THE ABSENCE OF ARBITRAGE PROPERTY IN MIXED FRACTIONAL BROWNIAN MOTION SETTING

HENNOUNE HALIMA\textsuperscript{1.a} and KANDOUCI ABDELDJEBBAR\textsuperscript{2,b}

\textsuperscript{1} Laboratory of Stochastic Models, Statistics and Applications, University of Saida Dr Moulay Tahar.
Algeria.
aE-mail: halima.henoune@univ-saida.dz

\textsuperscript{2} Laboratory of Stochastic Models, Statistics and Applications, University of Saida Dr Moulay Tahar.
Algeria.
bE-mail: abdeldjebbar.kandouci@univ-saida.dz

Abstract

In this paper, we investigate the condition full support property (CFS) for the stochastic process 

\[ S_t = R_t + \int_0^t \phi_s dM^H_s, \]

where \( M^H_s \) is a mixed fractional Brownian motion, \( R_t \) a continuous adapted process, and \((\phi_t)\) an elementary predictable process. The problem of the absence of arbitrage for this process is treated.

1. Introduction

The conditional full support (CFS) property which is a mere condition on asset prices indicating that at any time the asset price path can continue arbitrarily close to any given path with positive conditional probability, was introduced by Guasoni et al. (2008, [7]) for the fractional Brownian motion, the only H-self similar Gaussian process with stationary increments with the arbitrary Hurst parameter \( H \in (0,1) \). This result was generalized in Cherny (2008, [3]), who proved that any Brownian moving average satisfies the (CFS) property with respect to ordinary Brownian motion. In 2009, Pakkanan [11] established the condition of the desired property for a general class of stochastic processes. After that, this notion was investigated for Gaussian processes with stationary increments by Gasbarra (2011, [6]).

Received March 18, 2021.

AMS Subject Classification: 34A08, 34K50, 93B05, 58J65.

Key words and phrases: Stochastic integral, mixed fractional Brownian motion, conditional full support.
In contrast to the considerable monographs that deal with fractional Brownian motion at a great expense, we find and see a rarity in those on the mixed fractional Brownian motion (MFBM) introduced by Patrick Cheridito (2001,[12]) and defined as a linear combination between the Brownian motion and the independent fractional Brownian motion of parameter $H \in (0,1)$. This process (MFBM) is not a semimartingale if $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},\frac{3}{4}]$, and it is not also a Markov process.

The purpose of this work is to study the properties and characteristics of the conditional full support property for asset prices driven by the Mixed fractional Brownian motion process.

The plan of this paper is organized as follows. In section 2, we collect some preliminaries of the consistent price system and the conditional full support needed to establish our main result. In section 3, we give some useful basic concepts on the mixed fractional Brownian motion. In section 4, we supply our main result on the conditional full support for the stochastic process $S_t = R_t + \int_0^t \phi_s dM_s$, where $M_s^H$ is a Mixed fractional Brownian motion, and we deal with the absence of arbitrage opportunities.

2. Preliminaries

Definition 1. A Wiener process $B_t$ (standard Brownian Motion) is a stochastic process with the following properties:

1. $B_0 = 0$.
2. Non-overlapping increments are independent: $\forall 0 \leq t < T \leq s < S$, the increments $B_T - B_t$ and $B_S - B_s$ are independent random variables.
3. $\forall 0 \leq t < s$ the increment $B_s - B_t$ is a normal random variable, with zero mean and variance $s - t$.
4. $w \in \Omega$, the path $t \to B_t(w)$ is a continuous function.

Definition 2. For $H \in (0,1)$, a fractional Brownian motion of Hurst parameter $H$ is a centered and continuous Gaussian process, denoted by $(B_t^H)_{t \geq 0}$ with the covariance function

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$
Remark 1. Trivially, when $H = \frac{1}{2}$, the fBm is the standard Brownian motion.

