Entangled Phase States via Quantum Beam Splitter

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Abstract

We study the entanglement effect of beam splitter on the temporally stable phase states. Specifically, we consider the eigenstates (phase states) of an unitary phase operator resulting from the polar decomposition of ladder operators of generalized Weyl–Heisenberg algebras possessing finite dimensional representation space. The linear entropy that measures the degree of entanglement at the output of the beam splitter is analytically obtained. We find that the entanglement is not only strongly dependent on the Hilbert space dimension but also quite related to strength the parameter ensuring the temporal stability of the phase states. Finally, we discuss the evolution of the entangled phase states.

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1 Introduction

Probably the entanglement phenomenon, often named quantum non-locality [1, 2], contains one of the most interesting features of quantum mechanics. Actually, it is at the heart of current development of quantum information processing such as quantum teleportation [3], superdense coding [4] and telecloning [5]. Entanglement also plays a key role in secure communication, for instance the Ekert protocol [6] based on entangled states is more robust than BB84 one [7]. It known that in quantum computation, the qubits are massively entangled. The preparation and characterization of entangled optical as well as atomic states has been studied extensively. In this respect and in a recent experimental advance, the polarization-entangled photons were generated using type I or type II parametric down conversion [8].

The production of entangled states belonging to an infinite dimensional Hilbert space can be achieved also by adopting the standard technique of parametric down conversion [9]. The beam splitter is also one of the few experimentally accessible devices, which may be used to generate entangled states. In this sense, different developments have been reported on the analysis of beam splitter as entangler [10, 11, 12, 13, 14, 15]. In particular, the effect of beam splitting on the spin (or SU(2)) coherent states for a single mode field was investigated in [14]. In the same spirit the entanglement via a beam splitter of SU(1,1) coherent states (Barut-Giradello [16] and Perelomov ones [17]) was discussed in [15]. The investigation of entanglement properties of coherent states [18, 19, 20, 21] is mainly motivated by the fact that the entangled nonorthogonal states also play an important role in the quantum cryptography [22] and quantum information processing [23]. Experimentally, the quantum optical systems are extensively investigated in order to generate, characterize and understand the entanglement properties.

In this work, we focus on the analysis of the entangled phase states of single modes of the electromagnetic field. It is well-known that the usual way to quantize these single modes is through the harmonic oscillator techniques with an infinity of states. In 1989, Pegg and Barnett [24] suggested to truncate up to some finite, but arbitrarily large, order the infinite dimensional representation space of the harmonic oscillator algebra. This was done to get rid of the difficulty related to the infinite dimensional character of the representation space, which constitutes a drawback in defining a phase operator in a consistent way [25, 26, 27]. Here we introduce a generalized version of the Weyl–Heisenberg algebra that allows us to achieve our goal. We particularly consider one algebra possessing finite dimensional bosonic representations. This is essential in defining the Hermitian phase operator and corresponding temporally stable phase states. For this purpose, we use a technique based on an approach developed in [28]. Furthermore, we show that the phase parameter ensuring the temporal stability of phase states plays a crucial role in the present analysis. Finally, we deal with the entanglement of phase states when passed trough a beam splitter.

The outline of the paper is as follows. In section 2, we introduce a generalized Weyl–Heisenberg algebra, which extends the dynamical symmetry of the usual harmonic oscillator. Subsequently, we discuss the corresponding finite dimensional Hilbertian representation and derive the temporally stable phase states, which are obtained to be dependent of $\varphi$ called phase parameter. This latter allows us to take into account the nature of the spectrum of the system. Because of the absence of $\varphi$, one can immediately notice that the SU(2) phase states derived in [29] are identical to those obtained by
Pegg and Barnett [24] for truncated harmonic oscillator. Furthermore, we analyze the basic features of the phase states. In section 3, we examine the entanglement resulting from the action of a beam splitter on the phase states. To investigate the degree of bipartite entanglement of the phase states, we determine the linear entropy. Finally, we close by some concluding remarks.

