A Branching Law for Subgroups Fixed by an Involution
and a Concompact Analogue of the Borel-Weil Theorem

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Abstract. We give a branching law for subgroups fixed by an involution. As an application we give a generalization of the Cartan-Helgason theorem and a noncompact analogue of the Borel-Weil theorem.

0. Introduction

0.1. Let $G_C$ be a simply-connected complex semisimple Lie group and let $\mathfrak{g} = \text{Lie} G_C$. Let $\mathfrak{g_0}$ be a real form of $\mathfrak{g}$ and let $G$ be the real semisimple Lie subgroup of $G_C$ corresponding to $\mathfrak{g_0}$. Using standard notation let $G = KAN$ be an Iwasawa decomposition of $G$ and let $M$ be the centralizer of $A$ in the maximal compact subgroup $K$ of $G$. The complexified Lie algebras of $K,M,A,$ and $N$ are denoted respectively by $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$. Let $h_m$ be a Cartan subalgebra of $\mathfrak{m}$ so that $\mathfrak{b} = h_m + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{m} = \mathfrak{m}_- + h_m + \mathfrak{m}_+$ be a triangular decomposition of $\mathfrak{m}$ so that $\mathfrak{b} = \mathfrak{b}_m + \mathfrak{a} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$ and

$$b = b_m + a + n$$

is a Borel subalgebra of $\mathfrak{g}$.

Let $\Lambda \subset h^*$ be the set of dominant integral linear forms on $h$, with respect to $b$ and, for each $\lambda \in \Lambda$, let $\pi : \mathfrak{g} \to \text{End} V_\lambda$ be an irreducible representation with highest weight $\lambda$. Let $0 \neq v_\lambda \in V_\lambda$ be a highest weight vector. It follows from the Lie algebra Iwasawa decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ that $U(\mathfrak{g}) = U(\mathfrak{t})U(\mathfrak{a} + \mathfrak{n})$ using the standard notation for universal enveloping algebras. However if $\lambda \in \Lambda$, then $Cv_\lambda$ is stable under $U(\mathfrak{a} + \mathfrak{n})$ and hence $V_\lambda$ is a cyclic $U(\mathfrak{t})$-module. In fact

$$V_\lambda = U(\mathfrak{t})v_\lambda$$

Let $L_\lambda(\mathfrak{t})$ be the (left ideal) annihilator of $v_\lambda$ in $U(\mathfrak{t})$. It is then an elementary fact that if $Z$ is any irreducible $\mathfrak{t}$-module, one has

$$\text{multiplicity of } Z \text{ in } V_\lambda = \dim Z^{L_\lambda(\mathfrak{t})}$$

(0.1)

where $Z^S = \{w \in Z \mid S \cdot w = 0\}$ for any subset $S \subset U(\mathfrak{t})$.

The equation (0.1) becomes a useful branching law as soon as one can explicitly determine generators of $L_\lambda(\mathfrak{t})$. It is one of the main results of this paper to explicitly exhibit such generators.

Let $\Delta \subset h^*$ be the set of roots for $\mathfrak{g}$ and let $\Delta_+$ be the set of positive roots corresponding to $b$. For any $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. Let $\ell = \text{rank} \mathfrak{g}$ and let $I = \{1, \ldots, \ell\}$ and let $\{h_i\}$, $i \in I$, be the basis of $h$ such that $h_i \in \mathfrak{g}_i$ where $\mathfrak{g}_i$ is the TDS generated by $e_{\alpha_i}$ and $e_{-\alpha_i}$ and
\[ [h_i, e_{\alpha_i}] = 2e_{\alpha_i}. \] Then there uniquely exists \( \lambda_i \in \Lambda, i \in I, \) such that \( \lambda_i(h_j) = \delta_{ij} \) for \( i, j \in I. \) In addition, every element \( \lambda \in \Lambda \) may be uniquely written

\[ \lambda = \sum_{i \in I} n_i(\lambda) \lambda_i \]

where \( n_i(\lambda) \in \mathbb{Z}_+. \)

**0.2.** Now let \( I = I_n \cup I_m \) be the partition defined so that \( i \) is in \( I_n \) or \( I_m \) according as \( e_{\alpha_i} \in \mathfrak{n} \) or \( e_{\alpha_i} \in \mathfrak{m}^+. \) Let \( \theta \) be the complex Cartan involution of \( \mathfrak{g} \) so that \( \mathfrak{t} \) is the space of \( \theta \) fixed points. For any \( i \in I_n \) let \( z_i = e_{-\alpha_i} + \theta(e_{-\alpha_i}). \) One of course has that \( z_i \in \mathfrak{t}. \) The set \( I_n \) partitions into a disjoint union \( I_n = I_s \cup I_{nil} \) where \( I_s = \{ i \in I_n \mid \theta(e_{-\alpha_i}) \in C e_{\alpha_i} \} \) and \( I_{nil} \) is the complement of \( I_s \) in \( I_n. \) One readily notes that \( z_i \) is semisimple if \( i \in I_s \) and \( z_i \) is nilpotent if \( i \in I_{nil}. \) In the latter case one easily proves that \( [e_{-\alpha_i}, \theta(e_{-\alpha_i})] = 0. \) For any \( \lambda \in \Lambda \) and \( i \in I_n \) let \( q_{\lambda,i}(t) \) be the polynomial of degree \( n_i(\lambda) + 1 \) defined by putting

\[ q_{\lambda,i}(t) = (t - n_j(\lambda))(t - n_j(\lambda) + 2) \cdots (t + n_i(\lambda)) \quad (0.2) \]

if \( i \in I_s \) and

\[ q_{\lambda,i}(t) = t^{n_i(\lambda) + 1} \quad (0.3) \]

if \( i \in I_{nil}. \) One readily proves that if \( i \in I_s, \) then \( e_{-\alpha_i} \) may be normalized so that, with respect to the Killing form inner product, \( (z_i, z_i) = (h_i, h_i). \) In that case \( q_{\lambda,i}(t) \) is the characteristic polynomial of \( \pi_\lambda(z_i) \) in the irreducible \( \mathfrak{g}_i \)-module \( U(\mathfrak{g}_i) \nu_\lambda. \) We assume that \( e_{-\alpha_i} \) is so normalized. No normalization is needed if \( i \in I_{nil}. \) The following is our first main result.

**Theorem 0.1.** Let \( \lambda \in \Lambda. \) Let \( \{ y_j \}, \) \( j = 1, \ldots, k, \) be a basis of \( \mathfrak{h}_m \) and let \( \mathcal{G}_\lambda(m) \) be the finite subset of \( U(\mathfrak{m}) \) defined so that

\[ \mathcal{G}_\lambda(m) = \left\{ e_{-\alpha_i}{ \lambda}+1 \mid i \in I_m \right\} \cup \left\{ y_j - \lambda(y_j) \mid j = 1, \ldots, k \right\} \cup \{ e_{\alpha_i} \mid i \in I_m \} \]

and if \( \mathfrak{n}_* \) is the Killing form complement of \( \mathfrak{m} \) in \( \mathfrak{t}, \) let \( \mathcal{G}_\lambda(\mathfrak{n}_*) \) be the finite set of polynomials of elements \( \mathfrak{n}_* \) defined so that

\[ \mathcal{G}_\lambda(\mathfrak{n}_*) = \{ q_{\lambda,i}(z_i) \mid i \in I_n \} \]

Put \( \mathcal{G}_\lambda = \mathcal{G}_\lambda(m) \cup \mathcal{G}_\lambda(\mathfrak{n}_*). \) Then the left ideal annihilator \( L_\lambda(\mathfrak{t}) \) in \( U(\mathfrak{t}) \) of the highest weight vector \( \nu_\lambda \) of \( V_\lambda \) is the left ideal generated by \( \mathcal{G}_\lambda. \) In particular if \( Z \) is any irreducible \( \mathfrak{t} \)-module, then

\[ \text{multiplicity of } Z \text{ in } V_\lambda = \dim Z^{\mathcal{G}_\lambda} \quad (0.4) \]

By separating the conditions imposed by \( \mathcal{G}_\lambda(m) \) and \( \mathcal{G}_\lambda(\mathfrak{n}_*) \) we may express (0.4) in a simpler form (see Theorem 0.5 below). Let \( \Lambda_m \subset \mathfrak{h}_m^* \) be the set of all dominant
integral linear forms on $h_m$ with respect to $b_m$ (see (178)). If $\lambda \in \Lambda$, then clearly $\Lambda|b_m \in \Lambda_m$. For any $\nu \in \Lambda_m$ let $\tau_\nu : m \to Aut Y_\nu$ be an irreducible $m$-module with highest weight $\nu$. If $Z$ is a finite dimensional irreducible $t$-module then, as a $m$-module, let $Z[\nu]$ be the primary $\tau_\nu$ component of $Z$. The subspace of $b_m$ highest weight vectors in $Z[\nu]$ is just $(Z[\nu])^{m+}$.

**Theorem 0.5.** Let $\lambda \in \Lambda$ and let $Z$ be any finite dimensional irreducible $t$-module. Then

\[
\text{multiplicity of } Z \text{ in } V_{\lambda} = \dim \{ w \in (Z[\lambda|h_m])^{m+} \mid q_{\lambda,i}(z_i)w = 0, \forall i \in I \} \tag{0.5}
\]

0.3. Another consequence of Theorem 0.1 is a generalization of the Cartan-Helgason theorem. In the generalization an arbitrary irreducible representation of $M$ replaces the trivial representation of $M$. For $j \in I_s$ let $\varepsilon_j = exp \pi i h_j$ and let $F_s$ be the (abelian) group generated by $\{ \varepsilon_j \}, j \in I_s$. The following structure theorem concerning the disconnectivity of $M$ does not appear to be in the literature. It however was known to David Vogan who points out that it is implicit in [V]. Theorem 0.6 follows from Theorem 0.1. Let $\ell_s = card I_s$ and let $M_e$ be the identity component of $M$.

**Theorem 0.6.** One has a group isomorphism

\[
F_s \cong \mathbb{Z}_\ell^s \tag{0.6}
\]

Furthermore $F_s \subset M$ and $M$ has the following product structure:

\[
M = F_s \times M_e \tag{0.7}
\]

Let $\widehat{F}_s$ be the character group of $F_s$. Let $\Lambda_{M_e}$ be the set of all $\nu \in \Lambda_m$ such that $\tau_\nu|m_v$ exponentiates to an (irreducible ) representation of $M_e$. If $\lambda \in \Lambda$ then clearly $\Lambda|b_m \in \Lambda_{M_e}$. Extending its previous use, let $\tau_\nu$, for $\nu \in \Lambda_{M_e}$, also denote the representation of $M_e$ which arises by exponentiating $\tau_\nu|m_v$. Then $\widehat{F}_s \times \Lambda_{M_e}$ parameterizes the set $\widehat{M}$ of all equivalences classes of irreducible representations (the unitary dual) of $M$. For each $(\zeta, \nu) \in \widehat{F}_s \times \Lambda_{M_e}$ let

\[
\tau_{\zeta,\nu} : M_e \to Aut Y_{\zeta,\nu}
\]

be the irreducible representation where, as a vector space, $Y_{\zeta,\nu} = Y_\nu$, but with the $M$-module structure defined so that

\[
(\varepsilon, a) y = \zeta(\varepsilon) \tau_\nu(a) y
\]

for $(\varepsilon, a) \in F_s \times M_e = M$ and $y \in Y_{\zeta,\nu}$. By abuse of notation we will take

\[
\widehat{M} = \{ \tau_{\zeta,\nu} \}, (\zeta, \nu) \in \widehat{F}_s \times \Lambda_{M_e}
\]

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For any $\lambda \in \Lambda$ let $\lambda | F_s \in \hat{F}_s$ be (uniquely) defined so that, for $j \in I_s$, one has $(\lambda | F_s)(\varepsilon_j) = 1$ if $n_j(\lambda)$ is even and $(\lambda | F_s)(\varepsilon_j) = -1$ if $n_j(\lambda)$ is odd.

If $\lambda \in \Lambda$ then $U(m) v_\lambda$ is an irreducible $M$-module with respect to $\pi_{\lambda|M}$. The significance of the pair $(\lambda | F_s, \lambda | h_m) \in \hat{F}_s \times \Lambda$ is that one has a module equivalence

$$Y_{\lambda | F_s, \lambda | h_m} \cong U(m) v_\lambda$$

(0.8)

of irreducible $M$-modules. One readily shows that the map

$$\Lambda \to \hat{F}_s \times \Lambda_{Me}, \quad \lambda \mapsto (\lambda | F_s, \lambda | h_m)$$

(0.9)

is surjective. For any $(\zeta, \nu) \in \hat{F}_s \times \Lambda_{Me}$, let $\mathcal{F}_{\zeta, \nu} \subset \Lambda$ be the fiber of (0.9) over $(\zeta, \nu)$. If $\zeta_1$ is the trivial character of $F_s$, so that $\tau_{\zeta_1,0}$ is the trivial representation of $M$, then the Cartan-Helgason theorem asserts that

$$\mathcal{F}_{\zeta_1,0} = \Lambda_{sph}$$

(0.10)

where

$$\Lambda_{sph} = \{ \lambda \in \Lambda \mid \pi_{\lambda|k} \text{ contains the trivial representation of } \mathfrak{k} \}$$

(0.11)

One may introduce a partial order in $\Lambda$ by declaring that $\lambda^b \gg \lambda^a$ if there exists a (necessarily unique and surjective) $\mathfrak{k}$-map $s : V_{\lambda^b} \to V_{\lambda^a}$ such that $s(v_{\lambda^b}) = v_{\lambda^a}$. In such a case one necessarily has

$$\text{multiplicity of } Z \text{ in } V^a \leq \text{multiplicity of } Z \text{ in } V^b$$

(0.12)

for all irreducible $K$-modules $Z$. We may express (0.11) by saying that $\mathcal{F}_{\zeta_1,0} = \{ \lambda \in \Lambda \mid \lambda \gg 0 \}$. Noting that $\lambda_{\zeta_1,0} = 0$ in the next theorem the generalization of the Cartan-Helgason theorem is the statement (see Theorem 3.12)

**Theorem 0.7.** Let $(\zeta, \nu) \in \hat{F}_s \times \Lambda_{Me}$. Then there exists a unique minimal element $\lambda_{\zeta, \nu} \in \mathcal{F}_{\zeta, \nu}$. In fact $\mathcal{F}_{\zeta, \nu} = \{ \lambda \in \Lambda \mid \lambda \gg \lambda_{\zeta, \nu} \}$. In particular

$$\text{multiplicity of } Z \text{ in } V_{\lambda_{\zeta, \nu}} \leq \text{multiplicity of } Z \text{ in } V_\lambda$$

(0.12)

for all irreducible $K$-modules $Z$ and all $\lambda \in \mathcal{F}_{\zeta, \nu}$. Furthermore

$$\mathcal{F}_{\zeta, \nu} = \lambda_{\zeta, \nu} + \Lambda_{sph}$$

(0.13)

Explicit formulas for $\lambda_{\zeta, \nu}$ and the elements in $\Lambda_{sph}$ are given in (193) and (187).

**0.4.** Let $D = \hat{F}_s \times \Lambda_{Me}$ so that $\hat{M} = \{ \tau_\delta \mid \delta \in D \}$. If $\phi$ is a smooth scalar or vector valued function on $G$ and $g \in G, z \in g$, then $g \cdot \phi$ and $z \cdot \phi$ is again such a function where the action is defined by (200) and (201). The set of principal series representations of $G$ are parameterized by $D \times \mathfrak{a}^*$. If $(\delta, \xi) \in D \times \mathfrak{a}^*$, then
the corresponding Harish-Chandra module, \( H(Y_\delta, \xi) \), is a \((g, K)\)-module of \( Y_\delta \) valued functions on \( G \) satisfying (202) and (203). Rather than dealing with vector valued functions on \( G \) we can, without loss, use scalar valued functions on \( G \), by use of the Borel-Weil theorem applied to \( M \). The space \( C^\infty(G) \) is a (left) module for \( U(g) \) when the latter operates as left invariant differential operators on \( G \). The action is denoted by \( f \cdot u \) for \( f \in C^\infty(G) \) and \( u \in U(g) \). If \( u \in g_\alpha \), then \( f \cdot u \) is given by (205). If \( (\delta, \xi) \in D \times a^* \), then the \((g, K)\)-module \( H(\delta, \xi) \) of scalar valued functions on \( G \) is defined in (218). A \((g, K)\)-equivalence of \( H(\delta, \xi) \) and \( H(Y_\delta, \xi) \) is established in Proposition 4.4.

A result of Casselman (see [C] or [BB]) asserts that any irreducible Harish-Chandra module \( H \) embeds in \( H(\delta, \xi) \) for some \((\delta, \xi)\). If \( H = V_\lambda \) for \( \lambda \in \Lambda \), then Wallach has shown (see [W], §8.5) that there exists a unique \((\delta, \xi) \in D \times a^* \) for which one has a \((g, K)\) injection

\[
V_\lambda \to H(\delta, \xi) \tag{0.14}
\]

and the map (0.14) is unique up to scalar multiplication. For completeness and to simplify the proof of Theorem 0.8 below we reprove, using our scalar valued functions instead of vector valued functions, the Wallach result. The explicit determination of \((\delta, \xi)\) is given in Propositions 4.16 and 4.17. Let \( V_\lambda \) be the image of (0.14). The functions in \( H(\delta, \xi) \) are of course determined by their restrictions to \( K \). Theorem 0.8 (proved in this paper as Theorem 4.19) is a noncompact analogue of the Borel-Weil theorem in the sense that \( V_\lambda \) will turn out to be a finite dimensional space in \( H(\delta, \xi) \) of solutions of certain differential equations on \( G \) arising from certain elements in \( U(\mathfrak{t}) \) operating as left invariant differential operators. If one was dealing with the compact form of \( G_\mathbb{C} \), the Borel-Weil theorem asserts the equations would be of Cauchy-Riemann type.

**Theorem 0.8.** Let \( \lambda \in \Lambda \) and let \( \delta \in D \) and \( \xi \in a^* \) be defined as in (0.14) (see (246) and (248)). Then \( V_\lambda \) may be given by

\[
V_\lambda = \{ f \in H(\delta, \xi) \mid f \cdot q_{\lambda^c, i}(z_i) = 0, \ \forall i \in I_n \} \tag{0.15}
\]

where \( z_i \in \mathfrak{t} \) and \( q_{\lambda^c, i}(t) \in \mathbb{C}[t] \), for \( i \in I_n \), are defined as in §0.2, and \( \lambda^c \in \Lambda \) is the highest weight of the dual module \( V_\lambda^* \).

1. The generators of the left ideal \( L_\lambda(\mathfrak{t}) \)

1.1. Let \( g \) be a complex semisimple Lie algebra and let \((x, y)\) be the Killing form \( B_g \) on \( g \). If \( t \) is any Lie subalgebra of \( g \), let \( U(t) \) be the universal enveloping algebra of \( t \). Of course we can regard \( U(t) \subset U(g) \).

Let \( \mathfrak{h} \subset \mathfrak{g} \) be, respectively, a Cartan subalgebra and a Borel subalgebra of \( g \). The restriction \( B_g|_{\mathfrak{h}} \) defines an isomorphism

\[
\eta : \mathfrak{h} \to \mathfrak{h}^* \tag{1}
\]

For \( \mu, \nu \in \mathfrak{h}^* \), the dual space to \( \mathfrak{h} \), let \((\mu, \nu)\) be the bilinear form on \( \mathfrak{h}^* \) defined so that \((\eta(x), \eta(y)) = (x, y)\) for any \( x, y \in \mathfrak{h} \). Let \( \ell = \text{dim} \mathfrak{h} \) and let \( \Delta \subset \mathfrak{h}^* \) be the
set of roots for the pair \((\mathfrak{h}, \mathfrak{g})\). Let a complete set of root vectors \(\{e_\varphi\}, \varphi \in \Delta\) be chosen. If \(\mathfrak{c} \subset \mathfrak{g}\) is any subspace which is stable under the action of \(\text{ad} \mathfrak{h}\), let \(\Delta(\mathfrak{c}) = \{\varphi \in \Delta \mid e_\varphi \in \mathfrak{c}\}\). A choice of a set \(\Delta_+\) of positive roots is made by putting \(\Delta_+ = \Delta(\mathfrak{b})\). Let \(\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta_+\) be the set of simple positive roots.

One knows there uniquely exists \(\lambda_i \in \Lambda, i = 1, \ldots, \ell\), such that

\[
2(\lambda_i, \alpha_j) / (\alpha_j, \alpha_j) = \delta_{ij}
\]

for \(i, j \in \{1, \ldots, \ell\}\) and that any \(\lambda \in \Lambda\) can be uniquely written

\[
\lambda = \sum_{i=1}^\ell n_i(\lambda) \lambda_i
\]

where \(n_i(\lambda) \in \mathbb{Z}_+\). For \(i = 1, \ldots, \ell\), let \(\mathfrak{g}_i\) be the TDS generated by \(e_{\alpha_i}\) and \(e_{-\alpha_i}\). Then there exists a unique scalar multiple, \(h_i\), of \([e_{\alpha_i}, e_{-\alpha_i}]\) such that \([h_i, e_{\alpha_i}] = 2e_{\alpha_i}\) and \([h_i, e_{-\alpha_i}] = -2e_{-\alpha_i}\). One has that \(\mathbb{C}h_i = \mathfrak{g}_i \cap \mathfrak{h}\) and

\[
\eta(h_i) = 2\alpha_i / (\alpha_i, \alpha_i)
\]

With respect to the pairing of \(\mathfrak{h}\) and \(\mathfrak{h}^*\), the equations (2) imply that

\[
(\lambda_i, h_j) = \delta_{ij}
\]

Let \(\mathfrak{b}_-\) be the Borel subalgebra of \(\mathfrak{g}\) which contains \(\mathfrak{h}\) and is such that \(\Delta(\mathfrak{b}_-) = \Delta_-\) where \(\Delta_- = -\Delta_+\). Let \(\mathfrak{u}\) and \(\mathfrak{u}_-\), respectively, be the nilradicals of \(\mathfrak{b}\) and \(\mathfrak{b}_-\). One then has the linear space direct sum \(\mathfrak{g} = \mathfrak{u}_- + \mathfrak{b}\). The Poincaré-Birkhoff-Witt theorem then implies that

\[
U(\mathfrak{g}) = U(\mathfrak{u}_-) U(\mathfrak{b})
\]

For each \(\lambda \in \Lambda\) let

\[
\pi_\lambda : U(\mathfrak{g}) \rightarrow \text{End} V_\lambda
\]

be an irreducible representation with highest weight \(\lambda\). Also let \(0 \neq v_\lambda \in V_\lambda\) be a highest weight vector. If \(u \in U(\mathfrak{g})\) and \(v \in V_\lambda\) and there is no danger of confusion, we will occasionally write \(u v\) or \(u \cdot v\) in \(V_\lambda\) for \(\pi_\lambda(u) v\).

Let \(\lambda \in \Lambda\). Since the 1 dimensional subspace \(\mathbb{C}v_\lambda\) is stable under \(U(\mathfrak{b})\), it follows from (5) that \(V_\lambda\) is a cyclic module for \(U(\mathfrak{u}_-)\) and in fact

\[
V_\lambda = U(\mathfrak{u}_-) v_\lambda
\]

Let \(L_\lambda(\mathfrak{u}_-)\) be the annihilator of \(v_\lambda\) in \(U(\mathfrak{u}_-)\) so that \(L_\lambda(\mathfrak{u}_-)\) is a left ideal in \(U(\mathfrak{u}_-)\) and, as \(U(\mathfrak{u}_-)\)-modules, one has the isomorphism

\[
V_\lambda = U(\mathfrak{u}_-) / L_\lambda(\mathfrak{u}_-)
\]
Now let $i \in \{1, \ldots, \ell\}$. Obviously $e_{\alpha_i} v_\lambda = 0$ and $h_i v_\lambda = n_i(\lambda)v_\lambda$ by (4) using the notation of §1.1. The following is well known and is a consequence of the representation theory of $Sl(2, \mathbb{C})$.

**Proposition 1.1.** The cyclic $U(g_i)$-submodule of $V_\lambda$ generated by $v_\lambda$ is $U(g_i)$-irreducible and has dimension $n_i(\lambda) + 1$. In addition $n_i(\lambda)$ is the smallest nonnegative integer $k$ such that $e^{k+1}_{-\alpha_i} v_\lambda = 0$.

It follows from Proposition 1 that $e^{n_i(\lambda)+1}_{-\alpha_i} \in L_\lambda(u_-)$. A proof of the following theorem may be found in [PRV]. See Lemma 2.1 in [PRV]. For more details about this result see Remark 44 (especially (170)), p. 315 in [K3].

