A REMARK ON GRADED COUNTABLE COHEN–MACAULAY
REPRESENTATION TYPE

NAOYA HIRAMATSU

Abstract. We show that there are only a finite number of isomorphism classes of graded
maximal Cohen–Macaulay modules with fixed Hilbert series over Cohen–Macaulay algebras of
graded countable representation type.

1. Introduction

In the representation theory of Cohen–Macaulay algebras, the algebras which are called
discrete (including finite, countable), tame and wild Cohen–Macaulay representation type are major
subjects. They are defined by the complexity of the classification of maximal Cohen–Macaulay
modules. For the graded Cohen–Macaulay algebras, many authors including Eisenbud and Herzog [6],
Stone [9], Drozd and Tovpyha [5] have studied. In this paper, we focus on a graded
Cohen–Macaulay algebra which is of countable Cohen–Macaulay representation type and give
a certain finite result about the maximal Cohen–Macaulay modules.

Let \( R = \oplus_{i=0}^{\infty} R_i \) be a commutative positively graded affine \( k \)-algebra with \( R_0 = k \) an alge-
braically closed uncountable field. Let \( S \) be a graded Noetherian normalization of \( R \). Then \( R \)
is a finitely generated graded \( S \)-module. A finitely generated graded \( R \)-module \( M \) is said to
be maximal Cohen–Macaulay if \( M \) is graded free as a graded \( S \)-module. We say that a graded
Cohen–Macaulay ring \( R \) is of graded countable Cohen–Macaulay representation type if there
are infinitely but only countably many isomorphism classes of indecomposable graded maximal
Cohen–Macaulay \( R \)-modules up to shift.

In the paper, we devote to prove the following result.

Theorem 1.1. Let \( R \) be of graded countable Cohen–Macaulay representation type. For each
graded free \( S \)-module \( F \) there are finitely many isomorphism classes of maximal Cohen–Macaulay
\( R \)-modules which are isomorphic to \( F \) as graded \( S \)-modules. In other words, there are only a
finite number of isomorphism classes of maximal Cohen–Macaulay \( R \)-modules with fixed Hilbert
series.

As an application, we show that the notion of graded countable Cohen–Macaulay representation
type coincides with the notion of graded discrete Cohen–Macaulay representation type in the sense of
Drozd and Tovpyha [5] provided that the algebra is with an isolated singularity (Corollary [1.1]).

To prove the theorem we consider the analogy of a module variety for finitely generated
modules over a finite dimensional algebra, which was introduced by Dao and Shipman [4].

2. Graded maximal Cohen–Macaulay modules of type \( V \)

Throughout the paper, \( k \) is an algebraically closed uncountable field of characteristic \( 0 \) and
\( R = \oplus_{i=0}^{\infty} R_i \) is a commutative positively graded affine \( k \)-algebra with \( R_0 = k \) and \( R_+ = \oplus_{i>0} R_i \).

2010 Mathematics Subject Classification. Primary 13C14; Secondary 14D06, 16G60.
Key words and phrases. graded countable Cohen–Macaulay representation type, maximal Cohen–Macaulay
modules, module variety.
NH was supported by JSPS KAKENHI Grant Number 18K13399.
Then we can take a graded Noetherian normalization $S$ of $R$. That is, $S = k[y_1, \ldots, y_n] \subseteq R$ where $n = \dim R$ such that $R$ is a finitely generated graded $S$-module (see [1] Theorem 1.5.17). Let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. For $i \in \mathbb{Z}$, $M(i)$ is defined by $M(i)_n = M_{n+i}$.

We denote by $\Hom_R(M, N)_i$, consisting of homogenous morphisms of degree $i$.

**Definition 2.1.** A finitely generated graded $R$-module $M$ is said to be maximal Cohen–Macaulay (abbr. MCM) if $M$ is graded free as an $S$-module:

$$M \cong \bigoplus_{i=1}^{m} S(l_i).$$

And then $M \cong S \otimes_k V$ for some finite dimensional graded $k$-module $V$, so that we call it a graded MCM $R$-module of type $V$.

We say that $R$ is of graded countable CM representation type if there are infinitely but only countably many isomorphism classes of indecomposable graded MCM modules up to shift.

**Example 2.2.** Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$. It is known that $R$ is of graded countable CM representation type whose indecomposable graded MCM $R$-modules are $I_n = (x, y^n)R$ for $n \geq 0$ up to shift (cf [3]). The graded Noetherian normalization of $R$ is $S = k[y]$. Set $V = V_0 \oplus V_n$ with $V_0 = V_n = k$. Then $I_n \cong S \otimes_k V$ as $S$-modules.

We state our main theorem of this paper, whose proof is given in Section 3.

**Theorem 2.3.** Let $R$ be of graded countable CM representation type. For each finite dimensional graded $k$-vector space $V$, there are finitely many isomorphism classes of graded MCM $R$-modules of type $V$.

