Improved Bounds for Distributed Load Balancing

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Abstract

In the load balancing problem, the input is an n-vertex bipartite graph \( G = (C \cup S, E) \)—where the two sides of the bipartite graph are referred to as the clients and the servers—and a positive weight for each client \( c \in C \). The algorithm must assign each client \( c \in C \) to an adjacent server \( s \in S \). The load of a server is then the weighted sum of all the clients assigned to it. The goal is to compute an assignment that minimizes some function of the server loads, typically either the maximum server load (i.e., the \( \ell_{\infty} \)-norm) or the \( \ell_p \)-norm of the server loads. This problem has a variety of applications and has been widely studied under several different names, including: scheduling with restricted assignment, semi-matching, and distributed backup placement.

We study load balancing in the distributed setting. There are two existing results in the CONGEST model. Czygrinow et al. [DISC 2012] showed a 2-approximation for unweighted clients with round-complexity \( O(\Delta^3) \), where \( \Delta \) is the maximum degree of the input graph. Halldórsson et al. [SPAA 2015] showed an \( O(\log n / \log \log n) \)-approximation for unweighted clients and \( O(\log^2 n / \log \log n) \)-approximation for weighted clients with round-complexity \( \text{polylog}(n) \).

In this paper, we show the first distributed algorithms to compute an \( O(1) \)-approximation to the load balancing problem in \( \text{polylog}(n) \) rounds:

- In the CONGEST model, we give an \( O(1) \)-approximation algorithm in \( \text{polylog}(n) \) rounds for unweighted clients. For weighted clients, the approximation ratio is \( O(\log n) \).
- In the less constrained LOCAL model, we give an \( O(1) \)-approximation algorithm for weighted clients in \( \text{polylog}(n) \) rounds.

Our approach also has implications for the standard sequential setting in which we obtain the first \( O(1) \)-approximation for this problem that runs in near-linear time. A 2-approximation is already known, but it requires solving a linear program and is hence much slower. Finally, we note that all of our results simultaneously approximate all \( \ell_p \)-norms, including the \( \ell_{\infty} \)-norm.

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1 Introduction

In this paper, we study the load balancing problem. The input is a bipartite graph $G = (C \cup S, E)$, where we refer to the sets $C$ and $S$ the clients and servers, respectively. The goal is to find an assignment of clients to servers such that no server is assigned too many clients. To be more precise, we define the load of a server in an assignment to be the number of clients assigned to it, and we are interested in finding an assignment that minimizes the maximum load of any server (or minimizes the $\ell_p$-norm of the server loads—we will discuss this objective more later in the introduction).

The load balancing problem has a rich history in the scheduling literature as the job scheduling with restricted assignment problem \cite{5, 12, 13, 18}, in the distributed computing literature as the backup placement problem \cite{2, 11, 21}, in sensor networks \cite{20, 22} and peer-to-peer systems \cite{17, 24, 25} as the load-balanced data gathering tree construction problem, and more generally as a relaxation of the bipartite matching problem known as the semi-matching problem \cite{7, 8, 12, 16}. We refer the reader to \cite{8, 11, 12} for more background.

Our primary focus in this paper is on the load balancing problem in a distributed setting, where clients and servers correspond to separate nodes in a network. Communication through the network happens in synchronous rounds, where in each round, every node can send $O(\log n)$ bits to its neighbors over any of its incident edges (formally, we work in the CONGEST model—see Section 2 for more details). In the distributed setting, the load balancing problem generalizes the distributed backup placement problem with replication factor one (introduced in \cite{11}), where the nodes (corresponding to clients) in a distributed network may have memory faults and therefore wish to store backup copies of their data at neighboring nodes (corresponding to servers). Since backup-nodes may incur faults as well, the number of nodes that select the same backup-node should be minimized. See Appendix A for the exact formulation of the distributed backup placement problem and for how some of our results extend to the more general version of the problem with arbitrary replication factor.

A simple distributed algorithm for the load balancing problem in which the nodes myopically re-assign themselves to a server with smaller load eventually converges to an $O(\log n \log \log n)$-approximation, where $n$ is the number of nodes in the network \cite{9, 15}, but as was shown in Halldórsson et al. \cite{11}, the algorithm requires $\Omega(\sqrt{n})$ rounds. The same paper \cite{11} shows a way of circumventing this costly process and gives a distributed algorithm that achieves the same $O(\log n / \log \log n)$-approximation in only $\text{polylog}(n)$ rounds. On the other hand, Czygrinow et al. \cite{7} show a distributed $O(1)$-approximation (precisely, a 2-approximation) that requires $O(\Delta^3)$ rounds, where $\Delta$ is the maximum degree of a node in the network. This algorithm is highly efficient for low-degree networks but is again too expensive for high-degree graphs.

This state-of-affairs is the starting point of our work: Can we obtain the best of both worlds, namely, an $O(1)$-approximation algorithm in polylog($n$) rounds?

Our first contribution. Our first main contribution in this paper is an affirmative answer to this question.

**Result 1** (Formalized in Theorem 1). We give an $O(1)$-approximate randomized distributed algorithm for load balancing in the CONGEST model that runs in $O(\log^5 n)$ rounds.

At the core of our algorithm is a new structural lemma for the load balancing problem. Informally speaking, we show that eliminating all “short augmenting paths” of length $O(\log n)$ is sufficient to assign all clients to servers with load a constant factor as much as the optimum (Lemma 3.1). In conjunction with ideas from \cite{11}, this effectively reduces the load balancing
problem to that of finding a matching with no short augmenting paths, which can be solved using the by-now standard algorithm of Lotker et al. [19].

**Our second contribution.** Next, we consider the *weighted* load balancing problem in which every client comes with a weight. The load of a server is then the total weight of the clients assigned to it. The goal, as before, is to minimize the maximum load of any server. Halldórsson et al. [11] also studied the weighted problem and gave an $O\left(\frac{\log^2 n}{\log \log n}\right)$-approximation in polylog($n$) rounds using a simple reduction to the unweighted case.

Using the same weighted-to-unweighted reduction, our algorithm in Result 1 also implies an $O(\log n)$-approximation for the weighted load balancing problem in polylog($n$) rounds of the CONGEST model. Our main technical contribution in this paper is a new algorithm for this problem that achieves an $O(1)$-approximation in the less constrained LOCAL model, in which communication over edges in each round is unbounded.

**Result 2** (Formalized in Theorem 4). *We give an $O(1)$-approximate randomized distributed algorithm for weighted load balancing in $\log^3 n$ rounds of the LOCAL model.*

Our LOCAL algorithm consists of two main components: a distributed algorithm for (approximately) solving a relaxed version of the problem where each client $c$ with weight $w(c)$ should be assigned to $w(c)$ adjacent servers with multiplicity—a *split assignment*—and a novel distributed rounding procedure. Using our structural result in Lemma 3.1, we can find a split assignment by approximately solving (or rather, eliminating short augmenting paths in) a generalized $b$-matching problem with edge capacities. We are not aware of any efficient algorithm for this problem in the CONGEST model, but we can show that a simple extension of the work of [19] can solve this problem in polylog($n$) rounds in the LOCAL model. The rounding step is also based on a new application of our Lemma 3.1 that allows us to circumvent the typical use of “cycle canceling” procedures for rounding fractional matching LP solutions into integral ones, which do not translate to efficient distributed algorithms.

We now turn to two important extensions of Results 1 and 2. The first is the more general problem of all-norm load balancing, and the second is a fast sequential algorithm.

