INTERPOLATION IN SEMIGROUPOID ALGEBRAS

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ABSTRACT. A seminal result of Agler characterizes the so-called Schur-Agler class of functions on the polydisk in terms of a unitary colligation transfer function representation. We generalize this to the unit ball of the algebra of multipliers for a family of test functions over a broad class of semigroupoids. There is then an associated interpolation theorem. Besides leading to solutions of the familiar Nevanlinna-Pick and Carathéodory-Fejér interpolation problems and their multivariable commutative and noncommutative generalizations, this approach also covers more exotic examples.

1. INTRODUCTION

The transfer function realization formalism for contractive multipliers of (families of) reproducing kernel Hilbert spaces and Agler-Pick interpolation has been, starting with the work of Agler [3], generalized from the classical setting of $H^\infty(D)$ (D the unit disk), to many other algebras of functions.

In this paper we pursue realization formulæ and Agler-Pick interpolation in two directions. First we consider an algebra of functions on a semigroupoid $G$. The precise definition of a semigroupoid is given below. In essence it can be thought of as an ordered unital semigroup, though perhaps with more than one unit. For now the salient point is that the semigroupoid structure means that the algebra product generalizes both the pointwise and convolution products. This setting has the advantage of being fairly concrete and amenable to study using reproducing kernel Hilbert space ideas and techniques while at the same time connecting with the theory of graph $C^*$-algebras.

Secondly, we view the norm on the algebra as being determined by a (possibly infinite) collection $\Psi$ of functions on $G$, referred to as test functions. Results on Agler-Pick interpolation (in the classical sense) for both finite and infinite collections of test functions with varying amounts of additional imposed structure can be found in [6, 8] and this point of view goes back at least to [4]. A collection of test functions determines a family of kernels, and vice versa. This duality between test functions and kernels will have a familiar feel to those acquainted with Agler’s model theory [2]. The advantage of such an approach is that it allows us to consider interpolation problems on, for example, polydisks and multiply connected domains [22].

We should mention that Kribs and Power [29, 30] introduce a somewhat more restrictive notion of a semigroupoid algebra. These are related to so-called quiver algebras of Muhly [38], and are the nonselfadjoint analogues of the higher rank graph algebras of Kumjian and Pask [31]. In these papers order is either imposed through the presence of a functor from the semigroupoid to $\mathbb{N}^d$, or by the assumption of freeness. In either case, the resulting object is cancellative, and there is a representation (related to our Toeplitz representation on characteristic functions $\chi_a$; see section 1.3) in terms of partial isometries and projections on a generalized Fock space with orthonormal basis labelled by the elements of the semigroupoid. The algebras of interest in these papers are obtained as...
the weak operator topology closure of the algebras coming from the left regular representation (i.e.,
the projections and partial isometries mentioned above), and so in a natural sense are the multiplier
algebras for these Fock spaces.

The Kribs and Power semigroupoid algebras include the noncommutative Toeplitz algebras first
introduced in [44]. Pick and Carathéodory interpolation has been considered in this context by Arias
and Popescu [11] and Davidson and Pitts [20] (with some earlier work by Popescu on these and related
interpolation problems to be found in [42, 43, 45, 46]), and somewhat more generally by Jury and
Kribs [27]. See also [28]. In fact, while the commutant lifting theorem unifies the classical Pick and
Carathéodory-Fejér interpolation problems, to our knowledge, Jury’s PhD dissertation [26] was the
first to do so in terms of the positivity of kernels, and also the first to give a concrete realization formula
for the case of the semigroup \( \mathbb{N} \). Recently, realization formulæ in a noncommutative setting have also
been investigated in [17]. Muhly and Solel [39] have considered Nevanlinna-Pick interpolation from
the vantage of what they call Hardy algebras, covering many of the examples mentioned above along
with the statement of a realization formula.

Interpolation problems on domains other than the unit disk in \( \mathbb{C} \) have been of long-standing interest.
On multiply connected domains, the seminal work is that of Abrahamse [1], with further contributions
to be found in [16, 21, 34, 35, 36, 48]. Regarding domains in \( \mathbb{C}^n \), the fundamental paper of Agler
[3] provides the foundation upon which most subsequent work is based. A sampling of papers of
particular interest in this direction includes [5, 6, 7, 8, 9, 14, 15, 18, 19, 23, 24, 37].

In this paper we have for clarity restricted our attention to scalar valued interpolation (although we
stray a bit in the examples in Section 8). We do not anticipate that the generalization to the matrix
case will provide any obstacles which cannot be overcome with what are by now standard techniques.
Indeed we have ensured that none of the proofs found below depend on the commutativity of the
coefficients of our functions, and it appears likely that the results will continue to hold when the
coefficients come from, say, a norm closed subalgebra of a \( C^* \)-algebra. This is left for later work.

A few words about the organization of the paper. The rest of Section 1 outlines the basic tools
used throughout: semigroupoids, \( \star \)-products, Toeplitz representations, test functions and reproduc-
ing kernels, the \( C^* \)-algebra generated by evaluations on the set of test functions and its dual, and
transfer functions. This is followed by a statement of the main results, which are the realization and
interpolation theorems.

In Section 2 we more closely study the \( \star \)-product, especially with regards to inverses and positivity.
Section 3 begins with a consideration of the semigroupoid algebra analogue of the Szegő kernel, and
highlights the close connection between positivity of these kernels and complete positivity of the
\( \star \)-product map (a generalization of the Schur product map). As noted earlier in the introduction,
multiplier algebras arising from a single reproducing kernel are too restrictive for us, so we detail how
we will handle families of kernels and the associated families of test functions. Cyclic representations
of the space of functions over certain finite sets (they should be “lower” with respect to the order on
the semigroupoid) which are contractive on test functions are shown to be connected to reproducing
kernels. This plays a crucial role in the Hahn-Banach separation argument in the realization theorem.

Given a positive object, an analyst’s immediate inclination is to factor. The fourth section is devoted
to a factorization result for positive kernels on the dual of the \( C^* \)-algebra from Section 1, as well as
making connections to representations of this algebra.

Two other key items needed in the proof of the realization theorem are taken up in Section 5. The
first is the cone of matrices \( C_F \). For the separation argument in the proof of the realization theorem to
work, we must know that \( C_F \) is closed and has nonempty interior. Closedness requires a surprisingly
delicate argument, and so occupies the bulk of the section. We also show that certain sets of kernels
in the dual of the \( C^* \)-algebra mentioned above are compact.
Sections 6 and 7 comprise the proof of the realization theorem and the interpolation theorem. The first implication of the proof of the realization theorem is essentially the most involved part, but due to all of the preparatory work in Sections 3–5, is dispensed with quickly. Other parts involve variations on themes which will be familiar to those acquainted with recent proofs of interpolation results. These include an application of Kurosh’s theorem, a lurking isometry argument, and a fair amount of tedious calculation. After the proof of the realization theorem, the proof of the interpolation theorem is almost an afterthought.

In Section 8 we turn briefly to a menagerie of examples, both old and new. Though we mention it in passing, we have postponed the application to Agler-Pick interpolation on an annulus to a separate paper for two reasons. First, the argument is fairly long and involves ideas and techniques unrelated to the rest of this paper; and second, the underlying semigroupoid structure is that of Pick semigroupoid (which is essentially trivial) and as such the version of Theorem 1.3 which is needed does not require the semigroupoid overhead. In any case, this section barely scratches the surface of what is possible!

We would like to thank Robert Archer for his careful reading of the paper, and the many useful comments and questions which have without a doubt improved it.

1.1. Semigroupoids. There is no standard name in the literature for the sort of object on which we want to define our function algebras. The names “small category” and “semigroupoid” are two commonly used terms, though our definition differs somewhat from that standardly given for either of these. We have opted for the latter.

The term “semigroupoid” was originally coined by Vagner, as far as we are aware. Similar notions are familiar from the theory of inverse semigroups (see for example, [32] or [40]), and have been explored in connection with the classification theory of \( C^* \)-algebras. The use of semigroupoids in the study of nonselfadjoint algebras originates with Kribs and Power [29], though again, their use of the terminology is a bit different from ours.

So let \( G \) be a set with a function \( X \subset G \times G \to G \), called a partial multiplication and written \( xy \) for \( (x,y) \in X \). We define idempotents as those elements \( e \) of \( G \) such that \( ex = x \) whenever \( ex \) is defined and \( ye = y \) whenever \( ye \) is defined. Note that these are commonly referred to as identities in the groupoid literature.

The following laws are assumed to hold:

1. (associative law) If either \((ab)c\) or \(a(bc)\) is defined, then so is the other and they are equal. Also if \(ab, bc\) are defined, then so is \((ab)c\).
2. (existence of idempotents) For each \(a \in G\), there exist \(e, f \in G\) with \(ea = a = af\).
   Furthermore if \(e \in G\) satisfies \(e^2 = e\), then \(e\) is idempotent.
3. (nonexistence of inverses) If \(a, b \in G\) and \(ab = e\) where \(e\) is idempotent, then \(a = b = e\).
4. (strong artinian law) For any \(a \in G\) the cardinality of the set \(\{z, b, w : zbw = a\}\) is finite.
   Moreover there is an \(N < \infty\) such that \(\sup_{a, c \in G} \text{card}\{b \in G : cb = a\} \leq N\).

Hereafter we refer to a set \(G\) with a partially defined multiplication with all of the properties so far listed as a semigroupoid.

Since we have associativity, we can mostly forget parentheses. If we were to reverse the third law (so that every element has an inverse), then the first three rules would comprise the definition of a groupoid. The strong artinian law is related to the (partial) order which we eventually impose on our semigroupoid. The first part of it ensures that the multiplication we will define for functions over the semigroupoid is well defined, while the second part guarantees the existence of at least one collection of test functions, or equivalently, that the associated collection of reproducing kernels is nontrivial. It does so by restricting how badly non-cancellative the semigroupoid can be. Alternately, the strong
artinian law could be replaced by the condition that for each \(a \in G\) the set \(\{z, b, w : zbw = a\}\) is finite and a hypothesis about the existence of a collection of test functions (see Section 1.4).

There is one other rule which it is useful to state, though it follows from those already given:

(5) **strong idempotent law** If \(zaw = a\), then \(z\) and \(w\) are idempotents.

To see that this is a consequence of our other laws, first note that \(zaw = a\) means that \(z^n aw^n = a\) for \(n \in \mathbb{N}\). The strong artinian law implies that only finitely many of the \(z^n\) are distinct. In particular, there is an \(M > 0\) such that \((z^{2M})^2 = z^{2j}\) for some \(j \leq M\). Let \(h_1 = z^{2j}, h_2 = h_1^2\), and so on, with \(h_m = h_{m-1} = z^{2M}\) (so \(h_1 = h_2^2\)). Clearly \(h_j h_k = h_k h_j\) for all \(j, k\). Hence

\[(h_1 h_2 \cdots h_m)^2 = h_1^2 \cdots h_m^2 = h_2 \cdots h_m h_1 = h_1 h_2 \cdots h_m,\]

and so \(h_1 h_2 \cdots h_m = z^{2j+\cdots+2M}\) is idempotent. Since there are no inverses, this implies that \(z\) is idempotent. Likewise \(w\) is idempotent. Note that (5) implies our assumption that \(e\) is idempotent if \(e^2 = e\).

If \(ea = a\) then \(e\) is unique, since if \(e'a = a\), then \(a = ea = e(e'a) = (ee'a)\), implying that \(ee'\) is defined. But then since \(e\) and \(e'\) are assumed to be idempotents, \(e = ee' = e'\).

From the definition we have \(e^2 = e\) means that \(e\) is idempotent. On the other hand, if \(a = ea\) then \(a = e(ea) = e^2 a\), and so \(e^2\) is defined, and by uniqueness, \(e^2 = e\). Also if \(e\) and \(f\) are idempotents and \(ef\) is defined, then \(e = ef = f\).

The product \(ab\) exists if and only if there is an idempotent \(f\) such that \(af, fb\) are defined. For if such an \(f\) exists, then by associativity, \((af)b = a(fb) = ab\), while conversely, if \(ab\) is defined, then there is an idempotent \(f\) such that \(af = a\) and \((af)b = a(fb)\) is defined and so \(fb\) is defined.

Based on these observations, it is common to view a set with a partial multiplication verifying the first two rules as a sort of generalized directed graph with the vertices representing the idempotents, though because we have not assumed any cancellation properties, this analogy is imperfect.

We define subsemigroupoids in the obvious way. In particular, a subset \(H\) of a semigroupoid \(G\) will be a subsemigroupoid if whenever \(a, b \in H\) and \(ab\) makes sense in \(G\) then \(ab \in H\), and for all \(a \in H\) the idempotents \(e, f\) such that \(ea = a = af\) are also in \(H\).

We put a partial order on a semigroupoid \(G\) as follows: say that \(b \leq a\) if there exist \(z, w \in G\) such that \(a = zbw\). By the existence of idempotents, \(a \leq a\). Transitivity is likewise readily verified. If \(a \leq b\) and \(b \leq a\) then \(a = zbw, b = z'aw'\) and so \(a = (zz')a(w'w)\). Then by the strong idempotent law \(zz'\) and \(w'w\) are idempotents. But then by the nonexistence of inverses, \(z, z', w\) and \(w'\) are idempotents and so \(a = b\). Other partial orders are considered in Section 2.4.

By this definition, and the existence of idempotents, if \(a = bc\), then both \(b\) and \(c\) are less than or equal to \(a\). Also, by the nonexistence of inverses and uniqueness of idempotents, the idempotents comprise the minimal elements of \(G\). We write \(G_e\) for the collection of idempotents.

We say that a set \(F \subseteq G\) is lower if \(a \in F\) and \(b \leq a\) then \(b \in F\). Observe that for a lower set \(F, F_e = F \cap G_e \neq \emptyset\). Note too that if \(H\) is a finite subset of \(G\), then there is a finite lower set \(F \supset H\): simply let \(F = \{a : \text{there exists a } b \in H \text{ such that } a \leq b\}\).

1.1.1. **Examples.** We list here several important examples of semigroupoids.

1. Let \(G\) be a set, and assume \(G_e = G\) (so all elements are idempotent). We refer to such semigroupoids as Pick semigroupoids.

2. Let \(G = \mathbb{N} = 0, 1, 2, \ldots \) with the product \(ab = a + b\). \(G\) is in fact a commutative cancellative semigroup with idempotent 0.

