Quantum squeezing of optical dissipative structures

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Abstract

We show that any optical dissipative structure supported by degenerate optical parametric oscillators contains a special transverse mode that is free from quantum fluctuations when measured in a balanced homodyne detection experiment. The phenomenon is not critical as it is independent of the system parameters and, in particular, of the existence of bifurcations. This result is a consequence of the spatial symmetry breaking introduced by the dissipative structure. Effects that could degrade the squeezing level are considered.

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Introduction. Vacuum quantum fluctuations constitute the ultimate noise source affecting any coherent radiator, like a laser. These fluctuations define the so-called standard quantum limit as they set the maximum precision attainable with classical optical techniques, even rendering the latter useless in some applications such as precision metrology \cite{1} and quantum information protocols \cite{2,3}. It is possible however to break this limit with the help of quantum states of light: Squeezed states \cite{2,3}, displaying fluctuations below the standard quantum limit in one of the field quadratures, play a prominent role in this regard, and are by now routinely generated, e.g., by single-mode optical parametric oscillators/amplifiers \cite{1,2,3,4}

In the last fifteen years a new branch of quantum optics has emerged under the general name of “quantum imaging” that studies the spatial aspects of the field quantum fluctuations and generalizes the study of squeezing to multimode beams \cite{5}. We consider in this context the squeezing properties of optical dissipative structures (DS) –transverse patterns–, which are self-sustained, stable spatial structures that form across the plane perpendicular to the axis of multi transverse-mode nonlinear resonators \cite{6}. We consider two outstanding classes of optical DS in the planar degenerate optical parametric oscillator (DOPO): periodic patterns and localised structures.

Localised structures of nonlinear optical resonators are also called cavity solitons (CS) for their resemblance with optical fibre solitons and with optical spatial solitons. However, in spite of their appealing resemblances, the latter are Hamiltonian objects resulting from perfect compensation between dispersion/diffraction and nonlinearity, while CS are also ruled by dissipation \cite{7} and feedback and are not true solitons in the mathematical sense. The squeezing properties of optical fibre solitons are well understood since long time ago \cite{8,9}, and those of optical spatial solitons have been considered more recently \cite{10}.

We show below that any stationary DOPO DS is phase-squeezed in a special transverse mode in the linear approximation. Importantly, the degree of squeezing is independent of the system parameter values and of the existence of bifurcation points. This result is a direct consequence of the free diffusion of the DS across the transverse plane, ruled by quantum noise. Although different in nature, the reported phenomenon resembles the quantum noise suppression on the difference of intensities (amplitude-squeezing) of a two–mode optical parametric oscillator above threshold \cite{11}, which is associated to the existence of a continuous diffusion of the phase difference between the two modes. In our case the fact that squeezing occurs in a transverse mode with a special spatial shape could be useful for some applications \cite{1}.

Model. We consider the model of \cite{12} for a DOPO with plane cavity mirrors. A plane wave coherent field of frequency $2\omega_0$ and amplitude $E_0$ pumps the resonator containing a $\chi^{(2)}$ crystal, which converts pump photons into signal photons (of frequency $\omega_s$) and vice versa. Only two longitudinal cavity modes, of frequencies $\omega_0$ (pump mode) and $\omega_1$ (signal mode), the closest to $2\omega_0$ and $\omega_s$, respectively, are assumed to be relevant. These modes are damped at rates $\gamma_n$ ($n = 0, 1$) and losses are assumed to occur at a single cavity mirror. The intracavity field envelope operators for pump and signal modes are denoted by $A_0(\mathbf{r}, t)$ and $A_1(\mathbf{r}, t)$, respectively, where $\mathbf{r} = (x, y)$ denotes the transverse coordinates, which obey standard equal-time commutation relations $[A_n(\mathbf{r}, t), A_m^\dagger(\mathbf{r}', t)] = \delta(\mathbf{r} - \mathbf{r}')$. Using the positive $P$-representation \cite{12,13,14}, which sets a correspondence between the quantum operators $A_n$ and $A_n^\dagger$ and the independent, stochastic c-number fields $A_n$ and $A_n^\dagger$, respectively, we derive the following model equations.
\[ \partial_t A_0 (r, t) = -\gamma_0 (1 + i\Delta_0) A_0 + E_{in} + \frac{\gamma_1}{2} l^2 \nabla^2 A_0 \]
\[ \partial_t A_1 (r, t) = -\gamma_1 (1 + i\Delta_1) A_1 + i\gamma_1 l^2 \nabla^2 A_1 + g A_0 A_1^* + \sqrt{g A_0} \eta (r, t), \]

