A Companion of Ostrowski inequality for the Stieltjes integral of monotonic functions

Research Article

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Abstract: Some companions of Ostrowski’s integral inequality for the Riemann-Stieltjes integral \( \int_{a}^{b} f(t) \, du(t) \), where \( f \) is assumed to be of \( r \)-Hölder type on \([a,b]\) and \( u \) is of bounded variation on \([a,b]\), are proved. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

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1. Introduction

In [10], Dragomir has proved an Ostrowski inequality for the Riemann-Stieltjes integral, as follows:

**Theorem 1.1.**

Let \( f : [a,b] \to \mathbb{R} \) be a \( r \)-Hölder type mapping, that is, it satisfies the condition

\[
|f(x) - f(y)| \leq H |x - y|^r, \quad \forall x, y \in [a,b],
\]

where, \( H > 0 \) and \( r \in (0,1) \) are given, and \( u : [a,b] \to \mathbb{R} \) is a mapping of bounded variation on \([a,b]\). Then we have the inequality

\[
\left| f(x)(u(b) - u(a)) - \int_{a}^{b} f(t) \, du(t) \right| \leq H \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right]^r \cdot \mathcal{V}_u(a, b)
\]

for all \( x \in [a,b] \), where, \( \mathcal{V}_u(a, b) \) denotes the total variation of \( u \) on \([a,b]\). Furthermore, the constant \( \frac{1}{2} \) is the best possible in the sense that it cannot be replaced by a smaller one, for all \( r \in (0,1) \).

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In [11], Dragomir has proved the dual case as follows:

**Theorem 1.2.**

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a mapping of bounded variation on \([a, b]\) and \( u : [a, b] \rightarrow \mathbb{R} \) be of \( r\)-Hölder type on \([a, b]\). Then we have the inequality

\[
\left| (u(b) - u(a)) f(x) - \int_a^b f(t) \, du(t) \right| \leq H \left[ (x-a)^r \cdot \sqrt[2]{\int_a^b (f)} + (b-x)^r \cdot \sqrt[2]{\int_a^b (f)} \right]
\]

\[
\leq H \left[ (x-a)^r + (b-x)^r \right] \left[ \frac{1}{2} \sqrt[4]{\int_a^b (f)} + \frac{1}{2} \left| \int_a^b (f) - \sqrt[2]{\int_a^b (f)} \right| \right]
\]

\[
\leq H \left[ (x-a)^r + (b-x)^r \right] \left[ \frac{1}{2} \sqrt[4]{\int_a^b (f)} \right]^1 + \left[ \frac{(b-a)^r}{2} + \left| x - \frac{a+b}{2} \right| \right] \cdot \sqrt[2]{\int_a^b (f)}
\]

In [5], Barnett et al. established some Ostrowski and trapezoid type inequalities for the Stieltjes integral \( \int_a^b f(t) \, du(t) \) in the case of Lipschitzian integrators for both Hölder continuous and monotonic integrals are obtained. The dual case is also analyzed. In [6], Cerone et al. proved some Ostrowski type inequalities for the Stieltjes integral where the integrand \( f \) is absolutely continuous while the integrator \( u \) is of bounded variation. For other results concerning inequalities for Stieltjes integrals, see [3, 7, 8, 14, 16, 18, 20].

Motivated by [17], Dragomir in [13], established the following companion of the Ostrowski inequality for mappings of bounded variation.

**Theorem 1.3.**

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a mapping of bounded variation on \([a, b]\). Then we have the inequalities:

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{3} + \left| x - \frac{a+b}{4} \right| \right] \cdot \sqrt[2]{\int_a^b (f)}
\]

for any \( x \in [a, \frac{a+b}{2}] \), where \( \sqrt[2]{\int_a^b (f)} \) denotes the total variation of \( f \) on \([a, b]\). The constant \( 1/4 \) is best possible.

For recent results concerning the above companion of Ostrowski’s inequality and other related results see [1, 2, 4, 13, 15, 19].