**Approximating all norms.** Recall that our goal in the load balancing problem has been to minimize the maximum load of any server. Assuming we denote the loads of servers under some assignment $A$ by a vector $L_A := [L_A(s_1), L_A(s_2), \ldots, L_A(s_n)]$ for all $s_i \in S$, minimizing the maximum server load is equivalent to minimizing $\|L_A\|_\infty$, i.e., the $\ell_\infty$-norm of $L_A$. Depending on the application, however, minimizing this norm may not be the most natural notion of a “balanced” assignment; if some server requires vastly more load than the other servers, an $\ell_\infty$-norm-minimizing assignment may put needlessly large load on those other servers.

As a result, it is natural to consider minimizing some other $\ell_p$-norm of $L_A$ for some $p \geq 1$. This is done, for instance, in [8, 12, 16], which considered $\ell_2$-norms. An even more general objective is the *all-norm* problem, studied in [1, 4, 6, 12], where the goal is to *simultaneously* optimize with respect to every $\ell_p$-norm. These results compute an assignment which is an $O(1)$-approximation (or even optimal) simultaneously with respect to all $\ell_p$-norms, including $p = \infty$ (*a priori*, even the existence of such an assignment is not clear).

All of our results extend to the all-norm problem without any increase in approximation factor or round-complexity. In particular, in the CONGEST model, we give randomized distributed $O(1)$- and $O(\log n)$-approximation algorithms for all-norm load balancing in polylog($n$) rounds, in the
unweighted and weighted variant of the problem, respectively (Theorem 2). In the LOCAL model, the approximation ratio for the weighted problem can be reduced to \(O(1)\) as well (Theorem 4).

**Faster sequential algorithms.** Finally, we show that our new approach to weighted load balancing can also be used to design a near-linear time algorithm for this problem in the sequential setting. We give a deterministic \(O(m \log^3(n))\) time algorithm for the \(O(1)\)-approximate all-norm load balancing problem in the sequential setting (Theorem 5).

Previously, a deterministic \(O(m \sqrt{n} \log n)\) time for the exact problem in case of unweighted graphs was given in [8]. The weighted variant of the problem is NP-hard [1]; 2-approximate algorithms were shown in [1] and [6], but they are based on solving, respectively, the linear and convex programming relaxations of the problem exactly using the ellipsoid algorithm, and thus are much slower than the algorithm we present.

## 2 Preliminaries

**Notation.** For any function \(f : A \to \mathbb{N}\) and \(B \subseteq A\), we use the notation \(f(B) = \sum_{b \in B} f(b)\) to sum \(f\) over all elements in \(B\). For any integer \(t \geq 1\), we denote \([t] := \{1, \ldots, t\}\).

Throughout, we assume \(G = (C \cup S, E)\) is a bipartite graph. We refer to \(C\) and \(S\) as the clients and the servers, respectively. We let \(uv\) denote the edge between vertices \(u\) and \(v\) and let \(\delta(v)\) denote the set of edges incident to a vertex \(v\). We use \(n\) as number of vertices in \(G\) and \(m\) as the number of edges in \(G\).

**Load balancing.** In the load balancing problem, the input is a bipartite graph \(G = (C \cup S, E)\) together with a client weight function \(w : C \to [W]\). The output is an assignment \(A : C \to S\) mapping every client to one of its adjacent servers. The load \(L_A(s)\) of a server \(s \in S\) under assignment \(A\) is the sum of the weights of the clients assigned to it: \(L_A(s) = w(A^{-1}(s))\). The maximum load of an assignment \(A\) is the maximum load of any server under \(A\). We refer to the problem of computing an assignment of minimum load as the (weighted) min-max load balancing problem.

As mentioned in the introduction, the min-max objective can be generalized by considering any \(\ell_p\)-norm of \(L_A\), defined as \(\|L_A\|_p = (\sum_{s \in S} (L_A(s))^p)^{1/p}\). For brevity, we also use the notation \(\|A\|_p := \|L_A\|_p\). In the language of norms, the min-max objective corresponds to minimizing the load vector’s \(\ell_\infty\)-norm. When the goal is to find an assignment \(A\) that simultaneously minimizes \(\|A\|_p\) for all \(p \geq 1\), including \(p = \infty\), the problem is called the (weighted) all-norm load balancing problem. Prior results in [1,4,6,12] show the existence of an assignment that can (approximately) minimize all these norms simultaneously. In particular, we use the following result due to Harvey et al. [12] in our proofs (see also [4]).

**Lemma 2.1** ([12]). Given any instance of the unweighted load balancing problem, there exists an assignment \(A^*\) that simultaneously minimizes \(\|A^*\|_p\) for all \(p \geq 1\), including \(p = \infty\).

**b-matchings.** In addition to assignments, we will also work with \(b\)-matchings. For a vertex capacity function \(b : V \to \mathbb{Z}^+\), a \(b\)-matching is an assignment \(x : E \to \mathbb{Z}^+\) of integer multiplicities to edges so that for every vertex \(v\), the sum \(x(\delta(v))\) of the multiplicities of the edges incident to \(v\) does not exceed \(b(v)\).

Since we will focus solely on the case when \(G\) is bipartite and \(V = C \cup S\), it will be convenient to split \(b\) into two separate capacity functions, one for the clients and one for the servers. We use \(\kappa : C \to \mathbb{Z}^+\) to denote the client capacities and \(\tau : S \to \mathbb{Z}^+\) to denote the server capacities. A
\((\kappa, \tau)-\text{matching}\) is then a function \(x : E \rightarrow \mathbb{Z}^+\) assigning multiplicities to edges such that
\[
\sum_{s \in N(c)} x(cs) \leq \kappa(c) \tag{1}
\]
for every client \(c\) and
\[
\sum_{c \in N(s)} x(cs) \leq \tau(s)
\]
for every server \(s\). A \((\kappa, \tau)\)-matching is client-perfect if (1) holds with equality for all \(c \in C\). We say that a server \(s\) (resp. client \(c\)) is \(x\)-saturated if \(x(\delta(s)) = \tau(s)\) (resp. \(x(\delta(c)) = \kappa(c)\)). If a vertex is not \(x\)-saturated, then it is \(x\)-unsaturated. An \(x\)-augmenting path is a path \(v_1, \ldots, v_{2k+1}\) such that \(v_1\) and \(v_{2k+1}\) are \(x\)-unsaturated and \(x(v_{2i+1}v_{2i+2}) > 0\) for all \(0 \leq i < k\).

We will make repeated use of the following simple remark.

**Remark 2.2.** When all client weights are one (the unweighted case), a client-perfect \((1, \tau)\)-matching induces an assignment of maximum load at most \(\max_{s \in S} \tau(s)\), and vice versa.

Note that the remark does not generalize to weighted clients; under a \((w, \tau)\)-matching, a client may be split across multiple servers, which does not correspond to a proper assignment.

**The LOCAL and CONGEST models.** In both the LOCAL and the CONGEST models of distributed computation, each vertex of the input graph hosts a processor that initially only knows its neighbors and its weight. Following a standard assumption, we assume that all vertices know \(n\) and the maximum weight \(W\). Computation proceeds in synchronous rounds; in each round, vertices may send messages to their neighbors and then receive messages from their neighbors in lockstep. Local computation is free—we are only interested in the round complexity, the number of rounds required by the algorithm.

The LOCAL and CONGEST models differ in that in the LOCAL model, vertices can send and receive arbitrarily large messages to and from their neighbors, while in the CONGEST model, the communication between adjacent vertices in each round is capped at \(O(\log n)\).

### 3 A Structural Lemma

A crucial component of our results is a structural observation about approximate \((\kappa, \tau)\)-matchings in the context of the load balancing problem, which is inspired by results from online load balancing [3, 10]: if a graph contains some client-perfect \((\kappa, \tau)\)-matching, then every \((\k, 2\tau)\)-matching is either client-perfect or can be augmented via an augmenting path of logarithmic length. Formally, and more generally, we have the following lemma.

