NOETHERIAN CRITERIA FOR DIMER ALGEBRAS

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Abstract. Let A be a nondegenerate dimer (or ghor) algebra on a torus, and let Z be its center. Using cyclic contractions, we show the following are equivalent: A is noetherian; Z is noetherian; A is a noncommutative crepant resolution; each arrow of A is contained in a perfect matching whose complement supports a simple module; and the vertex corner rings $e_iAe_i$ are pairwise isomorphic.

1. Introduction

In this article, all dimer quivers are on a real two-torus $T^2$, and all algebras are over an algebraically closed base field $k$. A prominent class of noncommutative crepant resolutions (NCCRs) and Calabi-Yau algebras are cancellative dimer algebras. In fact, a dimer algebra is an NCCR and Calabi-Yau if and only if it is cancellative [MR, D, Br]. Cancellativity has been shown to be equivalent to certain combinatorial conditions on the dual graph of the quiver [HV, IU, KS, G]. The main objective of this article is to characterize cancellative dimer algebras in terms of noetherianity, vertex corner ring structure, and perfect matchings whose complements support simple modules.

Dimer algebras were introduced in string theory in 2005 to describe a class of quiver gauge theories [HK, F-K]. A dimer algebra is a quiver algebra $A = kQ/I$ of a quiver $Q$ whose underlying graph $\overline{Q}$ embeds in a torus $T^2$ (or more generally, a compact surface), such that each connected component of $T^2 \setminus \overline{Q}$ is simply connected and bounded by an oriented cycle, called a unit cycle. The relations of $A$ are given by the ideal

$$I := \langle p - q \mid \exists a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ,$$

where $p$ and $q$ are paths. Since $I$ is generated by certain differences of paths, we may refer to a path modulo $I$ as a path in the dimer algebra $A$.

A distinguishing property of dimer algebras is cancellativity. Two paths $p, q$ in a dimer algebra $A$ are said to form a non-cancellative pair if $p \neq q$, and there is a path $r \in A$ such that

$$rp = rq \neq 0 \quad \text{or} \quad pr = qr \neq 0.$$

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For the definition of an NCCR and a Calabi-Yau algebra, see for example [V, Br].
If such a pair exists, then $A$ and $Q$ are called non-cancellative; otherwise they are cancellative. We will show that a dimer algebra is cancellative if and only if it is noetherian.

We will also characterize noetherianity for ghor algebras. The ghor algebra of a dimer quiver $Q$ on a torus is (isomorphic to) the quotient of the dimer algebra $A = kQ/I$ by non-cancellative pairs $^2$

$$\Lambda = A/\langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle.$$ 

Ghor algebras were introduced in [B2], and often have nicer algebraic and homological properties than do their dimer counterparts [B5, B7, BB]. Note that a dimer algebra coincides with its ghor algebra if and only if it is cancellative.$^3$

The primary tool we use to study noetherianity is the notion of a cyclic contraction, also introduced in [B2]. A cyclic contraction is map of dimer algebras $\psi : kQ/I \to kQ'/I'$, such that $Q'$ is obtained from $Q$ by contracting a set of arrows to vertices whilst preserving the cycles of $Q$ in a suitable sense (see Section 2.3). Cyclic contractions enable non-cancellative and cancellative dimer algebras to be related to each other. An example of a cyclic contraction is given in Figure 1. In this example, $A = kQ/I$ is non-cancellative and $A' = kQ'/I'$ is cancellative: Let $a, b, c$ be the respective red, blue, and green arrows in $Q$, as shown in the figure. Then the paths $ab, ba \in A$ form a non-cancellative pair since $ab \neq ba$ and $cab = cba \neq 0$.

Our main theorem is the following.

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$^2$ In [B5], we called ghor algebras ‘homotopy algebras’ since their relations identify homotopic paths in the quiver if the surface is a torus. However, in [BB] we showed that on higher genus surfaces homologous cycles in the quiver are also identified, and so the prefix ‘homotopy’ became less suitable. The word ‘ghor’ is Klingon for surface.

$^3$ Using the definition of a ghor algebra given in [B2 Introduction], the statement that a dimer algebra (on a torus) coincides with its ghor algebra if and only if it is cancellative is nontrivial and is shown in [B2 Theorem 4.31].
Theorem 1.1. (Theorems 3.11 and 3.21.) Let $A = kQ/I$ be a nondegenerate dimer algebra on a torus with center $Z$, and let $\Lambda$ be the corresponding ghor algebra with center $R$. The following are equivalent:

1. $A$ is cancellative.
2. $A$ is noetherian.
3. $Z$ is noetherian.
4. $A$ is a finitely generated $Z$-module.
5. The vertex corner rings $e_i A e_i$ are pairwise isomorphic.
6. Each vertex corner ring $e_i A e_i$ is isomorphic to $Z$.
7. Each arrow annihilates a simple $A$-module of dimension vector $1 Q_0$.
8. Each arrow is contained in a simple matching.
9. The center $R$ of $\Lambda$ equals the cycle algebra $S$ of $A$.
10. If $\psi : A \to A'$ is a cyclic contraction, then $\psi$ is trivial.

Furthermore, these conditions are equivalent to each condition (2) – (7) with $A$ and $Z$ replaced by $\Lambda$ and $R$.

More precisely, in (9) we mean that $R$ is generated by the monomial images of all the cycles of $Q$ under the canonical quotient map $kQ \to \Lambda$. The implications (1) $\Rightarrow$ (2) - (6) are well known (e.g., [D, MR, B2, Proposition 4.28]); one of the main objectives of this article is to prove the converse implications, which are new and much more involved. Some of the equivalences do not hold in general for dimer or ghor algebras on surfaces other than the torus, and are thus unexpected. For example, the equivalences (2) $\Leftrightarrow$ (8) $\Leftrightarrow$ (6) do not hold on higher genus surfaces by [BR, Proposition 3.9, Lemma 4.3].

2. Preliminaries

Given a quiver $Q$, we denote by $kQ$ the path algebra of $Q$, and by $Q_\ell$ the paths of length $\ell$. The idempotent at vertex $i \in Q_0$ is denoted $e_i$, and the head and tail maps are denoted $h, t : Q_1 \to Q_0$. Multiplication of paths is read right to left, following the composition of maps. By module we mean left module; and by infinitely generated module, we mean a module that is not finitely generated. The ‘support’ of a module $V$ over a quiver algebra $kQ/I$ is the largest subquiver $Q' \subseteq Q$ for which no vertex or arrow of $Q'$ annihilates $V$. Finally, we denote by $e_{ji} \in M_n(k)$ the $n \times n$ matrix with a 1 in the $ji$-th slot and zeros elsewhere.

2.1. Covering quivers and perfect matchings. Let $A = kQ/I$ be a dimer algebra. If $p, q$ are paths in $Q$ that are equal modulo $I$, then we will write $p \equiv q$; if $p, q$ are regarded as paths in $A$, then we will write $p = q$.