**Proposition 1.2.** Let $\lambda \in \Lambda$. Then the elements $e^{n_i(\lambda)+1}_{-\alpha_i}$, $i = 1, \ldots, \ell$, generate $L_\lambda(u_-)$. That is,

$$L_\lambda(u_-) = \sum_{i=1}^{\ell} U(u_-) e^{n_i(\lambda)+1}_{-\alpha_i} \quad (8)$$

1.2. Let $g_o$ be a real form of $g$ and let

$$g_o = \mathfrak{t}_o + \mathfrak{p}_o \quad (9)$$

be a Cartan decomposition of $g_o$. Thus $Ad_{g_o} \mathfrak{t}_o$ is a maximal compact subgroup of $Ad g_o$ and $\mathfrak{p}_o$ is the $B_o$-orthocomplement of $\mathfrak{t}_o$ in $g_o$. Let $\theta$ be the involution on $g$ such that $\theta|_{g_o}$ is the Cartan involution of $g_o$ corresponding to (9). Let $a_o$ be a maximal abelian subalgebra contained in $p_o$ and let $m_o$ be the centralizer of $a_o$ in $t_o$. Then one knows there exists $w \in a_o$ whose centralizer in $g_o$ is $m_o + a_o$. The operator $ad w|_{g_o}$ is diagonalizable (and hence has real eigenvalues). Let $n_o$ be the span of eigenvectors of $ad w$ in $g_o$ corresponding to positive eigenvalues. Then $g_o = \mathfrak{t}_o + \mathfrak{a}_o + \mathfrak{n}_o$ is, infinitesimally, an Iwasawa decomposition of $g_o$. We will denote the complexification of the Lie algebras introduced above by dropping the subscript $o$. In particular,

$$g = \mathfrak{t} + \mathfrak{a} + \mathfrak{n} \quad (10)$$

is, infinitesimally, a complexified Iwasawa decomposition of $g$. Also $q = m + a + n$ is a parabolic Lie subalgebra of $g$ and $m + a, n$ are, respectively, a Levi factor and nilradical of $q$. Also if $g^w$ is the centralizer of $w$ in $g$, then

$$g^w = m + a \quad (11)$$

Let $(h_m)_o$ be a Cartan subalgebra of the reductive Lie algebra $m_o$ and let $h_m$ be the complexification of $(h_m)_o$. Then $h_m + a$ is a Cartan subalgebra of $g$. We will fix the Cartan subalgebra $h$ of $g$ in §1.1 so that

$$h = h_m + a \quad (12)$$

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Next the Borel subalgebra $b$ of §1.1 will be fixed so that $h \subset b$ (an assumption in §1) and such that $b \subset q$. It follows immediately that

$$a + n \subset b$$  \hspace{1cm} (13)

But then as a consequence of (10), (13) and the Poincaré-Birkhoff-Witt theorem one has

$$U(g) = U(t)U(b)$$  \hspace{1cm} (14)

A Lie subalgebra of $g$ is called symmetric if it is the set of fixed vectors for an involutory automorphism of $g$. In particular $t$ is a symmetric subalgebra of $g$. In fact $t$ is the most general symmetric subalgebra of $g$. Let $\lambda \in \Lambda$ and let $L_{\lambda}(t)$ be the (left ideal) annihilator in $U(t)$ of $v_{\lambda}$ in $V_{\lambda}$. The equality (14) readily implies

**Proposition 1.3.** Any finite dimensional irreducible $g$-module, where $g$ is a complex semisimple Lie algebra, is a cyclic module for any symmetric subalgebra of $g$. More specifically, in the notation above, let $\lambda \in \Lambda$. Then $V_{\lambda}$ is a cyclic $U(t)$-module and $v_{\lambda}$ is a cyclic generator. In particular, as $U(t)$-modules, one has an isomorphism

$$V_{\lambda} \cong U(t)/L_{\lambda}(t)$$  \hspace{1cm} (15)

**1.3.** If $Z$ is any $U(t)$-module and $S \subset U(t)$ is any subset, let $Z^S = \{z \in Z \mid Sz = 0\}$. Let $Z$ be any $U(t)$-module and let $\lambda \in \Lambda$. If $\sigma \in Hom_{\mathbb{k}}(V_{\lambda}, Z)$, then obviously $\sigma(v_{\lambda}) \in Z^{L_{\lambda}(t)}$. Conversely if $z \in Z^{L_{\lambda}(t)}$ then by (15) there clearly exists a unique element $\sigma \in Hom_{\mathbb{k}}(V_{\lambda}, Z)$ such that $\sigma(v_{\lambda}) = z$. That is, one has

**Lemma 1.4.** Let $\lambda \in \Lambda$ and let $Z$ be a $U(t)$-module. Then the map

$$Hom_{\mathbb{k}}(V_{\lambda}, Z) \to Z^{L_{\lambda}(t)}, \ \sigma \mapsto \sigma(v_{\lambda})$$  \hspace{1cm} (16)

is a linear isomorphism.

Of course $t$ is a reductive Lie subalgebra of $g$. The assumption that a finite dimensional $t$-module $Z$ is completely reducible reduces to the assumption that $Z$ is completely reducible as a $\text{Cent} t$-module. Lemma 1.4. leads to the following branching principle for symmetric subalgebras of semisimple Lie algebras.

**Proposition 1.5.** Let $Z$ be a finite dimensional completely irreducible $t$-module. Let $\lambda \in \Lambda$. Then there is a nonsingular pairing of $Hom_{\mathbb{k}}(V_{\lambda}, Z)$ and $Hom_{\mathbb{k}}(Z, V_{\lambda})$ so that (recalling (16)) one has

$$\dim Hom_{\mathbb{k}}(Z, V_{\lambda}) = \dim Z^{L_{\lambda}(t)}$$  \hspace{1cm} (17)

In particular if $Z$ is $t$-irreducible, then the

multiplicity of $Z$ in $\pi_{\lambda} \mid t = \dim Z^{L_{\lambda}(t)}$  \hspace{1cm} (18)
Proof. We may identify $\text{Hom}(V_\lambda, Z)$ with $(V_\lambda^* \otimes Z)^k$. On the other hand, the dual space to $V_\lambda^* \otimes Z$ may be written as $Z^* \otimes V_\lambda$. The latter however identifies with $\text{Hom}(Z, V_\lambda)$. Since the tensor product action of $\mathfrak{k}$ is clearly completely reducible, the nonsingular pairing of $V_\lambda^* \otimes Z$ and $Z^* \otimes V_\lambda$ restricts to a nonsingular pairing of $(V_\lambda^* \otimes Z)^\mathfrak{k}$ with $(Z^* \otimes V_\lambda)^\mathfrak{k} = \text{Hom}_\mathfrak{k}(Z, V_\lambda)$. QED

Remark 1.6. Proposition 1.5 is useful to establish branching laws for symmetric Lie subalgebras $\mathfrak{t}$ of $\mathfrak{g}$ if one can determine explicit generators of $L_\lambda(\mathfrak{t})$ for any $\lambda \in \Lambda$. Such a determination will be the one of the main results in this paper.

1.4. The bilinear form $B_\mathfrak{g}|(m+a)$ is nonsingular since $m+a$ is a Levi factor of a parabolic subalgebra $(\mathfrak{q})$ of $\mathfrak{g}$. Let $s$ be the $B_\mathfrak{g}$-orthocomplement of $m+a$ in $\mathfrak{g}$ so that

$$\mathfrak{g} = m + a + s$$

(19)

Obviously $[m+a,s] \subset s$. In particular $[\mathfrak{h},s] \subset s$, since $\mathfrak{h} \subset m+a$, so that

$$s = \sum_{\varphi \in \Delta(s)} \mathbb{C} e_\varphi$$

It follows from (11) that

$$\Delta(s) = \{ \varphi \in \Delta \mid \langle \varphi, w \rangle \neq 0 \}$$

(20)

In particular

$$\Delta(s) = -\Delta(s)$$

(21)

Obviously $\mathfrak{n} = s \cap \mathfrak{q}$. But in fact

$$\mathfrak{n} = s \cap \mathfrak{b}$$

(22)

This follows easily from the fact that $\mathfrak{n}$ must be contained in $\mathfrak{b}$ since $\mathfrak{b}$ is maximal solvable and $\mathfrak{b}$ normalizes $\mathfrak{n}$ because $\mathfrak{b} \subset \mathfrak{q}$. It follows from (22) that

$$\mathfrak{n} = \sum_{\varphi \in \Delta_+ \cap \Delta(s)} \mathbb{C} e_\varphi$$

(23)

But (21) and (22) also imply that one has a linear direct sum

$$s = n + n_-$$

(24)

where

$$n_- = \sum_{\varphi \in \Delta_+ \cap \Delta(s)} \mathbb{C} e_{-\varphi}$$

(25)

Let $q_- = m + a + n_-$. 

9
**Proposition 1.7.** The subspace \( q^- \) is a parabolic subalgebra of \( g \) with nilradical \( n^- \) and Levi factor \( m + a \).

**Proof.** Recalling the definition of \( w \in a_o \) in §1.2 it follows from (11), (21) and (25) that (a) \( q \) is the \( \mathbb{C} \)-span of all eigenvectors of \( adw \) corresponding to nonpositive eigenvalues, (b) \( m + a \) is the kernel of \( adw \) and (c) \( n^- \) is the \( \mathbb{C} \)-span of all eigenvectors of \( adw \) corresponding to negative eigenvalues. QED

The inclusion \( b \subset q \) implies of course that \( u \subset q \) using the notation of §1.2. But \( n \subset u \) by (23). However \( u \cap (m + a) = u \cap m \) since \( a \) is central in the Levi factor \( m + a \). Let \( m_+ = u \cap m \) so that one has the Lie algebra semi-direct sum

\[
u = m_+ + n
\]

and hence one has the disjoint union

\[
\Delta_+ = \Delta(m_+) \cup \Delta(n)
\]

On the other hand, \( \Delta(m) = -\Delta(m) \) since \( B_g|m \) is clearly nonsingular. Thus if \( m_- = u_- \cap m \) one has

\[
\Delta(m_-) = -\Delta(m_+)
\]

In addition

\[
m = m_- + b_m + m_+
\]

is a “triangular decomposition” of the reductive Lie algebra \( m \). Indeed

\[
g = u_+ + h + u
\]

is a “triangular decomposition” of \( g \) and the three components on the right side of (29) are the respective intersections of \( m \) with the three components on the right side of (30). One notes that \( m \) is the sum of the respective three intersections since \([a, m] = 0\) and all three components on the right side of (30) are stable under \( ad a \). Let \( b_m = b_m + m_+ \). Then \( b_m \) is a Borel subalgebra of \( m \) since (29) is a triangular decomposition of \( m \). Furthermore

\[
b_m = b \cap m
\]

since (1) \( b_m \) is obviously contained in \( b \cap m \) and (2) a Borel subalgebra is maximal solvable and \( b \cap m \) is solvable. Taking the negative of the roots in (27) it follows from (23), (25) and (28) that one has the disjoint union

\[
\Delta_- = \Delta(m_-) \cup \Delta(n_-)
\]

and consequently the linear space direct sum

\[
u_- = m_- + n_-
\]
Recalling Proposition 1.7 let \( \mathfrak{v} \) be the Lie subalgebra of \( \mathfrak{q}^- \) defined by putting \( \mathfrak{v} = \mathfrak{m} + \mathfrak{n}^- \). Then by (24) one has the linear direct sum

\[
\mathfrak{g} = \mathfrak{v} + \mathfrak{a} + \mathfrak{n}
\]  

(33)

On the other hand

**Proposition 1.8.** One has the linear space direct sum

\[
\mathfrak{v} = \mathfrak{u}^- + \mathfrak{b}_m
\]  

(34)

and the map

\[
U(\mathfrak{u}^-) \otimes U(\mathfrak{b}_m) \to U(\mathfrak{v}), \quad u \otimes v \mapsto uv
\]  

(35)

is a linear isomorphism.

**Proof.** The fact that the right side of (34) is a direct sum is immediate since \( \mathfrak{b}_m \subset \mathfrak{b} \). On the other hand, by (29) and (32)

\[
\mathfrak{u}^- + \mathfrak{b}_m = \mathfrak{n}^- + \mathfrak{m}^- + \mathfrak{b}_m
\]

\[
= \mathfrak{n}^- + \mathfrak{m}
\]

\[
= \mathfrak{v}
\]

The statement (35) is an immediate consequence of the Poincaré-Birkhoff-Witt theorem. QED

1.5. Let \( \lambda \in \Lambda \). Comparing (10) and (33) the argument leading to Proposition 1.3 applies as well to \( U(\mathfrak{v}) \). Namely, one has that \( V_\lambda \) is a cyclic module for \( U(\mathfrak{v}) \) with cyclic generator \( v_\lambda \). Let \( L_\lambda(\mathfrak{v}) \) be the annihilator of \( v_\lambda \) in \( U(\mathfrak{v}) \). As a preliminary step in determining the left ideal \( L_\lambda(\mathfrak{k}) \) in \( U(\mathfrak{k}) \) we will now determine \( L_\lambda(\mathfrak{v}) \). It follows from (27) that the set, \( \Pi \), of simple positive roots admits the following partition:

\[
\Pi = \Pi_n \cup \Pi_m
\]  

(36)

where \( \Pi_n = \Pi \cap \Delta(\mathfrak{n}) \) and \( \Pi_m = \Pi \cap \Delta(\mathfrak{m}_+) \). Let \( \{y_j\}, j = 1, \ldots, k \), be a basis of \( \mathfrak{h}_m \). Let

\[
\omega_\lambda : U(\mathfrak{b}_m) \to \mathbb{C}
\]

be the character on \( U(\mathfrak{b}_m) \) defined so that \( u v_\lambda = \omega_\lambda(u) v_\lambda \) for all \( u \in U(\mathfrak{b}_m) \). It is clear that the ideal \( \text{Ker} \omega_\lambda \) of codimension one in \( U(\mathfrak{b}_m) \) is generated by \( \{y_j - \lambda(y_j)\}, j = 1, \ldots, k \), and \( \{e_\alpha\}, \alpha \in \Pi_m \).

**Proposition 1.9.** Let \( \lambda \in \Lambda \). Then

\[
L_\lambda(\mathfrak{v}) = \sum_{i=1}^{\ell} U(\mathfrak{v}) e_{-\alpha_i}^{n_i(\lambda)+1} + \sum_{j=1}^{k} U(\mathfrak{v})(y_j - \lambda(y_j)) + \sum_{\alpha \in \Pi_m} U(\mathfrak{v}) e_\alpha
\]  

(37)
Proof. Now one has the direct sum

$$U(b_m) = \text{Ker} \omega \lambda \oplus \mathbb{C} 1 \quad (38)$$

But, by abuse of notation in Proposition 1.8, we can then write

$$U(v) = U(u_-) \otimes U(b_m) \quad (39)$$

Thus one has the direct sum

$$U(v) = U(u_-) \oplus U(u_-) \text{Ker} \omega \lambda \quad (40)$$

But obviously $$U(u_-) \text{Ker} \omega \lambda \subset J \lambda$$ so that

$$L \lambda (v) = (U(u_-) \cap L \lambda (v)) \oplus U(u_-) \text{Ker} \omega \lambda \quad (41)$$

On the other hand, clearly $$U(u_-) \cap L \lambda (v) = L \lambda (u_-)$$ using the notation of Proposition 1.2 so that

$$U(u_-) \cap L \lambda (v) = \sum_{i=1}^{\ell} U(u_-)c_{i-\alpha}^{(\lambda) + 1} \quad (42)$$

However, as noted above, $$\text{Ker} \omega \lambda$$ is the ideal in $$U(b_m)$$ generated by $$\{y_j - \lambda(y_j)\}, j = 1, \ldots, k, \{e_{\alpha}\}, \alpha \in \Pi_m$$. But then by (39) the left ideal in $$U(v)$$ generated by these elements is the second summand in (41). Hence the second summand on the right side of (41) is the same as the sum of the last two sums on the right side of (37). But then if $$L' \lambda (v)$$ is the left ideal given by the right side of (37), one has $$L \lambda (v) \subset L' \lambda (v)$$ by (42). But obviously $$L' \lambda (v) \subset L \lambda (v)$$. QED

1.6. Now if $$\tau \subset g$$ is a subspace stable under the involution $$\theta$$ let $$\tau^\theta$$, (see §1.2) be the space of $$\theta$$ fixed vectors in $$\tau$$. Of course $$g^\theta = \tau$$ and $$\tau^\theta = \tau \cap \tau$$. It is immediate from the orthogonal direct sum (19) that $$m + a$$ and $$s$$ are stable under $$\theta$$, and hence if $$n_* = s^\theta$$ then (19) implies

$$\tau = m + n_* \quad (43)$$

One has $$\theta(w) = -w$$ for the element $$w \in a$$ which defines $$n$$ and also $$n_-$$ in terms of the spectrum of $$ad \, w$$ (see §1.2, (23), and (25)). But then clearly

$$\theta(n_-) = n \quad (44)$$

But then, by (24),

$$n_* = \{z + \theta(z) \mid z \in n_-\} \quad (45)$$

But now $$h = h_m + a$$ is stable under $$\theta$$. Hence $$\theta$$ carries root spaces to root spaces. Thus we may define an involution $$\theta : \Delta \rightarrow \Delta$$, preserving root addition when the sum
is a root, such that for any \( \varphi \in \Delta \) one has \( \Theta(\mathbb{C}e_\varphi) = \mathbb{C}e_{\Theta(\varphi)} \). More explicitly, since \( \theta = \theta^{-1} \), one readily has

\[
\langle \theta(\varphi), x \rangle = \langle \varphi, \theta(x) \rangle
\]

for any \( \varphi \in \Delta \) and \( x \in h \). With respect to the action of \( \theta \) on \( \Delta_- \), it follows from (31) and (44) that

\[
\theta(\Delta(n_-)) = \Delta(n)
\]

\( \theta \) reduces to the identity map on \( \Delta(m_-) \) (47).

**Remark 1.10.** If \( \varphi \in \Delta(n_-) \) then \( \theta(\varphi) \in \Delta(n) \) by (47). However it is not necessarily true that \( \theta(\varphi) = -\varphi \). A condition that \( \theta(\varphi) = -\varphi \) is given in the next proposition.

For any \( \varphi \in \Delta \) let \( h_\varphi \) be the unique element in the TDS generated by \( e_\varphi \) and \( e_{-\varphi} \) such that \( [h_\varphi, e_\varphi] = 2e_\varphi \) and \( [h_\varphi, e_{-\varphi}] = -2e_{-\varphi} \). One of course has \( h_\varphi \in h \).

**Proposition 1.11.** Let \( \varphi \in \Delta_- \). Then

\[
[e_\varphi, \theta(e_\varphi)] \in a
\]

Furthermore \( [e_\varphi, \theta(e_\varphi)] \neq 0 \) if and only if \( \theta(\varphi) = -\varphi \), in which case \( [e_\varphi, \theta(e_\varphi)] \) is a nonzero multiple of \( h_\varphi \), noting that \( h_\varphi \in a \) by (48). In any case \( \varphi + \theta(\varphi) \) is never a root.

**Proof.** By the definition of \( \theta(\varphi) \) one has \( 0 \neq \theta(e_\varphi) \in \mathbb{C}e_{\theta(\varphi)} \). Let \( x \in a \). Then

\[
[x, \theta(e_\varphi)] = -\langle \varphi, x \rangle \theta(e_\varphi)
\]

by (46) since \( \theta(x) = -x \). Thus if we put \( z = [e_\varphi, \theta(e_\varphi)] \), then

\[
[x, z] = 0
\]

(49)

On the other hand \( \theta(z) = -z \) since \( \theta \) is involutory. Thus \( z \in p \). But then (49) proves (48) since \( a \) maximally commutative in \( p \) (because \( a_o \) is maximally commutative in \( p_o \) and \( g_o \) is a real form of \( g \)). Assume \( \theta(\varphi) \neq -\varphi \). But then \( z \neq 0 \) if and only if \( \varphi + \theta(\varphi) \) is a root. Furthermore in this case \( z \) is a nonzero multiple of \( e_{\varphi + \theta(\varphi)} \). This is impossible since all elements in \( a \) are semisimple. Thus \( z = 0 \) and \( \varphi + \theta(\varphi) \) is not a root. If \( \theta(\varphi) = -\varphi \), then from the structure of a TDS, it follows that \( z \) is a nonzero multiple of \( h_\varphi \). In addition \( h_\varphi \in a \) by (48). QED

Let \( u_* = n_* + m_- \). But then

\[
\mathfrak{k} = u_* + b_m
\]

is a linear direct sum by (29) and (43).

1.7. Now simply order \( \Delta_+ \) so that \( \Delta_+ = \{\varphi_1, \ldots, \varphi_r\} \). Then \( \{e_{-\varphi_1}, \ldots, e_{-\varphi_r}\} \) is a basis of \( u_- \). Recall (27) and (28). If \( \varphi \in \Delta(n) \) put \( z_{-\varphi} = e_{-\varphi} + \theta(e_{-\varphi}) \) and if \( \varphi \in \Delta(m) \) put \( z_{-\varphi} = e_{-\varphi} \).
Lemma 1.12. The set \( \{ z_{-\varphi_1}, \ldots, z_{-\varphi_r} \} \) is a basis of \( u_* \).

Proof. This is obvious from (32), (45) and the fact that \( m_- \subset \mathfrak{t} \). QED

1.7. Now simply order the roots in \( \Delta(m_+) \) so that we can write \( \Delta(m_+) = \{ \mu_1, \ldots, \mu_d \} \). Hence \( \{ e_{\mu_1}, \ldots, e_{\mu_d} \} \) is a basis of \( m_+ \). Consequently, using the notation of Proposition 1.9 the following lemma follows from (34) and (50).

Lemma 1.13. The set \( \{ e_{-\varphi_1}, \ldots, e_{-\varphi_r}, y_1, \ldots, y_k, e_{\mu_1}, \ldots, e_{\mu_d} \} \) is a basis of \( \mathfrak{v} \) and the set \( \{ z_{-\varphi_1}, \ldots, z_{-\varphi_r}, y_1, \ldots, y_k, e_{\mu_1}, \ldots, e_{\mu_d} \} \) is a basis of \( \mathfrak{t} \).

The elements \( y_1, \ldots, y_k \), are algebraically independent generators of the polynomial ring \( U(\mathfrak{h}_m) \). The same is true for elements \( w_i = y_i - c_i, i = 1, \ldots, k \), where \( c_i \in \mathbb{C}, i = 1, \ldots, k \), are arbitrary fixed constants. Let \( n = r + k + d \) and regard \( \mathbb{Z}_+^n = \mathbb{Z}_+^r \times \mathbb{Z}_+^k \times \mathbb{Z}_+^d \). Furthermore if we write \( (p, q, s) \in \mathbb{Z}_+^n \) it will mean that \( p = (p_1, \ldots, p_r) \in \mathbb{Z}_+^r, q = (q_1, \ldots, q_k) \in \mathbb{Z}_+^k, \) and \( s = (s_1, \ldots, s_d) \in \mathbb{Z}_+^d \). Furthermore if \( (p, q, s) \in \mathbb{Z}_+^n \) let

\[
e(p, q, s) = e_{-\varphi_1}^{p_1} \cdots e_{-\varphi_r}^{p_r} w_1^{q_1} \cdots w_k^{q_k} e_{\mu_1}^{s_1} \cdots e_{\mu_d}^{s_d}
\]

and let

\[
z(p, q, s) = z_{-\varphi_1}^{p_1} \cdots z_{-\varphi_r}^{p_r} w_1^{q_1} \cdots w_k^{q_k} e_{\mu_1}^{s_1} \cdots e_{\mu_d}^{s_d}
\]

so that by the Poincaré-Birkhoff-Witt theorem one has

Proposition 1.14. The set \( \{ e(p, q, s) \}, (p, q, s) \in \mathbb{Z}_+^n, \) is a basis of \( U(\mathfrak{v}) \) and the set \( \{ z(p, q, s) \}, (p, q, s) \in \mathbb{Z}_+^n, \) is a basis of \( U(\mathfrak{t}) \).

1.8. Let \( R \) be the span of \( \{ z(p, 0, 0) \}, p \in \mathbb{Z}_+^r \). Then it is immediate from Proposition 1.14 that (by abuse of notation) we may write

\[
U(\mathfrak{t}) = R \otimes U(\mathfrak{h}_m) \tag{51}
\]

where \( \otimes \) identifies with multiplication.

Remark 1.15. Note that the set \( \{ e(p, 0, 0) \}, p \in \mathbb{Z}_+^r, \) is a basis of \( U(\mathfrak{u}_-). \)

Now let \( h \in \mathfrak{h} \) be the unique element such that \( \alpha(h) = 1 \) for all \( \alpha \in \Pi \). Then the eigenvalues of the diagonalizable operator \( ad \ h \) on \( \mathfrak{g} \) are integers. Extend the action of \( ad \ h \) to \( U(\mathfrak{g}) \) by commutation. Then the eigenvalues of \( ad \ h \) on \( U(\mathfrak{g}) \) are still integers and \( U(\mathfrak{g}) \) is still completely reducible under the action of \( ad \ h \). Thus one has the direct sum

\[
U(\mathfrak{g}) = \sum_{i \in \mathbb{Z}} U_i(\mathfrak{g}) \tag{52}
\]

where \( U_i(\mathfrak{g}) \) is the eigenspace of \( ad \ h \) corresponding to the eigenvalue \( i \). For any \( u \in U(\mathfrak{g}) \) let \( u_i \) be the component of \( u \) in \( U_i(\mathfrak{g}) \) with respect to the decomposition
If \( u \neq 0 \) let \( \delta(u) \in \mathbb{Z} \) be the minimal value of \( i \) such that \( u_i \neq 0 \) and put \( u_\delta = u_{\delta(u)} \) so that \( u_\delta \neq 0 \). Put \( 0_\delta = 0 \) and \( \delta(0) = \infty \). One readily verifies the following properties: If \( u, v \in U(g) \) then
\[
(\delta(u) + \delta(v)) = \delta(u) + \delta(v)
\]
and if \( \delta(u) < \delta(v) \) one has
\[
(\delta(u) + \delta(v)) = \delta(u)
\]
If \( \delta(u) = \delta(v) \) and \( u_\delta + v_\delta \neq 0 \), then
\[
(\delta(u) + \delta(v)) = \delta(u) = \delta(v)
\]
If \( \delta(u) = \delta(v) < \infty \) and \( u_\delta + v_\delta = 0 \), then
\[
\delta(u + v) > \delta(u) = \delta(v)
\]
Now any root vector in \( g \) is of course an eigenvector of \( adh \). It follows then from (47) that for \( i = 1, \ldots, r \),
\[
(z_{-\varphi_i}) = e_{-\varphi_i}
\]
Also for \( i = 1, \ldots, d \),
\[
(e_{\mu_i}) = e_{\mu_i}
\]
Any element in \( U(h_m) \) is an \( adh \) eigenvector with eigenvalue 0. Thus for \( i = 1, \ldots, k \), one has
\[
(w_i) = w_i
\]
But now (53) implies

**Proposition 1.16.** Let \( (p, q, s) \in \mathbb{Z}_+ \) be arbitrary. Then
\[
z(p, q, s) = e(p, q, s)
\]
Now \( U(u_-) \) is clearly stable under \( adh \). Let \( U_i(u_-) = U_i(g) \cap U(u_-) \). One notes that \( U_i(u_-) = 0 \) if \( i \) is positive so that one has the direct sum
\[
U(u_-) = \bigoplus_{i=0}^\infty U_i(u_-)
\]
More explicitly, let \( E = \sum_{i=1}^\ell c_i e_{-\alpha_i} \). Obviously \( E \) generates \( U(u_-) \). Put \( E^0 = \mathbb{C} 1 \), and for \( j \in \mathbb{Z}_+ \) we define \( E^{1-j} \) inductively so that if \( j > 0 \), then \( E^{1-j} = E E^{1-j} \). The following proposition is immediate.
Proposition 1.17. One has $E^{-j} = U_{-j}(u_-)$ for any $j \in \mathbb{Z}_+$. In particular $U_{-j}(u_-)$ is finite dimensional. Also the maximal possible value of $\delta(u)$ for $0 \neq u \in U(u_-)$ is 0.

