REPRESENTATIONS OF GROUPS WITH CAT(0) FIXED POINT PROPERTY

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Abstract. We show that certain representations over fields with positive characteristic of groups having CAT(0) fixed point property $\text{FB}_{\tilde{A}_n}$ have finite image. In particular, we obtain rigidity results for representations of the following groups: the special linear group over $\mathbb{Z}$, $\text{SL}_k(\mathbb{Z})$, the special automorphism group of a free group, $\text{SAut}(F_k)$, the mapping class group of a closed orientable surface, $\text{Mod}(\Sigma_g)$, and many other groups.

In the case of characteristic zero we show that low dimensional complex representations of groups having CAT(0) fixed point property $\text{FB}_{\tilde{A}_n}$ have finite image if they always have compact closure.

1. Introduction and statements

In this paper we study linear representations of groups with certain CAT(0) fixed point property.

Let $\mathcal{B}_{\tilde{A}_n}$ be the class of buildings of type $\tilde{A}_n$. Roughly speaking, a building of type $\tilde{A}_n$ is a highly symmetrical $n$-dimensional CAT(0) simplicial complex with a vertex coloring. We say that a group $G$ has property $\text{FB}_{\tilde{A}_n}$ if any simplicial color-preserving action of $G$ on any member of $\mathcal{B}_{\tilde{A}_n}$ has a fixed point. Our investigations on property $\text{FB}_{\tilde{A}_n}$ are motivated by Serre’s property $\text{FA}$. Recall that a group $G$ is said to satisfy Serre’s property $\text{FA}$ if every action, without inversions, of $G$ on a simplicial tree has a fixed point. Note that if a group $G$ has Serre’s property $\text{FA}$ it has also property $\text{FB}_{\tilde{A}_1}$, since buildings of type $\tilde{A}_1$ are trees without leaves.

We show that property $\text{FB}_{\tilde{A}_n}$ strongly affects the representation theory of groups. The following theorem illustrates this fact.

**Theorem A.** Let $G$ be a finitely generated group with property $\text{FB}_{\tilde{A}_n}$ and $\rho : G \to \text{GL}_{n+1}(K)$ a linear representation over a field $K$ with positive characteristic. Then $\rho(G)$ is finite.

As an application of Theorem A we obtain the following results.

**Corollary B.** Let $K$ be a field with positive characteristic. Then every linear representation

(i) $\text{SL}_k(\mathbb{Z}) \to \text{GL}_n(K)$ for $k \geq 3$ and all $n \in \mathbb{N}$,

(ii) $\text{SL}_k(\mathbb{Z}[\frac{1}{p}]) \to \text{GL}_n(K)$ for $p$ prime number, $k \geq 3$ and $n \leq k - 1$,

(iii) $\text{SAut}(F_k) \to \text{GL}_n(K)$ for $k \geq 3$ and $n \leq k - 1$,

(iv) $\text{Mod}(\Sigma_g) \to \text{GL}_n(K)$ for $g \geq 2$ and $n \leq g$,

(v) $W \to \text{GL}_d(K)$ for $d \leq n$ and $W$ is a f.g. Coxeter group such that every special parabolic subgroup of rank $n$ is finite.

has finite image.

Results of similar flavor were proved in [Var18] where it is shown that low dimensional representations of $\text{SAut}(F_k)$ over fields of characteristic not equal 2 are trivial. Concerning the mapping class group of a closed orientable surface of genus $g$, $\text{Mod}(\Sigma_g)$, it is proved in [But16] that this group has no faithful linear representation in any dimension over any field of positive characteristic.

Let us further mention that Theorem A is not true for fields of characteristic zero, as we can consider the group $\text{SL}_n(\mathbb{Z})$ which has property $\text{FB}_{\tilde{A}_{n-1}}$ but the canonical embedding $\text{SL}_n(\mathbb{Z}) \to \text{GL}_n(\mathbb{R})$ has infinite image.

In the case of characteristic zero we prove:

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Theorem C. Let \( G \) be a finitely generated group with property \( \text{FB}_{\tilde{A}_n} \) and \( \rho : G \to \text{GL}_{n+1}(\mathbb{Q}) \) a linear representation. If there exists a positive definite quadratic form on \( \mathbb{Q}^{n+1} \) which is invariant under \( \rho(G) \), then \( \rho(G) \) is finite.

