Multilevel Picard approximations of high-dimensional semilinear partial differential equations with locally monotone coefficient functions

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Abstract

The full history recursive multilevel Picard approximation method for semilinear parabolic partial differential equations (PDEs) is the only method which provably overcomes the curse of dimensionality for general time horizons if the coefficient functions and the nonlinearity are globally Lipschitz continuous and the nonlinearity is gradient-independent. In this article we extend this result to locally monotone coefficient functions. Our results cover a range of semilinear PDEs with polynomial coefficient functions.

1 Introduction

High-dimensional second-order partial differential equations (PDEs) are abundant in many important areas including financial engineering, economics, quantum mechanics, statistical physics, etc; see e.g. the surveys [22, 1]. The challenge in the numerical approximation of high-dimensional nonlinear PDEs lies in the possible curse of dimensionality by which we mean that the complexity of the problem goes up exponentially as a function of the dimension or of the inverse prescribed accuracy. Most approximation methods of nonlinear PDEs suffer from this curse, including sparse grid methods (e.g., [62]), sparse polynomial approximation (e.g., [14]), and BSDE-methods (e.g., [2, 11, 28, 9, 24, 63, 10, 12, 30]; see also the literature discussion in [19]). Branching diffusion approximations do not suffer from the curse; see, e.g., [61, 33, 36, 35]. However, these approximations are only applicable for small time horizons and small terminal conditions; see also the discussion in [19, Section 4.7]. Recently various deep learning-based methods have been proposed for numerical approximations of PDEs; see, e.g., [3, 4, 8, 13, 18, 21, 23, 25, 29, 34, 37, 55, 56, 57, 58, 59, 60] or the overview article [7]. There is empirical evidence that these deep learning-based methods work well at least for medium prescribed accuracies; see, e.g., the simulations in [13, 18, 32, 4, 3]. However, stochastic optimization methods may get trapped in local minima and there exists no theoretical convergence result; cf., e.g., [17].

Key words and phrases: curse of dimensionality, high-dimensional PDEs, multilevel Picard approximations, multilevel Monte Carlo method, locally monotone, tamed Euler-type approximation

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To the best of our knowledge the only approximation method which has been mathematically proved to overcome the curse of dimensionality for certain semilinear PDEs is the full history recursive multilevel Picard (MLP) method introduced in [20] and analyzed, e.g., in [45, 48, 53, 46, 51, 6, 50, 27, 6]. In this article we extend the analysis of MLP approximations to the case of semilinear PDEs with locally monotone coefficient functions and globally Lipschitz continuous, gradient-independent nonlinearities. The case of locally monotone coefficients is particularly important since many equations from applications satisfy such a condition; see, e.g., [15, Section 4]. Building on the analysis of the case of Lipschitz coefficients in [48], the nonlinearity (f in (5)) is not difficult to deal with. So we focus now on linear PDEs (5) with \( f \equiv 0 \). Linear PDEs with locally monotone coefficient functions have been approximated in the literature with essentially optimal rate; see, e.g., [41] in combination with the multilevel Monte Carlo method in [26]. However, the analysis in [41] is not explicit in the dimension and it remained unclear under which conditions it is possible to approximate linear PDEs with locally monotone coefficients without curse. The key contribution of this article is to derive explicit error bounds so that dependencies on the dimension become clear. In particular, we observe that on the right-hand side of the one-sided linear growth condition (2) below the additive part may grow polynomially in the dimension whereas it is sufficient to assume that the prefactor of \(|x|^2\) is bounded in the dimension. The following theorem illustrates our main result, Theorem 3.1 below, in the case of coefficient functions which satisfy the global monotonicity condition.

**Theorem 1.1.** Consider the notation in Subsection 1.1, let \( T, \delta \in (0, \infty) \), \( b, c, \beta, \eta \in [1, \infty) \), let \( f : \mathbb{R} \to \mathbb{R} \) be globally Lipschitz continuous, for every \( d \in \mathbb{N} \) let \( \mu_d \in C(\mathbb{R}^d, \mathbb{R}^d) \), \( \sigma_d = (\sigma_{d,1}, \ldots, \sigma_{d,d}) \in C(\mathbb{R}^d, \mathbb{R}^{d \times d}) \), \( g_d \in C(\mathbb{R}^d, \mathbb{R}) \), \( u_d \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \), for every \( d \in \mathbb{N} \), \( h \in (0, T] \) let \( D_h^d \subseteq \mathbb{R}^d \) satisfy that \( D_h^d = \{ x \in \mathbb{R}^d : h \leq 0.5, d_{BA}(\|x\|^2 + d^2)^{1/2} \leq \exp((\text{ln}(h))^{1/2}) \} \), assume for all \( d \in \mathbb{N} \) that \( \sup_{t \in [0,T], x \in \mathbb{R}^d} \frac{|\partial_t u_d(t,x)|}{\sqrt{\text{Var}(u_d(t,x))}} < \infty \), assume for all \( d \in \mathbb{N} \), \( x, y \in \mathbb{R}^d \), \( t \in [0, T], h \in (0, T) \) that

\[
\begin{align*}
&\langle x - y, \mu_d(x) - \mu_d(y) \rangle + 3 \| \sigma_d(x) - \sigma_d(y) \|^2 \leq c \| x - y \|^2 \\
&\langle x, \mu_d(x) \rangle + \frac{16\beta - 1}{2} \| \sigma_d(x) \|^2 \leq c \| x \|^2 + d^2 \\
&\| g_d(x) - g_d(y) \| + \| \mu_d(x) - \mu_d(y) \| + \| \sigma_d(x) - \sigma_d(y) \| \leq c d^\beta \| x - y \| \left( \| x \|^2 + \| y \|^2 + d^2 \right)^\beta, \\
&\max \{ \| g_d(x) \|, \| \mu_d(x) \|, \| \sigma_d(x) \| \} \leq c d^\beta \| x \|^2 + d^2 \beta, \\
&(\frac{2}{\sqrt{d}} u_d(t,x) + \langle \mu_d(x), (\nabla_x u_d)(t,x) \rangle) + \frac{1}{2} \text{tr} (\sigma_d(x)(\sigma_d(x))^*(\text{Hess}(u_d))(t,x)) = -f(u_d(t,x)),
\end{align*}
\]

and \( u_d(T,x) = g_d(x) \), let \( \Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n \), let \( (\Omega, \mathcal{F}, \mathbb{F}, (\mathbb{F}_t)_{t \in [0,T]}) \) be a filtered probability space which satisfies the usual conditions\(^\dagger\), let \( \mathbf{r}^d : \Theta \to [0, 1] \), \( \theta \in \Theta \), be independent random variables which are uniformly distributed on \([0, 1]\), let \( W^d, \Theta : [0, T] \times \Theta \to \mathbb{R}^d \), \( d \in \mathbb{N} \), \( \theta \in \Theta \), be independent standard \((\mathbb{F}_t)_{t \in [0,T]}\)-Brownian motions with continuous sample paths, assume that \( \sigma(\{\mathbf{r}^d : \theta \in \Theta\}) \) and \( \sigma(\{W_{t,\delta}^d : d \in \mathbb{N}, \theta \in \Theta, t \in [0, T]\}) \) are independent, for every \( d \in \mathbb{N} \), \( \theta \in \Theta \), \( t \in (0, T) \), \( x \in \mathbb{R}^d \) let \( (Y_{t,s}^d,N,\Theta(x,\omega))_{s \in [t,T], \omega \in \Omega} : [t, T] \times \Theta \to \mathbb{R}^d \) satisfy for all \( k \in \{0, 1, \ldots, N\} \), \( s \in (\frac{t}{N}, (k+1)\frac{T}{N}] \cap (t, T) \) that

\[
Y_{t,s}^{d,N,\Theta}(x) = Y_{t,\max\{t,\frac{kT}{N}\}}^{d,N,\Theta}(x) + \mathbb{1}_{D_{\delta}^d}(\mathbf{r}^d_{t,max\{t,\frac{kT}{N}\}}) \sigma_d(Y_{t,\max\{t,\frac{kT}{N}\}}^{d,N,\Theta}(x)) + \sqrt{d} \sigma_d(Y_{t,\max\{t,\frac{kT}{N}\}}^{d,N,\Theta}(x)) \left( W_{t,\delta}^d - W_{t,\delta}^d \right)_{\max\{t,\frac{kT}{N}\}} \right]
\]

\[
+ \frac{\sigma_d(Y_{t,\max\{t,\frac{kT}{N}\}}^{d,N,\Theta}(x)) (W_{t,\delta}^d - W_{t,\delta}^d \max\{t,\frac{kT}{N}\})^2}{1 + \left( \sigma_d(Y_{t,\max\{t,\frac{kT}{N}\}}^{d,N,\Theta}(x)) (W_{t,\delta}^d - W_{t,\delta}^d \max\{t,\frac{kT}{N}\})^2 \right)^\beta},
\]

let \( V_{n,M} : [0, T] \times \mathbb{R}^d \times \Theta \to \mathbb{R} \), \( d, n, M \in \mathbb{N}_0 \), \( \theta \in \Theta \), satisfy for all \( d, M \in \mathbb{N} \), \( n \in \mathbb{N}_0 \), \( \theta \in \Theta \),

\(^\dagger\)Let \( T \in (0, \infty) \) and let \( \Omega = (\Omega, \mathcal{F}, \mathbb{F}, (\mathbb{F}_t)_{t \in [0,T]}) \) be a filtered probability space. Then we say that \( \Omega \) satisfies the usual conditions if and only if it holds for all \( t \in [0, T] \) that \( \{ A \in \mathcal{F} : \mathbb{P}(A) = 0 \} \subseteq \mathbb{F}_t \subseteq \cap_{s \in [t,T]} \mathbb{F}_s \).
$t \in [0, T]$, $x \in \mathbb{R}^d$

$$V_{n,M}^{d,\theta}(t, x) = \frac{n}{M} \sum_{i=1}^M g_d(\gamma_{t,T}^{d, M, \theta, 0, -i, x})$$

$$+ \sum_{\ell=0}^{n-1} \left[ \frac{(T-t)}{M} \sum_{i=1}^{M^n-\ell} \left( f \circ V_{\ell,M}^{d, \theta, \ell, i} - \mathbb{I}_N(\ell) f \circ V_{\max\{\ell+1, 0\}, M}^{d, \theta, \ell, i} \right) \right] \left( t + (T-t) \gamma_{(T-t), \ell, i}^{d, \theta, \ell, i} \right),$$

and for every $d, n, M \in \mathbb{N}$ let $\text{FE}_{d,n,M} \in \mathbb{N}$ be the number of function evaluations of $(f, g_d, \mu_d, \sigma_d)$ needed to compute one realization of $V_{n,M}^{d, 0}(0, 0): \Omega \to \mathbb{R}^d$ (cf. (115) for a precise definition). Then there exist $\varepsilon \in \mathbb{R}$, $\mathbf{n} : \mathbb{N} \times (0, 1] \to \mathbb{N}$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$ it holds that

$$\left( \mathbb{E}[\|u_d(0, 0) - V_{n,M}^{d, 0}(0, 0)\|^2] \right)^{1/2} \leq \varepsilon$$

and $\text{FE}_{d,n,M} \leq c \cdot d^{6 \gamma + (4+\delta)} \cdot \varepsilon^{-(4+\delta)}$.

Theorem 1.1 follows from Theorem 3.1, Lemma 2.6, and the Feynman-Kac formula. Next, we comment on the statement of Theorem 1.1. For all $\delta \in (0, 1)$ the computational effort of this method to achieve an accuracy of size $\varepsilon \in (0, \infty)$ grows at most like $\varepsilon^{-4-\delta}$ times a polynomial of the dimension $d \in \mathbb{N}$. Here, we define the errors as the $L^2$-distances between the exact solutions of the PDE (5) and the MLP approximations (6) at $(0, 0)$ and we measure computational effort through the number of function evaluations of all parameter functions of the PDE (5) needed to compute an MLP approximation at a fixed space-time point. The MLP approximation method is based on the idea

(a) to reformulate the PDE (5) as stochastic fixed-point equation $u_d = \Phi_d(u_d)$ with a suitable function $\Phi_d$,

(b) to approximate the fixed point $u_d$ through Picard iterates $(u_d^{(n)})_{n \in \mathbb{N}}$,

(c) to write $u_d$ as telescoping series

$$u_d = u_d^{(0)} + \sum_{n=1}^{\infty} (u_d^{(n+1)} - u_d^{(n)}) = u_d^{(0)} + \sum_{n=1}^{\infty} (\Phi_d(u_d^{(n)}) - \Phi_d(u_d^{(n-1)})),$$

and

(d) to approximate the series by a finite sum and the temporal and spatial integrals in the sums by Monte Carlo averages with fewer and fewer independent samples as $n$ increases;

see, e.g., [46] for more details. The approximations (6) of the stochastic differential equation associated with the linear part of the PDE (5) are named Euler-type approximations as proposed in [43, 38, 52]. We note that classical Euler-Maruyama approximations cannot be used if the coefficients $\sigma_d, \mu_d, g_d$ and the terminal condition $g_d$ are assumed to satisfy the local Lipschitz condition (3) and the polynomial growth condition (4), and the nonlinearity $f$ is assumed to be globally Lipschitz continuous. A central observation of this article is that the constant $c$ in (1) and in (2) appears in the exponent of our upper bounds (cf. Theorem 3.1). Therefore, to avoid the curse of dimensionality we have to assume that $c$ grows at most logarithmically in the dimension or does not depend on the dimension as in Theorem 1.1. Furthermore, due to the term $d^\alpha$ in (2)–(4), our result includes, e.g., the case of additive noise where $\sigma_d = I_{d \times d}$. We note that Theorem 3.1 below assumes much

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2 applied for every $d \in \mathbb{N}$ with $m \wedge d, b \wedge d, c \wedge d, \alpha \wedge 0, L \wedge \sup_{p \in \mathbb{R}^d} \|W_p\|_2, \gamma \wedge 0, \Omega \wedge (0, T) \times \mathbb{R}^d \times \mathbb{R}$, $\kappa \wedge 1, \varphi \wedge \mathbb{R}^d \ni x \mapsto ||x||^2 + d^\beta$, $\beta \in [1, \infty]$, $U \wedge (8Tc) \wedge (0, T) \times \mathbb{R}^d \times \mathbb{R}$, $f \wedge ([0, T] \times \mathbb{R}^d \times \mathbb{R} \ni (t, x, w) \mapsto f(w) \in \mathbb{R})$, $\mu \wedge \mu_d, \sigma \wedge \sigma_d, (D_{a_0})_{a \in (0, T)} \wedge (D^a_{a_0})_{a \in (0, T)}, (W_{a_0})_{a \in (0, T)} \wedge (W^a_{a_0})_{a \in (0, T)}$ in the notation of Theorem 3.1.

3 applied for every $d \in \mathbb{N}$ with $m \wedge d, p \wedge 8$, $a \wedge d^0$ in the notation of Lemma 2.6.
weaker conditions than (1)–(5). In particular, the restrictive global monotonicity condition (1) is weakened to the local monotonicity condition (105).

The remainder of this article is organized as follows. Section 2 focuses on the linear PDE-part. Lemma 2.1 establishes moment estimates, Lemma 2.3 provides exponential moment estimates, and Lemma 2.4 proves strong error estimates for tamed Euler-type approximations. Finally, our main result, Theorem 3.1 below, establishes an error analysis for MLP approximations of PDEs with locally monotone coefficient functions.

1.1 Notation

Let $\| \cdot \| : \mathbb{R}^{m \times n} \to [0, \infty)$ satisfy for all $m, n \in \mathbb{N}$, $b = (b_{ij})_{i \in [1, m], j \in [1, n]} \in \mathbb{R}^{m \times n}$ that $\| b \|^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |b_{ij}|^2$, let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy for all $n \in \mathbb{N}$, $x = (x_i)_{i \in [1, n]} \in \mathbb{R}^n$ that $\langle x, y \rangle = \sum_{i=1}^{n} x_iy_i$, and for every $m, n, \ell \in \mathbb{N}$ let $L^{(m)}(\mathbb{R}^n, \mathbb{R}^\ell)$ be the set of $m$-linear mappings from $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ $(m$ times) to $\mathbb{R}^\ell$ and let $\| \cdot \|_{L^{(m)}(\mathbb{R}^n, \mathbb{R}^\ell)} : L^{(m)}(\mathbb{R}^n, \mathbb{R}^\ell) \to \mathbb{R}$ satisfy for all $A \in L^{(m)}(\mathbb{R}^n, \mathbb{R}^\ell)$ that

$$\| A \|_{L^{(m)}(\mathbb{R}^n, \mathbb{R}^\ell)} = \sup \{ \| A(x_1, x_2, \ldots, x_m) \| : \forall i \in [1, m] \cap \mathbb{N}: x_i \in \mathbb{R}^n \text{ and } \| x_i \| = 1 \}. \quad (9)$$

2 Strong approximation theory for SDEs with locally monotone coefficient functions

Results in the literature on moments and strong convergence rates of implementable approximations of SDEs do not clarify the dependence of upper bounds on the dimension. It is a key contribution of this article to provide upper bounds for moments, exponential moments, and errors of approximations of SDEs; see Lemma 2.1, Lemma 2.3, and Lemma 2.4 below for details.