2 Finite Fock space for generalized Weyl–Heisenberg algebra

A basic ingredient that will be used in the forthcoming analysis is the generalized Weyl–Heisenberg algebra. This is generated algebraically by three elements denoted by \{a^+, a^-, N\}, which are satisfying the commutation relations

\[
\begin{align*}
[N, a^-] &= -a^-, & [N, a^+] &= +a^+, & [a^-, a^+] &= G(N)
\end{align*}
\]

where \(G(N) = [G(N)]^\dagger\) is a Hermitian function of the number operator \(N\). Clearly, by requiring that \(G(N) = I\), where \(I\) is the unity operator, we end up with the usual harmonic oscillator algebra. Also, for \(G(N) = aN + b\), with two real parameters \(a\) and \(b\), (1) reproduces the \(W_k\) algebra discussed in the context of fractional supersymmetric quantum mechanics [30]. It is also important to stress that this algebra covers the extended harmonic oscillator worked out in [31, 32].

Let us consider the abstract Fock representation of the above algebra through a complete set of orthonormal states \{\(|n\rangle, n \in \mathbb{N}\)\} those are eigenstates of the number operator \(N, N|n\rangle = n|n\rangle\). In this representation, the vacuum state defined as \(a^-|0\rangle = 0\) and the orthonormalized eigenstates constructed by successive applications of the creation operator \(a^+\). Indeed, we define the actions of creation and annihilation operators as

\[
\begin{align*}
a^-|n\rangle &= \sqrt{F(n)} e^{i(F(n) - F(n-1))\varphi} |n-1\rangle \\
a^+|n\rangle &= \sqrt{F(n+1)} e^{-i(F(n+1) - F(n))\varphi} |n+1\rangle
\end{align*}
\]

where the structure function \(F(.)\) is an analytic function, with the properties \(F(0) = 0\) and \(F(n) > 0\) for \(n = 1, 2, \ldots\). The phase parameter \(\varphi\) will be discussed in the next by emphasizing its play a crucial role in constructing the phase states. In what follows we shall denote the Fock space as \(\mathcal{F}\) where the operators \(a^+\) and \(a^-\) are mutually adjoint, \(a^+ = (a^-)^\dagger\) on \(\mathcal{F}\). It is easy to check that \(F(.)\) satisfies the recursion relation

\[
F(n+1) - F(n) = G(n)
\]

which gives by simple iteration the form

\[
F(n) = \sum_{m=0}^{n-1} G(m).
\]

In the forthcoming analysis, we restrict ourselves to generalized oscillator algebra defined through the structure functions those fulfilling the condition

\[
F(2s + 1) = 0
\]

where \(2s\) is a positive integer value. It this case, it follows that the creation and annihilation operators satisfy the nilpotency relations \((a^-)^{2s+1} = (a^+)^{2s+1} = 0\). This means that the corresponding
representation is $2s+1$-dimensional. It is interesting to note that by using (4), one can write the condition (5) as

$$\text{Tr } G = 0. \quad (6)$$

where the trace is over the $(2s+1)$-dimensional Fock space. This new algebra is covering the following results:

- (i)- The truncated harmonic oscillator introduced by Pegg-Barnett [24]:
  $$F(N) = N, \quad G(N) = I - (2s + 1)|2s\rangle\langle 2s|. \quad (7)$$

- (ii)- The finite dimensional oscillator algebra $A_\kappa$ ($\kappa = -1/2s < 0$) defined in [28]:
  $$F(N) = N \left[1 + \kappa(N - 1)\right], \quad G(N) = I + 2\kappa N. \quad (8)$$

- (iii)- The truncated generalized oscillator algebra $A_\kappa$ ($\kappa > 0$) [28]:
  $$F(N) = N \left[1 + \kappa(N - 1)\right], \quad G(N) = I + 2\kappa N - F(2s + 1)|2s\rangle\langle 2s|. \quad (9)$$

The eigenvalues of the operators $G(N)$, corresponding to the special cases (i), (ii) and (iii) are respectively given by $G(n) = 1 - (2s + 1)\delta_{n,2s}$, $G(n) = 1 + 2\kappa n$ and $G(n) = 1 + 2\kappa n - F(2s + 1)\delta_{n,2s}$ for $n = 0, 1, \cdots, 2s$. At this level, we have different comments in order. It is easily seen that for $\kappa = 0$, the case (iii) reduces to (i). As mentioned before, the algebra (1) is more general and covers many other variants of generalized harmonic oscillators. The particular cases (i), (ii) and (iii) are interesting in the context of quantum optics. Indeed, the case (i) was introduced by Pegg and Barnett [24] to define, in a consistent way, the phase operator. The others cases (ii) and (iii) constitute an extension of the ideas developed in [24] to define phase operator for systems with nonlinear spectrum [28].

