Local and Global Properties of $p$-Laplace Hénon Equation

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Abstract
We first give some apriori estimates of positive radial solutions of $p$-Laplace Hénon equation. Then we study the local and global properties of those solutions. Finally, we generalize some radial results to the nonradial case.

Keywords $p$-Laplace Hénon equation, singular solutions, removable singularity.

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1 Introduction and Main Results
In this paper, we study local and global properties of $p$-Laplace Hénon equation

$$-\Delta_p u = |x|^\alpha u^q,$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$. Local properties refer to local behavior of solutions near a certain point, like removable singularity and the order of isolated singularity. Global properties refer to properties of solutions in $\mathbb{R}^N$.

When $p = 2$, this is the usual Hénon equation

$$-\Delta u = |x|^\alpha u^q.$$

Equation (2) was proposed by astrophysicist Hénon in [8]. The first mathematical study about this equation was by [14]. After that, a lot of results such as existence, nonexistence, and symmetry breaking were studied, see [13, 15, 18].

When $p = 2, \alpha = 0$, equation (1) is the well known Lane-Emden equation

$$-\Delta u = u^q,$$

which was studied in [5]. Lions studied its isolated singularity in [11].
Because the \( p \)-Laplacian operator is lack of linearity when \( p \neq 2 \), equation (1) is more difficult than equation (2). Nevertheless, in [16] Serrin generalized the Carleson’s result about the harmonic function [4] and obtained the well-known local properties of general quasilinear equations. Based on his work, the local and global properties of \( p \)-Laplace Lane-Emden equation

\[-\Delta_p u = u^q,\]

were studied in [1, 7].

Usually, the right hand side of equation (1) is called a source term, see [22]. When the right hand side is changed to negative, like

\[-\Delta_p u = -|x|^\alpha u^q,\]

it is called an absorption term. Results about local and global properties of elliptic partial differential equations with an absorption term can be found in [2, 3, 6, 12, 20, 21].

Before stating our main results, we first give some definitions. Define \( \mu \) as the fundamental solution of \(-\Delta_p u = \delta_0\) in distributional sense,

\[
\mu(x) = \mu(|x|) = \begin{cases} \frac{p-1}{N-p} (\omega_{N-1})^{\frac{p-N}{p-1}} |x|^{\frac{p-N}{p-1}}, & \text{if } 1 < p < N; \\ (\omega_{N-1})^{\frac{p-1}{p}} \ln \left( \frac{1}{|x|} \right), & \text{if } p = N. \end{cases}
\]

where \( \omega_{N-1} \) is the area of unit sphere \( S^{N-1} \) and \( \delta_0 \) is the Dirac delta function.

The concept of continuous solution was introduced by Serrin [17]. \( u \) is called a continuous solution of (1) in \( \Omega \) if \( u \) is continuous in \( \Omega \) with \( \nabla u \in L^p_{\text{loc}}(\Omega) \) and \( u \) satisfies

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi = \int_{\Omega} |x|^\alpha u^q \varphi, \quad \forall \varphi \in C^\infty_c(\Omega).
\]

All the solutions referred in the following are continuous solutions.

This paper is organized as follows. First, in Section 2 we give some lemmas which will be used in the following sections. Then in Section 3 we study the local properties of (1) in the radial case.

**Theorem 1.1.** Assume \( 1 < p < N \), \( \Omega \) is an open domain containing \( \{0\} \) in \( \mathbb{R}^N \), and let \( u \) be a positive radial solution of (1) in \( \Omega' = \Omega \setminus \{0\} \).

(i) In the subcritical case \( p-1 < q \leq \frac{(N+\alpha)(p-1)}{N-p} \), either \( u \) can be extended to \( \Omega \) as a \( C^1 \) solution of (1) or there exists some constant \( C > 0, \tilde{C} > 0 \) such that \( \lim_{x \to 0} u(x) = C \). Furthermore, \( u \) satisfies \(-\Delta_p u - |x|^\alpha u^q = \tilde{C} \delta_0\) in the distributional sense.

