Multifractal analysis of dimension spectrum and the set of irregular points in non-uniformly hyperbolic systems

GUAN-ZHONG MA\textsuperscript{1}, YAO XIAO\textsuperscript{2*}

\textsuperscript{1}, Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China  
\textsuperscript{2}, Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China  
mgz09@mails.tsinghua.edu.cn, yaox11@mails.tsinghua.edu.cn

Abstract

We study the multifractal analysis of dimension spectrum for almost additive potential in a class of one dimensional non-uniformly hyperbolic dynamic systems and prove that the irregular set has full Hausdorff dimension.

Key words: multifractal analysis; non-uniformly hyperbolic; measure concatenation  
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1 Introduction

Given a compact metric space $X$, and $T$ a continuous transformation from $X$ to itself, we call the pairs $(X, T)$ a topological dynamical systems. A sequence $\Phi = (\phi_n)_{n=1}^{\infty}$ is said to be almost additive if every $\phi_n$ is continuous from $X$ to $\mathbb{R}$ and there is a positive constant $C(\Phi) > 0$ such that

$$-C(\Phi)+\phi_n(x)+\phi_p(T^n x) \leq \phi_{n+p}(x) \leq \phi_n(x)+\phi_p(T^n x)+C(\Phi), \forall n, p \in \mathbb{N}, \forall x \in X.$$ 

We denote by $C_{aa}(X, T)$ the collection of almost additive potentials on $X$. The almost additive potential arise naturally in the study of non-conformal repellers \cite{P} and topological pressure of product of positive matrices \cite{R}.

If $\Phi = (\Phi^1, \cdots, \Phi^d)$ and $\Phi^j \in C_{aa}(X, T)$ for each $j$, we call $\Phi$ a vector-valued almost additive potential and write $\Phi \in C_{aa}(X, T, d)$. For $\Phi \in C_{aa}(X, T, d)$, we have $\Phi = (\phi_n)_{n=1}^{\infty}$ with $\phi_n = (\phi^1_n, \cdots, \phi^d_n)$.
Given any \( \Phi \in C_{aa}(X, T, d) \), by subadditivity we have \( \Phi_*(\mu) := \lim_{n \to \infty} \int_X \frac{\Phi_n}{n} \, d\mu \) exists for every \( \mu \in \mathcal{M}(X, T) \). We define the set \( \mathcal{L}_\Phi = \{ \Phi_*(\mu) : \mu \in \mathcal{M}(X, T) \} \), which is compact and convex. Given \( \Phi \in C_{aa}(X, T, d) \) and \( \alpha \in \mathbb{R}^d \), one can define the level set as \( X_\alpha := \{ x \in X : \lim_{n \to \infty} \frac{\Phi_n(x)}{n} = \alpha \} \). It is well known that if \( (X, T) \) satisfies specification condition, then \( X_\alpha \neq \emptyset \) if and only if \( \alpha \in \mathcal{L}_\Phi \). Roughly speaking, the level sets \( X_\alpha \) forms multifractal decomposition and the map \( \alpha \to \dim_H X_\alpha \) forms a multifractal spectrum.

We also define the set \( X_{irr} = \{ x \in X : \lim_{n \to \infty} \Phi_n(x) \text{ does not exist} \} \).

The theory of multifractal analysis for uniformly hyperbolic conformal dynamic system is well developed in the aspects of entropy spectrum and Birkhoff spectrum and local dimension of Gibbs measure \([4, 12, 6, 13]\). In the case of sub-shift of finite type, the multifractal analysis for the level sets of almost additive potential or quotient almost additive potential has been well understood \([1, 2]\). However there is still not a complete picture for the multifractal analysis of non-uniform hyperbolic dynamic systems. In the recent years, people become more and more interested in the multifractal analysis of non-uniform hyperbolic dynamic systems \([8, 10]\). In this note we proved that irregular set in non-uniform hyperbolic dynamic system carries full of Hausdorff dimension unless it is an empty set. The corresponding part in uniform hyperbolic dynamic systems was proved in \([3, 4]\).

We start with an introduction about the basic settings. Let \( T : \bigcup_{i=1}^m I_i \to [0, 1] \) be a piecewise \( C^1 \) map satisfies the following condition:

- \( I_i \subset [0, 1], i = 1, \ldots, m \) such that \( I_i \) and \( I_j \) do not overlap for \( i \neq j \).
- \( T|_{I_j} : I_j \to [0, 1] \) is onto and \( C^1 \) map, for all \( 1 \leq j \leq m \). There is a unique \( x_j \in I_j \) such that \( T(x_j) = x_j \).
- \( T'(x) > 1 \) for \( x \notin \{ x_1, \ldots, x_m \} \).

We remark that since the map \( T \) is \( C^1 \), we have \( T'(x_j) \geq 1 \) for \( j = 1, \ldots, m \). If for some \( j \), \( T'(x_j) = 1 \), we call \( x_j \) a parabolic fixed point.

Define the attractor of \( T \) as

\[
\Lambda = \left\{ x \in \bigcup_{j=1}^m I_j | T^n(x) \in [0, 1], \forall n \geq 0 \right\}.
\]

It is known that \( \Lambda \) is invariant under \( T \) and we get a dynamic system \( T : \Lambda \to \Lambda \).

