Large System Analysis for Amplify & Forward SIMO Multiple Access Channel with Ill-conditioned Second Hop

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Abstract—Relaying has been extensively studied during the last decades and has found numerous applications in wireless communications. The simplest relaying method, namely amplify and forward, has shown potential in MIMO multiple access systems, when Gaussian fading channels are assumed for both hops. However, in some cases ill conditioned channels may appear on the second hop. For example, this impairment could affect cooperative BS systems with microwave link backhauling, which involve strong line of sight channels with insufficient scattering. In this paper, we consider a large system analysis of such a model focusing on both optimal joint decoding and joint MMSE filtering receivers. Analytical methods based on free probability are presented for calculating the ergodic throughput, the MMSE error and the average SINR. Furthermore, the performance degradation of the system throughput is evaluated considering second hop impairments such as ill-conditioning and rank deficiency, while high- and low-SNR limits are calculated for the considered performance metrics. Finally, the cooperative BS system is compared to a conventional channel resource division strategy and suitable operating points are proposed.

Index Terms—Amplify and Forward, Multiuser Detection, Ill-conditioned Channel, Rank-deficient Channel.

I. INTRODUCTION

The Dual Hop (DH) Amplify-and-Forward (AF) relay channel has attracted a great deal of attention mainly due to its low complexity and its manyfold benefits, such as coverage extension and decreased outage probability. Although the DH AF channel has been extensively studied in the literature [1]–[3], the effect of second hop condition number on its performance is not well quantified yet.

Assuming Gaussian channel matrices in both hops, authors in [1] approached the problem asymptotically using Silverstein’s fixed-point equation and found closed-forms expressions for the Stieltjes transform. Under similar assumptions, a finite analysis was recently performed by [2]. On the other hand, authors in [3] following a replica analysis tackled the problem of Kronecker correlated Gaussian matrices.

In addition, the MIMO MAC has been studied heavily during the last decades since it comprises a fundamental channel model for multiuser uplink cellular [4] and multibeam return link communications [5], [6]. The work in [7], [8] has combined AF relaying with a MAC and has performed a free-probabilistic analysis for channel capacity. Furthermore, the work in [9] has combined AF relaying with cooperative Base Stations and has performed a replica analysis for channel capacity and MMSE throughput.

In our scenario, we study a DH AF SIMO MAC modelling cooperative BSs with microwave link backhauling and we focus on the impact of ill-conditioned or rank-deficient MIMO channel matrices in the second hop. The paradigm of BS cooperation (also known as multicell joint decoding and network MIMO) was initially proposed almost three decades ago and its performance gain over conventional cellular systems was demonstrated in two seminal papers [10], [11]. The main assumption is the existence of a central processor (CP) which is interconnected to all the BSs through a backhaul of wideband, delayless and error-free links. In addition, the central processor is assumed to have perfect Channel State Information (CSI) about all the wireless links of the system. These assumptions enable the central processor to jointly decode all the UTs of the system, rendering the concept of intercell interference void. Since then, there has been an ongoing research activity extending and modifying the initial results for more practical propagation environments, transmission techniques and backhaul infrastructures in an attempt to better quantify the performance gain.

More specifically, it was demonstrated in [12] that Rayleigh fading promotes multiuser diversity which is beneficial for the ergodic capacity performance. Subsequently, realistic path-loss models and user distribution were investigated in [13], [14] providing closed-form capacity expressions based on the cell size, path loss exponent and user spatial p.d.f. The beneficial effect of MIMO links was established in [15], [16], where a linear scaling with the number of BS antennas was proven. However, correlation between multiple antennas has an adverse effect as shown in [4], especially when correlation affects the BS-side.

Regarding backhauling, the ideal assumptions of previous studies can only be satisfied by fiber connectivity between all BSs and the central processor. However, in current backhaul infrastructure microwave links are often used, especially in rural environments where the cable network is unavailable. Recent studies have tried to alleviate the perfect backhaul assumption by focusing in finite-rate errorless links to the CP [17], finite-rate errorless links between adjacent BSs [18] and finite-sum-rate backhaul with imperfect CSI [19]. Contrary to these approaches, this paper assumes microwave backhauling from all BSs to the CP, operating over the same frequency. The BSs amplify and forward the received signals to an antenna array at the CP and thus the backhaul rate is limited by the
system geometry, the relaying power and the impairments of the second hop MIMO channel.

In this direction, the main contributions of this paper are:

- the derivation of the ergodic capacity and a lower bound on the average Minimum Mean Square Error (MMSE) for AF SIMO MAC with ill-conditioned second hop
- the derivation of high and low SNR limits for channel capacity and MMSE performance
- the evaluation of the condition number and normalized rank of the second hop channel matrix on the system performance
- the performance comparison to a conventional system which employs resource division access to eliminate multiuser interference.

