Liouville Theorems for critical points of the $p$-Ginzburg-Landau type functional

Tian Chong, Bofeng Cheng, Yuxin Dong and Wei Zhang

Abstract. In this paper, we consider the smooth map from a Riemannian manifold to the standard Euclidean space and the $p$-Ginzburg-Landau energy ($p \geq 1$). Under suitable curvature conditions on the domain manifold, some Liouville type theorems are established by assuming either growth conditions of the $p$-Ginzburg-Landau energy or an asymptotic condition at the infinity for the maps. In the end of paper, we obtain the unique constant solution of the constant Dirichlet boundary value problems on starlike domains.

1 Introduction

One of the important problems for harmonic maps or generalized harmonic maps is to study their Liouville type results. (cf. [5, 10, 11, 19, 13]). It is well known that the stress-energy tensor is a useful tool to investigate the energy behavior and some vanishing results of related energy functional. Most Liouville results have been established by assuming either the finiteness of the energy of the map or the smallness of the whole image of the domain manifold under the map. In [13], Z.R. Jin has shown several interesting Liouville theorems for harmonic maps from complete manifolds with assumptions on the asymptotic behavior of the maps at infinity.

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded simply connected domain. Consider the following functional defined for maps $u \in H^1(\Omega, \mathbb{C})$:

$$ E_\epsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\epsilon^2} \int_\Omega (|u|^2 - 1)^2. $$

Ginzburg-Landau introduced this Functional in the study of phase transition problems and it plays an important role ever since, especially in superconductivity, superfluidity and XY-magnetism (see details for [15, 17, 20]). A lot of papers devote to the
asymptotic behavior of minimizers $u_\epsilon$ of $E_\epsilon(u, \Omega)$ in $H$ as $\epsilon \to 0$. It was shown in those cases that $u_\epsilon$ converges strongly to a harmonic map $u_0$ on any compact subset away from the zeros. Readers can refer to [2, 3, 4, 21] for the progress in this field. In the past decades, $p$-Ginzburg-Landau functionals have been introduced. In [12, 16], the authors investigated the convergence of a $p$-Ginzburg-Landau type functional when the parameter goes to zero.

In this paper, we consider a smooth map $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$ from a Riemannian manifold to the standard Euclidean space and the following $p$-Ginzburg-Landau energy

$$E_{GL}^p(u) = \int_M \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g,$$

where $p \geq 1$ and $\epsilon$ is any small positive number. To generalize the Liouville type results for harmonic maps to the critical points of $p$-Ginzburg-Landau energy functional, we introduce the stress energy tensor $S_{u,p}^{GL}$ associated with the $p$-Ginzburg-Landau functional $E_{GL}^p(u)$. It is easy to show that any critical point of the $p$-Ginzburg-Landau functional satisfies the conservation law, that is, $\text{div} S_{u,p}^{GL} = 0$. Using a basic integral formula linked naturally to the conservation law enables us to establish some monotonicity formulae for these critical points of the $p$-Ginzburg-Landau energy functional. Consequently, several Liouville type results can be deduced from these monotonicity formulae under suitable growth conditions on the energy. We also build a Liouville type result under the condition of slowly divergent energy.

Next we want to generalize Jin’s results in [13] to the critical point of the $p$-Ginzburg-Landau energy functional. The methods we use for proving the result is very similar to Jin’s. Firstly, we may use the stress-energy tensor to establish the monotonicity formula which gives a lower bound for the growth rates of the energy. Secondly, we use the asymptotic assumption of the map at infinity to obtain the upper energy growth rates. Under suitable conditions on $u$ and the Hessian of the distance functions of the domain manifolds, one may show that these two growth rates are contradictory unless the critical point is constant. In this way, we establish some Liouville theorems for the critical points of the $p$-Ginzburg-Landau energy functional with the asymptotic property at infinity from some complete manifolds.

In addition to establishing Liouville type results, the monotonicity formulae may be used to investigate the constant Dirichlet boundary value problem as well. We obtain the unique constant solution of the constant Dirichlet boundary value problem on starlike domains for the critical point of $p$-Ginzburg-Landau energy functional.
2 \( p \)-Ginzburg-Landau energy functional and stress-energy tensor

Let \( u : (M^n, g) \to (R^n, h) \) be a smooth map from a Riemannian manifold to the standard Euclidean space. We consider the following \( p \)-Ginzburg-Landau energy

\[
E_{GL}^p(u) = \int_M |du|^p + \frac{1}{4\epsilon^n}(1 - |u|^2)^2dv_g,
\]

where \( p \geq 1 \) and \( \epsilon \) is any small positive number. Let \( \{u_t\}(|t| < \kappa) \) with \( u_0 = u \) and \( v = \frac{\partial u}{\partial t}|_{t=0} \) be a one parameter variation, we have the following lemma.

**Lemma 2.1.** The first variation formula for \( p \)-Ginzburg-Landau energy functional

\[
\frac{d}{dt}|_{t=0}E_{GL}^p(u_t) = -\int_M \langle \text{div}(|du|^{p-2}du) + \frac{1}{\epsilon^n}(1 - |u|^2)u, v \rangle dv_g.
\]

**Proof.** Let \( \{e_i\}_{i=1}^m \) be a local orthonormal frame of \( TM \). Since the target manifold is Standard Euclidean space, we can perform the following calculations

\[
\frac{d}{dt}|_{t=0}E_{GL}^p(u_t) = \int_M \frac{\partial}{\partial t}|_{t=0}(\frac{|du_t|^p}{p})dv_g + \int_M \frac{\partial}{\partial t}|_{t=0}(\frac{1}{4\epsilon^n}(1 - |u_t|^2)^2)dv_g
\]

\[
= \int_M |du|^p - \frac{1}{2\epsilon^n} \int_M (1 - |u|^2)^2 \frac{\partial}{\partial t}|_{t=0}u_t dv_g
\]

\[
= \int_M (1 - |u|^2) |du|^{p-2} \sum_{i=1}^m \langle \nabla_{e_i} du_t(e_i), du_t(e_i) \rangle|_{t=0} dv_g - \frac{1}{\epsilon^n} \int_M (1 - |u|^2) v \cdot udv_g
\]

\[
= -\int_M \langle \text{div}(|du|^{p-2}du) , v \rangle dv_g - \frac{1}{\epsilon^n} \int_M (1 - |u|^2) v \cdot udv_g
\]

\[
= -\int_M \langle \text{div}(|du|^{p-2}du) , v \rangle dv_g + \frac{1}{\epsilon^n}(1 - |u|^2)u, v \rangle dv_g.
\]

**Definition 2.1.** \( u \) is called a critical point of \( p \)-Ginzburg-Landau energy functional if

\[
\text{div}(|du|^{p-2}du) + \frac{1}{\epsilon^n}(1 - |u|^2)u = 0.
\]

When \( p = 2 \), above equation is reduced to \( \Delta u + \frac{1}{\epsilon^n}(1 - |u|^2)u = 0 \).

In [BE], Baird-Eells introduced the stress-energy tensor associated with the usual energy and proved that harmonic maps satisfy the conservation law. We can also define the stress-energy tensor \( S_{u,p}^{GL} \) associated with the \( p \)-Ginzburg-Landau energy functional \( E_{GL}^p(u) \) and prove that the critical points satisfy the conservation law, i.e. \( \text{div}S_{u,p}^{GL} = 0 \).
**Definition 2.2.** Let $u : (M^m, g) \to (R^n, h)$ be a smooth map from a Riemannian manifold to the standard Euclidean space. The stress-energy tensor of $u$ is the symmetric 2-tensor on $M$ given by

$$ S_{u,p}^{GL} = \left[\frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2\right]g - |du|^{p-2}u^*h. $$

**Theorem 2.1.** $\text{div} S_{u,p}^{GL}(X) = -\langle \text{div}(|du|^{p-2}du) + \frac{1}{e^n}(1 - |u|^2)u, du(X)\rangle$, for any $X \in \Gamma(TM)$.