In this paper, we establish a companion of Ostrowski’s integral inequality for the Riemann-Stieltjes integral \( \int_a^b f(t) \, du(t) \), where \( f \) is assumed to be of \( r\)-Hölder type on \([a, b]\) and \( u \) is of bounded variation on \([a, b]\), are given. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.
2. The Results

The following companion of Ostrowski’s inequality for Riemann-Stieltjes integral holds.

**Theorem 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be a \( r-H \)-Hölder type mapping, where, \( H > 0 \) and \( r \in (0, 1] \) are given, and \( u : [a, b] \to \mathbb{R} \) is a monotonic nondecreasing function on \([a, b]\). Then we have the inequality

\[
\left| f(x) \left[ u \left( \frac{a + b}{2} \right) - u(a) \right] + f(a + b - x) \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] - \int_a^b f(t) \, du(t) \right| \leq 2H \left\{ \left( \frac{a+b}{2} - x \right)^r \left[ u \left( \frac{a + b}{2} \right) - u(x) \right] + (x-a)^r \left[ u(x) - \frac{u(a) + u(b)}{2} \right] \right\}
\]

\[
\leq 2H \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^r \left[ u \left( \frac{a + b}{2} \right) - \frac{u(a) + u(b)}{2} \right] ,
\]

for all \( x \in [a, \frac{a+b}{2}] \).

**Proof.** Using the integration by parts formula for Riemann–Stieltjes integral, we have

\[
\int_a^{\frac{a+b}{2}} [f(x) - f(t)] \, du(t) = f(x) \left[ u \left( \frac{a + b}{2} \right) - u(a) \right] - \int_a^{\frac{a+b}{2}} f(t) \, du(t),
\]

and

\[
\int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)] \, du(t) = f(a + b - x) \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] - \int_{\frac{a+b}{2}}^b f(t) \, du(t).
\]

Adding the above equalities, we have

\[
\int_a^{\frac{a+b}{2}} [f(x) - f(t)] \, du(t) + \int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)] \, du(t)
\]

\[
= f(x) \left[ u \left( \frac{a + b}{2} \right) - u(a) \right] + f(a + b - x) \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] - \int_a^b f(t) \, du(t).
\]

It is well known that if \( p : [c, d] \to \mathbb{R} \) is continuous and \( \nu : [c, d] \to \mathbb{R} \) is monotonic nondecreasing, then the Stieltjes integral \( \int_c^d p(t) \, d\nu(t) \) exists and the following inequality holds:

\[
\left| \int_c^d p(t) \, d\nu(t) \right| \leq \int_c^d |p(t)| \, d\nu(t).
\]
Making use of this property and the fact that \( f \) is of \( r \)-Hölder type on \([a, b]\), we can state that

\[
|f(x)\left[u\left(\frac{a + b}{2}\right) - u(a)\right] + f(a + b - x)\left[u(b) - u\left(\frac{a + b}{2}\right)\right] - \int_a^b f(t)\,du(t) |
\]

\[
= \left| \int_a^{a + b/2} [f(x) - f(t)]\,du(t) + \int_{a + b/2}^b [f(a + b - x) - f(t)]\,du(t) \right|
\]

\[
\leq \left| \int_a^{a + b/2} [f(x) - f(t)]\,du(t) \right| + \left| \int_{a + b/2}^b [f(a + b - x) - f(t)]\,du(t) \right|
\]

\[
\leq \int_a^{a + b/2} |f(x) - f(t)|\,du(t) + \int_{a + b/2}^b |f(a + b - x) - f(t)|\,du(t)
\]

\[
\leq H \int_a^{a + b/2} |x - t|^r\,du(t) + H \int_{a + b/2}^b |a + b - x - t|^r\,du(t).
\]  

(6)

