Critical collapse of a massive vector field

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We perform numerical simulations of the critical gravitational collapse of a massive vector field. The result is that there are two critical solutions. One is equivalent to the Choptuik critical solution for a massless scalar field. The other is periodic.

I. INTRODUCTION

Critical gravitational collapse was first found by Choptuik in simulations of a spherically symmetric massless scalar field. A natural question to pose is then how critical collapse behaves when the scalar field has a mass, since this will introduce a characteristic length that destroys the scale invariance of the field equations. This question was studied by Brady et al. The results of reference show that there are two critical solutions: one which is essentially the Choptuik critical solution for the massless scalar field, and another which is a periodic solution first found by Seidel and Suen. At first it might seem puzzling that the Choptuik solution can be a critical solution for both the massless and massive scalar field. The resolution of this conundrum is that as the singularity is approached in the Choptuik critical solution, the amplitude of the scalar field remains bounded while its gradient diverges. In the stress energy tensor, the mass terms are associated with the amplitude of the field, while other terms are associated with its gradient. So as the singularity is approached the mass terms in the stress energy become negligible.

Given the results of reference one might conjecture that similar behavior occurs in the case of a spherically symmetric massive vector field: i.e. that there is a critical solution for which the mass of the vector field becomes negligible. However, this conjecture involves a paradox: a massless vector field is just a Maxwell field, and a spherically symmetric Maxwell field has no degrees of freedom. Therefore there is no gravitational collapse (and thus no critical solution) of a spherically symmetric massless vector field. What then is the critical behavior of a massive vector field?

In this paper we consider this question. We perform numerical simulations of the collapse of a spherically symmetric massive vector field. The equations and numerical methods are presented in section 2. Results are given in section 3 and conclusions in section 4.

II. EQUATIONS AND NUMERICAL METHODS

A massive vector field is described by the Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} \mu^2 A_a A^a$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ and $\mu$ is a constant. Note: throughout, we use the conventions of Wald and in particular use a metric with signature $(-, +, +, +)$. Note that were we to employ the opposite signature, the sign of one of the terms in the Proca Lagrangian would have to be changed (see e.g.). The equation of motion that follows from equation is

$$\nabla_a F^{ab} - \mu^2 A^b = 0$$

from which it follows that

$$\nabla_a A^a = 0$$
It also follows from equation (11) that the Einstein field equation is
\[ G_{ab} = 2F_{ac}F^c_b + 2\mu^2 A_a A_b - g_{ab} \left( \frac{1}{2} F_{cd} F^{cd} + \mu^2 A_c A^c \right) \] (4)

We now specialize to spherical symmetry. We employ two different methods to simulate the Einstein-Proca system: a Cauchy method using polar-radial coordinates and a characteristic method using the coordinates used by Christodoulou [6] to treat the Einstein-scalar system. The metric in polar-radial coordinates takes the form
\[ ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (5)

Define the quantities \( X \) and \( W \) by
\[ X \equiv \frac{a}{\alpha} A_t \] (6)
\[ W \equiv \frac{1}{\alpha a} (\partial_t A_r - \partial_r A_t) \] (7)

From the definition of \( W \) we find
\[ \partial_t A_r = \alpha a W + \partial_r \left( \frac{\alpha}{a} X \right) \] (8)

Then from equation (5) we have
\[ \partial_t X = \frac{1}{r^2} \partial_r (r^2 W) + \mu^2 X \] (9)

From equation (2) we find the following constraint equation for \( W \)
\[ \frac{1}{r^2} \partial_r (r^2 W) + \mu^2 X = 0 \] (10)

Equations (8), (9) and (10) provide the evolution equations for the matter field. In spherical symmetry the metric has no degrees of freedom. Therefore the metric functions \( \alpha \) and \( a \) are given by “constraint” equations once the matter fields are known. To find the appropriate constraint equations, note that for a metric of the form given in equation (5) the \( tt \) and \( rr \) components of the Einstein tensor are
\[ G_{tt} = \frac{\alpha^2}{r^2 a^3} \left[ -a + a^3 + 2r \frac{\partial a}{\partial r} \right] \] (11)
\[ G_{rr} = \frac{1}{r^2 \alpha} \left[ (1 - a^2) \alpha + 2r \frac{\partial \alpha}{\partial r} \right] \] (12)

