Identifying Deficiencies of Standard Accretion Disk Theory: Lessons from a Mean-Field Approach

A. Hubbard$^{1,2}$, E.G. Blackman$^{1,2}$

$^1$ Dept. of Physics and Astronomy, Univ. of Rochester, Rochester, NY 14627; $^2$ Laboratory for Laser Energetics Univ. of Rochester, Rochester NY, 14623

ABSTRACT

Turbulent viscosity is frequently used in accretion disk theory to replace the microphysical viscosity in order to accommodate the observational need for instabilities in disks that lead to enhanced transport. However, simply replacing the microphysical transport coefficient by a single turbulent transport coefficient hides the fact that the procedure should formally arise as part of a closure in which the hydrodynamic or magnetohydrodynamic equations are averaged, and correlations of turbulent fluctuations are replaced by transport coefficients. Here we show how a mean field approach leads quite naturally two transport coefficients, not one, that govern mass and angular momentum transport. In particular, we highlight that the conventional approach suffers from a seemingly inconsistent neglect of turbulent diffusion in the surface density equation. We constrain these new transport coefficients for specific cases of inward, outward, and zero net mass transport. In addition, we find that one of the new transport terms can lead to oscillations in the mean surface density which then requires a constant or small inverse Rossby number for disks to maintain a monotonic power-law surface density.

Subject headings: accretion discs–planetary systems:protoplanetary discs–turbulence

1. Introduction

Accretion disks are ubiquitous around stars and compact objects ($^{[\text{Frank et al. 2002}]}$). Those around black holes produce some of the most luminous objects in the universe, while those around young stars provide the material from which planets form. Thin astrophysical accretion disks are primarily angular momentum supported and thus to be accreting, require a mechanism to export angular momentum. Observations suggest that micro-physical viscosity acting on the differential rotation in a near-Keplerian disk is too weak to accommodate
needed time scales and luminosities, given mass density constraints (Pringle 1981) and so attention has focused on turbulence to enhance the needed transport. While outflows are also likely important, the focus on turbulent transport raises three issues: (1) What are the minimum properties that turbulence must have to transport angular momentum outward? (2) What causes the turbulence? (3) What practical set of equations captures the correct physics?

Turbulence was first incorporated into a practical analytic formalism by Shakura-Sunyaev (Shakura & Syunyaev 1973). There, the only role of turbulence emerges as a simple replacement of the micro-physical viscosity in the momentum equation with an enhanced turbulent viscosity. One way that this immediately falls short is that the current leading candidate for turbulent transport is the magneto-rotational instability (MRI) which involves magnetic fields not considered in Shakura-Sunyaev formalism. That being said, the MRI drives turbulence and one might hope that the resulting transport could be modeled via a Shakura-Sunyaev formalism. Evidence is emerging that this is not the case (Pessah et al. 2007, Pessah et al. 2008, Blackman et al. 2008) and there presently remains a disconnect between the insights gained from numerical simulations and a practical formalism that can both accommodate the MRI physics and be of value to phenomenological modelers (e.g. Pessah et al. 2007, Ogilvie 2003).

As part of a long term effort to develop a formalism which captures the minimum properties that turbulence must have to transport angular momentum outwards it is instructive to take a step back from the MRI and rethink the basic meaning of accretion disk equations. Any axisymmetric model that includes turbulent transport is implicitly a mean field theory in that locally, turbulence breaks axisymmetry. Therefore equations that evolve quantities presumed to be only a function of radius $R$ are only relevant if they can be derived from a plausible (even if crude) mean field formalism.

Here we revisit the hydrodynamic accretion disk equations from a mean field point of view and evaluate whether the standard presence of turbulent transport as merely an enhanced viscosity emerges from plausible assumptions from a mean field theory. We find that it does not. Instead we find transport terms in the surface equation and in the angular momentum transport equation that are missing from the standard $\alpha$ viscosity formalism. This leads to conditions on the properties of these transport terms that allow outward transport of angular momentum. That only turbulence with specific properties can transport angular momentum outward is suggested by the simple fact that specific angular momentum in a Keplerian disk increases with radius and therefore one might initially expect turbulent diffusion to transport angular momentum inwards. In at least some systems this expectation is confirmed, as studies of convective turbulence show this effect (Ryu & Goodman 1992).
Kley et al. (1993) and indeed one of the virtues of MRI is that it is expected to drive accretion (Balbus & Hawley 1998).

