LONG TIME DECAY TO THE LEI-LIN SOLUTION OF 3D NAVIER-STOKES EQUATIONS

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Abstract. In this paper we prove, if \( u \in C([0, \infty), X^{-1}(\mathbb{R}^3)) \) is global solution of 3D Navier-Stokes equations, then \( \|u(t)\|_{X^{-1}} \) decays to zero as time goes to infinity. Fourier analysis and standard techniques are used.

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1. Introduction

In this paper we deal with the following 3-D incompressible Navier-Stokes equations:

\[
\begin{cases}
\partial_t u - \nu \Delta u + (u, \nabla) u = -\nabla p, & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u|_{t=0} = u^0 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( \nu > 0 \) is the viscosity of the fluid, and \( u = u(t, x) = (u_1, u_2, u_3) \) and \( p = p(t, x) \) denote respectively the unknown velocity and the unknown pressure of the fluid at the point \( (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \). Here, \( u^0 = (u^0_1, u^0_2, u^0_3) \) is a given initial velocity. If the condition is fairly regular, one can express the pressure using the speed. The study of local existence is studied by several researchers, Leray [11, 12], Kato [8], etc.... The global existence of weak solutions goes back to Leray [11] and Hopf [7]. The global well-posedness of strong solutions for small initial data is due to Fujita and Kato [5] in the critical Sobolev space \( \dot{H}^{1/2} \) also Chemin [3] has prove the case of \( \dot{H}^s \), \( s > 1/2 \), Kato [9] in the Lebesgue space \( L^3 \), and Koch and Tataro [10] in the space...
BMO$^{-1}$ (see also [2, 11, 14]). It should be noted, in all these works, that the norms in corresponding spaces of the initial data are assumed to be very small, smaller than the viscosity coefficient $\nu$ multiplied by tiny positive constant $c$. For further results and details can consult the book by Cannone [1]. In [13], the authors consider a new critical space that is contains in BMO$^{-1}$, where they show it is sufficient to assume the norms of initial data are less that the viscosity coefficient $\nu$. Then the space used in [13] is the following

$$\mathcal{X}^{-1}(\mathbb{R}^3) := \{ f \in \mathcal{D}'(\mathbb{R}^3), \int_{\mathbb{R}^3} |\hat{f}(\xi)| d\xi < \infty \},$$

with the norm

$$\| f \|_{\mathcal{X}^{-1}} = \int_{\mathbb{R}^3} |\hat{f}(\xi)| d\xi.$$

We will also use the notation, for $i = 0, 1,$

$$\mathcal{X}^i(\mathbb{R}^3) := \{ f \in \mathcal{D}'(\mathbb{R}^3), \int_{\mathbb{R}^3} |\xi|^i |\hat{f}(\xi)| d\xi < \infty \}.$$  

For the small initial data, the authors proved the global existence, precisely:

**Theorem 1.1.** [13] Let $u^0 \in \mathcal{X}^{-1}(\mathbb{R}^3)$, such that $\| u^0 \|_{\mathcal{X}^{-1}} < \nu$, then there is a unique $u \in C(\mathbb{R}^+, \mathcal{X}^{-1})$ such that $\Delta u \in L^1(\mathbb{R}^+, \mathcal{X}^{-1})$. Moreover, for all $t \geq 0$

$$\sup_{0 \leq t < \infty} \left( \| u(t) \|_{\mathcal{X}^{-1}} + (\nu - \| u^0 \|_{\mathcal{X}^{-1}}) \int_{0}^{t} \| \nabla u(t) \|_{L^\infty} \right) \leq \| u^0 \|_{\mathcal{X}^{-1}}. \hspace{1cm} (1.1)$$

To show this theorem, the authors used a method of regularization of the initial data $u^0_\lambda = \zeta^\lambda \ast u^0$, in order to use the standard local existence theory of the Navier-Stokes equations. They obtain uniform estimates in suitable spaces, and pass to the weak limit as $\lambda$ tends towards zero. If we change this method, and by using Fixed Point Theorem on $C([0, T], \mathcal{X}^{-1}(\mathbb{R}^3)) \cap L^1([0, T], \mathcal{X}^1(\mathbb{R}^3))$ and Lemma [2, 11] we can deduce the following: Let $u^0 \in \mathcal{X}^{-1}(\mathbb{R}^3)$, such that $\| u^0 \|_{\mathcal{X}^{-1}} < \nu$, then there is a unique $u \in C(\mathbb{R}^+, \mathcal{X}^{-1})$ such that $\Delta u \in L^1(\mathbb{R}^+, \mathcal{X}^{-1})$. Moreover, for all $t \geq 0$

$$\| u(t) \|_{\mathcal{X}^{-1}} + (\nu - \| u^0 \|_{\mathcal{X}^{-1}}) \int_{0}^{t} \| u(z) \|_{\mathcal{X}^1} dz \leq \| u^0 \|_{\mathcal{X}^{-1}}. \hspace{1cm} (1.2)$$

