 Scalar \( \sigma \) meson via chiral and crossing dynamics

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Abstract

We show that the non-strange scalar \( \sigma \) meson, as now reported in the 1996 PDG tables, is a natural consequence of crossing symmetry as well as chiral dynamics for both strong interaction low energy \( \pi\pi \) scattering and also \( K \to 2\pi \) weak decays.

1 Introduction

The 1996 Particle Data Group (PDG) tables\[1\] now includes a broad non-strange \( I=0 \) scalar \( \sigma \) resonance referred to as \( f_0 \) (400-1200). This is based in part on the Törnqvist-Roos\[2\] re-analysis of low energy \( \pi\pi \) scattering, finding a broad non-strange \( \sigma \) meson in the 400-900 MeV region with pole position \( \sqrt{s_0} = 0.470 - \imath 0.250 \) MeV. Several later comments in PRL\[3-5\] all stress the importance of rejecting\[3\] or confirming\[4,5\] the above Törnqvist-Roos\[2\] \( \sigma \) meson analysis based on (t-channel) crossing symmetry of this \( \pi\pi \) process.

In this brief report we offer such a \( \sigma \) meson-inspired crossing symmetry model in support of Refs. \[2,4,5\] based on chiral dynamics for strong interaction \( \pi\pi \) scattering (Sect. II). This in turn supports the recent \( s \)-wave \( \pi\pi \) phase shift analyses\[6\] in

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Sect. III using a negative background phase obtaining a broad $\sigma$ resonance in the 535-650 MeV mass region. This is more in line with the prior analysis of Ref. [5] and with the dynamically generated quark-level linear $\sigma$ model (L$\sigma$M) theory of Ref. [7] predicting $m_\sigma \approx 650$ MeV. Section IV looks instead at processes involving two final-state pions where crossing symmetry plays no role, such as for the DM2 experiment[8] $J/\Psi \to \omega \pi \pi$ and for $\pi N \to \pi \pi N$ polarization measurements[9]. Section V extends the prior crossing-symmetric strong interaction chiral dynamics to the non-leptonic weak interaction $\Delta I = \frac{1}{2}$ decays $K^0 \to 2\pi$. We give our conclusions in Sect. VI.

2 Strong Interactions, Crossing Symmetry and the $\sigma$ Meson

It has long been understood[10-12] that the non-strange isospin I=0 $\sigma$ meson is the chiral partner of the I=1 pion. In fact Gell-Mann-Lévy’s[10,11] nucleon-level L$\sigma$M requires the meson-meson couplings to satisfy (with $f_\pi \approx 93$ MeV)

$$g_{\sigma\pi\pi} = \frac{m_\sigma^2 - m_\pi^2}{2f_\pi} = \lambda f_\pi,$$  \hspace{1cm} (1)

where $g_{\sigma\pi\pi}$ and $\lambda$ are the cubic and quartic meson couplings respectively. On the other hand, the $\sigma$ meson pole for the $\pi\pi$ scattering amplitude at the soft point $s = m_\pi^2$ using (1) becomes

$$M^\sigma_{\pi\pi} \rightarrow \frac{2g_{\sigma\pi\pi}}{s - m_\sigma^2} \rightarrow \frac{2g_{\sigma\pi\pi}}{m_\pi^2 - m_\sigma^2} = -\lambda = -M_{\pi\pi}^{contact}. \hspace{1cm} (2)$$

The complete tree-level L$\sigma$M $\pi\pi$ amplitude is the sum of the quartic contact amplitude $\lambda$ plus $\sigma$ poles added in a crossing symmetric fashion from the s, t and u-
channels. Using the chiral symmetry soft-pion limit (2) combined with the (non-soft) Mandelstam relation $s + t + u = 4m_\pi^2$, the lead $\lambda$ contact $\pi\pi$ amplitude miraculously cancels[11]. Not surprisingly, the resulting net $\pi^a\pi^b \to \pi^c\pi^d$ amplitude in the L$\sigma$M is the low energy model-independent Weinberg amplitude[13].