3. The last example obviously generalizes to \(\mathbb{S}_n\), the free (noncommutative) monoid on \(n\) generators. This in turn is a special case of what we term the Kribs-Power semigroupoids [29].
which are defined as follows. Let $\Lambda$ be a countable directed graph. The semigroupoid $\mathfrak{S}^+(\Lambda)$ determined by $\Lambda$ comprises the vertices of $\Lambda$, which act as idempotents, and all allowable finite paths in $\Lambda$, with the natural concatenation of allowable paths in $\Lambda$ defining the partial multiplication. In particular, $\mathfrak{S}^+(\Lambda) = \mathfrak{S}_n$ when $\Lambda$ is a directed graph with one vertex and $n$ distinct loops.

1.2. **The convolution products.** The product on $G$ naturally leads to a product on functions over lower sets $F \subset G$ in one or more variables.

1.2.1. **The $\ast$-product for functions of one variable.** Let $F$ be a lower subset of $G$. There is a natural algebra structure on the set $P(F)$ of functions $f : F \to \mathbb{C}$ which we call the semigroupoid algebra of $F$ over $\mathbb{C}$. Addition of $f, g \in P(F)$ is the usual pointwise addition of functions and the product is defined by

$$ (f \ast g)(a) = \sum_{rs = a} f(r)g(s), $$

which makes sense because of the artinian hypothesis on $G$ and the assumption that $F$ is lower.

The multiplicative unit of $P(F)$ is given by

$$ \delta(x) = \begin{cases} 1 & x \in F_e, \\ 0 & \text{otherwise}. \end{cases} $$

The distributive and associative properties are readily checked, so we have an algebra. A function $f$ is invertible if and only if $f(x)$ is invertible for all $x \in F_e$. The proof follows the same lines as in the matrix case given below, so we do not give it here.

If $a \in F' \subset F$ and $F'$ is itself lower, then

$$ (f_{|F'} \ast g_{|F'})(a) = (f \ast g)(a). $$

Hence, we can be lax in specifying our lower set and usually act as if it is finite.

Later we have need for powers of functions with respect to the $\ast$-product. To avoid confusion, for a function $\varphi$ on $G$, we let $\varphi^{n\ast}$ denote the $n$-fold $\ast$-product of $\varphi$ with itself.

As it happens, it is unimportant that a function over $F$ map into $\mathbb{C}$. For instance, the $\ast$-product clearly generalizes to functions $f, g : F \to \mathfrak{C}$, where $\mathfrak{C}$ is a $C^\ast$-algebra.

There will be times when we will want to interchange $r$ and $s$ in the definition of the convolution product. Over $\mathbb{C}$ or, more generally, any commutative $C^\ast$-algebra $\mathfrak{C}$ this simply changes $f \ast g$ into $g \ast f$. But in the noncommutative case this will not work. Hence we introduce the notation

$$ (f \hat{\ast} g)(a) = \sum_{rs = a} f(s)g(r). $$

For the $\hat{\ast}$-product the multiplicative unit remains $\delta$, the associative and distributive laws continue to hold, and $f$ is invertible with respect to this product if and only if $f(x)$ is invertible for all $x \in F_e$. We write $f^{-1\hat{\ast}}$ and $f^{-1\ast}$ for the $\ast$-inverse and $\hat{\ast}$-inverse of $f$, respectively. By considering $f^{-1\hat{\ast}} \hat{\ast} f \ast f^{-1\hat{\ast}}$, we see that $f^{-1\hat{\ast}} = f^{-1\ast}$.

Another useful and easily checked property relating the two products is that

$$ (f \ast g)^\ast = g^\ast \ast f^\ast. \quad (1.1) $$

Consequently $(f \hat{\ast} g)(g^\ast \ast f^\ast) \geq 0$. We also have $(f^{-1\hat{\ast}})^\ast = f^{-1\ast}$.

In the examples listed above, the $\ast$-product is just pointwise multiplication for Pick semigroupoids. For the second example, it is the usual convolution.
1.2.2. The $\star$-product for matrices. The following bivariate version of the convolution product is the canonical generalization of Jury’s product \cite{26} to semigroupoids.

For a lower set $F$, let $M(F)$ denote the set of functions $A : F \times F \to \mathbb{C}$. When $F$ is finite, thinking of elements of $M(F)$ as matrices (indexed by $F$), the notation $A_{a,b}$ is used interchangeably with $A(a,b)$. The set of functions from $F \times F$ to $X$ will be denoted $M(F,X)$.

**Definition 1.1.** Let $F$ be a lower set and suppose $A, B \in M(F)$. Define $A \star B$ by

$$(A \star B)(a,b) = \sum_{pq=a} \sum_{rs=b} A(p,r)B(q,s).$$

Once again, the artinian hypothesis on $G$ guarantees the product is defined. Further, $(A \star B)(a,b)$ does not actually depend upon the lower set $F$ which contains $a$ and $b$. In particular, since there is always a finite lower set containing $a$ and $b$ (just take the union of the set of elements less than or equal to $a$ and those less than or equal to $b$), this product can and will be interpreted as a $\star$-product of matrices.

The assumption that the entries of $A$ and $B$ are in $\mathbb{C}$ is not important, and we will at times use the $\star$-product when the entries are in other algebras. The $\star$ notation should cause no confusion, since in essence the $\star$-product is the bivariate analogue of the convolution product. Indeed, it is clear that the $\star$-product could be defined for functions of three or more variables as well, though we have no need for this here.

Unlike Jury’s $\star$-product, ours will not necessarily be commutative (though this will be the case if $G$ is commutative). In the special example of the Pick semigroupoids, the $\star$-product is just the matrix Schur product.

As with functions we can also define the $\hat{\star}$-product of matrices:

$$(A \hat{\star} B)(a,b) = \sum_{pq=a} \sum_{rs=b} A(q,s)B(p,r).$$

Over $\mathbb{C}$ and any other commutative algebra, $A \hat{\star} B = B \hat{\star} A$. However we will need both products in a noncommutative setting.

Define $[1]$ (or $[1]_F$ if we want to make absolutely clear the lower set involved) to be the matrix defined by $[1]_{a,b} = 1$ for $a, b$ both elements of $F$, and zero otherwise. It is easy to see that for $A \in M(F)$, $A \star [1] = [1] \hat{\star} A = A$. Note too that we can factor $[1] = \delta \delta^\star$.

1.3. The Toeplitz representation. Let $\varphi$ be a function on a (finite) lower set $F$. Define the associated \textit{(left) Toeplitz representation} $\overline{\Xi}(\varphi)$ by

$$[\overline{\Xi}(\varphi)]_{a,b} = \begin{cases} \sum_c \varphi(c), & cb = a; \\ 0, & \text{otherwise}. \end{cases}$$

We drop the “left” hereafter, though we could also consider a right Toeplitz representation with $bc = a$ rather than $cb = a$. It is expedient to assume $F$ is finite to ensure that $\overline{\Xi}(\varphi)$ is bounded, though the definition makes sense formally when $F$ is not finite and in many interesting cases yields a bounded operator.

As defined, $\overline{\Xi}(\varphi)$ is a mapping of $F \times F$ into $\mathbb{C}$. Let $\mathbb{C}^F$ denote the column vector space (of dimension equal to the cardinality of $F$) with positions labelled by the entries of $F$, which is naturally isomorphic to $P(F)$ as a vector space. Then for a function $f \in P(F)$, viewed as an element of $\mathbb{C}^F$,

$$(\overline{\Xi}(\varphi)f)(a) = \sum_b [\overline{\Xi}(\varphi)]_{a,b}f(b) = \sum_{cb=a} \varphi(c)f(b) = (\varphi \hat{\star} f)(a). \quad (1.2)$$
In this way $\mathcal{T}(\varphi)$ is an operator in $B(\mathbb{C}^F)$ and the mapping $P(F) \ni \varphi \mapsto \mathcal{T}(\varphi) \in B(\mathbb{C}^F)$ is a representation. Indeed, it essentially acts as the left regular representation of $P(F)$. The use of the notation $M(F)$ to denote either $B(\mathbb{C}^F)$ or functions from $F \times F \to \mathbb{C}$ should be clear from the context. Further, since the Toeplitz representation depends upon the lower set $F$ in a consistent way, it should cause no serious harm that the notation $\mathcal{T}(\varphi)$ makes no reference to $F$. On occasions when we need to make the dependence on $F$ explicit, we will write $\mathcal{T}^F$ for $\mathcal{T}$.

In the case that $G$ is the semigroupoid $\mathbb{N}$, $\mathcal{T}(\varphi)$ is precisely the Toeplitz matrix associated with the sequence $\{\varphi(j)\}$. At the other extreme, when $G$ is a Pick semigroupoid, $\mathcal{T}(\varphi)$ is the diagonal matrix with diagonal entries $\varphi(a)$ for $a \in G$ which, despite our terminology, seems very un-Toeplitz like!

When $G$ is the Kribs-Power semigroupoid $\mathfrak{S}^+(\Lambda)$ determined by a countable directed graph $\Lambda$, a lower set $F$ of $\mathfrak{S}^+(\Lambda)$ is closed under taking left and right subpaths. The vector space $P(F)$ may be regarded as a subspace of the generalized Fock space $\mathcal{H}_\Lambda$ over $\Lambda$ (see [27]). In this interpretation $\mathcal{T}^F(\chi_w)$ is the compression to $P(F)$ of the partial creation operators indexed by $w \in F$. More generally, $\{\chi_v : v \in \mathfrak{S}^+(\Lambda)\}$ may be thought as an orthonormal basis of $\mathcal{H}_\Lambda$ and $\mathcal{T}$ behaves as a representation $\mathfrak{S}^+(\Lambda) \to B(\mathcal{H}_\Lambda)$. Hence, the weak operator topology closed subalgebra generated by the family $\{\mathcal{T}(\chi_v)\}$ is the free semigroupoid algebra of Kribs and Power [29], which includes, as particular case, the noncommutative Toeplitz algebra [10, 11, 20]. The set of generators can be restricted, as we see later.

Even when $F$ is not necessarily finite, $\mathcal{T}$ still behaves formally as a representation, but of course it need not be the case that $\mathcal{T}(\varphi)$ is bounded.

It is also possible to work with the $\hat{\star}$-product. Presumably, there is a distinction between a collection of test functions, defined below with respect to the $\star$-product, and those with respect to the $\hat{\star}$-product, though we do not develop this.

1.4. Test functions. For a function $\varphi$ on $G$, recall that $\varphi^n$ denotes the $n$-fold $\star$-product of $\varphi$ with itself, $n = 1, 2, \ldots$.

**Definition 1.2.** A collection $\Psi$ of functions on $G$ into $\mathbb{C}$ is a collection of test functions if

(i) for each finite lower set $F \subset G$ and $\psi \in \Psi$, $\|\mathcal{T}(\psi)\| \leq 1$;

(ii) for each $a \in G_e$,

$$\lim_{n \to \infty} \psi^n(a) = \lim_{n \to \infty} \psi^n(a) = 0,$$

uniformly in $\psi$; and

(iii) for each finite lower set $F$, the algebra generated by $\Psi|_F = \{\psi|_F : \psi \in \Psi\}$ is all of $P(F)$.

The condition $\Psi|_F$ generates $P(F)$ is not essential. It does however simplify statements of results. Given $x \in G$, let $f$ be the unique idempotent so that $xf = x$. Since,

$$[\mathcal{T}(\psi)]_{x,f} = \sum_{c,f=x} \psi(c) = \psi(x),$$

item (i) says that $|\psi(x)| \leq 1$ for each $x \in G$. By the same reasoning, if $\psi_1, \psi_2 \in \Psi$, then

$$|\psi_1(x) - \psi_2(x)| \leq \|\mathcal{T}(\psi_1) - \mathcal{T}(\psi_2)\|.$$

Item (ii) says that for each $a \in G_e$ and $\epsilon > 0$ there is an $N$ so that for all $n \geq N$ and $\psi \in \Psi$, $|\psi^n(a)| < \epsilon$, and so for fixed $a \in G_e$, $\sup_{\psi \in \Psi} |\psi(a)| < 1$. Furthermore, for any $a \in G$, we automatically obtain $\lim_{n \to \infty} \psi^n(a) = 0$. This follows from a straightforward counting argument estimating the maximum number of ways of writing $a \in G$ as a product of $n$ elements. Assume $\text{card}\{b : b \leq a\} = r$ (which is finite since $G$ is artinian) and that $n \gg r$. Let $c = \max_{b \leq a} |\psi(b)|$, and $c_e$ be the maximum of $|\psi(b)|$ over all idempotents less than or equal to $a$. As noted above, $c_e < 1$. In
the product of \( n \) terms, there are at most \( \binom{n}{r} \) ways of choosing which of the at most \( r \) terms are not idempotent, and then at most \( r' \) ways of choosing these terms. The nonidempotents act as separators between at most \( r + 1 \) blocks of idempotents. Within each block of idempotents, each term must be the same idempotent (since the product of unequal idempotents is not defined). So there are at most \( r^{r+1} \) ways of choosing which idempotent is in each block. Consequently

\[
|\psi^n_s(a)| \leq \binom{n}{r} r^r c_e^{r+1} c_e^{n-r} \leq r^{2r+1} c_e^n,
\]

which clearly goes to zero as \( n \to \infty \).

For a given semigroupoid \( G \) it is legitimate to wonder if there actually exists any family of test functions. It so happens that the strong artinian condition in the definition of a semigroupoid ensures this. Let \( \kappa = \sup_{a,c \in G} \text{card}\{b \in G : cb = a\} \), which we have assumed is finite. Let \( \psi_0 : G \to \mathbb{D} \) with \( \psi_0|_{G_e} \) injective and \( \psi_0|_{G \setminus G_e} = 0 \). (This assumes the cardinality of \( G_e \) is less than or equal to that of the continuum — it is only slightly more trouble to handle the more general case.) Let \( \Psi_s = \{ \frac{1}{n} \chi_e : c \in G \setminus G_e \} \cup \{ \psi_0 \} \). Then \( \Psi_s \) can be shown to be a collection of test functions (here \( \chi_e(x) \) equals 1 if \( x = e \) and zero otherwise). In particular, the condition \( \kappa < \infty \) for all \( c \in G \) will hold if \( G \) is right cancellative (so in particular, for Kribs-Power semigroupoids). In the case \( G = G_e \), this choice of test functions will ultimately correspond to \( B(G) \), the normed algebra of all bounded functions on \( G \).

1.5. Test functions and reproducing kernel Hilbert spaces. Let \( F \subset G \) be a lower set. A function \( k : F \times F \to \mathbb{C} \) is a positive kernel if for each finite subset \( A \subset F \) the matrix \( [k(a,b)]_{a,b \in A} \) is positive (i.e., positive semidefinite).

More generally, it makes sense to speak of a kernel with values in the dual of a \( C^* \)-algebra. If \( \mathfrak{B} \) is a \( C^* \)-algebra with Banach space dual \( \mathfrak{B}^* \), then a function \( \Gamma : F \times F \to \mathfrak{B}^* \) is positive if for each finite subset \( A \subset F \) and each function \( f : A \to \mathfrak{B} \),

\[
\sum_{x,y \in A} \Gamma(x,y)(f(x)f(y)^*) \geq 0.
\]

In the sequel, unless indicated otherwise, kernels take their values in \( \mathbb{C} \).

