WEIGHTED BLOCH SPACES AND QUADRATIC INTEGRALS

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Abstract. Let $B_\omega(B_d)$ denote the $\omega$-weighted Bloch space in the unit ball $B_d$ of $\mathbb{C}^d$, $d \geq 1$. We show that the quadratic integral
\[ \int_x^1 \frac{\omega^2(t)}{t} \, dt, \quad 0 < x < 1, \]
governs the radial divergence and integral reverse estimates in $B_\omega(B_d)$.

1. Introduction

Let $H(B_d)$ denote the space of holomorphic functions on the unit ball $B_d$ of $\mathbb{C}^d$, $d \geq 1$.

1.1. Weighted Bloch spaces. Given a gauge function $\omega : (0, 1] \to (0, +\infty)$, the weighted Bloch space $B_\omega(B_d)$ consists of those $f \in H(B_d)$ for which
\[ \|f\|_{B_\omega(B_d)} = |f(0)| + \sup_{z \in B_d} \frac{|Rf(z)(1-|z|)}{\omega(1-|z|)} < \infty, \]
where
\[ Rf(z) = \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}(z), \quad z \in B_d, \]
is the radial derivative of $f$. $B_\omega(B_d)$ is a Banach space with respect to the norm defined by $\|f\|_{B_\omega(B_d)}$. If $\omega \equiv 1$, then $B_\omega(B_d)$ is the classical Bloch space $B(B_d)$. Usually we suppose that the gauge function $\omega$ is increasing; hence, we have $B_\omega(B_d) \subset B(B_d)$.

The above notation is not completely standard: often the weight $t/\omega(t)$ is attributed to $B_\omega(B_d)$. Assuming that $\omega$ is sufficiently regular, we show in the present paper that the quadratic integral
\[ I(x) = I_\omega(x) = \int_x^1 \frac{\omega^2(t)}{t} \, dt, \quad 0 < x < 1, \]
governs the radial divergence and integral reverse estimates in $B_\omega(B_d)$. In both cases, the solutions are based on the classical Hadamard gap series.

1.2. Radial divergence. Given $f \in H(B_d)$ and $\zeta \in \partial B_d$, we say that $f$ has a radial limit at $\zeta$ if there exists a finite limit $f^*(\zeta) = \lim_{r \to 1^-} f(r\zeta)$.

Let $\sigma_d$ denote the normalized Lebesgue measure on the unit sphere $\partial B_d$. The radial convergence or divergence in $B_\omega(B_d)$ is described in terms of $I(0+)$ by the following dichotomy:

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Proposition 1.1. Let \( \omega : (0,1] \to (0, +\infty) \) be an increasing function.

(i) Let \( I(0+) < \infty \). If \( f \in \mathcal{B}^\omega(B_d) \), then \( f \) has radial limits \( \sigma_d \)-almost everywhere.

(ii) Let \( I(0+) = \infty \) and let \( \omega(t)/t^{1-\varepsilon} \) be decreasing for some \( \varepsilon > 0 \). Then the space \( \mathcal{B}^\omega(B_d) \) contains a function with no radial limits \( \sigma_d \)-almost everywhere.

Remark that the condition \( I(0+) = \infty \) was previously used by Dyakonov \cite{8} to construct a non-BMO function lying in \( \mathcal{B}^\omega(B_1) \) and in all Hardy spaces \( H^p(B_1) \), \( 0 < p < \infty \).

1.3. Reverse estimates. Given an unbounded decreasing function \( \upsilon : (0,1] \to (0, +\infty) \), typical reverse estimates are obtained in the growth space \( \mathcal{A}^\upsilon(B_d) \), which consists of \( f \in H(B_d) \) such that
\[
|f(z)| \leq C\upsilon(1 - |z|) \quad \text{for all } z \in B_d.
\]
for all \( z \in B_d \) (see, for example, \cite{1} and references therein).

