Influence of temperature dependent shear viscosity on elliptic flow at back- and forward rapidities in ultrarelativistic heavy-ion collisions

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We explore the influence of a temperature-dependent shear viscosity over entropy density ratio \( \eta/s \) on the azimuthal anisotropies \( v_2 \) and \( v_4 \) of hadrons at various rapidities. We find that in Au+Au collisions at full RHIC energy, \( \sqrt{s_{NN}} = 200 \) GeV, the flow anisotropies are dominated by hadronic viscosity at all rapidities, whereas in Pb+Pb collisions at the LHC energy, \( \sqrt{s_{NN}} = 2760 \) GeV, the flow coefficients are affected by the viscosity both in the plasma and hadronic phases at midrapidity, but the further away from midrapidity, the more dominant the hadronic viscosity is. We find that the centrality and rapidity dependence of the elliptic and quadrangular flows can help to distinguish different parametrizations of \( (\eta/s)(T) \). We also find that at midrapidity the flow harmonics are almost independent of the decoupling criterion, but show some sensitivity to the criterion at back- and forward rapidities.

I. INTRODUCTION

Determining the transport properties of the quark-gluon plasma (QGP) formed in ultrarelativistic nuclear collisions is nowadays one of the main goals in high-energy nuclear physics. Fluid dynamical models indicate a very low shear viscosity to entropy density ratio \( \eta/s \) when tuned to reproduce the azimuthal anisotropies of the transverse momentum distributions of observed hadrons. For recent reviews see for example Refs. [2,4]. The values favored by state-of-the-art calculations are in the vicinity of the conjectured lower limit for shear viscosity, \( \eta/s = 1/(4\pi) \), based on the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [5]. For example, the values found in Ref. [6] are \( \eta/s = 0.12 \) for collisions at the Relativistic Heavy-Ion Collider (RHIC) at Brookhaven National Laboratory, and \( \eta/s = 0.2 \) at the Large Hadron Collider (LHC) at CERN.

The values quoted above were obtained using a constant \( \eta/s \) ratio during the entire evolution of the system. For a physical system \( \eta/s \) depends at least on temperature [5], and on baryon density [6]. A constant value of \( \eta/s \) represents only an effective average over the entire space-time evolution of the system. The slightly larger effective \( \eta/s \) obtained for collisions at the LHC, i.e., at larger collision energy, thus may be interpreted as an indication of the temperature dependence of \( \eta/s \) [9,10]. Unfortunately extracting the temperature dependence of \( \eta/s \) from the experimental data is a challenging problem.

In our previous works [11–13], we have studied the consequences of relaxing the assumption of a constant \( \eta/s \). We found that the relevant temperature region where the shear viscosity affects the elliptic flow most varies with the collision energy. At RHIC the most relevant region is around and below the QCD transition temperature, while for higher collision energies the temperature region above the transition becomes more and more important. To constrain the temperature dependence of \( \eta/s \) better, it would thus be necessary to find observables which are sensitive to the shear viscosity at different stages of the evolution of a single collision.

In this work we relax the assumption of boost-invariance of our earlier works, solve the evolution equations numerically in all three dimensions, and study whether the azimuthal anisotropies have similar dependence on \( (\eta/s)(T) \) at all rapidities. If not, the measurements of \( v_n \) at back- and forward rapidities could bring further constraints to \( (\eta/s)(T) \).

We also approach the problem of extracting the temperature dependence of \( \eta/s \) in a fashion similar to Ref. [6]: We tune different parametrizations to reproduce the anisotropies at one collision energy and centrality, and check whether anisotropies at different centralities, rapidities, and collision energies can distinguish between these parametrizations.

Furthermore, we check the sensitivity of our results to different decoupling criteria. To this end we carry out the calculations using a dynamical freeze-out criterion, i.e., freeze-out at constant Knudsen number [14,15], and compare the results to those obtained using the conventional freeze-out at constant temperature.

In the following we describe the structure, and freeze-out in our 3+1 dimensional dissipative fluid dynamical model in Sec. III and the parameters in our calculations in Sec. IV. The comparison of our results with experimental data, while in Secs. V and VI we discuss whether it is possible to distinguish the details of 1 In this work \( \eta_n \) denotes the coefficient of shear-viscosity, \( \eta_{ch} \) the pseudo-rapidity and \( \eta \) the space-time rapidity.
different parametrizations of \((\eta_s/s)(T)\), as well as the effects of a dynamical freeze-out criterion. We summarize our results in Sec. V.\(\text{VI}\)

Specific details of the fluid dynamical equations are relegated to Appendix A. The numerical algorithm and details of our implementation, and the numerical accuracy of our code, are discussed in the Appendices B and C respectively.

In this work we use natural units \(h = c = k = 1\).

II. FLUID DYNAMICS

A. Equations of motion

Relativistic fluid dynamics corresponds to the local conservation of energy-momentum and net-charge currents (if any),

\[
\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N_i^\mu = 0,
\]

where \(T^{\mu\nu}\) is the energy-momentum tensor, and \(N_i^\mu\) are the net-charge four-currents.

These macroscopic fields can be decomposed with respect to the fluid flow velocity defined by Landau and Lifshitz,\(\text{[17]}\), as

\[
T^{\mu\nu} = eu^\mu u^\nu - P \delta^{\mu\nu} + \pi^{\mu\nu},
\]

\[
N_i^\mu = n_i u^\mu + V_i^\mu,
\]

where \(e = T^{\mu\nu} u_\mu u_\nu\), and \(n_i = N_i^{\mu} u_\mu\) are the energy and net-charge densities in the local rest frame, respectively. \(P = -T^{\mu\nu} \Delta_{\mu\nu}/3\) is the isotropic pressure, and \(V_i^\mu = N_i^{\nu} \Delta_{\mu\nu}\) are the charge diffusion currents. The shear-stress tensor, \(\pi^{\mu\nu} = T^{(\mu\nu)} - T^{\mu\nu}/3\), is the traceless and orthogonal part of the energy-momentum tensor. With the \((+, - , - , -)\) convention for the metric tensor \(g^{\mu\nu}\), the projection tensor is \(\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu\). The angular brackets \(\langle\rangle\) denote an operator leading to the symmetric, traceless, and orthogonal to the flow velocity, part of a tensor:

\[
T^{(\mu\nu)} = \frac{1}{2} (\Delta_{\mu}^{\alpha} \Delta_{\nu}^{\beta} + \Delta_{\nu}^{\alpha} \Delta_{\mu}^{\beta}) - \frac{1}{3} \Delta^{\alpha\beta} \Delta_{\mu\nu} T^{\alpha\beta}.
\]

Landau’s matching condition allows one to associate the rest frame densities with their equilibrium values, \(e = e_0(T, \{\mu_i\})\), and \(n_i = n_i(0, \{\mu_j\})\). The difference between the isotropic and equilibrium pressures defines so-called bulk viscosity, \(\Pi = P - P_0\).

The equations \(\text{[2]}\) and \(\text{[3]}\) can be closed by providing an equation of state (EoS), together with the equations determining the evolution of dissipative quantities \(\pi^{\mu\nu}\), \(\Pi\), and \(V_i^{\mu}\). These quantities represent the dissipative forces in the system as well as deviations from the local thermal equilibrium. In the Navier-Stokes approximation they are linearly proportional to the gradients of velocity and temperature, with proportionality coefficients for shear viscosity \(\eta_s(T, \{\mu_i\})\), bulk viscosity \(\zeta(T, \{\mu_i\})\) and charge diffusion \(\kappa(T, \{\mu_j\})\) quantifying the transport properties of the matter.

It is well known that the bulk viscosity coefficient of a relativistic gas is about three orders of magnitude smaller than its shear viscosity coefficient, and vanishes in the ultrarelativistic limit \(\text{[18]}\). However, it is still important for relativistic systems around phase-transitions, therefore even if the bulk viscosity is negligible in the QGP-phase, it may be large near and below the phase-transition \(\text{[19]}\). A large bulk viscosity at those stages may or may not have a significant effect on the observables \(\text{[20, 21]}\). Since disentangling the effects of shear and bulk on the observed spectra is difficult, and beyond the scope of this work, we adopt the approach of Ref. \(\text{[21]}\). We assume that bulk viscosity is large only in the vicinity of the QCD phase transition but due to the critical slowing down its effect is so small that it can be safely ignored.

At midrapidity the matter formed in ultrarelativistic collisions at RHIC and at the LHC is to a good approximation net-baryon free, and thus in boost-invariant calculations it has been an excellent approximation to neglect all conserved charges. Since in this study we want to investigate the back- and forward rapidity regions of the system where net-baryon density is finite, in principle we should include the net-baryon current and baryon charge diffusion in the description of the system. However, the baryon charge diffusion in a QGP as well as in a hadron gas is largely unknown at the moment. Also, at low values of net-baryon density where the lattice QCD results \(\text{[26, 27]}\) can be used, the effect of the finite density on the EoS is small \(\text{[28]}\). Therefore to simplify the description of the system, and to allow us to concentrate solely on the effects of shear viscosity on the spectra, we ignore the finite baryon charge in the fluid as well. Thus we are left with the shear stress tensor \(\pi^{\mu\nu}\) as the only dissipative quantity in the system.

In so-called second order or causal fluid dynamical theories by Müller, and Israel and Stewart \(\text{[29, 31]}\) the dissipative quantities fulfill certain coupled relaxation equations. Here we recall the relaxation equation for the shear-stress tensor obtained from the relativistic Boltzmann equation \(\text{[32, 34]}\)

\[
\tau_D \pi^{\mu\nu} = 2 \eta_s \sigma^{\mu\nu} - \pi^{\mu\nu} - \tau_T \theta \left( \pi^{\mu\nu} u^\nu + \pi^{\lambda\nu} u^\lambda \right) D u^\lambda
\]

\[
- \delta \pi^{\mu\nu} \theta + \tau_T \left( \pi^{\lambda\nu} \sigma^{\mu\lambda} \right)
\]

\[
+ 2 \tau_T \left( \omega^{\mu\nu} \right) + \varphi_T \left( \pi^{\mu\nu} \right) \lambda .
\]

Here \(\tau_D\) is the shear-stress relaxation time, \(D\pi^{\mu\nu} = u^\alpha \sigma_{\alpha\beta}^{\mu\nu}\) denotes the time derivative, \(\theta\) the expansion rate, \(\sigma^{\mu\nu}\) the shear-stress tensor and \(\omega^{\mu\nu}\) the vorticity. The other coefficients can be calculated self-consistently from microscopic theory and for example in case of an ultrarelativistic massless Boltzmann gas we obtain in the 14 moment approximation, \(\tau_D = \frac{2}{3} \lambda_{\text{mf}, p}, \delta \pi = (4/3) \tau_T, \tau_D = (10/7) \tau_T\), while \(\varphi_D = (9/70)/\tau_D\), where \(\lambda_{\text{mf}, p}\) is the mean free path between the collisions. For QCD these coefficients are mostly unknown, however for a high temperature QCD matter the coefficients given above may be acceptable as first approximation.

