Modular transformation and twist between trigonometric limits of $sl(n)$ elliptic R-matrix

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Abstract

We study the modular transformation of $\mathbb{Z}_n$-symmetric elliptic R-matrix and construct the twist between the trigonometric degeneracy of the elliptic R-matrix.

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1 Introduction

Universal twists connecting quantum (super) algebras to elliptic (super) algebras have been constructed in [1, 2, 3, 4, 5]. Recently, universal twists connecting double Yangian $DY(sl(2))_c$ to deformed double Yangian $DY_\xi(sl(2))_c$ (or $\tilde{A}_{\hbar,\eta}(\hat{\mathfrak{sl}}_2)$ [6, 7]), has been given in [8]. They show the quasi-Hopf structure of elliptic (super) algebras and deformed Yangian.

If $(\mathcal{A}, \Delta, \mathcal{R})$ defines a quasi-triangular Hopf (super) algebra and the universal twist $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfies the cocycle-like relations (For the details we will refer the reader to...
\begin{align}
\Delta^F(\cdot) &= \mathcal{F}\Delta(\cdot)\mathcal{F}^{-1}, \\
\mathcal{R}^F &= F_{21}\mathcal{R}F_{12}^{-1},
\end{align}

defines a quasi-triangular quasi-Hopf (super) algebra. As for the R-matrices interpreted as evaluation representation of universal ones, it becomes

\begin{align}
\mathcal{R}^F &= F_{21}R F_{12}^{-1},
\end{align}

where the particular matrix \( F \) is the evaluation representation of \( \mathcal{F} \).

It is well-known that there exist two kind trigonometric degeneracy limits of elliptic R-matrix: scaling limit which is the R-matrix of deformed double Yangian \( DY_\xi(sl(n))_c \) \([6]\) and ordinary limit which is also related to another type deformed Yangian (e.g \( DY_\xi V^6 \)). In this letter, we will construct the twist between these two trigonometric R-matrices. In special case \( n = 2 \), our result coincides with the result of Arnaudon’s \([9]\).

2 Trigonometric limits of elliptic R-matrix

2.1 The \( sl(n) \) elliptic R-matrix

Let \( n \) be an integers and \( n \leq 2 \), \( w \in \mathcal{C} \) and \( Imw > 0 \), \( \xi \) take real value and \( \xi > 0 \), \( \tau \in \mathcal{C} \) and \( Im\tau > 0 \). Set \( V \) be a n-dimensional vector space with standard basis \( \{e_j\}_{j\in\mathbb{Z}^n} \). Introduce \( n \times n \) matrices \( h \), \( g \) and \( I_\alpha \) with \( \alpha \in \mathbb{Z}^n \otimes \mathbb{Z}^n \) by

\[
he_j = e_{j+1}, \quad ge_j = \omega^j e_j, \quad j \in \mathbb{Z}^n,
\]

\[
I_\alpha \equiv I_{\alpha_1,\alpha_2} = h^{\alpha_1}g^{\alpha_2},
\]

where \( \omega = exp\left\{ \frac{2i\pi}{n} \right\} \). Define the elliptic functions

\[
\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{m \in \mathbb{Z}} exp\{i\pi[(m + a)^2 \tau + 2(m + a)(z + b)]\},
\]

\[
\sigma_\alpha(z, \tau) = \sigma_{(\alpha_1,\alpha_2)}(z, \tau) = \theta \begin{bmatrix} \frac{1}{2} + \frac{\alpha_1}{n} \\ \frac{1}{2} + \frac{\alpha_2}{n} \end{bmatrix} (z, \tau).
\]

The \( \mathbb{Z}_n \)-symmetric R-matrix can be defined as

\begin{align}
\mathcal{S}(z, w, \tau) &= \frac{\sigma_0(w, \tau)}{\sigma_0(z + w, \tau)} \sum_\alpha W_\alpha(z, \tau)I_\alpha \otimes I_\alpha^{-1},
\end{align}

\[2\]
where

\[ W_\alpha(z, \tau) = \frac{\sigma_\alpha(z + \frac{w}{n}, \tau)}{n \sigma_\alpha(\frac{w}{n}, \tau)}. \]

The elements of \( \mathbf{Z}_n \)-symmetric R-matrix can be expressed explicitly\[11\]

\[ \{ \mathbf{S}(z, w, \tau) \}_{i,j}^{kl} = \frac{\sigma_0(w, \tau)}{\sigma_0(z + w, \tau)} \prod_{j=0}^{n-1} \theta \left[ \frac{1}{2} + \frac{j}{n} \right] (z, n\tau) \prod_{j=1}^{n-1} \theta \left[ \frac{1}{2} + \frac{j}{n} \right] (0, n\tau) \]

\[ \times \frac{\theta \left[ \frac{1}{2} + \frac{l-k}{n} \right] (z + w, n\tau) \theta \left[ \frac{1}{2} + \frac{l-i}{n} \right] (w, n\tau)}{\theta \left[ \frac{1}{2} + \frac{l-k}{n} \right] (w, n\tau) \theta \left[ \frac{1}{2} + \frac{l-i}{n} \right] (z, n\tau)}. \]

