Intersecting random graphs and networks with multiple adjacency constraints: A simple example

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Abstract—When studying networks using random graph models, one is sometimes faced with situations where the notion of adjacency between nodes reflects multiple constraints. Traditional random graph models are insufficient to handle such situations. A simple idea to account for multiple constraints consists in taking the intersection of random graphs. In this paper we initiate the study of random graphs so obtained through a simple example. We examine the intersection of an Erdős-Rényi graph and of one-dimensional geometric random graphs. We investigate the zero-one laws for the property that there are no isolated nodes. When the geometric component is defined on the unit circle, a full zero-one law is established and we determine its critical scaling. When the geometric component lies in the unit interval, there is a gap in that the obtained zero and one laws are found to express deviations from different critical scalings. In particular, the first moment method requires a larger critical scaling than in the unit circle case in order to obtain the one law. This discrepancy is somewhat surprising given that the zero-one laws for the absence of isolated nodes are identical in the geometric random graphs on both the unit interval and unit circle.

Index Terms—Random graphs, zero-one laws, node isolation, wireless ad hoc networks.

I. INTRODUCTION

Graphs provide simple and useful representations for networks with the presence of an edge between a pair of nodes marking their ability to communicate with each other. Thus, for some set $V$ of nodes, an undirected graph $G \equiv (V, E)$ with edge set $E$ is defined such that an edge exists between nodes $i$ and $j$ if and only if these nodes can establish a communication link. This adjacency between nodes in the graph representation may depend on various constraints, both physical and logical. In typical settings, only a single adjacency constraint is considered. Here are some examples.

(i) In wireline networks, an edge between two nodes signifies the existence of a physical point-to-point communication link (e.g., fiber link) between the two nodes;
(ii) Imagine a wireless network serving a set of users distributed over a region $\mathbb{D}$ of the plane. A popular model, known as the disk model, postulates that nodes $i$ and $j$ located at $x_i$ and $x_j$ in $\mathbb{D}$ are able to communicate if $\|x_i - x_j\| \leq r$ where $r$ is the transmission range;
(iii) Eschenauer and Gligor [8] have recently proposed a key pre-distribution scheme for use in wireless sensor networks: Each node is randomly assigned a small set of distinct keys from a large key pool. These keys form the key ring of the node, and are inserted into its memory. Nodes can establish a secure link between them when they have at least one key in common in their key rings.

Sometimes in applications there is a need to account for multiple adjacency constraints to reflect the several criteria that must be satisfied before communication can take place between two users. For instance, consider the situation where the Eschenauer-Gligor scheme is used in a wireless sensor network whose nodes have only a finite transmission range (as is the case in practice). Then, in order for a pair of nodes to establish a secure link, it is not enough that the distance between them does not exceed the transmission range. They must also have at least one key in common.

Such situations can be naturally formalized in the following setting: Suppose we have two adjacency constraints, say as in the example above, modeled by the undirected graphs $G_1 \equiv (V, E_1)$ and $G_2 \equiv (V, E_2)$. The intersection of these graphs is the graph $(V, E)$ with edge set $E$ given by

$$E := E_1 \cap E_2,$$

and we write $G_1 \cap G_2 := (V, E_1 \cap E_2)$. Through the intersection graph $G_1 \cap G_2$, we are able to simultaneously capture two different adjacency constraints. Of course the same approach can be extended to an arbitrary number of constraints, but in the interest of concreteness we shall restrict the discussion to the case of two constraints.

In an increasing number of contexts, random graph models have been found to be more appropriate. For instance, in wireless networking several classes of random graphs have been proposed to model the effects of geometry, mobility and user interference on the wireless communication link, e.g., geometric random graphs (also known as disk models) [11], [13], [17] and signal-to-interference-plus-noise-ratio (SINR) graphs [5], [6]. See Sections II-A and II-B for a description of the two classes of random graphs considered here.

When random graphs are used, we can also define their intersection in an obvious manner: Given two random graphs
with vertex set $V$, say $G_1 \equiv (V, E_1)$ and $G_2 \equiv (V, E_2)$, their intersection is the random graph $(V, E)$ where
\[ E := E_1 \cap E_2. \]

For simplicity assume the random graphs $G_1$ and $G_2$ to be independent. A natural question to ask is the following: How are the structural properties of the random graph $G_1 \cap G_2$ shaped by those of the random graphs $G_1$ and $G_2$? Here we are particularly interested in zero-one laws for certain graph properties – More on that later.

Intersecting graphs represents a modular approach to building more complex models. It could be argued that this approach is of interest only if the known structural properties for the component random graphs can be leveraged to gain a better understanding of the resulting intersection graphs. As we shall see shortly through the simple example developed here, successfully completing this program is not as straightforward as might have been expected.

In this paper we consider exclusively the random graph obtained by intersecting Erdős-Rényi graphs with certain geometric random graphs in one dimension. We were motivated to consider this simple model for the following reasons:

(i) The disk model popularized by the work of Gupta and Kumar [11] assumes simplified pathloss, no user interference and no fading, and the transmission range is a proxy for transmit power to be used by the users. One crude way to include fading is to think of it as link outage. Thus an edge is present between a pair of nodes if and only if they are within communication range (so that there is a communication link in the sense of the usual disk model) and that link between them is indeed active (i.e., not in outage). This simple model is simply obtained by taking the intersection of the disk model with an Erdős-Rényi graph.

(ii) Both Erdős-Rényi graphs and geometric random graphs are well understood classes of random graphs with an extensive literature devoted to them; see the monographs [2], [15], [17] for Erdős-Rényi graphs and the text [17] for geometric random graphs. Additional information concerning one-dimensional graphs can be found in the references [9] [10] [12] [14] [16]. It is hoped that this wealth of results will prove helpful in successfully carrying out the program outlined earlier.

(iii) Furthermore, this simple model is a trial balloon for the study of more complicated situations. In particular, we have in mind the study of wireless sensor networks employing the Eschenauer-Gligor scheme to establish secure links. In that case the resulting random intersection graphs, the so-called random key graphs under partial visibility [3] [4] [18], share some similarity with the models discussed here, but have far greater complexity due to lack of independence in the link assignments in the non-geometric component; see comments in Section XII. Our ability to successfully complete the study of the models considered in this paper would provide some measure of comfort that the more complicated cases are indeed amenable to analysis, with pointers to possible results.

We would like to draw attention to a similar problem which has been studied recently. In [19], the authors consider a geometric random graph where the nodes become inactive independently with a certain probability. In contrast, we are interested in the situation where the edges in the geometric random graph can become inactive.

In the context of our simple model we investigate the zero-one laws for the property that there are no isolated nodes; particular emphasis is put on identifying the corresponding critical scalings. This is done with the help of the method of first and second moments. Even this simple and well-structured situation gives rise to some surprising results: When the geometric component is defined on the unit circle, a full zero-one law is established and we determine its critical scaling. When the geometric component lies in the unit interval, there is a gap in the results in that the obtained zero and one laws are found to express deviations from different critical scalings. In particular, we encounter a situation where the first moment method requires a larger critical scaling than in the unit circle case in order to obtain the one law. This discrepancy is somewhat surprising given that the zero-one laws for the absence of isolated nodes are identical in the geometric random graphs on both the unit interval and unit circle. Thus one is led to the (perhaps naive) expectation that the boundary effects of the geometric component play no role in shaping the zero-one laws in the random intersection graphs. Therefore, it appears that this discrepancy between the zero and one laws is an artifact of the method of first moment, and a different approach is needed to bridge this gap.

The analysis given here provides some insight into classical results. This is done by developing a new interpretation of the critical scalings (for the absence of isolated nodes) in terms of the probability of an edge existing between a pair of nodes. This interpretation seems to hold quite generally. In fact, it is this observation which enabled us to guess the form of the zero-one law for the random intersection graphs and may find use in similar problems.

It is natural to wonder here what form take the zero-one laws for the property of graph connectivity. We remark that this is now a more delicate problem for contrary to what occurs with one-dimensional graphs [9] [10] [12] [14] [16], the total ordering of the line cannot be used to advantage, and new approaches are needed. But not all is lost: In some sense the property that there are no isolated nodes can be viewed as a “first-order approximation” to the property of graph connectivity – This is borne out by the fact that for many classes of random graphs these two properties are asymptotically equivalent under the appropriate scaling; see the monographs [2] and [17] for Erdős-Rényi graphs and geometric random graphs, respectively. In that sense the preliminary results obtained here constitute a first step on the road to establish zero-one laws for the property of graph connectivity.

A word on the notation and conventions in use: Throughout $n$ will denote the number of nodes in the random graph and all limiting statements, including asymptotic equivalences, are understood with $n$ going to infinity. The random variables (rvs) under consideration are all defined on the same probability triple $(\Omega, \mathcal{F}, P)$. Probabilistic statements are made with respect to this probability measure $P$, and we denote the corresponding expectation operator by $E$. Also, we use the notation $\rightarrow_{\text{st}}$ to indicate distributional equality. The indicator function of an
event $E$ is denoted by $\mathbb{1} [E]$. 

