Elliptic curves and birational representation of Weyl groups

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Abstract

Some Weyl group acts on a family of rational varieties obtained by successive blow-ups at \( m \) (\( m \geq n + 2 \)) points in the projective space \( \mathbb{P}^n(\mathbb{C}) \). In this paper we study the case where all the points of blow-ups lie on a certain elliptic curve in \( \mathbb{P}^n \). Investigating the action of Weyl group on the Picard groups on the elliptic curve and on rational varieties, we show that the action on the parameters can be written as a group of linear transformations on the \((m + 1)\)-st power of a torus.

1 Introduction

By the works of Coble [1] and Dolgachev-Ortland [2], it has been known that some Weyl group behaves as pseudo-isomorphisms (isomorphisms excluding sub-varieties of codimension 2 or higher) and acts on a family of rational varieties obtained by successive blow-ups at \( m \) (\( m \geq n + 2 \)) points in \( \mathbb{P}^n(\mathbb{C}) \). Here, the Weyl group is given by the Dynkin diagram in Fig. 1 and denoted by \( W(n, m) \).

![Figure 1: W(n, m) Dynkin diagram](image)

Recently, one of the authors (TT) [7] introduced dynamical systems defined by translations of affine Weyl groups (§6.5 in [3]) with the symmetric Cartan matrices included in \( W(n, m) \). For example, if \( m \geq n + 7 \), \( W(n, m) \) includes the affine Weyl group of type \( E_8^{(1)} \). In the case of \( n = 2, m = 9 \), this dynamical system coincides with the elliptic difference Painlevé equation proposed by Sakai [6], from which all the discrete and continuous

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Painlevé equations are obtained by degeneration. For \( n \geq 3 \), however, the time evolution of the parameters may not be solved in general, thus, the systems should be considered not to be \( n \)-dimensional but to be higher dimensional.

On the other hand, Kajiwara et al. \cite{4} has proposed a birational representation of the Weyl group \( W(n, m) \), in which all the points of blow-ups lie on a certain elliptic curve in \( \mathbb{P}^n \). This representation is a special case of \cite{1, 2}. In this case, the action on the parameters can be written as a group of linear transformations on the \((m + 1)\)-st power of a torus. The calculation was carried out in a rather heuristic manner in \cite{4}.

In this paper, we recover the birational representation of Kajiwara et al. geometrically, by investigating the actions of the Weyl group on the Picard groups on the elliptic curve and on rational varieties. Our method corresponds to the “linear map” or the “period map” in 2-dimensional case \cite{3, 4}.

This article is organized as follows. In Section 2, we review the relationship between rational varieties and groups of Cremona transformations. In Section 3, it is proved that general two elliptic curves in \( \mathbb{P}^n \) of degree \( n + 1 \) are translated to each other by a projective linear transformation. In Section 4, we investigate how the birational representation of the Weyl group is restricted onto elliptic curves of degree \( n + 1 \) geometrically. In Section 5, we present some examples of calculation, and recover the birational representation of Section 4.

2 The birational representation of Weyl groups by Coble and Dolgachev-Ortland

Let \( m \geq n + 2 \). Let \( X_{n, m} \) be the configuration space of ordered \( m \) points in \( \mathbb{P}^n \):

\[
X_{n, m} = \frac{\text{PGL}(n+1)\left\{ \begin{pmatrix} a_{01} & \cdots & a_{0m} \\ a_{11} & \cdots & a_{1m} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \mid \text{the determinant of every } (n+1) \times (n+1) \text{ sub-matrix is nonzero} \right\}}{(\mathbb{C}^\times)^m},
\]

which is a quasi-projective variety of dimension \( n(m - n - 2) \). We also consider \( X^1_{n, m} \simeq X(n, m + 1) \) with a natural projection \( \pi : X^1_{n, m} \to X_{n, m} \):

\[
\begin{pmatrix} a_{01} & \cdots & a_{0m} & x_0 \\ a_{11} & \cdots & a_{1m} & x_1 \\ \vdots & \cdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & x_n \end{pmatrix} \mapsto \begin{pmatrix} a_{01} & \cdots & a_{0m} \\ a_{11} & \cdots & a_{1m} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix},
\]

where each fiber is \( \mathbb{P}^n \) and \( X_{n, m} \) is referred to as the parameter space.

Let \( A \in X_{n, m} \) and let \( X_A \) be the rational variety obtained by successive blow-ups at the points \( P_i = P_i(A) = (a_{0i} : \cdots : a_{ni}) \) \((i = 1, 2, \ldots, m)\) from \( \mathbb{P}^n \). We denote the family of rational projective varieties \( X_A \) \((A \in X_{n, m})\) by \( \tilde{X}^1_{n, m} \), which also has the natural fibration \( \tilde{\pi} : \tilde{X}^1_{n, m} \to X_{n, m} \).

Let \( E = E(A) \) be the divisor class on \( X_A \) of the total transform of a hyper-plane in \( \mathbb{P}^n \) and let \( E_i = E(A) \) be the exceptional divisor class generated by blow-up at the point
The group of divisor classes of $X_A$: $\text{Pic}(X_A) \cong H^1(X_A, \mathcal{O}_X) \cong H^2(X_A, \mathbb{Z})$ (the second equivalence comes from the fact that $X_A$ is a rational projective variety), is described as the lattice

$$\text{Pic}(X_A) = ZE \oplus ZE_1 \oplus ZE_2 \oplus \cdots \oplus ZE_m.$$  

(1)

Notice that this cohomology group is independent of $A$, while $X_A$ is not isomorphic to $X_{A'}$ for $A' \neq A \in X_{n,m}$ in general.

