Isotopic Dependence of the Casimir Force

Dennis E. Krause\textsuperscript{1,2} and Ephraim Fischbach\textsuperscript{2}

\textsuperscript{1}Physics Department, Wabash College, Crawfordsville, IN 47933-0352
\textsuperscript{2}Physics Department, Purdue University, West Lafayette, IN 47907-1396

(November 2, 2018)

Abstract

We calculate the dependence of the Casimir force on the isotopic composition of the interacting objects. This dependence arises from the subtle influence of the nuclear masses on the electronic properties of the bodies. We discuss the relevance of these results to current experiments utilizing the iso-electronic effect to search at very short separations for new weak forces suggested by various unification theories.
The Casimir effect \cite{1} has been the subject of intense study recently, both experimentally and theoretically \cite{2,3,4,5,6,7,8,9}. Not only is it of interest in its own right, as a novel and fundamental quantum effect, but the increasing precision of experimental tests of the Casimir effect has led to its use in searching for new macroscopic forces acting over sub-millimeter scales \cite{10,11,12}. This follows from the fact that the Casimir force becomes the dominant background to any new macroscopic force over short distance scales after electrostatic and magnetic effects have been eliminated. If we consider, for example, the Casimir force per unit area $P_C(d)$ between two perfectly conducting parallel plates at temperature $T = 0$ separated by a distance $d$, then \cite{1}

$$P_C(d) = -\frac{\pi^2hc}{240} \frac{1}{d^4} = -\frac{0.013}{(d/\mu m)^2} \text{dyn/cm}^2.$$ \hspace{1cm} (1)

It is instructive to compare $P_C(d)$ to the gravitational force: For two Cu plates having dimensions $1 \text{ cm}^2 \times 1 \text{ mm}$ a distance $d$ apart, the Casimir force exceeds the Newtonian gravitational force when $d \lesssim 14 \mu m$. (This distance becomes slightly larger at room temperature \cite{13}.) However, this is the very distance scale over which recent extra-dimensional theories suggest the possibility of new short-range gravitational forces \cite{14,15,16}. If follows that detecting such forces requires understanding the (presumably larger) Casimir background.

For Casimir experiments using real (rather than idealized) conductors, the effects of finite conductivity, finite temperature, and surface roughness become important, and considerable progress has been made in recent years in dealing with these effects \cite{2,17,18,19}. Still, the inherent difficulty in calculating the Casimir force for real materials to high precision limits the use of Casimir force experiments to constrain new forces. The object of the present paper is to implement a recent proposal \cite{12,13,20} which aims to sidestep these problems when searching for new very short-ranged macroscopic forces. The approach relies on the fact that the Casimir force depends primarily on the *electronic* properties of the interacting bodies, while the proposed new forces, including those arising from new spatial dimensions, depend on the test bodies’ *nuclear*, as well as on their electronic, properties. One should therefore be able to set limits on new forces at sub-micron separations by measuring the
differences in forces between test bodies composed of different isotopes of the same element (the iso-electronic effect), since the Casimir force should be independent of isotope to a good approximation. Any observed differences between the test bodies could then be attributed to new physics after other effects (differences in sample preparation, etc.) have been accounted for. Furthermore, this method does not require a detailed calculation of the Casimir force, if this force is known to be the same for the isotopes being compared. However, before one can extract reliable limits on new forces from an experiment based on the iso-electronic effect, one must be confident that any small isotopic dependence of the Casimir force will produce a force difference that is less than the force resolution of the experiment. In what follows we present the first calculation of the isotopic dependence of the Casimir force.

This calculation is of interest for two reasons: First, it is directly relevant to an experiment currently underway \[21\] to search for new short-range forces by comparing the Casimir forces for two isotopes of the same element. Secondly, our results reveal new characteristics of the Casimir force, the dependence on lattice spacing and nuclear masses, which may be utilized in future applications.

