QUANTUM FLUCTUATIONS AND INERTIA
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Abstract

Vacuum field fluctuations exert a radiation pressure which induces mechanical effects on scatterers. The question naturally arises whether the energy of vacuum fluctuations gives rise to inertia and gravitation in agreement with the general principles of mechanics. As a new approach to this question, we discuss the mechanical effects of quantum field fluctuations on two mirrors building a Fabry-Perot cavity. We first put into evidence that the energy related to Casimir forces is an energy stored on field fluctuations as a result of scattering time delays. We then discuss the forces felt by the mirrors when they move within vacuum field fluctuations, and show that energy stored on vacuum fluctuations contributes to inertia in conformity with the law of inertia of energy. As a further consequence, inertial masses exhibit quantum fluctuations with characteristic spectra in vacuum.

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1 Introduction

Fundamental problems are raised by the mechanical effects associated with radiation pressure fluctuations in vacuum. The instability of motions when radiation reaction is taken into account, and the existence of ”runaway solutions” [1], can be avoided for mirrors by recalling that they are actually transparent to high frequencies of the field [2]. However, partially transmitting mirrors, and cavities, introduce scattering time delays which result in a temporary storage of part of the scattered vacuum fluctuations [3]. In particular, the energy related to Casimir forces [4] identifies with the energy of field fluctuations stored in the cavity [3]. This revives the questions of the contribution of vacuum fluctuations to inertia and gravitation [5], and of its consistency with the general principles of equivalence and of inertia of energy.

Vacuum fluctuations of quantum fields have been known for long to correspond to an infinite energy density [6], or at least to a problematically high energy density, if only frequencies below Planck frequency are considered [7]. A common way to escape the problems raised by consequent gravitational effects, exploits the fact that only differences of energy are involved in all other interactions. Vacuum energy is set to zero by definition, a prescription which is embodied in normal ordering of quantum fields. In such a scheme, variations of vacuum energy, like the energy associated with Casimir forces [4], hardly give rise to inertia and gravitation. Furthermore, normal ordering cannot be implemented as a covariant prescription and leads to ambiguities in defining the gravitational effects of quantum fields [8]. Then, the question naturally arises of the compatibility of the mechanical effects induced by quantum field fluctuations with the general principles which govern the laws of mechanics.

As a new approach to this question, we discuss the mechanical effects of quantum field fluctuations on two mirrors building a Fabry-Perot cavity. We first put into evidence that the energy related with Casimir forces is an energy stored on field fluctuations as a result of scattering time delays. We then discuss the forces felt by the mirrors when they move within vacuum field fluctuations, and in particular the contribution of Casimir energy to inertia.

2 Casimir energy

As a result of the radiation pressure of field quantum fluctuations in which they are immersed, two mirrors at rest in vacuum feel a mean Casimir force which depends on their distance $q$. For partially transmitting mirrors, characterised by their frequency dependent reflection coefficients ($r_1$ and $r_2$), the Casimir force takes a simple form (written here for a cavity in two-dimensional space-time immersed in the vacuum of a scalar field; similar
expressions hold in four-dimensional space-time, and for electromagnetic and also thermal fields) [3]:

\[
F_c = \int_0^\infty \frac{d\omega \hbar \omega}{2\pi c} \{1 - g[\omega]\}
\]

\[
g[\omega] = \frac{1 - |r[\omega]|^2}{1 - r[\omega]e^{2i\omega q/c}}
\]

\[
r[\omega] = r_1[\omega]r_2[\omega]
\]

The first part of this expression corresponds to the energy-momentum of incoming vacuum field fluctuations (\(\hbar\) is Planck constant, and \(c\) the light velocity). The second part describes the effect of the cavity on the modes: \(g[\omega]\) describes an enhancement of energy density for modes inside the resonance peaks of the cavity, and an attenuation for modes outside.

This mean force can be seen as the variation of a potential energy, more precisely, as the length dependent part of the energy of the cavity immersed in field fluctuations:

\[dE_c = F_c dq\]

One easily derives the well-known phase-shift representation of Casimir energy [4], whose expression in the present case takes the simple following form:

\[
E_c = \int_0^\infty \frac{d\omega \hbar}{2\pi} \{-\delta[\omega]\}
\]

\[
2\delta[\omega] = i\text{Log} \frac{1 - r[\omega]e^{2i\omega q/c}}{1 - r[\omega]^*e^{-2i\omega q/c}}
\]

\[
detS = detS_1detS_2e^{2i\delta}
\]

\(\delta[\omega]\) is the frequency dependent phase-shift introduced by the cavity on the propagation of field modes, as given by the scattering matrix \((S)\) of the cavity (more precisely, its definition divides by the individual scattering matrices of the mirrors, whose contributions to the total energy are length independent).

