The instability of a black hole with $f(R)$
global monopole under extended uncertainty
principle

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Abstract

We consider the evolution of black hole involving an $f(R)$ global monopole based on the Extended Uncertainty Principle (EUP). The black hole evolutions refer to the instability due to the Parikh-Kraus-Wilczeck tunneling radiation or fragmentation. It is found that the EUP corrections make the entropy difference larger to encourage the black hole to radiate more greatly. We also show that the appearance of the EUP effects result in the black hole’s division. The influence from global monopole and the revision of general relativity can also adjust the black hole evolution simultaneously, but can not change the final result that the black hole will not be stable because of the EUP’s effects.

PACS number(s): 03.65.Bz, 03.65.Ta, 04.60.Bc

Keywords: EUP, $f(R)$ theory, global monopole black hole, uncertainty principle

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I. Introduction

According to the fact of accelerated expansion of the universe, Buchdahl proposed the $f(R)$ theory as a kind of modified gravity [1] and further the theory has been applied to explain the accelerated-inflation problem without dark matter or dark energy [2-4]. The $f(R)$ gravity generalizes the general relativity and the generalization certainly arises in the description of the background around the gravitational sources. The universe evolves with decreasing temperature. In the process of the vacuum phase transition in the early stage of the universe, several types of topological defects such as domain walls, cosmic strings and monopoles may have arisen [5, 6]. These topological defects formed in favour of a breakdown of local or global gauge symmetries [7]. For example, a global monopole as a spherically symmetric topological defect appears in the phase transition of a system composed of a self-coupling triplet of a scalar field whose original global O(3) symmetry is spontaneously broken to U(1) [6, 7]. It was shown that the structure of the metric outside a monopole has a solid angle leading all light rays to be deflected at the same angle [8]. In the case of a massive source involving a global monopole in the universe with accelerated expansion, its metric with terms associated with the monopole and $f(R)$ issue are necessary [9, 10]. The nonvanishing modified parameter $\psi_0$ from $f(R)$ theory belonging to the metric components bring a cosmological horizon as a boundary of the universe to the spacetime limited by the $f(R)$ global monopole [9]. More efforts have been contributed to the model. it was found that the parameter subject to the modification of gravity provides stable circular orbits for massive test particles in the gravitational field of an $f(R)$ global monopole [9, 10]. The quasinormal modes for this kind of black holes were calculated with WKB approximation [11-13]. The thermodynamics of the black hole with a global monopole within the frame of $f(R)$ gravity was investigated [14, 15]. The strong gravitational lensing for the same models was discussed analytically [16]. The corrections from the global monopole and the gravity modification in the $f(R)$ theory to the dominant term in the scattering absorption cross section were computed in view of low frequency and small angles [17].

In the research on black holes, there are more significant measurement and theoretical predictions recently. One key experiment is the Event Horizon Telescope (EHT) [18]. The EHT data show the gravitational physics at the event horizon where no light escapes from, which opens a window to probe the details of the black hole core. The regions surrounding black holes are known as the black hole shadows [18]. The black holes could be fundamentally quantum objects regardless of their size, so the quantum gravity effects can not be neglected for the physics of microscopic black holes such as their evaporation profile and singularity removal [19]. The quantum characteristics relate to the horizon of black hole like the metric fluctuations [20-23] and the quantum structures around the black hole [24-26]. It should be pointed out that the quantum effects have something to do with the Uncertainty Relation [24-26]. It is impossible to omit the gravitational influence, so the terms with the Newtonian constant would appear in the Heisenberg uncertainty principle [27-33]. These terms can be functions of momentum difference or distance interval. The General Uncertainty Principle (GUP) with a series of terms based on momentum difference is shown as a
quantum gravitational correction to the standard Heisenberg relation [28, 34]. One of its simple forms is chosen as,

$$\Delta x \Delta p \geq 1 + \beta l^2_p \Delta p^2$$

(1)

where $\beta$ is a constant of order unit and the natural units $\hbar = c = 1$ are utilized [28, 34]. Within the tiny region, the momentum difference is large, so the deviations are obvious according to the inequality (1). The GUP is used to study the quantum gravity phenomenology of black holes and to cure the divergence from states density near the black hole horizon while relating the entropy of black hole to a minimal length as quantum gravity scale [35-38]. The GUP also modifies the black hole horizon and further changes the black hole entropy [35-38]. In an anti-de Sitter spacetime, the Heisenberg uncertainty principle should be deformed with a suitably chosen parameterization [39]. The Extended Uncertainty Principle (EUP) is shown as a position-uncertainty correction to the Heisenberg inequality [28, 39]. We can select [28, 39],

$$\Delta x \Delta p \geq 1 + \alpha \frac{\Delta x^2}{L_*^2}$$

(2)

where $\alpha$ is a constant of order unit and $L_*$ is thought as a large fundamental distance scale. The additional terms involve the ratio of distance difference and distance scale according to Eq. (2). It is manifest that the EUP introduces the quantum effects over the macroscopic distances [28, 39]. The EUP will redefine the horizon to revise the entropy [19].

