Stationarity of Switching VAR and Other Related Models

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Abstract

Switching ARMA models greatly enhance the standard linear models to the extent that different ARMA model is allowed in a different regime, and the regime switching is typically assumed a Markov chain on the finite states of potential regimes. Although statistical issues have been the subject of many recent papers, there is few systematic study of the probabilistic aspects of this new class of nonlinear models. This paper discusses some basic issues concerning this class of models including strict stationarity, influence of initial conditions, and second-order property by studying SVAR models. A number of examples are given to illustrate the theory and the variety of applications. Extensions to other models such as mean-shifting, and inhomogeneous transition probabilities are discussed.

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1 Introduction

Switching ARMA models belong to a new class of time series models which are capable of capturing various nonlinear aspects of time series data such as nonnormality, asymmetry, irreversibility, and variable predictability [e.g. Hamilton 1989; Huges and Guttord 1994; Krolzig 1997; Lu and Berliner 1997]. This class of models extends the ARMA linear system to the extent that different ARMA model is allowed in a different regime, and the regime switching is typically assumed a Markov chain on the finite states of potential regimes. While statistical aspects of fitting these models have been much discussed as summarized by Krolzig (1997); there is, however, few systematic study of the probabilistic aspects of switching ARMA models, such as stationarity or ergodicity.

This paper discusses some general conditions that ensure stationarity and other probabilistic properties such as existence of moments. A general theory due to Brandt (1986) is reviewed (Section 2.2). A theory of stability (or, of the noninfluence of initial conditions) of switching vector autoregressive models (SVAR) is developed (Section 2.3). Some interesting examples are given to illustrate the subtle generality of the developed stationarity conditions and the variety of applications of the switching vector autoregressive models. For example, we exhibit (as in Holst et al (1994)) that unstable subprocesses and stable processes can be mixed to produce a stationary process (Example 2), two unstable subprocesses can still be mixed to be stationary (Example 4), and stable subprocesses may not always produce stationary mixed process, and a counter-example is given (Example 3). The second-order theory of switching AR models is developed (Section 4). We also discuss the mean shifting models (Section 3.3), switching moving average, and switching ARMA models (Section 5).
2 General theory

2.1 Switching vector AR models

A general model is the following vector stochastic difference equation

\[ X_n = A_n X_{n-1} + E_n, \quad n \in \mathbb{Z}, \]  

(2.1)

where \( X_n \in \mathbb{R}^p \) and \( A_n \) is a \( p \times p \) matrix and \( E_n \in \mathbb{R}^p \) is a noise vector. Various additional structure will be imposed on \( A_n, E_n \) later. For example, an AR(p) process can be represented as (2.1) in which \( A_n \) is a constant matrix assuming a special structure. When \( \{(A_n, E_n)\} \) is iid, (2.1) is called the Random Coefficient Autoregressive (RCA) model (Nicholls and Quinn, 1992). Since in large part such a system is used for modelling stationary time series data, stationarity property is a priority in the study of probabilistic aspects of such random dynamical systems. A theory for the general stochastic equations (2.1) is reviewed in Section 2.2. However, one of our objectives is to study the so-called Markov switching vector AR(1) model (SVAR(1)):

Suppose there are \( r \) potential regimes, say \( S = \{1, 2, \ldots, r\} \) and \( I_n \) is a Markov chain taking on values in \( S \). Define

\[ A_n = \sum_{i=1}^{r} B_i 1\{I_n=i\}, \]  

(2.2)

where \( B_1, \ldots, B_r \) are \( r \) unknown or partially unknown \( p \times p \) matrices; and

\[ E_n = \sum_{i=1}^{r} \Sigma i \varepsilon_{ni} 1\{I_n=i\} \]  

(2.3)

where \( \{\varepsilon_{ni}\} \) are independent processes, each subsequence is iid within itself, having zero mean and identity covariance matrix. In addition, we make the assumption of independence, that \( \{I_n\} \) is independent of noise processes \( \{\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nr}\} \). We also assume that \( \{I_n\} \) is irreducible and aperiodic, thus ergodic.

First we ask the question whether there exists a strict stationary solution for (2.1)? Since a strict stationary process may not have any moment existing, this is a fairly weak assumption. Though necessary and sufficient stationarity conditions for
RCA models are available [e.g. Nicholls and Quinn (1992)], necessary and sufficient stationarity conditions when \( \{A_n\} \) is dependent has not yet been given (see, however, Bougerol and Picard 1992). However, very general sufficient condition that ensure stationarity can be formulated from the proof of Brandt (1986), and is first made known in Bougerol and Picard (1992). For convenience of later use, we will restate a general theorem related to this theory. Here it will be assumed that the super-process \( \{(A_n, E_n)\}_{n=-\infty}^{\infty} \) are (jointly) stationary matrices and vectors. It appears that all known results in this area make this convenient assumption, though more can be said in our setup (later).

2.2 Brandt’s result

We first state a general result giving sufficient conditions for strict stationarity. Here, we do not need to assume that \( A_n \) takes on discrete values \( B_i \)'s. In the case of SVAR(1), stationarity of \( \{A_n, E_n\} \) is equivalent to assuming that the ergodic chain \( \{I_n\} \) starts from the remote past or \( I_0 \) takes on the stationary distribution.

The tool is the theory of Lyapunov exponents or product of random matrices. A technical assumption that ensures existence of Lyapunov exponents for a stationary sequence of random matrices \( A_1, A_2, \ldots, A_n, \ldots \) is

\[
E \max(\log \|A_1\|, 0) < \infty.
\]  

(2.4)

This is obviously satisfied if \( A_1 \) takes on only finite number of values as in the case of SVAR(1).

Under (2.4) the (largest) Lyapunov exponent is defined as

\[
\lambda = \lim_{n\to\infty} (1/n) \log \|A_n \ldots A_1\|
\]  

(2.5)

which holds almost surely.

Furthermore, if the process is ergodic, the Lyapunov exponent is constant and

\[
\lambda = \inf \{(1/n)E\log \|A_n \ldots A_1\|, n \geq 1\}.
\]  

(2.6)
The existence of the limit in (2.5) and (2.6) can be justified using Kingman’s subadditive ergodic theorem. Similarly, the following limit theorem holds with the same Lyapunov exponent as a consequence of (2.6):

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log \|A_0 A_{-1} \ldots A_{-n+1}\|. \quad (2.7)$$

Note that $\lambda$ is defined independent of the particular matrix norm used.