For the weighted Bloch space \( \mathcal{B}^\omega(B_d) \), the following result provides integral reverse estimates related to the function \( \Phi_{1/2}(1 - |z|) \), \( z \in B_d \), where
\[
\Phi(x) = \Phi_{\omega}(x) = 1 + \int_x^1 \frac{\omega(t)}{t} \, dt, \quad 0 < x < 1.
\]

Theorem 1.2. Let \( d \in \mathbb{N} \) and let \( 0 < p < \infty \). Assume that \( \omega : (0,1] \to (0, +\infty) \) increases and \( \omega(t)/t^{1-\varepsilon} \) decreases for some \( \varepsilon > 0 \). Then there exists a constant \( \tau_{d,p,\omega} > 0 \) and functions \( F_y \in \mathcal{B}^\omega(B_d) \), \( 0 \leq y \leq 1 \), such that
\[
\|F_y\|_{\mathcal{B}^\omega(B_d)} \leq 1 \quad \text{and} \quad (1.2) \quad \int_0^1 |F_y(z)|^{2p} \, dy \geq \tau_{d,p,\omega}\Phi^p(1 - |z|)
\]
for all \( z \in B_d \).

For \( \omega \equiv 1 \) and for logarithmic functions \( \omega \), the above estimates were obtained in \cite{6} and \cite{14}, respectively.

1.4. Organization of the paper. Section 2 is devoted to the radial divergence problem. In Section 3 we prove Theorem 1.2 and we show that estimate (1.2) is sharp, up to a multiplicative constant. Applications of Theorem 1.2 are presented in Section 4.

2. Radial divergence

Proposition 1.1(i) is a known fact. Indeed, if \( I(0+) < \infty \) and \( f \in \mathcal{B}^\omega(B_d) \), then \( |Rf(z)|^2(1 - |z|) \) is a Carleson measure, hence, \( f \in \text{BMOA}(B_d) \). In particular, \( f \) has radial limits \( \sigma_d \)-a.e.

2.1. Proof of Proposition 1.1(ii) for \( d = 1 \). Put
\[
f(z) = \sum_{k=0}^{\infty} \omega(2^{-k})z^{2^k}, \quad z \in B_1.
\]
Standard arguments guarantee that \( f \in \mathcal{B}^\omega(B_1) \). For example, let \( t \in (0, 1] \) and let \( \tau = \frac{1}{t} \geq 1 \). Observe that

\[
\frac{\tau \omega(\frac{1}{\tau})}{\tau} \quad \text{is a decreasing function of } \tau \geq 1,
\]

because \( \omega(t) \) is increasing. Also, \( \frac{\tau \omega(\frac{1}{\tau})}{\tau^\varepsilon} \) is an increasing function of \( \tau \geq 1 \), because \( \omega(t)/t^{1-\varepsilon} \) is decreasing. Therefore, \( \tau \omega(\frac{1}{\tau}), \tau \geq 1 \), is a normal weight in the sense of \([17]\). The derivative \( f' \) is represented by a Hadamard gap series, hence, \( f \in \mathcal{B}^\omega(B_1) \) (see, e.g., \([13]\)).

Since \( \omega \) is increasing, we have

\[
\sum_{k=0}^{\infty} \omega^2(2^{-k}) \geq I(0+) = \infty.
\]

Thus, \( f \) has no radial limits \( \sigma_1 \)-a.e. by \([21]\) Chapter V, Theorem 6.4).

**2.2. Proof of Proposition 1.1(ii) for \( d \geq 2 \).** Fix a Ryll–Wojtaszczyk sequence \( \{W[n]\}_{n=1}^{\infty} \) (see \([16]\)). By definition, \( W[n] \) is a holomorphic homogeneous polynomial of degree \( n \), \( \|W[n]\|_{L^\infty(\partial B_d)} = 1 \) and \( \|W[n]\|_{L^2(\partial B_d)} \geq \delta \) for a universal constant \( \delta > 0 \). In particular, (2.1) guarantees that \( \sum_{k=0}^{\infty} \|\omega(2^{-k})W[2^k]\|_{L^2(\partial B_d)}^2 = \infty. \)

