Dirac cohomology on manifolds with boundary and spectral lower bounds

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Abstract
Along the lines of the classic Hodge–De Rham theory a general decomposition theorem for sections of a Dirac bundle over a compact Riemannian manifold is proved by extending concepts as exterior derivative and coderivative as well as elliptic absolute and relative boundary conditions for both Dirac and Dirac Laplacian operators. Dirac sections are shown to be a direct sum of harmonic, exact and coexact spinors satisfying alternatively absolute and relative boundary conditions. Cheeger’s estimation technique for spectral lower bounds of the Laplacian on differential forms is generalized to the Dirac Laplacian. A general method allowing to estimate Dirac spectral lower bounds for the Dirac spectrum of a compact Riemannian manifold in terms of the Dirac eigenvalues for a cover of 0-codimensional submanifolds is developed. Two applications are provided for the Atiyah–Singer operator. First, we prove the existence on compact connected spin manifolds of Riemannian metrics of unit volume with arbitrarily large first non zero eigenvalue, which is an already known result. Second, we prove that on a degenerating sequence of oriented, hyperbolic, three spin manifolds for any choice of the spin structures the first positive non zero eigenvalue is bounded from below by a positive uniform constant, which improves an already known result.

Keywords
Dirac operator · Spectral lower bounds · Dirac cohomology

Mathematics Subject Classification
58J50 · 53C27

1 Introduction
When dealing with direct and indirect spectral theory on Riemannian manifolds, the following question naturally arises. Given a formally selfadjoint operator of Laplace type (or Laplacian for short) over a compact Riemannian manifold, consider a finite open cover or a decomposition into 0-codimensional submanifolds with boundary and add an appropriate
elliptic boundary condition. Is there a general principle allowing to find lower bounds of the spectrum of the manifold in terms of the spectra of the pieces?

To our knowledge the only answer to this question known so far is a dissection principle, known also as domain monotonicity, which was originally formulated for the Laplacian on functions on domains in \( \mathbb{R}^m \) by Courant and Hilbert [14, 19]. The remarkable fact is that it still holds for any formally selfadjoint operator of Laplace type under Neumann boundary conditions, as recognized for the first time by Bär [2] for the Dirac Laplacian.

The main contribution of this paper is a new technique allowing to estimate the lower spectral part of a general Dirac operator in terms of the spectra of a finite cover under the appropriate boundary conditions. The original idea in the case of differential forms is due to Cheeger but unpublished, based on the Mayer–Vietoris scheme, was carried out in [40]. In order to extend it to the set up of Dirac bundles, a Dirac complex as in non commutative differential geometry is introduced, as well as appropriate elliptic local boundary conditions for both Dirac and Dirac Laplacian. Concepts like derivation and coderivation and boundary conditions like absolute and relative ones can be extended from the context of differential forms to that of Dirac sections.

If \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) is a Dirac bundle over the Riemannian manifold \((M, g)\), where \(M\) is compact with boundary, and if there exists a bundle isomorphism \(T\) on \(V\) anticommuting with \(\gamma\) and with the Dirac operator \(Q\), for which \(T^2 = 1\), then, the tuple \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma, \nabla)\) where \(\nabla := iT\gamma\) defines a \((1, 1)\)-Dirac bundle structure with corresponding Dirac operators \(Q\) and \(\overline{Q}\). The operators

\[
d := \frac{1}{2} (Q - i\overline{Q}) = \frac{1 + T}{2} Q \quad \text{and} \quad \delta := \frac{1}{2} (Q + i\overline{Q}) = \frac{1 - T}{2} Q
\]

(1)
can be seen derivative and coderivative on \(M\), while the zero-order boundary operators

\[
B_{\pm} := \frac{1 \mp T\gamma(\nu)}{2}
\]

(2)
play the role of the absolute \((B_-)\) and relative \((B_+)\) boundary conditions on \(\partial M\) for the Dirac operator \(Q\). The Dirac Laplacian can be decomposed as

\[
Q^2 = d\delta + \delta d
\]

(3)
and the corresponding first order boundary operators read as:

\[
B_- \oplus B_- d \quad \text{(absolute)} \quad B_+ \oplus B_+ \delta \quad \text{(relative)}.
\]

(4)

**Theorem 1** (Orthogonal decomposition of Dirac sections) *Let \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) be a Dirac bundle over the compact Riemannian manifold with boundary \((M, g)\) admitting a bundle isomorphism \(T\) anticommuting with \(\gamma\) and with the Dirac operator \(Q\), such that \(T^2 = 1\) holds. Let \(C^\infty(M, V)\) denote the Dirac spinors, i.e. the differentiable sections of \(V\), \(H_{B_{\pm}}(M, V)\) the harmonic, \(\Omega^d_{B_{\pm}}(M, V)\) the exact and \(\Omega^\delta_{B_{\pm}}(M, V)\) the coexact Dirac sections satisfying the absolute \((B_-)\) and the relative \((B_+)\) boundary conditions. Then, the following orthogonal decomposition holds:

\[
C^\infty(M, V) = H_{B_{\pm}}(M, V) \oplus \Omega^d_{B_{\pm}}(M, V) \oplus \Omega^\delta_{B_{\pm}}(M, V).
\]

(5)

This theorem generalizes Morrey’s Theorem (cf. [41, 43]) for differential forms on manifold under the relative or absolute boundary condition. By using this Hodge–De Rham-like decomposition theorem a variational characterization of the Dirac spectrum in terms of the Dirac spectrum on exact Dirac sections can be derived. This is the technical result needed to prove the following:
Theorem 2 (Spectral lower bounds by dissection) Let $(V, \langle \cdot, \cdot \rangle, \nabla, \gamma)$ be a Dirac bundle over the compact Riemannian manifold without boundary $(M, g)$ with Dirac operator $Q$. Assume the existence of a bundle isomorphism $T$ on $V$ anticommuting with $\gamma$ and with $Q$, such that $T^2 = 1$ holds. Let $(U_j)_{j=0}^K$ be a collection of closed sets whose interiors cover $M$. Choose and fix $(\rho_j)_{j=0}^K$ a subordinate partition of unity and set

\[ U_{a_0, a_1, \ldots, a_k} := \bigcap_{i \in \{a_0, \ldots, a_k\}} U_i \]

\[ N_1 := \sum_{i,j=0}^K \dim \mathcal{H}_{B_-(U_{i,j}, V)} \]

\[ N_2 := \sum_{i,j,k=0}^K \dim \mathcal{H}_{B_-(U_{i,j,k}, V)} \]

\[ N := N_1 + N_2 + 1 \]

\[ m_i := |\{j \neq i \mid U_j \cap U_i \neq \emptyset\}| \]

\[ C_\rho := \frac{1}{2} \max_{0 \leq i \leq K} \sup_{x \in U_i} |\nabla \rho_i(x)|^2. \]

For any closed set $U \subset M$ let $\lambda(U)$ denote the smallest positive eigenvalue of the Dirac operator on exact Dirac sections satisfying the absolute boundary condition $B_-$ on $\partial U$. Then, the $N$-th positive eigenvalue of the Dirac operator over $M$ has the following positive lower bound:

\[ \lambda_N(Q) \geq \frac{1}{\sqrt{\sum_{i=0}^K \left( \frac{1}{\lambda^2(U_i)} + 4 \sum_{j=0}^{m_i} \left( \frac{C_\rho}{\lambda^2(U_{ij})} + 1 \right) \left( \frac{1}{\lambda^2(U_i)} + \frac{1}{\lambda^2(U_j)} \right) \right)}}. \]

This is the generalization of Cheeger’s technique for the Laplacian on differential forms (cf. [40]). The lower spectral bound method found can be applied to prove the new results introduced by the following two subsections.

1.1 Large first eigenvalues

Let $(M, g)$ be a compact, connected $n$ dimensional Riemannian manifold, and $\lambda_1(\Delta^g)$ the smallest positive eigenvalue of the Laplacian on $p$ forms. Hersch [32] proved, that for functions on the sphere $S^2$ we have

\[ \lambda_1(\Delta^g) \text{Vol}(S^2, g) \leq 8\pi \]

for every Riemannian metric $g$. In connection with this result, Berger [8] asked whether there exists a constant $k(M)$ such that

\[ \lambda_1(\Delta^g) \text{Vol}(M, g)^{\frac{2}{m}} \leq k(M) \]

for any Riemannian metric $g$ on a manifold $M$ of dimension $m$. Yang and Yau [51] proved that the inequality above holds for a compact surface $S$ of genus $\Gamma$ with $k(S) = 8\pi(\Gamma + 1)$. Later, Bleecker [10], Urakawa [48] and others constructed examples of manifolds of dimension $m > 3$ for which the inequality (9) is false. Xu [50], and Colbois and Dodziuk [18] showed...
that inequality (9) is false for every Riemannian manifold of dimension $m > 3$. Tanno [46] posed the analogous question for forms of degree $p$, if there exist a constant $k(M)$ such that

$$\lambda_1(\Delta^g_p) \text{Vol}(M, g)^{\frac{2}{m}} \leq k(M)$$

(10)

for any Riemannian metric $g$ on $M$. Pagliara and Gentile [23] showed that inequality (10) is false for $m > 4$ and $2 < p < m - 2$. We can adapt now their proof to show

**Theorem 3** Every compact connected spin manifold $M$ of dimension $m \geq 2$ without boundary admits for a given spin structure $s$ metrics $g$ of volume one with arbitrarily large first non zero Atiyah–Singer operator eigenvalue $\lambda_1(D^{(M, g)}_s)$.

When the proof of this theorem was written, the author was unaware that Amman and Jammes [1] had proved this result in the context of conformally covariant elliptic operators.

### 1.2 Lower spectrum of degenerating hyperbolic three manifolds

According to Thurston’s cusp closing Theorem (cf. [47]), every complete, non compact, hyperbolic, three manifold $M$ of finite volume is the limit in the sense of pointed Lipschitz of a sequence of compact, hyperbolic, three manifolds $(M_j)_{j \geq 0}$.

The Laplace–Beltrami operator on $p$-forms is selfadjoint and non negative. Its spectrum is contained in $[0, \infty[$ and can be seen as the disjoint union of pure point spectrum i.e. eigenvalues and continuous spectrum i.e. approximate eigenvalues or, alternatively, as the disjoint union of non essential spectrum i.e. isolated eigenvalues of finite multiplicity and essential spectrum i.e. cluster points of the spectrum and eigenvalues of infinite multiplicity.

On the basis of Thurston’s Theorem, we expect the eigenvalues of $\Delta_p$ on $M_j$ to accumulate at points of the spectrum of $\Delta_p$ on $M$.

In three dimensions the spectra of functions and coexact 1-forms fully determines the spectra of forms in all degree. In the case of functions, the results of Donnelly [21] implied $\text{ess spec}(\Delta_0) = [1, \infty[$ and a sharp estimate for the number of eigenvalues of $M_j$ in any interval $[1, 1 + x^2]$ was given by Chavel and Dodziuk [15]. In the case of 1-forms, Mazzeo and Phillips [39] proved $\text{spec}(\Delta_1) = [0, \infty[$ and the accumulation rate near 0 was estimated by McGowan [40]. Later on, these results were extended by Dodziuk and McGowan [20], who gave an asymptotic formula for the number of 1-form eigenvalues in an arbitrary interval $[0, x]$.

**Theorem 4** (Dodziuk, Mc Gowan) On a degenerating sequence of hyperbolic compact three manifolds without boundary $(M_j, g_j)_{j \geq 0}$ the lower eigenvalues of the Laplace–Beltrami operator acting on 1-forms accumulate near zero as the inverse of the square of the diameter. More precisely, there exists an integer $N_0 \in \mathbb{N}_0$ such that

$$\lambda_{N_0}(\Delta_1^{(M_j, g_j)}) = \frac{O(1)}{\text{diam}^2(M_j, g_j)} \quad (j \to \infty).$$

(11)

Recall that $c_j = O(1)$ ($j \to \infty$) if and only if $(c_j)_{j \geq 0}$ is a bounded sequence and that for a degenerating sequence of hyperbolic manifolds $\text{diam}(M_j, g_j) \uparrow +\infty$ ($j \to \infty$).

An explicit lower bound for the first eigenvalue with respect to the diameter has been recently provided by Jammes (cf. [35]).
Theorem 5 (Jammes) For any real $V > 0$, there exists a constant $c(V) > 0$ such that, if $M$ is a three dimensional hyperbolic compact without boundary manifold of volume smaller than $V$, whose thin part has $k$ components, then
\[
\lambda_1(\Delta^{(M,g)}) \geq \frac{c(V)}{\text{diam}^3(M,g) \exp(2k \text{diam}^3(M,g))}
\]
(12)
\[
\lambda_{k+1}(\Delta^{(M,g)}) \geq \frac{c(V)}{\text{diam}^2(M,g)}.
\]

Theorem 6 (Jammes) For every non compact three dimensional hyperbolic manifold $M$ of bounded volume, there exists a constant $c > 0$ and a degenerating sequence of hyperbolic three compact without boundary manifolds $(M_j, g_j)_{j \geq 0}$ converging to $M$ such that for all $j \geq 0$
\[
\lambda_1(\Delta^{(M_j,g_j)}) \geq \frac{c}{\text{diam}^2(M_j,g_j)}.
\]
(13)

In two dimensions the spectrum of $\Delta_0$ fully determines the spectra of forms of all degree. The analogous questions for surfaces were studied by Wolpert [49], Hejahl [31] and Ji [36] and a sharp estimate for the accumulation rate was obtained by Ji and Zworski [37]. In addition Colbois and Courtois [16, 17] proved that the eigenvalues below the bottom of the essential spectrum are limits of eigenvalues of $M_j$ for both Riemann surfaces and hyperbolic three manifolds.

Problems of this kind don’t arise in dimensions greater than or equal to four (cf. [29]), because the number of complete hyperbolic manifolds of volume less than or equal to a given constant is finite in this case.

In the case of the classical Dirac operator Bär [5] proved:

Theorem 7 (Bär) On a degenerating sequence of oriented, hyperbolic, three compact without boundary manifolds $(M_j, g_j)_{j \geq 0}$ for any spin structure $(s_j)_{j \geq 0}$ on $M_j$ the lower eigenvalues of the Atiyah–Singer operator $D^{(M_j,g_j)}_{s_j}$ do not accumulate. More precisely, there exists an integer $N_0 \in \mathbb{N}$ such that
\[
|\lambda_{N_0}(D^{(M_j,g_j)}_{s_j})| = O(1) \quad (j \to \infty).
\]
(14)

The different behaviour of spin Laplacian and Laplacian on forms is due to topological reasons. We can improve Theorem 7 proving, by means of Theorem 2, that in Theorem 8 $N_0 = 1$ can be chosen and providing an explicit lower bound for the first non zero eigenvalue of the Dirac operator.

