Modular Algorithm for Computing Cohomology:
Lie Superalgebra of Special Vector Fields on
(2|2)-dimensional Odd-Symplectic Superspace

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Abstract. We describe an essential improvement of our recent algorithm for computing cohomology of Lie (super)algebra based on partition of the whole cochain complex into minimal subcomplexes. We replace the arithmetic of rational numbers or integers by a much cheaper arithmetic of a modular field and use the inequality between the dimensions of cohomology $H$ over any modular field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and over $\mathbb{Q}$: $\dim H(\mathbb{F}_p) \geq \dim H(\mathbb{Q})$. With this inequality we can, by computing over arbitrary $\mathbb{F}_p$, quickly find the (usually, rare) subcomplexes for which $\dim H(\mathbb{F}_p) > 0$ and then carry out the full computation over $\mathbb{Q}$ within these subcomplexes.

We also present the results of application of the corresponding C program to the Lie superalgebra of special vector fields preserving an “odd-symplectic” structure on the (2|2)-dimensional supermanifold. For this algebra, we found some new basis elements of the cohomology in the trivial module.

1 Introduction

Recently we proposed a new algorithm for computation of cohomology of a wide class of Lie superalgebras. This algorithm reduces the computation for the whole cochain complex to a number of smaller tasks within smaller subcomplexes. One can demonstrate that if $T$ is the computation time for the whole complex, then partition of the complex into $N$ subcomplexes reduces the computation time roughly to the value $T/N^2$. Thus, the approach appeared to be efficient enough to cope with several difficult tasks in computing cohomology for particular Lie (super)algebras [1–5]. More detailed experiments with the C implementation of the algorithm, including profiling, reveal that arithmetic operations over $\mathbb{Q}$ take the main part of computation time (usually more than 90% for large tasks). The same is true if $\mathbb{Q}$ is replaced by $\mathbb{Z}$ (though computation becomes somewhat faster).

A standard way to reduce negative influence of this bottleneck is to compute several modular images of the problem with subsequent restoring the result over $\mathbb{Z}$ or $\mathbb{Q}$ by the Chinese remaindering or an algorithm for recovering a rational number from its modular residue [6, 7]. Though, as is clear, the sum of sizes of modules used for constructing images can not be less than the size of maximum integer in the final result, the modular approach allows the intermediate swelling of coefficients to be avoided. Moreover the use of modular images is much more advantageous in the case of (co)homology computation than in the traditional problems of linear algebra. As we demonstrate further, the overwhelming part of computation can be accomplished using only one modular image. Recall that the Gauss elimination, the basic constituent of algorithms for computing (co)homology, over $\mathbb{F}_p$ has only cubic computational complexity in contrast to the exponential one over $\mathbb{Q}$ or $\mathbb{Z}$.

Note that the approach presented here can be applied not only for the Lie superalgebras, but in more general case of computation of homology or cohomology, especially when there is a practical method of splitting (co)chain complex into smaller subcomplexes.

To demonstrate the power of the new algorithm and program, we present the results of computation of cohomology in the trivial module for the algebra $\text{SLe}(2)$. This is an example (for $n = 2$) of the Lie superalgebra of special (divergence free) vector fields on the $(n|n)$-dimensional supermanifold preserving the odd version of symplectic (periplectic, as A. Weil called it) structure [8, 9]. These superalgebras, being super counterparts of the Lie algebras of Hamiltonian vector fields, are vital in the Batalin–Vilkovisky formalism, see [10].
2 Combining Splitting Algorithm with Modular Search

The $k$th cohomology is defined as the quotient group

$$H^k = Z^k/B^k = \text{Ker} \ d^k/\text{Im} \ d^{k-1}$$

for the cochain complex

$$0 \rightarrow C^0 \xrightarrow{d^0} \cdots \xrightarrow{d^k} C^{k-1} \xrightarrow{d^k} C^k \xrightarrow{d^{k+1}} C^{k+1} \xrightarrow{d^{k+1}} \cdots . \quad (1)$$

Here, the $C^k$ are abelian groups of cochains, graded by the integer $k$ (called dimension or degree); the $d^k$ are differentials ($d^k \circ d^{k-1} = 0$); the $Z^k = \text{Ker} \ d^k$ and $B^k = \text{Im} \ d^{k-1}$ are the subgroups of cocycles and coboundaries, respectively (see [11] for details). In order to apply without restrictions the linear algebra algorithms, we assume that the groups of cochains are additive groups of certain linear spaces or modules and we shall use the corresponding terms in the subsequent text.

2.1 Splitting Algorithm

To compute the $k$th cohomology, it suffices to consider the following part of (1):

$$C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} . \quad (2)$$

First of all we split (2) using the $\mathbb{Z}$-grading in the cochain spaces induced by the gradings in the Lie (super)algebra (and the module over this algebra) involved in the construction of the cochain spaces:

$$\left( C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \right) = \bigoplus_{g \in G} \left( C_g^{k-1} \xrightarrow{d_g^{k-1}} C_g^k \xrightarrow{d_g^k} C_g^{k+1} \right) .$$

Here, $G \subseteq \mathbb{Z}$ is a grading subset.

It appears that, as a rule, any subcomplex in a given degree $g$ can be split, in turn, into smaller subcomplexes:

$$\left( C_g^{k-1} \xrightarrow{d_g^{k-1}} C_g^k \xrightarrow{d_g^k} C_g^{k+1} \right) = \bigoplus_{s \in S} \left( C_{g,s}^{k-1} \xrightarrow{d_{g,s}^{k-1}} C_{g,s}^k \xrightarrow{d_{g,s}^k} C_{g,s}^{k+1} \right) . \quad (3)$$

Here $S$ is a finite or infinite set of subcomplexes.

