The power of private likelihood-ratio tests for
goodness-of-fit in frequency tables

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Abstract

Privacy-protecting data analysis investigates statistical methods under privacy con-
straints. This is a rising challenge in modern statistics, as the achievement of confiden-
tiality guarantees, which typically occurs through suitable perturbations of the data,
may determine a loss in the statistical utility of the data. In this paper, we consider
privacy-protecting tests for goodness-of-fit in frequency tables, this being arguably
the most common form of releasing data, and present a rigorous analysis of the large
sample behaviour of a private likelihood-ratio (LR) test. Under the framework of (ε, δ)-
differential privacy for perturbed data, our main contribution is the power analysis of
the private LR test, which characterizes the trade-off between confidentiality, measured
via the differential privacy parameters (ε, δ), and statistical utility, measured via the
power of the test. This is obtained through a Bahadur-Rao large deviation expansion
for the power of the private LR test, bringing out a critical quantity, as a function of
the sample size, the dimension of the table and (ε, δ), that determines a loss in the
power of the test. Such a result is then applied to characterize the impact of the sample
size and the dimension of the table, in connection with the parameters (ε, δ), on the
loss of the power of the private LR test. In particular, we determine the (sample) cost
of (ε, δ)-differential privacy in the private LR test, namely the additional sample size
that is required to recover the power of the Multinomial LR test in the absence of
perturbation. Our power analysis rely on a non-standard large deviation analysis for
the LR, as well as the development of a novel (sharp) large deviation principle for sum
of i.i.d. random vectors, which is of independent interest.

Keywords: Bahadur-Rao large deviation expansion; convolutional-type exponential mecha-
nism; differential privacy; Edgeworth expansion; likelihood-ratio test; non-standard large deviation
analysis; power analysis; truncated Laplace exponential mechanism

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1 Introduction

Privacy-protecting data analysis is a rising subject in modern statistics, building upon the following challenge: for given data, say $D$, how to determine a transformation $\mathcal{M}$, called (perturbation) mechanism, such that if $\mathcal{M}(D)$ is released then confidentiality will be protected and also the value of $D$ for statistical analysis, called utility, will be preserved in $\mathcal{M}(D)$? Measuring utility is common in statistics, and the decrease of utility arising from releasing $\mathcal{M}(D)$ rather than $D$ may be measured as the loss in the accuracy of a statistical method applied to the undertaken data analysis. Measuring confidentiality has been attracting much attention in computer science, where differential privacy (DP) has been put forth as a mathematical framework to quantify privacy guarantees (Dwork, 2006; Dwork et al., 2006). Roughly speaking, DP requires that the distribution of $\mathcal{M}(D)$ remains almost unchanged when an individual is included or removed from $D$, thus ensuring that nothing can be learnt about individuals. In a recent work, Rinott et al. (2017) provided a comprehensive, and practically oriented, treatment of DP in the dissemination of frequency tables, this being arguably the most common form of releasing data. Consider data to be arranged in a list of $k > 1$ cells $a = (a_1, \ldots, a_k)$, with $\sum_{1 \leq i \leq k} a_i = n > k$, where $a_i$ is the number of individuals taking the attribute values corresponding to cell $i$, for $i = 1, \ldots, k$. Under the curator framework of DP, or global DP, Rinott et al. (2017) introduced a class of truncated exponential mechanisms (EMs) and showed how they allow to increase utility of the perturbed list $b = M(a)$ at the cost of relaxing DP to the $(\varepsilon, \delta)$-DP (Dwork et al., 2006; Dwork and Roth, 2013). The parameters $\varepsilon$ and $\delta$ control the level of privacy against intruders: privacy guarantees become more stringent as $\varepsilon$ and $\delta$ tend to zero, with the DP corresponding to $\delta = 0$. Then, an empirical analysis of truncated EMs is presented for the problem of testing goodness-of-fit in frequency tables of dimension $k = 2$, showing the effect of the perturbation in the power of Pearson’s chi-squared and likelihood-ratio (LR) tests. See Wang et al. (2015) and Kifer and Roger (2017) for similar analyses under DP.

1.1 Our contributions

In this paper, we present a rigorous analysis of the large sample behaviour of the LR test for goodness-of-fit under the $(\varepsilon, \delta)$-DP framework of Rinott et al. (2017). We focus on the popular truncated Laplace EM, though our results can be easily extended to a broad class of truncated EMs discussed in Rinott et al. (2017). The perturbed list $b$ is assumed to be modeled as the convolution between a Multinomial distribution with parameter $(n, p)$ and the distribution of i.i.d truncated (discrete) Laplace random variables on $\{-m, \ldots, m\}$ with location 0 and scale $\varepsilon^{-1}$, in such a way that the sample size of $b$ is $n$. Under such a model, which is referred to as the “true” model, we consider the LR test to assess goodness-of-fit in the form: $H_0 : p = p_0$, for a fixed $p_0 \in \Delta_{k-1}$, against $H_1 : p \neq p_0$, where $\Delta_{k-1} = \{ p \in [0,1]^{k-1} : |p| \leq 1 \}$ and $|p| = \sum_{1 \leq i \leq k-1} p_i$. First, we establish an Edgeworth expansion of the distribution the “true” LR, providing a “private”, and more accurate, version of the classical chi-squared limit of the LR (Wilks, 1938). Then, our main contribution provides a quantitative characterization of the trade-off between confidentiality, measured via the DP parameters $\varepsilon$ and $m$, and utility, measured via the power of the LR test. In particular, we rely on non-standard large deviation analysis (Petrov, 1975; Saulis and Statulevicius, 1991) to establish a Bahadur-Rao large deviation expansion for the power of the “true” LR test. This result provides a “private” version of a theorem by Hoeffding (1965, 1967) on the power of the Multinomial LR test. See also Bahadur (1960); Rao (1962); Bahadur (1967); Efron (1967); Efron and Truax (1968). Our large deviation expansion brings out a critical quantity, as a function of $n$, $k$, $\varepsilon$ and $m$, which determines a loss in the power of the “true” LR test. This leads to characterize the impact of $n$ and $k$, in connection with the parameters $(\varepsilon, \delta)$, on the loss of the power of the
private LR test. Concretely, we determine the (sample) cost of \((\varepsilon, \delta)\)-DP under the “true” LR test, namely the additional sample size required to recover the power of the Multinomial LR test in the absence of perturbation.

As a complement to our power analysis of the “true” LR test, we investigate the well-known problem of releasing negative values in frequency tables under the \((\varepsilon, \delta)\)-DP. The “true” model allows for negative values in the perturbed list \(b\), which may be questionable in the context of frequency tables. If publishing data with negative values is not acceptable for some reason, then the common policy is to post-process \(b\) by reporting negative values as zeros, which preserves \((\varepsilon, \delta)\)-DP (Dwork and Roth, 2013). However, as a matter of fact, releasing lists that have an appearance similar to that of original lists may lead to ignoring the perturbation and analyze data as if they were not perturbed, which is known empirically to provide unreliable conclusions (Fienberg et al., 2010; Rinott et al., 2017). Following Rinott et al. (2017), we consider a “naïve” model for the perturbed list \(b\), that is a statistical model that does not take the exponential EM into account. In particular, if \(b^+ = (b_1^+, \ldots, b_k^+)\) denotes the post-processed list, which is obtained from the \((\varepsilon, \delta)\)-DP list \(b\) by setting negative values to be equal to zeroes, then the “naïve” model assumes \(b^+\) to be modeled as the Multinomial distribution with parameter \(\sum_{1 \leq i \leq k} b_i^+\) and \(p\). Under the “naïve” model, we consider the LR test to assess goodness-of-fit, and we establish an Edgeworth expansion of the distribution of the “naïve” LR. This result allows to identify a critical quantity, as a function of \(k\), \(n\) and the variance of the truncated Laplace EM, which determines a loss in the statistical significance of the “naïve” LR test with respect to the “true” LR test. Our analysis thus shows the importance of taking the perturbation into account, and presents a first rigorous evidence endorsing the release of negative values when the \((\varepsilon, \delta)\)-DP is adopted.

1.2 Related literature

Under the curator framework of \((\varepsilon, \delta)\)-DP, some recent works have considered the problem of privacy-protecting tests for goodness-of-fit in frequency tables. Our work is the first to adopt a LR approach for an arbitrary dimension \(k \geq 2\) and a two-sided alternative hypothesis. The work of Awan and Slavković (2018) is closely related to ours, as they consider a truncated EM and adopt a LR approach. However, Awan and Slavković (2018) assume \(k = 2\) and, motivated by the study of uniformly most powerful private tests, they consider pointwise and one-sided alternative hypotheses. The works of Gaboardi et al. (2016) and Kifer and Roger (2017) are also related to our work, as they assume \(k \geq 2\) and they consider a two-sided alternative hypothesis. However, besides not considering truncated EMs, these works do not adopt a LR approach, introducing private tests through suitable perturbations of Pearson’s chi-squared test. Other recent works, though less related to ours, consider a minimax analysis for a class of identity tests in the form \(H_0 : p = p_0\) against \(H_1 : d_{TV}(p, p_0) \geq \rho\), for some choice of \(\rho > 0\), with \(d_{TV}\) being the total variation distance (Cai et al., 2017; Acharya et al., 2018; Aliakbarpour et al., 2018; Cummings et al., 2018; Canonne et al., 2019, 2020). In general, to the best of our knowledge, our work is the first to make use of the power of the test to quantify the trade-off between confidentiality and utility. This is achieved through the use of non-standard large deviation analysis, which is known to be challenging in a setting such as ours, where the statistical model is discrete, multidimensional and not belonging to the exponential family. In particular, our Bahadur-Rao large deviation expansion relies on the development of a novel (sharp) large deviation principle for sum of i.i.d. random vectors, which is of independent interest.
1.3 Organization of the paper

The paper is structured as follows. In Section 2 we recall the curator framework of \((\varepsilon,\delta)\)-DP in the context of frequency tables, define the class of truncated EMs, and introduce some related terminology and notation. In Section 3 we introduce the “true” model and establish a Bahadur-Rao large deviation expansion for the power of the “true” LR test, quantifying the trade-off between confidentiality and utility. In Section 4 we introduce the “naïve” model and establish an Edgeworth expansion for the distribution of the “naïve” LR test with respect to the “true” LR test. Section 5 contains some concluding remarks and directions for future work. Proofs are deferred to Appendix A for \(k = 2\) and to Appendix B for \(k > 2\).

2 The curator framework of \((\varepsilon,\delta)\)-DP

Before presenting our main results in Section 3, it is helpful to recall the definitions of \((\varepsilon,\delta)\)-DP and truncated EM, and to introduce some related terminology and notation (Dwork, 2006; Dwork and Roth, 2013; Rinott et al., 2017). Under the curator framework of \((\varepsilon,\delta)\)-DP for frequency tables, or global \((\varepsilon,\delta)\)-DP, the list \((a_1,\ldots,a_k)\) is centrally stored and a trusted curator is responsible for its perturbation. This is different from local \((\varepsilon,\delta)\)-DP, under which individual data points are perturbed (Dwork and Roth, 2013). For notational convenience, let \(a = (a_1,\ldots,a_{k-1})\) and \(a_k = n - |a|\), with \(|a| = \sum_{1 \leq i < k-1} a_i\). We consider a class of (randomized) perturbation mechanisms \(\mathcal{M}\) on a universe \(A\) and with range \(B\), and we denote by \(B(a)\) the range of the perturbed list \(b = \mathcal{M}(a)\) such that \(B(a) \subseteq B\); here, it is assumed that \(A = B\), that is \(b\) has the same structure as \(a\). As we have recalled in the introduction, a privacy loss occurs when an intruder can learn from the perturbed list \(b\) about an individual contributing to the original list \(a\). To quantify such a privacy loss, it is useful to consider two neighbouring lists \(a, a' \in A\), denoted by \(a \sim a'\), meaning that \(a'\) can be obtained from \(a\) by adding or removing exactly one individual. Then, the \((\varepsilon,\delta)\)-DP provides a suitable measure on how much can be learnt about any individual by taking the ratio between the likelihood of the perturbed list \(b = \mathcal{M}(a)\) and the likelihood of the neighbouring perturbed list \(b' = \mathcal{M}(a')\) (Dwork, 2006; Dwork and Roth, 2013). The LR may be alternatively viewed as a posterior odds ratio, or Bayes factor, from a Bayesian perspective. Placing an upper bound on such a LR motivates the definition of \((\varepsilon,\delta)\)-DP, and then leads to the following definition of \((\varepsilon,\delta)\)-DP mechanism.

**Definition 1.** (Dwork and Roth, 2013) For every \(\varepsilon,\delta \geq 0\) we say that a mechanism \(\mathcal{M}\) satisfies \((\varepsilon,\delta)\)-DP if for all \(a, a' \in A\) such that \(a \sim a'\) and all \(S \subseteq B\),

\[
\Pr[\mathcal{M}(a) \in S] \leq e^\varepsilon \Pr[\mathcal{M}(a') \in S] + \delta.
\] (1)

The definition of \(\varepsilon\)-DP by Dwork (2006) arises from Definition 1 by setting \(\delta = 0\). For small values of \(\varepsilon\), the \(\varepsilon\)-DP guarantees that the distribution of the perturbed list \(b\) is not affected by the data of any single individual. This leads to protect individuals' confidentiality against intruders, in the sense that the data of any single individual is not reflected in the released data \(b\). The definition of \((\varepsilon,\delta)\)-DP has been proposed as a relaxation of \(\varepsilon\)-DP to reduce confidentiality protection in a controlled way, and hence to increase the utility of the released data (Dwork and Roth, 2013). According to Definition 1, the parameter \(\delta\) adds flexibility to \(\varepsilon\)-DP by allowing \(b\) to have a probability \(\delta\) of having an undesirable LR with a higher associated disclosure risk. That is, \(\delta\) may be interpreted as the probability of data accidentally being leaked, thus suggesting that \(\delta\) should
be small. An implication of (1) is that with probability \( \delta \) the data may be released unperturbed, though the \((\varepsilon, \delta)\)-DP mechanism described in this paper never releases the whole unperturbed data set. In general, the choice of \( \varepsilon \) and \( \delta \) should take into account a balance between confidentiality and utility of the released data. See Dwork and Roth (2013) and Steinke and Ullman (2016) for a discussion on the choice of \( \delta \) in connection with the sample size \( n \) and the utility of the data. We refer to Rinott et al. (2017) for a comprehensive discussion of \((\varepsilon, \delta)\)-DP, as well as of other relaxations of \( \varepsilon \)-DP, in the context of frequency tables.

As a perturbation mechanism \( \mathcal{M} \), in this paper we consider a truncated version of the Laplace EM, which is arguably the most popular EM (Rinott et al., 2017). However, our results can be easily extended the broad class of truncated convolutional-type EMs. For \( \mathbf{a} \in \mathcal{A} \) and \( \mathbf{b} \in \mathcal{B}(\mathbf{a}) \) consider an additive utility function of the form \( u(\mathbf{a}, \mathbf{b}) = \sum_{1 \leq i \leq k} v(a_i, b_i) \) for some function \( v \), which enables us to specify a mechanism that perturbs the cells of a list \( \mathbf{a} \) independently, and impose that \( |a_i - b_i| \leq m \) for a truncation level \( m \in \mathbb{N}_0 \), for any \( i = 1, \ldots, k - 1 \). If \( w(\mathbf{b}; \mathbf{a}) \) is the conditional probability that the list \( \mathbf{a} \) is perturbed to the list \( \mathbf{b} \), then a truncated EM is defined as follows

\[
w(\cdot; \mathbf{a}) \propto \exp \left\{ \frac{u(\mathbf{a}, \cdot)}{\Delta u(\mathbf{a})} \right\},
\]

where \( \eta \) is a value that depends on \( \varepsilon \), and \( \Delta u \) is defined as \( \Delta u(\mathbf{a}) = \max_{\mathbf{b} \in \mathcal{B}(\mathbf{a}')} \max_{\mathbf{a} \in \mathcal{A}} |u(\mathbf{a}, \mathbf{b}) - u(\mathbf{a}', \mathbf{b})| \). According to (2), a truncated EM attaches higher probability to perturbed lists with higher utility. Truncated convolutional-type EMs correspond to the choice \( v(a_i, b_i) = g(a_i - b_i) \), for some function \( g \). In particular, the truncated Laplace EM is a truncated convolutional-type EM for the choice \( v(a_i, b_i) = -|a_i - b_i| \). The next theorem states that truncated convolutional-type truncated EMs are \((\varepsilon, \delta)\)-DP mechanism, with \( \delta > 0 \) depending on \( m \) and the utility function. We refer Rinott et al. (2017, Section 4) for details on the calculation of \( \delta \) as a function of \( \varepsilon \) and \( m \).

**Theorem 2.** (Rinott et al., 2017) Let \( u \) be an utility function of the form \( g(\mathbf{a} - \mathbf{b}) \) for some function \( g \), and let \( \mathcal{M} \) be a mechanism such that \( Pr[\mathcal{M}(\mathbf{a}) = \mathbf{b}] \propto \exp\{\varepsilon u(\mathbf{a}, \mathbf{b})/\Delta u(\mathbf{a})\} \) for all lists \( \mathbf{a} \in \mathcal{A} \) and all \( \mathbf{b} \in \mathcal{B}(\mathbf{a}) \) such that \( |a_i - b_i| \leq m \leq +\infty \) for any \( i = 1, \ldots, k - 1 \). Assume that for all \( \mathbf{a}, \mathbf{a}' \in \mathcal{A} \) such that \( \mathbf{a} \sim \mathbf{a}' \) it holds that \( Pr[\mathcal{M}(\mathbf{a}') = \mathbf{b}] = 0 \) implies \( Pr[\mathcal{M}(\mathbf{a}) = \mathbf{b}] < \delta \). Then the mechanism \( \mathcal{M} \) is \((\varepsilon, \delta)\)-DP, with \( \delta = 0 \) when \( m = +\infty \).

### 3 LR tests for goodness-of-fit under \((\varepsilon, \delta)\)-DP

For any \( n, k \in \mathbb{N} \) with \( k < n \), let \( (a_1, \ldots, a_k) \) be the list to be released, such that \( n = \sum_{1 \leq i \leq k} a_i \).

We consider the Multinomial model for \( \mathbf{a} \), that is the list \( \mathbf{a} \) is assumed to be the realization of the random variable \( \mathbf{A}_n = (A_{1,1}, \ldots, A_{k-1,n}) \) distributed as a Multinomial distribution with parameter \( n \in \mathbb{N} \) and \( \mathbf{p} = (p_1, \ldots, p_{k-1}) \in \Delta_{k-1} = \{ \mathbf{p} \in [0,1]^{k-1} : |\mathbf{p}| \leq 1 \} \), where \( |\mathbf{p}| = \sum_{1 \leq i \leq k-1} p_i \). Then,

\[
Pr[\mathbf{A}_n = \mathbf{a}] = \frac{n!}{(n - |a|)!} \left( \prod_{i=1}^{k-1} \frac{p_i^{a_i}}{a_i!} \right) (1 - |\mathbf{p}|)^{n - |a|} \mathbb{1}_{\mathcal{P}_{n,k-1}(\mathbf{a})},
\]

where \( \mathcal{P}_{n,k-1} := \{ \mathbf{a} \in \mathbb{N}_0^{k-1} : |\mathbf{a}| \leq n \} \). We assume that \( \mathbf{a} \) is perturbed by means of the truncated Laplace EM. More precisely, for \( \varepsilon \geq 0 \) and \( m \in \mathbb{N}_0 \) the cells of \( \mathbf{a} \) are perturbed independently through the conditional distribution

\[
w(b_i; a_i) = \frac{1}{c_{\varepsilon, m}} e^{-\varepsilon |b_i - a_i|} \mathbb{1}_{\{-m, \ldots , 0, \ldots , m\}}(|b_i - a_i|)
\]
for \(i = 1, \ldots, k-1\), where \(c_{\varepsilon,m} = \sum_{-m \leq l \leq m} \exp\{-\varepsilon|l|\}\), and \(b_k = n - |b|\). The resulting perturbed list \(\mathbf{b}\) has the same sample size \(n\) as the original list \(\mathbf{a}\), i.e. \(n = \sum_{1 \leq i \leq k} a_i = \sum_{1 \leq i \leq k} b_i\). According to Theorem 2, the truncated Laplace EM is \((\varepsilon, \delta)\)-DP with respect to the utility function \(u(\mathbf{a}, \mathbf{b}) = -\sum_{1 \leq i \leq k-1} |a_i - b_i|\), with \(\delta = c_{\varepsilon,m}^{-1}\exp\{-\varepsilon m\}\) (Rinott et al., 2017, Section 5). Throughout this paper, we assume \(m < n\), thus excluding the case \(m = +\infty\) that corresponds to \(\delta = 0\), i.e. the \(\varepsilon\)-DP. See Lemma 11 for details.

The “true” or natural model for \(\mathbf{b}\) is defined as a statistical model that takes into account the truncated Laplace EM. Because of the form of the distribution (3), the truncated Laplace EM may be viewed as adding, independently for each cell of \(\mathbf{a}\), random variables that are i.i.d. as a truncated (discrete) Laplace distribution. In particular, let \(\mathbf{L} = (L_1, \ldots, L_{k-1})\) be a random variable independent of \(\mathbf{A}_n\), and such that the \(L_i\)'s are i.i.d. according to

\[
\Pr[L = l] = \frac{1}{c_{\varepsilon,m}} \exp\{-\varepsilon|l|\} \mathbb{1}_{(-m, \ldots, m)}(l),
\]

i.e. the \(m\)-truncated Laplace distribution with location 0 and scale \(\varepsilon^{-1}\). Then, the “true” model assumes that \(\mathbf{b}\) is the realization of a random variable whose distribution is the convolution between the distributions of \(\mathbf{A}_n\) and \(\mathbf{L}\), i.e. the distribution of \(\mathbf{B}_n = (B_{1,n}, \ldots, B_{k-1,n})\) with \(B_{i,n} = A_{i,n} + L_i\) for \(i = 1, \ldots, k-1\), and \(B_{k,n} = n - |\mathbf{B}_n|\). The corresponding likelihood function is

\[
L_{n,T}(\mathbf{p}; \mathbf{b}) \propto \left( \prod_{i=1}^{k-1} \sum_{l_i=-m \vee (b_i-n)}^{m \wedge b_i} \frac{p_i^{b_i-l_i} \exp\{-\varepsilon|l_i|\}}{(b_i-l_i)!} \right) (1 - |\mathbf{p}|)^{n-|\mathbf{b}|}.
\]

Under the “true” model, we make use of the LR test to assess goodness-of-fit in the form: \(H_0 : \mathbf{p} = \mathbf{p}_0\), for a fixed \(\mathbf{p}_0 \in \Delta_{k-1}\), against \(H_1 : \mathbf{p} \neq \mathbf{p}_0\). Without loss of generality, we assume that \(\mathbf{p}_0\) belongs to the interior of \(\Delta_{k-1}\). We present a careful large sample analysis, in terms of Edgeworth expansions, of the distribution of the LR and of the power of the corresponding LR test. Such an analysis is new for likelihood functions in the convolutional form (5), and it is definitely a challenging task due to the fact that the “true” model is discrete, multidimensional and not belonging to the exponential family.

While we focus on the popular truncated Laplace EM, our analysis and results can be easily extended to any truncated convolutional-type EMs. An example is the truncated Gaussian EM, such that for any \(\varepsilon \geq 0\) and \(m \in \mathbb{N}_0\) the cells of the list \(\mathbf{a}\) are perturbed independently through the conditional distribution

\[
w(b_i; a_i) = \frac{1}{d_{\varepsilon,m}} \exp\{-\varepsilon(l^2/(2m+1))\} \mathbb{1}_{(-m, \ldots, a_m)}(b_i - a_i)
\]

for \(i = 1, \ldots, k-1\), where \(d_{\varepsilon,m} = \sum_{-m \leq l \leq m} \exp\{-\varepsilon l^2/(2m+1)\}\), and \(b_k = n - |b|\). According to Theorem 2, the truncated Gaussian EM is \((\varepsilon, \delta)\)-DP with respect to the utility function \(u(\mathbf{a}, \mathbf{b}) = -\sum_{1 \leq i \leq k-1} (a_i - b_i)^2\), with \(\delta = d_{\varepsilon,m}^{-1}\exp\{-\varepsilon m^2/(2m+1)\}\). The truncated Gaussian EM corresponds to add, independently for each cell of \(\mathbf{a}\), random variables that are i.i.d. as the \(m\)-truncated (discrete) Gaussian distribution with location 0 and (squared) scale \((2m+1)/2\varepsilon\), i.e.

\[
\Pr[G = g] = \frac{1}{d_{\varepsilon,m}} \exp\{-\varepsilon g^2\} \mathbb{1}_{(-m, \ldots, m)}(g).
\]

The resulting “true” model for \(\mathbf{b}\) has a likelihood function in a convolutional form similar to (5), and therefore our analysis and results under the truncated Laplace EM can be easily adapted to the truncated Gaussian EM. We refer to Rinott et al. (2017) for other examples of truncated convolutional-type EMs.
3.1 The “true” LR test, and a “private” Wilks theorem

For any parameter $p \in \Delta_{k-1}$, we denote by $\hat{p}_{n,T}$ the maximum likelihood estimator of $p$ under the “true” model for the perturbed list $b$. Such an estimator is not available in a closed-form expression. Then, we define the “true” LR as follows

$$
\Lambda_{n,T}(p_0) = 2 \log \left( \frac{L_{n,T}(\hat{p}_{n,T}; B_n)}{L_{n,T}(p_0; B_n)} \right),
$$

and denote by $\Pr[\Lambda_{n,T}(p_0) \in \cdot; p_0]$ the distribution of $\Lambda_{n,T}(p_0)$ computed with respect to $A_n$, distributed as a Multinomial distribution with parameter $(n, p_0)$. For any $p_0$ in the interior of $\Delta_{k-1}$, let $\Lambda_n(p_0)$ be the Multinomial LR, namely the LR in the absence of perturbation, which is obtained from $\Lambda_{n,T}(p_0)$ by setting $m = 0$ and/or $\varepsilon \to +\infty$. If $F_n(t) = \Pr[\Lambda_n(p_0) \leq t; p_0]$ and $K_k$ is the cumulative distribution function of a chi-squared distribution with $k \geq 1$ degrees of freedom, then a classical result by Wilks (1938) shows that for any $t > 0$

$$
\lim_{n \to +\infty} F_n(t) = K_{k-1}(t).
$$

See also Ferguson (2002, Chapter 4), and references therein, for a detailed account on Wilks’ theorem and generalizations thereof. The next theorem establishes an Edgeworth expansion, with respect to $n \to +\infty$, of the distribution of $\Lambda_{n,T}(p_0)$. Such a result provides a “private” and refined version of (8).

**Theorem 3.** Let $F_{n,T}(t) = \Pr[\Lambda_{n,T}(p_0) \leq t; p_0]$ be the cumulative distribution function of $\Lambda_{n,T}(p_0)$. For any $n, k \in \mathbb{N}$ and any $t > 0$ it holds true that

$$
F_{n,T}(t) = K_{k-1}(t) + \frac{c_{1,k,p_0}(t)}{n^{1/2}} + \frac{c_{s,k,p_0}(t; m, \varepsilon)}{\sqrt{2\pi n}} + \frac{c_{2,k,p_0}(t)}{n} + R_{(k,n),p_0,T}(t),
$$

with

$$
|R_{(k,n),p_0,T}(t)| \leq \frac{C_{k,p_0,T}(t; m, \varepsilon)}{n^{3/2}},
$$

where the functions $c_{1,k,p_0}$ and $c_{2,k,p_0}$ are independent of $n$ and of the distribution of $L$, i.e. independent of $\varepsilon$ and $m$, the function $c_{s,k,p_0}$ is independent of $n$ and such that $|c_{s,k,p_0}| \leq (k - 1)$, and the function $C_{k,p_0,T}$ is independent of $n$.