**Lemma 3.1.** If \(G\) contains a client-perfect \((\kappa, \tau)\)-matching and \(x\) is a \((\kappa, \alpha\tau)\)-matching for \(\alpha > 1\), then either \(x\) is client-perfect or there is an \(x\)-augmenting path of length at most \(2[\log_\alpha \tau(S)] + 1\).

**Proof.** Suppose \(G\) contains a client-perfect \((\kappa, \tau)\)-matching \(x^*\). To simplify the discussion, we define a directed multigraph \(D\) on \(V(G)\) whose arcs are oriented edges in the support of \(x\) and \(x^*\) as follows. For every \(cs \in E(G)\) with \(c \in C\) and \(s \in S\), \(D\) has \(x(cs)\) copies of the arc \((s, c)\) and \(x^*(cs)\) copies of the arc \((c, s)\) and no other arcs. Notice that every directed path in \(D\) starting at an \(x\)-unsaturated client and ending at an \(x\)-unsaturated server corresponds to an \(x\)-augmenting path in \(G\).

Suppose \(x\) is not client-perfect and let \(c \in C\) be an \(x\)-unsaturated client. Let \(k \in \mathbb{N}\) be fixed and define \(U_k\) to be the set of vertices reachable via a walk of length \(k\) from \(c\) in \(D\). Call \(U_k\) full if \(u\) is \(x\)-saturated for all \(u \in U_k\).
The lemma follows from two simple claims:

1. if $U_{2k+1}$ is not full, then $G$ contains an $x$-augmenting path of length at most $2k + 1$; and
2. if $U_{2k+1}$ is full, then $\tau(U_{2k+3}) \geq \alpha \tau(U_{2k+1})$.

The first claim follows from the fact that a directed walk contains a directed path with the same endpoints and from the correspondence noted earlier between directed paths with unsaturated endpoints in $D$ and augmenting paths in $G$.

We proceed to the second claim. If $s \in U_{2k+1}$ and $U_{2k+1}$ is full, then $s$ is $x$-saturated and the out-degree of $s$ is $\alpha \tau(s)$. Thus, the total out-degree of $U_{2k+1}$—and also the total in-degree of $U_{2k+2}$—is $\alpha \tau(U_{2k+1})$. Now we use the fact that the out-degree of a client $c \in U_{2k+2}$ is at least as large as its in-degree. This follows simply from the fact that the in-degree must be at most $\kappa(c)$, and since $x^*$ is client-perfect, the out-degree is exactly $\kappa(c)$. Following the arcs once more, the total in-degree of $U_{2k+3}$ is at least $\alpha \tau(U_{2k+1})$. Finally, since the in-degree of $U_{2k+3}$ is also point-wise less than $\tau$, we have $\alpha \tau(U_{2k+1}) \leq \tau(U_{2k+3})$.

Now we show how the two claims together imply the lemma. If any $U_{2i+1}$ for $i \leq \lceil \log_\alpha(\tau(S)) \rceil$ is not full, we are done by the first claim. Otherwise, the sums of capacities grow exponentially starting with $\tau(U_1) \geq 1$. Inductively, $|U_{2i+1}| \geq \alpha^i$ for all $i \in \mathbb{N}$. For $k = \lceil \log_\alpha \tau(S) \rceil$, therefore, we have $\tau(U_{2k+3}) \geq \alpha \tau(S)$, a contradiction. Thus, not all $\{U_{2i+1}\}$ are full.

4 Unweighted Load Balancing

Assuming an algorithm to eliminate augmenting paths up to a certain length efficiently, the structural lemma from the previous section almost immediately implies an algorithm for the unweighted load balancing problem. To obtain an algorithm for eliminating short augmenting paths, we use the following lemma which is implied by Lemma 24 in [11].

**Lemma 4.1** ([11]). There exists an $O(k^3 \log n)$-round randomized algorithm in the CONGEST model that, with high probability, given a graph $G = (C \cup S, E)$, a positive integer $k$, and server capacity function $\tau$, computes a $(1, \tau)$-matching with no augmenting paths of length less than $k$.

The proof of Lemma 4.1 combines two existing results. The algorithm of Lotker et al. [19] computes a $(1,1)$-matching with no augmenting paths of length $\leq k$ in $O(k^3 \log n)$ rounds. Halldórsson et al. [11] then show a black-box extension from $(1,1)$-matching to $(1, \tau)$-matching which does not increase the round-complexity; see [11] for more details.

**Remark 4.2.** Both our algorithm and the algorithm of [11] use the above lemma as a starting point. But the algorithm of [11] only removes short augmenting paths to ensure that the $(1,\tau)$-matching is approximately optimal. Since a near-optimal matching is still not an assignment (it is not client-perfect), they then use a different set of tools to convert an approximate $(1,\tau)$-matching to an $O(\log n/ \log \log n)$-approximate assignment.

Our analysis, by contrast, directly exploits the non-existence of short augmenting paths via Lemma 3.1. We thus avoid the additional conversion of [11], which leads to a better approximation ratio, as well as a simpler algorithm.

Approximating the $\ell_\infty$-norm (the min-max load balancing problem). Let $B^*$ be the optimum $\ell_\infty$-norm. We will first describe an algorithm that assumes as input some $B \geq 2B^*$. The algorithm begins by using Lemma 4.1 to compute a $(1,B)$-matching $x$ with no augmenting paths of
length \(4\lfloor \log_2 n \rfloor + 1\). The sum of the server capacities is at most \(nB\), and since clients have unitary weight, we can assume \(B \leq n\). Therefore, by Lemma 3.1, since there are no \(x\)-augmenting paths of length \(2\lfloor \log_2 (nB) \rfloor + 1 \leq 4\lfloor \log_2 n \rfloor + 1\), we know that \(x\) is necessarily client-perfect. A client \(c\) can now assign itself to the vertex it is matched to under \(x\).

To remove the assumption that we are given a \(B \geq 2B^*\), we run the algorithm above \(\log n\) times with \(B = 1, 2, 4, \ldots, n\). For every run where \(B \geq 2B^*\), the algorithm will successfully assign every client. Note, however, that in the distributed setting, there is no efficient way for the clients to determine the smallest \(B\) for which the algorithm successfully matched every client. Instead, each client \(c\) locally assigns itself according to the run with smallest \(B\) that succeeded—i.e., according to the first run in which \(c\) was matched. We show that the resulting assignment has maximum load at most \(8B^*\). See Algorithm 1 for a concise treatment.

