Transport in vortex state of $d$-wave superconductors at zero temperature: Wiedemann-Franz violation

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We show that the Wiedemann-Franz law is violated at zero temperature in the vortex state of a $d$-wave BCS superconductor with isotropic impurity scattering. We use a semiclassical approach to include the Doppler shift experienced by the quasiparticles due to the circulating supercurrents and consider as well the Andreev scattering from an array of vortices assumed to be randomly distributed. We also show that the vertex corrections to the electric conductivity, which can be large when there is significant anisotropy in the impurity scattering, become unimportant as the magnetic field is increased. For the thermal conductivity, the corrections remain negligible as in the absence of a magnetic field.

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I. INTRODUCTION

The Wiedemann-Franz (WF) law is a hallmark of metallic behavior. This law states that the ratio of the thermal conductivity ($\kappa$) to the dc optical conductivity ($\sigma$), is given by a universal number times the temperature ($T$). The Lorenz number is defined by $L \equiv \kappa/(T\sigma)$, and is expected to equal $\pi^2/(3e^2)$ (we use $\hbar \equiv k_B \equiv 1$). This law is well obeyed in most metals, and can be theoretically justified well beyond the case of a simple metal.

Recently several experiments have been designed to test the validity of the WF law, particularly at very low temperature, in the cuprate superconductors. The law is well obeyed in overdoped Tl$_2$Ba$_2$CuO$_6+\delta$ which is forced into its normal state through application of a magnetic field. A similar result was also found in optimally doped Bi$_{2+\delta}$Sr$_{2-\delta}$CuO$_6+\delta$ which is forced into its normal state through application of a magnetic field. All these results clearly favor the existence of ordinary quasiparticles in these materials and do not require the spin-charge separation picture of the Luttinger liquid. In this picture collective modes replace the quasiparticles with spinons carrying the spins and holons the charges, and the WF law does not hold. On the other hand the normal state of a slightly underdoped sample of Pr$_2-x$Ce$_x$CuO$_4-y$ studied by Hill et al. revealed serious violations of the WF law, which indicates the existence of an exotic normal state. The data puts severe constraints on the state involved. For example, formation of a competing hidden ordered state such as the $d$-density wave (DDW) state is not expected to lead to a violation of the WF law.

In this paper we will confine our discussion to the superconducting state with a $d$-wave gap symmetry. In particular we will study the mixed state where vortices are created by the application of an external magnetic field $H$ perpendicular to the two dimensional CuO$_2$ plane.

Near the critical temperature $T_c$, the cuprates exhibit strong inelastic scattering. The inelastic scattering rate is of order a few times $T_c/\hbar$. The large peak seen in the microwave conductivity as a function of temperature is widely interpreted as evidence for the collapse of the inelastic scattering as $T$ is reduced below $T_c$. The idea is that the development of a superconducting gap suppresses the low frequency density of bosons which causes the superconductivity. This suppression is a general characteristic of any electronic mechanism, in which the boson is a collective mode of the superconducting electrons. Consequently for $T \ll T_c$ elastic impurity scattering will dominate even in the purest samples in which the quasiparticle mean free path can be of order one micron. Therefore in the low temperature regime, which is the region of interest here, it is sufficient, as a first approximation, to ignore the specifics of the mechanism involved in the inelastic scattering. A $d$-wave BCS formulation which includes impurity scattering will suffice.

For a $d$-wave gap symmetry both the quasiparticle density of states and the effective impurity scattering rate acquire an important frequency ($\omega$) dependence. These essential frequency dependences can lead to new phenomena. For example impurities modify, in an essential way, the quasiparticle density of states $N(\omega)$ in the limit $\omega \rightarrow 0$. In the pure case the density of states approaches zero linearly; however, with impurity scattering it takes on a finite value at zero frequency which leads directly to the concept of the universal value of the conductivity at $\omega = 0$ and $T = 0$, independent of impurity content. This behavior also holds in the case of the thermal conductivity and has been verified in many experiments. However, the WF law still holds in this limit.

If finite temperature is considered, the WF law is violated. Violations can be either positive or negative (the Lorenz number is greater or less than one), dependent on the details of the impurity potential. In the unitary limit, the Lorenz number can increase rapidly with temperature while in the Born limit (weak scattering) it decreases. In pure samples it drops to one half its original ($T = 0$) value within a few degrees Kelvin. In the simplest approximation, $\kappa$ and $\sigma$ both depend on the overlap of thermal factors with a product of the density of quasiparticle states $N(\omega)$ and the scattering time $\tau(\omega)$.
The thermal factors for the electrical case are peaked about $\omega = 0$ while for the thermal case they peak at higher frequencies of order $T$. This means that a different frequency range of $N(\omega)\tau(\omega)$ is most importantly sampled in the two cases and the WF law no longer holds at finite temperature.\(^{25}\) The Lorenz number can be smaller or greater than its conventional value depending on the behavior of $N(\omega)\tau(\omega)$ as a function of $\omega$. For the superconducting state this important function $N(\omega)\tau(\omega)$ is different for the electrical and thermal conductivity because of differences in coherence factors but the physics described above for the normal state continues to hold.

Recent extensions of the theory at $T = 0$ to include vertex corrections, which is necessary when the impurity potential is anisotropic, have led to modifications of the above picture. Durst and Lee (DL) found that in these circumstances there can be important corrections to the electrical conductivity while the thermal conductivity is almost unaltered.\(^{25}\) Consequently the WF law no longer holds even at $T = 0$ and the Lorenz number is found to be reduced. The amount of reduction depends on the impurity potential $V_i$ (Ref.\(^\text{25}\)) and is largest for the unitary limit ($V_i \to \infty$). The physics underlying this phenomenon is well described in Ref.\(^\text{25}\) and has its origin in the fact that the thermal conductivity depends on the group velocity of the quasiparticles while its electrical counterpart is sensitive to the Fermi velocity.

In this paper we want to consider the effect that a magnetic field ($H$) applied perpendicular to the two dimensional $d$-wave superconducting plane will have on the WF law and on the vertex corrections for the in-plane transport. It is well known\(^{27,28,29,30}\) that the application of $H$ will increase the density of states at zero frequency $N(0)$ and thus increase both $\sigma$ and $\kappa$. When $H$ is larger than the lower critical field $H_c1$ but still much less than the upper critical field $H_c2$, vortices form and these will provide supercurrents as well as an added scattering mechanism for the quasiparticles. Here we treat effects of the supercurrents semiclassically and this additional scattering in a phenomenological model as Andreev scattering\(^{31,32,33}\) which can be different for charge and heat transport. We found the WF law is violated at $T = 0$ in the mixed state of a BCS $d$-wave superconductor. This result holds even when the vertex corrections for anisotropy are not taken into account, i.e. at the level of the bare bubble diagrams in the treatment of the correlation functions involved. We have also studied the influence of an external magnetic field on the vertex corrections. For the case of the electrical conductivity, it is found that the vertex corrections present at zero $H$ can be rapidly suppressed as $H$ increases. For the thermal conductivity, the vertex corrections are never important.

