Operator spaces with the WEP, the OLLP and the Gurarii property

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Abstract

We construct non-exact operator spaces satisfying the Weak Expectation Property (WEP) and the Operator space version of the Local Lifting Property (OLLP). These examples should be compared with the example we recently gave of a C*-algebra with WEP and LLP. The construction produces several new analogues among operator spaces of the Gurarii space, extending Oikhberg’s previous work. Each of our “Gurarii operator spaces” is associated to a class of finite dimensional operator spaces (with suitable properties). In each case we show the space exists and is unique up to completely isometric isomorphism.

The main goal of this note is the construction of examples of operator spaces $X \subset B(H)$ that have both the weak expectation property (in short WEP) and the operator space analogue of the local lifting property (in short OLLP) but are not exact. The WEP implies that the bidual $X^{**}$ is the range of a completely contractive projection on $B(H)^{**}$, and hence can be identified with a ternary ring of operators (TRO) (see Remark 4.41). This construction was known to us before we obtained an example of a C*-algebra with both WEP and LLP (see [39]). While the operator space case seems much less significant, certain features of the various spaces we obtain, in particular their Gurarii property, may be of independent interest for the theory of triple operator systems or ternary rings of operators (in short TROs).

The origin of the latter theory goes back to Choi and Effros [7] (for unital projections) and Youngson [12], who proved that the range of a completely contractive projection $P$ on a C*-algebra $A$ is (completely isometrically) isomorphic to a triple subsystem of another C*-algebra $B$ with the triple product defined on $P(A)$ by $[a,b,c] = P(ab^*c)$. A ternary ring of operators (TRO in short) is a closed subspace of $B(H, K)$ ($H, K$ being Hilbert spaces) stable by the triple product $(a,b,c) \mapsto ab^*c$. A typical example of TRO is the subspace $V = pAp \subset A$ where $p, q$ are (self-adjoint) projections in a C*-algebra $A$. Then $V$ is a left (resp. right) submodule with respect to the C*-subalgebras $pAp$ (resp. $qAg$). TROs, which were introduced by Hestenes in 1961, now have a rather developed theory, we refer to the paper [18] and the book [4] for a detailed exposition including history and references.

An operator space $X \subset B(H)$ has the WEP if and only if there is a completely contractive projection $P : B(H)^{**} \to X^{**}$ onto $X^{**}$, and $P$ can always be chosen normal. Thus in this case $X^{**}$ is always completely isometric to a TRO by Youngson’s theorem. We will produce examples of such spaces with several surprising properties. The main one being the “Gurarii property”. In Banach space theory the Gurarii space is a separable Banach space $X$ with the property that for any finite dimensional (f.d. in short) Banach space $F$ and any subspace $E \subset F$, for any $\varepsilon > 0$ any linear embedding $f : E \to X$ such that $\|f\|\|f^{-1}\|_{f(E)} < 1 + \varepsilon$ admits an extension $\tilde{f} : F \to X$ still...
satisfying the strict inequality $\|f\|\|f^{-1}|_{f(E)}\| < 1 + \varepsilon$. Equivalently, assuming $\|f\|\|f^{-1}|_{f(E)}\| \leq 1 + \varepsilon$, for any $\delta > 0$ we can find an extension $\tilde{f}: F \to X$ satisfying $\|\tilde{f}\|\|\tilde{f}^{-1}|_{\tilde{f}(E)}\| \leq (1 + \varepsilon)(1 + \delta)$. Gurarii\cite{14} showed that such a space exists and is unique up to $\varepsilon$-isometric isomorphism, whence the terminology “Gurarii space”. We will denote this space by $\mathbb{G}$. Later on Lusky\cite{29} showed its uniqueness up to isometric isomorphism. More recently in\cite{23} Kubiś and Solecki (see also\cite{11}) gave a much simpler proof that greatly influenced 5 of the present paper.

We refer to\cite{11, 2} for more recent results and references on generalizations of Gurarii spaces, which Gurarii originally called spaces of almost universal disposition.

In\cite{31} Oikhberg investigated a non-commutative version of the Gurarii space in the operator space framework. Operator spaces are defined as closed linear subspaces of $B(H)$, between which the completely bounded maps are the natural morphisms, in contrast with the bounded ones used in classical Banach space theory (see the books\cite{3, 37} for details). The immediate difficulty that arises in this context is that in sharp contrast with the Banach space case the metric space of finite dimensional operator spaces (o.s. in short) is not separable for the natural analogue of the Banach-Mazur distance. In particular there is no separable o.s. $X$ containing, for any $\varepsilon > 0$, a completely $(1 + \varepsilon)$-isomorphic copy of any f.d.o.s. $F$, and this is even impossible when we restrict to $\dim(F) \leq 3$. This phenomenon was discovered by Junge and the author in\cite{17} to disprove a conjecture of Kirchberg. In sharp contract, the characteristic property of $\mathbb{G}$ obviously implies that it contains a $(1 + \varepsilon)$-isomorphic copy of any f.d. Banach space $F$. Observing this difficulty, Oikhberg chose to restrict to the class of 1-exact o.s. that is those o.s. $F$ admitting for any $\delta > 0$ a completely $(1 + \delta)$-isomorphic embedding into $B(H)$ for some finite dimensional $H$, or equivalently into $M_N$ for some $N < \infty$. The latter class is clearly separable. Oikhberg then proved the analogue of Gurarii’s result and Lupini later on proved the analogue of Lusky’s. Actually Lupini\cite{26, 27} (following Henson and Ben Yaacov\cite{3}) developed a much more general theory in which spaces with a Gurarii property are described as Fraïssé limits. The examples we present in this paper can be viewed as new illustrations of this idea. We consider a class of f.d.o.s. $\mathcal{E}$ that is stable under the operations of passing to the quotient (in the o.s. sense) and of direct sum $(F_1, F_2) \mapsto F_1 \oplus_1 F_2$. We also assume that any space completely isometric to one in $\mathcal{E}$ is also in $\mathcal{E}$. (Such a class is what we call a “league”.) Lastly, we assume that $\mathcal{E}$ is separable for the distance $d_{cb}$. For any such $\mathcal{E}$ we show that there is a unique (up to completely isometric isomorphism) separable o.s. $X$ satisfying the cb-analogue of the Gurarii property restricted to $F$’s in $\mathcal{E}$ and also such that $X$ locally embeds in $\mathcal{E}$ in the following sense: for any f.d. $E \subset X$ and any $\varepsilon > 0$ there is $F \in \mathcal{E}$ containing $E$ completely $(1 + \varepsilon)$-isometrically. We will denote this unique X by $\mathbb{G}_E$. We show that any separable o.s. $Y$ that locally embeds in $\mathcal{E}$ embeds completely isometrically in $\mathbb{G}_E$.

While there are many possibilities, our interest focuses on 3 main examples of such $\mathcal{E}$’s. The first one is the class $\mathcal{E}_{\text{max}}$ of all the f.d. “maximal” o.s. in the sense of\cite{3} (see also\cite{3, 37}). This is the class of those f.d. $F \subset B(H)$ such that $\|u\|_{cb} = \|u\|$ for any $u : F \to Y$ where $Y$ is an arbitrary o.s. This means that $F$ is a f.d. quotient of $\ell_1$ itself equipped with its maximal o.s. structure. Equivalently, in place of $\mathcal{E}_{\text{max}}$ we may consider the class $\cup_n Q_{\ell_1^n}$ where $Q_{\ell_1^n}$ denotes the class of all quotients of $\ell_1^n$. Indeed, the latter class is suitably dense in $\mathcal{E}_{\text{max}}$.

The second example is the class $\mathcal{E}_S = \cup_n Q S_1^n$ where $Q S_1^n$ denotes the class of all quotients of $S_1^n$ and $S_1^n = M_n^*$ (o.s. dual). Note that in sharp contrast with $\ell_1$ for $\mathcal{E}_{\text{max}}$ any f.d.o.s. is a quotient of $S_1$ (see e.g.\cite{4} p. 27), so here we cannot replace $\{S_1^n \ | \ n \geq 1\}$ by $S_1$.

The third one is the class $\mathcal{E}_{S_F}$ formed of all the f.d. subspaces of the C$^*$-algebra $\mathcal{G} = C^*(F_\infty)$ of the free group $F_\infty$ (with countably infinitely many generators). Its stability under $\oplus_1$, under quotients as well as under duality were observed in\cite{17}. In the latter two cases the space $\mathbb{G}_E \subset B(H)$ has the WEP, whence a projection $P$ from $B(H)^{**}$ onto $\mathbb{G}_E^{**}$ with $\|P\|_{cb} = 1$, while for $\mathcal{E} = \mathcal{E}_{\text{max}}$ we
only obtain a $P$ with $\|P\| = 1$. In the case $\mathcal{E} = \mathcal{E}_1$, the space $\mathcal{G}_\mathcal{E}$ has the o.s. version of the LLP (in short OLLP), which is extensively studied by Ozawa in [32]. No infinite dimensional $C^*$-algebra has the OLLP. Thus $\mathcal{G}_\mathcal{E}_1$ is an o.s. analogue of the $C^*$-algebra in [39].

The space $\mathcal{G}_{\mathcal{E}_{\text{max}}}$ and its bidual might be relevant to tackle some important open questions about maximal operator spaces that we describe in Remark 4.3.

For completeness, we should add to the preceding list the class $\mathcal{E}_{\text{min}}$ formed of all f.d. o.s. that are “minimal” in the sense of [5] (see also [8, 37]). This is the class of those f.d. $F \subset B(H)$ such that $\|u\|_{cb} = \|u\|$ for any $u : Y \to F$ where $Y$ is an arbitrary o.s. But this example goes no further than the classical case. Indeed, it turns out (see Remark 4.2) that the space $\mathcal{G}_{\mathcal{E}_{\text{min}}}$ is completely isometric (and a fortiori isometric) to the classical $\mathcal{G}$ equipped with its minimal o.s. structure. More interesting variations on the same theme are presented in §3 we describe the operator space variant of the so-called “push out” construction.

We now briefly describe the contents of this paper. In §1 (resp. §2) we review the WEP (resp. the OLLP). In §3 we describe the operator space variant of the so-called “push out” construction. Our main results are in §4 where we produce various examples of o.s. with the Gurarii property, in particular a non-exact one with both the WEP and the OLLP. In §5 we prove the uniqueness (up to complete isometry) of the Gurarii space associated to any given league. The proof is a simple adaptation of the proof given by Kubiś and Solecki in [23] for the classical Gurarii case. In §6 we relate our non-exact examples to Oikhberg’s exact Gurarii space.

**Notation and general background**

As often, we abbreviate completely bounded by c.b., completely positive by c.p. and completely contractive by c.c. We will also abbreviate operator space (or operator spaces) by o.s. and finite dimensional by f.d.

An o.s. is a closed subspace of a $C^*$-algebra or of $B(H)$. The duality of o.s. is a consequence of Ruan’s characterization (see e.g. [8]) of the sequences of norms on $(M_n(E))_{n \geq 1}$ (where $M_n(E)$ denotes the space of $n \times n$-matrices with entries in a vector space $E$) that come from an embedding of $E$ into $B(H)$ for some $H$. Given an o.s. $E \subset B(H)$ (with Banach space sense) dual $E^*$ there is an $\mathcal{H}$ and an isometric embedding $j : E^* \to B(\mathcal{H})$ that induces isometric isomorphisms $M_n(j(E^*))) \simeq CB(E,M_n)$ for all $n$. The embedding $j$ allows one to consider $E^*$ as an o.s. This is what is called the dual o.s. structure on $E^*$. We will often refer to it as the o.s. dual of $E$. We refer to [8, 37, 4] for more background on operator spaces. Let $\{(E_i)_{i \in I}\}$ be a family of o.s. Assuming $E_i \subset B(H_i)$, the direct sum in the $\ell_\infty$-sense of $(E_i)_{i \in I}$ can be realized as embedded “block diagonally” in $B(\oplus_{i \in I} H_i)$. The resulting operator space with be denoted by $\oplus \sum_{i \in I} E_i$, and if all the $E_i$’s coincide with a single space $E$ we denote it by $\ell_\infty(I; E)$. When $(E_i)_{i \in I}$ is a sequence of spaces $(E_n)$ we use the lighter notation $\ell_\infty(\{E_n\})$ instead of $\oplus \sum_{n \in \mathbb{N}} E_n$. Moreover, we denote by $c_0(\{E_n\})$ the subspace formed of the sequences $x = (x_n) \in \ell_\infty(\{E_n\})$ such that $\lim \|x_n\| = 0$. We will use the analogous notion of direct sum with respect to $\ell_1$. The space $\oplus \sum_{i \in I} E_i$ is the space of families $(x_i)_{i \in I} \in \prod_{i \in I} E_i$ with $\sum \|x_i\|_{E_i} < \infty$, equipped with the o.s. structure associated to the embedding $J : (\oplus \sum_{i \in I} E_i)_{1} \subset B(H)$ defined by

$$J((x_i)_{i \in I}) = \oplus_{u \in C} \sum_{i \in I} u_i(x_i)$$

where $C$ is the collection of all $u = (u_i)_{i \in I} \in \prod_{i \in I} \beta_i$ with $\beta_i$ denoting the unit ball of $CB(E_i, \mathcal{H})$, and $\mathcal{H}$ being a suitably large Hilbert space (say with dimension equal to the cardinal of $\oplus \sum_{i \in I} E_i$). Note that

$$\|J((x_i)_{i \in I})\| = \sup_{\|u_i\|_{cb} \leq 1} \| \sum_{i \in I} u_i(x_i) \|.$$
Therefore $\|J((x_i)_{i \in I})\| = \sum_{i \in I} \|x_i\|$ so that $J$ is an isometric embedding of the $\ell_1$-sense (Banach space theoretic) direct sum of $(E_i)_{i \in I}$ into $B(H)$. Similarly for any $(x_i)_{i \in I} \in M_N((\oplus_{i \in I} E_i)_1)$ we have

$$\|(Id_{MN} \otimes J)((x_i)_{i \in I})\|_{M_N(B(H))} = \sup_{\|u_i\|_{cb} \leq 1} \|\sum_{i \in I} (Id_{MN} \otimes u_i)(x_i)\|_{M_N(B(H))},$$

but the latter supremum is less easy to describe than in the case $N = 1$. When $(E_i)_{i \in I} = \{E, F\}$ we denote $(\oplus_{i \in I} E_i)_1$ simply by $E_1 F$.

When $E_i = C$ for any $i \in I$ we set $\ell_1(I) = (\oplus_{i \in I} E_i)_1$. In that case it is easy to see that

$$\|u\|_{cb} = \|u\|$$

for any $u : \ell_1(I) \to B(H)$, which means that $\ell_1(I)$ is a maximal o.s.

In particular, we set

$$\ell_1^n = \ell_1(\{1, \ldots, n\}).$$

Thus for us $\ell_1^n$ is an o.s.

