Numerical Integration over arbitrary Tetrahedral Element by transforming into standard 1-Cube

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Abstract. In this paper, we are using two different transformations to transform the arbitrary linear tetrahedron element to a standard 1-Cube element and obtain the numerical integration formulas over arbitrary linear tetrahedron element implementing generalized Gaussian quadrature rules, with minimum computational time and cost. We also obtain the integral value of some functions with singularity over arbitrary linear tetrahedron region, without discretizing the tetrahedral region into P3 tetrahedral regions. It may be noted the computed results are converging faster than the numerical results in referred articles and are exact for up to 15 decimal values with minimum computational time. In a tetrahedral sub-atomic geometry, a focal particle is situated at the middle with four substituents that are situated at the sides of a tetrahedron. The bond edges are \( \cos^{-1}(-\frac{1}{3}) = 109.4712206\ldots^\circ \approx 109.5^\circ \) when each of the four substituents are the same, as in methane(CH4) and in addition its heavier analogs. The impeccably symmetrical tetrahedron has a place with point amass Td, yet most tetrahedral particles have brought down symmetry. Tetrahedral atoms can be chiral. Mathematically the problem is to evaluate the volume integral over an arbitrary tetrahedron transforming the triple integral over arbitrary linear tetrahedron into the integrals over a standard 1-cube using two different parametric transformations.

1. Introduction

Numerical methods date back to Archimedes who tried to calculate the area of a circle, which happened even before the field of integral calculus was formed. Finite element method (FEM), is an ideal numerical method for a wide range of engineering, biomedical, etc., problems due to its rich mathematical formulation and the ability to model complex geometries using unstructured meshes and employing elements that can be individually tagged makes the method unique. It stems from properly-posed functional minimization principles. Numerical integration is an essential tool to evaluate integrals over iso-parametric tetrahedral elements, as in ‘figure1’.

In a tetrahedral sub-atomic geometry, a focal particle is situated at the middle with four substituents that are situated at the sides of a tetrahedron. The bond edges are \( \cos^{-1}(-\frac{1}{3}) = 109.4712206\ldots^\circ \approx 109.5^\circ \) when each of the four substituents are the same, as in methane(CH4) and in addition its heavier analogs. The impeccably symmetrical tetrahedron has a place with point amass Td, yet most tetrahedral particles have brought down symmetry. Tetrahedral atoms can be chiral.

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Innumerable formulae have been developed for quadrature like Newton cotes formulae and the classical Gauss quadrature formulae, because of the simplicity of the problem and its practical value. The traditional Gaussian quadrature rules are greatly productive when the capacities to be coordinated are very much approximated by polynomials; however in the event that the integrands are unique in relation to polynomials they don’t perform well. Standard practice has been to use Gauss integration, Stroud [1], Cattani [2,3] and Rathod [4,5], since such principles utilize a minimal number of test focuses to accomplish a coveted level of exactness. This economy is imperative for productive firmness lattice estimations. The generalization, of the traditional Gauss quadrature rules, called as generalized-Gaussian quadrature (GGQ), are introduced by Karlin and Studden [6], and numerical scheme for the construction of GGQ was presented by Ma et. al.[7]. Also generalized Gaussian quadrature method over (0, 1) introduced by Ma et al., as in [7], has been verified to give more precise results than the traditional Gauss Legendre quadrature rules over (-1, 1), as in [8, 9, 10, 11], and also it is applicable for functions with end point singularities. In reference [8, 9], Rathod et al. proposed Gauss-Legendre quadrature rule for the numerical integration of an arbitrary function over the standard tetrahedron by transforming the standard tetrahedron into a standard 2-cube. As in [10, 11], he has used a product rule based on zeros and weight coefficients of the Gauss Legendre-Gauss Jacobi quadrature rules, using one of the parametric transformations to transform a standard tetrahedron into a standard 1-cube, as in ‘figure 2’. Also, as in [12], Rathod et al. has proposed another parametric transformation to transform a standard tetrahedron into a standard 1-cube and have applied the product rule for numerical integration which has a higher precision.