See Appendix A and Appendix B for the proof of Theorem 3 with $k = 2$ and $k > 2$, respectively. The functions $c_{1,k,p_0}$, $c_{s,k,p_0}$ and $c_{2,k,p_0}$ in Theorem 3, as well as $C_{k,p_0,T}$, can be made explicit by gathering some equations in the proof. In particular, $c_{1,k,p_0}$ and $c_{2,k,p_0}$ are the same that would appear in the Edgeworth expansion of the distribution of $\Lambda_n(p_0)$, which ensues as a corollary of (9) by setting $m = 0$ and/or $\varepsilon \to +\infty$. For a fixed (reference) level of significance $\alpha \in (0, 1)$, the “true” LR test to assess goodness-of-fit has a rejection region \{ $\Lambda_{n,T} > \lambda_T(\alpha)$\}, with the critical point $\lambda_T(\alpha)$ being determined in such a way that $F_{n,T}(\lambda_T(\alpha)) = 1 - \alpha$. According to Theorem 3, if $n$ increases then the distribution $F_{n,T}$ of $\Lambda_{n,T}$ becomes close to $K_{k-1}$, and for a fixed $n \geq 1$ the parameters $\varepsilon$ and $m$ interact with $k$ and $n$ to control such a closeness. For a fixed $n \geq 1$, Theorem 3 allows us to determine $\lambda_T(\alpha)$ as a function of $\varepsilon$ and $m$, thus quantifying the contribution of the truncated Laplace EM, in terms $\varepsilon$ and $m$, to the rejection region of the “true” LR test. From (9), the truncated Laplace EM gives its most significant contribution in the term of order $n^{-3/2}$, whereas in the limit its contribution vanishes. Hence, we write

$$
\lambda_T(\alpha) = \lambda(\alpha) + \frac{\tilde{c}_{1,k,p_0}(\alpha)}{n^{1/2}} + \frac{\tilde{c}_{2,k,p_0}(\alpha)}{n} + \frac{\tilde{c}_{3,k,p_0}(n; \alpha; \varepsilon, m)}{n^{3/2}},
$$

where $\lambda(\alpha)$ is the critical point of the “true” LR test, and $\tilde{c}_{1,k,p_0}(\alpha)$, $\tilde{c}_{2,k,p_0}(\alpha)$, and $\tilde{c}_{3,k,p_0}(n; \alpha; \varepsilon, m)$ are functions of $\alpha$ and $m$ that quantify the contribution of the truncated Laplace EM.
where \( \lambda, \tilde{c}_{1,k,p_0}, \) and \( \tilde{c}_{2,k,p_0} \) are positive constants independent of \( n, \varepsilon \) and \( m \), while \( \tilde{c}_{3,k,p_0} \) is, for fixed \( \alpha, \varepsilon \) and \( m \), a bounded function of \( n \). Along the same lines of the proof of Theorem 3, it is easy to show that an analogous result holds true for the truncated Gaussian EM, and in general for any truncated convolutional-type EMs. Theorem 3 provides a preliminary result to the study of the power of the “true” LR. A further discussion of Theorem 3 is deferred to Section 4, with respect to a comparison between the rejection regions defined under the “true” model and the “naive” model, the latter being a statistical model that does not take into account the truncated Laplace EM.

### 3.2 The power of the “true” LR test, and a “private” Bahadur-Rao theorem

For a fixed \( \alpha \in (0,1) \), Theorem 3 provides the critical point \( \lambda_T(\alpha) \) of the “true” LR test. Then assuming \( p_0 \) and \( p_1 \) in the interior of \( \Delta_{k-1} \), and such that \( p_1 \neq p_0 \), the power of the “true” LR test with respect to \( p_1 \) is defined as

\[
\beta_{n,T}(p_1; \alpha) = \Pr[\Lambda_{n,T}(p_0) > \lambda_T(\alpha); p_1],
\]

with \( \Pr[\Lambda_{n,T}(p_0) \in \cdot; p_1] \) being computed with respect to \( A_n \) distributed as a Multinomial distribution with parameter \( (n, p_1) \). The Kullback-Leibler divergence between two Multinomial distributions of parameters \( (1, p) \) and \( (1, q) \) is

\[
D_{KL}(p \parallel q) = E_p \left\{ \sum_{i=1}^{k} \log \left( \frac{M_i}{q_i} \right) + \log \left( \frac{p_i}{q_i} \right) \right\} = \sum_{i=1}^{k} p_i \log \frac{p_i}{q_i},
\]

where \( p_k = 1 - |p|, q_k = 1 - |q| \) and \( M_k = 1 - \sum_{1 \leq i \leq k-1} M_i \). We denote by \( \beta_n(p_1; \alpha) \) the power of the Multinomial LR test with rejection region \( \{\Lambda_n(p_0) > \lambda(\alpha)\} \). A classical result from Rao (1962) and Bahadur (1967) shows that

\[
\lim_{n \to +\infty} -\frac{1}{n} \log(1 - \beta_n(p_1; \alpha)) = D_{KL}(p_0 \parallel p_1). \tag{10}
\]

See Hoeffding (1965, 1967) for refinements of (10) in terms of Bahadur-Rao large deviation expansions of \( -n^{-1} \log(1 - \beta_n(p_1; \alpha)) \), which introduce the dependence on \( \alpha \) in the right-hand side of (10) (Efron, 1967; Efron and Truax, 1968).

We show that the large sample behaviour (10) holds true under the “true” model for \( b \). Precisely, for a fixed \( p_1 \in \Delta_{k-1} \) such that \( p_1 \neq p_0 \) we show that

\[
\lim_{n \to +\infty} -\frac{1}{n} \log(1 - \beta_{n,T}(p_1; \alpha)) = D_{KL}(p_0 \parallel p_1). \tag{11}
\]

Note that Equation (11) does not show explicitly the contribution of the truncated Laplace EM to the power \( \beta_{n,T}(p_1; \alpha) \) of the “true” LR test, hiding the parameters \( \varepsilon \) and \( m \), as well as it does not show any contribution of the (reference) level of significance \( \alpha \). In other terms, the right-hand side of (11) does not depend on \( \varepsilon, m \) and \( \alpha \). To bring out the contribution of the truncated Laplace EM to \( \beta_{n,T}(p_1; \alpha) \), we introduce a refinement of (11) in the sense of Hoeffding (1965, 1967). In particular, in the next theorem we rely on non-standard large deviation analysis (Petrov, 1975; Saulis and Statulevicius, 1991; von Bahr, 1967) in order to establish a Bahadur-Rao large deviation expansion of \( n^{-1} \log(1 - \beta_{n,T}(p_1; \alpha)) \). Our result thus provides a “private” and refined version of (10). This is a critical tool, as it leads to a quantitative characterization of the tradeoff between
confidentiality, measured via $\varepsilon$ and $m$, and utility of the data, measured via the power of the “true” LR test. As a corollary, the next theorem allows us to define the (sample) cost of $(\varepsilon, \delta)$-DP for the “true” LR test, that is the additional sample size that is required in order to recover the power of the Multinomial LR test.

**Theorem 4.** For a fixed (reference) level of significance $\alpha$, let $\lambda(\alpha)$ be the critical point such that $K_{k-1}(\lambda(\alpha)) = 1 - \alpha$. For any $k \in \mathbb{N}$, it holds true that

$$- \frac{1}{n} \log(1 - \beta_{n,T}(p_1; \alpha))$$

$$= D_{KL}(p_0 \parallel p_1) + \frac{c_{1,k,(p_0,p_1)}(\lambda(\alpha))}{\sqrt{n}} + \frac{k}{4} \left( \log \frac{n}{\alpha} \right) + \frac{c_{2,k,(p_0,p_1)}(\lambda(\alpha))}{n}$$

as $n \to +\infty$, where $\mathcal{M}_L(z) = E[\exp \{ Lz \}] \geq 1$, i.e. the moment generating function of $L$ evaluated in $z$, and $c_{1,k,(p_0,p_1)}$ and $c_{2,k,(p_0,p_1)}$ are constants independent of $n$.

See Appendix A and Appendix B for the proof of Theorem 4 with $k = 2$ and $k > 2$, respectively. The constants $c_{1,k,(p_0,p_1)}$ and $c_{2,k,(p_0,p_1)}$ in Theorem 4 can be made explicit by gathering equations in the proof. The proof of (12) relies on a novel (sharp) large deviation principle for sum of i.i.d. random vectors, which is of independent interest. In particular, consider a $C^2$-regular compact and convex subset $\mathcal{E} \subset \mathbb{R}^d$, such that there exists a function $h \in C^2(\mathbb{R}^d)$ for which: i) $\mathcal{E} = \{ x \in \mathbb{R}^d \mid h(x) \leq 0 \}$; ii) $\partial \mathcal{E} = \{ x \in \mathbb{R}^d \mid h(x) = 0 \}$ is a $C^2$-hyperplane; iii) $\nabla h$ does not vanish on $\partial \mathcal{E}$; iv) the hypersurface $\partial \mathcal{E}$ is oriented by the normal field $|\nabla h(x)|^{-1} \nabla h(x)$. Let $\{X_i\}_{i \geq 1}$ be a sequence of i.i.d. $d$-dimensional random variables such that $E[e^{\mathbf{x}X_1}] < +\infty$ for all $\mathbf{z}$ with $|\mathbf{z}| < H$ for some $H > 0$, and such that $E[X_1] = 0$. Set $L(\mathbf{z}) := \log E[e^{\mathbf{z}X_1}]$. If, for any given a vector $\mathbf{\xi} \in \mathbb{R}^d$, there exists a unique $\hat{\mathbf{z}} \in \mathbb{R}^d$ with $|\hat{\mathbf{z}}| < H$ for which $\nabla L(\hat{\mathbf{z}}) = \mathbf{\xi}$, then it holds true that

$$\Pr \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \in \sqrt{n} \mathbf{\xi} + \mathcal{E} \right]$$

$$= \exp \left\{ -n[\mathbf{z}\nabla L(\hat{\mathbf{z}}) - L(\hat{\mathbf{z}})] + \sqrt{n} \min_{\mathbf{v} \in \mathcal{E}} \mathbf{z} \mathbf{v} \right\} n^{-(d+1)/4}[1 + o(1)],$$

as $n \to +\infty$, where $d$ denotes the dimension of the $X_i$’s. See Appendix B for the proof of (13). Analogous large deviation principles, though not useful in our specific context, are in Aleshkyavichene (1983), Osipov (1982), Saulis (1983), von Bahr (1967). See also Saulis and Statulevicius (1991) and references therein.

**Remark 5.** Under the Multinomial LR test, Theorem 4 provides a refinement of (10). This is obtained from (12) by setting $L = (0, \ldots, 0)$, which implies

$$\log \mathcal{M}_L(\nabla p_0 D_{KL}(p_0 \parallel p_1)) = 0.$$

The resulting expression is a novel Bahadur-Rao large deviation expansion of $n^{-1} \log(1 - \beta_n(p_1; \alpha))$, which improves on the results of Hoeffding (1965, 1967).

Theorem 4 shows that, as $n$ increases, $-n^{-1} \log(1 - \beta_n(p_1; \alpha))$ becomes close to $D_{KL}(p_0 \parallel p_1)$, and, for a fixed $n \geq 1$, the parameters $\varepsilon$ and $m$ interact with $k$ and $n$ in order to control such a closeness. Empirical analyses in Rinott et al. (2017) show that, for a fixed $n \geq 1$ and $\alpha \in (0, 1)$,
the power $\beta_{n,T}(p_1;\alpha)$ decreases as $\varepsilon$ decreases and/or $m$ increases; such a behaviour agrees with intuition, as decreasing $\varepsilon$ and/or increasing $m$ leads to increase the perturbation in the data. Theorem 4 provides a theoretical guarantee to the analyses in Rinott et al. (2017) by quantifying the contribution of the truncated Laplace EM, in terms of $\varepsilon$ and $m$, to the power of the “true” LR test. According to (12), the truncated Laplace EM gives its most significant contribution in the term of order $n^{-1}$, whereas in the limit its contribution vanishes. That is,

$$L(p_0, p_1) (\varepsilon, m) = \log \mathcal{M}_L (\nabla p_0 \mathcal{D}_{KL}(p_0 \parallel p_1))$$

determines the loss in the power of the “true” LR test. Since $L(p_0, p_1) (\varepsilon, m)$ increases as $\varepsilon$ decreases and/or $m$ increases, then the power of the “true” LR test decreases under such a behavior of $\varepsilon$ and $m$. See Figure 1 for an illustration of such a behaviour in the problem of testing a Uniform ($U$) $p_0$, with $k = 2$ and $k = 4$, versus some alternatives for $p_1$’s. Along the same lines of the proof of Theorem 4, it is easy to show that an analogous result holds true for the truncated Gaussian EM, with $L$ being replaced by $G = (G_1, \ldots, G_{k-1})$ such that the $G_i$’s are random variables i.i.d. according to (6). For fixed values of $\varepsilon$ and $\delta$, that is fixed levels of $(\varepsilon, \delta)$-DP, for the same tests considered in Figure 1, Table 1 shows that the truncated Laplace EM produces a smaller decrease in the power than the truncated Gaussian EM.

Figure 1: Values of $L(p_0, p_1) (\varepsilon, m)$ as a function of $\varepsilon \geq 0$ for $m = 5, 10, 15, 20$: $H_0 : p = 0.5$ vs $H_1 : p = 0.1$ (top-left panel); $H_0 : p = 0.5$ vs $H_1 : p = 0.4$ (top-right panel); $H_0 : p = (0.25, 0.25, 0.25)$ vs $H_1 : p = (0.45, 0.45, 0.05)$ (bottom-left panel); $H_0 : p = (0.25, 0.25, 0.25)$ vs $H_1 : p = (0.85, 0.05, 0.05)$ (bottom-right panel).
Remark 6. Theorem 4 shows that the contribution of the truncated Laplace EM to the power of the “true” LR test vanishes in the limit $n \to +\infty$, leading to (11). In particular, given a constant $C > 0$, a value of $(\varepsilon, m)$ that solve

$$\mathcal{L}_{(p_0, p_1)}(\varepsilon, m) = nC,$$

say $(\varepsilon_n, m_n)$, provides the correct scaling for the parameters $\varepsilon$ and $m$, with respect to the sample size $n$, so that the contribution of the truncate Laplace EM to the power of the “true” LR test does no longer vanish in the large $n$ limit.

| $(\varepsilon, \delta)$ | $m$ | $\mathcal{L}_{(U,0.1)}(\varepsilon, m)$ | $\mathcal{L}_{(U,0.4)}(\varepsilon, m)$ | $\mathcal{L}_{(U,(0.45,0.45,0.05)}(\varepsilon, m)$ | $\mathcal{L}_{(U,(0.85,0.05,0.05)}(\varepsilon, m)$ |
|-------------------------|------|---------------------------------|----------------------------------|---------------------------------|----------------------------------|
| $(0.025, 0.09)$         | 5 5  | 8.652 8.674 0.698 0.707 17.303 17.349 11.773 11.796 |
| $(0.025, 0.04)$         | 10 10 | 18.927 18.972 2.037 2.069 37.854 37.944 25.228 25.274 |
| $(0.05, 0.08)$          | 5 5  | 8.596 8.642 0.6789 0.697 17.191 17.284 11.715 11.762 |
| $(0.05, 0.04)$          | 10 10 | 18.802 18.899 1.962 2.029 37.605 37.795 25.101 25.197 |
| $(0.075, 0.08)$         | 5 5  | 8.538 8.610 0.660 0.687 17.076 17.219 11.656 11.729 |
| $(0.075, 0.03)$         | 10 11 | 18.672 20.902 1.885 2.281 37.345 41.803 24.970 27.836 |
| $(0.1, 0.07)$           | 5 6  | 8.479 10.573 0.641 0.903 16.958 21.147 11.595 14.327 |
| $(0.1, 0.03)$           | 10 12 | 18.536 22.893 1.807 2.525 37.073 45.786 24.833 30.462 |
| $(0.125, 0.07)$         | 5 6  | 8.419 10.531 0.622 0.888 16.837 21.063 11.533 14.283 |
| $(0.125, 0.02)$         | 10 12 | 18.396 22.795 1.727 2.470 36.792 45.591 24.690 30.362 |
| $(0.150, 0.06)$         | 5 6  | 8.357 10.489 0.603 0.874 16.713 20.978 11.469 14.240 |
| $(0.150, 0.02)$         | 10 12 | 18.250 22.696 1.646 2.414 36.499 45.391 24.542 30.261 |
| $(0.175, 0.06)$         | 5 6  | 8.293 10.446 0.584 0.859 16.587 20.892 11.404 14.195 |
| $(0.175, 0.02)$         | 10 12 | 18.098 22.595 1.565 2.358 36.197 45.189 24.389 30.158 |
| $(0.2, 0.05)$           | 5 6  | 8.228 10.403 0.565 0.845 16.457 20.801 11.337 14.150 |
| $(0.2, 0.02)$           | 10 13 | 17.942 24.536 1.483 2.566 35.884 49.072 24.231 32.734 |

Table 1: Values of $\mathcal{L}_{(p_0, p_1)}(\varepsilon, m)$ for fixed levels of $(\varepsilon, \delta)$-DP under the truncated Laplace EM (L) and the truncated Gaussian EM (G); $H_0 : p = 0.5$ vs $H_1 : p = 0.1$ (third column); $H_0 : p = 0.5$ vs $H_1 : p = 0.4$ (fourth column); $H_0 : p = (0.25, 0.25, 0.25)$ vs $H_1 : p = (0.45, 0.45, 0.05)$ (fifth column); $H_0 : p = (0.25, 0.25, 0.25)$ vs $H_1 : p = (0.85, 0.05, 0.05)$ (sixth column).

Theorem 4 allows us to define the loss of power of the “true” LR test, which admits a natural formulation as the additional sample size that is required in the “true” LR test in order to recover the power of the Multinomial LR test. According to Theorem 4, for any fixed value $\beta > 0$ of the power, there exist: i) an integer $\tilde{n}_{(p_0,p_1)}(\alpha, \beta)$ such that, for $n \geq \tilde{n}_{(p_0,p_1)}(\alpha, \beta)$, the LR test in the absence of perturbation has eventually power not less than $\beta$; ii) an integer $\tilde{n}_{(p_0,p_1),T}(\alpha, \beta)$ such that, for $n \geq \tilde{n}_{(p_0,p_1),T}(\alpha, \beta)$, the “true” LR test has eventually power not less than $\beta$. That is, $\tilde{n}_{(p_0,p_1)}(\alpha, \beta)$ and $\tilde{n}_{(p_0,p_1),T}(\alpha, \beta)$ provide lower bounds for the sample size $n$ required to obtain a power (at least equal to) $\beta$. Accordingly, for any $p_1 \in \Delta_{k-1}$ such that $p_1 \neq p_0$, we can make use of the large $n$ asymptotics of $\beta_n(p_1, \alpha)$ and $\beta_n,T(p_1, \alpha)$ in Theorem 4 to quantify $\tilde{n}_{(p_0,p_1)}(\alpha, \beta)$ and $\tilde{n}_{(p_0,p_1),T}(\alpha, \beta)$ and, above all, the gap between these lower bounds. In particular, from (12) we can write

$$\tilde{n}_{(p_0,p_1),T}(\alpha, \beta) \simeq \tilde{n}_{(p_0,p_1)}(\alpha, \beta) + R_{(p_1,p_0)}(\varepsilon, m)$$
with
\[ R(p_1, p_0)(\varepsilon, m) = \tilde{n}(p_0, p_1)(\alpha, \beta) \frac{\mathcal{L}(p_0, p_1)(\varepsilon, m)}{D_{KL}(p_0 \| p_1)}, \]
where \( \simeq \) means that \( \tilde{n}(p_0, p_1), \tau(\alpha, \beta) \) and \( \tilde{n}(p_0, p_1)(\alpha, \beta) \) are obtained after dropping the \( o(n^{-1}) \)-term on the right-hand side of (12). Therefore, the gap \( R(p_1, p_0)(\varepsilon, m) \) between \( \tilde{n}(p_0, p_1), \tau(\alpha, \beta) \) and \( \tilde{n}(p_0, p_1)(\alpha, \beta) \) takes on the interpretation of the (sample) cost of \( (\varepsilon, \delta) \)-DP under the “true” LR test. That is, the additional sample size required to recover the power of the Multinomial LR test.

We conclude with a brief discussion on the effect of the dimension \( k > 1 \) on the power of the “true” LR test. Theorem 4 shows how the sample size \( n \) interacts with the truncated Laplace EM. However, according to Theorem 4, also \( k \) interacts with the truncated Laplace EM, still in terms of the parameters \( \varepsilon \) and \( m \), thus having a direct effect on the power of the “true” LR test. By maintaining the sample size \( n \) large enough, we assume that \( k \) increases moderately. Differently from the sample size \( n \), any increase of the dimension \( k \) determines a corresponding change of \( p_0 \) and \( p_1 \), as these probabilities are elements of \( \Delta_{k-1} \). Therefore, instead of the sole \( k \), it is critical to consider a suitable function of \( k \), \( p_0 \) and \( p_1 \), say \( \nu_k(p_0, p_1) \), and then show how such a function interacts with the truncated Laplace EM. From (12), it is clear that the interaction between \( \nu_k(p_0, p_1) \) and the parameters \( \varepsilon \) and \( m \) is determined by \( -n^{-1} \mathcal{L}(p_0, p_1)(\varepsilon, m) \). The behaviour of such a term depends on the specific form of the moment generating function of the Laplace distribution (4), which turns out to be a convex, even function with a global minimum at 0. The larger \( m \) the flatter the \( U \)-shape of this function around its minimum. Since \( \mathcal{L}(p_0, p_1)(\varepsilon, m) \) is the sum of \( k \) logarithmic terms, it is critical to count the number of terms in the sum that are not of the form \( \log(1 + \omega) \), for some small \( \omega \). That is, the number of “large-scale terms”:

\[ \nu_k(p_0, p_1)(\varepsilon, \eta) := \# \left\{ i = 1, \ldots, k : \left| \frac{\partial}{\partial p_0,i} D_{KL}(p_0 \| p_1) \right| \geq \varepsilon(1 + \eta) \right\}. \]

By a suitable tuning of \( \eta \), the quantity \( \nu_k(p_0, p_1)(\varepsilon, \eta) \) gives the number of large-scale terms. The next proposition shows how \( \nu_k(p_0, p_1)(\varepsilon, \eta) \) interacts with the truncated Laplace EM, and its impact on the power of the “true” LR test.

**Proposition 7.** Under the assumptions of Theorem 4, for a given \( \eta > 0 \), it holds that

1) if \( \nu_k(p_0, p_1)(\varepsilon, \eta) > 0 \) then the contribution of \( \mathcal{L}(p_0, p_1)(\varepsilon, m) \) to the power of the “true” LR test decreases at least proportionally to the quantity

\[ \nu_k(p_0, p_1)(\varepsilon, \eta) \log E[\exp \{ L_1 \varepsilon(1 + \eta) \}]; \]

2) if \( \nu_k(p_0, p_1)(\varepsilon, \eta) = 0 \) and \( \varepsilon^2(1 + \eta)^2 \text{Var}[L] < 1 \) then the contribution of \( \mathcal{L}(p_0, p_1)(\varepsilon, m) \) to the power of the “true” LR decreases proportionally to

\[ k\varepsilon^2(1 + \eta)^2 \text{Var}[L]. \]

See Appendix B for the proof of Proposition 7. According to Proposition 7, the choice of the parameter \( \eta > 0 \) determines a “large scale” and a “small scale” for the terms that are involved in the sum defined through \( \mathcal{L}(p_0, p_1)(\varepsilon, m) \). Under the large scale behaviour of \( \mathcal{L}(p_0, p_1)(\varepsilon, m) \), i.e. \( \nu_k(p_0, p_1)(\varepsilon, \eta) > 0 \), Proposition 7 shows that there exists an interaction between \( \nu_k(p_0, p_1)(\varepsilon, \eta) \) and the parameters \( \varepsilon \) and \( m \), and that such an interaction is with respect to \( \log E[\exp \{ L_1 \varepsilon(1 + \eta) \}] \). Under the small scale behaviour of \( \mathcal{L}(p_0, p_1)(\varepsilon, m) \), i.e. \( \nu_k(p_0, p_1)(\varepsilon, \eta) = 0 \), Proposition 7 shows that the exists an interaction between \( k \) and the parameters \( \varepsilon \) and \( m \), and that such an interaction is
with respect to $\text{Var}[L]$. The small scale behaviour thus leads to an interaction directly in terms of the dimension $k$, thus showing clearly the effect of $k$ on the power of the “true” LR test. In particular, under the small scale behaviour, the contribution of $\mathcal{L}_{(p_0,p_1)}(\varepsilon,m)$ to the power of the “true” LR test decreases proportionally to $k\varepsilon^2(1+\eta)^2\text{Var}[L]$. That is, the smaller $k$ and the smaller $\text{Var}[L]$, i.e. the larger $\varepsilon$ and/or the smaller $m$, the smaller is the contribution of $-n^{-1}\mathcal{L}_{(p_0,p_1)}(\varepsilon,m)$ to the power of the “true” LR test.

4 A “naïve” LR test for goodness-of-fit under $(\varepsilon,\delta)$-DP

As in Section 3, we assume that $a = (a_1,\ldots,a_k)$ is perturbed by means of the truncated Laplace EM, in such a way that the resulting perturbed list $b$ has the same sample size $n$ as the original list $a$. Then, to avoid the introduction of negative values, we assume that $b$ is post-processed in such a way that the negative values of $b$ are replaced by zeros (Rinott et al., 2017, Section 6). That is, we define $b^+ = (b_1^+,\ldots,b_k^+)$, where $b_i^+ = \max\{0, b_i\} \geq 0$ for $i = 1,\ldots,k-1$ and $b_{k,n} = \max\{0, n - \sum_{1 \leq i \leq k-1} b_i^+\}$. Note that the sample size $|b^+|$ of $b^+$ may be different from the sample size $n$ of $a$. Assuming that $b$ is modeled as the “true” model described in Section 3, i.e. $b$ is the realization of the random variable $B = (B_{1,n},\ldots,B_{k-1,n})$ and $B_{k,n} = n - |B_n|$, it is natural to model $b^+$ as the realization of the random variable $B^+ = (B_{1,n}^+,\ldots,B_{k,n}^+)$ with $B_{i,n} = \max\{0, B_{i,n}\} \geq 0$ for $i = 1,\ldots,k-1$ and $B_{k,n}^+ = \max\{0, n - \sum_{1 \leq i \leq k-1} B_{i,n}^+\}$. Therefore, differently from $B$, the sample size $|B^+|$ of $B^+$ is random. Following the work of Rinott et al. (2017), we consider a “naïve” model for the perturbed list $b^+$, that is a statistical model that does not take into account the truncated Laplace EM. The “naïve” model assumes that $b^+$ is modeled as the distribution of the random variable $B^+ = (B_{1,n}^+,\ldots,B_{k,n}^+)$ such that the conditional distribution of $B^+$ given $|B^+| = n^+$ is a Multinomial distribution with parameter $(p,n^+)$. Under the “naïve” model for $b^+$, we make use of the LR test to assess goodness-of-fit in the form: $H_0 : p = p_0$, for a fixed $p_0 \in \Delta_{k-1}$, against $H_1 : p \neq p_0$. As in Section 3, without loss of generality, we assume that $p_0$ belongs to the interior of $\Delta_{k-1}$.

Remark 8. A straightforward application of classical concentration inequalities for the Multinomial distribution entails that, for any fixed $n \geq 1$, the probability of the event $\{B^+ \neq B\}$ can be bounded by an exponential term of the form $e^{-\tau n}$, for a suitable choice of $\tau > 0$. This result shows that the additional randomness carried by the sample size $|B^+|$ does not contribute in any significant way to the Edgeworth expansions of the distribution of the “naïve” LR in Theorem 9 and Theorem 10 below, where only terms of $O(n^{-1/2})$ and $O(n^{-1})$-type are relevant. We refer to Remark 17 for further details.