In Sec. 2 we derive our mean field formalism and derive the evolution equations for the mean surface density and mean orbital angular momentum. We compare the results to the conventional versions of these equations in Sec. 3. In Sec. 4 we discuss the implications of the new equations for outward angular momentum transport and steady state disks. We conclude in Sec. 5.

2. Mean Field Formalism

We distinguish three relevant time scales for the accretion disk: (1) a long global time-scale \( t_g \) corresponding to accretion of the entire disk; (2) a dynamical time-scale \( t_d \leq t_g \) corresponding to the local accretion time-scale; (3) a short time-scale \( \tau \ll t_d \) corresponding to the energy containing eddy turnover time. We assume that that the disk is thin (height \( h \ll R \)) and that the time averaged disk is equivalent to an axially symmetric and vertically integrated disk. While full vertical averaging is appropriate for extracting fluxes, some physics is lost (e.g. Blackman 2001). Accordingly, we adopt cylindrical coordinates \((R, \phi, z)\). We assume that the largest turbulent eddies are small compared to the radial scale at all radii, that is, \( \lambda \ll R \), where \( \lambda \) is the eddy scale. We then decompose all quantities into fluctuating and mean values such that \( q = q + \tilde{q} \) where \( q \) is a flow quantity where \( \bar{q} = \langle q \rangle \) represents the mean and \( \tilde{q} \) is the fluctuation about the mean. The fluctuations average to zero over time-scales longer than \( \tau \) and the means have temporal variation scales much larger than \( \tau \) (Moffatt 1978).

In addition, we make the important simplifying anzatz that fluctuating quantities can be separated into a factor that varies on spatial scale \( R \) times an isotropic homogeneous fluctuation. For the surface density and the velocity in particular, we write

\[
\tilde{\Sigma} = \Sigma_* u_\Sigma \quad \text{and} \quad \tilde{v} = v_* u_v,
\]

where \( v_* \equiv \langle \tilde{v}^2 \rangle^{1/2} \), \( u_v \) is a statistically isotropic and homogeneous vector, \( \Sigma_* \equiv \langle \tilde{\Sigma}^2 \rangle^{1/2} \) and \( u_\Sigma \) is a statistically homogeneous scalar. This decomposition is similar to a WKB approximation and will prove useful in section 3.

Using the above decomposition, we now produce a mean-field derivation of the evolution of a conserved quantity with areal density \( q \), the relevant examples of which are surface density \( \Sigma \) or orbital angular momentum density \( L_z \). For such a conserved quantity, we have

\[
\frac{\partial}{\partial t} q + \vec{V} \cdot \langle q \vec{V} \rangle = 0 = \frac{\partial}{\partial t} \bar{q} + \frac{\partial}{\partial t} \tilde{q} + \vec{V} \cdot (\bar{q} \vec{V} + \tilde{q} \vec{V} + \bar{\vec{V}} + \tilde{\vec{V}}).
\]
Taking the time-average (as described above) of Eq. (2) we find
\[ \frac{\partial}{\partial t} \bar{q} + \bar{\nabla} \cdot (\bar{q} \bar{v}) + \bar{\nabla} \cdot \langle \tilde{q} \tilde{v} \rangle = 0, \] (3)
where transport terms akin to diffusion or viscosity derive from the term $\bar{\nabla} \cdot \langle \tilde{q} \tilde{v} \rangle$. Subtracting Eq. (3) from Eq. (2) we find:
\[ \frac{\partial}{\partial t} \tilde{q} = -\bar{\nabla} \cdot (\bar{q} \bar{v} + q \tilde{v} + \tilde{q} \tilde{v} - \langle \tilde{q} \tilde{v} \rangle). \] (4)

