Scalar particle with Darwin-Cox structure in external Coulomb field

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Abstract. Generalized Klein-Fock-Gordon equation for a scalar particle with Darwin-Cox structure, which takes into account distribution of electric charge of the particle inside a finite spherical region is studied in external Coulomb field. Corresponding radial equation has two irregular singular points, $r = 0$ of the rank 3, $r = \infty$ of the rank 2, and four regular singular points. In the case of minimal angular momentum, $l = 0$, the structure of singularities becomes simpler: the points $r = 0, r = \infty$ are both of the rank 2, and four regular points remain the same. There are constructed formally exact Frobenius type solutions of the derived equations, convergence of relevant power series, with 8-term and 7-term recurrent relations respectively, is studied. As analytical quantization rule is taken so-called transcendence conditions. It provides us with 4-th order algebraic equation with respect to energy values, which has four sets of roots. Only one set of roots, $0 < E_{l,k} < 1$, depending on angular momentum $l = 0, 1, 2, \ldots$ and main quantum number $n = 0, 1, 2, \ldots$ may be interpreted as corresponding to some bound states of the particle in the Coulomb field. In the same manner, a generalized nonrelativistic Schrödinger equation for such a particle was studied, the final results are similar.

1. Introduction, Cox equation

In the frames of the theory of generalized relativistic wave equations, a special model for a spin zero particle was proposed by W. Cox [1]. Cox constructed the wave equation for a scalar particle with a larger set of tensor functions than it exists in the Proca’s approach. He used the set of a scalar, 4-vector, antisymmetric and (irreducible) symmetric tensor, thus starting with the 20-component wave function. In particular, the Cox electromagnetic structure in presence of external electric field may be associated with the known C.G. Darwin interaction term in nonrelativistic Schrödinger equation, this interaction is related to non-point-like distribution of the electric charge in the finite volume inside the particle see.

In the present paper we start with Cox covariant system (let $D_\alpha = i\nabla_\alpha - \frac{e}{\hbar c}A_\alpha$):

$$
\begin{align*}
\lambda_1 D^\beta \Phi_\beta - \frac{Mc}{\hbar} \Phi &= 0, \\
\lambda_1 D_\beta \Phi + \lambda_2 D^\alpha \Phi_{[\alpha\beta]} - \lambda_3 D^\alpha \Phi_{(\alpha\beta)} - \frac{Mc}{\hbar} \Phi_\beta &= 0, \\
\lambda_2^2 (D_\alpha \Phi_\beta - D_\beta \Phi_\alpha) - \frac{Mc}{\hbar} \Phi_{[\alpha\beta]} &= 0, \\
\lambda_2^2 (D_\alpha \Phi_\beta + D_\beta \Phi_\alpha) - \frac{1}{2} g_{\alpha\beta} D^\rho \Phi_\rho - \frac{Mc}{\hbar} \Phi_{(\alpha\beta)} &= 0,
\end{align*}
$$

(1)
auxiliary numerical parameters $\lambda_1, \lambda_2, \lambda_3$ subject to the constraints

$$\lambda_2 \lambda_2^* - \lambda_3 \lambda_3^* = 0, \quad \lambda_1 \lambda_1^* - \frac{3}{2} \lambda_3 \lambda_3^* = 1. \quad (2)$$

With the help of the third and the fourth equations in (1), we can exclude tensor components, $\Phi_{(\alpha\beta)}, \Phi_{[\alpha\beta]}$, so deriving a generalized Proca system

$$D_\beta \Phi_\beta - \frac{Mc}{\hbar} \Phi = 0, \quad D_\beta \Phi - \lambda (F_{\beta\alpha} + \frac{\hbar}{e} R_{\beta\alpha}) \Phi^\alpha - \frac{Mc}{\hbar} \Phi_\beta = 0, \quad (3)$$

note that $\lambda$ is an imaginary number

$$\lambda = i \frac{\hbar^2}{Mc \hbar} (2 \lambda_3 \lambda_3^*) . \quad (4)$$

In (3) we can see a non-minimal interaction term of the Cox’s scalar particle with external electromagnetic field and with geometric background through the Ricci tensor.