Under the “naïve” model for the post-processed list $b^+$, the likelihood function is

$$L_{n^+,N}(p;b^+) \propto \left(\prod_{i=1}^{k-1} \frac{P_i^j}{b_i^+!}\right) (1 - |p|)^{b_k^+}.$$  \hspace{1cm} (14)

Now, for the sake of simplicity, we start developing an asymptotic analysis under the assumption that $k = 2$. That is, $A_n$ has a Binomial distribution with parameter $(n,p)$, $L$ is independent of $A_n$ and distributed according to (4), and $B_n = A_n + L$. Under the “naïve” model with $k = 2$, we make use of the LR test to assess goodness-of-fit in the form: $H_0 : p = p_0$, for a fixed $p_0 \in (0,1)$, against $H_1 : p \neq p_0$. Recall that the LR under the “true” model is given by (7) with $k = 2$. We denote by
\[ \hat{p}_{n+} \text{ the maximum likelihood estimator of } p \text{ under the “naïve” model for } b^+, \text{ that is the estimator obtained through the maximization of the likelihood function (14) with } k = 2. \text{ Then, we define} \]

\[ \Lambda_{n+,N}(p_0) = 2 \log \left( \frac{L_{n+,N}(\hat{p}_{n+,N}; B_n)}{L_{n+,N}(p_0; B_n)} \right), \]

which is referred to as the “naïve” LR, and we denote by \( \Pr[\Lambda_{n+,N}(p_0) \leq t; p_0] \) the distribution of \( \Lambda_{n+,N}(p_0) \) computed with respect to \( A_n \) distributed as a Binomial distribution with parameter \((n, p_0)\). The next theorem establishes an Edgeworth expansion, with respect to \( n \to +\infty \), for the distribution of \( \Lambda_{n+,N}(p_0) \). To emphasize the comparison with respect to the “true” model, we also include the Edgeworth expansion of the distribution of \( \Lambda_{n,T}(p_0) \), which follows from Theorem 3 with \( k = 2 \). We denote by \( K \) the cumulative distribution function of a chi-squared distribution with 1 degree of freedom.

**Theorem 9.** Let \( F_{n,T}(t) = \Pr[\Lambda_{n,T}(p_0) \leq t; p_0] \) be the cumulative distribution function of \( \Lambda_{n,T}(p_0) \), and let \( F_{n+,N}(t) = \Pr[\Lambda_{n+,N}(p_0) \leq t; p_0] \) be the cumulative distribution function of \( \Lambda_{n+,N}(p_0) \). For any \( n \in \mathbb{N} \) and any \( t > 0 \)

\[ F_{n,T}(t) = K(t) + \frac{c_{1,p_0}(t)}{\sqrt{n}} + \frac{c_{2,p_0}(t; m, \varepsilon)}{\varepsilon t/\sqrt{2\pi n}} + \frac{c_{2,p_0}(t)}{n} + R_{n,p_0,T}(t) \]

with

\[ |R_{n,p_0,T}(t)| \leq \frac{C_{p_0,N}(t; m, \varepsilon)}{n^{3/2}}, \]

and

\[ F_{n+,N}(t) = K(t) + \frac{c_{1,p_0}(t)}{\sqrt{n}} + \frac{c_{2,p_0}(t)}{n} - \left( \frac{t e^{-t}}{2\pi} \right)^{1/2} \frac{\text{Var}[L]}{n p_0(1 - p_0)} + R_{n,p_0,N}(t), \]

with

\[ |R_{n,p_0,N}(t)| \leq \frac{C_{p_0,N}(t; m, \varepsilon)}{n^{3/2}}, \]

where: the functions \( c_{1,p_0} \) and \( c_2 \) are independent of \( n \) and of the distribution of \( L \), that is independent of \( \varepsilon \) and \( m \); the function \( c_{2,p_0} \) is independent of \( n \) and such that \( |c_{2,p_0}| \leq 1 \); the functions \( C_{p_0,T} \) and \( C_{p_0,N} \) are independent of \( n \).

See Appendix A for the proof of Theorem 9. The functions \( c_{1,p_0}, c_{2,p_0} \) and \( c_{2,p_0} \) in Theorem 9, as well as the functions \( C_{p_0,T} \) and \( C_{p_0,N} \), can be made explicit by gathering some equations in the proof of the theorem. To assess goodness-of-fit by means of \( \Lambda_{n,T}(p_0) \) and \( \Lambda_{n+,N}(p_0) \), we fix a (reference) level of significance \( \alpha \in (0, 1) \) and find \( \lambda_T(\alpha) > 0 \) and \( \lambda_N(\alpha) > 0 \) such that \( K(\lambda_T(\alpha)) = 1 - \alpha \) and \( K(\lambda_N(\alpha)) = 1 - \alpha \), respectively. Then, we define the rejection regions \( \{\Lambda_{n,T}(p_0) > \lambda_T(\alpha)\} \) and \( \{\Lambda_{n+,N}(p_0) > \lambda_N(\alpha)\} \). Some empirical analyses in Rinott et al. (2017) show that the “true” LR test has statistical significance at level \( \alpha \), with a power that varies with \( \varepsilon \) and \( m \), whereas the “naïve” LR test has no statistical significance at the same level \( \alpha \). Theorem 9 provides a theoretical guarantee to such empirical analyses, showing that the term \( (np_0(1 - p_0))^{-1}\text{Var}[L] \) determines the loss in the statistical significance of the “naïve” LR test. For a fixed \( n \geq 1 \), the variance of the truncated Laplace EM is critical to control the closeness between \( F_{n+,N} \) and \( K \): the larger \( (np_0(1 - p_0))^{-1}\text{Var}[L] \) the less close \( F_{n+,N} \) to \( K \), and hence the larger \( (np_0(1 - p_0))^{-1}\text{Var}[L] \) the less the statistical significance of the “naïve” LR test. Note that \( \text{Var}[L] = c_{e,m} \sum_{-m \leq j \leq m} \left| \pi - \varepsilon \right| \) increases as \( \varepsilon \) decreases and/or \( m \) increases. Then, the loss in the statistical significance of the “naïve” LR test is driven by \( \varepsilon \) and \( m \), through \( \text{Var}[L] \), in combination with \( n \). Along the same
lines of the proof of Theorem 3, it is easy to show that an analogous result holds true for any convolutional-type EM. In particular, for the truncated Gaussian EM, Theorem 3 holds true with the variance of $L$ replaced by the variance of $G$ with distribution (6). For some fixed values of $\varepsilon$ and $\delta$, that is fixed levels of $(\varepsilon, \delta)$-DP, Table 2 shows that $\text{Var}[L] \leq \text{Var}[G]$, and hence the truncated Laplace EM produces a smaller loss of statistical significance in the “naïve” LR test than the truncated Gaussian EM.

| $(\varepsilon, \delta)$ | $m$ | $L$ | $G$ | $\text{Var}(L)$ | $\text{Var}(G)$ |
|-------------------------|-----|-----|-----|-----------------|-----------------|
| (0.025, 0.09)           | 5   | 5   | 9.660| 9.824           |
| (0.025, 0.04)           | 10  | 10  | 34.288| 35.411          |
| (0.05, 0.08)            | 5   | 5   | 9.650| 9.324           |
| (0.05, 0.04)            | 10  | 10  | 31.965| 34.187          |
| (0.075, 0.08)           | 5   | 5   | 8.991| 9.478           |
| (0.075, 0.03)           | 10  | 11  | 29.711| 39.189          |
| (0.1, 0.07)             | 5   | 6   | 8.662| 12.850          |
| (0.1, 0.03)             | 10  | 12  | 27.541| 43.925          |
| (0.125, 0.07)           | 5   | 6   | 8.337| 12.574          |
| (0.125, 0.02)           | 10  | 12  | 25.465| 42.084          |
| (0.150, 0.06)           | 5   | 6   | 8.019| 12.303          |
| (0.150, 0.02)           | 10  | 12  | 23.493| 40.317          |
| (0.175, 0.06)           | 5   | 6   | 7.706| 12.037          |
| (0.175, 0.02)           | 10  | 12  | 21.631| 38.624          |
| (0.2, 0.05)             | 5   | 6   | 7.399| 11.776          |
| (0.2, 0.02)             | 10  | 13  | 19.884| 42.001          |

Table 2: Variance of the truncated Laplace EM ($L$) and the truncated Gaussian EM for fixed levels of $(\varepsilon, \delta)$-DP.

We conclude by extending Theorem 9 to an arbitrary dimension $k \geq 2$. Note that it is sufficient to present such an extension under the “naïve” mode, as the extension under the “true” model is precisely Theorem 3. Recall that $A_n$ is distributed as a Multinomial distribution with parameter $(n, p)$, $L = (L_1, \ldots, L_{k-1})$ is independent of $A_n$ and with the $L_i$’s being random variables i.i.d. according to (4), and $B_n = (B_{1,n}, \ldots, B_{k-1,n})$ with $B_{i,n} = A_{i,n} + L_i$ for $i = 1, \ldots, k - 1$ and $B_{k,n} = n - |B_n|$. Under the “naïve” model, we make use of the LR test to assess goodness-of-fit in the form: $H_0 : p = p_0$, for a fixed $p_0 \in \Delta_{k-1}$, against $H_1 : p \neq p_0$. The LR under the “true” model is given by (7). We denote by $\hat{p}_{n+,N}$ be the maximum likelihood estimator of $p$ under the “naïve” model for $b^+$, that is the estimator obtained through the maximization of the likelihood function (14). Then, we define the LR

$$\Lambda_{n+,N}(p_0) = 2 \log \left( \frac{L_{n+,N}(\hat{p}_{n+,N}; B_n)}{L_{n+,N}(p_0; B_n)} \right),$$

which is referred to as the “naïve” LR, and we denote by $\Pr[\Lambda_{n+,N}(p_0) \in \cdot; p_0]$ the distribution of $\Lambda_{n+,N}(p_0)$ computed with respect to $A_n$ distributed as a Multinomial distribution with parameter $(n, p_0)$. As $n \to +\infty$, the theorem establishes an Edgeworth expansion for the distribution of
"naïve" LR, $\Lambda_{n^+,N}(p_0)$. With respect to $k = 2$, there is a new element to take into account, that is the $(k-1) \times (k-1)$ covariance matrix $\Sigma(p)$ whose entry $\sigma_{r,s}$ is $\sigma_{r,s} = -p_r p_s$ for $1 \leq r \neq s \leq k-1$ and by $\sigma_{r,r} = p_r (1 - p_r)$ for $1 \leq r \leq k-1$.

Theorem 10. Let $\mathbb{I}(p)$ denote the Fisher information matrix of the Multinomial model with parameter $(n,p)$, and let $F_{n^+,N}(t) = \Pr[\Lambda_{n^+,N}(p_0) \leq t; p_0]$ be the cumulative distribution function of $\Lambda_{n^+,N}(p_0)$. For any $n, k \in \mathbb{N}$ and any $t > 0$

$$F_{n^+,N}(t) = \mathcal{K}_{k-1}(t) + \frac{c_{1,k,p_0}(t)}{n^{1/2}} + \frac{c_{2,k,p_0}(t)}{n} - \frac{(1/2)^{(k+1)/2} e^{-t/2} t^{(k-1)/2}}{\Gamma(k+1/2)} \frac{\text{tr}(\mathbb{I}(p_0))}{n} + R_{(k,n),p_0,N}(t),$$

with

$$|R_{(k,n),p_0,N}(t)| \leq \frac{C_{k,p_0,N}(t; m, \varepsilon)}{n^{3/2}},$$

where the functions $c_{1,k,p_0}$ and $c_{2,k,p_0}$ are independent of $n$ and of the distribution of $L$, i.e. independent of $\varepsilon$ and $m$, and the function $C_{k,p_0,N}$ is independent of $n$.

See Appendix B for the proof of Theorem 10. Theorem 10 leads to the same conclusions as Theorem 9, showing the critical role of the Laplace perturbation mechanism. In general, Theorem 9 and Theorem 10 highlight the importance of taking the perturbation into account in privacy-protecting LR tests for goodness-of-fit, and therefore the importance of negative values in the released list $b$. Official statistical agencies are typically reluctant to disseminate perturbed tables with negative frequencies, and hence a common policy to preserve $(\varepsilon, \delta)$-DP consists in reporting negative values as zeros. However, as a matter of fact, releasing tables that have an appearance similar to that of original tables may lead to ignoring the perturbation and to analyzing data as they were not perturbed, i.e. under the “naïve” model (Rinott et al., 2017). Our analysis shows the importance of taking the perturbation into account, i.e. the importance of the “true” model versus the “naïve”, showing a loss in the statistical significance of the test when perturbed data are treated as they were not perturbed. Such a loss provides an evidence of the importance of taking the perturbation into account in the statistical model, thus endorsing the release of negative values if $(\varepsilon, \delta)$-DP is adopted.

5 Discussion

Under the framework of $(\varepsilon, \delta)$-DP for frequency table, we developed a rigorous analysis of the large sample behaviour of the “true” LR test. Our main contributions are with respect to the power analysis of the “true” LR test, and they built upon a Bahadur-Rao large deviation expansion for the power of the “true” LR test. By relying on a novel (sharp) large deviation principle or sum of i.i.d. random vectors, such an expansion brought out the critical quantity $\mathcal{L}_{(p_0,p_1)}(\varepsilon,m)$, which determines a loss in the power of the “true” LR test. This result has then been applied to characterize the impact of the sample size $n$ and the dimension $k$ of the table, in connection with the parameters $(\varepsilon, \delta)$, on the loss of the power of the private LR test. In particular, we determined the (sample) cost of $(\varepsilon, \delta)$-DP under the “true” LR test, namely the additional sample size required to recover the power of the Multinomial LR test in the absence of perturbation. As a complement to our power analysis of the “true” LR test, we investigated the well-known problem of releasing
negative values in frequency tables under the \((\varepsilon, \delta)\)-DP. By comparing the Edgworth expansions for the distribution of the “true” LR test and the “naïve” LR test, we showed the importance of taking the perturbation into account in private LR tests for goodness-of-fit, thus providing the first rigorous evidence to endorse the release of negative values when \((\varepsilon, \delta)\)-DP is adopted. Our work provides the first rigorous treatment of privacy-protecting LR tests for goodness-of-fit in frequency tables and, in particular, it is the first work to make use of the power of the test to quantify the trade-off between confidentiality and utility. This is achieved through a non-standard large deviation analysis of the LR test, which is known to be challenging in a setting such as ours, where the statistical model is multidimensional, discrete, and not belonging to the exponential family.

Our power analysis of the “true” LR test can be easily extended to any truncated convolutional-type EM. As an example, we considered the truncated Gaussian EM, comparing it with the truncated Laplace EM. A numerical comparison showed how the truncated Laplace EM produces a smaller decrease in the power of the “true” LR test than the truncated Gaussian EM. Such a finding leads to the natural problem of identifying the optimal truncated convolutional-type EM, that is the truncated convolutional-type EM that leads to the smallest decrease in the power of the “true” LR test. In particular, is the truncated Laplace EM the optimal convolutional-type EM? This is an interesting open problem, whose rigorous solution requires an extension of Theorem 4 to deal with truncated EMs such that \(v(a_i, b_i) = g(a_i - b_i)\) for a general function \(g\). Based on the proof of Theorem 4, we conjecture that the resulting Bahadur-Rao large deviation expansion is still in the form (12), with a critical term at the order \(n^{-1}\) that depends on \(g\). Given that, one has to deal with a challenging optimization problem with respect to \(g\), that is finding \(g\) that leads to the smallest decrease in the power of the “true” LR test. Along similar lines, one may consider the more general problem of identifying the optimal truncated EM that leads to the smallest decrease in the power of the “true” LR test with respect to the sample size \(n\), say \((\varepsilon_n, \delta_n)\), and then identify the right scaling for which the contribution of the truncated Laplace EM to the power of the “true” LR test does no longer vanish in the large \(n\) limit.

Among other directions for future research, the study of optimal properties of the “true” LR is of special interest. In particular, for a fixed (reference) level of significance \(\alpha\), let \(\beta^*_n(p_1; \alpha)\) and \(\beta_n(p_1; \alpha)\) denote the powers of an arbitrary test statistic and of the LR test, respectively, for testing testing \(H_0: p = p_0\) against \(H_1: p = p_1\), for fixed \(p_0, p_1 \in \Delta_{k-1}\). It is known from Bahadur (1960) that

\[
\liminf_{n \to +\infty} \frac{1}{n} \log(1 - \beta^*_n(p_1; \alpha)) \geq -D_{KL}(p_1 \| p_0) \tag{18}
\]

and

\[
\lim_{n \to +\infty} \frac{1}{n} \log(1 - \beta_n(p_1; \alpha)) = -D_{KL}(p_1 \| p_0).
\]

That is, the LR test attains the lower bound (18). This is referred to as Bahadur efficiency of the LR test, and it provides a well-known optimal property of the LR test (Bahadur, 1960, 1967). An open problem emerging from our work is to establish a “private” version of the lower bound (18), and investigate the Bahadur efficiency of the “true” LR with respect to such a lower bound. Establishing such a property of optimality for the “true” LR test would provide a framework to compare goodness-of-fit tests under \((\varepsilon, \delta)\)-DP.
A Proofs for $k = 2$

In this section we will prove Theorems 9 and 4 in the case that our data set is a table with two cells, i.e. when $k = 2$. Recall that, in this case, the statement of Theorem 3 is already included in that of Theorem 9. Thus, the observable variable reduces to the counting $b$ contained in the first cell. At the beginning, some preparatory steps are needed to analyze the likelihood, the MLE and the LR relative to the null hypothesis $H_0 : p = p_0$, under the true model (5).

A.1 Preparatory steps

First of all, let us state a proposition that fixes the exact expression of the likelihood under the true model (5).

**Lemma 11.** Let $m \in \mathbb{N}$ and $\varepsilon > 0$ be fixed to define the Laplace distribution (4). Then, if $n > m$ and $b \in R(n, m) := \{m, m + 1, \ldots, n - m\}$, we have

$$L_{n,T}(p; b) = \frac{1}{c_{\varepsilon,m}} \sum_{l=-m}^{m} e^{-\varepsilon|l|} \binom{n}{b-l} p^{b-l}(1-p)^{n-b+l} \quad (p \in [0,1]).$$

Moreover, under the same assumption, for any $p \in (0,1)$ we can write

$$L_{n,T}(p; b) = L_{n,0}^{(0)}(p; b) \cdot H_{n,T}^{(\varepsilon,m)}(p; b)$$

with

$$L_{n,0}^{(0)}(p; b) := \binom{n}{b} p^{b}(1-p)^{n-b}$$

$$H_{n,T}^{(\varepsilon,m)}(p; b) := \frac{1}{c_{\varepsilon,m}} \sum_{l=-m}^{m} e^{-\varepsilon|l|} \rho(n, b, l) \left(\frac{1-p}{p}\right)^l$$

$$\rho(n, b, l) := \frac{bl(n-b)!}{(b-l)!(n-b+l)!}.$$  

**Proof.** Start from

$$B_n = A_n + L$$

where the random variables $A_n$ and $L$, defined on the probability space $(\Omega, \mathcal{F}, \Pr)$, are independent, $A_n \sim Bin(n, p)$ and $L$ has the Laplace distribution (4). By definition $L_{n,T}(p; b) := \Pr[B_n = b]$, so that, under the assumption of the Lemma, the independence of $A_n$ and $L$ entails

$$\Pr[B_n = b] = \sum_{l=-m}^{m} \Pr[A_n = b-l] \cdot \Pr[L = l]$$

proving (19). Finally, the decomposition (20) ensues from straightforward algebraic manipulations. \[\square\]

The next step aims at providing a large $n$ asymptotic expansion of the true likelihood. This expansion can be obtained by considering the observable quantity $b$ as itself dependent by $n$, according to the following
Lemma 12. Let $\xi \in (0, 1)$ be a fixed number. Under the assumption that $b = n(\xi + \epsilon_n)$ with \(\lim_{n \to 0} \epsilon_n = 0\), there exists $n_0 = n_0(\xi; \epsilon, m)$ such that

\[
\rho_n(\xi, l) := \rho(n, b, l) = \left(\frac{\xi}{1 - \xi}\right)^l \times \\
\times \left\{1 + \alpha_1(\xi, l)\epsilon_n + \alpha_2(\xi, l)\epsilon_n^2 + \alpha_3(\xi, l)\frac{1}{n} + \alpha_4(\xi, l)\frac{\epsilon_n}{n} + R_n(\xi, l)\right\}
\]

holds for any $n \geq n_0$ and $l \in \{-m, \ldots, m\}$, where

\[
\begin{align*}
\alpha_1(\xi, l) &:= \frac{l}{\xi(1 - \xi)} \\
\alpha_2(\xi, l) &:= \frac{l(l - 1 + 2\xi)}{2\xi^2(1 - \xi)^2} \\
\alpha_3(\xi, l) &:= -\frac{l(l - 1 + 2\xi)}{2\xi(1 - \xi)} \\
\alpha_4(\xi, l) &:= -\frac{l(l^2 - 2l + 1 + 2\xi^2 + 4l\xi - 2\xi)}{2\xi^2(1 - \xi)^2}
\end{align*}
\]

for some constant $C(\xi, l)$. Therefore, under the same assumption, there holds

\[
H_{n,T}^{(\epsilon, m)}(p; b) = \mathcal{H}_0^{(\epsilon, m)}(p; \xi) + \mathcal{H}_1^{(\epsilon, m)}(p; \xi)\epsilon_n + \mathcal{H}_2^{(\epsilon, m)}(p; \xi)\epsilon_n^2
\]

\[
+ \mathcal{H}_3^{(\epsilon, m)}(p; \xi)\frac{1}{n} + \mathcal{H}_4^{(\epsilon, m)}(p; \xi)\frac{\epsilon_n}{n} + \mathcal{R}_n(p; \xi)
\]

for any $n \geq n_0$, where

\[
\begin{align*}
\mathcal{H}_0^{(\epsilon, m)}(p; \xi) &:= \frac{1}{c_{\epsilon, m}} \sum_{l=-m}^{m} e^{-\epsilon[l]} \left(\frac{\xi(1 - p)}{p(1 - \xi)}\right)^l \\
\mathcal{H}_i^{(\epsilon, m)}(p; \xi) &:= \frac{1}{c_{\epsilon, m}} \sum_{l=-m}^{m} e^{-\epsilon[l]} \alpha_i(\xi, l) \left(\frac{\xi(1 - p)}{p(1 - \xi)}\right)^l \quad (i = 1, 2, 3, 4) \\
|\mathcal{R}_n(p; \xi)| &\leq C(p; \xi) \left|\epsilon_n\right|^3 + \frac{1}{n^2} + \frac{\epsilon_n^2}{n}
\end{align*}
\]

with $C(p; \xi) := \frac{1}{c_{\epsilon, m}} \sum_{l=-m}^{m} e^{-\epsilon[l]} C(\xi, l) \left(\frac{\xi(1 - p)}{p(1 - \xi)}\right)^l$.

Proof. We start by dealing with (25). First, we find $n_0 = n_0(\xi; \epsilon, m)$ in such a way that the assumptions of Lemma 11 are fulfilled for any $n \geq n_0$. Now, if $l = 0$, then $\rho_n(\xi, 0) = 1$ and the thesis follows trivially. Also, if $l = 1$, then $\rho_n(\xi, 1) = (\xi + \epsilon_n)/(1 - \xi - \epsilon_n + \frac{1}{n})$ and the validity of the thesis can be checked by direct computation. Then, if $l \in \{2, \ldots, m\}$, we use (23) to get

\[
\rho(n, b, l) = \frac{(b)_l}{(n - b + 1)_l} = \frac{(b)_l}{(n - b)_l(l + 1)} = \frac{\sum_{k=1}^{l} s(l, k) b^k}{\sum_{k=1}^{l+1} s(l + 1, k)(n - b)^{k-1}}
\]

where $s(l, k)$ denotes the Stirling number of first kind. Whence,

\[
\rho_n(\xi, l) = \frac{\sum_{k=1}^{l} s(l, k)n^{k-l}(\xi + \epsilon_n)^k}{\sum_{k=1}^{l+1} s(l + 1, k)n^{k-l-1}(1 - \xi - \epsilon_n)^{k-1}} = \frac{\mathcal{R}_n}{\mathcal{D}_n}.
\]
Proof. First, we get a large expansion for the log-likelihood, as follows

\[ \ell_n(T) = \log L_n(T) = \log \left( \frac{n}{b} \right) + n(\xi + \epsilon_n) \log p + n(1 - \xi - \epsilon_n) \log (1 - p) \]

for suitable expressions of \( R_n(\xi, l) \) (possibly different from line to line) satisfying, in any case, the relation

\[ |R_n(\xi, l)| \leq C(\xi, l) \left[ |\epsilon_n|^3 + \frac{1}{n^2} + \frac{\epsilon_n^2}{n} \right]. \]

To proceed further, we exploit that \( \frac{1}{1+t} = 1 - t + t^2 + o(t^2) \) as \( t \to 0 \), to obtain

\[ \mathcal{D}_n^{-1} := \frac{1}{(1 - \xi)^l} \left\{ 1 - l \left( \frac{\epsilon_n}{1 - \xi} + \frac{l(l + 1)}{2} \left( \frac{\epsilon_n^2}{(1 - \xi)^2} - \frac{l(l + 1)}{2n} \frac{1}{1 - \xi} \right) \right\}. \]

The thesis now follows by multiplying the last expression by that of \( \mathcal{G}_n \), neglecting all the terms which are comparable with \( R_n(\xi, l) \). Thus, (25) is proved also for all \( l \in \{2, \ldots, m\} \).

For \( l \in \{-m, \ldots,-1\} \) the argument can be reduced to the previous case. In fact, since \( \rho(n, b, -l) = \rho(n, n - b, l) \), we can put \( h := -l > 0, \eta := 1 - \xi \) and \( \delta_n := -\epsilon_n \) to obtain

\[ \rho_n(\xi, l) = \left( \frac{\eta}{1 - \eta} \right)^h \left\{ 1 + \alpha_1(\eta, h)\delta_n + \alpha_2(\eta, h)\delta_n^2 + \alpha_3(\eta, h) \frac{1}{n} \right. \\
\left. + \alpha_4(\eta, h) \frac{\delta_n}{n} + R_n(\eta, h) \right\}. \]

This completes the proof of (25) since \( \left( \frac{\eta}{1 - \eta} \right)^h = \left( \frac{\xi}{1 - \xi} \right)^l \) and \( -\alpha_1(\eta, h) = \alpha_1(\xi, l), \alpha_2(\eta, h) = \alpha_2(\xi, l), \alpha_3(\eta, h) = \alpha_3(\xi, l) \) and \( -\alpha_4(\eta, h) = \alpha_4(\xi, l) \).

Finally, (26) follows immediately from the combination of (22) with (25).

