SOME EXAMPLES OF 1-CONVEX NON-EMBEDDABLE THREEFOLDS

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Abstract. We construct a family of 1-convex threefolds, with exceptional curve $C$ of type $(0, -2)$, which are not embeddable in $\mathbb{C}^m \times \mathbb{CP}_n$. In order to show that they are not Kähler we exhibit a real 3-dimensional chain $A$ whose boundary is the complex curve $C$.

1. Introduction

In general a 1-convex manifold with 1-dimensional exceptional set is embeddable, that is it can be realized as an embedded subvariety of $\mathbb{C}^m \times \mathbb{CP}_n$, for suitable $m$ and $n$. The only possible exceptions are given by the following theorem:

THEOREM 1.1. (see Theorem 3 in [C]) Let $X$ be a non-embeddable, 1-convex manifold whose exceptional set $C$ has dimension 1. Then $\dim_{\mathbb{C}} X = 3$ and $C$ has an irreducible component which is a rational curve of type $(-1, -1)$, $(0, -2)$ or $(1, -3)$.

As regards the existence of the quoted exceptions, in [C] there is an example of type $(-1, -1)$. In [V1] (p. 242 B) there is an example of type $(0, -2)$, but the argument is dubious (see [C], Remark 4, but see also [V2]). For the case $(1, -3)$ nothing is known.

In fact the first two cases are easier: as it is well known (see [L]), there is a family $\{W_k\}_{k \in \mathbb{N}}^*$ of fiber bundles over $\mathbb{CP}_1$ which give a local model: this means that if $X$ is a 1-convex threefold whose exceptional set is a smooth rational curve $C$ of type $(-1, -1)$ (respectively $(0, -2)$) then there is a neighbourhood of $C$ biholomorphic to a neighbourhood of the null section of $W_1$ (resp. of $W_k$, for a suitable $k \geq 2$) (see Definition 2.1 and Proposition 2.2). Moreover the sequence of the normal bundles associated to $C$ is

$$(0, -2), \ldots, (0, -2), (-1, -1).$$

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Starting from this remark we shall show:

**THEOREM 1.2.** For every integer \( k \geq 1 \) there is a non-embeddable 1-convex threefold \( \tilde{X}_k \) whose exceptional set is a smooth rational curve \( C \) for whose the sequence of normal bundles is

\[
(0, -2), \ldots, (0, -2), (-1, -1).
\]

In particular \( N_{C|\tilde{X}_1} = \mathcal{O}(-1) \oplus \mathcal{O}(1) \) and, for \( k \geq 2 \), \( N_{C|\tilde{X}_k} = \mathcal{O}(0) \oplus \mathcal{O}(-2) \).

For \( k = 1 \) we get the above quoted Coltoiu’s example. We point out that our construction is explicit and elementary; in order to see that \( \tilde{X}_k \) is not Kähler we shall exhibit a real 3-chain \( A \) whose boundary is the exceptional curve \( C \).

2. **The proof of Theorem 1.2**

**Definition 2.1.** Let \( k \geq 1 \) be an integer. The equations:

\[
\begin{align*}
 w &= \frac{1}{z} \\
 y_1 &= z^2 x_1 + z x_2^k \\
 y_2 &= x_2
\end{align*}
\]

in the coordinates \((z, x_1, x_2)\) and \((w, y_1, y_2)\), define a fiber bundle on \( \mathbb{C}P_1 \), with fiber \( \mathbb{C}^2 \), which will be denoted by \( W_k \).

As we just said in the Introduction, these manifolds \( W_k \) are local models for 1-convex threefolds, as the following Proposition states.

**Proposition 2.2.** (see [P], p. 234) In any 1-convex threefold whose exceptional set is a smooth rational curve \( C \) of type \((-1, -1)\) (resp. \((0, -2)\)) there exists a neighbourhood of \( C \) biholomorphic to a neighbourhood of the null section of \( W_1 \) (resp. of \( W_k \), for a suitable \( k \geq 2 \)).