Given a set of test functions \( \Psi \) let \( K_\Psi \) denote the collection of positive (i.e., positive semidefinite) kernels \( k \) on \( G \) such that for each \( \psi \in \Psi \), the kernel

\[
G \times G \ni (x,y) \mapsto k_\psi(x,y) = (\Gamma(1 - \psi\psi^*) * k)(x,y)
\]

is positive. Here \( 1 - \psi\psi^* \) is the function defined on \( G \times G \) by \( (1 - \psi\psi^*)(p,q) = [1]_{p,q} - \psi(p)\psi(q)^* \) so that the right hand side of equation (1.3) is the \( \ast \)-product of the functions (or matrices indexed by \( G \)) \( 1 - \psi\psi^* \) and \( k \), evaluated at \( (x,y) \in G \times G \).

The set \( K_\Psi \) is nonempty, since it at least contains \( k = 0 \). More importantly, from the hypothesis that \( \Psi \) is a family of test functions and the strong artinian law, it also contains the kernel \( s : G \times G \to \mathbb{C} \) given by \( s(x,y) = 1 \) if \( x = y \) and 0 otherwise, which is strictly positive definite. We call \( s \) the Toeplitz kernel.
Let us verify that \( s \in \mathcal{K}_\Psi \), for the collection of test functions \( \Psi_s \) constructed in the last subsection. For the test function \( \psi_0 \),

\[
(\psi_0 \psi_0^s \ast s)(a, b) = \sum_{pq = a} \sum_{rt = b} \psi_0(p) \psi_0^s(r) s_{q,t} = \sum_{p, r \in G_e \atop pq = a \atop rt = b} \psi_0(p) \psi_0^s(r) s_{q,t}
\]

\[
= \begin{cases} 
\psi_0(p) \psi_0^s(p), & p \in G_e, a = b, pa = a \\
0 & \text{otherwise.}
\end{cases}
\]

Hence since \( [1] \ast s = s \), \( ([1] - \psi_0 \psi_0^s) \ast s \) is a diagonal matrix with entries of the form \( 1 - \psi_0(p) \psi_0^s(p) \geq 0 \), and so is positive. On the other hand suppose \( \psi_c = \frac{1}{\kappa} \chi_c \), \( c \in G \backslash G_e \). Then

\[
(\psi_c \psi_c^s \ast s)(a, b) = \sum_{pq = a} \sum_{rt = b} \psi_c(p) \psi_c^s(r) s_{q,t} = \frac{1}{\kappa} \left( \sum_{cq = a = b} 1 \right)^2 \in [0, 1],
\]

and so \( ([1] - \psi_c \psi_c^s) \ast s \) is a positive diagonal matrix.

The kernels determined by a family of test functions \( \Psi \) in turn give rise to a normed algebra of functions on \( G \). Let \( H^\infty(\mathcal{K}_\Psi) \) denote those functions \( \varphi : G \to \mathbb{C} \) such that there exists a \( C > 0 \) such that for each \( k \in \mathcal{K}_\Psi \), the kernel

\[
G \times G \ni (x, y) \mapsto ((C^2 [1] - \varphi \varphi^*) \ast k)(x, y)
\]

is positive. The infimum of all such \( C \) is the norm of \( \varphi \). With this norm \( H^\infty(\mathcal{K}_\Psi) \) is a Banach algebra under the convolution product. By construction \( \Psi \) is a subset of the unit ball of \( H^\infty(\mathcal{K}_\Psi) \).

There is a duality between kernels and test functions in Agler’s model theory [2,4]. Roughly, the idea is, given a collection \( \mathcal{K} \) of positive kernels on \( G \), to let \( \Psi = \mathcal{K}^\perp \) denote those functions \( \psi \in G \) such that for each \( k \in \mathcal{K} \), the kernel

\[
G \times G \ni (x, y) \mapsto (([1] - \psi \psi^s) \ast k)(x, y)
\]

is positive. In the case that Agler considers, where the semigroupoid consists solely of idempotents (i.e., a Pick semigroupoid), mild additional hypotheses on \( \mathcal{K} \) guarantee that \( \Psi \) is a family of test functions, in which case \( \mathcal{K}_\Psi = \mathcal{K}^{\perp \perp} \).

1.6. The evaluation \( E \) and \( C^*\)-algebra \( \mathfrak{B} \). Let \( \Psi \) be a given collection of test functions and \( C_b(\Psi) \) the continuous functions on \( \Psi \), where \( \Psi \) is compact in the bounded pointwise topology. Define \( E \in B(G, C_b(\Psi)) \) (the bounded functions from \( G \) to \( C_b(\Psi) \)) by

\[
E(x)(\psi) = \psi(x), \quad \psi \in \Psi,
\]

with

\[
\|E(x)\| = \sup_{\psi \in \Psi} \{|E(x)(\psi)|\}.
\]

So \( E(x) \) is the evaluation map on \( \Psi \), \( \|E(x)\| < 1 \) for each \( x \in G_e \) and \( \|E(x)\| \leq 1 \) otherwise.

Since evidently the collection \( \{E(x) : x \in G\} \) separates points and we include the identity, the smallest unital \( C^*\)-algebra containing all the \( E(x) \) is \( C_b(\Psi) \). For convenience, we denote this algebra as \( \mathfrak{B} \).
1.7. **Colligations.** Following [8] we define a \( \mathcal{B} \)-unitary colligation \( \Sigma \) to be a triple \( \Sigma = (U, \mathcal{E}, \rho) \) where \( \mathcal{E} \) is a Hilbert space,

\[
U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{E} \oplus \mathcal{E} \to \mathcal{C} \oplus \mathcal{C}
\]

is unitary, and \( \rho : \mathcal{B} \to B(\mathcal{E}) \) is a unital \( * \)-representation. The transfer function associated to \( \Sigma \) is

\[
W_{\Sigma}(x) = (D\delta + C(\rho(E)) \ast (\delta - A\rho(E))^{-1} \ast (B\delta))(x).
\]

Observe that this looks like the standard transfer function over a Pick semigroupoid.

1.8. **The main event.** We now state the realization theorem for elements of the unit ball of \( H^\infty(\mathcal{K}_\Psi) \) and a concomitant interpolation theorem.

**Theorem 1.3 (Realization).** If \( \Psi \) is a collection of test functions for the semigroupoid \( G \), then the following are equivalent:

(i) \( \varphi \in H^\infty(\mathcal{K}_\Psi) \) and \( \|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} \leq 1 \);

(iiF) for each finite lower set \( F \subset G \) there exists a positive kernel \( \Gamma : F \times F \to \mathcal{B}^* \) so that for all \( x, y \in F \)

\[
([1] - \varphi\varphi^*)(x, y) = (\Gamma \ast ([1] - EE^*))(x, y);
\]

(iiG) there exists a positive kernel \( \Gamma : G \times G \to \mathcal{B}^* \) so that for all \( x, y \in G \)

\[
([1] - \varphi\varphi^*)(x, y) = (\Gamma \ast ([1] - EE^*))(x, y);
\]

(iii) there is a \( \mathcal{B} \)-unitary colligation \( \Sigma \) so that \( \varphi = W_\Sigma \).

**Theorem 1.4 (Agler-Jury-Pick Interpolation).** Let \( F \) be a finite lower set and suppose \( f \in P(F) \).

The following are equivalent:

(i) There exists \( \varphi \in H^\infty(\mathcal{K}_\Psi) \) so that \( \|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} \leq 1 \) and \( \varphi|_F = f \);

(ii) for each \( k \in \mathcal{K}_\Psi \), the kernel

\[
F \times F \ni (x, y) \mapsto (([1] - ff^*) \ast k)(x, y)
\]

is positive;

(iii) there is a positive kernel \( \Gamma : F \times F \to \mathcal{B}^* \) so that for all \( x, y \in F \)

\[
([1] - ff^*)(x, y) = (\Gamma \ast ([1] - EE^*))(x, y).
\]

**Remark 1.5.** Of course, item (i) of Theorem 1.4 combined with item (iii) of Theorem 1.3 says that \( \varphi \) in Theorem 1.4 has a \( \mathcal{B} \)-unitary transfer function representation.

The hypothesis that \( \Psi|_F \) generates all of \( P(F) \) means that the representation \( \pi : H^\infty(\mathcal{K}_\Psi) \to P(F) \) which sends \( \varphi \) to \( \varphi|_F \) is onto and identifies \( P(F) \) with the quotient \( H^\infty(\mathcal{K}_\Psi)/\ker(\pi) \). Theorem 1.4 can be interpreted as identifying the quotient norm.

2. **Further properties of the \( * \)-products**

2.1. **The convolution products.** The convolution products over finite lower sets \( F \) can be related to the tensor product of matrices as follows. Take \( V : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n \), where \( n = \text{card}(F) \), such that \( Ve_a = \sum_{pq=aa} e_p \otimes e_q \), \( \{e_k\} \) the standard basis for \( \mathbb{C}^n \) labelled with the elements of \( F \), and extending by linearity. Then \( f \ast g = V^*(f \otimes g) \) and \( f \star g = (g \otimes f)V \). Note that \( V \) is an isometry only in the case that \( F = F_x \), in which case the convolution products become the pointwise product. In all other cases it still has zero kernel and in fact maps orthogonal basis vectors \( e_a \) and \( e_b \) to orthogonal vectors, though it generally acts expansively.
2.2. The matrix $\star$-product—basic properties and an alternate definition. Straightforward calculations show that the various associative and distributive laws hold for the bivariate $\star$-product. Here, for example, is the proof that $C \star (A \star B) = (C \star A) \star B$:

$$[C \star (A \star B)]_{\mu,\nu} = \sum_{lm=\mu} \sum_{jk=\nu} C_{l,j} [A \star B]_{m,k}$$

$$= \sum_{lm=\mu} \sum_{jk=\nu} C_{l,j} \sum_{pq=m} \sum_{rs=k} A_{p,r} B_{q,s}$$

$$= \sum_{l,p,q} \sum_{j,r,s} C_{l,j} A_{p,r} B_{q,s},$$

while

$$[(C \star A) \star B]_{\mu,\nu} = \sum_{iq=\mu} \sum_{ns=\nu} [C \star A]_{i,n} B_{q,s}$$

$$= \sum_{iq=\mu} \sum_{ns=\nu} \sum_{lp=i} \sum_{jr=n} C_{l,j} A_{p,r} B_{q,s}$$

$$= \sum_{l,p,q} \sum_{j,r,s} C_{l,j} A_{p,r} B_{q,s},$$

and since $l(pq) = (lp)q$, $j(rs) = (jr)s$, the two are equal.

There is an alternate equivalent definition of the $\star$-product, just as with the convolution products. Take $V$ defined as in the last subsection. Then it is easy to check that

$$A \star B = V^* (A \otimes B) V.$$  \hspace{1cm} (2.1)

The Schur product is the matrix analogue of the pointwise product of functions in which case $V$ is isometric, though otherwise it will not be. From this formulation it is clear that the $\star$-product is continuous.

Another important property which the $\star$-product shares with the Schur product is that if $A,B \in \mathcal{M}(F)$ are positive, then so is $A \star B$. This follows immediately from the fact that $A \otimes B \geq 0$ if $A,B \geq 0$. Similarly, since the tensor product of selfadjoint matrices is selfadjoint, the $\star$-product of selfadjoint matrices is selfadjoint.

2.3. Positivity and the $\star$-product. It should be emphasized that unlike with ordinary matrix multiplication, the inverse with respect to the $\star$-product of a positive matrix need not be positive. This is already clear when considering Schur products, but we illustrate with another simple example. Suppose that $e, a \in G$ with $e$ idempotent and $eae = a$. Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$ where $c > 0$ and the first row and column is labelled by $e$ while the second is labelled by $a$. An easy calculation shows that $A^{-1,\star} = \begin{pmatrix} 1 & 0 \\ 0 & -c \end{pmatrix}$.

The $\star$-product behaves somewhat unexpectedly with respect to adjoints (at least if you forget its connection to the tensor product). Using the formulation of the $\star$-product given in (2.1), we see that $(A \star B)^* = V^* (A \otimes B)^* V = V^* (A^* \otimes B^*) V = A^* \star B^*$. However with regard to inverses and adjoints, $[1] = [1]^* = (A \star A^{-1,\star})^* = V^* (A^* \otimes (A^{-1,\star})^*) V$, and so by uniqueness of the inverse, $A^*$ is invertible if $A$ is and $(A^*)^{-1,\star} = (A^{-1,\star})^*$. Consequently we see that if $A$ is selfadjoint and invertible, then $A^{-1,\star}$ is selfadjoint.
Let $F$ be a finite lower set. An $A \in M(F)$ gives rise to the $\star$-product operator $S_A : M(F) \rightarrow M(F)$ given by $S_A(B) = A \star B = V^*(A \otimes B)V$. The argument in Paulsen’s book ([41], Theorem 3.7) which shows that Schur product with a positive matrix gives a completely positive map carries over with the obvious modifications to show that $S_A$ is completely positive. In particular, the cb-norm of $S_A$ is given by $\| A \star 1 \|$, where $1 \in M(F)$ is the identity (not the $\star$-product identity).

All of the above carries over in total to the $\star$-product, with a small change in the definition in terms of the tensor product, where we have

$$A \star B = V^*(B \otimes A)V.$$

### 2.4. More on order on semigroupoids.

The following lemmas give general properties of an artinian order on a semigroupoid $G$; i.e., a partial order $\leq$ such that for any $a \in G$, the set $\{b \in G : b \leq a\}$ is finite. Since $\leq$ is a partial order, it is permissible to use the notation $y < x$ to mean $y \leq x$, but $y \neq x$.

As before, a set $F$ is lower if for all $a \in F$, $\{b : b \leq a\} \subset F$. Clearly the intersection of lower sets is again lower. For $z \in G$, let $S_z = \{x \leq z\}$. This is a lower set. Furthermore, if $H$ is any subset of $G$, $x$ is minimal in $H$ is equivalent to $S_x \cap H = \{x\}$. By the artinian assumption, $S_x$ is finite. Note that $b \leq a$ is equivalent to $S_b \subset S_a$. (In fact there is an equivalence between artinian partial orders $(G, \preceq)$ and functions $\lambda : G \rightarrow \mathcal{G}_G$, $\mathcal{G}_G$ the set of all finite subsets of $G$, $\lambda$ injective and $\lambda(\lambda(x)) = \lambda(x)$, where $a \preceq b$ if and only if $\lambda(a) \subseteq \lambda(b)$.)

