Quantum Relaxation of the Cosmological Constant **

R. Jackiw*¹, Carlos Núñez*² and S.-Y. Pi †³

* Center for Theoretical Physics, Massachusetts Institute of Technology Cambridge, MA 02139, USA
†Department of Physics
Boston University, Boston, MA 02215, USA

ABSTRACT: We describe a mechanism that drives the Cosmological Constant to zero value. This mechanism is based on the quantum triviality of $\lambda \phi^4$ field theory and works in $AdS$ space. Some subtleties of the model are discussed.

** Einstein Memorial Issue, Physics Letters A.
Presented at the Kummerfest, Vienna, January 2005.

BUHEP-05-04
MIT-CPT 3609
hep-th/0502215

¹jackiw@lns.mit.edu
²nunez@lns.mit.edu
³soyoung@bu.edu
1 Introduction

Astrophysical observations, which have been interpreted as evidence for a cosmological constant $\Lambda$ [1], have moved the “cosmological constant problem” from explaining a vanishing value, $\Lambda = 0$, to explaining a non-vanishing but tiny positive value, (for reviews see [2], [3], [4]). In this note we remain with the original problem. We discuss a possible mechanism that could drive $\Lambda$ to zero, in the belief that once a vanishing cosmological constant is secured, raising it to its tiny but non-vanishing value is a milder problem.

The mechanism for driving $\Lambda$ to zero, to which here we call attention, is a quantum effect encountered in the $\lambda \phi^4$ theory: while classically $\lambda$ can take any value, in the quantized theory only $\lambda = 0$ is possible [5]. Of course, a regularized quantum field theory may possess any value for $\lambda$ but this value vanishes as the regulator is removed.

Before stating the proposal let us first set the conventions that we shall be using. The metric signature will be mostly minus $\eta_{\mu\nu} = (+, -, -, -)$. The Christoffel symbol is given by

$$\Gamma^\lambda_{\mu\nu} = \frac{g^{\lambda\alpha}}{2} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}).$$

The Riemann and Ricci tensors are defined as

$$R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\beta\nu} - \partial_\beta \Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\alpha\sigma} \Gamma^\sigma_{\beta\nu} - \Gamma^\mu_{\beta\sigma} \Gamma^\sigma_{\alpha\nu}, \quad R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \quad (2)$$

With these conventions, the equations of motion derived from the Einstein-Hilbert action

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda),$$

read

$$R_{\mu\nu} = g_{\mu\nu} \Lambda, \quad (4)$$

and consequently

$$R = 4\Lambda. \quad (5)$$

For $\Lambda > 0$ ($\Lambda < 0$) this corresponds to Anti de Sitter, AdS (de Sitter, dS) space. The units of the quantities above are $[G] = m^{-2}$, $[R] = m^2$, $[\Lambda] = m^2$, $[x] = m^{-1}$.

2 The proposal

We consider the Einstein-Hilbert action

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} (R - 2\Lambda),$$

and propose that $\Lambda$, which is arbitrary in the classical theory, will be driven to zero by quantum effects.
Owing to the non-renormalizability of (6), it is not possible for us to asses our proposal convincingly in General Relativity. Nevertheless, the following calculation supports the proposal. In eq. (6), rescale the metric tensor as
\[ g_{\mu\nu} = \phi^2 \hat{g}_{\mu\nu}, \quad ds^2 = \phi^2 \hat{ds}^2. \] (7)

In four dimensional spacetime, this will scale the volume factor as \( \sqrt{-g} = \phi^4 \sqrt{-\hat{g}} \) and the Ricci scalar as
\[ R(g) = \phi^{-2} R(\hat{g}) - 6 \phi^{-3} \hat{D}^2 \phi. \] (8)

All quantities on the right, including the covariant derivative \( \hat{D} \) involve the rescaled metric \( \hat{g}_{\mu\nu} \).

