Λ²-contribution to the condensate in lattice gauge theory

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Abstract

We present some evidence that in lattice gauge theory the condensate contains a non-perturbative contribution proportional to Λ², the square of the physical scale. This result is based on an analysis of the Wilson loop plaquette expectation from Monte Carlo simulations and its perturbative expansion computed to eight loops. The analysis is not fully conclusive since the calculations are done on a finite lattice and one needs an extrapolation to infinite lattice. It has been recently suggested that in the gluon condensate a Λ²-contribution could be present coming from the large momentum behaviour of the running coupling and not connected to operator product expansion.

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1 Introduction

The Wilson \cite{1} operator product expansion (OPE) has been applied \cite{2} to the study of non-perturbative contributions of physical observables in asymptotically free theories. In these studies an important rôle is played by the gluon condensate \langle \alpha_s \text{Tr} \ F^2 \rangle. This quantity depends on the regularization scheme and on a momentum scale \( Q \), a subtraction point or an ultraviolet (UV) cutoff. According to OPE the gluon condensate has the expansion

\[ W \equiv \frac{\langle \alpha_s \text{Tr} \ F^2 \rangle}{Q^4} = W_0 + \left( \frac{\Lambda^4}{Q^4} \right) W_4 + \cdots, \tag{1} \]

where \( \Lambda \) is the physical scale of the theory related to the running coupling \( \alpha_s = \alpha_s(Q^2) \) at the scale \( Q \). For large \( Q \) one has \( \Lambda^2/Q^2 \sim \exp\{-1/4\pi\beta_0\alpha_s\} \), with \( \beta_0 \) the first beta function coefficient. Since \( \Lambda \) has no expansion in \( \alpha_s \), perturbative contributions are present only in \( W_0 \), which from power counting is quartically divergent in the UV region. The term with the power \( \Lambda^2/Q^2 \) is absent since there are no gauge invariant operators of dimension two. The term \( W_4 \) is the “genuine” vacuum expectation of the gluon condensate of dimension four.

The fact that the perturbative expansion of \( W \) in \( \alpha_s \) is non-convergent \cite{3}, implies that the various terms \( W_0, W_4, \cdots \) cannot be identified simply by collecting contributions with a given power of \( \Lambda \). Indeed one expects that \( W_0 \) itself contains non-perturbative terms proportional to \( \Lambda^4/Q^4 \). These contributions, needed to regularize the non-convergent expansion in \( \alpha_s \), have a simple origin. According to the ITEP formulation, \( W_0 \) is obtained from contributions of Feynman diagrams in which the virtual (Euclidean) momenta \( k \) are in the perturbative region, i.e. \( k^2 \gg \Lambda^2 \). The presence of this lower integration bound introduces in \( W_0 \) terms proportional to \( \Lambda^4/Q^4 \) which should be canceled \cite{4} by similar terms present in the coefficient of the unit operator contribution in the OPE of a physical observable.

In this paper we address the question whether the non-perturbative contributions in \( W_0 \) are only the ones discussed above which are proportional to \( \Lambda^4/Q^4 \). The gluon condensate is particularly suitable for discussing this question since no contributions proportional to \( \Lambda^2/Q^2 \) are predicted by OPE. Then one may ask whether a non-perturbative term of order \( \Lambda^2/Q^2 \) could be present. We perform the analysis in lattice gauge theory in which, due to the presence of an UV cutoff, the gluon condensate has no UV renormalon \cite{5}.

By using the results on \( W \) in \( SU(3) \) lattice gauge theory obtained by Monte Carlo simulations \cite{6} and by higher order perturbative calculations \cite{7} we present some evidence that in the lattice regularization \( W \) contains terms proportional to \( \Lambda^2/Q^2 \). The analysis is not fully conclusive since the calculations are done on lattices of finite size and one needs an extrapolation to infinite lattice.

If the presence of terms of order \( \Lambda^2/Q^2 \) in \( W \) is confirmed, one may worry that OPE could be violated.

Recently Grunberg \cite{4} and Akhoury and Zakharov \cite{8} pointed out that the gluon condensate could contain a \( \Lambda^2/Q^2 \) contribution not accounted for by OPE but originating from power corrections in the high frequency part of the running coupling. Since the perturbative Feynman diagrams for the gluon condensate are quartically divergent in the UV region, power contributions in the running coupling, which are highly subleading at large
momentum, could generate a $\Lambda^2/Q^2$ term in $W_0$. Power corrections in the running coupling at large momentum are naturally expected in physical schemes for the coupling definition such as the dispersive method [9] and the V-scheme [10]. A similar observation has been given in [11].