**Remark 2.4.** Theorem 1.1 is immediate from Theorem 2.3. Let $V$ be a finite dimensional graded $k$-vector space and $M$ and $N$ be graded MCM $R$-modules of type $V$, that is, $M, N \cong S \otimes_k V$. Then we see that $\dim_k M_i = \dim_k N_i$ for all $i$, so that $M$ and $N$ have the same Hilbert series.

It is natural to ask what happens if we fix the Hilbert polynomial instead of the Hilbert series. We have the following example.

**Example 2.5.** Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$ and $I_n = (x, y^n)R$. Then Hilbert polynomials of $I_n$ are 2 for all $n$ and $I_n \not\cong I_m$ if $n \neq m$.

3. A variety of graded MCM modules

In this section we give the proof of Theorem 2.3. For this, we recall the notion of a variety of graded MCM modules, which plays a key role in our results.

Given a graded MCM $R$-module $M$, since $M \cong S \otimes_k V$, there exists a degree 0 graded $S$-algebra homomorphism $\alpha : R \to \End_S(S \otimes_k V)$:

$$\alpha \in \Hom_{S-\text{alg}}(R, \End_S(S \otimes_k V))_0.$$  

Then $\Hom_{S-\text{alg}}(R, \End_S(S \otimes_k V))_0$ is an algebraic variety over $k$. We denote it by $\text{Rep}_S(R, V)(k)$.

**Example 3.1.** Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$ and $S = k[y]$ a graded Noetherian normalization of $R$. Set $V = V_0 \oplus V_1$ where $V_0 = V_1 = k$. Then giving a graded MCM $R$-module which is isomorphic to $S \otimes_k V = S \oplus S(-1)$ is equivalent to giving a $\mu \in \End_S(S \otimes_k V)_1$ with $\mu^2 = 0$. Note that

$$\End_S(S \otimes_k V)_1 = \left\{ \begin{pmatrix} ay & bx^2 \\ c & dy \end{pmatrix} | a, b, c, d \in k \right\}.$$  

Hence one can show that

$$\text{Rep}_S(R, V)(k) = \Hom_{S-\text{alg}}(k[a, b, c, d]/(a^2 + bc + ab + bd, ac + bc, bc + d^2), k).$$
Remark 3.2. Dao and Shipman [4] introduced a functor \( \text{Rep}_S(R, V) \) from the category of commutative \( k \)-algebras to sets. For a commutative \( k \)-algebra \( T \), they define the notion of \( T \)-flat family of \( V \)-framed \( R \)-modules (\[4\] Definition 2.1]), and \( \text{Rep}_S(R, V)(T) \) is a set of the modules. They show that \( \text{Rep}_S(R, V) \) is represented by an affine variety of finite type (\[4\] Proposition 2.2).

Remark 3.3. (1) The algebraic group \( G_V = \text{Aut}_S(S \otimes_k V)_0 \) acts on \( \text{Rep}_S(R, V)(k) \) by conjugation, and we have 1-1 correspondence;

\[
\{G_V\text{-orbits in } \text{Rep}_S(R, V)(k)\} \xrightarrow{1-1} \{M|M \cong S \otimes_k V \text{ as graded } S\text{-modules }\} \cong .
\]

(2) Note that \( \text{Rep}_S(R, V)(k) \) parameterizes graded MCM \( R \)-modules with fixed Hilbert series (Remark 2.3).

Example 3.4. Let \( R = k[x, y]/(x^2) \) with \( \deg x = \deg y = 1 \) and \( V = V_0 \oplus V_1 \) where \( V_0 = V_1 = k \). Then \( \text{Rep}_S(R, V)(k) \) contains the following three orbits:

\[
R \cong \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), (x, y^2)R(1) \cong \mathcal{O}\left(\begin{pmatrix} 0 & y^2 \\ 0 & 0 \end{pmatrix}\right), R/(x)(1) \oplus R/(x) \cong \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right),
\]

where we denote by \( \mathcal{O}(\mu) \) the \( G_V \)-orbit of \( \mu \in \text{Rep}_S(R, V)(k) \).

Now let us prove our main theorem. First we mention a lemma.

Lemma 3.5. Let \( X \subseteq \mathbb{A}^n(k) \) be an algebraic set and let \( X_i \subseteq X \) be closed subsets with \( \dim X_i \leq \dim X \). Then \( X \) can be never represented by a countable union of \( X_i \).

