BIFURCATION FROM INFINITY FOR AN ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the asymptotically linear Schrödinger equation (1.1) and show that if \( \lambda_0 \) is an isolated eigenvalue for the linearization at infinity, then under some additional conditions there exists a sequence \((u_n, \lambda_n)\) of solutions such that \( \|u_n\| \to \infty \) and \( \lambda_n \to \lambda_0 \). Our results extend those by Stuart [21]. We use degree theory if the multiplicity of \( \lambda_0 \) is odd and Morse theory (or more specifically, Gromoll-Meyer theory) if it is not.

1. Introduction

In this paper we consider the Schrödinger equation

\[
-\Delta u + V(x)u = \lambda u + f(x, u), \quad x \in \mathbb{R}^N,
\]

where \( \lambda \) is a real parameter, \( V \in L^\infty(\mathbb{R}^N) \), \( f(x,u)/u \to m(x) \) as \( |u| \to \infty \), \( m \in L^\infty(\mathbb{R}^N) \) and \( \lambda_0 \) is an isolated eigenvalue of finite multiplicity for \( L := -\Delta + V(x) - m(x) \). \( L \) will be considered as an operator in \( L^2(\mathbb{R}^N) \). It is well known (see e.g. [18]) that \( L \) is selfadjoint and its domain \( D(L) \) is the Sobolev space \( H^2(\mathbb{R}^N) \). We shall show that if the distance from \( \lambda_0 \) to the essential spectrum \( \sigma_e(L) \) of \( L \) is larger than the Lipschitz constant of \( f - m \) (with respect to the \( u \)-variable), then there exists a sequence of solutions \((u_n, \lambda_n) \subset H^2(\mathbb{R}^N) \times \mathbb{R} \) such that \( \|u_n\| \to \infty \) and \( \lambda_n \to \lambda_0 \). See Theorems 1.3 and 1.4 for more precise statements. We shall say that these solutions bifurcate from infinity or that \( \lambda_0 \) is an asymptotic bifurcation point. Our results extend those by Stuart [21] who has shown using degree theory that if \( f(x,u) = f(u) + h(x) \), then asymptotic bifurcation occurs if \( \lambda_0 \) is of odd multiplicity and the bifurcating set contains a continuum.

Both here and in [21] (see also [20]) the result is first formulated in terms of an abstract operator equation. Let \( E \) be a Hilbert space, \( L : D(L) \to E \) a selfadjoint linear operator and let \( N : E \to E \) be a continuous nonlinear operator which is asymptotically linear in the sense of Hadamard (\( H \)-asymptotically linear for short, see Definition 2.1(i)). We show that if \( \lambda_0 \) is an isolated eigenvalue of odd multiplicity for \( L \) and if the distance \( \text{dist}(\lambda_0, \sigma_e(L)) \) from \( \lambda_0 \) to the essential spectrum of \( L \) is larger than the asymptotic Lipschitz constant of \( N \) (introduced in Definition 2.1(ii)), then \( \lambda_0 \) is an asymptotic bifurcation point for the equation

\[
Lu = \lambda u + N(u), \quad u \in D(L).
\]

Here we have assumed for notational simplicity that the asymptotic derivative \( N'(\infty) \) of \( N \) is 0, see Theorem 1.1 for the full statement. This theorem slightly extends some results in [20, 21].
where the distance condition on $\lambda_0$ was somewhat stronger. If $N$ is the gradient of a $C^1$-functional and $\lambda_0$ is an isolated eigenvalue of finite (not necessarily odd) multiplicity, we show that under an additional hypothesis $\lambda_0$ is an asymptotic bifurcation point for (1.2). The exact statement is given in Theorem 1.2. Existence of asymptotic bifurcation when the multiplicity of $\lambda_0$ is even seems to be new and is the main abstract result of this paper. A related problem was then shown that each eigenvalue 1/$\lambda_0$ of $A$ with $|\lambda_0k| < 1$ is an asymptotic bifurcation point. However, the arguments there seem to break down in our case.

The proofs in [20, 21] were effected by first making the inversion $u \mapsto u/\|u\|^2$ (an idea that goes back to Rabinowitz [16] and Toland [22]). In this way the problem is transformed to that of looking for bifurcation from 0 instead of infinity. In the next step a finite-dimensional reduction is performed and finally it is shown that since $\lambda_0$ has odd multiplicity, the Brouwer degree for the linearization of the reduced operator at $u = 0$ changes as $\lambda$ passes through $\lambda_0$. This forces bifurcation, and an additional argument which goes back to [15] and uses degree theory in an essential way, shows that there is a continuum bifurcating from $(0, \lambda_0)$. Since the degree does not change if the multiplicity of $\lambda_0$ is even, in Theorem 1.2 we use Morse theory instead, and therefore we need the assumption that $N$ is the gradient of a functional. Morse theory can only assert that there exists a sequence, and not necessarily a continuum, bifurcating from infinity. Let us also point out that in [20] a more general operator equation of the form $F(\lambda, u) = 0$ has been considered (2

The fact that $\text{dist}(\lambda_0, \sigma_e(L))$ is larger than the Lipschitz constant of $N$ at infinity is needed in order to perform a finite-dimensional reduction of Liapunov-Schmidt type. As we shall see, if the distance condition is satisfied, then one can find an orthogonal decomposition $E = Z \oplus W$, where $\dim Z < \infty$, such that writing $u = z + w \in Z \oplus W$, it is possible to use the contraction mapping principle in order to express $w$ as a function of $z$ and $\lambda$. Although one may think this is only a technical condition, it has been shown by Stuart [21] that there exist examples where asymptotic bifurcation does not occur at eigenvalues of odd multiplicity (and in Section 5.3 there one finds an example where asymptotic bifurcation occurs when $\lambda_0$ is not an eigenvalue). So the above condition, or some other, is needed.

The reason for requiring $N$ to be $H$-asymptotically and not just asymptotically linear (in the sense of Fréchet) is that, in contrast to the situation when (1.1) is considered for $x$ in a bounded domain, we cannot expect the Nemyskii operator $N$ induced by $f$ to be asymptotically linear. Indeed, it has been shown in [19] that if $f(u)/u \to m$ as $|u| \to \infty$, then $N$ is always $H$-asymptotically linear, and it is asymptotically linear if and only if $f(u) = mu$. In the proof of Theorem 1.3 we show that also the Nemyskii operator corresponding to $f(x, u)$ is $H$-asymptotically linear if $f(x, u)/u \to m(x)$ as $|u| \to \infty$. The related concept of $H$-differentiability in the context of elliptic equations in $\mathbb{R}^N$ has been introduced in a series of papers by Evéquoz and Stuart, see e.g. [7].
Now we can state our main results. The symbols $N'(\infty)$ and $\text{Lip}_\infty$ (denoting asymptotic $H$-derivative and asymptotic Lipschitz constant) which appear below are introduced in Definition 2.1

**Theorem 1.1.** Let $E$ be a Hilbert space and suppose that $L : D(L) \to E$ is a selfadjoint linear operator. Suppose further that

(i) $N$ is $H$-asymptotically linear and $N'(\infty) : E \to E$ is selfadjoint,

(ii) $\lambda_0$ is an isolated eigenvalue of odd multiplicity for $L - N'(\infty)$ and

$$\text{Lip}_\infty(N - N'(\infty)) < \text{dist}(\lambda_0, \sigma_e(L - N'(\infty))).$$

Then $\lambda_0$ is an asymptotic bifurcation point for equation (1.2). Moreover, there exists a continuum bifurcating from infinity at $\lambda_0$.

By a continuum bifurcating from infinity at $\lambda_0$ we mean a closed connected set $\Gamma \subset E \times \mathbb{R}$ of solutions of (1.2) which contains a sequence $(u_n, \lambda_n)$ such that $\|u_n\| \to \infty$, $\lambda_n \to \lambda_0$. This theorem should be compared with Theorem 4.2 and Corollary 4.3 in [21] (see also Theorem 6.3 in [20]) where the distance condition was somewhat stronger than in (ii) above. The main ingredient in the proof is a finite-dimensional reduction which roughly speaking goes as follows. Let $W$ be an $L$-invariant subspace of $E$ such that $\text{codim} W < \infty$ and $Z := W^\perp \subset D(L)$. Let $P : E \to W$ be the orthogonal projection and write $w = Pu$, $z = (I - P)u$. Then (1.2) is equivalent to the system

$$Lw - \lambda w = PN(w + z),$$

$$Lz - \lambda z = (I - P)N(w + z).$$

Choosing an appropriate $W$, $\delta > 0$ small enough and $R > 0$ large enough, one can uniquely for $w$ in the first equation provided $|\lambda - \lambda_0| \leq \delta$ and $\|z\| \geq R$. In this way we obtain $w = w(\lambda, z)$ which inserted in the second equation gives a (finite-dimensional) problem on $Z \setminus B_R(0)$. See Proposition 3.4 for more details. Now the proof of Theorem 1.1 is completed by a well-known argument using Brouwer’s degree.

If $N$ is a potential operator, then the reduced problem has variational structure. More precisely, suppose $N(u) = \nabla \psi(u)$ for some $\psi \in C^1(E, \mathbb{R})$ and let $\Phi_\lambda(u) := \frac{1}{2} \langle Lu - \lambda u, u \rangle - \psi(u)$. Then the functional $\varphi_\lambda$ given by $\varphi_\lambda(z) = \Phi_\lambda(w(\lambda, z) + z)$ is of class $C^1$ and $z \in Z \setminus B_R(0)$ is a critical point of $\varphi_\lambda$ if and only if $u = w(\lambda, z) + z$ is a solution of (1.2), see Proposition 3.6. Recall that a functional $\varphi$ is said to satisfy the Palais-Smale condition ((PS) for short) if each sequence $(z_n)$ such that $\varphi(z_n)$ is bounded and $\varphi'(z_n) \to 0$ contains a convergent subsequence.