Theorem 8 On a degenerating sequence of oriented, hyperbolic, three spin compact without boundary manifolds for any choice of the spin structures the lower eigenvalues of the Atiyah–Singer operator do not accumulate and the first positive non zero eigenvalue is bounded from below by a positive uniform constant $c > 0$
\[
\lambda_1(D^{(M_j,g_j)}_{s_j}) \geq c.
\]
(15)

2 Dirac bundles

The purpose of this chapter is to recall some basic definitions concerning the theory of Dirac operators, establishing the necessary self contained notation and introducing the standard examples. The general references are [2, 9, 11, 38].
2.1 Dirac bundle

Definition 1  (Dirac bundle) The quadruple \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\), where

(i) \(V\) is a complex (real) vector bundle over the Riemannian manifold \((M, g)\) with Hermitian (Riemannian) structure \(\langle \cdot, \cdot \rangle\),

(ii) \(\nabla : C^\infty(M, V) \to C^\infty(M, T^*M \otimes V)\) is a connection on \(M\),

(iii) \(\gamma : \text{Cl}(M, g) \to \text{Hom}(V)\) is a real algebra bundle homomorphism from the Clifford bundle over \(M\) to the real bundle of complex (real) endomorphisms of \(V\), i.e. \(V\) is a bundle of Clifford modules,

(iv) \(\gamma(v)^* = -\gamma(v), \forall v \in TM\) i.e. the Clifford multiplication by tangent vectors is fiberwise skew-adjoint with respect to the Hermitian (Riemannian) structure \(\langle \cdot, \cdot \rangle\).

(v) \(\nabla \langle \cdot, \cdot \rangle = 0\) i.e. the connection is Leibnizian (Riemannian). In other words it satisfies the product rule:

\[
d\langle \varphi, \psi \rangle = \langle \nabla \varphi, \psi \rangle + \langle \varphi, \nabla \psi \rangle, \quad \forall \varphi, \psi \in C^\infty(M, V). \tag{16}
\]

(vi) \(\nabla \gamma = 0\) i.e. the connection is a module derivation. In other words it satisfies the product rule:

\[
\nabla(\gamma(w)\varphi) = \gamma(\nabla g w)\varphi + \gamma(w)\nabla \varphi, \quad \forall \varphi, \psi \in C^\infty(M, V),
\]

\[
\forall w \in C^\infty(M, \text{Cl}(M, g)). \tag{17}
\]

Among the different geometric structures on Riemannian Manifolds satisfying the definition of a Dirac bundle (cf. [26]) the canonical example is the spinor bundle.

Definition 2  (Spin manifold) \((M, g, s)\) is called a spin manifold if and only if

1. \((M, g)\) is a \(m\)-dimensional oriented Riemannian manifold.
2. \(s\) is a spin structure for \(M\), i.e. for \(m \geq 3\) \(s\) is a Spin\((m)\) principal fibre bundle over \(M\), admitting a double covering map \(\pi : s \to \text{SO}(M)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
s \times \text{Spin}(m) & \longrightarrow & s \\
\downarrow \pi \times \Theta & & \quad \downarrow \pi \\
\text{SO}(M) \times \text{SO}(m) & \longrightarrow & \text{SO}(M)
\end{array}
\]

where \(\text{SO}(M)\) denotes the \(\text{SO}(m)\) principal fiber bundle of the oriented basis of the tangential spaces, and \(\Theta : \text{Spin}(m) \to \text{SO}(m)\) the canonical double covering. The maps \(s \times \text{Spin}(m) \to s\) and \(\text{SO}(M) \times \text{SO}(m) \to \text{SO}(M)\) describe the right action of the structure groups \(\text{Spin}(m)\) and \(\text{SO}(m)\) on the principal fibre bundles \(s\) and \(\text{SO}(M)\) respectively.

When \(m = 2\) a spin structure on \(M\) is defined analogously with \(\text{Spin}(m)\) replaced by \(\text{SO}(2)\) and \(\Theta : \text{SO}(2) \to \text{SO}(2)\) the connected two-sheet covering. When \(m = 1\) \(\text{SO}(M) \cong M\) and a spin structure is simply defined to be a two-fold covering of \(M\).

The vector bundle over \(M\) associated to \(s\) w.r.t the spin representation \(\rho\) i.e.

\[
\Sigma M := s \times \mathbb{C}^l \quad l := 2^{\left\lceil \frac{m}{2} \right\rceil}
\]

is called spinor bundle over \(M\), see [2] page 18.
Example 2.1 (Spinor bundle as a Dirac bundle) Let $(M, g, s)$ be a spin manifold of dimension $m$. We can make the spinor bundle into a Dirac bundle by the following choices:

- $V := \Sigma M$ : spinor bundle, rank$(V) = l$
- $(\cdot, \cdot)$: Riemannian structure induced by the standard Hermitian product in $\mathbb{C}^l$ (which is Spin$(m)$-invariant) and by the representation $\rho$.
- $\nabla = \nabla^\Sigma$: spin connection = lift of the Levi-Civita connection to the spinor bundle.

\[
\gamma : T M \rightarrow \text{Hom}(V)
\]

where $\gamma(v) := v \cdot \varphi$ ($\cdot$ is the Clifford product)

We identified $TM$ with $SO(M)$ $\times \mathbb{R}^m$ ($\alpha$ is the standard representation of $\mathbb{R}^m$) and $\Sigma M$ with $s \times \mathbb{R}^l$. Since $\gamma^2(v) = -g(v, v)\mathbb{I}$, by the universal property, the map $\gamma$ extends uniquely to a real algebra bundle endomorphism $\gamma : \text{Cl}(M, g) \rightarrow \text{Hom}(V)$.

Example 2.2 (Exterior algebra bundle as a Dirac bundle) Let $(M, g)$ be a $C^\infty$ Riemannian manifold of dimension $m$. The tangent and the cotangent bundles are identified by the $\flat$-map defined by $v\flat(w) := g(v, w)$. Its inverse is denoted by $\sharp$. The exterior algebra can be seen as a Dirac bundle after the following choices:

- $V := \Lambda(T^*M) = \bigoplus_{j=0}^m A^j(T^*M)$ : exterior algebra over $M$
- $(\cdot, \cdot)$: Riemannian structure induced by $g$
- $\nabla$: (lift of the) Levi-Civita connection

\[
\gamma : T M \rightarrow \text{Hom}(V)
\]

where $\text{ext}(v) \varphi := v^\flat \wedge \varphi$ and $\text{int}(v) \varphi := \varphi(v, \cdot)$. Since $\gamma^2(v) = -g(v, v)\mathbb{I}$, by the universal property, the map $\gamma$ extends uniquely to a real algebra bundle endomorphism $\gamma : \text{Cl}(M, g) \rightarrow \text{Hom}(V)$.

2.2 Dirac operator and Dirac Laplacian

Definition 3 Let $(V, (\cdot, \cdot), \nabla, \gamma)$ be a Dirac bundle over the Riemannian manifold $(M, g)$. The Dirac operator $Q : C^\infty(M, V) \rightarrow C^\infty(M, V)$ is defined by

\[
\begin{array}{ccc}
C^\infty(M, V) & \xrightarrow{\nabla} & C^\infty(M, T^*M \otimes V) \\
\downarrow & & \downarrow \\
\mathcal{Q} := \gamma \circ (\otimes \mathbb{I}) \circ \nabla & \xleftarrow{\gamma} & C^\infty(M, TM \otimes V)
\end{array}
\]  

(19)

The square of the Dirac operator $P := Q^2 : C^\infty(M, V) \rightarrow C^\infty(M, V)$ is called the Dirac Laplacian.

Remark 1 The Dirac operator $Q$ depends on the Riemannian metric $g$ and on the homomorphism $\gamma$. If different metrics or homomorphisms are considered, then the notation $Q = Q_g^\gamma = Q^g = Q_\gamma$ is utilized to avoid ambiguities.
**Proposition 1** The Dirac operator is a first order differential operator over $M$. Its leading symbol is given by the Clifford multiplication:

$$\sigma_L(Q)(x, \xi) = i \gamma(\xi^\sharp)$$

(20)

where $i := \sqrt{-1}$. The Dirac operator has the following local representation:

$$Q(\varphi|_U) = \sum_{j=1}^m \gamma(e_j) \nabla_{e_j}(\varphi|_U)$$

(21)

for a local orthonormal frame $\{e_1, \ldots, e_m\}$ for $T_M|_U$ and a section $\varphi \in C^\infty(M, V)$.

The Dirac Laplacian is a second order partial differential operator over $M$. Its leading symbol is given by the Riemannian metric:

$$\sigma_L(Q^2)(x, \xi) = g_x(\xi^\sharp, \xi^\sharp) 1_{V_x} \forall x \in M, \xi \in T^*_x M.$$  

(22)

**Example 2.3** (Atiyah–Singer operator and spin Laplacian) The Dirac operator in the case of spin manifolds $(M, g, S)$ is the Atiyah–Singer operator $D^S_g$ on the sections of the spinor bundle $\Sigma M$. The Dirac Laplacian $\Delta^S_g := (D^S_g)^2$ is the spin Laplacian.

**Example 2.4** (Euler and Laplace–Beltrami operators) The Dirac operator in the case of the exterior algebra bundle over Riemannian manifolds $(M, g)$ is the Euler operator $d + \delta$ on forms on $M$. The Dirac Laplacian $\Delta := (d + \delta)^2 = d\delta + \delta d$ is the Laplace–Beltrami operator.

### 2.3 Dirac complexes

**Definition 4** (Normalized orientation) Let $(V, \langle \cdot, \cdot \rangle, \nabla, \gamma)$ be a Dirac bundle over the oriented Riemannian manifold $(M, g)$. We consider a positively oriented local orthonormal frame $\{e_1, \ldots, e_m\}$ for $T M$. Then the product

$$S := i^{\frac{m+1}{2}} \gamma(e_1) \cdots \gamma(e_m) \in \text{Hom}(V)$$

(23)

is called the normalized orientation of the Dirac bundle.

**Proposition 2** The normalized orientation $S$ is well defined and independent of the choice of the positively oriented local orthonormal frame. Moreover, it has the following properties:

1. $S^2 = 1$
2. $\nabla S = 0$
3. $QS = (-1)^{m-1} SQ$.

**Definition 5** (Dirac complex) Let $Q$ be an operator of Dirac type for the vector bundle $V$ over the Riemannian manifold $(M, g)$ and $T \in \text{Hom}(V)$. $(Q, T)$ is called a complex of Dirac type if and only if

1. $T^2 = 1$
2. $QT = -TQ$.

**Notation 9**

$$\Pi_{\pm} := \frac{1 \mp T}{2} \quad V_{\pm} := \Pi_{\pm}(V) \quad Q_{\pm} := Q|_{C^\infty(M, V_{\pm})}.$$  

(24)

**Proposition 3** $Q_{\pm} : C^\infty(M, V_{\pm}) \to C^\infty(M, V_{\mp})$

\[ Springer \]
2. \( Q = \begin{bmatrix} 0 & Q_+ \\ Q_+ & 0 \end{bmatrix} : C^\infty(M, V_+ \oplus V_-) \longrightarrow C^\infty(M, V_+ \oplus V_-) \)

\[
0 \longrightarrow C^\infty(M, V_+) \xrightarrow{Q_+} C^\infty(M, V_-) \xrightarrow{Q_-} C^\infty(M, V_+) \longrightarrow 0
\]

is a complex i.e. \( Q \cdot Q_+ = 0 \).

3. \( (\Pi\pm Q)^2 = 0 \) \( \Pi_+ Q + \Pi_- Q = Q \) \( \Pi_+ Q \Pi_- Q + \Pi_- Q \Pi_+ Q = Q^2 \).

**Example 2.5** (Exterior algebra in even dimensions)

\( T := S \): normalized orientation.

\((d + \delta, S)\): signature complex.

**Example 2.6** (Exterior algebra in any dimensions)

\( T \) defined as \( T|_{\Lambda^k(T^*M)} := (-1)^k \Lambda^k(T^*M) \)

\((d + \delta, T)\): (rolled up) De Rham complex.

**Example 2.7** (Spinor bundle in even dimension)

\( T := S \): normalized orientation.

\((D, S)\): spin complex.

By Proposition 2 any Dirac bundle over an even dimensional manifold can be made into a complex of Dirac type by means of the normalized orientation. In odd dimensions this is not possible, because normalized orientation and Dirac operator commute.

The restriction of a Dirac bundle to a one codimensional submanifold is again a Dirac bundle, as following theorem (cf. [4, 24]) shows.

**Theorem 10** Let \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) be a Dirac bundle over the Riemannian manifold \((M, g)\) and let \( N \subset M \) be a one codimensional submanifold with normal vector field \( v \). Then \((N, g|_N)\) inherits a Dirac bundle structure by restriction. We mean by this that the bundle \( V|_N \), the connection \( \nabla|_{C^\infty(N, V|_N)} \), the real algebra bundle homomorphism \( \gamma_N := -\gamma(V)\gamma|_{C(N, \mathbb{R}|_N)} \), and the Hermitian (Riemannian) structure \( \langle \cdot, \cdot \rangle|_N \) satisfy the defining properties (iv)–(vi). The quadruple \((V|_N, \langle \cdot, \cdot \rangle|_N, \nabla|_{C^\infty(N, V|_N)}, \gamma_N)\) is called the **Dirac bundle structure induced on** \( N \) by the Dirac bundle \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) on \( M \).

For a spin manifold of arbitrary dimension we will now construct a vector bundle isomorphism \( T \) which anticommutes with the Atiyah–Singer operator \( Q \) making a generic spin bundle to a complex of Dirac type \((Q, T)\). Inspired by [6], we embed a given manifold into a cylinder.

**Definition 6** *(Generalized cylinder)* Let \((M, g, \text{Spin}(M))\) be a spin manifold of dimension \( n \), Riemannian metric \( g \) and spin structure \( \text{Spin}(M) \). The manifold \( Z := I \times M \), where \( I \) denotes an interval of the real line, equipped with the Riemannian metric \( g^Z(u, x) := du^2 \otimes g(x) \) and with the spin structure \( \text{Spin}(Z) := \text{Spin}(I) \times \text{Spin}(M) \), with double covering map

\[
\pi : \text{Spin}(Z) = \text{Spin}(I) \times \text{Spin}(M) \rightarrow SO(Z) = SO(I) \times SO(M), \quad (\pi|_{\text{Spin}(I)}, \pi|_{\text{Spin}(M)})
\]

is a spin manifold \((Z, g^Z, \text{Spin}(Z))\) termed generalized cylinder, and \( i : SO(M) \rightarrow SO(Z), (e_1, \ldots, e_n) \mapsto (v, e_1, \ldots, e_n) \) denotes the canonical embedding.

It can easily be proved (cf. [6, Chapter 5] and [33, Chapter 2]) that
Proposition 4 The original spin manifold and the generalized cylinder satisfy the following properties:

1. \( \text{Spin}(M) = \pi^{-1}(i(SO(M))) \).
2. \( \gamma^M \) and \( T := i\gamma^Z \left( \frac{\partial}{\partial u} \right) \) anticommute. In fact, for all \( v \in TM \)
   \[
   \gamma^M(v)T = -T\gamma^M(v).
   \tag{26}
   
3. \((\tilde{Q}^M, T)\) is a complex of Dirac type, where \( \tilde{Q}^M := Q^M \) if \( n \) is even, and \( \tilde{Q}^M := \text{diag}(Q^M, Q^M) \) if \( n \) is odd, is termed the extrinsic Dirac operator. In this context \( Q^M \) is termed intrinsic Dirac operator.
4. \( \nabla^M T = 0 \).