Proof. We prove by induction on \( n \). Suppose that \( n = 1 \). Then \( \dim X_i = 0 \), so that \( X_i \) is a finite set of points. Assume that \( X = \cup_{i \geq 1} X_i \), and then \( X \) contains infinitely but countably many points. This is a contradiction. Since \( k \) is an uncountable filed \( X \) must contain uncountably many points. Suppose that \( n \geq 2 \). Considering the irreducible decomposition of \( X \), we may assume that \( X \) is irreducible. Then \( X \) is represented by \( V(\mathfrak{p}) \) for some prime ideal \( \mathfrak{p} \) in \( k[x_1, x_2, \cdots, x_n] \). We also put \( I_i \) ideals with \( X_i = V(I_i) \). After renumbering, we may assume that \( X \) is not contained in \( V(x_1 - c) \) for all \( c \in k \). For each minimal prime ideal \( \mathfrak{q} \) of \( I_i \), there are a finite number of elements \( c \in k \) such that \( x_1 - c \in \mathfrak{q} \). Note that a number of minimal prime ideals are finite. Recall that \( k \) is an uncountable field. Thus we can take \( \lambda \in k \) such that \( x_1 - \lambda \) is neither contained in \( \mathfrak{p} \) nor all minimal prime ideals of \( I_i \) for all \( i \). Consider the quotient by \( x_1 - \lambda \). Then we can reduce the case that \( X = \cup_{i \geq 1} X_i \) is a closed subset of \( \mathbb{A}^{n-1}_k \) preserving \( \dim X_i \leq \dim X \) for all \( i \). By the induction hypothesis, we obtain the assertion.

Theorem 3.6. Let \( X \subseteq \mathbb{A}^n(k) \) be an algebraic set. Suppose that \( X \) is represented by a countable disjoint union of locally closed subsets, that is \( X = \cup_{i \geq 1} Y_i \) where \( Y_i \) are locally closed and \( Y_i \cap Y_j = \emptyset \) for \( i \neq j \). Then \( X \) is a “finite” union.

Proof. Let \( X_1, X_2, \ldots, X_m \) be irreducible components of \( X \). For each component \( X_k \), we have

\[
X_k = X_k \cap X = X_k \cap (\cup_{i \geq 1} Y_i) = \cup_{i \geq 1} (X_k \cap Y_i).
\]

Note that \( X_k \cap Y_i \) are locally closed for all \( i \). By Lemma 3.5 there exists \( j \) such that \( \dim X_k = \dim X_k \cap Y_j \), so that \( X_k = X_k \cap Y_j \). Since \( X_k \cap Y_j \) is open in \( X_k \cap Y_j = X_k \),

\[
X_{k,2} := X_k \setminus (X_k \cap Y_j)
\]

is closed and \( \dim X_{k,2} \leq \dim X_k \). We decompose \( X_{k,2} \) into its irreducible components \( X_{k,2,1}, X_{k,2,2}, \ldots, X_{k,2,m'} \) and apply the above argument for each components \( X_{k,2,j} \). Then we also obtain the closed subsets \( X_{k,3,j'} \) with \( \dim X_{k,3,j'} \leq \dim X_{k,2,j'} \) for all \( j' \). Repeating this at most \( \dim X \) times, we achieve the case that each components are of dimension 0. Since a closed subset of dimension 0 is a finite set of points, these subset can be represented by a finite union of (locally) closed subsets which derive from \( Y_i \). A number of \( Y_i \) which are appeared in this arguments is finite, so that we obtain the assertion.
Remark 3.7. Let $X$ be a $G$-variety. Namely $X$ is a variety equipped with an action of the group $G$. For $x \in X$, we denote by $O(x)$ the $G$-orbit of $x$. Then one can show the following statements hold.

(1) $O(x)$ is locally closed. See [10] Proposition 21.4.3(i).

(2) $O(x) = O(y)$ if and only if $O(x) = O(y)$ for $x, y \in X$. See [8] Proposition 3.5 for instance.

The proof of Theorem 2.3 is based on our Theorem 3.6.

Proof of Theorem 2.3. According to Remark 3.3, it is enough to show that $\text{Rep}_S(R, V)(k)$ consists of finitely many orbits. Since $R$ is of graded countable CM representation type, $\text{Rep}_S(R, V)(k)$ consists of infinitely but countably many orbits. It follows from Theorem 3.6 and Remark 3.7 that $\text{Rep}_S(R, V)(k)$ is a finite union of orbits. \hfill $\Box$

4. An application

At the end of this paper, we investigate the relation between graded countable CM representation type and graded discrete CM representation type. The following definition is taken from [5].

Definition 4.1. [5 Definition 1.1.] We say that $R$ is of graded discrete CM representation type if, for any fixed $r > 0$, there are only finitely many isomorphism classes of indecomposable graded MCM $R$-modules with rank $r$ up to shift. Here the rank is taken over $S$.

One can show that if $R$ is of discrete CM representation type then $R$ is of countable CM representation type. Because there is only a countable set of graded MCM $R$-modules up to isomorphism and shift if $R$ is of discrete. The converse does not hold in general.