More generally, under stationarity and (2.4), one can apply Oseledec’s multiplicative ergodic theorem to define a spectrum of Lyapunov exponents $\lambda_1 = \lambda \geq \lambda_2 \geq \ldots \geq \lambda_p$:

$$\lambda_i = \lim_{n \to \infty} \frac{1}{n} \log \delta_i(n), \text{ holds almost surely for } 1 \leq i \leq p, \quad (2.8)$$

where $\delta_1(n) \geq \ldots \geq \delta_p(n)$ are the singular values of $A_n A_{n-1} \ldots A_1$. Under ergodicity, $\lambda_i$’s are constants, independent of the particular realization in $\{A_n\}$.

**Proposition 1** Given that the super-process $\{A_n, E_n\}$ is stationary and ergodic. Suppose that $P(A_0 = 0) > 0$ or the following conditions are met: (2.4) holds and the Lyapunov exponent for $\{A_n\}$ is negative; that is

$$(NL): \quad \lambda = \lim_{i \to \infty} \frac{1}{i} \log \|A_0 A_{-1} \ldots A_{-i+1}\| < 0 \quad (2.9)$$

and the noise satisfies

$$\mathbb{E} \max(\log \|E_1\|, 0) < \infty. \quad (2.10)$$

Then (i)

$$W_n = E_n + \sum_{i=0}^{\infty} A_n A_{n-1} \ldots A_{n-i} E_{n-i-1} \quad (2.11)$$

is the only proper stationary solution of (2.1) for the given $\{A_n, E_n\}$. (ii) The sum on the right-hand side of (2.11) converges absolutely almost surely. (iii) Furthermore,

$$P(\lim_{n \to +\infty} |X_n(x) - W_n| = 0) = 1, \quad (2.12)$$

for arbitrary random variable $X_{-m-1} = x$ at time $-m-1$ (defined on the same probability space as $\{A_n, E_n\}$), in particular

$$X_n(x) \xrightarrow{d} W_0, \text{ as } n \to +\infty. \quad (2.13)$$
Part (i) of this result is first given in Bougerol and Picard (1992). The proof is similar to the one-dimensional case as proved by Brandt (1986) under a stronger assumption. See Bougerol and Picard (1992) for more details and a necessary condition.

For convenience of readers, we prove the following lemma for part (ii).

**Lemma 1** If the stationary super-process \( \{A_n, E_n\} \) satisfies (2.9) and (2.10), the RHS of (2.11) converges absolutely almost surely.

**Proof.** First, by (2.9) and (2.10)

\[
\limsup_{i \to \infty} (1/(i+1)) \log \|A_n A_{n-1} \ldots A_{n-i} E_{n-i-1}\| \\
\leq \limsup_{i \to \infty} (1/(i+1)) \log \|A_n A_{n-1} \ldots A_{n-i}\| + (1/(i+1)) \log \|E_{n-i-1}\| \\
\leq \lambda + 0 < 0, \text{ a.s.}
\]

which implies that

\[
\limsup_{i \to \infty} \|A_n A_{n-1} \ldots A_{n-i} E_{n-i-1}\|^{1/i} < 1 \text{ a.s.}
\]

Thus, the RHS of (2.11), which is bounded by

\[
\|E_n\| + \sum_{i=0}^{\infty} \|A_n A_{n-1} \ldots A_{n-i} E_{n-i-1}\|
\]

is absolutely convergent almost surely by virtue of Cauchy’s root criterion. \( \square \)

Since the process \( W_n \) defined by (2.11) is a well-defined moving average function of ergodic stationary process \( \{A_n, E_n\} \), it follows that it is stationary and ergodic. Thus, \( W_n \) is a \( MA(\infty) \) process with random coefficients.

A key idea in the proof of Proposition 1 is based on the following expansion which holds for any integers \( m \) and \( n \) as implied by the recursive nature of (2.1)

\[
X_n(x) = A_n A_{n-1} \ldots A_1 A_0 A_{-1} \ldots A_{-m} x \\
+ \sum_{i=0}^{n+m-1} A_n A_{n-1} \ldots A_{n-i} E_{n-i-1} + E_n \quad (2.14)
\]
where $X_n(x)$ can be interpreted as the state at time $n$ of the system governed by (2.1) if it starts at time $-m - 1$ with the random initial state $X_{-m-1} = x$. Thus, $W_n$ can be regarded as the limit of $X_n(x)$ starting from the remote past.

Further, part (iii) of the theorem says that $X_n(x)$ converges to $W_n$ forward in time as time $n$ tends to the future. This follows from that

$$X_n(x) - W_n = A_n A_{n-1} ... A_1 A_0 A_{-1} ... A_{-m} x + \sum_{i=n+m}^{\infty} A_n A_{n-1} ... A_{n-i} E_{n-i-1}$$

which tends to zero almost surely under condition (NL) thanks to Lemma 1.

Remark 1. Since for any positive random variable $X$, by Jensen’s inequality $E \log X \leq \log E X$ holds whenever $EX < \infty$, it follows that whenever $EX^\alpha < \infty$ for any $\alpha > 0$ we have $E \log X < \infty$ and hence $E \max(0, \log X) < \infty$. (Note that $\max(0, \log X)$ represents the positive part of $\log X$.)

Next, we consider the more realistic situation that a Markov switching process starts from a finite time in the past and discuss when such a process can be stationary and ergodic.

### 2.3 Stability of SVAR models

Under (2.4) the (largest) Lyapunov exponent is defined as in 2.5

$$\lambda = \lim_{n \to \infty} (1/n) \log \|A_n ... A_1\| \text{ almost surely.} \quad (2.15)$$

Now consider the situation that the SVAR(1) process starts at some fixed time, say time 0, with some arbitrary starting value $X_0$ and the regime process $\{I_n\}$ starts from an arbitrary distribution $I_0$. Let $X_n(X_0, I_n(I_0))$ denote the process evolved according to (2.1) with starting value $X_0$ and starting regime $I_0$ at time 0. The question arises as to what’s the influence of the initial condition or the transient effect. Naturally, one would hope that the initial effect will eventually be washed out or vanish. It is
indeed so. We prove it in the next theorem after illustrating a lemma. The result of this lemma is well known (e.g., Bhattacharya and Waymire (1990), p.197) however we put it here for the sake of completeness.

