EXPANDING SOLITONS TO THE HERMITIAN CURVATURE FLOW ON COMPLEX LIE GROUPS

MATTIA PUJIA

Abstract. We investigate the algebraic structure of complex Lie groups equipped with left-invariant metrics which are expanding semi-algebraic solitons to the Hermitian curvature flow (HCF). We show that the Lie algebras of such Lie groups decompose in the semidirect product of a reductive Lie subalgebra with their nilradicals. Furthermore, we give a structural result concerning expanding semi-algebraic solitons on complex Lie groups. It turns out that the restriction of the soliton metric to the nilradical is also an expanding algebraic soliton and we explain how to construct expanding solitons on complex Lie groups starting from expanding solitons on their nilradicals.

1. Introduction

In 2011 Streets and Tian introduced a new flow of Hermitian metrics called Hermitian curvature flow (HCF) [21]. The flow evolves an initial metric in the direction of a Ricci-type tensor of the Chern curvature modified with some first order terms in the torsion. The defining equation is strictly parabolic and when the initial metric is Kähler the HCF reduces to the Kähler-Ricci flow.

The flow is defined as follow. Let \((M, g)\) be a Hermitian manifold of complex dimension \(n\), with Chern connection \(\nabla\) and Chern curvature tensor \(\Omega\). Let \(S_{ij} = g^{lk}\Omega_{ki}g_{lj}\) be the \((1, 1)\)-tensor obtained by contracting \(\Omega\) in the first two entries and let

\[ K(g) := S(g) - Q(g), \]

where \(Q\) is a \((1, 1)\)-tensor quadratic in the torsion components (see [21] for the precise definition of \(Q\)). Then HCF is defined by

\[ \partial_t g_t = -K(g_t), \quad g_{t=0} = g_0, \]

where \(g_0\) is an initial Hermitian metric on \(M\). In [21] the tensor \(Q\) is chosen in order to make the flow satisfying a gradient-type equation. Nevertheless, since \(Q(g)\) contains only first order terms in \(g\), different choices of \(Q\) do not affect the parabolicity of the flow and these lead to different interesting flows (see e.g. [17], [18], [19], [20], [22], [23], [24], [25], and the references therein).

In this paper we focus on soliton solutions to the HCF, i.e. Hermitian metrics satisfying

\[ K(g) = cg + \mathcal{L}_X g, \]

for some \(c \in \mathbb{R}\) and a complete holomorphic vector field \(X\), where \(\mathcal{L}\) denotes the Lie derivative. By definition, \(K\) is both scale invariant and diffeomorphisms equivariant. Therefore, if \(g_0\) satisfies (2) then the solution to (1) satisfies \(g_t = s(t)\varphi_t^*g_0\), where \(s(t) > 0\) and \(\varphi_t : M \to M\) are respectively a smooth scaling function and a one-parameter family of biholomorphisms.

Date: March 6, 2019.
2010 Mathematics Subject Classification. Primary 53C15; Secondary 53B15, 53C30, 53C44.
This work was supported by G.N.S.A.G.A. of I.N.d.A.M.
In complex Lie groups context it is quite natural to focus on semi-algebraic solitons, which are left-invariant metrics $g_0$ such that the solutions to (1) have the form $g_t = s(t) \varphi_t^* g_0$ and $\varphi_t$ is a Lie group automorphism for every $t$. If further $\partial_t \varphi_t^{-1}|_{t=0}$ is $g$-self-adjoint, then $g$ is called an algebraic soliton. Semi-algebraic and algebraic solitons of other flows have been studied in [1], [5], [14], [15], [16].

Now we state the main result of the paper. Let $(G, g)$ be a complex Lie group equipped with a left-invariant Hermitian metric and consider the orthogonal splitting of its Lie algebra $g = r \oplus n$, where $n$ is the nilradical of $g$. Denote by $g_n$ the pull-back of $g$ to the Lie group $N$ of $n$. Then we have

**Theorem 1.1.** The metric $g$ is an expanding (i.e. $c < 0$) semi-algebraic soliton to HCF if and only if $g_n$ is an expanding algebraic soliton to HCF on $N$, $r$ is a reductive Lie subalgebra, $\sum [ad_{r_i}|n, ad_{\bar{r}_i}|n] = 0$ for any unitary basis $\{r_i\}$ of $r$, and

$$K(g_t)(X, \bar{Y}) = cg_t(X, \bar{Y}) + \frac{1}{2} \text{tr}(ad_X|n ad_{\bar{Y}}|n) - \frac{1}{2} \text{tr} ad_X \cdot \text{tr} ad_{\bar{Y}},$$

for any $X, Y \in r$, where $g_t$ is the pull-back of $g$ to the Lie group of $r$.

Note that if $G$ is unimodular, then the expression of $K(g_t)$ in Theorem 1.1 simplifies to

$$K(g_t)(X, \bar{Y}) = cg_t(X, \bar{Y}) + \frac{1}{2} \text{tr}(ad_X|n ad_{\bar{Y}}|n).$$

Our interest in expanding algebraic solitons on complex unimodular Lie groups comes from [7], where it is proved that expanding algebraic solitons on such Lie groups are limit points to the normalized HCF. Indeed, when $(G, g_0)$ is a complex unimodular Lie group equipped with a left-invariant metric, the solution $g_t$ to the HCF starting from $g_0$ is defined for every positive $t$ and $(G, (1 + t)^{-1}g_t)$ converges in Cheeger-Gromov sense to $(\bar{G}, \bar{g})$, where $\bar{G}$ is a complex unimodular Lie group and $\bar{g}$ is an algebraic soliton.

Next we observe that in the solvable case we can improve Theorem 1.1 by giving an explicit description of $g_t$.

**Corollary 1.2.** Assume $G$ unimodular and solvable. Then, $g$ is an expanding algebraic soliton to HCF if and only if $g_n$ is an expanding algebraic soliton to HCF on $N$, the Lie group $G$ is standard (i.e. $g = r \oplus n$ with $r$ abelian), $\sum [ad_{r_i}|n, ad_{\bar{r}_i}|n] = 0$ for any unitary basis $\{r_i\}$ of $r$, and

$$g_t(X, \bar{Y}) = -\frac{1}{2c} \text{tr}(ad_X|n ad_{\bar{Y}}|n),$$

for any $X, Y \in r$.

The proof of Theorem 1.1 is mainly based on real geometric invariant theory (GIT), in the same fashion as in [12].