Recall the subspace $R \subset U(\mathfrak{t})$. See (51).

Proposition 1.18. One has $v_\delta \in U(u_-)$ for any $v \in R$. Furthermore if $j \in \mathbb{Z}_+$ and $u \in U_{-j}(u_-)$ there exists $v \in R$ such that $v_\delta = u$.

Proof. Let $0 \neq v \in R$. Then there exists a nonempty finite subset $P \subset \mathbb{Z}_+^r$ and a set of nonzero scalars $\{c_p\}$, $p \in P$, such that $v = \sum_{p \in P} c_p z(p, 0, 0)$. Let $m$ be the minimal value of $\delta(z(p, 0, 0))$ for $p \in P$ and let $P' = \{p \in P \mid \delta(z(p, 0, 0)) = m\}$. Obviously $P'$ is not empty and hence $0 \neq v' \in U(u_-)$ by Remark 1.15 where $v' = \sum_{p \in P'} c_p e(p, 0, 0)$. But $v_\delta = v'$ by (54), (55) and (59). This proves the first statement of the proposition.

For the second statement we can assume $0 \neq u \in U_{-j}(u_-)$. Since any $e(p, 0, 0)$ is an ad$h$-eigenvector it follows from Remark 1.15 and Proposition 1.17 that there exists a finite subset $P_j \subset \mathbb{Z}_+^r$ such that $\{e(p, 0, 0)\}$, $p \in P_j$, is a basis of $U_{-j}(u_-)$. But then there exists a nonempty subset $P'_j \subset P_j$ and nonzero scalars $\{c_p\}$, $p \in P'_j$, such that $u = \sum_{p \in P'_j} c_p e(p, 0, 0)$. Let $v = \sum_{p \in P'_j} c_p z(p, 0, 0)$ so that $0 \neq v \in R$. But then $v_\delta = u$ by (55) and (59). QED

Proposition 1.19. Let $\Gamma$ be some countable index set. Assume $\{b_\gamma\}$, $\gamma \in \Gamma$, is a subset of $R$ with the property that $\{(b_\gamma)\}_\gamma$, $\gamma \in \Gamma$, is a basis of $U(u_-)$. Then $\{b_\gamma\}, \gamma \in \Gamma$, is a basis of $R$.

Proof. We first prove linear independence. Let $\Gamma_o$ be a finite nonempty subset of $\Gamma$ and let $\{c_\gamma\}, \gamma \in \Gamma_o$, be a set of nonzero scalars. Put $v = \sum_{\gamma \in \Gamma_o} c_\gamma b_\gamma$. We must show that $v \neq 0$. Let $m$ be the minimal value of $\delta(b_\gamma)$ for $\gamma \in \Gamma_o$. (Clearly $m$ is finite since all $(b_\gamma)_\delta$ are not zero.) Let $\Gamma'_o = \{\gamma \in \Gamma_o \mid \delta(\gamma) = m\}$. Obviously $\Gamma'_o$ is not empty. Put $u = \sum_{\gamma \in \Gamma'_o} c_\gamma (b_\gamma)_\delta$. Then $u \neq 0$. But $v = u$ by (54) and (55). Thus $v = 0$.

Let $R^*$ be the span of $\{b_\gamma\}, \gamma \in \Gamma$. Put $R_{\infty} = \{0\}$ and for $j \in \mathbb{Z}_+$ let $R_{-j} = \{v \in R \mid \delta(v) = -j\}$. But then

$$R = R_{\infty} \cup \bigcup_{j \in \mathbb{Z}_+} R_{-j} \quad (61)$$

is a disjoint union by (60) and Proposition 1.18. Obviously $R_{\infty} \subset R^*$. Assume $j \in \mathbb{Z}_+$ and assume inductively that $R_i \subset R^*$ for all $i > -j$ in $-\mathbb{Z}_+ \cup \infty$. Let $v \in R_{-j}$. But now since all the elements of the basis $\{(b_\gamma)\}_\gamma$, $\gamma \in \Gamma$ of $U(u_-)$ are, by definition, eigenvectors of $ad\ h$, there exists a nonempty finite subset $\Gamma_j \subset \Gamma$ such that $\{(b_\gamma)\}_\gamma$, $\gamma \in \Gamma_j$ is a basis of $U_{-j}(u_-)$. But then there exists a nonempty subset $\Gamma'_j$ of $\Gamma_j$ and nonzero scalars $\{c_\gamma\}$, $\gamma \in \Gamma'_j$ such that if $u = \sum_{\gamma \in \Gamma'_j} c_\gamma (b_\gamma)_\delta$, then $u = v_\delta$.

Put $w = \sum_{\gamma \in \Gamma'_j} c_\gamma b_\gamma$. But $w \in R^*$ and $w_\delta = u$ by (55). Hence $w_\delta = v_\delta$. Put
$v - w = z \in R$. But then $\delta(z) > -j$ by (56). But then $z \in R^*$ by induction. Hence $w + z = v \in R^*$. Thus $R^* = R$. QED

1.9. Recall the definition of $z_{-\varphi}$ for $\varphi \in \Delta_+$. See the paragraph preceding Lemma 1.12. Let $\Pi_s = \{ \alpha \in \Pi \mid \theta(-\alpha) = \alpha \}$. One has

$$\Pi_s \subset \Pi_n$$

recalling (47) and the notation of (36).

**Proposition 1.20.** Let $\alpha \in \Pi$. Then $z_{-\alpha}$ is either a semisimple element or a nilpotent element. In addition in the latter case $z_{-\alpha}$ is also nilpotent as an element of $\mathfrak{g}$ (i.e. $z_{-\alpha} \in [\mathfrak{t}, \mathfrak{t}]$). Furthermore the following conditions are equivalent:

(a) $\alpha \in \Pi_s$

(b) $z_{-\alpha}$ is semisimple

(c) $(z_{-\alpha}, z_{-\alpha}) \neq 0$

**Proof.** Recall (36). If $\alpha \in \Pi_m$, then $\alpha \notin \Pi_s$ by (62) and $z_{-\alpha} = e_{-\alpha}$ is nilpotent. The element $z_{-\alpha}$ is also nilpotent as an element of $\mathfrak{g}$ since $\mathfrak{g}_i \subset [\mathfrak{t}, \mathfrak{t}]$ where $i \in \{1, \ldots, \ell, \}$ is such that $\alpha = \alpha_i$. Clearly $z_{-\alpha}$ does not satisfy either (a), (b) or (c). Next assume $\alpha \in \Pi_n$ but $\alpha \notin \Pi_s$. Then $e_{-\alpha}$ commutes with $\theta(e_{-\alpha})$ by Proposition 1.11 and hence $z_{-\alpha} = e_{-\alpha} + \theta(e_{-\alpha})$ is nilpotent. Thus $(ad z_{-\alpha})^j = 0$. To show that $z_{\alpha}$ is nilpotent as an element of $\mathfrak{g}$, it suffices then to prove $z_{-\alpha} \in [\mathfrak{t}, \mathfrak{t}]$. But the Killing form $B_\mathfrak{g}$ is nonsingular on $\mathfrak{t}$ and certainly nonsingular on the semisimple Lie algebra $[\mathfrak{t}, \mathfrak{t}]$. It follows immediately that $B_\mathfrak{g}|\text{Cent} \mathfrak{t}$ is nonsingular and $\text{Cent} \mathfrak{t}$ is the $B_\mathfrak{g}$-orthocomplement of $[\mathfrak{t}, \mathfrak{t}]$ in $\mathfrak{t}$. But if $w \in \text{Cent} \mathfrak{t}$, then $(w, z_{-\alpha}) = trad z_{-\alpha} ad w$. But $(ad z_{-\alpha} ad w)^j = (ad z_{-\alpha})^j (ad w)^j = 0$. Thus $(w, z_{-\alpha}) = 0$ so that $z_{-\alpha} \in [\mathfrak{t}, \mathfrak{t}]$. In this present case where $\alpha \in \Pi_n$ but $\alpha \notin \Pi_s$, it is obvious that (a), (b) and (c) are not satisfied. Finally assume $\alpha \in \Pi_s$. Then (c) is satisfied since $(e_{-\alpha}, e_{\alpha}) \neq 0$. But $\alpha = \alpha_i$ for some $i \in \{1, \ldots, \ell, \}$, using the notation of §1.1. But then (c) implies that $z_{-\alpha}$ is a semisimple element of the TDS $\mathfrak{g}_i$. QED

We have not normalized our choice of root vectors. We wish to do so now to the extent that $\theta(e_{-\alpha_i}) = e_{\alpha_i}$ for $\alpha_i \in \Pi_s$. Hence for $\alpha_i \in \Pi_s$ we may assume

$$z_{-\alpha_i} = e_{-\alpha_i} + e_{\alpha_i}$$

Furthermore by replacing $e_{-\alpha}$ by a suitable complex multiple of $e_{-\alpha_i}$, if necessary, we can also assume that

$$(z_{-\alpha_i}, z_{-\alpha_i}) = (h_i, h_i)$$

using the notation of (4).
1.10. Let $\lambda \in \Lambda$. For $i \in \{1, \ldots, \ell\}$ we will now define a polynomial $q_{\lambda,i}(t) \in \mathbb{C}[t]$ of degree $n_i(\lambda) + 1$. See §1.1. If $\alpha_i \notin \Pi_s$, put $q_{\lambda,i}(t) = t^{n_i(\lambda) + 1}$. If $i \in \Pi_s$, let $q_{\lambda,i}(t)$ be the monic polynomial of degree $n_i(\lambda) + 1$ with (multiplicity 1) roots $\{n_i(\lambda) - 2j\}$, $j = 0, \ldots, n_i(\lambda)$. Thus in this case
\[
q_{\lambda,i}(t) = (t - n_j(\lambda))(t - n_j(\lambda) + 2) \cdots (t + n_i(\lambda))
\] (65)
Now recall that $L_\lambda(\mathfrak{t})$ is the (left ideal) annihilator in $U(\mathfrak{t})$ of the $U(\mathfrak{t})$-cyclic vector $v_\lambda \in V_\lambda$.

**Proposition 1.21.** For any $i \in \{1, \ldots, \ell\}$ one has
\[
q_{\lambda,i}(z_{-\alpha_i}) \in L_\lambda(\mathfrak{t})
\] (66)
Furthermore
\[
q_{\lambda,i}(z_{-\alpha_i}) = e_{-\alpha_i}^{n_i(\lambda) + 1}
\] (67)

**Proof.** Recall (36). If $\alpha_i \in \Pi_n$, then by definition $z_{-\alpha_i} = e_{-\alpha_i}$ and hence (66) and (67) follow from Proposition 1.1. Next assume $\alpha_i \in \Pi_n$ but $\alpha_i \notin \Pi_s$. Then $e_{-\alpha_i}$ commutes with $\theta(e_{-\alpha_i})$ by Proposition 1.11 and $\theta(-\alpha_i) \in \Delta_+$ by (47). Thus $\theta(e_{-\alpha_i})v_\lambda = 0$. This proves (66) by Proposition 1.1. The equation (67) follows from (57) and (53). Finally assume $\alpha_i \in \Pi_s$. The ring of polynomial adjoint invariants on $\mathfrak{g}_i$ is generated by the quadratic form on $\mathfrak{g}_i$ defined by $B_{\mathfrak{g}_i}$. Any element of this ring therefore takes the same value on the semisimple elements $z_{-\alpha_i}$ and $h_i$ by (64). Consequently $z_{-\alpha_i}$ and $h_i$ are conjugate in $\mathfrak{g}_i$. But, as one knows, $q_{\lambda,i}(t)$ is the characteristic polynomial of $h_i$ operating on the irreducible $U(\mathfrak{g}_i)$-module $U(\mathfrak{g}_i)v_\lambda$. See Proposition 1.1. By conjugacy this proves (66). The equation (67) follows from the factorization (65), with $z_{-\alpha_i}$ replacing $t$, together with (53) and (57). QED

Recall (51). By definition $R \subset U(\mathfrak{t})$ is the span of $\{z(p, 0, 0)\}, p \in \mathbb{Z}_+^\ell$. On the other hand $\{e(p, 0, 0)\}, p \in \mathbb{Z}_+^\ell$ is a clearly a basis of $U(u_-)$ as noted in Remark 1.15.

Let $I = \{1, \ldots, \ell\}$. For each $(p, i) \in \mathbb{Z}_+^\ell \times I$, let $z(p, i) = z(p, 0, 0)q_{\lambda,i}(z_{-\alpha_i})$ and let $e(p, i) = e(p, 0, 0)^{n_i(\lambda) + 1}$. Obviously $z(p, i) \in U(\mathfrak{t})$ and $e(p, i) \in U(u_-)$. Recall that $L_\lambda(u_-)$ is the annihilator of $v_\lambda$ in $U(u_-)$. See (7).

**Proposition 1.22.** Let $(p, i) \in \mathbb{Z}_+^\ell \times I$. Then $z(p, i) \in L_\lambda(\mathfrak{t})$ and $e(p, i) \in L_\lambda(u_-)$.

Furthermore
\[
z(p, i) = e(p, i)
\] (68)

**Proof.** The first statement follows from (66) and (8). The second statement follows from (59), (67) and (53). QED

Now recalling (38) and (51) one has
\[
U(\mathfrak{t}) = R \oplus RKer \omega_\lambda
\] (69)
Note that (69) implies
\[ R \ker \omega_\lambda = U(\mathfrak{t}) \ker \omega_\lambda \] (70)
But now obviously
\[ R \ker \omega_\lambda \subset L_\lambda(\mathfrak{t}) \]
Hence if \( R_\lambda = R \cap L_\lambda(\mathfrak{t}) \) one has
\[ L_\lambda(\mathfrak{t}) = R_\lambda \oplus R \ker \omega_\lambda \] (71)
For any \((p, i) \in \mathbb{Z}_+^r \times I\) let (recalling Proposition 1.22) \( b(p, i) \in R_\lambda \) and \( d(p, i) \in R \ker \omega_\lambda \) be the components of \( z(p, i) \) with respect to the decomposition (71) so that
\[ z(p, i) = b(p, i) + d(p, i) \] (72)

Lemma 1.23. Let \((p, i) \in \mathbb{Z}_+^r \times I\). Then
\[ b(p, i) = e(p, i) \] (73)

Proof. If \( d(p, i) = 0 \) the result follows from directly from (68). Hence we can assume \( d(p, i) \neq 0 \). In §1.7 following Lemma 1.13 we defined \( w_i = y_i - c_i, i = 1, \ldots, k \), where \( \{y_i\}, i = 1, \ldots, k \), is a basis of \( \mathfrak{h}_m \) (see paragraph preceding (37)) and \( \{c_i\}, i = 1, \ldots, k \), are arbitrary complex scalars. We now fix the \( c_i \) so that \( c_i = \lambda(y_i) \). One then immediately has
\[ \{z(0, q, s)\}, (q, s) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^d - (0, 0) \] is a basis of \( \ker \omega_\lambda \) (74)
But then if \((\mathbb{Z}_+^n)^* = \{(p', q, s) \in \mathbb{Z}_+^n \mid (q, s) \neq (0, 0)\}\) one has
\[ \{z(p', q, s), (p', q, s) \in (\mathbb{Z}_+^n)^*\}, \text{ is a basis of } R \ker \omega_\lambda \] (75)
But then there exists a finite nonempty subset \( F \subset (\mathbb{Z}_+^n)^* \) and a set \( \{c_f\}, f \in F \), of nonzero scalars, such that \( d_{p,i} = \sum_{f \in F} c_f z(f) \). Let \( m \) be the minimal value of \( \delta(z(f)) \) for \( f \in F \) and let \( F_m = \{f \in F \mid \delta(f) = m\} \). Then \( F_m \) is not empty and
\[ (d(p, i)) = \sum_{f \in F_m} c_f e(f) \] (76)
by (54), (55) and (59). But if \( f \in F_m \) and \( f = (p', q, s) \) one has \( (q, s) \neq (0, 0) \). Thus \( 0 \neq (d(p, i)) \in U(u_-) \ker \omega_\lambda \). But \( (b(p, i)) \in U(u_-) \) by Proposition 1.18. However one has the direct sum
\[ U(\mathfrak{v}) = U(u_-) \oplus U(u_-) \ker \omega_\lambda \] (77)
by (35). Thus
\[(b(p, i))_\delta + (d(p, i))_\delta \neq 0\]
However
\[(b(p, i) + d(p, i))_\delta = z(p, i)_\delta = e(p, i) \in U(u_-)\]
Thus we cannot have \(m < \delta(b(p, i))\) or even \(m = \delta(b(p, i))\). Consequently \(m > \delta(b(p, i))\). Thus \(b(p, i)_\delta = z(p, i)_\delta\). This together with (68) proves (73). QED

1.11. We can now state and prove our key result.

Theorem 1.24. Let \(\mathfrak{f}\) be any symmetric Lie subalgebra of a complex semisimple Lie algebra \(\mathfrak{g}\) (i.e. \(\mathfrak{f}\) is the set of fixed elements in \(\mathfrak{g}\) for some involutory automorphism of \(\mathfrak{g}\)). Let \(V_\lambda\) be a finite dimensional irreducible \(\mathfrak{g}\)-module with arbitrary highest weight \(\lambda\) and let \(0 \neq v_\lambda \in V_\lambda\) be a highest weight vector. Then \(V_\lambda\) is a cyclic \(U(\mathfrak{f})\)-module with cyclic generator \(v_\lambda\). Let \(L_\lambda(\mathfrak{f})\) be the annihilator (and hence left ideal) of \(v_\lambda\) in \(U(\mathfrak{f})\). Then in the proceeding notation

\[
L_\lambda(\mathfrak{f}) = \sum_{i=1}^{\ell} U(\mathfrak{f}) q_{\lambda,i}(z_{-\alpha}) + \sum_{j=1}^{k} U(\mathfrak{f})(y_j - \lambda(y_j)) + \sum_{\alpha \in \Pi_m} U(\mathfrak{f}) e_\alpha \quad (78)
\]

Proof. The statement about cyclicity is just Proposition 1.3. For notational convenience let \(J_\lambda = L_\lambda(u_-)\) so that \(J_\lambda\) is the annihilator of \(v_\lambda\) in \(U(u_-)\) (see §1.1) Obviously \(J_\lambda\) is stable under \(ad h\) so that, recalling (60), one has

\[
J_\lambda = \bigoplus_{j=0}^\infty (J_\lambda)_{-j} \quad (79)
\]

where \((J_\lambda)_{-j} = J_\lambda \cap U_{-j}(u_-)\). If \(Y\) is a vector space and \(X \subset Y\) is a subspace of \(Y\) we will write \(\text{codim}(X, Y)\) for the codimension of \(X\) in \(Y\). Thus \(\text{codim}(X, Y)\) will take values in \(\mathbb{Z}_+ \cup \{\infty\}\). Let \(d_\lambda = \dim V_\lambda\). By (7) one has

\[
\sum_{j=0}^\infty \text{codim}((J_\lambda)_{-j}, U_{-j}(u_-)) = d_\lambda \quad (80)
\]

Obviously all but a finite number of summands on the left side of (80) are equal to 0. Let \((p, i) \in \mathbb{Z}_+^r \times I\). Then \(e(p, i) \in J_\lambda\) by Proposition 1.2 and furthermore \(e(p, i)\) is an \(ad h\) eigenvector. Thus

\[
e(p, i) \in (J_\lambda)_{-j} \quad (81)
\]

for some \(j\). Let \((\mathbb{Z}_+^r \times I)_{-j} = \{(p, i) \in \mathbb{Z}_+^r \times I \mid e(p, i) \in (J_\lambda)_{-j}\}\). But \(\delta(e(p, i)) = \delta(e(p, 0, 0)) + n_i(\lambda) + 1\) by (53). Thus \((\mathbb{Z}_+^r \times I)_{-j}\) is a finite set by Proposition 1.17 and Remark 1.15. But the set \(\{e(p, i) \mid (p, i) \in (\mathbb{Z}_+^r \times I)_{-j}\}\) spans \((J_\lambda)_{-j}\) by Proposition 1.2 and Remark 1.15. Let \(\Gamma'_j\) be a subset of \((\mathbb{Z}_+^r \times I)_{-j}\) such that \(\{e(p, i) \mid (p, i) \in \Gamma'_j\}\) is
a basis of $(J_{\lambda})_{-j}$. Now let $\Gamma''$ be an index set of cardinality $\text{codim}((J_{\lambda})_{-j}, U_{-j}(u_-))$ and choose for each $\gamma \in \Gamma''$ an element $e(\gamma) \in U_{-j}(u_-)$ such that if $\Gamma_j$ is the disjoint union of $\Gamma'_j$ and $\Gamma''_j$, then \{e(\gamma) \mid \gamma \in \Gamma_j\} is a basis of $U_j(u_-)$. Let $\Gamma, \Gamma'$ and $\Gamma''$, be, respectively, the (disjoint) union over all $j \in \mathbb{Z}_+$ of $\Gamma_j, \Gamma'_j$ and $\Gamma''_j$. Thus

$$\Gamma = \Gamma' \cup \Gamma'' \quad (82)$$

is disjoint union. Also, by (80),

$$\text{card} \ \Gamma'' = d_{\lambda} \quad (83)$$

and

$$\{e(\gamma) \mid \gamma \in \Gamma\} \text{ is a basis of } U(u_-) \quad (84)$$

and

$$\Gamma' \subset Z^*_+ \times I \quad (85)$$

Now let $L_{\lambda}(t)^{(1)}$ be the left ideal of $U(t)$ defined by the right side of (78). Clearly $L_{\lambda}(t)^{(1)} \subset L_{\lambda}(t)$ by (66). However $\text{codim}(L_{\lambda}(t), U(t)) = d_{\lambda}$ by (15). Thus to prove Theorem 22 it suffices to prove

$$\text{codim}(L_{\lambda}(t)^{(1)}, U(t)) \leq d_{\lambda} \quad (86)$$

But now the sum of the last two sums on the right side of (78) is clearly $U(t) \text{ Ker } \omega_{\lambda}$. Thus

$$L_{\lambda}(t)^{(1)} = \sum_{i=1}^{\ell} U(t) q_{\lambda,i}(z_{-\alpha_i}) + R \text{ Ker } \omega_{\lambda} \quad (87)$$

by (70). But then if

$$L_{\lambda}(t)^{(2)} = \sum_{i=1}^{\ell} R q_{\lambda,i}(z_{-\alpha_i}) + R \text{ Ker } \omega_{\lambda}$$

one has $L_{\lambda}(t)^{(2)} \subset L_{\lambda}(t)^{(1)}$, since of course $R \subset U(t)$, and hence to prove the theorem it suffices to prove

$$\text{codim}(L_{\lambda}(t)^{(2)}, U(t)) \leq d_{k} \quad (88)$$

But $\sum_{i=1}^{\ell} R q_{\lambda,i}(z_{-\alpha_i})$ is the span of \{z(p, i)\}, \((p, i) \in Z^*_+ \times I\) by definition of $R$ (see §1.8). Recall (72). One has $b(p, i) \in R$. Let $R^{(2)}$ be the span of \{b(p, i)\}, \((p, i) \in Z^*_+ \times I\). Since $d(p, i) \in R \text{ Ker } \omega_{\lambda}$ it then follows that

$$L_{\lambda}(t)^{(2)} = R^{(2)} \oplus R \text{ Ker } \omega_{\lambda}$$

Hence by (69) to prove the theorem it suffices to prove

$$\text{codim} (R^{(2)}, R) \leq d_{\lambda} \quad (89)$$
Recall (85). Let $R^{(3)}$ be the span of $\{b(\gamma)\}$, $\gamma \in \Gamma'$. One has $R^{(3)} \subset R^{(2)}$ by (85). Thus it suffices to prove

$$\text{codim} (R^{(3)}, R) = d_\lambda$$

(90)

But for each $\gamma \in \Gamma''$ there exists $b(\gamma) \in R$ such that $b(\gamma)_\delta = e(\gamma)$ by Proposition 1.18. Thus $b(\gamma) \in R$ is defined for all $\gamma \in \Gamma = \Gamma' \cup \Gamma''$. But also for any $\gamma \in \Gamma'$ one has $b(\gamma)_\delta = e(\delta)$ by (73). Then $\{b(\gamma), \gamma \in \Gamma\}$ is a basis of $R$ by Proposition 1.19 and (84). But then if $R''$ is the span of all $b(\gamma)$ for $\gamma \in \Gamma''$, one has that $\dim R'' = d_\lambda$ and

$$R = R'' \oplus R^{(3)}$$

This proves (90). QED

2. The branching law for $\mathfrak{t}$ and the precise structure of $M$

2.1. We may regard the dual $\mathfrak{h}^*_m$ to $\mathfrak{h}_m$ as the subspace of $\mathfrak{h}^*$ that is orthogonal to $\mathfrak{a}$. In particular $\Delta(m) \subset \mathfrak{h}^*_m$. Let $\Lambda_m \subset \mathfrak{h}^*_m$ be the set of dominant (with respect to $\mathfrak{b}_m$) integral linear forms on $\mathfrak{m}$. That is, if $\nu \in \mathfrak{h}^*_m$ then $\nu \in \Lambda_m$ if and only if

$$2(\nu, \alpha)/(\alpha, \alpha) \in \mathbb{Z}_+, \forall \alpha \in \Pi_m$$

For each $\nu \in \Lambda_m$ there exists an irreducible representation

$$\tau_\nu : U(\mathfrak{m}) \rightarrow \text{End} Y_\nu$$

(91)

with highest weight $\nu$ (with respect to $\mathfrak{b}_m$). If $X$ is any completely reducible finite dimensional $\mathfrak{m}$-module and $\nu \in \Lambda_m$, we will denote the primary $\tau_\nu$-submodule of $X$ by $X[\nu]$. The space of highest weight vectors in $X[\nu]$ is of course $X[\nu]^{\mathfrak{m}^+}$. Since $\mathfrak{m}$ is reductive in $\mathfrak{k}$ and in $\mathfrak{g}$, any finite dimensional completely reducible $\mathfrak{t}$-module or any finite dimensional $\mathfrak{g}$-module is completely reducible with respect to the action of $\mathfrak{m}$. Obviously $\lambda|\mathfrak{m} \in \Lambda_m$. The following is fairly well known. For $\lambda \in \Lambda$ let $V^n_\lambda = \{v \in V_\lambda \mid n v = 0\}$.