3 Spectral properties of the Dirac operator

We consider Dirac bundles over compact manifolds, possibly with boundary. The aim of this section is to summarize “the state of the art” concerning the generic results about spectral results, especially in connection with boundary conditions. The existence of a regular discrete spectral resolution for both Dirac and Dirac Laplacian operators under the appropriate boundary conditions is a special case of the standard elliptic boundary problems theory developed by Seeley [44, 45] and Greiner [27, 28]. The general references are [30, 34]. See [11, 26] for the specific case of the Dirac and Dirac Laplacian operators.

3.1 Dirac and Dirac Laplacian spectra on manifolds without boundary

The Dirac operator \( Q \) and the Dirac Laplacian \( P \) for a Dirac bundle \( V \) over a compact Riemannian manifold without boundary are easily seen by Green’s formula to be symmetric operators for the \( C^\infty \)-sections of Dirac bundle. Taking the completion of the differentiable sections of \( V \) in the Sobolev \( H^1 \) – and respectively \( H^2 \)-topology, leads to two selfadjoint operators in \( L^2(V) \).

Theorem 11 The Dirac \( Q \) and the Dirac Laplacian \( P \) operators of a Dirac bundle over a compact Riemannian manifold \( M \) without boundary have a regular discrete spectral resolution with the same eigenspaces. It exists a sequence \( (\varphi_j, \lambda_j)_{j \in \mathbb{Z}^*} \) such that \( (\varphi_j)_{j \in \mathbb{Z}^*} \) is an orthonormal basis of \( L^2(V) \) and that for every \( j \in \mathbb{Z}^* \) it must hold \( Q\varphi_j = \lambda_j\varphi_j \) \( P\varphi_j = \lambda_j^2\varphi_j \) and \( \varphi_j \in C^\infty(V) \). The eigenvalues of the Dirac operator \( (\lambda_j)_{j \in \mathbb{Z}^*} \) are a monotone increasing real sequence converging to \( \pm \infty \) for \( j \to \pm \infty \). The eigenvalues of the Dirac Laplacian are the squares of the eigenvalues of the Dirac operator and hence not negative.

Therefore, for Dirac bundle over a manifold without boundary the knowledge of the spectrum for the Dirac operator and the Dirac Laplacian are equivalent. Moreover, in the case of a Dirac complex the spectrum of the Dirac operator is symmetric with respect to the origin.

Proposition 5 If there is an isomorphism \( T \) for the Dirac bundle \( V \) anticommuting with the Dirac operator \( Q \), then the discrete spectral resolution of Theorem 11 can be chosen such that the equalities \( \lambda_{-j} = -\lambda_{+j} \) and \( \varphi_{-j} = T\varphi_{+j} \) hold for every \( j \in \mathbb{N}^* \). In particular, the dimension of the space of harmonic sections is always even.
Remark 2 An interesting consequence of Proposition 5 and of Proposition 4 is that the spectrum of the extrinsic classical Dirac operator is symmetric with respect to the origin in any dimension. The spectrum of the intrinsic classical Dirac operator is always symmetric in even dimensions. In odd dimensions nothing can be said a priori: there are cases, where the spectrum of the intrinsic Dirac operator is not symmetric as Berger’s spheres in dimension \( \equiv 3 \mod 4 \) (cf. [4]), or some of the three dimensional compact Bieberbach manifolds beside the torus (see [42] for details), and cases where it is symmetric as Berger’s spheres in dimension \( \equiv 1 \mod 4 \).

3.2 Dirac and Dirac Laplacian spectra on manifolds with boundary

The case of manifolds with boundary is more complex and the spectra of the Dirac and Dirac Laplacians are no more equivalent as they are in the boundaryless case. Moreover, while for the Dirac Laplacian it is always possible to find local elliptic boundary conditions allowing for a discrete spectral resolution, this is not always true for the Dirac operator. The Dirac Laplacian \( P \) for a Dirac bundle \( V \) over a compact Riemannian manifold with boundary is easily seen by Green’s formula to be a symmetric operator for the \( \mathcal{C}^\infty \)-sections of Dirac bundle if we impose the Dirichlet boundary condition

\[
\text{BD} \varphi := \varphi |_{\partial M} = 0 \quad \text{or the Neumann boundary condition} \quad \text{BN} \varphi = \nabla_v \varphi |_{\partial M} = 0.
\]

Taking the completion of the differentiable sections of \( V \) satisfying the boundary conditions in the Sobolev \( H^2 \)-topology, leads to a selfadjoint operator in \( L^2(V) \).

Theorem 12 The Dirac Laplacian \( P \) of a Dirac bundle over a compact Riemannian manifold \( M \) with boundary under the Neumann or the Dirichlet condition has a regular discrete spectral resolution \((\varphi_j, \lambda_j)_{j \geq 0}\). This means that \((\varphi_j)_{j \geq 0}\) is an orthonormal basis of \( L^2(V) \) and that for every \( j \geq 0 \) it must hold \( P \varphi_j = \lambda_j \varphi_j \), \( \varphi_j \in \mathcal{C}^\infty(V) \), and \( B \varphi_j = 0 \) for either \( B = B_D \) or \( B = B_N \). The eigenvalues \((\lambda_j)_{j \geq 0}\) are a monotone increasing real sequence bounded from below and converging to infinity. The Dirichlet eigenvalues are all strictly positive. The Neumann eigenvalues are all but for a finite number strictly positive.

The situation for the Dirac operator is more subtle. Although it is -again by Green’s formula- a symmetric operator under the Dirichlet boundary condition, it is not selfadjoint. As a matter of fact the Dirichlet boundary condition is elliptic for the Dirac Laplacian but not for the Dirac operator. If we are looking for local elliptic boundary conditions for the Dirac operator, we need to introduce the following

Definition 7 Let \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) be a Dirac bundle over a manifold \( M \) with boundary \( \partial M \). The isomorphism \( \chi \in \text{Hom}(V|_{\partial M}) \) is called boundary chirality operator for the Dirac bundle if satisfies \( \chi^2 = 1 \) and anticommutes with the Clifford multiplication, i.e. \( \chi \gamma(v) + \gamma(v) \chi = 0 \) for any \( v \in T_M|_{\partial M} \). The corresponding boundary condition operator is given by \( B_\pm := \frac{1}{2}(1 \mp \chi \gamma(v)) \).

In the even dimensional case one can always find boundary chirality operators for any Dirac bundle: it suffices to choose \( \chi := S|_{\partial M} \), where \( S \) denotes the normalized orientation. For the special case of the exterior algebra bundle in any dimension the choice \( \chi := \text{ext}(v) + \text{int}(v) \) leads to the absolute and relative boundary conditions for differential forms which are elliptic for the Euler operator \( d + \delta \).

In the odd dimensional case there are obstructions to the existence of local boundary chirality operators for Dirac bundles. As a matter of fact, if there exist a local elliptic boundary condition for the Dirac operator, then \( \text{tr}(S) = 0 \). The non vanishing of the trace of
the normalized orientation, is therefore the topological obstruction, termed the Atiyah–Bott obstruction, for the existence of local elliptic boundary conditions for the full Dirac Operator. In even dimension this obstruction always vanishes because the full Dirac operator and the normalized orientation always anticommute. In odd dimensions the obstruction for the full Dirac operator can or cannot vanish. It vanishes for the classic Dirac operator. For the chiral Dirac operator, defined on the sections of the eigenbundles of the normalized orientation the situation is complementary. In odd dimension the obstruction vanishes, while in even ones it does not, see [25] page 248 and [26] page 102.

An elliptic boundary condition for both full and chiral Dirac operator always exists in any dimension, but it is defined by mean of a zero order pseudodifferential operator, the spectral projections of the Dirac operator on the boundary. This is the famous Atiyah–Patodi–Singer boundary condition (see [11, 44]). In a neighbourhood of the boundary \( \partial M \) it is possible to decompose the Dirac operator as

\[
Q = \gamma(v)(\nabla_v + A).
\]

(27)

Remark that it is not necessary to assume that the geometric structures are a product on this neighbourhood. The operator \( A_{\partial M} \) is an operator of Dirac type for \( V|_{\partial M} \) over the boundaryless manifold \( \partial M \). The operator \( A_{APS} := A_{\partial M} + \frac{i}{2}H\mathbb{1} \), where \( H \) denotes the mean curvature of the boundary, is an operator of Dirac type for \( \partial M \) and, by Theorem 11, it has a discrete regular spectral resolution \((\psi_j, \mu_j)_{j \geq 0}\). The subspace of \( L^2(V) \) defined by \( E_\mu(A_{APS}) := \ker(A_{APS} - \mu \mathbb{1}) \) is the eigenspace of \( A_{APS} \) if \( \mu \) is in the spectrum of \( A_{APS} \) and the zero subspace otherwise.

**Definition 8** Let \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) be a Dirac bundle over the oriented Riemannian manifold \((M, g)\) with Dirac operator \( Q \) and normalized orientation \( S \). The generalized Atiyah–Patodi–Singer boundary condition for \( Q \) is given by \( B_{APS}(\varphi|_{\partial M}) = 0 \), where \( B_{APS} \) denotes the orthogonal projection in \( L^2(V|_{\partial M}) \) onto

\[
\bigoplus_{\mu < 0} E_\mu(A_{APS}) \oplus \frac{1}{2}(\mathbb{1} - S)(E_0(A_{APS})).
\]

(28)

The Dirac operator \( Q \) for a Dirac bundle \( V \) over a compact Riemannian manifold with boundary is easily seen by Green’s formula to be a symmetric operator for the \( C^\infty \)-sections of Dirac bundle if we impose the boundary conditions \( B_\pm \) induced by a boundary chirality operator or by the generalized APS boundary condition. Taking the completion of the differentiable sections of \( V \) satisfying the boundary condition \( B \) in the Sobolev \( H^1 \)-topology, leads to a selfadjoint operator in \( L^2(V) \). Of course, the associated first order boundary conditions for the Dirac Laplacian are elliptic as well and lead to a self adjoint operator with pure point spectrum if we define the domain of \( P \) as the completion of the differentiable sections of \( V \) satisfying the boundary conditions \( B \oplus BQ \) in the Sobolev \( H^2 \)-topology. In [22] it is given an elementary proof (with no reference to the calculus of elliptic pseudodifferential operators as in [13, 34]) of the following result:

**Theorem 13** The Dirac \( Q \) and the Dirac Laplacian \( P \) operators of a Dirac bundle over a compact Riemannian manifold \( M \) with boundary have under the boundary conditions \( B \) and \( B \oplus BQ \) respectively, for either \( B = B_\pm \), (if a boundary chirality operator exists), or \( B = B_{APS} \) a regular discrete spectral resolution with the same eigenspaces. It exists a sequence \((\varphi_j, \lambda_j)_{j \in \mathbb{Z}^*} \) such that \((\varphi_j)_{j \in \mathbb{Z}^*} \) is an orthonormal basis of \( L^2(V) \) and that for every \( j \in \mathbb{Z}^* \) it must hold \( Q\varphi_j = \lambda_j \varphi_j \) \( P\varphi_j = \lambda_j^2 \varphi_j \), \( \varphi_j \in C^\infty(V) \), \( B(\varphi_j|_{\partial M}) = 0 \) and \( B((Q\varphi_j)|_{\partial M}) = 0 \). The eigenvalues of the Dirac operator \((\lambda_j)_{j \in \mathbb{Z}^*} \) under the boundary conditions...
condition B are a monotone increasing real sequence converging to \(±\infty\) for \(j\to±\infty\). The eigenvalues of the Dirac Laplacian are the squares of the eigenvalues of the Dirac operator and hence not negative.

Remark that the Dirac operator under the complementary APS-boundary condition, that is, the orthogonal projection from \(L^2(V)\) onto

\[
\bigoplus_{\mu>0} E_\mu(A_{\text{APS}}) \oplus \frac{1}{2}(\mathbb{1} + S)(E_0(A_{\text{APS}})), \tag{29}
\]

is only symmetric but not selfadjoint and thus must not have a discrete real spectrum. The complementary APS-boundary condition is not elliptic.

Extending the result in the boundaryless case, for a Dirac complex \((Q, T)\) preserving the boundary condition \(B\), that is, where \(T\) anticommutes with \(B\), the spectrum of the Dirac operator is symmetric with respect to the origin.

**Proposition 6** If there is an isomorphism \(T\) for the Dirac bundle \(V\) anticommuting with the Dirac operator \(Q\) and commuting with the boundary condition \(B\) for either \(B = B_+\) or \(B = B_-\) or \(B = B_{\text{APS}}\), then the discrete spectral resolution of Theorem 13 can be chosen such that the equalities \(\lambda_{-j} = -\lambda_{+j}\) and \(\varphi_{-j} = T\varphi_{+j}\) hold for every \(j \in \mathbb{N}^*\). In particular, the dimension of the space of harmonic sections satisfying the boundary condition \(B\) is always even.

The local boundary conditions \(B_\pm\) defined by mean of a boundary chirality operator \(\chi \in \text{Hom}(V|_{\partial M})\) are preserved by the Dirac complex \((Q, T)\) on \(V\) if and only if \(\gamma(v)\chi\) and \(T|_{\partial M}\) commute. This is always the case for Dirac bundles in even dimensions, if we choose \(T := S\) and \(\chi := S|_{\partial M}\), where \(S\) is the normalized orientation of the Dirac bundle. A special case, where Proposition 6 in any dimension for local elliptic boundary conditions, is the De Rham complex with either the absolute or relative boundary conditions.

The global boundary condition \(B_{\text{APS}}\) defined by mean of the projection onto the eigenspaces of the non positive eigenvalues of \(A_{\text{APS}}\) are preserved by the Dirac complex \((Q, T)\) on \(V\) if and only if \(A_{\text{APS}}\) and \(T|_{\partial M}\) commute. This is always the case for Dirac bundles in any dimension, if we choose \(T := S\), where \(S\) is the normalized orientation of the Dirac bundle.

### 4 Dirac cohomology and Hodge theory under boundary conditions

In this section we will prove Theorem 1. We will have to introduce for Dirac bundles concepts which mimick the situation for differential forms like derivation, coderivation, absolute and relative boundary conditions.

**Proposition 7** Let \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) be a Dirac bundle over the Riemannian manifold \((M, g)\) with a bundle isomorphism \(T\) on \(V\) such that \(\nabla := iT\gamma\) anticommutes with \(\gamma\) and with the Dirac operator \(Q\). The tuple \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma, \nabla)\) defines a \((1, 1)\)-Dirac bundle structure with corresponding Dirac operators \(Q\) and \(\bar{Q}\). The operators

\[
d := \frac{1}{2}(Q - i\bar{Q}) = \frac{1}{2}(\mathbb{1} + T)Q \quad \text{and} \quad \delta := \frac{1}{2}(Q + i\bar{Q}) = \frac{1}{2}(\mathbb{1} - T)Q \tag{30}
\]

are called **derivative and coderivative operators** on \(M\) and have following properties
1. The derivative defines a complex: \( d^2 = 0 \).
2. The coderivative defines a complex: \( \delta^2 = 0 \).
3. The Dirac operator can be decomposed as \( Q = d + \delta \).
4. The Dirac Laplacian can be decomposed as \( P := Q^2 = d\delta + \delta d \).

