A solution to the problem of elastic half-plane with a cohesive edge crack

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Abstract. This paper considers the problem of extension of an elastic half-plane slackened by a rectilinear edge crack. The opposite edges of the crack are attracted to each other. The intensity of attracting forces – the forces of cohesion – depends on displacements of the edges; this dependence is nonlinear in the general case. External load and cohesive forces are related to each other by the condition of finite stresses at the crack tip. The authors apply Picard’s method of successive approximation. In each iteration, Irwin’s method is used to solve the problem of a half-plane with a crack, the edges of which are subjected to irregularly distributed load. The solution of the resulting integral equation is found by Galerkin’s method. The paper includes examples of calculations and their results. Some of them are compared with the data of previous studies.

1. Introduction
Cracks usually appear on the surface of the material in the stress concentration zone. Therefore, studying the conditions of surface crack growth is of undoubted interest. This work considers this problem in a plane formulation, as it is customary in fracture mechanics. The calculation model is an elastic half-plane with a rectilinear crack perpendicular to the half-plane boundary (Figure 1).

![Figure 1](image-url)

**Figure 1.** A half-plane with an edge cohesive crack: 1 – the half-plane boundary; 2 – a crack with the length $a$; 3 – a cohesive zone with the length $b$. 
Self-balanced distributed normal load is applied to the edges of the crack. Thus, the crack under consideration belongs to the cracks in mode I. The problem is symmetric relative to the abscissa axis; so, it is reasonable to consider the deformation of only the upper half of the half-plane. The load applied to the upper edge of the crack is represented as

\[ p_1 = 0; \quad p_2(x_i) = p(x) = p_0(x) - g(x); \quad p_0, g \geq 0; \quad x = x_i \]  

(1)

where \( p_0(x) \) is external load and \( g(x) \) is cohesive forces. Cohesive forces are non-zero only in the interval \( x \in [a - b, a] \) (in the cohesive zone). The equation of state for them is usually written as

\[ g = g[v(x)] \]

(2)

where \( v(x) = u_z(x, 0) \) is displacement of the upper edge of the crack along the ordinate axis.

If the length of the cohesive zone is much smaller than the length of the crack, the solution of a fracture mechanics problem is not dependent on the function form (2) [1]. However, in the general case considered in the present study, it is not true.

It is possible to distinguish three types of relationships (2). The first type is the case when the function (2) is reduced to the constant \( g = \text{const} \). This problem for a half-plane with an edge crack was solved by Howard and Otter [2]. The second type is a linear ratio:

\[ g = \begin{cases} g_M \left(1 - \frac{v}{v_M}\right), & v \leq v_M \\ 0, & v > v_M \end{cases} \]

(3)

where \( g_M, v_M \) are certain constants. A solution to the case with the allocation (3) was found by Wang and Dempsey [3]. The third type is a non-linear relationship, i.e. the general case of the formula (2). A solution to this problem is presented in this paper.

2. Mathematical model

2.1. Iterative process

A similar but simpler problem of a cohesive crack in a plane with nonlinear defining relationship (1) was considered in other works [4, 5]. To solve the nonlinear problem, Huang [4] applied Newton’s method. Ungsuwarungsri and Knauss [5] showed that Picard’s method of successive approximation is even more effective and much simpler in this case. This paper follows the recommendations of the work [5]. In each iteration of the solution process of the nonlinear problem, we should solve the problem of the linear elasticity for the model shown in Figure 1. The difference between the iterations lies in different distribution of forces \( p(x) \). In this paper, the solution to this linear problem is obtained by Irwin’s method [6, 7]. Its main point is that the initial problem is represented as a composition of two basic problems: the problem of a crack with length \( 2a \) in a plane (problem 1) and the problem of a half-plane loaded along the boundary contour (problem 2). These problems are connected by introducing additional fictitious forces that ensure the fulfillment of boundary conditions.

2.2. Solution to the problem of a crack in a half-plane

We consider an elastic half-plane with an edge rectilinear crack of length \( a \) (Figure 2); a distributed normal load of intensity \( p(x_i) \) is symmetrically applied to the edges of the crack. It is necessary to calculate the stress intensity factor \( K_i \) and the stresses at interior points of the half-plane. The procedure of this problem solution will be as follows [6, 7]. Let us consider a plane with two intersecting cracks (Figure 3): a crack 1 with length \( 2a \) and an unloaded crack 2 with length \( 2b \). At
$b \rightarrow \infty$, the crack 2 is transformed into boundary contours of two half-planes; the design model shown in Figure 3 becomes equivalent to the design model shown in Figure 2. Thus, the initial problem is represented [6, 7] as a composition of two problems, whose solutions are known [8]. The first of these problems (hereinafter, Problem 1) is about a crack of length $2a$ in a plane and the second one (Problem 2) is about the right half-plane loaded along the boundary contour.