This paper is organized as follows: In Section II, we describe the formalism for $d$-wave superconductors with impurities in the vortex state. As a first approximation, Matthiessen’s rule is applied to consider the impurity and vortex scattering. We also study self-consistently, in Section III, effects of the magnetic field on the impurity scattering, which is beyond Matthiessen’s rule. In Section IV, magnetic field effects on the vertex corrections to $\sigma$ and $\kappa$ is presented. Section V is devoted to conclusions.

## II. THEORETICAL APPROACH

The calculation of the thermal and dc conductivity proceeds in the standard way through the evaluation of the appropriate current-current correlation functions for heat and charge current, respectively. Formally, the required correlation function is written in Nambu notation (2×2 matrix notation) and an analytic continuation from the Matsubara to real frequencies is needed before the zero frequency limit is taken. The superconducting matrix Green’s function in the Nambu notation takes the form $G (\mathbf{k}, i\omega_n) = (\omega \tau_0 + \Delta_k \tau_1 + \xi_k \tau_3)/(\omega^2 - \xi_k^2 - \Delta_k^2)$, where $\tau_i$ are the Pauli matrices in spin space, $\Delta_k$ a $d$-wave order parameter, and $\xi_k$ is the electronic energy dispersion in the normal state. The correlation function can be expanded into a product of two Green’s functions and a vertex function. At the lowest level of approximation, the vertex is neglected and only the bare bubble is retained. While we start with this approximation later we will generalize our work to include the vertex corrections as in Ref.\(^\text{25}\); these vanish when the impurity scattering is isotropic, but are required when the scattering is anisotropic.

To include the effects of an external magnetic field $H$, for $H_c1 \lesssim H \ll H_c2$, we employ the semiclassical approximation which includes, in the single particle Green’s functions, the Doppler shift\(^{27,28,29,30,34}\) associated with the circulating supercurrent around a vortex core. This is accomplished formally by changing the Matsubara frequencies $i\omega_n$ in the single particle Green’s function to $i\omega_n - v_s \cdot \mathbf{k}$, where $v_s$ is the velocity field of the supercurrent and $\mathbf{k}$ is the momentum of a quasiparticle on the Fermi surface. Now the new Green’s function depending on $v_s \cdot \mathbf{k}$ is inhomogeneous in space variable $\mathbf{r}$ through the dependence of the velocity field $v_s(\mathbf{r})$ on position $\mathbf{r}$ measured from the core of a vortex. Thus thermal and transport coefficients become local quantities which need to be averaged over a vortex unit cell, assumed to be a circle of radius $R$ for simplicity. The radius $R$ is related to the coherence length $\xi_0$ and $H_c2$ through $2R = \xi_0 \sqrt{\frac{\pi a}{\Phi_0}} (H_c2/H)^{1/2}$, where $a$ is a geometrical factor of order unity. The average of any quantity $F(\mathbf{r})$ is then

$$F(H) = \frac{1}{\pi R^2} \int d\mathbf{r} F(\mathbf{r}) .$$

We will assume that $\xi_0 \ll R$ and that the circulating current velocity field can be taken to drop as $1/r$ throughout the vortex unit cell. For a magnetic field applied perpendicular to the CuO$_2$ plane, the Doppler shift $v_s \cdot \mathbf{k}$ takes the form $E_H/\rho$ with $\rho = r/R$ and $E_H = av_f/2 \sqrt{\pi H/\Phi_0}$, where $v_f$ is the Fermi velocity and $\Phi_0$ is the flux quantum related to $H_c2$ by $\Phi_0 = 2\pi \xi_0^2 H_c2$. 
Impurities can be introduced into the calculations through the impurity self energy term which is to be evaluated self-consistently in a t-matrix approximation. The retarded self energy \( \Sigma_{i,ret} \) is given by

\[
\Sigma_{i,ret}(\tilde{\omega}) = \frac{\Gamma G_0(\tilde{\omega})}{\epsilon^2 - G_0^2(\tilde{\omega})},
\]

where \( G_0(\omega) = [2\pi N(0)]^{-1} \sum_k \text{Tr}[\hat{G}_{ret}(k, \omega)] \) with \( \hat{G}_{ret} \) the retarded Green’s function. The summation over \( k \) is to be carried out over the two dimensional Brillouin zone. In Eq. 2 \( \Gamma \) is related to the impurity density \( n_i \) by \( \Gamma = n_i/\pi N(0) \) and \( c \) is the strength of the impurity potential \( V_i \); specifically \( c = 1/(\pi N(0)V_i) \). The unitary limit corresponds to \( V_i \to \infty \), i.e. \( c \to 0 \), while the Born limit corresponds to \( V_i \to 0 \) \((c \to \infty)\). As we have done in our previous work on the thermal conductivity, in the mixed state we assumed as a first approximation that \( \Sigma_{i,ret} \) remains unmodified by an external magnetic field. That is, appealing to Matthiessen’s rule, we add on to the term \( \text{Im} \Sigma_{i,ret} \equiv \gamma(\omega) \) a term which describes phenomenologically the Andreev scattering of the quasi-particles off the vortex core. A simple expression for the quasi-particle-vortex scattering has been given by Yu et al. Examination of their formula for the geometry used here \((H \perp \text{CuO}_2 \text{ plane})\) reveals that it is proportional to the magnetic energy \( E_H \) as also argued through dimensional analysis in Ref. The constant of proportionality \( b \) is to be determined by comparison with experimental data as done in our previous work. This linear dependence of the vortex scattering on \( E_H \) implies that the vortex cores behave like scattering centers. This means that we consider two effects due to a vortex: the Doppler shift due to the supercurrent as well as the vortex scattering due to the core. The total scattering rate denoted by \( \gamma_{tot}(\omega) \) is then given by

\[
\gamma_{tot}(\omega) = \gamma(\omega) + bE_H ,
\]

where the first term \( \gamma(\omega) \) is due to impurities and the second term \( bE_H \) is due to vortex scattering. Later we will include the effect of \( H \) on the impurity scattering itself.

