Monge–Ampère Type Equations on Almost Hermitian Manifolds

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Abstract In this paper we consider the Monge–Ampère type equations on compact almost Hermitian manifolds. We derive $C^\infty$ a priori estimates under the existence of an admissible $C$-subsolution. Finally, we obtain an existence result if there exists an admissible supersolution.

Keywords Monge–Ampère type equations, almost Hermitian manifold, a priori estimates, subsolution

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1 Introduction

Let $(M, J, \omega)$ be a compact almost Hermitian manifold of real dimension $2n$. Suppose $\chi$ is a real $(1, 1)$ form on $M$. For each $u \in C^2(M, \mathbb{R})$, we will use the shorthand

$$\chi_u = \chi + \sqrt{-1} \partial \bar{\partial} u,$$

which is still a real $(1, 1)$ form on $M$. Let

$$\mathcal{H}(M) = \{u \in C^2(M, \mathbb{R}) : \chi_u > 0\}$$

be the admissible set with respect to $\chi$. In this paper we wish to consider the Monge–Ampère type equations for $u$ which can be written in the form

$$\begin{cases}
  u \in \mathcal{H}(M), \\
  \chi_u^n = \psi \chi_u^{n-m} \wedge \omega^m \text{ on } M,
\end{cases}$$

(1.1)

where $\psi \geq c > 0$ is a smooth real valued function on $M$ and $m \in [1, n]$ is a fixed integer.

In contrast to the complex Monge–Ampère equation [8], we need to further assume there exists at least one $C$-subsolution (see Definition 2.2) for (1.1), which was also specified in [18, 36]. We have the following theorem.

**Theorem 1.1** Let $(M, J, \omega)$ be a compact almost Hermitian manifold of real dimension $2n$. Suppose there exists an admissible $C$-subsolution $\underline{u}$ and $u$ is the smooth solution for (1.1). Then we have estimates

$$\|u\|_{C^{k, \alpha}} < C,$$

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where $C$ is a constant depending on $(M, J, \omega)$, $\chi$, $k$, $\alpha$, $\psi$ and $u$.

We say $v \in C^2(M)$ is an admissible supersolution for (1.1) if $v \in \mathcal{H}(M)$ and
\[ \chi_u^n \leq \psi \chi_v^{n-m} \wedge \omega^m. \] (1.2)

Analogous to the Hermitian setting in [32], we can also obtain the following existence theorem under an extra condition.

**Theorem 1.2** Let $(M, J, \omega)$ be a compact almost Hermitian manifold of real dimension $2n$. Suppose there exist an admissible $C$-subsolution $u$ and an admissible supersolution for the equation (1.1). Then there exists a pair $(u, b) \in C^\infty(M) \times \mathbb{R}$ such that
\[
\begin{cases}
u \in \mathcal{H}(M), & \sup_M v = 0, \\
\chi_u^n = e^b \psi \chi_u^{n-m} \wedge \omega^m & \text{on } M.
\end{cases}
\] (1.3)

Equations of this sort have been well-studied in the past several decades, which play an important role in the study of fully nonlinear second order elliptic PDEs. In the Kähler case, when $M$ is compact, the complex Monge–Ampère equation
\[ \chi_u^n = \psi \omega^n \] (1.4)
was studied by Yau in his famous paper [43], which is a fundamental tool in the study of certain theories of Kähler manifolds. Cao [2] provided a new proof by using the parabolic approach (also called as Kähler Ricci flow). To be more precise, he investigated the parabolic Monge–Ampère equation
\[ \frac{\partial u_t}{\partial t} = \log \frac{\chi_u^n}{\omega^n} - \log \psi, \quad u_0 = 0, \] (1.5)
where we further require $\chi_{u_t} > 0$. He proved that the solutions of (1.5) exist for all time and convergence to the solution of (1.4). The complex Hessian equations
\[ \chi_u^m \wedge \omega^{n-m} = \psi \omega^n, \quad 1 \leq m \leq n - 1, \] (1.6)
were completed by Dinew–Kolodziej [10], by using the second order estimate of Hou–Ma–Wu [24] to obtain the gradient estimate.

The equation (1.1) could also be regarded as a natural extension of Donaldson’s equation [11] where $M$ is Kähler. Specifically, when $\psi$ is a constant and $m = 1$, i.e.,
\[ \chi_u^n = c \chi_u^{n-1} \wedge \omega, \]
where $\chi$ is closed and $c$ is a constant depending on the classes $[\chi]$ and $[\omega]$. Chen [4, 5] also found it in the study of Mabuchi energy. Further related works see also [14, 21, 31, 34, 41, 42] and references therein. For overview of recent development for complex Monge–Ampère equations, we refer the reader to the comprehensive survey paper [27].

If $M$ is a compact Hermitian manifold, Tosatti–Weinkove [39] had considered the equation (1.4) and gave a complete proof, together with the work of Zhang [46]. The parabolic proof was given by Gill [16]. Different from Kähler case, when $\chi$ is only a smooth real $(1, 1)$ form, Sun [33] obtained the $C^\infty$ a priori estimates for equations
\[ \chi_u^k \wedge \omega^{n-k} = \psi \chi_u^l \wedge \omega^{n-l}, \quad 1 \leq l < k \leq n \] (1.7)
with an extra $C$-subsolution, namely,

$$k\chi'^{k-1} \wedge \omega^{n-k} > l\psi \chi'^{n-1} \wedge \omega^{n-l}$$

for some $k$-positive (with respect to $\omega$) $\chi' \in [\chi]$. More general fully nonlinear second order elliptic equations were investigated by Székelyhidi [36]. The Dirichlet problems were considered by Guan–Li [19] and Guan–Sun [20] under the existence of subsolution, which extended the Guan’s works [17, 18]. One can also refer to [32–34, 44] and references therein for recent developments of this topic. The Hessian equations on Hermitian manifolds were considered in [30, 45] for compact case and by Collins–Picard [9] for Dirichlet problem.

The complex Monge–Ampère equation on almost Hermitian manifolds was studied by Chu–Tosatti–Weinkove [8]. Chu [6] also provided a parabolic proof, analogous to (1.5). Chu–Huang–Zhu [7] had considered the $\sigma_2$ equation and gave the second order estimate. On the strictly pseudoconvex domains of almost complex manifolds, the Dirichlet problem was studied by Harvey–Lawson [22] for continuous weak solution, and by Plis [28] for smooth solution by using a similar idea of Caffarelli–Kohn–Nirenberg–Spruck [3] on $\mathbb{C}^n$.

Analogous to (1.5), it is also of interest to consider the parabolic version of the equation (1.1). More precisely, we can consider the following heat equations

$$\frac{\partial u_t}{\partial t} = \log \frac{\chi^m}{\chi^{n-m} \wedge \omega^m} - \log \psi, \quad u(0, x) \in \mathcal{H}(M)$$

with $\chi_{u_t} > 0$. This result has studied in the Hermitian case by Sun [35] and in Kähler case by [4, 5, 12–15, 41, 42] and references therein. We shall prove the existence and convergence of (1.8) in elsewhere.

**Structure of the paper:** In §2 we will give a brief introduction to the almost Hermitian manifolds and reviews some lemmas from [36].

In §3 we will prove the oscillation estimate. As in [8], we need to use the modified Alexandroff–Bakelman–Pucci maximum principle in [1, 36]. In the case of Monge–Ampère equation, once we have proved $\chi_u \geq \frac{1}{2} \omega$, then the upper bound for $\lambda_s(\tilde{g}_{ij})$ is easily obtained from a Laplacian inequality. But the Laplacian inequality might not hold for our Monge–Ampère type equations, in order to bound $\lambda_s(\tilde{g}_{ij})$ from above, we need to further use the condition of $C$-subsolution.

In §4 we prove the $C^1$ estimate by using the maximum principle. We will heavily rely on the properties of $C$-subsolution to derive the Corollary 2.6.

In §5 we will give the $C^2$ estimate, which is the most troublesome part in the whole paper. Inspired by the method of Hou–Ma–Wu [24] and Székelyhidi [36], we use a similar argument of [8] together with the properties of $C$-subsolution to estimate the largest eigenvalue of the real Hessian of $u$. On the lower bound for $L(Q)$ (see Lemma 5.2), it is a critical point for us to carefully use the term involving $-C^{ik, jl}$ to control the bad negative third order terms.

Once we have proved them, higher order estimates can be obtained by applying the well-known Evans–Krylov theory (see [25, 38] for instance). We omit the proof of the standard steps here. In the last section we will give the proof of existence theorem.
2 Preliminaries

On an almost Hermitian manifold \((M,g,J)\) with real dimension \(2n\), for each \((p,q)\)-form, we can define \(\partial\) and \(\bar{\partial}\) operators as in \([8,22]\). Denote \(A^{1,1}(M)\) by the set of \(C^\infty\) real \((1,1)\)-forms on \(M\). Let \(e_1,\ldots,e_n\) be a local \(g\)-orthonormal frame of \(T_{1,0}M\). For \(v \in C^2(M,\mathbb{R})\), let

\[
\chi_{ij}(v) = \chi_{ij} + e_i \bar{e}_j v - [e_i, \bar{e}_j]^{0,1} v.
\]

Then in this local frame we have

\[
\chi_v = \sqrt{-1} \sum_{i,j} \chi_{ij}(v) \theta_i \wedge \bar{\theta}_j,
\]

where \(\theta_1,\ldots,\theta_n\) is the corresponding local \(g\)-orthonormal coframe of \(T^*M\).

In what follows, we will let \(\bar{g}_{ij}, \bar{g}^{ij}\) be \(\chi_{ij}(u), \chi_{ij}(\bar{u})\) respectively for simplicity. Under this notation, our equation in (1.1) can be expressed as

\[
F(\bar{g}^{ij}) = \left( \frac{\sigma_n(\lambda^*(\bar{g}^{ij}))}{\sigma_{n-m}(\lambda^*(\bar{g}^{ij}))} \right)^{\frac{1}{m}} = h,
\]

where \(h\) is defined by \(\psi = \binom{n}{m} h^m\), or the inverse Hessian equation

\[
G(\bar{g}^{ij}) = -\sigma_m(\lambda^*(\bar{g}^{ij})) = -\binom{n}{m} \psi^{-1} =: \hat{h}.
\]

Here \(\lambda^*(A)\) (resp., \(\lambda^*(A)\)) denotes the eigenvalues of Hermitian matrix \(A\) with respect to \(g\) (resp., \(g^{-1}\)) and \(\sigma_k\) \((1 \leq k \leq n)\) denotes the \(k\)-th elementary symmetric polynomial defined by

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda \in \mathbb{R}^n.
\]

For each \(\{i_1,\ldots,i_s\} \subset \{1,\ldots,n\}\) and each \(1 \leq k \leq n-1\), let

\[
\sigma_{k;i_1\cdots i_s}(\lambda) = \sigma_k(\lambda|\lambda_{i_1}=\cdots=\lambda_{i_s}=0).
\]

In what follows, we also let

\[
F^{ij} = \frac{\partial F}{\partial \bar{g}_{ij}}, \quad G^{ij} = \frac{\partial G}{\partial \bar{g}_{ij}}, \quad G^{ik,jl} = \frac{\partial^2 G}{\partial \bar{g}_{ik} \partial \bar{g}_{jl}}.
\]

It is not hard to see

\[
G^{ij} = -\frac{mG}{F} F^{ij} = \frac{m\hat{h}}{h} F^{ij}.
\]

This pro rata implies that \(G\) and \(F\) share same certain properties in some lemmas below.