For the sake of simplicity we ignore the last two terms in Eq. \(\text{[4]}\). This is justified since the relative contribution of the \(\varphi_T\) coefficient was shown to be negligible compared
to the others \[34\]. Similarly, we have observed that the term proportional to the vorticity has little effect on the overall evolution of the system, and is omitted from the final calculations shown here.

### B. The freeze-out stage

During the fluid dynamical evolution the system cools and dilutes due to the expansion, and consequently the microscopic rescattering rate of particles, \( \Gamma \sim n\sigma \approx \lambda_{mfp}^{-1} \), decreases, until the rescatterings cease and particles stream freely towards detectors. The transition from (almost) equilibrated fluid to free streaming particles is a gradual process, but since implementing such a gradual transition to a fluid dynamical description is very complicated \[35, 36\], it is usually assumed to take place on an infinitesimally thin space-time layer, on the so-called freeze-out surface. Therefore the total number of particles crossing the surface \( \Sigma \), with a normal vector \( d^3\Sigma_n(x) \), pointing outwards, leads to the following invariant distribution of particles emitted from the fluid, known as the Cooper-Frye formula \[37\]:

\[
E \frac{d^3N}{dp} = \int d^3\Sigma_n(x) p^\mu f(x,p), \tag{5}
\]

where \( p^\mu = (E, p) \) denotes the four-momentum, while \( f(x,p) \) is the phase-space distribution function of particles on the surface.

To apply the Cooper-Frye formula, we need an appropriate criterion for choosing the surface \( \Sigma \). Since scattering rates strongly depend on temperature, the usual approach is to assume the freeze-out to take place on a surface of constant temperature or energy-density. However, it has been argued that it would be more physical to assume that the freeze-out happens when the average scattering rate is roughly equal to the expansion rate of the system \[38\].

This latter, so-called dynamical freeze-out, criterion can be expressed in terms of the Knudsen number, \( Kn \), which is the ratio of a characteristic microscopic time or length scale, like \( \lambda_{mfp} \), and a characteristic macroscopic scale of the fluid, such as inverse of the local gradients, \( \tilde{L}^{-1} \approx \partial_i \). In terms of the Knudsen number the dynamical freeze-out criterion is \( Kn \approx 1 \), which has occasionally been used in ideal fluid calculations \[13, 15, 39, 40\], but for viscous fluids it is more appropriate to use the relaxation time(s) of dissipative quantity(ies) as the microscopic scale, since they appear naturally in the evolution equations for dissipative quantities \[32\].

In most of our calculations we use the conventional constant temperature freeze-out, but to evaluate how sensitive our results are to the particular freeze-out criterion, and the freeze-out description in general, we also do the calculations assuming freeze-out at constant Knudsen number. We take the relaxation time of shear-stress, \( \tau_s \), as the microscopic scale, and the inverse of the expansion rate of the system, \( \theta^{-1} \), as the macroscopic scale. Thus we get a local Knudsen number:

\[
Kn = \tau_s\theta. \tag{6}
\]

Since the Knudsen number can be evaluated in many different ways \[12\], we do not insist on freeze-out at \( Kn = 1 \), but treat the freeze-out Knudsen number as a free parameter chosen to reproduce rapidity and \( p_T \)-distributions of experimental data. To avoid pathologies encountered in Refs. \[14, 15\], we also require that the dynamical freeze-out takes place below \( T = 180 \) MeV and above \( T = 80 \) MeV temperature.

To evaluate the distributions on the freeze-out surface, we assume that the distribution of particles for each species \( i \), i.e., \( f_i(x,p) \), is given by the well known Grad’s 14-moment ansatz which includes corrections \( \delta f_i \) (shear-viscosity only) to the local equilibrium distribution function as,

\[
f_i(x,p) \equiv f_{0i} + \delta f_i = f_{0i} \left[ 1 + \frac{p_i^\mu p_i^\nu \pi_{\mu\nu}}{2T^2 (e + p)} \right], \tag{7}
\]

where \( f_{0i} \) is the local equilibrium distribution function,

\[
f_{0i}(x,p) = \frac{g_i}{\left(2\pi^3\right)^2} \exp \left( \frac{p_i^\mu u_\mu - m_i}{T} \right) \pm 1^{-1}. \tag{8}
\]

We also include the contribution from all strong and electromagnetic two- and three-particle decays of the hadronic resonances up to 2 GeV mass to the final particle distributions.

The flow anisotropies are defined from a Fourier decomposition of the particle spectra as,

\[
E \frac{d^3N}{d^3p} = \frac{d^2N}{2\pi p_T dp_T dy_p} \left( 1 + 2 \sum_{n=1}^{\infty} v_n \cos n(\phi - \Psi_n) \right), \tag{9}
\]

where \( y_p = \frac{1}{2} \ln \left( (p_T^0 + p_T^z)/(p_T^0 - p_T^z) \right) \) is the rapidity of the particle, \( p_T = \sqrt{p_T^x^2 + p_T^y^2} \) its transverse momenta, and \( \Psi_n \) is the event plane for coefficient \( v_n \). The Fourier coefficients \( v_n = v_n(p_T, y_p) \) are the differential flow components. In this work the differential and integrated \( v_n \) is calculated using the event plane method.

### III. PARAMETERS

We mostly implement the parametrization used in Refs. \[11, 12\], but retune the parameter values, and generalize it for a 3+1 dimensional non-boost invariant case.

#### A. Equation of state

For the EoS we use the s95p-PCE-v1 parametrization of lattice QCD results at zero net-baryon density \[41\]. The high-temperature part of the EoS is given by the
FIG. 1. (Color online) Different parametrizations of \( \eta/s \) as a function of temperature. The LH-LQ line has been shifted downwards and the HH-HQ upwards for better visibility.

hotQCD collaboration [42, 43] and it is smoothly connected to the low-temperature part described as a hadron resonance gas, where resonances up to mass of 2 GeV are included. The hadronic part includes a chemical freeze-out at \( T_{\text{chem}} = 150 \text{ MeV} \) where all stable particle ratios are fixed [44–46]. Since the construction of the EoS assumes that the entropy per particle is conserved after chemical freeze-out, the small (approximately 1%) entropy increase below \( T_{\text{chem}} \) leads to a small increase in particle yields too.

B. Transport coefficients

As in our earlier works [11–13], we use four different parametrizations of the temperature-dependent shear viscosity over entropy ratio, see Fig. 1.

- \( \text{LH-LQ} \): \( (\eta/s)(T) = 0.08 \) for all temperatures,
- \( \text{LH-HQ} \): \( (\eta/s)(T) = 0.08 \) for the hadronic phase, while above \( T_{\text{tr}} \), the viscosity to entropy ratio increases according to
  \[
  (\eta/s)(T)_{\text{QGP}} = -0.289 + 0.288 \frac{T}{T_{\text{tr}}} + 0.0818 \left( \frac{T}{T_{\text{tr}}} \right)^2, \tag{10}
  \]
- \( \text{HH-LQ} \): in the hadronic phase below \( T_{\text{tr}} \),
  \[
  (\eta/s)(T)_{\text{HRG}} = 0.681 - 0.0594 \frac{T}{T_{\text{tr}}} - 0.544 \left( \frac{T}{T_{\text{tr}}} \right)^2, \tag{11}
  \]
  while in the QGP-phase \( \eta/s = 0.08 \),
- \( \text{HH-HQ} \): we use \( (\eta/s)(T)_{\text{QGP}} \) and \( (\eta/s)(T)_{\text{HRG}} \) for the hadronic and QGP phases respectively.

Unless stated otherwise, the value of \( \eta/s \) at the transition temperature, \( T_{\text{tr}} = 180 \text{ MeV} \), is \( (\eta/s)(T_{\text{tr}}) = 0.08 \). This is a close approximation to the lower bound conjectured in the framework of the AdS/CFT correspondence [2]. For all parametrizations the relaxation time for shear-stress tensor is

\[
\tau_\pi = \frac{5\eta}{c + p} \tag{12}
\]

For the sake of comparison, we also do the calculations using zero shear viscosity, i.e., ideal fluid.

C. The initial state

In this work we ignore the effects of event-by-event fluctuations [17,18], and generalize a simple optical Glauber model [19] for non-boost invariant initial state. In different variants of the Glauber model the initial energy density in the transverse plane at midrapidity and at initial time \( \tau_0 \) is given as a function of the density of binary collisions, \( n_{\text{BC}}(x, y, b) \), wounded nucleons, \( n_{\text{WN}}(x, y, b) \), or both:

\[
e_T(\tau_0, x, y, b) = C_\varepsilon(\tau_0) f(n_{\text{BC}}, n_{\text{WN}}), \tag{13}
\]

where the normalization constant \( C_\varepsilon(\tau_0) \) is selected to reproduce the multiplicity measured in central collisions, and \( b \) is the impact parameter of the collision. In the following we use our BCfit parametrization [11–13], where the energy density depends solely on the number of binary collisions:

\[
f_{\text{BC}}(n_{\text{BC}}, n_{\text{WN}}) = n_{\text{BC}} + c_1 n_{\text{BC}}^2 + c_2 n_{\text{BC}}^3, \tag{14}
\]

and the coefficients \( c_1 \) and \( c_2 \) are chosen to reproduce the observed centrality dependence of multiplicity.

