The Kontsevich Connection on the Moduli Space of FZZT Liouville Branes

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Abstract

We point out that insertions of matrix fields in (connected amputated) amplitudes of (generalized) Kontsevich models are given by covariant derivatives with respect to the Kontsevich moduli. This implies that correlators are sections of symmetric products of the (holomorphic) tangent bundle on the (complexified) moduli space of FZZT Liouville branes. We discuss the relation of Kontsevich parametrization of moduli space with that provided by either the (p, 1) or the (1, p) boundary conformal field theories. It turns out that the Kontsevich connection captures the contribution of contact terms to open string amplitudes of boundary cosmological constant operators in the (1, p) minimal string models. The curvature of the connection is of type (1,1) and has delta-function singularities at the points in moduli space where Kontsevich kinetic term vanishes. We also outline the extension of our formalism to the c = 1 string at self-dual radius and discuss the problems that have to be understood to reconcile first and second quantized approaches in this case.

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1 Introduction

Kontsevich matrix models \cite{1,2,3,4,5,6} have been invented to compute correlators of observables in closed non-critical string theory coupled to either minimal or \( c = 1 \) conformal matter. After the discovery of the AdS/CFT correspondence and the understanding of open/closed duality in the topological context \cite{7,8}, it might have been natural to suspect that Kontsevich actions should be interpreted as “world volume” actions for some system of branes. Only a few years ago, however, advances in Liouville boundary conformal field theory \cite{9,10} have allowed to identify precisely which branes are described by Kontsevich models: they are stable branes of Liouville theory coupled to minimal matter, also known as FZZT branes. A derivation of the Kontsevich model as the Witten open string field theory on these branes has been given recently \cite{11} and only in the case of \((1,2)\) matter — which is equivalent to pure topological gravity. So, curiously enough, the open string interpretation of Kontsevich models, which should represent, “microscopically”, their very definition, has been very little explored: the present paper concentrates on this aspect.

For closed string computations, the observables of interest are \( gl(N) \) invariant composite operators of the Kontsevich matrix field \( X_{ij} \); with \( i, j = 1, \ldots, N \). The physics of open string stretching between \( N \) branes is captured instead by “colored” correlators of the elementary field \( X_{ij} \). \( X \) is the string field corresponding to a primary vertex operator of the boundary conformal field theory, that, as we will see, is the boundary cosmological constant operator \( B \). \( B \) is a marginal operator and this manifests itself as the fact that the Kontsevich action also depends on an external matrix source \( Z_{ij} \). The eigenvalues of \( Z \) are the variables by which the string field theory parametrizes the moduli space of FZZT branes. Naively, insertions of \( B \) in world-sheet correlators are given by derivatives with respect to the moduli \( Z \). One of the lessons from topological string theories \cite{12} is that in general this is true only up to contact terms, which in our case come from the points in moduli space where open vertex operators collide between themselves or with a node of the surface. The contribution of contact terms is to convert simple derivatives with respect to the moduli into covariant ones. In this paper we show that this is exactly what happens also in the case of the Kontsevich model: the connected amputated \( n \)-point correlator of the matrix fields \( X \) is given by the \( Z_{ij} \)-covariant derivative of the \( n - 1 \)-point correlator. We deduce the connection which defines the covariant derivative from a differential equation.
satisfied by the effective potential of the model, as a function of the matrix $Z$ and of a matrix field source $\phi$. We do this in Section 2 for the case of the $(1, 2)$ model and in Section 4 for the general $(1, p)$ models and $c = 1$. Extending the matrix $Z$ to the complex domain, one finds that the curvature of this connection has vanishing $(2, 0)$ part, as required by the symmetry of the correlators with respect to exchange of two fields $X$. The $(1, 1)$ part of the curvature, however, is non-zero and has delta-functions with support at the points in $Z$ moduli space at which the Kontsevich kinetic term vanishes. At these points the Kontsevich matrix theory becomes singular, perhaps signaling the fact that more degrees of freedom are needed to describe the physics in these regions of the moduli space. Such a possibility is not unlikely since the Kontsevich matrix models are reductions of the full-fledged Witten open string field theory and should be thought of as effective theories.

From the point of view of the boundary conformal field theory, the computation of contact terms is generally a difficult problem. The Kontsevich connection, encoding the information on these contacts, thus provides predictions that are quite non-trivial. In Section 3, a comparison between the Kontsevich results and the world-sheet computations is carried out, in the case of a single brane and at string tree level. This comparison, apart from elucidating some conceptual issues regarding the consistency between string field theory and conformal field theory descriptions, also provides non-trivial support for the conjecture, not yet explicitly proven beyond $p = 2$, that Kontsevich theory for $(1, p)$ minimal strings is the (effective) string field theory of FZZT branes coupled to minimal matter.

In order to compare with boundary conformal field theory results, one first has to understand the relation between the string field theory parameter $z$ and the conformal field theory modulus, which, in our case, is the boundary cosmological constant $\mu_B$. In general \cite{13, 14}, string field theory moduli and conformal field theory moduli give different parametrization of the open string moduli space. In this context, we find it convenient to relate our analysis to the recent results presented in \cite{15}, which have shown that disk string amplitudes of boundary cosmological operators in $(p, q)$ minimal models are encoded in their closed ground rings. Since the closed ground ring equations also naturally appear in Kontsevich models as the equations of motion, the picture of \cite{15} is the natural one to prove the equivalence between Kontsevich open string field theory and open minimal strings. The upshot of the discussion of Section 3 is the following: The Kontsevich parameter $z$ is a non-trivial function of the boundary cosmological constant of the $(p, 1)$ model.
theory — it is the dual boundary cosmological constant. Correlators of $n$ boundary operators have to be interpreted as sections of the $n$-th symmetric product of the tangent space to the moduli space. The Kontsevich connection defines a covariant derivative on this bundle. This connection vanishes in the parametrization of moduli space associated to the conformal $(p, 1)$ model but not in the parametrization associated to the Kontsevich model. The reason why this is possible is that Kontsevich theory, unlike the conformal $(p, 1)$ theory, picks up the globally defined parameter on the moduli space. Thus, the example we are considering confirms the belief \cite{13, 14} that open string field theory always provides a one-to-one parametrization of moduli space. Since there is a non-trivial reparametrization between the moduli of the open string field theory and those of the boundary conformal field theory, correlators computed in the two theories are equal only up to a coordinate transformation on moduli space.

The existence of a non-trivial Kontsevich connection also elucidates the equivalence of the string theories based on the $(p, 1)$ and $(1, p)$ conformal models. The parametrization of moduli space of the $(1, p)$ model is the globally defined one: therefore, the corresponding connection does not vanish, unlike the one of the $(p, 1)$ model. This means that amplitudes of the $(1, p)$ model must receive non-trivial contributions from contact terms: it is only by taking these into account that one can establish the equality — up to reparametrization — of these amplitudes with those of the $(p, 1)$ model.

Finally, in Section 4.2, we extend the analysis to the Kontsevich model \cite{6} for the $c = 1$ non-critical strings at self-dual radius and we conjecture that also in this case the matrix field correlators, that we compute, should be identified with amplitudes of boundary cosmological constant operators. In this case, however, a comparison with worldsheet computations appears at present problematic: The naive limit of correlators of Liouville boundary cosmological constant operators for $c = 1$ is divergent and no univocal prescription to “renormalize” these divergences has been understood so far.

2 The Kontsevich Connection

(Generalized) Kontsevich matrix models are defined by the following matrix integral

$$e^{F(g_s, Z)} = \int [dM] e^{-\frac{1}{g_s} S_K(M, Z)}$$

(1)
where both $M$ and $Z$ are hermitian $N \times N$ matrices and the action
\[ S_K(M, Z) \equiv \text{Tr}[V(M) - V(Z) - V'(Z)(M - Z)] \quad (2) \]
depends on the potential $V(M)$.

In the early nineties it was understood that when the potential $V(M)$ is the polynomial
\[ V(M) = \frac{M^{q+1}}{q+1} \quad (3) \]
with $q$ integer greater than 1, the associated Kontsevich matrix model is related to \textit{closed} bosonic string theory coupled to minimal conformal matter of the $(p, q)$ type. Kontsevich originally considered the model with $q = 2$:
\[ e^{F_2(g_s, Z)} = e^{-\frac{2}{g_s} \text{Tr} Z^3} \int [dM] e^{-\frac{1}{g_s} \text{Tr} \left[ \frac{M^3}{3} - Z^2 M \right]} = \int [dX] e^{-\frac{1}{g_s} \text{Tr} \left[ Z X^2 + \frac{X^3}{3} \right]} \quad (4) \]
where we have put \[ M = Z + X \quad (5) \]
He showed that the matrix model encodes the correlators at all genus of pure topological gravity. This latter theory had been understood to be equivalent to the double scaled one-matrix models which describe bosonic closed strings coupled to $(p, 2)$ conformal matter.

