Abstract. Flocculation is the process whereby particles (i.e., flocs) in suspension reversibly combine and separate. The process is widespread in soft matter and aerosol physics as well as environmental science and engineering. We consider a general size-structured flocculation model, which describes the evolution of flocs in an aqueous environment. Our work provides a unified treatment for many size-structured models in the environmental, industrial, medical, and marine engineering literature. In particular, our model accounts for basic biological phenomena in a population of microorganisms including growth, death, sedimentation, predation, renewal, fragmentation and aggregation. Our central goal in this paper is to rigorously investigate the long-term behavior of this generalized flocculation model. Using results from fixed point theory we derive conditions for the existence of continuous, non-trivial stationary solutions. We further apply the principle of linearized stability and semigroup compactness arguments to provide sufficient conditions for local exponential stability of stationary solutions as well as sufficient conditions for instability.

The end results of this analytical development are relatively simple inequality-criteria which thus allows for the rapid evaluation of the existence and stability of a non-trivial stationary solution. To our knowledge, this work is the first to derive precise existence and stability criteria for such a generalized model. Lastly, we also provide an illustrating application of this criteria to several flocculation models.

Key words. Flocculation model, nonlinear evolution equations, principle of linearized stability, spectral analysis, structured populations dynamics, semigroup theory

AMS subject classifications. 35Q02, 35P02, 45C02, 45G02, 45K02, 92B05

1. Introduction. Flocculation is the process whereby particles (i.e., flocs) in suspension reversibly combine and separate. The process is widespread in soft matter and aerosol physics as well as environmental science and engineering. A popular model for flocculation is a 1D nonlinear partial integro-differential equation which describes the time-evolution of the particle size number density. In the engineering literature, this equation can be derived using a so-called population balance equation (PBE) framework and we direct the interested reader to the book by Ramkrishna [50] for more on this framework.

Previous analytical work on these models focused on classes of flocculation equations that did not allow for the vital dynamics (i.e., birth and death) of individual particles. These phenomena are obviously critical features in the modeling of microbial flocculation. Accordingly, in this work we consider a general size-structured flocculation model which accounts for growth, aggregation, fragmentation, surface erosion and sedimentation. The variable \( p(t,x) \) denotes the number density of flocs of size \( x \) at time \( t \), and for a given interval \( U \subseteq (x_0, x_1) \), the function \( \chi_U \) represents the characteristic function of the interval \( U \). A floc is assumed to have a minimum and maximum size \( x_0 \) and \( x_1 \) such that \( 0 \leq x_0 < x_1 \leq \infty \). The equations for the microbial flocculation model can be written as

\[
\partial_t p = \mathcal{F}[p] \tag{1.1}
\]

where

\[
\mathcal{F}[p] := \mathcal{G}[p] + \mathcal{A}[p] + \mathcal{B}[p],
\]

\( \mathcal{G} \) denotes growth

\[
\mathcal{G}[p] := -\partial_x (gp) - \mu(x)p(t,x), \tag{1.2}
\]

\( \mathcal{A} \) denotes aggregation

\[
\mathcal{A}[p] := \frac{1}{2} \chi_{[2x_0,x_1]}(x) \int_{x_0}^{x-x_0} k_a(x-y,y)p(t,x-y)p(t,y) \, dy
\]

\[
- \chi_{[x_0,x_1-x_0]}p(t,x) \int_{x_0}^{x_1-x} k_a(x,y)p(t,y) \, dy \tag{1.3}
\]
and $\mathcal{B}$ denotes breakage
\[
\mathcal{B}[p] := \chi_{(x_0, x_1-x_0)}(x) \int_{x_0}^{x_1} \Gamma(x; y)k_f(y)p(t, y) \, dy - \frac{1}{2} \chi_{[2x_0, x_1]}k_f(x)b(t, x).
\] (1.4)

The boundary condition is traditionally defined at the smallest size $x_0$ and the initial condition is defined at $t = 0$
\[
g(x_0)p(t, x_0) = \int_{x_0}^{x_1} q(x)p(t, x) \, dx,
\]
\[
p(0, x) = p_0(x) \in L^1(I).
\]

Note that in vivo, there are no flocs of size zero and the flocs cannot grow indefinitely, so the only biologically realistic case is $0 < x_0 < x_1 < \infty$. However, when $x_0 > 0$, the characteristic functions appearing in equations (1.3) and (1.4) make our theoretical development rather cumbersome. Hence, for the sake of convenience, in this paper we consider the case $x_0 = 0 < x_1 < \infty$, and postpone the analysis of the case with $0 < x_0 < x_1 = \infty$ for our future papers. Therefore, we will denote the closed interval by $I := [0, x_1]$ and will make extensive use of this interval in our development. We carry out the analysis of this work (unless otherwise specified) on the space of absolutely integrable functions on $I$, denoted by $L^1(I)$. We also note that well-posedness of the flocculation model on this space has been established by Banasiak and Lamb [9].

1.1. Background and model terms. In this section, we will provide a brief background and overview of the individual terms in the general flocculation equation above.

To begin, the Sinko-Streifer [51] terms in (1.2) correspond to the growth and removal of flocs, respectively. The function $g(x)$ represents the average growth rate of the flocs of size $x$ due to mitosis, and the coefficient $\mu(x)$ represents a size-dependent removal rate due to gravitational sedimentation and cell death. Specifically, when an individual cell in the floc of size $x$ divides into daughter cells, the new cells can remain with the floc, contributing in an increase in its total size. Conversely, a daughter cell can also leave the floc to form a new single-cell floc. This second case is modeled by McKendrick-von Foerster type renewal boundary conditions,
\[
g(x_0)p(t, x_0) = \int_{x_0}^{x_1} q(x)p(t, x) \, dx,
\] (1.5)
where the renewal rate $q(x)$ represents the number of new cells that leave a floc of size $x$ and enter the single cell population. We note that this boundary condition could also be used to model the surface erosion of flocs, where single cells are eroded off the floc and enter single cell population. The well-posedness and stability of equilibrium solutions of the Sinko-Streifer equations has been extensively studied by many researchers using a wide variety of mathematical conditions [23, 34, 10, 49, 48, 18]. For numerical simulation of the model, a convergent numerical scheme has been proposed in [10], and inverse problems for estimation of the parameters of the model have been discussed in [11, 12, 29].

The aggregation of flocs into larger ones is modeled in (1.3), by the Smoluchowski coagulation equation. The function $k_a(x, y)$ is the aggregation kernel, which describes the rate with which the flocs of size $x$ and $y$ agglomerate to form a floc of size $x + y$. This equation has been widely used, e.g., to model the formation of clouds and smog in meteorology [47], the kinetics of polymerization in biochemistry [50], the clustering of planets, stars and galaxies in astrophysics [10], and even schooling of fish in marine sciences [45]. The equation has also been the focus of considerable mathematical analysis. For the aggregation kernels satisfying the inequality $k_a(x, y) \leq 1 + x + y$, existence of mass conserving global in time solutions were proven [21, 30, 42] (for some suitable initial data). Conversely, for aggregation kernels satisfying $(xy)^{\gamma/2} \leq k_a(x, y)$ with $1 < \gamma \leq 2$, it has been shown that the total mass of the system blows up in a finite time (referred as a gelation time) [23]. For a review of further mathematical results, we refer readers to review articles by Aldous [5], Menon and Pego [43], and Wattis [54] and the book by Dubovskii [20]. Lastly, although the Smoluchowski equation has received substantial theoretical work, the derivation of analytical solutions for many realistic aggregation kernels has proven elusive. Towards this end, many discretization schemes for numerical simulations of the Smoluchowski equations have been proposed, and we refer interested readers to the review by Bortz [14, §6].

The breakage of flocs due to fragmentation is modeled by the terms in (1.4), where the fragmentation kernel $k_f(x)$ calculates the rate with which a floc of size $x$ fragments. The integrable function $\Gamma(x; y)$
represents the post-fragmentation probability density of daughter flocs for the fragmentation of the parent
flocs of size \( y \). The post-fragmentation probability density function \( \Gamma \) is one of the least well-understood
terms in the flocculation model. Many different forms are used in the literature, among which normal and
log-normal densities are the most common \[55, 53\]. Recent modeling and computational work suggests that
normal and log-normal forms for \( \Gamma \) are not correct and that a form closer to an \( \arcsin(x; y) \) density would
be more accurate \[15, 17\]. However, in this work we do not restrict ourselves to any particular form of \( \Gamma \); and instead simply assume that the function \( \Gamma \) satisfies the mass conservation requirement. In other words,
all the fractions of daughter flocs formed upon the fragmentation of a parent floc sum to unity,
\[
\int_{x_0}^{y} \Gamma(x; y) \, dx = 1 \quad \text{for all } y \in (x_0, x_1].
\]

1.2. Overview of model assumptions. The flocculation model, presented in \[1.1\], is a generalization
of many mathematical models appearing in the size-structured population modeling literature and has broad
applications in environmental, industrial, medical, and marine sciences. For example, when the fragmentation
kernel is omitted, \( k_f \equiv 0 \), the flocculation model reduces to algal aggregation model used to describe evolution
of phytoplankton community \[3\]. When the removal and renewal rates are set to zero, the flocculation model
simplifies to a model used to describe the proliferation of \textit{Klebsiella pneumoniae} in a bloodstream \[16\].
Furthermore, the flocculation model, with only growth and fragmentation terms, was used to investigate the
elongation of prion polymers in infected cells \[31, 19\].