**Lemma 2** Let $I^1(\cdot)$ and $I^2(\cdot)$ be two independent replicas of an irreducible and aperiodic Markov chain $I(\cdot)$ with finite state space $S$ (with $r$ number of elements), having the same transition probability ($P = ((p_{ij}))$). Define,

$$\tau = \inf\{k \geq 0 : I^1_k = I^2_k\}.$$  

Then, for any $i, j \in S$, $P(\tau > n \mid I^1_0 = i, I^2_0 = j)$ converges to zero, exponentially fast, as $t \to \infty$.

**Proof.** Define,

$$p(r_0) = \max_{k,l} P(I^1_m \neq I^2_m, 1 \leq m \leq r_0, \mid I^1_0 = k, I^2_0 = l)$$

Since the state space is finite, under the condition of irreducibility and aperiodicity, it is clear that, there exists an $r_0 \geq 1$ such that $p^{(r_0)}_{ij} > 0$ for all $i, j \in S$. Let \(\alpha_0 = \min_{i,j} p^{(r_0)}_{ij}\). Then $\alpha_0 > 0$ and $p(r_0) \leq \max_{k,l} P(I^1_{r_0} \neq I^2_{r_0}, \mid I^1_0 = k, I^2_0 = l) = \max_{k,l}(1 - \sum_i P(I^1_1 = i \mid I^1_0 = k))P(I^2_1 = i \mid I^2_0 = l) = \max_{k,l}(1 - \sum_i p_{ki}p_{li}) \leq (1 - r\alpha_0^2) < 1$. Let $n \geq 1$ be an integer. Then, using Markov property and stationarity of the joint Markov chain $(I^1, I^2)$ we obtain,
\[ P(I_m^1 \neq I_m^2, 1 \leq m \leq nr_0 \mid I_0^1 = i, I_0^2 = j) \]
\[ = \sum_{k \neq l} P(I_m^1 \neq I_m^2, 1 \leq m < (n-1)r_0, I_{(n-1)r_0}^1 = k, I_{(n-1)r_0}^2 = l \mid I_0^1 = i, I_0^2 = j) \times P(I_m^1 \neq I_m^2, (n-1)r_0 < m \leq nr_0 \mid I_{(n-1)r_0}^1 = k, I_{(n-1)r_0}^2 = l) \]
\[ = \sum_{k \neq l} P(I_m^1 \neq I_m^2, 1 \leq m < (n-1)r_0, I_{(n-1)r_0}^1 = k, I_{(n-1)r_0}^2 = l \mid I_0^1 = i, I_0^2 = j) \times P(I_m^1 \neq I_m^2, 1 \leq m < (n-1)r_0 \mid I_{(n-1)r_0}^1 = k, I_{(n-1)r_0}^2 = l) \]
\[ \leq \sum_{k \neq l} P(I_m^1 \neq I_m^2, 1 \leq m < (n-1)r_0, I_{(n-1)r_0}^1 = k, I_{(n-1)r_0}^2 = l \mid I_0^1 = i, I_0^2 = j) \times p(r_0) \]
\[ = P(I_m^1 \neq I_m^2, 1 \leq m \leq (n-1)r_0 \mid I_0^1 = i, I_0^2 = j) \times p(r_0) . \]

Using the above argument recursively we get
\[ P(I_m^1 \neq I_m^2, 1 \leq m \leq nr_0 \mid I_0^1 = i, I_0^2 = j) \leq p^n(r_0) . \]

Consequently, we obtain, for any \( n \geq r_0, \)
\[ P(\tau > n \mid I_0^1 = i, I_0^2 = j) \]
\[ = P(I_m^1 \neq I_m^2, 1 \leq m \leq n \mid I_0^1 = i, I_0^2 = j) \]
\[ \leq P(I_m^1 \neq I_m^2, 1 \leq m \leq [n/r_0]r_0 \mid I_0^1 = i, I_0^2 = j) \]
\[ \leq p^{[n/r_0]}(r_0) , \]

where \([t]\) = the largest integer that is less than or equal to \( t. \) Hence the result. \( \square \)

**Theorem 1** As in the condition (NL) assume that under (2.4) the (largest) Lyapunov exponent \( \lambda, \) defined as,
\[ \lambda := \lim_{n \to \infty} (1/n) \log \| A_n \ldots A_1 \| < 0 \quad \text{almost surely.} \quad (2.16) \]

Under this assumption the SVAR process is stable, i.e., it has unique asymptotic distribution that is free from the influence of the initial distribution.
Proof. Let us assume first that \( \{I_n\} \) starts at \( I_0 \) which is the stationary distribution for the ergodic chain. Then, it follows that, \( \{A_n, E_n\} \) are stationary. Hence

\[
X_n(X_0, I_n(I_0)) = A_n A_{n-1} \ldots A_1 X_0 + \sum_{i=0}^{n-1} A_n A_{n-1} \ldots A_{n-i} E_{n-i-1} + E_n
\]

\[
\begin{align*}
&= A_n A_{n-1} \ldots A_1 X_0 + \sum_{i=0}^{n-1} A_{i+1} A_i \ldots A_1 E_0 + E_0 \text{ in distribution.}
\end{align*}
\]

Then for any fixed \( i \geq 0 \),

\[
\limsup_{n \to \infty} \frac{1}{i+1} \log \| A_{i+1} A_i \ldots A_1 E_0 \|
\leq \limsup_{n \to \infty} \frac{1}{i+1} \log \| A_{i+1} A_i \ldots A_1 \| + \frac{1}{i+1} \log \| E_0 \|
\leq \lambda + 0 < 0 \text{ a.s. (2.18)}
\]

which implies that

\[
\limsup_{i \to \infty} \| A_{i+1} A_i \ldots A_1 E_0 \|^\frac{1}{i+1} < 1 \text{ a.s.}
\]

Thus, the RHS of (2.17) is bounded by

\[
\| E_0 \| + \sum_{i=0}^{\infty} \| A_{i+1} A_i \ldots A_1 E_0 \|,
\]

which is absolutely convergent almost surely by Cauchy’s root criterion and \( \| A_n A_{n-1} \ldots A_1 X_0 \| \to 0 \) as \( n \to \infty \) for any \( X_0 \) as in (2.18). Therefore, \( X_n(X_0, I_n(I_0)) \) converges in distribution as \( n \to \infty \) whenever \( I_0 \) starts from the stationary distribution.

