Inflation from Supersymmetric Quantum Cosmology

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We derive a special scalar field potential using the anisotropic Bianchi type I cosmological model from canonical quantum cosmology under determined conditions in the evolution to anisotropic variables $\beta_{\pm}$. In the process, we obtain a family of potentials that has been introduced by hand in the literature to explain cosmological data. Considering supersymmetric quantum cosmology, this family is scanned, fixing the exponential potential as more viable in the inflation scenario $V(\phi) = V_0 e^{-\sqrt{3}\phi}$.

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I. INTRODUCTION

One of the main problems of inflationary cosmology is to find a mechanism to derive in a natural way the appropriate scalar field potential in order to develop enough e-foldings of inflation. By natural, we understand a mechanism from which some theory provides a scalar field potential that offers the convenient features of inflation. In this work we derive a scalar field potential from supersymmetric quantum cosmology that gives these conditions.

In a previous work, we determined scalar potentials from an exact solution to the Wheeler-DeWitt (WDW) equation in the quantum cosmology scenario [1], using as a toy model an homogeneous and isotropic cosmological model. There we focus on solutions that may be relevant for the early universe constructed within of WKB approximation. Recently, these scalars potentials were

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obtained using a local supersymmetric scheme \cite{2}. Nowadays it is a common issue in cosmology to make use of scalar fields $\phi$ as the responsible agents of some of the most intriguing aspects of our universe \cite{3,11}, such as inflation \cite{12,13}, dark matter and dark energy \cite{14}. The natural derivation of a scalar potential is a challenge, posing the following question: What physical processes provide the adequate scalar field potentials that govern the universe in determined epoch? To answer this question, we use the ideas of quantum cosmology to solve the Wheeler-DeWitt equation with a particular ansatz for the Bianchi type I universe wave-function. In this scheme, we obtain two possible scenarios, the first one with an scalar exponential potential $V(\phi) = V_0 e^{\lambda \phi}$, and the second one giving a family of potentials, similar to those obtained in our previous work \cite{1}. It is interesting that in the first scenario the $\lambda$ parameter is not fixed by the quantum scheme, remaining as a free parameter of the theory. To fix it we invoke supersymmetric scale, using the tools of supersymmetric quantum cosmology in order to find most viable scalar potential for the inflationary epoch, in this scale. To do this, we applied supersymmetry as a square root of general relativity \cite{15,18}, in which the Grassmann variables are only auxiliary and can not identified as the supersymmetric partners of the cosmological bosonic variables. Therefore, we construct a family of scalar potentials treating the quantum solutions to anisotropic Bianchi type I cosmological model in the anisotropic variables $\beta_+$ and $\beta_-$. The conditions we use give us a special structure for the scalar potential; By simplifying the Wheeler-DeWitt equation we obtain two cases: one in which both parameters $\beta_\pm$ have hyperbolic trigonometric functions as solutions, and another where $\beta_- (\beta_+)$ have a trigonometric (hyperbolic trigonometric) behavior. This potential is also a good candidate, depending on the parameter value, in order to study inflation, dark matter, dark energy or tachyon models \cite{19}. The transform Wheeler-DeWitt equation can be solved using a particular ansatz in the WKB approximation (Bohmian representation \cite{20}). This method has been used in the literature \cite{21} to solve the cosmological Bianchi class A models, and in a particular, our result in the second case is similar to the one found in reference \cite{1} for the isotropic Friedmann-Robertson-Walker (FRW) cosmological model. On the other hand, the best candidates quantum solutions become these that have a damping behavior with respect to the scale factor, in sense that we obtain a good classical solution using the WKB approximation in any scenario in the evolution of our universe \cite{22,23}. The supersymmetric scheme have the particularity that is very restrictive because there is more constraints equations applied to the wavefunction. So, in this work we found that exist a tendency for supersymmetric vacua to remain close to their semiclassical limits, because the exact solutions found are also the lowest-order WKB approximation, and not corresponds to the full quantum solutions found previously.
II. THE WHEELER-DEWITT EQUATION

On the Wheeler-DeWitt equation there are a lot of papers dealing with different problems, for example, Gibbons and Grishchuk [24] asked the question of what a typical wave function for the universe is. In reference [25] appears one excellent recopilation of paper on quantum cosmology where the problem of how the universe emerged from Big Bang singularity can no longer be neglected. Also, an important approach to this problem is the wave function proposal in which the universe would be completely self-contained without any singularities and without any edges. Our goal in this paper deals with the problem to built the appropriate scalar potential in the inflationary scenario.

