On spectral measures and convergence rates in von Neumann’s Ergodic Theorem
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Abstract
We show that the power-law decay exponents in von Neumann’s Ergodic Theorem (for discrete systems) are the pointwise scaling exponents of a spectral measure at the spectral value 1. In this work we also prove that, under an assumption of weak convergence, in the absence of a spectral gap, the convergence rates of the time-average in von Neumann’s Ergodic Theorem depend on sequences of time going to infinity.

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1 Introduction
1.1 Contextualization and main results
Let $U$ be a unitary operator on a separable complex Hilbert space $\mathcal{H}$ and let $P^U$ be its resolution of the identity (recall details from definition of $P^U$ in Subsection 1.2). For each $\psi \in \mathcal{H}$, set
$$\psi^* := P^U(\{1\})\psi.$$
Note that $P^U(\{1\})$ is the orthogonal projection onto the closed subspace $I(U) := \{ \varphi \mid U\varphi = \varphi \}$.

Now, recall the following classical result.

**Theorem 1** (von Neumann’s Ergodic Theorem). Let $U$ be a unitary operator on $\mathcal{H}$. Then, for each $\psi \in \mathcal{H}$,

$$
\lim_{K \to \infty} \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi = \psi^*.
$$

The study of convergence rates for ergodic theorems is a long-established subject in spectral theory and dynamical systems (see [8, 10, 16–20, 22, 25] and references therein). In the particular case of the von Neumann’s Ergodic Theorem (vNET), some authors have obtained (power-law) convergence rates for the time-average in relation (1) in the absence of a spectral gap at $z = 1$ (that is, in case $z = 1$ is not an isolated point of the spectrum of $U$); most of these works are motivated by possible applications to Koopman operators (see [16–20] and references therein; see also [8] for a discussion involving continuous dynamical systems).

On the other hand, if there is a spectral gap at $z = 1$, it is well known that the convergence rates for the time-average in relation (1) are uniform in $\psi$ and at least of order $K^{-1}$; in what follows, $\sigma(U)$ stands for the spectrum of $U$, a subset of $\partial \mathbb{D} := \{ z \in \mathbb{C} \mid |z| = 1 \}$.

**Theorem 2** (von Neumann’s Ergodic Theorem with a spectral gap at $z = 1$). Let $U$ be a unitary operator on $\mathcal{H}$. If there exists $0 < \gamma < \pi$ such that $\sigma(U) \subset \{1\} \cup \{ e^{i\theta} \mid \theta \in (-\pi, -\gamma] \cup (\gamma, \pi) \}$, then for every $K \in \mathbb{N}$,

$$
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j - P^U(\{1\}) \right\|^2 \leq \frac{4}{\gamma K}.
$$

Although the proof of Theorem 2 is a consequence of Lemma 1 below, for the convenience of the reader, we present it in Appendix A. In what follows, $\mu_{\psi-\psi^*}^U$ stands for the spectral measure of $U$ with respect to $\psi - \psi^*$ on $\partial \mathbb{D}$.

**Lemma 1** (von Neumann’s Lemma). Let $U$ be a unitary operator on $\mathcal{H}$. Then, for each $\psi \in \mathcal{H}$ and each $K \in \mathbb{N}$,

$$
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = \frac{1}{K^2} \int_{\partial \mathbb{D}} \left| \frac{z^K - 1}{z - 1} \right|^2 d\mu_{\psi-\psi^*}^U(z).
$$

Lemma 1 is a consequence of the Spectral Theorem; its proof is also presented in Appendix A.

**Remark 1.** Relation (2) indicates that the convergence rates of the time-average in (1) depend only on local scale properties of $\mu_{\psi-\psi^*}^U$ at $z = 1$. In this context, this relation arises from an argument due to von Neumann, presented in [25] to prove the Quasi-Ergodic Hypothesis, and it was used more recently by Kachurovskii, Podvigin...
and Sedalishchev [16, 17, 19] to relate the local scale properties of \( \mu^U_{\psi - \psi^*} \) at \( z = 1 \) to the power-law convergence rates in (1) (see Theorem 3 ahead).

Next, we recall the result due to Kachurovskii [16], mentioned in Remark 1, that relates the (power-law) convergence rates in \( \text{vNET} \) to the local scale properties of \( \mu^U_{\psi - \psi^*} \) at \( z = 1 \), in case there is no spectral gap.

In what follows, for each \( 0 < \epsilon \leq \pi \), set

\[
A_\epsilon := \{ e^{i\theta} | -\epsilon < \theta \leq \epsilon \}.
\]

**Theorem 3** (Theorem 3. in [16]). Let \( U \) be a unitary operator on \( \mathcal{H} \), \( \psi \in \mathcal{H} \) and \( 0 < \alpha < 2 \).

i) If there exists \( C_\psi > 0 \) so that, for every \( 0 < \epsilon \leq \pi \), \( \mu^U_{\psi - \psi^*}(A_\epsilon) \leq C_\psi \epsilon^\alpha \), then there exists \( \tilde{C}_\psi > 0 \) such that, for every \( K \in \mathbb{N} \),

\[
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 \leq \frac{\tilde{C}_\psi}{K^\alpha}.
\]

ii) Conversely, if there exists \( C_\psi > 0 \) such that for every \( K \in \mathbb{N} \),

\[
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 \leq \frac{C_\psi}{K^\alpha},
\]

then there exists \( \tilde{C}_\psi > 0 \) so that for every \( 0 < \epsilon \leq \pi \), \( \mu^U_{\psi - \psi^*}(A_\epsilon) \leq \tilde{C}_\psi \epsilon^\alpha \).

**Remark 2.** Although Theorem 3 was originally stated for Koopman operators, its proof holds for any unitary operator. Namely, the proof is essentially a consequence of Lemmas 1. and 2. in [19] (see also Lemma 2 ahead), which also hold for abstract operators. It is worth mentioning that a refinement of Theorem 3. in [16], with optimal constants for which the result is valid, is presented by Theorem 1 in [19].

It is clear from Lemma 1 and Theorem 3 that, in this setting, the behavior of the power-law convergence rates in \( \text{vNET} \) depends only on the local scale properties of \( \mu^U_{\psi - \psi^*} \) at \( z = 1 \). Taking this into account, our first result in this work shows (under some mild assumptions) that these convergence rates are indeed ruled by the lower and upper pointwise exponents of \( \mu^U_{\psi - \psi^*} \) at \( z = 1 \).