The frequency dependent phase-shift corresponds to time delays in the propagation of fields through the cavity:

\[
\tau[\omega] = \partial_\omega \delta[\omega]
\]

This time delay [9] describes the time lag undergone by a wave packet around frequency \(\omega\) and is the sum of several contributions:

\[
\tau[\omega] = -\{1 - g[\omega]\}\{\frac{q}{c} + \frac{1}{2} \partial_\omega \varphi\} + g[\omega] \sin(2\omega \frac{q}{c} + \varphi) \frac{\partial_\omega \rho}{1 - \rho^2}
\]

\[
r[\omega] = \rho[\omega]e^{i\varphi[\omega]}
\]
The main contribution identifies with the length of the cavity (divided by \( c \)), modified by the function \( g \) describing energy densities within the cavity. Other contributions are corrections due to the frequency dependence of the mirrors’ reflection coefficients, i.e. delays introduced during reflection on the mirrors themselves.

Casimir energy can be rewritten in terms of these scattering time delays, integrating by parts and noting that boundary terms vanish in particular because of high frequency transparency:

\[
E_c = \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega \tau[\omega]
\]

The result takes a simple form, as an integral over all modes of the product of the spectral energy density of quantum field fluctuations by the corresponding time delay. In particular, the length dependent part of Casimir energy is negative, corresponding to a binding energy, so that negative time delays contribute in majority \([10]\). As time delays are indeed relative to free propagation, i.e. in absence of cavity, the retardation effect of the cavity on resonant modes is thus dominated by the opposite effect on modes outside resonance peaks.

It can be shown that the same expressions remain valid for Casimir force and energy of a cavity immersed in thermal fields, provided the spectral energy density for thermal quantum fluctuations is substituted (\( T \) is the temperature) \([3]\):

\[
F_c = \int_0^\infty \frac{d\omega}{2\pi} \frac{\hbar \omega}{c} \left\{ \frac{1}{2} + \frac{1}{e^{\hbar \omega/T} - 1} \right\} \{1 - g[\omega]\}
\]

\[
E_c = \int_0^\infty \frac{d\omega}{2\pi} 2\hbar \omega \left\{ \frac{1}{2} + \frac{1}{e^{\hbar \omega/T} - 1} \right\} \tau[\omega]
\]

To the contribution of zero-point fluctuations, one must add the contribution due to the mean number of photons as given by Planck’s formula. In all cases, Casimir energy appears as part of the energy of quantum field fluctuations which is stored inside the cavity, as a consequence of scattering time delays.

### 3 Motional Casimir forces

The Casimir forces felt by two mirrors at rest result from the radiation pressure exerted by the fluctuating quantum fields in which they are immersed. Hence, these forces also fluctuate and their fluctuations can be characterised by their correlations (\( i, j = 1, 2 \) label the two mirrors):

\[
< F_i(t)F_j(0) > - < F_i > < F_j > = C_{F_iF_j}(t)
\]
For a stationary state of the field, correlations are equivalently characterised by spectral functions \(11\):

\[
C_{F_i F_j}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \chi_{F_i F_j}[\omega]
\] (2)

The fluctuating forces induce random motions of the mirrors around their mean positions which can be described as quantum Brownian motions. As a consequence of general principles governing motion in a fluctuating environment \(12\), when set into motion mirrors feel additional forces which depend on their motions. For small displacements \(\delta q_i\), these forces are conveniently described by motional susceptibilities:

\[
< \delta F_i[\omega] > = \sum_j \chi_{F_i F_j}[\omega] \delta q_j[\omega]
\] (3)

The motional forces can be obtained using motion dependent scattering matrices \(13\). The scattering matrix of a mirror in its rest frame leads to a scattering matrix in the original frame which depends on the mirror’s motion, and can easily be obtained up to first order in the mirror’s displacement. Radiation pressures and forces exerted on the mirrors are thus obtained up to the same order \(11\). (For perfect mirrors, forces have been obtained exactly for arbitrary motions of the mirrors \(14\)).

