Linear quadratic problems (On “linear” approaches in nonlinear system theory)

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Abstract. The term “linear quadratic problem” refers to the problems of dynamics where linear controlled systems are associated with a certain quadratic functional of performance. “Linear” approaches are the approaches that are usually applied to linear systems, e.g. the methods of the integral transforms (Laplace, Fourier), the linear functional analysis and the frequency domain methods. The paper considers some cases where these approaches are applied to nonlinear systems.

1. Introduction. Two classes of problems
1.1 Optimal control Tracing back the early history of control, it is possible to observe, among various approaches, most of them displaying pioneering character, the presence of the integral quality criteria. It is worth mentioning here the early book [1] where even in its first chapter, among the performance requirements for servomechanism design, the authors mention the minimum of a mean square criterion with reference to [2] and to the following Chapter VII (Analysis of the servomechanisms based on mean square criterion). An even earlier reference is [3].

The first outcome of this approach is given by the so called parameter optimization: the process dynamics and the controller dynamics are linear and structurally known but the controller parameters are to be chosen from the minimization of the quadratic index. A technique was developed for a rational choice of this index, following two aims: a) an acceptable transient process defined by the “extremal curve” of the index; b) an achievable extremal that is of a closed loop transient process sufficiently close to the aforementioned extremal. Good references for this approach are the books that are considered classics now [4, 5].

In a more contemporary way the problem is described as follows. Consider the controlled dynamics integrating the process dynamics and the controller having the controlled error \( \varepsilon(t) \) as regulated output

\[
\dot{x} = Ax, \quad \varepsilon = c^*x
\]  

Here \( x \in \mathbb{R}^n \) is the state vector of the deviations with respect to a steady state imposed by a reference signal. The integral performance index is usually given by a quadratic form in \( \varepsilon \) and its derivatives (up to \( n - 1 \)). Taking into account (1), this index appears as a quadratic form in \( x \)

\[
\eta(0, \infty) = \int_0^\infty x(t)^TQx(t)dt
\]
with $Q > 0$. Since the closed loop system is assumed to be exponentially stabilized by the controller, the integral is convergent along the solutions of (1) and its value is written as $x(0)^* P x(0)$ where $P > 0$ solves a Lyapunov matrix equation

$$A^* P + PA = -Q$$

(here the asterisk denotes transposition). The parametric optimization means such a choice of the controller parameters to minimize $x(0)^* P x(0)$; this minimization must be viewed over some important transient processes: for instance, if a step reference signal is applied starting from a zero steady state, then the initial state vector results from $\epsilon(0) = -1, \epsilon'(0) = \epsilon''(0) = \ldots = \epsilon^{n-1}(0) = 0$.

The “second” optimization problem is in fact a structural synthesis, with minimization of a quadratic criterion. The controlled process is described by

$$\dot{x} = Ax + b\mu(t)$$

and its performance by the integral index

$$\eta(0,T) = x(t)^* H x(t)|_0^T + \int_0^T F(x(t),\mu(t))dt$$

where $F(x,\mu)$ is a quadratic form in both arguments

$$F(x,\mu) = \bar{\mu} \kappa \mu + \bar{l} x^* x + x^* l \mu + x^* M x$$

Here complex coefficients are assumed, hence the asterisk denotes transposition and complex conjugation; the bar is just complex conjugation.

It is required to find a linear feedback device to minimize the quadratic index. For $T \to \infty$ the minimization must be done over a convergent integral hence the feedback device should be stabilizing.

This problem can be solved in two directions. If one remains at the input/output framework, then one finds the Wiener optimal synthesis. An excellent early reference is [6]. In the last decades the input/output framework approach gave the $H_\infty$ control e.g. [7, 8]. Also at the end of the 50ies of the previous century R. E. Kalman started the analysis of the optimal synthesis within the state space framework aiming towards a synthesis of an optimal state feedback for (4)-(5). Again in the case $T \to \infty$ feedback stabilization appeared as essential and it led to the so called analytical design of the controllers [9, 10, 11, 12, 13] or optimal stabilization [14]. A comprehensive reference about a certain state of the art is the “Kalman issue” of IEEE Transactions on Automatic Control of December 1971 (Issue 6 of the volume AC-16). It is worth mentioning that robustness requirements upon the optimal feedback led to the subsequent development including $H_\infty$ control with its nonlinear version [15].

