A LEFT TOPOLOGICAL MONOID ASSOCIATED TO A TOPOLOGICAL GROUPOID

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Abstract. This paper presents a functor $S$ from the category of groupoids to the category of semigroups. Indeed, a monoid $S_G$ with a right zero element is related to a topological groupoid $G$. The monoid $S_G$ is a subset of $C(G, G)$, the set of all continuous functions from $G$ to $G$, and with the compact-open topology inherited from $C(G, G)$ is a left topological monoid. The group of units of $S_G$, which is denoted by $H(1)$, is isomorphic to a subgroup of the group of all bijection maps from $G$ to $G$ under composition of functions. Moreover, it is proved that $H(1)$ is embedded in the group of all invertible linear operators on $C(G)$, the set of all complex continuous functions on $G$.

2010 Mathematics Subject Classification: 20D60, 18B40

Keywords: Topological Groupoid; Semigroup

1. Introduction

If $G$ is a topological group, then each element $x$ of $G$ defines two translation operators $L_x$ and $R_x$ on $C(G)$, the set of all complex valued continuous functions on $G$, by

$$L_x f(y) = f(x^{-1}y), \ R_x f(y) = f(yx).$$

The maps $x \mapsto L_x$ and $x \mapsto R_x$ are two monomorphisms, injective homomorphism, from $G$ into $L(C(G))$, the semigroup of all linear operators on $C(G)$ under composition of operators. In the case where $G$ is a groupoid, since $x \in G$ is not composable with each element, it is not possible to define the operators $L_x$ and $R_x$ on $C(G)$ as in the group case. Of course, $x \in G$ defines $L_x : C(G^{d(x)}) \to C(G^{r(x)})$ by $L_x f(y) = f(x^{-1}y)$ for $f \in C(G^{d(x)})$. Similarly, $x \in G$ defines $R_x : C(G^{r(x)}) \to C(G^{d(x)})$ by $R_x f(y) = f(yx)$ for $f \in C(G^{r(x)})$. It is easy to check that if $(x, y)$ is a composable pair, then $(L_xL_y)(f) := L_x(L_y f), (R_xR_y)(f) := R_x(R_y f)$ are well defined and $L_xL_y = L_{xy}, R_xR_y = R_{xy}$.

For a topological groupoid $G$, we introduce two monoids, semigroups with identity, $S_G$ and $S'_G$. Indeed, the map $G \mapsto S_G$ is a functor from the category of groupoids to the category of semigroups. The elements of $S_G$ and $S'_G$ are chosen from $C(G, G)$, the set of continuous functions from $G$ to $G$. The range and domain maps, $r$ and $d$, on the groupoid


and have a common idempotent element \(j\), the inverse map from \(G\) to \(G\), which is also a right zero for them. We will show that \(S_G\) with the compact-open topology inherited from \(C(G,G)\) is a left topological monoid. The group of units, the maximal subgroup containing the identity element, of \(S_G\) is obtained and is denoted by \(H(1)\). The group \(H(1)\) is isomorphic to a subgroup of the group of all bijection map from \(G\) to \(G\) under composition of functions. Also there exists an monomorphism \(f \mapsto L_f\) from \(S_G\) into \(\mathcal{L}(C(G))\), where \((L_f(g))(x) = g(f(x)x)\) for \(g \in C(G)\) and \(x \in G\). By this monomorphism the group \(H(1)\) is embedded in the group of all invertible linear operators on \(C(G)\). The monoid \(S_G\) has a left cancellative submonoid \(T_G\) which is embedded in the submonoid of all injective linear operators on \(C(G)\).

2. Definition and Notation

The following definition is the definition of a groupoid given by P. Hahn in [3].

**Definition 2.1.** A groupoid is a set \(G\) endowed with a product map \((x, y) \mapsto xy : G^2 \to G\) where \(G^2\) is a subset of \(G \times G\) is called the set of composable pairs, and an inverse map \(x \mapsto x^{-1} : G \to G\) such that the following relations are satisfied:

1. For every \(x \in G\), \((x^{-1})^{-1} = x\).
2. If \((x, y), (y, z) \in G^2\), then \((xy, z)\), \((x, yz) \in G^2\) and \((xy)z = x(yz)\).
3. For all \(x \in G\), \((x^{-1}, x) \in G^2\) and if \((x, y) \in G^2\), then \(x^{-1}(xy) = y\). Also for all \(x \in G\), \((x, x^{-1}) \in G^2\) and if \((z, x) \in G^2\), then \((zx)x^{-1} = z\).

The maps \(r\) and \(d\) on \(G\) defined by the formulae \(r(x) = xx^{-1}\) and \(d(x) = x^{-1}x\) are called the range and domain maps. It follows easily from the definition that they have a common image called the unit space of \(G\) which is denoted by \(\mathcal{G}^0\). The pair \((x, y)\) is composable if and only if \(r(y) = d(x)\).