The remainder of this paper is structured as follows: Section II introduces the system model, while section III describes the derivation of high and low SNR limits for channel performance

A. Notation

Throughout the formulations of this paper, normal x, lower-case boldface x and upper-case boldface X font is used for scalars, vectors and matrices respectively. $\mathbb{E}[-]$ denotes the expectation, $(\cdot)^H$ denotes the conjugate transpose matrix, and $\odot$ denotes the Hadamard product. The Frobenius norm of a scalar is denoted by $|\cdot|$, the absolute value of a scalar is denoted by $|\cdot|$, and the delta function is denoted by $\delta(\cdot)$. $(\cdot)^+$ is equivalent to $\max(0, \cdot)$, $\mathbb{I}\{\cdot\}$ is the indicator function and $\Rightarrow$ denotes almost sure (a.s.) convergence.

II. SYSTEM MODEL

Figure 1 is a conceptual illustration of the input-output model, which included $M$ users, $K$ BSs and a CP equipped with a $K$-antenna array. It can be seen that the BS-CP (Central Processor) microwave links (second hop) form an ill-conditioned SIMO MAC, whereas the user-BS-CP links can be modelled as SIMO AF MAC. Gaussian input is considered at the user-side, while neither users nor relays are aware of the Channel State Information (CSI). On the other hand, the CP is assumed to have perfect knowledge of system-wide CSI. The described channel model can be expressed as follows:

$$y_1 = H_1x_1 + z_1$$
$$y_2 = H_2\sqrt{\nu}y_1 + z_2 \Leftrightarrow$$
$$y_2 = \sqrt{\nu}H_2H_1^*x_1 + \sqrt{\nu}H_2^*z_1 + z_2,$$

(1)

where the $M \times 1$ vector $x_1$ denotes the user transmitted symbol vector with individual Signal to Noise Ratio (SNR) $\mu (\mathbb{E}[x_1^Hx_1] = \mu I)$, $y_1$ denotes the $K \times 1$ received symbol vector by the BSs and the $K \times 1$ vector $z_1$ denotes AWGN at BS-side with $\mathbb{E}[z_1] = 0$ and $\mathbb{E}[z_1^H] = I$. The received signal $y_1$ is amplified by $\nu$ and forwarded as a result $y_2$ denotes the $K \times 1$ received symbol vector by the CP and the $K \times 1$ vector $z_2$ denotes AWGN at CP-side with $\mathbb{E}[z_2] = 0$ and $\mathbb{E}[z_2^H] = I$. It should be noted that for the remainder of this document $\mu$ and $\nu$ will be referred to as First Hop Power (FHP) and Second Hop Power (SHP) respectively.

The $K \times M$ channel matrix $H_1$ and the $K \times K$ channel matrix $H_2$ represent the concatenated channel vectors for the user-BS and BS-CP links respectively. The first hop Rayleigh fading channel $H_1 \sim CAV(0, I)$ can be modelled as a Gaussian matrix with independent identically distributed (i.i.d.) complex circularly symmetric (c.c.s.) elements. The BSs-CP channel $H_2$ under line of sight suffers from correlation due to lack of scattering and thus it can be modelled as an ill-conditioned deterministic channel with variable condition number $\zeta^2 = \lambda_{\text{max}}(H_2^HH_2)/\lambda_{\text{min}}(H_2^HH_2)$ or as a rank-deficient deterministic channel with variable normalized rank $\alpha = \text{rank}(H_2^HH_2)/K$. The exact matrix models for $H_2$ are described in detail in sections III-B and III-D.

B. Performance Metrics

The performance metrics considered in this work are the channel capacity achieved by successive interference cancellation at the CP and the average Minimum Mean Square Error (MMSE) achieved by joint MMSE filtering at the CP followed by single-user decoding. It should be noted that both of these receiver structures require multiuser processing at the CP. On the other hand, section III-C considers a conventional system where Frequency or Time Division Multiple Access is used in combination with single-user interference-free decoding at the CP.

The capacity per receive antenna of this channel model is given by (20)–(23):

$$C = \frac{1}{K} \mathbb{E} \left[ \log \det \left( I + \mu H_2H_1^HH_2^H \left( I + \nu H_2^HH_2^H \right)^{-1} \right) \right]$$

(2)

$$= \frac{1}{K} \mathbb{E} \left[ \log \det \left( I + \nu H_2^HH_2^H \right) \right]$$

(3)

$$\approx \frac{1}{K} \mathbb{E} \left[ \log \det \left( I + \nu H_2^HH_2^H \left( I + \mu H_1^HH_1^H \right) \right) \right]$$

(4)

where step $(a)$ uses the property $\log \det (I + AB) = \log \det (I + BA)$. It can be observed that the positive term $C_1$...
corresponds to the mutual information due to relaying, while the negative term $C_2$ represents the performance loss due to noise amplification.