In reference [9, 11], Rathod et al. have evaluated the numerical integration of arbitrary functions applying Gauss Legendre quadrature rules, over the p3 tetrahedral regions using one of the parametric transformations to transform a standard tetrahedron into a standard 1-cube,by discretization of standard tetrahedral region Vi into p3 tetrahedral regions each of which has volume equal to units.

In reference [13], Shivaram presents a Generalised Gaussian quadrature method for the evaluation of volume integral of arbitrary functions over the volume of tetrahedral region \( \{(x, y, z)/0 \leq x, y, z \leq a; x + y + z \leq a\} \) by using Gauss divergence theorem to covert volume integral over a tetrahedral region to surface integral as a sum of four integrals over the triangle region with various values of a governed by the proposed method.

In reference, Mamatha and Venkatesh [14], have evaluated the numerical integration of arbitrary integrands by decomposing standard tetrahedral region T(0,1) into four hexahedral element regions H(-1,1) using Gauss quadrature rules. With minimum number of divisions of the tetrahedral region they have shown the convergence of integral values to exact solutions, and also the number of computations and errors are reduced drastically.
Mamatha T. M. and B. Venkatesh [15], have evaluated integrals over tetrahedron using generalized Gaussian quadrature rules and in [16] over an ellipsoid discretized into 10-node tetrahedrons.

In this paper we try to obtain the numerical integration formulas over arbitrary linear tetrahedron by using two different transformations to transform the integral over a linear tetrahedron to a standard 1-cube element, implementing generalized Gaussian quadrature rules and also obtain the integral values of some arbitrary functions without discretization of arbitrary linear tetrahedron region into many tetrahedral regions. Integral values obtained using these quadrature nodes and weights for different integrands are tabulated in this paper and it is established that these quadrature rules give better accuracy for the integrands, including with end point singularities, with minimum computational time and cost.

The problem can be defined mathematically as evaluation of the integral defined ‘as in equation (1)’ by transforming the triple integral over linear tetrahedron into the integrals over a standard 1-cube.

\[ I = \iiint_{0}^{1} f(x, y, z) \, dz \, dy \, dx \]

(1)

2. Volume integration over an arbitrary tetrahedron by transforming into a standard 1-cube

In the reference article by Rathod[2005], the author has applied Gauss-Legendre quadrature rule for numerical integration over the standard tetrahedron by transforming the standard tetrahedron into a standard 2-cube. And in [2007] they have applied the same method by discretizing the tetrahedral region into p3 tetrahedral regions. In another article Rathod[2007, 2011], he has applied the Gauss Legendre- Gauss Jacobi quadrature product rules to evaluate integrals over arbitrary tetrahedral regions by an affine-transformation to transform an arbitrary tetrahedron in \((x, y, z)\) space into an orthogonal tetrahedron \((\xi_1, \xi_2, \xi_3)\), and then transforming the tetrahedral region into a standard 1-Cube \(0 \leq \xi_1, \xi_2, \xi_3 \leq 1\).

In this paper, we use generalized Gaussian quadrature rules to evaluate integrals over arbitrary tetrahedral regions by transforming the orthogonal tetrahedron into a standard 1-cube \(0 \leq \xi_1, \xi_2, \xi_3 \leq 1\) by using two parametric representations ‘as in equations (5), (8)’ and show that the numerical evaluation of integrands converges faster with minimum computational time and cost.