**Proposition 2.1.** Let $\lambda \in \Lambda$. Then $U(\mathfrak{m})v_\lambda$ is an $\mathfrak{m}$-irreducible component of $V_\lambda[\lambda|\mathfrak{m}]$ with highest weight vector $v_\lambda$ and

$$U(\mathfrak{m})v_\lambda = U(\mathfrak{m}-)v_\lambda$$

(92)

Furthermore if one regards $V_\lambda$ as an $\mathfrak{a}$-module, then $U(\mathfrak{m})v_\lambda$ is the weight space for $\mathfrak{a}$ of weight $\lambda|\mathfrak{a}$. In addition

$$U(\mathfrak{m})v_\lambda = V^n_\lambda$$

(93)

**Proof.** The first statement (including (92)) follows from the equation $u v_\lambda = \omega_\lambda(u) v_\lambda$ for all $u \in U(\mathfrak{b}_m)$. Furthermore if $V_\lambda^{\lambda|\mathfrak{a}}$ is the $\mathfrak{a}$-weight space in $V_\lambda$ with weight $\lambda|\mathfrak{a}$ one has

$$U(\mathfrak{m})v_\lambda \subset V_\lambda^{\lambda|\mathfrak{a}}$$
since \( m \) centralizes \( a \). But

\[
U(u_-) = U(m_-) \oplus U(u_-)n_-
\]

by (32) so that

\[
V_\lambda = U(m)v_\lambda + U(u_-)n_- v_\lambda
\]

by (7) and (92). Recalling the definition of \( w \in a \) in §1.2 one notes that the spectrum of \( adw \) on \( U(u_-)n_- \) is strictly negative. Thus the \( a \)-weight space for the weight \( \lambda | a \) in the second summand on the right side of (95) vanishes. This proves the second statement of the proposition and incidentally the fact that (95) is a direct sum.

Let \( V \) equal the right side of (93). Obviously \( v_\lambda \in V \). But since \( n \) is normalized by \( m \) it follows that \( V \) is stabilized by \( m \) and in particular one must have that \( U(m)v_\lambda \subset V \). But, by (26), any highest vector \( v \) for \( b_m \) in \( V \) is annihilated by \( u \). Thus \( v \in Cv_\lambda \). Hence \( V \) is \( m \)-irreducible. This proves (93). QED

Recall \( I = \{1, \ldots, \ell\} \). Let \( I_m = \{i \in I \mid \alpha_i \in \Pi_m\} \). Note that if \( i \in I_m \) then \( h_i \in b_m \) (see (4)) since \( h_i \in \mathbb{C}[e_{\alpha_i}, e_{-\alpha_i}] \), in particular \( g_i \subset m \). (See Proposition 1.1.)

**Remark 2.2.** Note that if \( X \) is any finite dimensional completely reducible \( m \)-module and \( \lambda \in \Lambda \), then for any \( i \in I_m \)

\[
e^{-\alpha_i}_- (X|\lambda|^m^+) = 0
\]

Indeed, as in the case of Proposition 1.1, this follows from the representation theory of \( SL(2, \mathbb{C}) \) since \( h_i \) reduces to the scalar \( \langle \lambda, h_i \rangle \) on \( X|\lambda|^m^+ \).

Let \( I_n \) be the complement of \( I_m \) in \( I \) so that \( j \in I_n \) if and only if \( \alpha_j \in \Pi_n \) (see (36)). We recall that \( \Pi_s = \{\alpha \in \Pi \mid \theta(\alpha) = -\alpha\} \) and that \( \Pi_s \subset \Pi_n \) (see (62)). Let \( I_s = \{i \in I \mid \alpha_i \in \Pi_s\} \) and let \( I_{n,l} \) be the complement of \( I_s \) in \( I_n \). For any \( i \in I \) and \( \lambda \in \Lambda \) we defined, in §1.9, a polynomial \( q_{\lambda,i}(t) \in \mathbb{C}[t] \) of degree \( u_i(\lambda) + 1 \). The branching law, Theorem 2.3, below, will require knowledge of the polynomials \( q_{\lambda,i} \) only for \( i \in I_n \). Also since the definition of \( z_{-\alpha_i} \in \mathfrak{t} \) for \( i \in I_n \) is different from its definition when \( i \in I_m \), it is convenient to simplify the notation and put \( z_i = z_{-\alpha_i} \) for \( i \in I_n \). Thus for \( i \in I_n \)

\[
z_i = e_{-\alpha_i} + \theta(e_{-\alpha_i})
\]

where, if \( i \in I_s \subset I_n \), then \( e_{-\alpha_i} \) is normalized so that \( \theta(e_{-\alpha_i}) = e_{\alpha_i} \) and hence

\[
z_i = e_{-\alpha_i} + e_{\alpha_i}
\]

and, in addition, \( e_{-\alpha_i} \) is normalized in this case so that

\[(z_i, z_i) = (h_i, h_i)\]
Of course if $i \in I_s$ then $z_i \in \mathfrak{g}_i$ and in fact

$$\mathfrak{g}_i \cap \mathfrak{k} = \mathbb{C}z_i$$

(100)

using the notation of Proposition 1.1.

2.2. If $i \in I$ and $\lambda \in \Lambda$, then we have defined a polynomial $q_{\lambda,i}(t) \in \mathbb{C}[t]$ of degree $n_i(\lambda) + 1$. If $i \in I_s$ then, recalling (65),

$$q_{\lambda,i}(t) = (t - n_j(\lambda))(t - n_j(\lambda) + 2) \cdots (t + n_i(\lambda))$$

and if $i$ is in the complement $I_{nil}$ of $I_s$ in $I_n$, then $q_{\lambda,i}(t) = t^{n_i(\lambda)+1}$.

If $Z$ is any finite dimensional irreducible $\mathfrak{k}$-module and $\lambda \in \Lambda$, let

$$Z^\lambda = \{ v \in Z|\lambda|m^{m^+} \mid q_{\lambda,i}(z_i)v = 0, \forall i \in I_n \}$$

(101)

Also let $\text{mult}_{V_{\lambda}}(Z)$ be the multiplicity of the irreducible representation $Z$ in $V_{\lambda}$, regarded as a $\mathfrak{k}$-module. The following branching law is one of the main theorems.

**Theorem 2.3.** Let $\mathfrak{t}$ be any symmetric Lie subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$ (i.e. $\mathfrak{t}$ is the set of fixed elements in $\mathfrak{g}$ for some involutory automorphism of $\mathfrak{g}$.) Using the notation of §1.1, let $\lambda \in \Lambda$ and let $V_{\lambda}$ be an irreducible (necessarily finite dimensional) $\mathfrak{g}$-module with highest weight $\lambda$. Let $Z$ be any finite dimensional irreducible $\mathfrak{k}$-module. Then in the notation of (101) one has

$$\text{mult}_{V_{\lambda}}(Z) = \dim Z^\lambda$$

(102)

**Proof.** Recalling the definition of $L_{\lambda}(\mathfrak{t})$ (see §1.2) one has

$$\text{mult}_{V_{\lambda}}(Z) = \dim Z^{L_{\lambda}(\mathfrak{t})}$$

(103)

by (18). But now using the notation of Theorem 1.23 one obviously has

$$Z[\lambda|m^{m^+}] = \{ v \in Z \mid (y_i - \lambda(y_i))v = 0, i = 1, \ldots, k, \text{ and } e_\alpha v = 0, \forall \alpha \in \Pi_m \}$$

However, by (96), one automatically has $q_{\lambda,i}(z_{-\alpha_i})v = 0$ for any $i \in I_m$ and $v \in Z[\lambda|m^{m^+}]$, using the notation of Theorem 1.24. But then

$$Z^{L_{\lambda}(\mathfrak{t})} = Z^\lambda$$

(104)

by (78). The theorem then follows from (103). QED

2.3. Now let $G_\mathbb{C}$ be a simply connected semisimple Lie group such that $\mathfrak{g} = \text{Lie } G_\mathbb{C}$ and let $G$ be the real form and subgroup of $G_\mathbb{C}$ corresponding to $\mathfrak{g}_o$. Thus the subgroup
$K$ of $G$ corresponding to $\mathfrak{k}$, is a maximal compact subgroup of $G$. Let $A$ be the subgroup of $G$ corresponding to $\mathfrak{a}_0$, and let $M$ be the centralizer of $A$ in $K$. Let $\ell_o$ be the (real) dimension of $A$ so that $\ell_o$ is the split rank of $G$. The compact group $M$ is not connected in general and we denote the identity component of $M$ by $M_e$. Since $G_C$ is simply connected, the involutory automorphism $\theta$ lifts to an automorphism of $G_C$ which we continue to denote by $\theta$. Of course $\theta$ is the identity on $K$. Let $A_C$ be the subgroup of $G_C$ which corresponds to $a$. One thus has

$$A_C \cong (\mathbb{C}^*)^{\ell_o}$$

In particular if $T$ is the group of all elliptic elements in $A_C$, then one has the group product structure

$$A_C = A \times T$$

In particular if $F = \{a \in A_C \mid a^2 = 1\}$ then $F \subset T$ and

$$F \cong \mathbb{Z}_{2}^{\ell_o} \quad (105)$$

Since $\theta = -1$ on $a$ one has $\theta(a) = a^{-1}$ for all $g \in A_C$. The finite 2-group $F$ can then also be characterized by

$$F = \{a \in A_C \mid \theta(a) = a\} \quad (106)$$

Let $F_M = F \cap K$. One notes that

$$F_M = A_C \cap K \quad (107)$$

Indeed any element on the right side of (107) is fixed under $\theta$ so that (107) follows from (106). The following is well known and is proved for completeness. Let

$$\kappa : M \to M/M_e \quad (108)$$

be the quotient homomorphism

**Lemma 2.4.** One has $F_M \subset M$ and in fact

$$F_M \subset \text{Cent } M \quad (109)$$

Furthermore $\kappa|F_M$ is surjective so that

$$M = F_M M_e \quad (110)$$

**Proof.** Clearly $F_M$ commutes with $A_0$, since $A_C$ is commutative, so that $F_M \subset M$. Furthermore if $b \in M$, then $Ad b$ fixes all elements in $a_0$, and, by complexification, fixes all elements in $a_C$. Thus $F_M$ centralizes $M$ so that $F_M \subset \text{Cent } M$. 

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Let \( g_u = t_o + ip_o \) so that \( g_u \) is a compact real form of \( g \). Let \( G_u \subset G_C \) be the group corresponding to \( g_u \) so that \( G_u \) is a maximal compact subgroup of \( G_C \) and furthermore \( G_u \) is simply connected. Of course \( K \subset G_u \). Let \( g \in M \). Then the centralizer \( G'_u \) of \( g \) in \( G^a \) is a reductive connected (since \( G_u \) is simply connected) compact subgroup of \( G_u \) and \( \ell = \text{rank} G'_u \) since \( g \) lies in some maximal torus of \( G_u \). But \( T \subset G'_u \) since \( T_i \) is a maximal torus of \( G_u \) and \( T \). Hence \( T \) is contained in a maximal torus \( T_u \) of \( G'_u \). Let \( t_u = \text{Lie} T_u \) and let \( t_t = t_u \cap t_o \) so that

\[
\text{Lie} T = i\alpha_o
\]

Thus \( T \) is contained in a maximal torus \( T_u \) of \( G'_u \). Let \( t_u = \text{Lie} T_u \) and let \( t_t = t_u \cap t_o \) so that

\[
t_t \subset m_o
\]

But now if \( z \in t_u \) we may write \( z = x + y \) where \( x \in t_o \) and \( y \in i\alpha_o \). But then \([x, i\alpha_o] \subset i\alpha_o \) and \([y, i\alpha_o] \subset k_o \). However \([z, i\alpha_o] = 0 \). Thus we must have \([y, i\alpha_o] = 0 \). However \( i\alpha_o \) is maximal abelian in \( i\alpha_o \). Thus \( y \in i\alpha_o \subset t_u \). Thus \( x \in t_u \cap k = t_t \). Consequently one has

\[
t_u = t_t + i\alpha_o
\]

and hence \( T_u = T_t T \) where \( T_t \) is the group corresponding to \( t_t \). But \( g \in T_o \) since of course \( g \in \text{Cent} G'_u \) and consequently \( g \) lies in every maximal torus of \( G'_u \). Consequently we may write \( g = g_t a \) where \( g_t \in T_t \) and \( a \in T \). But \( T_t \subset M_e \) by (112) so that \( g \) is congruent to \( a \) modulo \( M_e \). But \( a \in T \cap K \) so that \( a \in F_M \) by (107). QED

Let \( j \in I \) and let \((G_C)_j \) be the subgroup of \( G \) which corresponds to the TDS \( g_j \) (see §1.1). One immediate question is whether \((G_C)_j \) is isomorphic to \( \text{SL}(2,\mathbb{C}) \) or \( \text{PSL}(2,\mathbb{C}) \). In fact, as noted in the next proposition, the former is always the case. Recalling the notation of §1.1 let \( \pi_j = \pi_{\lambda_j} \), \( V_j = V_{\lambda_j} \) and \( v_j = v_{\lambda_j} \) so that \( \pi_j : g \to \text{End} V_j \) is a fundamental representation of \( g \). If \( i \in I \) and \( j \neq i \) then, as a consequence of (4) and Proposition 1.1,

\[
U(g_i)v_j = \mathbb{C}v_j
\]

whereas

\[
dim U(g_j)v_j = 2
\]

**Proposition 2.5.** Let \( j \in I \). Then \((G_C)_j \) is isomorphic to \( \text{SL}(2,\mathbb{C}) \).

**Proof.** It suffices to show that there exists a representation \( \pi \) of \( G \) such that one of the irreducible components of \( \pi u(G_C)_j \) is 2-dimensional. But if \( \pi = \pi_j \) this is immediate from (115) and Proposition 1.1. QED

Of course \( \text{Cent} \text{SL}(2,\mathbb{C}) \) is isomorphic to \( \mathbb{Z}_2 \). Thus

\[
\text{Cent} (G_C)_j \cong \mathbb{Z}_2
\]

for any \( j \in I \) by Proposition 2.5. Consequently for any \( j \in I \) there exists a unique element \( \varepsilon_j \in (G_C)_j \) such that \( 1 \neq \varepsilon_j \in \text{Cent} (G_C)_j \). Also of course

\[
\varepsilon_j^2 = 1
\]
We are really only concerned about the case where \( j \in I_s \).

**Lemma 2.6.** Let \( j \in I_s \). Then using the notation of (4) and (98)

\[
\varepsilon_j = \exp \pi i h_j = \exp \pi i z_j \tag{118}
\]

**Proof.** Under any isomorphism of \((G_C)_j\) with \( SL(2, \mathbb{C}) \) the element \( \varepsilon_j \) corresponds to minus the identity. But the eigenvalues of \( h_j \) on \( U(g_j) \) are \( \pm 1 \). This proves the first line in (118). But as observed in the proof of Proposition 1.21, the element \( z_j \) is conjugate to \( h_j \) in \( g_j \). This proves the second line in (118) since \( \varepsilon_j \) is central in \((G_C)_j\).

QED

**Lemma 2.7.** Let \( j \in I_s \). Then \( \varepsilon_j \in F_M \).

**Proof.** It is obvious from the first line in (118) that \( \varepsilon_j \in F \). It suffices then only to prove that \( \varepsilon_j \in K \). But from the first line in (118) this would follow if one proves \( i z_j \in \mathfrak{t}_o \). Obviously \( i z_j \in \mathfrak{t} \). Let \( \theta_o \) be the conjugation involution of \( g \) whose fixed set is \( g_o \). It suffices then to prove that \( i z_j \) is fixed by \( \theta_o \) or equivalently

\[
\theta_o(z_j) = -z_j \tag{119}
\]

But \( \theta_o \) stabilizes the Cartan subalgebra \( \mathfrak{h} = \mathfrak{h}_m + \mathfrak{a} \). Hence \( \theta_o(e_{-\alpha_j}) \) is a root vector corresponding to some \( \beta \in \Delta \). But \( \theta_o \) stabilizes both \( \mathfrak{h}_m \) and \( \mathfrak{a} \). However \( -\alpha_j \) vanishes on \( \mathfrak{h}_m \) since \( [e_{-\alpha_j}, e_{\alpha_j}] \in \mathfrak{h} \). But \( \theta_o = 1 \) on \( \mathfrak{a}_o \) and since \( -\alpha_j | a_o \) is a real linear functional one must have \( \beta = -\alpha_j \). Thus \( e_{-\alpha_j} \) is an eigenvector for \( \theta_o \). But \( \pm 1 \) are the only eigenvalues of \( \theta_o \). Thus \( \theta_o(e_{-\alpha_j}) = c e_{-\alpha_j} \) where \( c \in \{ \pm 1 \} \). But \( \theta_o \) clearly commutes with \( \theta \) so that \( \theta_o(e_{\alpha_j}) = c e_{\alpha_j} \) (see (98)). Hence \( \theta_o(z_j) = cz_j \). However one cannot have \( c = 1 \) since otherwise \( z_j \in \mathfrak{t}_o \), and this contradicts (99) because \( B_{\mathfrak{g}}(\mathfrak{t}_o) \) is negative definite. Hence (119) is established. QED

Let \( H \) be the (Cartan) subgroup of \( G \) corresponding to \( \mathfrak{h}_o = \mathfrak{h} \cap \mathfrak{g}_o \) and let \( H_C \) be the (Cartan) subgroup of \( G_C \) corresponding to \( \mathfrak{h} \). For \( j \in I \), let \( H_j \) and \( (H_j)_C \) be the subgroups of \( H_C \) corresponding, respectively, to \( \mathbb{R} h_j \) and \( \mathbb{C} h_j \). In particular \( (H_j)_C \) is a Cartan subgroup of \((G_C)_j\). The equations (114) and (115) immediately imply that \( H_C \) is a direct product

\[
H_C = (H_1)_C \times \cdots \times (H_\ell)_C \tag{120}
\]

Let \( \ell_s \) be the cardinality of \( \Pi_s \) (see (62)). Let \( a_s \) be the complex span of \( \{ h_j \}, j \in I_s \). Let \( A_s \) and \( (A_s)_C \) be subgroups of \( G \) and \( G_C \), respectively, which correspond to \( a_s \cap a_o \) and \( a_s \). It is clear that

\[
(A_s)_C \cong (\mathbb{C}^\ast)^{\ell_s}
\]

In particular if \( F_s = F \cap (A_s)_C \) then

\[
F_s \cong (\mathbb{Z}_2)^{\ell_s} \tag{121}
\]

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For \( j \in I_s \) let \( C_j = \text{Cent} (G_C)_j \) so that
\[
C_j = \{1, \varepsilon_j\} \quad \text{and} \quad C_j \subset (H_j)C
\] (122)

Clearly \( C_j \subset F_s \).

**Proposition 2.8.** \( F_s \) is the group generated by \( \{C_j\}, j \in I_s \). In fact
\[
F_s = \prod_{j \in I_s} C_j
\] (123)

and (123) is a direct product.

**Proof.** Let \( F'_s \) be the group generated by \( \{C_j\}, j \in I_s \). But \( F'_s \) is a direct product
\[
F'_s = \prod_{j \in I_s} C_j
\] (124)

by (120). However (124) implies that \( F'_s \) and \( F_s \) have the same order and hence \( F'_s = F_s \). QED

2.4. Let \( \varphi \in \Delta \). Then, (see (19)), \( \varphi|a \neq 0 \) if and only if \( \varphi \in \Delta(s) \) An element \( \nu \in a^* \) is called a restricted root if there exists \( \varphi \in \Delta(g) \) such that \( \nu = \varphi|a \). Let \( \Delta^{res} \subset a^* \) be the set of restricted roots. By definition one has a surjection
\[
p : \Delta(s) \rightarrow \Delta^{res}
\] (125)

where \( p(\varphi) = \varphi|a \). It is well known that \( \Delta^{res} \) is a root system (see e.g. [A]) and a system, \( \Delta^{res}_+ \), of positive restricted roots is chosen (see (23) and (24)) by putting
\[
\Delta^{res}_+ = p(\Delta(n))
\] (126)

For any \( \gamma \in \Delta^{res} \) let \( g(\gamma) \) be the \( ad \ a \) weight space for the weight \( \gamma \). It is obvious that \( g(\gamma) \) is stable under \( ad h \) and
\[
g(\gamma) = \sum_{\varphi \in \Delta(g(\gamma))} \mathbb{C}e_\varphi
\] (127)

where clearly
\[
\Delta(g(\gamma)) = p^{-1}(\gamma)
\] (128)

Furthermore clearly \( [m, g(\gamma)] \subset g(\gamma) \) so that the adjoint action of \( m \) defines a representation
\[
\sigma_\gamma : U(m) \rightarrow \text{End} g(\gamma)
\] (129)

of \( m \) on \( g(\gamma) \). The space \( g(\gamma) \) is clearly also stable under \( Ad M \). We will use \( \sigma_\gamma \) to also denote the corresponding group representation of \( M \) on \( g(\gamma) \).
Proposition 2.9. Let $\gamma \in \Delta^{res}$. Then $e_\varphi$, for $\varphi \in p^{-1}(\gamma)$, is an $m$-weight vector of $\sigma_\gamma$ with respect to $h_m$ and $\varphi|h_m$ is the corresponding weight. Furthermore all weights of $\sigma_m$ have multiplicity one.

Proof. If $\varphi, \varphi' \in p^{-1}(\gamma)$ and $\varphi|h_m = \varphi'|h_m$ then one has $\varphi = \varphi'$ since of course $\varphi|a = \varphi'|a = \gamma$. This proves the last statement of the proposition. The remainder is obvious. QED

The Killing form $B_g$ is negative definite on $t_o$ and positive definite on $p_o$. Thus if $B_\theta$ is the symmetric bilinear form $(x,y)_{\theta}$ on $g$ defined by putting

$$(x,y)_\theta = -(\theta(x),y)$$

then $B_\theta$ is positive definite on $g_o$. Furthermore the adjoint action of $K$ on $g$ is orthogonal with respect to $B_\theta$ since this action commutes with $\theta$.

Let $\gamma \in \Delta^{res}$. Since $\text{ad } x|g(\gamma)$ has a real spectrum for any $x \in a_o$, it follows that $g(\gamma)$ is stable under the conjugation involution (over $g_o$) $\theta_o$. Thus $g(\gamma)$ is the complexification of $g(\gamma)_o$ where $g(\gamma)_o = g(\gamma) \cap g_o$. Let $O(g(\gamma)_o)$ be the orthogonal group on $g(\gamma)_o$ with respect to the positive definite bilinear form $B_\theta|g(\gamma)_o$. Obviously

$$\sigma_\gamma(M) \text{ stabilizes } g(\gamma)_o \neq \sigma_\gamma(M)|g(\gamma)_o \subset O(g(\gamma)_o)$$

(130)

Note that since $\theta = -1$ on $a$ it follows that $\theta(g(\gamma)) = g(-\gamma)$. But since $\theta$ commutes with $\theta_o$ one also has

$$\theta(g(\gamma)_o) = g(-\gamma)_o$$

(131)

Clearly $[g(\gamma)_o, g(-\gamma)_o]$ is centralized by $a_o$ so that one has

$$[g(\gamma)_o, g(-\gamma)_o] \subset m_o + a_o$$

(132)

We may regard $a^*$ as the subspace of $h^*$ orthogonal to $h_m$ so that $\Delta^{res} \subset h^*$. Recalling (1) let $w_\gamma = \eta^{-1}(\gamma)$ so that $(w_\gamma, x) = \gamma(x)$ for any $x \in a$. Since $B_g|a_o$ is positive definite clearly

$$w_\gamma \in a_o \text{ and } \gamma(w_\gamma) > 0$$

(133)

Let $S(\gamma)$ be the unit sphere in $g(\gamma)_o$ with respect to $B_\theta|g(\gamma)_o$.