(ii) In the critical case \( q = \frac{(N+\alpha)(p-1)}{N-p} \), either \( u \) can be extended to \( \Omega \) as a \( C^1 \) solution of (1) or

\[
\lim_{x \to 0} |x|^{\frac{N-p}{(p+\alpha)(p-1)}} \left( \frac{1}{|x|} \right)^{\frac{N-p}{p+\alpha}} \left[ \frac{\left( \frac{N-p}{p+\alpha} \right)^{(p-1)}}{\left( \frac{N-p}{p+\alpha} \right)^{(p-1)}} \right] u(x) = \left[ \frac{\left( \frac{N-p}{p+\alpha} \right)^{(p-1)}}{\left( \frac{N-p}{p+\alpha} \right)^{(p-1)}} \right] \cdot \frac{\tilde{C} \delta_0}{\left( \frac{N-p}{p+\alpha} \right)^{(p-1)}}.
\]
In the supercritical case \( \frac{(N+\alpha)(p-1)}{N-p} < q < \frac{(N+\alpha)p}{N-p} - 1 \), either \( u \) can be extended to \( \Omega \) as a \( C^1 \) solution of (1) or

\[
\lim_{x \to 0} |x|^\frac{p + \alpha}{q + 1 - p} u(x) = \lambda \left[ \left( \frac{p + \alpha}{q + 1 - p} \right)^{p-1} \left( \frac{N - pq + \alpha(p - 1)}{q + 1 - p} \right) \right]^{\frac{1}{p-1}}.
\]

**Theorem 1.2.** Assume \( p = N, q > p - 1, \Omega \) is an open domain containing \( \{0\} \) in \( \mathbb{R}^N \), and let \( u \) be a positive radial solution of (1) in \( \Omega' = \Omega \setminus \{0\} \). Then either \( u \) can be extended to \( \Omega \) as a \( C^1 \) solution of (1) or there exists some constant \( C > 0, \tilde{C} > 0 \) such that \( \lim_{x \to 0} \frac{u(x)}{\mu(x)} = C \), and \( u \) satisfies \(-\Delta_p u - |x|^\alpha u^q = \tilde{C}\delta_0 \) in the distributional sense.

Next, in Section 4 we obtain the following global property by dealing with the exterior problem.

**Theorem 1.3.** Assume \( 1 < p < N \) and \( p - 1 < q < \frac{(N+\alpha)p}{N-p} - 1 \). Then equation (1) has no positive radial solution in \( \mathbb{R}^N \).

Finally, in Section 5 we prove some of the radial results can be generalized to the nonradial case, such as the subcritical case in Theorem 1.1 and Theorem 1.3.

**Theorem 1.4** (Subcritical Case). Assume \( 1 < p < N, p - 1 < q < \frac{(N+\alpha)(p-1)}{N-p} \), and let \( u \) be a positive solution of (1) in \( \Omega' = \Omega \setminus \{0\} \) satisfying \( \frac{u(x)}{\mu(x)} \leq C \) for some constant \( C > 0 \). Then

(i) either \( u \) can be extended to \( \Omega \) as a \( C^1 \) solution of (1), or

(ii) there exists some constant \( a > 0 \) such that \( \lim_{x \to 0} \frac{u(x)}{\mu(x)} = a \).

**Theorem 1.5.** Assume \( 1 < p < N, p - 1 < q < \frac{(N+\alpha)p}{N-p} - 1 \) and let \( u(x) \) be a nonnegative solution of (1) in \( \mathbb{R}^N \) satisfying \( |x|^\frac{p + \alpha}{q + 1 - p} u(x) \leq C \) for some constant \( C > 0 \), then \( u \equiv 0 \).

As far as we know, there are several papers studying local and global properties of \( p \)-Laplace Lane-Emden equation, but there are few about \( p \)-Laplace Hénon equation. Some results such as Theorem 1.5 are new even for \( p \)-Laplace Lane-Emden equation.

## 2 Preliminaries

In this section, we list some lemmas which will be used later. For quasilinear partial differential equations

\[
\text{div} A(x, u, \nabla u) = B(x, u, \nabla u),
\]

(3)
where $A$ is a given vector function and $B$ is a measurable function satisfying
\[
|A(x, u, \xi)| \leq a|\xi|^{p-1} + b|u|^{p-1} + e,
\]
\[
|B(x, u, \xi)| \leq c|\xi|^{p-1} + d|u|^{p-1} + f,
\]
\[
\xi \cdot A(x, u, \xi) \geq |\xi|^{p-2} - d|u|^{p-2} - g,
\]
we assume that $1 < p \leq N$, $a \in (0, \infty)$ and coefficients $b$ through $g$ are measurable functions in the respective Lebesgue classes
\[
b, e \in L^{\frac{N}{p-1}-\epsilon}; c \in L^{\frac{N}{1-\epsilon}}; d, f, g \in L^{\frac{N}{p-\epsilon}}, \; \epsilon > 0.
\]

The following lemma comes from Theorem 1 in [17].