We can now provide a large \( n \) asymptotic expansion for the MLE relative to the true likelihood \( L_{n,T} \), contained in the following

**Lemma 13.** Under the same assumption of Lemma 12, there holds

\[ \hat{\rho}_{n,T} = \xi + \epsilon_n + O \left( \frac{\epsilon_n^3}{n} \right). \]

**Proof.** First, we get a large \( n \) expansion for the log-likelihood, as follows

\[ \ell_{n,T}(p; b) := \log L_{n,T}(p; b) = \log \left( \frac{n}{b} \right) + n(\xi + \epsilon_n) \log p + n(1 - \xi - \epsilon_n) \log (1 - p) + h_{0}(\epsilon_n,p;\xi) + h_{1}(\epsilon_n,p;\xi)\epsilon_n + h_{2}(\epsilon_n,p;\xi)\epsilon_n^2 \]

20
where

\[
\begin{align*}
& h_0^{(e,m)}(p; \xi) := \log \mathcal{H}_0^{(e,m)}(p; \xi) \\
& h_1^{(e,m)}(p; \xi) := \frac{\mathcal{H}_1^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} \\
& h_2^{(e,m)}(p; \xi) := \frac{\mathcal{H}_2^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} - \frac{1}{2} \left( \frac{\mathcal{H}_1^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} \right)^2 \\
& h_3^{(e,m)}(p; \xi) := \frac{\mathcal{H}_3^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} - \frac{\mathcal{H}_1^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} \frac{\mathcal{H}_3^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} \\
& h_4^{(e,m)}(p; \xi) := \frac{\mathcal{H}_4^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} - \frac{\mathcal{H}_1^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} \frac{\mathcal{H}_4^{(e,m)}(p; \xi)}{\mathcal{H}_0^{(e,m)}(p; \xi)} \\
|r_n(p; \xi)| & \leq C(p; \xi) \left[ \epsilon_n^3 + \frac{1}{n^2} + \frac{\epsilon_n^2}{n} \right]
\end{align*}
\]

for some constant \( C(p; \xi) \). Then, to find the maximum point of the likelihood, we study the equation

\[
\frac{d}{dp} \ell_{n,r}(p; b) = 0,
\]

which reads

\[
\begin{align*}
& n \frac{p - \xi - \epsilon_n}{p(1-p)} = [\partial_p h_0^{(e,m)}(p; \xi)] + [\partial_p h_1^{(e,m)}(p; \xi)] \epsilon_n + [\partial_p h_2^{(e,m)}(p; \xi)] \epsilon_n^2 \\
& + [\partial_p h_3^{(e,m)}(p; \xi)] \frac{1}{n} + [\partial_p h_4^{(e,m)}(p; \xi)] \frac{\epsilon_n}{n} + r_n(p; \xi).
\end{align*}
\]

The solution of such an equation can be obtained by inserting the expression \( \xi + \epsilon_n + \Gamma_0(\xi) \frac{1}{n} + \Gamma_1(\xi) \frac{\epsilon_n}{n} + \Gamma_2(\xi) \frac{\epsilon_n^2}{n} + \Delta_n(\xi; \epsilon_n) \) in the place of \( p \), and then expanding both members. For the left-hand side we get

\[
\frac{1}{\xi(1-\xi)} \left( \Gamma_0(\xi) + \left( \Gamma_1(\xi) + \frac{2\xi - 1}{\xi(1-\xi)} \Gamma_0(\xi) \right) \epsilon_n \\
+ \left( \Gamma_2(\xi) + \frac{2\xi - 1}{\xi(1-\xi)} \Gamma_1(\xi) + \frac{1 - 4\xi + 4\xi^2}{\xi^2(1-\xi)^2} \Gamma_0(\xi) \right) \epsilon_n^2 + \omega_n \right)
\]

where \( \omega_n = O(\epsilon_n^3 + \frac{1}{n}) \). On the other hand, a Taylor expansion around \( p = \xi \) yields for the right-hand side

\[
\begin{align*}
[\partial_p h_0^{(e,m)}(p; \xi)]_{p=\xi} + \left( [\partial_p^2 h_0^{(e,m)}(p; \xi)]_{p=\xi} + [\partial_p h_1^{(e,m)}(p; \xi)]_{p=\xi} \right) \epsilon_n \\
+ \left( \frac{1}{2} [\partial_p^2 h_0^{(e,m)}(p; \xi)]_{p=\xi} + [\partial_p^2 h_1^{(e,m)}(p; \xi)]_{p=\xi} + [\partial_p h_2^{(e,m)}(p; \xi)]_{p=\xi} \right) \epsilon_n^2 + \omega_n.
\end{align*}
\]

Here, it is important to notice that we have disregarded all the terms of \( O(\frac{1}{n}) \)-type, considering them of lower order with respect to the terms of \( O(\epsilon_n^2) \)-type. This is due to the law of iterated logarithm, by which \( \frac{1}{n} A_n - p \sim \sqrt{(\log \log n)/n} \), when \( A_n \sim Bin(n,p) \). Thus, requiring identity between the above expansions yields

\[
\Gamma_0(\xi) = \xi(1-\xi)[\partial_p h_0^{(e,m)}(p; \xi)]_{p=\xi}
\]

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\[ \Gamma_1(\xi) = \xi (1 - \xi) \left( [\partial_p^2 h_0^{(e,m)}(p; \xi)]_{p=\xi} + [\partial_p h_1^{(e,m)}(p; \xi)]_{p=\xi} \right) + (1 - 2\xi) [\partial_p h_0^{(e,m)}(p; \xi)]_{p=\xi} \]

\[ \Gamma_2(\xi) = \xi (1 - \xi) \left( \frac{1}{2} [\partial_p^3 h_0^{(e,m)}(p; \xi)]_{p=\xi} + [\partial_p^2 h_1^{(e,m)}(p; \xi)]_{p=\xi} + [\partial_p h_2^{(e,m)}(p; \xi)]_{p=\xi} \right) - \frac{1 - 4\xi + 4\xi^2}{\xi (1 - \xi)} [\partial_p h_0^{(e,m)}(p; \xi)]_{p=\xi} + (1 - 2\xi) \left( [\partial_p^2 h_0^{(e,m)}(p; \xi)]_{p=\xi} + [\partial_p h_1^{(e,m)}(p; \xi)]_{p=\xi} \right) + (1 - 2\xi)^2 [\partial_p h_0^{(e,m)}(p; \xi)]_{p=\xi} . \]

At this stage, to complete the proof it remains to evaluate the various terms \([\partial_p h_i^{(e,m)}(p; \xi)]_{p=\xi}\).

We start with the terms involving \(h_0^{(e,m)}(p; \xi)\). First, we notice that

\[ \partial_p h_0^{(e,m)}(p; \xi) = \frac{\partial_p h_0^{(e,m)}(p; \xi)}{H_0^{(e,m)}(p; \xi)} \]

\[ \partial_p^2 h_0^{(e,m)}(p; \xi) = \frac{H_0^{(e,m)}(p; \xi) \partial_p^2 H_0^{(e,m)}(p; \xi) - [\partial_p H_0^{(e,m)}(p; \xi)]^2}{[H_0^{(e,m)}(p; \xi)]^2} \]

\[ \partial_p^3 h_0^{(e,m)}(p; \xi) = \frac{[H_0^{(e,m)}(p; \xi)]^2 \partial_p^2 H_0^{(e,m)}(p; \xi) + 2[\partial_p H_0^{(e,m)}(p; \xi)]^3}{[H_0^{(e,m)}(p; \xi)]^3} - \frac{3\partial_p H_0^{(e,m)}(p; \xi) \partial_p^2 H_0^{(e,m)}(p; \xi)}{[H_0^{(e,m)}(p; \xi)]^2} \]

After putting \(\phi(p; \xi) := \log \frac{\xi(1-p)}{p(1-\xi)}\) and noticing that

\[ H_0^{(e,m)}(p; \xi) = \frac{1}{c_{e,m}} \sum_{l=-m}^{m} \exp \{-\varepsilon |l| + l \phi(p; \xi)\} , \]

we get

\[ \partial_p H_0^{(e,m)}(p; \xi) = \partial_p \phi(p; \xi) \frac{1}{c_{e,m}} \sum_{l=-m}^{m} l \exp \{-\varepsilon |l| + l \phi(p; \xi)\} \]

\[ \partial_p^2 H_0^{(e,m)}(p; \xi) = \partial_p^2 \phi(p; \xi) \frac{1}{c_{e,m}} \sum_{l=-m}^{m} l \exp \{-\varepsilon |l| + l \phi(p; \xi)\} + [\partial_p \phi(p; \xi)]^2 \frac{1}{c_{e,m}} \sum_{l=-m}^{m} l^2 \exp \{-\varepsilon |l| + l \phi(p; \xi)\} \]

\[ \partial_p^3 H_0^{(e,m)}(p; \xi) = \partial_p^3 \phi(p; \xi) \frac{1}{c_{e,m}} \sum_{l=-m}^{m} l \exp \{-\varepsilon |l| + l \phi(p; \xi)\} + 3\partial_p \phi(p; \xi) \partial_p^2 \phi(p; \xi) \frac{1}{c_{e,m}} \sum_{l=-m}^{m} l^2 \exp \{-\varepsilon |l| + l \phi(p; \xi)\} \]

\[ + [\partial_p \phi(p; \xi)]^3 \frac{1}{c_{e,m}} \sum_{l=-m}^{m} l^3 \exp \{-\varepsilon |l| + l \phi(p; \xi)\} . \]

Now, when \(p = \xi\), we have

\[ \phi(\xi; \xi) = 0 \]
\[
[\partial_p \phi(p; \xi)]_{p=\xi} = -\frac{1}{\xi(1-\xi)} \\
[\partial_p^2 \phi(p; \xi)]_{p=\xi} = \frac{1 - 2\xi}{\xi^2(1-\xi)^2}
\]

and, hence,

\[
\mathcal{H}_0^{(e,m)}(p; \xi)_{p=\xi} = 1 \\
[\partial_p \mathcal{H}_0^{(e,m)}(p; \xi)]_{p=\xi} = 0 \\
[\partial_p^2 \mathcal{H}_0^{(e,m)}(p; \xi)]_{p=\xi} = \frac{\text{Var}(L)}{\xi^2(1-\xi)^2} \\
[\partial_p^3 \mathcal{H}_0^{(e,m)}(p; \xi)]_{p=\xi} = -3\frac{(1-2\xi)\text{Var}(L)}{\xi^3(1-\xi)^3}.
\]

Then, we consider the terms containing \(h_1^{(e,m)}(p; \xi)\). We start from the identities

\[
\partial_p h_1^{(e,m)}(p; \xi) = \frac{\mathcal{H}_0^{(e,m)}(p; \xi)\partial_p \mathcal{H}_1^{(e,m)}(p; \xi) - \mathcal{H}_1^{(e,m)}(p; \xi)\partial_p \mathcal{H}_0^{(e,m)}(p; \xi)}{[\mathcal{H}_0^{(e,m)}(p; \xi)]^2} \\
\partial_p^2 h_1^{(e,m)}(p; \xi) = \frac{\mathcal{H}_0^{(e,m)}(p; \xi)\partial_p^2 \mathcal{H}_1^{(e,m)}(p; \xi) - \mathcal{H}_1^{(e,m)}(p; \xi)\partial_p^2 \mathcal{H}_0^{(e,m)}(p; \xi)}{[\mathcal{H}_0^{(e,m)}(p; \xi)]^2} \\
- \frac{2}{\partial_p \mathcal{H}_0^{(e,m)}(p; \xi)}\frac{[\mathcal{H}_0^{(e,m)}(p; \xi)]^3}{[\mathcal{H}_0^{(e,m)}(p; \xi)]^3} \partial_p^3 \mathcal{H}_1^{(e,m)}(p; \xi) - \mathcal{H}_1^{(e,m)}(p; \xi)\partial_p \mathcal{H}_0^{(e,m)}(p; \xi)]
\]

and

\[
\partial_p \mathcal{H}_1^{(e,m)}(p; \xi) = \frac{\partial_p \phi(p; \xi)}{\xi(1-\xi)} \sum_{l=-m}^m l^2 \exp\{-\varepsilon |l| + l\phi(p; \xi)\} \\
\partial_p^2 \mathcal{H}_1^{(e,m)}(p; \xi) = \frac{\partial_p^2 \phi(p; \xi)}{\xi(1-\xi)} \sum_{l=-m}^m l^2 \exp\{-\varepsilon |l| + l\phi(p; \xi)\} \\
+ \frac{[\partial_p \phi(p; \xi)]^2}{\xi(1-\xi)} \sum_{l=-m}^m l^3 \exp\{-\varepsilon |l| + l\phi(p; \xi)\}.
\]

Now, when \(p = \xi\), we have

\[
\mathcal{H}_1^{(e,m)}(p; \xi)_{p=\xi} = 0 \\
[\partial_p \mathcal{H}_1^{(e,m)}(p; \xi)]_{p=\xi} = -\frac{\text{Var}(L)}{\xi^2(1-\xi)^2} \\
[\partial_p^2 \mathcal{H}_1^{(e,m)}(p; \xi)]_{p=\xi} = \frac{(1-2\xi)\text{Var}(L)}{\xi^3(1-\xi)^3}.
\]

Finally, we consider the terms containing \(h_2^{(e,m)}(p; \xi)\). We start from the identities

\[
\partial_p h_2^{(e,m)}(p; \xi) = \frac{\mathcal{H}_0^{(e,m)}(p; \xi)\partial_p \mathcal{H}_2^{(e,m)}(p; \xi) - \mathcal{H}_2^{(e,m)}(p; \xi)\partial_p \mathcal{H}_0^{(e,m)}(p; \xi)}{[\mathcal{H}_0^{(e,m)}(p; \xi)]^2}
\]
and
\[
\partial_p \mathcal{H}_2^{(e,m)}(p; \xi) = \frac{\partial_p \phi(p; \xi)}{2\xi^2(1 - \xi)^2} \sum_{l=-m}^m l^3 \exp\{-\varepsilon|l| + l\phi(p; \xi)\} - \frac{(1 - 2\xi)\partial_p \phi(p; \xi)}{2\xi^2(1 - \xi)^2} \sum_{l=-m}^m l^2 \exp\{-\varepsilon|l| + l\phi(p; \xi)\}.
\]

Now, when \( p = \xi \), we have
\[
\mathcal{H}_2^{(e,m)}(p; \xi)|_{p=\xi} = \frac{\text{Var}(L)}{2\xi^2(1 - \xi)^2}
\]
\[
[\partial_p \mathcal{H}_2^{(e,m)}(p; \xi)|_{p=\xi} = \frac{(1 - 2\xi)\text{Var}(L)}{2\xi^3(1 - \xi)^3}.
\]

At the end of all these computations, we are in a position to conclude that \( \Gamma_0(\xi) = \Gamma_1(\xi) = \Gamma_2(\xi) = 0 \), thus proving (27).

The way is now paved to analyze the LR relative to the null hypothesis \( H_0 : p = p_0 \), by means of the following

**Lemma 14.** Under the same assumption of Lemma 12 with \( \xi = p_0 \), there hold
\[
\Lambda_{n,T}(p_0; b) := -2\log \left( \frac{L_{n,T}(p_0; b)}{L_{n,T}(\hat{p}_n,T; b)} \right) = 2\log \left( \frac{L_n^{(0)}(\hat{p}_n,T; b)}{L_n^{(0)}(p_0; b)} \right) + 2\log \left( \frac{H_n^{(e,m)}(\hat{p}_n,T; b)}{H_n^{(e,m)}(p_0; b)} \right)
\]
(28)
along with
\[
2\log \left( \frac{L_n^{(0)}(\hat{p}_n,T; b)}{L_n^{(0)}(p_0; b)} \right) = n \left\{ \frac{1}{p_0(1 - p_0)} \epsilon_n^2 + \frac{2p_0 - 1}{3p_0^2(1 - p_0)^2} \epsilon_n^3 + \frac{1 - 3p_0 + 3p_0^2}{6p_0^3(1 - p_0)^3} \epsilon_n^4 \right\} + O(\epsilon_n^5)
\]
(29)
and
\[
2\log \left( \frac{H_n^{(e,m)}(\hat{p}_n,T; b)}{H_n^{(e,m)}(p_0; b)} \right) = -\frac{\text{Var}(L)}{p_0^2(1 - p_0)^2} \epsilon_n^2 + \frac{(1 - 2p_0)\text{Var}(L)}{p_0^2(1 - p_0)^2} \epsilon_n + O(\epsilon_n^3).
\]
(30)
Therefore, putting \( \zeta_n := \sqrt{n}\epsilon_n \), we get
\[
\Lambda_{n,T}(p_0; b) = \mathfrak{A}_n^{(e,m)}(p_0) \zeta_n^2 + \mathfrak{B}_n^{(e,m)}(p_0) \frac{\zeta_n^3}{\sqrt{n}} + \mathfrak{C}_n^{(e,m)}(p_0) \frac{\zeta_n^4}{n} + O(n^{-3/2})
\]
(31)
with
\[
\mathfrak{A}_n^{(e,m)}(p_0) = \frac{1}{p_0(1 - p_0)} - \frac{\text{Var}(L)}{np_0^2(1 - p_0)^2} + O \left( \frac{1}{n^2} \right)
\]
(32)
\[
\mathfrak{I}^{(\epsilon, m)}_n(p_0) = \frac{2p_0 - 1}{3p_0^2(1 - p_0)^2} + \mathfrak{I}^{(\epsilon, m)}_n(p_0) + O\left(\frac{1}{n^2}\right) \tag{33}
\]
\[
\mathfrak{C}^{(\epsilon, m)}_n(p_0) = \frac{1 - 3p_0 + 3p_0^3}{6p_0^3(1 - p_0)^3} + \mathfrak{C}^{(\epsilon, m)}_n(p_0) + O\left(\frac{1}{n^2}\right) \tag{34}
\]

for suitable terms \( \mathfrak{I}^{(\epsilon, m)}_n(p_0) \) and \( \mathfrak{C}^{(\epsilon, m)}_n(p_0) \).

**Remark 15.** Notice that the expression inside the brackets in (29) coincides with the Taylor polynomial of order 4 of the map \( \epsilon_n \mapsto D_{KL}(p_0 + \epsilon_n\|p_0) \) where

\[
D_{KL}(p\|p_0) := p \log \left(\frac{p}{p_0}\right) + (1 - p) \log \left(\frac{1 - p}{1 - p_0}\right)
\]
denotes the Kullback-Leibler divergence relative to the Bernoulli model. In particular, we have that

\[
\frac{1}{p_0(1 - p_0)} = \frac{\partial^2}{\partial p^2} D_{KL}(p\|p_0)|_{p = p_0}
\]

coincides with the Fisher information of the Bernoulli model. Finally, it is worth noticing that, in the Bernoulli model, the Fisher information just coincides with the inverse of the variance.

**Proof.** Since

\[
\Lambda_{n,T}(p_0, b) := -2 \log \left(\frac{\sup_{p \in [0,1]} L_{n,T}(p; b)}{\sup_{p \in (p_0, 1]} L_{n,T}(p; b)}\right) = -2 \log \left(\frac{L_{n,T}(p_0; b)}{L_{n,T}^{(0)}(p_0; b)}\right)
\]

holds by definition, identity (28) follows immediately from (20). Next, we derive (29) by combining (21) with (27). In fact, we have

\[
2 \log \left(\frac{L_{n,T}^{(0)}(\hat{p}_{n,T}; b)}{L_{n,T}^{(0)}(p_0; b)}\right) = 2n \left\{ (p_0 + \epsilon_n) \log \left(1 + \frac{\epsilon_n}{p_0} \right) + (1 - p_0 - \epsilon_n) \log \left(1 - \frac{\epsilon_n}{1 - p_0} \right) \right\}
\]

where \( \Delta_n(p_0) = \frac{1}{n}[\Gamma_3(p_0)\epsilon_n^2 + \Gamma_4(p_0)\epsilon_n^4 + O(\epsilon_n^5)] \), according to (27). Therefore, using the Taylor expansion of the function \( z \mapsto \log(1 + z) \), we can further specialize the last expression as

\[
2n \left\{ (p_0 + \epsilon_n) \left[ \frac{\epsilon_n + \Delta_n(p_0)}{p_0} - \frac{(\epsilon_n + \Delta_n(p_0))^2}{2p_0^2} + \frac{(\epsilon_n + \Delta_n(p_0))^3}{3p_0^3} - \frac{(\epsilon_n + \Delta_n(p_0))^4}{4p_0^4} \right] \right.
\]
\[
- (1 - p_0 - \epsilon_n) \left[ \frac{\epsilon_n + \Delta_n(p_0)}{1 - p_0} + \frac{(\epsilon_n + \Delta_n(p_0))^2}{2(1 - p_0)^2} + \frac{(\epsilon_n + \Delta_n(p_0))^3}{3(1 - p_0)^3} + \frac{(\epsilon_n + \Delta_n(p_0))^4}{4(1 - p_0)^4} \right] \right\}
\]
\[
+ O(\epsilon_n^5).
\]

At this stage, careful algebraic computations based on the Newton binomial formula lead to (29). Incidentally, it is interesting to notice that the terms \( \Gamma_3(p_0) \) and \( \Gamma_4(p_0) \), depending on the perturbation, do not appear in the expansion (29).

Coming to (30), we start by noticing that

\[
\frac{1 - \hat{p}_{n,T}}{\hat{p}_{n,T}} = \frac{1 - p_0}{p_0} \left[ 1 - \frac{\epsilon_n}{p_0(1 - p_0)} + \frac{\epsilon_n^2}{p_0^2(1 - p_0)} + O(\epsilon_n^3) \right]
\]

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which entails
\[
\left( \frac{1 - \hat{p}_n}{\hat{p}_n} \right)^l = \left( \frac{1 - p_0}{p_0} \right)^l \left[ 1 - \frac{l}{p_0(1 - p_0)} \epsilon_n + \frac{l}{p_0^2(1 - p_0)} \epsilon_n^2 + \frac{l(l - 1)}{2p_0^2(1 - p_0)^2} \epsilon_n^2 + O(\epsilon_n^3) \right].
\]
This identity, combined with (22) and (25), yields
\[
H_{\hat{n}}^{(\epsilon, m)}(\hat{p}_n; b) = 1 - \frac{\text{Var}(L)}{2np_0(1 - p_0)} + \frac{(1 - 2p_0)\text{Var}(L)}{2p_0^2(1 - p_0)^2} \epsilon_n + O(\epsilon_n^3).
\]
Moreover, another combination of (22) and (25) gives
\[
H_{\hat{n}}^{(\epsilon, m)}(p_0; b) = 1 + \frac{\text{Var}(L)}{2p_0^2(1 - p_0)^2} \epsilon_n^2 - \frac{\text{Var}(L)}{2np_0(1 - p_0)} + \frac{(1 - 2p_0)\text{Var}(L)}{p_0^2(1 - p_0)^2} \epsilon_n + O(\epsilon_n^3).
\]
Then, (30) is a straightforward consequence of the last two identities.
Finally, identities (31)-(32)-(33)-(34) follows immediately from (28)-(29)-(30).

The last preparatory result provides an equivalent reformulation of the event
\[
\{ b : H_0 \text{ is rejected} \} = \{ \Lambda_{n,T}(p_0; b) > \lambda_{\alpha} \}
\]
for some $\lambda_{\alpha} > 0$, to be determined after assessing the level $\alpha$ of the test.

**Lemma 16.** Under the same assumption of Lemma 12 with $\xi = p_0$, the event (35) is equivalent to the event
\[
\{ b : \zeta_n > 3^{(\epsilon, m)}_{n,+}(p_0; \lambda_{\alpha}) \} \cup \{ b : \zeta_n < 3^{(\epsilon, m)}_{n,-}(p_0; \lambda_{\alpha}) \}
\]
where
\[
3^{(\epsilon, m)}_{n,+}(p_0; \lambda_{\alpha}) := \sqrt{\lambda_{\alpha} p_0(1 - p_0)} + \frac{(1 - 2p_0)\lambda_{\alpha}}{6\sqrt{n}} + \frac{\sqrt{\lambda_{\alpha}}}{2n\sqrt{p_0(1 - p_0)}} \{ \text{Var}(L) - \lambda_{\alpha}(1 + 2p_0 - 2p_0^2) \} + O(n^{-3/2}),
\]
\[
3^{(\epsilon, m)}_{n,-}(p_0; \lambda_{\alpha}) := -\sqrt{\lambda_{\alpha} p_0(1 - p_0)} + \frac{(1 - 2p_0)\lambda_{\alpha}}{6\sqrt{n}} - \frac{\sqrt{\lambda_{\alpha}}}{2n\sqrt{p_0(1 - p_0)}} \{ \text{Var}(L) - \lambda_{\alpha}(1 + 2p_0 - 2p_0^2) \} + O(n^{-3/2}).
\]

**Proof.** Since $\mathcal{A}^{(\epsilon, m)}_n(p_0) > 0$ and $\mathcal{C}^{(\epsilon, m)}_n(p_0) > 0$ eventually, we first notice that, always eventually, the graphic of the function
\[
\Xi_n : x \mapsto \mathcal{A}^{(\epsilon, m)}_n(p_0)x^2 + \mathcal{B}^{(\epsilon, m)}_n(p_0)x^3 + \mathcal{C}^{(\epsilon, m)}_n(p_0)x^4/n
\]
goes to $+\infty$ as $x \to \pm\infty$, decreases for $x < 0$, increases for $x > 0$, and has a unique absolute minimum at $x = 0$. Thus, for any $\lambda_{\alpha} > 0$ (to be determined later on), the equation $\Xi_n(x) = \lambda_{\alpha}$
admits two real solutions, that we just denote as \( \xi^{(\varepsilon,m)}_{n,+}(p_0; \alpha) \) and \( \xi^{(\varepsilon,m)}_{n,-}(p_0; \alpha) \). To evaluate them, we put

\[
\xi^{(\varepsilon,m)}_{n,+}(p_0; \lambda_\alpha) := \sqrt{\frac{\lambda_\alpha}{2\mathfrak{B}_n^{(\varepsilon,m)}(p_0)}} + \frac{\xi^{(\varepsilon,m)}_+(p_0; \lambda_\alpha)}{\sqrt{\frac{n}{n}}} + \frac{\eta^{(\varepsilon,m)}_+(p_0; \lambda_\alpha)}{n} + O(n^{-3/2})
\]

\[
\xi^{(\varepsilon,m)}_{n,-}(p_0; \lambda_\alpha) := -\sqrt{\frac{\lambda_\alpha}{2\mathfrak{B}_n^{(\varepsilon,m)}(p_0)}} + \frac{\xi^{(\varepsilon,m)}_-(p_0; \lambda_\alpha)}{\sqrt{\frac{n}{n}}} + \frac{\eta^{(\varepsilon,m)}_-(p_0; \lambda_\alpha)}{n} + O(n^{-3/2})
\]

After inserting these expression into the function \( \Xi_n \), we set equal to zero the coefficients of \( \frac{1}{\sqrt{n}} \) and \( \frac{1}{n} \), obtaining

\[
\xi^{(\varepsilon,m)}_+(p_0; \lambda_\alpha) = \xi^{(\varepsilon,m)}_-(p_0; \alpha) = -\frac{\mathfrak{B}_n^{(\varepsilon,m)}(p_0)\lambda_\alpha}{2[\mathfrak{B}_n^{(\varepsilon,m)}(p_0)]^2}
\]

\[
\eta^{(\varepsilon,m)}_+(p_0; \lambda_\alpha) = -\eta^{(\varepsilon,m)}_-(p_0; \alpha) = \frac{\lambda_\alpha^{3/2}}{2[\mathfrak{B}_n^{(\varepsilon,m)}(p_0)]^{5/2}} \left[ \frac{5[\mathfrak{B}_n^{(\varepsilon,m)}(p_0)]^2}{4\mathfrak{B}_n^{(\varepsilon,m)}(p_0)} - \sigma_n^{(\varepsilon,m)}(p_0) \right].
\]

Finally, to prove (37)-(38), it is enough to use (32)-(33)-(34) of Lemma 24, showing that

\[
\sqrt{\frac{\lambda_\alpha}{2\mathfrak{B}_n^{(\varepsilon,m)}(p_0)}} = \sqrt{\lambda_\alpha p_0(1-p_0)} \left( 1 + \frac{\operatorname{Var}(L)}{2np_0(1-p_0)} \right) + O \left( \frac{1}{n^2} \right)
\]

and

\[
\frac{\mathfrak{B}_n^{(\varepsilon,m)}(p_0)\lambda_\alpha}{2[\mathfrak{B}_n^{(\varepsilon,m)}(p_0)]^2} = \frac{\lambda_\alpha}{2} [p_0(1-p_0)]^2 \frac{2p_0-1}{3p_0^2(1-p_0)^2} \]

\[
\frac{\lambda_\alpha^{3/2}}{2[\mathfrak{B}_n^{(\varepsilon,m)}(p_0)]^{5/2}} \left[ \frac{5[\mathfrak{B}_n^{(\varepsilon,m)}(p_0)]^2}{4\mathfrak{B}_n^{(\varepsilon,m)}(p_0)} - \sigma_n^{(\varepsilon,m)}(p_0) \right] = \frac{\lambda_\alpha^{3/2}}{2} [p_0(1-p_0)]^{5/2} \frac{5}{4} p_0(1-p_0) \times \]

\[
\left. \left[ \frac{2p_0-1}{3p_0^2(1-p_0)^2} \right] ^2 - \frac{1-3p_0+3p_0^2}{6p_0^2(1-p_0)^3} \right],
\]

which concludes the proof.

\[\square\]

## A.2 Proof of Theorem 9

We start again from (24) where the random variables \( A_n \) and \( L \), defined on the probability space \( (\Omega, \mathcal{F}, \operatorname{Pr}) \), are independent, \( A_n \sim Bin(n, p_0) \) and \( L \) has the Laplace distribution (4). Thus, under the validity of \( H_0 \), we define the event \( E_{0,n} \subset \Omega \) as

\[
E_{0,n} := \left\{ \omega \in \Omega : \left| \frac{A_n(\omega)}{n} - p_0 \right| \leq c(p_0) \left( \frac{\log n}{n} \right)^{1/2} \right\}
\]

for some \( c(p_0) > 0 \). By a standard large deviation argument (see, e.g., Theorem 1 of Okamoto (1959)), there holds

\[
\Pr[E_{0,n}] \geq 1 - 2n^{-c(p_0)^2}.
\]

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which implies that the probability $E_{0,n}^c$ can be made arbitrarily small after a suitable choice of $c(p_0)$. See also Proposition 2.1 of Dolera and Regazzini (2019) for more refined bounds. In particular, it can be chosen so that

$$\Pr[E_{0,n}^c] \leq 2n^{-3/2}. \quad (40)$$

In this way, for $\ell = T, N$, we can write

$$F_{n,\ell}(t) = \Pr[\Lambda_{n,\ell}(p_0) \leq t, E_{0,n}] + \Pr[\Lambda_{n,\ell}(p_0) \leq t, E_{0,n}^c]$$

with

$$\Pr[\Lambda_{n,\ell}(p_0) \leq t, E_{0,n}^c] \leq 2n^{-3/2}.$$

The advantage of such a preliminary step is that, on $E_{0,n}$, the assumptions of all the Lemmata stated in the previous subsection are fulfilled and, by resorting to (36), we can write

$$\{\Lambda_{n,T}(p_0) \leq t, E_{0,n}\} = \left\{3_{n,\ell}^{(\varepsilon,m)}(p_0; t) \leq Z_n^+ \leq 3_{n,\ell}^{(\varepsilon,m)}(p_0; t), E_{0,n}\right\}$$

where

$$Z_n := \frac{B_n - np_0}{\sqrt{n}} = \sqrt{p_0(1 - p_0)}W_n + \frac{1}{\sqrt{n}}L \quad (41)$$

with

$$W_n := \frac{A_n - np_0}{np_0(1 - p_0)}.$$

**Remark 17.** This very same argument, based on straightforward application of concentration inequalities for the Binomial distribution, makes clear that the additional randomness that emerges in the “na"ïve” model, due to the presence of the random sample size $n^+$, is irrelevant in our asymptotic analysis. Indeed, after fixing $p_0 \in (0, 1)$, we get that the probability that $B_n \neq B_n^+$ (that is, the first cell contains a negative value) is bounded by a term of the form $e^{-\tau n}$ for some $\tau > 0$. Since terms of this form are irrelevant in the Edgeworth expansion (16), we can work from the beginning under the assumption that $B_n = B_n^+$ (that is, the first cell does not contain a negative value) which entails, in particular, that $n^+ = n$. An analogous consideration holds in larger dimensions, that is when $k > 2$, if $p_0$ is fixed in the interior of $A_{k-1}$.