The zero-order boundary operators

\[
B_\pm := \frac{1}{2} \mp T \gamma(v)
\]  

(31)

**define the absolute** \( B_- \) **and relative** \( B_+ \) **boundary conditions** on \( \partial M \) for the Dirac operator \( Q \) and have following properties

1. \( B_+ \oplus B_- = \mathbb{1} \).
2. \( B_+^2 = B_+ = B_-^* \).
3. \( B_-^2 = B_- = B_+^* \).
4. \( \gamma(v)B_\pm = B_\mp \gamma(v) \) and \( \gamma(v) : \ker(B_+) \oplus \ker(B_-) \to \ker(B_-) \oplus \ker(B_+) \).

The following Green’s formula holds for all smooth sections \( \varphi, \psi \) of the Dirac bundle

\[
(d\varphi, \psi) - (\varphi, \delta\psi) = -\int_{\partial M} \text{dvol}_{\partial M} \langle \gamma(v)B_-\varphi, \psi \rangle = -\int_{\partial M} \text{dvol}_{\partial M} \langle \gamma(v)\varphi, B_+\psi \rangle.
\]  

(32)

For the Dirac Laplacian the corresponding first order boundary operators are \( C_- := B_- \oplus B_-d \) (absolute boundary condition) and \( C_+ := B_+ \oplus B_+\delta \) (relative boundary condition). In fact

1. The absolute boundary condition is preserved by the derivative operator: \( B_-\varphi|_{\partial M} = 0 \Rightarrow B_-d\varphi|_{\partial M} = 0 \).
2. The relative boundary condition is preserved by the coderivative operator \( B_+\varphi|_{\partial M} = 0 \Rightarrow B_+\delta\varphi|_{\partial M} = 0 \).

**Proof** The properties of derivative and coderivative are a direct consequence of their definition where an isomorphism \( T \) such that \( (Q, T) \) is a Dirac complex was utilized. The properties of the boundary conditions follows from the fact that \( (i\gamma(v)\overline{\gamma(v)})^2 = \mathbb{1} \). The Green’s formula (32) follows from the corresponding Green’s formulae for the Dirac operators \( Q \) and \( \overline{Q} \). To prove the preservation of the absolute boundary condition by the derivative operator, we note that, by Green’s formula

\[
(d\varphi, \psi) = (\varphi, \delta\psi),
\]  

(33)

for a \( \varphi \) satisfying \( B_-\varphi|_{\partial M} = 0 \) and any \( \psi \). Applying Green’s formula to \( d\varphi \) and \( \psi \) we obtain

\[
(dd\varphi, \psi) - (d\varphi, \delta\psi) = -\int_{\partial M} \text{dvol}_{\partial M} \langle \gamma(v)B_-d\varphi, \psi \rangle
\]  

(34)

The left hand side of (34) vanishes because of (33) and the fact that \( d^2 = 0 \). Thus, the boundary integral vanishes for all \( \psi \) and so does \( B_-d\varphi|_{\partial M} \). The proof of the preservation of the relative boundary condition under the coderivative operator reads analogously. \( \square \)

After having introduced operators and boundary condition we would like to study the spectrum.
Proposition 8 Let $H^1(M, V)$, $H^1_0(M, V)$ and $H^1_{B_{\pm}}(M, V)$ be the domain of definitions of $d$, $d_0$, $d_{B_{\pm}}$ and $\delta$, $\delta_0$, $\delta_{B_{\pm}}$ and $Q$, $Q_0$, $Q_{B_{\pm}}$, respectively. Let $H^2(M, V)$, $H^2_0(M, V)$, $H^2_{B_{\pm}}(M, V)$ the domain of definitions of $P$, $P_0$ and $P_{B_{\pm}}$. They satisfy following properties:

1. $d \subset \delta^*_0$, $\delta \subset d^*_0$, $Q_0 \subset Q^*_0$ and $P_0 \subset P^*_0$.
2. $d^*_{B_{\pm}} = \delta_{B_{\pm}}$ and $\delta^*_{B_{\pm}} = d_{B_{\pm}}$.
3. $(Q, B_{\pm})$ are elliptic boundary value problems and $Q^*_{B_{\pm}} = Q_{B_{\pm}}$ are selfadjoint operators. If $M$ is compact, the operators $Q_{B_{\pm}}$ have discrete spectra and the corresponding eigensections are smooth.
4. $(P, C_{\pm})$ are elliptic boundary value problems and $P^*_{B_{\pm}} = P_{B_{\pm}}$ are selfadjoint operators. If $M$ is compact, the operators $P_{B_{\pm}}$ have non negative discrete spectra and the corresponding eigensections are smooth.

Proof The proof is based on the Green's formula (32) and standard elliptic operator theory.

Theorem 14 (Orthogonal decomposition of Dirac sections) Let $(V, \langle \cdot, \cdot \rangle, \nabla, \gamma)$ be a Dirac bundle over the compact Riemannian manifold $(M, g)$ admitting a bundle isomorphism $T$ anticommuting with $\gamma$ and with the Dirac operator $Q$, such that $T^2 = 1$ holds, and

\[
\begin{align*}
\mathcal{O}(M, V) &:= C^\infty(M, V) \text{ be the smooth sections of the Dirac bundle on } M, \\
\mathcal{H}_{B_{\pm}}(M, V) &\text{ be the harmonic sections of the Dirac bundle on } M \text{ satisfying the absolute or relative, respectively, boundary condition,} \\
\mathcal{O}^d_{B_{\pm}}(M, V) &:= \{ \varphi \in \mathcal{O}_{B_{\pm}}(M, V) | \exists \psi \in \mathcal{O}(M, V) : d\psi = \varphi \} \text{ be the smooth exact Dirac sections on } M \text{ satisfying the absolute or relative, respectively, boundary condition,} \\
\mathcal{O}^\delta_{B_{\pm}}(M, V) &:= \{ \varphi \in \mathcal{O}^d_{B_{\pm}}(M) | \exists \psi \in \mathcal{O}(M, V) : \delta\psi = \varphi \} \text{ be the smooth coexact Dirac sections on } M \text{ satisfying the absolute or relative, respectively, boundary condition.}
\end{align*}
\]

Then, the following orthogonal decomposition holds:

\[
C^\infty(M, V) = \mathcal{H}_{B_{\pm}}(M, V) \oplus \mathcal{O}^d_{B_{\pm}}(M, V) \oplus \mathcal{O}^\delta_{B_{\pm}}(M, V). \tag{35}
\]

Proof The proof is based on standard elliptic operator theory and the fact that derivative and coderivative operators preserve the absolute and the relative, respectively, boundary condition.

Definition 9 (Dirac cohomology) The group

\[
\mathbb{H}_{B_{\pm}}(M, V) := \{ \omega \in \mathcal{O}_{B_{\pm}}(M, V) | d\omega = 0 \} / d\mathcal{O}^d_{B_{\pm}}(M, V) \tag{36}
\]

is called absolute, respectively, relative Dirac cohomology of the Dirac bundle.

Since we will not need it going forward, we mention without proof the following result

Theorem 15 The mappings

\[
I_{\pm} : \mathcal{H}_{B_{\pm}}(M, V) \to \mathbb{H}_{B_{\pm}}(M, V), \omega \mapsto I(\omega) := [\omega] \tag{37}
\]

are a natural isomorphisms between harmonic Dirac sections and Dirac cohomologies.

Remark 3 Of course decomposition (35) is a variation of the famous Hodge's Theorem and the isomorphisms (37) provide a similar result to De Rham's Theorem. The Dirac Cohomology is a Riemannian but not a topological invariant.
To motivate the terminology introduced so far, we prove that in the case of the Euler operator, for a particular choice of the bundle isomorphism $T$ for the exterior algebra bundle, the derivative and coderivative operators are the classical exterior and interior differentiation for forms, the Dirac Cohomologies are the De Rham cohomologies under the absolute and relative boundary conditions and Theorem 35 the classical Hodge decomposition theorem for differential forms on a manifold with boundary.

**Proposition 9** Let $(M, g)$ be an $m$ dimensional Riemannian manifold and $\{e_i\}_{i=1,\ldots,m}$ be a local orthonormal field of $TM$. Let $T_i := \text{int}(e_i) \otimes \text{ext}(e_i) - \text{ext}(e_i) \otimes \text{int}(e_i)$, and $T := \sum_{i=1}^{m} T_i P_i$, where the operator $P_i$ be the orthogonal projection onto $W_i := \{\text{ext}(e_i) \varphi | \varphi \text{ is a local section of } \Lambda(T^*M)\}$. The operator $T$ can be extended to $M$ by a partition of unit argument and satisfies the following properties:

1. $T^2 = 1$,
2. $\frac{1}{2} T (d + \delta) = d$
3. $\frac{1}{2} T (d + \delta) = \delta$,
4. $T (d + \delta) = -(d + \delta) T$,
5. Absolute boundary condition: $\text{int}(\nu) (\varphi)|_{\partial M} = 0 \Leftrightarrow B_+ (\varphi)|_{\partial M} = 0$,
6. Relative boundary condition $\text{ext}(\nu) (\varphi)|_{\partial M} = 0 \Leftrightarrow B_- (\varphi)|_{\partial M} = 0$,

where $B_\pm := \frac{1}{1+T} \gamma(\nu)$ for $\gamma(\nu) := \text{ext}(\nu) - \text{int}(\nu)$.

**Proof** This can be verified by a direct computation. \hfill \Box

5 Mayer–Vietoris’s scheme and generalization of Cheeger’s spectral estimate

In this section we will prove Theorem 2. We first have to introduce several technicalities. Let $M$ be a compact manifolds with boundary. If we impose the absolute boundary condition $B_\phi|_{\partial M} = 0$ on all Dirac eigensections considered, Theorem 14 and the preservation of the first order absolute boundary condition under the derivative $d$ will allow for a special variational characterization of the spectra for Dirac and Dirac Laplacian. Inspired by results for Laplace–Beltrami operator on forms (cf. [20]) and using Theorem 14, one can prove

**Lemma 1** Let $\lambda \in \text{spec}(P_{C_\pm})$ be a non zero eigenvalue of the Dirac Laplacian under absolute or relative boundary conditions, and

- $E_{B_\pm}(\lambda) := \{\varphi \in \Omega_{B_\pm}(M, V) | P \varphi = \lambda \varphi\}$ be Dirac eigensections with eigenvalue $\lambda$,
- $E_{B_\pm}^d(\lambda) := E_{B_\pm}(\lambda) \cap \Omega_{B_\pm}^d(M, V)$ be exact Dirac eigensections with eigenvalue $\lambda$,
- $E_{B_\pm}^s(\lambda) := E_{B_\pm}(\lambda) \cap \Omega_{B_\pm}^s(M, V)$ be coexact Dirac eigensections with eigenvalue $\lambda$.

Then:

1. $E_{B_\pm}(\lambda) = E_{B_\pm}^d(\lambda) \oplus E_{B_\pm}^s(\lambda)$
2. $d : E_{B_\pm}^s(\lambda) \longrightarrow E_{B_\pm}^d(\lambda)$ and $\delta : E_{B_\pm}^d(\lambda) \longrightarrow E_{B_\pm}^s(\lambda)$ are isomorphisms between finite dimensional subspaces of $L^2(M, V)$.
3. $E_{B_\pm}^d(\lambda) = d E_{B_\pm}^s(\lambda)$ and $E_{B_\pm}^s(\lambda) = \delta E_{B_\pm}^d(\lambda)$.

This lemma has an important consequence. The knowledge of the spectrum of the Dirac Laplacian on all exact (or coexact) Dirac sections implies the knowledge of the spectrum of the Dirac Laplacian on all sections namely.
Corollary 1  The spectrum of the Dirac Laplacian can be decomposed as
\[
\text{spec}(P_{B_\pm}) = \{0\} \cup \text{spec}(P_{B_\pm}|_{\Omega^2(M,V)}) \cup \text{spec}(P_{B_\pm}|_{\Omega^0(M,V)})
\] (38)

The multiplicity of zero is the dimension of the absolute or relative, respectively, Dirac Cohomology. The multiplicity of an eigenvalue \(\lambda > 0\) is the sum of its multiplicities as exact and coexact eigenvalue.

Thus, to study the Dirac Laplacian and hence the Dirac spectrum, it suffices to study the spectrum of exact Dirac sections, whose eigenvalues allow for the following minimax characterization.

Proposition 10  If \((\lambda_i^d)_{i\geq 0} := \text{spec}(P|_{\Omega^0_{B_-}(M,V)})\) are the eigenvalues of the Dirac Laplacian on exact sections, then
\[
\lambda_i^d = \inf_{L} \sup_{\eta \in L \eta \neq 0} \left\{ \frac{(\eta, \eta)}{(\psi, \psi)} \mid \phi \in \Omega_{B_-}(M, V), \ d\phi = \eta \right\}
\] (39)

where \(L\) varies over all \(i\)-dimensional subspaces of \(\Omega_{B_-}(M, V)\).