The way in which the model computes the correlators of topological gravity is the following: Let us denote by $O_n$, with $n = 1, 2, \ldots$ the observables of closed topological gravity. The correlators of the $O_n$’s are encoded in the generating function
\[ Z_{\text{top.grav.}}(g_s, t_n) \equiv \langle e^{\sum_n t_n O_n} \rangle \quad (6) \]
which depends on both the closed string coupling constant $g_s$ and the infinite number of variables $t_n$. Let also $z_i$, with $i = 1, \ldots, N$, be the $N$ eigenvalues of the hermitian matrix $Z$. Kontsevich showed that
\[ Z_{\text{top.grav.}}(g_s, t_n) \bigg|_{t_n = t_n(z_i)} = \mathcal{N}(Z)^{-1} e^{F_2(g_s, Z)} \quad (7) \]
where the functions $t_n(z_i)$ are defined by means of the the Frobenius-Miwa-Kontsevich transform \[ t_n(z_i) \equiv -\frac{g_s}{n} \text{Tr} Z^{-n} \quad (8) \]
\[ ^1 \text{Standard arguments of matrix field theory show that the free energy } F_2(g_s, Z) \text{ on the R.H.S. of Eq. } 7 \text{ does indeed depend on } g_s \text{ and the combinations } \text{ of the eigenvalues } z_i. \]
and the normalization factor $\mathcal{N}(Z)$ is given by the quadratic matrix integral

$$
\mathcal{N}(Z) = \int [dX] e^{-\frac{1}{g_s} \text{Tr} Z X^2}
$$

(9)

Recently it was proposed [11] that Kontsevich matrix integral (4) be interpreted as Witten open string field theory on $N$ stable branes of non-critical bosonic strings in the background of (1,2) conformal minimal matter. These branes have boundary conditions of the FZZT type along the Liouville direction. FZZT boundary conditions are parametrized by the boundary cosmological constant $\mu_B$. The parameter $\mu_B$ shows up in the Liouville action on the world-sheet $\Sigma$ with boundary $\partial \Sigma$ as the coupling constant that multiplies the boundary cosmological constant vertex operator $B_b(\phi) \equiv e^{b \phi}$

$$
S_{\text{boundary}} = \mu_B \int_{\partial \Sigma} e^{b \phi}
$$

(10)

where $\phi$ is the Liouville field, and the parameter $b$ is related to the Liouville Virasoro central charge

$$
c_{\text{Liouville}} = 1 + 6Q^2 \quad Q = b + \frac{1}{b}
$$

(11)

When the matter is represented by the $(p, q)$ minimal models,

$$
b^2 = \frac{p}{q}
$$

(12)

According to [11], the matrix $X$ in the Kontsevich integral (4) is to be identified with the second quantized field corresponding to the boundary cosmological constant vertex operator $B_b(\phi)$. The eigenvalues $z_i$ of the matrix $Z$ which appears in the Kontsevich action define coordinates on the moduli space of FZZT branes. In the boundary conformal field theory the same moduli space is parametrized by the boundary cosmological constants $\mu^{(i)}_B$ on the $i$-th brane. In general [13] [14], open string field theory and conformal field theory coordinates are non-trivially related. In [11], by comparing Kontsevich and FZZT amplitudes for the $(1, 2)$ theory, it was found that this relation is simply given by

$$
\mu^{(i)}_B = z_i
$$

(13)

Since, however, the $(1, 2)$ and $(2, 1)$ theory are physically equivalent, Kontsevich field theory must provide predictions which are consistent with both. In
the following we explore the implications of the identifications of \[11\] for the string amplitudes of $n$ boundary cosmological constant operators $B_b(\phi)$: We will explain, among other things, how the $(1, 2)$ and $(2, 1)$ model are related and in which way Kontsevich theory gives a consistent description of both.

Connected correlators of $n$ matrix fields $X_{ij}$

$$\langle X_{i_1j_1} \cdots X_{i_nj_n} \rangle$$

are given by differentiating $n$ times the generating function $F_X(g_s, Z, J)$

$$e^{F_X(g_s, Z, J)} \equiv \int [dX] e^{-\frac{1}{g_s} \text{Tr} [Z X^2 + \frac{X^3}{3} + \text{Tr} J X]}$$

with respect to the matrix source $J_{ji}$ and then setting $J$ to zero. (Connected) string amplitudes equal *amputated* connected correlators of the second quantized theory. Therefore, string amplitudes with $n$ boundary cosmological integrated vertex operators $B_b(\phi)$ are given by differentiating $n$ times the function

$$F_X(g_s, Z, \frac{1}{g_s} \{Z, \phi\})$$

with respect to $\phi_{ji}$ and then setting $\phi$ to zero. It is useful to introduce the effective potential $H(Z, \phi)$ which is defined by subtracting to the generating function of the *amputated* connected correlators its free part:

$$H(Z, \phi) \equiv F_X(g_s, Z, \frac{1}{g_s} \{Z, \phi\}) - \log N(Z) - \frac{1}{g_s} \text{Tr} Z \phi^2$$

We now want to derive an equation that allows to express insertions of fields $X$ in amputated connected amplitudes in terms of derivatives with respect to $Z$. The existence of such an equation might be expected a priori, given the relation (13) and the fact that insertions of integrated vertex operators $B_b(\phi)$ in string amplitudes should be given — up to contact terms — by derivatives with respect to $\mu_B$ (see Eq. (10)). We will see that Kontsevich theory implies specific values of the contact terms.

To derive this equation, let us go back to the integration variable $M$ in Eq. (15):

$$e^{F_X(g_s, Z, J)} = e^{\frac{-Z^3}{3g_s} - \text{Tr} J Z} \int [dM] e^{-\frac{1}{g_s} \text{Tr} [M^3 + Z^2 + g_s J M]} =
\begin{align*}
&= e^{\frac{-Z^3}{3g_s} - \text{Tr} J Z + F_2(g_s, (Z^2 + g_s J)^{\frac{3}{2}}) + \frac{2}{3g_s} \text{Tr} (Z^2 + g_s J)^{\frac{3}{2}}} 
\end{align*}$$

(18)
and thus
\[ H(Z, \phi) = F_2(g_s, [Z^2 + \{Z, \phi\}]^{\frac{3}{2}}) - \log N(Z) + \]
\[ + \frac{2}{3g_s} [\text{Tr} (Z^2 + \{Z, \phi\})^{\frac{3}{2}} - \text{Tr} Z^3 - 3 \text{Tr} Z^2 \phi - \frac{3}{2} \text{Tr} Z \phi^2] = \]
\[ = f_2(Y) - \log N(Z) - \frac{2}{3g_s} [\text{Tr} Z^3 + 3 \text{Tr} Z^2 \phi + \frac{3}{2} \text{Tr} Z \phi^2] \]  (19)
where we introduced the hermitian \( N \times N \) matrix
\[ Y \equiv Z^2 + \{Z, \phi\} \]  (20)
and the invariant function \( f_2(Y) \) of \( Y \)
\[ f_2(Y) \equiv F_2(g_s, Y^{\frac{3}{2}}) + \frac{2}{3g_s} \text{Tr} Y^{\frac{3}{2}} \]  (21)
It follows from Eq. (4) that \( f_2(Y) \) satisfies
\[ e^{f_2(g_s, Y)} = \int [dM] e^{-\frac{1}{2g_s} [\text{Tr} M^3 - YM]} \]  (22)
Eq. (19) essentially states that the generating function \( H(Z, \phi) \) depends on \( Z \) and \( \phi \) only through the combination \( Y \). In the following we want to translate this property of \( H(Z, \phi) \) into a differential equation.