(2.2)

Introduce \( sl(n) \) elliptic R-matrix

\[ S(z) \equiv S(z, w, \tau) = x^{\frac{z}{w}(\frac{1}{n} - 1)} \frac{g_1(z)}{g_1(-\frac{z}{w})} \mathbf{S}(z, w, \tau), \]

(2.3)

with \( x = e^{i\pi w} \) and

\[ g_1(v) = \frac{\{ x^{2v}x^2 \} \{ x^{2v}x^{2n+2\xi-2} \}}{\{ x^{2v}x^{2n} \} \{ x^{2v}x^{2\xi} \}}, \]

(2.4)

where \( \{ z \} = (z; e^{2i\pi \tau}, x^{2n}) \) and \( (z; p_1, \cdots, p_m) \equiv \prod_{\{m\}}^\infty (1 - z p_1^{m_1} \cdots p_m^{m_m}). \)

The \( sl(n) \) elliptic R-matrix \( S(v) \) satisfies the following properties\[12\]

Yang – Baxter equation : \[ S_{12}(v_1 - v_2)S_{13}(v_1 - v_3)S_{23}(v_2 - v_3) = S_{23}(v_2 - v_3)S_{13}(v_1 - v_3)S_{12}(v_1 - v_2), \]

Unitarity : \[ S_{12}(v)S_{21}(-v) = 1, \]

Crossing – Unitarity : \[ S_{12}(v)^{\ell_2}S_{21}(-v - n)^{\ell_2} = 1, \]

(2.5)

2.2 Scaling limit of the \( sl(n) \) elliptic R-matrix

The scaling limit of the R-matrix (2.3) is taken by

\[ \frac{z}{\tau} = \frac{i\beta}{\hbar \xi}, \quad \frac{w}{\tau} = \frac{1}{\zeta}, \quad w \rightarrow 0^+, \quad \text{with } \beta, \xi \text{ and } \hbar \text{ fixed}. \]

(2.5)
The above limit should be understood as $w$ go to 0 with the $Imw$ from 0+. Noting the properties of elliptic functions under the scaling limit (2.5)

$$\begin{align*}
\theta \left[ \frac{1}{2} + a \right] \left( \frac{i\beta w}{h}, n\xi w \right) & \quad w \to 0^+ \\
\theta \left[ \frac{1}{2} + b \right] \left( \frac{i\beta w}{h}, n\xi w \right) & \quad w \to 0^+ \\
\theta \left[ \frac{1}{2} + a \right] \left( \frac{i\beta w}{h}, \xi w \right) & \quad w \to 0^+ \\
\theta \left[ \frac{1}{2} + b \right] \left( \frac{i\beta w}{h}, \xi w \right) & \quad w \to 0^+
\end{align*}
$$

we can obtain that in the scaling limit the matrix elements of (2.3) become

$${\{ R^{DY}_{i,j} \}}_{kl} \overset{def}{=} \lim_{w \to 0^+} S^{kl}_{ij} \left( \frac{i\beta w}{h}, w, \xi w \right)$$

$$= \kappa(\beta) \left( \prod_{\alpha=1}^{n-1} \frac{\sin(\frac{i\beta}{nh} + \frac{\alpha}{n})}{\sin(\frac{\alpha}{n})} \right) \sin(\frac{1}{\xi}) \sin\pi\left( \frac{i\beta}{nh} + \frac{1}{n} + \frac{\xi}{n} \right)$$

$$\sin\pi\left( \frac{i\beta}{nh} + \frac{1}{n} + \frac{\xi}{n} \right) \sin\pi\left( \frac{i\beta}{nh} + \frac{1}{n} + \frac{\xi}{n} + \frac{\xi}{n} \right) \sin\pi\left( \frac{i\beta}{nh} + \frac{1}{n} + \frac{\xi}{n} \right)$$

$$= \exp\{-2 \int_{0}^{\infty} \frac{sh(n-1)ht \ sh(\xi-1)ht \ sh2i\beta t dt}{shh \xi t \ shnht} \},$$

and $\kappa(\beta)$ can be re-expressed in terms of double sine function\[[6]\]. In the deriving, we have used the identity

$$\prod_{j=1}^{n-1} \frac{\sin(x + j\pi/n)}{\sin(\frac{\pi}{n})} = \frac{\sin x}{n \sin x}$$
In the particular case \( n = 2 \), one recovers explicitly \([13, 14]\):