We start by recalling the canonical formulation of the ADM formalism to the diagonal Bianchi Class A cosmological models. The metrics have the form

\[ \text{ds}^2 = -(N^2 - N_iN_j)\text{dt}^2 + e^{2\Omega(t)}e^{2\beta(t)}\omega^i\omega^j, \]  

where \( N \) and \( N_i \) are the lapse and shift functions respectively, \( \Omega(t) \) is a scalar and \( \beta_{ij}(t) \) a 3x3 diagonal matrix, \( \beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \), \( \omega^j \) are one-forms that characterize each cosmological Bianchi type model, and that obey \( d\omega^j = \frac{1}{2}C_{ijk}\omega^i \wedge \omega^k \), \( C_{ijk} \) the structure constants of the corresponding invariance group [26]. The metric for the Bianchi type I, takes the form

\[ \text{ds}_I^2 = -N^2\text{dt}^2 + e^{2\Omega}e^{2\beta_+ + 2\sqrt{3}\beta_-}dx^2 + e^{2\Omega}e^{2\beta_+ - 2\sqrt{3}\beta_-}dy^2 + e^{2\Omega}e^{-4\beta_+}dz^2, \]  

the total lagrangian density function is given by

\[ \mathcal{L}_{\text{Total}} = \mathcal{L}_g + \mathcal{L}_\Lambda + \mathcal{L}_{\text{matter,}\phi} = \sqrt{-g}(R - 2\Lambda) + \mathcal{L}_{\text{matter,}\phi}, \]  

we use as a first approximation a perfect fluid and a scalar field as the matter content, in a comoving frame [26],

\[ \mathcal{L}_{\text{Total}} = \sqrt{-g}(R - 2\Lambda + 16\pi G\rho + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)), \]  

and using [2] we have

\[ \mathcal{L}_{\text{Total}} = e^{3\Omega} \left[ \frac{6}{N}\dot{\Omega}^2 - \frac{6}{N}\dot{\beta}_+^2 - \frac{6}{N}\dot{\beta}_-^2 - \frac{6}{N}\dot{\varphi}^2 + 16\pi GN\rho - 2\Lambda N + \frac{V(\phi)N}{2} \right], \]  

where we redefined the original scalar field as \( \phi = \sqrt{12}\varphi \).
The corresponding momenta are calculated in the usual way

\[ \Pi_\Omega = \frac{\partial L}{\partial \dot{\Omega}} = \frac{12 \dot{\Omega}}{N} e^{3\Omega} \rightarrow \dot{\Omega} = \frac{e^{-3\Omega}}{12} N \Pi_\Omega \]

\[ \Pi_+ = \frac{\partial L}{\partial \beta_+} = -\frac{12 \beta_+}{N} e^{3\Omega} \rightarrow \dot{\beta}_+ = -\frac{e^{-3\Omega}}{12} N \Pi_+ \]

\[ \Pi_- = \frac{\partial L}{\partial \beta_-} = -\frac{12 \beta_-}{N} e^{3\Omega} \rightarrow \dot{\beta}_- = -\frac{e^{-3\Omega}}{12} N \Pi_- \]

\[ \Pi_\varphi = \frac{\partial L}{\partial \varphi} = -\frac{12 \varphi}{N} e^{3\Omega} \rightarrow \dot{\varphi} = -\frac{e^{-3\Omega}}{12} N \Pi_\varphi \]

now writing (5) in canonical form \( L_{\text{can}} = \Pi_\Omega \dot{\Omega} - N \mathcal{H} \) where \( \mathcal{H} \) is the hamiltonian density function,

\[ L_{\text{can}} = \Pi_\Omega \dot{\Omega} + \Pi_+ \dot{\beta}_+ + \Pi_- \dot{\beta}_- + \Pi_\varphi \dot{\varphi} - \frac{N e^{-3\Omega}}{24} \left( \Pi_\Omega^2 - \Pi_+^2 - \Pi_-^2 - \Pi_\varphi^2 - e^{6\Omega} \left[ 384 \pi G \rho - 48 \Lambda + 12 V(\varphi) \right] \right) \]

we obtain the corresponding Hamiltonian density function

\[ \mathcal{H} = \frac{e^{-3\Omega}}{24} \left( \Pi_\Omega^2 - \Pi_+^2 - \Pi_-^2 - \Pi_\varphi^2 - e^{6\Omega} \left[ 384 \pi G \rho - 48 \Lambda + 12 V(\varphi) \right] \right) \]  