Recall that the lower and upper pointwise exponents of a finite (positive) Borel measure on \( \partial \mathbb{D} \), say \( \mu \), at \( z = 1 \) are defined, respectively, by

\[
d^-_\mu(1) := \liminf_{\epsilon \downarrow 0} \frac{\ln \mu(A_\epsilon)}{\ln \epsilon} \quad \text{and} \quad d^+_\mu(1) := \limsup_{\epsilon \downarrow 0} \frac{\ln \mu(A_\epsilon)}{\ln \epsilon},
\]

if, for all small enough \( \epsilon > 0 \), \( \mu(A_\epsilon) > 0 \); one sets \( d^\mu_+(w) := \infty \), otherwise.

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Theorem 4. Let $U$ be a unitary operator on $\mathcal{H}$, $0 \neq \psi \in \mathcal{H}$ and suppose that $d_{\mu_{\psi, \psi^*}}^+(1) \leq 2$. Then,

$$\liminf_{K \to \infty} \frac{\ln \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2}{-\ln K} = d_{\mu_{\psi, \psi^*}}^-(1),$$

(3)

$$\limsup_{K \to \infty} \frac{\ln \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2}{-\ln K} = d_{\mu_{\psi, \psi^*}}^+(1).$$

(4)

Remark 3. Note that if $d_{\mu_{\psi, \psi^*}}^-(1) = d_{\mu_{\psi, \psi^*}}^+(1) = d \leq 2$, then it follows from Theorem 4 that for each $\epsilon > 0$ and each sufficiently large $K$, one has

$$K^{-\epsilon - d} \leq \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 \leq K^{\epsilon - d}.$$ 

We note that if $d = 2$, then (in general) one cannot guarantee that for all sufficiently large $K$,

$$\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 \leq K^{-2};$$

see, e.g., Remarks 2 and 3 in [16]. Examples where $d_{\mu_{\psi, \psi^*}}^-(1) = d_{\mu_{\psi, \psi^*}}^+(1) = 2$ can be found on page 661 in [16] (see also [11]).

Remark 4. It is well known that for Koopman operators, for every $\psi \in \mathcal{H}$ such that $\psi - \psi^* \neq 0$, the respective time-average presented in vNET does not converge to $\psi^*$ faster than $K^{-2}$ (see, for instance, Corollary 5 in [14]). In this sense, since most of the possible applications of Theorem 4 are related to Koopman operators, the hypothesis that $d_{\mu_{\psi, \psi^*}}^+(1) \leq 2$ is quite reasonable.

Remark 5. It is worth underlying that the problem of obtaining the actual values of the decaying exponents of time-averages in terms of the dimensional properties of measures in dynamical systems and quantum dynamics is very natural and recurring in the literature; see, for instance, [2, 4–6, 10, 15]. In this context, Theorem 4 does add something to the body of knowledge on power-law convergence rates in vNET; namely, it establishes an explicit relation between the power-law convergence rates of such time-average and the local scale spectral properties of the unitary operator $U$ at $z = 1$.

By Theorem 4, if for some $\psi \in \mathcal{H}$ one has

$$0 \leq d_{\mu_{\psi, \psi^*}}^-(1) < d_{\mu_{\psi, \psi^*}}^+(1) \leq 2,$$
then the power-law convergence rates (related to $\psi$) in vNET actually depend on sequences of time going to infinity.

Write $L := UP^U(\partial \mathbb{D} \setminus \{1\})$; in this work, we also prove that the condition

$$d^{-}_{\mu_{\psi - \varphi}, \epsilon} < d^{+}_{\mu_{\psi - \varphi}, \epsilon}$$

is satisfied for a unitary operator $U$ and for a $G_{\delta}$ dense set of vectors $\psi \in \mathcal{H}$, if $L^j$ converges weakly to zero (i.e., for each pair $\psi, \varphi \in \mathcal{H}$, $(L^j\psi, \varphi) \to 0$ as $j \to \infty$) and if 1 is an accumulation point of $\sigma(U)$.

**Theorem 5.** Let $U$ be a unitary operator on $\mathcal{H}$ such that 1 is an accumulation point of its spectrum $\sigma(U)$. If $L^j$ converges to zero in the weak operator topology, then the set $\mathcal{G}(U)$ of $\psi \in \mathcal{H}$ such that, for all $\epsilon > 0$,

\[
\limsup_{K \to \infty} K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = \infty \tag{5}
\]

and

\[
\liminf_{K \to \infty} K^{2-\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = 0, \tag{6}
\]

is a dense $G_{\delta}$ set in $\mathcal{H}$.

**Corollary 1.** Under the same hypotheses of Theorem 5, if $\psi \in \mathcal{G}(U)$, then

$$0 = d^{-}_{\mu_{\psi - \varphi}, \epsilon} < 2 \leq d^{+}_{\mu_{\psi - \varphi}, \epsilon}.$$  

**Remark 6.** The results presented in this work are analogous to those obtained by the first three authors in [2] on the convergence rates of normal semigroups (see also [1, 9, 10]). The main technical difference is that, for normal semigroups, it is enough to analyze the (self-adjoint) real part of the generator in order to evaluate the power-law convergence rates, whereas here one needs to analyze spectral properties of unitary operators.

We note that an adaptation of the arguments in [2] could be employed to study the convergence rates in vNET for continuous dynamical systems, since this case leads naturally to self-adjoint generators.

### 1.2 Notations and organization of the text

Some words about notation: $U$ will always denote a unitary operator acting on the separable complex Hilbert space $\mathcal{H}$; we denote its spectrum by $\sigma(U)$.

Recall that, by the Spectral Theorem [24] (see also [13]), there is a unique map that associates with each bounded Borel measurable function $f$ on the unit circle $\partial \mathbb{D}$
a bounded normal operator, that is,
\[ P_U(f) = \int_{\partial D} f(z) dP_U(z), \]
that satisfies, for each \( \varphi \in \mathcal{H} \),
\[ \|P_U(f)\varphi\|^2 = \left\| \int_{\partial D} f(z) dP_U(z)\varphi \right\|^2 = \int_{\partial D} |f(z)|^2 d\mu_U^\varphi(z); \]
here, \( \mu_U^\varphi \) is a positive finite Borel measure supported on \( \partial D \), the so-called spectral measure of \( U \) with respect to \( \varphi \).