The stresses in the plane problem of the elasticity theory are determined by the formulas [3]:

\[
\begin{align*}
\sigma_{11} + \sigma_{22} &= 2 \left[ \Phi(z) + \Phi(z) \right] \\
\sigma_{22} - \sigma_{11} + 2i \sigma_{12} &= 2 \left[ \Phi'(z) + \Psi(z) \right]
\end{align*}
\]  

(4)

where $\sigma_{\text{res}}$ are stresses in Cartesian coordinates $x_i$; $z = x_1 + i x_2$ is a complex variable ($i$ is an imaginary unit); $\Phi(z)$, $\Psi(z)$ are holomorphic functions determined from the boundary conditions of the problem; the bar over the symbol denotes complex conjugation, the prime denotes a derivative. Due to the linearity of the problem, these functions are represented as

\[
\Phi(z) = \Phi_{(1)}(z) + \Phi_{(2)}(z); \quad \Psi(z) = \Psi_{(1)}(z) + \Psi_{(2)}(z)
\]  

(5)

where the subscript in parentheses indicates the problem number.

The first problem, taking into account its symmetry (Figure 3), has the following form [8]:

\[
\Phi_{(1)}(z) = -\frac{z}{\pi(z^2 - a^2)^{1/2}} \int_0^z \left(a^2 - t^2\right)^{1/2} \frac{p(t) dt}{i t^2 - z^2};
\]

\[
\Psi_{(1)}(z) = -z\Phi_{(1)}'(z); \quad p_{(1)}(t) = p(t) + q(t)
\]  

(6)

where $q(t)$ is unknown additional load compensating for the impact of the cut along the ordinate axis.

In view of symmetry, the solution to the second problem is represented as [8]:

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Figure 2. A half-plane with an edge crack: $a$ – the length of the crack. Equal oppositely directed loads are applied to the edges of the crack, $p_1 = 0$, $p_2 = p(x_i)$.

Figure 3. A plane with two intersecting cracks. The load intensity $p(x_i)$ is an even function.
\[ \Phi_{(2)}(z) = -z \int_0^\infty \frac{p_{(2)}(\eta)}{\eta^2 + z^2} \, d\eta; \quad \Psi_{(2)}(z) = z \Phi'_{(2)}(z) \]  

(7)

where the load \( p_{(2)}(\eta) \), being applied to the boundary contour of the half-plane in the direction of the abscissa axis, compensates for the impact of the cut (crack).

According to formulas (6) and (4), the load coming from the left half-plane affects the right half-plane; this load is equal to \( -\sigma_{(0)11}(0, y) \), where \( y = x_2 \). It is compensated by the distributed force \( p_{(2)}(y) \). It follows from relations (6) and (4) that

\[ p_{(2)}(y) = \sigma_{(0)11}(0, y) = -\frac{2}{\pi} \frac{d}{dy} \left[ \frac{y^2}{(y^2 + x^2)^2} \int_0^\infty \frac{(a^2 - t^2)^{1/2}}{t^2 + y^2} p_{(0)1}(t) \, dt \right] \]

(8)

Solution (7) and formulas (4) suggest that the upper edge of the crack has a load coming from the lower edge. This load is equal to \( -\sigma_{(2)22}(x, 0) \), where \( x = x_1 \). It is compensated by the load \( q(x) \).

From formulas (7) and (4), we have:

\[ q(x) = \sigma_{(2)22}(x, 0) = -\frac{4x}{\pi} \int_0^\infty \frac{\eta^2 p_{(2)}(\eta) \, d\eta}{(\eta^2 + x^2)^2}; \quad x \in [0, a] \]

(9)

The system of integral equations (8) and (9) with respect to the functions \( q(x) \), \( p_{(2)}(y) \) is identical to the system obtained by Irwin [6, 7] with the help of integral transformations.