**Proof.** For any 2-tensor field $W \in \Gamma(T^*M \otimes T^*M)$, the divergence of $W$ is defined by

$$ (\text{div}W)(X) = \sum_{i=1}^m (\nabla_{e_i}W)(e_i, X), $$

where $\{e_i\}_{i=1}^m$ is an local orthonormal basis of $M$. Then we have

$$ \text{div}S_{u,p}^{GL}(X) = \sum_{i=1}^m [\nabla_{e_i}S_{F,u}^{GL}(e_i, X)] - S_{F,u}^{GL}(e_i, \nabla_{e_i}X) $$

$$ = \sum_{i=1}^m e_i\left\{\frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2\right\}\langle e_i, X \rangle - \sum_{i=1}^m e_i\{|du|^{p-2}\langle du(e_i), du(X)\rangle\} $$

$$ - \left[\frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2\right]\sum_{i=1}^m \langle e_i, \nabla_{e_i}X \rangle + |du|^{p-2}\sum_{i=1}^m \langle du(e_i), du(\nabla_{e_i}X) \rangle $$

$$ = X\frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2 - \sum_{i=1}^m e_i\langle du|^{p-2}\rangle\langle du(e_i), du(X)\rangle $$

$$ - \sum_{i=1}^m |du|^{p-2}\langle \nabla_{e_i}du(e_i), du(X)\rangle - \sum_{i=1}^m |du|^{p-2}\langle du(e_i), \nabla_{e_i}du(X) \rangle $$

$$ + |du|^{p-2}\sum_{i=1}^m \langle du(e_i), du(\nabla_{e_i}X) \rangle $$

$$ = -\langle \text{div}(|du|^{p-2}du) + \frac{1}{e^n}(1 - |u|^2)u, du(X)\rangle $$

\[\blacksquare\]

**Definition 2.3.** We say that $u$ satisfies the conservation law if $\text{div}S_{u,p}^{GL} = 0$.

**Corollary 2.1.** If $u : (M^m, g) \to (R^n, h)$ is a critical point of $p$-Ginzburg-Landau energy functional, then $u$ satisfies the conservation law, i.e., $\text{div}S_{u,p}^{GL} = 0$.

For any vector field $X \in \Gamma(TM)$, let $\theta_X$ denote the dual one form of $X$, that is,

$$ (2.2) \quad \theta_X(Y) = g(X,Y), \quad \forall Y \in \Gamma(TM). $$
The covariant derivative of $\theta_X$ is given by

$$(\nabla^M\theta_X)(Y, Z) = (\nabla^M_Y \theta_X)(Z) = g(\nabla^M_Y X, Z),$$

for any $X, Y, Z \in \Gamma(TM)$. If $X = \nabla^M\psi$ is the gradient of some smooth function $\psi$ on $M$, then $\theta_X = d\psi$ and $\nabla^M\theta_X = \text{Hess}_g(\psi)$.

Let $W \in \Gamma(T^*M \otimes T^*M)$ be any symmetric 2-tensor. By a direct computation, we have

$$\text{div}(i_X W) = (\text{div}W)(X) + \langle W, \nabla^M\theta_X \rangle = (\text{div}W)(X) + \frac{1}{2}\langle W, L_X g \rangle,$$

where $i_X W \in A^1(M)$ denotes the interior product by any $X \in \Gamma(TM)$.

In terms of the Stoke’s formula we get

**Lemma 2.2.** Let $D$ be any bounded domain of $M$ with $C^1$ boundary. Denote by $\nu$ the unit outward normal vector field along $\partial D$. For any symmetric 2-tensor $W \in \Gamma(T^*M \otimes T^*M)$ and any vector field $X \in \Gamma(TM)$, we have

$$\int_{\partial D} (i_X W)(\nu) ds_g = \int_D \langle W, \nabla^M\theta_X \rangle + (\text{div}W)(X) dv_g$$

and

$$\int_{\partial D} (i_X W)(\nu) ds_g = \int_D \frac{1}{2}\langle W, L_X g \rangle + (\text{div}W)(X) dv_g.$$

Applying Lemma 2.2 to $S_{u,p}^{GL}$, we immediately obtain the following integral formulae:

$$\int_{\partial D} S_{u,p}^{GL}(X, \nu) ds_g = \int_D \langle S_{u,p}^{GL}, \nabla\theta_X \rangle + (\text{div}S_{u,p}^{GL})(X) dv_g$$

and

$$\int_{\partial D} S_{u,p}^{GL}(X, \nu) ds_g = \int_D \frac{1}{2}\langle S_{u,p}^{GL}, L_X g \rangle + (\text{div}S_{u,p}^{GL})(X) dv_g.$$

If $u$ is a critical point of $p$-Ginzburg-Landau energy functional, by Corollary 2.1 we obtain

$$\int_{\partial D} S_{u,p}^{GL}(X, \nu) ds_g = \int_D \langle S_{u,p}^{GL}, \nabla\theta_X \rangle dv_g,$$

$$= \int_D \frac{1}{2}\langle S_{u,p}^{GL}, L_X g \rangle dv_g. \quad (2.3)$$

For applications of the stress-energy tensor, the readers may refer to [6, 7, 8, 22]. In next section, we will use the similar method to establish the monotonicity formulae.
3 Monotonicity formulae and Liouville type results under growth conditions.

From now on, we always assume that \((M^m, g)\) is a complete Riemannian manifold with a pole \(x_0\). A pole \(x_0 \in M\) is a point such that the exponential map from the tangent space to \(M\) at \(x_0\) into \(M\) is a diffeomorphism. We will establish monotonicity formulae on these manifolds.

Denote by \(r(x)\) the distance function relative to the pole \(x_0\). For any \(x \in M\), let \(\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_m(x)\) be the eigenvalues of \(\text{Hess}_g(r^2)\) at \(x\).

**Theorem 3.1.** Assume that \(u : (M^m, g) \rightarrow (\mathbb{R}^n, h)\) is the critical point of \(p\)-Ginzburg-Landau energy functional. If there exists a constant \(\sigma > 0\) such that

\[
(P_1) \quad \frac{1}{2} \sum_{i=1}^{m} \lambda_i - p\lambda_m \geq \sigma,
\]

then

\[
\int_{B_{\rho_1}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \leq \int_{B_{\rho_2}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g,
\]

for any \(0 < \rho_1 \leq \rho_2\).

**Proof.** Set \(D = B_R(x_0) = \{x \in M | r(x) \leq R\}\) and \(X = \frac{1}{2} \nabla r^2 = r \frac{\partial}{\partial r}\). Since \(u\) is a critical point, use (2.3) we have

\[
(3.1) \quad \int_{\partial B_R(x_0)} S_{u,P}^{GL}(X, \nu) ds_g = \int_{B_R(x_0)} \frac{1}{2} \langle S_{u,P}^{GL}, L_X g \rangle dv_g,
\]

where \(\nu = \frac{\partial}{\partial r}\) is the unit outward normal vector field of \(B_R(x_0)\).