2.1 Moment estimates for tamed Euler-type approximations

A key observation of this article is that the functions $V \cap (\mathbb{R}^d \ni x \mapsto (\| x \|^2 + d)^p)$, $d, p \in \mathbb{N}$, satisfy the condition (11) below with dimension-independent $c$ and satisfy the Lyapunov-type condition (12) below for a number of interesting SDEs. Note that the functions $V \cap (\mathbb{R}^d \ni x \mapsto (\| x \|^2 + 1)^p)$, $d, p \in \mathbb{N}$, do not satisfy (12) in the case of $\mu = 0$ and $\sigma(x) = \mathbb{1}_{x \in d}$ with dimension-independent $c$. Our proof of Lemma 2.1 below adapts a number of arguments from [39, Chapter 2].

**Lemma 2.1 (Moment estimates). Consider the notation in Subsection 1.1, let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, $p \in [3, \infty)$, $b, c \in [1, \infty)$, $\beta \in (0, \infty)$, $\kappa \in (0, p/(3\beta + 4)]$, $V \in C^1(\mathbb{R}^d)$, for every $s \in [0, T)$ let $\mathcal{P}(s, T)$ be the set given by $\mathcal{P}(s, T) = \{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : n \in \mathbb{N}, s = t_0 < t_1 < \ldots < t_n = T \}$, for every $\theta \in \bigcup_{t \in [0, T]} \mathcal{P}(s, T)$, $\theta \in \mathbb{N}$ with $\theta = (t_0, t_1, \ldots, t_n)$ let $\mathcal{L} \ni \gamma^t : [t_0, t_n] \to \mathbb{R}$ satisfy for all $t \in [t_0, t_n]$ that $\mathcal{L} \ni \gamma^t = \text{sup} \{ \{ t_0, t_1, \ldots, t_n \} \cap [t_0, t) \} \lor t_0 \land \gamma^t = t_0$, let $\mu : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma = (\sigma_1, \ldots, \sigma_m) : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be Borel measurable, let $(D_h)_{h \in [0, T]} \subseteq \mathcal{B}(\mathbb{R}^d)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W : [0, T] \times \Omega \to \mathbb{R}^d$ be a standard $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian motion with continuous sample paths, for every $t \in [0, T)$, $x \in \mathbb{R}^d$, $\theta \in \mathcal{P}(t, T)$ let $Y^\theta_{t, s}(\omega)_{s \in [t, T], \omega \in \Omega} : [t, T] \times \Omega \to \mathbb{R}^d$ satisfy for all $s \in (t, T)$ that

$$Y^\theta_{t, s} = x \quad \text{and} \quad Y^\theta_{t, s} = [z + \mathbb{1}_{D_h} \gamma^t(z) \left( \mu(z)(s - \gamma^t) + \frac{\sigma(z)(W_s - W_{\gamma^t})}{1 + \| \sigma(z)(W_s - W_{\gamma^t}) \|^2} \right)]_{z = Y^\theta_{t, t \lor \gamma^t}} \quad (10)$$
assume for all \( \ell \in \{1, 2, 3\} \), \( x \in \mathbb{R}^d \) that
\[
\|(D^\ell V)(x)\|_{L^{(\ell)}(\mathbb{R}^d, \mathbb{R})} \leq c|V(x)|^{1-\frac{\ell}{2}},
\]
(11)
\[
(DV)(\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} (D^2 V)(\sigma_k(x), \sigma_k(x)) \leq cV(x),
\]
(12)
\[
\|\mu(x)\| \leq b|V(x)|^{\frac{\ell+1}{p}}, \quad \text{and} \quad \|\sigma(x)\|^2 \leq b|V(x)|^{\frac{\ell+2}{p}},
\]
(13)
assume for all \( h \in (0, T] \), \( x \in D_h \) that \( V(x) \leq c(b^h)^{-\kappa} \), and let \( \rho \in \mathbb{R} \) satisfy that
\[
\rho = (5c^{\ell+1/\kappa}p)^{3p}.
\]
(14)
Then it holds for all \( t \in [0, T) \), \( s \in [t, T] \), \( x \in \mathbb{R}^d \), \( \theta \in \mathcal{P}(s, T) \) that \( \mathbb{E}[V(Y_{t,s}^\theta)] \leq e^{\rho(s-t)}V(x) \).

**Proof of Lemma 2.1.** First, [39, Lemma 2.12] proves for all \( x, y \in \mathbb{R}^d \) that
\[
V(x + y) \leq V(x) + e^{p^{2p-1}}\|y\|[V(x)]^{1-\frac{1}{p}} + e^{p^{p-1}}\|y\|^p.
\]
(15)
Furthermore, the chain rule and (11) yield for all \( x, y \in \mathbb{R}^d \), \( t \in [0, 1] \) that
\[
\left| \frac{d}{dt}(V(x + ty)) \right| = \left| (DV)(x + ty) \right| y
\leq \|(DV)(x + ty)\|_{L^{(1)}(\mathbb{R}^d, \mathbb{R})} \|y\| \leq c\|V(x + ty)\|^{1 - \frac{1}{p}}\|y\|.
\]
(16)
This and [39, Lemma 2.11] (applied for \( x, y \in \mathbb{R}^d \) with \( T \land 1, c \land c\|y\|, y \land ([0, 1] \ni s \mapsto V(x + sy) \in \mathbb{R}) \) in the notation of [39, Lemma 2.11]) imply for all \( x, y \in \mathbb{R}^d \) that
\[
V(x + y) \leq \left[ |V(x)|^{1/p} + \frac{c\|y\|}{p} \right]^p.
\]
(17)
This, the fundamental theorem of calculus, (11), and the fact that \( \forall a_1, a_2 \in [0, \infty), r \in [0, \infty): (a_1 + a_2)^r \leq 2^{(r-1)\log_2(|a_1|^r + |a_2|^r)} \) imply for all \( i \in \{1, 2\}, x, y \in \mathbb{R}^d \) that
\[
\|(D^iV)(x) - (D^iV)(y)\|_{L^{(i)}(\mathbb{R}^d, \mathbb{R})} \leq \int_0^1 \|(D^{i+1}V)(x + ty - x)\|_{L^{(i+1)}(\mathbb{R}^d, \mathbb{R})} \|y - x\| dt
\leq \int_0^1 c\|V(x + ty - x)\|^\frac{p-1}{p} \|y - x\| dt \leq \int_0^1 c \left[ |V(x)|^\frac{1}{p} + \frac{c\|y - x\|}{p} \right]^p \|y - x\| dt
\leq (2c)^{p-1} \left[ |V(x)|^\frac{p-1}{p} \|y - x\| + \frac{\|y - x\|^{p-1}}{p} \right].
\]
(18)
Hence, (9), (13), and the triangle inequality show for all \( x, y \in \mathbb{R}^d \) that
\[
\left| [(DV)(x) - (DV)(y)](\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} [(D^2 V)(x) - (D^2 V)(y)](\sigma_k(x), \sigma_k(x)) \right|
\leq \|(DV)(x) - (DV)(y)\|_{L^{(1)}(\mathbb{R}^d, \mathbb{R})} \|\mu(x)\| + \frac{1}{2} \left[ \|(D^2 V)(x) - (D^2 V)(y)\|_{L^{(2)}(\mathbb{R}^d, \mathbb{R})} \|\sigma(x)\|^2 \right]
\leq \sum_{i=1}^{2} (2c)^{p-1} \left[ |V(x)|^\frac{p-1}{p} \|y - x\| + \frac{\|y - x\|^{p-1}}{p} \right] b|V(x)|^{\frac{\ell+1}{p}}
\leq (2c)^{p-1} \left[ |V(x)|^\frac{p-1}{p} \|y - x\| + \frac{\|y - x\|^{p-1}}{p} \right].
\]
(19)
Next, Jensen’s inequality, the Burkholder-Davis-Gundy inequality (see, e.g., [16, Lemma 7.7], applied with \( r \wedge (r \vee 2)/2 \) and \( \Phi \wedge (0, T) \ni s \mapsto \sigma(x) \in \mathbb{R}^{d \times m} \) in the notation of [16, Lemma 7.7]) imply for all \( r \in [1, \infty) \), \( t \in (0, T) \), \( x \in \mathbb{R}^d \) that

\[
\left( \mathbb{E} \left[ \|\sigma(x)W_t\|^r \right] \right)^{1/r} \leq \left( \mathbb{E} \left[ \|\sigma(x)W_t\|^{r \vee 2} \right] \right)^{1/(r \vee 2)} \leq \sqrt{\frac{(r \vee 2)((r \vee 2) - 1)}{2}} \sqrt{t} \|\sigma(x)\| \leq r \sqrt{t} \|\sigma(x)\|.
\]
(20)

The fact that \( \kappa \leq p/(3\beta) \) shows that for all \( t \in (0, T) \), \( x \in \mathbb{R}^d \) with \( V(x) \leq c(b^\beta t)^{-\kappa} \) it holds that

\[
c^{-1/\kappa}b^\beta t |V(x)|^{\frac{\alpha}{p}} \leq c^{-\frac{1}{\kappa}} \left| \frac{c}{V(x)} \right|^{\frac{\alpha}{p}} |V(x)|^{\frac{\alpha}{p}} = |V(x)|^{\frac{\alpha}{p} - \frac{1}{\kappa}} \leq |V(x)|^{\frac{2\alpha}{p} - 1} \leq 1.
\]
(21)

This, the triangle inequality, (20), the assumption that \( c \geq 1 \), and (13) imply that for all \( r \in [1, \infty) \), \( t \in (0, T) \), \( x \in \mathbb{R}^d \) with \( V(x) \leq c(b^\beta t)^{-\kappa} \) it holds that

\[
\left( \mathbb{E} \left[ \|\mu(x)t + \sigma(x)W_t\|^r \right] \right)^{1/r} \leq t \|\mu(x)\| + \left( \mathbb{E} \left[ \|\sigma(x)W_t\|^r \right] \right)^{1/r} \leq bt |V(x)|^{\frac{\alpha}{p} + r} r \sqrt{\mathbb{E} \left[ |V(x)|^{p \vee 2} \right]}^{\frac{1}{p \vee 2}} \leq c^{\frac{1}{\kappa}} |V(x)|^{\frac{\alpha}{p}} \left[ c^{-\frac{1}{\kappa}} bt |V(x)|^{\frac{\alpha}{p}} + r \left( c^{-\frac{1}{\kappa}} bt |V(x)|^{\frac{\alpha}{p}} \right)^{\frac{1}{2}} \right] \leq c^{\frac{1}{\kappa}} |V(x)|^{\frac{\alpha}{p}} \left[ (1 + r) \left( c^{-\frac{1}{\kappa}} bt |V(x)|^{\frac{\alpha}{p}} \right)^{\frac{1}{2}} \right] \leq \frac{c^{\frac{1}{\kappa}}}{b} |V(x)|^{\frac{\alpha}{p}} (1 + r) |V(x)|^{\frac{2\alpha}{p} - 1} = c^{\frac{1}{\kappa}} (1 + r) |V(x)|^{\frac{2\alpha}{p} - 1}.
\]
(22)

Next, the assumption that \( p \geq 3 \) shows for all \( i \in \{1, 2\} \) that

\[
\frac{p + \beta - 1}{p} + \frac{1 - \beta}{p} = 1 \quad \text{and} \quad \frac{\beta + i}{p} + \frac{(1 - \beta)(p - i)}{p} = \frac{(i + 1)\beta + p - \beta p}{p} \leq 1.
\]
(23)

Combining (19), (22) (applied for \( q \in \{1, p - 1, p - 2\} \) with \( r \wedge q \) in the notation of (22)), and the fact that \( p \geq 3 \) then shows that for all \( t \in (0, T) \), \( x \in \mathbb{R}^d \) with \( V(x) \leq c(b^\beta t)^{-\kappa} \) it holds that

\[
\mathbb{E} \left[ (DV)(y) - (DV)(y)(\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} \left[ (D^2 V)(x) - (D^2 V)(y) \right] (\sigma_k(x), \sigma_k(x)) \right] \left| y = x + \mu(x)t + \sigma(x)W_t \right| \leq (2c)^{p} \left[ |V(x)|^{\frac{p + \beta - 1}{p} - \frac{1}{r}} \mathbb{E} \left[ \|\mu(x)t + \sigma(x)W_t\|^{r \vee 2} \right] + \left[ \frac{1}{2p} \sum_{i=1}^{2} |V(x)|^{\frac{p + \beta - 1}{p} - \frac{1}{r}} \mathbb{E} \left[ \|\mu(x)t + \sigma(x)W_t\|^{p - i} \right] \right] \right] \leq (2c)^{p} \left[ |V(x)|^{\frac{p + \beta - 1}{p} - \frac{1}{r}} \mathbb{E} \left[ \|\mu(x)t + \sigma(x)W_t\|^{r \vee 2} \right] + \left[ \frac{1}{2p} \sum_{i=1}^{2} |V(x)|^{\frac{p + \beta - 1}{p} - \frac{1}{r}} \left[ \frac{c^{\frac{1}{\kappa}}}{b} (p - i + 1) \right] |V(x)|^{\frac{(1 - \beta)(p - i)}{p}} \right] \right] \leq (2c)^{p} c^{\frac{\alpha}{p}} p^{p} \left( \frac{2}{p^p} + \frac{p}{2p} + \frac{p - 1}{2p} \right) |V(x)| \leq (2c^{1+\frac{\alpha}{p}} p) |V(x)|.
\]
(24)

Itō’s formula, a telescoping sum argument, the triangle inequality, and (12) then ensure that for all \( t \in (0, T) \), \( x \in \mathbb{R}^d \) with \( V(x) \leq c(b^\beta t)^{-\kappa} \) it holds that

\[
\mathbb{E} \left[ V(x + \mu(x)t + \sigma(x)W_t) \right] = V(x) + \int_0^t \mathbb{E} \left[ (DF)(y)(\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} (D^2 F)(y)(\sigma_k(x), \sigma_k(x)) \right] \left| y = x + \mu(x)s + \sigma(x)W_s \right| ds
\]
\[
\begin{align*}
\text{Next, the fact that } \kappa \in [0, p/(3\beta + 4)] \text{ implies that}
\frac{1}{2} - \kappa \left[ \frac{3(\beta + 2)}{2p} - \frac{1}{p} \right] \geq \frac{1}{2} - \frac{p}{3\beta + 4} \left( \frac{3\beta + 4}{2p} \right) = 0.
\end{align*}
\]

The fact that \( \forall x \in \mathbb{R}^d: \|x - x/(1 + \|x\|^2)\| \leq \|x\|^2 \), (22), the fact that \( 1 + 3p \leq 3.5p \), the fact that \( 1 \leq V \), the fact that \( b \geq 1 \), and the fact that \( \kappa \in (0, p/(3\beta + 4)] \), then prove that for all \( t \in (0, T], x \in \mathbb{R}^d \) with \( V(x) \leq c(b^3t)^{-\kappa} \) it holds that \( c^{-1/\kappa}b^3t \leq 1 \) and

\[
\begin{align*}
\left( \mathbb{E} \left[ \left\| \frac{\sigma(x)W_t - \sigma(x)W_i}{1 + \|\sigma(x)W_t\|^2} \right\|^p \right] \right)^{1/p} \leq \left( \mathbb{E} \left[ \|\sigma(x)W_t\|^{3p} \right] \right)^{1/3} = \left[ (1 + 3p) \frac{c^3}{b^3} \left( c(b^3t)^{-\kappa} \right)^{3(\beta + 2)/2p} \right]^{1/3} \leq [3.5p]^3 c^2t \cdot (c^{-1/\kappa}b^3t)^{-1} \left( \mathbb{E} \left[ \left\| \frac{\sigma(x)W_t}{1 + \|\sigma(x)W_t\|^2} \right\|^p \right] \right)^{1/p}.
\end{align*}
\]

This, Hölder’s inequality, and (25) imply that for all \( t \in (0, T], x \in \mathbb{R}^d \) with \( V(x) \leq c(b^3t)^{-\kappa} \) it holds that

\[
\begin{align*}
\mathbb{E} \left[ \left\| \frac{\sigma(x)W_t - \sigma(x)W_i}{1 + \|\sigma(x)W_t\|^2} \right\| \right] \leq \left[ \mathbb{E} \left[ \left\| \frac{\sigma(x)W_t}{1 + \|\sigma(x)W_t\|^2} \right\|^p \right] \right]^{1/p} \leq [3.5p]^3 c^2t \cdot (c^{-1/\kappa}b^3t)^{-1} \left( \mathbb{E} \left[ \left\| \frac{\sigma(x)W_t}{1 + \|\sigma(x)W_t\|^2} \right\|^p \right] \right)^{1/p}.
\end{align*}
\]