Hence, by \([15]\) Lemma 7.2.7, there exists a sequence \( \{U_k\}_{k=1}^{\infty} \) of unitary operators on \( \mathbb{C}^d \) such that

\[
\sum_{k=0}^{\infty} \omega^2(2^{-k})\|W[2^k] \circ U_k(\zeta)\|^2 = \infty.
\]

for \( \sigma_d \)-almost all \( \zeta \in \partial B_d \). Put

\[
f(z) = \sum_{k=0}^{\infty} \omega(2^{-k})W[2^k] \circ U_k(z), \quad z \in B_d.
\]

First, fix a point \( \zeta \in \partial B_d \) with property (2.2). Consider the slice-function \( f_\zeta(\lambda) = f(\lambda \zeta), \lambda \in B_1 \). Note that

\[
f_\zeta(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^{2k}, \quad \lambda \in B_1,
\]

where \( a_k = \omega(2^{-k})W[2^k] \circ U_k(\zeta) \). By (2.2), we have \( \{a_k\}_{k=1}^{\infty} \notin \ell^2 \), thus, \( f_\zeta \) has no radial limits \( \sigma_1 \)-a.e. by \([21]\) Chapter V, Theorem 6.4). Since the latter property holds for \( \sigma_d \)-almost all \( \zeta \in \partial B_d \), Fubini’s theorem guarantees that \( f \) has no radial limits \( \sigma_d \)-a.e.

Second, recall that \( \|W[2^k] \circ U_k\|_{L^\infty(\partial B_d)} = 1 \). So, we deduce that \( f \in \mathcal{B}^\omega(B_d) \), applying the argument from Section 2.1 to the slice-functions \( f_\zeta, \zeta \in \partial B_d \). This ends the proof of Proposition 1.1.

**2.3. Comments.**
2.3.1. **Radial divergence everywhere.** If \( \omega(0+) > 0 \), then \( \mathcal{B}^{\omega}(B_d) \) coincides with \( \mathcal{B}(B_d) \), hence, \( \mathcal{B}^{\omega}(B_d) \) contains a function with no radial limits everywhere (see \( [19, 20] \)). However, if \( \omega(0+) = 0 \), then Proposition 1.1(ii) is not improvable in this direction. Indeed, if \( \omega(0+) = 0 \) and \( f \in \mathcal{B}^{\omega}(B_1) \), then \( f \) has radial limits on a set of Hausdorff dimension one (see \( [12] \)).

2.3.2. **Hyperbolic setting.** To obtain the hyperbolic analog of \( \mathcal{B}^{\omega}(B_d) \), replace \( Rf(z) \) by \( \frac{R\varphi(z)}{1 - |\varphi(z)|^2} \), where \( \varphi: B_n \to B_m, m, n \in \mathbb{N} \), is a holomorphic mapping. The radial limit \( \varphi^*(\zeta) \) is defined at \( \sigma_n \)-almost every point of \( \partial B_n \), hence, it is natural to replace the radial divergence condition by the following property:

\[ |\varphi^*(\zeta)| = 1 \quad \text{\( \sigma_d \)-a.e.}, \text{ that is, } \varphi \text{ is inner.} \]

While the problem in the hyperbolic setting is more sophisticated, the following analog of Proposition 1.1 is known, at least for \( n = m = 1 \).

**Theorem 2.1** ([18, Theorem 1.1], [4, Theorem 5.1]). Let \( \omega: (0, 1] \to (0, +\infty) \) be an increasing function.

(i) Assume that \( I(0+) < \infty \) and \( \varphi: B_1 \to B_1 \) is a holomorphic function such that

\[ \frac{|\varphi'(z)|(1 - |z|)}{1 - |\varphi(z)|} \leq \omega(1 - |z|), \quad z \in B_1. \]

Then \( \varphi \) is not inner.

(ii) Assume that \( I(0+) = \infty \) and \( \omega(t)/t^{1-\varepsilon} \) decreases for some \( \varepsilon > 0 \). Then there exists an inner function \( \varphi: B_1 \to B_1 \) such that

\[ \frac{|\varphi'(z)|(1 - |z|)}{1 - |\varphi(z)|} \leq \omega(1 - |z|), \quad z \in B_1. \]

In Section 4.2, we apply Theorem 1.2 to obtain quantitative versions of Theorem 2.1(i).