This special class of non-uniform hyperbolic maps includes the famous example of Manneville-Pomeu map and Farey map \([14]\).
The above system has a symbolic coding which can be defined as follows. Let \( T_i \) be the inverse map of \( T|_{I_i} : I_i \to [0, 1] \) for \( i = 1, \ldots, m \). Let \( \mathcal{A} = \{1, \ldots, m\} \) and \( \Sigma = \mathcal{A}^\mathbb{N} \). There is a shift map \( \sigma : \Sigma \to \Sigma \) defined by \( \sigma((\omega_n)_{n\geq1}) = (\omega_n)_{n\geq2} \). Define a projection \( \Pi : \Sigma \to [0, 1] \) as
\[
\Pi(\omega) = \lim_{n \to \infty} T_{\omega_1} \circ T_{\omega_2} \circ \cdots \circ T_{\omega_n}([0, 1])
\]
Then \( \Pi(\Sigma) = \Lambda \) and moreover
\[
\Pi \circ \sigma(\omega) = T \circ \Pi(\omega).
\]
Obviously, we have that \( \Pi \) is a bijection except for at most countable points.

In this paper, we concern with \( \Lambda_\alpha \) and \( X_\alpha \) respectively for \( \Phi \in C_{aa}(\Lambda, T, d) \) and \( \Psi \in C_{aa}(\Sigma, \sigma, d) \). Two kinds of level set are related in the following way. Given \( \Phi \in C_{aa}(\Lambda, T, d) \). Define \( \Psi = \Phi \circ \Pi \), then \( \Psi \in C_{aa}(\Sigma, T, d) \) and \( \Pi(X_\alpha) = \Lambda_\alpha \).

Define \( g(\omega) := -\log T'_{\omega_1} \Pi(\sigma \omega) \) and let
\[
\tilde{\Sigma} = \left\{ \omega \in \Sigma : \liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(\sigma^j \omega) > 0 \right\}.
\]
Let \( h(\mu, \sigma), \lambda(\mu, \sigma) \) be the metrical entropy and Lyapunov exponent of \( \mu \). We have the following theorem:

**Theorem 1.** [9] Given \( \Phi \in C_{aa}(\Sigma, T, d) \), then for \( \alpha \in \mathcal{L}_\Phi \),
\[
\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) = \sup_{\mu \in \mathcal{M}(\Sigma, \sigma)} \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} \left| \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(\sigma^j \omega) > 0 \right\} \right\}.
\]

If we take \( d = 1 \) and \( \phi_n = nc \) where \( c \) is a real constant. We have the following result.

**Corollary 1.** Let \( T : \Lambda \to \Lambda \) is non-uniform hyperbolic, we have
\[
\dim_H \Pi(\tilde{\Sigma}) = \sup_{\mu \in \mathcal{M}(\Sigma, \sigma)} \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} \left| \lambda(\mu, \sigma) > 0 \right\} \right\}.
\]

Roughly speaking \( \Pi\tilde{\Sigma} \) can be seen as the hyperbolic part of non-uniform hyperbolic attractor. Of course this corollary implies the following theorem in the uniform hyperbolic setting, for which \( \tilde{\Sigma} = \Sigma \). One has
Theorem 2. Assume that $T : \Lambda \to \Lambda$ is uniformly hyperbolic, then
\[
\dim_H \Pi(\Sigma) = \sup_{\mu \in \mathcal{M}(\Sigma, \sigma)} \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} \mid \lambda(\mu, \sigma) > 0 \right\}.
\]

In the non-uniform hyperbolic dynamic systems, it is really subtle that whether the hyperbolic part of attractor $\Pi \tilde{\Sigma}$ carries the full Hausdorff dimension of the attractor of $\Pi \Sigma$. This has been verified in [16] for the case that each inverse branch of $T$ is $C^{1+\alpha}$ together with some geometric conditions. In [8], it was proved that $\dim_H \Pi \tilde{\Sigma} = \dim_H \Pi \Sigma$ under $C^{1+\text{Lip}}$ condition. However, as pointed in [10], it is still unknown whether it is true for $C^1$ condition in non-uniform hyperbolic dynamic systems. The following assumptions implied that $\dim_H \Pi \tilde{\Sigma} = \dim_H \Pi \Sigma$, which was first proposed in [10].

Assumptions: For any $\epsilon > 0$, there exists $\nu \in \mathcal{M}(\Lambda, T)$ such that $\lambda(\nu, T) > 0$ and $\frac{h(\nu, T)}{\lambda(\nu, T)} > \dim_H \Lambda - \epsilon$.

Consider the system $T : \Lambda \to \Lambda$. Let $I \subset \{x_1, \cdots, x_m\}$ be the set of parabolic fixed points. Given $\Phi \in \mathcal{C}_{\text{aa}}(\Lambda, T, d)$, we define $A = \text{Co} \left\{ \lim_{n \to \infty} \frac{\phi_n(x)}{n} : x \in I \right\}$, which is the convex hull of $\left\{ \lim_{n \to \infty} \frac{\phi_n(x)}{n} : x \in I \right\}$.