Moreover, in [15] Zhang and Yin prove the local existence for large initial data and blow up criteria if the maximal time is finite, precisely:

**Theorem 1.2.** Let $u^0$ be in $\mathcal{X}^{-1}(\mathbb{R}^3)$. There exists time $T$ such that the system $(NS)$ has unique solution $u$ in $L^2([0, T], \mathcal{X}^{-1}(\mathbb{R}^3))$ which also belongs to $C([0, T]; \mathcal{X}^{-1}(\mathbb{R}^3)) \cap L^1([0, T], \mathcal{X}^1(\mathbb{R}^3)) \cap L^\infty([0, T]; \mathcal{X}^{-1}(\mathbb{R}^3))$.

Let $T^*$ denote the maximal time of existence of such solution. Then:

(i) If $\| u^0 \|_{\mathcal{X}^{-1}} < \nu$, then $T^* = \infty$.

(ii) If $T^*$ is finite, then

$$\int_{0}^{T^*} \| u(t) \|_{\mathcal{X}^0}^2 dt = \infty.$$
Our main result is to prove non-blowup at large time and the norm of the global solution in \( \mathcal{X}^{-1}(\mathbb{R}^3) \) goes to zero at infinity.

**Theorem 1.3.** Let \( u \in C(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3)) \) be a global solution of \((NS)\), then
\[
\limsup_{t \to \infty} \|u(t)\|_{\mathcal{X}^{-1}} = 0.
\]

In the following we give a natural application of Theorem 1.3, it is the stability of global solutions of \((NS)\) system.

**Theorem 1.4.** Let \( u \in C(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3)) \) be a global solution of \((NS)\), then for all \( v^0 \in \mathcal{X}^{-1}(\mathbb{R}^3) \) such that
\[
\|v^0 - u(0)\|_{\mathcal{X}^{-1}} < \nu \frac{2}{3} \int_0^\infty \|\widehat{u}(s)\|_{L^1}^2 ds,
\]
then, Navier-Stokes system starting by \( v^0 \) has a global solution. Moreover, if \( v \) is the corresponding global solution; then, for all \( t \geq 0 \),
\[
\|v(t) - u(t)\|_{\mathcal{X}^{-1}} + \frac{\nu}{2} \int_0^t \|v(s) - u(s)\|_{\mathcal{X}^1} ds \leq \|v^0 - u(0)\|_{\mathcal{X}^{-1}} e^{\frac{2}{3} \int_0^\infty \|\widehat{u}(s)\|_{L^1}^2 ds}.
\]

The remainder of this paper is organized in the following way: In section 2 we give some notations and important preliminaries results. Section 3 is devoted to prove the principle result. In section 4 we prove the stability result for global solutions.

**2. Notations and Preliminaries Results**

2.1. **Notations.** In this short section we collect some notations and definitions that will be used later on.

- The Fourier transformation is normalized as
  \[
  \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix.\xi)f(x)dx, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
  \]

- The inverse Fourier formula is
  \[
  \mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi.x) f(\xi)d\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.
  \]

- For \( s \in \mathbb{R} \), \( H^s(\mathbb{R}^3) \) denotes the usual non homogeneous Sobolev space on \( \mathbb{R}^3 \) and \( \langle ., . \rangle_{H^s(\mathbb{R}^3)} \) denotes the usual scalar product on \( H^s(\mathbb{R}^3) \).

- For \( s \in \mathbb{R} \), \( \dot{H}^s(\mathbb{R}^3) \) denotes the usual homogeneous Sobolev space on \( \mathbb{R}^3 \) and \( \langle ., . \rangle_{\dot{H}^s(\mathbb{R}^3)} \) denotes the usual scalar product on \( \dot{H}^s(\mathbb{R}^3) \).

