STONE DUALITY AND QUASI-ORBIT SPACES FOR GENERALISED C*-INCLUSIONS

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Abstract. Let $A$ and $B$ be C*-algebras with $A \subseteq M(B)$. Exploiting Stone duality and a Galois connection between restriction and induction for ideals in $A$ and $B$, we identify conditions that allow to define a quasi-orbit space and a quasi-orbit map for $A \subseteq M(B)$. These objects generalise classical notions for group actions. We characterise when the quasi-orbit space is an open quotient of the primitive ideal space of $A$ and when the quasi-orbit map is open and surjective. We discuss applications of these results to cross section C*-algebras of Fell bundles over locally compact groups, regular C*-inclusions, tensor products, relative Cuntz–Pimsner algebras, and crossed products for actions of locally compact Hausdorff groupoids and quantum groups.

1. Introduction

There are many different ways to build a C*-algebra $B$ as a crossed product for a C*-algebra $A$ with some kind of dynamics. The dynamics may be, for instance, an action of a locally compact group, groupoid, an inverse semigroup, a semigroup or a quantum group. The theory for each type of crossed product aims at understanding the structure of $B$ using the dynamics on $A$. Here we are interested in the ideal lattice $\mathcal{I}_p B$ and the primitive ideal space $\hat{B}$ of $B$. For a locally compact amenable group $G$ and a separable C*-dynamical system $(A, G, \alpha)$, the primitive ideal spaces of $A$ and of the crossed product $B := A \rtimes_G G$ are linked by a quasi-orbit map $\rho: \hat{B} \to \hat{A}/\sim$. Its target is the quasi-orbit space $\hat{A}/\sim$, where $p_1 \sim p_2$ if and only if $G \cdot p_1 = G \cdot p_2$. Both $\rho$ and the quotient map $\hat{A} \to \hat{A}/\sim$ are open, continuous and surjective. These are well known results. But even the existence of the quasi-orbit map is non-trivial. Quasi-orbit spaces are a key ingredient in the Effros–Hahn Conjecture, see [16, 26] (quasi-orbit spaces seem somewhat implicit in the groupoid version of this conjecture in [49]). And they are interesting objects in their own right, compare [25]. Moreover, under some freeness assumptions, the quasi-orbit map is a homeomorphism $\hat{B} \cong \hat{A}/\sim$; see [16,26,27,39,53,56,59] for the classical case of group actions, [24, Theorem 3.2] for partial group actions, [48, Theorem 6.8] for Fell bundles over discrete groups, or [6, Theorem 3.17] for a recent result for groupoid C*-algebras of étale groupoids.

In this paper, we provide a general framework for existence and properties of quasi-orbit spaces and quasi-orbit maps that are indispensable in the study of primitive ideal spaces of various C*-algebraic constructions. Our main technical tool is Theorem 3.2 which characterises open and surjective maps in lattice-theoretical terms. This is a result in pointless topology. It uses Stone duality and is interesting in its own right. Its proof is inspired by the proof that $\hat{A} = \text{Prim}(A)$ for a separable C*-algebra $A$.

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For an ordinary C*-inclusion $A \subseteq B$, it is customary to call $J \in \mathbb{I}(B)$ induced from $I \in \mathbb{I}(A)$ if $J = BI_B$ and to call $I$ the restriction of $J$ if $I = J \cap A$. We allow the more general situation of a *-homomorphism $\varphi: A \to \mathcal{M}(B)$ to the multiplier algebra of $B$. We speak of a generalised C*-inclusion if $\varphi$ is injective. Let $I \in \mathbb{I}(A)$, $J \in \mathbb{I}(B)$. The induction map $i: \mathbb{I}(A) \to \mathbb{I}(B)$ is defined by $i(I) = B\varphi(I)B$ as expected. The restriction map $r: \mathbb{I}(B) \to \mathbb{I}(A)$ is defined so that $r$ and $i$ form a Galois connection, that is, $I \subseteq r(J)$ holds if and only if $i(I) \subseteq J$. The Galois connection property has many useful consequences. For instance, the partially ordered sets $\mathbb{I}^B(A) := r(\mathbb{I}(B)) \subseteq \mathbb{I}(A)$ and $\mathbb{I}^A(B) := i(\mathbb{I}(A)) \subseteq \mathbb{I}(B)$ of restricted and induced ideals are complete lattices, and the maps $i$ and $r$ restrict to mutually inverse isomorphisms that yield $\mathbb{I}^B(A) \cong \mathbb{I}^A(B)$. Examples suggest that everything that can be said about the ideal structure of $A$ and $B$ in this generality follows from the Galois connection property (see Section 2).

Our main results need $\text{Prime}^B(A)$ to be first countable, and we assume this for the rest of the introduction. A sufficient condition for this is that $\tilde{A}$ be second countable. In Section 3, we characterise when the quasi-orbit map and the quasi-orbit space exist and when the quasi-orbit map is open and surjective. This involves the following lattice-theoretic conditions:

\begin{itemize}
  \item[(JR)] joins of restricted ideals remain restricted;
  \item[(C1)] $I \cap r \circ i(J) = r \circ i(I \cap J)$ for all $I \in \mathbb{I}^B(A)$ and $J \in \mathbb{I}(A)$;
  \item[(MI)] finite meets of induced ideals are again induced;
  \item[(MII)] arbitrary meets of induced ideals are again induced;
  \item[(C2)] $I \cap F(J) = F(I \cap J)$ for every $I \in \mathbb{I}^A(B)$ and $J \in \mathbb{I}(B)$, where $F(J)$ is the meet of all induced ideals that contain $J$.
\end{itemize}

We need condition [JR] to define the quasi-orbit space $\tilde{A}/\sim$ of $\varphi: A \to \mathcal{M}(B)$. Namely, [JR] says that the inclusion $\mathbb{I}^B(A) \hookrightarrow \mathbb{I}(A)$ is a morphism of locales. By Stone duality, this is equivalent to the existence of a continuous map $\pi: \tilde{A} \to \text{Prime}^B(A)$, and we let $A/\sim$ be the resulting quotient space. Theorem 3.2 shows that [C1] holds if and only if $\pi$ is an open surjection, that is, if and only if $\tilde{A}/\sim \cong \text{Prime}^B(A)$ for an open equivalence relation on $\tilde{A}$. We say that $A$ separates ideals in $B$ if $r: \mathbb{I}(B) \to \mathbb{I}(A)$ is injective (i.e. if all ideals in $B$ are induced). Then $\text{Prime}^B(A) \cong \text{Prime}(B)$, which is equal to $\tilde{B}$ under our countability assumption. Accordingly, $r$ induces a homeomorphism $\tilde{B} \cong A/\sim$ for an open equivalence relation $\sim$ on $A$ if and only if $A$ separates ideals in $B$ and [JR] and [C1] hold.

Condition [MII] is needed for the quasi-orbit map $\varphi: \tilde{B} \to \tilde{A}/\sim$ to exist. It holds if and only if the inclusion $\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)$ is a morphism of locales. If $\tilde{A}/\sim \cong \text{Prime}^B(A)$, this is equivalent to the existence of a continuous map $\varphi: \tilde{B} \to \tilde{A}/\sim$ inducing the inclusion $\mathbb{I}^B(A) \hookrightarrow \mathbb{I}(B)$. Theorem 3.2 shows that the conditions [MII] and [C2] characterise when $\varphi$ is open and surjective. We separate the conditions [MII] and [C2] because [MII] is far easier to check. Moreover, as we show below, already for cross products by group actions it may happen that [MII] fails, even though [C2] holds.

In Section 6 we identify an easily checkable condition for $\varphi: A \to \mathcal{M}(B)$ that implies [C1] and [MII]. It also implies [JR] for an ordinary inclusion $A \subseteq B$. Namely, we call $\varphi: A \to \mathcal{M}(B)$ symmetric, if every restricted ideal $I \in \mathbb{I}^B(A)$ is symmetric in the sense that $\varphi(I) \cdot B = B \cdot \varphi(I)$.

The remaining sections consider examples and applications. We begin with crossed products for group actions, section C*-algebras of Fell bundles over groups, and regular inclusions in Section 6. These are the prototypical examples for our theory. All these cases lead to a symmetric inclusion where conditions [JR], [C1] and [MII] hold.
There is, however, a quasi-orbit space for the gauge action of the quasi-orbit space of $O$.

In Section 7.2, we consider a relative Cuntz–Pimsner algebra following construction from [5] to restrict ideals along $\phi$. A pond correspondence $X$ is contained in all $J$ prime ideal space of the lattice of $I$. Let $I$ be a family of ideals. Their $\bigvee_{s \in S} I_s$ is the smallest ideal that contains all $I_s$; it is equal to the closed linear span of the ideals $I_s$. Their $\bigwedge_{s \in S} I_s$ is the largest ideal that is contained in all $I_s$; it is equal to the intersection $\bigcap_{s \in S} I_s$.

Let $B$ be a C*-algebra and $M(B)$ its multiplier algebra. Let $\varphi : A \to M(B)$ be a *-homomorphism. If $\varphi$ is injective, we call it a generalised inclusion. We use the following construction from [5] to restrict ideals along $\varphi$. For any $J \in \mathbb{I}(B)$, we let $M(B, J) := \{ m \in M(B) : m \cdot B + B \cdot m \subseteq J \}$.

**Lemma 2.1.** Let $J \in \mathbb{I}(B)$.

1. $M(B, J)$ is the largest ideal $I$ in $M(B)$ with $I \cap B \subseteq J$.
2. $M(B, J) = \{ m \in M(B) : m \cdot B \subseteq J \} = \{ m \in M(B) : B \cdot m \subseteq J \} = \{ m \in M(B) : B \cdot m, B \cdot m \subseteq J \}$.
3. $M(B, J)$ is the kernel of the canonical *-homomorphism $M(B) \to M(B/J)$.

**Proof.** The subset $M(B, J)$ is a closed, two-sided ideal in $M(B)$ because $J$ is a closed, two-sided ideal in $B$ and $m \cdot B + B \cdot m \subseteq B$ for all $m \in M(B)$. We have $M(B, J) \cap B = M(B, J) \cdot B \subseteq J$. Let $I \in \mathbb{I}(M(B))$ satisfy $I \cap B \subseteq J$. Then $m \cdot B + B \cdot m \subseteq I \cap B \subseteq J$ for all $m \in I$, that is, $I \subseteq M(B, J)$. Thus $M(B, J)$ is the largest ideal in $M(B)$ that intersects $B$ in $J$.
Let $m \in \mathcal{M}(B)$. Let $N$ be a directed set and let $(e_n)_{n \in N}$ be an approximate unit for $B$. Since $B \cdot m \subseteq B$, we have $b \cdot m = \lim b \cdot m \cdot e_n$ for all $b \in B$. Hence $B \cdot m \cdot B \subseteq B$. Similarly, it implies $m \cdot B \subseteq J$. Conversely, $m \in \mathcal{M}(B, J)$ implies both $m \cdot B \subseteq J$ and $B \cdot m \subseteq J$, and these imply $B \cdot m \cdot B \subseteq B$. If $m \in \mathcal{M}(B)$, then $m \cdot J + J \cdot m \subseteq J$, so that $m$ descends to a multiplier of $B/J$. This is the canonical *-homomorphism $\mathcal{M}(B) \to \mathcal{M}(B/J)$ in [3]. Its kernel consists of those maps $m \in \mathcal{M}(B)$ with $m \cdot B \subseteq B$. This is $\mathcal{M}(B, J)$ by [2].

**Definition 2.2.** The restriction of $J \in \mathcal{I}(B)$ is

$$r(J) := \varphi^{-1}(\mathcal{M}(B, J)).$$

The induced ideal of $I \in \mathcal{I}(A)$ is

$$i(I) := B\varphi(I)B = \text{span}\{b_1\varphi(x)b_2 : b_1, b_2 \in B, x \in I\}.$$

**Remark 2.3.** The maps $r$ and $i$ also appear in [27] Proposition 9.(i). They coincide with the maps denoted by $\text{Res}_\varphi$ and $\text{Ex}_\varphi$ in [27] Lemma 1.1.

We have $i(I) \subseteq \mathcal{I}(B)$ for all $I \in \mathcal{I}(A)$ by construction. Let $J \in \mathcal{I}(B)$. Lemma 2.1 implies that $r(J)$ is the kernel of the composite *-homomorphism $A \to \mathcal{M}(B) \to \mathcal{M}(B/J)$. So $r(J) \subseteq \mathcal{I}(A)$, and $\varphi$ induces an injective *-homomorphism

$$\varphi_J : A/r(J) \to \mathcal{M}(B/J).$$

In particular, $r(0) = \ker \varphi$, which is 0 if and only if $\varphi$ is injective.

**Definition 2.5.** Let $\varphi : A \to \mathcal{M}(B)$ be a *-homomorphism. We call $I \subseteq \mathcal{I}(A)$ restricted if $I = r(J)$ for some $J \in \mathcal{I}(B)$. We call $J \subseteq \mathcal{I}(B)$ induced if $J = i(I)$ for some $I \subseteq \mathcal{I}(A)$. Let $\mathcal{I}^B(A) \subseteq \mathcal{I}(A)$ and $\mathcal{I}^A(B) \subseteq \mathcal{I}(B)$ be the subsets of restricted and induced ideals.

**Remark 2.6.** The *-homomorphism $\varphi_{r(0)} : A/r(0) \to \mathcal{M}(B)$ as in (2.4) is injective and we have natural order isomorphisms $\mathcal{I}^B(A) \cong \mathcal{I}^B(A/r(0))$ and $\mathcal{I}^A(B) \cong \mathcal{I}^{A/r(0)}(B)$. Thus in all statements concerning only induced and restricted ideals one may assume that $\varphi : A \to \mathcal{M}(B)$ is injective, that is, a generalised $C^*$-inclusion.

**Lemma 2.7.** Let $I \subseteq \mathcal{I}(A)$, $J \subseteq \mathcal{I}(B)$. Then $I \subseteq r(J)$ if and only if $i(I) \subseteq J$.

**Proof.** Since $I \subseteq A$, we have $I \subseteq r(J)$ if and only if $\varphi(I) \subseteq \mathcal{M}(B, J)$. By Lemma 2.1 this is equivalent to $b_1\varphi(x)b_2 \in J$ for all $x \in I$, $b_1, b_2 \in B$. And this is equivalent to $i(I) \subseteq J$.

The relationship between the two maps

$$r : \mathcal{I}(B) \to \mathcal{I}(A), \quad i : \mathcal{I}(A) \to \mathcal{I}(B)$$

in Lemma 2.7 says that they form a (monotone) Galois connection, see [12] Definition 7.23. This was already noticed by Green in [27] Proposition 9.(i). It has several useful consequences, which we will list in Proposition 2.8. Before that, we stress that the maps $i$ and $r$ in a Galois connection determine each other; $r$ is the upper adjoint of $i$ and $i$ is the lower adjoint of $r$. More precisely, the Galois connection property dictates that $r(J) \subseteq \mathcal{I}(A)$ is the join of all $I \subseteq \mathcal{I}(A)$ with $i(I) \subseteq J$, whereas $i(I) \in \mathcal{I}(B)$ is the meet of all $J \in \mathcal{I}(B)$ with $I \subseteq r(J)$.

**Proposition 2.8.** Let $\varphi : A \to \mathcal{M}(B)$ be a *-homomorphism. Then

1. The maps $r : \mathcal{I}(B) \to \mathcal{I}(A)$ and $i : \mathcal{I}(A) \to \mathcal{I}(B)$ are monotone.
2. If $I \subseteq \mathcal{I}(A)$, then $r \circ i(I) \supseteq I$ and $i \circ r \circ i(I) = i(I)$.
3. If $J \subseteq \mathcal{I}(B)$, then $i \circ r(J) \subseteq J$ and $r \circ i \circ r(J) = r(J)$.
4. The maps $i$ and $r$ restrict to isomorphisms of partially ordered sets between $\mathcal{I}^B(A)$ and $\mathcal{I}^A(B)$ that are inverse to each other.
(5) The map $i$ preserves joins and $r$ preserves meets.
(6) $i(0) = 0$ and $r(B) = A$.
(7) Meets of restricted ideals in $A$ remain restricted and joins of induced ideals in $B$ remain induced.
(8) The isomorphic partially ordered sets $\mathcal{I}(A) \cong \mathcal{I}(B)$ are complete distributive lattices.
(9) The inclusion $\mathcal{I}(B) \hookrightarrow \mathcal{I}(A)$ and the retraction $r \circ i : \mathcal{I}(A) \to \mathcal{I}(B)$ form a Galois connection, that is, $I \in \mathcal{I}(A)$ and $J \in \mathcal{I}(B)$ satisfy $I \subseteq J$ if and only if $r \circ i(I) \subseteq J$.
(10) The retraction $i \circ r : \mathcal{I}(B) \to \mathcal{I}(A)$ and the inclusion $\mathcal{I}(A) \hookrightarrow \mathcal{I}(B)$ form a Galois connection, that is, $I \in \mathcal{I}(B)$ and $J \in \mathcal{I}(A)$ satisfy $J \subseteq I$ if and only if $J \subseteq i \circ r(I)$.

Proof. Statements (1)–(3) are [12, Lemma 7.26]. And (2) and (3) imply (4). Statement (5) is [12, Proposition 7.31]. This contains (6) because the minimal and maximal elements are the join and the meet of the empty family of ideals. All meets and joins exist in $\mathcal{I}(A)$ and $\mathcal{I}(B)$, and restriction preserves meets and induction preserves joins by (5). This implies (7). Statement (8) follows from (4) and (7).

We prove (9). Let $I \in \mathcal{I}(A)$ and $J \in \mathcal{I}(B)$, that is, $J = r(K)$ for some $K \in \mathcal{I}(B)$. If $r \circ i(I) \subseteq J$, then $I \subseteq J$ because $J \subseteq r \circ i(I)$. Conversely, if $I \subseteq J = r(K)$, then $i(I) \subseteq K$ and hence $r \circ i(I) \subseteq r(K) = J$. This proves (9).

We prove (10). Let $J \in \mathcal{I}(B)$ and $I \in \mathcal{I}(A)$, that is, $J = i(K)$ for some $K \in \mathcal{I}(A)$. If $J \subseteq i \circ r(I)$, then $J \subseteq I$ because $i \circ r(I) \subseteq I$. Conversely, if $J = i(K) \subseteq I$, then $K \subseteq r(I)$ and hence $J = i(K) \subseteq i \circ r(I)$. This proves (10).

The statements (9) and (10) say that $r \circ i(I)$ for $I \in \mathcal{I}(A)$ is the smallest restricted ideal that contains $I$, whereas $i \circ r(J)$ for $J \in \mathcal{I}(B)$ is the largest induced ideal contained in $J$. An inclusion with a lower adjoint such as $\mathcal{I}(B) \hookrightarrow \mathcal{I}(A)$ is also called an upper Galois insertion, whereas an inclusion with an upper adjoint such as $\mathcal{I}(A) \hookrightarrow \mathcal{I}(B)$ is called a lower Galois insertion.

Remark 2.9. The subset $\mathcal{I}(B) \subseteq \mathcal{I}(A)$ need not be closed under joins and $\mathcal{I}(A) \subseteq \mathcal{I}(B)$ need not be closed under meets. Nevertheless, the map $r \circ i : \mathcal{I}(A) \to \mathcal{I}(B)$ preserves joins and $i \circ r : \mathcal{I}(B) \to \mathcal{I}(A)$ preserves meets by [12, Proposition 7.31]. This is no contradiction because here joins and meets are taken in the respective sublattices. By definition, the join of $(I_{\alpha})_{\alpha \in S}$ in $\mathcal{I}(A)$ is the smallest element of $\mathcal{I}(A)$ that contains $I_{\alpha}$ for all $\alpha \in S$. This is indeed equal to $r \circ i(\bigvee I_{\alpha})$.

Definition 2.10. Let $\varphi : A \to \mathcal{M}(B)$ be a $*$-homomorphism. We say that $A$ detects ideals in $B$ if $r(J) = r(0)$ for $J \in \mathcal{I}(B)$ implies $J = 0$, and $A$ separates ideals in $B$ if $r(J_1) = r(J_2)$ for $J_1, J_2 \in \mathcal{I}(B)$ implies $J_1 = J_2$.

Proposition 2.11. Let $\varphi : A \to \mathcal{M}(B)$ be a $*$-homomorphism. Then $A$ detects ideals in $B$ if and only if any non-zero ideal in $B$ contains a non-zero induced ideal. And $A$ separates ideals in $B$ if and only if all ideals in $B$ are induced, if and only if $A/r(J)$ embedded into $\mathcal{M}(B)/J$ using the induced injective $*$-homomorphism $A/r(J) \hookrightarrow \mathcal{M}(B)/J$ in [26], detects ideals for all induced ideals $J \in \mathcal{I}(B)$.

Proof. By definition, $A$ detects ideals in $B$ if $J \neq 0$ implies $r(J) \neq r(0)$. The statements (2) and (3) in Proposition 2.5 say that $i \circ r(J) \subseteq J$ and $r \circ i \circ r = r$. Assume first that $A$ detects ideals in $B$. For each $J \neq 0$, the induced ideal $i \circ r(J)$ is contained in $J$, and it is non-zero because $r(i \circ r(J)) = r(J) \neq r(0)$. Conversely, assume that any $J \in \mathcal{I}(B)$ contains a non-zero induced ideal $i(I)$. Then $r(J) \supseteq r \circ i(I)$, and $r \circ i(I) \neq r(0)$ because $i(r \circ i(I)) = i(I)$ and $i \circ r(0) = 0$ differ. This proves the first statement about detection of ideals.
By definition, $A$ separates ideals in $B$ if and only if $r$ is injective. If $r$ is not injective, then neither is $i \circ r$. Conversely, if $i \circ r$ is not injective, then neither is $r \circ i \circ r$, which is equal to $r$ by Proposition 2.3. Thus $A$ separates ideals in $B$ if and only if $i \circ r$ is injective. Since $i \circ r$ is a retraction from $\ll B \gg$ onto $\ll A \gg(B)$, it is injective if and only if $\ll A \gg(B) = \ll B \gg$.

So there is $J \subseteq \ll B \gg$ with $i \circ r(J) \neq J$ if (and only if) $A$ does not separate ideals in $B$. Proposition 2.3 implies $i \circ r(J) \subseteq J$ and $r(J) = r(i \circ r(J))$. So $J$ is mapped to a non-zero ideal in $B/i \circ r(J)$, which is not detected by the map $A/r(J) \mapsto M(B/r \circ i(I))$. Conversely, if $A/r(J)$ does not detect ideals in $B/J$ for some induced ideal $J \subseteq \ll B \gg$, then there is an ideal $J' \subseteq \ll B \gg$ with $J \subseteq J'$ and $r(J') = r(J)$. So $A$ does not separate ideals in $B$.