- Consider a covering map $\pi : \mathbb{R}^2 \to T^2$ such that for some $i \in Q_0$,

$$\pi (\mathbb{Z}^2) = i.$$
Denote by $Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$ the infinite covering quiver of $Q$. For each path $p$ in $Q$, denote by $p^+$ the unique path in $Q^+$ with tail in the unit square $[0, 1) \times [0, 1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$. If two paths $p^+$ and $q^+$ in $Q^+$ have coincident tails and coincident heads, then we denote the compact region they bound by $R_{p,q}$ (or $R_{p+I,q+I}$ if the representatives $p$ and $q$ are clear from context.)

- We will consider the following sets of arrows:
  - A perfect matching $x \subset Q_1$ is a set of arrows such that each unit cycle contains precisely one arrow in $x$.
  - A simple matching $x \subset Q_1$ is a perfect matching such that $Q \setminus x$ supports a simple $A$-module of dimension 1.

Denote by $P$ and $S$ the set of perfect and simple matchings of $Q$, respectively. $A$ and $Q$ are said to be nondegenerate if each arrow is contained in a perfect matching.

- By a cyclic subpath of a path $p$, we mean a subpath of $p$ that is a nontrivial cycle.

We will consider the following sets of cycles in $A$:

- Denote by $C$ the set of cycles in $A$ (i.e., cycles in $Q$ modulo $I$).
- For $u \in \mathbb{Z}^2$, denote by $C^u$ the set of cycles $p \in C$ such that $h(p^+) = t(p^+) + u \in Q_0$.
- For $i \in Q_0$, denote by $C_i$ the set of cycles in the vertex corner ring $e_i Ae_i$.
- Denote by $\hat{C}$ the set of cycles $p \in C$ such that the lift of each cyclic permutation of each representative of $p$ does not have a cyclic subpath.

We decorate $C$ so as to specify a set of cycles; e.g., $\hat{C}^u := \hat{C} \cap C^u \cap C_i$. Note that $\hat{C}^0 = Q_0$.

It is well-known that if $\sigma_i, \sigma'_i$ are two unit cycles at $i \in Q_0$, then $\sigma_i = \sigma'_i$ in $A$; we will denote by $\sigma_i \in A$ the unique unit cycle at $i$. Furthermore, the sum $\sum_{i \in Q_0} \sigma_i$ is in the center of $A$.

2.2. Matrix ring homomorphisms. Let $A = kQ/I$ be a dimer algebra. By [B2, Lemma 2.1], there are algebra homomorphisms from $A$ to the matrix rings

\[ \tau : A \rightarrow M_{|Q_0|}(k[S]) \quad \text{and} \quad \eta : A \rightarrow M_{|Q_0|}(k[P]) \]

defined on $i \in Q_0$ and $a \in Q_1$ by

\[ \tau(e_i) = e_{ii}, \quad \eta(e_i) = e_{ii}, \]

\[ \tau(a) = e_{h(a),t(a)} \prod_{x \in S : x \geq a} x, \quad \eta(a) = e_{h(a),t(a)} \prod_{x \in P : x \geq a} x, \]

and extended multiplicatively and $k$-linearly to $A$.

For each $i, j \in Q_0$, denote by

\[ \bar{\tau} : e_j Ae_i \rightarrow k[S] \quad \text{and} \quad \bar{\eta} : e_j Ae_i \rightarrow k[P] \]
the respective $k$-linear maps defined on $p \in e_jAe_i$ by
\[
\tau(p) = \bar{\tau}(p)e_{ji} \quad \text{and} \quad \eta(p) = \bar{\eta}(p)e_{ji}.
\]
Denote by $\sigma$ the product of all the variables,
\[
\sigma := \bar{\tau}(\sigma_i) = \prod_{x \in S} x \quad \text{or} \quad \sigma := \bar{\eta}(\sigma_i) = \prod_{x \in P} x.
\]

We will make use of the following results from [B2]. See Figure 2 for reference.

**Proposition 2.1.** Let $A$ be a nondegenerate dimer algebra. Given a path $p$, denote $p$ either $\bar{\eta}(p)$ or $\bar{\tau}(p)$.

- First suppose $p, q \in e_jAe_i$ are distinct paths satisfying $t(p^+) = t(q^+)$ and $h(p^+) = h(q^+)$. Then:
  1. There is an $m \in \mathbb{Z}$ such that $p = q\sigma_m$.
  2. $p, q$ is a non-cancellative pair if and only if $\bar{\eta}(p) = \bar{\eta}(q)$.

- Now suppose either $A$ is cancellative, or there is some $u \in \mathbb{Z}^2 \setminus 0$ such that $\hat{C}_i \neq \emptyset$ for each $i \in Q_0$.
  3. Let $p \in C$. Then $p \in \hat{C}$ if and only if $\sigma \nmid p$.
  4. If $p, q \in \hat{C}_u$, then $p = q$.

- Finally, suppose $A$ is cancellative. Then:
  5. For each $u \in \mathbb{Z}^2$ and $i \in Q_0$, the set $\hat{C}_i^u$ is nonempty.
  6. Let $p \in \hat{C}_u$. Then $u = 0$ if and only if there is some $\ell \geq 0$ such that $p = \sigma^\ell$.
  7. For each $i, j \in Q_0$, we have
    \[
e_iAe_i = Ze_i \cong Z \cong \bar{\tau}(e_iAe_i) = \bar{\tau}(e_jAe_j).
    \]
  8. $A$ is a finitely generated $\mathbb{Z}$-module, and $Z \cong \bar{\tau}(e_iAe_i)$ is a finitely generated $k$-algebra, generated by $\sigma$ and a finite set of monomials in $k[S]$ not divisible by $\sigma$.

**Proof.** The claims are respectively (1), (2): [B2] Lemma 4.3; (3), (4): [B2] Lemma 4.8, Proposition 4.21; (5): [B2] Proposition 4.11; (6): [B2] Lemma 4.29; and (7), (8): [B2] Propositions 4.28, 5.14, Theorem 5.9.3. □

2.3. **Cyclic contractions.** Let $A = kQ/I$ and $A' = kQ'/I'$ be dimer algebras, and suppose $Q'$ is obtained from $Q$ by contracting a set of arrows $Q_1^* \subset Q_1$ to vertices. This operation defines a $k$-linear map of path algebras
\[
\psi : kQ \to kQ'.
\]
If $\psi(I) \subseteq I'$, then $\psi$ induces a $k$-linear map of dimer algebras, $\psi : A \to A'$, called a contraction. Consider the algebra homomorphism
\[
\tau : A' \to M_{|Q_0^*|}(k[S^*])
\]
Figure 2. Examples for Proposition 2.1. All paths shown are paths of positive length in the cover $Q^+$ (with the superscripts $+$ omitted). In (i), the paths $p, q \in e_jAe_i$ satisfy $t(p^+) = t(q^+)$ and $h(p^+) = h(q^+)$. In (ii), the path $p = p_3p_2p_1$ is in $C_i \setminus \hat{C}_i$, since the cyclic permutation $(p_1p_3p_2)^+$ of $p^+$ has a nontrivial cyclic subpath. Furthermore, $p_3p_1$ is in $C_0^i$, and therefore $p_3p_1 = \sigma^\ell$ for some $\ell \geq 1$.