It is a well known fact that the dual o.s. of $(\oplus_{i \in I} E_i)_1$ can be identified completely isometrically with $(\oplus_{i \in I} E_i)_\infty$. In particular we have completely isometrically

$$\ell_1^{n*} = \ell_\infty^n.$$

Let $S^n_1$ (resp. $S_1$) be the trace class on $\ell_1^n$ (resp. $\ell_2$). Viewing $S_1^n$ (resp. $S_1$) as a space formed of $n \times n$-matrices (resp. bi-infinite matrices), we have natural embeddings $S_1^n \subset S_1^{n+1} \subset S_1$. Thus $S_1$ appears as the closure of $\cup_n S^n_1$. Recall $\|x\|_{S_1} = \text{tr}(\|x\|)$ for all $x \in S_1$. Let $E$ be an o.s. In [38] a vector valued (meaning here $E$-valued) version of the trace class is considered. The space $S^n_1[E]$ is defined as the completion of the algebraic tensor product $S_1^n \otimes E$ equipped with the norm defined as follows

$$\forall x \in S^n_1 \otimes E \quad \|x\|_{S^n_1[E]} = \inf\{\|a\|_{S^n_1} \|y\|_{M_n(E)} \|b\|_{S^n_1} : x = (a \otimes I) \cdot y \cdot (b \otimes I)\},$$

where the $S^n_1$-norm is the Hilbert-Schmidt norm and $M_n(E)$ is viewed as $M_n \otimes E$. One can check that $S^n_1[E] \subset S^{n+1}_1[E]$ isometrically for any $n$ and hence we may define $S_1[E]$ simply by setting

$$S_1[E] = \cup_n S^n_1[E].$$

This space coincides isometrically with the o.s.projective tensor product of $S_1$ and $E$ when $S_1$ is equipped with its o.s. structure as the dual of the space $K$ of all compact operators on $\ell_2$, with the duality defined by $x(k) = \text{tr}(kx)$ for $x \in S_1, k \in K$ (see [37] §4 [37] p. 140 for more details).

A mapping $u : E \to F$ between o.s. is called completely isometric if it is injective and if $\|u\|_{cb} = \|u|_{u(E)}|_{cb}^{-1}\|_{cb} = 1$. More generally, let $\epsilon > 0$, we will say that $u$ is completely $(1 + \epsilon)$-isometric if $\|u\|_{cb} \leq 1 + \epsilon$ and $\|u|_{u(E)}|_{cb}^{-1}\|_{cb} \leq 1 + \epsilon$. In [39] for short we call such maps $\epsilon$-embeddings. By an embedding we mean just a linear embedding, or equivalently an injective map between f.d. spaces.

Let $E, F$ be completely isomorphic o.s. We denote by $d_{cb}(E, F)$ the “multiplicative distance” defined by

$$d_{cb}(E, F) = \inf\{\|u\|_{cb} \|u^{-1}\|_{cb}\}$$

where the infimum runs over all isomorphisms $u : E \to F$. We set $d_{cb}(E, F) = \infty$ if $E, F$ are not completely isomorphic.

Let $c \geq 1$. An operator space $X$ is called $c$-exact if for any $\epsilon > 0$ and any f.d. $E \subset X$ there is an integer $N$ and $F \subset M_N$ such that $d_{cb}(E, F) \leq c + \epsilon$. Any exact (in particular any nuclear) $C^*$-algebra is 1-exact. See [37] p. 285 for more on exact o.s.
We refer the reader to the books [37, 8] for the standard notions of o.s. theory, in particular the notions of dual and quotient o.s. that we use freely throughout this paper.

We end this section with two technical points that are very useful when dealing with local (meaning finite dimensional) questions.

**Remark 0.1 (About perturbation).** We refer repeatedly to perturbation arguments. By this we mean the following easily verified assertion (see [37, p. 69]). Let $\varepsilon > 0$. Assume $x_1, \ldots, x_n$ are linearly independent vectors in an o.s. $X \subset B(H)$. Let $y_1, \ldots, y_n \in X$ be such that $\sup_j \|x_j - y_j\| \leq \varepsilon$. Then for all small enough $\varepsilon > 0$ there is a complete isomorphism $w : X \to X$ such that $w(x_j) = y_j$ and such that $\|w\|_{cb}\|w^{-1}\|_{cb} \leq 1 + \delta(\varepsilon)$ with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. In fact $w$ is a perturbation of the identity on $X$. (Indeed, if $\xi_j \in X^*$ is biorthogonal to $(x_j)$ we may take $w = Id_X + \sum \xi_j \otimes (y_j - x_j).$

**Remark 0.2 (About inductive limits).** Let $(\delta_n)$ be a summable sequence of positive numbers. Consider a sequence $(E_n)$ of o.s. given together with a sequence of embeddings $\varphi_n : E_n \to E_{n+1}$ such that $\|\varphi_n\|_{cb} \leq 1 + \delta_n$ and $\|\varphi_n^{-1}\|_{cb} \leq 1 + \delta_n$. The inductive limit $\mathcal{L}(\{E_n, \varphi_n\})$ of the system $(E_n, \varphi_n)$ is defined as follows. Let $\mathcal{L} = \ell_\infty(\{E_n\})/c_0(\{E_n\})$ and let $j_n : E_n \to \mathcal{L}$ be defined for $x \in E_n$ by $j_n(x) = Q(\langle x_k \rangle)$ where $Q : \ell_\infty(\{E_n\}) \to \mathcal{L}$ is the quotient map and where $(x_k)$ is defined by $x_k = 0$ for all $k < n$, $x_n = x$ and $x_k = \varphi_{k-1} \cdots \varphi_n(x)$ when $k > n$. Note that in the latter case

$$(1 + \delta_{k-1}) \cdots (1 + \delta_n)^{-1} \|x\| \leq \|x_k\| \leq (1 + \delta_{k-1}) \cdots (1 + \delta_n)\|x\|,$$

and $\|j_n(x)\| = \limsup_k \|x_k\|$. Since we assume $(\delta_n)$ summable $j_n : E_n \to X$ is completely $(1 + \delta_n')$-isometric for some $\delta_n' \to 0$ and $d_{cb}(E_n, j_n(E_n)) \to 1$. Note $j_n(E_n) \subset j_{n+1}(E_{n+1}) \subset \mathcal{L}$ for all $n$.

We then define the inductive limit as

$$\mathcal{L}(\{E_n, \varphi_n\}) = \bigcup_{\varepsilon > 0} j_n(E_n) \subset \mathcal{L}.$$

The construction shows that the following diagram commutes.

Let $\mathcal{L}(\{F_n, \psi_n\})$ be another similar inductive limit associated to maps $\psi_n : F_n \to F_{n+1}$. Suppose given for each $n$ a map $\psi_n : F_n \to E_n$ with $\sup \|\psi_n\| < \infty$ (resp. $\sup \|\psi_n\|_{cb} < \infty$). The sequence $(\psi_n)$ defines a bounded (resp. c.b.) “diagonal” map from $\ell_\infty(\{F_n\})$ to $\ell_\infty(\{E_n\})$, and hence also from $\ell_\infty(\{F_n\})/c_0(\{F_n\})$ to $\ell_\infty(\{E_n\})/c_0(\{E_n\})$. The latter map has norm (resp. c.b. norm) $\leq \limsup \|\psi_n\|$ (resp. $\leq \limsup \|\psi_n\|_{cb}$). Moreover if $(\psi_n)$ is such that $\psi_{n+1} \psi_n = \varphi_n \psi_n$ for all $n$ (“intertwining”), then $(\psi_n)$ defines a bounded (resp. completely isometric) map from $\mathcal{L}(\{F_n, \psi_n\})$ to $\mathcal{L}(\{E_n, \varphi_n\})$.

Lastly, if each $\psi_n$ is isometric (resp. completely isometric) then so is the latter map.

## 1 The WEP

The weak expectation property (in short WEP) for $C^*$-algebras was introduced by Lance who gave several equivalent characterizations for it. One of them matches the following natural generalization to operator spaces, which was explicitly considered in [4] p. 270).

**Definition 1.1.** An operator space $X \subset B(H)$ has the WEP if there is $T : B(H) \to X^{**}$ with $\|T\|_{cb} = 1$ such that $T|_X = i_X : X \to X^{**}$, where $i_X : X \to X^{**}$ denotes the canonical inclusion.
Equivalently, the bitransposed embedding $X^{**} \subset B(H)^{**}$ admits a projection $P : B(H)^{**} \to X^{**}$ such that $\|P\|_{cb} = 1$.

This notion does not depend on the (completely isometric) embedding $X \subset B(H)$, as well as its analogue for merely contractive projections considered in the next statement.

**Proposition 1.2.** Let $X \subset B(H)$ be an o.s. The following are equivalent.

(i) There is a map $T : B(H) \to X^{**}$ with $\|T\|_{cb} \leq 1$ such that $T(x) = x$ for any $x \in X$.

(ii) There is a contractive projection $P : B(H)^{**} \to X^{**}$ onto $X^{**}$.

(ii)' Same as (ii) with a weak* to weak* continuous projection $P : B(H)^{**} \to X^{**}$.

(iii) For any $n \geq 1$ and any subspace $S \subset \ell_1^n$, any linear map $u : S \to X$ admits for each $\varepsilon > 0$ an extension $\tilde{u} : \ell_1^n \to X$ with

$$\|\tilde{u}\|_{cb} \leq (1 + \varepsilon)\|u\|_{cb}.$$  

**Proof.** (i) $\Rightarrow$ (ii)' follows by considering $P = (T^*|_{X^*})^*$, and (ii)' $\Rightarrow$ (ii) is trivial. Note that (ii) $\Rightarrow$ (i). Assume (i). Let $u$ be as in (iii). Using the injectivity of $B(H)$ we find $\tilde{u} : \ell_1^n \to X^{**}$ extending $u$ with $\|\tilde{u}\|_{cb} = \|u\|_{cb}$. By the local reflexivity principle (which here boils down to the weak* density of $B_X$ in $B_X^{**}$), there is a net of maps $\tilde{u}(i) : \ell_1^n \to X$ with $\|\tilde{u}(i)\|_{cb} \leq 1$ tending weak* to $\tilde{u}$ so that $\tilde{u}(i)|_S$ tends to $u$ for $\sigma(X, X^*)$. By Mazur’s theorem, passing to convex combinations we get a similar net such that $\tilde{u}(i)|_S$ converges pointwise in norm to $u$. Then by perturbation (see Remark [11]), (iii) follows. (iii) $\Rightarrow$ (i) is a well known application of Hahn-Banach. See e.g. [39, Prop. 2.1].

In the o.s. context we have to distinguish between contractive and completely contractive projections $P : B(H)^{**} \to X^{**}$, as follows.

Let $S_1^n = M_n^*$ be the operator space dual of $M_n$, which is a non-commutative analogue of $\ell_1^n$.

**Proposition 1.3.** Let $X \subset B(H)$ be an o.s. The following are equivalent.

(i) There is a map $T : B(H) \to X^{**}$ with $\|T\|_{cb} \leq 1$ such that $T(x) = x$ for any $x \in X$.

(ii) There is a completely contractive projection $P : B(H)^{**} \to X^{**}$ onto $X^{**}$.

(ii)' Same as (ii) with a weak* to weak* continuous projection $P : B(H)^{**} \to X^{**}$.

(iii) For any $n \geq 1$ and any subspace $S \subset S_1^n$, any linear map $u : S \to X$ admits for each $\varepsilon > 0$ an extension $\tilde{u} : S_1^n \to X$ with

$$\|\tilde{u}\|_{cb} \leq (1 + \varepsilon)\|u\|_{cb}.$$  

**Proof.** The proofs that (i) $\Rightarrow$ (ii)' (again with $P = (T^*|_{X^*})^*$), (ii)' $\Rightarrow$ (ii) $\Rightarrow$ (iii) are essentially identical to the ones for the preceding proposition. (iii) $\Rightarrow$ (i) also follows by Hahn-Banach using the o.s. version of the projective tensor product norm (see e.g. [39, Prop. 2.1]). We give the details for the reader’s convenience. Assume (iii). It follows that for any o.s. $Y$ we have and isometric inclusion

$$Y^* \otimes_A X \subset Y^* \otimes_A B(H)$$

where $\otimes_A$ stands for the o.s. projective tensor product. Indeed, this follows easily from the factorization of the finite rank linear map $u : Y \to X$ such that the associated tensor $t_u \in Y \otimes B(H)$ satisfies $\|t_u\|_{\otimes A} < 1$. When applied to $Y = X$ this implies that for any $t \in X^* \otimes X$ we have

$$\forall t \in X^* \otimes X \quad |\text{tr}(t)| \leq \|t\|_{X^* \otimes_A B(H)}.$$  

The Hahn-Banach extension of the linear map $t \mapsto \text{tr}(t)$, which is of unit norm on $X^* \otimes_A B(H)$, defines a complete contraction $T : B(H) \to X^{**}$ such that $T|_X = i_X$.
Corollary 1.4. An o.s. $X$ has the WEP if (and only if) for any $n \geq 1$ and any subspace $S \subset S^n_1$ any linear map $u : S \to X$ admits for each $\varepsilon > 0$ an extension $\tilde{u} : S^n_1 \to X$ with

$$\|\tilde{u}\|_{cb} \leq (1 + \varepsilon)\|u\|_{cb}.$$

## 2 The OLLP

**Definition 2.1.** We define the LLP for a $C^*$-algebra $A$ by the equality $A \otimes_{\min} \mathcal{B} = A \otimes_{\max} \mathcal{B}$, where $\mathcal{B} = B(\ell_2)$.

Kirchberg showed that this property is equivalent to a certain local lifting property (analogous to that of $L_1$ in Banach space theory), which has several equivalent forms, one of which as follows:

**Proposition 2.2.** A $C^*$-algebra $A$ satisfies $A \otimes_{\min} \mathcal{B} = A \otimes_{\max} \mathcal{B}$ iff for any $*$-homomorphism $\pi : A \to C/I$ into a quotient $C^*$-algebra, for any f.d. subspace $E \subset A$ and any $\varepsilon > 0$ the restriction $\pi|_E$ admits a lifting $v : E \to C$ with $\|v\|_{cb} \leq (1 + \varepsilon)$.

**Remark 2.3.** Actually when $A$ has the LLP the preceding local lifting even holds with $\varepsilon = 0$. Moreover, the conclusion holds for any decomposable $\pi : A \to C/I$ with dec-norm equal to 1 (see [35, Th. 9.38]).

**Remark 2.4.** According to Kirchberg [20] a unital $C^*$-algebra $A$ has the lifting property (LP) if any unital $*$-homomorphism $\pi : A \to C/I$ as above admits a (global) unital c.p. lifting. Then this lifting holds for any unital c.p. $\pi$. If $A$ is not unital, by definition $A$ has the LP if its unitization does. Kirchberg [20] proved that the (full) $C^*$-algebra of any free group $C^*(\mathbb{F})$ has the LP. Clearly (see Proposition 2.2) the LP implies the LLP.

Let $X$ be an operator space, $C$ a $C^*$-algebra, $I \subset C$ a (closed, self-adjoint, 2-sided) ideal in $A$, with quotient $C^*$-algebra $C/I$. Consider a completely contractive map $\varphi : X \to C/I$. We say that $\varphi$ locally lifts (or admits a local lifting) if for any finite dimensional (f.d. in short) subspace $E \subset X$, there exists a completely contractive map $\psi : E \to C$ such that $q \circ \psi = \varphi|_E$, where $q : C \to C/I$ is the quotient map. The OLLP was defined by Ozawa in [32]:

**Definition 2.5.** An o.s. $X$ has the OLLP if any complete contraction $\varphi : X \to C/I$ into an arbitrary quotient $C^*$-algebra locally lifts. We say $X$ has the OLP if one can take $E = X$ in the above situation.

Let $C^*(X)$ denote the universal $C^*$-algebra of $X$, characterized by the fact that it contains $X$ as a generating subspace and any complete contraction on $X$ into $B(H)$ uniquely extends to a $*$-homomorphism on $C^*(X)$ (see e.g. [37]).

**Proposition 2.6 ([32]).** The operator space $X$ has OLLP (resp. OLP and is separable) if and only if $C^*(X)$ has LLP (resp. LP and is separable).

**Remark 2.7.** By [32, Prop. 2.9] a separable o.s. $X$ has the OLLP if and only if any complete contraction into the Calkin algebra admits a completely contractive lifting.