Theorem 3. Let $(\Lambda, T)$ be a system defined as above. Given $\Phi \in \mathcal{C}_{\text{aa}}(\Lambda, T, d)$ and define $A$ as above. Under the assumption above, then for any $\alpha \in \mathcal{L}_\Phi \setminus A$, we have
\[
\dim_H \Lambda_\alpha = \sup_{\mu \in \mathcal{M}(\Lambda, T)} \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \Phi_* (\mu) = \alpha \right\},
\]
and for all $\alpha \in A$ we have $\dim_H \Lambda_\alpha = \dim_H \Lambda$.

Finally we consider the Hausdorff dimension of $\Lambda_{\text{irr}}$. By Kingman’s sub-additive ergodic Theorem, we have $\mu(\Lambda_{\text{irr}}) = 0$ for any $\mu \in \mathcal{M}(\Lambda, T)$. However, this set carries full topological entropy and full Hausdorff dimension in most cases [4], especially in uniformly hyperbolic dynamics. Motivated by the method in [10, 5], we can proved it is also true for non-uniform hyperbolic dynamic systems in a simple way.

Theorem 4. Under the assumption in Theorem 3 and assume that $\sharp \mathcal{L}_\Phi \geq 2$, then $\dim_H \Lambda_{\text{irr}} = \dim_H \Lambda$.

Remark 1. In [9], Theorem 1 and Theorem 3 are proved for the case of additive potential in higher dimension. The skills there can be completely extended to the almost additive potential.
Remark 2. There has been great interest in the study of irregular set in recent trend [1] [11] [4] [5] [15]. The full dimension of irregular set has been verified in subshift of finite type [3] [4], conformal repellers [1] [4]. It is interesting to ask the corresponding question in the non-uniform hyperbolic dynamic systems. It is possible to follow the line in [5] and use the approximation skills as [8] to give a proof for the Hausdroff dimension of irregular set. Here we combine some ideas in [3] [5] [10] to give a short and direct proof.

The rest of this note is organized as follows. In Section 2 we give some preliminary results and lemmas which are needed for the proof. In Section 3, we prove Theorem 4.

2 Preliminaries

In this section, we will give the notations and the lemmas needed in the proof.

Assume that $T : X \rightarrow X$ is a topological dynamical system. Denote by $\mathcal{M}(X,T)$ the set of all invariant probability measures and $\mathcal{E}(X,T)$ the set of all ergodic probability measures. Given $\mu \in \mathcal{M}(X,T)$, let $h(\mu, T)$ be the metric entropy of $\mu$.

Recall that $A = \{1, 2 \ldots m\}$ and $\Sigma = A^\mathbb{N}$. Write $\Sigma_n = \{w = w_1 \cdots w_n : w_i \in A\}$. For $\omega = (\omega_n)^\infty_{n=1} \in \Sigma$, write $\omega|_n = \omega_1 \cdots \omega_n$. For $w \in \Sigma_n$ define the cylinder $[w] := \{\omega : \omega|_n = w\}$.

If $\phi : \Sigma \rightarrow \mathbb{R}^d$ is continuous, we define the $n$-th variation of $\Phi$ as

$$||\phi||_n := \sup_{\omega|_n = \tau|_n} |\phi(\omega) - \phi(\tau)|.$$ 

For $\Phi \in C_{aa}(\Sigma, T, d)$, we define $||\Phi||_n := ||\phi_n||_n$.

where $| \cdot |$ is the Euclidean norm in $\mathbb{R}^d$. Given $f : \Sigma \rightarrow \mathbb{R}^d$ continuous, let $||f|| := \sup_{\tau \in \Sigma} |f(\tau)|$. For $f : \Lambda \rightarrow \mathbb{R}^d$ continuous we define $||f||$ similarly.

We have the following standard result:

Lemma 1. If $\Phi = \{\phi_n\}^\infty_{n=1} \in C_{aa}(\Sigma, T, d)$, then $\lim_{n \rightarrow \infty} \frac{1}{n}||\Phi||_n = 0$.

Consider the projection $\Pi : \Sigma \rightarrow \Lambda$. Let $\tilde{\Lambda} := \{x \in \Lambda : \#(\Pi^{-1}(x)) = 2\}$. In other words $\tilde{\Lambda}$ is the set of such $x$ with two codings. By our assumption on $I_j$, we know that both $\tilde{\Lambda}$ and $\Pi^{-1}\tilde{\Lambda}$ are at most countable. Moreover

$$\Pi^{-1}\tilde{\Lambda} \subset \{\omega : \omega = w\infty \text{ or } \bar{w}1\infty\}. \quad (1)$$

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Then it is seen that
\[ \Pi : \Sigma \setminus \Pi^{-1} (\tilde{\Lambda}) \to \Lambda \setminus \tilde{\Lambda} \]
is a bijection. We will need this fact in the proof of the lower bound of Theorem \(\Pi\).