Let $e \in H_2(X_A, \mathbb{Z})$ be the class of a generic line in $\mathbb{P}^n$ and let $e_i$ be the class of a generic line in the exceptional divisor of the blow-up at the point $P_i$. Then, $e, e_0, e_1, \ldots, e_m$ consist a basis of $H_2(X_A, \mathbb{Z}) \cong (H^2(X_A, \mathbb{Z}))^*$ (the Poincaré duality) and the intersection numbers are given by

$$\langle E, e \rangle = 1, \quad \langle E, e_j \rangle = 0, \quad \langle E_i, e \rangle = 0, \quad \langle E_i, e_j \rangle = -\delta_{ij}.$$ 

Following Dolgachev-Ortland [2], we take the root basis \{\alpha_0, \ldots, \alpha_{m-1}\} $\subset H^2(X_A, \mathbb{Z})$ and the co-root basis \{\alpha_0^\vee, \ldots, \alpha_{m-1}^\vee\} $\subset H_2(X_A, \mathbb{Z})$ as

$$\alpha_0 = E - E_1 - E_2 - \cdots - E_{n+1}, \quad \alpha_i = E_i - E_{i+1} \quad (i > 0),$$

$$\alpha_0^\vee = (n-1)e - e_1 - e_2 - \cdots - e_{n+1}, \quad \alpha_i^\vee = e_i - e_{i+1} \quad (i > 0),$$

then, $\langle \alpha_i, \alpha_j^\vee \rangle = -2$ holds for any $i$ and these root bases define the Dynkin diagram of type $T_{2,n+1,m-n-1}$ by assigning a root $\alpha_i$ to every vertex $\alpha_i$ and connecting two distinct vertices $\alpha_i$ and $\alpha_j$ if $\langle \alpha_i, \alpha_j^\vee \rangle = 1$ (in our case $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ or 1 for $i \neq j$) (Fig. 1).

Let us define the root lattice $Q = Q(n, m) \subset H^2(X_A, \mathbb{Z})$ and the co-root lattice $Q^\vee = Q^\vee(n, m) \subset H_2(X_A, \mathbb{Z})$ as $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{m-1}$ and $Q^\vee = \mathbb{Z}\alpha_0^\vee \oplus \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_{m-1}^\vee$ respectively. For every $\alpha_i$ the formulae

$$r_{\alpha_i}(D) = D + \langle D, \alpha_i^\vee \rangle \alpha_i \quad \text{for any } D \in Q$$

$$r_{\alpha_i^\vee}(d) = d + \langle \alpha_i, d \rangle \alpha_i^\vee \quad \text{for any } d \in Q^\vee$$

(2)

define linear involutions (called simple reflections) of the bi-lattice $(Q, Q^\vee)$ and they generate the Weyl group $W$ of type $T_{2,n+1,m-n-1}$, which we denote by $W_{n}(n, m)$.

These simple reflections correspond to certain birational transformations on the fiber space $\pi : X_{n,m}^1 \rightarrow X_{n,m}$. Let us define birational transformations $r_{i,j}$ ($1 \leq i < j \leq m$) and $r_{i_0,i_1,\ldots,i_n}$ ($1 \leq i_0 < \cdots < i_n \leq m$) on the fiber space as:

$$r_{i,j} : \begin{pmatrix} \cdots & a_i & \cdots & a_j & \cdots & x \end{pmatrix} \mapsto \begin{pmatrix} \cdots & a_j & \cdots & a_i & \cdots & x \end{pmatrix}$$

$$r_{i_0,i_1,\ldots,i_n} : \begin{pmatrix} \cdots & a_{i_0} & \cdots & a_{i_1} & \cdots & \cdots & a_{i_n} & \cdots & x \end{pmatrix} \mapsto \begin{pmatrix} \cdots & \cdots & \cdots & a_{i_0} & \cdots & a_{i_1} & \cdots & \cdots & \cdots & a_{i_n} & \cdots & \cdots & x \end{pmatrix}$$

(3)

and $r_{i_0,i_1,\ldots,i_n}$ is the standard Cremona transformation with respect to the points $P_{i_0}, P_{i_1}, \ldots, P_{i_n}$, i.e. for example, $r_{1,2,\ldots,n+1}$ is the composition of a projective transformation and the standard Cremona transformation with respect to the origins $(0 : \cdots : 0 : 1 : 0 \cdots : 0)$ as
\[ r_{1,2,\ldots,n+1}: (A \mid x) = (A_{1,\ldots,n+1} \mid A_{n+2,\ldots,m} \mid x) \]

\[ \mapsto A_{1,\ldots,n+1}^{-1}(A \mid x) =: \left( \begin{array}{ccccc} I_{n+1} & \cdots & a''_{i,j} & \cdots & x''_i \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{array} \right) \]

where \( A_{j_1,j_2,\ldots,j_k} \) denotes the \((n + 1) \times k\) matrix \((a_{j_1} \mid \cdots \mid a_{j_k})\) and \( I_k \) denotes the \(k \times k\) identity matrix. (In section 5, \( a''_{i,j}^{-1} \) and \( x''_i^{-1} \) are denoted as \( a_{i,j}' \) and \( x_i' \), respectively.)