Similar results, concerning the Ricci flow on different homogeneous spaces, can be found in [6] and [16]. However, as pointed out by Lafuente and Lauret in [6], for the Ricci flow there is a limitation given by Alekseevskiis conjecture. Indeed, if Alekseevskiis conjecture is confirmed, then any Ricci flow expanding algebraic soliton $(G/H, g)$ should be diffeomorphic to an Euclidean space [5] and thus, accordingly, only solvmanifolds could admit expanding algebraic solitons to the Ricci flow. In the HCF case such a limitation does not exist. As shown in [7], also semisimple
complex Lie groups admit soliton metrics. Specifically, a complex Lie group $G$ admits a left-invariant static Hermitian metric, i.e. a metric satisfying the Einstein-type equation

$$K(g) = cg,$$

for some constant $c \in \mathbb{R}$, if and only if the group is semisimple and the static metric is the ‘canonical metric’ induced by the Killing form. Hence, we have a wider set of expanding algebraic solitons for the HCF, with algebraic structures completely classified by Theorem 1.1 in the case of complex Lie groups.

The paper is organized as follows. In Section 2 we recall main results about HCF on complex Lie groups and GIT on Lie groups. In Section 3 we prove Theorem 1.1 and its corollary. Finally, in Section 4 we apply Corollary 1.2 to construct explicit examples of expanding algebraic solitons to HCF on 4-dimensional solvable complex unimodular Lie groups.

Notation and conventions. By a complex Lie group we mean a Lie group endowed with a bi-invariant complex structure (i.e. the multiplication is a holomorphic map).

Acknowledgments. The research of the present paper was originated by some conversations with Jorge Lauret, during a visiting period of the author at FaMAF (Cordoba). The author is very grateful to Lauret for many useful suggestions and insights on the problems studied in the paper, and to Luigi Vezzoni for his comments on a preliminary version of the paper. The author would like to thank the referee for him/her constructive comments, which helped to improve the paper.

2. HCF and GIT results

In this section we recall some results on the HCF on complex Lie groups and GIT which will be useful in the sequel.

2.1. HCF on complex Lie groups. The following proposition characterizes the HCF tensor on complex Lie groups.

Proposition 2.1. [7] Let $G$ be a complex Lie group equipped with a left-invariant Hermitian metric $g$. Then

$$K(g) = \text{Ric}^{1,1} + \hat{Q},$$

where $\text{Ric}^{1,1}$ is the $(1,1)$-part of the Ricci tensor of $g$ and

$$\hat{Q}(Z, W) := \frac{1}{2} \text{tr} \text{ad}_Z \cdot \text{tr} \text{ad}_W.$$

Here, $Z, W$ are left-invariant vector fields of type $(1,0)$.

It is well known (see e.g. [3]) that the Ricci tensor of a left-invariant metric $g$ on a Lie group $G$ can be written as

$$\text{Ric} = M - \frac{1}{2} B - S(\text{ad}_H),$$

where, for any $X, Y$ in the Lie algebra $(\mathfrak{g}, \mu)$ of $G$,

$$M(X, Y) = -\frac{1}{2} \sum_k g(\mu(X, X_k), \mu(Y, X_k)) + \frac{1}{4} \sum_{k,j} g(\mu(X_k, X_j), X)g(\mu(X_k, X_j), Y).$$
Here \( \{X_r\} \) denotes an orthonormal basis of \( g \); \( H \) is the mean curvature vector given by the relation \( g(H, X) = \text{tr} \text{ad}_X \), for any \( X \in g \), and
\[
S(\text{ad}_H)(X, Y) = \frac{1}{2}(g(\mu(H, X), Y) + g(\mu(H, Y), X));
\]
\( B(X, Y) = \text{tr}(\text{ad}_X \text{ad}_Y) \) is the Killing form of \( g \). If further \( G \) is a complex Lie group and \( g \) is a left-invariant Hermitian metric \( g \), then the \((1,1)\)-part of the Ricci tensor satisfies
\[
\text{Ric}^{1,1} = M - S(\text{ad}_H).
\]
Finally, when the Lie group \( G \) is unimodular \( \text{Ric}^{1,1} = M \), since the \( S(\text{ad}_H) \)-term vanishes.

Although our goal is to study solutions to the HCF on complex Lie groups, our results hold true for left-invariant solutions \( g_t \) to the \( K \)-flow
\[
\partial_t g_t = -K(g_t), \quad g_{t=0} = g_0,
\]
on Lie groups, where
\[
K(g) := M - S(\text{ad}_H) + \hat{Q}
\]
and \( \hat{Q}(X, Y) := \frac{1}{2} \text{tr} \text{ad}_X \cdot \text{tr} \text{ad}_Y \). From now on we focus on this more general setting and we obtain the results stated in the introduction as special cases.

**Definition 2.2.** A left-invariant metric \( g \) on a Lie group \( G \) is a semi-algebraic \( K \)-soliton if its \( K \)-tensor satisfies
\[
K(g) = cg + \frac{1}{2}(g(D, \cdot) + g(\cdot, D)) \quad c \in \mathbb{R}, \quad D \in \text{Der}(g).
\]
If further \( D^t \in \text{Der}(g) \), then the soliton is algebraic.

Note that, we can regard \( K(g) \) as an endomorphism
\[
K_g : g \to g
\]
of the Lie algebra of \( G \) via \( g(K_g \cdot, \cdot) = K(g)(\cdot, \cdot) \). Thus, the semi-algebraic \( K \)-soliton condition can be written in terms of \( K_g \) as
\[
K_g = c I + \frac{1}{2}(D + D^t), \quad c \in \mathbb{R}, \quad D \in \text{Der}(g).
\]

**Remark 2.3.** Every semi-algebraic \( K \)-soliton is a soliton in the usual sense. Indeed, if \( G \) is a simply-connected Lie group, then the solution to (5) starting from a semi-algebraic \( K \)-soliton \( g \) is \( g_t = (-ct + 1)\varphi_t^*g \), where \( \varphi_t \in \text{Aut}(G) \) is the unique automorphism such that \( d\varphi_t|_e = e^{-tD/2} \in \text{Aut}(g) \).

### 2.2. GIT on Lie groups.

Let \( N \) be a Lie group with Lie algebra \( (n, \mu_0) \). The Lie bracket of \( n \) is an element of the variety of Lie algebras (see e.g. \cite{9, 11, 14})
\[
\mathcal{C} = \{ \mu \in \Lambda^2 n^* \otimes n : \mu \text{ satisfies the Jacobi identity} \}.
\]
By changing \( \mu \in \mathcal{C} \) we obtain all the possible structures of Lie algebra on the vector space \( n \). The Lie group \( \text{GL}(n) \) acts canonically on
\[
V := \Lambda^2 n^* \otimes n
\]
by
\[
A \cdot \mu(\cdot, \cdot) = A \mu(A^{-1} \cdot, A^{-1} \cdot).
\]
The action induces the Lie algebra representation \( \pi : \text{End}(n) \rightarrow \text{End}(\Lambda^2 n^* \otimes n) \) given by
\[
(\pi(E)\mu)(X,Y) := E(\mu(X,Y)) - \mu(E(X,Y)) - \mu(X,E(Y)), \quad X,Y \in n, \quad E \in \text{End}(n),
\]
and it satisfies \( \pi(D)\mu = 0 \), for any derivation \( D \in \text{Der}(n) \).