The flocculation model in this form \(1.1\) was first considered by Banasiak and Lamb in \[9\], where they
employed the flocculation model to describe the dynamical behavior of phytoplankton cells. The authors
showed that under some conditions the flocculation model is well-posed, i.e., there exist a unique, global in
time, positive solution for every absolutely integrable initial distribution. For the case \( x_1 = \infty \), Banasiak
\[8\] establishes that for certain range of parameters, the solutions of the flocculation model blow up in finite
time. Nevertheless, to the best of our knowledge, for the case \( x_1 < \infty \) the long-term behavior of this model
has not been considered. This is mainly due to nonlinear nature of Smoluchowski coagulation equations
used for modeling aggregation. Hence, our main goal in this paper is to study the long-term behavior of the
broad class of flocculation models described in \(1.1\). For the remainder of this work, we make the following
assumptions

\begin{align*}
(A1) \quad & g \in C^1(I) \quad g(x) > 0 \quad \text{for } x \in I \\
(A2) \quad & k_a \in W^{1, \infty}(I \times I), \quad k_a(x, y) = k_a(y, x) \\
& \text{and } k_a(x, y) = 0 \quad \text{if } x + y \geq x_1, \\
(A3) \quad & \mu \in C(I) \quad \text{and } \mu \geq 0 \quad \text{a.e. on } I, \\
(A4) \quad & q \in L^\infty(I) \quad \text{and } q \geq 0 \quad \text{a.e. on } I, \\
(A5) \quad & k_f \in C(I) \quad k_f(0) = 0 \quad \text{and } k_f \geq 0 \quad \text{a.e. on } I, \\
(A6) \quad & \Gamma(x, y) \in W^{1, \infty}(I), \quad \Gamma(x, y) \geq 0 \quad \text{for } x \in (0, y]; \\
& \text{and } \Gamma(x, y) = 0 \quad \text{for } x \in (y, x_1) .
\end{align*}

Assumption (A1) states that the floc of any size has strictly positive growth rate. This in turn implies
that flocs can grow beyond the maximal size \( x_1 \), i.e., the model ignores what happens beyond the maximal
size \( x_1 \) (as many authors in the literature have done \[28, 3, 1, 26\]). We also note that the Assumption (A1)
generates biologically unrealistic condition \( g(0) > 0 \), i.e., the flocs of size zero also have positive growth rate.
However, this assumption is crucial for our work, and thus we postpone the analysis of the case \( g(0) = 0 \) for
our future papers. Assumption (A2) states that for the aggregates of size \( x \) and \( y \) the aggregation rate is
zero if the combined size of the aggregates is larger than the maximal size. Lastly, Assumption (A3) on \( \mu(x) \)
enforces continuous dependence of the removal on the size of a floc and ensures that every floc is removed
with a non-negative rate.

When the long-term behavior of biological populations is considered, many populations converge to
a stable time-independent state. Thus, identifying conditions under which a population converges to a
stationary state is one of the most important applications of mathematical population modeling. It is trivially
true that a zero stationary solution exists, but we are also interested in non-trivial stationary solutions of
the flocculation model. Hence, in Section \[2\] we first show that under some suitable conditions on the model
parameters the flocculation equation has at least one non-trivial (non-zero and non-negative) stationary solution. Once a stationary solution to a model is shown to exist, the next natural question is whether it is stable or unstable. When the associated evolution equation of a population model is linear, many of stability properties can be deduced from the spectral properties of this linear operator \([18,32]\). However, almost no information about the operator can be deduced from the spectrum of a nonlinear operator \([7]\). Moreover, there is no general consensus among mathematicians on how to define spectrum of a nonlinear operator. Thus, our stability analysis in this work is based on the principle of linearized stability for nonlinear evolution equations \([55,37]\). Hence, in Section \(3\) we summarize the principle of linearized stability and linearize the flocculation model around its stationary solutions. In Section \(3\) we also derive conditions for the regularity of the linearized flocculation model. Next, in Sections \(4\) and \(5\) we derive sufficiency conditions for the linearized stability and instability of zero and non-zero stationary solutions. In Section \(6\) we illustrate our results with several examples. Finally, in Section \(7\) we summarize and discuss the conclusions of this work.

2. Existence of a positive stationary solution. The flocculation model under our consideration \((1.1)\), accounts for physical mechanisms such as growth, removal, fragmentation, aggregation and renewal of microbial flocs. Thus, under some conditions, which balance these mechanisms, one could reasonably expect that the model possesses a non-trivial stationary solution. Hence, our main goal in this section is to derive sufficient conditions for the model terms such that the equation \((1.1)\) engenders a positive stationary solution.

Recall that at a steady state we should have

\[ p_t = 0 = F[p]. \tag{2.1} \]

By Assumption \((A1)\), we know that \(1/g \in C(I)\) and thus we can define \(p = f/g\) for some \(f \in C(I)\). The substitution of this \(f\) into \((2.1)\), integration between 0 and an arbitrary \(x\), and rearrangement of the terms yields

\[ f(x) = \int_0^x \frac{q(y)}{g(y)} f(y) \, dy - \int_0^x \frac{k(y)/2 + \mu(y)}{g(y)} f(y) \, dy + \int_0^x \int_z^x \frac{\Gamma(z; y)k_f(y)}{g(y)} f(y) \, dy \, dz + \frac{1}{2} \int_0^x \int_0^z \frac{k_a(z-y, y)}{g(z-y)g(y)} f(z-y) f(y) \, dy \, dz - \int_0^x f(z) \int_0^z \frac{k_a(z, y)}{g(y)} f(y) \, dy \, dz. \]

We now define the operator \(\Phi\) as

\[ \Phi[f](x) := \int_0^x \frac{q(y)}{g(y)} f(y) \, dy - \int_0^x \frac{k_f(y)/2 + \mu(y)}{g(y)} f(y) \, dy + \int_0^x \int_z^x \frac{\Gamma(z; y)k_f(y)}{g(y)} f(y) \, dy \, dz + \frac{1}{2} \int_0^x \int_0^z \frac{k_a(z-y, y)}{g(z-y)g(y)} f(z-y) f(y) \, dy \, dz - \int_0^x f(z) \int_0^z \frac{k_a(z, y)}{g(y)} f(y) \, dy \, dz. \tag{2.2} \]

and will use a fixed point theorem to prove the existence of a fixed point \(f\) of \(\Phi\). This in turn will allow us to claim that equation \((2.1)\) has at least one non-trivial positive solution.

The use of fixed point theorems for showing existence of non-trivial stationary solutions is not new in size-structured population modeling. For example, fixed point theorems, based on Leray-Schauder degree theory, have been used to find stationary solutions of linear Sinko-Streifer type equations \([28,48]\). Moreover, the Schauder fixed point theorem has been used to establish the existence of steady state solutions of nonlinear coagulation-fragmentation equations \([39]\). For our purposes we will use the following fixed point theorem, and refer readers to \([8]\) for the full discussion of the proof.

**Theorem 2.1.** Let \(X\) be a Banach space, \(K \subset X\) a closed convex cone, \(K_r = K \cap B_r(0), \Phi : K_r \to K\) continuous such that \(\Phi(K_r)\) is relatively compact. Assume that

1. \(\Phi[x] \neq \lambda x\) for all \(\|x\| = r\) and \(\lambda > 1\).
2. There exists a \(\rho \in (0, r)\) and \(k \in K \setminus \{0\}\) such that

\[ x - \Phi[x] \neq \lambda k \quad \text{for all} \quad \|x\| = \rho \quad \text{and} \quad \lambda > 0. \]

\[ \text{Hereafter, we refer to the following theorem as “the fixed point theorem”} \]
Then $\Phi$ has at least one fixed point $x_0 \in K$ such that $\rho < \|x_0\| < r$.

Next, we show that the operator $\Phi$ defined in (2.2) satisfies the assumptions of the above theorem, which in turn implies existence of a positive stationary solution of the operator $\mathcal{F}$. Since we have been working on the space of absolutely integrable functions on $I$, a natural candidate for the Banach space $\mathcal{X}$ would be $L^1(I)$. However, to obtain sufficient regularity for stability analysis of a stationary solution, we choose $\mathcal{X} = C(I)$ with usual uniform norm $\|\cdot\|_u$ on $I$. We also denote the usual essential supremum of a function by $\|\cdot\|_{\infty}$. Since the positive cone in $C(I)$, denoted by $(C(I))_+$, is closed and convex, we choose $K$ to be $(C(I))_+$. Then $K_r = K \cap B_r(0)$, where $B_r(0) \subset \mathcal{X}$ is an open ball of radius $r$ and centered at zero, and $r$ has yet to be chosen. We are now in a position to state the main result of this section in the following theorem.

**Theorem 2.2.** Assume that the conditions

$$0 < q(x) + \frac{1}{2}k_f(x) - \mu(x), \quad (C1)$$

and

$$\int_x^{x_1} \frac{1}{g(y)} \left( k_f(y) \int_0^y \Gamma(z; y) \, dz + q(y) \right) \, dy + \int_0^x \frac{1}{g(y)} \left( q(y) + \frac{1}{2}k_f(y) - \mu(y) \right) \, dy \leq 1, \quad (C2)$$

hold true for all $x \in I$. Then the operator $\Phi$ defined in (2.2) has at least one non-zero fixed point, $f_+ \in K$ satisfying

$$0 < \eta < \|f_+\|_u < r \quad (2.3)$$

for some $\eta, r > 0$. Moreover, the non-zero and non-negative function

$$p_+ = \frac{f_+}{g} \in C(I) \quad (2.4)$$

is the stationary solution of the flocculation model defined in (1.1).