One can also define, for each Borel set \( \Lambda \subset \partial D \), the spectral projection
\[ P_U(\Lambda) := P_U(\chi_\Lambda); \]
the family \( \{P_U(\Lambda)\}_\Lambda \) is called the resolution of the identity of \( U \) (see [12, 24]).

The work is organized as follows. In Section 2, we illustrate our general results by presenting an explicit application of Theorem 5 to Koopman operators. The proof of Theorems 4 and 5 and of Corollary 1 are left to Section 3. In Appendix A, we prove Lemma 1 and Theorem 2.

2 Application to Koopman operators

Let \((X, \mathcal{A}, m)\) be a Lebesgue space with continuous measure \(m\) (i.e., a non-atomic probability space obtained by the completion of a Borel measure on a complete separable metric space); for every automorphism \(T\) on \(X\), one defines the Koopman operator related to \(T\) by the law
\[ U_T : L^2_m(X) \to L^2_m(X), \quad (U_T f)(x) := f(Tx), \quad x \in X. \]

We say that \(\sigma(U_T)\) is purely absolutely continuous on \(\partial D \setminus \{1\}\) if \(U_T\) is purely absolutely continuous on
\[ \{f \in L^2_m(X) \mid \int_X f(x) \, dm(x) = 0\}; \]
observe that 1 is always an eigenvalue of the Koopman operator, since \(U_T \varphi = \varphi\) for every constant function \(\varphi\).

**Theorem 6.** Let \(T\) be an automorphism on \(X\) with purely absolutely continuous spectrum on \(\partial D \setminus \{1\}\) such that 1 is an accumulation point of \(\sigma(U_T)\). Then, the set of vectors \(\psi \in L^2_m(X)\) so that, for all \(\epsilon > 0\),
\[ \limsup_{K \to \infty} K^{\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U_T^j \psi - \psi^* \right\|^2 = \infty \]

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and

$$\liminf_{K \to \infty} K^{2-\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U_{j}^{*} \psi - \psi^{*} \right\|^{2} = 0,$$

is a dense $G_{\delta}$ set in $L_{m}^{2}(X)$.

**Remark 7.**

i) The result stated in Theorem 6 applies to $K$-automorphisms, in particular to Bernoulli shifts [23].

ii) Note that under the assumptions of Theorem 6, it follows from Corollary 1 that for a typical $\psi \in L_{m}^{2}(X)$, $d_{\mu_{\psi}} - \mu(T \psi - \psi^{*}) (1) = 0$ and $d_{\mu_{\psi}} + \mu(T \psi - \psi^{*}) (1) \geq 2$.

The proof of Theorem 6 is a direct consequence of Theorem 5 and of the next result.

**Proposition 7.** Let $U$ be a unitary operator on $H$ with absolutely continuous spectrum in $\partial D \setminus \{1\}$ and $L = U P_{\mu_{\psi}}(\partial D \setminus \{1\})$. Then, $L^{j}$ converges to zero in the weak operator topology as $j \to \infty$.

**Proof.** For each $\phi \in \mathcal{H}$, let

$$\mathcal{H}_{\phi} := \{ P_{\mu_{\psi}}^{U}(g) \mid g \in L_{\mu_{\psi}}^{2}(\partial D) \}$$

be the cyclic subspace generated by $\phi$, and so $P_{\phi}$ be the corresponding orthogonal projection onto $\mathcal{H}_{\phi}$ and $F : \mathcal{H}_{\phi} \to L_{\mu_{\psi}}^{2}(\partial D)$ be the unitary operator given by the law $F(P_{\mu_{\psi}}^{U}(g) \phi) = g$ (note that $L = P_{\mu_{\psi}}^{U}(g \chi_{\partial D \setminus \{1\}})$, with $g(z) = z$).

Fix $\psi, \varphi \in \mathcal{H}$ and let $h \in L_{\mu_{\psi}}^{2}(\partial D)$ be such that $P_{\psi} \varphi = P_{\mu_{\psi}}^{U}(h) \psi$. Let also $\{g_{n}\}$ be a sequence in the space of bounded Borel functions in $\partial D$, which is a dense subspace of $L_{\mu_{\psi}}^{2}(\partial D)$, with $\|g_{n} - h\|_{L_{\mu_{\psi}}^{2}(\partial D)} \to 0$ as $n \to \infty$.

Now, since $P_{\mu_{\psi}}^{U}(\cdot \cap \partial D \setminus \{1\})$ is absolutely continuous, it follows from Radon-Nikodym Theorem that there exists $f \in L^{1}(\partial D \setminus \{1\})$ such that $d\mu_{\psi}^{U}(z) = f(z)dz$. Therefore, $g_{n} f \in L^{1}(\partial D \setminus \{1\})$ for each $n \in \mathbb{N}$, and so, it follows from Riemann-Lebesgue Lemma that for every $n \in \mathbb{N}$,

$$\lim_{j \to \infty} \langle L_{j}^{1} \psi, P_{\mu_{\psi}}^{U}(g_{n}) \psi \rangle = \lim_{j \to \infty} \int_{\partial D \setminus \{1\}} z^{j} g_{n}(z) d\mu_{\psi}^{U}(z) = \lim_{j \to \infty} \int_{\partial D \setminus \{1\}} z^{j} g_{n}(z) f(z) dz = \lim_{j \to \infty} \int_{0}^{1} e^{2i\pi \theta} g_{n}(e^{2i\pi \theta}) f(e^{2i\pi \theta}) d\theta = 0; \quad (7)$$

we have used the fact that the Lebesgue measure on $\partial D$ is the pushforward, with respect to the map $h : [0, 1) \to \partial D$, $h(\theta) = e^{2i\pi \theta}$, of the Lebesgue measure on $[0, 1)$. 

