Non-compact Calabi-Yau Spaces and other Non-Trivial Backgrounds for Four-dimensional Superstrings

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ABSTRACT

A large class of new 4-D superstring vacua with non-trivial/singular geometries, spacetime supersymmetry and other background fields (axion, dilaton) are found. Killing symmetries are generic and are associated with non-trivial dilaton and antisymmetric tensor fields. Duality symmetries preserving N=2 superconformal invariance are employed to generate a large class of explicit metrics for non-compact 4-D Calabi-Yau manifolds with Killing symmetries.

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1 Introduction

Ricci-flat Kähler spaces [1], so-called Calabi-Yau spaces, provide consistent backgrounds [2] for the propagation of superstrings or heterotic strings. These backgrounds lead to target space supersymmetry and, consequently, the perturbative vacuum is guaranteed to be stable. Moreover, during recent years many of the Calabi-Yau backgrounds were shown to correspond to exact \( N = 2 \) superconformal field theories [3]. In the past the discussion mainly concentrated on flat four-dimensional Minkowski space-time tensored with a six-dimensional internal compact Kähler space without torsion and with constant dilaton field. We would like however, to construct supersymmetric string vacua with, in addition to the metric background, more general (non-constant) background fields. Moreover, to address certain important questions in quantum gravity one has to consider string backgrounds which describe four-dimensional curved and non-compact space-times. In particular, one is interested to construct exact superconformal field theories which correspond to four-dimensional black-hole backgrounds, cosmological or supersymmetric instanton type of solutions.

In this contribution we will report about a relatively systematic discussion [4] on supersymmetric string backgrounds with \( N = 2 \) or \( N = 4 \) superconformal symmetry, based on compact as well as non-compact spaces plus non-trivial antisymmetric tensor-field and non-constant dilaton. Thus, we will extend in a more systematic way the exact \( N=4 \) solution constructed recently, [5, 6]. In contrast to the compact Calabi-Yau spaces, almost all backgrounds with non-trivial dilaton field will possess Killing symmetries. Many such backgrounds exhibit singularities on some hypersurface in spacetime and can be regarded as a higher dimensional generalization of the two-dimensional black-hole considered in [1].

A key to the proper understanding of string propagation on curved spaces is provided by duality symmetries present in curved backgrounds [5, 7 10 11 12]. Duality symmetries relate different backgrounds which nevertheless correspond to the same (super)conformal field theory. The duality symmetries become manifest after a rearrangement in the Hilbert-space of the superconformal field theory and mean that stringy probes, when excited in different modes, see different geometries or topologies. Therefore, the concept of geometry or topology is not well defined in string theory. For the case of compact (Calabi-Yau) backgrounds, several very intriguing examples of stringy duality equivalences were found. The simplest example of duality symmetries is the \( R \rightarrow 1/R \) transformation for toroidal type of backgrounds [13] where \( R \) is the characteristic length scale of the compact space. Here the duality symmetry originates from the exchange of internal momentum and winding modes. A second, very interesting example where duality plays a central role is mirror symmetry for a general class of Calabi-Yau compactifications [14]. Mirror symmetry exchanges compact spaces of different topology but the same string physics. Duality symmetries also persist in several non-compact backgrounds, where momentum states are exchanged by oscillator type of excitations [1]. We expect that for non-compact spaces, the existence of the duality symmetries will
radically modify our understanding of space-time at least in regions of very strong curvature. In this contribution we will embark into finding four-dimensional backgrounds for superstring propagation. We will demand that the worldsheet theory has at least N=2 superconformal invariance in order to hope for spacetime supersymmetry (due to the presence of spectral flow which will pair the bosonic and fermionic spectrum). We will then analyze the one-loop $\beta$-function equations and find many interesting solutions (duality will be one of our tools). Among other things we will see that some interesting non-Kählerian $N=4$ solutions, which describe four-dimensional axionic instantons, are dual-equivalent to four-dimensional, non-compact Ricci-flat Kähler spaces.