A simplification in the required algebra results when the limit of low \( T \) is considered. For \( T \ll \Delta_0 \) (amplitude of a \( d \)-wave gap), it is appropriate to make a nodal approximation for the summation over the Brillouin zone. Application of this approximation to \( G_0(\omega) \) in Eq. 2 leads to

\[
G_0(\tilde{\omega}) \simeq \frac{2}{\pi} \left[ \frac{\tilde{\omega}}{\Delta_0} \ln \frac{\tilde{\omega}}{4\Delta_0} - i \frac{\pi}{2} \frac{\tilde{\omega}}{\Delta_0} \right],
\]

where a cut-off of \( 4\Delta_0 \) has been applied to the real part of \( G_0 \) in Eq. 1. In a similar way, application of the nodal approximation to the expression for the thermal conductivity \( \kappa(T, H) \) leads to

\[
\frac{\kappa(T, H)}{T} = \frac{1}{\pi^2 T^2} \left( \frac{v_f}{v_g} + \frac{v_g}{v_f} \right) \int d\epsilon \int d\omega \omega^2 \left( -\frac{\partial f}{\partial \omega} \right) A_\kappa(\omega, \epsilon),
\]

where \( v_{f(g)} \) is the Fermi (gap) velocity and \( f(\omega) = [1 + \exp(\omega/T)]^{-1} \) is the Fermi-Dirac thermal factor. The thermal conductivity in the plane \( \kappa_{xx} = \kappa/2 \) in the nodal approximation. This relation also holds for the electrical conductivity. In deriving Eq. 3 we have introduced a vortex distribution function \( \mathcal{P}(\epsilon) \) through the equation

\[
\mathcal{P}(\epsilon) = \frac{1}{\pi R^2} \int d\mathbf{r} \delta (\epsilon - v_s(\mathbf{r}) \cdot \mathbf{k}) ,
\]

where the integration is over the vortex unit cell. The actual thermal conductivity is obtained by averaging the local quantity over the unit cell. In Eq. 4, the function \( A_\kappa(\omega, \epsilon) \) can be written as

\[
A_\kappa(\omega, \epsilon) = 1 + \left( \frac{\omega - \epsilon}{\gamma_\ell(\omega)} + \frac{\gamma_\ell(\omega)}{\omega - \epsilon} \right) \arctan \left( \frac{\omega - \epsilon}{\gamma_\ell(\omega)} \right) ,
\]

where \( \omega = \omega - \text{Re}[\Sigma_{ret}] \). Several choices can be made for the vortex distribution function and results depend somewhat on the choice. However, qualitative results are not sensitive to the choice. To be specific we report here the results based on the Gaussian distribution

\[
\mathcal{P}(\epsilon) = \frac{1}{\sqrt{\pi} E_0} \exp \left( -\frac{\epsilon^2}{E_0^2} \right) .
\]

Note that for \( H = 0 \) we can take \( \epsilon = 0 \) in Eq. 7, namely, \( A_\kappa(\omega, 0) \).

If we further take \( T \to 0 \), we obtain

\[
\frac{\kappa_00}{T} = \frac{2}{3} \left( \frac{v_f}{v_g} + \frac{v_g}{v_f} \right).
\]

This value is the well-known expression for the universal limit of the thermal conductivity independent of the impurity scattering rate at zero frequency \( (\gamma_{00}) \). If instead we had taken \( T \to 0 \) first in Eq. 3, we would have

\[
\lim_{T \to 0} \frac{\kappa(T, H)}{T} = \frac{1}{3} \left( \frac{v_f}{v_g} + \frac{v_g}{v_f} \right) \int d\epsilon \mathcal{P}(\epsilon) A_\kappa(0, \epsilon) ,
\]

where

\[
A_\kappa(0, \epsilon) = 1 + \left( \frac{\epsilon}{\gamma_\ell(0)} + \frac{\gamma_\ell(0)}{\epsilon} \right) \arctan \left( \frac{\epsilon}{\gamma_\ell(0)} \right) .
\]
Note that in this case only the zero frequency limit of \( \gamma_t \) enters. At any finite \( T \), however, the \( \omega \) dependence of \( \gamma_t(\omega) \) becomes very important, and the \( T \) dominated region \( T \gg \gamma_t(0) \) can be quite different from the impurity dominated region for which \( T \ll \gamma_t(0) \). Finally we note that taking \( H \to 0 \) in Eq. (10) again gives the value of the universal limit as it should.

Similar algebra gives the expression for the zero frequency value of the optical conductivity \( \sigma(T, H) \), which takes the form

\[
\sigma(T, H) = \frac{e^2}{\pi^2} \left( \frac{v_f}{v_g} \right) \int dk P(e) \int d\omega \left( -\frac{\partial f}{\partial \omega} \right) A_\sigma(\omega, e),
\]

(12)

where

\[
A_\sigma(\omega, e) = 2 \left[ 1 + \frac{\omega - e}{\gamma_t(\omega)} \arctan \left( \frac{\omega - e}{\gamma_t(\omega)} \right) \right].
\]

(13)

In Eq. (12), \( e \) is the electron charge. We note that the difference between \( A_\sigma \) and \( A_e \) is traced to the different coherence factors that enter, respectively, the electrical and thermal conductivity. In terms of the spectral functions \( A(k, \omega) \) and \( B(k, \omega) \) associated, respectively, with the diagonal and off-diagonal part of Green’s function, the optical (thermal) case involves \( A^2(k, \omega) \pm B^2(k, \omega) \), respectively. As \( T \to 0 \), we obtain

\[
\sigma(0, H) = \frac{e^2}{\pi^2} \left( \frac{v_f}{v_g} \right) \int dk P(e) A_\sigma(0, e).
\]

(14)

If we also take \( H = 0 \), we recover the well-known result of the universal value of the dc conductivity

\[
\sigma_{00} = 2 \frac{e^2}{\pi^2} \left( \frac{v_f}{v_g} \right).
\]

(15)

Now for \( H = 0 \) and \( T \to 0 \), the Lorenz number \( L = \kappa/T\sigma \) can be evaluated to test the WF law. Using Eqs. (10) and (11), we obtain \( L_{00} = \kappa_{00}/(T\sigma_{00}) \approx \pi^2/(3e^2) \) assuming \( v_g \ll v_f \). This is the universal constant normally associated with the WF law.