(6)

when we include the energy-momentum tensor for a barotropic perfect fluid \( p = \gamma \rho \), we have

\[ \mathcal{H} = \frac{e^{-3\Omega}}{24} \left( \Pi_\Omega^2 - \Pi_+^2 - \Pi_-^2 - \Pi_\varphi^2 + 48 \Lambda e^{6\Omega} - 384 \pi G M_\gamma e^{-3(\gamma-1)\Omega} - 12 e^{6\Omega} V(\varphi) \right) . \]  

(7)

Imposing the quantization condition and applying this hamiltonian to the wave function \( \Psi \), we obtain the WDW equation for these models in the minisuperspace by the usual identification \( p^\mu \) by \(-i\partial_{q^\mu}\) in (7), with \( q^\mu = (\Omega, \beta_+, \beta_-, \varphi) \), and following Hartle and Hawking [22] we consider a semi-general factor ordering which gives

\[ \hat{H} \Psi = \left[ -\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2} + \frac{\partial^2}{\partial \varphi^2} + Q \frac{\partial}{\partial \Omega} + 48 \Lambda e^{6\Omega} - 384 \pi G M_\gamma e^{-3(\gamma-1)\Omega} - 12 e^{6\Omega} V(\varphi) \right] \Psi = 0, \]

where \( Q \) measures the ambiguity in the factor ordering between the scalar function \( \Omega \) and its corresponding momenta. This equation is not easy to solve, first because we do not have the structure of scalar potential, and second, it depends strongly on the class of scenario we analyze with barotropic equation. In the follow, for simplicity we shall use the inflationary case, \( \gamma = -1 \)

Using the following ansatz for the wavefunction \( \Psi(\Omega, \varphi, \beta_\pm) = e^{\pm a_1 \beta_+ \pm a_2 \beta_-} \Xi(\Omega, \varphi) \), we obtain a reduced WDW

\[ \left[ -\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \varphi^2} + Q \frac{\partial}{\partial \Omega} + e^{6\Omega} \left( 48 \Lambda - 384 \pi G M_\gamma - 12 V(\varphi) \right) + c^2 \right] \Xi = 0, \]  

(8)
where the constant $c^2 = a_1^2 + a_2^2$.

Eqn (8) can be written in compact form as

$$
\Box \Xi + Q \frac{\partial \Xi}{\partial \Omega} - U(\Omega, \varphi, \lambda_{eff}) \Xi = 0,
$$

(9)

where the d’Alambertian in two dimensions is redefined as $\Box \equiv -\partial^2_\Omega + \partial^2_{\varphi}$, $\lambda_{eff} = 48\Lambda - 384\pi GM - 1$ is the effective cosmological constant, and the potential $U(\Omega, \varphi, \Lambda) = e^{6\Omega} [12V(\varphi) - \lambda_{eff}] - c^2$.

To solve (8) we take the ansatz, which is similar to the one used in the Bohmian formalism into quantum mechanics

$$
\Xi(\Omega, \varphi) = W(\Omega, \varphi) e^{-S(\Omega, \varphi)},
$$

(10)

where $S(\Omega, \varphi)$ is the superpotential function. Eq (9) can be written as the following set of partial differential equations

$$
(\nabla S)^2 - U = 0,
$$

(11a)

$$
W \left( \Box S + Q \frac{\partial S}{\partial \Omega} \right) + 2\nabla W \cdot \nabla S = 0,
$$

(11b)

$$
\Box W + Q \frac{\partial W}{\partial \Omega} = 0,
$$

(11c)

where the first equation is the classical Hamilton-Jacobi equation, which plays an important role in this work. The different terms in this equation are

$$
\nabla W \cdot \nabla S \equiv - (\partial_\Omega W) (\partial_\Omega S) + (\partial_\varphi W) (\partial_\varphi S), \quad (\nabla)^2 \equiv - (\partial_\Omega)^2 + (\partial_\varphi)^2.
$$

Any exact solution complying with the set of equations (11a, 11b, 11c) will also be an exact solution of the original WDW equation. Following reference [1], first we shall choose to solve eqns (11a) and (11b), whose solutions at the end will have to fulfill with eqn (11c) which play the role of a constraint equation.