Example 4.2. Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$ and $I_n = (x, y^n)R$. It is known that $I_n$ is an indecomposable graded MCM $R$-modules for $n \geq 0$. Note that $\text{rank}_S I_n = 2$ since $I_n \cong S(-1) \oplus S(-n)$ where $S = k[y]$. But $I_n \not\cong I_m$ if $n \neq m$, so that $R$ is not of graded discrete CM representation type.

The following theorem is due to Dao and Shipman. We say that $R$ is with an isolated singularity if each graded localization $R_{(p)}$ is regular for each graded prime ideal $p$ with $p \neq R_+$.

Theorem 4.3. [4 Theorem 3.1] Assume that $R$ is with an isolated singularity. For each $r > 0$ there exists $\alpha_r > 0$ such that if $M$ is an indecomposable graded MCM $R$-module with rank $r$ then

$$g_{\text{max}}(M) - g_{\text{min}}(M) < \alpha_r,$$

where $g_{\text{max}}(M) = \max \{m \mid (M/S_+M)m \neq 0\}$ and $g_{\text{min}}(M) = \min \{m \mid (M/S_+M)m \neq 0\}$.

One can deduce from the theorem that for a MCM graded $R$-module of rank $r$ up to shift, only finitely many finite dimensional graded $k$-vector space $V$ come into question. Indeed, for an indecomposable graded MCM modules $M$ with rank $r$, we shift $M$ to normalize it, and then $g_{\text{min}}(M) = 0$. Thus $0 \leq g_{\text{max}}(M) < \alpha_r$. Note that $\alpha_r$ depends on only $r$. Since $M/S_+M \cong V$ as $k$-modules, we obtain the claim.

Corollary 4.4. Let $R$ be of graded countable CM representation type. Suppose that $R$ is with an isolated singularity. Then $R$ is of graded discrete CM representation type.

Proof. We identify an orbit of a point in $\text{Rep}_S(R, V)(k)$ with the corresponding isomorphism classes of a graded MCM $R$-module (Remark 4.2.1). For each $r > 0$, we have the inclusion of
A REMARK ON GRADED COUNTABLE COHEN–MACAULAY REPRESENTATION TYPE

sets:

\[ \{ M : \text{indecomposable graded MCM } R\text{-modules with rank } r \}/ \langle \cong, \text{shift} \rangle \subseteq \bigcup_{V: \text{finite dimensional graded } k\text{-vector space}} \Rep_S(R,V)(k). \]

According to Theorem 4.3, the right hand side is a finite union of \( \Rep_S(R,V)(k) \). Since \( \Rep_S(R,V)(k) \) consists of finitely many orbits (Theorem 2.3), the left hand side is a finite set. (Compare with [4, Corollary A]. See also [7, Theorem 3.16].)

□

Remark 4.5. As mentioned in [2, Definition 8.13.], for a CM \( k \)-algebra \( R \), the CM representation type of \( R \) is said to be discrete if the set of isomorphism classes of indecomposable graded MCM modules is infinite but countable. Usually, we consider the type in this sense.

REFERENCES

[1] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993. xii+403 pp. Revised edition, 1998.
[2] I. Burban and Yu. Drozd, Maximal Cohen–Macaulay modules over non–isolated surface singularities, Memoirs of the AMS Memoirs of the AMS 248, no. 1178 (2017).
[3] R.-O. Buchweitz, G.-M. Greuel and F.-O. Schreyer, Cohen–Macaulay modules on hypersurface singularities. II. Invent. Math. 88 (1987), no. 1, 165-182.
[4] H. Dao and I. Shipman, Representation schemes and rigid maximal Cohen-Macaulay modules, Selecta Math. (N.S.) 23 (2017), no. 1, 1-14.
[5] Y. Drozd and O. Tovpyha, Graded Cohen-Macaulay rings of wild Cohen-Macaulay type, J. Pure Appl. Algebra 218 (2014), no. 9, 1628-1634.
[6] D. Eisenbud and J. Herzog, The classification of homogeneous Cohen-Macaulay rings of finite representation type, Math. Ann. 280 (1988), no. 2, 347-352.
[7] N. Karroum, MCM-einfache moduln, PhD dissertation - Ruhr-Universität Bochum (2009).
[8] H. Kraft, Geometric methods in representation theory, Representations of algebras (Puebla, 1980), pp. 180-258, Lecture Notes in Math., 944, Springer, Berlin-New York, 1982.
[9] B. Stone, Super-stretched and graded countable Cohen-Macaulay type, J. Algebra 398 (2014), 1-20.
[10] P. Tauvel and R.W.T. Yu, Lie algebras and algebraic groups, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. xvi+653 pp.

General Education Program, National Institute of Technology, Kure College, 2-2-11, Agami-nami, Kure Hiroshima, 737-8506 Japan

E-mail address: hiramatsu@kure-nct.ac.jp