SEIBERG-WITTEN INVARIANTS OF 4-MANIFOLDS WITH FREE CIRCLE ACTIONS

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1. Introduction

The main result of this paper describes a formula for the Seiberg-Witten invariant of a 4-manifold $X$ which admits a nontrivial free $S^1$-action. A free circle action on $X$ is classified by its orbit space, a 3-manifold $M$, and its Euler class $\chi \in H^2(M; \mathbb{Z})$. If $\chi = 0$, then $X = M \times S^1$, and it is well-known that the Seiberg-Witten invariants of $X$ are equal to the 3-dimensional Seiberg-Witten invariants of $M$.

Our result expresses the Seiberg-Witten invariants of $X$ in terms of the Seiberg-Witten invariants of $M$ and the Euler class $\chi$:

**Theorem 1.** Let $X$ be a smooth 4-manifold with $b_+ \geq 2$ and a free circle action. Let $M^3$ be the smooth orbit space and suppose that the Euler class $\chi \in H^2(M; \mathbb{Z})$ of the free circle action is not torsion. Let $\xi$ be a spin$c$ structure over $X$. If $\xi$ is not pulled up via $\pi : X \to M$, then $SW_X(\xi) = 0$. Otherwise, let $\xi^* \equiv \xi \pmod{\chi}$ be a spin$c$ structure on $M$ such that $\xi \equiv \pi^*(\xi^*) \pmod{\chi}$, then

$$SW_X^4(\xi) \equiv \sum_{\xi' \equiv \xi^* \pmod{\chi}} SW_M^3(\xi').$$

The difference of two spin$c$ structures gives rise to a well-defined element $\xi' - \xi \in H^2(X; \mathbb{Z})$. For more information, see section (4.1). Because $\chi$ is nontorsion, the equivalence relation in the above theorem is well-defined. The pullback of a spin$c$ structure is discussed in section (4.2).

As an application of this theorem we shall produce a nonsymplectic 4-manifold with a free circle action whose orbit space fibers over $S^1$. This example runs counter to intuition since there is a well-known conjecture of Taubes that $M^3 \times S^1$ admits a symplectic structure if and only if $M^3$ fibers over the $S^1$. Furthermore, there is evidence [FGM] which suggests that many such 4-manifolds are, in fact, symplectic. As another application of our formula, we construct a 3-manifold which...
is not the orbit space of any symplectic 4-manifold with a free circle action. A corollary of the main theorem is a formula for the Seiberg-Witten invariant of the total space of a circle bundle over a surface. This formula can be thought of as the 3 dimensional analog of the 4 dimensional formula.

2. Classifying free circle actions

Let $X$ be an oriented connected 4-manifold carrying a smooth free $S^1$-action. Its orbit space $M$ is a 3-manifold whose orientation is determined, so that, followed by the natural orientation on the orbits, the orientation of $X$ is obtained. Choose a smooth connected loop $l$ representing the the Poincaré dual $PD(\chi) \in H_1(M; \mathbb{Z})$. Remove a tubular neighborhood $N \cong D^2 \times l$ of $l$ from $M$, and set $X_0 = (M \setminus N) \times S^1$. View $X_0$ as an $S^1$-manifold whose action is given by rotation in the last factor. Let $m$ be the meridian of $l$, and let $t$ be an orbit in $X_0$. We then have:

**Lemma 2.** The manifold $X$ is diffeomorphic (by a bundle isomorphism) to the manifold

$$X(l) = X_0 \cup \varphi \cdot D^2 \times T^2$$

where $\varphi : T^3 \to \partial X_0$ is an equivariant diffeomorphism which evaluates $\varphi_*([\partial(D^2 \times pt)]) = [m + t]$ in homology.

When gluing $D^2 \times T^2$ into the boundary of a manifold, the resulting closed manifold is determined up to diffeomorphism by the image in homology of $[\partial(D^2 \times pt)]$. (For example, see [MMS].)

**Proof.** The manifold $X$ is a principal $S^1$-bundle. Since $\chi$ evaluates on any 2-cycle in $M \setminus N$ by intersecting that 2-cycle against $l$, it follows that the restriction of the Euler class $\chi$ restricts trivially to $M \setminus N$. Therefore, the $S^1$-bundle is trivial over $M \setminus N$, and $\pi^{-1}(M \setminus N)$ is diffeomorphic to $X_0$. Similarly, $\pi^{-1}(N)$ is diffeomorphic to $D^2 \times S^1 \times S^1$. Let $m'$, $l'$, and $t'$ be the circles which correspond to the factors in $D^2 \times S^1 \times S^1$ respectively.