Let us substitute the function \( p_{(2)}(y) \), which is defined by expression (8), into equation (9). In the obtained expression, we perform integration by parts for the variable \( \eta \) and then change the order of integration. The resulting equation is:

\[ q(x) = \frac{16x}{\pi^2} \int_0^\infty \frac{\eta^2}{(\eta^2 + x^2)^2} \int_0^x \frac{(a^2 - t^2)^{1/2} p_{(0)1}(t) \, dt}{(\eta^2 + t^2)^2(\eta^2 + a^2)^{1/2}} \, d\eta \]

(10)

The inner integral can be expressed in terms of elementary functions by the substitution \( \eta = (\xi^2 - a^2)^{1/2} \):

\[ I(x, t) = \int_0^\infty \frac{\eta^2}{(\eta^2 + x^2)^2(\eta^2 + t^2)^2(\eta^2 + a^2)^{1/2}} \, d\eta = \int_a^\infty \frac{(\xi^2 - a^2)^2}{(\xi^2 - (a^2 + x^2))^2(\xi^2 - (a^2 + t^2))^2} \, d\xi = \frac{t^2(t^2 + x^2)[F_2(x) - F_2(t)] - x^2(3t^2 - x^2)}{(t^2 - x^2)^2} \]

(11)

where
\[ F_1(x) = \frac{1}{(a^2-x^2)^{\frac{3}{2}}} \ln \left[ \frac{a + \left( a^2 - x^2 \right)^{\frac{1}{2}}}{x} \right]; \quad F_2(x) = \frac{1}{2(a^2-x^2)} \left[ a - F_1(x) \right]; \]

\[ F_3(x) = \frac{1}{4(a^2-x^2)} \left[ \frac{a}{x^2} - 3F_2(x) \right]. \]  \hfill (12)

Taking into account expressions (6), equation (10) can be represented in the standard form:

\[ q(x) - \lambda \int_0^a K(x,t) q(t) dt = f(t); \quad \lambda = 16/\pi^2; \]

\[ K(x,t) = x(a^2-t^2)^{\frac{1}{2}} I(x,t); \quad f(t) = \lambda \int_0^a K(x,t) p(t) dt \]  \hfill (13)

Equation (13) will become a Fredholm equation under the following condition:

\[ \int_0^a \left[ K(x,t) \right]^2 dx dt < M \]  \hfill (14)

where \( M \) is an arbitrarily large real number. It follows from the relations (11) and (12) that the kernel \( K(x,t) \) is continuous for all possible arguments, except for the case when they both tend to zero. Let us find an asymptotic expression for the kernel at \( x,t \to 0 \), resorting to polar coordinates

\[ x = r \cos \varphi; \quad t = r \sin \varphi; \quad \varphi \in [0, \pi/2] \]  \hfill (15)

and letting \( r \) tend to zero. Formulas (11) and (12) allow making an asymptotic expression for the kernel \( K(x,t) \):

\[ \frac{1}{r^2} \left[ \ln \tan \varphi - \frac{\cos \varphi \sin^2 \varphi}{r \cos^2 2\varphi} \left( \frac{\ln \tan \varphi}{\cos 2\varphi} + 1 \right) \right] \]  \hfill (16)

Using L’Hospital rule, it is easy to show that at \( r \neq 0 \), this expression is finite at \( \varphi = 0 \), as well as at \( \varphi = \pi/4 \) or \( \varphi = \pi/2 \). Because of the singularity at the point \( r = 0 \), the kernel does not satisfy inequality (14). However, it follows from expression (16) that the integral

\[ \int_0^a \int_0^a K(x,t) \psi(x,t) dx dt \]  \hfill (17)

where \( \psi(x,t) \) is a continuous function, converges. This gives grounds to use Galerkin’s method [9] for the solution of equation (13). The solution can be found by the finite number of terms in the series of coordinate functions [9]:

\[ q(\zeta) = \sum_{k=0}^{N-1} c_k P_k(\zeta) \]  \hfill (18)

where \( c_k \) are desired expansion coefficients. In this study, the coordinate functions \( P_k(\zeta) \) are Legendre polynomials [10], which are orthogonal in the interval \([-1,1]\) and form a complete system of functions [10]. Here, Galerkin’s method gives a solution that tends to be exact at \( N \to \infty \) [9]. To
convert the integration interval in equation (13) to the interval \([-1,1]\), we introduce new variables \(\zeta, \tau\):

\[
x = 0.5a(1+\zeta); \quad t = 0.5a(1+\tau); \quad \zeta, \tau \in [-1,1]
\]

Multiplying equation (13) by \(P_j(\zeta)\) and integrating over the interval \([-1,1]\), we obtain a system of \(N\) linear algebraic equations for \(c_k\):

\[
\sum_{k=0}^{N-1} \mu_{jk} c_k = b_j; \quad j = 0,1,\ldots,N-1;
\]

\[
\mu_{jk} = \frac{1}{1} \int_{-1}^{1} P_j(\zeta) P_j(\zeta) d\zeta - 0.5 \lambda \int_{-1}^{1} K(\zeta, \tau) P_j(\zeta) d\zeta d\tau
\]

\[
b_j = 0.5 \lambda \int_{-1}^{1} K(\zeta, \tau) p(\tau) P_j(\zeta) d\zeta d\tau
\]

The formula can be simplified for \(\mu_{jk}\), considering the property of Legendre polynomials [10]:

\[
\int_{-1}^{1} P_j(\zeta) P_j(\zeta) d\zeta = \frac{2\delta_{jk}}{2j+1}
\]

where \(\delta_{jk}\) is Kronecker delta.