We call a contraction $\psi : A \to A'$ cyclic if $A'$ is cancellative and

$$S := k[\bigcup_{i \in Q_0^i} \tau \psi(e_i Ae_i)] = k\left[ \bigcup_{i \in Q_0^i} \tau(e_i A'^i) \right].$$

The algebra $S$, called the cycle algebra of $A$, is independent of the choice of cyclic contraction $\psi$ [B3, Theorem 3.14]. Furthermore, every nondegenerate dimer algebra admits a cyclic contraction [B1, Theorem 1.1].

Finally, we remark that if $\psi : A \to A'$ is a cyclic contraction, then the ghor algebra of $A'$ of $A$ is simply $A'$ itself since $A'$ is cancellative,

$$A' = A' / \langle p - q \mid p, q \text{ a non-cancellative pair} \rangle = A'.$$

Furthermore, the cycle algebra $S$ and the algebra generated by the intersection

$$R = k[\bigcap_{i \in Q_0^i} \tau \psi(e_i Ae_i)]$$

are the centers of the ghor algebras of $A'$ and $A$ respectively [B2, Theorem 5.9.3].

3. Proof of main theorem

Throughout, let $A$ be a nondegenerate dimer algebra.

**Definition 3.1.** We say paths $p, q \in Q_{\geq 0}$ are a non-cancellative pair if they are representatives of a non-cancellative pair $p + I, q + I \in A$. We say a non-cancellative pair $p, q \in Q_{\geq 0}$ is minimal if, given any non-cancellative pair $s, t \in Q_{\geq 0}$ satisfying $R_{s,t} \subseteq R_{p,q}$, we have $\{s,t\} = \{p,q\}$. Let $r \in Q_{\geq 0}$ be a path for which $rp \equiv rq$ (resp. $pr \equiv qr$). We say $r$ is minimal if there is no proper subpath $r'$ of $r$ satisfying $r'p \equiv r'q$ ($pr' \equiv qr'$).
Lemma 3.2. Let \( p, q \in Q_{\geq 0} \) be a minimal non-cancellative pair, and let \( r \) be a minimal path satisfying \( rp \equiv rq \) (resp. \( pr \equiv qr \)). If the rightmost (leftmost) arrow subpath of \( r^+ \) lies in \( R_{p,q} \), then \( r^+ \) lies wholly in \( R_{p,q} \), and only meets \( p^+ \) or \( q^+ \) at its tail (head).

Proof. Set \( \sigma := \prod_{x \in P} x \), and for a path \( s \), set \( \tilde{s} := \eta(s) \).

Suppose that \( rp \equiv rq \), and the rightmost arrow subpath of \( r^+ \) lies in \( R_{p,q} \). Assume to the contrary that \( r^+ \) meets \( p^+ \) at a vertex other than its tail; see Figure 3. Then \( p \) and \( r \) factor into paths

\[
p = p_2p_1 \quad \text{and} \quad r = r_3r_2r_1,
\]

where \( p_2, r_1, r_3 \in Q_{\geq 0} \) are paths (of unspecified length), \( r_2 \in Q_1 \) is an arrow, and \( h(p_1) = h(r_2) \).

Since \( r_2 \) is an arrow, the cycle \( r_3r_1p_2 \in C^0 \) is nontrivial. Therefore there is some \( m \geq 1 \) such that \( r_2r_1p_2 = \sigma^m \), by Proposition 2.1.1. In particular, the paths \( p_1\sigma^m_i \) and \( r_2r_1q \) satisfy

\[
p_1\sigma^m = r_2r_1p_2 = r_2r_1p = r_2r_1q = r_2r_1q,
\]

where (i) holds by Proposition 2.1.2. Consequently, either \( p_1\sigma^m_i \) and \( r_2r_1q \) are equal modulo \( I \), or they form a non-cancellative pair, by Proposition 2.1.2.

(i) First suppose \( p_1\sigma^m_i \) and \( r_2r_1q \) form a non-cancellative pair. By choosing a representative of the unit cycle \( \sigma^+_i(p) \) that lies in \( R_{p_1,r_2r_1q} \), there is proper containment

\[
R_{p_1\sigma^m_i(p),r_2r_1q} \subset R_{p,q}.
\]

But then the non-cancellative pair \( p, q \) is not minimal, contrary to assumption.

(ii) So suppose \( p_1\sigma^m_i \equiv r_2r_1q \). Since \( r_2 \) is an arrow, there is a path \( s \) for which \( (sr_2)^+ \) is a unit cycle contained in \( R_{p_1,r_2r_1q} \). By Proposition 2.1.1, there is an \( n \in \mathbb{Z} \) such that

\[
(2) \quad r_1p_2 = s\sigma^n.
\]

Since \( s \) is a subpath of a unit cycle, we have \( n \geq 0 \). Whence,

\[
(3) \quad \overline{s\sigma^n_i(p)} = r_1p_2 = r_1p = r_1q,
\]

where (i) holds by Proposition 2.1.2.

Now (2) implies that either \( r_1p_2 \) and \( s\sigma^n_i \) are equal modulo \( I \), or they form a non-cancellative pair, by Proposition 2.1.2. Since \( p, q \) is minimal and \( s^+ \) lies in \( R_{p_1,r_2r_1q} \), we have

\[
(4) \quad r_1p_2 \equiv s\sigma^n_i(p_1).
\]

Thus

\[
(5) \quad s\sigma^n_i(p_1) \equiv r_1p_2 \equiv r_1p \not\equiv r_1q.
\]
Figure 3. Setup for Lemma 3.2. Here, $p_1, p_2, q, r_1, r_3, s \in Q \geq 0$ are paths (of unspecified length); $r_2 \in Q_1$ is an arrow; $r_2 s$ is a unit cycle; and $r_2 r_1 p_2$ lifts to a nontrivial cycle in the cover. The paths $p = p_2 p_1$, $q, r = r_3 r_2 r_1$, are drawn in red, blue, and green respectively.

where (i) holds by (4); and (ii) holds since $r$ has minimal length and $r_2$ is a nontrivial path. Therefore (3) and (5) together imply that the paths $s = p_2 p_1$, $q, r = r_3 r_2 r_1$, are drawn in red, blue, and green respectively.

\[ Q_1^S := \{ a \in Q_1 \mid a \notin x \text{ for all } x \in S \}. \]

**Proposition 3.3.** Let $p, q \in Q \geq 0$ be a minimal non-cancellative pair, and let $r$ be a minimal path such that $r p \equiv r q$ (resp. $p r \equiv q r$). If the rightmost (leftmost) arrow subpath of $r^+$ lies in $R_{p,q}$, then each arrow subpath of $r$ is in $Q_1^S$.

**Proof:** Set $\sigma := \prod_{x \in p} x$, and for a path $s$, set $s := \bar{\eta}(s)$.

Suppose $r p \equiv r q$ and the rightmost arrow subpath of $r^+$ lies in $R_{p,q}$. Assume to the contrary that there is a simple matching $x$ that contains an arrow subpath of $r$; then $r$ factors into paths $r = r_3 r_2 r_1$, where $r_2 \in Q_1 \setminus Q_1^S$ and $r_1, r_3 \in Q \geq 0$.