2 The $N=2$ ($N=4$) Background and $U(1)$ Duality Transformations

The most general $N=2$ superspace action \[ S = \frac{1}{2\pi\alpha'} \int d^2x D_+ D_- D_- K(U, \bar{U}, V, \bar{V}). \] for $m$ chiral superfields $U_i$ ($i = 1, \ldots, m$) and $n$ twisted chiral superfields $V_p$ ($p = 1, \ldots, n$) in two dimensions is determined by a single real function $K(U, \bar{U}, V, \bar{V})$ (which we will henceforth call the quasi-Kähler potential):

\[ S = \int d^2x D_+ D_- D_- K(U, \bar{U}, V, \bar{V}). \] (2.1)

The fields $U_i$ and $V_p$ obey a chiral or twisted chiral constraint

\[ \bar{D}_\pm U_i = 0, \quad \bar{D}_\pm V_p = D_- V_p = 0. \] (2.2)

The action (2.1) is invariant, up to total derivatives, under quasi-Kähler gauge transformations

\[ K \rightarrow K + f(U, V) + g(U, \bar{V}) + \bar{f}(\bar{U}, \bar{V}) + \bar{g}(\bar{U}, V). \] (2.3)

To see the background interpretation of the theory it is convenient to write down the purely bosonic part of the superspace action (2.1):

\[ S = -\frac{1}{2\pi\alpha'} \int d^2x \left[ K_{u_i\bar{u}_j} \partial^a u_i \partial_a \bar{u}_j - K_{v_p\bar{v}_q} \partial^a v_p \partial_a \bar{v}_q \right. \]

\[ \left. + \epsilon_{ab} \left( K_{u_i\bar{u}_j} \partial_a u_i \partial_b \bar{u}_j + K_{v_p\bar{v}_q} \partial_a v_p \partial_b \bar{v}_q \right) \right]. \] (2.4)

where $K_{u_i\bar{u}_j} = \frac{\partial^2 K}{\partial U_i \partial \bar{U}_j}$, etc. Here $u_i$ is the lowest component of the superfield $U_i$ and so on. Thus, one recognizes that the first two terms in above equation describe the in general non-Kählerian metric background of the model (the metric is Kähler only when $m = 0$ or $n = 0$). The $\epsilon_{ab}$-term in (2.4) provides the antisymmetric tensor field background.

It follows that the field strength $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$ can also be expressed entirely in terms of the function $K$: $H_{u_i\bar{u}_j v_p} = \frac{\partial^3 K}{\partial U_i \partial \bar{U}_j \partial V_p}$, etc.

Of course, in order that these backgrounds provide consistent string solutions, they have to satisfy the string equation of motion, i.e. the vanishing of the $\beta$-function equations
Including also the dilaton background $\Phi(u_i, v_p)$, we obtain the following equations of motion for the background fields,

\begin{equation}
0 = \beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H^\lambda_{\mu\sigma} H_{\nu\lambda\sigma} + 2 \nabla_\mu \nabla_\nu \Phi + O(\alpha')
\end{equation}

\begin{equation}
0 = \beta^B_{\mu\nu} = \nabla_\lambda H^\lambda_{\mu\nu} - 2(\nabla_\lambda \Phi) H^\lambda_{\mu\nu} + O(\alpha').
\end{equation}

These equations will lead to some differential equations for the two functions $K$ and $\Phi$ as we will discuss in the following. Moreover, the central charge deficit $\delta c$ of the background is determined by the $\beta$-function of the dilaton field as

\begin{equation}
\delta c \equiv c - \frac{3D}{2} = \frac{3}{2} \alpha'[4(\nabla \Phi)^2 - 4 \nabla^2 \Phi - R + \frac{1}{12} H^2] + O(\alpha'^2).
\end{equation}

We must emphasize here that in the presence of N=4 superconformal symmetry the solution to the lowest order in $\alpha'$ is exact to all orders in a specific scheme, and $\delta c$ remains zero to all orders \cite{17}.

