MINIMAL PERIOD ESTIMATES FOR BRAKE ORBITS OF
NONLINEAR SYMMETRIC HAMILTONIAN SYSTEMS

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Abstract. In this paper, we consider the minimal period estimates for brake orbits
of nonlinear symmetric Hamiltonian systems. We prove that if the Hamiltonian
function $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ is super-quadratic and convex, for every number
$\tau > 0$, there exists at least one $\tau$-periodic brake orbit $(\tau, x)$ with minimal period $\tau$ or $\tau/2$
provided $H(Nx) = H(x)$.

§1 Introduction and main result

Let $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ with $I$ being the identity of $\mathbb{R}^n$. Suppose
$H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ satisfying

$$H(Nx) = H(x), \quad \forall x \in \mathbb{R}^{2n}. \quad (1.1)$$

We consider the following problem

$$\begin{cases}
\dot{x}(t) = JH'(x(t)), \\
x(-t) = Nx(t), \\
x(\tau + t) = x(t), \quad \forall t \in \mathbb{R}.
\end{cases} \quad (1.2)$$

A solution $(\tau, x)$ of (1.2) is a special periodic solution of the Hamiltonian system
in (1.2), we call it a brake orbit and $\tau$ the brake period of $x$.

The existence and multiplicity of brake orbits on a given energy hypersurface
was studied by many Mathematicians. In 1987, P. Rabinowitz in [26] proved that if
$H$ satisfies (1.1), $\Sigma = H^{-1}(1)$ is star-shaped, and $x \cdot H'(x) \neq 0$ for all $x \in \Sigma$, then
there exists a brake orbit on $\Sigma$. In 1987, V. Benci and F. Giannoni gave a different
proof of the existence of one brake orbit on $\Sigma$ in [1]. In 1989, A. Szulkin in [27]
proved that there exist at least \( n \) brake orbits on \( \Sigma \), if \( H \) satisfies conditions in [26] of Rabinowitz and the energy hypersurface \( \Sigma \) is \( \sqrt{2} \)-pinched. Long, Zhang and Zhu in [23] proved that there exist at least 2 geometrically distinct brake orbits on any central symmetric strictly convex hypersurface \( \Sigma \). Recently, Z. Zhang and the author in [16] proved that there exist at least \( \lfloor n/2 \rfloor + 1 \) geometrically distinct brake orbits on any central symmetric strictly convex hypersurface \( \Sigma \), furthermore, there exist at least \( n \) geometrically distinct brake orbits on \( \Sigma \) if all brake orbits on \( \Sigma \) are non-degenerate.

In his pioneering work [24], P. Rabinowitz proposed a conjecture on whether a superquadratic Hamiltonian system possesses a periodic solution with a prescribed minimal period. This conjecture has been deeply studied by many mathematicians. For the strictly convex case, i.e., \( H''(x) > 0 \), Ekeland and Hofer in [6] proved that Rabinowitz’s conjecture is true. We refer to [3]-[6],[8], [10], [17]-[19], and reference therein for further survey of the study on this problem.

For Rabinowitz’ conjecture on the second order Hamiltonian systems, similar results under various convexity conditions have been proved (cf. [5] and reference therein). In [17] and [19], under precisely the conditions of Rabinowitz, Y. Long proved that for any \( \tau > 0 \) the second order system
\[
\ddot{x} + V'(x) = 0
\]
possesses a \( \tau \)-periodic solution \( x \) whose minimal period is at least \( \tau/(n+1) \). Similar result for the first order system (1.1) is still unknown so far.

It is natural to ask the Rabinowitz’s question for the brake orbit problem: for a superquadratic Hamiltonian function \( \dot{H} \) satisfying condition (1.1), whether the problem (1.2) possesses a solution \((\tau, x)\) with prescribed minimal period \( \tau \) for any \( \tau > 0 \) (brake orbit minimal periodic problem in short).

In this paper we first consider the brake orbit minimal periodic problem for the nonlinear Hamiltonian systems. From Section 3, we have the following result.

**Theorem 1.1.** Suppose the Hamiltonian function \( H \) satisfies the conditions:

(H1) \( H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \) satisfying \( H(Nx) = H(x) \), \( \forall x \in \mathbb{R}^{2n} \).

(H2) there are constants \( \mu > 2 \) and \( r_0 > 0 \) such that
\[
0 < \mu H(x) \leq H'(x) \cdot x, \quad \forall |x| \geq r_0.
\]

(H3) \( H(x) = o(|x|^2) \) near \( x = 0 \).

