CURVATURE INHERITANCE SYMMETRY ON M-PROJECTIVELY FLAT SPACETIMES

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Abstract. The paper aims to investigate curvature inheritance symmetry in M-projectively flat spacetimes. It is shown that the curvature inheritance symmetry in M-projectively flat spacetime is a conformal motion. We have proved that M-projective curvature tensor follows the symmetry inheritance property along a vector field $\xi$, when spacetime admits the conditions of both curvature inheritance symmetry and conformal motion or motion along the vector field $\xi$. Also, we have derived some results for M-projectively flat spacetime with perfect fluid following the Einstein field equations with a cosmological term and admitting the curvature inheritance symmetry along the vector field $\xi$. We have shown that an M-projectively flat perfect fluid spacetime obeying the Einstein field equations with a cosmological term and admitting the curvature inheritance symmetry along a vector field $\xi$ is either a vacuum or satisfies the vacuum-like equation of state. We have also shown that such spacetimes with the energy momentum tensor of an electromagnetic field distribution do not admit any curvature symmetry of general relativity. Finally, an example of M-projectively flat spacetime has been exhibited.

1. Introduction

A pseudo-Riemannian manifold with a non-zero $(0,2)$ type Ricci tensor $R_{ij}$ is known as Einstein manifold if $R_{ij}$ is proportional to the metric tensor $g_{ij}$. In other words, we can simply say that Einstein manifolds form a natural subclass of the manifold of constant curvature [9]. In the field of general relativity [16] and Riemannian geometry, Einstein manifolds play a significant role. A spacetime is considered as a four dimensional connected semi-Riemannian manifold $(V_4,g)$ with Lorentzian metric $g$ of signature $(-,+,+,+)$. Due to the casual character of vectors, the Lorentzian manifolds [9] are the practical option for the studying general theory of relativity.

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With metric tensor $g_{ij}$ and Riemannian connection $\nabla$, let $(V_4, g)$ be a 4-dimensional differentiable manifold of class $C^\infty$. M-projective curvature tensor of the manifold $V_4$ was defined by Pokhariyal and Mishra [24] in 1971 which can be written in the following form
\[
\bar{W}^h_{ijk} = R^h_{ijk} + \frac{1}{6}(\delta^h_j R^i_k - \delta^h_k R^i_j + R^h_{ij} g_{ik} - R^h_{ik} g_{ij}),
\]
where $R^h_{ijk}$ is the Riemannian curvature, $g_{ij}$ is the metric and $R_{ij}$ is Ricci tensor in $V_4$, the tensor field $\bar{W}^h_{ijk}$ is the M-projective curvature tensor. We note that the M-projective curvature tensor is a generalized curvature tensor, i.e., it satisfies the symmetry and skew-symmetry properties like Riemann curvature and satisfies Bianchi identity. If $\bar{W}^h_{ijk} = 0$ on $V_4$ identically, then that manifold is known as M-projectively flat manifold. We mention that any M-projectively flat semi-Riemannian manifold is an Einstein manifold and a space of constant curvature. Many authors have studied about spacetimes and their geometries and physics such as: Abdussattar and Dwivedi [1] studied conharmonic symmetries in fluid spacetimes, Zengin [30] studied M-Projectively flat spacetimes, pseudo Z-symmetric spacetimes by Mantica and Suh [21], concircular curvature tensor and fluid spacetimes by Ahsan et al. [2–7] and many more. The properties of M-projective curvature tensor were studied by Ojha [22,23] and M-projective curvature tensor on various structures has been studied by several authors (see, [11,20,25,26,28,31]).

The present paper is dedicated to certain investigations in general theory of relativity for M-projectively flat case. We have explored the condition when energy momentum tensor follow symmetry inheritance property and its Lie derivative vanishes in M-projectively flat spacetimes.