**Algorithm 1:** Approximate unweighted load balancing in the CONGEST model.

```plaintext
for \(B \in \{1, 2, 4, \ldots, n\}\) do
    compute a \((1, B)\)-matching \(x_B\) with no augmenting paths of length \(4\lfloor \log n \rfloor + 1\)
end

each client \(c\) locally finds the minimum \(B\) such that \(c\) is matched in \(x_B\) and assigns itself to the server it is matched to in \(x_B\)
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**Theorem 1.** In the CONGEST model, there is a randomized algorithm (Algorithm 1) that with high probability computes an 8-approximation to the min-max load balancing problem in \(O(\log^5 n)\) rounds.

**Proof.** First observe that since augmenting paths with respect to a \((1, n)\)-matching have length at most 1, all clients are assigned in \(x_n\). Therefore the algorithm always outputs an assignment of all clients.

Let \(B^*\) be the optimal maximum load and \(B\) be the smallest power of 2 that is at least \(2B^*\) (we have \(B \leq 4B^*\)). The \((1, B)\)-matching \(x_B\) computed in the main loop of Algorithm 1 will be client-perfect by Lemma 3.1, and so no client will assign itself according to \(x_B'\) for \(B' > B\). A single server is only assigned at most \(i\) clients from \(x_i\), and since \(x_1, x_2, x_4, \ldots, x_B\) are the only assignments in play, the total load of a server under any combination of these assignments is at most \(1 + 2 + 4 + \cdots + B < 2B \leq 8B^*\). Thus every server has load at most \(8B^*\).

Finally, each \(x_B\) is computed in \(O(\log^4 n)\) rounds with high probability by Lemma 4.1. We compute them sequentially, resulting in total round complexity of \(O(\log^5 n)\).

**Remark 4.3.** In the LOCAL model, the round complexity of Algorithm 1 is \(O(\log^3 n)\). One log factor is shaved off of Lemma 4.1 because the algorithm of [19] for finding a \((1,1)\)-matching is faster in the LOCAL model. The second log factor is shaved off by running the for-loop of Algorithm 1 in parallel for every \(B\).

**Approximating all \(\ell_p\)-norms simultaneously (the all-norm load balancing problem).**

The assignment produced by Algorithm 1 in fact does more than approximate the optimal \(\ell_\infty\)-norm; it also simultaneously approximates every \(\ell_p\)-norm for \(p \geq 1\), as we will now show.

Recall that by Lemma 2.1, when clients are unweighted, there is an all-norm optimal assignment that simultaneously minimizes the \(\ell_p\)-norm for all \(p \geq 1\), including \(p = \infty\). Let \(\mathcal{A}^*\) be the set of all all-norm optimal assignments; by Lemma 2.1, the set \(\mathcal{A}^*\) is non-empty. We need the following key definition.
Definition 4.4. The level \( \ell(s) \) of a server \( s \) is the maximum load of \( s \) over all assignments in \( A^* \), i.e., \( \ell(s) := \max_{A \in A^*} L_A(s) \). The level \( \ell(c) \) of a client \( c \) is \( \ell(c) := \max_{A \in A^*} L_A(A(c)) \).

Lemma 4.5. For \( i \in [n] \), let \( C_i \subseteq C \) be the set of clients at level at most \( i \) and let \( S_i \subseteq S \) be the set of servers at level \( i \). There are no edges in \( G \) from \( C_i \) to \( S_j \) for any \( j < i \).

Proof. Fix a client \( c \in C_i \) and let \( A \) be an all-norm optimal assignment such that \( L_A(A(c)) = i \). Suppose to obtain a contradiction that \( N(c) \cap S_j \neq \emptyset \) and let \( s \in N(c) \cap S_j \). Consider the assignment \( A' \) which maps \( c \) to \( s \) and is otherwise identical to \( A \). Comparing the load vector of \( A \) to \( A' \), one entry of the load vector moves from \( i \) to \( i - 1 \), another entry moves from \( j \) to \( j + 1 \), and the rest remain unchanged. Since \( j < i \), it follows that \( \|A'\|_p \leq \|A\|_p \) for all \( p \). If \( \|A'\|_p < \|A\|_p \) for some \( p \), then \( A \) is not all-norm optimal, a contradiction. Otherwise, the norms are equal for all \( p \). But then there is an all-norm optimal assignment, namely \( A' \), in which \( s \) has level \( j + 1 \), contradicting that the level of \( s \) is \( j \).

Lemma 4.6. If a client \( c \in C \) has level at most \( \ell \) and \( x \) is a \((1, B)\)-matching with \( B \geq 2\ell \) such that there are no \( x \)-augmenting paths of length at most \( 4 \lfloor \log n \rfloor + 1 \) in the graph, then \( c \) is matched under \( x \).

Proof. Consider the graph \( G' \) constructed by removing all servers of load larger than \( \ell \) from \( G \) and let \( x' \) be \( x \) restricted to \( G' \). Since every augmenting path in \( G' \) is an augmenting path in \( G \), there are no \( x' \)-augmenting paths of length \( \leq 4 \lfloor \log n \rfloor + 1 \) in \( G' \). Note that any optimal assignment in \( G \) restricted to \( G' \) has maximum load at most \( \ell \). Since \( x' \) is a \((1, B)\)-matching, \( G' \) has a client-perfect \((1, \ell)\)-matching, and \( B \geq 2\ell \), it follows that \( x' \) is client-perfect in \( G' \) by Lemma 3.1. As \( x' \) is the restriction of \( x \) to \( G' \), it follows that \( x \) matches all clients of level at most \( \ell \) in \( G \).

Theorem 2. In the CONGEST model, there is a randomized algorithm (Algorithm 1) that with high probability computes an 8-approximation to the all-norm load balancing problem in \( O(\log^5 n) \) rounds.

Proof. For each client \( c \), let \( B_c \) be the smallest power of two that is at least \( 2\ell(c) \). By Lemma 4.6, each client \( c \) will be assigned in \( x_{B_c} \). Fix a server \( s \). Lemma 4.5 implies that the load of \( s \) is only determined by clients whose level is at most \( \ell(s) \) as no other clients can be adjacent to \( s \). Since each \( x_{B_c} \) contributes at most \( B_c \) to the load of \( s \) and only contributes for \( B \leq 4\ell(s) \), we obtain that \( L_A(s) \leq 1 + 2 + 4 + \cdots + 4\ell(s) \leq 8\ell(s) \). Thus,

\[
\|A\|_p = \left( \sum_{s \in S} (L_A(s))^p \right)^{1/p} \leq \left( \sum_{s \in S} (8\ell(s))^p \right)^{1/p} = 8 \|A^*\|_p. \]

5 Weighted Load Balancing

In this section, we describe our algorithms for the weighted load balancing problem. We start by showing that with the simple reduction in [11] from unweighted to weighted load balancing, our unweighted algorithm (Algorithm 1) also implies an \( O(\log n) \)-approximate polylog\((n)\)-round CONGEST algorithm for weighted instances. We then turn to the main result of this section: an \( O(1) \)-approximate polylog\((n)\)-round LOCAL algorithm for the weighted load balancing problem. We conclude this section with an \( O(1) \)-approximate sequential algorithm that runs in near-linear time.
As our goals in this section are to obtain, at best, an $O(1)$-approximation, we may assume that all client weights are powers of two. If not, rounding weights up to the nearest power of two will at most double the approximation ratio. We can assume similarly that the maximum weight $W \leq n$. Indeed, clients with load less than $W/n$ can collectively distribute at most $W$ weight across the servers and can therefore be assigned arbitrarily. Thus, when $W > n$, clients can simply rescale their own weight by $n/W$ (and round it up to the nearest integer).

Throughout this section, we denote by $C_i$ the set of clients whose weight is exactly $2^i$. (By our previous assumption, the sets $\{C_i\}$ partition $C$.) We let $G_i := G[C_i, \mathcal{N}(C_i)]$ be the induced graph on $C_i$ and its neighborhood.