II. Model and Assumptions

In this paper we are only concerned with undirected graphs. As usual, a graph $G \equiv (V, E)$ is said to be connected if every pair of nodes in $V$ can be linked by at least one path over the edges (in $E$) of the graph. We say a node is isolated if no edge exists between the node and any of the remaining nodes. Also, let $\mathcal{E}(G)$ refer to the set of edges of $G$, namely $\mathcal{E}(G) = E$. We begin by recalling the classical random graph models used in the definition of the model analyzed here.

A. The geometric random graphs

Two related geometric random graphs are introduced. Fix $n = 2, 3, \ldots$ and $r > 0$, and consider a collection $X_1, \ldots, X_n$ of i.i.d. rvs which are distributed uniformly in the interval $[0, 1]$ (referred to as the unit interval). We think of $r$ as the transmission range and $X_1, \ldots, X_n$ as the locations of $n$ nodes (or users), labelled $1, \ldots, n$, in the interval $[0, 1]$. 

Nodes $i$ and $j$ are said to be adjacent if $|X_i - X_j| \leq r$, in which case an undirected edge exists between them. The indicator r.v $\chi_{ij}^{(L)}(r)$ that nodes $i$ and $j$ are adjacent is given by

$$\chi_{ij}^{(L)}(r) := \mathbb{1} \left[ |X_i - X_j| \leq r \right].$$

This notion of edge connectivity gives rise to an undirected geometric random graph on the unit interval, thereafter denoted $\mathbb{G}^{(L)}(n; r)$. The number of isolated nodes in $\mathbb{G}^{(L)}(n; r)$ is then given by

$$I_n^{(L)}(r) := \sum_{i=1}^{n} \chi_{i,i}^{(L)}(r).$$

We also consider the geometric random graph obtained by locating the nodes uniformly on the circle with unit circumference (thereafter referred to as the unit circle) – This corresponds to identifying the end points of the unit interval. In this formulation, we fix some reference point on the circle and the node locations $X_1, \ldots, X_n$ are given by the length of the clockwise arc with respect to this reference point. We measure the distance between any two nodes by the length of the smallest arc between the nodes, i.e., the distance between nodes $i$ and $j$ is given by

$$||X_i - X_j|| := \min(|X_i - X_j|, 1 - |X_i - X_j|).$$

As we still think of $r$ as the transmission range, nodes $i$ and $j$ are now said to be adjacent if $||X_i - X_j|| \leq r$. The indicator r.v $\chi_{ij}^{(C)}(r)$ that nodes $i$ and $j$ are adjacent is given by

$$\chi_{ij}^{(C)}(r) := \mathbb{1} \left[ ||X_i - X_j|| \leq r \right].$$

This notion of adjacency leads to an undirected geometric random graph on the unit circle, thereafter denoted $\mathbb{G}^{(C)}(n; r)$. This model is simpler to analyze as the boundary effects have been removed.

The indicator r.v $\chi_{n,i}^{(C)}(r)$ that node $i$ is an isolated node in $\mathbb{G}^{(C)}(n; r)$ is again defined by

$$\chi_{n,i}^{(C)}(r) := \prod_{j=1, j \neq i}^{n} \left( 1 - \chi_{ij}^{(C)}(r) \right).$$

The number of isolated nodes in $\mathbb{G}^{(C)}(n; r)$ is then given by

$$I_n^{(C)}(r) := \sum_{i=1}^{n} \chi_{n,i}^{(C)}(r).$$

Throughout, it will be convenient to view the graphs $\mathbb{G}^{(L)}(n; r)$ and $\mathbb{G}^{(C)}(n; r)$ as coupled in that they are constructed from the same rvs $X_1, \ldots, X_n$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Note that the two models differ only in the manner in which the distance between two users is defined. To take advantage of this observation, we shall write

$$d(x, y) := \begin{cases} |x - y| & \text{on the unit interval} \\ ||x - y|| & \text{on the unit circle} \end{cases}$$

for all $x, y \in [0, 1]$ as a compact way to capture the appropriate notion of “distance”. Also, in the same spirit, as a way to lighten the notation, we omit the superscripts $(L)$ and $(C)$ from the notation when the discussion applies equally well to both cases.

B. The Erdős-Rényi graphs

Fix $n = 2, 3, \ldots$ and $p \in [0, 1]$. In this case, $p$ corresponds to the probability that an (undirected) edge exists between any pair of nodes. We start with rvs $\{B_{ij}(p)\}$, $1 \leq i < j \leq n$, which are i.i.d. $\{0, 1\}$-valued rvs with success probability $p$. Nodes $i$ and $j$ are said to be adjacent if $B_{ij}(p) = 1$. This notion of edge connectivity defines the undirected Erdős-Rényi (ER) random graph, thereafter denoted $\mathbb{G}(n; p)$.

For each $i = 1, \ldots, n$, node $i$ is isolated in $\mathbb{G}(n; p)$ if $B_{ij}(p) = 0$ for $i < j \leq n$ and $B_{ji}(p) = 0$ for $1 \leq j < i$. The indicator $\chi_{n,i}(p)$ that node $i$ is an isolated node in $\mathbb{G}(n; p)$ is then given by

$$\chi_{n,i}(p) := \prod_{i<j \leq n} \left( 1 - B_{ij}(p) \right) \cdot \prod_{1 \leq j < i} \left( 1 - B_{ji}(p) \right).$$

The number of isolated nodes in $\mathbb{G}(n; p)$ is the rv $I_n(p)$ given by

$$I_n(p) := \sum_{i=1}^{n} \chi_{n,i}(p).$$

C. Intersecting the geometric and Erdős-Rényi graphs

The random graph model studied in this paper is parametrized by the number $n$ of nodes, the transmission range $r > 0$ and the probability $p$ ($0 \leq p \leq 1$) that a link is active (i.e., not in outage). To lighten the notation we often group the parameters $r$ and $p$ into the ordered pair $\theta \equiv (r, p)$. 

...
Throughout we always assume that the collections of rvs
\(\{X_i, i = 1, \ldots, n\}\) and \(\{B_{ij}(p), 1 \leq i < j \leq n\}\) are mutually independent. With the convention introduced earlier, the intersection of the two graphs \(G(n; r)\) and \(G(n; p)\) is the graph
\[
G(n; \theta) := G(n; r) \cap G(n; p)
\]
defined on the vertex set \(\{1, \ldots, n\}\) with edge set given by
\[
\mathcal{E}(G(n; \theta)) = \mathcal{E}(G(n; r)) \cap \mathcal{E}(G(n; p)).
\]
We refer to \(G(n; \theta)\) as the intersection graph on the unit interval (resp. unit circle) when in this definition, \(G(n; r)\) is taken to be \(G^{(k)}(n; r)\) (resp. \(G^{(C)}(n; r)\)).

The nodes \(i\) and \(j\) are adjacent in \(G(n; \theta)\) if and only if they are adjacent in both \(G(n; r)\) and \(G(n; p)\). The indicator rv \(\chi_{ij}(\theta)\) that nodes \(i\) and \(j\) are adjacent in \(G(n; \theta)\) is given by
\[
\chi_{ij}(\theta) = \begin{cases} 
\chi_{ij}(r)B_{ij}(p) & \text{if } i < j \\
\chi_{ij}(r)B_{ji}(p) & \text{if } j < i.
\end{cases}
\]

For each \(i = 1, \ldots, n\), node \(i\) is isolated in \(G(n; \theta)\) if either it is not within transmission range from each of the \((n - 1)\) remaining nodes, or being within range from some nodes, the corresponding links all are inactive. The indicator rv \(\chi_{n,i}(\theta)\) that node \(i\) is an isolated node in \(G(n; \theta)\) can be expressed as
\[
\chi_{n,i}(\theta) := \prod_{i=1, j \neq i}^{n} (1 - \chi_{ij}(\theta)).
\]

As expected, the number of isolated nodes in \(G(n; \theta)\) is similarly defined as
\[
I_n(\theta) := \sum_{i=1}^{n} \chi_{n,i}(\theta).
\]

### D. Scalings

Some terminology: A scaling for either of the geometric graphs is a mapping \(r : \mathbb{N}_0 \to \mathbb{R}_+\), while a scaling for ER graphs is simply a mapping \(p : \mathbb{N}_0 \to [0, 1]\). A scaling for the intersection graph combines scalings for each of the component graphs, and is defined as a mapping \(\theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1]\).

The main objective of this paper can be stated as follows: Given that
\[
\mathbb{P}[G(n; \theta_n)\text{ has no isolated nodes}] = \mathbb{P}[I_n(\theta_n) = 0]
\]
for all \(n = 2, 3, \ldots\), what conditions are needed on the scaling \(\theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1]\) to ensure that
\[
\lim_{n \to \infty} \mathbb{P}[I_n(\theta_n) = 0] = 1 \quad \text{(resp. } 0).\]

In the literature such results are known as zero-one laws. Interest in them stems from their ability to capture the threshold behavior of the underlying random graphs.

### III. Classical Results

#### A. Erdős-Rényi graphs

There is no loss of generality in writing a scaling \(p : \mathbb{N}_0 \to [0, 1]\) in the form
\[
p_n = \frac{\log n + \alpha_n}{n}, \quad n = 1, 2, \ldots
\]
for some deviation function \(\alpha : \mathbb{N}_0 \to \mathbb{R}\). The following result is well known [2], [15].