The receiver complexity in order to achieve the channel capacity is quite high since it involves successive interference cancellation [24]. In this direction, we consider a less complex receiver which involves multiuser MMSE filtering followed by single-user decoding. Since, this is a linear operation we assume that $K = M$. The performance of the MMSE receiver is dependent on the achieved MSE averaged over users and channel realizations and is given by:

$$\text{mmse}_{\text{avg}} = \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^{M} \text{mmse}_m \right]$$

$$= \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^{M} \left( I + \mu \mathbf{H}^H \mathbf{H}^{-1} \right) \right]$$

$$= \mathbb{E} \left[ \frac{1}{M} \text{tr} \left\{ \left( I + \mu \mathbf{H}^H \mathbf{H}^{-1} \right) \right\} \right]$$

$$= \mathbb{E} \left[ \frac{1}{M} \text{tr} \left\{ \left( I + \mu \mathbf{H}^H \mathbf{H}^{-1} \right) \right\} \right]$$

The average SINR and the achieved throughput per receive antenna using LMMSE is given by:

$$\text{SINR}_{\text{avg}} = \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^{M} \text{mmse}_m \right] - 1$$

$$C_{\text{mmse}} = \log (1 + \text{SINR}_{\text{avg}}) \geq - \log (\text{mmse}_{\text{avg}})$$

$$= - \log \left( \frac{1}{M} \mathbb{E} \left[ \text{tr} \left\{ \left( I + \mu \mathbf{H}_2 \left( I + \mu \mathbf{H}_1 \mathbf{H}^H \right) \mathbf{H}^{-1} \right) \right\} \right] \right).$$

Compared to existing literature, our work starts from eq. [4] since the original problem in eq. [3] yields quite involved solutions [11–13]. In addition, by decomposing the problem in two components, deeper insights can be acquired. We follow a free probabilistic analysis as in [4], [23], [25]–[27] to derive the channel capacity, but we extend it for the described DH AF SIMO MAC including the noise amplification terms and ill-conditioned second hop modelling. More importantly, we consider the MMSE filtering receiver and we obtain a lower bound on the average MMSE performance.

To simplify the notations during the mathematical analysis, the following auxiliary variables are defined:

$$\mathbf{M} = I + \mu \mathbf{H}_1 \mathbf{H}_1^H$$

$$\tilde{\mathbf{M}} = I + \mu \mathbf{H}_2 \mathbf{H}_2^H$$

$$\mathbf{N} = \mathbf{H}_1 \mathbf{H}_1^H$$

$$\tilde{\mathbf{N}} = \mathbf{H}_2 \mathbf{H}_2^H$$

$$\mathbf{K} = \mathbf{H}_2 \mathbf{H}_2 \left( I + \mu \mathbf{H}_1 \mathbf{H}_1^H \right) = \tilde{\mathbf{N}} \mathbf{M}$$

$$\tilde{\mathbf{K}} = \mathbf{H}_2 \left( I + \mu \mathbf{H}_1 \mathbf{H}_1^H \right) \mathbf{H}_2^H$$

$$\beta = \frac{M}{K}$$

where $\beta \geq 1$ is the ratio of horizontal to vertical dimensions of matrix $\mathbf{H}_1$ (users/BS).

### C. Conventional System

In a conventional cellular system, the available resources (frequency or time) would have to be split in $K$ pieces in order to avoid multiuser interference from neighboring BSs. This entails that only $K$ out of $M$ users could be served simultaneously, namely one user per cell ($\beta = 1$). Moreover, this is the usual approach employed by current standards in order to avoid co-channel interference. On the plus side, each user or BS relay could concentrate its power on a smaller portion of the resource using $K\mu$ and $K\nu$ respectively. Assuming a single user per cell ($K = M$), the conventional channel model for a single user-BS-CP link can be written as:

$$y_1 = h_1 x_1 + z_1$$

$$y_2 = h_2 \sqrt{K \nu} y_1 + z_2 \Leftrightarrow$$

$$y_2 = \sqrt{K \nu} h_2 x_1 + \sqrt{K \nu} h_2 z_1 + z_2 \quad (8)$$

with $x_1$ Gaussian input with $\mathbb{E}[x_1^2] = K\mu$ and $z_1, z_2$ AWGN with $\mathbb{E}[z_1^2] = \mathbb{E}[z_2^2] = 1$. In this case, the per-antenna capacity at the CP would be:

$$C_{\text{co}} = \mathbb{E} \left[ \log (1 + \text{SNR}) \right] = \mathbb{E} \left[ \log \left( 1 + \frac{K^2 \nu h_2^2 \mu_1^2}{1 + K \nu h_2^2} \right) \right],$$

where $h_1$ and $h_2$ are the channel coefficients of the first and second hop respectively. The first and second hop are modelled as Rayleigh fading and AWGN channels respectively and thus we can assume that $h_1 \sim \mathcal{CN}(0,1)$ and $h_2 = 1$. The performance of the conventional and proposed transmission schemes are compared in section [IV-D].

### III. Performance Analysis

In order to calculate the system performance analytically, we resort to asymptotic analysis which entails that the dimensions of the channel matrices grow to infinity assuming proper normalization. It has already been shown in many occasions that asymptotic analysis yields results which are also valid for finite dimensions [22], [23], [29]. In other words, the expressions of interest converge quickly to a deterministic value as the number of channel matrix dimensions increases.

