Simplification Rules for Birdtrack Operators

J. Alcock-Zeilinger\textsuperscript{1} and H. Weigert\textsuperscript{1}

\textsuperscript{1}University of Cape Town; Dept. of Physics, Private Bag X3, Rondebosch 7701, South Africa

Abstract: This paper derives a set of easy-to-use tools designed to simplify calculations with birdtrack operators comprised of symmetrizers and antisymmetrizers. In particular, we present cancellation rules allowing one to shorten the birdtrack expressions of operators, and propagation rules identifying the circumstances under which it is possible to propagate symmetrizers past antisymmetrizers and vice versa. We exhibit the power of these simplification rules by means of a short example in which we apply the tools derived in this paper on a typical operator that can be encountered in the representation theory of SU(\(N\)) over the product space \(V^{\otimes m}\). These rules form the basis for the construction of compact Hermitian Young projection operators and their transition operators addressed in companion papers \[1,\ 2\].

Contents

1 Introduction 2

2 Notation, conventions and known results 3
  2.1 Tableaux .......................................................... 3
  2.2 Birdtracks ......................................................... 4

3 Cancellation rules 9
  3.1 Cancellation of wedged Young projectors .................. 9
  3.2 Cancellation of factors between bracketing sets .......... 11

4 Propagation rules 17
  4.1 Proof of Theorem 2 (propagation rules) .................. 22
    4.1.1 Unpacking the Theorem conditions: .................. 23
    4.1.2 Strategy of the proof: .................................. 24
    4.1.3 Propagating transpositions: .......................... 24
    4.1.4 Propagating antisymmetrizers: ....................... 28
  4.2 Proof of Theorem 3 (generalized propagation rules) .... 29

5 Conclusion 30
1 Introduction

In the 1970’s Penrose [3, 4] developed a graphical method of dealing with objects typically encountered in the representation theory of semi-simple compact Lie groups, as is used in QFT. This new formalism was subsequently applied in a collaboration with MacCallum [5]. It is clear from Penrose’s work that these graphical tools found their inspiration in Feynman diagrams and thus allow visually intuitive calculations of quantities in the QCD context, since $SU(N)$ is the gauge group of QCD.

Penrose’s graphical formalism obtained a more modern treatment by Cvitanović [6] in early part of the 21st century. It is Cvitanović who dubbed the diagrams birdtracks.

Birdtracks are gaining in their popularity as a computational tool for a modern treatment of group theory, in particular the representation theory of semi-simple Lie groups, and their applications to QFT. There however do not exist any practical tools that allow the easy manipulation of birdtracks in the literature. The authors suspect that this is the reason why birdtracks are not yet as widely used as they ought to be. This paper aims to narrow this gap by providing several easy-to-use rules that greatly simplify dealing with birdtrack operators.

We will lay our focus on operators that are derived from Young projection operators [7], and the simplification rules presented in this paper are thus best suited for such operators. The reason for this is the authors’ interest in the applications of these tools in a QCD context where factorization invariably involves color singlet projections of Wilson line correlators (see e.g. [8–11] for a varied set of fields with possible applications). Since $SU(N)$ is the gauge group of QCD, Young projection operators come into play through the theory of invariants, which relates the irreducible representations of $SU(N)$ over $V^\otimes n$ to the Young tableaux of size $n$, see [12, 13] and other standard textbooks. However, the lack of Hermiticity of Young projection operators disqualifies them from the application to QCD calculations [14].

Keppeler and Sjödahl made a first step towards overcoming this problem in [15], where they present an iterative algorithm to construct Hermitian versions of Young projection operators in the birdtrack formalism. However, the KS-operators soon become unwieldy and thus impractical to work with in automated calculations owing to computing time and memory resources necessary in their construction and application.

Using the simplification rules presented in this paper, the KS-operators can be simplified drastically; an example of this is given in Figure 5.2.

This direct application, however, is not where these simplification rules exhaust their usefulness. Further applications are presented in a list of companion papers:

1. In [1] we present an alternative construction algorithm for Hermitian Young projection operators, which directly leads to significantly more compact and explicitly Hermitian expressions of the operators.

2. The simplification rules are a crucial prerequisite for an algorithm that allows us to construct transition operators between (Hermitian) Young projection operators corresponding to equivalent irreducible representations of $SU(N)$ [2] and an orthogonal basis for the algebra of invariants on $V^\otimes m$.

3. This orthogonal basis can then be used to form a basis for the singlet states necessary to determine all color neutral Wilson line correlators [14] which find direct applications in many branches of QCD. First applications (in a context that can be covered with direct calculations) can be found in [8, 16].

In this paper, we present two classes of simplification rules, they form the foundation for all three companion papers.
1. rules that determine whether certain symmetrizers or antisymmetrizers can be cancelled from an operator (section 3), and

2. rules describing when it is possible to propagate sets of (anti-) symmetrizers through certain parts of the operator (section 4).

Each result in these sections is accompanied by an example. In section 5, Fig. 5.2, we exhibit the applicability of these rules.

Before we set out to describe the simplification rules, we need to lay the groundwork by summarizing the conventions used in this paper in the following section.

2 Notation, conventions and known results

There exists a multitude of (sometimes contradicting) nomenclature and conventions in the literature with regards to Young tableaux, birdtracks, and related objects. This section serves to clarify the conventions used in this paper, as well as to collect a list of previously known results that are needed for this paper.

2.1 Tableaux

Consider an arrangement of \( m \) boxes filled with unique integers between 1 and \( k \) (for \( k \geq m \)) for example,

\[
\begin{array}{cccc}
1 & 10 & 3 \\
6 & 5 & 4 \\
7 \\
9 & 2 & 8
\end{array}
\]

(1)

In this paper, we will refer to such a construct as a semi-standard irregular tableau. In particular, the term “semi-standard” will refer to the requirement that each number appears at most once within a tableau. A special case of such a tableau is a Young tableau, in which we require \( k = m \) and the boxes to be top-aligned and left-aligned, as well as the numbers in the boxes to increase within each row from left to right and within each column from top to bottom, see [13, 17, 18] and many other standard textbooks.\(^1\) For example

\[
\begin{array}{cccc}
1 & 3 & 5 & 6 \\
2 & 4 & 7 \\
8
\end{array}
\]

(2)

is a Young tableau of size 8. In this paper, we shall denote a Young tableau by an upper case Greek letter, usually \( \Theta \) or \( \Phi \), and a semi-standard irregular tableau by \( \tilde{\Theta} \) or \( \tilde{\Phi} \). Furthermore, we will denote the set of all Young tableaux of size \( n \) by \( Y_n \).\(^2\) For example,

\[
Y_3 := \left\{ \begin{array}{ccc}
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}, & \begin{array}{ccc}
1 & 3 & 2 \\
2 & 1 & 3
\end{array}, & \begin{array}{ccc}
1 & 2 & 3
\end{array} \right\}.
\]

(3)

\(^1\)In some references, the presently described tableau may also be referred to as a standard Young tableau, for example [17, 18].

\(^2\)The size of the set \( Y_n \) is finite for any integer \( n \), as is shown in [19].
For a particular Young tableau $\Theta \in \mathcal{Y}_n$, we refer to $\Theta^{(m)} \in \mathcal{Y}_{n-m}$ (for $m < n$) as the \textit{ancestor tableau} of $\Theta$ $m$ generataions back if $\Theta^{(m)}$ is obtained from $\Theta$ by removing the boxes $\begin{array}{c} n \end{array}, \begin{array}{c} n-1 \end{array}, \ldots, \begin{array}{c} n-m \end{array}$ from $\Theta$. For example, if

$$\Theta := \begin{array}{ccccc} 1 & 2 & 4 & 5 \\ 3 & 6 & 8 \\ 7 & 9 \end{array} \quad \text{and} \quad \Phi := \begin{array}{ccccc} 1 & 2 & 4 & 5 \\ \end{array},$$

then $\Phi$ is the ancestor tableau of $\Theta$ four generations back and we write $\Phi = \Theta^{(4)}$.

In this paper, we will need another kind of tableau, namely the \textit{amputated tableau}, as is described in the following definition:

\textbf{Definition 1 (Amputated Tableaux)} Let $\tilde{\Theta}$ be a tableau.\textsuperscript{3} Furthermore, let $R$ be a particular row in $\tilde{\Theta}$ and $C$ be a particular column in $\tilde{\Theta}$. Then, we form the column-amputated tableau of $\tilde{\Theta}$ according to the row $R$, $\tilde{\Theta}^c[R]$, by removing all columns of $\tilde{\Theta}$ which do not overlap with the row $R$. Similarly, we form the row-amputated tableau of $\tilde{\Theta}$ according to the column $C$, $\tilde{\Theta}^r[C]$, by removing all rows of $\tilde{\Theta}$ which do not overlap with the column $C$.

It should be noted that if $\tilde{\Theta}$ is semi-standard, then $\tilde{\Theta}^c[R]$ and $\tilde{\Theta}^r[C]$ will also be semi-standard. As an example, consider the semi-standard irregular tableau

$$\tilde{\Theta} := \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 6 & 5 \\ 7 & 8 \end{array},$$

where we have marked the row $R := (1,2,3,4)$ in yellow, and the column $C := (3,6)^t$ in green. Then, the column- and row-amputated tableaux according to $R$ and $C$ respectively are given by

$$\tilde{\Theta}^c[R] = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 6 & 5 \\ \end{array},$$

where the column $(8)^t$ was removed since it does not have an overlap with the row $R = (1,2,3,4)$, $(1,2,3,4) \cap (8)^t = \emptyset$,\textsuperscript{4} and

$$\tilde{\Theta}^r[C] = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 6 & 5 \end{array},$$

where the row $(7,8)$ was removed from $\tilde{\Theta}$, as it does not have an overlap with the column $C = (3,6)^t$, $(3,6)^t \cap (7,8) = \emptyset$.

\subsection{2.2 Birdtracks}

As is clear by the title of this paper, we aim to provide simplification rules for birdtrack operators. In particular, this paper focuses on operators comprised of symmetrizers and antisymmetrizers. In this section, we

\textsuperscript{3}We do not require $\tilde{\Theta}$ to be a Young tableau for this definition, a more general kind of tableau (e.g. a semi-standard irregular tableau) will suffice.

\textsuperscript{4}Where we transferred the familiar set-notation to rows of tableaux.
give a short overview of the birdtrack notation [6] and its correspondence to Young projection operators [13].

For a more extensive introduction to birdtracks, readers are referred to [6], which also serves as the main resource for this section.

For each semi-standard tableau $\tilde{\Theta}$ (be it irregular or Young), one may construct the corresponding sets of symmetrizers $S_{\tilde{\Theta}}$ and antisymmetrizers $A_{\tilde{\Theta}}$ – this is in fact a generalization to the standard construction principle of symmetrizers and antisymmetrizers corresponding to Young tableaux [6, 7, 13]. Each row $\mathcal{R}$ of the tableau will correspond to a symmetrizer over the numbers appearing in $\mathcal{R}$, and each column $\mathcal{C}$ corresponds to an antisymmetrizer over the numbers in $\mathcal{C}$. For example, the symmetrizer over elements 1 and 2, $S_{12}$, corresponds to the tableau

$$\begin{bmatrix} 1 & 2 \end{bmatrix}.$$  \hspace{1cm} (6)

This symmetrizer $S_{12}$ is given by $\frac{1}{2} \left( \text{id} + (12) \right)$, where $\text{id}$ is the identity and $(12)$ denotes the transposition swapping elements 1 and 2. For example, $S_{12}$ acts on a tensor $T^{ab}$ as

$$S_{12} T^{ab} = \frac{1}{2} \left( T^{ab} + T^{ba} \right).$$  \hspace{1cm} (7)

Graphically, we denote the symmetrizer $S_{12} = \frac{1}{2} \left( \text{id} + (12) \right)$ as

$$S_{12} = \frac{1}{2} \left( \begin{array}{c} \begin{array}{c} \text{id} \end{array} \\ \begin{array}{c} (12) \end{array} \end{array} \right).$$  \hspace{1cm} (8)

This operator is read from right to left,\footnote{If the tableau $\tilde{\Theta}$ consists of $m$ boxes filled with unique integers between 1 and $k$ for $k > m$, we will draw an empty index line for each integer $\leq k$ not appearing in the tableau $\tilde{\Theta}$ in birdtrack notation.} as it is viewed to act as a linear map from the space $V \otimes V$ into itself.