Therefore, in view of (40), we can conclude that

$$F_{n,T}(t) = \Pr\left[3_{n,\ell}^{(\varepsilon,m)}(p_0; t) \leq Z_n \leq 3_{n,\ell}^{(\varepsilon,m)}(p_0; t)\right] + O(n^{-3/2})$$

$$= G_n\left(3_{n,\ell}^{(\varepsilon,m)}(p_0; t)\right) - \lim_{\tau \to 3_{n,\ell}^{(\varepsilon,m)}(p_0; t)^-} G_n(\tau) + O(n^{-3/2}) \quad (42)$$

where $G_n$ denotes the distribution function of $Z_n$. Moreover, the same analysis developed in the previous section shows that

$$F_{n,N}(t) = \Pr\left[\Psi_{n,\ell}^{(\varepsilon,m)}(p_0; t) \leq Z_n \leq \Psi_{n,\ell}^{(\varepsilon,m)}(p_0; t)\right] + O(n^{-3/2})$$

$$= G_n\left(\Psi_{n,\ell}^{(\varepsilon,m)}(p_0; t)\right) - \lim_{\tau \to \Psi_{n,\ell}^{(\varepsilon,m)}(p_0; t)^-} G_n(\tau) + O(n^{-3/2}) \quad (43)$$

with

$$\Psi_{n,\ell}^{(\varepsilon,m)}(p_0; \lambda_\alpha) := \sqrt{\lambda_\alpha p_0(1 - p_0)} + \frac{(1 - 2p_0)\lambda_\alpha}{6\sqrt{n}}$$
\[-\frac{\lambda^3/2}{2n\sqrt{p_0(1-p_0)}} (1 + 2p_0 - 2p_0^2) + O(n^{-3/2}) \]  
(44)

\[\mathfrak{V}_{n, -}^{(\varepsilon, m)}(p_0; \lambda_\alpha) := -\sqrt{\lambda_\alpha p_0(1-p_0)} + \frac{(1 - 2p_0)\lambda_\alpha}{6\sqrt{n}} + \frac{\lambda^3/2}{2n\sqrt{p_0(1-p_0)}} + O(n^{-3/2}) .\]  
(45)

Therefore, it is evident from the previous identities that the proof of Theorem 9 can be carried out after providing an explicit expansion of Berry-Esseen type of \(G_n\), which is contained in the following

**Proposition 18.** Let \(\Phi\) denote the cumulative distribution function of the standard Normal distribution, and let, for any \(m \in \mathbb{N}\),

\[S_{2m+1}(x) := 2 \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{(2\pi k)^{2m+1}} \quad \text{and} \quad S_{2m}(x) := 2 \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{(2\pi k)^{2m}} .\]

In addition, for any \(j \in \mathbb{N}\), let \(Q_j\) denote the function

\[Q_j(x) := -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4}} \sum_{(*)} H_{j+2s-1}(x) \prod_{m=1}^j \frac{1}{k_m!} \left( \frac{\gamma_m}{(m+2)!p_{\frac{m}{2}+1}(1-p)^{\frac{m}{2}+1}} \right)^{k_m}\]

where \((*)\) means that the sum ranges over all the \(j\)-tuples \((k_1, \ldots, k_j) \in \mathbb{N}_0^j\) such that \(k_1 + 2k_2 + \cdots + jk_j = j\), with \(s = k_1 + k_2 + \cdots + k_j\), \(H_m\) denotes the \(m\)th Chebyshev-Hermite polynomial, i.e.

\[H_m(x) := (-1)^m e^{\frac{x^2}{2}} \left( \frac{d^m}{dx^m} e^{-\frac{x^2}{2}} \right)\]

and \(\gamma_m\) stands for the \(m\)th cumulant of the Bernoulli distribution of parameter \(p\). Then, for any \(x \in \mathbb{R}\), there holds

\[G_n(x) = \Phi \left( \frac{x}{\sqrt{n}p_0(1-p_0)} \right) + \sum_{j=1}^{2} \left( \frac{1}{\sqrt{n}} \right)^j Q_j \left( \frac{x}{\sqrt{n}p_0(1-p_0)} \right)\]

\[\frac{\text{Var}(L)}{2np_0(1-p_0)} \Phi'' \left( \frac{x}{\sqrt{n}p_0(1-p_0)} \right)\]

\[+ \frac{1}{np_0(1-p_0)} S_1(np_0 + \sqrt{n}x) \left[ \Phi' \left( \frac{x}{\sqrt{n}p_0(1-p_0)} \right) + \frac{1}{\sqrt{n}} Q'_1 \left( \frac{x}{\sqrt{n}p_0(1-p_0)} \right) \right]\]

\[+ \frac{1}{np_0(1-p_0)} S_2(np_0 + \sqrt{n}x) \Phi'' \left( \frac{x}{\sqrt{n}p_0(1-p_0)} \right) + R_n(x)\]

with the remainder term \(R_n\) satisfying an inequality like

\[\sup_{x \in \mathbb{R}} |R_n(x)|(1 + |x|)^5 \leq \frac{C(p_0, m, \varepsilon)}{n^{3/2}}\]

for some suitable constant \(C(p_0, m, \varepsilon)\) independent of \(n\).
Proof. After denoting by $F_n$ the distribution function of $W_n$, invoke well-known results (see, e.g., Theorem 6 in Chapter VI of Petrov Petrov (1975), or Dolera and Favaro (2020)) to write (in the same notation adopted by Petrov)

$$F_n(y) = U_5(y) + \frac{1}{\sqrt{np_0(1-p_0)}}S_1(np_0 + y\sqrt{np_0(1-p_0)})U_5'(y) + \frac{1}{np_0(1-p_0)}S_2(np_0 + y\sqrt{np_0(1-p_0)})U_5''(y) + T_n(y)$$

where $U_5(y) := \Phi(y) + \sum_{j=1}^3 \left(\frac{1}{\sqrt{n}}\right)^j Q_j(y)$ and

$$|T_n(y)| \leq \frac{J(p_0)}{n^{3/2}(1+|y|)^5}$$

for some positive constant $J(p_0)$. Now, in view of the definition of $V_n$, its distribution function $G_n$ is given by

$$G_n(x) = \sum_{l=-m}^m \frac{1}{c_{\varepsilon,m}} e^{-|l|} F_n \left( \frac{x - l/\sqrt{n}}{\sqrt{np_0(1-p_0)}} \right) .$$

To obtain an expansion for $G_n$, notice that, for $y = \frac{x-l/\sqrt{n}}{\sqrt{np_0(1-p_0)}}$, there holds

$$S_j(np_0 + y\sqrt{np_0(1-p_0)}) = S_j(np_0 + x\sqrt{n} - l) = S_j(np_0 + x\sqrt{n})$$

thanks to the periodic character of $S_j$. Hence, the effect of putting $y = \frac{x-l/\sqrt{n}}{\sqrt{np_0(1-p_0)}}$ is mainly observed in the terms $U_5$, $U_5'$, and $U_5''$: by resorting to the Taylor expansion, for any $k \in \mathbb{N}_0$, write

$$U_5^{(k)} \left( \frac{x - l/\sqrt{n}}{\sqrt{np_0(1-p_0)}} \right) = U_5^{(k)} \left( \frac{x}{\sqrt{np_0(1-p_0)}} \right) - \frac{l}{\sqrt{np_0(1-p_0)}} U_5^{(k+1)} \left( \frac{x}{\sqrt{np_0(1-p_0)}} \right) + \frac{l^2}{2np_0(1-p_0)} U_5^{(k+2)} \left( \frac{x}{\sqrt{np_0(1-p_0)}} \right) + T_{n,k}(x)$$

where the remainder term satisfies

$$|T_{n,k}(x)| \leq \frac{J_k(p_0)|l|^3}{n^{3/2}(1+|x|)^5}$$

for some positive constant $J_k(p_0)$. Thus, exploiting the symmetry of the Laplace distribution, obtain

$$\sum_{l=-m}^m \frac{1}{c_{\varepsilon,m}} e^{-|l|} U_5^{(k)} \left( \frac{x - l/\sqrt{n}}{\sqrt{np_0(1-p_0)}} \right) = U_5^{(k)} \left( \frac{x}{\sqrt{np_0(1-p_0)}} \right)$$

$$+ \frac{\text{Var}(L)}{2np_0(1-p_0)} U_5^{(k+2)} \left( \frac{x}{\sqrt{np_0(1-p_0)}} \right) + T_{n,k}^*(x)$$

where the remainder term satisfies

$$|T_{n,k}^*(x)| \leq \frac{C_k(p_0, m, \varepsilon)}{n^{3/2}(1+|x|)^5}$$

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for some suitable constant $C_k(p_0, m, \varepsilon)$ independent of $n$. Finally, gathering the previous identities, conclude that
\[
G_n(x) = U_5 \left( \frac{x}{\sqrt{p_0(1 - p_0)}} \right) + \frac{\text{Var}(L)}{2np_0(1 - p_0)} U_5'' \left( \frac{x}{\sqrt{p_0(1 - p_0)}} \right) + T_{n,0}^*(x) \\
+ \frac{1}{\sqrt{np_0(1 - p_0)}} S_1(np_0 + x\sqrt{n}) \times \left[ U_5' \left( \frac{x}{\sqrt{p_0(1 - p_0)}} \right) \right] \\
+ \frac{\text{Var}(L)}{2np_0(1 - p_0)} U_5'' \left( \frac{x}{\sqrt{p_0(1 - p_0)}} \right) + T_{n,1}^*(x) \\
+ \frac{1}{np_0(1 - p_0)} S_2(np_0 + x\sqrt{n}) \times \left[ U_5'' \left( \frac{x}{\sqrt{p_0(1 - p_0)}} \right) \right] \\
+ \frac{\text{Var}(L)}{2np_0(1 - p_0)} U_5''' \left( \frac{x}{\sqrt{p_0(1 - p_0)}} \right) + T_{n,2}^*(x) \\
+ \sum_{t = -m}^{m} \frac{1}{c_{\varepsilon,m} e^{-\varepsilon|t|}} T_n \left( \frac{x - l/\sqrt{n}}{\sqrt{p_0(1 - p_0)}} \right).
\]

Incorporating all the terms of type $O(n^{-3/2})$ in the remainder yields the thesis to be proved. \qed

The way is now paved to complete the proof of Theorem 9. Indeed, (15) follows by combining the thesis of Proposition 18 with (42) and (37)-(38). To see this, we start from the analysis of the quantity
\[
\Phi \left( \frac{3^{(\varepsilon,m)}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right) + \sum_{j=1}^{2} \left( \frac{1}{\sqrt{n}} \right)^j Q_j \left( \frac{3^{(\varepsilon,m)}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right) \\
+ \frac{\text{Var}(L)}{2np_0(1 - p_0)} \Phi'' \left( \frac{3^{(\varepsilon,m)}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right) \Phi \left( \frac{3^{(\varepsilon,m)}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right) \\
- \sum_{j=1}^{2} \left( \frac{1}{\sqrt{n}} \right)^j Q_j \left( \frac{3^{(\varepsilon,m)}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right) - \frac{\text{Var}(L)}{2np_0(1 - p_0)} \Phi'' \left( \frac{3^{(\varepsilon,m)}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right)
\]
which represents the “regular” part of the expression of (42). Taking account of (37)-(38), a straightforward Taylor expansion shows that the above quantity is equal to
\[
\Phi(\sqrt{t}) + \Phi'(\sqrt{t}) \left[ \frac{(1 - 2p_0)t}{6np_0(1 - p_0)} \right] + \frac{\sqrt{t}}{2np_0(1 - p_0)} \left\{ \text{Var}(L) - t(1 + 2p_0 - 2p_0^2) \right\} \\
+ \frac{1}{2} \Phi''(\sqrt{t}) \left( \frac{1 - 2p_0)^2 t^2}{36np_0(1 - p_0)} \right) + Q_1(\sqrt{t}) \frac{1}{\sqrt{n}} + Q_1'(\sqrt{t}) \frac{1 - 2p_0)t}{6n \sqrt{p_0(1 - p_0)}} + Q_2(\sqrt{t}) \frac{1}{n} \\
+ \frac{\text{Var}(L)}{2np_0(1 - p_0)} \Phi''(\sqrt{t}) - \Phi(\sqrt{t}) - \Phi'(\sqrt{t}) \left[ \frac{(1 - 2p_0)t}{6np_0(1 - p_0)} \right] \\
- \frac{\sqrt{t}}{2np_0(1 - p_0)} \left\{ \text{Var}(L) - t(1 + 2p_0 - 2p_0^2) \right\} - \frac{1}{2} \Phi''(\sqrt{t}) \left( \frac{1 - 2p_0)^2 t^2}{36np_0(1 - p_0)} \right) \\
- Q_1(-\sqrt{t}) \frac{1}{\sqrt{n}} - Q_1'(-\sqrt{t}) \frac{(1 - 2p_0)t}{6n \sqrt{p_0(1 - p_0)}} - Q_2(-\sqrt{t}) \frac{1}{n} - \frac{\text{Var}(L)}{2np_0(1 - p_0)} \Phi''(\sqrt{t}).
\]
Exploiting that \( x \Phi'(x) + \Phi''(x) = 0 \), we show that the terms containing \( \text{Var}(L) \) cancel out. Moreover, 
\[
\Phi(\sqrt{t}) - \Phi(-\sqrt{t}) = K(t) \text{ and } \Phi'(-\sqrt{t}) = \Phi'(-\sqrt{t}) \text{ hold for any } t \geq 0.
\]
These considerations lead to the following equivalent reformulation of the above term
\[
K(t) - \Phi'(\sqrt{t}) \frac{t^{3/2}(1 + 2p_0 - 2p_0^2)}{np_0(1 - p_0)} + \Phi''(\sqrt{t}) \frac{(1 - 2p_0)^2 t^2}{36np_0(1 - p_0)} + Q_1(\sqrt{t}) \frac{2}{\sqrt{n}}.
\]
This is the main part of the proof. Then, we deal with the “irregular” part of the expression of (42). The argument is essentially the same as above, even if we have now to take care of the expressions

\[
\text{Exploiting that } \Phi(\sqrt{t}) - \Phi(-\sqrt{t}) = K(t) \text{ and } \Phi'(-\sqrt{t}) = \Phi'(-\sqrt{t}) \text{ hold for any } t \geq 0.
\]

The only term which is not caught by this technique is 
\[
\frac{\Phi'(\sqrt{t})}{\sqrt{np_0(1 - p_0)}} \left[ S_i(np_0 + \sqrt{n} \mathcal{J}^{(e,m)}_{n, \pm}(p_0; t)) - S_i(np_0 + \sqrt{n} \mathcal{J}^{(e,m)}_{n, \pm}(p_0; t)) \right]
\]
which corresponds to the terms \( \frac{c_1(t; p_0, c_0, m)}{\sqrt{2\pi n}} e^{-t/2} \) in (15). Therefore, we can finally set

\[
c_1(t; p_0) := 2Q_1(\sqrt{t})
\]
\[
c_2(t; p_0) := -\Phi'(\sqrt{t}) \frac{t^{3/2}(1 + 2p_0 - 2p_0^2)}{p_0(1 - p_0)} + \Phi''(\sqrt{t}) \frac{(1 - 2p_0)^2 t^2}{36p_0(1 - p_0)} + I_n(t; p_0).
\]

Then, we pass to the analysis of the quantity

\[
\Phi \left( \frac{\mathcal{J}^{(e,m)}_{n, \pm}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right) + \sum_{j=1}^{2} \left( \frac{1}{\sqrt{n}} \right)^j Q_j \left( \frac{\mathcal{J}^{(e,m)}_{n, \pm}(p_0; t)}{\sqrt{p_0(1 - p_0)}} \right)
\]
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C holds for some \( \delta \) so that \( \delta \) is a fixed quantity, we resort once again on a large deviation argument to choose \( \delta \) sufficiently small. Then, recalling that \( \Phi \) is the cumulative distribution function of the standard normal distribution, we get the equivalent expression
\[
\frac{\text{Var}(L)}{2np_0(1-p_0)} \Phi'' \left( \frac{\mathcal{Y}^{(\varepsilon,m)}_{n_+}(p_0; t)}{\sqrt{p_0(1-p_0)}} \right) - \Phi \left( \frac{\mathcal{Y}^{(\varepsilon,m)}_{n_+}(p_0; t)}{\sqrt{p_0(1-p_0)}} \right)
- \sum_{j=1}^{2} \left( \frac{1}{\sqrt{n}} \right)^j Q_j \left( \frac{\mathcal{Y}^{(\varepsilon,m)}_{n_+}(p_0; t)}{\sqrt{p_0(1-p_0)}} \right) - \frac{\text{Var}(L)}{2np_0(1-p_0)} \Phi'' \left( \frac{\mathcal{Y}^{(\varepsilon,m)}_{n_+}(p_0; t)}{\sqrt{p_0(1-p_0)}} \right)
\]

which represents the “regular” part of the expression of (43). Again expanding by the Taylor formula, we get the equivalent expression
\[
K(t) - \Phi'(\sqrt{t}) \frac{t^{3/2}(1+2p_0-2p_0^2)}{np_0(1-p_0)} + \Phi''(\sqrt{t}) \frac{(1-2p_0)^2t^2}{36np_0(1-p_0)} + Q_1(\sqrt{t}) \frac{2}{\sqrt{n}}\]

Since \( \Phi''(\sqrt{t}) = -\left( \frac{t_{1/2}}{2\pi} \right)^{1/2} \) and the “irregular” part contained in \( I_n(t;p_0) \) is the same as above, we conclude the validity of (16), proving the theorem.

### A.3 Proof of Theorem 4 for \( k = 2 \)

We start again from (24) where the random variables \( A_n \) and \( L \), defined on the probability space \((\Omega, \mathcal{F}, \text{Pr})\), are independent, \( A_n \sim \text{Bin}(n,p_1) \) and \( L \) has the Laplace distribution (4). After fixing \( \lambda_\alpha \) such that \( K(\lambda_\alpha) = 1-\alpha \), according to Theorem 9, we have that
\[
1 - \beta_n(p_1; \alpha) = \text{Pr}[A_{n,T}(p_0) \leq \lambda_\alpha].
\]

First, for \( \delta \in (0,1/2) \), we define the event \( E_n(\delta) \subset \Omega \) as
\[
E_n(\delta) := \left\{ \omega \in \Omega : \frac{A_n(\omega)}{n} \in [\delta, 1-\delta] \right\}. \tag{46}
\]

Then, recalling that
\[
\mathcal{D}_{KL}(p_0 \parallel p_1) := p_0 \log \left( \frac{p_0}{p_1} \right) + (1-p_0) \log \left( \frac{1-p_0}{1-p_1} \right)
\]
is a fixed quantity, we resort once again on a large deviation argument to choose \( \delta \) sufficiently small so that
\[
\text{Pr}[E_n(\delta)^c] \leq e^{-nC(\delta)} \tag{47}
\]
holds for some \( C(\delta) > \mathcal{D}_{KL}(p_0 \parallel p_1) \). In this way, we can write
\[
1 - \beta_n(p_1; \alpha) = \text{Pr}[A_{n,T}(p_0) \leq \lambda_\alpha, E_n(\delta)] + \text{Pr}[A_{n,T}(p_0) \leq \lambda_\alpha, E_n(\delta)^c]
\]
with
\[
\text{Pr}[A_{n,T}(p_0) \leq \lambda_\alpha, E_n(\delta)^c] = o \left( e^{-n\mathcal{D}_{KL}(p_0 \parallel p_1)} \right).
\]

Then, for \( M > 0 \), we define another event \( E'_n(M) \subset \Omega \) as
\[
E'_n(M) := \left\{ \omega \in \Omega : \left| \frac{A_n(\omega)}{n} - p_0 \right| \leq \frac{M}{\sqrt{n}} \right\}. \tag{48}
\]

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and we write
\[
\Pr[\Lambda_n,\mathcal{T}(p_0) \leq \lambda_\alpha, E_n(\delta)] = \Pr[\Lambda_n,\mathcal{T}(p_0) \leq \lambda_\alpha, E_n(\delta) \cap E'_n(M)] \\
+ \Pr[\Lambda_n,\mathcal{T}(p_0) \leq \lambda_\alpha, E_n(\delta) \cap E'_n(M)^c] .
\]

The advantage of such a preliminary step is that, on \( E_n(\delta) \cap E'_n(M) \), the assumptions of all the Lemmata stated in the previous subsection are fulfilled and, by resorting to (36), we can write
\[
\{ \Lambda_n,\mathcal{T}(p_0) \leq \lambda_\alpha, E_n(\delta) \cap E'_n(M) \} = \begin{cases} 
3^{(\varepsilon,m)}_{n,-}(p_0; \lambda_\alpha) \leq Z_n \leq 3^{(\varepsilon,m)}_{n,+}(p_0; \lambda_\alpha), E_n(\delta) \cap E'_n(M) \end{cases}
\]
where \( Z_n \) is the same random variable as in (41). Moreover, if \( M > \lambda_\alpha \), as we will choose, we have eventually that
\[
\{ \Lambda_n,\mathcal{T}(p_0) \leq \lambda_\alpha, E_n(\delta) \cap E'_n(M)^c \} = \emptyset
\]
eventually, yielding that
\[
1 - \beta_n(p_1; \alpha) = \Pr[3^{(\varepsilon,m)}_{n,-}(p_0; \lambda_\alpha) \leq Z_n \leq 3^{(\varepsilon,m)}_{n,+}(p_0; \lambda_\alpha)] + o\left(e^{-nD_{KL}(p_0 \parallel p_1)}\right).
\]

The proof of (49), along with the proper choice of \( M \), is contained in the following

**Lemma 19.** If \( b/n \in [\delta, 1-\delta] \) and \( |b/n - p_0| > \sqrt{\lambda_\alpha + S(p_0, \delta; \epsilon, m)}/n \), with
\[
S(p_0, \delta; \epsilon, m) := 2 \log \left( \max_{x \in [\delta, 1-\delta]} \mathbb{E} \left[ \exp \left\{ \frac{x(1-p_0)}{(1-x)p_0} L \right\} \right] \right),
\]
then, eventually, there holds
\[
2 \log \left( \frac{L_n,\mathcal{T}(\hat{p}_n,\mathcal{T}; b)}{L_n,\mathcal{T}(p_0; b)} \right) > \lambda_\alpha .
\]

**Proof.** Since \( L_n,\mathcal{T}(\hat{p}_n,\mathcal{T}; b) \geq L_n,\mathcal{T}(b/n; b) \), by definition of MLE estimator, we recall (20) to write
\[
2 \log \left( \frac{L_n,\mathcal{T}(\hat{p}_n,\mathcal{T}; b)}{L_n,\mathcal{T}(p_0; b)} \right) \geq 2 \log \left( \frac{L_n^{(0)}(b/n; b)}{L_n^{(0)}(p_0; b)} \right) + 2 \log \left( \frac{H_n^{(\varepsilon,m)}(\hat{p}_n,\mathcal{T}; b)}{H_n^{(\varepsilon,m)}(p_0; b)} \right).
\]

Then, recalling (21), we have
\[
2 \log \left( \frac{L_n^{(0)}(b/n; b)}{L_n^{(0)}(p_0; b)} \right) = 2nD_{KL}(b/n \parallel p_0) \geq n(b/n - p_0)^2 > \lambda_\alpha + S(p_0, \delta; \epsilon, m) .
\]

Finally, exploiting that \( \rho(n, b, l) \sim \left( \frac{b/n}{1-b/n} \right)^l \) and recalling (22), we conclude that
\[
2 \log \left( \frac{H_n^{(\varepsilon,m)}(b/n; b)}{H_n^{(\varepsilon,m)}(p_0; b)} \right) \geq -S(p_0, \delta; \epsilon, m)
\]
holds eventually, completing the proof. \( \square \)
At this stage, we come back to (50) by writing

\[
Z_n = \sqrt{p_1(1-p_1)} V_n + \frac{L}{\sqrt{n}} - \sqrt{n} \Delta
\]

with

\[
V_n := \frac{A_n - np_1}{\sqrt{np_1(1-p_1)}}
\]

and \(\Delta := p_0 - p_1\). Thus, the event considered in (50) can be rewritten in terms of the random variable \(V_n\) as follows

\[
\{ 3_{n,-}^{(e,m)}(p_0; \lambda_\alpha) \leq Z_n \leq 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha) \}
= \left\{ -\frac{L}{\sqrt{n}} + \sqrt{n} \Delta + 3_{n,-}^{(e,m)}(p_0; \lambda_\alpha) \leq \sqrt{p_1(1-p_1)} V_n \leq -\frac{L}{\sqrt{n}} + \sqrt{n} \Delta + 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha) \right\}.
\]