Let \( w \) denotes the reflection \( r_{i,j} \) or \( r_{i_0,i_1,\ldots,i_n} \). The reflection \( w \) acts on the parameter space \( X_{n,m} \) and preserves the fibration \( \pi : X_{n,m} \to X_{n,m} \). Recall that \( H^2(X_A, \mathbb{Z}) \) is independent of \( A \in X_{n,m} \). Hence, \( w \) defines an action on this co-homology group. Moreover, the induced birational map \( w : X_A \to X_{w(A)} \) for generic \( A \in X_{n,m} \) is a pseudo-isomorphism, i.e. an isomorphism except sub-manifolds of co-dimension 2 or higher, and the lines corresponding to the classes \( e \) and \( e_i \) can be chosen so that they do not meet the excepted part. Since \( H_2(X_A, \mathbb{Z}) \) is also independent of \( A \in X_{n,m}, w \) defines an action on this homology group and preserves the intersection form \( \langle \cdot, \cdot \rangle : H^2(X_A, \mathbb{Z}) \times H_2(X_A, \mathbb{Z}) \to \mathbb{Z} \).

The birational maps \( r_{i,i+1} \) and \( r_{1,2,\ldots,n+1} \) correspond to the simple reflections \( r_{\alpha_i} \) (\( 1 \leq i \leq m - 1 \)) and \( r_{\alpha_0} \) respectively. Indeed, their push-forward actions \( H^2(X_A, \mathbb{Z}) \to H^2(X_{w(A)}, \mathbb{Z}) \) \((w = r_{i,i+1} \text{ or } r_{1,2,\ldots,n+1})\) and \( H_2(X_A, \mathbb{Z}) \to H_2(X_{w(A)}, \mathbb{Z}) \) are given by the formulæ:

\[ r_{i,i+1}(D) = D + \langle D, \alpha_i \rangle \alpha_i \]
\[ r_{i,i+1}(d) = d + \langle \alpha_i, d \rangle \alpha_i \]
\[ r_{1,2,\ldots,n+1}(D) = D + \langle D, \alpha_0 \rangle \alpha_0 \]
\[ r_{1,2,\ldots,n+1}(d) = d + \langle \alpha_0, d \rangle \alpha_0 \]

for any \( D \in H^2(X_A, \mathbb{Z}) \) and any \( d \in H_2(X_A, \mathbb{Z}) \). Recall that root lattice \( Q(n,m) \) and co-root lattice \( Q^\vee(n,m) \) are subsets of \( H^2(X_A, \mathbb{Z}) \) and \( H_2(X_A, \mathbb{Z}) \) respectively and hence the formulæ \( \text{(5)} \) are extensions of \( \text{(2)} \) onto these (co)-homology groups.

**Definition.** A hyper-surface in \( X_A \) is called nodal if its class is \( w_s(\alpha_0) \) for some \( w_s \in W_s(n,m) \). Let \( N_{n,m} \) denote the set of \( A \in X_{n,m} \) such that \( X_A \) admits a nodal hyper-surface. We also write \( N^1_{n,m} \) and \( \tilde{N}^1_{n,m} \) as \( \pi^{-1}(N_{n,m}) \) and \( \tilde{\pi}^{-1}(N_{n,m}) \) respectively.

We define \( W(n,m) \) as the group generated by \( r_{i,i+1} \) \((i = 1, 2, \ldots, m-1)\) and \( r_{1,2,\ldots,n+1} \).

**Proposition 2.1 (Coble, Dolgachev-Ortland).** Let \( m \geq n + 2 \).

i) \( W(n,m) \simeq W_s(n,m) \) holds. \( W(n,m) \) acts on \( \tilde{X}^1_{n,m} \setminus \tilde{N}^1_{n,m} \)
ii) Each element \( w \in W(n, m) \) defines an action on \( H^2(X_A, \mathbb{Z}) \) and \( H_2(X_A, \mathbb{Z}) \), and preserves the intersection form \( \langle \cdot, \cdot \rangle : H^2(X_A, \mathbb{Z}) \times H_2(X_A, \mathbb{Z}) \rightarrow \mathbb{Z} \).

iii) The birational maps \( r_{i, i+1} \) and \( r_{1, 2, \ldots, n+1} \) correspond to the simple reflections \( r_{\alpha_i} \) (\( 1 \leq i \leq m-1 \)) and \( r_{\alpha_0} \) respectively. For the reflection \( r_{\alpha} = w \circ r_{\alpha_i} \circ w^{-1} \), where \( \alpha = w(\alpha_i) \) is a real root, the formulae

\[
\begin{align*}
r_{\alpha}(D) &= D + \langle D, \alpha^\vee \rangle \alpha \\
r_{\alpha}(d) &= d + \langle \alpha, d \rangle \alpha^\vee
\end{align*}
\]

hold for any \( D \in H^2(X_A, \mathbb{Z}) \) and any \( d \in H_2(X_A, \mathbb{Z}) \).

iv) Every \( r_{i,j} \) or \( r_{i,0,\ldots, i_n} \) is an element of \( W(n, m) \).

3 On the embedding of elliptic curve to \( \mathbb{C}\mathbb{P}^n \)

Let \( T \) be an elliptic curve \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \), and let \( \iota : T \rightarrow \mathbb{P}^n \) be an embedding. We define the degree of \( \iota \) as that of the pull-back of the line bundle \( O_{\mathbb{P}^n}(1) \cong E \) by \( \iota \).