**Theorem 3.1:** For any scaling \(p : \mathbb{N}_0 \to [0, 1]\) in the form (1), we have the zero-one law
\[
\lim_{n \to \infty} \mathbb{P}[I_n(p_n) = 0] = \begin{cases} 
0 & \text{if } \lim_{n \to \infty} \alpha_n = -\infty \\
1 & \text{if } \lim_{n \to \infty} \alpha_n = +\infty
\end{cases}
\]
where the deviation function \(\alpha : \mathbb{N}_0 \to \mathbb{R}\) is determined through (1).

This result identifies the scaling \(p^* : \mathbb{N}_0 \to [0, 1]\) given by
\[
p_n^* = \frac{\log n}{n}, \quad n = 1, 2, \ldots
\]
as the critical scaling for the absence of isolated nodes in ER graphs.

#### B. Geometric random graphs

Any scaling \(r : \mathbb{N}_0 \to \mathbb{R}_+\) can be written in the form
\[
r_n = \frac{\log n + \alpha_n}{2n}, \quad n = 1, 2, \ldots
\]
for some deviation function \(\alpha : \mathbb{N}_0 \to \mathbb{R}\). The following result can be found in [1], [17].

**Theorem 3.2:** For any scaling \(r : \mathbb{N}_0 \to \mathbb{R}_+\) written in the form (2) for some deviation function \(\alpha : \mathbb{N}_0 \to \mathbb{R}\), we have the zero-one law
\[
\lim_{n \to \infty} \mathbb{P}[I_n(r_n) = 0] = \begin{cases} 
0 & \text{if } \lim_{n \to \infty} \alpha_n = -\infty \\
1 & \text{if } \lim_{n \to \infty} \alpha_n = +\infty
\end{cases}
\]

Theorem 3.2 identifies the scaling \(r^* : \mathbb{N}_0 \to [0, 1]\) given by
\[
r_n^* = \frac{\log n}{2n}, \quad n = 1, 2, \ldots
\]
as the critical scaling for the absence of isolated nodes in geometric random graphs.

For reasons that will become apparent shortly, we now develop an equivalent version of Theorem 3.2 that bears a striking resemblance with the zero-one law of Theorem 3.1 for ER graphs.

To that end, define
\[
\ell(r) := \min(1, 2r), \quad r \geq 0.
\]

Intuitively, \(\ell(r)\) is akin to the probability that an edge exists between any pair of nodes in \(G(n; r)\) — In fact it has exactly that meaning for \(G^{(C)}(n; r)\) while it is true approximately (when boundary conditions are ignored) for \(G^{(L)}(n; r)\).

With this in mind, for any scaling \(r : \mathbb{N}_0 \to \mathbb{R}_+\) write
\[
\ell(r_n) = \frac{\log n + \beta_n}{n}, \quad n = 1, 2, \ldots
\]
for some deviation function $\beta : \mathbb{N}_0 \to \mathbb{R}$. The representations (2) and (4) together require

$$\beta_n = \min(\alpha_n, n - \log n), \quad n = 1, 2, \ldots$$

It is easily verified that $\lim_{n \to \infty} \beta_n = -\infty$ (resp. $\lim_{n \to \infty} \beta_n = +\infty$) if and only if $\lim_{n \to \infty} \alpha_n = -\infty$ (resp. $\lim_{n \to \infty} \alpha_n = +\infty$). This implies the following equivalent rephrasing of Theorem 3.2.

**Theorem 3.3:** For any scaling $r : \mathbb{N}_0 \to \mathbb{R}_+$ written in the form (4) for some deviation function $\beta : \mathbb{N}_0 \to \mathbb{R}$, we have the zero-one law

$$\lim_{n \to \infty} \mathbb{P}[I_n(r_n) = 0] = \begin{cases} 0 & \text{if } \lim_{n \to \infty} \beta_n = -\infty \\ 1 & \text{if } \lim_{n \to \infty} \beta_n = +\infty. \end{cases}$$

By Theorem 3.3 any scaling $r^* : \mathbb{N}_0 \to \mathbb{R}_+$ such that

$$\ell(r^*_n) = \frac{\log n}{n}, \quad n = 1, 2, \ldots \quad (5)$$

is a critical scaling for the absence of isolated nodes in geometric random graphs. As expected, it is easy to see that any scaling is critical under the definition (5) if and only if it is under the definition (3).

**IV. THE BASIC DIFFICULTY**

**A. Intersecting Erdős-Rényi graphs**

As a détour consider intersecting two independent ER graphs. This results in another ER graph, i.e.,

$$\mathcal{G}(n; p) \cap \mathcal{G}(n; p') =_{st} \mathcal{G}(n; pp'), \quad 0 \leq p, p' \leq 1.$$  

It is therefore a simple matter to select scalings $p, p' : \mathbb{N}_0 \to [0, 1]$ such that the intersection graph $\mathcal{G}(n; p) \cap \mathcal{G}(n; p')$ exhibits a zero-one law for the absence of isolated nodes. By Theorem 3.1, it suffices to take these scalings such that

$$p_n p'_n = \frac{\log n + \alpha_n}{n}, \quad n = 1, 2, \ldots$$

for some appropriate deviation function $\alpha : \mathbb{N}_0 \to \mathbb{R}$.

Despite its simplicity, this result has some interesting implications: For instance, select the two scalings such that

$$p_n p'_n = \frac{\log n}{2 n}, \quad n = 1, 2, \ldots$$

with

$$p_n = p'_n = \sqrt{\frac{\log n}{2}} n, \quad n = 1, 2, \ldots$$

In that case, upon writing

$$\frac{\log n}{2 n} = \frac{\log n + \left(-\frac{1}{2} \log n\right)}{n}, \quad n = 1, 2, \ldots,$$

we conclude

$$\lim_{n \to \infty} \mathbb{P}[(\mathcal{G}(n; p_n) \cap \mathcal{G}(n; p'_n) \text{ has no isolated nodes}) = 0$$

by the zero law of Theorem 3.1. Yet, we also have

$$\lim_{n \to \infty} \mathbb{P}[(\mathcal{G}(n; p_n) \text{ has no isolated nodes}) = 1$$

and

$$\lim_{n \to \infty} \mathbb{P}[(\mathcal{G}(n; p'_n) \text{ has no isolated nodes}) = 1$$

by the one law of Theorem 3.1 as we note that

$$\frac{\sqrt{\log n}}{\sqrt{\frac{\log n}{2}}} = \frac{\log n + \alpha_n}{n}$$

for all $n = 1, 2, \ldots$ with the choice

$$\alpha_n := \sqrt{n \log n - \log n}.$$

Thus, even when the individual graphs $\mathcal{G}(n; p_n)$ and $\mathcal{G}(n; p'_n)$ contain no isolated nodes with a probability close to one, it is possible for the intersection graph $\mathcal{G}(n; p_n) \cap \mathcal{G}(n; p'_n)$ to contain isolated nodes with a probability very close to one. The reason for this is quite simple: A node that is isolated in $\mathcal{G}(n; p_n)$ may not be isolated in either of the component graphs $\mathcal{G}(n; p_n)$ and $\mathcal{G}(n; p'_n)$. For ER graphs, the answer, although very simple, fails to give much insight into how the individual graphs interact with each other and how this affects the overall behavior of the intersection graph.

**B. Intersecting an Erdős-Rényi graph with a geometric random graph**

With this in mind, note that with $0 < r < 1$ for the unit interval (resp. $0 < r < 0.5$ for the unit circle) and $0 < p < 1$, the intersection graph $\mathcal{G}(n; r) \cap \mathcal{G}(n; p)$ is not stochastically equivalent to either a geometric random graph or an Erdős-Rényi graph, i.e., it is not possible to find parameters $r' = r'(n; r, p)$ and $p' = p'(n; r, p)$ in $\mathbb{R}_+$ and $[0, 1]$, respectively, such that

$$\mathcal{G}(n; r) \cap \mathcal{G}(n; p) =_{st} \mathcal{G}(n; r')$$

and

$$\mathcal{G}(n; r) \cap \mathcal{G}(n; p) =_{st} \mathcal{G}(n; p').$$

Consequently, results for either ER or geometric random graphs (as given in Section III) cannot be used in a straightforward manner to determine the zero-one laws for the intersection graphs.

On the other hand, it is obvious that if either $\mathcal{G}(n; r)$ or $\mathcal{G}(n; p)$ contains isolated nodes, then $\mathcal{G}(n; r) \cap \mathcal{G}(n; p)$ must contain isolated nodes. Therefore, a zero law for the intersection graph should follow by combining the zero laws for the ER and geometric random graphs. However, as will become apparent from our main results, such arguments are too loose to provide the best possible zero law.

A direct approach is therefore required with the difficulty mentioned earlier remaining, namely that a node isolated in $\mathcal{G}(n; r) \cap \mathcal{G}(n; p)$ may not be isolated in either $\mathcal{G}(n; r)$ or $\mathcal{G}(n; p)$. Nevertheless the corresponding zero-one laws do provide a basis for guessing the form of the zero-one law for the intersection graphs. This is taken on in the next section.