It is perhaps helpful to picture a groupoid as a collection of points with various arrows connecting the points. For example, we can write \((x \xrightarrow{g} y)\) to indicate that \(g\) is an arrow with source \(x\) and target \(y\). By this notation, if \((g, h) \in G^2\), then \((x \xrightarrow{g} y \xrightarrow{h} z) \Rightarrow (x \xrightarrow{g \circ h} z)\) by the product map.

Condition (iii) implies that \(r(x)x = x, xd(x) = x\). For \(u, v \in G^0\), \(G^n = r^{-1}(u), G_v = d^{-1}(v)\) and \(G^0_v = G^n \cap G_v\). A groupoid \(G\) is called a principal groupoid if and only if the map \((r, d)\) from \(G\) into \(G^0 \times G^0\) denoted by \((r, d)(x) = (r(x), d(x))\) is one-to-one.

If \(G, H\) are two groupoids. A map \(\phi : G \to H\) is an homomorphism if and only if \((x, y) \in G^2\) implies \((\phi(x), \phi(y))\) is in \(H^2\), and in this case, \(\phi(xy) = \phi(x)\phi(y)\). Also a map \(\psi : G \to H\) is an antihomomorphism if and only if \((x, y) \in G^2\) implies \((\psi(y), \psi(x))\) is in \(H^2\), and in this case, \(\psi(xy) = \psi(y)\psi(x)\). It is easy to see that groupoid homomorphisms
and groupoid antihomomorphisms between two groupoids map units to units and inverses to inverses.

The reader interested in groupoids is referred to the books [4] and [5].

A topological groupoid consists of a groupoid $G$ and a topology compatible with the groupoid structure. That is the inverse map $x \mapsto x^{-1} : G \to G$ is continuous, as well as the product map $(x, y) \mapsto xy : G^2 \to G$ is continuous where $G^2$ has the induced topology from $G \times G$. We are concerned with topological groupoids whose topology is Hausdorff and locally compact. We call them locally compact Hausdorff groupoids.

If $S$ is a semigroup, then a nonempty subset $T$ of $S$ is called a left ideal of $S$ if $S.T \subset T$ and it is called a right ideal of $S$ if $T.S \subset T$. If $T$ is both a left ideal and a right ideal of $S$, then $T$ is called an ideal of $S$. A left ideal(right ideal, ideal) of a semigroup $S$ is said to be minimal if it properly contains no left ideal (right ideal, ideal, respectively) of $S$. An element $e$ of $S$ is said to be an idempotent if $e^2 = e = e$. An element $z$ in a Semigroup $S$ is a right zero if $s.z = z$ for all $s \in S$. By Theorem 2.8 of [1] for an idempotent $e$ in $S$, the left ideal $Se$ is minimal left ideal if and only if $eS$ is a minimal right ideal and equivalently $eSe$ is a group. An injective homomorphism between two semigroups is called a monomorphism and a semigroup with identity element is called a monoid.

3. A MONOID RELATED TO A TOPOLOGICAL GROUPOID

For a topological groupoid $G$, we denote the set of all continuous function from $G$ into itself by $C(G, G)$.

**Definition 3.1.** For a topological groupoid $G$, set

$S_G = \{ f \in C(G, G) \mid f(x) \in G_{r(x)} \text{ for all } x \in G \}$,

$S'_G = \{ f \in C(G, G) \mid f(x) \in G_{d(x)} \text{ for all } x \in G \}$.

Note that if $G$ is a topological group, then $S_G = S'_G = C(G, G)$.

**Proposition 3.2.** Two sets $S_G$ and $S'_G$ with the following binary operations are two isomorphic monoid.

$(f \ast g)(x) = g(f(x)x) f(x) \quad f, g \in S, x \in G$,

$(h \ast k)(x) = h(x)k(xh(x)) \quad h, k \in S', x \in G$.

The range and domain maps, $r$ and $d$, on the groupoid $G$ are the identity element of $S_G$ and $S'_G$, respectively.

**Proof.** Note that if $f, g \in S_G$, $h, k \in S'_G$ and $x \in G$, then $(f(x), x, h(x)) \in G^2$ and

$d\left(g(f(x)x)\right) = r(f(x)x) = r(f(x))$. 
\[ d(h(x)) = d(xh(x)) = r(k(xh(x))). \]

Hence \((g(f(x)), f(x)), (h(x), k(xh(x))) \in G^2\), consequently \(f * g\) and \(h * k\) are well-defined function from \(G\) to \(G\) and by the continuity assumption of \(f\) and \(g\), are continuous.

Also,
\[
\begin{align*}
\quad & d((f * g)(x)) = d(g(f(x))) = d(f(x)) = r(x), \\
\quad & r((h * k)(x)) = r(h(x)) = d(x).
\end{align*}
\]

Therefore \(f * g \in S_G\) and \(h * k \in S'_G\). So \(S_G\) and \(S'_G\) are closed under these binary operations.