2.1. Basic properties of fractional Brownian motion

According to [2], the fractional Brownian motion, $B^H_t$, of Hurst parameter $H \in (0, 1)$ has the following properties:

- Selfsimilarity: for all $a > 0$, $(B^H_{at}) \overset{d}{=} (a^H B^H_t)$.
- Stationary increments: for all $h > 0$, $(B^H_{t+h} - B^H_h) \overset{d}{=} B^H_t$.
- Hölder Continuity: For $H \in (0, 1)$, the sample paths of fBm are a.s. Hölder continuous of an order strictly less than $H$.
- Differentiability: For $H \in (0, 1)$, the fBm sample path $B^H_t$ is not differentiable. Indeed, for every $t_0 \in [0, \infty]$,
  $\mathbb{P} \left( \lim_{t \to t_0} \sup \left| \frac{B^H_{t} - B^H_{t_0}}{t - t_0} \right| = \infty \right) = 1$.
- Bounded variation: The fBm is of an unbounded variation, i.e.
  $\sup_{t_i} \sum_{i} |B^H_{t_{i+1}} - B^H_{t_{i}}| = \infty$.
- The fBm is not a semimartingale for $H \neq \frac{1}{2}$: the fact that the fBm is not a semimartingale implies that we are not able to integrate with respect to it, as we usually do in the classical stochastic calculus. Effectively, the most general class of integrators are semimartingales.
- Long-range dependence: The fractional Brownian motion $B^H_t$ is long-range dependent for $H \in (\frac{1}{2}, 1)$.

2.2. Representation of fBm on a finite interval

There are some representations of the fBm as a Wiener integral defined on an interval, e.g. commonly taken as $[0, T]$:

For the representation on the real line, for a one-sided fBm $(B^H_t)_{0 \leq t \leq T}$, we
have a general formula

$$B_t^H = \int_0^t K_H(t, s) dB_s, \quad t \in [0, T],$$

where $\{B_t\}_{0 \leq t \leq T}$ is a one-sided standard Brownian motion.

Let $B^H$ be a fractional Brownian motion with a parameter $H \in (0, 1)$. According to [8], the fBm admits a representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t, s) dB_s, \quad \text{Levy-Hida representation},$$

where $B = \{B_t\}_{t \in T}$ is a Wiener process, and $K_H(t, s)$ is the kernel

$$K_H(t, s) = d_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1 \left( \frac{1}{s} \right),$$

d_H being a constant and

$$F_1(z) = d_H \left( \frac{1}{2} - H \right) \int_0^{z-1} \theta^{H-\frac{3}{2}} (1 - (\theta + 1)^{H-\frac{1}{2}}) d\theta.$$  

If $H > \frac{1}{2}$, the kernel $K_H$ has the simpler expression

$$K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $t > s$ and $c_H = \left( \frac{H(H-1)}{\beta(2-2H,H-H)} \right)^{\frac{1}{2}}$. As the process $B^H$ is a fBm, then

$$\int_0^{t \wedge s} K_H(t, u) K_H(s, u) du = R_H(t, s).$$

Further, the kernel $K_H$ satisfies the condition:

$$\frac{\partial K_H}{\partial t}(t, s) = d_H \left( \frac{1}{2} \right) s^{\frac{1}{2} - H} (t-s)^{H-\frac{3}{2}}.$$  

2.3. Representations of the fBm on $\mathbb{R}$

According to [8], we present some representations of the fBm as a Wiener
integral (i.e. w.r.t Brownian motion):

\[ B_t^H = C \int_{\mathbb{R}} K_H(t, u) dB_u, \]

where \( C \) is a standardized constant.

- Moving average representation

The fBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let \((B_s)_{s \in \mathbb{R}}\) be a standard Brownian motion. Then

\[ B_t^H = \frac{1}{C(H)} \int_{\mathbb{R}} [(t - s)^{H - \frac{1}{2}} - (-s)^{H - \frac{1}{2}}] dB_s, \]

where \( C(H) > 0 \) an explicit normalizing constant.