Now we fix a inner product \( g \) on \( g \) and for \( A, B \in \text{End}(n) \) we denote by
\[
\langle A, B \rangle := \text{tr} AB^t
\]
the scalar product induced on \( \text{End}(n) \), where the transpose is given with respect to \( g \). In order to simplify the notation we still denote with \( \langle \cdot, \cdot \rangle \) the scalar product induced on \( \Lambda^2 n^* \otimes n \). The pair \((\mu_0, g)\) induces a tensor \( M \) via (4). Using the metric, we can regard \( M \) as an endomorphism \( M_g : n \rightarrow n \).

By fixing \( g \) and changing \( \mu \) we obtain a different \( M \)-endomorphism in \( \text{End}(n) \) which we denote by \( M_{\mu} \). In this way, we have a map from \( C \) to \( \text{End}(n) \), \( \mu \mapsto M_{\mu} \). Note that by definition \( M_{\mu_0} = M_g \).

**Proposition 2.4.** The map
\[
\mu \mapsto \frac{4}{\|\mu\|^2} M_{\mu}
\]
from \( \Lambda^2 n^* \otimes n \setminus \{0\} \) to \( \text{End}(n) \) is a moment map, in the sense of GIT, i.e.
\[
\langle M_{\mu}, E \rangle = \frac{1}{4} \langle \pi(E)\mu, \mu \rangle,
\]
for any \( E \in \text{End}(n) \) and \( \mu \in \Lambda^2 n^* \otimes n \setminus \{0\} \).

Next, we recall a stratification theorem involving \( V \) proved in [12]. Fix a basis in \( n \) and for any element \( \mu \in V \) denotes with \( \mu_{ij}^k \) its components. Moreover, let
\[
\mathcal{N} := \{ \mu \in C : \mu \text{ is nilpotent} \}
\]
be the variety of nilpotent Lie algebras,
\[
t^+ := \{ \beta = \text{diag}(a_1, \ldots, a_n) \in t : a_1 \leq \ldots \leq a_n \}
\]
and \( \alpha_{ij}^k := E_{kk} - E_{ii} - E_{jj} \), where \( E_{ij} \) is the zero matrix with 1 in the \( ij \)-entry. Here, \( t \) denotes the maximal torus algebra in \( \mathfrak{gl}_n(\mathbb{R}) \) given by the \( n \times n \) diagonal matrices.

**Theorem 2.5.** There exists a finite subset \( B \subset t^+ \), such that every \( \beta \in B \) satisfies
\[
\text{tr } \beta = -1 \quad \text{and} \quad V \setminus \{0\} = \bigcup_{\beta \in B} S_{\beta} \quad (\text{disjoint union}),
\]
where \( \{S_{\beta}\}_{\beta \in B} \) is a family of \( \text{GL}_n(\mathbb{R}) \)-invariant subset of \( V \). Given \( \mu \in S_{\beta} \)
\[
\beta + \|\beta\|^2 I \quad \text{is positive definite} \quad \forall \beta \in B \quad \text{such that} \quad S_{\beta} \cap \mathcal{N} \neq \emptyset,
\]
\[
\langle [\beta, D], D \rangle \geq 0, \quad \forall D \in \text{Der}(\mu) \quad (\text{equality holds} \iff [\beta, D] = 0)
\]
and
\[
\|\beta\| \leq \frac{4}{\|\mu\|^2 \|M_{\mu}\|}.
\]

The equality in Moreover, if \( \mu \in S_{\beta} \) satisfies
\[
\text{min}\{\langle \beta, \alpha_{ij}^k \rangle : \mu_{ij}^k \neq 0\} = \|\beta\|^2,
\]
and
\[
\beta + \|\beta\|^2 I \quad \text{is positive definite} \quad \forall \beta \in B \quad \text{such that} \quad S_{\beta} \cap \mathcal{N} \neq \emptyset,
\]
\[
\langle [\beta, D], D \rangle \geq 0, \quad \forall D \in \text{Der}(\mu) \quad (\text{equality holds} \iff [\beta, D] = 0)
\]
and
\[
\|\beta\| \leq \frac{4}{\|\mu\|^2 \|M_{\mu}\|}.
\]

The equality in Moreover, if \( \mu \in S_{\beta} \) satisfies
\[
\text{min}\{\langle \beta, \alpha_{ij}^k \rangle : \mu_{ij}^k \neq 0\} = \|\beta\|^2,
\]
then
\[(14)\quad \langle \pi (\beta + \|\beta\|^2 I)\mu, \mu \rangle \geq 0\]
and
\[(15)\quad \text{tr} \beta D = 0, \quad \forall D \in \text{Der}(\mu).\]

The equality in (14) holds if and only if \(\beta + \|\beta\|^2 I \in \text{Der}(\mu)\).

Remark 2.6. Note that condition (13) is always satisfied by some element in the \(O(n)\)-orbit of \(\mu\). If condition (13) is satisfied and \(\mu \in S_\beta\), then
\[\beta = \text{mcc}\{\alpha^k_{ij} : \mu^k_{ij} \neq 0\}.\]

Here with \(\text{mcc}(X)\) we mean the unique element of minimal norm in the convex hull \(\text{CH}(X)\) of a subset \(X \subset \mathfrak{t}\).

3. Structure of solitons on Lie groups

Let \((G, g)\) be a Lie group equipped with a left-invariant metric. Let \((\mathfrak{g}, [\cdot, \cdot])\) be the Lie algebra of \(G\) and \(\langle \cdot, \cdot \rangle\) the inner product induced by \(g\) on \(\mathfrak{g}\). Let
\[\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}\]
be the orthogonal decomposition of \(\mathfrak{g}\), where \(\mathfrak{n}\) is the nilradical of \(\mathfrak{g}\), and
\[\lambda := [\cdot, \cdot]|_{\mathfrak{r} \times \mathfrak{r}}, \quad \sigma := [\cdot, \cdot]|_{\mathfrak{r} \times \mathfrak{n}}, \quad \mu := [\cdot, \cdot]|_{\mathfrak{n} \times \mathfrak{n}}.\]

Note that \(\lambda\) can be further decomposed in \(\lambda_0 : \mathfrak{r} \times \mathfrak{r} \to \mathfrak{r}\) and \(\lambda_1 : \mathfrak{r} \times \mathfrak{r} \to \mathfrak{n}\).

Let \(\beta\) such that \(\mu \in S_\beta\) and define \(E_\beta \in \text{End}(\mathfrak{g})\) by
\[E_\beta|_\mathfrak{r} = 0, \quad E_\beta|_\mathfrak{n} = \beta + \|\beta\|^2 I,\]
where \(I\) is the identity of \(\mathfrak{n}\). Moreover, we denote by \(M_\mathfrak{n} : \mathfrak{n} \to \mathfrak{n}\) the endomorphism of \(\mathfrak{n}\) defined by using (4) and, when \(\mathfrak{r}\) is a subalgebra of \(\mathfrak{g}\), we denote by \(M_\mathfrak{r} : \mathfrak{r} \to \mathfrak{r}\) the endomorphism of \(\mathfrak{r}\).