*Proof.* For $f \in K_r$ we have

$$\Phi[p] \geq \int_0^{x_1} \frac{q(y)}{g(y)} f(y) \, dy - \int_0^x \frac{k_f(y)/2 + \mu(y)}{g(y)} f(y) \, dy$$

$$+ \int_0^x \int_0^y \Gamma(z; y) k_f(y) f(y) \, dz \, dy - \int_0^x \frac{f(z)}{g(z)} \int_0^x \frac{k_a(z, y)}{g(y)} f(y) \, dy \, dz$$

$$\geq \int_0^x \frac{f(z)}{g(z)} \left[ q(z) - \frac{1}{2}k_f(z) - \mu(z) - \int_0^{x-z} \frac{k_a(z, y)}{g(y)} f(y) \, dy \right] \, dz + \int_0^x \frac{k_f(y)f(y)}{g(y)} \int_0^y \Gamma(z; y) \, dz \, dy$$

$$\geq \int_0^x \frac{f(z)}{g(z)} \left[ q(z) - \frac{1}{2}k_f(z) - \mu(z) - r \cdot \|k_a\|_{\infty} \left\| \frac{1}{g} \right\|_1 \right] \, dz,$$

where $\|\cdot\|_1$ represents the usual $L^1$ norm on $I$. The first condition of the theorem (C1) guarantees that

$$q(z) + \frac{1}{2}k_f(z) - \mu(z) > 0 \text{ for all } z \in I,$$

so we can choose $r$ in $K_r$ sufficiently small such that $\Phi[f] \geq 0$, i.e., $\Phi : K_r \rightarrow K$. On the other hand, using the assumptions (A1)-(A6), it is straightforward to show that $\Phi(K_r) \subset C^1(I)$.

This in turn, from Arzelà-Ascoli theorem, implies that the operator $\Phi$ has relatively compact image.

Next we prove the second assumption of the theorem. For the sake of contradiction, suppose that there exist $f \in K_r$ with $\|f\|_u = r$ and $\lambda > 1$ such that

$$\Phi[f] = \lambda f.$$

\[\text{Recall that continuous differentiability implies uniform continuity, and thus equicontinuity}\]
Then it follows that
\[
\lambda f(x) \leq \int_0^x \frac{g(y)}{g(y)} f(y) dy - \int_0^x \frac{k_f(y)/2 + \mu(y)}{g(y)} f(y) dy + \int_0^x \frac{\Gamma(z; y)k_f(y)}{g(y)} f(y) dy dx \\
- \int_0^x \int_0^x \frac{\Gamma(z; y)k_f(y)}{g(y)} f(y) dy dz + \frac{1}{2} \int_0^x \frac{\int_0^x k_a(z - y, y)}{g(z - y)g(y)} f(y) dy dz \\
\leq \int_0^x \frac{1}{g(y)} \left( k_f(y) \int_0^x \Gamma(z; y) dz + q(y) \right) f(y) dy \\
+ \int_0^x \frac{1}{g(y)} \left( q(y) + \frac{1}{2} k_f(y) - \mu(y) \right) f(y) dy + \|f\|_u \cdot \|k_a\|_\infty \cdot \left\| \frac{1}{g} \right\|_1^2
\]
which yields that
\[
\lambda \leq \sup \left\{ \int_0^x \frac{1}{g(y)} \left( k_f(y) \int_0^x \Gamma(z; y) dz + q(y) \right) dy + \int_0^x \frac{1}{g(y)} \left( q(y) + \frac{1}{2} k_f(y) - \mu(y) \right) dy \right\} \\
+ r \|k_a\|_\infty \left\| \frac{1}{g} \right\|_1^2.
\]
From the second condition of the theorem (C2) it follows that
\[
\sup \left\{ \int_0^x \frac{1}{g(y)} \left( k_f(y) \int_0^x \Gamma(z; y) dz + q(y) \right) dy + \int_0^x \frac{1}{g(y)} \left( q(y) + \frac{1}{2} k_f(y) - \mu(y) \right) dy \right\} \leq 1.
\]
Then, we can choose \( r \) sufficiently small such that it contradicts the first assumption of the fixed point theorem, \( \lambda > 1 \).

Next we will derive conditions for the second condition of the fixed point theorem. For the sake of contradiction, let us choose \( k \equiv 1 \in K \setminus \{0\} \) and assume that there exists \( f \in K_r \) with \( \|f\|_u = \eta < r \) and \( \lambda > 0 \) such that
\[
f - \Phi[f] = \lambda k = \lambda.
\]
This equation in turn can be written as
\[
\lambda k = \lambda = f(x) - \Phi[f](x) \\
\leq \|f\|_u - \int_0^x \frac{g(y)}{g(y)} f(y) dy + \int_0^x \frac{k_f(y)/2 + \mu(y)}{g(y)} f(y) dy \\
- \int_0^x \int_0^x \frac{\Gamma(z; y)k_f(y)}{g(y)} f(y) dy dz + \int_0^x \frac{f(z)}{g(z)} \int_0^x \frac{k_a(z, y)}{g(y)} f(y) dy dz \\
\leq \eta - \int_0^x \frac{g(y)}{g(y)} f(y) dy + \int_0^x \frac{\mu(y) - k_f(y)/2}{g(y)} f(y) dy + \eta^2 \cdot \|k_a\|_\infty \cdot \left\| \frac{1}{g} \right\|_1^2 \\
\leq \int_0^x \frac{\mu(y) - k_f(y)/2 - q(y)}{g(y)} f(y) dy + \eta^2 \cdot \|k_a\|_\infty \cdot \left\| \frac{1}{g} \right\|_1^2 + \eta,
\]
which should hold for all \( x \in I \). Thus, provided that the condition (C1) holds, we can choose \( \eta \in (0, r) \) sufficiently small such that we get a contradiction to the second assumption of the fixed point theorem, \( \lambda > 0 \). Hence, the fixed point theorem guarantees the existence of a positive fixed point of \( \Phi \) satisfying the bounds (2.3).

Therefore, the function \( p_\ast = f_\ast/g \) is a stationary solution of the flocculation equations (1.1). Moreover, from the assumption (A1) and the continuity of the fixed point \( f_\ast \) it follows that \( p_\ast \) is non-zero, non-negative and continuous on \( I \).
3. Principle of linearized stability and regularity properties of the linearized semigroup.

In this section we summarize the principle of linearized stability as it applies to semigroups in general and our flocculation equation in particular.

For a given autonomous ordinary differential equation,

$$\dot{u} = f(u),$$

the method for determining the local asymptotic behavior of a stationary solution $u_\ast$, $f(u_\ast) = 0$, by the eigenvalues of the Jacobian $Jf(u_\ast)$ is quite well-known. In semigroup theory this method is known as the principle of linearized stability and was developed in the context of semilinear partial differential equations in [36, 52, 55]. Later, Kato [37] extended this principle to a broader range of nonlinear evolution equations. Before presenting the principle of linearized stability we introduce some terminology, which can be found in many functional analysis books (see [13] for instance).

The *growth bound* $\omega_0(A)$ of a strongly continuous semigroup $(S(t))_{t \geq 0}$ with an infinitesimal generator $A$ is defined as

$$\omega_0(A) := \inf \left\{ \omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ such that } \|S(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0 \right\}.$$

The operator $DA(f)$ denotes the Fréchet derivative of an operator $A$ evaluated at $f$, which is defined as

$$DA(u)h = A[u + h] - A[u] + o(h), \quad \forall u \in D(A),$$

where $o$ is little-o operator satisfying $\|o(h)\| \leq b(r)\|h\|$ with increasing continuous function $b : [0, \infty) \to [0, \infty)$, $b(0) = 0$.

The *discrete spectrum* $\sigma_D(A)$ of an arbitrary operator $A$ on a Banach space $X$, is the subset of the point spectrum of $A$,

$$\sigma_p(A) = \{ \lambda \in \mathbb{C} : \exists \phi \neq 0 \in X \text{ s.t. } A\phi = \lambda \phi \},$$

such that $\lambda \in \sigma_D(A)$ is an isolated eigenvalue of finite multiplicity, i.e., the dimension of the set

$$\{ \psi \in X : A\psi = \lambda \psi \}$$

is finite and nonzero. Let $(T(t))_{t \geq 0}$ be a $C_0$ semigroup on the Banach space $X$ with its infinitesimal generator $A$. Then the limit $\omega_1(A) = \lim_{t \to \infty} t^{-1} \log \|T(t)\|$ is well-defined and called the $\alpha$-growth bound of $(T(t))_{t \geq 0}$. The function $\alpha[M] = 0$ implies that $M$ (closure of $M$) is a compact set. Analogously, for a semigroup $(T(t))_{t \geq 0}$, $\alpha[T(t)] = 0$ indicates that the semigroup is eventually compact.

With the above definitions, we are now ready to present the principle of linearized stability in the form of the following proposition (see [55] for the complete discussion of the proof of the following proposition).

**Proposition 3.1.** Define the nonlinear operator $N : D(F) \subset L^1(I) \to L^1(I)$ and let $f_\ast \in D(N)$ be a stationary solution of (1.1), i.e., $N[f_\ast] = 0$. If $N$ is continuously Fréchet differentiable on $L^1(I)$ and the linearized operator $L = DN(f_\ast)$ is the infinitesimal generator of a $C_0$-semigroup $T(t)$, then the following statements hold:

1. If $\omega_0(L) < 0$, then $f_\ast$ is locally asymptotically stable in the following sense: There exists $\eta, C \geq 1$, and $\alpha > 0$ such that if $\|f - f_\ast\| < \eta$, then a unique mild solution $T(t)f$, satisfies $\|T(t)f - f_\ast\| \leq Ce^{-\alpha t}\|f - f_\ast\|$ for all $t \geq 0$.