We consider without loss of generality the simplest case of a single $U(1)$ isometry (compatible with complex structure) by assuming that the potential $K$ has one Killing symmetry, $R = Z + \bar{Z}$:

\begin{equation}
K = K(Z + \bar{Z}, Y_i, \bar{Y}_i, V_p, \bar{V}_p)
\end{equation}

where $Z$ and $Y_i$ are chiral fields, whereas $V_p$ are twisted chiral fields. (Of course the discussion holds in the same way if $Z$ is a twisted chiral field.) In \cite{10,15} a duality transformation was described in which twisted superfields are interchanged with untwisted ones. Concretely, consider the ‘dual’ potential

\begin{equation}
\tilde{K}(R, Y_i, \bar{Y}_i, V_p, \bar{V}_p, \Psi + \bar{\Psi}) = K(Z + \bar{Z}, Y_i, \bar{Y}_i, V_p, \bar{V}_p) - R(\Psi + \bar{\Psi}),
\end{equation}

where $Z$ is a chiral field and $\Psi$ a twisted chiral field. Varying the action with respect to $\Psi$ gives back the original theory. On the other hand one can equally well consider the constraint coming from the variation with respect to $Z$, \cite{10}

\begin{equation}
\frac{\delta S}{\delta Z} = 0 \rightarrow \frac{\partial K}{\partial r} - (\psi + \bar{\psi}) = 0,
\end{equation}

and the dual theory is obtained as a Legendre transform of $K$. Now the independent variable are $\psi$, $y_i$ and $v_p$. It follows that the dual metric has the following form:

\begin{equation}
\tilde{G}_{\mu\nu} = \begin{pmatrix}
0 & -\tilde{K}_{\psi\bar{\psi}} & 0 & 0 & 0 & -\tilde{K}_{\psi\bar{v}_q} \\
-\tilde{K}_{\psi\bar{\psi}} & 0 & 0 & 0 & -\tilde{K}_{v_p\bar{\psi}} & 0 \\
0 & 0 & 0 & \tilde{K}_{y_i\bar{y}_j} & 0 & 0 \\
0 & 0 & \tilde{K}_{y_i\bar{y}_j} & 0 & 0 & 0 \\
0 & -\tilde{K}_{v_p\bar{\psi}} & 0 & 0 & 0 & -\tilde{K}_{v_p\bar{v}_q} \\
-\tilde{K}_{\psi\bar{v}_q} & 0 & 0 & 0 & -\tilde{K}_{v_p\bar{v}_q} & 0
\end{pmatrix}.
\end{equation}
Similarly, the dual antisymmetric tensor field is obtained as

\[ B_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & \tilde{K}_{\psi\bar{y}_i} & 0 & 0 \\
0 & 0 & \tilde{K}_{y_i\bar{\psi}} & 0 & 0 & 0 \\
0 & -\tilde{K}_{y_i\bar{\psi}} & 0 & 0 & 0 & \tilde{K}_{y_i\bar{v}_p} \\
-\tilde{K}_{\psi\bar{y}_i} & 0 & 0 & 0 & \bar{K}_{v_p\bar{y}_i} & 0 \\
0 & 0 & 0 & -\tilde{K}_{v_p\bar{y}_i} & 0 & 0 \\
0 & 0 & -\tilde{K}_{y_i\bar{v}_p} & 0 & 0 & 0
\end{pmatrix}. \quad (2.11) \]

Moreover, the dual dilaton field has the form

\[ 2\tilde{\Phi} = 2\Phi - \log 2K_{rr}. \quad (2.12) \]

It can be shown \([10, 4]\) that this N=2 duality transformation via Legendre transform is the same as the usual abelian duality transformation \([8]\).

### 3 Kähler Spaces without Torsion and their Duals

It is already well known that if the torsion vanishes and there is no dilaton field the condition that a \(\sigma\)-model has N=2 supersymmetry is that the target space is Kähler, \([4]\). Thus, for the time being, we start with a Kähler manifold specified (locally) by its Kähler potential \(K(u_i, \bar{u}_i)\) and a dilaton field \(\Phi\). The metric is given in terms of the Kähler potential by the standard formula

\[ G_{ij} = G_{i\bar{j}} = 0 \quad , \quad G_{i\bar{j}} = K_{u_i\bar{u}_j}. \quad (3.1) \]

It is obvious that the metric is invariant under the so called Kähler transformations of the potential

\[ K(u_i, \bar{u}_i) \rightarrow K(u_i, \bar{u}_i) + \Lambda(u_i) + \bar{\Lambda}(\bar{u}_i) \quad (3.2) \]

Then the Ricci-tensor takes its well-known form

\[ R_{u_i\bar{u}_j} = -\partial_{u_i}\partial_{\bar{u}_j}U \quad , \quad R_{u_iu_j} = R_{\bar{u}_i\bar{u}_j} = 0 \quad (3.3) \]

with \(U = \log \det K_{u_i\bar{u}_j} = \frac{1}{2} \log \det G\).