We now introduce the distribution function \(H_n\) of \(V_n\), and we notice that (50) becomes

\[
1 - \beta_n(p_1; \alpha) = \frac{1}{c_{\varepsilon,m}} \sum_{l=-m}^{m} e^{-\varepsilon |l|} \left[ H_n \left( \frac{\sqrt{n} \Delta + l/\sqrt{n} + 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha)}{\sqrt{p_1(1-p_1)}} \right) - H_n \left( \frac{\sqrt{n} \Delta + l/\sqrt{n} + 3_{n,-}^{(e,m)}(p_0; \lambda_\alpha)}{\sqrt{p_1(1-p_1)}} \right) \right] + o(e^{-nD_{KL}(p_0 \parallel p_1)}).
\]

(51)

At this stage, we are in a position to apply Theorem 10 in Chapter VIII of Petrov (1975). Supposing, for instance that \(\Delta < 0\), we have that

\[
H_n \left( \frac{\sqrt{n} \Delta + l/\sqrt{n} + 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha)}{\sqrt{p_1(1-p_1)}} \right) \sim \Phi \left( \frac{\sqrt{n} \Delta + l/\sqrt{n} + 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha)}{\sqrt{p_1(1-p_1)}} \right) \times \exp \left\{ \frac{[\sqrt{n} \Delta + l/\sqrt{n} + 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha)]^3}{\sqrt{n} p_1(1-p_1)^{3/2}} \right\} L \left( \frac{\Delta + l/n + 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha)/\sqrt{n}}{\sqrt{p_1(1-p_1)}} \right)
\]

(52)

where \(L\) stands for the so-called Cramér-Petrov series relative to the Bernoulli distribution of parameter \(p_1\). Thus, putting \(\mathcal{X}_{n,\pm} := \frac{\Delta + l/n + 3_{n,\pm}^{(e,m)}(p_0; \lambda_\alpha)/\sqrt{n}}{\sqrt{p_1(1-p_1)}}\), we proceed by resorting to the well-known Mills approximation to write

\[
\Phi \left( \frac{\sqrt{n} \Delta + l/\sqrt{n} + 3_{n,+}^{(e,m)}(p_0; \lambda_\alpha)}{\sqrt{p_1(1-p_1)}} \right) \sim \frac{1}{\sqrt{n}} \exp \left\{ -\frac{n}{2} \mathcal{X}_{n,\pm}^2 \right\}.
\]

Therefore, (52) can be simplified as follows

\[
H_n \left( \frac{\sqrt{n} \Delta + l/\sqrt{n} + 3_{n,\pm}^{(e,m)}(p_0; \lambda_\alpha)}{\sqrt{p_1(1-p_1)}} \right) \sim \frac{1}{\sqrt{n}} \exp \left\{ -n \left( \frac{1}{2} \mathcal{X}_{n,\pm}^2 - \mathcal{X}_{n,\pm}^3 L(\mathcal{X}_{n,\pm}) \right) \right\}.
\]

At this stage, to complete the proof, we need a technical result that characterizes the expression \(\frac{1}{2} t^2 - t^3 L(t)\) in an exponential model parametrized by the mean.
Lemma 20. Let \((X, \mathcal{X})\) be a measurable space, endowed with a \(\sigma\)-finite reference measure \(\mu\). Let \(t : X \to \mathbb{R}^d\) be a measurable map such that the set 
\[
\Gamma := \left\{ y \in \mathbb{R}^d \mid \int_X e^{y \cdot t(x)} \mu(dx) < +\infty \right\}
\]
is open and convex. Putting \(M(y) := \log \left( \int_X e^{y \cdot t(x)} \mu(dx) \right)\), \(V(y) := \nabla M(y)\) and \(\Theta := V(\Gamma)\), we have that \(\Theta\) is open and \(V\) is a smooth diffeomorphism between \(\Gamma\) and \(\Theta\). Moreover, the family of \(\mu\)-densities \(\{f_\theta\}_{\theta \in \Theta}\) given by
\[
f_\theta(x) := \exp \{V^{-1}(\theta) \cdot t(x) - M(V^{-1}(\theta))\} \quad (x \in X)
\]
defines a regular exponential family parametrized by the mean. Moreover, we have 
\[
D_{KL}(\tau \parallel \theta) := \int_X \log \left( \frac{f_\tau(x)}{f_\theta(x)} \right) f_\tau(x) \mu(dx)
\]
\[
= \tau \cdot [V^{-1}(\tau) - V^{-1}(\theta)] - [M(V^{-1}(\theta)) - M(V^{-1}(\tau))]
\]
\[
= \Psi_0^\tau(\tau) := \sup\{\tau \cdot y - \Psi_\theta(\tau)\}
\]
where \(\Psi_\theta(\tau) := \log \left( \int_X e^{\tau \cdot t(x)} f_\theta(x) \mu(dx) \right)\). For \(d = 1\), putting
\[
\sigma^2(\theta) := \int_X [t(x) - \theta]^2 f_\theta(x) \mu(dx),
\]
we have that 
\[
\frac{1}{2} \left( \frac{\tau - \theta}{\sigma(\theta)} \right)^2 - \left( \frac{\tau - \theta}{\sigma(\theta)} \right)^3 \mathcal{L} \left( \frac{\tau - \theta}{\sigma(\theta)} \right) = D_{KL}(\tau \parallel \theta) \tag{53}
\]
for every \(\tau, \theta \in \Theta\), where \(\mathcal{L}\) denotes the Cramér-Petrov series relative to the distribution of \(t(X) - \theta\), with \(X \sim f_\theta\). Finally, for a generic dimension \(d\), putting \(L(z) := \Psi_\theta(z) - z \cdot \theta\), we have that 
\[
\hat{z} \cdot \nabla L(\hat{z}) = D_{KL}(\tau \parallel \theta) \tag{54}
\]
if \(\hat{z}\) is defined as the solution of the equation \(\nabla L(\hat{z}) = \tau - \theta\).

Proof. For the main part of the lemma, we just quote any good reference on exponential families, such as Barndorff-Nielsen (1978). Here, we only prove (53). Putting \(\eta := \frac{\tau - \theta}{\sigma(\theta)}\), we have by definition that
\[
\frac{1}{2} \eta^2 - \eta^3 \mathcal{L}(\eta) := \bar{z}L'(\bar{z}) - L(\bar{z})
\]
where \(L(z) := -z\theta + \Psi_\theta(z)\) and \(\bar{z}\) is the solution of the equation \(\sigma(\theta)\eta = L'(\bar{z})\). Thus, the last equation can be rewritten as \(\tau = \Psi_\theta'(\bar{z})\). Whence,
\[
\bar{z}L'(\bar{z}) - L(\bar{z}) = \bar{z}(-\theta + \Psi_\theta'(\bar{z})) + \bar{z}\theta - \Psi_\theta(\bar{z}) = \tau\bar{z} - \Psi_\theta(\bar{z}) = \Psi_\theta(\tau)
\]
completing the proof of (53). The same computation in generic dimension yields (54). \(\square\)

At this stage, noticing that the family of Bernoulli distributions is a member of the regular exponential family parametrized by the mean, we can apply (53) with \(\theta = p_1\) and \(\tau = p_0 + l/n + 3^{(\varepsilon, m)}_{n, \pm}(p_0; \lambda_0)\) to conclude that 
\[
H_n \left( \frac{\sqrt{n} \Delta + l/\sqrt{n} + 3^{(\varepsilon, m)}_{n, \pm}(p_0; \lambda_0)}{\sqrt{p_1(1 - p_1)}} \right)
\]
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\[ \sim \frac{1}{\sqrt{n}} \exp \left\{ -n \mathcal{D}_{KL} \left( p_0 + l/n + 3^{(\varepsilon,m)}(p_0; \lambda_\alpha)/\sqrt{n} \parallel p_1 \right) \right\}. \]

Thus, if \( \Delta < 0 \), we have
\[ \mathcal{D}_{KL} \left( p_0 + l/n + 3^{(\varepsilon,m)}(p_0; \lambda_\alpha)/\sqrt{n} \parallel p_1 \right) < \mathcal{D}_{KL} \left( p_0 + l/n + 3^{(\varepsilon,m)}(p_0; \lambda_\alpha)/\sqrt{n} \parallel p_1 \right) \]
and we conclude that
\[ 1 - \beta_n(p_1; \alpha) \sim \frac{1}{\sqrt{n}} c_{\varepsilon,m} \frac{1}{\sqrt{n}} \sum_{l=-m}^m e^{-|l|} \times \]
\[ \times \exp \left\{ -n \mathcal{D}_{KL} \left( p_0 + l/n + 3^{(\varepsilon,m)}(p_0; \lambda_\alpha)/\sqrt{n} \parallel p_1 \right) \right\}. \]

As a last step of the proof, we expand the above expression containing \( \mathcal{D}_{KL} \) by the Taylor formula, obtaining
\[ \mathcal{D}_{KL} \left( p_0 + l/n + 3^{(\varepsilon,m)}(p_0; \lambda_\alpha)/\sqrt{n} \parallel p_1 \right) \]
\[ = \mathcal{D}_{KL} (p_0 \parallel p_1) + \partial_{p_0} \mathcal{D}_{KL} (p_0 \parallel p_1) \left( l/n + 3^{(\varepsilon,m)}(p_0; \lambda_\alpha)/\sqrt{n} \right) \]
\[ + \frac{1}{2} \partial^2_{p_0} \mathcal{D}_{KL} (p_0 \parallel p_1) \left[ 3^{(\varepsilon,m)}(p_0; \lambda_\alpha) \right]^2/n + O(n^{-3/2}). \]

In conclusion, we get
\[ 1 - \beta_n(p_1; \alpha) \sim \frac{1}{\sqrt{n}} \mathcal{L}_n (\partial_{p_0} \mathcal{D}_{KL} (p_0 \parallel p_1)) \times \]
\[ \times \exp \left\{ -n \mathcal{D}_{KL} (p_0 \parallel p_1) + \partial_{p_0} \mathcal{D}_{KL} (p_0 \parallel p_1) 3^{(\varepsilon,m)}(p_0; \lambda_\alpha)/\sqrt{n} \right. \]
\[ + \left. \frac{1}{2} \partial^2_{p_0} \mathcal{D}_{KL} (p_0 \parallel p_1) \left[ 3^{(\varepsilon,m)}(p_0; \lambda_\alpha) \right]^2/n \right\} \]
which entails the thesis of the theorem.

\section{Proofs for \( k > 2 \)}

In this section we will prove Theorems 3, 10 and 4 in the case that our data set is a table with more than two cells, i.e. when \( k > 2 \). Thus, the observable variable reduces to the countings \( b = (b_1, \ldots, b_{k-1}) \) contained in the first \( k - 1 \) cells. Here, we state some preparatory results as in Appendix A.

\subsection{Preparatory steps}

First of all, let us state a proposition that fixes the exact expression of the likelihood under the true model (5).

\textbf{Lemma 21.} Let \( m \in \mathbb{N} \) and \( \varepsilon > 0 \) be fixed to define the Laplace distribution (4). Then, if \( n > m \) and \( b_1 \in R(n,m) := \{m, m+1, \ldots, n-m\} \) for \( i = 1, \ldots, k-1 \), we have
\[ L_{n,T}(p; b) = \left( \frac{1}{c_{\varepsilon,m}} \right)^{k-1} \prod_{l_1=-m}^m \prod_{l_{k-1}=-m}^m e^{-\varepsilon l_1} \left( b_1 - l_1, \ldots, b_{k-1} - l_{k-1} \right) \times \] \[ (55) \]

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\[ \times p_1^{b_1-l_1} \cdots p_k^{b_k-l_k} (1 - |p|)^{n-|b|} \quad (p \in \Delta_{k-1}). \]

Moreover, under the same assumption, for any \( p \) in the interior \( \Delta_{k-1}^o \) of \( \Delta_{k-1} \) we can write

\[ L_{n,T}(p; b) = L_n^{(0)}(p; b) \cdot H_{n,T}^{(\varepsilon,m)}(p; b) \quad (56) \]

with

\[ L_n^{(0)}(p; b) := \left( \frac{n}{b_1, \ldots, b_{k-1}} \right) \prod_{i=1}^{b_k-1} (1 - |p|)^{n-|b|} \quad (57) \]

\[ H_{n,T}^{(\varepsilon,m)}(p; b) := \left( \frac{1}{c_{\varepsilon,m}} \right)^{k-1} \sum_{l_1=-m}^{m} \cdots \sum_{l_{k-1}=-m}^{m} e^{-\varepsilon|l|} \rho(n, b, l) \times \]

\[ \times \left( \frac{1-|p|}{p_1} \right)^{l_1} \cdots \left( \frac{1-|p|}{p_{k-1}} \right)^{l_{k-1}} \quad (58) \]

\[ \rho(n, b, l) := \frac{b_1! \cdots b_k! (n-|b|)!}{(b_1-l_1)! \cdots (b_k-l_{k-1})! (n-|b+l|)!}. \quad (59) \]

**Proof.** Start from

\[ B_n = A_n + L \quad (60) \]

where the random variables \( A_n \) and \( L \), defined on the probability space \( (\Omega, \mathcal{F}, Pr) \), are independent, \( A_n \sim Mult(n, p) \) and \( L = (L_1, \ldots, L_{k-1}) \) is a random vector with independent components with each \( L_i \) having the Laplace distribution \( (4) \). By definition \( L_{n,T}(p; b) := Pr[B_n = b] \), so that, under the assumption of the Lemma, the independence of \( A_n \) and \( L \) entails

\[ Pr[B_n = b] = \sum_{l_1=-m}^{m} \cdots \sum_{l_{k-1}=-m}^{m} Pr[A_n = b - l] \cdot Pr[L = l] \]

proving \( (55) \). Finally, the decomposition \( (56) \) ensues from straightforward algebraic manipulations.

The next result is a multidimensional analogue of Lemma \( 12 \), that can be obtained by considering the observable quantity \( b \) as itself dependent by \( n \). This task proves very cumbersome for a general dimension \( k \) so that, in the remaining part of the subsection, we will confine ourselves to dealing with the case \( k = 3 \).

**Lemma 22.** Let \( \xi \in \Delta_{k}^o \) be a fixed vector. Under the assumption that \( b = n(\xi + \epsilon_n) \) with \( \lim_{n \to 0} \epsilon_n = 0 \), there exists \( n_0 = n_0(\xi; \epsilon, m) \) such that

\[ \rho_n(\xi, l) := \rho(n, b, l) := \left( \frac{\xi_1}{1 - \xi_1 - \xi_2} \right)^{l_1} \left( \frac{\xi_2}{1 - \xi_1 - \xi_2} \right)^{l_2} \times \]

\[ \times \left\{ 1 + \alpha_1(\xi, l) \cdot \epsilon_n + \epsilon_n \alpha_2(\xi, l) \epsilon_n + \alpha_3(\xi, l) \frac{\epsilon_n}{n} + \alpha_4(\xi, l) \cdot \frac{\epsilon_n}{n} + R_n(\xi, l) \right\} \quad (61) \]

holds for any \( n \geq n_0 \) and \( l \in \{-m, \ldots, m\}^2 \), where

\[ \alpha_1(\xi, l) := \left( \frac{l_1}{\xi_1} + \frac{l_1 + l_2}{1 - \xi_1 - \xi_2}, \frac{l_2}{\xi_2} + \frac{l_1 + l_2}{1 - \xi_1 - \xi_2} \right) \]

\[ \alpha_2(\xi, l) := \begin{pmatrix} a_{1,1}(\xi, l) & a_{1,2}(\xi, l) \\ a_{2,1}(\xi, l) & a_{2,2}(\xi, l) \end{pmatrix} \]

\[ \alpha_3(\xi, l) := \begin{pmatrix} \frac{\xi_1}{1 - \xi_1 - \xi_2} & \frac{\xi_1}{1 - \xi_1 - \xi_2} \\ \frac{\xi_2}{1 - \xi_1 - \xi_2} & \frac{\xi_2}{1 - \xi_1 - \xi_2} \end{pmatrix} \]

\[ \alpha_4(\xi, l) := \begin{pmatrix} \frac{\xi_1}{1 - \xi_1 - \xi_2} & \frac{\xi_1}{1 - \xi_1 - \xi_2} \\ \frac{\xi_2}{1 - \xi_1 - \xi_2} & \frac{\xi_2}{1 - \xi_1 - \xi_2} \end{pmatrix} \]

\[ R_n(\xi, l) := \begin{pmatrix} \frac{\xi_1}{1 - \xi_1 - \xi_2} & \frac{\xi_1}{1 - \xi_1 - \xi_2} \\ \frac{\xi_2}{1 - \xi_1 - \xi_2} & \frac{\xi_2}{1 - \xi_1 - \xi_2} \end{pmatrix} \]

\[ \lim_{n \to 0} \frac{\epsilon_n}{n} = 0 \quad (62) \]
\[a_{1,1}(\xi, 1) := \left( \frac{1}{2} \right) \frac{1}{\xi_1^2} + \left( \frac{1}{2} \right) \frac{1}{\xi_1 - \xi_2} + \frac{1}{\xi_1(1 - \xi_1 - \xi_2)}\]

\[a_{1,2}(\xi, 1) := \left( \frac{1}{2} \right) \frac{1}{\xi_2^2} + \left( \frac{1}{2} \right) \frac{1}{\xi_1 - \xi_2} + \frac{1}{\xi_2(1 - \xi_1 - \xi_2)}\]

\[a_{2,1}(\xi, 1) := \left( \frac{1}{2} \right) \frac{1}{\xi_1^2} + \left( \frac{1}{2} \right) \frac{1}{\xi_1 - \xi_2} + \frac{1}{\xi_1(1 - \xi_1 - \xi_2)}\]

\[a_{2,2}(\xi, 1) := \left( \frac{1}{2} \right) \frac{1}{\xi_2^2} + \left( \frac{1}{2} \right) \frac{1}{\xi_1 - \xi_2} + \frac{1}{\xi_2(1 - \xi_1 - \xi_2)}\]

\[\alpha_3(\xi, 1) := \left[ \left( \frac{1}{2} \right) \frac{1}{\xi_1} + \left( \frac{2}{2} \right) \frac{1}{\xi_2} + \left( \frac{1}{2} \right) \frac{1}{\xi_1 - \xi_2} \right] \]

\[\alpha_4(\xi, 1) := -\frac{\lambda(2 - \lambda + 2\xi^2 + 4\xi - 2\xi)}{2\xi^2(1 - \xi)^2}\]

\[|R_n(\xi, 1)| \leq C(\xi, 1) \left[ |\xi|^3 + \frac{1}{n^2} + \frac{|\xi|^2}{n} \right]\]

for some constant \(C(\xi, 1)\). Therefore, under the same assumption, there holds

\[H_{n, \lambda}^{(\epsilon, m)}(p; b) = H_0^{(\epsilon, m)}(p; \xi) + H_1^{(\epsilon, m)}(p; \xi) \cdot \epsilon_n + \epsilon_n H_2^{(\epsilon, m)}(p; \xi) \cdot \frac{1}{n} + R_n(p; \xi)\]

(62)

for any \(n \geq n_0\), where

\[H_{i}^{(\epsilon, m)}(p; \xi) := \left( \frac{1}{c_{\epsilon, m}} \right)^2 \sum_{l_1 = -m}^{m} \sum_{l_2 = -m}^{m} e^{-\epsilon(|l_1| + |l_2|)} \alpha_i(\xi, 1) \times \]

\[\times \left( \frac{\xi_1(1 - p_1 - p_2)}{p_1(1 - \xi_1 - \xi_2)} \right)^{l_1} \left( \frac{\xi_2(1 - p_1 - p_2)}{p_2(1 - \xi_1 - \xi_2)} \right)^{l_2} \quad \text{for} \quad i = 0, 3\]

\[H_{i}^{(\epsilon, m)}(p; \xi) := \left( \frac{1}{c_{\epsilon, m}} \right)^2 \sum_{l_1 = -m}^{m} \sum_{l_2 = -m}^{m} e^{-\epsilon(|l_1| + |l_2|)} \alpha_i(\xi, 1) \times \]

\[\times \left( \frac{\xi_1(1 - p_1 - p_2)}{p_1(1 - \xi_1 - \xi_2)} \right)^{l_1} \left( \frac{\xi_2(1 - p_1 - p_2)}{p_2(1 - \xi_1 - \xi_2)} \right)^{l_2} \quad \text{for} \quad i = 1, 4\]

\[H_{2}^{(\epsilon, m)}(p; \xi) := \left( \frac{1}{c_{\epsilon, m}} \right)^2 \sum_{l_1 = -m}^{m} \sum_{l_2 = -m}^{m} e^{-\epsilon(|l_1| + |l_2|)} \alpha_2(\xi, 1) \times \]

\[\times \left( \frac{\xi_1(1 - p_1 - p_2)}{p_1(1 - \xi_1 - \xi_2)} \right)^{l_1} \left( \frac{\xi_2(1 - p_1 - p_2)}{p_2(1 - \xi_1 - \xi_2)} \right)^{l_2} \]

\[|R_n(p; \xi)| \leq C(p; \xi) \left[ |\xi|^3 + \frac{1}{n^2} + \frac{|\xi|^2}{n} \right]\]

with \(\alpha_0(\xi, 1) \equiv 1\) and

\[C(p; \xi) := \left( \frac{1}{c_{\epsilon, m}} \right)^2 \sum_{l_1 = -m}^{m} \sum_{l_2 = -m}^{m} e^{-\epsilon(|l_1| + |l_2|)} C(\xi, 1) \times \]

\[\times \left( \frac{\xi_1(1 - p_1 - p_2)}{p_1(1 - \xi_1 - \xi_2)} \right)^{l_1} \left( \frac{\xi_2(1 - p_1 - p_2)}{p_2(1 - \xi_1 - \xi_2)} \right)^{l_2} \]
Proof. We start by dealing with (61). First, we find \( n_0 = n_0(\xi; \varepsilon, m) \) in such a way that the assumptions of Lemma 11 are fulfilled for any \( n \geq n_0 \). Then, as in the proof of Lemma 12, the thesis can be checked by direct computation if either \( l_1 \in \{0, 1\} \) or \( l_2 \in \{0, 1\} \). Now, if \( l_1, l_2 \in \{2, \ldots, m\} \), we use (59) to get

\[
\rho(n, b, l) = \frac{(b_1)_{j_1} (b_2)_{j_2}}{(n - b_1 - b_2 + 1)!} = \frac{(b_1)_{j_1} (b_2)_{j_2} (n - b_1 - b_2)}{(n - b_1 - b_2 + 1)!}
\]

where again \( s(l, k) \) denotes the Stirling number of first kind. Whence,

\[
\rho_n(\xi, l) = \frac{(\sum_{k=1}^l s(l_1, k_1) n^{k_1-l_1} (\xi + \epsilon_{n,1})^{k_1}) \cdot (\sum_{l_2}^s l_2 (l_2, k_2) n^{k_2-l_2} (\xi + \epsilon_{n,2})^{k_2})}{\sum_{k=1}^{l_1+l_2} s(l_1, l_2 + 1, k) n^{k-l_1-l_2 - 1} (1 - \xi - \xi - \epsilon_{n,1} - \epsilon_{n,2})^{k-1}}
\]

At this stage, recalling that \( s(l, l) = 1 \) and \( s(l, l - 1) = -(\frac{l}{2}) \), and exploiting the binomial formula, we have

\[
\mathcal{N}_n := \xi_1^{l_1} \left\{ 1 + l_1 \frac{\epsilon_{n,1}}{\xi_1} + \frac{l_1 (l_1 - 1)}{2} \frac{\epsilon_{n,1}^2}{\xi_1^2} - \frac{l_1 (l_1 - 1)}{2} \frac{\epsilon_{n,1}^2}{\xi_1^2} + R_n(\xi_1, l_1) \right\} \times \xi_2^{l_2} \left\{ 1 + l_2 \frac{\epsilon_{n,2}}{\xi_2} + \frac{l_2 (l_2 - 1)}{2} \frac{\epsilon_{n,2}^2}{\xi_2^2} - \frac{l_2 (l_2 - 1)}{2} \frac{\epsilon_{n,2}^2}{\xi_2^2} + R_n(\xi_2, l_2) \right\}
\]

\[
\mathcal{D}_n := (1 - \xi - \xi) l_1 + l_2 \left\{ 1 - (l_1 + l_2) \frac{\epsilon_{n,1} + \epsilon_{n,2}}{1 - \xi - \xi} + \frac{(l_1 + l_2)(l_1 + l_2 - 1)}{2} \frac{(\epsilon_{n,1} + \epsilon_{n,2})^2}{(1 - \xi - \xi)^2} \right\}
\]

for suitable expressions of \( R_n(\xi, l) \) (possibly different from line to line) which are, in any case, bounded by an expression like \( C(\xi, l) \left[ \epsilon_{n,1}^3 + \frac{1}{m^2} + \frac{\epsilon_{n,2}^2}{m} \right] \). To proceed further, we exploit that \( \frac{1}{1 + t} = 1 - t + t^2 + o(t^2) \) as \( t \to 0 \), to obtain

\[
\mathcal{D}_n^{-1} := \frac{1}{(1 - \xi - \xi) l_1 + l_2} \left\{ 1 + (l_1 + l_2) \frac{\epsilon_{n,1} + \epsilon_{n,2}}{1 - \xi - \xi} + \frac{(l_1 + l_2)(l_1 + l_2 + 1)}{2} \frac{(\epsilon_{n,1} + \epsilon_{n,2})^2}{(1 - \xi - \xi)^2} \right\}
\]

The thesis now follows by multiplying the last expression by that of \( \mathcal{N}_n \), neglecting all the terms which are comparable with \( R_n(\xi, l) \). Thus, (25) is proved also for all \( l \in \{2, \ldots, m\} \). If either \( l_1 \in \{-m, \ldots, -1\} \) or \( l_2 \in \{-m, \ldots, -1\} \) the argument can be reduced to the previous case, as in the proof of Lemma 12. This completes the proof of (61).

Finally, (62) follows immediately from the combination of (58) with (61). \( \square \)
The way is now paved to state a result on the expansion of the MLE which is analogous to Lemma 13

**Lemma 23.** Under the same assumption of Lemma 22, there holds

\[ \hat{p}_{n,T} = \xi + \epsilon_n + R_{n,T}(\xi; \epsilon_n) \]  

(63)

where \( R_{n,T}(\xi; \epsilon_n) \) is a remainder term satisfying \( |R_{n,T}(\xi; \epsilon_n)| = O(|\epsilon_n|^3/n) \).