Next we show that the binary operation of \(S_G\) is associative. The associativity of the binary operation of \(S'_G\) is similar. For \(f, g, h \in S_G\) and \(x \in G\),
\[
(f * (g * h))(x) = (g * h)(f(x))f(x) = \left[h\left(g(f(x))f(x)g(f(x))\right)\right]f(x).
\]

On the other hand
\[
(f * (g * h))(x) = \left[h\left(f(g(x))x\right)\right](f * g)(x) = \left[h\left(g(f(x))f(x)\right)\right]g(f(x))f(x).
\]

So \(f * (g * h) = (f * g) * h\).

The range map \(r(x) = xx^{-1}\) belongs to \(S_G\) and for \(g \in S_G\),
\[
\quad & (r * g)(x) = g(r(x))r(x) = g(x)r(x) = g(x)d(g(x)) = g(x), \\
\quad & (g * r)(x) = r(g(x))g(x) = r(g(x))g(x) = g(x).
\]

so \(r * g = g * r = g\), and therefore \(r\) is the identity of \(S_G\). Similarly, the domain map \(d(x) = x^{-1}x\) is the identity of \(S'_G\). Therefore \(S_G\) and \(S'_G\) are two monoids.

If for \(f \in S_G\) we define \(f^*(x) = (f(x^{-1}))^{-1}\), then \(f^*\) is continuous and for each \(x \in G\),
\[
\quad & r(f^*(x)) = r((f(x^{-1}))^{-1}) = d(f(x^{-1})) = r(x^{-1}) = d(x).
\]

Therefore \(f^* \in S'_G\). Note that \((f^*)^* = f\) for every \(f \in S_G\). We show that the map \(f \mapsto f^*\) from \(S_G\) to \(S'_G\) is a semigroup isomorphism.

The proof will be completed if we prove that \((f * g)^* = f^* * g^*\) for every \(f, g \in S_G\).

If \(f, g \in S_G\), then for each \(x \in G\),
\[
\begin{align*}
(f * g)^*(x) &= \left((f * g)(x^{-1})\right)^{-1} \\
&= \left(g(f(x^{-1})x^{-1})f(x^{-1})\right)^{-1} \\
&= (f(x^{-1}))^{-1}\left(g(f(x^{-1})x^{-1})\right)^{-1} \\
&= f^*(x)\left[g((x^*f^*)^{-1})\right]^{-1} \\
&= f^*(x)g^*(xf^*(x)) \\
&= (f^* * g^*)(x).
\end{align*}
\]

\(\square\)
Proposition 3.3. The following assertions hold for the two monoids.

1. The inverse map \( j(x) = x^{-1} \) from \( G \) to \( G \) is belong to \( S_G \cap S'_G \) and is an idempotent of these two semigroups which is also a right zero element for them.

2. The element \( j \) is left zero for \( S_G \) if and only if \( f(u) = u \) for every \( f \in S_G \) and \( u \in G^0 \). Similar assertion holds for the semigroup \( S'_G \).

3. \( S_G \cap S'_G \) is a left ideal of \( S_G \) and \( S'_G \).

4. \( jS_G \) is a minimal ideal of \( S_G \), and \( jS'_G \) is a minimal ideal for \( S'_G \).

5. If \( \phi \in S_G \) is a bijection, then \( \phi^{-1} \) is an element of \( S'_G \), where \( \phi^{-1}(\phi(x)) = x = \phi(\phi^{-1}(x)) \) for every \( x \in G \).

6. The only injective antihomomorphism element of \( S_G \) which is also an idempotent is the element \( j \).

7. The only antihomomorphism element of \( S_G \) which is also a right zero element is the element \( j \).

8. If \( f \in S_G \cap S'_G \) is an antihomomorphism, then \( f \ast f = f \ast f \).

Proof. (1) For every \( x \in G \), \( j(x) = x^{-1} \in G_{r(x)}^d \). So \( j \) is an element of \( S_G \cap S'_G \). Also

\[
(j \ast j)(x) = j(j(x)xj(x) = x^{-1}(j(x))^{-1}j(x) = x^{-1} = j(x),
\]

that is \( j \ast j = j \). Since \( j^* = j \), therefore \( j \ast j = (j \ast j)^* = j^* = j \). For \( f \in S_G \), \( g \in S_G \) and \( x \in G \),

\[
(f \ast j)(x) = j(f(x)xj(x) = x^{-1}(f(x))^{-1}f(x) = x^{-1} = j(x).
\]

Consequently \( g \ast j = (g^* \ast j)^* = j \). Hence \( j \) is a right zero for \( S_G \) and \( S'_G \).