Let us now observe,

\[
X_n(X_0, I_n(I_0)) - X_n(X'_0, I_n(I_0)) = A_n(X_{n-1}(X_0, I_n(I_0)) - X_{n-1}(X'_0, I_n(I_0)))
\]

\[
= \cdots = A_n A_{n-1} \ldots A_1 (X_0 - X'_0)
\]

(2.19)

Thus, we obtain,

\[
(1/n) \log(|X_n(X_0, I_n(I_0)) - X_n(X'_0, I_n(I_0))|)
\]
\[(1/n) \sum_{j=1}^{n} \log(\|A_j\|) + (1/n) \log(\|(X_0 - X_0')\|)\]

and hence by strong law for \(\{A_j\}\)'s, and under the condition (2.16), we obtain that the distance between \(X_n(X_0, I_n(I_0))\) and \(X_n(X'_0, I_n(I_0))\) converges to zero, almost surely, exponentially fast regardless of \(I_0\) as \(n\) tends to infinity.

To see that \(X_n(X_0, I_n(I_0))\) and \(X_n(X'_0, I_n(I_0))\) have same asymptotic distribution, it is important to notice that, for \(I_n(I_0)\) and \(I_n(I'_0)\) two independent finite state ergodic Markov chain starting at \(I_0\) and \(I'_0\) respectively, will meet at some finite stopping time, say \(\tau\), (whose all moments are also finite) with probability one.

Define,

\[
\tilde{I}_n(I'_0) = \begin{cases} 
I_n(I'_0), & \text{for } n < \tau \\
I_n(I_0), & \text{for } n \geq \tau,
\end{cases}
\]

i.e., \(\tilde{I}_n(I'_0)\) follows the chain \(I_n(I'_0)\) in the beginning and switches to \(I_n(I_0)\) at the stopping time \(\tau\) moves along the same path thereafter. Since \(I_n(I'_0)\) and the \(\tilde{I}_n(I'_0)\) have same initial distribution and the transition law and hence they have the same distribution. Hence for any bounded and Lipschitzian \(f\) we get,

\[
|Ef(X_n(X_0, I_n(I_0))) - Ef(X_n(X'_0, I_n(I'_0)))| \\
= |Ef(X_n(X_0, I_n(I_0))) - Ef(X_n(X_0, \tilde{I}_n(I'_0)))| \\
= |E([f(X_n(X_0, I_n(I_0))) - f(X_n(X'_0, \tilde{I}_n(I'_0))))|_{\tau \leq m}) \\
+ E([f(X_n(X_0, I_n(I_0))) - f(X_n(X'_0, \tilde{I}_n(I'_0))))|_{\tau > m})| \\
\leq |E[E([f(X_n(X_0, I_n(I_0))) - f(X_n(X'_0, \tilde{I}_n(I'_0))))|_{\tau \leq m} | F_{\tau_m}]| \\
+ 2\|f\|P(\tau > m), 
\]

where \(\tau_m = \tau \wedge m\) and \(F_j\) is an appropriate filtration. with respect to which \(\{I_n s, X_n s\}\) are adapted. We restrict the class of \(f\) such that the lipschitzian constant is bounded by one and the \(\|f\| \leq 1\) and call that restricted class as BL. Then by Markov property we get, for \(m < n\),

\[
|E([f(X_n(X_0, I_n(I_0))) - f(X_n(X'_0, \tilde{I}_n(I'_0))))|_{\tau \leq m} | F_{\tau_m}]|
\]
\[ |E[f(X_{n-\tau_m}(z, I_{n-\tau_m}(J))) - f(X_{n-\tau_m}(z', I_{n-\tau_m}(J)))]| \leq E(|X_{n-\tau_m}(z, I_{n-\tau_m}(J)) - X_{n-\tau_m}(z', I_{n-\tau_m}(J))| \wedge 2) \] (2.21)

conditionally on \( z = X_{\tau_m}(X_0, I_{\tau_m}(I_0)), z' = X_{\tau_m}(X'_0, I_{\tau_m}(I'_0)) \) and \( J = I_{\tau_m}(I_0) \). Since, by earlier argument, for each \( z, z', J \), \( |X_{n-\tau_m}(z, I_{n-\tau_m}(J)) - X_{n-\tau_m}(z', I_{n-\tau_m}(J))| \) goes to zero almost surely, exponentially fast, as \( n \to \infty \), by Lebesgue’s dominated convergence theorem \( E(|X_{n-\tau_m}(z, I_{n-\tau_m}(J)) - X_{n-\tau_m}(z', I_{n-\tau_m}(J))| \wedge 2) \to 0 \), as \( n \to \infty \), almost surely, for each fixed \( m \geq 1 \). Therefore, again using Lebesgue’s dominated convergence theorem and the fact that \( \tau \) is finite with probability one (by Lemma 2), we obtain, first by taking limit \( n \to \infty \) and then \( m \to \infty \),

\[ |Ef(X_0, I_0) - Ef(X_0', I_0')| \leq |E[|f(X_0, I_0) - f(X'_0, I'_0)|I_{\tau \leq m} | \mathcal{F}_{\tau_m}]| + 2\|f\|P(\tau > m) \to 0, \] (2.22)

uniformly over bounded Lipschitzian \( f \) in \( BL \). Since the class of \( BL \) characterizes the weak convergence, and hence the theorem (for an analogous result in continuous time, see Basak, Bisi and Ghosh (1999)).

**Corollary 1** Under a useful and simpler condition where the random matrix \( A_1 \) satisfies

\[ (CB): \quad E \log \|A_1\| < 0. \] (2.23)

for a given norm \( \| \cdot \| \), the SVAR process is stable, i.e., it has unique asymptotic distribution that is free from the influence of the initial distribution.