It then follows from equations (11) and (4) that
\[ \frac{1}{\alpha} \frac{\partial \alpha}{\partial r} + \frac{a^2 - 1}{2r} = \frac{r}{2} \left[ \mu^2 (X^2 + A_r^2) \right] \] (13)

Then using equations (11), (12) and (4) we find
\[ \frac{1}{\alpha} \frac{\partial \alpha}{\partial r} + \frac{1}{a} \frac{\partial a}{\partial r} = \mu^2 (X^2 + A_r^2) \] (14)

To implement these equations numerically, we replace spatial derivatives with centered differences and implement the time evolution using the iterated Crank-Nicholson method. We also put in Kreiss-Oliger dissipation for added stability. Initial data for this system is \( X \) and \( A_r \) at the initial time. Given these initial data, equations (10), (13) and (14) are then integrated in turn to obtain \( W \) and the metric functions. Finally equations (8) and (9) are used to produce \( A_r \) and \( X \) at the next time step.

In addition to this unigrid code, we also perform simulations with an adaptive mesh code. Note that our equations are quite similar to those used by Choptuik et al [9] to simulate critical collapse in the Einstein-Yang-Mills system. Our adaptive code is produced by modifying the code of reference [9] to simulate our system.

We now present the characteristic method using the coordinates of reference [6]. Here the metric takes the form
\[ ds^2 = -e^{2\nu} dt^2 - 2e^{\nu+\lambda} dudr + r^2 d\Omega^2 \] (15)
We introduce the null vectors
\[ l^a = e^{-\lambda} \left( \frac{\partial}{\partial r} \right)^a \]
\[ n^a = e^{-\nu} \left( \frac{\partial}{\partial u} \right)^a - \frac{1}{2} e^{-\lambda} \left( \frac{\partial}{\partial r} \right)^a \]
(16)

The matter in this coordinate system is determined by the components \( A_u \), \( A_r \) and \( F_{ur} = e^{\nu+\lambda} W \). Note that \( W \) defined in this way is the same as in equation (17) as can be seen by the fact that \( F^{ab}F_{ab} = -2W^2 \) in both cases. We also introduce the quantities \( g \equiv e^{\nu+\lambda} \) and \( \bar{g} \equiv e^{\nu-\lambda} \).

For a metric of the form of equation (15) we have
\[ G_{ab}l^al^b = 2\bar{g} rg^2 \frac{\partial g}{\partial r} \]
(18)
\[ G_{ab}n^al^b = -\frac{1}{gr^2} \left[ \frac{\partial}{\partial r} (rg) - g \right] \]
(19)

From equation (18) it then follows that the corresponding Einstein equations become
\[ \frac{2\bar{g}}{rg^2} \frac{\partial g}{\partial r} = 2\mu^2 \frac{g}{g} (A_r)^2 \]
(20)
\[ -\frac{1}{gr^2} \left[ \frac{\partial}{\partial r} (rg) - g \right] = W^2 \]
(21)

We now define a scalar field \( \phi \) by \( \partial_r \phi = \mu A_r \). Note that this defines \( \phi \) up to addition of an overall constant, since smoothness of \( \phi \) implies that \( \partial_r \phi = \partial_u \phi \) at \( r = 0 \). We also define the quantities \( h \) and \( \bar{h} \) by
\[ h \equiv \frac{\partial}{\partial r} (r\phi) \]
\[ \bar{h} \equiv \frac{1}{r} \int_0^r h dr \]
(22)
(23)