**Remark 2.8.** The question whether any separable maximal operator space $X$ has the OLP is equivalent to the well known open problem whether any separable quotient $C^*$-algebra $C/I$ admits a contractive lifting $C/I \to C$. It is known (see [1]) that this holds if $X$ has the metric approximation property. See [40] for a more recent account on this topic. A fortiori, all maximal operator spaces with the metric approximation property have the OLLP. Whether this holds without the metric approximation property is an important open question put forward by Ozawa, who shows in [32, Prop. 3.1] that this question is equivalent to asking whether any maximal o.s. is locally reflexive (see also [30] and [15]).
Remark 2.9. Examples of o.s. with the OLLP include all preduals of von Neumann algebras (the so-called non-commutative $L_1$-spaces). When separable these have the OLP. Moreover the Hardy space $H_1$, and the space $H_1(M_\alpha)$ for any von Neumann algebra $M$, have the OLLP (see \cite[Th.4.2(ii)]{32}).

We will use the following characterization of the OLLP from \cite{32}.

Theorem 2.10. An operator space $X \subset B(H)$ has the OLLP iff for any $\varepsilon > 0$ and any f.d. subspace $E \subset X$ there are an integer $n$, a quotient $F$ of $S^n_1$ and a factorization $E \xrightarrow{v} F \xrightarrow{w} X$ of the inclusion $E \subset X$ with $\|v\|_{cb}\|w\|_{cb} < 1 + \varepsilon$.

Definition 2.11. An o.s. $X$ has the strong OLLP if for any $\varepsilon > 0$ and any f.d. subspace $E \subset X$ there is a f.d. subspace $E_1$ with $E \subset E_1 \subset X$ such that there are an integer $N$ and a quotient $F$ of $S^n_N$ with $d_{cb}(E_1, F) < 1 + \varepsilon$.

By perturbation arguments (see Remark 0.1), Theorem 2.10 implies:

Corollary 2.12. Assume $X = \bigcup E_n$ where $(E_n)$ is an increasing sequence of f.d. subspaces, such that all the inclusions $E_n \rightarrow X$ satisfy the factorization in Theorem 2.10. Then $X$ has the OLLP. In particular, this holds if, for each $n$, $E_n$ is completely isometric to a quotient of $S^n_1$ for some $N = N(n)$. In the latter case, $X$ has the strong OLLP.

We do not know whether the OLLP implies the strong OLLP.

It seems worthwhile to enlarge the preceding definitions like this:

Definition 2.13. A map $u : X \rightarrow Y$ between o.s. will be called OLLP if for any $\varepsilon > 0$ and any f.d. subspace $E \subset X$ there is an integer $n$, a quotient $F$ of $S^n_1$ and a factorization $E \xrightarrow{u|_E} F \xrightarrow{w} Y$ of the inclusion $u|_E$ with $\|u|_{cb}\|w\|_{cb} < 1 + \varepsilon$. (When $u = Id_X$ this means that $X$ has the OLLP.)

Remark 2.14. It is easy to see that if a map $u : X \rightarrow Y$ is OLLP then for any complete contraction $\varphi : Y \rightarrow C/\mathcal{I}$ (into an arbitrary quotient $C^*$-algebra) the composition $\varphi u : X \rightarrow C/\mathcal{I}$ is locally liftable. The converse can also be checked by the same argument as the one used by Ozawa in \cite{32} for the case $u = Id_X$.

Let us denote by $QS^n_1$ the class of o.s. quotients of $S^n_1$. We denote

$$\mathcal{E}_1 = \cup_n QS^n_1.$$ 

Remark 2.15. Let $X \subset B(H)$ be an o.s. Then the inclusion $X \subset B(H)$ (or any completely isometric embedding into $B(H)$) is an OLLP map if and only if for any f.d. subspace $E \subset X$ there are an integer $n$ and $F \in \cup_n QS^n_1$ such that $E$ embeds completely $(1 + \varepsilon)$-isometrically into $F$. Indeed, this follows from the injectivity of $B(H)$.

Anticipating a bit over Definition 4.14 we will say that $X$ locally embeds in $\mathcal{E}_1 = \cup_n QS^n_1$ if it satisfies the property in Remark 2.16.

Proposition 2.16. For an o.s. $X$ that locally embeds in $\mathcal{E}_1$, the WEP implies the OLLP.

Proof. Let $E \subset X$ be a f.d. subspace. Let us assume $E \subset F$ (completely isometrically) for some $F \in \mathcal{E}_1$. If $X$ has the WEP the inclusion $v : E \subset X$ extends to $\tilde{v} : F \rightarrow X$** with $\|\tilde{v}\|_{cb} \leq 1$. Since $F^* \subset M_n$ for some $n$, (in particular $F^*$ is 1-exact), this guarantees that $F^* \otimes_{\min} X^{**} = (F^* \otimes_{\min} X)^{**}$ isometrically. It follows that there is a net of maps $v_i : F \rightarrow X$ with $\|v_i\|_{cb} \leq 1$ that tends weak* to $\tilde{v}$. Recall that $\tilde{v}(E) \subset X$. Passing to convex suitable convex combinations we find a similar net such that $v_i|_E$ tends in norm to $\tilde{v}|_E = v$. By norm-perturbation (see Remark 0.1), we obtain $w : F \rightarrow X$ with $\|w\|_{cb} \leq 1 + \varepsilon$ such that $w|_E = u$. Whence a factorization $E \xrightarrow{v} F \xrightarrow{w} X$ as required in Theorem 2.10. Here we simplified slightly by assuming $v : E \rightarrow F$ is completely isometric, but it is easy to adapt the argument to cover the completely $(1 + \varepsilon)$-isometric case. \qed
The preceding statement is analogous to the fact that for a $C^*$-algebra that locally embeds in $\mathcal{C} = C^*(\mathbb{F}_\infty)$ (or equivalently locally embeds in the class $\mathcal{E}_{\mathcal{S}\mathcal{E}}$ appearing in Remark 4.31) the WEP implies the LLP (see [39, Prop. 3.7]).

3 Main tools

The following key statement is called the “push out lemma” in [23]. It seems to have deep roots in category theory, as emphasized in Pedersen’s [33]. In the Banach space context, this same construction was first used by Kisliakov [22] (without any name for it), and later became the basic building block for the author’s construction in [35] (see also [6]). It was adapted to the operator space context by Oikhberg in [31]. Instead of reproducing Oikhberg’s argument in [31] (which uses [0.2]) we choose a presentation based on the space $S_1[E]$ described in [0.3] that (although less self-contained) emphasizes more clearly the parallelism between the Banach and operator space cases.

**Lemma 3.1.** Let $E, L$ be o.s. Let $S \subset L$ be a subspace and let $u : S \to E$ be a c.b. map. Let $G_u = \{(s, -us) \mid s \in S\}$. Let $E_1 = [L \oplus_1 E]/G_u$, let $Q : [L \oplus_1 E] \to [L \oplus_1 E]/G_u$ denote the quotient map and let $j : E \to E_1$ and $\tilde{u} : L \to E_1$ be defined by

$$\forall e \in E \quad j(e) = Q(0 \oplus e) \quad \text{and} \quad \forall x \in L \quad \tilde{u}(x) = Q(x \oplus 0).$$

Then $j$ is injective and $\tilde{u}$ extends $u$ in the sense that $\tilde{u}|_S = ju$ and $\|\tilde{u}\|_{cb} \leq 1$.

Moreover, if $\|u\|_{cb} \leq 1$ then $j$ is a complete isometry.

Lastly, if $u$ is completely isometric then so is $\tilde{u}$.

More precisely, if $u$ is injective with $\|u\|_{cb} \leq 1$ then

$$\|\tilde{u}^{-1} \mid_{\tilde{u}(L)}\|_{cb} = \|u^{-1} \mid_{u(S)}\|_{cb}.$$  

**Proof.** We start by quickly reviewing the Banach space case, by which we mean the same statement but with ordinary norms instead of the cb-ones everywhere.

$$\begin{array}{ccc}
L & \xrightarrow{\tilde{u}} & E_1 \\
\downarrow & & \downarrow \\
S & \xrightarrow{u} & E
\end{array}$$

The injectivity of $j$ and the assertions $\tilde{u}|_S = u$ and $\|\tilde{u}\| \leq 1$ are immediate. For $e \in E$ we have $\|je\| = \inf_{s \in S} \{\|s\| + ||e - us||\}$ and hence (take $s = 0$) $\|je\| \leq \|e\|$, but also $\|s\| + ||e - us|| \geq \|s\| + \|e\| - \|us\|$ and hence $\|je\| \geq \|e\|$ if $\|u\| \leq 1$. Thus $j$ is isometric. We have $\|\tilde{u}(x)\| = \inf_{s \in S} \{\|x + s\| + \|us\|\}$ for all $x \in L$. Assuming $u$ injective, let $c = \|u^{-1} \mid_{u(S)}\|$. Assume $\|u\| \leq 1$ so that $c \geq 1$. Then,

$$\|x + s\| + \|us\| \geq \|x + s\| + c^{-1}\|s\| \geq c^{-1} (\|x + s\| + \|s\|) \geq c^{-1} \|x\|.$$  

Therefore $\|\tilde{u}(x)\| \geq c^{-1}\|x\|$. Equivalently $\|\tilde{u}^{-1} \mid_{\tilde{u}(L)}\| \leq c$. Since $u$ is the restriction of $\tilde{u}$ (meaning $\tilde{u}|_S = ju$) we must have $c \leq \|\tilde{u}^{-1} \mid_{\tilde{u}(L)}\|$. This gives us (3.1) for the norms.

To deduce from this Lemma 3.1 as stated above it will be convenient to use the “vector valued trace class” defined in [36], as follows. Let $K$ denote the space of compact operators on $\ell_2$. Let $S_1 = K^*$ (the trace class) equipped with its dual o.s. structure. For any o.s. $X$ we denote $S_1[X]$ the (o.s. sense) projective tensor product of $S_1$ with $X$, described in [0.3]. We will use the following facts. Let $u : X \to Y$ be a map between o.s. Then $\|u\|_{cb} = \|Id \otimes u : S_1[X] \to S_1[Y]\|$. Moreover $u$ is completely
isometric if and only if \(Id \otimes u : S_1[X] \to S_1[Y]\) is isometric. We have \(S_1[X \oplus Y] = S_1[X] \oplus S_1[Y]\) (isometrically) and for any subspace \(S \subset X\) we have \(S_1[X/S] = S_1[X]/S_1[S]\) (isometrically). Actually these identities are all completely isometric but we do not need that here. For all these facts we refer to our astérisque memoir [36] or [37, p. 142].

Using this it suffices to apply the first part of the proof to \(Id \otimes u : S_1[S] \to S_1[E]\) with \(S_1[S] \subset S_1[L]\). Observe that by the preceding facts \([S_1[L] \oplus_1 S_1[E]]/S_1[G_u]\) can be identified naturally with \(S_1[E_1]\). Then \(Id \otimes u : S_1[L] \to S_1[E_1]\) can be identified with \(Id \otimes \tilde{u}\) and the associated isometric embedding \(S_1[E] \to S_1[E_1]\) can be identified with \(Id \otimes j\). With these identifications, Lemma 3.1 becomes immediate by the Banach space case.

Remark 3.2. The space \(E_1\) is the solution of a universal problem. Given \(S \subset L\) and \(u : S \to E\), assume that we have an o.s. space \(Z\) and operators \(v : L \to Z\) and \(w : E \to Z\) such that \(v|_S = wu\). Then there is a unique operator \(T : E_1 \to Z\) such that \(v = Tu\) and \(w = Tj\); moreover, we have \(\|T\|_{cb} \leq \max\{\|v\|_{cb}, \|w\|_{cb}\}\). It is easy to see by diagram chasing that if there exists an \(E_1\) satisfying this, then it is unique up to complete isometry, and it does exist since the space \(E_1\) in Lemma 3.1 clearly has this universal property.

Remark 3.3 \((E_1/E \simeq L/S)\). As a supplement to Lemma 3.1 the space \(E_1/E\) is completely isometrically isomorphic to \(L/S\). More precisely, there is a completely isometric isomorphism \(\tilde{u} : L/S \to E_1/E\) induced by the map \(\tilde{u} : L \to E_1\). The verification is an easy exercise. The point is that the possible metric lifting properties of \(L \to L/S\) are all inherited by \(E_1 \to E_1/E\).

Lemma 3.1 admits the following immediate generalization:

Lemma 3.4. Let \(E\) be an o.s. Let \((L_i)_{i \in I}\) be a family of o.s. with subspaces \(S_i \subset L_i\) and c.b. maps \(u_i : S_i \to E\). There is an o.s. \(E_1\) and a completely isometric embedding \(j : E \subset E_1\) such that for all \(i \in I\) there is an operator \(\tilde{u}_i : L_i \to E_1\) with \(\|\tilde{u}_i\|_{cb} = \|u_i\|_{cb}\) such that \(\tilde{u}_i|_{S_i} = j u_i\) for all \(i \in I\). Moreover, whenever \(u_i\) is injective, so is \(\tilde{u}_i\) and we have

\[
\|\tilde{u}_i^{-1}|_{\tilde{u}_i(L_i)}\|_{cb} = \|u_i^{-1}|_{u_i(S_i)}\|_{cb}. \tag{3.3}
\]

In addition, \(E_1/E\) is completely isometrically isomorphic to \((\oplus L_i/S_i)\).

Proof. Let \(u_i' = \|u_i\|_{cb}^{-1} u_i\). Let \(S = (\oplus_{i \in I} S_i)\), \(L = (\oplus_{i \in I} L_i)\) and let \(u : S \to E\) be defined for all \((s_i) \in S\) by \(u((s_i)) = \sum u_i'(s_i)\), so that \(\|u\|_{cb} = 1\). Let \(E_1, j : E \to E_1\) (completely isometric) and \(\tilde{u} : L \to E_1\) be as in Lemma 3.1. Let \(v_i : L_i \to E_1\) be the restriction of \(\tilde{u}\) on the \(i\)-th factor. We have \(\|\tilde{u}\|_{cb} = 1\) and hence \(\|v_i\|_{cb} \leq 1\) for all \(i \in I\). Let \(\tilde{u}_i = \|u_i\|_{cb} v_i : L_i \to E_1\) and hence \(\|\tilde{u}_i\|_{cb} \leq \|u_i\|_{cb}\) Note \(\tilde{u}_i|_{S_i} = j u_i\) so that \(\|\tilde{u}_i\|_{cb} = \|u_i\|_{cb}\). To check (3.3) we essentially repeat the argument in (3.2): for \(x_i \in L_i\) and \(s \in S\) we have

\[
\|x_i + s\| + \sum_{j \neq i} \|s_j\| + \|u_i' s_i\| + \sum_{j \neq i} u_j' s_j \geq \|x_i + s\| + \|u_i' s_i\|
\]

and hence \(\|x_i\| \geq \|u_i'^{-1}|_{u_i(S_i)}^{-1}\|s_i\| = \|u_i^{-1}|_{cb}\|u_i^{-1}|_{u_i(S_i)}^{-1}\||x_i||\). This implies \(\|\tilde{u}_i^{-1}|_{\tilde{u}_i(S_i)}\| \leq \|u_i^{-1}|_{cb}\|u_i^{-1}|_{u_i(S_i)}^{-1}\||x_i||\) and since \(\tilde{u}_i\) extends \(u_i\) we obtain (3.3) for the norms. Using the spaces \(S_1[E]\) as before, we obtain the same with c.b. norms.

Remark 3.5. One can also deduce Lemma 3.4 from Lemma 3.1 by iterated applications of the latter (as in Remark 3.3) and using transfinite induction after well ordering the set \(I\).

