Influence of compressibility on the foam fracture modeling

A V Porubov\textsuperscript{1,2} and I D Antonov\textsuperscript{1,2}

\textsuperscript{1} Peter the Great St.Petersburg Polytechnic University (SPbPU), Polytechnicheskaya st., 29, Saint Petersburg 195251, Russia
\textsuperscript{2} Institute of Problems in Mechanical Engineering, Bolshoy 61, V.O., Saint Petersburg 199178, Russia
E-mail: alexey.porubov@gmail.com

Abstract. The two-dimensional nonlinear model equations for hydraulic fracturing with foams are obtained. Both multi-phased compressibility and non-Newtonian rheological viscosity of the foam are taken into consideration. A local relationship between the effective pressure on the lateral surface of the fracture is obtained using a dynamic statement of the problem.

1. Introduction

Using foam for the hydraulic fracturing minimizes the volume of liquid phase injected to the formation, is very effective for proppant transport in the fracture due to its high viscosity and has other advantages. The modeling of the fracture propagation during hydraulic fracturing with foam is significantly different from the classical one. It is important that the foam contains gas part, so the compressibility of the fracturing fluid should be taken into account. Also it is necessary to account for non-Newtonian character of the foam viscosity in order to obtain the proper model of hydraulic fracturing with foam.

The modeling of the process of fracture propagation is a complicated task, and various simplifications are employed. In particular, it concerns compressibility of the fracturing fluid and its viscosity [1–5]. Also some factors affecting dynamics behavior are phenomenologically employed from the static modeling, e.g., the relationship between the fluid pressure in the fracture and the opening of the fracture [6, 7]. In this work we develop the model equations for the hydraulic fracturing by the foam basing on the dynamic statement of the problem and taking into account both compressibility and viscosity of the foam.

2. Statement of the problem

Consider a vertical infinite in the vertical direction $y$ fracture which is developing in the horizontal direction $x$. The varying width of the fracture is $w(x, t)$. The foam inside the fracture is considered as a compressible and non-Newtonian fluid. The density, pressure, and velocities along $x$ and $y$ axes are $\rho(x, t), p(x, t), u(x, y, t), v(x, y, t)$ respectively.

The Navier-Stokes equations are used to describe the foam motion. The equations consist of the continuity equation,

$$\rho_t + (\rho u)_x + (\rho v)_y = 0,$$

(1)
and the equations of motion,

\[
\rho (u_t + uu_x + vv_y) = - p_x + 2(\nu u_x)_x + (\nu v_y)_y, \tag{2}
\]

\[
\rho (v_t + uu_x + vv_y) = - p_y + 2(\nu v_y)_y + (\nu u_x)_x, \tag{3}
\]

where \( \nu \) is viscosity of the foam.

### 2.1. Compressibility

The compressibility of the foam is caused by the presence of two phases in it, gas and liquid. To characterise the multiphase nature of the foam, the quality of the foam \( \Gamma \) is introduced \([8,9]\),

\[
\Gamma = \frac{V_g}{V_g + V_l}, \tag{4}
\]

where \( V_g, V_l \) are the volumes of the gas and liquid phases of the foam respectively. The isothermic Boyle’s law is,

\[
p_0 V_{0g} = p V_g, \tag{5}
\]

where \( p_0 \) is the injection pressure, allows us to get from Eq. (4)

\[
\frac{p}{p_0} = \frac{\Gamma_0 (1 - \Gamma) V_{0l}}{(1 - \Gamma_0) \Gamma V_l} \tag{6}
\]

where \( \Gamma_0 \) is the foam injection quality. The density of the foam is

\[
\rho = \frac{V_l \rho_l + V_g \rho_g}{V + g + V_l} = (1 - \Gamma) \rho_l + \Gamma \rho_g, \tag{7}
\]

where \( \rho_l \) and \( \rho_g \) is density of liquid phase and gas phase respectively. Equation (6) allows us to express the quality through the pressure \( p \). Substitution of the resulting expression into Eq. (7) results in

\[
\rho = \frac{(1 - \Gamma_0) \rho_l p_0}{\Gamma_0 p_0 + (1 - \Gamma_0) p} + \frac{\Gamma_0 \rho_l \rho_g}{\Gamma_0 p_0 + (1 - \Gamma_0) p}. \tag{8}
\]

The last relationship is substituted further in the Navier-Stokes equations. It should be noticed that the second term in equation (7) can be neglected due to \( \rho_g/\rho_l \ll 1 \) and \( \rho_l \) is assumed to be constant due to high incompressibility of liquid phase.