Then the solution of equations (20) and (21) become
\[ g = \exp \left[ \int_0^r \frac{1}{r} (h - \bar{h})^2 dr \right] \]
(24)
\[ \bar{g} = \frac{1}{r} \int_0^r g(1 - W^2) dr \]
(25)

Next we find an expression for the matter variable \( W \) in terms of \( h \). Contracting equation (22) with \( l_a \) we find
\[ \frac{\partial}{\partial r} (r^2 W) + \mu^2 r^2 A_r = 0 \]
(26)
for which the solution is
\[ W = -\frac{\mu}{r^2} \int_0^r r(h - \bar{h}) dr \]
(27)

We now find an evolution equation for \( h \). From \( F_{ab} = 2\partial_{[a}A_{b]} \) it follows that
\[ \partial_a A_r - \partial_r A_a = gW \]
(28)
Which from the definition of \( \phi \) leads to
\[ \partial_u \partial_r \phi = \partial_r (\mu A_a) + \mu gW \]
(29)

Now from this equation, its integral with respect to \( r \) and equation (22) we find
\[ \partial_u h = \partial_r (r\mu A_u) + \mu g r W + \int_0^r \mu g W dr \]
(30)
Thus to find the evolution equation for \( h \) we must find an expression for \( \partial_r(r\mu A_u) \) in terms of \( h \). To do this, we note that from equation (3) it follows that

\[
\partial_u A_r + \frac{1}{r^2} \partial_r \left[ r^2 (A_u - \bar{g} A_r) \right] = 0 \tag{31}
\]

Then subtracting equation (28) from equation (31) we obtain

\[
\partial_r (\mu r A_u) = -\frac{1}{2} \mu g W + \frac{1}{2} \partial_r (r^2 \bar{g} \partial_r \phi) \tag{32}
\]

Note that equation (32) can be integrated to yield an expression for the remaining matter variable \( A_u \) in terms of \( h \). Thus, given \( h \) at a time \( u \), all matter and metric variables at that time can be determined by integrals. Now using equation (23) and equation (28) in equation (30) we obtain

\[
Dh = \frac{1}{2r} (h - \bar{h}) (g - \bar{g}) - \frac{1}{2} gr W^2 + \frac{1}{2} \mu g r W + \int_0^r \mu g W dr \tag{33}
\]

Here \( D = \partial_u - (\bar{g}/2) \partial_r \) is derivative along the ingoing null direction.

The numerical method used is the same as that used in reference [14] for scalar field collapse. Initial data is given for \( h \) at an initial \( u \). Equations (26), (27), (24) and (25) are then integrated to find the other matter and metric variables. The integration method is Simpson’s rule for unequally spaced points, but near the origin a Taylor series is used. Then equation (33) is used to find \( h \) at the next value of \( u \). Each grid point is an ingoing light ray, and both \( h \) and \( r \) are evolved along the grid points. When a grid point reaches \( r = 0 \), it is removed from the computational grid and when half of the points have been lost, they are put back in between the remaining points by using interpolation.

The scale invariance \((A_a, g_{ab}) \rightarrow (k A_a, k^2 g_{ab})\) of the Einstein-Maxwell system extends to the Einstein-Proca system. Specifically, if \((A_a, g_{ab}, \mu)\) is a solution of equations (2) and (4) then \((kA_a, k^2g_{ab}, \mu/k)\) is also a solution of these equations, where \( k \) is any positive constant. This allows us to set \( \mu = 1 \) without loss of generality, which we do in all runs. Note that large \( k \) can render \( \mu \) negligible while retaining the effect of the additional Proca terms. We will encounter this effect in our investigation of the critical behavior of this system.

III. RESULTS

Runs were done on various Unix and Linux workstations and on PCs. To test universality, we tried initial data of several forms, including a gaussian shape and a sech shape. However, here we will present only the results of runs with the gaussian initial data. Specifically, for the Cauchy codes we use the following form of initial data: \( X = 0 \) and \( A_r = pr \exp[-(r - r_0)^2/\sigma^2] \) where \( p, r_0 \) and \( \sigma \) are constants. For the characteristic code, the initial data is \( \phi = p r^2 \exp[-(r - r_0)^2/\sigma^2] \). In simulations, we fix \( r_0 \) and \( \sigma \) and have \( p \) as the parameter that is varied. The critical value of \( p \) (denoted \( p^* \)) is found by a binary search.

