The Geodesic Light-Cone Coordinates, an Adapted System for Light-Signal-Based Cosmology

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Most of cosmological observables are light-propagated. I will present coordinates adapted to the propagation of null-like signals as observed by a geodesic observer. These “geodesic light-cone (GLC) coordinates” are general, adapted to calculations in inhomogeneous geometries, and their properties make them useful for a large spectrum of applications, from the estimation of the distance-redshift relation, the average on our past light cone, the effect of the large-scale structure on the Hubble diagram, to weak lensing calculations.

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1. Motivations

The more observational precision increases, the more inhomogeneous our Universe looks. Supernova\(^1\) lensing\(^2-4\), and other light-propagated observables will soon encounter associated complications\(^5-13\). We present coordinates first developed to simplify averages of scalars on our past light-cone\(^14\), next used to estimate the effect of inhomogeneities on luminosity distance\(^15-18\), Hubble diagram\(^19,20\), and recently applied to lensing quantities\(^21\) (and illustrated in a Lemaître-Tolman-Bondi model), following this historical order.

2. The geodesic light-cone coordinates

We define a light-cone adapted metric (close to “observational coordinates”\(^22,23\), but different\(^24\)) composed of 6 arbitrary functions (\(\Upsilon, U^a, \gamma_{ab}\)) and totally gauged fixed:

\[
d^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw) .
\]  

This metric uses a null coordinate \(w\) defining past light cones, the proper time of a geodesic observer \(\tau\), and angles \(\tilde{\theta}\) that photons keep along their path orthogonal to a 2-spheres \(\Sigma(w, \tau)\) of constant time in our past light-cone (see Fig. 1). In the FLRW limit: \(w = \eta + r\) (conformal time, radius), \(\tau = t\) (cosmic time), \((\tilde{\theta}^1, \tilde{\theta}^2) = (\theta, \phi)\), \(\Upsilon = a(t)\), \(U^a = 0\), \(\gamma_{ab} = a^2 r^2 (1, \sin^2 \theta)\). In general\(^23\) \(\Upsilon\) is like an inhomogeneous scale factor (lapse function), \(U^a\) like a shift-vector and \(\gamma_{ab}\) is the metric inside \(\Sigma(w, \tau)\). We can notice two
3. Simplification of light-cone averages

The light-cone average (3) of a scalar $S$ (e.g. $d_L \cdot d_L^{-2}$) is in general given by $(S) (V_0, A_0) = I(S; V_0, A_0)/I(1; V_0, A_0)$ and we define the average integral to be gauge invariant, invariant under $(A, V) \rightarrow (\tilde{A}(A), V(V))$ and general coordinate transformations:

$$I(S; V_0, A_0) = \int_{A_0} d^4x \sqrt{-g} D(V_0 - V) D(A - A_0) N(V, A, \partial_\mu) S(x) \ ,$$

where $N(V, A, \partial_\mu)$ is a normalization, $D(X) = \delta_D(X)$ or $\Theta(X)$ (Heaviside function):

| Average $\delta_D$ | Illustration | $(S)_{V_0, A_0}$ | $(S)_{V, A_0}$ | $(S)_{V_0, A_0}$ |
|-------------------|-------------|-----------------|---------------|-----------------|
| $N(V, A, \partial_\mu)$ | $\sqrt{-\partial_\mu A \partial^\mu A}$ | $[\partial_\mu V \partial^\mu A]$ | $\sqrt{-\partial_\mu A \partial^\mu A}$ | $[\partial_\mu V \partial^\mu A]$ |

Among these 3 types of averages, $(S)_{V_0, A_0}$ is closer to physical observables as it averages over the deformed 2-sphere embedded in the light-cone $V = V_0$ and a spatial hypersurface $A = A_0$. In GLC coordinates (where $V \rightarrow w$, $A \rightarrow \tau$) we can simplify the average and use Eq. (2) to get (with $\tau_z \equiv \tau(z_s, w_0, \theta^a)$):

$$(S)_{w_0, z_s} = \left( \int d^2\tilde{\theta} \sqrt{\gamma(w_0, \tau_z, \tilde{\theta}^3)} S(w_0, \tau_z, \tilde{\theta}^3) \right) / \left( \int d^2\tilde{\theta} \sqrt{\gamma(w_0, \tau_z, \tilde{\theta}^3)} \right) \ ,$$

allowing us to average scalars on the sky, at a certain redshift.