The optimization problem was given such an extension in the Introduction because no more reference to it follows throughout the paper.

1.2 Absolute stability This stability property appeared in science and engineering from the papers of B. V. Bulgakov [16, 17], see also [18]; its genuine spread is connected to the seminal paper of A. I. Lurie and V. N. Postnikov [19]. The absolute stability is defined for a feedback system enclosing a linear dynamical block and a static nonlinear block (figure 1). While the information on the linear block is “complete”, the information on the nonlinear one is defined as “poor” - see the report of A. M. Letov and A. I. Lurie [20]; this means that the nonlinear function is uncertain - all that is known is its confinement to a sector e.g.

$$-\sigma \varphi \sigma^2 < \varphi(\sigma) \sigma < \sigma \varphi \sigma^2$$

(see also figure 1, b)).

Consequently the stability of the zero solution of the system will have two properties: global asymptotic (stability) and validity for all nonlinear functions of the considered class (robustness).

In what is left of this paper we shall point out some features of the absolute stability as a linear quadratic problem where linear methods are applied to a nonlinear system.
2. First stage of the absolute stability theory

As pointed out by A. A. Voronov [21], page 144, the papers of B. V. Bulgakov [16, 17] displayed mainly the way of introducing uncertain nonlinear functions by considering a whole class of functions defined by the sector inequalities (7) while the paper of A. I. Lurie and V. N. Postnikov [19] introduced the most popular, up to 1959 unique, method of tackling it - the Lyapunov function approach. This approach relied on the construction of a special Lyapunov function - a quadratic form in the state variables plus the integral of the nonlinear function. From here it started what S. Lefschetz called in [22] the “pre-Popov period” of the theory development that is the “USSR period”, when the most important results of the Lyapunov approach were obtained and published by the outstanding scientists of the century: A. I. Lurie, I. G. Malkin, M. A. Aizerman, N. N. Krasovskii, A. M. Letov, Ya. Z. Tsypkin, V. A. Yakubovich and others. It is neither the time, nor the place to repeat here all the results of the Lyapunov function period. All we want is to point out the linear quadratic type aspects of the problem which were transparent enough.

If we start from the fact that the Lyapunov theory is the generalization of the “couple” dynamical system/energy and, for the linear systems, the kinetic energy is a quadratic form of the velocities, then the things become more “at hand”. Consider the case of the feedback system of figure 1a and the linear dynamical block to have lumped parameters i.e. being described by Ordinary Differential Equations

\[ \dot{x} = Ax - b\varphi(c^*x) \]  

which can be separated in two blocks in negative feedback connection

\[ \dot{x} = Ax + bu(t) \quad \sigma = c^*x \]  
\[ u(t) = -\varphi(\sigma(t)) \quad \varphi(\sigma^2) \leq \varphi(\sigma)\sigma \leq \bar{\varphi}\sigma^2 \]  

Clearly the sector restrictions are quadratic restrictions. The Lyapunov function approach emerging from the seminal paper [19] relies on functions of the form

\[ V(x) = x^*Hx + \beta \int_0^{c^*x} \varphi(\lambda) \, d\lambda \]  

with \( H \) a Hermitian (symmetric) matrix and \( \beta \) a real parameter. Assume for a while that \( \beta \geq 0 \). Then the sector conditions will imply

\[ x^*(H + \beta\bar{\varphi}c^*)x \leq V(x) \leq x^*(H + \bar{\varphi}c^*)x \]
Now, if \( H + \beta \varphi cc^* \) is positive definite, \( H + \beta \varphi cc^* \) is also and the Kamke Massera functions bounding the Lyapunov functions are quadratic.

If \( V^*(t) := V(x(t)) \) is differentiated along the solutions of (8), the following derivative function of state is obtained

\[
W(x) = x^*H(Ax - b\varphi(c^*x)) + (Ax - b\varphi(c^*x))^*Hx + \beta \varphi(c^*x)c^*(Ax - b\varphi(c^*x))
\]

which can be written as a quadratic form in \( x \) and \( \varphi \) as follows

\[
W(x) = (x^* \varphi) \begin{pmatrix} HA + A^*H & \frac{1}{2}\beta A^*c - Hb \\ \frac{1}{2}\beta c^*A - b^*H & \frac{1}{2}\beta (c*b + b*c) \end{pmatrix} \begin{pmatrix} x \\ \varphi \end{pmatrix}
\]