Proposition 2.10. Let $\gamma \in \Delta^{res}$ and let $e \in S(\gamma)$. Then

$$[\theta(e), e] = w_\gamma$$

(134)

Proof. Let $w = [\theta(e), e]$. Then $\theta(w) = -w$ and hence $w \in a_o$ by (132). On the other hand, if $x \in a$ then since $e \in S(\gamma)$,

$$(w, x) = ([\theta(e), e], x)$$

$$= -(\theta(e), [x, e])$$

$$= -\theta(e), e \gamma(x)$$

$$= \gamma(x)$$
This proves \( w = w_{\gamma} \). QED

**Remark 2.11.** Note that, as a consequence of Proposition 2.10, if \( \gamma \in \Delta^\text{res} \) and \( e \in S(\gamma) \), then \( e, \theta(e) \) and \( w_{\gamma} \) span a TDS.

**Lemma 2.12.** Let \( \gamma \in \Delta^\text{res} \) and \( e \in S(\gamma) \). Assume \( 0 \neq f \in \mathfrak{g}(\gamma)_o \) is \( B_\theta \)-orthogonal to \( e \). Then \( [e, \theta(f)] \) is a nonzero element of \( m_o \).

**Proof.** Let \( z = [e, \theta(f)] \). Then \( z \in m_o + a_o \) by (132). However if \( x \in a_o \) then \( (z, x) = (\theta(f), [x, e]) \). But \( [x, e] = \gamma(x) e \). Hence \( (z, x) = 0 \) since \( (\theta(f), e) = 0 \). Thus one must have \( z \in m_o \). But \( [w_{\gamma}, \theta(f)] = -\gamma(w_{\gamma})\theta(f) \). Thus if \( z = 0 \), then \( \theta(f) \) is a highest weight vector for the adjoint action, on \( \mathfrak{g} \), of the TDS in Remark 2.11. This contradicts a well-known result in the representation theory of a TDS since the corresponding weight is negative by (133). QED

2.5. The following result in the split rank 1 case \( (\ell_o = 1) \) was proved as Theorem 2.1.7 in [K1]. The proof given here in the general case is essentially the same as the proof in [K1]. In fact even in the general case the statement really reduces to the split rank 1 case. Using the notation of Proposition 2.10 note that, by (130), one has an action of \( M \) on the sphere \( S(\gamma) \)

\[
M \times S(\gamma) \to S(\gamma)
\]

via the representation \( \sigma_\gamma \)

**Theorem 2.13.** Let \( \gamma \in \Delta^\text{res} \). Assume \( \text{dim} \mathfrak{g}(\gamma)_o > 1 \). Then \( M_e \) operates transitively on \( S(\gamma) \) with respect to the action (135).

**Proof.** Let \( e \in S(\gamma) \). Let \( W \) be the \( B_\theta \) orthocomplement of \( \mathbb{R} e \) in \( \mathfrak{g}(\gamma)_o \). Then \( [m_o, e] \subset W \) by (130). To prove the theorem it suffices to prove

\[
[m_o, e] = W
\]

Indeed (136) implies that the \( M_e \)-orbit of \( e \) is open in \( S(\gamma) \). But the orbit is certainly closed since \( M_e \) is compact. Thus (136) implies \( \sigma_\gamma(M_e)(e) = S(\gamma) \). Assume (136) is false. Then there exists \( 0 \neq f \in W \) such that \( (f, [y, e])_\theta = 0 \) for any \( y \in m_o \). But \( (f, [y, e])_\theta = -(\theta(f), [y, e]) \) and \( -(\theta(f), [y, e]) = -([e, \theta(f)], y) \). But \( [e, \theta(f)] \) is a nonzero element of \( m_o \) by Lemma 2.12. Hence \(-([e, \theta(f)], y) \) cannot vanish for all \( y \in m_o \) since \( B_\theta m_o \) is negative definite. QED

Let \( \gamma \in \Delta^\text{res} \). Then a root \( \psi \in \Delta(\mathfrak{g}(\gamma)) \) will be called a lowest weight (relative to (29)) in case \( \sigma_\gamma(m_-) e_\psi = 0 \). Let \( \Delta_{\text{low}}(\gamma) \) be the set of lowest weights in \( \Delta(\mathfrak{g}(\gamma)) \). Since it is understood that \( \mathfrak{g}(\gamma) \) is a \( U(m) \)-module with respect to \( \sigma_\gamma \), we will, when convenient, suppress the symbol \( \sigma_\gamma \).
Proposition 2.14. Let $\gamma \in \Delta^{res}$ and let $\psi \in \Delta_{low}(\gamma)$. Then $U(m_+)e_\psi$ is an irreducible component of $\sigma_\gamma$ and

$$g(\gamma) = \sum_{\psi \in \Delta_{low}(\gamma)} U(m_+)e_\psi$$

is a unique decomposition of $g(\gamma)$ into irreducible components of $\sigma_\gamma$. In particular $\text{card } \Delta_{low}(\gamma)$ is the number of such components. Furthermore all the components define inequivalent irreducible representations of $m$.

Proof. This is an immediate consequence of Proposition 2.9, and especially the multiplicity one statement in Proposition 2.9. QED

As noted in the proof of Lemma 2.7, the conjugation involution $\theta_o$ carries root vectors to root vectors and consequently induces a bijection map $\theta_o : \Delta \to \Delta$. On the other hand, clearly $\theta_o = -\theta$ on the space $i(h_m)_o + a_o$ of hyperbolic elements in the Cartan subalgebra $h$. Thus

$$\theta_o = -\theta \text{ on } \Delta$$

(138)

Proposition 2.15. The conjugation involution $\theta_o$ interchanges $m_+$ and $m_-$. Also $\theta_o$ stabilizes $n$.

Proof. The first statement follows from (138) since $\theta$ is clearly the identity on $\Delta(m)$. The second statement follows from (138) and (47). QED

We will say that an element $\gamma \in \Delta^{res}$ is irreducible in case $\sigma_\gamma$ is an irreducible representation of $m$, i.e. in case (see Proposition 2.14) $\Delta_{low}(\gamma)$ only has one element. Otherwise $\gamma$ will be called reducible.

Remark 2.16. If $W$ is a real finite dimensional module for a group $\Gamma$ and $W$ is real irreducible, then the commuting ring $\text{End}_\Gamma W$ is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$. One knows that in the first case ($\mathbb{R}$) the complexification $W_\mathbb{C}$ is complex irreducible. In the second and third cases ($\mathbb{C}$ or $\mathbb{H}$) $W_\mathbb{C}$ decomposes into a direct sum of two complex irreducible components,

$$W_\mathbb{C} = W'_\mathbb{C} \oplus W''_\mathbb{C}$$

(139)

The cases $\mathbb{C}$ and $\mathbb{H}$ are distinguished by the fact that, in the $\mathbb{H}$, case the two components of (139) are equivalent and the decomposition (139) is not unique. In the $\mathbb{C}$ case the two components in (139) are inequivalent, the decomposition is unique, and $W''_\mathbb{C}$ is the conjugate (with respect to $W$) image of $W'_\mathbb{C}$. Furthermore in case there exists a nonsingular symmetric $\Gamma$-invariant bilinear form on $W_\mathbb{C}$ which is real on $W$ then of course, in the $\mathbb{R}$ case, the irreducible $\Gamma$-module is self-contragredient. In the $\mathbb{C}$ and $\mathbb{H}$ cases both components of (139) are isotropic with respect to $B_W$ and $W''_\mathbb{C}$ has the structure of the $\Gamma$-module which is contragredient to $W'_\mathbb{C}$.
Theorem 2.17. Let \( \gamma \in \Delta^{res} \). Then the real representation of \( \sigma_{\gamma}|_{M_e} \) on \( g(\gamma)_o \) is real irreducible so that (see Remark 2.16) if \( \gamma \) is reducible, then \( g(\gamma) \) decomposes into a sum of only two irreducible components. That is, in any case, \( \Delta_{low}(\gamma) \) has either one or two elements according as \( \gamma \) is irreducible or reducible. Furthermore if \( \gamma \) is reducible then, in the notation of Remark 2.16, one is always in the \( \mathbb{C} \) case (i.e. the \( \mathbb{H} \) case never occurs). In particular if \( \Delta_{low}(\gamma) = \{ \psi, \psi' \} \) then

\[
\theta_o(U(m_+)e_{\psi}) = U(m_+)e_{\psi'} \tag{140}
\]

(see (137)). In addition if \( \gamma \) is irreducible, then

\[
\Delta(g(\gamma))|h_m = -\Delta(g(\gamma))|h_m \tag{141}
\]

and if \( \gamma \) is reducible

\[
\Delta(U(m_+)e_{\psi'})|h_m = -\Delta(U(m_+)e_{\psi})|h_m \tag{142}
\]

in the notation of (140).

Proof. The transitivity of \( M_e \) on \( S(\gamma) \), given in Theorem 2.13, obviously implies the real representation \( \sigma_{\gamma}|_{g(\gamma)_o} \) is real irreducible. If \( \gamma \) is reducible we are in the \( \mathbb{C} \) case of Remark 2.16 by the inequivalence statement (last statement) in Proposition 2.14. The existence of the \( M \)-invariant bilinear form \( B_{\theta}|g(\gamma) \) puts into effect the last two statements in Remark 2.16. This yields (141) and (142) since the set of weights of a representation of a reductive group is the negative of the set of weights of its contragredient representation. QED

Remark 2.18. It is suggestive from (140) that \( \theta_o(\psi) \) might be \( \psi' \). This, however, is not true in general. In fact by the first statement in Proposition 2.15, \( \theta_o(e_{\psi}) \) is in fact the highest (and not the lowest weight vector) in \( U(m_+)e_{\psi'} \).

2.6. Let \( \Pi^{res} \) be the set of simple positive restricted roots. One knows (see [A]) that \( \Pi^{res} \) is a basis of \( \mathfrak{a}^* \) and hence we can write

\[
\Pi^{res} = \{ \beta_1, \ldots, \beta_{\ell_o} \}
\]

Let \( J = \{ 1, \ldots, \ell_o \} \).

Theorem 2.19. Recall the map (126). Then

\[
p(\Pi_n) = \Pi^{res} \tag{143}
\]

In fact

\[
\Pi_n = \bigcup_{j \in J} \Delta_{low}(\beta_j) \tag{144}
\]
where of course (144) is a disjoint union.

**Proof.** Let \( j \in J \) and let \( \psi \in \Delta_{\text{low}}(\beta_j) \) so that \( \psi \in \Delta(n) \) and \( p(\psi) = \beta_j \). Assume that \( \psi \) is not simple. Then there exists \( \varphi_1, \varphi_2 \in \Delta_+ \) such that \( \psi = \varphi_1 + \varphi_2 \). However we cannot have \( \varphi_1, \varphi_2 \in \Delta(n) \) since otherwise \( \beta_j = p(\varphi_1) + p(\varphi_2) \) and this contradicts the simplicity of \( \beta_j \). Hence without loss we may assume \( \varphi_2 \in \Delta(m+) \). But then obviously \( \varphi_1 \in \Delta(n) \) and \( p(\varphi_1) = \beta_j \). Also one has

\[
e_{\psi} \in \mathbb{C} \sigma_{\beta_j}(e_{\varphi_2})(e_{\varphi_1})
\]  

(145)

Hence \( e_{\varphi_1} \) and \( e_{\psi} \) lie in the same irreducible \( m \)-submodule of \( g(\beta_j) \) by Proposition 2.9. But then (145) contradicts the minimality of \( \psi \). We have proved that the right side of (144) is contained in the left side.

Now let \( i \in I_n \) (see §2.1) and let \( \beta = p(\alpha_i) \). Assume \( \beta \notin \Pi^{\text{cs}} \). Then we may write \( \beta = \gamma + \gamma_1 \) for some \( \gamma, \gamma_1 \in \Delta^{\text{cs}}_+ \). But then there must exist \( \varphi \in \Delta(g(\gamma)) \) and \( \varphi_1 \in \Delta(g(\gamma_1)) \) such that \( \varphi + \varphi_1 = \varphi_2 \) is a root. Otherwise, if \( g_\gamma \) is the TDS of Remark 2.11, one must have \([e, g(\gamma_1)] = 0\) and \([\theta(e), \gamma(\beta)] = 0\). But \( \beta(w_1) > \gamma_1(w_1) \) by (133). But this contradicts either the statement that a lowest weight vector for this TDS corresponds to a nonpositive weight or the statement that a highest weight vector corresponds to a nonnegative weight. Hence choices can be made so that \( \varphi_2 \in \Delta(n) \). Clearly \( p(\varphi_2) = \beta \). Without loss we may assume that \( e_{\varphi_2} \) and \( e_{\alpha_i} \) lie in the same \( m \)-irreducible component of \( g(\beta) \). Otherwise, recalling (140), (138) and the last statement of Proposition 2.15, we replace \( \varphi \) and \( \varphi_1 \) by \( \theta_o(\varphi) \) and \( \theta_o(\varphi_1) \). But then \( e_{\varphi_2} \in \sigma_{\beta}(U(m))e_{\alpha_i} \) so that \( \varphi + \varphi_1 \) differs from \( \alpha_i \) by a linear combination of elements in \( \Pi_m \). This is a contradiction of the simplicity of \( \alpha_i \) since both \( \varphi \) and \( \varphi_1 \) have positive coefficients of some elements in \( \Pi_n \) when they are expressed as a positive integral combination of elements of \( \Pi \). This proves (143) and we can assume \( \beta = \beta_j \) for some \( j \in J \). We assert \( \alpha_i \in \Delta_{\text{low}}(\beta_j) \), thereby establishing (144). Indeed this is immediate from the fact that \( \alpha_i - \alpha_i' \) is not a root for any \( i' \in I \) and in particular for \( i' \in \Pi_m \). QED

2.7. Let \( c = \text{Cent} m \) so that \( c \subset h_m \) and if \( h_m^s = [m, m] \cap h_m \), then clearly

\[ h = h_m^s + c + a \]  

(146)

is a \( B_a \)-orthogonal direct sum. Let \( p_c \) and \( p_a \) be the respective projections of \( h \) on \( c \) and \( a \) with respect to the decomposition (146). Let \( p_{c+a} = p_c + p_a \). Obviously \( card \, \Pi_m = dim \, h_m^s \) and \( \{ h_i \} \), \( i \in I_m \), is a basis of \( h_m^s \) (see §2.1). This observation and of course (36) clearly proves

**Proposition 2.20.** One has

\[ card \, \Pi_n = \ell_o + dim \, c \]  

(147)

and in fact (see §2.1)

\[ \{ p_{c+a}(h_i) \}, i \in I_n, \text{ is a basis of } c + a \]  

(148)
Hence if \( \beta \) is the negative of \( \theta \). It suffices by (138) to show that \( \beta \) is irreducible. One scalar is the negative of the other.

Next we assert that \( \beta \) is irreducible. However one scalar is the negative of the other by (141). Thus \( \beta \) is irreducible. QED

For any \( i \in I_2 \) let \( i' \in I_2 \) be the unique element where \( i \neq i' \) and \( p(\alpha_i) = p(\alpha_{i'}) \). From the inverse point of view given \( j \in J_2 \), let \( i_j \in I_2 \) be the unique element such that \( i_j < i_j' \) and such that \( \{\alpha_{i_j}, \alpha_{i_j'}\} = p^{-1}(\beta_j) \cap \Pi_n \). One has the disjoint union

\[
I_2 = \bigcup_{j \in J_2} \{i_j, i_j'\} \tag{149}
\]

**Theorem 2.22.** One has

\[
\text{card } \Pi_2^{res} = \dim \mathfrak{c} \tag{150}
\]

Furthermore for any \( j \in J_2 \),

\[
p_c(h_{i_j}) = -p_c(h_{i_j'}) \tag{151}
\]

and

\[
\{p_c(h_{i_j} - h_{i_j'}), j \in J_2, \text{ is a basis of } \mathfrak{c} \tag{152}
\]

In fact

\[
\{h_i \mid i \in I_m\} \cup \{\{h_{i_j} - h_{i_j'}, j \in J_2\} \text{ is a basis of } \mathfrak{h}_m \tag{153}
\]

**Proof.** One has \( \text{card } I_n = \text{card } I_1 + \text{card } I_2 \). But \( \text{card } I_1 = \text{card } J_1 \) and \( \text{card } I_2 = 2 \text{ card } J_2 \) by (149). However \( \text{card } J_1 + \text{card } J_2 = \ell_o \). Thus \( \text{card } I_n = \ell_o + \text{card } J_2 \). But then (150) follows from (147).

Let \( j \in J_2 \) so that \( g(\beta_j) = U(m_\gamma)e_{\alpha_{i_j}} + U(m_\gamma)e_{\alpha_{i_j'}} \) is the decomposition of \( g(\beta_j) \) into a direct sum of irreducible \( m \)-modules. But if \( x \in \mathfrak{c} \), then \( \sigma_{\beta_j}(x) \) is a scalar operator in each of these two modules. However one scalar is the negative of the other by (142). Thus

\[
\alpha_{i_j} | \mathfrak{c} = -\alpha_{i_j'} | \mathfrak{c} \tag{154}
\]

Next we assert that \( \alpha_{i_j} \) and \( \alpha_{i_j'} \) have the same length. That is

\[
(\alpha_{i_j}, \alpha_{i_j}) = (\alpha_{i_j'}, \alpha_{i_j'}) \tag{155}
\]

It suffices by (138) to show that \( \theta_o(\alpha_{i_j}) \) and \( \alpha_{i_j'} \) have the same length. But the restriction of these two roots to \( \mathfrak{a} \) are the same, namely \( \beta_j \). It suffices then to show that the
restriction of these roots to \( \mathfrak{h}_m \) have the same length. But these restrictions are the
highest and lowest weights of the same irreducible \( m \)-module by Remark 2.18. Consequently these restrictions have the same length since they are Weyl group conjugate
for the Weyl group of the pair \((\mathfrak{h}_m, m)\). But then (154) and (3) proves (151).

Since \( c \subset \mathfrak{h} \) it follows that \( c \) is spanned by \( \{p_i(h_i)\} \), \( i \in I \). But obviously \( p_i(h_i) = 0 \) for \( i \in I_m \). But also \( p_i(h_i) = 0 \) for \( i \in I_1 \) by Lemma 2.21. Hence \( c \) must be spanned by all \( p_i(h_i) \) for \( i \in I_2 \). But, by (151), \( c \) must be spanned by \( \{p_i(h_i)\} \), \( j \in J_2 \). But then \( \{p_i(h_i)\} \), \( j \in J_2 \), must be a basis of \( c \) by (150). Consequently (152) follows from (151).

But now if \( j \in J_2 \) then \( \alpha_i | a = \alpha_i' | a = \beta_j \). Thus \( p_a(h_i) = p_a(h_i') \) by (3) and (155). This proves
\[
  h_{ij} - h_{ij'} \in \mathfrak{h}_m
\]
(156)
But obviously \( \{h_i \mid i \in I_m\} \) is a basis of \( \mathfrak{h}_m \) (see (146)). But then (153) follows from (152), (156) and the obvious fact that \( \mathfrak{h}_m = \mathfrak{h}_m + c \) is an orthogonal direct sum. QED

Let \( \lambda \in \Lambda \). We will say that \( \lambda \) is \( m \)-trivial in case \( \lambda | \mathfrak{h}_m = 0 \).

**Remark 2.23.** Note that Proposition 2.1 implies that \( \lambda \) is \( m \)-trivial if and only
if
\[
  a v_\lambda = v_\lambda
\]
(157)
for all \( a \in M_e \).

**Theorem 2.24.** Let \( \lambda \in \Lambda \). Then, in the notation of §1.1 and Theorem 2.22, \( \lambda \) is \( m \)-trivial if and only if the following two conditions are satisfied:
\[
  \begin{align*}
    (a) & \quad n_i(\lambda) = 0, \quad \forall i \in I_m \\
    (b) & \quad n_j(\lambda) = n_{j'}(\lambda), \quad \forall j \in J_2
  \end{align*}
\]
(158)
where we recall that if \( \ell_o \) is the split rank of \( \mathfrak{g}_o \), \( J = \{1, \ldots, \ell_o\} \), and \( \Pi^{\text{red}} = \{\beta_1, \ldots, \beta_\ell_o\} \)
is the set of simple positive restricted roots, then \( J_2 = \{j \in J \mid \beta_j \text{ is reducible}\} \). We recall also that if \( j \in J_2 \), then \( \{i_j, i'_j\} \subset I_2 \) where \( I_n = I_1 \cup I_2 \) is the disjoint union
defined, after (148), and \( I_n \) is the subset of \( I = \{1, \ldots, \ell\} \) which indexes the set, \( \Pi_n \),
of simple positive roots in \( \Delta(n) \).

**Proof.** This is immediate from (4) and (153). QED

### 2.8.
We can now deal with the full group \( M \). Recall the subset \( I_s \subset I \) defined in §2.1.

**Lemma 2.25.** Let \( i \in I_s \). Then \( \lambda_i \) is \( m \)-trivial.

**Proof.** By Theorem 2.24 we have only to show that \( I_s \) is disjoint from \( I_2 \) and \( I_m \). But \( I_s \subset I_1 \) by Lemma 2.21 and hence \( I_s \) is disjoint from \( I_2 \). But \( I_1 \subset I_n \) and hence \( I_1 \) is certainly disjoint from \( I_m \). QED
We have put $\ell_s = \text{card } I_s$ (see (120)). Recall the 2-group $F_s$ of order $2^{\ell_s}$, defined in (121). One has $F_s \subset F_M$ by Lemma 2.1 where $F_M \subset M$ is defined by (107). Finally recall the quotient map $\kappa : M \to M/M_e$ (see (108)).

**Lemma 2.26.** The restriction

$$\kappa : F_s \to M/M_e$$

is injective.

**Proof.** Let $\varepsilon \in F_s \cap M_e$. We must prove that $\varepsilon = 1$. Assume not. Then there exists a nonempty subset $I' \subset I_s$ such that $\varepsilon = \prod_{i \in I'} \varepsilon_i$. Let $j \in I'$. Then using the notation of (114) and (115) one has, by (114), (115) and (118), that $\varepsilon v_j = -v_j$. However $\lambda_j$ is m-trivial by Lemma 2.25. Thus $\varepsilon v_j = v_j$ by Remark 2.23. This is a contradiction. QED

Let $j \in I_s$ (see (62) and §2.1. Then as noted in the proof of Lemma 2.7, one has $iz_j \in \mathfrak{k}$. If $S_j \subset K$ is the subgroup corresponding to $Riz_j$, then $S_j$ is isomorphic to the unit circle in $C^*$ by (117) and (118).

**Lemma 2.27.** Assume that $Z$ is a finite dimensional $K$- (and hence $\mathfrak{k}$-) module. Let $j \in I_s$ and assume that $0 \neq v \in Z$ is a $\varepsilon_j$-eigenvector (see (117) and (118)) so that $\varepsilon_j v = cv$ where $c \in \{-1, 1\}$. Let $Z_j(v) = U(Cz_j)v$. Then $z_j$ is diagonalizable on $Z_j(v)$. Furthermore the set of eigenvalues of $z_j|Z_j(v)$ is a set of odd integers or a set of even integers according as $c = -1$ or $c = 1$.

**Proof.** Since $S_j$ is compact the $U(Cz_j)$-module $Z_j(v)$ is completely reducible and hence $z_j|Z_j(v)$ is diagonalizable and the eigenvalues are all integers by (117) and (118). On the other hand, by commutativity, $\varepsilon$ must operate as the scalar operator defined by $c$ on $Z_j(v)$. The result then follows from (117) and (118). QED

The following result does not appear to be in the literature. It was, however, known to David Vogan who points out that it is implicit in [V].