We approximate the implications of Eq. (4) at a point $p$ by estimating the time integrated fluctuating flux of $q$ through the surface $S$ composed of those points a distance $\lambda = v_\ast \tau$ from the point $p$ and approximating it as constant for a time $\tau$. The distance $\lambda$ is determined at the surface $S$ and is chosen because it encloses the volume $V$ that the point $p$ can exchange material with over a time $\tau$. This allows us to write Eq. (4) as:
\[ \tilde{q} \simeq -\tau \frac{\lambda}{V} \int_S (\bar{q} \bar{v} + q \tilde{v} + \tilde{q} \tilde{v} - \langle \tilde{q} \tilde{v} \rangle) \cdot d\vec{S} \simeq -\frac{\tau \lambda}{V} \vec{\nabla} \cdot \lambda^{n-1} (\bar{q} \bar{v} + q \tilde{v} + \tilde{q} \tilde{v} - \langle \tilde{q} \tilde{v} \rangle). \] (5)

In the above Eq. (5) $\vec{F}_q$ is the fluctuating flux of $q$ as given in Eq. (4) and $n$ is the number of spatial dimensions, with $n = 2$ for our thin disk. The factor $\lambda^{n-1}$ in the right-most term in Eq. (5) comes from the fact that under our definitions, $S \propto \lambda^{n-1}$. In producing Eq (4), our use of a correlation time $\tau$ has therefore lead us to construct an effective divergence given by $\vec{\nabla}_{\text{fluc}} \equiv \lambda^{-(n-1)} \vec{\nabla} \cdot \lambda^{n-1}$ that is centered on $p$ not in units of distance, but rather in units of correlation lengths. Note that this generalized divergence reduces to the basic divergence only if $\lambda$ is independent of position. In the case of an accretion disk, $\lambda$ does depend on position and varies on the scale $R$. Upon applying $n = 2$ to Eq. (5) and using $\lambda = v_\ast \tau$ we then obtain:
\[ \tilde{q} \simeq -\frac{\tau}{\lambda} \vec{\nabla} \cdot \lambda (\bar{q} \bar{v} + q \tilde{v} + \tilde{q} \tilde{v} - \langle \tilde{q} \tilde{v} \rangle) = -\frac{1}{v_\ast} \vec{\nabla} \cdot \tau \bar{v}_\ast (\bar{q} \bar{v} + q \tilde{v} + \tilde{q} \tilde{v} - \langle \tilde{q} \tilde{v} \rangle). \] (6)

We can use Eq. (6) to find
\[ \vec{\nabla} \cdot \langle \tilde{q} \tilde{v} \rangle = -\vec{\nabla} \cdot \left[ \bar{v}_\ast \vec{\nabla}_{\text{inner}} \left[ \tau \bar{v}_\ast \left( \bar{q} \bar{v} + q \tilde{v} + \tilde{q} \tilde{v} \right) \right] \right] \] (7)
where we have not written the term $\vec{\nabla} \cdot \langle \bar{u}_\ast \vec{\nabla} \cdot \langle \tilde{q} \tilde{v} \rangle \rangle$ which vanishes as it is the average of fluctuation times a mean. We now discuss how (7) simplifies for our quantities of interest: surface mass density $\Sigma$ and angular momentum surface density $L = L_z = \Sigma R^2 \Omega$.

For the case $q = \Sigma$, we first note that $\Sigma_\ast \ll \Sigma$ because turbulent density fluctuations do not dominate for a non-self gravitating disk with $\tau \ll t_d$, and accretion disks have inner
boundaries of no return. More specifically, in a near-steady state for which $t_d \ll t_g$, Eq. (3) implies that
\[ \vec{\nabla} \cdot (\Sigma \vec{\nabla}) \sim \vec{\nabla} \cdot \langle \tilde{\Sigma} \tilde{\vec{v}} \rangle. \] (8)

Only the radial derivatives contribute so only the radial mean velocity contributes to the divergence, and thus $\tilde{v}_R / v_s \sim \Sigma_s / \Sigma \ll 1$. This implies term (i) $\ll$ term (ii) and also that both can be dropped, as they are small with respect to term (iii) in (7) when $q = \Sigma$. Then, upon using the chain rule for the inner derivative on term (iii) the contribution that comes from the derivative operating on the fluctuating $\tilde{\vec{v}}$ vanishes because it involves a vector correlation of fluctuating quantities (products of $\vec{u}$ vectors) which vanish in our decomposition. Only the contributions to term (iii) with the derivative operating on $\Sigma$ contributes. Finally then, we can write:
\[ \vec{\nabla} \cdot \langle \tilde{\Sigma} \tilde{\vec{v}} \rangle \simeq -\vec{\nabla} \cdot (\vec{\nabla} \tau v^2 \Sigma) \simeq -\nabla^2 (\nu_1 \Sigma), \] (9)

where $\nu_1 \sim \tau v^2_s$ is the diffusion coefficient. The diffusion coefficient $\nu_1$ is due to the correlations of fluctuating values.