Construct a manifold $X(l)$ as above using a bundle isomorphism $\varphi : \partial(D^2 \times S^1) \times S^1 \to X_0$. Bundle isomorphisms covering the identity are classified up to vertical equivariant isotopy by homotopy classes of maps in $[\partial(D^2 \times S^1), S^1] = \mathbb{Z} \oplus \mathbb{Z}$. Explicitly, an equivariant map $\varphi$ inducing $1_{\partial(D^2 \times S^1)}$ is classified by integers $(r,s)$ where $\varphi_*[m'] = [m] + r[t]$ and $\varphi_*[l'] = [l] + s[t']$. A bundle automorphism $\Phi$ of $(D^2 \times S^1) \times S^1$ can be constructed such that $\Phi_*[m'] = [m']$ and $\Phi_*[l'] = [l'] + s[t']$ for any $s \in \mathbb{Z}$. These bundle automorphisms are just the equivariant maps
classified by $[D^2 \times S^1, S^1] = H^1(D^2 \times S^1; \mathbb{Z})$. Therefore the resulting bundle $X(l)$ depends only on the integer $r$ and the homology class $[l]$. In particular, the obstruction to extending the constant section

$$M \setminus N \to X_0 = (M \setminus N) \times S^1$$

over $D^2 \times S^1$ lies in $H^2(D^2 \times S^1, \partial(D^2 \times S^1); \mathbb{Z})$ and is given by $r$. The Euler class of $X(l)$ is then $PD(r[l]) = r\chi$. Taking $r = 1$ produces the desired bundle.

From now on we shall work with $X(l)$ and refer to it as $X$. Furthermore, it is clear from the construction above that the map $\varphi$ can be chosen so that in homology,

$$\varphi_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

with respect to the basis $\{[m], [l], [t]\}$.

### 3. Gluing along $T^3$

Since we have $X = X_0 \cup_\varphi (D^2 \times T^2)$ we may apply the gluing theorem of Morgan, Mrowka, and Szabó [MMS]. Recall that $\varphi_*([m']) = [m + t]$.

**Theorem 3** (Morgan, Mrowka, and Szabó). *If the spin$^c$ structure $\xi$ over $X$ restricts nontrivially to $D^2 \times T^2$, then $SW_X(\xi) = 0$. For each spin$^c$ structure $\xi_0 \to X_0$ that restricts trivially to $\partial X_0$, let $V_X(\xi_0)$ denote the set of isomorphism classes of spin$^c$ structures over $X$ whose restriction to $X_0$ is equal to $\xi_0$. Then we have*

$$\sum_{\xi \in V_X(\xi_0)} SW_X(\xi) = \sum_{\xi \in V_{M \times S^1}(\xi_0)} SW_{M \times S^1}(\xi) + \sum_{\xi \in V_{X_0/1}(\xi_0)} SW_{X_0/1}(\xi),$$

*where the manifold $X_{0/1} = X_0 \cup_{\varphi_{0,1}} D^2 \times T^2$ is defined by the map $\varphi_{0,1}$ which maps $[m'] \mapsto [t]$ in homology.*

In our situation, this formula simplifies significantly. Let $i$ denote the inclusion of $\partial X_0$ into $X_0$. A study of the long exact sequences in homology shows that the left hand side consists of a single term when $i_*[m + t]$ is indivisible. Since $i_*[t]$ is independent of $i_*[m]$ and $i_*[t]$ is a primitive class in $H_1(X_0; \mathbb{Z})$, $i_*[m + t]$ is such a class. Therefore, the formula enables the calculation of the SW invariants of $X$ in terms of the SW invariants of $M \times S^1$ and a manifold $X_{0/1}$.

The manifold $X_{0/1}$ admits a semi-free $S^1$-action whose fixed point set is a torus. Its orbit space is $M \setminus N$, and $\partial(M \setminus N) = \partial N$ is the
image of the fixed point set. The condition \( b_+(X) \geq 2 \) of the main theorem implies that \( b_+(X_{0/1}) > 1 \) and that

\[
\text{rank } H_1(M \setminus N, \partial(M \setminus N); \mathbb{Z}) > 1.
\]

The two statements are proved as follows. The Gysin sequence

\[
\xymatrix{ H^2(M; \mathbb{Z}) \ar[r]^-{\pi^*} & H^2(X; \mathbb{Z}) \ar[r]^-{\cdot \chi} & H^1(M; \mathbb{Z}) \ar[r]^-{\partial X} & H^3(M; \mathbb{Z}) }
\]

implies

\[
H^2(X; \mathbb{Z}) \cong \left( H^2(M; \mathbb{Z})/ \langle \chi \rangle \right) \\
\quad \oplus \ker \left( \bigcup_{\chi} : H^1(M; \mathbb{Z}) \to H^3(M; \mathbb{Z}) \right).
\]