- Harmonizable representation

This is another representation which uses the complex-valued Brownian motion (but the fBm is real-valued). In fact, for a fBm \((B_t^H)_{t \in \mathbb{R}}\), we have

\[ B_t^H = \frac{1}{C_1(H)} \int_{\mathbb{R}} e^{ixt} \frac{1}{ix} \left| x \right|^{-(H - \frac{1}{2})} d\tilde{B}_x, \quad t \in \mathbb{R}, \]

where \((\tilde{B}_t)_{t \in \mathbb{R}}\) is a complex Brownian measure and

\[ C_1(H) = \left( \frac{\Pi}{H \Gamma(2H) \sin(H \Pi)} \right)^{1/2}. \]

Let us note that the complex Brownian measure on \(\mathbb{R}\) can be splitted as \( \tilde{B} = B_1 + iB_2 \) and is such that \( B_1(A) = B_1(-A), B_2(A) = -B_2(-A) \) and \( \mathbb{E}(B_1(A))^2 = \frac{|A|^2}{2}, \forall A \in \mathcal{B}(\mathbb{R}). \) We also call this representation, the spectral representation.

3. Condition Full Support and Stochastic Integral

**Definition 3.** Let \( \mathbf{F} \) be a separable metric space and \( \mu : \mathcal{B}(\mathbf{F}) \to [0, 1] \) be a Borel probability measure; the support of \( \mu \), noted by \( \text{supp}(\mu) \), is the unique minimal closed set \( A \subset \mathbf{F} \) such that \( \mu(A) = 1 \).
Let \((X_t)_{t \in [0,T]}\) be a continuous stochastic process defined on a complete probability space \((\Omega, \mathcal{F}, P)\), and let \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}\) be its natural filtration.

Moreover, let \(C_x([u,v], I)\) be the space of function \(f \in C([u,v], I)\) such that \(f(u) = x \in I\).

As usual, we equip the spaces \(C([u,v], I)\) and \(C_x([u,v], I), x \in I\), with the uniform topologies.

**Definition 4.** \([10]\) We say that the process \(X\) has Conditional Full Support (CFS) with respect to the filtration \(\mathbb{F}\), or briefly \(\mathbb{F}\)-CFS, if

1. \(X\) is adapted to \(\mathbb{F}\).
2. For all \(t \in [0, T]\) and \(P\)-almost all \(\omega \in \Omega\),
   \[
   \text{supp} \left( \text{law} \left[ (X_u)_{u \in [t,T]} \mid \mathcal{F}_t \right] (\omega) \right) = C_{X_t}(\omega)([t,T],I).
   \]

**Definition 5** \([10]\). Let \(\varepsilon > 0\). An \(\varepsilon\)-consistent price system to \(X\) is a pair \((\tilde{X}, Q)\), where \(Q\) is a probability measure equivalent to \(P\) and \(\tilde{X}\) is a \(Q\)-martingale in the filtration \(\mathcal{F}\), such that 
\[
\frac{1}{1+\varepsilon} \leq \frac{\tilde{X}_i(t)}{X_i(t)} \leq 1 + \varepsilon,
\]
almost surely for all \(t \in [0, T]\) and \(i = 1, ..., n\).

The result on absence of arbitrage is presented in the following Theorem:

**Theorem 1** \([17]\). Let \(X_t\) be an \(\mathbb{R}^d_+\)-valued, continuous adapted process satisfying CFS. Then \(X\) admits an \(\varepsilon\)-consistent pricing system for all \(\varepsilon > 0\).