Proposition 3.1. (Algebraic version) Let \( \iota \) and \( \iota' \) be embeddings of \( T \) to \( \mathbb{P}^n \) of degree \( n+1 \) s.t. both \( \iota(T) \) and \( \iota'(T) \) are not contained in any hyper-plane. Then, there exists a translation \( \sigma : T \rightarrow T \) and a projective linear transformation \( G \in \text{PGL}(n+1) \) s.t. \( G \circ \iota = \iota' \circ \sigma \) holds.

\[
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{G} & \mathbb{P}^n \\
\iota & \circ & \iota' \\
T & \xrightarrow{\sigma} & T
\end{array}
\]

This proposition can be also stated as follows.

Let \( f(u) \) be a holomorphic function on \( \mathbb{C} \). We say that \( f(u) \) has the quasi-periodicity if there exist constants \( l_1, l_\tau, c_1, c_\tau \) in \( \mathbb{C} \) s.t. the formulae

\[
\begin{align*}
f(u + 1) &= f(u) \exp\{2\pi \sqrt{-1}(l_1 u + c_1)\} \\
f(u + \tau) &= f(u) \exp\{2\pi \sqrt{-1}(l_\tau u + c_\tau)\}
\end{align*}
\]

hold, and we refer such a function to as a theta function for \( T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \).

Proposition 3.2. (Analytic version) Let \( \iota \) be a holomorphic map \( \iota : \mathbb{C} \rightarrow \mathbb{P}^n : u \mapsto (f_0(u) : f_1(u) : \ldots : f_n(u)) \) s.t. (i) \( f_i \)'s have the same quasi-periodicity, (ii) \( f_i \)'s are linearly independent, and (iii) each of them has \( n+1 \) zero points in the fundamental domain. Let \( \iota' : \mathbb{C} \rightarrow \mathbb{P}^n : u \mapsto (f'_0(u) : f'_1(u) : \ldots : f'_n(u)) \) be also a holomorphic map s.t. \( f'_i \)'s satisfy (i), (ii), (iii) with the same \( \tau \) (the quasi-periodicity need not be the same with that of \( f_i \)'s). Let \( u_0, u_1, \ldots, u_n \) be zeros of \( f_0(u) \) and let \( u'_0, u'_1, \ldots, u'_n \) be zeros of \( f'_0(u) \). Then, there exists a projective linear transformation \( G \in \text{PGL}(n+1) \) s.t. \( \iota(u)G = \iota'(u + a) \) holds, where \( a \) is determined by

\[
a = \frac{1}{n+1}(u'_0 + u'_1 + \cdots + u'_n - u_0 - u_1 - \cdots - u_n).
\]
Proof. The sum of zeros of \( f'_0(u+a) \) is \( (u'_0-a)+(u'_1-a)+\cdots+(u'_n-a) = u_0+u_1+\cdots+u_n \), then \( f_0(u) \) and \( f'_0(u+a) \) coincide up to a trivial theta function, i.e. the quasi-periods of \( f_0(u) \) and \( f'_0(u+a) \) \( \exp(\alpha_2u^2+\alpha_1u) \) coincide for some \( \alpha_1, \alpha_2 \in \mathbb{C} \). Thus, the quasi-periods of \( f_i(u)'s \) and \( f'_i(u+a) \) \( \exp(\alpha_2u^2+\alpha_1u)'s \) also coincide. Hence, we can assume \( f_i's \) and \( f'_i's \) have the same quasi-periodicity.

From the Riemann-Roch theorem
\[
\dim \mathcal{H}^0(T, \mathcal{O}(D)) - \dim \mathcal{H}^1(T, \mathcal{O}(D)) = 1 - g + \deg D,
\]
where \( D \) is a divisor on \( T \), and the Serre duality \( \mathcal{H}^1(T, \mathcal{O}(D))^* \simeq \mathcal{H}^0(T, \Omega(-D)) \), we have
\[
\dim \mathcal{H}^0(T, \mathcal{O}(D)) - \dim \mathcal{H}^0(T, \Omega(-D)) = \deg D.
\]
Put \( D = u_0 + u_1 + \cdots + u_n \) (summation of divisors), then we have \( \mathcal{H}^0(T, \Omega(-D)) = 0 \) and therefore \( \dim \mathcal{H}^0(T, \mathcal{O}(D)) = n + 1 \). Hence, meromorphic functions \( f_i(u)/f_0(u) \) \( (i = 0, 1, \ldots, n) \) on \( T \) consist a basis of \( \mathcal{H}^0(T, \mathcal{O}(D)) \) and \( f'_i(u)/f_0(u) \) \( (i = 0, 1, \ldots, n) \) also consist a basis of \( \mathcal{H}^0(T, \mathcal{O}(D)) \). Hence, there exists a linear transformation \( G \) on \( \mathbb{C}^{n+1} \) s.t.
\[
\left( \frac{f'_0(u)}{f_0(u)}, \ldots, \frac{f'_n(u)}{f_0(u)} \right) = \left( \frac{f_0(u)}{f_0(u)}, \ldots, \frac{f_n(u)}{f_0(u)} \right) G
\]
holds, and therefore,
\[
(f'_0(u), \ldots, f'_n(u)) = (f_0(u), \ldots, f_n(u)) G
\]
holds. \( \square \)

4 Elliptic curves and birational representation of Weyl groups

Let \( X_A \) be the rational variety obtained by successive blow-ups of \( \mathbb{P}^n \) at points \( P_i \):
\( \rho_A : X_A \rightarrow \mathbb{P}^n \). Assume that there exist \( \tau, \, \text{Im} \tau > 0 \), and an embedding \( \iota_A : T = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \rightarrow \mathbb{P}^n \) s.t. (i) the degree of \( \iota_A \) is \( n+1 \); (ii) all \( P_i \) are on \( \iota_A(T) \) (thus, \( \iota_A(T) \) is not contained in any hyper-plane). The embedding \( \iota_A \) can be lifted to an embedding uniquely \( \tilde{i}_A : T \rightarrow X_A \) s.t. \( \rho_A \circ \tilde{i}_A = \iota_A \) holds. We denote by \( \tilde{i}_A' \) the pull-back \( \tilde{i}_A' : \text{Pic}(X_A) \rightarrow \text{Pic}(T) \).