The following examples show that, in general, restriction and induction of ideals do not have more properties than those in Proposition 2.3. Examples 2.12 and 2.13 are very easy cases of a crossed product by the finite group $\mathbb{Z}/2$. Example 2.15 is related to graph $C^*$-algebras, see also Example 7.9.

**Example 2.12.** Let $A = \mathbb{C}$ be embedded unitally into $B = \mathbb{C} \oplus \mathbb{C}$. Here $r(J) = A \cap J$, and $r(\mathbb{C} \oplus 0) = 0 = r(0 \oplus \mathbb{C})$, but $r(\mathbb{C} \oplus 0 + 0 \oplus \mathbb{C}) = r(\mathbb{C} \oplus \mathbb{C}) = \mathbb{C}$. So $r$ does not commute with finite joins.

**Example 2.13.** Let $A = \mathbb{C} \oplus \mathbb{C}$ be embedded diagonally into $B = \mathbb{M}_2(\mathbb{C})$. Here $i(\mathbb{C} \oplus 0) = B = i(0 \oplus \mathbb{C})$, but $i(\mathbb{C} \oplus 0 \cap 0 \oplus \mathbb{C}) = i(0 \oplus 0) = 0$. So $i$ does not commute with finite meets.

**Example 2.14.** Since $r(0) = \ker \varphi$, the map $r$ preserves the minimal elements if and only if $\varphi$ is injective. Similarly, $i$ preserves the maximal elements if and only if $i(A) = B$, that is, $B \cdot \varphi(A) \cdot B = B$. This happens, for instance, if $\varphi$ is non-degenerate.

We will give an example of a meet of two induced ideals that is no longer induced in Example 7.6. The following example shows a join of two restricted ideals that is no longer restricted:

**Example 2.15.** Let $B : = \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$ and let $A \subseteq B$ be the commutative $C^*$-subalgebra spanned by the orthogonal diagonal projections $(E_{00}, 0), (0, E_{00})$, and $(E_{11}, E_{11})$. Let $J_1 = \mathbb{M}_2(\mathbb{C}) \oplus 0$ and $J_2 = 0 \oplus \mathbb{M}_2(\mathbb{C})$. Then $I_1 := J_1 \cap A = \mathbb{C} \cdot (E_{00}, 0)$ and $I_2 := J_2 \cap A = \mathbb{C} \cdot (0, E_{00})$ are restricted ideals in $A \cong \mathbb{C}^3$. The only other restricted ideals are $\{0\}$ and $A$. So the join $I_1 + I_2 \neq A$ of the two restricted ideals $I_1$ and $I_2$ is not restricted.

### 3. Stone Duality and Open Subjective Maps

We briefly recall Stone duality, which is the key to turn information about ideal lattices into information about prime and primitive ideal spaces (see [30][41]).

The category of locales has complete, distributive lattices as objects and maps that preserve arbitrary joins and finite meets as arrows. If $X$ is a topological space, then the partially ordered set of open subsets $\mathcal{O}(X)$ is a locale. And if $f : X \rightarrow Y$ is a continuous map, then $f^{-1} : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is a morphism of locales. Thus $\mathcal{O}$ is a contravariant functor from topological spaces to locales. **Stone duality** says that it has an adjoint functor $P$. That is, the functor $P$ maps a locale $L$ to a topological space $P(L)$ in such a way that continuous maps $X \rightarrow P(L)$ for a topological space $X$ are in natural bijection with locale morphisms $L \rightarrow \mathcal{O}(X)$. Here $P$ may stand for “points” or for “prime elements.” We recall the equivalence between characters and prime elements. For a character $\chi : L \rightarrow \{0, 1\}$, there is a largest element $p_\chi \in L$ with $\chi(p_\chi) = 0$. Thus $\chi(I) = 0$ for $I \leq p_\chi$ and $\chi(I) = 1$ otherwise by maximality.
of $p_x$. So $p_x$ is (meet) prime, that is, if $I_1, I_2 \in L$ satisfy $I_1 \wedge I_2 \leq p_x$, then $I_1 \leq p_x$ or $I_2 \leq p_x$. Conversely, if $p \in L$ is prime, then $\chi_p(I) := 0$ for $I \leq p$ and $\chi_p(I) := 1$ otherwise defines a character $\chi_p$. Thus characters are equivalent to prime elements. For $I \in L$, let

$$U_I := \{ \chi : L \to \{0,1\} \text{ character} : \chi(I) = 1 \} \cong \{ p \in L \text{ prime} : I \not\subset p \}.$$  

The map $I \mapsto U_I$ preserves joins and finite meets by the definition of a character. Hence $(U_I : I \in L)$ is a topology on $P(L)$. Any locale morphism $G : R \to L$ between locales $R$ and $L$ induces a continuous map $G^* : P(L) \to P(R)$ by $G^*(\chi) := \chi \circ G$. Equivalently, if $p \in L$ is prime, then $G^*(p)$ is the unique element of $R$ with

$$I \leq G^*(p) \text{ if and only if } G(I) \leq p.$$  

The map $G^*(p)$ is the largest element in $R$ with $G(G^*(p)) \leq p$.

The functor $P$ is adjoint to $\mathcal{O}$. The units of the adjunction are the natural maps $U^L : L \to \mathcal{O}(P(L))$, $I \mapsto U_I$, for a locale $L$ and $\delta^X : X \to P(\mathcal{O}(X))$, $x \mapsto \delta_x$, for a topological space $X$, where $\delta_x$ is the character

$$\delta_x : \mathcal{O}(X) \to \{0,1\}, \quad U \mapsto \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \not\in U. \end{cases}$$

The character $\delta_x$ corresponds to the prime element $X \setminus \{x\}$ of $\mathcal{O}(X)$. The induced continuous map $(U^L)^* : P(\mathcal{O}(P(L))) \to P(L)$ is a homeomorphism with inverse $U^{P(L)}$, and the induced locale morphism $(\delta^X)^{-1} : \mathcal{O}(P(\mathcal{O}(X))) \to \mathcal{O}(X)$ is a lattice isomorphism with inverse $U^{\mathcal{O}(X)}$. This implies the adjunction between $\mathcal{O}$ and $P$. A locale morphism $G : L \to \mathcal{O}(X)$ and its adjunct $\pi : X \to P(L)$ are related by $G(I) = \pi^{-1}(U_I)$ for all $I \in L$ and $\pi(x) = \delta_x \circ G^*$ for all $x \in X$.

The map $U^L$ is always surjective. A locale is called spatial if $U^L$ is injective. Equivalently, the characters separate elements of $L$. Let $X$ be a topological space. The locale $\mathcal{O}(X)$ is spatial. The space $X$ is $T_0$ if and only if $\delta^X$ is injective. Since $(\delta^X)^{-1} : \mathcal{O}(P(\mathcal{O}(X))) \to \mathcal{O}(X)$ is a lattice isomorphism, $\delta^X$ is a homeomorphism once it is bijective. The space $X$ is called sober in this case. This holds if and only if every irreducible closed set in $X$ is the closure of a singleton. Spaces of points of locales are always sober.

Let $A$ be a C*-algebra. Let $\hat{A}$ be the primitive ideal space of $A$. Then $\mathcal{I}(A) \cong \mathcal{O}(\hat{A})$, see [40, Theorem 4.1.3]. Hence $\mathcal{I}(A)$ is a spatial locale. By definition, $P(\mathcal{I}(A))$ is the set Prime($A$) of prime ideals in $A$ with the hull-kernel topology. The space $\hat{A}$ is $T_0$, see [13, 3.3.8]. Hence $\hat{A} \subseteq \text{Prime}(A)$. The inclusion $\hat{A} \subseteq \text{Prime}(A)$ is a homeomorphism if and only if $\hat{A}$ is sober. A C*-algebra where this fails is built in [55]. However, $\hat{A} = \text{Prime}(A)$ when $A$ is separable, see [46, Proposition 4.3.6]. In fact, the proof in [46] works under the assumption that $\hat{A}$ be second countable. Moreover, since $\hat{A}$ is always locally quasi-compact, $T_0$ and Baire (see [13, 3.1.3, 3.4.13]), [28, Proposition 1] implies that $\hat{A}$ is sober if $\hat{A}$ is second countable or, more generally, if Prime($A$) is first countable (see also Corollary 3.3 below).

**Theorem 3.2.** Let $X$ be a topological space, let $L$ be a locale, and let $G : L \to \mathcal{O}(X)$ be a locale morphism. Let the continuous map $\pi : X \to P(L)$ be the adjunct of $G$. If $\pi$ is an open surjection and $L$ is spatial, then $G$ is an upper Galois insertion and its lower adjoint $F : \mathcal{O}(X) \to L$ satisfies

$$(3.3) \quad I \wedge F(V) = F(G(I) \cap V) \quad \text{for all } I \in L, V \in \mathcal{O}(X).$$

Conversely, let $G$ be an upper Galois insertion, $F$ its lower adjoint, and assume (3.3). Then $L$ is spatial. And $\pi$ is an open surjection if

1. $P(L)$ is first countable or $X$ is second countable, and
2. all closed subspaces of $X$ are Baire spaces.
Proof. First we prove the easy assertions in the first paragraph. So assume $L$ to be spatial and $\pi$ to be an open surjection. Since $L$ is spatial, the map $I \mapsto U_I$ is an isomorphism $L \cong \mathbb{O}(P(L))$. The adjunction between $G$ and $\pi$ says that $G(I) = \pi^{-1}(U_I)$ for all $I \in L$. Since $\pi$ is open, it defines a map $F: \mathbb{O}(X) \mapsto \mathbb{O}(P(L)) \cong L$, $V \mapsto \pi(V)$. Since $\pi(V) \subseteq U_I$ if and only if $V \subseteq \pi^{-1}(U_I) = G(I)$, the map $F$ is the lower adjoint of $G$. Since $\pi$ is surjective, the map $G$ is injective. And $U_I \cap \pi(V) = \pi(\pi^{-1}(U_I) \cap V)$ for all $I \in L$, $V \in \mathbb{O}(X)$. This proves \ref{lem:2.2} and finishes the proof of the assertions in the first paragraph.

In the converse direction, $L$ is spatial once $G: L \mapsto \mathbb{O}(X)$ is injective because then the points of $X$ separate the elements of $L$. The proof that $\pi$ is an open surjection is more interesting. Most ideas needed for it appear already in the proof of \cite{H6} Proposition 4.3.6] in a more concrete setting. We first discuss our assumptions.

Lemma 3.4. If $X$ is second countable, then so is $P(L)$.

Proof. Let $(V_n)_{n \in \mathbb{N}}$ be a countable basis for the topology on $X$. Let $I \in L$. Then we may write $G(I) \in \mathbb{O}(X)$ as $G(I) = \bigvee_{k \in S} V_k$ for some $S \subseteq \mathbb{N}$. Then $G(I) = G \circ F \circ G(I) = G(\bigvee_{k \in S} F(V_k))$ because $F$ preserves joins as it is a lower Galois adjoint. Since $G$ is injective, this implies $I = \bigvee_{k \in S} F(V_k)$. Thus the open subsets $U_{F(V_k)}$ form a countable basis for $P(L)$.

Hence our Assumption $[1]$ ensures that $P(L)$ is first countable, that is, any point has a countable neighbourhood basis. We assume this in the following.

Since $G$ is a locale morphism, $G(L)$ is a topology on $X$. By the adjunction between $G$ and $\pi$, the topology $G(L)$ consists of all subsets of the form $\pi^{-1}(U_I)$ for $I \in L \cong \mathbb{O}(P(L))$. Thus $G(L)$ is equal to the topology on $X$ induced by $\pi$.

Lemma 3.5. Let $I \in L$. Then $G(I)$ is dense for the topology $G(L)$ if and only if it is dense for the topology $\mathbb{O}(X)$.

Proof. The subset $G(I)$ is dense for the topology $\mathbb{O}(X)$ if and only if $G(I) \cap V \neq \emptyset$ for all $V \in \mathbb{O}(X)$ with $V \neq \emptyset$. This is clearly stronger than being dense for the topology $G(I)$. Now assume that $G(I)$ is not dense for $\mathbb{O}(X)$. We claim that $G(I)$ is not dense for $G(L)$ either. By assumption, there is $V \in \mathbb{O}(X)$ with $V \neq \emptyset$ and $G(I) \cap V = \emptyset$. Then \ref{lem:2.2} implies $I \cap F(V) = F(G(I) \cap V) = F(\emptyset)$. Since $F$ is a lower adjoint, it commutes with arbitrary joins. So does $G$ as a locale morphism. In particular, both preserve minimal elements. So $G(F(\emptyset)) = \emptyset$. Since $G$ preserves finite meets,

$$G(I) \cap G \circ F(V) = G(I \cap F(V)) = G(F(\emptyset)) = \emptyset.$$ 

Since $G \circ F(V) \supseteq V \neq \emptyset$, this shows that $G(I)$ is not dense for the topology $G(L)$.

Lemma 3.6. The set $X$ with the topology $G(L)$ is a Baire space.

Proof. Let $I_n \in L$ for $n \in \mathbb{N}$ be such that $G(I_n)$ is dense for the topology $G(L)$ on $X$. By the previous lemma, $G(I_n)$ is dense for $\mathbb{O}(X)$. Since $X$ is a Baire space by assumption \ref{lem:2.2} the intersection $\bigcap G(I_n)$ is dense for $\mathbb{O}(X)$. Then it is also dense for $G(L)$.

Now we turn to the key step in the proof. We assume that the minimal element of $L$, which we denote here by $0$, is prime. We are going to show that $0 = \pi(x)$ for some point $x \in X$. The point $0 \in P(L)$ is dense, that is, $0 \in U$ for all non-empty open subsets $U \subseteq P(L)$. Equivalently, this holds for $U_I$ for all $I \in L \setminus \{0\}$. Since $P(L)$ is first countable, $0$ has a countable neighbourhood basis. This is the same as a decreasing sequence $(J_n)_{n \in \mathbb{N}}$ in $L \setminus \{0\}$ such that for every $I \in L \setminus \{0\}$ there is
some $n \in \mathbb{N}$ with $J_n \leq I$. Then $J_m \cap I \geq J_{\max(n,m)} \neq \emptyset$ for all $m \in \mathbb{N}$. Since $G$ is injective, it follows that $G(J_m)$ is dense in $X$ for the topology $G(L)$. Since $X$ with this topology is a Baire space by Lemma 3.4, the intersection

$$\bigcap G(J_n) = \bigcap \pi^{-1}(U_{J_n}) = \pi^{-1}\left(\bigcap U_{J_n}\right)$$

is dense and hence non-empty. Let $x$ be an element of it. Then $\pi(x) \in U_{J_n}$ for all $n \in \mathbb{N}$. So $\pi(x) \in U_I$ for all $I \subseteq L \setminus \{\emptyset\}$. Then $\pi(x) = 0$ because $P(L)$ is $T_0$ and no open subset of $P(L)$ separates 0 and $\pi(x)$.

Now let $p$ be an arbitrary prime element of $L$. We are going to find $x \in X$ with $\pi(x) = p$. Let

$$L_{\geq p} := \{J \subseteq L : J \geq p\}.$$

This subset of $L$ has the minimal element $p$ and is closed under arbitrary joins and non-empty meets. So $L_{\geq p}$ is a locale. An element of $L_{\geq p}$ is prime if and only if it is prime in $L$. And the prime elements that do not belong to $L_{\geq p}$ are precisely those in $U_p$. Thus $P(L_{\geq p}) = P(L) \setminus U_p$. The topology on $P(L_{\geq p})$ is the subspace topology from $P(L)$, and $I \subseteq L_{\geq p}$ corresponds to the (relatively) open subset $U_I \setminus U_p \subseteq P(L) \setminus U_p$. Since $\pi^{-1}(U_p) = G(p) \subseteq X$, the map $\pi$ restricts to a continuous map

$$\pi_p : X \setminus G(p) \to P(L_{\geq p}),$$

where $X \setminus G(p) \subseteq X$ also carries the subspace topology. This map is the adjunct of

$$G_p : L_{\geq p} \to \mathcal{O}(X \setminus G(p)), \quad I \mapsto G(I) \setminus G(p),$$

because $\pi_p^{-1}(U_I \setminus U_p) = \pi^{-1}(U_I) \setminus G(p) = G(I) \setminus G(p)$. The map $G_p$ is injective. Since $G(p)$ is open, a subset $V \subseteq X \setminus G(p)$ is relatively open if and only if $V \cup G(p) \subseteq \mathcal{O}(X)$. Then $F(V \cup G(p)) \supseteq F(G(p)) = p$, that is, $F(V \cup G(p)) \subseteq L_{\geq p}$. Define

$$F_p : \mathcal{O}(X \setminus G(p)) \to L_{\geq p}, \quad V \mapsto F(V \cup G(p)).$$

If $I \subseteq L_{\geq p}$, $V \subseteq \mathcal{O}(X \setminus G(p))$, then

$$I \cap F_p(V) = I \cap F(V \cup G(p)) = F(G(I) \cup (V \cup G(p))) = F(G(p) \cup (G(I) \cup V)) = F(G_p(I) \cup V).$$

Thus $F_p$ is a lower adjoint for $G_p$, and \textit{[3.5]} holds for $G_p$ and $F_p$.

If $P(L)$ is first countable, then so is the subspace $P(L_{\geq p})$. And the closed subset $X \setminus G(p) \subseteq X$ is a Baire space by the assumption \textit{[2]} So $X \setminus G(p), L_{\geq p}, G_p, \pi_p, F_p$ satisfy all the assumptions that were used to prove that the minimal element is in the image of $\pi$ when it is prime. So the special case treated above gives $x \in X \setminus G(p)$ with $\pi(x) = p$. This finishes the proof that $\pi : X \to P(L)$ is surjective.

To prove that $\pi$ is open, let $V \subseteq \mathcal{O}(X)$. We claim that

$$(3.7) \quad \pi(V) = U_{F(V)},$$

which is open in $P(L)$. So the proof of this claim will finish the proof of the theorem.

The subset $L_{\leq F(V)} \subseteq L$ has the maximal element $F(V)$ and is closed under joins and non-empty meets. So it is a locale. If $p$ is a character on $L$, then $p$ restricted to $L_{\leq F(V)}$ is either a character or the constant function 0. Thus the prime elements in $L_{\leq F(V)}$ are exactly those of the form $p \cap F(V)$ for a prime $p \in L$ with $F(V) \leq p$.

That is,

$$P(L_{\leq F(V)}) \cong \{p \in P(L) : F(V) \leq p\} = U_{F(V)},$$

which is an open subset of $P(L)$. Give $V$ the subspace topology from $X$. Define

$$G_V : L_{\leq F(V)} \to \mathcal{O}(V), \quad I \mapsto G(I) \cap V.$$
The map $G_V$ is injective because $F(G_V(I)) = F(G(I) \cap V) = I \cap F(V) = I$ for $I \leq F(V)$ by (3.3). The adjunctions between $F$ and $G$ and between $G$ and $\pi$ imply $V \subseteq G(F(V)) = \pi^{-1}(U_{F(V)})$. That is, $\pi(V) \subseteq U_{F(V)} = P(L_{\leq F(V)})$. And $G_V(I) = \pi^{-1}(U_I) \cap V = (\pi|_V)^{-1}(U_I)$ for all $I \in L_{\leq F(V)}$. So $G_V$ and $\pi|_V : V \to P(L_{\leq F(V)})$ are adjoints of each other. We compute

$$I \cap F(W) = F(G(I) \cap W) = F(G(I) \cap V \cap W) = F(G(V)(I) \cap W)$$

for all $I \in L_{\leq F(V)}$ and $W \in \mathcal{O}(V)$. The subspace $P(L_{\leq F(V)}) \subseteq P(L)$ inherits first countability. The Baire property is hereditary for open subsets. Since any closed subset in $V$ is open in a closed subset of $X$, it inherits the Baire property. Thus $\pi|_V : V \to P(L_{\leq F(V)}) = U_{F(V)}$ is surjective by what we have already proved. Thus $\pi(V) = U_{F(V)}$, which is open in $P(L)$. □

**Corollary 3.8.** Let $A$ be a $C^*$-algebra. If $\text{Prime}(A)$ is first countable or $\hat{A}$ is second countable, then $\text{Prime}(A) = \hat{A}$.

**Proof.** Apply the second part of Theorem 3.2 with $X = \hat{A}$, $L = \mathbb{I}(A)$, and $G = \text{Id}_{\mathbb{I}(A)}$. □

## 4. The quasi-orbit space

Throughout this section, we fix a $^*$-homomorphism $\varphi : A \to \mathcal{M}(B)$. Then $\mathbb{I}(A)$, $\mathbb{I}(B)$, $\mathbb{I}^B(A)$ and $\mathbb{I}^A(B)$ are locales, and $\mathbb{I}^B(A) \cong \mathbb{I}^A(B)$ are isomorphic as locales by Proposition 2.3. Let $\text{Prime}^B(A)$ and $\text{Prime}^A(B)$ be the topological spaces of prime elements in $\mathbb{I}^B(A)$ and $\mathbb{I}^A(B)$, respectively. Then $\text{Prime}^B(A) \cong \text{Prime}^A(B)$. This space is a candidate for the “quasi-orbit space” of $\varphi$. Usually, however, the quasi-orbit space is defined as a quotient of $\hat{A}$, see [6, 16, 24–27, 38, 39, 53, 56, 59]. We shall define a quotient space $\hat{A}/\sim$ related to $\text{Prime}^B(A) \cong \text{Prime}^A(B)$ when the inclusion $\mathbb{I}^B(A) \hookrightarrow \mathbb{I}(A)$ is a locale morphism.

**Lemma 4.1.** The inclusion map $\mathbb{I}^B(A) \hookrightarrow \mathbb{I}(A)$ is a locale morphism if and only if (JR) joins of restricted ideals remain restricted.

Then the induced continuous map $\pi : \hat{A} \subseteq \text{Prime}(A) \to \text{Prime}^B(A)$ is given by

$$\pi(p)$$ is the largest restricted ideal in $A$ that is contained in $p$.

**Proof.** The inclusion of a sublattice is a locale morphism if and only if the sublattice is closed under joins and finite meets. Since $\mathbb{I}^B(A) \subseteq \mathbb{I}(A)$ is closed under meets by Proposition 2.3, $\mathbb{I}^B(A) \hookrightarrow \mathbb{I}(A)$ is a locale morphism if and only if joins of restricted ideals are again restricted. Then there is a unique continuous map $\pi : \text{Prime}(A) \to \text{Prime}^B(A)$ that verifies (4.1), that is, $I \subseteq \pi(p)$ if and only if $I \subseteq p$ for $I \in \mathbb{I}^B(A)$, $p \in \text{Prime}(A)$. This and $\pi(p) \in \text{Prime}^B(A)$ imply (4.2). □

**Definition 4.3.** Let $\varphi : A \to \mathcal{M}(B)$ be such that $\text{[JR]}$ holds. Then the map $\pi : \hat{A} \to \text{Prime}^B(A)$ in (4.2) is defined. The quasi-orbit space of $\varphi$ is $\hat{A}/\sim$ with the quotient topology, where $p \sim q$ if and only if $\pi(p) = \pi(q)$.