Equation (3) means that the spaces $C_g^i$ split into the direct sum of subspaces

$$C_g^i = \bigoplus_{s \in S} C_{g,s}^i ,$$

and the matrices of the linear mappings $d^i_g$ can be represented in the block-diagonal form

$$d^i_g = \bigoplus_{s \in S} d^i_{g,s} .$$

The construction of these subcomplexes is the central part of the splitting algorithm. Thus, the whole task reduces to a collection of easier tasks of computing

$$H^k_{g,s} = \text{Ker} \ d^k_{g,s}/\text{Im} \ d^{k-1}_{g,s} . \quad (4)$$

As a basis of the cochain space $C^k_{g,s}$, we choose the set of super skew-symmetric monomials of the form

$$c(e_{i_1}, \ldots, e_{i_k}; a_\alpha) \equiv e^{i_1} \wedge \cdots \wedge e^{i_k} \otimes a_\alpha . \quad (5)$$

Here, $e_i$ and $a_\alpha$ are basis elements of the algebra and module, respectively, and $e^i$ is the dual element to $e_i$, that is, $e^i(e_j) = \delta^i_j$. The degrees of factors in (5) satisfy the relation

$$\text{gr}(e^{i_1}) + \cdots + \text{gr}(e^{i_k}) + \text{gr}(a_\alpha) = g .$$
Notice that $\text{gr}(e^t) = -\text{gr}(e_c)$ and this is a serious obstacle to extraction of finite-dimensional subcomplexes for infinite-dimensional Lie (super)algebras when computing cohomology in the adjoint module (important in the deformation theory). We also assume that $i_1 \leq \cdots \leq i_k$.

To construct a subcomplex

$$C_{g,s}^{k-1} \xrightarrow{d_{g,s}^{k-1}} C_{g,s}^k \xrightarrow{d_{g,s}^k} C_{g,s}^{k+1}$$

(6)

from the sum in right hand side of (3), we begin with choosing somehow an arbitrary starting monomial $m_{g,s}^{k,\text{start}}$ of the form (5). There are various choices of the starting monomial and the time and space efficiency of computation depends on the choice. Having no better idea, we use at present the following three strategies: choice of a lexicographically minimal, or lexicographically maximal, or random monomial. We call these strategies bottom, top and random, respectively. Among these, the top strategy seems to be most efficient (see experimental data in Tables 1 and 2) and it is used by default. Nevertheless, other strategies help sometimes to push through difficult tasks when the top strategy fails.

Then we construct the three sets $M_{g,s}^{k-1}, M_{g,s}^k$ and $M_{g,s}^{k+1}$ of basis monomials for $C_{g,s}^{k-1}, C_{g,s}^k$ and $C_{g,s}^{k+1}$, respectively, by the procedure ConstructSubcomplex presented on page 11.

The function TakeMonomialFromSet called within ConstructSubcomplex takes the current monomial from a set of monomials.

The function InverseImageMonomials generates the set of $(q-1)$-monomials whose images with respect to the mapping $d_{g,s}^{q-1}$ contain a given $q$-monomial.

The function ImageMonomials generates the set of $(q+1)$-monomials whose inverse images with respect to the $d_{g,s}^q$ contain a given $q$-monomial.

In the finite-dimensional case, the loops in the procedure ConstructSubcomplex are finite and we obtain in the end a minimal subcomplex of the form (6). This is the unique minimal subcomplex involving the starting monomial $m_{g,s}^{k,\text{start}}$.

### 2.2 Modular Search

Let us consider in more detail the procedure of computation of cohomology within the subcomplex in accordance with formula (4). From now on we assume that $C_{g,s}^{k-1}, C_{g,s}^k$ and $C_{g,s}^{k+1}$ in (6) are finite-dimensional spaces over $\mathbb{Q}$ or $\mathbb{F}_p$ or modules over $\mathbb{Z}$.

Since important in mathematics and physics fields $\mathbb{R}$ and $\mathbb{C}$ are, in principle, non-algorithmic objects, our main interest will be focused on the cohomology over the field $\mathbb{Q}$ (or its algebraic extensions). In accordance with a general theorem in the homological algebra, called the universal coefficient theorem [12], (co)homology with coefficients from an arbitrary abelian group $G$ can be expressed in terms of (co)homology with coefficients in $\mathbb{Z}$. Thus, we can carry out the computation over $\mathbb{Z}$ and then go to the coefficient group we are interested in. Let us consider now the connection between $H_{g,s}^k(\mathbb{Q})$ and $H_{g,s}^k(\mathbb{Z})$.

In the finite-dimensional case, the group $H_{g,s}^k(\mathbb{Z})$ is a finitely generated abelian group having the following canonical representation

$$H_{g,s}^k(\mathbb{Z}) \simeq \bigoplus_{\beta} \bigoplus_{i_1, \ldots, i_r} (\mathbb{Z}/\mathbb{Z}_{t_{i_1}} \oplus \cdots \oplus \mathbb{Z}/\mathbb{Z}_{t_r})^\beta.$$  

(7)

Here, $\beta^k$, the number of copies of the integer group $\mathbb{Z}$, is called the rank of the abelian group $H_{g,s}^k(\mathbb{Z})$ or the Betti number. The cyclic groups $\mathbb{Z}_{t_i}$ are called the torsion subgroups and their orders $t_i$, having the property $t_i > 1$, $t_1|t_2, t_2|t_3, \ldots$ and so on, are called the torsion coefficients.