Next, we present some results about CFS for the Brownian motion:

\[
Z_t := R_t + \int_0^t \phi_s dB_s, \quad t \in [0,T],
\]

where \(R\) is a continuous process, the integrator \(B\) is a Brownian motion, and the integrand \(\phi\) satisfies some varying assumptions. We take into consideration two cases:

1. Independent integrands and Brownian integrators.

**Theorem 2** \([17]\). Let us define
\[
Z_t := R_t + \int_0^t \phi_s dB_s, \quad t \in [0,T].
\]
Suppose that

(1) \((R_t)_{t \in [0,T]}\) is a continuous process,

(2) \((\phi_t)_{t \in [0,T]}\) is a measurable process s.t. \(\int_0^T \phi_s^2 \, ds < \infty\) a.s,

(3) \((B_t)_{t \in [0,T]}\) is a standard Brownian motion independent of \(R\) and \(\phi\).

If we have,

\[
\text{meas}(t \in [0,T] : \phi_t = 0) = 0 \quad \mathbf{P} - \text{a.s},
\]

then \(Z\) has CFS.

2. Progressive integrands and Brownian integrators.

Let’s note that the assumption about independence between \(B\) and \((R, \phi)\) cannot be dispensed with, in general, without imposing additional conditions. Namely, if,

\[
R_t = 1; \phi_t := e^{B_t - \frac{1}{2} t}; t \in [0,T],
\]

then \(Z = \phi = \xi(B)\), the Dooleans exponential of \(B\), which is strictly positive, does not have CFS, if the process is considered in \(\mathbb{R}\).

**Theorem 3** \([7]\). Suppose that

(1) \((X_t)_{t \in [0,T]}\) is a continuous process,

(2) \(R\) and \(\phi\) are progressive \([0,T] \times C([0,T])^2 \to \mathbb{R}\),

(3) \(\varepsilon\) is a random variable,

(4) and \(\mathcal{F}_t = \sigma(\varepsilon, X_s, B_s : s \in [0,t], t \in [0,T])\).

If \(W\) is an \(\mathcal{F}_t \in [0,T]\)-Brownian motion and

(1) \(E[e^\lambda \int_0^T \phi_s^{-2} ds] < \infty\) for all \(\lambda > 0\),

(2) \(E[e^{2 \int_0^T \phi_s^{-2} h_s^2 ds}] < \infty\), and

(3) \(\int_0^T \phi_s^2 \, ds \leq \bar{K} \text{ a.s for some constant } \bar{K} \in (0, \infty)\),

then the process

\[
Z_t = \varepsilon + \int_0^t R_s \, ds + \int_0^t \phi_s \, dB_s, \quad t \in [0,T]
\]

has CFS.
4. Mixed fractional Brownian motion

The mixed fractional Brownian motion of parameter $H$ is a stochastic process that has been introduced by [12], used in mathematical finance, in the modelling of some arbitrage-free and complete markets. In this part, we present some stochastic properties of this process.

It is well-known that the fractional Brownian motion of Hurst parameter $H \in ]0; 1[$ is a centered Gaussian process $B^H = B^H_t, t \geq 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance function

$$\mathbb{E}(B^H_t B^H_s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

If $H = 1/2$, $B^H$ is the ordinary Brownian motion denoted by $B = B_t, t \geq 0$.

Let us take $a$ and $b$ as two real constants such that $(a, b) \neq (0, 0)$.

**Definition 6.** A mixed fractional Brownian motion (MFBM) of parameters $a, b$, and $H$ is a process $M^H = M^H_t(a, b); t \geq 0 = M^H_t, t \geq 0$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\forall t \in \mathbb{R}_+, \quad M^H_t(a, b) = aB_t + bB^H_t,$$

where the two processes $(B^H_t)_{t \in \mathbb{R}_+}$ and $(B_t)_{t \in \mathbb{R}_+}$ are independent.