Let \(\{e_i\}_{i=1}^{m}\) be an orthonormal frame of \((M, g)\). Moreover, we can assume that \(\text{Hess}_g(r^2)\) becomes a diagonal matrix with respect to \(\{e_i\}_{i=1}^{m}\).

\[
\langle S_{u,P}^{GL}, \frac{1}{2} L_X g \rangle = \frac{1}{2} \langle S_{u,P}^{GL}, \text{Hess}_g(r^2) \rangle
\]

\[
= \frac{1}{2} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \langle g, \text{Hess}_g(r^2) \rangle \right) - \frac{1}{2} |du|^{p-2} \langle u^* h, \text{Hess}_g(r^2) \rangle
\]

\[
= \frac{1}{2} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \sum_{i,j=1}^{m} g(e_i, e_j) \cdot \text{Hess}_g(r^2)(e_i, e_j) \right)
\]

\[
- \frac{1}{2} |du|^{p-2} \sum_{i,j=1}^{m} u^* h(e_i, e_j) \cdot \text{Hess}_g(r^2)(e_i, e_j)
\]

\[
geq \frac{1}{2} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \sum_{i=1}^{m} \lambda_i - \frac{1}{2} |du|^p \lambda_m
\]
≥ \frac{1}{2} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \sum_{i=1}^{m} \lambda_i - \frac{1}{2} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) p\lambda_m \\
(3.2)
= \frac{1}{2} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \left( \sum_{i=1}^{m} \lambda_i - p\lambda_m \right).

On the other hand,
\int_{\partial B_R(x_0)} S_{u,p}^{GL}(r \frac{\partial}{\partial r}, \nu) ds_g \leq \int_{\partial B_R(x_0)} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) g(r \frac{\partial}{\partial r}, \nu) ds_g \\
= R \int_{\partial B_R(x_0)} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) ds_g \\
= R \frac{d}{dR} \int_{B_R(x_0)} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g. \\
(3.3)

It follows from (3.1), (3.2) and (3.3) that
\int_{B_R(x_0)} \frac{1}{2} \left( \sum_{i=1}^{m} \lambda_i - p\lambda_m \right) \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g.
(3.4)

By condition (P_1), we obtain
\int_{B_R(x_0)} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g \geq \sigma \int_{B_R(x_0)} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g.

Integrating the above formula on \([\rho_1, \rho_2]\), finally, we can get
\frac{\int_{B_{\rho_1}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g}{\rho_1^p} \leq \frac{\int_{B_{\rho_2}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g}{\rho_2^p}.

Next we list some vanishing results which are immediate applications of the monotonicity formulae.

**Theorem 3.2.** Under the same condition of Theorem 3.1 and
\int_{B_r(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g = o(r^\sigma),
then \( du = 0 \), that is, \( u \) is a constant.
Proof. By Theorem 3.1 for any $0 < \rho < r$, we have the following inequality
\[
\frac{1}{\rho^p} \int_{B_\rho(x_0)} |du|^p \frac{1}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \leq \frac{1}{r^\sigma} \int_{B_r(x_0)} |du|^p \frac{1}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g.
\]
Letting $r \to +\infty$, under the assumption, we may conclude
\[
\int_{B_\rho(x_0)} |du|^p \frac{1}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g = 0.
\]
Since $\rho$ is arbitrary, then $du = 0$. ■

Next, we will introduce some comparison theorems in Riemannian geometry.

**Lemma 3.1.** (cf. [8, 9, 18]) Let $(M, g)$ be a complete Riemannian manifold with a pole $x_0$ and let $r$ be the distance function relative to $x_0$. Denote by $K_r$ the radial curvature of $M$.

(i) If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$, then
\[
\beta \coth(\beta r)[g - dr \otimes dr] \leq \text{Hess}_g(r) \leq \alpha \coth(\alpha r)[g - dr \otimes dr].
\]
(ii) If $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$, $0 \leq B < 2\varepsilon$, then
\[
\frac{1 - r^2 B}{r^2} \leq \text{Hess}_g(r) \leq \frac{e^{\frac{A}{r^2}}}{r}[g - dr \otimes dr].
\]
(iii) If $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$, then
\[
\frac{1 + \sqrt{1 - 4b^2}}{2r} [g - dr \otimes dr] \leq \text{Hess}_g(r) \leq \frac{1 + \sqrt{1 + 4a^2}}{2r} [g - dr \otimes dr].
\]

**Proof.** The case (i) is standard (cf. [9]). The case (ii) is discussed in [8]. For (iii), see [9, 14, 18] for details. ■

**Lemma 3.2.** Let $(M, g)$ be a complete Riemannian manifold with a pole $x_0$ and let $r$ be the distance function relative to $x_0$. Assume that there exist two positive functions $h_1(r)$ and $h_2(r)$ such that
\[
h_1(r)[g - dr \otimes dr] \leq \text{Hess}_g(r) \leq h_2(r)[g - dr \otimes dr] \quad \text{and} \quad rh_2(r) \geq 1,
\]
then
\[
\sum_{i=1}^{m} \lambda_i - p\lambda_m \geq 2 \{1 + (m - 1)rh_1(r) - prh_2(r)\}.
\]

Combing Lemma 3.1 and Lemma 3.2, we can obtain the following.
Lemma 3.3. Let $(M, g)$ be a complete Riemannian manifold with a pole $x_0$ and let $r$ be the distance function relative to $x_0$. Denote by $K_r$ the radial curvature of $M$.

(i) If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(m-1)\beta - p\alpha > 0$, then

$$\sum_{i=1}^{m} \lambda_i - p\lambda_m \geq 2(m - \frac{p\alpha}{\beta}).$$

(ii) If $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq -\frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$, $0 \leq B < 2\varepsilon$ and $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - pe^{\frac{A}{2\varepsilon}} > 0$, then

$$\sum_{i=1}^{m} \lambda_i - p\lambda_m \geq 2[1 + (m-1)(1 - \frac{B}{2\varepsilon}) - e^{\frac{A}{2\varepsilon}}].$$

(iii) If $-\frac{a^2}{1+r^2} \leq K_r \leq -\frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ and $2 + (m-1)(1 + \sqrt{1-4b^2}) - p(1 + \sqrt{1+4a^2}) > 0$, then

$$\sum_{i=1}^{m} \lambda_i - p\lambda_m \geq 2[1 - \frac{p}{2} + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - \frac{p}{2}\sqrt{1+4a^2}].$$

Corollary 3.1. Let $u : (M, g) \to (R^n, h)$ be a critical point of $p$-Ginzburg-Landau energy functional from a Riemannian manifold with a pole $x_0$ to a standard Euclidean space. Assume that the radial curvature $K_r$ of $M$ satisfies one of the following three conditions:

(i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(m-1)\beta - p\alpha > 0$;

(ii) $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq -\frac{B}{(1+r^2)^{1+\varepsilon}} > 0$ with $\varepsilon, A \geq 0$, $0 \leq B < 2\varepsilon$ and $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - pe^{\frac{A}{2\varepsilon}} > 0$;

(iii) $-\frac{a^2}{1+r^2} \leq K_r \leq -\frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ and $2 + (m-1)(1 + \sqrt{1-4b^2}) - p(1 + \sqrt{1+4a^2}) > 0$.

Then

$$\int_{B_{\rho_1}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \leq \int_{B_{\rho_2}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g,$$

for any $0 < \rho_1 \leq \rho_2$, where

$$\sigma = \begin{cases} (m - \frac{p\alpha}{\beta}); & \text{for } K_r \text{ satisfies (i)} \\
1 + (m-1)(1 - \frac{B}{2\varepsilon}) - pe^{\frac{A}{2\varepsilon}}; & \text{for } K_r \text{ satisfies (ii)} \\
2 - p + (m-1)(1 + \sqrt{1-4b^2}) - p\sqrt{1+4a^2}. & \text{for } K_r \text{ satisfies (iii)} \end{cases}$$
Corollary 3.1 yields immediately the following vanishing result.