(2) Suppose that \( j \) is a left zero for \( S_G \), for \( f \in S_G \) and \( x \in G \)

\[
x^{-1} = j(x) = (j \ast f)(x) = f(j(x)xj(x) = f(d(x))x^{-1}.
\]

Therefore \( f(d(x)) = d(x) \) for every \( x \in G \). Conversely, if \( f(u) = u \) for every \( u \in G^0 \), then for every \( x \in G \)

\[
(j \ast f)(x) = f(j(x)xj(x) = f(d(x))x^{-1} = d(x)x^{-1} = x^{-1} = j(x).
\]

(3) Let \( g \in S_G \cap S'_G \) and \( f \in S_G \), then

\[
r f(g(x)) = r(g(f(x)x)) = r(g^d(f(x))) = d(f(x)) = d(x)
\]

and

\[
d f(g(x)) = d(g(f(x)x)) = d(f(x)) = r(x).
\]

So \( f \ast g \in S_G \cap S'_G \). Similarly, we can show that \( S_G \cap S'_G \) is a left ideal of \( S'_G \).
Remark 3.4. We consider \( jS_G \) is a right ideal of \( S_G \) and since \( j \) is a right zero for \( S_G \), \( jS_G \) is also a left ideal of \( S_G \). Now let \( f \in S_G \). If \( S_G(j \ast f)S_G = jS_G \), then by Proposition 2.4 of [1], \( jS_G \) is a minimal ideal of \( S_G \). However, since \( j \) is a right zero for \( S_G \), \( S_G(j \ast f)S_G = (j \ast f)S_G \subset jS_G \). If \( g \in S_G \), then \( j \ast g = j \ast (j \ast f) \ast j \ast g \in S_G(j \ast f)S_G \). So \( jS_G \subset S_G(j \ast f)S_G \).

(5) Let \( \psi \) be the inverse of \( \phi \), \( y \in G \) and \( \psi(y) = x \). Then \( r(\psi(y)) = r(\psi(\phi(x))) = r(x) = d(\phi(x)) = d(y) \), that is \( \psi \in S'_G \).

(6) Let \( \phi \) be an idempotent which is also an injection antihomomorphism element of \( S_G \). Then \( \phi \ast \phi = \phi \), that is, \( \phi(\phi(x))\phi(x) = \phi(x) \) for all \( x \in G \), so \( \phi(\phi(x)x) = r(\phi(x)) = \phi(d(x)) \). The injectivity of \( \phi \) implies that \( \phi(x)x = d(x) \), so \( \phi(x) = x^{-1} = j(x) \).

(7) By definition of a right zero element in a semigroup, \( \phi \ast \psi = \psi \) for all \( \phi \in S_G \). Therefore in a special case \( j \ast \psi = \psi \) and consequently \( \psi(j(x)x)j(x) = \psi(x) \). Hence \( j(x) = [\psi(j(x)x)]^{-1} \psi(x) = [\psi(d(x))]^{-1} \psi(x) = [r(\psi(x))]^{-1} \psi(x) = r(\psi(x)) \psi(x) = \psi(x) \).

(8) By the definition of the binary operations of \( S_G, S'_G \), it is straightforward.

\[ \square \]

Remark 3.4. We consider \( C(G,G) \) with the compact-open topology. Recalling that the compact-open topology is the topology generated by the base consisting of all sets \( \cap_{i=1}^{k} M(C_i, U_i) \), where \( C_i \) is a compact subset of \( G \) and \( U_i \) is an open subsets of \( G \) for \( i = 1, 2, \ldots, k \) and where, \( M(A,B) = \{ f \in C(G,G) : f(A) \subset B \} \) for \( A, B \subset G \). The reader is referred to [2] for more details about this topology.

In the following we will show that \( S_G \) with the compact-open topology inherited from \( C(G,G) \) is a left topological monoid. It is easy to check that the isomorphism \( f \mapsto f^* \) is continuous from \( S_G \) into \( S'_G \), with compact-open topology, so \( S'_G \) is left topological.

Proposition 3.5. The monoid \( S_G \) with the compact-open topology is a left topological semigroup.

Proof. Let \( f \in S_G \) and let \( \{ g_\alpha \}_{\alpha \in \Sigma} \) be a net in \( S_G \) converging to \( g \in S_G \) in compact-open topology. We will show that \( f \ast g_\alpha \to f \ast g \) in compact-open topology. Suppose that \( \cap_{i=1}^{k} M(C_i, U_i) \) is a neighborhood of \( f \ast g \) in compact-open topology. So \( g(f(x)x)f(x) \in U_i \) for every \( x \in C_i \) and \( i = 1, 2, \ldots, k \). Since \( G \) is a topological groupoid, for \( x \in C_i \), there exists tow open sets \( U'_x \) and \( V'_x \) with \( g(f(x)x) \in U'_x \), \( f(x) \in V'_x \) and \( U'_x V'_x \subset U_i \). Let \( U_x, V_x \) be two open set in \( G \) with \( g(f(x)x) \in U_x \), \( f(x) \in V_x \) and \( \overline{U_x} \subset U'_x \) and \( \overline{V_x} \subset V'_x \). The set \( \{ (g(f(x)x), f(x)) : x \in C_i \} \) is a compact subset of \( G^2 \), hence there exist \( x_1, x_2, \ldots, x_n \),
in $C_i$ such that
\[
\left\{ \left( g(f(x)x), f(x) \right) : x \in C_i \right\} \subset \bigcup_{j=1}^{n_i} \left( U_{x_j} \times V_{x_j} \right) \cap G^2.
\]
So
\[
\left\{ g(f(x)x)f(x) : x \in C_i \right\} \subset \bigcup_{j=1}^{n_i} U_{x_j}V_{x_j} \subset U_i.
\]