Fixed a point \(x \in M\), we choose a local frame around \(x\) such that \(g_{ij} = \delta_{ij}\) and the matrix \((\bar{g}^{ij})\) is diagonal, then \((G^{ij})\) is also diagonal at \(x\) and \(G^{ii} = \sigma_m-1;1;(\bar{g}^{ii})^2\) for each \(i\). We denote the (second order) linearization of \(G\) by

\[
L = \sum_{i,j} G^{ij}(e_i \bar{e}_j - [e_i, \bar{e}_j]^{0,1}).
\]

2.1 \(C\)-subsolution

Let us denote

\[
\Gamma_n = \{ \lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n : \lambda_i > 0, 1 \leq i \leq n \}.
\]

The following lemma is well-known and one can refer \([26]\) for the proof.
Lemma 2.1 Let $f(\lambda) = \left[ \frac{\sigma_{\lambda}}{\sigma_{n-m(\lambda)}} \right]^{\frac{1}{m}}$ be a function defined on $\Gamma_n$. We have

1. $f_i = \frac{\partial f}{\partial \lambda_i} > 0$ with $f$ is 1-homogeneous and concave;
2. $f|_{\partial \Gamma_n} = 0$;
3. For each $\sigma < \infty$ and $\lambda \in \Gamma_n$, we have $\lim_{t \to \infty} f(t\lambda) > \sigma$.

Proof When $\Gamma_n^\sigma$ is not empty, therefore, $\partial \Gamma_n^\sigma = f^{-1}(\sigma)$ is a smooth hypersurface in $\Gamma_n$.

Definition 2.2 We say $u \in C^2(M)$ is a $C$-subsolution for (1.1) if at each $x \in M$,

\[ (\lambda_s(\bar{g}_{ij}) + \Gamma_n) \cap \partial \Gamma_n^{h(x)} \]  

(2.4)

is a bounded set.

Therefore, at each $x \in M$, there are uniform constants $\delta, R > 0$ such that

\[ (\lambda_s(\bar{g}_{ij}) - \delta I_n + \Gamma_n) \cap \partial \Gamma_n^{h(x)} \subset B_R(0), \]  

(2.5)

where $B_R(0)$ is an $R$-radius ball in $\mathbb{R}^n$ with center 0.

Remark 2.3 For equation (1.1), the set $(\lambda_s(\bar{g}_{ij}) + \Gamma_n) \cap \partial \Gamma_n^{h(x)}$ is bounded implies

\[ \lim_{t \to +\infty} f(\lambda_s(\bar{g}_{ij}) + te_s) > h \]

for each $s$, where $e_s$ is the $s$-th standard basis vector.

Let $u \in C^2(M)$ be an admissible $C$-subsolution, then there is a uniform constant $0 < \tau < 1$ such that

\[ \tau^{-1} \omega \geq \chi_{\mathbb{R}} \geq \tau \omega. \]  

(2.6)

The following lemma is useful to us (see e.g. [18, 36]).

Lemma 2.4 Let $[a, b] \subset (0, \infty)$ and for some $\delta, R > 0$. There exists a constant $\theta$ (depending only on $\delta, R$) such that the following holds. Suppose there exist a Hermitian matrix $B$ and a constant $\sigma \in [a, b]$ satisfying

\[ (\lambda(B) - \delta I_n + \Gamma_n) \cap \partial \Gamma_n^\sigma \subset B_R(0). \]

Then for $\lambda(A) \in \partial \Gamma_n^\sigma$, we have either

\[ \sum_{i,j} F^{ij}(A)(B_{ij} - H_{ij}) \geq \theta \sum_i F^{ii}(A) \]

or

\[ F^{kk}(A) \geq \theta \sum_i F^{ii}(A), \quad \text{for each } k. \]

Proof When $|\lambda(A)| > R$, the conclusion follows from [36, Proposition 6]. So we may assume $|\lambda(A)| \leq R$. Consider the set

\[ S_{R,\sigma} = \{ N \in H : \lambda(N) \in \overline{B_R(0)} \cap \partial \Gamma^\sigma \}, \]

where $H$ denotes the set of Hermitian matrices. It is clear that $S_{R,\sigma}$ is compact, then there exists a constant $C$ such that for $A \in S_{R,\sigma}$,

\[ C^{-1} \leq F^{kk}(A) \leq C, \quad \text{for each } k. \]
Decreasing $\theta$ if necessary,

$$F^{kk}(A) > \theta \sum_p F^{p\bar{p}}(A), \quad \text{for each } k.$$ 

For the $f$ defined in (2.1), the following result is also important for us.

**Lemma 2.5** ([14, 18, 29, 36]) *For each $\sigma \in (0, \infty)$,*

1. There is a large constant $N_0$ depending on $\sigma$, such that for $N \geq N_0$,

$$\Gamma_n + NI_n \subset \Gamma_n^\sigma.$$ 

2. There is a constant $\kappa$ depending on $\sigma$ such that for any $\lambda \in \partial \Gamma_n^\sigma$,

$$\sum_i f_i(\lambda) > \kappa.$$ 

We will use the following corollary frequently in the next sections.

**Corollary 2.6** *Assume $u$ is an admissible $C$-subsolution and $u$ is the solution for (1.1). For some $\delta, R > 0$ such that*

$$(\lambda_*(\bar{g}_{ij}) - \delta I_n + \Gamma_n) \cap \partial \Gamma_n^h \subset B_R(0),$$

*there exists a constant $\theta$ (depending on $\delta, R$), such that we have either*

$$L(u - u) = \sum_{i,j} G^{ij}(\bar{g})(\bar{g}_{ij} - \bar{g}_{ij}) \geq \theta \sum_i G^{ii}(\bar{g})$$

*or*

$$G^{kk}(\bar{g}) \geq \theta \sum_i G^{ii}(\bar{g}), \quad \text{for each } k$$

*if $\lambda_*(\bar{g}_{ij}) \in \partial \Gamma_n^h$. In addition, there is a constant $\Theta > 0$ depending on $h$ such that*

$$G = \sum_i G^{ii}(\bar{g}) > \Theta, \quad \text{if } \lambda_*(\bar{g}_{ij}) \in \partial \Gamma_n^h.$$ 

### 3 Oscillation Estimate

The following $L^1$-estimate is well-known (see e.g. [8]).

**Proposition 3.1** ([8]) *There exists a constant $C$ depending only on $(M, \omega, J)$ such that for each $u \in \mathcal{H}(M)$ with $\sup_M (u - \underline{u}) = 0$. Then*

$$\int_M |u - \underline{u}|^\omega \leq C.$$  

(3.1)

*The main result of this section is the following proposition.*

**Proposition 3.2** *Let $u$ be the solution for (1.1) with $\sup_M (u - \underline{u}) = 0$. Then*

$$\text{osc}_M(u - \underline{u}) \leq C$$  

(3.2)

*for some constant $C$ depending on $(M, J, \omega), \chi, k, \alpha, \psi$ and $\underline{u}$.*

**Proof** From the hypothesis, it suffices to estimate $m_0 = \inf_M (u - \underline{u})$, we may assume $m_0$ is attained at the origin. Choose a local coordinate chart $(x^1, \ldots, x^{2n})$ in a neighborhood of the origin containing the unit ball $B_1 \subset \mathbb{R}^{2n}$.

Consider the test function

$$v = u - \underline{u} + \varepsilon \sum_{i=1}^{2n} (x^i)^2$$
for a small $\varepsilon > 0$ to be determined later. Then

$$v(0) = m_0 \quad \text{and} \quad v \geq m_0 + \varepsilon \quad \text{on} \partial B_1.$$ 

We define the lower contact set of $v$ by

$$S = \left\{ x \in B_1 : |Dv(x)| \leq \frac{\varepsilon}{2}, v(y) \geq v(x) + Dv(x) \cdot (y-x), \forall y \in B_1 \right\}.$$ 

By the modified Alexandroff–Bakelman–Pucci maximum principle (see e.g. [1, 36]), there is constant $c_0 = c_0(n) > 0$ such that

$$c_0 \varepsilon^{2n} \leq \int_S \det(D^2v). \quad (3.3)$$

In what follows, the constant $C$ below in this section may be changed from line to line, but it depends on the allowed data. Note that $0 \in S$ and $D^2v \geq 0$ on $S$. Moreover,

$$(D^2(u - \bar{u}))^J(x) \geq (D^2v)^J(x) - C \varepsilon I_d \geq -C \varepsilon I_d.$$ 

Using the fact $|D(u - \bar{u})| \leq \frac{\varepsilon}{2}$ on $S$ by definition, which implies

$$H(u) - H(u) = (D^2 u + E(Du))^J - (D^2 u + E(Du))^J = (D^2(u - \bar{u}))^J + (E(D(u - \bar{u})))^J \geq -C \varepsilon I_d.$$ 

So, $\chi_u - \chi_{\bar{u}} \geq -C \varepsilon \omega$. So, if we choose $\varepsilon$ sufficient small such that $C \varepsilon \leq \delta$, then $\lambda_s(\bar{g}_{ij}) \in \lambda_s(\bar{g}_{ij}) - \delta I_n + \Gamma_n$. On the other hand, the equation (2.1) implies $\lambda_s(\bar{g}_{ij}) \in \partial \Gamma_n^{h(x)}$. Thus,

$$\lambda_s(\bar{g}_{ij}) \in (\lambda_s(\bar{g}_{ij}) - \delta I_n + \Gamma_n) \cap \partial \Gamma_n^{h(x)} \subset B_R(0)$$

for some $R > 0$ as (2.5). This gives an upper bound for $H(u)$ and hence also for $H(v) - E(Dv)$ on $S$. The rest of the proof is analogous to [8], we present it on the below for completeness.