### 2.2. Rheology

The experimental data reveal non-constant viscosity of the foam. It depends on the shear rate tensor \( \dot{\gamma} = (1/2)(\nabla v + \nabla v^T) \). In our natations it is

\[
\dot{\gamma} = \begin{pmatrix}
u_x \\
(1/2) (u_y + v_x) \\
v_y
\end{pmatrix}
\]

However, the scalar viscosity coefficient cannot depend directly on the tensor. Instead, it can be the function of the scalar characteristic of the shear rate tensor, or on its the invariants,

\[
I = \text{Tr} \dot{\gamma} = u_x + v_y, \tag{9}
\]

\[
II = \text{Tr} \dot{\gamma}^2 = u_x^2 + v_y^2 + u_y v_x + \frac{1}{2} u_y^2 + \frac{1}{2} v_x^2 \tag{10}
\]
\[ \text{III} = \text{Tr} \gamma^3 = \frac{3}{2} u_y (u_x v_x + v_x v_y) + v_y^3 + u_x^3 + \frac{3}{4} u_y^2 (u_x + v_y + v_x^2 v_y + u_x v_x^2) \]  

(11)

For an incompressible fluid \( I = 0 \). Moreover, for shear flows \( \text{III} \) is also zero [10], and the dynamic viscosity coefficient can be the function of only on the second invariant. We assume such dependence for the compressible foams.

The conventional power law relationship is

\[
\nu = K |\sqrt{\text{II}}/2|^{n-1},
\]

where \( K \) is a constant coefficient, \( n \) depends on the quality of foam and the temperature, usually its values lies between 0.5 and 1 [11]. In order to evaluate the dynamic coefficient of the foam, the known relationships are used for \( K(\Gamma) \) and \( n(\Gamma) \) [12].

3. Boundary conditions

The boundary conditions consist of the condition of impermeability of a solid surface \( y = w(x, t)/2 \),

\[ w_t + u w_x = 2v, \]

(13)

the no-slip boundary condition at \( y = w(x, t)/2 \),

\[ 4u + 2v w_x = w_x w_t, \]

(14)

Also the conditions for the normal and tangential stresses hold,

\[ p = 2\nu \frac{2v}{4 + w_x^2} (4v_y + u_x w_x^2 - 2w_x (u_y + v_x)) = p^*, \]

(15)

\[ (4 - w_x^2)(u_y + v_x) + 4w_x(v_y - u_x) = f^*, \]

(16)

where \( p^* \), \( f^* \) are the boundary stresses in the solid surrounded fracture at \( y = w(x, t)/2 \). Moreover, one has to assume \( v = 0 \) at \( y = 0 \) from the point of view of the symmetry of the problem.

The remaining boundary conditions are zero fracture opening at \( x = L \),

\[ w = 0, \]

and the condition for the input pressure at \( x = 0 \),

\[ p = p_0. \]

The solid medium surrounding the fracture mainly affects it via the normal stresses \( p^* \). Usually the relationship for it is obtained in the framework of the static problem. Recently we developed a model of nonlinear strain waves propagation in an elastic rod surrounded by another elastic medium [13]. In this paper the method of finding the dynamic response of the medium has been described in the framework of a dynamic problem. Now we extend it to the problem of an influence of the surrounded medium on the fracture surface.

Consider a problem of a linearly elastic isotropic medium (which models the soil surrounding the fracture) whose elastic constants are \( \lambda \) and \( \mu \) and the constant density is \( \rho \). We approximate the shape of the fracture as a round one, thus the cylindrical geometry can be used where axis \( x \) is directed along the fracture while the medium occupies the area \( 0 < r < \infty \). No variations along the polar angle are assumed. The displacements along \( x \) and \( r \) are \( u(x, r, t) \) and \( v(x, r, t) \) respectively. The equations of motion for the medium are

\[
\rho u_{tt} - (\lambda + 2\mu) u_{xx} - (\lambda + \mu) \left( v_{rx} + \frac{v_x}{r} \right) - \lambda \left( u_{rr} + \frac{u_r}{r} + v_{rx} + \frac{v_x}{r} \right) = 0
\]