3. **Reverse estimates**

3.1. **Auxiliary results.**

**Lemma 3.1.** Let \( \omega: (0, 1] \to (0, +\infty) \) be an increasing function. Put

\[ \Psi(r) = \sum_{k=0}^{\infty} \omega^2(2^{-k})r^{2^k-1}, \quad 0 \leq r < 1. \]

Then \( \Psi(r) \geq C\Phi(1-r) \) for a constant \( C = C_\omega > 0 \).

**Proof.** Let \( 2^{-n-1} \leq 1 - r < 2^{-n} \) for some \( n \in \mathbb{Z}_+ \). Then

\[ 2\Psi(r) \geq 2\omega^2(1) + \sum_{k=1}^{n} \omega^2(2^{-k}) (1 - 2^{-n})^{2^k-1} \]

\[ \geq \omega^2(1) + \frac{1}{e} \sum_{k=0}^{n} \omega^2(2^{-k}) \geq C\Phi(2^{-n-1}) \geq C\Phi(1-r), \]

since \( \omega \) is increasing and \( \Phi \) is decreasing. \( \square \)

Also, we need the following improvement of the Ryll–Wojtaszczyk theorem used in Section 2.2.
Theorem 3.2 ([3] Theorem 4)). Let \( d \in \mathbb{N} \). Then there exist \( \delta = \delta(d) \in (0,1) \) and \( J = J(d) \in \mathbb{N} \) with the following property: For every \( n \in \mathbb{N} \), there exist holomorphic homogeneous polynomials \( W_j[n] \) of degree \( n \), \( 1 \leq j \leq J \), such that
\[
\|W_j[n]\|_{L^\infty(\partial B_d)} \leq 1 \quad \text{and} \\
\max_{1 \leq j \leq J} |W_j[n](\xi)| \geq \delta \quad \text{for all } \xi \in \partial B_d.
\]

Probably, it is worth mentioning that \( J(1) = 1 \).

3.2. Proof of Theorem 3.2 Let the constant \( \delta \in (0,1) \) and the polynomials \( W_j[n], 1 \leq j \leq J, n \in \mathbb{N} \), be those provided by Theorem 3.2.

For each non-dyadic \( y \in (0,1] \), consider the following functions:
\[
F_{j,y}(z) = \sum_{k=0}^{\infty} R_k(y)\omega(2^{-k})W_j[2^k - 1](z), \quad z \in B_d, \quad 1 \leq j \leq J,
\]
where
\[
R_k(y) = \text{sgn} \sin(2^{k+1}\pi y), \quad y \in [0,1],
\]
is the Rademacher function.

First, arguing as in Section 2 and using estimate (3.1), we deduce that
\[
\|F_{j,y}\|_{\mathcal{W}^p(B_d)} \leq C.
\]

Second, we obtain
\[
C_p \int_0^1 |F_{j,y}(z)|^{2p} dy \geq \left( \sum_{k=0}^\infty |\omega(2^{-k})W_j[2^k - 1](z)|^2 \right)^p
\]
by [21 Chapter V, Theorem 8.4]. Given positive numbers \( a_j, 1 \leq j \leq J = J(d), \)
we have
\[
\left( \sum_{j=1}^J a_j \right)^p \leq C_{d,p} \sum_{j=1}^J a_j^p.
\]
Hence,
\[
C_{d,p} \sum_{j=1}^J \int_0^1 |F_{j,y}(z)|^{2p} dy \geq \left( \sum_{k=0}^\infty \sum_{j=1}^J \omega^2(2^{-k})|W_j[2^k - 1](z)|^2 \right)^p
\]
Since \( W_j[2^k - 1], 1 \leq j \leq J, \) are homogeneous polynomials of degree \( 2^k - 1, \) we obtain
\[
\sum_{k=0}^\infty \sum_{j=1}^J \omega^2(2^{-k})|W_j[2^k - 1](z)|^2 \geq \delta^2 \sum_{k=0}^\infty \omega^2(2^{-k})|z|^{2^{k+2}-2}
\]
\[
\geq \delta^2 C_{d,\omega} \Phi(1 - |z|^2), \quad z \in B_d,
\]
by (3.2) and Lemma 3.1 with \( r = |z|^2 \). So,
\[
C_{d,p} \sum_{j=1}^J \int_0^1 |F_{j,y}(z)|^{2p} dy \geq \left( \delta^2 C_{d,\omega} \Phi(1 - |z|^2) \right)^p, \quad z \in B_d.
\]
Changing the indices of the functions \( F_{j,y} \) and using a new variable of integration, we may reduce the above sum of integrals to one integral over \([0,1]\). So, it remains to verify that \( C\Phi(1 - r^2) \geq \Phi(1 - r), 0 \leq r < 1 \).
First, if $0 \leq r \leq \frac{2}{3}$, then $\Phi(1 - r) \leq C_\omega \leq C_\omega \Phi(1 - r^2)$ for a constant $C_\omega > 0$.