For \(w = w_1 \cdots w_n\), write \(I_w = T_{w_1} \circ \cdots \circ T_{w_n} [0,1]\). Especially for \(\omega \in \Sigma\), we write \(I_n(\omega) = I_{\omega|_n}\). Let \(D_n(\omega) = \text{diam}(I_n(\omega))\). Recall that we have defined \(g(\omega) := -\log T'_{\omega_1} \Pi(\sigma \omega)\) and
\[ D_n(\omega) \]
can be estimated via \(A_n g(\omega)\) by the following lemma:

**Lemma 2** ([16] [10]). Under the assumption on \(T\), \(D_n(\omega)\) converges to 0 uniformly. Moreover
\[ \lim_{n \to \infty} \sup_{\omega \in \Sigma} \left\{ -\frac{1}{n} \log D_n(\omega) - A_n g(\omega) \right\} = 0. \]

By this lemma we can understand that \(\tilde{\Sigma}\) is the set of such points \(\omega\) such that the length of \(I_n(\omega)\) tends to 0 exponentially. To simplify the notation we write \(\tilde{\lambda}_n(\omega) = -\log D_n(\omega)/n\).

Given \(\mu \in \mathcal{M}(\Sigma, \sigma)\), let \(\lambda(\mu, \sigma) := \int g d\mu\) be the Lyapunov exponent of \(\mu\). Similarly given \(\mu \in \mathcal{M}(\Lambda, T)\), let \(\lambda(\mu, T) := \int \log |T'| d\mu\) be the Lyapunov exponent of \(\mu\). For a \(\mu \in \mathcal{M}(\Sigma, \sigma)\), we denote the image of \(\mu\) under \(\Pi\) by \(\Pi_* \mu\).

The following lemma, which is a combination of Lemma 2 and Lemma 3 in [10], is very useful in our proof.

**Lemma 3.** For any \(\mu \in \mathcal{M}(\Sigma, \sigma)\), there exists a sequence of ergodic measures \(\{\mu_n : n \geq 1\}\) such that \(\mu_n \to \mu\) in the weak star topology and
\[ h(\mu_n, \sigma) \to h(\mu, \sigma); \quad \lambda(\mu_n, \sigma) \to \lambda(\mu, \sigma). \]

We remark that from their proof each ergodic measure \(\mu_n\) is continuous, i.e. \(\mu_n\) has no atom.

**Lemma 4.** Assume that \(\Phi \in \mathcal{C}_{aa}(\Sigma, \sigma, d)\), and given a sequence of measures \(\{\mu_n\}_{n=1}^\infty\), such that \(\lim_{n \to \infty} \mu_n = \mu \in \mathcal{M}(\Sigma, \sigma)\), then \(\lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \int \phi_n d\mu_n = \lim_{m \to \infty} \frac{1}{m} \int \phi_n d\mu\), i.e. \(\lim_n \Phi_n(\mu_n) = \Phi(\mu)\).

**Proof.** Let \(C\) be a constant vector with each coordinate positive such that
\[ -C + \phi_n(T^p x) + \phi_p(x) \leq \phi_{n+p}(x) \leq C + \phi_n(T^p x) + \phi_p(x) \]
∀n, p ∈ N, x ∈ X. By sub-additivity of the families \( \{φ_m(x) + C\}_{m=1}^{∞} \) and \( \{φ_m(x) - C\}_{m=1}^{∞} \), we have

\[
\lim_{m \to ∞} \frac{1}{m} \int (φ_m - C)dμ_n = \lim_{m \to ∞} \frac{1}{m} \int φ_m dμ_n = \lim_{m \to ∞} \frac{1}{m} \int (φ_m + C)dμ_n
\]

and

\[
\sup_m \frac{1}{m} \int (φ_m - C)dμ_n = \lim_{m \to ∞} \frac{1}{m} \int φ_m dμ_n = \inf_m \frac{1}{m} \int (φ_m + C)dμ_n.
\]

Thus we get

\[
\frac{1}{m} \int (φ_m - C)dμ_n = \lim_{m \to ∞} \frac{1}{m} \int φ_m dμ_n = \frac{1}{m} \int (φ_m + C)dμ_n.
\]

Then taking \( n \) goes to infinity, and \( m \) goes to infinity, we get the desired result.

3 Proof of Irregular set Theorem \( \text{[4]} \)

For \( Φ ∈ C_{aa}(Λ, T, d) \), we define \( Ψ = Φ \circ Π \). It is rather easy to check \( L_Φ = L_Ψ \). Then Theorem \( \text{[4]} \) is a immediately consequence of the following Lemma.

Lemma 5. For any \( µ, ν ∈ M(σ, Σ) \) with \( λ(µ, σ) > 0, λ(ν, σ) > 0 \) and \( Ψ_*(µ) ≠ Ψ_*(ν) \), we have

\[
\dim_H Λ_{irr} ≥ \min \left\{ \frac{h(µ, σ)}{λ(µ, σ)}, \frac{h(ν, σ)}{λ(ν, σ)} \right\}.
\]

Proof of Theorem \( \text{[4]} \). Under the assumption of Theorem \( \text{[3]} \)

\[
\dim_H Λ = \sup_{µ ∈ M(Σ, σ)} \left\{ \frac{h(µ, σ)}{λ(µ, σ)} : λ(µ, σ) > 0 \right\}.
\]

For any \( ε > 0 \), there exists \( µ ∈ M(σ, Σ) \) such that \( \frac{h(µ, σ)}{λ(µ, σ)} ≥ \dim_H Λ - ε \).