**Theorem 3.3.** Suppose that \( u : (M, g) \to (\mathbb{R}^n, h) \) is a critical point of \( p \)-Ginzburg-Landau energy functional. Let \( r \) be the distance function relative to \( x_0 \). If the radial curvature \( K_r \) of \( M \) satisfies one of the three conditions in Corollary 3.1 and

\[
\int_{B_r(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g = o(r^\sigma),
\]

where \( \sigma \) is given by Corollary 3.1. Then \( du = 0 \), that is, \( u \) is constant.

**Definition 3.1.** \( E_p^{GL}(u) \) is said to have slowly divergent energy, if there exists a positive continuous function \( \psi(r) \) such that

\[
\int_{R_1}^{+\infty} \frac{dr}{r\psi(r)} = +\infty
\]

for some \( R_1 > 0 \), and

\[
(3.6) \quad \lim_{R \to \infty} \int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \frac{dv_g}{\psi(r(x))} < \infty.
\]

**Theorem 3.4.** Let \( u : (M, g) \to (\mathbb{R}^n, h) \) be the critical point of \( p \)-Ginzburg-Landau energy functional. If \( r(x) \) satisfies the condition \( (P_1) \) and \( E_p^{GL}(u) \) has slowly divergent energy, then \( u \) is a constant map and \( u(M) \subseteq S^{n-1} \).

**Proof.** From Theorem 3.1 we obtain

\[
R \frac{d}{dR} \int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \\
\geq \int_{B_R(x_0)} \frac{1}{2} \sum_{i=1}^m \lambda_i - p\lambda_{\max} \left( \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g.
\]

If \( u \) is not a constant map contained in \( S^{n-1} \), there exists constants \( R_0 > 0 \) and \( C_0 > 0 \) such that

\[
\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \geq C_0
\]

for any \( R \geq R_0 \). Thus

\[
\int_{\partial B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 ds_g \geq \frac{\sigma C_0}{R}, \quad \forall R \geq R_0.
\]

Since \( E_p^{GL}(u) \) has slowly divergent energy, then

\[
\lim_{R \to \infty} \int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \frac{dv_g}{\psi(r(x))} = \int_0^\infty \frac{dR}{\psi(R)} \int_{\partial B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 ds_g.
\]
It is in contradiction to (3.6). \(\blacksquare\)

4 Liouville theorem under the asymptotic conditions

In [13], Jin established several Liouville theorems for harmonic maps between some Riemannian manifold under some asymptotic condition of the maps at infinity. In particular, he proved that for any harmonic map \(f : (R^m, g_0) \rightarrow (N^n, h)\) \((m \geq 3)\), if \(f(x) \rightarrow P_0 \in N^n\) as \(|x| \rightarrow +\infty\), then \(f\) must be a constant map. In this section, using a similar technique or idea, we can derive a Liouville theorem for the critical points of the \(p\)-Ginzburg-Landau energy. To generalize this result to our case it is necessary to give more strictly asymptotic condition at infinity. We begin with evaluating the lower bounder of the energy.

**Proposition 4.1.** Assume that \(u : (M^m, g) \rightarrow (R^n, h)\) is a critical point of the \(p\)-Ginzburg-Landau energy functional from a Riemannian manifold with a pole \(x_0\) to a standard Euclidean space. \(r(x)\) is the distance function relative to the pole \(x_0\). If it satisfies the condition (\(P_1\)) and \(u(M)\) is not contained in \(S^{n-1}\), then

\[
\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2 dv_g \geq C(u) R^\sigma, \quad R \rightarrow \infty
\]

where \(C(u)\) is a positive constant only depending on \(u\).

**Proof.** Since \(u\) satisfies the conditions in Theorem 3.1, we obtain

\[
\int_{B_\rho(x_0)} \frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2 dv_g \leq \int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2 dv_g
\]

for any \(0 < \rho < R\). Note that \(u(M)\) is not contained in \(S^{n-1}\), there exist some \(\rho > 0\) such that

\[
\int_{B_\rho(x_0)} \frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2 dv_g > 0.
\]

Denoted by \(C(u) = \frac{\int_{B_\rho(x_0)} \frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2 dv_g}{\rho^\sigma}\), then

\[
\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4e^n}(1 - |u|^2)^2 dv_g \geq C(u) R^\sigma.
\]
By Corollary 3.1 and using the above proposition, we easily obtain the following corollary.

**Corollary 4.1.** Let \( u : (M, g) \rightarrow (R^n, h) \) be a critical map of the p-Ginzburg-Landau energy functional from a Riemannian manifold with a pole \( x_0 \) to a standard Euclidean space. Assume that the radial curvature \( K_r \) of \( M \) satisfies one of the following three conditions:

(i) \(-\alpha^2 \leq K_r \leq -\beta^2 \) with \( \alpha > 0, \beta > 0 \) and \((m - 1)\beta - p\alpha > 0\);

(ii) \(-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq -\frac{B}{(1+r^2)^{1+\varepsilon}} > 0 \) with \( A \geq 0, 0 \leq B < 2\varepsilon \) and \( 1 + (m - 1)(1 - \frac{B}{2\varepsilon}) - pe^{\frac{A}{2\varepsilon}} > 0 \);

(iii) \(-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2} \) with \( a \geq 0, b^2 \in [0, \frac{1}{4}] \) and \( 2 + (m - 1)(1 + \sqrt{1 - 4b^2}) - p(1 + \sqrt{1 - 4a^2}) > 0 \).

If \( u(M) \) is not contained in \( S^{n-1} \), then

\[
\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\varepsilon^n}(1 - |u|^2)^2dv_g \geq C(u)R^\sigma, \quad \text{as } R \to \infty
\]

where \( C(u) \) is a positive constant only depending on \( u \), and

\[
\sigma = \begin{cases} 
(m - \frac{\alpha}{\beta}); & \text{for } K_r \text{ satisfies (i)} \\
1 + (m - 1)(1 - \frac{B}{2\varepsilon}) - pe^{\frac{\alpha}{2\varepsilon}}; & \text{for } K_r \text{ satisfies (ii)} \\
\frac{2 - p + (m - 1)(1 + \sqrt{1 - 4b^2}) - p\sqrt{1 + 4a^2}}{2}. & \text{for } K_r \text{ satisfies (iii)}
\end{cases}
\]

Next, we will show that if \( u \) is the critical map of the 2-Ginzburg-Landau energy functional and it is uniformly bounded, the condition \((P_1)\) may be replaced by \((P_1')\): The left hand side of the inequality in \((P_1)\) is nonnegative on the whole \( M \) and there exists an \( R_0 > 0 \) such that \((P_1)\) holds for \( r(x) \geq R_0 \).

To prove this assertion, we start with the following lemmas.

**Lemma 4.1.** Suppose that \( u : (M^m, g) \rightarrow (R^n, h) \) is a critical map of the 2-Ginzburg-Landau energy functional and \( u \) is uniformly bounded. If \( u \) is constant in an open set of \( M \), then \( u \) is constant on \( M \).

**Proof.** From the Euler-Lagrange equation, we have

\[
\Delta u + \frac{1}{\varepsilon^n}(1 - |u|^2)u = 0.
\]

Since \( u \) is bounded, using unique continuation theorem in \( \mathbb{P} \), one can deduce that \( u \) is constant on \( M \).
Lemma 4.2. Assume that \( u \) satisfies \((\tilde{P}_1)\). If \( du \) is not identically zero, then \( E_2^{GL}(u) = +\infty \).