4. Distance-redshift relation at $O(2)$

The GLC metric enables the computation of $d_L(z)$ at $O(2)$ in the Newtonian gauge (NG):

$$ds_{NG}^2 = a^2(\eta) \left[ -(1 + 2\Phi) d\eta^2 + (1 - 2\Psi)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) \right]$$

with $\Phi = \psi + \frac{1}{2} \phi^{(2)}$, $\Psi = \psi + \frac{1}{2} \psi^{(2)}$ the gauge invariant Bardeen potentials (no matter shear at $O(1)$, see Ref. [22]). Establishing the full transformation between GLC and NG coordinates at $O(2)$ in (scalar) perturbations, we get $(\tau, w, \theta^3, \tilde{\theta}^2) = \text{func}(\eta, r, \theta, \phi)$ and $(\Upsilon, U^\alpha, \gamma^{ab}) = \text{func}(\psi, \psi^{(2)}, \phi^{(2)})$ which allows us to compute $d_L$ to $O(2)$ (using Eq. (3)):

$$d_L(z_s, \theta^a) = d_L^{FLRW}(z_s) \left( 1 + \delta_S^{(1)}(z_s, \theta^a) + \delta_S^{(2)}(z_s, \theta^a) \right) \ .$$

The first order in scalar perturbations is given by:

$$\delta_S^{(1)}(z_s, \theta^a) = \left( 1 - \frac{1}{H_s \Delta \eta} \right) J - \frac{Q_s}{\Delta \eta} - \psi_{(1)}^{(s)} - \text{Lensing}^{(1)} \ .$$
with $\mathcal{H}_s = a'({\eta}_s)/a({\eta}_s)$, $\Delta \eta = \eta_0 - \eta_s^{(0)}$, containing (Integrated) Sachs-Wolfe ([I]SW), Doppler, and lensing (convergence) effects:

$$J = \left( [\partial_\eta Q]_s - [\partial_\eta P]_s \right) - \left( [\partial_\eta P]_s^{(0)} - [\partial_\eta P]_s^{(0)} \right) \sim \text{SW} + \text{ISW} + \text{Doppler}, \quad (9)$$

$$\text{Lensing}^{(1)} = \frac{1}{2} \nabla_\eta \tilde{\theta}^{(1)}(\eta) = \int_{\eta_0}^{\eta_0} d\eta_0 \frac{\eta_0 - \eta_{s_s}^{(0)}}{\eta_0 - \eta} \Delta_2 \psi(\eta, \eta_0 - \eta, \theta^a),$$

where ‘$o$’ (‘$s$’) denote quantities evaluated at the observer (source), and we defined:

$$Q(\eta, \eta_0, \theta^a) = \int_{\eta_0}^{\eta} dx \psi(\eta, x, \theta^a). \quad P(\eta, r, \theta^a) = \int_{\eta_0}^{\eta} d\eta_0 \frac{\eta_0}{a(\eta_0)} \psi(\eta, r, \theta^a). \quad (10)$$

Similarly obtained $\mathcal{O}(2)$ corrections contain (see Refs. [23, 24]):

- Two dominant terms: (Doppler)$^2$, (Lensing)$^2$,
- Combinations of $\mathcal{O}(1)$-terms: $\psi^2$, ([I]SW)$^2$, [I]SW × Doppler, ($\psi_s, Q_s$) × (Lensing, [I]SW, Doppler),
- Genuine $\mathcal{O}(2)$-terms: $\psi_s^2$, $Q_s^2$. $\text{Lensing}^{(2)} = \frac{1}{2} \nabla_\eta \tilde{\theta}^{(2)}$,
- New integrated effects: $\frac{1}{2 \tau_{\eta_0}} \int_{\eta_0}^{\eta_0} d\eta_0 \int_{\eta_0}^{\eta_0} dx \left[ 4 \psi \partial_\eta Q \gamma^2 \partial_\eta Q \right] (\eta_0^0 + x, \theta^a),$
- Angle deformations: $(\gamma_0)_{ab} \partial_+ \tilde{\theta}^{(1)} (\eta), \gamma_0 \tilde{\theta}^{(2)} (\eta), \partial_+ \partial_{+} \tilde{\theta}^{(2)} (\eta),$
- Redshift perturbations from Eq. (2), involving transverse peculiar velocities: $\gamma^2 \partial_\eta P \partial_\eta Q \partial_\eta Q \partial_\eta Q \partial_\eta P \partial_\eta P,$
- Other important terms: Lens-Lens coupling, corrections to Born approximation.