\[
M_{\pi\pi} = \frac{s - m_\pi^2}{f_\pi^2} \delta^{ab}\delta^{cd} + \frac{t - m_\pi^2}{f_\pi^2} \delta^{ac}\delta^{bd} + \frac{u - m_\pi^2}{f_\pi^2} \delta^{ad}\delta^{bc},
\]

due to partial conservation of axial currents (PCAC) applied crossing-consistently to all three $s$, $t$, $u$-channels. Recall that the underlying PCAC identity $\partial A^i = f_\pi m_\pi^2 \phi_\pi$, upon which the Weinberg crossing-symmetric PCAC relation (3) is based, was originally obtained from the L$\sigma$M lagrangian[10,11].

Although the above (L$\sigma$M) Weinberg PCAC $\pi\pi$ amplitude (3) predicts an $s$-wave $I=0$ scattering length[13] $a^{(0)}_{\pi\pi} = 7m_\pi/32\pi f_\pi^2 \approx 0.16 m_\pi^{-1}$ which is $\sim 30\%$ less than first obtained from $K_{\ell4}$ data[14], more precise experiments are now under consideration. Moreover a simple chiral-breaking scattering-length correction $\Delta a^{0}_{\pi\pi}$ follows from the L$\sigma$M using a Weinberg-like crossing-symmetric form[15]

\[
M_{\pi\pi}^{abcd} = A(s, t, u)\delta^{ab}\delta^{cd} + A(t, s, u)\delta^{ac}\delta^{bd} + A(u, t, s)\delta^{ad}\delta^{bc},
\]

\[
A^{L\sigma M}(s, t, u) = -2\lambda \left[1 - \frac{2\lambda f_\pi^2}{m_\sigma^2 - s}\right] = \left(\frac{m_\sigma^2 - m_\pi^2}{m_\sigma^2 - s}\right) \left(\frac{s - m_\pi^2}{f_\pi^2}\right),
\]

where the L$\sigma$M Eq. (1) has been used to obtain the second form of (5). Then the $I=0$ $s$-channel amplitude $3A(s, t, u) + A(t, s, u) + A(u, t, s)$ predicts the $s$-wave scattering length at $s = 4m_\pi^2$, $t = u = 0$ using the L$\sigma$M amplitude (5) with $\varepsilon = m_\pi^2/m_\sigma^2 \approx 0.046$ for the L$\sigma$M mass[7] $m_\sigma \approx 650$ MeV:

\[
a^{(0)}_{\pi\pi}|_{L\sigma M} \approx \left(\frac{7 + \varepsilon}{1 - 4\varepsilon}\right) \frac{m_\pi}{32\pi f_\pi^2} \approx (1.23) \frac{7m_\pi}{32\pi f_\pi^2} \approx 0.20 m_\pi^{-1}.
\]
This simple 23% $L\sigma M$ enhancement of the Weinberg PCAC prediction\[13\] agrees in magnitude with the much more complicated one-loop order chiral perturbation theory approach\[16\] which also predicts an $s$-wave scattering length correction of order $\Delta a_{\pi\pi}^0 \sim 0.04 m$\(\pi\)^{-1}. This indirectly supports a $\sigma(650)$ scalar meson mass scale as used in (6).

The above “miraculous (chiral symmetry) cancellation”, due to Eqs. (1) and (2) has been extended to final-state pionic processes $A_1 \to \pi(\pi\pi)_{s\text{-wave}}$\[17\], $\gamma\gamma \to 2\pi^0$\[18\] and $\pi^- p \to \pi^- \pi^+ n$. In all of these cases the above $L\sigma M$ “miraculous cancellation” is simulated by a (non-strange) quark box – quark triangle cancellation due to the Dirac-matrix identity\[17,18\]

\[
\frac{1}{\gamma.p - m} \frac{2m\gamma_5}{\gamma.p - m} = -\gamma_5 \frac{1}{\gamma.p - m} - \frac{1}{\gamma.p - m} \gamma_5 ,
\]

combined with the quark-level Goldberger relation (GTR) $f_\pi g_{\pi qq} = m_q$ and the $L\sigma M$ meson couplings in (1).