Using the algebraic structure of the generalized oscillator algebra (1), one can introduce an operator that generalizes the Hamiltonian $a^+a^-$ for the one-dimensional harmonic oscillator. Starting from

$$a^+a^-|n\rangle = F(n)|n\rangle \quad (10)$$

it is easy to realize the required Hamiltonian as

$$H(N) \equiv F(N) = a^+a^- \quad (11)$$

where $H(N)$ can be regarded as the Hamiltonian corresponding to the algebra (1). As usual, one can proceed with the eigenvalue equation

$$H(N)|n\rangle = F(n)|n\rangle \quad (12)$$

to get the solutions of the energy spectrum of a quantum dynamical system described by $F(N)$. As illustration of the realized Hamiltonian, we give the explicit forms for different cases mentioned above. Indeed, we have for case (i)

$$H(N) = \sum_{n=0}^{2s} n|n\rangle\langle n| \quad (13)$$
and case (ii)

\[ H(N) = \sum_{n=0}^{2s} \frac{n}{2s} (2s + 1 - n)|n\rangle\langle n| \]  \hspace{1cm} (14)

as well as case (iii)

\[ H(N) = \sum_{n=0}^{2s} n [1 + \kappa(n - 1)] |n\rangle\langle n|. \]  \hspace{1cm} (15)

### 3 Temporally phase states

We show how to build the phase states, which are temporally stable under time evolution. For this, recall that our Hilbert space \( F \) has \((2s + 1)\)-dimensions where the actions of the operators \( a^- \) and \( a^+ \) on \( F \) are given in (2). These are supplemented by the condition

\[ a^+|2s\rangle = 0 \]  \hspace{1cm} (16)

which can easily be deduced from the calculation of expectation value \( \langle 2s|a^-a^+|2s\rangle \).

As \( F(N) \) is a positive definite operator on the finite dimensional Fock space, let us consider the decomposition of the annihilation \( a^- \) and creation \( a^+ \) operators given by \[28\]

\[ a^- = E \sqrt{F(N)}, \quad a^+ = \sqrt{F(N)} E^\dagger \]  \hspace{1cm} (17)

and one can show that the operator \( E \) satisfies

\[ E|n\rangle = e^{i[F(n)-F(n-1)]}\phi|n-1\rangle, \quad n = 1, 2, \ldots, 2s \]  \hspace{1cm} (18)

which is valid modulo \( 2s + 1 \). For \( n = 0 \), we have

\[ E|0\rangle = e^{i[F(0)-F(2s)]}\phi|2s\rangle. \]  \hspace{1cm} (19)

It follows that for \( E^\dagger \), one can obtain

\[ E^\dagger|n\rangle = e^{-i[F(n+1)-F(n)]}\phi|n + 1\rangle \]  \hspace{1cm} (20)

where \( n + 1 \) should be understood modulo \( 2s + 1 \). Note that from above relations, one can easily check that \( E \) is an unitary operator. Therefore, it is clear now that (17) constitutes a polar decomposition of the operators \( a^- \) and \( a^+ \).

To deduce the phase states associated with the finite dimensional algebra defined in the previous subsection, let us in the beginning look at the eigenstates of the operator \( E \). In doing so, we solve the eigenvalue equation

\[ E|z\rangle = z|z\rangle, \quad |z\rangle = \sum_{n=0}^{2s} C_n z^n |n\rangle \]  \hspace{1cm} (21)

where \( z \in \mathbb{C} \). According to the method developed in [28], it is easy to see that the eigenvalues \( z \) should satisfy the discretization condition

\[ z^{2s+1} = 1 \]  \hspace{1cm} (22)
and therefore the complex variable $z$ is a root of unity, such as

$$z = q^m \quad m = 0, 1, \ldots, 2s$$

where the $q$ parameter is defined as

$$q := e^{2\pi i/(2s + 1)}.$$  \hspace{1cm} (24)