Each component of the direct sum above has rank \( b_1(M) - 1 \). The bilinear form of \( X \) is the direct sum of hyperbolic pairs which implies that \( b_+(X) = b_1(M) - 1 \). Since \([l]\) is not a torsion element, removing \( N \) from \( M \) implies the rank of \( H_1(M \setminus N, \partial(M \setminus N); \mathbb{Z}) \) is also \( b_1(M) - 1 \). The second statement now follows because \( b_1(M) - 1 = b_+(X) > 1 \). The first statement requires the following Mayer-Vietoris sequence

\[
H_3(T^3; \mathbb{Z}) \to H_2(X_0; \mathbb{Z}) \oplus H_2(D^2 \times T^2; \mathbb{Z}) \to H_2(X_{0/1}; \mathbb{Z}) \to H_1(T^3; \mathbb{Z}).
\]

The rank of \( H_2(X_0; \mathbb{Z}) \) is \( 2b_1(M) - 1 \) and the rank of the image of the first map is 2. Therefore \( b_2(X_{0/1}) = 2b_1(M) - 2 \). Since the bilinear form of \( X_{0/1} \) is also a direct sum of hyperbolic pairs, \( b_+(X_{0/1}) > 1 \).

**Proposition 4.** Let \( X \) be a smooth closed oriented 4-manifold with a smooth semi-free circle action and \( b_+(X) > 1 \). Let \( X^* = X/S^1 \) be its orbit space. Suppose that \( X^* \) has a nonempty boundary and rank \( H_1(X^*, \partial X^*; \mathbb{Z}) > 1 \). Then \( SW_X \equiv 0 \).

**Proof.** Let \( F \) denote the fixed point set of \( X \) and \( F^* \) its image in \( X^* \). Then \( \partial X^* \subset F^* \). The restriction of the circle action to \( X \setminus F \) defines a principal \( S^1 \)-bundle whose Euler class lies in \( H^2(X^* \setminus F^*; \mathbb{Z}) \). Let \( \chi' \in H_1(X^*, F^*; \mathbb{Z}) \) denote its Poincaré dual. Consider the exact sequence

\[
0 \to H_1(X^*, \partial X^*; \mathbb{Z}) \xrightarrow{i_*} H_1(X^*, F^*; \mathbb{Z}) \to \\
\quad \to H_0(F^*, \partial X^*; \mathbb{Z}) \to H_0(X^*, \partial X^*; \mathbb{Z}).
\]

Since the rank of \( H_1(X^*, \partial X^*; \mathbb{Z}) \) is greater than 1, there is a class in \( i_*(H_1(X^*, \partial X^*; \mathbb{Z})) \) which is primitive and not a multiple of \( \chi' \). This class may be represented by a path \( \alpha \) in \( X^* \) which starts and ends on \( \partial X \) but is otherwise disjoint from \( F^* \).
The preimage \( S = \pi^{-1}(\alpha) \) is a 2-sphere of self-intersection 0 in \( X \). The Gysin sequence gives:

\[
H_3(X^*, F^*, \mathbb{Z}) \to H_1(X^*, F^*, \mathbb{Z}) \xrightarrow{i_*} H_2(X, F, \mathbb{Z}) \to H_2(X^*, F^*, \mathbb{Z})
\]

where \( \rho_*(i_*[\alpha]) = [S] \). The image of \( H_3(X^*, F^*, \mathbb{Z}) \) in \( H_1(X^*, F^*, \mathbb{Z}) \) is generated by \( \chi' \). Since \( i_*[\alpha] \) is primitive and not a multiple of \( \chi' \), the class \([S] \in \text{Im} \rho \subset H_2(X, F, \mathbb{Z})\) is not torsion; hence \([S] \) is nontorsion as an element of \( H_2(X; \mathbb{Z}) \).

It now follows from [FS1] that \( \text{SW}_X \equiv 0 \). \( \square \)

This type of vanishing theorem is quite common for 4-manifolds with circle actions. For instance, it follows from [F] that Seiberg-Witten invariants vanish for simply connected 4-manifolds which have \( b_+ > 1 \) and a smooth circle action.

Proposition 4 implies that the formula (4) simplifies to

\[
(7) \quad \text{SW}_X(\xi) = \sum_{\xi' \in V_{M \times S^1}(\xi|_{X_0})} \text{SW}_{M \times S^1}(\xi').
\]

4. Understanding the \( \text{spin}^c \) structures

In this section we shall prove that all basic classes of \( X \) come from \( \text{spin}^c \) structures that are pulled up from \( M \) (in a suitable sense). We shall also identify the \( \text{spin}^c \) structures in the set \( V_{M \times S^1}(\xi|_{X_0}) \) coming from the gluing theorem.