The Bianchi identities describe the interaction between the matter and free gravitational parts of the gravitational field, which is characterized by the curvature tensor in general theory of relativity. In gravitational physics, the main objective of all investigations is the construction of the gravitational potential, which satisfies the Einstein field equations (EFEs) with a cosmological constant. Consider the EFEs with cosmological term as
\[
R_{ij} - \frac{1}{2} g_{ij} \Lambda g_{ij} = k T_{ij},
\]
where $g_{ij}, R_{ij}, T_{ij}, \Lambda$ and $k$ are respectively denote the metric, Ricci tensor, energy-momentum tensor, cosmological term and the gravitational constant. The energy-momentum tensor for a perfect fluid is defined by the following
\[
T_{ij} = (\mu + p)u_i u_j + pg_{ij},
\]
here $\mu$, the energy density, $p$ the isotropic pressure and $u^i$ is the velocity vector field of the flow satisfying $g_{ij}u^j = u^i$ for all $i$, also $u_iu^i = -1$.

Geometrical symmetries of the spacetime are represented by the following equation \[29\]

$$\mathcal{L}_\xi A = 2\Omega A$$ \hspace{1cm} (1.4)

where $\mathcal{L}_\xi$ stands for the Lie derivative along the vector field $\xi^i$. The vector field $\xi^i$ may be timelike ($\xi^i\xi_i < 0$), spacelike ($\xi^i\xi_i > 0$) or null ($\xi^i\xi_i = 0$), ‘$A$’ denotes a geometrical/physical quantity and $\Omega$ is a scalar function.

A simple example can be provided as the metric inheritance symmetry in particular for $g_{ij}$, it is conformal motion (Conf M) \[19\] along a vector field $\xi^i$, so

$$\mathcal{L}_\xi g_{ij} = 2\psi g_{ij}, \hspace{1cm} (1.5)$$

where $\psi$ is a conformal function, equation (1.5) implies motion/isometry for $\psi = 0$. In this case $\xi$ is called a Killing vector \[29\]. More than 30 geometric symmetries have been found in literature till date. For detailed study of symmetry inheritance, see \[2, 13–15, 18\].

In 1992, Duggal \[14\] introduced the concept of curvature inheritance (CI) symmetry. Curvature inheritance is the generalization of curvature collineation (CC), which was defined by Katzin in 1969 \[19\].

**Definition 1.1.** \[14\] A spacetime $(V_4, g)$ admits curvature inheritance symmetry along a smooth vector field $\xi^i$, if it satisfies

$$\mathcal{L}_\xi R^h_{ijk} = 2\Omega R^h_{ijk}, \hspace{1cm} (1.6)$$

where $\Omega = \Omega(x^i)$ is known as inheriting factor or inheritance function.

If $\Omega = 0$, then $\mathcal{L}_\xi R^h_{ijk} = 0$ and $\xi^i$ is said to follow a curvature symmetry on $V_4$ or simply to write it generate a curvature collineation. The CI equation (1.6) can be written in component form as

$$R^h_{ijkl}\xi_l^i - R^l_{ijk}\xi^h_i + R^h_{ijk}\xi^i_l + R^h_{ilk}\xi^k_j + R^h_{ijl}\xi^i_k = 2\Omega R^h_{ijk}. \hspace{1cm} (1.7)$$

**Definition 1.2.** \[14\] A spacetime $(V_4, g)$ admits Ricci inheritance (RI) symmetry along a smooth vector field $\xi^i$, if it satisfies the following equation

$$\mathcal{L}_\xi R_{ij} = 2\Omega R_{ij}. \hspace{1cm} (1.8)$$
Contraction of (1.6) gives (1.8). Thus, in general, every curvature inheritance is Ricci inheritance symmetry (i.e., CI ⇒ RI), but the converse may not hold. Also, RI reduces to RC when $\Omega = 0$ and for $\Omega \neq 0$, it is called as proper RI.