In the case of cohomology, the universal coefficient theorem is expressed by the following split short exact sequence

$$0 \to H_{g,s}^k(\mathbb{Z}) \otimes G \to H_{g,s}^k(G) \to \text{Tor}(H_{g,s}^{k+1}(\mathbb{Z}), G) \to 0,$$

(8)

where the operation Tor is the periodic product of abelian groups. In our context

$$\text{Tor}(H_{g,s}^{k+1}(\mathbb{Z}), G) = (\text{torsion } H_{g,s}^{k+1}(\mathbb{Z})) \otimes (\text{torsion } G).$$

The term “split”, in application to sequence (8), means the possibility to construct the isomorphism

$$H_{g,s}^k(G) \simeq H_{g,s}^k(\mathbb{Z}) \otimes G \oplus \text{Tor}(H_{g,s}^{k+1}(\mathbb{Z}), G).$$

(9)

Replacing $G$ by $\mathbb{Q}$ in (9) and taking into account that $\text{Tor}(A, \mathbb{Q}) = 0$ for any abelian group $A$, we have $H_{g,s}^k(\mathbb{Q}) \simeq H_{g,s}^k(\mathbb{Z}) \otimes \mathbb{Q}$. Since $\mathbb{Z}_m \otimes \mathbb{Q} = 0$ for arbitrary $m$ and $\mathbb{Z} \otimes \mathbb{Q} \simeq \mathbb{Q}$, the dimension of $H_{g,s}^k(\mathbb{Q})$, interpreted as vector space over $\mathbb{Q}$, coincides with the rank (Betti number) $\beta^k$ of the group $H_{g,s}^k(\mathbb{Z})$.

Our modular approach is based on the following
Theorem 1.
\[ \dim H^k_{g,s}(\mathbb{F}_p) \geq \dim H^k_{g,s}(\mathbb{Q}). \] (10)

Remarks:

1. Inequality (10) means that non-trivial cohomology classes computed over the field of rational numbers \( \mathbb{Q} \) can exist only in the subcomplexes with non-trivial cohomology classes computed over the finite field \( \mathbb{F}_p \) with arbitrary prime \( p \).

2. H. Khudaverdian turned author’s attention to the fact that inequality (10) can be deduced immediately from the universal coefficient theorem: considering the product \( H^k_{g,s}(\mathbb{Z}) \otimes \mathbb{F}_p \) and taking into account representation (7) and isomorphism \( \mathbb{Z} \otimes \mathbb{F}_p \simeq \mathbb{F}_p \), we see that the dimension of \( H^k_{g,s}(\mathbb{F}_p) \), as a vector space over \( \mathbb{F}_p \), can not be less than \( \beta^k \) (only additional dimensions may appear, if the torsions in \( H^k_{g,s}(\mathbb{Z}) \) or \( H^{k+1}_{g,s}(\mathbb{Z}) \) contain cyclic groups of the form \( \mathbb{Z}_{p^\infty} \)). Nevertheless, we give here a direct constructive proof in order to demonstrate in parallel the main ideas of (co)homology computation.

Proof. To prove inequality (10), in such a way as to avoid cancellations of integers and apply the modular homomorphism \( \phi_p : \mathbb{Z} \to \mathbb{F}_p \), at the end of computation. Thus, it is convenient to consider (4) over \( \mathbb{Z} \) instead of \( \mathbb{Q} \).

We assume that \( p \) is odd and use a symmetric representation of \( \mathbb{F}_p \), i.e.,
\[ \mathbb{F}_p = \left\{ -\frac{p-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{p-1}{2} \right\}. \]

We will also apply \( \phi_p \) component-wise to multicomponent objects over \( \mathbb{Z} \), like vectors and matrices. We begin with the following setup:
- \( C^{k-1}_{g,s}, C^k_{g,s} \) and \( C^{k+1}_{g,s} \) are represented as finite-dimensional modules \( M^- = \mathbb{Z}^n, M = \mathbb{Z}^m, M^+ = \mathbb{Z}' \), respectively, i.e., \( \dim C^{k-1}_{g,s} = n, \dim C^k_{g,s} = m, \dim C^{k+1}_{g,s} = l \).
- the differentials \( d^{k-1}_{g,s} \) and \( d^k_{g,s} \) are represented (in the monomial bases of the form (5), in our case) as integer \( m \times n \) and \( l \times m \) matrices
\[ D' \equiv mD' = \begin{bmatrix} (d')^1_1 & \cdots & (d')^1_n \\ \vdots & \ddots & \vdots \\ (d')^m_1 & \cdots & (d')^m_n \end{bmatrix} \quad \text{and} \quad D \equiv lD = \begin{bmatrix} d_1^1 & \cdots & d^1_m \\ \vdots & \ddots & \vdots \\ d_l^1 & \cdots & d^l_m \end{bmatrix}, \]

respectively. We write \( i_A \) to indicate that matrix \( A \) has \( i \) rows and \( j \) columns.

- the matrices \( D \) and \( D' \) satisfy the relation \( DD' = 0 \).

The computation of cohomology, i.e., construction of quotient module, can be reduced to the construction of so-called \( (co)\)homology decomposition \[ \mathbb{Z} \] based on the computation of the Smith normal forms \[ \mathbb{Z} \] of the matrices representing differentials.

First of all let us determine the cocycle submodule, i.e., \( \text{Ker} \ D \subseteq M \), by reducing the matrix \( D \) to the integer Smith normal form \( S = UDV \). The matrix \( S \) has the form
\[ S = iS = \begin{bmatrix} rS_{\overline{r}} & rO_{\overline{r}} \\ lO_{\overline{l}} & lO_{\overline{l}} \end{bmatrix}, \] (11)

where, \( iO \) is the \( i \times j \) zero matrix, \( r = \text{rank} \mathbb{Z} D, rS = \text{diag} (s_1, \ldots, s_r), s_1, \ldots, s_r \) are positive integers called the invariant factors of \( D \). These invariant factors have the property \( s_i | s_{i+1} \) for all \( i \). Note that there is connection between the invariant factors and the torsion coefficients from formula (7), namely, the prime divisors of the invariant factors are also divisors of some torsion coefficients. The transformation matrices \( U = iU \) and \( V = mV \) are unimodular integer matrices, i.e., \( \det U = \pm 1, \det V = \pm 1 \). With such determinants, these matrices are invertible and their inverses \( U^{-1} \) and \( V^{-1} \) are obviously integer matrices too.