Further, we generalize a theorem of Alperin [Alp90], who proved the following result for \( n = 1 \).

Theorem D. Let \( G \) be a finitely generated group with property \( \text{FB}_{\tilde{A}_n} \). If every \((n+1)\)-dimensional representation \( \rho : G \to \text{GL}_{n+1}(\mathbb{C}) \) has image with compact closure, then the image \( \rho(G) \) is finite for every \((n+1)\)-dimensional representation.

The general strategy of the proofs consists of three steps. First, as Farb noted in [Far09, 1.7] one can generalize Serre’s result [Ser03, 6.2.22] as follows.

Proposition E. Let \( G \) be a finitely generated group with property \( \text{FB}_{\tilde{A}_n} \) and let \( \rho : G \to \text{GL}_{n+1}(K) \) be a linear representation over a field \( K \).

(i) If \( K \) has positive characteristic, then the eigenvalues of \( \rho(g) \) for \( g \in G \) are roots of unity.

(ii) If \( K \) has characteristic zero, then the coefficients of the characteristic polynomial of \( \rho(g) \) for \( g \in G \) and its roots are integral over \( \mathbb{Z} \).

We include a proof of this proposition here which uses the same ideas as the proof in [Ser03, 6.2.22]. Then we show that each element in the image of a linear representation \( \rho \) has finite order. Hence, the image of \( \rho \) is a finitely generated torsion linear group which is by Schur’s Theorem finite ([Sch11], [Lam01] 9.9)).

2. General results of property \( \text{FB}_{\tilde{A}_n} \)

Definitions and properties concerning Coxeter groups and buildings of type \( \tilde{A}_n \) can be found in [AB08], [Hum90], [Ron09].

We need the following crucial definition.

Definition 2.1. A group \( G \) has property \( \text{FB}_{\tilde{A}_n} \) if any simplicial type preserving action on any building of type \( \tilde{A}_n \) has a fixed point.

An important fact about buildings of type \( \tilde{A}_n \) is that its geometric realisation has a natural metric that satisfies the CAT(0) curvature condition, for more details see ([AB08 11.16]) and [BH99].

Let us compare property \( \text{FB}_{\tilde{A}_n} \) to the fixed point property \( \text{FA}_n \) which was defined by Farb in [Far09].

Definition 2.2. A group \( G \) has property \( \text{FA}_n \) if any simplicial action of \( G \) on any \( n \)-dimensional simplicial CAT(0) complex has a fixed point (in the geometric realisation).

Since the geometric realisation of a building of type \( \tilde{A}_n \) is a \( n \)-dimensional CAT(0) simplicial complex, we immediately obtain the following result.

Lemma 2.3. If a group \( G \) has property \( \text{FA}_n \), then it has property \( \text{FB}_{\tilde{A}_n} \).

There are many groups known to have property \( \text{FA}_n \), hence by Lemma 2.3 these groups also have property \( \text{FB}_{\tilde{A}_n} \).

Proposition 2.4. (i) For \( n \geq 3 \) and all \( k \in \mathbb{N} \) the group \( \text{SL}_n(\mathbb{Z}) \) has property \( \text{FA}_k \), ([Far09] p. 3).

(ii) For \( n \geq 3 \) and \( p \) prime number the group \( \text{SL}_n(\mathbb{Z}^{\frac{1}{p}}) \) has property \( \text{FA}_{n-2} \), ([Far09] 1.2).

(iii) For \( n \geq 3 \) the group \( \text{SAut}(F_n) \) has property \( \text{FA}_{n-2} \), (private communication with Bridson).

(iv) For \( g \geq 2 \) the group \( \text{Mod}(\Sigma_g) \) has property \( \text{FA}_{g-1} \), ([Bri12] Theorem A).

(v) Let \( W \) be a Coxeter group, such that each special parabolic subgroup of rank \( n \) is finite, then \( W \) has property \( \text{FA}_{n-1} \) ([Bar06] 1.1)].
The above proposition shows that Corollary B follows by Theorem A.
For a proof of Proposition E we will need the following result.

**Lemma 2.5.** Let $G$ be a group with property $\text{FB}_{\tilde{\Delta}_n}$. Then $G$ has no quotient isomorphic to $\mathbb{Z}$.