- The convolution product of a suitable pair of functions \( f \) and \( g \) on \( \mathbb{R}^3 \) is given by
  \[
  (f * g)(x) := \int_{\mathbb{R}^3} f(y)g(x-y)dy.
  \]

- If \( f = (f_1, f_2, f_3) \) and \( g = (g_1, g_2, g_3) \) are two vector fields, we set
  \[
  f \otimes g := (g_1f, g_2f, g_3f),
  \]
  and
  \[
  \text{div}(f \otimes g) := (\text{div}(g_1f), \text{div}(g_2f), \text{div}(g_3f)).
  \]
For any subset $X$ of a set $E$, the symbol $1_X$ denote the characteristic function of $X$ defined by

$$1_X(x) = 1 \text{ if } x \in X, \quad 1_X(x) = 0 \text{ elsewhere}.$$  

2.2. Preliminaries Results.

**Lemma 2.1.** (i) If $f, g \in X^0(\mathbb{R}^3)$, then $fg \in X^0(\mathbb{R}^3)$ and

$$\|fg\|_{X^0} \leq \|f\|_{X^0} \|g\|_{X^0}.$$  

(ii) If $f \in X^{-1}(\mathbb{R}^3) \cap X^1(\mathbb{R}^3)$, then $f \in X^{-1}(\mathbb{R}^3)$ and

$$\|f\|_{X^0} \leq \|f\|_{X^{-1}}^{1/2} \|f\|_{X^1}^{1/2}.$$

**Proof of lemma 2.1.** (i) is a given by direct application of Young inequality. To prove (ii), we can write

$$\|f\|_{X^0} = \int_\xi |\hat{f}(\xi)|d\xi = \int_\xi |\xi|^{1/2} |\hat{f}(\xi)|^{1/2} |\xi|^{1/2} |\hat{f}(\xi)|^{1/2} d\xi.$$  

Cauchy-Schwartz inequality gives the desired result.

**Lemma 2.2.** If $s > 1/2$, we have $H^s(\mathbb{R}^3) \hookrightarrow X^{-1}(\mathbb{R}^3)$ and

$$\|f\|_{X^{-1}} \leq C_s \|f\|_{H^s} \|f\|_{L^2}^{1/2} \|f\|_{H^s}^{1/2}.$$  

**Proof of lemma 2.2.** For $R > 0$, we have

$$\|f\|_{X^{-1}} \leq \|f1_{\{|D|<R\}}\|_{X^{-1}} + \|f1_{\{|D|>R\}}\|_{X^{-1}}.$$  

Cauchy-Schwartz inequality gives

$$\|f1_{\{|D|<R\}}\|_{X^{-1}} = \int_{|\xi|<R} |\hat{f}(\xi)| \frac{1}{|\xi|^s} d\xi \leq \left( \int_{|\xi|<R} \frac{1}{|\xi|^s} d\xi \right)^{1/2} \|f\|_{L^2} \leq \sqrt{4\pi R^{2-s}} \|f\|_{H^s},$$  

and

$$\|f1_{\{|D|<R\}}\|_{X^{-1}} = \int_{|\xi|>R} \frac{1}{|\xi|^{s+1}} |\xi|^s |\hat{f}(\xi)| d\xi \leq \left( \int_{|\xi|>R} \frac{1}{|\xi|^{2s+2}} d\xi \right)^{1/2} \|f\|_{H^s} \leq \sqrt{4\pi} R^{2-s} \|f\|_{L^2},$$  

To conclude, it suffices to take $R = (\|f\|_{L^2})^{1/s}$. 
Remark 2.3. In the case $s = 1/2$ there is no comparison between $H^{1/2}(\mathbb{R}^3)$ and $\mathcal{X}^{-1}(\mathbb{R}^3)$. It suffices to consider the functions $f$ and $g$ defined as follows

$$f = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^{3/2}}1_{\{\xi|<1\}}\right) \quad \text{and} \quad g = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^{7/4}}1_{\{|\xi|>1\}}\right).$$

Indeed:

$$\|f\|_{\mathcal{X}^{-1}} = 4\pi \int_0^1 \frac{1}{r^{1/2}} dr = 8\pi, \quad \|f\|_{H^{1/2}}^2 = 4\pi \int_0^1 \frac{1}{r} dr = \infty$$

and

$$\|g\|_{\mathcal{X}^{-1}} = 4\pi \int_1^\infty \frac{1}{r^{3/4}} dr = \infty, \quad \|g\|_{H^{1/2}}^2 = 4\pi \int_1^\infty \frac{1}{r^{3/2}} dr = 8\pi.$$ 