Thus, these tell us that the normalized eigenstates $|z\rangle \equiv |m, \varphi\rangle$ of $E$ are of the form

$$|m, \varphi\rangle = \frac{1}{\sqrt{2s + 1}} \sum_{n=0}^{2s} e^{-iF(n)\varphi} q^{mn} |n\rangle$$  \hspace{1cm} (25)

where the parameters $m \in \mathbb{Z}/(2s + 1)\mathbb{Z}$ and $\varphi \in \mathbb{R}$. It is easy to see that

$$E|m, \varphi\rangle = e^{i\theta_m} |m, \varphi\rangle, \quad \theta_m = m \frac{2\pi}{2s + 1}$$  \hspace{1cm} (26)

which reflects that $E$ is indeed a phase operator and therefore $|m, \varphi\rangle$ are the required phase states. Note that, in the particular case $\varphi = 0$, the states $|m, 0\rangle$ correspond to an ordinary discrete Fourier transform of the basis $\{|n\rangle : n = 0, 1, \ldots, 2s\}$ of the $(2s + 1)$-dimensional space $\mathcal{F}$. We show that the phase states $|m, \varphi\rangle$ are temporally stable under “time evolution”, namely

$$e^{-iH(N)t}|m, \varphi\rangle = |m, \varphi + t\rangle$$  \hspace{1cm} (27)

for any value of the real parameter $t$. The parameter $\varphi$ plays a key role to ensure the temporal stability of the states $|m, \varphi\rangle$. For fixed $\varphi$, they satisfy the equiprobability relation

$$|\langle n|m, \varphi\rangle| = \frac{1}{\sqrt{2s + 1}}, \quad n, m \in \mathbb{Z}/(2s + 1)\mathbb{Z}.$$  \hspace{1cm} (28)

They constitute an orthonormal set

$$\langle m, \varphi|m', \varphi'\rangle = \delta_{m,m'} \quad m, m' \in \mathbb{Z}/(2s + 1)\mathbb{Z}$$  \hspace{1cm} (29)

for fixed $\varphi$ and satisfy the closure property

$$\sum_{m=0}^{2s} |m, \varphi\rangle\langle m, \varphi| = I.$$  \hspace{1cm} (30)

Finally, it is interesting to note that the temporally stable phase states are not all orthogonal. Indeed, the overlap between two phase states $|m', \varphi'\rangle$ and $|m, \varphi\rangle$ reads as

$$\langle m, \varphi|m', \varphi'\rangle = \frac{1}{2s + 1} \sum_{n=0}^{2s} q^{\rho(m-m', \varphi-\varphi', n)}$$  \hspace{1cm} (31)

where the function $\rho$ is

$$\rho(m - m', \varphi - \varphi', n) = -(m - m')n + \frac{2s + 1}{2\pi}(\varphi - \varphi')F(n)$$  \hspace{1cm} (32)

and $q$ is given in (24).
At this level, we would like to emphasize the key role of the phase parameter $\varphi$ introduced starting from relations (2) that defines the actions of creation and annihilation operators. Indeed, firstly $\varphi$ ensures the temporal stability of the phase states $|m, \varphi\rangle$ under time evolution. Secondly, if one ignores $\varphi$, i.e. $\varphi = 0$, $|m, \varphi\rangle$ reduce to those derived by Begg and Barnett using the truncated harmonic oscillator [24]. This means that for two generalized Weyl–Heisenberg algebras characterized by different structure functions $F(N)$, we end up with the same phase states. Thus, to differentiate between these states, the presence of $\varphi$ is necessary in their forms. In this sense, it must be noticed that the $SU(2)$ phase states obtained in [29] are similar to those of Pegg and Barnett despite the fact that the involved symmetries are different. Hence, to avoid such problem we introduced $\varphi$ that allows us to keep the trace of the symmetry and dynamics of the system under consideration. Note also that $\varphi$ plays a key role in relating phase states and mutually unbiased bases [28].