(H4) \( H(x) \geq 0 \), \( \forall x \in \mathbb{R}^{2n} \).

(H5) \( H''(x) > 0 \), \( \forall x \in \mathbb{R}^{2n} \).

Then there exists a brake orbit \((\tau, x)\) of problem (1.2) with minimal brake period \( \tau \) or \( \tau/2 \).

In fact, in Section 3 a more general theorem is proved (see Theorem 3.1) where the superquadratic condition (H2) is relaxed to
\[
\dot{H}(x) = \frac{1}{2} (B x, x) + \tilde{H}(x)
\]
with \( \tilde{H} \) satisfying condition (H2), and the convexity condition (H5) is relaxed to
\[
H''(x(t)) \geq 0, \quad \forall t \in \mathbb{R} \text{ and } \int_0^{\tau/2} H''(x(t)) \, dt > 0 \text{ for all brake orbit } (\tau, x).
\]

We also prove some results about the brake orbit minimal periodic problems for the second order Hamiltonian systems in Section 3.
§2 Iteration Inequalities of the $L_0$-Index Theory

We observe that the problem (1.2) can be transformed to the following Lagrangian boundary value problem

$$\begin{cases}
\dot{x}(t) = JH'(x(t)), \\
x(0) \in L_0, \quad x(\tau/2) \in L_0,
\end{cases} \tag{2.1}$$

where $L_0 = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n}$.

An index theory suitable for the study of problem (2.1) was established in [13] for any Lagrangian subspace $L$. As usual, we denote

$$\text{Sp}(2n) = \{M \in \mathcal{L}(\mathbb{R}^{2n}) | M^TJ = J\},$$

$$\mathcal{P}(2n) = \{\gamma \in C([0, 1], \text{Sp}(2n)) | \gamma(0) = I_{2n}\}$$
and

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], \text{Sp}(2n)) | \gamma(0) = I_{2n}\}.$$

For a symplectic path $\gamma \in \mathcal{P}(2n)$, its Maslov-type index associated with a Lagrangian subspace $L$ is assigned to a pair of integers $(i_L(\gamma), \nu_L(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, n\}$. We call it the $L$-index of $\gamma$ in short. In [23], the index $\mu_j(\gamma)$, $j = 1, 2$ was defined for $\gamma \in \mathcal{P}(2n)$, the $\mu_j$-indices are essentially the special $L$-indices for $L = L_0$ and $L = L_1 = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$ up to a constant $n$, respectively. In order to estimate the period of a brake orbit, we need to estimate the $L_0$-index of the iteration path $\gamma^k$ associated to the iterated brake orbit $x^k$.

For reader’s convenience, we recall the definition of the $L_0$-index which was first established in [13]. Some properties for this index theory are listed in the appendix below. For $L_0 = \{0\} \oplus \mathbb{R}^n$, we define the following two subspaces of $\text{Sp}(2n)$ by

$$\text{Sp}(2n)_{L_0}^+ = \{M \in \text{Sp}(2n) | \det V > 0\},$$

$$\text{Sp}(2n)_{L_0}^0 = \{M \in \text{Sp}(2n) | \det V = 0\},$$
for $M = \begin{pmatrix} S & V \\ T & U \end{pmatrix}$.

Since the space $\text{Sp}(2n)$ is path connected, and the $n \times n$ non-degenerated matrix space has two path connected components, one with $\det V > 0$, and another with $\det V < 0$, the space $\text{Sp}(2n)^*_L$ has two path connected components as well. We denote by

$$\text{Sp}(2n)_{L_0}^{\pm} = \{M \in \text{Sp}(2n) | \pm \det V > 0\}$$

then we have $\text{Sp}(2n)_{L_0}^* = \text{Sp}(2n)_L^+ \cup \text{Sp}(2n)_L^-$. We denote the corresponding symplectic path space by

$$\mathcal{P}(2n)_{L_0}^* = \{\gamma \in \mathcal{P}(2n) | \gamma(1) \in \text{Sp}(2n)_{L_0}^*\}$$

and

$$\mathcal{P}(2n)_{L_0}^0 = \{\gamma \in \mathcal{P}(2n) | \gamma(1) \in \text{Sp}(2n)_{L_0}^0\}.$$

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Definition 2.1. We define the $L_0$-nullity of any symplectic path $\gamma \in \mathcal{P}(2n)$ by

$$\nu_{L_0}(\gamma) \equiv \dim \ker_{L_0}(\gamma(1)) := \dim \ker V(1) = n - \text{rank}V(1)$$

with the $n \times n$ matrix function $V(t)$ defined in (2.1).