Since $x$ is simple, there is a path $t'$ from $t(p)$ to $h(r_2)$ whose arrow subpaths are not contained in $x$. By Lemma 3.2, the head of $r_2^+$ lies in the interior of $R_{p,q}$. Therefore there is a leftmost nontrivial subpath $t^+$ of $t^+$ contained in $R_{p,q}$ with tail on $p^+$ or $q^+$; suppose $t^+$ has tail on $p^+$. Then $p$ factors into paths $p = p_2 p_1$, where $h(p_1^+) = t(t^+)$, as shown in Figure 4.

Consider the path $s$ such that $(r_2 s)^+$ is a unit cycle that lies in $R_{t,r_2 r_1 p_2}$. (Note that $s$ and $r_1 p$ may share common subpaths.) Then $x \vdash \hat{s}$ since $r_2 \in x$. Furthermore, $x \vdash \hat{t}$ since no arrow subpath of $t$ is contained in $x$. Thus

\[ x \vdash \hat{s} \hat{t} = \hat{s} \hat{t}. \]

Therefore

\[ \sigma \vdash \hat{s} \hat{t}. \]
Figure 4. Setup for Proposition 3.3. Here, $p_1, p_2, q, r_1, r_3, s \in Q_{\geq 0}$ are paths; $r_2 \in Q_1 \setminus Q^S_1$ is an arrow; and $r_2s$ is a unit cycle. The paths $p = p_2p_1$, $q = r_2r_1$, are drawn in red, blue, and green respectively.

Since $t^+$ lies in $R_{p,q}$ and $h(t^+)$ lies in the interior of $R_{p,q}$, we have proper containment

$$R_{r_1p_2, st} \subset R_{p,q}.$$ 

Furthermore, by (6) and Proposition 2.1.1, there is some $m \geq 0$ such that

$$\sigma^m_{st} = \tilde{r}_1 \tilde{p}_2.$$ 

Thus, by the minimality of $p, q$ and Proposition 2.1.2,

$$\sigma^m_{h(s)st} \equiv r_1p_2.$$ 

Whence

(7)

$$\sigma^m_{h(s)st} p_1 \equiv r_1p.$$ 

Therefore

$$\sigma^m_{h(s)st} p_1 = \tilde{r}_1 \tilde{p} \equiv r_1q,$$

where (i) holds by Proposition 2.1.2. Furthermore, since $t^+$ lies in $R_{p,q}$ and $h(t^+)$ lies in the interior of $R_{p,q}$, there is proper containment

$$R_{stp_1, r_1q} \subset R_{p,q}.$$ 

Thus, again by the minimality of $p, q$ and Proposition 2.1.2,

(8)

$$\sigma^m_{h(s)st} p_1 \equiv r_1q.$$ 

Consequently,

$$r_1p \equiv r_1q,$$

where (i) holds by (7), and (ii) holds by (8). But $r_2$ is an arrow. Therefore $r$ is not minimal. □

Remark 3.4. If $p, q \in Q_{\geq 0}$ is a non-cancellative pair that is not minimal, and $r$ is a minimal path satisfying $rp \equiv rq$, then it is possible that no representative of $r^+$ lies in $R_{p,q}$, and no arrow subpath of $r$ is in $Q^S_1$. Such an example is given in Figure 5.

Corollary 3.5. If $A$ is non-cancellative, then $Q^S_1 \neq \emptyset$. 

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**Figure 4**: Diagram showing the setup for Proposition 3.3. Paths $p_1, p_2, q, r_1, r_3, s$ are drawn in red, blue, and green respectively, with $r_2 \in Q_1 \setminus Q^S_1$ as an arrow and $r_2s$ as a unit cycle. The paths $p = p_2p_1$ and $q = r_2r_1$ are highlighted.
Figure 5. Setup for Remark 3.4. In case (i), the paths $p, q$, drawn in red and blue, form a non-cancellative pair that is not minimal. The green path $r$ is a minimal path satisfying $rp \equiv rq$ and lies outside of $R_{p,q}$. In case (ii), the paths $p', q'$, again drawn in red and blue, form a non-cancellative pair that is minimal. The green path $r'$ is a minimal path satisfying $r'p' \equiv r'q'$ and, in contrast to case (i), necessarily lies inside of $R_{p', q'}$.

Proof. For any non-cancellative pair $p, q \in A$, there is some $m \geq 1$ such that $\sigma_{h(p)}^m p = \sigma_{h(q)}^m q$. Furthermore, there is a representative of $\sigma_{h(p)}$ whose lift lies in $R_{p,q}$. The corollary then follows from Proposition 3.3. \hfill $\square$

Corollary 3.5 is used in [B1] to show that every nondegenerate dimer algebra admits a cyclic contraction. Our standing assumption that $A$ is nondegenerate thus implies that $A$ admits a cyclic contraction. For the remainder of the article, let $\psi : A \to A'$ be a cyclic contraction on a set of arrows $Q_1^*$ unless stated otherwise. Set $\sigma := \prod_{x \in S'}$, and for each $s \in e_j A e_i$, $t \in e_t A' e_k$, set

$$(9) \quad \bar{s} := \bar{\tau}_\psi(s) := \bar{\tau}_\psi(s), \quad \bar{t} := \bar{\tau}(t).$$

We will use the following results from [B7].

Lemma 3.6. Let $p, q \in e_i A e_i$ be cycles.

1. If $\bar{p} = \bar{q}$ and $p - q \in Ze_i$, then

   $$p^2 = pq = qp = q^2.$$

2. There is an $N \geq 1$ such that for each $n \geq 1$, $p^n \sigma_i^N$ is in $Ze_i$.

3. If $p$ is in $R$, then there is an $n \geq 1$ such that $p^n$ is in $Ze_i$.

4. The nilradical of $Z$ consists of the central elements annihilated by $\psi$:

   $$\text{nil } Z = \ker \psi \cap Z.$$

Proof. The claims are respectively (1): [B7 Proposition 3.4]; (2), (3): [B7 Proposition 5.4] ; and (4): [B7 Theorem 3.5]. \hfill $\square$
Theorem 3.7. If $\psi : A \to A'$ is a cyclic contraction, then no contracted arrow is contained in a simple matching of $A$,

$$Q'_1 \subseteq Q^S_1.$$  

Proof. Assume to the contrary that there is an arrow $\delta \in Q'_1 \setminus Q^S_1$; let $x \in S$ be a simple matching containing $\delta$. Let $s \in Q_{\geq 0}$ be a path for which $s\delta$ is a unit cycle. Since $\delta$ is contracted to a vertex, $\psi(s)\delta$ is a unit cycle in $Q'$. Whence

$$s = \bar{\tau}(\psi(s)) = \sigma.$$  

(10)

Since $x$ is simple, there is a cycle $p$ of the subquiver $Q \setminus x$ that passes through each vertex of $Q$ and contains $s$ as a subpath. Since $s$ is a subpath of $p$, $\psi(s)$ is a subpath of $\psi(p)$. In particular, $s \mid p$. Therefore (10) implies

$$\sigma \mid \bar{p}.$$  

(11)