2. If there exists $\lambda_0 \in \sigma(L)$ such that $Re\lambda > 0$ and

$$\max \left\{ \omega_1(L), \sup_{\lambda \in \sigma_D(L) \setminus \{\lambda_0\}} Re\lambda \right\} < Re\lambda_0,$$

then $f_\ast$ is an unstable equilibrium in the sense that there exists $\varepsilon > 0$ and sequence $\{f_n\}$ in $X$ such that $f_n \to f_\ast$ and $\|T(n)f_n - f_\ast\| \geq \varepsilon$ for $n = 1, 2, \ldots$. 


Having the explicit statement of the principle of linearized stability in hand, we now show that the nonlinear operator \( F \) defined in (1.4) satisfies all the conditions of Proposition 3.1. Towards this end, we first establish the elementary assumption of Proposition 3.1 in the following lemma.

**Lemma 3.2.** The nonlinear operator \( F \) defined in (1.4) is continuously Fréchet differentiable on \( L^1(I) \).

*Proof.* The Fréchet derivative of the nonlinear operator \( F \) is given explicitly as

\[
DF(\phi)[h(x)] = -\partial_x [gh](x) - \left( \mu(x) + \frac{1}{2}k_f(x) \right) h(x) + \int_x^{x_1} \Gamma(x; y)k_f(y)h(y) \, dy \\
+ \frac{1}{2} \int_0^x k_a(x - y, y) [\phi(y)h(x - y) + h(y)\phi(x - y)] \, dy \\
- h(x) \int_0^{x_1-x} k_a(x, y)\phi(y) \, dy - \phi(x) \int_0^{x_1-x} k_a(x, y)h(y) \, dy.
\]

For the arbitrary functions \( u_1, u_2 \in L^1(I) \) we have

\[
|DF(u_1)h(x) - DF(u_2)h(x)| \leq \frac{1}{2} \|k_a\|_{\infty} \int_0^x |u_1(y) - u_2(y)||h(x - y)| \, dy \\
+ \frac{1}{2} \|k_a\|_{\infty} \int_0^x |h(y)||u_1(x - y) - u_2(x - y)| \, dy \\
+ |h(x)||k_a\|_{\infty} \int_0^{x_1} |u_1(y) - u_2(y)| \, dy \\
+ |u_1(x) - u_2(x)||k_a\|_{\infty} \int_0^{x_1} |h(y)| \, dy.
\]

Consequently, taking the integral of both sides with respect to \( x \) and an application of Young’s inequality for convolutions (see [1, Theorem 2.24]) to the first two integrals yields

\[
\|DF(u_1)h(x) - DF(u_2)h(x)\|_1 \leq \|k_a\|_{\infty} \|u_1 - u_2\|_1 \|h\|_1 + \|k_a\|_{\infty} \|u_1 - u_2\|_1 \|h\|_1 + \|u_1 - u_2\|_1 \|k_a\|_{\infty} \|h\|_1
\]

for all \( h \in L^1(I) \). Then it follows that

\[
\|DF(u_1) - DF(u_2)\|_1 \leq 3 \|k_a\|_{\infty} \|u_1 - u_2\|_1,
\]

which in turn implies that the nonlinear operator \( F \) is continuously Fréchet differentiable on \( L^1(I) \).

In the previous section we have shown that the nonlinear operator \( F \) has at least one non-trivial stationary solution, \( p_* \) (in addition to trivial zero stationary solution). To derive stability results for this stationary solutions we first linearize the equation (1.1) around \( p_* \). A simple calculation yields that the Fréchet derivative of the nonlinear operator \( F \) evaluated at a stationary solution \( p_* \) (see Theorem 2.2) is given explicitly by

\[
\mathcal{L}[h](x) = DF(p_*)[h](x) = -\partial_x [g(x)h(x)] - A(x)h(x) + \int_x^{x_1} \Gamma(x; y)k_f(y)h(y) \, dy \\
- \int_0^{x_1-x} E(x, y)h(y) \, dy + \int_0^x E(x, y)h(y) \, dy,
\]

where

\[
E(x, y) = k_a(x, y)p_*(x)
\]

and

\[
A(x) = \frac{1}{2}k_f(x) + \mu(x) + \int_0^{x_1-x} E(y, x) \, dy.
\]

We first prove that the linear operator \( \mathcal{L} \) is an infinitesimal generator of a strongly continuous semigroup \( \mathcal{S} = (T(t))_{t \geq 0} \). Consequently, we will prove two regularity results for the semigroup \( \mathcal{S} \), which will prove
useful in the spectral analysis of the operator $\mathcal{L}$. Particularly, we will show that under some conditions on the model ingredients the semigroup $\mathcal{S}$ is positive and eventually compact. The main implication of eventual compactness is that the Spectral Mapping Theorem holds (see [22]) for the semigroup $\mathcal{S}$,

$$\sigma(T(t)) \setminus \{0\} = \exp(\sigma(L)), \quad t \geq 0.$$ 

Consequently, we will use the positivity of the semigroup $\mathcal{S}$ in Section 5.2 where we employ the positive perturbation method introduced in [28].

**Lemma 3.3.** If we define the domain of the linearized operator $\mathcal{L}$ as

$$\mathcal{D}(\mathcal{L}) = \{ \phi \in L^1(I) \mid (g\phi)' \in L^1(I), (g\phi)(0) = K[\phi] \},$$

then the operator $\mathcal{L}$ generates a $C_0$ semigroup on $\mathcal{D}(\mathcal{L})$.

**Proof.** The linear operator $\mathcal{L}$ can be written as the sum of an unbounded operator

$$\mathcal{L}_1[g](x) = -\partial_x (g(x)h(x)) - A(x)h(x)$$

and bounded operators

$$\mathcal{L}_2[g](x) = \int_x^{x_1} \Gamma(x; y)k_f(y)h(y) dy - \int_0^{x_1-x} E(x, y)h(y) dy, \quad \mathcal{L}_3[g](x) = \int_0^z E(x - y, y)h(y) dy.$$ 

From the fact that $g(x), A(x) \in C(I)$ and from the Lemma 2.4 of [3] it follows that $\mathcal{L}_1$ generates a $C_0$ semigroup on $\mathcal{D}(\mathcal{L})$. Consequently, the bounded perturbation theorem of [46, §3, Theorem 1.1] yields that the operator $\mathcal{L}$ is also an infinitesimal generator of a $C_0$ semigroup.

**Lemma 3.4.** For a given stationary solution $p_* \in C(I)$ the operators $\mathcal{L}_2 : \mathcal{D}(\mathcal{L}) \to L^1(I)$ and $\mathcal{L}_3 : \mathcal{D}(\mathcal{L}) \to L^1(I)$ defined in (3.3) are compact operators.

**Proof.** We first prove that the operator $\mathcal{L}_2$ is compact. Then compactness of the operator $\mathcal{L}_3$ follows from analogous arguments. Let us denote a unit ball centered at zero in $L^1(I)$ by $B = \{ \phi \in L^1(I) \mid ||\phi||_1 \leq 1 \}$. Recall that an operator is compact if it maps a unit ball into a relatively compact set. Consequently, observe that the assumptions (A2) and (A6) together imply that the operator

$$\partial_x \mathcal{L}_2[g](x) = k_a(x, x_1 - x)p_*(x)h(x_1 - x) - \int_0^{x_1 - x} \partial_x (k_a(x, y)p_*(x))h(y) dy \\
+ \int_x^{x_1} \partial_x \Gamma(x; y)k_f(y)h(y) dy - \Gamma(x; x_1)k_f(x)h(x)$$

is also bounded. Hence $\mathcal{L}_2[B] \subset W^{1,1}(I)$ and from the Rellich-Kondrachov embedding theorem (see [4, Theorem 6.3] for a statement of the theorem) it follows that the set $\mathcal{L}_2[B]$ is relatively compact.

**Lemma 3.5.** The operator $\mathcal{L}$ defined in (3.3) generates an eventually compact $C_0$ semigroup. And thus, the spectrum of the operator $\mathcal{L}$ consists of isolated eigenvalues of a finite multiplicity only, i.e., $\sigma(\mathcal{L}) = \sigma_D(\mathcal{L})$.

**Proof.** The operator $\mathcal{L}_1$ defined in (3.4) is well-known operator in size-structured dynamics literature. If $g \in C^1(I)$ and $A \in C(I)$, then in Farkas and Hagen [24, Theorem 3.1] it has been shown that the $C_0$ semigroup generated by the operator $\mathcal{L}_1$ is compact for $t > 2 \int_0^{x_1} \frac{1}{g(y)} dy$. The condition $g \in C^1(I)$ follows from our main assumption (A1), and continuity of the function

$$A(x) = \frac{1}{2} k_f(x) + \mu(x) + \int_0^{x_1-x} k_a(x, y)p_*(y) dy$$

follows from the assumptions (A1)- (A6). Thus the semigroup generated by $\mathcal{L}_1$ is eventually compact. Conversely, in Lemma 3.4 we have shown that the operators $\mathcal{L}_2$ and $\mathcal{L}_3$ are compact. Hence, the $C_0$ semigroup generated by the operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ is also compact for $t > 2 \int_0^{x_1} \frac{1}{g(y)} dy$.

Therefore, the eventual compactness of the semigroup $\mathcal{S}$ (generated by $\mathcal{L}$) combined with Theorem 3.3 of [46, §2.3] and Corollary 1.19 of [22, §4] together imply that the spectrum of $\mathcal{L}$ consists of isolated eigenvalues of finite multiplicity.