**Theorem 2.28.** The restriction

$$\kappa : F_s \to M/M_e$$

of the quotient map $\kappa$ (see (108)) is an isomorphism. Furthermore $F_s \subset \text{Cent } M$ and in fact $M$ is a group direct product

$$M = F_s \times M_e$$

of groups so that

$$M \cong \mathbb{Z}_{2}^{\ell_s} \times M_e$$
Proof. Let $M' = F_s M$. But

$$F_s \subset F_M$$

by Lemma 2.7. Hence $F_s \subset \text{Cent} M$ by (109). But then $M' = F_s \times M_e$ by (159). Recalling (121) we have only to prove that $M' = M$. But $M'$ is normal in $M$ and by Lemma 2.4. Assume $M' \neq M$. Then $M/M' \cong \mathbb{Z}^d_+$ for some positive integer $d$ by Lemma (2.4). But then there exists a nontrivial 1-dimensional representation $\tau : M \to \mathbb{C}^*$ such that $\tau|M'$ is trivial. Since $M \subset K$ and both $M$ and $K$ are compact there exists an irreducible representation $\sigma : K \to \text{Aut} V$ such that $\sigma|M$ contains $\tau$ with positive multiplicity. Let $0 \neq v \in Z$ be such that

$$a v = \tau(a) v$$

for all $a \in M$. But then if $j \in I_s$ one has $\varepsilon_j v = v$. Hence if $Z_j(v)$ is defined as in Lemma 2.27 the spectrum of $z_j|Z_j(v)$ is a set $X_j$ of even integers. Let $n_j \in 2\mathbb{Z}_+$ be such that $n_j \geq |q|$ for all $q \in X_j$. But then if $\lambda \in \Lambda$ is such that $n_j = n_j(\lambda)$, for all $j \in I_s$, using the notation of §1.1, one has

$$f_{\lambda,j}(z_j)v = 0$$

for all $j \in I_s$, using the notation of (65). Recall (see §2.1) $I_{nil}$ is, by definition, the complement of $I_s$ in $I_n$. But then, recalling (97), if $i \in I_{nil}$, $z_i$ is a nilpotent element of $t$ by Proposition 1.20. We further fix $\lambda$ by choosing $n_i(\lambda)$, for $i \in I_s$ so that

$$z_i^{n_i(\lambda)+1} v = 0$$

Recall that $I_2 \subset I_{nil}$ since $I_s \subset I_1$ by Lemma 2.21. For $i \in I_2$ we further choose $n_i(\lambda)$ so that in addition to (166) one has

$$n_i(\lambda) = n_i'(\lambda)$$

To completely determine $\lambda$ it remains only to fix $n_i(\lambda)$ for $i \in I_m$. We do so by choosing

$$n_i(\lambda) = 0$$

for $i \in I_m$. But then, by (4), (153), and (167) one has $\lambda | h_m = 0$. But $v \in Z[0]^m$. Hence $v \in Z^\lambda$ by (101). But by (104) one has $Z^\lambda = Z^{L_\lambda(t)}$ so that $v \in Z^{L_\lambda(t)}$. But then, by Proposition 1.3, there exists a unique

$$s \in \text{Hom}_K(V_\lambda, Z)$$

such that $s(v_\lambda) = v$. On the other hand, $\tau|M_M$ is nontrivial by (110) since $\tau|M$ is nontrivial but $\tau|M_e$ is trivial. But $F_M$ is contained in the Cartan subgroup $H_C$ by (107). Hence

$$a v_\lambda = \tau(a) v_\lambda$$
for all $a \in F$. Now let $Y = \mathbb{C}y$ be the 1-dimensional trivial $K$-module. Then, in the notation of (101), where $Y$ replaces $Z$, one clearly has $Y = Y[0]^{m+}$. But in fact

$$Y = Y^\lambda$$

(170)

That is, $q_{\lambda,i}(z_i)v = 0$ for all $i \in I_a$. This is immediate if $i \in I_{nil}$. But if $i \in I_s$, then $n_i(\lambda)$ is even and hence $t$ is one of the factors of $q_{\lambda,i}(t)$. This proves (170). But then the argument yielding (168) implies that there exists $s_o \in Hom_K(V_\lambda, Y)$ such that $s_o(v_\lambda) = y$. But $F_M$ operates trivially on $y$ and nontrivially on $v_\lambda$, by (169). This is a contradiction. Hence $M' = M$. QED

3. The minimal $G$-module $V_{\lambda,\zeta}$ and a generalization of the Cartan-Helgason Theorem

3.1. Recalling the notation of (91), let $\Lambda_M$ be the set of all $\nu \in \Lambda_m$ such that $\tau_{\nu}|m_o$ exponentiates to an (irreducible) representation of $M_e$. Extending the use of the notation of (91), let $\tau_{\nu}$, for $\nu \in \Lambda_M$, also denote the representation of $M_e$ which arises by exponentiating $\tau_{\nu}|m_o$. Let $\hat{F}_s$ be the character group of $F_s$ so that $\hat{F}_s$ is a 2-group of order $2^{\ell_s}$. Then, by Theorem 2.28, $\hat{F}_s \times \Lambda_M$ parameterizes the set $\hat{M}$ of all equivalences classes of irreducible representations (the unitary dual) of $M$. For each $(\zeta, \nu) \in \hat{F}_s \times \Lambda_M$, let

$$\tau_{\zeta,\nu} : M_e \to Aut Y_{\zeta,\nu}$$

be the irreducible representation where, as a vector space, $Y_{\zeta,\nu} = Y_\nu$, but with the $M$-module structure defined so that

$$(\varepsilon, a) y = \zeta(\varepsilon) \tau_{\nu}(a) y$$

(171)

for $(\varepsilon, a) \in F_s \times M_e = M$ and $y \in Y_{\zeta,\nu}$. By abuse of notation we will take

$$\hat{M} = \{\tau_{\zeta,\nu}\}, (\zeta, \nu) \in \hat{F}_s \times \Lambda_M$$

(172)

For any $\lambda \in \Lambda$ let $\lambda|F_s \in \hat{F}_s$ be (uniquely) defined so that, for $j \in I_s$, one has $(\lambda|F_s)(\varepsilon_j) = 1$ if $n_j(\lambda)$ is even and $(\lambda|F_s)(\varepsilon_j) = -1$ if $n_j(\lambda)$ is odd.

**Proposition 3.1.** Let $\lambda \in \Lambda$. Then $\lambda|h_m \in \Lambda_M$ so that $(\lambda|F_s, \lambda|h_m) \in \hat{F}_s \times \Lambda_M$ and hence one has a map

$$\Lambda \to \hat{F}_s \times \Lambda_M,$$

$$\lambda \mapsto (\lambda|F_s, \lambda|h_m)$$

(173)

Furthermore (see Proposition 2.1) the irreducible $m$-module $U(m)v_{\lambda} \subset V_{\lambda}$ is stable under $M$ and as an irreducible $M$-module, $U(m)v_{\lambda} \subset V_{\lambda}$, transforms according to $\tau_{\lambda|F_s, \lambda|h_m} \in \hat{M}$.
Proof. Since \( M \subset G \) it is obvious, by Proposition 2.1, that \( U(\mathfrak{m})v_\lambda \) is an irreducible \( \mathcal{M}_c \)-module with highest weight \( \lambda|\mathfrak{h}_m \). Thus \( \lambda|\mathfrak{h}_m \in \Lambda_{\mathcal{M}_c} \). On the other hand, if \( j \in I_s \), then \( \varepsilon_j v_\lambda = (-1)^{a_j(\lambda)} v_\lambda \) by (4) and the first line in (118). Since \( F_s \) centralizes \( \mathcal{M}_c \) it follows that \( \varepsilon_j \) stabilizes \( U(\mathfrak{m})v_\lambda \) and reduces to the scalar operator \( (\lambda|F_s)(\varepsilon_j) \) on \( U(\mathfrak{m})v_\lambda \). QED

3.2. Let \( H_m \) be the subgroup of \( G \) corresponding to \( (\mathfrak{h}_m)_o \) (see §1.2) so that \( H_m \) is a maximal torus of \( \mathcal{M}_c \). Let

Let \( J_2 \) be as in §2.7 and for any \( j \in J_2 \), let \( h(j) = h_{ij} - h_{ij}' \), using the notation of (153), so that \( h(j) \in \mathfrak{h}_m \) by (153). The element \( ih(j) \) is clearly an elliptic element of \( \mathfrak{h}_m \) by (120), and hence \( ih(j) \in (\mathfrak{h}_m)_o \). Let \( H_{(j)} \) be the one parameter subgroup of \( H_m \) corresponding to \( \mathbb{R} ih(j) \). Let \( (\mathfrak{h}_m)_o \subset (\mathfrak{h}_m)_o \) be the real span of \( \{h(j)\}, j \in J_2 \), and let \( H_{(m)} \) be the subgroup of \( H_m \) corresponding to \( (\mathfrak{h}_m)_o \). One has

\[
dim \mathfrak{h}_{(m)} = \dim \mathfrak{c} \tag{174}
\]

For notational simplicity let \( q = \dim \mathfrak{c} \).

Proposition 3.2. For any \( j \in J_2 \) the group \( H_{(j)} \) is isomorphic to the unit circle in \( \mathbb{C}^* \). In fact if \( t \in \mathbb{R} \), then \( \exp tih_{ij} = 1 \) if and only if \( t \in 2\pi \mathbb{Z} \). Furthermore \( H_{(m)} \) is a closed and hence compact subgroup of the torus \( H_m \). In fact \( H_{(m)} \) has the product structure

\[
H_{(m)} = H_{(1)} \times \cdots \times H_{(q)} \tag{175}
\]

Proof. It is immediate from Proposition 2.5 that if \( j \in I \), then \( \exp tih_j = 1 \) if and only if \( t \in 2\pi \mathbb{Z} \). But then same statement is true for if \( j \in J_2 \) and \( (j) \) replaces \( j \), by the product formula (120). But then, recalling (149), the product formula (120) also yields (175). QED

For any \( \lambda \in \Lambda \) let \( \chi_\lambda : H_\mathbb{C} \to \mathbb{C}^* \) be the character defined by \( \lambda \) so that if \( a \in H_\mathbb{C} \) then

\[
av_\lambda = \chi_\lambda(a) v_\lambda \tag{176}
\]

Let \( \tilde{H}_{(m)} \) be the character group of \( H_{(m)} \).

Lemma 3.3. The group \( \tilde{H}_{(m)} \) may be parameterized by \( \mathbb{Z}^q \) in such a fashion that if \( m \in \mathbb{Z}^q \) and \( m = (m_1, \ldots, m_q) \) then

\[
\Phi_m(\exp \sum_{j=1}^q t_j i h_{ij}) = e^{i \sum_{j=1}^q t_j m_j} \tag{176}
\]

where \( \Phi_m \) is the character corresponding to \( m \). Moreover given \( m \in \mathbb{Z}^q \) there exists \( \lambda \in \Lambda \) such that

\[
\chi_\lambda|H_{(m)} = \Phi_m \tag{177}
\]

39
In fact this is the case if and only if

\[ n_i(\lambda) - n_i'_{\lambda} = m_j \]  

(178)

for all \( j \in J_2 \).

**Proof.** This is immediate from (4), Proposition 3.2 and the definition above of \( h_{(j)} \). QED

Recall (see above (91)) the definition of the set \( \Lambda_m \subset h^*_m \) of integral dominant linear forms on \( h_m \). That is, if \( \nu \in h^*_m \) then, by (3), \( \nu \in \Lambda_m \) if and only if

\[ \nu(h_i) \in \mathbb{Z}_+, \forall i \in I_m \]  

(179)

The subset \( \Lambda_{Me} \) of \( \Lambda_m \) was defined in §3.1 by the condition that \( \nu \in \Lambda_{Me} \) if the representation \( \tau_{\nu}|_{Me} \) (see (91)) integrates to a representation (also denoted by \( \tau_{\nu} \)) of \( Me \). Recall the basis of \( h_m \) given by (153).

**Theorem 3.4.** Let \( \nu \in h^*_m \). Then \( \nu \in \Lambda_{Me} \) if and only if the following 2 conditions are satisfied:

\begin{align*}
1) \ &\nu(h_i) \in \mathbb{Z}_+, \forall i \in I_m \\
2) \ &\nu(h_{ij} - h_{ij}') \in \mathbb{Z}, \forall j \in J_2
\end{align*}

(180)

Furthermore if \( \lambda \in \Lambda \), then \( \lambda|_{h_m} \in \Lambda_{Me} \) and the map

\[ \Lambda \to \Lambda_{Me}, \lambda \mapsto \lambda|_{h_m} \]  

(181)

is surjective and in fact if \( \nu \in \Lambda_{Me} \) satisfies (180), then \( \lambda|_{h_m} = \nu \) if and only if

\begin{align*}
1) \ &\nu(h_i) = \nu(h_i), \forall i \in I_m \text{ and} \\
2) \ &\nu(h_{ij} - h_{ij}') = \nu(h_{ij} - h_{ij}') \in \mathbb{Z}, \forall j \in J_2
\end{align*}

(182)

In particular if \( \lambda^a, \lambda^b \in \Lambda \), then \( \lambda^a|_{h_m} = \lambda^b|_{h_m} \) if and only if

\begin{align*}
1) \ &\lambda^a(h_i) = \lambda^b(h_i), \forall i \in I_m \text{ and} \\
2) \ &\nu(h_{ij} - h_{ij}') = \nu(h_{ij} - h_{ij}') \in \mathbb{Z}, \forall j \in J_2
\end{align*}

(183)

**Proof.** Assume \( \nu \in \Lambda_{Me} \). Then \( \nu \in \Lambda_m \) so that (1) of (180) is satisfied. But \( \nu|_{(h_m)_o} \) integrates to \( H_m \) since (recalling (91)) if \( 0 \neq y_{\nu} \) is the highest weight vector of \( Y_{\nu} \), then one must have \( \exp x y_{\nu} = e^{\nu(x)} y_{\nu} \) for any \( x \in (h_m)_o \). But then (2) of (180) is satisfied by Lemma 3.3 since \( (h_{(m)})_o \subset (h_m)_o \). Conversely if (1) and (2) of (180) are satisfied there clearly exists \( \lambda \in \Lambda \) which satisfies (1) and (2) of (182). But then
π_\lambda|M_\nu restricted to U(m)v_\lambda (see Proposition 2.1) defines an integration of τ_\nu|m_\nu to M_\nu. Hence ν ∈ Λ_{M_\nu}.

If λ ∈ Λ then λ|h_m satisfies (1) and (2) of (180) so that λ ∈ Λ ∈ Λ_{M_\nu}. The argument above establishes that the map (181) is surjective and additionally (182) and (183) are immediate. QED

3.3. We now consider the full group M = F_s × M_\nu. If \( \hat{G} \) here denotes the set of all equivalence classes of finite dimensional irreducible representations of G then (abusing notation) we may identify \( \hat{G} \) with \{π_\lambda | λ ∈ Λ\}. Thus by Proposition 3.1 one has a map

\[ \hat{G} \to \hat{M}, \quad π_\lambda \mapsto τ_\lambda|F_s,λ|h_m \] (184)

**Theorem 3.5.** The map (185) is surjective. Furthermore if (recalling the notation of (172)) if (ζ, ν) ∈ \( \hat{F}_s × Λ_{M_\nu} \) is arbitrary, then π_λ is in the inverse image of \( τ_\zeta,ν \) in (184) if and only if λ satisfies (1) and (2) of (182) and in addition

\[ (-1)^{n_i(λ)} = ζ(ε_i), \quad ∀ \ i ∈ I_s \] (185)

(recalling the notation of §2.1). Furthermore if λ^a, λ^b ∈ Λ, then π_λ^a and π_λ^b have the same image under the map (184) if and only if (1) and (2) of (183) are satisfied and in addition

\[ n_i(λ^a) ≡ n_i(λ^b) \mod 2, \quad ∀ \ i ∈ I_s \] (186)

**Proof.** This follows immediately from (118) and the fact that \( I_s \) is disjoint from both \( I_2 \) (see (149) and Lemma 2.21) and \( I_m \) (see (136)). QED

Let \( ζ_1 \) be the the trivial character of \( F_s \) so that τ_\zeta_1,0 is the 1-dimensional trivial representation of \( M \). Let

\[ Λ_{[sph]} = \{ λ ∈ Λ | π_\lambda ↦ τ_ζ,0, \text{ with respect to (184)} \} \]

**Remark 3.6.** The Cartan-Helgason theorem asserts that, for λ ∈ Λ, π_λ|K has the trivial representation of K as a component, i.e. π_λ is a spherical representation of G, if and only if λ ∈ Λ_{[sph]}. The following result characterizes Λ_{[sph]} using the notation of §1.1. As noted in Remark 3.8 below this characterization yields the Cartan-Helgason theorem. It will later yield a generalization of the Cartan-Helgason theorem wherein the trivial representation of M is replaced by any irreducible representation of M.

As an immediate corollary of Theorem 3.5 one has
Theorem 3.7. Let $\lambda \in \Lambda$. Then $\lambda \in \Lambda_{\text{sph}}$ if and only if the following three conditions are satisfied:

(a) $n_i(\lambda) \in 2\mathbb{Z}_+$, $\forall \ i \in I_s$

(b) $n_i(\lambda) = 0$, $\forall \ i \in I_m$

(c) $n_{ij}(\lambda) = n_{ij}(\lambda)$, $\forall \ j \in J_2$

(187)

Remark 3.8. Let $\lambda \in \Lambda$ and let $Z$ be the 1-dimensional trivial representation of $K$. To see that Theorem 3.7 yields an algebraic proof of the Cartan-Heiligson theorem it suffices by Theorem 2.3 and (103) to show that

$$Z^\lambda = Z$$

(188)

if and only if $\lambda$ satisfies (a), (b) and (c) of Theorem 3.7. If $\lambda$ satisfies (a), (b) and (c) of Theorem 3.7 then $\lambda \| h_m = 0$ by Theorem 3.7 and and hence obviously $Z[0] = Z$. But, then recalling (101), $Z^\lambda = Z[0] = Z$ since $q_{\lambda,i} (z_i)$ clearly vanishes on $Z[0]$ for all $i \in I_a$. The only really significant observation to be made here is that, for $i \in I_s$, $n_i(\lambda)$ is even and hence $t$ is a factor of $q_{\lambda,i}(t)$. This proves (188). Conversely if (188) is satisfied one must have $Z[\lambda] = Z$ by (101). But then obviously $\lambda \| h_m = 0$ so that $\lambda$ satisfies (b) and (c) of Theorem 3.7. However the equality (188) implies that $q_{\lambda,i}(z_i)$ vanishes on $Z$ for $i \in I_s$. Hence $t$ must be a factor of $q_{\lambda,i}(t)$ for $i \in I_s$. Consequently $\lambda$ also satisfies (a).

3.4. If $V^a$ and $V^b$ are two finite dimensional $K$-modules we will say that the $K$-spectrum of $V^b$ dominates the $K$-spectrum of $V^a$ in case

$$\text{multiplicity of } Z \text{ in } V^a \leq \text{multiplicity of } Z \text{ in } V^b$$

for all irreducible $K$-modules $Z$. That is, if there exists an injective map

$$r \in Hom_K(V^a, V^b)$$

(189)

Theorem 3.5 asserts that the map (184) and hence equivalently the map (173) are surjective. We will refer to the inverse images of elements in $\widehat{M}$, in the case of (184), or the inverse images of elements in $\widehat{F}_s \times \Lambda_{M_e}$, in the case (173), as fibers of the respective maps (184) and (173).

Theorem 3.9. Let $\lambda^a \in \Lambda$ and $\lambda^b \in \Lambda_{\text{sph}}$. Put $\lambda^b = \lambda^a + \lambda'$. Then $\lambda^a$ and $\lambda^b$ lie in the same fiber of (173). Furthermore, recalling Proposition 1.3, one has

$$L_{\lambda^b}(\mathfrak{k}) \subset L_{\lambda^a}(\mathfrak{k})$$

(190)

so that there exists a unique surjective map $K$-map

$$s : V_{\lambda^b} \to V_{\lambda^a}$$

(191)
such that \(s(v_{\lambda^k}) = v_{\lambda^s}\). In particular the \(K\)-spectrum of \(V_{\lambda^k}\) dominates the \(K\)-spectrum of \(V_{\lambda^s}\).

**Proof.** The statement about lying in the same fiber is an immediate consequence of Theorems 3.5 and 3.7. But now (190) follows from Theorem 1.24 since the polynomial \(q_{\lambda^s,i}(t)\) is clearly a factor of the polynomial \(q_{\lambda^k,i}(t)\) for any \(i \in I\). By reductivity the kernel of the map \(s\) has a \(K\)-stable complement which is clearly \(K\)-equivalent to \(\pi_{\lambda^s}\). This proves the final statement of the theorem. QED

**Remark 3.10.** The final statement of Theorem 3.9 can be proved much more easily using the fact that \(\pi_{\lambda^k}\) is the Cartan product of \(\pi_{\lambda^s}\) and \(\pi_{\lambda^t}\). The spherical vector in \(V_{\lambda^s}\) tensor \(V_{\lambda^t}\) has a faithful projection on the Cartan product.

3.5. We will now find that the “fibration” (173) (or equivalently (184)) has a natural cross-section. We recall \(I = \{1, \ldots, \ell\}\). From (36) (and recalling §2.1) one has the disjoint union \(I = I_m \cup I_n\). Recalling §2.7 one has the disjoint union \(I_n = I_1 \cup I_2\). By Lemma 2.21 one has \(I_s \subset I_1\). Let \(I'_s\) be the complement of \(I_s\) in \(I_1\) so that

\[I = I_m \cup I_s \cup I'_s \cup I_2\]  

(192)

is a disjoint union. We recall also that \(J = \{1, \ldots, \ell_o\}\) where \(\ell_o\) is the split rank of \(g_o\) and that \(J\) parameterizes the restricted simple positive roots (see §2.6). Recall also the disjoint union \(J = J_1 \cup J_2\) (see §2.7) and that \(J_2\) parameterizes the “pair” decomposition (149) of \(I_2\). Now for any \((\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_+}\), let \(\lambda_{\zeta, \nu} \in \Lambda\) be defined so that

1. For \(i \in I_s\), \(n_i(\lambda_{\zeta, \nu}) = 0\), if \(\zeta(\varepsilon_i) = 1\), and \(n_i(\lambda_{\zeta, \nu}) = 1\), if \(\zeta(\varepsilon_i) = -1\)
2. For \(i \in I_m\), \(n_i(\lambda_{\zeta, \nu}) = \nu(h_i)\)
3. For \(j \in J_2\) and \(\nu(h_{j_2} - h_{j_2'}) \geq 0\), \(n_{j_2}(\lambda_{\zeta, \nu}) = \nu(h_{j_2} - h_{j_2'})\) and \(n_{j_2'}(\lambda_{\zeta, \nu}) = 0\)
4. For \(j \in J_2\) and \(\nu(h_{j_2} - h_{j_2'}) \leq 0\), \(n_{j_2}(\lambda_{\zeta, \nu}) = 0\) and \(n_{j_2'}(\lambda) = -\nu(h_{j_2} - h_{j_2'})\)
5. For \(i \in I'_s\), \(n_i(\lambda_{\zeta, \nu}) = 0\)

(193)

For any \((\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_+}\), let \(F_{\zeta, \nu}\) be the fiber of the map (173) over \((\zeta, \nu)\). Obviously

\[\Lambda = \bigcup_{(\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_+}} F_{\zeta, \nu}\]  

(194)

is a disjoint union. It is immediate from (193) and Theorem 3.5 that, for all \((\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_+}\),

\[\lambda_{\zeta, \nu} \in F_{\zeta, \nu}\]  

(195)

We also recall that (see §3.3) for the trivial pair \((\zeta_1, 0) \in \hat{F}_s \times \Lambda_{M_+}\),

\[F_{\zeta_1, 0} = \Lambda_{[\text{sph}]}\]  

(196)

and that \(\Lambda_{[\text{sph}]}\) is characterized in Theorem 3.7. Note that

\[\lambda_{\zeta_1, 0} = 0\]  

(197)

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Introduce a partial ordering in \( \Lambda \) by defining, for \( \lambda^a, \lambda^b \in \Lambda \), \( \lambda^a < \lambda^b \), in case \( \lambda^b - \lambda^a \in \Lambda \). The following theorem asserts that the most general fiber in (194) is just a translate of \( \Lambda_{[\text{sph}]} \).

**Theorem 3.11.** Let \( (\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_e} \). Then \( \lambda_{\zeta, \nu} \) is the unique minimal element in \( F_{\zeta, \nu} \) and that furthermore

\[
F_{\zeta, \nu} = \lambda_{\zeta, \nu} + \Lambda_{[\text{sph}]} \quad (198)
\]

**Proof.** The proof is an immediate observation using (193), (195), Theorem 3.5 and Theorem 3.7. QED

As a corollary of Theorem 3.11 one has

**Theorem 3.12.** Let \( (\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_e} \) so that \( \tau_{\zeta, \nu} \) is the most general (up to equivalence) irreducible representation of \( M \). Let \( \lambda \in F_{\zeta, \nu} \) so that \( \pi_{\lambda} \), up to equivalence, is the most general finite dimensional irreducible representation of \( G \) such that \( U(\mathfrak{m})v_{\lambda} \) transforms under \( M \) according to \( \tau_{\zeta, \nu} \). Then there exists a unique \( K \)-map

\[
s: V_{\lambda} \to V_{\lambda_{\zeta, \nu}} \quad (199)
\]

such that \( s(v_{\lambda}) = v_{\lambda_{\zeta, \nu}} \). In particular, as \( K \)-modules, \( V_{\lambda} \) dominates \( V_{\lambda_{\zeta, \nu}} \). Conversely given \( \lambda \in \Lambda \) and a \( K \)-map (199) such that \( s(v_{\lambda}) = v_{\lambda_{\zeta, \nu}} \), then \( \lambda \in F_{\zeta, \nu} \).

**Proof.** The first conclusion follows from Theorems 3.9 and 3.11. The last statement follows from the fact that the condition on \( s \) implies the equivalence of \( U(\mathfrak{m})v_{\lambda} \) and \( U(\mathfrak{m})v_{\lambda_{\zeta, \nu}} \) as \( M \)-modules. QED

**Remark 3.13.** Note that Theorem 3.12 is a generalization of the Cartan-Helgason theorem. The latter is the case where \( \tau_{\zeta, \nu} \) is the trivial representation of \( M \). Indeed if \( \tau_{\zeta, \nu} \) is the trivial representation of \( M \) then \( \pi_{\lambda_{\zeta, \nu}} \) is the trivial representation of \( G \), by (197), and hence \( V_{\lambda_{\zeta, \nu}} \) as a \( K \)-module is the trivial representation of \( K \).