\[
R^{DY}_{\xi}(\beta, \xi; \hbar) = \kappa(\beta) \begin{pmatrix}
\frac{\cos \frac{\pi}{2} \cos \frac{\pi}{2n}}{\cos \pi (\frac{1}{2n} + \frac{1}{2})} & 0 & 0 & -\frac{\sin \frac{\pi}{2} \sin \frac{\pi}{2n}}{\cos \pi (\frac{1}{2n} + \frac{1}{2})} \\
0 & \frac{\cos \frac{\pi}{2} \sin \frac{\pi}{2n}}{\sin \pi (\frac{1}{2n} + \frac{1}{2})} & \frac{\sin \frac{\pi}{2} \cos \frac{\pi}{2n}}{\sin \pi (\frac{1}{2n} + \frac{1}{2})} & 0 \\
0 & \frac{\sin \frac{\pi}{2} \cos \frac{\pi}{2n}}{\cos \pi (\frac{1}{2n} + \frac{1}{2})} & \frac{\cos \frac{\pi}{2} \sin \frac{\pi}{2n}}{\cos \pi (\frac{1}{2n} + \frac{1}{2})} & 0 \\
-\frac{\sin \frac{\pi}{2} \sin \frac{\pi}{2n}}{\cos \pi (\frac{1}{2n} + \frac{1}{2})} & 0 & 0 & \frac{\cos \frac{\pi}{2} \cos \frac{\pi}{2n}}{\cos \pi (\frac{1}{2n} + \frac{1}{2})}
\end{pmatrix}.
\tag{2.9}
\]

### 2.3 The ordinary trigonometric limit of \( sl(n) \) elliptic R-matrix

The ordinary trigonometric limit of the R-matrix \([2, 3]\) is taken by: \( \tau \rightarrow +i\infty \) with \( z \) and \( w \) fixed \([11, 14]\). Noting the properties

\[
\begin{align*}
\theta \left[ \frac{1}{2} + \frac{a}{n} \right] (z, \tau) \rightarrow & +i\infty \quad \{ \exp\{2i\pi(\frac{a}{n} - \frac{1}{2})(z-w)\}, \quad 0 < a < n \\
\theta \left[ \frac{1}{2} + \frac{a}{n} \right] (w, \tau) \rightarrow & +i\infty \quad \{ \exp\{2i\pi(\frac{a}{n} + \frac{1}{2})(z-w)\}, \quad -n < a < 0 \}
\end{align*}
\]

we have that in ordinary trigonometric limit the matrix elements of \([2, 3]\) become \([14]\):

\[
\{R^Q\}_{ij}^{kl}(\beta, \xi; \hbar) \overset{\text{def}}{=} \kappa(\beta) \times \lim_{\tau \rightarrow +i\infty} S_{ij}^{kl}(\frac{1}{\hbar \xi}, \frac{1}{\xi} + \frac{1}{\hbar \xi}, \tau) \tag{2.10}
\]

\[
\begin{cases}
\{R^Q\}_{ij}^{ij}(\beta, \xi; \hbar) = \kappa(\beta) \frac{\sin \frac{\pi}{2} \sin \frac{\pi}{2n}}{\sin \pi (\frac{1}{2n} + \frac{1}{2})} \exp\{2i\pi(\frac{1}{n} - \frac{1}{2}) - \frac{1}{2} i \beta \hbar \}, & i < j \\
\{R^Q\}_{ij}^{ij}(\beta, \xi; \hbar) = \kappa(\beta) \frac{\sin \frac{\pi}{2} \sin \frac{\pi}{2n}}{\sin \pi (\frac{1}{2n} + \frac{1}{2})} \exp\{2i\pi(\frac{1}{n} + \frac{1}{2}) - \frac{1}{2} i \beta \hbar \}, & j < i \\
\{R^Q\}_{ij}^{ij}(\beta, \xi; \hbar) = \kappa(\beta) \frac{\sin \frac{\pi}{2} \sin \frac{\pi}{2n}}{\sin \pi (\frac{1}{2n} + \frac{1}{2})} \exp\{2i\pi(-\frac{1}{2} + \frac{1}{n}) - \frac{1}{2} i \beta \hbar \}, & i < j \\
\{R^Q\}_{ij}^{ij}(\beta, \xi; \hbar) = \kappa(\beta) \frac{\sin \frac{\pi}{2} \sin \frac{\pi}{2n}}{\sin \pi (\frac{1}{2n} + \frac{1}{2})} \exp\{2i\pi(\frac{1}{2} + \frac{1}{n}) - \frac{1}{2} i \beta \hbar \}, & j < i \\
\{R^Q\}_{ij}^{ij}(\beta, \xi; \hbar) = 0, & \text{otherwise}
\end{cases}
\]
In the particular case \( n=2 \), one recovers
\[
R^Q(\beta, \xi; \bar{h}) = \kappa(\beta) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\sin \frac{i\pi \beta}{2\bar{h}}} {\sin \frac{i\pi \beta}{2\bar{h}} + \frac{1}{2}\xi} & \frac{-\sin \frac{i\pi \beta}{2\bar{h}}}{\sin \frac{i\pi \beta}{2\bar{h}} + \frac{1}{2}\xi} & 0 \\
0 & \frac{\sin \frac{i\pi \beta}{2\bar{h}}}{\sin \frac{i\pi \beta}{2\bar{h}} + \frac{1}{2}\xi} & \frac{-\sin \frac{i\pi \beta}{2\bar{h}}}{\sin \frac{i\pi \beta}{2\bar{h}} + \frac{1}{2}\xi} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (2.11)