**Theorem 4.4.** Let $\varphi : A \to \mathcal{M}(B)$ be such that $\text{[JR]}$ holds. If the continuous map $\pi : \hat{A} \to \text{Prime}^B(A)$ defined in (4.2) is open and surjective, then

(C1) $I \cap r \circ i(J) = r \circ i(I \cap J)$ for all $I \in \mathbb{I}^B(A)$ and $J \in \mathbb{I}(A)$.

Conversely, if (C1) holds and $\text{Prime}^B(A)$ is first countable or $\hat{A}$ is second countable, then the continuous map $\pi$ is surjective and open and induces a homeomorphism $\hat{A}/\sim \cong \text{Prime}^A(B)$. 
Then the continuous map \( \pi \) only if \( i \) equal to the restriction of the conditions restricts to a homeomorphism. Identify \( \pi \). Thus \( \pi \) is the adjunct of \( G \). The condition \([C1]\) is equivalent to condition \([3.6]\). The locale \( \mathcal{I}^B(A) \) is spatial because it is contained in the spatial locale \( \mathcal{I}(A) \). So Theorem \([3.2]\) implies all assertions except \( \hat{A}/\sim \cong \text{Prime}^B(A) \). This follows from the definition of the equivalence relation \( \sim \) whenever \( \pi \) is an open surjection.

**Proposition 4.5.** If \( A \) separates ideals in \( B \) and \( \text{Prime}^B(A) \) is first countable, then \( \hat{B} \) is a homeomorphism.

**Proof.** If \( A \) separates ideals in \( B \), then \( \mathcal{I}(B) = \mathcal{I}(A) \) by Proposition \([2.11]\). Thus \( i \) and \( r \) identify \( \mathcal{I}(B) \) with \( \mathcal{I}^B(A) \) by Proposition \([2.8](4)\). So \( \text{Prime}^B(A) \cong \text{Prime}^B(A) \) is first countable. Hence \( \hat{B} = \text{Prime}(B) \) by Corollary \([3.8]\). □

**Corollary 4.6.** Let \( \varphi : A \to \mathcal{A}(B) \). Suppose that \( \text{Prime}^B(A) \) is first countable. Identify \( \mathcal{I}(A) \) and \( \mathcal{I}(B) \) with \( \mathcal{O}(A) \) and \( \mathcal{O}(B) \), respectively. There is a continuous open surjection \( \pi : \hat{A} \to \hat{B} \) with \( r = \pi^{-1} \) if and only if \( A \) separates ideals in \( B \) and the conditions \([JR]\) and \([CT]\) hold. Then \( \hat{B} \cong \text{Prime}^B(A) \cong \hat{A}/\sim \).

**Proof.** If \( r = \pi^{-1} \) for a continuous open surjection \( \pi : \hat{A} \to \hat{B} \), then \( r \) is injective, that is, \( A \) separates ideals in \( B \). Then Proposition \([1.5]\) implies \( \hat{B} \cong \text{Prime}^B(A) \). Now all assertions follow from Lemma \([4.4]\) and Theorem \([4.4]\). □

**Proposition 4.7.** The inclusion map \( \mathcal{I}^A(B) \to \mathcal{I}(B) \) is a lattice morphism if and only if
\[(M) \text{ finite meets of induced ideals are again induced.} \]

Then the continuous map \( \text{Prime}(B) \to \text{Prime}^A(B) \) induced by \( \mathcal{I}^A(B) \to \mathcal{I}(B) \) is equal to the restriction of \( i \circ r : \mathcal{I}(B) \to \mathcal{I}^A(B) \) to \( \text{Prime}(B) \). So \( r : \text{Prime}(B) \to \text{Prime}^B(A) \) is a well defined continuous map.

**Proof.** The inclusion \( \mathcal{I}^A(B) \to \mathcal{I}(B) \) is a lower Galois insertion by Proposition \([2.8](10)\). The sublattice \( \mathcal{I}^A(B) \to \mathcal{I}(B) \) is closed under joins by Proposition \([2.8](7)\). So the inclusion is a locale morphism if and only if \( \mathcal{I}^A(B) \) is closed under finite meets in \( \mathcal{I}(B) \). Then the inclusion induces a continuous map \( \text{Prime}(B) \to \text{Prime}^A(B) \). By \([3.1]\), it maps \( p \in \text{Prime}(B) \) to the largest induced ideal contained in \( p \). This is exactly \( i \circ r(p) \). □

**Lemma 4.8.** Let \( \varphi : A \to \mathcal{A}(B) \). Then \( A \) separates ideals in \( B \) if and only if \( r \) restricts to a homeomorphism \( r : \text{Prime}(B) \to \text{Prime}^B(A) \).

**Proof.** If \( A \) separates ideals in \( B \) then \( r : \mathcal{I}(B) \to \mathcal{I}^B(A) \) is a lattice isomorphism. Hence it restricts to a homeomorphism \( r : \text{Prime}(B) \to \text{Prime}^B(A) \). Conversely, if \( r : \text{Prime}(B) \to \text{Prime}^B(A) \) is a homeomorphism, then \( r : \mathcal{I}(B) \to \mathcal{I}^B(B) \) is a locale isomorphism because \( \mathcal{I}(B) \cong \mathcal{O}(\text{Prime}(B)) \) and \( \mathcal{I}^B(A) \cong \mathcal{O}(\text{Prime}^B(A)) \). Then \( A \) separates ideals in \( B \). □
If we assume conditions \([\text{JR}]\) and \(\text{(MI)}\), then we have the following commutative diagram of continuous maps:

\[
\begin{array}{ccc}
\hat{B} & \xrightarrow{i} & \text{Prime}(B) \\
\downarrow & & \downarrow r
\end{array}
\begin{array}{ccc}
\text{Prime}(A) & \xrightarrow{\pi} & \text{Prime}^B(A) \\
\uparrow & & \uparrow \pi_x
\end{array}
\begin{array}{c}
\hat{A} \\
\uparrow \pi_x
\end{array}
\]

(4.9)

Here \(i \circ \pi = i\) if and only if \(\pi = r \circ i\) if and only if \(\mathbb{I}(A) = \mathbb{I}(B)\). So there is, in general, no natural map between \(\text{Prime}(B)\) and \(\text{Prime}(A)\). The best substitute seems to be the map \(\hat{B} \to \hat{A}/\sim\) that exists when the map \(\pi_x : \hat{A}/\sim \to \text{Prime}^B(A)\) is a homeomorphism:

**Definition 4.10.** Let \(\varphi : A \to \mathcal{M}(B)\) be such that \(\text{[JR]}\), \(\text{[CL]}\), \(\text{(MI)}\) are satisfied and \(\text{Prime}^B(A)\) is first countable. The quasi-orbit map for \(\varphi\) is the continuous map \(\varphi : \hat{B} \to \hat{A}/\sim\) defined by \(\pi_x \circ i\), where \(\pi_x : \hat{A}/\sim \to \text{Prime}^B(A)\) is a homeomorphism by Theorem 4.13 and \(r : \hat{B} \to \text{Prime}^B(A)\) is continuous by Proposition 4.7.

**Lemma 4.11.** Let \(\varphi : A \to \mathcal{M}(B)\) be such that \(\text{(MI)}\) holds. The inclusion map \(\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)\) is an upper Galois insertion if and only if

(\(\text{MI}\)) arbitrary meets of induced ideals are again induced.

Then its lower Galois adjoint \(F : \mathbb{I}(B) \hookrightarrow \mathbb{I}^A(B)\) is given on \(J \in \mathbb{I}(B)\) by

\[
F(J) = \text{the meet of all induced ideals that contain } J.
\]

**Proof.** The inclusion \(\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)\) is an upper Galois insertion if and only if for every \(J \in \mathbb{I}(B)\) there is a least induced ideal \(F(J)\) with \(J \subseteq F(J)\). If \(\text{(MI)}\) holds, then \(F(J)\) is the meet of all induced ideals that contain \(J \in \mathbb{I}(B)\). Conversely, assume that the inclusion \(\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)\) has a lower Galois adjoint \(F : \mathbb{I}(B) \hookrightarrow \mathbb{I}^A(B)\). Then the inclusion preserves all meets by \([12]\) Proposition 7.31. Equivalently, the meet of a family of induced ideals is induced.

**Theorem 4.13.** Let \(\varphi : A \to \mathcal{M}(B)\) be a \(*\)-homomorphism. If \(r : \text{Prime}(B) \to \text{Prime}^B(A)\) is a well defined continuous, open and surjective map, then \(\text{(MI)}\) holds and the map \(F : \mathbb{I}(B) \hookrightarrow \mathbb{I}^A(B)\) given by (4.12) satisfies

(\(\text{C2}\)) \(I \cap F(J) = F(I \cap J)\) for every \(I \in \mathbb{I}^A(B)\) and \(J \in \mathbb{I}(B)\).

Conversely, if \(\text{(MI)}\) and \(\text{(C2)}\) hold and \(\text{Prime}^B(A)\) is first countable – which follows if \(\hat{A}\) is second countable – then \(r : \hat{B} \subseteq \text{Prime}(B) \to \text{Prime}^B(A)\) is a well defined continuous, open surjection.

**Proof.** Assume first that \(r : \text{Prime}(B) \to \text{Prime}^B(A)\) is a well defined continuous open surjection. Then \(i \circ r : \text{Prime}(B) \to \text{Prime}^A(B)\) is a well defined continuous open surjection. Since it is adjoint to the inclusion map \((i \circ r)^{-1} : \mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)\), the inclusion map is a locale morphism. Hence \(\text{(MI)}\) holds by Proposition 4.7 and \(\mathbb{I}^A(B)\) is spatial (because \(\mathbb{I}(B)\) is spatial). By the first part of Theorem 4.12 the inclusion \(\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)\) is an upper Galois insertion and its lower adjoint \(F : \mathbb{I}(B) \hookrightarrow \mathbb{I}^A(B)\) satisfies \(\text{(C2)}\). Lemma 4.11 shows that \(F\) is given by (4.12) and that \(\text{(MI)}\) holds. Conversely, assume \(\text{(MI)}\) and \(\text{(C2)}\) and that \(\text{Prime}^B(A)\) is first countable or \(\hat{B}\) is second countable. By Lemma 4.11 \(F : \mathbb{I}(B) \hookrightarrow \mathbb{I}^A(B)\) is lower Galois adjoint to \(\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)\). The second part of Theorem 4.12 applied to the upper Galois insertion \(\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)\) shows that the induced continuous map \(\hat{B} \subseteq \text{Prime}(B) \to \text{Prime}^B(A)\) is surjective and open. This gives the assertion by Proposition 4.7.\(\square\)
Corollary 4.14. Suppose that \((\text{JR})\), \((\text{C1})\), and \((\text{MI})\) are satisfied and \(\text{Prime}^B(A)\) is first countable. The quasi-orbit map \(\varphi: B \to \bar{A}/\sim\) is open and surjective if and only if \((\text{MI})\) and \((\text{C2})\) are satisfied.

Corollary 4.15. Suppose that \((\text{JR})\) and \((\text{C1})\) are satisfied and \(\text{Prime}^B(A)\) is first countable. The quasi-orbit map \(\varphi: B \to \bar{A}/\sim\) exists and is a homeomorphism if and only if \(A\) separates ideals in \(B\).

5. Symmetric ideals

Our next task is to verify the assumptions \((\text{JR})\), \((\text{C1})\), and \((\text{MI})\) in interesting cases. Then the quasi-orbit map exists. As we shall see, restricted ideals often satisfy the equivalent conditions in the following lemma:

Lemma 5.1. Let \(\varphi: A \to \mathcal{M}(B)\) and \(I \in \mathcal{I}(A)\). The following are equivalent:

1. the map \(I \to \mathcal{M}(\varphi(I))\) induced by \(\varphi: A \to \mathcal{M}(B)\) is non-degenerate;
2. \(i(I) = \varphi(I)B\varphi(I)\);
3. \(i(I) = \varphi(I)B\);
4. \(i(I) = B\varphi(I)\);
5. \(\varphi(I)B = B\varphi(I)\).

If \(\varphi: A \to B\) is injective and the above equivalent conditions hold, then \(I \in \mathcal{I}^B(A)\).

Proof. If \(I \to \mathcal{M}(\varphi(I))\) is non-degenerate, then \(i(I) = \varphi(I)i(I)\varphi(I) \subseteq \varphi(I)B\varphi(I)\). We claim that the reverse inclusion \(\varphi(I)B\varphi(I) \subseteq B\varphi(I)B = i(I)\) always holds. Let \((e_n)_{n \in N}\) be an approximate unit in \(B\). Then \(\lim e_n \cdot x = x\) for all \(x \in \varphi(I)B\varphi(I)\) because \(\varphi(I)B\varphi(I) \subseteq B\). Thus \(x\) belongs to the norm closure of \(B\varphi(I)B\varphi(I)B \subseteq B\varphi(I)B\) as asserted. Thus (1) implies (2).

If \(i(I) = \varphi(I)B\varphi(I)\), then \(i(I) \subseteq \varphi(I)B\). The reverse inclusion \(\varphi(I)B \subseteq B\varphi(I)B = i(I)\) always holds, the proof is the same approximate unit argument as above. Thus (2) implies (3) and (4) are equivalent. Clearly, the equivalent conditions (3) and (4) imply (5) if \(\varphi(I)B = B\varphi(I)\), then \(i(I) = B\varphi(I)B = B\varphi(I)\varphi(I)B = \varphi(I)B\varphi(I)B = \varphi(I)i(I)\). Hence (5) implies (1). This proves the first part of the assertion.

Now assume that \(\varphi: A \to B\) is injective and let \(I \in \mathcal{I}(A)\) satisfy the equivalent conditions (1)–(5). The inclusion \(I \subseteq r(i(I))\) always holds by Proposition 2.3. We show the reverse inclusion. Let \((e_n)_{n \in N}\) be an approximate unit for \(I\). By assumption, \(i(I) = \varphi(I)B\). So \(\lim \varphi(e_n)x = x\) holds for all \(x \in i(I)\). Since \(\varphi\) is injective and \(\varphi(r \circ i(I)) \subseteq i(I)\), this implies \(\lim e_nx = x\) for all \(x \in r(i(I))\). Thus \(r(i(I)) \subseteq I\).

Definition 5.2. We call an ideal \(I \in \mathcal{I}(A)\) symmetric for \(\varphi: A \to \mathcal{M}(B)\) if it satisfies the equivalent conditions in Lemma 5.1. We call the \(^*\)-homomorphism \(\varphi: A \to \mathcal{M}(B)\) symmetric if all restricted ideals are symmetric.

The name “symmetric” is suggested by condition (5) in Lemma 5.1.

Remark 5.3. For a usual inclusion \(A \subseteq B\), all symmetric ideals are restricted by Lemma 5.4. There are, however, important symmetric generalised C*-inclusions \(A \subseteq \mathcal{M}(B)\) that admit symmetric ideals that are not restricted, see Example 5.4.

Lemma 5.4. If \(I, J \in \mathcal{I}(A)\) and \(I\) is symmetric, then \(i(I \cap J) = i(I) \cap i(J)\) and hence \(r \circ i(I \cap J) = r \circ i(I) \cap r \circ i(J)\). This is equal to \(I \cap r \circ i(J)\) if \(I \in \mathcal{I}^B(A)\).

The set of symmetric ideals is closed under joins.

Proof. If \(\varphi(I)B = B\varphi(I)\), then
\[
i(I \cap i(J)) = i(I)i(J) = (\varphi(I)B)(B\varphi(J)B) = \varphi(I)B\varphi(J)B = B\varphi(I)\varphi(J)B = B\varphi(I \cap J)B = i(I \cap J).
\]
Thus $r \circ i(I \cap J) = r \circ i(I) \cap r \circ i(J)$ because $r$ preserves meets by Proposition 2.8(5).

Let $(I_\alpha)_{\alpha \in \Sigma}$ be symmetric ideals. Then $\varphi(\bigvee I_\alpha)B$ is the closed linear span of $\varphi(I_\alpha)B = B\varphi(I_\alpha)$, and this is equal to $B\varphi(\bigvee I_\alpha)$. So $\bigvee I_\alpha$ is symmetric as well. □

**Corollary 5.5.** If $A \subseteq B$ is a symmetric inclusion, then conditions [(JR), (C1)] and [(MI)] are satisfied. If $\varphi: A \to \mathcal{M}(B)$ is a symmetric $*$-homomorphism, then conditions [(C1)] and [(MI)] hold.

**Proof.** Any induced ideal is induced by a restricted ideal by Proposition 2.8(2). Therefore, if all restricted ideals are symmetric, then Lemma 5.4 implies conditions [(C1)] and [(MI)]. If the sets of restricted and symmetric ideals are equal, then the last part of Lemma 5.4 also yields [(JR)]. The latter is satisfied when $A \subseteq B$ by the last part of Lemma 5.1. □

It is unclear whether all symmetric generalised $C^*$-inclusions satisfy [(JR)] because there may be symmetric ideals that are not restricted (see Remark 5.3). Condition [(MI)] may fail for symmetric inclusions, see Example 7.2 below.

### 6. Group actions and regular inclusions

The theory above suggests the following programme to study any type of $C^*$-inclusion $A \hookrightarrow \mathcal{M}(B)$. First verify the locale-theoretic conditions [(JR), (C1)] and [(MI)], a good Ansatz for this is showing that the inclusion is symmetric. This requires a good characterisation of the restricted ideals. Usually, they are “invariant” in a suitable sense. Then the assumptions needed for Theorem 4.4 and the existence of the quasi-orbit map $\varphi: \hat{B} \to \hat{A}/\sim$ are in place, except for the first or second countability assumptions, which remain assumptions in all the following theorems. Secondly, one may try to verify the conditions [(MI)] and [(C2)] which are then equivalent to openness and surjectivity of the quasi-orbit map, see Corollary 4.13. This in turn requires a good understanding of induced ideals, which are usually “invariant” in some dual sense.

In this section, we apply the above programme to some prototypical examples. We begin with the crossed product for actions of locally compact groups by automorphisms. Here we recover classical results that played a crucial role in the study of the Effros–Hahn Conjecture and related problems. We generalise these results to Fell bundles over locally compact groups. We finish this section with the case of regular inclusions, which we treat by relating them to Fell bundles over inverse semigroups.

#### 6.1. Crossed products over locally compact groups

Let $G$ be a locally compact group and let $\alpha: G \to \text{Aut}(A)$ be a continuous group action. A crossed product is a $C^*$-algebra $B$ with surjective maps $A \rtimes_\alpha G \to B \to A \rtimes_{\alpha,\lambda} G$ whose composition is the regular representation $\lambda: A \rtimes_\alpha G \to A \rtimes_{\alpha,\lambda} G$. The canonical $*$-homomorphism $A \to \mathcal{M}(A \rtimes_\alpha G)$ gives a generalised $C^*$-inclusion $\varphi: A \to \mathcal{M}(B)$ because the canonical $*$-homomorphism $A \to \mathcal{M}(A \rtimes_{\alpha,\lambda} G)$ is injective.

**Proposition 6.1.** Let $\varphi: A \to \mathcal{M}(B)$ be the canonical generalised $C^*$-inclusion of $A$ into a crossed product $B$.

1. $\mathcal{I}^B(A) = \mathcal{I}^A(A)$ is the lattice of $\alpha$-invariant ideals in $A$;
2. an ideal $I \lt A$ is induced if and only if it is the image of $I \rtimes_{\alpha,|A|}$, $G$ in $B$ for some $\alpha$-invariant ideal $I \lt A$.

The $C^*$-inclusion $\varphi: A \to \mathcal{M}(B)$ is symmetric and conditions [(JR), (C1)] and [(MI)] are satisfied.
Proof. Statements [1] and [2] are well known to experts, and we prove them in greater generality in the proof of Proposition 6.3 below. Since the canonical homomorphisms \( I \to \mathcal{M}(I \times \alpha G) \) are non-degenerate, statements [1] and [2] imply that \( \varphi \) is symmetric. Thus conditions [CI] and [MI] follow from Corollary 6.3. The closed linear span of a family of \( \alpha \)-invariant ideals is again \( \alpha \)-invariant. So [1] implies condition [JR]. □

If \( G \) is discrete, then all symmetric ideals are restricted by Lemma 5.1. This fails for locally compact \( G \):

Example 6.2. Let \( G \) be a locally compact group that is not discrete. Let \( A = C_0(G) \) and let \( \alpha \) be the translation action. So \( B := A \times G \cong \mathbb{K}(L^2(G)) \). Let \( e \in G \) be, say, the unit element. The ideal \( I := C_0(G \setminus \{e\}) \) is not invariant and not restricted. The closed right ideal \( I \cdot B \) in \( B = \mathbb{K}(L^2(G)) \) consists of those compact operators whose image is contained in the closure of \( I \cdot L^2(G) \). Now \( I \cdot L^2(G) \) is dense in \( L^2(G) \) because \( \{e\} \) is a set of measure zero. Thus \( I \cdot B = B \). This implies \( B \cdot I = B \) by taking adjoints. So \( I \) is a symmetric ideal that is not restricted.

Let \( \varphi: A \to \mathcal{M}(B) \) be the canonical generalised inclusion into a crossed product \( B \). By Proposition 6.3 and Lemma 6.1 there is a continuous map
\[
\pi: \text{Prime}(A) \to \text{Prime}(\mathcal{I}^0(A))
\]
that maps \( p \in \text{Prime}(A) \) to the largest invariant ideal contained in \( p \), that is,
\[
\pi(p) = \bigcap_{g \in G} \alpha_g(p).
\]
The open subset of \( \text{Prime}(A) \) corresponding to this intersection is the complement of the closure of the orbit \( G \cdot p = \{\alpha_g(p) : g \in G\} \). Thus \( p_1 \sim p_2 \) for \( p_1, p_2 \in \text{Prime}(A) \) if and only if \( G \cdot p_1 = G \cdot p_2 \). This is the usual definition of the quasi-orbit space for a group action, compare [53]; when \( A \) is commutative or separable, we may replace \( \text{Prime}(A) \) by \( \hat{A} \). So our definition of the quasi-orbit space for a generalised C*-inclusion extends the usual definition for group actions. In particular, our results yield the canonical quasi-orbit map \( \hat{B} \to \hat{A} / \sim \) (see, for instance, [53, (3.2)], [27, pp. 221–223], [25, p. 290], or [25, p. 620]).