This, the assumption that \( V \geq 1, (15), (25), (27) \), the fact that \( \forall a_1, a_2, t \in [0, \infty): (1 + a_1t)(1 + a_2t) \leq e^{(a_1 + a_2)t} \), the fact that \( p \geq 3 \), the fact that \( 3^p + 2^p(3.5)^{3p} \leq 3 + 2(3.5)^3 \leq 5^{3p} \), and the definition of \( \rho \) in (14) demonstrate that for all \( t \in (0, T], x \in \mathbb{R}^d \) with \( V(x) \leq c(b^3t)^{-\kappa} \) it holds that \( t \leq c^{1/\kappa} \) and

\[
\begin{align*}
&\mathbb{E} \left[ (\mathbb{E} \left[ \left\| \frac{\sigma(x)W_t}{1 + \|\sigma(x)W_t\|^2} \right\|^p \right] \right)^{1/p} \right] \leq \mathbb{E} \left[ \left\| \frac{\sigma(x)W_t}{1 + \|\sigma(x)W_t\|^2} \right\|^\rho \right] \leq [3.5p]^3 c^2t W(x) \left( 1 + (3c^{1/\kappa}p)^\rho t \right).
\end{align*}
\]
\begin{align}
= V(x) \left( 1 + (3c^{1+\frac{1}{p}}p)t \right) \left( 1 + 2p^{p+\frac{2}{p}}[3.5p]^{3p} t \right) \leq V(x) \exp \left( (3c^{1+\frac{1}{p}}p) + 2p^{p+\frac{2}{p}}[3.5p]^{3p} t \right) \\
\leq V(x) \exp \left( (5c^{1+\frac{1}{p}}p)^{3p} t \right) = V(x)e^{pt}.
\end{align}

This, the tower property, the Markov property of \( W \), and the fact that \( \forall s \in [0, T], t \in [s, T], B \in \mathcal{B}(\mathbb{R}^d) \); \( \mathbb{P}(W_t - W_s) = \mathbb{P}(W_{t-s} \in B) \) imply for all \( x \in \mathbb{R}^d, t \in [0, T], s \in [t, T], \theta \in \mathcal{P}(t,T) \) that

\begin{align}
\mathbb{E}[V(Y_{t,s}^{\theta,x}) \mathbb{1}_{D_{|t|}}(Y_{t,s}^{\theta,x})] &= \mathbb{E}\left[ \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{D_{|t|}}(Y_{t,s}^{\theta,x}) \Big| \mathcal{F}_{L,s,\theta} \right] \right] \\
= \mathbb{E}\left[ \mathbb{E}\left[ V\left( z + \mu(z)(s - L,s,\theta) \right) + \frac{\sigma(z)(W_s - W_{L,s,\theta})}{1 + ||\sigma(z)(W_s - W_{L,s,\theta})||^2} \mathbb{1}_{D_{|t|}}(z) \right] \right]_{z = Y_{t,s}^{\theta,x}}^{Y_{t,s}^{\theta,x}} \mathcal{F}_{L,s,\theta} \\
= \mathbb{E}\left[ \mathbb{E}\left[ V\left( z + \mu(z)(s - L,s,\theta) \right) + \frac{\sigma(z)(W_s - W_{L,s,\theta})}{1 + ||\sigma(z)(W_s - W_{L,s,\theta})||^2} \mathbb{1}_{D_{|t|}}(z) \right] \right]_{z = Y_{t,s}^{\theta,x}}^{Y_{t,s}^{\theta,x}} \mathcal{F}_{L,s,\theta} \\
\leq e^{\theta(s - L,s,\theta)} \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{D_{|t|}}(Y_{t,s}^{\theta,x}) \right].
\end{align}

Next, (10) shows for all \( x \in \mathbb{R}^d, t \in [0, T], s \in [t, T], \theta \in \mathcal{P}(t,T) \) that

\begin{align}
\{ \omega \in \Omega: Y_{t,s}^{\theta,x}(\omega) \in \mathbb{R}^d \setminus D_{|t|} \} \subseteq \{ \omega \in \Omega: Y_{t,s}^{\theta,x}(\omega) = Y_{t,L,s,\theta}(\omega) \}.
\end{align}

This and (31) imply for all \( x \in \mathbb{R}^d, t \in [0, T], s \in [t, T], \theta \in \mathcal{P}(t,T) \) that

\begin{align}
\mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \right] = \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{D_{|t|}}(Y_{t,s}^{\theta,x}) \right] + \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{\mathbb{R}^d \setminus D_{|t|}}(Y_{t,s}^{\theta,x}) \right] \\
= \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{D_{|t|}}(Y_{t,s}^{\theta,x}) \right] + \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{\mathbb{R}^d \setminus D_{|t|}}(Y_{t,s}^{\theta,x}) \right] \\
\leq e^{\theta(s - L,s,\theta)} \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{D_{|t|}}(Y_{t,s}^{\theta,x}) \right] + \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{\mathbb{R}^d \setminus D_{|t|}}(Y_{t,s}^{\theta,x}) \right] \\
= e^{\theta(s - L,s,\theta)} \mathbb{E}\left[ V(Y_{t,s}^{\theta,x}) \mathbb{1}_{D_{|t|}}(Y_{t,s}^{\theta,x}) \right].
\end{align}

An induction argument and (10) then show for all \( x \in \mathbb{R}^d, t \in [0, T], s \in [t, T], \theta \in \mathcal{P}(t,T) \) that \( \mathbb{E}[V(Y_{t,s}^{\theta,x})] \leq e^{\theta(s - t)} V(x) \). This completes the proof of Lemma 2.1.

\begin{flushright}
\[ \square \]
\end{flushright}

\section{2.2 Strong error estimates for approximations of SDEs}

We establish strong error estimates in Lemma 2.4 below. First we introduce the setting, Setting 2.2, in which we work in the rest of this section and then we establish exponential moment estimates which are uniform in the dimension.

\begin{setting}
Consider the notation in Subsection 1.1, let \( d, m \in \mathbb{N}, T \in (0, \infty), b, c, \beta, \gamma, r \in [1, \infty), \alpha \in [0, \infty), p \in [4r\beta, \infty), \bar{U} \in C(\mathbb{R}^d, [0, \infty)), \kappa \in (0, (3\beta + 1)], \varphi \in C^3(\mathbb{R}^d, [0, \infty)), U \in C^3([0, \infty)), \mu: \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma = (\sigma_1, \ldots, \sigma_m): \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m} be Borel measurable functions, let \( \rho \in \mathbb{R} \) satisfy that

\begin{align}
\rho = (5c^{2+\frac{1}{p}}p)^{3p},
\end{align}

for every \( t \in [0, T] \) let \( \mathcal{P}(t,T) \) be the set given by \( \mathcal{P}(t,T) = \{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1}: n \in \mathbb{N}, t = t_0 < t_1 < \ldots < t_n = T \} \), for every \( \theta \in \cup_{t \in [0,T]} \mathcal{P}(t,T), n \in \mathbb{N}, (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \) with \( \theta = (t_0, t_1, \ldots, t_n) \) let \( |\theta| \in [0, T], \omega_{t-s,\theta} = [t_0, t_n] \rightarrow \mathbb{R} \) for all \( s \in (t_0, t_n) \) that

\begin{align}
|\theta| = \max_{i \in \{0, n-1\} \cap \mathbb{N}_0} |t_{i+1} - t_i|, \quad \omega_{t_0-\theta} = t_0, \quad \text{and} \quad \omega_{t-s,\theta} = \sup \{ \{t_0, t_1, \ldots, t_n\} \cap [t_0, s) \},
\end{align}

let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) \) be a filtered probability space which satisfies the usual conditions, let \( D: (0, T) \rightarrow \mathcal{B}(\mathbb{R}^d) \), let \( W: [0, T] \times \Omega \rightarrow \mathbb{R}^m \) be a standard \( (\mathcal{F}_t)_{t \in [0,T]} \)-Brownian motion with
continuous sample paths, for every $t \in [0, T]$, $x \in \mathbb{R}^d$ let $(X^x_{t,s}(\omega))_{t \in [s,T],\omega \in \Omega} : [s, T] \times \Omega \to \mathbb{R}^d$ be adapted stochastic processes with continuous sample paths which satisfy that for all $s \in [t, T]$ it holds $\mathbb{P}$-a.s. that

$$
\int_t^s \|\mu(X^x_{t,r})\| + \|\sigma(X^x_{t,r})\|^2 \, dr < \infty \quad \text{and} \quad X^x_{t,s} = x + \int_t^s \mu(X^x_{t,r}) \, dr + \int_t^s \sigma(X^x_{t,r}) \, dW_r,
$$

(36) for every $t \in [0, T)$, $x \in \mathbb{R}^d$, $\theta \in \mathcal{P}(t, T)$ let $(Y^{\theta,x}_{t,s}(\omega))_{s \in [t,T],\omega \in \Omega} : [t, T] \times \Omega \to \mathbb{R}^d$ satisfy that for all $s \in (t, T]$ it holds that $Y^{\theta,x}_{t,t} = x$ and

$$
Y^{\theta,x}_{t,s} = Y^{\theta,x}_{t,s}|_{s \neq t} = Y^{\theta,x}_{t,s} + 1_{D_{\theta t}}(Y^{\theta,x}_{t,s}|_{s \neq t}) \left[ \mu(Y^{\theta,x}_{t,s}|_{s \neq t})(s - l_{s,t}) + \frac{\sigma(Y^{\theta,x}_{t,s}|_{s \neq t})(W_s - W_{s,t})}{1 + \|\sigma(Y^{\theta,x}_{t,s}|_{s \neq t})(W_s - W_{s,t})\|^2} \right],
$$

(37) assume for all $\ell \in \{1, 2, 3\}$, $x, y \in \mathbb{R}^d$, $t \in [0, T)$, $s \in [t, T]$ that

$$
\| (D^{\ell} \varphi)(x) \|_{L^{(0)}(\mathbb{R}^d, \mathbb{R})} \leq c |\varphi(x)|^{1-\tau},
$$

(38)

$$
(D\varphi(x))(\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} (D^2 \varphi(x))(\sigma_k(x), \sigma_k(x)) \leq c \varphi(x),
$$

(39)

$$
\max \left\{ \|\mu(x)\|, \|\sigma(x)\|, \|x\| \right\} \leq b |\varphi(x)|^{\frac{\alpha}{\beta}},
$$

(40)

$$
\|\mu(x) - \mu(y)\| \vee \|\sigma(x) - \sigma(y)\| \leq b \|x - y\| \left[ |\varphi(x)|^{\frac{\alpha}{\beta}} + |\varphi(y)|^{\frac{\alpha}{\beta}} \right],
$$

(41)

$$
\langle x - y, \mu(x) - \mu(y) \rangle + (2r - 1) \|\sigma(x) - \sigma(y)\|^2 \leq \|x - y\|^2 \left[ \frac{U(x) + U(y)}{8r T} + \frac{\bar{U}(x) + \bar{U}(y)}{8r} \right],
$$

(42)

$$
\| (D^{(h)} U)(x) \|_{L^{(0)}(\mathbb{R}^d, \mathbb{R})} \leq c |U(x)|^{1-\frac{\alpha}{\beta}}, \quad |\bar{U}(x)| \leq c (1 + |U(x)|^\gamma),
$$

(43)

$$
|\bar{U}(x) - \bar{U}(y)| \leq c \left[ 1 + |U(x)|^\gamma + |U(y)|^\gamma \right] \|x - y\|, \quad \text{and}
$$

(44)

$$
(DU(x))(\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} (D^2 U(x))(\sigma_k(x), \sigma_k(x))
$$

(45)

$$
+ \frac{1}{2} e^{\alpha T} \|\sigma(x)^\ell (\nabla U)(x)\|^2 + \bar{U}(x) \leq \alpha U(x),
$$

assume for all $h \in (0, T)$, $x \in D_h$ that

$$
\max \left\{ \|\mu(x)\|, \|\sigma(x)\|, U(x), b^{\beta^c} \varphi(x) \right\} \leq \min \left\{ c h^{-\frac{1}{1-\gamma}}, c h^{-\frac{1}{\beta^c + \gamma}}, c h^{-\gamma}, r \log \left( \frac{1}{r} \right) \right\},
$$

(46)

and let $c_0 : \mathbb{R} \to [1, \infty)$ satisfy for all $a \in \mathbb{R}$ that

$$
c_0(a) = \exp \left( 2 \left[ 720 \max\{T, \alpha, 1\} (ce^{\alpha T})^3 (720(ce^{\alpha T})^3 \max\{T, 1\} + 7)\gamma \right] \min\{|a|, 1\}^{1/8} \right).
$$

(47)

The following lemma, Lemma 2.3, suitably applies [54, Proposition 2.14] to obtain exponential moment estimates.

**Lemma 2.3** (Exponential moments). Assume Setting 2.2 and let $t_0 \in (0, T)$, $x_0 \in \mathbb{R}^d$, $\theta \in \mathcal{P}(t_0)$. Then it holds for all $s \in [t_0, T]$ that

$$
\mathbb{E} \left[ \exp \left( e^{\alpha(T-t)} U(Y_{t_0,s}^{\theta,x_0}) + \int_{t_0}^s 1_{D_{\theta t}}(Y_{t_0,l,s}^{\theta,x_0}) e^{\alpha(T-t)} \bar{U}(Y_{t_0,l,s}^{\theta,x_0}) \, dt \right) \right] \leq c_0(|\theta|) \exp \left( U(x_0) e^{\alpha(T-t_0)} \right),
$$

(48)

and

$$
\mathbb{E} \left[ \exp \left( e^{\alpha(T-t)} U(X_{t_0,s}^{x_0}) + \int_{t_0}^s e^{\alpha(T-t)} \bar{U}(X_{t_0,l,s}^{x_0}) \, dt \right) \right] \leq \exp \left( U(x_0) e^{\alpha(T-t_0)} \right).
$$

(49)
Proof of Lemma 2.3. Throughout this proof let $\delta, \varsigma \in \mathbb{R}$, $n \in \mathbb{N}$, $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, $\tilde{\theta} \in \mathcal{P}(0, T-t_0)$, $\tilde{Y} = (\tilde{Y}_t(\omega))_{t \in [0, T-t_0], \omega \in \Omega}$: $[0, T - t_0] \times \Omega \to \mathbb{R}$ satisfy for all $t \in [0, T-t_0]$ that 

$$
\delta = \frac{1}{28}, \quad \varsigma = \frac{1}{8 + 8\gamma}, \quad \theta = (t_0, t_1, \ldots, t_n),
$$

then it holds that

$$
\tilde{\theta} = (0, t_1 - t_0, \ldots, t_n - t_0), \quad \text{and} \quad \tilde{Y}_t = Y_{t_0, t_0 + t}. 
$$

(50)

Then it holds that

$$
\frac{1 - 14\delta}{2 + 2\gamma} = \frac{1}{4 + 4\gamma}, \quad \varsigma \in \left(0, \frac{1 - 14\delta}{2 + 2\gamma}\right), \quad \text{and} \quad \varsigma + \varsigma \gamma + 7\delta - \frac{1}{2} = \frac{1}{8} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{8},
$$

(51)

and it holds for all $t \in [0, T - t_0]$ that

$$
[\theta] = [\tilde{\theta}], \quad t_0 + \iota t - j \varphi = t_0 + \iota t_{-j}, \quad \text{and} \quad Y_{t_0, t_0 + t_0 + t_{-j}}^{\theta, x_0} = Y_{t_0, t_0 + t_0 + j_{-j}}^{\theta, x_0} = \tilde{Y}_{t_0, t_0 + t_{-j}}.
$$

(52)

Then (37) implies for all $t \in [0, T - t_0]$ that

$$
\tilde{Y}_t = Y_{t_0, t_0 + t}^{\theta, x_0} = Y_{t_0, t_0 + t_{-j}}^{\theta, x_0} + \mathbb{I}_{D|\theta}(Y_{t_0, t_0 + t_{-j}}^{\theta, x_0}) + \mu(Y_{t_0, t_0 + t_{-j}}^{\theta, x_0})(t_0 + t - \iota t_{-j}) + \sigma(Y_{t_0, t_0 + t_{-j}}^{\theta, x_0})(W_{t_0 + t} - W_{t_0 + t_{-j}}) \\
= \tilde{Y}_{t_0, t_0 + t_{-j}} + \mathbb{I}_{D|\theta}(\tilde{Y}_{t_0, t_0 + t_{-j}}) + \mu(\tilde{Y}_{t_0, t_0 + t_{-j}})(t - \iota t_{-j}) + \sigma(\tilde{Y}_{t_0, t_0 + t_{-j}})(W_{t_0 + t} - W_{t_0 + t_{-j}})
$$

(53)

Next, (46) implies for all $h \in (0, T]$, $x \in D_h$ that

$$
\max \{\|\mu(x)\|, \|\sigma(x)\|, U(x)\} \leq \min \left\{ch^{-\frac{1}{2}}, ch^{-\frac{1}{2}}, ch^{-\frac{1}{2}}\right\} = c \min \left\{h^{1/2}, h^{1/2}, h^{1/2}\right\}.
$$

(54)