**Lemma 2.1** (Isolated Singularity). Assume $\Omega$ is an open domain containing $\{0\}$, and let $u$ be a continuous solution of (3) in the $\Omega' = \Omega \setminus \{0\}$. Suppose that $u \geq L$ for some constant $L$. Then either $u$ has removable singularity at 0, or else
\[
u \approx \left\{ \begin{array}{ll}
|\xi|^{\frac{N}{p-1}}, & \text{if } p < N, \\
\ln \left( \frac{1}{|\xi|} \right), & \text{if } p = N,
\end{array} \right.
\]
in the neighborhood of the origin, where “$\approx$” means “has the same order with”.

We call $u$ is a $p$-harmonic function if $\Delta_p u = 0$. The following Comparison Principle comes from Theorem 2.15 in [10].

**Lemma 2.2** (Comparison Principle). Suppose that $u$ and $v$ are $p$-harmonic functions in a bounded domain $\Omega$. If at each $\zeta \in \partial \Omega$
\[
\limsup_{x \to \zeta} u(x) \leq \liminf_{x \to \zeta} v(x),
\]
excluding the situation $\infty \leq \infty$ and $-\infty \leq -\infty$, then $u \leq v$ in $\Omega$.

The following Strong Comparison Principle comes from Proposition 1.5.2 in [22].

**Lemma 2.3** (Strong Comparison Principle). Let $\Omega \subset \mathbb{R}^N$ be a domain, $p > 1$ and $c \in L^{\infty}(\Omega)$. Assume $u$ and $v$ belong to $C^1(\Omega)$, satisfy
\[-\Delta_p u + cu \leq 0 \text{ and } -\Delta_p v + cv \geq 0 \text{ in } \mathcal{D}'(\Omega)
\]
and $\nabla v$ never vanishes in $\Omega$. If $u \leq v$ in $\Omega$ and there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$, then $u \equiv v$ in $\Omega$.

### 3 Local Properties in the Radial Case

We first give some apriori estimates which will be used in the proof of Theorem 1.1.
Lemma 3.1 (Apriori Estimate). Assume $1 < p < N$, $\Omega$ is an open domain containing $\{0\}$, and let $u$ be a positive radial solution of $\text{II}$ in $\Omega' = \Omega \setminus \{0\}$.

(i) If $p - 1 < q$, then there exists some constant $C > 0$ such that $\frac{u(x)}{\mu(x)} \leq C$ near $0$.

(ii) If $q = \frac{(N+\alpha)(p-1)}{N-p}$, then there exists some constant $C > 0$ such that
$$\frac{u(x)}{\mu(x)} \cdot \left(\ln \frac{1}{|x|}\right)^{\frac{N-p}{p-1}} \leq C \text{ near } 0.$$  

(iii) If $q > \frac{(N+\alpha)(p-1)}{N-p}$, then there exists some constant $C > 0$ such that
$$|x|^{\frac{N+\alpha}{p} - p} u(x) \leq C \text{ near } 0.$$

Proof. The idea of proof comes from [7]. Without loss of generality, we assume $B_1(0) \subset \Omega$. Let
$$\beta = \frac{p-N}{p-1}, s = r^\beta, v(s) = u(r),$$
then we obtain
$$(p-1)|v'|^{p-2}v'' + s^{\frac{1-\beta}{p-1}} \frac{v'^q}{s} = 0, \quad s \in [1, \infty) \tag{4}$$
As a result, $v'' < 0$, which means $v'$ is decreasing and bounded in $[1, \infty)$.

(i) When $v(s)$ is bounded, we can automatically get the conclusion. So we only need to consider the unbounded case, which means $v(s) \to \infty$ as $s \to \infty$ because of the concavity of $v$. It is easy to check $v' \geq 0$. According to L’Hospital’s rule,
$$\lim_{s \to \infty} \frac{v(s)}{s} = \lim_{s \to \infty} v'(s).$$ So $v(s)$ is bounded when $s$ is large, which means $\frac{v(s)}{s}$ is bounded near $0$. This proves the first part.