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On the other hand, for each \( j \geq 0 \), one has
\[
\left| \langle L^j \psi, \varphi \rangle - \langle L^j \psi, P^U(g_n)\psi \rangle \right| = \left| \langle P_\psi L^j \psi, \varphi \rangle - \langle L^j \psi, P^U(g_n)\psi \rangle \right|
\]
\[= \left| \langle L^j \psi, P_\psi \varphi \rangle - \langle L^j \psi, P^U(g_n)\psi \rangle \right|
\]
\[= \left| \langle L^j \psi, P^U(h)\psi - P^U(g_n)\psi \rangle \right|
\]
\[\leq \|\psi\| \|P^U(h)\psi - P^U(g_n)\psi\|
\]
\[= \|\psi\| \|h - g_n\|_{L^2_\psi(\partial \mathbb{D})},
\]
and so
\[
\lim_{n \to \infty} \left| \langle L^j \psi, \varphi \rangle - \langle L^j \psi, P^U(g_n)\psi \rangle \right| = 0;
\]
note that this convergence is uniform on \( j \). Therefore, by combining this result with relation (7) and Moore-Osgood Theorem, one gets
\[
\lim_{j \to \infty} \langle L^j \psi, \varphi \rangle = \lim_{j \to \infty} \lim_{n \to \infty} \langle L^j \psi, P^U(g_n)\psi \rangle
\]
\[= \lim_{n \to \infty} \lim_{j \to \infty} \langle L^j \psi, P^U(g_n)\psi \rangle = 0.
\]

\[\Box\]

3 Proofs

3.1 Proof of Theorem 4

We begin with some preparation. In what follows, for each \( K \in \mathbb{N} \), set
\[
S_K := \{e^{i\theta} \mid -\frac{\pi}{K} < \theta \leq \frac{\pi}{K}\}.
\]

The lemma below is the main tool in the proof of Theorem 4. Let \( f : (-\pi, \pi] \to \partial \mathbb{D} \) denote the map \( f(\theta) = e^{i\theta} \) and let \( f^*\mu \) be the pull-back, with respect to \( f \), of the positive Borel measure \( \mu \) on \( \partial \mathbb{D} \).

**Lemma 2** (Lemma 2. in [19]). For every \( K \in \mathbb{N} \), the following inequality holds:
\[
\frac{1}{K^2} \int_{-\pi}^{\pi} \left| \frac{\sin(\theta K/2)}{\sin(\theta/2)} \right|^2 \frac{\mu_{\psi^*}(S_1)}{K^2} \leq \frac{\mu_{\psi^*}(S_1)}{K^2} + \frac{1}{K^2} \sum_{j=1}^{K-1} (2j + 1) \mu_{\psi^*}(S_j)
\]
(8)
Let us proceed to the proof of Theorem 4. It follows from Lemma 1 that, for each $K \in \mathbb{N}$,

$$
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|_2^2 = \frac{1}{K^2} \int_{\mathbb{D}} \frac{z^K - 1}{z - 1}^2 \, d\mu_{\psi^*}(z)
$$

$$
= \frac{1}{K^2} \int_{-\pi}^{\pi} \frac{\sin(\theta K/2)}{\sin(\theta/2)}^2 \, d(f^* \mu_{\psi^*})(\theta)
$$

$$
= \frac{1}{K^2} \int_{-\epsilon}^{\epsilon} \frac{\sin(\theta K/2)}{\sin(\theta/2)}^2 \, d(f^* \mu_{\psi^*})(\theta)
$$

$$
+ \frac{1}{K^2} \int_{(-\pi,-\epsilon)\cup(\epsilon,\pi]} \frac{\sin(\theta K/2)}{\sin(\theta/2)}^2 \, d(f^* \mu_{\psi^*})(\theta)
$$

$$
\geq \frac{1}{K^2} \int_{-\epsilon}^{\epsilon} \frac{\sin(\theta K/2)}{\sin(\theta/2)}^2 \, d(f^* \mu_{\psi^*})(\theta).
$$

Since for every $-\pi/2 \leq \theta \leq \pi/2$, $\frac{1}{2} \leq |\sin(\theta)| \leq |\theta|$, it follows that for each $K \in \mathbb{N}$ and each $0 < \epsilon < 1/K$,

$$
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|_2^2 \geq \frac{(f^* \mu_{\psi^*})(-\epsilon, \epsilon)}{4} = \frac{\mu_{\psi^*}(A_\epsilon)}{4}.
$$

Hence,

$$
\liminf_{K \to \infty} \frac{\ln \left\{ \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|_2^2 \right\} - \ln K}{- \ln K} \leq d^-_{\mu_{\psi^*}} \quad (9)
$$

$$
\limsup_{K \to \infty} \frac{\ln \left\{ \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|_2^2 \right\} - \ln K}{- \ln K} \leq d^+_{\mu_{\psi^*}} \quad (10)
$$

Now, we prove the complementary inequalities in (9) and (10).

**Case** $d^-_{\mu_{\psi^*}} (1)$: If $d^-_{\mu_{\psi^*}} (1) = 0$, the complementary inequality in (9) follows readily. So, assume that $d^- := d^-_{\mu_{\psi^*}} (1) > 0$. It follows from the definition of $d^-$ that for every $0 < \epsilon < d^-$, there exists $K_\epsilon \in \mathbb{N}$ such that for each $K \in \mathbb{N}$ with $K > K_\epsilon$, one has

$$
\mu_{\psi^*}(S_K) \leq K^{\epsilon - d^-}.
$$

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Now, it follows from relation (8) that for each \( K \in \mathbb{N} \),
\[
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = \frac{1}{K^2} \int_{\partial D} \left| \frac{z^{K-1}}{z-1} \right|^2 d\mu_{\psi - \psi^*}(z) \\
= \frac{1}{K^2} \int_{-\pi}^{\pi} \left| \frac{\sin(\theta K/2)}{\sin(\theta/2)} \right|^2 d(f^* \mu_{\psi - \psi^*})(\theta) \\
\leq \frac{\mu_{\psi - \psi^*}(S_1)}{K^2} + \frac{4}{K^2} \sum_{j=1}^{K-1} j \mu_{\psi - \psi^*}(S_j). \tag{12}
\]

Hence, by combining relations (11) and (12), one concludes that there exists \( C = C(\epsilon) > 0 \) such that for each \( K \in \mathbb{N} \) with \( K > K_\epsilon \),
\[
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 \leq \frac{C}{K^2} + \frac{4}{K^2} \sum_{j=1}^{K-1} j \mu_{\psi - \psi^*}(S_j) \\
\leq \frac{C}{K^2} + \frac{4}{K^2} \sum_{j=1}^{K-1} j^{1-(d^- - \epsilon)}. \tag{13}
\]