Remaining at \( T = 0 \) we can consider the effects of a magnetic field \( H \) on the WF law at the level of bare bubble \( i.e. \) ignoring the vertex corrections. The normalized thermal \( (\kappa/\kappa_{00}) \) and dc \( (\sigma/\sigma_{00}) \) conductivity can be written as

\[
\frac{\kappa}{\kappa_{00}} = \frac{1}{\sqrt{\pi}} \int_0^\infty dx \; e^{-x^2} \left[ 1 + \left( \frac{1}{\Gamma_t} + \frac{1}{\Gamma} \right) \arctan \left( \frac{x}{\sqrt{\Gamma}} \right) \right],
\]

(16)

and

\[
\frac{\sigma}{\sigma_{00}} = \frac{2}{\sqrt{\pi}} \int_0^\infty dx \; e^{-x^2} \left[ 1 + \frac{x}{\Gamma_t} \arctan \left( \frac{x}{\sqrt{\Gamma}} \right) \right],
\]

(17)

where \( \Gamma_t = \gamma_t(0)/E_H = \gamma_{00}/E_H + b \) with \( b = b_\kappa \) for the thermal conductivity and \( b = b_\sigma \) for the dc conductivity. These two quantities are not the same for Andreev scattering off a vortex core. On physical grounds we expect \( b_\kappa > b_\sigma \) because Andreev scattering is more effective for heat transport than for charge transport. Evaluation of these parameters requires a microscopic mechanism for the quasiparticle-vortex scattering, which is beyond the scope of this work. Here we effectively treat them as phenomenological parameters and present results for various values of \( b \). The value of \( b_\kappa \) has been determined for two samples in our previous work\(^a\) on the thermal conductivity. In the ultra pure sample of Hill \textit{et al.}\(^a\) its value is about \( b_\kappa = 0.3 \) while in the impure sample discussed by these same authors we found that Andreev scattering is not important compared with the impurity scattering so that the value of \( b_\kappa \) is effectively zero.

A first result can be obtained analytically from Eqs. (16) and (17) assuming no vortex scattering \( (b_\kappa = b_\sigma = 0) \). In the high field limit \( E_H \gg \gamma_{00} \) or \( \Gamma_t \ll 1 \) we can expand \( \kappa/\kappa_{00} \) and \( \sigma/\sigma_{00} \) in terms of \( \gamma_{00}/E_H \) and obtain

\[
\frac{\kappa}{\kappa_{00}} = \frac{\sqrt{\pi}}{4\Gamma_t} - \frac{\sqrt{\pi}}{4} (2\ln\Gamma_t + \gamma_e - 1) \Gamma_t
\]

(18)

where \( \gamma_e \approx 0.577 \) is Euler's number, and

\[
\frac{\sigma}{\sigma_{00}} = \frac{\sqrt{\pi}}{2\Gamma_t} + \frac{\sqrt{\pi}}{4} \Gamma_t.
\]

(19)

Note that \( E_H \sim \sqrt{\gamma_{00}} \) indicating that \( \kappa/\kappa_{00} \sim \sqrt{\gamma_{00}} + \mathcal{O}(\ln(H)/\sqrt{\gamma_{00}}) \) while \( \sigma/\sigma_{00} \sim \sqrt{\gamma_{00}} + \mathcal{O}(1/\sqrt{\gamma_{00}}) \). The Lorenz number is given by \( (\Gamma_t < < 1) \):

\[
L = \frac{\kappa}{T\sigma} = \frac{1}{2} L_{00}
\]

(20)

where \( L_{00} \) is the universal value of the Lorenz number. This means that the Lorenz number at a high field is half its universal value, which is a serious violation of the WF law at zero temperature. We should emphasize that to our knowledge this law has always been obeyed in the past at \( T = 0 \), even in an exotic state such as the DDW state.\(^3,17\) Our result represents a significant exception.

For finite but small vortex scattering \( (b_\kappa < 1 \text{ and } b_\sigma < 1) \), \( \kappa/\kappa_{00} \) and \( \sigma/\sigma_{00} \) both saturate to values determined only by \( b_\kappa \) and \( b_\sigma \), respectively, in the limit \( E_H >> \gamma_{00} \),

\[
\frac{\kappa}{\kappa_{00}} = \frac{\sqrt{\pi}}{4b_\kappa} - \frac{\sqrt{\pi}}{4} [2\ln(b_\kappa) + \gamma_e - 1] b_\kappa
\]

(21)
The Lorenz number $L$ normalized to its universal value $L_{00} \simeq \pi^2/(3e^2)$ as a function of a magnetic field $H$ applied perpendicular to the CuO$_2$ plane. The top frame is for a pure sample (BZO) with $\gamma_{00} = 0.3K$ while the bottom frame is for a dirtier sample (YSZ) with $\gamma_{00} = 13K$. (See Refs. [35,36] for details.) Various choices of the vortex scattering parameters $b_c$ and $b_\sigma$ are considered: $(b_c, b_\sigma) = (0.3, 0.3)$ for the solid curves in both frames, $(0, 0)$, $(0.3, 0.1)$, and $(0.3, 0)$ for the dashed, dot-dashed, and dotted curves, respectively.

\[ \frac{\sigma}{\sigma_{00}} = \frac{\sqrt{\pi}}{2b_\sigma} + \frac{\sqrt{\pi}}{4} b_\sigma. \]  

As $b_c$ and $b_\sigma \rightarrow 0$, we obtain

\[ \frac{L}{L_{00}} = \frac{b_\sigma}{2b_c}. \]  

Since we expect $b_\sigma$ to be less than $b_c$, the Lorenz number should be smaller than its universal value in this regime and Eq. (23) could be used to find the ratio $b_\sigma/b_c$ of the two parameters associated with the vortex scattering. Note that for $b_c = b_\sigma \rightarrow 0$, we recover the previous result of $L/L_{00} = 1/2$. In our previous work, we were able to determine $b_\sigma$ from the saturated value of the thermal conductivity as a function of a magnetic field, and so Eq. (23) would give $b_\sigma$ directly from the Lorenz number.