(ii) According to the Mean Value Theorem,
$$\frac{v(s) - v(1)}{s - 1} = v'(\theta) \geq v'(s), \quad 1 < \theta < s.$$ 
So $v(s) \geq sv'(s) + o(s)$ when $s$ is large. As a result, we induce from (4) that there exists some $c > 0$ such that when $s$ is large
$$(v')^{p-1} + cs^{\frac{1-\beta}{p-1}} + q v'^q \leq 0.$$ 
Let $\gamma = \frac{1-\beta}{p} + \frac{\alpha}{p} + q$, $\psi(s) = (v'(s))^{p-1}$. We study the following two cases in the situation when $s$ is large.

When $q = \frac{(N+\alpha)(p-1)}{N-p}$, $\gamma = -1$, $\psi(s)$ satisfies
$$\psi' + cs^{-1} \psi^\gamma \leq 0.$$ 

After integration, we get $\psi(s) \leq c(\log s)^{-\frac{p-1}{p+1}}$, which means

$$v'(s) \leq c(\log s)^{-\frac{1}{p+1}} = c(\log s)^{-\frac{N}{p(N-p)}}.$$  

As a consequence, $\frac{\psi(s)}{\log s}^{\frac{N}{p(N-p)}}$ is bounded when $s$ is large. This means $\frac{u(x)}{\log x}^{\frac{1}{p+1}}$ is bounded near 0.

(iii) When $q > \frac{(N+\alpha)(p-1)}{N-p}$, $\gamma > -1$, $\psi(s)$ satisfies

$$\psi' + cs^\gamma \psi < 0.$$  

After integration, we get $\psi(s) \leq cs^{-\frac{\gamma+1}{p+1}}$, which means

$$v'(s) \leq cs^{-\frac{\gamma+1}{p+1}} = cs^{-\frac{p+1}{p(N-p)}}.$$  

As a consequence, $v(s)s^{\frac{p+1}{p(N-p)}}$ is bounded when $s$ is large. This means $u(x)|x|^{\frac{p+1}{p}}$ is bounded near 0.

Now we are ready to prove Theorem 1.1.

**Proof.** (i) **the Subcritical Case**

$|x|^\alpha u^q = |x|^\alpha u^{q+1-p}u^{p-1}$. From Lemma 3.1, $\frac{v}{p} \leq C$ near 0. In addition, when $p - 1 < q < \frac{(N+\alpha)(p-1)}{N-p}$, we have $q + 1 - p < \frac{(p+\alpha)(p-1)}{N-p}$. So

$$|x|^\alpha u^{q+1-p} \in L^{\frac{N}{p-1}}(\Omega)$$  

for some $\epsilon > 0$. According to Lemma 2.1, either $u$ can be extended to $\Omega$ as a $C^1$ solution of (1) or there exists constants $c_1, c_2 > 0$ such that $c_1 \mu(x) \leq u(x) \leq c_2 \mu(x)$ near 0. Using the notation in Lemma 3.1, we have $\frac{u(x)}{\mu(x)} \geq c_1$ which means $v(s) \geq c_1$ when $s$ is large. From the proof of Lemma 3.1, we know $v'(s)$ is decreasing in $[1, \infty)$. So there exists a constant $C$ such that $\lim_{s \to \infty} \frac{v(s)}{s} = \lim_{s \to \infty} v'(s) = C$, which means

$$\lim_{x \to 0} \frac{u(x)}{\mu(x)} = C, \lim_{x \to 0} \frac{u(x)}{\mu(x)} = \tilde{C},$$

where $\tilde{C} = (C \cdot \frac{N-p}{p-1})^{p-1} \cdot \omega_{N-1}$.