Now we should consider the coboundary submodule
\[ \text{Im} \ D' \subseteq \text{Ker} \ D \subseteq M. \]
Combining the relation
\[ SV^{-1}D' = UDD' = 0 \]
with the structure of the matrix \( S \) (see formula (11)), we can reduce the matrix \( D' \) determining coboundaries to the matrix \( \widetilde{D}' \) acting in the submodule of cocycles:

\[
V^{-1}D' = \begin{bmatrix}
  r^O_m \\
  r^{n-r'} \\
  m-r \widetilde{D}'
\end{bmatrix}.
\]

Computing the Smith normal form \( S' = \widetilde{U}' \widetilde{D}'V' \) for the reduced coboundary matrix we get

\[
S' = m-rS' = \begin{bmatrix}
  r' \\
  r' \widetilde{S}' \\
  m-r' \widetilde{S}'
\end{bmatrix},
\]

where \( r' = \text{rank}_Z \widetilde{D}' \) and \( r' \widetilde{S}' = \text{diag}(s'_1, \ldots, s'_r) \).

We can extend the transformation matrix \( \widetilde{U}' = m-r \widetilde{U}' \) acting in the submodule of cocycles to the transformation matrix acting in the whole module \( M : \)

\[
U' = mU' = \begin{bmatrix}
  r I \\
  r O \\
  m-r O
\end{bmatrix}.
\]

Here, \( r I \) is the \( r \times r \) identity matrix. Using the transformation matrices \( U' \) and \( V \) we can transform the initial (monomial in our case) basis \( e = (e_1, \ldots, e_m) \) in the module \( M \) into the basis \( a = (a_1, \ldots, a_m) = eV(U')^{-1} \) making the cohomology decomposition explicit

\[
M = (a_1, \ldots, a_r) \oplus (a_{r+1}, \ldots, a_{r+r'}) \oplus (a_{r+r'+1}, \ldots, a_m).
\] (12)

In this decomposition we have

\[
\ker d^k_{g,s} = \langle a_{r+1}, \ldots, a_m \rangle
\]

and

\[
\text{Im } d^{k-1}_{g,s} = \langle a_{r+1}, \ldots, a_{r+r'} \rangle.
\]

The formula for the dimension of cohomology (Betti number) follows from decomposition (12)

\[
\dim H^k_{g,s}(\mathbb{Q}) = \beta^k = m - r - r'.
\] (13)

Now let us consider how (13) changes under \( \phi_p \). The image of (13) takes the form

\[
\dim H^k_{g,s}(\mathbb{F}_p) = m_p - r_p - r'_p.
\] (14)

Since \( \phi_p \) is a ring homomorphism, we have for arbitrary unimodular matrix \( A \) with integer entries

\[
\det \phi_p(A) = \phi_p(\det A) = \phi_p(\pm1) = \pm1,
\]

that is \( \phi_p \) maps the above transformation matrices into invertible matrices. Hence the number of elements in the decomposition basis \( a \) remains unchanged, \( m_p = m \). On the other hand, the invariant factors \( s'_1, \ldots, s'_r, \) and \( s_1, \ldots, s_r \) of the matrices \( S' \) and \( S \) divisible by \( p \) vanish, hence \( \text{rank } \phi_p(S) = r_p \leq r \) and \( \text{rank } \phi_p(S') = r'_p \leq r' \) and inequality (10) is proved by comparing (13) and (14). \( \square \)
2.3 Implementation

An algorithm based on the above ideas was implemented in the C language. The program called LieCo-

homologyModular has the following structure:

1. Input Lie (super)algebra $A$, module $X$ over $A$, cohomology degree (dimension) $k$ and grade $g$. $A$ and $X$ should be defined over (some algebraic extension of) $\mathbb{Z}$ or $\mathbb{Q}$.
2. Construct the full set $M^k_g$ of $k$-cochain monomials in grade $g$.
3. Choose a prime $p$ for searching subcomplexes with non-trivial cocycles by computing over $\mathbb{F}_p$.
4. Choose an element $m^k_g \in M^k_g$ (the starting monomial).
5. Construct a minimal subcomplex $s : C^{k-1}_{g,s} \xrightarrow{\partial^{k-1}_{g,s}} C^k_{g,s} \xrightarrow{\partial^k_{g,s}} C^{k+1}_{g,s}$ such that $m^k_g \in C^k_{g,s}$.
6. Compute $n = \dim H^k_{g,s}(\mathbb{F}_p)$.
7. If $n > 0$, then compute $H^k_{g,s}$ over $\mathbb{Z}$ or $\mathbb{Q}$ (or their extensions). We can use here the Chinese remain-
dering or the rational recovery algorithm as more efficient procedures than direct computation over $\mathbb{Z}$ or $\mathbb{Q}$.
8. Delete all basis monomials of $C^k_{g,s}$ from $M^k_g$.
9. If $M^k_g$ is empty, then stop computation, otherwise go to Step 4.