We note that $\text{rank}\left(\begin{pmatrix} V(t) \\ U(t) \end{pmatrix}\right) = n$, so the complex matrix $U(t) \pm \sqrt{-1}V(t)$ is invertible. We define a complex matrix function by

$$Q(t) = [U(t) - \sqrt{-1}V(t)][U(t) + \sqrt{-1}V(t)]^{-1}.$$ 

It is easy to see that the matrix $Q(t)$ is a unitary matrix for any $t \in [0, 1]$. We denote by

$$M_+ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad M_- = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \quad J_n = \text{diag}(-1, 1, \cdots, 1).$$

It is clear that $M_\pm \in \text{Sp}(2n)_{L_0}^\pm$.

For a path $\gamma \in \mathcal{P}(2n)_{L_0}^\ast$, we first adjoin it with a simple symplectic path starting from $J = -M_+$, i.e., we define a symplectic path by

$$\bar{\gamma}(t) = \begin{cases} I \cos \left(\frac{(1-2t)\pi}{2}\right) + J \sin \left(\frac{(1-2t)\pi}{2}\right), & t \in [0, 1/2]; \\ \gamma(2t - 1), & t \in [1/2, 1]. \end{cases}$$

then we choose a symplectic path $\beta(t)$ in $\text{Sp}(2n)_{L_0}^\ast$ starting from $\gamma(1)$ and ending at $M_+$ or $M_-$ according to $\gamma(1) \in \text{Sp}(2n)_{L_0}^+$ or $\gamma(1) \in \text{Sp}(2n)_{L_0}^-$, respectively. We now define a joint path by

$$\bar{\gamma}(t) = \beta * \bar{\gamma} := \begin{cases} \bar{\gamma}(2t), & t \in [0, 1/2], \\ \beta(2t - 1), & t \in [1/2, 1]. \end{cases}$$

By the definition, we see that the symplectic path $\bar{\gamma}$ starting from $-M_+$ and ending at either $M_+$ or $M_-$. As above, we define

$$\bar{Q}(t) = [\bar{U}(t) - \sqrt{-1}\bar{V}(t)][\bar{U}(t) + \sqrt{-1}\bar{V}(t)]^{-1}.$$ 

for $\bar{\gamma}(t) = \begin{pmatrix} \bar{S}(t) \\ \bar{T}(t) \end{pmatrix}$, $\bar{V}(t)$, $\bar{U}(t)$. We can choose a continuous function $\bar{\Delta}(t)$ in $[0, 1]$ such that

$$\det \bar{Q}(t) = e^{2\sqrt{-1}\bar{\Delta}(t)}.$$ 

By the above arguments, we see that the number $\frac{1}{\pi}(\bar{\Delta}(1) - \bar{\Delta}(0)) \in \mathbb{Z}$ and it does not depend on the choice of the function $\bar{\Delta}(t)$. We note that there is a positive continuous function $\rho : [0, 1] \to (0, +\infty)$ such that

$$\det(\bar{U}(t) - \sqrt{-1}\bar{V}(t)) = \rho(t) e^{\sqrt{-1}\bar{\Delta}(t)}.$$
Definition 2.2. For a symplectic path $\gamma \in \mathcal{P}(2n)_L^*$, we define the $L_0$-index of $\gamma$ by
\[
i_{L_0}(\gamma) = \frac{1}{\pi} (\Delta(1) - \Delta(0)).
\] (2.2)

For a $L_0$-degenerate symplectic path $\gamma \in \mathcal{P}(2n)_L^0$, its $L_0$-index is defined by the infimum of the indices of the nearby nondegenerate symplectic paths.

Definition 2.3. For a symplectic path $\gamma \in \mathcal{P}(2n)_L^0$, we define the $L_0$-index of $\gamma$ by
\[
i_{L_0}(\gamma) = \sup_{U \in \mathcal{N}(\gamma)} \inf \{i_{L_0}(\tilde{\gamma}) | \tilde{\gamma} \in U \cap \mathcal{P}(2n)_L^*\},
\]
where $\mathcal{N}(\gamma)$ is the set of all open neighborhood of $\gamma$ in $\mathcal{P}(2n)_L$.