In the optical Glauber model, the density of binary collisions on the transverse plane is calculated from

\[
n_{\text{BC}}(x, y, b) = \sigma_{\text{NN}} T_A(x + b/2, y) T_B(x - b/2, y), \tag{15}
\]

where \( \sigma_{\text{NN}} \) is the total nucleon-nucleon inelastic cross section, and \( T_{A/B} \) is the nuclear thickness function. As a cross section we use \( \sigma_{\text{NN}} = 42 \text{ mb} \) at RHIC [49, 50], and \( \sigma_{\text{NN}} = 64 \text{ mb} \) at the LHC [51]. As usual, we define the thickness function as

\[
T_A(x, y) = \int_{-\infty}^{\infty} dz \rho_A(x, y, z), \tag{16}
\]

where \( \rho_A \) is the Woods-Saxon nuclear density distribution

\[
\rho_A(r) = \frac{\rho_0}{1 + \exp[(r - R_A)/d]}, \tag{17}
\]

and \( \rho_0 = 0.17 \text{ fm}^{-3} \) is the ground state nuclear density, and \( d = 0.54 \text{ fm} \) is the surface thickness. The nuclear radii \( R_A \) are calculated from \( R_A = 1.12 A^{1/3} - 0.86/A^{1/3} \),
which gives $R_{Au} \simeq 6.37$ fm and $R_{Pb} \simeq 6.49$ fm ($A_{Au} = 197$ and $A_{Pb} = 208$).

Unfortunately there are very few theoretical constraints for the longitudinal structure of the initial state, since even the most sophisticated approaches to calculate the initial state from basic principles are restricted to midrapidity. Here we follow the simple approaches shown in Refs. [54–55], and in a similar fashion assume longitudinal scaling flow, $v_\perp = z/t$, and a constant energy density distribution around midrapidity [57], followed by exponential tails in both back- and forward directions. We parametrize the longitudinal energy density distribution as

$$e_L(\eta) = \exp \left( -2c_0 \sqrt{1 + \frac{(|\eta| - \eta_0)^2}{2c_0^2\sigma_\eta^2}} \Theta(|\eta| - \eta_0) + 2c_\eta \right),$$

(18)

where $\eta = \frac{1}{2} \ln [(t + z)/(t - z)]$ is the space-time rapidity, and $\Theta(x)$ the Heaviside step function. Thus the normalized energy density distribution is

$$e(\tau_0, x, y, \eta, b) = e_T(\tau_0, x, y, b) e_L(\eta).$$

(19)

We are aware that there are more sophisticated approaches in the literature [22, 58–60], but since attempts to create more plausible longitudinal structures easily lead to a rapidity distribution of $v_\perp$ which strongly deviates from the observed one [54], we leave the detailed study of the longitudinal structures for a later work.

Due to entropy production in dissipative fluids, the different parametrizations of $N_\eta/s$ lead to different entropy production and therefore different final multiplicities of hadrons. Because most of the entropy is produced during the early stages of the expansion when the longitudinal gradients are largest [61], it is sufficient to adjust initial densities according to the entropy produced in the partonic phase. Further entropy production during the hadronic evolution turns out to represent only a small contribution in the final multiplicities and it is not corrected in our calculations.

At RHIC, we used as maximum energy density, $e_0 = e(\tau_0, 0, 0, 0)$, for

- ideal fluid: $e_0 = 17.0$ GeV/fm$^3$,
- LH-LQ and HH-LQ: $e_0 = 15.8$ GeV/fm$^3$,
- LH-HQ and HH-HQ: $e_0 = 14.9$ GeV/fm$^3$,

while at the LHC we used for

- ideal fluid: $e_0 = 57.5$ GeV/fm$^3$,
- LH-LQ and HH-LQ: $e_0 = 54.5$ GeV/fm$^3$,
- LH-HQ and HH-HQ: $e_0 = 49.5$ GeV/fm$^3$.

Note that these values are smaller than the ones given in Refs. [11,13]. Main reason is that we used different data to fit the centrality dependence, and chose to fit the multiplicity as a function of centrality class, not as a function of number of participants as was done in Refs. [11,13]. This lead to different values of $c_1$ and $c_2$ parameters, and, consequently, the maximum density in a head on collision (which practically never happens) is different even if the energy density at midrapidity at impact parameters $b > 2$ fm is almost identical.

The parameters controlling the centrality dependence, $c_1$ and $c_2$ in Eq. (14), are $c_1 = -0.035$ fm$^{-2}$ and $c_2 = 0.00034$ fm$^{-4}$ at RHIC, and $c_1 = -0.02$ fm$^{-2}$ and $c_2 = 0.000175$ fm$^{-4}$ at the LHC. Parameters in Eq. (15) defining the longitudinal structure are $\sigma_\eta = 4$ at RHIC and $\sigma_\eta = 2$ at the LHC, while $\eta_0 = 2.0$ for the constant rapidity plateau for both. The width of the rapidity distribution is $\sigma_\eta = 1.0$ at RHIC and $\sigma_\eta = 1.8$ at the LHC. The average impact parameter in each centrality class is given in Table I.

If not stated otherwise the fluid dynamical evolution is started at $\tau_0 = 1$ fm/c proper time. The initial value for the transverse fluid velocity and shear-stress tensor are always set to zero. The value of the decoupling temperature or Knudsen number is indicated in the figures.

To obtain the final particle distributions we use the framework described in Ref. [62]. Thus we sample particle distributions to create “events” even if we are not doing event-by-event calculations, but use conventional averaged initial states. The particle spectra and other measurables at RHIC are obtained as an average over $N_{ev} = 100$ events where the sampling is done over $p_T = (0, 5.4)$ GeV and $N_{ch} = (6, 6, 6, 6)$ with $N_{pT} = 36$ and $N_{Nch} = 22$ bins. At the LHC the particle multiplicity is about $\approx 2.5$ larger than at RHIC, hence we average over $N_{ev} = 40,000$ events.

| Centrality [%] | RHIC $b$ [fm] | LHC $b$ [fm] |
|----------------|----------------|----------------|
| 0-5            | 2.24           | 2.32           |
| 5-10           | 4.09           | 4.24           |
| 10-20          | 5.78           | 5.99           |
| 20-30          | 7.49           | 7.76           |
| 30-40          | 8.87           | 9.19           |
| 40-50          | 10.06          | 10.43          |

### Table I. The average impact parameter $b$ in each centrality class at RHIC and the LHC.

IV. RESULTS AND COMPARISONS TO DATA

#### A. Au+Au at $\sqrt{s_{NN}} = 200$ GeV at RHIC

We fix the parameters characterizing the initial state, Eqs. (15), (16), and (15), by comparison to the PHOBOS charged particle pseudorapidity distribution, $dN_{ch}/d\eta_{ch}$, at various centralities. [65]. We present our results in Fig. 3 where the calculations are shown for $0 - 5\%$ centrality, and for the average of $10 - 20\%$ and $20 - 30\%$ as well as $30 - 40\%$ and $40 - 50\%$ centralities. This is in order to facilitate a comparison to the data taken at
0–6%, 15–25% and 35–45% centralities. As required, the final multiplicity and pseudorapidity distribution is well reproduced at all centralities for all parametrizations of the temperature-dependent shear viscosity to entropy density ratio. Here we once again stress the importance of fixing the initial energy density to compensate for the entropy production for different $\eta/s$ parametrizations. Otherwise, for fixed initial densities, the larger the effective viscosity, the larger the entropy production and thus the final multiplicity.

The kinetic freeze-out temperature, $T_{\text{dec}}$, affects the charged particle pseudorapidity distribution very weakly. We have chosen $T_{\text{dec}} = 100$ MeV by comparison to the pion, kaon and proton $p_T$-spectra measured by the PHENIX collaboration [64], and checked that if we use $T_{\text{dec}} = 140$ MeV, the pseudorapidity distributions are still within error bars, and the change is on the same level than the differences due to different viscosities shown in Fig. 2. Such a weak dependence is not surprising; It is well known that in a chemically frozen system pion $p_T$-distributions are weakly sensitive to the kinetic freeze-out temperature [65]. We now observe similar behavior in the longitudinal direction.

In Figs. 3, 4 and 5 we present the $p_T$-spectra of positive pions, kaons and protons, respectively, corresponding to centrality classes, 0–5%, 10–20%($\times 10^{-1}$), 20–30%($\times 10^{-2}$), 30–40%($\times 10^{-3}$) and 40–50%($\times 10^{-4}$). Here the multiplicative factors are applied (to both theoretical and experimental points) for better visibility. The experimental data is by the PHENIX collaboration [64].

As seen before in viscous calculations (e.g. in Ref. 13), the slopes of pion spectra are reasonably well reproduced up to $p_T \simeq 1.5$ GeV for semicentral collisions, but the agreement recedes with increasing impact parameter. The kaon yields are over-predicted at all centralities, whereas the fit to proton spectra is slightly better than the fit to kaons. Since we do not include finite baryochemical potential in our calculation, we are consistently overestimating the yields of heavy particles, which
might imply the need for even lower chemical freeze-out temperature.

The pion spectra become flatter with increasing freeze-out temperature, hence for example for $T_{\text{dec}} = 140$ MeV the theoretical calculations are in a better agreement at larger momenta, but then we overestimate the spectra around $p_T \sim 1$ GeV. The slope of the proton spectra become steeper with increasing freeze-out temperature as well, and thus $T_{\text{dec}} = 100$ MeV provides the best compromise.

As expected, after the initial densities are fixed to reproduce the yield, the slopes are practically unaffected by the different $\eta_\text{s}/\rho$ parametrizations, and the corresponding $\delta f_i$ in each case represents only a small correction compared to the thermal spectra.

In Figs. 6 and 7 the elliptic flow coefficient $v_2$ at various centralities is shown as a function of transverse momentum $p_T$, and pseudorapidity $\eta_{\text{ch}}$. In Fig. 6 the experimental data is by the STAR collaboration, whereas in Fig. 7 the average of 0 – 5% and 10 – 20% and of 10 – 20% and 20 – 30% events are compared to the data by the PHOBOS collaboration for 3 – 15% and 15 – 25% centrality classes, and to the STAR collaboration data in the 15 – 25% centrality class.

As expected, the $p_T$-differential elliptic flow coefficient shows the behavior reported in Refs. 11, 13: at RHIC the elliptic flow coefficient is very sensitive to viscosity in the hadronic phase but independent of the high-temperature parametrization of the viscosity. The same observation also holds for the rapidity dependent elliptic flow coefficient at all centrality classes. The dissipative reduction of $v_2$ is quite independent of rapidity, and thus we cannot reproduce the shape of $v_2(\eta_{\text{ch}})$ very well. On the other hand, slightly larger hadronic viscosity would further reduce $v_2$, and our result would be very close to the ideal fluid + UrQMD hybrid calculation of Ref. 69.