**Proof.** Let us assume that $G$ has a quotient isomorphic to $\mathbb{Z}$. Then there exists a surjective homomorphism $\Phi : G \to \mathbb{Z}$. Let $(W, I)$ be the Coxeter system of type $\tilde{\Delta}_n$ and $\Sigma(W, I)$ the associated Coxeter complex (thin building of type $\tilde{\Delta}_n$). Since $W$ is an affine Coxeter group, there exists an element $w_0 \in W$ with infinite order. The action

$$
\Psi \circ \Phi : G \to \mathbb{Z} \cong \langle w_0 \rangle \to \text{Aut}(\Sigma(W, I))
$$

where $\Psi$ acts via left multiplication is type preserving. Let $wW$ where $\Psi$ acts via left multiplication is type preserving. Let $wW$ where $\Psi$ acts via left multiplication is type preserving. Let $wW$ for $J \subset I$ be an arbitrary simplex in $\Sigma(W, I)$. Then $\text{stab}(wW_J) = wW_Jw^{-1}$. Since $J \neq I$, the special parabolic subgroup $W_J$ is finite and therefore $wW_Jw^{-1}$ is finite. Thus the action $\Psi \circ \Phi$ has no fixed point which is a contradiction. \(\square\)

We now state an useful elementary fact.

**Proposition 2.6.** If a finitely generated group $G$ has property $\text{FB}_{\tilde{\Delta}_n}$, then the abelianization $G^{ab} = G/[G, G]$ of $G$ is finite.

**Proof.** Let us consider the following natural map:

$$
\pi : G \to G^{ab}
$$

Hence $G$ is a finitely generated group, its abelianization is a finitely generated abelian group. Therefore by the fundamental theorem of finitely generated abelian groups we have the following decomposition

$$
G^{ab} \cong \mathbb{Z}^k \times \text{Tor}(G),
$$

where $\text{Tor}(G)$ is a finite group. Assume that $k \geq 1$, then we get a surjective map

$$
G \to G^{ab} \cong \mathbb{Z}^k \times \text{Tor}(G) \to \mathbb{Z}
$$

which is by Lemma 2.5 a contradiction. \(\square\)

3. Proofs

3.1. **Proof of Proposition E.** Before we can prove Proposition E we need a result by Grothendieck. Recall that a field extension $L/K$ is said to be finitely generated over $K$ if there exist finitely many elements $a_1, \ldots, a_l$ in $L$ such that the smallest field extension $K(a_1, \ldots, a_l)$ containing $a_1, \ldots, a_l$ equals $L$.

For finitely generated field extensions $L/\mathbb{F}_p$, where we denote by $\mathbb{F}_p$ the prime field of $L$, there is a good description on integral (algebraic) closure of $\mathbb{F}_p$ in $L$, and similarly for fields extensions of $\mathbb{Q}$.

**Proposition 3.1.** \([\text{Gro61}, \text{p.140 7.1.8}]\)

(i) Let $L$ be an infinite field of characteristic $p$. If $L$ is finitely generated over $\mathbb{F}_p$, then the integral closure of $\mathbb{F}_p$ in $L$ is equal to the intersection of all discrete valuation rings in $L$.

(ii) Let $L$ be a field of characteristic 0. If $L$ is finitely generated over $\mathbb{Q}$, then the integral closure of $\mathbb{Z}$ in $L$ is equal to the intersection of all discrete valuation rings in $L$.

Now we are ready to prove Proposition E.

**Proof.** Let $\rho : G \to \text{GL}_{n+1}(K)$ be a linear representation over a field $K$. Let $\{g_1, \ldots, g_l\}$ be a finite generating set of $G$ and let $K_{\rho}$ be the subfield of $K$ generated by the coefficients of the matrices $\rho(g_i)$ for $i = 1, \ldots, l$. We get $\rho(G) \subseteq \text{GL}_{n+1}(K_{\rho})$. Further, the field $K_{\rho}$ is finitely generated over its prime field $\mathbb{F}$. Therefore there exist elements $a_1, \ldots, a_m \in K_{\rho}$ such that $K_{\rho} = \mathbb{F}(a_1, \ldots, a_m)$. 

- \(\square\)
If the characteristic of $K$ is positive and the elements $a_1, \ldots, a_m$ are algebraic over $\mathbb{F}$, then $K_\rho$ is a finite field and hence $\rho(G)$ is finite.