We have the following lemma.

Lemma 3.1. \[\text{Assume that } (\mathfrak{n}, \mu) \text{ satisfies } (13). \text{ Then,}\]
\[\langle \pi(E_\beta)[\cdot, \cdot][\cdot, \cdot] \rangle \geq 0\]
and
\[\langle \pi(E_\beta)[\cdot, \cdot][\cdot, \cdot] \rangle = \langle \pi(\beta + \|\beta\|^2 I)\mu, \mu \rangle + \sum \langle \beta + \|\beta\|^2 I[r_i, r_j], [r_i, r_j] \rangle + \sum 2\langle [\beta, \text{ad } r_i|\mathfrak{n}], \text{ad } r_i|\mathfrak{n} \rangle,\]
with \(\{r_i\}\) orthonormal basis of \(\mathfrak{r}\). Moreover, each term is non-negative.

Henceforth, when confusion cannot occur, we identify tensor \(K\) with its associated endomorphism \(K_g\). Also \(K\)-tensor components will be identify with their associated endomorphisms. The following lemma (whose proof is a direct computation) will be useful in the sequel.
Lemma 3.2. Assume $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}$. Then, for any $A, B \in \mathfrak{r}$ and $Z, W \in \mathfrak{n}$,

$$\langle MZ, W \rangle = \langle M_n Z, W \rangle + \frac{1}{2} \sum \langle [\text{ad}_{r_i}], \text{ad}_{r_i}^t \rangle Z, W,$$

$$\langle MA, B \rangle = \langle M_r A, B \rangle - \frac{1}{2} \text{tr}(\text{ad}_A \text{ad}_B^t),$$

$$\langle MA, W \rangle = -\frac{1}{2} \text{tr}(\text{ad}_A \text{ad}_W^t),$$

where $\{ r_i \}$ is an orthonormal basis of $\mathfrak{r}$.

Remark 3.3. Note that under the assumptions of Lemma 3.2 in matrix notation we have

$$M_g = \frac{1}{2} \begin{bmatrix} 2M_n - \tilde{B} & -\tilde{B} \\ -\tilde{B} & 2M_n + \sum [\text{ad}_{r_i}], \text{ad}_{r_i}^t] \end{bmatrix},$$

where $\tilde{B}$ is the operator given by $\langle \tilde{B} X, Y \rangle = \text{tr}(\text{ad}_X \text{ad}_Y^t)$, for all $X, Y \in \mathfrak{g}$, and the blocks are in terms of $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$.

From now on we assume that the metric $g$ satisfies the semi-algebraic expanding soliton equation

$$K_g = c I + \frac{1}{2} (D + D^t), \quad c < 0, \quad D \in \text{Der}(\mathfrak{g}),$$

and we set

$$F := S(\text{ad}_H + D),$$

where $S(A)$ is the symmetrization of $A \in \text{End}(\mathfrak{g})$.

Lemma 3.4. We have

$$c \text{tr} F + \text{tr} F^2 = 0.$$ 

Proof. Let $E := \text{ad}_H + D$, then $E \in \text{Der}(\mathfrak{g})$ and

$$\text{tr}(c I - \hat{Q} + F)E = \text{tr} M_g E = \frac{1}{4} \langle \pi(E) \rangle = 0,$$

from (9). Since $\hat{Q}$ is invariant under automorphisms of $\mathfrak{g}$, it follows

$$e^{-t \hat{D}} \hat{Q} e^{-t \hat{D}} = \hat{Q},$$

for any derivation $\hat{D} \in \text{Der}(\mathfrak{g})$. Differentiating at $t = 0$, we have $D^t \hat{Q} + \hat{Q} D = 0$, which implies

$$0 = \text{tr}(D^t \hat{Q} + \hat{Q} D) = 2 \text{tr} \hat{Q} D,$$

and the claim follows. 

□

Now we have

Proposition 3.5. The orthogonal complement $\mathfrak{r}$ of the nilradical $\mathfrak{n}$ is a reductive Lie subalgebra of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{n}.$$ 

Proof. Without loss of generality we can suppose that condition (13) holds, since the claim condition is preserved by the $O(n)$-action on $(\mathfrak{n}, \mu)$ (see [6] for more details).

To prove the statement, we study separately the case when either $\mathfrak{n}$ is abelian or not. In the former case, i.e. $\mu = 0$, let $E \in \text{End}(\mathfrak{g})$ be given by $E|_k = 0$ and $E|_n = I$. Since $\text{tr} F = \text{tr} F|_n$ ([6], Lemma 2.6), by [6] we have

$$cn + \text{tr} F = \text{tr}(c I - \hat{Q} + F)E = \text{tr} M_g E = \frac{1}{4} |\lambda_1|^2,$$

(18)
where \( n := \dim(n) \). Clearly, if \( n = 0 \) the claim follows. Otherwise, from [17] and [18] we have
\[
c n + \text{tr } F \geq 0 \quad \text{and} \quad \text{tr } F^2 \leq n^{-1}(\text{tr } F)^2,
\]
which force \( \lambda_1 = 0, F|_r = 0 \) and \( F|_n = tI \), for some \( t \geq 0 \).

Now assume \( n \) non-abelian and recall that [13] holds. Then, in view of Lemma 3.1 we have
\[
(19) \quad \langle \pi(E_\beta)[\cdot,\cdot],\cdot,\cdot \rangle = \langle \pi(E_\beta)\lambda_0,\lambda_0 \rangle + \langle \pi(E_\beta)\lambda_1,\lambda_1 \rangle + 2\langle \pi(E_\beta)\sigma,\sigma \rangle + \langle \pi(E_\beta)\mu,\mu \rangle \geq 0,
\]
which implies
\[
(20) \quad c \text{tr } E_\beta + \text{tr } FE_\beta = \text{tr}(cI - \hat{Q} + F)E_\beta = \text{tr } M_\beta E_\beta \geq 0,
\]
since [9] holds and \( \text{tr } \hat{Q}E_\beta = 0 \). Hence, the following equalities hold (since \( \text{tr } \beta = -1 \)):
\[
\text{tr } E_\beta^2 = \|\beta\|^2 \text{tr } E_\beta \quad \text{and} \quad \text{tr } FE_\beta = \|\beta\|^2 \text{tr } F,
\]
and using the above formulae we have
\[
\text{tr } F^2 \text{tr } E_\beta^2 \leq (\text{tr } FE_\beta)^2 \leq (\text{tr } F^2 \text{tr } E_\beta^2),
\]
which implies
\[
F = t E_\beta, \quad \text{for some } t \geq 0.
\]
Moreover, since [17] and [19] hold, we have
\[
c \text{tr } E_\beta + \text{tr } FE_\beta = 0
\]
and \( \lambda_1 = 0 \). Hence, the claim follows. \( \square \)

From the proof of Proposition 3.5 we can easily deduce the following result.