We find two different critical solutions depending on the value of \( \sigma \). One is a periodic type I critical solution. Figure 1, produced using the unigrid Cauchy code, shows \( X \) at \( r = 0 \) as a function of \( t \) for this solution. For this run, we have \( \sigma = 1.5, r_0 = 3.0 \) and \( p^* = 0.104135195147191 \).

The other solution is a type II DSS critical solution which appears to be identical to the Choptuik critical solution for a massless scalar field. Figure 2 shows a plot of \( \ln M vs \ln (p - p^*) \) for solutions above but near the critical one. (Here \( M \) is the mass of the black hole). This plot was produced using the adaptive Cauchy code. Here, \( r_0 = 3.0, \sigma = 0.5 \) and \( p^* = 0.134075353579 \). For the Choptuik critical solution, the results of [11, 12] show that the graph of \( \ln M vs \ln (p - p^*) \) is a straight line with a periodic wiggle. Here the slope of the line is called \( \gamma \) and the period of the wiggle is \( T_w = \Delta/(2\gamma) \) where \( \Delta \) is the period of the DSS critical solution. The simulations of [12] give \( \gamma = 0.374 \) and \( \Delta = 3.4453 \) which yields \( T_w = 4.61 \). We fit the data of figure 2 to a straight line plus a sine wave. We find that the slope of the line is \( \gamma = 0.379 \) and the period of the wiggle is \( T_w = 4.63 \). Thus, to the accuracy of our simulation, our DSS critical solution gives values of \( \gamma \) and \( T_w \) in agreement with those of the Choptuik critical solution.

Similarly figure 3 shows a plot of \( \ln R_{max} vs \ln (p^* - p) \) for solutions below but near the critical one. Here \( R_{max} \) is the maximum value of the scalar curvature at the center. This figure was produced using the characteristic code. Here, \( r_0 = 2.0, \sigma = 0.5 \) and \( p^* = 0.05018050202078927 \). (Note though that due to the different type of data, these parameters have different meaning than in the Cauchy case). As shown in [13] this sort of plot should also be a straight line with a periodic wiggle. Here the slope of the line should be \( -2\gamma \) and the period of the wiggle should be \( T_w = \Delta/(2\gamma) \). A fit of the data of this figure to a straight line plus a sine wave yield that the slope of the line is \( -2\gamma = -0.727 \) which gives rise to \( \gamma = 0.363 \) while the period of the wiggle is \( T_w = 4.64 \). These values are again comparable to the values of the Choptuik critical solution.
We now make a direct comparison between this critical solution and the Choptuik critical solution for a massless scalar field. The simplest way to do this is to note that our characteristic equations (22, 24), (27) and (33) formally go over to the corresponding equations for a massless scalar field if we set the parameter $\mu$ to zero. We will return to this point in the next section. Thus we can find the Choptuik critical critical solution with our code by performing a binary search with $\mu = 0$. Figure 4 contains a comparison of the two critical solutions. What is plotted is $h$ at $r = 0$ as a function of $T$ where $e^{-T} = u* - u$ and $u*$ is the value of $u$ at which the singularity forms. We use the invariances of the two systems to choose offsets in $T$ and $h$ so that the two solutions coincide at a maximum of $h$. Note that the
FIG. 3: $\ln R_{\text{max}}$ vs $\ln(p^* - p)$ near the DSS critical solution

FIG. 4: $h(0)$ vs $T$ for the Proca DSS critical solution and the Choptuik critical solution

two solutions (after an initial transient has died away) are the same.