According to [10], the mixed fractional Brownian motion $(M^H_t(a, b))_{t \in \mathbb{R}_+}$ has the following properties:

- $M^H$ is a centered Gaussian process.
- For all $t \in \mathbb{R}_+$, $\mathbb{E}((M^H_t(a, b))^2) = a^2 t + b^2 t^{2H}$.
- $\forall s \in \mathbb{R}_+, \forall t \in \mathbb{R}_+$,

$$\text{Cov}(M^H_t(a, b), M^H_s(a, b)) = a^2 (t \wedge s) + \frac{1}{2}[b^2(t^{2H} + s^{2H} - |t - s|^{2H})],$$

where $t \wedge s = 1/2(t + s - |t - s|)$.
- The increments of the MFBM are stationary.
- For all $H \in ]0; 1[\{1/2\}, a \in \mathbb{R}$ and $b \in \mathbb{R}\{0\}, (M^H_t(a, b))_{t \in \mathbb{R}_+}$ is not a Markovian process.
- For any $h > 0$, $M^H_{t+h}(a,b) \triangleq M^H_t(a h^{1/2}, b h^H)$. This property is called the mixed-self-similarity.

- For all $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$, the increments of $(M^H_t(a,b))_{t \in \mathbb{R}_+}$ are long-range dependent if and only if $H > 1/2$.

- $\forall s \in \mathbb{R}_+, \forall t \in \mathbb{R}_+, \forall h \in \mathbb{R}_+, 0 < h \leq t - s$, the correlation coefficient $\rho(M^H_{t+h} - M^H_t, M^H_{s+h} - M^H_s)$ is given as

$$
\rho(M^H_{t+h} - M^H_t, M^H_{s+h} - M^H_s) = \frac{b^2}{2(a^2 h + b^2 h^{2H})}[(t - s + h)^{2H} - 2(t - s)^{2H} + (t - s - h)^{2H}].
$$

For all $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$, the increments of $(M^H_t(a,b))_{t \in \mathbb{R}_+}$ are positively correlated if $1/2 < H < 1$, uncorrelated if $H = 1/2$, and negatively correlated if $0 < H < 1/2$.

- For all $T > 0$ and $\gamma < 1/2 \land H$, the MFBM has a modification which sample paths have a Hölder-continuity, with order $\gamma$, on the interval $[0,T]$.

- For all $\alpha \in ]0, 1/2 \land H[\land, the sample paths of the MFBM are almost surely $\alpha$-differentiable at every $t_0 \geq 0$, and

$$
\forall t_0 \geq 0, \; \mathbb{P}\{d^\alpha M^H_{t_0} = 0\} = 1.
$$

- For all $\alpha \in ]1/2 \land H; 1[\land, the sample paths of the MFBM are nowhere $\alpha$-differentiable, almost surely.

5. Main Result

The main goal of this work is to present conditions that imply the conditional full support (CFS) property, introduced by [7], for the stochastic process:

$$
S_t = R_t + \int_0^t \phi_s dM^H_s,
$$

where $M^H_s$ is a mixed fractional Brownian motion.

Our main result is given in the following Theorem.
Theorem 4. Let us consider the process

\[ S_t = R_t + \int_0^t \phi_s dM^H_s, \]

where

- \((R_t)_{t \in [0,T]}\) is a continuous adapted process,
- \((\phi_t)_{t \in [0,T]}\) is an elementary predictable s.t. \(\int_0^T \phi_s^2 ds < \infty\) a.s.,
- \((M^H_t)_{t \in [0,T]}\) is a mixed fractional Brownian motion independent of \(R\) and \(\phi\).

If we have

\[ \text{meas}(t \in [0,T] : \phi_t = 0) = 0, \quad P - a.s. (\text{meas : Lebesgue measure}), \]

then \(S\) has CFS.

**Proof.** As the mixed fractional Brownian motion is a Gaussian process, then for the proof of our theorem, we can follow the same steps of Proposition 4.2 given in [7].

Let

\[ S(t) = \int_0^t \phi_s dM^H_s. \]

It is sufficient to prove that the conditional law \(P(S_{[v,T]} | F_v)\) has full support on \(C_{S_v}([v,\mu], \mathbb{R})\) almost surely. Then, it will be enough to show this property on an interval, where \(\phi\) is constant with respect to time (and thus continuous). Therefore, we can take \(T\) small enough such that \(\phi\) has the form \(\phi(t) = \zeta\) on \([v,T]\), where \(\zeta \neq 0\) and it is \(F_v\)-measurable.