Then the $u, d$ quark box graph in Fig. 1a for $A_1 \to 3\pi$ in the chiral limit (miraculously) cancels the quark triangle graph of Fig. 1b coupled to the $\sigma$ meson because of the GTR and the $L\sigma M$ chiral meson identity (1) along with the minus signs on the right-hand-side (rhs) of (7):

\[
M_{A_1,3\pi}^{\text{box}} + M_{A_1,3\pi}^{\text{tri}} \to -\frac{1}{f_\pi} M(A_1 \to \sigma\pi) + \frac{1}{f_\pi} M(A_1 \to \sigma\pi) = 0 .
\]

This soft pion theorem\[17\] in (8) is compatible with the PDG tables\[1\] listing the decay rate $\Gamma[A_1 \to \pi(\pi\pi)_{sw}] = 1 \pm 1$ MeV.

Similarly, the $\gamma\gamma \to 2\pi^0$ quark box graph suppresses the quark triangle $\sigma$ resonance.
graph in the 700 MeV region, also compatible with $\gamma\gamma \rightarrow 2\pi^0$ cross section data[18].

Finally, the peripheral pion in $\pi^- p \rightarrow \pi^- \pi^+ n$ sets up an analogous $\pi\pi$ or quark box – quark triangle $s$-wave soft pion cancellation which completely suppresses any such $\sigma$ resonance – also an experimental fact for $\pi^- p \rightarrow \pi^- \pi^+ n$.

3 $\pi\pi$ Phase Shifts

The above miraculous (chiral) cancellation in $\pi\pi \rightarrow \pi\pi, A_1 \rightarrow 3\pi, \gamma\gamma \rightarrow 2\pi^0$ and $\pi^- p \rightarrow \pi^- \pi^+ n$ amplitudes and in data lends indirect support to the analyses of Refs. [2,4,5]. Reference [3] claims instead that the I=0 and I=2 $\pi\pi$ phase shifts require t-channel forces due to “exotic”, crossing-asymmetric resonances in the I=$\frac{3}{2}$ and 2 cross-channels rather than due a broad low-mass scalar $\sigma$ meson (in the $s$-channel). We suggest that this latter picture in Ref. [3] does not take account of the crossing-symmetric extent of the chiral $\pi\pi$ forces in all three $s, t$ and $u$-channels, leading to the above miraculous chiral cancellation.

Specifically the recent $\pi\pi$ phase shift analyses in Refs. [6] use a negative background phase approach compatible with unitarity. This background phase has a hard core of size $r_c \approx 0.63 \text{ fm}$ (the pion charged radius) such that $\delta^{BG} = -p_{\text{CM}}r_c$. Combining this background phase with the observed $\pi\pi$ phase shifts (e.g., of CERN-Munich or Cason et al.), the new I=0 phase shift goes through 90$^\circ$ resonance in the range 535-650 MeV, while the I=2 phase shift does not resonate but remains negative as observed. References [6] justify this background phase approach because of the “compensating $\lambda\phi^4$ contact (L$\sigma$M) interaction”. From our Sect. II we rephrase this as due
to the crossing symmetric $L\sigma M$ chiral ‘miraculous cancellation’$[11]$ which recovers Weinberg’s$[13]$ PCAC $\pi\pi$ amplitude in our Eq. (3).

Then Refs. $[6]$ choose a slightly model-dependent form factor $F(s)$ (designed to fit the lower energy region below 400 MeV) along with the best-fitted $\sigma \rightarrow \pi\pi$ effective coupling (double the $L\sigma M$ field theory coupling (1)). This gives the resonant $\sigma$ width$[6]$

$$\Gamma_R(s) = \frac{p^\text{CM}_\pi}{8\pi s} [g_R F(s)]^2 \approx 340 \text{ MeV} \quad \text{at} \quad \sqrt{s}_R \approx 600 \text{ MeV}, \quad g_R \approx 3.6 \text{ GeV}, \quad (9)$$