After partial integration, with surface terms ignored, the action (6) becomes
\[ S = -\frac{3}{4\pi G} \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{12} R(\hat{g}) \phi^2 + \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\Lambda}{6} \phi^4 \right]. \] (9)

In order to have canonical units for fields and couplings, we define
\[ \varphi = \frac{\phi}{\sqrt{G}}, \quad \Lambda = \frac{\lambda}{4G}, \] (10)
and finally the action (6) reads,
\[ S = -\frac{3}{4\pi} \int d^4x \sqrt{-\hat{g}} \left( \frac{1}{12} R(\hat{g}) \varphi^2 + \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{\lambda}{4!} \varphi^4 \right). \] (11)

Let us now, use (11) in two ways. First working in a theoretical laboratory with mini-superspace variables, we set \( \hat{g}_{\mu\nu} \) to be the Minkowski metric, and retain only the conformal

\[ R(g) = \phi^{-2k} \left( R(\hat{g}) - 2k(d-1)\phi^{-1} \hat{D}^2 \phi + [2k(d-1) - k^2(d-1)(d-2)] \phi^{-2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right), \] (i)

and the Einstein-Hilbert action reads, after integrations by parts,
\[ \int d^d x \sqrt{|g|} R = \int d^d x \sqrt{|\hat{g}|} \phi^{k(d-2)} [R(\hat{g}) + k^2(d-1)(d-2)\phi^{-2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi]. \] (ii)

In the exponent factor, \( k \) is set to \( k = \frac{d}{d-2} \) by requiring that the scalar kinetic term be conventional. Then eq. (i) implies
\[ \frac{(d-2)}{8(d-1)} \int d^d x \sqrt{|g|} (R(g) - \frac{8(d-1)}{d-2} \lambda) = \int d^d x \sqrt{|\hat{g}|} \left[ \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{(d-2)}{8(d-1)} \phi^2 R(\hat{g}) - \lambda \phi^{2d/(d-2)} \right]. \] (iii)

This shows that (in any \( d > 2 \)) the Weyl invariant action for a conformally coupled scalar field with self interaction, as on the right in (iii), is equal to the Einstein-Hilbert action with a cosmological constant and a rescaled metric, as on the left in (iii). This identity is a consequence of Weyl invariance which may be used to set \( \phi = 1 \) on the right hand side of eq. (iii), therefore achieving the left side.
factor $\varphi$. Eq. (11) shows that the field $\varphi$ follows the dynamics of a $\lambda \varphi^4$ theory, which according to K. Wilson [5], makes $\lambda$ vanish in the quantum field theory. Second, we make the physical argument that it does make sense, for our present day universe, to take a flat background metric. We are then again left with a $\lambda \varphi^4$ theory, which is suppressed quantum mechanically.

Guided by these observations, we suppose that Wilson’s argument for $\lambda \varphi^4$ theory can be extended from flat space-time [5] so that it holds for (9) and (11). For example a constant curvature background gives to the $\varphi$ field a mass and changes the kinetic term, but this should not modify the short distance behavior needed for Wilson’s argument.

Let us note the issues that arise because of the signs in (11). In the flat limit, we need $\lambda$ in (11), to be positive relative to a positive kinetic term. Thus this relaxation mechanism works in AdS space. The overall sign of the action in (11) is negative, compared to the usual matter action (in our conventions). This would render the $\varphi$ dynamics unstable if the field $\varphi$ couples to other matter fields. We evade this problem by assuming that the dynamics of matter fields is Weyl invariant, hence independent of $\varphi$. In the Standard Model, this is true, except for the potential energy in the Higgs sector; but little is certain about Higgs dynamics. Indeed, the action of the gauge fields scales as (the gauge potential does not scale)

$$S_{\text{gauge}} = \int d^4x \sqrt{-g} g^\mu\nu g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \rightarrow \int d^4x \sqrt{-\hat{g}} \hat{g}^{\mu\nu} \hat{g}^{\alpha\beta} \hat{F}_{\mu\alpha} \hat{F}_{\nu\beta}. \quad (12)$$

For the fermions, which scale as $\psi = \phi^{-3/2} \hat{\psi}$, the action (for a charged fermion) in curved space is also Weyl invariant.

$$S_{\text{fermion}} = \int d^4x \sqrt{-\hat{g}} \hat{\psi} \gamma^\mu D_\mu(\omega, A) \psi = \int d^4x \sqrt{-\hat{g}} \hat{\psi} \gamma^\mu (\partial_\mu + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} + A_\mu) \psi \rightarrow \int d^4x \sqrt{-\hat{g}} \hat{\psi} \hat{\gamma}^\mu \hat{D}_\mu(\hat{\omega}, A) \hat{\psi}. \quad (13)$$

We emphasize that the quantum suppression of the quartic self coupling (and therefore of the cosmological constant) is not driven by the perturbative renormalization group, because perturbation theory is inapplicable to $\lambda \varphi^4$ theory when the regularization is removed. Indeed, the perturbative renormalization group with the lowest order beta function $c\lambda^2, c = 3/16\pi^2$ gives a running