By homogeneity, Lemma 3.1 implies the following:
Lemma 3.6. Let $S \subset L$, $E$ be as in Lemma 3.1. Let $v : S \to E$ be an injective c.b. map with $\|v^{-1}\|_{cb} < \infty$ and let $u = v^* v^{-1}$. Let $E_1$ and $j : E \to E_1$ (completely isometric) be associated to $u$ as in Lemma 3.1. There is $\tilde{v} : L \to E_1$ extending $v$ such that

$$\|\tilde{v}\|_{cb} \|\tilde{v}|_{L}\|_{cb} = \|v\|_{cb} \|v^{-1}\|_{cb}.$$ (3.4)

Proof. Indeed, we have $\|\tilde{u}\|_{cb} = \|u\|_{cb} = 1$, so that if we set $\tilde{v} = \tilde{u}\|v\|_{cb}$ then $\tilde{v}$ extends $v$ and (3.4) follows from (3.1).

\[ \Box \]

4 The Gurarii property for operator spaces

Let $OS_{fd}$ denote the (complete) metric space of all f.d. operator spaces equipped with the cb-analogue of the Banach-Mazur distance $d_{cb}$ defined by (0.4). Two elements $E, F \in OS_{fd}$ are considered the same if they are completely isometrically isomorphic. This holds if and only if $d_{cb}(E, F) = 1$ (or equivalently $\log d_{cb}(E, F) = 0$). We will often identify an $E \in OS_{fd}$ with a “concrete” representative $E \subset B(H)$. Although slightly abusive these conventions should not lead to any confusion.

Let $\mathcal{E} \subset OS_{fd}$ be a set (or “a class”) of finite dimensional o.s. We will always assume implicitly that $\mathcal{E}$ is non-void and $\mathcal{E} \neq \{\{0\}\}$.

A class $\mathcal{E}$ will be called separable if it is so with respect to the metric $d_{cb}$, i.e. if there is a countable $\mathcal{E}' \subset \mathcal{E}$ such that for any $E \in \mathcal{E}$ and any $\varepsilon > 0$ there is $E' \in \mathcal{E}'$ with $d_{cb}(E, E') < 1 + \varepsilon$.

For any f.d.o.s. $E \subset B(H)$ we denote

$$d_{cb}(E, \mathcal{E}) = \inf\{d_{cb}(E, F) \mid F \in \mathcal{E}\}. $$ (4.1)

Let $X$ be an o.s. We denote

$$d_{SX}(E) = \inf\{d_{cb}(E, F) \mid F \subset X\}. $$ (4.2)

Notation: Let $u : S \to E$ be a map that defines a complete isomorphism into its image. We define the “distortion” $D(u)$ of $u$ by setting

$$D(u) = \|u\|_{cb} \|u^{-1}\|_{cb}. $$ (4.3)

Obviously if $v : E \to F$ is a complete isomorphism from $E$ to $v(E)$ we have

$$D(vu) \leq D(v)D(u) \quad \text{and also} \quad D(u) = D(u^{-1} |_{u(S)}). $$ (4.4)

Note $D(tu) = D(u)$ for all $t > 0$. Let us denote

$$D'(u) = \max\{\|u\|_{cb}, \|u^{-1}\|_{u(S)}\|_{cb}\}, $$ (4.5)

so that

$$D'(vu) \leq D'(v)D'(u) \quad \text{and also} \quad D'(u) = D'(u^{-1} |_{u(S)}). $$ (4.6)

Note $D(u) \leq D'(u)^2$. Note also that $D'(u) = D(u) = \|u^{-1}\|_{u(S)}\|_{cb}$ when $\|u\|_{cb} = 1$.

Definition 4.1. A map $u$ as above such that $\max\{\|u\|_{cb}, \|u^{-1}\|_{u(S)}\|_{cb}\} \leq 1 + \varepsilon \quad \text{(i.e.} \quad D'(u) \leq 1 + \varepsilon \text{)}$ will be called an $\varepsilon$-embedding.

When $u$ is surjective we say that $u$ is an $\varepsilon$-isomorphism.
Definition 4.2. Let $\mathcal{E}$ be a class of f.d. operator spaces. We will say that $\mathcal{E}$ is a Gurarii class or simply is Gurarii if it satisfies the following property.

Consider an injective operator $u : S \to E$ where $S \subset L$ and $L, E \in \mathcal{E}$. Then for any $\delta > 0$ there is a space $E_1 \in \mathcal{E}$, a completely isometric embedding $\varphi : E \to E_1$ and a map $\tilde{u} : L \to E_1$ extending $u$ (i.e. $\tilde{u}|_S = \varphi u$) such that

\[
D(\tilde{u}) \leq (1 + \delta)D(u).
\]

We will say that $\mathcal{E}$ is tightly Gurarii if this holds with $\delta = 0$. We will say that $\mathcal{E}$ is loosely Gurarii if for any $\varepsilon > 0$ this holds for some $\varepsilon$-embedding $\varphi : E \to E_1$.

Obviously: tightly Gurarii $\Rightarrow$ Gurarii $\Rightarrow$ loosely Gurarii.

Remark 4.3. Lemma 3.1 shows that the class of all f.d. operator spaces is tightly Gurarii, just like the class of all f.d. Banach spaces in the Banach space setting.

Remark 4.4 (On iteration). Let $\mathcal{E}$ be a Gurarii class. Fix $E \in \mathcal{E}$. The Gurarii property behaves nicely with respect to iteration. Indeed, suppose given a finite set of injective maps $u_i : S_i \to E$ with $S_i \subset L_i$ ($1 \leq i \leq n$). We claim that there is $E_n \in \mathcal{E}$ and a complete isometry $\Phi_n : E \to E_n$ for which there are maps $\tilde{u}_i$ such that $\tilde{u}_i|_{S_i} = \Phi_n u_i$ for any $1 \leq i \leq n$ and (4.7) holds for any $u \in \{u_1, \ldots, u_n\}$.

Indeed, let $E_1$ and $\varphi_1 : E \to E_1$ be as in Definition 4.2 applied to $u = u_1$, and let $\tilde{u} : L_1 \to E_1$ be the corresponding map. We may reapply the property in Definition 4.2 this time with $E$ replaced by $E_1$ and $u$ replaced by $\varphi_1 u_2 : S_2 \to E_1$. Thus we find $E_2$, a complete isometry $\varphi_2 : E_1 \to E_2$ and $\tilde{u}_2 : L_2 \to E_2$ such that $\tilde{u}_2|_{S_2} = \varphi_2 \varphi_1 u_2$. We set $\tilde{u}_1 = \varphi_2 \tilde{u} : L_1 \to E_2$. Then the map $\Phi_2 = \varphi_2 \varphi_1$ satisfies the claim for $n = 2$. Moreover, (4.7) holds for any $u \in \{u_1, u_2\}$. Continuing in this way, we obtain complete isometries $\varphi_i : E_{i-1} \to E_i$ so that $\Phi = \varphi_n \cdots \varphi_1$ satisfies the announced property.

Remark 4.5. We have trivially $\|u\|_{cb} \leq \|\tilde{u}\|_{cb}$ and $\|u^{-1}|_{u(E)}\|_{cb} \leq \|\tilde{u}^{-1}|_{\tilde{u}(E)}\|_{cb}$ (since $\tilde{u}|_S = u$). Therefore (4.7) implies

\[
\|\tilde{u}\|_{cb} \leq (1 + \delta)\|u\|_{cb}\text{ and }\|u^{-1}|_{u(S)}\|_{cb} \leq (1 + \delta)\|\tilde{u}^{-1}|_{\tilde{u}(S)}\|_{cb}.
\]

Conversely, (4.8) implies (4.7) with $(1 + \delta)\|u\|_{cb}$ in place of $(1 + \delta)\|u\|_{cb}$. A fortiori

\[
D(\tilde{u}) \leq (1 + \delta)D(u) \Rightarrow D'(\tilde{u}) \leq (1 + \delta)D'(u).
\]

The goal of the next lemma is to show that we may apply the Gurarii extension property even when the inclusion $S \subset L$ is replaced by an $\varepsilon$-isometric map $T : Y \to L$.

Lemma 4.6. Let $L, E, E_1$ be f.d. operator spaces. Let $\delta > 0$. Let $\varphi : E \to E_1$ be an $\varepsilon$-embedding for some $\varepsilon > 0$. Assume that for any f.d. subspace $S \subset L$ and for any injective $u : S \to E$ there is $\tilde{u} : L \to E_1$ injective extending $u$ in the sense that $\tilde{u}|_S = \varphi u$ and such that $D'(\tilde{u}) \leq (1 + \delta)D'(u)$. Then for any f.d. $Y$ and any injective map $T : Y \to L$ the following generalization holds:

For any injective $v : Y \to E$ there is $\tilde{u} : L \to E_1$ injective extending $v$ in the sense that $\tilde{u}T = \varphi v$ and such that $D'(\tilde{u}) \leq (1 + \delta)D'(v)D'(T)$.
Proof. Consider $S = T(Y) \subset L$ and let $u = vT^{-1}|_S : S \to E$. Then by (4.6) we have $D'(\tilde{u}) \leq (1 + \delta)D'(vT^{-1}|_S) \leq (1 + \delta)D'(v)D'(T)$.

The next definition is a simple modification of a notion considered by Oikhberg in [31] to extend the Gurarii space from the Banach space framework to the operator space one.

**Definition 4.7.** We will say that an operator space $X$ has the $E$-Gurarii property if

for any $\varepsilon > 0$ and for any pair of spaces $S \subset L$ with $L \in E$ the following holds:

for any injective linear map $u : S \to X$ there is an injective map $\tilde{u} : L \to X$ extending $u$ such that

$$D(\tilde{u}) \leq (1 + \varepsilon)D(u).$$

Consequently, if $S \subset L$ with $L \in E$, for any $\varepsilon, \delta > 0$ any $\varepsilon$-embedding $u : S \to X$ extends to an $(\varepsilon + \delta)$-embedding $\tilde{u} : L \to X$.

**Definition 4.8.** We will say that an o.s. $X$ is $E$-injective if for any $\varepsilon > 0$, any $E \in E$ and any $S \subset E$, any map $u : S \to X$ admits an extension $\tilde{u} : E \to X$ with $\|\tilde{u}\|_{cb} \leq (1 + \varepsilon)\|u\|_{cb}$.

**Remark 4.9.** For instance when $E$ is the class (denoted below by $E_{SC}$) of all the f.d. subspaces of $C^*(\mathbb{F}_\infty)$, then any WEP o.s. $X$ is $E$-injective (see [38, Remark 21.5]).

**Lemma 4.10.** For an operator space $X$, the $E$-Gurarii property implies $E$-injectivity.

Proof. By Remark 4.5 the extension property in Definition 4.8 is satisfied whenever the map $u : S \to X$ is injective. But since dim$(S) < \infty$ the set of injective maps is dense in the space $CB(S, X)$. By the open mapping theorem (applied to the map $u \mapsto \tilde{u}|_S$) this shows that $X$ is $E$-injective.

**Remark 4.11.** Let $SE$ be the collection formed of all the o.s. that are (completely isometrically isomorphic to) subspaces of spaces in $E$. It is obvious that $E$ (resp. loosely) Gurarii implies $SE$ (resp. loosely) Gurarii. Moreover the $E$-Gurarii property implies (and hence is equivalent to) the $SE$-Gurarii property by an immediate restriction argument.

**Remark 4.12.** Let $\overline{E}$ be the closure of $E$ for the $d_{cb}$-distance (i.e. $E \in \overline{E}$ if and only if there is a sequence $(E_n)$ in $E$ such that $d_{cb}(E, E_n) \to 1$). Then the $\overline{E}$-Gurarii property and the $E$-Gurarii property are obviously equivalent. By the preceding remark they are also equivalent to the $SE$-Gurarii property.

**Remark 4.13.** For instance when $E = \{\ell^\infty_n\}$ as in Remark 4.23 the class $\overline{SE}$ is equal to the class of all f.d. minimal o.s. In the Banach space analogue this is the class of all f.d. spaces. When $E = \{M_n\}$ the class $\overline{SE}$ is equal to the class of all 1-exact o.s. When $E = \{R_n\}$ (row matrices) or $E = \{C_n\}$ (column matrices) or when $E$ is the collection of all the f.d. subspaces of a 1-homogeneous 1-Hilbertian o.s. in the sense of [37, p.172], then $E = \overline{SE}$ (exercise).
**Definition 4.14.** We will say that an o.s. $Y$ locally embeds in $\mathcal{E}$ if $F \in \overline{SE} \mathcal{E}$ for any f.d. $F \subset Y$. In other words for any f.d. $F \subset Y$ and any $\varepsilon > 0$ and there is $F_1 \in \mathcal{E}$ and a subspace $\tilde{F} \subset F_1$ such that $d_{cb}(F, \tilde{F}) \leq 1 + \varepsilon$.

**Remark 4.15.** Note that, by perturbation, a separable o.s. $Y$ locally embeds in $\mathcal{E}$ if and only if $Y$ is the closure of an increasing union of f.d. subspaces $Y_n$ such that $d_{cb}(Y_n, \mathcal{E}) \to 1$, and in that case actually $d_{cb}(Y_n, \mathcal{E}) = 1$ for all $n$. In particular, when $Y = \bigcup Y_n$, if all $Y_n$’s locally embed in $\mathcal{E}$ then so does $Y$.

**Remark 4.16.** Let $Y$ be as in Remark 4.15. Then $Y$ is completely isometric to an inductive limit of a system $(E_n, \varphi_n)$ as in Remark 4.12 with $E_n \in \mathcal{SE}$ for all $n$. Indeed, let $(Y_n)$ be as in Remark 4.15. Let $\delta_n > 0$ be such that $\sum \delta_n < \infty$. For each $n$ there is $E_n \in \mathcal{SE}$ and a $\delta_n$-isomorphism $w_n : Y_n \to E_n$. Let $\varphi_n : E_n \to E_{n+1}$ be defined by $\varphi_n = w_{n+1}w_n^{-1}$, and let $X$ be the inductive limit in the sense of Remark 4.12. Then the sequence $(w_n)$ defines a completely isometric embedding $Y \to X$.

Conversely, if $d_{\mathcal{SE}}(E_n) \to 1$ then any inductive limit $X$ as in Remark 4.12 locally embeds in $\mathcal{E}$.

**Definition 4.17.** We will say that an operator space $X$ is an $\mathcal{E}$-Gurarii space if it is separable, has the $\mathcal{E}$-Gurarii property and locally embeds in $\mathcal{E}$.

**Remark 4.18 (On the non-separable case).** Let $E$ be an arbitrary o.s. We may apply Lemma 6.3 to the case when $I$ is equal to the collection formed of all the maps $u : S \to E$ where $L$ runs over the set of all possible f.d. subspaces of $B(\ell_2)$ and $S \subset L$ runs over all possible subspaces of $L$. Let us denote the latter collection by $I(E)$. This gives us $E_1 \supset E$. Applying the same to $E_1$ and $I(E_1)$, we obtain $E_2 \supset E_1$ and so on. Let $X = \bigcup E_n$. It is easy to see that $X$ has the $\mathcal{E}$-Gurarii property with respect to the class $\mathcal{E}$ of all f.d.o.s. A priori this looks like a good non-commutative analogue of the classical Gurarii space. Note however that such a space cannot be separable since $OS_{fd}$ is not separable by [17].

Although many arguments below make sense in the non-separable case (like the preceding one), we restrict to the separable case, which seems to be the main case of interest for us.

**Remark 4.19.** Let $X$ be an $\mathcal{E}$-Gurarii space and assume that $Y$ locally embeds in $\mathcal{E}$. Let $E \subset F \subset Y$ and let $u : E \to X$ be an injective map. We observe that for any $\delta > 0$ there is an injective map $\tilde{u} : F \to X$ extending $u$ such that $D'(\tilde{u}) \leq (1 + \delta)D'(u)$ holds. Indeed, since $F \in \overline{SE}$ this follows from Remarks 4.11 and 4.12.

It is rather easy to see that any $\mathcal{E}$-Gurarii space $X$ is universal-but only up to $\varepsilon$-for the class of spaces that locally embed in $\mathcal{E}$, meaning that if a separable o.s. $Y$ locally embeds in $\mathcal{E}$ then $Y$ embeds (globally) completely $(1 + \varepsilon)$-isometrically into $X$ for any $\varepsilon > 0$. This is easy to deduce from Remark 4.19. In Theorem 4.22 below we show that this even holds for $\varepsilon = 0$ when $\mathcal{E}$ is a league. For this we will use the two lemmas that follow.