Proposition 4.1. For \( w \in W(n, m) \), \( w \) induces an isomorphism from \( \tilde{i}_A(T) \) to \( w \circ \tilde{i}_A(T) \).

Proof. Since \( P_{i_0}, P_{i_1}, \ldots, P_{i_m} \) \( (1 \leq i_0 < i_1 < \cdots < i_n \leq m) \) are not on any hyper-plane, the elliptic curve \( \iota_A(T) \subset \mathbb{P}^n \) is not contained in a hyper-plane; therefore the generators \( r_{1,2,\ldots,n+1}, r_{i,i+1} \) act \( \iota_A(T) \) birationally. Further, they can be extended to an isomorphism of elliptic curves. Thus, by composition and lifting, the assertion follows. \( \square \)

Lemma 4.1. The homology class of \( \tilde{i}_A(T) \) in \( H_2(X_A, \mathbb{Z}) \) is \( (n+1)e - e_1 - e_2 - \cdots - e_m \).
Proof. The intersection numbers of $\tilde{i}_A(T)$ and the basis of $H^2(X_A, \mathbb{Z})$ are
\[
\langle E, \tilde{i}_A(T) \rangle = n + 1 \quad \text{(the degree of } i_A(T))
\]
\[
\langle E_i, \tilde{i}_A(T) \rangle = 1 \quad (1 \leq i \leq m).
\]

Proposition 4.2. The degree of $w \circ i_A : T \to \mathbb{P}^n$ is $n + 1$.

Proof. From (5), we have
\[
\deg(w \circ i_A(T)) = H_2(X_{w(A)}, \mathbb{Z}) \langle E, w \circ \tilde{i}_A(T) \rangle
\]
\[
= \langle E, w_*(n + 1)e - e_1 - e_2 - \cdots - e_m) \rangle
\]
\[
= \langle E, (n + 1)e - e_1 - e_2 - \cdots - e_m) \rangle
\]
\[
= n + 1.
\]

Notation For $u \in T$, we denote the divisor on $T$ corresponding to $u$ by $u$ again and we denote its class by $[u]$. We denote the addition of divisor classes $[u_1]$ and $[u_2]$ by $[u_1] + [u_2]$.

Let $Y_{n,m}$ denote a subset of $X_{n,m}$:
\[
\{ A \in X_{n,m} \setminus N_{n,m} : \exists \tau, \Im \tau > 0, \exists i_A : T \to \mathbb{P}^n: \text{an embedding s.t. } (*) \}\]
where (*) is (i) the degree of $i_A$ is $n + 1$; (ii) all $P_i$ are on $i_A(T)$. From Prop. 4.1 and Prop. 4.2 the action of $W(n, m)$ can be restricted on the fiber space over $Y_{n,m}$. For an embedding of an elliptic curve $i_A : T \to \mathbb{P}^n$ and $w \in W(n, m)$, we write $w \circ i_A : T \to \mathbb{P}^n$ as $i_{w(A)}$. It should be noted that $i_A$ and $i_{w(A)}$ are not determined only by $A$ and $w(A)$, respectively, e.g., both $i_A = (1, \varphi(u), \varphi'(u), \ldots, \varphi^{(n-1)}(u))$ and $i_{A} = (1, \varphi(-u), \varphi'(-u), \ldots, \varphi^{(n-1)}(-u))$ satisfy (i) and (ii).

In this section, we consider the action of $W(n, m)$ on $Y_{n,m}$ via the orbit of an embedding of an elliptic curve $i_A : T \to \mathbb{P}^n$ for a parameter $A \in Y_{n,m}$.

We use the following notation (**):
$u$: a point on $T$;
$P_i$: a point in $\mathbb{P}^n$ determined by the $i$-th column of the parameter $A$;
$u_i$ ($1 \leq i \leq m$): the point on $T$ s.t. $i_A(u_i) = P_i$;
$E$: the total transform of the class of a hyper-plane in $X_A$;
therefore, there exists such a point \( v \in T \);  

\[ A' := w(A); \]

\( P_i' \): a point in \( \mathbb{P}^n \) determined by the \( i \)-th column of the parameter \( A' \);  

\( u'_i := \iota^{-1}_A \circ w \circ \iota_A(u) = u \).

\( u'_i (1 \leq i \leq m) \): the point on \( T \) s.t. \( \iota_A(u'_i) = P_i' \);

\( E' \): the total transform of the class of a hyper-plane in \( X_{A'} \);

\( v' \): a point in \( T \) s.t. \( (n+1)[v] = \iota_{A'}^*(E') \);

\[ X_A \twoheadrightarrow \{ v \} \rightarrow X_{A'} \]

\[ \iota_A \quad \iota_{A'} \]

\[ \downarrow \quad \downarrow \]

\[ T \quad \Pic(T) \]

\[ \iota_A \quad \iota_{A'} \]

\[ \downarrow \quad \downarrow \]

\[ \Pic(X_A) \quad \Pic(X_{A'}) \]

Remark 4.1. In the above diagrams, \( P_i' \neq w(P_i) \) may occur, and therefore \( u'_i \neq \iota^{-1}_A \circ w \circ \iota_A(u_i) \) also may occur. For example, we have \( r_{ij}(P_i) = P_i \) and \( P'_i = P_j \) (\( P'_i \) is the \( i \)-th column of \( r_{ij}(A) \)).