By the integration by parts formula for the Stieltjes integral, we have

\[
\int_a^{a + b/2} |x - t|^r\,du(t) = \int_a^x (x - t)^r\,du(t) + \int_x^{a + b/2} (t - x)^r\,du(t)
\]

\[
= \left(\frac{a + b}{2} - x\right)^r u\left(\frac{a + b}{2}\right) - (x - a)^r u(a)
\]

\[
+ r \int_a^x \frac{u(t)}{(x - t)^{1-r}}\,dt - \int_x^{a + b/2} \frac{u(t)}{(t - x)^{1-r}}\,dt
\]

and

\[
\int_{a + b/2}^b |a + b - x - t|^r\,du(t) = \int_{a + b/2}^{a + b - x} (a + b - x - t)^r\,du(t) + \int_{a + b - x}^b (t + x - a - b)^r\,du(t)
\]

\[
= \left(\frac{a + b}{2} - x\right)^r u\left(\frac{a + b}{2}\right) - (x - a)^r u(b)
\]

\[
+ r \int_{a + b - x}^b \frac{u(t)}{(t + x - a - b)^{1-r}}\,dt - \int_{a + b/2}^{a + b - x} \frac{u(t)}{(a + b - x - t)^{1-r}}\,dt
\]

Now, by the monotonicity property of \( u \) we have

\[
\int_a^x \frac{u(t)}{(x - t)^{1-r}}\,dt \leq u(x) \int_a^x \frac{dt}{(x - t)^{1-r}} = \frac{1}{r} (x - a)^r u(x),
\]

\[
\int_x^{a + b/2} \frac{u(t)}{(t - x)^{1-r}}\,dt \geq u(x) \int_x^{a + b/2} \frac{dt}{(t - x)^{1-r}} = \frac{1}{r} \left(\frac{a + b}{2} - x\right)^r u(x),
\]

\[
\int_{a + b - x}^b \frac{u(t)}{(t + x - a - b)^{1-r}}\,dt \leq u(x) \int_{a + b - x}^b \frac{dt}{(t + x - a - b)^{1-r}} = \frac{1}{r} (x - a)^r u(x),
\]

\[
\int_{a + b/2}^{a + b - x} \frac{u(t)}{(a + b - x - t)^{1-r}}\,dt \geq u(x) \int_{a + b/2}^{a + b - x} \frac{dt}{(a + b - x - t)^{1-r}} = \frac{1}{r} \left(\frac{a + b}{2} - x\right)^r u(x),
\]
The following inequalities are hold:

\[
\int_{\frac{a+b}{2}}^{a+b-x} \frac{u(t)}{(a+b-x-t)^{1-r}} dt \geq u(x) \int_{\frac{a+b}{2}}^{a+b-x} \frac{dt}{(a+b-x-t)^{1-r}} = \frac{1}{r} \left( \frac{a+b}{2} - x \right)^r u(x),
\]

which follows that

\[
\int_{a}^{x} \frac{u(t)}{(x-t)^{1-r}} dt - \int_{x}^{\frac{a+b}{2}} \frac{u(t)}{(t-x)^{1-r}} dt \leq \frac{1}{r} \left[ (x-a)^r - \left( \frac{a+b}{2} - x \right)^r \right] u(x)
\]

and

\[
\int_{a+b-x}^{b} \frac{u(t)}{(t+x-a-b)^{1-r}} dt - \int_{\frac{a+b}{2}}^{a+b-x} \frac{u(t)}{(a+b-x-t)^{1-r}} dt
\]

\[
\leq \frac{1}{r} \left[ (x-a)^r - \left( \frac{a+b}{2} - x \right)^r \right] u(x),
\]

which implies that

\[
\left( \frac{a+b}{2} - x \right)^r \left[ u\left( \frac{a+b}{2} \right) - (x-a)^r u(a) \right] + \frac{1}{r} \left[ \int_{a}^{x} \frac{u(t)}{(x-t)^{1-r}} dt - \int_{x}^{\frac{a+b}{2}} \frac{u(t)}{(t-x)^{1-r}} dt \right]
\]

\[
\leq \left( \frac{a+b}{2} - x \right)^r \left[ u\left( \frac{a+b}{2} \right) - (x-a)^r u(a) \right] + \frac{1}{r} \left[ (x-a)^r - \left( \frac{a+b}{2} - x \right)^r \right] u(x)
\]