Put $F_j = \{ f(x)x : x \in C_i \}$ and $g(f(x)x) \in U_{x_j}$, then $F_j$ is a compact set for $j = 1, 2, \ldots, n_i$ and $g(F_j) \subset U'_{x_j}$. So $\bigcap_{j=1}^{n_i} \{ F_j \} \subset U'_{x_j}$ is a neighborhood of $g$ in compact-open topology. Therefore there exists $\beta \in \Sigma$ with $g_{\alpha}(F_j) \subset U'_{x_j}$ for every $\alpha \geq \beta$ and $j = 1, 2, \ldots, n_i$ and $i = 1, 2, \ldots, k$. Now if $x \in C_i$, then there exists $j \in \{1, 2, \ldots, n_i\}$ with $f(x)x \in F_j$, then $f(x) \in V'_{x_j}$. Therefore $(f * g_{\beta})(x) = g_{\alpha}(f(x)x)f(x) \in g_{\alpha}(F_j)f(x) \subset U'_{x_j} \subset U_i$ for every $\alpha \geq \beta$, that is $(f * g_{\beta})(C_i) \subset (f * g)(U_i)$ for $\alpha \geq \beta$, and $i = 1, 2, \ldots, k$. Therefore $f * g_{\alpha} \to f * g$ in compact-open topology. \qed

**Proposition 3.6.** The map $G \mapsto S_G$ from the category of groupoids to the category of semigroups is a functor. Therefore if $G$ and $H$ are two isomorphic groupoids, then $S_G$ and $S_H$ are two isomorphic monoids.

**Proof.** Suppose that $\psi : G \to H$ is a groupoid homomorphism. Define $\Psi : S_G \to S_H$ by $(\Psi(f))(x) = (\psi \psi^{-1})(x)$. We have
\[
\begin{align*}
d\left( \Psi(f)(x) \right) &= d\left( \psi\left( f(\psi^{-1}(x)) \right) \right) \\
&= \psi d\left( f(\psi^{-1}(x)) \right) \\
&= \psi r(\psi^{-1}(x)) \\
&= \psi (\psi^{-1}(r(x))) \\
&= r(x).
\end{align*}
\]
Therefore $\Psi(f) \in S_H$. We will show that the map $\Psi$ is a semigroup homomorphism. Let $f, g \in S_G$ and $x \in H$,
\[
\begin{align*}
(\Psi(f * g))(x) &= \psi\left( f * g(\psi^{-1}(x)) \right) \\
&= \psi g\left( f(\psi^{-1}(x))\psi^{-1}(x) \right) f(\psi^{-1}(x)) \\
&= \psi g\left( f(\psi^{-1}(x))\psi^{-1}(x) \right) \psi f(\psi^{-1}(x)) \\
&= \psi g\left( \psi^{-1}\left( \psi f(\psi^{-1}(x)) \right) \psi^{-1}(x) \right) f(\psi^{-1}(x)) \\
&= \left( \psi g(\psi^{-1}) \right) \left( \psi f(\psi^{-1})(x)x \right) \left( \psi f(\psi^{-1})(x) \right) \\
&= \Psi(g) \left( \Psi(f)(x) \right) (\Psi(f))(x) \\
&= (\Psi(f) * \Psi(g))(x).
\end{align*}
\]
So the map $f \mapsto \Psi(f)$ is a semigroup homomorphism. It is obvious that if $id_G : G \to G$ is the identity, then $\Psi(id_G) : S_G \to S_G$ is the identity. So the proof is completed. \qed
Lemma 3.7. The monoid $S_G$ is isomorphic to a submonoid of the semigroup $C(G,G)$ under the binary operation $(f \circ g)(x) = g(f(x))$ for $f, g \in C(G,G)$ and $x \in G$.

Proof. For $\phi \in S_G$ define, $L_\phi : G \rightarrow G$ by $L_\phi(x) = \phi(x)x$. Then

$L_{\phi \circ \psi}(x) = (\phi \ast \psi)(x)x = \psi(\phi(x)x)\phi(x)x = L_\psi(\phi(x)x) = L_\psi(L_\phi(x))$.

That is, the map $\phi \mapsto L_\phi$ is an homomorphism from $S_G$ into $C(G,G)$. It is obvious that this homomorphism is injective. So $S_G$ is isomorphic to the subsemigroup $\{L_\phi : \phi \in S_G\}$ of the semigroup $(C(G,G), \circ)$.