**Proof.** By definition (2.6), condition (CB) implies the negative Lyapunov condition (NL) in Proposition 1 for any norm. Hence the proof.

**Remark.** Brandt (1986) focuses mainly on (CB). However, being independent of a matrix norm, condition (NL) of Proposition 1 is more natural in multidimensional systems.
**Remark.** It is clear that, if the assumption of irreducibility is dropped then one needs to restrict attentions within the irreducible subclasses. Within each irreducible subclass the above result is true under aperiodicity. Also, it is easy to see, if the assumption of aperiodicity is dropped then the above theorem fails, i.e., asymptotic distribution would have the influence of initial distribution.

Importance of Theorem 1 is in realizing the fact that in practice, we don’t have data that starts from \(-\infty\) or follows a nice initial distribution (such as the stationary distribution), rather we have data which starts from a finite time in the past and with an arbitrary initial distribution, usually unknown. In such a case, having a common limiting distribution in forward time is a necessity in making inference of the data.

Certainly, the question remains in determining the rate of convergence to the limiting distribution. A more interesting and challenging problem is to check for stability using the Lyapunov exponent approach. For this, a theoretical question arises: whether the analogue of Kingman’s subadditive ergodic theorem or more Oseledec’s multiplicative ergodic theorem is true when the sequence of random matrices \(\{A_n\}\) follows a Markov chain and the initial value is arbitrary? We think this is likely the case (recall the law of large numbers for Markov chain) but haven’t seen any known result on this.

### 3 Examples

Proposition 1 gives a general criterion for checking stationarity of switching autoregressive models via negativity of the largest Lyapunov exponent. Theorem 1 proves the more relevant stability property under a stronger condition. Technique for calculating Lyapunov exponents for a sequence of random matrices becomes very important in checking for stationarity. Unfortunately, it is extremely difficult to have explicit formula of Lyapunov exponents except in very special cases, and in the general case we may have to resort to numerical method.
3.1 Cases when $A_i$'s commute

In the special cases when formula for Lyapunov exponents is available, condition for stationarity follows immediately. Some situations are discussed next. Let $A_1, A_2, \ldots$ be an ergodic stationary sequence of $p \times p$ random matrices and denote $A_k = (a_{ij}(k))$.

**Lemma 3** (i) If $A_k$'s are upper triangular, i.e. $a_{ij} = 0$ for any $i > j$, and assume that $E \max(0, \log |a_{ii}|) < \infty$ for all $1 \leq i \leq p$. Then the Lyapunov exponents exist, and they correspond to the ordered sequence of the $r$ quantities defined by

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log |a_{ii}(k)| = E \log |a_{ii}(1)|, \text{ for } i = 1, \ldots, p.
$$

(ii) If any pairs of matrices $A_k$'s commute, let $\delta_1(1) \geq \ldots \geq \delta_p(1)$ be the ordered eigenvalues of $A_1$ and assume $E \max(0, \log |a_{11}|) < \infty$. Then, the Lyapunov exponents exist and are given by $\lambda_i = E \log |\delta_i(1)|$ for $i = 1, \ldots, p$.

We now specialize the preceding theory to the switching AR model (2.1) when $A_n$ takes on one of the $r$ possible matrices $B_1, \ldots, B_r$. Obviously, if the sequence $A_1, A_2, \ldots$ is stationary, Lyapunov exponents always exist, because (2.4) holds automatically. In particular, let the stationary distribution of $I_n$ be $\rho$ such that $P(I_n = i) \to \rho_i$ for $1 \leq i \leq r$ and $\rho_1 + \ldots + \rho_r = 1$. Let $E$ denote the expectation over the joint product space of $\{I_n\}$ and $\{\varepsilon_{ni}, i = 1, \ldots, r\}$ under $\rho$. Then, (2.6) implies that

$$
\lambda \leq \sum_{i=1}^{r} \rho_i \log \|B_i\|.
$$

(3.1)

Thus, if there exists a norm such that $\|B_i\| \leq 1$ for $1 \leq i \leq r$ where inequality holds for at least one $i$, then the negative Lyapunov condition is satisfied. If

$$
E \max(0, \log \|\varepsilon_{1i}\|) < \infty \text{ for } 1 \leq i \leq \infty,
$$

(3.2)

by Proposition 1, the Markov switching AR model with at most random walk type nonstationarity in subprocesses and at least one stable subprocess is stationary. By now we have used the term stable process or stability in several places. What we mean
is the processes starting from different initial conditions converge. In the case of a vector AR(1) process, this is equivalent to the coefficient matrix $A$ having eigenvalues whose norms are all less than one. And the latter coincides with the stationarity condition (cf. Example 1).

**Example 2.** In the one-dimensional case, negative Lyapunov condition reduces to $E \log |a_n| < 0$. In particular, if $a_n$ takes on finite numbers $b_1, \ldots, b_r$, this is

$$
\sum_{i=1}^{r} \log |b_i| Pr(a_n = b_i) < 0.
$$

(3.3)

This is satisfied if one $|b_i| < 1$ and all other $|b_j| \leq 1 (j \neq i)$. That is, under (3.2) a switching autoregressive model is stable as long as it has a positive probability of being in a stable regime while all other regimes are either stationary or random-walk type nonstationary. Obviously, explosive behavior ($|b_i| > 1$) in some regimes is also allowed as long as (3.3) is satisfied.

The conclusion of Example 2 in the one-dimensional case, though benign and reasonable, cannot be extended to multi-dimensional case, except in trivial cases such as Lemma 3 when $B_i$’s are either triangular or commutable. Initially, we thought that the mixture of two stable processes is always stable. This turns out not to be true in the multidimensional case. A counterexample (Example 3) is given to show that two stable subprocesses can be mixed to produce an unstable switching process. On the other hand, two unstable subprocesses can be mixed to produce a stable switching process (Example 4).