Write \( α = Ψ_*(µ) \). Since \( 2L_Ψ ≥ 2 \), we can choose \( ν ∈ M(σ, Σ) \) such that \( Ψ_*(ν) = β ≠ α \).

Define \( ν_s = sµ + (1 - s)ν \), where \( s ∈ [0, 1] \). We have \( Ψ_*(µ_s) = sα + (1 - s)β ≠ α \) for any \( s ∈ [0, 1] \). By Lemma \( \text{[3]} \)

\[
\dim_H Λ_{irr} ≥ \min \left\{ \frac{h(µ, σ)}{λ(µ, σ)}, \frac{h(µ_s, σ)}{λ(µ_s, σ)} \right\} = \min \left\{ \frac{h(µ, σ)}{λ(µ, σ)}, \frac{sh(µ, σ) + (1 - s)h(ν, σ)}{sλ(µ, σ) + (1 - s)λ(ν, σ)} \right\}
\]
for all $s \in [0, 1)$. Taking $s$ goes to 1, we get $\dim_H \Lambda_{irr} \geq \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} \geq \dim_H \Lambda - \epsilon$.

By the arbitrary of $\epsilon$, we get the desired result.

**Proof of Lemma 5**. By Lemma 3 and Lemma 4, we can choose a decreasing sequence $\epsilon_i \downarrow 0$ such that for all $n \geq 2i - 1$,

$$\frac{1}{n}||\Psi||_n < \epsilon_{2i-1}, \quad \text{var}_n A_n g < \epsilon_{2i-1} \quad \text{and} \quad |\tilde{\lambda}_n(\omega) - A_n g(\omega)| < \epsilon_{2i-1} (\forall \omega \in \Sigma). \quad (2) \tag{*}$$

By Lemma 3 and Lemma 4, we can choose a sequence of $\mu_{2i-1} \in \mathcal{E}(\Sigma, \sigma)$, such that

$$|\Psi(\mu_{2i-1}) - \alpha| < \epsilon_{2i-1}, \quad |h(\mu_{2i-1}, \sigma) - h(\mu, \sigma)| < \epsilon_{2i-1} \quad \text{and} \quad |\lambda(\mu_{2i-1}, \sigma) - \lambda(\mu, \sigma)| < \epsilon_{2i-1}. \quad (3) \tag{*}$$

Since $\mu_{2i-1}$ is ergodic, for $\mu_{2i-1}$ a.e. $\omega$,

$$\frac{1}{n}\Psi_n(\omega) \to \Psi(\mu_{2i-1}), \quad A_n g(\omega) \to \lambda(\mu_{2i-1}, \sigma) \quad \text{and} \quad \frac{-\log \mu_{2i-1}[\omega|n]}{n} \to h(\mu_{2i-1}, \sigma). \quad (4) \tag{*}$$

Fix $\delta > 0$. Since $\mu_{2i-1}$ is continuous as we remarked after Lemma 3, there exists $\ell_{2i-1} \geq 2i - 1$ such that $\mu_{2i-1}(\bigcup_{j=1}^m [j^{\ell_{2i-1}}]) \leq \delta/2$. By Egorov’s theorem, there exists $\Omega'(2i-1) \subset \Sigma$ such that $\mu_{2i-1}(\Omega'(2i-1)) > 1 - \delta/2$ and (3) holds uniformly on $\Omega'(2i-1)$. Then there exists $\ell_{2i-1} \geq \ell_{2i-1} \geq 2i - 1$ such that for all $n \geq \ell_{2i-1}$ and $\omega \in \Omega'(2i-1)$, we have

$$\begin{cases}
|\frac{1}{n}\Psi_n(\omega) - \Psi(\mu_{2i-1})| < \epsilon_{2i-1} \\
|A_n g(\omega) - \lambda(\mu_{2i-1}, \sigma)| < \epsilon_{2i-1} \\
|\frac{-\log \mu_{2i-1}[\omega|n]}{n} - h(\mu_{2i-1}, \sigma)| < \epsilon_{2i-1}
\end{cases} \quad (5) \tag{*}$$

Let

$$\Sigma(2i-1) = \{\omega|_{\ell_{2i-1}} \mid \omega \in \Omega'(2i-1)\} \setminus \{1^{\ell_{2i-1}}, \ldots, m^{\ell_{2i-1}}\}.$$ 

Let $\Omega(2i-1) = \bigcup_{\omega \in \Sigma(2i-1)}[\omega]$. Then

$$\mu_{2i-1}(\Omega(2i-1)) \geq \mu_{2i-1}(\Omega'(2i-1)) - \mu_{2i-1}(\bigcup_{j=1}^m [j^{\ell_{2i-1}}]) \geq 1 - \delta/2 - \delta/2 = 1 - \delta.$$

Similarly for all $n \geq 2i$, we have

$$\frac{1}{n}||\Psi||_n < \epsilon_{2i}, \quad \text{var}_n A_n g < \epsilon_{2i} \quad \text{and} \quad |\tilde{\lambda}_n(\omega) - A_n g(\omega)| < \epsilon_{2i} (\forall \omega \in \Sigma). \quad (6) \tag{*}$$