Proposition 3.8. For $\phi \in S_G$ denote the map $x \mapsto \phi(x)x$ by $L_\phi$, then the set

$H(1) = \{\phi \in S_G : \text{the map } L_\phi \text{ is a bijection}\}$

is the group of units of $S_G$, the maximal subgroup of $S_G$ which containing the identity element $r$.

Proof. Obviously the map $r$ which is the identity of $S_G$ belong to $H(1)$. Also by the previous Lemma the map $\phi \mapsto L_\phi$ from the monoid $(S_G, \ast)$ to the monoid $(C(G,G), \circ)$ is a homomorphism. Since the composition of two bijection map is bijective, $H(1)$ is a submonoid of $S_G$. Now let $\phi \in H(1)$ and define $\psi(y) = (\phi(x))^{-1}$, where $y = L_\phi(x) = \phi(x)x$, $\psi$ is well-defined. We will show that $\psi$ is the inverse of $\phi$, that is $\psi \ast \phi = \phi \ast \psi = r$. By definition of $\psi$, $(\phi \ast \psi)(x) = \psi(\phi(x)x)\phi(x) = (\phi(x))^{-1}\phi(x) = d(\phi(x)) = r(x)$. To prove that $\psi \ast \phi = r$, note that $y = \phi(x)x$ implies that $\psi(y)y = \psi(y)\phi(x)x = \psi(\phi(x)x)\phi(x)x = (\phi \ast \psi)(x)x = r(x)x = x$. Therefore $(\psi \ast \phi)(y) = \phi(\psi(y)y)\psi(y) = (\psi(y))^{-1}\psi(y) = d(\psi(y)) = r(y)$ for every $y \in G$. To complete the proof we just need to show that $H(1)$ is maximal. Let $K$ be a subgroup of $S_G$ which containing $r$. To prove that $K \subset H(1)$, it is enough to show that if $\phi \in K$, then the map $L_\phi$ is a bijection. Suppose that $\psi$ is the inverse of $\phi$, then $\phi \ast \psi = \psi \ast \phi = r$. Therefore $L_\phi \circ L_\phi = L_\phi \circ L_\psi = h_x = I$ the identity map on $G$. So $L_\phi$ is a bijection.

For $\phi \in S_G$, the set of all fixed point of $\phi$ is denoted by $\text{Fix}(\phi)$.

Proposition 3.9. Let $A$ be a subset of $G$ and put $I_A = \{\phi \in C(G,G) : \phi(A) \subset A\}$, then the following assertion hold,

If $A$ is a subgroupoid of $G$, then $S_A = I_A \cap S_G$ is a subsemigroup of $S_G$.

Proof. It is easy to see that for a subgroupoid $A$ of $G$, $\phi \in I_A$ if and only if $L_\phi \in I_A$. Now let $\phi, \psi \in S_A$, then by Lemma 3.7, $L_{\phi \ast \psi}(A) = L_\psi(L_\phi(A)) \subset L_\psi(A) \subset A$. So $S_A$ is a subsemigroup of $S_G$.

Remark 3.10. If $\phi : G \rightarrow H$ is a groupoid homomorphism, then $d(\phi(x)) = \phi(d(x))$ and $r(\phi(x)) = \phi(r(x))$ for all $x \in G$. Note that the conditions $\phi \circ r = r \circ \phi$ and $\phi \circ d = d \circ \phi$ does
not imply that \( \phi \) is a homomorphism. For example in the case where \( G \) and \( H \) are two groups, every function \( \phi: G \to H \) which preserves the identity element, satisfies in the two conditions. Similarly \( d(\psi(x)) = \psi(r(x)) \) and \( r(\psi(x)) = \psi(d(x)) \) for all \( x \in G \) does not imply that \( \psi \) is an antihomomorphism. Now let \( \phi \in C(G,G) \) and \( \text{Fix}(\phi) \) be the fixed-point set of \( \phi \). If \( \psi \in S_G \) with \( d(\psi(x)) = \psi(r(x)) \) for all \( x \in G \), then \( \psi(r(x)) = d(\psi(x)) = r(x) \) and therefore \( G^0 \subset \text{Fix}(\psi) \). Conversely, if \( \psi \circ r = d \circ \psi \) and \( G^0 \subset \text{Fix}(\psi) \), then \( \psi \in S_G \).

Therefore for an element \( \psi \) of \( C(G,G) \) with \( \psi \circ r = d \circ \psi \), we have \( \psi \in S_G \) if and only if \( G^0 \subset \text{Fix}(\psi) \).

In a special case if \( \phi \) is a continuous antihomomorphism, then \( \phi \in S_G \) if and only if \( G^0 \subset \text{Fix}(\phi) \). In the following we obtain this result when the condition \( G^0 \subset \text{Fix}(\phi) \) is replaced by \( \phi(G^u) \cap G_u \neq \emptyset \) for every \( u \in G^0 \), where \( \phi(G^u) \) is the image of \( G^u \) under the map \( \phi \).