Let \( L \in \mathbb{N} \). If \( d^- - \epsilon \in (0, 1) \), then (see page 1111 in [19])
\[
\frac{2}{L^2} \sum_{j=1}^{L-1} j^{1-(d^- - \epsilon)} \leq \frac{2}{2-(d^- - \epsilon)} L^{\epsilon-d^-} + \frac{2}{2-(d^- - \epsilon)} L^{-2}.
\]

If \( d^- - \epsilon = 1 \), then
\[
\frac{2}{L^2} \sum_{j=1}^{L-1} j^{1-(d^- - \epsilon)} = 2L^{-1} - 2L^{-2}.
\]

If \( d^- - \epsilon \in (1, 2) \), then (see page 1112 in [19])
\[
\frac{2}{L^2} \sum_{j=1}^{L-1} j^{1-(d^- - \epsilon)} \leq \frac{2}{2-(d^- - \epsilon)} L^{\epsilon-d^-} \\
+ \left(2 - \frac{2}{2-(d^- - \epsilon)} \right) L^{-2} + 2L^{-2-(d^- - \epsilon)}.
\]

Therefore, by combining the discussion above with relation (13), one concludes that there exists \( D = D(\epsilon, d^-) > 0 \) such that for each \( K \in \mathbb{N} \) with \( K > \tilde{K} = \tilde{K}(\epsilon, d^-) \),
\[
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 \leq \frac{C}{K^2} + DK^{\epsilon-d^-}. \tag{14}
\]
Finally, since $0 < d^- - \epsilon < 2$ by hypothesis, it follows from (14) that for each sufficiently large $K$,

$$\left\| \frac{1}{K} \sum_{j=0}^{K-1} U_j^* - \psi^* \right\|^2 \leq \max\{C, D\} K^{d^- - d^+}.$$  

Hence,

$$\liminf_{K \to \infty} \ln \left\| \frac{1}{K} \sum_{j=0}^{K-1} U_j^* - \psi^* \right\|^2 - \ln K \geq d^- - \epsilon = d^-_{\mu_{\bar{\psi} - \psi^*}} - (1 - \epsilon).$$

Since $0 < \epsilon < d^-_{\mu_{\bar{\psi} - \psi^*}} - (1)$ is arbitrary, the complementary inequality in (9) follows.

**Case** $d^+_{\mu_{\bar{\psi} - \psi^*}} - (1)$: Let $\epsilon > 0$. It follows from the definition of $d^+_{\mu_{\bar{\psi} - \psi^*}} - (1)$ that there exists a monotonically increasing sequence $\{K_l\}$ of natural numbers such that for each $l \in \mathbb{N}$,

$$\mu_{\bar{\psi} - \psi^*}(S_{K_l}) \leq K_l^{d^+_{\mu_{\bar{\psi} - \psi^*}}}.$$  

(15)

On the other hand, it also follows from the definition of $d^+_{\mu_{\bar{\psi} - \psi^*}} - (1) \leq 2$ that there exists $K_\epsilon \in \mathbb{N}$ such that for each $K \in \mathbb{N}$ with $K > K_\epsilon$,

$$\frac{1}{K^{2 + \epsilon}} \leq \mu_{\bar{\psi} - \psi^*}(S_K).$$  

(16)

In particular,

$$\liminf_{K \to \infty} K^{2 + \epsilon} \mu_{\bar{\psi} - \psi^*}(S_K) \geq \liminf_{K \to \infty} K^{\epsilon} = \infty.$$  

So, we can extract from $\{K_l\}$ a monotonically increasing subsequence $\{K_{l_m}\}$ such that

$$\max_{1 \leq j \leq K_{l_m}} \{j^{2 + \epsilon} \mu_{\bar{\psi} - \psi^*}(S_j)\} = K_{l_m}^{2 + \epsilon} \mu_{\bar{\psi} - \psi^*}(S_{K_{l_m}}).$$  

(17)

Thus, by combining (8), (15), (16) and (17), it follows that for each sufficiently large $m$,

$$\left\| \frac{1}{K_{l_m}} \sum_{j=0}^{K_{l_m}-1} U_j^* - \psi^* \right\|^2 \leq \frac{\mu_{\bar{\psi} - \psi^*}(S_1)}{K_{l_m}^2} + \frac{4}{K_{l_m}^2} \sum_{j=1}^{K_{l_m}-1} j \mu_{\bar{\psi} - \psi^*}(S_j) \leq \frac{\mu_{\bar{\psi} - \psi^*}(S_1)}{K_{l_m}^2} + \frac{4}{K_{l_m}^2} \sum_{j=1}^{K_{l_m}} j^{-(1 + \epsilon)} j^{2 + \epsilon} \mu_{\bar{\psi} - \psi^*}(S_j) \leq \frac{\mu_{\bar{\psi} - \psi^*}(S_1)}{K_{l_m}^2} + 4K_{l_m}^{\epsilon} \mu_{\bar{\psi} - \psi^*}(S_{K_{l_m}}) \sum_{j=1}^{\infty} j^{-(1 + \epsilon)} \leq C_\epsilon K_{l_m}^{\epsilon} \mu_{\bar{\psi} - \psi^*}(S_{K_{l_m}}) \leq C_\epsilon K_{l_m}^{d^+_{\mu_{\bar{\psi} - \psi^*}}}.  

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with 
$$C_{\epsilon} = \max\{\mu_{\psi^-\psi^*}(S_1), 4 \sum_{j=1}^{\infty} j^{-(1+\frac{3}{2})}\}.$$ Hence,
$$\limsup_{K \to \infty} \frac{\ln \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2}{-\ln K} \geq d^+ - \epsilon = d^+_{\psi^-\psi^*} (1) - \epsilon,$$
and since \(\epsilon > 0\) is arbitrary, the complementary inequality in (10) follows.

### 3.2 Proof of Theorem 5

Again, we begin with some preparation.