In our paper, we have \((M^H_t)_{t \in [0,T]}\) is a mixed fractional Brownian motion, then

\[ M^H_t = M^H_t(a, b) = aB_t + bB^H_t. \]

Without loss of generality, we can study the property of CFS for the mBm for \(a = b = 1\); we have

\[ S(t) = \int_0^t \phi_s dM^H_s \]
\[
\int_0^t \phi_s d[B^H_s] + B_s \quad (B^H_s \text{ and } B_s \text{ are independent})
\]
\[
\int_0^t \phi_s dB^H_s + \int_0^t \phi_s dB_s.
\]

**Remark 2.** The independence does not concern the components of the Gaussian process \(S^H\) but it concerns the couple \((B^H, B)\).

Next, we have to prove that
\[
S(t) = \int_v^t \phi_s K_H(t, s) dB_s + \int_v^t \phi_s dB_s
\]
has full support on \(C_0([v, T], \mathbb{R})\).

Theorem 3 in [5] states that the topological support of a continuous Gaussian process is equal to the norm closure of its reproducing kernel Hilbert space.

In our case, the support of \(S(t)\) is
\[
\mathbb{H} := \left\{ f \in C_0([v, T], \mathbb{R}) : f(t) = \int_v^t \phi(s) K_H(t, s) g(s) ds + \int_v^t \phi(s) g(s) ds, \right. \\
\left. \text{for some } g \in L^2[v, T] \right\}.
\]

Thus, it is sufficient to show that \(\mathbb{H}\) is norm-dense in \(C_0([v, T], \mathbb{R})\).

To ensure this condition, we define the Liouville fractional integral operator for any \(f \in L^1[a, b]\) and \(\alpha > 0\),
\[
(I_{a}^\alpha g)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t g(s)(t - s)^{\alpha - 1} ds, \quad a \leq t \leq b,
\]
and represent the kernel operator \(K_H\) for the fractional Brownian motion:
\[
(K_H g)(t) := \int_0^t K_H(t, s) g(s) ds, \quad t \in [0, T].
\]

We can treat this problem following these two cases:
1. Case $H < \frac{1}{2}$. Let

$$(K_H(g\phi))(t) := \zeta \int_v^t K_H(t, s)g(s)ds, \quad g \in L^2[0, T], \; t \in [0, T].$$

Using Theorem 2.1 in [5], we have

$$(K_H(g\phi)) = I_{0^+}^{2H} \left( s^{\frac{1}{2}} I_{0^+}^{\frac{1}{2} - H} \left( s^{H - \frac{1}{2}} (g\phi)(s) \right) \right).$$

For any $v$, the argument needs to be split into two steps:

**Step 1.** We rely on the following Lemma:

**Lemma 1** ([7]). If $g \in C_0[v, T]$, then $L_1g \in C_0([v, T])$, where

$$L_1g(t) = (I_{0^+}^{\frac{1}{2} - H} (s^{H - \frac{1}{2}} g(s)))(t).$$

Moreover, $L_1 : C_0[v, T] \to C_0[v, T]$ is continuous and has a dense range (with respect to the uniform norm).

As we have $\varphi \in C_0[v, T]$, then $L_1\varphi \phi \in C_0([v, T])$, where

$$(L_1\varphi \phi)(t) = (I_{0^+}^{\frac{1}{2} - H} (s^{H - \frac{1}{2}} (g\phi)(s)))(t).$$

Let us recall the identity for $a, b > 0$,

$$\int_0^t (t - u)^{a-1}u^{b-1}du = C(a, b)t^{a+b-1},$$

where $C(a, b) \neq 0$ is a constant.