Proof  We take any \(\phi \in \Omega_{B_-}(M, V)\) such that \(d\phi = \eta\). By Theorem 14, any Dirac section \(\phi\) splits into the orthogonal sum \(\phi = h \oplus d\alpha \oplus \delta\beta\), where \(h\) is an harmonic section, and \(\alpha, \beta\) Dirac sections. Set \(\psi := \delta\beta \in \Omega^0_{B_-}(M, V)\). By the orthogonality of the decomposition
\[
(\phi, \phi) \geq (\psi, \psi)
\]
and, by Green’s formula and the coexactness of \(\psi\):
\[
(d\phi, d\phi) = (d\psi, d\psi) = (d\delta\psi, \psi) = (P\psi, \psi).
\]
So,
\[
\frac{(\eta, \eta)}{(\psi, \psi)} = \frac{(d\phi, d\phi)}{(\phi, \phi)} \leq \frac{(P\psi, \psi)}{(\psi, \psi)}
\]
and
\[
\inf_{L} \sup_{\eta \in L \eta \neq 0} \left\{ \frac{(\eta, \eta)}{(\psi, \psi)} \right\} = \inf_{R} \sup_{\psi \neq 0} \frac{(P\psi, \psi)}{(\psi, \psi)}
\]
where \(L\) varies over all \(i\) dimensional subspaces of \(\Omega^0_{B_-}(M, V)\) and \(R\) over all \(i\) dimensional subspaces of \(\Omega^0_{B_-}(M, V)\). The right hand side of this equation is the standard minimax characterization of \(\lambda_i^d\), the \(i\)-th eigenvalue of coexact Dirac sections, which by conclusion 2 of Lemma 1 is equal to the \(i\)-th eigenvalue \(\lambda_i^d\) of exact sections. \(\square\)

After having proved the variational characterization of Dirac Laplacian eigenvalues on exact sections satisfying the absolute boundary conditions, we possess now the technical tools to prove

Proposition 11  Let \((V, \langle \cdot, \cdot \rangle, \nabla, \gamma)\) be a Dirac bundle over a compact Riemannian manifold \((M, g)\). We assume the existence of an isomorphism \(T\) anticommuting with \(\gamma\) and with the Dirac operator \(Q\). Let \(\mu(U)\) be the smallest positive eigenvalue of the Dirac Laplacian \(P\) on exact Dirac sections satisfying the absolute boundary condition on \(U\). Moreover, for an open cover of \(M\) denoted by \(\{U_i\}_{i=0, \ldots, K}\) we introduce the following notation:
Let \( \psi := \Phi, \psi \) for any pair of points during the proof to make specific choices for the coefficients \( a_i \)'s. Let us consider the following diagram

\[
\begin{array}{c}
0 \rightarrow \Omega(\mathcal{M}) \xrightarrow{r} \Pi_i \Omega_{B^{-}}(U_i) \xrightarrow{s} \Pi_{i,j} \Omega_{B^{-}}(U_{i,j}) \xrightarrow{s} \ldots \\
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow \Omega(\mathcal{M}) \xrightarrow{r} \Pi_i \Omega_{B^{-}}(U_i) \xrightarrow{s} \Pi_{i,j} \Omega_{B^{-}}(U_{i,j}) \xrightarrow{s} \ldots \\
\end{array}
\]

Thereby, we utilize the notation:
– $r$: the restriction operator, which restricts global Dirac sections on $M$ to each open set of the cover according to

$$ r(\omega) := \{\omega|_{U_i}\}. $$ (44)

– $s$: the difference operator, which maps $\omega \in \prod_{\alpha_0,\ldots,\alpha_p} \Omega(U_{\alpha_0,\ldots,\alpha_p})$ with components $\omega_{\alpha_0,\ldots,\alpha_p} \in \Omega(U_{\alpha_0,\ldots,\alpha_p})$ is defined as

$$ (s\omega)_{\alpha_0,\ldots,\alpha_p} := \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_p}, $$ (45)

where $\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_p$ means that the index $\alpha_i$ has been dropped from the index sequence $\alpha_0, \ldots, \alpha_p$.

The rows in diagram (43) are exact but the columns are not (in general). Since we are interested in lower bounds for exact Dirac sections, we will pick $\Phi \in \text{Span}\{\Phi_i\}_{i=0,\ldots,N}$, the first $N$ exact eigensections. We restrict now $\Phi$ by means of $r$ to get $\{\Phi_i\}_i \in \prod_i \Omega(U_i)$. Since $\Phi$ is exact, we can choose $\{\psi_i\}_i \in \prod_i \Omega(U_i)$ so that $d\psi_i = \Phi_i$. Now, we can use the fact that we have a lower eigenvalue bound for exact sections on $U_i$ for all $i$ to choose $\psi_i$'s with bounded $L^2$ norm. We will then piece together these $\psi_i$'s into a section defined on all $M$. It is in general not true that $\psi_i = \psi_j$ on $U_{i,j}$, i.e. that $s\{\psi_i\} = 0$. Therefore, we set $\{\omega_{i,j}\} = s\{\psi_i\}$, where $\omega_{i,j} := \psi_i - \psi_j$ on $U_{i,j}$. Notice that

$$ d\omega_{i,j} = d\psi_i - d\psi_j = \Phi - \Phi = 0, $$ (46)

so that, by Theorem 14, we can write

$$ \omega_{i,j} = h_{i,j} \oplus \eta_{i,j}, $$ (47)

where $h_{i,j}$ is harmonic. We can choose appropriate coefficients $a_i$'s for $\Phi = \sum_{i=0}^{N} a_i \Phi_i \neq 0$ so that $h_{i,j} = 0$. The dimension of the space of such $\Phi$'s will be at least $N - N_1 = N_2 + 1$.

We pick the unique coexact $\eta_{i,j}$ such that $\omega_{i,j} = d\eta_{i,j}$. Therefore, by Proposition 10

$$ \frac{(d\eta_{i,j}, d\eta_{i,j})}{(\eta_{i,j}, \eta_{i,j})} \geq \mu(U_{i,j}) $$ (48)

Next, let us consider $\{v_{i,j,k}\} = s\{\eta_{i,j}\} = ((\eta_{j,k} - \eta_{i,k} + \eta_{i,j})|_{U_{i,j,k}})$ for which

$$ dv_{i,j,k} = d\eta_{j,k} - d\eta_{i,k} + d\eta_{i,j} = \omega_{j,k} - \omega_{i,k} + \omega_{i,j} = \psi_k - \psi_j - \psi_k + \psi_j + \psi_j - \psi_i = 0, $$ (49)

and therefore

$$ \{\Phi_i\} \xrightarrow{s} \{0\} $$ (50)
We want to replace the $\psi_i$’s with some $\overline{\psi}_i$’s which are restrictions of a globally defined section on $M$ and such that on $U_i$

$$d\overline{\psi}_i = d\psi_i = \Phi_i. \quad (51)$$

the exactness of the rows of diagram (43) would allow us, if all the $v_{i,j,k}$s were zero, to find $\{\tau_i\} \in \Pi_i \Omega(U_i)$ so that $s\{\tau_i\} = \{\eta_{i,j}\} = \{\tau_j - \tau_i|_{U_{ij}}\}$. An explicit choice is given by

$$\tau_i := \sum_{j=1}^{K} \rho_j \eta_{i,j}, \quad (52)$$

where $\{\rho_j\}_{j=0,\ldots,K}$ is the partition of unity subordinate to the cover $\{U_j\}_{j=0,\ldots,K}$. However, so far we can only claim that $d\nu_{i,j,k} = 0$, i.e. that $v_{i,j,k}$ is closed. On the other hand $v_{i,j,k}$ is coexact, i.e. $v_{i,j,k} = \delta \alpha_{i,j,k}$. The mapping

$$\Phi \rightarrow \psi_i(\Phi) \rightarrow \omega_{i,k}(\Phi) \rightarrow v_{i,j,k}(\Phi) \quad (53)$$

is linear in $\Phi$, which is in a space of dimension at least $N^2 + 1$. Therefore, we can choose $\Phi = \sum_{i=0}^{N} a_i \Phi_i \neq 0$ such that

$$v_{i,j,k}(\Phi) = 0 \quad \text{for all } i, j, k. \quad (54)$$

As a matter of fact condition (54) represents $N^2$ linear equations in $N^2 + 1$ unknowns. So,

$$ds\{\tau_i\} = sd\{\tau_i\} = \{\omega_{i,j}\}, \quad (55)$$

and, if we take $\overline{\psi}_i := \psi_i - d\tau_i$, then

$$s\{\overline{\psi}_i\} = \{\psi_j - \psi_i - d(\tau_j - \tau_i)\} = \{\psi_j - \psi_i - \omega_{i,j}\} = \{0\} \quad (56)$$

Therefore, $\overline{\psi}_i = \overline{\psi}|_{U_i}$, where $\overline{\psi}$ is a globally defined section. Notice that $d\overline{\psi}_i = d\psi_i = \Phi_i$ on $U_i$. Since

$$(\overline{\psi}, \overline{\psi}) \leq \sum_{i} (\overline{\psi}_i, \overline{\psi}_i), \quad (57)$$

it follows

$$\frac{(\Phi, \Phi)}{\sum_{i}(\overline{\psi}_i, \overline{\psi}_i)} \leq \frac{(\Phi, \Phi)}{(\overline{\psi}, \overline{\psi})} \quad (58)$$

A lower bound on the left hand side of inequality (58) will give a lower eigenvalue bound for exact sections on $M$. Note that all norms are $L^2$-norms unless otherwise indicated, and are computed on the appropriate open sets.

Being $\Phi_i$, the restriction of $\Phi$ to $U_i$, the variational characterization of the eigenvalues in Proposition 10 implies

$$\frac{\|\Phi\|^2}{\|\psi_i\|^2} \geq \frac{\|\Phi_i\|^2}{\|\psi_i\|^2} \geq \mu(U_i), \quad (59)$$

so that

$$\|\psi_i\|^2 \leq \frac{\|\Phi\|^2}{\mu(U_i)}. \quad (60)$$

Both operators $Q$ and $\tilde{Q}$ satisfy the product rule for all smooth functions $f$ and sections $\varphi$

$$Q(f \varphi) = \gamma(\text{grad} f)\varphi + f Q\varphi \quad \text{and} \quad \tilde{Q}(f \varphi) = \tilde{\gamma}(\text{grad} f)\varphi + f \tilde{Q}\varphi, \quad (61)$$

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so that the operator $d := \frac{1}{2}(Q - i\tilde{Q})$

$$d(f\varphi) = \frac{\gamma - i\tilde{\gamma}}{2} (\text{grad} f)\varphi + f d\varphi.$$  

(62)

This formula allows to estimate $\|d\tau_i\|:

$$\|d\tau_i\|^2 = \|d(\sum_j \rho_j \eta_{i,j})\|^2 = \left\|\sum_j \frac{\gamma - i\tilde{\gamma}}{2} (\text{grad} \eta_{i,j} + \rho_j d\eta_{i,j})\right\|^2 \leq 2 \sum_j \left(\left\|\frac{\gamma - i\tilde{\gamma}}{2} (\text{grad} \eta_{i,j})\right\|^2 + \|\rho_j d\eta_{i,j}\|^2\right) \leq 2 \sum_j (C_j \|\eta_{i,j}\|^2 + \|d\eta_{i,j}\|^2).$$  

(63)

Since $\eta_{i,j}$ fulfilling the condition (48) was chosen, we have

$$\|\eta_{i,j}\|^2 \leq \frac{\|d\eta_{i,j}\|^2}{\mu(U_{i,j})} = \frac{\|\psi_i - \psi_j\|^2}{\mu(U_{i,j})} \leq \frac{2(\|\psi_i\|^2 - \|\psi_j\|^2)}{\mu(U_{i,j})}. \quad (64)$$

Assembling the inequalities (64), (63) and (60) into the definition of $\overline{\psi}_i$, we obtain

$$\|\overline{\psi}_i\|^2 \leq \|\psi_i\|^2 + \|d\tau_i\|^2 \leq \|\psi_i\|^2 + \|d(\sum_j \rho_j \eta_{i,j})\|^2 \leq \frac{\|\Phi\|^2}{\mu(U_i)} + 4 \sum_j \left(C\frac{\|\psi_i\|^2 + \|\psi_j\|^2}{\mu(U_{i,j})} + \|\psi_i\|^2 + \|\psi_j\|^2\right). \quad (65)$$

and therefore

$$\frac{\|\overline{\psi}_i\|^2}{\|\Phi\|^2} \leq \frac{1}{\mu(U_i)} \frac{1}{4 \sum_j \left(C\frac{\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_{i,j})}}{\mu(U_{i,j})} + \frac{1}{\mu(U_i)} + \frac{1}{\mu(U_{i,j})}\right)}. \quad (66)$$

Because of inequality (57) we finally obtain

$$\frac{\|\Phi\|^2}{\|\psi_i\|^2} \leq \frac{1}{\mu(U_i)} \frac{1}{4 \sum_j \left(C\frac{\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_{i,j})}}{\mu(U_{i,j})} + \frac{1}{\mu(U_i)} + \frac{1}{\mu(U_{i,j})}\right)}. \quad (67)$$

which completes the proof.

Theorem 2 is now a direct consequence of Proposition 11, Lemma 1 and Corollary 1.
6 Large first Dirac eigenvalue: proof of the result

Proof of Theorem 3 We apply Theorem 2 to the extrinsic Dirac operator as in Proposition 4 noting that the spectral bound holds true for the intrinsic Dirac operator as well, because the spectra of both extrinsic and intrinsic Dirac Laplacians, possibly under the absolute boundary condition, are the same.

We take a topological sphere $S^m$ and choose a metric $g_0$ on it, such that $S^n$ looks like a cigar, where the middle part has length 3. In particular this middle part is a product for the metric $g_0$, i.e. a cylinder $I \times S^{m-1}$. We then remove the half-sphere $H_2$ at one end of the cigar and form a connected sum with $M$. The resulting manifold is diffeomorphic to $M$ and has a submanifold $N$, with smooth boundary, naturally identified with $S^m \setminus H_2$. Let $g_1$ be an arbitrary metric on $M$ whose restriction to $N$ is equal to $g_0|_N$. The manifold $N$ contains an open cylinder of length 3. We subdivide this cylinder into 3 cylinders $Z_1, Z_2, Z_3$ of length 1. Let $g_t$ be a metric on $M$ such that $g_t|_{M \setminus Z_2} = g_1|_{M \setminus Z_2}$ and such that $Z_3 = I \times S^{m-1}$ becomes a cylinder of length $t$. This is accomplished by replacing the unit interval by the interval $[0, t]$ and using the product metric on $Z_2$. Now $\text{Vol}(M, g_t) = a + bt$, where $a$ and $b$ are positive real constants. We take the following open cover of $M$:

1. $U_1 = H_1 \cup Z_1$,
2. $U_2 = M \setminus H_1 \cup Z_1 \cup Z_2$,
3. $U_3 = Z_1 \cup Z_2 \cup Z_3$,

which has the property that $U_1 \cap U_2 = \emptyset, U_1 \cap U_3 = Z_1, U_2 \cap U_3 = Z_3$ and $U_1 \cap U_2 \cap U_3 = \emptyset$. Let $\mu_1(M_t)$ be the first positive eigenvalue of the Dirac Laplacian on exact sections on $M_t = (M, g_t)$ for the given spin structure. To estimate $\mu_1(M)$ we apply Theorem 2 to $M_t$ and the cover $\{U_1, U_2, U_3\}$. The eigenvalues $\mu(U_1), \mu(U_2), \mu(U_{1,3})$ and $\mu(U_{2,3})$ are independent of $t$. Let $\lambda_k(O)$ be the $k$-th eigenvalue of the Dirac Laplacian $P$ on $O$ under the absolute boundary condition. If $0$ is an eigenvalue for the absolute boundary conditions, we denote it by $\lambda_0(O)$. By using the Künneth’s formula for $m \geq 2$, that is $P^{I \times S^{m-1}} = P^I \otimes 1 + 1 \otimes P^{S^{m-1}}$, and noting that $\lambda_i(I)$ depends on $t$, but is positive for all $i \geq 0$, we get the following inequality for $\mu(U_3)$:

$$
\mu(U_3) \geq \lambda_1(U_3) = \lambda_1(I \times S^{m-1}) \\
\geq \min_{i,j} \{\lambda_i(I) + \lambda_j(S^{m-1})\} \geq \min_{j} \{\lambda_j(S^{m-1})\} =: C,
$$

where $C$ is a constant independent of $t$. If $m = 3$, then $S^2$ has no harmonic spinors (cf. [2], [3]). In other dimensions if the Riemannian metric on $S^{m-1}$ allows for non trivial harmonic spinors, a small perturbation of the metric reduces the harmonic spinors to the zero section (cf. [12]). Therefore, it is always possible to find a Riemannian metric for which $\lambda_1(S^{m-1}) > 0$. Therefore, the constant $C$ is strictly positive. From Theorem 2 we get that

$$
\mu_1(M_t) \geq \epsilon > 0
$$

for an $\epsilon$ independent of $t$. The volume of $M_t$ is given by $\text{Vol}(M, g_t) = a + bt$ with constants $a, b > 0$. Set $h_t = (a + bt)^{\frac{2}{n}}$. For $(M, h_t)$ we have that $\text{Vol}(M, h_t) = 1$ and $\lambda_1^2(D_s^{(M, h_t)}) = (a + bt)^{\frac{2}{n}} \lambda_1^2(D_s^{(M, h_t)})$. This implies that

$$
\lambda_1^2(D_s^{(M, h_t)}) > \epsilon (a + bt)^{\frac{2}{n}}.
$$

Therefore $\lambda_1^2(D_s^{(M, h_t)}) \to +\infty$ as $t \to +\infty$. The proof is completed. $\square$
7 Lower Dirac eigenvalues on degenerating hyperbolic three dimensional manifolds

7.1 The geometry of three hyperbolic manifolds

A very readable survey of the geometry of compact, hyperbolic, three manifolds and their degenerations is contained in Gromov [29]. A very thorough discussion of this topic can be found in Thurston [47] or in Benedetti and Petronio [7].