These $\mathcal{O}(2)$ results were confirmed recently [25], working directly in terms of GLC coordinates rather than NG. An independent derivation [26] lead to similar results, but a rigorous comparison with Ref. [18] is still lacking. Results were also given for vector/tensor perturbations [27] (Poisson gauge) and extended to the case with $\mathcal{O}(1)$ anisotropic stress [28, 29].

5. Effects of large-scale structure on the Hubble diagram

In Sec. 4 we expressed $d_L$ in terms of $(\psi, \psi^2), \phi^2)$, hence it needs a description of the Bardeen potentials at $\mathcal{O}(1, 2)$. We decompose the first order gravitational potential $\psi$ in Fourier modes and denote by $(\ldots)$ the ensemble (or stochastic) average over perturbations. The $\mathcal{O}(2)$ potentials can be related to $\psi$ by $\xi_{\psi}, \psi, \phi^2 \propto \nabla^2 (Q \partial_\psi \bar{\psi}) + \partial_\eta \partial_\psi \bar{\psi}.$

Hence we can combine light-cone and stochastic averages in order to study the effect of statistical perturbations on the whole sky. For example, the (trivial) average of $\psi^2$ gives:

$$\langle \psi^2 \rangle = \int_0^{\infty} \frac{dk}{2\pi^2} P_{\psi}(k) \langle \psi_k^2 \rangle = \left( 3/5 \right)^2 A(k, k_0)^{\eta_s - 1} T(k)^2 \psi^2,$$

is the power spectrum describing perturbations. At linear order, with $\Lambda, n_s, k_0$ taken from WMAP, $T(k)$ is a transfer function including a baryonic component (Silk damping), and $g(z)$ is the growth factor describing the recent time evolution of perturbations. In CDM we get exactly the spectral coefficients coming from each correction of $d_L (z, \theta^a)$ described in Sec. 4. $\langle d_L^2 \rangle = \int_0^{\infty} \frac{dk}{2\pi^2} P_{\psi}(k) C(k, \Delta \eta).$ We do the same in $\Lambda$CDM, with reasonable assumptions to simplify integrations [29] and also with a non-linear power spectrum [30].

Like $d_L$, one can also average the flux $\Phi = L/(4\pi \ell P)$ $\sim \Phi_0 + \Phi_1 + \Phi_2$. We get:

- $\langle d_L^2 \rangle \equiv \langle d_L^{FLRW} \rangle^{-2} \left[ 1 + f_\Phi(z) \right]$ where $f_\Phi(z) \simeq f(z) \int_0^\infty \frac{dk}{2\pi^2} \left( \frac{k}{\eta_s} \right)^2 P_{\psi}(k)$. 


\[ \langle d_L(z) \rangle = d_L^{FLRW} [1 + f_d(z)] \] with \( f_d \) = \(-\frac{1}{2} f_{\Phi} + \frac{3}{8} \langle (\Phi_1/\Phi_0)^2 \rangle \).

Corrections to \( d_L \) involve a flux variance dominated by peculiar velocity and lensing:
\[ \langle (\Phi_1/\Phi_0)^2 \rangle \approx \langle \left( \frac{1 - H_s \Delta \eta}{2} \right)^2 \rangle + \langle \text{Lensing}(z) \rangle^2 \]

as shown in Fig. 2 for realistic (non-)linear power spectra. It turns out that the luminosity flux is minimally affected by lensing w.r.t. other scalar observables at large redshift. This calculation can be seen as a check at \( O(2) \) of Weinberg’s argument of flux conservation and has also been confirmed by recent papers through different approaches.

![Fractional Corrections](image)

\[ \langle \mu \rangle = \mu^{FLRW} - 1.25(\log_{10} e) \left[ f_{\Phi} - \langle (\Phi_1/\Phi_0)^2 \rangle \right], \quad \sigma_{\mu} = 2.5(\log_{10} e) \sqrt{\langle (\Phi_1/\Phi_0)^2 \rangle} \] (11)

Compared to the Union 2 data and using a non-linear power spectrum in \( \Lambda \)CDM (Fig. 3, Left), we find that peculiar velocities explain well the scatter at small \( z \) and that lensing explains only part of the scatter at large \( z \). Finally, we can compare our dispersion on the Hubble diagram with the experimental estimations coming from lensing (Fig. 3, Right). We find that the total effect is well fitted by Doppler \( (z \leq 0.2) \) + lensing \( (z > 0.3) \) effects and that the lensing prediction is in great agreement with experiments so far.