**Theorem 1.2.** Let $E$ be a Hilbert space and suppose that $L : D(L) \to E$ is a selfadjoint linear operator. Suppose further that

(i) $N$ is a potential operator, i.e. there exists a functional $\psi \in C^1(E, \mathbb{R})$ such that $\nabla \psi(u) = N(u)$ for all $u \in E$,

(ii) $N$ is $H$-asymptotically linear and $N'(\infty) : E \to E$ is selfadjoint,

(iii) $\lambda_0$ is an isolated eigenvalue of finite multiplicity for $L - N'(\infty)$ and

$$\text{Lip}_\infty(N - N'(\infty)) < \text{dist}(\lambda_0, \sigma_e(L - N'(\infty))).$$

If $\varphi_{\lambda_0}$ satisfies (PS), then $\lambda_0$ is an asymptotic bifurcation point for equation (1.2).
Note that here we do not assume \( \lambda_0 \) is of odd multiplicity. In Theorem 1.4 below we shall give sufficient conditions for \( f \) in order that such \( \lambda_0 \) be an asymptotic bifurcation point for (1.1).

To formulate our results for equation (1.1) we introduce the following assumptions on \( f \):

1. \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) satisfies the Carathéodory condition, i.e., it is continuous in \( s \) for almost all \( x \in \mathbb{R}^N \) and measurable in \( x \) for all \( s \in \mathbb{R} \), and there exist \( \alpha \in L^2(\mathbb{R}^N), \beta \in \mathbb{R}^+ \) such that \( |f(x,s)| \leq \alpha(x) + \beta|s| \) for all \( x \in \mathbb{R}^N, s \in \mathbb{R} \);
2. \( f \) is Lipschitz continuous in the second variable, with Lipschitz constant \( \text{Lip}(f) := \inf \{ C : |f(x,s) - f(x,t)| \leq C|s-t| \text{ for all } x \in \mathbb{R}^N, s, t \in \mathbb{R} \} \);
3. \( g(x,s) := f(x,s) - f(x,0) \) is bounded by a constant independent of \( x \in \mathbb{R}^N \) and \( s \in \mathbb{R} \);
4. Assume the limits \( g_{\pm}(x) := \lim_{s \to \pm \infty} g(x,s) \) exist and either \( \pm g_{\pm} \geq 0 \) a.e. or \( \pm g_{\pm} \leq 0 \) a.e.
   In addition, there exists a set of positive measure on which none of \( g_{\pm} \) vanishes;
5. Assume the limits \( h_{\pm}(x) := \lim_{s \to \pm \infty} g(x,s) \) exist, \( h_{\pm} \in L^\infty(\mathbb{R}^N) \) and either \( g(x,s) \geq 0 \) or \( g(x,s) \leq 0 \) for all \( x \in \mathbb{R}^N, s \in \mathbb{R} \). In addition, there exists a set of positive measure on which none of \( h_{\pm} \) vanishes.

Note that if \( f(x,s) = \alpha(x) + f_0(s) \) and \( |f_0(s)| \leq \beta|s| \), where \( \alpha \in L^2(\mathbb{R}^N), \beta > 0 \) and \( f_0 \) is continuous, then \( f \) satisfies (f1). As we have already mentioned, such functions \( f \) have been considered in [21].

**Theorem 1.3.** Suppose that \( V \in L^\infty(\mathbb{R}^N) \) and \( f \) satisfies (f1)-(f3). Let \( g(x,s) := f(x,s) - m(x)s \).
If \( \lambda_0 \) is an isolated eigenvalue of odd multiplicity for \( -\Delta + V - m \) and \( \text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m)) \), then \( \lambda_0 \) is an asymptotic bifurcation point for equation (1.1). Moreover, there exists a continuum bifurcating from infinity at \( \lambda_0 \).

This strengthens some of the results of [21, Theorem 5.2]. Using examples in [21, Theorems 5.4, 5.6] and the remarks following them we shall show in Remark 5.1 that the condition on \( \text{Lip}(g) \) above is sharp in the sense that if \( \text{Lip}(g) > \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m)) \), then there may be no bifurcation at a simple eigenvalue.

**Theorem 1.4.** Suppose that \( V \in L^\infty(\mathbb{R}^N) \) and \( f \) satisfies (f1)-(f4) and either (f5) or (f6). If \( \lambda_0 \) is an isolated eigenvalue of finite multiplicity for \( -\Delta + V - m \) and \( \text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m)) \), then \( \lambda_0 \) is an asymptotic bifurcation point for equation (1.1).

To our knowledge there are no earlier results on asymptotic bifurcation for (1.1) if the multiplicity of \( \lambda_0 \) is even.

The rest of the paper is organized as follows. Section 2 contains some preliminary material. In Section 3 a finite-dimensional reduction is performed. In Section 4 we prove Theorems 1.1 and 1.2 and Section 5 is concerned with the proofs of Theorems 1.3 and 1.4.

**Notation.** \( \langle \cdot, \cdot \rangle \) denotes the inner product in a (real) Hilbert space \( E \) and \( \| \cdot \| \) is the corresponding norm. If \( \Phi \in C^1(E, \mathbb{R}) \), then \( \Phi'(u) \in E^* \) is the Fréchet derivative of \( \Phi \) at \( u \) and \( \nabla \Phi(u) \) (the gradient of \( \Phi \) at \( u \)) is the corresponding element in \( E \), i.e., \( \langle \nabla \Phi(u), v \rangle = \Phi'(u)v \). The graph norm
corresponding to a linear operator \( L \) will be denoted by \( \| \cdot \|_L \). The symbol \( B_r(a) \) will stand for the open ball centered at \( a \) and having radius \( r \), and we denote the \( L^p \)-norm of \( u \) by \( \| u \|_p \).

### 2. Preliminaries

Let \( X, Y \) be (real) Banach spaces and let \( N : X \setminus B_R(0) \rightarrow Y \).

**Definition 2.1.**

(i) We say that \( N \) is **asymptotically linear in the sense of Hadamard** (\( H \)-asymptotically linear for short) if there is a bounded linear operator \( B : X \rightarrow Y \) such that

\[
\lim_{n \to \infty} \frac{N(t_n u_n)}{t_n} = Bu
\]

for all sequences \( (t_n) \subset \mathbb{R}, (u_n) \subset X \) such that \( u_n \rightarrow u \) and \( \|t_n u_n\| \rightarrow \infty \). The operator \( B \) is called the **asymptotic \( H \)-derivative** and is denoted by \( N'(\infty) \).

(ii) We say that \( N \) is **Lipschitz continuous at infinity** if

\[
\text{Lip}_\infty(N) := \lim_{R \to \infty} \sup \left\{ \frac{\|N(u) - N(v)\|}{\|u - v\|} : u \neq v, \|u\|, \|v\| \geq R \right\} < \infty.
\]

Note that the limit is well defined because the supremum above decreases as \( R \) increases.

**Remark 2.2.**

(i) The definition of \( H \)-asymptotic linearity given in [19] is in fact a little different but the one formulated above is somewhat more convenient and is equivalent to the original one as has been shown in [19, Theorem A.1].

(ii) Recall that \( N \) is **asymptotically linear** (in the sense of Fréchet) if there is a bounded linear operator \( B \) such that

\[
\lim_{\|u\| \to \infty} \frac{\|N(u) - Bu\|}{\|u\|} = 0.
\]

It is clear that if \( N \) is asymptotically linear, then it is \( H \)-asymptotically linear and \( N'(\infty) = B \). If, however, \( \dim X < \infty \), then \( H \)-asymptotic linearity is equivalent to asymptotic linearity and (2.1) above holds for \( B = N'(\infty) \), see [19, Remark 2].

Recall that a linear operator \( L : D(L) \subset X \rightarrow Y \) is called a **Fredholm operator** if it is densely defined, closed, \( \dim N(L) < \infty \) (where \( N(L) \) is the kernel of \( L \)), the range \( R(L) \) is closed and \( \text{codim} R(L) < \infty \). The number

\[
\text{ind} (L) := \dim N(L) - \text{codim} R(L)
\]

is the **index of \( L \)** (cf. [17, Section 1.3]).

Suppose that \( E \) is a real Hilbert space and let \( L : D(L) \subset E \rightarrow E \) be a selfadjoint Fredholm operator. Then \( \text{ind} (L) = 0, E = N(L) \oplus R(L) \) (orthogonal sum) and \( S := L|_{R(L) \cap D(L)} \) is invertible with bounded inverse. Hence, in view of [9, Problem III.6.16],

\[
\|S^{-1}\| = r(S^{-1}) = \frac{1}{\text{dist} (0, \sigma(S))} = \frac{1}{\text{dist} (0, \sigma(L) \setminus \{0\})},
\]

where \( r(S^{-1}) \) denotes the spectral radius of \( S^{-1} \). The first equality holds since \( S^{-1} \) is selfadjoint, see [9, (V.2.4)]. Recall that a selfadjoint operator is necessarily densely defined and closed.
It is clear that if $W$ is a closed subspace of $R(L)$, invariant with respect to $L$ (i.e. $L(W \cap D(L)) \subseteq W$), then $L_W := L|_{W \cap D(L)}$ is also invertible and
\[ \|L_W^{-1}\| = \frac{1}{\text{dist}(0, \sigma(L_W))}. \]

**Remark 2.3.** Keeping the above notation observe that $L_W^{-1} : W \to W \cap D(L)$ is bounded with respect to the graph norm $\| \cdot \|_L$ in $W \cap D(L)$ (recall that $\|u\|_L := \|u\| + \|Lu\|$ for $u \in D(L)$). In fact,
\[ \|L_W^{-1}w\|_L = \|L_W^{-1}w\| + \|w\| \leq \left(1 + \frac{1}{\text{dist}(0, \sigma(L_W))}\right) \|w\|, \quad w \in W. \]

**Definition 2.4.** For a selfadjoint Fredholm operator $L : D(L) \to E$, let us put
\[ (2.2) \quad \gamma(L) := \inf\{\|(L|_{W \cap D(L)})^{-1}\| : W \in \mathcal{W}\}, \]
where $\mathcal{W}$ denotes the family of closed $L$-invariant linear subspaces of $R(L)$ such that $\text{codim} W < \infty$ and $W \perp \subseteq D(L)$.