Let $u \in \mathbb{Z}^2$ be such that $\psi(p) \in C_u$. Since $A'$ is cancellative, there is a cycle $t \in \hat{C}_u(\psi(p))$, by Proposition 2.1.5. Furthermore, there is an $\ell \in \mathbb{Z}$ such that

$$p = \bar{\tau}\psi(p)(t) \equiv \bar{\tau}(t)\sigma^\ell = \ell\sigma^\ell,$$

where (i) holds by Proposition 2.1.1. But $\sigma \nmid \ell$, by Proposition 2.1.3. Whence, $\ell \geq 0$. Thus, $\ell \geq 1$ by (11). Therefore

$$\bar{p}\sigma^{-1} = \bar{l}\sigma^{\ell-1} = \bar{\tau}(t\sigma^{\ell-1}) \in S.$$

In particular, there is a cycle $q$ in $Q$ satisfying

$$\bar{q} = \bar{p}\sigma^{-1},$$

since the contraction $\psi : A \to A'$ is cyclic.

Set $i := t(q)$. Since $p$ contains each vertex in $Q$, the monomial $\bar{p} = \bar{q}\sigma_i$ is in $R$, and so we may assume $t(p) = i$. Thus there is some $n_1, n_2 \geq 1$ such that the cycles $p^{n_1}$ and $(q\sigma_i)^{n_2}$ are in $Ze_i$, by Lemma 3.6.3. Set $n := n_1n_2$. Then

$$p^n = (q\sigma_i)^n \in Ze_i.$$  

This, together with $\bar{p}^n = (\bar{q}\sigma_i)^n$, implies

$$p^{2n} = (q\sigma_i)^{2n},$$  

(12)

by Lemma 3.6.1.

Finally, let $V$ be an $A$-module with support $Q \setminus x$ and dimension $1^{Q_0}$. Then $p^{2n}$ does not annihilate $V$ since $p^{2n}$ is cycle of the subquiver $Q \setminus x$. However, $\sigma_i$ contains an arrow in $x$ since $x$ is a perfect matching. Thus $(q\sigma_i)^{2n}$ annihilates $V$. But this contradicts (12). □

Lemma 3.8. If an arrow annihilates a simple $A$-module of dimension $1^{Q_0}$, then it is contained in a simple matching of $A$.  

Proof. Let $V_\rho$ be a simple $A$-module of dimension $1^{Q_0}$, and suppose $\rho(a) = 0$. Let $i \in Q_0$. Since $V_\rho$ is simple of dimension $1^{Q_0}$, there is a path $p$ from $t(a)$ to $i$ such that $\rho(p) \neq 0$. Furthermore, $\sigma_i p = p\rho t(a)$ since $\sum_{j \in Q_0} \sigma_j$ is central. Thus, since $a$ is a subpath $\sigma_{t(a)}$ (modulo $I$), we have
\[
\rho(\sigma_i)\rho(p) = \rho(\sigma_i p) = \rho(p\rho t(a)) = 0.
\]
Whence
\[
\rho(\sigma_i) = 0.
\]
Thus each unit cycle contains at least one arrow that annihilates $V_\rho$. Therefore there are perfect matchings $x_1, \ldots, x_m \in \mathcal{P}$ such that $V_\rho$ has support $Q \setminus (x_1 \cup \cdots \cup x_m)$. Moreover, since $\rho(a) = 0$, there is some $1 \leq \ell \leq m$ such that $x_\ell$ contains $a$.

Since $V_\rho$ is simple of dimension $1^{Q_0}$, the subquiver $Q \setminus (x_1 \cup \cdots \cup x_m)$ contains a path $r$ that passes through each vertex of $Q$. But since $r$ is a path in $Q \setminus (x_1 \cup \cdots \cup x_m)$, $r$ is also a path in $Q \setminus x_\ell$. Therefore $x_\ell$ is a simple matching containing $a$. \hfill \Box

Lemma 3.9. Suppose $A$ is cancellative. If $p \in \mathcal{C}^u$, $q \in \mathcal{C}^v$, and $\bar{\tau}(p) = \bar{\tau}(q)$, then $u = v$.

Proof. For a path $s$, set $\bar{s} := \bar{\tau}(s)$. Suppose the hypotheses hold, and assume to the contrary that $u \neq v$.

(i) First suppose $u = mv$ for some $m \in \mathbb{Z}_{\geq 1}$. Let $s^+$ and $t^+$ be paths in $Q^+$ such that
\[
t(s^+) = t(p^+), \quad h(s^+) = t(q^+), \quad t(t^+) = h(q^+), \quad h(t^+) = h(p^+).
\]
Then $ts = \pi(t^+)\pi(s^+)$ is a cycle in $Q$. Furthermore, there is some $\ell \in \mathbb{Z}$ for which
\[
\bar{t}p\bar{s} \equiv (i) \bar{t}q\bar{s} \equiv \bar{q}\bar{s} \equiv (u) \bar{p}\sigma^\ell,
\]
where (i) holds by assumption, and (ii) holds by Proposition 2.1.1. Whence,
\[
\bar{t}s = \bar{t}s = \sigma^\ell.
\]
Thus, since $A$ is cancellative, $(ts)^+$ is a cycle in $Q^+$, by Proposition 2.1.6. Therefore $m = 1$.

(ii) Now suppose $u = mv$ for some $m \in \mathbb{Z}_{\leq 0}$. Let $s^+$ and $t^+$ be paths in $Q^+$ such that
\[
t(s^+) = h(p^+), \quad h(s^+) = t(q^+), \quad t(t^+) = h(q^+), \quad h(t^+) = t(p^+).
\]
As in case (i), $ts$ is a cycle in $Q$ satisfying
\[
\bar{t}s = \bar{t}s = \sigma^\ell
\]
for some $\ell \geq 0$. Thus, there is some $n \in \mathbb{Z}$ such that
\[
\bar{p}^2\sigma^\ell \equiv (i) \bar{p}\sigma^\ell \equiv (u) \bar{p}\bar{q}\bar{s} \equiv \bar{p}\bar{t}\bar{q}\bar{s} \equiv (u) \sigma^n,
\]
where (i) holds by assumption; (ii) holds by (14); and (iii) holds by Proposition 2.1.1. Whence, $\bar{p}^2 = \sigma^{n-\ell}$. Consequently, $(p^+)^+$, hence $p^+$, is a cycle in $Q^+$, by Proposition 2.1.6. Similarly, $q^+$ is a cycle in $Q^+$. Therefore $u = (0,0) = v$. 

(iii) Finally, suppose \( u \) is not a multiple of \( v \). Then, since any two non-parallel lines in \( \mathbb{R}^2 \) intersect, there are lifts of \( p \) and \( q \) to \( Q^+ \subset \mathbb{R}^2 \) that intersect at a vertex \( i^+ \in Q_0^+ \). By cyclically permuting \( p \) and \( q \), we may assume that \( t(p) = t(q) = i \).