From Eq. (20) one obtains
\[ \frac{\partial Y_{ij}}{\partial Z_{kl}} = \frac{\partial Y_{ij}}{\partial \phi_{kl}} + \delta_{ik} \phi_{lj} + \phi_{ik} \delta_{lj} \]  (23)
Therefore
\[ \frac{\partial f_2(Y)}{\partial Z_{kl}} = \frac{\partial f_2(Y)}{\partial \phi_{kl}} + \{\phi, \frac{\partial f_2(Y)}{\partial Y}\}_{lk} \]  (24)
Moreover
\[ \frac{\partial f_2(Y)}{\partial \phi_{kl}} = \sum_{i,j} (Z_{ik} \delta_{lj} + \delta_{ik} Z_{lj}) \frac{\partial f_2(Y)}{\partial Y_{ij}} = \sum_{i,j} \Delta_{ik;ji}(Z) \frac{\partial f_2(Y)}{\partial Y_{ij}} \]  (25)
where we introduced \( \Delta_{ik;ji}(Z) \), the inverse of the propagator of Kontsevich matrix theory. Going to the basis in which \( Z \) is diagonal,
\[ \Delta_{ik;ji}^{-1}(Z) = \frac{\delta_{ij} \delta_{kl}}{z_i + z_j} \]  (26)
Therefore, in such basis one has
\[
\frac{\partial f_2(Y)}{\partial Y_{kl}} = \frac{1}{z_k + z_l} \frac{\partial f_2(Y)}{\partial \phi_{kl}} \tag{27}
\]
Plugging this into (24) we obtain
\[
\frac{\partial f_2(Y)}{\partial Z_{kl}} = \frac{\partial f_2(Y)}{\partial \phi_{kl}} + \sum_i \phi_{li} \frac{\partial f_2(Y)}{\partial \phi_{ki}} + \frac{\phi_{ik}}{z_i + z_l} \frac{\partial f_2(Y)}{\partial \phi_{il}} \tag{28}
\]
Hence, from (19) we deduce
\[
\left[ \frac{\partial}{\partial Z_{kl}} - \frac{\partial}{\partial \phi_{kl}} - \sum_i \frac{\phi_{li}}{z_i + z_k} \frac{\partial}{\partial \phi_{ki}} - \frac{\phi_{ik}}{z_i + z_l} \frac{\partial}{\partial \phi_{il}} \right] H(Z, \phi) = \\
- \left[ \frac{\partial}{\partial Z_{kl}} - \frac{\partial}{\partial \phi_{kl}} - \sum_i \frac{\phi_{li}}{z_i + z_k} \frac{\partial}{\partial \phi_{ki}} - \frac{\phi_{ik}}{z_i + z_l} \frac{\partial}{\partial \phi_{il}} \right] \\
\left[ \log N(Z) + \frac{2}{3} g_s [\text{Tr} Z^3 + 3 \text{Tr} Z^2 \phi + \frac{3}{2} \text{Tr} Z \phi^2] \right] \tag{29}
\]
Since
\[
\frac{\partial}{\partial Z_{kl}} \frac{2}{3} [\text{Tr} Z^3 + 3 \text{Tr} Z^2 \phi + \frac{3}{2} \text{Tr} Z \phi^2] = (2 Z^2 + 2 \{Z, \phi\} + \phi_2)_{lk} \tag{30}
\]
and
\[
\frac{\partial}{\partial \phi_{kl}} \frac{2}{3} [\text{Tr} Z^3 + 3 \text{Tr} Z^2 \phi + \frac{3}{2} \text{Tr} Z \phi^2] = (2 Z^2 + \{Z, \phi\})_{lk} \tag{31}
\]
it follows that
\[
\left[ \frac{\partial}{\partial Z_{kl}} - \frac{\partial}{\partial \phi_{kl}} - \sum_i \frac{\phi_{li}}{z_i + z_k} \frac{\partial}{\partial \phi_{ki}} - \frac{\phi_{ik}}{z_i + z_l} \frac{\partial}{\partial \phi_{il}} \right] \frac{2}{3} g_s \text{Tr} Z^3 + \\
3 \text{Tr} Z^2 \phi + \frac{3}{2} \text{Tr} Z \phi^2] = - \frac{1}{g_s} (\phi^2)_{lk} \tag{32}
\]
Moreover
\[
\frac{\partial \log N(Z)}{\partial Z_{kl}} = - \delta_{lk} \sum_i \frac{1}{z_i + z_k} \tag{33}
\]
Eq. (29) becomes therefore
\[
\left[ \frac{\partial}{\partial Z_{kl}} - \frac{\partial}{\partial \phi_{kl}} - \sum_i \frac{\phi_{li}}{z_i + z_k} \frac{\partial}{\partial \phi_{ki}} - \frac{\phi_{ik}}{z_i + z_l} \frac{\partial}{\partial \phi_{il}} \right] H(Z, \phi) = \\
= \delta_{lk} \sum_i \frac{1}{z_i + z_k} + \frac{1}{g_s} (\phi^2)_{lk} \tag{34}
\]
Introducing the generating function of the (both connected and non-connected) amputated correlators of $X$

$$Z(Z, \phi) \equiv e^{H(Z, \phi)} \quad (35)$$

we obtain the homogeneous linear equation

$$\left[ \frac{\partial}{\partial Z_{kl}} \phi_{kl} \sum_{i} \frac{\phi_{li}}{z_i + z_k} \frac{\partial}{\partial \phi_{ki}} - \frac{\phi_{ik}}{z_i + z_l} \frac{\partial}{\partial \phi_{il}} + \right.$$  

$$\left. - \delta_{lk} \sum_{i} \frac{1}{z_i + z_k} - \frac{1}{g_s} \phi^2_{lk} \right] Z(Z, \phi) = 0 \quad (36)$$

Eq. (34) relates $X$ insertions in correlators to derivatives with respect to $Z_{kl}$. Let us see explicitly how this works. Let us introduce the covariant $\phi$-derivative

$$D_{lk}^\phi \equiv \frac{\partial}{\partial \phi_{kl}} + \sum_{i} \left( \frac{\phi_{li}}{z_i + z_k} \frac{\partial}{\partial \phi_{ki}} + \frac{\phi_{ik}}{z_i + z_l} \frac{\partial}{\partial \phi_{il}} \right) \quad (37)$$

Since

$$\left[ \frac{\partial}{\partial \phi_{mn}}, D_{lk}^\phi \right] = \frac{\delta_{lm}}{z_n + z_k} \frac{\partial}{\partial \phi_{kn}} + \frac{\delta_{nk}}{z_m + z_l} \frac{\partial}{\partial \phi_{ml}} \quad (38)$$

derivation of Eq. (34) with respect $\phi_{mn}$ gives

$$\left[ \frac{\partial}{\partial Z_{kl}}, \frac{\partial}{\partial \phi_{mn}} \right] \frac{\partial H(Z, \phi)}{\partial \phi_{mn}} = \frac{\delta_{lm}}{z_n + z_k} \frac{\partial H(Z, \phi)}{\partial \phi_{kn}} \frac{\delta_{nk}}{z_m + z_l} \frac{\partial H(Z, \phi)}{\partial \phi_{ml}}$$

$$= D_{lk}^\phi \frac{\partial H(Z, \phi)}{\partial \phi_{mn}} + \frac{1}{g_s} \left( \delta_{lm} \phi_{nk} + \phi_{lm} \delta_{nk} \right) \quad (39)$$

Setting now $\phi_{ij} = 0$, one expresses the connected amputated 2-point function of $X$ as $Z$ covariant derivative of the 1-point function:

$$\frac{\partial^2 H(Z, \phi)}{\partial \phi_{kl} \partial \phi_{mn}} = \langle X_{lk} X_{nm} \rangle_{c.a.} - \frac{1}{g_s} (\delta_{nk} Z_{lm} + Z_{nk} \delta_{lm}) =$$

$$= \frac{\partial}{\partial Z_{kl}} \langle X_{nm} \rangle_{c.a.} - \frac{\delta_{lm} \langle X_{nk} \rangle_{c.a.}}{z_n + z_k} - \frac{\delta_{nk} \langle X_{lm} \rangle_{c.a.}}{z_m + z_l} =$$

$$= \frac{\partial}{\partial Z_{kl}} \langle X_{nm} \rangle_{c.a.} - \sum_{i,j} \delta_{lm} \delta_{ki} \delta_{nj} + \delta_{nk} \delta_{mi} \delta_{lj} \langle X_{ji} \rangle_{c.a.} =$$

$$= \frac{D}{D Z_{kl}} \langle X_{nm} \rangle_{c.a.} \quad (40)$$
The two terms that make up the connection piece of the covariant derivative \( \frac{D}{DZ_{kl}} \) have a clear physical origin: they are the two contact terms that are generated when the operator \( X_{lk} \) collides with the operator \( X_{nm} \) either from the left or from the right (see Figure 1). By taking more \( \phi \) derivatives of (39) one obtains the generalization of the recursion relation (40) for higher point functions. The case of the 3-point function is somewhat special, due to the term linear in \( \phi \) that appears in the R.H.S. of Eq. (39): taking the derivative of this equation with respect to \( \phi_{pq} \) gives

\[
\langle X_{lk} X_{qp} X_{nm} \rangle_{c.a.} = \frac{D^{(2)}}{DZ_{kl}} \frac{\partial^2 H(Z, \phi)}{\partial \phi_{pq} \partial \phi_{mn}} - \frac{1}{g_s} (\delta_{lm} \delta_{np} \delta_{qk} + \delta_{lp} \delta_{qm} \delta_{nk}) \quad (41)
\]
where now the connection in the covariant derivative $\frac{D}{DZ_{lk}}$ with respect to $Z_{lk}$ includes the terms associated with the contacts between $X_{lk}$ with two operators, $X_{nm}$ and $X_{qp}$:

$$
\frac{D}{DZ_{kl}} \frac{\partial^2 H(Z, \phi)}{\partial\phi_{pq} \partial\phi_{mn}} = \frac{\partial}{\partial Z_{kl}} \frac{\partial^2 H(Z, \phi)}{\partial\phi_{pq} \partial\phi_{mn}} - \frac{\delta_{lm}}{z_n + z_k} \frac{\partial^2 H(Z, \phi)}{\partial\phi_{pq} \partial\phi_{kn}} - \frac{\delta_{nk}}{z_m + z_l} \frac{\partial^2 H(Z, \phi)}{\partial\phi_{pq} \partial\phi_{ml}} - \frac{\delta_{lp}}{z_q + z_k} \frac{\partial^2 H(Z, \phi)}{\partial\phi_{pq} \partial\phi_{mn}}
$$

(42)

$n$-point functions of $X$ with $n > 3$ are given by covariant $Z$-derivative of the $n - 1$-point functions:

$$
\langle X_{j_1i_1} X_{j_2i_2} \cdots X_{j_ni_n} \rangle_{\text{c.a.}} = \frac{D}{DZ_{i_1j_1}} \langle X_{j_2i_2} \cdots X_{j_ni_n} \rangle_{\text{c.a.}} \quad \text{for } n > 3 \quad (43)
$$

where the covariant derivative $\frac{D}{DZ_{i_1j_1}}$ includes the connection terms which correspond to the contacts of $X_{j_1i_1}$ with the $n - 1$ operators $X_{j_2i_2}, \cdots, X_{j_ni_n}$.

