On the Improved No-Local-Collapsing Theorem of Ricci Flow

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Abstract
In this note we derive an improved no-local-collapsing theorem of Ricci flow under the scalar curvature bound condition along the worldline of the basepoint. It is a refinement of Perelman’s no-local-collapsing theorem.

Keywords Ricci flow · Volume non-collapsing · Nash entropy

Mathematics Subject Classification 53E20

1 Introduction

In 1982, Hamilton [4] introduced the Ricci flow equation, which is defined by

\[
\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)).
\]

Since then, Ricci flow has become a very powerful tool in understanding the geometry and topology of Riemannian manifolds. In Perelman’s groundbreaking work [5–7], the Ricci flow was used to give the proof of the Poincaré and geometrization conjecture.

The no-local-collapsing theorem of Perelman is important in his work of Ricci flow. In his work, whenever we need to take a Gromov–Hausdorff limit, we need to apply the no-local-collapsing theorem. Perelman introduced two approaches to prove the no-local-collapsing theorem: \(W\)-entropy and spacetime geometry, and both of them have become very useful tools in the study of Ricci flow theory.

Let us first recall Perelman’s no-local-collapsing theorem.
Theorem 1.1 (Perelman [5]) For any $A < \infty$ and dimension $n$, there exists $\kappa = \kappa(n, A) > 0$ such that the following holds: Let $(M, (g(t))_{t \in [-r_0^2, 0]})$ be a Ricci flow on compact, $n$-dimensional manifold that satisfies:

1. $|\mathbf{Rm}| \leq r_0^{-2}$ on $P(x_0, -r_0^2; r_0, r_0^2)$;
2. $\operatorname{Vol}_{g(-r_0^2)}(B_{g(-r_0^2)}(x_0, r_0)) \geq A^{-1}r_0^n$.

Then for any other metric ball $B_{g(0)}(x, r) \subset B_{g(0)}(x_0, Ar_0)$ with $r \leq r_0$ and $|\mathbf{Rm}| \leq r^{-2}$ on $P(x, 0; r, -r^2)$, we have

$$\operatorname{Vol}_{g(0)}(B_{g(0)}(x, r)) \geq \kappa r^n.$$ (1.2)

See Sect. 2 for the definition of the parabolic neighborhood $P(x_0, -r_0^2; r_0, r_0^2)$.

Next, let us recall some extensions of this no-local-collapsing theorem.

In [13], Zhang localized Perelman’s $\mathcal{W}$-entropy and the differential Harnack inequality, and proved a uniform local Sobolev inequality for the metric ball $B_{g(0)}(x_0, Ar_0)$ (see [13, Theorem 6.3.2]). Then he showed that the curvature condition (1.1) in Theorem 1.1 can be relaxed to scalar curvature bound at the final time slice, namely

$$R \leq r^{-2} \text{ on } B_{g(0)}(x, r).$$ (1.3)

Then, Wang [12] gave an independent proof of this improved no-local-collapsing theorem, where he can relax the curvature condition (1) in Theorem 1.1 to the Ricci curvature bound

$$|\mathbf{Ric}| \leq r_0^{-2} \text{ on } P(x_0, -r_0^2; r_0, r_0^2),$$ (1.4)

and Wang also found its application to Kähler–Ricci flow on smooth minimal models of general type.

Lately, under the same curvature condition (1.4) Tian–Zhang [11] established a relative volume comparison theorem of Ricci flow, that is, they remove the non-collapsing condition (2) in Theorem 1.1 for the initial metric, and obtain

$$\frac{\operatorname{Vol}_{g(0)}(B_{g(0)}(x, r))}{r^n} \geq \kappa \frac{\operatorname{Vol}_{g(-r_0^2)}(B_{g(-r_0^2)}(x_0, r_0))}{r_0^n}. $$ (1.5)

Using this estimate, Song–Tian–Zhang [10] obtained the diameter bound for the long time solution of the normalized Kähler–Ricci flow under the assumption of a local Ricci curvature bound.