**Lemma 4.20.** Let $\mathcal{E}$ be a loosely Gurarii class. Let $E, L_0, \ldots, L_n \in \mathcal{E}$ ($n \geq 0$). For any $\delta > 0$ and $\delta' > 0$ there is $E_1 \in \mathcal{E}$ and a $\delta'$-embedding $\varphi : E \to E_1$ such that for any $L \in \{L_0, \ldots, L_n\}$, any $S \subset L$ and any $u : E \to S$ there is $\tilde{u} : L \to E_1$ such that $\tilde{u}|_S = \varphi u$ and $D(\tilde{u}) \leq (1 + \delta)D(u)$.

If $\mathcal{E}$ is Gurarii we can achieve this with a completely isometric $\varphi : E \to E_1$ (i.e. with $\delta' = 0$).

**Proof.** Assume first that the set $\{L_0, \ldots, L_n\}$ is reduced to a single $L \in \mathcal{E}$. The claim in this lemma is clear for a single $S$ and a single $u : S \to E$, and hence by iteration (as in Remark 6.4) for any finite set of $(S, u)$’s. A compactness argument will show that this suffices. Indeed, by perturbation (see Remark 6.11), it suffices to show the claim for a finite set of subspaces $S \subset L$ generated say by points in a $\delta''$-net in the unit ball of $L$ for $\delta'' > 0$ chosen suitably small. By homogeneity we
may restrict to \( u \) such that \( \|u\|_{cb} = 1 \). Furthermore, if the claim holds for all \( u \) in a finite \( \delta'' \)-net in the unit sphere of \( CB(S, E) \) with \( \delta'' > 0 \) small enough, we get the same result for all \( u : S \to E \). We are thus reduced to checking the claim for a finite set of \( (S, u) \)'s for some \( \delta'' \) chosen suitably small. This completes the proof for a single \( L \), say for \( L = L_0 \). But actually the discretization we just used can be used identically for each space in the finite set \( \{L_0, \cdots, L_n\} \), so the claim follows as stated in the lemma. An alternative for the final argument: applying the same result with \( E \) replaced by \( E_1 \) and \( L_0 \) replaced by \( L_1 \), we obtain the claim in the lemma for \( L \in \{L_0, L_1\} \), and iterating further we obtain it for any \( L \in \{L_0, \cdots, L_n\} \).

Lemma 4.21. Let \( E, L_0, \cdots, L_n \) be as in Lemma 4.20. Assume that we are given f.d. spaces \( Y_0, Y_1 \in SE \), an embedding \( T_0 : Y_0 \to Y_1 \), and an embedding \( v_0 : Y_0 \to E \). Then there is \( \hat{E}_1 \in E \), an embedding \( \varphi_0 : E \to \hat{E}_1 \) with \( D'(\varphi_0) \leq (1 + \delta')^3 D'(T_0) D'(v_0) \) and a \( \delta' \)-embedding \( \hat{v}_1 : Y_1 \to \hat{E}_1 \) such that \( v_1T_0 = \varphi_0 v_0 \). In addition, for any \( L \in \{L_0, \cdots, L_n\} \), any \( S \subset L \) and any \( u : S \to E \) there is \( \hat{u} : L \to \hat{E}_1 \) such that \( \hat{u}|_S = \varphi_0 u \) and \( D(\hat{u}) \leq (1 + \delta) D(u) \frac{(1 + \delta')^2 D'(T_0) D'(v_0)}{2} \).

Proof. Having obtained \( E_1 \) as in Lemma 4.20 we apply the same lemma again but with \( E \) replaced by \( Y_0 \) and with the augmented family \( \{L_0, \cdots, L_n\} \cup \{E_1\} \). Moreover, since \( \delta > 0 \) is arbitrarily small we may take \( \delta = \delta' \) and \( \delta' \) as before for this new application. This gives us a space \( \hat{E}_1 \) and a \( \delta' \)-embedding \( \hat{\psi} : Y_1 \to \hat{E}_1 \) such that for any \( L \in \{L_0, \cdots, L_n\} \cup \{E_1\} \), any \( S \subset L \) and any \( u : S \to Y_1 \) there is \( \hat{u} : L \to \hat{E}_1 \) such that \( \hat{u}|_S = \psi u \) and \( D(\hat{u}) \leq (1 + \delta') D(u) \), which by 129 implies \( D'(\hat{u}) \leq (1 + \delta') D'(u) \). In particular, applying this to \( u = T_0 \), \( S = Y_0 \), \( L = E_1 \) with the embedding \( \varphi v_0 : Y_0 \to E_1 \) and recalling Lemma 4.6 there is \( \tilde{T}_0 : E_1 \to \tilde{E}_1 \) such that \( \tilde{T}_0 \varphi v_0 = \psi T_0 \) and

\[
D'(\tilde{T}_0) \leq (1 + \delta') D'(T_0) D'(\varphi v_0) \leq (1 + \delta') D'(T_0) D'(\varphi) D'(v_0) \leq (1 + \delta')^2 D'(T_0) D'(v_0).
\]

We now set \( \varphi_0 = \tilde{T}_0 \varphi : E \to \hat{E}_1 \) and \( v_1 = \psi \). We have

\[
D'(\varphi_0) \leq D'(\tilde{T}_0) D'(\varphi) \leq (1 + \delta') D'(\tilde{T}_0) \leq (1 + \delta')^3 D'(T_0) D'(v_0).
\]

Also \( v_1T_0 = \varphi_0 v_0 \) and \( D'(v_1) = D'(\psi) \leq \delta' \). As for the last assertion, if \( \tilde{u} \) is as in Lemma 4.20 we set \( \hat{u} = \tilde{T}_0 \tilde{u} \), so that

\[
D(\hat{u}) \leq D(\tilde{u}) D(\tilde{T}_0) \leq (1 + \delta) D(u) D'(\tilde{T}_0)^2.
\]

This settles the last assertion.

The next result shows in particular (say when \( Y = \{0\} \)) that \( E \)-Gurarii spaces always exist whenever \( E \) is separable and loosely Gurarii. Our proof is similar to Oikhberg’s one for the “exact” analogue [31, Th. 1.1].

Theorem 4.22. Let \( E \) be a separable loosely Gurarii class. For any separable o.s. \( Y \) that locally embeds in \( E \) there is a separable \( E \)-Gurarii o.s. \( X \) containing \( Y \) completely isometrically. The space \( X \) can be written as \( X = \bigcup E_n \) for some increasing sequence \( (E_n) \) of f.d. subspaces of \( X \) such that \( d_{cb}(E_n, E) \to 1 \).

If \( E \) is a separable Gurarii class, there is an \( E \)-Gurarii o.s. of the form \( X = \bigcup E_n \) with \( E_n \in E \) for all \( n \) such that for any \( \varepsilon > 0 \) there is an \( \varepsilon \)-embedding of \( Y \) in \( X \). 

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Proof. By Remark \ref{4.16} we may view \( Y \) as the limit of \( \mathcal{L}(\{Y_n, T_n\}) \) of a sequence \( Y_n \in \mathcal{S}E \) with respect to \( \varepsilon_n \)-embeddings \( T_n : Y_n \to Y_{n+1} \), with \( \sum \varepsilon_n < \infty \). Let \( \delta'_n \) be decreasing and such that \( \sum \delta'_n < \infty \). Obviously there is a positive sequence \( (\delta_n) \) such that \( \sum \delta_n < \infty \) and
\[
(1 + \delta'_n)^5 (1 + \varepsilon_n)^2 (1 + \delta'_n)^2 \leq 1 + \delta_n \quad \forall n.
\]
Starting from \( E_0 \in \mathcal{E} \) chosen arbitrarily, we will construct \( X \) as a limit of a sequence \( E_n \in \mathcal{E} \) with respect to \( \delta'_n \)-embeddings \( \varphi_n : E_n \to E_{n+1} \). When \( \mathcal{E} \) is Gurarii and we drop the requirement on \( Y \) we can take \( \delta'_n = 0 \) so that \( \varphi_n \) is completely isometric. Let \( \mathcal{E'} = \{L_n\} \) be a dense sequence in \( \mathcal{E} \). Assume first that \( \mathcal{E} \) is Gurarii. We will construct \( (E_n) \) by induction so that for any \( n \geq 0 \) we have \( E_n \in \mathcal{E} \) and a complete isometry \( \varphi_n : E_n \to E_{n+1} \) satisfying the following condition:

\((\ast)_n\) For any \( L \in \{L_0, \cdots, L_n\} \), any \( S \subset L \) and any \( u : S \to E_n \) there is \( \tilde{u} : L \to E_{n+1} \) such that \( \tilde{u}_S = \varphi_n u \) and \( D(\tilde{u}) \leq (1 + \delta_n)D(u) \).

The verification of the induction step is a consequence of Lemma \ref{4.20}. Let then \( X = \mathcal{L}(\{E_n, \varphi_n\}) = \bigcup j(E_n) \) as in Remark \ref{0.2}. Clearly, \( j |_{E_n} : E_n \to j(E_n) \) is completely isometric, so that the inclusion \( j(E_n) \subset j(E_{n+1}) \) has the same property as \( \varphi_n \) in the preceding diagram. To check the Gurarii property, we must consider \( L \in \mathcal{E}, \varepsilon > 0 \) and \( u : S \to X \) as in \((\ast)_n\). By the density of \( \mathcal{E'} \) it suffices to consider \( L \in \mathcal{E'} \). By the density of \( \bigcup j(E_n) \) and perturbation we may assume that \( u(S) \subset j(E_m) \) for some \( m \) such that \( \delta_m < \varepsilon \). A fortiori \( u(S) \subset j(E_n) \) for all \( n \geq m \), and hence we may assume that \( L \in \{L_0, \cdots, L_n\} \) and \( \delta_n < \varepsilon \). Then the preceding property of the inclusion \( j(E_n) \subset j(E_{n+1}) \) gives us the desired Gurarii property. At this point in the proof, the Gurarii property of the space \( X \) and an easy induction argument shows that for any \( \varepsilon > 0 \) there is an \( \varepsilon \)-embedding of \( Y \) in \( X \). But to produce an \( X \) for which this holds for \( \varepsilon = 0 \) some more care is needed. Curiously, the argument below seems to require to modify \( X \) (although by Theorem \ref{4.29} in many cases we have uniqueness). We will need to allow \( \varphi_n \)’s that are \( \delta_n \)-isometric instead of c.i.

Now if \( \mathcal{E} \) is only loosely Gurarii, we claim that there is a sequence \( (E_n) \in \mathcal{E} \) and \( \delta_n \)-embeddings \( \varphi_n : E_n \to E_{n+1} \) such that the pair \( (E_n, \varphi_n) \), satisfies \((\ast)_n \) for all \( n \geq 0 \), together with embeddings \( v_n : Y_n \to E_n \) such that \( D'(v_n) \leq 1 + \delta'_n \) and such that \( v_{n+1}T_n = \varphi_nv_n \) for all \( n \geq 0 \). Now we set \( X = \mathcal{L}(\{E_n, \varphi_n\}) = \bigcup j(E_n) \) as in Remark \ref{0.2}. Now \( j |_{E_n} : E_n \to j(E_n) \) is no longer c.i. but we still have \( D(j |_{E_n}) \to 1 \) so that the same argument leads to the Gurarii property for \( X \). Moreover, since \( D'(v_n) \to 1 \), the sequence \( (v_n) \) defines a c.i. embedding of \( Y = \mathcal{L}(\{Y_n, T_n\}) \) into \( X \).

To prove our claim, we use induction again. Since \( Y_0 \in \mathcal{S}E \) we find \( E_0 \in \mathcal{E} \) and \( v_0 : Y_0 \to E_0 \) with \( D'(v_0) \leq 1 + \delta'_0 \). Now let \( n \geq 0 \), assume that we have found \( E_k, \varphi_{k-1}, v_k \) satisfying the required bounds and \( (\ast)_{k-1} \) for all \( k < n \). We must produce \( E_{n+1}, \varphi_n, v_{n+1} \) satisfying similar bounds and \( (\ast)_n \). In particular we need \( D'(\varphi_n) \leq 1 + \delta_n, D'(v_{n+1}) \leq 1 + \delta'_n+1 \) and \( v_{n+1}T_n = \varphi_n v_n \). When \( n = 0 \) we assume given only \( E_0, v_0 \), the argument that follows will produce \( E_1, \varphi_0, v_1 \). To produce the case \( k = n + 1 \), we first invoke Lemma \ref{4.21} with \( \delta, \delta' > 0 \) to be specified : taking \( E = E_n \), this gives us a space \( \tilde{E}_1 \in \mathcal{E} \) and an embedding \( \tilde{\varphi}_n : E_n \to \tilde{E}_1 \) such that for any \( L \in \{L_0, \cdots, L_n\} \), any \( S \subset L \) and any \( u : S \to E \) there is \( \tilde{u} : L \to \tilde{E}_1 \) such that
\[
\tilde{u}_S = \varphi_n u \text{ and } D(\tilde{u}) \leq (1 + \delta)D(u)[(1 + \delta')^2 D'(T_n)D'(v_n)]^2.
\]
We also have \( D'(\varphi_n) \leq (1 + \delta')^3 D'(T_n)D'(v_n) \). Moreover, we have an embedding \( v_{n+1} : Y_{n+1} \to \tilde{E}_1 \) such that \( v_{n+1}T_n = \varphi_n v_n \). We have \( D'(v_{n+1}) \leq 1 + \delta' \).

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It remains to check $(*)_n$. From (4.12) one finds the constant in $(*)_n$ majorized by

$$(1 + \delta)((1 + \delta')^2D'(T_n)D'(v_n))^2.$$ 

To conclude, we choose $\delta = \delta' = \delta'_{n+1}$. This ensures $D'(v_{n+1}) \leq 1 + \delta'_{n+1}$. Then by (4.11) we have

$$(1 + \delta)((1 + \delta')^2D'(T_n)D'(v_n))^2 \leq (1 + \delta'_{n+1})^3(1 + \delta_n)^2 \leq 1 + \delta_n$$

which yields the desired constant in $(*)_n$. Moreover,

$$D'(\varphi_n) \leq (1 + \delta')^3D'(T_n)D'(v_n) \leq (1 + \delta'_{n+1})^3(1 + \varepsilon_n)(1 + \delta_n) \leq 1 + \delta_n.$$

This completes the induction step and the proof.

\[\square\]

\textbf{Remark 4.23} (Basic example 1). Consider the class $\mathcal{E} = \{\ell^p_n \mid n \geq 1\}$. We claim that it is Gurarii. To check this let $u : S \to E$ and $L$ with $L,E \in \mathcal{E}$. By the Banach space version described at the beginning of Lemma 3.1 we know that the class of all f.d. spaces is tightly Gurarii, so there is a f.d. o.s. $D_1$, an isometric embedding $E \subset D_1$ and $\hat{u} : L \to D_1$ extending $u$ and such that $D(\hat{u}) = D(u)$. For any $\delta > 0$ there is $G \in \mathcal{E}$ and an embedding $v : D_1 \to G$ such that $\|v\| \leq 1$ and $\|v^{-1}\|_{v(D_1)} \leq 1 + \delta$. Since $E$ is injective, there is a projection $P : D_1 \to E$ with $\|P\| = 1$. Let $E_1 = G \oplus_\infty E$, let $w : D_1 \to E_1$ be defined by $w(x) = v(x) \oplus P(x)$, let $\tilde{u} = w\hat{u}$ and $j = w|_E$. Then $j$ is isometric and $\tilde{u}|_S = wu = ju$. Moreover, we have $\|\tilde{u}^{-1}_{|\tilde{u}(L)}\| \leq \|v^{-1}\|\|\tilde{u}^{-1}_{|\tilde{u}(L)}\| \leq (1 + \delta)\|\tilde{u}^{-1}_{|\tilde{u}(L)}\|$. Thus, we have

$$D(\tilde{u}) \leq (1 + \delta)D(u) = (1 + \delta)D(u).$$

Clearly $E_1 = G \oplus_\infty E \in \mathcal{E}$. This gives us (4.17) with ordinary norms in place of the c.b. ones. But if we equip $L, E_1$ and $E$ with their minimal o.s. structure we obtain (4.17) with c.b. norms and $j$ is completely isometric, so that indeed $\mathcal{E}$ is Gurarii.