Let \( r^+ \) be a path from \( h(p^+) \) to \( h(q^+) \); then \( r \) is a cycle at \( i \). Since \( A \) is cancellative, we may choose \( r \) to be in \( \hat{C}_i \), by Proposition 2.1.5. Whence
\[
\sigma \nmid r,
\]
by Proposition 2.1.3. Moreover, there is an \( m \in \mathbb{Z} \) such that
\[
\bar{r}p = \bar{r}p^{(i)} \equiv q\sigma^m \equiv \bar{q}\sigma^m,
\]
where \( (i) \) holds by Proposition 2.1.1, and \( (ii) \) holds by assumption. Thus \( r = \sigma^m \).

Hence \( \bar{r}p \) is a cycle in \( Q_0^+ \), by Proposition 2.1.6. Therefore \( h(p^+) = h(q^+) \). But then \( u = v \), contrary to assumption. □

Lemma 3.10. Let \( \psi : A \to A' \) be a cyclic contraction. Then \( R = S \) if and only if \( \hat{C}^u_i \neq \emptyset \) for each \( u \in \mathbb{Z}^2 \) and \( i \in Q_0 \).

Proof. \((\Rightarrow)\) Suppose \( R = S \). Fix \( u \in \mathbb{Z}^2 \setminus 0 \) and \( i \in Q_0 \); we claim that \( \hat{C}^u_i \neq \emptyset \).

Since \( A' \) is cancellative, there is a cycle \( q \in \hat{C}^u_i \), by Proposition 2.1.5. In particular, \( \sigma \nmid q \) by Proposition 2.1.3. Since \( R = S \), there is a cycle \( p \in \hat{C}_i \) such that \( \bar{p} = \bar{q} \).

Furthermore, since \( \sigma \nmid \bar{p} \), \( p \) is in \( \hat{C}_i \), by Proposition 2.1.3. Thus to show that \( \hat{C}^u_i \neq \emptyset \), it suffices to show that \( p \) is also in \( C^u \).

Now \( \psi(p) \) is in \( C^u \) since \( q \) is in \( C^u \) and \( \overline{\psi(p)} = \overline{\bar{q}} \), by Lemma 3.9. Furthermore, \( \psi \) cannot contract a cycle in the underlying graph of \( Q \) to a vertex \([B2, \text{Lemma 3.6}]\). Therefore \( p \) is in \( C^u \), proving our claim.

\((\Leftarrow)\) Conversely, suppose \( \hat{C}^u_i \neq \emptyset \) for each \( u \in \mathbb{Z}^2 \) and \( i \in Q_0 \). To show that \( S \subseteq R \), it suffices to show that each monomial in \( S \) not divisible by \( \sigma \) is in \( R \), by Proposition 2.1.8. So let \( g \in S \) be such that \( \sigma \nmid g \). Since \( \sigma \nmid g \), there is a cycle \( p \in \hat{C}_i \) for which \( \bar{p} = g \), by Proposition 2.1.3. Let \( u \in \mathbb{Z}^2 \) be such that \( p \in \hat{C}^u_i \). If \( q \) is another cycle in \( \hat{C}^u_i \), then \( \bar{q} = \bar{p} \), by Proposition 2.1.4. Therefore \( g = \bar{p} \) is in \( R \) since \( \hat{C}^u_i \neq \emptyset \) for each \( i \in Q_0 \). □

Theorem 3.11. Let \( A = kQ/I \) be a nondegenerate dimer algebra. The following are equivalent:

(1) \( A \) is cancellative.
(2) Each arrow annihilates a simple \( A \)-module of dimension \( 1^{Q_0} \).
(3) Each arrow is contained in a simple matching, \( Q_1^S = \emptyset \).
(4) The center \( R \) of \( \Lambda \) equals the cycle algebra \( S \).
(5) If \( \psi : A \to A' \) is a cyclic contraction, then \( \psi \) is trivial, \( Q_1^* = \emptyset \).

Proof. \((2) \iff (3)\): Lemma 3.8
\((3) \implies (1)\): Proposition 3.3
(4) ⇒ (3): Suppose $R = S$ (recall that $S$ is independent of the choice of cyclic contraction). Then $C_i^u \neq \emptyset$ for each $i \in Q_0$ and $u \in \mathbb{Z}^2$, by Lemma 3.10. Therefore $Q_1^S = \emptyset$ by [B2, Theorem 4.25].

(1) ⇒ (4): If $A$ is cancellative, then $R = S$ by Proposition 2.1.7.

(1) ⇒ (5): Suppose $A$ is cancellative, and $\psi: A \to A'$ is a cyclic contraction. Then $Q_1^*(i) \subseteq Q_1^S \equiv \emptyset$, where (i) holds by Theorem 3.7, and (ii) holds by the implication (1) ⇒ (2). Therefore $Q_1^* = \emptyset$.

(5) ⇒ (4): Clear. □

We will use the following definition in the proof of Proposition 3.13.

**Definition 3.12.** [B2, Definition 2.6] Let $A$ be a finitely generated $k$-algebra and let $Z$ be its center. An impression of $A$ is an algebra monomorphism $\tau: A \to M_d(B)$ to a matrix ring over a commutative finitely generated $k$-algebra $B$, such that

- for generic $b \in \text{Max } B$, the composition $A \xrightarrow{\tau} M_d(B) \xrightarrow{1} M_d(B/b) \cong M_d(k)$ is surjective; and
- the morphism $\text{Max } B \to \text{Max } \tau(Z)$, $b \mapsto b \cap \tau(Z)$, is surjective.

**Proposition 3.13.** Suppose $A$ is non-cancellative. The ghor algebra $\Lambda$ is nonnoetherian and an infinitely generated module over its center $R$.

**Proof.** Consider the $k$-linear map $\tau_\psi: A \to M_{|Q_0|}(k[S'])$ defined for each $i,j \in Q_0$ and $p \in e_j Ae_i$ by

$p \mapsto \bar{\tau}_\psi(p)e_{ji}.$

It is shown in [B2, Theorem 5.9.1] that this map induces an impression of $\Lambda$, (16) $\tau_\psi: \Lambda \to M_{|Q_0|}(k[S'])$.

Furthermore, it is shown in [B6, Theorem 3.18] that on the algebraic variety $\text{Max } S$, the following loci coincide:

$U_{S/R}^* := \{ n \in \text{Max } S \mid R_{n \cap R} \text{ is noetherian} \} = \{ n \in \text{Max } S \mid R_{n \cap R} = S_n \} =: U_{S/R}.$

But $R \neq S$, by Theorem 3.11. Therefore $\Lambda$ is nonnoetherian and an infinitely generated $R$-module by [B4, Theorem 4.1.2] 4

Recall (9): for $p \in e_j Ae_i$, $i,j \in Q_0$, set $\bar{\tau}_\psi(p) := \bar{\tau}_\psi(p)$.