In this paper, the elements of $S_n$ (the permutation group of $n$ objects) and linear combinations thereof will always be interpreted as elements of $\text{Lin} (V^\otimes n)$ (the space of linear maps over $V^\otimes n$). Following [6], we will refer to the permutations of $S_n$ as the primitive invariants (of SU($N$) over $V^\otimes n$), and thus denote the real subalgebra of $\text{Lin} (V^\otimes n)$ that is spanned by these primitive invariants by $\text{API} (SU(N), V^\otimes n) \subset \text{Lin} (V^\otimes n)$.

Following [6], we denote a symmetrizer over an index-set $\mathcal{N}$, $S_{\mathcal{N}}$, by an empty (white) box over the index lines in $\mathcal{N}$. Thus, the symmetrizer $S_{12}$ is denoted by $\phantom{}\begin{array}{c} \text{empty box} \end{array}$.

Similarly, an antisymmetrizer over an index-set $\mathcal{M}$, $A_{\mathcal{M}}$, is denoted by a filled (black) box over the appropriate index lines. For example,

$$A_{12} = \phantom{}\begin{array}{c} \text{filled box} \end{array}$$  \hspace{1cm} (9)

since antisymmetrizers correspond to columns of tableaux. It should be noted that (sets of) (anti-) symmetrizers are Hermitian with respect to the canonical scalar product on $V^\otimes m$ (inherited from $V$), that is,

$$S^\dagger_{\tilde{\Theta}} = S_{\tilde{\Theta}} \quad \text{and} \quad A^\dagger_{\tilde{\Theta}} = A_{\tilde{\Theta}}.$$  \hspace{1cm} (10)

This is easiest seen in the birdtrack formalism where Hermitian conjugation (with respect to the canonical scalar product) of an operator $A$ corresponds to flipping $A$ about its vertical axis and reversing the arrows (followed, in general by complex conjugation, which plays no role in the real algebra $\text{API} (SU(N), V^\otimes m)$ of interest to us here) [6].

\footnote{This is no longer strictly true for birdtracks representing primitive invariants of SU($N$) over a mixed product $V^\otimes m \otimes (V^*)^\otimes n$, where $V^*$ is the dual vector space of $V$. A more informative discussion on this is out of the scope of this paper; readers are referred to [6].}
For each tableau \( \tilde{\Theta} \), one can then define an operator \( \tilde{Y}_{\tilde{\Theta}} \) as the product of \( S_{\tilde{\Theta}} \) and \( A_{\tilde{\Theta}} \)

\[
\tilde{Y}_{\tilde{\Theta}} := S_{\tilde{\Theta}}A_{\tilde{\Theta}} ; \tag{11}
\]

this is in fact the generalization of Young operators [6, 7, 13] (c.f. eq. (13)) to semi-standard tableaux. As an example, the operator corresponding to the tableau (5) is given by

As already alluded to in the previous paragraph, Young projection operators are merely a special kind of the operators discussed so far, namely that where \( \tilde{\Theta} = \Theta \) is a Young tableau. One aspect that makes Young projection operators special is that there exists a unique constant \( \alpha_{\Theta} \neq 0 \) such that

\[
Y_{\Theta} := \alpha_{\Theta} \cdot S_{\Theta}A_{\Theta} \tag{13}
\]

is idempotent; the object \( Y_{\Theta} \) is referred to as the Young projection operator corresponding to \( \Theta \). For an operator \( \bar{Y}_{\Theta} \) corresponding to a semi-standard irregular tableau \( \Theta \), it is not necessarily true that a non-zero constant \( c \) can be found that would yield \( Y_{\Theta} := c \cdot \bar{Y}_{\Theta} \) idempotent.\(^8\) Therefore, we adapt the following notation: \( \bar{Y}_{\Theta} \) shall denote the operator corresponding to a (semi-standard irregular or Young) tableau \( \Theta \) according to (11), while the symbol \( Y_{\Theta} \) will refer to the unique Young projection operator corresponding to \( \Theta \) that is furnished with the appropriate constant \( \alpha_{\Theta} \) yielding \( Y_{\Theta} \) to be idempotent, c.f. eq. (13).

Let us now summarize the most important properties of Young projection operators [7, 13]:

1. **Idempotency**: The Young projection operator \( Y_{\Theta} \) corresponding to a Young tableau \( \Theta \) in \( Y_n \) satisfies

\[
Y_{\Theta} \cdot Y_{\Theta} = Y_{\Theta} \quad \text{for all } n. \tag{14a}
\]

2. **Orthogonality**: If \( \Theta \) and \( \Phi \) are two Young tableaux in \( Y_n \), then the corresponding Young projection operators \( Y_{\Theta} \) and \( Y_{\Phi} \) are mutually orthogonal as projectors,

\[
Y_{\Theta} \cdot Y_{\Phi} = \delta_{\Theta\Phi} Y_{\Theta} \quad \text{for } n = 1, 2, 3, 4 , \tag{14b}
\]

and more generally (for all \( n \)) if \( \Theta \) and \( \Phi \) have different shapes.

3. **Completeness**: The Young projection operators corresponding to all Young tableaux in \( Y_n \) sum up do the identity operator on \( V^\otimes n \),

\[
\sum_{\Theta \in Y_n} P_{\Theta} = 1_n \quad \text{for } n = 1, 2, 3, 4 \tag{14c}
\]

but not beyond.

\(^7\)\( \alpha_{\Theta} \) is a combinatorial constant involving the Hook length of the tableau \( \Theta \) [6, 17, 18].

\(^8\)This is easiest seen by means of an example: It can be verified via direct calculation that the operator corresponding to

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
6 & 5 & & \\
7 & 8 & &
\end{array}
\]

is not a projection operator.
Generalizations of the Young projection operators that remove the restrictions on \( n \) on the latter two of these three properties allow one to fully classify the irreducible representations of SU\((N)\) over \( V^\otimes n \) via Young tableaux in \( Y_n \) [6, 12, 20, 21]. All these generalization build on the generally valid idempotency property of Young projectors, which will also be the only property we will rely on in this paper.

The Hermitian conjugate of a Young projection operator (13) is given by\(^9\)

\[
Y_\Theta^\dagger = (\alpha_\Theta \cdot S_\Theta \cdot A_\Theta)^\dagger = \alpha_\Theta^\dagger \cdot A_\Theta^\dagger \cdot S_\Theta^\dagger = \alpha_\Theta \cdot A_\Theta \cdot S_\Theta.
\]  

(15)

In general, sets of symmetrizers and antisymmetrizers corresponding to a Young tableau do not commute,

\[
[S_\Theta, A_\Theta] \neq 0 \tag{16}
\]

implying that Young projection operators are not Hermitian; the lack of Hermiticity of Young projection operators and the implications thereof is discussed in [1].

As a last example, we construct the birdtrack Young projection operator corresponding to the following Young tableau

\[
\Theta = \begin{array}{cccc}
1 & 3 & 4 \\
2 & 5 
\end{array}
\]  

(17)

Since \( Y_\Theta \) must be comprised of symmetrizers corresponding to the rows of \( \Theta \) and antisymmetrizers corresponding to the columns of \( \Theta \), we find that

\[
Y_\Theta = 2_{\alpha_\Theta} S_{134} S_{25} A_{12} A_{35},
\]

(18)

where \( \alpha_\Theta = 2 \) ensures the idempotency of \( Y_\Theta \). In birdtrack notation, this Young projection operator becomes

\[
Y_\Theta = 2_{\alpha_\Theta} \bar{Y}_\Theta,
\]

(19)

where we have used the bar-notation introduced previously, c.f. eqs. (11) and (13). The benefit of the bar-notation is that it allows one to ignore additional scalar factors: Let \( O \) be a birdtrack operator comprised of symmetrizers and antisymmetrizers. Then, \( \bar{O} \) denotes the graphical part of \( O \) only, and we have that

\[
\omega \cdot \bar{O} = \bar{O} \quad \text{but} \quad \omega \cdot O \neq O
\]

(20)

for any non-zero scalar \( \omega \).

In expression (19) for \( Y_\Theta \) we were able to draw the two symmetrizers underneath each other since they are disjoint, and similarly for the two antisymmetrizers. In fact, the symmetrizers (resp. antisymmetrizers) corresponding to a semi-standard tableau will always be disjoint, since each number can occur at most once by the definition of semi-standard tableaux.

Any operator \( O \in \text{Lin} (V^\otimes n) \) can be embedded into \( \text{Lin} (V^\otimes m) \) for \( m > n \) in several ways, simply by letting the embedding act as the identity on \( (m-n) \) of the factors; how to select these factors is a matter of what one plans to achieve. The most useful convention for our purposes is to let \( O \) act on the first \( n \) factors and

\(^9\)Using the fact that \( \alpha_\Theta \) is a real constant, see [6, 17, 18] and other standard textbooks.
operate with the identity on the remaining last \((m - n)\) factors. We will call this the \textit{canonical embedding}. On the level of birdtracks, this amounts to letting the index lines of \(O\) coincide with the top \(n\) index lines of \(\text{Lin}(V^\otimes m)\), and the bottom \((m - n)\) lines of the embedded operator constitute the identity birdtrack of size \((m - n)\). For example, the operator \(\bar{Y}_{123}\) is canonically embedded into \(\text{Lin}(V^\otimes 5)\) as

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\implies
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\end{array}
\]  

(21)

Furthermore, we will use the same symbol \(O\) for the operator as well for its embedded counterpart. Thus, \(\bar{Y}_{123}\) shall denote both the operator on the left as well as on the right hand side of the embedding (21).

Lastly, if a \textit{Hermitian} projection operator \(A\) projects onto a subspace completely contained in the image of a projection operator \(B\), then we denote this as \(A \subset B\), transferring the familiar notation of sets to the associated projection operators. In particular, \(A \subset B\) if and only if

\[
A \cdot B = B \cdot A = A
\]  

(22)

for the following reason: If the subspaces obtained by consecutively applying the operators \(A\) and \(B\) in any order is the same as that obtained by merely applying \(A\), then the subspaces onto which \(A\) and \(B\) project not only need to overlap (as otherwise \(A \cdot B = B \cdot A = 0\)), but the subspace corresponding to \(A\) must be completely contained in the subspace of \(B\) - otherwise the last equality of (22) would not hold.

Hermiticity is crucial for these statements: since we have seen that sets of symmetrizers and anitsymmetrizers individually are Hermitian, (22) does hold for such sets: a symmetrizer \(S_N\) can be absorbed into a symmetrizer \(S_{N'}\), as long as the index set \(N\) is a subset of \(N'\), and the same statement holds for antisymmetrizer [6]. For example,

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\implies
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\end{array}
\]  

(23)

Thus, by the above notation, \(S_{N'} \subset S_N\), if \(N \subset N'\). Or, as in our example,

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\subset
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\]  

(24)

In this sense, eq. (24) is a simplification rule in its own right, as it allows us to “cancel” (anti-) symmetrizers that can be absorbed into longer (anti-) symmetrizers. In particular, (24) implies that the image of any (anti-) symmetrizer is contained in the image of its ancestor (anti-) symmetrizers!\\\(^{10}\) This nested inclusion of ancestor operators breaks down for the standard Young projection operators whenever they are not Hermitian [1], as for example

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\implies
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\end{array}
\]  

(25)

but

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\implies
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {
\begin{tikzpicture}
\draw [ultra thick] (0,1) -- (1,1);
\draw [ultra thick] (1,0) -- (1,1);
\draw [ultra thick] (0,0) -- (0,1);
\end{tikzpicture}
};
\end{tikzpicture}
\end{array}
\end{array}
\]  

(26)

\\\(^{10}\)Where we transfer the nomenclature of ancestor-tableaux to the corresponding (anti-) symmetrizers.
which can be verified by direct calculation. On the other hand, the image of a Hermitian Young projection operators is contained in the images of its ancestor Hermitian projectors [1].