In this direction, the components of eq. [4] can be written asymptotically as:

$$C_1 = \frac{1}{K} \lim_{K \to \infty} \mathbb{E} \left[ \log \text{det} \left( I + \nu \mathbf{H}_2 \mathbf{H}_2 \left( I + \mu \mathbf{H}_1 \mathbf{H}_1^H \right) \right) \right]$$

$$= \lim_{K \to \infty} \mathbb{E} \left[ \frac{1}{K} \sum_{i=1}^{K} \log (1 + \nu \lambda_i (\mathbf{K})) \right]$$

$$\to \int_{0}^{\infty} \log (1 + \nu x) f_K (x) \, dx,$$

1In reality, higher frequency reuse can be used in order to exploit spatial separation of cells. However, frequency reuse cannot be exploited in the considered system without creating multiuser interference in the CP through the AF relaying.
where $\lambda_i(\mathbf{X})$ is the $i$th ordered eigenvalue of matrix $\mathbf{X}$ and $f_\infty^X$ is the asymptotic eigenvalue probability density function (a.e.p.d.f.) of $\mathbf{X}$. It should be noted that while the channel dimensions $K, M$ grow to infinity, the matrix dimension ratio $\beta$ is kept constant.

Using a similar approach, the average MMSE when $\beta = 1$ can be expressed as:

$$\text{mmse}_{\text{avg}} = \lim_{K,M \to \infty} \mathbb{E} \left[ \frac{1}{M} \text{tr} \left( (1 + \nu \mathbf{M})^{-1} \mathbf{K} \right) \right]$$

$$\geq \lim_{K,M \to \infty} \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^{M} \lambda_{m-1} \left( \mathbf{M} \right) \right]$$

$$\rightarrow \int_0^1 \frac{F_{\mathbf{X}}^{-1}(1 - x)}{1 + \nu F_{\mathbf{X}}^{-1}(x)} \, dx,$$

where step (a) follows from property $\text{tr} \{ \mathbf{A} \mathbf{B} \} \geq \sum_{m=1}^{M} \lambda_m(\mathbf{A}) \lambda_{m-1}(\mathbf{B})$ in [30] and $F_{\mathbf{X}}^{-1}$ denotes the inverse function of the asymptotic eigenvalue cumulative density function (a.e.c.d.f.). From the last step follows from the fact that the ordered eigenvalues can be obtained by uniformly sampling the inverse c.d.f. in the asymptotic regime [5].

To calculate the expression of eq. (10), (11), (13), it suffices to derive the asymptotic densities of $\mathbf{K}, \mathbf{N}, \mathbf{K}, \mathbf{M}$, which can be achieved through the principles of free probability theory [31–34] as described in sections III-A and III-B. Free probability (FP) has been proposed by Voiculescu [31] and has found numerous applications in the field of wireless communications. More specifically, FP has been applied for capacity derivations of variance profiled [13], correlated Rayleigh channels, as well as Rayleigh product channels [25]. Furthermore, it has been used for studying cooperative relays [8], interference channels [25] and interference alignment scenarios [26]. The advantage of FP methodology compared to other techniques, such as Stieltjes method, replica analysis and deterministic equivalents, is that the derived formulas usually require just a polynomial solution instead of fixed-point equations. However, the condition for these simple solutions is that the original aepdfs can be expressed in polynomial form [36]. For completeness, some preliminaries of Random Matrix Theory have been included in appendix A in order to facilitate the comprehension of derivations in sections III-A, III-B and III-D.

A. Fading First Hop

The first hop from users to BSs can be modelled as a Rayleigh fading channel, namely $\mathbf{H}_i \sim \mathcal{CN}(0,1)$.

Definition III.1. Considering a Gaussian $K \times M$ channel matrix $\mathbf{H}_i \sim \mathcal{CN}(0,1)$, the a.e.p.d.f. of $\frac{1}{\sqrt{K}} \mathbf{H}_i \mathbf{H}_i^H$ converges almost surely (a.s.) to the non-random limiting eigenvalue distribution of the Marčenko-Pastur law [37], whose density functions are given by

$$f_{\text{MP}}^X(\mathbf{x}, \beta) \rightarrow f_{\text{MP}}(x, \beta)$$

$$f_{\text{MP}}(x, \beta) = (1 - \beta)^+ \delta(x) + \frac{\sqrt{(x - a)^+ (b - x)^+}}{2\pi x}$$

where $a = (1 - \sqrt{\beta})^2, b = (1 + \sqrt{\beta})^2$ and $\eta$-transform, $\Sigma$-transform and Shannon transform are given by

$$\eta_{\text{MP}}(x, \beta) = 1 - \frac{\phi(x, \beta)}{4x}$$

$$\phi(x, \beta) = \left(\sqrt{x (1 + \sqrt{\beta})^2 + 1} - \sqrt{x (1 - \sqrt{\beta})^2 + 1}\right)^2$$

$$\Sigma_{\text{MP}}(x, \beta) = \frac{1}{\beta + x}$$

$$\nu_{\text{MP}}(x, \beta) = \beta \log \left(1 + x - \frac{1}{4} \phi(x, \beta)\right) + \log \left(1 + x\beta - \frac{1}{4} \phi(x, \beta)\right) - \frac{1}{4x} \phi(x, \beta).$$

Lemma III.1. The cumulative density function of the Marčenko-Pastur law for $\beta = 1$ is given by:

$$F_{\text{MP}}(x) = \sqrt{-x(x-4)} + 2 \arcsin \left(-1 + x/2\right) + \pi$$

Proof: The c.d.f. follow from eq. (14) after integration for $\beta = 1$.