2. Setting the problem

In flat space, equations can be presented as

$$(mc/\hbar) (\delta_\alpha^\beta + \frac{\lambda}{mc/\hbar} F_\alpha^\beta) \Phi_\beta = D_\alpha \Phi, \quad D^\alpha \Phi_\alpha = \frac{mc}{\hbar} \Phi, \quad (5)$$

where $D_\alpha = i \nabla_\alpha + (e/\hbar c) A_\alpha$ (note that parameter $e$ is positive). Shortly, they read ($\Lambda$ stands for $4 \times 4$ matrix)

$$\frac{mc}{\hbar} \Lambda^\beta_\alpha \Phi_\beta = D_\alpha \Phi, \quad D^\alpha \Phi_\alpha = \frac{mc}{\hbar} \Phi, \quad (6)$$

whence it follows a generalized Klein-Fock-Gordon equation for scalar function $\Phi(x)$:

$$[D_\rho (\Lambda^{-1})^\rho\alpha (x) D_\alpha \Phi - \frac{m^2 c^2}{\hbar^2}] \Phi = 0. \quad (7)$$

Explicit form of eq. (7) in space with metric $g_{\alpha\beta}(x)$ is as follows

$$\left[ \left( \frac{i}{\sqrt{g}} \frac{\partial}{\partial x^\rho} \sqrt{-g} + \frac{e}{\hbar c} A_\rho \right) (\Lambda^{-1}(x))^{\rho\alpha} \left( \frac{i}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} + \frac{e}{\hbar c} A_\alpha \right) - \frac{m^2 c^2}{\hbar^2} \right] \Phi = 0. \quad (7)$$

3. Separating the variables

We consider the particle in external Coulomb field, using spherical coordinates $(r, \theta, \phi)$:

$$A_0 = \frac{e}{r}, \quad e > 0, \quad F_{0\theta} = -\frac{e}{r^2}. \quad (8)$$

The relevant matrix $\Lambda^\alpha_\beta$ has the structure

$$\Lambda = (\Lambda^\alpha_\beta) = \begin{pmatrix} 1 & \mu e & 0 & 0 \\ \mu e & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mu = -\frac{\lambda e}{mc/\hbar}; \quad (9)$$
its inverse matrix is given by expression (let $\Lambda^{-1} = (K_\beta^\rho)$)

\[
(K_\beta^\rho) = \begin{vmatrix}
\frac{r^4}{r^4 - \mu^2} & -\frac{\mu^2}{r^4 - \mu^2} & 0 & 0 \\
-\frac{\mu^2}{r^4 - \mu^2} & \frac{r^4}{r^4 - \mu^2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{vmatrix},
\]

\[
(K^\beta_\rho) = \begin{vmatrix}
\frac{r^4}{r^4 - \mu^2} & -\frac{\mu^2}{r^4 - \mu^2} & 0 & 0 \\
+\frac{\mu^2}{r^4 - \mu^2} & -\frac{r^4}{r^4 - \mu^2} & 0 & 0 \\
0 & 0 & 0 & -1/r^2 \\
0 & 0 & 0 & -1/r^2 \sin^2 \theta
\end{vmatrix}.
\]

Explicitly, the wave equation looks as (let $\alpha = e^2/hc$)

\[
\left\{ \left( i\partial_\theta + \frac{\alpha}{r} \right)^2 \frac{r^4}{r^4 - \mu^2} - \left( i\partial_\theta + \frac{\alpha}{r} \right) \frac{\mu^2}{r^4 - \mu^2} i\partial_r \\
+ \frac{i}{r^2} \partial_r r^2 \frac{\mu^2}{r^4 - \mu^2} \left( i\partial_\theta + \frac{\alpha}{r} \right) - \frac{i}{r^2} \partial_r r^2 \frac{r^4}{r^4 - \mu^2} i\partial_r \\
+ \frac{1}{r^2} \left( \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi \partial_\phi \right) - \frac{m^2 c^2}{r^2}\right\} \Phi = 0.
\] (8)