**Theorem 4.1.** Suppose that \( w^*(E') \) and \( w^*(E'_i) \) are represented as \( b_i^0E + \sum_{j=1}^m b_j^iE_j \) and \( b_i^0E + \sum_{j=1}^m b_j^iE_j \), respectively. The points \( u_i' \in T \) (1 \( \leq i \leq m \)) and \( u' \in T \) are given by the formulae:

\[ u'_i = (n+1)b_i^0v + \sum_{j=1}^m b_j^i u_j \quad (7) \]

\[ u' = u. \quad (8) \]

Moreover,

\[ (n+1)v' = (n+1)b_i^0v + \sum_{j=1}^m b_j^i u_j \quad (9) \]

holds.

**Proof.** From the definition of \( \iota_w(A) \), (8) is trivial. From the relation \( \iota_A^* \circ w^* = \iota_{A'}^* : \Pic(X_{A'}) \rightarrow \Pic(T) \), the divisor class

\[ \iota_A^* \circ w^*(E_i) = \iota_A^*(b_i^0E + \sum_{j=1}^m b_j^iE_j) = (n+1)b_i^0[v] + \sum_{j=1}^m b_j^i[u_j] \]

coincides with the class

\[ \iota_{A'}^*(E_i) = [u'_i] \]

therefore, we have (9) by Abel’s theorem. Similarly, since the divisor class

\[ \iota_A^* \circ w^*(E) = \iota_A^*(b_i^0E + \sum_{j=1}^m b_j^iE_j) = (n+1)b_i^0[v] + \sum_{j=1}^m b_j^i[u_j] \]
coincides with the class 
\[ \iota_A^*(E) = (n + 1)[v'], \]
we have (9).

\[ \square \]

5 Representatives

In order to investigate the actions of the Weyl group, we should choose a suitable representative of the parameter \( A \). In this section, we consider realizations of \( A \) and \( \iota_A \), and study normalizations of \( w(A) \) by \( \text{PGL}(n + 1) \).

Let \( A \in Y_{n,m} \), then, there exist \( \tau \) (\( \text{Im} \tau > 0 \)) and an embedding \( \iota_A : T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \rightarrow \mathbb{P}^n \) s.t. (i) the degree of \( \iota_A \) is \( n + 1 \); (ii) \( \iota_A(T) \) contains \( P_i \). We use the notation \((**)\) in the previous section.

Theorem 5.1. Suppose that \( w^*(E') \) is represented as \( b_0^0 E + \sum_{j=1}^m b_0^j E_j \). Then, there exists a translation \( \sigma : T \rightarrow T : \sigma(u) = u - s \) and a projective linear transformation \( G \in \text{PGL}(n + 1) \) s.t.

\[ \iota_A \circ \sigma = G \circ \iota_{w(A)} \tag{10} \]

holds, where \( s \in T \) satisfies

\[ (n + 1)s = (n + 1)b_0^0 v + \sum_{j=1}^m b_0^j u_j - (n + 1)v. \tag{11} \]

Remark 5.1. By equality (11) the following diagram commutes

\[ \begin{array}{ccc}
X_A & \xrightarrow{w} & X_{w(A)} \\
\iota_A & \nearrow & \downarrow \iota_{w(A)} \\
T & \searrow & T \\
& \iota_A \circ \sigma & \end{array} \]

Proof. The relation (10) follows from Prop.3.1. Thus, we have

\[ (n + 1)[v'] = \iota_{w(A)}^*(E') = (G^{-1} \circ \iota_A \circ \sigma)^*(E') = (\iota_A \circ \sigma)^*(E') = \sigma^*((n + 1)[v]) = (n + 1)[v + s]; \]

therefore, \( (n + 1)(v' - v) = (n + 1)s \in T \) holds. Further, Theorem 4.1 implies the equality (11).

\[ \square \]

In order to compute the action of \( w \) on a general point in \( \mathbb{P}^n \), we have to determine \( G \in \text{PGL}(n + 1) \) explicitly. For that purpose, it is sufficient to compute \( n + 2 \) points in \( \mathbb{P}^n \) in general position.
i) The points \( G(P_i') (i = 1, 2, \ldots, m) \) are calculated as
\[
G(P_i') = G \circ \iota_w(A)(u_i') = \iota_A \circ \sigma(u_i') = \iota_A(u_i' - s),
\]
where \( s \) and \( u_i' \) are given by (11) and (7), respectively.

ii) The point \( G \circ \iota_w(A)(u') (u \neq u_i) \) is calculated as
\[
G \circ \iota_w(A)(u') = \iota_A \circ \sigma(u') = \iota_A(u' - s) = \iota_A(u - s).
\]
The last equality follows from (8).