And define, for a fixed $\alpha > 0$ :

$$\varphi(s) := \frac{(s - v)^{\alpha}}{\zeta s^{H - \frac{1}{2}}}.$$ 

Then, we obtain, for $t \in [v, T]$

$$(L_1\varphi \phi)(t) = \frac{\zeta}{\Gamma(\frac{1}{2} - H)} \int_v^t (t - s)^{-H - \frac{1}{2}} \varphi(s) s^{H - \frac{1}{2}} ds + \int_v^t \zeta \varphi(s) ds.$$
\[
\begin{align*}
\frac{1}{\Gamma\left(\frac{1}{2} - H\right)} \int_{v}^{t} (t - s)^{-\frac{1}{2}} (s - v)^{\alpha} ds + \int_{v}^{t} (s - v)^{\alpha} ds \\
\leq \int_{0}^{t - v} u^{\alpha} (t - v - u)^{-\frac{1}{2}} du + \int_{0}^{t - v} u^{\alpha} du \\
= C\left(\frac{1}{2} - H, \alpha + 1\right) (t - v)^{\alpha - \frac{3}{2} + \frac{1}{2}} + C\left(1, \alpha + 1\right) (t - v)^{\alpha + 1}.
\end{align*}
\]

By varying \(\alpha\), we find that \((t - v)^{n} \in \text{Im}(L_{1})\) for \(n \geq 1\) and the Stone-Weierstrass theorem guarantees that \(\text{Im}(L_{1})\) is dense in \(C_{0}[v, T]\).

**Step 2.** For this step, the following lemma is needed:

**Lemma 2** ([10]). If \(g \in C_{0}[v, T]\), then \(L_{2}g \in C_{0}([v, T])\), where

\[
(L_{2}g)(t) = (I_{0}^{2H} (s^{\frac{1}{2} - H} g(s)))(t),
\]

and \(L_{2} : C_{0}[v, T] \rightarrow C_{0}[v, T]\) is continuous and has a dense range.

Since the restriction of \(K_{H}\) to \(C_{0}[v, T]\) is exactly \(L_{2} \circ L_{1}\), we may conclude that \(K_{H} : C_{0}[v, T] \rightarrow C_{0}[v, T]\) has a dense range and, a fortiori, \(\mathbb{H}\) is norm-dense in \(C_{0}[v, T]\).

2. **Case** \(H \geq \frac{1}{2}\).

Following [5], we have the following representation:

\[
K_{H}(g\phi) = I_{0}^{1} (s^{H - \frac{1}{2} - \frac{1}{2}} (s^{\frac{1}{2} - H} (g\phi))).
\]

Then, for the proof, two steps are considered

**Step 1.** We have \(g \in C_{0}[v, T]\), then \(L_{3}g\phi \in C_{0}([v, T])\), where

\[
(L_{3}(g\phi))(t) = \zeta(I_{0}^{1} (s^{H - \frac{1}{2}} g(s)))(t).
\]

Moreover, \(L_{1} : C_{0}[v, T] \rightarrow C_{0}[v, T]\) is continuous and has a dense range (with respect to the uniform norm), where \(\mathcal{C}(a, b) \neq 0\) is a constant.

Define, for a fixed \(\alpha > 0\),

\[
g(s) := \frac{(s - v)^{\alpha}}{\zeta s^{\frac{1}{2} - H}},
\]
Then, we obtain, for \( t \in [v, T] \),

\[
(L_3 g \phi) = \frac{\zeta}{\Gamma(H - \frac{1}{2})} \int_v^t (t - s)^{H - \frac{3}{2}} g(s)s^{\frac{1}{2} - H} ds + \int_v^t \zeta g(s)ds \\
= \frac{1}{\Gamma(H - \frac{1}{2})} \int_v^t (t - s)^{H - \frac{3}{2}} (s - v)\alpha ds + \int_v^t (s - v)\alpha ds \\
\leq \int_0^{t-v} u^\alpha(t - v - u)^{H - \frac{3}{2}} du + \int_0^{t-v} u^\alpha du \\
= \mathcal{C}(H - \frac{1}{2}, \alpha + 1)(t - v)^{\alpha + H - \frac{1}{2}} + \mathcal{C}(1, \alpha + 1)(t - v)^{\alpha + 1}.
\]