**Lemma 3.6.** For a given steady state solution $p_*$ let us choose the functions $k_a, k_f$ and $\Gamma$ such that

$$\partial_x (k_a(x, y)p_*(x)) \leq 0 \quad \text{for all } x \in I \text{ and } y \in (0, x)$$

(3.6)
\[ \Gamma(x; y)k_f(y) \geq k_a(x, y)p_*(x) \text{ for all } x \in I \text{ and } y \in [x, x_1) \]  

Then the operator \( \mathcal{L} \) generates a positive \( C_0 \) semigroup.

**Proof.** In Furukawa and Hagen [26] Theorem 3.3] it has been shown that the operator \( \mathcal{L}_1 : \mathcal{D}(\mathcal{L}) \to L^1(I) \) generates a positive \( C_0 \) semigroup under the main assumptions (A1)-(A6). On the other hand, from the conditions (3.6) and (3.7) it follows that

\[
\mathcal{L}_2[h](x) + \mathcal{L}_3[h](x) \geq \int_0^x k_a(x - y, y)p_*(x - y)h(y) \, dy + \int_x^{x_1} \Gamma(x; y)k_f(y)h(y) \, dy
\]

\[
- \int_0^{x_1} k_a(x, y)p_*(x)h(y) \, dy
\]

\[
\geq \int_0^x [k_a(x - y, y)p_*(x - y) - k_a(x, y)p_*(x)] h(y) \, dy
\]

\[
+ \int_x^{x_1} [\Gamma(x; y)k_f(y) - k_a(x, y)p_*(x)] h(y) \, dy \geq 0,
\]

which in turn ensures that the operator \( \mathcal{L}_2 + \mathcal{L}_3 \) is a positive operator. Since the positivity of a semigroup is invariant under a bounded and positive perturbation of its generator (see [22 §6, Corollary 1.11]), the result follows immediately. \( \square \)

**Remark 1.** Lemma [3.6] has very important consequence. Specifically, if the positivity conditions (3.6) and (3.7) hold and the spectral bound \( s(\mathcal{L}) = \sup \{ \mathcal{R} \lambda \mid \lambda \in \sigma(\mathcal{L}) \} \) is not equal to \(-\infty\), then \( s(\mathcal{L}) \) belongs to the spectrum \( \sigma(\mathcal{L}) \) Engel and Nagel [22 §Theorem 1.10]. Moreover, the positivity and eventual compactness of the semigroup \( \mathcal{T} \) together imply that the spectral bound \( s(\mathcal{S}) \) is one of the eigenvalues of \( \mathcal{L} \) with finite multiplicity.

### 4. Linearized stability and instability criteria for the zero stationary solution.

In this section we will derive linearized stability results for the zero stationary solution of the flocculation equation. In contrast to non-trivial stationary solutions, zero stationary solution always exists (provided that the well-posedness assumptions (A1)-(A6) hold true). As we have discussed in Section 3, the stability of the steady states depends on the spectral properties of the linear operator \( \mathcal{L} \) defined in (3.2). We define the operator \( \mathcal{M} \) as the linear operator \( \mathcal{L} \) evaluated at the trivial stationary solution, \( p_* \equiv 0 \).

\[
\mathcal{M}[h](x) = -\partial_x[gh](x) - \left( \mu(x) + \frac{1}{2}k_f(x) \right) h(x) + \int_x^{x_1} \Gamma(x; y)k_f(y)h(y) \, dy.
\]  

(4.1)

The assumptions (A1)-(A6) also ensure that the regularity conditions of Section 3 are all satisfied. Hence, the operator \( \mathcal{M} \) generates a positive, eventually compact and strongly continuous semigroup. By Remark 1 we know that the spectral bound \( s(\mathcal{M}) \) of the operator \( \mathcal{M} \) is a dominant eigenvalue of \( \mathcal{M} \) with finite multiplicity. Then, by the principle of linearized stability (Proposition 3.1), the stability of the zero stationary solution depends on the sign of this dominant eigenvalue. Thus, in the subsequent two subsections we derive conditions which guarantee positivity and negativity of the spectral bound \( s(\mathcal{M}) \), respectively.

For a more thorough discussion of the stability of zero stationary solution we refer readers to [23].

#### 4.1. Instability of the trivial stationary solution.

The operator \( \mathcal{M} \) can be written as the sum of an unbounded operator

\[
\mathcal{M}_1[h](x) = -\partial_x[gh](x) - \left( \mu(x) + \frac{1}{2}k_f(x) \right) h(x)
\]

and a bounded operator

\[
\mathcal{M}_2[h](x) = \int_x^{x_1} \Gamma(x; y)k_f(y)h(y) \, dy.
\]

In [20], authors have shown that the operator \( \mathcal{M}_1 \) generates a positive, eventually compact semigroup. Moreover, authors have shown that the spectral bound of \( \mathcal{M}_1 \) is positive if

\[
\int_0^{x_1} \frac{q(x)}{g(x)} \exp \left( - \int_0^x \frac{\mu(s) + \frac{1}{2}k_f(s)}{g(s)} \, ds \right) \, dx > 1.
\]

(4.2)
On the other hand, we note that $\mathcal{M}_2$ is a positive operator. Then, Corollary 1.11 of [22, §6] yields that the operator $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ also generates a positive, eventually compact semigroup. Furthermore, the following inequality holds for spectral bound of $\mathcal{M}_1$ and $\mathcal{M}$,

$$s(\mathcal{M}_1) \leq s(\mathcal{M}_1 + \mathcal{M}_2) = s(\mathcal{M}). \quad (4.3)$$

Consequently, this implies that the operator $\mathcal{M}$ also has a positive spectral bound provided that the condition (4.2) is satisfied. At this point, in Proposition 3.1 choosing $\lambda_0$ equal to the eigenvalue of $\mathcal{M}$ corresponding to $s(\mathcal{M})$ and using Lemma 3.5 yields

$$\max \left\{ \omega_1(\mathcal{M}), \sup_{\lambda \in \sigma_D(\mathcal{M}) \setminus \{\lambda_0\}} \Re \lambda \right\} = \sup_{\lambda \in \sigma_D(\mathcal{M}) \setminus \{\lambda_0\}} \Re \lambda < \Re \lambda_0.$$ 

Then, the operator $\mathcal{M}$ satisfies all the conditions of Proposition 3.1 and thus results of this section can be summarized in the form of the following condition.

**Condition 1.** Assume that the assumptions (A1)-(A6) hold true. Moreover, assume that

$$\int_0^{x_1} \frac{q(x)}{g(x)} \exp \left( - \int_0^x \frac{\mu(s) + \frac{1}{2}k_f(s)}{g(s)} ds \right) dx > 1,$$

then the zero stationary solution of the flocculation equation is unstable.

### 4.2. Stability of the trivial stationary solution

In this section we will prove that under certain condition on model parameters we can ensure that the spectral bound of $\mathcal{M}$ is strictly negative. Since the positivity arguments that we used in the previous section cannot guarantee negativity of $s(\mathcal{M})$, we use a direct approach to prove that growth bound of $\mathcal{M}$ is strictly negative, $\omega_0(\mathcal{M}) < 0$. To achieve our goal we use the following version of the well-known Lumer-Philips theorem (see for instance [22, §2, Corollary 3.6] and [13, Theorem 2.22]).

**Theorem 4.1.** (Lumer-Philips) Let a linear operator $A$ on a Banach space $(X, \|\cdot\|)$ the following are equivalent:

1. $A$ is closed, densely defined. Furthermore, $A - \lambda I$ is surjective for some $\lambda > 0$ (and hence for all $\lambda > 0$) and there exists a real number $\omega$ such that $A - \omega I$ is dissipative, i.e.,

$$\|f - \lambda (A - \omega I)f\| \geq \|f\| \quad \text{for all } \lambda > 0 \text{ and } f \in D(A).$$

2. Then, $(A, D(A))$ generates a strongly continuous quasicontractive semigroup $(T(t))_{t \geq 0}$ satisfying

$$\|T(t)\| \leq e^{\omega t} \quad \text{for } t \geq 0.$$

In the following lemma, we show the operator satisfies the first part of the Lumer-Philips theorem. Particularly, we establish that there exist a strictly negative real number $\omega < 0$ such that the operator $\mathcal{M} - \omega I$ is dissipative.