We now consider the case $q = L = \Sigma R^2 \Omega$ in (7). First we note that term (i) can again be neglected with respect to term (ii) as this does not depend on the choice of $q$. To foster evaluation of the contributions from (ii) and (iii) we note that the mean and fluctuating values of $L$ are given by
\[ L = \langle \Sigma \Omega R^2 \rangle = \overline{\Sigma \Omega} R^2 + \langle \tilde{\Sigma} \tilde{v}_\phi R \rangle \simeq \overline{\Sigma \Omega} R^2 \] (10)

and
\[ \tilde{L} = \overline{(\tilde{\Sigma} R^2 \Omega)} = \overline{\tilde{\Sigma} R^2 \Omega} + \overline{\tilde{\Sigma} \tilde{v}_\phi R} + \overline{\tilde{\Sigma} (\tilde{\tilde{v}}_\phi - \langle \tilde{\Sigma} \tilde{v}_\phi \rangle)} \simeq \overline{\Sigma R^2 \Omega} + \overline{\Sigma \tilde{v} \phi} \] (11)

respectively. Note that the fourth term in (10) is smaller than the third term for the same reason that in (11) term (c) $\ll$ (a) and term (b) $\ll$ (a), namely because $\tilde{v} \ll R \Omega$ (thin disk) and $\tilde{\Sigma} \ll \Sigma$ as noted in the previous paragraph. Then, by analogy to Eq. (9), term (iii) of Eq. (7) becomes
\[ -\nabla^2 (\nu_1 L). \] (12)

Because the transport coefficient deriving from term (iii) of Eq. (7) is independent of the transported mean quantity $\overline{\Sigma}$, it is plausible to conclude that the transport coefficients of Eqs (9) and (12) are the same, explaining why we have used $\nu_1$ in both equations.

Unlike the case of surface density, term (ii) of Eq. (7) can be significant for angular momentum. Inserting the fluctuating angular momentum density (Eq. (11)) into term (ii) of Eq. (7) we find
\[ -\vec{\nabla} \cdot \langle \underline{\vec{u}} \cdot \vec{v}_{\text{inner}} \cdot \tau v_s (\overline{\Sigma R^2 \Omega} + \overline{\Sigma \tilde{v}_\phi}) \tilde{\vec{v}} \rangle. \] (13)
The dependence of term (b) of Eq. (13) on $\tilde{v}_\phi$ provides a preferred direction, so the averaging does not eliminate the term in which the inner divergence operates on $\tilde{v}_\phi$ in (b). In fact this is the dominant contribution from (b) since fluctuating quantities vary on scale $\lambda \ll R$. Using $\vec{\nabla} \sim \frac{1}{\tilde{v}_\phi}$, we can write this contribution from term (b) as

$$\vec{\nabla} \cdot \langle u \mu \tau v^2 R \frac{1}{v_* \tau} \Sigma \rangle,$$

(14)

where we have defined $\mu \equiv \tilde{v}_\phi / \tilde{v}_*$ which parameterizes the dependence on the angular distribution of the velocity fluctuations. The significance of the parameter $\mu$ can be seen by contemplating the difference between radial and azimuthal turbulent forcing: in a Keplerian disk radial forcing will move material to positions where it has below Keplerian angular momentum while azimuthal forcing, combined with orbital mechanics, will move material to positions where it has above Keplerian angular momentum. The averaged quantity in Eq. (14), like all mean values, depends only on $R$ so only the radial component of $u$ survives. However, $\mu$ depends on $\tilde{v}_\phi$. Since orbital motion converts $\tilde{v}_\phi$ into a radial velocity on a timescale $\Omega^{-1}$, only a fraction $\sim \tau \Omega \tilde{R}$ of $u$ contributes to the correlation. Note that $\Omega \tau$ is the inverse Rossby number for the disk. Using this in (14), term (b) of Eq. (13) can therefore be written

$$\vec{\nabla} \cdot \langle \mu \tau v^2 R \frac{1}{v_* \tau} \Sigma \rangle = \vec{\nabla} \cdot \left( \frac{\nu_2 R}{R} \right),$$