The integrals in the formulas for the equation system coefficients can be determined numerically. In this study we use Gaussian quadrature. The next steps after solving the system of equations are calculation of the stress intensity factor

\[
K_I = 2 \left( \frac{a}{\pi} \right)^{1/2} \int_0^{a/2} \left( \frac{p(t)}{\sqrt{a^2 - t^2}} \right) dt
\]

and determination of the stress – by the formulas (4) – (8).

The case of constant load calculation can serve as an example. The dimensionless stress intensity factor for this problem is determined by Wigglesworth’s formula [11]:

\[
K_I^* = \frac{K_I}{p\sqrt{\pi a}} = \exp \left[ -\frac{1}{\pi} \int_{0}^{\infty} \frac{\eta^2}{\sinh^2(0.5\pi\eta)} d\eta \right]
\]

Equation (23) determines \(K_I^*\) as a value, which is accurate to six significant digits and is equal to 1.12152. Table 1 shows the values of \(K_I^*\) depending on the number of coordinate functions \(N\); they were obtained with the developed method. Table 1 reflects acceptable accuracy and sufficiently rapid convergence of the solution with an increase in the number \(N\).

| \(N\) | 2     | 20    | 200   |
|------|-------|-------|-------|
| \(K_I^*\) | 1.07785 | 1.12102 | 1.12152 |
2.3. The algorithm for solving the cohesive crack problem

The distribution of cohesive forces and the actual load are related by the condition of absent stress singularity at the tip of the crack [1]:

\[ K_I = 0 \] (24)

When constructing an iterative process algorithm, it is convenient to represent the external load in the following form:

\[ p_0(x) = Cq_0(x) \] (25)

where \( q_0(x) \) is a given function, \( C \) is a constant varying from iteration to iteration. For the given distribution of cohesive forces, equation (24) is a linear equation for determining \( C \).

To calculate the cohesive forces, it is necessary to find the displacement of the upper edge of the crack along the ordinate axis according to formula (1). Since the crack is a great discontinuity of displacement field and the displacements in the basic problem 2 are continuous, the above displacement of the upper edge of the crack is completely determined by the solution to the basic problem 1 – the problem of a crack in a plane. Thus, it is defined (for the plain strain case) by the following formula [8]:

\[
\begin{align*}
\Phi_{\beta}(z) &= \frac{z}{\pi} \left[ \frac{1}{(z^2-a^2)^{1/2}} \int_0^z \frac{p_0(t)dt}{(a^2-t^2)^{1/2}} + \frac{z^2-a^2}{2} \int_0^z \frac{p_0(t)dt}{(t^2-z^2)(a^2-t^2)^{1/2}} \right] \\
\end{align*}
\] (27)

It follows from (22) and (24) that the first summand is zero. Integration of the second summand in (27) with respect to \( z \) and simple transformations result in formula (26) in the following form:

\[
\begin{align*}
v(x) &= \frac{2(1-v^2)}{\pi E} \int_0^x \left[ \frac{a^2-t^2}{(a^2-x^2)^{1/2}} + \frac{(a^2-x^2)^{1/2}}{(a^2-t^2)^{1/2}-(a^2-x^2)^{1/2}} \right] p_0(t)dt \\
\end{align*}
\] (28)

It is natural to assume that the cohesive forces along the crack length change continuously. Since they are non-zero only in the cohesive zone, expression (1) takes the form

\[
g = \begin{cases} 
g_{\beta} f(v), & v \leq v_{\beta} \\
0, & v > v_{\beta} 
\end{cases} \quad f(0) = 1; \quad f(1) = 0 \] (29)

From equation (29) and the problem formulation, it follows that \( v_{x=a-b} = v_{\beta} \).