2.1. Volume Co-ordinates for an Arbitrary Linear Tetrahedron

For transforming the tetrahedral region into a standard 1-Cube, the limits of integration have to be mapped to \(0 \leq \xi_1, \xi_2, \xi_3 \leq 1\) in \((\xi_1, \xi_2, \xi_3)\) space. This is achieved by transforming the \((x, y, z)\) to the natural coordinates \((\xi_1, \xi_2, \xi_3)\) using the equation

\[
\begin{align*}
x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \\
y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \\
z &= N_1 z_1 + N_2 z_2 + N_3 z_3 + N_4 z_4 \\
N &= N_1 + N_2 + N_3 + N_4
\end{align*}
\]

(2)

The ratio of the volumes of tetrahedron based on an internal point C, as in ‘figure 3’, in the total volume element is identified as the physical nature of the coordinates.

\[
\begin{align*}
N_1 &= \frac{\text{Volume of } C_{234}}{\text{volume of } C_{1234}}, \quad N_2 = \frac{\text{Volume of } C_{134}}{\text{volume of } C_{1234}}, \quad N_3 = \frac{\text{Volume of } C_{124}}{\text{volume of } C_{1234}}, \quad N_4 = \frac{\text{volume of } C_{132}}{\text{volume of } C_{1234}}
\end{align*}
\]
The \((x, y, z)\) coordinates transformed to natural coordinates \((\xi_1, \xi_2, \xi_3)\) using the equations ‘as in equation (2)’
\[
x = N_1 x_i ; y = N_2 y_i ; z = N_3 z_i ;
\]
\(N_1 + N_2 + N_3 + N_4 = 1, \) where \((x_i, y_i, z_i)\) are the coordinates of vertex of the tetrahedron.

\[\text{Figure 3 Volume co-ordinates}\]

### 2.1.1 Affine Transformation for an Arbitrary Linear Tetrahedron

The transformation between the \((x, y, z)\) to the natural coordinates \((\xi_1, \xi_2, \xi_3)\) ‘as in equation (3)’ is rewritten by substituting
\[
x = \frac{\xi_1}{N_1} x_{14} + \frac{\xi_2}{N_2} x_{24} + \frac{\xi_3}{N_3} x_{34} + x_4
\]
\[
y = \frac{\xi_1}{N_1} y_{14} + \frac{\xi_2}{N_2} y_{24} + \frac{\xi_3}{N_3} y_{34} + y_4
\]
\[
y = \frac{\xi_1}{N_1} z_{14} + \frac{\xi_2}{N_2} z_{24} + \frac{\xi_3}{N_3} z_{34} + z_4
\]
with \(x_{ij} = x_i - x_j ; y_{ij} = y_i - y_j ; z_{ij} = z_i - z_j\)

By the parametric representation ‘as in equation (4)’ we have for the volume element ‘as in equation (1)’ represented as
\[
I = \iiint_T f(x, y, z) \, dx \, dy \, dz
\]
\[
= \iiint_{\tau} f(x(\xi_1, \xi_2, \xi_3), y(\xi_1, \xi_2, \xi_3), z(\xi_1, \xi_2, \xi_3)) |J| \, d\xi_1 \, d\xi_2 \, d\xi_3
\]
Where
\[
|J| = \begin{vmatrix}
x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\
y_1 - y_4 & y_2 - y_4 & y_3 - y_4 \\
z_1 - z_4 & z_2 - z_4 & z_3 - z_4
\end{vmatrix}
\]
\(= 6\) (volume of tetrahedron)

\(V\) is the unit orthogonal tetrahedron spanned by the vertices \(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\).

### 2.2. Transformation of Integrals over an Arbitrary Tetrahedron into Integrals over a Standard 1-Cube

In this section, we transform the triple integral of the form ‘as in equation (1)’ into the integrals over the standard 1-Cube \((0 \leq \xi_1, \xi_2, \xi_3 \leq 1)\) by using two parametric transformations.