4 Beam splitting of phase states and entanglement

We briefly describe the effect of beam splitting on a known input state with the vacuum at the second port. We assume that the vertical input beam is always prepared in the Fock ground state and the state of interest is the input state in the horizontal beam, which by convention precedes that of vertical beam. Algebraically, the beam splitting transformation is described as follows. The input field described by the usual harmonic oscillator operator $a_1^\pm$ is superposed on the other input field with operator $a_2^\pm$ by a lossless symmetric beam splitter, with the transmission $t$ and reflection $r$ coefficients. The output field operators $a_3^\pm$ and $a_4^\pm$ are given by

$$a_3^\pm = B(\theta)a_1^\pm B^\dagger(\theta), \quad a_4^\pm = B(\theta)a_2^\pm B^\dagger(\theta)$$

where the unitary beam splitter operator of angle $\theta$ is

$$B(\theta) = \exp \left[ \frac{i}{2} \theta (a_1^+ a_2^- + a_1^- a_2^+) \right].$$

The action of $B(\theta)$ on the state $|n, 0\rangle$ reads as

$$B(\theta)|n, 0\rangle = \sum_{p=0}^{n} \sqrt{\frac{n!}{p!(n-p)!}} t^p r^{n-p} |p, n-p\rangle$$

where the transmission and reflection coefficients are defined by

$$t = \cos \frac{\theta}{2}, \quad r = \sin \frac{\theta}{2}.$$

It is well known that among pure states of a single mode field, only harmonic oscillator or Glauber coherent states do not become entangled upon beam splitting. Indeed, for the 50:50 beam splitter with an ordinary Glauber coherent state incident on one port and a vacuum on the other, the beam splitting result is a product of two Glauber states. This procedure do not provides us with an entangled system. In general, any other pure state at the input results in an entangled state at the output. This explains how the beam splitter acts as entangler and why it is used in experiments as device to generate entangled states.
Now, let us proceed in our case to rewrite the action of the beam splitter operator on the phase states (25) as

\[
B(\theta)|m, \varphi\rangle \otimes |0\rangle = \frac{1}{\sqrt{2s+1}} \sum_{n=0}^{2s} \sum_{p=0}^{n} \sqrt{\frac{n!}{(n-p)!p!}} q^{mn} p^p (ir)^{n-p} e^{-iF(n)\varphi} |p, n-p\rangle.
\] (37)

The corresponding output density is then given by

\[
\rho_{1,2} = B(\theta)|m, \varphi\rangle \otimes |0\rangle \langle m, \varphi| \otimes \langle 0|B^\dagger(\theta)
\] (38)

It follows that one can write the bipartite reduced density matrix \(\rho_{12}\), which is obtained by tracing out the second system. This is

\[
\rho_1 = \text{Tr}_2 \rho_{1,2}
\] (39)

where label 2 is reflecting the trace on second states. It is easy to see that the reduced density matrix is given by

\[
\rho_1 = \sum_{n=0}^{2s} \sum_{n'=0}^{2s} \sum_{l=0}^{\min(2s-n, 2s-n')} c(n, l)c(n', l)|n\rangle\langle n'|
\] (40)

where the coefficients \(c(n, l)\) have the form

\[
c(n, l) = \frac{1}{\sqrt{2s+1}} \sqrt{\frac{(n+l)!}{n!!}} q^{m(n+l)} p^n (ir)^l e^{-iF(n+l)\varphi}.
\] (41)

Next, we are interested in the amount of entanglement of the beam splitter output states. In doing so, we examine the entanglement of phase states when passed through a beam splitter by using the linear entropy [33]

\[
S = 1 - \text{Tr}(\rho_1^2).
\] (42)

Clearly, it goes to zero for a pure state. After some algebra, we show that [42] rewrites

\[
S = 1 - \sum_{n=0}^{2s} \sum_{n'=0}^{2s} \sum_{l=0}^{\min(2s-n, 2s-n')} \sum_{l'=0}^{\min(2s-n, 2s-n')} s(n, n', l, l')
\] (43)

where \(s(n, n', l, l')\) is given by

\[
s(n, n', l, l') = \frac{1}{(2s+1)^2} e^{-i\phi(n, n', l, l')} \sqrt{(n+l)!(n'+l')!(n+l')!(n+l)!} \frac{2^{n+n'}}{n!n'!} \frac{2^{l+l'}}{l!l'!}
\] (44)

and the phase term reads as

\[
\phi(n, n', l, l') = \left[F(n+l) + F(n'+l') - F(n'+l) - F(n+l)\right] \varphi.
\] (45)