By Lemma 3 we can pick a sequence of $\nu_{2i} \in \mathcal{E}(\Sigma, \sigma)$, such that

$$|\Psi(\nu_{2i}) - \alpha| < \epsilon_{2i}, \quad |h(\nu_{2i}, \sigma) - h(\nu, \sigma)| < \epsilon_{2i} \quad \text{and} \quad |\lambda(\nu_{2i}, \sigma) - \lambda(\nu, \sigma)| < \epsilon_{2i}. \quad (7) \tag{*}$$
Since \( \nu_{2i} \) is ergodic, for \( \nu_{2i} \) a.e. \( \omega \),

\[
\frac{1}{n} \Psi_n(\omega) \to \Psi_*(\nu_{2i}), \quad A_{n}g(\omega) \to \lambda(\nu_{2i}, \sigma) \quad \text{and} \quad -\frac{\log \nu_{2i}[\omega|_n]}{n} \to h(\mu_{2i}, \sigma). \tag{8}
\]

Similarly for all \( n \geq 2i \), we have

\[
\begin{cases}
|\frac{1}{n} \Psi_n(\omega) - \Psi_*(\nu_{2i})| < \epsilon_{2i} \\
|A_{n}g(\omega) - \lambda(\nu_{2i}, \sigma)| < \epsilon_{2i-1} \\
|n - \log \nu_{2i}[\omega|_n]/n - h(\nu_{2i}, \sigma)| < \epsilon_{2i}
\end{cases} \tag{9}
\]

Let

\[
\Sigma(2i) = \{ \omega|_{l_{2i}} | \omega \in \Omega'(2i) \} \setminus \{1^{2i}, \ldots, m^{2i}\}.
\]

Let \( \Omega(2i) = \bigcup_{w \in \Sigma(2i)} [w] \). Then

\[
\nu_{2i}(\Omega(2i)) \geq \nu_{2i}(\Omega'(i)) = \nu_{2i}(\bigcup_{j=1}^{m} [j^{2i}]) \geq 1 - \delta/2 - \delta/2 = 1 - \delta.
\]

It is seen that we can take \( l_i \) such that \( l_i \uparrow \infty \) and still satisfies all the above property. Let \( N_0 = 1 \), \( N_i = 2^{l_{i+2} + N_{i-1}} \), \( i \geq 1 \). Let

\[
M = \prod_{i=1}^{\infty} \prod_{j=1}^{N_i} \Sigma(i).
\]

By the definition of \( \Sigma(i) \) and (1), it is ready to see that \( M \cap \Pi^{-1} \Lambda = \emptyset \). In the following we will show that \( \Pi M \subset \Lambda_{irr} \). To be precise, we will check the following result:

**Lemma 6.** Let \( n_j = \sum_{i=1}^{j} l_i N_i \) and fix \( \omega \in M \), then we have

\[
\lim_{j \to \infty} \frac{\Psi_{n_{2j+1}}(\omega)}{n_{2j+1}} = \alpha, \quad \lim_{j \to \infty} \frac{\Psi_{n_{2j}}(\omega)}{n_{2j}} = \beta.
\]

**Proof of Lemma 6**

\[
\Psi_{n_{2j+1}}(\omega) - n_{2j+1} \alpha
\]
\[\begin{align*}
&\leq \sum_{i=1}^{2j+1N_i-1} \sum_{k=0}^{N_{2i-1}-1} [\Psi_{l_i}(\sigma^{n_{i-1}+kl_i}\omega) - l_i\alpha + C] \\
&= \sum_{i=1}^{j+1N_{2i-1}-1} \sum_{k=0}^{N_{2i-1}-1} [\Psi_{l_{2i-1}}(\sigma^{n_{2i-2}kl_{2i-1}}\omega) - l_{2i-1}\alpha + C] + \sum_{i=1}^{j} \sum_{k=0}^{N_{2i-1}-1} [\Psi_{l_{2i}}(\sigma^{n_{2i-1}kl_{2i}}\omega) - l_{2i}\alpha + C] \\
&\leq \sum_{i=1}^{j+1} 3l_{2i-1}N_{2i-1}\epsilon_{2i-1}\bar{\chi} + \sum_{i=1}^{j} [3l_{2i}N_{2i}\epsilon_{2i}\bar{\chi} + l_{2i}N_{2i}(\beta - \alpha)] + \sum_{i=1}^{j} N_iC \\
&= \sum_{i=1}^{j} N_i(3l_i\epsilon_i\bar{\chi} + C) + \sum_{i=1}^{j} l_{2i}N_{2i}(\beta - \alpha).
\end{align*}\]

where for the second inequality we use (2) (3) (5) (6) (7) (9) and similar method used in the proof of lower bound of Theorem 1. Similarly we have

\[\Psi_{n_{2j+1}}(\omega) - n_{2j+1}\alpha \geq - \sum_{i=1}^{2j+1} N_i(3l_i\epsilon_i\bar{\chi} + C) + \sum_{i=1}^{j} l_{2i}N_{2i}(\beta - \alpha).\]

Noting that

\[\lim_{j \to \infty} \frac{l_2N_2 + l_4N_4 + \cdots + l_{2j}N_{2j}}{l_1N_1 + l_2N_2 + \cdots + l_{2j+1}N_{2j+1}} = 0,\]

we have

\[\lim_{j \to \infty} \frac{\Psi_{n_{2j-1}}(\omega)}{n_{2j-1}} = \alpha.\]

Similarly we can also get

\[\lim_{j \to \infty} \frac{\Psi_{n_{2j}}(\omega)}{n_{2j}} = \beta.\]

This implies that \(\Pi M \subset \Lambda_{irr}\).