After separating the variables with the help of the usual substitution

\[
\Phi(x) = e^{-i\epsilon x^0} Y_{lm}(\theta, \phi) R(r), \quad \epsilon = \frac{E}{\hbar c}, \quad |\epsilon| = \frac{1}{\text{meter}}
\]

we derive the radial equation

\[
\frac{d^2 R}{dr^2} + \left[ \frac{2}{r} + \frac{4\gamma^2}{r(r^4 + \gamma^2)} \right] \frac{dR}{dr} + \left[ \epsilon^2 - M^2 + \frac{2\alpha \epsilon}{r} + \frac{\alpha^2 - l(l + 1)}{r^2} \right. \\
+ \frac{4\alpha \epsilon}{r^5} \right. \\
+ \frac{3\alpha \gamma - \gamma^2 M^2}{r^4} - \frac{l(l + 1)\gamma^2}{r^6} - \frac{4(\gamma \epsilon r + \gamma \alpha)}{r^4 + \gamma^2} \left\} R = 0,
\] (9)

where $i\mu = \gamma$ is real-valued. Here we have four regular points and two irregular ones:

\[
r^4 + \gamma^2 = (r - e^{+i\pi/4} \sqrt{\gamma})(r + e^{+i\pi/4} \sqrt{\gamma})(r - e^{-i\pi/4} \sqrt{\gamma})(r + e^{-i\pi/4} \sqrt{\gamma}) ,
\]

\[
r = 0, \quad \text{Rank} = 3, \quad r = \infty, \quad \text{Rank} = 2;
\]

note identities

\[
\sigma \equiv (-\gamma^2)^{1/4}, \quad \sigma^2 = \sqrt{-\gamma^2} = i\gamma, \quad \sigma^4 = -\gamma^2,
\]

\[
\frac{1}{r^4 + \gamma^2} = \frac{1}{4\sigma^3} \left( \frac{1}{r - \sigma} - \frac{1}{r + \sigma} + \frac{i}{r - i\sigma} - \frac{i}{r + i\sigma} \right).
\]

Behavior of solutions near four regular points is trivial:

\[
r \to \pm \sigma, \quad R \sim (r \mp \sigma)^\rho, \quad \rho = 0, 2; \quad (10)
\]

\[
r \to \pm i\sigma, \quad R \sim (r \mp i\sigma)^\rho, \quad \rho = 0, 2. \quad (11)
\]

Substitution for local Frobenius solutions of Eq. (9) in vicinity of the point $r = 0$ is as follows

\[
l = 0, \quad R(r) = r^C e^{(Ae^B + D)/r} f(r); \quad l = 1, 2, ..., \quad R(r) = r^C e^{(Ae^B + D)/r^2} f(r). \quad (12)
\]
4. States with minimal orbital momentum

Let us consider the case of $l = 0$, the relevant radial equation becomes simpler

$$\frac{d^2 R}{dr^2} + \left( \frac{6}{r} - \frac{4r^3}{r^4 + \gamma^2} \right) \frac{dR}{dr}$$

$$+ \left( \epsilon^2 - M^2 + \frac{2\alpha\epsilon}{r} + \frac{\alpha^2}{r^2} + \frac{4\gamma\epsilon}{r^3} + \frac{3\alpha\gamma - \gamma^2 M^2}{r^4} - \frac{4\gamma(\epsilon r + \alpha)}{r^4 + \gamma^2} \right) R = 0.$$  

Here we have four regular points, and two irregular points, $r = 0$ and $r = \infty$, both of Rank 2. The main equation may be presented in more symmetrical form

$$\frac{d^2 R}{dr^2} + \left( \frac{6}{r} - \frac{1}{r - \sigma} - \frac{1}{r - i\sigma} - \frac{1}{r + \sigma} - \frac{1}{r + i\sigma} \right) \frac{dR}{dr}$$

$$+ \left[ \epsilon^2 - M^2 + \frac{2\alpha\epsilon}{r} + \frac{\alpha^2}{r^2} + \frac{4\gamma\epsilon}{r^3} + \frac{3\alpha\gamma - \gamma^2 M^2}{r^4} \right.$$

$$- \frac{\gamma(\epsilon + \alpha)}{(r - \sigma)\sigma^2} + \frac{\gamma(\epsilon + i\alpha)}{(r + i\sigma)\sigma^2} + \frac{\gamma(-\epsilon + \alpha)}{(r + \sigma)\sigma^2} - \frac{\gamma(-\epsilon + i\alpha)}{(r - i\sigma)\sigma^2} \left. \right] R = 0.$$  