(15)

where $\nu_2 \sim \tau v^2$ and includes the system’s dependence on the angular distribution of the velocity fluctuations (through the dependence of $\nu_2$ on $\mu$, which is not linear because of the averaging process). The quantity $\nu_2$ can become negative if the velocity fluctuations along the orbital motion correlate sufficiently strongly with inwards fluctuating motion. Further, while we expect $|\nu_2| \sim |\nu_1|$, we do not expect $\nu_1 = \nu_2$. In §3 we will explore the consequences of different relative values of $\nu_1$ and $\nu_2$.

We have yet to evaluate term (a) of (13), but we now argue that it is much smaller than term (b), the latter contributing Eq. (15) to our eventual conservation equations. We need only compare $\vec{\nabla} \cdot \vec{\Sigma} R^2 \Omega \vec{v}$, (a) to $\vec{\nabla} \cdot \vec{\Sigma} R \tilde{v}_\phi \vec{v}$, (b). We note from (1) that the divergence on (a) contributes only a surviving term that pulls out a factor of $\sim 1/R$. Further, we estimate that $\tilde{v} \sim \nu_2 R \sim \tilde{v}_* \lambda$. We can then use Eq. (8) to find $\tilde{\Sigma} \sim \tilde{\Sigma} R / \tilde{v}_* \sim \tilde{\Sigma} \lambda / R$. We find for (a):

$$\frac{\Sigma R^2 \Omega}{R} \sim \frac{\Sigma R \Omega}{v_*} \sim \frac{\Sigma \lambda R \Omega}{R} \sim \Sigma \lambda \Omega.$$

(16)

To estimate (b) we note that the inner divergence pulls out a factor $\frac{\Omega}{v_* \tau}$ where the top comes from the azimuthal to radial velocity conversion factor and the denominator comes from the scale over which an azimuthal turbulent speed varies; note here that this term results from
which picks out a preferred direction violating isotropy and keeping the spatial scale of variation \( v_* \tau \) rather than \( R \) as in term (a). We thus have for term (b)

\[
\frac{\Sigma R v_* \tau \Omega}{v_* \tau} \sim \Sigma R \Omega.
\] (17)

Since \( \lambda \ll R \), (17) dominates (16) so that term (b) dominates term (a) of Eq. (13).

We can now finally combine Eqs (12) and (15) to find that Eq. (7) with \( q = L \) gives

\[
\vec{\nabla} \cdot \langle \vec{L} \vec{v} \rangle \simeq -\nabla^2 (\nu_1 L) + \vec{\nabla} \cdot \left( \frac{\nu_2 L}{R} \hat{R} \right),
\] (18)

where only radial derivatives are non-vanishing.

3. Conservation Equations and Comparison to Conventional Form

We can use Eqs (9) and (15) to write the full, mean-field conservation equations for mean surface density \( \Sigma \) and orbital angular momentum surface density \( L = \Sigma R^2 \Omega \):

\[
\frac{\partial}{\partial t} \Sigma + \frac{1}{R} \frac{\partial}{\partial R} (R v_R \Sigma) = \frac{1}{R} \frac{\partial}{\partial R} (\nu_1 \Sigma) \quad (19)
\]

\[
\frac{\partial}{\partial t} L + \frac{1}{R} \frac{\partial}{\partial R} (R v_R L) = \frac{1}{R} \frac{\partial}{\partial R} (\nu_1 L - \nu_2 L). \quad (20)
\]

The mass flow can be characterized as a turbulent flux (right side of Eq. 19) and a fall-back flux due to orbital angular momentum mismatches (\( v_R \)). For \( \Omega \propto R^{-3/2} \) and \( \partial \Omega / \partial t = 0 \), we eliminate \( v_R \) by combining the above two equations to find the net outwards mass flux

\[
F_{M,net} = \frac{\partial}{\partial R} (\nu_1 \Sigma) - v_R \Sigma = \frac{\partial}{\partial R} (\nu_1 \Sigma) + \frac{\nu_1 \Sigma}{2R} - \frac{2}{R^2 \Omega} \frac{\partial}{\partial R} (\nu_2 \Sigma). \quad (21)
\]

A finite \( \nu_2 > 0 \) is important in reducing angular momentum mismatches between turbulently transported material and the local material at the new position.