for $p^\text{CM}_\pi = \sqrt{s/4 - m^2_\pi} \approx 260 \text{ MeV}$. However, the decay width in (9) accounts only for $\sigma \rightarrow \pi^+\pi^-$ decay. To include as well the $\sigma \rightarrow \pi^0\pi^0$ decay mode, one must scale up (9) by a factor of $3/2$:

$$\Gamma_{\sigma \rightarrow 2\pi} = \frac{3}{2} \Gamma_R(s) \approx 510 \text{ MeV}, \quad (10)$$

not incompatible with Refs. $[1,2,5]$ but still slightly below Weinberg’s recent mended chiral symmetry (MCS) prediction$[19]$

$$\Gamma_{\sigma \rightarrow 2\pi}^{\text{MCS}} = \frac{9}{2} \Gamma_\rho \approx 680 \text{ MeV}, \quad (11a)$$

or the $L\sigma M$ decay width$[15]$

$$\Gamma_{L\sigma M}^{\sigma \rightarrow 2\pi} = \frac{3}{2} \frac{p^\text{CM}_\pi}{8\pi} \frac{(2g_{\sigma\pi\pi})^2}{m^2_\sigma} \approx 580 \text{ MeV}, \quad (11b)$$

for $m_\sigma \approx 600 \text{ MeV}$. Note too that the best fit $\sigma \rightarrow \pi^+\pi^-$ effective coupling in Refs. $[6]$ of 3.60 GeV is close to the $L\sigma M$ value in (1) at $m^R_\sigma \approx 600 \text{ MeV}$:

$$g_R \rightarrow 2g_{\sigma\pi\pi} = (m^2_\sigma - m^2_\pi)/f_\pi \approx 3.66 \text{ GeV}. \quad (12)$$
4 Crossing-Asymmetric Determinations of $\sigma$ (600-750)

With hindsight, the clearest way to measure the $\sigma \rightarrow \pi \pi$ signal is to avoid $\pi \pi \rightarrow \pi \pi, \gamma \gamma \rightarrow 2\pi^0, \pi^- p \rightarrow \pi^- \pi^+ n$ scatterings or $A_1 \rightarrow \pi(\pi \pi)_{sw}$ decay, since these processes are always plagued by the $\pi \pi$ miraculous chiral cancellation in (2) or an underlying quark box – triangle cancellation due to (7) as in (8). First consider the 1989 DM2 experiment\[8] $J/\Psi \rightarrow \omega \pi \pi$. Their Fig. 13 fits of the $\pi^+\pi^-$ and $\pi^0\pi^0$ distributions clearly show the known non-strange narrow $f_2(1270)$ resonance along with a broad $\sigma(500)$ “bump” (both bumps are non-strange and the accompanying $\omega$ is 97% non-strange). Moreover, DM2 measured the (low mass) $\sigma$ width as\[8]

$$\Gamma_{\sigma \rightarrow \pi \pi}^{DM2} = 494 \pm 58 \text{ MeV},$$

very close to the modified Ref. \[6\] $\sigma$ width fit of 510 MeV in Eq. (10).

Finally, this Fig. 13 of DM2\[8] clearly shows that the nearby $f_0(980)$ bump in the $\pi \pi$ distribution is only a “pimple” by comparison. This suggests that the observed\[1\] $f_0(980) \rightarrow \pi \pi$ decay mode proceeds via a small $\sigma - f_0$ mixing angle and that $f_0(980)$ is primarily an $\Omega s$ meson, compatible with the analyses of Refs. \[2,20\]. However, such a conclusion is not compatible with the $qqqq$ or $K\bar{K}$ molecule studies noted in Ref. \[3\].

Lastly, polarization measurements are also immune to the (spinless) miraculous chiral cancellation\[11\] in $\pi \pi \rightarrow \pi \pi$. This detailed polarization analysis of Ref. \[9\] approximately obtains the $\rho (770)$ mass and 150 MeV decay width. While the resulting $\sigma$ mass of 750 MeV is well within the range reported in the 1996 PDG\[1\] and closer
to the $\sigma$ mass earlier extracted from $\pi\pi \rightarrow K\bar{K}$ studies in Ref. [21], the inferred $\sigma$ width of $\Gamma_\sigma \sim 200 - 300$ MeV in Ref. [9] is much narrower than reported in Refs. [1, 2, 8, 21] or in our above analysis.