Similarly the $v_4(p_T)$ and $v_4(\eta_{\text{ch}})$ of charged hadrons in different centrality classes are compared to the experimental data by the STAR collaboration in Figs. 8 and 9. The $v_4$ coefficient, both as a function of transverse momentum $p_T$ and pseudorapidity $\eta_{\text{ch}}$, shows a similar behavior.
momentum or as a function of pseudorapidity, complies with the previously made observations about the elliptic flow coefficient. As we have reported earlier [12, 13], \( v_4 \) is sensitive to viscosity at even later stages of the evolution than \( v_2 \), and a large hadronic viscosity is sufficient to turn \( v_4(p_T) \) negative at quite low \( p_T \). The comparison of Figs. 7 and 9 also shows the well known fact that the larger the \( n \), the stronger the viscous suppression of \( v_n \) [70, 71]. Viscosity has only a weak effect on the shapes of \( v_2(\eta) \) and \( v_4(\eta) \), but quite interestingly the effect on the shapes is different for different coefficients: Increasing viscosity makes the (approximate) plateau in \( v_2(\eta) \) narrower, but wider in \( v_4(\eta) \).

From Fig. 8 it is apparent that the \( v_4(p_T) \) data favors the parametrizations with low hadronic viscosity unlike \( v_2(p_T) \). However, we have to remember that the experimental data was obtained using different methods for the \( v_2 \) and \( v_4 \), i.e., four-particle cumulant and the mixed harmonic event-plane methods, whereas we use event plane method to evaluate all the harmonics. Another uncertainty is that event-by-event fluctuations cause a sizable fraction of \( v_4 \), but they are not included in our study. Thus we advice against drawing any conclusions about the favored \((\eta/s)(T)\) from this particular result.

B. Pb+Pb at \( \sqrt{s_{NN}} = 2760 \) GeV at the LHC

Alike at RHIC, we use the pseudorapidity distribution of charged particles to fix the initialization, and the \( p_T \)-distributions of identified particles to fix the kinetic freeze-out temperature.

In Fig. 10 the charged particle pseudorapidity distribution \( dN_{ch}/d\eta_{ch} \) for different centrality bins are compared to the experimental data by the ALICE collaboration [72]. The pseudorapidity distribution of charged particles reasonably matches the data for all centrality classes given in the figure. Similarly as for RHIC we slightly overshoot the experimental results at the LHC for the most central collisions while undershoot the peripheral ones. Moreover, as observed before, the pseudo-
rapidity distributions of charged particles are insensitive to the chosen freeze-out temperature.

In Figs. 11, 12 and 13 we show the $p_T$-spectra of positive pions, positive kaons and protons corresponding to centrality classes, with multiplicative factors applied for better visibility. The experimental data is from the ALICE collaboration [73]. These distributions behave in a similar way than the RHIC results, and are thus unaffected by the different $\eta_s/s$ parametrizations. We note that as in many other calculations [73, 74], the low $p_T$ part of the pion distribution turned out to be very difficult to reproduce.

In Figs. 14 and 15 the elliptic flow coefficient $v_2$ is shown as functions of transverse momentum and pseudo-rapidity, respectively. In both figures the experimental data is from the ALICE collaboration [75]. At the LHC viscous suppression of the elliptic flow is less dominated by the hadronic viscosity than at RHIC. In central collisions at midrapidity, both QGP and hadronic viscosities affect $v_2$ equally: large QGP viscosity may be compensated with a low hadronic viscosity and vice versa (compare LH-HQ with HH-LQ for $10^{-20\%}$ and $20^{-30\%}$ up to $p_T \leq 2$ GeV or $\eta_{ch} \leq 2$). In peripheral collisions and at large rapidities $v_2$ loses its sensitivity to QGP viscosity, and the system behaves like at RHIC. Thus measuring $v_2$ at large rapidities at the LHC would provide an additional handle to the temperature dependence of the $\eta_s/s$ ratio.

Finally in Figs. 16 and 17 we present the $v_4$ coefficients as functions of $p_T$ and $\eta_{ch}$. As discussed in Refs. [12, 13], $v_4$ is sensitive to viscosity at lower temperatures than $v_2$. Therefore the behavior of $v_4$ at the LHC is similar to the behavior of $v_2$ and $v_2$ at RHIC: the curves are grouped according to their hadronic viscosity, and show no sensitivity to QGP viscosity. The suppression of $v_4$ at both the LHC and RHIC is clearly sensitive to the hadronic viscosity, (compare Fig. [8] with Fig. [10] and Fig. [9] with [17]), and to the minimum value of $\eta/s$.  

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**FIG. 11.** (Color online) Transverse momentum spectra of positive pions at the LHC. Experimental data from the ALICE collaboration [73].

**FIG. 12.** (Color online) Transverse momentum spectra of positive kaons at the LHC. Experimental data from the ALICE collaboration [73].

**FIG. 13.** (Color online) Transverse momentum spectra of protons at ALICE. Experimental data for protons and antiprotons from the ALICE collaboration [73].
V. THE DISTINGUISHABILITY OF THE $\eta/s$ PARAMETRIZATIONS

In the previous section we described how the sensitivity of $v_2$ and $v_4$ to QGP and hadronic shear viscosities depends on centrality, transverse momentum $p_T$, and pseudorapidity $\eta_{ch}$. Now we use this observation to distinguish between different parametrizations of $(\eta/s)(T)$. We rescale our existing parametrizations in such a way that they all lead to almost identical $p_T$-differential $v_2$ in central collisions, and see whether the calculated $v_2$ and $v_4$ differ at other centralities and rapidities. Note that this procedure also tests the sensitivity of the flow coefficients to the minimum value of $\eta/s$, and not only to its values above and below the transition temperature.

The new scaled parametrizations are shown in Fig. 18. At RHIC energies the value of the viscosity to entropy ratio for LH-LQ and LH-HQ is increased uniformly with $\Delta \eta/s = 0.1$ for all temperatures, while the other two parametrizations remain unchanged. Since the sensitivity to the temperature dependence of $\eta/s$
is more complicated at the LHC, the required changes in parametrizations are: \( \Delta \eta/s = 0.1 \) for the LH-LQ, \( \Delta \eta/s = 0.06 \) for the LH-HQ and \( \Delta \eta/s = 0.04 \) for the HH-LQ. The increase in \( \eta/s \) leads to larger entropy production, and thus to larger final multiplicities, which we have counteracted by rescaling the initial densities accordingly.

Note that since the LH-HQ and HH-LQ parametrizations require different rescalings at RHIC and the LHC, they can be distinguished already by comparing the \( v_2(p_T) \) in central collisions at different energies, but LH-LQ and HH-HQ cannot. Furthermore, we want to check whether it is possible to distinguish LH-HQ and HH-LQ in collisions at the same energy by varying the centrality and rapidity.

In Figs. [19][20][21] we present \( v_2(p_T), v_2(\eta,ch) \), and
$v_4(p_T)$ at RHIC using these new parametrizations. As required, in central collisions all parametrizations lead to similar $v_2(p_T)$—the differences due to different hadronic viscosity at very late stages of the evolution are compensated by the larger viscosity at and after the QCD transition region. However, when one moves to larger centralities, and thus to smaller systems, the region where $v_2$ is most sensitive to shear viscosity moves towards lower temperatures, and the parametrizations with different hadronic viscosities can be identified, see Fig. 19. The same, although weaker, phenomenon happens when we move to larger rapidities, see Fig. 20. Most of the sensitivity comes from the change in centrality, but as seen in the 15 – 25% centrality class (Fig. 20b), the difference at large rapidities increases faster than at midrapidity. On the other hand, the $v_4$ coefficient shows larger sensitivity than $v_2$: in central collisions all parametrizations are equal, but the difference increases with increasing fraction of cross section faster than for $v_2$. Note that none of the observables is sensitive to the plasma viscosity, but we have to study the collisions at the LHC to be able to distinguish, say, HH-LQ and HH-HQ parametrizations.

At the LHC we see slightly different behavior. In central collisions $v_2(p_T)$ is again the same for all parametrizations by construction, but the differences appear slowly and stay modest when we move towards more peripheral collisions, see Fig. 22. Again, in more peripheral collisions, the system is most sensitive to viscosity in lower temperatures, and $v_2(p_T)$ curves are ordered according to hadronic viscosity—the larger viscosity at freeze-out, the lower $v_2(p_T)$. In Fig. 15 the pseudorapidity distribution of $v_2$ showed clear sensitivity to shear viscosity. In that figure different parametrizations caused different $v_2$ already at midrapidity in central collisions. Now viscosity is scaled to remove this difference, and the sensitivity of the shape of $v_2(\eta_{ch})$ to the viscosity is more visible. As one can see from Fig. 23 larger hadronic viscosity causes $v_2(\eta_{ch})$ to drop slightly faster with increasing rapidity. Strongest difference is seen in $v_4(p_T)$ which is able to distinguish the new parametrizations at the LHC, see Fig. 24 but its resolving power at the LHC is weaker than at RHIC (Fig. 21). Thus we conclude that differential measurements of the flow anisotropies as function of transverse momentum, pseudorapidity, and
centrality can provide constraints for the temperature
dependence of $\eta/s$, but the measurements at various
energies are essential to constrain the parametrizations
properly.

VI. DYNAMICAL FREEZE-OUT

To test the sensitivity of our results to the freeze-out criterion and the freeze-out description in general, we redo some of the calculations using the dynamical freeze-out criterion [38]. In these calculations we use only our HH-LQ and HH-HQ parametrizations for the shear viscosity, since the low value of $\eta/s$ in hadron gas leads to very slowly increasing relaxation time and thus to unrealistically low temperatures, $\langle T \rangle \ll 80$ MeV on the freeze-out surface when $Kn_{dec} \sim 1$. Since the Knudsen number can be based on many quantities [14], and since we do not know when exactly the hydrodynamical description should break down, we use the freeze-out Knudsen number as a free parameter chosen to fit the rapidity and $p_T$-distributions.