The author and Song [9] gave an alternative proof of Tian–Zhang’s [11] relative volume comparison theorem by using the Nash entropy; using the idea of the proof, we obtained the diameter bound for the long time solution of the normalized Kähler–Ricci flow without assuming local Ricci curvature bound.
The Nash entropy used in [9] was introduced by Hein–Naber [8], where they obtained heat kernel bounds and \( \varepsilon \)-regularity theorem for Ricci flow. Later in [1], Bamler established systematic results on the Nash entropy and heat kernel bounds on Ricci flow background. With these bases, in [2, 3], Bamler established the compactness theory of Ricci flow and the structure theory of non-collapsed limits of Ricci flows.

Our main result in this paper is the following theorem which improves Perelman’s no-local-collapsing theorem.

**Theorem 1.2** For any \( A < \infty \) and dimension \( n \), there exists \( \kappa = \kappa(n, A) > 0 \) such that the following holds:

Let \((M, (g(t)))_{t \in [-4r_0^2, 0]}\) be a Ricci flow on compact, \( n \)-dimensional manifold, with a base point \( x_0 \) that satisfies:

1. \( R(x_0, t) \leq \frac{A}{|t|} \) for \( t \in [-r_0^2, 0] \);
2. \( \text{Vol}_{g(-r_0^2)}(B_{g(-r_0^2)}(x_0, r_0)) \geq A^{-1}r_0^n \).

Then for any other metric ball \( B_{g(0)}(x, r) \subset B_{g(0)}(x_0, Ar_0) \) with \( r \leq r_0 \) and

\[
R \leq r^{-2} \quad \text{on } B_{g(0)}(x, r),
\]

we have

\[
\text{Vol}_{g(0)}(B_{g(0)}(x, r)) \geq \kappa r^n.
\]

In the proof of the theorem, we shall also use the Nash entropy and heat kernel bounds on Ricci flow background, which are established by Bamler in [1]. The ingredient of the proof is that we need to estimate the \( W_1 \)-distance of two points on the same worldline by using the heat kernel upper bound of Ricci flow.

**2 Preliminaries**

We recall some basic definitions and results of Ricci flow in this section. This is the framework established by Bamler in [1].

Let \((M, (g(t)))_{t \in I}\) be a smooth Ricci flow on a compact \( n \)-manifold with \( I \subset \mathbb{R} \) being an interval. Letting \((x_0, t_0) \in M \times I\) with \( A, T^-, T^+ \geq 0 \), the parabolic neighborhood is defined as

\[
P(x_0, t_0; A, -T^-, T^+) := B(x_0, t_0, A) \times ([t_0 - T^-, t_0 + T^+] \cap I),
\]

where we may omit \( -T^- \) or \( T^+ \) if it is zero.

We consider the heat operator

\[
\Box := \partial_t - \Delta_{g(t)},
\]
and the conjugate heat operator

$$\Box^* := -\partial_t - \Delta_{g(t)} + R,$$

where $R$ denotes the scalar curvature at time $t$.

For any $x, y \in M$ and $s, t \in I$ with $s \leq t$, we denote by $K(x, t; y, s)$ the heat kernel along the Ricci flow. That is, for fixed $(y, s)$, the function $K(\cdot, t; \cdot, s)$ is a heat kernel based at $(y, s)$:

$$\Box K(x, t; y, s) = 0; \lim_{t \searrow s} K(x, t; y, s) = \delta_y,$$

and for fixed $(x, t)$, the function $K(x, t; \cdot, \cdot)$ is a conjugate heat kernel based at $(x, t)$:

$$\Box^* K(x, t; \cdot, \cdot) = 0; \lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x.$$

**Definition 2.1** For $(x, t) \in M \times I$ and $s \in I$ with $s \leq t$, we denote by $\nu_{x,t;s}$ the conjugate heat kernel measure based at $(x, t)$ as follows:

$$d \nu_{x,t;s} := K(x, t; \cdot, \cdot)dg(t) =: (4\pi \tau)^{-\frac{n}{2}} e^{-f} dg(t),$$

where $\tau = t_0 - t$ and $f \in C^\infty(M \times (I \cap (\infty, t)))$ is called the potential.