Proof. By co-area formula, we have

\[
\int_M \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g = \int_0^{+\infty} \frac{dr}{r} \int_{\partial B_r(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \frac{1}{|\nabla r|} ds_g = \int_0^{+\infty} \frac{dr}{r} \int_{\partial B_r(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 ds_g.
\]

If \( E_2^{GL}(u) < +\infty \), we can choose a sequence \( \{r_i\} \) such that

\[
(4.1) \quad \lim_{r_i \to +\infty} r_i \int_{\partial B_{r_i}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 ds_g = 0.
\]

Since \( r(x) \) satisfies \((\tilde{P}_1)\), by (3.4) we obtain

\[

r \frac{d}{dr} \int_{B_r(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \geq \sigma \int_{B_r(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g,
\]

that is

\[
r \int_{\partial B_r(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \geq \sigma \int_{B_r(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g.
\]

Let \( r = r_i \) tend to infinity in above inequality, it follows from (4.1) that

\[
\int_{M \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g = 0.
\]

It follows from Lemma 4.1 that \( du = 0 \) on \( M \) which contradicts with the condition. Therefore \( E_2^{GL}(u) = +\infty \).

Proposition 4.2. Assume that \( u : (M^m, g) \to (R^n, h) \) is a critical point of the 2-Ginzburg-Landau energy functional and \( u \) is uniformly bounded. If \( r(x) \) satisfies \( \tilde{P}_1 \) and \( u(M) \) is not contained in \( S^{n-1} \), then

\[
\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \geq C(u) R^2,
\]

where \( R \) is sufficiently large and \( C(u) \) is a positive constant only depending on \( u \).

Proof. Taking \( D = B_R(x_0) \setminus B_{R_0}(x_0) \) and \( X = r^2 \frac{d}{dr} = \frac{1}{2} \nabla r^2 \), by (2.3) we get

\[
\int_{\partial B_R(x_0)} S_{2,u}^{GL}(X, \nu) ds_g - \int_{\partial B_{R_0}(x_0)} S_{2,u}^{GL}(X, \nu) ds_g = \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \nabla^{GL} \cdot \nabla X > dv_g.
\]
Denoting \(\int_{\partial B_{R_0}(x_0)} S^H_{p=2, u}(X, \nu) ds_g \) by \(H(R_0)\), then (3.3) and condition \((\tilde{P}_1)\) yield

\[
R \int_{\partial B_{R}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 ds_g - 2H(R_0) \geq \sigma \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g.
\]

It also can be written as

\[
R \frac{d}{dR} \left\{ \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} \right\} \geq \sigma \left\{ \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} \right\}.
\]

From Lemma 4.2 we know that \(E_2^G(u) = +\infty\). Therefore, when \(R\) is sufficiently large, we get

\[
\int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} > 0.
\]

Then

\[
\frac{d}{dR} \left\{ \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} \right\} \geq \frac{\sigma}{R}.
\]

Fixing some \(R_0 < \tilde{R} < R\) and integrating the above formula on \([\tilde{R}, R]\), we get

\[
\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g \geq \left\{ H(R_0, \tilde{R}) - \frac{2H(R_0)}{\sigma R^\sigma} \right\} R^\sigma,
\]

where \(H(R_0, \tilde{R}) = \int_{B_\tilde{R}(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} \). When \(R \to +\infty\), \(\frac{2H(R_0)}{\sigma R^\sigma}\) can be controlled by \(H(R_0, \tilde{R})\). Consequently

\[
\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\sigma_n} (1 - |u|^2)^2 dv_g \geq C(u) R^\sigma,
\]

where \(C(u)\) is a constant depending on the map \(u\).

Next we will use the assumption for the map at infinity to derive an upper bound for the growth rate. The condition that we will assume for \(u\) is as follow:

(P2) There exists a positive constant \(\tilde{\sigma}\) less than \(\sigma\) in \((P_1)\) such that

\[
\max_{r(x)=r} h^2(u(x), P_0) \leq r^{\tilde{\sigma}} \int_r^{+\infty} \frac{ds}{\text{vol}(\partial B_s(x_0))} \quad \text{for} \quad r(x) \gg 1.
\]
Theorem 4.1. Let $u : (M, g) \to (R^n, h)$ be a critical point of $p$-Ginzburg-Landau energy functional. Suppose that $|du|^{p-2}$ is uniformly bounded and $r(x)$ satisfies the condition $(P_1)$. If $u(x) \to P_0 \in S^{n-1}$ and $u$ satisfies the condition $(P_2)$, the $u$ must be a constant map.

Proof. Suppose the critical point $u$ is not constant, then by Proposition[4.1] the energy of $u$ must be infinite. That is, \[
\int_{B_R(x_0)} |du|^p + \frac{1}{|x|^2} (1 - |u|^2)^2 dv_g \to +\infty \text{ as } r(x) \to +\infty.
\]

Since $P_0 = (c_1, c_2, \cdots, c_n) \in S^{n-1}$, then $\sum_{\alpha=1}^{n} c_\alpha^2 = 1$. It is clear that we can choose a orthogonal matrix $A$ such that $AP_0 = \tilde{P}_0 = (\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_n)$, $\tilde{c}_\alpha \neq 0$, for each $\alpha = 1, 2, \cdots, n$. Clearly if $u$ is the critical point of $p$-Ginzburg-Landau energy functional, then $Au$ is also the critical point. Hence without loss of generality we may assume that $u(x) \to p_0 \in S^{n-1}$, where $p_0 = (c_1, c_2, \cdots, c_n), c_\alpha \neq 0, \alpha = 1, 2, \cdots, n$.

Now the assumption that $u(x) \to P_0$ as $r(x) \to +\infty$ implies that there exists an $R_1 > 0$ and a neighbourhood $U$ of $P_0$ such that for $r(x) > R_1$, $u(x) \in U$ and $u_\alpha \neq 0$ for any $\alpha = 1, 2, \cdots, n$.

For $\omega \in C_0^\infty (M \setminus B_{R_1}(x_0), U)$, we consider the variation $u + t \omega : M \to R^n$ defined as follows:
\[
(u + t \omega)(q) = \begin{cases} 
q & q \in B_{R_1}(x_0), \\
(u + t \omega)(q) & q \in M^m \setminus B_{R_1}(x_0)
\end{cases}
\]
for sufficiently small $t$. Since $u$ is the critical point of $p$-Ginzburg-Landau energy functional, we have
\[
\frac{d}{dt}|_{t=0} E_p^{GL}(u + t \omega) = 0
\]
that is,
\[
\int_{M^m \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \partial u_k \partial \omega_k - \frac{1}{\epsilon^n} (1 - \sum_{k=1}^{n} u_k^2) \sum_{k=1}^{n} u_k \omega_k dv_g = 0. \tag{4.2}
\]

Choose $\omega(x) = \phi(r(x)) \tilde{u}(x)$ in (4.2) for $\phi(t) \in C_0^\infty (R_1, \infty)$, $\tilde{u}_k = \frac{u_k^2 - c_k^2}{u_k}$, we obtain
\[
\int_{M^m \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \partial u_k \partial \tilde{u}_k \phi(r(x)) - \frac{1}{\epsilon^n} (1 - \sum_{k=1}^{n} u_k^2) \phi(r(x)) \sum_{k=1}^{n} u_k \tilde{u}_k dv_g = 0.
\]
\[
\int_{M^m \setminus B_{R_1}(x_0)} (du)^{p-2} \sum_{k=1}^{n} g^{ij} \partial u_k \partial \frac{\phi(r(x))}{\epsilon^{n}} \sum_{k=1}^{n} u_k \tilde{u}_k dv_g \tag{4.3}
\]
By a standard approximation argument, (4.3) holds for Lipschitz functions $\phi$ with compact support.