The only condition for conformal invariance is \(\beta^G_{\mu\nu} = 0\) which here implies

\[ \Phi = \frac{1}{2}U + f(u_i) + \bar{f}(\bar{u}_i), \quad (3.4) \]

and

\[ \nabla_{u_i}\partial_{u_j}\Phi = \nabla_{u_i}\partial_{\bar{u}_j}\Phi = 0. \quad (3.5) \]

where \(f\) is an arbitrary holomorphic function. In addition we get

\[ \delta c = \frac{3}{2} \alpha K^{u_i\bar{u}_j}(8\partial_{u_i}\Phi\partial_{\bar{u}_j}\Phi - 4\partial_{u_i}\partial_{\bar{u}_j}\Phi). \quad (3.6) \]
If one demands for enlarged $N = 4$ world-sheet supersymmetry, this implies that the Kählerian space has to be hyper-Kähler. If the theory is also positive, then $\delta c = 0$ from CFT arguments. In such a case the Riemann tensor is self-dual and therefore the space is Ricci-flat. Ricci flatness and $\delta c = 0$ implies constant dilaton. However, we note that the hyper-Kähler condition is not the only way to obtain $\delta c = 0$; in fact we will provide $N = 4$ examples which are non-Ricci-flat and have non-constant dilaton field. These examples will be presented later.

As described in [4] it is not difficult to show that the vanishing of the holomorphic double derivative on the dilaton implies, for non-trivial dilaton, that there is a generic Killing symmetry in the Kähler metric as well as in the dilaton. Then, in a special coordinate system the compatibility of the equations (3.4) and (3.3) along with our freedom to perform Kähler transformations implies that

$$K = K(z + \bar{z}, y_i, \bar{y}_i), \quad \Phi = \partial_z K = \partial_{\bar{z}} K$$

and

$$\Phi = \frac{1}{2} U + C(z + \bar{z})$$

where $C$ is any real number. We can take (3.7) as the equation specifying the dilaton in terms of the metric and then (3.8) becomes a non-linear differential equation for the Kähler potential

$$\det[K_{u_i\bar{u}_j}] = \exp[-2C(z + \bar{z}) + K + K]$$

generalizing the CY condition.

In the same coordinate system we can also compute the central charge deficit:

$$\delta c = 12\alpha' C.$$  

Let us consider a special class of solutions which can be regarded as the generalization of the two-dimensional black hole backgrounds found in [7]. Specifically, assume that the model has a $U(N)$ isometry, i.e.

$$K = K(x), \quad \Phi = \Phi(x) \quad x = \sum_{i=1}^{N} |u_i|^2.$$  

The general form of the metric is then

$$K_{u_i\bar{u}_j} = K''\delta_{ij} + K''\bar{u}_i u_j, \quad K' = \frac{\partial K}{\partial x}.$$  

For $N > 1$, the linear term in the dilaton, eq. (3.8), is not allowed by the $U(N)$ isometry and the dilaton field becomes

$$\Phi = \frac{1}{2} U = \frac{1}{2} \log[(K')^N - (K' + xK'')]$$

Let us define the following function:

$$Y(x) = xK'(x).$$  

Now we have to insert the ansatz eqs. (3.12, 3.13) into the field equation (3.5), and the solution of this equation takes the following form:

\[ e^Y \sum_{m=0}^{N-1} \frac{(-1)^m Y^m}{m!} = A + Bx^N. \]  

(3.15)

Here A and B are arbitrary parameters. From this we immediately obtain

\[ Y' = BN!(-1)^{N-1}e^{-Y}Y^{1-N}x^{N-1}. \]  

(3.16)

Then the dilaton, eq. (3.13), can be also expressed entirely of Y as

\[ \Phi = \frac{1}{2}U = -\frac{1}{2}Y + \text{const.} \]  

(3.17)

The Ricci tensor becomes

\[ R_{u\bar{u}j} = -\partial_u \partial_{\bar{u}} U = Y' \delta_{ij} + Y'' \bar{u}_i u_j. \]  