Solutions are constructed with the use of substitution $R(r) = r^C e^{Ar} e^{B/r} f(r)$; further we get four sets of parameters and an equation for function $f(r)$:

$$A = \pm \sqrt{M^2 - \epsilon^2}, \quad B = \delta \sqrt{\gamma (\gamma M^2 - 3\alpha)}, \quad C = -2 + \frac{2\gamma \epsilon}{\sqrt{\gamma (\gamma M^2 - 3\alpha)}},$$  

where $\delta = \pm 1$. To describe bound states, we should use negative values for $A$: $A = -\sqrt{M^2 - \epsilon^2}$. As for $B$, first let us consider the possibility of imaginary values for $B$:

$$e^{B/r} = e^{\pm i R/r}, \quad R > 0, \quad \gamma \in (0, \frac{3\alpha}{M^2}).$$  

For this case, we have divergent and oscillating behavior for solutions near the point $r = 0$:

$$r \to 0, \quad R \sim \frac{1}{r^2} e^{\pm i \frac{2\gamma}{3\alpha} \ln r} e^{\pm i R/r};$$  

hardly this behavior may be appropriate for bound states, By this reason, in the following we will assume that $B$ is real-valued, this results in

$$B = -\sqrt{\gamma (\gamma M^2 - 3\alpha)}, \quad \gamma < 0.$$  

Taking in mind these restrictions for parameters, we get the following equation for $f(r)$ (its short form is written down):

$$f'' + \left( \frac{a_1}{r} + \frac{a_2}{r^2} - \frac{1}{r + \sigma} + \frac{1}{r - \sigma} + \frac{1}{r - i\sigma} + \frac{1}{r + i\sigma} \right) f'$$

$$+ \left( \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{\beta_1}{r + \sigma} + \frac{\beta_2}{r - \sigma} + \frac{\beta_3}{r + i\sigma} + \frac{\beta_4}{r - i\sigma} \right) f = 0.$$  

Solutions are searched as power series: $f(r) = \sum_{n=0}^{\infty} c_n r^n$. Then we derive 7-term recurrent relations for coefficients:

$$k = 5, 6, 7, ..., \quad [a(k-5) + (b_1 + \beta_1 + \beta_2 + \beta_3 + \beta_4)]c_{k-5}$$
\[ +[(k - 4)(k - 5) + (a_1 - 4)(k - 4) + \{(-\beta_1 + \beta_2 - i\beta_3 + i\beta_4)\}c_{k-4} + a_2(k - 3) + \sigma^2(\beta_1 + \beta_2 - \beta_3 - \beta_4)|c_{k-3} + (-\beta_1 + \beta_2 + i\beta_3 - i\beta_4)\sigma^3c_{k-2} + [-\sigma^4a(k - 1) - b_1\sigma^4]c_{k-1} + [-\sigma^4k(k - 1) - \sigma^4a_1k - b_2\sigma^4]c_k - \sigma^4a_2(k + 1)c_{k+1} = 0. \quad (19) \]

In accordance with Poincaré-Peronne method, we should divide the last relationship by \(k^2c_{k-5}\):

\[
\frac{1}{k^2} \left[ a(k - 5) + (b_1 + \beta_1 + \beta_2 + \beta_3 + \beta_4) \right]c_{k-4} + \frac{1}{k^2} [(k - 4)(k - 5) + (a_1 - 4)(k - 4) + \{(-\beta_1 + \beta_2 - i\beta_3 + i\beta_4)\}c_{k-4} + a_2(k - 3) + \sigma^2(\beta_1 + \beta_2 - \beta_3 - \beta_4)]c_{k-3} \frac{c_{k-4}}{c_{k-5}} \frac{c_{k-4}}{c_{k-5}} + \frac{1}{k^2} (-\beta_1 + \beta_2 + i\beta_3 - i\beta_4)\sigma^3c_{k-2} \frac{c_{k-3} \frac{c_{k-4}}{c_{k-5}}}{c_{k-5}} + \frac{1}{k^2} [-\sigma^4a(k - 1) - b_1\sigma^4]c_{k-1} \frac{c_{k-2} \frac{c_{k-3} \frac{c_{k-4}}{c_{k-5}}}{c_{k-5}}}{c_{k-5}} \frac{c_{k-2}}{c_{k-3}} \frac{c_{k-4}}{c_{k-5}} + \frac{1}{k^2} [-\sigma^4k(k - 1) - \sigma^4a_1k - b_2\sigma^4]c_k \frac{c_{k-1} \frac{c_{k-2} \frac{c_{k-3} \frac{c_{k-4}}{c_{k-5}}}{c_{k-5}}}{c_{k-5}}}{c_{k-5}} \frac{c_{k-1}}{c_{k-2}} \frac{c_{k-3}}{c_{k-4}} \frac{c_{k-4}}{c_{k-5}} = 0,
\]