Absolute stability is ensured provided \( V(x) > 0 \) and \( W(x) \leq 0 \). Consequently the existence of a Lyapunov function of the form (10) subject to \( V(x) > 0, W(x) \leq 0 \) means finding \( H \) and \( \beta \) such that \( H + \beta \varphi cc^* > 0 \) for \( \varphi \leq h \leq \bar{\varphi} \) and the matrix of the quadratic form (13) – negative semi-definite. These are called Lurie solution equations and they foreshadow the Linear Matrix Inequalities [23] (together with the Lyapunov matrix inequality). We shall not enumerate here all development connected to this first stage of absolute stability analysis but send the reader to the monographs [24, 25, 26]. It is worth mentioning however that inequality \( W(x) \leq 0 \) needs not be fulfilled on the entire state space \( \mathbb{R}^n \) but only on the set defined by the quadratic restrictions of the sector which are defined by

\[
(\varphi(\sigma) - \underline{\varphi}c^*x)(\varphi(\sigma) - \bar{\varphi}c^*x) \leq 0
\]

By the so called \( \mathcal{S} \)-procedure, the inequality on the restricted set is replaced by another one, on the entire space - easier to investigate; in the case of a single nonlinear function the \( \mathcal{S} \)-procedure is lossless, i.e. the domain of the parameters ensuring existence of the solutions for the Lurie equations remains unchanged [27].

We end this section by mentioning an early Romanian contribution due to V. M. Popov [28, 29]. It dealt with the so called “indirect control” in absolute stability - the simplest critical case with a simple zero root. For the system written under the form

\[
\dot{x} = Ax - b\varphi(\sigma), \quad \dot{\sigma} = c^*x - \rho\varphi(\sigma)
\]

the usual Lyapunov function

\[
V(x, \sigma) = x^*Hx + 2(h^*x)\sigma + \alpha\sigma^2 + \beta\int_0^\sigma \varphi(\sigma)d\sigma
\]

is replaced by the modified one

\[
V_{\text{mod}}(x, \sigma) = V(x, \sigma) + \frac{\gamma}{2(p - c^*A^{-1}b)}(c^*A^{-1}x - \sigma)^2
\]

allowing a certain relaxation of the sufficient conditions for absolute stability.

3. The turning point of V. M. Popov (1959 and after)

In order to understand the rather surprising approach of V. M. Popov (published in 1959 [30] and spread mainly through the seminal paper [31]), it is worth starting with the quadratic restrictions - in the form known today due to references [32, 33, 34, 35]. We shall discuss here some quadratic restrictions which are tightly connected with the Popov approach. Referring again to figure 1b, inequality (14) can be re-written as

\[
(u(t) + \varphi e^*x(t))(u(t) + \phi e^*x(t)) \leq 0, \quad \forall t \in \mathbb{R}
\]
in fact for all pairs \((u(t), x(t))\) with
\[
    u(t) = -\varphi(\sigma(t)) = -\varphi(e^x(t)) \tag{19}
\]
x\((t)\) being a solution of the nonlinear system \((8)\). A quadratic restriction like \((18)\) is called local. At the same time it is known (elementary geometric argument) that
\[
    \int_0^\sigma (\varphi(\eta) - \varphi_\eta) d\eta = \psi(\sigma) \geq 0 ; \int_0^\sigma (\varphi_\eta - \varphi(\eta)) d\eta = \psi(\sigma) \geq 0 \tag{20}
\]
This allows introducing the non-local restrictions
\[
    \int_0^\sigma (u(\tau) + \varphi c^x x(\tau)) c^x (Ax(\tau) + bu(\tau)) d\tau = \int_0^\sigma (u(\tau) + \varphi \sigma(\tau)) \sigma(\tau) d\tau = \tag{21}
\]
\[
    = - \int_{\sigma(0)}^\sigma (\varphi(\eta) - \varphi_\eta) d\eta = \psi(\sigma(0)) - \psi(\sigma(t)) \leq \psi(\sigma(0))
\]
and
\[
    - \int_0^\sigma (u(\tau) + \varphi c^x x(\tau)) c^x (Ax(\tau) + bu(\tau)) d\tau = \psi(\sigma(0)) - \psi(\sigma(t)) \leq \psi(\sigma(0)) \tag{22}
\]
It is clear that these restrictions are quadratic. References \([32, 33, 34]\) contain extensive lists of local and non-local quadratic restrictions but we shall focus on \((18), (21), (22)\). It is quite straightforward that these inequalities can be “embedded” in a larger class defined by
\[
    W_j(u, \sigma) \leq 0 , \int_0^T W_k(u(\tau), \sigma(\tau), \sigma(\tau)) d\tau \leq \gamma_k(\sigma(0)) \tag{23}
\]
A straightforward consequence of \((23)\) reads as follows: the nonlinear block of figure 1a defined by the sector restrictions \((7)\) - see figure 1b - can be described by \((23)\) which however are much general than \((7)\). Moreover, as it will appear in the following development, inequalities \((23)\) serve to define an integral index
\[
    \eta(0, T) = \int_0^T \left[ \sum \tau_j W_j(u(\tau), \sigma(\tau)) + \sum \theta_k W_k(u(\tau), \sigma(\tau), \sigma(\tau)) \right] d\tau \tag{24}
\]
with \(\tau_j \geq 0, \theta_k \geq 0\) being freely chosen parameters. From \((23)\) it follows that
\[
    \eta(0, T) \leq \sum \theta_k \gamma_j(\sigma(0)) \tag{25}
\]
Inequality \((25)\) was called later a generalized feedback \([36]\); even later \([37]\) it served to define a new stability concept called “bounded index – bounded state”.