We performed numerical calculations of the Lorenz number for various values of $b_c$ and $b_\sigma$ at $T = 0$. In Fig. 1, we plot results for $L/L_{00}$ as a function of $H$ in two specific cases. The top frame is for $\gamma_{00} = 0.3K$, a value of the impurity scattering rate deduced from consideration of the thermal conductivity of an ultra pure sample$^{35,36}$ of YBCO$_{6.99}$ denoted by BZO and the bottom frame is for a dirty sample denoted by YSZ with $\gamma_{00} = 13K$. The values of $(b_c, b_\sigma)$ are shown as labels to the curves. For example, $(b_c, b_\sigma) = (0.3, 0.3)$ for the solid curve. For the dashed, dot-dashed and dotted curve $(b_c, b_\sigma) = (0, 0)$, $(0.3, 0.1)$, and $(0.3, 0)$, respectively. Note that the analytic result Eq. (20) is verified by the dashed curve with $(0, 0)$ in the top frame. In fact, in this case, the value $1/2$ is reached already around $H = 1$ Tesla. For the bottom frame (YSZ sample) the reduction to $1/2$ is not yet seen even at 10 Tesla because the saturation is controlled not by $H$ but by the ratio $E_H/\gamma_{00}$, which is needed to be much larger than 1 in order to realize Eq. (20). Estimates of $E_H/\gamma_{00}$ are about $30\sqrt{H}$ (Tesla$^{-1}$) for the top frame which is much larger than 1 for $H = 1$ Tesla. But the bottom sample is dirtier by a factor of approximately 40. Eq. (23) is also verified in our numerical work. Next we turn to the consideration of the effects of a magnetic field on the impurity scattering. This takes us beyond the simple Matthiessen’s rule which we relied on in this section.

III. MAGNETIC FIELD EFFECT ON IMPURITY SCATTERING

We now return to the impurity self energy given by Eq. (2). So far, for simplicity, we have treated the self energy approximately, by appealing to Matthiessen’s rule; we simply added the Andreev scattering effect $bE_H$ to the impurity scattering $\gamma$ which is itself left unaltered by the magnetic field. Here $b = b_c(\sigma)$ for the thermal (dc) conductivity as we mentioned. Of course, in reality the existence of a vortex lattice, which introduces spatial inhomogeneities, will modify the impurity term itself in Eq. (4). We need to return to Eq. (2) for the retarded self energy and include the effects of the magnetic field on $G_0$ itself. In obtaining Eq. (2) an impurity average has been performed in the standard way to restore translational invariance. After this procedure is carried out, the momentum becomes a good quantum number again and one can use the semiclassical approximation to take into account the Doppler shift. This re-introduces spatial inhomogeneities. A second average over a random vortex distribution is then taken. The legitimacy behind such separate averaging procedures is an assumption that the impurities and vortices are uncorrelated.

To deal with both impurity and vortex averaging in a manageable way, what we propose here is to replace $G_0(\omega)$ in Eq. (2) with its vortex averaged counterpart. Returning to Eq. (4) valid at low $T$, let us introduce the Doppler shift and take an average over, for example, the Gaussian distribution. Then in the Born limit, we obtain for the total scattering rate $\gamma_{tot}(\omega)$ at $\omega = 0$ using $\Sigma_{i,ret}(E_H) = -i\gamma(E_H)$.
\[ \gamma_{\text{tot}} = 4\eta_B \int d\varepsilon \mathcal{P}(\varepsilon) \left[ \frac{\gamma_{\text{tot}}}{\kappa} \ln \left( \frac{p_0}{\sqrt{\varepsilon^2 + \gamma_{\text{tot}}^2}} \right) + \frac{\varepsilon}{\gamma_{\text{tot}}} + bE_H \right], \]  

(24)

where \( \eta_B = n_i V_i^2 / (2\pi v_f v_g) \). The factor 4 in Eq. (24) appears for isotropic impurity scattering. When we consider the anisotropic scattering, this factor will be changed to reflect the scattering anisotropy. Eq. (24) is to be solved self-consistently for the effective total scattering rate \( \gamma_{\text{tot}}(E_H) = \gamma(E_H) + bE_H \). Results for the impurity scattering \( \gamma(E_H) \) are plotted in Fig. 2, where \( \gamma(E_H) \) is normalized to the zero frequency and zero field scattering rate \( \epsilon(0,0) = \gamma_{00} \). The magnetic energy in Fig. 2 is also normalized to \( \gamma_{00} \). The top frame applies to the Born limit and the bottom to the unitary limit. Three cases are considered. The solid curve is for \( \eta_B = 0.04 \) and \( b = 0.1 \), the dashed curve for \( \eta_B = 0.08 \) and \( b = 0.1 \). The dotted curve is for \( b = 0 \) and applies to all values of \( \eta_B \), as is shown below. After some simple algebra we find from Eq. (24)

\[ 4\eta_B \int d\varepsilon \mathcal{P}(\varepsilon) \left[ \frac{\gamma_{\text{tot}}}{\kappa} \ln \left( \frac{1}{\sqrt{\varepsilon^2 + \gamma_{\text{tot}}^2}} \right) + \frac{\varepsilon}{\gamma_{\text{tot}}} \right] + bE_H = 0, \]  

(25)

where \( \varepsilon = \epsilon/\gamma_{00} \), \( \gamma_{\text{tot}} = \gamma_{\text{tot}}/\gamma_{00} \), and \( E_H = E_H / \gamma_{00} \). It is clear from Eq. (24) that for \( b = 0 \), the resulting equation does not depend on the value \( \eta_B \). Consequently, the dotted curve in the top frame of Fig. 2 results regardless of a specific value of \( \eta_B \). For any finite \( b \), however, the result does depend on \( \eta_B \). We note that in all cases in the Born limit, the effect of the magnetic field \( H \) is to increase the scattering rate, and that for a given value of \( E_H / \gamma_{00} \) the increase is largest for the smallest \( \eta_B \) value. These results confirm the expectation based on the following argument. Let \( \Gamma_N \) be the impurity scattering rate in the normal state. For the Born limit, the zero frequency scattering rate in the superconducting state is much less than \( \Gamma_N \) while for the unitary limit it is larger. In fact, for the Born limit the decrease in \( \gamma_{00} \) is exponentially dependent on \( \Gamma_N \). As we introduce more vortices by increasing the magnetic field, more normal regions appear at the expense of superconducting ones and, hence, we would expect that, on average, the effective scattering rate would increase towards \( \Gamma_N \). This is precisely what we have found. On the other hand, in the unitary limit \( \gamma_{00} \) is very much larger than \( \Gamma_N \). A useful and well-known approximate formula\(^{15,16}\) for a relation between these two quantities in this limit is \( \gamma_{00} \approx 0.63 \sqrt{\Gamma \Delta_0} \). Thus, as \( H \) is increased we expect now that \( \gamma(E_H) \) will decrease because it should tend towards its normal state value. This expectation is confirmed in the bottom frame of Fig. 2, where we present numerical results for two values of \( b \); namely, \( b = 0 \) and \( b = 0.1 \), and two values of \( \eta_B = 2\pi n_i v_f v_g / \gamma_{00}^2 \). One can estimate \( \eta_B \) using \( \Gamma = \gamma_{00} / \pi N(0) \), \( \gamma_{00} \approx 0.63 \sqrt{\Gamma \Delta_0} \), and \( \Delta_0 \approx n_i / \pi N(0) v_f v_g \) in the nodal approximation. In all cases, as expected \( \gamma(E_H) \) decreases and the decrease is larger when \( b \neq 0 \). The results in the unitary limit are based on the following equation:

\[ \frac{\eta_U}{4 \int d\varepsilon \mathcal{P}(\varepsilon) \left[ \gamma_{\text{tot}} \ln \left( \frac{p_0}{\sqrt{\varepsilon^2 + \gamma_{\text{tot}}^2}} \right) + \frac{\varepsilon}{\gamma_{\text{tot}}} \right] + bE_H }, \]  

(26)

One can see that, in this case, even for \( b = 0 \), \( \gamma(E_H) \) still depends on \( \eta_U \). These expressions are used to obtain results at the bare bubble level for the dc and thermal conductivity as a function of \( E_H \) (see Figs. 3 and 4 below for \( b = 0 \) and \( b = 0.3 \), respectively).

Even when Andreev scattering from randomly distributed vortices is not considered, we still see, in Fig. 3, that including the effect of \( H \) in \( \gamma_{\text{tot}} \) increases both \( \kappa \) (top frame) and \( \sigma \) (middle frame) over the values (dotted curve) it would have if \( \gamma_{00} \) was used instead of \( \gamma(E_H) \) (i.e. no effect of \( H \) is included in \( \gamma \)) in the unitary limit while the opposite holds in the Born limit. This is as ex-
expected from Fig. 2, where it was shown that $H$ reduces $\gamma(E_H)$ in the unitary limit while it increases $\gamma(E_H)$ in the Born limit. Further, in Fig. 4, where a non-zero value of $b = 0.3$ is considered, the effects remain as in Fig. 3 but all variations with $E_H$ are much reduced. This means that Andreev scattering off vortices offsets to a large degree the increased conductivity due to the increase of quasiparticles created by the Doppler shift. Returning to the bottom frames of Figs. 3 and 4, we note that in all cases the Lorenz number is reduced below its universal value $L_{00}$ as the field is increased. In particular for $b = 0$, the decrease is very rapid, and at $E_H/\gamma_{00} = 2$ the unitary case has dropped by more than 30% while in the Born limit it is by more than 20%. Also for this ratio, including the effect of $H$ on the impurity scattering is not important in the unitary limit but is more important in the Born case. This also holds for $b = 0.3$ shown in the bottom frame of Fig. 4. There is no qualitative difference between the $b = 0$ and $b = 0.3$ cases but the variations are not as rapid when Andreev scattering is included in the calculations.

IV. VERTEX CORRECTIONS TO CONDUCTIVITY

We now consider the effect of vertex corrections to the dc and thermal conductivity of a $d$-wave superconductor in the vortex state. This problem was treated in detail by DL in the case when the impurity scattering is
anisotropic but no magnetic field is applied to the sample. The necessary mathematics is rather lengthy so that here we will deal only with the essential modifications to the formalism of Ref.\[sup]\textsuperscript{25}\] that are required when a magnetic field is applied. Following Durst and Lee, we also introduce impurity scattering potentials to the same node (V\(_1\)), the adjacent node (V\(_2\)), and the opposite node (V\(_3\)). Thus the anisotropy can be parametrized by \(2R_2/V_1\) and \(2R_3/V_1\). If \(R_2 = R_3 = 1\), the impurity scattering is isotropic. Our aim here will not be to give the most general formulation but rather consider a simplified case which can nevertheless provide a preliminary understanding of the effects of the magnetic field on the vertex corrections to the conductivity. The most important modification in the formalism is to replace, for all relevant calculations, the Green’s function by its average over the vortex distribution. This replacement may limit the generality of our results but allows us to get a concrete numerical evaluation of the resulting vertex corrections. A central function in the calculations is \(F(z)\) as in Ref.\[sup]\textsuperscript{25}\]

\[
F(z) = -4\pi v_f v_g \int_0^{p_0} \frac{d^2p}{(2\pi)^2 v_f v_g} \hat{G}(p, z)
\]  

(27)

which replaces Eq. (3.13) of DL. In the above Eq.\[sup]\textsuperscript{25}\],

\[
f_1^{(0)}(\epsilon) = \frac{\partial F(z)}{\partial z} \bigg|_{z = z_0}
\]  

(29)

and

\[
f_2^{(0)}(\epsilon) = \frac{F(z) - F(z^*)}{z - z^*} \bigg|_{z = z_0}
\]  

(30)

with \(z_0 = -\epsilon + i\gamma_s(0)\) and

\[
F(z) = 2 \int_0^{p_0} \frac{pd\epsilon}{p^2 - z^2} \simeq 2z \ln \frac{|\epsilon|}{z}.
\]  

(31)

The parameters \(A_1^{(0)}\) and \(A_2^{(0)}\) are due to the scattering anisotropy. They will be specified later but for isotropic scattering they vanish. In this instance, we obtain

\[
\frac{\sigma}{\sigma_0} = \frac{1}{2} \int d\epsilon \mathcal{P}(\epsilon) \Re \left[ I_2^{(0)}(\epsilon) - I_1^{(0)}(\epsilon) \right],
\]  

(32)

which needs to reduce to the expression we obtained in the previous section where the vertex corrections were neglected. To see if we recover the bare bubble results we need to calculate \(I_1^{(0)}\) and \(I_2^{(0)}\):

\[
I_1^{(0)}(\epsilon) = 2 \ln \frac{|\epsilon|}{z_0} - 2
\]  

(33)

and

\[
I_2^{(0)}(\epsilon) = \frac{2}{\gamma_s(0)} \ln \left\{ z_0 \ln \frac{|\epsilon|}{z_0} \right\}.
\]  

(34)

Now we obtain

\[
I_2^{(0)}(\epsilon) - I_1^{(0)}(\epsilon) = 2 - 2\frac{\epsilon}{\gamma_s(0)} \arg \left( \frac{ip_0}{z_0} \right) - 2i \arg \left( \frac{ip_0}{z_0} \right),
\]  

(35)

where

\[
\arg \left( \frac{ip_0}{z_0} \right) = -\arctan \left( \frac{\epsilon}{\gamma_s(0)} \right).
\]  

(36)

Thus the dc conductivity becomes

\[
\frac{\sigma}{\sigma_0} = \int d\epsilon \mathcal{P}(\epsilon) \left[ 1 + \frac{\epsilon}{\gamma_s(0)} \arctan \left( \frac{\epsilon}{\gamma_s(0)} \right) \right],
\]  

(37)

which is what we had before at the bare bubble level (Eqs.\[sup]\textsuperscript{14} and \[sup]\textsuperscript{15}).