\textbf{Remark 4.24}. When $\mathcal{E} = \{\ell^p_n \mid n \geq 1\}$ the space $X$ given by Theorem 4.22 is clearly a minimal o.s. (i.e. an operator subspace of a commutative C*-algebra) and the underlying Banach space possesses the characteristic property of the classical Gurarii space $G$. By the uniqueness of the latter (that we will reprove in §3) $X \cong G$ isometrically and hence $X \cong MIN(G)$ completely isometrically where (following the notation in §5) $MIN(G)$ denotes $G$ equipped with its minimal o.s. structure. Note that by Theorem 4.22 we have $G = \overline{\bigcup_n E_n}$ for subspaces $\cdots E_n \subset E_{n+1} \subset \cdots$ such that $E_n \cong \ell^p_{k(n)}$ isometrically. The latter fact is a well known property of $G$, see e.g. [24] for more on this theme.

\textbf{Remark 4.25} (Basic example 2). Similarly the class $\mathcal{E} = \{M_n \mid n \geq 1\}$ is Gurarii. Since $M_n$ is injective the same argument as in Remark 4.23 works. The space $X$ given by Theorem 4.22 is then completely isometric to Oikhberg’s exact Gurarii space (but our argument to show that $\{M_n \mid n \geq 1\}$ is Gurarii seems simpler). See [30] for more details.

\textbf{Remark 4.26} (An easy exercise). Let $\mathcal{E} = \{R_n\}$ (resp. $\mathcal{E} = \{C_n\}$) or let $\mathcal{E}$ be the collection of all the f.d. subspaces of a 1-homogeneous 1-Hilbertian o.s. $\mathcal{H}$ in the sense of [37] p.172]. Then $\mathcal{E}$ is tightly Gurarii and the space given by Theorem 4.22 is $X = R$ (resp. $X = C$) or $X = \mathcal{H}$.

In practice, the notion of Gurarii class seems too “abstract”. It will be convenient to work with classes $\mathcal{E}$ satisfying certain basic stability properties, that we call “leagues”:

\textbf{Definition 4.27}. A class $\mathcal{E}$ will be called a league if it is stable under $\oplus_1$-direct sums and quotients. Assuming $\{0\} \in \mathcal{E}$, this means that for any $E_1, E_2 \in \mathcal{E}$ and for any subspace $N \subset E_1 \oplus_1 E_2$ we have $(E_1 \oplus_1 E_2)/N \in \mathcal{E}$. (We also recall that by convention any space that is completely isometric to one in $\mathcal{E}$ is also in $\mathcal{E}$.)
Proposition 4.28. If $\mathcal{E}$ is a league then $\mathcal{E}$ is tightly Gurarii.

Proof. This follows from Lemma 3.6.

Our next observation follows well known ideas going back to the Banach version of the Gurarii space: by a result due to Lusky [29] (improving on Gurarii’s initial $(1 + \epsilon)$-isometrical uniqueness theorem) the latter space is unique up to isometric isomorphism. Our next statement says that the $\mathcal{E}$-Gurarii space—if it exists—is unique up to isometric isomorphism. In [32], Oikhberg proved the analogue of Gurarii’s uniqueness result in his framework involving exact operator spaces, and in [26] Lupini proved the analogue of Lusky’s result. Imitating the argument of Kubiś and Solecki from [23] we will prove in the next section the following uniqueness:

Theorem 4.29. Assume that $\mathcal{E}$ is a separable league. Let $X,Y$ be $\mathcal{E}$-Gurarii spaces. Then $X \simeq Y$ completely isometrically.

Proof. This follows from Corollary 5.8 and Lemma 5.3.

Remark 4.30 (Notation $G_{\mathcal{E}}$). By Theorem 4.22 with any $Y \in \mathcal{E}$ (or even with $Y = \{0\}$) there is an $\mathcal{E}$-Gurarii o.s. and it is unique by Theorem 4.29. We will denote it by $G_{\mathcal{E}}$.

Remark 4.31 (Some examples of leagues). Let us denote by $Q\ell_1^n$ (resp. $QS_1^n$) the class of o.s. quotients of $\ell_1^n$ (resp. $S_1^n$). By Proposition 4.28 the following 3 natural classes satisfy the assumptions of Theorem 4.22:

(i) $\mathcal{E}_{\text{max}} = \bigcup_{n \geq 1} Q\ell_1^n$.
(ii) $\mathcal{E}_1 = \bigcup_{n \geq 1} QS_1^n$.
(iii) $\mathcal{E}_{S\mathcal{E}} = \{E \text{ f.d.o.s.} \mid d_{S\mathcal{E}}(E) = 1\}$, where $d_{S\mathcal{E}}$ is as in (4.2) and where

$$\mathcal{C} = C^*(\mathbb{F}_\infty).$$

Note that any f.d. $E$ with $d_{S\mathcal{E}}(E) = 1$ embeds completely isometrically in $\mathcal{C}$ (see [37, p. 352] or [38, Remark 20.8]), so $\mathcal{E}_{S\mathcal{E}}$ is simply the class of f.d. subspaces of $\mathcal{C}$.

That (i) and (ii) are leagues is obvious.

In cases (i) and (ii) the space $G_{\mathcal{E}}$ has the strong OLLP by Corollary 2.12.

For (iii) the stability under quotient and dual is proved in [17]. It is an immediate consequence of the following lemma. Indeed, by duality, since the dual of $E \oplus_1 F$ is $E^* \oplus_\infty F^*$, (4.13) implies:

$$\forall E,F \text{ f.d.o.s.} \quad d_{S\mathcal{E}}(E \oplus_1 F) = \max\{d_{S\mathcal{E}}(E),d_{S\mathcal{E}}(F)\},$$

and also for any $N \subset E$ (since $(E/N)^* \subset E^*$):

$$d_{S\mathcal{E}}(E/N) \leq d_{S\mathcal{E}}(E).$$

Lemma 4.32 ([17]). Let $E \subset B(\ell_2)$ be a f.d.o.s. and let $E^* \subset B(\ell_2)$ be any completely isometric embedding of the dual o.s. Then

$$d_{S\mathcal{E}}(E) = \sup\{|t|_{B(\ell_2) \otimes_{\max} B(\ell_2)} \mid t \in E^* \otimes B(\ell_2), \|t\|_{B(\ell_2) \otimes_{\min} B(\ell_2)} = 1\}.\quad (4.13)$$

Remark 4.33. By Theorem 4.22 $\ell_1$ (resp. $S_1$) embeds completely isometrically in $G_{\mathcal{E}}$ when $\mathcal{E} = \mathcal{E}_{\text{max}}$ (resp. when $\mathcal{E} = \mathcal{E}_1$). Since $\mathcal{C}$ (trivially) locally embeds in $\mathcal{E}_{S\mathcal{E}}$, it embeds completely isometrically in $G_{\mathcal{E}}$ when $\mathcal{E} = \mathcal{E}_{S\mathcal{E}}$. Moreover, we have completely isometric embeddings

$$G_{\mathcal{E}_{\text{max}}} \subset G_{\mathcal{E}_1} \subset G_{\mathcal{E}_{S\mathcal{E}}}.$$
These are strict inclusions since the three corresponding classes are distinct. Indeed, for instance the column space $C_n$ is in $\mathcal{E}_1 \setminus \overline{\mathcal{E}^{\text{max}}_1}$ (see [37, Th. 10.5, p. 222]) for all $n > 1$; moreover $M_n$ and $\ell_1^n$ are in $\mathcal{E}_{\mathcal{S}E} \setminus \overline{\mathcal{E}^{\text{max}}_1}$ for all $n > 2$. The latter because if an injective space $E$ is in $\overline{\mathcal{E}^{\text{max}}_1}$ its o.s. dual $E^*$ must be 1-exact, and this fails when $n > 2$ by [37, Th. 21.5, p. 336]. Since $M_n$ and $\ell_1^n$ have the LP (they are nuclear!) they are in $\mathcal{E}_{\mathcal{S}E}$.

The class $\mathcal{E}_{\text{max}}$ is actually the smallest league containing $\mathbb{C}$. We chose to denote it by $\mathcal{E}_{\text{max}}$ because the $d_{\text{ap}}$-closure of $\mathcal{E}_{\text{max}}$ is the class of f.d. maximal o.s. meaning f.d. spaces equipped with their maximal o.s. structures in the sense of Blecher-Paulsen in [5] (see also e.g. [37, §3]), that can be defined as follows: an o.s. $X$ is maximal if and only if $\|u\|_{cb} = \|u\|$ for all $u : X \to B(H)$ and all $H$. The class of f.d. maximal o.s. reappears below in Remarks 4.39 and 6.2 as the class $\mathcal{E}_{\text{max}}^{[1]}$, among more general variations.

We say that $X$ is strongly maximal if, for any $\varepsilon > 0$, any f.d. $E \subset X$ is contained in a larger f.d. $F \subset X$ such that $\forall H, \forall u : F \to B(H) \quad \|u\|_{cb} \leq (1 + \varepsilon)\|u\|$. When $X$ is separable this means there is an increasing sequence of f.d. subspaces $X_n \subset X$ with dense union such that for some $\varepsilon_n \to 0$ we have

$$\forall H, \forall u : X_n \to B(H) \quad \|u\|_{cb} \leq (1 + \varepsilon_n)\|u\|.$$

Remark 4.34 (Some important questions). Clearly, strongly maximal implies maximal. The converse is apparent even (open if we only require for $(\varepsilon_n)$ to be a bounded sequence). We conjecture that the converse fails for lack of a suitable approximation property in $X$. Actually it is unclear whether any maximal space locally embeds in $\mathcal{E}_{\text{max}}$ (this was raised already in [30]). Even whether it locally embeds in $\mathcal{E}_{\mathcal{S}E}$ is unclear to us. In fact we do not know if the class of f.d. subspaces of maximal o.s. (equivalently the class of all f.d. subspaces of $B(\ell_2)$ equipped with its maximal o.s. structure) is separable. A negative answer would be a very significant strengthening of the non separability of $OS_{fd}$ proved in [17]. The failure of the approximation property for $B(\ell_2)$ (see [41]) seems to play a role here behind the scene. These queries are related to some of Ozawa’s questions in §3 (see also §6).

Corollary 4.35. There is a separable o.s. $\mathcal{X}_{\text{max}} \subset B(H)$ that is $\mathcal{E}_{\text{max}}$-Gurarii, maximal and even strongly maximal, with the strong OLLP and for which there is a (normal) projection $P : B(H)^* \to \mathcal{X}_{\text{max}}^{**}$ with $\|P\| = 1$.

Proof. By Proposition 4.28 we may apply Theorem 4.22 to obtain our $\mathcal{E}_{\text{max}}$-Gurarii space. Let $\mathcal{X}_{\text{max}} = \mathcal{G}_E$ for $E = \mathcal{E}_{\text{max}}$. Since $\mathcal{X}_{\text{max}} = \cup E_n$ with $E_n$ f.d. and maximal strong maximality follows. The last assertion follows from (iii)’ in Proposition 1.2.

Remark 4.36. One might be tempted to guess that the classical Gurarii space (which is a $\mathcal{L}_\infty$-space, see Remark 4.21) equipped with its maximal o.s. structure could have the properties in Corollary 4.35 but it is not so. Indeed, by Theorem 4.22 when $E = \mathcal{E}_{\text{max}}$ the space $\mathcal{G}_E$ contains $\ell_1$ completely isometrically. But if a maximal o.s. $X$ contains $\ell_1$ completely isomorphically, then $X$ has quotients uniformly isomorphic (in the Banach space sense) to $M_n$ $(n \geq 1)$, and hence cannot be isomorphic to a $\mathcal{L}_\infty$-space. Indeed, any metric surjection $\ell_1 \to M_n$ is a fortiori a complete contraction, and by the injectivity of $M_n$ it extends to a (complete) contraction from $X$ onto $M_n$, which is also a metric surjection. The fact that the spaces $\{M_n \mid n \geq 1\}$ are not uniformly isomorphic to quotients of $\mathcal{L}_\infty$-spaces, or equivalently that $\{M_n^* \mid n \geq 1\}$ does not uniformly embed in $\mathcal{L}_1$-spaces goes back to Gordon and Lewis [13] (see also [34]). The same argument shows that $\mathcal{G}_{\mathcal{E}_{\text{max}}}$ cannot have local unconditional structure in the sense of [13].

Remark 4.37. Friedman and Russo [9] (see also [10]) proved that the range of a contractive projection $P$ on a $C^*$-algebra is a Jordan triple system for the triple product $(a,b,c) \mapsto P(ab^*c + cb^*a)/2$.
that has a faithful representation as a \( J^* \)-algebra. A \( J^* \)-algebra is closed subspace of \( B(H,K) \) stable by the mapping \( x \mapsto xx^*x \). This applies to \( \mathcal{X}^{**} \).

Remark 4.38. Note that, by Ozawa’s Theorem 2.10 a f.d.o.s. \( E \) has the OLLP if and only if \( E \in \mathcal{E}_1 \). Since \( \ell^n_\infty \not\in \mathcal{E}_1 \) when \( n > 2 \) (see [37] p. 336) and \( \ell^n_\infty \) is injective, an o.s. with OLLP cannot contain \( \ell^n_\infty \) completely isometrically. This shows that no infinite dimensional \( C^* \)-algebra has the OLLP. In particular the \( \mathcal{E} \)-Gurarii spaces for \( \mathcal{E} \in \mathcal{E}_1 \) are not \( C^* \)-algebras.

Remark 4.39. Fix an integer \( N \geq 1 \). Let \( \mathcal{E}^{[N]}_{\text{max}} \) be the class formed of all f.d. quotients of o.s. of the form \( \ell_1(S^N_1) \). This is the class of f.d. \( M_N \)-maximal o.s. in the sense of Lehner [25]. See [30] for more details on \( M_N \)-spaces.

Then \( \mathcal{E}^{[N]}_{\text{max}} \subset \mathcal{E}^{[N]}_{\text{max}} \subset \mathcal{E}_1 \). Moreover it is easy to check that \( \mathcal{E}^{[N]}_1 = \bigcup_{N} \mathcal{E}^{[N]}_{\text{max}} \), and the latter class coincides with that of f.d.o.s. \( E \) such that \( E^* \) is 1-exact. This should be compared with Remark 6.2. The space \( \mathcal{G}_{\mathcal{E}^{[N]}_{\text{max}}} \subset B(H) \) has the OLLP and is such that there is a projection \( P: B(H)^{**} \to \mathcal{G}_{\mathcal{E}^{[N]}_{\text{max}}} \) with \( Id_{M_N} \otimes P \) contractive. The proof is similar to that of Proposition 1.3.

By Proposition 1.3 whenever \( \mathcal{E} \supset \mathcal{E}_1 \) the space \( \mathcal{G}_{\mathcal{E}} \) is an \( \mathcal{E} \)-Gurarii space with the WEP. In particular:

Corollary 4.40. There is a separable o.s. \( X_1 \subset B(H) \) with the strong OLLP that also has the WEP. In particular there is a (normal) projection \( P: B(H)^{**} \to X_{1}^{**} \) with \( \|P\|_{cb} = 1 \).