3. Proof of Theorem 1.3

This proof is inspired from the work of Gallagher-Iftimie-Planchon in [6]. Let $\varepsilon > 0$, a sufficient condition on $\varepsilon$ is as follows

$$\varepsilon \leq \frac{\nu}{2}.$$ 

For $k \in \mathbb{N}$, put

$$\mathcal{A}_k = \{\xi \in \mathbb{R}^3; \quad |\xi| \leq k \quad \text{and} \quad |\hat{u}^0(\xi)| \leq k\}$$

Clearly $\mathcal{F}^{-1}(1_{\mathcal{A}_k}\hat{u}^0)$ converges to $u^0$ in $\mathcal{X}^{-1}(\mathbb{R}^3)$. Then, there is $k \in \mathbb{N}$ such that

$$\|u^0 - \mathcal{F}^{-1}(1_{\mathcal{A}_k}\hat{u}^0)\|_{\mathcal{X}^{-1}} < \varepsilon/2.$$ 

Put $v^0_k$ and $w^0_k$ as follows

$$v^0_k = \mathcal{F}^{-1}(1_{\mathcal{A}_k}\hat{u}^0) \quad \text{and} \quad w^0_k = u^0 - v^0_k.$$ 

Then $\|w^0_k\|_{\mathcal{X}^{-1}} < \varepsilon/2$ and $v^0_k \in \mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Now, consider the following system

$$(NS_k) \quad \left\{ \begin{array}{l}
\partial_t w - \nu \Delta w + w \cdot \nabla w = -\nabla p_{1,k}, \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div} \ w = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
w|_{t=0} = w^0_k \quad \text{in} \quad \mathbb{R}^3.
\end{array} \right.$$ 

As $\|w^0_k\|_{\mathcal{X}^{-1}} < \varepsilon/2 < \nu$ and by using Theorem 1.1 and inequality 1.2, we get a unique global solution $w_k$ of $(NS_k)$ such that $w_k \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+, \mathcal{X}^1(\mathbb{R}^3))$. Moreover,

$$\|w_k(t)\|_{\mathcal{X}^{-1}} + \frac{\nu}{2} \int_0^t \|w_k(z)\|_{\mathcal{X}^{-1}} dz \leq \|w^0_k\|_{\mathcal{X}^{-1}}, \quad \forall t \geq 0.$$ 

Put $v_k = u - w_k$, clearly $v_k \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3))$ and satisfies

$$\left\{ \begin{array}{l}
\partial_t v_k - \nu \Delta v_k + v_k \cdot \nabla v_k + w_k \cdot \nabla v_k + v_k \cdot \nabla w_k = -\nabla p_{2,k}, \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div} \ v_k = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
v_k|_{t=0} = v^0_k \quad \text{in} \quad \mathbb{R}^3.
\end{array} \right.$$ 

Taking the inner product in $L^2(\mathbb{R}^3)$ with $v_k$, we get

$$\frac{1}{2} \frac{d}{dt} \|v_k\|^2_{L^2} + \nu \|\nabla v_k\|^2_{L^2} \leq |\langle w_k \cdot \nabla v_k / v_k \rangle_{L^2}|.$$
To estimate the RHT,

\[
\langle w_k \cdot \nabla v_k / v_k \rangle_{L^2} = \langle \text{div} (w_k \otimes v_k) / v_k \rangle_{L^2} = \langle w_k \otimes v_k / \nabla v_k \rangle_{L^2} \leq \|w_k \otimes v_k\|_{L^2} \|\nabla v_k\|_{L^2} \leq \|\mathcal{F}(w_k \otimes v_k)\|_{L^2} \|\nabla v_k\|_{L^2} \leq \|w_k * \hat{v}_k\|_{L^2} \|\nabla v_k\|_{L^2}.
\]

Young inequality and Lemma 2.1 give

\[
\langle w_k \cdot \nabla v_k / v_k \rangle_{L^2} \leq \|\hat{w}_k\|_{L^1} \|\hat{v}_k\|_{L^2} \|\nabla v_k\|_{L^2} \leq \|w_k\|_{X^{-1}}^{1/2} \|w_k\|_{X^1}^{1/2} \|v_k\|_{L^2} \|\nabla v_k\|_{L^2}.
\]

Using inequality \(ab \leq \frac{a^2}{2} + \frac{b^2}{2}\), we get

\[
\langle w_k \cdot \nabla v_k / v_k \rangle_{L^2} \leq \frac{1}{2\nu} \|w_k\|_{X^{-1}} \|w_k\|_{X^1} \|v_k\|_{L^2}^2 + \frac{\nu}{2} \|\nabla v_k\|_{L^2}^2.
\]

and

\[
\frac{d}{dt} \|v_k\|_{L^2}^2 + \nu \|\nabla v_k\|_{L^2}^2 \leq \frac{1}{\nu} \|w_k\|_{X^{-1}} \|w_k\|_{X^1} \|v_k\|_{L^2}^2.
\]