**Proposition 3.6.** Assume \( \mu \neq 0 \) and satisfying [13]. Then
\begin{enumerate}[i.]
\item \( [\beta, \text{ad}_r|_n] = 0 \),
\item \( \beta + \|\beta\|^2 I \in \text{Der}(n) \),
\item \( F = t E_\beta \), where \( t = \frac{\text{tr } F|_n}{-1 + \|\beta\|^2 \dim n} \).
\end{enumerate}

While, for \( \mu = 0 \) it follows \( F|_r = 0 \) and \( F|_n = tI \), where \( t = \frac{\text{tr } F|_n}{\dim n} \).

**Proof.** Items (i) and (ii) respectively follow from [11] and [14], since \( \langle \pi(E_\beta)[\cdot,\cdot],\cdot,\cdot \rangle = 0 \). The other claims follow directly by the previous proof. \( \square \)

**Remark 3.7.** Let \( a \) be the center of \( \mathfrak{r} \). In view of Proposition 3.5 \( \mathfrak{r} \) is a reductive Lie algebra and consequently it decomposes as
\[
\mathfrak{r} = \mathfrak{h} \oplus a,
\]
where \( \mathfrak{h} := \lambda(\mathfrak{r},\mathfrak{r}) \) is a semisimple Lie algebra. Hence, we can write \( \mathfrak{g} \) as
\[
\mathfrak{g} = (\mathfrak{h} \oplus a) \ltimes_\theta \mathfrak{n},
\]
where \( \theta(X) := \text{ad}_X|_a \), for all \( X \in \mathfrak{r} \). However, since \( a \) is an abelian subalgebra of \( \mathfrak{g} \), we can also write
\[
\mathfrak{g} = \mathfrak{h} \ltimes_\theta (a \ltimes_\theta \mathfrak{n}),
\]
where \( \theta(X) := \text{ad}_X|_{a \subset \mathfrak{n}} \) and \( \theta(X)A = 0 \), for any \( X \in \mathfrak{h} \) and \( A \in a \).

With the notations of Proposition 3.5 in mind, we have the following
Lemma 3.8. We have 
\[ \text{ad}_X^t|_n \in \text{Der}(n), \]
for any \( X \in \mathfrak{r}, \) and
\[ \sum [\text{ad}_{r_i}|_n, \text{ad}_{t_i}|_n] = 0, \]
where \( \{r_i\} \) is an orthonormal basis of \( \mathfrak{r}. \)

Proof. If \( n \) is abelian, i.e. \( \mu = 0, \) then the claims trivially follow. Let’s assume \( \mu \neq 0 \) and satisfying (13). It follows from Propositions 3.5 and 3.6 that \( F = tE_\beta, \) for some \( t \geq 0. \) Since \( \text{tr} F^2|_n = \text{tr} F^2, \) we have
\[ t = -\frac{c}{\|\beta\|^2} \quad \text{and} \quad F|_n = -c I - \frac{c}{\|\beta\|^2} \beta. \]
Thus, from Lemma 3.2 and \( K|_n = c I + \frac{1}{2}(D|_n + D^t|_n) \) it follows
\[ (21) \quad M_n + \frac{1}{2} \sum [\text{ad}_{r_i}|_n, \text{ad}_{t_i}|_n] + \frac{c}{\|\beta\|^2} \beta = 0. \]

By tracing the left-hand side of (21) and taking into account \( \text{tr} \beta = -1 \) we obtain
\[ c = -\frac{1}{4}\|\beta\|^2\|\mu\|^2. \]
Moreover, since \( \pi \) is a Lie algebra morphism and \( \pi(\text{ad}_X)^t = \pi(\text{ad}_X^t), \) for all \( X \in \mathfrak{g}, \) we have
\[ \text{tr} M_n[\text{ad}_{r_i}|_n, \text{ad}_{t_i}|_n] = \frac{1}{4}(\pi(\text{ad}_{r_i}|_n)\pi(\text{ad}_{r_i}|_n)\mu, \mu) \]
\[ = \frac{1}{4}(\pi(\text{ad}_{t_i}|_n)\mu, \pi(\text{ad}_{r_i}|_n)^t|_n \mu) \]
\[ = 4\|\pi(\text{ad}_{r_i}|_n)\mu\|^2, \]
for any \( r_i \in \{r_i\}, \) and multiplying (21) by \( M_n \)
\[ 0 = \text{tr} M_n^2 + \frac{1}{8} \sum \|\pi(\text{ad}_{r_i}|_n)\mu\|^2 + \frac{c}{\|\beta\|^2} \text{tr} M_n \beta \]
\[ = \frac{1}{8} \sum \|\pi(\text{ad}_{r_i}|_n)\mu\|^2 + \|\mu\|^2 \left( \frac{4}{\|\mu\|^2} \|M_n\|^2 - \langle M_n, \beta \rangle \right). \]

Then, by (12) we have
\[ \langle M_n, \beta \rangle \leq \frac{4}{\|\mu\|^2} \|M_n\|^2 \]
and
\[ \sum \|\pi(\text{ad}_{r_i}|_n)\mu\|^2 = 0, \]
which implies \( \text{ad}_{r_i}|_n \in \text{Der}(n), \) for all \( i, \) and the first claim follows.

To prove the second claim it is enough to observe that \( M_n \) and \( \beta \) are orthogonal to any derivation of \( n, \) and applying (21)
\[ \sum [\text{ad}_{r_i}|_n, \text{ad}_{r_i}|_n] = 0. \]
Remark 3.9. By \([22]\), given a metric Lie algebra \(g\),
\[
\sum [\text{ad}_{r_i}|_n, \text{ad}_{r_i}'|_n] = 0, \quad \text{for any orthonormal basis } \{r_i\} \text{ of } \mathfrak{r},
\]
implies
\[
\text{ad}_{r_i}'|_n \in \text{Der}(n), \quad \text{for any } X \in \mathfrak{r}.
\]

3.1. Proof of the main results. The next proposition implies our Theorem 1.1.

**Proposition 3.10.** Let \((G, g)\) be a Lie group equipped with a left-invariant metric and \(g\) its Lie algebra. Let \(g = \mathfrak{r} \oplus n\) be the orthogonal decomposition of \(g\), where \(n\) is the nilradical of \(g\), and let \(g_n\) be the pull-back of \(g\) to the Lie group \(N\) of \(n\). Then, \(g\) is an expanding semi-algebraic \(K\)-soliton if and only if

(i) \(g = \mathfrak{r} \ltimes n\), with \(\mathfrak{r}\) reductive Lie subalgebra and \(n\) nilradical of \(g\);

(ii) \(g_n\) is an expanding algebraic \(K\)-soliton on \(N\);

(iii) \(\sum [\text{ad}_{r_i}|_n, \text{ad}_{r_i}'|_n] = 0\), where \(\{r_i\}\) is an orthonormal basis of \(\mathfrak{r}\);

(iv) for any \(X, Y \in \mathfrak{r}\)
\[
\mathcal{K}(g_n)(X, Y) = c g_n(X, Y) + \frac{1}{2} \text{tr}(\text{ad}_X|_n \text{ad}_Y'|_n) - \frac{1}{2} \text{tr} \text{ad}_X \cdot \text{tr} \text{ad}_Y,
\]
where \(g_n\) is the pull-back of \(g\) to the Lie group of \(\mathfrak{r}\).