Proof. By Proposition 4.28 and Theorem 4.22 we may just set \( X_1 = \mathcal{G}_{\mathcal{E}_1} \). \( \square \)

Remark 4.41. By a result due to Youngson [12] the range of a completely contractive projection \( P \) on a \( C^* \)-algebra \( B \) is (completely isometric to) a triple subsystem of another \( C^* \)-algebra \( C \) with triple products given respectively by \( [a,b,c] = P(ab^*c) \ (a,b,c \in P(B)) \) and \( \{x,y,z\} = xy^*z \ (x,y,z \in C) \). Moreover, for any \( x,y,z \in B \) we have

\[
P(P(x)P(y)^*P(z)) = P(xP(y)^*P(z)) = P(P(x)y^*P(z)) = P(P(x)P(y)^*z).
\]

This implies that \( P(B) \) is completely isometric to a ternary ring of operator (TRO). By definition, a TRO is a subspace of \( B(H,K) \) (\( H,K \) Hilbert spaces) (or simply of a \( C^* \)-algebra) that is stable under the triple product \( (x,y,z) \mapsto xy^*z \). Kirchberg in [21] uses the term \( C^* \)-triple system. By [21] Prop. 4.2 (see also [18]) a TRO can be identified (as a TRO and completely isometrically) with \( pA(1-p) \) where \( A \) is its “linking” \( C^* \)-algebra for some projection \( p \in A \). In the weak* closed case \( A \) is a von Neumann algebra. This applies in particular to the space \( X_{1}^{**} \) in Corollary 4.40. Perhaps the underlying linking von Neumann algebra deserves more investigation. We refer the reader to [4] for more results on triple systems and TROs.

## 5 Uniqueness of \( \mathcal{E} \)-Gurarii spaces

Recall that a linear map \( f: E \to F \) between o.s. is called an \( \varepsilon \)-embedding if it is injective and

\[
\max\{\|f\|_{cb}, \|f^{-1}\|_{f(E)}\|_{cb}\} \leq 1 + \varepsilon.
\]

To prove the uniqueness up to complete isometry of Theorem 4.29 we will use the same idea as in [23].

### Definition 5.1

We will say that a Gurarii class \( \mathcal{E} \) is perturbative if there is a function \( \delta: (0,\infty) \to (0,1) \) with \( \lim_{\varepsilon \to 0} \delta(\varepsilon) = 0 \) such that for any \( \varepsilon > 0 \) the following holds: for any \( L \in \mathcal{E}, S \subset L, E \in \mathcal{E} \) and any \( \varepsilon \)-isometric \( u: S \to E \), there is for any \( \delta' > 0 \) a space \( \mathcal{E}_1 \in \mathcal{E} \), and \( \delta' \)-isometric maps \( \varphi: E \to \mathcal{E}_1 \) and \( \widetilde{u}: L \to \mathcal{E}_1 \) such that \( \|\widetilde{u}|_{S} - \varphi u\|_{cb} \leq \delta(\varepsilon) \).

\[\]
Remark 5.2. Again it is easy to check that if $\mathcal{E}$ is perturbative then so are $\mathcal{SE}$ and $\overline{\mathcal{SE}}$.

This property expresses roughly that if the range is suitably enlarged (within $\mathcal{E}$) any $\varepsilon$-isometric $u : S \to \overline{E}$ is the restriction of a $\delta(\varepsilon)$-perturbation of a map (namely $\widetilde{u}$) that is almost isometric. Since $\widetilde{u}$ is “more isometric” than $u$, of course there has to be a compensation and $\delta(\varepsilon)$ has to be essentially larger than $\varepsilon$, but still if $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$, as we will see, this leads to some strong consequences.

Lemma 5.3. If $\mathcal{E}$ is a league then for all $\varepsilon > 0$, all $E, F \in \mathcal{E}$ and all $\varepsilon$-isometric $f : E \to F$ there is $Z \in \mathcal{E}$ and completely isometric maps $i : E \to Z$ and $j : F \to Z$ such that

$$\|jf - i\|_{cb} \leq \varepsilon.$$  

Proof. The proof is similar to the one given by Lupini in [26, Lemma 3.1]. Consider $Z = E \oplus F \oplus E$ with norm

$$\|(x, y, z)\| = \|x\| + \|y\| + \varepsilon\|z\|.$$  

We will quotient this by

$$N = \{(-e, f(e), e) \mid e \in E\}.$$  

Let $q : Z \to Z/N$ denote the quotient map. We set $Z = Z/N$. Then we set $i(x) = q((x, 0, 0))$ ($x \in E$) and $j(y) = q((0, y, 0))$ ($y \in F$). Note

$$j(f(x)) - i(x) = q((-x, f(x), 0)) = q((-x, f(x), x)) + q((0, 0, -x)) = q((0, 0, -x))$$  

and hence

$$\|jf - i\| \leq \varepsilon.$$  

We have clearly $\|i\| \leq 1$ and $\|j\| \leq 1$. Moreover, for any $e \in E$

$$\|(x, 0, 0) + (-e, f(e), e)\| = \|x - e\| + \|f(e)\| + \varepsilon\|e\| \geq \|x\| - \|e\| + \|f(e)\| + \varepsilon\|e\|$$  

$$\geq \|x\| - \|e\| + \|e\|(1 + \varepsilon)^{-1} + \varepsilon\|e\| \geq \|x\|,$$

and hence $\|i(x)\| \geq \|x\|$. Similarly

$$\|(0, y, 0) + (-e, f(e), e)\| = \|-e\| + \|y + f(e)\| + \varepsilon\|e\| \geq \|e\| + \|y\| + \|f(e)\| + \varepsilon\|e\| \geq \|y\|,$$

and hence $\|j(y)\| \geq \|y\|$. This gives us isometries $i, j$ and Lemma 5.3 with the usual norms instead of cb-ones. To pass to cb-norms we use as earlier the identity $\|f\|_{cb} = \| IdS_1 \otimes f : S_1[E] \to S_1[F]\|$ valid for any $f : E \to F$. We equip $Z$ with the o.s.s. of the direct sum in the sense of $\ell_1$ (with the third factor weighted by $\varepsilon$) so that $S_1[Z] \simeq S_1[E] \oplus S_1[F] \oplus S_1[E]$ and for any $(x, y, z) \in S_1[E] \oplus S_1[F] \oplus S_1[E]$ we have

$$\|(x, y, z)\|_{S_1[Z]} = \|x\|_{S_1[E]} + \|y\|_{S_1[F]} + \varepsilon\|z\|_{S_1[E]}.$$  

Moreover,

$$S_1[N] \simeq \{(-e, (IdS_1 \otimes f)e, e) \mid e \in S_1[E]\}.$$  

Thus if we run the preceding argument with $f$ replaced by $IdS_1 \otimes f : S_1[E] \to S_1[F]$ and $Z$ replaced by $S_1[Z]$ we obtain the announced statement with cb-norms and with $\delta(\varepsilon) = \varepsilon$. □

Lemma 5.4. If $\mathcal{E}$ is a league then it is a perturbative Gurarii class.
Definition 5.1 with \( \delta \parallel f \parallel \) \( E \) \( \| \).

Proof. Let \( S \subset L \) and \( u : S \to E \) be as in Definition 5.1. We know by Proposition 4.28 and Remark 4.5 that there is \( E_1 \in \mathcal{E} \), a complete isometry \( k : E \to E_1 \) and an \( \varepsilon \)-isometry \( f : L \to E_1 \) such that \( f_{|S} = ku \). Applying the preceding lemma we find \( Z \in \mathcal{E} \) with \( i : L \to Z \) and \( j : E_1 \to Z \) such that \( \| jf - i \|_\infty \leq \varepsilon \). Let \( \hat{E}_1 = Z, \varphi = jk \) and \( \hat{u} = i \), so that \( \varphi u = jf_{|S} \). We obtain the property in Definition 5.1 with \( \delta' = 0 \) and \( \delta(\varepsilon) = \varepsilon \).

There is a quite different sort of perturbative Gurarii class that we describe next.

Lemma 5.5. Let \( \mathcal{E} \) be a class of f.d.o.s. Assume that

(i) \( \mathcal{E} \) is stable by \( \oplus_\infty \), meaning that for any \( L, E \in \mathcal{E} \) we have \( L \oplus_\infty E \in \mathcal{E} \).

(ii) Each \( E \in \mathcal{E} \) is injective (meaning that there is a completely isometric embedding \( E \subset B(H) \) and a projection \( P : B(H) \to E \) with \( \| P \|_\infty = 1 \)).

Then \( \mathcal{E} \) is a perturbative tightly Gurarii class.

Proof. Let us first show that \( \mathcal{E} \) is tightly Gurarii. Consider \( L, E \in \mathcal{E} \) and \( u : S \to E \) as in Definition 5.2. By Lemma 3.1 there is a f.d.o.s. \( E_1, \bar{u} : L \to E_1 \) and a complete isometry \( \varphi : E \to E_1 \) such that \( \bar{u}_{|S} = \varphi u \) and \( D(\bar{u}) = D(u) \). We claim that there is a space \( \hat{E}_1 \in \mathcal{E} \) and an injective map \( \psi : E_1 \to \hat{E}_1 \) with \( \| \psi \|_\infty = 1 \) such that the maps \( \hat{u} : L \to \hat{E}_1 \) and \( \hat{\varphi} : E \to \hat{E}_1 \) defined by \( \hat{u} = \psi \bar{u} \) and \( \hat{\varphi} = \psi \varphi \) (and hence \( \bar{u}_{|S} = \hat{\varphi} u \)) satisfy the requirements to show that \( \mathcal{E} \) is tightly Gurarii.

The space \( \hat{E}_1 \) is defined simply as \( \hat{E}_1 = L \oplus_\infty E \). The map \( \psi : E_1 \to \hat{E}_1 \) is defined by setting \( \psi(x) = \| w \|^{-1}_\infty w(x) \oplus \| v \|^{-1}_\infty v(x) \) where \( w \) and \( v \) are as follows. Consider \( \hat{u}^{-1} : \hat{u}(L) \to L \). Since \( L \) is injective the latter map admits an extension \( w : E_1 \to L \) with \( \| w \|_\infty = \| \hat{u}^{-1} \|_\infty \). Similarly, since \( E \) is injective the map \( \varphi^{-1} : \varphi(E) \to E \) admits an extension \( v : E_1 \to E \) with \( \| v \|_\infty = 1 \). Note that \( \hat{u}(y) = \psi \hat{u}(y) = (\| w \|^{-1}_\infty y, *) \) \((y \in L) \) and \( \hat{\varphi}(e) = \psi \varphi(e) = (*, e) \) \((e \in E) \). So we have

\[
\| \hat{u}^{-1} : \hat{u}(L) \to L \|_\infty \leq \| w \|_\infty = \| \hat{u}^{-1} \|_\infty \leq \| \hat{u}^{-1} \|_\infty,
\]

and hence

\[
D(\hat{u}) \leq D(\bar{u}) = D(u).
\]

Moreover \( \hat{\varphi} \) is completely isometric. By our assumptions on \( \mathcal{E} \) we know that \( \hat{E}_1 \in \mathcal{E} \). Thus we conclude that \( \mathcal{E} \) is tightly Gurarii. The same argument can be repeated to show that \( \mathcal{E} \) is perturbative.

Lemma 5.6. Assume that \( \mathcal{E} \) is perturbative with associated function \( \varepsilon \mapsto \delta(\varepsilon) \). Let \( f : E \to F \) be a complete \( \varepsilon \)-embedding \((\varepsilon > 0)\) with \( E, F \in \mathcal{E} \). Assume that \( E \) is a f.d. subspace of an \( \mathcal{E} \)-Gurarii space \( X \). Then for any \( 0 < \varepsilon' < 1 \) there is a subspace \( E_1 \subset X \) containing \( E \) and a complete \( \varepsilon' \)-embedding \( g : F \to E_1 \) such that

\[
\| (gf - Id_{E_1})_{|E} \|_\infty \leq 2(1 + \varepsilon)\delta(\varepsilon).
\]

Proof. Let \( i_E : E \to X \) denote the inclusion. We apply the perturbative Gurarii property with \( L = F, S = f(E) \subset L \) and \( u = f^{-1}_{|f(E)} \). For any \( \varepsilon'' > 0 \) this gives us \( \hat{E}_1 \in \mathcal{E} \), and \( \varepsilon'' \)-isometric maps \( \varphi : E \to \hat{E}_1 \) and \( \hat{u} : F \to \hat{E}_1 \) such that \( \| \hat{u} - \varphi u \| \leq \delta(\varepsilon) \). By the \( \mathcal{E} \)-Gurarii property of \( X \) applied to \( \varphi^{-1} : \varphi(E) \to E \), there is a map \( w : \hat{E}_1 \to X \) such that \( w_{|\varphi(E)} = i_E \varphi^{-1} \) or equivalently \( w\varphi = i_E \) and (recall \( 4.9 \))

\[
D'(w) \leq (1 + \varepsilon'')D'(\varphi) \leq (1 + \varepsilon'')^2.
\]
We then set \( g = w u : F \to X \). We have \( D'(g) \leq (1 + \varepsilon'')^3 \) and
\[
\| (g f - i_E) \|_{cb} \leq \| u \|_{cb} \| \bar{u}_{|S} - \varphi u \|_{cb} \| f \|_{cb} \leq (1 + \varepsilon'')^2 \delta(\varepsilon)(1 + \varepsilon).
\]

Let \( E_1 \subset X \) be any f.d. subspace such that \( g(F) + E \subset E_1 \), we may then view \( g \) and \( i_E \) as having range in \( E_1 \) and it only remains to choose \( \varepsilon'' \) small enough with respect to \( \varepsilon' \) to obtain the announced result.

 Mimicking the approach of [23] we will prove

**Theorem 5.7.** Assuming that \( E \) is a perturbative Gurarii class with associated function \( \varepsilon \mapsto \delta(\varepsilon) \). Let \( X, Y \) be \( E \)-Gurarii spaces, let \( \varepsilon > 0 \), and let \( E \subset X \) be a f.d. subspace in \( E \). Let \( u : E \to Y \) be a complete \( \varepsilon \)-embedding. Then for any \( \delta' > 0 \) there is a completely isometric isomorphism \( f : X \to Y \) such that \( \| f_{|E} - u \|_{cb} \leq 8 \delta(\varepsilon) + \delta' \).

In particular, \( X \) and \( Y \) are completely isometrically isomorphic.

**Proof.** The proof can be completed exactly as in [23, p. 453] but using our Lemma 5.6 in place of [23, Lemma 2.2]. Unfortunately, our notation clashes with that of [23] so we repeat the argument of [23] for the reader’s convenience.

Fix \( \varepsilon_0 = \varepsilon \). Let \( (\varepsilon_n)_{n>0} \) be a sequence of numbers such that \( 0 < \varepsilon_n < 1 \) for all \( n > 0 \) and \( \sum \delta(\varepsilon_n) < \infty \) (to be specified). By induction, following [23] we construct sequences \( (E_n) \) and \( (F_n) \) of f.d. subspaces respectively of \( X \) and \( Y \), together with maps \( (f_n) \) and \( (g_n) \) satisfying the following conditions:

1. \( E_0 = E, F_0 = u(E) \),
2. \( f_n : E_n \to F_n \) is an \( \varepsilon_n \)-embedding,
3. \( g_n : F_n \to E_{n+1} \) is an \( \varepsilon_{n+1} \)-embedding,
4. \( \| (g_n f_n - Id_{E_{n+1}}) |_{E_n} \|_{cb} \leq 4 \delta(\varepsilon_n) \| x \| \) and \( E_n \subset E_{n+1} \),
5. \( \| (f_{n+1} g_n - Id_{F_{n+1}}) |_{F_n} \|_{cb} \leq 4 \delta(\varepsilon_{n+1}) \| y \| \) and \( F_n \subset F_{n+1} \),
6. \( X = \overline{\cup E_n}, Y = \overline{\cup F_n} \).