Gronwall Lemma yields

\[
\|v_k\|_{L^2}^2 + \nu \int_0^t \|\nabla v_k\|_{L^2}^2 \leq \|v_k^0\|_{L^2}^2 e^{\frac{\nu}{2} \int_0^t \|w_k\|_{X^{-1}} \|w_k\|_{X^1}}.
\]

Using inequality (3.2), we get

\[
\|v_k\|_{L^2}^2 + \nu \int_0^t \|\nabla v_k\|_{L^2}^2 \leq \|v_k^0\|_{L^2}^2 e^{\frac{\nu}{2} \int_0^t \|w_k\|_{X^{-1}}^2}.
\]

Combining the above inequality and Lemma 2.2, we can deduce that \(v_k \in L^4(\mathbb{R}^+, X^{-1}(\mathbb{R}^3))\), and

\[
\int_0^\infty \|v_k\|_{X^{-1}}^4 \leq \int_0^\infty \|v_k\|_{L^2}^2 \|\nabla v_k\|_{L^2}^2 \leq \nu^{-1} \|v_k^0\|_{L^2}^2 e^{\frac{\nu}{2} \int_0^t \|w_k\|_{X^{-1}}^2}.
\]

By continuity of \(v_k\) in \(X^{-1}(\mathbb{R}^3)\), there is a time \(t_0\) such that \(\|v_k(t_0)\|_{X^{-1}} < \varepsilon/2\). Using equation (3.2), we get

\[
\|u(t_0)\|_{X^{-1}} \leq \|v_k(t_0)\|_{X^{-1}} + \|w_k(t_0)\|_{X^{-1}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Now, consider the Navier-Stokes system starting at \(t = t_0\) and using the global existence for the small initial data, we get

\[
\|u(t)\|_{X^{-1}} + (\nu - \varepsilon) \int_{t_0}^t \|\Delta u(\tau)\|_{X^{-1}} d\tau \leq \varepsilon, \forall t \geq t_0.
\]

Then, the desired result is proved.
4. Stability of global solutions

In this section we prove Theorem 1.4. This proof is done in two steps.

**Step 1:** Beginning by proving the following property: If $u$ is a maximal solution of $(NS)$ system with $u^0 \in \mathcal{X}^{-1}(\mathbb{R}^3)$ and $T^*$ is the maximal time of existence. We know that $u \in C([0, T^*); \mathcal{X}^{-1}(\mathbb{R}^3)) \cap L^1_{loc}([0, T^*), \mathcal{X}^1(\mathbb{R}^3))$. We have, if $T^* < \infty$ then

\[ \int_0^{T^*} \|u(t)\|_{\mathcal{X}^1} dt = \infty. \]

Indeed: Suppose that $\int_0^{T^*} \|u(t)\|_{\mathcal{X}^1} dt < \infty$. Let a time $T \in (0, T^*)$ such that $\int_T^{T^*} \|u(t)\|_{\mathcal{X}^1} dt < 1/2$. Lemma 2.1 gives, for all $t \in [T, T^*)$,

\[
\|u(t)\|_{\mathcal{X}^1} \geq \|u(T)\|_{\mathcal{X}^1} + \int_T^t \|u(s)\|_{\mathcal{X}^1} ds \\
\leq \|u(T)\|_{\mathcal{X}^1} + \frac{1}{2} \sup_{z \in [T, t]} \|u(z)\|_{\mathcal{X}^1}. \\
\]

We can deduce

\[ \|u(s)\|_{\mathcal{X}^1} \leq 2\|u(T)\|_{\mathcal{X}^1}, \forall s \in [T, T^*). \]

Let $M = \sup_{z \in [T, T^*)} \|u(z)\|_{\mathcal{X}^1} < \infty$. We have

\[ u(t') - u(t) = \nu \int_t^{t'} \Delta u - \int_t^{t'} \text{div} (u \otimes u) \]