Using the fact $\det(A + B) \geq \det(A) + \det(B)$ for positive definite Hermitian matrices $A, B$, on $S$, we have

$$\det(D^2v) \leq 2^{2n-1} \det((D^2v)^J) = 2^{2n-1} \det(H(v) - E(Dv)) \leq C.$$ 

Plugging it into (3.3) gives us

$$c_0 \varepsilon^{2n} \leq C|S|.$$ 

For each $x \in S$, we have $D^2v(x) \geq 0$, then

$$m_0 = v(0) \geq v(x) - |Dv(x)||x| \geq v(x) - \frac{\varepsilon}{2}.$$ 

We may assume $m_0 + \varepsilon < 0$ (otherwise we are done), then, on $S$,

$$-v \geq |m_0 + \varepsilon|.$$ 

It follows from Proposition 3.1 that

$$c_0 \varepsilon^{2n} \leq C|S| \leq C \frac{\int_S (-v) \omega^n}{|m_0 + \varepsilon|} \leq \frac{C}{|m_0 + \varepsilon|}.$$ 

Then this gives a uniform lower bound for $m_0$. \qed
4 First Order Estimate

Proposition 4.1 Let $u$ (resp., $\underline{u}$) be the solution (resp., $C$-subsolution) for (1.1) with $\sup_M (u - \underline{u}) = 0$. Then

$$|\partial u| \leq C$$

(4.1)

for some constant $C$ depending on $(M, J, \omega, \chi, k, \alpha, \psi)$ and $u$.

Proof Let $w = \frac{1}{3} e^{B\eta}$ for $\eta = u - u - \inf_M (u - u)$, where $B$ is a certain large constant to be chosen soon.\(^1\)

Consider the function

$$V = e^w |\partial u|^2.$$ 

Suppose $V$ achieves maximum at the origin. Then near the origin, we can choose a local $g$-unitary frame still denoted by $e_1, \ldots, e_n$ such that $g_{ij} = \delta_{ij}$ and the matrix $(\hat{g}_{ij})$ is diagonal.

By the maximum principle, at the origin, it follows that

$$0 \geq \frac{L(V)}{Bwe^w |\partial u|^2} = \frac{L(e^w)}{Bwe^w} + \frac{L(|\partial u|^2)}{Bwe^w} + 2G^{ij}\text{Re}\left\{e_i (w) \hat{e}_j (|\partial u|^2)\right\}$$

$$= L(\eta) + B(1 + w)G^{ij}|\eta_i|^2 + \frac{L(|\partial u|^2)}{Bwe^w}$$

$$+ \frac{2}{|\partial u|^2} \sum_j G^{ij}\text{Re}\{e_i (\eta)\hat{e}_j (u)\hat{e}_j (u) + e_i (\eta)\hat{e}_i (j) e_j (u)\}. \quad (4.2)$$

By direct calculation,

$$L(|\partial u|^2) = G^{ij} (e_i \hat{e}_j (|\partial u|^2) - [e_i, \hat{e}_i]^0,1 (|\partial u|^2)) = I + II + III,$$

(4.3)

where

$$I = G^{ij} (e_i \hat{e}_j u - [e_i, \hat{e}_i]^0,1 e_j u),$$

$$II = G^{ij} (e_i \hat{e}_j u - [e_i, \hat{e}_i]^0,1 e_j u),$$

$$III = G^{ij} ([e_i, \hat{e}_j]^0,1) u.$$ 

Differentiating (2.2) once, since $g$ is almost Hermitian, we have

$$G^{ij} (e_j e_i e_i u - e_j e_i e_i u) = -\hat{h}_j.$$ 

(4.4)

Note that

$$G^{ij} (e_j \hat{e}_i e_i u - [e_i, \hat{e}_i]^0,1 e_j u)$$

$$= G^{ij} (e_j \hat{e}_i e_i u + e_i [e_i, \hat{e}_i] u + [e_i, \hat{e}_i] e_i u - [e_i, \hat{e}_i]^0,1 e_j u)$$

$$= -\hat{h}_j + G^{ij} (e_i [e_i, \hat{e}_j]^0,1 u + G^{ij} (e_i [\hat{e}_i, e_i] u + [e_i, e_j] e_i u - [e_i, \hat{e}_i]^0,1 e_j u)$$

$$=-\hat{h}_j + G^{ij} (e_i [\hat{e}_i, e_j] u + [e_i, e_j] e_i u + [e_i, \hat{e}_i] e_i u - [e_i, \hat{e}_i]^0,1 e_j u).$$

We may and do assume $|\partial u| \gg 1$. Therefore,

$$I + II \geq -2 \sum_j \text{Re}\{\hat{h}_j u_j\} - C |\partial u| \sum_j G^{ij} (e_i e_j u + e_i e_j u) - C |\partial u|^2 G$$

\(^1\) The $C, C_0$ below in this section denote the constants that may change from line to line, where $C_0$ is a constant depending on all the allowed data, but $C$ does not depend on $B$ that we are yet to choose.
\[
\geq -2 \sum_j \text{Re}\{\tilde{h}_j u_j\} - \frac{C}{\varepsilon} |\partial u|^2 G - \varepsilon \sum_j G^{ii} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \tag{4.5}
\]

Plugging (4.5) into (4.3),
\[
\frac{L(|\partial u|^2)}{Bw|\partial u|^2} \geq -2 \sum_j \text{Re}\{\tilde{h}_j u_j\} + (1 - \varepsilon) \sum_j G^{ii} \frac{|e_i e_j u|^2 + |e_i \bar{e}_j u|^2}{Bw|\partial u|^2} - \frac{C G}{Bw\varepsilon}. \tag{4.6}
\]

Now we estimate the last term of (4.2). On the one hand, using Cauchy–Schwarz inequality, for each \(0 < \varepsilon \leq \frac{1}{2}\) we have
\[
2 \sum_j G^{ii} \text{Re}\{e_i(\eta)\bar{e}_i e_j(u)\bar{e}_j(u)\} = 2 \sum_j G^{ii} \text{Re}\{e_i \eta u_i - [e_j, \bar{e}_i],[e_j, \bar{e}_i]^0,1(u) - [e_j, \bar{e}_i]^{1,0}(u)\} \geq 2G^{ii}\bar{g}_{ii} - 1)\text{Re}\{\eta u_i\} - 2 \sum_j G^{ii} \text{Re}\{e_i(\eta)\bar{e}_i e_j(u)[e_j, \bar{e}_i]^{1,0}(u)\} \geq 2G^{ii}\bar{g}_{ii} \text{Re}\{\eta u_i\} - \varepsilon Bw|\partial u|^2 G^{ii} |\eta_i|^2 - \frac{C}{Bw\varepsilon}|\partial u|^2 G.
\]

On the other hand, \(0 < \varepsilon \leq \frac{1}{2}\) implies \(1 \leq (1 - \varepsilon)(1 + 2\varepsilon)\). Using Cauchy–Schwarz inequality again, we derive
\[
2 \sum_j G^{ii} \text{Re}\{e_i(\eta)\bar{e}_i e_j(u)\bar{e}_j(u)\} \geq \frac{(1 - \varepsilon)}{Bw} \sum_j G^{ii} |\bar{e}_i e_j(u)|^2 - (1 + 2\varepsilon) Bw|\partial u|^2 G^{ii} |\eta_i|^2.
\]

Therefore,
\[
2G^{ii}\text{Re}\left\{e_i(w)\bar{e}_i(\frac{\partial u(\cdot)}{Bw}|\partial u|^2)\right\} \geq 2G^{ii}\bar{g}_{ii} \text{Re}\{\eta u_i\} - (1 + 3\varepsilon) Bw G^{ii} |\eta_i|^2 \geq \frac{2C}{Bw\varepsilon} G - (1 - \varepsilon) \sum_j G^{ii} \frac{|\bar{e}_i e_j(u)|^2}{Bw|\partial u|^2}. \tag{4.7}
\]

Combining (4.2), (4.6)–(4.7), we obtain
\[
0 \geq L(\eta) + B(1 + w)G^{ii} |\eta_i|^2 - \frac{2C}{Bw\varepsilon} G - \frac{2 \sum_j \text{Re}\{\tilde{h}_j u_j\}}{Bw|\partial u|^2} + 2G^{ii}\bar{g}_{ii} \text{Re}\{\eta u_i\} - (1 + 3\varepsilon) Bw G^{ii} |\eta_i|^2 \geq L(\eta) + B(1 - 3\varepsilon w)G^{ii} |\eta_i|^2 - \frac{2C}{Bw\varepsilon} G - \frac{C}{Bw|\partial u|^2} + 2G^{ii}\bar{g}_{ii} \text{Re}\{\eta u_i\}.
\]

Thus, we can choose \(\varepsilon = \frac{1}{Bw(0)} = \frac{1}{2} e^{-Bn(0)} \leq \frac{1}{2}\), which gives us
\[
L(\eta) + G^{ii}\bar{g}_{ii} \frac{2 \text{Re}\{\eta u_i\}}{|\partial u|^2} + \frac{B}{2} G^{ii} |\eta_i|^2 \leq \frac{C}{Bw|\partial u|^2} + \frac{C}{B} G. \tag{4.8}
\]

**Case 1** Suppose (2.7) holds, we divide the proof into two parts.

(i) Assume \(G^{ii} \geq D\) for some \(j\), where \(D > 0\) is a constant to be chosen soon. Then
\[
L(\eta) \geq \theta G \geq \frac{D\theta}{2} + \frac{\theta}{2} G.
\]
Now we may assume $|\partial u| \geq |\partial \bar{u}|$, then $|\partial \eta| \leq |\partial (\bar{u} - u)| \leq 2|\partial u|$. Hence
\[ G^{\bar{u}} \tilde{g}_{\bar{u}} \text{Re}\{\eta_{\bar{u}} u_{\bar{u}}\} \geq -4G^{\bar{u}} \tilde{g}_{\bar{u}} = -4m\hat{h}. \tag{4.9} \]

Plugging it into (4.8) gives us
\[ \frac{D\theta}{2} - 4m\hat{h} + \left( \frac{\theta}{2} - \frac{C}{B} \right) \hat{G} \leq \frac{C}{B w|\partial u|}. \]

Choose $B, D$ sufficiently large such that the third term above can be cancelled and $\frac{D\theta}{2} \geq 1 + 4m \sup_M \hat{h}$. Therefore, $|\partial u| \leq C_0$.