Taking the ansatz

$$
S(\Omega, \varphi) = \frac{1}{\mu} e^{3\Omega} g(\varphi) + c(b_1 \Omega + b_2 \Delta \varphi),
$$

(12)

with $\Delta \varphi = \varphi - \varphi_0$, $\varphi_0$ is a constant scalar field, $b_i$ arbitrary constants, Eq (11a) is transformed as

$$
\left[ -\frac{9}{\mu^2} g^2 + \frac{1}{\mu^2} \left( \frac{dg}{d\varphi} \right)^2 - 12V(\varphi) + \lambda_{eff} \right] e^{6\Omega} + c^2 \left[ 1 - b_1^2 + b_2^2 \right] + \frac{6c}{\mu} \left[ \frac{b_2}{3} \frac{dg}{d\varphi} - b_1 g \right] e^{3\Omega} = 0.
$$

(13)

This equation is more difficult to solve, at this point we introduce the main idea of the paper to obtain the scalar potential family, which are strongly dependents to solutions for the anisotropic variables $\beta_\pm$. We include two steps to solve equation (13):
1. First, consider that the second and third parenthesis are null, but maintaining that $c \neq 0$. The first condition implies that $b_1 = \sqrt{1 + b_2^2}$, and the constants $a_1$ and $a_2$ are real, the solutions for the anisotropic variables $\beta_{\pm}$ can be considered as hyperbolic trigonometric functions. The second condition becomes an ordinary differential equation for the unknown function $g(\varphi)$, yielding

$$g(\varphi) = g_0 e^{\pm \Delta \varphi},$$

(14)

with $g_0$ an integration constant and $\alpha = \frac{6b_1}{b_2} = \pm 6\sqrt{1+b_2^2}$ with $b_2 \neq 0$. The scalar potential function become, when we take the first parenthesis in eqn (13),

$$V(\varphi) = (4\Lambda - 32\pi GM_{-1}) + V_0 e^{\alpha \Delta \varphi},$$

(15)

with $V_0 = \frac{3g_0^2}{4b_2^2 \mu^2}$. With these results, the superpotential function (12) is

$$S(\Omega, \varphi) = \frac{g_0}{\mu} e^{3\Omega} e^{\frac{2}{3} \Delta \varphi} + c \left( \pm \sqrt{1 + b_2^2 \Omega + b_2 \Delta \varphi} \right).$$

(16)

Substituting (16) into (15), the corresponding solutions for the function $W$ in the form $e^{q\Omega + \eta \varphi}$ become

$$W = \text{Exp} \left[ \frac{1}{2} (Q - 3) \Omega - \frac{1}{4} \alpha \Delta \varphi \right],$$

we develop the following wavefunction

$$\Psi = \text{Exp} \left[ a_1 \beta_+ + a_2 \beta_- + \frac{1}{2} (Q - 3) \Omega - \frac{1}{4} \alpha \Delta \varphi \right] e^{-e^{3\Omega + \frac{2}{3} \Delta \varphi} \mu},$$

(17)

with the constraint on the parameter $\alpha \leq \pm 6$.

In the literature [28], the scalar potential type $V(\phi) = e^{\lambda \phi}$ gives a power law in the classical scale factor, considering the flat FRW cosmological model when $\lambda < -\sqrt{2}$. If we consider the extreme values for the $\alpha$ parameter ($Q=0$) and the corresponding transformation between $\varphi \rightarrow \phi$, we obtain the special scalar potential $V(\phi) = V_0 e^{\pm \sqrt{3} \phi}$, for standard inflationary model, this class of potential has the advantage that classical analytical solutions can be found and for appropriate values of the parameters, inflation can be obtained.

2. In the second step, we consider that the constant $c=0$, implying that $a_1 = \pm ia_2$, then the solutions for the wave function for the anisotropic variables $\beta_{\pm}$ are considered trigonometric functions for the variable $\beta_+$ and hyperbolic trigonometric function for the variable $\beta_-$. Thus the superpotential term (12) has the simple form

$$S(\Omega, \varphi) = \frac{1}{\mu} e^{3\Omega} g(\varphi),$$

(18)
and equation (13) becomes an ordinary differential equation for the unknown function \( g(\varphi) \) in terms of the scalar potential \( V(\varphi, \lambda_{\text{eff}}) = V(\varphi) - \frac{\lambda_{\text{eff}}}{12} \),

\[
\left( \frac{dg}{d\varphi} \right)^2 - 9g^2(\varphi) = 12\mu^2 \left( V(\varphi) - \frac{\lambda_{\text{eff}}}{12} \right) = 12\mu^2 V(\varphi, \lambda_{\text{eff}}),
\]

(19)

this equation is similar to the one obtained in reference [1]. It is not surprising that this equation is similar to eqn (12) in reference [1], because the anisotropic Bianchi type I cosmological model is the generalization of the flat FRW model. The last equation has several exact solutions, which can be generated in the following way. Consider that \( V = g^2 F(g) \), where \( F(g) \) is an arbitrary function of its argument. So, eq. (19) can be written in cuadratures as

\[
\Delta \varphi = \pm \frac{1}{2\sqrt{3}} \int \frac{d\ln g}{\sqrt{\frac{3}{4} + \mu^2 F(g)}}.
\]