(17)
\[ \rho v_{tt} - (\lambda + 2\mu) \left( v_{rr} + \frac{v_r}{r} - \frac{v}{r^2} \right) - \mu w_{xx} - (\lambda + \mu) w_{rx} = 0 \]  

(18)

The conditions corresponding to the fracture are imposed at \( r = R \).

\[ v = w(x, t), \quad \text{at} \ r = R, \]  

(19)

the fracture is modeled by a continuity condition for the normal stresses and zero tangential stresses,

\[ \sigma_{rr} = p^*, \quad \text{at} \ r = R, \]  

(20)

\[ \sigma_{rx} = 0, \quad \text{at} \ r = R, \]  

(21)

\[ u \to 0, \ w \to 0 \quad \text{at} \ r \to \infty. \]  

(22)

where \( \sigma_{rr} = (\lambda + 2\mu) v_r + \lambda \frac{v}{r} + \lambda u_x, \sigma_{rx} = \mu (u_r + w_x) \).

Assume that all the variables depend upon the phase variable \( \theta = x - c t \), and the radius \( r \) where \( c \) is the phase velocity of the wave of fracture. The unknown functions \( u, v \) are sought as

\[ u = \Phi_\theta + \frac{\Psi}{r}, \quad v = \Phi_r - \Psi_\theta, \]  

(23)

then \( \Phi \) and \( \Psi \) satisfy the equations:

\[ \Phi_{rr} + \frac{1}{r} \Phi_r + \left(1 - \frac{c_l^2}{c_l^2}\right) \Phi_{\theta\theta} = 0, \]  

(24)

\[ \Psi_{rr} + \frac{1}{r} \Psi_r - \frac{1}{r^2} \Psi + \left(1 - \frac{c_\tau^2}{c_\tau^2}\right) \Psi_{\theta\theta} = 0, \]  

(25)

where \( c_l \) and \( c_\tau \) are the velocities of the bulk longitudinal and shear linear waves in the medium, respectively. They depend on the density and the Lamé coefficients, \( c_l^2 = (\lambda + 2\mu)/\rho \), and \( c_\tau^2 = \mu/\rho \).

To solve equations (24), (25) we introduce the Fourier transforms of \( \Phi \) and \( \Psi \):

\[ \tilde{\Phi} = \int_{-\infty}^{\infty} \Phi \exp(-k \theta) \, d\theta, \quad \tilde{\Psi} = \int_{-\infty}^{\infty} \Psi \exp(-k \theta) \, d\theta \]

that reduces Eqs. (24), (25) to the Bessel equations :

\[ \tilde{\Phi}_{rr} + \frac{1}{r} \tilde{\Phi}_r - k^2 \tilde{\Phi} = 0, \]  

(26)

\[ \tilde{\Psi}_{rr} + \frac{1}{r} \tilde{\Psi}_r - \frac{1}{r^2} \tilde{\Psi} - k^2 \tilde{\Psi} = 0, \]  

(27)

with \( \alpha = 1 - c_l^2/c_l^2 \), and \( \beta = 1 - c_\tau^2/c_\tau^2 \). The ratios between \( c, c_l \) and \( c_\tau \) define the signs of \( \alpha \) and \( \beta \), hence three possible sets of solutions to the equations (26), (27) appear, vanishing at infinity due to b.c. (22). Using the boundary conditions (19), (21), we obtain the following relationships for the Fourier images of normal stresses at the lateral surface \( r = R \):

1) when \( 0 < c < c_\tau \),

\[ \tilde{\sigma}_{rr} = \frac{\mu \tilde{w}}{1 - \beta} \left( \frac{2(\beta - 1)}{R} + \frac{k(1 + \beta)^2 K_0(\sqrt{\alpha}kR)}{\sqrt{\alpha} K_1(\sqrt{\alpha}kR)} - \frac{4k\sqrt{\beta} K_0(\sqrt{\beta}kR)}{K_1(\sqrt{\beta}kR)} \right), \]  