Second, if $0 < \varepsilon < \frac{1}{3}$, then $\Phi(\varepsilon) - \Phi(2\varepsilon) \leq \omega^2(2\varepsilon) \leq 3\Phi(2\varepsilon)$, because $\omega$ is increasing. Thus $\Phi(1 - r) \leq 4\Phi(1 - r^2)$ for $\frac{2}{3} < r < 1$.

The proof of Theorem 1.2 is finished.

### 3.3. Integral means

To show that inequality (1.2) is sharp, we estimate the integral means

$$M_p(f, r) = \left( \int_{\partial B_d} |f(r\zeta)|^p \, d\sigma_d(\zeta) \right)^{\frac{1}{p}}, \quad 0 < r < 1,$$

for the functions $f \in \mathcal{B}^\omega(B_d)$.

For $\omega \equiv 1$, the following result was obtained in [5] and [11].

**Proposition 3.3.** Let $0 < p < \infty$ and let $f \in \mathcal{B}^\omega(B_d)$. Then

$$M_p(f, r) \leq C \|f\|_{\mathcal{B}^\omega(B_d)} \Phi^\frac{1}{p}(1 - r), \quad 0 < r < 1,$$

for a constant $C > 0$.

**Proof.** For $f \in H(B_d)$ and $0 < r < 1$, we have

$$M_p(f, r) \leq C|f(0)| + C \left( \int_{\partial B_d} \left( \int_0^1 r^2 |\Re f(r\zeta)|^2 (1 - t) \, dt \right)^{\frac{p}{2}} \, d\sigma_d(\zeta) \right)^{\frac{1}{p}}$$

for a constant $C > 0$; see, for example, [2] Theorem 3.1. If $f \in \mathcal{B}^\omega(B_d)$, then, using the defining property (1.1), we obtain

$$\int_0^1 r^2 |\Re f(r\zeta)|^2 (1 - t) \, dt = \int_0^r |\Re f(t\zeta)|^2 (r - t) \, dt$$

$$\leq \|f\|_{\mathcal{B}^\omega(B_d)}^2 \int_0^r \frac{\omega^2(1 - t)}{1 - t} \, dt \leq \|f\|_{\mathcal{B}^\omega(B_d)}^2 \Phi(1 - r).$$

Since $|f(0)| \leq \|f\|_{\mathcal{B}^\omega(B_d)}$, in sum we obtain the required estimate. \qed

Comparing Proposition 3.3 and Theorem 1.2, we conclude that the direct estimate (3.3) and the reverse estimate (1.2) are not improvable, up to multiplicative constants.

### 3.4. Hardy–Bloch spaces

Given a gauge function $\omega$, the weighted Hardy–Bloch space $\mathcal{B}^\omega_p(B_d)$, $0 < p < \infty$, consists of those $f \in H(B_d)$ for which

$$\|f\|_{\mathcal{B}^\omega_p(B_d)} = |f(0)| + \sup_{0 < r < 1} \frac{M_p(\Re f, r)(1 - r)}{\omega(1 - r)} < \infty.$$

Clearly, we have $\mathcal{B}^\omega(B_d) \subset \mathcal{B}^\omega_p(B_d)$, $0 < p < \infty$. So, it is interesting that estimate (3.3) is sharp for $f \in \mathcal{B}^\omega_p(B_d)$ and holds for all $f \in \mathcal{B}^\omega_p(B_d)$ with $p \geq 2$. Namely, we have the following proposition that was proved in [9] for $\omega \equiv 1$.