and then tend \(k\) to infinity \(k \to \infty\). This results in an algebraic equation which determines possible convergence radii \(R_{\text{conv}} = \frac{1}{|\sigma|}\):

\[
\lim_{k \to \infty} \frac{c_{k-4}}{c_{k-5}} = r, \quad r - \sigma^4r^5 = 0, \quad R_{\text{conv}} = \frac{1}{|\sigma|} = |\sqrt{3}|, \infty.
\]

The problem is how to quantize the energy values. To this end, we will apply the constraint which separates from all Frobenius solutions the so-called transcendental solutions. To clarify this matter, we are to consider the above recurrent formula (19)

\[ P_{k-5}c_{k-5} + P_{k-4}c_{k-4} + P_{k-3}c_{k-3} + P_{k-2}c_{k-2} + P_{k-1}c_{k-1} + P_kc_k + P_{k+1}c_{k+1} = 0, \]

and impose a requirement to have vanished the coefficient at \(c_{k-5}\) (assuming \(k \geq 5\)):

\[ P_{k-5} = a(k - 5) + (b_1 + \beta_1 + \beta_2 + \beta_3 + \beta_4) = 0. \quad (20) \]

If additionally, we assume the vanishing the coefficients

\[ c_{k-4} = 0, \quad c_{k-3} = 0, \quad c_{k-2} = 0, \quad c_{k-1} = 0, \quad c_k = 0, \quad (21) \]

then due to recurrent formula all remaining coefficients \(c_{k+1}, \ldots\) will vanish identically, and the series will becomes polynomials. Evidently, the last (polynomial) method is worked only if the system of algebraic equations (20) and (21) has joint and physically admissible roots. However, numerical study shows that here it is not possible.
5. Energy quantization for states with \( l = 0 \)

Before numerical study, it is convenient to transform all relations to dimensionless quantities:

\[
Mr = x, \quad \frac{\epsilon}{M} = \frac{E'}{c\hbar} = \frac{E'}{mc^2} = E, \quad \gamma M^2 = \Gamma, \tag{22}
\]

and the radial equation (for states with \( l = 0 \)) takes the form

\[
\frac{d^2 R}{dx^2} + \left( \frac{6}{x} - \frac{4x^3}{x^4 + \Gamma^2} \right) \frac{dR}{dx} + \left( E^2 - 1 + \frac{2\alpha E}{x} + \frac{\alpha^2}{x^2} + \frac{4\Gamma E}{x^3} + \frac{3\alpha \Gamma - \Gamma^2}{x^4} - \frac{4\Gamma(Ex + \alpha)}{x^4 + \Gamma^2} \right) R = 0.
\]

Substitution for Frobenius solutions is

\[
R(x) = x^C e^{A x} e^{B/x} f(x),
\]

where

\[
A = -\sqrt{1 - E^2}, \quad B = -\sqrt{-\Gamma(3\alpha - \Gamma)}, \quad C = \frac{-2\Gamma E}{\sqrt{-\Gamma(3\alpha - \Gamma)}} - 2, \quad \Gamma < 0,
\]

the differential equation for \( f(r) \) is

\[
f'' + \left( A + \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{1}{x + \Sigma} - \frac{1}{x - \Sigma} - \frac{1}{x + i\Sigma} - \frac{1}{x - i\Sigma} \right) f' + \left( \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{\beta_1}{x + \Sigma} + \frac{\beta_2}{x - \Sigma} + \frac{\beta_3}{x + i\Sigma} + \frac{\beta_4}{x - i\Sigma} \right) f = 0. \tag{23}
\]

Now we are to specify the above transcendency condition \( k \geq 5, P_{k-5} = 0 \). In explicit form this constraint reads

\[
2A k - 8A + 2\alpha E + 2AC = 0,
\]

whence we obtain a 4-th order algebraic equation with respect to the variable \( E \):

\[
\frac{4\Gamma^2 E^4}{\Gamma(-3\alpha + \Gamma)} - 4 \frac{(k - 6) \Gamma E^3}{\sqrt{\Gamma(-3\alpha + \Gamma)}} + \left[ (k - 6)^2 + \alpha^2 - \frac{4\Gamma^2}{\Gamma(-3\alpha + \Gamma)} \right] E^2 + 4 \frac{(k - 6) \Gamma E}{\sqrt{\Gamma(-3\alpha + \Gamma)}} - (k - 6)^2 = 0.
\]