This, the substitution rule, (51)–(53), (43)–(45), the fact that $\gamma \geq 1$, [54, Proposition 2.14] (applied with $H \subset \mathbb{R}^d$, $U \subset \mathbb{R}^m$, $T \subset (T - t_0)$, $\rho \subset \alpha$, $\delta \subset \delta$, $c \subset c^{\alpha(T-t_0)}$, $\gamma \subset \gamma$, $\varsigma \subset \varsigma$, $F \subset \mu$, $B \subset \sigma$, $V \subset c^{\alpha(T-t_0)}U$, $V \subset c^{\alpha(T-t_0)}U$, $S \subset ((0, T - t_0) \times s \rightarrow \mathcal{D}_{m, \beta}(\mathbb{R}^d, \mathbb{R}^d))$ where $\mathcal{D}_{m, \beta}$ is the identity function on $\mathbb{R}^d$, $(D_h)_{h \in [0, T]} \subset \mathcal{D}_{m, \beta}$, $F_{t_0, t_0 + t} = \mathcal{D}_{m, \beta} \subset \mathcal{D}_{m, \beta}$, $W_t \subset (W_{t_0 + t} - W_{t_0}) \subset [0, T - t_0], \quad \theta \subset \theta$, $(Y_{t_0, t_0 + t})_{t \in [t_0, T] \subset [0, T - t_0]}$ in the notation of [54, Proposition 2.14]), the fact that $Y_{t_0, t_0 + t} = x_0$, and (47) show for all $s \in [t_0, T]$ that

$$
\mathbb{E}
\left[
\begin{array}{c}
(\frac{\text{exp}(a(t-s))U(Y_{t_0, s}) + \int_{t_0}^{s} 1_{D|\theta}(Y_{t_0, t_0 + t_{-j}}) e^{a(T-t)}U(Y_{t_0, t}) dt)}{e^{a(s-t_0)}} + \int_{0}^{s-t_0} 1_{D|\theta}(Y_{t_0, t_0 + t_{-j}}) e^{a(T-t)}U(Y_{t_0, t_0 + t}) dt)\\
(\frac{\text{exp}(a(T-t_0))U(Y_{t_0, s-t_0})}{e^{a(s-t_0)}} + \int_{0}^{s-t_0} 1_{D|\theta}(Y_{t_0, t_0 + t_{-j}}) e^{a(T-t_0)}U(Y_{t_0, t_0 + t}) dt)\\
\end{array}
\right]
\leq \mathbb{E}
\left[
\begin{array}{c}
\text{exp}(e^{a(T-t_0)}U(Y_{t_0, s-t_0}))\\
\end{array}
\right]
\cdot \exp
\left[
\begin{array}{c}
\text{min}\{\frac{720}{\min\{\tilde{\theta}, 1\}\varsigma + \varsigma \gamma + 7\delta - \frac{1}{2}}
\end{array}
\right]
\leq \text{exp}(U(x_0)e^{a(T-t_0)}). 
$$

(55)
This implies (48). Next, the substitution rule, (43)–(45), the fact that \( \gamma \geq 1 \), and [15, Corollary 2.4] (applied for \( s \in [t_0, T] \) with \( d \cap d, m \cap m, T \cap T - t_0, O \cap \mathbb{R}^d, \mu \cap \mu, \sigma \cap \sigma, \) \( (\mathcal{F}_t)_{t \in (0,T)} \cap (\mathcal{F}_{t_0+t})_{t \in (0,T-t_0)}, (\mathcal{W}_t)_{t \in [0,T]} \cap (\mathcal{W}_{t_0+t} - \mathcal{W}_{t_0})_{t \in (0,T-t_0)}; \alpha \cap \alpha, U \cap U^\alpha(T-t_0), U \cap U^\alpha(T-t_0), \tau \cap (\Omega \ni \omega \mapsto s - t_0 \in [0, T - t_0]), (X_t)_{t \in [0,T]} \cap (X_{t_0+t})_{t \in [0,T-t_0]} \) in the notation of [15, Corollary 2.4]) imply for all \( s \in [t_0, T] \) that

\[
\mathbb{E}\left[\exp\left(e^{\alpha(T-t_0)}U(x_0)\right)\right] = \exp\left(e^{\alpha(T-t_0)}U(x_0)\right).
\]

(55)

This implies (49). The proof of Lemma 2.3 is thus completed.

The following lemma, Lemma 2.4, establishes strong error estimates and implies strong convergence with rate 1/2. This strong convergence rate is well known in the literature; see, e.g., [41]. The main contribution of Lemma 2.4 is to derive an explicit upper bound which allows to exploit its dependence on the dimension.

**Lemma 2.4 (Strong error).** Assume Setting 2.2 and let \( x_0 \in \mathbb{R}^d, t_0 \in [0, T), \theta \in \mathcal{P}(t_0, T) \). Then it holds that

\[
\sup_{s \in [t_0, T]} \left( \mathbb{E}\left[\left\|X_{t_0, s} - Y_{t_0, s}^{\beta, x_0}\right\|^r\right]\right)^{1/r} 
\leq 74\sqrt{2}(T + 1)^{r/2}b^2 \exp\left(\frac{(2 + \|\theta\|)(T - t_0)}{2}\right) \left[c_0(|\theta|) \exp\left(e^{\alpha(T-t_0)}U(x_0)\right) \varphi(x_0)\right]^{1/2} \sqrt{|\theta|}.
\]

(57)

**Proof of Lemma 2.4.** Throughout this proof let \( \text{Id}_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^d \) be the identity on \( \mathbb{R}^d \), let \( \psi : \mathbb{R}^d \to \mathbb{R}^d \) be the function which satisfies for all \( y \in \mathbb{R}^d \) that

\[
\psi(y) = y/\left(1 + ||y||^2\right),
\]

(58)

let \( e_k \in \mathbb{R}^{m \times 1}, k \in [1, m] \cap \mathbb{N} \), be the \( m \)-dimensional standard basis vectors of \( \mathbb{R}^{m \times 1} \), let \( e_k^T \), \( k \in [1, m] \cap \mathbb{N} \), satisfy for all \( k \in [1, m] \cap \mathbb{N} \) that \( e_k^T \in \mathbb{R}^{1 \times m} \) is the transposed matrix of \( e_k \), let \( Z : [t_0, T] \times \Omega \to \mathbb{R}^d \) be the function which satisfies for all \( t \in [t_0, T] \) that

\[
Z_t = \sigma(Y_{t_0, t, \omega}^{\beta, x_0})(W_t - W_{t, \omega}),
\]

(59)

let \( \tau : \Omega \to [t_0, T] \) satisfy that

\[
\tau = \inf\{\{s \in [t_0, T], Y_{t_0, s, \omega}^{\beta, x_0} \notin D_{|\theta|}\} \cup \{T\},
\]

(60)

and let \( a : [t_0, T] \times \Omega \to \mathbb{R}^d, b : [t_0, T] \times \Omega \to \mathbb{R}^{d \times m} \) be the functions which satisfy for all \( t \in [t_0, T] \) that

\[
a_t = \mu(Y_{t_0, t, \omega}^{\beta, x_0}) + \sum_{k=1}^{m} \left[\frac{1}{2}([D^2 \psi](Z_s)) \left(\sigma(Y_{t_0, t, \omega}^{\beta, x_0}) e_k, \sigma(Y_{t_0, t, \omega}^{\beta, x_0}) e_k\right)\right]
\]

and

\[
b_t = \sum_{k=1}^{m} ([D \psi](Z_s)) \left(\sigma(Y_{t_0, t, \omega}^{\beta, x_0}) e_k\right) e_k^T.
\]

(61)

The fact that \( \forall a \in \mathbb{R}^{d \times m} : a = \sum_{k=1}^{m} a e_k e_k^T \) and the fact that

\[
\forall s \in [t_0, T], t \in (\mathbb{R}_{\omega} \ni \omega) : \mathbb{R}_{\omega} = \mathbb{R}_{\omega}
\]

(62)
then show that for all \( s \in [t_0, T] \) it holds \( \mathbb{P} \)-a.s. that
\[
Z_s = \sum_{k=1}^{m} \int_{s-L_s}^{s} (\sigma(Y_{t_0,s+L_s},e_k)e_k dW_t \quad \text{and} \quad \mu(Y_{t_0,s+L_s}) (s-L_s)_i = \int_{s-L_s}^{s} \mu(Y_{t_0,s+L_s}) dt. \tag{63}
\]
Furthermore, (60) and (62) imply for all \( t \in [t_0, T) \) that
\[
\left\{ Y_{t_0,s+L_s}^{\theta_0} \in D_{[\theta]} \right\} = \{ t < \tau \}. \tag{64}
\]
This, (37), (60), (58), Itô’s formula, (59), (61), and (63) show that for all \( s \in [t_0, T] \) it holds \( \mathbb{P} \)-a.s. that
\[
Y_{t_0,s}^{\theta, x_0} = Y_{t_0,L_s}^{\theta, x_0} + \sum_{k=1}^{m} \int_{s-L_s}^{s} \left( \sigma(Y_{t_0,L_s},e_k) \right) \frac{1}{2} \sum_{k=1}^{m} \left( D^2 \psi)(Z_i) \right) (\sigma(Y_{t_0,L_s},e_k) \right) \frac{1}{2} \sum_{k=1}^{m} \left( D^2 \psi)(Z_i) \right) dt
\]
\[
= Y_{t_0,s}^{\theta, x_0} + \sum_{k=1}^{m} \int_{s-L_s}^{s} \left( \sigma(Y_{t_0,L_s},e_k) \right) \frac{1}{2} \sum_{k=1}^{m} \left( D^2 \psi)(Z_i) \right) \frac{1}{2} \sum_{k=1}^{m} \left( D^2 \psi)(Z_i) \right) dt
\]
An induction argument then shows that for all \( s \in [t_0, T] \) it holds \( \mathbb{P} \)-a.s. that
\[
Y_{t_0,s}^{\theta, x_0} = x_0 + \int_{t_0}^{s} \mathbb{1}_{[s-t]} dt + \int_{t_0}^{s} \mathbb{1}_{[s-t]} b dt. \tag{66}
\]
Next, (40), Jensen’s inequality, a Lyapunov-type estimate (see, e.g., [15, Lemma 2.2]) combined with (36) and (39), the fact that \( c \leq \rho \), and Lemma 2.1 (applied with \( \beta \cap \beta - 1 \) in the notation of Lemma 2.1) combined with (38)–(40), (34), the assumption that \( \kappa \in [0, p/(3\beta + 1)] \), and the fact that \( \forall h \in (0, T), x \in D_h: \varphi(x) \leq c(b^h - \kappa) \) (see (46)) imply that for all \( t \in [t_0, T] \), \( q \in [1, \infty) \) with \( \beta q \leq p \) it holds that
\[
\left( E \left[ \left\| \mu(Y_{t_0,t}^{\theta, x_0}) \right\|^q \right] + \left( E \left[ \left\| \varphi(Y_{t_0,t}^{\theta, x_0}) \right\|^q \right] \right) \right] \leq b e^{\rho \beta (t-t_0)/p} (\varphi(x_0))^{\beta/p}. \tag{67}
\]
Next, (37) and the fact that Brownian motions have independent increments prove for all \( t \in [t_0, T] \) that \( \sigma(Y_{t_0,L_s},e_k) \) and \( W_t - W_{t-L_s} \) are independent. This, (59), the Burkholder-Davis-Gundy inequality (see, e.g., Lemma 7.7 in [16]), and (67) combined with the fact that \( \forall t \in \mathbb{R}: |t| \leq |t|^2 \vee 1 \) imply that for all \( t \in [t_0, T] \), \( q \in [2, \infty) \) with \( \beta q \leq p \) it holds that
\[
\left( E \left[ \left\| Y_{t_0,L_s}^{\theta, x_0} (W_t - W_{t-L_s}) \right\|^q \right] \right] \leq \left( E \left[ \left\| Y_{t_0,L_s}^{\theta, x_0} \right\|^q \right] \right] \leq b e^{\rho \beta (t-t_0)/p} (\varphi(x_0))^{\beta/p} \sqrt{|t|} \sqrt{q(q-1)/2}. \tag{68}
\]
This and the fact that \( \forall y \in \mathbb{R}^d : \| (D^2 \psi)(y) \|_{L^2(\mathbb{R}^d)} \leq 14(1 \wedge \|y\|) \) (see [40, Lemma 3.1]) imply that for all \( t \in [t_0, T] \), \( q \in [2, \infty) \) with \( \beta q \leq p \) it holds that

\[
\left( \mathbb{E} \left[ \left( D^2 \psi \right)(Z_t)^q \right] \right)^\frac{1}{q} \leq 14b \rho^{\beta(t-t_0)/p}(\varphi(x))^{\beta/p} \sqrt{\theta} \sqrt{q(q-1)/2}. \tag{69}
\]

This, the fact that for all \( m \in L^2(\mathbb{R}^d, \mathbb{R}^d) \), \( a \in \mathbb{R}^{d \times m} \) it holds that

\[
\sum_{k=1}^{m} \| m(a e_k, a e_k) \| \leq \sum_{k=1}^{m} \left( \| m \|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} \| a e_k \| \right)^2 = \| m \|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} \| a \|^2, \tag{70}
\]

Hölder’s inequality, and (67) then prove that for all \( t \in [t_0, T] \), \( q \in [1, \infty) \) with \( 2\beta q \leq p \) it holds that

\[
\left( \mathbb{E} \left[ \left( \frac{1}{2} \sum_{k=1}^{m} \left( D^2 \psi \right)(Z_t) \right) \left( \sigma(Y_{t_0,t}) e_k, \sigma(Y_{t_0,t}) e_k \right) \right]^q \right)^\frac{1}{q} \leq \frac{1}{2} \left( \mathbb{E} \left[ \left( D^2 \psi \right)(Z_t)^q \right] \left( \sigma(Y_{t_0,t}) \right)^{2q} \right)^\frac{1}{q} \tag{71}
\]

\[
\leq \frac{1}{2} \left( \mathbb{E} \left[ \left( D^2 \psi \right)(Z_t)^q \right] \left( \sigma(Y_{t_0,t}) \right)^{2q} \right)^\frac{1}{q} \leq 7b^2 e^{2\rho^{\beta(t-t_0)/p}(\varphi(x))^{\beta/p}} \sqrt{\theta} \sqrt{q(q-1)/2}.
\]

Next, (37), (58), (59), the fact that \( \forall y \in \mathbb{R}^d : \| \psi(y) \| \leq \| y \| \), (67), and (68) show that for all \( t \in [t_0, T] \), \( q \in [1, \infty) \) with \( \beta q \leq p \) it holds that

\[
\left( \mathbb{E} \left[ \left( y_{t_0,t} - Y_{t_0,t} \right)^q \right] \right)^\frac{1}{q} \leq \left( \mathbb{E} \left[ \left( \mu(Y_{t_0,t}) \right)^q \right] \right)^\frac{1}{q} \leq \left( \mathbb{E} \left[ \left( \mu(Y_{t_0,t}) \right)^q \right] \right)^\frac{1}{q} \leq 2e^{\rho^{\beta(t-t_0)/p}(\varphi(x))^{\beta/p}} \sqrt{\theta} \sqrt{q(q-1)/2} \tag{72}
\]

This, (41), Hölder’s inequality, the triangle inequality, and (67) imply that for all \( t \in [t_0, T] \), \( q \in [1, \infty) \) with \( 2\beta q \leq p \) it holds that

\[
\left( \mathbb{E} \left[ \left( \mu(Y_{t_0,t}) \right)^q \right] \right)^\frac{1}{q} \leq \left( \mathbb{E} \left[ \left( \mu(Y_{t_0,t}) \right)^q \right] \right)^\frac{1}{q} \leq \left( \mathbb{E} \left[ \left( \mu(Y_{t_0,t}) \right)^q \right] \right)^\frac{1}{q} \leq 4b^2 (T + 1) e^{\rho^{\beta(t-t_0)/p}(\varphi(x))^{\beta/p}} \sqrt{\theta} \sqrt{q(q-1)}.
\]

This, (61), the triangle inequality, and (71) show that for all \( t \in [t_0, T] \), \( q \in [1, \infty) \) with \( 2\beta q \leq p \) it holds that

\[
\left( \mathbb{E} \left[ \left( y_{t_0,t} - \mu(Y_{t_0,t}) \right)^q \right] \right)^\frac{1}{q} \leq 11(T + 1) e^{2\rho^{\beta(t-t_0)/p}(\varphi(x))^{\beta/p}} \sqrt{\theta} \sqrt{q(q-1)}.
\]

\[
\left( \mathbb{E} \left[ \left( y_{t_0,t} - \mu(Y_{t_0,t}) \right)^q \right] \right)^\frac{1}{q} \leq 11(T + 1) e^{2\rho^{\beta(t-t_0)/p}(\varphi(x))^{\beta/p}} \sqrt{\theta} \sqrt{q(q-1)}.
\]
Next, for all $m \in L^1(\mathbb{R}^d, \mathbb{R}^d)$, $a \in \mathbb{R}^{d \times m}$ it holds that

$$\sum_{k=1}^{m} [m(ae_k)e_k^T] = [m(ae_1) \ m(ae_2) \ldots m(ae_m)] \in \mathbb{R}^{d \times m}. \quad (75)$$

This shows for all $m \in L^1(\mathbb{R}^d, \mathbb{R}^d)$, $a \in \mathbb{R}^{d \times m}$ that

$$\left\| \sum_{k=1}^{m} m(ae_k)e_k^T \right\|^2 = \sum_{k=1}^{m} \left\| m(ae_k) \right\|^2 \leq \sum_{k=1}^{m} \left\| m \left\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)} \right\|^2 = \left\| m \right\|^2_{L^1(\mathbb{R}^d, \mathbb{R}^d)} \| a \|^2. \quad (76)$$

This, (61), the fact that $\forall a \in \mathbb{R}^{d \times m}: \sum_{k=1}^{m} ae_k e_k^T = a$, the fact that

$$\forall y \in \mathbb{R}^d: \| D\psi(y) - Id_{\mathbb{R}^d} \|_{L^1(\mathbb{R}^d, \mathbb{R}^d)} \leq 3 (\| y \| \wedge 1)^2 \leq 3 \| y \|$$