For $\forall \varphi \in C_c^\infty(B_1(0))$, if we multiply both sides of (1) by $\varphi$ and integrate by parts over $B_1(0) \setminus B_1(0)$, then we get

$$\int_{\partial B_1(0)} |\nabla u|^p \nabla \cdot \nabla \varphi + \int_{B_1(0) \setminus B_1(0)} |\nabla u|^p - 2 \nabla u \cdot \nabla \varphi = \int_{B_1(0) \setminus B_1(0)} |x|^\alpha u^q \varphi,$$

(5)
where \( \mathbf{n} = \frac{|x|}{r} \). It is known that the fundamental solution \( \mu(x) \) satisfies

\[
- \int_{\partial B_r(0)} |\nabla \mu|^{p-2} \nabla \mu \cdot \mathbf{n} \varphi + \int_{B_r(0)} |\nabla \mu|^{p-2} \nabla \mu \cdot \nabla \varphi = \varphi(0).
\]

Sending \( \epsilon \to 0 \) in (5), we have

\[
- \tilde{C} \varphi(0) + \int_{B_1(0)} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_{B_1(0)} |x|^\alpha u^q \varphi,
\]

which implies

\[-\Delta u - |x|^\alpha u^q = \tilde{C} \delta_0\]

in the distributional sense.

(ii) **the Critical Case**

Let

\[
w(s) = r^\delta u(r), s = -\ln r, \delta = \frac{N - p}{p - 1} = \frac{p + \alpha}{q + 1 - p}
\]

Then (11) can be transformed into

\[
[|w'| + \delta w|^{p-2}(w' + \delta w)]' + w^q = 0.
\]

As a result,

\[
|w'| + \delta w|^{p-2}(w' + \delta w)]_t^n = - \int_t^\infty w^q.
\]

From Lemma 3.1, \( w(s) \leq Cs^{\frac{N-p}{p-\alpha}} \), so for \( \forall t > 0, w^q \in L^1(t, \infty) \). As a consequence,

\[
|w'(t) + \delta w(t)|^{p-2}(w'(t) + \delta w(t)) = \int_t^\infty w^q.
\]

This means \( w'(t) + \delta w(t) > 0, t \in (0, \infty) \),

\[
(w'(t) + \delta w(t))^{p-1} = \int_t^\infty w^q.
\]

Using the same procedure as in [7] in the critical case, we claim that

(a) \( w(t) \) is decreasing in (1, \( \infty \)),

(b) \( \frac{w'(t)}{w(t)} \to 0, \) or \( \frac{w'(t)}{w(t)} \to -\delta \) as \( t \to \infty \).

If \( \frac{w'(t)}{w(t)} \to -\delta \) as \( t \to \infty \), we can choose \( \epsilon_0 \) small such that \( w(t) \leq ce^{(-\delta + \epsilon_0)t} \)

for some \( c > 0 \) when \( t \) is large, which means \( u(r) \leq c \frac{1}{r^{\epsilon_0}} \) as \( r \to 0 \). As in the subcritical case, we can prove \( |x|^\alpha u^{q+1-p} \in L^{\frac{N}{p}+\epsilon} \) for some \( \epsilon > 0 \). So
by Lemma 2.1, $u$ is regular or have the same order with $\mu(x)$. The later is impossible.

If $\frac{w'(t)}{w(t)} \to 0$ as $t \to \infty$, we have when $t$ is large,

$$
(\delta w(t))^{p-1} \left[ 1 + o \left( \frac{w'(t)}{w(t)} \right) \right] = \int_t^\infty w^q.
$$

So

$$
w(s) \to \left[ \frac{p-1}{q+1-p} \left( \frac{1}{\delta^{p-1}} s - c \right)^{-1} \right]^{\frac{N-p}{(p+\alpha)(p-1)}},
$$

$$
s^{\frac{N-p}{(p+\alpha)(p-1)}} w(s) \to \left[ \left( \frac{N-p}{p+\alpha} \right) \left( \frac{N-p}{p-1} \right)^{p-1} \right]^{\frac{N-p}{(p+\alpha)(p-1)}}
$$

as $s \to \infty$.

(iii) the Supercritical Case

As in the proof of the critical case, we set $\delta = \frac{p + \alpha}{q + 1 - p}, w(s) = r^\delta u(r)$ where $s = -\ln r$. Then equation (1) becomes

$$
|w'|^{p-2} (a_1 w + a_2 w' + a_3 w'') + w^q = 0
$$

where

$$
a_1 = \delta[(p-1)(\delta + 1) - (N-1)] = \delta c_1, a_2 = (p-1)\delta + c_1, a_3 = p-1.
$$

It is the same equation as obtained in [7] except that $\delta$ is different. Regarding [6] as an autonomous system with variables $(w(s), w_s(s))$, we can prove $(\lambda, 0)$ is asymptotically stable when

$$
\frac{(N+\alpha)(p-1)}{N-p} < q < \frac{(N+\alpha)p}{N-p} - 1.
$$

By the phase plane analysis, as in [7] we assert either $(w(s), w_s(s)) \to (\lambda, 0)$ which means $\lim_{x \to 0} |x|^{\frac{1}{p-1}} u(x) = \lambda$, or $w(s) \leq ce^{-\delta s}$ which means $u(x)$ is regular. This completes the proof.