Otherwise $K_\rho$ is infinite. Since $K_\rho$ is finitely generated over $\mathbb{F}$ and infinite, there exists a discrete valuation $\nu$ on $K_\rho$. Let $O_\nu$ be the corresponding valuation ring and $\Delta(K_\rho^{n+1}, \nu)$ be the associated building of type $\widetilde{A}_n$, for details see [Ron09 chap. 9]. Let $GL_{n+1}(K_\rho)^{o}$ be the kernel of the homomorphism

$$\nu \circ \det : GL_{n+1}(K_\rho) \to \mathbb{Z}.$$ 

Since $\rho(G)$ has property $FB_{\widetilde{A}_n}$, by Lemma 2.5 this group has no quotient isomorphic to $\mathbb{Z}$. This shows that $\rho(G)$ is contained in $GL_{n+1}(K_\rho)^{o}$, which acts by type preserving automorphisms on $\Delta(K_\rho^{n+1}, \nu)$. Since $\rho(G)$ has property $FB_{\widetilde{A}_n}$, there is a vertex $x$ of $\Delta(K_\rho^{n+1}, \nu)$ which is invariant under $\rho(G)$. Hence $\rho(G)$ is contained in the stabilizer of $x$. All vertex stabilizers are conjugate to $GL_{n+1}(O_\nu)$. Thus for each $g \in G$ the coefficients of the characteristic polynomial of $\rho(g)$ belong to $O_\nu$. We can do this construction for each discrete valuation on $K_\rho$, hence the coefficients of the characteristic polynomial of $\rho(g)$ lie in the intersection $\bigcap_{\nu \text{ disc. val. on } K_\rho} O_\nu$.

If the characteristic of $K$ is zero, then by Proposition 3.1(ii) this intersection is equal to the integral closure of $Z$ in $K_\rho$ and hence the coefficients of the characteristic polynomial and his roots are integral over $\mathbb{Z}$.

If the characteristic of $K$ is $p$, then by Proposition 3.1(i) the intersection $\bigcap_{\nu \text{ disc. val. on } K_\rho} O_\nu$ is equal to the integral closure of $\mathbb{F}_p$ in $K_\rho$ and hence the coefficients of the characteristic polynomial and his roots are integral (algebraic) over $\mathbb{F}_p$. Let $\lambda$ be an eigenvalue of $\rho(g)$. Then there exists a monic polynomial $f$ in $\mathbb{F}_p[X]$ with $f(\lambda) = 0$. Thus the field extension $[\mathbb{F}_p(\lambda) : \mathbb{F}_p] \leq \deg(f)$ is finite, therefore $\mathbb{F}_p(\lambda)$ is a finite field of order $q = p^k$ for suitable $k \in \mathbb{N}$. Since $\lambda \in \mathbb{F}_p(\lambda)^*$, it follows by Lagrange’s Theorem that $\lambda^{q-1} = 1$. Hence $\lambda$ is a root of unity. \(\square\)

Before we can prove the main theorems we need Schur’s result about linear torsion groups.

**Theorem 3.2.** ([Lam01], 9.9]) Let $K$ be an arbitrary field and $n \in \mathbb{N}$. Then every finitely generated torsion subgroup of $GL_n(K)$ is finite.

### 3.2. Proof of Theorem A.

**Proof.** Let $\rho : G \to GL_{n+1}(K)$ be a linear representation over a field $K$ with characteristic $p$.

Let $\{g_1, \ldots, g_l\}$ be a finite generating set of $G$ and let $K_\rho$ be the subfield of $K$ generated by the coefficients of the matrices $\rho(g_i)$ for $i = 1, \ldots, l$. We get $\rho(G) \subseteq GL_{n+1}(K_\rho)$. Our first goal is to show that the order of $\rho(g)$ for $g \in G$ is finite.