**Lemma 4.2.** Assume that the assumptions (A1)-(A6) hold true. Then the linear operator $\mathcal{M}$ defined in (4.1) is closed, densely defined operator on the Banach space $L^1(I)$, and for sufficiently large $\lambda > 0$ the operator $\mathcal{M} - \lambda I : D(\mathcal{M}) \mapsto L^1(I)$ is surjective. Furthermore, if

$$\mu(x) - q(x) - \frac{1}{2}k_f(x) > 0 \quad (4.4)$$

for all $x \in I$, then there exists $\alpha > 0$ such that the semigroup $(T(t))_{t \geq 0}$ generated by $\mathcal{M}$ satisfies the estimate

$$\|T(t)\|_1 \leq e^{-\alpha t} \quad \text{for all } t \geq 0.$$

**Proof.** Since the operator $\mathcal{M}$ generates a strongly continuous semigroup (Lemma 3.3), the first argument of the lemma is an immediate consequence of the Generation Theorem of [22, §2.3]. We now prove that there exist $\alpha > 0$ such that $\mathcal{M} + \alpha I$ is dissipative. For a given $f \in D(\mathcal{M}) = D(\mathcal{L})$ and some $h \in H$ and $\lambda > 0$ we have

$$f - \lambda (\mathcal{M} + \alpha I)f = h.$$
Consequently, multiplying both sides by the sign function of \( f \) yields

\[
|f(x)| = f(x) \operatorname{sgn}(f(x))
\]

\[
= -\lambda [g(x) f(f(x))]' \operatorname{sgn}(f(x)) \, dx - \lambda \left[ \frac{1}{2} k_f(x) + \mu(x) \right] f(x) \operatorname{sgn}(f(x))
\]

\[
+ \lambda \operatorname{sgn}(f(x)) \int_x^{x_1} \Gamma(x; y) k_f(y) f(y) \, dy \, dx + \lambda \alpha f(x) + h(x) \operatorname{sgn}(f(x)) ,
\]

where function \( \operatorname{sgn}(f(x)) \) is defined as usual with \( \operatorname{sgn}(0) = 0 \). For a given \( f \in \mathcal{D} (\mathcal{M}) \) the set of points for which \( f \) does not vanish can be written as a finite union of disjoint open sets \( I_j = (a_j, b_j) \), i.e., \( f(x) \neq 0 \) for all \( x \in \bigcup_{j=1}^{n} I_j = (0, x_1) \). On each interval \( I_j \) the function \( f \) can be either strictly positive or strictly negative. Moreover, on the boundaries we have \( f(a_j) = 0 \) and \( f(b_j) = 0 \) unless \( a_j = 0 \) or \( b_j = x_1 \). Then, integrating both sides of (4.5) on a given interval \( I_j = (a_j, b_j) \) we have

\[
\int_{a_j}^{b_j} |f(x)| \, dx \leq -\lambda g(b_j) |f(b_j)| + \lambda g(a_j) |f(a_j)| - \lambda \int_{a_j}^{b_j} \left[ -\alpha + \frac{1}{2} k_f(x) + \mu(x) \right] |f(x)| \, dx
\]

\[
+ \lambda \int_{a_j}^{b_j} \int_x^{x_1} \Gamma(x; y) k_f(y) |f(y)| \, dy \, dx + \int_{a_j}^{b_j} |h(x)| \, dx .
\]

Consequently, by summing (4.6) for \( j = 1, \ldots, n \) we get

\[
\int_0^{x_1} |f(x)| \, dx \leq -\lambda g(x_1) |f(x_1)| + \lambda g(0) |f(0)| - \lambda \int_0^{x_1} \left[ -\alpha + \frac{1}{2} k_f(x) + \mu(x) \right] |f(x)| \, dx
\]

\[
+ \lambda \int_0^{x_1} \int_x^{x_1} \Gamma(x; y) k_f(y) |f(y)| \, dy \, dx + \int_0^{x_1} |h(x)| \, dx
\]

\[
\leq -\lambda \int_0^{x_1} \left[ -\alpha - q(x) + \frac{1}{2} k_f(x) + \mu(x) \right] |f(x)| \, dx
\]

\[
+ \lambda \int_0^{x_1} k_f(y) |f(y)| \int_0^y \Gamma(x; y) \, dx \, dy + \int_0^{x_1} |h(x)| \, dx
\]

\[
= -\alpha - q(x) - \frac{1}{2} k_f(x) + \mu(x) \right] |f(x)| \, dx + \int_0^{x_1} |h(x)| \, dx .
\]

Hence, provided that we have

\[
-\alpha - q(x) - \frac{1}{2} k_f(x) + \mu(x) > 0
\]

for all \( x \in I \), it follows that

\[
\|f\|_1 \leq \|h\|_1 = \|f - \lambda (\mathcal{M} + \alpha I) f\|_1 .
\]

In fact, if (4.4) holds true, then there exists \( \alpha > 0 \) such that \( \mathcal{M} + \alpha I \) is dissipative. Consequently, the result follows immediately from the Lumer-Philips theorem. As a direct consequence of Proposition 3.1 and the above lemma, we summarize the results of this section in form of the following condition.

**CONDITION 2.** Assume that the assumptions (A1)-(A6) hold true. Moreover, assume that

\[
q(x) + \frac{1}{2} k_f(x) - \mu(x) < 0
\]

for all \( x \in I \), then the zero stationary solution of the flocculation equation is locally exponentially stable.

5. **Linearized instability and stability criteria for non-trivial steady states.** In this section we present linearized stability results for the non-trivial stationary solution \( p_* \neq 0 \). We first derive conditions for instability (Section 5.1) and then derive conditions for linear stability (Section 5.2).
5.1. Linearized instability. Recall that, from Proposition 3.1, instability of the non-trivial stationary solution depends on the spectral properties of the operator $L$. Specifically, the spectrum of $L$ contains at least one point $\lambda_0 \in \sigma(L)$ satisfying the instability condition (3.1). Towards this end (as we did in Section 4.1), we first show that the operator $L$ has a positive spectral radius.

Lemma 5.1. Assume that the positivity conditions (3.6)–(3.7) hold. Moreover, if the model parameters satisfy the following condition
\[
\int_0^{\infty} q(x) \exp \left( -\int_0^x \mu(s) + \frac{1}{2} k_f(s) + \int_0^{x-s} k_n(s, y) p_*(y) \frac{dy}{g(s)} \right) \, dx > 1,
\]
then the operator $L$ has a positive spectral radius.

Proof. Recall that the operator $L$ can be written as the sum of the operators $L_1$, $L_2$ and $L_3$. Moreover, in Lemma 3.6 we have shown that the operator $L_1$ generates a positive semigroup and the positivity assumptions (3.6) and (3.7) ensure the positivity of the operator $L_2 + L_3$. Therefore, from Corollary 1.11 of [22, §6] it follows that the spectral radius of $L$ is always greater than the spectral radius of the operator $L_1$, i.e.,
\[
s(L_1) \leq s(L_1 + L_2 + L_3) = s(L). \tag{5.2}
\]
Conversely, provided that the condition (5.1) holds, the arguments of [26, Theorem 5.1] can be used to show that the spectral radius of the operator $L_1$ is strictly positive. This result, combined with the inequality (5.2), implies that the spectral radius of $L$ is strictly positive. □

Condition 3. Under the main assumptions (A1)-(A6) and the positivity conditions (3.6)–(3.7) the non-trivial steady state solution of the nonlinear evolution equation defined in (1.1) is unstable if
\[
\int_0^{\infty} q(x) \exp \left( -\int_0^x \mu(s) + \frac{1}{2} k_f(s) + \int_0^{x-s} k_n(s, y) p_*(y) \frac{dy}{g(s)} \right) \, dx > 1. \tag{5.3}
\]

Proof. Recall that from the proof of Lemma 5.1 it follows that
\[
s(L_1) \leq s(L),
\]
where the operators $L_1$ and $L$ are defined in (3.2). Note that Farkas and Hagen [26] have shown that the operator $L_1$ has a positive spectral radius provided that the condition (5.3) holds true. Consequently, if the condition (5.3) holds true, it follows that the operator $L$ has a positive spectral radius. Then from Proposition 3.5 and Remark 4.8 it follows that $s(L) \in \sigma_D(L)$. Moreover, Proposition 3.5 together with Remark 4.8 imply that $\alpha$-growth bound of $L$ is equal to negative infinity, $\omega_1(L) = -\infty$. Therefore, in Proposition 3.1 choosing $\lambda_0$ equal to the eigenvalue corresponding to $s(L)$ yields
\[
\max \left\{ \omega_1(L), \sup_{\lambda \in \sigma_D(L) \setminus \{\lambda_0}\} \mathbf{Re} \lambda \right\} = \sup_{\lambda \in \sigma_D(L) \setminus \{\lambda_0}\} \mathbf{Re} \lambda < \mathbf{Re} \lambda \leq \mathbf{Re} \lambda_0,
\]
and implies that the non-trivial stationary solution $p_*$ of the nonlinear evolution equation defined in (1.1) is unstable. □

Remark 2. Let $S \subset L^1(I)$ be the set of non-trivial stationary solutions and $S_1$ (subset of $S$) denote the set of non-trivial stationary solutions existence of which guaranteed by Theorem 2.2. For a stationary solution $p_* \in S_1$ the modeling terms need to satisfy the conditions (C1) and (C2). Consequently, plugging $x = 0$ into (C2) yields the inequality
\[
\int_0^{x_1} q(x) \, dx \leq 1.
\]
Conversely, the instability condition (5.3) implies that
\[
1 < \int_0^{x_1} \frac{q(x)}{g(x)} \exp \left( -\int_0^x \mu(s) + \frac{1}{2} k_f(s) + \int_0^{x-s} k_n(s, y) p_*(y) \frac{dy}{g(s)} \right) \, dx \leq \int_0^{x_1} \frac{q(x)}{g(x)} \, dx,
\]
which contradicts the existence condition (C2). This in turn implies that stationary solutions in the set $S_1$ do not satisfy the instability condition. However, we note that $S_1$ is only subset of $S$, and thus the results of this subsection is only valid for non-trivial stationary solutions in the set $S \setminus S_1$. 

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5.2. Linearized stability. In Section 3.4 we have shown that the spectrum of the operator \( L \) is not empty. This result, together with Proposition 3.6 and Remark 1 imply that the spectral radius of \( L \) is one of the eigenvalues of the operator \( L \), so it is sufficient to show that all the eigenvalues of \( L \) have a negative real part. However, to the best of our knowledge, the eigenvalue problem

\[
L[\phi] = \lambda \phi
\]

does not have an explicit solution. This forces us to utilize the positive perturbation method Farkas and Hinow [28] to locate the dominant eigenvalue of \( L \). This method relies on the fact that compact perturbations do not change the essential spectrum of a semigroup. Towards this end we will perturb the operator \( L = L_1 + L_2 + L_3 \) (the operators \( L_1, L_2 \) and \( L_3 \) are defined in (3.4)-(3.5)) by a positive compact operator so that we can identify the point spectrum of the resulting operator.