Suppose the continuous symplectic path $\gamma : [0, 1] \to \text{Sp}(2n)$ is the fundamental solution of the following linear Hamiltonian system
\[
\dot{z}(t) = JB(t)z(t)
\] (2.3)
with $B(t)$ satisfying $B(t + 2) = B(t)$ and $B(1 + t)N = NB(1 - t))$. We define the $L_0$-iteration paths $\gamma^k : [0, k] \to \text{Sp}(2n)$ of $\gamma$ by
\[
\gamma^1(t) = \gamma(t), \ t \in [0, 1],
\]
\[
\gamma^2(t) = \begin{cases} 
\gamma(t), \ t \in [0, 1] \\
N\gamma(2 - t)\gamma(1)^{-1}N\gamma(1), \ t \in [1, 2], 
\end{cases}
\]
and in general, for $j \in \mathbb{N}$
\[
\gamma^{2j - 1}(t) = \begin{cases} 
\gamma(t), \ t \in [0, 1], \\
N\gamma(2 - t)\gamma(1)^{-1}N\gamma(1), \ t \in [1, 2], \\
\cdots \\
N\gamma(2j - 2 - t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2j-5}, \ t \in [2j - 3, 2j - 2], \\
\gamma(t - 2j + 2)\gamma(2)^{2j-4}, \ t \in [2j - 2, 2j - 1], 
\end{cases}
\] (2.4)
\[
\gamma^{2j}(t) = \begin{cases} 
\gamma(t), \ t \in [0, 1], \\
N\gamma(2 - t)\gamma(1)^{-1}N\gamma(1), \ t \in [1, 2], \\
\cdots \\
\gamma(t - 2j + 2)\gamma(2)^{2j-4}, \ t \in [2j - 2, 2j - 1], \\
N\gamma(2j - t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2j-3}, \ t \in [2j - 1, 2j], 
\end{cases}
\] (2.5)

We note that if $\tilde{\gamma}(t), \ t \in \mathbb{R}$ is the fundamental solution of the linear system (2.3), then there holds $\gamma^k = \tilde{\gamma}|_{[0,k]}$. For the iteration path $\gamma^k$, the following Bott-type iteration formulas were proved in [16].
Proposition 2.4. Suppose $\omega_k = e^{\pi \sqrt{-1}/k}$. For odd $k$ we have

\[ i_{L_0}(\gamma) = i_{L_0}(\gamma^1) + \sum_{i=1}^{[k/2]} i_{\omega_k^i}(\gamma^2), \]

\[ \nu_{L_0}(\gamma) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{[k/2]} \nu_{\omega_k^i}(\gamma^2), \]

for even $k$, we have

\[ i_{L_0}(\gamma) = i_{L_0}(\gamma^1) + i_{L_0}^{k/2}(\gamma^1) + \sum_{i=1}^{k/2-1} i_{\omega_k^i}(\gamma^2), \]

\[ \nu_{L_0}(\gamma) = \nu_{L_0}(\gamma^1) + \nu_{L_0}^{k/2}(\gamma^1) + \sum_{i=1}^{k/2-1} \nu_{\omega_k^i}(\gamma^2), \]

where the $(L_0 - \omega)$ index $(i_{L_0}^\omega(\gamma), \nu_{L_0}^\omega(\gamma))$ of $\gamma$ for $\omega \in U := \{z \in \mathbb{C} : |z| = 1\}$ was defined in [16], and the $\omega$-index $(i_{\omega}(\gamma^2), \nu_{\omega}(\gamma^2))$ of $\gamma^2$ for $\omega \in U$ was defined in [22](cf.[21]).

We note that $\omega_k^{k/2} = \sqrt{-1}$. For any two $2k_1 \times 2k_1$ matrices of square block form, $M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ with $i = 1, 2$, the $\diamond$-product of $M_1$ and $M_2$ is defined to be the $2(k_1 + k_2) \times 2(k_1 + k_2)$ matrix

\[
M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.
\]