Next we prove Theorem 1.2.

Proof. Let $s = -\ln r, v(s) = u(r)$, then

$$
(N - 1)|v'|^{N-2}v'' + e^{-s(N+\alpha)}v'^q = 0
$$

Using the same procedure as in the proof of the subcritical case when $p < N$, we can obtain the conclusion.
4 Global Properties in the Radial Case

Lemma 4.1 (Exterior Problem). Assume $1 < p < N$, and $G = \{x \in \mathbb{R}^N : |x| \geq 1\}$.

(i) If $p - 1 < q \leq \frac{(N+\alpha)(p-1)}{N-p}$, then (1) has no positive radial solution in $G$.

(ii) If $\frac{(N+\alpha)(p-1)}{N-p} < q$, and $u$ is a positive radial solution of (1) in $G$, then there exists some constants $c_1, c_2 > 0$ such that

$$c_1\mu(x) \leq u(x) \leq c_2|x|^{-\frac{p+\alpha}{p+1-p}}$$

when $|x|$ is large.

Proof. (i) We argue by contradiction, assuming that equation (1) has a positive radial solution $u$ in $G$. Using the same transformation as in the proof of Lemma 3.1,

$$v(s) = u(r), s = r^\beta, \beta = \frac{p-N}{p-1},$$

we obtain

$$((|v'|^{p-2}v')' + \frac{1}{|\beta|^p s^\frac{1}{p}} (p+\alpha) - p)v' = 0, \ s \in (0,1].$$

Because $v''(s) < 0$, there exists a constant $c$ such that $v(s) \to c$ as $s \to 0$. As in [7], we can prove $c = 0$ and when $s$ is small

$$v'(s) \leq \begin{cases} 
  cs^{-\frac{1}{p+1-p} - 1}, & q < \frac{(N+\alpha)(p-1)}{N-p}, \\
  c \left( \ln \frac{1}{s} \right)^{-\frac{1}{p+1-p}}, & q = \frac{(N+\alpha)(p-1)}{N-p}.
\end{cases}$$

This means $v'(s) \to 0$ as $s \to 0$. It is a contradiction to $v'(0) > 0$.

(ii) Because $v(s)$ is concave, $\frac{v(s)}{s} \geq v'(s)$. In addition, $v'(s) > c_1 > 0$ when $s$ is small, so $\frac{v(s)}{s} \geq c_1$ which proves $u(x) \geq c_1\mu(x)$.

When $q > \frac{(N+\alpha)(p-1)}{N-p}$, and $s$ is small, there exists constant $c_2 > 0$ such that $v'(s) \leq c_2 s^{-\frac{1}{p+1-p} - 1}$. So

$$\frac{v(s)}{s} \leq 2c_2 s^{-\frac{1}{p+1-p} - 1},$$

which means

$$u(x) \leq 2c_2 |x|^{-\frac{p+\alpha}{p+1-p}}.$$

This completes the proof.

Proof of Theorem 1.3
Proof. We argue by contradiction, assuming that $u$ is a positive radial solution of \( (\text{I}) \) in $\mathbb{R}^N$. By Lemma 2.1 we only need to prove the case when \( \frac{(N+\alpha)(p-1)}{N-p} < q < \frac{N+\alpha}{N-p} - 1 \). Multiply both sides of \( (\text{I}) \) by $u$ and integrate by parts in $B_R = \{ x \in \mathbb{R}^N : |x| \leq R \}$,

\[
\int_{\partial B_R} |\nabla u|^{p-2}\nabla u \cdot \vec{n}uds + \int_{B_R} |\nabla u|^p dx = \int_{B_R} |x|^\alpha u^q dx.
\]

