THE WEIGHTED MEAN MATRIX WITH WEIGHT SEQUENCE $w_n = 2n + 1$ IS A HYPONORMAL OPERATOR ON $\ell^2$

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Abstract. Posinormality is used to demonstrate that the weighted mean matrix associated with the sequence of odd positive integers is a hyponormal operator on $\ell^2$.

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1. Introduction

If $B(H)$ is the set of all bounded linear operators on a Hilbert space $H$, then $A \in B(H)$ is said to be is posinormal (see [1]) if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, called the interrupter, and $A$ is hyponormal if $A^*A - AA^* \geq 0$. Hyponormal operators are necessarily posinormal.

A lower triangular infinite matrix $M = [m_{ij}] \in B(\ell^2)$ is factorable if its entries are of the form

$$m_{ij} = \begin{cases} a_ic_j & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}$$

where $a_i$ depends only on $i$ and $c_j$ depends only on $j$. A weighted mean matrix is a lower triangular matrix with entries $w_j/W_i$, where $\{w_j\}$ is a nonnegative sequence with $w_0 > 0$, and $W_i = \sum_{j=0}^i w_j$. A weighted mean matrix is factorable, with $a_i = 1/W_i$ and $c_j = w_j$ for all $i,j$.

Under consideration here will be the weighted mean matrix $M$ associated with the weight sequence $w_n = 2n + 1$. This example is not easily seen to be hyponormal directly from the definition, and it fails to satisfy the sufficient conditions for hyponormality given in [3, Corollary 1]. However, it survives the necessary condition given in [3, Corollary 2].

Since this example does not satisfy the key lemma used in [1] for the case $w_n = n + 1$, a somewhat different approach will be required. The calculations here are more complex than those in [1]; they have been aided by the Sage computer software package [5].

2. Main Result

The main tool – an expression for the interrupter $P$ associated with the matrix $M$ – is provided by the following theorem.
Theorem 2.1. Suppose $M = [a,c]$ is a lower triangular factorable matrix that acts as a bounded operator on $\ell^2$ and that the following conditions are satisfied:

(a) both $\{a_n\}$ and $a_n/c_n$ are positive decreasing sequences that converge to 0, and

(b) the matrix $B$ defined by $B = [b_{ij}]$ by

$$b_{ij} = \begin{cases} 
  c_i \left( \frac{1}{a_i} - \frac{1}{c_{i+1}} \right) & \text{if } i < j; \\
  \frac{1}{a_i} & \text{if } i = j + 1; \\
  0 & \text{if } i > j + 1.
\end{cases}$$

is a bounded operator on $\ell^2$.

Then $M$ is posinormal with interruper $P = B^*B$. The entries of $P = [p_{ij}]$ are given by

$$p_{ij} = \begin{cases} 
  \frac{c_j^2}{c_{j+1}^2} - \frac{c_{j+1}^2}{c_j^2} + \left( \sum_{k=0}^{j-1} c_k^2 \right) \left( c_{j+1} c_j - c_{j+1} c_j \right)^2 & \text{if } i = j; \\
  \frac{c_j^2}{c_{j+1}^2} - \frac{c_{j+1}^2}{c_j^2} \left( \sum_{k=0}^{j-1} c_k^2 \right) & \text{if } i > j; \\
  0 & \text{if } i < j.
\end{cases}$$

Proof. See [2].

We are now ready for the main result.

Theorem 2.2. The weighted mean matrix $M$ associated with the weight sequence $w_n = 2n + 1$ is a hyponormal operator on $\ell^2$.

Proof. One easily verifies that the weighted mean matrix $M$ associated with $w_n = 2n + 1$ satisfies the hypotheses of Theorem 2.1. For $M$ to be hyponormal, it must be true that

$$\langle (M^*M - MM^*)f, f \rangle = \langle (M^*M - M^*PM)f, f \rangle = \langle (I - P)Mf, Mf \rangle \geq 0$$

for all $f$ in $\ell^2$. Consequently, we can conclude that $M$ will be hyponormal when $Q := I - P \geq 0$; we note that the range of $M$ contains all the $e_n$’s from the standard orthonormal basis for $\ell^2$.

Using the given weight sequence, we determine that the entries of $Q = [q_{mn}]$ are given by

$$q_{mn} = \begin{cases} 
  \frac{12n^4 + 60n^3 + 104n^2 + 66n + 7}{3(n+2)(2n+1)(2n+3)} & \text{if } m = n; \\
  \frac{-1}{6n^2 + 16m + 11} & \text{if } m > n; \\
  \frac{3}{(n+2)^2(2n+1)(2n+3)} & \text{if } m < n.
\end{cases}$$