(20)

In this way, we can solve the \( g(\varphi) \) function, and then use the expression for the potential term \( V = g^2 F(g) \) back again to find the corresponding scalar potential that leads to an exact solution to the Hamilton-Jacobi equation (11a). Some examples are shown in Table I.

| \( F(g) \) | \( g(\varphi) \) | \( V(\varphi, \lambda_{\text{eff}}) \) |
|---|---|---|
| 0 | \( \text{Exp} \left[ \pm 3\Delta \varphi \right] \) | 0 |
| \( V_0 g^{-2} \) | \( \sqrt{\frac{4\mu^2 V_0}{3}} \sinh(\pm 3\Delta \varphi) \) | \( V_0 \exp (\alpha \Delta \varphi), \alpha = \pm 4\sqrt{3}\sqrt{\frac{3}{4} + \mu^2 V_0} \) |
| \( V_0 \) | \( e^{\pm \Delta \varphi} \) | \( \frac{e^{\pm \Delta \varphi}}{2\sqrt{3}} \left[ e^{\Delta \varphi} - 4\mu^2 V_0 \right]^{2(2-n)/n}, \eta = \frac{3n}{2} \) |
| \( V_0 g^{-n} \ (n \neq 2) \) | \( e^{u(\varphi)} \) | \( \frac{ue^{2u}}{2\sqrt{3}}, u = \left( \mu \sqrt{3} \Delta \varphi \right)^2 - \frac{3}{2\mu^2} \) |
| \( \ln g \) | \( e^{r(\varphi)} \) | \( r^2 e^{2r}, r = \frac{1}{2} \left[ e^{u(\varphi)} - \frac{3}{4\mu^2} e^{-u(\varphi)} \right], u = 2\sqrt{3} \mu \Delta \varphi \) |
| \( (\ln g)^2 \) | | |

In this way, the superpotential \( S(\Omega, \varphi) \) is known.

For solve (11b) we assume

\[
W = e^{[\varepsilon(\Omega)+\omega(\varphi)]},
\]

(21)
we arrive to a set of ordinary differential equations for the functions \(z(\Omega)\) and \(\omega(\varphi)\)

\[
2 \frac{dz}{d\Omega} - Q = k, \quad \Rightarrow \quad z(\Omega) = \frac{Q + k}{2} \Omega
\]

\[
\frac{d^2 g}{d\varphi^2} + 2 \frac{dg}{d\varphi} \frac{d\omega}{d\varphi} = 3(k + 3)g, \quad \Rightarrow \quad \omega(\varphi) = \frac{3k}{2} \int \frac{d\varphi}{\partial_\varphi(\ln g)} - 3\mu^2 \int \frac{d[V(\varphi, \lambda_{eff})]}{(\partial_\varphi g)^2}.
\]

Thus, the explicit form for the function \(W\) becomes

\[
W = \exp \left\{ \frac{3k}{2} \left[ \frac{\Omega}{3} + \int \frac{d\varphi}{\partial_\varphi(\ln g)} \right] + \frac{Q}{2} \Omega - 3\mu^2 \int \frac{d[V(\varphi, \lambda_{eff})]}{(\partial_\varphi g)^2} \right\}.
\]

The constraint (11C) can be written as

\[
\partial_\varphi^2 \omega + \left( \partial_\varphi \omega \right)^2 - \frac{k^2 - Q^2}{4} = 0,
\]

and

\[
\partial_\varphi \omega = \frac{3k}{2\partial_\varphi(\ln g)} - 3\mu^2 \frac{\partial_\varphi [V(\varphi, \lambda_{eff})]}{(\partial_\varphi g)^2}.
\]

Taking into account table I, we present the corresponding wave function in each case, in table II.