Consistency of the recursion relations (43) requires the Kontsevich covariant $Z$-derivative to be flat

$$
\left[ \frac{D}{DZ_{kl}}, \frac{D}{DZ_{ij}} \right] = 0 \quad (44)
$$

The flatness relation (44) follows from the fact that the Kontsevich covariant $Z$-derivative can be rewritten as

$$
\frac{D}{DZ_{kl}} \langle X_{j_1i_1} \cdots X_{j_ni_n} \rangle_{\text{c.a.}} = \Delta_{j_1i_1:a_1b_1}(Z) \cdots \Delta_{j_ni_n:a_nb_n}(Z) \times
$$

$$
\times \frac{\partial}{\partial Z_{kl}} \left[ \Delta_{a_1b_1;c_1d_1}(Z) \cdots \Delta_{a_nb_n;c_nd_n}(Z) \langle X_{c_1d_1} \cdots X_{c_ni_n} \rangle_{\text{c.a.}} \right] \quad (45)
$$

and from the following symmetry of the Kontsevich propagator

$$
\frac{\partial}{\partial Z_{i_1j_1}} \Delta_{j_2i_2:ab}(Z) = \frac{\partial}{\partial Z_{i_2j_2}} \Delta_{j_1i_1:ab}(Z) \quad (46)
$$
3 The disk

The tree approximation of the Kontsevich matrix theory computes disk correlators of Liouville boundary cosmological constant operators. At tree level the one-point function vanishes

\[ \langle X_{nm} \rangle_{\text{tree}}^{\text{c.a.}} = 0 \]  

Therefore (40) and (41) imply that

\[ \frac{\partial^2 H_{\text{tree}}(Z, \phi)}{\partial \phi_{kl} \partial \phi_{mn}} = 0 \]  

and

\[ \langle X_{lk} X_{qp} X_{nm} \rangle_{\text{tree}}^{\text{c.a.}} = -\frac{1}{g_s} (\delta_{lm} \delta_{np} \delta_{qk} + \delta_{lp} \delta_{qm} \delta_{nk}) \]  

Since this is \( Z \)-independent, it follows from the recursion relations (43) that the “bulk” contributions to the string amplitudes with three or more boundary cosmological constant operators vanish. The only non-vanishing contributions to the four and higher-point functions come from the Kontsevich connection. From the point of view of the Liouville boundary conformal field theory this means that the bulk contributions to string amplitudes with \( n \) boundary cosmological constant operators vanish when \( n \geq 4 \):

\[ \langle e^{b \phi(x_1)} \ldots e^{b \phi(x_n)} \rangle = 0 \]  

Disk string amplitudes with \( n \geq 4 \) are given only by contact terms, coming from the boundary of the (open) string moduli space, i.e. from the regions where some of the points \( x_i \) and \( x_j \) collide. This is the paradigmatic situation of topological string models. Typically, contact contributions to string amplitudes are difficult to compute from the world-sheet (first quantized) point of view: we have just shown that the Feynman rules of Kontsevich open string field theory precisely encode the information about such contacts, very much like Chern Simons field theory computes the contacts of open topological non-linear sigma models \[16\]. In the following of this Section we will discuss how Kontsevich prediction for the disk amplitudes with \( n \) boundary cosmological constant operators, contained in Eqs. (49) and (43), compares with the computations \[15, 17\] of the same amplitudes which are based on the results about Liouville boundary conformal field theory on FZZT branes obtained in \[9, 10\].
Disk string amplitudes of boundary cosmological constant operators are precisely the focus of the work [15], where it was observed that they are completely captured by the (closed) ground ring equations for the corresponding minimal string theory. The ground ring of the \((p, q)\) theory is generated by two elements, \(x\) and \(y\) subject to the relation

\[
T_p(y/C) - T_q(x) = 0 \tag{51}
\]

where \(T_n(x)\) are the Chebyshev polynomials of the first kind, and a \(C\) is a computable constant. Let \(y = y_{p,q}(x)\) be the solution of the relation (51), and define the primitive \(F_{p,q}(x)\) of \(y_{p,q}(x)\):

\[
F_{p,q}(x) = \int x^{y_{p,q}(x')}
\]

It was remarked in [15] that the disk string amplitudes with \(n\) boundary operators \(B_b(\phi)\) for the case of \(N = 1\) FZZT brane are given by the \(n\)-fold derivative of \(F_{p,q}(x)\) with respect to \(x\),

\[
\langle B_b^n \rangle_{p,q} = \frac{\partial^n}{\partial x^n} F_{p,q}(x) \tag{53}
\]

upon the identification

\[
x = \frac{\mu B}{\sqrt{\mu}} \tag{54}
\]

where \(\mu\) is the bulk Liouville cosmological constant\(^2\).

Before comparing Liouville theory result (53) for \((p, q) = (2, 1)\) to the Kontsevich prediction let us first make a side remark. Both \((p, q)\) and \((q, p)\) correspond to the same conformal theory, and therefore the \((p, q)\) and the \((q, p)\) string theories should be identical. On the other hand the ground ring (51) is invariant under the exchange of \(p\) with \(q\) only if one simultaneously exchange \(x\) with \(y\). It is therefore natural to consider, together with the \(F_{p,q}(x)\), the primitive \(\tilde{F}_{p,q}(y)\) of \(x_{p,q}(y)\), the function inverse of \(y_{p,q}(x)\):

\[
\tilde{F}_{p,q}(y) = \int y^{x_{p,q}(y')}
\]

\(^2\)If \(B_b\) is the vertex operator that multiplies \(\mu B\) in the Liouville action, its \(n\)-point amplitude is actually given by Eq. (53) times a dimensionfull factor \(\sqrt{\mu^{1/b^2 + 1 - n}}\). In this section we set, for clarity, \(\mu = 1\): the \(\mu\) dependence can be easily reconstructed from dimensional arguments.
As pointed out in [15], $F_{p,q}(x)$ and its “dual” $\tilde{F}_{p,q}(y)$ are related by a Legendre transformation

$$F_{p,q}(x) = x y - \tilde{F}_{p,q}(y)$$  \hspace{1cm} (56)

Moreover, $y$ can be identified with the dual boundary cosmological constant of Liouville theory

$$y = \frac{\bar{\mu}_B}{\sqrt{\mu}}$$  \hspace{1cm} (57)

where $\bar{\mu} \equiv \mu^{1/2}$ is the dual bulk cosmological constant.

It is then clear that, because of the invariance of (51) under the simultaneous exchange of $p$ with $q$ and $x$ with $y$, one has

$$F_{p,q}(x) = \tilde{F}_{q,p}(x)$$  \hspace{1cm} (58)

and

$$\tilde{F}_{p,q}(y) = F_{q,p}(y)$$  \hspace{1cm} (59)

The identity (58) shows that one can compute the disk string amplitude with $n$ boundary cosmological constant operators either by differentiating $F_{p,q}(x)$ $n$ times with respect to the boundary cosmological constant $\mu_B$ (at fixed $\mu$) or differentiating the dual partition function $\tilde{F}_{q,p}(y)$ $n$ times with respect to the dual boundary cosmological constant $\bar{\mu}_B$ (at fixed $\bar{\mu}$):

$$\langle B^n_b \rangle_{p,q} = \frac{\partial^n}{\partial x^n} F_{p,q}(x) = \frac{\partial^n}{\partial y^n} \tilde{F}_{q,p}(y) |_{y=x}$$  \hspace{1cm} (60)

Note, however, that $F_{q,p}(x) \neq F_{p,q}(x) = \tilde{F}_{q,p}(x)$: one may, therefore, wonder what do the derivatives of $\tilde{F}_{p,q} = F_{q,p}$ compute. Since $b^2 = p/q$, one might suspect that they compute $n$-point functions of the dual cosmological constant operator $B^1_\frac{1}{2}(\phi)$

$$\langle B^n_b \rangle_{p,q} = \frac{\partial^n}{\partial y^n} \tilde{F}_{p,q}(y) = \frac{\partial^n}{\partial x^n} F_{q,p}(x) |_{x=y}$$  \hspace{1cm} (61)

We are going to show that in fact this is not the case: actually if (61) were true, one would face a puzzle. Indeed, since $F_{q,p} \neq F_{p,q}$ no direct relation between the amplitudes of the $(p,q)$ and the $(q,p)$ models would hold. This would be surprising since the two models should correspond to the same string theory.
Going back to the topological theory \((p, q) = (2, 1)\), the ground ring curve (51) reduces in this case to

\[ y^2 - 1 - x = 0 \]  

(62)

Thus

\[ F_{2,1}(x) = \int x' \frac{1}{\sqrt{2x' + 1}} = \frac{2}{3} (x + 1)^{\frac{3}{2}} \]  

(63)

and

\[ \tilde{F}_{2,1}(y) = x(y) y - F_{2,1}(x(y)) = \frac{1}{3} y^3 - y \]  

(64)

Thus, Liouville world-sheet computation for the the string amplitude with \(n\) boundary cosmological constant operators of the \((2, 1)\) theory gives

\[ \langle B^n_b \rangle_{2,1} = \frac{\partial^n}{\partial x^n} 2 (x + 1)^{\frac{3}{2}} \]  

(65)

Note however that had we chosen \((p, q) = (1, 2)\) we would have obtained

\[ \langle B^3_b \rangle_{1,2} = \frac{\partial^3}{\partial y^3} F_{1,2}(y) = \frac{\partial^3}{\partial y^3} \tilde{F}_{2,1}(y) = 2 \]  

\[ \langle B^n_b \rangle_{1,2} = \frac{\partial^n}{\partial y^n} F_{1,2}(y) = \frac{\partial^n}{\partial y^n} \tilde{F}_{2,1}(y) = 0 \quad \text{for} \quad n \geq 4 \]  

(66)

To compare (65) and (66) with Kontsevich prediction, let us specialize the general formula Eq. (43) to the case of one single \((N = 1)\) brane on the disk

\[ \langle B^n_b \rangle_{Kontsevich} = D_z^{(n-1)} \langle B^3_b \rangle_{Kontsevich} \quad \text{for} \quad n \geq 3 \]  

(67)

where the covariant derivative is

\[ D_z^{(n-1)} = \frac{\partial}{\partial z} - \frac{n - 1}{z} \]  

(68)

Taking into account the value of the tree 3-point function (49)

\[ \langle B^3_b \rangle_{Kontsevich} = -2 \]  

(69)

(having set for clarity \(g_s = 1\)), one has

\[ \langle B^n_b \rangle_{Kontsevich} = D_z^n F_K(z) \equiv D_z^{(n-1)} D_z^{(n-2)} \cdots D_z^{(1)} D_z^{(0)} F_K(z) \]  

(70)
with
\[ F_K(z) = \frac{2}{3}z^3 \] (71)

In (70), \( F_K(z) \) should be considered a scalar function from the point of view of the covariant derivative, while its \( n-1 \)-th covariant derivative is a section of the \( n-1 \)-th power of the holomorphic tangent to \( z \).