The paper is organized as follows. In Sect. 2 we recall the expected behaviour of the perturbative coefficients for $W_0$, a quartically divergent quantity. In Sect. 3 we recall lattice gauge theory results on the high order perturbative coefficients of $W$ and recall that they are consistent with the factorial growth predicted for a quartically divergent quantity. In Sect. 4 we analyze the remainder of the expansion and show the indication for the presence of a $\Lambda^2/Q^2$ contribution in the gluon condensate. We show that finite size effects should not be too important. In Sect. 5 we discuss this result and its compatibility with OPE.

## 2 Factorial growth of perturbative coefficients

In the lattice theory all frequencies are bounded by the UV cutoff $Q = \pi/a$ with $a$ the lattice spacing. The condensate $W$ can be written in the general form (we assume an infinite lattice for the moment)

$$W = \int_0^{Q^2} \frac{k^2}{Q^4} \frac{dk^2}{Q^4} f(k^2/\Lambda^2),$$

where $k$ is the softest virtual momentum. This expression is based on the fact that the associated observable has dimension four and is renormalization group invariant [4] so that the function $f(k^2/\Lambda^2)$, which does not depend on $Q$, for large $Q$, can be expressed in terms of a running coupling at the scale $k^2$.

We now consider the contribution $W_0$ which is given by the large frequency part of this integral. At large $k^2$ the running coupling can be approximated by the perturbative coupling $\alpha_s(k^2)$ obtained by taking into account the first few terms of the beta function. For instance at two loops one finds $\alpha_s(k^2)$ given in terms of the physical scale $\Lambda$ by

$$\Lambda^2 \simeq k^2 \left( \frac{1}{4\pi b_0 \alpha_s(k^2)} \right)^{b_1/b_0^2} e^{-1/4\pi b_0 \alpha_s(k^2)}, \quad b_0 = 11/(4\pi)^2, \quad b_1 = 102/(4\pi)^4.$$  

For simplicity we may limit the discussion to the case in which at large $k^2$ the function $f(k^2/\Lambda^2)$ is proportional to the perturbative running coupling. We then consider the contribution to $W_0$ given by

$$W_0^{\text{ren}} = C \int_{\rho \Lambda^2}^{Q^2} \frac{k^2}{Q^4} \frac{dk^2}{Q^4} \alpha_s(k^2),$$

with $\rho \gg 1$ to make $\alpha_s(k^2)$ within the perturbative region. Terms with higher power of $\alpha_s(k^2)$ will give similar contributions (see later). $W_0^{\text{ren}}$ depends on the parameter $\rho$ which sets the separation of the low and high frequency part of Eq. 2). Clearly the lower bound $\rho \Lambda^2$ contributes to this function by terms proportional to $\Lambda^4/Q^4$.

Notice that, since the integral is quartically divergent for $Q^2 \to \infty$, the precise form of $\alpha_s(k^2)$ at large $k^2$ is very important. In the following we assume that at large $k^2$ the running
coupling $\alpha_s(k^2)$ is given by the beta function at two loops. We introduce the variable

$$z \equiv z_0 \left(1 - \frac{\alpha_s}{\alpha_s(k^2)}\right), \quad \alpha_s = \alpha_s(Q^2), \quad z_0 = \frac{1}{3b_0}, \quad (5)$$

and by using for $\alpha_s(k^2)$ the two-loop form one finds

$$\frac{k^2 \, dk^2}{Q^4} \alpha_s(k^2) \sim dz \, e^{-\beta z} \left(z_0 - z\right)^{-1-\gamma}, \quad 4\pi \alpha_s = \frac{6}{\beta}, \quad \gamma = \frac{2b_1}{b_0}, \quad (6)$$

where higher orders in $\alpha_s(k^2)$ have been ignored. We have introduced the inverse coupling $\beta$ for the comparison with the lattice theory. The integration region in (4) is mapped into the region $0 < z < z_0 - \bar{z} = z_0 (1 - \bar{\beta}/\beta)$ with $6/\bar{\beta} = 4\pi \alpha_s(\rho \Lambda^2)$, and one finds the regularized “renormalon” expansion