To understand how the isotopic dependence of the Casimir force comes about we note that for two infinitely thick parallel plates composed of real dielectrics at \( T \neq 0 \), the expression for the Casimir force \( F_C(d, T) \) is more complicated than Eq. (1), and is given by the Lifshitz formula \[22,23,24,17\]:

\[
F_C(d, T) = -\frac{k_B T A}{\pi \epsilon^3} \sum_{l=0}^{\infty} \xi_l^3 \int_1^{\infty} p^2 dp \\
\times \left\{ \left( \frac{K(i\xi_l, T) + \epsilon(i\xi_l, T) p}{K(i\xi_l, T) - \epsilon(i\xi_l, T) p} \right)^2 e^{2d(\xi_l/c)p} - 1 \right\}^{-1} \\
+ \left\{ \left( \frac{K(i\xi_l, T) + p}{K(i\xi_l, T) - p} \right)^2 e^{2d(\xi_l/c)p} - 1 \right\}^{-1}.
\]

Here \( A \) is the area of the plates, \( k_B \) is Boltzmann’s constant, \( \omega = i\xi_l, \xi_l = 2\pi k_B T l / \hbar \), and

\[
K(i\xi_l, T) = \left[ p^2 - 1 + \epsilon(i\xi_l, T) \right]^{1/2},
\]
where \( \varepsilon(\omega, T) \) is the frequency and temperature dependent dielectric constant. The prime on the summation sign indicates that the \( l = 0 \) term should be multiplied by \( 1/2 \). The dielectric properties of the interacting media determine the Casimir force acting between real metallic plates. In practice, one obtains \( \varepsilon(\omega, T) \) from a Drude or plasma model for low frequencies and from tables of optical data for higher frequencies.

We note from Eq. (2) that the Casimir force depends on the temperature \( T \), as well as on \( d \), and that this \( T \)-dependence between real metals enters in two different ways [18]. First, the quantum electromagnetic field is at temperature \( T \) and this contributes a thermal pressure from real photons on the plates which becomes significant for larger plate separations. When \( d \gtrsim \hbar c/2k_B T \), the energy spacing between the field modes decreases allowing thermal energy to more easily excite higher modes. [Note that for a room temperature experiment (\( T = 300 \) K), \( \hbar c/2k_B T = 3.8 \) \( \mu \)m, while for an experiment operating at liquid helium temperatures (\( T = 4 \) K), \( \hbar c/2k_B T = 0.29 \) mm.] If \( d \ll \hbar c/2k_B T \), there is a sufficiently large gap between the ground state and first excited states to prevent significant thermal excitation of higher modes in which case \( F(d, T) \) reduces to [22, 23, 24, 17]

\[
F_T(d) = -\frac{\hbar A}{2\pi^2 c^3} \int_0^\infty d\xi \int_1^\infty p^2 dp \left\{ \left( \frac{K(i\xi, T) + \varepsilon(i\xi, T)p}{K(i\xi, T) - \varepsilon(i\xi, T)p} \right)^2 e^{2d(\xi/c)p} - 1 \right\}^{-1} \left( \frac{K(i\xi, T) + p}{K(i\xi, T) - p} \right)^2 e^{2d(\xi/c)p} - 1 \right\}^{-1}.
\]

As was observed recently [18], we can see from Eq. (4), that there remains another temperature dependence to the Casimir force. The dielectric constant \( \varepsilon(\omega, T) \) is also temperature dependent as discussed below, even when one can neglect the temperature fluctuations of the field. Therefore, following Ref. [18], we define \( F(d, T = 0) \equiv F_T(d) \), were the subscript denotes this implicit \( T \)-dependence. Thus, it is important to check that the tabulated data for the dielectric constant are appropriate for the temperature at which the experiment is conducted. This has not been a problem in previous experiments since they have all been...
conducted at room temperature at which most of the tabulated optical data are obtained. However, these data may be inappropriate for experiments conducted at low temperatures.

For the case of two infinite plates, we see that the isotopic dependence of the Casimir force must enter through $\varepsilon(i\xi, T)$, which depends mainly upon the electronic properties of the material. Optical data for the isotopes of interest, at temperatures relevant for an experiment, are difficult to find in the literature. Furthermore, experience from room temperature Casimir force experiments indicates that, ideally, one should obtain optical data directly from the actual samples used, since there is sufficient variation from sample to sample.

In the absence of relevant experimental data, we can estimate the isotopic dependence of the Casimir force from theoretical considerations. As we have discussed, the Casimir force is determined through the Lifshitz formula by the dielectric constant $\varepsilon(\omega, T)$. For metals which can be described by the plasma model, $\varepsilon(\omega, T)$ is characterized by a single parameter, the plasma frequency $\omega_p$, which depends, in turn, on the lattice constant $a$. In the presence of an anharmonic potential, the lattice spacing $a$ will be different for two isotopes of the same element, since the zero point motion of the isotopes at $T = 0$ depends on the respective isotopic masses. Thus, the isotopic dependence of the Casimir force arises from the dependence of the lattice constant on mass, and this dependence affects the dielectric constant, and eventually the Casimir force, through the Lifshitz formula.