Let \( \Gamma = -0.001 \), and \( k = 5, 6, ..., 30 \); then the needed roots are (only one series is physically interpretable as corresponding to energies of bound states: \( 0 < E_{l=0,k}^1 < 1 \), also see Fig. 1)

| Table 1. Energy values at \( l = 0 \). |
|--------------------------------------|
| \( E = 0.9999213823 \) & \( E = 0.9998474908 \) & \( E = 0.999867506 \) |
| \( E = 0.9999954435 \) & \( E = 0.999997197 \) & \( E = 0.999998352 \) |
| \( E = 0.9999990925 \) & \( E = 0.9999993533 \) & \( E = 0.9999995159 \) |
| \( E = 0.9999996241 \) & \( E = 0.9999996997 \) & \( E = 0.9999997545 \) |
| \( E = 0.9999997957 \) & \( E = 0.9999998272 \) & \( E = 0.9999998520 \) |
| \( E = 0.9999998718 \) & \( E = 0.9999998879 \) & \( E = 0.9999999012 \) |
| \( E = 0.9999999122 \) & \( E = 0.9999999215 \) & \( E = 0.9999999293 \) |
| \( E = 0.9999999361 \) & \( E = 0.9999999419 \) & \( E = 0.9999999470 \) |
6. Energy quantization for states with $l = 0, \Gamma = -10^{-3}$

Analogous study of the radial equation for states with higher values of $l$ gives similar results. We have 8-term recurrent relations for coefficients of series. In this case, the relevant transcendency condition $P_{k-6} = 0$ takes on the form of quadratic equation with respect to the variable $E$:

$$(2k - 13)\sqrt{1 - E^2} = 2\alpha E + \frac{\sqrt{1 - E^2}(3\alpha + \gamma)}{\sqrt{l(l + 1)}}. \quad (24)$$

At the given parameters $\alpha = \frac{1}{137}$, $\Gamma = -10^{-3}$, $l = 1$, $k = 6, \ldots, 30$ we numerically get the following values for energy levels:

| $E$         | $E$         | $E$         |
|------------|------------|------------|
| 0.9998965381495 | 0.9998902389493 | 0.9999880428546 |
| 0.999978161294 | 0.999986801326 | 0.9999957123741 |
| 0.999993680399 | 0.999995254716 | 0.9999978161294 |
| 0.999997580299 | 0.9999997306431 | 0.999999935867 |
| 0.999998293044 | 0.999999536688 | 0.9999998293044 |
| 0.999998890011 | 0.999999902062 | 0.9999999221009 |
| 0.9999999423299 | 0.9999999743438 | 0.9999999473438 |

which are illustrated by Fig. 2.

Figure 2. Energy levels, $l = 1, \Gamma = -10^{-3}$

Similarly, for parameters $\alpha = \frac{1}{137}$, $\Gamma = -10^{-3}$, $l = 2$, $k = 6, \ldots, 30$ we get the following energy levels (also see Fig. 3)
**Table 3.** Energy values at $l = 2$.

| $E$           | $E$           | $E$           | $E$           | $E$           |
|---------------|---------------|---------------|---------------|---------------|
| 0.9998952529767 | 0.999916171901 | 0.999880927290 | 0.999957230969 | 0.9999978200294 |
| 0.999993686474 | 0.99995285669  | 0.9999981730969 | 0.999999218163 | 0.9999993686474 |
| 0.9999986819655 | 0.9999995258669 | 0.9999996309146 | 0.9999997045584 | 0.9999997581739 |
| 0.99999998890558 | 0.9999999990293 | 0.9999999993686474 | 0.999999999525867 | 0.9999999996309146 |
| 0.9999999997045584 | 0.9999999998092729 | 0.9999999999218163 | 0.9999999999818163 | 0.9999999999995258669 |
| 0.99999999997045584 | 0.99999999999990293 | 1.0000000000000000 | 1.0000000000000000 | 1.0000000000000000 |

**Figure 3.** Energy levels, $l = 2$, $\Gamma = -10^{-3}$

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