**Proof.** Let \((G, g)\) be an expanding semi-algebraic \(K\)-soliton with \(\mathcal{K}_g = c I + \frac{1}{2}(D + D')\), for some \(D \in \text{Der}(g)\), and denote with \(\tilde{B} : g \to g\) the endomorphism defined by
\[
\langle \tilde{B} X, Y \rangle = \text{tr}(\text{ad}_X|_n \text{ad}_Y'|_n).
\]

Items (i) and (iii) follow from Proposition 3.5 and Lemma 3.8 respectively. Item (iv) follows from Proposition 3.5 and Lemma 3.2 since
\[
M|_\mathfrak{r} + \tilde{Q}|_\mathfrak{r} = c I|_\mathfrak{r} \quad \text{and} \quad M|_\mathfrak{r} = M|_\mathfrak{r} + \frac{1}{2} \tilde{B}|_\mathfrak{r}.
\]

Finally, item (ii) follows from Lemma 3.2 and Lemma 3.8. Indeed,
\[
(c I + S(D))|_n = M|_n - S(\text{ad}_H)|_n = M|_n - S(\text{ad}_H)|_n = \mathcal{K}_g|_n - S(\text{ad}_H)|_n,
\]
where \(\mathcal{K}_g|_n\) denotes the \(K\)-operator of the Lie algebra \(n\). Thus, the claim follows and it turns out that the derivation associated to \(g_n\) is given by \(D_1 = S(\text{ad}_H + D)|_n\).

Viceversa, suppose that (i)-(iv) hold. Let \(n = n_1 \oplus \ldots \oplus n_r\) be an orthogonal decomposition of \(n\) such that
\[
[n, n] = n_2 \oplus \ldots \oplus n_r, \quad [n, [n, n]] = n_3 \oplus \ldots \oplus n_r
\]
and so on. Since \(\text{ad}_X|_n\) and \(\text{ad}_X'|_n\) are both derivations by Remark 3.9, we have \(\text{ad}_X(n_i) \subset n_i\) and \(\text{ad}_Z(n_i) \subset n_{i+1}\), for any \(X \in \mathfrak{r}\) and \(Z \in \mathfrak{n}\). Thanks to Lemma 3.2 and (iii), under these assumptions, we have
\[
M = \begin{bmatrix} M_r - \frac{1}{2} \tilde{B} & 0 \\ 0 & M_n \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix},
\]
where the block representations are with respect to \(g = \mathfrak{r} \oplus n\).

Now, let \(D_1\) be the derivation characterizing \(g_n\) and \(D := -\text{ad}_H + \begin{bmatrix} 0 & 0 \\ 0 & D_1 \end{bmatrix}\). Since \(\mathfrak{r}\) is reductive and (iv) holds, we have
\[
\mathcal{K}|_\mathfrak{r} = M|_\mathfrak{r} - S(\text{ad}_H)|_\mathfrak{r} + \tilde{Q}|_\mathfrak{r} = M|_\mathfrak{r} - \frac{1}{2} \tilde{B}|_\mathfrak{r} - S(\text{ad}_H)|_\mathfrak{r} + \tilde{Q}|_\mathfrak{r},
\]
which implies $K|_n = cI - S(\text{ad}_H)$. Similarly,

$$K|_n = M|_n - S(\text{ad}_H)|_n = M_n - S(\text{ad}_H)|_n$$

and $K|_n = cI + S(-\text{ad}_H + D_1)$, since (ii) holds.

It only remains to show that $D \in \text{Der}(g)$. To prove the claim it is enough that \( \begin{bmatrix} 0 & 0 \\ 0 & D_1 \end{bmatrix} \in \text{Der}(g) \), or equivalently \( D_1, \text{ad}_X|_n = 0 \), for any $X \in \mathfrak{r}$. However, since $K|_n = M_n = cI + D_1$ and $M_n$ commutes with any derivation of $n$ whose transpose is also a derivation (see Remark 2.5), the claim follows.

Corollary 1.2 follows since in the solvable case $\mathfrak{r}$ is abelian and, consequently, $K(\mathfrak{g}_\mathfrak{r}) = 0$.

**Remark 3.11.** When $G$ is unimodular the derivation $D := K_g - cJ$ only acts on the nilradical $n$ of $g$, since $H = 0$ (and therefore $\text{ad}_H = 0$) by definition.

### 4. Applications

In this section we use our results to construct explicit examples of expanding algebraic solitons to HCF on complex Lie groups.

We work on 4-dimensional solvable (non-nilpotent) complex unimodular Lie algebras, which are classified by the following list (see e.g. [4]):

- $\mathfrak{s}_3(-1) \oplus \mathbb{C}$, with structure equations
  $$[Z_1, Z_2] = Z_2, \quad [Z_1, Z_3] = Z_3;$$

- $\mathfrak{g}_1(-2)$, with structure equations
  $$[Z_1, Z_2] = Z_2, \quad [Z_1, Z_3] = Z_3, \quad [Z_1, Z_4] = -2Z_4;$$

- $\mathfrak{g}_4$, with structure equations
  $$[Z_1, Z_2] = Z_3, \quad [Z_1, Z_3] = Z_4, \quad [Z_1, Z_4] = Z_2;$$

- $\mathfrak{g}_7$, with structure equations
  $$[Z_1, Z_2] = Z_3, \quad [Z_1, Z_3] = Z_2, \quad [Z_2, Z_3] = Z_4;$$

- $\mathfrak{g}_3(\alpha)$, with structure equations
  $$[Z_1, Z_2] = Z_3, \quad [Z_1, Z_3] = Z_4, \quad [Z_1, Z_4] = \alpha(Z_2 + Z_3), \quad \alpha \in \mathbb{C}^*.$$

We show that in the first four cases ($\mathfrak{s}_3(-1) \oplus \mathbb{C}, \mathfrak{g}_1(-2), \mathfrak{g}_4, \mathfrak{g}_7$) there exists a soliton to HCF on the corresponding Lie group (in view of [7], Theorem 1.2), a complex unimodular Lie groups has at most one algebraic soliton to HCF up to homotheties). In the last case the existence of a soliton remains an open question.