**Lemma 2.1.** Each nonempty subset $H$ of $G$ contains a minimal element with respect to an artinian partial order $\preceq$.

**Proof.** Clearly any finite subset has a least element. Suppose $H$ is any nonempty set and choose $z \in H$. Now $S_z \cap H$ is a nonempty finite set, so it has a minimal element $x \in H$. Since $S_x \subseteq S_z$, $S_x \cap H = S_x \cap (S_z \cap H) = \{x\}$; that is, $x$ is minimal in $H$. □

For a semigroupoid $G$ with artinian order $\preceq$, we define a stratification of $G$ as follows. Set $G_0 = G_e$. For natural numbers $n$, define

$$G_n = \{x \in G : y < x \Rightarrow y \in G_m \text{ for some } m < n \text{ and } y < x \text{ for some } y \in G_{n-1}\}.$$

We call $G_n$ the $n$th stratum with respect to the order $\preceq$ and $\{G_n\}$ where $G_n$ is nonempty a stratification of $G$ with respect to the order $\preceq$.

**Lemma 2.2.** For every $g \in G$ there is a unique $n \in \mathbb{N}$ such that $g \in G_n$.

**Proof.** Suppose $H = G \setminus \bigcup_0^\infty G_n$ is nonempty. From Lemma 2.1 $H$ has a minimal element $z$ (with respect to $\preceq$). In particular, $z \in S_z \subset \bigcup_0^\infty G_n$, a contradiction. □

For a lower set $F \subset G$ with respect to the order $\preceq$ we define the stratification $\{F_n\}$ of $F$ with strata $F_n = F \cap G_n$ where $F_n \neq \emptyset$.

The order $\preceq$ which we originally introduced on semigroupoids (where $b \leq a$ if and only if $a = zbw$ for some $z, w \in G$) is artinian by definition of a semigroupoid. Hence the above lemmas apply to $G$ with this order. There is another artinian order which will be useful in proving the existence of inverses with respect to the $\star$-product.

Define the left order $\leq_L$ on $G$ by declaring $y \leq_L x$ if there is an $a$ so that $x = ay$.

**Lemma 2.3.** The relation $\leq_L$ is a partial order on $G$ which is more restrictive than the order $\preceq$ on $G$; that is, if $y \leq_L x$, then $y \preceq x$. 12
Proof. The existence of idempotents implies that $x \leq \ell$ $x$. If $z \leq \ell y \leq \ell x$, then there exist $a, b$ so that $x = ay$ and $y = bz$. Hence, $x = a(bz) = (ab)z$ by the associative law and thus $z \leq \ell x$. Finally, choosing $x = z$ above gives $x = (ab)x$. By the strong idempotent law, it follows that $ab$ is idempotent; and then by nonexistence of inverses $a = b = e$ where $e$ is the idempotent so that $ex = x$. Thus $x = ey$. But by what it means to be idempotent, $ey = y$. Hence if $x \leq \ell y$ and $y \leq \ell x$, then $x = y$. This proves that $\leq \ell$ is an order on $G$.

If $y \leq \ell x$, then $x = ay$ for some $a$. There is always an idempotent $f$ so that $xf = x$. Thus, $x = xf = ayf$ (by associativity) and $y \leq x$. Hence $\{y : y \leq \ell x\} \subseteq \{y : y \leq x\}$. The latter set is finite, so both are finite.

We use the notation $\{F^\ell_n\}$ for the stratification of a lower set $F$ with respect to the left order, and for $z \in G$, we write $S^\ell_z$ for $S_z$ with respect to the left order.

2.5. \textit{$*$-inverses.} We next prove the statement about inverses of matrices with respect to the $*$-product made in the introduction. A similar (and in fact easier) proof works for inverses of functions with respect to the $*$-product. The arguments in the proof also apply to matrices over any $C^*$-algebra, though the theorem is stated for matrices over $\mathbb{C}$.

**Theorem 2.4.** Let $F$ be a lower set. A matrix $A \in M(F)$ is $*$-invertible if and only if $A_{ab}$ is invertible for all $a, b \in F_e$. Furthermore the inverse is unique.

**Proof.** For $a, b \in F_e = F \cap G_e$, the term $(B \star A)(a, b)$ is essentially the Schur product (that is, $(B \star A)(a, b) = B_{ab}A_{ab}$ if $a, b \in F_e$) and if $B$ is a $*$-inverse of $A$, then $(B \star A)(a, b) = 1$ in this case. Thus, $A_{ab}$ is invertible.

The proof of the converse proceeds as follows. Under the hypotheses of the theorem, a left $*$-inverse $B$ for $A$ is constructed which itself satisfies the hypotheses of the theorem. By what has already been proved, $B$ then has a left $*$-inverse $C$. Associativity of the $*$-product guarantees that $C = A$ and thus $B$ is also a right $*$-inverse for $A$. Uniqueness of the $*$-inverse similarly follows from the construction.

So assume $A_{ab}$ is invertible for all $a, b \in F^\ell_0 = F \cap G_e$. Let $\{F^\ell_n\}$ be the left stratification of $F$. Define $P_{jk} = \{(a, b) : a \in F^j_k, b \in F^\ell_k\}$, and $Q_N = \bigcup_{j,k \leq N} P_{jk}$. The proof proceeds by induction on $N$.

We require

$$
(B \star A)(a, b) = \sum_{pq=a \atop rs=b} B_{pr}A_{qs} = \begin{cases} 1 & \text{both } a, b \in G_e \\ 0 & \text{otherwise.} \end{cases}
$$

(2.2)

In the case $N = 0$ (so that $a, b \in F \cap G_e$), the choice $B_{ab} = A_{ab}^{-1}$ is the unique solution to this equation.

Now suppose that $B_{ab}$ have been defined for $(a, b) \in Q_N$ satisfying equation (2.2) and suppose $(a, b) \in Q_{N+1}\setminus Q_N$. Isolating the $(a, b)$ term in equation (2.2) gives

$$
0 = B_{ab}A_{ef} + \sum_{pq=a \atop rs=b} B_{pr}A_{qs},
$$

where $e, f \in G_e$ with $ae = a$ and $bf = b$. In the second term on the right hand side, $p < \ell a$ and $r < \ell b$. In particular, $p, r \in Q_N$ and the matrix $B_{pr}$ is already defined. Since $A_{ef}$ is invertible, $B_{ab}$ is uniquely determined.

**Lemma 2.5.** Let $L, F$ be lower sets in $G$ with $L \supset F$. Suppose $A \in M(L)$ is $*$-invertible. Then $A|_F$ is $*$-invertible and $(A|_F)^{-1,*} = A^{-1,*}|_F$.

**Proof.** This follows by observing that $A|_F \star A^{-1,*}|_F = (A \star A^{-1,*})|_F = [1]_F$. □
3. Reproducing kernels

3.1. Generalized Szegő kernels. In this section we investigate those kernels which play the role over semigroupoids of Szegő kernels. Recall, for a function $\varphi$ defined on a lower set $F$, the $n$-fold $\star$-product of $\varphi$ with itself is denoted $\varphi^n$. We use $A^n$ similarly when $A$ is a matrix.

**Theorem 3.1.** Let $A \in M(F)$ be positive, and suppose $\|A^n\| \to 0$ as $n \to \infty$. Then $[1] - A$ is invertible (with respect to the $\star$-product) and $([1] - A)^{-1} \geq 0$. In particular, the result holds if $\|A\| < 1$.

**Proof.** Observe that under the hypotheses, $[A^n]_{e,e} = A^n_{e,e} \to 0$ as $n \to \infty$ for $e \in F_e$. Hence $|A_{e,e}| < 1$ for all $e \in F_e$. Positivity of $A$ then implies that $|A_{e,f}| < 1$ for all $e, f \in F_e$. Consequently $[1] - A$ is invertible.

It is easily seen that

$$1 + A + A^2 + \cdots + A^n = ([1] - A)^{-1} \star ([1] - A^{(n+1)}).$$

But $[1] - A^{(n+1)} \to [1]$ as $n \to \infty$ and $1 + A + A^2 + \cdots + A^n$ is an increasing sequence of positive operators, and so converges strongly to $([1] - A)^{-1}$. Thus $([1] - A)^{-1} \geq 0$. The last part of the theorem follows from the submultiplicativity of the operator norm. 

It is not difficult to verify that the above arguments also work if we instead consider matrices over a unital $C^*$-algebra.

**Corollary 3.2.** If $\mathcal{C}$ is a unital $C^*$-algebra, $F$ a finite lower set, and $\varphi \in \mathcal{P}(F, \mathcal{C})$. Suppose that

$$\lim_{n \to \infty} \varphi^n(a) = 0$$

for each $a \in F_e$. Then $([1] - \varphi \varphi^*)^{-1} \in M(F, \mathcal{C})$ is well defined and positive. In particular, if $\|\mathcal{T}(\varphi)\| < 1$, the result follows (and in this case $F$ need not be finite).

**Proof.** Let $A(a, b) = \varphi(a)\varphi(b)^*$. Since for $e \in F_e$, $\|A^n(e, e)\| = \|A^n(e, e)\| \to 0$ as $n \to \infty$, it follows that $\|A(e, f)\| < 1$ for all $e, f \in F_e$. A counting argument in the same vein as that following Definition 1.2 then shows that $\|A^n(a, b)\| \to 0$ as $n \to \infty$, and so since $F$ is assumed to be finite, $\lim_{n \to \infty} \|A^n\| = 0$. The conditions for Theorem 3.1 hold and the result follows directly.

If $\|\mathcal{T}(\varphi)\| < 1$ then since

$$(A \star 1)_{a,b} = \sum_{pq=a} \sum_{rs=b} \varphi(p)\varphi(r)^* \langle 1_q, 1_s \rangle$$

and

$$(\mathcal{T}(\varphi)\mathcal{T}(\varphi)^*)_{a,b} = \sum_q \mathcal{T}(\varphi)_{a,q}[\mathcal{T}(\varphi)^*]_{q,b}$$

the result is then a consequence of the last statement of Theorem 3.1, since $\|A \star 1\| = \|A\|_{cb}$ dominates the operator norm. 

\[ \square \]
In what follows the theorem will be applied to test functions \( \psi \) and more generally the evaluation \( E \). That \( E \) satisfies the hypothesis of Corollary 3.2 is equivalent to item (ii) in the definition of test functions (Subsection 3.4).

**Lemma 3.3.** For each \( a \in G_e \), the sequence \( E^n(\psi) \) from \( \mathcal{B} \) converges to 0.

3.2. **The multiplier algebra for a single kernel.** Let \( k : G \times G \rightarrow \mathbb{C} \) be a positive kernel. For \( b \in G \), the function \( k_b : G \rightarrow \mathbb{C} \) defined by \( k_b = k(\cdot, b) \) is point evaluation at \( b \). In the usual way we form a sesquilinear form \( \langle \cdot, \cdot \rangle \) on linear combinations of kernel functions by setting \( \langle k_b, k_a \rangle = k(a, b) \) and modding out by the kernel. We then complete to get a Hilbert space, \( H^2(k) \).

On \( H^2(k) \) addition is defined term-wise. The multiplier algebra \( H^\infty(k) \) consists of the collection of operators \( T_\varphi : f \mapsto \varphi \ast f \) for functions \( \varphi : G \rightarrow \mathbb{C} \) satisfying \( \varphi \ast f \in H^2(k) \) for each \( f \in H^2(k) \). (The product is well defined by the assumption that \( G \) is artinian.) Note that \( H^\infty(k) \) is nonempty, since it contains \( T_\delta \), the \( \ast \)-product identity for functions on \( G \). The closed graph theorem implies that the elements of \( H^\infty(k) \) are bounded.

Observe that for \( f \in H^2(k) \),

\[
\langle T_\varphi f, k_a \rangle = (\varphi \ast f)(a) = \sum_{bc=a} \varphi(b)f(c) = \sum_{bc=a} \varphi(b)\langle f, k_c \rangle = \sum_{bc=a} \langle f, \varphi(b)^*k_c \rangle = \left\langle f, \sum_{bc=a} \varphi(b)^*k_c \right\rangle,
\]

which gives the formula \( T_\varphi^*k_a = \sum_{bc=a} \varphi(b)^*k_c \).

For a lower set \( F \), if we set \( \mathcal{M}(F) \) to the closed linear span of kernel functions \( k_a, a \in F \), then the usual sort of argument gives \( \mathcal{M}(F) \) invariant for the adjoints of multipliers \( T_\varphi \).

The \( \ast \)-product is useful in characterizing multipliers. Indeed, \( \|T_\varphi^*\mathcal{M}(F)\| \leq C \) is equivalent to \( 0 \leq (\langle (C^2 - T_\varphi T_\varphi^*)k_a, k_b \rangle) \) which by the previous calculation is

\[
\|T_\varphi^*\mathcal{M}(F)\| \leq C \iff C^2k - \varphi^* \ast k \ast \varphi \geq 0.
\]

In the above \( \varphi^* \ast k \ast \varphi \) stands for \( (\varphi^* \ast k)(k^* \ast \varphi) \) where \( k(x) = k_x \) in the factorization \( k(x, y) = k_xk_y^* \) for \( x, y \in F \).

3.3. **The Toeplitz kernel.** A special case of interest is the kernel \( s : F \times F \rightarrow \mathbb{C} \) given by \( s(x, y) = 1 \) if \( x = y \) and 0 if \( x \neq y \). This kernel is evidently positive and, as noted earlier is referred to as the Toeplitz kernel. It arises naturally by declaring \( \langle x, y \rangle = s(x, y) \) for \( x, y \in F \) and extending by linearity. That is, the Hilbert space \( H^2(s) \) is nothing more than the Hilbert space with orthonormal basis indexed by \( F \), i.e., \( \mathbb{C}^F \). The Toeplitz representation of \( \varphi : F \rightarrow \mathbb{C} \) determined by \( s \) as in the previous subsection is thus the Toeplitz representation \( \mathcal{T}(\varphi) \) of \( \varphi \).

Note that \( \langle s(x, y) \rangle_{x,y \in F} = 1 \in M(F) \), the usual identity matrix.

3.4. **Kernels and representations.** The results of Subsection 3.2 have an alternate interpretation. Let \( k \) be a reproducing kernel on \( G \). Recall that we use \( P(F) \) to denote the complex valued functions on the finite lower set \( F \subset G \), which under the \( \ast \)-product is an algebra. If we now compress \( k \) to \( F \), it is
still a positive kernel (on F) which we continue to call k (or k_F if it is not absolutely clear from the context). Furthermore, since F is lower
\[
F \times F \ni (x, y) \mapsto ([1] - \psi^* \psi) \ast k(x, y)
\]  
(3.2)
is positive for each \( \psi \in \Psi|_F \). In this case, any \( \varphi \in P(F) \) is a multiplier of \( H^2(k_F) \) since the algebra generated by \( \Psi|_F \) is all of \( P(F) \) and \( \pi : P(F) \to B(H^2(k_F)) \) defined by \( \pi(\varphi) = \mathcal{T}^F(\varphi) \) is a representation of \( P(F) \). Further, the assumption that (3.2) is positive implies \( \| \pi(\psi) \| \leq 1 \) for each \( \psi \in \Psi|_F \).