5 $K^\circ \rightarrow 2\pi$ Weak Decays and the $\sigma(600-700)$ Meson

To show that the $\sigma(600-700)$ scalar meson also arises with chiral crossing-symmetric weak forces, we consider the $\Delta I=1/2$ – dominant $K^\circ \rightarrow 2\pi$ decays. To manifest such a $\Delta I=1/2$ transition, we first consider the virtual $K^\circ I = \frac{1}{2}$ meson t-channel tadpole graph of Fig. 2. Here the weak tadpole transition $< 0|H_w|K^\circ >$ clearly selects out the $\Delta I=1/2$ part of the parity-violating component of $H_w$, while the adjoining strong interaction $K^\circ K^\circ \rightarrow \pi\pi$ is the kaon analogue of the t-channel $\pi\pi \rightarrow \pi\pi$, with Weinberg-type PCAC[22] amplitude $(t - m_\pi^2)/2f_\pi^2$ for $t = (p_K - 0)^2 = m_K^2$. Then the $\Delta I=1/2$ amplitude magnitude is[23]

$$| < \pi\pi|H_w|K^\circ > | = \frac{| < 0|H_w|K^\circ > |}{2f_\pi^2} (1 - m_\pi^2/m_K^2) . \quad (14)$$

A crossed version of this $\Delta I=1/2$ transition (14) is due to the s-channel $I=0$ $\sigma$ meson pole graph of Fig. 3 at $s = m_K^2$[24]. This leads to the $\Delta I=1/2$ amplitude magnitude

$$| < \pi\pi|H_w|K^\circ > | = | < \pi\pi|\sigma > \frac{1}{m_K^2 - m_\sigma^2 + im_\sigma\Gamma_\sigma} < \sigma|H_w|K^\circ > | . \quad (15a)$$

Applying chiral symmetry $< \sigma|H_w|K^\circ > = < \pi^\circ|H_w|K^\circ >$ (converting the former parity-violating to the latter parity-conserving transition) along with the LσM values

$$| < \pi\pi|\sigma > |$$

8
\[ m_\pi^2 / f_\pi \] from (1) and \( \Gamma_\sigma \approx m_\sigma \) to (15a), one sees that the \( \sigma \) mass scale cancels out of (15a), yielding\[25\]

\[ | < \pi \pi | H_w | K^\circ \rangle | \approx | < \pi^0 | H_w | K^\circ \rangle / f_\pi |. \tag{15b} \]

Not only has (15b) been derived by other chiral methods\[26\], but (15b) also is equivalent to (14) in the \( m_\pi = 0 \) chiral limit because weak chirality \([Q, H_w] = -[Q_5, H_w]\) for V-A weak currents and PCAC clearly require \[ | < \pi^0 | H_w | K^\circ \rangle | \approx | < 0 | H_w | K^\circ \rangle / 2 f_\pi |, \]
as needed.

Thus, we see that the existence of an \( I=0 \) scalar \( \sigma \) meson below 1 GeV manifests crossing symmetry (from the \( t \) to the \( s \)-channel) for the dominant \( \Delta I=1/2 \) equivalent amplitudes (14) and (15b). Further use of the quark model and the GIM mechanism\[27\] converts the \( K_{2\pi}^0 \) amplitudes in (14) or (15b) to the scale\[23\] \( 24 \times 10^{-8} \) GeV, close to the observed \( K_{2\pi}^0 \) amplitudes\[1\].

While the \( \Delta I=1/2 \) \( K^\circ \to 2\pi \) decays are controlled by the tadpole diagram in Fig. 2 (similar to \( \Delta I=1 \) Coleman-Glashow tadpole for electromagnetic (em) mass splittings\[28,29\]), the smaller \( \Delta I=3/2 \) \( K^+ \to 2\pi \) amplitude is in fact suppressed by “exotic” \( I=3/2 \) meson cross-channel Regge trajectories\[30\] (in a manner similar to the \( I=2 \) cross-channel exotic Regge exchange for the \( \pi^+ - \pi^0 \) em mass difference\[31\]).