Now it is interesting to point out how the whole set of new concepts introduced by V. M. Popov along more than three decades are transparent from the very first paper \([30]\). Consider the quadratic constraints \((18), (21), (22)\) allowing the definition of the integral index
\[
    \eta(0, T) = \int_0^T \left[ \tau_1 W_1(u(\tau), \sigma(\tau)) + \theta_2 W_2(u(\tau), \sigma(\tau), \sigma(\tau)) + \theta_3 W_3(u(\tau), \sigma(\tau), \sigma(\tau)) \right] d\tau \tag{26}
\]
\(W_1, W_2, W_3\) being those quadratic forms from \((18), (21), (22)\) respectively. Along the solutions of \((8)\) and taking \(u(t) = -\varphi(\sigma(t))\) the following equality is obtained
\[
    \theta_2 \psi(\sigma(T)) + \theta_3 \psi(\sigma(T)) + \eta(0, T) = \theta_2 \psi(\sigma(0)) + \theta_3 \psi(\sigma(0)) \tag{27}
\]
Later this equality will suggest hyperstability \([38]\) and dissipativeness \([39, 40]\). V. M. Popov himself used to call \((27)\) direct (that is easy available or a priori) information.
In the next stage the Cauchy formula of variations of constants was applied to (8) leading to the nonlinear integral equation

$$\sigma(t) = c^* e^{At} y(0) - \int_0^t c^* e^{A(t-\tau)} b\phi(\sigma(\tau)) d\tau$$  (28)

easily “embedded” in the more general integral equation

$$\sigma(t) = \rho(t) - \int_0^t \kappa(t-\tau) \phi(\sigma(\tau)) d\tau$$  (29)

Now, if $A$ has all its eigenvalues in $\mathbb{C}^-$, the functions $\rho(t)$, $\kappa(t)$, $\phi(t)$ - also their higher derivatives - are exponentially decreasing to 0 for $t \to \infty$. It is worth mentioning that precisely this general case was considered in [30]. In what follows we shall present in brief what is necessary to obtain a second inequality for the index $\eta(0,T)$; together with (27) it will lead to the main stability inequality. Following V. M. Popov we shall take $\phi = 0$ and re-write $\eta(0,T)$ as follows

$$\eta(0,T) = -\frac{1}{2} \theta_3 \hat{\phi}(t) \hat{\phi}(t) \left|_{1}^{T} \right. - \int_0^T \phi(\sigma(t)) (\tau_1 \hat{\phi} \rho(t) + (\theta_2 - \theta_3) \hat{\rho}(t)) dt + \int_0^T u(t) (\tau_1 u(t) + \theta_2 \hat{\phi} w(t) + (\theta_2 - \theta_3) \hat{w}(t)) dt$$  (30)

At this point another pioneering idea (within the field of absolute stability) due to V. M. Popov occurs: the use of the truncated signals

$$u_T(t) = \begin{cases} -\phi(\sigma(t)), & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases}$$  (31)

$$w_T(t) = \int_0^t \kappa(t-\tau) u_T(\tau) d\tau ; \sigma_T(t) = \rho(t) + w_T(t)$$