Now we consider the vertex corrections to the dc conductivity in the Born limit. In this case, following DL we obtain after the appropriate generalization \(A_1^{(0)} = \alpha_1^{(0)} \langle I_1^{(0)} \rangle\) and \(A_2^{(0)} = \alpha_2^{(0)} \langle I_2^{(0)} \rangle\) with \(\alpha_1^{(0)} = a_1^{(0)}\), where \(\langle I_i^{(0)} \rangle = \int d\epsilon \mathcal{P}(\epsilon) I_i^{(0)}(\epsilon) \) (\(i = 1, 2\)) and

\[
\alpha_i^{(0)} = \frac{n_i V_1^2}{4\pi v_f v_g} (1 - R_i^3).
\]  

(38)

Note in this case,

\[
\ln \frac{p_0}{\gamma(0)} = \frac{1}{\eta_B \left( 1 + 2R_2^3 + R_3^3 \right)}
\]  

(39)

where \((1 + 2R_2^3 + R_3^3) \to 4\) for the isotropic scattering \((R_2 = R_3 = 1)\) as we mentioned for Eq. \[sup]\textsuperscript{24}\] in the previous section. The expression for the dc conductivity,
including the vertex corrections $\sigma_{VC}$, can be simplified and becomes

$$\frac{\sigma_{VC}}{\sigma_{00}} = \frac{1}{2} \left[ \frac{\langle I_2^{(0)} \rangle - \langle I_1^{(0)} \rangle}{\left( 1 - \alpha_1^{(0)} \langle I_1^{(0)} \rangle \right) \left( 1 - \alpha_1^{(0)} \langle I_2^{(0)} \rangle \right)} \right]$$  \hspace{1cm} (40)$$

Since $\frac{\sigma_{VC}}{\sigma_{00}} = \frac{1}{2} \left[ \langle I_2^{(0)} \rangle - \langle I_1^{(0)} \rangle \right]$, we finally obtain

$$\frac{\sigma_{VC}}{\sigma} = \frac{1}{\left( 1 - \alpha_1^{(0)} \langle I_1^{(0)} \rangle \right) \left( 1 - \alpha_1^{(0)} \langle I_2^{(0)} \rangle \right)}.$$  \hspace{1cm} (41)$$

For the numerical calculations we choose $R_2 = 0.9$ and $R_3 = 0.8$. Since it is assumed that $V_1$ is small, we retain only terms of order $V_1^2$. In Fig. 5, we show numerical results with $\eta_B = n_i V_1^2 / 2 \pi v_f v_g = 0.05$. In a rigorous sense $\eta_B \to 0$ since $V_1 \to 0$ in the Born limit; however, this value is unrealistic for an actual sample. In the top frame of Fig. 5, we plot $\sigma_{VC}/\sigma_{00}$ as a function of $E_H/\gamma_0$ for $b_0 = 0$ (solid curve) and $b_0 = 0.1$ (dashed curve). At small values of $E_H$, the Andreev scattering does not have a large effect on $\sigma_{VC}/\sigma_{00}$. For large values of $E_H$, however, the difference can be much larger. The conductivity also rises significantly above its universal value. This rise is not to be assigned to the vertex corrections. In the bottom frame we plot $\sigma_{VC}/\sigma$, where $\sigma$ is the dc conductivity at the bare bubble level. We see now that in both cases, including ($b_0 \neq 0$) and ignoring ($b_0 = 0$) Andreev scattering, the application of a magnetic field decreases the effect of the vertex corrections although the decrease is not large in magnitude. Note that for $H = 0$ one can get analytically $\sigma_{VC}/\sigma = \left( 1 + 2 R_2^2 + R_3^2 \right)^2 / \left( 2 R_2^2 + 2 R_3^2 \right)^2$ as in Ref.25 For $R_2 = 0.9$ and $R_3 = 0.8$, $\sigma_{VC}/\sigma \simeq 1.26$ for $H = 0$. In our numerical calculation, $\sigma_{VC}/\sigma \simeq 1.24$ at $H = 0$ is a little less than 1.26 because, as we mentioned, $\eta_B$ is finite. However, when we reduce $\eta_B$ to values smaller than 0.05, our numerical results approach this analytic value; for example, for $\eta_B = 0.01$ we obtain $\sigma_{VC}/\sigma \simeq 1.26$.

A more interesting case is the unitary scattering limit, where the vertex corrections are much larger at zero field. The parameters $A_1^{(0)}$ and $A_2^{(0)}$ are now, $A_1^{(0)} = \langle \beta_1^{(0)} \rangle / \langle I_1^{(0)} \rangle$ and $A_2^{(0)} = \langle \beta_2^{(0)} \rangle / \langle I_2^{(0)} \rangle$ with $\langle \beta_1^{(0)} \rangle = |\langle \beta_1^{(0)} \rangle|$, where $\langle \beta_1^{(0)} \rangle = \int d\epsilon P(\epsilon) \beta_1^{(0)}(\epsilon)$. For $R_2 = 0.9$ and $R_3 = 0.8$, we obtain

$$\ln \frac{\gamma_0}{\gamma(0)} = \frac{3 n_i 2 \pi v_f v_g}{4} (\gamma(0))^2$$  \hspace{1cm} (42)$$

and

$$\beta_1^{(0)} = \frac{n_i}{4} \frac{2 \pi v_f v_g}{z_0^2} \ln^2 (i p_0 / z_0)$$  \hspace{1cm} (43)$$

where $z_0 = -\epsilon + i \gamma_0(0)$ as before. Our numerical results are given in Fig. 6. The format is the same as Fig. 5. The impurity parameter $\eta_B = 10$ and $\eta_B = 0.01$. Referring to the bottom frame we see that the application of a magnetic field reduces the vertex corrections rapidly as the magnetic field increases up to a few times $\gamma_0$.