Figs. 25 and 26 show the charged particle pseudorapidity distributions at RHIC and the LHC, respectively. As expected, the pseudorapidity distributions are only weakly dependent on the precise value of $Kn_{dec}$, but it turned out that our choice of Knudsen number and relaxation time lead to weak sensitivity of $p_T$-distributions to the value of $Kn_{dec}$ too. Nevertheless, we found that decoupling at constant Knudsen number $Kn_{dec} = 0.8$ lead to basically same rapidity and $p_T$-distributions than the conventional decoupling at $T_{dec} = 100$ MeV.

The $p_T$-differential $v_2$ of charged hadrons at RHIC and the LHC is shown in Figs. 27 and 28 respectively. Unlike in Ref. [16], where both $p_T$-distributions and anisotropies depended on the freeze-out criterion, see that once the freeze-out parameters are fixed to produce similar $p_T$-distributions, the anisotropies become very similar. This is especially clear at the LHC. Below $p_T \sim 2$ GeV both criteria lead to identical $v_2(p_T)$, and the difference seen in the plots is due to the shear viscosity parametrization. At RHIC both parametrizations lead to identical $v_2(p_T)$, and a weak sensitivity to the freeze-out criterion appears around $p_T \sim 1$ GeV. However, this sensitivity is too weak to be significant.

As a function of pseudorapidity $v_2$ shows more sensitivity to the freeze-out criterion, see Figs. 29 and 30. Both at RHIC and LHC $v_2(\eta_{ch})$ drops faster with increasing rapidity, when the dynamical freeze-out criterion is used. Also, with both freeze-out criteria the sensitivity to plasma viscosity disappears at large rapidities even at the LHC. This is again a manifestation of previously seen behavior: At large rapidities at the LHC, the system behaves like the system at RHIC.

The rather weak dependence of anisotropies on the decoupling criterion means that at midrapidity fluid dynamical results are surprisingly robust against variations in the decoupling procedure. As well, this gives a reason to expect that the hybrid model results are sensitive only to the value of the switching criterion from fluid to cascade, not to the criterion itself. Since the fluid dynamical results concerning the viscosity of QGP are based on...
the analysis of anisotropies at midrapidity, this means that those results are not compromised by the freeze-out criterion. On the other hand, our results indicate that in smaller systems, i.e., at lower collision energies, the fluid dynamical results may be sensitive to the freeze-out criterion. Thus one has to pay extra attention to the freeze-out description in a fluid dynamical description of the collisions at $\sqrt{s_{NN}} = 3 - 9$ GeV in the future FAIR and NICA facilities.
VII. CONCLUSIONS

We have studied the effects of temperature dependent $\eta/s$ on the azimuthal anisotropies of hadron transverse momentum spectra using genuinely $3+1$ dimensional viscous hydrodynamics. We have extended our previous studies to back- and forward rapidities and explored the resolving power of differential measurements of $v_2$ and $v_3$ to distinguish between different parametrizations of $(\eta/s)(T)$.

In close to central collisions at the LHC energy, $\sqrt{\text{S}_{\text{NN}}} = 2.76$ TeV, viscous suppression of elliptic flow at midrapidity is affected by both hadronic and QGP viscosities, but when one moves towards back- and forward rapidities, hadronic viscosity becomes more and more dominant—the system becomes effectively smaller, and begins to behave like in collisions at RHIC, $\sqrt{\text{S}_{\text{NN}}} = 200$ GeV. Therefore with large hadronic viscosity $v_2$ tends to drop slightly faster with increasing rapidity, the effect being stronger in peripheral collisions. At both energies and at all rapidities $v_2$ is mostly suppressed by hadronic viscosity, but if we simultaneously change the minimum value of $\eta/s$, hadronic, and QGP viscosities, it is difficult to predict which coefficient at which collision energy is most sensitive to the changes. Nevertheless, the differential measurements of $v_n$ as function of transverse momentum, rapidity, centrality and collision energy provide a way to distinguish between different parametrizations of $(\eta/s)(T)$, and thus constrain the temperature dependence of the $\eta/s$ ratio.

We also studied how sensitive our results are to the freeze-out criterion, and found that once the freeze-out parameters are fixed to reproduce $p_T$-distributions, both decoupling at constant temperature and at constant Knudsen number lead to very similar anisotropies at midrapidity. Towards the large rapidities $v_2$ tends to drop faster with the dynamical freeze-out criterion. This indicates that uncertainties in the decoupling description do not affect the present fluid dynamical results regarding the anisotropies, but at lower collision energies the results may be more sensitive to the freeze-out criterion.

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Appendix A: Equations in $(3+1)$–dimensions

In the following the components of four-vectors and tensors of rank-2 in 4-dimensional space-time are denoted by Greek indices that take values from 0 to 3 while Roman indices from 1 to 3. If not stated otherwise the Einstein summation convention for both Greek and Roman indices is implied.

First we recall the definitions of the covariant derivative of contravariant four-vectors and tensors of rank-2,

$$A_{\alpha} = \partial_{\alpha} A^\mu + \Gamma_{\alpha\mu\nu} A^\nu ,$$

$$A_{\alpha\mu} = \partial_{\alpha} A_{\mu} + \Gamma_{\alpha\mu\nu} A_{\nu} ,$$

where $\Gamma_{\alpha\mu\nu} = \frac{1}{2} g^{\alpha\nu} (\partial_{\mu} g_{\nu\beta} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu})$ denotes the Christoffel symbol of the second kind and $\partial_{\alpha} = \partial/\partial x^\alpha$ denotes the four-derivative. For scalar quantities the covariant derivative reduces to the ordinary four-derivative, i.e., $(A^\mu A_{\mu})_{\alpha} = \partial_{\alpha} (A^\mu A_{\mu})$.

Applying the definition of the transverse projection operator $\Delta_{\mu\nu} = g_{\mu\nu} - u^\mu u^\nu$ we can decompose the covariant derivative as the sum of the covariant time derivative $D_{\alpha}$ and spatial gradient $\nabla_{\alpha}$,

$$D_{\alpha} A^{\mu_1 \cdots \mu_n} = v^\beta A^{\mu_1 \cdots \mu_n}_{\beta} ,$$

$$\nabla_{\alpha} A^{\mu_1 \cdots \mu_n} = A^{\mu_1 \cdots \mu_n}_{\alpha} ,$$

hence $A^{\mu_1 \cdots \mu_n} = u_{\alpha} D A^{\mu_1 \cdots \mu_n} + \nabla_{\alpha} A^{\mu_1 \cdots \mu_n}$, while for later use we also introduce the comoving or convective time derivative,

$$d A^{\mu_1 \cdots \mu_n} = v^\beta \partial_{\beta} A^{\mu_1 \cdots \mu_n} .$$

In the following we summarize the equations of relativistic dissipative fluid dynamics in hyperbolic coordinates (i.e. $(\tau, x, y, z)$–coordinates) [53] where $\tau = (t^2 - z^2)^{-1/2}$ is the longitudinal proper time and $\eta = 1/2 \ln[(t + z)/(t - z)]$ is the space-time rapidity. The proper metric tensors are $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $g_{\mu\nu} = \text{diag}(1, 1, 1, 1)$. Thus the only non-vanishing Christoffel symbols are $\Gamma^\eta_{\tau\tau} = \Gamma^\eta_{\tau y} = \tau^{-1}$ and $\Gamma^\tau_{\eta\eta} = \tau$, and the gradient is $\partial_{\eta} = (\partial_{\tau}, \partial_{\tau}, \partial_{y}, \partial_{\eta})$, while $\partial_{\mu} \equiv g^{\mu\nu} \partial_{\nu} = (\partial_{\tau}, -\partial_{\tau}, -\partial_{y}, -\partial_{\eta})$. The inverse transformations to Minkowski coordinates with $g_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, 1, -1, -1)$, are $t = \tau \cos \eta$ and $z = \tau \sin \eta$. Note that the hyperbolic coordinates are similar to the Milne coordinates that are spherically symmetric, i.e., $r \equiv \sqrt{x^2 + y^2 + z^2} = \tau \sinh \eta$.

The contravariant flow velocity is

$$u^\mu = \gamma (1, v_x, v_y, v_y) ,$$

hence the covariant flow velocity is $u_{\mu} \equiv g_{\mu\nu} u^\nu = \gamma (1, -v_x, -v_y, -v_y)$, where the normalization condition $u^\mu u_\mu = 1$ leads to $\gamma = (1 - v_x^2 - v_y^2 - \tau^2 v_y^2)^{-1/2}$ as well as to $u^\mu u_{\mu} u^\nu u_{\nu} = 0$.

The energy-momentum conservation equation in general coordinates is

$$T_{\mu\nu}^{\alpha} \equiv \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} T^{\mu\nu}) + \Gamma^{\mu}_{\rho\delta} T^{\rho\nu} = 0 .$$

Notes:

- The Christoffel symbol of the second kind $\Gamma_{\alpha\mu\nu}$ is given by $\Gamma_{\alpha\mu\nu} = \frac{1}{2} g^{\alpha\nu} (\partial_{\mu} g_{\nu\beta} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu})$.
- The covariant derivative $\nabla_{\alpha}$ can be expressed as $\nabla_{\alpha} = \partial_{\alpha}$ in a coordinate system where $\partial_{\alpha} = \partial/\partial x^\alpha$.
- The transverse projection operator $\Delta_{\mu\nu} = g_{\mu\nu} - u^\mu u^\nu$ is defined as $\Delta_{\mu\nu} = g_{\mu\nu} - u^\mu u^\nu$.
- The energy-momentum conservation equation in general coordinates is $T_{\mu\nu}^{\alpha} = \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} T^{\mu\nu}) + \Gamma^{\mu}_{\rho\delta} T^{\rho\nu} = 0$.
where \( g \equiv -\det(g_{\mu\nu}) \) is the negative determinant of the metric tensor, which in hyperbolic coordinates leads to \( g = \tau^2 \).