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - c\lambda(\mu_0) \log(\frac{\mu}{\mu_0})}. \quad (14)$$

This perturbative results, holds only for very small (compared to 1) values of $\lambda$. For the ‘original’ cosmological constant $\Lambda$ this implies

$$\Lambda(\mu) = \frac{\Lambda(\mu_0)}{1 - 4c\Lambda(\mu_0)G \log(\frac{\mu}{\mu_0})}. \quad (15)$$

which does not provide sufficient running from a sizeable cosmological constant at a high energy $\mu$ to its small value at $\mu_0 = 2.3K$ in energy units. Perturbation theory, even when improved by the renormalization group, does not address our proposal.
However, one can encounter within perturbation theory indications that the renormalized \( \lambda \) must vanish. This plausibility argument for triviality of \( \lambda \varphi^4 \) is found within the lowest order, dimensionally regulated renormalization group analysis [7]. Setting dimensionality \( \epsilon = (4 - d) \), we recognize that (14) and (15) arise in the limit \( \epsilon \to 0 \) from the equality

\[
\lambda_0 = \mu_1^\epsilon \frac{\lambda(\mu_1)}{1 - \lambda(\mu_1)^{2/\epsilon}} = \mu_2^\epsilon \frac{\lambda(\mu_2)}{1 - \lambda(\mu_2)^{2/\epsilon}},
\]

where \( \lambda_0 \) is the bare coupling. The theory should be well defined for \( d = 4 - \epsilon \) with \( \epsilon > 0 \). To reach physical 4 dimensions we let \( \epsilon \to 0^+ \). But before reaching \( d = 4 \) we encounter the singularity at \( \epsilon = c\lambda(\mu) > 0 \) where, as stressed above, the theory should be well defined. This causes the bare coupling \( \lambda_0 \) to diverge, So, the problem is avoided by supposing that the renormalized coupling vanishes, and only a non-interacting theory remains.

The idea then is that a field theory, which is not well defined for large values of the energy cut-off solves its problems by becoming non-interacting when the problematic energy scale is reached. Our proposal for the cosmological constant relies on such a non-perturbative suppression of the coupling constant.

There exist numerous studies of triviality of \( \lambda \varphi^4 \) theory. Most of them are motivated by the fact that the Higgs potential in the standard model should show a behavior like the one described above. These studies include lattice simulations that support the result (see for example [8] and references cited in [9]. Also there are proofs of triviality for space times with dimension greater than four. In four dimensions there are arguments that are very suggestive of triviality, but no definitive proof [10].

After formulating this argument, we were reluctant to put it forward publicly, because of the many lacunae. But then we were delighted and encouraged to discover that a similar proposal for suppressing the cosmological constant was made by A. Polyakov in his Oscar Klein Memorial Lecture [11].

3 Acknowledgments:

We thank colleagues for discussions and useful comments. The work of R. Jackiw, C. Nuñez and S.-Y. Pi was supported by by funds provided by the U.S Department of Energy (DOE) under cooperative research agreements DE-FC02-94ER-40818 and DE-FG02-91-ER-40676. C. Nuñez is also supported by a Pappalardo Fellowship.

References

[1] D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003) [arXiv:astro-ph/0302209].
[2] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989).

[3] T. Padmanabhan, Phys. Rept. 380, 235 (2003) [arXiv:hep-th/0212290].

[4] S. Nobbenhuis, “Categorizing different approaches to the cosmological constant problem,” arXiv:gr-qc/0411093.

[5] K. Wilson, Phys. Rev. B 4, 3184 (1971).

[6] R. M. Wald, “General Relativity” (University of Chicago Press, Chicago 1984)

[7] L. Brown, “Quantum Field Theory” (Cambridge University Press, Cambridge UK 1992).

[8] J. Kuti, L. Lin and Y. Shen, Phys. Rev. Lett. 61, 678 (1988).

[9] D. J. E. Callaway, Phys. Rept. 167, 241 (1988).

[10] J. Fröhlich, Nucl. Phys. B 200, 281 (1982).

[11] A. M. Polyakov, Yad. Fiz. 64, 594 (2001) [English translation: Phys. Atom. Nucl. 64, 540 (2001)] [arXiv:hep-th/0006132].