Because of the trivialization of the Kontsevich connection (45), one can rewrite (70) as
\[ \langle B_n^b \rangle_{\text{Kontsevich}} = (\Delta(z))^n \frac{\partial^n}{\partial x^n} F_K(z(x)) \] (72)
where \( \Delta(z) = 2z \) is the Kontsevich inverse propagator and \( z(x) \) satisfies
\[ \frac{\partial x}{\partial z} = \Delta(z) = 2z \] (73)

This equation reproduces the ring relation (62) once we identify the Kontsevich parameter \( z \) with the dual boundary cosmological constant \( y \)
\[ z = y \] (74)

With this identification, \( F_K(z(x)) = F_{2,1}(x) \) (75)

and
\[ \langle B_n^b \rangle_{\text{Kontsevich}} = \left( \frac{\partial x}{\partial y} \right)^n \langle B_n^b \rangle_{2,1} \] (76)

Finally, note that Kontsevich \( n \)-point amplitude can also be written as follows
\[ \langle B_n^b \rangle_{\text{Kontsevich}} = D_y^{n-3} \left[ -\frac{\partial^3 \tilde{F}_{2,1}(y)}{\partial y^3} \right] \] (77)

Let us sum up what we learnt from the comparison of Kontsevich and Liouville computations.

First, it is clear that Kontsevich theory really computes correlators of the \((1,2)\) minimal theory, rather than those of the \((2,1)\) theory. This, in a sense, does not come as a surprise since the authors of [11] did derive Kontsevich theory from Witten open string field theories on Liouville branes with \( b^2 = p/q = 1/2 \) (rather than \( b^2 = 2 \)).

Moreover, Kontsevich theory shows that the string amplitudes of the \((1,2)\) theory are not correctly computed in the Liouville conformal field theory.
approach (66), with the exception of the 3 point function. The reason of course is that the conformal field theory computation misses the contact terms which are instead correctly encoded in the Kontsevich connection.

Last — and most important — we understood the relation between amplitudes of the \((2,1)\) and those of the \((1,2)\) theory: they are related simply by a coordinate transformation on their common moduli space. Indeed, since the boundary cosmological constant operator is a (holomorphic) vector tangent to the moduli space of FZZT branes, the \(n\)-point amplitudes should transform as (holomorphic) tensors with \(n\) indices. This is precisely the content of Eq. (66). This is good news: after all the \((2,1)\) and the \((1,2)\) model should correspond to the same string theory. It is therefore comforting to verify that the amplitudes of the two models do contain the same physical information, and they merely parametrize the same moduli space with different coordinates. Introducing the Kontsevich connection also nicely restores the symmetry between the partition function \(F_{1,2}(x)\) and (minus) its dual \(\tilde{F}_{1,2}(y)\): one can start with the 3-point functions computed with either partition functions, and then compute higher-point functions with the appropriate (covariant) derivative, using either Eq. (53) or Eq. (77)\(^3\).

The relation between the ground ring equation and the Kontsevich model can be straightforwardly extended to the \((p,1)\) minimal string models. For these theories the ground ring becomes

\[
T_p(y/C) - x = 0
\]

(78)

Following the steps\(^4\) discussed above, the corresponding Kontsevich model should be described by the action (2) with a potential \(V(M)\) given by

\[
V'(M) = T_p(M/C)
\]

(79)

and the eigenvalues \(z\) of the matrix \(Z\) identified with the dual boundary cosmological constants \(z = y\) on the FZZT branes. One is lead therefore to conjecture that the Kontsevich model with potential (79) represents the

\(^3\)It should be kept in mind that the string amplitudes that are well defined on the disk are those with \(n \geq 3\). The dual of (53) is therefore (77), but not \(-D_y\tilde{F}_{2,1}\).

\(^4\)The computation that connects the curve (78) with the Kontsevich model with potential (79) is essentially the same performed in [18]. That paper, however, deals with topological strings propagating in a non-compact Calabi-Yau space. In that context, Eq. (78), rather than being interpreted as the ground ring equation of non-critical bosonic strings, represents the locus where the fibration which defines the Calabi-Yau degenerates.
(effective) open string field theory on the FZZT branes of the \((1, p)\) model: the Kontsevich field \(X = M - Z\) should be identified with the boundary cosmological constant vertex operator \(\mathcal{B}_b(\phi)\). Hopefully, this could be proven along the lines of \([11]\), starting from the microscopic Witten open string field theory. Even if the topological localization of \([11]\) would work for this case as well, the matrix theory that one would obtain upon localization would still involve \(q - 1\) matrix fields. One can further conjecture, following the suggestion of \([11]\), that, by integrating out all but the matrix field corresponding to the vertex operator \(\mathcal{B}_b(\phi)\), one obtains precisely the Kontsevich model \((79)\).

Verifying directly these conjectures is a task that remains to be accomplished. One can however compare formulas for the disk amplitudes of boundary cosmological constants that one obtains from the ground ring \((78)\) with those predicted by the Kontsevich model \((79)\). Our previous discussion made clear that agreement between these two computations has to hold only up to the appropriate coordinate transformation on the moduli space. Therefore the conjectured relation between amplitudes evaluated in the two approaches should be just as in Eq. \((76)\):

\[
\mathcal{B}_b^{(1)} = [V''_p(y)]^n \mathcal{B}_b^{(n)}(p, 1)
\]

where

\[
V_p(y) = \int dy' T_p(y'/C) = \tilde{F}_{p, 1}(y)
\]

is the potential of the Kontsevich model\(^5\). It is immediate to verify that \((80)\) holds if and only if the Kontsevich correlators for one single brane are given by covariant derivatives, that, when acting on correlators with \(n\) vertex operators \(\mathcal{B}_b(\phi)\), write as follows

\[
D^{(n)} = [V''_p(y)]^n \partial_y \frac{1}{[V''_p(y)]^n} = \partial_y - n \frac{V'''_p(y)}{V''_p(y)}
\]

Following the steps of Section 2 one can indeed easily prove the validity of Eq. \((82)\) for the Kontsevich model with a generic potential \(V(y)\) and \(N = 1\). The generalization of this computation to the case with \(N\) branes will be presented in the next Section, where, for sake of clarity, we will restrict

\(^5\)Note that \(V_2(y)\) defined by Eq. \((81)\) is \(V_2(y) = y^3/3 - y\) (as in Eq. \((64)\)), rather than the purely cubic potential of Section 2. However, since in general \(F_K(z) = V(z) - z V'(z)\), the Kontsevich free energy \(F_K(z)\) does not depend on the linear part of the potential.
ourself to the case of monomial potentials

\[ V(M) = \frac{M^{p+1}}{p+1} \tag{83} \]

The case with a generic polynomial potential is not significantly more complicated.

Actually, it had been already suggested in the nineties [2], [3], [4], [5] that the generalized Kontsevich models with monomial potentials (83) are related to closed minimal strings of the \((p, 1)\) type with zero bulk cosmological constant \(\mu\). This also agrees with the ground ring approach. In fact, taking into account the definitions (54) and (57) of the parameters \(x\) and \(y\), one has that in the limit \(\mu \rightarrow 0\) the ring (78) reduces to

\[ y^p - x = 0 \tag{84} \]

in agreement with (83).

Let us conclude this Section by commenting on the implications of the Kontsevich connection for the geometry of the moduli space of FZZT branes. From the point of view of the conformal field theory, this moduli space is parametrized by the boundary cosmological constant \(x\), which is the coupling constant of the marginal operator \(B_b\). It is natural to analytically continue \(x\) (and thus \(y\)) to the complex plane. Since at tree level \(x\) and \(y\) satisfy the ground ring equation (78), the semi-classical moduli space is identified with the genus zero Riemann surface \(M_{p,1}\) whose equation is (78). The uniformizing parameter for \(M_{p,1}\) is \(y\), i.e. the complex \(y\) plane covers the Riemann surface once and only once: We just understood that \(y\) is the same as \(z\), the parameter that parametrizes the moduli space in the Kontsevich open string field theory. Thus we notice two things: First, as expected on general grounds [13, 14], there is a non-trivial reparametrization between the conformal field theory and the open string field theory moduli, given by the relation \(z = y(x)\) found in (74). Moreover, in the case we are considering, string field theory provides a one-to-one parametrization of the quantum moduli space: This supports the proposal advanced in [13, 14] that this might be a general property of Witten open string field theory.