The Kazhdan–Margulis decomposition gives a simple insight of the geometrical structure of hyperbolic three manifolds, particularly where the injectivity radius is small. There exists a universal (i.e. depending only on the dimension) positive constant $\mu$, called the Kazhdan–Margulis constant, for which the following construction can always be carried out. Any hyperbolic manifold $M$ of finite volume splits into two parts:

$$M = M_{[0, \mu]} \cup M_{[\mu, \infty]}.$$ (71)

$M_{[0, \mu]}$ is called the thin part and contains all points of $M$, whose injectivity radius is smaller than or equal to $\mu$. The thin part is found to be a finite union of tubes and cusps. A tube $T$ is a tubular neighbourhood of a closed geodesic. A cusp $C$ is the warped product $[0, +\infty] \times F$, equipped with the metric $du^2 + e^{-2u}ds^2$, where $F$ is a 2-dimensional torus and $ds^2$ a flat metric on $F$. The points of $M$, where the injectivity radius is bigger than $\mu$, form the so-called thick part $M_{[\mu, \infty]}$. The thick part is non empty and connected.

The following theorem, due to Thurston (cf. [7] page 197), states that any complete hyperbolic three manifold of finite volume, observed from its thick part, looks on its bounded part like a compact hyperbolic three manifold.

**Theorem 16** (Thurston) Let $M$ be a complete, hyperbolic, three manifold with $p$ cusps, $p \geq 1$, and of finite volume $\text{vol}(M)$. Then, there is a sequence $(M_j)_{j \geq 0}$ of compact, hyperbolic, three manifolds having $p$ simple closed geodesics, whose lengths go to zero as $j \to \infty$, such that $(M_j, x_j)$ converges to $(M, x)$ in the sense of pointed Lipschitz, for appropriate $x_j$ and $x$ belonging to the thick part of $M_j$ and $M$, respectively. In particular, $\text{vol}(M_j) \uparrow \text{vol}(M)$, $\text{diam}(M_{j, \text{thick}}) \to \text{diam}(M_{\text{thick}})$ and if $M$ is non compact, then $\text{diam}(M_j) \uparrow \infty$.

**Definition 10** (Pointed Lipschitz convergence) The dilatation of a map $f : M \to N$ between two metric spaces $M$ and $N$ is defined as

$$\text{dil}(f) := \sup_{x, y \in M \atop x \neq y} \frac{d(f(x), f(y))}{d(x, y)} \in [0, +\infty].$$ (72)

The Lipschitz distance between $M$ and $N$ is the defined as

$$d_L(M, N) := \inf \{| \log \text{dil}(f)| + | \log \text{dil}(f^{-1})| \}$$ (73)

where the infimum is taken over all Lipschitz homeomorphisms $f : M \to N$. The sequence $(M_j, x_j)_{j \geq 0}$ of metric spaces $M_j$ with distinct points $x_j \in M_j$ is said to converge to $(M, x)$ in the sense of pointed Lipschitz, if and only if the following condition is satisfied:

for every $r > 0$ there exists a sequence $(\varepsilon_j)_{j \geq 0}$ of positive real numbers $\varepsilon_j \to 0^+$ ($j \to +\infty$), such that

$$d_L \left( B^M_j(x_j, r + \varepsilon_j), B^M(x, r) \right) \to 0 \quad (j \to +\infty)$$ (74)

where $B^M(x, r)$ denotes the ball of radius $r$ in $M$ centered at $x$. 

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As a matter of fact, Thurston shows that the compact manifolds $M_j$, obtained by closing the cusps of an hyperbolic, complete, non compact manifold $M$ using Dehn’s surgery, support for all but for a finite number of exceptions an hyperbolic metric and approximate $M$.

**Definition 11** If the limit manifold $M$ is non compact, then the sequence $(M_j)_{j \geq 0}$ described above is called a **degenerating family of hyperbolic three manifolds**.

A brief review of Riemannian metrics on tubes and cusps is needed for the following. We refer to [7] for more details. To keep the notation simple, the manifold $M$ in Thurston’s Theorem is assumed without loss of generality to have only one cusp. There is a positive $R_j$ for which the component of the thin part $(M_j)_{|0, \mu]}$ of $M_j$ containing the closed simple geodesic $\gamma_j$, whose length $\varepsilon_j \to 0$ as $j \to \infty$, is the solid torus

$$T_j := \{ x \in M_j \mid \text{dist}(x, \gamma_j) \leq R_j \}. \quad (75)$$

This torus is the quotient of a solid hyperbolic cylinder $\tilde{T}_j$ in the universal cover $\mathbb{H}^3$ of $M_j$ by the action of an infinite cyclic group of isometries generated by an hyperbolic twist of length $\varepsilon_j$ and angle $\rho_j \in [0, \pi[$. Some non trivial facts about hyperbolic geometry accounted for example in Colbois and Courtois ([16]) or in Dodziuk and McGowan ([20]) force the distinguished constants $R_j, \varepsilon_j, \rho_j$ to satisfy the following inequalities:

$$D_1 e^{-2R_j} \leq \varepsilon_j \leq D_2 e^{-2R_j}$$
$$E_1 e^{-R_j} \leq \rho_j \leq E_2 e^{-R_j} \quad (76)$$

where $D_j$ and $E_j \ (j = 1, 2)$ are positive constants. In terms of Fermi coordinates $(r, t, \theta)$ with respect to the geodesic $\tilde{\gamma}_j$, the lift of $\gamma_j$ in $\mathbb{H}^3$, we can write the twist as

$$A_{\gamma_j} : (r, t, \theta) \to (r, t + \varepsilon_j, \theta + \rho_j) \quad (77)$$

and the metric on $\tilde{T}_j$ as

$$\tilde{g}_j = dr^2 + \cosh^2 r \, dt^2 + \sinh^2 r \, d\theta^2, \quad (78)$$

where $r \in [0, R_j], \ t \in [0, \varepsilon_j]$ and $\theta \in [0, 2\pi[$. If we change the radial coordinate by $u := R_j - r \in [0, R_j[$ and introduce the following auxiliary functions

$$\varphi_j(u) := \frac{1}{4} (e^{2u} - 1) \cosh^{-2} R_j \left( e^{-2R_j} (1 + e^{2u}) + 2 \right)$$
$$\psi_j(u) := \frac{1}{4} (e^{2u} - 1) \sinh^{-2} R_j \left( e^{-2R_j} (1 + e^{2u}) - 2 \right), \quad (79)$$

the metric on $\tilde{T}_j$ becomes in the new coordinates

$$\tilde{g}_j' = du^2 + e^{-2u} \left\{ (1 + \varphi_j(u)) \cosh^2 R_j \, dt^2 + (1 + \psi_j(u)) \sinh^2 R_j \, d\theta^2 \right\}, \quad (80)$$

from which the similarity with the warped product metric

$$\tilde{g}_j' = du^2 + e^{-2u} \left\{ \cosh^2 R_j \, dt^2 + \sinh^2 R_j \, d\theta^2 \right\} \quad (81)$$

is evident. As a matter of fact $\varphi_j = o(1)$ and $\psi_j = o(1)$ pointwise on $[0, R_j]$ and, in view of Thurston’s Theorem, $T_j$ is expected to become a cusp in the limit $j \to +\infty$.

We conclude by some observations about the fibers of the tubes and the cusp. The warped product metric on $T_j$ writes as

$$g_j' = du^2 + e^{-2u} ds_j^2$$
where \((F_j, ds_j^2)\) is a flat torus. More exactly: \(\tilde{F}_j = \mathbb{R}^2\) and \(F_j = \tilde{F}_j/\sim\) w.r.t. the identifications in polar coordinates \((t, \theta) \sim (t + \epsilon, \theta + \rho_j)\) and \((t, \theta) \sim (t, \theta + 2\pi)\) for all \((t, \theta)\) and the metric in the universal cover \(\mathbb{R}^2\) is given by
\[
\tilde{ds}_j^2 = \cosh^2 R_j dt^2 + \sinh^2 R_j d\theta^2.
\]

Colbois-Courtois \([16]\) proved:

**Proposition 12** A subsequence of \((F_j, ds_j^2)\) converges in the sense of Lipschitz to the flat torus \((F, ds^2)\), where \(C = [0, +\infty) \times F\) is the cusp in the limit manifold \((M, g)\).

### 7.2 Spectrum of the tube

We want to compute the eigenvalues of the Dirac Laplacian under the absolute boundary condition for \(U := \{x \in M | r_0 \leq \text{dist}(x, \gamma) \leq R_0\}\), a piece of tube \(T\) of a compact hyperbolic spin three manifold \(M\). To do so we introduce a local o.n. frame for the spinor bundle over \(U\). Recall from 7.1 that the points of the tube \(T\) at geodesic distance \(u\) from \(\partial T\) form a flat torus \(F_u\), whose metric in its universal cover \(\tilde{F}_u\) is \(\tilde{ds}^2 = f^2(u)dt^2 + h^2(u)d\theta^2\), where \(f(u) := \cosh(R - u)\) and \(h(u) := \sinh(R - u)\). The Dirac bundle structure on the odd dimensional manifold \(T\) induces on each 1-codimensional submanifold \(F_u\) a unique Dirac bundle structure (see Theorem 10).

**Proposition 13** Let us denote with \(\partial_u, \partial_t, \partial_\theta\) the partial derivatives w.r.t. \(u, t, \theta\) on the local frame for \(TM|_{U}\) corresponding to these coordinates. It exist a local o.n. frame \(\{s_1, \ldots, s_l\}\) for the spinor bundle \(\Sigma M|_{U}\), whose rank is \(l\), with the following properties:

(i) \(\nabla^{F_u}s_k = 0\) \((1 \leq k \leq l, u \in [r_0, R_0])\) i.e. the spinors are parallel in each fiber,

(ii) \(\nabla^M_{\partial_u}s_k = 0\) \((1 \leq k \leq l, u \in [r_0, R_0])\) i.e. the spinors are parallel to the radial direction.

(iii) \(B_{-s_1} = s_1, \ldots, B_{-s_{\frac{l}{2}}} = s_{\frac{l}{2}}\) and \(B_{-s_{\frac{l}{2}+1}} = 0, \ldots, B_{-s_l} = 0\), where \(B_{\pm} := \frac{\mp T\gamma^M(\partial_u)}{2}\) and \(T\) as in Proposition 4.

**Proof** Being \(F_u\) flat and the spin connection the lift of the Levi-Civita connection, the parallel transport on \(F_u\) doesn’t depend (locally!) on the path. We consider \(x_0 = (u_0, t_0, \theta_0) \in F_{u_0}\) and an o.n. basis \(s_1(t_0, t_0, \theta_0), \ldots, s_l(t_0, t_0, \theta_0)\) of \(V_{x_0}\) where \(V := \Sigma M\). There exist a neighbourhood of \(x_0\) in \(F_{u_0}\), where, without being worried about paths, we can set for any \(1 \leq k \leq l\)
\[
s_k(u_0, t, \theta) := \Pi_{F_{u_0}}^{(t, \theta)\rightarrow (t, \theta)} s_k(u_0, t_0, \theta_0),
\]
where \(\Pi_{F_{u_0}}\) denotes the parallel transport on \(F_{u_0}\). Since the parallel transport is an isometry, the frame \(\{s_k(u_0, \cdot, \cdot)\}_{1 \leq k \leq l}\) is a local o.n. frame for \(V|_{F_{u_0}}\) satisfying by definition the property (i) for \(u = u_0\).

Now we set
\[
s_k(u, t, \theta) := \Pi_{M}^{u_0\rightarrow u} s_k(u_0, t, \theta),
\]
where \(\Pi_{M}^{u_0\rightarrow u}\) denotes the parallel transport on the tube along the \(u\)-lines. Since the parallel transport is an isometry, the frame \(\{s_k\}_{1 \leq k \leq l}\) is a local o.n. frame for \(V|_{U}\), satisfying by definition property (ii). We choose \(u_0 := r_0\) and \(u\) can vary in \([r_0, R_0]\).