**V. THE MAIN RESULTS**

**A. Guessing the form of the results**

Upon comparing the zero-one laws of Theorems 3.1 and 3.3, the following shared structure suggests itself: For the random graphs of interest here (as well as for others, e.g., random key graphs [18]), it is possible to identify a quantity which gives

$$\mathbb{P}[\text{Edge exists between two nodes}],$$
either exactly (e.g., \( p \) in ER graphs or \( \ell(r) \) in geometric random graphs on the circle) or approximately (e.g., \( \ell(r) \) in geometric random graphs on the interval). Critical scalings for the absence of isolated nodes are then determined through the requirement
\[
P[\text{Edge exists between two nodes}] = \frac{\log n}{n}.
\tag{6}
\]
In particular the zero-one law requires scalings satisfying
\[
P[\text{Edge exists between two nodes}]
\]
“much smaller” than \( \frac{\log n}{n} \),
while the one-one law deals with scalings satisfying
\[
P[\text{Edge exists between two nodes}]
\]
“much larger” than \( \frac{\log n}{n} \).
The exact technical meaning of “much smaller” and “much larger” forms the content of results such as Theorems 3.1 and 3.3.

With this in mind, for the random intersection graphs studied here it is natural to take
\[
P[\text{Edge exists between two nodes}] := p\ell(r)
\tag{7}
\]
under the enforced independence assumptions. We expect that a critical scaling \( \theta^* : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \) for the random intersection graphs should be determined by
\[
p_n\ell(r_n^*) = \frac{\log n + \alpha_n}{n}, \quad n = 1, 2, \ldots
\tag{8}
\]
The exact form taken by the results is discussed in Sections V-B and V-C. We start with the model on the circle for which we have obtained the most complete results.

B. Intersection graphs on the unit circle

With a scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \), we associate the sequence \( \alpha : \mathbb{N}_0 \to \mathbb{R} \) through
\[
p_n\ell(r_n) = \frac{\log n + \alpha_n}{n}, \quad n = 1, 2, \ldots
\tag{8}
\]
In the case of the intersection graph on the circle we get a full zero-one law.

**Theorem 5.1 (Unit circle):** For any scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \), we have the zero-one law
\[
\lim_{n \to \infty} \Pr\left[I_n^{(C)}(\theta_n) = 0\right] = \begin{cases} 0 & \text{if } \lim_{n \to \infty} \alpha_n = -\infty \\ 1 & \text{if } \lim_{n \to \infty} \alpha_n = +\infty \end{cases}
\]
where the sequence \( \alpha : \mathbb{N}_0 \to \mathbb{R} \) is determined through (8).

C. Intersection graphs on the unit interval

With a scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \), we also associate the sequence \( \alpha' : \mathbb{N}_0 \to \mathbb{R}_+ \) through
\[
p_n\ell(r_n) = \frac{2(\log n - \log \log n) + \alpha'}{n}, \quad n = 1, 2, \ldots
\tag{9}
\]
For the intersection graph on the unit interval there is a gap between the zero and one laws.

**Theorem 5.2 (Unit interval):** For any scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \), we have the zero-one law
\[
\lim_{n \to \infty} \Pr\left[I_n^{(L)}(\theta_n) = 0\right] = \begin{cases} 0 & \text{if } \lim_{n \to \infty} \alpha_n = -\infty \\ 1 & \text{if } \lim_{n \to \infty} \alpha'_n = +\infty \end{cases}
\]
where the sequences \( \alpha, \alpha' : \mathbb{N}_0 \to \mathbb{R} \) are determined through (8) and (9), respectively.

An elementary coupling argument shows that for any particular realization of the rvs \( \{X_i, \quad i = 1, \ldots, n\} \) and \( \{B_{ij}(p), \quad 1 \leq i < j \leq n\} \), the graph on the circle contains more edges than the graph on the interval. As a result, the zero law for the unit circle automatically implies the zero law for the unit interval, and only the former needs to be established.

VI. METHOD OF FIRST AND SECOND MOMENTS

The proofs rely on the method of first and second moments [15, p. 55], an approach widely used in the theory of Erdős–Rényi graphs: Let \( Z \) denote an \( \mathbb{N} \)-valued rv with finite second moment. The method of first moment [15, Eqn. (3.10), p. 55] relies on the inequality
\[
1 - \mathbb{E}[Z] \leq \Pr[Z = 0],
\tag{10}
\]
while the method of second moment [15, Remark 3.1, p. 55] uses the bound
\[
\Pr[Z = 0] \leq 1 - \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}.
\tag{11}
\]
Now, pick a scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \). From (10) we see that the one law
\[
\lim_{n \to \infty} \Pr[I_n(\theta_n) = 0] = 1
\]
is established if we show that
\[
\lim_{n \to \infty} \mathbb{E}[I_n(\theta_n)] = 0.
\tag{12}
\]
On the other hand, it is plain from (11) that
\[
\lim_{n \to \infty} \Pr[I_n(\theta_n) = 0] = 0
\]
if
\[
\lim \inf_{n \to \infty} \left( \frac{\mathbb{E}[I_n(\theta_n)]^2}{\mathbb{E}[I_n(\theta_n)^2]} \right) \geq 1.
\tag{13}
\]
Upon using the exchangeability and the binary nature of the rvs involved in the count variables of interest, we can obtain simpler characterizations of the convergence statements (12) and (13). Indeed, for all \( n = 2, 3, \ldots \) and every \( \theta \) in \( \mathbb{R}_+ \times [0, 1] \), the calculations
\[
\mathbb{E}[I_n(\theta)] = \sum_{i=1}^{n} \mathbb{E}[\chi_{n,i}(\theta)] = n\mathbb{E}[\chi_{n,1}(\theta)]
\]
and
\[
\mathbb{E}[I_n(\theta)^2] = \mathbb{E}\left[\left(\sum_{i=1}^{n} \chi_{n,i}(\theta)\right)^2\right] = \sum_{i=1}^{n} \mathbb{E}[\chi_{n,i}(\theta)] + \sum_{i=1, i \neq j}^{n} \mathbb{E}[\chi_{n,i}(\theta)\chi_{n,j}(\theta)] = n\mathbb{E}[\chi_{n,1}(\theta)] + (n-1)\mathbb{E}[\chi_{n,1}(\theta)\chi_{n,2}(\theta)]
\]
are straightforward, so that
\[
\frac{\mathbb{E} \left[ I_n(\theta)^2 \right]}{\mathbb{E} \left[ I_n(\theta) \right]^2} = \frac{1}{n \mathbb{E} \left[ \chi_{n,1}(\theta) \right]} + \frac{n-1}{n} \cdot \frac{\mathbb{E} \left[ \chi_{n,1}(\theta) \chi_{n,2}(\theta) \right]}{\mathbb{E} \left[ \chi_{n,1}(\theta) \right]^2}.
\]

Thus, for the given scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \), we obtain the one law by showing that
\[
\lim_{n \to \infty} n \mathbb{E} \left[ \chi_{n,1}(\theta_n) \right] = 0,
\]
while the zero law will follow if we show that
\[
\lim_{n \to \infty} n \mathbb{E} \left[ \chi_{n,1}(\theta_n) \right] = \infty
\]
and
\[
\lim_{n \to \infty} \sup \left( \frac{\mathbb{E} \left[ \chi_{n,1}(\theta_n) \chi_{n,2}(\theta_n) \right]}{\mathbb{E} \left[ \chi_{n,1}(\theta_n) \right]^2} \right) \leq 1.
\]

The bulk of the technical discussion therefore amounts to establishing (14), (15) and (16) under the appropriate conditions on the scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \).

To that end, in the next two sections we derive expressions for the quantities entering (14), (15) and (16). Throughout we denote by \( X, Y \) and \( Z \) three mutually independent \( \{0, 1\} \)-valued rv's which are uniformly distributed on \([0, 1]\), and by \( B, B' \) and \( B'' \) three mutually independent \( \{0, 1\} \)-valued rv's with success probability \( p \). The two groups of rv's are assumed to be independent.

**VII. First moments**

Fix \( n = 2, 3, \ldots \) and \( \theta \) in \( \mathbb{R}_+ \times [0, 1] \). For both the unit circle and unit interval, the enforced independence assumptions readily imply
\[
\mathbb{E} \left[ \chi_{n,1}(\theta) \right] = \mathbb{E} \left[ \prod_{i=1, j \neq i}^n (1 - \chi_{ij}(\theta)) \right]
= \mathbb{E} \left[ (1 - p\alpha(X; r))^n \right]
= \int_0^1 (1 - p\alpha(x; r))^n \, dx
\]
where we have set
\[
\alpha(x; r) := \mathbb{P} \left[ d(x, Y) \leq r \right], \quad 0 \leq x \leq 1, \quad r > 0.
\]

Closed-form expressions for (18) depend on the geometric random graph being considered.