4The assumption that $k$ is uncountable in [B4, Theorem 4.1] is only used in 4.1.1, and not in 4.1.2.
Proposition 3.14. Let \( p \in e_i A e_i \) be a cycle. If \( p \not\in \bar{\tau}_\psi(e_j A e_j) \) and \( \sigma \nmid p \), then for each \( n \geq 1 \),
\[
\bar{p}^n \not\in \bar{\tau}_\psi(e_j A e_j).
\]
Consequently, if \( p \not\in R \) and \( \sigma \nmid p \), then for each \( n \geq 1 \), \( p^n \not\in R \). Moreover, if \( A \) is non-cancellative, then such a cycle exists.

Proof. (i) Assume to the contrary that there is a cycle \( p \in e_i k Q e_i \) such that \( p \not\in R \), \( \sigma \nmid p \), and \( p^n \in R \) for some \( n \geq 2 \). Let \( u \in \mathbb{Z}^2 \) be such that \( p \in C_u \).

Since \( p \) is not in \( R \), there is a vertex \( j \in Q_0 \) such that
\[
(17) \quad p \not\in \bar{\tau}_\psi(e_j A e_j).
\]
Furthermore, since \( p^n \) is in \( R \), \( p^n \) homotopes to a cycle \( q \in e_i k Q e_i \) that passes through \( j \),
\[
(18) \quad q \equiv p^n.
\]
For \( v \in \mathbb{Z}^2 \), denote by \( q^+_v \in \pi^{-1}(q) \) the preimage with tail
\[
t(q^+_v) = t(q^+) + v \in Q_0^+.
\]
Since \( Q^+ \) embeds in the plane \( \mathbb{R}^2 \), there is a path \( r^+ \) from \( j^+ \) to \( j^+ + u \) that is constructed from subpaths of \( q^+, q^+_u, \) and \( q^+_m \), for some \( m \in \mathbb{Z} \); see Figure 6. In particular, the cycle \( r := \pi(r^+) \in e_j k Q e_j \) is in \( C^u_j \).

Since \( \sigma \nmid p \), there is a simple matching \( x \) such that \( x \nmid p \). Whence \( x \nmid q \) by (18). Thus \( x \nmid r \). We therefore have
\[
(19) \quad \sigma \nmid p, \quad \sigma \nmid r, \quad \text{and} \quad p, r \in C^u.
\]
But if \( s, t \) are cycles in \( C^u \), then there is an \( \ell \in \mathbb{Z} \) such that \( s = t \sigma^\ell \) \[\text{[B2, Lemma 4.19]}. Consequently, (19) implies \( r = p \). Therefore
\[
\bar{p} = \bar{r} \in \bar{\tau}_\psi(e_j A e_j),
\]
contrary to (17).

(ii) Now suppose no cycle \( p \) exists for which \( \bar{p}^n \not\in R \) for each \( n \geq 1 \). Then by Claim (i), if \( q \) is a cycle satisfying \( \bar{q} \not\in R \), then \( \sigma \nmid \bar{q} \). By the contrapositive of this assumption, for each cycle \( q \) satisfying \( \sigma \nmid \bar{q} \), we have \( \bar{q} \in R \). But \( S \) is generated by \( \sigma \) and a set of monomials in \( k[S'] \) not divisible by \( \sigma \), by Proposition 2.1.8. Whence \( S \subseteq R \) since \( \sigma \in R \). Thus \( S = R \). Therefore \( A \) is cancellative by Theorem 3.11.

Lemma 3.15. For each \( i \in Q_0 \), there is an inclusion \( \bar{\tau}_\psi(Z e_i) \subseteq R \).

Proof. For \( i \in Q_0 \), we have
\[
\bar{\tau}_\psi(Z e_i) = \bar{\tau}_\psi((Z/(\ker \psi \cap Z)) e_i) \overset{(i)}{=} \bar{\tau}_\psi((Z/\text{nil } Z) e_i) \overset{(ii)}{\subseteq} R,
\]
where (i) holds by Lemma 3.6.4, and (ii) holds by [B7, Theorem 4.1].
Theorem 3.16. Suppose \( A \) is non-cancellative. Then \( A \), its center \( Z \), its reduced center \( \hat{Z} := Z / \text{nil} \, Z \), and its ghor center \( R \), are all nonnoetherian algebras.

Proof. (i) We first claim that \( A \) is nonnoetherian. Indeed, since \( A \) is non-cancellative, there is a cycle \( p \in e_i A e_i \) and vertex \( j \in Q_0 \) such that for each \( n \geq 1 \),

\[
\tag{20} p^n \notin \bar{\tau}_\psi(e_j A e_j),
\]

by Proposition 3.14. Furthermore, there is an \( N \geq 1 \) such that for each \( n \geq 1 \),

\[
\bar{p}^n \sigma^N \in \bar{\tau}_\psi(Z e_j) \subseteq \bar{\tau}_\psi(e_j A e_j).
\]

by Lemma 3.6.2. Let \( q_n \in Z e_j \) be such that

\[
q_n := \bar{p}^n \sigma^N.
\]

Consider the ascending chain of (two-sided) ideals of \( A \),

\[
\langle q_1 \rangle \subseteq \langle q_1, q_2 \rangle \subseteq \langle q_1, q_2, q_3 \rangle \subseteq \cdots.
\]

Assume to the contrary that the chain stabilizes; then for some \( m \geq 1 \) there are elements \( a_1, \ldots, a_m \in A \) satisfying

\[
\tag{21} q_m = \sum_{n=1}^{m-1} a_n q_n.
\]

Since each \( q_n \) is in \( e_j A e_j \), we have

\[
q_m = e_j q_m = e_j \sum_{n<m} a_n e_j q_n = \sum_{n<m} (e_j a_n e_j) q_n.
\]

In particular, we may take each \( a_n \) to be in \( e_j A e_j \). Furthermore, (21) implies

\[
\bar{p}^m \sigma^N = \bar{q}_m = \sum_{n<m} a_n \bar{q}_n = \sum_{n<m} \bar{a}_n \bar{q}_n = \sum_{n<m} \bar{a}_n \bar{p}^n \sigma^N.
\]
Thus, since $k[S']$ is an integral domain, we have

\[(22) \quad p^m = \sum_{n<m} a_n p^n,\]

with each $a_n$ in $\bar{\tau}(e_j A e_j)$.

Now the subalgebra $\bar{\tau}(e_j A e_j)$ is generated by a set of monomials in the polynomial ring $k[S']$. Thus each polynomial $\bar{a}_n$ is a $k^\times$-linear combination of distinct monomials $\alpha_{n,\ell}$ in $\bar{\tau}(e_j A e_j)$. Furthermore, (22) implies that for some $0 \leq n < m$, there is an $\ell$ for which $\alpha_{n,\ell} = p^{m-n}$. But then $p^{m-n}$ is in $\bar{\tau}(e_j A e_j)$, contrary to (20). Therefore $A$ is nonnoetherian.

(ii) We claim that $Z$ is nonnoetherian. The cycles $q_n$ were chosen to be in $Ze_j$, so we may consider the chain of ideals of $Ze_j$,

\[(q_1)Z \subseteq (q_1, q_2)Z \subseteq (q_1, q_2, q_3)Z \subseteq \cdots.\]

This chain does not stabilize by the argument in Claim (i) since $Ze_j \subseteq e_j A e_j$. Thus $Ze_j$ is nonnoetherian. Consequently, $Z$ is nonnoetherian since the vertex idempotents are orthogonal.