(ii) Assume $G^{\bar{u}} \geq D$ for each $j$. Recall that $|\partial u| \geq \max\{1, |\partial \bar{u}|\}$, then by Cauchy–Schwarz inequality,
\[ \sum_i G^{\bar{u}} \tilde{g}_{\bar{u}} \text{Re}\{\eta_{\bar{u}} u_{\bar{u}}\} \geq -B \frac{4}{4} \sum_i G^{\bar{u}} |\eta_{\bar{u}}|^2 - \frac{4}{B |\partial u|^2} \sum_i G^{\bar{u}} \tilde{g}_{\bar{u}}. \]

It follows that
\[ \frac{1}{2} \theta \hat{\Theta} + \frac{\theta}{2} \hat{G} \leq \frac{C}{B w|\partial u|} + \frac{C}{B} \hat{G} + \frac{4}{B |\partial u|^2} \sum_i G^{\bar{u}} \tilde{g}_{\bar{u}}. \]

For the choice of $B$ as before, we have
\[ \frac{1}{2} \theta \hat{\Theta} \leq \frac{C}{B w|\partial u|} + \frac{4}{B |\partial u|^2} \sum_i G^{\bar{u}} \tilde{g}_{\bar{u}}. \]

It suffices to prove for each $i$,
\[ \sigma_{m-1,i}(\lambda^*(\tilde{g}^{ij})) = G^{\bar{u}} \tilde{g}_{\bar{u}} \leq C. \tag{4.10} \]

We may assume $\tilde{g}^{11} \geq \cdots \geq \tilde{g}^{n\bar{n}}$ at the origin. Hence
\[ \prod_{i=1}^m \tilde{g}^{\bar{u}} \geq \sigma_m(\lambda^*(\tilde{g}^{ij})) = \psi^{-1}. \]

Therefore,
\[ \psi^{-1} \tilde{g}^{11} \leq (\tilde{g}^{11})^2 \prod_{i=2}^m \tilde{g}^{\bar{u}} \leq \sigma_{m-1:1} \cdot (\tilde{g}^{11})^2 = G^{11} \leq D. \]

It follows maximality of $\tilde{g}^{11}$ among all the $\{\tilde{g}^{\bar{u}}\}$, we see that
\[ \sigma_{m-1:1}(\lambda^*(\tilde{g}^{ij})) \leq C(\tilde{g}^{11})^{m-1} \leq C(D\psi)^{m-1} \leq C_0 \]
and this proves (4.10).

**Case 2** Suppose (2.8) holds, then we have
\[ G^{kk} \geq \theta \hat{G} \geq \theta \hat{\Theta}, \quad \text{for each } k. \tag{4.11} \]

Using (2.6), it gives us
\[ L(\eta) = G^{\bar{u}}(\tilde{g}_{\bar{u}} - \tilde{g}_{\bar{u}}) \geq \tau \hat{G} - G^{\bar{u}} \tilde{g}_{\bar{u}}. \tag{4.12} \]

Plug (4.9) and (4.12) into (4.8) implies
\[ \left( \tau - \frac{C}{B} \right) \hat{G} + \frac{B}{2} G^{\bar{u}} |\eta_{\bar{u}}|^2 \leq 5G^{\bar{u}} \tilde{g}_{\bar{u}} + \frac{C}{B w|\partial \eta|} \leq C + \frac{C}{B w|\partial \eta|}. \]
Choosing $B$ large enough once again such that $B\tau \geq C$, the first term on the left hand can be cancelled. Hence by (4.11) we have

$$\frac{B\theta\Theta}{2} |\partial \eta|^2 \leq C + \frac{C}{Bw|\partial \eta|},$$

which gives an upper bound for $|\partial \eta|$ and hence $|\partial u| \leq C_0$. \hfill \Box

5 Second Order Estimate

Theorem 5.1 Let $u$ (resp. $u$) be the solution (resp. $C$-subsolution) for (1.1) with $\sup_M (u - \underline{u}) = 0$. Then there exists a constant $C$ depending on $(M, J, \omega)$, $\chi$, $k$, $\alpha$, $\psi$ and $u$ such that

$$\|\nabla^2 u\|_{C^0(M)} \leq C. \quad (5.1)$$

Our proof of the above theorem is modeled on the arguments of [8] for the Monge–Ampère equation. The mainly difference is that we will use the concavity of $G$ carefully to obtain some good terms (see Lemma 5.5 below), and those terms can be used to control the bad third order terms. Analogous to the arguments in [8], it suffices to bound the largest eigenvalue $\lambda_1(\nabla^2 u)$ of the real Hessian $\nabla^2 u$ with respect to $g$ from above. We consider the test function

$$Q = \log \lambda_1(\nabla^2 u) + \phi(|\partial u|^2) + \varphi(\eta)$$

on $\Omega = \{\lambda_1(\nabla^2 u) > 0\} \subset M$. Here $\varphi$ is a function defined by

$$\varphi(\eta) = e^{B\eta}, \quad \eta = \underline{u} - u + \sup_M (u - \underline{u}) + 1$$

for a real constant $B > 0$ to be determined later, and $\phi$ is defined by

$$\phi(s) = -\frac{1}{2} \log(1 + \sup_M |\partial u|^2 - s).$$

Set $K = 1 + \sup_M |\partial u|^2$. Note that

$$\frac{1}{2K} \leq \phi'(|\partial u|^2) \leq \frac{1}{2}, \quad \phi'' = 2(\phi')^2.$$

We may assume $\Omega$ is a nonempty (relative) open set (otherwise we are done). When $z$ approaches to $\partial \Omega$, then $Q(z) \to -\infty$. Suppose $Q$ achieves a maximum at the origin in $\Omega$ (after a translation), we can choose a proper local frame $x^1, \ldots, x^{2n}$ as in [8] such that

$$g_{ij} = \delta_{ij}, \quad \partial g_{\alpha\beta} = 0$$

and the matrix $(\tilde{g}_{ij})$ is diagonal at the origin. \hfill (5.2)

We may further assume at the origin, $\tilde{g}_{11} \geq \cdots \geq \tilde{g}_{nn}$. Consider the perturbation $\Phi$ for $\nabla^2 u$ defined by $\Phi^\gamma_\beta = \sum \gamma g^{\alpha\gamma}(\nabla^2 u - S_{\gamma\beta})$ for some smooth section $S$ on $T^*M \otimes T^*M$ such that $\lambda_1(\Phi) \leq \lambda_1(\nabla^2 u)$ on $\Omega$ with equality only at the origin, but also $\lambda_1(\Phi) \in C^2(\Omega)$ (see e.g. [8, 36]). Let $V_1, \ldots, V_{2n}$ be the eigenvectors for $\Phi$ at the origin with eigenvalues $\lambda_1(\Phi), \ldots, \lambda_2n(\Phi)$ respectively. We (locally) define the test function

$$Q = \log(\lambda_1(\Phi)) + \phi(|\partial u|^2) + \varphi(\eta).$$

For simplicity, we denote $\lambda_\beta = \lambda_\beta(\Phi)$.

In what follows, we will use the Einstein summation convention, and all the following calculations are done at the origin. The $C, C_0$ below in this section denotes positive constants.
those may change from line to line, where \( C_0 \) is a constant depending all the allowed data, but \( C \) does not depend on \( B \) that we are yet to choose.

By the maximum principle, at the origin, for each \( i \), we have

\[
\frac{1}{\lambda_i} e_i(\lambda_1) = -\phi' e_i(|\partial u|^2) - Be^{B\eta} e_i(\eta), \quad (5.3)
\]

\[
0 \geq L(Q) = \frac{L(\lambda_1)}{\lambda_i} - G^{i\alpha} \frac{|e_i(\lambda_1)|^2}{\lambda_1} + \frac{\phi''}{\varepsilon} \sum_j G^{ji} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \frac{\phi'}{2} \sum_j G^{ji} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + \frac{\phi''}{\varepsilon} |e_i(\partial u|^2)|^2 + Be^{B\eta} L(\eta) + B^2 e^{B\eta} G^{ij} |e_i(\eta)|^2.
\]

5.1 Lower Bound for \( L(Q) \)

The main result of this subsection is the following lemma.

**Lemma 5.2** For each \( \varepsilon \in (0, \frac{1}{2}] \), at the origin, we have

\[
L(\lambda_1) \geq 2 \sum_{\alpha > 1} G^{i\alpha} \frac{|e_i(u\nabla_i V_j)|^2}{\lambda_1 - \lambda_\alpha} - \frac{1}{\lambda_i} G^{ik,jl} V_k(\tilde{g}_{ik}) V_l(\tilde{g}_{jl}) - \frac{C}{\varepsilon} \sum_j G^{ji} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + \frac{\phi''}{\varepsilon} |e_i(\partial u|^2)|^2 + Be^{B\eta} L(\eta) + B^2 e^{B\eta} G^{ij} |e_i(\eta)|^2.
\]

First, we calculate \( L(\lambda_1) \). Let \( u_{ij} = e_i e_j u - (\nabla e_i e_j) u \) and \( u V_i V_j = u_{kl} V_k V_j \). The first and second derivative of \( \lambda_1 \) are well-known (see e.g. [8, 29, 36]), we see that

\[
L(\lambda_1) \geq 2 \sum_{\alpha > 1} G^{i\alpha} \frac{|e_i(u\nabla_i V_j)|^2}{\lambda_1 - \lambda_\alpha} + G^{ij} (e_i \bar{e}_i - [e_i, \bar{e}_i]^{0,1})(u\nabla_i V_j) - C\lambda_1 G.
\]

Similar to (4.4), we get

\[
G^{ij} V_1 \{e_i \bar{e}_i (u) - [e_i, \bar{e}_i]^{0,1}(u)\} = -V_1(\dot{h}).
\]

Differentiating with \( V_1 \) again, we obtain

\[
G^{ij} V_1 V_1(\tilde{g}_{ij}) = -G^{ik,jl} V_k(\tilde{g}_{ik}) V_l(\tilde{g}_{jl}) - V_1 V_1(\dot{h}).
\]

**Claim 1** If \( \lambda_1 \gg 1 \), then

\[
G^{ij} (e_i \bar{e}_i - [e_i, \bar{e}_i]^{0,1})(\lambda_1) \geq -G^{ik,jl} V_k(\tilde{g}_{ik}) V_l(\tilde{g}_{jl}) - C\lambda_1 G - 2G^{ij} \{[V_1, \bar{e}_i] V_1 e_i (u) + [V_1, e_i] V_1 \bar{e}_i (u)\}.
\]

**Proof** By direct calculation,

\[
G^{ij} (e_i \bar{e}_i - [e_i, \bar{e}_i]^{0,1})(u\nabla_i V_j)
\]

\[
= G^{ij} e_i \bar{e}_i (V_1 V_j(u) - (\nabla V_i V_j) u) - G^{ij} [e_i, \bar{e}_i]^{0,1} V_1 V_j(u) - (\nabla V_i V_j) u
\]

\[
\geq G^{ij} V_1 V_j(e_i \bar{e}_i(u) - [e_i, \bar{e}_i]^{0,1}(u)) - 2G^{ij} \{[V_1, \bar{e}_i] V_1 e_i (u) + [V_1, e_i] V_1 \bar{e}_i (u)\}
\]

\[
- G^{ij} (\nabla V_i V_j) e_i \bar{e}_i (u) + G^{ij} (\nabla V_i V_j) [e_i, \bar{e}_i]^{0,1}(u) - C\lambda_1 G
\]

\[
\geq G^{ij} V_1 V_1(\tilde{g}_{ij}) - 2G^{ij} \{[V_1, \bar{e}_i] V_1 e_i (u) + [V_1, e_i] V_1 \bar{e}_i (u)\} + (\nabla V_i V_j)(\dot{h}) - C\lambda_1 G.
\]