### 3.2 Calculating Lyapunov exponents in a nontrivial case

For Example 3, we need a result on an explicit formula for Lyapunov exponent in a nontrivial case due to Pincus (1985), see Lima and Rahibe (1994). Consider the case $r = 2$ and two $2 \times 2$ real matrices $B_1$ and $B_2$, where $B_1$ is singular. Denote the transition probability matrix of $\{I_n\}$ by $P(I_n = j|I_{n-1} = i) = p_{ij}, i, j = 1, 2$ and initial distribution $P(I_0 = i) = p_i, i = 1, 2$. 

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By change of basis, we can assume that \( B_1 \) takes on the form
\[
B_1 = \begin{pmatrix}
\delta & 0 \\
0 & 0 \\
\end{pmatrix}
\]
(Another form of \( A 
B_1 = \begin{pmatrix}
0 & \delta \\
0 & 0 \\
\end{pmatrix}
\)
is not interesting because \( B_1^2 = 0 \).)

We write \( B_2^n \) in the form
\[
B_2^n = \begin{pmatrix}
b_{11}(n) & b_{12}(n) \\
b_{21}(n) & b_{22}(n) \\
\end{pmatrix}
\]
then a result due to Pincus (1985) and Lima and Rahibe (1994) says that the Lyapunov exponent is given by
\[
\lambda = \frac{p_{21}}{p_{21} + p_{12}} \log |\delta| + \sum_{i=1}^{\infty} \frac{p_1 p_{21} p_{12} p_{22}^{i-1}}{p_{12} + p_{21}} \log |b_{11}(n)|. \quad (3.4)
\]
In the case that \( B_2 \) is singular, we consider the case that
\[
B_2 = Q^{-1} \begin{pmatrix}
\delta_2 & 0 \\
0 & 0 \\
\end{pmatrix} Q
\]
where \( Q \) is an invertible matrix. (By a simple argument, in the other case \( B_2 = Q^{-1} \begin{pmatrix}
0 & \delta_2 \\
0 & 0 \\
\end{pmatrix} Q \), we have \( \lambda = -\infty \). Not what we want.)

Then, from Lima and Rahibe (3.2),
\[
\lambda = \frac{p_{21}}{p_{21} + p_{12}} \log |\delta| + \frac{p_{12}}{p_{21} + p_{12}} \log |\delta_2| + \frac{p_{12} p_{21}}{p_{12} + p_{21}} \log \left| \frac{b_{11}}{\text{Tr}(B_2)} \right|. \quad (3.5)
\]

**Example 3.** Consider \( B_1 = \begin{pmatrix}
\delta_1 & 0 \\
0 & 0 \\
\end{pmatrix} \) and \( B_2 = \begin{pmatrix}
b_1 & -c b_1 \\
b_2 & -c b_2 \\
\end{pmatrix} \). The eigenvalues for \( B_2 \) are 0 and \( \delta_2 = b_1 - c b_2 \). The Lyapunov exponent is given by
\[
\lambda = \frac{p_{21}}{p_{21} + p_{12}} \log |\delta_1| + \frac{p_{12}}{p_{12} + p_{21}} \log |b_1 - c b_2| + \frac{p_{12} p_{21}}{p_{12} + p_{21}} \log \left| \frac{b_1}{b_1 - c b_2} \right|. \quad (3.6)
\]
We want to choose \( b_1, b_2, c, \delta_1 \) and \( p_{ij} \)’s so that \(|\delta_1| < 1, |\delta_2| < 1\) and \( \lambda > 0 \). Since the first two terms in (3.6) are negative, we need to make the third term as large as possible. Thus, \( b_1/(b_1 - cb_2) \) should be large. For example, if we choose

\[
b_1 = 100, \quad c = 10, \quad b_2 = 9.99, \quad \delta = 0.1.
\]

Then in order \( \lambda > 0 \) we require

\[
-p_{21} \log |\delta_1| + p_{12} \log 10 < 3p_{21}p_{12} \log 10.
\]

This is satisfied if e.g. \( \delta_1 = 0.1, p_{21} = p_{12} = 0.8 \).

If one subprocess is stable, the other is unstable, in most situations there always exists a switching strategy to make the mixing process stable. Consider the situation that there exists a subordinate matrix norm such that \( \|B_1\| < 1, \|B_2\| > 1 \). Then, \( E \log \|A_1\| = \rho_1 \log \|B_1\| + \rho_2 \log \|B_2\| \) can be made less than 0 if \( \rho_2 \) is small enough.

We call this strategy the preferred switching, to denote the phenomenon that a mixture process with less frequent unstable regime can still be stable.

Now we give an example that two unstable vector processes can give rise to a stable mixing process.

**Example 4.** Consider an extension of Example 2 to multidimensional case when \( B_I \)’s commute. For example, let

\[
B_1 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}.
\]

The two Lyapunov exponents associated with the switching between \( B_1 \) and \( B_2 \) are given by

\[
\lambda_1 = \rho_1 \log 2 - \rho_2 \log 3, \quad \lambda_2 = -\rho_1 \log 2 + \rho_2(\log 3 - \log 2).
\]

We require that \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \). Let \( \rho = \rho_1 \). This is true if and only if

\[
\frac{\log 3 - \log 2}{\log 3} < \rho < \frac{\log 3}{\log 2 + \log 3}.
\]

\( \square \)
3.3 Mean shifting models

Consider the mean shifting model given by

\[ X_n = M_n + A_n X_{n-1} + E_n \]  

(3.7)

where \( A_n \) and \( E_n \) as before and \( M_n \) is the shifting mean, defined as \( \mu_i \) when \( I_n = i \) for \( i = 1, 2, \ldots, r \) or \( M_n = \sum_{i=1}^{r} \mu_i 1(I_n = i) \).

The mean-shifting model can be regarded as a more general case of SAR when \( E_n \) may be allowed to take nonzero mean as well, such as, \( \mu_i \) when \( I_n = i \) for some \( i \). An interesting case is when \( A_n \) is a constant and only the mean or variance of \( E_n \) varies among different regimes.