(iii) We claim that $R$ is nonnoetherian. By Lemma 3.15 the monomials $q_n$ are in $R$. Thus we may consider the chain of ideals of $R$,

\[(q_1)R \subseteq (q_1, q_2)R \subseteq (q_1, q_2, q_3)R \subseteq \cdots.\]

This chain does not stabilize by the argument in Claim (i) since $R \subseteq \bar{\tau}(e_j A e_j)$. Therefore $R$ is nonnoetherian.

(iv) Finally, we claim that the reduced center $\hat{Z}$ is nonnoetherian. There are no cycles in $(\text{nil } Z)e_j$ since $\text{nil } Z = \ker \psi \cap Z$, by Lemma 3.6.4. In particular, the cycles $q_n \in Ze_j$ are not in $(\text{nil } Z)e_j$, and so are nonzero in $\hat{Ze}_j$. The claim then follows from Claim (ii).}

Although $Z$, $\hat{Z}$, and $R$ are nonnoetherian, it is shown in [B6, Theorem 1.1] that they each have Krull dimension 3 and are generically noetherian. In particular, though they do not satisfy the ascending chain condition on all ideals, they do satisfy the ascending chain condition on prime ideals.

**Lemma 3.17.** If $A$ is a finitely generated $Z$-module, then $S$ is a finitely generated $R$-module.

**Proof.** Suppose $A$ is finite over $Z$. Then there are elements $a_1, \ldots, a_N \in A$ such that

\[A = \sum_{n=1}^{N} Z a_n.\]
Whence, for each $i \in Q_0$,
\[
\bar{\tau}_\psi (e_i A e_i) = \bar{\tau}_\psi \left( e_i \sum_n Z a_n e_i \right) = \bar{\tau}_\psi \left( \sum_n Z e_i (e_i a_n e_i) \right) = \sum_n \bar{\tau}_\psi (Z e_i) e_i a_n e_i \subseteq \sum_n R e_i a_n e_i,
\]
where (1) holds by Lemma 3.15. Thus $\bar{\tau}_\psi (e_i A e_i)$ is finite over $R$. But then
\[
S = k \left( \prod_{i \in Q_0} \bar{\tau}_\psi (e_i A e_i) \right) = R \left( \prod_{i \in Q_0} \bar{\tau}_\psi (e_i A e_i) \right) = R \left( \prod_{i \in Q_0} e_i a_{n(i)} e_i \mid 1 \leq n(i) \leq N \right).
\]
Therefore $S$ is finite over $R$. \hfill \qed

**Theorem 3.18.** Suppose $A$ is non-cancellative. Then $A$ is an infinitely generated $Z$-module.

*Proof.* The cycle algebra $S$ is a finitely generated $k$-algebra by Proposition 2.1.8, whereas $R$ is an infinitely generated $k$-algebra by Theorem 3.16. Thus $S$ is infinite over $R$ by the Artin-Tate lemma. Therefore $A$ is infinite over $Z$ by Lemma 3.17. \hfill \qed

An immediate question following Theorem 3.18 is whether non-cancellative dimer algebras satisfy a polynomial identity. We note that ghor algebras are always PI since they are subalgebras of matrix rings over commutative rings [B2, Introduction].

**Proposition 3.19.** Suppose $\psi : A \to A'$ is a cyclic contraction. If the head (or tail) of each arrow in $Q_1^*$ has indegree 1, then $A$ contains a free subalgebra. In particular, $A$ is not PI.

*Proof.* Let $a \in Q_1^*$; then the indegree of $h(a)$ is 1 by assumption. It suffices to suppose the indegree of $t(a)$ is at least 2. Consider the paths $p, q$ and the path $b$ of maximal length such that $b a p$ and $b a q$ are unit cycles. Let $b' \in Q_1'$ be a leftmost arrow subpath of $b$. Since $b$ has maximal length, the indegree of $h(b') = h(b)$ is at least 2. In particular, $b'$ is not contracted. Furthermore, no vertex in $Q'$ has indegree 1 since $A'$ is cancellative. Whence

\[
\psi(b a p) = b' p \quad \text{and} \quad \psi(b a q) = b' q
\]

are unit cycles in $Q'$. Therefore $p, q$ is a non-cancellative pair and $\bar{p} = \bar{q}$. 
Since \( A' \) is cancellative, there is a simple matching \( x \in S' \) which contains \( b' \), by Theorem 3.11. Furthermore, since \( x \) is a simple matching, there is a path \( s \) in \( Q' \) from \( h(p) \) to \( t(p) \) which is a path of \( Q' \setminus x \). In particular,

\[
x \notin \bar{\tau}(sp) = \bar{\tau}(sq).
\]

Since the indegree of the head of each contracted arrow is 1, \( \psi \) is surjective. Thus there is path \( r \) in \( Q \) from \( h(p) \) to \( t(p) \) satisfying \( \psi(r) = s \). Whence

\[
x \notin \bar{\tau}(sp) = \bar{\tau}_\psi(rp) = \bar{\tau}_\psi(rq).
\]

But then \( b \) is not a subpath of \( rp \) or \( rq \) (modulo \( I \)) since \( x \notin \bar{\tau}_\psi(b') \). Consequently, there are no relations between the cycles \( rp \) and \( rq \). Therefore

\[
k\langle rp, rq \rangle
\]

is a free subalgebra of \( A \).

**Remark 3.20.** An example of a (nondegenerate) dimer algebra, with no vertex of indegree 1 and with a free subalgebra, is given in Figure 1. Indeed, let \( a, b \) be the red and blue arrows in \( Q \). Then \( k\langle a, b \rangle \) is a free subalgebra of \( A \), and so \( A \) is not PI.

**Theorem 3.21.** Let \( A \) be a non-degenerate dimer algebra. The following are equivalent:

1. \( A \) is cancellative.
2. \( A \) is noetherian.
3. \( Z \) is noetherian.
4. \( A \) is a finitely generated \( Z \)-module.
5. The vertex corner rings \( e_iAe_i \) are pairwise isomorphic algebras.
6. Each vertex corner ring \( e_iAe_i \) is isomorphic to \( Z \).

Furthermore, each condition is equivalent to each of the conditions (2) – (6) with \( A \) and \( Z \) replaced by \( \Lambda \) and its center \( R \).

**Proof.** First suppose \( A \) is cancellative. Then \( A \) is finite over \( Z \) and \( Z \) is noetherian, by Propositions 2.1.8. Therefore \( A \) is noetherian. Furthermore, the vertex corner rings are pairwise isomorphic and isomorphic to \( Z \), by Propositions 2.1.7 and 2.1.8. Finally, since \( A \) is cancellative, we have \( A = \Lambda \).

Conversely, suppose \( A \) is non-cancellative. Then \( A, Z, \) and \( R \) are nonnoetherian by Theorem 3.16; \( A \) is infinite over \( Z \) by Theorem 3.18; \( A \) is nonnoetherian and infinite over \( R \) by Proposition 3.13 and the vertex corner rings are not all isomorphic by Proposition 3.14.

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