Although the black holes are perceived as perfect absorbers classically, their evolutions including the tunneling radiation and fragmentation depending on the quantum mechanics and thermodynamics respectively [40-47]. The tunneling formalism for black holes subject to the imaginary part of action for classically forbidden region of emission across the horizon is of great concern [48-54]. With the help of the semi-classical tunneling put forward by Kraus et.al., a lot of efforts have been paid to the Hawking radiation of many kinds of objects such as BTZ black holes [55-58], Taub-NUT black holes [59], Kerr-Newman black holes [60-62], Godel black holes [63], etc.. The fragmentation of black holes as a kind of evolution has also attracted more attentions [47]. This thermodynamic instability was discussed under non-perturbation [47]. The fragmentation issue was used to probe the final fates of a series of black holes like the rotating anti-de Sitter black holes [64], black holes with a Gauss-Bonnet term [65] and charged anti-de Sitter black holes [66]. It should be pointed out that the black holes tunneling radiation and fragmentation both have something to do with their entropy associated with their horizons [40-42]. The standard Heisenberg uncertainty principle governs the horizons [27-34, 39]. As mentioned above that the generalizations of the principle undoubtedly modify the horizons and further the entropy, the GUP has influence over the tunneling radiation and fragmentation of black holes [28, 34-36, 39]. In the context of GUP modifying the quantum mechanics [27-34, 67-72], the authors of Ref. [35, 36] derived and estimate the relation between the Hawking tunneling radiation of black holes and a minimal length as quantum gravity scale in the higher dimensional spacetime by means of the tunneling formalism. Under the GUP,
we calculated the Parikh-Kraus-Wilczeck tunneling radiation of black hole involving an \( f(R) \) global monopole to show that the square of momentum difference term advances the emission of this kind of black holes while the global monopole and the revision of general relativity both hinder the black hole from emitting the photons [73]. We also discovered that the same black hole keeps stable instead of splitting without the GUP corrections [74]. Having researched on the fragmentation of the black holes with \( f(R) \) global monopole in virtue of the second law of thermodynamics, we showed that the influence from GUP leads the black hole to break into two parts with larger mass and smaller ones respectively [74].

It is necessary to consider the tunneling radiation and fragmentation of a Schwarzschild black hole with global monopole under EUP within the frame of \( f(R) \) scheme. As a kind of generalization of Heisenberg uncertainty principle, the EUP is significant and its quantum effects could appear at extremely large scale [19, 28, 29]. The EUP revises the relation between the horizon and the mass of black hole to correct the matter orbits and innermost stable circular orbits (ISCO), size of the photosphere [19]. The contribution of EUP corresponds to the dark matter effects because the EUP correction fits the Milky Way’s rotation curve [19]. The thermodynamic properties of the Schwarzschild black hole and the modified Unruh effect governed by the EUP were discussed [75]. There must exist the massive objects containing global monopoles in the spacetime with description of \( f(R) \) gravity as mentioned above. The EUP brings about the effects on the tunneling radiation and fragmentation of black holes through the EUP-corrected horizon [28, 34-36, 39]. During our research on the evolution of the black holes, we should not omit the corrections from EUP. To the best of our knowledge, few efforts have been made to the investigation of the EUP influence on the black hole stability due to the radiation and the fragmentation. Under the EUP, we are going to derive and calculate the entropy of a black hole swallowing \( f(R) \) global monopole to discuss the possibilities that the black hole radiates and breaks into two sections with the help of the techniques of Ref. [43-46] and Ref. [47] respectively. We wonder how the EUP affects the possibilities. We list our results and compare the results with those under the GUP finally.