4.1. \( \text{Spin}^c \) structures. First recall some basic facts about \( \text{spin}^c \) structures. The set of \( \text{spin}^c \) structures lifting the frame bundle of a 4-manifold \( X \) is a principal homogeneous space over \( H^2(X; \mathbb{Z}) \): given two \( \text{spin}^c \) structures \( \xi_1, \xi_2 \) their difference \( \delta(\xi_1, \xi_2) \) is a well-defined element of \( H^2(X; \mathbb{Z}) \). For details, see [FM] or [R].

Likewise, if \( \xi \) is a \( \text{spin}^c \) structure and \( e \in H^2(X; \mathbb{Z}) \) is a 2-dimensional cohomology class, there is a new \( \text{spin}^c \) structure \( \xi + e \). Let \( W_\xi \) be spinor bundle associated with \( \xi \), then the new spinor bundle is \( W_\xi \otimes L_e \) where \( L_e \) is the unique line bundle with first Chern class \( e \).

For all \( \text{spin}^c \) structures, a line bundle \( L_\xi \) can be associated to \( \xi \) called the determinant line bundle. Let \( (\xi, L_\xi) \) be a pair consisting a \( \text{spin}^c \) structure \( \xi \) whose determinant line bundle is \( L_\xi \). Given two \( \text{spin}^c \) structures \( (\xi_1, L_1), (\xi_2, L_2) \), the difference of their determinant line bundles is \( c_1(L_1) - c_1(L_2) = 2e \) for some element \( e \in H^2(X; \mathbb{Z}) \). If \( H^2(X; \mathbb{Z}) \) has no 2-torsion, then \( e \) is well-defined and \( c_1(L_\xi) \) determines the \( \text{spin}^c \) structure for \( (\xi, L_\xi) \). When \( H^2(X; \mathbb{Z}) \) has 2-torsion, one has a choice of two or more possible square roots of \( 2e \) and it seems that \( e \) is not well-defined. However, the difference element \( \delta(\xi_1, \xi_2) \) satisfies
\[ c_1(L_1) - c_1(L_2) = 2\delta(\xi_1, \xi_2) \]

and so there is a unique element in \( H^2(X; \mathbb{Z}) \) which determines the difference of two spin\(^c\) structures even in the presence of 2-torsion. So while \( c_1(L_\xi) \) does not determine \( \xi \) in this case, the difference between two spin\(^c\) structures is still well-defined.

### 4.2. Pullbacks of spin\(^c\) structures.

The spin\(^c\) structures on a 3-manifold \( M \) are defined by a pair \( \xi = (W, \rho) \) consisting of a rank 2 complex bundle \( W \) with a hermitian metric (the spinor bundle) and an action \( \rho \) of 1-forms on spinors,

\[ \rho : T^* M \to \text{End}(W), \]

which satisfies the following property

\[ \rho(v)\rho(w) + \rho(w)\rho(v) = -2 < v, w > \text{Id}_W. \]

For a 4-manifold the definition is similar, but consists of a rank 4 complex bundle with an action on the cotangent space that satisfies the same property. There is a natural way to define the pullback of a spin\(^c\) structure. Let \( \eta \) denote the connection 1-form of the circle bundle \( \pi : X \to M \), and let \( g_M \) be a metric on \( M \), then we can endow \( X \) with the metric \( g_X = \eta \otimes \eta + \pi^*(g_M) \). Using this metric, there is an orthogonal splitting

\[ T^* X \cong \mathbb{R}\eta \oplus \pi^*(T^*M). \]

If \( \xi = (W, \rho) \) is a spin\(^c\) structure over \( M \), define the pullback of \( \xi \) to be \( \pi^*(\xi) = (\pi^*(W) \oplus \pi^*(W), \sigma) \) where the action

\[ \sigma : T^* X \to \text{End}(\pi^*(W) \oplus \pi^*(W)) \]

is given by

\[ \sigma(b\eta + \pi^*(a)) = \begin{pmatrix} 0 & \pi^*(\rho(a)) + b\text{Id}_{\pi^*(W)} \\ \pi^*(\rho(a)) - b\text{Id}_{\pi^*(W)} & 0 \end{pmatrix}. \]

One can easily check that this defines a spin\(^c\) structure on \( X \). Note that the first Chern class of \( \pi^*(\xi) \) is just \( \pi^*(c_1(L_\xi)) \). The other pulled back spin\(^c\) structures are now obtained by the addition of classes \( \pi^*(e) \) for \( e \in H^2(M; \mathbb{Z}) \).