This latter duality nature of crossing symmetry for exotic \( I=3/2 \) and \( I=2 \) channels was invoked in Ref. [3] to reject the low mass \( \sigma \) meson scheme reported in the 1996 PDG tables [1] based in part on the data analysis of Ref. [2]. That is, for exotic \( I=2 \) and \( I=3/2 \) (t-channel) dual exchanges, the dynamical dispersion relations thus generated are unsubtracted, so that one can then directly estimate the observed \( \Delta I=2 \) em mass
differences[32] and also the $\Delta I=3/2$ weak $K^+_{2\pi}$ decay amplitude[33]. However, for $I=1$ and $I=1/2$ dual exchanges, the resulting dispersion relations are once-subtracted, with subtraction constants corresponding to contact $\Delta I=1$ and $\Delta I=1/2$ tadpole diagrams for em and weak transitions, respectively. Contrary to Ref. [3], we instead suggest that these duality pictures for exotic $I=3/2$ and $I=2$ channels of Refs. [30,31] in fact help support the existence of the $I=0$ chiral $\sigma$ meson in Refs. [2,4-7].

6 Summary

We have studied both strong and weak interactions involving two final-state pions at low energy, using chiral and crossing symmetry to reaffirm the existence of the low-mass $I=0$ scalar $\sigma$ meson below 1 GeV. This supports the recent phenomenological data analyses in Refs. [2,4-6] and the quark-level linear $\sigma$ model [L$\sigma$M] theory of Ref. [7].

In Sect. II we focussed on $\pi\pi$ scattering and the crossing symmetry miraculous chiral cancellation[11] in the L$\sigma$M and its extension to the quark box - quark triangle soft pion cancellation[17,18]. Such chiral cancellations in $\pi\pi \rightarrow \pi\pi, A_1 \rightarrow 3\pi, \gamma\gamma \rightarrow 2\pi^0, \pi^-p \rightarrow \pi^-\pi^+n$ in turn suppress the appearance of the $\sigma(600-700)$ meson. Then in Sect. III we supported the recent re-analyses[6] of $\pi\pi$ phase shift data invoking a negative background phase. This led to an $I=0$ $\sigma$ meson in the 535-650 MeV region, but with a broader width $\Gamma_\sigma \sim 500$ MeV than found in Refs. [6] (but not incompatible with the 1996 PDG $\sigma$ width[1]).

In Sect. IV we briefly reviewed two different crossing-asymmetric determinations
of the I=0 $\sigma(600-750)$ which circumvent the above crossing-symmetric ‘miraculous’
chiral suppression of the $\sigma$ meson. Finally, in Sect. V we reviewed how the low mass
I=0 $\sigma$ meson s-channel pole for $\Delta I=1/2 \ K^0 \rightarrow 2\pi$ decays is needed to cross over to
the t-channel $\Delta I=1/2$ tadpole graph (which in turn fits data). This $\Delta I=1/2$ crossing-
symmetry $K \rightarrow \pi\pi$ picture was also extended by crossing duality to justify why the
(much smaller) $\Delta I=3/2 \ K_2^\pi$ decay is controlled by exotic I=3/2 t-channel Regge
trajectories[30], while the above I=1/2 dispersion relation has a (tadpole) non-exotic
Regge subtraction constant.

7 Acknowledgements

The author is grateful for hospitality and partial support at TRIUMF.
8 References

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9 Figure Captions

Fig. 1 Quark box (a) and quark triangle (b) graphs for $A_1 \rightarrow 3\pi$.

Fig. 2 $\Delta I=1/2$ t-channel $K^\circ$ tadpole graph for $K^\circ \rightarrow 2\pi$.

Fig. 3 $\Delta I=1/2$ s-channel $\sigma$ pole graph for $K^\circ \rightarrow 2\pi$. 