#### 4.1. $\mathfrak{s}_3(-1) \oplus \mathbb{C}$

Let $g$ be a Hermitian inner product on $\mathfrak{s}_3(-1) \oplus \mathbb{C}$. We can find a $g$-unitary basis \( \{W_i\} \) such that

$$W_1 \in \langle Z_1, Z_2, Z_3, Z_4 \rangle, \quad W_2 \in \langle Z_2, Z_3, Z_4 \rangle, \quad W_3 \in \langle Z_3, Z_4 \rangle, \quad W_4 \in \langle Z_4 \rangle.$$  

With respect to this new basis, we have

$$[W_1, W_2] = pW_2 + qW_3 + rW_4, \quad [W_1, W_3] = -pW_3 + sW_4,$$

for some $p, q, r, s \in \mathbb{C}$ with $p \neq 0$, and

$$\mathfrak{s}_3(-1) \oplus \mathbb{C} = \mathfrak{r} \oplus \mathfrak{n},$$

where $\mathfrak{r} = \langle W_1 \rangle$ and $\mathfrak{n} = \langle W_2, W_3, W_4 \rangle$. 

Since the nilradical $\mathfrak{n}$ is an abelian ideal, $g_\mathfrak{n}$ trivially induces an expanding algebraic soliton to HCF on the Lie group of $\mathfrak{n}$. Therefore, by Corollary 1.2, $g$ induces an expanding algebraic soliton to HCF on the Lie group of $\mathfrak{s}_{3, -1} \oplus \mathbb{C}$ if and only if

$$[\text{ad}_{W_1}^n, \text{ad}_{W_1}^t |_{\mathfrak{n}}] = 0 \quad \text{and} \quad g(W_1, \bar{W}_1) = -\frac{1}{2c} \text{tr}(\text{ad}_{W_1} |_{\mathfrak{n}} \text{ad}_{W_1}^t |_{\mathfrak{n}}).$$

It is straightforward to show that the first condition holds if and only if $q, r, s = 0$; while, since $\{W_i\}$ is a $g$-unitary basis, we have

$$1 = g(W_1, \bar{W}_1) = -\frac{1}{2c} \text{tr}(\text{ad}_{W_1} |_{\mathfrak{n}} \text{ad}_{W_1}^t |_{\mathfrak{n}}) = -\frac{|p|^2}{c},$$

which implies $c = -|p|^2$. Thus in matrix notation, with respect to $\{W_i\}$, we have

$$K_g = -|p|^2 I + D,$$

where $D := \text{diag}(0, |p|^2, |p|^2, |p|^2)$.

Finally, we note that

$$g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0 \iff q = r = s = 0,$$

and we have the following result:

**Proposition 4.1.** A Hermitian inner product $g$ on $\mathfrak{s}_{3, -1} \oplus \mathbb{C}$ induces an expanding algebraic soliton to HCF on the corresponding (simply connected) Lie group if and only if $g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0$.

4.2. $\mathfrak{g}_1(-2)$. Given a Hermitian inner product $g$ on $\mathfrak{g}_1(-2)$, there exists a $g$-unitary basis satisfying

$$W_1 \in \langle Z_1, Z_2, Z_3, Z_4 \rangle, \quad W_2 \in \langle Z_2, Z_3, Z_4 \rangle, \quad W_3 \in \langle Z_3, Z_4 \rangle, \quad W_4 \in \langle Z_4 \rangle.$$

With respect to this new basis, we have

$$[W_1, W_2] = pW_2 + qW_3 + rW_4, \quad [W_1, W_3] = sW_3 + tW_4, \quad [W_1, W_4] = uW_4,$$

for some $p, q, r, s, t, u \in \mathbb{C}$, where $p + s + u = 0$ and $p, s, u \neq 0$. Then, $\mathfrak{g}_1(-2)$ splits in $\mathfrak{g}_1(-2) = \mathfrak{r} \oplus \mathfrak{n}$, where $\mathfrak{r} = \langle W_1 \rangle$ and $\mathfrak{n} = \langle W_2, W_3, W_4 \rangle$, and $g_\mathfrak{n}$ gives rise to an expanding algebraic soliton to HCF on the Lie group of $\mathfrak{n}$, since $\mathfrak{n}$ is an abelian ideal.

Now, a direct computation yields that

$$[\text{ad}_{W_1} |_{\mathfrak{n}}, \text{ad}_{W_1}^t |_{\mathfrak{n}}] = 0 \quad \text{if and only if} \quad q, r, t = 0;$$

while

$$1 = g(W_1, \bar{W}_1) = -\frac{1}{2c} \text{tr}(\text{ad}_{W_1} |_{\mathfrak{n}} \text{ad}_{W_1}^t |_{\mathfrak{n}}) = -\frac{|p|^2 + |s|^2 + |u|^2}{2c},$$

since $\{W_i\}$ is a $g$-unitary basis. Therefore, if $q, r, t = 0$ and $c = -(|p|^2 + |s|^2 + |u|^2)/2$, the assumptions in Corollary 1.2 are satisfied and, in matrix notation with respect to $\{W_i\}$, we have

$$K_g = cI + D,$$

where $D := -\text{diag}(0, c, c, c)$.

Noting that

$$g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0 \iff q = r = t = 0,$$

we obtain the following result.
Proposition 4.2. A Hermitian inner product \( g \) on \( \mathfrak{g}_1(-2) \) induces an expanding algebraic soliton to HCF on the corresponding (simply connected) Lie group if and only if \( g(Z_2, \bar{Z}_3) = g(Z_3, \bar{Z}_4) = 0 \).

4.3. \( \mathfrak{g}_4 \). Let \( \tilde{g} \) be a Hermitian inner product on \( \mathfrak{g}_4 \) such that \( Z_2, Z_3, Z_4 \) are orthogonal to each other. Let \( \{W_i\} \) be a \( \tilde{g} \)-unitary basis satisfying

\[
W_1 \in \langle Z_1, Z_2, Z_3, Z_4 \rangle, \quad W_2 \in \langle Z_2 \rangle, \quad W_3 \in \langle Z_3 \rangle, \quad W_4 \in \langle Z_4 \rangle.
\]

Then, we have

\[
[W_1, W_2] = pW_3, \quad [W_1, W_3] = qW_4, \quad [W_1, W_4] = rW_2,
\]

and we assume \( p, q, r \in \mathbb{R}^+ \setminus \{0\} \). Hence, \( \mathfrak{g}_4 \) splits as

\[
\mathfrak{g}_4 = \mathfrak{r} \oplus \mathfrak{n},
\]

where \( \mathfrak{r} = \langle W_1 \rangle \) and \( \mathfrak{n} = \langle W_2, W_3, W_4 \rangle \).