The iterative process is made as follows. By the beginning of the \( k \)-th iteration, we know the displacement of the upper edge of the crack \( v_{k-1}(x) \) and, consequently, the boundary of the cohesive zone \( b_{k-1} \). Formula (29) defines the distribution of the cohesive forces along the edge of the crack \( g_{k-1}(x) \). The next step is to solve the linear problem of a half-plane with an edge crack and to define
the constant \( C_{k-1} \) from equation (24). Next, we find the displacements \( v_k(x) \) by formula (28) and, basing on the condition \( v|_{x=a-b} = v_M \), a new value for the boundary of the cohesive zone \( b_k \). The iterative process is repeated until convergence, which is determined by the criterion

\[
|b_k - b_{k+1}| / a < \varepsilon
\]

where \( \varepsilon > 0 \) is a small given number.

To solve the problem by Picard’s method of successive approximation, it is necessary to find the initial approximation \( v_0(x) \), which should be sufficiently close to the solution. If the initial approximation is unsuccessful, the iterative process converges to some infinite set of trivial solutions – solutions for a fully cohesive crack, which are characterized by the condition

\[
b = a; \quad v(0) \in [0, v_M]
\]

In this case, the second constraint (29) on the function \( f(v/v_M) \) is not satisfied.

In accordance with earlier recommendations [5], the initial approximation is chosen as:

\[
v_0(x) = v_M \left( \frac{a-x}{b_0} \right)^{1.5}
\]

Changing the value \( b_0 \), it is possible to find an initial approximation that ensures convergence of the iterative process to a nontrivial solution.

The effect of the cohesive zone extent is manifested in the magnitude of the \( J^- \)-integral [12]. Let us introduce a cohesive force potential:

\[
W(v) = \int_0^v g(\eta) d\eta
\]

Using Rice’s formula [12], we obtain:

\[
J = -2 \int_{a-b}^a g(x) \frac{dv}{dx} dx = 2W(v_M)
\]

This value can naturally be compared with the value of the \( J^- \)-integral for a non-cohesive crack:

\[
J^- = (1 - \nu^2) K_I^2 / E
\]

where \( K_I \) is the stress intensity factor calculated by formula (22) provided that \( p(x) = p_0(x) \), i.e. in the absence of cohesive forces.

### 3. Calculation results

In the examples discussed below, the external load was assumed constant along the length of the crack, i.e. \( q_0 = 1 \). The chosen number of coordinate functions of Galerkin’s method was 60. We consequently obtained \( K_I = 1.12147 \), which differs from the exact solution by 0.00446 %. The permissible relative error of the solution was assumed equal to \( \varepsilon = 10^{-7} \); the Poisson’s ratio was \( \nu = 0.3 \).

Let us compare the results of the calculations via the developed method with the results of the work [3] obtained for the linear law (3). Figure 4 shows the graphs of external load intensity. They indicate that the results of the present paper and work [3] become significantly different when the cohesive zone extends almost over the entire crack, i.e. in the most important applied case of short cracks.
The difference is caused by imperfection of the method used in the work [3], where trivial but meaningful for physics solutions are out of the problem consideration – when the cohesive zone covers the entire crack, but the crack opening is not equal to zero.

\begin{equation}
    M_{\rho} = \frac{g_{\rho}}{g_{M}} = \begin{cases} 
        1 - 3(v/v_{M})^2 + 2(v/v_{M})^3, & v \leq v_{M}; \\
        0, & v > v_{M}
    \end{cases}
\end{equation}

\begin{equation}
    f(0) = 1; \quad f(1) = 0; \quad f'(0) = 0; \quad f'(1) = 0
\end{equation}

Some calculation results are represented in Figure 5 and 6. Figure 5 shows the dependence of external load on cohesive zone length, Figure 6 – the dependence of the ratio of the $J$-integral values calculated by formulas (34) and (35).
4. Conclusion
Formulation and solution of problems on a crack with a cohesive zone commensurable with the crack are associated with attempts to simulate the anomalous properties of short cracks [13-15]. Within the framework of the model considered in this paper, a short crack differs from a long one only in the fact that the cohesive zone length in a long crack is negligible compared to the length of the crack. Therefore, all Barenblatt postulates are valid for a long crack; its growth is determined by the magnitude of the stress intensity factor from external load. A short break should be justified by other laws; it is evidently presented in Figure 6. Their construction should be based on experimental data (for example, see [13-15]). Most likely, all those laws will be non-linear. Thus, when creating methods for solving specific problems, it is necessary to provide for the possibility to use nonlinear defining relations for cohesive forces. This study develops a numerical method for solving such a problem – the problem of a cohesive crack in a half-plane. It is based on the progressive approximation method combined with Irwin’s method of solving the elastic problem of a crack in a half-plane. Numerical examples demonstrate the effectiveness of the method.

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