**Definition 2.5.** By the **essential spectrum** $\sigma_e(L)$ of a selfadjoint linear operator $L : E \supset D(L) \to E$ we understand the set
\[ \{ \lambda \in \mathbb{C} : L - \lambda I \text{ is not a Fredholm operator} \} \]
(see [17, §1.4]).

It follows immediately from this definition that $\sigma_e(L) \subseteq \sigma(L)$ and $\sigma(L) \setminus \sigma_e(L)$ consists of isolated eigenvalues of finite multiplicity.

**Theorem 2.6.** Let $L : E \supset D(L) \to E$ be a selfadjoint linear operator and let $\lambda_0 \in \sigma(L) \setminus \sigma_e(L)$. Then $L - \lambda_0 I$ is a Fredholm operator and
\[ \gamma(L - \lambda_0 I) = \frac{1}{\text{dist}(\lambda_0, \sigma_e(L))}. \]
If $\sigma_e(L) = \emptyset$ (this is the case e.g. if $L$ is resolvent compact), then $\gamma(L - \lambda_0 I) = 0$.

**Proof.** Since $\sigma_e(L) - \lambda_0 = \sigma_e(L - \lambda_0 I)$ and hence
\[ \text{dist}(\lambda_0, \sigma_e(L)) = \text{dist}(0, \sigma_e(L - \lambda_0 I)), \]
we may assume without loss of generality that $\lambda_0 = 0$ and we will show that
\[ \gamma(L) = \frac{1}{\text{dist}(0, \sigma_e(L))}. \]
If $W \in \mathcal{W}$ and $Z := W \perp$, then $\text{dim} Z < \infty$ and $Z \subseteq D(L)$ is $L$-invariant. Hence $\sigma(L) = \sigma(L|_{W \cap D(L)}) \cup \sigma(L|_Z)$. Obviously, any $\lambda \in \sigma(L|_Z)$ is an isolated eigenvalue of finite multiplicity; thus $\sigma_e(L) \subseteq \sigma(L|_{W \cap D(L)})$. This implies that
\[ \|(L|_{W \cap D(L)})^{-1}\| = \frac{1}{\text{dist}(0, \sigma(L|_{W \cap D(L)}))} \geq \frac{1}{\text{dist}(0, \sigma_e(L))} \quad \text{and therefore} \quad \gamma(L) \geq \frac{1}{\text{dist}(0, \sigma_e(L))}. \]

Take any $0 < d < \text{dist}(0, \sigma_e(L))$ and let
\[ D = [-d, d] \cap \sigma(L), \quad B := \sigma(L) \setminus D. \]
Clearly $D$ is finite: if $\lambda \in D$, then $\lambda \in \sigma(L) \setminus \sigma_e(L)$, i.e., $\lambda$ is an isolated eigenvalue of finite multiplicity. Therefore $B$ is closed and $\sigma_e(L) \subset B$. Obviously, $\sigma(L) = D \cup B$. Let $Z$ be the subspace spanned by the eigenfunctions corresponding to the eigenvalues in $D$ and let $W = Z^\perp$. Then $Z \subset D(L)$, $W \subset R(L)$, $Z,W$ are invariant with respect to $L$, $L|_Z$ is bounded, $D = \sigma(L|_Z)$ and $B = \sigma(L|_{W \cap D(L)})$. Clearly, $W \in W$ since $\dim Z < \infty$. Now

$$
\| (L|_{W \cap D(L)})^{-1} \| = r((L|_{W \cap D(L)})^{-1}) = \frac{1}{\dist(0, \sigma(L|_{W \cap D(L)}))} \leq \frac{1}{\dist(0, B)} \leq \frac{1}{d}.
$$

This implies the assertion. Note that if $\sigma_e(L) = \emptyset$, we can choose any $d > 0$. Hence $\gamma(L) = 0$. □

**Remark 2.7.** Let $L$ be a Fredholm operator of index 0 and let $\mathcal{P}(L)$ denote the collection of all bounded operators $K$ of finite rank and such that $L + K$ is invertible. Clearly, $\mathcal{P}(L) \neq \emptyset$. Put

$$
\tilde{\gamma}(L) := \inf\{\| (L + K)^{-1} \| : K \in \mathcal{P}(L) \}.
$$

Then $\tilde{\gamma}(L)$ corresponds to the notion of *essential conditioning number* in [20, Section 5.1], see also [21, Section 3.1] where the definition above appears explicitly.

We claim that if $L$ is a selfadjoint Fredholm operator, then $\tilde{\gamma}(L) = \gamma(L)$. For $K \in \mathcal{P}(L)$, $\sigma_e(L) = \sigma_e(L + K) \subset \sigma(L + K)$, hence

$$
\| (L + K)^{-1} \| \geq r((L + K)^{-1}) = \frac{1}{\dist(0, \sigma(L + K))} \geq \frac{1}{\dist(0, \sigma_e(L))}.
$$

So $\tilde{\gamma}(L) \geq \gamma(L)$ according to the definition of $\tilde{\gamma}$ and Theorem 2.6. On the other hand, take any $W \in W$ and let $Z := W^\perp$. As before, write $u = z + w \in Z \oplus W$ and let $Ku := \alpha z - Lz$, where

$$
\alpha := \inf\{\|Lw\| : w \in W \cap D(L), \|w\| = 1\} > 0.
$$

Then $K$ has finite rank and, for $u \in D(L)$, $Lu + Ku = Lw + \alpha z$. Hence $L + K$ is invertible and it is easy to see that

$$
\inf\{\|Lu + Ku\| : u \in D(L), \|u\| = 1\} \geq \alpha.
$$

So

$$
\tilde{\gamma}(L) \leq \| (L + K)^{-1} \| \leq \frac{1}{\alpha} = \| (L|_{W \cap D(L)})^{-1} \|
$$

and $\tilde{\gamma}(L) \leq \gamma(L)$. We have shown that $\tilde{\gamma}(L) = \gamma(L)$. Therefore Theorem 2.6 may be considered as a refinement of [20, Theorem 5.5 and Corollary 5.6].

3. The problem and finite-dimensional reduction

Let $E$ be a real Hilbert space and $L : E \supset D(L) \to E$ a selfadjoint operator. We shall study the existence of solutions to the eigenvalue problem (1.2), i.e.,

$$
Lu = \lambda u + N(u), \quad u \in D(L), \quad \lambda \in \mathbb{R},
$$

or, more precisely, the existence of asymptotic bifurcation of solutions to (1.2). Recall that $\lambda_0 \in \mathbb{R}$ is an *asymptotic bifurcation point* for (1.2) if there exist sequences $\lambda_n \to \lambda_0$ and $(u_n) \subset D(L)$ such that $\|u_n\| \to \infty$ and $Lu_n - N(u_n) = \lambda_n u_n$.

By $X$ we denote the domain $D(L)$ furnished with the graph norm

$$
\|u\|_L := \|u\| + \|Lu\|, \quad u \in D(L).
$$
Then $X$ is a Banach space, $L$ is bounded as an operator from $X$ to $E$ and the inclusion $i : X \hookrightarrow E$ is continuous.

If $N$ is a potential operator, i.e. there exists $\psi \in C^1(E, \mathbb{R})$ such that $N = \nabla \psi$, then along with (1.2) we can consider the existence of critical points of the functional $\Phi_\lambda : X \to \mathbb{R}$, $\lambda \in \mathbb{R}$, given by

$$\Phi_\lambda(u) := \frac{1}{2}\langle Lu - \lambda u, u \rangle - \psi(u), \quad u \in X.$$  

Since $|\langle Lu, u \rangle| \leq \|Lu\|\|u\| \leq \|u\|^2_L$, $\Phi_\lambda \in C^1(X, \mathbb{R})$ and

$$\Phi'(\lambda)u = \langle Lu - \lambda u, v \rangle - \langle N(u), v \rangle, \quad u,v \in X.$$  

It is clear that if $u \in X$ solves (1.2) for some $\lambda \in \mathbb{R}$, then $\Phi'(\lambda)u = 0$ for all $v \in X$, i.e., $u$ is a critical point of $\Phi_\lambda$. Conversely, if $u \in X$ and $\Phi'(\lambda)u = 0$, then $u$ solves (1.2) since $D(L)$ is dense in $E$. Note that if $L$ is unbounded, then $\Phi_\lambda$ is defined on $D(L)$ and is not $C^1$ with respect to the original norm $\| \cdot \|$ of $E$ on $D(L)$.

In what follows we assume:

**3.1.** $N$ is $H$-asymptotically linear with $N'(\infty) = 0$;

**3.2.** $N$ is Lipschitz continuous at infinity;

**3.3.** $\lambda_0 = 0 \in \sigma(L) \setminus \sigma_e(L)$ and $\text{Lip}_\infty(N) < \text{dist}(0, \sigma_e(L))$.

Observe that these assumptions cause no loss of generality in Theorems 1.1 and 1.2 since if $N'(\infty) \neq 0$ is selfadjoint and $\lambda_0 \neq 0$, then we may replace $L$ by $L - N'(\infty) - \lambda_0 I$ and $N$ by $N - N'(\infty)$.