where \( \Delta_0 = (\omega_0 - 2\omega_b) / \gamma_0 \) and \( \Delta_1 = (\omega_1 - \omega_b) / \gamma_1 \) are cavity detunings, \( l_1 = c / \sqrt{\omega_0 \gamma_1} \) is a characteristic (diffraction) length, \( c \) is the speed of light in the crystal,\( \nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) accounts for diffraction, and \( g \) is the (real) coupling coefficient proportional to the relevant second-order nonlinear susceptibility of the crystal. As everywhere along the rest of this Letter the equations for \( A_n \) are to be complemented by those for the "hermitian conjugate" fields \( A_n^* \), which are obtained from those for \( A_n \) by complex-conjugating the parameters and by doing the replacements \( A_n \leftrightarrow A_n^* \) and \( \eta \leftrightarrow \eta^* \). Finally \( \eta \) and \( \eta^* \) are independent, real white Gaussian noises of zero mean and correlations

\[ \langle \eta (r, t) \eta (r', t') \rangle = \langle \eta^+ (r, t) \eta^+ (r', t') \rangle = \delta (r - r') \delta (t - t'). \]

Thus \( A_0 \) and \( A_1^* \) are not complex conjugate, although the stochastic average of any function of them, \( \langle f (A_0, A_1^*, A_1^* A_1^*) \rangle \), yields the normally ordered quantum expectation value \( \langle f (A_0, A_1^*, A_1^* A_1^*) \rangle \). For example, the signal photon density is calculated as

\[ N_1 (r, t) = \langle A_1^* (r, t) A_1 (r, t) \rangle = \langle A_1^* (r, t) A_1 (r, t) \rangle. \]

In the limit of large pump detuning \( (\gamma_0 |\Delta_0| \gg \gamma_1 |\Delta_1|, \gamma_0, \gamma_1) \) pump diffraction plays a negligible role and pump fields can be adiabatically eliminated \[16\] as \( A_0 = \gamma_1 (\mu + i \frac{\sigma}{\kappa^2} A_1^2) / g \), where \( \kappa = \gamma_1 \sqrt{2|\Delta_0|/g} \), \( \mu = g |E_{in}| / (\gamma_1^2 |\Delta_0|) > 0 \) is the dimensionless pump parameter, \( \sigma = \text{sign} \Delta_0 \), and we wrote \( E_{in} = i \sigma |E_{in}| \) without loss of generality. Then eq. \[12\] becomes

\[ \partial_t A_1 = \gamma_1 \left[ -(1 + i\Delta_1) A_1 + \mu A_1^* + i l^2 \nabla^2 A_1 + i \frac{\sigma}{\kappa^2} A_1^2 A_1^* \right] + \sqrt{\gamma_1 (\mu + i \frac{\sigma}{\kappa^2} A_1^2) \eta}. \]

\[ e^{2i\sigma \theta} = \mu^{-1} \left( 1 + i \sigma \sqrt{\mu^2 - 1} \right), \]
\[ (\sigma l^2 \nabla^2 - \beta^2 + F^2) F = 0, \beta^2 = \sigma \Delta_1 + \sqrt{\mu^2 - 1}, \]

where \( r_1 = (x_1, y_1) \) is arbitrary due to the translation invariance and \( F \) is real. Note that the only parameters defining the DS are \( \{ \sigma, \mu, \Delta_1 \} \) as \( \kappa \) and \( l_1 \) merely act as scale factors.

**Dynamics of quantum fluctuations.** Fluctuations around any classical DS, eq. \[10\], are studied by setting

\[ A_1 (r, t) = \tilde{A}_1 (r - r_1) + a_1 (r - r_1, t), \]
\[ A_1^* (r, t) = \tilde{A}_1^* (r - r_1) + a_1^* (r - r_1, t), \]

where, given the translation invariance, the position of the classical DS, \( r_1 (t) \), is let to vary in time as it is an undamped variable that is excited by noise. Expressing the fluctuations as \( a = (a_1, a_1^*)^T \) (T denotes transposition) and linearizing the resulting eq. \[5\], stochastic equations for the quantum fluctuations are obtained:

\[ -\kappa (G_x dx_1 / dt + G_y dy_1 / dt) + \partial_t a = \gamma_1 \mathbf{C} a + \sqrt{\gamma_1} \mathbf{h}, \]

with \( L_1 = (l^2 \nabla^2 - \Delta_1) \). The spectra of the linear operators \( L \) and \( L^* \), which we introduce as \( \mathbf{L}_1 = \lambda_i \mathbf{v}_i \), \( \mathbf{L}_1^* \mathbf{w}_i = \lambda_i^* \mathbf{w}_i \), are clearly relevant \[19\]. In order to deal with stable DS, it is assumed that \( \text{Re} \lambda_i \leq 0 \) for any \( i \).