**An $O(\log n)$-approximation in the CONGEST model.** We begin with an easy corollary of our unweighted algorithm following a simple reduction in [11].

**Theorem 3.** In the CONGEST model, an $O(\log n)$-approximation to the all-norm weighted load balancing problem can be computed with high probability in $O(\log^5 n)$ rounds.

**Proof.** Consider the following algorithm: For each weight class $i$, compute an assignment $A_i$ of $G_i$ using Algorithm 1 by treating all clients as having weight 1. Then, have each client in $C_i$ assign itself according to $A_i$.

Since all $A_i$’s can be computed in parallel (as the graphs $G_i$ are edge-disjoint, only one of the parallel copies need to communicate over an edge), the algorithm runs in $O(\log^5 n)$ rounds. We now show that the resulting assignment $A$ is $O(\log n)$-approximate for all norms.

Fix any $p \geq 1$ including $p = \infty$; let $A^*$ be an assignment for $G$ with minimum $\ell_p$-norm, and let $A_i^*$ be an assignment for $G_i$ with minimum $\ell_p$-norm. Clearly $\|A_i^*\|_p \leq \|A^*\|_p$ for all $i$. By Theorem 2, $\|A_i\|_p \leq 8 \|A_i^*\|_p \leq 8 \|A^*\|_p$. It follows that

$$\|A\|_p = \|A_1 + \cdots + A_{\log n}\|_p \leq \|A_1\|_p + \cdots + \|A_{\log n}\|_p \leq 8 \log(n) \|A^*\|_p.$$ 

**Preliminaries for the weighted algorithms.** Though they use entirely different techniques, the LOCAL and sequential algorithms of the next two subsections both follow the same high-level approach: first compute a split assignment, then round it into an integral one.

**Definition 5.1.** Let $G = (C \cup S, E)$ be a bipartite graph with client weights $w : C \to \mathbb{Z}^+$. A split assignment $y_f$ in $G$ is a client-perfect $(w, \infty)$-matching (so servers have unbounded capacity). For every server $s$, the load $L_{y_f}(s)$ is the sum of edge-multic和平icities incident to $s$.

Notice that split assignments are a relaxation of standard assignments by allowing clients to be assigned to several different servers at once, contributing an integral load to each server, provided that the total load distributed by the client does not exceed its weight.

We will also need the following notion. Define the client-expanded graph $\tilde{G}$ of $G$ as the graph formed by making $w(c)$ copies of each client $c$. Formally, for each $c \in C$, the client-expanded graph has vertices $c_1, \ldots, c_{w(c)}$ and an edge between $c_i$ and $s$ for all $i \in [w(c)]$ if and only if $G$ has an edge between $c$ and $s$.

**Observation 5.2.** A split assignment $y_f$ in $G$ corresponds to an integral assignment in the client-expanded graph $\tilde{G}$ with the same server loads. Thus, since $\tilde{G}$ is unweighted, by Lemma 2.1 there exists an all-norm optimal split assignment $y^*_f$. 

8
5.1 An $O(1)$-approximation in the LOCAL model

Our main result in this section is the following theorem.

**Theorem 4.** In the LOCAL model, there is a randomized algorithm (Algorithm 2) that with high probability computes an $O(1)$-approximation to the weighted all-norm load balancing problem in $O(\log^3 n)$ rounds.

We will need the next rounding lemma to describe our algorithm; the proof is standard.

**Lemma 5.3.** If $G = (C \cup S, E)$ contains a client-perfect $(\kappa, \tau)$-matching $x$, then there exists an assignment $A : C \to S$ such that for all servers $s \in S$,

$$L_A(s) \leq \tau(s) + \max_{c \in A^{-1}(s)} \kappa(c).$$

**Proof of Lemma 5.3.** Consider the set of edges $F$ in the support of $x$. If $C \subseteq F$ is a cycle, we can alternately increase and decrease the value of $x(e)$ on each edge $e$ of the cycle by $\min_{f \in C} x(f)$ to break the cycle without changing $x(\delta(v))$ for any $v \in V$ (this cycle can only be of even length as the input graph is bipartite). Thus, we may assume that the support of $x$ has no cycles and thus is a forest.

We can next turn $F$ into a collection of stars centered on servers. This done by rooting each tree $T$ in the support of $F$ arbitrarily, picking each server $s$ which has a client parent-node $c$, and setting the edge $x(cs) = \kappa(c)$ and $x(cs') = 0$ for all other $s' \in N(c)$. This clearly satisfies the requirement of client $c$ and the load on server $s$ can only ever be increased by $\max_{c \in A^{-1}(s)} \kappa(c)$ as each server can only have one parent client. At this point, in $F$, any client is assigned to exactly one server and thus we obtain an integral solution in which the load of any server $s$ is at most $\tau(s) + \max_{c \in A^{-1}(s)} \kappa(c)$, finalizing the proof.

Our LOCAL algorithm consists of two main parts, an algorithm for solving the split load balancing problem and a rounding procedure, which we describe now in turn.

**Computing a split assignment.** The first step of the LOCAL algorithm is to compute an assignment $\tilde{A}$ in the client-expanded graph $\tilde{G}$ of $G$ using Algorithm 1. Note that in the LOCAL model, each client $c$ can simulate all “new” clients $c_1, \ldots, c_{w(c)}$ in Algorithm 1 without any overhead in the round complexity.\(^1\) As mentioned in Observation 5.2, the assignment $\tilde{A}$ corresponds to a split assignment with the same server loads. To limit the amount of notation in the algorithm description, we will sometimes refer to $\tilde{A}$ as a split assignment in $G$, although formally it is an assignment in $\tilde{G}$.

The guarantees of Algorithm 1 tell us that $\tilde{A}$ has small $\ell_p$-norm. The next step is to use $\tilde{A}$ to find an integral assignment without much loss in the norm.

**A “rounding” procedure.** We would now ideally round the split assignment $\tilde{A}$ into an integral assignment, but even in the LOCAL model we cannot afford to run such a procedure directly. The fact that a good rounding exists, however, is enough for us to apply Lemma 3.1 to obtain a similarly good assignment, as we show below.

For each $i$, let $\tilde{A}_i$ be $\tilde{A}$ restricted to $G_i$. Lemma 5.3 states that there is a way to round $\tilde{A}_i$ into an assignment with load $\tau_i(s) = L_{\tilde{A}_i}(s) + 2^i$ for servers $s$ assigned to by $\tilde{A}_i$ and $\tau_i(s) = 0$ for the

\(^1\)We remark that computing this assignment is the only step of our weighted algorithm that does not run efficiently in the CONGEST model, precisely because this simulation not possible in the CONGEST model in polylog($n$) rounds.
ALGORITHM 2: Approximate weighted (all-norm) load balancing in the LOCAL model.

1 emulate Algorithm 1 on $\tilde{G}$ to compute an assignment $\tilde{A}$
2 for $i \in \{1, 2, 4, \ldots, n\}$ in parallel do
3 \hspace{1em} let $A_i$ be $\tilde{A}$ restricted to $G_i$
4 \hspace{1em} let $\tau_i(s) = \begin{cases} L_{\tilde{A}_i}(s) + 2^i, & \text{if } L_{\tilde{A}_i}(s) > 0 \\ 0, & \text{otherwise} \end{cases}$
5 \hspace{1em} \Comment{by Lemma 5.3, an assignment with load vector point-wise less than $\tau_i$ exists in $G_i$}
6 \hspace{1em} \Comment{therefore, scaling clients in $G_i$ to weight 1, a $[1, 2^{-i}\tau_i]$-matching exists}
7 \hspace{1em} treating $G_i$ as unweighted, compute a $(1, 2[2^{-i}\tau_i])$-matching $x_i$ in $G_i$ with no augmenting paths of length $4\lceil \log n \rceil + 1$
8 \hspace{1em} \Comment{by Lemma 3.1, $x_i$ is client-perfect}
9 \hspace{1em} let $A_i$ be the assignment induced by $x_i$
10 assign each $c \in C_i$ to $A_i(c)$
11 end