(see [40, Lemma 3.1]), Hölder’s inequality, (68), and (67) imply that for all $t \in [t_0, T]$, $q \in [1, \infty)$ with $2\beta q \leq p$ it holds that

$$\left( E \left[ \left\| b_t - \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^q \right] \right)^{\frac{1}{q}} \leq \left( E \left[ \left\| (D\psi)(Z_t) - Id_{\mathbb{R}^d} \right\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)} \left\| \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^q \right] \right)^{\frac{1}{q}} \leq \left( E \left[ \left\| (D\psi)(Z_t) - Id_{\mathbb{R}^d} \right\||Z_t|| \left\| \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^q \right] \right)^{\frac{1}{q}} \leq 3 \left( E \left[ \left\| Z_t \right\|^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \left\| \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^{2q} \right] \right)^{\frac{1}{2q}} \left( E \left[ \left\| (D\psi)(Z_t) - Id_{\mathbb{R}^d} \right\|_2^2 \right] \right)^{\frac{1}{2}} \leq 3 \left( \| b_{t_0} \|^2 \right)^{\frac{1}{2}} \left( \| \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \|^2 \right)^{\frac{1}{2}} \leq 3 \left( b_{t_0} \right)^{\frac{1}{2}} \left( \| \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \|^2 \right)^{\frac{1}{2}} \leq \left( E \left[ \left\| b_t - \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^q \right] \right)^{\frac{1}{q}} \leq \left( E \left[ \left\| b_t - \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^q \right] \right)^{\frac{1}{q}} + \left( E \left[ \left\| \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) - \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^q \right] \right)^{\frac{1}{q}} \leq 7(T \vee 1)^{\frac{3}{2}} e^{2\beta q (t-t_0)/p (\varphi(x_0))^\beta/p} \sqrt{\theta} \sqrt{q(2q-1)} \left( E \left[ \left\| \sigma(Y_{t_0, t \wedge \cdot}^{\theta, x_0}) \right\|^2 \right] \right)^{\frac{1}{2}} \leq 7(T \vee 1)^{\frac{3}{2}} e^{2\beta q (t-t_0)/p (\varphi(x_0))^\beta/p} \sqrt{\theta} \sqrt{q(2q-1)}. \quad (79)$$

Next, observe that Jensen’s inequality and Tonelli’s theorem imply for all $s \in (t_0, T]$, $X \in \{(X_{t_0, t})_{t \in [t_0, T]}: (Y_{t_0, t}^{\theta, x_0})_{t \in [t_0, T]} \}$ that

$$E \left[ \exp \left( \int_{t_0}^{s \wedge \tau} \frac{U(X_t)}{T} dt \right) \right] \leq \left( E \left[ \exp \left( \frac{1}{s-t_0} \int_{t_0}^{s} U(X_t) dt \right) \right] \right)^{\frac{1}{p}} \leq \frac{1}{s-t_0} \int_{t_0}^{s} E \left[ \frac{U(X_t)}{T} \right] dt \leq \sup_{t \in [t_0, T]} \frac{E \left[ \exp \left( U(X_t) \right) \right]}{t}. \quad (80)$$

This, (42), Hölder’s inequality, (64), the fact that $\min\{U, \bar{U} \} \geq 0$, and Lemma 2.3 imply that
Next, (64) and (37) imply that \((\mathbb{I}_{[t<\tau]}\mathbb{I})\in\mathbb{E}[\mathbb{I}_{[t<\tau]}\mathbb{I}]\)-predictable. Combining this, (36), (42), (66), the fact that \((a_t)\in\mathbb{E}[\mathbb{I}_{[t<\tau]}\mathbb{I}]\), \((b_t)\in\mathbb{E}[\mathbb{I}_{[t<\tau]}\mathbb{I}]\), and \((\mathbb{I}_{[\tau<\tau]}\mathbb{I})\in\mathbb{E}[\mathbb{I}_{[\tau<\tau]}\mathbb{I}]\)-predictable (see (61)), the fact that \((X_{t_0})\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I}]\) and \((Y_{t_0})\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I}]\)-adapted, and [40, Corollary 2.12] applied for \(s \in [t_0, T]\) with \(H \cap \mathbb{R}^d, U \cap \mathbb{R}^m, O \cap \mathbb{R}^d, O \cap \mathbb{R}^d, T \cap (T - t_0)\), \((\mathcal{F}_t)\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I} \cap (\mathbb{F}_{t_0+})\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I}]\), \((\mathbb{W}_t)\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I} \cap (\mathbb{W}_{t_0+})\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I}]\), \((\mathcal{E}_k)\in\mathbb{E}[\mathbb{I}_{[1,m]}\mathbb{I} \cap (\mathbb{E}_k)\in\mathbb{E}[\mathbb{I}_{[1,m]}\mathbb{I}]\), \(\mu \cap \sigma \cap \sigma, \varepsilon \cap 1, p \cap 2r, \tau \cap ((s \wedge \sigma) - t_0)\), \((\mathcal{X}_t)\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I} \cap (\mathbb{X}_{t_0+})\in\mathbb{E}[\mathbb{I}_{[t_0]}\mathbb{I}]\), \((\delta \cap 1, \rho \cap 1, r \cap r, q \cap 2r)\) in the notation of [40, Corollary 2.12], (81), (74) (applied with \(q \cap 2r\) in the notation of (74)), (79) (applied with \(q \cap 2r\) in the notation of (79)), the assumption that \(4r \beta \leq p\) imply that

\[
\sup_{s \in [t_0, T]} \left( \mathbb{E}\left[ \left\| \mathbb{I}_{D_{t_0}}(Y_{t_0}) - Y_{t_0} \right\| \right] \right)^{1/\beta}
\leq \sup_{s \in [t_0, T]} \left( \mathbb{E}\left[ \left( \int_{t_0}^s \left( x - \mu(t) \sigma(x) \right) \right)^2 dt \right] \right)^{1/\beta}
\leq \sup_{s \in [t_0, T]} \left( \mathbb{E}\left[ \left( \int_{t_0}^s \left( x - \mu(t) \sigma(x) \right) \right)^2 dt \right] \right)^{1/\beta}
\leq \mathbb{E}\left[ \left( \int_{t_0}^s \left( x - \mu(t) \sigma(x) \right) \right)^2 dt \right]^{1/\beta}
\leq e^{(T-t_0)(3/2-1/r)} \left[ e^{(T-t_0)} \mathbb{E}\left[ \left( \int_{t_0}^s \left( x - \mu(t) \sigma(x) \right) \right)^2 dt \right] \right]^{1/\beta}
\leq 72\sqrt{2} (T \vee 1)^{3/2} e^{(2 + 2\beta/p)(T-t_0)} \left[ \mathbb{E}\left[ \left( \int_{t_0}^s \left( x - \mu(t) \sigma(x) \right) \right)^2 dt \right] \right]^{1/\beta} \mathbb{E}\left[ \left( \int_{t_0}^s \left( x - \mu(t) \sigma(x) \right) \right)^2 dt \right]^{1/\beta}
\]
Hölder’s inequality, the triangle inequality, and a combination of (67) (applied with $q \prec 2r$ in the notation of (67)) and of the assumption that $4r \beta \leq p$ therefore prove for all $s \in [t_0, T]$ that
\[
\left( \mathbb{E} \left[ \left\| \mathbb{I}_{D[\theta]} (Y_{t_0, s}^{\theta, x_0}) (X_{t_0, s}^{x_0} - Y_{t_0, s}^{\theta, x_0}) \right\|^r \right] \right)^{\frac{1}{r}}
\leq \left( \mathbb{P} \left[ Y_{t_0, s}^{\theta, x_0} \notin D[\theta] \right] \right)^{\frac{1}{r}} \left[ \left( \mathbb{E} \left[ \left\| X_{t_0, s}^{x_0} \right\|^{2r} \right] \right)^{\frac{1}{2r}} + \left( \mathbb{E} \left[ \left\| Y_{t_0, s}^{\theta, x_0} \right\|^{2r} \right] \right)^{\frac{1}{2r}} \right]^{\frac{1}{r}}
\leq \left[ c_0(\theta) \exp \left( e^{c_0(T-t_0)} U(x_0) \right) \right]^{\frac{1}{r}} \left[ 2b e^{c_0(T-t_0)\beta/p} (\varphi(x_0))^{\beta/p} \right] + 72\sqrt{2} (T \vee 1)^{3/2} b^2 e^{c_0(T-t_0)} \left( \exp \left( e^{c_0(T-t_0)} U(x_0) \right) \right)^{\frac{1}{p}} (\varphi(x_0))^{\beta/p} \left[ c_0(\theta) \right]^{\frac{1}{p}} \sqrt{\theta}
\leq 74\sqrt{2} (T \vee 1)^{3/2} b^2 \exp \left( (2 + \frac{p}{2r})(T - t_0) \right) \left[ c_0(\theta) \exp \left( e^{c_0(T-t_0)} U(x_0) \right) \varphi(x_0) \right]^{\frac{p}{r}} \sqrt{\theta}.
\] (84)

This completes the proof of Lemma 2.4.

The following corollary extends Lemma 2.4 to a formulation which we later need to apply [47, Corollary 3.12].

**Corollary 2.5.** Assume Setting 2.2, let $t_0 \in [0, T)$, $x_0 \in \mathbb{R}^d$, $t \in [t_0, T]$, $s \in [t, T]$, and let $\varepsilon: \mathbb{R} \to [0, \infty)$ be the function which satisfies for all $a \in \mathbb{R}$ that
\[
\varepsilon(a) = 74\sqrt{2} (T \vee 1)^{3/2} b^2 \exp \left( (2 + \frac{p}{2r})(T - t_0) \right) \left[ c_0(a) \exp \left( e^{c_0(T-t_0)} U(x_0) \right) \varphi(x_0) \right]^{\frac{p}{r}} \sqrt{|a|}.
\] (86)

Then it holds for all $\theta \in \mathcal{P}(t_0, T)$ that
\[
\left( \mathbb{E} \left[ \left\| X_{t_0, s}^{\theta, x_0} - X_{t_0, s}^{\theta, x_0} \right\|^r \right] \right)^{\frac{1}{r}} \leq \varepsilon(\theta).
\] (87)

**Proof of Corollary 2.5.** First, we consider the trivial cases $t = t_0$ and $t = T$. Observe that (36) and (37) imply that $Y_{t_0, t_0} = X_{t_0, t_0} = x_0$. This shows in the case $t = t_0$ that (87) is trivially true. Next, Lemma 2.4 and (86) imply for all $\theta \in \mathcal{P}(t_0, T)$ that
\[
\left( \mathbb{E} \left[ \left\| Y_{t_0, s}^{\theta, x_0} - X_{t_0, s}^{x_0} \right\|^r \right] \right)^{\frac{1}{r}} \leq \varepsilon(\theta).
\] (88)

This shows in the case $t = T$ that (87) holds. For the rest of the proof we assume that $t \in (t_0, T)$. The fact that the Brownian motion has independent increments, (36), and (37) imply that
\[
\sigma \left( \{ X_{t, s}^{\eta, x_0} : \eta \in \mathcal{P}(t, T), x \in \mathbb{R}^d \} \right) \text{ and } \sigma \left( \{ Y_{t, s}^{\eta, x_0} : \eta \in \mathcal{P}(t, T), x \in \mathbb{R}^d \} \right)
\]
are independent.

This combined with disintegration (see, e.g., [49, Lemma 2.2]), (35), the flow property, (88), and the fact that $\varepsilon_{|_{0, \infty}}$ is non-decreasing imply that for all $\theta, \eta \in \mathcal{P}(t_0, T), \eta \in \mathcal{P}(t, T), n \in \mathbb{N}$,
lem

= \left( \mathbb{E} \left[ \left\| Y_{t,s}^\eta - X_{t,s}^\eta \right\|_\bullet \right] \right)^{1/r} 
  \leq \left( \mathbb{E} \left[ \left\| Y_{t,s}^\mu - X_{t,s}^\mu \right\|_\bullet \right] \right)^{1/r}

(90)

This, (89) combined with a basic result on disintegration (see, e.g., Lemma 2.2 in [49]), (88), Fatou’s lemma, continuity of \( \varepsilon \), and the fact that \( \varepsilon(0) = 0 \) imply for all \( \theta \in \mathcal{P}(t_0, T) \) that

\[
\left( \mathbb{E} \left[ \left\| Y_{t,s}^\mu - X_{t,s}^\mu \right\|_\bullet \right] \right)^{1/r}
  \leq \liminf_{v \in \mathcal{P}(t_0, T)} \mathbb{E} \left[ \left\| Y_{t,s}^\mu - X_{t,s}^\mu \right\|_\bullet \right] \leq \varepsilon(\theta \cup |v|) = \varepsilon(\theta).
\]

This implies (87) and completes the proof of Corollary 2.5.

\[\square\]

2.3 A Lyapunov-type function

The following lemma, Lemma 2.6, shows that the functions \( V \) \( \cap (\mathbb{R}^d \ni x \mapsto (\| x \|^2 + d)^{p+1}, d, p \in \mathbb{N}, \) are suitable Lyapunov-type functions.

Lemma 2.6. Consider the notation in Subsection 1.1, let \( d, m \in \mathbb{N}, p \in [3/2, \infty), a, c \in (0, \infty), \) \( \) and let \( \mu : \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) : \mathbb{R}^{d \times m} \rightarrow \mathbb{R}, \) \( \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \) satisfy for all \( x \in \mathbb{R}^d \) that

\[
\varphi(x) = (a + \| x \|^2)^p \quad \text{and} \quad \langle \mu(x), x \rangle + \frac{1}{2}(2p - 1)\| \sigma(x) \|^2 \leq c\varphi(x)^{1/p}.
\]

(92)

Then it holds for all \( x \in \mathbb{R}^d, i \in \{1, 2, 3\} \) that \( \| (D^i\varphi)(x) \|_{L^1(\mathbb{R}^d, \mathbb{R})} \leq (2p)^i\varphi(x)^{1 - \frac{i}{p}} \) and

\[
((D\varphi)(x)(\mu(x) + \frac{1}{2}\sum_{k=1}^m (D^2\varphi)(x)(\sigma_k(x), \sigma_k(x)) \leq 2cp\varphi(x).
\]

Proof of Lemma 2.6. First, (92) and the Cauchy-Schwarz inequality show that for all \( x, u, v, w \in \mathbb{R}^d \) it holds that \( \| x \| \leq (\varphi(x))^{-1/p}, \)

\[
(D\varphi)(x)(u, v, w) = 4p(p - 1)(p - 2) \left[ a + \| x \|^2 \right]^{p-3} 2(w, v) \left\| 2(x, u) \right\| + 4p(p - 1) \left[ a + \| x \|^2 \right]^{p-3} 2(w, v) \left\| x, u \right\|
\]

\[
\leq 8p(p - 1)(p - 2)(\varphi(x))^{-2} \left\| x \right\|^3 \left\| w \right\| \left\| v \right\| \left\| u \right\|
\]

(95)

+ 4p(p - 1)(\varphi(x))^{-2} \left( \left\| w \right\| \left\| u \right\| \left\| x \right\| \left\| v \right\| \right) + 8p(p - 1)(\varphi(x))^{-2} 2\left\| x \right\| \left\| w \right\| \left\| u \right\| \left\| v \right\|.
\]
This, (9), the triangle inequality, and the assumption that $p \geq 3/2$ imply for all $x \in \mathbb{R}^d$ that
\[
\| (D\varphi)(x) \| \leq 2p(\varphi(x))^{\frac{2}{p-1}} \| x \| \leq 2p(\varphi(x))^{1-\frac{1}{p}} ,
\]
\[
\| (D^2\varphi)(x) \|_{L^p(\mathbb{R}^d, \mathbb{R})} \leq (4p(p-1) + 2p)(\varphi(x))^{\frac{2}{p-1}} \leq 2p(2p-1)(\varphi(x))^{1-\frac{1}{p}} , \quad \text{and}
\]
\[
\| (D^3\varphi)(x) \|_{L^p(\mathbb{R}^d, \mathbb{R})} \leq [8p(p-1)((p-2) \lor 0) + 12p(p-1)](\varphi(x))^{1-\frac{1}{p}} \leq p(p-1)[8(p-2) \lor 0 + 12](\varphi(x))^{1-\frac{1}{p}} \leq (2p)^3(\varphi(x))^{1-\frac{1}{p}} .
\]  
(96)

This, (93), and (92) prove for all $x \in \mathbb{R}^d$ that
\[
((D\varphi)(x))(\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} ((D^2\varphi)(x))(\sigma_k(x), \sigma_k(x)) \leq 2p(\varphi(x))^{\frac{2}{p-1}} \| \mu(x) \| + \frac{1}{2}\| (D^2V)(x) \|_{L^p(\mathbb{R}^d, \mathbb{R})} \sum_{k=1}^{m} \| \sigma_k(x) \|^2 
= 2p(\varphi(x))^{\frac{2}{p-1}} \left[ \| \mu(x) \| + \frac{1}{2}(2p-1)|\sigma(x)|^2 \right] \leq 2p(\varphi(x))^{\frac{2}{p-1}} c(\varphi(x))^{\frac{1}{p}} = 2c_p \varphi(x) .
\]
This and (96) complete the proof of Lemma 2.6. \qed 

3 Error analysis for MLP approximations

A central assumption of Theorem 3.1 below is the local monotonicity condition (105) where $U, \tilde{U}$ satisfy the exponential integrability condition (108). These conditions are satisfied by many interesting SDEs from applications; see, e.g., [15, Chapter 4]. The two main steps of the proof of Theorem 3.1 are to (a) apply [47, Corollary 3.12] with the forward process given by (113) to obtain (131) and to (b) apply [47, Lemma 2.3] to obtain the distance (130) between the exact PDE solution and the solution of the stochastic fixed-point equation with respect to (113).