Multiply both sides of \( (\text{I}) \) by $\nabla \cdot x$ and integrate by parts in $B_R$,

\[
\left( \frac{N}{p} - 1 - \frac{N+\alpha}{q+1} \right) \int_{B_R} |x|^\alpha u^{q+1} dx + \left( \frac{N}{p} - 1 \right) \int_{\partial B_R} |\nabla u|^{p-2}\nabla u \cdot \vec{n}uds = 0
\]

\[
+ \int_{\partial B_R} |\nabla u|^{p-2}\nabla u \cdot \vec{n}u \cdot xds - \frac{1}{p} \int_{\partial B_R} |\nabla u|^p (x \cdot \vec{n})ds \tag{7}
\]

\[
+ \frac{1}{q+1} \int_{\partial B_R} |x|^\alpha u^{q+1} (x \cdot \vec{n})ds = 0
\]

According to Lemma 4.1 $v' \leq 2c_2s^{-\frac{1}{p+1} \frac{p+\alpha}{q+1} - 1}$, so $|\nabla u(R)| \leq CR^{-\frac{q+1+\alpha}{q+1} - 1}$ when $R$ is large. Because $u$ is positive in $\mathbb{R}^N$, the first part of \( (7) \) is negative when $q < \frac{(N+\alpha)p}{N-p} - 1$. The boundary parts has the same order with $R^{N-p-\frac{(p+\alpha)p}{q+1} - 1}$, so they tend to 0 as $R \to \infty$. This is a contradiction, which implies \( (\text{I}) \) has no positive radial solution in $\mathbb{R}^N$ when $p-1 < q < \frac{(N+\alpha)p}{N-p} - 1$. \qed

5 Nonradial Case

5.1 Proof of Theorem 1.4

Proof. Because $\frac{u(x)}{\mu(x)} \leq C$, by Lemma 2.1 either $u$ can be extended to $\Omega$ as a $C^1$ solution of \( (\text{I}) \) or there exists constants $c_1, c_2 > 0$ such that

\[
c_1 \mu(x) \leq u(x) \leq c_2 \mu(x).
\]

Assume $\limsup_{x \to 0} \frac{u(x)}{\mu(x)} = a$, then there exists a sequence $\{ x_n \}$ satisfying

\[
\lim_{n \to \infty} x_n = 0, \quad \lim_{n \to \infty} \frac{u(x_n)}{\mu(x_n)} = a.
\]

Denote $x_n = r_n \xi_n$, where $r_n = |x_n|$. Define $u_{r_n}(\xi) = \frac{u(r_n \xi)}{\mu(r_n)}$, $\xi \in (0, \frac{1}{r_n})$. It can be easily checked that $u_{r_n}$ satisfies

\[
div\xi(|\nabla u_{r_n}|^{p-2}\nabla u_{r_n} + r_n^{p+\alpha} \mu(r_n)^{1-p}|\xi|^\alpha) u_{r_n} = 0 \tag{8}
\]

As $q < \frac{(N+\alpha)(p-1)}{N-p}$, and $q + 1 - p < \frac{(p+\alpha)(p-1)}{N-p}$, so

\[
r_n^{p+\alpha} \mu(r_n)^{q+1-p} = \mu(1) r_n^{(1-\theta)(p+\alpha)}
\]

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for some $0 < \theta < 1$. In addition, according to definition, $u_{\infty}(\xi) \leq C\frac{\mu(\xi)}{\mu(1)}$. So $u_{\infty}$ is bounded in $[\frac{1}{2}, \frac{3}{2}]$, and
\[ r_n^{p+\alpha}\mu(r_n)^{q+1-p}u_{r_n}^q \to 0 \]
as $n \to \infty$. By [9, 19], $u_{r_n}$ is Hölder continuous in $[\frac{1}{2}, \frac{3}{2}]$. As a result, by Arzela-Ascoli theorem, there exists a subsequence still denoted by $u_{r_n}(\xi)$ such that $u_{r_n}(\xi) \to v(\xi)$ in $[\frac{1}{2}, \frac{3}{2}]$ for some $p$-harmonic function $v(\xi)$. Clearly,
\[ v(\xi) \leq a\frac{\mu(\xi)}{\mu(1)}, \quad v(\xi) = \frac{\mu(\xi_n)}{\mu(1)}, \]
so by Lemma 2.2 $v(\xi) = a\frac{\mu(\xi)}{\mu(1)}$. For any $\epsilon > 0$ we can choose $n$ large such that
\[ u_{r_n}(\xi) \geq (a - \epsilon)\frac{\mu(\xi)}{\mu(1)} \]
when $\epsilon$ is small, that is
\[ u(r_n\xi) \geq (a - \epsilon)\mu(r_n\xi). \]
In particular, $u(x) \geq (a - \epsilon)\mu(x)$ on the boundary of annuli $r_{n+1} \leq |x| \leq r_n$, by Lemma 2.2 $u(x) \geq (a - \epsilon)\mu(x)$ in $r_{n+1} \leq |x| \leq r_n$. Letting $n \to \infty$, we have $u(x) \geq (a - \epsilon)\mu(x)$ in $B_{r_{n_0}} \setminus \{0\}$ for some big $n_0$. As a consequence, \[ \liminf_{x \to 0} \frac{u(x)}{\mu(x)} \geq a - \epsilon. \] Letting $\epsilon \to 0$, we complete the proof.