**Theorem 8** (Theorem 1 in [3]). Let \(L\) be a bounded linear operator on \(H\) such that \(L^j\) converges to zero in the weak operator topology. Suppose that \(1 \in \sigma(L)\). Then, for each sequence \((a_j)_{j \geq 0}\) of positive numbers satisfying \(\lim_{j \to \infty} a_j = 0\) and for each \(\delta > 0\), there exists \(\eta \in H\) such that \(\|\eta\| < \sup_j \{a_j\} + \delta\) and

\[
\Re \langle L^j \eta, \eta \rangle > a_j, \quad \forall j \geq 0.
\]

**Lemma 3.** Let \(U\) be a unitary operator on \(H\), \(L = UP^U(\partial D \setminus \{1\})\), and suppose that \(L^j\) converges to zero in the weak operator topology. If \(1\) is an accumulation point of \(\sigma(U)\), then for each \(0 < \epsilon < 1\), there exists \(\eta \in H\) with \(\|\eta\| \leq 1\) so that

\[
\liminf_{K \to \infty} K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\|^2 = \infty.
\]

**Proof.** Observe that:

i) \(1 \in \sigma(L)\), since \(1\) is an accumulation point of \(\sigma(U)\).

ii) \(L^j\) converges to zero in the weak operator topology.

iii) By the spectral functional calculus, \(UP^U(\partial D \setminus \{1\}) = P^U(\partial D \setminus \{1\})U\), and so, for \(j \in \mathbb{N}\),

\[
L^j = U^j (1 - P^U(\{1\}))
= U^j - U^j P^U(\{1\}) = U^j - P^U(\{1\}).
\]

Let \(0 < \epsilon' < 1\) be such that \(0 < 2\epsilon' < \epsilon < 1\). By Theorem 8, there exists \(\eta \in H\) with \(\|\eta\| \leq 1\) such that, for each \(j \geq 0\),

\[
\Re \langle L^j \eta, \eta \rangle > \frac{1}{(j+2)^{\epsilon'}}.
\]
thus, for $K \gg 1$, 
\[
\text{Re} \left( \frac{1}{K} \sum_{j=0}^{K-1} L^j \eta, \eta \right) > \frac{1}{K} \sum_{j=0}^{K-1} \frac{1}{(j+2)\epsilon'} \geq \frac{1}{K} \frac{1}{(K+2)\epsilon'} \geq \frac{1}{2K\epsilon'}
\]

. By iii) above, for $K \gg 1$,
\[
\text{Re} \left( \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^*, \eta \right) = \text{Re} \left( \left( \frac{1}{K} \sum_{j=1}^{K-1} L^j \eta, \eta \right) + \left( \frac{1}{K} (\eta - \eta^*), \eta \right) \right) > \frac{1}{2K\epsilon'} = \text{Re} \left( \frac{\eta^*}{K}, \eta \right).
\]

By Cauchy-Schwarz inequality, for $K \gg 1$,
\[
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\|^2 > \frac{1}{2K\epsilon'} - \frac{1}{K} \geq \frac{1}{4K\epsilon'}.
\]

Hence,
\[
\liminf_{K \to \infty} K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\|^2 = \infty,
\]
and we are done. \qed

Let us proceed to the proof of Theorem 5. Let $\epsilon > 0$; since for each $K \in \mathbb{N}$, the map
\[
\mathcal{H} \ni \psi \mapsto K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\| = K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \left\{ I - P^U(\{1\}) \right\} \psi \right\|
\]
is continuous (namely, it consists of finite sums and compositions of continuous functions), it follows that
\[
G^+(\epsilon) := \left\{ \psi \mid \limsup_{K \to \infty} K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\| = \infty \right\}
\]
\[
= \bigcap_{n \geq 1} \left\{ \psi \mid \limsup_{K \to \infty} K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\| > n \right\}
\]
\[
= \bigcap_{n \geq 1} \bigcup_{t \geq 1} \bigcup_{K \geq t} \left\{ \psi \mid K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\| > n \right\}
\]

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is a $G_δ$ set in $\mathcal{H}$. The proof that

$$G^-(\epsilon) := \left\{ \psi \mid \liminf_{K \to \infty} \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\} = 0$$

is also a $G_δ$ set in $\mathcal{H}$ is completely analogous, and so we omit it.

**Claim:** For each $\epsilon > 0$, $G^+(\epsilon)$ and $G^-(\epsilon)$ are dense sets in $\mathcal{H}$.

It follows from **Claim** and Baire Category Theorem that $G := \bigcap_{\epsilon \in \mathbb{Q} \cap (0, \infty)} (G^+(\epsilon) \cap G^-(\epsilon))$ is also a dense $G_δ$ set in $\mathcal{H}$. (Note that if $0 < \epsilon < \delta$, so $G^+(\epsilon) \subset G^+(\delta)$; therefore, is enough take just the intersection at $\mathbb{Q}$). Since for each $\psi \in G$ and each $\epsilon > 0$,

$$\limsup_{K \to \infty} K^\epsilon \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = \infty$$

and

$$\liminf_{K \to \infty} K^{2-\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = 0,$$

this concludes the proof. Therefore, one just has to prove the **Claim**.

**Proof of the Claim.** Fix $\epsilon > 0$.

• $G^+(\epsilon)$ is dense in $\mathcal{H}$. Let $0 \neq \psi \in \mathcal{H} \setminus G^+(\epsilon)$; so, there exists $C_\psi > 0$ such that for sufficiently large $K$,

$$\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\| \leq \frac{C_\psi}{K^{\epsilon/2}} \quad (18)$$

(otherwise, $\psi \in G^+(\epsilon)$). It follows from Lemma 3 that there exists $\eta \in \mathcal{H}$, with $\|\eta\| \leq 1$, such that

$$\liminf_{K \to \infty} K^{\epsilon/3} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\|^2 = \infty. \quad (19)$$

Note that, by Cauchy-Schwarz inequality and triangle inequality,

$$-\text{Re} \left\langle \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \mid \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\rangle \leq \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\| \|(I - P_U\{\{\}\})\eta\| \leq \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|. \quad (20)$$
Finally, for each $m \in \mathbb{N}$, set $\psi_m := \psi + \frac{\eta}{m}$; naturally, $\psi_m \to \psi$ as $m \to \infty$. Moreover, by (18), (19) and (20), one has for each $m \in \mathbb{N}$ and each sufficiently large $K$,

$$K^{\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi_m - \psi^* \right\|^2 = K^{\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2$$

$$+ \frac{2K^{\epsilon}}{m} \Re \left\{ \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \cdot \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\}$$

$$+ \frac{K^{\epsilon}}{m^2} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\|^2$$

$$\geq \frac{2K^{\epsilon}}{m} \Re \left\{ \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \cdot \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\}$$

$$+ \frac{K^{\epsilon}}{m^2} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \eta - \eta^* \right\|^2$$

$$\geq \frac{2K^{\epsilon}}{m} \cdot \frac{C^{\psi}}{K^{\epsilon/2}} + \frac{K^{\epsilon}}{m^2} \cdot \frac{1}{K^{\epsilon/3}}.$$