**Proof.** We provide only a sketch of the proof, since the ensuing computations are too heavy to be fully reproduced. In any case, we start by providing a large \( n \) expansion for the log-likelihood, as follows

\[
\ell_{n,T}(p; b) := \log L_{n,T}(p; b) = \log \left( \frac{n}{b_1, b_2} \right) + n(\xi_1 + \epsilon_n, 1) \log p_1 + n(\xi_2 + \epsilon_n, 2) \log p_2 \\
+ n(1 - \xi_1 - \xi_2 - \epsilon_n, 1 - \epsilon_n, 2) \log(1 - p_1 - p_2) \\
+ h_0^{(\epsilon, m)}(p; \xi) + h_1^{(\epsilon, m)}(p; \xi) \cdot \epsilon_n \\
+ t \epsilon_n \| \epsilon_n \|_2^{(\epsilon, m)}(p; \xi) \epsilon_n \\
+ h_3^{(\epsilon, m)}(p; \xi) \frac{1}{n} + h_4^{(\epsilon, m)}(p; \xi) \cdot \frac{\epsilon_n}{n} + r_n(p; \xi)
\]

where, upon denoting by \( \otimes \) the outer product between vectors,

- \( h_0^{(\epsilon, m)}(p; \xi) := \log H_0^{(\epsilon, m)}(p; \xi) \)
- \( h_1^{(\epsilon, m)}(p; \xi) := \frac{H_1^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} \)
- \( H_2^{(\epsilon, m)}(p; \xi) := \frac{H_2^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} - \frac{1}{2} \left( \frac{H_1^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} \right) \otimes \left( \frac{H_1^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} \right) \)
- \( h_3^{(\epsilon, m)}(p; \xi) := \frac{H_3^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} \)
- \( h_4^{(\epsilon, m)}(p; \xi) := \frac{H_4^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} - \frac{H_3^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} \frac{H_3^{(\epsilon, m)}(p; \xi)}{H_0^{(\epsilon, m)}(p; \xi)} \)

for some constant \( C(p; \xi) \). Then, to find the maximum point of the likelihood, we study the equation \( \nabla_p \ell_{n,T}(p; b) = 0 \), which reads

\[
\begin{cases}
\frac{n p_1 - \xi_1 - \epsilon_n, 1 + p_2(\xi_1 + \epsilon_n, 1) - p_1(\xi_2 + \epsilon_n, 2)}{p_1(1 - p_1 - p_2)} &= \left[ \partial_{p_1} h_0^{(\epsilon, m)}(p; \xi) \right] + \left[ \partial_{p_1} h_1^{(\epsilon, m)}(p; \xi) \right] \cdot \epsilon_n \\
&+ t \epsilon_n \left[ \partial_{p_1} H_2^{(\epsilon, m)}(p; \xi) \right] \epsilon_n + \left[ \partial_{p_1} h_3^{(\epsilon, m)}(p; \xi) \right] \frac{1}{n} \\
&+ \left[ \partial_{p_1} h_4^{(\epsilon, m)}(p; \xi) \right] \cdot \frac{\epsilon_n}{n} + \left[ \partial_{p_1} r_n(p; \xi) \right] \\
\frac{n p_2 - \xi_2 - \epsilon_n, 2 + p_1(\xi_2 + \epsilon_n, 2) - p_2(\xi_1 + \epsilon_n, 1)}{p_2(1 - p_1 - p_2)} &= \left[ \partial_{p_2} h_0^{(\epsilon, m)}(p; \xi) \right] + \left[ \partial_{p_2} h_1^{(\epsilon, m)}(p; \xi) \right] \cdot \epsilon_n \\
&+ t \epsilon_n \left[ \partial_{p_2} H_2^{(\epsilon, m)}(p; \xi) \right] \epsilon_n + \left[ \partial_{p_2} h_3^{(\epsilon, m)}(p; \xi) \right] \frac{1}{n} \\
&+ \left[ \partial_{p_2} h_4^{(\epsilon, m)}(p; \xi) \right] \cdot \frac{\epsilon_n}{n} + \left[ \partial_{p_2} r_n(p; \xi) \right].
\end{cases}
\]
The solution of such an equation can be obtained by inserting the expression \( \xi_i + \epsilon_{n,i} + \Gamma_{0,i}(\xi) \frac{1}{n} + \Gamma_{1,i}(\xi) \cdot \frac{\epsilon_{n}}{n} + \frac{1}{n} \epsilon_{n} \nabla_{v_2,i}(\xi) \epsilon_{n} + \Delta_{n,i}(\xi; \epsilon_{n}) \) in the place of \( p_i, \ i = 1, 2 \), and then expanding both members. By carefully carrying out all of these computations, we are in a position to conclude that \( \Gamma_{0,i}(\xi) = 0, \ \Gamma_{1,i}(\xi) = 0 \) and \( \nabla_{v_2,i}(\xi) = 0 \), thus completing the proof.

The last preparatory result is the multidimensional analogous of Lemma 24.

**Lemma 24.** Under the same assumption of Lemma 22 with \( \xi = p_0 \), there hold

\[
\Lambda_{k,n,T}(p_0; b) := -2 \log \left( \frac{L_{n,T}(p_0; b)}{L_{n,T}(p_0; b_0)} \right)
= 2 \log \left( \frac{L_n(0)(\hat{p}_{n,T}; b)}{L_n(0)(p_0; b)} \right) + 2 \log \left( \frac{H_n(\epsilon_{n})(\hat{p}_{n,T}; b)}{H_n(\epsilon_{n})(p_0; b)} \right).
\tag{64}
\]

Moreover, we have

\[
2 \log \left( \frac{L_n(0)(\hat{p}_{n,T}; b)}{L_n(0)(p_0; b)} \right) = n \left\{ t \epsilon_{n} \mathbb{I}(p_0) \epsilon_{n} + O(|\epsilon_{n}|^3) \right\}
\tag{65}
\]

where \( \mathbb{I}(p_0) \) denotes the Fisher information matrix of the Multinomial model which, for \( k = 3 \), reads

\[
\begin{pmatrix}
\frac{1}{p_{0,1}} + \frac{1}{1 - p_{0,1} - p_{0,2}} & \frac{1}{1 - p_{0,1} - p_{0,2}} \\
\frac{1}{1 - p_{0,1} - p_{0,2}} & \frac{1}{p_{0,2}} + \frac{1}{1 - p_{0,1} - p_{0,2}}
\end{pmatrix}
\]

and

\[
2 \log \left( \frac{H_n(\epsilon_{n})(\hat{p}_{n,T}; b)}{H_n(\epsilon_{n})(p_0; b)} \right) = -2 \left( t \epsilon_{n} \mathbb{I}_2(\epsilon_{n})(p_0; p_0) \epsilon_{n} + O(|\epsilon_{n}|^3) \right)
= -\text{Var}[L_1] \left( t \epsilon_{n} \mathbb{I}(p_0) \right)^2 \epsilon_{n} + O(|\epsilon_{n}|^3) .
\tag{66}
\]

Therefore, putting \( \zeta_{n} := \sqrt{n}\epsilon_{n} \), we get

\[
\Lambda_{k,n,T}(p_0; b) = \left( t \epsilon_{n} \mathbb{I}(p_0) - \frac{\text{Var}[L_1]}{n} \mathbb{I}(p_0)^2 \right) \zeta_{n} + O \left( \frac{|\zeta_{n}|^3}{\sqrt{n}} \right)
\tag{67}
\]

\[
\Lambda_{k,n,N}(p_0; b) = \left( t \epsilon_{n} \mathbb{I}(p_0) \right) \zeta_{n} + O \left( \frac{|\zeta_{n}|^3}{\sqrt{n}} \right).
\tag{68}
\]

**Proof.** Since

\[
\Lambda_{k,n,T}(p_0; b) := -2 \log \left( \frac{\sup_{p \in \{p_0\}} L_{n,T}(p; b)}{\sup_{p \in \Delta_{k-1}} L_{n,T}(p; b)} \right) = -2 \log \left( \frac{L_{n,T}(p_0; b)}{L_{n,T}(\hat{p}_{n,T}; b)} \right)
\]

holds by definition, identity (64) follows immediately from (56).

Next, we derive (65) by combining (57) with (63). In any case, a quicker argument can be based on Remark 15, according to which the right-hand side of (65) coincides with the Taylor expansion of the map \( \epsilon_{n} \mapsto D_{KL}(p_0 + \epsilon_{n}||p_0) \). Thus, the result to be proved boils down to the application of the well-known relationship between the Kullback-Leibler divergence and the Fisher information matrix in a regular parametric model.
Coming to (66), we start by noticing that
\[
\left(1 - \hat{p}_{n,T,1}\right)^{l_1} = \left(1 - \frac{p_{0,1} - p_{0,2}}{p_{0,1}}\right)^{l_1} \left[1 - l_1 \left(\frac{\epsilon_{n,1}}{p_{0,1}} + \frac{\epsilon_{n,1} + \epsilon_{n,2}}{1 - p_{0,1} - p_{0,2}}\right)\right]
+ l_1 \left(\frac{\epsilon_{n,1}^2}{p_{0,1}^2} + \frac{\epsilon_{n,1}(\epsilon_{n,1} + \epsilon_{n,2})}{p_{0,1}(1 - p_{0,1} - p_{0,2})}\right)
+ \left(\frac{l_1}{2}\right) \left(\frac{\epsilon_{n,1}}{p_{0,1}} + \frac{\epsilon_{n,1} + \epsilon_{n,2}}{1 - p_{0,1} - p_{0,2}}\right)^2 + O(|\epsilon_n|^3)
\]
\[
\left(1 - \hat{p}_{n,T,2}\right)^{l_2} = \left(1 - \frac{p_{0,1} - p_{0,2}}{p_{0,2}}\right)^{l_2} \left[1 - l_2 \left(\frac{\epsilon_{n,2}}{p_{0,2}} + \frac{\epsilon_{n,1} + \epsilon_{n,2}}{1 - p_{0,1} - p_{0,2}}\right)\right]
+ l_2 \left(\frac{\epsilon_{n,2}^2}{p_{0,2}^2} + \frac{\epsilon_{n,2}(\epsilon_{n,1} + \epsilon_{n,2})}{p_{0,2}(1 - p_{0,1} - p_{0,2})}\right)
+ \left(\frac{l_2}{2}\right) \left(\frac{\epsilon_{n,2}}{p_{0,2}} + \frac{\epsilon_{n,1} + \epsilon_{n,2}}{1 - p_{0,1} - p_{0,2}}\right)^2 + O(|\epsilon_n|^3)
\] .

These identities, combined with (58) and (61), yield
\[
H_{n}^{(\epsilon,m)}(\hat{p}_{n,T}; b) = 1 + O \left(\frac{1}{n} + |\epsilon_n|^3\right)
\]

The combination of this identity with (62), in which \(p = \xi = p_0\), yields (66). Incidentally, it is crucial in what follows to notice that \(H_{n}^{(\epsilon,m)}(p_0; p_0) = \text{Var}[L_1]\|\xi\|^2\), but this is only a matter of direct computations.

Finally, identity (67) follows immediately from (64)-(65)-(66). \(\square\)

### B.2 Proof of Theorems 3 and 10

We start again from (60) where the random variables \(A_n\) and \(L\), defined on the probability space \((\Omega, \mathcal{F}, \text{Pr})\), are independent, \(A_n \sim \text{Mult}(n, p)\) and \(L = (L_1, \ldots, L_{k-1})\) is a random vector with independent components with each \(L_i\) having the Laplace distribution (4). Thus, under the validity of \(H_0\), we define the event \(E_{0,n} \subset \Omega\) as
\[
E_{0,n,k} := \left\{ \omega \in \Omega : \left|\frac{A_n(\omega)}{n} - p_0\right| \leq c(p_0) \left(\frac{\log n}{n}\right)^{1/2}\right\}
\]
for some \(c(p_0) > 0\). After recalling that each component of \(A_n\) is a Binomial random variable, we can reduce the problem to the analogous one already treated in Section A.2, and we can find \(c(p_0)\) so that
\[
\text{Pr}[E_{0,n,k}^c] \leq 2(k - 1)n^{-3/2}.
\]

In this way, for \(\ell = T, N\), we can write
\[
\mathcal{F}_{k,n,\ell}(t) = \text{Pr}[A_{k,n,\ell}(p_0) \leq t, E_{0,n,k}] + \text{Pr}[A_{k,n,\ell}(p_0) \leq t, E_{0,n,k}^c]
\]
with
\[
\text{Pr}[A_{k,n,\ell}(p_0) \leq t, E_{0,n,k}^c] \leq 2(k - 1)n^{-3/2}.
\]

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The advantage of such a preliminary step is that, on $E_{0,n}$, the assumptions of all the Lemmata stated in the previous subsection are fulfilled and, by resorting to (67), we can write

$$\{A_{k,n,T}(p_0) \leq t, E_{0,n,k}\}$$

$$= \left\{ tZ_n \left( I(p_0) - \frac{\text{Var}[L_1]}{n}I(p_0)^2 \right) Z_n + O\left( \frac{|Z_n|^3}{\sqrt{n}} \right) \leq t, E_{0,n,k} \right\}$$

where

$$Z_n := \frac{B_n - np_0}{\sqrt{n}} = \Sigma(p_0)^{1/2}W_n + \frac{1}{\sqrt{n}}L$$

with

$$W_n := \Sigma(p_0)^{-1/2}A_n - np_0$$

and $\Sigma(p_0)$ is the covariance matrix of $A_n$.

As in the preliminary part of this second Appendix, we confine ourselves to dealing with the case of $k = 3$. The following proposition represents the multidimensional analogous of the Berry-Esseen expansion already given in Proposition 18.

**Proposition 25.** Let $\phi_{0,\Sigma}$ denote the density of the 2-dimensional Normal distribution with mean equal to 0 and covariance matrix equal to $\Sigma(p_0)$, and let $\Phi_{0,\Sigma}$ stand for the associated distribution function. In addition, for any multi-index $\nu$, denote by $\chi_\nu$ the $\nu$-cumulant of the random vector $(A_{n,1} - p_0,1, A_{n,2} - p_0,2, A_{n,3} - p_0,3)$, and put $\chi_m(u) := \sum_{|\nu|=m} \chi_{\nu}^m u^\nu$ to set

$$\tilde{P}_s(u : \{\chi_\nu\}) := \sum_{(s)} \prod_{m=1}^s \frac{1}{k_m!} (\chi_{m+2}(u))^{k_m}$$

where $(s)$ is a shorthand to indicate that the summation is extended to all the s-uples $(k_1, \ldots, k_s)$ such that $k_1 + 2k_2 + \cdots + sk_s = s$. Afterwords, define

$$P_s(-\Phi_{0,\Sigma} : \{\chi_\nu\})(y) := \left( \frac{1}{2\pi} \right)^k \int_{\mathbb{R}^k} e^{-i\int u \tilde{P}_s(u : \{\chi_\nu\})} du$$

and $P_s(-\Phi_{0,\Sigma} : \{\chi_\nu\})(x) := \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} P_s(-\Phi_{0,\Sigma} : \{\chi_\nu\})(y) dy$. Finally, set

$$\Xi_1(y) := P_1(-\Phi_{0,\Sigma} : \{\chi_\nu\})(y) - S_1(np_{0,1} + \sqrt{n}y_1)\partial_1\Phi_{0,\Sigma}(y) - S_1(np_{0,2} + \sqrt{n}y_2)\partial_2\Phi_{0,\Sigma}(y)$$

and

$$\Xi_2(y) := P_2(-\Phi_{0,\Sigma} : \{\chi_\nu\})(y) - S_1(np_{0,1} + \sqrt{n}y_1)\partial_1P_1(-\Phi_{0,\Sigma} : \{\chi_\nu\})(y) - S_1(np_{0,2} + \sqrt{n}y_2)\partial_2P_1(-\Phi_{0,\Sigma} : \{\chi_\nu\})(y)$$

$$+ S_2(np_{0,1} + \sqrt{n}y_1)\partial_1^2\Phi_{0,\Sigma}(y) + S_2(np_{0,2} + \sqrt{n}y_2)\partial_2^2\Phi_{0,\Sigma}(y) + S_1(np_{0,1} + \sqrt{n}y_1)S_1(np_{0,2} + \sqrt{n}y_2)\partial_1^2\partial_2\Phi_{0,\Sigma}(y)$$

where $S_1$ and $S_2$ are the same as in Proposition 18. Then, for any $x \in \mathbb{R}^2$, there holds

$$G_n(x) = \Phi_{0,\Sigma}(x) + \frac{1}{\sqrt{n}}\Xi_1(x) + \frac{1}{2n}\text{Var}(L_1) \left[ \partial_1^2\Phi_{0,\Sigma}(x) + \partial_2^2\Phi_{0,\Sigma}(x) \right] + \frac{1}{n}\Xi_2(x) + R_n(x)$$

with the remainder term $R_n$ satisfying an inequality like

$$\sup_{x \in \mathbb{R}^2} \{|R_n(x)|\} \leq \frac{C(p_0, m, \varepsilon)}{n^{3/2}}$$

for some suitable constant $C(p_0, m, \varepsilon)$ independent of $n$. 44
Proof. We start by putting
\[ F_n(x) := \Pr \left[ \sqrt{n} \left( \frac{1}{n} A_n - p_0 \right) \leq x \right] = \Pr \left[ \frac{1}{\sqrt{n}} (A_n - np_0) \leq x \right]. \]
By independence, we have
\[ G_n(x) = \left( \frac{1}{e^{x,m}} \right)^2 \sum_{l_1=-m}^{m} \sum_{l_2=-m}^{m} e^{-\varepsilon(\|l_1\|+\|l_2\|)} F_n \left( x_1 - \frac{l_1}{\sqrt{n}}, x_2 - \frac{l_2}{\sqrt{n}} \right) \]
\[ = \mathbb{E} \left[ F_n \left( x - \frac{1}{\sqrt{n}} L \right) \right]. \]
At this stage, we exploit the well-known asymptotic expansions of \( F_n \) displayed, e.g., in Section 23 of Bhattacharya and Rao (2010). We put \( s = 5 \) in Theorem 23.1 to obtain
\[ F_n(y) = \Phi_{0,\Sigma}(y) + \frac{1}{\sqrt{n}} \Xi_1(y) + \frac{1}{n} \Xi_2(y) + R_n(x) \]
where \( R_n \) is a remainder term satisfying an inequality like
\[ \sup_{x \in \mathbb{R}^2} |R_n(x)| (1 + |x|)^5 \leq \frac{C(p_0, m, \varepsilon)}{n^{3/2}} \]
for some suitable constant \( C(p_0, m, \varepsilon) \) independent of \( n \). The key remark is about the modification of the terms of the type \( S_k(np_0 + \sqrt{n} y) \) after the substitution \( y_i = x_i - \frac{l_i}{\sqrt{n}} \). Indeed, we have
\[ S_k(np_0 + \sqrt{n} y) = S_k(np_0 + \sqrt{n} x_i - l_i) = S_k(np_0 + \sqrt{n} x_i) \]
because of the periodicity of the \( S_k \)'s. Therefore, the substitution \( y_i = x_i - \frac{l_i}{\sqrt{n}} \) affects the functions \( \Xi_1 \) and \( \Xi_2 \) only in the terms involving \( \Phi_{0,\Sigma}, P_1(-\Phi_{0,\Sigma} : \{\chi\nu\}), P_2(-\Phi_{0,\Sigma} : \{\chi\nu\}), P_3(-\Phi_{0,\Sigma} : \{\chi\nu\}) \)
and their derivatives. Since these functions are smooth, we can apply the Taylor formula to get
\[ D^\alpha \Phi_{0,\Sigma} \left( x - \frac{1}{\sqrt{n}} L \right) = D^\alpha \Phi_{0,\Sigma}(x) - \frac{l_1}{\sqrt{n}} \partial_1 D^\alpha \Phi_{0,\Sigma}(x) - \frac{l_2}{\sqrt{n}} \partial_2 D^\alpha \Phi_{0,\Sigma}(x) \]
\[ + \frac{l_1^2}{2n} \partial_1^2 D^\alpha \Phi_{0,\Sigma}(x) + \frac{l_2^2}{2n} \partial_2^2 D^\alpha \Phi_{0,\Sigma}(x) + \frac{l_1 l_2}{n} \partial_1 \partial_2 D^\alpha \Phi_{0,\Sigma}(x) \]
\[ + O \left( \frac{1}{n^{3/2}} \right) \]
for any multi-index \( \alpha \), and analogous expansions for \( D^\alpha P_k(-\Phi_{0,\Sigma} : \{\chi\nu\}) \left( x - \frac{1}{\sqrt{n}} L \right) \). Now, the assumption on the distribution of \( L \) entails that \( \mathbb{E}[L^\alpha] = \mathbb{E}[L_1^{\alpha_1} L_2^{\alpha_2}]=0 \) as soon as either \( \alpha_1 \) or \( \alpha_2 \) is odd. Whence,
\[ \mathbb{E} \left[ D^\alpha \Phi_{0,\Sigma} \left( x - \frac{1}{\sqrt{n}} L \right) \right] = D^\alpha \Phi_{0,\Sigma}(x) + \frac{1}{2n} \text{Var}(L_1) \left[ \partial_1^2 D^\alpha \Phi_{0,\Sigma}(x) + \partial_2^2 D^\alpha \Phi_{0,\Sigma}(x) \right] + O \left( \frac{1}{n^{3/2}} \right) \]
for any multi-index \( \alpha \), and analogous expansions for the term
\[ \mathbb{E} \left[ D^\alpha P_k(-\Phi_{0,\Sigma} : \{\chi\nu\}) \left( x - \frac{1}{\sqrt{n}} L \right) \right]. \]
This completes the proof. □
At this stage, like in the previous proof of Theorem 9, the thesis of Theorems 3 and 10 follow from the combination of the Berry-Esseen expansion (72) with identity (67). Indeed, (72) shows that the probability distribution of \( Z_n \) has both an absolutely continuous part as well as a singular part. As before, the core of the proof hinges on the study of the absolutely continuous part, which is significantly affected by the Laplace perturbation because of the term \( \frac{1}{2n} \text{Var}[L_1] [\partial^2 \Phi_0, \Sigma(x) + \partial^2 \Phi_0, \Sigma(x)] \).

Thus, combining the last identities, we get

\[
\Phi_0, \Sigma(x) + \frac{1}{2n} \text{Var}[L_1] [\partial^2 \Phi_0, \Sigma(x) + \partial^2 \Phi_0, \Sigma(x)]
\]
yields the following density (with respect to the 2-dimensional Lebesgue measure)

\[
g_n(x) = \varphi\Sigma(p_0)(x) + \frac{\text{Var}[L_1]}{2n} \Delta_x \varphi\Sigma(p_0)(x)
\]

where \( \varphi\Sigma \) denotes the density of the 2-dimensional Normal distribution with mean \( 0 \) and covariance matrix \( \Sigma \), \( \Sigma(p_0) \) is the covariance matrix of the random vector \( A_n \sim Mult(n, p_0) \), \( \Delta_x \) stands for the Laplacian operator, and \( \text{tr} \) is the trace operator. At this stage, it is enough to notice that the principal term of the expression of \( F_{k,n,T}(t) \) is given by the following integral

\[
\int_{\{x : (I(p_0) - \frac{\text{Var}[L_1]}{n} I(p_0)^2)^{-1/2} x \leq t\}} g_n(x) \, dx
\]

\[
= \int_{\{|y|^2 \leq t\}} g_n \left( \left( I(p_0) - \frac{\text{Var}[L_1]}{n} I(p_0)^2 \right)^{-1/2} y \right) \frac{dy}{\sqrt{\det \left( I(p_0) - \frac{\text{Var}[L_1]}{n} I(p_0)^2 \right)}}.
\]

To handle the last integral, we start by noticing that

\[
\left( I(p_0) - \frac{\text{Var}[L_1]}{n} I(p_0)^2 \right)^{-1/2} = I(p_0)^{-1/2} + \frac{\text{Var}[L_1]}{2n} I(p_0)^{1/2} + O \left( \frac{1}{n^2} \right).
\]

Thus, combining the last identities, we get

\[
g_n \left( \left( I(p_0) - \frac{\text{Var}[L_1]}{n} I(p_0)^2 \right)^{-1/2} y \right)
\]

\[
= g_n \left( I(p_0)^{-1/2} y + \frac{\text{Var}[L_1]}{2n} I(p_0)^{1/2} y \right) + O \left( \frac{1}{n^2} \right)
\]

\[
= \varphi\Sigma(p_0) \left( I(p_0)^{-1/2} y + \frac{\text{Var}[L_1]}{2n} I(p_0)^{1/2} y \right) \times
\]

\[
\times \left[ 1 - \frac{\text{Var}[L_1]}{2n} \text{tr} (\Sigma(p_0)^{-1}) + \frac{\text{Var}[L_1]}{2n} I(p_0)^{-1} \Sigma(p_0)^{-2} \left( I(p_0)^{-1/2} y \right) \right] + O \left( \frac{1}{n^2} \right).
\]

In view of the identity \( \nabla_x \varphi\Sigma(x) = -\varphi\Sigma(x) \Sigma^{-1} x \), we can use the Taylor formula to write

\[
\varphi\Sigma(p_0) \left( I(p_0)^{-1/2} y + \frac{\text{Var}[L_1]}{2n} I(p_0)^{1/2} y \right)
\]

\[
= \varphi\Sigma(p_0) \left( I(p_0)^{-1/2} y \right) + \frac{\text{Var}[L_1]}{2n} I(p_0)^{1/2} y \cdot \nabla \varphi\Sigma(p_0) \left( I(p_0)^{-1/2} y \right) + O \left( \frac{1}{n^2} \right).
\]
\[= \varphi \Sigma(p_0) \left( \mathbb{I}(p_0)^{-1/2} y \right) \left[ 1 - \frac{\text{Var}[L_1]}{2n} \mathbb{I}(p_0)^{1/2} y \cdot \left( \Sigma^{-1}(p_0) \mathbb{I}(p_0)^{-1/2} y \right) \right].\]

Now, it is crucial to observe that, for the Multinomial model, it holds \(\mathbb{I}(p_0)^{-1} = \Sigma(p_0)\). Whence,

\[
\varphi \Sigma(p_0) \left( \mathbb{I}(p_0)^{-1/2} y \right) = \frac{1}{\sqrt{\det(\Sigma(p_0))}} \varphi(y) \\
\mathbb{I}(p_0)^{-1/2} y \cdot \left( \Sigma^{-1}(p_0) \mathbb{I}(p_0)^{-1/2} y \right) = y^\top \mathbb{I}(p_0) y
\]

where \(\varphi\) stands for the density of the standard Normal distribution. In conclusion, we have

\[
g_n \left( \left( \mathbb{I}(p_0) - \frac{\text{Var}[L_1]}{n} \mathbb{I}(p_0)^2 \right)^{-1/2} y \right) \\
= \frac{\varphi(y)}{\sqrt{\det(\Sigma(p_0))}} \left[ 1 - \frac{\text{Var}[L_1]}{2n} y^\top \mathbb{I}(p_0) y \right] \times \left[ 1 - \frac{\text{Var}[L_1]}{2n} \text{tr} \left( \Sigma(p_0)^{-1} \right) + \frac{\text{Var}[L_1]}{2n} y^\top \mathbb{I}(p_0) y \right] + O \left( \frac{1}{n^2} \right) \\
= \frac{\varphi(y)}{\sqrt{\det(\Sigma(p_0))}} \left[ 1 - \frac{\text{Var}[L_1]}{2n} \text{tr} \left( \Sigma(p_0)^{-1} \right) \right] + O \left( \frac{1}{n^2} \right).
\]

Then, invoking the Taylor formula for the determinant operator, we get

\[
\frac{1}{\sqrt{\det \left( \mathbb{I}(p_0) - \frac{\text{Var}[L_1]}{n} \mathbb{I}(p_0)^2 \right)}} = \sqrt{\det(\Sigma(p_0))} \left[ 1 + \frac{\text{Var}[L_1]}{2n} \text{tr} \left( \Sigma(p_0)^{-1} \right) \right] + O \left( \frac{1}{n^2} \right)
\]

which entails

\[
\int_{\{x \in \mathbb{I}(p_0) - \frac{\text{Var}[L_1]}{n} \mathbb{I}(p_0)^2 \mid x \leq t\}} g_n(x) \, dx = \int_{\{y^2 \leq t\}} \varphi(y) \, dy + O \left( \frac{1}{n^2} \right) \\
= K_2(t) + O \left( \frac{1}{n^2} \right).
\]

This argument proves rigorously the presence of the terms \(K_{k-1}(t)\) in (9). Now, we take cognizance that the expressions of \(c_{k,1}(t; p_0)\), \(c_{k,2}(t; p_0; m, \varepsilon)\), \(c_{k,2}(t; p_0)\) and \(R_{k,n,T}(t)\) in (9) ensue from the various terms in (72), according to the above line of reasoning. In particular, \(c_{k,1}(t; p_0)\) ensues from the term \(P_1(-\Phi_0 \Sigma : \{\chi_\nu\})\) that figures in the expression of \(\Xi_1\), while \(c_{k,2}(t; p_0; m, \varepsilon)\) corresponds to the manipulation of the quantity \(-S_1(n \eta p_0 + \sqrt{n} y_1) \partial_1 \Phi_0 \Sigma(y) - S_1(n \eta p_0 + \sqrt{n} y_2) \partial_2 \Phi_0 \Sigma(y)\) which also figures in the expression of \(\Xi_1\). Furthermore, \(c_{k,3}(t; p_0)\) ensues from the expression of \(\Xi_2\). However, what is crucial is just to remark once again that the additional term

\[
\frac{1}{2n} \text{Var}[L_1] \left[ \partial_1^2 \Phi_0 \Sigma(x) + \partial_2^2 \Phi_0 \Sigma(x) \right]
\]

that appears in (72), which depends significantly on the Laplace perturbation, is no more present in (9).
It remains to justify the expression of $\Gamma_{k,n,N}$, which follows from the combination of the Berry-Esseen expansion (72) with identity (68). Indeed, the core of the argument consists in the study of the integral $\int_{\{x : (p_0)x \leq t\}} g_n(x) dx$, which equal to

$$
\int_{\{y : y^2 \leq t\}} g_n \left( (p_0)^{-1/2}y \right) \frac{dy}{\sqrt{\det((p_0))}} = \int_{\{y : y^2 \leq t\}} \varphi \Sigma(p_0) \left( (p_0)^{-1/2}y \right) \left[ 1 - \frac{\text{Var}[L_1]}{2n} \text{tr} \left( \Sigma(p_0)^{-1} \right) \right] + O \left( \frac{1}{n^2} \right)
$$

Then, we can pass to polar coordinates and notice that

$$
\int_{\{y : y^2 \leq t\}} \varphi(y) \left[ 1 - \frac{\text{Var}[L_1]}{2n} \text{tr} \left( \Sigma(p_0)^{-1} \right) + \frac{\text{Var}[L_1]}{2n} t \Sigma(p_0)^{-1} y \right] dy + O \left( \frac{1}{n^2} \right).
$$

which completes the proof of (17).