Henceforth the energy conservation equation leads to
\[
\partial_t T^{\tau\tau} + \partial_x (v_x T^{\tau\tau}) + \partial_y (v_y T^{\tau\tau}) + \partial_\theta (v_\theta T^{\tau\tau}) = -\partial_x (v_x P - v_x \pi^{\tau\tau} + \pi^{x\tau}) - \partial_y (v_y P - v_y \pi^{\tau\tau} + \pi^{y\tau}) - \partial_\theta (v_\theta P - v_\theta \pi^{\tau\tau} + \pi^{\theta\tau}) - \frac{1}{\tau} (T^{\tau\tau} + \tau T^{\eta\eta}), \tag{A8}
\]
while the momentum-conservation equation leads to
\[
\partial_t T^{x\tau} + \partial_x (v_x T^{x\tau}) + \partial_y (v_y T^{x\tau}) + \partial_\theta (v_\theta T^{x\tau}) = -\partial_x (v_x P - v_x \pi^{x\tau} + \pi^{xx}) - \partial_y (v_y P - v_y \pi^{x\tau} + \pi^{yx}) - \partial_\theta (v_\theta P - v_\theta \pi^{x\tau} + \pi^{\theta x}) - \frac{1}{\tau} (T^{x\tau} + \tau T^{\eta\eta}), \tag{A9}
\]
\[
\partial_t T^{y\tau} + \partial_x (v_x T^{y\tau}) + \partial_y (v_y T^{y\tau}) + \partial_\theta (v_\theta T^{y\tau}) = -\partial_x (v_x P - v_x \pi^{y\tau} + \pi^{yx}) - \partial_y (v_y P - v_y \pi^{y\tau} + \pi^{yy}) - \partial_\theta (v_\theta P - v_\theta \pi^{y\tau} + \pi^{\theta y}) - \frac{1}{\tau} (T^{y\tau} + \tau T^{\eta\eta}), \tag{A10}
\]
\[
\partial_t T^{\theta\tau} + \partial_x (v_x T^{\theta\tau}) + \partial_y (v_y T^{\theta\tau}) + \partial_\theta (v_\theta T^{\theta\tau}) = -\partial_x (v_x P - v_x \pi^{\theta\tau} + \pi^{\theta x}) - \partial_y (v_y P - v_y \pi^{\theta\tau} + \pi^{\theta y}) - \partial_\theta (v_\theta P - v_\theta \pi^{\theta\tau} + \pi^{\theta\theta}) - \frac{1}{\tau} (T^{\theta\tau} + \tau T^{\eta\eta}). \tag{A11}
\]

The corresponding tensor components are defined according to the general definition of the energy-momentum tensor Eq. (2),
\[
T^{\tau\tau} = (e + \bar{P}) \gamma^2 - g^{\tau\tau} P + \pi^{\tau\tau}, \tag{A12}
\]
\[
T^{\tau i} \equiv (e + \bar{P}) \gamma^2 \bar{v}_i - g^{\tau\tau} P + \pi^{\tau i}, \tag{A13}
\]
\[
T^{ij} \equiv (e + \bar{P}) \gamma^2 \bar{v}_i \bar{v}_j - g^{ij} \bar{P} + \pi^{ij}, \tag{A14}
\]
\[
\bar{v}_i = \frac{v_i}{M}, \tag{A21}
\]
\[
\bar{v}_1 = \frac{\bar{v}_1}{M}. \tag{A22}
\]

For example choosing \( \pi^{xx}, \pi^{yy}, \pi^{\theta\theta} \) as independent components, the other four components of shear-stress tensor follow from the orthogonality \( \pi^{\mu\nu} g_{\mu\nu} = 0 \),
\[
\pi^{xx} = \pi^{xx} v_x + \pi^{yy} v_y + \pi^{\theta\theta} v_\theta, \tag{A23}
\]
\[
\pi^{yy} = \pi^{yy} v_x + \pi^{yy} v_y + \pi^{\theta\theta} v_\theta, \tag{A24}
\]
\[
\pi^{\theta\theta} = \pi^{\theta\theta} v_x + \pi^{\theta\theta} v_y + \pi^{\theta\theta} v_\theta, \tag{A25}
\]
\[
\pi^{\eta\eta} \equiv \tau^{-2} (\pi^{xx} - \pi^{yy}) v_x + \pi^{yy} (v_y^2 - 1) + 2\pi^{xx} v_x v_y + 2\pi^{yy} v_x v_y + 2\pi^{\theta\theta} v_\theta v_\eta, \tag{A26}
\]
whereas the last unknown component is available from the tracelessness condition \( \pi^{\mu\nu} g_{\mu\nu} = 0 \),
Where according to Eqs. (A3) [A5] the proper time derivatives are given by $Du_\mu = dU_\mu - \Gamma^\nu_{\mu\beta} u^\alpha u_\beta$ hence

$$Du_\tau \equiv D_\tau = \gamma \left[ \partial_\tau \gamma + v_\tau \partial_x \gamma \right. + v_y \partial_y \gamma \left. + v_\theta \partial_\theta \gamma \right] + \tau \gamma^2 v_\tau^2 ,$$

(A43)

$$Du_x \equiv -D_{-x} = -\gamma \left[ \partial_x (\gamma v_x) + v_y \partial_y (\gamma v_x) \right. + v_\theta \partial_\theta (\gamma v_x) \left. \right] + v_\tau \partial_\tau (\gamma v_x) + v_y \partial_y (\gamma v_x) + v_\theta \partial_\theta (\gamma v_x) ,$$

(A44)

$$Du_y \equiv -D_{-y} = -\gamma \left[ \partial_y (\gamma v_y) + v_x \partial_x (\gamma v_y) \right. + v_\theta \partial_\theta (\gamma v_y) \left. \right] + v_\tau \partial_\tau (\gamma v_y) + v_x \partial_x (\gamma v_y) + v_\theta \partial_\theta (\gamma v_y) ,$$

(A45)

$$Du_\theta \equiv \tau D_\theta = -\tau^2 \gamma^2 \left[ \partial_\theta (\gamma v_\theta) + v_x \partial_x (\gamma v_\theta) \right. + v_y \partial_y (\gamma v_\theta) \left. \right] + v_\tau \partial_\tau (\gamma v_\theta) + v_x \partial_x (\gamma v_\theta) + v_y \partial_y (\gamma v_\theta) - 2 \tau \gamma^2 v_\theta .$$

(A46)

Note that $Du_\tau \equiv dU_\tau + \tau^2 \gamma^2 \equiv d_\tau + \gamma^2 \gamma v_\tau^2, Du_x \equiv \gamma dU_x + d_\tau, Du_y \equiv -dU_y, Du_\theta \equiv \gamma dU_\theta, Du_\theta \equiv \gamma dU_\theta \neq dU$, since $Du_\theta \equiv dU_\theta + 2 \tau^{-1} \gamma^2 v_\theta = -\tau^2 \gamma v_\theta$.

The $I_3$ terms are

$$I_3^{xx} = 2 \left( \pi^{x\tau} \omega^x_\tau + \pi^{x\eta} \omega^x_\eta + \pi^{x\gamma} \omega^x_\gamma \right),$$

(A47)

$$I_3^{yy} = 2 \left( \pi^{y\tau} \omega^y_\tau + \pi^{y\eta} \omega^y_\eta + \pi^{y\gamma} \omega^y_\gamma \right),$$

(A48)

$$I_3^{yz} = \pi^{x\tau} \omega^y_\tau + \pi^{y\tau} \omega^x_\tau + \pi^{x\gamma} \omega^y_\gamma + \pi^{y\gamma} \omega^x_\gamma + \pi^{y\eta} \omega^x_\gamma,$$

(A49)

$$I_3^{zy} = \pi^{x\gamma} \omega^x_\gamma + \pi^{y\gamma} \omega^y_\gamma + \pi^{y\tau} \omega^x_\tau + \pi^{x\tau} \omega^y_\tau + \pi^{x\eta} \omega^y_\eta + \pi^{y\eta} \omega^x_\eta,$$

(A50)

$$I_3^{yz} = \pi^{y\tau} \omega^y_\tau + \pi^{y\eta} \omega^y_\eta + \pi^{y\gamma} \omega^y_\gamma + \pi^{y\eta} \omega^x_\gamma,$$

(A51)

where the vorticities are defined most generally as,

$$\omega^\mu_{\nu} = \frac{1}{2} \Delta^\mu_{\nu} \Delta^\beta_{\alpha} \left( \partial_\alpha u_\beta - u_\beta \partial_\alpha \right),$$

$$\omega^\mu_{\nu} = \frac{1}{2} \left( g^{\mu\alpha} \partial_\nu u_\alpha - u_\nu \partial_\alpha \right) - g^{\mu\beta} \left( \partial_\nu u_\beta - u_\nu \partial_\beta \right) \right],$$

(A52)

Here we used that the Christoffel symbols of second kind are symmetric $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$, with respect to the interchange of the two lower indices.

The different components of the vorticity are given as,

$$\omega^x_{\tau} \equiv \omega^x_{\tau} = \left[ \partial_\tau (\gamma v_\tau) + \partial_x \right],$$

(A53)

$$\omega^y_{\tau} \equiv \omega^y_{\tau} = \left[ \partial_\tau (\gamma v_\tau) + \partial_y \right],$$

(A54)

$$\omega^\tau_{\gamma} \equiv \omega^\tau_{\gamma} = \left[ \partial_\tau (\gamma v_\gamma) + \partial_\gamma \right],$$

(A55)
and
\[
\omega^x_y \equiv -\omega^y_x = \frac{1}{2} \left[ \partial_y (\gamma v_x) - \partial_x (\gamma v_y) \right] \\
+ \frac{1}{2} [2 \gamma v_x d(\gamma v_x) - \gamma v_x d(\gamma v_y)] , \quad \text{(A56)}
\]
\[
\omega^x_\eta \equiv -\tau^x \omega^\eta_x = \frac{1}{2} \left[ \partial_\eta (\gamma v_x) - \partial_x (\tau^x \gamma v_\eta) \right] \\
+ \frac{1}{2} [\tau^2 \gamma v_x d(\gamma v_x) - \gamma v_x d(\tau^x \gamma v_y)] , \quad \text{(A57)}
\]
\[
\omega^y_\eta \equiv -\tau^y \omega^\eta_y = \frac{1}{2} \left[ \partial_\eta (\gamma v_y) - \partial_y (\tau^x \gamma v_\eta) \right] \\
+ \frac{1}{2} [\tau^2 \gamma v_y d(\gamma v_y) - \gamma v_y d(\tau^y \gamma v_\eta)] . \quad \text{(A58)}
\]

Note that the general expression of the vorticity given in Eq. (10) in Ref. [77] is missing the contribution of the Christoffel symbols compared to Eq. (A52) in this work. Therefore, the values for \( \omega^x, \omega^y, \text{ and } \omega^\eta \) given in Eqs. (C.22, C.23) and Eq. (C.24) in Ref. [77] are also incorrect compared to these formulas.