The direction of the arrow on the index lines of the birdtrack encode whether the line acts on the vector space \( V \) (arrow pointing from right to left) or its dual \( V^* \) (arrow pointing from left to right) [6]. In this paper, we will only consider birdtracks acting on a space \( V^\otimes m \) (never on the dual) and thus only encounter birdtracks with arrows pointing from right to left. To reduce clutter, we will therefore suppress the arrows and (for example) simply write

\[
\begin{array}{c}
\end{array}
\]

when we mean

\[
\begin{array}{c}
\end{array}
\]

(27)

We are now in a position to discuss the main result of this paper: We describe two classes of simplification rules for birdtrack operators \( O \) comprised of symmetrizers and antisymmetrizers, namely

1. **Cancellation rules**: These describe a set of rules to cancel certain symmetrizers and antisymmetrizers within an operator \( O \). The usefulness of these rules is that they can make a long expression significantly shorter, and thus more practical and less computationally expensive to work with. These rules are described in section 3.

2. **Propagation rules**: These describe the circumstances under which it is possible to commute a particular symmetrizer through a (set of) antisymmetrizer(s), and vice versa. These rules can be used to create a situation in which the cancellation rules (see part 1) can be used, or to make certain features of a particular operator \( O \) (for example its Hermiticity) explicit. These rules can be found in section 4.

These simplification rules come into their own when they are applied to birdtrack operators in group-theoretic calculations. For examples, we extensively used these rules in our papers on a compact construction of Hermitian Young projection operators [1] and transition operators [2]. A further example is given in section 5 (Fig. 5.2).

### 3 Cancellation rules

#### 3.1 Cancellation of wedged Young projectors

We begin by presenting two main cancellation rules, Theorem 1 and Corollary 2. The benefit of these rules is that they can be used to shorten the birdtrack-expressions of certain operators (sometimes inducing a constant factor), and thus make the resulting expression more useful for practical calculations.

**Theorem 1 (cancellation of wedged Young projectors)** Consider an operator \( O \) consisting of an alternating product of altogether four symmetrizers and anti-symmetrizers, with the middle pair being proportional to a Young projection operator

\[
O = A_{\Phi_1} S_{\Theta} A_{\Theta} S_{\Phi_2} = A_{\Phi_1} Y_{\Theta} S_{\Phi_2}
\]

such that \( S_{\Theta} \supset S_{\Phi_2} \) and \( A_{\Theta} \supset A_{\Phi_1} \), i.e. \( S_{\Theta} S_{\Phi_2} = S_{\Phi_2} S_{\Theta} \) and \( A_{\Theta} A_{\Theta} = A_{\Phi_1} = A_{\Phi_1} A_{\Theta} \) (c.f. eq. (22)). Then, we can drop \( Y_{\Theta} \) while acquiring a scalar factor \( 1/\alpha_{\Theta} \):

\[
A_{\Phi_1} Y_{\Theta} S_{\Phi_2} = \frac{1}{\alpha_{\Theta}} A_{\Phi_1} S_{\Phi_2}.
\]

(29)
Corresponding cancellations apply if all symmetrizers are exchanged for antisymmetrizers and vice versa.

Using $Y_\Theta$ instead $\tilde{Y}_\Theta$ removes the constant. The form presented here is that usually encountered in practical calculations.

Before looking at a general proof for this statement, we will develop the strategy for it through an example. To this end take $O$ to be

$O = A_{\phi_1} A_\Theta S_{\Theta} A_\phi S_{\phi_1}$. \hspace{1cm} (30)

The central sets of symmetrizers and antisymmetrizers correspond to the Young tableau

$\Theta = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$, \hspace{1cm} (31)

embedded into Lin $(V \otimes^5)$. The inclusion criterion can be verified in multiple ways:

- Thinking in terms of image inclusions we note that $S_{\Theta} \supset S_{\phi_2}$ (since $S_{\Theta} = \{S_{12}\} \supset \{S_{125}\} = S_{\phi_2}$) and $A_\Theta \supset A_{\phi_1}$ (since $A_\Theta = \{A_{13}\} \supset \{A_{13}, A_{24}\} = A_{\phi_1}$)
- Equivalently, in terms of birdtracks we see that

$S_{\Theta} \supset S_{\phi_2}$ and $A_\Theta \supset A_{\phi_1}$.

(32)

Let us explore how the cancellation of eq. (29) comes about in example (30): First note that due to eq. (32) we may rewrite $O$ as

$O = \frac{1}{\alpha_\Theta} \cdot \sum S_{\phi_1} \rightarrow A_\phi S_{\Theta} \rightarrow A_\phi S_{\phi_2}$. \hspace{1cm} (33)

Idempotency of $Y_\Theta$ implies $Y_\Theta^\dagger Y_\Theta^\dagger = 1/\alpha_\Theta Y_\Theta^\dagger$ so that

$O = \frac{1}{\alpha_\Theta} \cdot \sum S_{\phi_1} \rightarrow A_\phi S_{\Theta} \rightarrow A_\phi S_{\phi_2}$. \hspace{1cm} (34)

The calculation exhibits a clear three step pattern that immediately furnishes the general proof:

1. Factor $S_{\Theta}$ from $S_{\phi_2}$ and $A_\Theta$ from $A_{\phi_1}$ to generate $Y_{\Theta_1}^\dagger Y_{\Theta_2}^\dagger$ (this is possible since $S_{\Theta_1} \supset S_{\phi_2}$ and $A_{\Theta_1} \supset A_{\phi_1}$ as required by the Theorem)

$O = A_{\phi_1} A_\Theta S_{\Theta} A_\phi S_{\phi_2}$. \hspace{1cm} (35)
2. use idempotency of $Y_\Theta$ so simplify $Y_\Theta Y_\Theta = \frac{1}{\alpha_\Theta} Y_\Theta$

\[
O = \frac{1}{\alpha_\Theta} \cdot A_\Phi, A_\Theta S_\Theta S_\Phi. \tag{36}
\]

3. reabsorb $S_\Theta$ into $S_\Phi$, and $A_\Theta$ into $A_\Phi$

\[
O = \frac{1}{\alpha_\Theta} \cdot A_\Phi, A_\Theta S_\Theta S_\Phi = \frac{1}{\alpha_\Theta} \cdot A_\Phi, S_\Phi. \tag{37}
\]

In some applications one finds the ingredients of Theorem 1 embedded into chains of Young projectors [1, 15], we thus explicitly formulate the following Corollary:

**Corollary 1 (cancellation of wedged ancestor-operators)** Consider two Young tableaux $\Theta$ and $\Phi$ such that they have a common ancestor tableau $\Gamma$. Let $Y_\Theta, Y_\Phi$ and $Y_\Gamma$ be their respective Young projection operators, all embedded in an algebra that encompasses all three. Then

\[
Y_\Theta Y_\Gamma Y_\Phi = Y_\Theta Y_\Phi. \tag{38}
\]

This Corollary immediately follows from Theorem 1 since the product $Y_\Theta Y_\Gamma Y_\Phi$ will be of the form

\[
Y_\Theta Y_\Gamma Y_\Phi = \alpha_\Theta \alpha_\Gamma \alpha_\Phi \cdot S_\Theta A_\Theta S_\Theta A_\Theta S_\Phi A_\Phi, \tag{39}
\]

where the marked factor constitutes $O$ as defined in equation (28) in Theorem 1.

### 3.2 Cancellation of factors between bracketing sets

In this section, we present another cancellation Theorem that allows us to significantly shorten certain operators. The results presented here follow immediately from a result given in [13, Lemma IV.5], which we paraphrase here in Lemma 1. Before we can give [13]'s result, we need to define horizontal and vertical permutations of a Young tableau:

**Definition 2 (horizontal and vertical permutations)** Let $\tilde{\Theta}$ be a semi-standard (Young or irregular) tableau such that $n$ is the largest integer appearing in $\tilde{\Theta}$. Then, $h_{\tilde{\Theta}}$ shall denote the subset of all permutations in $S_n$ that only operate within the rows of $\tilde{\Theta}$; i.e. that do not swap numbers across rows. We call this the set the horizontal permutations of $\tilde{\Theta}$. Similarly, we define the set of vertical permutations of $\tilde{\Theta}$, $v_{\tilde{\Theta}}$, to be the subset of permutations in $S_n$ that only operate within the columns of $\tilde{\Theta}$, i.e. those that do not swap numbers across columns.

By definition of semi-standard tableaux (which requires each integer to appear exactly once within the tableau $\tilde{\Theta}$), it is clear that

\[
h_{\tilde{\Theta}} \cap v_{\tilde{\Theta}} = \{id\}, \tag{40}
\]

where id is the identity permutation in $S_n$. 

---

11
For example, if
\[ \Theta = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 \end{bmatrix}, \]  
then
\[ h_\Theta = \{ \text{id}, (13), (25), (13)(25) \} \]  
and
\[ v_\Theta = \{ \text{id}, (12), (14), (24), (124), (35), (12)(35), (14)(35), (24)(35), (124)(35), (142)(35) \}. \]

With these definitions we can restate Lemma IV.5 of [13]:

**Lemma 1 (Tung's Lemma IV.5)** Let \( \Theta \in \mathcal{Y}_n \) be a Young tableau and let \( \rho \) be a (linear combination of) permutation(s) in \( S_n \). If \( \rho \) satisfies
\[ h_\Theta \rho v_\Theta = \text{sign}(v_\Theta) \rho \]  
for all \( h_\Theta \in h_\Theta \) and for all \( v_\Theta \in v_\Theta \), then \( \rho \) is proportional to the Young projection operator corresponding to \( \Theta \),
\[ \rho = \lambda \cdot Y_\Theta. \]

Furthermore, if we write \( \rho \) as a sum of permutations,
\[ \rho = \sum_{\sigma \in S_n} a_\sigma \sigma, \]
where the \( a_\sigma \) are constants, then \( \lambda \) is proportional to the coefficient of the identity in the series expansion of \( \rho \),
\[ \lambda = \frac{H_\Theta}{b_O} a_{\text{id}}, \]
where \( H_\Theta \) denotes the hook length of \( \Theta \) [17, 18] and \( b_O \) is the product of (length of (anti-) symmetrizer)! for all symmetrizers and antisymmetrizers in \( O \) [6].

The last statement is not included in the original version shown in [13], but follows from the proof presented there.

It should be noted that in [13], symmetrizers and antisymmetrizers are not normalized: for example, we define \( S_{12} := \frac{1}{2}(\text{id} + (12)) \) while [13] defines \( S_{12} := \text{id} + (12) \). Thus, the constant \( b_O \) arises in our statement of the Lemma 1, but is not present in [13]. Furthermore, [13]'s statement of this Lemma compares the algebra element \( \rho \) with the irreducible symmetrizer \( e_\Theta \), which differs from \( Y_\Theta \) by the constant \( H_\Theta \), (keeping in mind the different normalizations of symmetrizers and antisymmetrizers used in this paper and in [13]). This leads to the constant \( H_\Theta \) in our rendition of the Lemma.