There is a geometrical description of these threefolds:

**Proposition 2.3.** Let \( k \geq 1 \) be an integer. Let \( N_k \xrightarrow{f_k} \mathbb{C}^4 \) be the blow-up with center at the complex smooth surface

\[
S_k := \{ z \in \mathbb{C}^4; z_1 - iz_2 = z_3 - z_4^k = 0 \}.
\]

Then \( W_k \) is the strict transform of the hypersurface

\[
Y_k := \{ z \in \mathbb{C}^4; z_1^2 + z_2^2 + z_3^2 - z_4^{2k} = 0 \}
\]

whose only one singular point is the origin \( P_k = 0 \); the null section of \( W_k \) is \( C_k = f_k^{-1}(P_k) \).
Proof. It is enough to follow the outline of [P], Example 2.14. □

Now we shall investigate in more detail this geometric construction and build a suitable commutative diagram

\[
\begin{array}{cccccccc}
N_0 & h_1 & N_1 & h_2 & \ldots & h_{k-1} & N_{k-1} & h_k & N_k \\
\downarrow f_0 & \downarrow f_1 & \downarrow f_2 & \downarrow f_3 & \downarrow f_{k-1} & \downarrow f_k \\
M_0 & g_1 & M_1 & g_2 & \ldots & g_{k-1} & M_{k-1} & g_k & M_k = \mathbb{C}^4
\end{array}
\]

Step 1. Applying the desingularization process to the hypersurface \(Y_k \subset \mathbb{C}^4 := M_k\) we the sequence

\[
M_0 \xrightarrow{g_1} M_1 \xrightarrow{g_2} \ldots \xrightarrow{g_{k-1}} M_{k-1} \xrightarrow{g_k} M_k = \mathbb{C}^4
\]

More precisely this sequence is defined by induction: \(M_{k-1} \xrightarrow{g_k} M_k = \mathbb{C}^4\) is the blow-up with center \(P_k := 0\). Then define the chart \((U_{k-1}; u_1, \ldots, u_4)\) of \(M_{k-1}\) saying that in these coordinates the map \(g_k\) has the following equations:

\[
\begin{cases}
z_j = u_j u_4, & \text{for } j = 1, 2, 3 \\
z_4 = u_4
\end{cases}
\]

Denoting by \(Y_{k-1} \subset M_{k-1}\) the strict transform of \(Y_k\), we get that the only singular point of \(Y_{k-1}\) is \(P_{k-1} := 0 \in U_{k-1}\) and

\[
Y_{k-1} \cap U_{k-1} = \{u_1^2 + u_2^2 + u_3^2 - u_4^{2(k-1)} = 0\}.
\]

Comparing (2.6) with (2.3), we see that we can iterate the process: the map \(M_{j-1} \xrightarrow{g_j} M_j\) is the blow-up of center \(P_j\) and \(Y_{j-1}\) is the strict transform of \(Y_j\). Finally, since

\[
Y_0 \cap U_0 = \{u_1^2 + u_2^2 + u_3^2 - 1 = 0\}
\]

\(Y_0\) is smooth, so that the process ends.

We need the following lemma:

**Lemma 2.4.** Let \(S\) be a smooth complex surface in a complex 4-fold \(M\) and let \(P \in S\). There is the following commutative diagram:

\[
\begin{array}{cccc}
N' & \xrightarrow{h} & N \\
\downarrow f' & & \downarrow f \\
M' & \xrightarrow{g} & M
\end{array}
\]

where: \(g\) is the blow-up with center \(P\), \(S'\) is the strict transform of \(S\) in \(M'\), \(f\) (resp. \(f'\)) is the blow-up of center \(S\) (resp. \(S'\)), \(h\) is the blow-up with center the curve \(C := f^{-1}(P)\).
Proof. The problem is local near $P$, thus we may assume that $M = \mathbb{C}^4$ and that $S$ is a plane. Choosing $S = \mathbb{C}^2 \times \{0\}$ the direct computation is easier.

Now, recalling Proposition 2.3 we finish our construction:

**Step 2.** Define $S_{j-1}$ as the strict transform of $S_j$ by means of the map $M_{j-1} \xrightarrow{g_j} M_j$, $1 \leq j \leq k$. Let $N_j \xrightarrow{f_j} M_j$ be the blow-up of center $S_j$, $0 \leq j \leq k - 1$. Moreover let $C_j := f_j^{-1}(P_j)$ and $N_{j-1} \xrightarrow{h_j} N_j$ be the blow-up with center $C_j$, $1 \leq j \leq k$.