The next term we need is given by
\[
I^{xx}_\tau = (\pi^{xx} \sigma^{xx} - \pi^{yy} \sigma^{yy}) + \frac{1}{3} [1 + \gamma^2 \nu^2 v^2 \pi^3_\alpha \sigma^\alpha \beta] , \quad \text{(A59)}
\]
\[
I^{yy}_\tau = -\pi^{yy} \sigma^{yy} + \pi^{xx} \sigma^{xx} + \frac{1}{3} [1 + \gamma^2 \nu^2 v^2 \pi^3_\alpha \sigma^\alpha \beta] , \quad \text{(A60)}
\]
\[
I^{xy}_\tau = \frac{1}{2} (\pi^{xx} \sigma^{xy} + \pi^{yy} \sigma^{yx} - \pi^{xy} \sigma^{xx} - \pi^{yx} \sigma^{yy}) \\
+ \frac{1}{3} [1 + \gamma^2 \nu^2 v^2 \pi^3_\alpha \sigma^\alpha \beta] , \quad \text{(A61)}
\]
\[
I^{xy}_\eta = \frac{1}{2} (\pi^{xx} \sigma^{xy} + \pi^{yy} \sigma^{yx} - \pi^{xy} \sigma^{xx} - \pi^{yx} \sigma^{yy}) \\
+ \frac{1}{3} [1 + \gamma^2 \nu^2 v^2 \pi^3_\alpha \sigma^\alpha \beta] , \quad \text{(A62)}
\]
\[
I^{yy}_\eta = \frac{1}{2} (\pi^{xx} \sigma^{xx} + \pi^{yy} \sigma^{yy}) - \frac{1}{2} [2 \gamma v_x d(\gamma v_x) + \gamma^2 \nu^2 v^2 \theta^3] , \quad \text{(A63)}
\]

The shear-tensor is most generally defined as
\[
\sigma^{x\nu} \equiv \nabla^\nu u^x = \frac{1}{2} \Delta^{m\beta} \Delta^{n\gamma} (u_{\alpha;\beta} + u_{\beta;\alpha}) - \frac{\theta}{3} \Delta^{x\nu} \\
= \frac{1}{2} [\gamma^{\mu\nu} (\partial^\nu u_{\mu} - \nu^x d u_{\nu}) + \gamma^{\nu\beta} (\partial^\nu u_{\mu} - \mu^x d u_{\nu})] \\
- \Delta^{x\beta} \Delta^{\beta\alpha} u_{\lambda} + \frac{\theta}{3} \Delta^{x\nu} . \quad \text{(A64)}
\]

whereas the expansion scalar is
\[
\theta \equiv \nabla_{\mu} u^\mu = \partial_{\mu} u^\mu + \Gamma^\lambda_{\mu} u^\mu = \frac{\gamma}{\tau} + \partial_x \gamma + \partial_y (\gamma v_x) + \partial_y (\gamma v_y) + \partial_\eta (\gamma v_\eta) . \quad \text{(A65)}
\]

The various shear tensor components that we need to use are
\[
\sigma^{x\tau} = -\tau \gamma^3 v^2 \gamma v_x + (\partial_x \gamma - \gamma d \gamma) + (\gamma^2 - 1) \frac{\theta}{3} , \quad \text{(A66)}
\]
\[
\sigma^{x\nu} = -\frac{1}{2} (\tau \gamma^3 v^2 \gamma v_x) + \frac{1}{2} [\partial_x (\gamma v_x) - \partial_x \gamma] \\
- \frac{1}{2} [\gamma v_x d \gamma + \gamma d (\gamma v_x)] + \gamma^2 \nu^2 v^2 \theta^3 , \quad \text{(A67)}
\]
\[
\sigma^{y\nu} = -\frac{1}{2} (\tau \gamma^3 v^2 \gamma v_y) + \frac{1}{2} [\partial_x (\gamma v_y) - \partial_y \gamma] \\
- \frac{1}{2} [\gamma v_y d \gamma + \gamma d (\gamma v_y)] + \gamma^2 \nu^2 v^2 \theta^3 , \quad \text{(A68)}
\]
\[
\sigma^{x\gamma} = -\frac{\gamma^3 v^2 \gamma v_x}{2 \tau} + (2 + \tau^x \gamma^2 \nu^2 v^2) \frac{1}{2} [\partial_x (\gamma v_x) - \frac{1}{\tau^2} \partial_\eta (\gamma v_\eta)] \\
- \frac{1}{2} [\gamma v_x d \gamma + \gamma d (\gamma v_x)] + \gamma^2 \nu^2 v^2 \theta^3 , \quad \text{(A69)}
\]
\[
\sigma^{y\gamma} = -\frac{\gamma^3 v^2 \gamma v_y}{2 \tau} + (2 + \tau^y \gamma^2 \nu^2 v^2) \frac{1}{2} [\partial_x (\gamma v_y) - \frac{1}{\tau^2} \partial_\eta (\gamma v_\eta)] \\
- \frac{1}{2} [\gamma v_y d \gamma + \gamma d (\gamma v_y)] + \gamma^2 \nu^2 v^2 \theta^3 , \quad \text{(A70)}
\]

and
\[
\sigma^{x\nu} = -\frac{1}{2} [\partial_x (\gamma v_x) + \gamma v_x d (\gamma v_x)] + (1 + \gamma^2 \nu^2 v^2) \frac{\theta}{3} , \quad \text{(A71)}
\]
\[
\sigma^{y\nu} = -\frac{1}{2} [\partial_x (\gamma v_y) + \gamma v_y d (\gamma v_y)] + (1 + \gamma^2 \nu^2 v^2) \frac{\theta}{3} , \quad \text{(A72)}
\]
\[
\sigma^{x\gamma} = -\frac{1}{2} [\partial_x (\gamma v_x) + \partial_y (\gamma v_y)] \\
- \frac{1}{2} [\gamma v_x d (\gamma v_x) + \gamma v_x d (\gamma v_y)] + \gamma^2 \nu^2 v^2 \theta^3 , \quad \text{(A73)}
\]
\[
\sigma^{y\gamma} = -\frac{1}{2} [\partial_x (\gamma v_y) + \partial_y (\gamma v_y)] \\
- \frac{1}{2} [\gamma v_y d (\gamma v_x) + \gamma v_y d (\gamma v_y)] + \gamma^2 \nu^2 v^2 \theta^3 , \quad \text{(A74)}
\]
\[
\sigma^{x\eta} = -\frac{\gamma^3 v^2 \gamma v_x}{\tau} - \frac{1}{2} [\partial_x (\gamma v_x) + \frac{1}{\tau^2} \partial_\eta (\gamma v_\eta)] \\
- \frac{1}{2} [\gamma v_x d (\gamma v_x) + \gamma v_x d (\gamma v_y)] + \gamma^2 \nu^2 v^2 \theta^3 , \quad \text{(A75)}
\]
The last contribution from Eq. (A32) are
\[ \begin{align*}
I_{5x}^{xx} &= (\pi^{xx})^2 - (\pi^{zz})^2 - (\pi^{yy})^2 - (\tau^{xx})^2 \\
+ \frac{1}{3} (1 + \gamma^2 v_y^2) \pi_0^2 \pi_0^2,
\end{align*} \tag{A76} \]
\[ \begin{align*}
I_{5y}^{yy} &= (\pi^{xx})^2 - (\pi^{yy})^2 - (\tau^{yy})^2 \\
+ \frac{1}{3} (1 + \gamma^2 v_y^2) \pi_0^2 \pi_0^2,
\end{align*} \tag{A77} \]
\[ \begin{align*}
I_{5y}^{xy} &= \pi^{xx} \pi^{yy} - \pi^{xx} \pi^{yy} - \pi^{yy} \pi^{yy} - \pi^{yy} \pi^{yy} \\
+ \frac{1}{3} (1 + \gamma^2 v_y^2) \pi_0^2 \pi_0^2,
\end{align*} \tag{A78} \]
\[ \begin{align*}
I_{5}^{xy} &= \pi^{xx} \pi^{yy} - \pi^{xx} \pi^{yy} - \pi^{yy} \pi^{yy} - \pi^{yy} \pi^{yy} \\
+ \frac{1}{3} (1 + \gamma^2 v_y^2) \pi_0^2 \pi_0^2,
\end{align*} \tag{A79} \]
\[ \begin{align*}
I_{5y}^{yy} &= \pi^{xx} \pi^{xx} - \pi^{xx} \pi^{xx} - \pi^{yy} \pi^{yy} - \pi^{yy} \pi^{yy} \\
+ \frac{1}{3} (1 + \gamma^2 v_y^2) \pi_0^2 \pi_0^2.
\end{align*} \tag{A80} \]

Furthermore, to evaluate the Cooper-Frye formula, Eq. [5], as well as the argument of the equilibrium distribution function, Eq. [5], we express the four-momenta of particles as,
\[ p^\mu = \left( m_T \cosh (y_\rho - \eta), p_x, p_y, \frac{m_T}{\tau} \sinh (y_\rho - \eta) \right), \tag{A81} \]
where \( m \) is the rest mass of the particle, \( m_T = \sqrt{m^2 + p_x^2 + p_y^2} \) denotes the transverse mass, while \( y_\rho \) is the rapidity of the particle. Therefore, the non-equilibrium corrections to the spectra from Eq. (4) are given with an argument of
\[ \begin{align*}
\pi^{\alpha\beta} p_\alpha p_\beta &= m_T^2 \left[ \cosh^2 (y_\rho - \eta) \pi^{\tau\tau} + \tau^2 \sinh^2 (y_\rho - \eta) \pi^{\eta\eta} \right] \\
+ \left( p_x^2 \pi^{xx} + 2 p_x p_y \pi^{xy} + p_y^2 \pi^{yy} \right) \\
- 2 m_T \cosh (y_\rho - \eta) \left( p_x \pi^{xx} + p_y \pi^{xy} \right) \\
+ 2 \pi^{xy} \cosh (y_\rho - \eta) \left( p_x \pi^{xy} + p_y \pi^{yy} \right) \\
- 2 m_T \pi^{xy} \sinh (y_\rho - \eta) \cos (y_\rho - \eta) \tau^{\tau}, \tag{A82} \end{align*} \]
while using Eq. (C3) we obtain,
\[ \begin{align*}
p^\mu d^3\Sigma_\mu &= \tau m_T \cosh (y_\rho - \eta) x d\bar{y} d\ell - p_x d\tau d\bar{y} d\ell \\
- p_y d\tau d\bar{y} d\ell - \frac{m_T}{\tau} \sinh (y_\rho - \eta) d\tau d\bar{y} d\ell. \tag{A83} \end{align*} \]

Appendix B: Numerical methods

The conservation laws as well as the relaxation equations are solved using the well known SHASTA "SHarp and Smooth Transport Algorithm" originally developed by Boris and Book [79], and later refined by Zalesak [76] and others [80]. This numerical algorithm solves equations of the conservation type with source terms,
\[ \partial_t U + \partial_i (v_i U) = S(t, \mathbf{x}), \tag{B1} \]
where \( U = U(t, \mathbf{x}) \) is for example \( T^{00} \) or \( T_{xx} \), while \( v_i \) is the \( i \)th component of three-velocity, and \( S(t, \mathbf{x}) \) is a source term, for more details see Refs. [77, 81, 82].