Lemma III.2. The a.e.p.d.f. of $\mathbf{M}$ converges almost surely (a.s.) to:

$$f_{\mathbf{M}}^X(x, \beta, \bar{\nu}) \rightarrow \frac{\sqrt{(x - 1 - \bar{\nu} + 2\beta \sqrt{\bar{\nu} - \bar{\nu}^2} (\beta + 2\bar{\nu} \sqrt{\bar{\nu} + \bar{\nu}^2 - x + 1}}}{2\beta \pi (x - 1)}.$$

where $\bar{\nu} = K \mu$.

Proof: The a.e.p.d.f. can be calculated considering the transformation $z(x) = (1 + K \mu) x$, where $z$ and $x$ represent the eigenvalues of $\mathbf{M}$ and $\frac{1}{\sqrt{K}} \mathbf{H}_i \mathbf{H}_i^H$ respectively:

$$f_{\mathbf{M}}(x) = \left| \frac{1}{z'(z^{-1}(x))} \right| \cdot f_{\text{MP}}^X(\mathbf{x}, \beta) (z^{-1}(x)) = \frac{1}{\bar{\nu}} f_{\text{MP}} \left( \frac{x - 1}{\bar{\nu}} \right).$$

Theorem III.1. The inverse $\eta$-transform of $\mathbf{M}$ is given by

This analysis can be straightforwardly extended for cases where variable received power is considered for each BS due to variable transmit powers or propagation paths across users. In this case, the channel can be modeled as a variance-profiled Gaussian matrix and it can be tackled using a scaling approximation as described in [3], [13].
Proof: See Appendix \[\Box\]

**Theorem III.2.** The a.e.c.d.f. of $M$ for $\beta = 1$ is given by:

$$F_{M}(x) = \frac{(x - 1)(4\mu - x + 1) - 2 \arcsin\left(\frac{2\mu + x - 1}{2\mu}\right)}{2\pi \mu}. \quad (22)$$

Proof: The c.d.f. follows from eq. (14) after integration for $\beta = 1$. \[\blacksquare\]

**Theorem III.3.** The inverse $\eta$-transform of $K$ is given by:

$$\eta^{-1}_{K}(x) = \Sigma_{N}(x - 1)\eta^{-1}_{M}(x) \quad (23)$$

Proof: Given the asymptotic freedom between deterministic matrix with bounded eigenvalues $\tilde{N}$ and unitarily invariant matrix $M$, the $\Sigma$-transform of $K$ is given by multiplicative free convolution:

$$\Gamma_{K}(x) = \Sigma_{K}(x)\Sigma_{M}(x) \quad (24)$$

$$\left(\frac{x + 1}{x}\right)^{-1} \eta^{-1}_{K}(x + 1) = \Sigma_{N}(x)\left(\frac{x + 1}{x}\right)\eta^{-1}_{M}(x + 1)$$

where step (a) combines Definition A.3 and eq. (16). The variable substitution $y = x + 1$ yields eq. (23). \[\blacksquare\]

B. Ill-conditioned Second Hop

Matrix $H_{2}$ is modelled as a deterministic matrix with power normalization $\text{tr}(H_{0}^{2}H_{2}) = K$. Due to the lack of scattering in line-of-sight environments, this matrix may be ill-conditioned. The simplest model would be to assume a uniform distribution of eigenvalues with support $[\zeta^{-1}, \zeta]$ and condition number $\zeta^2$. The a.e.p.d.f. and transforms for uniform eigenvalue distribution with variable condition number are given by:

$$f_{\mathcal{N}}(x) = \frac{\zeta}{\zeta^2 - 1} \{\zeta^{-1} \ldots \zeta\} \quad (25)$$

$$\eta_{\mathcal{N}}(x) = \frac{\ln(\zeta) - \ln(x + \zeta + 1 + (1 + x))}{(\zeta^2 - 1)x} \quad (26)$$

$$S_{\mathcal{N}}(x) = \frac{\zeta}{\zeta^2 - 1} \{\zeta^{-1} \ldots \zeta\} \quad (27)$$

However, this model results in exponential expressions for the $R$- and $\Sigma$-transforms which yields complex closed form expressions. To construct an analytically tractable problem, we consider the tilted semicircular law distribution which can accommodate a variable condition number and more importantly its $\Sigma$-transform is given by a first degree polynomial [38].