We have some remarks in order. Firstly, according to last equations we notice that the linear entropy is \(m\)-independent. Secondly, one can verify that the functions \(\phi(n, n', l, l')\) satisfy some symmetries with respect to interchange of their quantum numbers. These are

\[
\phi(n, n', l, l') = -\phi(n, n', l', l), \quad \phi(n, n', l, l') = \phi(n', n, l, l')
\] (46)

which can be used to express the linear entropy in terms of \(\cos \phi(n, n', l, l')\) instead of \(e^{-i\phi(n, n', l, l')}\) together with the symmetry of the summations in [43].
For the temporally stable phase states considered in section 3, we study numerically the behavior of the first-order entropy as a function of the relevant parameters $s$, $\varphi$ and the reflection coefficient $r$ as well. We shall focus on the phase states associated with the generalized Weyl–Heisenberg algebra defined through the structure function

$$F(N) = \frac{N}{2s} (2s + 1 - N).$$  

(47)

We first plot the entanglement against $\varphi$ and $r^2$ for different finite dimensional Hilbert of $(2s+1)$-dimensions.

Figure 1: Linear entropy of phase states as a function of $R = r^2$ and $\varphi$ for $s = 1/2$. 
According to the above figures, we note that for all value taking by $s$ there is a maximum entanglement at the point $r^2 = 1/2$, which corresponds to a 50:50 beam splitter. Furthermore, we have two interesting features in order. The first is that for qubits $s = 1/2$, the degree of entanglement is $\varphi$-independent as shown in Figure 1. However, for qutrits ($s = 1$) and fixed reflection coefficient, the behavior of the linear entropy is Gaussian. The maximum is reached for $r^2 = 1/2$ and $\varphi = \pi$, see
Figure 2. This changes completely for higher dimensional Hilbert spaces. Indeed, for \( s = 3/2 \), for fixed value of \( r^2 \), the linear entropy undergoes an initial rapid increase to reach the maximum entropy for \( \varphi = \pi \) followed by a slower drop and then increases when \( \varphi \) does go to \( 2\pi \) as it shown in Figure 3.

As we explained above, the parameter \( \varphi \) plays a key role in defining the phase states and ensuring the temporal stability of phase states. It follows that it is interesting to examine the linear entropy in term of \( \varphi \) for \( s \) fixed and \( r^2 = 1/2 \) for which one has the maximum of entanglement of phase states.

![Figure 4: Linear entropy of phase states after being passed through a 50:50 beam splitter against \( \varphi \) for \( s = 1/2, 1, \ldots, 5 \).](image)

The behavior of linear entropy as function of the parameter \( \varphi \) is represented in Figure 4. It is easily seen that for \( s = 1/2 \), the linear entropy is constant. For qutrits, i.e. \( s = 1 \), the entropy looks like a gaussian with a maximum around \( \varphi = 2\pi \). For higher levels quantum systems, the linear entropy behaves differently. For instance, with phase states of quartits, i.e. \( s = 3/2 \), the degree of entanglement increases for \( \varphi < \pi \), decreases for \( \pi < \varphi < 3\pi/2 \) and increases after. As the parameter \( \varphi \) is deeply related to time evolution of phase states, the figure 4 can be viewed as representing the temporal evolution of the degree of entanglement of phase states.

In Figure 5 we plot the linear entropy of phase states versus the parameter \( s \) for different values of \( \varphi \) to underline its basic behavior.
Following this figure, we notice that the linear entropy initially rapidly rises for lower $s$ but increases slowly for $s \geq 10$. However for higher $s$, the linear entropy does not approaches zero. This shows that the degree of entanglement is increasing when the dimension of the generalized Weyl–Heisenberg becomes large.

5 Concluding remarks

The main idea of the present work is the investigation of the degree of entanglement of temporally phase states. The latter are constructed as eigenstates of a unitary phase operator resulting from the polar decomposition of ladder operators of finite dimensional Weyl–Heisenberg algebra. Using the beam splitter as entangler, we computed the degree of entanglement of phase states by mean of linear entropy. We investigated the behavior of this quantity as function of the dimension of the Hilbert space, the transmission-reflection coefficients of the beam splitter and the parameter ensuring the temporal stability of the phase states.

It is clearly shown that the maximal entanglement is provided by 50:50 beam splitter. Also, the entanglement increases with increasing dimension of the system. For instance the phase states for qutrits are more entangled than qubits. The entanglement of phase states when passed through a beam splitter is strongly dependent on the parameter $\varphi$ which play an essential role in defining the eigenstates of the unitary phase operator and ensure their temporal stability.