5.1 Example 1

Let us calculate \( G \) for the embedding \( \iota_A(u) = \iota(1, \varphi(u), \varphi'(u), \ldots, \varphi^{(n-1)}(u)) \). In this case, we can choose \( v \) as \( v = 0 \).

i) For \( r_{1,2,\ldots,n+1} \). From (10), the action is given by:
\[
\begin{pmatrix}
I_{n+1} & \cdots & a''_{ij} & \cdots & x''_i \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\xrightarrow{\text{SCT}}
\begin{pmatrix}
I_{n+1} & \cdots & a''_{ij} & \cdots & x''_i \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\xrightarrow{G}
\begin{pmatrix}
A' | x' \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix}
= \begin{pmatrix}
\iota_A(u'_1 + s), \iota_A(u'_2 + s), \ldots, \iota_A(u'_m + s) | \varpi
\end{pmatrix},
\]
where \( u'_i \) and \( s \) are calculated by Theorem 4.1 and (11) as
\[
u'_i = \begin{cases}
u_i - \sum_{j=1}^{n+1} u_j & (1 \leq i \leq n+1) \\
u_i & (n+2 \leq i \leq m)
\end{cases},
\]
\[
s = -\frac{n-1}{n+1} \sum_{j=1}^{n+1} u_j.
\]

Thus, \( G \in \text{PGL}(n+1) \) is determined by
\[
G(I_{n+1}, (\iota_A(0))') = (\iota_A(u'_1 + s), \iota_A(u'_2 + s), \ldots, \iota_A(u'_{n+1} + s), \iota_A(s)),
\]

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where \((\nu_A(0))'\) is

\[
(\nu_A(0))' = 
\begin{pmatrix}
0 & -1 & 0 & -1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\nu_A(u_2) \cdots \nu_A(u_{n+1}) & \nu_A(u_1) & \cdots & \nu_A(u_{n+1}) \\
0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1
\end{pmatrix},
\]

Here, \(G\) can be decomposed as \(G_2 \circ G_1\), where

\[
G_1(I_{n+1}, (\nu_A(0))') = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}.
\]

Thus, \(G_1\) and \(G_2\) are explicitly written as

\[
G_1 = 
\begin{pmatrix}
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
\nu_A(u_2) \cdots \nu_A(u_{n+1}) & \nu_A(u_1) & \cdots & \nu_A(u_{n+1}) \\
0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1
\end{pmatrix},
\]

and \(G_2^{-1} = (\text{diag}(A^{-1}\nu_A(s)))^{-1}A^{-1}\); thus, \(G_2 = A\text{ diag}(A^{-1}\nu_A(s))\).

ii) For \(r_{i,j}\) \([3]\), we have \(u_i' = u_j\), \(u_j' = u_i\), \(u_k' = u_k\ (k \neq i,j)\), \(s = 0\), and \(G\) is the identity.

### 5.2 Example 2

Let us calculate \(G\) for the embedding proposed by Kajiwara \textit{et al.} \([4]\):

\[
\nu_A(u) = \left\{ \frac{u - u_1 - \varepsilon}{u - u_1}, \ldots, \frac{u - u_{n+1} - \varepsilon}{u - u_{n+1}} \right\},
\]

where \([\varepsilon]\) is a theta function whose zero points are \(\mathbb{Z} + \mathbb{Z}\tau\) with the order 1, and \(\varepsilon\) is an extra-parameter. It should be noted that \(A_1, \ldots, n+1 = (\nu_A(u_1), \ldots, \nu_A(u_{n+1})) = I_{n+1}\) holds. In this case, we chose \(G\) in a manner that \(\nu := G \circ w \circ \nu_A\) is written in the form

\[
\nu(u) = \left\{ \frac{u - u_1' - \varepsilon}{u - u_1'}, \ldots, \frac{u - u_{n+1}' - \varepsilon}{u - u_{n+1}'} \right\}.
\]

From the diagram

\[
\begin{aligned}
X_A & \xrightarrow{\nu_A} X_{w(A)} \xrightarrow{G} X_{G \circ w(A)} \\
T & \xrightarrow{\nu_A} T
\end{aligned}
\]

\[
\begin{aligned}
X_A & \xrightarrow{\nu} X_{w(A)} \xrightarrow{G} X_{G \circ w(A)} \\
T & \xrightarrow{\nu} T
\end{aligned}
\]

\[
\begin{aligned}
X_A & \xrightarrow{\nu} X_{w(A)} \xrightarrow{G} X_{G \circ w(A)} \\
T & \xrightarrow{\nu} T
\end{aligned}
\]

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we have
\[(n + 1)[v'] = \iota_w^*(A)(E')\]
\[= (G^{-1} \circ \iota_{\Pi} \circ \sigma)^*(E')\]
\[= (\iota_{\Pi} \circ \sigma)^*(E')\]
\[= \sigma^*((n + 1)[\overline{v}]\]
\[= (n + 1)[\overline{v} + s],\]

and \(v, v'\) and \(\overline{v}\) are given by
\[(n + 1)v = \varepsilon + \sum_{i=1}^{n+1} u_i\]
\[(n + 1)\overline{v} = \varepsilon + \sum_{i=1}^{n+1} u_i'\]
\[(n + 1)v' = (n + 1)b_0^n v + \sum_{j=1}^{m} b_0^j u_j.\]

i) For \(r_{1,\ldots,n+1}\). From theorem 4.1 we have
\[u_i' = \begin{cases} 
  u_i + \varepsilon & (1 \leq i \leq n+1) \\
  u_i & (n+2 \leq i \leq m) 
\end{cases},\]
\[(n + 1)s = -\varepsilon - \overline{v}.\]

Here, we can choose \(\varepsilon\) as \(\varepsilon = -\varepsilon\), then we have \(s = 0\).