Denote by $M^\diamond k$ the $k$-fold $\diamond$-product of $M$. Let $N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$ for $\lambda = \pm 1$ and $b = \pm 1$, or 0. Denote by $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. We remind that the unit circle in the complex plane is defined by $U = \{z \in \mathbb{C} : |z| = 1\}$, and the upper(lower) semi closed unit circle $U^+(U^-)$ is defined by $U^\pm = \{z \in U : z = e^{\pm \sqrt{-1}}, 0 \leq \pm \theta \leq \pi\}$. In [22], for any $M \in \text{Sp}(2n)$, Long defined the homotopy set of $M$ in $\text{Sp}(2n)$ by

\[ \Omega(M) = \{N \in \text{Sp}(2n) : \sigma(N) \cap U = \sigma(M) \cap U \text{ and } \dim_{\mathbb{C}} \ker_{\mathbb{C}}(N - \lambda I) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - \lambda I), \forall \lambda \in \sigma(M) \cap U\}. \]

The path connected component of $\Omega(M)$ which contains $M$ is denoted by $\Omega_0(M)$, and is called the homotopy component of $M$ in $\text{Sp}(2n)$.

In [15], the following result was proved (cf. Theorem 10.1.1 of [21]).
Proposition 2.5. 1° For any $\gamma \in \mathcal{P}(2n)$ and $\omega \in \cup \{1\}$, there always holds

$$i_1(\gamma) + \nu_1(\gamma) - n \leq i_\omega(\gamma) \leq i_1(\gamma) + n - \nu_\omega(\gamma). \quad (2.6)$$

2° The left equality in (2.6) holds for some $\omega \in \cup^+ \{1\}$ (or $\cup^- \{1\}$) if and only if there holds $I_{2p} \circ N_1(1, -1)^{\omega} \circ K \in \Omega_0(\gamma(\tau))$ for some non-negative integers $p$ and $q$ satisfying $0 \leq p + q \leq n$ and $K \in \text{Sp}(2(n - p - q))$ with $\sigma(K) \subset \cup \{1\}$ satisfying that all eigenvalues of $K$ located within the arc between 1 and $\omega$ including $\omega$ in $\cup^+ \ (or \ \cup^-)$ possess total multiplicity $n - p - q$. If $\omega \neq -1$, all eigenvalues of $K$ are in $\cup \mathbb{R}$ and those in $\cup^+ \mathbb{R}$ (or $\cup^- \mathbb{R}$) are all Krein negative (or positive) definite. If $\omega = -1$, it holds that $-I_{2s} \circ N_1(1, -1)^{\omega} \circ H \in \Omega_0(K)$ for some non-negative integers $s$ and $t$ satisfying $0 \leq s + t \leq n - p - q$, and some $H \in \text{Sp}(2(n - p - q - s - t))$ satisfying $\sigma(H) \subset \cup \mathbb{R}$ and that all elements in $\sigma(H) \cap \cup^+ \ (or \ \sigma(H) \cap \cup^-)$ are all Krein-negative (or Krein-positive) definite.

3° The left equality in (2.6) holds for all $\omega \in \cup \{1\}$ if and only if $I_{2p} \circ N_1(1, -1)^{(n - p)} \in \Omega_0(\gamma(\tau))$ for some integer $p \in [0, n]$. Specifically in this case, all the eigenvalues of $\gamma(\tau)$ equal to 1 and $\nu_\tau(\gamma) = n + p \geq n$.

4° The right equality in (2.6) holds for some $\omega \in \cup^+ \{1\}$ (or $\cup^- \{1\}$) if and only if there holds $I_{2p} \circ N_1(1, 1)^{\omega} \circ K \in \Omega_0(\gamma(\tau))$ for some non-negative integers $p$ and $r$ satisfying $0 \leq p + r \leq n$ and $K \in \text{Sp}(2(n - p - r))$ with $\sigma(K) \subset \cup \{1\}$ satisfying the condition that all eigenvalues of $K$ located within the closed arc between 1 and $\omega$ in $\cup^+ \{1\}$ (or $\cup^- \{1\}$) possess total multiplicity $n - p - r$. If $\omega \neq -1$, all eigenvalues in $\sigma(K) \cap \cup^+$ (or $\sigma(K) \cap \cup^-$) are all Krein positive (or negative) definite; if $\omega = -1$, there holds $-I_{2s} \circ N_1(-1, 1)^{\omega} \circ H \in \Omega_0(K)$ for some non-negative integers $s$ and $t$ satisfying $0 \leq s + t \leq n - p - r$, and some $H \in \text{Sp}(2(n - p - r - s - t))$ satisfying $\sigma(H) \subset \cup \mathbb{R}$ and that all elements in $\sigma(H) \cap \cup^+$ (or $\sigma(H) \cap \cup^-$) are all Krein positive (or negative) definite.