Define the functions
\[
\chi_a(x) = \begin{cases} 
1 & x = a, \\
0 & \text{otherwise}. 
\end{cases}
\]
Routine calculation verifies \( (\chi_a \ast \chi_b)(x) = \sum_{pq=x} \chi_a(p)\chi_b(q) = \chi_{ab}(x) \), where for convenience we take \( \chi_{ab} = 0 \) if the product \( ab \) is not in our partial multiplication.

Clearly the set \( \{ \pi(\chi_a)\delta \} \) forms a spanning set for \( H^2(k) \), and, since \( \pi(\varphi)\delta = \varphi = 0 \) if and only if \( \varphi = 0 \), it is in fact a basis. Indeed it is a dual basis to \( \{ k_a \} \), since
\[
\langle k_a, \pi(\chi_b)\delta \rangle = \langle \mathcal{T}^F(\chi_b)^*k_a, \delta \rangle = \sum_{pq=a} \langle \chi_b^*(p)k_q, \delta \rangle = \begin{cases} 
1 & p = b = a, \ q \in F_e \\
0 & \text{otherwise.} 
\end{cases}
\]

In some cases it is possible to reverse the above, obtaining a kernel from a representation \( \mu : P(F) \to B(\mathcal{H}) \). For instance, suppose \( F \) is a finite lower set and assume that \( \mu \) is cyclic with dimension equal to the cardinality of \( F \). Write \( \gamma \) for the cyclic vector for \( \mu \), so that \( \mathcal{H} \) is spanned by \( \{ \ell_a = \mu(\chi_a)\gamma : a \in F \} \). Since by assumption the dimension of \( \mu \) is the cardinality of \( F \), this set is in fact a basis for \( F \).

If \( \mu \) is to come from a kernel \( k \), we require that for any function \( \varphi \) on \( F \),
\[
\mu(\varphi)^*k_a = \sum_{pq=a} \varphi(p)^*k_q.
\]
It suffices to have this for the functions \( \chi_b \), in which case we need
\[
\mu(\chi_b)^*k_a = \sum_{pq=a} \chi_b(p)k_q = \sum_{bq=a} k_q.
\]
Choose \( \{ k_a : a \in F \} \) to be a dual basis to \( \ell_a \). Then compute,
\[
\langle \mu(\chi_b)^*k_a, \ell_c \rangle = \langle k_a, \mu(\chi_b)\ell_c \rangle
\]
\[
= \langle k_a, \mu(\chi_b)\mu(\chi_c)\gamma \rangle
\]
\[
= \langle k_a, \mu(\chi_b \ast \chi_c)\gamma \rangle
\]
\[
= \langle k_a, \ell_{bc} \rangle
\]
\[
= \begin{cases} 
1 & bc = a \\
0 & bc \neq a 
\end{cases}
\]
\[
= \langle \sum_{bq=a} k_q, \ell_c \rangle.
\]
Since this is true for all \( c \in F \), it follows that
\[
\mu(\chi_b)^*k_a = \sum_{bq=c} k_q.
\]
as desired.

It is worth considering the example where \( \mu(\varphi) = \mathcal{T}(\varphi) \), the Toeplitz representation. The function \( \delta(x) \), which is 1 if \( x \in F_e \) and zero otherwise is a cyclic vector for \( \mu \). Moreover,

\[
\mu(\chi_a)(x,y) = \mathcal{T}(\chi_a)(x,y) = \sum_{py=x} \chi_a(p) = \chi ay(x).
\]

Therefore

\[
\ell_a(x) = \sum_y \mu(\chi_a)(x,y)\delta(y) = \sum_y \chi ay(x)\delta(y) = \chi a(x),
\]

which is just the standard basis, and so the assumption that \( \{\ell_a : a \in F\} \) is a basis is automatically met. In this case we choose \( k_a = \chi_a \), and the kernel is the Toeplitz kernel \( s \).

If \( F \) is infinite, this construction fails, since it need not be the case that \( \chi a \in H^\infty(\mathcal{K}^F_\psi) \).

3.5. \( \mathcal{P}(F) \) as a normed algebra. Given a finite lower set \( F \), let \( \pi_F : H^\infty(\mathcal{K}_\psi) \to \mathcal{P}(F) \) denote the mapping \( \pi_F(\varphi) = \varphi|_F \). The hypothesis on the collection of test functions \( \Psi \) imply that this mapping is onto and so \( \ker(\pi_F) = \{ \varphi \in H^\infty(\mathcal{K}_\psi) : \varphi|_F = 0 \} \). Thus \( \mathcal{P}(F) \) is naturally identified with the quotient of \( H^\infty(\mathcal{K}_\psi) \) by \( \ker(\pi_F) \) and this gives \( \mathcal{P}(F) \) a norm for which \( \pi_F \) is contractive. There is an alternate candidate for a norm on \( \mathcal{P}(F) \) constructed in much the same way as the norm on \( H^\infty(\mathcal{K}_\psi) \), called the \( H^\infty(\mathcal{K}_F^F) \)-norm. Let \( \mathcal{K}^F_\psi \) denote the kernels \( k \) defined on \( F \) for which

\[
F \times F \ni (x,y) \mapsto (|[1] - \psi|_F \psi^*) \star k(x,y)
\]

is a positive kernel and, for \( \varphi \in \mathcal{P}(F) \), say that \( ||\varphi|| \leq C \) (here \( C \geq 0 \)) provided for each \( k \in \mathcal{K}^F_\psi \), the kernel

\[
F \times F \ni (x,y) \mapsto ((C^2[1] - \varphi \varphi^*) \star k)(x,y)
\]

is positive.

The following lemma ultimately implies that the quotient norm dominates the \( H^\infty(\mathcal{K}_F^F) \)-norm. Theorem 1.4 then says that these norms are the same.

**Lemma 3.4.** Suppose \( \mu : \mathcal{P}(F) \to B(\mathcal{H}) \) is a cyclic unital representation of the finite lower set \( F \) and let \( \varphi \in H^\infty(\mathcal{K}_\psi) \) be given. Let \( \pi_F : H^\infty(\mathcal{K}_\psi) \to \mathcal{P}(F) \) be the restriction map, \( \mu_F = \mu \circ \pi_F \). If \( ||\mu_F(\psi)|| \leq 1 \) for each \( \psi \in \Psi \), but \( ||\mu_F(\varphi)|| > 1 \), then there exists a \( k \in \mathcal{K}_\psi \) so that the kernel

\[
F \times F \ni (x,y) \mapsto (([1] - \varphi \varphi^*) \star k)(x,y)
\]

is not positive. In particular, \( ||\varphi|| > 1 \).

**Proof.** Let \( \gamma \) denote a cyclic vector for the representation \( \mu \). Choose \( f \in \mathcal{P}(F) \) so that \( ||\mu(f)\gamma|| = 1 \) but \( ||\mu_F(\varphi)(f)\gamma|| = 1 + \eta > 1 \), and \( \epsilon \) so that \( (1 + \epsilon^2 f||f||^2_{H^2(s)}) (1 + \eta/2)^2 = (1 + \eta)^2 \), where \( ||f||_{H^2(s)} \) is the norm of \( f \) in the space with the Toeplitz kernel \( s \).

Recall that for a finite lower set \( L \), \( \mathcal{T}^L \) denotes the Toeplitz representation with its cyclic vector \( \delta^L \). If \( L \supseteq F \), let \( \pi^L_F \) be the restriction of \( \mathcal{P}(L) \) to \( \mathcal{P}(F) \) and set \( \mu^L_F = \mu \circ \pi^L_F \). As above, define \( \pi_L : H^\infty(\mathcal{K}_\psi) \to \mathcal{P}(L) \) to be the restriction map.

For \( L \supseteq F \) lower, there is a finite dimensional Hilbert space given by \( \mathcal{H}_L = \{ \mu^L_F(h)\gamma + \epsilon \mathcal{T}^L(h)\delta^L : h \in \mathcal{P}(L) \} \). Recall \( \mathcal{T}^L(h)\delta^L = h \). Define a representation \( \rho_L : \mathcal{P}(L) \to B(\mathcal{H}_L) \) by

\[
\rho_L(g)(\mu^L_F(h)\gamma + \epsilon \mathcal{T}^L(h)\delta^L) = \mu^L_F(g \star h)\gamma + \epsilon g \star h
\]
Since for \( \psi \in \Psi, h \in P(L), \)
\[
\| \rho_L(\pi_L(\psi))(\mu(h)\gamma + \epsilon h) \| = \| \mu(\pi_F(\psi))\mu(h)\gamma + \mathcal{T}^L(\pi_L \psi) \epsilon h \| \\
\leq \max\{\| \mu(\pi_F(\psi)) \|, \| \mathcal{T}^L(\pi_L \psi) \| \} \| \mu(h)\gamma + \epsilon h \|
\]
\[
\leq \| \mu(h)\gamma + \epsilon h \|,
\]
\[
\| \rho_L(\psi) \| \leq 1, \text{ and in particular, taking } L = F \text{ we have } \| \rho_F(\psi) \| \leq 1.
\]
From the discussion in Subsection 3.4, there is a kernel \( k^F \) on \( F \) which implements the representation \( \rho_F \). In particular, since
\[
\| \rho_F(\varphi) - \mu(f)\gamma \epsilon f \sqrt{1 + \epsilon^2 \| f \|_H^2(s)} \|^2 = \frac{1}{1 + \epsilon^2 \| f \|_H^2(s)} (\| \mu(\varphi)\mu(f)\gamma \|^2 + \epsilon^2 \| \varphi \ast f \|^2)
\]
\[
\geq \frac{1}{1 + \epsilon^2 \| f \|_H^2(s)} (1 + \eta)^2
\]
\[
= (1 + \eta/2)^2,
\]
the kernel
\[
F \times F \ni (x, y) \mapsto (\{1\} - \varphi \varphi^*) \ast k^F(x, y)
\]
is not positive.

Define \( k : G \times G \to \mathbb{C} \) by
\[
k(a, b) = \begin{cases} 
  k^F(a, b) & \text{if } (a, b) \in F \times F \\
  \frac{1}{2} s(a, b) & \text{if } (a, b) \notin F \times F.
\end{cases}
\]
In particular, if \( a \in F \) and \( b \notin F \) (or vice-versa), then \( k(a, b) = 0 \). We will complete the proof by showing \( k \in \mathcal{K}_\Psi \).

The representation \( \rho_L \) defined as above is non-degenerate for any \( L \supseteq F \), in the sense of the discussion in Subsection 3.4. In particular, for any such \( L \) there is a reproducing kernel \( k^L \) which implements this representation. Consequently, for each \( \psi \in \Psi, \)
\[
L \times L \ni (x, y) \mapsto (\{1\} - \psi \psi^*) \ast k^L(x, y)
\]
is positive. Our goal now is to show that \( k^L(x, y) = k(x, y) \) for \( x, y \in L \) from which it will follow that \( k \in \mathcal{K}_\Psi \).

For this, once again recall the construction of \( k^L \) from \( \rho_L \). Let \( \ell^L_a = \rho_L(\chi_a)h \), where \( h = \gamma \epsilon \delta = \mu_F(\delta^L) \gamma \epsilon \mathcal{T}^L(\delta^L) \delta^L \) is the cyclic vector for the representation \( \rho_L \). Next, let \( k^L_b \) denote a dual basis to the basis \( \ell^L_a \) and define \( k^L \) by \( k^L(a, b) = (k^L_b, k^L_a) \).

We calculate
\[
\ell^L_a = \rho_L(\chi_a)h = \mu_F(\chi_a \ast \delta^L) \gamma \epsilon \chi_a \ast \delta^L = \mu_F(\chi_a) \epsilon \chi_a,
\]
which reduces to \( \{0\} \oplus \epsilon \chi_a \) if \( a \notin F \), and which equals \( \ell^F_a \oplus \{0\} \) if \( a \in F \). Hence the dual basis is
\[
k^L_a = \begin{cases} 
  k^F_a \oplus \{0\} & a \in F, \\
  \{0\} \oplus \epsilon \chi_a & \text{otherwise},
\end{cases}
\]
via which we immediately verify that \( k^L = k \).
3.6. **Toeplitz representation for \( C^* \)-algebra-valued functions.** The notion of the Toeplitz representation naturally generalizes to functions \( f : F \to \mathfrak{C} \), where \( F \) is a lower set and \( \mathfrak{C} \) is a \( C^* \)-algebra with \( [\mathfrak{T}(\varphi)]_{a,b} \in \mathfrak{C} \) and \( \mathfrak{T}(\varphi) \in M(F, \mathfrak{C}) \), the \( \mathfrak{C} \)-valued matrices labelled by elements of \( F \).

**Lemma 3.5.** Suppose that \( \mathfrak{C} \) is another \( C^* \)-algebra. If \( \rho : \mathfrak{C} \to \mathfrak{C} \) is a unital \(*\)-representation, then

\[
(1 \otimes \rho)(\mathfrak{T}(f)) = \mathfrak{T}(\rho \circ f)
\]

and moreover, \( \|\mathfrak{T}(f)\| \geq \|\mathfrak{T}(\rho \circ f)\| \).

**Proof.** Simply compute

\[
(1 \otimes \rho)(\mathfrak{T}(f)) = \left[ \rho \left( \sum_{cb=a} f(c) \right) \right]_{a,b} = \sum_{cb=a} \rho(f(c)) = \left[ [\mathfrak{T}(\rho \circ f)]_{a,b} \right]_{a,b}.
\]

The norm estimate follows since \( \rho \) is completely contractive. \( \square \)

In our applications of this lemma \( f \) will be the function \( E : F \to \mathfrak{B} \) and \( \rho : \mathfrak{B} \to B(\mathcal{E}) \) will be the representation arising in a \( \mathfrak{B} \)-unitary colligation.

4. **Factorization**

**Proposition 4.1.** If \( \Gamma : G \times G \to \mathfrak{B}^* \) is positive, then there exists a Hilbert space \( \mathcal{E} \) and a function \( L : G \to B(\mathfrak{B}, \mathcal{E}) \) such that

\[
\Gamma(x,y)(fg^*) = \langle L(x)f, L(y)g \rangle
\]

for all \( f, g \in \mathfrak{B} \).