Obviously \( M_n \) is a stationary sequence if \( I_n \) is. Using an expansion similar to (2.14) and Proposition 1, it can be shown that the proper stationary solution of (3.7) is given by

\[ W_n' = (M_n + \sum_{i=0}^{\infty} A_n A_{n-1} \cdots A_{n-i} M_{n-i-1}) + (E_n + \sum_{i=0}^{\infty} A_n A_{n-1} \cdots A_{n-i} E_{n-i-1}). \]  

(3.8)

That is, the stationary solution of (3.7) is given by the sum of two stationary processes

\[ \bar{M}_n = M_n + \sum_{i=0}^{\infty} A_n A_{n-1} \cdots A_{n-i} M_{n-i-1} \]  

(3.9)

and \( W_n \) of (2.11). Note that (3.9) is in general well-defined under the negative Lyapunov exponent assumption [cf. (2.9)] and

\[ \text{E max}(\log \|M_1\|, 0) < \infty \]

(cf. Proof of Lemma 1). In particular, above condition is satisfied if \( M_n \) takes on values from a finite set.

Example 5. Hamilton (1989)'s model for business cycle uses a fourth-order autoregression and mean-shifting model with two regimes. Writing in our state space rep-
presentation (3.7), this corresponds to $A_n$ taking on a fixed $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

and $M_n$ taking on $(\mu_i, 0, 0, 0)^T$ depending on $I_n = i, i = 1, 2$. By our theory, this model has a stationary and stable solution as long as $A$ is stable. In particular, the empirical model of Krolzig (1997, Sec.11.3.1) for German business cycle with $a_1 = 0.2932, a_2 = 0.1055, a_3 = 0.0026, a_4 = 0.3812$ clearly has a stationary solution because $a_1 + a_2 + a_3 + a_4 < 1$ and $a_i$’s are positive.

**Example 6.** We discuss another mean-shifting model which is a simplified version of Lu and Berliner (1997)’s model for a riverflow time series $y_n$. Their model consists of mixture of AR(1), ARX(1), and AR(1) models with different means at each of the three regimes, corresponding to normal (0), rising (1), or falling (2) of the riverflow, where in the rising regime the past rainfall $x_{n-1}$ series is included linearly. We assume here that the regime switching process is independent of both $\{x_n\}, \{y_n\}$ and follows a Markov chain. This model can be easily embedded in our formulation (3.7) with $p = 1$ and $M_n$ taking on fixed values except in the rising regime when $M_n = \mu_1 + ax_{n-1}$. Extending slightly the argument used in this section, if the rainfall series $\{x_n\}$ is stationary and the regime switching process is ergodic, the riverflow series $\{y_n\}$ is stationary if the AR(1) processes are either stationary or nonstationary of the random walk type (cf. Example 2).

4 **Existence of moments**

Existence of moments is often assumed in time series analysis, notably for the second-order theory (cf. Brockwell and Davis, 1991). For a general stochastic difference equation, Karlsen (1990) gives some general conditions for checking the existence of finite moments. He also gives some examples where more explicit results can
be derived. In this section, by exploiting the Markovian structure in the hidden state process, we derive directly some explicit conditions for existence of second-order moment of SVAR models and the related autocorrelation property.

We make the following assumption.
\[(A) \lim_{n \to \infty} E[\|A_n \ldots, A_1\||I_0 = i] = 0 \text{ for any } i = 1, \ldots, R.\]

By the ergodicity of \(\{I_n\}\), one can easily show that \((A)\) is equivalent to the condition that
\[(A') \lim_{n \to \infty} E[\|A_n \ldots, A_1\|] = 0.\]

Consider the property of the quantity defined by
\[\Phi_{ni}(I_i) = E[\|A_n \ldots, A_{i+1}\||I_i]\]
for any \(n, i < n\). Then, since \(\{A_n\}\) is an induced matrix-valued FMC defined in terms of \(I_n\). It shares the usual Markov property, and in particular \(\Phi_{ni}(I_i)\) is independent of \(i\) and depends only on \(n - i\). If we write
\[\Phi_\ell(I_0) = E[\|A_\ell \ldots, A_1\||I_0]\]
then
\[\Phi_{ni}(I_i) = \Phi_{n-i}(I_i).\]
We use \(\Phi_\ell\) or \(\Phi_{ni}\) to denote their unconditional analogues.

We have the following proposition on \(\Phi_\ell(I_0)\).

**Proposition 2**

\[\Phi_n(I_0) \to 0 \text{ if and only if } \Phi_n(I_0) \text{ tends to 0 geometrically.}\]

Proof: Since \(\Phi_n(I_0) \to 0\) uniformly over \(I_0\). Then, there exist an integer \(\ell\) and constant \(\gamma < 1\) such that \(\Phi_\ell(i) \leq \gamma\) for all \(i\).

\[\Phi_{2\ell}(I_0) \leq E[\Phi_\ell(I_\ell)\|A_\ell \ldots A_1\||I_0]\]
There exists a constant $C$ such that $\Phi_n \leq C\gamma^{[n/\ell]}$ for any $n$. That is, $\Phi_n$ tends to 0 at a geometric rate. The sufficient part of the proof is easy to establish. 

**Theorem 2** The SVAR process has a stationary solution whose second-order moment exists if (A) is satisfied.

Proof: Consider the expansion for SVAR in (2.1):

$$X_n = A_n \ldots A_1 X_0 + A_n \ldots A_2 E_1 + \cdots + A_n E_{n-1} + E_n.$$ 

Then,

$$E\|X_n\| \leq E\|A_n \ldots A_1\| \cdot \|X_0\| + E\|A_n \ldots A_2\| \cdot \|E_1\|$$

$$+ \cdots + E\|A_n\| \cdot \|E_{n-1}\| + E\|E_n\|$$

$$= E\{E[\|A_n \ldots A_1\| I_0] \cdot \|X_0\|\} + E\{E[\|A_n \ldots A_2\| I_1]\} \cdot \|\Sigma I_1 \epsilon I_1\|\}$$

$$+ \cdots + E\{E[\|A_n\| I_{n-1}] \cdot \|\Sigma I_{n-1} \epsilon I_{n-1} I_{n-1}\|\} + E\|E_n\|$$

$$\leq \max \Phi_n(i) E\|X_0\| + \max \Phi_{n-1}(i) \cdot E\|E_1\|$$

$$+ \cdots + \max \Phi_1(i) \cdot E\|E_{n-1}\| + E\|E_n\|$$

which is convergent if $E\|X_0\| < \infty$, by Proposition 2 and ergodicity of $\{I_n\}$. Here assumptions on $\{E_n\}$ and independence of $\{I_n\}$ and $\{\epsilon_{ni}\}$ are used. 