**A. The unit circle**

As there are no border effects, we get
\[
\alpha^{(C)}(x; r) = \ell(r), \quad 0 \leq x \leq 1, \quad r > 0
\]
and with the help of (17) this yields
\[
\mathbb{E} \left[ \chi_{n,1}^{(C)}(\theta) \right] = (1 - p\ell(r))^n, \quad r > 0, \quad p \in [0, 1].
\]

**B. The unit interval**

For \( r \geq 1 \), it is plain that
\[
a^{(I)}(x; r) = 1, \quad 0 \leq x \leq 1.
\]

On the other hand, when \( 0 < r < 1 \), elementary calculations show that
\[
a^{(I)}(x; r) = \begin{cases} 
  x + r & \text{if } 0 < r \leq 0.5, 0 \leq x \leq r \\
  \ell(r) & \text{if } 0 < r < 0.5, 0 \leq x \leq 1 - r \\
  1 - x + r & \text{if } 0 < r \leq 0.5, 1 - r \leq x \leq 1 \\
  1 - x & \text{if } 0 < r < 0.5, 1 - r \leq x \leq 1.
\end{cases}
\]

Reporting this information into (17), we obtain the following expressions in a straightforward manner:

(i) For \( 0 < r \leq 0.5 \) and \( 0 < p \leq 1 \),
\[
\mathbb{E} \left[ \chi_{n,1}^{(I)}(\theta) \right] = (1 - 2r)(1 - 2pr)^{n-1} + \frac{2}{np} ((1 - pr)^n - (1 - 2pr)^n).
\]

(ii) For \( 0.5 < r < 1 \) and \( 0 < p \leq 1 \),
\[
\mathbb{E} \left[ \chi_{n,1}^{(I)}(\theta) \right] = (2r - 1)(1 - p)^{n-1} + \frac{2}{np} ((1 - pr)^n - (1 - p)^n).
\]

(iii) For \( r \geq 1 \) and \( 0 < p \leq 1 \),
\[
\mathbb{E} \left[ \chi_{n,1}^{(I)}(\theta) \right] = (1 - p)^{n-1}.
\]

(iv) For \( r > 0 \) and \( p = 0 \),
\[
\mathbb{E} \left[ \chi_{n,1}^{(I)}(\theta) \right] = 1.
\]

The expressions (21) and (22) can be combined into the single expression
\[
\mathbb{E} \left[ \chi_{n,1}^{(I)}(\theta) \right] = |2r - 1| (1 - p\ell(r))^{n-1} + \frac{2}{np} ((1 - pr)^n - (1 - p\ell(r))^n)
\]
on the range \( 0 < r < 1 \) and \( 0 < p \leq 1 \). Collecting (23), (24) and (25) we get the upper bound
\[
\mathbb{E} \left[ \chi_{n,1}(\theta) \right] \leq (1 - p\ell(r))^{n-1} + \frac{2}{np} \left( 1 - \frac{1}{2} p\ell(r) \right)^n
\]
for any fixed \( n = 2, 3, \ldots \), and \( \theta \) in \( \mathbb{R}_+ \times [0, 1] \).

**VIII. Second moments**

Again fix \( n = 2, 3, \ldots \), and \( \theta \) in \( \mathbb{R}_+ \times [0, 1] \). The same arguments apply for both the unit circle and unit interval: For \( x, y \) in \([0, 1]\), write
\[
b(x, y; \theta) := \mathbb{E} \left[ (1 - B' \mathbf{1}[d(x, Z) \leq r]) (1 - B'' \mathbf{1}[d(y, Z) \leq r]) \right] = 1 - pa(x; r) - pa(y; r) + p^2 u(x, y; r)
\]
with
\[ u(x, y; r) := \mathbb{P}[d(x, Z) \leq r, d(y, Z) \leq r]. \]

We then proceed with the decomposition
\[
\chi_{n,1}(\theta) \chi_{n,2}(\theta) = \prod_{j=2}^{n} (1 - \chi_{1j}(\theta)) \prod_{k=1, k \neq 2}^{n} (1 - \chi_{2k}(\theta)) = (1 - \chi_{12}(\theta)) \prod_{j=3}^{n} (1 - \chi_{1j}(\theta))(1 - \chi_{2j}(\theta)).
\]
Under the enforced independence assumptions, an easy conditioning argument (with respect to the triple \(X_1, X_2\) and \(B_{12}\)) based on this decomposition now gives
\[
\mathbb{E}[(\chi_{n,1}(\theta) \chi_{n,2}(\theta))] = \mathbb{E}[(1 - B_{1} [d(X, Y) \leq r]) b(X, Y; \theta)^{n-2}].
\]
As mentioned earlier we need only consider the unit circle as we do from now on: From (19) it is plain that
\[
\tilde{u}^{(C)}(x, y; \theta) = 1 - 2\ell(r) + p^2 u^{(C)}(x, y; r)
\]
for all \(x, y \in [0, 1]\), where we note that
\[
u^{(C)}(x, y; r) = \mathbb{P}[\|x - Z\| \leq r, \|y - Z\| \leq r] = u^{(C)}(0, \|x - y\|; r)
\]
by translation invariance. Thus, writing
\[
\tilde{b}^{(C)}(z; \theta) := 1 - 2\ell(r) + p^2 \tilde{u}^{(C)}(z; r), \quad z \in [0, 0.5)
\]
with
\[
\tilde{u}^{(C)}(z; r) := u^{(C)}(0, z; r),
\]
we get
\[
b^{(C)}(x, y; \theta) = \tilde{b}^{(C)}(\|x - y\|; \theta), \quad x, y \in [0, 1].
\]
Taking advantage of these facts we now find
\[
\mathbb{E}[(\chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta))]
= \mathbb{E}[(1 - B_{1} [\|X - Y\| \leq r]) \tilde{b}^{(C)}(\|X - Y\|; \theta)^{n-2}]
= \mathbb{E}((1 - p_{1} [\|X - Y\| \leq r]) \tilde{b}^{(C)}(\|X - Y\|; \theta)^{n-2}]
= 2 \int_{0}^{0.5} (1 - p_{1} [z \leq r]) \tilde{b}^{(C)}(z; \theta)^{n-2} dz
\]
by a straightforward evaluation of the double integral
\[
\int_{0}^{1} dx \int_{0}^{1} dy (1 - p_{1} [\|x - y\| \leq r]) \tilde{b}^{(C)}(\|x - y\|; \theta)^{n-2}.
\]
Consequently,
\[
\mathbb{E}[(\chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta))] \leq 2 \int_{0}^{0.5} \tilde{b}^{(C)}(z; \theta)^{n-2} dz.
\]
It is possible to compute the value of \(\tilde{u}^{(C)}(z; r)\) for various values for \(z, r\): For \(0 < r < 0.5\), we find
\[
\tilde{u}^{(C)}(z; r) = \begin{cases}
2r - z & \text{if } 0 < r < 0.25, 0 \leq z \leq 2r \\
0 & \text{if } 0 < r < 0.25, 2r < z \leq 0.5 \\
2r - z & \text{if } 0.25 \leq r < 0.5, 0 \leq z \leq 1 - 2r \\
4r - 1 & \text{if } 0.25 \leq r < 0.5, 1 - 2r < z \leq 0.5.
\end{cases}
\]
Details are outlined in Appendix A. Obviously, if \(r \geq 0.5\), then \(\tilde{u}^{(C)}(z; r) = 1\) for every \(z\) in \([0, 0.5]\). Thus, for \(0 \leq p \leq 1\), through (28) we obtain
\[
b^{(C)}(z; \theta) = \begin{cases}
1 - 4pr + p^2 (2r - z) & \text{if } 0 < r < 0.25, 0 \leq z \leq 2r \\
1 - 4pr & \text{if } 0 < r < 0.25, 2r < z \leq 0.5 \\
1 - 4pr + p^2 (2r - z) & \text{if } 0.25 \leq r < 0.5, 0 \leq z \leq 1 - 2r \\
1 - 4pr + p^2 (4r - 1) & \text{if } 0.25 \leq r < 0.5, 1 - 2r < z \leq 0.5.
\end{cases}
\]
Using this fact in (29) and evaluating the integral, we obtain the following upper bounds, see Appendix B for details:

(i) For \(0 < r < 0.25\) and \(0 < p \leq 1\),
\[
\mathbb{E}[(\chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta)) \leq (1 - 4r)(1 - 4pr)^{n-2} + 2(1 - 4pr)^{n-1} \left( \frac{1}{(n-1)p^2} \right)^{n-1}.
\]
(ii) For \(0.25 \leq r < 0.5\) and \(0 < p \leq 1\),
\[
\mathbb{E}[(\chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta)) \leq (4r - 1)(1 - 2pr)^{2(n-2)} + (2 - 4r)(1 - 4pr + 2p^2 r)^{n-2}.
\]
(iii) For \(r \geq 0.5\) and \(0 < p \leq 1\),
\[
\mathbb{E}[(\chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta)) = (1 - p)^{2n-3}.
\]
(iv) For \(r > 0\) and \(p = 0\),
\[
\mathbb{E}[(\chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta)) = 1.
\]
Furthermore, combining these bounds with (20), we obtain the following upper bound on
\[
R_n(\theta) := \frac{\mathbb{E}[(\chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta))] \mathbb{E}[(\chi_{n,1}^{(C)}(\theta)^2) - \mathbb{E}[(\chi_{n,1}^{(C)}(\theta)\chi_{n,2}^{(C)}(\theta))^2].
\]
in the various cases listed below.
(i) For $0 < r < 0.25$ and $0 < p \leq 1$,
\[ R_n(\theta) \leq \frac{1 - 4r}{1 - 4pr} + \frac{2}{(n-1)p^2} \left( 1 + \frac{2p^2r}{1 - 4pr} \right)^{n-1}. \tag{30} \]

(ii) For $0.25 \leq r < 0.5$ and $0 < p \leq 1$,
\[ R_n(\theta) \leq \frac{4r - 1}{(1 - 2pr)^2} + (2 - 4r) \frac{(1 - 4pr + 2p^2r)^{n-2}}{(1 - 2pr)^{2(n-1)}}. \tag{31} \]

(iii) For $r \geq 0.5$ and $0 < p \leq 1$,
\[ R_n(\theta) = \frac{1}{1 - p}. \tag{32} \]

(iv) For $r > 0$ and $p = 0$,
\[ R_n(\theta) = 1. \tag{33} \]

IX. PROOF OF THE ONE LAWS

As discussed in Section VI, the one law will be established if we show that (14) holds. Below we consider separately the unit circle and the unit interval. In that discussion we repeatedly use the elementary bound
\[ 1 - x \leq e^{-x}, \quad x \geq 0. \tag{34} \]

A. One law over the unit circle

The one law over the unit circle reduces to showing the following convergence.