**Proposition 3.4.** Let $2 \leq p < \infty$ and let $f \in \mathcal{B}^\omega_p(B_d)$. Then

$$M_p(f, r) \leq C \|f\|_{\mathcal{B}^\omega_p(B_d)} \Phi^\frac{1}{p}(1 - r), \quad 0 < r < 1,$$

for a constant $C > 0$. 
Proof. For \( f \in H(B_d) \) and \( 0 < r < 1 \), we have

\[
M_p(f, r) \leq C |f(0)| + C \left( \int_0^1 \left( \int_{\partial B_d} |Rf(r\zeta)|^p \, d\sigma_d(\zeta) \right)^{\frac{1}{p}} \, r^2(1-t) \, dt \right)^{\frac{1}{2}}
\]

for a constant \( C > 0 \) (see [10] for \( d = 1 \); integration by slices gives the result for \( d \geq 2 \)). Now, we argue as in the proof of Proposition 3.3. Namely, for \( f \in B_\omega(B_d) \), the defining property (3.4) guarantees that

\[
\int_0^1 \left( \int_{\partial B_d} |Rf(r\zeta)|^p \, d\sigma_d(\zeta) \right)^{\frac{1}{p}} \, r^2(1-t) \, dt = \int_0^r M_p^2(Rf, t)(r-t) \, dt \leq \| f \|_{B_\omega(B_d)}^2 \int_0^r \omega^2(1-t) \, dt \leq \| f \|_{B_\omega(B_d)}^2 \Phi(1-r).
\]

Since \( |f(0)| \leq \| f \|_{B_\omega(B_d)} \), the proof is finished. \( \square \)

4. Applications of Theorem 1.2

In this section, we assume that \( \omega : (0, 1] \to (0, +\infty) \) is an increasing function.

4.1. Carleson measures. Given a space \( X \subset H(B_d) \) and \( 0 < q < \infty \), recall that a positive Borel measure \( \mu \) on \( B_d \) is called \( q \)-Carleson for \( X \) if \( X \subset L^q(B_d, \mu) \).

Suppose that \( \omega(t)/t^{1-\varepsilon} \) decreases for some \( \varepsilon > 0 \). A direct application of Theorem 1.2 gives the following result:

**Corollary 4.1.** Let \( 0 < q < \infty \) and let \( \mu \) be a \( q \)-Carleson measure for \( B_\omega(B_d) \). Then

\[
\int_{B_d} \Phi^\frac{q}{2}(1-|z|) \, d\mu(z) < \infty.
\]

If \( \mu \) is a radial measure, then the above corollary is reversible. Moreover, the corresponding result holds for all spaces \( B_\omega^p(B_d), p \geq 2 \).

**Proposition 4.2.** Let \( 0 < q < \infty \) and let \( \rho \) be a positive measure on \([0, 1)\). Then the following properties are equivalent:

\[
\int_0^1 \int_{\partial B_d} |f(r\zeta)|^q \, d\sigma_d(\zeta) \, d\rho(r) < \infty \quad \text{for all } f \in B_\omega^p(B_d), \quad p \geq 2;
\]
\[
\int_0^1 \int_{\partial B_d} |f(r\zeta)|^q \, d\sigma_d(\zeta) \, d\rho(r) < \infty \quad \text{for all } f \in B_\omega(B_d);
\]
\[
\int_0^1 \Phi^\frac{q}{2}(1-r) \, d\rho(r) < \infty.
\]