Then Claim 1 follows if \( \lambda_1 \gg 1 \).
Combining the equalities (5.6) and (5.9) together, it follows that
\[
L(\lambda_1) \geq 2 \sum_{\alpha > 1} G^{i\bar{i}} \left| \frac{e_i(uv_{\alpha}v_1)}{\lambda_1 - \lambda_\alpha} \right|^2 - G^{i\bar{k},j\bar{l}} V_1(\bar{g}_{i\bar{k},j\bar{l}}) V_1(\bar{g}_{j\bar{l},i\bar{k}}) - 2G^{i\bar{i}} \text{Re}\{[V_1, e_i]V_\bar{i}(u) + [V_1, \bar{e}_i]V_\bar{i}(u)\} - C\lambda_1 G. \tag{5.10}
\]

Since by (4.6) it gives us
\[
L(|\partial u|^2) \geq \frac{1}{2} \sum_j G^{i\bar{i}}(\text{Re} |e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - CG. \tag{5.11}
\]

Hence we have
\[
L(Q) \geq 2 \sum_{\alpha > 1} G^{i\bar{i}} \left| \frac{e_i(uv_{\alpha}v_1)}{\lambda_1 - \lambda_\alpha} \right|^2 - \frac{1}{\lambda_1} G^{i\bar{k},j\bar{l}} V_1(\bar{g}_{i\bar{k},j\bar{l}}) V_1(\bar{g}_{j\bar{l},i\bar{k}}) + B^2 e^{Bn} G^{i\bar{i}} |e_i(\eta)|^2 + B e^{Bn} L(\eta) - 2G^{i\bar{i}} \text{Re}\{[V_1, e_i]V_\bar{i}(u) + [V_1, \bar{e}_i]V_\bar{i}(u)\} - C G

- G^{i\bar{i}} \left| \frac{e_i(\lambda_1)}{\lambda_1^2} \right|^2 + \frac{g'}{2} \sum_j G^{i\bar{i}}(\text{Re} |e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + \phi'' G^{i\bar{i}} |e_i(|\partial u|^2)|^2. \tag{5.12}
\]

Now we deal with the third derivatives of the right hand.

**Claim 2** For any \( \varepsilon \in (0, \frac{1}{2}] \), we have
\[
2G^{i\bar{i}} \text{Re}\{[V_1, e_i]V_\bar{i}(u) + [V_1, \bar{e}_i]V_\bar{i}(u)\} \leq \varepsilon G^{i\bar{i}} \left| \frac{e_i(\lambda_1)}{\lambda_1^2} \right|^2 + \varepsilon \sum_{\alpha > 1} G^{i\bar{i}} \left| \frac{e_i(uv_{\alpha}v_1)}{\lambda_1 - \lambda_\alpha} \right|^2 + C G. \tag{5.13}
\]

**Proof** We may find \( \mu_{i\beta} \in \mathbb{C} \) such that
\[
[V_1, e_i] = \sum_{\beta=1}^{2n} \mu_{i\beta} V_\beta, \quad [V_1, \bar{e}_i] = \sum_{\beta=1}^{2n} \mu_{i\bar{\beta}} \bar{V}_\beta.
\]

Therefore,
\[
\text{Re}\{[V_1, e_i]V_\bar{i}(u) + [V_1, \bar{e}_i]V_\bar{i}(u)\} \leq C \sum_{\beta=1}^{2n} |V_\beta V_1 e_i(u)|.
\]

Then it suffices to estimate \( \sum_{\beta} G^{i\bar{i}} \frac{|V_\beta V_1 e_i(u)|}{\lambda_1} \). Since
\[
|V_\beta V_1 e_i(u)| = |e_i V_\beta V_1(u) + V_\beta [V_1, e_i](u) + [V_\beta, e_i] V_1(u)| = |e_i(uv_{\beta}v_1) + e_i(\nabla v_\beta v_1)(u) + V_\beta [V_1, e_i](u) + [V_\beta, e_i] V_1(u)| \leq |e_i(uv_{\beta}v_1)| + C\lambda_1,
\]

it follows that
\[
\sum_{\beta} G^{i\bar{i}} \frac{|V_\beta V_1 e_i(u)|}{\lambda_1} \leq G^{i\bar{i}} \left| \frac{e_i(uv_{\beta}v_1)}{\lambda_1} \right| + C G
\[
\leq G^{i\bar{i}} \left| \frac{e_i(\lambda_1)}{\lambda_1} \right| + \sum_{\alpha > 1} G^{i\bar{i}} \left| \frac{e_i(uv_{\alpha}v_1)}{\lambda_1} \right| + C G.
\]
By Cauchy–Schwarz inequality, for \( \varepsilon \in (0, \frac{1}{2}] \), we derive
\[
G^{i\bar{i}} \left| \frac{e_i(\lambda_1)}{\lambda_1} \right| \leq \varepsilon G^{i\bar{i}} \left| \frac{e_i(\lambda_1)}{\lambda_1^2} \right| + \frac{C}{\varepsilon} \mathcal{G} \tag{5.14}
\]
and
\[
\sum_{\beta > 1} G^{i\bar{i}} \left| \frac{e_i(u v_\beta v_{\bar{1}})}{\lambda_1} \right| \leq \varepsilon \sum_{\beta > 1} G^{i\bar{i}} \left| \frac{e_i(u v_\beta v_{\bar{1}})}{\lambda_1} \right| + \frac{\lambda_1 - \lambda_{\beta}}{\varepsilon \lambda_1} \mathcal{G} \leq \varepsilon \sum_{\beta > 1} G^{i\bar{i}} \left| \frac{e_i(u v_\beta v_{\bar{1}})}{\lambda_1} \right| \frac{C}{\varepsilon} \mathcal{G}, \tag{5.15}
\]
where we used
\[
\sum_{\beta = 1}^{2n} \lambda_\beta = \Delta u = \Delta^C u + \tau(du) \geq -C + \tau(du) \geq -C
\]
(see [8, Eq. (2.5)]) for the last inequality. Here \( \tau \) is the torsion vector field of \((\omega, J)\) (the dual of its Lee form, see [37, Lemma 3.2]). By (5.14)–(5.15), we have
\[
\sum_{\beta = 1}^{2n} G^{i\bar{i}} \left| V_{\beta} V_{\bar{1}} e_i(u) \right| \leq \varepsilon G^{i\bar{i}} \left| \frac{e_i(\lambda_1)}{\lambda_1^2} \right| + \varepsilon \sum_{\beta > 1} G^{i\bar{i}} \left| \frac{e_i(u v_\beta v_{\bar{1}})}{\lambda_1} \right| + \frac{C}{\varepsilon} \mathcal{G}.
\]
Then these prove (5.13).

Consequently, Lemma 5.2 follows from (5.12) and (5.13).

5.2 Continued Proof of Theorem 5.1
The proof can be divided into two cases.

Case 1 At the origin, either

Subcase 1.1
\[
G^n \leq B^3 e^{2B\eta} G^{1\bar{1}}, \text{ or} \tag{5.16}
\]

Subcase 1.2
\[
\frac{\rho'}{4} \sum_{j} G^{i\bar{i}} \left( |\varepsilon_i e_j u|^2 + |e_i \bar{e}_j u|^2 \right) > 6 \sup_{M} (|\partial \eta|^2) B^2 e^{2B\eta} \mathcal{G} \tag{5.17}
\]

In this case we choose \( \varepsilon = \frac{1}{2} \). Using the elemental inequality \( |a + b|^2 \leq 4|a|^2 + 4|b|^2 \) for (5.3), we get
\[
-(1 + \varepsilon) G^{i\bar{i}} \left| \frac{e_i(\lambda_1)}{\lambda_1^2} \right| \geq -6 \sup_{M} (|\partial \eta|^2) B^2 e^{2B\eta} \mathcal{G} - 2(\rho')^2 G^{i\bar{i}} |\varepsilon_i (|\partial u|^2)|^2. \tag{5.18}
\]

Plugging it into (5.5) gives us
\[
L(Q) \geq (2 - \varepsilon) \sum_{\alpha > 1} G^{i\bar{i}} \left| \frac{e_i(u v_\alpha v_{\bar{1}})}{\lambda_1(1 - \lambda_\alpha)} \right| - \frac{1}{\lambda_1} G^{i\bar{k},j\bar{j}} V_{\bar{1}} (\tilde{g}_{i\bar{k}}) V_{\bar{1}} (\tilde{g}_{j\bar{j}})
\]
\[
- \left( \frac{C}{\varepsilon} + 6 \sup_{M} (|\partial \eta|^2) B^2 e^{2B\eta} \right) \mathcal{G} + \frac{\rho'}{2} \sum_{j} G^{i\bar{i}} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2)
\]
\[
+ Be^{B\eta} L(\eta) + B^2 e^{2B\eta} G^{i\bar{i}} |\varepsilon_i(\eta)|^2.
\]
5.2.1 Proof of Subcase 1.1
In this subcase using the facts of concavity of $G$, $L(\eta)$ has uniform lower bound and $\mathcal{G} \geq \Theta$, we have
\[ 0 \geq L(Q) \geq \frac{\phi'}{2} \sum_{i,j} G^{\bar{i}\bar{j}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - C_B \mathcal{G}. \] (5.19)
Where and hereafter $C_B$ are positive constants depending on $B$. Since $\{G^{\bar{i}\bar{j}}\}$ are pairwise comparable (up to a multiplier $B^3 e^{2Bn}$) by (5.16), then
\[ \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq C_B K. \]
Then the complex covariant derivatives $u_{ij} = e_i e_j u - (\nabla_{e_i} e_j) u$, $u_{\bar{i}\bar{j}} = e_i \bar{e}_j u - (\nabla_{e_i} \bar{e}_j) u$ satisfy
\[ \sum_{i,j} (|u_{ij}|^2 + |u_{\bar{i}\bar{j}}|^2) \leq C_B K, \]
and this proves (5.1).