**Theorem 3.1.** Consider the notation in Subsection 1.1, let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, $b, c, \beta, \gamma \in [1, \infty)$, $L, \alpha \in [0, \infty)$, $p \in [8\beta, \infty)$, $\tilde{U} \in C(\mathbb{R}^d, [0, \infty))$, $\kappa \in (0, p/(3\beta + 1)]$, $\varphi \in C^2(\mathbb{R}^d, [1, \infty))$, $U \in C^3(\mathbb{R}^d, [0, \infty))$, $g \in C(\mathbb{R}^d, \mathbb{R})$, let $f: [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be $\mathcal{B}([0, T] \times \mathbb{R}^d \times \mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, let $\mu \in C(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in C(\mathbb{R}^d, \mathbb{R}^{d \times m})$, let $F: \mathbb{R}^{[0,T] \times \mathbb{R}^d} \rightarrow \mathbb{R}^{[0,T] \times \mathbb{R}^d}$ satisfy for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}^{[0,T] \times \mathbb{R}^d}$ that
\[
(F(w))(t, x) = f(t, x, w(t, x))
\]
(98)

let $(D_h)_{h \in (0,T]} \subseteq \mathcal{B}(\mathbb{R}^d)$, let $\rho \in \mathbb{R}$ satisfy that $p = (5e^{1+\frac{1}{p}})^{3p}$, assume for all $\ell \in \{1, 2, 3\}$,
\[
\| (D^\ell \varphi)(x) \|_{L^p(\mathbb{R}^d, \mathbb{R})} \leq c|\varphi(x)|^{1-\frac{1}{p}} ,
\]
(99)

\[
(D\varphi)(x)(\mu(x)) + \frac{1}{2} \sum_{k=1}^{m} (D^2\varphi)(x)(\sigma_k(x), \sigma_k(x)) \leq c\varphi(x) ,
\]
(100)

\[
\max \left\{ \| Tf(t, x, 0) \|, \| g(x) \|, \| \mu(x) \|, \| \sigma(x) \|, \| x \| \right\} \leq b(\varphi(x))^{\frac{\alpha}{p}} ,
\]
(101)

\[
| f(t, x, w_1) - f(t, y, w_2) | \leq L | w_1 - w_2 | + bT^{-\gamma/2}(\varphi(x) + \varphi(y))^{\frac{\alpha}{p}} \| x - y \| ,
\]
(102)

\[
| g(x) - g(y) | \leq bT^{-\gamma/2}(\varphi(x) + \varphi(y))^{\frac{\alpha}{p}} \| x - y \| ,
\]
(103)

\[
\max \left\{ \| \mu(x) - \mu(y) \|, \| \sigma(x) - \sigma(y) \| \right\} \leq b \| x - y \| \left[ (\varphi(x))^{\frac{\alpha}{p}} + (\varphi(y))^{\frac{\alpha}{p}} \right] ,
\]
(104)
\[
\langle x - y, \mu(x) - \mu(y) \rangle + 3\|\sigma(x) - \sigma(y)\|^2 \\
\leq \|x - y\|^2 \left[ \frac{U(x) + U(y)}{16T} + \frac{\bar{U}(x) + \bar{U}(y)}{16} \right],
\]
(105)

\[
\|\langle D^i U(x) \rangle \|_{L^q(R^d \times R)} \leq c\|U(x)\|^{1 - \frac{1}{2}}, \quad |\bar{U}(x)| \leq c(1 + |U(x)|^\gamma),
\]
(106)

\[
|\bar{U}(x) - \bar{U}(y)| \leq c[1 + |U(x)|^\gamma + |U(y)|^\gamma]\|x - y\|, \quad \text{and}
\]
(107)

\[
\langle D U(x) \rangle (\mu(x)) + \frac{1}{2} \sum_{k=1}^m \langle D^2 U(x) \rangle (\sigma_k(x), \sigma_k(x)) \\
+ \frac{1}{2} e^{\alpha T} \|\sigma(x)^* (\nabla U)(x)\|^2 + \bar{U}(x) \leq \alpha U(x),
\]
(108)

assume for all \( h \in (0, T) \), \( x \in D_h \) that

\[
\max \left\{ \|\mu(x)\|, \|\sigma(x)\|, U(x), \theta^{\infty} \varphi(x) \right\} \leq \min \left\{ \text{ch}^{-\frac{\alpha}{\gamma}}, \text{ch}^{-\frac{1}{\gamma + \gamma}}, \text{ch}^{-\infty}, 2 \log \left( \frac{1}{\gamma} \right) \right\},
\]
(109)

let \( P(0, T) \) be the set which satisfies that

\[
P(0, T) = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : n \in \mathbb{N}, 0 = t_0 < t_1 < \ldots < t_n = T \},
\]
(110)

for every \( \delta \in P(0, T) \), \( n \in \mathbb{N} \), \( (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \) with \( \delta = (t_0, t_1, \ldots, t_n) \) let \( |\delta| \in (0, T] \), \( \langle \cdot, \cdot \rangle : [t_0, t_n] \to \mathbb{R} \) satisfy for all \( t \in (t_0, t_n] \) that

\[
|\delta| = \max_{i \in [0, n-1] \cap \mathbb{N}_0} |t_{i+1} - t_i|, \quad \langle t_0, \delta \rangle = t_0, \quad \text{and} \quad \langle t, \delta \rangle = \sup \{(t_0, t_1, \ldots, t_n) \cap [t_0, t)\},
\]
(111)

let \((\delta_n)_{n \in \mathbb{N}} \subseteq P(0, T) \) satisfy for all \( n \in \mathbb{N} \) that

\[
\delta_n = (0, 1, \ldots, n^n) \frac{F}{n^n},
\]
(112)

let \( \Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n \), \( \{(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]}) \} \) be a filtered probability space which satisfies the usual conditions, let \( v^\theta : \Theta \to [0, 1], \theta \in \Theta \), be independent random variables which are uniformly distributed on \([0, 1]\), let \( \mathcal{R}^\theta : [0, T] \times \Omega \to [0, T], \theta \in \Theta \), satisfy for all \( t \in [0, T] \), \( \theta \in \Theta \) that \( \mathcal{R}^\theta_t = t + (T - t)^{v^\theta}, \) let \( W^\theta : [0, T] \times \Omega \to \mathbb{R}^m, \theta \in \Theta \), be standard \((\mathbb{F}_t)_{t \in [0, T]}\)-Brownian motions with continuous sample paths, assume that \( \sigma(\{v^\theta : \theta \in \Theta \}) \) and \( \sigma(\{W^\theta : \theta \in \Theta, t \in [0, T]\}) \) are independent, for every \( \theta \in \Theta \), \( t \in [0, T], \) \( x \in \mathbb{R}^d \), \( \delta \in P(0, T) \) let \( Y_{t, s}^\delta(x, \omega) \) for all \( s \in (t, T] \) that \( Y_{t, s}^\delta(x) = x \) and

\[
Y_{t, s}^\delta(x) = \left[ z + 1_{D_{|\delta|}(t)}(z) \left( \mu(z) s - \mathbb{1}_{s \leq |\delta|}(t) + \frac{\sigma(z)(W^\theta_{s} - W^\theta_{s \leq |\delta|}(t))}{1 + \|\sigma(z)(W^\theta_{s} - W^\theta_{s \leq |\delta|}(t))\|^2} \right) \right] z = Y_{t, s, \delta, \theta}(x),
\]
(113)

let \( V_{n, M}^\delta : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}, \ n \in \mathbb{N}, M \in \mathbb{Z}, \theta \in \Theta, \delta \in P(0, T) \), \( V_{n, M}^\delta(t, x) \) satisfy for all \( n, M \in \mathbb{N}, \theta \in \Theta, \delta \in P(0, T), t \in [0, T], x \in \mathbb{R}^d \) that \( V_{n, M}^\delta(t, x) = V_{n, M}^\delta(t, x) = 0 \) and

\[
V_{n, M}^\delta(t, x) = \frac{1}{M^n} \sum_{i=1}^{M^n} g^\delta(\langle t, T,M \rangle - i) Y_{t, t, M}^{\delta, \theta}(x) \\
+ \sum_{i=1}^{n-1} \left( \frac{T - t}{M^{n-\ell}} \right) \sum_{i=1}^{M^{n-\ell}} \left( F(V_{t, M}^{\delta, \theta, \ell,i}) - 1_N(\ell) F(V_{t-1, M}^{\delta, \theta, \ell,i}) \right) \left( \mathcal{R}_t^{(\theta, i)}, Y_{t, M}^{\delta, \theta, \ell,i}(x) \right),
\]
(114)
let \((\text{FE}_{n,M}^\delta)_{\delta \in \mathcal{P}(0,T), n \in \mathbb{N}, M \in \mathbb{N}}\) satisfy for all \(\delta \in \mathcal{P}(0,T), n, M \in \mathbb{N}\) that \(\text{FE}_{n,M}^\delta = 0\) and

\[
\text{FE}_{n,M}^\delta \leq \left(2 + \left\lceil \frac{T}{\delta} \right\rceil\right) M^n + \sum_{\ell = 0}^{n-1} \left[ M^{n-\ell} \left(3 + \left\lceil \frac{T}{\delta} \right\rceil + \text{FE}_{\ell,M}^\delta + 1_{N}(\ell)\text{FE}_{\ell-1,M}^\delta\right)\right],
\]

(115)

and let \(c_0 : \mathbb{R} \to \mathbb{R}\) satisfy for all \(a \in \mathbb{R}\)

\[
c_0(a) = \exp\left(\exp\left(2 \left[720 \max\{T, \alpha, 1\}\right] e^{\alpha T} \right)^3 \left[720\left(e^{\alpha T}\right)^3 \max\{T, 1\} + \gamma\right] \right) \min\{|a|, 1\}^{1/8}.
\]

(116)

Then

i) for every \(t \in [0, T], x \in \mathbb{R}^d, \theta \in \Theta\) there exists an \((\mathbb{F}_s)_{s \in [t,T]}\)-adapted stochastic process \((X^\theta_{t,s}(x, \omega))_{s \in [t,T], \omega \in \Omega} : [t, T] \times \Omega \to \mathbb{R}^d\) with continuous sample paths which satisfies that for all \(s \in [t, T]\) it holds \(\mathbb{P}\)-a.s. that

\[
X^\theta_{t,s}(x) = x + \int_t^s \mu(X^\theta_{t,r}(x)) \, dr + \int_t^s \sigma(X^\theta_{t,r}(x)) \, dW^\theta_{r},
\]

(117)

ii) there exists a unique \(\mathcal{B}([0, T] \times \mathbb{R}^d)/\mathcal{B}(\mathbb{R})\)-measurable function \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) which satisfies that \(\sup_{t \in [0,T], x \in \mathbb{R}^d}[u(t, x)(\varphi(x))^{-\beta/|\rho|}] < \infty\) and which satisfies for all \(t \in [0, T], x \in \mathbb{R}^d\) that \(\mathbb{E}[|\varphi(X^0_{t,T}(x))|] + \int_t^T \mathbb{E}[|f(s, X^0_{t,s}(x), u(s, X^0_{t,s}(x)))|] \, ds < \infty\) and

\[
u(t, x) = \mathbb{E}\left[g(X^0_{t,T}(x)) + \int_t^T f(s, X^0_{t,s}(x), u(s, X^0_{t,s}(x))) \, ds\right],
\]

(118)

iii) it holds for all \(t \in [0, T], x \in \mathbb{R}^d, \delta \in \mathcal{P}(0,T), n, M \in \mathbb{N}\)

\[
\begin{align*}
\left(\mathbb{E}\left[V^\delta_{n,M}(t, x) - u(t, x)\right]^2\right)^{1/2} & \leq \frac{e^{M/2} e^{2nLT}}{M^{n/2}} + \left|\delta\right|^{1/2} \right) 1184b^3 e^{3T+\rho T + 3LT(c_0(|\delta|))} T^{1/2} \left(\varphi(x)\right)^{1/2},
\end{align*}
\]

(119)

and

iv) there exist \((n(\varepsilon, x))_{x \in \mathbb{R}^d, \varepsilon \in (0,1)} \subseteq \mathbb{N}\) such that for all \(\varepsilon, \gamma \in (0, 1), n \in [n(\varepsilon, x), \infty) \cap \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d\) it holds that \(\mathbb{E}\left[V^\delta_{n,n}(t, x) - u(t, x)\right]^2 < \varepsilon^2\) and

\[
\begin{align*}
\sum_{k=1}^{n(\varepsilon, x)} \text{FE}_{k,k}^\delta & \leq \varepsilon^{4+\gamma}
\end{align*}
\]

\[
\leq \sup_{n \in \mathbb{N}} \left[\frac{n(\varepsilon, x)}{5^{n(\varepsilon, x)} e^{2nT} T^{1/2}}\right] \left[10^4 b^3 e^{3T+\rho T + 3LT(c_0(1))} T^{1/2} \left(\varphi(x)\right)^{1/2}\right]^{1/4}.
\]

(120)

**Remark 3.2.** We discuss the computational effort of the approximation method in (114). For every \(n \in \mathbb{N}_0, M \in \mathbb{N}, \delta \in \mathcal{P}(0,T), t \in [0,T], x \in \mathbb{R}^d, \theta \in \Theta\) we think of \(\text{FE}_{n,M}^\delta\) as an upper bound for the number of function evaluations which are required to compute one realization of \(V^\delta_{n,M}(t, x)\) in (114). Then (115) can be explained as follows. For every \(n, M \in \mathbb{N}, i \in \{1, 2, \ldots, M^n\}, \delta \in \mathcal{P}(0,T), t \in [0,T], x \in \mathbb{R}^d, \theta \in \Theta\) we need no more than \((1 + [T/|\delta|])\) function evaluation to get one realization of \(Y^\delta_{t,T}(x)\) and therefore we need no more than \((2 + [T/|\delta|])\) function evaluations to get one realization of \(g(Y^\delta_{t,T}(x))\). Next, for every \(n, M \in \mathbb{N}, \ell \in \{0, 1, \ldots, n\}, i \in \{1, 2, \ldots, M^{n-\ell}\}, \delta \in \mathcal{P}(0,T), t, s \in [0,T], x, y \in \mathbb{R}^d, \theta \in \Theta\) we need (see (113)) no more than \(1 + [T/|\delta|]\) function evaluations to get one realization of
\[ Y_{t,\ell,\mathrm{m}}^{\delta, (\theta, \ell, i)}(x) \] we need (see (114)) no more than \( FE_{\delta} \) to get one realization of \( V_{t,\ell,\mathrm{m}}^{\delta, (\theta, \ell, i)}(s,y) \), we need (see (114)) no more than \( {}_{1\mathbb{N}}(\ell) FE_{\ell,\mathrm{m}}^{\delta} \) function evaluations to get one realization of \( V_{t,\ell,\mathrm{m}}^{\delta, (\theta, \ell, i)}(s,y) \), and therefore we need no more than \( 3 + [T/|\delta|] + \mathbb{I}_{\mathbb{N}}(\ell) FE_{\ell,\mathrm{m}}^{\delta} \) function evaluations to get one realization of \( (F(V_{t,\ell,\mathrm{m}}^{\delta, (\theta, \ell, i)}) - \mathbb{I}_{\mathbb{N}}(\ell) F(V_{t,\ell,\mathrm{m}}^{\delta, (\theta, \ell, i)}))(R_{t,\ell,\mathrm{m}}^{(\theta, \ell, i)}, Y_{t,\ell,\mathrm{m}}^{\delta, (\theta, \ell, i)}(x)) \).

**Proof of Theorem 3.1.** Throughout this proof let \( \Delta \subseteq [0,T]^2 \) be the set which satisfies that \( \Delta = \{(t,s) \in [0,T]^2 : t \leq s\} \) and let \( \psi \in C([0,T] \times \mathbb{R}^d, [0,\infty)) \) be the function which satisfies for all \( t \in [0,T] \), \( x \in \mathbb{R}^d \) that

\[
\psi(t,x) = \left[ \exp \left( e^{\alpha (T-t)} U(x) \right) \right]^{\frac{1}{2}} \left[ e^{\rho (T-t)} \varphi(x) \right]^{\frac{1}{2}}. \tag{121}
\]

First, (104) and the assumption that \( \phi \in C(\mathbb{R}^d, [0,\infty)) \) prove that \( \mu \) and \( \sigma \) are locally Lipschitz continuous. This, (100), (101), and a standard result on SDEs with locally Lipschitz continuous coefficients (see, e.g., [31, Corollary 2.6]) imply (i).