With the usual definition of scalar product \( \langle \mathbf{b} | \mathbf{c} \rangle = \int d^2 r \mathbf{b}^\dagger (\mathbf{r}) \cdot \mathbf{c} (\mathbf{r}) \), the relation \( \langle \mathbf{w}_i | \mathbf{L} \mathbf{c} \rangle = \lambda_i \langle \mathbf{w}_i | \mathbf{c} \rangle \) holds. We assume that all eigenvectors are suitably orthonormalised as \( \langle \mathbf{w}_i | \mathbf{v}_j \rangle = \delta_{ij} \). In general the spectra must be computed numerically; nevertheless two general properties of the discrete spectra can be stated: (i) \( \mathbf{G}_{x(y)} \) are Goldstone modes as \( \mathbf{L} \mathbf{G}_{x(y)} = 0 \) we write \( \mathbf{v}_{1x(1y)} = \mathbf{G}_{x(y)} \) and denote by \( \mathbf{w}_{1x(1y)} \) the associated adjoint eigenvectors: \( \mathbf{L}^\dagger \mathbf{w}_{1x(1y)} = 0 \); and (ii)

\[ \mathbf{L}^\dagger \mathbf{w}_{2x(2y)} = -2 \mathbf{w}_{2x(2y)}, \mathbf{w}_{2x(2y)} = \left( w_{2x(2y)}, w_{2x(2y)}^* \right)^T, \]

\[ w_{2x(2y)} = i G_{x(y)}. \]

Properties (i) and (ii) occur independently of the set of parameters \( \{ \sigma, \mu, \Delta_1 \} \). Property (i) is a mere consequence of the translational invariance of the problem.
Property (ii) is the key for our analysis. Note that these eigenvectors exist as the classical DS breaks the spatial symmetry: for a spatially homogeneous solution ($F = \text{cst}$) $G_{x(y)} = 0$.

Before solving eq. (11), we consider its projections onto the eigenvectors $w_{1x(y)}$ and $w_{2x(2y)}$ introduced above:
\[
dx_1/dt = -\sqrt{\gamma_1^2 - \xi_{1x}}, \quad dc_{2x}/dt = -2\gamma_1 c_{2x} + \sqrt{\gamma_2}\xi_{2x},
\]
where $\xi_{i}(t) = \langle w_i|\mathbf{h} \rangle$ are noise sources, $c_{2x}(t) = \langle w_{2x}|a\rangle$ is the projection of the fluctuations onto the eigenmode $w_2$, and corresponding expressions for $d\rho_1/dt$ and $d\rho_2/dt$. Note that the equation for $x_1$ is diffusive, as anticipated, because Goldstone modes are excited without cost (20). We further notice that the equation for $c_{2x}$ is analogous to that derived in (21) for the hexagonal mode stationary phase in a Kerr cavity, which is later interpreted as the hexagonal pattern transverse momentum in (22).

**Squeezing via optical homodyning.** We consider the squeezing properties of a DS as measured in a balanced homodyne detection experiment (23): The outgoing quantum field, $A_{1,\text{out}}(r,t)$, is combined in a beam splitter with a local oscillator field (LOF) that lies in an intense (multimode) coherent state of transverse complex envelope $\alpha_L(r - r_L(t))$, which is allowed to be dynamically shifted. In the detection of squeezing one measures the normally ordered part of the fluctuation spectrum of the intensity difference between the two output ports of the beam splitter, $S$, which can be computed as (24)
\[
S(\omega) = 2\gamma_1 \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \langle \delta E_H(t + \tau)\delta E_H(t) \rangle,
\]
\[
\langle \delta E_H(t) \rangle = \frac{1}{\sqrt{\int d^2r |\alpha_L|^2}} \langle \alpha_L(r + \rho(t))|a(r,t)\rangle,
\]
\[
\alpha_L = (\alpha_L, \alpha_L^*)^T, \quad \rho = r_1 - r_L.
\]
When $A_{1,\text{out}}(r,t)$ is in a (multimode) coherent state $S(\omega) = 0$, the standard quantum limit. On the other hand $S(\omega_{ks}) = -1$ signals complete absence of quantum fluctuations at $\omega = \omega_{ks}$.