**Lemma 9.1:** For any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1]$, we have
\[ \lim_{n \to \infty} n \mathbb{E} \left[ \chi_n(\theta) \right] = 0 \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = +\infty \]
where the sequence $\alpha : \mathbb{N}_0 \to \mathbb{R}_+$ is determined through (8).

**Proof.** Fix $n = 1, 2, \ldots$ and in the expression (20) substitute $(r, p)$ by $(r_n, p_n)$ according to the scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1]$. We get
\[ n \mathbb{E} \left[ \chi_{n,1}(\theta_n) \right] = n \left( 1 - p_n \ell(r_n) \right)^{n-1} \leq n e^{-\frac{n}{n} \left( \log n + \alpha_n \right)} = n e^{-\frac{n}{n} \left( \log n + \alpha_n \right)} \]
where the bound (34) was used. Letting $n$ go to infinity we get the desired conclusion since $\lim_{n \to \infty} \alpha_n = +\infty$.

B. One law over the unit interval

A similar step is taken for the random intersection graph over the unit interval.

**Lemma 9.2:** For any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1]$, we have
\[ \lim_{n \to \infty} n \mathbb{E} \left[ \chi_n(\theta) \right] = 0 \quad \text{if} \quad \lim_{n \to \infty} \alpha'_n = +\infty \]
where the sequence $\alpha' : \mathbb{N}_0 \to \mathbb{R}_+$ is determined through (9).

**Proof.** Fix $n = 1, 2, \ldots$ and in the upper bound (26) substitute $(r, p)$ by $(r_n, p_n)$ according to the scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1]$. We get
\[ n \mathbb{E} \left[ \chi_n(\theta) \right] \leq n \left( 1 - p_n \ell(r_n) \right)^{n-1} + 2 p_n \left( 1 - \frac{1}{2} p_n \ell(r_n) \right)^n. \]
As in the proof of Lemma 9.1, we can show that
\[ \lim_{n \to \infty} n \left( 1 - p_n \ell(r_n) \right)^{n-1} = 0 \]
under the condition $\lim_{n \to \infty} \alpha'_n = +\infty$. To do so, fix $n = 1, 2, \ldots$ sufficiently large so that $\alpha'_n \geq 0$.

On that range we note that
\[ \frac{1}{p_n} \left( 1 - \frac{1}{2} p_n \ell(r_n) \right)^n \leq \frac{1}{p_n} \left( \frac{1 - \frac{1}{2} p_n \ell(r_n)}{n} \right)^n \leq \frac{1}{n} \left( \frac{2 \log n - \log \log n + \alpha'_n}{n} \right)^{-1} e^{-\frac{1}{2} \alpha'_n} \]
\[ = \frac{\log n}{2 \log n - \log \log n} e^{-\frac{1}{2} \alpha'_n} \]
upon using the fact $\ell(r_n) \leq 1$ and the bound (34). Letting $n$ go to infinity we obtain (35).

X. PROOF OF THE ZERO LAWS

As observed earlier, when dealing with the zero law we need only concern ourselves with the unit circle case. Throughout this section, we take $\theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1]$ and associate with it the sequence $\alpha : \mathbb{N}_0 \to \mathbb{R}_+$ through (8). We now show (15) and (16) under the condition $\lim_{n \to \infty} \alpha_n = -\infty$. This will complete the proof of the zero laws.

In the discussion we shall make use of the following elemental fact: For any sequence $a : \mathbb{N}_0 \to \mathbb{R}_+$, the asymptotic equivalence
\[ (1 - a_n)^n \sim e^{-n a_n} \tag{36} \]
holds provided $\lim_{n \to \infty} a_n = \lim_{n \to \infty} n a_n^2 = 0$. 

\[ \]
A. Establishing (15)

The first step is contained in the following zero-law complement of Lemma 9.1.

**Lemma 10.1:** For any scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \), we have

\[
\lim_{n \to \infty} nE \left[ \chi_{n,1}^{(C)}(\theta_n) \right] = \infty \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = -\infty
\]

where the sequence \( \alpha : \mathbb{N}_0 \to \mathbb{R}_+ \) is determined through (8).

**Proof.** Fix \( n = 1, 2, \ldots \) and in the expression (20) substitute \((r, p)\) by \((r_n, p_n)\) according to the scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \). As in the proof of Lemma 9.1 we start with the expression

\[
nE \left[ \chi_{n,1}^{(C)}(\theta_n) \right] = n \left( 1 - p_n \ell(r_n) \right)^{n-1}.
\]

Under the condition \( \lim_{n \to \infty} \alpha_n = -\infty \) we note that \( \alpha_n = -|\alpha_n| \) for all \( n \) sufficiently large, say for all \( n \geq n^* \) for some finite integer \( n^* \). Using (8) we get \( |\alpha_n| \leq \log n \) on that range by the non-negativity condition \( p_n \ell(r_n) \geq 0 \). Therefore,

\[
p_n \ell(r_n) \leq \frac{\log n}{n} \quad \text{and} \quad n \left( p_n \ell(r_n) \right)^2 \leq \left( \frac{\log n}{n} \right)^2
\]

for all \( n \geq n^* \), and the equivalence (36) (with \( a_n = p_n \ell(r_n) \)) now yields

\[
n \left( 1 - p_n \ell(r_n) \right)^{n-1} \sim n e^{-n p_n \ell(r_n)}
\]

with

\[
n e^{-n p_n \ell(r_n)} = n e^{-\left( \log n + \alpha_n \right)} = e^{-\alpha_n}, \quad n = 1, 2, \ldots
\]

Finally, letting \( n \) go to infinity in (37) and using (39)-(40), we find

\[
\lim_{n \to \infty} n \left( 1 - p_n \ell(r_n) \right)^{n-1} = \lim_{n \to \infty} e^{-\alpha_n} = \infty
\]

as desired under the condition \( \lim_{n \to \infty} \alpha_n = -\infty \). \( \blacksquare \)

B. Establishing (16)

The proof of the one-law will be completed if we establish the next result.

**Proposition 10.2:** For any scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \), we have

\[
\lim_{n \to \infty} \sup_n R_n(\theta_n) \leq 1 \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = -\infty
\]

where the sequence \( \alpha : \mathbb{N}_0 \to \mathbb{R}_+ \) is determined through (8).

The proof of Proposition 10.2 is organized around the following simple observation: Consider a sequence \( a : \mathbb{N}_0 \to \mathbb{R} \) and let \( N_1, \ldots, N_K \) constitute a partition of \( \mathbb{N}_0 \) into \( K \) subsets, i.e., \( N_k \cap N_{k'} = \emptyset \) for distinct \( k, k' = 1, \ldots, K \), and \( \bigcup_{k=1}^{K} N_k = \mathbb{N}_0 \). In principle, some of the subsets \( N_1, \ldots, N_K \) may be either empty or finite. For each \( k = 1, \ldots, K \) such that \( N_k \) is non-empty, we set

\[
\alpha_k := \limsup_{n \to \infty} a_n = \inf_{n \in N_k} \left( \sup_{m \in N_k : m \geq n} a_m \right)
\]

with the natural convention that \( \alpha_k = -\infty \) when \( N_k \) is finite. In other words, \( \alpha_k \) is the limsup for the subsequence \( \{ a_n, \ n \in N_k \} \). It is a simple matter to check that

\[
\limsup_{n \to \infty} a_n = \max^* (\alpha_k, \ k = 1, \ldots, K)
\]

with \( \max^* \) denoting the maximum operation over all indices \( k \) such that \( N_k \) is non-empty.

**Proof.** As we plan to make use of this fact with \( K = 4 \), we write

\[
R_k := \limsup_{n \to \infty} R_n(\theta_n), \quad k = 1, \ldots, 4
\]

with

\[
N_1 := \{ n \in \mathbb{N}_0 : 0 < r_n < 0.25, \ 0 < p_n \leq 1 \},\ 
N_2 := \{ n \in \mathbb{N}_0 : 0.25 \leq r_n < 0.5, \ 0 < p_n \leq 1 \},\ 
N_3 := \{ n \in \mathbb{N}_0 : 0.5 \leq r_n, \ 0 < p_n \leq 1 \}
\]

and

\[
N_4 := \{ n \in \mathbb{N}_0 : \ r_n > 0, \ p_n = 0 \}.
\]

Therefore, we have

\[
\limsup_{n \to \infty} R_n(\theta_n) = \max^*(R_k, \ k = 1, \ldots, 4)
\]

and the result will be established if we show that

\[
R_k \leq 1, \quad k = 1, \ldots, 4.
\]

In view of the convention made earlier, we need only discuss for each \( k = 1, \ldots, 4 \), the case when \( N_k \) is countably infinite, as we do from now on.