II. The tunneling radiation of a black hole with an \( f(R) \) global monopole under extended uncertainty principle (EUP)

We are going to investigate the entropy of a black hole with a global monopole in the \( f(R) \) theory. Under the corrections from \( f(R) \) statement, the spherically symmetric solution to the gravitational field equation coupled to the matter field with a spontaneously broken \( O(3) \) symmetry was found [9, 10, 15],

\[
\begin{align*}
    ds^2 &= A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \\
    A(r) &= B^{-1}(r) = 1 - 8\pi G\eta^2 - \frac{2GM}{r} - \psi_0 r
\end{align*}
\] (3)  

where

\[
A(r) = B^{-1}(r) = 1 - 8\pi G\eta^2 - \frac{2GM}{r} - \psi_0 r
\] (4)
and $G$ is the Newton constant. As a monopole parameter in a typical grand unified theory, $\eta$ is of order $10^{18} GeV$ to lead $8\pi G \eta^2 \approx 10^{-5}$ [6, 8]. $M$ is mass parameter. The factor $\psi_0$ represents the extension of the standard general relativity. The roots of the equation $A(r) = 0$ from metric (3) are [9, 10, 15],

$$r_{\pm} = \frac{1 - 8\pi G \eta^2 \pm \sqrt{(1 - 8\pi G \eta^2)^2 - 8GM\psi_0}}{\psi_0}$$

(5)

Here $r_+$ and $r_-$ stand for the outer radius and inner ones respectively. It is obvious that the outer horizon will disappear if the modified parameter $\psi_0$ vanishes.

The entropy of a black hole has something to do with the horizon [40-43]. The corrected horizon radius certainly generalize the expression of the entropy [19, 35, 36, 74, 75]. According to the scheme of Ref. [19], the distance interval can be estimated as,

$$\Delta x' = \frac{\Delta x}{1 + \alpha L^2 \Delta x^2}$$

(6)

where $\Delta x$ is the original size of black hole. If the influence from EUP disappears like $\alpha = 0$, the distance difference $\Delta x'$ will recover to be original ones. The $\alpha$-term from EUP shortens the black hole size according to Eq. (6). Based on the approaches of Ref. [40-42], the Hawking temperature for the black hole with the descriptions (3) and (4) is a function of variables like $\eta$ and $\psi_0$,

$$T_H = \frac{1}{2\pi} \left( \frac{1 - 8\pi G \eta^2}{\Delta x} - \psi_0 \right)$$

(7)

Here the interval can be let $\Delta x = 2r_H$ and $r_H = r_-$ is the black hole horizon without the extension of Heisenberg’s relation. The Bekenstein-Hawking entropy may be obtained from the Hawking temperature (7) with the help of the following thermodynamic relation [40-42, 46],

$$T_H = \frac{dE}{dS} \approx \frac{dM}{dS}$$

(8)

In the emission process [46], the comparison between the initial and the final values of the entropy of the black hole with the solid deficit angle and $f(R)$ correction can be approximated as,

$$\Delta S \approx -\frac{4\pi G}{(1 - 8\pi G \eta^2)^2} [M^2 - (M - \hbar \omega)^2] - \frac{16\pi G^2 \psi_0}{(1 - 8\pi G \eta^2)^4} [M^3 - (M - \hbar \omega)^3]$$

(9)

where $\omega$ is a shell of energy moving along the geodesics towards the black hole with metric (3) [46, 56]. The black hole’s tunneling probability can be expressed as [46, 56],

$$\Gamma \sim e^{\Delta S}$$

(10)

From Eq. (9), the higher order of typical grand unified theory like increasing $8\pi G \eta^2$ will lead larger absolute value of negative entropy difference, so will the farther away from the Einstein’s general relativity like increasing the magnitude of $\psi_0$. It can be argued that the existence of global monopole in the black hole or the deviation from general relativity damps the emission of the black
hole. The same topics were considered under GUP as in Eq. (1). The emission of this kind of black hole is promoted in favour of the greater parameter \( \beta \) as a coefficient of a quadratic term in the momentum difference [73, 74].

It is significant to wonder how the EUP affects the entropy difference associated with the tunneling probability of the black hole described by the \( f(R) \) global monopole metric. Following the procedure of Ref. [40-42, 74], we choose the distance interval in the temperature (7) as \( \Delta x' \) shown in Eq. (6) to obtain the EUP-corrected Hawking temperature,

\[
T'_H = \frac{1}{2\pi} \left( \frac{1 - 8\pi G \eta^2}{\Delta x'} - \psi_0 \right)
\]

while we also let \( \Delta x = 2r_H \) as above. The corrections from EUP make the Hawking temperature higher. By means of the thermodynamic relation (8) [40-42, 46], we derive the corrected entropy difference of the radiating black hole involving \( f(R) \) global monopole as follow,

\[
\Delta S' = \Delta S'(\eta, \psi_0, \alpha)
= \frac{2\pi}{G} \int_{r_H}^{r_H'} \frac{-2\psi_0 r_H^2 + (1 - 8\pi G \eta^2) r_H}{\psi_0 (1 - 8\pi G \eta^2) r_H^2 - 2\psi_0 r_H + (1 - 8\pi G \eta^2)} \, dr_H
\]