\[
= \left( \frac{a+b}{2} - x \right)^r \left[ u\left( \frac{a+b}{2} \right) - u(x) \right] + (x-a)^r [u(x) - u(a)],
\]

similarly,

\[
\left( \frac{a+b}{2} - x \right)^r \left[ u\left( \frac{a+b}{2} \right) - (x-a)^r u(b) \right]
\]

\[
+ \frac{1}{r} \left[ \int_{a+b-x}^{b} \frac{u(t)}{(t+x-a-b)^{1-r}} dt - \int_{\frac{a+b}{2}}^{a+b-x} \frac{u(t)}{(a+b-x-t)^{1-r}} dt \right]
\]

\[
\leq \left( \frac{a+b}{2} - x \right)^r \left[ u\left( \frac{a+b}{2} \right) - (x-a)^r u(b) \right] + \frac{1}{r} \left[ (x-a)^r - \left( \frac{a+b}{2} - x \right)^r \right] u(x)
\]

\[
= \left( \frac{a+b}{2} - x \right)^r \left[ u\left( \frac{a+b}{2} \right) - u(x) \right] + (x-a)^r [u(x) - u(b)],
\]

which together with (6) proves the first inequality in (4). The second inequality is obvious by the property of max function and we omit the details here.

The following inequalities are hold:

**Corollary 2.1.**

Let \( f \) and \( u \) as in Theorem 2.1. In (4) choose
1. \( x = a \), then we get the following trapezoid type inequality
\[
\left| f(a) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] + f(b) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_a^b f(t) \, du(t) \right| 
\leq H \left( \frac{b-a}{2} \right)^r \cdot \sqrt[4]{u(b)}.
\] (7)

2. \( x = \frac{a+b}{2} \), then we get the following mid-point type inequality
\[
\left| (u(b) - u(a)) f \left( \frac{a+b}{2} \right) - \int_a^b f(t) \, dt \right| \leq H \left( \frac{b-a}{2} \right)^r \cdot \sqrt[4]{u(b)}.
\] (8)

We may state the following Ostrowski type inequality:

**Corollary 2.2.**
Let \( f \) and \( u \) as in Theorem 2.1. Additionally, if \( f \) is symmetric about the \( x \)-axis, i.e., \( f(a+b-x) = f(x) \), then we have
\[
\left| (u(b) - u(a)) f \left( \frac{x}{2} \right) - \int_a^b f(t) \, dt \right| \leq H \left( \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right)^r \cdot \sqrt[4]{u(b)}.
\] (9)

for all \( x \in [a, a+b/2] \).

**Corollary 2.3.**
Let \( u \) as in Theorem 2.1, and \( f : [a,b] \to \mathbb{R} \) be an \( L \)-Lipschitzian mapping on \([a,b]\), that is,
\[
|f(x) - f(y)| \leq L |x - y|, \quad \forall x, y \in [a,b],
\]
where, \( L > 0 \) is fixed. Then, for all \( x \in [a, a+b/2] \), we have the inequality
\[
\left| f \left( \frac{x}{2} \right) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] + f(a+b-x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_a^b f(t) \, du(t) \right| 
\leq L \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \sqrt[4]{u(b)}.
\] (10)

The constant \( \frac{1}{4} \) is the best possible in the sense that it cannot be replaced by a smaller one.

**Corollary 2.4.**
In Theorem 2.1, if \( u \) is monotonic on \([a, b]\), and \( f \) is of \( r \)-Hölder type. Then, for all \( x \in [a, a+b/2] \), we have the inequality
\[
\left| f \left( \frac{x}{2} \right) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] + f(a+b-x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_a^b f(t) \, du(t) \right| 
\leq H \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^r \cdot |u(b) - u(a)|.
\] (11)
Corollary 2.5.
Let $f$ be of $r$-Hölder type and $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then we have the inequality

$$
\left| f(x) \int_a^{a+b} g(s) ds + f(a+b-x) \int_{a+b-x}^{b} g(s) ds - \int_{a}^{b} f(t) g(t) dt \right| 
\leq H \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^r \| g \|_1, \tag{12}
$$

for all $x \in [a, \frac{a+b}{2}]$, where $\| g \|_1 = \int_a^b |g(t)| dt$.