Due to the fact that for smooth solutions (like in our case) the multidimensional antidiffusion limiter suffers from instabilities around the boundary caused by small ripples propagating into the interior [53], we further stabilize SHASTA by letting the antidiffusion coefficient \( A_{ad} \), which controls the amount of numerical diffusion, to be proportional to
\[ A_{ad} = \frac{A_{ad}^S}{(k/c)^2 + 1}. \tag{B2} \]
where \( A_{ad}^S = 0.125 \) is the default value for the antidiffusion coefficient [75], \( c \) is the energy density in the local rest frame, and \( k = 6 \times 10^{-5} \text{ GeV/fm}^3 \) is a numerical parameter. In this way we increase the amount of numerical diffusion in the low density region and \( A_{ad} \) goes smoothly to zero near the boundaries of the grid. In our cases of interest this neither affects the solution nor produces more entropy inside the decoupling surface.

It is also important to mention that in the 3+1D case both the conservation and the relaxation equations are solved using SHASTA, employing the above mentioned modification for the antidiffusion coefficient. Earlier, for the 2+1D boost invariant case, we used a simple centered second-order difference algorithm to solve the relaxation equations [12]. However doing so in the 3+1D case does not always lead to stable solutions.

To further stabilize the numerical calculations (also for ideal fluids) we use a smaller value for the antidiffusion coefficient in the transverse directions \( A_{ad}^{xx} = 0.105 \), but kept \( A_{ad}^{yy} = 0.125 \) in the \( \eta \)-direction. Decreasing the antidiffusion coefficient produces smoother solutions inside the decoupling hypersurface but also increases the numerical diffusion, which in turn may decrease the numerical accuracy. The reason we used a different coefficient in the longitudinal direction is to increase the accuracy, see the next section for more details.

The numerical calculations are done on a discretized spatial grid (including 4 boundary points in each direction as required by the algorithm) of \( N_x \times N_y \times N_\eta \) cells with \( N_x = N_y = 180 \) while \( N_\eta = 2 \times 120 \) with \( \Delta x = \Delta y = \Delta \eta = 0.15 \text{ fm cell sizes. The time step is given from } \Delta \tau = \Delta x. \) For further details, the system is symmetric around the \( x \)- and \( y \)-directions, with exponentially interpolated boundary conditions for the conserved quantities (e.g. for Glauber type initial conditions) and linearly interpolated boundary conditions for the shear-stress tensor (the shear-stress tensor may change sign).

Finally, the freeze-out hypersurface is constructed at time intervals \( \Delta t_{CF} \equiv 5 \Delta \tau = 0.3 \text{ fm/c} \). The space is sampled uniformly both in the transverse direction and longitudinal directions, at \( \Delta x = \Delta \tau = 0.06 \text{ fm/c. Furthermore, the system is symmetric } \). The freeze-out hypersurface is calculated using the CORNELIUS++ subroutine presented in Ref. [84] and its
source code can be obtained from the Open Standard Codes and Routines (OSCAR) website [3].

Appendix C: Remarks on the numerical accuracy

SHASTA solves the fluid dynamical equations up to some finite numerical accuracy. In most cases this means that in Cartesian coordinates the particle number and energy are conserved up to $O_5$ accuracy. However in $(τ, x, y, η)$ coordinates the expressions for the conserved quantities as well as the equations of motion change with additional source terms resulting from the non-vanishing Christoffel symbols.

As an example let us evaluate a conserved quantity at a given time or proper time, hence comparing this initial value with one at a later time we can follow the accuracy of the fluid-dynamical solver during this time interval.

The total conserved charge $N_{tot}$ across any given hypersurface is

$$N_{tot}(τ) = \int N^μ d^3Σ^μ = \int N^0 d^3Σ^0 + \int N^i d^3Σ^i.$$  \hspace{1cm} (C1)

Here the hypersurface element $d^3Σ^μ$ can be specified in any coordinate system according to the following general formula,

$$d^3Σ^μ = -ε_{μλκ} \frac{∂Σ^λ}{∂τ} \frac{∂Σ^κ}{∂ν} \frac{∂Σ^ν}{∂ζ} dτdνdζ.$$  \hspace{1cm} (C2)

where $ε_{μλκ}$ is the Levi-Civita symbol.

For example in Cartesian coordinates the hypersurface normal vector is $Σ^{τ(x,y,z)} (τ, x, y, z)$ where $t = t (x, y, z)$, hence

$$d^3Σ^{τ(x,y,z)} ≡ (dxdydz, -dt, -dx, -dy) = \left( \frac{∂τ}{τ} sinh η + cosh η, -\frac{∂τ}{x}, -\frac{∂τ}{y}, -\frac{∂τ}{η} sinh η - cosh η \right) dxdydz,$$  \hspace{1cm} (C3)

while in $(τ, x, y, η)$ coordinates for $Σ_μ^{(τ,η)} (τ, x, y, η)$ and $τ = τ (x, y, η)$ we obtain,

$$d^3Σ^{(τ,η)} = τ (dxdydz, -dx, -dy, -dη).$$  \hspace{1cm} (C4)

If we are interested in the conserved current across constant time or proper time hypersurfaces then $d^3Σ^{τ(x,y,z)} = d^3Σ_τ = 0$, hence in Cartesian coordinates we get,

$$N_{tot}(τ) = \int N^μ d^3Σ^{τ(x,y,z)} = \gamma n_0 ∫ dxdydz,$$  \hspace{1cm} (C5)

where $N^μ ≡ n_0 u^μ = γ n_0 (1, vx, vy, vz)$ is the conserved charge current. Similarly, Eq. (C1) leads to the total conserved charge at any proper time hypersurface in hyperbolic coordinates,

$$N_{tot}(τ) = \int N^μ d^3Σ^{(τ,η)} = γ n_0 ∫ τ dxdydz.$$  \hspace{1cm} (C6)

To calculate how the total energy-momentum changes between two closed hypersurfaces, first we define the energy-momentum current across a hypersurface

$$E_{tot}^μ ≡ \int T^{μν} d^3Σ_ν = \int T^{00} d^3Σ^0 + \int T^{μi} d^3Σ^i.$$  \hspace{1cm} (C7)

In Cartesian coordinates $E_{tot}^μ = (E_{tot}^0, E_{tot}^ν, E_{tot}^ξ)$, such that $E_{tot}^0$ denotes the energy current while $E_{tot}^ν$ the momentum current trough the hypersurface. Therefore the total energy current across a constant $t$-hypersurface is,

$$E_{tot}^0 (τ) = \int T^{0ν} d^3Σ^{(t,ν)} = \int T^{00} dxdydz.$$  \hspace{1cm} (C8)

The energy-momentum current across a constant $τ$-hypersurface in $(τ, x, y, η)$ coordinates can also be calculated from Eq. (C7), together with the general transformation rules $E_{tot}^μ = (δτ, δx^ν, δζ^ξ)$ $E_{tot}^ν$, where the position vectors are $x^μ ≡ (t, x, y, z) = (τ sinh η, x, y, τ sinh η)$, and $x^μ ≡ (τ, x, y, η)$. Thus the total energy across constant $τ$-hypersurface is given by,

$$E_{tot}^0 (τ) = \int cosh η T^{ττ} d^3Σ^{(τ,η)} + \int τ sinh η T^{τν} d^3Σ^{(τ,η)} = \int (cosh η T^{ττ} + τ sinh η T^{τν}) τ dxdydz.$$  \hspace{1cm} (C9)

Using the latter formulas we can check the energy conservation from the initial time to the end using,

$$ΔE_{tot}^0 (τ) = E_{tot}^0 (τ_{end}) - E_{tot}^0 (τ_{ini})$$  \hspace{1cm} (C10)

It turns out that solving the fluid dynamical equations in Cartesian coordinates we can achieve $ΔE_{tot}^0 (τ) ≈ O_5$ numerical accuracy, while in hyperbolic coordinates $ΔE_{tot}^0 (τ) ≈ O_1$. This behavior is due to two different reasons.

First of all, the numerical algorithm is accurate only to finite precision, meaning that $T^{00}$ or $T^{ττ}$ is calculated correctly only up to the first six digits. However, due to the hyperbolic functions in Eq. (C3), the total energy of the system is given by a differently weighted sum over all cells (compared to Cartesian coordinates). These hyperbolic weights increase very rapidly as a function of $η$, hence even though the numerical error of the solver is acceptable small for SHASTA, the weighted sum over all cells in hyperbolic coordinates shows otherwise.

We have checked that for RHIC energies $ΔE_{tot}^0 (τ) < 2\%$ while at the LHC energies this number can be as much as 20%. This is due to fact the $f (η)$ is much narrower at RHIC than at the LHC. Similar results were also obtained in Ref. [3] using a different computational fluid dynamical algorithm.

We also verified the energy conservation inside the constant temperature freeze-out hypersurface, and we found that in that case the energy is conserved at 1% accuracy, both at RHIC and the LHC. This was expected since inside the $T = 100$ MeV freeze-out hypersurface the space-time rapidity of matter is $η < 10$. 

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