In the current implementation we obtain the relations determining cocycles and coboundaries (in fact, the rows of matrices of differentials) within the procedure ConstructSubcomplex. These relations are generated one by one as by-product of the functions InverseImageMonomials and ImageMonomials. To prevent unnecessary memory consumption, every newly arising relation is reduced modulo the system of relations existing to the moment and, if the result is not zero, the new relation is added to the system. Thus, we automatically have the matrices of differentials in the normal form just after completion of the procedure ConstructSubcomplex. This process is obviously equivalent to the Gauss elimination method, the most standard method for the computation of the Smith normal form of a matrix.

In recent years, a number of new fast algorithms for the determination of Smith normal form have been elaborated [15, 16]. These algorithms appear to be well suited to the (co)homology computation. It is worth to study the possibility to incorporate these algorithms in our implementation. We could, for example, remove the generation of relations from the functions InverseImageMonomials and ImageMonomials making them as fast as possible. Then, after construction of subcomplex with the help of these modified functions, we should generate the matrices of differentials separately and apply the fast algorithms to these matrices. Of course, this modification should be done if the total computation time decreases without substantial increase in the memory consumption. The works [15, 16] contain a detailed analysis of the properties of the sets of primes most appropriate for application of modular algorithms to a given matrix.

Here we give only a few comments concerning the choice of prime $p$ in our algorithm. These comments are based mainly on experiments with the program.

From the practical point of view, we should use only primes $p$ for which all operations in $\mathbb{F}_p$ can be done within one machine word. Thus, for 32bit architecture we should choose $p$ from the set of 8951 primes $(3, 5, \ldots, 92681)$. A good choice should not produce excessive cocycles. Of course, such cocycles will be removed at Step 7 anyway, but at the expense of additional work. In our context, an “unlucky” prime is the one which divides the invariant factors of matrices of differentials (or, in other words, the torsion coefficients of cohomology over $\mathbb{Z}$) and, as is clear, the probability for a given prime to be unlucky diminishes as the prime grows. On the other hand, there is an increase of time (in the examples we have computed, up to factor 2 or 3) and space expenditures with increase of $p$ within the set $(3, 5, \ldots, 92681)$, so it makes sense not to use too large primes for searching subcomplexes with potentially non-trivial cohomology. In our practice, we use, as a rule, a compromise: a prime near the half of 32bit word, namely, $p = 65537 = 2^{24} + 1$, i. e., the 4th Fermat number.

However, quite satisfactory results can be obtained even with much smaller primes, as is illustrated in Table 3. The symbols $n_p(\mathbb{Q})$ in the boxes of this table mean that (non-zero) $n = \dim_{\mathbb{F}_p}(\mathbb{Q}) H(H(2))^{k}_g$, whereas $H(H(2))^{k}_g$ means the cohomology with coefficients in the trivial module for the Lie algebra $H(2)$ of Hamiltonian vector fields on the 2-dimensional symplectic manifold. We performed computation over all modular fields $\mathbb{F}_3$ through $\mathbb{F}_{17}$. As is seen in Table 3, the results for $\mathbb{F}_{17}$ fully coincide with those for $\mathbb{Q}$ for all computed grades $g \in [-2, \ldots, 8]$ (and for all cohomology degrees $k$). The table also illustrates Theorem 1, i.e., all boxes containing non-zero dimensions for the field $\mathbb{Q}$ contain also non-zero ($\geq$ same for $\mathbb{Q}$) dimensions for all fields $\mathbb{F}_p$ considered.
In Tables 1 and 2 we present (considering both algebra and superalgebra cases) the running times for computation over \( \mathbb{F}_{17} \) of cohomology \( H^k_g(H(2)) \) (for \( k = 7, 4 \leq g \leq 8 \)) and \( H^k_g(SLe(2)) \) (for \( k = 6, 0 \leq g \leq 4 \)). The columns presented in these tables are: the dimensions of cochain spaces, i.e., the sizes of matrices of differentials; the running times in seconds for the top strategy of the choice of the starting monomial and comparison of the bottom and top strategies.

The times in both tables were obtained on a 1133MHz Pentium III PC with 512Mb. Note that the maximum memory consumption is near 46Mb and near 14Mb for the tasks in Table 1 and in Table 2, respectively.

**Table 1.** Timing for \( H^k_g(H(2), \mathbb{F}_{17}), k = 7 \)

| \( g \) | \( \dim C^k_{g-1} \) | \( \dim C^k_{g} \) | \( \dim C^k_{g+1} \) | \( T_{top} \) | \( \frac{T_{bottom}}{T_{top}} \) |
|----|----|----|----|----|----|
| 4  | 1580 | 1128 | 479 | < 1 | 2.0 |
| 5  | 3382 | 2730 | 1388 | 4 | 2.8 |
| 6  | 6734 | 6132 | 3606 | 27 | 3.3 |
| 7  | 12766 | 12818 | 8546 | 214 | 3.7 |
| 8  | 23074 | 25488 | 18963 | 1128 | 4.5 |

**Table 2.** Timing for \( H^k_g(SLe(2), \mathbb{F}_{17}), k = 6 \)

| \( g \) | \( \dim C^k_{g-1} \) | \( \dim C^k_{g} \) | \( \dim C^k_{g+1} \) | \( T_{top} \) | \( \frac{T_{bottom}}{T_{top}} \) |
|----|----|----|----|----|----|
| 0  | 1867 | 6605 | 22119 | 1 | 2.2 |
| 1  | 3528 | 12162 | 39796 | 4 | 4.1 |
| 2  | 6546 | 22102 | 70817 | 21 | 4.7 |
| 3  | 11878 | 39652 | 124768 | 87 | 5.7 |
| 4  | 21073 | 70110 | 217696 | 413 | 6.0 |

3 Computing \( H^k_g(SLe(2)) \)

In this section we present the results of application of the program \texttt{LieCohomologyModular} to the Lie superalgebra of special vector fields preserving periplectic structure on (2|2)-dimensional superspace. Periplectic supermanifolds with a fixed volume element play an important role in the geometrical formulation of the Batalin-Vilkovisky formalism [17, 18], an efficient method for quantizing gauge theories.