Proof. Define the mapping $u : [a, b] \to \mathbb{R}$, $u(t) = \int_a^t g(s) ds$. Then $u$ is differentiable on $(a, b)$ and $u'(t) = g(t)$. Using the properties of the Riemann-Stieltjes integral, we have

$$
\int_a^b f(t) du(t) = \int_a^b f(t) g(t) dt,
$$

and

$$
\nabla_a^b (u) = \int_a^b |u'(t)| dt = \int_a^b |g(t)| dt.
$$

Remark 2.1.
In Corollary 2.5, if $f$ is symmetric about the $x$-axis, i.e., $f(a+b-x) = f(x)$, then we have

$$
\left| f(x) \int_a^{a+b} g(s) ds - \int_{a}^{b} f(t) g(t) dt \right| 
\leq H \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^r \| g \|_1, \tag{13}
$$

for all $x \in [a, \frac{a+b}{2}]$. For instance, choose $x = \frac{3a+b}{4}$, then we get

$$
\left| f \left( \frac{a+b}{2} \right) \int_a^{a+b} g(s) ds - \int_{a}^{b} f(t) g(t) dt \right| 
\leq H \left( \frac{b-a}{2} \right)^r \| g \|_1. \tag{14}
$$

Theorem 2.2.
Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$ and $u : [a, b] \to \mathbb{R}$ be of $r$-Hölder type on $[a, b]$, $r \in (0, 1]$. Then we have the inequality

$$
\left| f(x) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] + f(a+b-x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_{a}^{b} f(t) du(t) \right| 
\leq H \left[ (x-a)^r \cdot \nabla_a^x (f) + \left( \frac{a+b-2x}{2} \right)^r \cdot \nabla_x^{a+b-x} (f) + (x-a)^r \cdot \nabla_{a+b-x}^b (f) \right]
\leq H \left[ 2(x-a)^r + \left( \frac{a+b}{2} - x \right)^r \right] \max \left\{ \nabla_a^x (f), \nabla_x^{a+b-x} (f) , \nabla_{a+b-x}^b (f) \right\}
\leq H \left[ 2^r (x-a)^{qr} + \left( \frac{a+b}{2} - x \right)^{qr} \right] \max \left\{ \nabla_a^x (f), \nabla_x^{a+b-x} (f) , \nabla_{a+b-x}^b (f) \right\}
\leq H \left[ \frac{1}{2} (b-a) + \left| x - \frac{3a+b}{4} \right|^r \right] \nabla_x^b (f)
\right|^{1/p}, \text{ if } \frac{1}{p} + \frac{1}{q} = 1, \ p > 1. \tag{15}
$$

for all $x \in [a, \frac{a+b}{2}]$, where, $\nabla_x^b (f)$ denotes the total variation of $f$ on $[a, b]$. 
Proof. As $u$ is continuous and $f$ is of bounded variation on $[a, b]$, the following Riemann-Stieltjes integrals exist and, by the integration by parts formula, we can state that

$$\int_a^x (u(t) - u(a)) \, df(t) = (u(x) - u(a)) f(x) - \int_a^x f(t) \, du(t),$$

$$\int_x^{a+b-x} (u(t) - u \left( \frac{a+b}{2} \right)) \, df(t) = \left( u(a + b - x) - u \left( \frac{a+b}{2} \right) \right) f(a + b - x) - \left( u(x) - u \left( \frac{a+b}{2} \right) \right) f(x) - \int_x^{a+b-x} f(t) \, du(t)$$

and

$$\int_{a+b-x}^b (u(t) - u(b)) \, df(t) = (u(b) - u(a + b - x)) f(a + b - x) - \int_{a+b-x}^b f(t) \, du(t).$$