Basically, in Bamler’s viewpoint, instead of considering a point $(x, t)$ in spacetime, we shall consider its corresponding conjugate heat kernel measure $d \nu_{x,t;s}$ as defined above. Then Bamler used the $W_1$-Wasserstein distance to define distance between the conjugate heat kernel measures based at different points. If $\mu_1$ and $\mu_2$ denote two probability measures on a complete manifold $M$ and $g$ is a Riemannian metric on $M$, then we define the $W_1$-distance between $\mu_1$ and $\mu_2$ by

$$d_{W_1}^g(\mu_1, \mu_2) := \sup_{f} \left( \int_M f d\mu_1 - \int_M f d\mu_2 \right),$$

where the supremum is taken over all bounded, 1-Lipschitz function $f : M \to \mathbb{R}$.

Bamler also used the important notion called variance, which in some sense is the $L^2$-norm of the probability measures on a Riemannian manifold. The definition is the following

$$\text{Var}(\mu_1, \mu_2) := \int_M \int_M d^2(\mu_1, \mu_2) d\mu_1(x_1) d\mu_2(x_2).$$

If $(M, (g(t))_{t \in I})$ is a Ricci flow and if there is chance of confusion, then we will also write $\text{Var}_t$ for the variance with respect to the metric $g(t)$. We have the following basic relation between the variance and the $W_1$-distance:

$$d_{W_1}^g(\mu_1, \mu_2) \leq \sqrt{\text{Var}(\mu_1, \mu_2)}, \quad (2.1)$$
for any two probability measures.

Now we can define the $H_n$-center of a given base point in Ricci flow.

**Definition 2.2** A point $(z, t) \in M \times I$ is called an $H_n$-center of a point $(x_0, t_0) \in M \times I$ if $t \leq t_0$ and

$$\text{Var}_t(\delta_z, \nu_{x_0, t_0}; t) \leq H_n(t_0 - t).$$

Given $(x_0, t_0) \in M \times I$ and $t \leq t_0$, there always exists at least one $z \in M$ such that $(z, t) \in M \times I$ is an $H_n$-center of $(x_0, t_0)$ (see [1, Proposition 3.12]), and for such $H_n$-center, using inequality (2.1), we always have

$$d_{W^1}(\delta_z, \nu_{x_0, t_0}; t) \leq \sqrt{\text{Var}(\delta_z, \nu_{x_0, t_0}; t)} \leq \sqrt{H_n(t_0 - t)}.$$

Next, we come to define the so-called Nash entropy which was introduced by Hein–Naber [8]. Let $(M, g)$ be a closed Riemannian manifold of dimension $n$, $\tau > 0$ and $\nu = (4\pi \tau)^{-n/2}e^{-f}dg$ is a probability measure on $M$, where $f \in C^\infty(M)$, then the Nash entropy is defined by

$$\mathcal{N}[g, f, \tau] = \int_M f d\nu - \frac{n}{2}.$$

Then we can define

**Definition 2.3** Consider a conjugate heat kernel measure

$$d\nu_{x_0, t_0; t} := K(x_0, t_0; \cdot, t)dg(t) = (4\pi \tau)^{-n/2}e^{-f}dg(t),$$

based at some point $(x_0, t_0) \in M \times I$, where $\tau = t_0 - t$. The pointed Nash entropy based at $(x_0, t_0)$ is defined as

$$\mathcal{N}_{x_0, t_0}(\tau) := \mathcal{N}[g_{t_0-\tau}, f_{t_0-\tau}, \tau].$$

We set $\mathcal{N}_{x_0, t_0}(0) = 0$. For $s < t_0$, $s \in I$, we will write

$$\mathcal{N}^s_{x_0, t_0}(t) := \mathcal{N}_{x_0, t_0}(t - s).$$

The pointed Nash entropy $\mathcal{N}_{x_0, t_0}(\tau)$ is non-increasing when $\tau \geq 0$ is increasing. For more basic properties of the pointed Nash entropy, We refer the readers to [1, Proposition 5.2]. (See also Hein–Naber’s work [8].)

The following lemma is proved in [1, Theorem 5.9, Corollary 5.11].