Lemma 1 immediately gives rise to the following special case:
Corollary 2 (Cancellation of parts of the operator) Let \( \Theta \in \mathcal{Y}_n \) be a Young tableau and \( M \) an element of the algebra of primitive invariants \( \in \text{API}(\text{SU}(N), V^\otimes n) \). Then, there exists a (possibly vanishing) constant \( \lambda \) such that
\[
O := S_\Theta M A_\Theta = \lambda \cdot Y_\Theta . \tag{48}
\]
If furthermore the operator \( O \) is non-zero, then \( \lambda \neq 0 \).

Imagine that \( M \) is exclusively constructed as a product of symmetrizers and antisymmetrizers as will be the case in our applications. Then \( \Theta \in \mathcal{Y}_n \) and \( M \in \text{API}(\text{SU}(N), V^\otimes n) \) ensures that \( A_\Theta \) is (in birdtrack parlance) the longest set of antisymmetrizers in \( O \), and \( S_\Theta \) is the longest set of symmetrizers in \( O \). This is illustrated by the following example:
\[
O := \begin{array}{c}
\text{S}_\Theta \\
\text{M} \\
\text{A}_\Theta \\
\end{array}
\]
where \( \Theta := \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & & & \\
\end{array} \) . \tag{49}

This observation is a key element in recognizing where eq. (48) is applicable.

Proof of Corollary 2: From the definition of horizontal and vertical permutations (Definition 2) it is clear that
\[
h_\Theta S_\Theta = S_\Theta \quad \text{for all } h_\Theta \in h_\Theta \\
A_\Theta v_\Theta = \text{sign}(v_\Theta) A_\Theta \quad \text{for all } v_\Theta \in v_\Theta ,
\]
where \( \text{sign}(\rho) \) denotes the signature of the permutation \( \rho \).
\(^{11}\) Since \( O := S_\Theta M A_\Theta \) (eq. (48)), it immediately follows that, for all \( h_\Theta \in h_\Theta \) and all \( v_\Theta \in v_\Theta 
\]
\[
h_\Theta O = \frac{v_\Theta S_\Theta}{S_\Theta} M A_\Theta = S_\Theta M A_\Theta = \text{sign}(v_\Theta) S_\Theta M A_\Theta = \text{sign}(v_\Theta) O .
\]

More compactly, these conditions become
\[
h_\Theta O v_\Theta = \text{sign}(v_\Theta) O \quad \text{for all } h_\Theta \in h_\Theta \text{ and } v_\Theta \in v_\Theta . \tag{50}
\]
However, according to Lemma 1 \([13, \text{Lemma IV}.5]\), relation (50) holds if and only if \( O \) is proportional to the Young projection operator \( Y_\Theta \); that is, there exists a constant \( \lambda \) such that
\[
O = \lambda \cdot Y_\Theta . \tag{51}
\]
From this, it follows immediately that \( \lambda \neq 0 \) if and only if \( O \neq 0 \), thus establishing our claim.

One of the main cases of interest is a situation where the structure of \( O \) (and thus \( M \)) is such that we know from the outset that it is nonzero. One such condition is that none of the antisymmetrizers contained in \( O \) may exceed the length \( N \); if this occurs we refer to it as a dimensional zero. We will re-visit this scenario at the end of this section.

Two further conditions ensuring \( O \neq 0 \) are presented below, conditions 1 and 2 (condition 3 is a combination of conditions 1 and 2). We do not claim that the conditions given in this section represent an exhaustive list of cases yielding \( O \neq 0 \), but rather that these cases occur most commonly in practical examples, \([1, 2]\).

\(^{11}\) \( \text{sign}(\rho) \) is \( \pm 1 \) depending on whether \( \rho \) decomposes into an even or odd number of transpositions. Tung in \([13]\) means the same when he writes \( (-1)^{\text{sign}(\rho)} \).
Condition 1 (inclusion of (anti-) symmetrizers) Let $O$ be of the form \((48)\), $O = S_{\Theta} \ M \ A_{\Theta}$, and $M$ be given by

\[ M = A_{\Phi_1}, S_{\Phi_2}, A_{\Phi_3}, S_{\Phi_4} \cdots A_{\Phi_{k-1}}, S_{\Phi_k}, \]  

\hfill(52)

such that $A_{\Phi_i} \supset A_{\Theta}$ for every $i \in \{1, 3, \ldots, k - 1\}$ and $S_{\Phi_j} \supset S_{\Theta}$ for every $j \in \{2, 4, \ldots, k\}$. Then $O$ is a non-zero element of $\text{API}(\text{SU}(N), V^{\otimes n}) \subset \text{Lin}(V^{\otimes n})$.

Proof: The operator $O = S_{\Theta} \ M \ A_{\Theta}$ with $M$ given in \((52)\) is defined to be a product of alternating symmetrizers and antisymmetrizers. In particular, the outermost sets of symmetrizers and antisymmetrizers, $S_{\Theta}$ and $A_{\Theta}$ respectively, correspond to a Young tableau $\Theta$. By the definition of Young tableaux, this implies that each symmetrizer in $S_{\Theta}$ has at most one common leg with each antisymmetrizer in $A_{\Theta}$ (this is the underlying reason why $\bar{Y}_{\Theta} = S_{\Theta}A_{\Theta} \neq 0$). Furthermore, since $S_{\Phi_j} \supset S_{\Theta}$ for every $j \in \{2, 4, \ldots, k\}$ and $A_{\Phi_i} \supset A_{\Theta}$ for every $i \in \{1, 3, \ldots, k - 1\}$, the same applies for every other (not necessarily neighbouring) pair $S_{\Xi_i}$ and $A_{\Xi_j}$ occurring in $O$. This guarantees that the operator $O$ as defined in \((52)\) is non-zero.

As an example of condition 1 consider the operator

\[ O = \begin{array}{c}
  S_{\Theta} \\
  A_{\Theta} \\
  S_{\Theta} \\
  A_{\Theta}
\end{array} \]  

\hfill(53)

In $O$, the sets $S_{\Theta}$ and $A_{\Theta}$ correspond to the Young tableau

\[ \Theta := \begin{array}{cccc}
  1 & 2 & 5 \\
  3 & 4
\end{array} \]  

\hfill(54)

The inclusion conditions are $A_{\Phi_1} = \{A_{13}\} \supset \{A_{13}, A_{24}\} = A_{\Theta}$ and $S_{\Phi_2} = \{S_{12}, S_{34}\} \supset \{S_{12}, S_{34}\} = S_{\Theta}$.

Then, according to Corollary 2, we may cancel the wedged sets $A_{\Phi_1}$ and $S_{\Phi_2}$ at the cost of a non-zero constant $\kappa$,

\[ Q = \kappa \cdot \begin{array}{c}
  S_{\Theta} \\
  A_{\Theta}
\end{array} = \kappa \cdot \bar{Y}_{\Theta}. \]  

\hfill(55)

The simplification is notable and nontrivial. It is useful in all situations where the end result is simple enough and we have an external criterion to constrain the product of any of the unknown proportionality factors $\kappa$ acquired in the possible repeated application of Corollary 2.

A second way of constructing non-zero operators is by relating symmetrizers and antisymmetrizers of different Young tableaux with a permutation. To this end, we require the following definition.

Definition 3 (tableau permutation) Consider two Young tableaux $\Theta, \Phi \in \mathcal{Y}_n$ with the same shape. Then, $\Phi$ can be obtained from $\Theta$ by permuting the numbers of $\Theta$; clearly, the permutation needed to obtain $\Phi$ from $\Theta$ is unique. Denote this permutation by $\rho_{\Theta \Phi}$,

\[ \Theta = \rho_{\Theta \Phi}(\Phi) \iff \Phi = \rho_{\Theta \Phi}^{-1}(\Theta) = \rho_{\Phi \Theta}(\Theta). \]  

\hfill(56)

\hfill\footnotesize{\text{\textsuperscript{12}In this particular case, one can even notice that the set $A_{\Phi_2}$ corresponds to the ancestor tableau $\Theta_{(2)}$ and the set $S_{\Phi_3}$ corresponds to the ancestor tableau $\Theta_{(1)}$ of $\Theta$. Hence, $Q$ can be written as $Q = S_{\Theta}A_{\Theta_{(2)}}S_{\Theta_{(1)}}A_{\Theta}$.}}
To construct $\rho_{\Theta \Phi}$ explicitly, write the Young tableau $\Theta$ and $\Phi$ next to each other such that $\Theta$ is to the left of $\Phi$ and then connect the boxes in the corresponding position of the two diagrams, such as

\[
\Theta \rightarrow \begin{array}{|c|c|c|}
\hline
1 & 2 & \text{} \\
\hline
3 & \text{} & \text{} \\
\hline
\end{array}
\quad \text{←} \quad \Phi.
\]  

(57)

Write two columns of numbers from 1 to $n$ next to each other in descending order; the left column represents the entries of $\Theta$ and the right column represents the entries of $\Phi$. Connect the entries in the left and the right column in correspondence to (57). The resulting tangle of lines is the birdtrack corresponding to $\rho_{\Theta \Phi}$ and thus determines the permutation.

As an example, the permutation $\rho_{\Theta \Phi}$ between the tableaux

\[
\Theta = \begin{array}{|c|c|c|}
\hline
1 & 2 & \text{} \\
\hline
3 & \text{} & \text{} \\
\hline
\end{array}
\quad \text{and} \quad \Phi = \begin{array}{|c|c|}
\hline
1 & 3 \\
\hline
2 & \text{} \\
\hline
\end{array}
\]

(58)

is given by

\[
\Theta \rightarrow \begin{array}{|c|c|c|}
\hline
1 & 2 & \text{} \\
\hline
3 & \text{} & \text{} \\
\hline
\end{array}
\quad \text{←} \quad \Phi \quad \Rightarrow \quad \rho_{\Theta \Phi} = \begin{array}{|c|c|c|}
\hline
\text{} & \text{} & 1 \\
\hline
\text{} & 2 & \text{} \\
\hline
\end{array}
\]

(59)

Let $\Theta$ and $\Phi$ be two Young tableaux of the same shape and construct the permutation $\rho_{\Theta \Phi}$. Furthermore, consider a general operator $K_{\Theta}$ comprised of sets of (anti-) symmetrizers which can be absorbed into $S_{\Theta}$ and $A_{\Theta}$ respectively, and let $H_{\Phi}$ be an operator comprised of sets of (anti-) symmetrizers which can be absorbed into $S_{\Phi}$ and $A_{\Phi}$ respectively. Except for isolated examples, the product $K_{\Theta} \cdot H_{\Phi}$ vanishes. However, it turns out that

\[
H_{\Phi} \cdot \rho_{\Theta \Phi}^{-1} K_{\Theta} \cdot \rho_{\Theta \Phi} \neq 0 \quad \text{for all } \Theta, \Phi \in \mathcal{Y}_n \text{ for all } n.
\]

(60)

To better understand this, we accompany the general argument with an example: Consider the Young tableaux

\[
\Theta = \begin{array}{|c|c|c|}
\hline
1 & 3 & 5 \\
\hline
2 & 4 & \text{} \\
\hline
6 & \text{} & \text{} \\
\hline
\end{array}
\quad \text{and} \quad \Phi = \begin{array}{|c|c|c|}
\hline
1 & 2 & 6 \\
\hline
3 & 5 & \text{} \\
\hline
4 & \text{} & \text{} \\
\hline
\end{array}
\]

(61)

The permutation $\rho_{\Theta \Phi}$ as defined in Definition 3 is given by

\[
\rho_{\Theta \Phi} = \begin{array}{|c|c|c|}
\hline
\text{} & \text{} & 1 \\
\hline
\text{} & 2 & \text{} \\
\hline
\text{} & \text{} & \text{} \\
\hline
\end{array}
\]

(62)

For a general Young tableau $\Psi \in \mathcal{Y}_n$, we denote the irregular tableau that is obtained from $\Psi$ by deleting the boxes with entries $a_1$ up to $a_m$ ($m \leq n$) by $\Psi \setminus \{a_1, \ldots, a_m\}$. Even though $\Psi \setminus \{a_1, \ldots, a_m\}$ is not a

---

\[13\]This is true since the product of (most!) Young projection operators corresponding to different Young tableaux of the same shape in $\mathcal{Y}_n$ vanishes [13, 20].
Young tableau in general, it remains semi-standard. Thus, the (anti-) symmetrizers in the sets $S_{\Psi \{a_1,\ldots,a_m\}}$ and $A_{\Psi \{a_1,\ldots,a_m\}}$ are disjoint and the sets themselves individually remain Hermitian projection operators. These sets can further be absorbed into $S_{\Psi}$ and $A_{\Psi}$ respectively since $S_{\Psi \{a_1,\ldots,a_m\}}$ is merely the set of symmetrizers $S_{\Psi}$ with the legs $a_1$ up to $a_m$ deleted, and similarly for $A_{\Psi \{a_1,\ldots,a_m\}}$. Thus, they satisfy the absorption relations

$$S_{\Psi \{a_1,\ldots,a_m\}} S_{\Psi} = S_{\Psi} = S_{\Psi \{a_1,\ldots,a_m\}} S_{\Psi}$$

and

$$A_{\Psi \{a_1,\ldots,a_m\}} A_{\Psi} = A_{\Psi} = A_{\Psi \{a_1,\ldots,a_m\}} A_{\Psi} ,$$

(63)

this is easiest seen via the birdtracks corresponding to the semi-standard irregular tableau $\Psi \{a_1,\ldots,a_m\}$.