As a first step towards showing that $\lambda_0 = 0$ is an asymptotic bifurcation point for (1.2) we perform a kind of a Liapunov-Schmidt finite-dimensional reduction near infinity. Put

$$L_\lambda u := Lu - \lambda u, \quad u \in D(L_\lambda) = D(L), \quad \lambda \in \mathbb{R}$$

and note that the norms $\| \cdot \|_L$ and $\| \cdot \|_{L_\lambda}$ are equivalent. Given $W \in W$, let $P : E \to W$ be the orthogonal projection and $Z := W^\perp$. Observe that $u = w + z \in D(L)$, where $w \in W$, $z \in Z$, solves (1.2) if and only if

$$L_\lambda w = PN(w + z), \quad L_\lambda z = (I - P)N(w + z).$$

**Proposition 3.4.** There are a subspace $W \in W$, numbers $\delta \in (0, \text{dist}(0, \sigma(L) \setminus \{0\}))$, $R > 0$ and a continuous map $w : [-\delta, \delta] \times (Z \setminus B_R(0)) \to W \cap D(L)$ such that (3.2) holds for $w = w(\lambda, z)$ and:

(i) For any $\lambda$ with $|\lambda| \leq \delta$ , $z, z' \in Z \setminus B_R(0)$ and some constant $c > 0$,

$$||w(\lambda, z) - w(\lambda, z')||_L \leq c||z - z'||.$$

In particular, $w(\cdot, \cdot)$ is continuous with respect to the graph norm.

(ii) $w(\lambda, \cdot)$ is $H$-asymptotically linear with $w'(\lambda, \infty) = 0$.

(iii) $z \in Z \setminus B_R(0)$ is a solution of (3.3) with $w = w(\lambda, z)$ if and only if $u = w(\lambda, z) + z$ is a solution of (1.2).
Note that the condition on \( \delta \) implies invertibility of \( L_\lambda \) for \( 0 < |\lambda| \leq \delta \).

**Proof.** (i) According to Definition 2.4 of \( \gamma(L) \), Theorem 2.6 and assumption 3.3, there is a closed subspace \( W \in \mathcal{W} \) for which

\[
\text{Lip}_\infty(N)(L|_{W \cap D(L)})^{-1} < 1.
\]

Hence we can find \( \delta \in (0, \text{dist}(0, \sigma(L) \setminus \{0\})) \) and \( R > 0 \) such that

\[
k := \sup_{|\lambda| \leq \delta} \|L|_{W \cap D(L)}\| : \beta < 1,
\]

where

\[
(3.5) \quad \beta := \sup \left\{ \frac{\|N(u) - N(v)\|}{\|u - v\|} : u \neq v, \|u\|, \|v\| \geq R \right\}.
\]

Let \( Z := W^\perp \) and let \( P : E \to W \) be the orthogonal projection. To facilitate the notation let us put

\[
M_\lambda(w + z) := (L|_{W \cap D(L)})^{-1}PN(w + z) \in W \cap D(L), \quad w \in W, \ z \in Z \text{ and } |\lambda| \leq \delta.
\]

Then (3.2) is equivalent to the fixed point equation

\[
(3.6) \quad w = M_\lambda(w + z).
\]

Fix \( \lambda \in [-\delta, \delta] \) and \( z \in Z, \|z\| \geq R \). If \( w, w' \in W \), then \( \|w + z\|, \|w' + z\| \geq \|z\| \geq R \), so taking into account that \( \|P\| = 1 \), we have

\[
\|M_\lambda(w + z) - M_\lambda(w' + z)\| \leq k\|w - w'\|.
\]

By the Banach contraction principle there is a unique \( w = w(\lambda, z) \in W \cap D(L) \), continuously depending on \( \lambda \) and \( z \), such that (3.6), and hence (3.2), holds. Moreover,

\[
\|w(\lambda, z) - w(\lambda, z')\| = \|M_\lambda(w(\lambda, z) + z) - M_\lambda(w(\lambda, z') + z')\| \leq k\|w(\lambda, z) - w(\lambda, z')\| + k\|z - z'\|
\]

for all \( |\lambda| \leq \delta, \ z, z' \in Z \setminus B_R(0) \). So \( \|w(\lambda, z) - w(\lambda, z')\| \leq k(1 - k)^{-1}\|z - z'\| \). Using this, (3.3) and arguing as above, we obtain

\[
\|L_\lambda(z) - L_\lambda(z')\| = \|PN(w(\lambda, z) + z) - PN(w(\lambda, z') + z')\|
\]

\[
\leq \beta\|w(\lambda, z) - w(\lambda, z')\| + \beta\|z - z'\| \leq \frac{\beta}{1 - k}\|z - z'\|.
\]

Since \( \|\cdot\|_L \) and \( \|\cdot\|_{L_\lambda} \) are equivalent norms, the second inequality in (3.4) follows (the first one is obvious).

(ii) To show the \( H \)-asymptotic linearity of \( w(\lambda, \cdot) \) with \( w'(\lambda, \infty) = 0 \), let \( (z_n) \subset Z \) and \( (t_n) \subset \mathbb{R} \) be sequences such that \( z_n \to z \) and \( \|t_n z_n\| \to \infty \). Then, for sufficiently large \( n \),

\[
\|w(\lambda, t_n z_n) + t_n z_n\| \geq \|t_n z_n\| \geq R
\]

and

\[
\|w(\lambda, t_n z_n)\| \leq \|M_\lambda(w(\lambda, t_n z_n) + t_n z_n) - M_\lambda(t_n z_n)\| + \|M_\lambda(t_n z_n)\| \leq k\|w(\lambda, t_n z_n)\| + \|M_\lambda(t_n z_n)\|.
\]

Thus, in view of assumption 3.1

\[
(3.7) \quad \frac{\|w(\lambda, t_n z_n)\|}{|t_n|} \leq \frac{1}{1 - k} \frac{\|M_\lambda(t_n z_n)\|}{|t_n|} \to 0.
\]
(iii) is an immediate consequence of (i).

\[ \square \]

**Remark 3.5.** Suppose that \( z_n \to z \) in \( Z \) and take a sequence \( (t_n) \subset \mathbb{R} \) such that \( \|t_n z_n\| \to \infty \). Then, again in view of the \( H \)-asymptotic linearity of \( N \) and (3.7), we have

\[
N(w(\lambda, t_n z_n) + t_n z_n) = N\left(t_n \left(\frac{w(\lambda, t_n z_n)}{t_n} + z_n\right)\right) \to 0
\]

for each fixed \( \lambda \in [-\delta, \delta] \).

If \( N = \nabla \psi \), then we let

\[
\varphi_\lambda(z) := \Phi_\lambda(w(\lambda, z) + z), \quad |\lambda| \leq \delta, \quad z \in Z \setminus \overline{B}_R(0).
\]

**Proposition 3.6.** Let \( |\lambda| \leq \delta \). Then \( \varphi_\lambda \in C^1(Z \setminus \overline{B}_R(0), \mathbb{R}) \) and

\[ \nabla \varphi_\lambda(z) = L_\lambda z - (I - P)N(w(\lambda, z) + z). \]

Therefore \( z \in Z \setminus \overline{B}_R(0) \) is a critical point of \( \varphi_\lambda \) if and only if \( u = w(\lambda, z) + z \) solves (1.2).

Moreover, \( \nabla \varphi_\lambda \) is asymptotically linear with \( (\nabla \varphi_\lambda)'(\infty) = L_\lambda|_Z \).

**Proof.** To show (3.10) we shall compute the derivative of \( \varphi_\lambda \) in the direction \( h \in Z, \ h \neq 0 \). For notational convenience we write \( w(z) \) for \( w(\lambda, z) \). Let \( t > 0 \),

\[ u := w(z) + z, \quad \text{and} \quad \xi := w(z + th) - w(z) + th. \]

Then we have

\[ \varphi_\lambda(z + th) - \varphi_\lambda(z) = \Phi_\lambda(u + \xi) - \Phi_\lambda(u) - \Phi'_\lambda(u)\xi. \]

Clearly, \( \xi \neq 0 \) as \( t \to 0 \). In view of (3.1), (3.2) and since \( w(z + th) - w(z) \in W \),

\[ \Phi'_\lambda(u)\xi = \langle L_\lambda u - N(u), \xi \rangle = \langle L_\lambda w(z) - PN(u), \xi \rangle + \langle L_\lambda z - (I - P)N(u), \xi \rangle \]

\[ = \langle L_\lambda z - N(u), th \rangle = t\Phi'_\lambda(u)h. \]

Hence

\[
\frac{\varphi_\lambda(z + th) - \varphi_\lambda(z)}{t} = \Phi'_\lambda(u)h + \frac{\|\xi\|L}{t} \cdot \frac{\Phi_\lambda(u + \xi) - \Phi_\lambda(u) - \Phi'_\lambda(u)\xi}{\|\xi\|L}.
\]

It follows from (3.3) that

\[ \|\xi\|L \leq td\|h\| \]

for some \( d > 0 \). This, together with the Fréchet differentiability of \( \Phi_\lambda \) on \( X \) (i.e., on \( D(L) \) with the graph norm) implies that the second term on the right-hand side of (3.11) tends to 0 as \( t \to 0 \). So

\[
\lim_{t \to 0^+} \frac{\varphi_\lambda(z + th) - \varphi_\lambda(z)}{t} = \Phi'_\lambda(u)h = \langle L_\lambda z, h \rangle - \langle (I - P)N(w(z) + z), h \rangle.
\]

Therefore \( \varphi_\lambda \) is continuously Gâteaux differentiable, hence continuously Fréchet differentiable as well, and the derivative is as claimed.

If \( z \in Z \setminus \overline{B}_R(0) \) is a critical point of \( \varphi_\lambda \), then (3.3) with \( w = w(\lambda, z) \) is satisfied; this together with (3.2) shows that \( u = w(\lambda, z) + z \) solves (1.2).
Since \( \dim Z < \infty \), in order to prove the last part of the assertion it suffices to show that \( \nabla \varphi_\lambda \) is \( H \)-asymptotically linear (see Remark 2.2(ii)). If \( z_n \to z \) in \( Z \), \( (t_n) \subset \mathbb{R} \) and \( \|t_nz_n\| \to \infty \), then, in view of (3.8),
\[
\frac{\nabla \varphi_\lambda(t_nz_n)}{t_n} = L_\lambda z_n - \frac{(I - P)N(w(t_nz_n) + t_nz_n)}{t_n} \rightarrow L_\lambda z.
\]
This concludes the proof. \( \square \)

**Remark 3.7.** (i) Using (3.4) and the fact that \( \beta \) in (3.5) is finite, it is easy to see that \( \nabla \varphi_\lambda \) is Lipschitz continuous on \( Z \setminus \overline{B}_R(0) \) and the Lipschitz constant may be chosen independently of \( \lambda \in [-\delta, \delta] \).