The easy cases are handled first: From (33) it is obvious that \( R_4 = 1 \). Next as observed before, (38) holds for all \( n \) sufficiently large under the condition \( \lim_{n \to \infty} \alpha_n = -\infty \). Since \( \ell(r_n) = 1 \) for all \( n \in N_3 \), we conclude that

\[
\lim_{n \to \infty} p_n = 0
\]

and the conclusion \( R_3 = 1 \) is now immediate from (32). We complete the proof by invoking Lemmas 10.3 and 10.4 given next which establish \( R_1 \leq 1 \) and \( R_2 \leq 1 \), respectively. \( \blacksquare \)

**Lemma 10.3:** Under the assumptions of Proposition 10.2, with \( N_1 \) countably infinite, we have \( R_1 \leq 1 \).

**Proof.** Fix \( n = 2, 3, \ldots \) and pick \((r, p)\) such that \( 0 < r < 0.25 \) and \( 0 < p \leq 1 \). With (30) in mind, we note that

\[
\frac{2}{(n-1)p^2} \left( \left( 1 - \frac{2p^2r}{1 - 4pr} \right)^{n-1} - 1 \right)
\]

\[
= \frac{2}{(n-1)p^2} \left( \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{2p^2r}{1 - 4pr} \right)^k - 1 \right)
\]

\[
= \frac{2}{(n-1)p^2} \sum_{k=2}^{n-1} \binom{n-1}{k} \left( \frac{2p^2r}{1 - 4pr} \right)^k
\]

\[
= \frac{4r}{1 - 4pr} + \frac{2}{(n-1)p^2} \sum_{k=2}^{n-1} \binom{n-1}{k} \left( \frac{2p^2r}{1 - 4pr} \right)^k
\]
and we can rewrite the right handside of (30) as
\[
\frac{1 - 4r}{1 - 4pr} + \frac{2}{(n-1)p^2} \left( \left( 1 + \frac{2p^2r}{1 - 4pr} \right)^{n-1} - 1 \right)
\]
\[
= \frac{1}{1 - 4pr} + \frac{2}{(n-1)p^2} \sum_{k=2}^{n-1} (n-1) \left( \frac{2p^2r}{1 - 4pr} \right)^k
\]
\[
\leq \frac{1}{1 - 4pr} + \frac{2}{(n-1)} \sum_{k=2}^{n-1} (n-1) \left( \frac{2pr}{1 - 4pr} \right)^k
\]
since \( p^k \leq p^2 \) for \( k = 2, \ldots, n - 1 \). Therefore,
\[
R_n(\theta) \leq \frac{1}{1 - 4pr} + \frac{2}{(n-1)} \left( 1 + \frac{2pr}{1 - 4pr} \right)^{n-1}
\]
and the desired result \( R_1 \leq 1 \) will follow if we show that
\[
\limsup_{n \to \infty} \left( \frac{1}{1 - 4pr} \right) = 1
\] (41)
and
\[
\limsup_{n \to \infty} \left( \frac{2}{(n-1)} \left( 1 + \frac{2pr}{1 - 4pr} \right)^{n-1} \right) = 0. \quad (42)
\]
To do so, under the condition \( \lim_{n \to \infty} \alpha_n = -\infty \) we once again use the fact that (38) holds for large \( n \) with \( p_n \ell(r_n) = 2p_n r_n \) for all \( n \) in \( N_1 \). Thus,
\[
\lim_{n \to \infty} p_n r_n = 0
\]
and the convergence (41) follows.

Next, since \( 1 + x \leq e^x \) for all \( x \in \mathbb{R} \), we note for all \( n \) in \( N_1 \) that
\[
\frac{2}{n-1} \left( 1 + \frac{p_n \ell(r_n)}{1 - 2p_n \ell(r_n)} \right)^{n-1}
\]
\[
= \frac{2}{n-1} \left( 1 + \frac{p_n \ell(r_n)}{1 - 2p_n \ell(r_n)} \right)^{n-1}
\]
\[
\leq \frac{2}{n-1} e^{\beta_n} \left( 1 - \frac{p_n \ell(r_n)}{2p_n \ell(r_n)} \right)^{n-1}
\]
with
\[
\beta_n := (n-1) \left( \frac{p_n \ell(r_n)}{1 - 2p_n \ell(r_n)} - \log(n-1) \right).
\]
Thus, (42) follows if we show that
\[
\lim_{n \to \infty} \beta_n = -\infty. \quad (43)
\]
From (38) we get
\[
\beta_n \leq \left( \frac{n-1}{n} \right) \log \frac{n}{1 - 2\log \frac{n}{n}} - \log(n-1)
\]
for large \( n \). It is now a simple exercise to check that
\[
\lim_{n \to \infty} \left( \frac{n-1}{n} \right) \log \frac{n}{1 - 2\log \frac{n}{n}} - \log(n-1) = 0
\]
and the conclusion (43) is obtained under the assumption \( \lim_{n \to \infty} \alpha_n = -\infty \).

\[
\text{Lemma 10.4: Under the assumptions of Proposition 10.2, with } N_2 \text{ countably infinite, we have } R_2 \leq 1.
\]

\[
\text{Proof. Fix } n = 2, 3, \ldots \text{ and pick } (r, p) \text{ such that } 0.25 < r \leq 0.5 \text{ and } 0 < p \leq 1. \text{ From (31) we get}
\]
\[
R_n(\theta) \leq \frac{4r}{(1 - 2p^2)^2} \left( 1 - \frac{(1 - 4pr + 2p^2r)_{n-2}}{(1 - 2pr)^2_{n-2}} \right)
\]
\[
+ \frac{4r}{(1 - 2pr)^2} \left( 1 - \frac{(1 - 4pr + 2p^2r)_{n-2}^2}{(1 - 2pr)(2n-2)} \right) + \frac{1}{(1 - 2pr)^2} \left( \frac{2(1 - 4pr + 2p^2r)_{n-2}}{(1 - 2pr)(2n-2)} - 1 \right).
\]

Now fix \( n \) in \( N_2 \) and substitute \( (r, p) \) by \( (r_n, p_n) \) according to the scaling \( \theta : \mathbb{N}_0 \to \mathbb{R}_+ \times [0, 1] \) in (31). As before, properties of the limsup operation yield
\[
R_2 \leq R_{2c} (R_{2a} + R_{2b}) \quad (44)
\]
with
\[
R_{2a} := \limsup_{n \to \infty} \left( \frac{4r_n}{1 - 2p_n r_n} \right) \left( 1 - \frac{(1 - 4p_n r_n + 2p_n^2 r_n)_{n-2}}{(1 - 2p_n r_n)(2n-2)} \right),
\]
\[
R_{2b} := \limsup_{n \to \infty} \left( \frac{2(1 - 4p_n r_n + 2p_n^2 r_n)_{n-2}}{(1 - 2p_n r_n)(2n-2)} - 1 \right)
\]
and
\[
R_{2c} := \limsup_{n \to \infty} \left( \frac{1}{1 - 2p_n r_n} \right)^2.
\]
As in the proof of Lemma 10.3, it is also the case here that \( R_{2c} \) exists as a limit and is given by
\[
\lim_{n \to \infty} \frac{1}{1 - 2p_n r_n} = 1;
\]
details are omitted in the interest of brevity. Next, we show that
\[
\lim_{n \to \infty} \frac{1 - 4p_n r_n + 2p_n^2 r_n}{(1 - 2p_n r_n)(2n-2)} = 1. \quad (45)
\]
Once this is done, we see from their definitions that \( R_{2a} = 0 \) and \( R_{2b} = 1 \), and the conclusion \( R_2 \leq 1 \) follows from (44).

To establish (45) we note that
\[
4p_n r_n - 2p_n^2 r_n = p_n \ell(r_n)(2 - p_n) \leq 2p_n \ell(r_n)
\]
and
\[
2p_n r_n = p_n \ell(r_n)
\]
for all \( n \) in \( N_2 \). Now making use of (38) we conclude that
\[
\lim_{n \to \infty} \left( \frac{4p_n r_n - 2p_n^2 r_n}{n-2} \right) = \lim_{n \to \infty} \left( \frac{n-2}{(4p_n r_n - 2p_n^2 r_n)^2} \right) = 0
\]
while

\[ \lim_{n \to \infty} 2p_n r_n = \lim_{n \to \infty} (n-2)(2p_n r_n)^2 = 0. \]

By the equivalence (36) used with \( a_n = 4p_n r_n - 2p_n^2 r_n \) and \( a_n = 2p_n r_n \), respectively, we now conclude that

\[
\begin{align*}
&\frac{(1 - 4p_n r_n + 2p_n^2 r_n)^{n-2}}{(1 - 2p_n r_n)^{2(n-2)}} \\
&\sim \frac{e^{-(n-2)(4p_n r_n - 2p_n^2 r_n)}}{(e^{-(n-2)(2p_n r_n)})^2} \\
&= e^{2(n-2)(p_n^2 r_n)}
\end{align*}
\]

as \( n \) goes to infinity in \( N_2 \).