For the proof of (ii) we are going to apply [47, Proposition 2.2] and first check the assumptions. Jensen’s inequality, the fact that \( p \in [8\beta, \infty), 1 \leq c \leq p, \beta \in [1,\infty), \) and a combination of the assumption that \( \kappa \in (0,p/(3\beta + 1)), (113), (99), (100), (101), \) the fact that \( p = (5c^{2+1/\kappa})^{3p} \), [15, Lemma 2.2] (applied for \( t \in [0,T], s \in [t,T], x \in \mathbb{R}^d, \theta \in \Theta \) with \( T \cap T - t, O \cap \mathbb{R}^d, V \cap ([0,T-t] \times \mathbb{R}^d \ni (s, x) \mapsto \varphi(x) \in [0,\infty)), \alpha \cap ((0,T-t) \ni s \mapsto c \in [0,\infty)), \tau \cap s - t, X \cap (X_{t,\ell,\mathrm{m}}(t))_{t \in [0,T]} \) in the notation of [15, Lemma 2.2]), and of Lemma 2.1 (applied for \( \delta \in \mathcal{P}(0,T), s \in [0,T], x \in \mathbb{R}^d, \theta \in \Theta \) with \( \beta \cap \beta - 1, V \cap \varphi, \theta \cap (\delta \cup \{s\})_{s \in [T]}, Y_{t,\ell,\mathrm{m}} \cap \delta \varphi(x) \) in the notation of Lemma 2.1) imply for all \( \theta \in \Theta, \delta \in \mathcal{P}(0,T), x \in \{ Y^{\delta,0}, X^0 \}, t \in [0,T], s \in [t,T], x \in \mathbb{R}^d \) that

\[
E \left[ \exp \left( e^{\alpha (T-t)} U(x) \right) \right] \leq c_0(|\delta|) \exp \left( e^{\alpha (T-t)} U(x) \right) \tag{123}
\]

and

\[
\left( E \left[ \left\| X_{t,\ell,\mathrm{m}}^{\delta,0} (X_{t,\ell,\mathrm{m}}^{\delta,0}(x)) - X_{s,t}^{\delta,0} (Y_{t,\ell,\mathrm{m}}^{\delta,0}(x)) \right\| ^2 \right] \right)^{1/2} \\
\leq 74 \sqrt[4]{2} (T \vee 1)^{2/\kappa + 2} e^{2T + \rho T/4} c_0(|\delta|) \exp \left( e^{\alpha (T-t)} U(x) \right) |\delta|^{1/2} \tag{124}\]

Next, continuity of \( \mu, \sigma \), path continuity of \( W^\theta, \theta \in \Theta \), the fact that \( \{D_\delta\}_{\delta \in [0,T]} \subseteq B(\mathbb{R}^d) \), and Fubini’s theorem imply for all \( B([0,T] \times \mathbb{R}^d)/B([0,\infty)) \)-measurable functions \( \eta : [0,T] \times \mathbb{R}^d \rightarrow [0,\infty) \) and all \( \delta \in \mathcal{P}(0,T), \theta \in \Theta \) that

\[
\Delta \times \mathbb{R}^d \ni (t,s,x) \mapsto E \left[ \eta(s, Y_{t,\ell,\mathrm{m}}^{\delta,\theta}(x)) \right] \in [0,\infty] \text{ is } \mathcal{B}(\Delta \times \mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d) \text{-measurable.} \tag{125}
\]

Moreover, local Lipschitz continuity of \( \mu, \sigma \), (100), (101), and [5, Lemma 3.7] (applied with \( \mathcal{O} \cap \mathbb{R}^d, V \cap ([0,T] \times \mathbb{R}^d \ni (t,s,x) \mapsto e^{\eta(t)} \varphi(x) \in (0,\infty)) \) in the notation of [5, Lemma 3.7])
imply for all \( \theta \in \Theta \) and all \( \mathcal{B}([0, T] \times \mathbb{R}^d)/\mathcal{B}([0, \infty)) \)-measurable functions \( \eta: [0, T] \times \mathbb{R}^d \to [0, \infty) \) that

\[
\Delta \times \mathbb{R}^d \ni (t, s, x) \mapsto \mathbb{E}[\eta(s, X^0_{t,s}(x))] \in [0, \infty] \text{ is } \mathcal{B}(\Delta \times \mathbb{R}^d)/\mathcal{B}([0, \infty)) \text{-measurable.} \quad (126)
\]

This, (125), (122), (102), (101), and [47, Proposition 2.2] (applied for \( t \in [0, T], x \in \mathbb{R}^d, \delta \in \mathbb{P}(0, T) \) with \( O \cup \mathbb{R}^d, (X^2_{t,s})_{t,s \in [0,T]} \cap (Y^{0,\delta}_{t,s})_{t,s \in [0,T]}, (X^0_{t,s})_{t,s \in [0,T]} \cap (X^0_{t,s}(x))_{t,s \in [0,T]}, V \cap (0, T) \times \mathbb{R}^d \) establish that

a) there exist unique \( \mathcal{B}([0, T] \times \mathbb{R}^d)/\mathcal{B}(\mathbb{R}) \)-measurable functions \( u, v_\delta: [0, T] \times \mathbb{R}^d \to \mathbb{R}, \delta \in \mathbb{P}(0, T) \), which satisfy that \( \sup_{t \in [0,T], x \in \mathbb{R}^d} \left[ |u(t, x)| + |v_\delta(t, x)| \right] \) for all \( \delta \in \mathbb{P}(0, T) \), \( t \in [0, T], x \in \mathbb{R}^d \) that

\[
\begin{align*}
\mathbb{E} \left[ \left| g(Y^{0,\delta_{t,T}}) \right| + \int_t^T \left| f(s, Y^{0,\delta_{t,s}}(x), v_\delta(s, Y^{0,\delta_{t,s}}(x))) \right| \, ds \right] &< \infty, \\
\mathbb{E} \left[ \left| g(X^{0}_{t,s}) \right| + \int_t^T \left| f(s, X^{0}_{t,s}(x), u(s, X^{0}_{t,s}(x))) \right| \, ds \right] &< \infty, \\
v_\delta(t, x) &= \mathbb{E} \left[ g(Y^{0,\delta_{t,T}}) + \int_t^T f(s, Y^{0,\delta_{t,s}}(x), v_\delta(s, Y^{0,\delta_{t,s}}(x))) \, ds \right], \quad \text{and} \\
u(t, x) &= \mathbb{E} \left[ g(X^{0}_{t,s}) + \int_t^T f(s, X^{0}_{t,s}(x), u(s, X^{0}_{t,s}(x))) \, ds \right]
\end{align*}
\] (127)

and

b) it holds for all \( \delta \in \mathbb{P}(0, T) \) that

\[
\sup_{t \in [0,T], x \in \mathbb{R}^d} \left[ \frac{|u(t, x)|}{\mathbb{E}[|\mathbb{E}[\eta(s, X^0_{t,s}(x))]|^{\beta/p}(\varphi(x))^{\beta/p}]} \right] \leq \sup_{t \in [0,T], x \in \mathbb{R}^d} \left( \frac{|g(x)|}{\mathbb{E}[|\mathbb{E}[\eta(s, X^0_{t,s}(x))]|^{\beta/p}(\varphi(x))^{\beta/p}]} \right) e^{LT} \leq 2be^{LT}. \quad (128)
\]

This proves (ii).

For the proof of (iii) we are going to apply [47, Proposition 3.12] and first check the assumptions. Note that (121), the Cauchy-Schwarz inequality, (123), and (122) show for all \( \delta \in \mathbb{P}(0, T), \mathcal{X} \in \{Y^{0,\delta}, X^0\}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d \) that

\[
\mathbb{E}[\psi(s, \mathcal{X}_{t,s}(x))] = \mathbb{E} \left[ \left( \exp \left( e^{(0,s)-t} \mathcal{X}_{t,s}(x) \right) \right)^{\frac{1}{2}} \left( e^{(0,s)-t} \varphi(\mathcal{X}_{t,s}(x)) \right)^{\frac{1}{2}} \right] \\
\leq \left( \mathbb{E} \left[ \exp \left( e^{(0,s)-t} \mathcal{X}_{t,s}(x) \right) \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ e^{(0,s)-t} \varphi(\mathcal{X}_{t,s}(x)) \right] \right)^{\frac{1}{2}} \\
\leq \left[ c_0(|\delta|) \right] \left( e^{(0,s)-t} \varphi(\mathcal{X}_{t,s}(x)) \right)^{\frac{1}{2}} = \left( c_0(|\delta|) \right)^{1/2} e^{LT} \psi(t, x).
\]

Lipschitz continuity of \( \mu, \sigma \) (see (104)) implies for all \( t \in [0, T], s \in [t, T], r \in [s, T], x \in \mathbb{R}^d \) that \( \mathbb{P} \circ (X^0_{t,r})^{-1} = \mathbb{P} \circ (X^0_{s,t})^{-1} \). This combined with (i), (113), (122), (129), (127), (101), (128), (102), and (103), and with [47, Lemma 2.3] (applied for \( \delta \in \mathbb{P}(0, T) \) with \( \eta \cap (c_0(|\delta|))^{1/2}, \delta \cap 296b^2e^{3LT+\beta/4}c_0(|\delta|)^{1/4} \delta^{1/2}, p \cap p/\beta, q \cap 2, \| \| \cap \| \| \cap \| \| \| NN \cap \left(X^0_{t,s}(x)\right) t \in [0,T], s \in [t,T], x \in \mathbb{R}^d \cap (X^0_{t,s}(x)) t \in [0,T], s \in [t,T], x \in \mathbb{R}^d \cap \left. (Y^{0,\delta}_{t,s}(x)) t \in [0,T], s \in [t,T], x \in \mathbb{R}^d \cap \left. (X^{0,\delta}_{t,s}(x)) t \in [0,T], s \in [t,T], x \in \mathbb{R}^d \right), V \cap (0, T) \times \mathbb{R}^d \) establish the fact that \( 1 + LT \leq e^{LT} \), a combination of the fact that \( c_0 \geq 1 \) and of the fact that \( \forall a \in [1, \infty): ae^{LT} \leq e^{aLT} \), (121), and a combination of the fact that \( \varphi \geq 1 \) and of the fact that \( \beta/p \leq 1/4 \) imply for all \( t \in [0, T], x \in \mathbb{R}^d, \delta \in \mathbb{P}(0, T) \) that

\[
|u(t, x) - v_\delta(t, x)| \leq 4(1 + LT)T^{-1/2} e^{3LT+\beta/4} c_0(|\delta|)^{1/4} \\
\leq 1184b^2 e^{3LT+\beta/4} c_0(|\delta|)^{1/4} |\delta|^{1/2} \\
\leq 1184b^2 e^{3LT+\beta/4} c_0(|\delta|)^{1/4} |\delta|^{1/2}.
\]

\[130\]

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Next, [47, Corollary 3.12] (applied for \( \delta \in \mathcal{P}(0, T) \), \( t \in [0, T] \) with \( \rho \rightarrow 2\beta \rho/p, (Y_{t,x})_{t \in [0,T], x \in \mathcal{E}}, \Theta \in \mathcal{E}_{t \in [0,T], x \in \mathcal{E}}, \Phi \in \mathcal{E}_{t \in [0,T], x \in \mathcal{E}}, (V_{\delta,\rho}^{\delta,\rho})_{n,M \in \Xi}, \theta \in \Xi, (V_{\theta,\rho}^{\theta,\rho})_{n,M \in \Xi}, \theta \in \Xi, u \leq v^\delta, \phi \leq v^{2\beta/p}, \tau \leq t \) in the notation of [47, Corollary 3.12]), (98), (102), (127), (101), (128), and (122) imply for all \( \delta \in \mathcal{P}(0, T), t \in [0, T], n, M \in \mathbb{N} \) that

\[
\sup_{x \in \mathbb{R}^d} \left[ \mathbb{E} \left[ \frac{|V_{n,M}^{\delta,\rho}(t, x) - v_{\delta}(t, x)|^2}{(\phi(x))^{2\beta/p}} \right] \right]^{1/2} \leq 2e^{M/2}M^{-n/2}(1 + 2T)^{n-1}e^\rho^{2\beta T/p} \leq 2e^{M/2}M^{-n/2}(1 + 2T)^n e^{2\beta T/p} \leq 2e^{M/2}M^{-n/2}e^{2\beta T/p}.
\]

This, the triangle inequality, (130), the fact that \( 2\beta \leq p \), the fact that \( b, c_0, \phi \geq 1 \), and the fact that \( \forall a \in \mathbb{R} : c_0(a) \leq c_0(1) \) (see (116)) show for all \( n, M, n, t \in [0, T], x \in \mathbb{R}^d, \delta \in \mathcal{P}(0, T) \) that

\[
\left( \mathbb{E} \left[ |V_{n,M}^{\delta,\rho}(t, x) - u(t, x)|^2 \right] \right)^{1/2} \leq \left( \mathbb{E} \left[ |V_{n,M}^{\delta,\rho}(t, x) - v_{\delta}(t, x)|^2 \right] \right)^{1/2} + |u(t, x) - v_{\delta}(t, x)|
\leq 2e^{M/2}M^{-n/2}e^{2\beta T/p} \left( \phi(x) \right)^{2\beta/p} + 1184b^2(1 + 2T)^n e^{2\beta T/p} \left( \phi(x) \right)^{2\beta/p}
\]

\[
\leq (e^{M/2}e^{2\beta T/p}M^{-n/2} + |\delta|^{1/2}(1 + 2T)) |\phi(x)|^{1/2} + 1184b^2(1 + 2T)^n e^{2\beta T/p} |\phi(x)|^{1/2}.
\]

This shows (iii).

To prove (iv) let \( (n(\varepsilon, x))_{x \in (0,1), x \in \mathbb{R}^d} \subseteq [0, \infty] \) satisfy for all \( \varepsilon \in (0, 1), x \in \mathbb{R}^d \) that

\[
n(\varepsilon, x) = \inf \left\{ n \in \mathbb{N} : \sup_{k \in [n, \infty) \cap \mathbb{N}, t \in [0, T]} \mathbb{E} \left[ |V_{k,k}^{\delta,\rho}(t, x) - u(t, x)|^2 \right] < \varepsilon^2 \} \cup \{ \infty \} \right. \}
\]

(133)

The fact that \( \lim_{n \to \infty} (e^{n/2}e^{2\beta T/p}n^{-n/2}) = 0 \) and (71) then show for all \( x \in \mathbb{R}^d, \varepsilon \in (0, 1) \) that

\[
n(\varepsilon, x) \in \mathbb{N}.
\]

(134)

Next, a combination of (115) and of [49, Lemma 3.6] (applied with \( d \leq (2 + [T/|\delta|]) \), \( \mathcal{F}_{n,M} \mathcal{M}, n, M \in \mathbb{E} \), the notations of [49, Lemma 3.6]) and the fact that \( \forall \delta \in \mathcal{P}(0, T) : 2 + [T/|\delta|] \leq 4T/|\delta| \) show for all \( \delta \in \mathcal{P}(0, T), n, M \in \mathbb{N} \) that \( \mathcal{F}_{n,M} \leq (2 + [T/|\delta|]) (5M)^n \leq 4T|\delta|^{-1}(5M)^n \). This, (42), and the fact that \( \forall n \in \mathbb{N} : n^3 \leq 3^n \) show for all \( n \in \mathbb{N}, k \in [1, n + 1] \cap \mathbb{N} \) that

\[
2n^3 \mathcal{F}_{k,k}^\delta \leq 8nT|\delta|^{-1}(5k)^k \leq 8n(5k)^k \leq 8n(20n^2)^{n+1} = 160n^320^n2^n \leq 160n^{2n}60^n.
\]

(135)
This implies for all \( \gamma \in (0, \infty) \), \( n \in \mathbb{N} \), \( t \in [0, T] \), \( x \in \mathbb{R}^d \) that \( \sum_{k=1}^{n+1} \mathbb{F}^{\delta_k}_{k,k} \leq 160 nt^{2n60} \) and

\[
\left( \sum_{k=1}^{n+1} \mathbb{F}^{\delta_k}_{k,k} \right) \left( \mathbb{E} \left[ \left| V_{n,n}^{\delta_n,0}(t, x) - u(t, x) \right|^2 \right] \right)^{\frac{1}{2}} \leq 160 nt^{2n60} \]

\[
\left( \left( e^{n/2} e^{2nLT} n^{-n/2} + |\delta_n|^{1/2} T^{-1/2} \right) 1184 b^3 e^{3T+r+3LT(c_1(1))^{1/2}+e^{nT}U(x)/4} (\varphi(x))^{1/2} \right)^{\gamma+4} = 160 n^{2n60} \left( e^{n/2} e^{2nLT} n^{-n/2} + n^{-n/2} \right) 1184 b^3 e^{3T+r+3LT(c_1(1))^{1/2}+e^{nT}U(x)/4} (\varphi(x))^{1/2} \right)^{\gamma+4} \leq 160 n^{-\gamma/2} 60n \left( e^{n/2} e^{2nLT} n^{-n/2} + n^{-n/2} \right) 1184 b^3 e^{3T+r+3LT(c_1(1))^{1/2}+e^{nT}U(x)/4} (\varphi(x))^{1/2} \right)^{\gamma+4} \leq n^{-\gamma/2} \left( 160 n^{1/2} e^{n/260} e^{2nLT} 1184 b^3 e^{3T+r+3LT(c_1(1))^{1/2}+e^{nT}U(x)/4} (\varphi(x))^{1/2} \right)^{\gamma+4} \leq n^{-\gamma/2} \left( 5^n e^{2nLT} 10^4 b^3 e^{3T+r+3LT(c_1(1))^{1/2}+e^{nT}U(x)/4} (\varphi(x))^{1/2} \right)^{\gamma+4} .
\]

This, (135), and (134) imply that for all \( \varepsilon, \gamma \in (0, 1) \), \( t \in [0, T] \), \( x \in \mathbb{R}^d \) it holds in the case \( n(\varepsilon, x) = 1 \) that \( \sum_{k=1}^{n(\varepsilon, x)} \mathbb{F}^{\delta_k}_{k,k} \leq \mathbb{F}^{\delta_1}_{1,1} \leq 160 \cdot 60 \) and it holds in the case \( n(\varepsilon, x) \geq 2 \) that

\[
\left( \sum_{k=1}^{n_{\varepsilon,x}} \mathbb{F}^{\delta_k}_{k,k} \right) \varepsilon^{1+\gamma} \leq \left( \sum_{k=1}^{n_{\varepsilon,x}} \mathbb{F}^{\delta_k}_{k,k} \right) \mathbb{E} \left[ \left| V_{n,n}^{\delta_n,0}(t, x) - u(t, x) \right|^2 \right] \right)^{\frac{1}{2}} \leq \sup_{n \in \mathbb{N}} \left[ n^{-\gamma/2} \left( 5^n e^{2nLT} \right)^{\gamma+4} \right] 10^4 b^3 e^{3T+r+3LT(c_1(1))^{1/2}+e^{nT}U(x)/4} (\varphi(x))^{1/2} \right)^{\gamma+4} .
\]

This, the fact that \( b, c, \varphi \geq 1 \), and the fact that \( 160 \cdot 60 \leq (10^4)^4 \) prove that

\[
\left( \sum_{k=1}^{n_{\varepsilon,x}} \mathbb{F}^{\delta_k}_{k,k} \right) \varepsilon^{1+\gamma} \leq \sup_{n \in \mathbb{N}} \left[ \left( 5^n e^{2nLT} \right)^{\gamma+4} \right] 10^4 b^3 e^{3T+r+3LT(c_1(1))^{1/2}+e^{nT}U(x)/4} (\varphi(x))^{1/2} \right)^{\gamma+4} .
\]

This, (133), and (134) imply (iv). The proof of Theorem 3.1 is thus completed.