(ii) In what follows we may (and will need to) assume that \( \varphi_\lambda \) is defined on \( Z \) and not only on \( Z \setminus \overline{B}_R(0) \). Such an extension of \( \varphi_\lambda \) can be achieved e.g. as follows. Let \( \chi \in C^\infty([\mathbb{R}, [0,1]]) \) be a cutoff function such that \( \chi(t) = 0 \) for \( t \leq R + 1 \) and \( \chi(t) = 1 \) for \( t \geq R + 2 \). Set \( \tilde{\varphi}_\lambda(z) := \chi(\|z\|)\varphi_\lambda(z) \). Then \( \tilde{\varphi}_\lambda \) is of class \( C^1 \), Lipschitz continuous and \( \tilde{\varphi}_\lambda(z) = \varphi_\lambda(z) \) for \( \|z\| > R + 2 \). In particular, \( z \in Z \setminus \overline{B}_R(0) \), where \( \tilde{R} := R + 2 \), is a critical point of \( \tilde{\varphi}_\lambda \) if and only if \( u = w(\lambda, z) + z \) solves (1.2).

4. Proofs of Theorems 1.1 and 1.2

In the proof of Theorem 1.1 we shall need the following version of Whyburn’s lemma which may be found in [1], Proposition 5:

**Lemma 4.1.** Let \( Y \) be a compact space and \( A, B \subset Y \) closed sets. If there is no connected set \( \Gamma \subset Y \setminus (A \cup B) \) such that \( \overline{\Gamma} \cap A \neq \emptyset \) and \( \overline{\Gamma} \cap B \neq \emptyset \) (\( \overline{\Gamma} \) stands for the closure of \( \Gamma \) in \( Y \)), then \( A \) and \( B \) are separated, i.e. there are open sets \( U, V \subset Y \) such that \( A \subset U, B \subset V, U \cap V = \emptyset \) and \( Y = U \cup V \) (clearly, \( U, V \) are closed as well).

**Proof of Theorem 1.1.** By Proposition 3.4 it suffices to consider equation (3.3) with \( w = w(\lambda, z) \) which we re-write in the form
\[
F_\lambda(z) := L_\lambda z - (I - P)N(w(\lambda, z) + z) = 0.
\]
As in assumptions 3.1-3.3 it causes no loss of generality to take \( \lambda_0 = 0 \) and \( N'(\infty) = 0 \). Although \( F_\lambda \) in Proposition 3.4 has been defined for \( |\lambda| \leq \delta \) and \( \|z\| \geq R \), we may (and do) extend it continuously to \( [-\delta, \delta] \times Z \). Since \( w'(\lambda, \infty) = 0 \) (see (ii) of Proposition 3.4) and asymptotic linearity coincides with \( H \)-asymptotic linearity on \( Z \) (because \( \dim Z < \infty \)), we have, setting \( K_\lambda(z) := (I - P)N(w(\lambda, z) + z) \) and using Remark 3.5
\[
\lim_{\|z\| \to \infty} \frac{\|K_\lambda(z)\|}{\|z\|} = 0.
\]
Suppose there is no asymptotic bifurcation at \( \lambda_0 = 0 \). Taking smaller \( \delta \) and larger \( R \) if necessary, \( F_\lambda(z) \neq 0 \) for any \( |\lambda| \leq \delta \) and \( \|z\| \geq R \). Therefore the Brouwer degree \( \deg(F_\lambda, B_R(0), 0) \) (see e.g. [2], Section 3.1) is well defined and independent of \( \lambda \in [-\delta, \delta] \). Since \( \delta < \text{dist}(0, \sigma(L) \setminus \{0\}) \), \( L_{\pm \delta} \) are invertible. It follows therefore from (4.2) that if \( R_0 \geq R \) is sufficiently large, then \( L_{\pm \delta}z - tK_{\pm \delta}(z) \neq 0 \) for
0 for any \( \|z\| \geq R_0, \ t \in [0,1] \). Hence by the excision property and the homotopy invariance of degree,
\[
k = \deg(F_{\pm \delta}, B_R(0), 0) = \deg(F_{\pm \delta}, B_{R_0}(0), 0) = \deg(L_{\pm \delta}|_Z, B_{R_0}, 0)
\]
for some \( k \in \mathbb{Z} \). Let \( d_1, d_2 \) be the number of negative eigenvalues (counted with their multiplicity) of respectively \( L_\delta|_Z \) and \( L_{-\delta}|_Z \). Then \( k = (-1)^{d_1} = (-1)^{d_2} \) [2] Lemma 3.3]. However, since \( d_1 = d_2 + \dim N(L) \) and \( \dim N(L) \) is odd, this is impossible. So we have reached a contradiction to the assumption that there is no bifurcation.

It remains to prove that there exists a bifurcating continuum. Usually this is done by first making the inversion \( u \mapsto u/\|u\|^2 \) and then showing there is a continuum bifurcating from 0 [16, 20, 22]. Here we give a slightly different argument avoiding inversion. Let
\[
\Sigma := \{(z, \lambda) \in (Z \setminus B_R(0)) \times [-\delta, \delta] : F_\lambda(z) = 0 \}.
\]
Compactify \( Z \) by adding the point at infinity and let \( A := \overline{B_R(0)} \times [-\delta, \delta], B := \{(\infty, 0)\}, Y := A \cup \Sigma \cup B \). Then \( Y \) is compact, \( A \) and \( B \) are closed disjoint. We claim that if \( R \) is large enough, there is a connected set \( \Gamma \subset \Sigma \) such that \( \{(\infty, 0)\} \in \Gamma \) (the closure taken in \( Y \)) and \( \overline{\Gamma} \cap A \neq \emptyset \). Otherwise we shall show \( \Sigma \) is compact and bounded, there exists a bounded open set \( \mathcal{O} \subset Z \times [-\delta, \delta] \) such that \( U \subset \mathcal{O} \) and \( \partial \mathcal{O} \cap \Sigma = \emptyset \). Letting \( \mathcal{O}_\lambda := \{z : (z, \lambda) \in \mathcal{O}\} \) for \( \lambda \in [-\delta, \delta] \), it follows from the excision property and the generalized version of the homotopy invariance property of degree [2] Theorem 4.1] that \( \deg(F_\delta, \mathcal{O}_\delta, 0) = \deg(F_{-\delta}, \mathcal{O}_{-\delta}, 0) \), a contradiction since by the same argument as above \( \deg(F_\delta, \mathcal{O}_\delta, 0) = (-1)^{k_1}, \deg(F_{-\delta}, \mathcal{O}_{-\delta}, 0) = (-1)^{k_2} \) and \( k_1, k_2 \) have different parity.

In the proof of Theorem 1.2 we shall use Gromoll-Meyer theory. Below we summarize some pertinent facts which are special cases of much more general results of [12] where functionals were considered in a Hilbert space \( E \) with filtration, i.e., with a sequence \( (E_n) \) of subspaces such that \( E_n \subset E_{n+1} \) for all \( n \) and \( \bigcup_{n=1}^\infty E_n \) is dense in \( E \). In the terminology of [12], here we have the trivial filtration (i.e., \( Z_n = Z \) for all \( n \)) which, together with the fact that \( \dim Z < \infty \), considerably simplifies the proofs. An alternative approach is via the Conley index theory, see e.g. [3, 4], in particular [3] Corollary 2.3 and [4] Theorem 2.

Let \( \varphi : Z \to \mathbb{R} \) be a function such that \( \nabla \varphi \) is locally Lipschitz continuous. Suppose also \( K = K(\varphi) := \{z \in Z : \nabla \varphi(z) = 0\} \) is bounded. A pair \( (\mathcal{W}, \mathcal{W}^-) \) of closed subsets of \( Z \) will be called admissible \( C^1 \)-manifold of codimension 1, \( V \) is transversal to \( \mathcal{W}^- \), the flow \( \eta \) of \( -V \) can leave \( \mathcal{W} \) only via \( \mathcal{W}^- \) and if \( z \in \mathcal{W}^- \), then \( \eta(t,z) \notin \mathcal{W} \) for any \( t > 0 \).

\( (i) \ K \subset \text{int}(\mathcal{W}) \) and \( \mathcal{W}^- \subset \partial \mathcal{W} \);

\( (ii) \ \varphi|_{\mathcal{W}} \) is bounded;

\( (iii) \ There exist a locally Lipschitz continuous vector field \( V \) defined in a neighbourhood \( N \) of \( \mathcal{W} \) and a continuous function \( \beta : N \to \mathbb{R}^+ \) such that \( \|V(z)\| \leq 1, \langle V(z), \varphi(z) \rangle \geq \beta(z) \) for all \( z \in N \), and \( \beta \) is bounded away from 0 on compact subsets of \( N \setminus K \) (we shall call \( V \) admissible for \( (\mathcal{W}, \mathcal{W}^-) \));

\( (iv) \ \mathcal{W}^- \) is a piecewise \( C^1 \)-manifold of codimension 1, \( V \) is transversal to \( \mathcal{W}^- \), the flow \( \eta \) of \( -V \) can leave \( \mathcal{W} \) only via \( \mathcal{W}^- \) and if \( z \in \mathcal{W}^- \), then \( \eta(t,z) \notin \mathcal{W} \) for any \( t > 0 \).
Let $H^*$ denote the Čech (or Alexander-Spanier) cohomology with coefficients in $\mathbb{Z}_2$ and let the critical groups $c^*(\varphi, K)$ of the pair $(\varphi, K)$ be defined by

$$c^*(\varphi, K) := H^*(\mathbb{W}, \mathbb{W}^-).$$

**Lemma 4.2.** Suppose $\varphi$ satisfies (PS).

(i) For each $R > 0$ there exists a bounded admissible pair $(\mathbb{W}, \mathbb{W}^-)$ for $\varphi$ and $K$ such that $B_R(0) \subset \mathbb{W}$.