Here \( r_H = r_- \) and \( r_H' = r_- |_{M \to M - h\omega} \). It can be checked that before the performance of integral \( \Delta S' \) (12) will recover to be \( \Delta S \) specified by Eq. (9) if \( \alpha = 0 \) [46]. Having performed the integral under \( \alpha \neq 0 \), we can show the change in entropy \( \Delta S' \) in unit of the difference for Schwarzschild black hole like \( \Delta S_0 \approx -8\pi G M h \omega \) [46],

\[
\frac{\Delta S'}{\Delta S_0} \approx -\frac{1}{\alpha \frac{\psi_0}{r^2}} \frac{1}{4GM(1 - 8\pi G \eta^2) \sqrt{(1 - 8\pi G \eta^2)^2 - 8GM \psi_0}}
\]

In the case of EUP, the tunneling probability of black hole in Eq. (10) should be revised as \( \Gamma' \sim e^{\Delta S'} \) [46]. The dependence of the entropy difference formulated for the black hole including the \( f(R) \) global monopole in Eq. (12) on the variables \( \alpha \) and \( \psi_0 \) corresponding to the EUP correction and the deviation of standard gravity respectively is plotted in Figure 1. The entropy change \( \Delta S' \) is a decreasing function of \( \alpha \) for \( \psi_0 \) with a series of definite values. The stronger influence from EUP leads the value of \( \Delta S' \) smaller, which retards the radiation of the black hole, which is opposite to the case of GUP appearing in Eq. (1). The considerable correction on the general relativity also causes the black hole to be unstable due to the tunnel process.

III. The fragmentation instability of a black hole with an \( f(R) \) global monopole under extended uncertainty principle

The fragmentation probability of a Schwarzschild black hole whose spacetime has a solid deficit angle owing to a global monopole dominated by \( f(R) \) gravity should be discussed under EUP. We
can investigate the entropy of black hole to explore its fate. In view of Ref. [41, 77], the Bekenstein-Hawking entropy of black hole is proportional to the horizon area,

\[ S = \frac{1}{4} A_H \]  

(14)

where

\[ A_H = 4\pi r_H^2 \]  

(15)

According to the second law of thermodynamics, the thermodynamic argument for the fragmentation of a black hole claimed that the black hole entropy must increase during its evolution [47]. Here we assume that the black hole with \( f(R) \) global monopole breaks into two parts involving the same kind of monopole. In the process of fragmentation, the original black hole can be thought as the initial state and the final state consists of two black holes under the conservation of mass. Subject to the second law of thermodynamics, we can compare the two entropies for the initial and final state respectively to wonder whether the fragmentation could happen. It was shown that the nature of the entropy difference for the \( f(R) \) global monopole black hole limited by the Heisenberg inequality remains negative no matter whether the general relativity has been generalized. The division of this kind of isolated black holes can not occur spontaneously [78]. When the GUP is introduced, the black hole containing the \( f(R) \) global monopole will split into two parts [78]. The stronger influence from GUP can lead the difference of the masses for the two fragmented black holes to be smaller [78]. As mentioned above, the EUP is also a generalization of the Heisenberg uncertainty principle. We should study the influence from EUP on the fragmentation instability of the black hole swallowing a global monopole governed by \( f(R) \) theory. In the initial case, the entropy of the isolated black hole can be obtained from Eq. (14),

\[ S_i = \pi r_H^2(M, \eta^2, \psi_0) \]  

(16)

where \( r_H(M, \eta^2, \psi_0) = r_- \) shown in Eq. (5). We estimate the black hole horizon amended by the EUP with the original horizon like \( \Delta x = 2r_H(M, \eta^2, \psi_0) \) from Eq. (6) [19],

\[ r'_H(M, \eta^2, \psi_0) = \frac{r_H(M, \eta^2, \psi_0)}{1 + \frac{4a}{L^2} r_H^2(M, \eta^2, \psi_0)} \]  

(17)

The correction will disappear with \( \alpha = 0 \). Under EUP, the corrected horizons of black holes lead to the corrected entropy difference,

\[ \Delta S' = S'_f - S'_i \]  

(18)

where

\[ S'_i = \pi r_H^2(M, \eta^2, \psi_0) \]  