**Proposition 3.11.** Let \( \phi \in C(G,G) \) and \( d \circ \phi = \phi \circ r \), then \( \phi \in S_G \) if and only if \( \phi(G^u) \cap G_u \neq \emptyset \) for all \( u \in G^0 \).

**Proof.** Suppose that \( \phi(G^u) \cap G_u \neq \emptyset \) for all \( u \in G^0 \). Therefore for \( z \in G \) there exists \( x \in G^r(z) \) with \( \phi(x) \in G^r(\phi(z)) \). So

\[
\begin{align*}
d(\phi(z)) &= \phi(r(z)) \\
          &= \phi(r(x)) \\
          &= d(\phi(x)) \\
          &= r(z).
\end{align*}
\]

That is, \( \phi \in S_G \). The converse is hold, since \( d(\phi(x)) = r(x) \) and \( x \in G^r(\phi(z)) \), that is \( \phi(G^r(x)) \cap G_d(x) \neq \emptyset \) for every \( x \in G \).

For \( \phi \in S_G \) and \( \psi \in S'_G \), define \( L_\phi(x) = \phi(x)x \) and \( R_\psi(x) = x\psi(x) \).

**Proposition 3.12.** Let \( S_G \) and \( S'_G \) be the monoids which are defined in Definition 3.1. Set

\[
\begin{align*}
T_G &= \{ \phi \in S_G : \{ L_\phi(x) : x \in G \} \text{ is dense in } G \}, \\
T'_G &= \{ \psi \in S'_G : \{ R_\psi(x) : x \in G \} \text{ is dense in } G \}.
\end{align*}
\]

Then \( T_G \) is a left cancellative submonoid of \( S_G \), \( T'_G \) is a left cancellative submonoid of \( S'_G \), and \( T_G \) is isomorphic to \( T'_G \).

**Proof.** It is obvious that \( r \in T_G \) and \( d \in T'_G \). Suppose that \( \phi, \psi \in T_G \). Since the map \( L_\phi \) is continuous by using the density of the sets \( \{ L_\phi(x) : x \in G \} \) and \( \{ L_\psi(x) : x \in G \} \) in \( G \), we obtain that the set \( \{ L_\phi(L_\phi(x)) : x \in G \} \), which is equal to \( \{ L_{\phi \psi}(x) : x \in G \} \), is dense in \( G \). So \( T_G \) is a subsemigroup of \( S_G \). If \( f, g, h \in S_G \) and \( f \star g = f \star h \). Therefore
Proposition 4.1. There exists a map \( \mathcal{C} \) of \( L_g(L_f(x)) = L_h(L_f(x)) \) for every \( x \in G \). The density of \( \{L_f(x) : x \in G\} \) in \( G \) and the continuity of \( L_g \) and \( L_h \) imply that \( L_h = L_g \) and therefore \( g = h \).

Similarly, we can show that \( T_G^G \) is a left cancellative subsemigroup of \( S_G^G \). To prove that \( T_G \) and \( T_G^G \) are isomorphic, using the proof of Proposition 3.2, it is enough to show that if \( f \in T_G \), then \( f^* \in T_G^G \). Let \( f \in T_G \), then

\[
\{f(x) : x \in G\}^{-1} = \{x^{-1}(f(x))^{-1} : x \in G\} = \{f(t^{-1})^{-1} : t \in G\} = \{tf'(t) : t \in G\}.
\]

The continuity of the inverse map from \( G \) to \( G \) implies that \( A \subset G \) is dense in \( G \) if and only if \( A^{-1} \) is dense in \( G \). Therefore \( f^* \in T_G^G \). \( \square \)

Example 3.13. (Transformation group groupoids [4, p.6]). Suppose that the group \( T \) acts on the space \( U \) on the right. The image of the point \( u \in U \) by the transformation \( t \in T \) denoted by \( u.t \). The set \( G = U \times T \) is a groupoid with the following groupoid structure: \( (u,t),(v,t') \) is composable if and only if \( v = u.t,(u,t)(u,t') = (u,tt') \) and \( (u,t)^{-1} = (u.t^{-1}) \). Then \( r(u,t) = (u,e) \) and \( d(u,t) = (u.e) \). If a locally compact group \( T \) acts on a locally compact space \( U \) then the transformation group groupoid \( U \times T \) with the product topology is a locally compact groupoid. Now recall that \( S_T = C(T,T) \), since \( T \) is a group. Let \( \varphi \in S_T \), define \( f_\varphi : G \rightarrow G \) by \( f_\varphi(u,t) = \left(u,\varphi(t)^{-1},\varphi(t)\right) \). Then it is easy to check that \( f_\varphi \) is an element of \( S_G \) and it is straightforward to check that \( f_\phi \circ f_\psi = f_{\phi \circ \psi} \) for every \( \phi, \psi \in S_T \), and the map \( \phi \mapsto f_\phi \) is injective, that is the monoid \( S_T \) is algebraically isomorphic to a submonoid of \( S_G \). Also it is easy to check that the group of units of \( S_T \) is embedded in the group of units of \( S_G \) by this monomorphism.