There are spin\(^c\) structures on \( X \) which do not arise from spin\(^c\) structures that are pulled up from \( M \). In the next section we show that the Seiberg-Witten invariants vanish for these spin\(^c\) structures.

### 4.3. Spin\(^c\) structures which are not pullbacks.

Fix a spin\(^c\) structure \( \xi_0 = (W_0, \rho) \) on \( M \) and consider its pullback \( \xi = \pi^*(\xi_0) \) over \( X \). Looking at the Gysin sequence \( (\mathbb{F}) \), if a class \( e \in H^2(X; \mathbb{Z}) \) is not in the image of \( \pi^* \), then \( \xi + e \) is not a spin\(^c\) structure which is pulled back from \( M \).
Lemma 5. If \((\xi, L_\xi)\) is a spin\(^c\) structure on \(X\) which is not pulled back from \(M\), then \(SW_X(\xi) = 0\).

Proof. We claim that there exists an embedded torus which pairs non-trivially with \(c_1(L_\xi)\). Then by the adjunction inequality [KM] the spin\(^c\) structure \(\xi\) has Seiberg-Witten invariant equal to zero. Let

\[
H = \ker(\cdot \cup \chi : H^1(M; \mathbb{Z}) \to H^3(M; \mathbb{Z}))
\]

in equation (3), and consider for a moment the projection of \(c_1(L_\xi)|_H \in H^2(M; \mathbb{Z})\) maps to a class \(\beta \in H^1(M; \mathbb{Z})\), \(\beta \cup \chi = 0\). Thus the Poincaré dual of \(\beta\) can be represented by a surface \(b\), and there is a 1-cycle \(\lambda\) in \(M \setminus N\) rel \(\partial\) such that \([\lambda] \cdot [b] \neq 0\). Since \(\partial N\) is connected, \([\lambda]\) is actually represented by a loop \(\lambda\) in \(M \setminus N\). The preimage \(\pi^{-1}(\lambda) = \lambda \times S^1\) in \(X\) is a torus, and \(c_1(L_\xi)|_H \cdot [\pi^{-1}(\lambda)] = [b] \cdot [\lambda] \neq 0\).

On the other hand, if \(A \in \pi^*H^2(M; \mathbb{Z})\) then it Poincaré dual is represented by a loop \(\alpha\) in \(M\) which may be chosen disjoint from \(\lambda\). Thus \(A \cdot [\pi^{-1}(\lambda)] = 0\). This means that \(c_1(L_\xi) \cdot [\pi^{-1}(\lambda)] \neq 0\), as required. \(\square\)

4.4. Identifying the set \(V_{M \times S^1}(\xi|_{X_0})\). According to the previous lemma, the only nontrivial Seiberg-Witten spin\(^c\) structures are those pulled up from \(M\). Thus far we have seen that for such a spin\(^c\) structure \(\xi = \pi^*(\xi^*)\) with \(\xi_0 = \xi|_{X_0}\), we have

\[
SW_X(\xi) = \sum_{\xi' \in V_{M \times S^1}(\xi_0)} SW_{M \times S^1}(\xi').
\]

Let \(\tilde{\pi} : M \times S^1 \to M\) be the projection. We identify the set \(V_{M \times S^1}(\xi_0)\) of isomorphism classes of spin\(^c\) structures over \(M \times S^1\) which restrict on \(X_0\) to \(\xi_0\).

Lemma 6. \(V_{M \times S^1}(\xi_0) = \{ \tilde{\pi}^* (\xi^* + n \cdot \chi) \mid n \in \mathbb{Z} \}\).
Proof. The diagram

\[
\begin{array}{c}
\xymatrix{ & X \ar[r]^{\text{inc}} & X_0 \ar[r]^{\text{inc}} & M \times S^1 \\
 & M \backslash N \ar[lu]^\pi \ar[ru]_{\tilde{\pi}} & \\
M \ar[ru]^\pi & & M \times S^1 \ar[l]_{\text{inc}} \ar[lu]_{\tilde{\pi}} \\
M \ar[ru]^\pi & \ & \ \\
& M & & \ \\
& M \ar[ru]^\pi & & M \times S^1 \ar[l]_{\text{inc}} \ar[lu]_{\tilde{\pi}} \\
& M \ar[ru]^\pi & & M \times S^1 \ar[l]_{\text{inc}} \ar[lu]_{\tilde{\pi}} \\
& M \ar[ru]^\pi & & M \times S^1 \ar[l]_{\text{inc}} \ar[lu]_{\tilde{\pi}} \\
& M & & M \\
\end{array}
\]

induces spin\textsuperscript{c} structures on \(X, X_0\), and \(M \times S^1\) which satisfy

\[
\text{inc}^*(\pi^*(\xi^*)) = \xi_0 = \text{inc}^*(\tilde{\pi}^*(\xi^*)).
\]