A quick look at our example elucidates how equation (63) comes about in general: In (61), we may remove boxes from $\Theta$ at will – consider for example

$$\Theta \{3,6\} \Theta \{4,5,6\} \Theta \{2\} ,$$

(64)

It is clear from this list that only some of the resulting tableaux will be Young tableaux, most will not. Using the tableaux (64), we construct an operator $K_{\Theta}$ consisting of (anti-) symmetrizers which can be absorbed into $S_{\Theta}$ and $A_{\Theta}$,

$$K_{\Theta} := S_{\Theta \{3,6\}} A_{\Theta \{2\}} S_{\Theta} A_{\Theta \{2\}} A_{\Theta \{3,6\}} S_{\Theta \{4,5,6\}} = .$$

(65)

Conjugating the operator $K_{\Theta}$ by the permutation $\rho_{\Theta \Phi}$ yields

$$ \rho_{\Theta \Phi}$$

(66)

Each of the sets of (anti-) symmetrizers in (66) corresponds to one of the tableaux

$$\Phi \{2,4\} \Phi \{4,5,6\} \Phi \{3\} .$$

(67)
The tableaux in (67) are obtained by superimposing the tableaux in (64) on Φ in a cookie cutter fashion. By construction, all the $S_{\Phi \setminus \{b_1, ..., b_m\}}$ (resp. $A_{\Phi \setminus \{b_1, ..., b_m\}}$) can be absorbed into $S_\Phi$ (resp. $A_\Phi$), as claimed in eq. (63).

The pattern is completely general and in no way restricted to the particular example used to demonstrate it. Let us summarize:

**Condition 2 (relating (anti-) symmetrizers across tableaux)** Let $O$ be of the form $O = S_\Theta M A_\Theta$, eq. (48). Let $\Theta, \Phi \in Y_n$ be two Young tableaux with the same shape and construct the permutation $\rho_{\Theta\Phi}$ between the two tableaux according to Definition 3. Furthermore, let $D_\Theta$ be a product of symmetrizers and antisymmetrizers, each of which can be absorbed into $S_\Theta$ and $A_\Theta$ respectively. If $M$ is of the form

$$M = \rho_{\Theta\Phi} D_\Theta \rho_{\Theta\Phi},$$

then the operator $O$ is non-zero.

It immediately follows that a combination of conditions 1 and 2 also renders the operator $O$ non-zero:

**Condition 3 (combining conditions 1 and 2)** Let $O$ be an operator of the form $O = S_\Theta M A_\Theta$ and let $M$ be given by

$$M = M^{(1)} M^{(2)} \ldots M^{(l)},$$

such that for each $M^{(i)}$ either condition 1 or 2 holds; this implies that each (anti-) symmetrizer in $M$ can be absorbed into $S_\Theta$ or $A_\Theta$ respectively. Then $O$ is nonzero.

**Dimensional zeroes:** Let us conclude this section with a short discussion on how the operator $O$ becomes dimensionally zero. Since in either of the three conditions presented in this section all sets of antisymmetrizers in $M$ can be absorbed into $A_\Theta$,

$$A_j A_\Theta = A_\Theta A_j = A_\Theta A_j,$$

for every $A_j$ in $M$, it follows immediately that the antisymmetrizer in $O$ that contains the most legs (i.e. the “longest” antisymmetrizer in $O$) must be part of the set $A_\Theta$, as otherwise eq. (70) could not hold. Thus, $O$ is not dimensionally zero if $A_\Theta$ is not dimensionally zero. Furthermore, since $Y_\Theta \propto S_\Theta A_\Theta$, it suffices to require that $N$ is large enough for the Young projection operator $Y_\Theta$ to be non-zero to ensure that the operator $O$ in any of the conditions 1–3 is not dimensionally zero. Thus, in cancelling parts of the operator $O$ (to give it the structural form of $Y_\Theta$), one does not remove any indication of it being dimensionally zero: dimensional zeroes of $O$ occur exactly when $Y_\Theta$ is zero.

The cancellation rules given in this section are of enormous practical use as they allow us to shorten birdtrack operators, often significantly so. In particular, we use Corollary 2 in the construction of compact Hermitian Young projection operators [1] and the construction of transition operators [2].

### 4 Propagation rules

In this section, we will present propagation rules that allow us to propagate certain symmetrizers through sets of antisymmetrizers, and vice versa. These rules are particularly useful to make the Hermiticity of certain birdtrack operators visually explicit. We demonstrate their effectiveness in [1] and with a specific example in our conclusions, see Fig. 5.2.
The structure of our proof of these propagation rules has been strongly inspired by an example presented in the Appendix of Keppeler and Sjödahl’s (KS) paper on Hermitian Young projection operators [15]. In this example, KS clearly realized that symmetrizers sometimes can be propagated through sets of antisymmetrizers, and vice versa, by “swapping” appropriate sets of antisymmetrizers around; however, the general conditions under which this is possible were not identified by KS, and a proof is also not present in [15].

**Theorem 2 (propagation of (anti-) symmetrizers)** Let \( \Theta \) be a Young tableau and \( O \) be a birdtrack operator of the form

\[
O = S_{\Theta} A_{\Theta} S_{\Theta \setminus \mathcal{R}},
\]

in which the symmetrizer set \( S_{\Theta \setminus \mathcal{R}} \) arises from \( S_{\Theta} \) by removing precisely one symmetrizer \( S_{\mathcal{R}} \). By definition \( S_{\mathcal{R}} \) corresponds to a row \( \mathcal{R} \) in \( \Theta \) such that

\[
S_{\Theta} = S_{\Theta \setminus \mathcal{R}} S_{\mathcal{R}} = S_{\mathcal{R}} S_{\Theta \setminus \mathcal{R}}.
\]  

(72)

If the column-amputated tableau of \( \Theta \) according to the row \( \mathcal{R} \), \( \Theta_c[\mathcal{R}] \), is **rectangular**, then the symmetrizer \( S_{\mathcal{R}} \) may be propagated through the set \( A_{\Theta} \) from the left to the right, yielding

\[
O = S_{\Theta} A_{\Theta} S_{\Theta \setminus \mathcal{R}} = S_{\Theta \setminus \mathcal{R}} A_{\Theta} S_{\Theta},
\]

(73)

which implies that \( O \) is Hermitian.\(^{14}\) We may think of this procedure as moving the missing symmetrizer \( S_{\mathcal{R}} \) through the intervening antisymmetrizer set \( A_{\Theta} \). Eq. (72) immediately allows us to augment this statement to

\[
S_{\Theta} A_{\Theta} S_{\Theta \setminus \mathcal{R}} = S_{\Theta \setminus \mathcal{R}} A_{\Theta} S_{\Theta} = S_{\Theta} A_{\Theta} S_{\Theta}.
\]

(74)

If the roles of symmetrizers and antisymmetrizers are exchanged, we need to verify that the row-amputated tableau \( \Theta_r[c] \) with respect to a column \( \mathcal{C} \) is rectangular to guarantee that

\[
A_{\Theta} S_{\Theta} A_{\Theta \setminus \mathcal{C}} = A_{\Theta \setminus \mathcal{C}} S_{\Theta} A_{\Theta} = A_{\Theta} S_{\Theta} A_{\Theta}.
\]

(75)

This amounts to moving the missing antisymmetrizer \( A_{\mathcal{C}} \) through the intervening symmetrizer set \( S_{\Theta} \).

To clarify the statement of the Propagation-Theorem 2, consider for example the operator \( O \)

\[
O := \begin{bmatrix}
S_{\Theta} & A_{\Theta} & S_{\Theta \setminus \mathcal{R}}
\end{bmatrix}.
\]

(76)

where the Young tableau \( \Theta \) is

\[
\Theta = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 \\
6 & 7
\end{bmatrix}.
\]

(77)

The operator (76) meets the conditions laid out in Theorem 2: The sets \( S_{\Theta} \) and \( S_{\Theta \setminus \mathcal{R}} \) differ only by one symmetrizer, namely \( S_{\mathcal{R}} = S_{67} \), which corresponds to the row \((6,7)\) of the tableau \( \Theta \). Indeed, we find that the amputated tableau \( \Theta_c[6,7] \) is rectangular,

\[
\Theta_c[(6,7)] = \begin{bmatrix}
1 & 2 \\
4 & 5 \\
6 & 7
\end{bmatrix},
\]

(78)

\(^{14}\)Recall the Hermiticity of (sets of) (anti-) symmetrizers, eq. (10).
where we have highlighted the row corresponding to the symmetrizer $S_{67}$ in blue. We therefore may commute the symmetrizer $S_{67}$ from the left of $O$ to the right in accordance with the Propagation-Theorem 2,

$$O := \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}$$

(79)

Furthermore, if we factor the symmetrizer $S_{67}$ out of the set $S_\Theta$ (i.e. if we write $S_\Theta = S_{67} S_\Theta$) before commuting it through, we obtain

$$O := \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} \xrightarrow{\text{Thm. 2}} \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}.$$

(80)

We thus kept $S_{67}$ on both sides of the operator, making the Hermiticity of $O$ explicit.

Having understood the statement of the Propagation-Theorem 2, we will foreshadow the strategy of the proof (which is given in section 4.1). Consider the operator

$$P := \begin{array}{c}
\end{array}.$$

(81)

which satisfies all conditions posed in the Propagation-Theorem 2. It thus immediately follows from the Theorem that

$$\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}.$$

(82)

We would however like to show how this comes about explicitly, thus alluding to the strategy used in the proof of Theorem 2. In particular, we will use a trick originally used by Keppeler and Sjödahl in the appendix of [15].