5.2.2 Proof of Subcase 1.2
In this subcase we also have
\[ 0 \geq \frac{C}{\varepsilon} \mathcal{G} - 6 \sup_{M} (|\partial \eta|^2) B^2 e^{2Bn} \mathcal{G} + \frac{\phi'}{2} \sum_{i,j} G^{\bar{i}\bar{j}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + Be^{Bn} L(\eta) \]
\[ \geq \frac{\phi'}{4} \sum_{i,j} G^{\bar{i}\bar{j}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \frac{C}{\varepsilon} \mathcal{G} + Be^{Bn} L(\eta), \] (5.20)
where we have used (5.17) in the last inequality.

**Case (i)** If (2.7) holds, then by (5.20) and the fact $L(\eta) \geq \theta \mathcal{G} \geq \frac{1}{2} \theta (\mathcal{G} + \Theta)$,
\[ 0 \geq \frac{\phi'}{4} \sum_{i,j} G^{\bar{i}\bar{j}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + \left( \frac{1}{2} \theta B e^{Bn} - \frac{C}{\varepsilon} \right) \mathcal{G} + \frac{1}{2} \theta Be^{Bn} \Theta. \]
This yields a contradiction if we further assume $B$ is large enough.

**Case (ii)** If (2.8) holds, then by (5.20) and (4.12) we obtain
\[ 0 \geq \frac{\phi'}{4} \sum_{i,j} G^{\bar{i}\bar{j}}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - C_B \]
if $B$ is large enough such that $\frac{C}{\varepsilon} \mathcal{G}$ in (5.20) can be discarded. Moreover, using the fact
\[ G^{kk} \geq \theta \mathcal{G} \geq \Theta, \quad \text{for each } k, \]
we also have
\[ \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq C_B K. \]
The rest of the proof is same as Subcase 1.1.

**Case 2** If Case 1 does not hold, define the index set
\[ I = \{1 \leq i \leq n : G^{n\bar{n}} \geq B^3 e^{2Bn} G^{\bar{n}\bar{n}}\}. \]
Observe that $1 \in I$, $n \notin I$. Hence we may assume $I = \{1, 2, \ldots, p\}$ for $p < n$. 

Lemma 5.3 ([8]) Assume $B \geq 6n \sup_M |\partial \eta|^2$. At the origin, we have

$$-(1 + \varepsilon) \sum_{i \in I} G^{ii} \left| \frac{e_i(\lambda_1)}{\lambda_1^2} \right|^2 \geq -G - 2(\phi')^2 \sum_{i \in I} G^{ii} |e_i(|\partial u|^2)|. \quad (5.21)$$

Proof Using (5.3) and fundamental inequality $|a + b|^2 \leq 4|a|^2 + \frac{4}{9}|b|^2$, we obtain

$$-(1 + \varepsilon) \sum_{i \in I} G^{ii} \left| \frac{e_i(\lambda_1)}{\lambda_1^2} \right|^2 = -(1 + \varepsilon) \sum_{i \in I} G^{ii} |\phi'_i(e_i(|\partial u|^2) + B e_{B \eta} e_i(\eta))|^2$$

$$\geq -6 \sup_M (|\partial \eta|^2) B^2 e^{2B \eta} \sum_{i \in I} G^{ii} - 2(\phi')^2 \sum_{i \in I} G^{ii} |e_i(|\partial u|^2)|^2$$

$$\geq -6n \sup_M (|\partial \eta|^2) B^{-1} G^{\alpha \alpha} - 2(\phi')^2 \sum_{i \in I} G^{ii} |e_i(|\partial u|^2)|^2$$

$$\geq -G - 2(\phi')^2 \sum_{i \in I} G^{ii} |e_i(|\partial u|^2)|^2,$$

where we used the hypothesis $B \geq 6n \sup_M |\partial \eta|^2$ in the last second inequality.

Define a new $(1, 0)$ vector field by

$$\tilde{\varepsilon}_1 = \frac{1}{\sqrt{2}} (V_1 - \sqrt{-1} JV_1).$$

At the origin, we can find a sequence of complex numbers $\nu_1, \ldots, \nu_n$ such that

$$\tilde{\varepsilon}_1 = \sum_{k=1}^n \nu_k e_k, \quad \sum_{k=1}^n |\nu_k|^2 = 1.$$

Lemma 5.4 ([8]) We have

$$|\nu_k| \leq C_B \frac{\lambda_1}{\lambda_1}, \quad \text{for all } k \notin I.$$

Once we proved it, now we can estimate the first three terms in Lemma 5.2. Since $JV_1$ is $g$-unit and $g$-orthogonal to $V_1$, then we can find real numbers $\mu_2, \ldots, \mu_{2n}$ such that

$$JV_1 = \sum_{\alpha=1}^{2n} \mu_\alpha V_\alpha, \quad \sum_{\alpha=1}^{2n} \mu_\alpha^2 = 1 \quad \text{at the origin.}$$

The following lemma is important to our estimate.

Lemma 5.5 For any constant $\gamma > 0$, we have

$$-2 - (1 + \varepsilon) \sum_{\alpha=1}^{2n} G^{ii} \left| \frac{e_i(u \nu_\alpha V_1)}{\lambda_1 - \lambda_\alpha} \right|^2$$

$$\geq (2 - \varepsilon) \sum_{\alpha=1}^{2n} G^{ii} \left| \frac{e_i(u \nu_\alpha V_1)}{\lambda_1 - \lambda_\alpha} \right|^2 + \sum_{k \in I, i \notin I} \frac{2}{\lambda_1} G^{ii} g^{kk} |V_1(\tilde{g}_{ik})|^2 - 2(1 - \varepsilon)(1 + \gamma) \frac{\lambda_i}{\lambda_1} \sum_{k \in I, i \notin I} \frac{G^{ii} \left| e_i(u \nu_\alpha V_1) \right|^2}{\lambda_1 - \lambda_\alpha}$$

$$- 2(1 - \varepsilon)(1 + \gamma) \frac{\lambda_i}{\lambda_1} \sum_{k \in I, i \notin I} \frac{G^{ii} \left| e_i(u \nu_\alpha V_1) \right|^2}{\lambda_1 - \lambda_\alpha}$$

$$- (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha=1}^{2n} \frac{\lambda_\alpha \mu_\alpha^2}{\lambda_1 - \lambda_\alpha} \right) \sum_{i \in I} \sum_{\alpha=1}^{2n} \frac{G^{ii} \left| e_i(u \nu_\alpha V_1) \right|^2}{\lambda_1 - \lambda_\alpha},$$

if we assume $\lambda_1 \geq \frac{\tilde{\gamma}^2 C_B}{\varepsilon}$, where $\tilde{g}_{i1} = \sum \tilde{g}_{i1} |\nu_i|^2$. 

Proof. We divide the proof into several steps.

Step 1. We can prove

\[ e_i(λ_1) = \sqrt{2} \sum_k \nu_k V_1(\tilde{g}_{ik}) - \sqrt{-1} \sum_{α>1} μ_α e_i(u_{V_1V_α}) + O(λ_1), \tag{5.22} \]

where \( O(λ_1) \) denotes the terms those can be controlled by \( λ_1 \). Indeed, since \( \tilde{e}_1 = \frac{1}{\sqrt{2}}(V_1 + \sqrt{-1}JV_1) \), therefore,

\[ e_i(λ_1) = \sqrt{2} e_i(u_{V_1\tilde{e}_1}) - \sqrt{-1} e_i(u_{V_1JV_1}). \]

The first term

\[ e_i(u_{V_1\tilde{e}_1}) = e_i(V_1\tilde{e}_1 u - (\nabla V_1\tilde{e}_1) u) = \tilde{e}_1 e_i V_1 u + O(λ_1) = \sum_k \nu_k V_1(\tilde{g}_{ik}) + O(λ_1). \tag{5.23} \]

The second term

\[ e_i(u_{V_1JV_1}) = e_i V_1 JV_1(u) + O(λ_1) = JV_1 e_i V_1(u) + O(λ_1) = \sum_{α>1} V_α e_i V_1(u) + O(λ_1) = \sum_{α>1} e_i(u_{V_αV_1}) + O(λ_1). \tag{5.24} \]

Thus, (5.22) follows from (5.23) and (5.24).

Step 2. Hence we have

\[- (1 + \varepsilon) \sum_{i \notin I} G_{ii} \frac{|e_i(λ_1)|^2}{λ_i^4} \]

\[ \geq -(1 - \varepsilon) \sum_{i \notin I} G_{ii} \frac{\sqrt{2} \sum_{k \in I} \nu_k V_1(\tilde{g}_{ik}) - \sqrt{-1} \sum_{α>1} μ_α e_i(u_{V_1V_α})^2}{λ_i^4} \]

\[ - 3\varepsilon \sum_{i \notin I} G_{ii} \frac{|e_i(λ_1)|^2}{λ_i^4} - \frac{C_B}{\varepsilon} \sum_{i \notin I, k \notin I} G_{ii} \frac{|V_1(\tilde{g}_{ik})|^2}{λ_i^4} - \frac{C}{\varepsilon} G_i, \]

where we used Lemma 5.4.

Step 3. By the Cauchy–Schwarz inequality, we have

\[ \left| \sum_{α>1} μ_α e_i(u_{V_1V_α}) \right|^2 \leq \sum_{α>1} (λ_1 - λ_α) \sum_{β>1} \frac{|e_i(u_{V_1V_β})|^2}{λ_1 - λ_β}. \]

\[ \left| \sum_{k \in I} \nu_k V_1(\tilde{g}_{ik}) \right|^2 \leq \left( \sum_i \tilde{g}_{ii} \nu_i \right)^2 \sum_{k \in I} \tilde{g}_{kk} |V_1(\tilde{g}_{ik})|^2, \]

where we used the fact \( \sum_{α>1} μ_α^2 = 1 \). Then for \( γ > 0 \), using the Cauchy–Schwarz inequality again to get

\[ (1 - \varepsilon) \sum_{i \notin I} G_{ii} \frac{\sqrt{2} \sum_{k \in I} \nu_k V_1(\tilde{g}_{ik}) - \sqrt{-1} \sum_{α>1} μ_α e_i(u_{V_1V_α})^2}{λ_i^4} \]

\[ \leq 2(1 - \varepsilon)(1 + γ) \sum_{i \notin I} G_{ii} \frac{\sum_{k \in I} \nu_k V_1(\tilde{g}_{ik})^2}{λ_i^2} + (1 - \varepsilon) \left( 1 + \frac{1}{γ} \right) \sum_{i \notin I} G_{ii} \frac{\sum_{α>1} μ_α e_i(u_{V_1V_α})^2}{λ_i^4} \]

\[ \leq 2(1 - \varepsilon)(1 + γ) \tilde{g}_{ii} \sum_{i \notin I, k \notin I} G_{ii} \frac{\sqrt{2} \tilde{g}_{kk} V_1(\tilde{g}_{ik})^2}{λ_i^2} \]
+ (1 - \varepsilon) \left(1 + \frac{1}{\gamma}\right) \left(\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha\right) \sum_{i \notin I} \sum_{\alpha > 1} \frac{G_{\alpha i}^i |e_i(u_{V_i}, V_i)|^2}{\lambda_1^2 - \lambda_\alpha}.