Recall that \(\xi\) is the only spin\textsuperscript{c} structure induced on \(X\) by \(\xi_0\) since \(i_*[m + t]\) is indivisible. Since \(\tilde{\pi}^*(\xi^*) \in V_{M \times S^1}(\xi_0)\), the set of spin\textsuperscript{c} structures on \(M \times S^1\) is \(\{\tilde{\pi}^*(\xi^*) + e \in H^2(M \times S^1; \mathbb{Z}\}\). Now \(\tilde{\pi}^*(\xi^*) + e\) lies in \(V_{M \times S^1}(\xi_0)\) if and only if \(\text{inc}^*(\pi^*(\xi^*) + e) = \xi_0\), i.e. if and only if \(\text{inc}^*(e) = 0\). Therefore,

\[
(8) \quad V_{M \times S^1}(\xi_0) = \{\tilde{\pi}^*(\xi^*) + e \mid \text{inc}^*(e) = 0\}.
\]

The kernel of \(\text{inc}^*\) is equal to the image of \(j^*\) in the diagram below.

\[
\begin{array}{c}
\xymatrix{ H^2(M \times S^1, (M \backslash N) \times S^1; \mathbb{Z}) \ar[r] \ar[d]^{PD} & H^2(M \times S^1; \mathbb{Z}) \ar[r]^{\text{inc}^*} \ar[d]^{PD} & H^2(X_0; \mathbb{Z}) \ar[d]^{PD} \\
H_2(D^2 \times T^2; \mathbb{Z}) \ar[r] & H_2(M \times S^1; \mathbb{Z}) \ar[r] & H_2(X_0, \partial X_0; \mathbb{Z}) \\
n[T^2] \ar[r]_{j_*} & n[l \times t] \ar[r] & 0 \\
} \end{array}
\]

However \(j_*[\text{pt} \times T^2] = [l \times t]\), and since \(\tilde{\pi}^*(\chi) = PD^{-1}[l \times t]\), the lemma follows. \(\square\)

4.5. Relationship between \(SW^3\) and \(SW^4\). The following is a well-known fact about the relationship between the 3-dimensional Seiberg-Witten invariants and the 4-dimensional invariants.

**Proposition 7** (cf. Donaldson [D]). After making a suitable choice of orientations for \(M\) and \(M \times S^1\), the following equality holds

\[
SW^3_M(\xi) = SW^4_{M \times S^1}(\tilde{\pi}^*(\xi))
\]

for a spin\textsuperscript{c} structure \(\xi\) over \(M\).
A natural choice of orientations for $M \times S^1$ and $M$ is induced by the orientation of the circle action on $X$. This completes the proof of Theorem 1.

5. APPLICATIONS AND EXAMPLES

5.1. An application. An immediate corollary to the main theorem is the calculation of the 3 dimensional Seiberg-Witten invariants for the total space of a circle bundle over a surface. The following corollary can also be derived from [MOY] using different techniques.

**Corollary 8.** Let $\pi : Y \rightarrow \Sigma_g$ be a smooth 3-manifold which is the total space of a circle bundle over a surface of genus $g > 0$. Let $c_1(Y) = n\lambda \in H^2(\Sigma_g; \mathbb{Z})$ where $\lambda$ is the generator. The only invariants which are not zero on $Y$ come from spin$^c$ structures which are pulled back $\pi : Y \rightarrow \Sigma_g$. Hence,

$$SW_Y(\pi^*(s\lambda)) = \sum_{t \equiv s \mod n} SW_{\Sigma_g \times S^1}(\tilde{\pi}^*(t\lambda))$$

where $\tilde{\pi} : \Sigma_g \times S^1 \rightarrow \Sigma_g$.

**Proof.** Let $\pi : Y \rightarrow \Sigma_g$ be the total space of a circle bundle over $\Sigma$ with Euler class $n\lambda$. Then the manifold $Y \times S^1$ can be thought of as a smooth 4-manifold with a free circle action which orbit space is $\Sigma_g \times S^1$. The Euler class of the action is $\tilde{\pi}^*(n\lambda)$. Applying the main theorem gives

$$SW^4_Y((\pi, id)^*(\tilde{\pi}^*(s\lambda))) = \sum_{\tilde{\pi}^*(t\lambda) \equiv \tilde{\pi}^*(s\lambda) \mod \tilde{\pi}^*(n\lambda)} SW^3_{\Sigma_g \times S^1}(\xi')$$

the right hand side of the equation. Applying Proposition 7 shows that $SW^4 = SW^3$ in this case. \qed

Combining the Seiberg-Witten polynomial for the product of a surface with a circle,

$$SW_{\Sigma_g \times S^1}(t) = (t^1 - t^{-1})^{2g-2},$$

with the previous results gives a formula for the Seiberg-Witten polynomial in terms of the Euler class and the genus of the surface.