(3.18)

The scalar curvature can be computed to be

\[ R = 2(N - xY') = 2(N - f(Y)). \]  

(3.19)

with

\[ f(Y) = Y^{1-N} \sum_{m=0}^{N-1} \frac{(-1)^m Y^m}{m!} - Ae^{-Y} \]  

(3.20)

Finally, the central can be expressed as

\[ \delta c = 3N\alpha'. \]  

(3.21)

The explicit form of the scalar curvature, eq. (3.19), allows us to discuss the asymptotic behavior and the singularity structure of our class of solutions. First we recognize that the 2N-dimensional Kähler space has zero scalar curvature for \( Y \to \infty \). Second, for all N, the scalar curvature possesses a generic singularity for \( Y \to -\infty \). Moreover, there is another singularity at \( Y = 0 \) for \( N > 1 \) if \( A \neq 1 \). On the other hand, if \( A = 1 \), the scalar curvature is regular at \( Y = 0 \) and becomes \( R(Y = 0, A = 1) = 2N \).

For the simplest case, namely two-dimensional backgrounds with \( N = 1 \), the solution (3.13) reproduces the well-known backrounds which correspond to the exact conformal field theories \( SL(2,R)/U(1) \) and \( SU(2)/U(1) \) respectively depending on the choice of the parameters A and B. Moreover the duality transformation (2.8) [18] exactly corresponds to axial versus vector gauging of \( U(1) \) in the corresponding conformal field theory [9, 19].

Let us now consider four-dimensional backgrounds which are not direct products of two-dimensional spaces. Specifically we consider solutions of the form eq. (3.15) with \( N = 2 \). One has to emphasize that so far it is not known to us which exact supercoformal
field theory might correspond to this type of backgrounds. Using $Y$ together with the overall phase $\theta$ as (real) coordinates, the metric then reads:

$$
\begin{align*}
\text{d}s^2 &= \frac{(\text{d}Y)^2}{4f(Y)} + \frac{f(Y)}{4} \left(\text{d}\theta - i \frac{\bar{y} \text{d}y - y \text{d}\bar{y}}{1 + y \bar{y}}\right)^2 \\
&\quad + \frac{Y}{(1 + y \bar{y})^2} \text{d}y \text{d}\bar{y}, 
\end{align*}
$$

(3.22)

This metric in the $(\theta, \psi, \phi)$ subspaces is a deformation of the fibration of $S^3$ over $S^2$, whose line element is manifest in (3.22).

Now we will analyse the structure of the Euclidean manifold as a function of $A$ and $B$. We need some asymptotics of the function $f(Y)$:

$$
\begin{align*}
f(\infty) &= 2, \quad f(-\infty) = -\text{Sign}[A] \times \infty \quad (3.23) \\
f(0^+) &= -f(0^-) = \text{Sign}[A - 1] \times \infty \quad (3.24)
\end{align*}
$$

Since $x^2$ must be positive, we are dealing with the following cases:

1) $A > 1, B < 0$. There are two manifolds, the first with $Y \geq 0, x^2 > (1 - A)/B$ with signature $(4,0)$ and a curvature singularity at $Y = 0$ and the second with $Y \leq 0, (1 - A)/B \leq x^2 \leq -A/B$ with signature $(0,4)$ and curvature singularities at $Y = 0, -\infty$.

2) $A = 1, B < 0$. There is a regular Euclidean $(4,0)$ manifold for $Y > 0, x^2 > 0$ and a singular (at $Y = -\infty$) Euclidean $(0,4)$ manifold for $Y < 0, 0 \leq x^2 \leq -1/B$.

3) $0 < A < 1$. In this case $f(Y)$ has a positive and a negative zero which we will denote by $Y_{\pm}$: $f(Y_{\pm}) = 0$. For $B < 0$ there is again a regular (finite curvature) Euclidean manifold for $Y > Y_+$ with signature $(4,0)$ and another with $Y < Y_-$ with signature $(0,4)$ and a curvature singularity at $Y = -\infty$. For $Y_- < Y < Y_+$ and $B > 0$ there is another singular manifold with signature $(2,2)$.