According to linear response theory \(15\), fluctuation-dissipation relations identify the imaginary (or dissipative) part of a susceptibility with the commutator of the corresponding quantity with the generator of the perturbation. In the case of mirrors’ displacements, the generators are the forces exerted on the mirrors:

\[
\chi_{F_i F_j}[\omega] - \chi_{F_j F_i}[-\omega] = \frac{i}{\hbar} \{ C_{F_i F_j}[\omega] - C_{F_j F_i}[-\omega] \}
\]

Thus, fluctuation-dissipation relations provide a check for the results one obtains independently for force fluctuations \(4\) and for motional susceptibilities \(3\).

Although rather complex in their total generality, explicit expressions for motional forces induced by vacuum fluctuations on partially transmitting mirrors satisfy some general interesting properties \(11\). As expected, the motional forces present mechanical resonances for frequencies which coincide with optical modes of the cavity:

\[
\omega = n\pi \frac{c}{q}
\]

Although motional Casimir forces are naturally small, much smaller than static Casimir forces, resonance properties might be used to compensate their smallness using cavities with very high quality factors, thus possibly leading to experimental evidence.
Other interesting properties of these forces appear at the quasistatic limit, i.e. at the limit of very slow motions [16]. For displacements which vary slowly in time, one can use a quasistatic expansion (expansion around zero frequency \( \omega \sim 0 \)) of the expressions for motional susceptibilities (3) (a dot stands for time derivative):

\[
< \delta F_i[\omega] >= \sum_j \{ \chi_{F_iF_j}[0]\delta q_j[\omega] + \frac{1}{2}\chi''_{F_iF_j}[0]\omega^2\delta q_j[\omega] + \ldots \}
\]

\[
< \delta F_i(t) >= -\sum_j \{ \kappa_{ij}\delta q_j(t) + \mu_{ij}\delta \ddot{q}_j(t) + \ldots \}
\]

The first term, described by \( \kappa_{ij} (-\chi_{F_iF_j}[0]) \), just reproduces the variations of the static Casimir force when the length of the cavity is changed. The further terms correspond to new forces which emerge when the mirrors are accelerated in vacuum and which exhibit peculiar features. These forces are proportional to the mirrors’ accelerations and are conveniently expressed under the form of a mass matrix \( \mu_{ij} \left( \frac{1}{2}\chi''_{F_iF_j}[0] \right) \). Diagonal terms are corrections to the mirrors’ masses. They show that each mirror’s mass is modified by the presence of the other mirror, with a correction which depends on the distance between the two mirrors. But non diagonal terms are also present, corresponding to the emergence of an inertial force for one mirror when the other mirror is accelerated. These properties of the inertial forces induced by vacuum fluctuations are reminiscent of Mach’s principle of relativity of inertia. They indeed satisfy the requirements that Einstein [17] stated in his analysis of Mach’s conception of inertia and in the context of gravity. They strongly suggest a relation between modifications of vacuum fields and gravitational effects [18].

Inertial forces acting on the cavity as a whole are related with global motions of the cavity, i.e. identical motions of the two mirrors (in linear approximation for displacements):

\[
\delta \ddot{q}_1(t) = \delta \ddot{q}_2(t) = \delta \ddot{q}(t)
\]

The total force acting on the cavity moving in vacuum fields then contains a component which dominates for slow motions and which is proportional to the cavity’s acceleration:

\[
< \delta F(t) >= < \delta F_1(t) + \delta F_2(t) >= -\{ \mu \delta \ddot{q}(t) + \ldots \}
\]

\[
\mu = \sum_{ij} \mu_{ij}
\]

Explicit computation [16] shows that the corresponding mass correction for the cavity is proportional to the length of the cavity and to the Casimir force between the two mirrors:

\[
\mu c^2 = -2F_c q
\]
This correction appears to be proportional to the contribution of the intracavity fields to the Casimir energy, i.e. the energy stored on vacuum fluctuations due to the propagation delay inside the cavity (see (1)). For a cavity built with perfect mirrors in particular, this corresponds to Casimir energy:

\[ E_c = -F_c q \]

Although not quite obvious at first sight, the factor 2 is in fact the correct one in the present case. Indeed, it was already shown by Einstein [19], that for a stressed rigid body Lorentz invariance implies a relation for the mass (\( \mu \)), i.e. the ratio between momentum and velocity, that not only involves the internal energy of the body (\( E_c \)) but also the stress (\( F_c \)) exerted on the body:

\[ \mu c^2 = E_c - F_c q \]

When comparing the total momentum with the velocity of the center of inertia of the whole system, i.e. taking into account not only the masses of the two mirrors but also the energy stored in the fields inside the cavity, this relation leads to the usual equivalence between mass and energy. Thus, the energy of vacuum field fluctuations stored inside the cavity contributes to inertia in conformity with the law of inertia of energy.