Theorem 6.3. Let \( \varphi: A \to \mathcal{M}(B) \) be the canonical generalised C*-inclusion of \( A \) into a crossed product \( B \). The quasi-orbit space for \( \varphi \) exists and coincides with the usual quasi-orbit space for the action \( \alpha \). If \( \text{Prime}(\mathcal{I}^0(A)) \) is first countable or \( \hat{A} \) is second countable, then the quasi-orbit map exists:

1. The map \( \pi: \hat{A} \to \text{Prime}(\mathcal{I}^0(A)), p \mapsto \bigcap_{g \in G} \alpha_g(p), \) is continuous, open and surjective. It descends to a homeomorphism \( \hat{A} / \sim \to \text{Prime}(\mathcal{I}^0(A)) \). The quasi-orbit space \( \hat{A} / \sim \) is a quotient of \( \hat{A} \) by an open equivalence relation.

2. There is a continuous map \( q: \hat{B} \to \hat{A} / \sim, p \mapsto \hat{\pi}^{-1}(r(p)) \). It identifies \( \hat{B} \cong \hat{A} / \sim \) if and only if \( \alpha \) separates ideals in \( B \). And then \( B = A \rtimes_{\alpha, \lambda} G \).

Proof. The assertions follow from Theorem 4.3, Proposition 4.7 and the discussion above. For the last part, compare with Corollary 4.6 and use that if \( A \) detects ideals in \( B \), then the kernel of the surjection \( B \to A \rtimes_{\alpha, \lambda} G \) has to be trivial. □

The existence of the quasi-orbit map \( q: \hat{B} \to \hat{A} / \sim \) is established in [25, 27, 53] if \( A \) is separable and \( G \) is second countable and amenable. In fact, under these assumptions Gootman and Lazar prove that \( q \) is open and surjective, see [25, Theorem 4.8]. As we show in Example 7.3 below (see also Corollary 7.5), the map \( q \) may fail to be open when \( G \) is not amenable and \( B \) is the full crossed product. Nevertheless, we manage to improve [25, Theorem 4.8] by applying Corollary 4.14.
to the reduced crossed product $B = A \rtimes_{\alpha,\lambda} G$. Thus our next goal is to prove conditions \([\text{MI}]\) and \([\text{C2}]\) for the inclusion $A \to \mathcal{M}(B)$. Here we use Imai–Takai Duality \([29]\). This allows us to translate conditions \([\text{MI}]\) and \([\text{C2}]\) for $A \to \mathcal{M}(B)$ to conditions \([\text{JR}]\) and \([\text{C1}]\) for $B \to \mathcal{M}(B \rtimes_\lambda \hat{G})$, which are easy to check.

Recall that $B = A \rtimes_{\alpha,\lambda} \hat{G}$ is equipped with a (reduced) coaction $\hat{\alpha} \colon B \to B \otimes C^*_{\lambda}(\hat{G})$, where $\otimes$ denotes the minimal $C^*$-tensor product. This coaction generates a crossed product $B \rtimes_\lambda \hat{G}$, which comes with a morphism $B \to \mathcal{M}(B \rtimes_\lambda \hat{G})$, and Imai–Takai Duality \([29]\) identifies

$$B \rtimes_\lambda \hat{G} \cong A \otimes \mathbb{K}(L^2(\hat{G})).$$

Following \([51]\) Definition 1.5 we call the action $\alpha$ exact if every $I \in \mathbb{I}^\circ(A)$ induces a short exact sequence

$$0 \to I \rtimes_{\alpha|I,\lambda} G \to A \rtimes_{\alpha,\lambda} G \to A/I \rtimes_{\alpha|A/I,\lambda} G \to 0.$$

If $G$ is exact, then all actions of $G$. Any action $\alpha$ with $A \rtimes_{\alpha,\lambda} G = A \rtimes_\alpha G$ is exact. So all actions of amenable groups are exact.

**Theorem 6.4.** Let $\alpha$ be an exact action. Let $B := A \rtimes_{\alpha,\lambda} G$. Build induction and restriction maps for the canonical $^*$-homomorphisms $\varphi \colon A \to \mathcal{M}(B)$ and $\psi \colon B \to \mathcal{M}(B \rtimes_\lambda \hat{G}) \cong \mathcal{M}(A \otimes \mathbb{K})$ with $\mathbb{K} := \mathbb{K}(L^2(\hat{G}))$. Then $\mathbb{I}^A(B) = \mathbb{I}^{A \otimes \mathbb{K}}(B)$ and the following diagram commutes:

\begin{equation}
\begin{array}{c}
\mathbb{I}^A(B) \\
\downarrow \cong \\
\mathbb{I}^{A \otimes \mathbb{K}}(B) \\
\downarrow \cong \\
\mathbb{I}^B(A) \otimes \mathbb{K} \\
\cong \\
\mathbb{I}^B(A) \otimes \mathbb{K} \\
\end{array}
\end{equation}

And $A \to \mathcal{M}(B)$ satisfies \([\text{MI}]\) and \([\text{C2}]\). Thus, if $\hat{A}$ or $\hat{B}$ is second countable or, more generally, Prime($\mathbb{I}^\circ(A)$) is first countable, then the quasi-orbit map $\rho : \hat{B} \to \hat{A}/\sim$ is open and surjective, the quasi-orbit space $\hat{B}/\sim$ for $B \to \mathcal{M}(B \rtimes_\lambda \hat{G})$ exists, and there is a homeomorphism $\hat{A}/\sim \cong \hat{B}/\sim$ such that the map $\hat{B} \overset{\rho}{\to} \hat{A}/\sim \cong \hat{B}/\sim$ is the quotient map $\hat{B} \to \hat{B}/\sim$.

**Remark 6.6.** It is known that a diagram similar to \([6.5]\) commutes when $i$ is replaced by the map $\text{Ind}$ given by kernels of the corresponding induced representations, see \([25]\) Remarks 2.8 or \([45]\) Propositions 2.7, 2.8. Thus we need to show that $\text{Ind}$ and $i$ coincide on restricted ideals. It is readily seen that $\text{Ind}$ and $i$ coincide on $\mathbb{I}^B(A) = \mathbb{I}^\circ(A)$ if and only if the action $\alpha$ is exact. Hence this assumption is necessary for our proof to work. That $\text{Ind}$ restricted to $\mathbb{I}^{B \rtimes_\lambda \hat{G}}(B)$ coincides with $i$ is proved in \([29]\) Proposition 3.14(iii)] if $G$ amenable, see also \([45]\) Proposition 3.1(ii)], where the full crossed products are considered.

The proof of Theorem \([6.4]\) is based on the following two lemmas.

**Lemma 6.7.** An ideal in $A \otimes \mathbb{K}$ is induced from $B$ if and only if it is of the form $I \otimes \mathbb{K}$ for an invariant ideal $I \in \mathbb{I}^\circ(A)$, if and only if it is induced from an ideal in $A$ along $\psi \circ \varphi : A \to \mathcal{M}(A \otimes \mathbb{K})$. (This holds for any action $\alpha$.)

**Proof.** Let $I \in \mathbb{I}^\circ(A)$. The ideal in $B$ induced by $I$ is $I \rtimes_\lambda G$, see Proposition \([6.1]\). Inducing further to an ideal in $A \otimes \mathbb{K}$ gives the double crossed product ideal $(I \rtimes_\lambda G) \rtimes \mathbb{K} \cong I \otimes \mathbb{K}$. So ideals of this form are induced. We claim that any ideal in $A \otimes \mathbb{K}$ induced from $A$ along $\psi \circ \varphi$ is of this form. Indeed, if $I \prec A$ is arbitrary, then induction along $\psi \circ \varphi$ has the same effect as first inducing along $\varphi$ and then along $\psi$. When we induce along $\varphi$, we get $i(I) = i(I)$, where $I \prec A$ is the $\alpha$-invariant ideal generated by $I$. So $I$ and $I$ induce the same ideal also along $\psi \circ \varphi$. 


Now let $J \prec B$ be any ideal. It remains to prove that the induced ideal in $A \otimes K$ is of the form $I \otimes K$ for an $\alpha$-invariant ideal $I \prec A$. The $^*$-homomorphism $\psi$ first maps $J$ to its image in $A \rtimes \alpha G$. The coaction crossed product $C := B \rtimes \hat{G}$ comes with non-degenerate, injective $^*$-homomorphisms $B \to \mathcal{M}(C) \leftarrow C_0(G)$ such that $B \cdot C_0(G) = C_0(G) \cdot B = C$.

And it carries a canonical dual action $\gamma: \hat{G} \to \text{Aut}(C)$. The Imai–Takai isomorphism is $G$-equivariant, that is, it intertwines the action $\gamma$ and the action $\alpha \otimes \text{Ad}(\lambda_g)$ on $A \otimes K = A \otimes K(L^2(G))$. So the ideal in $C$ induced by $J \prec A \rtimes \alpha G$ has the desired form if and only if it is $\gamma$-invariant. The dual action $\gamma$ is built as follows. If $x \in A \rtimes \alpha G$, $f \in C_0(G)$, then $\gamma_g(x \cdot f) = x \cdot \lambda_g(f)$ with the automorphism $\lambda_g \in \text{Aut}(G)$ defined by the left regular representation, $(\lambda_g f)(x) = f(g^{-1}x)$. The ideal in $C$ induced by $J$ is $C \cdot J \cdot C = C_0(G) \cdot B \cdot J \cdot B \cdot C_0(G) = C_0(G) \cdot J \cdot C_0(G)$.

If $f_1, f_2 \in C_0(G)$, $x \in J$, then $\gamma_g(f_1 \cdot x \cdot f_2) = \lambda_g(f_1) \cdot x \cdot \lambda_g(f_2)$ again belongs to $C \cdot J \cdot C$. Thus $C \cdot J \cdot C$ is invariant. □

**Lemma 6.8.** Suppose that the action $\alpha$ is exact. An ideal in $B = A \rtimes \alpha G$ is induced from $A$ if and only if it is restricted from $A \otimes K$. And the generalised inclusion $\psi: B \to \mathcal{M}(A \otimes K)$ is symmetric.

**Proof.** Let $J \prec B$ be restricted from $A \otimes K$. Then it is the restriction of an induced ideal by Proposition 2.8.3. By Lemma 6.7, it is the restriction of $I \otimes K$ for an invariant ideal $I \in \text{I}_1^0(A)$. This is equal to the kernel of the canonical $^*$-homomorphism $B = A \rtimes \alpha G \to \mathcal{M}(A \otimes K) \to \mathcal{M}(A \otimes K / I \otimes K)$. This factors through the canonical injective $^*$-homomorphism $(A/I) \rtimes \alpha G \hookrightarrow \mathcal{M}(A/I) \otimes K = \mathcal{M}(A \otimes K / I \otimes K)$.

So the restricted ideal from $I \otimes K$ is the kernel of the $^*$-homomorphism $A \rtimes \alpha G \to (A/I) \rtimes \alpha G$. Since we assume that the action is exact, this is equal to $I \rtimes \alpha G$. The ideals of this form for $I \in \text{I}_1^0(A)$ are exactly the induced ideals by Proposition 6.1. Thus an ideal in $A \rtimes \alpha G$ is induced from $A$ if and only if it is restricted from $A \otimes K$. It follows that the generalised inclusion $\psi$ is symmetric because $(I \rtimes \alpha G) \cdot (A \otimes K) = I \otimes K = (A \otimes K) \cdot (I \rtimes \alpha G)$. □

**Proof of Theorem 6.4.** Lemmas 6.7 and 6.8 show that the diagram (6.5) commutes. Hence the restricted ideals for $\psi$ are the same as the induced ideals for $\varphi$. So conditions [M1] and [C2] for $\varphi$ are equivalent to conditions [JR] and [C1] for $\psi$. Moreover, $\psi$ satisfies condition [JR] because joins of induced ideals remain induced by Proposition 2.8.3 and $\psi$ satisfies condition [C1] because $\psi$ is symmetric, see Lemma 6.8 and Corollary 5.5.

Accordingly, the second parts of both Theorems 4.4 and 4.13 apply to $\varphi$. So $\varphi$ is open and surjective if $\text{Prime}(\text{I}_1^0(A)) = \text{Prime}^0(\hat{A})$ is first countable. And this follows if $\hat{A}$ or $\hat{B}$ is second countable. By Lemma 6.1, the quasi-orbit space $\hat{B}/\sim$ for $\varphi$ exists, and $p_1 \sim p_2$ for $p_1, p_2 \in \hat{B}$ if and only if the largest restricted-along-$\varphi$ ideals in $p_1$ and $p_2$ coincide. By (6.5), this holds if and only if the largest induced-along-$\varphi$ ideals contained in $p_1$ and $p_2$ coincide. Thus $\varphi(p_1) = \varphi(p_2) \iff r(p_1) = r(p_2) \iff (i \circ r)(p_1) = (i \circ r)(p_2) \iff p_1 \sim p_2$.

Hence the continuous open surjection $\varphi: \hat{B} \to \hat{A}/\sim$ factors through a homeomorphism $\hat{B}/\sim \cong \hat{A}/\sim$. □
6.2. Fell bundles over locally compact groups. Fell bundles over a locally compact group \( G \), introduced in [23], are the most general kinds of “actions” of \( G \) on \( C^* \)-algebras. These contain twisted partial actions (see [17]) as a special case. We may even allow measurable twists for global actions by [21], but this result seems not to have been extended to partial actions with a measurable twist yet. Twisted partial actions contain both partial actions and twisted actions and thus ordinary group actions by automorphisms. The full and reduced crossed products for a (twisted, partial) action are naturally isomorphic to the full and reduced section \( C^* \)-algebras of the corresponding Fell bundle.

We now explain how to generalise our results above to a Fell bundle \( \mathcal{A} = (A_g)_{g \in G} \) over \( G \); it comes with multiplication maps \( A_g \times A_h \to A_{gh} \) for \( g, h \in G \), involutions \( A_g \to A_{g^{-1}} \) for \( g \in G \) with certain properties, and a topology on \( \bigcup_{g \in G} A_g \), see [17] Definitions 2.2 and 3.9 or [13][15]. In particular, \( A := A_e \) is a \( C^* \)-algebra and each \( A_g \) becomes a Hilbert \( A \)-bimodule. The set \( C_c(G, \mathcal{A}) \) of continuous, compactly supported sections of the bundle \( \mathcal{A} = (A_g)_{g \in G} \) over \( G \) carries a \(*\)-algebra structure. The full section \( C^* \)-algebra \( C^*(\mathcal{A}) \) of \( \mathcal{A} \) is defined as a completion of \( C_c(G, \mathcal{A}) \) in the maximal \( C^* \)-norm. For locally compact \( G \), the reduced section \( C^* \)-algebra \( C^*_r(\mathcal{A}) \) is defined in [22] as the range of a regular representation \( \lambda : C^*(\mathcal{A}) \to \mathcal{B}(L^2(G)) \), where \( L^2(G) \) is a canonical Hilbert \( \mathcal{A} \)-module. A cross section \( C^* \)-algebra is any \( C^* \)-algebra \( B \) with surjective \(*\)-homomorphisms \( C^*(\mathcal{A}) \to B \to C^*_r(\mathcal{A}) \) whose composition is the regular representation \( \lambda : C^*(\mathcal{A}) \to C^*_r(\mathcal{A}) \).

An ideal \( I \vartriangleleft \mathcal{A} \) is called \( \mathcal{A} \)-invariant if it is invariant for every Hilbert bimodule \( A_g, g \in G \), that is, if \( I \cdot A_g = A_g \cdot I \) for all \( g \in G \). This is equivalent to \( A_g \cdot I \cdot A_{g^{-1}} \subseteq I \) for all \( g \in G \). Let \( I \) be \( \mathcal{A} \)-invariant. We may restrict the Fell bundle structure on \( (A_g)_{g \in G} \) to one on \( I_g := I \cdot A_g = A_g \cdot I \). And it induces a Fell bundle structure on the quotients \( (A_g/I_g)_{g \in G} \) as well. We denote these induced Fell bundles by \( \mathcal{A}/I \) and \( \mathcal{A}/A/I \), respectively.

**Proposition 6.9.** Let \( \varphi : \mathcal{A} \to \mathcal{M}(B) \) be the canonical generalised \( C^* \)-inclusion of a cross section algebra \( B \) of a Fell bundle \( \mathcal{A} = (A_g)_{g \in G} \).

1. \( \mathcal{M}(A) \) is the subset \( I^\mathcal{A}(A) \) of \( \mathcal{A} \)-invariant ideals in \( \mathcal{A} \);
2. an ideal \( I \vartriangleleft \mathcal{A} \) is induced if and only if it is the image of \( C^*(\mathcal{A}/I) \) in \( \mathcal{M}(B) \) for some \( \mathcal{A} \)-invariant ideal \( I \vartriangleleft \mathcal{A} \).

The \( C^* \)-inclusion \( \varphi : \mathcal{A} \to \mathcal{M}(B) \) is symmetric, and [JR], [CT] and [MI] hold.

**Proof.** First we show that restricted ideals are \( \mathcal{A} \)-invariant. To this end, let \( J \subseteq \mathcal{M}(B) \) and put \( I := r(J) \). We use the canonical maps from the spaces \( A_g \) to the multiplier algebra of \( C^*(\mathcal{A}) \). First, \( x \in A_g \) defines a multiplier on the \( * \)-algebra \( C_c(G, \mathcal{A}) \) by \( (x \cdot f)(h) = x \cdot (f(g^{-1}h)) \) and \( (f \cdot x)(h) = (f(hg^{-1})) \cdot x \) for \( h \in G \) and \( f \in C_c(G, \mathcal{A}) \). This extends to a multiplier on the \( C^* \)-completion (see [22] Lemma 1.1)). The maps \( A_g \to \mathcal{M}(C^*(\mathcal{A})) \to \mathcal{M}(B) \) form a Fell bundle representation. Since \( \mathcal{M}(B, J) \) is an ideal in \( \mathcal{M}(B) \), an element \( x \in A_g \) belongs to \( \mathcal{M}(B, J) \) if and only if \( x \cdot x^* \in \mathcal{M}(B, J) \) if and only if \( xx^* \in \mathcal{M}(B, J) \). Now \( xx^* \) and \( xx^* \) belong to \( A_g^* \cdot A_g = A_g \cdot A_{g^{-1}} \subseteq A_g \). An element of \( A_g \subseteq A \) is in \( \mathcal{M}(B, J) \) if and only if it belongs to \( I_g \). So \( xx^* \in r(J) \) if and only if \( xx^* \in r(J) \) if and only if \( xx^* \in r(J) \) for all \( x \in A_g \). During the proof of the Rieffel correspondence for the Hilbert bimodule \( A_g \), it is shown that \( xx^* \in I_g \) if and only if \( xx^* \in I_g \) and that \( xx^* \in I \) if and only if \( xx^* \in I_g \cdot I_g \). So \( I \cdot A_g = A_g \cdot I \) for all \( g \in G \). This finishes the proof that restricted ideals in \( \mathcal{A} \) are invariant.

Now let \( I \vartriangleleft A_e \) be \( \mathcal{A} \)-invariant. We claim that \( I \) is symmetric. Indeed, the standard formula for the convolution in \( C_c(G, \mathcal{A}) \) shows that \( I \cdot C_c(G, \mathcal{A}) \) and \( C_c(G, \mathcal{A}) \cdot I \) are contained in \( C_c(G, \mathcal{A}/I) \), the \( * \)-algebra of continuous, compactly supported sections of \( \mathcal{A}/I \). Moreover, \( C_c(G, \mathcal{A}/I) \) is non-degenerate as a left or right \( I \)-module.
Since $C_c(G,A)$ is a dense *-subalgebra in $C^*(A)$, we may view it also as a dense *-subalgebra of $B$. Thus taking the closures in the crossed product $B$, we see that both $I:B$ and $B:1$ are equal to the closure of $C_c(G,A)[-\gamma]$ in $B$. Thus $I:B = B:1$, that is, $I$ is symmetric.

Next we show that $I \in \mathcal{I}^\lambda(A)$ is restricted. We need to prove that $r \circ i(I) = I$. The inclusion $r \circ i(I) \supseteq I$ is already contained in Proposition 2.8(2). To see the reverse inclusion, note that the canonical map from $C^*(A)$ to $C^*\lambda(A)[-\gamma]$ annihilates $C_c(G,A)[-\gamma]$, which is dense in $i(I)$ by the proof above that $I$ is symmetric. Therefore, every element $a \in r \circ i(I)$ induces the zero multiplier on $C^*\lambda(A)[-\gamma]$. The canonical map from $A/I$ to the multiplier algebra of $C_c\lambda(A)[-\gamma]$ is injective. So $a \in r \circ i(I)$ is mapped to 0 in $A/I$, that is, $a \in I$. Hence $r \circ i(I) \subseteq I$, and $I$ is restricted.

This finishes the proof of (1) and shows that $\varphi$ is symmetric. In particular, conditions (CI) and (MI) follow from Corollary 5.5. It is readily seen that the closed linear span of a family of $A$-invariant ideals is again $A$-invariant. So (1) implies condition (JR).

Any induced ideal in $B$ is obtained by inducing a restricted ideal because $i(I) = i(r \circ i(I))$ for all $I \in \mathcal{I}(A)$. Restricted ideals are invariant ideals. The proof above that invariant ideals are symmetric also shows that the ideal in $B$ induced by an invariant ideal $I \in \mathcal{I}^\lambda(A)$ is the closure of the image of $C_c(G,A)[-\gamma]$ in $B$. This closure of $C_c(G,A)[-\gamma]$ in the full cross section algebra $C^*(A)$ is isomorphic to $C^*(A)[-\gamma]$. Thus $i(I)$ is the image of $C^*(A)[-\gamma] \hookrightarrow C^*(A)$ in $B$. This proves (2). □

Let $A = (A_g)_{g \in G}$ be a Fell bundle and recall that each $A_g$, $g \in G$, is a Hilbert $A$-bimodule over $A := A_e$. By the Rieffel correspondence, $A_g$ induces a homeomorphism $\tilde{A}_g$ between two open subsets of $\text{Prime}(A)$, which we view as a partial homeomorphism of $\tilde{A}$. The associativity of the multiplication in $\tilde{A}$ implies that $(\tilde{A}_g)_{g \in G}$ is a partial action of $G$ on $\tilde{A}$ or Prime$(A)$, see [3](p5). This action is continuous by [3](Proposition 5.5). The orbit of $p \in \tilde{A}$ under this action is the set $G \cdot p$ of those $p' \in \tilde{A}$ which lie in the domain of $\tilde{A}_g$ for some $g \in G$ with $\tilde{A}_g(p') = p$.