The fact that property (i) holds for any \(u \in [r_0, R_0]\), that is
\[
\nabla^{F_u}\Pi^{u_0\rightarrow u}_{M} s_k(u_0, t, \theta) = 0,
\]

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follows from the formulae

\[ \nabla^F_{u \frac{1}{f(u)} \partial_t(u)} \Pi^{u_0 \rightarrow u}_M = \frac{f(u_0)}{f(u)} \Pi^{u_0 \rightarrow u}_M \nabla^F_{u \frac{1}{f(u_0)} \partial_t(u_0)} \]  \hspace{1cm} (85) \]

and

\[ \nabla^F_{u \frac{1}{h(u)} \partial_\theta(u)} \Pi^{u_0 \rightarrow u}_M = \frac{h(u_0)}{h(u)} \Pi^{u_0 \rightarrow u}_M \nabla^F_{u \frac{1}{h(u_0)} \partial_\theta(u_0)} \]  \hspace{1cm} (86) \]

Since \((T \gamma(\partial_u))^2 = 1\), we can choose \(s_1(u_0, t_0, \theta_0), \ldots, s_i(u_0, t_0, \theta_0)\) satisfying (iii) in \(x_0\). Since \(\nabla^M\) commutes with \(T\) (see Proposition 4) and with \(\gamma(\partial_u)\), property (iii) holds for all \(s_1, \ldots, s_i\) over \(U\), which are obtained by parallel transport. Therefore, property (iii) holds true.

\[
\text{Remark 4} \quad \text{The domain of definition } O \subset U \text{ of such a local o.n. frame } \{s_1, \ldots, s_i\} \text{ is typically the image of an open subset } U_{u_0} \subset F_{u_0} \text{ under the exponential flow in the piece of the tube normal to the fiber } F_{u_0}.
\]

\[
\text{Lemma 2} \quad \text{Let } D^F := D^F_{u_0} \text{ and } \Delta_s^F := (D^F)^2 \text{ denote the Dirac operator and, respectively, the spin Laplacian on } F_u, \text{ and } H = -\frac{1}{2} \partial_u \log f(u) \text{ the mean curvature of } F_u \text{ in } M. \text{For any spinor } \sigma \text{ over the piece of the tube } U, \text{ the spin Laplacian writes as}
\]

\[
\Delta_s^M \sigma = \Delta_s^F \sigma + [D^F, \nabla^M_{\partial_u}] \sigma - (\nabla^M_{\partial_u})^2 \sigma + (\partial_u H - H^2) \sigma + 2H \nabla^M_{\partial_u} \sigma.
\]

\[
\text{Proof} \quad \text{According to Bär [4] the Dirac operator on the tube writes as}
\]

\[
D^M \sigma = \gamma(\partial_u) D^F \sigma - H \gamma(\partial_u) \sigma + \gamma(\partial_u) \nabla^M_{\partial_u} \sigma.
\]

So, for the spin Laplacian we have

\[
\Delta_s \sigma = D^M D^M \sigma = D^M (\gamma(\partial_u) D^F \sigma - H \gamma(\partial_u) \sigma + \gamma(\partial_u) \nabla^M_{\partial_u} \sigma)
\]

\[
= \gamma(\partial_u) D^F (\gamma(\partial_u) D^F \sigma - H \gamma(\partial_u) \sigma + \gamma(\partial_u) \nabla^M_{\partial_u} \sigma)
\]

\[
- H \gamma(\partial_u) (\gamma(\partial_u) D^F \sigma - H \gamma(\partial_u) \sigma + \gamma(\partial_u) \nabla^M_{\partial_u} \sigma)
\]

\[
+ \gamma(\partial_u) \nabla^M_{\partial_u} (\gamma(\partial_u) D^F \sigma - H \gamma(\partial_u) \sigma + \gamma(\partial_u) \nabla^M_{\partial_u} \sigma)
\]

\[
= (D^F)^2 \sigma + (D^F \nabla^M_{\partial_u} - \nabla^M_{\partial_u} D^F) \sigma - (\nabla^M_{\partial_u})^2 \sigma + (\partial_u H - H^2) \sigma
\]

\[
+ 2H \nabla^M_{\partial_u} \sigma,
\]

which is the assertion of the lemma. We used of course that

\[
\nabla^M_{\partial_u} \partial_u = 0
\]

and that

\[
- \gamma(\partial_u) \nabla^M_{\partial_u} (H \gamma(\partial_u) \sigma) = \partial_u H \sigma + H \nabla^M_{\partial_u} \sigma.
\]

\[
\text{Proposition 14} \quad \text{Let } c_{ri} := \{\gamma^F (f^{-1} \partial_i) s_r, s_i\} \text{ and } d_{ri} := \{\gamma^F (h^{-1} \partial_i) s_r, s_i\} \text{ for } 1 \leq r, i \leq l. \text{Let } \{s_1, \ldots, s_i\} \text{ be the local o.n. frame for the spinor bundle over the piece of tube } U \text{ defined in Proposition 13. Under the decomposition } \sigma = \sum_{k=1}^l a^k s_k, \text{ we obtain the following equivalences}
\]

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(i) Eigenvalue equation:

\[(\Delta_s - \lambda)\sigma = 0 \Leftrightarrow (\Delta_0 - \lambda)\sigma^k - \partial_u^2 \sigma^k + (\partial_u H - H^2)\sigma^k + 2H\partial_u \sigma^k - \sum_{i \neq k} \left( (f \partial_u (f^2) + \partial_u f)c_{ik}\partial_t \sigma^i \right) \]

\[-(h\partial_u (h^2) + \partial_u h)d_{ik}\partial_\theta \sigma^i \]  

(92)

\[1 \leq k \leq l). \]

(ii) Absolute boundary condition:

\[(B_\sigma)|_{\partial U} = 0, \quad (B_D^M \sigma)|_{\partial U} = 0 \]

\[\Leftrightarrow \sigma^k = 0, \quad (\partial_u - H)\sigma^{k+\frac{l}{2}} = 0 \quad (u = r_0, R_0 \text{ and } 1 \leq k \leq \frac{l}{2}). \]

\[(93)\]

**Proof** We insert the decomposition \(\sigma = \sum_{k=1}^{l} \sigma^k s_k\), where \(\sigma^k = \sigma^k(u, t, \theta)\), in the equation \(\Delta_s \sigma = \lambda \sigma\). We represent \(\Delta_s\) using Lemma 2:

\[\Delta^M_s \sigma = \Delta^F_s \sigma + (\nabla^M_{\partial_u})\sigma - (\nabla^M_{\partial_u})^2 \sigma + (\partial_u H - H^2)\sigma + 2H\nabla^M_{\partial_u} \sigma. \]

Using the properties of the local o.n. frame \(s_1, \ldots, s_l\) described in the Proposition 13, we find:

\[D^F \sigma = \sum_{k=1}^{l} \gamma^F (\text{grad}^F \sigma^k)s_k, \]

\[\nabla^M_{\partial_u} \sigma = \sum_{k=1}^{l} (\partial_u \sigma^k)s_k, \]

\[\Delta^F_s \sigma = \sum_{k=1}^{l} (\Delta_0 \sigma^k)s_k, \]

\[(\nabla^M_{\partial_u})^2 \sigma = \sum_{k=1}^{l} (\partial_u^2 \sigma^k)s_k, \]

\[[D^F, \nabla^M_{\partial_u}]\sigma = \sum_{k=1}^{l} \gamma^F ([\text{grad}^F, \nabla^M_{\partial_u}]\sigma^k)s_k, \]

and for any function \(\varphi\)

\[[\text{grad}^F, \nabla^F_{\partial_u}]\varphi = -(f \partial_u (f^2) + \partial_u f)\partial_t \varphi \varphi^{-1}\partial_t \]

\[-(h\partial_u (h^2) + \partial_u h)\partial_\theta \varphi \varphi^{-1}\partial_\theta. \]

Therefore,

\[[D^F, \nabla^M_{\partial_u}]\sigma = -\sum_{i,k=1}^{l} \left( (f \partial_u (f^2) + \partial_u f) \left\{ \gamma^F (f^{-1}\partial_t)s_i, s_k \right\} \partial_t \sigma^i \right)_{=c_{ik}} \]

\[+ (h\partial_u (h^2) + \partial_u h) \left\{ \gamma^F (h^{-1}\partial_\theta)s_i, s_k \right\} \partial_\theta \sigma^i \]  

\[=d_{ik} \]  

\[(97)\]
Remark that $c_{rr} = d_{rr} = 0$. In fact, $\nabla^F s_r = 0$ and thus $D^F s_r = 0$. Therefore for any $u_0 \in [r_0, R_0]$ and any open $G \subset U_{u_0} \subset F_{u_0}$:

$$
(D_{u_0}^F s_r, s_r) = (s_r, D_{u_0}^F s_r) - \int_{\partial G} \langle \gamma^F (v) s_r, s_r \rangle \text{dvol}_{\partial G}.
$$

(98)

If we denote by $\alpha := g(v, f^{-1} \partial_t)$ and by $\beta := g(v, h^{-1} \partial_\theta)$, we obtain for all $G$:

$$
\int_{\partial G} (\alpha c_{rr} + \beta d_{rr}) \text{dvol}_{\partial G} = 0
$$

(99)

Therefore, $c_{rr} = d_{rr} = 0$. Thus, the statement (i) follows. Statement (ii) follows by direct insertion and the properties of the frame $\{s_1, \ldots, s_l\}$. □

Since we are primarily interested in the first few eigenvalues, and the equations and the boundary conditions are linear, we can choose for any $i \neq k \sigma^i := 0$. Therefore we obtain for the lower absolute eigenvalues:

$$
\begin{cases}
(\Delta_0 - \lambda)\sigma^k - \partial_u^2 \sigma^k + (\partial_u H - H^2)\sigma^k + 2H \partial_u \sigma^k = 0. & (l + 1 \leq k \leq l) \\
(\partial_u - H)\sigma^k = 0 & (u = r_0, R_0)
\end{cases}
$$

(100)

or

$$
\begin{cases}
(\Delta_0 - \lambda)\sigma^k - \partial^2_u \sigma^k + (\partial_u H - H^2)\sigma^k + 2H \partial_u \sigma^k = 0. & (1 \leq k \leq \frac{l}{2})
\end{cases}
$$

(101)

We are going now to explicitly determine a regular discrete resolution for the function Laplacian $\Delta_0$ on $\mathbb{R}^2$ with metric given in polar coordinates by

$$
f^2(u) d\alpha^2 + h^2(u) d\beta^2
$$

under the periodicity conditions $a(t, \theta) = a(t + \epsilon, \theta + \rho)$ and $a(t, \theta) = a(t, \theta + 2\pi)$ for all $(t, \theta)$.

**Lemma 3** A regular spectral decomposition of $\Delta_0$ on $F_u$ is given by $(g_i, \kappa_i)_{i \in \mathbb{Z}^2}$ i.e. $\Delta_0 g_i = \kappa_i g_i$ for all $i \in \mathbb{Z}^2$ and $(g_i)_{i \in \mathbb{Z}^2}$ is an o.n.b. of $L^2(F_u, \mathbb{C})$, where for $i = (r, s) \in \mathbb{Z}^2$:

$$
\begin{align*}
g_i &= g_i(u, t, \theta) = \frac{e^{i[(2\pi s + r\rho)] \frac{1}{2} \theta}}{\sqrt{2\pi \varepsilon f h(u)}} \\
\kappa_i &= \kappa_i(u) = \frac{(2\pi s + r\rho)^2}{f^2(u) \varepsilon^2} + \frac{r^2}{h^2(u)}.
\end{align*}
$$

(102)

**Proof** We have to solve the partial differential equation

$$
\Delta_0 a = \kappa a
$$

(103)

for an unknown function of two variables $a = a(t, \theta)$, satisfying the periodicity conditions

$$
a(t, \theta) = a(t + \epsilon, \theta + \rho) \quad \text{and} \quad a(t, \theta) = a(t, \theta + 2\pi) \quad \text{for all} \ (t, \theta).
$$

(104)

The Ansatz $a(t, \theta) = A(t) B(\theta)$ inserted in $\Delta_0 a = \kappa a$ leads to two ordinary differential equations

$$
\begin{align*}
A'' + f^2 \mu A &= 0 \\
B'' + h^2 \nu B &= 0,
\end{align*}
$$

(105)

and $a := AB$ is then a solution of the original PDE with $\kappa = \mu + \nu$. In view of the periodicity conditions, we ignore the cases where $\mu < 0$ or $\nu < 0$ and find that $A(t) = e^{i f \sqrt{\mu t}}$ and
$B(\theta) = e^{ih\sqrt{\nu} \theta}$ are solutions. We insert $a(t, \theta) := A(t)B(\theta)$ in the second periodicity condition to obtain

$$v = \frac{r^2}{h^2} \text{ for } r \in \mathbb{Z}$$

and in the first

$$\mu = \frac{(2\pi s + r\rho)^2}{e^2 f^2} \text{ for } a s \in \mathbb{Z}.$$  

(107)

The eigenvalues are therefore, setting $i := (r, s) \in \mathbb{Z}^2$,

$$\kappa_i = \kappa_i(u) = \mu + v = \frac{(2\pi s + r\rho)^2}{f^2(u)e^2} + \frac{r^2}{h^2(u)}$$

and the eigenfunctions

$$a_i = a_i(t, \theta) = e^{i((2\pi s + r\rho)\frac{t}{e} - r\theta)}.$$  

(109)

To get an o.n. sequence we normalize as follows:

$$g_i(u, t, \theta) := \frac{a_i}{\|a_i\|}.$$  

(110)

Since

$$\|a_i\|^2_{L^2(F_u, C)} = 2\pi f h(u),$$

we find $g_i$ as claimed. The sequence $(g_i)_{i \in \mathbb{Z}^2}$ is an o.n.b. in $L^2(F_u, C)$.

By direct verification we obtain the following

**Lemma 4** The eigenfunctions $(g_i)_{i \in \mathbb{Z}^2}$ of $\Delta_0$ have the following properties under derivation:

$$\begin{align*}
\partial_u g_i &= -\frac{1}{2} \partial_u (\log(f h)) g_i + \frac{1}{2} \partial_u (\log(f h))^2 g_i \\
\partial_t g_i &= i \frac{2\pi s + r\rho}{e} g_i \\
\partial_\theta g_i &= -ir g_i \\
\partial^2_{tt} g_i &= \left(\frac{2\pi s + r\rho}{e}\right)^2 g_i \\
\partial^2_{\theta\theta} g_i &= -r^2 g_i.
\end{align*}$$

(113)

Now we decompose the $k$-th coordinate function of the spinor $\sigma$ in its Fourier serie w.r.t. the o.n.b $(g_i)_{i \in \mathbb{Z}^2}$ of $L^2(F_u, C)$ found in Lemma 3

$$\sigma^k = \sum_{i \in \mathbb{Z}^2} a_i^k g_i.$$  

(114)

where $a_i$ is the $u$-dependent $i$th Fourier coefficient. We insert this decomposition in the boundary value problems found at the end of the preceding subsection and drop the $k$ superscript, because they all have the same form, independently of what $k \in \{1, \ldots, l\}$ we consider.

Using the properties of all the $g_i$s under derivation, listed in Lemma 4, we obtain

$$\begin{align*}
\sum_{i \in \mathbb{Z}^2} [-a^\prime_i + (\kappa_i - \lambda)a_i] g_i &= 0 \\
\sum_{i \in \mathbb{Z}^2} [a^\prime_i - (\log f h) a_i] g_i &= 0 \quad (u = r_0, R_0)
\end{align*}$$

(115)
or
\[
\begin{align*}
\sum_{i \in \mathbb{Z}^{2}} [-d''_i + (\kappa_i - \lambda)d_i]g_i &= 0 \\
\sum_{i \in \mathbb{Z}^{2}} a_i g_i &= 0 \quad (u = r_0, R_0).
\end{align*}
\] (116)

All the equations are satisfied, if and only if all the Fourier coefficients vanish. This leads to
the following two families of 1-dimensional boundary value problems:
\[
\begin{align*}
-d''_i + (\kappa_i - \lambda)d_i &= 0 \\
a'_i - (\log f h)'a_i &= 0 \quad (u = r_0, R_0)
\end{align*}
\] (i \in \mathbb{Z}^{2}).
\] (117)
and
\[
\begin{align*}
-d''_i + (\kappa_i - \lambda)d_i &= 0 \\
a_i &= 0 \quad (u = r_0, R_0)
\end{align*}
\] (i \in \mathbb{Z}^{2}).
\] (118)

**Remark 5** All the eigenvalues of the original absolute eigenvalue equations for the piece of
the tube are eigenvalues of these two families of 1-dimensional boundary value problems,
but not vice versa. In fact to get the eigenspinors for \( \lambda \), on each \( O \) (cf. Remark 4), it suffices to
take the restrictions of an eigenspinor for \( \lambda \) on all \( U \) to \( O \). The converse procedure does not
work in general, because eigenspinors for the same eigenvalue \( \lambda \) on different open subsets
of \( U \) do not necessarily need to match on the overlaps.