In the case \( \bar{h} = \pi \), the above \( R \)-matrix coincides with the \( S \)-matrix of sine-Gordon model [15].

### 3 Modular transformation and the twist from \( R^Q \) to \( R^{DY}_\xi \)

#### 3.1 Modular transformation of elliptic functions

The elliptic \( \sigma_0 \)-function has the following modular transformation property [16]
\[
\theta \left[ \frac{1}{2} \right] (z, \frac{1}{\tau}, -\frac{1}{\tau}) = \exp\{i\pi \frac{z^2}{\tau}\} \theta \left[ \frac{1}{2} \right] (z, \tau) \times \text{const.}
\]
where the \text{const.} only depends on \( \tau \). Noting that
\[
\theta \left[ \frac{1}{2} + a \right] (z, \tau) = \exp\{2i\pi [z \frac{1}{2} + b + \frac{a\tau}{2}]\} \theta \left[ \frac{1}{2} \right] (z + b + a\tau, \tau),
\]
we have the following properties under modular transformation
\[
\theta \left[ \frac{1}{2} + a \right] (\frac{z}{\tau}, -\frac{1}{\tau}) = \exp\{i\pi [\frac{z^2}{\tau} + a - b + 2ab]\} \theta \left[ \frac{1}{2} + b \right] (z, \tau) \times \text{const.}.
\] (3.1)

From (3.1) and the definition of \( Z_n \)-symmetric \( R \)-matrix, we can derive the following relations [12]
\[
(M \otimes M)S(\frac{z}{\tau}, w, -\frac{1}{\tau})(M^{-1} \otimes M^{-1}) = x^{\frac{2sw(1-s)}{n\tau}} P S(z, w, \tau) P.
\] (3.2)

where \( P \) is the permutation operator acting on tensor space \( V \otimes V \) as: \( P(e_i \otimes e_j) = e_j \otimes e_i \), and the \( n \times n \) matrix \( M \) with the elements defined as \( (M)_{jk} = \omega^{-jk} \). The matrix \( M \) enjoys in the following properties
\[
MgM^{-1} = h^{-1}, \quad MhM^{-1} = g.
\] (3.3)

6
3.2 Twist from $R^Q$ to $R^{DY}_\xi$

In this subsection, we shall calculate the twist from $R^{DY}_\xi$ defined in (2.8) to $R^Q$ given in (2.10). From (2.8), we have

$$R^{DY}_\xi(\beta, \xi; \hbar) = \kappa(\beta) \lim_{w \to 0^+} S\left(\frac{i\beta w}{\hbar}, w, \xi w\right) = \kappa(\beta) (M \otimes M) P \lim_{w \to 0^+} \{S\left(\frac{i\beta}{\hbar\xi}, \frac{1}{\xi w}\right)\} P(M^{-1} \otimes M^{-1}) = (M \otimes M) P R^Q(\beta, \xi; \hbar) P(M^{-1} \otimes M^{-1}).$$

In the second line we have used (3.2). Therefore, the trigonometric degeneracy limits R-matrices $R^{DY}_\xi$ and $R^Q$ are related by

$$R^{DY}_\xi(\beta, \xi; \hbar) = F_{12} R^Q(\beta, \xi; \hbar) F_{12}^{-1},$$

where the twist $F_{12}$ is

$$F_{12} = M \otimes MP_{12}.$$ (3.5)

In the following, let us consider the special case $n = 2$. In this case,

$$M = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -\sqrt{2} V \sigma_z,$$

where

$$\sigma_z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Thanks to that $R^Q$ (2.11) satisfies the following properties

$$P R^Q(\beta, \xi; \hbar) P = R^Q(\beta, \xi; \hbar) = (\sigma_z \otimes \sigma_z) R^Q(\beta, \xi; \hbar)(\sigma_z \otimes \sigma_z),$$

our twist is equivalent to that given by Arnaudon et al \[9\].
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