Our analysis of Kontsevich string field theory also shows that on the quantum moduli space parametrized by \(z\) plane there is a connection

\[ \Gamma_p = V_p'' \partial_z (V_p'')^{-1} \tag{85} \]
whose curvature\(^6\)

\[ R_p = \partial_z \Gamma_p \tag{86} \]

has a delta function with support at the points where

\[ V''_p(z) = T'_p(z) = pU_{p-1}(z) = 0 \tag{87} \]

\((U_p(z)\) is the Chebyshev polynomial of the second kind). As \(U_{p-1}(z)\) has degree \(p - 1\), the Kontsevich connection for \((p, 1)\) minimal models defines a non-trivial line bundle of first Chern class equal to \(p - 1\).

In this context, it is interesting to note that recently the authors of [19] analyzed the structure of the quantum moduli space of FZZT branes \(\mathcal{M}_{p,q}\) in the general case \((p, q)\). At the semi-classical level \(\mathcal{M}_{p,q}\) has singularities in the non-topological case \(q > 1\): these are singularities of the *differential structure* on the moduli space which correspond to points \((x, y)\) on \(\mathcal{M}_{p,q}\) that satisfy both

\[ U_{p-1}(y) = 0 \tag{88} \]

and

\[ U_{q-1}(x) = 0 \tag{89} \]

The physical interpretation of these singularities is related to ZZ branes. It was found in [19] that these singularities disappear when non-perturbative effects are taken into account. What we have found here is that in the \(q = 1\) case — when (89) has no solutions — the quantum moduli space has *curvature* singularities located precisely at the points that are solutions of (89). At these points in the moduli space the Kontsevich matrix theory is singular. This means that the topological localization of [11] fails precisely at these points of the moduli space: around these points therefore the physics is described by the full fledged Witten open string field theory and not by its Kontsevich reduction. It would be interesting to understand the physical interpretation of this phenomenon.

### 3.1 Geometrical origin of the Kontsevich connection

In this subsection we will give a geometric interpretation of the Kontsevich connection, by showing that it is the Levi-Civita connection associated with a (singular) metric on the moduli space.

---

\(^6\)The curvature is associated with an holomorphic anomaly of the Kontsevich string field theory.
We have seen that the curve in $C^2$ associated with the Kontsevich model
with potential $V(y)$ is
\[ V'(y) = x \]  
(90)
Consider the family of flat metrics on $C^2$ given by
\[ (ds)^2 = dy \otimes d\bar{y} + \lambda^2 dx \otimes d\bar{x} \]  
(91)
parametrized by $\lambda^2 > 0$. This metric induces the following metric on the
Kontsevich curve (90)
\[ (ds)^2_{V,\lambda} = \left( \frac{1}{|V''(y(x))|^2} + \lambda^2 \right) dx \otimes d\bar{x} = \left( 1 + \lambda^2 |V''(y)|^2 \right) dy \otimes d\bar{y} \]  
(92)
Hence the associated Levi-Civita connection in local coordinates $(x, \bar{x})$ writes as follows
\[ \Gamma^x_{xx} = -\frac{V'''(y(x))}{\left( 1 + \lambda^2 |V''(y(x))|^2 \right)^2} V''(y) \]  
(93)
In local coordinates $(y, \bar{y})$ the connection is instead
\[ \Gamma^y_{yy} = \frac{\bar{V}''(\bar{y}) V'''(y)}{\bar{x} + |V''(y)|^2} \]  
(94)
The limit of $\lambda \rightarrow \infty$ of such a connection is
\[ \Gamma^x_{xx} = 0 \]  
(95)
in $x$ coordinates and
\[ \Gamma^y_{yy} = \frac{V'''(y)}{V''(y)} \]  
(96)
in $y$ coordinates. In conclusion the connection that appear in the Kontsevich
 correlators is the connection induced by the flat metric on $C^2$ (92) in the
limit $\lambda \rightarrow \infty$.

4 Generalizations to $q \neq 2$

4.1 $q > 2$

As recalled in Section 2, Kontsevich original model admits the following generalization when $q$ in (3) is greater then 2
\[ e^{F_q(g_s, Z)} = e^{-\frac{q}{(q+1)g_s} \text{Tr} Z^{q+1}} \int [dM] e^{-\frac{1}{g_s} \text{Tr} M^{q+1}} - Z^q M \int [dX] e^{-\frac{1}{g_s} \Gamma_{q+1}(X; Z)} \]  
(97)
where the action $\Gamma_{q+1}(X;Z)$ is the following invariant polynomial of order $q+1$ in $X$:

$$\Gamma_{q+1}(X;Z) \equiv \text{Tr} \frac{(X+Z)^{q+1} - Z^{q+1} - (q+1)Z^qX}{q+1} \quad (98)$$

The quadratic part of the action (98) is

$$\Gamma^{(2)}_q = \frac{1}{2} \sum_{ij,kl} \Delta^{(q)}_{lk;ji}(Z) X_{ji} X_{kl} \quad (99)$$

where we introduced the inverse propagator $\Delta^{(q)}_{lk;ji}(Z)$ of the generalized Kontsevich theory

$$\Delta^{(q)}_{lk;ji}(Z) = \sum_{a=0}^{q-1} Z_{ij}^a Z_{ik}^{q-1-a} \quad (100)$$

In the gauge in which $Z$ is diagonal the propagator becomes therefore

$$[\Delta^{(q)}_{lk;ji}(Z)]^{-1} = \frac{\delta_{ik} \delta_{jl}}{P^{(q-1)}(z_i, z_j)} \quad (101)$$

where $P^{(q-1)}(z_i, z_j)$ is the homogeneous symmetric polynomial of $z_i$ and $z_j$ of degree $q-1$

$$P^{(q-1)}(z_i, z_j) \equiv \sum_{a=0}^{q-1} z_i^a z_j^{q-1-a} = \frac{z_i^q - z_j^q}{z_i - z_j} \quad (102)$$

Introduce the analog of the invariant function defined in Eq. (22)

$$e^{f_q(g_s,Y)} = \int [dM] e^{-\frac{1}{g_s} \text{Tr} \left[ \frac{M}{q+1} - Y M \right]} \quad (103)$$

Then the effective potential $H(Z, \phi)$ is given by

$$H(Z, \phi) = f_q(g_s, Y) - \log \mathcal{N}_q(Z) - \frac{1}{g_s} \left[ \frac{q}{q+1} \text{Tr} Z^{q+1} + q \text{Tr} Z^q \phi + \frac{1}{2} \text{Tr} \sum_{a=0}^{q-1} \phi^a \phi Z^{q-1-a} \right] \quad (104)$$
where the matrix variable \( Y_{ij} \) is

\[
Y_{ij} = (Z^q)_{ij} + \sum_{a=0}^{q-1} (Z^a \phi Z^{q-1-a})_{ij} = (Z^q)_{ij} + \phi_{kl} \Delta^{(q)}_{ik;ji} \equiv (Z^q)_{ij} + L_Z(\phi)_{ij}
\]

and \( L_Z(\phi) = 0 \) are the linearized equations of motion for \( \phi \). In Eq. (104) we have also introduced the normalization factor

\[
\mathcal{N}_q(Z) = \int [dX] e^{-\frac{1}{g_s} r_q^{(2)}} = (\text{Det}_{(k_l;j_i)} \Delta^{(q)}_{ik;ji})^{-1/2}
\]

Thus

\[
\frac{\partial f_q(Y)}{\partial Z_{kl}} - \frac{\partial f_q(Y)}{\partial \phi_{kl}} - \left( [\Delta^{(q)}(Z)]_{nm,j}^{-1} \frac{\partial}{\partial Z_{kl}} \Delta^{(q)}_{qp;ji}(Z) \right) \phi_{pq} \frac{\partial}{\partial \phi_{mn}}
\]

and the differential equation satisfied by \( H(Z, \phi) \) becomes

\[
\frac{\partial}{\partial Z_{kl}} - \frac{\partial}{\partial \phi_{kl}} - \left( [\Delta^{(q)}(Z)]_{nm,ji}^{-1} \frac{\partial}{\partial Z_{kl}} \Delta^{(q)}_{qp;ji}(Z) \right) \phi_{pq} \frac{\partial}{\partial \phi_{mn}}
\]

\[
\left[ \log \mathcal{N}_q(Z) + \frac{1}{g_s} \left( \frac{q}{q+1} \text{Tr} Z^{q+1} + q \text{Tr} Z^q \phi + \frac{1}{2} \text{Tr} \sum_{a=0}^{q-1} \phi Z^a \phi Z^{q-1-a} \right) \right] =
\]

\[
= \frac{1}{2} [\Delta^{(q)}(Z)]_{qp;ji} \frac{\partial}{\partial Z_{kl}} \Delta^{(q)}_{jic;p} + \frac{1}{g_s} \sum_{a=0}^{q-1} \sum_{b=0}^{a-1} [Z^b \phi Z^{q-1-a} \phi Z^{a-1-b}]_{lk}
\]

Consequently the covariant \( \phi \)-derivative is now

\[
D^\phi_{ik} \equiv \frac{\partial}{\partial \phi_{kl}} + \left( [\Delta^{(q)}(Z)]_{nm,ji}^{-1} \frac{\partial}{\partial Z_{kl}} \Delta^{(q)}_{qp;ji}(Z) \right) \phi_{pq} \frac{\partial}{\partial \phi_{mn}} =
\]

\[
= \frac{\partial}{\partial \phi_{kl}} + \sum_{ij} \frac{1}{P(q-1)(z_i, z_j)} \frac{\partial}{\partial Z_{kl}} \left[ \sum_{a=0}^{q-1} Z^a_{im} \phi Z^{q-1-a} \phi Z^{a-1-b} \right]_{ij}
\]