(ii) If $(\mathbb{W}_1, \mathbb{W}_1^-)$ and $(\mathbb{W}_2, \mathbb{W}_2^-)$ are two admissible pairs for $\varphi$ and $K$, then $H^*(\mathbb{W}_1, \mathbb{W}_1^-) \cong H^*(\mathbb{W}_2, \mathbb{W}_2^-)$ (i.e., $c^*(\varphi, K)$ is well defined).

(iii) Suppose $\{\varphi_\lambda\}_{\lambda \in [0,1]}$ is a family of functions satisfying (PS) and such that $\nabla \varphi_\lambda$ is locally Lipschitz continuous, $\lambda \mapsto \nabla \varphi_\lambda$ is continuous, uniformly on bounded subsets of $Z$, and $K(\varphi_\lambda) \subset B_R(0)$ for some $R > 0$ and all $\lambda \in [0,1]$. Then $c^*(\varphi_\lambda, K(\varphi_\lambda))$ is independent of $\lambda$.

This lemma corresponds to Lemma 2.13 and Propositions 2.12, 2.14 in [12]. Note that condition (PS)* there is in our setting (i.e., for trivial filtration) equivalent to (PS).

**Outline of proof.** (i) Choose $R, a, b$ so that $K \subset B_R(0)$, $a < \varphi(z) < b$ for all $z \in B_R(0)$ and let

$$V(z) := \frac{\nabla \varphi(z)}{1 + \|\nabla \varphi(z)\|}.$$  

Clearly, the flow $\eta$ given by

$$\frac{d\eta}{dt} = -V(\eta), \quad \eta(0, z) = z$$

is defined on $\mathbb{R} \times Z$. Let

$$\mathbb{W} := \{\eta(t, z) : t \geq 0, \ z \in B_R(0), \ \varphi(\eta(t, z)) \geq a\}, \quad \mathbb{W}^- := \mathbb{W} \cap \varphi^{-1}(a).$$

Then $(\mathbb{W}, \mathbb{W}^-)$ is an admissible pair. The proof follows that of [12] Lemma 2.13 but is simpler - there is no need for using cutoff functions. Note that (here and below) the Palais-Smale condition rules out the possibility that $\varphi(\eta(t,z)) > a$ and $||\eta(t, z)|| \to \infty$ as $t \to \infty$, hence $t \mapsto \eta(t, z)$ either approaches $K$ as $t \to \infty$ or hits $\mathbb{W}^- = \varphi^{-1}(a)$ in finite time.

(ii) Assume that $\varphi$ is unbounded below and above (the other cases are simpler but somewhat different). Let $(\mathbb{W}_0, \mathbb{W}_0^-)$ be an admissible pair and $V_0$ a corresponding admissible vector field. As $\varphi|_{\mathbb{W}_0}$ is bounded, we may choose $a, b$ so that $a < \varphi(z) < b$ for all $z \in \mathbb{W}_0$. Since $(\mathbb{W}_1, \mathbb{W}_1^-) := (\varphi^{-1}([a,b]), \varphi^{-1}(a))$ is an admissible pair, it suffices to show that $H^*(\mathbb{W}_0, \mathbb{W}_0^-) \cong H^*(\mathbb{W}_1, \mathbb{W}_1^-)$. Put $V(z) := \chi_0(z)V_0(z) + \chi_1(z)V_1(z)$, where $V_1$ is given by (4.3) and $\{\chi_0, \chi_1\}$ is a Lipschitz continuous partition of unity such that $\chi_0(z) = 1$ on $\mathbb{W}_0$ and $\chi_1(z) = 1$ in a neighbourhood of $\partial \mathbb{W}_1$. Denote the flow of $-V$ by $\eta$. Let $A := \{\eta(t, z) : t \geq 0, \ z \in \mathbb{W}_0^-\} \cap \mathbb{W}_1$ and $\mathbb{W} = \mathbb{W}_0 \cup A$, $\mathbb{W}^- := \mathbb{W} \cap \mathbb{W}_1^-$. Then $(\mathbb{W}, \mathbb{W}^-)$ is a an admissible pair and using $\eta$ one obtains a strong deformation retraction of $A$ onto $\mathbb{W}^-$. So $H^*(A, \mathbb{W}^-) = 0$ and by exactness of the cohomology sequence of the triple $(\mathbb{W}, A, \mathbb{W}^-)$ and the strong excision property we have $H^*(\mathbb{W}, \mathbb{W}^-) \cong H^*(\mathbb{W}, A) \cong H^*(\mathbb{W}_0, \mathbb{W}_0^-)$. We also have, by excision again, $H^*(\mathbb{W}, \mathbb{W}^-) \cong H^*(\mathbb{W} \cup \mathbb{W}_1^-, \mathbb{W}_1^-)$. Finally, using the flow $\eta$ once more, we obtain a deformation of $(\mathbb{W}_1, \mathbb{W}_1^-)$ into $(\mathbb{W} \cup \mathbb{W}_1^-, \mathbb{W}_1^-)$ which leaves $\mathbb{W} \cup \mathbb{W}_1^-$ and $\mathbb{W}_1^-$ invariant. Hence $(\mathbb{W} \cup \mathbb{W}_1^-, \mathbb{W}_1^-)$ and $(\mathbb{W}_1, \mathbb{W}_1^-)$ are homotopy equivalent and thus have the same
cohomology. Putting everything together gives $H^*(\mathbb{W}_0,\mathbb{W}_0^-) \cong H^*(\mathbb{W}_1,\mathbb{W}_1^-)$. More details of the proof may be found in [12, Propositions 2.12 and 2.7].

(iii) Let $\lambda_0 \in [0,1]$. It suffices to show that $c^*(\varphi_\lambda, K(\varphi_\lambda))$ is constant for $\lambda$ in a neighbourhood of $\lambda_0$. Denote the vector field for $\varphi_\lambda$ given as in (1.3) by $V_\lambda$ and choose an admissible pair $(\mathbb{W}_{\lambda_0}, \mathbb{W}_{\lambda_0}^-)$ for $\varphi_{\lambda_0}$ and $K(\varphi_{\lambda_0})$ such that $B_{R_1}(0) \subset \mathbb{W}_{\lambda_0}$, where $R_1 > R$. By the construction in (i), we may assume $V_{\lambda_0}$ is admissible for this pair. Let $\tilde{V}(z) := \chi_1(z)V_{\lambda_0}(z) + \chi_2(z)V_{\lambda_0}(z)$, where $\{\chi_1, \chi_2\}$ is a partition of unity subordinate to the sets $B_{R_1}(0)$ and $\mathbb{W}_{\lambda_0} \setminus \overline{B}_R(0)$. It is easy to see that if $|\lambda - \lambda_0|$ is small enough, then $(\mathbb{W}_{\lambda_0}, \mathbb{W}_{\lambda_0}^-)$ is an admissible pair for $\varphi_\lambda$, $K(\varphi_\lambda)$ and $\tilde{V}$ is a corresponding admissible field. Note in particular that

$$\|\nabla \varphi_\lambda(z)\| \geq \|\nabla \varphi_{\lambda_0}(z)\| - \|\nabla \varphi_\lambda(z) - \nabla \varphi_{\lambda_0}(z)\| > 0$$

for $z \in \mathbb{W}_{\lambda_0} \setminus \overline{B}_R(0)$, so indeed $\tilde{V}$ is admissible. Hence $c^*(\varphi_\lambda, K(\varphi_\lambda)) \cong c^*(\varphi_{\lambda_0}, K(\varphi_{\lambda_0})).$ 

Proof of Theorem 1.2: Let $\varphi_\lambda$ be given by (3.9) and extend it to the whole space $Z$ according to Remark 3.7. If $\lambda_0 = 0$ is not an asymptotic bifurcation point for (1.2), then it follows from Proposition 4.6 that $\nabla \varphi_\lambda(z) \neq 0$ for $\lambda \in [-\delta, \delta]$ and $\|z\| > R$, possibly after choosing a smaller $\delta$ and larger $R$. By assumption, $\varphi_0$ satisfies (PS) and since $L_\lambda$ has bounded inverse if $0 < |\lambda| \leq \delta$, we see using (1.2) that $\nabla \varphi_\lambda$ is bounded away from 0 as $\|z\| \to \infty$. Hence all $\varphi_\lambda$, $|\lambda| \leq \delta$, satisfy (PS). By Lemma 4.2, $c^*(\varphi_\lambda, K(\varphi_\lambda))$ is independent of $\lambda \in [-\delta, \delta]$. For $\lambda = \delta$, let $Z = Z_\delta^+ \oplus Z_\delta^-$, and $z = z^+ + z^- \in Z_\delta^+ \oplus Z_\delta^-$, where $Z_\delta^\pm$ are the maximal $L_\delta$-invariant subspaces of $Z$ on which $L_\delta$ is respectively positive and negative definite. Choose $\epsilon > 0$ such that $\langle \pm L_\delta z^\pm, z^\pm \rangle \geq \epsilon \|z^\pm\|^2$ and let

$$\mathbb{W} := \{z \in Z : \|z^+\| \leq R_0, \|z^-\| \leq R_0\}, \quad \mathbb{W}^- := \{z \in \mathbb{W} : \|z^-\| = R_0\}.$$ 