The literature on study of spacetimes exhibits, a deep interest towards the research of different symmetries (in particular, curvature, Ricci, projective, matter, semi-conformal symmetry, conharmonic curvature inheritance). These geometrical symmetries are much helpful for obtaining exact solutions of Einstein field equations (1.2). Such work of researches [13, 27] motivates us to inquire the curvature inheritance symmetry in the spacetime, admitting M-projective curvature tensor. The plan of this paper is as follows:

After Section 1, the introduction and preliminaries, we study curvature inheritance in an M-projectively flat spacetime. Section 3 deals with M-projectively flat perfect fluid spacetime admitting curvature inheritance symmetry and examination of some properties of such a spacetime. Equation of state is derived in Section 4 which is physically significant in cosmology. Finally, we obtained some interesting results considering a purely electromagnetic distribution. In the last section, we provide an example in the form of a metric which is M-projectively flat.

2. Curvature Inheritance in M-Projectively Flat Spacetimes

Consider an M-projectively flat Lorentzian manifold $(V, g)$, therefore (1.1) under condition $\tilde{W}^h_{ijk} = 0$, implies

$$R^h_{ijk} = \frac{1}{6}(\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}).$$

(2.1)

Contracting with respect to $h$ and $k$, we get

$$R_{ij} = \frac{R}{4} g_{ij},$$

(2.2)

here $R$ is scalar curvature. Thus, we can state that “An M-projectively flat spacetime is an Einstein manifold.” From equations (2.1) and (2.2), we have

$$R^h_{ijk} = \frac{R}{12}(g^h_k g_{ij} - g^h_j g_{ik}).$$

(2.3)

Thus, a result by Zengin [30] is mentioned as “An M-projectively flat spacetime is of constant curvature”. Also, M-projectively flat Lorentzian manifold is Ricci symmetric, i.e. $\nabla_k R_{ij} = 0$.

For $(V, g)$ admitting Ricci inheritance, (1.8) and (2.2) lead to $\mathcal{L}_\xi g_{ij} = 2\Omega g_{ij}$. Thus, we have the following:
Theorem 2.1. Every Ricci inheritance in an M-projectively flat spacetime is a conformal motion.

Since, every M-projectively flat spacetime is Einstein [3], thus from Theorem 2.1 we can state the following:

Corollary 2.1. Every Ricci inheritance in an Einstein manifold is a conformal motion.

We know that a space with non-zero constant curvature or every harmonic space is Einstein. Then it yields the following:

Corollary 2.2. Every Ricci inheritance in a space of constant curvature is a conformal motion.

Corollary 2.3. Every Ricci inheritance in a harmonic space is a conformal motion.

We know that in a semi-Riemannian space, every curvature inheritance is a Ricci inheritance [14]. Hence, we have

Corollary 2.4. Every curvature inheritance in an M-projectively flat spacetime is a conformal motion with the conformal function $\Omega$.

On similar lines of Corollaries (2.1 - 2.3), we simply write the following:

Corollary 2.5. Every curvature inheritance in an Einstein spacetime is a conformal motion.

Corollary 2.6. Every curvature inheritance in a space of constant curvature is a conformal motion.

Corollary 2.7. Every curvature inheritance in a harmonic space is a conformal motion.

Now, we consider the Lorentzian space of two dimensional. Since, every $V_2$ is an Einstein space, where scalar curvature $R$ is a non-zero function of coordinates, (1.8) and (2.2) lead to (assuming that $V_2$ admits a Ricci inheritance)

$$4(2\Omega R_{ij}) = R(\mathcal{L}_\xi g_{ij}) + (\mathcal{L}_\xi R)g_{ij}. \quad (2.4)$$

Or

$$\mathcal{L}_\xi g_{ij} = \left[2\Omega - \frac{\mathcal{L}_\xi R}{R}\right] g_{ij}, \quad (2.5)$$

$$\Rightarrow \mathcal{L}_\xi g_{ij} = \left[2\Omega - g \left(\frac{\text{grad} R}{R}, \xi\right)\right] g_{ij}$$
In particular if \( \Omega = \frac{1}{2} g \left( \frac{\text{grad} R}{R} , \xi \right) \), then \( \xi \) is a Killing vector.