**Lemma 5.2.** Let us define the operator \( C \) as

\[
C[f](x) = c_1 \int_0^{x_1} f(y) dy,
\]

where \( c_1 = ||k_a \cdot p_s||_\infty + ||\Gamma \cdot k_f||_\infty \). Then the operator \( C - L_2 - L_3 \) is positive and compact.

**Proof.** It is easy to see that \( C - L_2 - L_3 \) is a positive operator, i.e.,

\[
C[f](x) - (C - L_2 - L_3)[f](x) \geq \int_0^x [||k_a \cdot p_s||_\infty - k_a(x - y, y) p_s(x - y)] f(y) dy
\]

\[
+ \int_x^{x_1} [||\Gamma \cdot k_f||_\infty - \Gamma(x; y) k_f(y)] f(y) dy \geq 0 \quad \forall f \in \left( L^1(1) \right)_+.
\]

Conversely, \( C \) is a bounded linear operator of rank one, hence it is compact. Then the compactness of \( C - L_2 - L_3 \) follows from compactness of the operators \( L_2 \) and \( L_3 \) (see Lemma 3.4). \( \square \)

Now define the perturbed operator \( P \) as \( P := L + C - L_2 - L_3 = L_1 + C \). Then the eigenvalue problem for the operator \( P \) reads as

\[
\lambda f - P[f] = \lambda f - L_1[f] - C[f] = 0.
\] (5.4)

This equation can be solved implicitly as

\[
f(x) = U_1 \frac{1}{T(\lambda; x)} g(x) + U_2 \frac{c_1}{g(x) T(\lambda; x)} \int_0^x T(\lambda; s) ds,
\] (5.5)

where

\[
U_1 = \int_0^{x_1} q(y) f(y) dy, \quad U_2 = \int_0^{x_1} f(y) dy
\]

and

\[
T(\lambda; x) = \exp \left( \int_0^x \frac{\lambda + A(y)}{g(y)} dy \right).
\]

Integrating the equation (5.5) on \( I \) yields one equation for solving for \( U_1 \) and \( U_2 \). Moreover, multiplying the equation (5.5) by \( q(x) \) and integrating over the interval \( I \) we obtain the second equation for solving for \( U_1 \) and \( U_2 \). Consequently, these two equations can be summarized in the following linear system,

\[
\begin{cases}
U_1 A_{11}(\lambda) + U_2 (A_{12}(\lambda) - 1) = 0 \\
U_1 (A_{21}(\lambda) - 1) + U_2 A_{22}(\lambda) = 0
\end{cases},
\] (5.6)

where

\[
A_{11}(\lambda) = \int_0^{x_1} \frac{1}{T(\lambda; x) g(x)} dx, \quad A_{12}(\lambda) = \int_0^{x_1} \frac{c_1}{g(x) T(\lambda; x)} \int_0^x T(\lambda; s) ds dx,
\]

\[
A_{21}(\lambda) = \int_0^{x_1} \frac{q(x)}{T(\lambda; x) g(x)} dx, \quad A_{22}(\lambda) = \int_0^{x_1} \frac{c_1 q(x)}{g(x) T(\lambda; x)} \int_0^x T(\lambda; s) ds dx.
\]

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If the eigenvalue problem (5.4) has a non-zero solution, then there is non-zero vector \((U_1, U_2)\) satisfying the linear system (5.6). On the other hand, if there is non-zero vector \((U_1, U_2)\) satisfying the linear system (5.6), then the eigenvalue problem has a non-zero solution. Hence \(\lambda \in \mathbb{C}\) is an eigenvalue value of the operator \(P\) if and only if

\[
K(\lambda) = \det \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{pmatrix} = A_{11}(\lambda)A_{22}(\lambda) - (1 - A_{12}(\lambda))(1 - A_{21}(\lambda)) = 0. \tag{5.7}
\]

In structured population dynamics the function \(K\) is often referred as a characteristic function of eigenvalues of an operator, and similar characteristic functions have been derived in [49, 24, 25, 26, 28]. The main advantage of having the characteristic function \(K\) is the task of locating the dominant eigenvalue value of the operator \(L\) reduces to locating the roots of the function \(K\). Hence, in the following lemma we show that under certain conditions on the model parameters all the roots of the characteristic function \(K\) lie in the left half of the complex plane.

**Lemma 5.3.** Under the conditions

\[
A_{12}(0) < 1, \quad A_{21}(0) < 1 \tag{5.8}
\]

and

\[
K(0) < 0 \tag{5.9}
\]

the function \(K\) does not have any roots with non-negative real part. Furthermore, the function \(K\) has at least one negative real root.

**Proof.** It is straightforward to see that

\[
A_{11}(\lambda) = A_{12}(\lambda) = A_{21}(\lambda) = A_{22}(\lambda) = 0 \text{ as } \lambda \to \infty,
\]

so

\[
\lim_{\lambda \to \infty} K(\lambda) = -1.
\]

Moreover, observe that for \(i = 1, 2\) and \(j = 1, 2\) the functions \(A_{ij} : \mathbb{R} \to \mathbb{R}_+\) are non-increasing, i.e.,

\[
\partial_\lambda A_{ij}(\lambda) \leq 0.
\]

Consequently, for \(\lambda \geq 0\) from (5.8) we have

\[
1 - A_{12}(\lambda) \geq 1 - A_{12}(0) > 0
\]

and

\[
1 - A_{21}(\lambda) \geq 1 - A_{21}(0) > 0.
\]

Conversely, differentiating \(K(\lambda)\) for \(\lambda \geq 0\) yields

\[
K'(\lambda) = A'_{11}A_{22} + A_{11}A'_{22} + A'_{12}(1 - A_{21}) + A'_{21}(1 - A_{12}) \leq 0.
\]

Thus the function \(K\) restricted to real numbers is non-increasing. This in turn together with the condition (5.9) implies that the function \(K\) does not have any positive real root.

Now for the sake of a contradiction, assume that there is \(\lambda_1 = a - bi \in \mathbb{C}\) with \(a \geq 0\) and \(b \neq 0\) such that

\[
K(\lambda_1) = 0. \tag{5.10}
\]

Let us define

\[
G(x) = \int_0^x \frac{1}{g(y)} \, dy,
\]

where
then for \( \lambda = a - bi \) we have
\[
A_{11}(\lambda) = \int_0^{x_1} \cos[bG(x)] \frac{1}{T(a, x)g(x)} \, dx + i \int_0^{x_1} \sin[bG(x)] \frac{1}{T(a, x)g(x)} \, dx.
\]
This in turn implies that
\[
-A_{11}(a) \leq \text{Re} \, A_{11}(\lambda_1) \leq A_{11}(a).
\] (5.11)
Analogous arguments yields similar inequalities for \( A_{12}, A_{21} \) and \( A_{22} \). On the other hand, if (5.10) holds true then using (5.11) it follows that
\[
A_{11}(a)A_{12}(a) \geq \text{Re} \, A_{12}(\lambda_1) (1 - A_{21}(\lambda_1)) \geq (1 - \text{Re} \, A_{12}(\lambda_1)) (1 - A_{21}(\lambda_1)) \geq (1 - A_{12}(a)) (1 - A_{21}(a)) \geq 0.
\] (5.12)
Since \( K(\lambda) \) is non-increasing for \( \lambda \geq 0 \), from (5.9) we have
\[
K(a) \leq K(0) < 0.
\] (5.13)
The equation (5.13) is equivalent to
\[
A_{11}(a)A_{12}(a) < (1 - A_{12}(a)) (1 - A_{21}(a)),
\]
which obviously contradicts the equation (5.12). Hence, the function cannot have a complex root with a non-negative real part.

To establish the last statement of the lemma, observe that the function \( 1 - A_{12}(\lambda) \) is continuous and non-decreasing with
\[
\lim_{\lambda \to -\infty} 1 - A_{12}(\lambda) = -\infty.
\]
Conversely, from the condition (5.8) we have \( 1 - A_{12}(\lambda) > 0 \) for \( \lambda \geq 0 \). Thus by Intermediate Value Theorem there is \( \lambda_0 < 0 \) such that \( 1 - A_{12}(\lambda_0) = 0 \). Consequently, evaluating \( K \) at \( \lambda = \lambda_0 \) yields
\[
K(\lambda_0) = A_{11}(\lambda_0)A_{22}(\lambda_0) - (1 - A_{12}(\lambda_0)) (1 - A_{21}(\lambda_0)) = A_{11}(\lambda_0)A_{22}(\lambda_0) \geq 0.
\]
Hence, the function \( K \) has at least one negative real root, which completes the proof of the lemma. With the above lemma in hand, we can now state the main result of this subsection in the form of the following condition.

**Condition 4.** Suppose that the conditions
\[
K(0) = A_{11}(0)A_{22}(0) - (1 - A_{12}(0)) (1 - A_{21}(0)) < 0,
\]
\[
A_{12}(0) = (\|k_0 \cdot p_*\|_\infty + \|\Gamma \cdot k_f\|_\infty) \int_0^{x_1} \frac{1}{g(x)} \int_0^x \exp \left( -\int_s^x \frac{A(s)}{g(s)} \right) ds \, dx < 1
\]
and
\[
A_{21}(0) = \int_0^{x_1} \frac{q(x)}{g(x)} \exp \left( -\int_0^x \frac{A(s)}{g(s)} ds \right) \, dx < 1
\]
hold true. Then, the non-trivial steady state solution \( p_* \) is linearly exponentially stable.