| \(g(\varphi)\) | wave function \(\Psi\) |
|-----------------|------------------|
| \(\exp[\pm 3\Delta \varphi]\) | \(\exp \left\{ a_2(\pm i\beta_+ + \beta_-) + \left( \frac{k+Q}{2} \right) \Omega \pm \frac{4\Delta \varphi}{2} \right\} e^{-\frac{3\Delta \varphi}{4} + \frac{\varphi}{\mu}} \) |
| \(\sqrt{\frac{4\mu^2 V_0}{3}} \sinh(3\Delta \varphi)\) | \(\cosh^\frac{3}{2}(-3\Delta \varphi) \exp \left\{ a_2(\pm i\beta_+ + \beta_-) + \left( \frac{k+Q}{2} \right) \Omega \right\} e^{-\frac{3\Delta \varphi}{4} + \frac{\varphi}{\mu}} \) |
| \(e^{\frac{3}{2} \Delta \varphi}\) | \(\exp \left\{ a_2(\pm i\beta_+ + \beta_-) + \left( \frac{k+Q}{2} \right) \Omega + \left( \frac{3k-12\mu^2}{\alpha} \right) \Delta \varphi \right\} e^{\frac{3\Delta \varphi}{4} + \frac{\varphi}{\mu}} \) |
| \(\left[ e^{\Delta \varphi} - 4\mu^2 V_0 e^{-\Delta \varphi} \right]^{\frac{1}{2}}\) | \(\omega(\varphi) = \frac{k}{2} \left[ -\Delta \varphi + \frac{2}{3n} \ln \left( 4\mu^2 V_0 + e^{2n\Delta \varphi} \right) - \frac{\mu^2 (2-n)}{3nV_0} \arctan \left( \frac{1}{4\mu^2 V_0} e^{2n\Delta \varphi} \right) \right] \) |
| \(e^{u(\varphi)}\) | \(\exp \left\{ a_2(\pm i\beta_+ + \beta_-) + \left( \frac{k+Q}{2} \right) \Omega + \frac{k^2 - \mu^2}{4\mu^2} \ln \Delta \varphi - \frac{4\mu^2 \Delta \varphi}{2} \right\} e^{-\frac{3\Delta \varphi}{4} + \frac{\varphi}{\mu}} \) |
| \(e^s(\varphi)\) | \(\omega(\varphi) = \frac{\sqrt{\Delta \varphi}}{6\mu} \arctan \left( \frac{2\varphi}{\sqrt{3} e^{u(\varphi)}} \right) \exp \left\{ a_2(\pm i\beta_+ + \beta_-) + \left( \frac{k+Q}{2} \right) \Omega + \omega(\varphi) \right\} e^{-\frac{3\Delta \varphi}{4} + \frac{\varphi}{\mu}} \) |

In special case, the third line in table II, have a damping term that correspond to \(e^{-S}\) and a plane wave type, similar to equation (17) obtained in the first step. The first line has this behaviour, but corresponds to null scalar potential.
In this way, using the quantum formalism in the sector of inflationary scenario, we found that the scalar potential becomes an exponential behaviour, however the coupling constant is undetermined. The question is, how can we fix the coupling constant?. The answer could be in the supersymmetric quantum cosmology using differential operators to the Grassmann variables, where we present the formalism in next section.

III. SUPERSYMMETRIC QUANTUM MECHANICS

In the following we shall apply the supersymmetric quantum formalism at the quantum structure obtained in the previous section, to obtain a closed value to the parameters that optimize the inflation scenario, i.e, we do an analysis to the family obtained in table I for the function $g(\varphi)$; or in other words, which is the constraint on the superpotential function that appears in the quantum level, equation (26), based in tables I or II?. In this order of ideas, we found one integrability condition on the $g(\varphi)$ function, which fix the coupling parameter of our problem.

For obtain these results, in this section we consider only a reduced supersymmetry in two bosonic variables $(\Omega, \varphi)$, without consider the anisotropic parameters, due that the full problem do not contemplate the initial condition on our original problem. For instance the decomposition of the wave function in the full expansion have 16 components, or $16 \times 16$ matrix components, and the solution is very complicated. In this sense, for solve our problem we use reduced bosonic hamiltonian, eqn (8), and in consequence, one reduced supersymmetry.

The idea of Witten [29] is to find the supersymmetric supercharges operators $Q, \bar{Q}$ that produce a superhamiltonian $\mathcal{H}_{\text{susy}}$, that satisfies the closed superalgebra

$$\mathcal{H}_{\text{susy}} = \frac{1}{2} [Q, \bar{Q}], \quad [Q, \bar{Q}] = [Q, Q] = 0,$$

where the superhamiltonian $\mathcal{H}_{\text{susy}}$ has the following form

$$\mathcal{H}_{\text{susy}} = \mathcal{H}_b + \frac{\partial^2 S}{\partial q^\mu \partial q^\nu} \left[ \psi^\mu, \psi^\nu \right],$$

with $\mathcal{H}_b$ is the bosonic hamiltonian (8) taking the constant $c=0$, then the supersymmetric approach will only be applied to reduced hamiltonian, and $S$ is the corresponding superpotential function that is related with the potential term that appear in the bosonic hamiltonian, i.e, have the same structure that in the quantum level, proposed in the last section. This idea was applied in reference [18] for all Bianchi Class A models without matter content, and in [30] to FRW cosmological model. For example in reference [31] it is explained that in particular, we can formulate a particle dynamics
in a potential $V(q^\mu)$ on a curved manifold and supersymmetry requires that the potential $V(q^\mu)$ is derivable from a globally defined superpotential $S(q^\mu)$ via $V(q^\mu) = \frac{1}{2}G_{\mu\nu}(q)\frac{\partial S(q)}{\partial q^\mu}\frac{\partial S(q)}{\partial q^\nu}$, where $G_{\mu\nu}(q)$ is the metric in the curved space. This equation is represented in the quantum level by eq. (11a).