For $0 < \theta \leq 1$, define
\[
\varphi_\theta(t) = \begin{cases} 
1 & t \leq 1; \\
1 + \frac{1-t}{\theta} & 1 < t < 1 + \theta; \\
0 & t \geq 1 + \theta.
\end{cases}
\]
In (4.3), choose the Lipschitz function $\phi(r(x))$ to be

$$\phi(r(x)) = \varphi_\theta(r(x)) (1 - \varphi_1(r(x))) = \frac{r(x)}{R_1} (1 - \varphi_1(r(x)))$$

$R > 2R_1$ and $R_2 = 2R_1$.

Then the first term on left hand side of (4.3) becomes

$$\int_{M^n \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} dv_g$$

$$= \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} dv_g$$

$$+ \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} dv_g$$

(4.4)

$$+ \int_{B_{R(1+\theta)}(x_0) \setminus B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} dv_g.$$ 

The second term on left hand side of (4.3) becomes

$$- \int_{M^n \setminus B_{R_1}(x_0)} \frac{1}{e^n} (1 - \sum_{k=1}^{n} u_k^2) \phi(r(x)) \sum_{k=1}^{n} u_k \tilde{u}_k dv_g$$

$$= - \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} \frac{1}{e^n} (1 - \sum_{k=1}^{n} u_k^2) \frac{1}{R_2} \sum_{k=1}^{n} u_k \tilde{u}_k dv_g$$

$$- \int_{B_{R}(x_0) \setminus B_{R_2}(x_0)} \frac{1}{e^n} (1 - \sum_{k=1}^{n} u_k^2) \sum_{k=1}^{n} u_k \tilde{u}_k dv_g$$

$$- \int_{B_{R(1+\theta)}(x_0) \setminus B_R(x_0)} \frac{1}{e^n} (1 - \sum_{k=1}^{n} u_k^2) \varphi_\theta \frac{r(x)}{R} \sum_{k=1}^{n} u_k \tilde{u}_k dv_g.$$ 

(4.5)

We can also compute the right hand side of (4.3) as follows

$$\int_{M^n \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} dv_g$$

$$= \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} dv_g$$

$$+ \frac{1}{R \theta} \int_{B_{R(1+\theta)}(x_0) \setminus B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial r(x)}{\partial x_j} dv_g.$$ 

(4.6)

Set

$$D(R_1) = \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} dv_g$$

$$= \frac{1}{R \theta} \int_{B_{R(1+\theta)}(x_0) \setminus B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial r(x)}{\partial x_j} dv_g.$$ 

16
\[-\int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} \frac{1}{e^n}(1 - \sum_{k=1}^{n} u_k^2)[1 - \varphi_1(\frac{r(x)}{R_1})] \sum_{k=1}^{n} u_k \tilde{u}_kdv_g \]

\[-\int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^{n} g^{ij}_k \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} - \frac{1}{e^n}(1 - \sum_{k=1}^{n} u_k^2) \sum_{k=1}^{n} u_k \tilde{u}_kdv_g + D(R_1) \]

(4.7)

Substitute (4.4), (4.5) and (4.6) into (4.3), then letting \( \theta \to 0 \), we have

\[
\int_{B_R(x_0)\setminus B_{R_2}(x_0)} |du|^{p-2} \sum_{k=1}^{n} \tilde{\mathcal{R}}_k \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k ds_g,
\]

where \( \nu^i = \frac{\partial r}{\partial x_j} \) and \( \nu = \nu^i \frac{\partial}{\partial x_i} \) is the outer normal vector field along \( B_{R_0}(x_0) \).

Note that \( \tilde{\mathcal{R}}_k = \frac{u_k^2 - c_k^2}{u_k} \). Thus \( \frac{\partial \tilde{\mathcal{R}}_k}{\partial x_j} = (1 + c_k^2u_k) \frac{\partial u_k}{\partial x_j} \). Then (4.7) becomes

\[
\int_{B_R(x_0)\setminus B_{R_2}(x_0)} |du|^{p-2} \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k ds_g,
\]

(4.8)

Indeed,

\[
\sum_{k=1}^{n} \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_i} dx^i \otimes \frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_j} \tilde{u}_k dx^i \otimes \frac{\partial}{\partial y_k}.
\]

Therefore

\[
\int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k ds_g
\]

\[
= \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_i} dx^i \otimes \frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_j} \tilde{u}_k dx^i \otimes \frac{\partial}{\partial y_k} ds_g
\]

\[
\leq \int_{\partial B_R(x_0)} |du|^{p-2} \frac{\partial u_k}{\partial x_i} \frac{\partial}{\partial y_k} \tilde{u}_k dx^i \otimes \frac{\partial}{\partial y_k} ds_g
\]

\[
\leq \int_{\partial B_R(x_0)} |du|^{p-2} \frac{\partial u_k}{\partial x_i} \frac{\partial}{\partial y_k} \tilde{u}_k ds_g \sqrt{\int_{\partial B_R(x_0)} |du|^{p-2} \frac{\partial}{\partial y_k} \tilde{u}_k ds_g}
\]

\[
= \sqrt{\int_{\partial B_R(x_0)} |du|^{p} ds_g \sqrt{\int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} \tilde{u}_k^2 ds_g}}
\]

(4.9)

\[
\leq \sqrt{\frac{\int_{\partial B_R(x_0)} |du|^{p} + \frac{1}{e^n}(1 - |u|^2)^2 ds_g \sqrt{\int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} \tilde{u}_k^2 ds_g}}{\int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^{n} \tilde{u}_k^2 ds_g}}.
\]
Next, for any $R \geq R_2$ we let
\[
G(R) = \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^p + \frac{1}{\epsilon^n}(1 - |u|^2)^2 dv_g + D(R_1).
\]
Then
\[
G'(R) = \int_{\partial B_R(x_0)} |du|^p + \frac{1}{\epsilon^n}(1 - |u|^2)^2 ds_g.
\]
Hence from (4.8), (4.9) and the fact that $1 + \frac{c^2}{u_\alpha} \geq 1$ for any $\alpha = 1, 2, \ldots, n$.
\[
G^2(R) \leq G'(R) \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \tilde{u}_k^2 ds_g.
\]
On the other hand, we have the following estimate.
\[
G(R) - D(R_1) = \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^p + \frac{1}{\epsilon^n}(1 - |u|^2)^2 dv_g
\]
\[= p \int_{\partial B_R(x_0) \setminus B_{R_2}(x_0)} \left[ \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \right]
\geq \min\{p, 4\} \int_{\partial B_R(x_0) \setminus B_{R_2}(x_0)} \left[ \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \right]
\]
\[= C_{p,4} \int_{\partial B_R(x_0) \setminus B_{R_2}(x_0)} \left[ \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \right].
\]
Since $E_p^{GL}(u)$ is infinity, there exists $\bar{R} \geq R_2$, $G(R) > 0$ for any $R > \bar{R}$.
Set $J(R) = \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \tilde{u}_k^2 ds_g$. Then
\[
G^2(R) \leq G'(R)J(R).
\]
For any $\bar{R} \leq R < R_4$, we have
\[
\int_{R}^{R_4} \frac{G'(r)}{G^2(r)} dr \geq \int_{R}^{R_4} \frac{dr}{J(r)},
\]
\[
\frac{1}{G(R)} - \frac{1}{G(R_4)} \geq \int_{R}^{R_4} \frac{dr}{J(r)}.
\]
When $R_4 \to +\infty$, we get $G(R) \leq \frac{1}{J(R)}$.