Then the diagram (2.4) is commutative.

Proof. By means of Lemma 2.4 it is enough to check that $P_j \in S_j$, $j = 0, \ldots, k$. But, as noted above, $P_j \in U_j$ and using the chart $U_j$, it is straightforward to check that

\[(2.7) \quad S_j \cap U_j = \{u_1 - iu_2 = u_3 - u_4\}. \square \]

**COROLLARY 2.5.** Let $X_j$ be the strict transform of $Y_j$ by means of the map $N_j \xrightarrow{f_j} M_j$, $0 \leq j \leq k$. Considering restrictions of maps, we get, from (2.4) the following commutative diagram:

\[
\begin{array}{ccccccc}
X_0 & \xrightarrow{h_0} & X_1 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_{k-1}} & X_{k-1} & \xrightarrow{h_k} & X_k = W_k \\
\downarrow f_0 & & \downarrow f_1 & & \cdots & & \downarrow f_{k-1} & & \downarrow f_k \\
Y_0 & \xrightarrow{g_1} & Y_1 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{k-1}} & Y_{k-1} & \xrightarrow{g_k} & Y_k
\end{array}
\]

Moreover:

(i) $X_j$ is smooth, the rational curve $C_j$ (which is the center of center of $h_j$) is contained in $X_j$ and there is a neighbourhood of $C_j$ in $X_j$ biholomorphic to a neighbourhood of the null section of $W_j$, $1 \leq j \leq k$;

(ii) $C_j = h_j(C_{j-1})$, $j \geq 2$;

(iii) $X_0 \xrightarrow{f_0} Y_0$ is a biholomorphism;

(iv) the exceptional divisor $E_0$ of $h_1$ is biholomorphic to $\mathbb{CP}_1 \times \mathbb{CP}_1$ and the induced map $E_0 \sim \mathbb{CP}_1 \times \mathbb{CP}_1 \xrightarrow{h_1} C_1 \sim \mathbb{CP}_1$ is one of the two canonical projections.

Proof. The diagram (2.4) is well defined, because diagram (2.4) is commutative. Comparing equations (2.3) and (2.6), (2.2) and (2.7) we may apply Proposition 2.3 for $j = 1, \ldots, k$. Thus we get $W_k = X_k$ and, for $1 \leq j \leq k - 1$, $W_j = X_j \cap f_j^{-1}(U_j)$ and $C_j$ is its null section. By Proposition 2.2 $N_{C_j|X_1} = (-1, -1)$, while $N_{C_j|X_j} = (0, -2)$, for $j \geq 2$. Thus the exceptional divisor $E_{j-1}$ of $h_j$ is a rational ruled surface: $E_0 = F_0$ (this proves (iv)) and $E_j = F_2$, for $j \geq 1$. Now the curve $C_j$ is not a
fiber of $F_2$, otherwise $N_{C_j|X_j} = (0, -1)$, thus $C_j$ is a section of $E_j \simeq F_2$; this shows (ii). Finally, since $S_0$ and $Y_0$ are smooth, $X_0 \xrightarrow{f_0} Y_0$ is a biholomorphism. □

Remark 2.6. In the rational ruled surface $F_2$ there is only one curve $C$ with negative self-intersection: $C.F_2C = -2$. Since $C_j$ is not a fiber of $E_j$, from the exact sequence

\[ 0 \to \mathcal{O}(C_j.E_jC_j) \to N_{C_j|X_j} \to \mathcal{O}(C_j.X_jE_j) \to 0 \]

it follows easily that $C_j$ is the curve of $E_j$ with negative self-intersection; this means that the sequence $X_0 \to \cdots \to X_k$ is the sequence of the blow-ups associated to the curve $C_k$.

Let us state the following elementary result

**Lemma 2.7.** Let $Q := \{ z \in \mathbb{CP}_3; z_0^2 + z_1^2 + z_2^2 - z_3^2 = 0 \}$. Every line of $Q$ has a real point.