**Proof.** The implication (4.1) \( \Rightarrow \) (4.2) is trivial, because \( B_\omega(B_d) \subset B_\omega^p(B_d) \). Next, (4.2) implies (4.3) by Corollary 4.1. Finally, Proposition 3.4 guarantees that (4.3) implies (4.2). \( \square \)
4.2. **Hyperbolic derivatives.** Let \( I_\omega(0+) < \infty \). As observed in [7], the conclusion of Theorem 2.1(i) remains true if the restriction
\[
\frac{|\varphi'(z)|(1-|z|)}{1-|\varphi(z)|} \leq \omega(1-|z|), \quad z \in B_1,
\]
is replaced by the following weaker assumption:
\[
\frac{|\varphi'(z)|(1-|z|)}{1-|\varphi(z)|} \Omega(1-|\varphi(z)|) \leq \omega(1-|z|), \quad z \in B_1,
\]
where \( \Omega : (0, 1] \to (0, +\infty) \) is a bounded measurable function such that
\[
I_\Omega = \int_0^1 \frac{\Omega^2(t)}{t} \, dt = \infty.
\]
To obtain quantitative results of the above type, we apply Theorem 1.2. Also, we make weaker assumptions about \( \varphi \).

So, suppose that \( \Omega \) is increasing and \( \Omega(t)/t^{1-\varepsilon} \) is decreasing for some \( \varepsilon > 0 \). Put
\[
\Phi_\Omega(x) = 1 + \int_x^1 \frac{\Omega^2(t)}{t} \, dt, \quad 0 < x < 1.
\]

**Corollary 4.3.** Let \( \varphi : B_1 \to B_1 \) be a holomorphic mapping and let \( 1 \leq p < \infty \). Assume that \( I_\omega(0+) < \infty \), \( I_\Omega = \infty \) and
\[
(1-r) \left( \int_{\partial B_1} \frac{|\varphi'(r\zeta)|}{1-|\varphi(r\zeta)|} \Omega(1-|\varphi(r\zeta)|) \right)^{2p} \, d\sigma_1(\zeta) \leq \omega(1-r)
\]
for \( 0 < r < 1 \). Then
\[
\sup_{0 < r < 1} \int_{\partial B_1} \Phi_\Omega^p(1-|\varphi(r\zeta)|) \, d\sigma_1(\zeta) < \infty.
\]
In particular, \( |\varphi^*| < 1 \) \( \sigma_1 \)-a.e.

**Proof.** Let the constant \( \tau = \tau_{1,p,\Omega} > 0 \) and the functions \( F_y \in B\Omega(1_{B_1}) \), \( 0 \leq y \leq 1 \),
be those provided by Theorem 1.2 for \( d = 1 \) and for \( \Omega \) in place of \( \omega \).

Since \( \|F_y\|_{B\Omega(1_{B_1})} \leq 1 \), we have
\[
|(F_y \circ \varphi)'(z)| \leq |F_y'(\varphi(z))||\varphi'(z)| \leq \frac{|\varphi'(z)|}{1-|\varphi(z)|} \Omega(1-|\varphi(z)|), \quad z \in B_1.
\]

So, using (4.4) and the hypothesis \( I_\omega(0+) < \infty \), we obtain
\[
\int_0^1 M_{2p}^2((F_y \circ \varphi)', t)(1-t) \, dt \leq \int_0^1 \omega^2(1-t) \, dt < \infty.
\]

We further observe that \( |F_y \circ \varphi(0)| \leq C \phi \|F_y\|_{B\Omega(1_{B_1})} \leq C \), and so estimate (3.6) guarantees that
\[
\int_{\partial B_1} |F_y \circ \varphi(r\zeta)|^{2p} \, d\sigma_1(\zeta) \leq C, \quad 0 \leq y \leq 1, \quad 0 < r < 1,
\]
for a universal constant \( C > 0 \). Hence, applying Fubini’s theorem and Theorem 1.2 we obtain
\[
C \geq \int_{B_1} \int_0^1 |F_y \circ \varphi(r\zeta)|^{2p} \, dy \, d\sigma_1(\zeta) \geq \int_{\partial B_1} \Phi_\Omega^p(1-|\varphi(r\zeta)|) \, d\sigma_1(\zeta),
\]
as required. \( \square \)
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