Further, there exists a unital \(*\)-representation \( \rho : \mathfrak{B} \to B(\mathfrak{E}) \) such that \( L(x)ab = \rho(a)L(x)b \) for all \( x \in G, a, b \in \mathfrak{B} \).

**Proof.** The proof is a variant on a usual proof of the factorization of positive semidefinite kernels. See the book [6] Theorem 2.53, Proof 1. The statement should be compared with a similar result in [8].

Let \( W \) denote a vector space with basis labelled by \( G \). On the vector space \( W \otimes \mathfrak{B} \) introduce the positive semidefinite sesquilinear form induced from

\[
\langle x \otimes f, y \otimes g \rangle = \Gamma(x,y)(fg^*),
\]

where \( x, y \in G \) and \( f, g \in \mathfrak{B} \), making \( W \otimes \mathfrak{B} \) into a pre-Hilbert space which is made into the Hilbert space \( \mathcal{E} \) by the standard modding out and completion.

One verifies that this is indeed positive as a consequence of the hypothesis that \( \Gamma \) is positive. Define \( L(x)a = x \otimes a \). Since for \( a \in \mathfrak{B} \),

\[
\|L(x)a\|^2 = \langle L(x)a, L(x)a \rangle = \Gamma(x,x)(a^*a) \leq \|\Gamma(x,x)\| \|a^*a\|
\]

\( L(x) \) does indeed define a bounded operator on \( \mathfrak{B} \) with \( \|L(x)\|^2 \leq \|\Gamma(x,x)\| \).
As for the ∗-representation, it is induced by the left regular representation of \( \mathcal{B} \). That is, define \( \rho : \mathcal{B} \to B(\mathcal{E}) \) by \( \rho(a)(x \otimes f) = x \otimes af \). To see that this is indeed bounded, first note that \( \|a\|^2 - a^*a \) is positive semidefinite in \( \mathcal{B} \) and hence there exists a \( b \) so that \( \|a\|^2 - a^*a = b^*b \). Thus,

\[
\|a\|^2 \left\| \sum x_j \otimes f_j \right\|^2 - \left\| \sum x_j \otimes af_j \right\|^2 \\
= \|a\|^2 \sum \Gamma(x_j, x_\ell)(f^*_\ell f_j) - \sum \Gamma(x_j, x_\ell)(f^*_\ell a^*af_j) \\
= \sum \Gamma(x_j, x_\ell)(f^*_\ell b^*bf_j) \geq 0
\]

where the inequality is a result of the assumption that \( \Gamma \) is positive. This shows at the same time that \( \rho \) is well defined.

We also have that \( \rho \) is unital, since \( \rho(1)(x \otimes f) = x \otimes 1f = x \otimes f \).

Finally,

\[
\langle \rho(a^*)(x \otimes f), y \otimes g \rangle = \langle x \otimes a^*f, y \otimes g \rangle \\
= \Gamma(x, y)(g^*a^*f) \\
= \langle x \otimes f, y \otimes ag \rangle \\
= \langle x \otimes f, \rho(a)(y \otimes g) \rangle \\
= \langle \rho(a)^*(x \otimes f), y \otimes g \rangle
\]

so that \( \rho(a^*) = \rho(a)^* \).

5. The cone \( \mathcal{C}_F \) and compact convex set \( \Phi_F \)

Given a finite subset \( F \subset G \), let \( M(F, \mathcal{B}^*)^+ \) denote the collection of positive kernels \( \Gamma : F \times F \to \mathcal{B}^* \) and define the cone

\[ \mathcal{C}_F = \{ (\Gamma \hat{\ast} ([1] - EE^*))_{x,y \in F} : \Gamma \in M(F, \mathcal{B}^*)^+ \} \]

5.1. The cone is closed.

**Theorem 5.1.** Let \( F \) be a finite lower set. The cone \( \mathcal{C}_F \) is closed in \( M(F) \).

**Proof.** Let \( M = \Gamma \hat{\ast} ([1] - EE^*) \in \mathcal{C}_F \), where \( \Gamma : F \times F \to \mathcal{B}^* \) is positive. Positivity of \( \Gamma \) means in particular that if \( \tilde{F} \) is a subset of \( F \), and \( \{f_q\}_{q \in \tilde{F}} \) is any collection of elements of \( \mathcal{B} \), then

\[
\sum_{p,q \in \tilde{F}} \Gamma(p, q) f_p f_q^* \geq 0. \quad (5.1)
\]

For convenience we define \( b \leq_r a \) to mean \( bc = a \) for some \( c \). As in Lemma 2.3, this can be shown to be an order on \( G \) and \( b \leq_r a \) implies \( b \leq a \).

Fix \( x \in F \), and suppose \( e \) is idempotent with \( xe = x \). Taking \( F_x = \{ y : y \leq_r x \} \), we get a finite subset of \( F \). For \( q \in F_x \), set

\[
\tilde{E}(q) = \sum_{p : qp=x} E(p).
\]
Observe that $\hat{E}(x) = E(e)$. With this notation, we have

$$M_{x,x} = \Gamma(x,x)(1 - \hat{E}(x)\hat{E}(x)^*) - \sum_{q < x} \Gamma(q,s)(\hat{E}(q)\hat{E}(s)^*)$$

$$- \sum_{q < r} \Gamma(q,x)(\hat{E}(q)\hat{E}(x)^*) - \sum_{s < r} \Gamma(x,s)(\hat{E}(x)\hat{E}(s)^*).$$

(5.2)

Choose $\epsilon > 0$ small enough that $1 - (1 + \epsilon^2)\hat{E}(x)\hat{E}(x)^* > 0$. This can be done since $1 - \hat{E}(x)\hat{E}(x)^* = 1 - E(e)E(e)^* > 0$ by property (ii) of Definition 12. Let $f_x = -\epsilon \hat{E}(x)$, $f_q = (e^{-i\theta_q}/\epsilon)\hat{E}(q)$ for $q \in F_x, q \neq x$, and $\theta_q = \arg(\sum_{q < x} \Gamma(q,x)\hat{E}(q)\hat{E}(x)^*)$. With this choice, (5.1) gives

$$2\sum_{q < x} \Gamma(q,x)(\hat{E}(q)\hat{E}(x)^*)| \leq (1/\epsilon)^2 \sum_{q < x} \sum_{s < x} \Gamma(q,s)(\hat{E}(q)\hat{E}(s)^*) + \epsilon^2 \Gamma(x,x)(\hat{E}(x)\hat{E}(x)^*).$$

(5.3)

Combining the inequality in (5.3) with (5.2) we have

$$\Gamma(x,x)(1 - (1 + \epsilon^2)\hat{E}(x)\hat{E}(x)^*) \leq M_{x,x} + (1 + 1/\epsilon^2) \sum_{q < x} \sum_{s < x} \Gamma(q,s)(\hat{E}(q)\hat{E}(s)^*).$$

(5.4)

Furthermore, positivity of $\Gamma$ and a calculation as for (5.3) yields for $g \in \mathfrak{B}$

$$2|\Gamma(x,y)g| \leq \Gamma(x,x) 1 + \Gamma(y,y) gg^* \leq \|\Gamma(x,x)\| + \|\Gamma(y,y)\| \|g\|^2,$$

and so

$$\|\Gamma(x,y)\| \leq \frac{1}{\epsilon} \left( \|\Gamma(x,x)\| + \|\Gamma(y,y)\| \right).$$

(5.5)

We show by induction on (right) strata that for each $p, q \in F$, there is a constant $c_{p,q}$, independent of $\Gamma$, such that $\|\Gamma(p,q)\| \leq c_{p,q} \|M\|$. By (5.5), it suffices to prove this for $p = q$. Since $F$ is assumed finite, it will then follow that $\|\Gamma\| \leq c\|M\|$ for some $c \geq 0$ and independent of $\Gamma$.

To begin with, if $e \in F$ is idempotent, then $M_{e,e} = \Gamma(e,e)(1 - E(e)E(e)^*)$, and since $1 - E(e)E(e)^* > 0$, we have that $c_{e,e}$ exists. Now suppose that we have $c_{p,q}$ for all $p, q$ in the $(n - 1)$st and lower strata. Let $x$ be in the $n$th stratum. Then by the induction hypothesis and (5.4), we find $c_{x,x}$.

Let $\{M_j\}$ be a bounded sequence from $C_F$, $M_j = \Gamma_j \ast (1 - EE^*)$, so that

$$M_j(x,y) = \Gamma_j(x,y)(1) - \sum_{pq=x \text{ or } rs=y} \sum_{j} \Gamma_j(q,s)(E(p)E(r)^*), \quad x, y \in F.$$

Then $\{\Gamma_j\}$ is a bounded sequence in $M(F, \mathfrak{B}^*)^+$; i.e., there is a uniform bound on the norm of the linear functional $\Gamma_j(x,y)$ independent of $x, y, j$. It follows from weak-$*$ compactness, that there exists $\Gamma \in M(F, \mathfrak{B}^*)$ and a subsequence $\{\Gamma_{j_{\ell}}\}$ of $\{\Gamma_j\}$ so that for each $x, y \in F$, the sequence $\{\Gamma_{j_{\ell}}(x,y)\}$ converges to $\Gamma(x,y)$ weak-$*$. In particular, $\{\Gamma_{j_{\ell}}(p,r)(E(q)E(s)^*)\}$ converges to $\Gamma(p,r)(E(q)E(s)^*)$ for each $p, q, r, s$ (and also with $E(q)E(s)^*$ replaced by 1). If now $\{M_j\}$ converges to some $M$, then

$$M = \lim_{\ell \to \infty} \left( \Gamma_{j_{\ell}} \ast (1 - EE^*)\right)_{x,y \in F} = \left( \Gamma \ast (1 - EE^*)\right)_{x,y \in F}.$$

If $f : F \to \mathfrak{B}$, then

$$0 \leq \sum_{x,y \in F} \Gamma_j(x,y)(f(x)f(y)^*) \to \sum_{x,y \in F} \Gamma(x,y)(f(x)f(y)^*),$$

which shows that $\Gamma$ is positive and completes the proof. □
5.2. The cone is big.

**Lemma 5.2.** Let $\Psi$ be a set of test functions for $G$. For each $\psi \in \Psi$ the function $\Gamma_\psi : G \times G \to \mathcal{B}^*$ given by

$$\Gamma_\psi(x, y)(f) = ([1] - \psi \psi^*)^{-1}(x, y) f(\psi), \quad f \in \mathcal{B} = C(\Psi),$$

is a positive kernel.

**Proof.** For each $x, y \in G$, the functional $\Gamma_\psi(x, y)$ is a multiple of evaluation at $\psi$ and hence does indeed define an element of $\mathcal{B}^*$.

For a finite lower set $F \subset G$ and a function $f : F \to \mathcal{B}$,

$$\sum_{x, y \in F} \Gamma_\psi(x, y)(f(x)f(y)^*) = \sum_{x, y \in F} ([1] - \psi \psi^*)^{-1}(x, y) (f(x)(\psi)f(y)(\psi)^*)$$

$$= \sum_{x, y \in F} ([1] - \psi \psi^*)^{-1}(x, y) (g(x)g(y)^*)$$

where $g : F \to \mathbb{C}$ is given by $g(x) = f(x)(\psi)$ and $g$ is the vector with $x$ entry $g(x)$. By Corollary 3.2

$$F \times F \ni (x, y) \mapsto ([1] - \psi \psi^*)^{-1}(x, y)$$

is a positive matrix in $M(F)$. The conclusion follows. □

**Lemma 5.3.** Suppose $F \subset G$ is a finite lower set. The cone $C_F$ contains all positive matrices. In particular, it contains $[1]$ and so has non-trivial interior.

**Proof.** Let $\Gamma_\psi$ denote the positive kernel from the previous lemma. Then

$$[1](x, y) = \Gamma_\psi \hat{\ast} ([1] - EE^*)(x, y) \in C_F.$$

On the other hand, if $P \in M(F)$ with $P \geq 0$, then $P \hat{\ast} \Gamma_\psi \geq 0$ and $P = P \hat{\ast} [1] = P \hat{\ast} (\Gamma_\psi \hat{\ast} ([1] - EE^*))$. Thus $C_F$ contains all positive $P \in M(F)$. □

**Lemma 5.4.** The cone $C_F$ is closed under conjugation; i.e., if $M = (M(x, y)) \in C_F$ and $c : F \to \mathbb{C}$, then $c \ast M \ast c^* \in C_F$, where $(c \ast M \ast c^*)(x, y) = \sum_{pq=x} \sum_{rs=y} c(p) M(q, s) c^*(r)$.

**Proof.** If $M = \Gamma \hat{\ast} ([1] - EE^*) \in C_F$, then $c \ast M \ast c^* = \tilde{\Gamma} \hat{\ast} ([1] - EE^*)$, where $\tilde{\Gamma} = c \ast \Gamma \hat{\ast} c^* \geq 0$. □

5.3. Separation.

**Lemma 5.5.** Let $F$ be a finite lower set and suppose $\varphi \in L^\infty(K_\psi)$. If

$$M_\varphi = (([1] - \varphi \varphi^*)(x, y))_{x, y \in F} \notin C_F,$$

then there exists a cyclic unital representation $\mu : P(F) \to B(H)$ such that $\|\mu(\psi)\| \leq 1$ for all $\psi \in \Psi|_F$, but $\|\mu(\varphi)\| > 1$.

**Proof.** By Theorem 5.1, the cone $C_F$ is closed (in the set of $F \times F$ matrices $M(F)$). As a consequence of the Hahn-Banach Theorem (see, for example, §12.F of [23]), there is a linear functional $\lambda$ on $M(F)$ such that $\lambda$ is nonnegative on $C_F$ and $\lambda(M_\varphi) < 0$. As $\|M_\varphi\| + M_\varphi \in C_F$ by Lemma 5.3, we have $\lambda(1) > 0$, where 1 is the identity in $M(F)$. So in particular, $\lambda$ is not identically zero on $C_F$.

Next define a scalar product on $P(F)$ by

$$\langle f, g \rangle = \lambda(fg^*).$$

(5.6)

For ease of notation, we will simply write “$f$” for the restriction $f|_F$ of $f$ to the lower set $F$. We then view $f, g \in \mathbb{C}^F$ as vectors so that $fg^* \in M(F)$ is the matrix with entries $fg^*(x, y) = f(x)g(y)^*$. 


Since, by Lemma 5.3, the cone $C_F$ contains all positive matrices and $\lambda$ is non-negative on $C_F$, the form in equation (5.6) is positive semi-definite.