IV. CONCLUSIONS

Given the results of reference[2] for a massive scalar field, it is not surprising that a massive vector field has a type I periodic critical solution. What does seem surprising is that it has a DSS critical solution that is identical to that of
a massless scalar field. For a DSS critical solution we would expect that since length scales are becoming arbitrarily small, that \( \mu \) is becoming negligible compared to the inverse of the relevant length scale (or rather, since we set \( \mu = 1 \), that the relevant inverse dimensionless length scale is becoming arbitrarily large).

Thus the DSS critical solution should in some sense also be a solution of the “\( \mu \to 0 \) limit” of the equations. We have already seen how to make sense of this limit in the case where spherical symmetry is imposed and the system is expressed in terms of variables chosen to be similar to those of reference [6]. We now show how to make sense of this limit more generally using equations (2 - 4). If \( A_a \) itself has a smooth \( \mu \to 0 \) limit, then the \( \mu \to 0 \) limit of equations \( (2 - 4) \) is simply the Einstein-Maxwell equations. Instead we assume that \( A_a \) takes the form

\[
A_a = \frac{1}{\mu} P_a + \mu Q_a
\]

(34)

Then in order that the stress-energy have a non-singular \( \mu \to 0 \) limit, we must have \( \nabla_{[a} P_{b]} = 0 \) and therefore there must be a scalar field \( \phi \) such that \( P_a = \nabla_a \phi \). Then in the \( \mu \to 0 \) limit equations (3) and (4) become respectively

\[
\nabla_a \nabla_a \phi = 0 \quad (35)
\]

\[
G_{ab} = 2 \nabla_a \phi \nabla_b \phi - g_{ab} \nabla_c \phi \nabla_c \phi \quad (36)
\]

In other words, the \( \mu \to 0 \) limit of the Einstein-Proca system becomes the Einstein-scalar system.

Consequently it is not surprising that these two systems possess the same DSS critical solution. We therefore see that the Einstein-Maxwell theory is not the \( \mu \to 0 \) limit of the Einstein-Proca system, a discontinuity reminiscent of that observed in pure gravitation [14]. Indeed, since gravitation couples to all forms of energy, it couples to the longitudinal mode of the Proca field, amplifying it during spherically symmetric critical collapse relative to the transverse modes which become negligible. The physics of the critical gravitational collapse of a Proca field therefore becomes indistinguishable from that of a massless scalar.

We close by noting that the type I critical solution we have found is essentially an analog for the Proca system of the soliton solution found by Seidel and Suen[3] for the massive scalar field. We therefore expect that our solution could be found directly using the methods of reference [3]. We will address this issue in a separate paper.

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[1] M. Choptuik, Phys. Rev. Lett. 70, 9 (1993)
[2] P. Brady, C. Chambers and S. Goncalves, Phys. Rev. D 56, 6057 (1997)
[3] E. Seidel and W. Suen, Phys. Rev. Lett. 66, 1659 (1991)
[4] R. Wald, General Relativity (University of Chicago Press, Chicago 1984)
[5] C. Itzykson and J. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980)
[6] D. Christodoulou, Commun. Math. Phys. 105, 337 (1986)
[7] M. Choptuik, in Deterministic Chaos in General Relativity, edited by D. Hobill, A. Burd and A. Coley (Plenum, New York, 1994), pp. 155-175
[8] H. Kreiss and J. Oliger, Methods for the Approximate Solution of Time Dependent Problems, Global Atmospheric Research Programme, Publication Series No. 10 (1973)
[9] M. Choptuik, T. Chmaj and P. Bizon, Phys. Rev. Lett. 77, 424 (1996)
[10] D. Garfinkle, Phys. Rev. D51, 5558 (1995)
[11] S. Hod and T. Prian, Phys. Rev. D55, 440 (1997)
[12] C. Gundlach, Phys. Rev. D55, 695 (1997)
[13] D. Garfinkle and G.C. Duncan, Phys. Rev. D58, 064024 (1998)
[14] H. van Dam and M.J.G. Veltman, Nucl.Phys. B22, 397 (1970)