To quantify the preceding discussion, we consider a Casimir force experiment utilizing conductors which can be described by the plasma model. Although this is a simple model of metals, it is sufficiently reliable for our present purposes. In the plasma model the dielectric constant $\varepsilon(\omega = i\xi)$ is given by

$$\varepsilon(i\xi) = 1 + \frac{\omega^2_p}{\xi^2};$$

(5)

where the plasma frequency $\omega_p$ is

$$\omega^2_p = \frac{4\pi Ne^2}{m_{\text{eff}} V}.$$
Here $N/V$ is the number of conduction electrons/volume and $m_{\text{eff}}$ is the effective electron mass. If Eq. (3) is substituted into Eq. (4), one finds in the limits $d \gg \frac{2\pi c}{\omega_p}$ and $T \ll \frac{\hbar c}{2k_Bd}$ \cite{25,2},

$$F(d) \simeq -\frac{\pi^2}{240} \frac{\hbar c A}{d^4} \left(1 - \frac{16}{3} \frac{c}{\omega_p d}\right).$$  

Equation (7)

In the simplest case, let $N_{\text{val}}/V$ be the number of valence electrons/volume and let $m_{\text{eff}} = m_e$, the free electron mass, in which case, Eq. (6) reduces to

$$\omega_p^2 = \frac{4\pi N_{\text{val}} e^2}{m_e V}.$$  

From Eq. (8), we see that in this case, all of the isotopic dependence must arise from $V$, the volume per atom, which is proportional to $a^3$.

The isotopic dependence of the lattice spacing has been a topic of interest for some time \cite{26,27}. It is well-known that the temperature dependence of $a$, which leads to thermal expansion of solids, arises from anharmonic terms in the interatomic potential \cite{28}. For example, in a one-dimensional lattice with a typical interatomic distance given by $x$, let an atom’s potential energy be approximated by $V(u) \simeq (1/2)ku^2 - (1/6)bu^3$, where $u = x - x_0$, $x_0$ is the equilibrium separation, $k$ is the effective spring constant, and $b$ characterizes the anharmonic contribution. At temperature $T$, the thermal average displacement depends on the anharmonic term so that the lattice constant in this model becomes temperature dependent and proportional to $b$ \cite{28}:

$$a(T) = x_0 + \langle u \rangle \simeq x_0 + \frac{b}{2k^2k_B T},$$

where $k_B$ is Boltzmann’s constant. This temperature dependence of the lattice spacing affects the plasma frequency, which leads to a temperature-dependent dielectric constant $\varepsilon(\omega, T)$ as mentioned earlier. If one replaces thermal vibrations, which allow atoms to sense the anharmonicity of the potential, with quantum zero-point motion ($kT \to \hbar\omega/2$), one finds that the lattice spacing in this model becomes dependent on the atomic mass $M$, since the vibrational frequency $\omega$ is proportional to $1/\sqrt{M}$ \cite{29}:


\[ a(T = 0) \simeq x_0 + \frac{b}{4k^2 \hbar \omega}. \tag{10} \]

It follows that the temperature and isotopic dependence of \( \varepsilon(\omega, T) \) are linked through the anharmonic term of the interatomic potential. At finite temperatures, both thermal and zero-point motions contribute, although the former dominate at higher temperatures. Hence the isotopic dependence of \( a \) is most significant at temperatures much less than the Debye temperature. For a current review of theoretical estimates and experimental results for the isotopic dependence of the lattice constant, see Plekhanov [27]. With the exception of nickel, which is one of the metals being considered for an experiment utilizing the iso-electronic effect [12], most of this effort has focused on materials which would not be appropriate for Casimir force experiments. Nonetheless, one finds (Table I) that \( \Delta a/a \sim 10^{-4} \) for those elements that have been studied, and this agrees with theoretical estimates.