All signals in (31) are in $L^1(0,\infty) \cap L^2(0,\infty)$ and have a Fourier transform. A rather straightforward computation leads to

$$\eta(0,T) = -\frac{1}{2} \theta_3 \hat{\phi}(t) \hat{\phi}(t) \left|_{1}^{T} \right. - \int_0^T \phi(\sigma(t)) (\tau_1 \hat{\phi} \rho(t) + (\theta_2 - \theta_3) \hat{\rho}(t)) dt + \int_0^T u(t) (\tau_1 u(t) + \theta_2 \hat{\phi} w(t) + (\theta_2 - \theta_3) \hat{w}(t)) dt$$  (32)

For the second integral the Plancherel equality can be applied to obtain

$$\int_0^\infty u_T(t) (\tau_1 u_T(t) + \theta_2 \hat{\phi} w_T(t) + (\theta_2 - \theta_3) \hat{w}(t)) dt =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re} [\hat{u}(\omega) (\tau_1 \hat{u}(\omega) + \theta_2 \hat{\phi} \hat{w}(\omega) + (\theta_2 - \theta_3) \omega \hat{\phi}(\omega))] d\omega$$  (33)

where the hatted variables are Fourier transforms. Observe that $\hat{w}(\omega) = \hat{\kappa}(\omega) \hat{u}(\omega)$. It is in (33) where the famous Popov frequency domain inequality occurs. If there exists $\beta \in \mathbb{R}$ such that

$$\frac{1}{2} \hat{\phi} + \text{Re} (1 + i\omega \beta) \hat{\kappa}(\omega) \geq 0, \forall \omega \geq 0$$  (34)

then (32) will imply

$$\eta(0,T) \geq -\frac{1}{2} \theta_3 \hat{\phi}(t) \hat{\phi}(t) \left|_{1}^{T} \right. - \int_0^T \phi(\sigma(t)) (\tau_1 \hat{\phi} \rho(t) + (\theta_2 - \theta_3) \hat{\rho}(t)) dt$$  (35)

Combining (35) with (27) will give (again, after some straightforward manipulation)

$$\left(\theta_2 + \theta_3\right) \psi(\sigma(t)) \left|_{1}^{T} \right. - \int_0^T \phi(\sigma(t)) (\tau_1 \hat{\phi} \rho(t) + (\theta_2 - \theta_3) \hat{\rho}(t)) dt \leq 0; \psi(\sigma) := \int_0^\sigma \phi(\eta) d\eta$$  (36)

Inequality (36) is precisely the main stability inequality. Obtaining the stability property is a matter of technique - see the papers of V. M. Popov [30, 31] or the monographs [26, 41, 42].
4. Further development

After the papers of V. M. Popov were published in 1959-1961, two approaches to the problem solution can be distinguished. The first one started with the aforementioned papers, dealing with systems described by (8), with the Lyapunov function of the form (10). Popov was able to prove that if a Lyapunov function of the form (10), having its derivative along the solutions of (8) non-positive, can be found, then a frequency domain inequality of the type (34) is fulfilled. The converse - a much more difficult task - was proven by V. A. Yakubovich [43] and R. E. Kalman [44]. It turned out later [45] that the aforementioned equivalence was a property of the so called positiveness, with reference to the Popov system consisting of the controlled differential equation (4) and the quadratic integral index (5). In fact Popov discussed the case of several input functions thus extending the results of Yakubovich and Kalman to the multivariable case. In this way the Main Lemma of the System Theory (Yakubovich-Kalman-Popov) was completely proven for the finite dimensional case.

The second approach to the problem followed directly from the seminal papers of Popov [30, 31] via the simple remark that the integral equations follow from the use of the Cauchy formula of the variations of constants. Consequently, systems with time delay and other classes of systems with distributed parameters were considered. For this last approach - where the quadratic index is also used - see[46, 42, 33, 47, 48].

Later the first approach was extended to Hilbert spaces, due to Yakubovich [49, 50] and his co-worker [51, 52, 53]; other contributions are [54, 55, 56]. It is also worth mentioning that the results of the first research direction mentioned above were given an applicable even commercial form of the Linear Matrix Inequalities [23] while the other research was aimed at finding necessary and sufficient conditions for absolute stability.

While the interest to the problem seems now decreasing, the future can still surprise us.

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