Further, \(G \in \text{PGL}(n + 1)\) is determined by
\[G \circ w(A_1,\ldots,n+2) = G(I_{n+1}, (\iota_A(u_{n+2}))') = (I_{n+1}, (\iota_{\Pi}(u_{n+2})).\]

and
\[(\iota_A(u_{n+2}))' = \left( \frac{[u_{n+2} - u_1]}{[u_{n+2} - u_1 - \varepsilon]} : \ldots : \frac{[u_{n+2} - u_{n+1}]}{[u_{n+2} - u_{n+1} - \varepsilon]} \right).\]

Hence, \(G\) is the identity.

ii) For \(r_{k,k+1}\), we have
\[u_i' = \begin{cases} 
  u_i & (i \neq k, k+1) \\
  u_{k+1} & (i = k) \\
  u_k & (i = k+1) 
\end{cases},\]

and
\[(n + 1)s = \begin{cases} 
  \varepsilon - \overline{v} & (k \neq n + 1) \\
  \varepsilon - \overline{v} + u_{n+1} - u_{n+2} & (k = n + 1) 
\end{cases}.\]
Here, we can choose \( \mathfrak{g} \) as

\[
\mathfrak{g} = \begin{cases} 
\varepsilon & (k \neq n + 1) \\
\varepsilon + u_{n+1} - u_{n+2} & (k = n + 1)
\end{cases},
\]

then we have \( s = 0 \).

For \( r_{k,k+1} \) \( (k \neq n + 1) \), \( G \) is the identity. We calculate \( G \) for \( r_{n+1,n+2} \). From

\[
G = \begin{pmatrix}
1 & \frac{u_{n+2} - u_1 - \varepsilon}{u_{n+2} - u_1} & \cdots & \frac{u_{n+2} - u_n - \varepsilon}{u_{n+2} - u_n} & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{u_{n+2} - u_n - \varepsilon}{u_{n+2} - u_n} & \cdots & 1 & \frac{u_{n+2} - u_{n+1} - \varepsilon}{u_{n+2} - u_{n+1}} & 0 \\
\frac{u_{n+2} - u_{n+1} - \varepsilon}{u_{n+2} - u_{n+1}} & \frac{u_{n+2} - u_{n+1} - \varepsilon}{u_{n+2} - u_{n+1}} & \cdots & 1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & \frac{u_{n+2} - u_1 - \varepsilon}{u_{n+1} - u_1} & \cdots & \frac{u_{n+2} - u_n - \varepsilon}{u_{n+1} - u_n} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{u_{n+2} - u_n - \varepsilon}{u_{n+1} - u_n} & \cdots & 1 & \frac{u_{n+2} - u_{n+1} - \varepsilon}{u_{n+1} - u_{n+1}} \\
\frac{u_{n+2} - u_{n+1} - \varepsilon}{u_{n+1} - u_{n+1}} & \frac{u_{n+2} - u_{n+1} - \varepsilon}{u_{n+1} - u_{n+1}} & \cdots & 1 & 1
\end{pmatrix},
\]

and by decomposing \( G \) into \( G_1 = G_2 \circ G_2 \) as example 1, we have

\[
G_1 = \text{diag} \left( \begin{pmatrix}
\frac{u_{n+2} - u_1}{u_{n+2} - u_1} & \cdots & \frac{u_{n+2} - u_n}{u_{n+2} - u_n} \\
\vdots & \ddots & \vdots \\
\frac{u_{n+2} - u_n}{u_{n+2} - u_n} & \cdots & 1
\end{pmatrix} \times \begin{pmatrix}
1 & \frac{u_{n+2} - u_1}{u_{n+2} - u_1} & \cdots & \frac{u_{n+2} - u_n}{u_{n+2} - u_n} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{u_{n+2} - u_n}{u_{n+2} - u_n} & \cdots & 1
\end{pmatrix}
\right)
\]

and

\[
G_2 = \begin{pmatrix}
[\frac{u_{n+2} - u_1 - \varepsilon}{u_{n+1} - u_1}] & \cdots & [\frac{u_{n+2} - u_n - \varepsilon}{u_{n+1} - u_n}] & [\frac{-\varepsilon}{u_{n+1} - u_{n+2}}] \\
\vdots & \ddots & \vdots & \vdots \\
[\frac{u_{n+2} - u_n - \varepsilon}{u_{n+1} - u_n}] & \cdots & [\frac{u_{n+2} - u_{n+1} - \varepsilon}{u_{n+1} - u_{n+1}}] & [\frac{-\varepsilon}{u_{n+1} - u_{n+2}}]
\end{pmatrix}.
\]

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References

1. Coble, A. B.: Algebraic geometry and theta functions, American Mathematical Society Colloquium Publications, vol. X, American Mathematical Society, Providence, 1929

2. Dolgachev, I. and Ortland, D.: Point sets in projective spaces and theta functions, Astérisque Soc. Math. de France 165, 1988

3. Kac, V.: Infinite dimensional lie algebras, 3rd ed., Cambridge University Press, Cambridge, 1990

4. Kajiwara, K., Masuda, T., Noumi, M., Ohta, Y. and Yamada, Y.: 10\( E_9 \) solution to the elliptic Painlevé equation, J. Phys. A 36 (2003), L263–L272
Point configurations, Cremona transformations and the elliptic difference Painlevé equation Preprint

\[\text{lin.SI/0411003}\]
5. Looijenga, E.: Rational surfaces with an anti-canonical cycle, *Annals of Math.* 114 (1981), 267–322

6. Sakai, H. Rational surfaces associated with affine root systems and geometry of the Painlevé equations, *Commun. Math. Phys.* 220 (2001), 165–229

7. Takenawa, T.: Integrability of \(n\)-dimensional dynamical systems of type \(E_7^{(1)}\) and \(E_8^{(1)}\), *Preprint* [nlin/0409051](https://arxiv.org/abs/nlin/0409051)