![Theoretical dispersion](image)

\[ \phi_0^{\text{new}} = (0.055 \pm 0.002) \]
\[ \phi_0^{\text{old}} = (0.05 \pm 0.002) \]
6. Jacobi map and weak lensing

We now consider lensing motivated by Sec. 3 and recent work on galaxy number counts in GLC. The relative separation of two neighbour light rays simultaneously emitted from a source S and converging to an observer O follows the geodesic deviation eq. (GDE) :

\[ \nabla^2 \Delta^\lambda = R^\mu_{\alpha\beta\nu} k^\alpha k^\nu \xi^\mu \text{ with } \nabla_\lambda \equiv D/\partial \lambda \equiv k^\mu \nabla_\mu, \text{ } k^\lambda \text{ the photon momentum, } \lambda \text{ an affine parameter along the photon path, and } \xi^\mu \text{ an orthogonal displacement w.r.t. to the rays.} \]

We project the GDE on the Sachs basis \( \{ s_{\alpha}^\mu \} \) (two zweibeins with flat index \( A = 1, 2 \) :

\[ g_{\mu\nu}s_{\alpha}^\mu s_{\beta}^\nu = \delta_{AB}, \quad s_{\alpha}^\mu u_\mu = 0, \quad s_{\alpha}^\mu k_\mu = 0, \quad \Pi^\lambda_{\alpha} \nabla_\lambda s_{\beta}^\mu = 0; \text{ with } u_\mu \text{ the peculiar velocity of comoving fluid (\( \vec{S}, \text{ O comoving too}, \)) \text{ and } \Pi^\lambda_{\alpha} \text{ a "screen" projector orthogonal to } u_\mu \text{ and } u_\mu + (u^k k_0)^{-1} k_\mu. \]

We define the Jacobi map \( J_B^A \), from the observed sky angle \( \bar{\theta}_b^A \) to \( \xi^\mu \xi^\nu, \), by \( \xi^A(\lambda) = J_B^A(\lambda, \bar{\theta}_b^A). \) Projected quantities \( \xi^A \) and \( R_B^A \equiv R_{\alpha\beta\nu\mu} k^\alpha k^\nu s_{B}^\beta s_{A}^\mu \) (optical tidal matrix) bring us the Jacobi equation (see e.g. Refs. [12, 13]):

\[ \frac{d^2}{d\xi^2} J_B^A(\lambda, \bar{\theta}_b^A) = R_C^A(\lambda) J_C^B(\lambda, \bar{\theta}_b^A), \]

with I.C. : \( J_B^A(\lambda, 0) = 0, \quad \frac{d}{d\xi} J_B^A(\lambda, \bar{\theta}_b^A) = (k^\mu u_\mu)_{\bar{\theta}_b}, \)

(12) A direct resolution of Eq. (12) gives the angular distance of the source \( d_A(\lambda) \equiv \sqrt{ds_\lambda/d^2\xi}, \text{ (convergence), } \bar{\omega} = |J_B^A - J_B^A|/2d_A \) (vorticity), \( |\gamma| \equiv \sqrt{(|\gamma_1|^2 + |\gamma_2|^2) = \sqrt{(1 - \kappa)^2 + \bar{\omega}^2 - \mu^{-1}} \text{ (shear) and } \mu \equiv 1/(\det A) \equiv \bar{d}_A^2/\det J_B^A \) (magnification).

Let us now turn to the GLC coordinates and express these lensing quantities in it. First, the zweibeins are written as \( s_B^\mu = (s_{1}^\mu, 0, s_{2}^\mu) \) and \( k^\mu \equiv \Omega^{-1} \bar{\gamma}_b^\mu \) (with \( \omega \) a pure constant). Second, the solution to Eqs. (12 and 13) is :

\[ J_B^A(\lambda, \bar{\theta}_b^A) = s_A^\mu(\lambda) \left[ 2u_\tau(\gamma_{ab})^{-1} \right]_a s_B^\mu(\lambda) \]

(15) where \( \left( \ldots \right) \equiv \partial_\tau(\ldots) \). The angular distance and the magnification become:

\[ d_A = 2u_\tau(\gamma_{ab})^{1/4}/\sqrt{\det \gamma_{ab}}, \quad \mu = (\bar{d}_A/d_A)^2 = \Phi/\bar{\Phi}, \]