**Lemma 2.4** If $R(\cdot, s) \geq R_{\text{min}}$ for some $s \in I$, then on $M \times (I \cap (s, \infty))$,

$$|\nabla \mathcal{N}^s_{x_0, t_0}| \leq \left(\frac{n}{2(t - s)} + |R_{\text{min}}|\right)^{1/2}. \quad (2.2)$$
Furthermore, if $s < t^* \leq \min\{t_1, t_2\}$ and $s, t_1, t_2 \in I$, then for any $x_1, x_2 \in M$,
\[
\mathcal{N}^*_s(x_1, t_1) - \mathcal{N}^*_s(x_2, t_2) \\
\leq \left( \frac{n}{2(t^* - s)} + |R_{\min}| \right) \frac{1}{2} d_{W_1}^g(t^*)^2 (v_{x_1, t_1}(t^*), v_{x_2, t_2}(t^*)) + \frac{n}{2} \log \left( \frac{t_2 - s}{t^* - s} \right). \tag{2.3}
\]

The following volume non-collapsing estimate is proved in [1] as a generalization of Perelman’s $\kappa$-non-collapsing theorem.

**Lemma 2.5** Let $(M, (g(t))_{t \in [-r^2, 0]})$ be a solution of the Ricci flow. If
\[
R \leq r^{-2} \quad \text{on } B_{g(0)}(x, r),
\]
then
\[
\Vol_{g(0)}(B_{g(0)}(x, r)) \geq c \exp \left( \mathcal{N}^*_r(x, 0) \right) r^n. \tag{2.4}
\]

The following volume non-inflation estimate is also proved in [1].

**Lemma 2.6** Let $(M, (g(t))_{t \in [-r^2, 0]})$ be a solution of the Ricci flow. If
\[
R \geq -r^{-2}
\]
on $M \times [-r^2, 0]$, then for any $A \geq 1$, there exist $C_0 = C_0(n), C = C(n) > 0$ such that
\[
\Vol_{g(0)}(x, Ar) \leq C \exp \left( \mathcal{N}^*_r(x, 0) + C_0 A^2 \right) r^n. \tag{2.5}
\]

The following heat kernel upper bound estimate is proved in [1, Theorem 7.2].

**Lemma 2.7** Let $(M, (g(t))_{t \in I})$ be a solution of the Ricci flow. Suppose that $[s, t] \subset I$ and $R \geq R_{\min}$ on $M \times [s, t]$. Let $(z, s) \in M \times I$ be an $H_n$-center of $(x, t) \in M \times I$. Then for any $\varepsilon > 0$ and $y \in M$, we have
\[
K(x, t; y, s) \leq \frac{C(R_{\min}(t - s), \varepsilon)}{(t - s)^{n/2}} \exp \left(-\mathcal{N}_{x,t}(t-s)\right) \exp \left(-\frac{d_s^2(z, y)}{C(\varepsilon)(t - s)}\right). \tag{2.6}
\]

For more results on the entropy and heat kernel bounds on Ricci flow, we refer the readers to [1].

### 3 Proof of the Improved No-Local-Collapsing Theorem

In this section, we come to prove Theorem 1.2. First, we need the following proposition.
Proposition 3.1  For any $T > 0$, consider a smooth Ricci flow $(M, (g(t))_{t \in (-T, 0)})$ on compact $n$-dimensional manifold, and choose $(s, t) \subset (-T, 0)$ with $s + T > \varepsilon > 0$. Consider a $C^1$ spacetime curve $\gamma : (0, t-s) \to M \times (-T, 0)$ with $\gamma(\tau) \in M \times \{t-\tau\}$ between points $\gamma(0) = x$ and $\gamma(t-s) = y$. Then we have

$$d_{W_1}^g(\delta_{y,s}, \nu_{x,t;s}) \leq C(\varepsilon) \left( 1 + \frac{L(\gamma)}{2\sqrt{t-s}} - \mathcal{N}_{x,t}(t-s) \right)^{\frac{1}{2}} \sqrt{t-s}. \quad (3.1)$$

Proof  This is obtained by Bamler in [3, Lemma 22.2]. We include the proof for the convenience of the readers.