remaining servers. Treating the clients as unweighted, $\tilde{A}_i$ corresponds to a $(1, [2^{-i}\tau_i])$-matching. We now compute a $(1, 2[2^{-i}\tau_i])$-matching $x_i$ with no augmenting paths of length $4\lceil \log n \rceil + 1$ or smaller. By Lemma 3.1, each $x_i$ is client-perfect, inducing an (integral) assignment $A_i$ in $G_i$. Lastly, each client in $C_i$ assigns itself in accordance with $A_i$ to produce the global assignment $A$. See Algorithm 2.

To formalize the logic of the algorithm, we make a few claims that together will imply the algorithm’s correctness. The first claim ensures that the algorithm produces a proper assignment.

Claim 5.4. Algorithm 2 assigns every client to some server.

**Proof.** We need to show that the matching $x_i$ computed in Line 5 of Algorithm 2 is client-perfect. Consider $\tau_i$ from Line 4 of Algorithm 2. Viewing $\tilde{A}_i$ as a client-perfect $(w, L_{\tilde{A}_i})$-matching, Lemma 5.3 guarantees that there is an assignment wherein each server $s$ has load at most $\tau_i(s)$.

Because all clients in $G_i$ have the same weight, we can interpret the assignment from Lemma 5.3 as a client-perfect $(1, [2^{-i}\tau_i])$-matching in the unweighted graph $G_i$. When treating clients as unweighted, server capacities are always bounded by $n$, and so by Lemma 3.1, if $x_i$ has no augmenting paths of length $\leq 4\lceil \log n \rceil + 1$, it follows that $x_i$ is client-perfect. □

The next claim shows that the assignment produced is $O(1)$-approximate.

Claim 5.5. There is a universal constant $C$ such that for all $p \geq 1$, including $p = \infty$, the assignment $A$ produced by Algorithm 2 satisfies $\|A\|_p \leq C\|A^*\|_p$, where $A^*$ is an $\ell_p$-norm-minimizing assignment.

**Proof.** Fix $p \geq 1$ (including $p = \infty$). Let $A^*$ and $\tilde{A}^*$ be assignments for $G$ and $\tilde{G}$, respectively, that minimize the $\ell_p$-norm. For brevity, we omit the subscript $p$ when writing norms with the understanding that all norms in this proof are $\ell_p$-norms. We will also treat the client weight function $w$ as a vector so that we can write its norm as $\|w\|$.

Our strategy is to decompose the final assignment $A$ into two parts and bound the norms of those parts separately. First, we decompose each assignment $A_i$ of Line 6. We define the first part, $\rho_i$, by $\rho_i(s) = 2^i$ if $s$ is assigned to by $A_i$ and $\rho_i(s) = 0$ otherwise. In other words, $\rho_i$ has
the same support as the load vector $L_{A_i}$ of $A_i$, but all of its nonzero entries are $2^i$. The second part, $\mu_i$, docks $2^{i+1}$ from the support of $L_{A_i}$: $\mu_i = L_{A_i} - 2\rho_i$. Letting $\mu = \sum_i \mu_i$ and $\rho = \sum_i \rho_i$, we have that $\|A\| = \|\mu + 2\rho\| \leq \|\mu\| + 2\|\rho\|$. It therefore suffices to show that $\|\mu\|$ and $\|\rho\|$ both $O(1)$-approximate $\|A^*\|$.

Let us first bound $\|\mu\|$. For any server $s$ assigned to by $A_i$, we have

$$
\mu_i(s) = L_{A_i}(s) - 2^{i+1} \\
\leq 2^{i+1}[2^{-i} \tau_i(s)] - 2^{i+1} \\
\leq 2^{i+1}[2^{-i}L_{A_i}(s) + 1] - 2^{i+1} \\
\leq 2^{i+1}2^{-i+1}L_{A_i}(s) + 2^{i+1} - 2^{i+1} \\
= \frac{1}{2^{i+1}} \rho_i(s).
$$

For any server $s$ not assigned to by $A_i$ we have $\mu_i(s) = 0$, and so trivially $\mu_i(s) \leq 4L_{A_i}(s)$ for such $s$. Therefore, $\mu(s) = \sum_i \mu_i(s) \leq \sum_i 4L_{A_i}(s) = 4L_{A_i}(s)$. Using Theorem 1, it follows that $\|\mu\| \leq \frac{1}{4} \|A\| \leq 32 \|A^*\| \leq 32 \|A^*\|$.

We now bound $\|\rho\|$. Define $\rho^*(s) = \max_{x \in \mathbb{A}_i(s)} w(c)$. Note that $\rho^*$ is the load vector of a “partial” assignment (not all clients are assigned) that assigns to each server at most once. Since $w$ can be interpreted as the load vector of an assignment that assigns every client to a unique server, we have $\|\rho^*\| \leq \|w\|$. Now observe that $\rho(s) = \sum_{i=1}^{\log \rho^*(s)} \rho_i(s) \leq 2\rho^*(s)$ simply because $\rho_i(s)$ is either 0 or $2^i$ for each $i$. To complete the bound, notice that $\|w\| \leq \|A^*\|$; the best (hypothetical) assignment would assign every client to a unique server, resulting in value $\|w\|$. Putting things together, we have shown that $\|\rho\| \leq 2 \|A^*\|$.

It remains to bound the round-complexity of the algorithm.

**Claim 5.6.** Algorithm 2 takes $O(\log^3 n)$ rounds in the LOCAL model.

**Proof.** In the LOCAL model, we can easily emulate Algorithm 1 (or any algorithm) on the client-expansion $G$ at no extra cost; any communication across an edge $c_i$ simply needs to specify which $c_i$ in the expansion the message is to/from. Since $W \leq n$, Algorithm 1 still runs in $O(\log^3 n)$ rounds in the LOCAL model (see Remark 4.3). The main for-loop is run in parallel, and so we only need to bound the round-complexity of its body. Line 5 is the only line inside the loop that requires (additional) communication, and this again only takes $O(\log^3 n)$ rounds. The total round-complexity is therefore $O(\log^3 n)$.

This concludes the proof of Theorem 4.

### 5.2 An $O(1)$-approximate $O(m \log^3 n)$-time sequential algorithm

We now show that our approach can also be used to compute an $O(1)$-approximation to the weighted all-norm load balancing problem in near-linear time in the standard sequential setting, proving the following theorem.

**Theorem 5.** In the standard sequential model, there is a deterministic algorithm to compute an $O(1)$-approximate solution to the weighted all-norm load balancing problem that runs in $O(m \log^3 n)$ time.
Previously, Azar et al. [1] showed a 2-approximate algorithm for this problem, which runs in two phases: (1) compute an optimal fractional assignment and (2) round the fractional assignment, which incurs a 2-approximation. But their algorithm computes the optimal fractional assignment using the ellipsoid method to solve a linear program with exponentially many constraints, and hence incurs a large polynomial runtime.

Our algorithm uses the same rounding procedure as [1], but instead of computing an exact fractional assignment, we compute an O(1)-approximate split assignment in near-linear time by simulating our distributed approach in the sequential setting. To this end, we will need the following subroutine:

**Lemma 5.7.** Given any bipartite graph \( G = (C \cup S, E) \) and capacity functions \( \kappa, \tau, \) it is possible to compute a \((\kappa, \tau)\)-matching with no augmenting paths of length \( \leq 8 \log(n) \) in \( O(m \log^2 n) \) time in the sequential setting.

**Proof.** Note that a \((\kappa, \tau)\)-matching corresponds to the following flow problem. Every edge in \( E \) gets infinite capacity; there is a dummy source \( v_s \) and for every client \( c \in C \) there is an edge \((v_s, c)\) of capacity \( \kappa(c) \); there is also a dummy sink \( v_t \) and for every server \( s \in S \) there is an edge from \( s \) to \( v_t \) of capacity \( \tau(s) \). It is immediate to verify that any \( v_s-v_t \) flow in this network corresponds to a \((\kappa, \tau)\)-matching and vice versa.

We now show how to compute a solution to this flow problem that contains no augmenting paths of length \( 9 \log(n) \geq 8 \log(n) + 2 \) which corresponds to the desired \((\kappa, \tau)\)-matching.

The algorithm simply runs \( 9 \log(n) \) successive iterations of blocking flow. A blocking flow in a capacitated graph can be computed in \( O(m \log n) \) time using the dynamic tree structure of Sleator and Tarjan [23].

We are now ready to show our algorithm to compute a split assignment.