**Corollary 9.** Let $\pi : Y \rightarrow \Sigma_g$ be the total space of a circle bundle over a genus $g$ surface. Assume $c_1(Y) = n\lambda$ where $\lambda \in H^2(\Sigma_g; \mathbb{Z})$ is the generator and $n$ is an even number $n = 2l \neq 0$, then the Seiberg-Witten polynomial of $Y$ is

$$SW_Y(t) = sign(n) \sum_{i=0}^{\lfloor l \rfloor - 1} \sum_{k=-2g+2}^{k=2g-2} (-1)^{(g-1)+i+k|l|} \left( \frac{2g-2}{(g-1) + i + k|l|} \right)^{2i}.$$
where \( t = \exp(\pi^*(\lambda)) \) and defining the binomial coefficient \( \binom{p}{q} = 0 \) for \( q < 0 \) and \( q > p \). For the formula where \( n \) is odd, replace \( l \) by \( n \) and \( t^{2i} \) by \( t^i \).

If one uses [MT] to calculate the Milnor torsion for a circle bundle \( Y \) over a surface, one finds that the invariant is identically 0. This is because all spin\(^c\) structures on \( Y \) with nontrivial invariants have torsion first Chern class. Turaev introduced another type of torsion in [Tu1, Tu2] and a combinatorially defined function on the set of spin\(^c\) structures \( T: \mathcal{S}(Y) \to \mathbb{Z} \) derived from this torsion, and showed that this function was the Seiberg-Witten polynomial up to sign. Hence, principal \( S^1 \)-bundles over surfaces provide simple examples which illustrate the difference between Milnor torsion and Turaev torsion.

5.2. A construction and a calculation. The following construction is similar to but simpler than the main construction in [FS2]. Let \( Y_K \) denote the manifold resulting from 0-surgery on a knot \( K \) in \( S^3 \). Let \( m \) be a meridian of the knot in \( Y_K \). Let \( m_1, m_2, m_3 \) be loops that correspond to the \( S^1 \) factors of \( T^3 \). Construct a new manifold

\[
M_K = T^3 \#_{m_1=m} Y_K = [T^3 \setminus (m_1 \times D^2)] \cup [Y_K \setminus (m \times D^2)]
\]

by removing tubular neighborhoods of \( m \) and \( m_1 \) and fiber summing the two manifolds along the boundary such that \( m = m_1 \) and such that \( \partial D^2 \) is sent to \( \partial D^2 \).

This is a familiar construction. If one forms a link \( L \) from the Borromean link by taking the composite of the first component with the knot \( K \) (see Figure 1), then \( M_K \) is the result of surgery on \( L \) with each surgery coefficient equal to 0. If \( K \) is a fibered knot, then the resulting manifold \( T^3 \#_{m_1=m} Y_K \) is a fibered 3-manifold.

Consider the formal variables \( t_\beta = \exp(PD(\beta)) \) for each \( \beta \in H_1(M; \mathbb{Z}) \) which satisfy the relation \( t_{\alpha+\beta} = t_\alpha t_\beta \). The Seiberg-Witten polynomial \( \mathcal{SW} \) of \( X \) is a Laurent polynomial with variables \( t_\beta \) and coefficients equal to the Seiberg-Witten invariant of the spin\(^c\) structure defined by \( t_\beta \).
Figure 2. $M_{K_1K_2}$ before surgery

**Theorem 10** (Meng and Taubes [MT]). In the situation above

\[ SW^3_{M_K} = \Delta_K(t_{m_1}^2) \]

where $\Delta_K$ is the symmetrized Alexander polynomial of $K$.

For example, the manifold $M_K$ in Figure 1, where $K$ is the trefoil knot, has Seiberg-Witten polynomial

\[ SW^3_{M_K}(t_{m_1}) = -t_{m_1}^{-2} + 1 - t_{m_1}^2. \]