Recall that a periplectic or an odd symplectic manifold is an \((n|n)\)-dimensional supermanifold equipped with an odd symplectic structure, that is an odd non-degenerate closed 2-form. In an analog of Darboux coordinates [19], it takes the shape

\[
\omega = \sum_{i=1}^n dx^i \wedge d\theta_i. \tag{15}
\]

Here, \( x^1, \ldots, x^n \) and \( \theta_1, \ldots, \theta_n \) are even and odd (Grassmann) variables, respectively. The vector fields preserving 2-form (15) form a Lie superalgebra denoted by Le(\( n \)). The elements of Le(\( n \)) can be expressed in terms of generating functions (also called hamiltonians) and these generating functions generate a nontrivial central extension of Le(\( n \)) called the Buttin algebra and denoted by B(\( n \)). The bracket for arbitrary two hamiltonians \( f \) and \( g \) is called the Buttin bracket or antibracket or odd Poisson bracket and takes the form

\[
\{ f, g \} = \sum_{i=1}^n \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta_i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial x^i} \right). \tag{16}
\]

Here, \( p(f) \) is parity of function \( f \).
The odd symplectic structure is a super version of the ordinary symplectic structure \( \omega = \sum_{i=1}^{n} dq_i \wedge dp_i \), where \( q_i \) and \( p_i \) are both even. In symplectic case there is an invariant volume form \( \rho_\omega = \omega^n \) (Liouville theorem) and all the vector fields preserving \( \omega \) are automatically divergence free. Contrariwise, in periplectic case the volume is not preserved (see [9], for more geometric consideration of the subject see [20]) and one can impose the divergence-free condition additionally.

Thus, we come to the special Buttin algebra \( SB(n) \). Its generating functions \( f \) satisfy the divergence free condition

\[ \Delta f = 0, \]  

(17)

where \( \Delta \) is the odd Laplacian

\[ \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^i \partial \theta_i}. \]  

(18)

This \( \Delta \)-operator is, actually, the Fourier transform with respect to the odd variables of the usual de Rham differential (see [9]). Together with its homology, \( \Delta \) plays the key role in the formulation of so-called Batalin-Vilkovisky “master equation”. We can slightly reduce the special Buttin algebra by removing constants from generating functions, i.e., taking the quotient algebra modulo the center \( Z \). The resulting algebra \( SLe(n) = SB(n) / Z \) is called the special Leites algebra.

Since the above algebras are infinite-dimensional, in order to compute cohomology, we introduce a grading by prescribing grades to the variables \( \{x^i\} \) and \( \{\theta_i\} \) with subsequent extension of the grading to the polynomial functions of these variables. Most natural grading can be provided by setting \( gr(x_i) = 1 \) and \( gr(\theta_i) = -1 \). Here the word “natural” means that, with this grading, the algebra \( B(n) \) (and \( Le(n) = B(n) / Z \)) contains the inner grading element

\[ \sum_{i=1}^{n} x^i \theta_i, \]  

(19)

guaranteeing that all non-trivial cohomology classes lie in the zero grade cochain subspaces (see [11]). Unfortunately, there is no good inner grading element\(^1\) in the divergence free algebra \( SB(n) \) (and \( SLe(n) \)). That is why the computation of cohomology for these algebras is much more difficult task than for the algebras without divergence free condition.

Let us now turn to \( SLe(2) \). Its basis elements up to grade 1 are

| grades | basis elements |
|--------|---------------|
| -2     | \{ \( O_1 = \theta \psi \), \( E_2 = \theta \), \( E_3 = \psi \), \( E_4 = y \theta \), \( E_5 = y \psi - x \theta \), \( E_6 = x \psi \), \( O_7 = y \), \( O_8 = x \), \( E_9 = y^2 \theta \), \( E_{10} = y^2 \psi - 2xy \theta \), \( E_{11} = xy \psi - \frac{1}{2} x^2 \theta \), \( E_{12} = x^2 \psi \), \} |

Here, \( x, y \) and \( \theta, \psi \) are even and odd variables of the \((2|2)\)-dimensional superspace, respectively; \( E_i \) and \( O_i \) are even and odd basis elements of the Lie superalgebra, respectively. Notice that odd generating function corresponds to the even element of superalgebra and vice versa. This is a property of Lie superalgebras with antibrackets called **parity shift**.

\(^1\) In a private communication I. Shchepochkina suggested a general view on the inner grading elements for the algebras with antibracket. She noticed that \( \sum_{i=1}^{n} g_i x^i \theta_i \) can be used as the inner grading element in \( B(n) \) or \( Le(n) \) as well as (19) assuming that \( g_i \) are arbitrary integers and \( gr(x^i) = g_i \) and \( gr(\theta_i) = - g_i \). This inner grading element can belong to a divergence-free subalgebra \( SB(n) \) or \( SLe(n) \) only if \( \sum_{i=1}^{n} g_i = 0 \) (in virtue of equation (17)), i.e., the set \( \{ g_i \} \) must contain non-zero integers with opposite signs. This leads to impossibility to construct finite-dimensional cochain subspaces of a given degree.
Let us present also the initial part of the multiplication table in $\text{SLe}(2)$ (given are only non-zero brackets)

\[
\begin{align*}
[E_2, E_5] &= E_2, \\
[E_2, E_6] &= -E_3, \\
[E_3, E_4] &= -E_2, \\
[E_3, E_5] &= -E_3, \\
[E_3, E_4] &= -2E_4, \\
[E_5, E_4] &= E_5, \\
[E_3, E_5] &= -E_3, \\
[O_1, O_7] &= -E_2, \\
[E_5, O_7] &= -O_7, \\
[E_6, O_7] &= -O_8, \\
[O_1, O_8] &= E_3, \\
[E_4, O_8] &= -O_7, \\
[E_5, O_8] &= O_8, \\