**Theorem III.4.** In the asymptotic regime preserving the power normalization, the tilted semicircular law converges to the following distribution:

$$f_{\mathcal{N}}(x) = \frac{2\zeta}{\pi (\zeta - 1)^{2}x^{2}} \sqrt{(\zeta x - 1)^{+} \left(1 - \frac{x}{\zeta}\right)^{+}} \quad (28)$$

with support $[\zeta^{-1}, \zeta]$. In this case, the transforms of the tilted semicircular law are given by:

$$\eta_{\mathcal{N}}(x) = \frac{1 + 2\zeta x + \zeta^2 - 2\zeta (x + \zeta + \zeta^2 x)}{(\zeta^2 - 1)^2} \quad (29)$$

$$S_{\mathcal{N}}(x) = \frac{-x + 2 \zeta - \zeta^2 x^2 + 2 \zeta (-x + \zeta^2 + \zeta - \zeta^2 x)}{x^2 (\zeta^2 - 1)^2} \quad (30)$$

$$R_{\mathcal{N}}(x) = \frac{2 \zeta - \sqrt{\zeta (\zeta + 2 \zeta x - x - \zeta^2 x)}}{x (\zeta^2 - 1)^2} \quad (31)$$

Proof: The closed-form expressions for the transforms are derived by integrating over the aepdf (17) using the definitions in app. \[\Box\]

**Theorem III.5.** The capacity term $C_{2}$ is given in closed form using the Shannon transform:

$$C_{2} = \mathcal{V}_{\mathcal{N}}(\nu) \quad (32)$$

and in the low SNR regime:

$$\lim_{\nu \rightarrow 0} C_{2} = \frac{4 \zeta - (\zeta^2 + 4 \zeta + 1) \log(4 \zeta) + 4 (\zeta^2 + 1) \ln(\zeta + 1)}{2(\zeta - 1)^2} \quad (33)$$

Proof: The first equation can be derived using eq. (11) and def. A.7. As a result, $\lim_{\nu \rightarrow 0} C_{2} = \mathcal{V}_{\mathcal{N}}(0)$. \[\blacksquare\]

**Theorem III.6.** The Stieltjes transform of $K$ is given by the solution of the cubic polynomial in \[\Box\]

Proof: The first step is to substitute eq. (21) and (30) into (23). Using prop. A.7 and applying suitable change of variables:

$$x \eta_{K}^{-1}(x) + 1 = 0 \quad (34)$$

The final form of the polynomial is derived through algebraic calculations. \[\blacksquare\]

**Remark III.1.** For $M = K$, the eigenvalues of $K$ and $\tilde{K}$ are identical. Thus, the a.e.p.d.f. of $\tilde{K}$ is given by eq. (35) and Lemma A.7 for $\beta = 1$. \[\Box\]

**Lemma III.3.** The quantity $C_{1}$ is given by eq. (10), where $f_{\mathcal{N}}(x)$ is given by lem. A.7 and eq. (35).

**Remark III.2.** The average MMSE $\text{mmse}_{\text{avg}}$ is given by eq. (13) where $F_{M}^{-1}(x)$ can be calculated using Theorem III.2 and $F_{M}^{-1}(x)$ using integration and inversion over the a.e.p.d.f. in Remark III.7.
can be written as:

\[ \nu \approx \exp \left( -\frac{1}{\nu} \log \left( \log \left( \frac{1}{\nu} \right) \right) \right) \]

In the asymptotic regime, the capacity converges to

\[ C \to \alpha \nu \left( -\frac{\mu \nu}{\nu + \alpha \nu} \right) \left( \beta \right) \to \alpha \nu \left( -\frac{\mu \nu}{\nu + \alpha \nu} \right) \left( \beta \right) \]
The a.e.p.d.f. of matrix $K$ follows a scaled version of the MP law:

$$ f_{\infty}^K = \frac{\alpha}{\mu} f_{\text{MP}} \left( \frac{\alpha x - 1}{\alpha \mu - \alpha} \frac{\beta}{\alpha} \right). \quad (43) $$

Proof: See Appendix C.

Remark III.3. For rank-deficient second-hop, the MMSE performance degrades rapidly since the equivalent receive dimensions are fewer than the number of users. As a result, the MMSE receiver could only be used if the channel rank is larger than the number of served users $\alpha \geq \beta$.

IV. Numerical Results

In order to verify the accuracy of the derived closed-form expressions and gain some insights on the system performance of the considered model, a number of numerical results are presented in this section.

A. A.e.p.d.f. Results

The accuracy of the derived closed-form expressions for the a.e.p.d.f. of matrices $K, M$ is depicted in Figures 2 and 3 for ill-conditioned second hop. The solid line in subfigure 2 is drawn using Theorem III.6 in combination with lem. A.1, in subfigure 3 using lem. III.3. The histograms denote the p.d.f. of matrices $K, M$ calculated numerically based on Monte Carlo simulations for $K = 10$. It can be seen that there is a perfect agreement between the two sets of results which verifies our analytic results.