**Transformation-I**

Consider the integral ‘as in equation (1)’ let us write the parametric representation for the transformation ‘as in equation (5),"
\[ x = l \times m (1-n) , y = m(1-l) , z = l(1-m) ; \] (5)

where 0 \leq l,m,n \leq 1, such that it completely describes the tetrahedral region \( x + y + z = 1, \)
\( x \geq 0 , y \geq 0 , z \geq 0. \)

Then the differential volume and the determinant of the Jacobian are
\[ dx \, dy \, dz = |J| \, dl \, dm \, dn = l^2 m^2 d l \, dm \, dn \]

and
\[
|J| = \frac{\partial (x,y,z)}{\partial(l,m,n)} = \frac{\partial x}{\partial l} \frac{\partial y}{\partial m} \frac{\partial z}{\partial n} + \frac{\partial x}{\partial m} \frac{\partial y}{\partial n} \frac{\partial z}{\partial l} + \frac{\partial x}{\partial n} \frac{\partial y}{\partial l} \frac{\partial z}{\partial m} = -l^2 m
\] (6)

Thus the volume integral ‘as in equation (1)’ is transformed over a standard 1-cube by using the
Transformation-I ‘as in equation (5)’ is given by the integral ‘as in equation (7)’.
\[
I = \iiint_{0 \leq l \leq 1, 0 \leq m \leq 1} f(l,m,n) d l \, dm \, dn
\] (7)

Transformation-II
Consider the integral ‘as in equation (1)’ let us write the parametric representation as
\[ x = l , y = m(1-l) , z = n(1-l)(1-m) ; \] (8)

where \( 1-x-y-z = (1-l)(1-m)(1-n) \), 0 \leq l,m,n \leq 1 such that it completely describes the region
\( x + y + z = 1, x \geq 0 , y \geq 0 , z \geq 0. \)

Then the differential volume and the determinant of the Jacobian are
\[ dx \, dy \, dz = |J| \, dl \, dm \, dn = (1-l)^2 (1-m) d l \, dm \, dn \]

and
\[
|J| = \frac{\partial (x,y,z)}{\partial(l,m,n)} = \frac{\partial x}{\partial l} \frac{\partial y}{\partial m} \frac{\partial z}{\partial n} + \frac{\partial x}{\partial m} \frac{\partial y}{\partial n} \frac{\partial z}{\partial l} + \frac{\partial x}{\partial n} \frac{\partial y}{\partial l} \frac{\partial z}{\partial m} = (1-l)^2 (1-m)
\] (9)

Hence the volume integral ‘as in equation (1)’ is transformed over a standard 1-cube by using the
transformation-2, ‘as in equation (8)’, is given by the integral ‘as in equation (10)’.
\[
I = \iiint_{0 \leq l \leq 1, 0 \leq m \leq 1} f(l,m,n) (1-l)^2 (1-m) d l \, dm \, dn.
\] (10)

2.3. Numerical Integration
We now apply generalized Gauss quadrature rule to evaluate integrals ‘as in equations (7), (10)’
\[ I = \iiint f(lm, l(1-n), l(1-m)) \, dldmdn = \sum_{m=1}^{N} a_m \, g(x_m, y_m, z_m) \quad (11) \]
\[ I = \iiint f(l, m(1-l), n(1-l)(1-m)^2(1-m)) \, dldmdn = \sum_{m=1}^{N} k_m \, h(x_m, y_m, z_m) \quad (12) \]

where \( N \) is the number of sampling points, the abscissas are as

\[ h(x_m, y_m, z_m) = f(l, m(1-l), n(1-l)(1-m)^2(1-m)), \]
\[ g(x_m, y_m, z_m) = f(lm, lm(1-n), l(1-m)) \]

and weight coefficients of generalized Gaussian quadrature rule are as

\[ k_m = (1-l)^2(1-m)w_i w_j w_k; \quad c_m = l^2 w_i w_j w_k. \]

Integral values as in equations (11), (12) obtained using these generalized Gaussian quadrature rules nodes and weights for different integrands based on availability of analytical solutions are tabulated and the effectiveness of the parametric transformation of the proposed method is tested.