If we assume that the entire isotopic dependence of the Casimir force is dominated by the isotopic dependence of the lattice constant through Eq. (8), and that the Casimir force is given by Eq. (7), then we find that a variation of the lattice constant \( \Delta a \) leads to a relative difference in the Casimir force for two different isotopes,

\[
\frac{\Delta F_{21}}{F} \simeq \left( \frac{16}{3} \frac{c}{\omega_p d} \right) \frac{\Delta \omega_p}{\omega_p} = -\left( \frac{8c}{\omega_p d} \right) \frac{\Delta a_{21}}{a}, \tag{11}
\]

where \( \Delta F_{21} = F_2 - F_1 \), \( \Delta a_{21} = a_2 - a_1 \), and we have used \( \Delta \omega_p/\omega_p = -(1/2)\Delta V/V = -(3/2)\Delta a/a \). Since Eq. (7) is valid when \( 2\pi c/\omega_p d \ll 1 \), and experimentally one finds (e.g., for nickel) \( \Delta a/a \sim 10^{-4} \), one expects

\[
\frac{\Delta F_{21}}{F} \ll 10^{-4}, \tag{12}
\]

under these conditions. This is several orders of magnitude below the current resolution of Casimir force experiments (\( \Delta F/F \sim 10^{-2} \)). However, since this problem remains unexplored experimentally, it may be possible to find situations in which the isotopic \( \Delta F/F \) is large enough to be observed.

To summarize, we have shown that for metals which can be described by the plasma model, the relative difference in the Casimir force between plates composed of different
isotopes separated by $d \gg 2\pi c/\omega_p$ is negligible in any current experiment [21] utilizing force differences to extract limits on new forces. For metals which are not well described by the plasma model, and for experiments with shorter separations, further analysis will be needed to ascertain how well our conclusions hold, particularly when more realistic models of the dielectric constants are used for the actual experimental samples. Additionally, other effects of an isotopic mass difference should be explored. These include the isotopic dependence of $m_{\text{eff}}$, and the possibility that the dielectric constant can depend on isotopic mass in other ways besides the lattice constant. However, these effects are not likely to alter the principal conclusion of our analysis, that current searches for new short-range forces using the iso-electronic effect [21] can ignore the isotopic dependence of the Casimir force.

ACKNOWLEDGMENTS

The authors thank G. Carugno, R. Decca, A. Lambrecht, D. López, V. M. Mostepanenko, A. W. Overhauser, A. K. Ramdas, S. Reynaud, G. Ruoso, and S. Rodriguez for helpful discussions. This work was supported in part by the U. S. Department of Energy under contract No. DE-AC02-76ER071428.
REFERENCES

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).

[2] M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rep. 353, 1 (2001).

[3] K. A. Milton, The Casimir Effect (World Scientific, New Jersey, 2001); V. M. Mostepa-

nenko and N. N. Trunov, The Casimir Effect and its Applications (Clarendon Press, Ox-

ford, 1997); P. W. Milonni, The Quantum Vacuum (Academic Press, San Diego, 1994);

S. K. Lamoreaux, Am. J. Phys. 67, 850 (1999).

[4] S. K. Lamoreaux, Phys. Rev. Lett. 78, 5 (1997).

[5] U. Mohideen and A. Roy, Phys. Rev. Lett. 81, 4549 (1998); A. Roy and U. Mohideen,

Phys. Rev. Lett. 82, 4380 (1999); A. Roy, C. -Y. Lin, and U. Mohideen, Phys. Rev. D

60, 111101(RT) (1999); F. Chen, U. Mohideen, G. L. Klimchitskaya, and V. M. Mostepa-

nenko, Phys. Rev. Lett. 88, 101801 (2002).

[6] B. W. Harris, F. Chen, and U. Mohideen, Phys. Rev. A 62, 052109 (2000).

[7] T. Ederth, Phys. Rev. A 62, 062104 (2000).

[8] H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop, and F. Capasso, Science 291,

1941 (2001); Phys. Rev. Lett. 87, 211801 (2001).

[9] G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso, Phys. Rev. Lett. 88, 041804 (2002).

[10] M. Bordag, B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. D 58,

075003 (1998); Phys. Rev. D 60, 055004 (1999); Phys. Rev. D 62, 011701(R) (2000).

[11] V. M. Mostepanenko and M. Novello, Phys. Rev. D 63 115003 (2001).