Note that, by the concave nature of $\log X$, the Jensen’s Inequality implies that the strict inequality

$$E \log \|A_n \ldots A_1\| < \log E\|A_n \ldots A_1\|$$

holds.

We denote $\limsup_{n \to \infty} (1/n) \log E\|A_n \ldots A_1\|$ by $\log(\gamma)$. Condition (A) is equivalent to $\gamma < 1$. By (4.1), this further implies that

$$\lambda < \log \gamma < 0.$$
This indicates that condition (A) or (A’) is stronger than negativity of the largest Lyapunov exponent $\lambda$, a potentially general condition for strict stationarity. However, the latter does not even ensure existence of second-order moment, see Bougerol and Picard for an example in the case of an GARCH process.

Using the fact that

$$X_{n+m} = A_{n+m} \ldots A_{n+1}X_n + A_{n+m} \ldots A_{n+2}E_{n+1} + \ldots A_{n+m}E_{n+m-1} + E_{n+m}$$

for any integers $m$ and $n$, we have

$$|E X_n^T X_{n+m}| = |E X_n^T A_{n+m} \ldots A_{n+1}X_n|$$

$$\leq |E| < X_n, A_{n+m} \ldots A_{n+1}X_n >|$$

$$\leq E \|A_{n+m} \ldots A_{n+1}\| \cdot \|X_n\|^2.$$

That is,

$$|E X_n^T X_{n+m}| \leq \Phi_mE\|X_1\|^2 \quad (4.3)$$

where we use the property that $\{X_n\}$ is causal and stationary, and $\{A_{n+i}\}$ is stationary. Thus, the autocovariance matrix at lag $m$ of the vector time series $\{X_n\}$ decays at a geometric rate, and is bounded by $\gamma^m$.

5 Switching ARMA models

We note some extensions of the switching autoregressive models. First, a switching moving average process of order $q$ (SMA(q)) can be defined as

$$X_n = E_n + C_{1n}E_{n-1} + C_{2n}E_{n-2} + \ldots + C_{qn}E_{n-q} \quad (5.1)$$

where $\{E_n\}$ is defined as before, and $E_{n-j} = \Sigma_{i=1}^r \Sigma_{i\in (n-j)}I_{I_n=i}$ for $j = 1, 2, \ldots, q$. The coefficient matrices $C_{jn}$ will take on member of a set of $r$ matrices depending on the value of $I_n$ for each $j$ between 1 and $q$.

We also assume that $\{(\varepsilon_1, \ldots, \varepsilon_{nr})^T\}$ is stationary as before. If $\{I_n\}$ is stationary, it follows that $\{X_n\}$ is stationary since it is a moving average function of stationary
processes. On the other hand, if \( \{ I_n \} \) is ergodic, for arbitrary starting regime, \( \{ I_n \} \) eventually converges to stationarity and thus \( \{ X_n \} \) is asymptotically stationary.

Similar to ARMA models, one can define switching ARMA (SARMA) models in which the coefficient matrices in both AR part and MA part take on different values depending on the \emph{current} regime. The stationarity condition for SVAR(1) models is also sufficient for SARMA(1,q) models. Since an AR(p) process can be represented as a vector AR(1) process, our theory applies to any switching ARMA(p,q) process.

Other extension is also possible. In particular, the transition probabilities of switching may be allowed to depend on past values of the process, or past values of another process. This interesting class of nonlinear time series models is closely related to some traditional state dependent nonlinear time series models (cf. Tong 1990). Not surprisingly, there are increasing interest in applying them in some real modelling situations such as security time series and high-frequency data. It is our hope that the present work may shed light on these more complex models.

References

[1] Basak, G. K., Bisi, A. and Ghosh, M. K. (1999). Stability and functional limit theorems for random degenerate diffusions. \emph{Sankhya Ser A}, 61 12-35.

[2] Bougerol, P. and Picard N. (1992). Strict stationarity of generalized autoregressive processes. \emph{The Annals of Probability}, Vol. 20, No. 4, 1714-1730.

[3] Brandt, A. (1986). The stochastic equation \( Y_{n+1} = A_n Y_n + B_n \) with stationary coefficients. \emph{Adv. Appl. Prob.}, 18, 211-220.

[4] Brockwell, P. J. and Davis, R. A. (1991). \emph{Time Series: Theory and Methods (2nd Ed)}. Springer-Verlag, New York.

[5] Hamilton, J. D. (1989). A new approach to the economics analysis of nonstationary time series and the business cycle. \emph{Econometrica}, 57, 357-384.
[6] Holst, U., Lindgren, G., Holst, J., and Thuvesholmen, M. (1994). Recursive estimation in switching autoregressive with a Markov regime. *Journal of Time Series Analysis*. 15 489-506.

[7] Karlsen, H. A. (1990). Existence of moments in a stationary stochastic difference equation. *Adv. Appl. Prob.*, 22, 129-146.

[8] Krolzig, H.-M. (1997). *Markov-Switching Vector Autoregressions: Modelling, Statistical Inference, and Application to Business Cycle Analysis*. Lecture Notes in Economics and Mathematical Systems, 454. Springer, Berlin.

[9] Lima, R. and Rahibe, M. (1994). Exact Lyapunov exponent for infinite products of random matrices. *J. Phys. A: Math. Gen.*, 27, 3427-3437.

[10] Lu, Z. Q. and Berliner, L. M. (1997). Markov switching time series models with application to a daily runoff series. *Water Resources Research*. 35, No. 2, 523-534.

[11] Nicholls, D. F. and Quinn, B. G. (1982). *Random Coefficient Autoregressive Models: An Introduction*. Lecture Notes in Statistics, Vol. 11. Springer, Berlin.

[12] Pincus, S. (1985). Strong law of large numbers for products of random matrices. *Transactions of The American Mathematical Society*, Volume 287, No. 1, 65-89.

[13] Tong, H. (1990). *Nonlinear Time Series, A Dynamical System Approach*. Oxford University Press.