2.3.1 Numerical Example

We consider some typical integrands whose integral values are known on standard tetrahedral domain and with the exact value of the integrals [2005, 2007].

\[ I_1 = \iiint_{T} \frac{1}{\sqrt{(1-x-y)^2 + z^2}} \, dx \, dy \, dz \quad (13) \]
\[ I_2 = \iiint_{T} (x + y + z) \frac{1}{2} \, dx \, dy \, dz \quad (14) \]
\[ I_3 = \iiint_{T} (x + y + z)^{1/2} \, dx \, dy \, dz \quad (15) \]
\[ I_4 = \iiint_{T} \sin(x + y^2 + z^4) \, dx \, dy \, dz \quad (16) \]
\[ I_5 = \iiint_{T} \frac{1}{1 + x + y + z^4} \, dx \, dy \, dz \quad (17) \]
\[ I_6 = \iiint_{T} x^3 \sin(y\pi) \sin(z\pi) \, dx \, dy \, dz \quad (18) \]

We evaluate the numerical solution to the integrals as in equations (11), (12) considering the arbitrary integrands as in equations (13)-(18) by applying generalized Gaussian quadrature (GGQM) integration method whose results are given in Appendix (Table I). It is observed that the computed results (GGQM) are converging faster than the Gauss-Legendre quadrature (GLQM) numerical method in [2005, 2007] and as compared to the Gauss-Legendre and Gauss-Jacobi quadrature (GLJQM) numerical method in [2007, 2011], without discretization of tetrahedral regions we are able to obtain the numerical results exact up to 15 decimal places. As in equation (18), we have computed the numerical solution to the integral with the singularity function, which is not computed in the referred articles.

**Table 4. Numerical Examples**
\[ I_1 = \iiint_{T} \frac{1}{\sqrt{(1-x-y)^2 + z^2}} \, dx \, dy \, dz = 0.4406867935 \, 0'9772 \]

**Transformation-1**

| \( N \) | \( \text{GGQ by Shivaram [2013]} \) | \( \text{GGQM} \) | \( \text{GLJQM}[11] \) | \( \text{GLQM by Rathod[2005]} \) |
|---|---|---|---|---|
| 5 | 0.44668900461523 | 0.40143409154618 | 0.43152943273086 | 0.438056466774589 | 0.438461566166572 | 0.43852199025403 |
| 10 | 0.44668900461526 | 0.429349205239327 | 0.432737424267085 | 0.43849153568623 | 0.43885702656758 | 0.438859152482304 |
| 15 | 0.435935540199289 | 9 | 0.43423418795857 | 0.43800119004454 | 0.439099456872214 | 0.43983654085687 |
| 20 | 0.437634725073964 | 10 | 0.43942502661705 | 0.439022570568174 | 0.439290122690003 | 0.439250549614584 |

**Transformation-2**

| \( N \) | \( \text{GGQ by Shivaram [2013]} \) | \( \text{GGQM} \) | \( \text{GLJQM by Rathod and Shafiqiu[2007]} \) | \( \text{GLQM by Rathod[2007]} \) |
|---|---|---|---|---|
| 5 | 0.440685900461525 | 0.404255520664723 | 0.440032854640855 | 0.440156655061396 | 0.44009086267002 |
| 10 | 0.440686793510626 | 0.43031208274682 | 0.440028320026023 | 0.440152657197482 | 0.44000515259131 |
| 15 | 0.440686793509740 | 9 | 0.43613569329094 | 0.44002572247437 | 0.440150367069788 | 0.44000436950297 |
| 20 | 0.440686793509771 | 10 | 0.440024154598790 | 0.4401489682863 | 0.440002234985244 |

\[ I_2 = \iiint_{T} \frac{1}{(x+y+z)^2} \, dx \, dy \, dz = 0.142857142857143 \]