$$W_0^{\text{ren}} = N \int_{z_0}^{z_0 - \bar{z}} dz \, e^{-\beta z} \left(z_0 - z\right)^{-1-\gamma} = \sum_{\ell=1} \beta^{-\ell} \left\{ c_\ell^{\text{ren}} + \mathcal{O}(e^{-z_0 \beta}) \right\}, \quad (7)$$

with $e^{-z_0 \beta} \sim \Lambda^4/Q^4$ and $c_\ell^{\text{ren}}$ the renormalon coefficients

$$c_\ell^{\text{ren}} = N' \, \Gamma(\ell + \gamma) \, z_0^{-\ell}. \quad (8)$$

In the coefficients of $\beta^{-\ell}$ in Eq. (4) the non-perturbative corrections of order $\Lambda^4/Q^4$ are essential to make the expansion convergent. They are coming from the lower cutoff $\rho \Lambda^2$ for the perturbative region and thus they depend on the parameter $\rho$. The perturbative coefficient $c_\ell^{\text{ren}}$ grows factorially due to an infrared (IR) renormalon of dimension four.

A similar factorial growth of the perturbative coefficients is found if one considers the contributions from higher powers of the coupling. Then the expression (4) gives a general form of the perturbative factorial growth with the numerical constant $N$ which takes into account higher order corrections.

### 3 Summary of $SU(3)$ lattice gauge theory results

We first recall the basic element of $SU(3)$ lattice gauge theory and then the results on the high order perturbative coefficient calculations. The action is

$$S[U] = -\frac{\beta_{\text{lat}}}{6} \sum_P \text{Tr} \left(U_P + U_P^\dagger\right), \quad 4\pi \alpha_s^{\text{lat}} = \frac{6}{\beta_{\text{lat}}}, \quad (9)$$

where the sum extends to all plaquettes $P$ in a hypercubic lattice in four dimensions. The plaquette field $U_P$ is obtained from the link variable $U_\mu(x) = \exp\{a \, A_\mu(x)/\sqrt{\beta_{\text{lat}}}\}$, given by an exponential map on $SU(3)$, where $a$ is the lattice spacing. In the continuum limit $a \to 0$, at the classical level, the plaquette tends to the Lagrangian density and Eq. (4) tends to the Yang-Mills continuous action. In the lattice regularization one has an ultraviolet (UV) cutoff $Q = \pi/a$, and $\alpha_s^{\text{lat}}$ is the coupling at this scale. If the lattice is finite one has also an infrared (IR) cutoff $Q_0 = 2\pi/Ma$, with $M$ the number of lattice points in each direction.
The condensate is given by the expectation value of the elementary plaquette

\[ W_{1x1} \equiv 1 - \frac{1}{3} \langle \text{Tr} \; U_P \rangle. \]  

We consider also the condensate \( W_{2x2} \), the expectation value of the double plaquette, that is a Wilson loop around a square of size 2. We denote in general by \( W \) these quantities.

\( W \) can be obtained numerically by Monte Carlo simulations. The lattice is finite and we denote by \( W(M) \) the quantity computed on a finite lattice of size \( M \). For the finite lattices considered in the present simulations the values of \( \beta_{\text{lat}} \) are taken into the range \( 6 - 6.5 \). In this region one expects that the renormalization group properties are satisfied: the finite lattice size is not crucial for \( \beta_{\text{lat}} \gtrsim 6; \) the lattice discretization artifacts are small for \( \beta_{\text{lat}} \gtrsim 6 \).

### 3.1 Factorial growth of lattice perturbative coefficients

The coefficients \( c^{\text{lat}}_\ell(M) \) of the perturbative expansion of \( W(M) \) on a lattice of size \( M \)

\[ W^{\text{pert}}(M) = \sum_{\ell \geq 1} c^{\text{lat}}_\ell(M) \beta_{\text{lat}}^{-\ell}, \]

are known up to eight loops. The first three terms have been computed analytically \[12\] for an infinite lattice. Eight terms for the expansion of both \( W_{1x1} \) and \( W_{2x2} \) have been computed numerically in Ref. \[6\] for a lattice with \( M = 8 \).

In Ref. \[6\] it has been shown that the growth with \( \ell \) of the first eight coefficients is consistent with the factorial behaviour described in the previous section for a quantity of dimension four like \( W \). In the following we recall the analysis done in \[6, 13\].