Let us assume momentarily that we can set $\rho = 0$ in eq. (19), which means that we can shift the LOF according to the DS movement. Let us choose a LOF with $\alpha_L = iG_{x(y)}$ (i.e., $\alpha_L = 2x_1$, eq. (10)) so that $\delta E_H(t) = c_{2x}(t)$, see after eq. (17). Standard techniques (13) applied to eq. (17) allow to compute the stochastic correlation $\langle \delta E_H(t + \tau)\delta E_H(t) \rangle = -\frac{1}{2}e^{-2\gamma_1|\tau|}$. Finally using eq. (18) we get
\[
S(\omega) = -\frac{4\gamma_1^2}{4\gamma_1^2 + \omega^2},
\]
which is the main result of this Letter: As $S(\omega = 0) = -1$, DOPO DS display perfect squeezing at $\omega = 0$ when probed with the appropriate LOF ($\alpha_L = iG_{x(y)}$). As eq. (21) is independent of the kind of DS and of the system parameters, the result is universal and independent of the existence of bifurcations. This LOF is, in principle, easily realisable as it is the $\pi/2$ phase-shifted gradient of the corresponding DS envelope, eq. (9), which can be easily synthesised by, e.g., Fourier filtering.

It is interesting to notice that in (21), a perfectly squeezed spatial mode was identified in the hexagonal pattern arising in a Kerr cavity and, as in our case, the result is independent of the parameter values. Although derived by different means from the ones used here, this result is very likely connected to the one we have just derived.

We note that the linearised approach is valid, in principle, when all eigenvalues are strictly negative, as then all fluctuations remain small. In our case however a null eigenvalue exists always—that associated with the Goldstone mode. Nevertheless that eigenvalue is just the responsible for the continuous diffusion of the position of the DS (similarly to the continuous diffusion of the phase difference in (11)), and does not entail an energetic divergence. Hence one can be confident that the linearised theory developed here represents quite an accurate description and that a nonlinear treatment (15) would not lead to dramatically different results.

Next we consider two effects that could degrade the measured squeezing level. Although everything to be said applies to any DOPO DS, we focus on the bright CS, which exists for $\sigma = +1$ (17, 18), for the sake of clarity.

**Influence of the CS movement.** Equation (21) is valid if we use a movable LOF which exactly follows the CS movement. This could be done by tracking the movement of $A_{0,\text{out}}$, which is correlated with $A_{1,\text{out}}$, without disturbing the subharmonic CS. The output of this continuous measurement would be then fed into a positioning system controlling $r_L(t)$, giving rise, in general, to a time delay $t_d$ so that $r_L(t) = r_1(t - t_d)$ yielding $\rho(t) = r_1(t) - r_1(t - t_d)$. The point is how much the CS position diffuses in time as compared with its width $\Delta x$. Standard techniques (15) applied to eq. (17) allow to obtain $\langle \rho^2(t) \rangle = D t_d$, where the diffusion constant $D \sim \gamma_1^2 - \gamma_2^2$ (24). Thus $\langle \rho^2(t) \rangle / \Delta x^2 \sim \gamma_1 t_d / N_1$, being $N_1 \sim (\kappa l)^2$ the number of intracavity signal photons in one CS and $\Delta x \sim l_1$ its width (25). We see that $N_1$ acts as an inertial mass. Using realistic values for the system parameters (20) one has $N_1 \sim 10^{12}$, and $\langle \rho^2(t) \rangle / \Delta x^2 \lesssim 5 \cdot 10^{-4}$ for a delay time $t_d \lesssim 1$ms. Thus the relative error existing between the location of the CS center and that of the LOF is very small as compared with the CS width. In order to assess the negligible influence of this effect we expand eq. (19) up to second order in $\rho$. The squeezing spectrum is then given by eq. (21) plus a correction proportional to $\langle \rho^2(t) \rangle / \langle \Delta x \rangle^2 \lesssim 3 \cdot 10^{-7}$, again for $t_d \lesssim 1$ms, which is absolutely negligible.
where the expansion excludes the Goldstone modes as
yields
\[ \langle \rho \rangle = \sum c_i \langle w_i | a \rangle, \]
where the expansion excludes the Goldstone modes as
\[ r_1 \] is to denote the position of the CS. By substituting
this expansion into eq. (11) and projecting, one obtains
\[ \frac{dc_i}{dt} = \gamma_1 \lambda_i c_i \] and projecting, one obtains
\[ d_{\text{cs}} / dt = \gamma_1 \lambda_i c_i + \sqrt{\gamma_1} (w_i | h). \]
The study is further facilitated by expressing a general LOF as
\[ \alpha = \sum \alpha_i w_i, \]
\[ \alpha_i = \langle w_i | a \rangle. \]
One obtains
\[ S(\omega) = \frac{2 \gamma_1}{\sqrt{\int d^2 \alpha \langle \alpha \rangle}} \sum_{p,q} \alpha_p^* \alpha_q S_{p,q}(\omega), \] (22)
which are easily evaluated by using standard methods [13]. In the end all we need is to compute the spectra of \( \mathcal{L} \) and \( \mathcal{L}' \), which was done by using a Fourier method [24]. We limit here our study to the 1D CS (\( \mathcal{F} = \sqrt{2} \beta \text{sech}(\beta x / l) \)) [16] for the sake of computational economy. The influence of the LOF shape was studied for a Gauss-Hermite mode of appropriate phase, \( \alpha, x = GH_1(x) = ie^{i \theta} x e^{-\frac{1}{2}v(x / \ell)^2} \), which similar to \( w_2 = iG_1(x) = ie^{i \theta} |d| \mathcal{F}. \] Figure 1(a) shows that quite high levels of squeezing can be reached even with this non ideal LOF. Finally the influence of a mispositioning was studied both for \( \alpha, x = w_2(x - \bar{x}) \) and \( \alpha, x = GH_1(x - \bar{x}), \) fig 1(b): Mispositionings as high as 15% of the CS width still yield quite good levels of squeezing as well.