**Theorem 6.10.** Let $A = (A_g)_{g \in G}$ be a Fell bundle over a locally compact group $G$. Let $\varphi : A \to \mathcal{M}(B)$ be the canonical generalised $C^*$-inclusion of $A := A_e$ into a cross section algebra $B$ of $A$. The relation $\sim$ that defines the quasi-orbit space $\tilde{A}/\sim$ for $\varphi$ is $p_1 \sim p_2$ if and only if $G \cdot p_1 = G \cdot p_2$ as closed subsets of $A$. Assume $\text{Prime}(\mathcal{I}^\lambda(A))$ to be first countable or $\tilde{A}$ to be second countable. Then the quasi-orbit map exists. More precisely:

1. The map $\pi : \tilde{A} \to \text{Prime}(\mathcal{I}^\lambda(A))$ that sends $p$ to the largest $A$-invariant ideal contained in $p$ is continuous, open and surjective. It descends to a homeomorphism $\tilde{\pi} : \tilde{A}/\sim \to \text{Prime}(\mathcal{I}^\lambda(A))$. The quasi-orbit space $\tilde{A}/\sim$ is a quotient of $\tilde{A}$ by an open equivalence relation.

2. There is a continuous map $\tilde{\varphi} : \tilde{B} \to \tilde{A}/\sim$, $p \mapsto \tilde{\pi}^{-1}(r(p))$. It identifies $\tilde{B} \cong \tilde{A}/\sim$ if and only if $A$ separates ideals in $B$. And then $B = A \rtimes_{\alpha,\lambda} G$.

**Proof.** Let $p \in \tilde{A}$. Let $\pi(p)$ be the largest $A$-invariant ideal contained in $p$. The open subset of Prime$(A)$ corresponding to $\pi(p)$ is the largest invariant open subset that does not contain $p$. So it is the complement of the closure of the orbit of $p$. Thus $p_1 \sim p_2$ for $p_1, p_2 \in \text{Prime}(A)$ if and only if $G \cdot p_1 = G \cdot p_2$. The remaining assertions follow from Proposition 6.9, Theorem 3.7 and Corollary 1.6. □

A Morita enveloping action for a Fell bundle is an ordinary group action that is equivalent to the Fell bundle in a suitable sense. It implies that the corresponding full, respectively reduced, $C^*$-algebras are Morita equivalent. For partial actions, these have been studied by Abadie [2]. For Fell bundles, they are built by Abadie,
Buss and Ferraro \cite{3,4}. They may be used for another proof of Theorem 6.10 by showing that the properties that we are interested in are preserved by Morita globalisations. We use this technique to generalise Theorem 6.3. We call a Fell bundle $\mathcal{A} = (A_g)_{g \in G}$ exact if for every $I \in \Pi^*(A)$ the induced sequence
\begin{equation}
0 \to C^*_R(A|_I) \to C^*_R(A) \to C^*_R(A|_{A/I}) \to 0
\end{equation}
is exact. For instance, Fell bundles with the approximation property introduced in \cite{22} are exact.

**Proposition 6.12.** Let $\mathcal{A} = (A_g)_{g \in G}$ and $\mathcal{B} = (B_g)_{g \in G}$ be Fell bundles that are (weakly) equivalent as in \cite[Definition 2.6]{3} or \cite[Definition 2.6]{4}.

(1) The $C^*$-algebras $C^*_R(A)$ and $C^*_R(B)$ are Morita equivalent.

(2) The induced Rieffel correspondence $\mathcal{I}(C^*_R(A)) \cong \mathcal{I}(C^*_R(B))$ restricts to an isomorphism $\mathcal{I}^A(C^*_R(A)) \cong \mathcal{I}^B(C^*_R(B))$ between the lattices of $G$-graded ideals.

(3) The inclusion $A_c \subseteq \mathcal{M}(C^*_R(A))$ satisfies conditions \([MI]\) and \([C2]\) if and only if $B_c \subseteq \mathcal{M}(C^*_R(B))$ satisfies them.

(4) $\mathcal{A}$ is exact if and only if $\mathcal{B}$ is exact.

**Proof.** Let $\mathcal{X} = (X_g)_{g \in G}$ be a Hilbert $\mathcal{A}$-$\mathcal{B}$-bundle. By \cite[Proposition 4.13]{3}, it yields a $C^*_R(A)$-$C^*_R(B)$-equivalence bimodule $C^*_R(\mathcal{X})$. This gives \cite[11]{11} Statement \([2]\) follows in essence from \cite[Corollary 4.3]{3}, as $G$-graded ideals are invariant with respect to dual coactions. In fact, \cite[Lemma 2.7(6)]{3} implies $B_g = \text{span}\{\langle X_r | X_{rg^{-1}} \rangle : r \in G\}$ for all $g \in G$. Let $I$ be an $\mathcal{A}$-invariant ideal in $A_c$. Let
\begin{equation}
J := \text{span}\{\langle X_r | IX_r \rangle : r \in G\}
\end{equation}
Let $r, g \in G$. The basic properties of Hilbert bundles imply

$B_g \cdot \langle X_r | IX_r \rangle_B \cdot B^{-1} = \langle X_r B_{g^{-1}} | IX_r B_{g^{-1}} \rangle_B \subseteq \langle X_{rg^{-1}} | IX_{rg^{-1}} \rangle_B \subseteq J.$

This implies that $J$ is a $\mathcal{B}$-invariant ideal in $B_c$. Since $I$ is $\mathcal{A}$-invariant, we may recover $I$ from $J$. Indeed, for $r, s \in G$, we get

$\langle X_r | IX_r \rangle_B \cdot \langle X_s | X_B \rangle_B = \langle X_r | IX_r X_s B \rangle_B = \langle X_r | I \cdot A < X_r | X_s \rangle B \rangle_B \subseteq \langle X_r | IA_{rs^{-1}} X_B \rangle_B = \langle X_r | A_{rs^{-1}} IX_r \rangle_B \subseteq \langle A_{rs^{-1}} X_r | IX_r \rangle_B \subseteq \langle X_s | IX_s \rangle_B.$

Hence $\mathcal{A}(\langle X_r | X_s \rangle_B) = \langle X_s | IX_s \rangle_B$. Thus

$\mathcal{A}(\langle X_r | J | X_s \rangle) = \mathcal{A}(\langle X_r | IX_r | X_s \rangle_B) = \mathcal{A}(\langle X_r | A_{rs^{-1}} IX_r \rangle_B) = \mathcal{A}(\langle A_{rs^{-1}} X_r | IX_r \rangle_B) \subseteq B_c.$

This implies $I = \text{span}\{\langle X_r | J | X_s \rangle : r \in G\}$. Let $r, s \in G$. Then

$\langle X_r | IX_s \rangle_B = \langle X_r | I \cdot A_{rs^{-1}} X_s \rangle_B = \langle X_r | A_{rs^{-1}} J | X_s \rangle_B = \langle X_r | X_s \rangle_B \cdot A_{rs^{-1}} J \subseteq B_{rs^{-1}} J.$

The latter implies that the Rieffel correspondence $R: \mathcal{I}(C^*_R(A)) \to \mathcal{I}(C^*_R(B))$ maps $C^*_R(A|_I)$ to an ideal $R(i(I))$ contained in $C^*_R(B|_J) = i(J)$. By symmetry, we get $R^{-1}(i(J)) = i(I)$. Hence $R(i(I)) = i(J)$. This proves \([2]\).

Statement \([3]\) follows immediately from \([2]\) because conditions \([MI]\) and \([C2]\) are phrased in terms of induced ideals.

To see \([4]\), let $C^*_R(A|_I)$ be the $G$-graded ideal in $C^*_R(A)$ corresponding to an $\mathcal{A}$-invariant ideal $I$. By \([2]\), the Rieffel correspondence maps $C^*_R(A|_I)$ to $C^*_R(B|_J)$ for the $\mathcal{B}$-invariant ideal $J$ defined in \([6,13]\). The Hilbert bundle $\mathcal{X}$ between $\mathcal{A}$ and $\mathcal{B}$
restricts to a Hilbert bundle $X^I$ between $A^I$ and $B^J$ with $(X^I)_g = I \cdot X_g$. And it induces a Hilbert bundle $X^I|A^I$ between $A^I|A^I$ and $B^J|B^J$ with $(X^I|A^I)_g = X_g/I \cdot X_g$. The restricted Hilbert bundles induce Morita equivalences $C^*_r(A^I) \sim C^*_r(B^J)$ and $C^*_r(A|A^I) \sim C^*_r(B|B^J)$. These are obtained by restricting the imprimitivity bimodule between $C^*_r(A)$ and $C^*_r(B)$. The sequence (6.11) is exact if and only if the primitive ideal space of $C^*_r(A)$ is the union of the primitive ideal spaces of $C^*_r(A^I)$ and $C^*_r(A|A^I)$. Since Morita equivalence implies an isomorphism between the ideal lattices, the Morita equivalence transfers this property from $A$ to $B$. This proves (4).

**Theorem 6.14.** Suppose that $A = (A_g)_{g \in G}$ is an exact Fell bundle over a locally compact group $G$. Let $\varphi : A \to \mathcal{M}(B)$ be the canonical generalised $C^*$-inclusion of $A$ into the reduced cross section algebra $B := C^*_\lambda(A)$ of $A$. If $\hat{A}$ or $\hat{B}$ are second countable or, more generally, Prime($\mathbb{P}(A)$) is first countable, then the quasi-orbit map $\varphi : \hat{B} \to \hat{A}/\sim$ is open and surjective.

**Proof.** By [3, Theorem 3.4], $A$ is equivalent to a Fell bundle $B$ associated to an action $\gamma : C \to \text{Aut}(C)$ on a $C^*$-algebra $C$. The Fell bundle $B$ is exact by Proposition 6.12.(4). Hence the inclusion $C \to \mathcal{M}(C \rtimes_{\lambda, \gamma} G)$ satisfies (MI) and (C2) by Theorem 6.13. Thus $A_e \to \mathcal{M}(C^*_\lambda(A))$ satisfies (MI) and (C2) by Proposition 6.12.(3). Hence the claims follow from Corollary 6.14. Here we may still describe the relation $\sim$ as in Theorem 6.10.

### 6.3. Regular inclusions and $C^*$-algebras graded by inverse semigroups.

The interest in regular $C^*$-inclusions started with the study of Cartan $C^*$-subalgebras, see [49,36,17,50]. They model a large class of examples, including crossed products of various sorts for actions of discrete groups, inverse semigroups, or étale groupoids. In order to apply our programme to these inclusions, we first introduce and discuss $C^*$-algebras graded by inverse semigroups. Then we translate the corresponding results to regular inclusions by showing that they are naturally graded by certain inverse semigroups.

**Definition 6.15.** Let $S$ be an inverse semigroup with unit element $e \in S$. An $S$-graded $C^*$-algebra is a $C^*$-algebra $B$ with a family of closed linear subspaces $(B_s)_{s \in S}$ such that $B_s^* = B_{s^{-1}}^*$, $B_g \subseteq B_h$ for all $g, h \in S$ and $B_g \subseteq B_h$ if $g \leq h$ in $S$, and $\sum B_s$ is dense in $B$. We call $A := B_e \subseteq B$ the unit fibre of the $S$-grading.

**Example 6.16.** Any discrete group $G$ may be viewed as an inverse semigroup with $g^* := g^{-1}$ for all $g \in S$. Then $g \leq h$ for $g, h \in G$ only happens for $g = h$. Thus in this case our notion of a $G$-graded $C^*$-algebra reduces to the standard one, see [20, Definition 16.2]. Crossed products for (partial or twisted) $G$-actions obviously have this structure, and so do the section $C^*$-algebras of Fell bundles over $G$. Here we may complete the $*$-algebra of sections of a Fell bundle in any $C^*$-seminorm for which the map from the unit fibre to the Hausdorff completion remains injective.

**Example 6.17.** Let $G$ be an étale groupoid with locally compact Hausdorff space of units $G^0$. A subset $U \subseteq G$ is called a bisection if it is open and the restrictions of the source and range maps to $U$ are injective. Bisections of $G$, with multiplication and inverse inherited from $G$, form an inverse semigroup $S(G)$. Let $A = (A_g)_{g \in G}$ be a Fell bundle over $G$ as in [8, Definition 2.6]. The section $C^*$-algebra $B = C^*_r(A)$ of $A$ is the completion of a certain convolution $*$-algebra $\mathcal{S}(A)$. For every $U \in \mathcal{S}(G)$, the space $A^*_U$ of continuous sections of $A$ vanishing outside $U$ embeds into $\mathcal{S}(A)$ and further into $B$, and these subspaces together span a dense subspace of $B$. These subspaces are closed because the $C^*$-norm on $B$ restricts to the supremum norm on $A^*_U$, and they satisfy $A^*_U = A^*_{U^*}$, $A^*_U \cdot A^*_V \subseteq A^*_{U \cap V}$ for all $U, V \in \mathcal{S}(G)$ and...
$A_U \subseteq A_V$ for $U \subseteq V$, see [8 Example 2.11]. Thus $(A_U)_{U \in S(G)}$ forms a grading of $B$ whose unit fibre $A$ is the $C_0(\mathbb{G})$-algebra $A_{C_0}$ of sections of the bundle $(A_x)_{x \in G^0}$. In fact, $B$ is graded by $(A_U)_{U \in S}$ for any (unital) inverse subsemigroup $S \subseteq S(G)$ such that the sets in $S$ form a basis of the topology in $G$. As in Example 6.16 the choice of the $C_*$-seminorm on $\mathfrak{G}(A)$ is immaterial: if $B$ is any quotient of the full section $C^*$-algebra, such that the unit fibre $A = A_{C^0} \hookrightarrow C^*(A)$ embeds into $B$ under the quotient homomorphism, then the grading $(A_U)_{U \in S}$ embeds into $B$.

The conditions $B^*_g = B_g^*$, $B_g \cdot B_h \subseteq B_{gh}$ and $B_g \leq B_h$ for $g \leq h$ in Definition 6.13 say that the spaces $B_g$ with the multiplication, involution, and norm inherited from $B$ form a Fell bundle over $S$ as in [19 Definition 2.1]. These conditions imply $B^*_g B^*_g = B^*_g B_g^* = B^*_g$ is a $B$ because $gg^* \leq e$. Since $AB_g + B_g A \subseteq B_g$, we see that $B_g$ for $g \in S$ is naturally a Hilbert $A$-bimodule with the inner products

$$\langle a \mid b \rangle := a \cdot b^* \in A$$ and $$\langle a \mid b \rangle := a^* \cdot b \in A$$ for $a, b \in B_g$.

The Fell bundle $(B_g)_{g \in S}$ is saturated if $B_g \cdot B_h = B_{gh}$ for every $g, h \in S$. Then $(B_g)_{g \in S}$ forms an action of $S$ on $B$ by Hilbert bimodules as in [10 Definition 4.7].

**Definition 6.18.** Let $\mathcal{B} = (B_g)_{g \in S}$ be an $S$-grading of a $C^*$-algebra $B$. We say that an ideal $I \in \mathfrak{I}(A)$ is $\mathcal{B}$-invariant if $B_g I B_g^* \subseteq I$ for all $g \in S$, see [10, Definition 4.3]. Let $\mathfrak{I}^\mathcal{B}(A)$ denote the set of $\mathcal{B}$-invariant ideals.

**Proposition 6.19.** Let $A := B_e \subseteq B$ be the unit fibre of an $S$-grading $(B_g)_{g \in S}$. For every $I \in \mathfrak{I}(A)$, the following are equivalent:

1. $I$ is restricted;
2. $I$ is $\mathcal{B}$-invariant;
3. $I$ is $B_g$-invariant, that is, $IB_g = B_g I$ for all $g \in S$;
4. $\sum_{g \in S} IB_g$ is an ideal in $B$;
5. $I$ is symmetric.

Thus $\mathfrak{I}^\mathcal{B}(A) = \mathfrak{I}^\mathcal{B}(A)$, $A \subseteq B$ is symmetric and conditions [JR], [C1] and [MI] hold.

**Proof.** Assume [1] so that $I = r \circ i(I) = BIB \cap A$. For every $g \in S$, $B_g I B_g^* \subseteq BIB \cap A = I$. Hence [1] implies [2]. Assume [2] since $B_g^* B_g^* = \langle B_g \mid B_g \rangle$ is a (closed two-sided) ideal in $A$, we get

$$B_g \cdot I = B_g B_g^* B_g \cdot I = B_g (B_g^* B_g \cdot I) = B_g IB_g^* B_g \subseteq I \cdot B_g.$$ 

The same computation for $B_g^*$ gives $I \cdot B_g \subseteq B_g \cdot I$. Thus $I$ is $B_g$-invariant. Hence [2] implies [3]. To see that [3] implies [4] note that $\sum_{g \in S} IB_g$ is a right ideal in $B$, and thus in the presence of [3] it is also a left ideal. If [4] holds, then $I \subseteq i(I) = \sum_{g \in S} IB_g$ is non-degenerate. Thus [4] implies [5]. That [5] implies [1] follows from Lemma 5.1. Conditions [JR], [C1] and [MI] follow from Corollary 5.5. 

**Proposition 6.20.** Let $A := B_e \subseteq B$ be the unit fibre of an $S$-grading $(B_g)_{g \in S}$. Let $J \in \mathfrak{I}(B)$ and put $J_g := J \cap B_g$ for $g \in S$. The spaces $(J_g)_{g \in S}$ form an $S$-grading of $i \circ r(J) = \sum_{g \in S} J_g$. In particular, $J$ is induced if and only if $J = \sum_{g \in S} J_g$ is $S$-graded.

**Proof.** Since $B_h J_g \subseteq J_{hg}$ and $J_h B_g \subseteq J_{gh}$ for $h,g \in S$, we conclude that $\sum_{g \in S} J_g$ is an ideal in $B$. It is easy to see that $(J_g)_{g \in S}$ is an $S$-grading of $\sum_{g \in S} J_g$. In particular, every $J_g$ is a Hilbert bimodule over $I := J_e = J \cap A$ and hence $J_g = I J_g = IB_g$. Thus $\sum_{g \in S} J_g = \sum_{g \in S} IB_g = i \circ r(J)$, see Proposition 6.19. 

□
**Theorem 6.21.** Let $A := B_e \subseteq B$ be the unit fibre of an $S$-grading $B = (B_t)_{t \in S}$. The quasi-orbit space $\tilde{A}/\sim$ of $A \subseteq B$ exists. If $\text{Prime}(I^B(A))$ is first countable or $\tilde{A}$ is second countable, then

1. there is a continuous, open and surjective map $\pi: \tilde{A} \to \text{Prime}(I^B(A))$, which maps $p$ to the largest $B$-invariant ideal contained in it. It descends to a homeomorphism $\tilde{\pi}: \tilde{A}/\sim \to \text{Prime}(I^B(A))$.

2. There is a continuous map $\varphi: \tilde{B} \to \tilde{A}/\sim$, $p \mapsto \tilde{\pi}^{-1}(r(p))$. It identifies $\tilde{B} \cong \tilde{A}/\sim$ if and only if $A$ separates ideals in $B$.

**Proof.** This follows from Theorem 6.17 and Propositions 6.19 and 6.19. □

**Remark 6.22.** If the grading $B = (B_t)_{t \in S}$ is saturated, then the induced partial homeomorphisms $(\tilde{B}_t)_{t \in S}$ form an action of $S$ by partial homeomorphisms on $\tilde{A}$, see [10, Lemma 6.12]. More specifically, for every $g \in S$, $D_g := B^*_g B_t$ is an ideal in $A$ and the Rieffel correspondence gives homeomorphisms $\tilde{B}_g: \tilde{D}_t \cong \tilde{D}_g$, where we treat $\tilde{D}_t$ and $\tilde{D}_g$ as open subsets of $\tilde{A}$. As in [9], we may associate to $(\tilde{B}_t)_{t \in S}$ the transformation groupoid $\tilde{A} \rtimes S$, compare [18, Section 4], where this groupoid is called the groupoid of germs. This is an étale topological groupoid with object space $\tilde{A}$. Arrows are equivalence classes of pairs $(t, p)$ for $p \in \tilde{D}_t \subseteq \tilde{A}$, where two pairs $(t, p)$ and $(t', p')$ are equivalent if $p = p'$ and there is $v \in S$ with $v \leq t, t'$ and $p \in \tilde{D}_v$. There is a unique topology on $\tilde{A} \rtimes S$ for which the source map $[t, p] \mapsto p$ is a partial homeomorphism onto $\tilde{D}_t$ for each $t \in S$. The subsets $U_t := \{[t, p] : p \in \tilde{D}_t\}$ form an open covering of $\tilde{A} \rtimes S$ by bisections, see [18, Corollary 4.16]. Then the range map $[t, p] \mapsto \tilde{B}_t(p)$ is also a local homeomorphism. As in the case of Fell bundles over groups, one concludes that for every $p_1, p_2 \in \tilde{A}$

$$p_1 \sim p_2 \text{ if and only if } (\tilde{A} \rtimes S) \cdot p_1 = (\tilde{A} \rtimes S) \cdot p_2.$$ 

Hence the quasi-orbit space of the $C^*$-inclusion $A = B_e \subseteq B$ is the quasi-orbit space of the transformation groupoid $\tilde{A} \rtimes S$, see also [9].

**Remark 6.23.** By Corollary 6.14 and Proposition 6.20, the quasi-orbit map $\varphi: \tilde{B} \to \tilde{A}/\sim$ defined above is open and surjective if and only if intersections of $S$-graded ideals are $S$-graded and the intersection of any $S$-graded ideal $I \in \mathfrak{I}(B)$ with the smallest $S$-graded ideal containing an ideal $J \in \mathfrak{I}(B)$ is the smallest $S$-graded ideal containing $I \cap J$.

Now we turn to regular $C^*$-inclusions. In the context of von Neumann algebras such inclusions were introduced by Dixmier in 1954. Here we will restrict our considerations to non-degenerate $C^*$-inclusions.

**Definition 6.24.** Let $A \subseteq B$ be a $C^*$-subalgebra. An element $b \in B$ normalises $A$ if $bAb^* \subseteq A$ and $b^*Ab \subseteq A$. Then we also call $b$ a normaliser of $A$. Let $N(A)$ be the subset of normalisers (see [38]). The inclusion $A \subseteq B$ is regular if it is non-degenerate, that is $AB = B$, and $N(A)$ generates $B$ as a $C^*$-algebra (see [50]).

**Lemma 6.25.** Let $A := B_e \subseteq B$ be the unit fibre of an $S$-grading $B = (B_t)_{t \in S}$. The $C^*$-inclusion $A \subseteq B$ is regular. In fact, $B_g \subseteq N(A)$ for every $g \in S$.