Using Lemma 2.1 we get

\[
\|u(t') - u(t)\|_{\mathcal{X}^1} \leq \nu \int_t^{t'} \|u(s)\|_{\mathcal{X}^1} ds + \int_t^{t'} \|u(s)\|_{\mathcal{X}^1} \|u(s)\|_{\mathcal{X}^1} ds \\
\leq (\nu + M) \int_t^{t'} \|u(s)\|_{\mathcal{X}^1} ds \\
\]

where the RHT goes to zero as $t$ and $t'$ tends to $T^*$. Then $u(t)$ is a Cauchy type at $T^*$. As $\mathcal{X}^{-1}(\mathbb{R}^3)$ is Banach space, then there is an element $u^*$ in $\mathcal{X}^{-1}(\mathbb{R}^3)$ such that $u(t) \to u^*$ in $\mathcal{X}^{-1}(\mathbb{R}^3)$ if $t$ goes to $T^*$. Now, consider the Navier-Stokes system starting by $u^*$, using Theorem 1.2 we get a unique solution which extend $u$ beyond to $T^*$ which is absurd.

**Step 2:** Let $v \in C([0, T^*), \mathcal{X}^{-1}(\mathbb{R}^3))$ be the maximal solution of $(NS)$ corresponding to the initial condition $v^0$. We want to prove $T^* = \infty$. Beginning by using Theorem 1.2 we get $v \in L^1_{loc}([0, T^*), \mathcal{X}^1(\mathbb{R}^3))$. Put $w = v - u$ and $w^0 = v^0 - u(0)$. We have

\[ \partial_t w - \nu \Delta w + w.\nabla w + u.\nabla w + w.\nabla u = -\nabla P \]

or

\[ \partial_t w - \nu \Delta w + \text{div} (w \otimes w) + \text{div} (u \otimes w) + \text{div} (w \otimes u) = -\nabla P. \]
Then, for $t \in [0, T^*)$
\[\|w(t)\|_{\chi^{-1}} + \nu \int_0^t \|w(t)\|_{\chi^1} \leq \|w^0\|_{\chi^{-1}} + (I) + (II)\]

where
\[(I) = \int_0^t \|\text{div} (w \otimes w)\|_{\chi^{-1}}\]
\[(II) = \int_0^t \|\text{div} (u \otimes w)\|_{\chi^{-1}} + \|\text{div} (w \otimes u)\|_{\chi^{-1}}.\]

Lemma 2.1 gives
\[(I) \leq \int_0^t \|w \otimes w\|_{\chi^0} \leq \int_0^t \|w\|_{\chi^{-1}} \|w\|_{\chi^1},\]
and
\[(II) \leq \int_0^t \|u \otimes w\|_{\chi^0} + \|w \otimes u\|_{\chi^0} \leq 2 \int_0^t \|u\|_{\chi^0} \|w\|_{\chi^{-1}} \|w\|_{\chi^1} \leq \frac{4}{\nu} \int_0^t \|u\|_{\chi^0}^2 \|w\|_{\chi^{-1}} + \frac{\nu}{4} \int_0^t \|w\|_{\chi^1}.\]

Then
\[\|w(t)\|_{\chi^{-1}} + \frac{3\nu}{4} \int_0^t \|w(t)\|_{\chi^1} \leq \|w^0\|_{\chi^{-1}} + \int_0^t \|w\|_{\chi^{-1}} \|w\|_{\chi^1} + \frac{2}{\nu} \int_0^t \|u\|_{\chi^0}^2 \|w\|_{\chi^{-1}}.\]

Put
\[T = \sup\{t \in [0, T^*), \sup_{z \in [0, t]} \|w(z)\|_{\chi^{-1}} < \frac{\nu}{4}\}.\]

For $t \in [0, T)$, we have
\[\|w(t)\|_{\chi^{-1}} + \frac{\nu}{2} \int_0^t \|w(t)\|_{\chi^1} \leq \|w^0\|_{\chi^{-1}} + \frac{2}{\nu} \int_0^t \|u\|_{\chi^0}^2 \|w\|_{\chi^{-1}}.\]

Gronwall Lemma yields
\[\|w(t)\|_{\chi^{-1}} + \frac{\nu}{2} \int_0^t \|w(t)\|_{\chi^1} \leq \|w^0\|_{\chi^{-1}} e^2 f_0^t \|g\|_{\chi^1}^2 \leq \|w^0\|_{\chi^{-1}} e^{\frac{2}{\nu} f_0^T \|g\|_{\chi^1}^2} < \frac{\nu}{8}.\]

Then $T = T^*$ and $\int_0^{T^*} \|w(t)\|_{\chi^1} < \infty$, therefore $T^* = \infty$ and the proof is finished.

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