\end{align*}
\]

Studying this multiplication table we can obtain some information about the structure of $\text{SLe}(2)$. For example, there are some subalgebras important in the construction of representations of $\text{SLe}(2)$:

- commutative negative grade subalgebra $A_{<0} = \langle O_1, E_2, E_3 \rangle$;
- semisimple zero grade subalgebra $A_0 = \langle E_4, E_5, E_6 \rangle \simeq \text{so}(3) \simeq \text{sl}(2) \simeq \text{sp}(2)$;
- non-positive grade subalgebra $A_{\leq 0} = A_{<0} \oplus A_0$, a semidirect sum of the semisimple algebra and the commutative ideal.

The results of computation of cohomology $H^k_g(\text{SLe}(2))$ are presented in Table 4. The boxes of this table contain either three numbers (with possible indication of non-trivial cohomology class) or right arrow. The three numbers from top downwards are

- $\dim C^k_g$, the dimension of the whole space of $k$-cochains in grade $g$;
- the number of minimal subcomplexes $C^{k-1}_{g,s} \xrightarrow{\text{\ for formula (3)}} C^k_{g,s} \xrightarrow{\text{\ for formula (3)}} C^{k+1}_{g,s}$ constituting the whole subcomplex in accordance with formula (3);
- $\max_{s \in S} \dim C^k_{g,s}$, maximum dimension of $(k,g)$-cochain subspaces among all the minimal subcomplexes.

The right arrow $\rightarrow$ means that all subsequent boxes in the row contain the same information. The reason for this is that all the relations defining cocycles and coboundaries for $k$-cochains coincide with those for $(k-1)$-cochains multiplied by the 1-cochain $c(O_1) \equiv c(\theta \psi)$.

In our computation we found four genuine cohomology classes, i.e., generators of the cohomology ring, $\alpha, \beta, \gamma, \delta$. They are parenthesized in the table. Their multiplicative consequences are underlined. Notice that the cohomology ring contains nilpotents and zerodivisors: there are arbitrary powers of the cocycle $\alpha = c(\theta \psi)$, but $\beta \alpha = 0$, $\gamma \alpha = 0$ and $\delta \alpha^2 = 0$.

4 Concluding Remark

When computing cohomology we start with the construction of the full set of $(k,g)$-monomials. At the moment we do not see how to avoid this in deterministic algorithms. To represent the set of monomials, we need to allocate $n = l \times \dim C^k_g$ elements of memory representing basis elements of algebra and module ($l = k$ for the trivial and $l = k + 1$ for any non-trivial module). In our implementation we represent basis elements by two-byte integers. For the last box in column 9 ($k = 9$) of Table 4 we have $\dim C^k_g = 648308$ and the set of monomials occupies near 12MB. The dimensions grow very rapidly, so, in fact, we are working on the brink of abilities of 32bit architecture and, even theoretically, we can add only a few rows to Table 4. But, as to arithmetical difficulties, we have some progress.
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Algorithm: **ConstructSubcomplex**

**Input:** \( m_{g, \text{start}} \), starting \((k, g)\)-monomial

**Output:** \( M^k_{g, s} \), \( M^k_{g, s} \), \( M^k_{g, s} \), monomial bases of cochain spaces in the current subcomplex \( s \) \( C^k_{g, s} \xrightarrow{d^k_{g, s}} C^k_{g, s} \xrightarrow{d^k_{g, s}} C^k_{g, s} \), such that \( m_{g, \text{start}} \in M^k_{g, s} \)

**Local:** \( M^k_{g, s} \subseteq M^k_{g, s} \), \( M^k_{g, s} \subseteq M^k_{g, s} \), \( M^k_{g, s} \subseteq M^k_{g, s} \), working subsets of currently “new” (not yet processed) monomials:

\( W_{g, s} \), \( W_{g, s} \), \( W_{g, s} \), working sets of monomials; \( m_{g, s} \), \( m_{g, s} \), \( m_{g, s} \), working monomials

**Initial setting:**

1. \( M^k_{g, s} := M^k_{g, s} := \emptyset \)
2. \( M^k_{g, s} := M^k_{g, s} := \{ m_{g, \text{start}} \} \)
3. \( M^k_{g, s} := M^k_{g, s} := \emptyset \)