**Proof.** That $A \subseteq B$ is non-degenerate follows from Proposition 6.19 applied to $I = A$. For every $g \in S$, $geg^* \subseteq e$ and so $B_g B_e B_g^* \subseteq B_e$. Thus $B_g \subseteq N(A)$. □

The spaces $B_g \subseteq N(A)$ in the above lemma have the special feature that they are bimodules over $A$. Exel calls such subspaces of $N(A)$ slices in [10] and proves a number of facts that allow us to show the converse to Lemma 6.25.
Proposition 6.26. Let \( A \subseteq B \) be a non-degenerate C*-subalgebra. Let
\[
S(A) := \{ M \subseteq N(A) : M \text{ is a closed linear space and } AM \subseteq M, \ MA \subseteq M \}
\]
and define
\[
M \cdot N := \operatorname{span}\{nm : n, m \in M\}, \quad M^* := \{ m^*: m \in M \}
\]
for \( M, N \in S(A) \). These operations turn \( S(A) \) into an inverse semigroup with unit \( A \in S(A) \). And \((M)_{\text{Mes}(A)} \) is a (saturated) \( S(A) \)-grading on the C*-algebra \( \sum_{\text{Mes}(A)} M \). The C*-inclusion \( A \subseteq B \) is regular if and only if \( B = \sum_{\text{Mes}(A)} M \).

Proof. Let \( M, N \in S(A) \). We have \( M \cdot N \in S(A) \) by [19] Proposition 13.1. Clearly, the multiplication \( \cdot \) is associative and \( M^* \in S(A) \). Proposition 10.2 implies \( M^* \cdot M \triangleleft A \) and \( M \cdot M^* \triangleleft A \). Hence \( M \) is naturally a Hilbert \( A \)-bimodule. So \( M \cdot M^* = M \) and \( A \cdot M = M \cdot A = M \). Thus \( S(A) \) is an inverse semigroup and \( A \) is a unit in \( S(A) \). This implies that \( \sum_{\text{Mes}(A)} M \) is a *-subalgebra of \( B \). Then \( \sum_{\text{Mes}(A)} M \) a C*-subalgebra. If \( M \leq N \in S(A) \), then \( M = M \cdot M^* \leq A \cdot N \leq N \). So \((M)_{\text{Mes}(A)} \) is a grading of \( \sum_{\text{Mes}(A)} M \). The Fell bundle \((M)_{\text{Mes}(A)} \) is saturated by definition. By [19] Proposition 10.5, every \( a \in N(A) \) lies in some \( M \in S(A) \). Thus \( B = \sum_{\text{Mes}(A)} M \) if and only if \( B \) is generated as a C*-algebra by \( N(A) \) if and only if \( A \subseteq B \) is regular. \( \square \)

Corollary 6.27. A C*-inclusion \( A \subseteq B \) is regular if and only if \( B \) is an \( S \)-graded C*-algebra with \( A \) as the unit fibre for the grading. Moreover, the grading may be chosen to be saturated.

Proof. Combine Lemma 6.25 and Proposition 6.26. \( \square \)

The above result has two advantages. Firstly, by passing to a larger inverse semigroup, every graded C*-algebra may be viewed as a C*-algebra with a saturated grading, see also [7]. Secondly, every regular C*-inclusion \( A \subseteq B \) may be studied as a graded C*-algebra by choosing any inverse subsemigroup \( S \subseteq S(A) \) with \( \sum_{\text{Mes}} M = B \). In fact, in certain cases we may even drop the assumption that \( S \) be a semigroup:

Proposition 6.28. Let \( A \subseteq B \) be a regular C*-subalgebra and let \( S \subseteq S(A) \) be any subset such that \( \sum_{\text{Mes}} M \) is dense in \( B \).

1. \( I \in \mathbb{I}(A) \) is restricted if and only if \( I \) is \( M \)-invariant, that is, \( IM = MI \) for all \( M \in S \);
2. \( J \in \mathbb{I}(B) \) is induced if and only if \( J = \sum_{\text{Mes}} J \cap M \).

Proof. Let \( \mathbb{S} \) be the semigroup generated by \( S \subseteq S(A) \). The ideal \( I \in \mathbb{I}(A) \) is \( M \)-invariant for every \( M \in S \) if and only if \( I \) is \( M \)-invariant for every \( M \in \mathbb{S} \). Hence (1) follows from Proposition 6.19 applied to \( \mathbb{S} \). Now let \( J \in \mathbb{I}(B) \). Clearly, \( J = \sum_{\text{Mes}} J \cap M \) implies \( J = \sum_{\text{Mes}} J \cap M \). Hence \( J \) is induced by Proposition 6.26. Conversely, if \( J \) is induced, then \( J = IB \) for \( I := r(J) = J \cap A \) because \( I \) is symmetric by Proposition 6.19. Hence \( J \) is equal to the closed linear span of \( I \cdot M \subseteq J \cap M \) for \( M \in S \). \( \square \)

We apply Theorem 6.21 to regular C*-inclusions using a notion of dual groupoid:

Definition 6.29. Let \( A \subseteq B \) be a regular inclusion. Let \( S(A) \) be its inverse semigroup of slices. We define the dual groupoid to the C*-inclusion \( A \subseteq B \) as the transformation groupoid \( \hat{A} \times S(A) \) associated to the dual action of \( S(A) \) on \( \hat{A} \), see Remark 6.22.
Remark 6.30. It follows from the construction that \( \hat{A} \times S(A) = \hat{A} \times S \) for every inverse subsemigroup \( S \subseteq S(A) \) with \( \sum_{\lambda \in S} \bar{M} = B \). In particular, if \( B \) is the full or reduced \( C^* \)-algebra \( C^*_r(G, \Sigma) \) associated to a twisted étale locally compact Hausdorff groupoid \((G, \Sigma)\) and \( A = C_0(G^0) \) is the subalgebra of functions on the space of units \( G^0 \) in \( G \), then the groupoid dual to \( A \subseteq B \) is \( G \). Indeed, we may take as \( S \) the spaces of functions living on bisections of \( G \) on which the twist \( \Sigma \) is trivial, see the proof of [2] Corollary 4.13. Then \( G \cong G^0 \times S \) by [18] Proposition 5.4.

Remark 6.31. Let \( A \subseteq B \) be an Abelian regular \( C^* \)-subalgebra. Renault introduced the Weyl pseudogroup \( G(A) \) and the Weyl groupoid \( G(A) \) for \( A \subseteq B \) in [50] Definition 4.11]. By construction, \( G(A) \subseteq (M)_{M \in S(A)} \) and the corresponding transformation groupoids coincide. Thus the Weyl groupoid \( G(A) \) is the groupoid of germs of the action of \( G \) on \( A \). [50] Proposition 3.2 gives a canonical surjective groupoid morphism \( \hat{A} \times S(A) \rightarrow G(A) \) which is an isomorphism if and only if \( \hat{A} \times S(A) \) is essentially principal. [50] Theorem 5.9 and Remark 6.30 imply that \( \hat{A} \times S(A) \cong G(A) \) when \( A \) is a Cartan subalgebra and \( B \) is separable. In general, \( \hat{A} \times S(A) \) carries more information about \( A \subseteq B \) than \( G(A) \).

Theorem 6.32. Let \( A \subseteq B \) be a regular inclusion and let \( G \) be its dual groupoid. The quasi-orbit space \( \hat{A}/\sim \) of \( A \subseteq B \) exists and it coincides with the quasi-orbit space of \( G \). That is, \( p_1 \sim p_2 \) for \( p_1, p_2 \in \hat{A} \) if and only if \( G \cdot p_1 = G \cdot p_2 \).

If \( \text{Prime}^B(A) \) is first countable which holds when \( \hat{A} \) is second countable then the quasi-orbit map exists:

1. There is a continuous, open surjection \( \pi: \hat{A} \rightarrow \text{Prime}^B(A) \), \( p \mapsto \bigcap_{p' \in \pi(p)} p' \).

   It descends to a homeomorphism \( \bar{\pi}: \hat{A}/\sim \rightarrow \text{Prime}^B(A) \).

2. There is a continuous map \( \varphi: \hat{B} \rightarrow \hat{A}/\sim \), \( p \mapsto \bar{\pi}^{-1}(r(p)) \). It identifies \( \hat{B} \cong \hat{A}/\sim \) if and only if \( A \) separates ideals in \( B \).

Proof. The assertion follows from Theorem 6.21 combined with Proposition 6.26 and Remark 6.22.

7. Further applications and examples

We now apply the theory developed above to several different situations, namely, commutative and skew-commutative tensor products in Section 7.1 relative Cuntz–Pimsner algebras of \( C^* \)-correspondences in Section 7.2 crossed products for groupoid actions in Section 7.3 and crossed products for quantum group coactions in Section 7.4.

7.1. Tensor products and \( C_0(X) \)-\( C^* \)-algebras.

If all ideals in \( A \) are symmetric, then Lemma 5.3 implies that \( i \) commutes with finite intersections. So \( i \) is a locale morphism and induces a continuous map \( \text{Prime}(B) \rightarrow \text{Prime}(A) \). We will exhibit two well known cases where this happens.

Example 7.1. Let \( B = A \otimes D \) be some \( C^* \)-tensor product with a \( C^* \)-algebra \( D \) and let \( \varphi: A \rightarrow M(B) \) be the canonical non-degenerate \( \ast \)-homomorphism. If \( I \in \mathbb{I}(A) \), then \( i(I) \) is the closure of the algebraic tensor product \( I \otimes D \) in \( A \otimes D \), which we also denote by \( I \otimes D \). Hence \( I \) is symmetric. If \( I, J \in \mathbb{I}(A) \), then

\[
I \otimes D \cap J \otimes D = (I \otimes D) \cdot (J \otimes D) = (I \cdot J) \otimes D = (I \cap J) \otimes D.
\]

So \( i \) commutes with finite meets. It always commutes with joins by Proposition 2.3 Hence \( i \) is a locale morphism from \( \mathbb{I}(A) \) to \( \mathbb{I}(B) \). So it induces a continuous map \( \text{Prime}(B) \rightarrow \text{Prime}(A) \).

We claim that any ideal \( I \in \mathbb{I}(A) \) is restricted. Let \( J \in \mathbb{I}(B) \) be the kernel of the canonical map from \( A \otimes D \) to the minimal \( C^* \)-tensor product \( A/I \otimes_{\text{min}} D \).
Then \( r(J) = I \) because the map from \( A \) to the multiplier algebra of \( A/I \otimes_{\min} D \) vanishes exactly on \( I \). Thus the quasi-orbit space is just the primitive ideal space \( \hat{A} \). The conditions \( [JR] \) and \( [CI] \) are trivial in this case, and \( [MI] \) holds because \( i \) commutes with finite meets. The quasi-orbit map \( \hat{B} \to \hat{A} \) may be constructed directly: it is the restriction of \( r \) to \( \hat{B} \subseteq \hat{B} \). Indeed, a representation \( \pi \) of \( A \otimes D \) on a Hilbert space \( \mathcal{H} \) is described by commuting representations \( \pi_A \) and \( \pi_D \) of \( A \) and \( D \) on \( \mathcal{H} \), respectively. Then \( r(\ker \pi) = \ker \pi_A \). A subspace of \( \mathcal{H} \) that is \( A \)-invariant is \( B \)-invariant as well. So the representation of \( A \) is irreducible if \( B \) acts irreducibly. If \( p \in B \) is a primitive ideal, that is, the kernel of an irreducible representation, then \( r(p) \in I(A) \) is primitive as well. The restriction \( r: \hat{B} \to \hat{A} \) of \( r \) is the quasi-orbit map. This map is continuous, and \( r^{-1}: \mathcal{O}(\hat{A}) \to \mathcal{O}(\hat{B}) \) becomes the map \( i \) when we identify \( \mathcal{O}(\hat{A}) \cong I(A) \) and \( \mathcal{O}(\hat{B}) \cong I(B) \).

Example 7.2. Let \( X \) be a locally compact space. A \( C^* \)-algebra over \( X \) or \( C_0(X) \)-\( C^* \)-algebra is a \( C^* \)-algebra \( B \) with a non-degenerate \(*\)-homomorphism from \( A := C_0(X) \) to the centre of the multiplier algebra of \( B \). Ideals in \( A \) are of the form \( C_0(U) \) for open subsets \( U \subseteq X \), and \( i(C_0(U)) = C_0(U) \cdot B = B \cdot C_0(U) \) because the image of \( A \) in \( M(B) \) is central. That is, all ideals in \( A \) are symmetric. So \( i \) is a locale morphism by Lemma 5.4 and Proposition 2.8.(5). Hence \( i \) induces a continuous map \( \text{Prime}(B) \to \text{Prime}(A) \cong X \). It restricts to a continuous map \( \pi: \hat{B} \to X \) (see also [40, Proposition 2.1]). Conversely, any continuous map \( \hat{B} \to X \) comes from a \( C_0(X) \)-\( C^* \)-algebra structure on \( B \) by the Dauns–Hofmann Theorem.

Let us identify ideals in \( A \) and \( B \) with open subsets in \( X \) and \( \hat{B} \), respectively. Then the induction and restriction maps have the form \( i(U) = \pi^{-1}(U) \) and \( r(V) = (X \setminus \pi(\hat{B} \cap V))^c \), for \( U \in \mathcal{O}(X) \), \( V \in \mathcal{O}(\hat{B}) \), where \( c \) stands for the interior of a given set. Since \( r \) preserves meets by Proposition 2.8(5) and \( i: \mathcal{O}(A) \to \mathcal{O}(B) \) is an isomorphism, we see that \( [MI] \) holds if and only if \( i \) preserves meets. This happens if and only if \( \pi \) is open (see [40, Lemma 2.9]). Thus \( [MI] \) holds if and only if \( \pi \) is a continuous \( C_0(X) \)-\( C^* \)-algebra. In particular, if \( [MI] \) holds, then \( i(U) = \pi^{-1}(U) \) is an upper Galois adjoint to \( F(V) := \pi(V) \), \( U \in \mathcal{O}(X) \), \( V \in \mathcal{O}(\hat{B}) \), and condition \( [MI] \) is satisfied. The map \( i \) is injective (is an upper Galois insertion) if and only if \( \pi \) is surjective, which agrees with Theorem 2.24.

As we have seen, condition \( [MI] \) fails in Example 7.2 for every \( C_0(X) \)-\( C^* \)-algebra which is not continuous. Now we show that \( [MI] \) may fail also in the situation of Example 7.1 for maximal tensor products.

Example 7.3. Suppose first that \( D \) is exact. We claim that for every \( C^* \)-algebra \( A \), the inclusion \( A \to M(A \otimes_{\min} D) \) satisfies \( [MI] \). Indeed, let \( \{I_A\}_{A \in A} \) be a family of ideals in \( A \). Clearly, \( \bigcap_{A \in A} I_A \otimes_{\min} D \cong \bigcap_{A \in A} (I_A \otimes_{\min} D) \). Let \( R_\psi: A \otimes_{\min} D \to A \) denote the slice map corresponding to a functional \( \psi \) on \( D \). Then

\[
R_\psi \left( \bigcap_{A \in A} (I_A \otimes_{\min} D) \bigcap_{A \in A} (I_A \otimes_{\min} D) = \bigcap_{A \in A} I_A.
\]

Hence \( \bigcap_{A \in A} (I_A \otimes_{\min} D) \subseteq \bigcap_{A \in A} I_A \otimes_{\min} D \) by [33] Theorem 1.1. Thus \( \otimes_{\min} D: I(A) \to I(A \otimes_{\min} D) \) commutes with intersections, that is, \( [MI] \) holds.

Now let us characterise condition \( [MI] \) for certain tensor products in terms of condition \( [MI] \) for \( C_0(X) \)-\( C^* \)-algebras. Let \( A \) be a continuous \( C_0(X) \)-\( C^* \)-algebra and let \( B = A \otimes D \) be a \( C^* \)-tensor product with some \( D \). The canonical homomorphism \( \varphi: A \to M(B) \) extends uniquely to a homomorphism \( M(A) \to M(B) \), which maps the centre of \( M(A) \) to the centre of \( M(B) \). Thus the composite \( C_0(X) \to M(A) \to M(B) \) gives a homomorphism \( C_0(X) \to M(B) \) that gives \( B \) a
C_0(X)\text{-C}^*\text{-algebra structure. Since } A \text{ is a continuous } C_0(X)\text{-algebra, } \text{(MI)} \text{ holds for } C_0(X) \to \mathcal{M}(A). \text{ Thus } A \to \mathcal{M}(B) \text{ satisfies } \text{(MI)} \text{ if and only if } C_0(X) \to \mathcal{M}(B) \text{ satisfies } \text{(MI)} \text{ which in turn holds if and only if } B \text{ is a continuous } C_0(X)\text{-C}^*\text{-algebra. If, in addition, the quasi-orbit map } \varphi: \hat{B} \to \hat{A} \text{ exists, then the composite } \hat{B} \overset{\varphi}{\to} \hat{A} \to X, \text{ where } \hat{A} \to X \text{ is the open map induced by } C_0(X) \to \mathcal{M}(A), \text{ is the map } \hat{B} \to X \text{ induced by } C_0(X) \to \mathcal{M}(B). \text{ Hence } A \to \mathcal{M}(B) \text{ satisfies } \text{(MI)} \text{ if and only if the quasi-orbit map } \varphi: \hat{B} \to \hat{A} \text{ is open.}

Finally, suppose that } D \text{ is a non-nuclear C}^*\text{-algebra. Let } X = \mathbb{N}^+ \text{ be the one-point compactification of } \mathbb{N}. \text{ By } [38, \text{ Theorem C}], \text{ there is a separable continuous } C(\mathbb{N}^+)\text{-C}^*\text{-algebra } A \text{ such that the } C(\mathbb{N}^+)\text{-C}^*\text{-algebra } A \otimes_{\max} D \text{ is not continuous. Hence, in view of the discussion above, } A \to \mathcal{M}(A \otimes_{\max} D) \text{ does not satisfy } \text{(MI)} \text{ and the quasi-orbit map } \varphi: \hat{B} \to \hat{A} \text{ exists but is not open. In particular, if } D = C^*(G) \text{ for a non-amenable (discrete) group, then } B = A \otimes_{\max} D = A \rtimes_{\text{triv}} G, \text{ where triv stands for the trivial action. Hence one cannot hope for counterparts of Theorems } 6.3 \text{ and } 6.4 \text{ for full crossed products.}

We state two corollaries of Example 7.4 (and Theorem 6.4).

**Corollary 7.4.** A C}^*\text{-algebra } D \text{ is nuclear if and only if for every C}^*\text{-algebra } A \text{ the generalised C}^*\text{-inclusion } A \to \mathcal{M}(A \otimes_{\max} D) \text{ satisfies } \text{(MI)}.

**Corollary 7.5.** A discrete group } G \text{ is amenable if and only if for every } G\text{-action on a separable C}^*\text{-algebra } A \text{ the quasi-orbit map } \varphi: \hat{B} \to \hat{A}/\sim \text{ for the full crossed product } B := A \rtimes G \text{ is open and surjective.}

Tensor products of C*algebras may be modified so that the tensor factors no longer commute. A rather general such construction using quantum group coactions on the tensor factors is introduced in [41]. Here we examine the simplest case – the skew-commutative tensor product of \mathbb{Z}/2-graded C*algebras where odd elements in the tensor factors anticommute, see [41, §2.6]. Our results are rather negative already in this case; that is, nothing beyond the Galois connection property seems to hold in general.

**Example 7.6.** Let } A \text{ and } D \text{ be } \mathbb{Z}/2\text{-graded C}^*\text{-algebras with grading involutions } \alpha, \delta \text{ and let } A = A_+ \oplus A_- \text{ and } D = D_+ \oplus D_- \text{ be the decompositions into even and odd elements, that is, the eigenspaces for } \alpha \text{ and } \delta.

The skew-commutative tensor product } A \hat{\otimes} D \text{ is a variant of the (minimal) C}^*\text{-tensor product where elements in } A_- \text{ and } D_- \text{ anti-commute. It is defined as a C}^*\text{-completion of the algebraic tensor product } A \otimes D. \text{ It comes with injective morphisms } A \to \mathcal{M}(A \hat{\otimes} D) \leftarrow D.

Let } I \in \mathcal{I}(A). \text{ Then } I \cdot (A \hat{\otimes} D) \text{ is the closure of } I \otimes D \text{ in } A \hat{\otimes} D. \text{ When we multiply on the left with } A \hat{\otimes} D, \text{ we get}

\[(A \otimes D) \cdot (I \otimes D) = I \otimes D + \alpha(I) \otimes D_- \cdot D\]

because } d \cdot (i \hat{\otimes} d_2) = \alpha(i) \otimes dd_2 \text{ if } d \in D_- \text{. So if } D_- \neq 0, \text{ then } I \text{ is symmetric if and only if } \alpha(I) = I.

If the ideal in } D_- \text{ generated by } D_- \cdot D_- \text{ is equal to } D_+, \text{ then we may rewrite } I \cdot (A \hat{\otimes} D) \text{ as the closure of } (I + \alpha(I)) \otimes D. \text{ So } i(I) = i(I + \alpha(I)). \text{ Then } \rho \circ i(I) = I + \alpha(I), \text{ which is the smallest } \mathbb{Z}/2\text{-invariant ideal containing } I. \text{ Therefore, the restricted ideals are exactly the } \mathbb{Z}/2\text{-invariant ideals, and these are also the same as the symmetric ideals. So our theory applies in this case, regardless whether } D \text{ is unital or not.}

If the ideal in } D_+ \text{ generated by } D_+ \cdot D_- \text{ is not equal to } D_+, \text{ however, then all this breaks down. Then } \rho \circ i(I) = I \text{ for all ideals } I \in \mathcal{I}(A), \text{ that is, all ideals in } A \text{ are restricted. But not all ideals are symmetric, unless } D_- = 0. \text{ So if } D_- \text{ is non-zero.
but not full as a $D_+$-module, then $i: A \to \hat{\otimes} D$ is not a locale morphism, unlike in the situation of commutative $C^*$-tensor products in Example 7.1.

An elementary case where this happens is $D = \mathbb{C} \oplus \hat{\mathbb{M}}_2$, where $\mathbb{C}$ is trivially graded and $\hat{\mathbb{M}}_2$ carries the usual inner grading where the off-diagonal entries are odd. Here the ideal generated by $D_+ \cdot D_-$ misses the first summand $\mathbb{C}$. Now take $A = \mathbb{C} \oplus \mathbb{C}$ with the flip grading. With this choice of $A$, we have $A \hat{\otimes} D \cong D \times_\mathbb{Z}/2$ for any $D$. In our case,

$$A \hat{\otimes} D \cong C^*(\mathbb{Z}/2) \otimes D \cong C^* \mathbb{C} \oplus \mathbb{C} \oplus \hat{\mathbb{M}}_2 \oplus \mathbb{M}_2$$

because the $\mathbb{Z}/2$-action on $D$ is inner. Here each ideal in $A$ is restricted from $A \hat{\otimes} D$. But only the invariant ideals $0$ and $A$ are symmetric. Since $D$ is unital, we have $A \subset A \hat{\otimes} D$ here. In this example, the ideals $i(\mathbb{C} \oplus 0)$ and $i(0 \oplus \mathbb{C})$ in $A \hat{\otimes} D$ are $\mathbb{C} \oplus 0 \oplus \mathbb{M}_2 \oplus \mathbb{M}_2$ and $0 \oplus \mathbb{C} \oplus \mathbb{M}_2 \oplus \mathbb{M}_2$, respectively. Their intersection $0 \oplus \mathbb{C} \oplus \mathbb{M}_2 \oplus \mathbb{M}_2$ is not induced, although it is the intersection of two induced ideals.