Now we will construct a measure \(\eta\) supported on \(M\) and show that for all \(x \in \Pi(M)\)

\[\liminf_{r \downarrow 0} \frac{\log \Pi_\ast \eta(B(x, r))}{\log r} \geq \min \left\{ h(\mu, \sigma), h(\nu, \sigma), \lambda(\mu, \sigma), \lambda(\nu, \sigma) \right\}.\]

Consequently, we have

\[\dim_H \Lambda_{irr} \geq \dim_H \Pi M \geq \min \left\{ h(\mu, \sigma), h(\nu, \sigma), \lambda(\mu, \sigma), \lambda(\nu, \sigma) \right\}.\]

Then the result follows.
For convenience we relabel the following sequence

\[ l_1 \cdots l_1, \ldots, l_i \cdots l_i, \ldots \]

as \( \{ l_i^* : i \geq 1 \} \). Relabel the following sequence

\[ \Sigma(1) \cdots \Sigma(1), \ldots, \Sigma(i) \cdots \Sigma(i), \ldots \]

as \( \{ \Sigma^*(i) : i \geq 1 \} \). Accordingly we get \( \{ \Omega^*(i) \}, \{ \nu^*_i \}, \{ \epsilon^*_i \} \). Let \( n_k = \sum_{i=1}^{k} l_i^* \). For any \( n > 0 \), there exists \( J(n) \in \mathbb{N} \) such that \( \sum_{i=1}^{J(n)} l_i^* \leq n < \sum_{i=1}^{J(n)+1} l_i^* \). There also exists \( r(n) \in \mathbb{N} \) such that \( \sum_{i=1}^{r(n)} N_i \leq J(n) < \sum_{i=1}^{r(n)+1} N_i \). It is seen that

\[ J(n) \leq J(n+1) \leq J(n) + 1, \quad l_{J(n)+1}^* = l_{r(n)+1} \quad \text{and} \quad l_{J(n)+2}^* \leq l_{r(n)+2}, \quad (10) \]

then, for \( j = 1, 2 \),

\[ \frac{l_{J(n)+j}}{\sum_{i=1}^{J(n)} l_i^*} \leq \frac{l_{r(n)+j}}{N_{r(n)}l_{r(n)}} = \frac{l_{r(n)+j}}{2^{N_{r(n)-1}+l_{r(n)+2}l_{r(n)}}}. \]

We have

\[ \sum_{i=1}^{J(n)+1} l_i^* / \sum_{i=1}^{J(n)} l_i^* \to 1 \quad \text{and} \quad l_{J(n)+j}^* / \sum_{i=1}^{J(n)} l_i^* \to 0, \quad j = 1, 2. \quad (11) \]

For convenience, define \( \eta_i = \mu_i \) if \( i \) is odd, and \( \eta_i = \nu_i \), if \( i \) is even. At first we define a probability \( m \) supported on \( M \). For each \( w \in \Sigma^*(i) \) define

\[ \rho_w^i = \frac{\eta_i^*[w]}{\eta_i^*(\Sigma^*(i))}. \]

It is seen that \( \sum_{w \in \Sigma^*(i)} \rho_w^i = 1 \). Write \( \mathcal{C}_n := \{ [w] : w \in \prod_{i=1}^{n} \Sigma^*(i) \} \).

It is seen that \( \sigma(\mathcal{C}_n : n \geq 1) \) gives the Borel-\( \sigma \) algebra in \( M \). For each \( w = w_1 \cdots w_n \in \mathcal{C}_n \) define

\[ \hat{\eta}(\{w\}) = \prod_{i=1}^{n} \rho_{w_i}^i. \]

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Let $\eta$ be the Kolmogorov extension of $\tilde{\eta}$ to all the Borel sets. By the construction it is seen that $\eta$ is supported on $M$.

Fix $\omega \in M$. At first we find a lower bound for $D_n(\omega)$. Define $n_0 = 0$, $n_i = \sum_{j=1}^i l^*_j$, for $i \geq 1$. Recall that $D_n(\omega) = e^{-n\lambda_n(\omega)}$. By the construction of $M$ we have $\sigma^{n_i-1}\omega \in [w]$ for some $w \in \Sigma^*(i)$, consequently there exists $\omega^i \in \Omega^*(i) \cap [w]$ such that (5) holds. By the similar method used in Theorem [1] we have

$$n\lambda_n(\omega) \leq n(A_n g(\omega) + \epsilon^*_J(n))$$
$$\leq \sum_{i=1}^{J(n)} l^*_i (A^*_i g(\sigma^{n_i-1}\omega) + \epsilon^*_i) + (n - n_{J(n)}) \left( A_{n-n_{J(n)}} g(\sigma^{n_{J(n)}} \omega) + \epsilon^*_J(n) \right)$$
$$\leq \sum_{i=1}^{J(n)} l^*_i \{ A^*_i g(\sigma^{n_i-1}\omega) - A^*_i g(\omega^i) + A^*_i g(\omega^i) - \lambda(\eta_i, \sigma) + \lambda(\eta_i, \sigma) + \lambda(\eta_i, \sigma) + \epsilon^*_i \} + l^*_J(n+1)(||g|| + \epsilon^*_J(n))$$
$$\leq \sum_{i=1}^{J(n)} l^*_i (\lambda(\eta_i, \sigma) + 4\epsilon^*_i) + l^*_J(n+1)(||g|| + \epsilon^*_J(n)) =: \rho(n).$$

Then $D_n(\omega) \geq e^{-\rho(n)}$. It is seen that $\rho(n)$ is increasing.