Hence, we have the following result

**Theorem 2.2.** Every Ricci inheritance in a 2-dimensional Riemannian space is a conformal motion.

**Corollary 2.8.** If in a 2-dimensional Lorentzian space admitting Ricci inheritance, \( \text{grad}(R) \) (i.e. the scalar potential) is orthogonal to \( \xi \), then \( \xi \) is a Killing vector.

Since, CI \( \Rightarrow \) RI, Theorem 2.2 gives the following result.

**Corollary 2.9.** Every curvature inheritance in a 2-dimensional Lorentzian space is a conformal motion.

If the scalar curvature \( R \) holds the inheritance symmetry property in a 2-dimensional Lorentzian space, i.e., \( \mathcal{L}_\xi R = 2\Omega R \) or \( \mathcal{L}_\xi R - 2\Omega R = 0 \), then equation (2.5) implies \( \mathcal{L}_\xi g_{ij} = 0 \). Thus, we have the following:

**Corollary 2.10.** If the scalar curvature of \( V_2 \) holds inheritance symmetry, then every curvature inheritance in \( V_2 \) is a motion.

Combining the equations (2.2) and EFEs (1.2), we get

\[
T_{ij} = \frac{1}{\kappa} (\wedge - \frac{R}{4})g_{ij}, \quad \kappa \neq 0. \tag{2.6}
\]

Thus, we have the following:

**Proposition 2.1.** An \( M \)-projectively flat spacetime, the energy-momentum tensor holds EFEs with a cosmological term is in the form of equation (2.6).

**Theorem 2.3.** If an \( M \)-projective curvature tensor in \((V_4, g)\) admits curvature inheritance along a Killing vector field \( \xi \), then \( M \)-projective curvature tensor follows the symmetry inheritance property along \( \xi \).

**Proof.** The Lie differentiation of equation (1.1) along to Killing vector field \( \xi \) leads to

\[
\mathcal{L}_\xi \bar{W}^{*h}_{ijk} = \mathcal{L}_\xi R^h_{ijk} + \frac{1}{6} \left[ \delta^h_j \mathcal{L}_\xi R_{ik} - \delta^h_k \mathcal{L}_\xi R_{ij} + \mathcal{L}_\xi (R^h_j) g_{ik} - \mathcal{L}_\xi (R^h_k) g_{ij} \right]. \tag{2.7}
\]

If we consider a spacetime \((V_4, g)\) admits curvature inheritance symmetry then from (2.7), we obtain
\( \mathcal{L}_\xi \tilde{W}^h_{ijk} = 2\Omega [ R^h_{ijk} + \frac{1}{6} (\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_{ij} g_{ik} - R^h_{ki} g_{ij})] \),

In view of (1.1), the above equation entails

\[ \mathcal{L}_\xi \tilde{W}^h_{ijk} = 2\Omega \tilde{W}^h_{ijk}. \]  

(2.8)

This leads the proof. \( \square \)

**Theorem 2.4.** If an M-projective curvature tensor on a \((V_4, g)\) with a CKV \(\xi\) satisfies the curvature inheritance property, then M-projective curvature tensor follows the symmetry inheritance property along \(\xi\).

**Proof.** The Lie differentiation of equation (1.1) along to Killing vector field \(\xi\) leads to

\[ \mathcal{L}_\xi \tilde{W}^h_{ijk} = \mathcal{L}_\xi R^h_{ijk} + \frac{1}{6} [\delta^h_j \mathcal{L}_\xi R_{ik} - \delta^h_k \mathcal{L}_\xi R_{ij} + \mathcal{L}_\xi (R^h_j) g_{ik} - \mathcal{L}_\xi (R^h_k) g_{ij} + 2\Omega (R^h_j g_{ik} - R^h_k g_{ij})]. \]  

(2.9)

If a spacetime \((V_4, g)\) admits curvature inheritance symmetry then from (2.9), we obtain

\[ \mathcal{L}_\xi \tilde{W}^h_{ijk} = 2\Omega [ R^h_{ijk} + \frac{1}{6} (\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_{ij} g_{ik} - R^h_{ki} g_{ij})]. \]

By virtue of the above equation, we obtain

\[ \mathcal{L}_\xi \tilde{W}^h_{ijk} = 2\Omega \tilde{W}^h_{ijk}. \]  

(2.10)

This gives the proof. \( \square \)

For M-projectively flat spacetimes we can simply obtain some results on curvature inheritance symmetry. Also, there are some results already in literature which can be proved for the M-projectively flat spacetime. For example, “every proper CI in an Einstein space \((R \neq 0)\) is a proper conf M with a conformal function \(\Omega\)” given by Duggal [14] is obvious in the M-projectively flat spacetime.