[12] E. Fischbach, D. E. Krause, V. M. Mostepanenko, and M. Novello, Phys. Rev. D 64,

075010 (2001).

[13] D. E. Krause and E. Fischbach, Gyros, Clocks, and Interferometers: Testing Relativistic
Gravity in Space, edited by C. Lämmerzahl, C.W.F. Everitt, F.W. Hehl (Springer-Verlag, 2001), pp. 292–307.

[14] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B 429, 263 (1998); Phys. Rev. D 59, 086004 (1999)

[15] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); Phys. Rev. Lett. 83, 4690 (1999).

[16] R. Randall, Science 296, 1422 (2002).

[17] G. L. Klimchitskaya and V. M. Mostepanenko, Phys. Rev. A 63, 062108 (2001).

[18] V. B. Bezerra, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. A 65, 052113 (2002).

[19] A. Lambrecht and S. Reynaud, Eur. Phys. J. D 8, 309 (2000); C. Genet, A. Lambrecht, and S. Reynaud, Phys. Rev. A 62, 012110 (2000).

[20] E. Fischbach, S. W. Howell, S. Karunatillake, D. E. Krause, R. Reifenberger, and M. West, Class. Quantum Grav. 18, 2427 (2001).

[21] R. Decca, E. Fischbach, D. E. Krause, and D. López, in preparation.

[22] E. M. Lifshitz, Zh. Éksp. Teor. Fiz. 29, 94 (1956) [Sov. Phys. JETP 2, 73 (1956)].

[23] I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Usp. Fiz. Nauk 73, 381 (1961) [Sov. Phys. Usp. 4, 153 (1961)].

[24] E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, Part 2 (Pergamon Press, Oxford, 1980), pp. 338–342.

[25] C. M. Hargreaves, Proc. K. Ned. Akad. Wet., Ser. B: Phys. Sci. 68, 231 (1965); J. Schwinger, L. L. ReRaad, Jr., and K. A. Milton, Ann. Phys. (N.Y.) 115, 1 (1978).

[26] H. London, Z. Phys. Chem. 16, 302 (1958).
[27] V. G. Plekhanov, *Isotope Effects in Solid State Physics* (Academic Press, San Diego, 2001), pp. 46–55.

[28] C. Kittel, *Introduction to Solid State Physics*, 6th edition (Wiley, New York, 1986), pp. 114–115.

[29] A. K. Ramdas, S. Rodriguez, M. Grimsditch, T. R. Anthony, and W. F. Banholzer, Phys. Rev. Lett. 71, 189 (1993).

[30] V. S. Kogan and A. S. Bulatov, Zh. Éksp. Teor. Fiz. 42, 1499 (1962) [Sov. Phys. JETP 15, 1041 (1962)].

[31] H. Holloway, K. C. Hass, M. A. Tamor, T. R. Anthony, and W. F. Banholzer, Phys. Rev. B 44 7123 (1991).

[32] E. J. Covington and D. J. Montgomery, J. Chem. Phys. 27, 1030 (1957).

[33] D. N. Batchelder, D. L. Losee, and R. O. Simmons, Phys. Rev. 173 873 (1968).

[34] E. Sozontov, L. X. Cao, A. Kazimirov, V. Kohn, M. Konuma, M. Cardona, J. Zegenhagen, Phys. Rev. Lett. 86, 5329 (2001).
| Isotopes           | $\Delta a/a$                      | Reference |
|-------------------|-----------------------------------|-----------|
| $^{58}$Ni, $^{64}$Ni | $1.4 \times 10^{-4}$ [T = 78 K]  | [30]      |
|                   | $5.7 \times 10^{-5}$ [T = 300 K] |           |
| $^{12}$C, $^{13}$C (Diamond) | $-1.5 \times 10^{-4}$ [T = 298 K] | [31]      |
| $^{6}$Li, $^{7}$Li  | $-2 \times 10^{-4}$ [T = 293 K]  | [32]      |
| $^{20}$Ne, $^{22}$Ne | $-1.9 \times 10^{-3}$ [T = 3 K]  | [33]      |
|                   | $-1.6 \times 10^{-3}$ [T = 24 K] |           |
| $^{70}$Ge, $^{76}$Ge | $-5.3 \times 10^{-5}$ [T = 30 K] | [34]      |
|                   | $-2.2 \times 10^{-5}$ [T = 300 K]|           |