By varying \( \alpha \), we find that \((t - v)^n \in \text{Im}(L_3)\), for \( n \geq 1 \), and the Stone-Weierstrass theorem guarantees that \( \text{Im}(L_3) \) is dense in \( C_0[v, T] \).

**Step 2.** We have \( g \in C_0[v, T] \), then \( L_4 g \phi \in C_0([v, T]) \), where

\[
(L_4(g \phi)(t) = \zeta(I_0^1(s^{H - \frac{1}{2}}(g \phi)(s)))(t),
\]

and \( L_4 : C_0[v, T] \rightarrow C_0[v, T] \) is continuous and has a dense range in \( C_0[v, T] \).

Since the restriction of \( K_H \) to \( C_0[v, T] \) is exactly \( L_4 \circ L_3 \), we may conclude that \( K_H : C_0[v, T] \rightarrow C_0[v, T] \) has a dense range and, a fortiori, \( \mathbb{H} \) is norm-dense in \( C_0[v, T] \).

Then we can check that there exists the property of CFS for the process \( (S_t) \) and there are the consistent price systems.

\( \Box \)

6. Conclusion

This article deals the absence of the arbitration opportunity by using the conditional full support property applied to processes with irregular trajectories such as the mixed fractional Brownian motion. As a perspective, we can study this property for Lévy fractional Brownian motion and mixed fractional Brownian motion on which the processes \( B_t^H \) and \( B_t \) are dependent.
Acknowledgments

We acknowledge funding from ATRST (Algeria) for the research results approved in this work.

References

1. F. Antonelli, Backward-forward stochastic differential equations, The Annals of Applied Probability, 3 (1993), No.3, 777-793.

2. B. Benoit. Mandelbrot and John W. Van Ness, Fractional Brownian Motions, Fractional Noises and Applications, Published By: Society for Industrial and Applied Mathematics, 10 (1968), No.4, 422-437.

3. A. Cherny, Brownian moving average have conditional full support, The Annals of Applied Probability, 18 (2008), No.5, 1825-1830.

4. Christoph Thäle, Further Remarks on Mixed Fractional Brownian Motion, Applied Mathematical Sciences, 3 (2009), No.38, 1885-1901.

5. L. Decreusefond and A. S. Ustunel, Stochastic analysis of the fractional Brownian motion, An International Journal Devoted to the Interactions between, 10 (1999), No.2, 177-214.

6. D. Gasbarra, T. Sottinen and H. Van Zanten, Conditional full support of Gaussian processes with stationary increments, Journal of Appl. Prob, 48 (2010), No.3, 650-667.

7. P. Guasoni, M. Rásonyi and W. Schachermayer, Consistent price systems and facelift pricing under transaction costs, The Annals of Applied Probability, 18 (2008), No.2, 491-520.

8. Joachim Y. Nahmani, Introduction to Stochastic Integration with respect to Fractional Brownian Motion, Institute of Mathematics A - Chair of Stochastic Processes Semester Project - Bachelor 6th semester, 2009.

9. Yu. A Kuznetsov, The absence of arbitrage in a model with fractal Brownian motion, Russian Mathematical Surveys, 54 (1999), No.4, 847-848.

10. MOUNIR Zili, On the mixed fractional brownian motion, International Journal of stochastic analysis, 2006.

11. M. S. Pakkanen, Stochastic integrals and conditional full support, Journal of Applied Probability, 47 (2010), No.3, 650-667.

12. Patrick Cheridito, Mixed fractional Brownian motion, Bernoulli, 7 (2001), No.6, 913-934.