It follows, from the same argument as in section 2, that the connected amputated 2-point function is the covariant \( Z \)-derivative of the 1-function

\[
\frac{\partial^2 H(Z, \phi)}{\partial \phi_{kl} \partial \phi_{mn}} = \langle X_{lk} X_{nm} \rangle_{c.a.} - \frac{1}{g_s} \Delta_{lk;nm}(Z) = \frac{D}{D Z_{kl}} \langle X_{nm} \rangle_{c.a.}
\]
where the covariant $Z_{kl}$-derivative acts on $X_{nm}$ as

\[
\frac{D}{D Z_{kl}} \langle X_{nm} \rangle_{c.a.} = \frac{\partial \langle X_{nm} \rangle_{c.a.}}{\partial Z_{kl}} - \sum_{ij} \frac{1}{P(q-1)(z_i, z_j)} \times \\
\times \frac{\partial}{\partial Z_{kl}} \left[ \sum_{a=0}^{q-1} Z_{im}^{q-1-a} Z_{nj}^{a} \right] \langle X_{ji} \rangle_{c.a.} = \\
= \frac{\partial \langle X_{nm} \rangle_{c.a.}}{\partial Z_{kl}} - \sum_{ij} \frac{1}{P(q-1)(z_i, z_j)} \sum_{a=1}^{q-2} \left[ \frac{\partial(Z_{im}^{a})}{\partial Z_{kl}} Z_{mj}^{q-1-a} + \frac{\partial(Z_{nj}^{a})}{\partial Z_{kl}} Z_{im}^{q-1-a} \right] \langle X_{ji} \rangle_{c.a.}
\]

(112)

Evaluating the $Z_{kl}$ derivatives in the gauge when $Z$ is diagonal, one obtains

\[
\frac{\partial(Z_{ij})}{\partial Z_{kl}} \bigg|_{z_{ij}=z_{kl}=1} = \delta_{ik} \delta_{ml} P^{(a-1)}(z_k, z_l)
\]

(113)

Thus, the covariant $Z$-derivative (112) writes as follows

\[
\frac{D}{D Z_{kl}} \langle X_{nm} \rangle_{c.a.} = \frac{\partial \langle X_{nm} \rangle_{c.a.}}{\partial Z_{kl}} - \sum_{ij} \frac{1}{P(q-1)(z_i, z_j)} \times \\
\times \sum_{a=1}^{q-2} \left[ \delta_{ik} \delta_{ml} \delta_{nj} z_n^{a} P^{(a-1)}(z_k, z_l) + \delta_{nk} \delta_{ij} \delta_{ml} z_m^{a} P^{(a-1)}(z_k, z_l) \right] \langle X_{ji} \rangle_{c.a.} = \\
= \frac{\partial \langle X_{nm} \rangle_{c.a.}}{\partial Z_{kl}} - \delta_{ml} \sum_{a=1}^{q-2} z_n^{a} P^{(a-1)}(z_k, z_l) \langle X_{nk} \rangle_{c.a.} - \delta_{nk} \sum_{a=1}^{q-2} z_m^{a} P^{(a-1)}(z_k, z_l) \langle X_{lm} \rangle_{c.a.}
\]

(114)

Now

\[
\sum_{a=1}^{q-2} z_n^{a} P^{(a-1)}(z_k, z_l) = \frac{P(q-1)(z_l, z_n) - P(q-1)(z_k, z_n)}{z_l - z_k}
\]

(115)

Therefore

\[
\frac{D \langle X_{nm} \rangle_{c.a.}}{D Z_{kl}} = \frac{\partial \langle X_{nm} \rangle_{c.a.}}{\partial Z_{kl}} - \delta_{ml} \frac{P(q-1)(z_l, z_n) - P(q-1)(z_k, z_n)}{(z_l - z_k) P(q-1)(z_k, z_n)} \langle X_{nk} \rangle_{c.a.} - \delta_{nk} \frac{P(q-1)(z_l, z_m) - P(q-1)(z_k, z_m)}{(z_l - z_k) P(q-1)(z_m, z_l)} \langle X_{lm} \rangle_{c.a.}
\]

(116)
Thus, in the general case as well, the Kontsevich connection in (116) includes two terms that correspond, in the first-quantized world-sheet picture, to the two contact terms that can arise when the operators \( X_{n_m} \) and \( X_{l_k} \) collide, as shown in Figure 1. It is worth noting that in the \( q > 2 \) case the factor associated with the contact that arises when an open string stretching from the brane \( z_l \) to the brane \( z_k \) collides with a string stretching from brane \( z_k \) to a third brane \( z_m \) depends not only on the final branes \( z_l \) and \( z_m \) but also on the intermediate brane \( z_k \).

Again one can verify that covariant \( Z \)-derivatives are flat since they can be written as

\[
\frac{D^{(n)}}{D Z_{kll}} (X_{j_{i_1}} \ldots X_{j_{n_{i_n}}} c.a.) = \Delta_{j_{i_1};a_1b_1}^{(q)}(Z) \ldots \Delta_{j_{n_{i_n};a_nb_n}}^{(q)}(Z) \times \quad (117)
\]

\[
\times \frac{\partial}{\partial Z_{kll}} \left[ \left[ \Delta_{j_{i_1};c_1d_1}^{(q)}(Z) \right]^{-1}_{a_1b_1;a_nb_n} \ldots \left[ \Delta_{j_{n_{i_n};c_n d_n}}^{(q)}(Z) \right]^{-1}_{a_nb_n;c_n d_n} (X_{c_1 d_1} \ldots X_{c_n d_n})_{c.a.} \right]
\]

where \( \left[ \Delta_{j_{i_1};c_1d_1}^{(q)}(Z) \right]^{-1}_{a_1b_1;a_nb_n} \) is the propagator (114) appropriate for the generalized Kontsevich model.

In the case of only one brane, \( N = 1 \), the previous formulas considerably simplify. For the Kontsevich model with generic potential \( V(M) \), the inverse propagator is

\[
\Delta^{(V)}(z) = V''(z) \quad (118)
\]

and thus the covariant \( Z \)-derivative given in Eq. (117) reduces to

\[
D^{(n)}_z = \partial_z - n \frac{V'''(z)}{V''(z)} \quad (119)
\]

which is the announced result (52).

### 4.2 \( c = 1 \)

In \[6\] the Kontsevich matrix model for \( c = 1 \) bosonic strings at self-dual radius was derived: it is defined by the following functional integral

\[
Z_K(t, \bar{t}) = \int [dM] e^{-\text{Tr} \left[ (N-\nu) \log M + \nu \sum_{k=1}^{\infty} \bar{t}_k M^k \right] + \nu \text{Tr} MA} \quad (120)
\]

where

\[
\nu = -i\mu \quad (121)
\]

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with $\mu$ the bulk cosmological constant. $t_n$ and $\bar{t}_k$, which are the moduli corresponding to left and right moving closed string tachyons, enter the Kontsevich $c = 1$ model \cite{120} in a quite asymmetric way: $\bar{t}_k$ are coupling constants of the invariant operators $\text{Tr} M^k$, while $t_n$ are expressed in terms of the external matrix source $A$ via the Frobenius-Miwa-Kontsevich transformation

$$t_k = \frac{1}{n\nu} \text{Tr} A^{-k}$$

(122)

The partition function $Z_K(t, \bar{t})$ encodes amplitudes between left moving and right moving closed string tachyon vertex operators $T_k$ and $T_{-k}$:

$$Z_K(t, \bar{t}) = (\det A)^{-\nu} \langle e^{\sum_k t_k T_k + \bar{t}_k T_{-k}} \rangle_{\text{closed}}$$

(123)

Following the lesson of non-critical minimal strings, one would like to conjecture \cite{11, 20, 21} that even the $c = 1$ Kontsevich model should be identified with the (effective) open string field theory of $N$ stable branes of Liouville theory coupled to $c = 1$ matter. Moreover, as in the cases studied so far, we propose to identify the $X$ matrix field with the cosmological boundary vertex operator. In the following we want to derive the implications of this identification for amplitudes of cosmological boundary vertex operators at $c = 1$.

The problem of computing these amplitudes directly in the boundary conformal field theory is a subtle one \cite{22} and has not been solved yet: This is due to the fact that taking the naive $b \to 1$ limit of Liouville correlators produces divergent answers. A step towards the solution of this problem has been taken in \cite{21}, where finite correlators of boundary tachyon operators of non-zero momentum has been obtained by a regularization and renormalization procedure. However the naive extension of the correlators of \cite{21} to zero-momentum boundary operators — which correspond to the boundary cosmological constant operators we are interested in — again produces infinite answers. It is possible that a further renormalization is needed to obtain a finite result. So, at the moment, the Kontsevich model for $c = 1$ provides a prediction for boundary cosmological constant operator amplitudes that cannot be independently verified.