In this approach, a supersymmetric state with $Q|\psi> = 0$ is automatically a zero energy ground state, in a similar way that in the quantum regime. This simplifies the problem of finding supersymmetric ground state because the energy is known as priori and the factorization of $H_{susy}|\psi> = 0$ into $Q|\psi> = 0$, $\bar{Q}|\psi> = 0$ often provides a simpler first order equation for the ground state wavefunction. The simplicity of this factorization is related to the solubility of certain bosonic hamiltonians. In this work, as in others, we find for the empty (+) and filled (-) sector of the expansion of the wavefunction in this approach, in the sector of the fermion Fock space zero energy solutions $|A_\pm> = e^{\pm S}|\pm>$ where $A_\pm$ are the corresponding components for the empty and filled fermionic sector.

The corresponding supercharges that satisfy the superalgebra, when we consider the bosonic hamiltonian given by equation (8) become

$$Q = \psi^\mu \left[ \frac{\partial}{\partial q^\mu} + \frac{\partial S}{\partial q^\mu} \right],$$

$$Q = \bar{\psi}^\nu \left[ \frac{\partial}{\partial q^\nu} - \frac{\partial S}{\partial q^\nu} \right].$$

Equation (28) are

$$Q = - \left[ \frac{\partial}{\partial q^0} + \frac{\partial S}{\partial q^0} \right] \frac{\partial}{\partial \theta^0} + \left[ \frac{\partial}{\partial q^1} + \frac{\partial S}{\partial q^1} \right] \frac{\partial}{\partial \theta^1},$$

$$Q = \theta^0 \left[ \frac{\partial}{\partial q^0} - \frac{\partial S}{\partial q^0} \right] + \theta^1 \left[ \frac{\partial}{\partial q^1} - \frac{\partial S}{\partial q^1} \right].$$

The decomposition of the wavefunction becomes

$$\Xi(\Omega, \varphi) = A_+ + B_0 \theta^0 + B_1 \theta^1 + A_- \theta^0 \theta^1,$$
The supersymmetric equations \( Q_\mathcal{E} > 0, \bar{Q}_\mathcal{E} > 0 \) are

\[
Q_\mathcal{E} = - \left[ \frac{\partial}{\partial q^0} + \frac{\partial S}{\partial q^0} \right] \frac{\partial}{\partial \theta^0} \left[ A_+ + B_0 \theta^0 + B_1 \theta^1 + A_- \theta^0 \theta^1 \right]
+ \left[ \frac{\partial}{\partial q^1} + \frac{\partial S}{\partial q^1} \right] \frac{\partial}{\partial \theta^1} \left[ A_+ + B_0 \theta^0 + B_1 \theta^1 + A_- \theta^0 \theta^1 \right],
\]

(33)

\[
\bar{Q}_\mathcal{E} = \theta^0 \left[ \frac{\partial}{\partial q^0} - \frac{\partial S}{\partial q^0} \right] \left[ A_+ + B_0 \theta^0 + B_1 \theta^1 + A_- \theta^0 \theta^1 \right]
+ \theta^1 \left[ \frac{\partial}{\partial q^1} - \frac{\partial S}{\partial q^1} \right] \left[ A_+ + B_0 \theta^0 + B_1 \theta^1 + A_- \theta^0 \theta^1 \right].
\]

(34)

Then (34) gives the following set of differential equations

\[
\theta^0 : \left[ \frac{\partial A_+}{\partial q^0} - A_- \frac{\partial S}{\partial q^0} \right] = 0,
\]

(35)

\[
\theta^1 : \left[ \frac{\partial A_+}{\partial q^1} - A_- \frac{\partial S}{\partial q^1} \right] = 0,
\]

(36)

\[
\theta^0 \theta^1 : \left[ \frac{\partial B_1}{\partial q^0} - B_1 \frac{\partial S}{\partial q^0} \right] - \left[ \frac{\partial B_0}{\partial q^1} - B_0 \frac{\partial S}{\partial q^1} \right] = 0,
\]

(37)

whose solutions to equations (35, 36) are

\[
A_+ = a_+ e^S,
\]