Next, we use the method in [11] to get nonexistence result in nonradial case.

**Theorem 5.1.** If $1 < p < N$, and $p - 1 < q \leq \frac{(N+\alpha)(p-1)}{N-p}$, then [11] has no positive solution in $G$, where $G = \{x \in \mathbb{R}^N : |x| \geq 1\}$.

**Proof.** We argue by contradiction, assuming that [11] has a positive solution in $G$. Denote $m = \min_{|x| = 2} u(x)$, define a sequence of radial functions as follows: $u_{n,0} \equiv 0$, when $k \geq 1$

\[
\begin{cases}
-\Delta_p u_{n,k} = |x|^\alpha u_{n,k-1}^q, & 2 < |x| < n, \\
u_{n,k} = m, & |x| = 2, \\
u_{n,k} = 0, & |x| = n.
\end{cases}
\]

Using Lemma 2.2 we can prove $u_{n,k-1}(x) \leq u_{n,k}(x) \leq u(x)$. In addition, by [9, 19] $u_{n,k}(x)$ is Hölder continuous. So there exists a subsequence still denoted by $u_{n,k}(x)$ such that $u_{n,k}(x) \to u_{n}(x)$ as $k \to \infty$ for some positive radial function $u_{n}(x)$ satisfying

\[
\begin{cases}
-\Delta_p u_n = |x|^\alpha u_{n}^q, & 2 < |x| < n, \\
u_n = m, & |x| = 2, \\
u_n = 0, & |x| = n.
\end{cases}
\]
un is increasing with n in each common domain, and un ≤ u(x), by the diagonal method, there exists a positive radial function v(x) such that un(x) → v(x) in |x| ≥ 2. In addition, v(x) satisfies

\[
\begin{cases}
-\Delta_p v = |x|^{\alpha} v^q, & |x| > 2, \\
v = m, & |x| = 2.
\end{cases}
\]

This is a contradiction to Theorem 4.1 which implies equation (1) has no positive radial solution in |x| ≥ 2.

5.2 Proof of Theorem 1.5

Proof. The idea comes from Lemma 1.1 in [6]. Let \( \phi_b(\xi) = b^{\frac{\alpha}{p+\alpha}} u(b\xi) \), where \( b \) is chosen such that \( |x| < b \), then \( \phi_b(\xi) \) satisfies

\[
div_\xi \left( |\nabla_\xi \phi_b|^{p-2} \nabla_\xi \phi_b \right) + |\xi|^\alpha \phi_b = 0.
\]

Let \( \Gamma = \{ 1 \leq |\xi| \leq 4 \}, \Gamma^* = \{ 2 \leq |\xi| \leq 3 \}. \) From \( |x|^{\frac{\alpha}{p+\alpha}} u(x) \leq C \), we get

\[
||\phi_b||_{L^\infty(\Gamma)} \leq b^{\frac{\alpha}{p+\alpha}} |b\xi|^{-\frac{\alpha}{p+\alpha}} \leq C,
\]

so \( ||\nabla \phi_b||_{C^0(\Gamma^*)} \leq C \), which means

\[
||\nabla u||_{C^0(\Gamma^*)} \leq C b^{\frac{\alpha}{p+\alpha}} \leq C |x|^{-\frac{\alpha}{p+\alpha}}.
\]

As we have the gradient estimate of u, we can use the same procedure as in the proof of Theorem 1.3 to prove \( u \equiv 0 \).

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