*Proof.* Lemma 5.3 implies that the operator \( L_1 + C \) has negative spectral radius,
\[
s(L_1 + C) < 0.
\]
Conversely, from Engel and Nagel [22] §6, Corollary 1.11], Proposition 3.6 and Lemma 5.2 it follows that
\[
s(L) = s(L_1 + L_2 + L_3) \leq s(L_1 + L_2 + C - L_2 - L_3) = s(L_1 + C) < 0.
\]
Consequently, from [22] §6, Theorem 1.15 it follows that
\[
\omega_0(L) = s(L) < 0.
\]
Hence, Proposition 3.1 yields that the non-trivial steady state solution \( p_* \) is linearly asymptotically stable. \( \square \)
6. Illustration of the results. In this section, we illustrate the theoretical development of the paper by giving explicit examples.

Example 1. First, observe that \( p(t, x) \equiv 0 \) is always a stationary solution of the flocculation model. Therefore, we only have to worry about the stability conditions derived in Section 4. For the maximal floc size, removal and renewal rates we choose terms similar to that of [2]

\[
\mu(x) = 1, \quad q(x) = b(x + 1)
\]

where \( b \) has yet to be chosen. Moreover, we assume that growth and fragmentation rates are proportional to the size of the floc,

\[
g(x) = x + 1 \quad \text{and} \quad k_f(x) = 2x.
\]

Since the stability of the zero stationary solution does not depend on the aggregation rate and the post-fragmentation density function, we only assume that the functions \( k_a \) and \( \Gamma \) satisfy the main assumptions \((A2)\) and \((A5)\), respectively. Consequently, plugging in this values into the instability condition (4.2) yields

\[
\int_0^1 q(x) g(x) \exp \left( - \int_0^x \frac{\mu(s) + \frac{1}{2} k_f(s)}{g(s)} \, ds \right) \, dx = b \int_0^1 \exp \left( - \int_0^x 1 \, ds \right) \, dx = b(1 - e^{-1}) > 1.
\]

Thus, provided that \( b > 2 \), the zero stationary solution of the flocculation model is unstable.

Conversely, plugging in the above parameters in the stability condition (4.4) one obtains

\[
q(x) + \frac{1}{2} k_f(x) - \mu(x) = bx + b - 1 < 0.
\]

Therefore, provided that we have \( 0 < b < \frac{1}{2} \), we can guarantee the local exponential stability of the zero stationary solution.

Example 2. In contrast to the zero stationary solution, positive stationary solutions do not always exist. Based on the analysis of Section 2 if the model parameters satisfy the existence conditions \((C1)\) and \((C2)\) of Section 2, then the flocculation model possesses at least one nontrivial stationary solution. Towards this end, we choose an exponential growth rate and a uniform probability distribution for the post-fragmentation density function,

\[
g(x) = \exp(-ax), \quad \Gamma(x; y) = \frac{\chi(0, y)}{y}.
\]

For other model parameters we choose linear rates,

\[
q(x) = b(x + 1), \quad k_f(x) = cx, \quad \mu(x) = \frac{c}{2} x.
\]

A straightforward computation shows that the above model parameters satisfy the first existence conditions \((C1)\). Furthermore, plugging in the above rates into the second existence condition \((C2)\) yields

\[
e^a + b \left( \frac{a + e^a - 1}{a^2} \right) \leq 1.
\]

(6.1)

Hence, provided that the constants \( a, b \) and \( c \) satisfy the inequality \((6.1)\), there exists a positive integrable stationary solution \( p_* \). Before we carry out the stability analysis for this stationary solution \( p_* \), the above functions should also satisfy the regularity conditions of Section 3. However, the regularity conditions of Section 3 depend heavily on the explicit form of the positive stationary solution in \( k_a(x, y)p_*(x) \). Since our analysis does not provide exact form of the positive stationary solution, we further assume that the stationary solution satisfies \( p_*(x) > 0 \) for all \( x \in I \). Consequently, we tailor the aggregation kernel such that it satisfies the positivity condition \((3.7)\) and thus we proceed by choosing the aggregation kernel as

\[
k_a(x, y) = \frac{d(1 - x)(1 - y)}{p_*(x)p_*(y)},
\]
where the constant $d$ has to be chosen sufficiently small such that
\[ c \min_{x \in I} p_*(x) > d. \]
This form of the aggregation kernel indeed solves the problem and all the regularity conditions of Section 3 are all satisfied.

We are now in a position to derive conditions for the stability (and instability) of the positive stationary solution $p_*(x)$. Towards this end, plugging the parameters into the instability conditions stated in Condition 3 and using the inequality (6.1) yields
\[ \int_0^1 q(x) \exp \left( - \int_0^x \mu(s) + \frac{1}{2} k_f(s) + \int_0^{1-s} k_a(s, y)p_*(y) \, dy \right) \, dx \leq \int_0^1 \frac{q(x)}{g(x)} \, dx = b \frac{a + e^a - 1 - 2ae^a}{a^2} \leq 1. \]
This in turn implies that these parameters do not satisfy the instability condition. However, since our instability conditions are only sufficient conditions, this does not imply that the positive stationary solution $p_*$ is always stable.

For the stability of the positive stationary solution, we plug in the above parameters to the stability conditions stated in Condition 4 and obtain
\[ A_{12}(0) \leq c_1 \frac{e^a(a - 1) + 1}{a^2} < 1, \quad A_{21}(0) \leq b \frac{a + e^a - 1 - 2ae^a}{a^2} < 1, \quad (6.2) \]
where
\[ c_1 = \| k_a \cdot p_* \|_\infty + \| \Gamma \cdot k_f \|_\infty = \frac{d}{\min_{x \in I} p_*(x)} + c < 2c. \]
Moreover, the last stability condition, $K(0) < 0$, yields
\begin{align*}
A_{11}(0) \cdot A_{22}(0) &< c_1 \frac{(e^a - 1)(a - 2 + (2a^2 - 3a + 2)e^a)}{a^4} \\
&< \left( 1 - c_1 \frac{e^a(a - 1) + 1}{a^2} \right) \left( 1 - b \frac{a + e^a - 1 - 2ae^a}{a^2} \right) \\
&< (1 - A_{12}(0))(1 - A_{21}(0)).
\end{align*}
Conversely, since $c_1 < 2c$ this yields another inequality for the parameters $a$, $b$ and $c$,
\[ 2c \frac{(e^a - 1)(a - 2 + (2a^2 - 3a + 2)e^a)}{a^4} < \left( 1 - 2c \frac{e^a(a - 1) + 1}{a^2} \right) \left( 1 - b \frac{a + e^a - 1 - 2ae^a}{a^2} \right). \quad (6.3) \]
Thus, provided that we choose the parameters $a$, $b$ and $c$ such that the inequalities (6.1)-(6.3) hold, we can guarantee local exponential stability of this positive stationary solution. For the convenience of the readers, in Figure 6.1 we have illustrated the stability region for the parameters $a$, $b$ and $c$. Observe that for larger values of the parameter $a$ the stability region for the parameters $b$ and $c$ significantly shrinks. As the value of the parameter $a$ increases the growth rate decreases. Thus, from biological point of view, in order to keep the non-trivial stationary solution stable the values of the parameters $b$ and $c$, associated to removal, fragmentation and renewal rates, should also decrease.

7. **Concluding remarks.** Our primary motivation in this paper is to investigate the ultimate behavior of solutions of a generalized size-structured flocculation model. The model accounts for a broad range of biological phenomena (necessary for survival of a community of microorganism in a suspension) including growth, aggregation, fragmentation, removal due to predation, and gravitational sedimentation. Moreover, the number of cells that erode from a floc and enter the single cell population is modeled with McKendrick-von Foerster type renewal boundary equation. Although it has been shown that the model has a unique positive solution, to the best of our knowledge, the large time behavior of those solutions has not been studied.

Using a fixed point theorem we showed that under relatively weak restrictions, which balance removal, growth, fragmentation and renewal rates, the flocculation model possesses a non-trivial stationary solution.
Figure 6.1. Stability region for the parameters $a$, $b$ and $c$. Positive $a$, $b$ and $c$ values lying inside the solid assure local exponential stability of the positive stationary solution.

(in addition to the trivial stationary solution). We used the principle of linearized stability for nonlinear evolution equations to linearize the problem around the stationary solution. This allowed us to infer stability of the stationary solutions by the spectral properties of the linearized problem. We then used the rich theory developed for semigroups, to derive the stability and instability conditions for the zero stationary solution. To derive instability conditions for the non-trivial stationary solution, we employed an eigenvalue localization method based on the well-known Krein-Rutman theorem. Lastly, we used compactness and positivity arguments to derive conditions for local stability of the non-trivial stationary solutions.

Lastly, even though we showed that the flocculation model has at least one non-trivial stationary solution, our analysis does not state these stationary solutions explicitly. The nonlinearity introduced to the model by Smoluchowski coagulation equations, makes the task of finding explicit stationary solutions challenging even for constant model parameters. On the other hand, when only Smoluchowski coagulation equation is considered, it has been shown that the model has closed form self-similar solutions for constant and additive aggregation kernels [41, 54]. Perhaps, under some conditions on the initial distribution and model parameters, solutions of the flocculation model also converge to self-similar profiles. Hence, as a future research, we plan to further investigate self-similar solutions of the flocculation model.

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