Step 4

By a direct calculation,

\[
\begin{align*}
-G^{i,j} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{jl})
&= \sum_{i \neq k} \sigma_{m-2;ik}(\tilde{g}_{ii}^2)(\tilde{g}_{kk}^2) \{V_1(\tilde{g}_{ii}) V_1(\tilde{g}_{kk}) - |V_1(\tilde{g}_{ik})|^2\} + 2 \sum_{i,k} G_{\alpha i}^i \tilde{g}_{kk}^i |V_1(\tilde{g}_{ik})|^2 \\
&= 2 \sum_{i \neq k} \sigma_{m-1;ik}(\tilde{g}_{ii}^2 |V_1(\tilde{g}_{ik})|^2) + 2 \sum_{i=1}^n \sigma_{m-1;ik}(\tilde{g}_{ii}^3 |V_1(\tilde{g}_{ii})|^2) \\
&\quad + \sum_{i \neq k} \sigma_{m-2;ik}(\tilde{g}_{ii}^2 |V_1(\tilde{g}_{ik})|^2) + 2 \sum_{i=1}^n \sigma_{m-1;ik}(\tilde{g}_{ii}^3 |V_1(\tilde{g}_{ii})|^2) \\
&\geq \sum_{i \neq k} \sigma_{m-1;ik}(\tilde{g}_{ii}^2 |V_1(\tilde{g}_{ik})|^2) + 2 \sum_{i=1}^n \sigma_{m-1;ik}(\tilde{g}_{ii}^3 |V_1(\tilde{g}_{ii})|^2) \\
&\quad + \sum_{i \neq k} \sigma_{m-2;ik}(\tilde{g}_{ii}^2 |V_1(\tilde{g}_{ik})|^2 V_1(\tilde{g}_{kk}),
\end{align*}
\]

(5.25)

where we used the following inequality (see [20])

\[
\begin{align*}
\sum_{i \neq k} (\sigma_{m-1;ik} - \sigma_{m-2;ik}\tilde{g}_{kk}^i)(\tilde{g}_{ii}^2 \tilde{g}_{kk}^i |V_1(\tilde{g}_{ik})|^2) \geq 0.
\end{align*}
\]

We also need the next inequality from [21] (see also [14, 20])

\[
\sum_{i} \frac{\sigma_{m-1;ik}(\tau)}{\tau_i} \xi_i \xi_k + \sum_{i \neq k} \sigma_{m-2;ik}(\tau) \xi_i \xi_k \geq \sum_{i,k} \frac{\sigma_{m-1;ik}(\tau) \sigma_{m-1;ik}(\tau)}{\sigma_m(\tau)} \xi_i \xi_k \geq 0
\]

for every \( \tau = (\tau_1, \ldots, \tau_n) \in \Gamma_n \) and \( (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \). Choosing \( \tau = (\tilde{g}_{11}^1, \ldots, \tilde{g}_{nn}^n) \) and \( \xi_i = V_1(\tilde{g}_{ii}) \), then

\[
\begin{align*}
\sum_{i=1}^n \sigma_{m-1;ik}(\tilde{g}_{ii}^2 |V_1(\tilde{g}_{ii})|^2) + \sum_{i \neq k} \sigma_{m-2;ik}(\tilde{g}_{ii}^2 |V_1(\tilde{g}_{ik})|^2 V_1(\tilde{g}_{kk}) \geq 0.
\end{align*}
\]

(5.26)

It follows (5.25)–(5.26) that

\[
\begin{align*}
-G^{i,j} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{jl}) \geq \sum_{i \neq k} G_{\alpha i}^i \tilde{g}_{kk}^i |V_1(\tilde{g}_{ik})|^2 + \sum_{i=1}^n G_{\alpha i}^i \tilde{g}_{ii}^i |V_1(\tilde{g}_{ik})|^2.
\end{align*}
\]

(5.27)

If \( \tilde{g}_{ii} \geq \tilde{g}_{kk} \), we have \( \sigma_{m-1;ik} \tilde{g}_{ii} \geq \sigma_{m-1;ik} \tilde{g}_{kk} \). Hence

\[
\begin{align*}
\sum_{i,k \in I} G_{\alpha i}^i \tilde{g}_{kk}^i |V_1(\tilde{g}_{ik})|^2 \geq \sum_{i \notin I, k \in I} G_{\alpha i}^i \tilde{g}_{kk}^i |V_1(\tilde{g}_{ik})|^2.
\end{align*}
\]

(5.28)

On the one hand, since \{\(i, k\) : \(1 \leq i, k \leq n, i \in I, k \notin I\}\) and \{\(i, k\) : \(1 \leq i \neq k \leq n, i, k \notin I, k \in I\}\) and \{\(i, k\) : \(1 \leq i \neq k \leq n, i, k \notin I\}\) are pairwise disjoint subsets of \{\(i, k\) : \(1 \leq i \neq k \leq n\}\), and by (5.28), to show

\[
\begin{align*}
\sum_{i \neq k} G_{\alpha i}^i \tilde{g}_{kk}^i |V_1(\tilde{g}_{ik})|^2 \geq 2 \sum_{i \notin I, k \in I} G_{\alpha i}^i \tilde{g}_{kk}^i |V_1(\tilde{g}_{ik})|^2 + \frac{C_B}{\varepsilon \lambda_1^3} \sum_{i \neq k, i, k \notin I} G_{\alpha i}^i |V_1(\tilde{g}_{ik})|^2;
\end{align*}
\]

we shall prove

\[
\tilde{g}_{kk}^i \geq \frac{C_B}{\varepsilon \lambda_1^3}
\]

(5.30)
for each $k$. Since $\tilde{g}_{i1}$ is comparable to $\lambda_1$ and $\lambda_1 \geq \frac{\alpha^2}{\bar{\alpha}_1}$, then $\tilde{g}^{11} \geq \frac{C_{\alpha}}{\varepsilon \lambda_1}$. This proves (5.30) because $\tilde{g}^{kk} \geq \tilde{g}^{11}$. On the other hand, it also implies that

$$\sum_{i=1}^{n} G^{ii} \tilde{g}^{ii} |V_1(\tilde{g}_{ii})|^2 \geq \frac{C_B}{\varepsilon \lambda_1^2} \sum_{i \notin I} G^{ii} |V_1(\tilde{g}_{ii})|^2. \quad (5.31)$$

It follows from (5.27), (5.29) and (5.31) that

$$- \frac{1}{\lambda_1} G^{ik,j} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{jl}) \geq \frac{2}{\lambda_1} \sum_{k \in I, i \notin I} G^{ii} \tilde{g}^{kk} |V_1(\tilde{g}_{ik})|^2 + \frac{C_B}{\varepsilon \lambda_1^2} \sum_{k \notin I, i \notin I} G^{ii} |V_1(\tilde{g}_{ik})|^2. \quad (5.32)$$

Consequently, the lemma follows from the previous several steps.

**Lemma 5.6** If we assume $\lambda_1 \geq C/\varepsilon^2$, then

$$(2 - \varepsilon) \sum_{\alpha > 1} G^{ii} |e_i(u V_{\alpha} V_i)|^2 - \frac{1}{\lambda_1} G^{ik,j} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{jl}) - (1 + \varepsilon) \sum_{i \notin I} G^{ii} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2}$$

$$\geq -6 \varepsilon B^2 \varepsilon^2 B \sum_i G^{ii} |e_i(\eta)|^2 - 6 \varepsilon (\phi')^2 \sum_i G^{ii} |e_i(|\partial u|^2)|^2 - \frac{C}{\varepsilon} G. \quad (5.33)$$

**Proof** It suffices to prove

$$\sum_{\alpha > 1} G^{ii} |e_i(u V_{\alpha} V_i)|^2 - \frac{1}{\lambda_1} G^{ik,j} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{jl}) - (1 + \varepsilon) \sum_{i \notin I} G^{ii} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2}$$

$$\geq -3 \varepsilon \sum_{\alpha > 1} G^{ii} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2} - (1 - \varepsilon)(\lambda_1 + \sum_{\alpha > 1}) \sum_{i \notin I} G^{ii} |V_1(\tilde{g}_{ik})|^2 \frac{|V_1(\tilde{g}_{ik})|^2}{\lambda_1^2}$$

$$- \frac{C}{\varepsilon} G - (1 - \varepsilon) \left(1 + \frac{1}{\gamma} \right) \left(\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \sum_{i \notin I} \sum_{\alpha > 1} G^{ii} |e_i(u V_{\alpha} V_i)|^2 \frac{\lambda_1^2}{\lambda_1 - \lambda_\alpha}. \quad (5.35)$$

We only need to choose $\gamma = \frac{\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2}{\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2}$. Then on the right hand of (5.35), the first term cancels the last term and the second term cancels the fourth term. This proves (5.33).

**Condition 1** Assume that

$$\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \geq 2(1 - \varepsilon) \tilde{g}_{i1} > 0. \quad (5.34)$$

**Proof of Condition 1** It follows from Lemma 5.5 and (5.34) that

$$\sum_{\alpha > 1} G^{ii} |e_i(u V_{\alpha} V_i)|^2 - \frac{1}{\lambda_1} G^{ik,j} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{jl}) - (1 + \varepsilon) \sum_{i \notin I} G^{ii} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2}$$

$$\geq \sum_{i \notin I} \sum_{\alpha > 1} \frac{G^{ii} |e_i(\lambda_1)|^2}{\lambda_1^2} \left(\frac{2 - \varepsilon}{\lambda_1} \right) + \sum_{k \notin I, i \notin I} \frac{2}{\lambda_1} G^{ii} \tilde{g}^{kk} |V_1(\tilde{g}_{ik})|^2 \frac{|V_1(\tilde{g}_{ik})|^2}{\lambda_1^2}$$

$$- 3 \varepsilon \sum_{\alpha > 1} \frac{G^{ii} |e_i(\lambda_1)|^2}{\lambda_1^2} - (1 + \gamma) \left(\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \sum_{k \notin I, i \notin I} \frac{G^{ii} \tilde{g}^{kk} |V_1(\tilde{g}_{ik})|^2}{\lambda_1^2}$$

$$- \frac{C}{\varepsilon} G - (1 - \varepsilon) \left(1 + \frac{1}{\gamma} \right) \left(\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \sum_{i \notin I} \sum_{\alpha > 1} \frac{G^{ii} |e_i(u V_{\alpha} V_i)|^2}{\lambda_1^2} \frac{\lambda_1^2}{\lambda_1 - \lambda_\alpha}. \quad (5.35)$$