However, for partially transmitting mirrors, the energy stored according to time delays due to reflection upon the mirrors (see (1)) is missing in the mass correction (4). The inertial forces obtained for a cavity moving in vacuum satisfy the law of inertia of energy for the energy of vacuum fluctuations stored inside the cavity, but not for the energy stored in the mirrors themselves. This result must be compared with a previous computation of the force exerted on a single, partially transmitting, mirror moving in vacuum fields, which appeared to vanish for uniformly accelerated motion [13]. This discrepancy with the general equivalence between mass and energy reflects a defect in the representation of the interaction of the mirror with the field. We shall now discuss, using an explicit model of interaction between mirror and field, how this representation can be improved.

4 Model of a pointlike scatterer

We consider the case of a scalar field \( \phi \) interacting with a pointlike mirror, located at \( q \), in two-dimensional space-time \( ((x^\mu)_{\mu=0,1} = (t, x)) \), described by the following manifestly relativistic Lagrangian (from now on, \( c = 1 \)) [20]:

\[
\mathcal{A} = \frac{1}{2} (\partial \phi)^2 \, d^2 x - \int m \sqrt{1 - \dot{q}^2} \, dt
\]

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - m \sqrt{1 - \dot{q}^2} \delta(x - q) \]

\[ m = m_b + \Omega \phi(q)^2 \]
The two terms are the usual Lagrangians for a free scalar field and a free particle, except that the mass of the particle is assumed to also contain a contribution which depends on the field. Such contribution generally describes a relativistically invariant interaction term for the field and the sources which are located on the mirror. In order to facilitate comparison with the simplified representation in terms of a $2 \times 2$ scattering matrix, the interaction is further assumed to be quadratic in the field. $\Omega$ is the inverse of a proper time characterising field scattering. Equations for the field involve the scatterer’s position and result in highly nonlinear coupling:

$$\partial^2 \phi = -2\sqrt{1 - \dot{q}^2} \Omega \phi \delta(x - q)$$  \hspace{1cm} (7)

However, if one considers as a first approximation that the mirror remains at rest at a fixed position $q$, then (7) becomes a linear equation describing propagation in presence of a pointlike source. The field on both sides of the scatterer decomposes on two components which propagate freely in opposite directions and which can be identified with incoming and outgoing fields. The scattering matrix which relates outgoing and incoming modes is obtained from equation (7), and identifies with a simple symmetric $2 \times 2$ matrix determined by the following frequency dependent diagonal ($s[\omega]$) and non diagonal ($r[\omega]$) elements:

$$s[\omega] = 1 + r[\omega] \hspace{1cm} r[\omega] = -\frac{\Omega}{\Omega - i\omega}$$ \hspace{1cm} (8)

This corresponds to the simple model of partially transmitting mirror, with a reflection time delay having a Lorentzian frequency dependence:

$$\tau[\omega] = \frac{\Omega}{\Omega^2 + \omega^2}$$ \hspace{1cm} (9)

Simple computation shows that the energy stored on field fluctuations due to this reflection time delay indeed identifies with the mass term describing the interaction with the field (6). The mean mass is determined by the correlations of the local field, which can be expressed in terms of incoming correlations and of the scattering matrix. For incoming fields in vacuum:

$$< \Omega \phi(q)^2 > = \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega \tau[\omega]$$ \hspace{1cm} (10)

Actually, the expression thus obtained for the mean value of the scatterer’s mass in vacuum is infinite, as a result of a diverging contribution of high frequency fluctuations. In fact, the approximation of a scatterer staying at rest, on which expression (9) for the time delay relies, cannot remain valid for sufficiently high frequencies. At field frequencies which become comparable with the scatterer’s mass, recoil of the scatterer cannot be neglected, so that the simplified $2 \times 2$ scattering matrix and its associated reflection time delay
fail to be good approximations. Although consistent with the approximation which neglects the scatterer’s recoil for all field frequencies, the result of an infinite stored energy does not correspond to the general case.