4) $A \leq 0$. In this case $f$ has a single positive zero, $Y_+$. For $B < 0$ and $Y > Y_+$ we have a regular manifold with signature $(4,0)$. For $B > 0$ and $Y < Y_+$ there is a singular manifold with signature $(2,2)$.

Applying eq. (2.8), the dual metric is given as

$$
\text{d}\tilde{s}^2 = \frac{\text{d}\psi \text{d}\bar{\psi}}{f(Y)} + \frac{Y}{(1 + y \bar{y})^2} \text{d}y \text{d}\bar{y},
$$

(3.25)

whereas the dual dilaton and antisymmetric tensor field are as follows $\tilde{\Phi} = -\frac{1}{2} \log[e^Y f(Y)]$, $\tilde{B}_{\psi \bar{y}} = 2y/(1 + y \bar{y})$. The dual scalar curvature becomes

$$
\tilde{R} = \frac{2Y^2 (f f'' - f'^2) + f^2 + 4Y f}{Y^2 f}
$$

(3.26)

Thus, for the dual space, $\tilde{R} = 0$ for $Y \to \infty$, and there are curvature singularities at $Y = -\infty, 0$ and, for generic values of $A$ at the zeros of $f(Y)$ (e.g. for $A = -1$, at $Y \approx 1.3$).
4 Four-dimensional Non-Kählerian Spaces with Torsion and their Duals

The discussion in the case of non-vanishing antisymmetric tensor fields will be restricted to the simplest non-trivial case, namely four-dimensional target spaces, i.e. $m = n = 1$. In that case it can be shown that the solutions fall into three mutually exclusive classes:

i) Solutions whose quasi-Kähler potential satisfies the ordinary Laplace equation,

$$ (\partial_a \partial_{\bar{a}} + \partial_{\bar{b}} \partial_b) K = 0. \tag{4.1} $$

and the dilaton field is simply given as

$$ 2\Phi = \log K_{u\bar{u}} + \text{constant}. \tag{4.2} $$

ii) Solutions with one isometry whose quasi-Kähler potential satisfies

$$ K_{ww} = K_{u\bar{u}} e^{-K_w + c_1 (w + \bar{w}) + c_2}. \tag{4.3} $$
in a special coordinate system and

$$ 2\Phi = \log(K_v\bar{v}) - c_1 (w + \bar{w}) - c_2 \tag{4.4} $$

ii) Solutions with two isometries whose quasi-Kähler potential satisfies

$$ K_{ww} e^{K_w + c_2 (w + \bar{w})} = K_{zz} e^{K_z + c_1 (z + \bar{z})}. \tag{4.5} $$
in a special coordinate system, and

$$ 2\Phi = \log K_{zz} - K_w - c_1 (z + \bar{z}) + \text{constant}. \tag{4.6} $$

In the above $c_1, c_2$ are constants.

In the following we will focus on the solutions of case (i). Eqs.(4.1) and (4.2) imply that $\delta c = 0$ and these backgrounds are expected to have $N = 4$ superconformal symmetry. This observation is consistent with the fact that eq.(4.1) is the generalization of the hyper-Kähler condition for spaces with antisymmetric tensor field. The form of the dilaton field has the important consequence that the four-dimensional metric in the Einstein frame is flat: $G_{\mu\nu}^{\text{Einstein}} = e^{-2\Phi} G^{\sigma}_{\mu\nu} = \delta_{\mu\nu}$. In fact, the solutions of the dilaton equation (4.2) are the type II versions of the axionic solutions of [20]:

$$ d\Phi = \pm \frac{1}{2} e^{-2\Phi} H^* \tag{4.7} $$

This relation is nothing else than the self-duality condition on the dilaton-axion field. Its solutions are known as axionic instantons. All these solutions leave spacetime supersymmetry unbroken. In particular it can be shown that one of the solutions in this class
(which in its heterotic version was identified with a magnetic monopole background \[21\]),
turns out to be a dual of flat space \[4\].