Eqs. (19,21) can be contrasted to the analogous equations presented in standard formulations for \( \Omega \propto R^{-3/2} \) [Frank et al. 2002]:

\[
\frac{\partial}{\partial t} \Sigma + \frac{1}{R} \frac{\partial}{\partial R} (R v_R \Sigma) = 0 \quad (22)
\]

\[
\frac{\partial}{\partial t} L + \frac{1}{R} \frac{\partial}{\partial R} (R v_R L) = \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{\nu L R}{\Omega} \frac{\partial \Omega}{\partial R} \right). \quad (23)
\]
and

\[ F_M \simeq -\frac{3\nu \Sigma}{2R} \]  \hspace{1cm} (24)

where \( \nu \) is the Shakura-Sunyaev turbulent viscosity. These equations differ from (19-21) in that there is no turbulent transport term on the right of (19) and there is only one transport coefficient \( \nu \) in the right of (23) as compared to the two transport coefficients in (20). The transport term in the surface density term (19) is particularly noteworthy. The presence of this term highlights that the mean radial velocity is not the only contributor to mass motion; there is also a turbulent diffusion of mass. This is generally expected in a turbulent flow and so its absence in the standard formalism raises concern.

What is the origin of the differences between (19-21) and (22-24) just presented? The usual derivation of the conservation equations does not naturally follow from a mean field derivation as we have shown by our derivation of (19-21). In particular, the standard approach to obtain (23) is to simply replace the micro-physical viscosity in the Navier-Stokes equation with a turbulent viscosity, assume axisymmetry and vertical integration and then derive the angular momentum conservation equation from the momentum density equation. Doing so does not produce any transport term in the surface density equation. A key point is that although we have written over-lined quantities \( \overline{\Sigma} \) and \( \overline{L} \) in (22-24) we do so only because it is only the mean quantities that have the axisymmetry and vertical independence usually assumed; formally a purely radial dependence requires a mean field formalism. We have therefore have identified an inconsistency with the standard accretion disk formalism.

Another expression of the inconsistency is to suppose that a replacement of micro-physical viscosity with a turbulent viscosity were a legitimate closure such that no other transport terms appeared and the equations remained the same. This would imply that the mean field Navier-Stokes equation and turbulent viscosity could be derived via integration of suitable Boltzmann equation by analogy to the derivation of the standard Navier-Stokes equation and micro-physical viscosity. However, the derivation of the usual Navier-Stokes equations comes from the truncation of an expansion in the mean free path divided by the macroscopic gradient scale, or a collision time divided by a macroscopic evolution time. For accretion disk turbulence, the latter ratio would be replaced by \( \tau \Omega \), a quantity not guaranteed to be small, and frequently assumed to be of order unity. This highlights why a simple replacement of the micro-physical viscosity by a turbulent viscosity is at best, incomplete.
4. Conditions for Outward Angular Momentum Transport and Steady-State

Here we investigate some consequences of \(19-21\). The equation for the net mass flux, Eq. \(21\), can be rewritten as

\[
F_{M,\text{net}} = \frac{1}{\sqrt{R}} \frac{\partial}{\partial R} (\sqrt{R} \nu_1 \Sigma) - \frac{2}{R^2 \Omega} \frac{\partial}{\partial R} (\nu_2 \Sigma R^2 \Omega). \tag{25}
\]

This form allows us to explore the consequences of different transport coefficients \(\nu_1\) and \(\nu_2\) and different radial density profiles.