**Proof.** Let $r \subset Q$ be a line and let $P \in r$. If $P$ is not real, consider the line $s$ passing through $P$ and $P$. Now $s$ is a real line and $s \cap \mathbb{R}^4$ is external to the real sphere $Q \cap \mathbb{R}^4$, therefore there are exactly two planes passing through $s$ tangent to $Q$ and the tangent points are real. One of these planes must be the plane $\alpha$ defined by the lines $r$ and $s$ (in fact $\alpha \cap Q$ contains $r$ and thus is a degenerate conic), therefore is tangent to $Q$ in a real point $Q$ which must belong to $r$. □

By means of the detailed description of the map $X_k \xrightarrow{f_k} Y_k$ given in (2.8), the following statement is a simple corollary.

**Corollary 2.8.** Let $B := \{ z \in Y_k \cap \mathbb{R}^4; z_4 > 0 \}$ and $A := f_k^{-1}(B)$. Then $A$ is a real threefold with boundary $\partial A = C_k$.

**Proof.** Let $D := (g_k \circ \cdots \circ g_1)^{-1}(B)$. Since the diagram (2.8) is commutative, $A = h_k \circ \cdots \circ h_1(f_0^{-1}(D))$. From (2.5) it follows that $g_k^{-1}(B) \subset U_{k-1}$, and iterating this argument we get that $D \subset U_0$, more precisely

\[ D = \{ x \in \mathbb{R}^4; x_1^2 + x_2^2 + x_3^2 - 1 = 0, \ x_4 > 0 \}. \]

Therefore the boundary $\partial D = \{ x \in \mathbb{R}^4; x_1^2 + x_2^2 + x_3^2 - 1 = x_4 = 0 \}$ is contained in $\{ z \in \mathbb{C}^4; z_1 + z_2^2 + z_3^2 - 1 = x_4 = 0 \} = E_0 \cap U_0$. By Corollary (2.5)iv the fibers of the map $E_0 \simeq f_0^{-1}(E_0) \xrightarrow{h_1} C_1$ are given by one of the two family of lines of the quadric $E_0$. By Lemma 2.7, each
of these lines intersect \( \partial D \), therefore \( h_1(f_0^{-1}(\partial D)) = C_1 \). Thus from Corollary 2.5(ii) \( \partial A = C_k \). □

In order to obtain our example \( \tilde{X}_k \) we must perturb \( Y_k \) outside the origin.

**LEMMA 2.9.** For every fixed integer \( k \geq 1 \) there exist an integer \( N > k \) and \( 0 < \varepsilon \leq 1 \) such that the origin is the only singular point of the hypersurface

\[
\tilde{Y}_k := \{ z_1^2 + z_2^2 + z_3^2 - z_4^{2k} + \varepsilon(z_1^{2N} + z_2^{2N} + z_3^{2N} + z_4^{2N}) = 0 \}.
\]

The equations \( w_j := z_j(1 + \varepsilon z_j^{2N})^{1/2}, 1 \leq j \leq 3 \) and \( w_4 = z_4(1 - \varepsilon z_4^{2N})^{1/2k} \) define a biholomorphic map \( V \xrightarrow{\Phi} \tilde{V} \) between two neighbourhood of the origin of \( \mathbb{C}^4 \). Thus we can define \( \tilde{X}_k \) gluing \( f_k^{-1}(V) \cap X_k \) and \( \tilde{Y}_k \setminus \{0\} \) by means of \( \Phi \). The maps \( \tilde{X}_{j-1} \to \tilde{X}_j \) and \( \tilde{Y}_{j-1} \to \tilde{Y}_j \) are defined as above since nothing is changed near \( P_j \) and \( C_j \), while the maps \( \tilde{X}_j \to \tilde{Y}_j \) are defined by a gluing process (these maps are not blow-ups).

**PROPOSITION 2.10.** \( \tilde{X}_k \) is not embeddable.

*Proof.* Let \( \tilde{B} := \mathbb{R}^4 \cap \tilde{Y}_k \cap \{ x_4 > 0 \} \). From (2.9) it follows that \( \tilde{B} \) is relatively compact. From the definition of \( \Phi \) it follows that \( \tilde{B} \cap \tilde{V} = \Phi^{-1}(B \cap V) \); thus the preimage \( \tilde{A} \) of \( \tilde{B} \) is again a 3-chain with boundary \( C_k \). Hence the exceptional curve \( C_k = \partial \tilde{A} \) is a boundary in \( \tilde{X}_k \), which is not Kähler. □

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