B. Capacity Results

Figures 4 and 5 depict the effect of condition number $\zeta^2$ and normalized rank $\alpha$ on the per-antenna channel capacity $C$ of the DH AF SIMO MAC and the per-antenna channel capacity $C_2$ which corresponds to an ill conditioned or rank-deficient single hop SIMO MAC respectively. The analytic

\[ \text{Fig. 2. A.e.p.d.f. plots of matrix } K. \text{ Parameters: } \beta = 1, \nu = \mu = 10\text{dB.} \text{ The solid analytic curves follow tightly the simulation-generated bars.} \]

\[ \text{Fig. 3. A.e.p.d.f. plots of matrix } M. \text{ Parameters: } \beta = 1, \nu = \mu = 10\text{dB.} \text{ The solid analytic curves follow tightly the simulation-generated bars.} \]

\[ \text{Fig. 4. Per-antenna capacity scaling vs. condition number } \zeta^2 \text{ and normalized rank } \alpha \text{ in dBS. Parameters: } \mu = \nu = \beta = 1. \]

\[ \text{Fig. 5. Per-antenna capacity scaling vs. condition number } \zeta^2 \text{ and second hop power } \nu \text{ in dBS. Parameters: } \mu = 10\text{dB}, \beta = 1. \]
solid curves are plotted using a) eq. (10) and eq. (11) for the ill-conditioned DH AF, b) eq. (42) for the rank deficient DH AF, c) eq. (32) for the ill-conditioned single hop and d) eq. (51) for the rank deficient single hop. It can be seen that the performance degrades much more steeply with normalized rank than condition number in all cases. Especially for the DH AF, it can be observed that rank deficiency is detrimental and quickly drives capacity to zero due to rank loss. On the other hand, the degradation with condition number is much smoother since the second hop channel matrix $H_2$ is still full-rank.

In addition, the per-channel capacity $C$ is plotted versus the second hop power $\nu$ and condition number $\zeta^2$. As it can be seen, it is possible to recover part of the lost performance due to ill-conditioning by increasing the amplification level $\nu$.

C. MMSE Results

Figure 6 depicts the effect of condition number $\zeta^2$ and second hop power $\nu$ on the average MMSE. As expected, the average MMSE increases with $\zeta^2$ but decreases with $\nu$. It can be seen that performance can be improved using stronger amplification but for high $\nu$ there is a saturation threshold which is governed by the first hop performance as described in sec. III-C2. Figures 7 and 8 depicts the accuracy of the proposed lower bound. The solid plots were calculated through Monte Carlo simulations of eq. (5), whereas the dashed plots represent our lower bound which was calculated using Remark III.2. It can be seen that the proposed bound is tight for low values of $\zeta^2$, but it progressively diverges as $\nu$ and $\zeta^2$ grow large.

D. Comparison

In this section, the performance of the proposed system is compared to the conventional system (as described in section II-C) by fixing the user and BS power at 10 dBs. As it can be seen in Fig. 9 while the condition number increases, the performance of the proposed system degrades and even falls below conventional performance for extremely ill-conditioned
BS-CP channels. There are two crossing points in 160 and 200 dBs for the MMSE throughput and channel capacity respectively. However, a two-fold performance gain can still be harnessed for condition numbers up to 120 dBs for MMSE receiver and up to 160 dBs for optimal receiver.

V. CONCLUSION

In this paper, we have investigated the performance of BS cooperation scenario with microwave backhauling to a CP, where multiple users and BSs share the same channel resources. The user signals are forwarded by the BSs to an antenna array connected to a CP which is responsible for multiuser joint processing. This system has been modelled as a DH AF SIMO MAC with a ill-conditioned or rank-deficient second hop due to lack of scattering in line-of-sight environments. Its performance in terms of channel capacity and MMSE performance has been analysed through a large-system free-probabilistic analysis. It can be concluded that performance degrades much more gracefully with condition number than with loss of rank. As a result, a performance gain can be achieved compared to conventional resource partitioning even for highly ill-conditioned second hop. Furthermore, performance degradation due to ill conditioning can be compensated through stronger amplification at BS-site until it reaches the first hop performance in the high amplification limit.

APPENDIX A

RANDOM MATRIX THEORY PRELIMINARIES

Let $f_X(x)$ be the eigenvalue probability distribution function of a matrix $X$.

**Definition A.1.** The Shannon transform of a positive semidefinite matrix $X$ is defined as

$$
\forall x \in \mathbb{R} \quad \text{Sh}(X)(\gamma) = \int_0^\infty \log(1 + \gamma x) f_X(x) dx.
$$

**Definition A.2.** The $\eta$-transform of a positive semidefinite matrix $X$ is defined as

$$
\eta_X(\gamma) = \int_0^\infty \frac{1}{1 + \gamma x} f_X(x) dx.
$$

**Definition A.3.** The $\Sigma$-transform of a positive semidefinite matrix $X$ is defined as

$$
\Sigma_X(x) = \frac{x + 1}{x} X^{-1}(x + 1).
$$

**Property A.1.** The Stieltjes-transform of a positive semidefinite matrix $X$ can be derived by its $\eta$-transform using

$$
S_X(x) = -\frac{\eta_X(-1/x)}{x}.
$$

**Lemma A.1.** The a.e.d.f. of $X$ is obtained by determining the imaginary part of the Stieltjes transform $S$ for real arguments

$$
f^\infty_X(x) = \lim_{y \to 0^+} \frac{1}{\pi} \mathcal{P} \{ S_X(x + iy) \}.
$$