If we add the above three identities, we obtain

$$\left[ u \left( \frac{a+b}{2} \right) - u(a) \right] f(x) + f(a + b - x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_a^b f(t) \, du(t)$$

$$= \int_a^x (u(t) - u(a)) \, df(t) + \int_x^{a+b-x} (u(t) - u \left( \frac{a+b}{2} \right)) \, df(t) + \int_{a+b-x}^b (u(t) - u(b)) \, df(t),$$

for all $x \in [a, \frac{a+b}{2}]$.

Now, using the properties of absolute value, we have:

$$|f(x)| \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] + f(a + b - x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_a^b f(t) \, du(t)$$

$$\leq \left| \int_a^x (u(t) - u(a)) \, df(t) \right| + \left| \int_x^{a+b-x} (u(t) - u \left( \frac{a+b}{2} \right)) \, df(t) \right|$$

$$+ \left| \int_{a+b-x}^b (u(t) - u(b)) \, df(t) \right|$$

$$\leq \int_a^x |u(t) - u(a)| \, df(t) + \int_x^{a+b-x} |u(t) - u \left( \frac{a+b}{2} \right)| \, df(t)$$

$$+ \int_{a+b-x}^b |u(t) - u(b)| \, df(t)$$

$$\leq H \int_a^x |t - a|^r \, df(t) + \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^r \, df(t) + \int_{a+b-x}^b |t - b|^r \, df(t)$$

$$\leq H \int_a^x (t - a)^r \, df(t) + \int_x^{a+b-x} \left( \frac{a+b}{2} - t \right)^r \, df(t) + \int_{a+b-x}^b \left( t - \frac{a+b}{2} \right)^r \, df(t)$$

$$+ \int_{a+b-x}^b (b - t)^r \, df(t)$$
and for the last inequality we have used the well-known property if $p: [c, d] \to \mathbb{R}$ is continuous and $\nu: [c, d] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t) \, d\nu(t)$ exists and the following inequality holds:

$$\left| \int_c^d p(t) \, d\nu(t) \right| \leq \sup_{t \in [c, d]} |p(t)| \sqrt{\nu}.$$  

As $u$ is of $r$-$H$-Hölder type on $[a, b]$, we can state that

$$\sup_{t \in [a, x]} |u(t) - u(a)| \leq \sup_{t \in [a, x]} [H(t - a)^r] = H(x - a)^r,$$

and

$$\sup_{t \in [x, a + b - x]} |u(t) - u(b)| \leq \sup_{t \in [a + b - x, b]} [H(b - t)^r] = H(x - a)^r.$$  

Now, using (17), we have

$$\left| \left( u\left( \frac{a + b}{2} \right) - u(a) \right) f(x) + \left( u(b) - u\left( \frac{a + b}{2} \right) \right) f\left( a + b - x \right) - \int_a^b f(t) \, du(t) \right| \leq H \left[ (x - a)^r \cdot \sqrt{\frac{x}{a}} (f) + \sqrt{\frac{a + b - x}{x}} (f) + (x - a)^r \cdot \sqrt{\frac{b}{a + b - x}} (f) \right] := M(x),$$

for all $x \in [a, \frac{a+b}{2}]$, and the first inequality in (15) is proved.  

### 3. An Approximation for the Riemann-Stieltjes Integral

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a, b]$, $h_i = x_{i+1} - x_i$, $(i = 0, 1, 2, \cdots, n - 1)$ and $\nu(h) := \max \{h_i | i = 0, 1, 2, \cdots, n - 1 \}$. Define the general Riemann-Stieltjes sum

$$S(f, u, I_n, \xi) \quad (16)$$

$$\quad = \sum_{i=0}^{n-1} f(\xi_i) \left[ u\left( \frac{x_i + x_{i+1}}{2} \right) - u(x_i) \right] + f(x_i + x_{i+1} - \xi_i) \left[ u(x_{i+1}) - u\left( \frac{x_i + x_{i+1}}{2} \right) \right]$$

In the following, we establish some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t) \, du(t)$ by its Riemann-Stieltjes sum $S(f, u, I_n, \xi)$.  