Note the fact that $|du|^{p-2}$ is uniformly bounded. Using the condition $(P_2)$ and $u(x) \to P_0$ as $r(x) \to +\infty$, we get
\[
J(R) = \int_{\partial B_r(x_0)} |du|^{p-2} \sum_{k=1}^n \tilde{u}_k^2 ds_g
\]

18
\[
\leq \int_{\partial B_r(x_0)} |du|^{p-2} \tau(r) ds_g \\
\leq \widetilde{C} \tau(r) \cdot vol(\partial B_r(x_0)),
\]
where \( \widetilde{C} \) is a constant only depending on \( u \) and \( \tau(r) \) is chosen in such a way that

1. \( \tau(r) \) is nonincreasing on \((\tilde{R}, +\infty)\) and \( \tau(r) \to 0 \) as \( r \to +\infty \);

2. \( \tau(r) \geq \max_{(\tilde{R}, +\infty)} \{ \sum_{k=1}^{n} \tilde{u}_k^2 \} \);

3. \( \tau(r) \leq C_{P_0} r^{3} \cdot \int_{r}^{\infty} \frac{ds}{\text{vol}(\partial B_s(x_0))} \),

where \( C_{P_0} \) is a constant only depending on \( P_0 \). Then we can derive

\[
\int_{R}^{\infty} \frac{dr}{J(r)} \geq \int_{R}^{\infty} \frac{dr}{C\tau(r) \cdot \text{vol}(\partial B_r(x_0))} \geq \frac{1}{C_1 \tau(R)} \int_{R}^{\infty} \frac{dr}{\text{vol}(\partial B_r(x_0))} \geq \frac{1}{C_1 \tilde{R}^{\tilde{\sigma}}},
\]

where \( C_1 = \widetilde{C} \cdot C_{P_0} \). Hence \( G(R) \leq C_1 R^{\tilde{\sigma}} \) for any \( R \geq \tilde{R} \). By the definition of \( G(R) \), we have

\[
\int_{B_{R}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^{n}} (1 - |u|^2)^2 dv_g \\
\leq \frac{C_{1}}{C_{p,4}} R^{\tilde{\sigma}} - \frac{D(R_1)}{C_{p,4}} + \int_{B_{R_2}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^{n}} (1 - |u|^2)^2 dv_g \\
= \{ C R^{\tilde{\sigma} - \sigma} + \frac{C(u)}{R^{\sigma}} \} R^{\sigma},
\]

where \( C(u) \) is constant only depending on \( u \). Since \( \tilde{\sigma} < \sigma \), it contradicts with Proposition 4.1.

In [18], the authors give the volume growth estimates under Ricci curvature conditions. Hence, applying the results to the following cases, the right side of the inequality in condition (P2) can be expressed as a polynomial.

**Corollary 4.2.** Let \( u : (M, g) \to (\mathbb{R}^n, h) \) be a critical point of \( p \)-Ginzburg-Landau energy functional. Suppose that \( |du|^{p-2} \) is uniformly bounded. Assume that the radial curvature \( K_r \) of \( M \) satisfies the following condition

\[
- \frac{A}{(1 + r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1 + r^2)^{1+\epsilon}} \quad \text{with} \quad \epsilon > 0,
\]

where \( A \geq 0, 0 \leq B < 2\epsilon \) and \( 1 + (m-1)(1 - \frac{B}{2\epsilon}) - pe^{\frac{A}{2\epsilon}} > 0 \). If \( u(x) \to P_0 \in S^{n-1} \) as \( r(x) \to +\infty \), and

\[
\max_{r(x) = r} h^2(u(x), P_0) \leq \frac{r^{\tilde{\sigma} - (m-2)}}{(m-2)\omega_m e^{\frac{(m-1)A}{2\epsilon}}}.
\]

then \( u \) must be a constant map. Here \( \omega_m \) is the \((m-1)\)-volume of the unit sphere in \( \mathbb{R}^m \) and \( \tilde{\sigma} \) is any positive constant such that \( \tilde{\sigma} < [1 + (m-1)(1 - \frac{B}{2\epsilon}) - e^{\frac{A}{2\epsilon}}] \).
Proof. For the condition on the radial curvature $K_r$, it follows that

$$Ric(x) \geq -\frac{(m-1)A}{(1+r^2(x))^{1+\varepsilon}},$$

for any $x \in M$. Then a direct calculation yields

$$\int_0^{+\infty} \frac{Ar}{(1+r^2)^{1+\varepsilon}} dr = \frac{A}{2\varepsilon}.$$  

By the volume comparison theorem (cf. Corollary 2.17 in [PRS]), we obtain

$$vol(\partial B_r(x_0)) \leq \omega_m e^{\frac{(m-1)A}{2\varepsilon}} r^{m-1},$$

where $\omega_m$ is the $(m-1)$-volume of the unit sphere in $\mathbb{R}^m$, and thus

$$(\int_R^{+\infty} \frac{dr}{vol(\partial B_r(x_0))})^{-1} \leq (m-2)\omega_m e^{\frac{(m-1)A}{2\varepsilon}} R^{m-2}$$

for $R \gg 1$. Using Corollary 4.1 and Theorem 4.1, we can get the result.

Corollary 4.3. Let $u : (M, g) \to (\mathbb{R}^n, h)$ be a critical point of $p$-Ginzburg-Landau energy functional. Suppose that $|du|^{p-2}$ is uniformly bounded. Assume that the radial curvature $K_r$ of $M$ satisfies the following condition

$$-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$$

with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ and $2 + (m-1)(1 + \sqrt{1-4b^2}) - p(1 + \sqrt{1+4a^2}) > 0$. If $u(x) \to P_0 \in S^{n-1}$ as $r(x) \to +\infty$ and

$$\max_{r(x)=R} h^2(u(x), P_0) \leq C\sigma^{-\frac{(m-1)A'}{2}} + 1,$$

then $u$ must be a constant map. Here $A' = \frac{1+\sqrt{1+4a^2}}{2}$ and $\sigma$ is any positive constant such that $\sigma < [1 - \frac{p}{2} + (m-1)\frac{1+\sqrt{1+4a^2}}{2} - \frac{p}{2}\sqrt{1+4a^2}].$

Proof. For the condition on the radial curvature $K_r$, it follows that

$$Ric(x) \geq -\frac{(m-1)a^2}{1+r^2(x)}$$

for any $x \in M$. We can also use the volume comparison theorem (cf. Corollary 2.17 in [PRS]), then

$$vol(\partial B_R(x_0)) \leq CR^{(m-1)A'},$$

where $C$ is suitable constant and $A' = \frac{1+\sqrt{1+4a^2}}{2}$. Thus

$$(\int_R^{+\infty} \frac{dr}{vol(\partial B_r(x_0))})^{-1} \leq CR^{(m-1)A'-1}$$

20
for $R \gg 1$. Using Corollary 4.1 and Theorem 4.1, we can get the result.

If the Riemannian manifold $(M, g)$ is the standard Euclidean space $(\mathbb{R}^m, h)$, the eigenvalues of $\text{Hess}g(r^2)$ are all 2. When $p = 2$, \( \frac{1}{2}(\sum_{i=1}^{m} \lambda_i - 2\lambda_m) = m - 2 \). Thus we have the following result.

**Corollary 4.4.** Let $u : (\mathbb{R}^m, h) \to (\mathbb{R}^n, h)$ be a critical point of $2$-Ginzburg-Landau energy functional. If $u(x) \to P_0 \in S^{n-1}$ as $r(x) \to +\infty$, $u$ must be a constant map contained in $S^{n-1}$.