**Loop over \( k \)-monomials:**

4. while \( M^k_{g, s} \neq \emptyset \) do
5. \( m^k_{g, s} := \text{TakeMonomialFromSet}(M^k_{g, s}) \)
6. Supplement the set \( M^k_{g, s} \):
7. \( W^k_{g, s} := \text{InverseImageMonomials}(d^k_{g, s}, m^k_{g, s}) \)
8. \( M^k_{g, s} := M^k_{g, s} \) \( \cup \) \( W^k_{g, s} \)
9. \( M^k_{g, s} := M^k_{g, s} \) \( \cup \) \( W^k_{g, s} \)
10. **Exclude processed monomial \( m^k_{g, s} \):**
11. **Loop over \((k + 1)\)-monomials:**
12. while \( M^{k+1}_{g, s} \neq \emptyset \) do
13. \( m^{k+1}_{g, s} := \text{TakeMonomialFromSet}(M^{k+1}_{g, s}) \)
14. Supplement the set \( M^{k+1}_{g, s} \):
15. \( W^{k+1}_{g, s} := \text{InverseImageMonomials}(d^{k+1}_{g, s}, m^{k+1}_{g, s}) \)
16. \( M^{k+1}_{g, s} := M^{k+1}_{g, s} \) \( \cup \) \( W^{k+1}_{g, s} \)
17. \( M^{k+1}_{g, s} := M^{k+1}_{g, s} \) \( \cup \) \( W^{k+1}_{g, s} \)
18. **Exclude processed monomial \( m^{k+1}_{g, s} \):**
19. \( M^{k+1}_{g, s} := M^{k+1}_{g, s} \) \( \setminus \) \( \{ m^{k+1}_{g, s} \} \)
20. **Loop over \((k - 1)\)-monomials:**
21. while \( M^{k-1}_{g, s} \neq \emptyset \) do
22. \( m^{k-1}_{g, s} := \text{TakeMonomialFromSet}(M^{k-1}_{g, s}) \)
23. Supplement the set \( M^{k-1}_{g, s} \):
24. \( W^{k-1}_{g, s} := \text{ImageMonomials}(d^{k-1}_{g, s}, m^{k-1}_{g, s}) \)
25. \( M^{k-1}_{g, s} := M^{k-1}_{g, s} \) \( \cup \) \( W^{k-1}_{g, s} \)
26. **Exclude processed monomial \( m^{k-1}_{g, s} \):**
27. **return** \( M^{k-1}_{g, s} \), \( M^{k}_{g, s} \), \( M^{k+1}_{g, s} \)
Table 3. dim $H^k_p(H(2), R)$ for $R = \mathbb{Q}$ and $\mathbb{F}_p$, $p = 3, 5, 7, 11, 13, 17$; $(k, g) \in [1, \ldots, \infty) \times [-2, \ldots, 8]$

| $g \backslash k$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|
| -2              | $1_3$ $1_5$ $1_7$ | $1_{11}$ $1_{13}$ | (1171Q) | $1_3$ $1_5$ $1_7$ | $1_{11}$ $1_{13}$ | (1171Q) | 23 | 25 | 23 | 23 | 23 |
| -1              | 23 | 23 | 25 | 23 | 25 | 23 | 23 | 23 | 23 | 23 | 23 |
| 0               | 13 | 13 | 33 | 33 | 13 | 13 | (1171Q) | 13 | 13 | 13 | 13 |
| 1               | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 |
| 2               | 13 | 43 | 33 | 13 | 35 | 83 | 45 | 113 | 15 | 43 | 13 | 13 |
| 3               | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 |
| 4               | 53 | 15 | 20 | 45 | 31 | 35 | 17 | 31 | 53 | 47 | 47 | 47 |
| 5               | 23 | 63 | 43 | 83 | 27 | 34 | 27 | 27 | 27 | 27 | 27 | 27 |
| 6               | 15 | 45 | 35 | 35 | 35 | 35 | 35 | 35 | 35 | 35 | 35 | 35 |
| 7               | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 |
| 8               | 33 | 83 | 83 | 13 | 13 | 17 | 83 | 13 | 17 | 83 | 13 | 13 |
| $g + 2k \mid k$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | $k > 9$ |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| 0              | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       |
|                | (α)   | α²    | α³    | α⁴    | α⁵    | α⁶    | α⁷    | α⁸    | α⁹    | α⁶      |
| 1              | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 2       |
| 2              | 3     | 4     | 4     | 1     | 1     | 1     | 1     | 1     | 1     | 1       |
| 3              | 6     | 6     | 8     | 8     | 8     | 8     | 8     | 8     | 8     | 8       |
| 4              | 1     | 4     | 4     | 13 (β) | 13 | 26 | 26 | 13 | 16 | 16       |
|                | 1     | 4     | 4     | 13 (β) | 13 | 26 | 26 | 13 | 16 | 16       |
| 5              | 10    | 16    | 18    | 7     | 7     | 7     | 7     | 7     | 7     | 7       |
| 6              | 12    | 22    | 26    | 15    | 15    | 15    | 15    | 15    | 15    | 15       |
| 7              | 14    | 26    | 23    | 26    | 23    | 23    | 23    | 23    | 23    | 23       |
| 8              | 16    | 30    | 42    | 42    | 42    | 42    | 42    | 42    | 42    | 42       |
| 9              | 18    | 34    | 52    | 52    | 52    | 52    | 52    | 52    | 52    | 52       |
| 10             | 20    | 38    | 65    | 65    | 65 (γ) | 65 (γ) | 65    | 65    | 65    | 65       |
| 11             | 22    | 42    | 72    | 72    | 72    | 72    | 72    | 72    | 72    | 72       |
| 12             | 24    | 46    | 84    | 84    | 84    | 84    | 84    | 84    | 84    | 84       |
| 13             | 26    | 50    | 92    | 92    | 92    | 92    | 92    | 92    | 92    | 92       |
| 14             | 28    | 54    | 108   | 108   | 108   | 108   | 108   | 108   | 108   | 108      |
| 15             | 30    | 58    | 130   | 130   | 130   | 130   | 130   | 130   | 130   | 130      |
| 16             | 32    | 62    | 154   | 154   | 154   | 154   | 154   | 154   | 154   | 154      |
| 17             | 34    | 66    | 178   | 178   | 178   | 178   | 178   | 178   | 178   | 178      |
| 18             | 36    | 70    | 202   | 202   | 202   | 202   | 202   | 202   | 202   | 202      |
| 19             | 38    | 74    | 226   | 226   | 226   | 226   | 226   | 226   | 226   | 226      |
| 20             | 40    | 78    | 250   | 250   | 250   | 250   | 250   | 250   | 250   | 250      |

Table 4. $H_0^g(\text{SLE}(2))$ for $(k, g) \in [1, \ldots, \infty) \times [-2k, \ldots, -2k + 20]$