We use the standard formula for the disk density scale height \(h = c_s/\Omega\), where \(c_s\) is the sound speed. We scale our transport coefficients to \(c_s h\) as in a Shakura-Sunyaev \(\alpha\) disk prescription \cite{ShakuraSunyaev1973}, \(\nu_i \sim \lambda t v_t = \alpha_i c_s h\), where the \(i\) refers to 1, 2 of the previous section. In a flared disk whose temperature profile is determined by stellar heating \((T \propto R^{-1/2})\), we have \(c_s \propto R^{-1/4}\) and \(h \propto R^{5/4}\). If all \(\alpha_i\) are constant, then \(\nu_i \propto R\).

From Eq. \(25\) we see that \(\Sigma \propto R^{-3/2}\) then results in no net mass transport (although the turbulent and fall-back \((\overline{\tau}_R)\) terms of e.g. \(21\) will in general be separately non-zero), and deviations from \(\log_R \Sigma = -3/2\) will result in differently signed mass fluxes (\(\log_R\) being the base \(R\) logarithm). For a non-zero steady state mass flux with \(\nu_1 \propto \nu_2 \propto c_s h\), Eq. \(25\) would imply \(\log_R (\nu_i \Sigma) = 0\). In that case, the mass flow (positive being outward) becomes

\[
\dot{M} = 2\pi RF_{M,\text{net}} = 2\pi \left( \frac{\nu_1 \Sigma}{2} - \nu_2 \Sigma \right) = 2\pi \left( \frac{\nu_1}{2} - \nu_2 \right) \Sigma, \tag{26}
\]

and we need \(\nu_2 > \nu_1/2 > 0\) to generate an inwards mass-flux. Finally, writing \(\nu_i \propto v_t \lambda t\), \(\lambda_t = v_t \tau\) and requiring that \(v_t \propto c_s\) the condition \(\log_R (\nu \Sigma) = 0\) becomes \(\log_R (\Sigma \tau \Omega) = \log_R (\Omega/c_s^2)\). For the aforementioned flared disk, \(\log_R (\Omega/c_s^2) = -1\).

We find another implication by noting that \(\nu_2 \neq 0\) implies the presence of orbital oscillations from \(20\). As long as the amplitudes are small, the oscillations can be treated as simple harmonic oscillators and we therefore expect oscillating quantities such as \(\overline{\Sigma}(R)\) to be proportional to \(\sin(\tau \Omega)\). For \(\overline{\Sigma}(R)\) to vary as a power law in \(R\), either a constant inverse Rossby number \(\tau \Omega\) (eddy life time scaling with orbit time) or a small \(\tau \Omega\) such that \(\sin(\tau \Omega) \sim \tau \Omega\) is then required. A short eddy lifetime is interesting as it lowers \(\nu\) for a given \(v_t\). (We note that \(\tau \Omega > 1\) need not be considered because any eddies with this property initially will be sheared such that such that \(\tau \to \Omega^{-1}\); shear makes an otherwise long lived eddy shred on a rotation time).
5. Conclusion

Any turbulent accretion disk model in which axisymmetry in surface density is assumed is unavoidably a mean field model and therefore the equations governing such a model should formally be derived from a mean field theory with a plausible closure. We have presented an explicit mean field approach for deriving the mean surface density and orbital angular momentum transport in accretion disks and found that two transport coefficients rather than just one emerge most naturally. Notably, we find that the conventional $\alpha$ formalism misses a turbulent diffusion term in the surface density equation.

In constraining the new transport coefficients for specific cases of inward, outward, and zero net mass transport, we find that: (1) $\log R(\nu \Sigma) = 0$ and $\nu_2 \neq \nu_1/2$ is the condition for a steady-state disk with non-zero net mass transport with $\nu_2 > \nu_1/2$ the condition for inwards net mass transport and (2) $\log R(\nu \Sigma) = -1/2$ or $\nu_2 = \nu_1/2$ are the conditions for a steady-state disk with zero net mass transport; (3) For a finite $\nu_2$, the inverse Rossby number $\tau \Omega$ must be either constant or small for disks to have monotonic power-law radial dependences of $\Sigma$. This emerges because a finite $\nu_2$ implies oscillations in the angular momentum equation. Such a condition on $\tau \Omega$ is not unexpected if one considers that these oscillations depend on $\sin(\tau \Omega)$ and since $\tau \Omega \sim 1$ blurs the distinction between fluctuating and orbital time scales, significant radial variations in $\tau \Omega$ would have implications for mean surface densities and fluxes.

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