### 7.2. Relative Cuntz–Pimsner algebras.

Let $X$ be a $C^*$-correspondence over a $C^*$-algebra $A$. That is, $X$ is a right Hilbert $A$-module with a $*$-homomorphism $\varphi_X: A \to B(X)$ which defines a left action of $A$ on $X$ by adjointable operators. We write $ax := \varphi_X(a)x$ for $a \in A, x \in X$.

A representation of the $C^*$-correspondence $A, X$ in a $C^*$-algebra $B$ is a pair of maps $(\psi_0, \psi_1)$ where $\psi_0: A \to B$ is a $*$-homomorphism and $\psi_1: X \to B$ is linear and $\psi_1(x^*\psi_1(y) = \psi_0(x^*y)_A$ and $\psi_0(a)\psi_1(x) = \psi_1(ax)$ for all $a \in A$ and $x, y \in X$.

The formula $\psi_1((x,y)) = \psi_1(x)\psi_1(y)^*$ for $x, y \in X$ defines a $*$-homomorphism $\psi_1: K(X) \to B$ on the $C^*$-algebra of compact operators on $X$.

Let $J(X) := \psi_0^{-1}(K(X))$ and $J_X := J(X) \cap (\ker \varphi_X)^\perp$.

The representation $(\psi_0, \psi_1)$ is called covariant on $J \triangleleft J(X)$ if $\psi_1((x, y)) = \psi_0(a)$ for all $a \in A$.

Let $J$ be an ideal in $J(X)$. There is a universal $C^*$-algebra $\mathcal{O}(J, X)$ generated by a representation $(j_0, j_1)$ that is covariant on $J$. We call $\mathcal{O}(J, X)$ the Cuntz–Pimsner algebra relative to $J$. The homomorphism $j_0: A \to \mathcal{O}(J, X)$ is injective if and only if $J \subseteq J_X$. Katsura’s Cuntz–Pimsner algebra of $X$ is $\mathcal{O}_X := \mathcal{O}(J, X)$. The Toeplitz algebra of $X$ is $\mathcal{T}_X := \mathcal{O}(0, X)$. The relative Cuntz–Pimsner algebra $\mathcal{O}(J, X)$ is equipped with a circle gauge action $\gamma: \mathbb{T} \to \text{Aut}(\mathcal{O}(J, X))$, defined by $\gamma_z(j_0(a)) = j_0(a)$ and $\gamma_z(j_1(x)) = z \cdot j_1(x)$ for all $z \in \mathbb{T}, a \in A, x \in X$.

Let $I$ be an ideal in $A$. Define

$$X(I) := \langle x \mid \varphi_X(x)Y \rangle_A = \text{span} \{ \langle x \mid a \cdot y \rangle_A \mid a \in I, x, y \in X \},$$

$$X^{-1}(I) := \{ a \in A : \langle x \mid a \cdot y \rangle_A \in I \text{ for all } x, y \in X \}.$$

We call $I$ positively invariant if $X(I) \subseteq I$. Given an ideal $I$ in $A$, we call $I$ $J$-negatively invariant or $J$-saturated if $X^{-1}(I) \cap J \subseteq I$. We call $I$ $J$-invariant if it is positively invariant and $J$-negatively invariant. The $J_X$-invariant ideals are called just invariant ideals in $[33]$.

**Proposition 7.7.** Let $B := \mathcal{O}(I, X)$ for an ideal $I \subseteq J(X)$. Let $\varphi = j_0: A \to B$ be the canonical $*$-homomorphism.

1. $\mathbb{L}^0(A)$ is equal to the lattice $\mathbb{L}^0_X(A)$ of $J$-invariant ideals in $A$.
2. An induced ideal $K \in \mathbb{L}^1(B)$, generated by $I := r(K)$, is naturally isomorphic to the relative Doplicher–Roberts algebra $\mathcal{O}_X(I, J \cap I)$ introduced in $[37]$ Definition 7.17. The ideal $K$ is gauge-invariant and Morita equivalent to the Cuntz–Pimsner algebra $\mathcal{O}(J \cap I, X)$. Moreover, $\mathcal{O}_X(I, J \cap I) = \mathcal{O}(J \cap I, X)$ if and only if $I$ is symmetric for $\varphi$.
3. Finite meets of induced ideals are induced, that is, condition [MI] holds.
(4) If $J + \ker(\varphi_X) = A$ or $X$ is a Hilbert bimodule and $J = J_X$, then $\mathcal{I}^A(B)$ coincides with the lattice $\mathcal{I}(B)$ of gauge-invariant ideals in $B$.

(5) If $\mathcal{I}^A(B) = \mathcal{I}(B)$ then conditions (1) and (2) hold.

Proof. Statements (1) and (2) follow from [37, Theorem 6.20]. In particular, the positive invariance of a restricted ideal is straightforward and well-known. Moreover, if $I \in \mathcal{I}^B(A)$ is $J$-invariant and we view $\mathcal{O}(I, J \cap I)$ and $\mathcal{O}(J \cap I, XI)$ as $\mathbb{C}^*$-subalgebras of $\mathcal{O}(J, X)$, then

$$\mathcal{O}(I, J \cap I) = \text{span}(j_n(I)j_0(I)j_m(X^{\otimes m})^*: n, m \in \mathbb{N}),$$

$$\mathcal{O}(J \cap I, XI) = \text{span}(j_n(I)j_0(I)j_m(X^{\otimes m})^*: n, m \in \mathbb{N}),$$

where $(j_0, j_n)$ is the representation of $(A, X^{\otimes n})$ induced by $(\hat{j}_0, j_1)$. This implies $\mathcal{O}(J \cap I, XI) \subseteq \mathcal{O}(I, J \cap I)$, and we have equality here if and only if $j_0(I)$ is a non-degenerate subalgebra of $\mathcal{O}(I, J \cap I)$, if and only if $I$ is symmetric.

Statement (1) follows from [37, Theorem 7.11]. We prove (5). Assume $\mathcal{I}^A(B) = \mathcal{I}(B)$. Let $F(J) := \sum_{z \in \mathbb{T}} \gamma_z(J)$ for $J \in \mathcal{I}(B)$. Then $F: \mathcal{I}(B) \to \mathcal{I}(B)$ satisfies (1) and (2). Thus (1) holds by Lemma 1.1. If $I \in \mathcal{I}(B)$ and $J \in \mathcal{I}(B)$, then $I \gamma_z(J) = \gamma_z(I) \gamma_z(J) = \gamma_z(IJ)$ for all $z \in \mathbb{T}$. Thus $I \cap J = I = \sum_{z \in \mathbb{T}} \gamma_z(IJ) = F(I \cap J)$. That is, (2) holds.

To prove (3), let $I_1, I_2 \in \mathcal{I}^B(A)$. Since $i$ is monotone, we have $i(I_1 \cap I_2) \subseteq i(I_1) \cap i(I_2)$. To see the reverse inclusion, recall that $i(I_1) \cap i(I_2) = i(I_1)\cap i(I_2)$ is spanned by elements in the sets

$$\left( j_n(X^{\otimes n})j_0(I_1)j_m(X^{\otimes m})^* \right) \left( j_i(X^{\otimes i})j_0(I_2)j_k(X^{\otimes k})^* \right),$$

where $a, m, l, k \in \mathbb{N}$. Assuming, for instance, that $m \geq l$ and using properties of the maps $j_m$ and $j_i$, we see that the above set is contained in

$$j_n(X^{\otimes n})j_0(I_1)j_m-i(X^{\otimes m-1})^*j_0(I_2)j_k(X^{\otimes k})^*.$$

This is contained in $j_n(X^{\otimes n})j_0(I_1)j_0(I_2)j_k(X^{\otimes k})^* \subseteq i(I_1 \cap I_2)$ because $I_2$ is positively invariant. Thus $i(I_1) \cap i(I_2) \subseteq i(I_1 \cap I_2)$.

Remark 7.8. Since arbitrary meets of restricted ideals are restricted by Proposition 2.3[7], statement (1) implies and strengthens [33, first part of Proposition 4.10 and Corollary 4.11].

It seems that the homomorphism $j_0: A \to \mathcal{O}(J, X)$ is never symmetric, unless $j_1(X)$ is a Hilbert bimodule over $j_0(A)$, when we are dealing with a Fell bundle over $Z$. Even if $A$ separates ideals in $\mathcal{O}_X$, condition (JR) usually fails.

Example 7.9 ([33, Example 4.12]). Let $X$ be the $\mathbb{C}^*$-correspondence over $A = \mathbb{C}^3$ built from the directed graph $\bullet \leftarrow \bullet \rightarrow \bullet$. Then $\mathcal{O}_X = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and

$$j_0(a, b, c) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}.$$ We have already met this inclusion in Example 2.1.

The join of the restricted ideals $I_1 = \mathbb{C} \oplus 0 \oplus 0$ and $I_2 = 0 \oplus \mathbb{C} \oplus 0$ is not restricted. Note that $A$ separates ideals in $B = \mathcal{O}_X$ and, in agreement with Lemma 4.8, there is a continuous map $r: B \rightarrow \text{Prime}(\mathcal{O}^B(A)) \subseteq \hat{A}$. There is, however, no surjective map from $\hat{A}$ onto $B$. This is no contradiction with Corollary 4.13 because (JR) is not satisfied.

The above discussion shows the following. In the context of Cuntz–Pimsner algebras, apart from the Galois connection, the only fact we get from our general theory is that $r: \text{Prime}(\mathcal{O}(J, X)) \to \text{Prime}(\mathcal{I}^B(A))$ is a well defined continuous map (combine Propositions 7.7 and 11.7). When we are not in the situation of statement (4) in Proposition 7.7, there are gauge-invariant ideals in $\mathcal{O}(J, X)$.
which are not induced, and then we cannot even apply Theorem 4.13 directly. Nevertheless, we may overcome this issue by using the following Lemma 7.10.

More specifically, for any positively invariant ideal $I$ there is a natural quotient $C^*$-correspondence $X/I := X/XI$ over $A/I$. Let $q_I^* : A 	o A/I$ be the quotient map and let

$$J_X(I) = \{ a \in A : \varphi_{X/I}(q_I(a)) \in \mathbb{K}(X_I), \ aX^{-1}(I) \subseteq I \}.$$ 

Let $J \ll J(X)$. A $J$-pair of $X$ is a pair $(I, I')$ of ideals $I, I'$ of $A$ such that $I$ is positively invariant and $J + I \subseteq I' \subseteq J_X(I)$. Let $\text{Pair}^J_X(A)$ be the set of $J$-pairs. Equip it with the natural pre-order coming from inclusion. Then the map $\Gamma^J_X(A) \rightarrow \text{Pair}^J_X(A), I \mapsto (I, I + J)$, is an order-preserving embedding. It is an isomorphism, for instance, in the situation of statement (4) in Proposition 7.7.

**Lemma 7.10.** Let $B := \mathcal{O}(J, X)$ for an ideal $J \subseteq J(X)$. Put $\tilde{A} := j_0(A) + j^{(1)}(\mathbb{K}(X))$ and consider the inclusion $\tilde{A} \subseteq B$.

1. We have a lattice isomorphism $\Gamma^B(\tilde{A}) \cong \text{Pair}^\tilde{Y}_X(\tilde{A})$, where a restricted ideal $I$ in $\tilde{A}$ is mapped onto the $J$-pair $(j_0^{-1}(I), j_0^{-1}(I + j^{(1)}(\mathbb{K}(X))))$ of $X$;
2. $\Gamma^B(\tilde{A}) = \Gamma^B(B)$, that is, induced and gauge-invariant ideals coincide.

**Proof.** The map $\Gamma^B(B) \ni I \mapsto (j_0^{-1}(I), j_0^{-1}(I + j^{(1)}(\mathbb{K}(X)))) \in \text{Pair}^\tilde{Y}_X(A)$ is a lattice isomorphism by [33 Proposition 11.9] (see also [37 Theorem 7.17]). This readily implies (2) and hence also (1). \qed

We take the opportunity to use a recent result of [11]. If $J \ll J(X)$ then the kernel $J_X$ of $j_0 : A \twoheadrightarrow \mathcal{O}(J, X)$ is the smallest $J$-invariant ideal containing 0. There is a canonical isomorphism $\mathcal{O}(J, X) \cong \mathcal{O}(q_{J_X}(J), X_{J_X})$, where $q_{J_X} : A \twoheadrightarrow A/J_X$ is the quotient map and $X_{J_X} = X/XJ_X$ is the quotient $C^*$-correspondence over $A/J_X$ (see [37 Theorem 6.23]).

**Definition 7.11.** Let $X$ be a $C^*$-correspondence over a liminal $C^*$-algebra $A$ and let $J \ll J(X)$. We call $X$ topologically free on $J$ if the graph dual to $X_{J_X}$ is topologically free on the set corresponding to $q_{J_X}(J)$ (see [11 Definitions 5.1 and 3.2]). We call $X$ residually topologically free on $J$ if, for every $(I, I') \in \text{Pair}^\tilde{Y}_X(A)$, the $C^*$-correspondence $X/I$ is topologically free on $q_I(I')$.

**Theorem 7.12.** Let $B := \mathcal{O}(J, X)$ for an ideal $J \subseteq J(X)$. Suppose that either $\text{Prime}(\text{Pair}^\tilde{Y}_X(A))$ is first countable or $B$ is second countable. The formula

$$r(p) := \left( j_0^{-1}(p), j_0^{-1}(p + j^{(1)}(\mathbb{K}(X))) \right)$$

defines a continuous open and surjective map $r : \tilde{B} \twoheadrightarrow \text{Prime}(\text{Pair}^\tilde{Y}_X(A))$. The following conditions are equivalent:

1. $r$ is a homeomorphism;
2. all ideals in $B$ are gauge-invariant;
3. $\tilde{A} := j_0(A) + j^{(1)}(\mathbb{K}(X))$ separates ideals in $\mathcal{O}(J, X)$.

If $A$ is liminal, the above are further equivalent to
4. $X$ is residually topologically free on $J$.

**Proof.** Since gauge-invariant ideals are closed under arbitrary meets, Lemma 7.10 implies that the $C^*$-inclusion $\tilde{A} \subseteq B$ satisfies the assumptions of the second part of Theorem 4.13. This theorem implies the first part of the present assertion. Since $\Gamma^B(\tilde{A}) = \Gamma^B(B)$, we readily see the equivalence between (1) and (3).

Assume now that $A$ is liminal. By [11 Theorem 8.3] (see also [11 Lemma 6.2]), the algebra $\tilde{A}$ detects ideals in $B$ if and only if $X$ is topologically free on $J$. For any
J ∈ I(A) = Γ(B), the quotient B/J is naturally isomorphic to the relative Cuntz–Pimsner algebra O(q_1(P), X_I), see [33] Theorem 7.17. Under this isomorphism, the algebra ˆA is mapped onto its counterpart in O(q_1(P), X_I). Hence Proposition 7.14 implies the equivalence between (3) and (4).

\[ \text{Corollary 7.13.} \] Let B := O(J, X) for an ideal J ⊆ J(X) with J + ker(φ_X) = A. Suppose that \text{Prime}(I_3^2(A)) is first countable or \text{second countable}. Then r(p) := \tilde{J}_0^{-1}(p) defines a continuous, open and surjective map r : \tilde{B} → \text{Prime}(I_3^2(A)), and r is a homeomorphism if and only if A separates ideals in B.

If A is liminal, then r induces a homeomorphism \tilde{B} ≅ \text{Prime}(I_3^2(A)) if and only if for every J-invariant ideal I the graph dual to X_I is topologically free on q_I(J) ≈ J/I.

Proof. If J + ker(φ_X) = A we may replace Pair_3^1(A) by I_3^2(A).

\[ \text{□} \]

7.3. Groupoid actions. Throughout this section, let G be a locally compact groupoid with Haar system. Let G^0 be its object space. We are going to generalise Proposition 4.1 and Theorem 4.3 to actions of G, see [33]. Such an action requires a C_0(G^0)-C*-algebra A. Let A_x for x ∈ G^0 be its fibres. The arrows in G act by isomorphisms α_g : A_{s(g)} (resp. A_{r(g)}) for all g ∈ G, which satisfy the usual algebraic condition α_g α_h = α_{gh} for composable g, h ∈ G and which are continuous in a suitable sense. Namely, if U ⊆ G is a Hausdorff, open subset, then applying α_g for g ∈ G pointwise gives an isomorphism of C_0(U)-C*-algebras s|_U(A) → r|_U(A).

We fix a C_0(G^0)-C*-algebra A with a continuous action α of G. The full crossed product A ×_α G is defined in [45], following [48]. The reduced crossed product A ×_α G is easier to define, using a particular family of “regular” representations of the convolution algebra that defines A ×_α G. The morphism A → M(A ×_α G) is well known to be injective. A crossed product for (A, G, α) is a C*-algebra B with surjections A ×_α G → B → A ×_α G. The following results apply to any crossed product.

Let I ∈ I(A). Let I_x ∈ I(A_x) for x ∈ G^0 be the image of I in the fibre A_x. These ideals determine I uniquely. We call I is α-invariant if α_g(I_x(g)) = I_r(g) for all g ∈ G. The ideal I inherits a C_0(G^0)-C*-algebra structure through the canonical morphism \text{M}(A) → \text{M}(I). Its fibres are canonically isomorphic to the ideals I_x ∈ I(A_x). Being invariant means that the action on A restricts to an action α|_I on I. Moreover, there is an induced action on the quotient A/I. It inherits a C_0(G^0)-C*-algebra structure through the canonical morphism \text{M}(A/I) → \text{M}(A/I). The fibre of A/I at x ∈ G^0 is the quotient A_x/I_x, and α_g for g ∈ G indeed induces an isomorphism α_g : A_{s(g)}/I_{s(g)} (resp. A_{r(g)}/I_{r(g)}) if I is invariant. The C_0(G^0)-C*-algebra structure on A is equivalent to a continuous map p : ˆA → G^0. The isomorphism α_g above induces a homeomorphism α^*_g : p^{-1}(s(g)) → p^{-1}(r(g)). This defines an action of the groupoid G on ˆA. This action is continuous, that is, the maps α^*_g above piece together to a homeomorphism G ×_{s,G^0,p} ˆA → G ×_{r,G^0,p} ˆA, regardless of whether ˆA is Hausdorff or not. An ideal I is G-invariant if and only if the corresponding open subset of A is G-invariant.

\[ \text{Lemma 7.14.} \] Let I ∈ I(A) be invariant. Then I is symmetric and restricted, and the induced ideal i(I) in B is the image in B of the ideal I × G = A × G.

Proof. Due to the surjective map A × G → B, it suffices to prove that I is symmetric for the inclusion A ⊆ M(A × G). By definition, A × G is the C*-completion of a certain convolution *-algebra Ω(G, A) of compactly supported functions G → A. Namely, Ω(G, A) is the linear span of the spaces of compactly supported continuous sections of the C_0(U)-C*-algebra r|_U^*(A) for Hausdorff, open, relatively compact...
subsets $U \subseteq G$. The left multiplication with elements of $A$ is simply pointwise multiplication. Therefore, $I \cdot \mathcal{S}(G, A) = \mathcal{S}(G, I)$. This is a *-subalgebra of $\mathcal{S}(G, A)$ because $I$ is invariant. So $\mathcal{S}(G, I)$ carries its own *-algebra structure, defined by the same formulas. Thus

$$\mathcal{S}(G, A) : I = (I^* \cdot \mathcal{S}(G, A))^* = (I \cdot \mathcal{S}(G, A))^* = \mathcal{S}(G, I).$$

Thus $I$ is symmetric. Moreover, the ideal in $A \rtimes G$ induced by $I$ is the closure of $\mathcal{S}(G, I)$. Passing to a quotient $B = A \rtimes G$, the ideal $i(I)$ becomes the image of $I \rtimes G$ in $B$ because it makes no difference whether we first close $\mathcal{S}(G, I)$ in $A \rtimes G$ and then project to $B$ or the other way around.

We are going to prove that $r(i(I)) = I$. We use the homomorphism $B/i(I) \to (A \rtimes \lambda G)/i(I) \to A/I \rtimes \lambda G$, where we use the canonical map $A \rtimes \lambda G \to A/I \rtimes \lambda G$, which clearly vanishes on $I \rtimes \lambda G$, the image of $I \rtimes G$ in $A \rtimes \lambda G$. The canonical map $A/I \to \mathcal{M}(A/I \rtimes \lambda G)$ is injective. Hence the map $A/I \to \mathcal{M}(B/i(I))$ is injective as well. Since $r(i(I))$ is the kernel of the map $A \to \mathcal{M}(B/i(I))$, this implies $I = r(i(I))$. 

\[ \square \]

**Lemma 7.15.** Joins of invariant ideals are again invariant.

**Proof.** The lattice $\mathbb{I}(A)$ is isomorphic to $\mathbb{O}(\bar{A})$, where joins are simply unions. Here invariant ideals correspond to $G$-invariant open subsets of $\bar{A}$. A union of $G$-invariant subsets of $\bar{A}$ is again $G$-invariant. \[ \square \]

**Proposition 7.16.** Let $\alpha$ be an action of a second countable groupoid $G$ on a separable $C^*$-algebra $A$. Let $\varphi : A \to \mathcal{M}(B)$ be the canonical generalised $C^*$-inclusion of $A$ into a cross crossed algebra $B$ of $\alpha$.

1. $\mathbb{I}^0(A)$ is the subset $\mathbb{I}^0(A)$ of $\alpha$-invariant ideals in $A$.
2. An ideal $J \prec B$ is induced if and only if it is the image of $I \rtimes G$ in $B$ for some $\alpha$-invariant ideal $I \prec A$.

The $C^*$-inclusion $\varphi : A \to \mathcal{M}(B)$ is symmetric, and $[\text{JR}]$ [CI] and $[\text{MI}]$ hold.

**Proof.** In view of Lemmas 5.14 and 7.15 and Corollary 5.15, it suffices to show that every restricted ideal is $\alpha$-invariant. To this end, let $J \in \mathbb{I}(B)$. We will prove that $r(J) \in \mathbb{I}(A)$ is $\alpha$-invariant.