**Proof.** Since $(M, g)$ is the standard Euclidean space, from the proofs in Theorem 4.1, we obtain

$$\int_{R}^{+\infty} \frac{dr}{J(r)} \geq \frac{C_m}{\tau(R)} \frac{1}{R^{m-2}}, \quad \text{for any } R \geq \tilde{R}$$

where $C_m$ is a positive constant only depending on $m$ and $\tau(r)$ satisfies the following conditions

1. $\tau(r)$ is nonincreasing on $(\tilde{R}, +\infty)$ and $\tau(r) \to 0$ as $r \to +\infty$;
2. $\tau(r) \geq \max_{r(x)=r}\{\sum_{k=1}^{n} \tilde{u}_k^2\}$.

Then

$$\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\pi}(1 - |u|^2)^2 dv_g \leq C(\tau(R) + \frac{C(u)}{R^{m-2}})R^{m-2}. $$

---

**5 Constant Dirichlet Boundary-value Problems**

**Definition 5.1.** A bounded domain $D \subseteq M$ with $C^1$ boundary $\partial D$ is called starlike if there exists an interior point $x_0 \in D$ such that

$$\langle \frac{\partial}{\partial r_{x_0}}, \nu \rangle |_{\partial D} \geq 0$$

where $\nu$ is the unit outer normal to $\partial D$, and the vector field $\frac{\partial}{\partial r_{x_0}}$ is the unit vector field such that for any $x \in D \setminus \{x_0\} \cup \partial D$, $\frac{\partial}{\partial r_{x_0}}$ is the unit vector tangent to the unique geodesic joining $x_0$ to $x$ and pointing away from $x_0$.

**Theorem 5.1.** Suppose $M$ satisfies the same condition of Theorem 5.1 and $D \subseteq M$ is a bounded starlike domain with $C^1$ boundary. If $u : (M, g) \to (\mathbb{R}^m, h)$ is a critical point of the $p$-Ginzburg-Landau energy functional and $u|_{\partial D} \subseteq S^{n-1}$ is constant, then $u|_D$ is constant.
Proof. Set $X = r \frac{\partial}{\partial r}$, where $r = r_{x_0}$. From the proof of Theorem 3.1, we have

$$\int_D \langle S_{u,p}^{GL}, \frac{1}{2} L_X g \rangle dv_g \geq \sigma \int_D \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g.$$  

(5.1)

Since $u|_{\partial D} \subseteq S^{n-1}$ is constant, then $|u|^2 = 1$ and for any $\eta \in T(\partial D)$, $du(\eta) = 0$. Thus

$$\int_{\partial D} S_{u,p}^{GL}(r \frac{\partial}{\partial r}, \nu) ds_g = \int_{\partial D} r \{ \frac{|du|^p}{p} \langle \frac{\partial}{\partial r}, \nu \rangle - |du|^{p-2} \langle du(\frac{\partial}{\partial r}), du(\nu) \rangle \} ds_g.$$  

(5.2)

$$= \int_{\partial D} r \{ |du|^p \langle \frac{\partial}{\partial r}, \nu \rangle - |du|^{p-2} \langle \frac{\partial}{\partial r}, \nu \rangle |du|^2 \} ds_g$$

Note that $D$ is starlike, by (2.3) and (5.2)

$$\int_D \langle S_{u,p}^{GL}, \frac{1}{2} L_X g \rangle dv_g \leq 0.$$  

(5.3)

From (5.1) and (5.3), we have

$$\int_D \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g = 0.$$  

Therefore $u$ is constant.  

References

[1] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities. J Math Pure Appl, 1957, 36:235-249.

[2] F. Bethuel, H. Brezis, F. Helein, Asymptotics for the minimization of a Ginzburg-Landau functional, Calc. Var. PDE 1(1993) 123-148.

[3] F. Bethuel, H. Brezis, F. Helein, Ginzburg-Landau Vortices, Birkhäuser, Boston, 1994.

[4] H. Brezis, F. Merle, T. Riviere, Quantization effects for $-\Delta u = u(1 - |u|^2)^2$, Arch. RationalMech. Anal. 126(1994) 35-58.

[5] S Y Cheng, Liouville theorem for harmonic maps. Inventiones Mathematicae, 1992, 108(1):1-10.

[6] Y.X. Dong and H.Z. Lin, Monotonicity formulae, vanishing theorems and some geometric applications, Q. J. Math. 65(2)(2014)365-397.
[7] Y.X. Dong, H.Z. Lin and G.L. Yang, Liouville theorems for F-harmonic maps and their applications, Results in Math. 69(2016), 105-127.

[8] Y.X. Dong, S.S. Wei, On vanishing theorems for vector bundle valued p-forms and their applications, Comm. Math. Phys. 304(2)(2011)329-368.

[9] R.E. Greene, H. Wu, Function theory on manifolds which possess a pole, Lecture Notes in Math, Springer, Berlin, 699(1979).

[10] W.D. Garber, S.N.M. Ruijsenaars, E. Seiler, D. Burns, On finite action solutions of the nonlinear -model, Ann. Phys. 119 (1979), 305-325.

[11] S. Hildebrandt, Liouville theorems for harmonic mappings, and an approach to Bernstein theorems, Ann. Math. Stud. 102 (1982), 107-131.

[12] M.C. Hong, Asymptotic behavior for minimizers of a GinzburgCLandau type functional in higher dimensions associated with n-harmonic maps, Adv. Diff. Equations 1 (1996) 611C634.

[13] Z.R. Jin, Liouville theorems for harmonic maps, Invent. Math. 108(1)(1992)1-10.

[14] M. Kassi, A Liouville theorem for F-harmonic maps with finite F-energy, Electron. J. Differential Equations. 15(2006)1-9.

[15] J. M. Kosterlitz, Thouless, D.J.: Two dimensional physics. In: Brewer, D.F. (ed.) Progress in low temperature physics, vol. VIIB. Amsterdam: North-Holland 1978.

[16] Y. Lei, Singularity analysis of a p-Ginzburg-Landau type minimizer. Bulletin Des Sciences Mathmatiques, 2010, 134(1):97-115.

[17] D. R. Nelson: Defect mediated phase transitions. In: Domb, C., Lebowitz, J.L. (eds) Phase transitions and critical phenomena, vol. 7. New York: Academic Press 1983.

[18] S. Pigola, M. Rigoli, A.G. Setti, Vanishing and finiteness results in geometric analysis: a generalization of the Bochner technique, Birkhäuser Verlag, Basel, 2008.

[19] R. Schoen, S.T. Yau, Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature, Comm. Math. Helv. 51 (1976), 333-341.

[20] D. Saint-James, G. Sarma, E. J. Thoma: Type II superconductivity. New York: Pergamon Press 1969.
[21] M. Struwe, On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2-dimensions, Diff. Int. Equations 7(1994) 1613C1624.

[22] Y.L. Xin, Differential forms, conservation law and monotonicity formula. Sci. Sinica Ser. A 29(1)(1986)40-50.

Tian Chong  
Department of Mathematics, Shanghai Second Polytechnic University  
Shanghai 201209, China  
E-mail: valery4619@sina.com

Bofeng Cheng  
School of Mathematical Science, Fudan University, Shanghai 200433, China.  
E-mail: 13641817752@163.com

Yuxin Dong  
School of Mathematical Science, Fudan University, Shanghai, 200433, China.  
E-mail: yxdong@fudan.edu.cn

Wei Zhang  
School of Mathematics, South China University of Technology, Guangzhou, 510641, China.  
E-mail: sczhangw@scut.edu.cn