We begin by factoring a transposition out of each symmetrizer on the left; this will not alter the operator $P$ in any way since

$$\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}.$$

(83)

We thus have that

$$P = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array},$$

(84)

where we have marked the top and bottom antisymmetrizer in $P$ as $\textcircled{1}$ and $\textcircled{2}$ respectively. It is important to notice that these two antisymmetrizers would be indistinguishable if it weren’t for the labelling. We may thus exchange them (paying close attention to which line enters and exits which antisymmetrizer), without changing the operator $P$,

$$\begin{array}{c}
\end{array} \xrightarrow{\text{Thm. 2}} \begin{array}{c}
\end{array}.$$

(85)
We have thus effectively commuted the transpositions marked in red through the set of antisymmetrizers from the left to the right. We may now absorb the transposition on top into the right symmetrizer,

\[
\begin{align*}
\begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) node[fill=red] {} -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array}
\end{align*}
\]

We therefore showed that

\[
\begin{align*}
P &= \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array}
\end{align*}
\]

It now remains to add up the two different expressions of \( P \) found in (87), and multiply this sum by a factor \( \frac{1}{2} \),

\[
\begin{align*}
\frac{1}{2} \left( \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) node[fill=red] {} -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} + \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} \right) &= \frac{1}{2} \left( \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} + \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} \right).
\end{align*}
\]

However, since

\[
\frac{1}{2} \left( \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} + \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} \right) = \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array},
\]

equation (88) simply becomes

\[
\begin{align*}
P &= \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array}.
\end{align*}
\]

Performing the above process in reverse then yields

\[
\begin{align*}
P &= \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} = \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} = \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array}.
\end{align*}
\]

as desired. That this strategy can be applied to the operator (76) can be seen via factoring out a symmetrizer of length 2 from each symmetrizer,

\[
\begin{align*}
O &= \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} = \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array} = \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (1,1);
\end{tikzpicture}}
\end{array}.
\end{align*}
\]

The part marked \( \tilde{O} \) in (92) can now be dealt with exactly as in the previous example, allowing one to commute the symmetrizer \( S_{67} \) from the left to the right. It remains to reabsorb the extra symmetrizers to obtain the
desired result (80),

\[
O = \begin{array}{c} \begin{array}{c} \text{absorb} \\
\text{absorb}
\end{array} \\
\end{array} = \begin{array}{c} \begin{array}{c} \text{absorb} \\
\text{absorb}
\end{array} \\
\end{array} = \begin{array}{c} \begin{array}{c} \text{absorb} \\
\text{absorb}
\end{array} \\
\end{array} .
\]  

(93)

In particular, the ability to factor out an operator \( \hat{O} \) within \( O \) is encoded in the requirement that the amputated tableau \( \tilde{\Theta}_{c[R]} \) be rectangular, as is discussed in section 4.1.3.

More generally, if \( \Theta = \tilde{\Theta} \) is a semi-standard irregular tableau, then the following set of conditions on the amputated tableau will determine whether certain symmetrizers can be propagated through antisymmetrizers and vice versa:

**Theorem 3 (generalized propagation rules)** A form of the Propagation-Theorem 2 holds also if \( \Theta = \tilde{\Theta} \) is a semi-standard irregular tableau:

\[
S_{\tilde{\Theta}} A_{\tilde{\Theta}} S_{\tilde{\Theta}\setminus R} = S_{\tilde{\Theta}\setminus R} A_{\tilde{\Theta}} S_{\tilde{\Theta}} = S_{\tilde{\Theta}} A_{\tilde{\Theta}} S_{\tilde{\Theta}}
\]  

if all rows in \( \tilde{\Theta}_{c[R]} \) have equal lengths and

\[
A_{\tilde{\Theta}} S_{\tilde{\Theta}} A_{\tilde{\Theta}\setminus C} = A_{\tilde{\Theta}\setminus C} S_{\tilde{\Theta}} A_{\tilde{\Theta}} = A_{\tilde{\Theta}} S_{\tilde{\Theta}} A_{\tilde{\Theta}}
\]  

if all columns in \( \tilde{\Theta}_{r[C]} \) have equal lengths.

Requiring the amputated tableaux to have rows (resp. columns) of equal lengths rather than them being rectangular allows for the fact that \( \tilde{\Theta} \) (for \( \tilde{\Theta} \) being a semi-standard irregular tableau) may contain disjoint pieces – this cannot happen for Young tableaux.\(^{15}\)

The proof of Theorem 2 only has to be altered in minor ways to become a proof of the generalized Propagation-Theorem 3. These alterations are given in section 4.2.

As an example of Theorem 3, consider the semi-standard irregular tableau

\[
\tilde{\Theta} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
7 & 5 & 6 & \\
8 & 9 \\
\end{array}
\]

with corresponding operator \( \bar{Y}_{\tilde{\Theta}} = \)

\[ \begin{array}{c} \begin{array}{c} \text{absorb} \\
\text{absorb}
\end{array} \\
\end{array} \]

(96)

cf. eq. (12). \( \tilde{\Theta} \) is neither a Young tableau nor does it uniquely specify \( \bar{Y}_{\tilde{\Theta}} \) as we could swap 3 and 4 around and still obtain an appropriate tableau for \( \bar{Y}_{\tilde{\Theta}} \). Let us now consider the operator

\[
O = \begin{array}{c} \begin{array}{c} \text{absorb} \\
\text{absorb}
\end{array} \\
\end{array} ,
\]  

(97)

\(^{15}\)It should be noted that missing boxes within a row/column reduce its length, for example the first row of the tableau

\[
\begin{array}{ccc}
\begin{array}{c} \text{absorb} \\
\end{array} \\
\end{array}
\]

has length 3 but the second row only has length 2, thus corresponding to symmetrizers of length 3 and 2 respectively.
where we would like to commute the symmetrizer $S_{\mathcal{R}} = S_{\Theta}$ from the left set $S_{\Theta}$ to the right set $S_{\Theta \setminus \mathcal{R}}$. This symmetrizer corresponds to the row $(8, 9)$ in $\tilde{\Theta}$.

$$\tilde{\Theta} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 \\ 7 & 8 & 9 \end{bmatrix};$$  

(98)

all rows in the column-amputated tableau of $\tilde{\Theta}$ according to the row $(8, 9)$ have equal lengths

$$\tilde{\Theta}_c[(8, 9)] = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 8 & 9 \end{bmatrix}. $$  

(99)

Thus, by Theorem 3 we may propagate the symmetrizer $S_{89}$ from the left to the right, yielding

$$O = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$  

(100)

The reason why the propagation rule also works in this general case is because an operator $\tilde{O}$ can be identified within $O$ in a similar way as was done in (92), see section 4.2.

### 4.1 Proof of Theorem 2 (propagation rules)

In this section, we provide a proof for eq. (74) of the Propagation-Theorem 2,

$$O = S_{\Theta} A_{\Theta} S_{\Theta \setminus \mathcal{R}} = S_{\Theta} A_{\Theta} S_{\Theta} = S_{\Theta \setminus \mathcal{R}} A_{\Theta} S_{\Theta}. $$  

(101)

The proof of eq. (75) (i.e. where the operator $O$ is of the form $O := A_{\Theta} S_{\Theta} A_{\Theta \setminus \mathcal{C}}$) only changes in minor ways; these differences are discussed in section 4.1.4.

The steps of the proof given in the present section can become rather abstract; we therefore chose to accompany it with several schematic drawings for clarification.

The strategy of this proof will be as follows: We start by understanding what the conditions posed in Theorem 2 (in particular the requirement that the amputated tableau be rectangular) imply for the operator $O$. Then, we use a trick originally given in [15] to propagate the constituent permutations of the symmetrizer $S_{\mathcal{R}}$ through the set $A_{\Theta}$ to the right of $O$; we recall that each symmetrizer is by definition the sum of its constituent permutations,

$$S_{\mathcal{R}} = \frac{1}{\text{length}(S_{\mathcal{R}})!} \sum_{\rho} \rho,$$  

(102)

where $\rho$ are the constituent permutations of $S_{\mathcal{R}}$, for example

$$\prod_{S_{123}} = \frac{1}{3!} \left( \frac{1}{3!} + \frac{1}{14} + \frac{1}{23} + \frac{1}{123} + \frac{1}{132} \right).$$  

(103)
The operators resulting from this propagation-process will then be summed up in the appropriate manner to recombine to the symmetrizer \( S_R \) on the right hand side of \( O \), yielding the desired result. Let us thus begin:

Let \( O := S_\Theta A_\Theta S_{\Theta \setminus R} \) be an operator as stated in the Propagation-Theorem 2, and let the sets \( S_\Theta \) and \( S_{\Theta \setminus R} \) only differ by one symmetrizer, \( S_R \), with \( S_R \) corresponding to the row \( R \) in the Young tableau \( \Theta \). We will represent \( O \) schematically as

\[
O = \begin{array}{c}
S_{\Theta \setminus R} \\
A_\Theta \\
S_{\Theta \setminus R}
\end{array},
\]

where we used the fact that \( S_\Theta = S_{\Theta \setminus R} S_R \), eq. (72).

### 4.1.1 Unpacking the Theorem conditions:

For the amputated tableau \( \Theta' [R] \) to be rectangular, we clearly require all columns that overlap with the row \( R \) to have the same length. However, this is equivalent to saying that every row in \( \Theta \) other than row \( R \) has to have length greater than or equal to \( \text{length}(R) \): Suppose \( R' \) is a row in \( \Theta \) with \( \text{length}(R') < \text{length}(R) \). Hence, by definition of Young tableaux, the row \( R' \) is situated below the row \( R \). Furthermore, by the left-alignedness of Young tableaux, this means that all the columns that overlap with \( R' \) also overlap with \( R \); let us denote this set of columns overlapping with the row \( R' \) by \( C_{R'} \). In addition, there will be at least one column that overlaps with \( R \) but does not overlap with \( R' \), since \( \text{length}(R) > \text{length}(R') \); let us denote this column by \( C \). Schematically, this situation can be depicted as

\[
\begin{array}{c}
C_{R'} \\
C \\
\end{array}
\]

It then follows by the top-alignedness of Young tableaux that \( C \) is strictly shorter than the columns in the set \( C_{R'} \), as is indicated in (105). This poses a contradiction, as we need all columns that overlap with \( R \) to be of the same length for the tableau \( \Theta' [R] \) to be rectangular. Hence, there cannot be a row in \( \Theta \) whose length is strictly less than the length of \( R \).

Let \( C_R \) denote the set of columns overlapping with the row \( R \). Since \( R \) is established to be (one of) the shortest row(s) in \( \Theta \), the top-alignedness and left-alignedness conditions of Young tableaux imply thus that every other row in \( \Theta \) also overlaps with every column in \( C_R \).

In the language of symmetrizers, the discussion given above can be formulated as:

1. \( S_R \) (corresponding to the row \( R \) of \( \Theta \)) is the shortest symmetrizer in the set \( S_\Theta \).
2. Each leg of \( S_R \) enters an antisymmetrizer in \( A_\Theta \) of equal length; let us denote this subset of antisymmetrizer by \( A' S_R \) (this set of antisymmetrizers correspond to the set of columns \( C_R \)).
3. Each symmetrizer in \( S_\Theta \) has one common leg with each antisymmetrizer in \( A' S_R \) (since each row in \( \Theta \) overlaps with each column in \( C_R \)).

\[\text{Similar to what was done in the example (88).}\]
4. Since, by the assumption of the Propagation-Theorem, $S_{\Theta \setminus \mathcal{R}}$ and $S_{\Theta}$ only differ by the symmetrizer $S_{\mathcal{R}}$, this means that each symmetrizer in the set $S_{\Theta \setminus \mathcal{R}}$ has a common leg with each antisymmetrizer in the set $A' S_{\mathcal{R}}$.

4.1.2 Strategy of the proof:

In this proof, we will use the fact that the symmetrizer $S_{\mathcal{R}}$ by definition is the sum of all permutations of the legs over which $S_{\mathcal{R}}$ is drawn. If $S_{\mathcal{R}}$ has length $k$, then this sum will consist of $k!$ terms, and there will be a constant prefactor $1/k!$; this was exemplified in (103). In particular, if $\lambda$ is a particular permutation in the series expansion of $S_{\mathcal{R}}$, then we will show that

$$O = \underbrace{\ldots}_{\lambda} =: O^\lambda.$$  \hspace{1cm} (106)

Since the constituent permutations of a symmetrizer over a subset of factors in $V^{\otimes n}$ form a sub-group of $S_n$ [13], it immediately follows that every constituent permutation of $S_{\mathcal{R}}$ can be written as a product of constituent transpositions of $S_{\mathcal{R}}$.\footnote{A proof that any permutation in $S_n$ can be written as the product of transpositions can be found in [22] and other standard textbooks.} It thus suffices to show that (106) holds for $\lambda$ being a constituent transposition of $S_{\mathcal{R}}$ (i.e. that we may propagate a transposition from the left symmetrizer $S_{\mathcal{R}}$ to the right), as then any other permutation can be produced by the successive propagation of transpositions.