Now fix $x \in \Pi(M)$ and some $r > 0$ small. Then there exists a unique $n = n_r$ such that

$$e^{-\rho(n+1)} \leq r < e^{-\rho(n)}. \quad (12) \ \footnotesize{\#C \leq 3}$$

Consider the set of $n$-cylinders

$$C := \{ I_n(\omega) : \omega \in M \text{ and } I_n(\omega) \cap B(x, r) \neq \emptyset \}. $$

By the bound $D_n(\omega) \geq e^{-\rho(n)}$, the above set consists of at most three cylinders, i.e. $\#C \leq 3$.

Choose $\omega \in M$ such that $I_n(\omega) \in C$. Write $\omega|_{n} = w_1 \cdots w_{J(n)} v$, then $w_i \in \Sigma^*(i)$ and $v$ is a prefix of some $\tilde{v} \in \Sigma^*(J(n) + 1)$. Then

$$\Pi_n \eta(I_n(\omega)) = \nu[\omega|_{n}] = \prod_{i=1}^{J(n)} \frac{\eta^*_i[w_i]}{\eta^*_i(\Sigma^*(i))} \cdot \frac{\eta^*_{J(n)+1}[v]}{\eta^*_{J(n)+1}(\Sigma^*(J(n) + 1))}$$
$$\leq (1 - \delta)^{-J(n)-1} \prod_{i=1}^{J(n)} \eta^*_i[w_i].$$

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Then we conclude that \( \Pi \ast \eta(B(x, r)) \leq 3(1 - \delta)^{-J(n)} \prod_{i=1}^{J(n)} \eta^*_i[w_i] \). For convenience, we define \( \tau_i \) be the measure which is \( \mu \) whenever \( i \) is odd is \( \nu \) whenever \( i \) even. Consequently

\[
\log \Pi \ast \eta(B(x, r)) \\
\leq - \sum_{i=1}^{J(n)} l_i^* \left( - \frac{\log \eta_i^*[w_i]}{l_i^*} \right) - (J(n) + 1) \log(1 - \delta) + \log 3 \\
\leq - \sum_{i=1}^{J(n)} l_i^* (h(\tau, \sigma) - 2\epsilon_i^*) - (J(n) + 1) \log(1 - \delta) + \log 3,
\]

where for the second inequality we use (7) and (9). Notice that \( r \to 0 \) if and only if \( n \to \infty \). By (10) we have \( J(n+1) \leq J(n) + 1 \). Together with (12) and (11) we get

\[
\liminf_{r \downarrow 0} \frac{\log \Pi \ast \eta(B(x, r))}{\log r} \\
\geq \liminf_{n \to \infty} \frac{\sum_{i=1}^{J(n)} l_i^* (h(\tau_i, \sigma) - 2\epsilon_i^*) + (J(n) + 1) \log(1 - \delta) - \log 3}{\sum_{i=1}^{J(n)} l_i^* (\lambda(\tau_i, \sigma) + 4\epsilon_i^*) + l_{J(n)+1}^* (\|g\| + \epsilon_{J(n)+1}^*)} \\
= \liminf_{n \to \infty} \frac{\sum_{i=1}^{J(n)} l_i^* (\lambda(\tau_i, \sigma) + 4\epsilon_i^*) \min\{h(\mu, \sigma) - 2\epsilon_i^*, h(\nu, \sigma) - 2\epsilon_i^*, \lambda(\mu, \sigma) + 4\epsilon_i^*, \lambda(\nu, \sigma) + 4\epsilon_i^*\}}{\sum_{i=1}^{J(n)} l_i^* (\lambda(\tau_i, \sigma) + 4\epsilon_i^*)} \\
\geq \lim_{n \to \infty} \frac{\sum_{i=1}^{J(n)} l_i^* (\lambda(\tau_i, \sigma) + 4\epsilon_i^*)}{\sum_{i=1}^{J(n)} l_i^* (\lambda(\tau_i, \sigma) + 4\epsilon_i^*)} \\
= \lim_{n \to \infty} \min \left\{ \frac{h(\mu, \sigma) - 2\epsilon_J(n)*}{\lambda(\mu, \sigma) + 4\epsilon_J(n)*}, \frac{h(\nu, \sigma) - 2\epsilon_J(n)*}{\lambda(\nu, \sigma) + 4\epsilon_J(n)*} \right\} \\
= \min \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)}, \frac{h(\nu, \sigma)}{\lambda(\nu, \sigma)} \right\}
\]

□
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