In summary we have shown that optical dissipative structures sustained by a DOPO always contain a transverse mode that is completely free from zero-frequency quantum fluctuations. Unlike single-mode cavity squeezing, which is perfect only at bifurcation points, our result does not depend on the parameter setting.

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**FIG. 1:** Squeezing level (at the labeled frequencies) displayed by the 1D CS when nonideal LOFs are used. In (a) a Gauss-Hermite LOF (GH_1) of width \( \xi \) is used. In (b) LOFs displaced \( \bar{x} \) from the CS center are considered \( (w_2 \) denotes a special LOF, see text). \( \Delta x = l_1 / \beta \) denotes the CS width. Parameters are \( \sigma = +1, \Delta_1 = 1, \mu = 1.2. \)

**Let us now consider how the squeezing properties of the CS are modified if the LOF is kept fixed, which corresponds to a simpler scheme. Due to the unbounded movement of the soliton, one must perform the heterodyning experiment in a short time (call it \( t_H \)) in order to obtain significant squeezing. If we assume that at \( t = 0 \) the LOF and the CS centers are made to coincide one has \( \rho(t) = \mathbf{r}_1(t) - \mathbf{r}_1(0) \) and \( \sqrt{\rho^2(t) / \Delta x} \sim \sqrt{\gamma_1 t / N} \sim 10^{-6} \sqrt{\gamma_1 t}. \) Then, if we take \( t_H \leq 1 \)ms (for the used parameters) we can again take \( \rho = 0 \) in eq. (14), which yields \( \delta S_{\eta}(t + \tau) \delta S_{\eta}(t) = -\frac{1}{2} e^{-2\gamma_1 |\tau|} \) as before. The squeezing spectrum is then given by eq. (15) with the limits of integration being replaced by \( \mp \frac{1}{H}. \) The result reads as eq. (21) plus a correction proportional to \( e^{-\gamma_1 t_H} \), which is virtually zero. Then the obtention of almost optimal levels of squeezing is not affected in practice by the existing CS movement. We note that this insensitivity contrasts with the issue of quantum images [27], whose squeezing properties are washed out by their jittering. This is a consequence of the strong inertia (\( \propto N_1^{3/2} \)) that the CS movement displays against fluctuations, as compared with the below- or close-to-threshold emission analysed in [27].

**Influence of the shape and positioning of the LOF.** Up to here we have dealt with a special LOF. However the use of a LOF with the exact form or precise positioning is not critical as we show next. In what follows we ignore the negligible influence of the CS movement and consider a static LOF. The study requires fully solving eq. (14), what was done by using the (biorthogonal) basis \( \mathbf{v}_i, \mathbf{w}_j \) formed by the eigenvectors of the linear operators \( \mathcal{L} \) and \( \mathcal{L}' \) in eq. (14) [28]. This technique is highly convenient as it allows to circumvent the numerical simulation of the Langevin eq. (11), which can be problematic and, in any case, extremely time consuming. We write the field fluctuations and the LOF vector as \( \mathbf{a}(r, t) = \sum c_i(t) \mathbf{v}_i(r), c_i = \langle \mathbf{w}_i | a \rangle, \) where the expansion excludes the Goldstone modes as \( r_1 \) is to denote the position of the CS. By substituting

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