In order to confirm that the computed eight coefficients growth factorially as the renormalon coefficients in \( (8) \) one has to take into account two facts: i) the numerical calculations are done on a finite lattice; ii) the coupling in \( (7) \) and in \( (9,11) \) are not necessarily within the same regularization scheme.

i) **Finite size lattice.** The numerical coefficients \( c^{\text{lat}}_\ell(M) \) have been computed on a finite lattice with \( M = 8 \) points in each direction. The effect of the presence of a finite volume can be estimated by putting the IR cutoff \( Q_0 = 2\pi/Ma \) in Eq. \( (3) \), i.e. \( z < z_{\text{ir}} = 4 \ln(M/2)/\beta \) in \( (3) \). This reduces the size of the perturbative coefficients and, for values of \( \beta \) with \( z_{\text{ir}} < z_0 \), makes the integral well defined and the perturbative expansion convergent. We then define

\[ W_0^{\text{ren}}(M) = N \int_0^{\min(z_{\text{ir}}, z_0 - \gamma)} dz \; e^{-\beta z} \; (z_0 - z)^{-1-\gamma} = \sum_{\ell \geq 1} c^{\text{ren}}_\ell(M) \beta^{-\ell}, \]

where \( c^{\text{ren}}_\ell(M) \) are given by incomplete Gamma functions. For \( M \to \infty \) one has \( z_{\text{ir}} \to \infty \) and then \( c^{\text{ren}}_\ell(M) \) tend to the infinite volume coefficients of Eq. \( (8) \). The reduction of the coefficients for finite \( M \) been studied in detail in \[13\]. The conclusion is that even for a small size lattice with \( M = 8 \), the factorial growth up to \( \ell = 8 \) is not tamed (see also later).

ii) **Continuous and lattice coupling.** The two coupling of eqs. \( (7, 11) \) are not necessarily within the same regularization scheme. Differences in the couplings do not modify the
factorial growth, but changes subasymptotic contributions. In [6] we have assumed that they are perturbatively related at two loop by

$$\beta = \beta_{\text{lat}} - r - \frac{r'}{\beta_{\text{lat}}}. \quad (13)$$

It is known that the relation between the lattice and continuum coupling involves large perturbative corrections. For instance if $\beta$ is the $\overline{\text{MS}}$ coupling one finds [14] the values $r = 1.8545$ and $r' = 1.667$. By using (13) we can connect the coefficients in (7) with the ones in (11) by computing the new coefficients $C_{\text{ren}}^{\ell}(r, r', M)$

$$W_{0}^{\text{ren}}(M) = \sum_{\ell \geq 1} c_{\text{lat}}^{\ell}(M) \beta^{-\ell} = \sum_{\ell \geq 1} C_{\text{ren}}^{\ell}(r, r', M) \beta^{-\ell}. \quad (14)$$

We find that for large $\ell$ the numerical coefficients of the single and double plaquette agree with $C_{\text{ren}}^{\ell}(r, r', M)$ for large values $r = 3.1$ and $r' = 2.0$. See Fig. 1. The fact that the resulting value of $r$ is larger than the one coming from $\overline{\text{MS}}$ can be simply explained by assuming that the scale entering in the running coupling in Eq. (4) is $s k^2 < k^2$, in analogy to the exact result of the sigma-model [14] (see also [13]).

![Figure 1: Coefficients of the perturbative expansion as function of the (loop) order $\ell$. The open squares represent $c_{\text{lat}}^{\ell}$ from Ref.6 for $\ell \leq 8$. The open triangles represent the renormalon coefficients $C_{\text{ren}}^{\ell}(r, r', M)$ in Eq. (14) for $r = 3.1$, $r' = 2.0$ and $M = 8$.](image)

Therefore the main conclusion of Ref. [3] was then that the factorial growth of the computed coefficients $c_{\text{lat}}^{\ell}$ is in agreement with a renormalon associated to dimension four, i.e. the the gluon condensate dimension.
4 Evidence of a $\Lambda^2/Q^2$ contribution

Here we present the evidence that in lattice gauge theory the condensate contains terms of order $\Lambda^2/Q^2 \sim e^{-z_0 \beta/2}$. We analyze $W - W_0$ as a function of $\beta$. The contribution $W$ is obtained by the Monte Carlo simulation. The contribution $W_0$ is constructed by adding to the computed eight-loop perturbative terms a remainder. Since for large orders the perturbative coefficients of $W_0^{\text{ren}}(M)$ approach the lattice ones (see Fig. 1), we use this function to obtain the remainder. In Eq. (12) we assumed that the running coupling $\alpha_s(k^2)$ has the two-loop asymptotic behaviour. Under this assumption the constructed $W_0$ contains, beside the perturbative terms, power terms of order $\Lambda^4/Q^4 \sim e^{-z_0 \beta}$. In this case the difference $W - W_0$ should be of order $\Lambda^4/Q^4 \sim e^{-z_0 \beta}$. We shall find instead that there is a contribution of order $\Lambda^2/Q^2 \sim e^{-z_0 \beta/2}$.