The assumptions imply that $B$ is separable. So there is a faithful representation $B/J \leadsto B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. We use the quotient map $A \rtimes G \to B$ to view it as a representation $\pi : A \rtimes G \to B(\mathcal{H})$. There are canonical morphisms $A \to \mathcal{M}(A \rtimes G)$ and $C^*(G) \to \mathcal{M}(A \rtimes G)$, which give us representations $\pi_A$ and $\pi_G$ of $A$ and $C^*(G)$ from $\pi$. The kernel of the morphism $A \to \mathcal{M}(B/J)$ is the kernel of $\pi_A$ because the extension of a faithful representation to the multiplier algebra remains faithful. We must show that the ideal $\ker \pi_A$ in $A$ is $\alpha$-invariant.

Our assumptions ensure that Renault’s Disintegration Theorem applies to $\pi$, see [13,48]. This gives us the following structure: first, a quasi-invariant measure $\nu$ on $G^0$ and a Borel field of Hilbert spaces $(\mathcal{H}_x)_{x \in G^0}$ over $G^0$ such that $\mathcal{H}$ is isomorphic to the Hilbert space $L^2(G^0, \nu, (\mathcal{H}_x)_{x \in G^0})$ of square-integrable sections of the field $(\mathcal{H}_x)_{x \in G^0}$ with respect to the measure $\nu$; secondly, a Borel representation $U$ of $G$ by unitary operators $U_g : \mathcal{H}_{\pi(g)} \to \mathcal{H}_{\pi(g)}$ for $g \in G$, such that the representation $\pi_G^0$ is obtained by integrating the representation $U$ of $G$; thirdly, a Borel family of representations $\pi_A^g$ of $\mathcal{A}_x$ on $\mathcal{H}_x$ for all $x \in G^0$, which is covariant with respect to the representation $U$, that is, $\pi_A^g(\alpha_g(a)) = U_g \pi_A^g(a) U_g^*$ for all $a \in A$, $g \in G$. The representation $\pi_A$ of $A$ on $\mathcal{H}$ is the pointwise application of $\pi_A^g$. So $\pi_A^g(a) = 0$ for $a \in A$ and only if the set of $x \in G^0$ with $\pi_A^g(a) \neq 0$ is a $\nu$-null set.

We identify $\ker(\pi_A)$ with an open subset $U$ of $\bar{A}$ and $\ker(\pi_A^g)$ with an open subset $U_x$ of $A_x := p^{-1}(x) \subseteq \bar{A}$ for each $x \in G^0$. Since $\pi_A^g$ is covariant with
respect to $U$, the subset $\tilde{U} := \bigcup_{x \in G^0} U_x \subseteq \tilde{A}$ is $G$-invariant. Given $a \in A$, let $V_a := \{ p \in A : a \notin p \}$. Then $\pi^A(a) = 0$ if and only if $V_a \cap \tilde{A} \subseteq U_x$. So $\pi^A(a) = 0$ if and only if $\nu(p(V_a \cap \tilde{A})) \subseteq G^0$ is a $\nu$-null set. The subsets $V_a \subseteq \tilde{A}$ are open. We have $V_a \subseteq U$ if $a \in \ker \pi^A$ and $U = V_a$ if $a \notin \ker(\pi^A)$, say, strictly positive. Thus $U$ is the largest open subset $V$ of $\tilde{A}$ with the property that $\nu(p(V \cap \tilde{A})) = 0$.

Let $\mu$ be the Haar system on $G$ and let $\nu \circ \mu$ be the measure on $G$ that first integrates along $\mu$ and then along $\nu$. Given a subset $T$ of $\tilde{A}$, let $r^a(T)$ and $s^a(T)$ be their pre-images in the arrow space $G \times_{r,G^0,p} \tilde{A}$ of the transformation groupoid $G \ltimes \tilde{A}$ under the range and source maps, respectively. Since the range map of a Fell bundle is open and each fibre of $\mu$ has full support, the largest open subset $V \subseteq G \ltimes \tilde{A}$ such that $\nu \circ \mu((\mathrm{Id} \times_{r,G^0,p} V) \setminus r^a(U)) = 0$ is $r^a(U)$. Since $\nu$ is quasi-invariant, the inversion in $G$ preserves the property of being a $\nu \circ \mu$-null set. Therefore, $r^a(U)$ and $s^a(U)$ are both the largest open subsets in $G \ltimes \tilde{A}$ with the same property. Hence they are equal. And this says that $U$ is $G$-invariant. □

**Theorem 7.17.** Let $\alpha$ be an action of a second countable groupoid $G$ on a separable C*-algebra $A$. Let $\varphi : A \to \mathcal{M}(B)$ be the canonical generalised C*-inclusion of $A$ into a crossed product algebra $B$ of $\alpha$. The quasi-orbit space $\tilde{A}/\sim$ and the quasi-orbit map $\varrho$ for $\varphi$ exist, and

1. $A/\sim$ is a quotient of $\tilde{A}$ by the open equivalence relation where $p_1 \sim p_2$ if and only if the orbit closures $\overline{G \cdot p_i}$ for $i = 1, 2$ are equal.

2. $\varrho : \tilde{B} \to A/\sim$ is given by $p \mapsto \tilde{\pi}^{-1}(r(p))$, where $\tilde{\pi} : A/\sim \to \text{Prime}(\mathbb{P}(A))$ is the homeomorphism induced by the continuous, open and surjective map $\pi : A \to \text{Prime}(\mathbb{P}(A))$, $p \mapsto \bigcap_{g \in \mathcal{L}(p)} \alpha_g^*(p)$.

**Proof.** By Proposition 7.16 we may apply Theorem 6.1 and the quasi-orbit map $\varrho : \tilde{B} \to A/\sim$ exists. Thus it suffices to note that, as in the group case, the map $\pi : \text{Prime}(A) \to \text{Prime}(\mathbb{P}(A))$ maps $p \in \text{Prime}(A)$ to the intersection $\bigcap_{p' \in \mathcal{L}(p) \cap \mathcal{L}(p')} \alpha_g^*(p)$. Hence $p_1 \sim p_2$ if and only if $\overline{G \cdot p_1} = \overline{G \cdot p_2}$. □

It is quite easy to generalise the result above to saturated Fell bundles over locally compact Hausdorff groupoids. The Packer–Raeburn Stabilisation Trick shows that any such Fell bundle is equivariantly Morita equivalent to a groupoid action in the usual sense. This equivariant Morita equivalence preserves all structure that we are interested in, that is, the reduced and full crossed products, the ideal lattices of the C*-algebras involved, and the restriction and induction maps between them (compare Proposition 6.12 about the group case). Thus all our results generalise to this situation. Invariant ideals for Fell bundles over groupoids are described as in the group case: an ideal $I \subseteq \mathcal{L}(A)$ is invariant with respect to a Fell bundle $(A_g)_{g \in G}$ over $G$ if and only if $I_{A_g} \cdot A_g = A_g \cdot I_{A_g}$ for all $g \in G$.

The case of Fell bundles over non-Hausdorff locally compact groupoids is a different matter: the Packer–Raeburn Stabilisation Trick fails in this case, even for rather important Fell bundles, see [10]. We cannot treat this case because the Disintegration Theorem has not yet been shown for Fell bundles over non-Hausdorff groupoids: Muhly and Williams [72] only treat Fell bundles over Hausdorff locally compact groupoids, which is the case when the Packer–Raeburn Stabilisation Trick allows to replace them by ordinary actions. It seems that non-saturated Fell bundles over groupoids have not yet been considered except possibly by Yamagami [58]. Renault’s original proof of the Disintegration Theorem in [48] covers Green twisted actions of non-Hausdorff groupoids on continuous fields of C*-algebras over $G^0$. So the results above also hold in this case.
7.4. Quantum group crossed products. Now we consider crossed products for C*-quantum groups. Here our general results are incomplete because all implications that we proved in our study of group and groupoid crossed products now require some technical assumptions.

Let $(C, \Delta)$ be a C*-quantum group as in [52, 57], that is, it is generated by a manageable multiplicative unitary $W \in U(H \otimes H)$ on some separable Hilbert space $H$. We are going to study restricted and induced ideals for reduced crossed products with $(C, \Delta)$. Throughout this section, a morphism from $A$ to $B$ is a non-degenerate *-homomorphism from $A$ to the multiplier algebra $\mathcal{M}(B)$. These extend uniquely to strictly continuous unital homomorphisms $\mathcal{M}(A) \to \mathcal{M}(B)$ and thus form a category.

A (right) coaction of $C$ on a C*-algebra $A$ (this is also called an action) is a faithful morphism $\alpha: A \to \mathcal{M}(A \otimes C)$ such that $\alpha(A) \cdot (1 \otimes C) = A \otimes C$ and the following coassociativity diagram commutes:

$$
\begin{array}{c}
A \\
\downarrow \alpha \\
\mathcal{M}(A \otimes C) \xrightarrow{\text{Id}_A \otimes \Delta} \mathcal{M}(A \otimes C \otimes C)
\end{array}
$$

Any C*-quantum group has a dual quantum group $(\hat{C}, \hat{\Delta})$. The multiplicative unitary $W$ generates faithful representations of $C$ and $\hat{C}$ on $H$, which we write down as morphisms to $K(H)$.

The reduced crossed product $B := A \rtimes_{\alpha, \lambda} \hat{C}$ is defined as the C*-subalgebra of $\mathcal{M}(A \otimes K(H))$ generated by $\alpha(A) \cdot (1 \otimes \hat{C})$, where we view $A \otimes C$ and $1 \otimes \hat{C}$ as non-degenerate C*-subalgebras of $\mathcal{M}(A \otimes K(H))$. In fact, the product $\alpha(A) \cdot (1 \otimes \hat{C})$ is already a C*-algebra. This follows if

$$\alpha(A) \cdot (1 \otimes \hat{C}) = (1 \otimes \hat{C}) \cdot \alpha(A).$$

(7.18)

It implies that there are canonical morphisms $A \to A \rtimes_{\alpha, \lambda} \hat{C} \leftarrow \hat{C}$. In the generality in which we are working, [7.18] is proved in [11]: the crossed product is an instance of the twisted tensor product $A \boxtimes \hat{C}$ introduced there using right coactions of $C$ and $\hat{C}$ on $A$ and $\hat{C}$. The assumption that $\alpha(A) \cdot (1 \otimes C) = A \otimes C$, not just $\alpha(A) \cdot (1 \otimes \hat{C}) \subseteq A \otimes C$, is crucial for this proof.

Definition 7.19. An ideal $I \triangleleft A$ is called $\alpha$-invariant if $\alpha(I) \cdot (1 \otimes C) \subseteq I \otimes C$. Let $I$ be $\alpha$-invariant. We say that the action $\alpha$ restricts to an action on $I$ if $\alpha(I) \cdot (1 \otimes C) = I \otimes C$. It descends to an action on $A/I$ if the map $\tilde{\alpha}: A/I \to \mathcal{M}(A/I \otimes C)$ induced by $\alpha: A \to \mathcal{M}(A \otimes C)$ is injective.

The conditions written in Definition 7.19 are exactly what is needed to get induced coactions on $I$ and $A/I$, respectively. Namely, $I$ is in the kernel of the composite map

$$A \xrightarrow{\alpha} \mathcal{M}(A \otimes C) \to \mathcal{M}(A/I \otimes C),$$

so that $\tilde{\alpha}$ makes sense. And $\tilde{\alpha}(A/I) \cdot (1 \otimes C) = A/I \otimes C$ and the coassociativity of $\alpha|_I$ and $\tilde{\alpha}$ follow from the corresponding properties of $\alpha$. If $I \triangleleft A$ is an $\alpha$-invariant ideal, then it is, in general, unclear whether $\alpha$ restricts to $I$ or descends to $A/I$. Our general theory only works well when this is the case for all invariant ideals.

Lemma 7.20. Let $I \triangleleft A$ be an invariant ideal.

1. If $\alpha$ restricts to $I$, then $I$ is symmetric and $i(I) = I \rtimes_{\alpha, \lambda} \hat{C}$.
2. If $\alpha$ descends to $A/I$ or if $\hat{C}$ is discrete and $\alpha$ restricts to $I$, then $I$ is restricted.
Proof. Assume that \( \alpha \) restricts to \( I \). Then

\[
\begin{align*}
I \cdot B &= \alpha(I) \cdot \alpha(A) \cdot (1 \otimes \hat{C}) = \alpha(I) \cdot (1 \otimes \hat{C}) = I \ast_{\alpha,\lambda} \hat{C}, \\
B \cdot I &= (1 \otimes \hat{C}) \cdot \alpha(A) \cdot \alpha(I) = (1 \otimes \hat{C}) \cdot \alpha(I) = I \ast_{\alpha,\lambda} \hat{C}.
\end{align*}
\]

Thus \( I \cdot B = I \ast_{\alpha,\lambda} \hat{C} = I \cdot B \), that is, the ideal \( I \) is symmetric and \( i(I) = I \ast_{\alpha,\lambda} \hat{C} \).

If \( \hat{C} \) is discrete, then \( A \subseteq B \). Then symmetric ideals are restricted by Lemma 5.1.

Now assume that \( \alpha \) descends to \( A/I \). Let \( p : B \to A/I \ast_{\hat{\alpha},\lambda} \hat{C} \) be the canonical quotient map and let \( J := \ker p \). We claim that \( r(J) = I \). The inclusion \( I \subseteq r(J) \) follows because \( p|_{r(J)} = 0 \). The canonical morphism \( A/I \to M(A/I \ast_{\hat{\alpha},\lambda} \hat{C}) \) is injective. Hence the kernel of the map \( A \to M(B/J) \) is contained in \( I \). That is, \( r(J) \subseteq I \).

\[ \square \]

Lemma 7.21. If \( C \) is an exact C*-algebra, then restricted ideals are invariant.

Proof. Let \( J \subseteq \mathfrak{I}(B) \). We want to argue as in the group case, using that the coaction on \( A \) extends to an inner coaction on \( B \). The coaction \( \Delta \) on \( C \) is implemented by the multiplicative unitary \( W \), that is, \( W(c \otimes 1)W^* = \Delta(c) \) holds as an operator on \( \mathcal{H} \otimes \mathcal{H} \) for all \( c \in C \). Hence the inner automorphism

\[
\text{Ad}_{1,J} : A \otimes \mathbb{K}(\mathcal{H}) \otimes C \to A \otimes \mathbb{K}(\mathcal{H}) \otimes C
\]

maps \( \alpha(a)_{1,2} \in M(A \otimes C \otimes C) \) to \( (\text{Id}_A \otimes \Delta)(\alpha(a)) = (\alpha \otimes \text{Id}_C)\alpha(a) \) by the coassociativity of \( \alpha \). Since \( W \in \mathcal{UM}(C \otimes C) \subseteq \mathcal{UM}(B \otimes C) \), the map \( \text{Ad}_W \) is an inner automorphism of \( B \otimes C \). Hence we get a morphism

\[
\beta : B \to M(B \otimes C), \quad b \mapsto W(b \otimes 1)W^*.
\]

The computations above show that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & M(B) \\
\downarrow{\beta} & & \downarrow{\beta} \\
M(A \otimes C) & \xrightarrow{\beta} & M(B \otimes C)
\end{array}
\]

commutes. (We do not claim and do not need that \( \beta \) is a coaction as defined above. It is indeed an injective and coassociative map with \( \beta(B) \cdot (1 \otimes C) \subseteq B \otimes C \), but equality here is unclear.) Since ideals are invariant under inner automorphisms, \( \beta(J \otimes C) \subseteq J \otimes C \). Hence we get an induced morphism

\[
\hat{\beta} : B/J \to M(B/J \otimes C), \quad [b] \mapsto W([b] \otimes 1)W^*.
\]

Let \( \varphi' : A \to M(B) \to M(B/J) \) be the canonical map. Its kernel is \( r(J) \), compare \([4.3]\). Let \( D \subseteq M(B/J) \) be its image. So we have a C*-algebra extension \( r(J) \to A \to D \). It induces another extension \( r(J) \otimes C \to A \otimes C \to D \otimes C \) because \( C \) is exact.

We are going to prove \( \alpha(a) \cdot (1 \otimes c) \in r(J) \otimes C \) for all \( a \in r(J), c \in C \). This means that the ideal \( r(J) \) in \( A \) is invariant. By the exact sequence above, it suffices to show that \( \alpha(a) \cdot (1 \otimes c) \) is mapped to \( 0 \) in \( D \otimes C \). The canonical map \( D \otimes C \to M(B/J \otimes C) \) is injective because \( D \subseteq M(B/J) \): this is a known property of the minimal tensor product \( \otimes \). Finally, the commuting diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & M(B) \longrightarrow M(B/J) \\
\downarrow{\beta} & & \downarrow{\hat{\beta}} \\
M(A \otimes C) & \xrightarrow{\beta} & M(B \otimes C) \longrightarrow M(B/J \otimes C)
\end{array}
\]

shows that

\[
(\varphi' \otimes \text{Id}_C)(\alpha(a) \cdot (1 \otimes c)) = \hat{\beta} \circ \varphi'(a) \cdot (1 \otimes c) = 0
\]

because \( a \in r(J) = \ker \varphi' \). \[ \square \]
Theorem 7.22. Let \((C, \Delta_C)\) be a C*-quantum group. Let \((A, \alpha)\) be a C*-algebra with a coaction of \(C\), and let \(B := A \rtimes_{\alpha} \hat{C}\) be the reduced crossed product. Assume Prime\(^B\)(\(A\)) to be first countable, \(C\) to be an exact C*-algebra, and one of the following:

1. the quantum group \(\hat{C}\) is discrete and \(\alpha\) restricts to all invariant ideals;
2. for any invariant ideal \(I \trianglelefteq A\), \(\alpha\) restricts to \(I\) and descends to \(A/I\).

Then an ideal in \(A\) is restricted if and only if it is invariant, and all restricted ideals are symmetric. There is a continuous, open surjection \(\pi: \hat{\Lambda} \to \text{Prime}^B(A)\), mapping \(p \in \hat{\Lambda}\) to the largest invariant ideal contained in \(p\). Define the equivalence relation \(\sim\) by \(p_1 \sim p_2\) if and only if the largest invariant ideals in \(p_1\) and \(p_2\) are equal. Then \(\pi\) induces a homeomorphism \(\hat{\Lambda}/\sim \cong \text{Prime}^B(A)\).

Proof. Since \(C\) is exact by assumption, Lemma 7.21 shows that restricted ideals are invariant. The converse holds by Lemma 7.20 which also shows that invariant ideals are symmetric. Since intersections of invariant ideals are again invariant, condition (JR) holds here. Since all restricted ideals are symmetric, Corollary 5.5 implies condition (C1). Now Theorem 4.4 gives most of the remaining assertions. The description of \(\pi\) is equivalent to (1.2) because an ideal is restricted if and only if it is invariant. \(\square\)

Next we show that induced ideals in the crossed product \(B := A \rtimes_{\alpha} \hat{C}\) are invariant for the dual coaction of \(\hat{C}, \hat{\Delta}\). In the generality of C*-quantum groups generated by manageable multiplicative unitaries, the dual coaction is defined in [11] through the functoriality of \(\otimes\) for covariant homomorphisms. It is the unique left coaction \(\gamma: B \to \hat{C} \otimes B\) with the following property. Let \(a \in A, c \in \hat{C}\). Then \(\gamma(\alpha(a) \cdot (1 \otimes c)) = \alpha(\alpha)_{23} \hat{\Delta}(c)_{13}\) in \(\mathcal{M}(\hat{C} \otimes B) \subseteq \mathcal{M}(\hat{C} \otimes A \otimes \mathbb{K}(H))\), where the subscripts are the leg numbering notation.

Lemma 7.23. Let \(I \in \mathbb{I}(A)\). Then the induced ideal \(i(I)\) in \(B\) is invariant under the dual coaction, and the dual coaction restricts to \(I\).

Proof. In leg numbering notation, we have
\[
i(I) = \hat{C}_2 \cdot \alpha(A) \cdot \alpha(I) \cdot \alpha(A) \cdot \hat{C}_2 = \hat{C}_2 \cdot \alpha(I) \cdot \hat{C}_2.
\]
The dual coaction maps this to \(\hat{\gamma}(i(I)) = \hat{\Delta}(\hat{C})_{13} \cdot \alpha(I)_{23} \cdot \hat{\Delta}(\hat{C})_{13}\). We must prove
\[
\hat{\gamma}(i(I)) \cdot \hat{C}_1 = \hat{C} \otimes B.
\]
Notice that we claim equality here, not just an inclusion. The proof uses that all C*-quantum groups are bisimplifiable, that is,
\[
\hat{\Delta}(\hat{C}) \cdot (\hat{C} \otimes 1) = \hat{C} \otimes \hat{C} = \hat{\Delta}(\hat{C}) \cdot (1 \otimes \hat{C}).
\]
So
\[
\hat{\gamma}(i(I)) \cdot \hat{C}_1 = \hat{\Delta}(\hat{C})_{13} \cdot \alpha(I)_{23} \cdot \hat{\Delta}(\hat{C})_{13} \cdot \hat{C}_1 = \hat{\Delta}(\hat{C})_{13} \cdot \alpha(I)_{23} \cdot \hat{C}_1 \cdot \hat{C}_3
\]
\[
= \hat{\Delta}(\hat{C})_{13} \cdot \hat{C}_1 \cdot \alpha(I)_{23} \cdot \hat{C}_3 = \hat{C}_1 \cdot \hat{C}_3 \cdot \alpha(I)_{23} \cdot \hat{C}_3 = \hat{C}_1 \otimes i(I).
\]
This says that \(i(I)\) is an invariant ideal and that the dual coaction restricts to it. \(\square\)

Example 7.24. Let \(C = C^*_\alpha(G)\) for a locally compact group \(G\) with the usual co-multiplication \(\Delta(\lambda_g) := \lambda_g \otimes \lambda_g\). This is a quantum group. If \(G\) is amenable, then any injective coaction \(\delta: A \to A \otimes C\) of \((C, \Delta)\) satisfies \(\delta(A) \cdot (1 \otimes C) = A \otimes C\) by [32 Proposition 6]. Thus \(\delta\) restricts to any \(\delta\)-invariant ideal \(I \trianglelefteq A\). Moreover, \(\delta\) descends to an action on \(A/I\) by [23 Proposition 3.14.(iii)]. (Note that Ind\(^\oplus\)\((I)\) in the notation of [25] coincides with the kernel of \(A \overset{\delta}{\to} \mathcal{M}(A \otimes C) \to \mathcal{M}(A/I \otimes C)\).)
Then we are in the situation of Theorem 7.22 (2). Its conclusions imply [25, Proposition 4.6].

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