(38)

On the other hand, eq. (33) gives

\[
\text{free term} : - \left[ \frac{\partial B_0}{\partial q^0} + B_0 \frac{\partial S}{\partial q^0} \right] + \left[ \frac{\partial B_1}{\partial q^1} + B_1 \frac{\partial S}{\partial q^1} \right] = 0,
\]

(39)

\[
\theta^1 : \left[ \frac{\partial A_-}{\partial q^0} + A_- \frac{\partial S}{\partial q^0} \right] = 0,
\]

(40)

\[
\theta^0 : \left[ \frac{\partial A_-}{\partial q^1} + A_- \frac{\partial S}{\partial q^1} \right] = 0,
\]

(41)

where (39) can be written as

\[
\eta^{\mu \nu} (\partial_\mu B_\nu + B_\nu \partial_\mu S) = 0,
\]

(42)

considering the following ansatz for the fields \( B_\mu \)

\[
B_\nu = e^S \partial_\nu f_+,
\]

(43)

(37) is satisfied identically, and (42) is

\[
\eta^{\mu \nu} (\partial_\mu \partial_\nu f_+ + \partial_\nu f_+ \partial_\mu S + \partial_\nu \partial_\mu f_+ S) = \eta^{\mu \nu} (\partial_\mu \partial_\nu f_+ + 2 \partial_\nu f_+ \partial_\mu S) = 0.
\]

(44)

where a possible solution is \( f_+ = h(\Omega \pm \varphi) \), and \( h \) is any function dependind to the argument, given the following constrain on the superpotential function

\[
\frac{\partial S}{\partial \Omega} = \pm \frac{\partial S}{\partial \varphi},
\]

(45)
and considering the structure of the superpotencial (18) we find one condition on integrability over the function \( g(\varphi) \), given
\[
g(\varphi) = g_0 e^{\pm 3\Delta \varphi}. \tag{46}
\]
and taking into account tables I or II, we obtain the following constraint in the parameter \( \alpha \) of the models when this last equation (45) is satisfied
\[
\frac{\alpha}{2} = \pm 3,
\]
so, only the exponential scalar potential can survive in table I, and the coupling constant become \( \alpha = \pm 6 \), given the scalar potential \( V(\phi) = V_0 e^{-\sqrt{3} \Delta \phi} \). In this way, supersymmetric quantum mechanics fix the values for the \( \alpha \) parameter, being valid the argument introduced in the quantum scheme.

Equations, (40, 41) can be written as
\[
\begin{align*}
\frac{\partial A}{\partial q^\mu} - \frac{\partial q^\mu}{\partial A} + A_+ \frac{\partial S}{\partial q^\mu} & = 0, \\
n\frac{1}{A_-} \frac{\partial A_-}{\partial q^\mu} & = - \frac{\partial S}{\partial q^\mu} \quad \rightarrow \quad \frac{\partial \ln A_-}{\partial q^\mu} = - \frac{\partial S}{\partial q^\mu},
\end{align*}
\tag{47}
\]
with solution
\[
A_- = a_- e^{-S}, \tag{48}
\]
then, the set of contributions for the supersymmetric wave functions are find,
\[
\begin{align*}
A_+ & = a_+ e^{\pm S} \\
B_0 & = e^S \partial_0 (f_+) \\
B_1 & = e^S \partial_1 (f_+)
\end{align*}
\]
It is interesting to note that supersymmetry is very restrictive because exist more constraints equations applied to the wave function. In this sense, we observe a tendency for supersymmetric vacua to remain close to their semi-classical limits, because the exact solutions found are also the lowest-order WKB approximation.

\section*{IV. CONCLUSIONS}

Using the quantum formalism in the inflationary scenario, we find that the scalar potential has an exponential behaviour as a good candidate. However, the coupling constant is undetermined. The question was, how can we fix the value of the coupling constant? The answer was in the
supersymmetric quantum cosmology using differential operators to the Grassmann variables, where
the coupling constant is found under one condition of integrability on the function \( g(\varphi) = g_0 e^{\pm 3 \Delta \varphi} \),
and taking into account tables I or II, \( \frac{\alpha}{\pi} = \pm 3 \). So, the main goal in this paper was to fix the
value for the coupling constant to the inflationary scenario \( \lambda = \frac{\alpha}{\pi} = \pm 3 \) using the supersymmetric
approach, when the quantum approach only gives the general structure for the scalar potential.
Also we find exact solutions in both regimes. In the quantum level, we found that the possible
solutions become the contributions to the empty(+) and filled (-) sector of decomposition to the
wavefunction in the supersymmetric approach.

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