The form of the solutions of the Laplace equation depends on the number of isometries
of the theory (which are compatible with the complex structure). In the case with two
translational \(U(1)\) Killing symmetries, i.e. \(K = K(u + \bar{u}, v + \bar{v})\) the most general solution
of (4.1) looks like

\[ K = iT(u + \bar{u} + i(v + \bar{v})) - i\bar{T}(u + \bar{u} - i(v + \bar{v})). \]  

(4.8)

In the case with one translational isometry the general solution becomes

\[ K(u + \bar{u}, v, \bar{v}) = i \int d\beta T(\beta, v + \beta(u + \bar{u}) - \beta^2 \bar{v}) + c.c. \]  

(4.9)

where in both (4.8), (4.9) \(T\) is an otherwise arbitrary function.

Let us now construct the dual spaces for the solutions of the Laplace equation with
one or two isometries, (4.8,4.9). We will perform a duality transformation on the chiral
\(U\)-field replacing it by a twisted chiral field \(\Psi\). The Legendre transformed potential \(\tilde{K}\)
will only contain twisted fields and will be therefore a true Kähler function leading to a
non-compact Kähler space without torsion.

Doing the Legendre transform we obtain the following line element

\[ ds^2 = \frac{1}{K_{uu}}(dz - K_{uv}dv)(d\bar{z} - K_{u\bar{v}}d\bar{v}) - K_{v\bar{v}}dv d\bar{v} \]  

(4.10)

where \(K(u + \bar{u}, v, \bar{v})\) is the original quasi-Kähler potential that satisfies the Laplace
equation \(K_{uu} + K_{v\bar{v}} = 0\) and \(z, \bar{z}\) are the dual coordinates defined via the Legendre
transform \(z + \bar{z} = K_u\). The coordinates \(v, \bar{v}, z, \bar{z}\) are now the Kähler coordinates. The
Laplace equation implies that the determinant of the Kähler metric (4.10) is constant so
we obtain a Ricci flat Kähler manifold. The dual dilaton is consequently constant.

The general solution to the 4-d Laplace equation with one isometry can be written
as in (4.9). Let us introduce the notation

\[ < T > \equiv \int d\beta T(\beta, v + \beta(u + \bar{u}) - \beta^2 \bar{v}). \]  

(4.11)

and the function

\[ Z(u + \bar{u}, v, \bar{v}) = K_u = i < \beta(T_v - \bar{T}_{\bar{v}}) > \]  

(4.12)

and we should remember that \(z + \bar{z} = Z(u + \bar{u}, v, \bar{v})\). Then the line element (4.10) can
be written in the form

\[ ds^2 = \frac{1}{G}(dz - A_v dv)(d\bar{z} - \bar{A}_{\bar{v}} d\bar{v}) + Gdv d\bar{v} \]  

(4.13)

where,

\[ G = \frac{\partial Z}{\partial u}, \quad A_v = \frac{\partial Z}{\partial v}, \quad \bar{A}_{\bar{v}} = \frac{\partial Z}{\partial \bar{v}} \]  

(4.14)
The interpretation of the metric (4.13) is as follows: The $Gdvd\bar{v}$ part describes the metric of a 2-d Riemann surface (generically non-compact). The metric depends also on $z + \bar{z}$. For fixed $z + \bar{z}$, $A_v, \bar{A}_\bar{v}$ describe a flat line bundle on the Riemann surface. The metric (4.13) is that of a flat complex line bundle on the Riemann surface. The functions $G, A_v, \bar{A}_{\bar{v}}$ are harmonic.

The metric (4.13) describes a large class of 4-d non-compact Calabi-Yau manifolds, which are also hyper-Kähler. The associated $\sigma$-models have $N=4$ superconformal symmetry and $c = 6$ ($\tilde{c} = 2$). The manifolds have generically asymptotically flat regions as well as curvature singularities.

Let us briefly display a simple example choosing
\[ T = -i\gamma(\beta)e^{u+\bar{u}+\bar{\beta}-\beta}. \] (4.15)
Then the potential becomes
\[ K = e^{u+\bar{u}}\phi(v, \bar{v}), \quad \phi(v, \bar{v}) = \int d\beta \gamma(\beta)[e^{\bar{\beta}-\beta} + e^{\bar{\beta}-\beta}]. \] (4.16)
In turn, the dual space is determined by the following Kähler potential:
\[ \tilde{K} = (z + \bar{z}) \log(\psi + \bar{\psi}) - (\psi + \bar{\psi}) \log \phi(v, \bar{v}). \] (4.17)
The integral in (4.16) can be explicitly performed if we choose $\gamma(\beta) = e^{\frac{\alpha}{\beta}} \beta^{-1}$:
\[ \phi(v, \bar{v}) = \text{constant} \left( \sqrt{\frac{A-v}{\bar{v}}} \right)^\nu K_\nu(2\sqrt{A-v}\bar{v}) + \text{h.c.} \] (4.18)
Here $K_\nu$ is the Bessel function with complex argument.