**Condition 2** Assume that

$$\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 < 2(1 - \varepsilon) \tilde{g}_{i1}. \quad (5.36)$$
Proof of Condition 2. By a similar calculation in [8], we have

\[ 0 < \bar{g}_{i1} \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \right) + C. \]  

(5.37)

Plugging it into (5.36), then \( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \geq -C \) and \( \bar{g}_{i1} \leq C/\varepsilon \). Hence,

\[ 0 < \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \leq 2\lambda_1 + C \leq (2 + 2\varepsilon^2)\lambda_1 \]

provided \( \lambda_1 \geq C/\varepsilon^2 \). Choosing \( \gamma = 1/\varepsilon^2 \), it follows that

\[ (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \right) \leq 2(1 - \varepsilon)(1 + \varepsilon^2)\lambda_1 \leq (2 - \varepsilon)\lambda_1. \]

Plugging it into Lemma 5.5 yields

\[
\begin{align*}
(2 - \varepsilon) \sum_{\alpha > 1} & G^{i\bar{a}} |e_i(u_{V_1} V_1)|^2_{\lambda_1(\lambda_1 - \lambda_\alpha)} - \frac{1}{\lambda_1} G^{k\bar{j}l} V_1(\bar{g}_{ik}) V_1(\bar{g}_{jl}) - (1 + \varepsilon) \sum_{i \notin I} G^{i\bar{a}} |e_i(\lambda_1)|^2_{\lambda_1^2} \\
& \geq 2 \sum_{k \in I, i \notin I} G^{i\bar{a}} g^{kk} |V_1(\bar{g}_{ik})|^2_{\lambda_1} - 3 \varepsilon \sum_{i \notin I} G^{i\bar{a}} |e_i(\lambda_1)|^2_{\lambda_1^2} \\
& - 2(1 - \varepsilon) \left( 1 + \frac{1}{\varepsilon^2} \right) \bar{g}_{i1}^2 \sum_{k \in I, i \notin I} G^{i\bar{a}} g^{kk} |V_1(\bar{g}_{ik})|^2_{\lambda_1^2} - C G \\
& \geq 2 \sum_{k \in I, i \notin I} G^{i\bar{a}} g^{kk} |V_1(\bar{g}_{ik})|^2_{\lambda_1} - 3 \varepsilon \sum_{i \notin I} G^{i\bar{a}} |e_i(\lambda_1)|^2_{\lambda_1^2} \\
& - (1 - \varepsilon) \left( 1 + \frac{1}{\varepsilon^2} \right) C \varepsilon \sum_{k \in I, i \notin I} G^{i\bar{a}} g^{kk} |V_1(\bar{g}_{ik})|^2_{\lambda_1^2} - C G \\
& \geq -3 \varepsilon \sum_{i \notin I} G^{i\bar{a}} |e_i(\lambda_1)|^2_{\lambda_1^2} - C \varepsilon G,
\end{align*}
\]

if we assume \( \lambda_1 \geq C/\varepsilon^3 \) in the last inequality. This proves (5.33) and hence the proof of the lemma is completely.

Now we are able to complete the proof of second order estimate. Plugging Lemma 5.6 into (5.5) we see that

\[
L(Q) \geq -6\varepsilon B^2 \varepsilon^2 B \eta G^{i\bar{a}} |e_i(\eta)|^2 - 6\varepsilon (\phi')^2 \sum_{i \notin I} G^{i\bar{a}} |e_i(\partial u)|^2 - \frac{C}{\varepsilon} G
\]

\[
+ \frac{\phi'}{2} \sum_j G^{i\bar{a}} (|e_i e_j u|^2 + |e_i e_j u|^2) + B^2 \varepsilon^2 B \eta G^{i\bar{a}} |e_i(\eta)|^2 + B \varepsilon^2 B \eta L(\eta)
\]

\[
+ \phi'' G^{i\bar{a}} |e_i(\partial u)|^2 - 2(\phi')^2 \sum_{i \notin I} G^{i\bar{a}} |e_i(\partial u)|^2.
\]  

(5.38)

Choosing \( \varepsilon < \min\{\frac{1}{6}, \frac{2}{\phi''}\} \) such that \( e B \eta(0) = \frac{1}{6\varepsilon} \), 2) and by \( \phi'' = 2(\phi')^2 \), we get

\[
\frac{B}{6\varepsilon} L(\eta) - \frac{C}{\varepsilon} G + \frac{\phi'}{2} \sum_j G^{i\bar{a}} (|e_i e_j u|^2 + |e_i e_j u|^2) \leq 0.
\]

Case (i) Suppose (2.7) holds. Then we have

2) This is possible if \( B \) is large enough.
\[
\left( \frac{B\theta}{6\varepsilon} - C \right) G + \frac{\phi'}{2} \sum_j G^{ii}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq 0.
\]

Choose \( B \) sufficiently large and \( \varepsilon < \frac{\theta}{6} \) is small enough such that \( \frac{B\theta}{6} - C \geq B\varepsilon \). We see that

\[
0 \geq BG + \frac{\phi'}{2} \sum_j G^{ii}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2).
\]

This yields a contradiction.

**Case (ii)** Suppose (2.8) holds. With the aid of (4.12), we have

\[
\frac{B}{6\varepsilon}(\tau G - C) - \frac{C}{\varepsilon} G + \frac{\phi'}{2} \sum_j G^{ii}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq 0.
\]

If we assume \( \frac{B}{6\varepsilon} > C \), then

\[
\frac{\phi'}{2} \sum_j G^{ii}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq CBe^{B\eta},
\]

Notice that \( G^{ii} \geq \theta G \geq \theta \Theta \) for each \( i \). Thus,

\[
\frac{\phi'\theta\Theta}{2} \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq CBe^{B\eta}. \tag{5.39}
\]

The rest of proof can be found in Subcase 1.1, so we omit it here. \( \square \)

### 6 Proof of Theorem 1.2

Our proof is close to the Hermitian setting which was given by Sun \[32\]. For an arbitrary \( v \in \mathcal{H}(M) \) satisfying (1.2). Define \( \tilde{\psi} = \frac{\chi^n}{\chi^{n-m} \omega^m} \leq \psi \). Consider the flow

\[
\chi^n_{v + u_t} = \psi^t \tilde{\psi}^{1-t} e^{b_t} \chi^{n-m} \omega^m, \quad \text{for } t \in [0, 1],
\]

with \( u_t \in \mathcal{H}(M, v) \). Here \( b_t \in \mathbb{R} \) for each \( t \) with \( b_0 = 0 \).

Denote \( \psi_t = \psi^t \tilde{\psi}^{1-t} e^{b_t} \). We claim that

\[
\psi_t \leq \psi, \quad \text{for } t \in [0, 1]. \tag{6.2}
\]

Indeed, at the maximum point of \( u_t \), \( \sqrt{-1} \partial \bar{\partial} u_t \leq 0 \). Using the monotonicity of \( F \) we have \( \psi_t \leq \psi \). That is, \( e^{b_t} \leq \psi^{-t} \tilde{\psi}^t \leq 1 \) since \( \psi \leq \tilde{\psi} \). This proves the claim.

Similarly, at the minimum point of \( u_t \), \( \sqrt{-1} \partial \bar{\partial} u_t \geq 0 \). We can also obtain a lower bound for \( b_t \), which yields a uniform bound for \( b_t \).

If we set \( h_t \) by \( \psi_t = (\frac{n}{m}) h_t^m \), by the previous claim we have \( h_t \leq h \). Then the hypersurface \( \partial \Gamma^h \) lies above \( \partial \Gamma^{h_t} \). Therefore, \( u_t \) is still the \( C \)-subsolution for the flow (6.1). Hence, we have uniform \( C^\infty \) estimates for \( u_t \).

Define a nonempty set (including 0)

\[
T = \{ t' \in [0, 1] : \exists u_t \in C^{3,\alpha}(M), b_t \text{ solves (6.1)} \} \quad \text{for } t \in [0, t'] \}. \tag{6.3}
\]

To solve the equation (1.1), it suffices to check \( T \) is both closed and open.

The closedness is easily from the uniform bounds for \( b_t \) and \( u_t \in C^{3,\alpha}(M) \).
For the openness, we shall prove for each $\hat{t} \in \mathcal{T}$, we have $[\hat{t}, \hat{t} + \delta) \subset \mathcal{T}$ for some $\delta > 0$. Set $G(u) = \frac{\chi_{u, +u}}{\chi_{u, -u} \wedge \omega^n}$ on $\mathcal{H}(M, \chi_v)$. Define an almost Hermitian metric
\[ \Omega = \sqrt{-1} \sum_{i,j} G_{ij}(u_t) \theta_i \wedge \bar{\theta}_j. \]
Here $G^{ij} = \frac{\partial \log G}{\partial u_{ij}}$ and $(G_{ij})$ is the inverse matrix of $(G^{ij})$. By Cauduchon’s work (see e.g. [8, Theorem 2.1]), there exists a potential function $\phi \in C^\infty(M)$ such that $e^{\phi} \Omega$ is a Gauduchon metric. We may normalize $\phi$ by adding a constant such that $\int_M e^{(n-1)\phi} \Omega^n = 1$.

Note that the flow (6.1) on $[\hat{t}, \hat{t} + \delta)$ is equivalent to
\[ G(u_t) = G(u_\hat{t}) e^{t-\hat{t}} \psi^{-t+\hat{t}+c_t} \left( \int_M G(u_t) e^{(n-1)\phi} \Omega^n \right) \] (6.4)
for some constant $c_t$ satisfying $\int_M e^{t-\hat{t}} \psi^{-t+\hat{t}+c_t} e^{(n-1)\phi} \Omega^n = 1$. Define a map $\Psi$ by
\[ \Psi(\eta) = \log \frac{G(u_\hat{t} + \eta)}{G(u_\hat{t})} - \log \left( \int_M G(u_\hat{t} + \eta) e^{(n-1)\phi} \Omega^n \right), \]
which maps $\eta \in C^3(M)$ with $\int_M e^{\Psi(\eta) + (n-1)\phi} \Omega^n = 0$ and $G^{\alpha \beta} = e^{\phi} \Omega$ to $C^1(M)$ with $\int_M e^{\Psi(\eta) + (n-1)\phi} \Omega^n = 1$. Observe that $\Psi(0) = 0$, and the linearization of $\Psi$ at $\eta = 0$ is given by
\[ D\Psi(0)(\xi) = \frac{n \Omega^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \xi}{\Omega^n} = \Delta_{\Omega} \xi, \]
for some $\delta > 0$ small. This completes the proof of the theorem.

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