In order to show that $Q$ is positive, it suffices to show that $Q_N$, the $N$th finite section of $Q$ (involving rows $m = 0, 1, 2, ..., N$ and columns $n = 0, 1, 2, ..., N$), has positive determinant for each positive integer $N$. For columns $n = 0, 1, ..., N - 1$, we multiply the $(n+1)^{st}$ column of $Q_N$ by $z_n := \frac{(n+1)(n+3)}{(n+2)^2}$ and subtract from the $n^{th}$ column. Call the new matrix $Q_N'$. Then we work with the rows of $Q_N'$. For $m = 0, 1, ..., N - 1$, we multiply the $(m+1)^{st}$ row of $Q_N'$ by $z_m$ and subtract from the $m^{th}$ row. This leads to the tridiagonal form
The weighted mean matrix with weight sequence $w_n = 2n + 1 \ldots$

\[
Y_N := \begin{pmatrix}
d_0 & s_0 & 0 & \ldots & 0 & 0 \\
s_0 & d_1 & s_1 & \ldots & 0 & 0 \\
0 & s_1 & d_2 & \ldots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & d_{N-1} & s_{N-1} \\
0 & 0 & 0 & \ldots & s_{N-1} & d_N
\end{pmatrix},
\]

where
\[
d_n = q_{nn} - z_n q_{n,n+1} - z_n (q_{n+1,n} - z_n q_{n+1,n+1}) = q_{nn} - 2z_n q_{n,n+1} + z_n^2 q_{n+1,n+1} = \frac{16n^2 + 200n^6 + 1024n^8 + 2768n^9 + 4226n^{10} + 4576n^{11} + 1481n + 197}{(n+2)^4(n+3)(2n+1)(2n+3)(2n+5)}
\]

and
\[
s_n = q_{n+1,n} - z_n q_{n+1,n+1} = \frac{(n+1)(2n^2 + 8n + 7)}{(n+2)^2(n+3)(2n+1)(2n+3)} \text{ when } 0 \leq n \leq N - 1; \text{ and}
\]
\[
d_N = q_{NN} = \frac{12N^4 + 60N^3 + 104N^2 + 66N + 7}{3(N+2)^4(2N+1)(2N+3)}.
\]

Note that $\det Y_N = \det Q_N = \det Q_N$. Next we transform $Y_N$ into a triangular matrix with the same determinant. The new matrix has diagonal entries $\delta_n$ which are given by the recursion formula: $\delta_0 = d_0$, $\delta_n = d_n - s_{n-1}^2 / \delta_{n-1}$ ($1 \leq n \leq N$).

An induction argument shows that
\[
\delta_n > \frac{4n + 10}{4n^2 + 20n + 37}
\]

for $0 \leq n \leq N - 1$. Since $d_N$ departs from the pattern set by the earlier $d_n$’s, $\delta_N$ must be handled separately:
\[
\delta_N \geq \frac{24N^8 + 140N^7 + 432N^6 + 140N^5 + 144N^4 + 144N^3 + 1070N^2 + 216N + 14}{6(N+1)^4(N+2)^4(2N+1)^2(2N+3)} > 0.
\]

Therefore $\det Q_N = \prod_{n=0}^{N} \delta_n > 0$, and the proof is complete. \hfill \square

We note that the verification of the induction step above reduces – using the Sage mathematics software system [5] – to the observation that $96n^{10} + 4944n^9 + 54624n^8 + 282288n^7 + 824294n^6 + 1447767n^5 + 1531563n^4 + 927504n^3 + 285132n^2 + 35022n + 178 \geq 0$ for all $n \geq 1$.

After seeing the above result for $w_n = 2n + 1$, one might reasonably wonder whether the calculations for the weighted mean operator associated with weight sequence $w_n = 3n + 1$ would be significantly more complex, and the answer is – Yes. To illustrate, we observe that for $w_n = 2n + 1$, the reduced form of $q_{nn}$ has a degree 4 numerator and a degree 5 denominator. In contrast, for $w_n = 3n + 1$, the reduced form of $q_{nn}$ turns out to have a degree 6 numerator and a degree 7 denominator – as well as larger coefficients for all the terms, so the calculations for that weight sequence are significantly more complicated.

References

[1] Kubrusly, C. S., Duggal, B. P. On posinormal operators, Adv. Math. Sci. Appl. 17 (2007), no. 1, 131-147.
[2] Rhaly Jr., H. C. Posinormal factorable matrices whose interrupter is diagonal, Mathematica 53 (76) (2011), no. 2, 181-188.
[3] Rhaly Jr., H. C., Rhoades, B. E., Conditions for factorable matrices to be hyponormal and dominant, Sib. Elektron. Mat. Izv. 9 (2012), 261-265.
[4] Rhaly Jr., H. C., Rhoades, B. E., The weighted mean operator on $\ell^2$ with the weight sequence $w_n = n + 1$ is hyponormal, New Zealand J. Math, in press.
[5] Stein, W. A. et al. Sage Mathematics Software (Version 6.2), The Sage Development Team, 2014, [http://www.sagemath.org](http://www.sagemath.org)
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