Now for \( z \in T \) define \( f_z(u,t) = (u,z^{-1},z) \) for all \( (u,t) \in G \). We have \( f_{za} \circ f_z = f_{za} \). Therefore the set \( \{f_z : z \in T\} \) with the pointwise topology is a subgroup of \( S_G \) which is topologically isomorphic to \( T \). In a special case if we let \( s \in T \) and define \( \varphi(t) = ts^{-1}t \) for every \( t \in T \), then \( f_{\varphi} \) by \( f_{\varphi}(u,t) = (u.t^{-1}st^{-1},ts^{-1}t) \) is an element of \( T_G \).

4. A REPRESENTATION OF THE ELEMENTS OF \( S_G \) AS LINEAR OPERATORS ON \( C(G) \)

In the following we will show that every \( f \in S_G \) is represented by a linear operator \( L_f \) on \( C(G) \). Also the map \( \Phi : S_G \rightarrow \mathcal{L}(C(G)) \) by \( \Phi(f) = L_f \) is a monomorphism, where \( \mathcal{L}(C(G)) \) is the monoid of all linear operators on \( C(G) \) under composition of operators. Moreover the group of units of \( S_G \) is embedded in the group of all invertible linear operators on \( C(G) \).

Proposition 4.1. There exists a map \( \phi : S_G \times C(G) \rightarrow C(G) \) with the following properties.

1. \( \phi(f_1 \circ f_2, g) = \phi(f_1, \phi(f_2, g)) \) for all \( g \in C(G) \), \( f_1, f_2 \in S_G \),
2. \( \phi(r, g) = g \) for all \( g \in C(G) \), where \( r(x) = xx^{-1} \).
Proposition 4.2. There exists a map \( f \) and therefore the map \( f \) ties.

Proof. For \( f \in S_G \) we have a function from \( G \) to \( G \), the map \( f \) is well defined and belongs to \( C(G) \). Therefore we have a function from \( S \times C(G) \) to \( C(G) \). Now let \( f_1, f_2 \in S_G \) and \( g \in C(G) \), then

\[
(g \circ L_{f_1} \circ f_2) = g \circ (L_{f_2} \circ L_{f_1}) = (g \circ L_{f_2}) \circ L_{f_1},
\]

and therefore the map \( f \mapsto g \) from \( S_G \) to \( L(C(G)) \) is an homomorphism. This complete the proof of part 1).

The proof of (2) is straightforward, since \( L_r = I \), the identity map on \( G \).

The proof of part (3) is obvious, since the topology of \( G \) is Hausdorff. For part (4), let \( f \in T_G \), \( g, h \in C(G) \) and \( L_f(g) = L_f(h) \). Therefore \( g(L_f(x)) = h(L_f(x)) \) for every \( x \in G \). The density of the set \( \{L_f(x) : x \in G\} \) in \( G \) and the continuity of \( g \) and \( h \) imply that \( g = h \). Finally (5) is proved by the part (2) and (3).

\[\square\]

There is a similar assertion on the monoid \( S_G' \), so we delete it’s proof.

Proposition 4.2. There exists a map \( \psi : C(G) \times S_G' \to C(G) \) with the following properties.

1. \( \psi(g, f_1 \ast f_2) = \psi(g, f_2, f_1) \) for all \( g \in C(G) \) and all \( f_1, f_2 \in S_G' \),
2. \( \psi(g, d) = g \) for all \( g \in C(G) \), where \( d(x) = x^{-1}x \).
3. For \( f \in S_G' \) the map \( R_f = \phi(., f) : C(G) \to C(G) \) is a linear operator, and the map \( f \mapsto R_f \) from \( S_G' \) to \( L(C(G)) \) is an injective homomorphism.
4. If \( f \in T_G' \), the map \( R_f \) is an injective linear operator.
5. The group \( H'(1), \) group of units of \( S_G' \), is embedded in the group of all invertible linear operators on \( C(G) \), under the composition of operators.

From (i) and (ii) we can say that the semigroup \( S_G' \) acts on \( C(G) \) on the right.

Corollary 4.3. The semigroup \( S_G \) acts on \( C(G) \) on the right. Similarly, the semigroup \( S_G' \) acts on \( C(G) \) on the left.

Proof. By application of the semigroup isomorphism \( f \mapsto f^* \) form \( G \) to \( S_G' \) and proposition 4.1, 4.2, it is obvious.

\[\square\]
It is easy to check that, if $G$ is a principal groupoid, then $S \cap S' = \{j\}$. The converse is probably true, but I don’t have a correct proof.

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