First we plot in Fig. 2 the quantity

$$\Delta_L W(M) = W(M) - \sum_{\ell=1}^{L} c_{\ell}^{\text{lat}}(M) \beta_{\text{lat}}^{-\ell},$$

in the range $\beta_{\text{lat}} = 6 - 7$ for various values of $L \leq 8$. The quantity $W(M)$ is obtained from the Monte Carlo simulation of Ref. [5] on a lattice with $M = 8$ as for the perturbative coefficients.

We observe that in this range of $\beta_{\text{lat}}$ the quantity $\Delta_L W(M)$ approaches for $L \to 8$ the behaviour of $\Lambda^2/Q^2$ instead than the expected behaviour of $\Lambda^4/Q^4$.

Before drawing a definite conclusion on this result one needs to analyze whether the $\Lambda^2/Q^2$ behaviour is modified by considering: 1) the remainder of the perturbative expansion; 2) the effects of finite volume.

1) Remainder of the perturbative expansion. As previously recalled, at large $\ell$ the perturbative coefficients $C_{\ell}^{\text{ren}}(r, r', M)$ in Eq. (14) reproduce the lattice coefficients $c_{\ell}^{\text{lat}}(M)$. We then estimate the remainder $\delta W_0$ by subtracting from $W_0^{\text{ren}}(M)$, given in Eq. (12), the first eight terms in (14)

$$\delta W_0(M) = W_0^{\text{ren}}(M) - \sum_{\ell=1}^{8} C_{\ell}^{\text{ren}}(r, r', M) \beta_{\text{lat}}^{-\ell}.$$  

We than obtain the following estimate for $W_0$

$$W_0(M) \equiv \sum_{\ell=1}^{8} c_{\ell}^{\text{lat}}(M) \beta_{\text{lat}}^{-\ell} + \delta W_0(M).$$

We plot in Fig. 2 the quantity

$$\Delta W(M) = W(M) - W_0(M),$$

for $M = 8$ in the region $\beta_{\text{lat}} = 6 - 7$. We see that the behaviour $\Lambda^2/Q^2$ is still maintained. The conclusion is that in the region considered for $\beta_{\text{lat}}$ the first eight perturbative terms give a reliable approximation of $W_0$, at least for $M = 8$.

2) Finite volume. This effect is quite difficult to estimate without performing a direct Monte Carlo simulation on lattices with $M$ sufficient large to have the IR cutoff below the Landau
Figure 2: (a) The subtracted MonteCarlo data $\Delta_L W$ of Eq. (15) compared to $\Lambda^2/Q^2$ and $\Lambda^4/Q^4$ for various values of $L$: upper curve for $L = 2$, lower curve for $L = 8$; (b) The subtracted MonteCarlo data $\Delta W$ of Eq. (18) after resummation of the renormalon contribution compared to $\Lambda^2/Q^2$ and $\Lambda^4/Q^4$.

singularity, i.e. $\ln(M/2) > \beta/12b_0$. For the contribution $W_0$ we can estimate the effect of the finite volume. For the first eight coefficients this has been done in [13] and, as already recalled, for $M = 8$ the factorial growth is still present. We can study the $M$ dependence of the remainder $\delta W_0(M)$ in (16) and we find that in the considered region of $\beta_{\text{latt}}$ the effect of finite size is small, i.e. less than 5%.

5 Discussion and conclusion

One has to consider the following two indications.

1) As shown in [6], the first eight perturbative coefficients of $W$ seem to agree with the factorial growth corresponding to a IR renormalon associated to an operator of dimension four, as required by OPE (see fig. 1).

2) Here we have studied the contribution $W_0$ in (1) obtained from the first eight terms
of the perturbative expansion and a remainder constructed on the hypothesis that only $\Lambda^4/Q^4$ corrections are present. By subtracting from $W$ the term $W_0$ we have found indications of an additional contribution proportional to $\Lambda^2/Q^2$ (see fig. 2).