The prominent example is \( \lambda = O \left( \frac{1}{R_0 - r_0} \right) \) corresponding to the choice \( i = 0 \in \mathbb{Z}^{2} \) and
\( \kappa_0 = 0 \), which is not an absolute eigenvalue of \( D^{M} \) on \( U \), because the restriction of the spin structure on \( M \) to the torus \( F_u \) is non trivial, i.e., it does not admit harmonic spinors, as explained in [5]

The boundary value problems for \( i \neq 0 \in \mathbb{Z}^{2} \) give rise to eigenvalues, which are bounded
away from 0 uniformly w.r.t. \( R \). We insert \( f(u) = \cosh(R-u) \) and \( h(u) = \sinh(R-u) \)
and set for any \( i \neq 0 \in \mathbb{Z}^{2} \)
\[
q_i(u) := \kappa_i(u) - \lambda = \frac{(2\pi s + r\rho)^2}{\cosh^2(R-u)\epsilon^2} + \frac{r^2}{\sinh^2(R-u)} - \lambda \quad \text{for } i = (r, s). \] (119)

Recall from Sect. 7.1 that \( \epsilon, \rho, R \) can’t be arbitrarily chosen but have instead to satisfy the
inequalities (76) for positive constants \( D_{1,2} \) and \( E_{1,2} \). This fact implies a certain behaviour for
the eigenvalues \( \kappa_i(u) \) of the function Laplacian of the tube fibers \( F_u \). There exists a positive
constants \( S \) such that for every \( R \geq S \) and every \( i \in \mathbb{Z}^{2} \)
\[
\kappa_i(u) \geq \left( \frac{E_1}{D_2} \epsilon^{\alpha_i} \right)^2 \quad \forall u \in [r_0, R]. \] (120)

If we choose \( r_0 \) big enough, then for any \( i \in \mathbb{Z}^{2} \)
\( q_i(u) \geq 5 - \lambda \) on \( [r_0, R_0] \). Let us choose \( R_0 := R - 1 \). If \( \lambda < 1 \) is an absolute eigenvalue, solution of (117), then there is a non trivial
solution \( a_i \) and \( q_i(u) > 4 \) on \( [r_0, R - 1] \). By Proposition 16 one has
\[
\lim_{R \to +\infty} \inf_{R \to +\infty} \frac{a'_i(R-1)}{a_i(R-1)} \geq 1 > 0. \] (121)

But this contradicts the absolute boundary condition at \( u = R - 1 \to +\infty \) (as \( R \to +\infty \)),
because
\[
\frac{a'_i(R-1)}{a_i(R-1)} = -(\tanh(1) + \tanh^{-1}(1)) < 0. \] (122)
If \( \lambda < 1 \) is an absolute eigenvalue, solution of (118), then there is a non trivial solution \( a_i \) and \( q_i(u) > 4 \) on \([r_0, R - 1]\). By Proposition 17 one has

\[
a_i(R - 1) \neq 0.
\] (123)

But this contradicts the absolute boundary condition at \( u = R - 1 \), because

\[
a_i(R - 1) = 0.
\] (124)

The conclusion is that there are two positive constants \( S \) and \( r_0 \) such that for all \( R \geq S \), any absolute eigenvalue must be greater than or equal to 1. In terms of the sequence of pieces of tubes converging to a cusp this means

**Proposition 15** There exist an integer \( j_0 \in \mathbb{N}_0 \) and a positive constant \( r_0 \) such that \( \forall j \geq j_0, \forall n \geq 0 \)

\[
\lambda_1((\Delta_{\gamma_j})_{B_x}) \geq 1,
\] (125)

where \( U_j := T([r_0, R_j - 1]) \) is the relevant piece of tube.

### 7.3 Proof of the lower bound inequality

We first sketch the structure of the proof of Theorems 8. We can assume without loss of generality that \( M \) has only one cusp. We apply Theorem 2 to the extrinsic Dirac operator as in Proposition 4 noting that the spectral bound holds true for the intrinsic Dirac operator as well, because the spectra of both extrinsic and intrinsic Dirac Laplacians, possibly under the absolute boundary condition, are the same.

1. For every \( j \geq 0 \), we cover its approximating manifold \( M_j \) with three 0-codimensional submanifolds with boundary:

   (a) \( W_j \supset (M_j)_{\mu, \infty} \cup \{ x \in (M_j)_{[0, \mu]} \mid R_j \geq \text{dist}(x, \gamma_j) \geq R_j - r_0 \} \): a compact neighborhood of the thick part of \( M_j \).

   (b) \( U_j \supset \{ x \in (M_j)_{[0, \mu]} \mid R_j - r_0 \geq \text{dist}(x, \gamma_j) \geq 1 \} \): a relevant piece of the tube (a solid annular torus).

   (c) \( V_j \supset \{ x \in (M_j)_{[0, \mu]} \mid 1 \geq \text{dist}(x, \gamma_j) \} \): a tubular neighborhood of the closed geodesic (a solid torus).

   The submanifolds can be chosen as the closure of a \( \varepsilon \) neighbourhood (for a fixed small \( \varepsilon \)) of the sets specified on the right hand side. The constant \( r_0 > 0 \) is chosen according to Proposition 15.

2. We compute the spectral bound given by Theorem 2.

3. We control the spectra of the bounded parts \( W_j \) and \( V_j \) under the absolute boundary conditions using spectral perturbation theory.

4. Since the metric of the tube converge to the metric on the cusp, the lower eigenvalues of \( P \) on the piece of cusp for the absolute boundary conditions converge to the lower eigenvalues of \( P \) on \( U_j \) under the absolute boundary condition.
**Proof of Theorem 8** Following the steps above we apply Theorem 2 for the cover of the manifold, for which we have \( N_1 = N_2 = 0, N = 1 \) to obtain

\[
\lambda_1^2(P) \geq C_1 \left\{ \frac{1}{\mu(W_j)} + \frac{1}{\mu(U_j)} + \frac{1}{\mu(V_j)} 
\right. \\
\left. + 4 \left( \frac{C_j}{\mu(W_j \cap U_j)} + 1 \right) \left( \frac{1}{\mu(W_j)} + \frac{1}{\mu(U_j)} \right) \right\}^{-1},
\]

for a \( C_1 > 0 \) and constants \( C_j > 0 \) depending on the \( C^1 \) norm of a partition of unity subordinate to the chosen cover (cf. Theorem 2). The constants \( C_j \) are bounded from above by a constant \( C_2 > 0 \).

Now, we examine the different eigenvalues involved:

- The eigenvalues \( \mu(W_j) \) and \( \mu(W_j \cap U_j) \) are bounded from below by a positive constant independent of \( j \) because \( W_j \) converges to a closed \( \epsilon \) neighbourhood of the thick part \( M_{\text{thick}} \), which is compact.
- By Proposition 15 there is a \( j_0 \in \mathbb{N}_0 \) such that the eigenvalue \( \mu(U_j) \geq 1 \) for all \( j \geq j_0 \).
- The eigenvalues \( \mu(V_j) \) and \( \mu(U_j \cap V_j) \) are uniformly bounded from below by a positive constant independent of \( j \), because \( V_j \) and \( U_j \cap V_j \) are bounded.

We conclude that there exist a positive constant \( c > 0 \) such that

\[
\lambda_1^2(P) \geq c,
\]

and the proof is completed.

\[\square\]

**Remark 6** We can mimick this proof for the Laplace–Beltrami operator on 1-forms and reobtain Jammes’s result stated in Theorem 6. The only essential difference is that the first non zero eigenvalue on the tube converges to zero as the inverse of the square of the diameter as the manifold degenerates.

**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**

**Conflict of interest** There are no data associated to this paper. The author states that there is no conflict of interest.

**Appendix: Some results about second order boundary value problems**

**Proposition 16** Let the function \( a = a(u) \) be a non trivial solution of the linear second order boundary value problem

\[
\begin{cases}
-a'' + qa = 0 \\
a'(m_0) + aa(m_0) = 0,
\end{cases}
\]

\[\square\]
where \( q \in C^\infty([m_0, m_1]) \) is a smooth function satisfying \( q > k^2 \) for constants \( k, \alpha \in \mathbb{R} \) such that \( k > 0 \) and \( \alpha \leq k \). Then, for the unique solution \( v \) of the initial value problem

\[
\begin{align*}
-v'' + k^2v &= 0 \\
v(m_0) &= a(m_0) \\
v'(m_0) &= a'(m_0)
\end{align*}
\]  

(129)

the following inequality holds on \([m_0, m_1]\):

\[
a' \geq \frac{v'}{v}.
\]  

(130)

In particular

\[
\liminf_{m_1 \to +\infty} \frac{a'(m_1)}{a(m_1)} \geq \frac{1}{2} - k.
\]  

(131)

**Proof** Without loss of generality we can prove the inequality on \([m_0, m_1]\) and choose \( m_0 := 0 \) and \( m_1 = +\infty \). We need to distinguish several cases:

CASE 0: \( a(0) = 0 \) never occurs. In fact, both cases \( a(0) = 0 \) and \( a'(0) = 0 \) are excluded by the assumption on the non triviality of \( a \) and by the existence and uniqueness theorem for the solutions of ordinary differential equations.

CASE 1: \( a(0) > 0 \).

Since \( a'(0) = -\alpha a(0) > -k a(0) \), we obtain \( v(u) = a(0) \cosh(ku) + \frac{a'(0)}{u} \sinh(ku) > 0 \) \( \forall u \in [0, +\infty[ \). With \( w := a'v - av' \) it follows \( w' = (q - k^2)av, w(0) = 0 \) and \( w'(0) = (q(0) - k^2)v(0) > 0 \). So, \( \epsilon_1 := \sup \{u \in ]0, +\infty[ \mid w' > 0 \text{ on } [0, u[\} \) must belong to \( ]0, +\infty[ \). If \( \epsilon_1 < +\infty \), then by continuity \( w'(\epsilon_1) = 0 \).

Analogously, since \( a(0) > 0 \), \( \epsilon_2 := \sup \{u \in ]0, +\infty[ \mid a > 0 \text{ on } [0, u[\} \) must be in \( ]0, +\infty[ \). If \( \epsilon_2 < +\infty \), then by continuity \( a(\epsilon_2) = 0 \). Set \( \epsilon := \min\{\epsilon_1, \epsilon_2\} \). On \( [0, \epsilon[ \) one has \( w \geq 0 \), i.e., \( \frac{a'}{a} \geq \frac{v'}{v} \), being \( a \) and \( v \) positive. Integrating both sides of this inequality, one gets \( a \geq v \) on \( [0, \epsilon[ \). So, on this interval one has \( w'(u) = (q - k^2)a(0)v(u) \geq (q(0) - k^2)v^2(u) = (q(0) - k^2)a(0)\cosh^2(ku) \) and \( a(u) \geq v = a(0) \cosh(ku) \). Assume now that \( \epsilon < +\infty \). There are two possibilities: if \( \epsilon = \epsilon_1 \), then by continuity \( w'(\epsilon_1) = (q(\epsilon_1) - k^2)a(0)\cosh(ke_1) > 0 \); if \( \epsilon = \epsilon_2 \), again by continuity \( a(\epsilon_2) \geq a(0) \cosh(ke_2) > 0 \). In both cases there is a contradiction, so it must be \( \epsilon = +\infty \). We therefore come to the conclusion that \( \frac{a'}{a} \geq \frac{v'}{v} \) on \( [0, +\infty[ \).

CASE 2: \( a(0) < 0 \).

We set \( \tilde{a} := -a \) and \( \tilde{v} := -v \). Case 1 leads to \( \frac{\tilde{a}'}{\tilde{a}} \geq \frac{\tilde{v}'}{\tilde{v}} \) on \( [0, +\infty[ \), which means \( \frac{a'}{a} \geq \frac{v'}{v} \) on the same interval.

By solving the initial value problem for \( v \), we can determine \( v \) and \( v' \) explicitly:

\[
v(u) = a(m_0) \cosh(k(u - m_0)) + \frac{a'(m_0)}{k} \sinh(k(u - m_0))
\]  

(132)

\[
v'(u) = ka(m_0) \sinh(k(u - m_0)) + a'(m_0) \cosh(k(u - m_0)).
\]

Since \( v(u) \neq 0 \) for \( u \in [m_0, m_1] \) we can write:

\[
\frac{v'(u)}{v(u)} = k \frac{a(m_0) \tanh(k(u - m_0)) + \frac{1}{k} a'(m_0)}{a(m_0) + \frac{1}{k} a'(m_0) \tanh(k(u - m_0))}.
\]  

(133)
We insert the boundary condition \( a'(m_0) + \alpha a(m_0) = 0 \) and simplify by \( a(m_0) \neq 0 \):

\[
\frac{v'(u)}{v(u)} = k \frac{\tanh(k(u - m_0)) - \frac{\alpha}{k}}{1 - \frac{\alpha}{k} \tanh(k(u - m_0))}.
\] (134)

Since \( k > \alpha \), we obtain

\[
\lim_{u \to +\infty} \frac{v'(u)}{v(u)} = k
\] (135)

and the inequality (131) follows from the estimate (130).

\[\square\]

**Proposition 17**  Let the function \( a = a(u) \) be a non trivial solution of the linear second order boundary value problem

\[
\begin{cases}
-a'' + qa = 0 \\
a(m_0) = 0,
\end{cases}
\] (136)

where \( q \in C^\infty([m_0, m_1]) \) is a smooth function satisfying \( q > k^2 \) for a constant \( k > 0 \). Then, for the unique solution \( v \) of the initial value problem

\[
\begin{cases}
-v'' + k^2 v = 0 \\
v(m_0) = 0 \\
v'(m_0) = a'(m_0)
\end{cases}
\] (137)

the following inequality holds on \([m_0, m_1] \):

\[
\frac{a'}{a} \geq \frac{v'}{v}.
\] (138)

There exist \( \delta > m_0 \) such that

\[
a(u) \geq a(\delta) e^{\frac{k(u-\delta)}{2}} > 0 \quad (a'(m_0) > 0)
\]

\[
a(u) \leq a(\delta) e^{\frac{k(u-\delta)}{2}} < 0 \quad (a'(m_0) < 0).
\] (139)

**Proof**  Without loss of generality we can prove the inequality on \([m_0, m_1] \) and choose \( m_0 := 0 \) and \( m_1 = +\infty \). We need to distinguish several cases:

CASE 0: \( a'(0) = 0 \) never occurs. Cf. CASE 0 in the proof of Proposition 16.

CASE 1: \( a'(0) > 0 \).

There exist a \( \delta > 0 \) small enough such that \( a'(\delta) > 0 \) and \( a(\delta) > 0 \). Note that \( \alpha := -\frac{a'(\delta)}{a(\delta)} < k \). We can continue by applying Proposition 16 and obtain the result stated.

CASE 2: \( a(0) < 0 \).

Analogously to CASE 2 in the proof of Proposition 16.

\[\square\]

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