4.1.3 Propagating transpositions:

The technique used to permute transpositions through the set of antisymmetrizers, as described in the previous paragraph, was inspired by an example presented in the Appendix of [15].

Suppose the set $A' S_{\mathcal{R}}$ introduced in condition 2 of the previous discussion contains $n$ antisymmetrizers. Then, by observations 1-4, the length of $S_{\mathcal{R}}$ will be exactly $n$, and each other symmetrizer in $S_{\Theta}$ (and thus also each symmetrizer in $S_{\Theta \setminus \mathcal{R}}$) will have length at least $n$. We may then factor “the symmetrizer $S_{\mathcal{R}}$” (i.e. a symmetrizer of length $n$) out of each symmetrizer in the sets $S_{\Theta}$ and $S_{\Theta \setminus \mathcal{R}}$, where we lumped together the antisymmetrizers $A' S_{\mathcal{R}}$ and the rest $(A_{\Theta} \setminus A' S_{\mathcal{R}})$. We will denote the left set of $S_{\mathcal{R}}$’s (which were factored out of $S_{\Theta}$) by $\{ S_{\mathcal{R}} \}$, and the right set of $S_{\mathcal{R}}$’s (which were factored out...
of $S_{\Theta \backslash \mathcal{R}}$ by $\{S_R\}_r$, see Figure 4.1. From now onwards, we will focus the part of the operator $O$ that is highlighted blue in Figure 4.1.

Figure 4.1: This diagram schematically depicts the operator $O$, (104), with a symmetrizer $S_R$ factored out of each symmetrizer in $S_\Theta$ and $S_{\Theta \backslash \mathcal{R}}$. The left set of $S_R$’s will be denoted by $\{S_R\}_l$, and the right set of $S_R$’s by $\{S_R\}_r$. In this proof, we will focus on the part of the operator that is highlighted in blue. This part will be denoted by $\tilde{O}$.

The significance of the operator $\tilde{O}$ in Fig. 4.1: The left part of $\tilde{O}$, namely $\{S_R\}_l \cdot A'_S$, by itself corresponds to a rectangular tableau, as each symmetrizer has the same length and each antisymmetrizer has the same length. This will be important, since we will need $\tilde{O}$ to stay unchanged under a swap of any pair of antisymmetrizers in $A'_S$ in order to commute the constituent permutations of $S_R$ (in analogy to what was done in example (85) – this will become evident below). Note that $\tilde{O}$ would not stay unchanged under such a swap if the antisymmetrizers in $A'_S$ had different length’s and would thus be distinguishable. In particular, the operator $\tilde{O}$ corresponds to the amputated tableau $\Theta_c[R]$, which is indeed rectangular by requirement of the Propagation-Theorem 2. This requirement therefore translates into the ability of finding an operator $\tilde{O}$ within the operator $O$, thus allowing the necessary propagation of permutations.

Suppose that $S_\Theta$ contains exactly $m$ symmetrizers (hence $S_{\Theta \backslash \mathcal{R}}$ contains $(m - 1)$ symmetrizers). Then $\{S_R\}_l$ will contain $m$ symmetrizers and $\{S_R\}_r$ contains $(m - 1)$ symmetrizers.

Furthermore, since each symmetrizer in $\{S_R\}_l$ has a common leg with each of the $n$ antisymmetrizer in $A'_S$, we may choose the $k^{th}$ leg exiting each symmetrizer in $\{S_R\}_l$ to enter the $k^{th}$ antisymmetrizer in $A'_S$.$^{18}$

$^{18}$ We may always choose to order index legs this way, since, within a symmetrizer, we may re-order index lines at will without changing the symmetrizer.
We may schematically draw this, as

\[
\tilde{O} = \quad \text{Figure 4.1}
\]

In (108), we have labelled the index lines for clarity; from Figure 4.1 it however should be noted that the \(i^{th}\) index in the above graphic is not necessarily the \(i^{th}\) index line in the operator \(O\). The part of the operator \(O\) highlighted in blue in Figure 4.1, operator \(\tilde{O}\), can then be represented as

\[
\tilde{O} = \quad \text{Figure 4.1}
\]

where the last symmetrizer in the set \(\{S_R\}_l\) is the symmetrizer \(S_R\) which we eventually wish to commute through to the right. In (109), we labeled the first and the second antisymmetrizer of the set \(A'S_n\) by \(\textcircled{1}\) and \(\textcircled{2}\) respectively for future reference.

As previously stated, we strive to commute constituent transpositions \((ij)\) of the symmetrizer \(S_R \in \{S_R\}_l\) through the set of anti-symmetrizers \(A'S_n\) to the right set \(\{S_R\}_r\). We achieve this goal in the following way: We first factor the transposition \((ij)\) out of each symmetrizer in \(\{S_R\}_l\). By doing so, the \(i^{th}\) leg of each symmetrizer now enters the \(j^{th}\) antisymmetrizer and vice versa; all the other legs stay unchanged. We may now “remedy” this change by swapping the \(i^{th}\) and \(j^{th}\) antisymmetrizer, similar to what we did in example (85). For instance, if \(i = 1\) and \(j = 2\), we factor the transposition \((12)\) out of each of the symmetrizers of \(\{S_R\}_l\),

\[
\tilde{O} = \quad \text{Figure 4.1}
\]
and then swap the first and second antisymmetrizer, which are marked as 1 and 2 respectively. The key observation to make is that the antisymmetrizers 1 and 2 would be indistinguishable if it weren’t for the labeling. Thus, the set $A'S_R$ remains unchanged even when the swap between antisymmetrizers $i$ (1) and $j$ (2) is carried out. This trick of swapping identical antisymmetrizers was initially used by KS in an example in the appendix of [15].

After we swapped the two antisymmetrizers, the $i^{th}$ leg of each symmetrizer in $\{S_R\}_l$ once again enters the $i^{th}$ antisymmetrizer and same is true for the $j^{th}$ leg. However, now the legs exiting the $i^{th}$ antisymmetrizer $A'S_R$ enter the symmetrizers in $\{S_R\}_r$ in the $j^{th}$ position, and the legs exiting the $j^{th}$ antisymmetrizer enter the symmetrizers in $\{S_R\}_r$ in the $i^{th}$ position. Thus, we have effectively commuted the transpositions $(ij)$ past the set $A'S_R$.

\[
\tilde{O} = \ldots
\]

(111)

All but one of the propagated transpositions $(ij)$ can then be absorbed into the symmetrizers of the set $\{S_R\}_r$. The bottom transposition will remain, as there is no symmetrizer in the set $\{S_R\}_r$ to absorb this transposition.

\[
\tilde{O} = \ldots
\]

(112)

We then re-absorb the sets $\{S_R\}_l$ and $\{S_R\}_l$ into $S_\Theta$ and $S_{\Theta\setminus R}$ respectively. This clearly leaves the transposition $(ij)$ un-absorbed. Thus, we have shown that

\[
O = \begin{array}{c}
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
\end{array} A_{\Theta} \begin{array}{c}
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
S_{w,x} \\
\end{array} = O^\lambda
\]

(113)

for $\lambda = (ij)$ being a transposition. We can repeat the above procedure with any constituent tranposition of $S_R$.

If $\lambda$ is a constituent permutation (not necessarily transposition) of $S_R$, we can also propagate $\lambda$ to the right hand side, since any such permutation $\lambda$ can be written as a product of constituent transpositions;
propagating the permutation $\lambda$ then corresponds to successively propagating the constituent transpositions through to the right, yielding

$$\tilde{O} =$$

for any constituent permutation $\lambda$ of $S_R$.

In order to obtain the missing symmetrizer on the right, it remains to add up all the terms $O^\lambda$ – since $S_R$ is assumed to have length $n$ there will be exactly $n!$ such terms. By relation (113), we know that each of these terms is equal to $O$, yielding the following sum,

$$\frac{1}{n!} \sum_1^{n!} \left( \begin{array}{c} S_{\Theta,C} \\ A_\Theta \\ S_{\Theta,C} \end{array} \right) = \frac{1}{n!} \sum_{\lambda \in S_n} \left( \begin{array}{c} S_{\Theta,C} \\ A_\Theta \\ S_{\Theta,C} \end{array} \right).$$

The left hand side of the above equation merely becomes $\frac{n!}{n!} O = O$. The right hand side yields the desired symmetrizer, such that

$$O = S_\Theta A_\Theta S_\Theta =$$

where we used the fact that $S_\Theta = S_{\Theta \setminus R} S_R = S_R S_{\Theta \setminus R}$ by assumption of Theorem 2 (c.f. eq. (72)). In particular, using the fact that $O$ as given in (116) is clearly Hermitian, $O^\dagger = O$, we find that

$$O = S_{\Theta,C} A_\Theta S_{\Theta,C} = S_{\Theta,C} A_\Theta S_{\Theta,C} = S_{\Theta,C} A_\Theta S_{\Theta,C} = O^\dagger,$$

as required.

### 4.1.4 Propagating antisymmetrizers:

The proof of the Propagation-Theorem 2 for an operator $Q$ of the form $Q := A_\Theta S_\Theta A_{\Theta \setminus C}$ is very similar to the proof given for the operator $O$, however there are some differences on which we wish to comment here: If we want to propagate an antisymmetrizer $A_C$ corresponding to a column $C$ in $\Theta$ from $A_\Theta$ to $A_{\Theta \setminus C}$, we first

19This was already exhibited in example (88).
20By the Hermiticity of (sets of) (anti-) symmetrizer, see (10).
check that the amputated tableau $\mathcal{G}_r[C]$ is rectangular. If so, we are able to isolate an operator $\bar{Q}$ within $Q$ the same way as we did for $\bar{O}$ within $O$, see Figure 4.1, where

$$\bar{Q} := \{A_C\}_l S'A_C \{A_C\}_r.$$  \hfill (118)

When we propagate a transposition $(ij)$ from the left to the right of $\bar{Q}$, we need to tread with care as this will induce a factor of $(-1)$; this factor however will be vital in the recombination-process where we recreate the antisymmetrizer $A_C$ by summing constituent permutations: Suppose the set $\{A_C\}_l$ contains $m$ antisymmetrizers, then the set $\{A_C\}_r$ contains $(m-1)$ antisymmetrizers. If we now factor a transposition $(ij)$ out of each antisymmetrizer in $\{A_C\}_l$, we obtain a factor of $(-1)^m$. Swapping the $i^{th}$ and $j^{th}$ antisymmetrizers will not induce an extra minus-sign, but absorbing the transpositions into the antisymmetrizers in the set $\{A_C\}_r$ will produce an extra factor of $(-1)^{m-1}$. Thus, for each transposition we commute through, we obtain a factor of $(-1)^{2m-1} = -1$, which is the signature of a transposition. In particular, each permutation $\lambda$ (consisting of a product of transpositions) will induce a pre-factor of $\text{sign}(\lambda)$ when commuted through, yielding

$$\bar{Q} = \text{sign}(\lambda)\bar{Q}^\lambda.$$  \hfill (119)

However, since an antisymmetrizer is by definition the sum of its constituent permutations weighted by their signatures, for example,

$$\prod = \frac{1}{3!} (-1 -1 +1 +1 +1),$$  \hfill (120)

equation (119) is exactly what we need in order to be able to reconstruct the antisymmetrizer $A_C$ on the right of the operator $\bar{Q}$ by summing up the terms $\text{sign}(\lambda)\bar{Q}^\lambda$. Re-absorbing $\{A_C\}_l$ into $A_{\Theta}$ and $\{A_C\}_r$ into $A_{\Theta\backslash C}$ yields the desired eq. (75).