First, from the definition of the reduced distance, we have

$$\ell_{(x,t)}(y,s) \leq \frac{L(\gamma)}{2\sqrt{t-s}}.$$

Hence from the estimate of Perelman [5, Corollary 9.5], we have the following lower bound on the heat kernel

$$K(x, t; y, s) \geq \frac{1}{(4\pi (t-s))^{n/2}} \exp \left( -\ell_{(x,t)}(y,s) \right) \geq \frac{1}{(4\pi (t-s))^{n/2}} \exp \left( -\frac{L(\gamma)}{2\sqrt{t-s}} \right). \quad (3.2)$$

But, since $s + T > \varepsilon$, we can apply the heat kernel upper bound estimate of Bamler, say Lemma 2.7, to obtain

$$K(x, t; y, s) \leq \frac{C(\varepsilon)}{(t-s)^{n/2}} \exp \left( -\mathcal{N}_{x,t}(t-s) \right) \exp \left( -\frac{d_s^2(z,y)}{C(\varepsilon)(t-s)} \right), \quad (3.3)$$

where $(z, s)$ is an $H_n$-center of $(x, t)$. Hence by the triangle inequality and (2.1), we have

$$\left| d_{W_1}^g(\delta_{y,s}, \nu_{x,t;s}) - d_s(y, z) \right|$$

$$= \left| d_{W_1}^g(\delta_{y,s}, \nu_{x,t;s}) - d_{W_1}^g(\delta_{y,s}, \delta_{z,s}) \right|$$

$$\leq d_{W_1}^g(\delta_{z,s}, \nu_{x,t;s}) \leq \sqrt{\text{Var} \left( \delta_{z,s}, \nu_{x,t;s} \right)}$$

$$\leq \sqrt{H_n(t-s)}. \quad (3.4)$$

Hence, using the trivial inequality $(a - b)^2 \geq \frac{a^2}{2} - 4b^2$ for all $a, b \in \mathbb{R}$, we obtain

$$H_n(t-s) \geq \frac{1}{2} d_{W_1}^g(\delta_{y,s}, \nu_{x,t;s})^2 - 4d_s^2(y, z).$$
Plugging this into inequality (3.3), we obtain

\[
K(x, t; y, s) \leq \frac{C(\varepsilon)}{(t-s)^{n/2}} \exp \left( -N_{x,t}(t-s) \right) \exp \left( -\frac{d_{W_1}^g(\delta_{y,s}, v_{x,t;s})^2}{C(\varepsilon)(t-s)} \right)
\]

\[
\leq \frac{C(\varepsilon)}{(t-s)^{n/2}} \exp \left( -N_{x,t}(t-s) - \frac{d_{W_1}^g(\delta_{y,s}, v_{x,t;s})^2}{C(\varepsilon)(t-s)} \right).
\]

(3.5)

Combining (3.5) with (3.2) we obtain

\[
\frac{1}{(4\pi(t-s)^{n/2})} \exp \left( -\frac{\mathcal{L}(\gamma)}{2\sqrt{t-s}} \right) \leq \frac{C(\varepsilon)}{(t-s)^{n/2}} \exp \left( -N_{x,t}(t-s) - \frac{d_{W_1}^g(\delta_{y,s}, v_{x,t;s})^2}{C(\varepsilon)(t-s)} \right).
\]

Rearranging this inequality, we obtain

\[
d_{W_1}^g(\delta_{y,s}, v_{x,t;s}) \leq C(\varepsilon) \left( 1 + \frac{\mathcal{L}(\gamma)}{2\sqrt{t-s}} - N_{x,t}(t-s) \right)^{\frac{1}{2}} \sqrt{t-s},
\]

which finishes the proof of the proposition. \(\square\)

Now we can prove Theorem 1.2.

**Proof of Theorem 1.2** After parabolic rescaling, we may assume without loss of generality that \(r_0 = 1\), and then \(r \leq 1\).