Since \( \mathfrak{n} \) is an abelian ideal, \( \tilde{g}_n \) induces an expanding algebraic soliton to HCF on the Lie group of \( \mathfrak{n} \). Moreover, by Corollary [1,2] \( \tilde{g} \) induces an expanding algebraic soliton to HCF on the Lie group of \( \mathfrak{g}_4 \) if and only if

\[
[\text{ad}_{W_1}|_n, \text{ad}_{W_1}^t|_n] = 0 \quad \text{and} \quad 1 = g(W_1, \bar{W}_1) = -\frac{1}{2c} \text{tr}(\text{ad}_{W_1}|_n \text{ad}_{W_1}^t|_n).
\]

The first condition is equivalent to require \( p = q = r \), while the second one is satisfied if and only if \( c = -\frac{3}{2}p^2 \). Hence, in matrix notation with respect to \( \{W_i\} \), we obtain

\[
K_{\tilde{g}} = -\frac{3}{2}p^2 I + D,
\]

where \( D := \frac{3}{4} \text{diag}(0, p^2, p^2, p^2) \).

Finally, we note that

\[
\tilde{g}(Z_2, \bar{Z}_2) = \tilde{g}(Z_3, \bar{Z}_3) = \tilde{g}(Z_4, \bar{Z}_4) \iff p = q = r,
\]

and we have following result:

Proposition 4.3. A Hermitian inner product on \( \mathfrak{g}_4 \) induces an expanding algebraic soliton to HCF on the corresponding (simply connected) Lie group if and only if it is homothetically equivalent to a Hermitian inner product \( g \) on \( \mathfrak{g}_4 \) satisfying \( g(Z_2, \bar{Z}_2) = g(Z_3, \bar{Z}_3) = g(Z_4, \bar{Z}_4) \) and \( g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0 \).

4.4. \( \mathfrak{g}_7 \). Let \( \tilde{g} \) be the standard Hermitian inner product on \( \mathfrak{g}_7 \). Then, \( \mathfrak{g}_7 \) splits in

\[
\mathfrak{g}_7 = \mathfrak{r} \oplus \mathfrak{n},
\]

where \( \mathfrak{r} = \langle Z_1 \rangle \) and \( \mathfrak{n} = \langle Z_2, Z_3, Z_4 \rangle \) is isomorphic to \( \mathfrak{h}_3(\mathbb{C}) \), the Lie algebra of the 3-dimensional complex Heisenberg Lie group \( \mathbb{H}_3(\mathbb{C}) \).

In view of [7, Proposition 4.1], any left-invariant Hermitian metric on \( \mathbb{H}_3(\mathbb{C}) \) is an expanding soliton to HCF. Therefore \( \tilde{g}_\mathfrak{n} \) induces an expanding algebraic soliton to HCF on the Lie group of \( \mathfrak{n} \), and a straightforward computation yields that

\[
[\text{ad}_{Z_1}|_n, \text{ad}_{Z_1}|_n^t] = 0 \quad \text{and} \quad \text{tr}(\text{ad}_{Z_1}|_n \text{ad}_{Z_1}|_n^t) = 2.
\]

Then, the assumptions in Corollary [1,2] are satisfied if and only if \( c = -1 \), and in such a case we have

\[
K_{\tilde{g}} = -I + D,
\]

where \( D := \text{diag}(0, 1, 1, 1, 1) \). Hence, we can claim the following proposition:
Proposition 4.4. A Hermitian inner product on \( g_7 \) induces an expanding algebraic soliton to \( HCF \) on the corresponding (simply connected) Lie group if and only if it is homothetically equivalent to \( \tilde{g} \).

References

[1] R. Arroyo, R. Lafuente, The long-time behaviour of the homogeneous pluriclosed flow, arXiv:1712.02075
[2] C. Böhm, R. Lafuente, Real geometric invariant theory. arXiv preprint arXiv:1701.00643v3 (2017).
[3] A. L. Besse, Einstein manifolds. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
[4] D. Burde, C. Steinhoff, Classification of Orbit Closures of 4-Dimensional Complex Lie Algebras, J. of Algebra 214, (1999) 729–739
[5] M. Jablonski, Homogeneous Ricci solitons. J. Reine Angew. Math. (Crelles Journal) 699, 159–182 (2015).
[6] R. Lafuente, J. Lauret, Structure of homogeneous Ricci solitons and the Alekseevskii conjecture, J. Differential Geom. 98(2), 315–347 (2014).
[7] R. Lafuente, M. Pujia, L. Vezzoni, Hermitian Curvature flow on Lie groups and static invariant metrics, arXiv:1807.00059
[8] J. Lauret, A canonical compatible metric for geometric structures on nilmanifolds. Annals of Global Analysis and Geometry, 30 (2006) no.2, 107–138.
[9] J. Lauret, The Ricci flow for simply connected nilmanifolds. Comm. Anal. Geom. 19(5), 831–854 (2011).
[10] J. Lauret, Convergence of homogeneous manifolds. Journal of the London Mathematical Society 86, no. 3 (2012): 701–727.
[11] J. Lauret, Ricci flow of homogeneous manifolds. Math. Z. 274, 373–403 (2013).
[12] J. Lauret, Einstein solvmanifolds are standard. Ann. of Math., 172 (2010), 1859–1877.
[13] J. Lauret, C.E. Will, Einstein solvmanifolds: existence and non-existence questions, Math. Annalen, 350 (2011), 199–225.
[14] J. Lauret, Geometric flows and their solitons on homogeneous spaces, Rend. Semin. Mat. Univ. Politec. Torino 74 (2016), no. 1–2, 55–93.
[15] J. Lauret, Laplacian flow of homogeneous \( G_2 \)-structures and its solitons, to appear in Proc. London Math. Soc.
[16] J. Lauret, Ricci soliton solvmanifolds, J. reine angew. Math., 650 (2011), 1-21.
[17] J. Streets, Pluriclosed flow on generalized Kähler manifolds with split tangent bundle, Journal für die reine und angewandte Mathematik (Crelles Journal), in press (2015).
[18] J. Streets, Pluriclosed flow, Born-Infeld geometry, and rigidity results for generalized Kähler manifolds, Commun. Part. Diff. Eq., vol. 41, no. 2 (2016), 318–374.
[19] J. Streets, Classification of solitons for pluriclosed flow on complex surfaces, arXiv:1802.00170
[20] J. Streets, G. Tian, A parabolic flow of pluriclosed metrics, Int. Math. Res. Not. IMRN 2010, no. 16, 3101–3133.
[21] J. Streets, G. Tian, Hermitian curvature flow, J. Eur. Math. Soc. (JEMS) 13 (2011), no. 3, 601–634.
[22] J. Streets, G. Tian, Generalized Kähler geometry and the pluriclosed flow, Nuc. Phys. B, Vol. 858, Issue 2, (2012) 366–376.
[23] J. Streets, G. Tian, Regularity results for the pluriclosed flow, Geom. & Top. 17 (2013) 2389–2429.
[24] Y. Ustinovskiy, The Hermitian curvature flow on manifolds with non-negative Griffiths curvature, arXiv:1604.04813
[25] Y. Ustinovskiy, Hermitian curvature flow on complex homogeneous manifolds, arXiv:1706.07023

Dipartimento di Matematica G. Peano, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: mattia.pujia@unito.it