### 3. Curvature Inheritance in M-Projectively flat perfect fluid spacetimes

In a perfect fluid spacetime,

\[ T_{ij} = (\mu + p) u_i u_j + pg_{ij}, \]  

(3.1)

where the symbols are explained after equation (1.3). On contraction of (2.6), we obtain

\[ T = \frac{1}{\kappa} (4 \land -R) \text{ or } R = (4 \land -\kappa T), \]  

(3.2)

and contraction of equation (3.1), leads to

\[ T = (3p - \mu). \]  

(3.3)
Use of equations (3.2) and (3.3), yields

\[ R = [4 \wedge + (\mu - 3p)\kappa]. \] (3.4)

Next, we use a dynamic result for a perfect fluid spacetime along a conformal Killing vector field \( \xi^i \), consider the equation [14]

\[ \mathcal{L}_\xi \mu = -2\Box \Omega - 2\Omega \mu - 2\Omega_{ij}u^iu^j, \] (3.5)

where \( \Box \Omega = -\frac{4}{3} \Omega R \) and \( \Omega_{ij} = -\frac{4}{3} \Omega R_{ij} \). In a perfect fluid spacetime, comparing with EFEs (1.2) and equation (3.1) with the condition \( u_iu^i = -1 \), we get

\[ R_{ij}u^iu^j = (\kappa \mu - \frac{R}{2} + \wedge) = [\kappa \left( \frac{3p + \mu}{2} \right) - \wedge]. \] (3.6)

If we set

\[ \Omega_{ij} = \frac{\Omega}{2} \left[ \frac{R}{3} g_{ij} - 2R_{ij} \right], \] (3.7)

then from [14], every CIV is also a CKV in an M-projectively flat spacetime. If we multiply both sides by \( u^iu^j \) in (3.7), using equation (3.6) and \( u^i \) to be timelike then we obtain

\[ \Omega_{ij}u^iu^j = \Omega \left( \frac{R}{3} - \kappa \mu - \wedge \right) = -\frac{\Omega}{3} \left[ (2\mu + 3p)\kappa - \wedge \right]. \] (3.8)

Now using \( \Box \Omega = -\frac{4}{3} \Omega R \) and (3.8) in (3.5), we get

\[ \mathcal{L}_\xi \mu = 2\Omega[(\kappa - 1)\mu + \wedge]. \] (3.9)

Next, we find \( \mathcal{L}_\xi p \) in M-projectively flat spacetime as follows:

\[ \mathcal{L}_\xi p = \frac{4}{3} \Box \Omega - 2\Omega p - \frac{2}{3} \Omega_{ij}u^iu^j. \] (3.10)

By using \( \Box \Omega = -\frac{4}{3} \Omega R \) and (3.8) in (3.10), we get

\[ \mathcal{L}_\xi p = 2\Omega[(\kappa - 1)p - \wedge]. \] (3.11)

Thus, we have the following result

**Theorem 3.1.** If an M-projectively flat spacetime \((V_4, g)\) with perfect fluid holds EFEs with a cosmological constant admits the curvature inheritance symmetry along a vector field \( \xi \), then we have the following

(a) \( \mathcal{L}_\xi \mu = 2\Omega[(\kappa - 1)\mu + \wedge] \)
Consider the following special cases:

**Case 1**: Under the hypothesis of Theorem 3.1, if \((V_4, g)\) is a perfect fluid hold the EFEs without a cosmological term (i.e. \(\Lambda = 0\)), then we have the following inheritance equations

\begin{align}
(a) \quad & \mathcal{L}_{\xi} \mu = 2\Omega(\kappa - 1)\mu, \\
(b) \quad & \mathcal{L}_{\xi} p = 2\Omega(\kappa - 1)p.
\end{align}

\[(3.12)\] \[(3.13)\]

**Case 2**: Under the hypothesis of Theorem 3.1, if \((V_4, g)\) is a perfect fluid spacetime satisfying the EFEs without a cosmological term (i.e. \(\Lambda = 0\) and \(\kappa = 1\)), then we have the following equations

\begin{align}
(a) \quad & \mathcal{L}_{\xi} \mu = 0, \\
(b) \quad & \mathcal{L}_{\xi} p = 0.
\end{align}

\[(3.14)\] \[(3.15)\]

A spacetime \((V_4, g)\) inherits symmetry along a homothetic vector field \(\xi\), if the following equations hold for a perfect fluid:

\begin{align}
(a) \quad & \mathcal{L}_{\xi} \mu = 2\Omega \mu \quad (b) \quad \mathcal{L}_{\xi} p = 2\Omega p \quad (c) \quad \mathcal{L}_{\xi} u^i = 2\Omega u^i.
\end{align}

Recently, the interest of many researcher has increased towards the study of the spacetime symmetry inheritance along a conformal Killing vector field \(\xi\). Coley and Tupper [12] have worked on special CKV. Now, it will be demonstrated that how the Theorem 3.1 is used to change the symmetry equations (3.12) and (3.13) for a curvature inheritance vector field in an M-projectively flat spacetime. Therefore, consider a conformal Killing vector field \(\xi\), such that from [13]

\[\mathcal{L}_{\xi} u^i = 2\Omega u^i + v^i\]

where \(v^i\) is a spacelike vector orthogonal to \(u^i\). Einstein field equations (1.2) for a perfect fluid, with (3.1), imply that \(u^i\) eigenvector of \(R_{ij}\), and \(u^i u_i = -1\)

\[R_{ij} u^j = [-\kappa(\frac{3p + \mu}{2}) + \Lambda] u_i.\]

\[(3.17)\]
If $\xi$ is the curvature inheritance vector or Ricci inheritance vector, then using (3.17) in (3.7), we get
\begin{equation}
\Omega_{,ij}u^iu^j = -\frac{\Omega}{3}(2\mu + 3p)\kappa - \Lambda.
\end{equation}
(3.18)
Using the above equation in $(\mu + p)v_i = 2[\Omega_{,ij}u^j + (\Omega_{,kl}u^k u^l)u_i]$, we find that $v_i = 0$ for $(\mu + p) \neq 0$. Then, equation (3.16) reduces to
\begin{equation}
\mathcal{L}_\xi u^i = 2\Omega u^i.
\end{equation}
(3.19)
The calculation of equations (3.12), (3.13) and (3.19) can be summarized in the following theorems.

**Theorem 3.2.** If an $M$-projectively flat spacetime with a perfect fluid obeying the Einstein field equations without a cosmological term admits a curvature inheritance symmetry along the vector field $\xi$, then symmetry with respect to $\xi$ is inherited by the spacetime.

**Theorem 3.3.** Let $(V_4, g)$ be an $M$-projectively flat spacetime with a perfect fluid obeying the Einstein field equation and admitting curvature inheritance symmetry along vector field $\xi$. If the perfect fluid denotes radiation era or stiff matter then this spacetime does not contain the cosmological constant.

**Proof.** Let us assume that the perfect fluid denotes the radiation era, i.e., $\mu = 3p$. By taking the Lie derivative of this equation and using (3.9) and (3.11), we can see that $\Lambda = 0$. Similarly, if the perfect fluid denotes stiff matter, i.e., $\mu = p$, then we also get $\Lambda = 0$. This completes the proof.