APPENDIX B

PROOF OF THEOREM III.1

Starting from eq. (29) and following def. A.2

$$
\eta_M(\psi) = \int_{-\infty}^{+\infty} \frac{1}{1 + \psi x} f_M^\infty(x) dx
$$

$$
= \frac{1}{\gamma} \int_{-\infty}^{+\infty} \frac{1}{1 + \psi x} f_N^\infty \left( \frac{x - 1}{\gamma} \right) dx
$$

$$
= \frac{1}{4i\pi} \oint_{|\zeta|=1} (\zeta(1 + \beta)\zeta + \sqrt{\beta}(\zeta^2 + 1))(1 + \psi(1 + \gamma + \gamma\beta)) \zeta \frac{d\zeta}{\sqrt{\beta} \sqrt{(\zeta^2 - 1)^2}}
$$

(49)

Step (a) requires the variable substitutions $x = w\gamma + 1$, $dx = \gamma dw$, followed by $w = 1 + \beta + 2\sqrt{\beta}\cos \omega$, $dw = 2\sqrt{\beta}(-\sin \omega) d\omega$ and finally $\zeta = e^{i\omega}$, $d\zeta = i\zeta d\omega$. Subsequently, a Cauchy integration is performed by calculating the poles $\zeta_i$ and residues $\rho_i$ of eq. (49).

$$
\zeta_0 = 0,
$$

$$
\zeta_{1,2} = \frac{-(1 + \beta) \pm (1 - \beta)}{2\sqrt{\beta}},
$$

$$
\zeta_{3,4} = \frac{-1 - \psi \gamma - \psi \beta \gamma \pm \left(\sqrt{1 + 2\psi + 2\psi \gamma + 2\psi \beta \gamma + \psi^2 + 2\psi^2 + 2\psi \beta \gamma}\right)}{2\sqrt{\beta}\psi \gamma},
$$

Using the residues which are located within the unit disk, the Cauchy integration yields:

$$
\eta_M(\psi) = -\frac{\beta}{2}(\rho_0 + \rho_1 + \rho_2 + \rho_4)
$$

Inversion yields eq. (21).

APPENDIX C

PROOF OF THEOREM III.7

The components of eq. (40) can be written as:

$$
C_1 = \frac{1}{K} \lim_{K,M \to \infty} \mathbb{E} \left[ \log \det \left( I_K + v \mathbf{H}^H \mathbf{H}_2 \left( I_K + \mu \mathbf{H}_1 \mathbf{H}_1^H \right) \right) \right]
$$

$$
= \frac{1}{K} \lim_{K,M \to \infty} \mathbb{E} \left[ \log \det \left( I_K + \frac{\nu}{\alpha} \mathbf{D} \left( I_K + \mu \mathbf{H}_1 \mathbf{H}_1^H \right) \right) \right]
$$

$$
= \frac{1}{K} \lim_{K,M \to \infty} \mathbb{E} \left[ \log \det \left( I_{\alpha K} + \frac{\alpha K \mu \nu}{\nu + \alpha} \mathbf{H}_1^H \mathbf{H}_1 \right) \right]
$$

$$
= \frac{1}{K} \lim_{K,M \to \infty} \mathbb{E} \left[ \log \det \left( I_{\alpha K} + \frac{\alpha K \mu \nu}{\nu + \alpha} \mathbf{H}_1^H \mathbf{H}_1 \right) \right]
$$

$$
\to \alpha \log \left( 1 + \frac{\nu}{\alpha} \right) + \alpha \mathcal{V}_{MP} \left( \alpha, \frac{\mu}{\nu + \alpha} \right),
$$

(50)

$$
C_2 = \frac{1}{K} \lim_{K \to \infty} \mathbb{E} \left[ \log \det \left( I_K + v \mathbf{H}_2 \mathbf{H}_2^H \right) \right]
$$

$$
= \frac{1}{K} \lim_{K \to \infty} \mathbb{E} \left[ \log \det \left( I_K + \frac{\nu}{\alpha} \mathbf{D} \right) \right]
$$

$$
= \frac{1}{K} \lim_{K \to \infty} \mathbb{E} \left[ \log \det \left( I_{\alpha K} \left( 1 + \frac{v}{\alpha} \right) \right) \right]
$$

$$
\to \alpha \log \left( 1 + \frac{v}{\alpha} \right),
$$

(51)

where $\mathbf{D}$ is a $K \times K$ zero matrix with $\alpha K$ ones across its diagonal and $\mathbf{H}_1$ is a $\alpha K \times M$ submatrix of $\mathbf{H}_1$. Substraction
yields the capacity expression. The aepdf follows from the equivalent matrix $K$:

$$K = \frac{1}{\alpha} \left( I_{\alpha K} + aK\mu \frac{\tilde{H} \tilde{H}^H}{aK} \right).$$  \tag{52}

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