Finally, we note that we did not consider the limiting case $s \to \infty$. In fact for infinite dimensional Hilbert space, one can not define unitary phase operator in a consistent way. Consequently the phase states associated with the ordinary Weyl–Heisenberg can not be constructed using the formalism developed in this work.

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References

[1] E. Schrödinger, Die gegenwärtige Situation in der Quantenmechanik, Naturwissenschaften 23 (1935) 807-812; 823-828; 844-849.

[2] E. Schrödinger, Discussion of Probability Relations Between Separated Systems, Proc. Camb. Phil. Soc. 31 (1935) 555-563.

[3] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W.K. Wootters, Phys. Rev. Lett. 70 (1993) 1895.
[4] C.H. Bennett and S.J. Wiesner, *Phys. Rev. Lett.* **69** (1992) 2881.

[5] M. Murao, D. Jonathan, M.B. Plenio and V. Vedral, *Phys. Rev.* **A59** (1999) 156.

[6] A.K. Ekert, *Phys. Rev. Lett.* **67** (1991) 661.

[7] C.H. Bennett and G. Brassard, Quantum cryptography: Public-key distribution and coin tossing, in Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India, 1984, (IEEE Press, 1984), pp. 175179.

[8] P.G. Kwiat, S. Barraza-Lopez, A. Stefanov and N. Gisin, *Nature* **409** (2001) 1014.

[9] A. Furusawa, J.L. Sørensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble and E.S. Polzik, *Science* **282** (1998) 706.

[10] S.M. Tan, D.F. Walls and M.J. Collett, *Phys. Rev. Lett.* **66** (1991) 252.

[11] B.C. Sanders, K.S. Lee and M.S. Kim, *Phys. Rev.* **A52** (1995) 735.

[12] M.G.A. Paris, *Phys. Rev.* **A59** (1999) 1615.

[13] M.S. Kim, W. Son, V. Buzek and P.L. Knight, *Phys. Rev.* **A65** (2002) 032323.

[14] D. Markham and V. Vedral, *Phys. Rev.* **A67** (2003) 042113.

[15] C.C. Gerry and A. Benmoussa, *Phys. Rev.* **A71** (2005) 062319.

[16] A.O. Barut and L. Girardello, *Communications in Mathematical Physics* **21** (1971) 41.

[17] A. Perelomov, Generalized Coherent States and their Applications, Texts and Monographs in Physics, (Springer-Verlag, 1986).

[18] B.C. Sanders, *Phys. Rev.* **A45** (1992) 6811.

[19] C.C. Gerry, *Phys. Rev.* **A59** (1999) 4095.

[20] A. Luis, *Phys. Rev.* **A64** (2001) 054102.

[21] X. Wang, B.C. Sanders and S.H. Pan, *J. Phys. A: Math. Gen.* **33** (2000) 7451.

[22] C.A. Fuchs, *Phys. Rev. Lett.* **79** (1997) 1162.

[23] H. Jeong, M.S. Kim and J. Lee, *Phys. Rev.* **A64** (2001) 052308.

[24] D.T. Pegg and S.M. Barnett, *Phys. Rev.* **A39** (1989) 1665.

[25] W.H. Louisell, *Phys. Lett.* **7** (1963) 60.

[26] L. Susskind and J. Glogower, *Physics (U S)* **49** (1964) 1.

[27] P. Carruthers and M.M. Nieto, *Rev. Mod. Phys.* **40** (1968) 411.

[28] M. Daoud and M. R. Kibler, *J. Phys. A: Math. Theor.* **43** (2010) 115303.
[29] A. Vourdas, *Phys. Rev. A* **41** (1990) 1653.

[30] M. Daoud and M. Kibler, *Phys. Lett. A* **321** (2004) 147; J. Math. Phys. **47** (2006) 122108.

[31] C. Quesne and N. Vansteenkiste, *Phys. Lett. A* **240** (1998) 21; *Int. J. Theor. Phys.* **39** (2000) 1175.

[32] C. Quesne, *Phys. Lett. A* **272** (2000) 313; erratum *Phys. Lett. A* **275** (2000) 313; *Mod. Phys. Lett. A* **18** (2003) 515.

[33] S. Bose and V. Vedral, *Phys. Rev. A* **61** (2000) 040101.