**B.3 Proof of (13)**

Upon putting $\mu_n(\cdot) := \Pr \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \in \cdot \right]$, we prove that

$$
\mu_n(\sqrt{n} \xi + \xi') = \exp \left\{-n[\dot{z} \cdot \nabla L(\dot{z}) - L(\dot{z})] + \sqrt{n} \min_{v \in \mathcal{E}} \dot{z} \cdot v \right\} n^{-(d+1)/4}[1 + o(1)]
$$

is valid as $n \to +\infty$, where $d$ is the dimension of the random vectors $X_i$. The proof is a small variation of a classical argument developed, e.g., in Chapter VIII of Petrov (1975), in von Bahr (1967), or in Aleshkovichane (1983).

Letting the distribution of $X_i$ be denoted by $\mu$, we introduce the new probability measure $\nu_{z}(A) := e^{-L(z)} \int_A e^{z \cdot y} \mu(dy)$ for some $|z| < H$ and, then, a sequence $\{Y_i\}_{i \geq 1}$ of i.i.d. random vectors with distribution $\nu_z$. We also put $m_z := \mathbb{E}[Y_1] = \nabla L(z)$ and $C_z := \text{Cov}(Y_1) = \text{Hess}[L](z)$. Hence, setting $\nu_{z,n}(\cdot) := \Pr[\sum_{i=1}^{n} Y_i \in \cdot]$, we find that

$$
\Pr \left[ \sum_{i=1}^{n} X_i \in A \right] = e^{nL(z)} \int_A e^{-z \cdot y} \nu_{z,n}(dy)
$$

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yielding in turn, after some manipulations, that
\[
\mu_n(\sqrt{n}\xi + \varepsilon) = e^{n L(z) - nz \cdot m_z} \int_{C_z^{-1/2}[\sqrt{n}(\xi - m_z) + \varepsilon]} e^{-\sqrt{n} \cdot z C_z^{1/2} y} \gamma_{z,n}(dy)
\]
(74)

with
\[
\gamma_{z,n}(\cdot) := \Pr \left[ \frac{1}{\sqrt{n}} C_z^{-1/2} \sum_{i=1}^{n} (Y_i - m_z) \in \cdot \right].
\]

At this stage, exploiting the assumption of the existence of \( \hat{z} \) for which \( \nabla L(\hat{z}) = m_z = \xi \), we choose \( z = \hat{z} \) so that (74) becomes
\[
\mu_n(\sqrt{n}\xi + \varepsilon) = e^{n L(\hat{z}) - n\hat{z} \cdot m_z} \int_{C_{\hat{z}}^{-1/2}\varepsilon} e^{-\sqrt{n} \cdot \hat{z} C_{\hat{z}}^{1/2} y} \gamma_{\hat{z},n}(dy).
\]

Therefore, to prove (73), it is enough to show that
\[
\int_{C_{\hat{z}}^{-1/2}\varepsilon} e^{-\sqrt{n} \cdot \hat{z} C_{\hat{z}}^{1/2} y} \gamma_{\hat{z},n}(dy) = \exp \left\{ \sqrt{n} \min_{v \in \varepsilon} \hat{z} \cdot v \right\} n^{-(d+1)/4}[1 + o(1)]
\]
as \( n \to +\infty \). Indeed, upon denoting by \( \gamma \) the standard Normal distribution on \( \mathbb{R}^d \), we can write
\[
\int_{C_{\hat{z}}^{-1/2}\varepsilon} e^{-\sqrt{n} \cdot \hat{z} C_{\hat{z}}^{1/2} y} \gamma_{\hat{z},n}(dy) = \left( \frac{1}{2\pi} \right)^{d/2} \int_{C_{\hat{z}}^{-1/2}\varepsilon} e^{-\sqrt{n} \cdot \hat{z} C_{\hat{z}}^{1/2} y - \frac{1}{2} |y|^2} dy
\]
\[
+ \int_{\mathbb{R}^d} \mathbf{1}_{C_{\hat{z}}^{-1/2}\varepsilon}(y) e^{-\sqrt{n} \cdot \hat{z} C_{\hat{z}}^{1/2} y} [\gamma_{\hat{z},n}(dy) - \gamma(dy)].
\]

For the former term on the right-hand side of (75), we have
\[
\left( \frac{1}{2\pi} \right)^{d/2} \int_{C_{\hat{z}}^{-1/2}\varepsilon} e^{-\sqrt{n} \cdot \hat{z} C_{\hat{z}}^{1/2} y - \frac{1}{2} |y|^2} dy
\]
\[
= \left( \frac{1}{2\pi} \right)^{d/2} \frac{1}{\sqrt{\det(C_{\hat{z}})}} \int_{\mathbb{R}^d} e^{-\sqrt{n} \hat{z} \cdot u - \frac{1}{2} |C_{\hat{z}}^{-1/2} u|^2} du
\]
\[
= C(d, \xi, \varepsilon) \exp \left\{ -\sqrt{n} \min_{u \in \varepsilon} \hat{z} \cdot u \right\} n^{-(d+1)/4}[1 + o(1)]
\]
where, in the last identity, \( C(d, \xi, \varepsilon) \) is a constant depending solely on \( (d, \xi, \varepsilon) \). For completeness, the validity of such an identity follows from a direct application of the multidimensional Laplace method displayed, e.g., in Theorem 46 of Breitung (1994). It remains to show that the absolute value of the latter term on the right-hand side of (75) is even less significant, as it can be bounded by an expression like
\[
C'(d, \xi, \varepsilon) \exp \left\{ -\sqrt{n} \min_{u \in \varepsilon} \hat{z} \cdot u \right\} n^{-(d+3)/4}[1 + o(1)]
\]
(67)

with some constant \( C'(d, \xi, \varepsilon) \) depending solely on \( (d, \xi, \varepsilon) \). This task can be carried out by resorting to the Plancherel identity, i.e.
\[
\int_{\mathbb{R}^d} \phi_n(y)[\gamma_{\hat{z},n}(dy) - \gamma(dy)] = \left( \frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} \hat{\phi}_n(t)[\hat{\gamma}_{\hat{z},n}(t) - \hat{\gamma}(t)] dt
\]
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well-known Berry-Esseen inequalities—shows that 
\( (8.23) \) of Bhattacharya and Rao (2010)—which constitute a multidimensional generalization of the
Thus, for the first integral on the right-hand side of (77), an application of inequalities (8.22)-
the former of the above identities following once again from the multidimensional Laplace method.

Therefore, we can write

\[
\left| \int_{\mathbb{R}^d} \phi_n(y) [\gamma_{\hat{z}, n}(dy) - \gamma(dy)] \right|
\]

\[
\leq \left( \frac{1}{2\pi} \right)^d \left\{ \|\phi_n\|_{L^1} \left( \int_{\{\lvert t \rvert \leq A(\hat{z})\sqrt{n} \}} \left| \gamma_{\hat{z}, n}(t) - \hat{\gamma}(t) \right| \, dt + \int_{\{\lvert t \rvert > A(\hat{z})\sqrt{n} \}} \left| \hat{\gamma}(t) \right| \, dt \right) + \left| \int_{\{\lvert t \rvert > A(\hat{z})\sqrt{n} \}} \phi_n(t) \hat{\gamma}_{\hat{z}, n}(t) \, dt \right| \right\} (77)
\]

with

\[
\|\phi_n\|_{L^1} := \int_{\mathbb{R}^d} \phi_n(y) \, dy = C'(d, \xi, \mathcal{E}) \exp \left\{ -\sqrt{n} \min_{u \in \mathcal{E}} u \cdot u \right\} n^{-(d+1)/4} \left[ 1 + o(1) \right]
\]

\[
A(\hat{z}) := (E[|Y_1| - \xi|^2])^{3/2} / E[|Y_1| - \xi]^3,
\]

the former of the above identities following once again from the multidimensional Laplace method. Thus, for the first integral on the right-hand side of (77), an application of inequalities (8.22)-(8.23) of Bhattacharya and Rao (2010)—which constitute a multidimensional generalization of the well-known Berry-Esseen inequalities—shows that

\[
\int_{\{\lvert t \rvert \leq A(\hat{z})\sqrt{n} \}} \left| \gamma_{\hat{z}, n}(t) - \hat{\gamma}(t) \right| \, dt \leq \frac{C''(d, \xi, \mathcal{E})}{\sqrt{n}}
\]

holds for some constant \( C''(d, \xi, \mathcal{E}) \) depending solely on \( (d, \xi, \mathcal{E}) \). Then, the second integral on the right-hand side of (77) is asymptotically equivalent to the integral

\[
\int_{\{\lvert t \rvert > A(\hat{z})\sqrt{n} \}} \left| \gamma_{\hat{z}, n}(t) - \hat{\gamma}(t) \right| \, dt \leq \frac{C''(d, \xi, \mathcal{E})}{\sqrt{n}}
\]

as \( n \to +\infty \). Lastly, the last integral on the right-hand side of (77) has different behaviors according on whether the distribution \( \mu \) is lattice or not. If \( \limsup_{|t| \to +\infty} |\hat{\mu}(t)| < 1 \), then the integral at issue is exponentially small like the second integral on the right-hand side of (77) described above. Otherwise, in the lattice case (which is of interest here), we deduce the expansion (76) by using the expression of \( \gamma_{\hat{z}, n} \) (explicitly available in the lattice case) and by resorting once again to the multidimensional Laplace method.

**B.4 Proof of Theorem 4 for \( k > 2 \)**

We start again from (60) where the random variables \( A_n \) and \( L \), defined on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), are independent, \( A_n \sim \text{Mult}(n, p_1) \) and \( L = (L_1, \ldots, L_{k-1}) \) is a random vector with
independent components with each \( L_i \) having the Laplace distribution (4). After fixing \( \lambda_\alpha \) such that \( K_{k-1}(\lambda_\alpha) = 1 - \alpha \), according to Theorem 3, we have that
\[
1 - \beta_n(p_1; \alpha) = \Pr[\Lambda_{k,n,T}(p_0) \leq \lambda_\alpha].
\]

First, for \( \delta \in (0, 1/2) \), we define the event \( E_{n,k}(\delta) \subset \Omega \) as
\[
E_{n,k}(\delta) := \left\{ \omega \in \Omega : \frac{A_{n,i}(\omega)}{n} \in [\delta, 1 - \delta], \text{for } i = 1, \ldots, k - 1 \right\}.
\]

Then, recalling that
\[
1 - \beta_n(p_1; \alpha) = \Pr[\Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta)] + \Pr[\Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta)^c]
\]
with
\[
\Pr[\Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta)^c] = o \left( e^{-nD_{KL}(p_0 \parallel p_1)} \right).
\]

Then, for \( M > 0 \), we define another event \( E'_{n,k}(M) \subset \Omega \) as
\[
E'_{n,k}(M) := \left\{ \omega \in \Omega : \left| \frac{A_n(\omega)}{n} - p_0 \right| \leq \frac{M}{\sqrt{n}} \right\}
\]
and we write
\[
\Pr[\Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta)] = \Pr[\Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta) \cap E'_{n,k}(M)] \\
+ \Pr[\Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta) \cap E'_{n,k}(M)^c].
\]

The advantage of such a preliminary step is that, on \( E_{n,k}(\delta) \cap E'_{n,k}(M) \), the assumptions of all the preparatory Lemmata contained in Appendix B are fulfilled and, by resorting to (67), we can write
\[
\{ \Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta) \cap E'_{n,k}(M) \} \\
= \left\{ tZ_n \left( \mathbb{E}(p_0) - \frac{\text{Var}(L_1)}{n} \mathbb{E}(p_0)^2 \right) Z_n + O \left( \frac{|Z_n|^3}{\sqrt{n}} \right) \leq \lambda_\alpha, E_{n,k}(\delta) \cap E'_{n,k}(M) \right\}
\]
where \( Z_n \) is the same random variable as in (83). Moreover, if \( M > \lambda_\alpha \), as we will choose, we have eventually that
\[
\{ \Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta) \cap E'_{n,k}(M) \} \\
= \left\{ tZ_n \left( \mathbb{E}(p_0) - \frac{\text{Var}(L_1)}{n} \mathbb{E}(p_0)^2 \right) Z_n + O \left( \frac{|Z_n|^3}{\sqrt{n}} \right) \leq \lambda_\alpha, E_{n,k}(\delta) \right\}.
\]

It remains to show that, for a suitable choice of \( M \), we have
\[
\{ \Lambda_{k,n,T}(p_0) \leq \lambda_\alpha, E_{n,k}(\delta) \cap E'_{n,k}(M)^c \} = \emptyset
\]
eventually, yielding that

\[ 1 - \beta_n(p_1; \alpha) = \Pr \left[ t \mathbf{Z}_n \left( \mathbb{I}(p_0) - \frac{\text{Var}[L_1]}{n} \mathbb{I}(p_0) \right)^2 \mathbf{Z}_n + O \left( \frac{|\mathbf{Z}_n|^3}{\sqrt{n}} \right) \leq \lambda_n \right] \]  

(82)

\[ + o \left( e^{-n D_{KL}(p_0 \parallel p_1)} \right) . \]

The proof of (81), along with the proper choice of \( M \), follows as an application of Lemma 19 component-wise.

Afterwords, we come back to (82) by writing

\[ \mathbf{Z}_n := \frac{B_n - np_0}{\sqrt{n}} = \Sigma(p_1)^{1/2} \mathbf{V}_n + \frac{1}{\sqrt{n}} \mathbf{L} - \sqrt{n} \Delta \]

(83)

where \( \Delta := p_0 - p_1 \),

\[ \mathbf{V}_n := \Sigma(p_1)^{-1/2} \frac{A_n - np_1}{\sqrt{n}} \]

and \( \Sigma(p_1) \) denotes the covariance matrix of \( A_n \). Exploiting the independence between \( A_n \) and \( \mathbf{L} \), upon putting

\[ \mathbb{I}_n(p_0) := \mathbb{I}(p_0) - \frac{\text{Var}[L_1]}{n} \mathbb{I}(p_0)^2 , \]

we get

\[ 1 - \beta_n(p_1; \alpha) = \left( \frac{1}{c_{\varepsilon,m}} \right)^{k-1} \sum_{l \in \{-m,\ldots,m\}^{k-1}} e^{-\varepsilon \|l\|_X} \]

\[ \times \Pr \left[ t \left( \Sigma(p_1)^{1/2} \mathbf{V}_n + \frac{1}{\sqrt{n}} \mathbf{L} - \sqrt{n} \Delta \right) \mathbb{I}_n(p_0) \left( \Sigma(p_1)^{1/2} \mathbf{V}_n + \frac{1}{\sqrt{n}} \mathbf{L} - \sqrt{n} \Delta \right)^3 \right] \]

\[ + O \left( \frac{\text{Var}[L_1]}{n} \right) \leq \lambda_n \]  

(84)

Now, we apply (13) to a sequence \( \{X_n\}_{n \geq 1} \) of i.i.d. random vectors taking values in \( \mathbb{X}_k := \{x = (x_1, \ldots, x_{k-1}) \in \{0,1\}^{k-1} \mid x_1 + \cdots + x_{k-1} \leq 1\} \), in such a way that

\[ \Pr \left[ X_1 = (0, \ldots, 0, 1, 0, \ldots, 0) \right] \]

\[ = p_{1,i} \quad (i = 1, \ldots, k - 1) . \]

(85)

Moreover, we can put \( \xi = \Delta - \frac{1}{n} \mathbf{1} \) and

\[ \mathcal{E} = \{x \in \mathbb{R}^{k-1} \mid t \mathbb{I}_n(p_0)x + O(|x|^3/\sqrt{n}) \lambda_n \} . \]

Actually, the equation that defines \( \mathcal{E} \) could be specified in a more precise way, as we have done for \( k = 2 \). In fact, exploiting the analogy with the case \( k = 2 \), we could write the big-O term as

\[ \frac{1}{\sqrt{n}} \sum_{\nu \in \mathbb{N}^{k-1}_{\nu} : |\nu| \leq 3} \mathbb{Z}_{n,\nu}^{(c,m)}(p_0)x^{\nu} + \frac{1}{n} \sum_{\nu \in \mathbb{N}^{k-1}_{\nu} : |\nu| \leq 4} \mathbb{E}_{n,\nu}^{(c,m)}(p_0)x^{\nu} + O(n^{-3/2}) \]
for suitable tensors \( B^{(\varepsilon,m)}_{\nu}(p_0) \) and \( C^{(\varepsilon,m)}_{\nu}(p_0) \) of third and fourth order respectively, both satisfying analogous expansions similar to (33)–(34). However, despite the cumbersome computation that would have needed to derive such quantities, we have already learnt from the case \( k = 2 \) that they do not play any active role in the final result encapsulated in (12). More precisely, we know that the 0-order terms in \( B^{(\varepsilon,m)}_{\nu}(p_0) \) and \( C^{(\varepsilon,m)}_{\nu}(p_0) \) would ensue from the Taylor expansion of the map \( \varepsilon_n \mapsto D_{KL}(p_0 + \varepsilon_n \parallel p_0) \), while the successive terms in their expansion—which explicitly depend on the Laplace perturbation—do not affect the expressions of \( c_1(p_0, p_1; \lambda_\alpha) \) and \( c_2(p_0, p_1; \lambda_\alpha) \) in (12).

Indeed, after defining the vector \( \hat{z} \) by means of the identity \( \nabla L(\hat{z}) = \xi \), where \( L(z) := \log E[e^{zX}] - p_1 \cdot z \), we are now able to derive also the term \( -\frac{1}{n} \log M_L(\nabla p_0 D_{KL}(p_0 \parallel p_1)) \) in (12). After noticing that the above distribution \( 85 \) is a member of the regular exponential family parametrized by the mean, we can resort to Lemma 20 and apply (54) with \( \theta = p_1 \) and \( \tau = p_0 + \frac{1}{n} 1 + \rho_n(p_0, p_1; \lambda_\alpha) \), where \( \rho_n(p_0, p_1; \lambda_\alpha) = O(1/n) \), to conclude that

\[
\hat{z} \cdot \nabla L(\hat{z}) - L(\hat{z}) = D_{KL}
(p_0 \parallel p_1) + \frac{1}{n} \cdot \nabla p_0 D_{KL}(p_0 \parallel p_1)
+ \rho_n(p_0, p_1; \lambda_\alpha) \cdot \nabla p_0 D_{KL}(p_0 \parallel p_1).
\]

Therefore, combining this last equation with (13) and (84) we get

\[
1 - \beta_n(p_1; \alpha) \sim n^{-k/4} M_L((\nabla p_0 D_{KL}(p_0 \parallel p_1))) \times
\exp\left\{-n [D_{KL}(p_0 \parallel p_1) + \rho_n(p_0, p_1; \lambda_\alpha) \cdot \nabla p_0 D_{KL}(p_0 \parallel p_1)]
+ \sqrt{n} \min_{v \in \delta} \hat{z} \cdot v\right\}
\]

which entails the thesis of the theorem, upon noticing that the quantity \( \min_{v \in \delta} \hat{z} \cdot v \) contains more explicit terms that depend on the Laplace perturbation only at the level \( O(1/n) \). This ends the proof.

### B.5 Proof of Proposition 7

First, we notice that

\[
\log M_L(z) = \sum_{i=1}^{k} \log E[\exp\{L_i z_i\}] \quad (z \in \mathbb{R}^k)
\]

and \( E[\exp\{L_i z\}] = \frac{1}{c_{\varepsilon,m}} \sum_{l=0}^{m} \exp\{-\varepsilon|l| + l z\} \). In particular, if \( z > 0 \), the main contribution comes from the sum over \( l \in \{0, \ldots, m\} \), so that \( E[\exp\{L_i z\}] = \frac{1}{c_{\varepsilon,m}} \sum_{l=0}^{m} \exp\{(z - \varepsilon)l\} + R_{\varepsilon,m}(z) \), where \( R_{\varepsilon,m}(z) \) denotes a small remainder term. In particular, if \( z \geq \varepsilon(1 + \eta) \), then

\[
E[\exp\{L_i z\}] \geq \frac{1}{c_{\varepsilon,m}} \sum_{l=0}^{m} e^{\varepsilon l} + R_{\varepsilon,m}(z)
\]

holds. If \( \nu_k(p_0, p_1; \varepsilon, \eta) > 0 \), then the relevant part of the sum on the right-hand side of (86) is given by the sum of \( \nu_k(p_0, p_1; \varepsilon, \eta) \) summands, each of which greater than \( \log E[\exp\{L_i \varepsilon(1 + \eta)\}] \).

Finally, if \( \nu_k(p_0, p_1; \varepsilon, \eta) = 0 \) and \( \varepsilon^2(1+\eta)^3 \text{Var}[L] < 1 \), we use the Taylor expansion of the logarithm to obtain that

\[
\log E[\exp\{L_i z\}] = \frac{1}{2} \text{Var}[L] z^2 + o(z^2)
\]
holds as $z \to 0$. Thus, if all the components of $\nabla_{p_0} \mathcal{D}_{KL}(p_0 \parallel p_1)$ are less than $\varepsilon(1 + \eta)$, we get that the main contribution in the sum on the right-hand side of (86) is given by $\frac{1}{2} \text{Var}[L](\varepsilon(1 + \eta))^2$. This completes the proof.

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**References**

Acharya, J., Sun, Z. and Zhang, H. (2018). Differentially private testing of identity and closeness of discrete distributions. In *Neural Information Processing Systems*.

Aleshkyavichene, A.K. (1983). Multidimensional integral limit theorems for large deviation probabilities. *Theory of Probability and its Applications* 28, 65–88.

Aliakbarpour, M., Diakonikolas, I. and Rubinfeld, R. (2018). Differentially private identity and equivalence testing of discrete distributions. In *Proceedings of the International Conference on Machine Learning*.

Awan, J. and Slavković (2018). Differentially private uniformly most powerful tests for Binomial data. In *Neural Information Processing Systems*.

Bahadur, R.R. (1960). Stochastic comparison of tests. *Annals of Mathematical Statistics* 31, 276–295.

Bahadur, R.R. (1967). An optimal property of the likelihood-ratio statistic. *Berkeley Symposium on Mathematical Statistics and Probability*, 13–26.

Barndorff-Nielsen, O. (1978). *Information and Exponential Families in Statistical Theory*. Wiley.

Bhattacharya, R.N. and Rao, R.R. (2010). *Normal approximation and asymptotic expansions*. Wiley.

Breitung, K.W. (1994). *Asymptotic approximations for probability integrals*. Springer.

Cai, B., Daskalakis, C. and Kamath, G. (2017). Priv’IT: private and sample efficient identity testing. In *Proceedings of the International Conference on Machine Learning*.

Canonne, C.L., Kamath, G., McMillan, A., Smith, A. and Ullman, J. (2019). The structure of optimal private tests for simple hypotheses. In *Proceedings of the ACM Symposium on the Theory of Computing*.
Canonne, C.L., Kamath, G., McMillan, A., Ullman, J. and Zakynthinou, L. (2020). Private identity testing for high-dimensional distributions. In Neural Information Processing Systems.

Cummings, R., Krehbiel, S., Mei, Y., Tuo, R. and Zhang, W. (2018). Differentially private change-point detection. In Neural Information Processing Systems.

Dolera, E. and Favaro, S. (2020). Rates of convergence in de Finetti’s representation theorem, and Hausdorff moment problem. Bernoulli 26, 1296–1322.

Dolera, E. and Regazzini, E. (2019). Uniform rates of the Glivenko–Cantelli convergence and their use in approximating Bayesian inferences. Bernoulli 25, 2982–3015.

Dwork, C. (2006). Differential privacy. In Proceedings of the International Colloquium on Automata, Languages and Programming.

Dwork, C., McSherry, F., Nissim, K. and Smith, A. (2006). Calibrating noise to sensitivity in private data analysis. In Proceeding of the Theory of Cryptography Conference.

Dwork, C. and Roth, A. (2013). The algorithmic foundations of differential privacy. Foundations and Trends in Theoretical Computer Science 9, 211–407.

Efron, B. (1967). The power of the likelihood-ratio test. Annals of Mathematical Statistics 38, 802–806.

Efron, B. and Truax, D. (1968). Large deviations theory in exponential families. Annals of Mathematical Statistics 39, 1402–1424.

Ferguson, T.S. (2002). A Course in large sample theory. Chapman and Hall/CRC.

Fienberg, S.E., Rinaldo, A. and Yang, G. (2010). Differential privacy and the risk-utility trade-off for multi-dimensional contingency tables. In Proceedings of the International Conference on Privacy in Statistical Databases.

Gaboardi, M., Lim, H., Roger, R. and Vdhan, S. (2016). Differentially private chi-squared hypothesis testing: goodness-of-fit and independence testing. In Proceedings of the International Conference on Machine Learning.

Hoeffding, W. (1965). Asymptotically optimal tests for multinomial distributions. Annals of Mathematical Statistics 36, 369–408.

Hoeffding, W. (1967). On probabilities of large deviations. In Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability.

Kifer, D. and Rogers, R. (2017). A new class of private chi-square tests. In Proceedings of the International Conference on Artificial Intelligence and Statistics.

Osipov, L.V. (1982). Large deviation probabilities of sums of independent random vectors for certain classes of sets. Mathematical Notes 31, 75–79.

Okamoto, M. (1959). Some inequalities relating to the partial sum of binomial probabilities. Annals of the Institute of Mathematical Statistics 10, 29–35.
Petrov, V.V. (1975). *Sum of independent random variables*. Springer.

Rao, C.R. (1962). Efficient estimates and optimum inference procedures in large samples. *Journal of the Royal Statistical Society Series B* **24**, 46–72.

Rinott, Y., O’Keefe, C.M., Shlomo, N. and Skinner, C. (2017). Confidentiality and differential privacy in the dissemination of frequency tables. *Statistical Science* **33**, 358–385.

Saulis, L. (1983). Large deviations for random vectors for certain classes of sets. *Lithuanian Mathematical Journal* **23**, 308–317.

Saulis, L. and Statulevicius, V.A. (1991). *Limit theorems for large deviations*. Kluwer.

Steinke, T. and Ullman, J. (2016). Between pure and approximate differential privacy. *Journal of Privacy and Confidentiality* **7**, 3–22.

von Bahr, B. (1967). Multi-dimensional integral limit theorems. *Arkiv för Matematik* **7**, 71–88.

Wang, Y., Lee, J. and Kifer, D. (2015). Revisiting differentially private hypothesis tests for categorical data. Preprint arXiv:1511.03376.

Wilks, S.S. (1938). The large-sample distribution of the likelihood-ratio for testing composite hypotheses. *Annals of Mathematical Statistics* **9**, 60–62.