(16) involving \( \bar{d}_A = d_A^2(\tau)r^2 \) with \( r = w - \int d\tau/a(\tau) \text{ measured from the observer and } \Phi (\bar{\Phi} \text{ the flux in the in(homogeneous) geometry). Expressions for the zweibeins can be obtained in the GLC coordinates[21], but it is more convenient to compute the squared lensing quantities, combined with \( s_A^\mu s_B^\mu = \gamma_{ab} \) and \( \epsilon_{AB} s_A^\mu s_B^\mu = \sqrt{\epsilon} \epsilon_{ab} (\epsilon \text{ the anti-sym. symbol}), \) to get:

\[ \left( 1 - \kappa + \bar{\omega}^2 \right) = \left( \frac{u_\tau}{d_A} \right)^2 \left[ \frac{\gamma_{ab} \bar{\gamma}_b^\mu \bar{\gamma}_d^\mu}{\det \gamma_{ab}} \right]_{b} \gamma_{ad} \pm \frac{2 \sqrt{\tau} \gamma_{ab}}{\det \gamma_{ab}} \right]_{a}, \]

We thus have general lensing quantities expressed with only 3 metric functions (\( \gamma_{ab} \)) showing the great advantage of working in GLC coordinates.
The Jacobi Eq. (12) can be rewritten as a first order differential equation for the so-called deformation matrix:

\[ \dot{S}_B^A = \frac{dJ_A^C}{d\lambda} (J^{-1})^C_B = \dot{\theta} \delta_B^A + \begin{pmatrix} \dot{\sigma}_1 & \dot{\sigma}_2 \\ \dot{\sigma}_2 & \dot{\sigma}_1 \end{pmatrix} , \]

(18)

involving the optical scalars, \( \dot{\theta} \) (expansion scalar) and \( \dot{\sigma} \equiv \dot{\sigma}_1 + i\dot{\sigma}_2 \) (shear scalar), and known as the Sachs equations:

\[ \frac{d\dot{\theta}}{d\lambda} + |\dot{\sigma}|^2 + \dot{\theta}^2 = \Phi_{00} , \quad \frac{d\dot{\sigma}}{d\lambda} + 2\dot{\theta}\dot{\sigma} = \Psi_0 . \]

(19)

The RHS terms are the Ricci and Weyl focusing and are defined as follows:

\[ \Phi_{00} = -\frac{1}{2} R^{\alpha\beta}_{\alpha\beta} , \quad \Psi_0 = \frac{1}{2} C^\mu_{\alpha\beta\mu\nu} k^\alpha k^\mu \Sigma^\beta \Sigma^\nu , \]

(20)

where \( R_{\alpha\beta} \) is the Ricci tensor, \( C^\mu_{\alpha\beta\mu\nu} \) the Weyl tensor, and \( \Sigma^\mu \equiv s_1^\mu + is_2^\mu \). As well as the amplification matrix, the deformation matrix simplifies in the GLC coordinates as:

\[ S_A^B = \frac{ds_A^a}{d\lambda} s_B^a = \frac{\omega}{2\Upsilon} s_A^a \gamma_{ab} . \]

(21)

Using \( s_A^a \gamma_{ab} = \gamma_{ab} \) we get the optical scalars:

\[ \dot{\theta} = \omega \frac{\gamma_{ab}}{4\Upsilon} , \quad |\dot{\sigma}|^2 = \left( \frac{\omega}{4\Upsilon} \right)^2 - \frac{\omega^2}{4\Upsilon^2} \frac{\det \gamma_{ab}}{\gamma} . \]

(22)

We also get the Ricci and Weyl focusing in GLC coordinates:

\[ \Phi_{00} = \frac{\omega^2}{4\Upsilon^2} \gamma_{ab} Y_{ab} , \quad |\Psi_0| = \frac{\omega^2}{4\Upsilon^2} \sqrt{\gamma_{ab} \gamma_{cd} - \gamma_{ac} \gamma_{bd} - \gamma_{ad} \gamma_{bc}} Y_{ab} Y_{cd} . \]

(23)

where \( Y_{ab} \equiv \gamma_{ab} - (\Upsilon/\Upsilon) \gamma_{ab} - (1/2) \gamma^{ac} \gamma_{cd} \gamma_{dh} \) depends only on \( \gamma_{ab} \), \( \Upsilon \) and their time derivatives. This proves again the usefulness of GLC coordinates for lensing and it was illustrated\(^{[21]}\) by the computation of lensing quantities in the case of an off-center observer in a Lemaître-Tolman-Bondi model (considering only the decaying mode).

7. Conclusions

We have shown that there are many advantages in using the GLC coordinates. They are indeed adapted to calculations involving light-propagation, they can also be used for weak lensing (where \( \gamma_{ab} \) acts as a screen), and may help to get new predictions on cosmology or to study other aspects of lensing (e.g. lensing statistics).

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