Recall $K_\lambda(z) = (I - P)N(w(\lambda, z) + z)$. Taking a sufficiently large $R_0$,

$$\langle \nabla \varphi_\delta(z), z^+ \rangle = \langle L_\delta z, z^+ \rangle - \langle K_\delta(z), z^+ \rangle \geq \epsilon \|z^+\|^2 - \frac{1}{4} \epsilon \|z\| \|z^+\| > 0, \quad z \in \mathbb{W}, \quad \|z^+\| = R_0.$$

Similarly,

$$\langle \nabla \varphi_\delta(z), z^- \rangle < 0, \quad z \in \mathbb{W}, \quad z^- \in \mathbb{W}^-.$$ 

So the flow of $-\nabla \varphi_\delta$ is transversal to $\mathbb{W}^-$ and can leave $\mathbb{W}$ only via $\mathbb{W}^-$. Hence $(\mathbb{W}, \mathbb{W}^-)$ is an admissible pair for $\varphi_\delta$ and $K(\varphi_\delta)$, and $V = \nabla \varphi_\delta$ is a corresponding admissible vector field. Note that this pair is also admissible for the quadratic functional $\Psi_\delta(z) := \frac{1}{2}\langle L_\delta z, z \rangle$. Since 0 is the only critical point of $\Psi_\delta$, it follows e.g. from [14, Corollary 8.3] that if $m$ is the Morse index of $\Psi_\delta$, then

$$c^\delta(\varphi_\delta, K(\varphi_\delta)) = c^\delta(\Psi_\delta, 0) = \delta_{q,m} Z_2.$$ 

A similar argument shows that $c^\delta(\varphi_{-\delta}, K(\varphi_{-\delta})) = \delta_{q,n} Z_2$, where $n$ is the Morse index of $\Psi_{-\delta}$. As the Morse index changes (by $\dim N(L)$) when $\lambda$ passes through 0, $m \neq n$ and $c^\delta(\varphi_\delta, K(\varphi_\delta)) \neq c^\delta(\varphi_{-\delta}, K(\varphi_{-\delta}))$. This is the desired contradiction. \qed
5. Proofs of Theorems 1.3 and 1.4

We assume throughout this section that $V \in L^\infty(\mathbb{R}^N)$ and $f$ satisfies $(f_1)$-$(f_3)$. We consider equation (1.1) which we re-write in the form

$$- \Delta u + V_0(x)u = \lambda u + g(x,u), \quad x \in \mathbb{R}^N,$$

where we have put $V_0(x) := V(x) - m(x)$ and $g(x,u) := f(x,u) - m(x)u$. Let $\lambda_0$ be an isolated eigenvalue of finite multiplicity for $-\Delta + V_0$. Replacing $V_0(x)$ by $V_0(x) - \lambda_0$ we may assume without loss of generality that $\lambda_0 = 0$.

Let $E := L^2(\mathbb{R}^N)$ and $Lu := -\Delta u + V_0(x)u$. As we have pointed out in the introduction, $L$ is a selfadjoint operator whose domain is the Sobolev space $H^2(\mathbb{R}^N)$ and the graph norm of $L$ is equivalent to the Sobolev norm. (A brief argument: using the Fourier transform one readily sees that $-\Delta + 1 : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is an isomorphism; hence the conclusion follows because $V \in L^\infty(\mathbb{R}^N)$.)

We define the operator $N$ (the Nemytskii operator) by setting

$$N(u) := g(\cdot,u(\cdot)), \quad u \in E.$$

It follows from $(f_1)$ and Krasnoselskii’s theorem [11] Theorems 2.1 and 2.3] that $N : E \to E$ is well defined and continuous. Let

$$G(x,s) := \int_0^s g(x,\xi) \, d\xi, \quad x \in \mathbb{R}^N, \quad s \in \mathbb{R}$$

and

$$\psi(u) := \int_{\mathbb{R}^N} G(x,u) \, dx, \quad u \in E.$$

Then $\psi \in C^1(E,\mathbb{R})$ and

$$\nabla \psi(u) = N(u),$$

see [11] Lemma 5.1. Furthermore, let

$$\Phi_\lambda(u) := \frac{1}{2} (Lu - \lambda u, u) - \psi(u), \quad u \in X := H^2(\mathbb{R}^N).$$

Then $\Phi_\lambda \in C^1(X,\mathbb{R})$ and $\Phi_\lambda'(u) = 0$ if and only if $u$ is a solution of (5.1).

Proof of Theorem 1.3 We verify the assumptions of Theorem 1.1. First we show that $N$ is $H$-asymptotically linear and $N'(\infty) = 0$. Let $u_n \to u$ and $\|t_n u_n\| \to \infty$ in $E$. Assume passing to a subsequence that $u_n \to u$ a.e. Since

$$\frac{g(x,t_n u_n)^2}{t_n^2} \leq \left( \frac{\alpha(x)}{t_n} + (\beta + \|m\|_\infty) |u_n| \right)^2,$$

and $g(x,s)/s \to 0$ as $|s| \to \infty$, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n \to \infty} \frac{\|N(u_n)\|^2}{t_n^2} = \int_{\mathbb{R}^N} \lim_{n \to \infty} \frac{g(x,t_n u_n)^2}{(t_n u_n)^2} \, u_n^2 \, dx = 0.$$

Hence (i) of Theorem 1.1 is satisfied. Since $\text{Lip}_\infty(N) = \text{Lip}_\infty(g) \leq \text{Lip}(g) < \text{dist}(0,\sigma_e(L))$ (where the second inequality follows by assumption), also (ii) of this theorem holds. This completes the proof. □
Remark 5.1. As we have mentioned in the introduction, the condition $\text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(L))$ is sharp in the sense that there may be no asymptotic bifurcation if $\text{Lip}(g) > \text{dist}(\lambda_0, \sigma_e(L))$ and other assumptions of Theorem III.3 are satisfied. Let $N = 1$ and suppose $V_0 \in C^1(\mathbb{R})$, $V_0'(x) \leq 0$ for $x$ large, $\lim_{|x| \to \infty} V_0(x) = V_0(\infty)$ exists and $\inf\{\|Lu, u\|_2 = 1\} < V_0(\infty)$. Then $\sigma_e(L) = [V_0(\infty), \infty)$ and $\lambda_0 := \inf \sigma(L) < \inf \sigma_e(L)$ is a simple eigenvalue. Assume without loss of generality that $\lambda_0 = 0$. Assume also that $g$ is independent of $x$, of class $C^1$, $g(0) = \lim_{|s| \to \infty} g(s)/s = 0$ and $\xi := V_0(\infty) + g'(0) < 0$. Given $\varepsilon > 0$, we may choose $g$ so that $\text{Lip}(g) = -g'(0) \in (V_0(\infty), V_0(\infty) + \varepsilon)$. So

$$\text{Lip}(g) - \varepsilon < \text{dist}(0, \sigma_e(L)) = V_0(\infty) < \text{Lip}(g),$$

and according to [21, Theorem 5.4] and the remarks following it, there is no asymptotic bifurcation at any $\lambda > \xi$, in particular, not at $\lambda_0 = 0$. See also the explicit Example 1 after the proof of Theorem 5.4 in [21]. A similar conclusion holds for $N \geq 2$, see [21, Theorem 5.6].

In the proof of Theorem III.3 we shall need an auxiliary result. Let $\lambda_0 = 0$ and write $w(z) = w(0, z)$. Then $w(z)$ satisfies equation (3.2), i.e. we have

$$Lw(z) = PN(w(z) + z).$$

Lemma 5.2. Suppose $(f_1)$-$(f_4)$ are satisfied. Then $\|w(z)\|_\infty \leq C$ for some constant $C > 0$ and all $\|z\| > R$.

Proof. Recall $L := -\Delta + V_0$, where $L : D(L) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. We also define $\tilde{L} := -\Delta + V_0$ when $-\Delta + V_0$ is regarded as an operator in $L^\infty(\mathbb{R}^N)$ (i.e., $\tilde{L} : D(\tilde{L}) \subset L^\infty(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$). By [3, Theorem 1], $\sigma(L) = \sigma(\tilde{L})$ and isolated eigenvalues of $L$ and $\tilde{L}$ have the same multiplicity. Since $Z$ is spanned by eigenfunctions of $-\Delta + V_0$ corresponding to isolated eigenvalues and such eigenfunctions decay exponentially [18, Theorem C.3.4], $Z \subset L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. It follows therefore from [19, Theorem III.6.17] that there is an $L$-invariant decomposition $L^\infty(\mathbb{R}^N) = Z \oplus \tilde{W}$, where $\tilde{Z} = Z$. Moreover, by [9, (III.6.19)],

$$Q := I - P = -\frac{1}{2\pi i} \int_\gamma (L - \lambda \mu)^{-1} d\lambda,$$

where $\gamma$ is a smooth simple closed curve (in $\mathbb{C}$) which encloses all eigenvalues corresponding to $Z$ and no other points in $\sigma(L)$. By [3, Proposition 2.1], $(L - \lambda \mu)^{-1}_{|L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)} = (\tilde{L} - \lambda \mu)^{-1}_{|L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)}$. Hence $Q|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)} = \tilde{Q}|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)}$, where $\tilde{Q}$ denotes the $\tilde{L}$-invariant projection of $L^\infty(\mathbb{R}^N)$ on $Z$, and the same equality holds for $P$ and $\tilde{P} := I - \tilde{Q}$. $\tilde{P}$ is a projection on a subspace of finite codimension, hence it is continuous and therefore $(f_1)$, $(f_4)$ imply $y = y(z) := PN(w(z) + z) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\|y\|_\infty \leq C_1$ for some $C_1$ independent of $z \in \mathbb{Z} - B_R(0)$. Since $\tilde{Q}|_{\tilde{W}}$ has bounded inverse, $\|\tilde{w}\|_\infty \leq C$, where $\tilde{w} = \tilde{w}(z) := \tilde{L}^{-1}y$ (note that for $w = w(z) = L^{-1}y$ we only have a $z$-dependent $L^2$-bound because $N(w + z)$ is not uniformly bounded in $L^2(\mathbb{R}^N)$).