For a finite mass scatterer, the scatterer’s recoil must be taken into account. This is described by the equations of motion for the scatterer which are derived from Lagrangian (3):

\[
\frac{dp^\mu}{dt} = F^\mu = 2\Omega \sqrt{1 - \dot{q}^2} \phi \phi^\mu \phi(q)
\]

\[
p^\mu = \left( \frac{m}{\sqrt{1 - \dot{q}^2}}, \frac{m\dot{q}}{\sqrt{1 - \dot{q}^2}} \right)
\]

These correspond to Newton equation, with a force depending on the local field. Recalling the equations of motion for the field (7), the force identifies with the radiation pressure exerted by the scattered field. An important feature of the equations characterising the scatterer’s recoil is that the mass involved in the relation between the force and the scatterer’s acceleration includes the mass correction (3), that is the energy stored by the scatterer on incoming field fluctuations. As exemplified by this simple model, a correct treatment of the interaction between field and a partially transmitting mirror leads to an energy stored on vacuum field fluctuations due to reflection time delays which also satisfies the universal equivalence between mass and energy.

As shown by equations (11), the energy and momentum of the scatterer satisfy the usual relations:

\[
p_0^2 - p_1^2 = m^2 \quad \quad p^1 = p^0 \dot{q}
\]

When submitted to the fluctuating radiation pressure of the field, the scatterer undergoes a relativistic stochastic process which remains causal, i.e. with a velocity never exceeding the light velocity. When fields with frequencies much smaller than the scatterer’s mass \((\hbar \omega \ll < m >)\) are reflected, recoil can be neglected and the scattering matrix is well approximated by the linear \(2 \times 2\) matrix (8). However, for frequencies of the order of the scatterer’s mass, recoil must be taken into account and the frequency dependence of scattering time delays differs significantly from the dependence at low frequencies (9).

A complete and accurate treatment should then consistently provide a finite stored energy for a finite mass scatterer.

Integration of the stored energy in the inertial mass in a consistent way leads to interesting new consequences. It directly results from their expressions in terms of quantum field fluctuations (for instance (11)), that stored energies not only possess a mean value but also fluctuations. Hence, the inertial mass is a fluctuating quantity, with a characteristic noise spectrum:

\[
<m(t)m(0)> - <m>^2 = \int \frac{d\omega}{2\pi} e^{-i\omega t} C_{mm}[\omega]
\]
For the pointlike scatterer just described, the inertial mass correction is quadratic in the local field, and mass fluctuations are derived from incoming field fluctuations and the scattering matrix. For frequencies well below the scatterer’s mass, recoil can be neglected and the mass noise spectrum in vacuum is readily obtained from (8):

\[ C_{mm}[\omega] = 2\hbar^2\theta(\omega) \int_0^\omega \frac{d\omega'}{2\pi} \omega' \tau[\omega'](\omega - \omega')\tau[\omega - \omega'] \]  

\( (\hbar\omega \ll < m >) \)

This spectrum shows the characteristic positive frequency domain of vacuum fluctuations. It also corresponds to a convolution (a direct product in time domain) of two expressions equal to the mean mass correction, a consequence of the gaussian property of local field fluctuations (at this level of approximation). Inertial mass thus exhibits properties of a quantum variable.

As expected, mass fluctuations become extremely small for ordinary time scales, i.e. for low frequencies. For frequencies below the reflection cut-off \( \Omega \), the mass noise spectrum grows like \( \omega^3 \):

\[ C_{mm}[\omega] \simeq \frac{\hbar^2}{6\pi\Omega^2} \theta(\omega)\omega^3 \]

The inertial mass remains practically constant in usual mechanical situations. For high frequencies however, mass fluctuations become important and cannot be neglected at very short time scales. As an illustration (of course, recoil should be accounted for at such frequencies), the same expression exhibits mass fluctuations which become comparable with the mean mass (for \( m_b = 0 \)):

\[ C_{mm}(t = 0) = < m^2 > - < m >^2 = \frac{2}{m} < m >^2 \]

5 Conclusion

Scattering time delays lead to a temporary storage of quantum field fluctuations by scatterers. Vacuum quantum field fluctuations induce stored energies and inertial masses which satisfy the universal equivalence between mass and energy, including for their fluctuations. Vacuum fluctuations result in mechanical effects which conform with general principles of mechanics. It can be expected that energies stored on quantum field fluctuations should also lead to gravitation, in conformity with the principle of equivalence. Moreover, mass fluctuations due to vacuum field fluctuations could play a significant role in a complete and consistent formulation of gravitational effects.
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