Mod out by the kernel and let $q(f)$ denote the image of $f$ in the quotient. (Since the space is finite dimensional there is no need to complete to get a Hilbert space.) The resulting Hilbert space, which we call $\mathcal{H}$, is nontrivial. In particular, $q(\delta^F) \neq 0$. To see this, first note that $[1] \in C_F$, so $\lambda([1]) \geq 0$. By assumption $\lambda(([1] - \varphi \varphi^*)) < 0$, which implies $\lambda(\varphi \varphi^*) > 0$. Since finite products of the test functions restricted to $F$ span $P(F)$, which is finite dimensional, we can write $\varphi = \sum_k c_k \xi_k$, for some finite collection of finite products of test functions $\{\xi_k\}$. Repeated use of the equality

$$[1] - \xi_k \xi_k^* = ([1] - \xi_k \xi_k^*) + \xi_k ([1] - \xi_j \xi_j^*)$$

and Lemma 5.4 shows that $[1] - \xi \xi^*$ is in $C_F$, and so $\lambda([1] - \xi \xi^*) \geq 0$, for any finite product of test functions $\xi$. By the Cauchy-Schwarz inequality, for any $j, k$, $|\lambda(\xi_j \xi_k^*)| \leq \lambda(\xi_j \xi_j^*) \lambda(\xi_k \xi_k^*)$, and so if for all $k$, $\lambda(\xi_k \xi_k^*)$ were zero, we would have $\lambda(\varphi \varphi^*) = 0$. Hence there is some product of test functions $\xi$ such that $\lambda(\xi \xi^*) > 0$. Consequently, $\lambda([1]) > 0$, and so $\|q(\delta^F)\| > 0$.

Let $\mu$ be the right regular representation of $P(F)$ on $\mathcal{H}$. That is, $\mu(g)q(f) = q(f \ast g)$ — provided of course that it is well defined. If $\psi \in \Psi$, then because of the definition of $C_F$,

$$\|q(f)\|^2 - \|q(f \ast \psi)\|^2 = \lambda(f \ast ([1] - \psi \psi^*) \ast f) \geq 0,$$

where the inequality follows from Lemma 5.4. Thus, $\mu(\psi)$ is well defined and since $\Psi|_F$ generates $P(F)$, $\mu$ is well defined.

Clearly $\mu$ is cyclic with cyclic vector $q(\delta^F)$. Finally,

$$\|q(\delta^F)\|^2 - \|\mu(\psi)q(\delta^F)\|^2 = \lambda([1] - \varphi \varphi^*) < 0$$

so that $\|\mu(\psi)\| > 1$. \qed

5.4. A compact set. Fix $\varphi : G \to \mathbb{C}$ and a collection of test functions $\Psi$. For $F \subset G$ a finite lower set, let

$$\Phi_F = \{\Gamma \in M(F, \mathcal{B}^*)^+ : ([1] - \varphi \varphi^*)(x, y) = (\Gamma \ast ([1] - EE^*))((x, y) + (x, y)) \text{ for } x, y \in F\}.$$  

The set $\Phi_F$ is naturally identified with a subset of the product of $\mathcal{B}$ with itself $|F|^2$ times. 

**Lemma 5.6.** The set $\Phi_F$ is compact.

**Proof.** Let $\Gamma_\alpha$ be a net in $\Phi_F$. Arguing as in the proof of Theorem 5.1 we find each $\Gamma_\alpha(x, x)$ is a bounded net and thus each $\Gamma_\alpha(x, y)$ is also a bounded net. By weak-* compactness of the unit ball in $\mathcal{B}^*$ there exists a $\Gamma$ and subnet $\Gamma_\beta$ of $\Gamma_\alpha$ so that for each $x, y \in F$, the net $\Gamma_\beta(x, y)$ converges to $\Gamma(x, y)$. \qed

6. Proof of the realization theorem, Theorem 1.3

6.1. Proof of (i) implies (iiF). Suppose that (iiF) does not hold. In this case there exists a finite lower set $F \subset G$ so that the matrix

$$M_\varphi = \{([1] - \varphi \varphi^*)(x, y)\}_{x, y \in F}$$

is not in the cone

$$C_F = \{((\Gamma \ast ([1] - EE^*))_{x, y \in F} : \Gamma \in M(F, \mathcal{B}^*)^+\}.$$  

Lemma 5.3 produces a representation $\mu : P(F) \to B(\mathcal{H})$ so that $\|\mu(\psi)\| \leq 1$ for all $\psi \in \Psi|_F$, but $\|\mu(\pi_F(\varphi))\| > 1$. Lemma 3.4 now implies $\|\varphi\| > 1$. 

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6.2. Proof of (iiF) implies (iiG). The proof here uses Kurosh’s Theorem and in much the same way as in [6].

The hypothesis is that for every finite lower set \( F \subset G \), \( \Phi_F \), as defined in Subsection [5.4] is not empty. The result in that section is that \( \Phi_F \) is compact. For a finite lower set \( F \) contained in a lower set \( H \), define \( \pi^H_F : \Phi_H \to \Phi_F \) by

\[
\pi^H_F(\Gamma) = \Gamma|_{F \times F}.
\]

Thus, with \( \mathcal{F} \) equal to the collection of all finite lower subsets of \( G \) partially ordered by inclusion, the triple \((\Phi_G, \pi^G_F, \mathcal{F})\) is an inverse limit of nonempty compact spaces. Consequently, by Kurosh’s Theorem ([6], p. 30), for each \( F \in \mathcal{F} \) there is a \( \Gamma_F \in \Phi_F \) so that whenever \( F, H \in \mathcal{F} \) and \( F \subset H \),

\[
\pi^H_F(\Gamma_H) = \Gamma_F.
\]

Define \( \Gamma : G \times G \to \mathcal{B}^* \) by \( \Gamma(x, y) = \Gamma_F(x, y) \) where \( F \in \mathcal{F} \) is any lower set so that \( x, y \in F \). This is well defined by the relation in equation (6.1). If \( F \) is any finite lower set and \( f : F \to \mathcal{B} \) is any function, then

\[
\sum_{x, y \in F} \Gamma(x, y)(f(x)f(y)^*) = \sum_{x, y \in F} \Gamma_F(x, y)(f(x)f(y)^*) \geq 0
\]

since \( \Gamma_F \in M(F, \mathcal{B}^*)^+ \). Any finite subset of \( G \) is contained in a finite lower set, and so it follows that \( \Gamma \) is positive.

6.3. Proof of (iiG) implies (iii). Let \( \Gamma \) denote the positive kernel in (iiG). Apply Proposition 4.1 to find \( \mathcal{E}, L : G \to B(\mathcal{B}, \mathcal{E}) \), and \( \rho : \mathcal{B} \to B(\mathcal{E}) \) as in the conclusion of the proposition.

Rewrite condition (iiG) as

\[
[1](x, y) + (\Gamma \ast (EE^*))((x, y) = (\varphi \varphi^*)(x, y) + (\Gamma \ast [1])(x, y)
\]

\[
[1](x, y) + \sum_{pq=x} \sum_{rs=y} \Gamma(q, s)(E(p)E(r)^*) = \varphi(x)(\varphi(y)^* + \sum_{pq=x} \sum_{rs=y} \Gamma(q, s)(\delta(p)\delta(r)^*)
\]

\[
\delta(x)(\varphi^* + \sum_{pq=x} \sum_{rs=y} \langle L(q)E(p)1, L(s)E(r)1 \rangle = \varphi(x)(\varphi(y)^* + \sum_{pq=x} \sum_{rs=y} \langle L(q)\delta(p), L(s)\delta(r) \rangle
\]

\[
\delta(x)(\varphi(y)^* + \langle (\rho(E) \ast L)(x)1, (\rho(E) \ast L)(y)1 \rangle = \varphi(x)(\varphi(y)^* + \langle (\rho(\delta) \ast L)(x)1, (\rho(\delta) \ast L)(y)1, \rangle
\]

where 1 is the identity in \( \mathcal{B} \). We have used the intertwining relation between \( L \) and \( \rho \) from Proposition 4.1. Notice that in doing so the \( \ast \)-product is replaced by the \( \ast \)-product.

From here the remainder of the proof is the standard lurking isometry argument.

Let \( \mathcal{E}_d \) denote finite linear combinations of

\[
\left( \begin{array}{c}
(\rho(E) \ast L)(x)1 \\
\delta(x)
\end{array} \right) \in \mathcal{E}
\]

and let \( \mathcal{E}_r \) denote finite linear combinations of

\[
\left( \begin{array}{c}
(\rho(\delta) \ast L)(x)1 \\
\varphi(x)
\end{array} \right) \in \mathcal{E}
\]

\[24\]
Define $V : \mathcal{E}_d \to \mathcal{E}_r$ by

$$V \left( \frac{(\rho(E) \ast L)(x)1}{\delta(x)} \right) = \left( \frac{(\rho(\delta) \ast L)(x)1}{\varphi(x)} \right)$$

and extending by linearity. Equation (6.2) implies

$$\left\| \sum c_j \left( \frac{(\rho(E) \ast L)(x_j)1}{\delta(x_j)} \right) \right\|^2 = \sum_{j,\ell} c_j c^*_\ell \left( \langle (\rho(E) \ast L)(x_j)1, (\rho(E) \ast L)(x_\ell)1 \rangle \right)$$

which shows simultaneously that $V$ is well defined and an isometry. Thus $V$ (the lurking isometry) extends to an isometry from the closure of $\mathcal{E}_d$ to the closure of $\mathcal{E}_r$. There exists a Hilbert space $\mathcal{H}$ containing $\mathcal{E}$ and a unitary map

$$U : \oplus \to \oplus$$

so that $U$ restricted to $\mathcal{E}_d$ is $V$; i.e., $U\gamma = V\gamma$ for $\gamma \in \mathcal{E}_d$.

Write

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6.3}$$

with respect to the decomposition $\mathcal{H} \oplus \mathbb{C}$. In particular,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \left( \frac{(\rho(E) \ast L)(x)1}{\delta(x)} \right) = \left( \frac{(\rho(\delta) \ast L)(x)1}{\varphi(x)} \right)$$

which gives the system of equations

$$A(\rho(E) \ast L)(x)1 + B\delta(x) = (\rho(\delta) \ast L)(x)1 \quad \text{(6.4)}$$

From the first equation in (6.4) we have

$$L(x)1 = ((\rho(\delta) - A\rho(E))^{-1} \ast (B\delta))(x), \quad \text{(6.5)}$$

where the inverse is with respect to the $\ast$-product. Plugging this into the second equation of (6.4) gives

$$\varphi(x) = D\delta(x) + C(\rho(E) \ast (\rho(\delta) - A\rho(E))^{-1} \ast (B\delta))(x), \quad \text{(6.6)}$$

which, using the fact that $\rho$ is unital, can be written

$$\varphi(x) = D\delta(x) + C(\rho(E) \ast (\delta - A\rho(E))^{-1} \ast (B\delta))(x), \quad \text{(6.7)}$$

as desired.
6.4. Proof of (iii) implies (i). Suppose \( \varphi = W_\Sigma \) as in equation \( \ref{eq:14} \) (equivalently, equation \( \ref{eq:6.7} \) above). We want to show that \( k - \varphi \star k \star \varphi^* \geq 0 \) for all \( k \in \mathcal{K}_W \). First, factor \( k(x, y) = k_y k_x^* \).

To make the notation consistent with that used in the calculations below, we write \( k(x) \) for \( k_x \). We compute the (four) terms in \( (k - \varphi \star k \star \varphi^*)(x, y) \), using the identities implied by \( U \) being unitary and the equality \( (\delta \star k)(k^* \star \delta) = \delta \star k \star \delta = k \). Recall that for functions \( f \) and \( g \), \( (f \star g)^* = g^* \star f^* \).

To begin with, \( CC^* = 1 - DD^* \) and so we have \( Dk(x, y)D^* = k(x, y) - CK(x, y)C^* \). Hence

\[
\begin{align*}
\text{the second term we consider is} \quad (C\delta - (\rho(E)A)^{-1}\delta) * (k^* \star (\rho(E)A)^* \star (\rho(E)A)^* \star (\rho(E)A)^* - 1 \mathcal{C}^*) (y).
\end{align*}
\]

For the next few terms it is useful to observe that

\[
\begin{align*}
\rho(E) * (\delta - Ap(E))^{-1} * \rho(E) = (\delta - Ap(E))^{-1} * \rho(E),
\end{align*}
\]

or equivalently,

\[
(\delta - Ap(E))^{-1} * \rho(E) * (\delta - Ap(E)) = \rho(E),
\]

which follows from

\[
\begin{align*}
\rho(E) * (\delta - Ap(E)) = \rho(E) * \delta - \rho(E) \star Ap(E) = (\delta - Ap(E)) * \rho(E).
\end{align*}
\]

The second term we consider is

\[
(\rho(E) * (\delta - Ap(E))^{-1} * (B\delta) \star k) (x)(k^* \star (\rho(E)A)^* \star (\rho(E)A)^* \star (\rho(E)A)^* - 1 \mathcal{C}^*) (y).
\]

For the third term,

\[
(Dk)(x) (k^* \star (\rho(E)A)^* \star (\delta - Ap(E))^{-1}) \star (B\delta) \star k) (y)^*
\]

Finally, the last term is

\[
(C\rho(E) * (\delta - Ap(E))^{-1} \star (B\delta) \star k) (x) (C\rho(E) * (\delta - Ap(E))^{-1} \star (B\delta) \star k) (y)^*
\]
Putting them together we have
\[
k - \varphi \ast k \ast \varphi^* = (C(\delta - \rho(E)A)^{-1} \ast \rho(E)^*) \ast k \ast (\delta - \rho(E)A)^{-1} \ast C^*
\]
which after a bit of algebra is seen to simplify to
\[
k - \varphi \ast k \ast \varphi^* = C(\delta - \rho(E)A)^{-1} \ast (k - \rho(E) \ast k \ast \rho(E)^*) \ast (\delta - \rho(E)A)^{-1} \ast C^*.
\]
Given a finite lower set \(F \subset G\), the matrix
\[
P = \left((k - E \ast k \ast E^*)(x, y)\right)_{x, y \in F} \in M(F, \mathbb{B})
\]
is a positive since its value at \(\psi \in \Psi\) is
\[
P(\psi) = \left((k - \psi \ast k \ast \psi^*)(x, y)\right).
\]
Consequently \(k - \varphi \ast k \ast \varphi^* \geq 0\) over \(F\), which completes the proof.

7. AGLER-JURY-PICK INTERPOLATION

We now turn to the proof of Theorem 1.4. Condition (i) implies condition (ii) simply by the definition of the norm on \(H^\infty(K_\Psi)\).

If condition (iii) does not hold, then an argument just as in the proof of Theorem 1.3 produces a kernel \(k \in K_\Psi\) so that the relevant kernel on \(F\) is not positive. Hence (ii) implies (iii).