**Lemma 5.8.** Let \( G = (C \cup S, E) \) be a bipartite graph with client-weights \( w(C) \). There exists a sequential algorithm that in \( O(m \log^3 n) \) time computes a split assignment \( y_f \) such that \( \|L_{y_f}\|_p \leq 8\|L_{\tilde{y}_f}\|_p \) for every \( p \geq 1 \) (including \( p = \infty \)).

**Proof.** Recall from Observation 5.2 that the optimal split assignment \( y_f^* \) corresponds to an optimal (integral) assignment \( \hat{A}^* \) in the client-expanded graph \( \tilde{G} \); server loads in the two solutions are the same, so \( \|L_{\tilde{y}_f}\|_p = \|L_{\hat{A}^*}\|_p \). We obtain our split assignment \( y_f \) by simulating Algorithm 1 on the graph \( \tilde{G} \): by Theorem 2, this yields the desired 8-approximation. We now describe how to execute the simulation in the sequential model and how to convert between the perspectives of split assignment in \( G \) and integral assignment in \( \tilde{G} \).

Firstly, in Line 2 of Algorithm 1, we need to a compute a \((1, B)\)-matching in \( \tilde{G} \) with no short augmenting paths. This is equivalent to a \((w, B)\)-matching in \( G \), which we compute in \( O(m \log^2 n) \) time using Lemma 5.7.

Secondly, in Line 4 of Algorithm 1, each client-copy \( \tilde{c} \) in \( \tilde{G} \) must find the minimum \( B \) such that \( \tilde{c} \) is matched in \( x_B \). We need to convert this line to the language of split assignments. In particular, note that in our sequential simulation, \( x_B \) is a \((w, B)\)-matching in \( G \) rather than a \((1, B)\)-matching in \( \tilde{G} \). It is easy to see that the following simulates Line 4. For each client \( c \) in \( G \), let \( s_B(c) \) be the set of servers incident to \( c \) in \( x_B \): if an edge \( cs \) has multiplicity \( \alpha \) in \( x_B \), then \( s \) appears \( \alpha \) times in \( s_B(c) \). To construct the split assignment \( y_f \), first assign \( c \) to the server in \( s_1(c) \) (if any). Then
assign $c$ to an arbitrary $|s_2(c)| - |s_1(c)|$ servers from $s_2(c)$, an arbitrary $|s_4(c)| - |s_2(c)|$ servers from $s_4(c)$, and more generally an arbitrary $|s_B(c)| - |s_{B/2}(c)|$ servers from $s_B(c)$. It is not hard to check that the resulting split assignment is equivalent to some integral assignment in $\tilde{G}$ formed by executing Line 4 of Algorithm 1 in $\tilde{G}$. It is also easy to see that for each $x_B$ the assignments can be performed in $O(m)$ time, for a total of $O(m \log n)$ time.

The running time of the algorithm is thus dominated by the time for computing matchings $x_B$. Each takes $O(m \log^2 n)$ time to compute (Lemma 5.7), and there are $O(\log(nW)) = O(\log n)$ values of $B$, so the total run-time is $O(m \log^3 n)$. \hfill \square

Finally, we round the split assignment to an integral assignment using the rounding procedure of [1], which is described in the proof of Lemma 5.3. The rounding procedure has two steps: cycle cancelling and computing a matching in a tree. The second can clearly be done in $O(m)$ sequential time. Cycle cancelling can be done deterministically in $O(m \log n)$ time (see, e.g., [14]). The total time for rounding is thus $O(m \log n)$. (Note that in the distributed setting we only relied on the existence of such a rounding procedure, because it is unclear how to implement cycle canceling efficiently in the LOCAL model.)

Following the exact same argument as in [1] or in the proof of Lemma 4 of this paper, since our split assignment was an 8-approximation (Theorem 5.8), the integral assignment formed by rounding yields a 9-approximation. This concludes the proof of Theorem 5.

References

[1] Y. Azar, L. Epstein, Y. Richter, and G. J. Woeginger. All-norm approximation algorithms. J. Algorithms, 52(2):120–133, 2004.

[2] L. Barenboim and G. Oren. Distributed backup placement in one round and its applications to maximum matching and self-stabilization. In Proc. 3rd Symposium on Simplicity in Algorithms (SOSA 2020), pages 99–105, 2020.

[3] A. Bernstein, J. Holm, and E. Rotenberg. Online bipartite matching with amortized $O(\log^2 n)$ replacements. J. ACM, 66(5):Art. 37, 23, 2019.

[4] A. Bernstein, T. Kopelowitz, S. Pettie, E. Porat, and C. Stein. Simultaneously load balancing for every $p$-norm, with reassignments. In Proc. 8th Innovations in Theoretical Computer Science Conference (ITCS 2017), volume 67 of LIPIcs. Leibniz Int. Proc. Inform., pages Art. No. 51, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017.

[5] J. Bruno, E. G. Coffman, Jr., and R. Sethi. Scheduling independent tasks to reduce mean finishing time. Comm. ACM, 17:382–387, 1974.

[6] D. Chakrabarty and C. Swamy. Simpler and better algorithms for minimum-norm load balancing. In Proc. 27th Annual European Symposium on Algorithms (ESA 2019), volume 144 of LIPIcs. Leibniz Int. Proc. Inform., pages Art. No. 27, 12. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019.

[7] A. Czygrinow, M. Hančkowiak, E. Szymańska, and W. Wawrzyniak. Distributed 2-approximation algorithm for the semi-matching problem. In Proc. 26th International Symposium on Distributed Computing (DISC 2012), volume 7611 of LNCS, pages 210–222. Springer, Heidelberg, 2012.
[8] J. Fakcharoenphol, B. Laekhanukit, and D. Nanongkai. Faster algorithms for semi-matching problems. *ACM Trans. Algorithms*, 10(3):Art. 14, 23, 2014. 1, 2, 3

[9] M. Gairing, T. Lücking, M. Mavronicolas, and B. Monien. The price of anarchy for restricted parallel links. *Parallel Process. Lett.*, 16(1):117–131, 2006. 1

[10] A. Gupta, A. Kumar, and C. Stein. Maintaining assignments online: matching, scheduling, and flows. In *Proc. 25th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2014)*, pages 468–479. ACM, 2014. 4

[11] M. M. Halldórsson, S. Köhler, B. Patt-Shamir, and D. Rawitz. Distributed backup placement in networks. *Distrib. Comput.*, 31(2):83–98, 2018. 1, 2, 5, 7, 8, 16, 17

[12] N. J. A. Harvey, R. E. Ladner, L. Lovász, and T. Tamir. Semi-matchings for bipartite graphs and load balancing. *J. Algorithms*, 59(1):53–78, 2006. 1, 2, 3

[13] W. A. Horn. Minimizing average flow time with parallel machines. *Oper. Res.*, 21(3), 1973. 1

[14] D. Kang and J. Payor. Flow rounding, 2015. arXiv:1507.08139. 13

[15] S. Köhler, V. Turau, and G. Mentges. Self-stabilizing local $k$-placement of replicas with minimal variance. In *Proc. 14th Stabilization, Safety, and Security of Distributed Systems (SSS 2012)*, pages 16–30, 2012. 1

[16] C. Konrad and A. Rosén. Approximating semi-matchings in streaming and in two-party communication. *ACM Trans. Algorithms*, 12(3):Art. 32, 21, 2016. 1, 2

[17] A. Kothari, S. Suri, C. D. Tóth, and Y. Zhou. Congestion games, load balancing, and price of anarchy. In *Proc. 1st Combinatorial and Algorithmic Aspects of Networking*, volume 3405 of *LNCS*, pages 13–27. Springer, Berlin, 2005. 1

[18] Y. Lin and W. Li. Parallel machine scheduling of machine-dependent jobs with unit-length. *European J. Oper. Res.*, 156(1):261–266, 2004. 1

[19] Z. Lotker, B. Patt-Shamir, and S. Pettie. Improved distributed approximate matching. *J. ACM*, 62(5):Art. 38, 17, 2015. 2, 5, 6, 17

[20] R. Machado and S. Tekinay. A survey of game-theoretic approaches in wireless sensor networks. *Comput. Networks*, 52(16):3047–3061, 2008. 1

[21] G. Oren, L. Barenboim, and H. Levin. Distributed fault-tolerant backup-placement in overloaded wireless sensor networks. In *Proc. 9th International Conference on Broadband Communications, Networks, and Systems (BROADNETS 2018)*, pages 212–224, 2018. 1