\subsection*{4.2 Proof of Theorem 3 (generalized propagation rules)}

Let $\tilde{\Theta}$ be a semi-standard irregular tableau and let $O$ be an operator of the form

$$O = S_{\tilde{\Theta}} A_{\tilde{\Theta}} S_{\tilde{\Theta}} \backslash R,$$  \hfill (121)

where $R$ denotes a particular row in $\tilde{\Theta}$. Having gone through the proof of Theorem 2, it is clear that the symmetrizer $S_R$ can be propagated through the set $A_{\tilde{\Theta}}$ if the following conditions are met:

1. Each leg in $S_R$ needs to enter an antisymmetrizer of the same length to perform the swapping procedure described in section 4.1.3. Call the set of antisymmetrizers sharing legs with $S_R$ $A'S_n$. If $S_R$ has length $n$, then $A'S_n$ contains exactly $n$ antisymmetrizers.

2. The remaining symmetrizers in $O$ may have

- 0 legs in common with the antisymmetrizers in $A'S_n$, i.e., they are not affected by swapping the antisymmetrizers in $A'S_n$ and thus do not contribute to $\tilde{O}$ (c.f. section 4.1.3), or
- $n$ legs in common with the antisymmetrizers in $A'S_n$ (one with each antisymmetrizer) in order to be able to absorb any permutation arising from swapping the antisymmetrizers in $A'S_n$.

\hfill 29
The requirement that $\tilde{\Theta}_c[\mathcal{R}]$ only contains rows of equal length ensures that the above conditions are met:

Differently to Young tableaux, if $\tilde{\Theta}$ is a semi-standard irregular tableau then removing columns in order to form $\tilde{\Theta}_c[\mathcal{R}]$ may remove whole rows in the process, for example

$$\tilde{\Theta} = \begin{bmatrix} 5 & 1 \\ 7 & 2 & 3 & 6 \\ 4 & 8 \end{bmatrix} \longrightarrow \tilde{\Theta}_c[(4, 8)] = \begin{bmatrix} 7 & 2 \\ 4 & 8 \end{bmatrix},$$

where the row $(5, 1)$ was removed. However, the symmetrizers corresponding to such rows have no common legs with the antisymmetrizers in $A'S_{\mathcal{R}}$, as is evident from the example ($S_{51}$ shares no legs with $A_{74}$ and $A_{28}$). This is also due to the fact that $\tilde{\Theta}$ is semi-standard, thus not allowing any of its entries to occur more than once.

Hence, the only symmetrizers that share index legs with the antisymmetrizers in $A'S_{\mathcal{R}}$ are those corresponding to the rows of $\tilde{\Theta}$ which have not been fully deleted (although maybe in part) in $\tilde{\Theta}_c[\mathcal{R}]$. Let us denote the set of these symmetrizers by $S'S_{\mathcal{R}}$. The requirement that each row in $\tilde{\Theta}_c[\mathcal{R}]$ has the same length ensures each symmetrizer in $S'S_{\mathcal{R}}$ shares exactly one leg with each antisymmetrizer in $A'S_{\mathcal{R}}$ and thus has length $\geq n$ (this was already argued in section 4.1.1).

Thus, all conditions required to perform the propagating procedure already explained in sections 4.1.2 and 4.1.3 are met, allowing us to propagate $S_{\mathcal{R}}$ through $A'S_{\mathcal{R}}$ at will,

$$O = S_{\tilde{\Theta}} A_{\tilde{\Theta}} S_{\tilde{\Theta}}\backslash\mathcal{R} = S_{\tilde{\Theta}} A_{\tilde{\Theta}} S_{\tilde{\Theta}} = S_{\tilde{\Theta}}\backslash\mathcal{R} A_{\tilde{\Theta}} S_{\tilde{\Theta}}.$$  

The proof for symmetrizers and antisymmetrizers exchanged follows similar steps and is thus left as an exercise to the reader.

5 Conclusion

We have established two classes of rules which allow us to simplify and manipulate birdtrack operators. The first such class are the cancellation rules, Theorem 1 and Corollary 2, allowing us to shorten the birdtrack expression of an operator. The simplification reached in this process is often very significant, as is exemplified in Figure 5.2. Shorter expressions of operators are desirable, as they are more practical to work with in that they allow for faster automated computation. Furthermore, short expressions offer a visual assessment of their action, making them more intuitive to work with.

The second class of rules are propagation rules, Theorems 2 and 3. Their use lies in the ability to make Hermitian birdtrack operators explicitly symmetric, thus exposing their innate Hermiticity. Since birdtracks are a graphical tool designed to make working with them more intuitive, it is desirable to visually expose the inherent properties of the birdtrack operators.

To illustrate how powerful these simplification rules are we show their effect on the Hermitian KS projector $[15]$ associated with the Young tableau

$$\Phi := \begin{bmatrix} 1 & 2 & 4 & 7 \\ 3 & 6 \\ 5 & 8 \\ 9 \end{bmatrix}.$$  

30
The recursive KS algorithm leads to Hermitian Young projection operator $P_\Phi$, with $\bar{P}_\Phi$ being of impressive length: It contains 127 sets of symmetrizers and antisymmetrizers and its Hermiticity is not visually apparent, see Figure 5.2. The cancellation rules achieve a tremendous simplification: the result contains only 13 sets. Furthermore, multiple applications of the propagation rules can be used to translate this into an explicitly symmetric form.

![Diagram](image)

Figure 5.2: For a size comparison, this figure shows the birdtrack arising from the iterative KS-construction in the top line, the much shortened version arising from the cancellation rule in the second line, and the explicitly symmetric version achieved via the propagation rules in the third line.

In fact, the shorter, explicitly symmetric result can be constructed directly, without first constructing the KS-operator and then applying the cancellation rules; we provide the tools to do so in a separate paper [1]. The algorithm allowing us to do this is the MOLD-algorithm [1]. When used to construct the Hermitian Young projection operators in Mathematica, the reduction in computational cost is impressive: On a modern laptop, the final result shown in Fig. 5.2 is obtained approximately 18600 times faster than the KS-equivalent even before simplification rules are implemented to shorten the result. Thus, the MOLD construction offers a significant improvement over the KS construction. The proof of the MOLD algorithm relies heavily on the manipulation rules laid out in this paper.

Augmenting the Hermitian Young projection operators with transition operators yields an alternative basis of the algebra of invariants of $SU(N)$ over $V^\otimes m$ [2]. The construction algorithm of the transition operators again is built upon the simplification rules presented in this paper. With a basis for $API(SU(N), V^\otimes m)$ consisting of projection and transition operators, one can construct a mutually orthogonal, complete basis for the singlet states of $SU(N)$ over $V^\otimes m \otimes (V^*)^\otimes m$ (and more generally over $V^\otimes m \otimes (V^*)^\otimes n$) [14]. These singlets are directly applicable to QCD, as they are needed to form Wilson line correlators used in the JIMWLK-framework [8] as well as a modern treatment of jet-evolution equations (see e.g. [9]), the infrared structure of QCD in form of gluon exchange webs (see e.g. [10]), GPD’s and TMD’s (see e.g. [11]). The references here only serve to mark a specific reference point we find intriguing and are by no means exhaustive: In fact, almost every branch of QCD in which the factorization Theorems apply makes use of Wilson line correlators and any attempt at completeness would be futile.

Besides their physics applications (which is the most appealing quality to the authors of this paper), birdtracks are also immensely useful in the study of the representation theory of semi-simple Lie groups, as is exhibited in [1, 6, 15, 23]. With the recent interest in Hermitian Young projection operators, birdtracks thus promise interesting further developement in this branch of mathematics. It is hoped that the simplification rules given here encourage the use of birdtracks as a viable tool for calculation.

**Acknowledgements:** H.W. is supported by South Africa’s National Research Foundation under CPRR grant nr 90509. J.A-Z. was supported (in sequence) by the postgraduate funding office of the University of Cape Town (2014), the National Research Foundation (2015) and the Science Faculty PhD Fellowship of the University of Cape Town (2016).
References

[1] J. Alcock-Zeilinger and H. Weigert, “Compact Hermitian Young Projection Operators,” arXiv:1610.10088 [math-ph].

[2] J. Alcock-Zeilinger and H. Weigert, “Transition Operators,” arXiv:1610.08802 [math-ph].

[3] R. Penrose, “Applications of negative dimension tensors,” in Combinatorial mathematics and its applications, D. Welsh, ed., pp. 221–244. Academic Press, New York, 1971.

[4] R. Penrose, “Angular momentum: An approach to combinatorical space-time,” in Quantum Theory and Beyond, T. Bastin, ed. Cambridge University Press, Cambridge, 1971.

[5] R. Penrose and M. A. H. MacCallum, “Twistor theory: An Approach to the quantization of fields and space-time,” Phys. Rept. 6 (1972) 241–316.

[6] P. Cvitanović, Group theory: Birdtracks, Lie’s and exceptional groups. Princeton, USA: Univ. Pr., 2008. http://birdtracks.eu.

[7] A. Young, “On Quantitative Substitutional Analysis - III,” Proc. London Math. Soc. 28 no. 2, (1928) 255–292.

[8] C. Marquet and H. Weigert, “New observables to test the Color Glass Condensate beyond the large-N_c limit,” Nucl. Phys. A843 (2010) 68–97, arXiv:1003.0813 [hep-ph].

[9] H. Weigert, “Nonglobal jet evolution at finite N(c),” Nucl. Phys. B685 (2004) 321–350, arXiv:hep-ph/0312050 [hep-ph].

[10] G. Falcioni, E. Gardi, M. Harley, L. Magnea, and C. D. White, “Multiple Glnon Exchange Webs,” JHEP 10 (2014) 10, arXiv:1407.3477 [hep-ph].

[11] J. Bomhof, P. J. Mulders, and F. Pijlman, “The Construction of gauge-links in arbitrary hard processes,” Eur. Phys. J. C47 (2006) 147–162, arXiv:hep-ph/0601171 [hep-ph].

[12] W. Fulton and J. Harris, Representation Theory - A First Course. Springer, USA, 2004.

[13] W. K. Tung, Group Theory in Physics. Singapore: World Scientific, 1985.

[14] J. Alcock-Zeilinger and H. Weigert, “A Construction Algorithm for the Singlets of SU(N) and what they tell us about Wilson Line Correlators.” (in preparation).

[15] S. Keppeler and M. Sjödahl, “Hermitian Young Operators,” J. Math. Phys. 55 (2014) 021702, arXiv:1307.6147 [math-ph].

[16] T. Lappi, A. Rammath, K. Rummukainen, and H. Weigert, “JIMWLK evolution of the odderon,” Phys. Rev. D94 (2016) 054014, arXiv:1606.00551 [hep-ph].

[17] W. Fulton, Young Tableaux. Cambridge University Press, Cambridge, 1997.

[18] B. Sagan, The Symmetric Group - Representations, Combinatorial Algorithms, and Symmetric Functions. New York, USA: Springer, 2nd ed., 2000.

[19] S. Chowla, I. N. Herstein, and W. K. Moore, “On recursions connected with the symmetric groups I,” Canad. J. Math. 3 (February, 1951) 328–334. http://cms.math.ca/10.4153/CJM-1951-038-3.
[20] D. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*. Clarendon, UK: Oxford Univ. Pr., 2nd ed., 1950.

[21] H. Weyl, *The Classical Groups: Their Invariants and Representations*. Princeton, USA: Univ. Pr., 2nd ed., 1946.

[22] M. Artin, *Algebra*. Prentice Hall, Boston, 2nd ed., 2011.

[23] S. Keppeler and M. Sjödahl, “Orthogonal multiplet bases in SU(Nc) color space,” *JHEP* **09** (2012) 124, arXiv:1207.0609 [hep-ph].