Finally, for \( n \) in \( N_2 \), because \( \ell(r_n) = 2r_n \geq 0.5 \), we get

\[
2(n-2)(p_n^2 r_n) = (n-2)\frac{(p_n\ell(r_n))^2}{\ell(r_n)} \\
\leq 2(n-2) \cdot (p_n \ell(r_n))^2 \\
= \frac{2(n-2)}{n} \cdot n(p_n \ell(r_n))^2
\]

so that

\[
\lim_{n \to \infty} 2(n-2)(p_n^2 r_n) = 0
\]

with the help of (38). The conclusion (45) now follows from (46), and the proof of Lemma 10.4 is complete.

\[ \square \]

XI. Simulation Results

In this section, we present some plots from simulations in Matlab which confirm the results in Theorem 5.1 and Theorem 5.2. For given \( n \) and \( r \), we estimate the probability that there are no isolated nodes by averaging over 1,000 instances of the random graphs \( G^{(C)}(n; \theta) \) and \( G^{(L)}(n; \theta) \).

In Figure 1(a), we have taken \( n = 100 \) and \( p = 0.25 \), and examine the threshold behavior of the probability that there are no isolated nodes by varying \( r \). Theorem 5.1 suggests that the critical range for the graph over the unit circle when \( n = 100 \) and \( p = 0.25 \) should be \( r^* = 0.09 \). This is confirmed by the simulation results. In the case of the unit interval, we expect from Theorem 5.2 that the critical range will be between \( r^* = 0.09 \) and \( r^{**} = 0.12 \); this is in agreement with the plot.

In Figure 1(b), we have taken \( n = 100 \) and \( r = 0.1 \), and repeat the analysis by choosing various values for \( p \). As expected from Theorem 5.1, the critical edge probability for the unit circle is found to occur at \( p^* = 0.23 \). It is also clear that for the unit interval, the critical edge probability is between \( p^* = 0.23 \) and \( p^{**} = 0.31 \) as predicted by Theorem 5.2.

XII. Concluding Remarks

Theorem 5.2 shows a gap between the zero and one laws in the case of the intersection graph on the unit interval: The zero law expresses deviations with respect to the scaling \( \theta^* : \mathbb{N}_0 \to \mathbb{R}_+ \times [0,1] \) determined through

\[
p_n^* \ell(r_n^*) = \frac{\log n}{n}, \quad n = 1, 2, \ldots
\]

as guessed. On the other hand, the one law reflects sensitivity with respect to the "larger" scaling \( \theta^{**} : \mathbb{N}_0 \to \mathbb{R}_+ \times [0,1] \) determined through

\[
p_n^{**} \ell(r_n^{**}) = \frac{2(\log n - \log \log n)}{n}, \quad n = 1, 2, \ldots
\]

Inspection of the proof readily shows that the method of first moment is not powerful enough to close the gap – To the best of our knowledge we are not aware of any other instance in the literature where this occurs. While we still believe that this gap can be bridged, it is clear that a different method of analysis will be needed.

The analysis given here also suggests the form of the zero-one law to expect when the geometric component lives in higher dimensions. Specifically, consider the case where the nodes are located in a region \( \mathbb{D} \subseteq \mathbb{R}^d \), without boundary, e.g., a torus or a spherical surface. Then it is easy to compute the probability of an edge between two nodes as

\[
p\ell(r) = p\mathbb{P} \left[d(x, Y) \leq r \right]
\]

where \( x \) is an arbitrary point in \( \mathbb{D} \), the rv \( Y \) is uniformly distributed over \( \mathbb{D} \) and \( d(\cdot, \cdot) \) is the appropriate notion of distance. As before, if we define the sequence \( \alpha : \mathbb{N}_0 \to \mathbb{R} \) through

\[
p_n \ell(r_n) = \frac{\log n + \alpha_n}{n}, \quad n = 1, 2, \ldots
\]

then the required dichotomy in the first moment (cf. Lemma 9.1 and Lemma 10.1) clearly holds even in higher dimensions. As a result, we expect the critical scaling for the absence of isolated nodes to be given through

\[
p_n^* \ell(r_n^*) = \frac{\log n}{n}, \quad n = 1, 2, \ldots
\]

Finally, similar inferences can be made for modeling wireless sensor networks which rely on the Eschenauer-Gligor scheme to secure their communication links: Power constraints restrict nodes to have a finite transmission range, a physical communication constraint which is captured by the disk model, the Eschenauer-Gligor scheme introduces a logical constraint which is well modeled by the random key graph [18]. Combining these two constraints amounts to taking the intersection of a geometric random graph with a random key graph [3] [4]. However, unlike Erdős-Rényi graphs, random key graphs exhibit dependencies between edges, and this renders the problem more complex. Nevertheless, we expect the determination of critical scalings through the probability of an edge between two nodes to take place here as well; see (6). This time, in (7) the probability \( p \) is replaced by the probability that two nodes share a common key in the Eschenauer-Gligor scheme.

\[ ^3 \text{The case when the transmission range is infinite is the so-called full visibility case [18].} \]
APPENDIX

A. Calculation of $\tilde{u}^{(C)}(z; r)$

Fix $0 < r < 0.5$. With $X$ still denoting a rv uniformly distributed over $[0, 1]$, we have

$$\tilde{u}^{(C)}(z; r) = P[\|X\| \leq r, \|X - z\| \leq r]$$

$$= 1 - P[\|X\| > r] - P[\|X - z\| > r]$$

$$+ P[\|X\| > r, \|X - z\| > r].$$

(47)

For the unit circle, the probability that a uniformly distributed node falls outside the range of a fixed node is independent of the node location, hence

$$P[\|X\| > r] = P[\|X - z\| > r] = 1 - 2r.$$

Next, consider

$$P[\|X\| > r, \|X - z\| > r] = P[\min(X, 1 - X) > r, \min(|X - z|, 1 - |X - z|) > r]$$

$$= P[E_1 \cap E_2 \cap (E_3 \cup E_4)]$$

$$= P[E_1 \cap E_2 \cap E_3] + P[E_1 \cap E_2 \cap E_4]$$

where

$$E_1 := [r < X < 1 - r],$$

$$E_2 := [z - (1 - r) < X < z + (1 - r)],$$

$$E_3 := [X > z + r]$$

and

$$E_4 := [X < z - r].$$

It is clear that

$$E_1 \cap E_2 \cap E_3 = [z + r < X < 1 - r]$$

and

$$E_1 \cap E_2 \cap E_4 = [r < X < z - r].$$

Consider the case $0 < r < 0.25$ and $0 \leq z \leq 2r$. Then, the inequality

$$z \leq \min(2r, 1 - 2r),$$

holds since $2r < 1 - 2r$ when $r < 0.25$. Therefore,

$$P[\|X\| > r, \|X - z\| > r] = 1 - 2r - z.$$

Using this fact in (47), we obtain for $0 < r < 0.25$ and $0 \leq z \leq 2r$

$$\tilde{u}^{(C)}(z; r) = 2r - z.$$

A similar calculation applies when $0.25 \leq r < 0.5$ and $0 \leq z \leq 1 - 2r$ since (48) holds in this case as well.

If $0 < r < 0.25$ and $2r < z \leq 0.5$, we obtain

$$P[\|X\| > r, \|X - z\| > r] = (1 - 2r - z) + (z - 2r) = 1 - 4r$$

and this implies

$$\tilde{u}^{(C)}(z; r) = 0$$

by substituting into (47).

On the other hand, if $0.25 \leq r < 0.5$ and $1 - 2r < z \leq 0.5$ we get

$$\tilde{u}^{(C)}(z; r) = 4r - 1,$$

since $P[\|X\| > r, \|X - z\| > r] = 0$ in this case.

B. Upper bound for $E\left[\chi^{(C)}_{n,1}(\theta)\chi^{(C)}_{n,2}(\theta)\right]$ 

The cases $(r \geq 0.5, 0 < p \leq 1)$ and $(p = 0, r > 0)$ are straightforward. If $0 < r < 0.25$ and $0 < p \leq 1$, we use (29)
to obtain
\[
\mathbb{E} \left[ \chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta) \right] \\
\leq 2 \int_0^{2r} (1 - 4pr + p^2 (2r - z))^{n-2} dz + 2 \int_0^{0.5} (1 - 4pr)^{n-2} dz \\
= \frac{2}{(n-1)p^2} \left( \left( 1 + 2p^2 r \right)^{n-1} - 1 \right) \\
+ (1 - 4r)(1 - 4pr)^{n/2}.
\]

For 0.25 \leq r < 0.5 and 0 < p \leq 1, we get
\[
\mathbb{E} \left[ \chi_{n,1}^{(C)}(\theta) \chi_{n,2}^{(C)}(\theta) \right] \\
\leq 2 \int_0^{1-2r} (1 - 4pr + p^2 (2r - z))^{n-2} dz \\
+ 2 \int_1^{0.5} (1 - 4pr + p^2 (4r - 1))^{n-2} dz \\
\leq (2 - 4r)(1 - 4pr + 2p^2 r)^{n/2} + (4r - 1)(1 - 2pr)^{(n-2)}.
\]

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