Theorem 3.1.  
Let \( u : [a, b] \to \mathbb{R} \) be a mapping of bounded variation on \([a, b]\) and \( f : [a, b] \to \mathbb{R} \) be of \( r\)-Hölder type on \([a, b]\). Then

\[
\int_a^b f(t) \, du(t) = S(f, u, I_n, \xi) + R(f, u, I_n, \xi)
\]

where, \( S(f, u, I_n, \xi) \) is given in (16) and the remainder \( R(f, u, I_n, \xi) \) satisfies the bound

\[
|R(f, u, I_n, \xi)| \leq H \left[ \frac{1}{4} \nu(h) + \max_{i=0,1,\ldots,n-1} \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right]^r \cdot \sqrt[n]{b-a}(u) \tag{17}
\]

Proof. Applying Theorem 2.1 on the intervals \([x_i, x_{i+1}]\), we may state that

\[
\left| f(\xi_i) \left[ u \left( \frac{x_i + x_{i+1}}{2} \right) - u(x_i) \right] + f(x_i + x_{i+1} - \xi_i) \left[ u(x_{i+1}) - u \left( \frac{x_i + x_{i+1}}{2} \right) \right] \right| - \int_{x_i}^{x_{i+1}} f(t) \, du(t) \leq H \left[ \frac{1}{4} h_i + \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right]^r \cdot \sqrt{\frac{x_{i+1}}{x_i}(u)},
\]

for all \( i \in \{0, 1, 2, \cdots, n-1\} \).

Summing the above inequality over \( i \) from 0 to \( n-1 \) and using the generalized triangle inequality, we deduce

\[
|R(f, u, I_n, \xi)| = \sum_{i=0}^{n-1} \left| f(\xi_i) \left[ u \left( \frac{x_i + x_{i+1}}{2} \right) - u(x_i) \right] + f(x_i + x_{i+1} - \xi_i) \left[ u(x_{i+1}) - u \left( \frac{x_i + x_{i+1}}{2} \right) \right] \right| - \int_{x_i}^{x_{i+1}} f(t) \, du(t) \leq H \sum_{i=0}^{n-1} \left[ \frac{1}{4} h_i + \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right]^r \cdot \sqrt{\frac{x_{i+1}}{x_i}(u)} \]

\[
\leq H \sup_{i=0,1,\ldots,n-1} \left[ \frac{1}{4} h_i + \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right]^r \cdot \sum_{i=0}^{n-1} \sqrt{\frac{x_{i+1}}{x_i}(u)}.
\]

However,

\[
\sup_{i=0,1,\ldots,n-1} \left[ \frac{1}{4} h_i + \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right]^r \leq \left[ \frac{1}{4} \nu(h) + \sup_{i=0,1,\ldots,n-1} \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right]^r,
\]

and

\[
\sum_{i=0}^{n-1} \sqrt{\frac{x_{i+1}}{x_i}(u)} = \sqrt{\frac{b}{a}(u)}.
\]
which completely proves the first inequality in (17).

For the second inequality, we observe that

$$\left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \leq \frac{1}{4} h_i$$

for all \(i \in \{0, 1, 2, \cdots, n-1\}\), which completes the proof. \(\square\)

**Corollary 3.1.**

In Theorem 3.1, additionally, if \(f\) is symmetric about the \(x\)-axis, then we have \(S(f, u, I_n, \xi)\) reduced to be

$$S(f, u, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi) [u(x_{i+1}) - u(x_i)].$$

(18)

Then

$$\int_a^b f(t) \, du(t) = S(f, u, I_n, \xi) + R(f, u, I_n, \xi)$$

where, \(S(f, u, I_n, \xi)\) is given in (18) and the remainder \(R(f, u, I_n, \xi)\) satisfies the bound in (17).

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