To prove that (iii) implies (i), first argue along the lines of the proof of (iiG) implies (iii) in Theorem 1.3, but work with the finite set \(F\) in place of \(G\). Next verify that the transfer function \(W_\Sigma\) so constructed and defined on all of \(G\) satisfies \(W_\Sigma(x) = f(x)\) for \(x \in F\) (since we worked with \(F\)). The implication (iii) implies (i) in Theorem 1.3 now says that \(\|W_\Sigma\| \leq 1\).

This leads to the following, which is reminiscent of results on left tangential Nevanlinna-Pick interpolation.

**Theorem 7.1.** Let \(F\) be a finite lower set in a semigroupoid \(G\). Suppose \(w(a), z(a) \in \mathbb{C}, a \in F\) are given. Then there is a function \(\varphi \in H^\infty(K_\Psi)\) with \(\|\varphi\|_{H^\infty(K_\Psi)} \leq 1\) such that
\[
(\varphi \ast z)(a) = w(a), \quad \text{for all } a \in F,
\]
if and only if
\[
(z^*z - w^*w) \ast k \geq 0, \quad \text{for all } k \in K_\Psi.
\]

**Proof.** If \(w = \varphi \ast z\) with \(\varphi \in H^\infty(K_\Psi)\) and \(\|\varphi\|_{H^\infty(K_\Psi)} \leq 1\), then for all \(k \in K_\Psi\),
\[
(z^*z - w^*w) \ast k = (z^* \ast (1 - \varphi^* \varphi) \ast z) \ast k
= ([1 - \varphi^* \varphi] \ast (z^*z) \ast k
= ([1 - \varphi^* \varphi] \ast (z^* \ast k \ast z)
\geq 0,
\]
since by Lemma 5.4, \(z^* \ast k \ast z \in K_\Psi\).
Now suppose \((zz^*-ww^*) \ast k \geq 0\) for all \(k \in K_\Psi\). Begin by assuming that \(z\) is an invertible function with respect to the \(\ast\) product (i.e., \(z(a)\) is invertible for all \(a \in F_e\)). Set \(f = w \ast z^{-1}\) on \(F\). Then restricting to \(F\),
\[
0 \leq (z^*z - w^*w) \ast k = (z^* \hat{k} \ ([1] - f^*f) \ast z) \ast k = ([1] - f^*f) \ast (z^* \ast k \ast \hat{k})
\]
for all \(k \in K_\Psi\). Again by Lemma 5.4, \(z^{-1} \ast k \ast z^{-1}\) is in \(K_\Psi\) if \(k \in K_\Psi\). Hence \(([1] - f^*f) \ast k \geq 0\) on \(F \times F\) for all \(k \in K_\Psi\), and so by Theorem 1.4 \(f\) extends to \(\varphi \in H^\infty(K_\Psi)\) with \(\|\varphi\|_{H^\infty(K_\Psi)} \leq 1\) such that \(([1] - \varphi^*\varphi) \ast k \geq 0\) on \(G \times G\). Pad \(z\) with zeros to make it a function in \(H^2(K)\) for all \(k \in K_\Psi\), and set \(w = \varphi \ast z\) (which agrees with the original definition of \(w\) on the lower set \(F\)).

If \(z\) is not \(\ast\)-invertible, then \(z(a) = 0\) for some \(a \in F_e\). This means that \(\{z(a) : a \in F\}\) (where \(z(a)\) is identified with the vector with this value in the \(a\)th position and zero elsewhere) is not a basis for \(C^F\). Choose a vector \(g\) with \(g(a) = 1\) for each \(a \in F_e\) where \(z(a) = 0\) and zero otherwise. Fix \(\epsilon > 0\). Let \(z' = g\) normalized so that \(\text{Re}(z, z') \geq 0\) and \(\|z'\| < \epsilon\). Then for \(z_\epsilon = z + z'\),
\[
(z_\epsilon^*z_\epsilon - w^*w) \ast k \geq (z^*z + z'^*z' - w^*w) \ast k \geq (z^*z - w^*w) \ast k \geq 0,
\]
and \(z_\epsilon\) is invertible, so we obtain by the last paragraph a corresponding \(f_\epsilon\) for which \(([1] - f_\epsilon^*f_\epsilon) \ast k \geq 0\) for all \(k \in K_\Psi\) and \(f_\epsilon \ast z_\epsilon = w\). Since we are on a finite dimensional space, the sequence \(\{f_1/n\}_{n=1,2,\ldots}\) converges to some \(f \in P(F)\) with \(([1] - f^*f) \ast k \geq 0\) for all \(k \in K_\Psi\). Also \(z_1/n \longrightarrow z\). Consequently, \(f \ast z = w\).

A right tangential problem could very easily be formulated and solved. One way to do this would be to replace \(\ast\) with \(\hat{}\) at appropriate points in the left interpolation theorem and proof, and then take adjoints. The details are left to the interested reader.

Finally note that taking \(z = \delta_F\) and \(w = f\) in the last theorem recovers the first two equivalences in Theorem 1.4.

8. Examples

8.1. The classical examples. View \(\mathbb{D}\) as a Pick semigroupoid. The partial multiplication is trivial and so each \(z \in \mathbb{D}\) is idempotent. Take \(\Psi = \{z\}\) (\(z\) meaning here the identity function) as the collection of test functions. The Agler-Jury-Pick interpolation theorem in this case is Pick interpolation.

Choose \(G = \mathbb{N}\) with the usual semigroup(oid) structure. Let \(\Psi = \{z\}\), where by \(z\) we mean the function \(z : \mathbb{N} \rightarrow \mathbb{C}\) given by \(z(j) = 0 \text{ if } j \neq 1\) and \(z(1) = 1\) (we think of \(z(j)\) as the derivatives of \(z\) at 0). In this case Agler-Jury-Pick interpolation is Carathéodory-Fejér interpolation.

For mixed Agler-Pick and Carathéodory-Fejér choose \(G = \mathbb{D} \times \mathbb{N}\) with the semigroupoid structure,
\[
(z, n)(w, m) = \begin{cases} 
\text{is not defined} & \text{if } z \neq w \\
(z, n + m) & \text{if } z = w
\end{cases}
\]
and let \(\Psi = \{z\}\) denote the function \(z(w, 0) = w, z(w, 1) = 1\) and \(z(w, m) = 0\) for \(m \geq 2\).

8.2. Agler-Pick interpolation on an annulus. Let \(A\) denote an annulus \(\{q < |z| < 1\}\), viewed as a Pick semigroupoid.

There is a family of analytic functions \(\psi : A \rightarrow \mathbb{D}\) which are unimodular on the boundary of \(A\) and have precisely two zeros in \(A\) (counting with multiplicity), normalized by \(\psi(\sqrt{q}) = 0\) and \(\psi(1) = 1\). If \(\varphi\) is any other analytic function on \(A\) which is unimodular on the boundary and has exactly two zeros (counting with multiplicity), then there is a Mőbius map \(m\) from the disk onto the disk such that \(m \circ \varphi \in \Psi\). There is a canonical parameterization of \(\Psi\) by the unit circle.
Theorem 8.1. The collection $\Psi$ is a family of test functions for $\mathcal{A}$ and the norm in $H^\infty(\mathcal{K}_\Psi)$ is the same as the norm on $H^\infty(\mathcal{A})$. Moreover, no proper subset of $\Psi$ is a set of test functions which gives the norm of $H^\infty(\mathcal{A})$.

In the case of Agler-Pick interpolation (on a finite set $F \subset \mathcal{A}$), the realization formula for a solution is in terms of a single positive measure on the unit circle.

Look for the details of this example in the forthcoming paper [22].

8.3. Carathéodory interpolation kernels. Let $\mathbb{N}$ denote the natural numbers with the usual semi-group(oid) structure. A kernel $k$ on $\mathbb{N}$ is a Carathéodory interpolation kernel [33] provided (by way of normalization) $k(0, 0) = 1, k(0, n) = 0$ for $n > 0$, and

$$b = [1] - k^{-1}$$

is positive.

For illustrative purposes, suppose $b$ has finite rank $d$ and so factors as $b = B^*B$, where $B : \mathbb{N} \to (\mathbb{C}^d)^*$. Although $B$ is not scalar-valued, $[1] - B(a)B(b)^* = [1] - b$ is scalar and moreover,

$$([1] - BB^*) * k = ([1] - b) * k = k^{-1} * k = [1] \geq 0.$$

Choosing $\Psi = \{B\}$, it turns out that $\varphi \in H^\infty(\mathcal{K}_\Psi)$ and $\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} \leq 1$ if and only if $([1] - \varphi\varphi^*) * k$ is positive.

8.4. NP kernels and Arveson-Arias-Popescu space. The situation for Nevanlinna-Pick (NP) kernels is similar to that for Carathéodory kernels. In particular, it requires a version of our results for vector valued test functions.

As a particular example, consider the semigroup $\mathbb{N}^g$ with the (single) vector valued test function $Z = (z_1, z_2, \ldots, z_g)^T$. This pair $(\mathbb{N}^g, Z)$ gives rise to symmetric Fock space; i.e., the space of multipliers of the space of analytic functions on the unit ball in $\mathbb{C}^g$ with reproducing kernel $k(z, w) = (1 - \langle z, w \rangle)^{-1}$ studied by Arveson ([12, 13], in the commutative case) and by Arias and Popescu ([10, 11], in both the commutative case and the noncommutative case discussed in the next subsection).

8.5. Noncommutative Toeplitz algebras. The following have been considered in the context of Nevanlinna-Pick and Carathéodory-Fejér interpolation by Davidson and Pitts [20] and Arias and Popescu [11], as well as by Popescu in [45, 46].

Let $\mathfrak{F} = \mathfrak{F}_g$ denote the free monoid on the $g$ letters $\{x_1, \ldots, x_g\}$. Let $\psi_j : \mathfrak{F} \to \mathbb{C}$ denote the function $\psi_j(x_j) = 1$ and $\psi(w) = 0$ if $w$ is any word other than $x_j$. The matrix $\Sigma(\psi_j)$ is a (truncated) shift on Fock space.

Given a word $w = x_{j_1}x_{j_2}\cdots x_{j_n}$, let

$$\psi^{w^*} = \psi_{j_1} * \psi_{j_2} * \cdots * \psi_{j_n}.$$

Since $\psi^{w^*}(v) = 1$ if $w = v$ and 0 otherwise, it follows that if $F$ is any finite subset of $\mathfrak{F}$, then $P(F)$ contains all functions on $F$.

Let $\psi = (\psi_1, \cdots, \psi_g)^T$ and consider $\psi$ as a (single) test function. We calculate

$$s(x, y) = ([1] - \psi^*\psi)^{-1} *$$

where $s$ is the Toeplitz kernel ($s(x, y) = 1$ of $x = y$ and $s(x, y) = 0$ if $x \neq y$). Then if $k$ is any kernel for which $([1] - \psi^*\psi) * k = Q$ is positive, we have $s * Q = k$. It follows that $\|\varphi\| \leq 1$ if and only if the kernel

$$\mathfrak{F} \times \mathfrak{F} \ni (x, y) \mapsto ([1] - \varphi\varphi^*) * s(x, y)$$
is positive, if and only if \( \| \Sigma(\varphi) \| \leq 1 \) The versions of the Nevanlinna-Pick theorem considered in the papers cited above coincide with Theorem 7.1 while Carathéodory-Fejér interpolation is given by Theorem 1.4.

As a final remark, note that each \( T(\psi_j) \) is an isometry and

\[
\sum T(\varphi) T(\varphi)^* = P_0 \geq 0.
\]

Here \( P_0 \) is the projection onto the span of the vacuum vector \( \emptyset \) in the Fock space.

8.6. The Polydisk. The semigroupoid \( \mathbb{N}^g \) (the \( g \)-fold product of the nonnegative integers) with the set of test functions \( z_j \), the characteristic function of \( e_j \) the vector with 1 in the \( j \)-th entry and 0 elsewhere, gives rise to the Schur-Agler class of the polydisk \( D^g \) returning us to the introduction and [3].

8.7. Semigroupoid algebras of Power and Kribs. Kribs and Power [29, 30] consider a generalization of the noncommutative Toeplitz algebras which they term a free semigroupoid algebra. Order arises from the assumption of freeness, the resulting semigroupoid is cancellative, and there is a representation (related to our Toeplitz representation on characteristic functions \( \chi_a \)) in terms of partial isometries and projections.

A notion of a generalized Fock space is developed, which is simply the Hilbert space with orthonormal basis labelled by the elements of the semigroupoid. The algebras of interest in these papers are obtained from the weak operator topology closure of the algebras generated by the left regular representations (i.e., the projections and partial isometries mentioned above).

The algebras are closely related to those in the present paper when \( G \) is a semigroupoid in this more restrictive sense and the collection of test functions consists of the characteristic functions of non-idempotent elements from the first stratum (to use our terminology).

It is assumed that for every idempotent \( e \in G \), there is a non-idempotent \( a \) such that \( ae \) is defined. Let \( G_1 \) be the first (left) stratum in \( G \), and assume that this set is countable. Then \( G \) is generated by \( G_1 \), in the sense that if \( x \) is in the \( n \)th stratum, then \( x = ay \), where \( y \) is in the \((n-1)\)st stratum and \( a \) is in the first stratum. Let \( P \) have the property that

\[
P(x, y) = \begin{cases} 1 & x = y \text{ and } x, y \notin G_e, \\ 0 & \text{otherwise}, \end{cases}
\]

and \( \tilde{s} = [1] + P \). Clearly \( \tilde{s} \) is invertible. Now mimic the proof in Section 8.5 by letting \( \psi_j(x_j) = 1 \) if \( x_j \in G_1 \). It is not difficult to verify that for \( \psi = (\psi_1 \ldots \psi_g)^T \),

\[
\tilde{s}(x, y) = ([1] - \psi^* \psi)^{-1*},
\]

and so just as in that subsection, if \( k \) is any kernel for which \( ([1] - \psi^* \psi) \ast k = Q \) is positive, \( \tilde{s} \ast Q = k \). It follows that the statements \( \| \varphi \| \leq 1 \),

\[
\tilde{S} \times \tilde{S} \ni (x, y) \mapsto ([1] - \varphi \varphi^*) \ast \tilde{s}(x, y)
\]

positive, and \( \| \Sigma(\varphi) \| \leq 1 \) are all equivalent.

A number of interesting algebras can be generated in this manner, including the noncommutative Toeplitz algebras above and the norm closed semicrossed product \( \mathbb{C}^n \times_\beta \mathbb{Z}_+ \) [30]. Indeed, the condition of being freely generated can be replaced by our more general conditions for a semigroupoid (again assuming though that for every idempotent \( e \in G \), there is a non-idempotent \( a \) such that \( ae \) is defined). Our results allow for interpolation in all of these algebras.
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