Let us study now the (more symmetric) special case of (4.13) with two isometries, i.e. $K(u + \bar{u}, v + \bar{v})$. If we parametrize, $u = r_1 + i\theta, v = r_2 + i\phi$ then K is of the form $K(r_1, r_2) = iT(r_1+ir_2) - i\bar{T}(r_1-ir_2)$. Introducing a new complex coordinate $z = r_1 + ir_2$, we can write the metric (4.13) in the following suggestive form
\[ ds^2 = \frac{ImT}{2}dzd\bar{z} + \frac{2}{ImT}(d\theta + Td\phi)(d\theta + \bar{T}d\phi) \] (4.19)
where T(z) is an arbitrary meromorphic function. It is crucial to note that the metric (4.19) is not written in Kähler coordinates. Such coordinates are $v, \bar{v}$ and $w, \bar{w}$ with $w + \bar{w} = iT'(r_1 + ir_2) - i\bar{T}'(r_1 - ir_2)$ and $w - \bar{w} = 2i\theta$.

Now the interpretation of the metric (4.19) is straightforward: If we take $\theta, \phi$ to be angular variables, then they parametrize a 2-d torus, with modulus $T(z)$ which depends holomorphically on the rest of the coordinates and conformal factor proportional to $1/ImT$. The zeros and poles of the Riemann tensor are determined by the zeros and poles (or essential singularities) of the function $T(z)$.

This solution (with a different interpretation) was found in [22], where some global issues were also addressed\footnote{Some generalizations of this idea to more dimensions were recently presented in [23].}. We should note that as in [22] a full invariance under the
torus modular group, \( T \to T + 1 \) and \( T \to -1/T \) can be implemented by a holomorphic
cordinate transformation in \( z \), which will modify \( ImT \) to a modular invariant in the
first part of (4.19). It was also argued that such a metric might receive higher order
corrections. However we have just shown that this metric is the dual of the family of
wormhole solutions which are absolutely stable as CFTs due to their N=4 superconformal
ymetry and it does possess a hyper-Kähler structure although not easily visible in this
ordinate system.

The 4-d non-compact CY manifolds presented in this section constitute a large class of
exact solutions to superstring theory with extended supersymmetry. A detailed analysis
of their structure as well as their potential Minkowski continuations is beyond the scope
of this work and is reserved for future study.

5 Conclusions

We have examined some four-dimensional superconformal theories with N=2 and N=4
uperconformal symmetry (classical solutions to superstring theory). We show that there
exists a plethora of such theories with non-trivial metric, dilaton and antisymmetric
tensor field.

Our solutions are classified in two classes: (i) Those that are based on a Kähler
manifold (when \( H_{\mu\nu\rho} = 0 \)). (ii) Non-Kählerian solutions with non-zero torsion. These
wo subclasses are related by \( Z_2 \) duality transformations (when isometries are present).
\( Z_2 \) duality interchanges the roles of untwisted and twisted chiral superfields and act in a
manif N=2 preserving fashion.

In the Kählerian case we show that the presence of a non-trivial dilaton field implies
the presence of an isometry in the background data (Kähler metric and dilaton). Among
the Kählerian solutions we find a large class of (non-compact) Ricci-flat (CY) manifolds
ith one isometry. This class of solutions generalizes the compact 4-d Ricci flat manifolds
(K3). A special case of the solutions above (with two isometries) is that of ref. [22] found
in a slightly different context. These CY manifolds are duals of non-zero torsion solutions
ith N=4 superconformal symmetry.

Let us finally emphasize that it is a very interesting problem to find the exact \( N = 2 \)
nd \( N = 4 \) superconformal field theories which correspond to our general solutions. Upon
alytic continuation of the Euclidean solutions we expect to obtain many cosmological
olutions to superstring theory whose spacetime properties deserve further study.

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