The proof can be illustrated by the following asymptotically approximately commuting diagram.

\[
\begin{array}{ccc}
\text{X} & \xrightarrow{f} & \text{Y} \\
\downarrow & & \downarrow \\
E_{n+1} & \xrightarrow{f_{n+1}} & F_{n+1} \\
\downarrow & & \downarrow \\
\text{E}_n & \xrightarrow{f_n} & \text{F}_n \\
& & \downarrow \\
& & \text{E}_{n+1} \\
\end{array}
\]

We start with \( E_0 = E, F_0 = u(E) \) and \( f_0 = u \).

Assume we know \( E_k, F_k, f_k \) for all \( k \leq n \). Then by Lemma 5.6 applied to \( f_n \) there is \( E_{n+1} \), that we can pick so that \( E_{n+1} \supset E_n \), and an \( \varepsilon_{n+1} \)-embedding \( g_n : F_n \to E_{n+1} \) such that (3) holds. With the latter map \( g_n : F_n \to E_{n+1} \) at hand, we apply Lemma 5.6 to it (now with \( Y \) in place of \( X \)), this gives us \( F_{n+1} \), that we can pick so that \( F_{n+1} \supset F_n \), and an \( \varepsilon_{n+1} \)-embedding \( f_{n+1} : E_{n+1} \to F_{n+1} \).
such that (4) holds. In particular, starting from $E_0, F_0, f_0$ we obtain $g_0$ as well as $E_1, F_1, f_1$, and then $g_1$ as well as $E_2, F_2, f_2$ and so on. The condition (5) can easily be ensured by enlarging if necessary the space $E_n$ (resp. $F_n$) at the $n$-th step so that it contains the first $n$ elements of a dense sequence in $X$ (resp. $Y$). Then (3) implies $\| (f_{n+1}g_n f_n - f_n) \|_{E_n} \leq 4\delta(\varepsilon_n)\| f_{n+1} \|_{cb} \leq 8\delta(\varepsilon_n)$ while (4) implies $\| f_{n+1}g_n f_n - f_n \|_{cb} \leq 4\delta(\varepsilon_n)\| f_{n+1} \|_{cb} \leq 8\delta(\varepsilon_n+1)$. Therefore by the triangle inequality

\[
\| (f_{n+1} - f_n) \|_{E_n} \leq \| (f_{n+1} - f_n) \|_{E_n} \| cb \leq 8\delta(\varepsilon_n) + 8\delta(\varepsilon_n+1).
\]

Since $(\delta(\varepsilon_n))$ is summable, $(f_n(x))$ converges in $Y$ to a limit $f(x)$ for any $x \in \bigcup E_n$. Clearly $f$ extends to a complete isometry from $X$ to $Y$. Arguing similarly with $g_{n+1}f_{n+1}g_n$ we find that $(g_n(y))$ converges in $X$ to a limit $g(y)$ for any $y \in \bigcup F_n$ that defines a complete isometry $g : Y \to X$. Clearly (3), (4) and (5) imply that $fg = Id_Y$ and $gf = Id_X$, whence the conclusion that $f$ is a bijective complete isometry. Since $f_0 = u$ and $E = E_0 \subset E_n$ the bound (5.2) gives us

$$\| f_{|E} - u \|_{cb} \leq \sum_0^\infty \| (f_{n+1} - f_n) \|_{E_n} \| cb \leq 8\delta(\varepsilon_n) + 8\delta(\varepsilon_n+1) = 8\delta(\varepsilon) + 16 \sum_1^\infty \delta(\varepsilon_n),$$

so that by a suitable choice of $(\varepsilon_n)_{n>0}$ we can ensure that $\| f_{|E} - u \|_{cb} \leq 8\delta(\varepsilon) + \delta'.$

Applying this for an arbitrary choice of $E$, we state for emphasis:

**Corollary 5.8 (Uniqueness).** If $\mathcal{E}$ is a perturbative Gurarii class, any two $\mathcal{E}$-Gurarii spaces are completely isometrically isomorphic.

**Remark 5.9 (More examples).** Let $\mathcal{E}$ be a family of injective f.d.o.s. stable by $\oplus_\infty$-direct sums. By Lemma 5.5, $\mathcal{E}$ is perturbative Gurarii and hence there is a unique space $\mathbb{G}_\mathcal{E}$.

Let $Z$ be a fixed injective f.d.o.s. Let $\mathcal{E}(Z)$ be the class formed of all the $\oplus_\infty$-direct sums of finitely many copies of $Z$. We can associate to $Z$ the space $\mathbb{G}_\mathcal{E}(Z)$. Perhaps the cases of $Z = R_n$ (row $n \times n$-matrices) or $Z = C_n$ (column $n \times n$-matrices) or more generally $Z = M_{p,q}$ (rectangular matrices of size $p \times q$) deserve further investigations.

We discuss the particular case $Z = M_N$ in the next section.

### 6 Oikhberg’s exact Gurarii space

In [31] Oikhberg proved the existence of an analogue of the Gurarii space among exact operator spaces. While the spaces of interest to us in this note are mainly non exact, we would like to indicate how the existence and uniqueness of the exact Gurarii space can be derived from Theorems 4.22 and 5.7. Note that, using [23], Lupini proved in [26] the uniqueness of Oikhberg’s space up to completely isometric isomorphism. Moreover, in [26, 27] (see also [12]) Lupini placed the whole subject of Gurarii operator spaces in the much broader context of Fraïssé limits, which were connected to Gurarii spaces by Ben Yaacov (see [3 §3.3]) inspired by Henson’s unpublished work.

In this framework (with which we confess we are not too familiar) our paper probably just boils down to providing more examples illustrating the applicability of Fraïssé limits in operator space theory, beyond Oikhberg’s exact space.

Let $\mathcal{E}_{\min}^{[N]}$ be the collection of all f.d. operator subspaces of $\ell_\infty(M_N)$ (or equivalently of $\ell_\infty \otimes_{\min} M_N$), and let $\mathcal{E}_{\min}^{[\infty]} = \bigcup_N \mathcal{E}_{\min}^{[N]}$. The set $\mathcal{E}_{\min}^{[\infty]}$ coincides with the collection of all f.d. 1-exact o.s.

**Lemma 6.1.** The classes $\mathcal{E}_{\min}^{[N]}$ and $\mathcal{E}_{\min}^{[\infty]}$ are perturbative Gurarii.
Proof. Let $\mathcal{E}[M_N]$ be as in Remark 5.9 the perturbative Gurarii class formed of all the $\oplus_\infty$-direct sums of finitely many copies of $M_N$. It is easy to check that $\mathcal{E}^\infty[M_N] = S\mathcal{E}[M_N]$. By Remark 5.2 the class $\mathcal{E}^\infty_{\min}$ is perturbative Gurarii. Since $(\mathcal{E}^\infty_{\min})$ is monotone increasing its union $\bigcup_N \mathcal{E}^\infty_{\min}$ is still perturbative Gurarii, and by Remark 5.2 so is its closure $\mathcal{E}^\infty_{\min}$.

Let $G_N$ and $G_\infty$ be respectively the unique $\mathcal{E}^\infty_{\min}$-Gurarii space and the unique $\mathcal{E}^\infty_{\min}$-Gurarii space (see Theorems 4.22 and 5.7).

Note that when $N = 1$ the space $G_1$ is just the classical Gurarii space equipped with its minimal o.s. structure (see Remark 4.24). The space $G_\infty$ is Oikhberg’s exact Gurarii space from [31], i.e. the unique $\mathcal{E}$-Gurarii space when $\mathcal{E}$ is the class of 1-exact f.d.o.s.

Remark 6.2. The class $\mathcal{E}^\infty_{\min}$ (resp. $\mathcal{E}^\infty_{\max}$) is the class of all f.d. $M_N$-minimal (resp. $M_N$-maximal) o.s. in the sense of Lehner [25]. When $N = 1$ these are just the f.d. minimal (resp. maximal) o.s. Note that for simplicity we previously denoted $\mathcal{E}^\infty_{\max}$ by $\mathcal{E}_{\max}$.

The notion of $M_N$-space in the sense of Lehner [25] provides a very clear picture of the properties of $G_N$, as an intermediate object between the classical Gurarii space and Oikhberg’s exact variant. To describe this we first give a quick review of $M_N$-space theory.

By an $M_N$-space we mean (following [25]) a vector space $V$ equipped with a norm $\| \cdot \|_{(N)}$ for which there is an embedding $V \subset B(H)$ with which $\| \cdot \|_{(N)}$ coincides with the norm induced by $M_N(B(H))$. The morphisms $u : V_1 \to V_2$ between $M_N$-spaces are now the maps such that $Id_{M_N} \otimes u : M_N(V_1) \to M_N(V_2)$ is bounded and the cb-norm is now replaced by

$$\|u\|_N := \|Id_{M_N} \otimes u : M_N(V_1) \to M_N(V_2)\|.$$ 

A map with $\|u\|_N \leq 1$ (resp. $\|u\|_N \leq 1$ and $\|u^{-1}\|_N \leq 1$) is called $N$-contractive (resp. $N$-isometric).

Just like operator spaces, $M_N$-spaces admit a duality, as well as natural notions of quotient space, of direct sums $\oplus_\infty$ and $\oplus_1$ and more.

When $N = 1$, we recover the usual Banach spaces, the latter notions become the usual ones and $\|u\|_N = \|u\|$.

When $N = \infty$ (where we replace $M_N$ by the algebra of compact operators on $\ell_2$) we recover operator space theory.

Given an $M_N$-space $V$, there are operator space structures $MIN_N$ and $MAX_N$ on $V$, with associated o.s. denoted by $MIN_N(V)$ and $MAX_N(V)$ such that the maps $MAX_N(V) \to V \to MIN_N(V)$ are $N$-isometric and these are extremal in the sense that for any o.s. $E,F,$ for any $u : V \to E$ and any $v : F \to V$ we have

$$\|u\|_N = \|u : MAX_N(V) \to E\|_N = \|u : MAX_N(V) \to E\|_{cb},$$

$$\|v\|_N = \|v : F \to MIN_N(V)\|_N = \|v : F \to MIN_N(V)\|_{cb}.$$ 

Given an o.s. $E \subset B(H)$, the norm on $M_N(E)$ determines the structure of an $M_N$-space on $E$; let us denote by $E[N]$ the latter $M_N$-space. Note that $M_n(E[N]) = M_n(E)$ for all $n \leq N$ and when passing from $E$ to $E[N]$ we roughly “forget” these norms for $n > N$. We then define a new o.s. $E[N]$ by setting

$$E[N] = MIN_N(E[N]).$$

Then $M_n(E[N]) = M_n(E)$ for all $n \leq N$, and for any o.s. $F$ and any $u : F \to E$ we have $\|u : F \to E\|_N = \|u : F \to E[N]\|_{cb}$. In particular the identity map satisfies $\|Id : E \to E[N]\|_{cb} = 1$. This means $\| \cdot \|_{M_n(E[N])} \leq \| \cdot \|_{M_n(E)}$ for all $n > N$ and the norms on $M_n(E[N])$ for $n > N$ are

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the largest possible ones (as a sequence satisfying the o.s. axioms) satisfying the preceding two properties. One way to realize an embedding $E^{[N]}\subset B(H)$ is to consider the embedding $\Phi_N$ defined below. Let $I_N = \{ u : E \to M_N \mid \|u\|_{cb} \leq 1 \}$. Note that by a well known lemma (due to Roger Smith, see e.g. [37, p. 26]) $\|u\|_N = \|u\|_{cb}$ for all $u \in I_N$. Let $\Phi_N : E \to \ell_\infty(I_N; M_N)$ be the mapping defined by

$$\forall x \in E \quad \Phi_N(x) = (u(x))_{u \in I_N}.$$  

Then $\Phi_N$ defines a completely isometric embedding of $E^{[N]}$ in $\ell_\infty(I_N; M_N)$ (and a fortiori in some $B(H)$).

This shows that

$$E \in \mathcal{E}_{\min}^{[N]} \iff E = E^{[N]}.$$

Note that if $F \in \mathcal{E}_{\min}^{[N]}$, any invertible $u : F \to E$ satisfies

$$\|u^{-1}_{|u(F)} : u(F) \to F\|_{cb} = \|u^{-1}_{|u(F)} : u(F) \to F\|_N.$$  

With these facts the following is easy to check.

**Lemma 6.3.** Let $S \subset L$, $u : S \to E$ and $E_1$ be as in Lemma [7.1]. Fix $N \geq 1$. If the spaces $L$ and $E$ both embed (completely isometrically) in $\ell_\infty(M_N)$, then the space $E_1^{[N]}$ satisfies all the properties of $E_1$ listed in Lemma [7.1].

**Proposition 6.4.** We have $\mathcal{G}_N \subset \mathcal{G}_{N+1}$ (completely isometrically) for all $N \geq 1$ and $\mathcal{G}_\infty = \overline{\bigcup \mathcal{G}_N}$.  

**Proof.** Using Lemma 6.3 one can easily show that Theorem 4.22 is valid for $\mathcal{E} = \mathcal{E}_{\min}^{[N]}$. Although the latter is not stable by the o.s. version of $\oplus_1$, it is stable by the $M_N$-space analogue of $\oplus_1$, which can be defined as in (0.2) but with the supremum over all $u$’s with $\|u\|_N \leq 1$. This explains transparently why the case of $\mathcal{E} = \mathcal{E}_{\min}^{[N]}$ is entirely analogous to those included in Theorem 4.22.

Let $\mathcal{G}_N$ be the $\mathcal{E}_{\min}^{[N]}$-Gurarii space. Since $\mathcal{E}^{[N]} \subset \mathcal{E}^{[N+1]}$, applying the $\mathcal{E}^{[N+1]}$-variant of Theorem 4.22 with $Y = \mathcal{G}_N$ and $X = \mathcal{G}_{N+1}$, we have $\mathcal{G}_N \subset \mathcal{G}_{N+1}$ (completely isometrically) and since $\mathcal{E}_{\min}^{[\infty]} = \bigcup \mathcal{E}_{\min}^{[N]}$ it is easy to check that $\overline{\bigcup \mathcal{G}_N}$ is an $\mathcal{E}_{\min}^{[\infty]}$-Gurarii space, and hence by uniqueness $\mathcal{G}_\infty = \overline{\bigcup \mathcal{G}_N}$.  

The next statement was already stated in [26, Remark 4.9] as a consequence of [3] Lemma 3.17.

**Lemma 6.5.** There are integers $(k(n))$ and an increasing sequence $E_1 \subset E_2 \subset \cdots$ of f.d. subspaces of $\mathcal{G}_\infty$ with dense union such that $E_n \simeq M_{k(n)}$ completely isometrically.

**Proof.** Indeed, since $\mathcal{E} = \{ M_n \mid n \geq 1 \}$ is Gurarii, there is an $\mathcal{E}$-Gurarii space $X$. By Theorem 4.22 and Remark 4.23 (ii) we may assume that there are integers $(k(n))$ and an increasing sequence $E_1 \subset E_2 \subset \cdots$ of f.d. subspaces of $X$ with dense union such that $E_n \simeq M_{k(n)}$ completely isometrically. By Remark 4.12 the space $X$ is automatically an $\overline{\mathcal{E}}$-Gurarii space. But since $\mathcal{E}_{\min}^{[\infty]} = \overline{\mathcal{E}_{\min}^{[N]}}$ (see Remark 4.13), we must have $X = \mathcal{G}_\infty$ by the uniqueness of $\mathcal{G}_\infty$.  

While the presentation in this section is arranged to illustrate our framework, in essence the proofs do not differ from Oikhberg’s ones from [31], updated with the contributions in [23] and [27].

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