5.3. **Example 1.** We first produce an example of a nonsymplectic 4-manifold which admits a free circle action whose orbit space is a 3-manifold which is fibered over the circle. Our construction generalizes easily to produce a large class of such manifolds with this property. Let $K_1$ and $K_2$ be any fibered knots. Form the fiber sum of the complements of $K_1$ and $K_2$ with neighborhoods of the first and second meridians of $T^3$, i.e.,

\[ M_{K_1K_2} = (S^3 \setminus K_1)\#_{m=m_1} T^3 \#_{m=m_2} (S^3 \setminus K_2) \]

where $m$ is the meridian of the corresponding knot. Since both $K_1$ and $K_2$ are fibered, the manifold $M_{K_1K_2}$ is a fibered 3-manifold. By Meng-Taubes theorem, the Seiberg-Witten polynomial of this manifold is

\[ SW^3_{M_{K_1K_2}}(t_{m_1}, t_{m_2}) = \Delta_{K_1}(t_{m_1}^2) \Delta_{K_2}(t_{m_2}^2). \]

Let $X_{K_1K_2}(l)$ be the 4-manifold with free circle action that has $M_{K_1K_2}$ for its orbit space and $PD[l]$ for the Euler class of the circle action. Taking both $K_1$ and $K_2$ to be the figure eight knot (see Figure 2), we get a manifold with Seiberg-Witten polynomial:

\[ SW^3_{M_{K_1K_2}} = t_{m_1}^{-2} t_{m_2}^{-2} - 3t_{m_2}^{-2} + t_{m_1}^2 t_{m_2}^{-2} - 3t_{m_1}^{-2} + 9 \]

\[ - 3t_{m_1}^2 + t_{m_1}^{-2} t_{m_2}^{-2} - 3t_{m_2}^2 + t_{m_1}^2 t_{m_2}^2. \]
The Seiberg-Witten polynomial of the manifold $X_{K_1,K_2}(4m_1)$ can be calculated from Theorem 1.

$$SW_{X_{K_1,K_2}(4m_1)} = 2t_{m_1}^{-2} - 3t_{m_2}^{-2} + 9 - 6t_{m_1}^2 + 2t_{m_1+m_2}^2 - 3t_{m_2}^2,$$

where $t_\beta = \exp(\pi^*(PD(\beta)))$ is the pullback of the spin$^c$ structure on $M_{K_1,K_2}$.

A theorem of Taubes [T] implies that the first Chern class $c_1$ of a symplectic 4-manifold must have Seiberg-Witten invariant $\pm 1$. We thus see that the manifold $X_{K_1,K_2}(4m_1)$ admits no symplectic structure with either orientation. This is not the only free $S^1$-manifold over $M_{K_1,K_2}$.

5.4. Example 2. Next we produce an example of a 3-manifold which is not the orbit space of any symplectic 4-manifold with a free circle action. Let $K_1 = K_2$ be the nonfibered knot 5_2 (see [3]). The Seiberg-Witten polynomial of $M_{K_1,K_2}$ is

$$SW_{M_{K_1,K_2}}^3 = 4t_{m_1}^{-2}t_{m_2}^{-2} - 6t_{m_2}^{-2} + 4t_{m_1}^2 t_{m_2}^{-2} - 6t_{m_1}^{-2} + 9 - 6t_{m_1}^2 + 4t_{m_1}^2 t_{m_2}^2 - 6t_{m_2}^2 + 4t_{m_1}^2 t_{m_2}^2.$$

One then needs to calculate as in Example 1. There are only finitely many free $S^1$ manifolds $X_{K_1,K_2}(l)$ which need to be checked because for all $l = am_1 + bm_2$ with $|a|, |b| > 2$ the Seiberg-Witten polynomial $SW^4$ is equal to the 3-dimensional polynomial (only the meaning of the variables will change). A calculation shows that the remaining free $S^1$-manifolds all have spin$^c$ structures with Seiberg-Witten invariant greater than one in absolute value. Therefore these manifolds are not symplectic. Therefore $M_{K_1,K_2}$ is not the orbit space of any symplectic 4-manifold with a free circle action.

5.5. Remarks. The above two examples show:

1. There exist nonsymplectic free $S^1$-manifolds with fibered orbit space.
2. There exists a 3-manifold which is not the orbit space of any symplectic 4-manifold with a free $S^1$-action.

We conclude with two questions.

Question 1. If $X$ is a free $S^1$-manifold which is symplectic, must its orbit space $M = X/S^1$ be fibered?

Taubes has conjectured this in case $X = M \times S^1$. Theorem 1 could be used to search for manifolds with free $S^1$-actions that had nonfibered orbit spaces and which do not have Seiberg-Witten obstructions to having symplectic structures. One would still need to prove that those
manifolds where symplectic. While a counter example may be obtainable, a proof to the affirmative is already at least as difficult as a proof of Taubes’ conjecture.

**Question 2.** Let $M$ be a 3-manifold with the property that every free $S^1$-manifold whose orbit space is $M$ is symplectic. Is $M$ fibered?

The 3-torus is an example of manifold with this property \[\text{[FGG]}\].

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