**Theorem 3.4.** Let $(V_4, g)$ be an $M$-projectively flat spacetime with a perfect fluid obeying the Einstein field equations and admitting curvature inheritance symmetry along vector field $\xi$. If the Lie derivative of $(\mu + p)$ is zero, then either the matter content of the spacetime satisfying the vacuum-like equation of state or $(\mu + p)$ is constant.

**Proof.** By using (3.9) and (3.11), we get
\begin{equation}
\mathcal{L}_\xi(\mu + p) = \mathcal{L}_\xi \mu + \mathcal{L}_\xi p = 0.
\end{equation}
(3.20)
Finally, we can say that either the matter content of the spacetime satisfy the vacuum like equation of state or $(\mu + p)$ is constant. Thus, the proof is completed.

□
4. Equation of State

Let \((V_4, g)\) be an M-projectively flat spacetime admitting curvature inheritance along a vector field \(\xi\). From Corollary 2.4, \(\xi\) is a CKV, satisfying the following equation

\[
(\xi^j R_{ij})_i = -3 \Box \psi.
\]  

(4.1)

Using Einstein field equations, (4.1) leads to

\[
([k T^{ij} - (\wedge - \frac{R}{2}) g^{ij}] \xi_j)_i = -3 \Box \psi.
\]  

(4.2)

As \(-3 \Box \psi = \alpha R\), therefore

\[
([k T^{ij} - (\wedge - \frac{R}{2}) g^{ij}] \xi_j)_i = \alpha R.
\]  

(4.3)

Further, for the perfect fluid spacetime with (3.1), we have

\[
[k p - \wedge + \frac{R}{2}] \xi^i_i = \alpha R.
\]  

(4.4)

Equation (4.4), on substitution \(\xi^i_i = 4 \alpha\) reduces to

\[
4(k p - \wedge) = -R.
\]  

(4.5)

Here \(\xi\) is CKV. For an M-projectively flat spacetime, the scalar curvature \(R = 4 \wedge - k(3p - \mu)\). Now comparing with (4.5), we have

\[
\mu + p = 0, \ (k \neq 0)
\]  

(4.6)

which may describe two cases (i) empty spacetime \(\mu = p = 0\) and (ii) perfect fluid spacetime holding vacuum like equation of state.

Thus, we state the result as follows

**Theorem 4.1.** Let an M-projectively flat spacetime with perfect fluid obeying the EFEs with cosmological constant be admit curvature inheritance symmetry along a vector field \(\xi\). Then \((V_4, g)\) is either a vacuum or satisfies the vacuum-like equation of state.

From [27], we note that \(\mu + p = 0\) implies that scalar curvature is equal to cosmological constant and Ojha [23] termed this as *Phantom Barrier*. Alan Guth in 1981 proposed the idea of *cosmic inflation* [17], and explained the similar conditions for the same in the universe. Amendola and Tsujikawa [8] explained the term inflation in their paper as rapid expansion of
the spacetime that might occurred just after the Big Bang. In the light of above discussion, we have

**Theorem 4.2.** Let a perfect fluid M-projectively flat spacetime \((V_4, g)\) obeying the EFEs with cosmological constant, admits curvature inheritance symmetry along a vector field \(\xi\). Then \((V_4, g)\) represents inflation.

From equations (3.1) and (4.6), we obtain

\[ T_{ij} = pg_{ij}. \tag{4.7} \]

Since, an M-projectively flat spacetime is Einstein, thus it is of constant scalar curvature. Further as \(\land\) and \(k\) are constants, therefore equation (4.5) confirms that \(p\) is constant. Hence \(\mu = -p\) is also constant. This condition has a special significance in the physics of spacetime. With this condition, the fluid starts behaving like a cosmological constant [27], which is also called Phantom Barrier [10]. This causes inflation of the universe [8].

Thus, we obtain the following theorem:

**Theorem 4.3.** The isotropic pressure and energy density of an M-projectively flat perfect fluid spacetime satisfying EFEs with cosmological constant admitting the curvature inheritance symmetry along a vector field \(\xi\) are constants.