Some caveat are in order. The reason for the unexpected behaviour $\Lambda^2/Q^2$ could be that our analysis is not complete. The major problem is the finiteness of the lattice size. However we have estimated its effects on the considered contributions to $W_0$ and they seem to be small. It may be that the Monte Carlo simulation for $W$ contains spurious finite size effects giving an effective $\Lambda^2/Q^2$ behaviour. Excluding this possibility would require an investigation on a very large lattice with $\ln(M/2) \gtrsim \beta$.

Recently it has been argued by Grunberg [7] and by Akhoury and Zakharov [8] that terms of order $\Lambda^2/Q^2$ can be present in the gluon condensate which are not accounted for by OPE, but are due to power corrections in the running coupling at high momentum. In physical schemes [8, 10] highly subleading power corrections at large momentum are naturally present in the running coupling. A similar observation has been given in [4]. These corrections could be responsible for the appearance of $\Lambda^2/Q^2$ terms in the condensate due to the fact that the integral for $W$ is quartically divergent. A $\Lambda^2/k^2$ contribution in $\alpha_s(k^2)$ in the integral (1) gives two terms. The first of order $\Lambda^2/Q^2$ is coming from the UV region ($k^2 \approx Q^2$), the second, of the canonical order $\Lambda^4/Q^4$, is coming from the IR region ($k^2 \approx \rho\Lambda^2$). These $\Lambda^2/Q^2$ terms are of “perturbative” nature and are then naturally associated to the contribution $W_0$ in the OPE. Moreover they should be process independent as the running coupling. An important question is whether these $\Lambda^2/Q^2$ are phenomenologically relevant (see [7, 8]).

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References

[1] K. Wilson, Phys. Rev. D 179 (1969) 1499.

[2] M.A. Shifman, A.I. Vainstein and V.I. Zakharov, Nucl. Phys. B147 (1979) 385, 448, 519; V.A. Novikov, M.A. Shifman, A.I. Vainstein and V.I. Zakharov, Phys. Rep. 116 (1984) 105; Vacuum Structure and QCD Sum Rules: Reprints, ed. M.A. Shifman (North-Holland, 1992: Current Physics, Sources and Comments, v. 10)

[3] For reviews and classic references see:
   V.I. Zakharov, Nucl. Phys. B385 (1992) 452;
   A.H. Mueller, in QCD 20 years later, vol. 1 (World Scientific, Singapore 1993).
[4] V.A. Novikov, M.A. Shifman, A.I. Vainstein and V.I. Zakharov, Nucl. Phys. B249 (1985) 445.

[5] M. Campostrini, A. DiGiacomo and V. Gündük, Phys. Lett. 223 (1989) 393 and new MC data from L. Scorzato.

[6] F. Di Renzo, E. Onofri, and G. Marchesini, Nucl. Phys. B457 (1995) 202.

[7] G. Grunberg, Power corrections and Landau singularity, Ecole Polytechnique preprint CPT h/S 505.0597, hep-ph/9705290.

[8] R. Akhoury and V.I. Zakharov, Renormalons and $1/Q^2$ corrections, University of Michigan preprint UM-TH-97-11, hep-ph/9705318. See also A.I. Vainstein and V.I. Zakharov, Phys. Rev. Lett. 73 (1994) 1207; Phys. Rev. D D54 (1996) 4039; K.K. Yamawaki and V.I. Zakharov, hep-ph/9406373.

[9] Yu.L. Dokshitzer, G. Marchesini and B.R. Webber, Nucl. Phys. B469 (1996) 93

[10] S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, Phys. Rev. D 28 (1983) 228.

[11] B. Ball, M. Beneke and V.M. Braun, Nucl. Phys. B452 (1995) 563.

[12] B. Allés, M. Campostrini, A. Feo and H. Panagopoulos, Phys. Lett. 324B (1994) 443 and references therein.

[13] F. Di Renzo, E. Onofri, and G. Marchesini, Nucl. Phys. B (to appear), hep-lat/9612016.

[14] M. Lüscher and P. Weisz, Nucl. Phys. B452 (1995) 234

[15] F. David, Nucl. Phys. B234 (1984) 237; Nucl. Phys. B263 (1986) 637; V.A. Novikov, M.A. Shifman, A.I. Vainstein and V.I. Zakharov, Phys. Rep. 116 (1984) 105; Nucl. Phys. B249 (1985) 445; M. Campostrini and P. Rossi, Phys. Lett. 242B (1990) 81, Phys. Lett. 242B (1990) 225, Riv. Nuovo Cim. 16 (1993) 1.