Let $K_\rho^{alg}$ be the algebraic closure of $K_\rho$ and

$$\iota : GL_{n+1}(K_\rho) \to GL_{n+1}(K_\rho^{alg})$$

be the canonical embedding. Since $K_\rho^{alg}$ is algebraically closed field, the matrix $\iota \circ \rho(g)$ is triangularizable. Therefore there exist $A, B \in GL_{n+1}(K_\rho^{alg})$ with

$$\iota \circ \rho(g) = A \cdot B \cdot A^{-1}$$

where $B$ is of the form

$$B = \begin{bmatrix}
\lambda_1 & * & * & \ldots \\
0 & \lambda_2 & * & \ldots \\
0 & 0 & \lambda_3 & \ldots \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$
where $\lambda_1, \ldots, \lambda_{n+1}$ are the eigenvalues of $\iota \circ \rho(g)$. The eigenvalues of $\iota \circ \rho(g)$ are by Proposition E(i) roots of unity, therefore the order of $\lambda_j$ is finite. Let $k = \text{lcm} \{ \text{o}(\lambda_j) \mid j = 1, \ldots, n+1 \}$. Then

$$B^k = \begin{bmatrix} 1 & * & * & \cdots \\ 0 & 1 & * & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & * & * & \cdots \\ 0 & 0 & * & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and by Frobenius homomorphism we get

$$B^{kp^l} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \text{Id}$$

for $l \in \mathbb{N}$ with $p^l \geq n+1$. Therefore the order of $B$ and hence of $\iota \circ \rho(g)$ and $\rho(g)$ is finite.

3.3. Proof of Theorem C.

Proof. Let $\rho : G \to \text{GL}_{n+1}(\mathbb{Q})$ be a linear representation of $G$. Since there exists a positive definite quadratic form on $\mathbb{Q}^{n+1}$ which is invariant under $\rho(G)$, the image of $\rho$ is contained in a conjugate of orthogonal matrices and therefore the eigenvalues of $\rho(g)$ for $g \in G$ have absolute value 1.

By Proposition E(ii) the coefficients of the characteristic polynomial of $\rho(g)$, denoted by $\chi_{\rho(g)}$, are integral over $\mathbb{Z}$ in $\mathbb{Q}$, hence $\chi_{\rho(g)} \in \mathbb{Z}[X]$. The set of all monic polynomials of degree $n+1$ with integer coefficients having roots with absolute value one is finite. To see this, we write

$$X^{n+1} + a_nX^n + \ldots + a_0 = \prod_{j=1}^{n+1}(X - \lambda_j)$$

where $a_j \in \mathbb{Z}$ and $\lambda_j$ are the roots of the polynomial.

Since the $a_j$ are integers, each $a_j$ is limited to at most $2 \cdot \binom{n}{j} + 1$ values and therefore the number of polynomial is finite and hence the set

$$E := \{ \lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } \rho(g) \text{ for } g \in G \}$$

is finite. Thus each $\lambda \in E$ has finite order. Let $g \in G$, then $\iota \circ \rho(g)$ is diagonalizable, where $\iota : \text{GL}_{n+1}(\mathbb{Q}) \to \text{GL}_{n+1}(\mathbb{C})$ is the canonical embedding. Since each eigenvalue has finite order the matrices $\iota \circ \rho(g)$ and $\rho(g)$ have finite order. So $\rho(G)$ is a finitely generated torsion group and by Theorem 3.2 this group is finite. \hfill \square

3.4. Proof of Theorem D.

Proof. Let $\rho : G \to \text{GL}_{n+1}(\mathbb{C})$ be a linear representation of $G$. Let $\{g_1, \ldots, g_l\}$ be a finite generating set of $G$ and let $\mathbb{C}_\rho$ be the subfield of $\mathbb{C}$ generated by the coefficients of the matrices $\rho(g_i)$ for $i = 1, \ldots, l$. We get $\rho(G) \subseteq \text{GL}_{n+1}(\mathbb{C}_\rho)$. Our goal is to show, that the eigenvalues of $\rho(g)$ for $g \in G$ are roots of unity.

It follows by Proposition E(ii) that the eigenvalues of any element of $\rho(G)$ are in some number field. Since $\rho$ has image with compact closure, the eigenvalues of $\rho(g)$ for $g \in G$ have absolute value 1 and since each representation of degree $n+1$ has image with compact closure it follows by Dirichlet Unit Theorem [Ash10, 6.1.6] that the eigenvalues of $\rho(g)$ for $g \in G$ are roots of unity.

Further $\rho(g)$ is diagonalizable and since each eigenvalue has finite order the matrix $\rho(g)$ has finite order. So $\rho(G)$ is a finitely generated torsion group and by Theorem 3.2 this group is finite. \hfill \square

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