For completeness we have also considered the vertex corrections to the thermal conductivity of a $d$-wave superconductor in the vortex state. It has been shown that the vertex corrections are negligible at zero field.25 In the presence of a magnetic field we obtain

$$\frac{\kappa_{VC}}{\kappa_{00}} = \frac{1}{2} \left[ 1 + (v_f / v_g)^2 \right]^{-1} \int d\epsilon P(\epsilon) \text{Re} \left[ \frac{I_2^{(3)}(\epsilon)}{1 - A_2^{(3)}} - \frac{I_1^{(3)}(\epsilon)}{1 - A_1^{(3)}} \right] + \frac{1}{2} \left[ 1 + (v_f / v_g)^2 \right]^{-1} \int d\epsilon P(\epsilon) \text{Re} \left[ \frac{I_1^{(1)}(\epsilon)}{1 - A_1^{(1)}} - \frac{I_1^{(3)}(\epsilon)}{1 - A_1^{(3)}} \right]$$  \hspace{1cm} (44)$$
\frac{\kappa}{\kappa_{00}} = \frac{1}{2} \int d\epsilon P(\epsilon) \text{Re} \left[ I_2^{(1)}(\epsilon) - I_1^{(1)}(\epsilon) \right].$$  \hspace{1cm} (47)$$

Since $I_1^{(1)}(\epsilon) = -1$ and

$$I_2^{(1)}(\epsilon) = \left[ \frac{\epsilon}{\gamma_0(0)} + \frac{\gamma_0(0)}{\epsilon} \right] \arctan \left( \frac{\epsilon}{\gamma_0(0)} \right),$$  \hspace{1cm} (48)$$

we reproduce the bare bubble result for the thermal conductivity.

In the Born limit, we obtain $A_1^{(3)} = \alpha_1^{(0)} \langle I_1^{(3)} \rangle$, $A_1^{(3)} = \alpha_1^{(0)} \langle I_1^{(3)} \rangle$, $A_2^{(3)} = -A_2^{(3)}$, and $A_1^{(3)} = -A_1^{(3)}$. Further

\begin{align*}
I_1^{(1)}(\epsilon) &= \frac{1}{2} \left[ \frac{\partial F(z)}{\partial z} - \frac{F(z)}{z} \right]_{z = z_0} \\
I_1^{(3)}(\epsilon) &= \frac{1}{2} \left[ \frac{\partial F(z)}{\partial z} - \frac{F(z)}{z} \right]_{z = z_0}.
\end{align*}  \hspace{1cm} (46)
simple analysis can show that $1 \pm \alpha_1(0) \langle I_2(3) \rangle \approx 1$ and $\pm \alpha_1(0) \langle I_1(3) \rangle \approx 1$ since $\alpha_1(0) \approx 1/\ln[p_0/\gamma(0)]$ and $\epsilon \ll p_0$ because of $\mathcal{P}(\epsilon)$. In the unitary limit, we obtain $A_2^{(3)} = A_2^{(1)} = \langle \beta_2(0) \rangle \langle I_2(3) \rangle$ and $A_1^{(3)} = A_1^{(1)} = \langle \beta_1(0) \rangle \langle I_1(3) \rangle$. Since $\langle \beta_1(0) \rangle \to 0$ as $p_0 \to \infty$, $1 - \langle \beta_1(0) \rangle \langle I_1(3) \rangle \approx 1$ and $1 - \langle \beta_2(0) \rangle \langle I_2(3) \rangle \approx 1$. We conclude therefore that the vertex corrections to the thermal conductivity remain negligible in the presence of a magnetic field; the ratio of its corrected value ($\sigma_{VC}/\sigma_0$) to the dc conductivity ($\sigma$) without vertex corrections. The solid (dashed) curves in both frames are for $b_\sigma = 0$ ($b_\sigma = 0.1$).

FIG. 5: The dc conductivity $\sigma_{VC}$ as a function of $E_H/\gamma_{00}$ including the vertex corrections in the Born limit ($\eta_B = 0.05$). The top frame shows $\sigma_{VC}/\sigma_0$ while the bottom frame gives $\sigma_{VC}/\sigma$ i.e. the ratio of its corrected value ($\sigma_{VC}$) to the dc conductivity ($\sigma$) without vertex corrections. The solid (dashed) curves in both frames are for $b_\sigma = 0$ ($b_\sigma = 0.1$).

FIG. 6: The dc conductivity $\sigma_{VC}$ as a function of $E_H/\gamma_{00}$ including in the unitary limit ($\eta_U \approx 12$). The top frame shows $\sigma_{VC}/\sigma_0$ while the bottom frame gives $\sigma_{VC}/\sigma$ i.e. the ratio of its corrected value ($\sigma_{VC}$) to the dc conductivity ($\sigma$) without vertex corrections. The solid (dashed) curves in both frames are for $b_\sigma = 0$ ($b_\sigma = 0.1$).

FIG. 7: The WF ratio, with vertex corrections as a function of $E_H/\gamma_{00}$, for the cases considered in Fig. 5 and Fig. 6 with $b_\alpha = b_\sigma$. The top (bottom) frame is for the Born (unitary) limit. In both cases a violation remains at all magnetic fields.

V. CONCLUSIONS

We have found that the Wiedemann-Franz law is violated in the mixed state of a $d$-wave superconductor at zero temperature at the bare bubble level. Recall that in this approximation there is no violation in the absence of a magnetic field. The deviation from the universal value of the Lorenz number $L_{00}$ increases with increasing magnetic field. When Andreev scattering from vortices is neglected, the high field saturated value of the Lorenz number can be as low as $L_{00}/2$. With Andreev scattering, it can be even smaller. We have also found that application of a magnetic field affects the effective impurity scattering rate $\gamma$ at zero temperature. In the Born limit, $\gamma$ increases while in the unitary limit, it decreases as is expected. Physically, in both cases, $\gamma$ should tend toward its normal state value as the field approaches $H_{c2}$.

In the superconducting state with zero field, the relevant scattering rate (as is well-known) is much smaller than its normal state value in the Born limit while the opposite holds in the unitary case.

We have also considered the vertex corrections arising from anisotropy in the impurity scattering. It has been shown that in the absence of a magnetic field the vertex corrections can be large for the zero frequency electrical...
conductivity at zero temperature while they are small for the thermal conductivity. We found that application of the magnetic field reduces the contribution of the vertex corrections to the electrical conductivity and they become unimportant. For the thermal conductivity, the corrections remain negligible as in the absence of a magnetic field.

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