4. The noncompact analogue of the Borel-Weil theorem

**4.1.** If \( G_1 \) is any Lie subgroup of \( G \) let \( C^\infty(G_1) \) be the space of all infinitely differentiable \( \mathbb{C} \)-valued functions on \( G_1 \). If \( Y \) is any finite dimensional \( \mathbb{C} \)-vector space then \( C^\infty(G_1) \otimes Y \) naturally identifies with the space of all infinitely differentiable \( Y \)-valued functions on \( G_1 \). Let \( \mathfrak{g}_1 = (\text{Lie } G_1)_\mathbb{C} \). The space \( C^\infty(G_1) \otimes X \) is a \( G_1 \) and a \( \mathfrak{g}_1 \)-module where if \( g \in G_1 \), \( z \in \mathfrak{g}_1 \) and \( \phi \in C^\infty(G_1) \otimes X \), then \( g \cdot \phi \) is defined so that for any \( h \in G_1 \)

\[
(g \cdot \phi)(h) = \phi(g^{-1} h) \quad (200)
\]
The map \( z \mapsto z \cdot \phi \) is complex linear and for (real) \( z \in \text{Lie}G_1 \),

\[
(z \cdot \phi)(h) = \frac{d}{dt} \phi(\exp(-t)zh)|_{t=0}
\]

(201)

For notational simplicity let \( D = \hat{F}_s \times \Lambda_{M_e} \), so that \( D \) parameterizes \( \hat{M} \) in that we can write \( \hat{M} = \{ \tau_\delta \delta \in D \} \). See (172). Let \( \delta \in D \) and let \( C^\infty(G, Y_\delta) \) be the space of all \( C^\infty \)-functions \( \phi \) on \( G \) with values in \( Y_\delta \) such that for any \( g \in G \) and any \( m \in M \) one has

\[
\phi(g m) = \tau_\delta(m^{-1}) \phi(g)
\]

(202)

The space \( C^\infty(G, Y_\delta) \) is a \( G \)- and a \( g \)-module where the action is defined by (200) and (201).

The (vector) subgroup \( A \) of \( G \) has been defined in §2.3. Let \( N \) be the (unipotent) subgroup of \( G \) corresponding to \( n_o \). For any \( (\delta, \xi) \in D \times a^* \) let \( H(Y_\delta, \xi) \) be the space of all \( \phi \in C^\infty(G, Y_\delta) \) such that

\[
\begin{align*}
(a) \text{ the span of } K \cdot \phi \text{ is finite dimensional} \\
(b) \phi(g a n) = e^{-\xi(\log a)} \phi(g), \quad \forall g \in G, a \in A \text{ and } n \in N
\end{align*}
\]

(203)

where \( \log a \) is the unique element in \( a_o \) such that \( a = \exp \log a \). It is simple to verify and is well known that, with respect to (200) and (201), \( H(Y_\delta, \xi) \) is stable under the action of \( K \) and \( g \). Furthermore using the Iwasawa decomposition \( G = K A N \) one notes that any \( \phi \in H(Y_\delta, \xi) \) is determined by its restriction to \( \hat{K} \). In addition as a \( K \)-module the Frobenius multiplicity theorem yields the equivalence

\[
H(Y_\delta, \xi) \cong \sum_{Z \in \hat{K}} \dim Z[\delta] Z
\]

(204)

where \( \hat{K} \) is a set of representatives of the unitary dual of \( K \) and, extending the notation of §2.1 so as to apply to \( M \) and not just \( m \), \( X[\delta] \), for \( \delta \in D \), is the primary \( M \)-component corresponding to \( \tau_\delta \) in any finite dimensional \( M \)-module \( X \).

Let \( \mathbb{C}[K] \) be the abstract group algebra of \( K \). Then the adjoint action of \( K \) on \( U(g) \) defines a Hopf algebra structure on the smash product

\[
\mathcal{A} = \mathbb{C}[K] \# U(g)
\]

A Harish-Chandra module is a module \( H \) for \( \mathcal{A} \) which, as a \( K \)-module, admits an equivalence

\[
H \cong \sum_{Z \in \hat{K}} m_Z Z
\]

where \( m_Z \in \mathbb{Z}_+ \), for any \( Z \in \hat{K} \), and is such that the \( t \)-action of \( H \) is the differential of the \( K \)-action. (This is well defined since \( K \cdot v \) spans a finite dimensional subspace
of $H$ for any $v \in H$.) It is clear that, in the notation of (204), that $H(Y_\delta, \xi)$ is a Harish-Chandra module. Conforming to common parlance we will also refer to a Harish-Chandra module as a $(\mathfrak{g}, K)$-module.

**Remark 4.1.** The $(\mathfrak{g}, K)$-module $H(Y_\delta, \xi)$ is the Harish-Chandra module (with respect to $K$) of the principal series representation of $G$ corresponding to $(\delta, \xi) \in D \times \mathfrak{a}^\ast$. As a $(\mathfrak{g}, K)$-module it is a well known result of Harish-Chandra that $H(Y_\delta, \xi)$ has a finite composition series and (the subquotient theorem) any irreducible Harish-Chandra module with respect to $K$ is equivalent to a subquotient of $H(Y_\delta, \xi)$ for some choice of $(\delta, \xi) \in D \times \mathfrak{a}^\ast$.

Now it is much more convenient for our purposes to replace $H(Y_\delta, \xi)$ with scalar valued functions on $G$. This may be done as follows using the Borel-Weil theorem. Assume $G_1$ is a Lie subgroup of $G$ and let $\mathfrak{g}_1 = (\text{Lie } G_1)_C$. If $z \in \mathfrak{g}_1$ and $f \in C^\infty(G_1)$ be defined so that $z \mapsto f \cdot z$ is complex linear and if $z \in \text{Lie } G_1$ (i.e. $z$ is real) then for any $g \in G_1$,

$$ (f \cdot z)(g) = \frac{d}{dt} f(g \exp t z) |_{t=0} \quad (205) $$

Let $\delta \in D$ and let $Y_\delta^*$ be the $M$- (and also $m$-) module which is contragredient to $Y_\delta$. The pairing of $y \in Y_\delta$ and $y^* \in Y_\delta^*$ is denoted by $\langle y, y^* \rangle$. Let $0 \neq y_\delta^* \in Y_\delta^*$ be a highest weight vector with respect to the Borel subalgebra $b_m$ (see §1.4) of $m$. We will regard any $\nu \in b_m^\ast$ as an element in $b_m^\ast$ with the property that $\nu|_{m_+} = 0$ (see (29)). Thus if $\delta = (\zeta, \nu) \in \hat{F}_s \times \Lambda_{Me}$ (see (172)) and $\nu^c \in \Lambda_{Me}$ is the highest weight of $Y_\delta^*$, then

$$ u y_\delta^* = \nu^c(u) y_\delta^*, \forall u \in b_m \quad (206) $$

Now if $G_1$ is any Lie subgroup of $G$ that contains $M$, let $C^\infty(G_1, \delta)$ be the space of all $f \in C^\infty(G_1)$ such that

$$ (a) \ f \cdot u = \nu^c(u) f, \forall u \in b_m \text{ and } \\
(b) \ f(g \varepsilon) = \zeta(\varepsilon) f(g), \forall g \in G_1, \varepsilon \in F_s \quad (207) $$

It is immediate that $C^\infty(G_1, \delta)$ is an $G_1$- and $\mathfrak{g}_1$-module with respect to the action defined by (200) and (201). Using an obvious extension of the Borel-Weil theorem to the (possibly disconnected) group $M = F_s \times M_e$, the Borel-Weil theorem asserts that the map

$$ Y_\delta \to C^\infty(M, \delta), \ y \mapsto \psi_y $$

is an equivalence of $M$ and $m$-modules where

$$ \psi_y(m) = \langle y, m g_\delta^* \rangle \quad (208) $$

**4.2.** Let

$$ Q_\delta : C^\infty(G, Y_\delta) \to C^\infty(G) $$

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be the map defined by

\[(Q_\delta(\phi))(g) = \langle \phi(g), y_\delta^* \rangle\] (209)

where \(g \in G\) and \(\phi \in C^\infty(G, Y_\delta)\).

**Proposition 4.2.** One has \(Q_\delta(\phi) \in C^\infty(G, \delta)\) for any \(\phi \in C^\infty(G, Y_\delta)\) and the map

\[Q_\delta : C^\infty(G, Y_\delta) \to C^\infty(G, \delta)\] (210)

is an isomorphism of \(G\)- and \(g\)-modules with respect to the action defined by (200) and (201).

**Proof.** Let \(\phi \in C^\infty(G, Y_\delta)\) and let \(g \in G, m \in M\). Then

\[Q_\delta(\phi)(gm) = \langle \phi(gm), y_\delta^* \rangle = \langle \tau_\delta(m^{-1})\phi(g), y_\delta^* \rangle = \langle \phi(g), my_\delta^* \rangle\] (211)

That is, if we put \(f = Q_\delta(\phi)\), then \(f(gm) = \langle \phi(g), my_\delta^* \rangle\). But then by differentiation and (206) one has \((f \cdot u)(g) = \nu^c(u) f(g)\) for any \(u \in b_m\) where \(\delta = (\zeta, \nu)\) in the notation of (206) and (207). Furthermore (211) implies that if \(\varepsilon \in F_\delta\), then

\[f(g \varepsilon) = \langle \phi(g), \varepsilon y_\delta^* \rangle = \langle \zeta(\varepsilon) \phi(g), y_\delta^* \rangle = \zeta(\varepsilon)f(g)\]

so that \(f\) satisfies (a) and (b) of (207). This proves that \(Q_\delta(\phi) \in C^\infty(G, \delta)\).

Now let \(0 \neq \phi \in C^\infty(G, Y_\delta)\). Then there exists \(g \in G\) such that \(0 \neq \phi(g) \in Y_\delta\). But, for \(m \in M\), \(\phi(gm) = \tau_\delta(m^{-1})\phi(g)\). But then the set \(\{\phi(gm) \mid m \in M\}\) spans \(Y_\delta\) since \(\tau_\delta\) is an irreducible representation of \(M\). Thus \(\phi(G)\) is not contained in the orthocomplement of \(y_\delta^*\). Consequently \(Q_\delta(\phi) \neq 0\). Thus (210) is injective.

Now let \(f \in C^\infty(G, \delta)\). For any \(g \in G\), let \(\Psi_f(g) \in C^\infty(M)\) be defined by putting \(\Psi_f(g)(m) = f(gm)\) for any \(m \in M\). It is immediate from the definition of \(C^\infty(G, \delta)\) that \(\Psi_f(g) \in C^\infty(M, \delta)\). By (208) there exists a unique \(y \in Y_\delta\) such that

\[\Psi_f(g) = \psi_y\] (212)

Let \(\phi : G \to Y_\delta\) be defined so that \(\phi(g) = y\). Since the function \(G \times M \to \mathbb{C}\) defined by \((g, m) \to f(gm)\) is \(C^\infty\), it is obvious that \(\phi\) is \(C^\infty\). On the other hand, if \(g \in G, m \in M\), then by (208) and (212)

\[(\Psi_f(g))(m) = \psi_y(m) = \langle y, my_\delta^* \rangle = \langle \phi(g), my_\delta^* \rangle\] (213)
On the other hand, if \( m = m_1 m_2 \) for \( m_1, m_2 \in M \), then
\[
(\Psi_f(g))(m) = f(gm)
= (\Psi_f(g m_1))(m_2)
= \langle \phi(g m_1), m_2 y_2^* \rangle
\]
Hence
\[
\langle \phi(g m_1), m_2 y_2^* \rangle = \langle \phi(g), m_1 m_2 y_2^* \rangle
= \langle \tau_{\delta}(m_1^{-1})\phi(g), m_2 y_2^* \rangle
\]
But \( m_1, m_2 \in M \) are arbitrary and, by irreducibility, \( Y_\delta^* \) is spanned by \( m_2 y_2^* \) over all \( m_2 \in M \). Hence (213) implies \( \phi(g m_1) = \tau_{\delta}(m_1^{-1})\phi(g) \). Thus \( \phi \in C^\infty(G, Y_\delta) \). But, for \( g \in G \), \((Q_\delta(\phi))(g) = (\phi(g), y_2^*) \) by definition (see (209)) of \( Q_\delta \). But then putting \( m = 1 \) in (213) one has \((Q_\delta(\phi))(g) = \Psi_f(g)(1) = f(g) \). Thus \( Q_\delta(\phi) = f \). Hence (210) is bijective. It is immediate from (209) that (210) is a \( G \)- and \( g \)-module map using the the action defined by (200) and (201). QED

4.3. Let \( \xi \in \mathfrak{a}^* \). Assume \( \psi \) is any scalar or vector space valued function on \( G \). We will say that \( \psi \) satisfies the \( AN - \xi \) condition if
\[
\psi(g a n) = e^{-\xi(\log a)} \psi(g)
\]
for any \( g \in G \), \( a \in A \) and \( n \in N \). Let \( \delta \in D \) and let \( C^\infty(Y_\delta, \xi) = \{ \phi \in C^\infty(G, Y_\delta) \mid \phi \) satisfies the \( AN - \xi \) condition\}. Similarly, for scalar valued functions, let \( C^\infty(\delta, \xi) = \{ f \in C^\infty(G, \delta) \mid f \) satisfies the \( AN - \xi \) condition\}. It is obvious that both \( C^\infty(Y_\delta, \xi) \) and \( C^\infty(\delta, \xi) \) are stable under the \( G \) and \( g \) action defined by (200) and (201).

**Proposition 4.3.** Let \( (\delta, \xi) \in D \times \mathfrak{a}^* \). Then \( Q_\delta(\phi) \in C^\infty(\delta, \xi) \) for any \( \phi \in C^\infty(Y_\delta, \xi) \) so that the isomorphism (see Proposition 4.2) \( Q_\delta \) restricts to a \( G \)- and \( g \)-module map
\[
Q_\delta : C^\infty(Y_\delta, \xi) \rightarrow C^\infty(\delta, \xi)
\]
with respect to the action defined by (200) and (201). Furthermore (216) is an isomorphism of \( G \)- and \( g \)-modules.

**Proof.** Let \( \phi \in C^\infty(G, Y_\delta) \) and put \( f = Q_\delta(\phi) \) so that \( f \in C^\infty(G, \delta) \) by Proposition 4.2. If \( g \in G \) then by definition
\[
f(g) = \langle \phi(g), y_2^* \rangle
\]
It is immediate from (217) that \( f \in C^\infty(\delta, \xi) \) in case \( \phi \in C^\infty(Y_\delta, \xi) \). This establishes the map (216). Also (216) is injective by Proposition 4.2. Now assume that \( f \in C^\infty(\delta, \xi) \). Let \( g \in G \), \( m \in M \), \( a \in A \) and \( n \in N \). Then since \( m \) normalizes \( N \) and
commutes with $A$ there exists $n' \in N$ such that

\[
\langle \phi(g an), my^*_\delta \rangle = \langle \tau_\delta(m^{-1})\phi(g an), my^*_\delta \rangle \\
= \langle \phi(g an m), y^*_\delta \rangle \\
= \langle \phi(g m a n'), y^*_\delta \rangle \\
= f(g m a n') \\
= e^{-\xi(log a)} f(g m) \\
= e^{-\xi(log a)} \langle \phi(g m), y^*_\delta \rangle \\
= e^{-\xi(log a)} \langle \phi(g), my^*_\delta \rangle
\]

Thus $\phi(g an) - e^{-\xi(log a)}\phi(g)$ is orthogonal to $my^*_\delta$ for all $m \in M$. By the $M$-irreducibility of $Y^*_\delta$ one has $\phi(g an) = e^{-\xi(log a)}\phi(g)$. Hence $\phi \in C^\infty(Y^*_\delta, \xi)$ so that (216) is bijective. QED

4.4. Let $(\delta, \xi) \in D \times a^*$. The Harish-Chandra module $H(Y^*_\delta, \xi)$ for the principal series representation of $G$, corresponding to $(\delta, \xi)$ has been defined in §4.1. In terms of the notation of Proposition 4.3 one has

\[
H(Y^*_\delta, \xi) = \{ \phi \in H(Y^*_\delta, \xi) \mid K \cdot \phi \text{ spans a finite dimensional vector space} \}
\]

Without loss we can now replace the vector (in general) valued functions on $G$ in $H(Y^*_\delta, \xi)$ by scalar valued functions on $G$. Let

\[
H(\delta, \xi) = \{ f \in C^\infty(\delta, \xi) \mid K \cdot f \text{ spans a finite dimensional vector space} \}
\]

In summary $H(\delta, \xi)$ is the set of all $f \in C^\infty(G)$ satisfying the following conditions:

\[
(a) \ f \cdot u = \nu^\xi(u) f, \ \forall u \in b_m, \ \text{using the notation of (207)} \\
(b) \ f(g \varepsilon) = \zeta(\varepsilon) f(g), \ \forall g \in G, \ \varepsilon \in F_s, \ \text{using the notation of (207)} \\
(c) \ f \text{ satisfies the } AN - \xi \text{ condition of (215)} \\
(d) \ K \cdot f \text{ spans a finite dimensional vector space}
\]

(218)

It is immediate that $H(\delta, \xi)$ is a $(g, K)$-submodule of $C^\infty(\delta, \xi)$ and one readily has the following consequence of Proposition 4.3.

**Proposition 4.4.** Let $(\delta, \xi) \in D \times a^*$. Then the restriction of (216) to $H(Y^*_\delta, \xi)$ defines an isomorphism

\[
Q_\delta : H(Y^*_\delta, \xi) \to H(\delta, \xi)
\]

of $(g, K)$-modules. We recall that if $\phi \in H(Y^*_\delta, \xi)$, $f = Q_\delta(\phi)$ and $g \in G$, then

\[
f(g) = \langle \phi(g), y^*_\delta \rangle
\]
Henceforth we will regard $H(\delta, \xi)$ as the Harish-Chandra module for the principal series representation of $G$ which corresponds to $(\delta, \xi)$. Now it follows immediately that the Borel subalgebra (see §1.1) $\mathfrak{b}$ of $\mathfrak{g}$ is given by (see §1.4)

$$\mathfrak{b} = \mathfrak{b}_m + \mathfrak{a} + \mathfrak{n}$$  \hspace{1cm} (221)

It is immediate by Lemma 2.6 that $\mathfrak{b}$ is normalized by $F_s$ and hence the smash product $B = \mathbb{C}[F_s] \# U(\mathfrak{b})$ is a Hopf subalgebra of $\mathcal{A} = \mathbb{C}[F_s] \# U(\mathfrak{g})$.

Let $(\delta, \xi) \in D \times \mathfrak{a}^*$. Write $\delta = (\zeta, \nu) \in \widehat{F}_s \times \Lambda_{M_k}$. Let $\chi_{\delta, \xi} : B \to \mathbb{C}$ be the unique 1-dimensional character on $B$ defined so that for $x \in F_s$, $\ell \in \mathfrak{b}_m$, $y \in \mathfrak{a}$ and $z \in \mathfrak{n}$, one has

\begin{align*}
(a) & \quad \chi_{\delta, \xi}(\varepsilon) = \zeta(\varepsilon) \\
(b) & \quad \chi_{\delta, \xi}(x) = \nu^\ell(x) \\
(c) & \quad \chi_{\delta, \xi}(y) = -\xi(y) \\
(d) & \quad \chi_{\delta, \xi}(z) = 0
\end{align*}

(222)

The Hopf algebra $\mathcal{A}$ has an antipode $s$ where for $k \in K$, $z \in \mathfrak{g}$, one has $s(k) = k^{-1}$ and $s(z) = -z$. Let $H$ be a $(\mathfrak{g}, K)$-module. Then the (full algebraic) dual space $H^*$ to $H$ has the structure of an $\mathcal{A}$-module where for $h \in H^*$, $v \in H$ and $a \in \mathcal{A}$ one has $(a \cdot h)(v) = h(s(a) \cdot v)$. Using the notation of (222) let

$$H_{\delta, \xi}^* = \{ h \in H^* \mid \forall b \in B \} \hspace{1cm} (223)$$

**Remark 4.5.** Note that any $0 \neq h \in H_{\delta, \xi}^*$ spans a 1 dimensional space which is stable under $U(\mathfrak{b})$ and hence, by definition, is a highest weight vector. If $H$ is finitely generated as a $U(\mathfrak{g})$-module (e.g. if $H$ is $(\mathfrak{g}, K)$-irreducible) it follows easily from Proposition 5.2, p. 158 in [K2] that $H$ is finitely generated as a $U(\mathfrak{n})$-module. It is immediate then that $H_n = H/(\mathfrak{n} \cdot H)$ is finite dimensional and has the structure of a $B$-module and also a completely reducible $M$-module. The dual space $H_n^*$ to $H_n$ naturally embeds as a finite dimensional $M$- and $B$-submodule of $H^*$. But clearly $H_{\delta, \xi}^* \subset H_n^*$ for any $(\delta, \xi) \in D \times \mathfrak{a}$ so that $H_{\delta, \xi}^* \neq 0$ for at most a finite number of $(\delta, \xi) \in D \times \mathfrak{a}$ and in any case $H_{\delta, \xi}^*$ is finite dimensional. On the other hand Casselman (see [C] or [BB]) has proved that if $H \neq 0$, then $H_n^* \neq 0$ and Lie’s theorem then implies that $H_{\delta, \xi}^* \neq 0$ for some $(\delta, \xi) \in D \times \mathfrak{a}^*$.

**4.5.** Now let $H$ be a nontrivial Harish-Chandra module and assume $0 \neq h \in H_{\delta, \xi}^*$ for some $(\delta, \xi) \in D \times \mathfrak{a}^*$ and $\delta = (\zeta, \nu) \in \widehat{F}_s \times \Lambda_{M_k}$. For any $v \in H$ one defines a scalar valued function $f_v$ on $G$ by putting

$$f_v(g) = e^{-\xi(log_a) h(k^{-1} \cdot v)} \hspace{1cm} (224)$$

where $g = k a n$ is the Iwasawa decomposition of $g$ with respect to $G = K A N$. Since any $K \cdot v$ spans a finite dimensional subspace of $H$, it is immediate that $f_v \in C^\infty(G)$ (in fact $f_v$ is clearly analytic on $G$). Also one has

$$f_v \text{ satisfies the } A N - \xi \text{ condition} \hspace{1cm} (225)$$

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See (215). This follows easily from the definition (224) and the fact that \( A \) normalizes \( N \). Clearly one has a linear map

\[
H \rightarrow C^{\infty}(G), \quad v \mapsto f_v
\]  

(226)

On the other hand, if \( k \in K, g \in G, \) and \( v \in H \) then \((k \cdot f_v)(g) = f_v(k^{-1}g)\). Hence if \( g = k_1 an \) is the Iwasawa decomposition of \( g \), then

\[
(k \cdot f_v)(g) = f_v(k^{-1} k_1 an) \\
= e^{-\xi(\log a)} f_v(k^{-1} k_1) \\
= e^{-\xi(\log a)} h(k_1^{-1} k \cdot v) \\
= e^{-\xi(\log a)} f_{k \cdot v}(k_1) \\
= f_{k \cdot v}(g)
\]

(227)

**Lemma 4.6.** Let the notation be as in (224). Then the map (226) is a map of \( C[K]\#U(\mathfrak{t}) \)-modules. Furthermore \( K \cdot f_v \) spans a finite dimensional subspace of \( C^{\infty}(G) \) for any \( v \in H \).

**Proof.** The statement that (226) is a map of \( C[K] \)-modules is established in (227). But this also proves the last statement of Lemma 4.6. One then has \( x \cdot f_v = f_{x \cdot v} \) for any \( x \in \mathfrak{t} \) since the derivatives involve differentiation in a finite dimensional space. QED

**Lemma 4.7.** Let \( \varepsilon \in F_s, g \in G \) and \( v \in H \). Then

\[
f_v(g \varepsilon) = \zeta(\varepsilon) f_v(g)
\]

(228)

**Proof.** Let \( g = k an \) be the Iwasawa decomposition of \( g \). Then there exists \( n_1 \in N \) such that \( g \varepsilon = k \varepsilon a n_1 \) is the Iwasawa decomposition of \( g \varepsilon \). But since \( \varepsilon \cdot h = \zeta(\varepsilon) h \) one has

\[
f_v(g \varepsilon) = e^{-\xi(\log a)} f_v(k \varepsilon) \\
= e^{-\xi(\log a)} h(k^{-1} \varepsilon v) \\
= e^{-\xi(\log a)} (\varepsilon \cdot h)(k^{-1} v) \\
= \zeta(\varepsilon) e^{-\xi(\log a)} h(k^{-1} v) \\
= \zeta(\varepsilon) f_v(g)
\]

This establishes (228). QED

Topologize \( H^* \) with the weak topology defined by the pairing of \( H \) and \( H^* \).
**Lemma 4.8.** Let $h_1 \in H^*$ and $x \in \mathfrak{k}_0$ (see (9)). Then \( \frac{d}{dt} \exp t \cdot h_1|_{t=0} \) exists and
\[
\frac{d}{dt} \exp t \cdot h_1|_{t=0} = x \cdot h_1
\] (229)

**Proof.** Let $v \in H$. Then \((\exp t \cdot h_1)(v) = h_1(\exp(-t) \cdot x \cdot v)\). But since there exists a finite dimensional $K$ and $U(\mathfrak{t})$-stable subspace $H_v$ of $H$ containing $v$, one has \( \frac{d}{dt} \exp -t \cdot x \cdot v|_{t=0} = -x \cdot v \). But this immediately implies (229). QED

**Lemma 4.9.** Let $u \in b_m$ and $v \in H$. Then
\[
f_v \cdot u = \nu^\mathfrak{c}(u) f_v
\] (230)
(see (205)).

**Proof.** Let $z \in \mathfrak{m}_o$ and $g \in G$. Let $g = k a n$ be the Iwasawa decomposition of $g$. For $t \in \mathbb{R}$ there exists $n_t \in N$ such that $g \exp t z = k \exp t a n_t$. Thus $f_v(g \exp t z) = e^{-\xi(\log a)} f_v(k \exp t z)$. Hence
\[
f_v(g \exp t z) = e^{-\xi(\log a)} h(\exp(-t z) k^{-1} \cdot v) = e^{-\xi(\log a)} (\exp t z \cdot h)(k^{-1} \cdot v)
\]
But then by Lemma 4.8 one has
\[
(f_v \cdot z)(g) = e^{-\xi(\log a)} (z \cdot h)(k^{-1} \cdot v)
\] (231)
By complexification (231) is clearly valid for any $z \in \mathfrak{m}$. In particular, one has (231) for $z = u \in \mathfrak{b}_m$. But $u \cdot h = \nu^\mathfrak{c}(u) h$. Replacing $z \cdot h$ by $\nu^\mathfrak{c}(u) h$ in (231) yields $(f_v \cdot u)(g) = \nu^\mathfrak{c}(u) f_v(g)$. This proves (230). QED

**4.6.** The relation of the map (226) to submodules of the principal series is established in

**Proposition 4.10.** Let $H$ be a nontrivial Harish-Chandra module. Let $(\delta, \xi) \in D \times \mathfrak{a}^*$. Assume there exists $0 \neq h \in H^*_\delta \xi$. For any $v \in H$ let $f_v$ be the scalar valued function on $G$ defined so that if $g \in G$ and $g = k a n$ is the Iwasawa decomposition of $g$, then $f_v(g) = e^{-\xi(\log a)} h(k^{-1} \cdot v)$. In addition $f_v \in H(\delta, \xi)$ (see (218)) so that one has a linear map
\[
T_h : H \to H(\delta, \xi)
\] (231)
where $T_v(h) = f_v$.