(19)
\[ S'_f = \pi r'^2_H(\varepsilon M, \eta^2, \psi_0) + \pi r'^2_H((1 - \varepsilon M)M, \eta^2, \psi_0) \]  

By means of Eq. (17), we obtain the corrected radii \( r'^H(\varepsilon M, \eta^2, \psi_0) \) and \( r'^H((1 - \varepsilon M)M, \eta^2, \psi_0) \) as follows,

\[ r'^H(\varepsilon M, \eta^2, \psi_0) = r'^H(M, \eta^2, \psi_0)|_{M \rightarrow \varepsilon M} \]  

\[ r'^H((1 - \varepsilon M)M, \eta^2, \psi_0) = r'^H(M, \eta^2, \psi_0)|_{M \rightarrow (1 - \varepsilon M)M} \]

It should be pointed out that \( r'^H(M, \eta^2, \psi_0) \) stands for the EUP-limited horizon of the initial black hole swallowing the \( f(R) \) global monopole. Under the EUP, \( r'^H(\varepsilon M, \eta^2, \psi_0) \) and \( r'^H((1 - \varepsilon M)M, \eta^2, \psi_0) \) are the horizon radii of the separated black holes belonging to the final state with masses \( \varepsilon M \) and \( (1 - \varepsilon M)M \) respectively. Following the same procedure as Ref. [66], we define the mass distribution and its region \( 0 \leq \varepsilon M \leq 1 \).

It is fundamental to explore the fragmentation possibility of an \( f(R) \) global monopole black hole under the EUP. The sign of the entropy difference during the evolution of the black hole helps us to determine whether the black hole can split because the entropy of a stable system cannot decrease in any spontaneous process [47, 79]. We ignore the generalization of the general relativity and show the entropy difference in Eq. (18) as a function of the ratio \( \varepsilon M \) graphically under EUP labelled as \( \alpha \) in Figure 2. We find that the entropy increases during the process that the black hole becomes two new ones with \( 0 \leq \varepsilon M \leq 1 \) and the existence of EUP. The coefficient \( \alpha \) can adjust the curves of entropy difference, but it can not change the natures of the difference. The Figure 3 indicates that the deviation from the general relativity can also adjust the entropy difference a little, but it can not let the sign of the difference negative. The EUP encourages the global monopole black hole under \( f(R) \)-generalized gravity to break into two parts spontaneously, no limit to the distribution of the mass of the black hole. The GUP also impels the same black hole to break up, but weaker influence of the generalized principle will cause the two new black holes to possess the different size, one larger and the other smaller.

IV. Discussion and Conclusion

We derive and compute the entropy difference of a black hole with a \( f(R) \) global monopole under the Extended Uncertainty Principle to investigate the black hole evolution such as the Parikh-Kraus-Wilczek tunneling radiation and fragmentation. The EUP adds a position-uncertainty term to the Heisenberg uncertainty principle to reflect the quantum corrections to gravity in the large scale [28, 39]. The EUP corrects the horizon of black hole to change the black hole entropy. In favour of the entropy difference modified by EUP in the process of emitting photons, the stronger influence from EUP weakens the tunneling radiation of the black hole with the description of \( f(R) \) gravity while the global monopole exists in the compact object, which is different from the case of
GUP. Under the Heisenberg inequality, the $f(R)$ global monopole black hole keeps stable instead of splitting. In the case of black hole fragmentation, the appearance of modification from EUP will cause the black hole to divide into two objects with arbitrary distributions of black hole mass contrary to the case under GUP.

Acknowledgement

This work is supported by NSFC No. 10875043.
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Figure 1: The solid, dotted and dashed curves of the dependence of entropy difference $\Delta S'$ on $\alpha$ for $\psi_0 = 0.01, 0.05, 0.08$ respectively and for simplicity $8\pi G \eta^2 = 0.1$ and $G = M = L_* = 1$
Figure 2: The solid, dotted and dashed curves of the dependence of entropy difference $\Delta S'$ on $\varepsilon_M$ for $\alpha = 2, 6, 10$ respectively and for simplicity $8\pi G\eta^2 = 0.1, \varepsilon_\eta = 0.5$ and $G = M = L_* = 1$.
Figure 3: The solid, dotted and dashed curves of the dependence of entropy difference $\Delta S'$ on $\alpha$ for $\psi_0 = 0.01, 0.06, 0.1$ respectively and for simplicity $8\pi G\eta^2 = 0.1$, $\alpha = 5$, $\varepsilon_0 = 0.5$ and $G = M = L_* = 1$