[22] N. Sadagopan, M. Singh, and B. Krishnamachari. Decentralized utility-based sensor network design. *Mobile Networks and Applications*, 11(3):341–350, 2006. 1

[23] D. D. Sleator and R. E. Tarjan. A data structure for dynamic trees. *J. Comput. Syst. Sci.*, 26(3):362–391, 1983. 12

[24] S. Suri, C. D. Tóth, and Y. Zhou. Uncoordinated load balancing and congestion games in P2P systems. In *Proc. 3rd International Workshop on Peer-to-Peer Systems (IPTPS 2004)*, pages 123–130, 2004. 1
[25] S. Suri, C. D. Tóth, and Y. Zhou. Selfish load balancing and atomic congestion games. *Algorithmica*, 47(1):79–96, 2007.
Appendix

A Distributed Backup Placement with Replication Factor

In the distributed backup placement problem, the input is a graph $G = (V, E)$. There are a set of nodes $C \subseteq V$ called the clients which host files that should be backed up on over a set of nodes $S \subseteq V$ called the servers. Unlike the load balancing problem, the clients and servers here need not partition $V$ or even be disjoint. Each client $c$ hosts a file of size $w(c)$ and may only backup the file on adjacent servers; we will refer to $w(c)$ as the weight of client $c$. When all of the client-weights are the same, we call the instance uniform. Finally, a replication factor $r$ specifies the number of distinct servers that each client must be backed up on; each client must be assigned to $r$ distinct adjacent servers. The goal is to minimize the maximum server load in the resulting assignment; as before, the load of a server is the sum of the client-weights assigned to it. Following the paper of Halldórsson et al. [11], we assume that every client has degree at least $r$, since otherwise there is no solution to the problem.

As was shown by Halldórsson et al. [11], when the replication factor is one, the distributed backup placement problem can be reduced to the load balancing problem in a natural way. Form a bipartite graph $G'$ of the clients and servers (a node for a server may appear on both sides of the partition). Clients and servers are adjacent in the new graph if and only if they were adjacent in $G$. A solution to the load balancing problem in $G'$ directly corresponds to a solution to the distributed backup placement problem in $G$. The results in our paper therefore immediately give improved bounds for the distributed backup placement problem with replication factor one.

We show that our approach can also be used to handle replication factor larger than 1. For simplicity, we only extend our results that are most close related to the existing state of the art by Halldórsson et al. [11]. Their paper shows that for any replication factor, in polylog($n$) rounds, it is possible to compute an $O(\log n / \log \log n)$-approximation to distributed backup placement with uniform client weights. They also show a simple reduction which gives a $O(\log^2 n / \log \log n)$-approximation for general client weights. Existing results on distributed backup placement focus only on minimizing maximum load (not general $\ell_p$-norm), so we will do the same.

In this section, we show that our Theorem 1 can be extended to the problem of distributed backup placement with arbitrary replication factor, thus improving upon the result of Halldórsson et al. [11]. In particular, we achieve the following:

**Theorem 6.** Given an instance of the backup placement problem with uniform client weights and replication factor $r$, an $8$-approximate solution can be found w.h.p. in the CONGEST model within $O(\log^5 n)$ rounds.

Using the same reduction from weighted clients to unweighted clients as in Theorem 3, we also obtain the following improvement over Corollary 27 in [11].

**Theorem 7.** Given an instance of the backup placement problem with non-uniform client weights and replication factor $r$, an $O(\log n)$-approximate solution can be found w.h.p. in the CONGEST model within $O(\log^5 n)$ rounds.

A.1 The Setup

Our Algorithm for Theorem 7 follows the same structure as Section 4 for load balancing. We just need small modifications to handle arbitrary replication factor $r$, rather than the replication factor 1 of standard load balancing.
To this end, we first generalize our notion of a \((1, B)\) matching

**Definition A.1.** Given any positive integers \(B, r\), we say that \(x \subseteq E\) is a \((1, B, r)\)-matching if every client has degree at most \(r\) in \(x\), and every server has degree at most \(B\). Note that in this definition, every edge has multiplicity at most 1, which is why a \((1, B, r)\)-matching is different from a \((r, B)\)-matching. We say that \(x\) is client-perfect if every client has degree exactly \(r\) in \(x\), and we say that a client is unsaturated if it has degree strictly less than \(r\). An \(x\)-augmenting path is defined the same way as before.

**Observation A.2.** A client-perfect \((1, B, r)\)-matching is a solution to backup placement with replication \(r\) that has maximum server load \(B\).

Our structural lemma for \((\kappa, \tau)\)-matchings (Lemma 3.1) can easily be extended to the case of \((1, B, r)\)-matchings; the proof is the same.

**Lemma A.3** (Extension of Lemma 3.1). If \(G\) contains a client-perfect \((1, B, r)\)-matching and \(x\) is a \((1, 2B, r)\)-matching, then either \(x\) is client-perfect or there is an \(x\)-augmenting path of length at most \(4\lceil \log n \rceil + 1\).

Finally, given any positive integers \(B, r\), we can generalize Lemma 4.1 to efficiently compute a \((1, B, r)\)-matching with no short augmenting paths:

**Lemma A.4** ([11]). (Extension of Lemma 4.1) There exists an \(O(k^3 \log n)\)-round randomized algorithm in the CONGEST model that, with high probability, given a graph \(G = (C \cup S, E)\), and positive integers \(B, r, k\), computes \((1, B, r)\)-matching with no augmenting paths of length less than \(k\).

The proof of the above lemma is the same as that of Lemma 4.1. In particular, the proof combines two existing results. The algorithm of Lotker et al. [19] computes a \((1,1)\)-matching with no augmenting paths of length \(\leq k\) in \(O(k^3 \log n)\) rounds. The paper of Halldórsson et al. [11] then shows a black-box extension from \((1,1)\)-matching to \((1, B, r)\)-matching which does not increase the round-complexity. In particular, they reduce from a problem called \(f\)-matching, which captures the setting where there can be multiple copies of both clients and servers, but where each edge can only be used once; by using \(r\) copies of each client and \(B\) copies of each server, we get a \((1, B, r)\)-matching. See [11] for more details.

### A.2 The Algorithm

Our algorithm is basically the same as our algorithm for load balancing (Algorithm 1). We now describe the changes we need to make to handle replication factor \(r\).

Firstly, in Line 2 of Algorithm 1, instead of computing a \((1, B)\)-matching \(x_B\) with no short augmenting paths, we invoke Lemma A.4 to compute a \((1, B, r)\)-matching \(x_B\) with no short augmenting paths.

Secondly, in Line 4 of Algorithm 1, each client \(c\) locally finds the minimum \(B\) such that \(c\) is matched \(r\) times in \(x_B\) and then assigns itself to those \(r\) servers. (Note that if \(c\) is assigned to \(< r\) servers in some \(x_{B'}\), then \(c\) simply ignores \(x_{B'}\).)

The round-complexity of the algorithm is clearly the same \(O(\log^5 n)\) as in Algorithm 1. The approximation analysis is exactly the same as in the proof of Theorem 1, so the algorithm computes an 8-approximate assignment for backup placement.