**Proof.** This follows immediately from the characterization of $H(\delta, \xi)$ given in (218) and the results (225), (226), Lemma 4.6, (228) and (230). QED
We have already seen that (231) is a map of \(C[K]\#U(t)\)-modules (see Lemma 4.6). We now wish to show that it is a map of \((g, K)\)-modules. We recall that in \(A\) one has the relation
\[
k u = Ad k(u) k
\]
(232)
The existence of an element \(h\) satisfying the condition of Proposition 4.10, when \(H\) is irreducible, is due to Casselman (see [C] or [BB]). Its significance is the following submodule theorem, due to Casselman.

**Theorem 4.11.** Let the notation be as in Proposition 4.10. Then \(T_h\) is a map of \((g, K)\)-modules.

**Proof.** Let \(v \in H\) so that \(f_v = T_h(v)\). Let \(u \in g\). Then we must show
\[
u \cdot f_v = f_{u \cdot v}
\]
(233)
By complex linearity we can assume \(u \in g_o\). Let \(g \in G\). Then
\[
(u \cdot f_v)(g) = \frac{d}{dt} f_v(\exp(-t)ug) |_{t=0}
\]
(234)
Let \(g = k a n\) be the Iwasawa decomposition of \(g\). Then
\[
f_v(\exp(-t)ug) = e^{-\xi(\log a)} f_v(\exp(-t)uk)
\]
Let \(w = Ad k^{-1}(u)\) so that, by Lemma 4.6,
\[
f_v(\exp(-t)ug) = e^{-\xi(\log a)} f_{k^{-1 \cdot v}}(\exp(-t)w)
\]
(235)
Let \(\exp(-t)w = k_t a_t n_t\) be the Iwasawa decomposition of \(\exp(-t)w\). Hence if we write \(w = x + y + z\) where \(x \in \mathfrak{k}_o, y \in \mathfrak{a}_o\) and \(z \in \mathfrak{n}_o\), then the tangent vectors to the curves \(k_t, a_t\) and \(n_t\) at \(t = 0\) are respectively \(-x, -y\) and \(-z\). But
\[
f_{k^{-1 \cdot v}}(\exp(-t)w) = e^{-\xi(\log a)} f_{k^{-1 \cdot v}}(k_t) = e^{-\xi(\log a)} f_{k^{-1 \cdot k^{-1 \cdot v}}(1)}
\]
However recall that since \(h \in H^*_{\delta, \xi}\) one has \(z \cdot h = 0\) and \(y \cdot h = -\xi(y) h\). Consequently differentiating (235) at \(t = 0\) one has
\[
(u \cdot f_v)(g) = e^{-\xi(\log a)}(\xi(y) f_{k^{-1 \cdot v}}(1) + f_{x k^{-1 \cdot v}}(1))
\]
Thus
\[
(u \cdot f_v)(g) = e^{-\xi(\log a)}(\xi(y) h(k^{-1 \cdot v}) + h(x k^{-1 \cdot v}))
\]
\[
= e^{-\xi(\log a)}(k(\xi(y) - x) \cdot h)(v)
\]
\[
= e^{-\xi(\log a)}(-k w \cdot h)(v)
\]
\[
= e^{-\xi(\log a)}(-u k \cdot h)(v)
\]
\[
= e^{-\xi(\log a)}h(k^{-1} u \cdot v)
\]
\[
= e^{-\xi(\log a)} f_{u \cdot v}(k)
\]
\[
= f_{u \cdot v}(g)
\]
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This proves (233). QED

Given the $T_h$ map (231) of $(g, K)$-modules, one recovers the linear functional $h$ by composing $T_h$ with the Dirac measure at the identity of $G$.

**Proposition 4.12.** Let the notation be as in Theorem 4.11. Then for any $v \in H$ one has

$$h(v) = (T_h(v))(1) \quad (236)$$

**Proof.** Let $v \in H$ so that in previous notation $T_h(v) = f_v$. But by definition (see (224)) one has $f_v(1) = h(v)$. QED

4.7. We now deal with the converse of Theorem 4.11 and Proposition 4.12. Let $(\delta, \xi) \in D \times a^*$ and let $0 \neq H$ be a Harish-Chandra module and assume that

$$T : H \to H(\delta, \xi)$$

is a nontrivial $(g, K)$-module map. Let $h \in H^*$ be defined by putting $h(v) = f_v(1)$ where $v \in H$ and $f_v = T(v)$. If $g \in G$ and $g = kan$ is the Iwasawa decomposition of $g$ then, since $f_v \in H(\delta, \xi)$, one has

$$f_v(g) = e^{-\xi(\log a)} f_v(k) \quad (237)$$

But since $T \neq 0$ there exists $v \in H$ such that $f_v \neq 0$. Thus there exists $g \in G$ such that $f_v(g) \neq 0$. Using the notation of (237) one has $f_v(k) \neq 0$. But $f_v(k) = (k^{-1} \cdot f_v)(1) = f_{k^{-1} \cdot v}(1)$. Thus $h(k^{-1} \cdot v) \neq 0$ and hence $h \neq 0$.

Now let $w \in g_o$ and let $v \in H$ and let the notation be as in the proof of Theorem 4.11. Then (237) implies

$$f_v(exp(-t)w) = e^{-\xi(\log a)} f_v(k_t) \quad (238)$$

Since $w \cdot f_v = f_{w \cdot v}$ one has, by differentiating (238) at $t = 0$, $f_{w \cdot v}(1) = \xi(y) f_v(1) + f_{x \cdot v}(1)$. That is $h(w \cdot v) = h(\xi(y)v + x \cdot v)$. Since $v \in H$ is arbitrary, this implies

$$-w \cdot h = \xi(y) h - x \cdot h \quad (239)$$

By complexification one has (239) for any $w \in g$ and $w = x + y + z$ where $x \in t$, $y \in a$ and $z \in n$. Now choosing $w = -u \in a + n$, it follows from (239) and recalling (222) one has

$$u \cdot h = \chi_{\delta, \xi}(u) h \quad (240)$$

for any $u \in a + n$. But now, recalling (238), note that $(w \cdot f_v)(1) = (f_v \cdot (-w))(1)$. But then (239) implies that, for $w = x \in t$, $(f_v \cdot (-x))(1) = (-x \cdot h)(v)$. But $f_v \cdot u = \nu^n(u) f_v$ for $u \in h_m$ and $\delta = (\zeta, \nu) \in F_o \times \Lambda_{M_0}$. Thus for $-x = u \in h_m$ one has $u \cdot h = \nu(u) h$. Hence (240) is established for all $u \in b$ (see (221)). Finally let $\varepsilon \in F_o$. 

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Then, by (218), one has \( f_v(\varepsilon) = \zeta(\varepsilon) f_v(1) \). But \( f_v(\varepsilon) = \varepsilon \cdot f_v(1) \) and \( \varepsilon \cdot f_v = f_{\varepsilon \cdot v} \). Thus \( h(\varepsilon \cdot v) = \zeta(\varepsilon) h(v) \). But this implies \( \varepsilon \cdot h = \zeta(\varepsilon) h \). Hence, recalling (218), \( h \in H^{*}_{\delta, \xi} \). This is the essential step in proving the following converse to Theorem 4.11 and Proposition 4.12.

**Theorem 4.13.** Let \( 0 \neq H \) be a Harish-Chandra module. Let \( (\delta, \xi) \in D \times \mathfrak{a}^* \) and assume \( T : H \rightarrow H(\delta, \xi) \) is a nontrivial map of \((g, K)\)-modules. Let \( h \in H^* \) be defined by putting \( h(v) = T(v)(1) \). Then \( 0 \neq h \in H_{\delta, \xi}^* \). Furthermore (recalling Theorem 4.11)

\[
T = T_h
\]

**Proof.** We established above that \( 0 \neq h \in H_{\delta, \xi}^* \). For any \( v \in H \) put \( f_v = T_h(v) \) and \( f'_v = T(v) \). By Proposition 4.12 one has \( f_v(1) = f'_v(1) \) for any \( v \in H \). Let \( g \in G \) and let \( g = k a u \) be the Iwasawa decomposition of \( g \). Then since \( f_v, f'_v \in H(\delta, \xi) \) one has \( f_v(g) = e^{-\xi(\log a)} f_v(k) \) and \( f'_v(g) = e^{-\xi(\log a)} f'_v(k) \). But since \( T \) and \( T_h \) are \( \mathbb{C}[K] \)-module maps one has \( f'_v(k) = f'_{v-1}(1) \) and \( f_v(k) = f_{v-1}(1) \). Thus \( f_v(g) = f'_v(g) \). Hence \( T = T_h \). QED

4.8. Any \( \mu \in \Lambda_{M_r} \) defines a linear functional on the Cartan subalgebra \( \mathfrak{h}^*_m \) of the semisimple Lie algebra \([m, m]\) (see (146)). The set of elements \( \{h_i\}, i \in I_m \), is a basis of \( \mathfrak{h}_m^* \) (see §2.1 and §2.7). For \( i \in I_m \) let \( n_i(\mu) \in \mathbb{Z}_+ \) be defined by putting \( n_i(\mu) = (\mu, h_i) \). Since \( \lambda|\mathfrak{h}_m \in \Lambda_{M_r} \) for any \( \lambda \in \Lambda \) note that \( n_i(\lambda) = n_i(\lambda|\mathfrak{h}_m) \) using the notation of §1.1 so that no ambiguity should arise.

Let \( (\delta, \xi) \in D \times \mathfrak{a}^* \) and let write \( \delta = (\zeta, \nu) \in F_\ast \times \Lambda_{M_r} \). One has \( \nu \) and \( \nu^c \in \Lambda_{M_r} \). See (206). We now observe that the functions in \( H(\delta, \xi) \subset C^\infty(G) \) satisfy certain differential equations on \( G \).

**Proposition 4.14.** Let \( f \in H(\delta, \xi) \) using the notation above. Then

\[
(a) \; f \cdot u = \nu^c(u) f \quad \text{and} \\
(b) \; f \cdot e^{-\alpha_i} = 0, \; \forall i \in I_m
\]

**Proof.** The statement (a) has already been established. See (a) in (218). Now recalling (216) and (218) there exists \( \phi \in H(Y_\delta, \xi) \) such that \( f = Q_\delta(\phi) \) where \( Q_\delta \) is defined by (209). Thus if \( g \in G \) and \( m \in M \) one has

\[
f(g m) = \langle \phi(g m), y^*_m \rangle \\
= \langle r_\delta(m^{-1}) \phi(g), y^*_m \rangle \\
= \langle \phi(g), m \cdot y^*_m \rangle
\]

Let \( x \in \mathfrak{m}_m \). Replacing \( m \) in (243) by \( \exp tx \) and differentiating at \( t = 0 \) yields

\[
(f \cdot x)(g) = \langle \phi(g), x \cdot y^*_m \rangle
\]
But then by complexification (244) is valid for any \( x \in \mathfrak{m} \). However, for \( i \in I_m \), one has \( e_i^{\lambda + 1} y^*_i = 0 \) by Proposition 1.1 applied to the case where \([\mathfrak{m}, \mathfrak{m}]\) replaces \( \mathfrak{g} \). But then iterating the argument which yields (244) readily establishes (242). QED

For any \( \lambda \in \Lambda \) let \( \lambda^c \) be the highest weight of the dual \( \mathfrak{g} \)-module \( V^*_\lambda \) and let \( 0 \neq \nu^*_\lambda \) be a highest weight vector of \( V^*_\lambda \). Let \( \kappa \) be the longest element in the Weyl group of \((\mathfrak{g}, \mathfrak{h})\) so that \( \kappa \lambda \) is the lowest weight in \( V_\lambda \). The pairing of \( V_\lambda \) and \( V^*_\lambda \) immediately implies

\[
\lambda^c = -\kappa \lambda
\]

(245)

Now obviously \( (V_\lambda)_n = V_\lambda / (n V_\lambda) \) and \( (V^*_\lambda)_n \) (see (93)) have the structure of \( M \)-modules.

**Proposition 4.15.** Let \( \lambda \in \Lambda \). Then \( (V_\lambda)_n \) is an irreducible \( M \)-module. Moreover the pairing of \( V_\lambda \) and \( V^*_\lambda \) induces a nonsingular \( M \)-invariant pairing (see Proposition 3.1) of \( (V_\lambda)_n \) and \( U(m) v^*_\lambda \).

**Proof.** Obviously the pairing of \( V_\lambda \) and \( V^*_\lambda \) induces a nonsingular \( M \)-invariant pairing of \( (V_\lambda)_n \) and \( (V^*_\lambda)_n \). But \( (V^*_\lambda)_n = U(m) v^*_\lambda \) by (93) and \( U(m) v^*_\lambda \) is an irreducible \( M \)-module by Proposition 3.1. Hence \( (V_\lambda)_n \) must be an \( M \)-irreducible module. QED

4.9. Let \( \lambda \in \Lambda \). Then by Proposition 4.15 there exists \( \delta \in D \) such that

\[
(V_\lambda)_n \cong Y_\delta
\]

(246)

Of course \( \delta = (\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_\lambda} \). It is easier to determine \( \nu^c \) than to determine \( \nu \) itself.

**Proposition 4.16.** Let \( \lambda \in \Lambda \) and let \( \delta = (\zeta, \nu) \in \hat{F}_s \times \Lambda_{M_\lambda} \) be defined by (246). Then in the notation of Proposition 3.1 one has \( \zeta = \lambda^c|F_s \) and

\[
\nu^c = \lambda^c|\theta_m
\]

(247)

See (245).

**Proof.** Let \( \delta^* \in \hat{M} \) be defined so that \( Y_\delta^* \cong Y_{\delta^*} \). Obviously \( \delta^* = (\zeta, \nu^c) \) as an element in \( \hat{F}_s \times \Lambda_{M_\lambda} \) since \( \zeta \) is self-contragredient. But \( Y_{\delta^*} \cong U(m) v^*_\lambda \) by Proposition 3.1. The result then follows from Proposition 3.1 since \( \lambda^c \), given by (245), is the highest weight of \( V^*_\lambda \). QED

Let \( \lambda \in \Lambda \). Since \( \mathfrak{a} \) normalizes \( \mathfrak{n} \) it follows that \( (V_\lambda)_n \) has the structure of an \( \mathfrak{a} \)-module. Furthermore since \( \mathfrak{a} \) commutes with \( \mathfrak{m} \), it follows that any element in \( \mathfrak{a} \) operates on \( (V_\lambda)_n \) by a scalar operator. Let \( \xi \in \mathfrak{a}^* \) be defined so that for any \( x \in \mathfrak{a} \),

\[
x \text{ operates as } \xi(x) \text{ on } (V_\lambda)_n
\]

(248)
Proposition 4.17. Let \( \lambda \in \Lambda \). Let \( \xi \in \mathfrak{a}^* \) be defined by (248). Then \( \xi = \kappa \lambda | \mathfrak{a} \) where \( \kappa \) is the long element of the Weyl group of \( (g, h) \).

Proof. The lowest weight of \( V_\lambda \) is \( \kappa \lambda \). Since any nonzero lowest weight vector clearly has a nonzero projection in \( (V_\lambda)_a \), the scalar defined by any \( x \in \mathfrak{a} \) on \( (V_\lambda)_a \) must be \( \xi(x) \). QED

We recover the following result of Wallach. See §8.5 in [W].

Proposition 4.18. Let \( \lambda \in \Lambda \) and let \( \delta \in D \) and \( \xi \in \mathfrak{a}^* \) be defined respectively by (246) and (248). Then regarding \( V_\lambda \) as the Harish-Chandra module \( H \) of Proposition 4.10, Theorems 4.11 and 4.13 the pair \( (\delta, \xi) \in D \times \mathfrak{a}^* \) is the unique pair such that \( H_{\delta, \xi}^* \neq 0 \) (see (223)). Moreover

\[
H_{\delta, \xi}^* = \mathbb{C} v_\lambda^*
\]

and hence, up to scalar multiple, \( h = v_\lambda^* \) is the unique element in \( H^* \) which satisfies Proposition 4.10. In particular (recalling Theorems 4.11 and 4.13) \( H(\delta, \xi) \) is the unique principal series Harish-Chandra module which has a finite dimensional \( (g, K) \)-submodule equivalent to \( V_\lambda \). Furthermore, the equivalence is defined by \( T_h \) (see (231)) which, in this case, is explicitly determined by

\[
T_h(v)(g) = \langle v, g v_\lambda^* \rangle
\]

for any \( v \in V_\lambda \) and \( g \in G \).

Proof. If \( (\delta', \xi') \in \mathfrak{d} \times \mathfrak{a}^* \) and there exists \( 0 \neq h \in H_{\delta', \xi'}^* \), then \( h \) is a highest weight vector in \( H^* \) by (223). But \( H^* = V_\lambda^* \) is irreducible and hence any highest weight vector is a multiple of \( v_\lambda^* \). To prove (249) it therefore suffices, by (223), to show that

\[
b \cdot v_\lambda^* = \chi_{\delta, \xi}(b) v_\lambda^*
\]

for any \( b \in \mathcal{B} \). As observed in (222) it suffices to take \( b = \varepsilon, z \) and \( x \) using the notation of (222). But \( \delta = (\zeta, \nu) \) using the notation of Proposition 4.16. But then (251) is satisfied for \( b = \varepsilon \) or \( b = z \) by Propositions 3.1 and 4.16 since \( v_\lambda^* \) is the highest weight vector of \( V_\lambda^* \) and the highest weight is \( \lambda^c \). But now \( x \cdot v_\lambda^* = \lambda^c(x) v_\lambda^* \) for \( x \in \mathfrak{a} \). However \( \lambda^c = -\kappa \lambda \) by (245) and hence

\[
-\xi = \lambda^c | \mathfrak{a}
\]

by Proposition 4.17. Recalling (222) this establishes (249). But now for \( h = v_\lambda^*, g \in G \) and \( v \in V_\lambda \) one has \( (T_h(v))(g) = e^{-\xi (log a)} \langle k^{-1}, \nu, v_\lambda^* \rangle \) by Proposition 4.10 where \( g = k a n \) is the Iwasawa decomposition of \( g \). But \( \langle k^{-1} \cdot v, v_\lambda^* \rangle = \langle v, k \cdot v_\lambda^* \rangle \) and \( a n \cdot v_\lambda^* = e^{-\xi (log a)} v_\lambda^* \) by (252). This establishes (250). The map \( T_h \) is clearly nontrivial and hence is injective since \( V_\lambda \) is \( g \)-irreducible. QED

4.10. Let \( \lambda \in \Lambda \) and let \( (\delta, \xi) \in D \times \mathfrak{a}^* \) be defined by (246) and (248). Let \( h \) be as in Proposition 4.18. Let \( V_\lambda = T_h(V_\lambda) \) so that, by Proposition 4.18, \( V_\lambda \) is a finite dimensional \( (g, K) \)-submodule, equivalent to \( V_\lambda \), of the principal series Harish-Chandra
module $H(\delta, \xi)$. The functions in $H(\delta, \xi)$ are of course determined by their restrictions to $K$. The following theorem characterizes the functions in the finite dimensional space $V_\lambda$ as solutions in $H(\delta, \xi)$ of differential equations which are associated with certain elements in $U(\mathfrak{t})$. This can viewed as analogous to the Borel-Weil theorem. In that case the differential equations are Cauchy-Riemann equations.

Let $\text{Diff } C^\infty(G)$ be the algebra of all differential operators on $C^\infty(G)$. One has an injective homomorphism

$$U(\mathfrak{g}) \rightarrow \text{Diff } C^\infty(G)$$

(253)

where the image is the algebra of all left $G$-invariant differential operators on $G$. If $f \in C^\infty(G)$ and $u \in U(\mathfrak{g})$ the action of $u$ on $f$ is denoted by $f \cdot u$. The homomorphism (253) is determined by the restriction of (253) to $\mathfrak{g}_0$. If $z \in \mathfrak{g}_0$ and $f \in C^\infty(G)$, then $f \cdot z$ is given in (205) where $G_1 = G$. The following noncompact analogue of Borel-Weil is one of the main results in the paper.

**Theorem 4.19.** Let $\lambda \in \Lambda$ and let $\delta \in D$ and $\xi \in a^*$ be defined by (246) and (248). Recall (see Proposition 4.18) that $H(\delta, \xi)$ is the unique principal series Harish-Chandra module which has a $(\mathfrak{g}, K)$-submodule (also unique) $V_\lambda$ that is $(\mathfrak{g}, K)$-equivalent to $V_\lambda$. Then $V_\lambda$ may be given by

$$V_\lambda = \{ f \in H(\delta, \xi) \mid f \cdot q_{\lambda^c, i}(z_i) = 0, \forall i \in I_n \}$$

(254)

where $z_i \in \mathfrak{k}$, for $i \in I_n$, is defined by (97) and $q_{\lambda^c, i}(t) \in C[t]$ is the polynomial defined in §2.2.

**Proof.** Let $f \in V_\lambda$. Then by (250) there exists $v \in V_\lambda$ such that, for any $g \in G$, $f(g) = \langle v, g v_\lambda^\ast \rangle$. Clearly then

$$(f \cdot u)(g) = \langle v, g u v_\lambda^\ast \rangle$$

(255)

for any $u \in U(\mathfrak{g})$. Now let $V$ be the right hand side of (254). To align the notation of (97) with that of Theorem 1.24 note that (see paragraph preceding Lemma 1.12)

$$z_{-\alpha_i} = z_i, \text{ for } i \in I_n$$

(256)

and

$$z_{-\alpha_i} = e_{-\alpha_i}, \text{ for } i \in I_m$$

(257)

But since $v_\lambda^\ast$ is the highest weight vector in $V_\lambda^\ast$ and the highest weight is $\lambda^c$, it follows that

$$q_{\lambda^c, i}(z_i)v_\lambda^\ast = 0, \forall i \in I_n$$

(258)

by Theorem 1.24. Thus $V_\lambda \subset V$ by (255). Hence to prove (254) it suffices to prove that

$$\dim V \leq d_\lambda$$

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Where \( d_\lambda = \text{dim} V_\lambda \). Now let \( \chi : U(a + n) \to \mathbb{C} \) be the character defined so that \( w v_\lambda^* = \chi(w) v_\lambda^* \) for all \( w \in U(a + n) \). Since \( U(g) \) is isomorphic, as a linear space, to \( U(\mathfrak{t}) \otimes U(a + n) \), it follows that \( U(g) = U(\mathfrak{t}) \oplus I \) where \( I = U(\mathfrak{t}) \text{Ker} \chi \). On the other hand, if \( L_{\lambda^c}(\mathfrak{t}) \) is the left ideal in \( U(\mathfrak{t}) \) defined as in Proposition 1.3, it follows from Proposition 1.3 that there exists a subspace \( W \subset U(\mathfrak{t}) \) of dimension \( d_\lambda \) such that \( U(\mathfrak{t}) = W \oplus L_{\lambda^c}(\mathfrak{t}) \). Hence one has the direct sum

\[ U(g) = W \oplus L_{\lambda^c}(\mathfrak{t}) \oplus I \]  

(259)

But now, by (218), any \( f \in H(\delta, \xi) \) satisfies the \( AN - \xi \) condition (see (215)). Thus in particular one has

\[ f \cdot I = 0, \ \forall f \in \mathcal{V} \]  

(260)

On the other hand we assert that

\[ f \cdot L_{\lambda^c}(\mathfrak{t}) = 0, \ \forall f \in \mathcal{V} \]  

(261)

To prove (261) first note that, using of course Theorem 1.24, we can write

\[ L_{\lambda^c}(\mathfrak{t}) = \sum_{i \in I_n} U(\mathfrak{t}) g_{\lambda^c, i} (z_i) + \sum_{i \in I_m} U(\mathfrak{t}) e_{n_i}^{\lambda^c + 1} + U(\mathfrak{t}) \text{Ker} \nu^c \]  

(262)

The first two sums in (262) replaces the first sum in (78) (recalling (255) and (256)) and the last summand in (262) rewrites the last two sums in (78) using (247) and our extension of the domain of \( \nu^c \) to \( b_m \). But now (261) follows from Proposition 4.14 and the definition of \( \mathcal{V} \).

Now assume that \( \text{dim} \mathcal{V} > d_\lambda \). Then there must exist \( 0 \neq f \in \mathcal{V} \) such that \( (f \cdot w)(1) = 0 \) for all \( w \in W \). But then by (259), (260) and (261) one has \( (f \cdot u)(1) = 0 \) for all \( u \in U(g) \). This is a contradiction since \( f \) is an analytic function and hence is uniquely determined by the set of values \( \{(f \cdot u)(1), \ u \in U(g)\} \). QED

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