(28)
II) when \( c_\tau < c < c_l \),
\[
\tilde{\sigma}_{rr} = \frac{\mu}{1-\beta} \left( \frac{2(\beta-1)}{R} + \frac{k(1+\beta)^2 K_0(\sqrt{\alpha k}R)}{\sqrt{\alpha} K_1(\sqrt{\alpha k}R)} - \frac{4k\sqrt{\beta} J_0(\sqrt{-\beta}kR)}{J_1(\sqrt{-\beta}kR)} \right),
\]
(29)

III) when \( c > c_l \),
\[
\tilde{\sigma}_{rr} = \frac{\mu}{1-\beta} \left( \frac{2(\beta-1)}{R} + \frac{k(1+\beta)^2 J_0(\sqrt{-\alpha k}R)}{\sqrt{-\alpha} J_1(\sqrt{-\alpha k}R)} - \frac{4k\sqrt{\beta} J_0(\sqrt{-\beta}kR)}{J_1(\sqrt{-\beta}kR)} \right),
\]
(30)

where \( J_i \) and \( K_i \) \((i = 0, 1)\) denote the corresponding Bessel functions.

The size of the fracture is small compared with the size of the medium, then all the Bessel functions can be expanded around \( R = 0 \). The inverse Fourier transform results in the approximate condition for the pressure of the local form
\[
p^* = k_1 w(x,t)
\]
(31)

where \( k_1 \) is defined for \( 0 < c < c_\tau \):
\[
k_1 = -2\mu,
\]
(32)

while for \( c_\tau < c < c_l \):
\[
k_1 = \frac{2\mu(4c_\tau^2 - c^2)}{c^2},
\]
(33)

and for \( c > c_l \):
\[
k_1 = \frac{2\mu[c^2(c_\tau^2 - c_1^2) + 3c_\tau^2 c_l^2 - 4c_\tau^4]}{c_\tau^2(c_\tau^2 - c_l^2)}
\]
(34)

One can see that the local dependence coefficient depends on the velocity of the strain wave propagating in the surrounding medium due to the development of the fracture.

4. Conclusion and future work
The model for the hydraulic fracturing by the foam is developed. It is suggested to describe the compressibility of the foam by the density-pressure relation (8), while its apparent viscosity is evaluated using equation (12). A local relation between the foam pressure in the fracture and the opening of the fracture (31) is obtained basing on the dynamic statement of the problem. An importance of considering the dynamic aspect of the hydraulic fracturing with foam should be verified in the future work using numerical simulations.

Acknowledgements
This work was supported by Ministry of Education and Science of the Russian Federation within the framework of the Federal Program Research and development in priority areas for the development of the scientific and technological complex of Russia for 2014–2020 (activity 1.2), grant No. 14.575.21.0146 of September 26, 2017, unique identifier: RFMEFI57517X0146. Industrial partner is Gazpromneft Science & Technology Centre.
References
[1] Economides M J et al. 2000 Reservoir Stimulation, 3rd Edition (Wiley)
[2] Nordgren R P 1972 Society of Petroleum Engineers Journal 12 306–314
[3] Esipov D V et al. 2014 Comput. Technologies 19 33–61
[4] Yew C H 2014 Mechanics of Hydraulic Fracturing (Gulf Professional Publishing)
[5] Spence D A and Sharp P 1985 Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 400 289–313
[6] Geertsma J and Klerk F D 1969 Journal of Petroleum Technology 21 1571–1581
[7] Protasov I I and Dontsov E V 2017 A comparison of non-local elasticity models for a bladelike hydraulic fracture 51st U.S. Rock Mechanics/Geomechanics Symposium, 25-28 June (San Francisco, California, USA: American Rock Mechanics Association)
[8] Ikoku C U 1978 Transient flow of non-Newtonian power-law fluids in porous media Ph.D. thesis Stanford University
[9] Escobar F H and Civan F 1996 Journal of Petroleum Science and Engineering 15 379–387
[10] Bird R B 1987 Dynamics of Polymeric Liquids, Volume 1: Fluid Mechanics (Wiley-Interscience)
[11] Kong X, McAndrew J and Cisternas P 2016 CFD study of using foam fracturing fluid for proppant transport in hydraulic fractures Abu Dhabi International Petroleum Exhibition & Conference (Society of Petroleum Engineers)
[12] Khade S D and Shah S N 2004 SPE Production & Facilities 19 77–85
[13] Porubov A V, Samsonov A M, Velarde M G and Bukhanovsky A V 1998 Physical Review E 58 3854–3864