**Transformation-1**

| \( N \) | \( \text{GGQ by Shivaram [2013]} \) | \( \text{GGQM} \) | \( \text{GLJQM by Rathod [2011]} \) | \( \text{GLQM by Rathod[2005]} \) |
|---|---|---|---|---|
| 5 | 0.142857116529783 | 0.142857129401889 | 0.143266558916461 | 0.142857143531164 | 0.142857144311217 | 0.142857143763835 |
| 10 | 0.142857142852557 | 0.142857142856268 | 8 | 0.142857149026250 | 0.142857143904410 | 0.142857143467725 | 0.142857143239767 |
| 15 | 0.142857142857140 | 9 | 0.142857169357884 | 0.142857143341263 | 0.142857143139396 | 0.142857143044018 |
| 20 | 0.142857142857143 | 10 | 0.142488864056452 | 0.142857143098771 | 0.142857142998081 | 0.142857142945423 |

**Transformation-2**

| \( N \) | \( \text{GGQ by Shivaram [2013]} \) | \( \text{GGQM} \) | \( \text{GLJQM by Rathod and Shafiqiu[2007]} \) | \( \text{GLQM by Rathod[2007]} \) |
|---|---|---|---|---|
| 5 | 0.142857116529783 | 0.142857116529782 | 7 | 0.142857173245610 | 0.142857143720264 | 0.142857143360363 | 0.142857143172488 |
| 10 | 0.142857142852557 | 0.142857142852559 | 8 | 0.142857152996523 | 0.142857143208842 | 0.142857143062192 | 0.142857142985637 |
| 15 | 0.142857142857117 | 9 | 0.142857152553564 | 0.142857143016253 | 0.142857142949907 | 0.142857142915274 |
| 20 | 0.142857142857143 | 10 | 0.142857142935285 | 0.142857142920270 | 0.142857142885692 |

\[ I_3 = \iiint_{T} \frac{1}{x+y+z} \, dx \, dy \, dz = 0.200000000000000 \]

**Transformation-1**

| \( N \) | \( \text{GGQ by Shivaram [2013]} \) | \( \text{GGQM} \) | \( \text{GLJQM by Rathod [2011]} \) | \( \text{GLQM by Rathod[2005]} \) |
|---|---|---|---|---|
| 5 | 0.200000275082717 | 0.1999999999961675 | 7 | 0.200673496303585 | 0.199999233473790 | 0.199999185415138 | 0.199999613466358 |
| 10 | 0.200000002496587 | 0.19999999999859248 | 8 | 0.1999992401298026 | 0.199999591454682 | 0.199999722109779 | 0.1999998091524 |
| 15 | 0.1999999999970998 | 9 | 0.199999517776173 | 0.199999764028885 | 0.199999839491058 | 0.199999885047250 |
| 20 | 0.1999999999999820 | 10 | 0.199607363698843 | 0.199999856139718 | 0.199999920147048 | 0.199999929920004 |

**Transformation-2**
\[ I_4 = \int \int \int \sin(x + y^2 + z^4) \, dx \, dy \, dz = 0.131902326890181 \]

Transformation-1

\[ I_5 = \int \int \int \frac{1}{(1 + x + y + z)^5} \, dx \, dy \, dz = 0.020833333333333 \]

Transformation-1

\[ I_6 = \int \int \int x^3 \sin(y \pi) \sin(z \pi) \, dx \, dy \, dz = 0.00118021542112911 \]
2.4. Conclusions

In this research article, we have evaluated the volume integral of an integrand over arbitrary tetrahedral region by first transforming into standard tetrahedron and later by using two different transformations we transform standard tetrahedron in $(x, y, z)$ space into a standard $1$-Cube in $(l, m, n)$ space. We have used generalized Gaussian quadrature method for the evaluation of the triple integral, as in (1). It may be noted that almost all the numerical results which we have obtained is exact for more than ten decimal values, converging faster than the numerical results in Rathod [2011] and without discretization of tetrahedral regions we are obtaining the exact integral values as compared to the numerical results in Rathod [10-13] by reducing the number of computations. We have also compared the computed results with the results obtained by K T Shivaram[16], the results are converging faster using the derived transformations than the results in [16]. We observed that the numerical results obtained with Transformation-2 for the integrands are more accurate with analytical solutions compared to the results obtained with Transformation-1.

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