We complete the proof by showing that $w = \tilde{w}$. Let $\mu_n \notin \sigma(L)$, $\mu_n \to 0$. By the resolvent equation [9, (I.5.5)] and §III.6.1,

$$w = L^{-1}y = (L - \mu_n \mu)^{-1}y - \mu_n L^{-1}(L - \mu_n \mu)^{-1}y.$$
and
\[ \tilde{w} = L^{-1}y = (\tilde{L} - \mu_n I)^{-1}y - \mu_n L^{-1}(\tilde{L} - \mu_n I)^{-1}y. \]

Let \( v_n := (L - \mu_n I)^{-1}y \). Since \( y \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), \( \| \cdot \|_{L^2 \cap L^\infty} := \| \cdot \|_2 + \| \cdot \|_\infty \) as well. As the last term on each of the right-hand sides above tends to 0, \((v_n)\) is a Cauchy sequence in \( L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) (with the norm \( \| \cdot \|_{L^2 \cap L^\infty} := \| \cdot \|_2 + \| \cdot \|_\infty \)) which yields \( w = \tilde{w} \).

**Proof of Theorem 1.4** We have already shown that assumptions (i)-(iii) of Theorem 1.2 are satisfied. Suppose first that \((f_4)\) and \((f_5)\) hold. We only need to verify that \( \varphi_0 \) satisfies (PS). Recall from (3.9) that for \( \|z\| > R \)
\[ \varphi_0(z) = \Phi_0(w(z) + z), \]

where we have put \( w(z) = w(0, z) \), and by Proposition 3.6, we have
\[ \langle \nabla \varphi_0(z), \zeta \rangle = \langle Lz, \zeta \rangle - \int_{\mathbb{R}^N} g(x, w(z) + z)\zeta dx \quad \text{for all } z, \zeta \in Z, \|z\| > R. \]

Let \( z = z^+ + z^- + z^0 \in Z^+ \oplus Z^- \oplus Z^0 \), where \( Z^+, Z^- \) respectively denote the subspaces of \( Z \) corresponding to the positive and the negative part of the spectrum of \( L|_Z \) and \( Z^0 := N(L) \subset Z \).

Let \((z_n) \subset Z\) be such that \( \nabla \varphi_0(z_n) \to 0 \). It suffices to consider \( z_n \) with \( \|z_n\| > R \), and we shall show that \((z_n)\) is bounded. Since \( Z \) is spanned by eigenfunctions of \(-\Delta + V_0\) and \( \dim Z < \infty \), it follows from [18, Theorem C.3.4] that there are constants \( \delta, C_0 > 0 \) such that \( |z(x)| \leq C_0 e^{-\delta|x|} \) for all \( x \in \mathbb{R}^N \) and all \( z \in Z \) with \( \|z\| \leq 1 \). In particular, such \( z \) are uniformly bounded in \( L^p(\mathbb{R}^N) \) for any \( p \in [1, \infty] \). Using this, \((f_4)\) and equivalence of norms in \( Z \), we obtain
\[ \left| \langle Lz_n^+, z \rangle \right| \leq \left| \int_{\mathbb{R}^N} g(x, w(z_n) + z_n)z dx + o(1)\|z\| \right| \leq c_1\|z\| \leq c_2 \quad \text{for all } z \in Z^+, \|z\| = 1. \]

Hence \((z_n^+)\) is bounded and a similar argument shows that so is \((z_n^-)\). Suppose \( \|z_n^0\| \to \infty \) and write \( z_n^0 = t_n z_n^0, |z_n^0| = 1 \). Passing to a subsequence, \( z_n^0 \to z^0 \in Z^0 \). Denote
\[ v_n := w(z_n) + z_n^+ + z_n^- . \]

We shall obtain a contradiction with the assumption \( \nabla \varphi_0(z_n) \to 0 \) by showing that
\[ \langle -\nabla \varphi_0(z_n), z_n^0 \rangle = \int_{\mathbb{R}^N} g(x, v_n + t_n z_n^0)z_n^0 dx \not\to 0. \]

By Lemma 5.2, the sequence \((w(z_n))\) is bounded in \( L^\infty(\mathbb{R}^N) \), and since so are the sequences \((z_n^+)\), \( v_n(x) + t_n z_n^0(x) \to \pm \infty \) for all \( x \in A_\pm := \{ x \in \mathbb{R}^N : \pm z^0(x) > 0 \} \).

Suppose \( \pm g_\pm \geq 0 \). Since \( g \) is bounded and \( z_n^0 \) is uniformly bounded in \( L^1(\mathbb{R}^N) \), we may use the Lebesgue dominated convergence theorem to obtain
\[ \lim_{n \to \infty} \int_{A_\pm} g(x, v_n + t_n z_n^0)z_n^0 dx = \int_{A_\pm} g_- z^0 dx \geq 0. \]

By the unique continuation property [5, Proposition 3 and Remark 2], \( z^0(x) \neq 0 \) a.e. Hence the measure of \( \mathbb{R}^N \setminus (A_+ \cup A_-) \) is 0 and thus
\[ \int_{A_+} g_+ z^0 dx + \int_{A_-} g_- z^0 dx > 0. \]
This implies \ref{5.3}. If $\pm g_{\pm} \leq 0$, the same argument remains valid after making some obvious changes.

Suppose now that $(f_4)$ and $(f_6)$ are satisfied. Here we do not know whether (PS) holds for $\varphi_0$, however, we will construct an admissible pair directly by adapting an argument in \[10\], see in particular the proof of Theorem 4.5 there. Suppose $g(x, s) s \geq 0$ in $(f_6)$ and let

$$\mathcal{W} := \{ z \in Z : \| z^\pm \| \leq R_0, \| z^0 \| \leq R_1 \}, \quad \mathcal{W}^- := \{ z \in \mathcal{W} : \| z^- \| = R_0 \text{ or } \| z^0 \| = R_1 \}$$

($R_0, R_1$ to be determined). Boundedness of $g$ and equivalence of norms in $Z$ yield

$$\left| \int_{\mathbb{R}^N} g(x, w(z) + z) z^\pm \, dx \right| \leq c_3 \| z^\pm \|.$$ 

Since $(\pm L z, z^\pm) \geq \varepsilon \| z^\pm \|^2$ for some $\varepsilon > 0$, $\langle \nabla \varphi_0(z), z^+ \rangle \geq \varepsilon \| z^\pm \|^2 - c_3 \| z^\pm \| > 0$ if $\| z^\pm \| = R_0$ and $\langle \nabla \varphi_0(z), z^- \rangle < 0$ if $\| z^- \| = R_0$ provided $R_0$ is large enough. We want to show that there exists a (large) $R_1$ such that $\langle \nabla \varphi_0(z), z^0 \rangle < 0$ for $z$ with $\| z^- \| = R_0$ and $\| z^0 \| = R_1$. Assuming the contrary, $\liminf_{n \to \infty} \langle \nabla \varphi_0(z_n), z_n^0 \rangle \geq 0$ for a sequence $(z_n)$ such that $\| z_n^0 \| \to \infty$. Below we use the same notation as in \ref{5.3}. We have

$$0 = \langle -\nabla \varphi_0(z_n), w(z_n) \rangle = \int_{\mathbb{R}^N} g(x, v_n + t_n z_n^0) w(z_n) \, dx,$$

$g(x, s) \to 0$ as $|s| \to \infty$ (because $h_{\pm} \in L^\infty(\mathbb{R}^N)$ by $(f_6)$) and $|g(x, v_n + t_n z_n^0) z_n^\pm| \leq c_4 e^{-\delta |x|}$. So according to the Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} g(x, v_n + t_n z_n^0) v_n dx = 0.$$ 

Hence Fatou’s lemma and $(f_6)$ give

$$\lim_{n \to \infty} \int_{A_{\pm}} g(x, v_n + t_n z_n^0) t_n z_n^0 dx = \liminf_{n \to \infty} \int_{A_{\pm}} g(x, v_n + t_n z_n^0) (v_n + t_n z_n^0) dx \geq \int_{A_{\pm}} h_{\pm} dx \geq 0.$$

Since by assumption at least one of the integrals on the right-hand side is positive (possibly infinite),

$$\liminf_{n \to \infty} \langle -\nabla \varphi_0(z_n), z_n^0 \rangle = \liminf_{n \to \infty} \int_{\mathbb{R}^N} g(x, v_n + t_n z_n^0) t_n z_n^0 dx > 0,$$

a contradiction. So $R_1$ exists as required and $(\mathcal{W}, \mathcal{W}^\pm)$ is an admissible pair. Now it is easy to see as in the proof of (iii) of Lemma \[1.2\] that this is also an admissible pair for $\varphi_{\pm \delta}$ if $\delta$ is small enough. As in the proof of Theorem \[1.4\] one shows that the critical groups for $\varphi_{\delta}$ and $\varphi_{-\delta}$ are different, and this forces bifurcation.

If $g(x, s) s \leq 0$, a similar argument shows that $\langle \nabla \varphi_0(z), z^0 \rangle > 0$ for some $R_1$, hence the exit set for the flow is $\mathcal{W}^- := \{ z \in \mathcal{W} : \| z^- \| = R_0 \}$.

\[\square\]

**Remark 5.3.** Note that \ref{5.4} is a variant of the Landesman-Lazer condition introduced in \[13\] and Theorem \[1.4\] remains valid if one assumes \ref{5.4} holds for all $z \in N(L)$. This is slightly less restrictive than $(f_5)$. The reason that we have chosen $(f_5)$ is that it is a general condition on $f$, with no reference to eigenfunctions corresponding to $\lambda_0$. $(f_6)$ is a kind of strong resonance condition because $g(x, s) \to 0$ as $|s| \to \infty$. Note also that our arguments show that under the assumptions of Theorem \[1.4\] there is a uniform bound for solutions of \[1.1\] with $\lambda = \lambda_0$. 


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