Thus, for every $m \in \mathbb{N}$,

$$\lim \sup_{K \to \infty} K^{\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi_m - \psi^* \right\|^2 = \infty,$$

from which follows that $G^+(\epsilon)$ is dense in $\mathcal{H}$.  \hfill \Box

- $G^-(\epsilon)$ is dense in $\mathcal{H}$. Let $\psi \in \mathcal{H}$ and set, for each $n \in \mathbb{N}$,

$$A_n := \{1\} \cup \left\{ e^{i\theta} : \theta \in \left( -\pi, -\frac{1}{n} \right] \cup \left( \frac{1}{n}, \pi \right) \right\}$$

and

$$\psi_n := P^U(A_n) \psi.$$  

Note that

$$\psi_n - \psi^*_n = P^U(A_n) \psi - P^U(\{1\})P^U(A_n) \psi$$

$$= P^U(A_n) \psi - P^U(A_n)P^U(\{1\}) \psi = P^U(A_n) (\psi - \psi^*),$$

and set
from which follows that \( \text{supp}(\mu_{\psi_n - \psi^*}) \subset A_n \). Moreover,
\[
\|\psi - \psi_n\|^2 = \int_{-\pi}^{\pi} |1 - \chi_{A_n}(\theta)|^2 d(f^* \mu_{\psi - \psi^*})(\theta) \to 0
\]
as \( n \to \infty \), by dominated convergence; so, \( \psi_n \to \psi \) as \( n \to \infty \).

Now, for each \( n \in \mathbb{N} \),
\[
\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi_n - \psi^* \|_2 \leq \frac{1}{K^2} \left( \sup_{z \in \{e^{it} : \theta \in (-\pi, -\frac{1}{n} \pi) \cup (\frac{1}{n} \pi, \pi)\}} \frac{z^K - 1}{z - 1} \right) \| \psi_n - \psi^* \|_2 \leq \frac{1}{(K \sin(1/2n))^2} \| (I - P^U \{1\}) \psi_n \|_2 \leq \frac{16n^2}{K^2} \| \psi_n \|_2,
\]
where we have used that \( 1/(4n) \leq \sin(1/(2n)). \) Hence, for each \( n \in \mathbb{N} \),
\[
\liminf_{K \to \infty} K^{-2-\epsilon} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi_n - \psi^* \right\|_2 = 0,
\]
from which follows that \( G^{-}(\epsilon) \) is dense in \( \mathcal{H} \).

### 3.3 Proof of Corollary 1

Let \( \psi \in \mathcal{G}(U) \). If \( d_{\mu_{\psi - \psi^*}}^{-1}(1) > 0 \), then it follows from the definition of \( d_{\mu_{\psi - \psi^*}}^{-1}(1) \) that for each \( 0 < \delta < d_{\mu_{\psi - \psi^*}}^{-1}(1) \), there exists \( \epsilon_\delta \) such that, for each \( 0 < \epsilon < \epsilon_\delta \),
\[
\mu_{\psi - \psi^*}^{-1}(A_\epsilon) \leq C_{\psi} d_{\mu_{\psi - \psi^*}}^{-1}(1-\delta) \leq C_{\psi} \min\{d_{\mu_{\psi - \psi^*}}^{-1}(1, 2) \delta \}.
\]

But then, it follows from Theorem 3 i) that
\[
\limsup_{K \to \infty} K \min\{d_{\mu_{\psi - \psi^*}}^{-1}(1, 2) \delta \} \frac{1}{K} \left\| \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|_2 = 0,
\]
which contradicts (5). Therefore, \( d_{\mu_{\psi - \psi^*}}^{-1}(1) = 0 \).
Now, if \( d_{\mu_{\psi - \psi^*}}^{+} \langle 1 \rangle < 2 \), let \( 0 < \delta < 2 - d_{\mu_{\psi - \psi^*}}^{+} \langle 1 \rangle \). Then, it follows from Theorem 4 that for every sufficiently large \( K \),

\[
K^{-d_{\mu_{\psi - \psi^*}}^{+} \langle 1 \rangle} \geq \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2,
\]

and so

\[
\liminf_{K \to \infty} K^{2-\frac{d}{K}} \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 \geq \liminf_{K \to \infty} K^{2-d_{\mu_{\psi - \psi^*}}^{+} \langle 1 \rangle} \geq \liminf_{K \to \infty} K^4 = \infty.
\]

Since this result contradicts (6), one concludes that \( d_{\mu_{\psi - \psi^*}}^{+} \langle 1 \rangle \geq 2 \).

**Appendix A  Appendix**

Here, we present the proofs of Lemma 1 and Theorem 2.

**A.1 Proof of Lemma 1**

For each \( K \in \mathbb{N} \) and each \( z \in \partial \mathbb{D} \setminus \{1\} \), recall that

\[
\sum_{j=0}^{K-1} z^j = \frac{z^K - 1}{z - 1}.
\]

Note that, for each \( \psi \in \mathcal{H} \), \( \mu_{\psi - \psi^*}^{U} \langle 1 \rangle = 0 \). Thus, by the Spectral Theorem, it follows that for each \( \psi \in \mathcal{H} \) and each \( K \in \mathbb{N} \),

\[
\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = \left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j (\psi - \psi^*) \right\|^2 = \left\| \frac{1}{K} \sum_{j=0}^{K-1} \int_{\partial \mathbb{D}} \frac{z^j}{z - 1} dP^U(z)(\psi - \psi^*) \right\|^2 = \left\| \frac{1}{K} \int_{\partial \mathbb{D}} \left| \frac{z^K - 1}{z - 1} dP^U(z)(\psi - \psi^*) \right|^2 = \frac{1}{K^2} \int_{\partial \mathbb{D}} \left| \frac{z^K - 1}{z - 1} \right|^2 d\mu_{\psi - \psi^*}^{U}(z).
\]
A.2 Proof of Theorem 2