Let \(\gamma : [0, 1] \to M \times [-1, 0]\) be the spacetime curve defined by

\[
\gamma(\tau) = (x_0, -\tau), \quad \tau \in [0, 1],
\]

which is usually called the worldline. Then from our assumption, we have

\[
R(\gamma(\tau)) \leq \frac{A}{\tau}, \quad \forall \tau \in (0, 1],
\]

and hence we can compute

\[
\mathcal{L}(\gamma) = \int_0^1 \sqrt{\tau} (|\gamma'(\tau)|^2 + R(\gamma(\tau), -\tau)) d\tau
\]

\[
= \int_0^1 \sqrt{\tau} R(\gamma(\tau), -\tau) d\tau
\]

\[
\leq \int_0^1 \sqrt{\tau} \cdot \frac{A}{\tau} d\tau
\]

\[
\leq 2A.
\]

(3.6)
Next, by the monotonicity of the Nash entropy, we have that \( 0 \geq N_{x_0,0}(1) \geq \mathcal{N}_{x_0,0}(2) \), and hence \( 0 \leq -\mathcal{N}_{x_0,0}(1) \leq -\mathcal{N}_{x_0,0}(2) \). Hence, we can apply Proposition 3.1 together with (3.6) to obtain

\[
d g_{W_1}^{(1)}(\delta_{x_0,-1}, \nu_{x_0,0;-1}) \\
\leq C \left( 1 + \frac{\mathcal{L}(\gamma)}{2\sqrt{0 - (-1)}} - \mathcal{N}_{x_0,0}(1) \right)^{\frac{1}{2}} \sqrt{0 - (-1)} \tag{3.7}
\]

\[
\leq C \left( 1 - \mathcal{N}_{x_0,0}(1) \right)^{\frac{1}{2}},
\]

where \( C = C(n, A) \). Hence, by applying the Harnack estimate of Nash entropy proved by Bamler, say Lemma 2.4, we have

\[
\left| \mathcal{N}^*_{-2}(x_0, -1) - \mathcal{N}^*_{-2}(x_0, 0) \right| \\
\leq \left( \frac{n}{2((-1) - (-2))} + n \right)^{\frac{1}{2}} d g_{W_1}^{(1)}(\delta_{x_0,-1}, \nu_{x_0,0;-1}) \\
+ \frac{n}{2} \log \left( \frac{0 - (-2)}{(-1) - (-2)} \right) \tag{3.8}
\]

\[
\leq C \left( C - \mathcal{N}^*_{-2}(x_0, 0) \right)^{\frac{1}{2}} + C \\
\leq 100C^2 + \frac{1}{100} \left( C - \mathcal{N}^*_{-2}(x_0, 0) \right) + C.
\]

Hence we obtain

\[
\frac{99}{100} \mathcal{N}^*_{-2}(x_0, 0) \geq \mathcal{N}^*_{-2}(x_0, -1) - C. \tag{3.9}
\]

But by the volume non-inflating estimate, say Lemma 2.6, since we have \( R \geq -C(n) \), we have

\[
A^{-1} \leq \text{Vol}_{g(-1)}(B_{g(-1)}(x_0, 1)) \leq C \exp (\mathcal{N}_{x_0,-1}(1)),
\]

which can be written as

\[
\mathcal{N}^*_{-2}(x_0, -1) = \mathcal{N}_{x_0,-1}(1) \geq -C(n, A).
\]

Combining this with (3.9), we obtain

\[
\mathcal{N}^*_{-2}(x_0, 0) \geq -C \tag{3.10}
\]

for some \( C = C(n, A) < \infty \).

Next, by the gradient estimate of the Nash entropy in Lemma 2.4, we have

\[
\left| \mathcal{N}^*_{-2}(x_0, 0) - \mathcal{N}^*_{-2}(x, 0) \right| \leq C(n)d_0(x_0, x) \leq C(n)A. \tag{3.11}
\]
Combining (3.10) and (3.11) we obtain

\[ \mathcal{N}_{x,0}(2) = \mathcal{N}^\ast_{-2}(x, 0) \geq -C. \]

Then we apply the monotonicity of the Nash entropy to obtain (note that \( r \leq 1 \))

\[ \mathcal{N}_{x,0}(r^2) \geq \mathcal{N}_{x,0}(2) \geq -C. \]

But we have assumed that \( R \leq r^{-2} \) on \( B_{g(0)}(x, r) \). We can apply the lower volume bound estimate, say Lemma 2.5, to obtain that:

\[ \text{Vol}_{g(0)}(B_{g(0)}(x, r)) \geq c(n) \exp \left( \mathcal{N}_{x,0}(r^2) \right) r^n \geq \kappa(n, A) r^n, \]

which finishes the proof of the theorem. \( \square \)

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