1 + dL \wedge d\theta\), and the Hamiltonian of the Wilberforce oscillator is

\[
H(p_1, q_1, L, \theta) = \frac{1}{2}(p_1^2 + q_1^2) + 2L + \varepsilon L q_1^2 \cos^2 \theta. \tag{27}
\]

Consider the restriction to the level set \(\Sigma_h = \{(p_1, q_1, L, \theta) | L = h\}\). Since this level set is foliated by orbits of \(X_{f_2}\), the Hamiltonian equations of (27) are

\[
\begin{cases}
\dot{\theta} = 2 + \varepsilon q_1^2 \cos^2 \theta, \\
\dot{p}_1 = -q_1 - 2\varepsilon h q_1 \cos^2 \theta, \\
\dot{q}_1 = p_1.
\end{cases} \tag{28}
\]

We now construct the cross section \(\sigma_0 = \{(p_1, q_1, h, \theta) \in \Sigma_h : \theta = 0\}\) on it. The integral curve of (28) through \(a\) is:

\[
\begin{cases}
\theta(t) = 2t + \varepsilon \int_0^t q_1^2 \cos^2 \theta \, d\tau, \\
p_1(t) = p_1^0 \cos t - q_1^0 \sin t - 2\varepsilon h \int_0^t q_1 \cos^2 \theta \, d\tau, \\
q_1(t) = p_1^0 \sin t + q_1^0 \cos t.
\end{cases} \tag{29}
\]

Let \(T(a, \varepsilon)\) be the time elapsed between two consecutive intersections of \(\sigma_0\). From equations (29), we get

\[
4\pi = 2T(a, \varepsilon) + \varepsilon \int_0^{T(a, \varepsilon)} q_1^2 \cos^2 \theta \, dt,
\]

so \(T(a, \varepsilon)\) has the form

\[
T(a, \varepsilon) = 2\pi - \varepsilon \frac{\pi}{2}(q_1^0)^2 + O(\varepsilon^2). \tag{30}
\]

Substituting (30) in (29), we obtain the following expression for the Poincaré map determined by \(\sigma_0\):

\[
\begin{align*}
p_1(T(a)) &= p_1^0 + \varepsilon \pi q_1 \left(\frac{1}{2}(q_1^0)^2 - 2h\right) + O(\varepsilon^2), \\
n_1(T(a)) &= q_1^0 + \varepsilon p_1 \left(\frac{\pi}{2}(q_1^0)^2 + O(\varepsilon^2)\right).
\end{align*}
\]
In order to prove that there exists periodic orbits for the Wilberforce oscillator in $\Sigma_h$, we must show that, for each $\varepsilon$ small enough, there exist $p^0_1(\varepsilon)$ and $q^0_1(\varepsilon)$ such that we get a fixed point:

$$p_1(T(p^0_1(\varepsilon), q^0_1(\varepsilon), -h, 0, \varepsilon)) = p^0_1(\varepsilon),$$

$$q_1(T(p^0_1(\varepsilon), q^0_1(\varepsilon), -h, 0, \varepsilon)) = q^0_1(\varepsilon).$$

To this end, we define the following function $F : \mathbb{R}^3 \to \mathbb{R}^2$,

$$F \left( \begin{array}{c} p_1 \\ q_1 \\ \varepsilon \end{array} \right) = \left( \begin{array}{c} \pi q_1 \left( \frac{1}{2} (q^0_1)^2 - 2h \right) + O(\varepsilon) \\ p_1 \pi h^2 + O(\varepsilon) \end{array} \right).$$

First, we note that $F(0, 2\sqrt{h}, 0) = 0$. A straightforward computation shows that

$$\det \left( \frac{\partial F}{\partial p_1 \partial q_1} \bigg|_{(0, 2\sqrt{h}, 0)} \right) = \det \left( \begin{array}{cc} \pi h & 0 \\ 0 & \pi h \end{array} \right) > 0.$$

By the implicit function theorem, there exists $\delta > 0$, an open neighborhood $U$ of $(0, 2\sqrt{h})$, and a function $g : (-\delta, \delta) \to U$, $g(\varepsilon) = (p_1(\varepsilon), q_1(\varepsilon))$, such that $g(0) = (0, 0)$ and $F(g(\varepsilon), \varepsilon) = 0$. Therefore,

$$p_1(T(g(\varepsilon), -h, 0, \varepsilon)) = p_1(\varepsilon),$$

$$q_1(T(g(\varepsilon), -h, 0, \varepsilon)) = q_1(\varepsilon).$$

This fact proves that for each sufficiently small $\varepsilon$, the Wilberforce oscillator has a unique stable periodic orbit $\gamma_\varepsilon$, with energy $h$, which branches off from the normal mode $\gamma$.

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