We will work at the “topological” point at which $\bar{t}_n = 0$. In this case, by setting

$$\nu Z = \nu' A^{-1}$$

(124)

with

$$\nu' = \nu - N$$

(125)
the function integral (120) reduces to one of the form (122) with potential

\[ V(M) = -\nu' \log M \] (126)

Thus, the partition function of the \( c = 1 \) Kontsevich model at the topological point is

\[ e^{f_{c=1}(\nu', Z)} = e^{\nu' \text{Tr} (1 - \log Z)} \int [dM] e^{\nu' \text{Tr} [\log M - Z^{-1} M]} = \int [dX] e^{-\nu' \Gamma_{c=1}(X; Z)} \] (127)

where \( \Gamma_{c=1}(X; Z) \) is the Penner-like action

\[ \Gamma_{c=1}(X; Z) \equiv -\text{Tr} [\log (1 + Z^{-1} X) - Z^{-1} X] = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \text{Tr} (Z^{-1} X)^n \] (128)

The quadratic part of the action and the inverse propagator are

\[ \Gamma_{c=1}^{(2)} = \frac{1}{2} \text{Tr} Z^{-1} X Z^{-1} X = \frac{1}{2} \sum_{ij,kl} \Delta^{(c=1)}_{lj;ik}(Z) X_{ji} X_{kl} \] (129)

\[ \Delta^{(c=1)}_{lj;ik}(Z) = Z_{ik}^{-1} Z_{lj}^{-1} \] (130)

The effective potential of the model is given by

\[ H(Z, \phi) = f_{c=1}(\nu', Y) - \log \mathcal{N}_{c=1}(Z) - \nu' \left[ \text{Tr} (\log Z - 1) + \text{Tr} Z^{-1} \phi + \frac{1}{2} \text{Tr} Z^{-1} \phi Z^{-1} \phi \right] \]

\[ e^{f_{c=1}(\nu', Y)} = \int [dM] e^{\nu' \text{Tr} [\log M + Y M]} \]

\[ Y = \nu' Z^{-1} (-1 + \phi Z^{-1}) \]

\[ \mathcal{N}_{c=1}(Z) = \int [dX] e^{-\nu' \Gamma_{c=1}^{(2)}} \] (131)

and it satisfies the following differential equation

\[
\left[ \frac{\partial}{\partial Z_{kl}} - \frac{\partial}{\partial \phi_{kl}} + \sum_i \left( z_i^{-1} \phi_{ik} \frac{\partial}{\partial \phi_{il}} + z_i^{-1} \phi_{il} \frac{\partial}{\partial \phi_{ki}} \right) \right] H(Z, \phi) = -N \delta_{kl} z_k^{-1} - (Z^{-1} \phi Z^{-1} \phi Z^{-1})_{lk} \] (132)
In the equation above we have used the fact that the Kontsevich connection is, in this case

\[ - \left[ \Delta^{(c=1)}(Z) \right]_{nm;ji}^{-1} \frac{\partial}{\partial Z_{kl}} \Delta^{(c=1)}(Z) = z_k^{-1} \delta_{kp} \delta_{lm} + z_i^{-1} \delta_{iq} \delta_{kn} \delta_{mp} \]  

(133)

From Eq. (131) we can read off the covariant Z-derivative acting \( X_{nm} \)

\[ \frac{D}{D Z_{kl}} \langle X_{nm} \rangle_{c.a.} = \frac{\partial \langle X_{nm} \rangle_{c.a.}}{\partial Z_{kl}} + z_i^{-1} \delta_{lm} \langle X_{nk} \rangle_{c.a.} + z_k^{-1} \delta_{nk} \langle X_{lm} \rangle_{c.a.} \]  

(134)

To make a prediction for amplitudes of boundary cosmological constant vertex operators on the disk we compute the Kontsevich effective potential \( H(Z, \phi) \) at tree level. Since, in the tree approximation,

\[ f_{c=1}(\nu', Y) = -\nu' \mathrm{Tr} (\log Y + 1) \]  

(135)

we find, from Eq. (131)

\[ H(Z, \phi)^{tree} = -\nu' \mathrm{Tr} \log(1 - \phi Z^{-1}) - \log \mathcal{N}_{c=1}(Z) - \nu' \left[ \mathrm{Tr} Z^{-1} \phi + \frac{1}{2} \mathrm{Tr} Z^{-1} \phi Z^{-1} \phi \right] \]  

(136)

By taking the third derivative with respect to \( \phi \) of \( H(Z, \phi)^{tree} \), we deduce that the connected amputated 3-point function at tree level is

\[ C^{(3)}(z) = \nu' \frac{\delta_{ij} \delta_{jq} \delta_{kp} + \delta_{ij} \delta_{jq} \delta_{kp}}{z_i z_k z_p} \]  

(137)

The two terms correspond to the two cyclically inequivalent orderings of the vertex operators. Higher point amplitudes are obtained from \( C^{(3)} \) via the covariant derivative of Eq. (134):

\[ C^{(n)}(z) = D_z^{n-3} C^{(3)}(z) \]  

(138)

In particular, in the case of only one brane, the \( n \)-point amplitude is

\[ C^{(n)}(z) = \nu' \frac{(n - 1)!}{z^n} \]  

(139)

As discussed above, the correlators \( C^{(n)} \) should be identified with the amplitudes of \( n \) boundary cosmological constant operators on the disk. In order to compare with boundary conformal field theory computations one
needs to know the relation between the string field theory parameter \( z \) and the conformal field theory parameter \( x = \mu_B/\sqrt{\mu} \). It is natural to extend the recipe valid for the \((p,1)\) minimal models to the \(c = 1\) strings as well: as the analog of the curve (78) for the logarithmic potential (123) is

\[ xy = 1 \] (140)

we propose the identification

\[ z = y = x^{-1} \] (141)

As we explained in section 3, the fact that amplitudes of boundary cosmological constant operators are given by the Kontsevich result (139), with the identification (141), is equivalent to the statement that, for \(c = 1\) non-critical strings, these amplitudes are captured by the curve (140) via the procedure of [15].

As a matter of fact, the curve (140) plays the same role for the topological B-model on the conifold that the curve (84) plays for the \((p,1)\) topological models [18]. However, we note that the conjecture (140), (141) appears to be somewhat problematic in the non-topological formulation: The curve (140) coincides with the ground ring of non-critical \(c = 1\) strings only in the case in which the scalar field representing the \(c = 1\) matter is non-compact. In the compact case, which is the one described by the Kontsevich-like model (120), the ground ring has two more generators, \(u\) and \(v\), and it is given by

\[ xy - uv = 1 \] (142)

It is not clear to us how the extra generators find their place in a suitable generalization — if it exists — of the mechanism discovered in [15] by which the closed ring computes open string amplitudes.

It is also not clear how the \(b \to 1\) limit of Liouville correlators would reproduce the prediction (139). As recalled before, the naive limit produce infinite expressions and an appropriate renormalization procedure, for the case of boundary cosmological constant operators, has not yet been defined.

5 Conclusions

We have shown that the interpretation of Kontsevich matrix models as open string field theory for FZZT branes coupled to minimal matter implies in-
teresting predictions for amplitudes of boundary cosmological constant operators. We have learned that Kontsevich theory provides both global coordinates for the open string moduli space and a connection on it: By expressing insertions of boundary cosmological constant operators as covariant derivatives with respect to this connection, Kontsevich theory captures contact term contributions to open string amplitudes, arising from the points in moduli space in which boundary operators collide either between themselves or with nodes of the Riemann surface.

We have shown that the consistency of Kontsevich theory with boundary conformal field theory formulas hinges on some rather subtle issues. We have analyzed these by comparing Kontsevich amplitudes with a representation of boundary cosmological constant correlators of \((p, 1)\) non-critical strings on the disk recently found in [15]. In order to prove the agreement between the first and the second quantized point of view two ingredients were essential: The determination of the change of variables between the open string field theory and the boundary conformal field theory coordinates on the brane moduli space and the realization that correlators of boundary operators transform, under such reparametrizations, as sections of the tensor product of the \((\text{holomorphic})\) tangent bundle to the moduli space. The agreement we found provides evidence for the conjecture, which had not yet been explicitly verified for the case \(p > 2\), that generalized Kontsevich models do indeed represent \((\text{effective})\) open string field theories of non-critical bosonic strings coupled to \((p, 1)\) minimal matter.

We have also presented an analysis of the Kontsevich model for \(c = 1\) non-critical strings along similar lines. In the \(c = 1\) case, however, the current limited understanding of Liouville theory in the \(b \to 1\) limit did not allow us to compare Kontsevich predictions with boundary conformal field theory results. It would, of course, be interesting to elucidate this issue. One possibility is that, once an appropriate renormalization procedure at \(b \to 1\) has been taken, amplitudes of boundary cosmological constant operators do indeed reduce to Kontsevich correlators. Another possibility is that the Kontsevich matrix field \(X\) is not, in this case, the string field corresponding to the cosmological constant operator but some more complicated composite field, originating from the process of integrating out all other fields in the full string field theory.

Perhaps one of the most intriguing results of our analysis is the fact that the Kontsevich connection is not flat and its curvature has delta function singularities at the points where the Kontsevich kinetic term degenerates.
The non-flatness of the connection is associated with a holomorphic anomaly of the Kontsevich matrix model, analogous to the one of closed topological strings [12]. The holomorphic anomaly of closed topological strings has deep consequences for the background independence of the models [23]. It would be interesting to see if a similar interpretation of the anomaly could be given in the Kontsevich case also. One should also elucidate the physical meaning of the singularities of the Kontsevich curvature. It has recently been shown in [19] that the quantum moduli space of open minimal string theory is smooth, as a differentiable manifold. The poles of the Kontsevich connection imply that the same moduli space, however, has curvature singularities. At the singularities topological localization fails and one presumably should reintroduce the full degrees of freedom of Witten open string field theory to correctly describe the physics around such points.

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