5. A Purely Electromagnetic Distribution Admitting Curvature Inheritance Symmetry

Section 5 is devoted to the study of few results for purely electromagnetic distribution. Contraction of (4.7), provides

\[ T = 4p. \tag{5.1} \]

Using the equations (5.1) and (3.3), we get \(p = -\mu = \frac{T}{4}\). Now since for a purely electromagnetic distribution, \(T = 0\). We have \(p = \mu = 0\), in other words “In a purely electromagnetic distribution, the isotropic pressure and the energy density of a perfect fluid spacetime satisfying Einstein field equations with cosmological constant vanish.”

Putting \(\mu = p = 0\) in (3.1), we get \(T_{ij} = 0\). This implies that the spacetime is vacuum. Taking contraction of (1.2), we get

\[ R = 4 \land -kT, \tag{5.2} \]
substituting (5.2) in EFEs (1.2) and for purely electromagnetic distribution, we get

\[ R_{ij} = \wedge g_{ij}. \] (5.3)

If M-projectively flat perfect fluid spacetime \((V_4, g)\) admits curvature inheritance symmetry along \(\xi\). Then Lie derivative of equation (5.3) leads to a conformal motion. Thus, we may state the following result:

**Theorem 5.1.** The curvature inheritance symmetry in an M-projectively flat perfect fluid spacetime for a purely electromagnetic distribution, which satisfies EFEs with a cosmological term is conformal motion.

In this case, we choose Einstein field equations without cosmological constant takes the form

\[ R_{ij} - \frac{R}{2} g_{ij} = k T_{ij}. \] (5.4)

Contracting (5.4) we get

\[ R = -kT. \] (5.5)

For purely electromagnetic distribution \(T = 0\), (5.5) reduces to \(R = 0\). Therefore, from (2.3), we obtain

\[ R^h_{ijk} = 0. \] (5.6)

This implies that \((V_4, g)\) is an Euclidean space. Thus, we have the following result:

**Corollary 5.1.** In a purely electromagnetic distribution, an M-projectively flat perfect fluid spacetime \((V_4, g)\) satisfying EFEs without a cosmological constant does not admit any curvature symmetry.

6. Example of an M-Projectively Flat Spacetime

Let \(\mathbb{R}^4\) be equipped with the following metrics in terms of coordinates \(x^1, x^2, x^3\) and \(x^4\) given as follows:

\[ ds^2 = -(dx^1)^2 + e^{x_1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \] (6.1)

\[ ds^2 = (dx^1)^2 + e^{x_1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]. \] (6.2)

We note that the metric in (6.1) (resp. (6.2)) is a Lorentzian (resp. Riemannian) metric. The non-vanishing components of second kind Christoffel symbols \(\Gamma^h_{ij}\) of the metric (6.1) are given
by
\[ \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{1}{2}, \quad \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = \frac{e^x}{2}. \]

The non-vanishing components of the Ricci tensor \( R_{ij} \) and Riemann-Christoffel curvature tensor \( R_{hijk} \) of the metric (6.1) are given by
\[ R_{1212} = R_{1313} = R_{1414} = -\frac{e^x}{4}, \quad R_{2323} = R_{2424} = R_{3434} = \frac{e^x}{4}, \]
\[ R_{11} = \frac{3}{4}, \quad R_{22} = R_{33} = R_{44} = -\frac{3e^x}{4}. \]

Also, for the metric (6.1), the scalar curvature is \( R = -3 \). It is easy to compute that all the components of the M-projective curvature tensor are zero. Thus we have the following result:

**Theorem 6.1.** Let \( \mathbb{R}^4 \) be equipped with the Lorentzian metric given in (6.1), then \( (\mathbb{R}^4, g) \) is an M-projectively flat spacetime.

**Remark 6.1.** We note that \( (\mathbb{R}^4, g) \) endowed with the Riemannian metric (6.2) is also M-projectively flat.

**Remark 6.2.** We mention that the metrics (6.1) and (6.2) are warped product \( M_1 \times_f M_2 \), where the warping function \( f = e^x \) with 1-dimensional base \( M_1 \) and 3-dimensional fibre \( M_2 \).

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