The error estimate (119) shows that the \( L^2 \)-distance between our approximations and the PDE solution at a fixed space-time point converges to 0 as \( n \to \infty \) for every fixed \( M > e^{2LT} \). To get an computational effort of order \( 4^+ \) as in (120), however, we need to let \( M \to \infty \) as \( n \to \infty \). We typically choose \( M = n \). In this case, the error decays up to exponential factors like \( n^{-n/2} \) (resp. like \( 1/\sqrt{n!} \)) and the number of function evaluations grows up to exponential factors like \( n^{2n} \) (resp. like \( (n!)^2 \)) as \( n \to \infty \); cf. (135).

In the following Example 3.3 we show that semilinear PDEs corresponding to competitive Lotka-Volterra equations can be approximated without curse of dimensionality under suitable assumptions on the parameters. We note that the coefficients of (139) are not globally Lipschitz continuous.

**Example 3.3.** Consider the notation from Subsection 1.1, let \( T \in (0, \infty) \), \( r = (r_i)_{i \in \mathbb{N}} \in [0, \infty)^N \), \( a = (a_{ij})_{i,j \in \mathbb{N}} \in [0, \infty)^{N \times N} \) satisfy that \( \sup_{i \in \mathbb{N}} r_i < \infty \) and \( \sup_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |a_{ij}|^2 < \infty \), for every \( x \in \mathbb{R} \) let \( x^+ = \max\{0, x\} \), and for every \( d \in \mathbb{N} \) let \( u_d : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) be a viscosity solution of the PDE

\[
\frac{\partial}{\partial t} u_d(t, x) + \sum_{i=1}^{d} r_i x_i \left( 1 - \sum_{j=1}^{d} a_{ij} x_j^+ \right) \left( \frac{\partial}{\partial x_i} u_d(t, x) \right) = \sum_{i=1}^{d} \frac{|x_i|^2 (\frac{\partial^2}{\partial x_i^2} u_d)(t, x)}{2(1 + \|x\|^2)^2} \tag{139}
\]

Then \( (u_d(0, 0))_{d \in \mathbb{N}} \) can be approximated without suffering from the curse of dimensionality.
To prove this statement we apply Theorem 3.1 and show that all assumptions are satisfied. For this fix $d \in \mathbb{N}$, and let $\alpha, r, \bar{a} \in [0, \infty)$, $U \in C(\mathbb{R}^d, [0, \infty))$, $f \in C([0, 1] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$, $\varphi \in C^3(\mathbb{R}^d, \mathbb{R})$, $f \in C([0, 1] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d) \in C(\mathbb{R}^d, \mathbb{R}^{d \times d})$ satisfy for all $i, j \in \{1, 2, \ldots, d\}$, $x = (x_1, x_2, \ldots, x_d)$, $t \in [0, T]$, $w \in \mathbb{R}$ that

$$
\mu_i(x) = r_ix_i \left(1 - \sum_{j=1}^{d} a_{ij}x_j^+\right), \quad \sigma(x) = \text{diag} \left(\frac{x_1}{1 + \|x\|^2}, \frac{x_2}{1 + \|x\|^2}, \ldots, \frac{x_d}{1 + \|x\|^2}\right),
$$

(140)

$$
f(t, x, w) = \sin(\|x\| + w), \quad g(x) = \|x\|^2,
$$

(141)

$$
\varphi(x) = (1 + \|x\|^2)^8, \quad U(x) = 8T \left(8T e^{\alpha T} r^2 \bar{a}^2 + 3 + \bar{r}\right) + e^{-\alpha T} \|x\|^2, \quad \bar{U}(x) = 0,
$$

(142)

$$
\bar{r} = \sup_{r \in \mathbb{N}} r_i, \quad \bar{a} = \left(\sup_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |a_{ij}|^2\right)^{\frac{1}{2}}, \quad \alpha = 2 \bar{r} + 3.
$$

(143)

First, (140) and (143) show for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that

$$
\langle x, \mu(x) \rangle = \sum_{i=1}^{d} \left(r_i x_i^2 \left(1 - \sum_{j=1}^{d} a_{ij}x_j^+\right)\right) \leq \bar{r}\|x\|^2 \quad \text{and} \quad \|\sigma(x)\| \leq \|x\|.
$$

(144)

This and (142) show for all $x \in \mathbb{R}^d$ that $(\mu(x), x) + 7.5\|\sigma(x)\|^2 \leq (\bar{r} + 7.5)\|x\|^2 \leq (\bar{r} + 7.5)(\varphi(x))^\frac{4}{3}$. This and Lemma 2.6 (applied with $m \cap d$, $p \cap 8$, $a \cap 1$, $c \cap \bar{r} + 7.5$ in the notation of Lemma 2.6) show for all $x \in \mathbb{R}^d$, $i \in \{1, 2, 3\}$ that

$$
\|\langle D^i \varphi(x) \rangle_{L^1(\mathbb{R}^d, \mathbb{R})} \leq 16^i \varphi(x)^{1 - \frac{1}{6^n}}
$$

(145)

and

$$
\langle \langle D \varphi(x) \rangle \rangle (\mu(x)) + \frac{1}{2} \sum_{k=1}^{d} \left(\langle D^2 \varphi(x) \rangle (\sigma_k(x), \sigma_k(x)) \right) \leq 16(\bar{r} + 7.5)\varphi(x).
$$

(146)

Furthermore, (141), the fact that $\forall x, y \in \mathbb{R}$: $|\sin(x) - \sin(y)| \leq |x - y|$, and the triangle inequality show for all $t \in [0, T]$, $x, y \in \mathbb{R}^d$, $w_1, w_2 \in \mathbb{R}$ that

$$
|f(t, x, w_1) - f(t, y, w_2)| = |\sin(\|x\| + w_1) - \sin(\|y\| + w_2)|
\leq |\|x\| + w_1 - (\|y\| + w_2)| \leq |w_1 - w_2| + \|x - y\|.
$$

(147)

Next, (141), the fact that $\forall x \in \mathbb{R}$: $x \leq 1 + \frac{x^2}{2}$, and (142) show for all $t \in [0, T]$, $x, y \in \mathbb{R}^d$ that

$$
|g(x) - g(y)| = |\|x\|^2 - \|y\|^2| = (\|x\| + \|y\|)(\|x\| - \|y\|)
\leq 0.5(1 + \|x\|^2 + 1 + \|y\|^2)\|x - y\| \leq 0.5 \left[ (\varphi(x))^\frac{4}{3} + (\varphi(y))^\frac{4}{3} \right] \|x - y\|.
$$

(148)

In addition, (140), the triangle inequality, (143), the Cauchy–Schwarz inequality, and the fact that $\forall x \in \mathbb{R}$: $|x^+| \leq |x|$ show for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that

$$
\|\mu(x)\| = \left(\sum_{i=1}^{d} r_i x_i \left(1 - \sum_{j=1}^{d} a_{ij}x_j^+\right)^2\right)^{\frac{1}{2}} \leq \bar{r}\|x\| + \left(\sum_{i=1}^{d} r_i x_i \sum_{j=1}^{d} a_{ij}x_j^+\right)^{\frac{1}{2}}
\leq \bar{r}\|x\| + r \left(\sum_{i=1}^{d} |x_i|^2\right)^{\frac{1}{2}} \left(\max_{i \in \mathbb{N}} \sum_{j=1}^{d} a_{ij}x_j^+\right) \leq \bar{r}\|x\| + \|x\| \left(\max_{i \in \mathbb{N}} \sum_{j=1}^{d} a_{ij}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{d} |x_j^+|^2\right)^{\frac{1}{2}}
\leq \bar{r}\|x\| + \bar{r}\|x\|^2.
$$

(149)
This, (140)–(142), and the fact that $\forall x \in \mathbb{R}: x \leq 0.5(1 + x^2)$ show for all $x \in \mathbb{R}^d$ that

$$
|Tf(t, x, 0)| + |g(x)| + \|\mu(x)\| + \|\sigma(x)\|^2 + \|x\|
\leq T + \|x\|^2 + (\bar{r}\|x\| + \bar{r}\|x\|^2) + 1 + \|x\| = T + 1 + (1 + \bar{r})\|x\| + (1 + \bar{r}\bar{a})\|x\|^2
$$

(150)

$$
\leq T + 0.5(3 + \bar{r}) + [0.5(1 + \bar{r}) + (1 + \bar{r}\bar{a})]\|x\|^2 \leq [T + 0.5(3 + \bar{r}) + (1 + \bar{r}\bar{a})](\varphi(x))^\frac{1}{2}.
$$

Next, the triangle inequality, the Cauchy–Schwarz inequality, (143), and the fact that $\forall x, y \in \mathbb{R}: |x^+ - y^+| \leq |x - y|$ show for all $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$ that

$$
\left( \sum_{i=1}^{d} \left| \left( x_i \sum_{j=1}^{d} a_{ij} x_j^+ - y_i \sum_{j=1}^{d} a_{ij} y_j^+ \right)^2 \right| \right)^\frac{1}{2}
= \left( \sum_{i=1}^{d} \left( x_i - y_i \right) \left( \sum_{j=1}^{d} a_{ij} x_j^+ + y_i \sum_{j=1}^{d} a_{ij} (x_j^+ - y_j^+) \right)^2 \right)^\frac{1}{2}
\leq \left( \sum_{i=1}^{d} (x_i - y_i)^2 \right)^\frac{1}{2} \left( \sup_{i \in \mathbb{N}} \left( \sum_{j=1}^{d} a_{ij} x_j^+ \right)^2 \right) + \left( \sum_{i=1}^{d} |y_i|^2 \right)^\frac{1}{2} \left( \sup_{i \in \mathbb{N}} \left( \sum_{j=1}^{d} a_{ij} (x_j^+ - y_j^+) \right)^2 \right)\frac{1}{2}
\leq \|x - y\| \left( \sup_{i \in \mathbb{N}} \sum_{j=1}^{d} |a_{ij}|^2 \right)^\frac{1}{2} \left( \sum_{j=1}^{d} x_j^+ |2^\frac{1}{2} \right) + \|y\| \left( \sup_{i \in \mathbb{N}} \sum_{j=1}^{d} |a_{ij}|^2 \right)^\frac{1}{2} \left( \sum_{j=1}^{d} x_j^+ - y_j^+ |2^\frac{1}{2} \right)
\leq \|x - y\| (\bar{a}\|x\| + \bar{a}\|y\|).
$$

(151)

Furthermore, (140) shows for all $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$ that

$$
\mu_i(x) - \mu_i(y) = r_i x_i \left( 1 - \sum_{j=1}^{d} a_{ij} x_j^+ \right) - r_i y_i \left( 1 - \sum_{j=1}^{d} a_{ij} y_j^+ \right)
= r_i (x_i - y_i) - r_i \left( x_i \sum_{j=1}^{d} a_{ij} x_j^+ - y_i \sum_{j=1}^{d} a_{ij} y_j^+ \right).
$$

(152)

This, the triangle inequality, (143), (151), the fact that $\forall x, y \in \mathbb{R}: xy \leq 0.5x^2 + 0.5y^2$, and (142) show for all $x, y \in \mathbb{R}^d$ that

$$
\|\mu(x) - \mu(y)\| \leq \bar{r}\|x - y\| + \bar{r}\|x - y\|(\bar{a}\|x\| + \bar{a}\|y\|)
= \bar{r}\|x - y\|((1 + \bar{a}\|x\| + \bar{a}\|y\|) \leq \bar{r}\|x - y\| \left( 1 + 0.5\bar{a} + 0.5\bar{a}\|x\|^2 + 0.5\bar{a} + 0.5\bar{a}\|y\|^2 \right)
\leq \bar{r}(1 + 0.5\bar{a})\|x - y\| \left[ 1 + \|x\|^2 + 1 + \|y\|^2 \right]
\leq \bar{r}(1 + 0.5\bar{a})\|x - y\| \left[ (\varphi(x))^\frac{1}{2} + (\varphi(y))^\frac{1}{2} \right].
$$

(153)

This, (152), the Cauchy–Schwarz inequality, and (151) show for all $x, y \in \mathbb{R}^d$ that

$$
\langle x - y, \mu(x) - \mu(y) \rangle = \sum_{i=1}^{d} \left[ r_i (x_i - y_i)^2 - r_i (x_i - y_i) \left( x_i \sum_{j=1}^{d} a_{ij} x_j^+ - y_i \sum_{j=1}^{d} a_{ij} y_j^+ \right) \right]
\leq \bar{r}\|x - y\|^2 + \bar{r}\|x - y\| \cdot \|x - y\|(\bar{a}\|x\| + \bar{a}\|y\|)
\leq \bar{r}(1 + \bar{a}\|x\| + \bar{a}\|y\|)\|x - y\|^2.
$$

(154)
This, the fact that \( \forall x, y \in \mathbb{R}: xy \leq \frac{1}{2}Sx^2 + \frac{1}{2}x^2, \) and (142) show for all \( x, y \in \mathbb{R}^d \) that

\[
\langle x - y, \mu(x) - \mu(y) \rangle + 3\|\sigma(x) - \sigma(y)\|^2 \leq \left[ 3 + \bar{r} + \bar{r}a\|x\| + \bar{r}a\|y\| \right]\|x - y\|^2
\]

\[
\leq \left[ 3 + \bar{r} + \frac{1}{2}Sx^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}Sx^2 + \frac{1}{2}\|y\|^2 \right]\|x - y\|^2
\]

\[
= \left[ \frac{1}{2} \left( 8Te^{\alpha T}x^2 + 3 + \bar{r} \right) + \frac{1}{2}\|x\|^2 + \frac{1}{2} \left( 8Te^{\alpha T}x^2 + 3 + \bar{r} \right) + \frac{1}{2}\|y\|^2 \right]\|x - y\|^2
\]

\[
= \frac{U(x) + U(y)}{16T}\|x - y\|^2.
\]

Next, (142) shows for all \( x, y, z \in \mathbb{R}^d \) that

\[
(\nabla U)(x) = \frac{2e^{-\alpha T}x}{\|x\|^2} \quad \text{and} \quad ((\nabla^2 U)(x))(y, z) = \frac{2e^{-\alpha T}(y, z)}{\|x\|^4}.
\]

This, (140), (144), (143), and (142) show for all \( x \in \mathbb{R}^d \) that

\[
(\mathbf{D}U(x))(\mu(x)) + \frac{1}{2} \sum_{k=1}^d (\nabla U)(x)(\sigma_k(x), \sigma_k(x)) + \frac{1}{2}e^{\alpha T}\|\sigma(x)\|^2(\nabla U)(x)\|^2 + \bar{U}(x)
\]

\[
\leq \langle 2e^{-\alpha T}x, \mu(x) \rangle + \frac{1}{2} \sum_{k=1}^d 2e^{-\alpha T}\|\sigma_k(x)\|^2 + \frac{1}{2}e^{\alpha T}4e^{-2\alpha T}\|x\|^2
\]

\[
\leq 2e^{-\alpha T}\bar{r}\|x\|^2 + e^{-\alpha T}\|x\|^2 + 2e^{-\alpha T}\|x\|^2 = e^{-\alpha T}(2\bar{r} + 3)\|x\|^2 \leq \alpha U(x).
\]

Combining this, (145), (146), (147), (148), (150), (153), (155), and (156) yields that the conditions (99)–(108) are satisfied (with \( p \bowtie 16, \beta \bowtie 2, L \bowtie 1, \gamma \bowtie 1, \) and suitably large enough \( b, c \) which do not depend on \( d \)).

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