Since, for each $\psi \in \mathcal{H}$, $\text{supp}(\mu_{\psi - \psi^*}^U) \subset \sigma(U) \subset \{1\} \cup \{e^{i\theta} \mid \theta \in (-\pi, -\gamma] \cup (\gamma, \pi]\}$ and $\mu_{\psi - \psi^*}^U(\{1\}) = 0$, it follows from Lemma 1 that for each $\psi \in \mathcal{H}$ and each $K \in \mathbb{N}$,

$$\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j \psi - \psi^* \right\|^2 = \frac{1}{K^2} \int_{\partial \mathcal{D}} \left| \frac{z^K - 1}{z - 1} \right|^2 \, d\mu_{\psi - \psi^*}^U(z)$$

$$\leq \frac{1}{K^2} \left( \sup_{z \in \{e^{i\theta} \mid \theta \in (-\pi, -\gamma] \cup (\gamma, \pi]\}} \left| \frac{z^K - 1}{z - 1} \right|^2 \right) \mu_{\psi - \psi^*}^U(\partial \mathcal{D})$$

$$= \frac{1}{K^2} \left( \sup_{\theta \in (-\pi, -\gamma] \cup (\gamma, \pi]} \left| \frac{\sin(K\theta/2)}{\sin(\theta/2)} \right|^2 \right) \|\psi - \psi^*\|^2$$

$$\leq \frac{1}{(K \sin(\gamma/2))^2} \left\| (I - P^U(\{1\})) \psi \right\|^2$$

$$\leq \frac{16}{\gamma^2 K^2} \|\psi\|^2,$$

where we have used that $\gamma/4 \leq \sin(\gamma/2)$ for each $0 < \gamma < \pi$. Hence, for every $K \in \mathbb{N}$,

$$\left\| \frac{1}{K} \sum_{j=0}^{K-1} U^j - P^U(\{1\}) \right\| \leq \frac{4}{\gamma K}.$$

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References

[1] Aloisio, M.: A note on spectrum and quantum dynamics. J. Math. Anal. Appl. 478, 595–603 (2019).

[2] Aloisio, M., Carvalho, S. L. and de Oliveira C. R.: Refined scales of decaying rates of operator semigroups on Hilbert spaces: Typical behavior. Proc. Amer. Math. Soc. 148, 2509–2523 (2020).

[3] Badea C. and Müller V.: On weak orbits of operators. Topology Appl. 156, 1381–1385 (2009).

[4] Barbaroux, J.-M., Combet J.-M. and Montcho, R.: Remarks on the relation between quantum dynamics and fractal spectra. J. Math. Anal. Appl. 213, 698–722 (1997).
[5] Barros V. and Rousseau J. Shortest distance between multiple orbits and generalized fractal dimensions. Ann. Henri. Poincaré 22, 1853–1885 (2021).

[6] Barros, V., Liao, L. and Rousseau, J.: On the shortest distance between orbits and the longest common substring problem. Adv. in Math. 344, 311–339 (2019).

[7] Bary, N. K.: A treatise on trigonometric series, Fizmatgiz, Moscow 1961; English transl., Pergamon Press, Oxford–London–New York–Paris–Frankfurt (1964).

[8] Ben-Artzi, J. and Morisse, B.: Uniform convergence in von Neumann’s ergodic theorem in the absence of a spectral gap. Ergod. Theory & Dyn. Syst. 41, 1601–1611 (2021).

[9] Carvalho, S. L. and Oliveira C. R.: Correlation wonderland theorems. J. of Math. Phys. 57, 063501 (2016).

[10] Carvalho, S. L. and Oliveira C. R.: Refined scales of weak-mixing dynamical systems: typical behaviour. Ergod. Theory & Dyn. Syst. 40, 3296–3309 (2021).

[11] Cornfeld, I. P., Fomin, S. V., Sinai Y. G. and Sossinskii A. B.: Ergodic theory. Springer, (1982).

[12] de Oliveira, C. R.: Intermediate spectral theory and quantum dynamics. Birkhäuser, (2009).

[13] Damanik, D., Fillman J. and Vance, R.: Dynamics of unitary operators. J. Fractal Geom. 1, 391–425 (2014).

[14] Gaposhkin, V. F.: Convergence of series connected with stationary sequences. Izv. Akad. Nauk SSSR Ser. Mat. 39, 1366–1392 (1975).

[15] Holschneider, M.: Fractal wavelet dimensions and localization. Commun. Math. Phys. 160, 457–473 (1994).

[16] Kachurovskii A. G.: The rate of convergence in ergodic theorems. Uspekhi Mat. Nauk 51 (4) (1996), 73–124. English transl. in Russian Math. Surveys 51 (4), 653–703 (1996).

[17] Kachurovskii A. G. and Podvigin I. V.: Estimates of the rate of convergence in the von Neumann and Birkhoff ergodic theorems. Trans. Moscow Math. Soc. 77, 1–53 (2016).

[18] Kachurovskii A. G. and Reshetenko A. V.: On the rate of convergence in von Neumann’s ergodic theorem with continuous time. Mat. Sb. 201:4 (2010), 25–32; English transl. in Sb. Math. 201:4, 493–500 (2010).

[19] Kachurovskii, A. G. and Sedalishchev V. V.: Constants in estimates for the rates of convergence in von Neumann’s and Birkhoff’s ergodic theorems. Matematicheski
Sbornik 202 (2011), 1105–1125.

[20] Kachurovskii, A. G. and Sedalishchev V. V.: On the constants in the estimates of the rate of convergence in von Neumann’s ergodic theorem. Mat. Zametki 87:5 (2010), 756–763. English Transl. in Math. Notes 87:5-6, 720–727 (2010).

[21] Klein, S. Liu, X.-C. and Melo, A.: Uniform convergence rate for Birkhoff means of certain uniquely ergodic toral maps. Ergod. Theory & Dyn. Syst. 41, 3363–3388 (2021).

[22] Podvigin, I. V.: Exponent of Convergence of a Sequence of Ergodic Averages. Mathematical Notes 112, 271-280 (2022).

[23] Rohlin, V.: Lectures on the entropy theory of transformations with invariant measure. Russian Math. Surveys. 22, 1–52 (1967).

[24] Rudin, W.: Functional analysis. McGraw-Hill, (1991).

[25] von Neumann, J.: Proof of the quasi-ergodic hypothesis. Proc. Nat. Acad. Sci. USA. 18, 70–82 (1932).