5° The right equality in (2.6) holds for all $\omega \in \cup \{1\}$ if and only if $I_{2p} \circ N_1(1, 1)^{(n - p)} \in \Omega_0(\gamma(\tau))$ for some integer $p \in [0, n]$. Specifically in this case, all the eigenvalues of $\gamma(\tau)$ must be 1, and there holds $\nu_\tau(\gamma) = n + p \geq n$.

6° Both equalities in (2.6) hold for all $\omega \in \cup \{1\}$ if and only if $\gamma(\tau) = I_{2n}$.

Combining Propositions 2.4 and 2.5, we have the following result.

Theorem 2.6. 1° For any $\gamma \in \mathcal{P}(2n)$ and $k \in \mathbb{N}$, there holds

$$i_{L_0}(\gamma^1) + \left[\frac{k}{2}\right] (i_1(\gamma^2) + \nu_1(\gamma^2) - n) \leq i_{L_0}(\gamma^k)$$

$$\leq i_{L_0}(\gamma^1) + \left[\frac{k}{2}\right] (i_1(\gamma) + n) - \frac{1}{2} \nu_1(\gamma^2) + \frac{1}{2} \nu_1(\gamma^2), \text{ if } k \in 2\mathbb{N} - 1, \quad (2.7)$$

$$i_{L_0}(\gamma^1) + i_{L_0}^{\sqrt{-1}}(\gamma^1) + \left(\frac{k}{2} - 1\right) (i_1(\gamma^2) + \nu_1(\gamma^2) - n) \leq i_{L_0}(\gamma^k) \leq i_{L_0}(\gamma^1) + i_{L_0}^{\sqrt{-1}}(\gamma^1)$$

$$+ \left(\frac{k}{2} - 1\right) (i_1(\gamma) + n) - \frac{1}{2} \nu_1(\gamma^2) + \frac{1}{2} \nu_1(\gamma^2) + \frac{1}{2} \nu_{-1}(\gamma^2), \text{ if } k \in 2\mathbb{N}. \quad (2.8)$$

The index $(i_{L_0}(\gamma), \nu_{L_0}(\gamma))$ is defined in [16] for $\omega \in \cup = \{z \in \mathbb{C} | |z| = 1\}$, see also Definition 4.9 in the appendix below.
2° The left equality of (2.7) holds for some $k \geq 3$ and of (2.8) holds for some $k \geq 4$ if and only if there holds $I_{2p} \circ N_1(1,-1)^o \circ K \in \Omega_0 (\gamma^2(2))$ for some non-negative integers $p$ and $q$ satisfying $p + q \leq n$ and some $K \in \text{Sp}(2(n-p-q))$ satisfying $\sigma(K) \subset U \setminus \mathbb{R}$. If $r = n - p - q > 0$, then $R(\theta_1) \circ \cdots \circ R(\theta_r) \in \Omega_0 (K)$ for some $\theta_j \in (0, \pi)$. In this case, all eigenvalues of $K$ on $U^+$ (on $U^-$) are located on the arc between 1 and $\exp(2\pi \sqrt{-1}/k)$ (and $\exp(-2\pi \sqrt{-1}/k)$) in $U^+$ (in $U^-$) and are all Krein negative (positive) definite.

3° The right equality of (2.7) holds for some $k \geq 3$ and of (2.8) holds for some $k \geq 4$ if and only if there holds $I_{2p} \circ N_1(1,1)^o \circ \in \Omega_0 (\gamma^2(2))$ for some non-negative integers $p$ and $r$ satisfying $p + r = n$.

4° Both equalities of (2.7), and also of (2.8), hold for some $k > 2$ if and only if $\gamma^2(2) = I_{2n}$.

Proof. By Proposition 2.4, summing the inequalities of (2.6) with $\omega = \omega_k^i, 1 \leq i < k/2, i \in \mathbb{N}$, we obtain the inequalities (2.7) for odd $k$ and (2.8) for even $k$. We remind that here we have used the Bott-type formula

$$\nu_1(\gamma^k) = \sum_{\omega^k=1} \nu_\omega(\gamma).$$

The equality conditions follow from 2° and 4° of Proposition 2.5 together with Corollary 9.2.8 and List 12 in P198 of [21]. We note that from List 12 in P198 of [21], no eigenvalue on $U^+$ is Krein positive definite.

Since we should consider the Bott-type iteration formulas in Proposition 2.4 in odd and even cases, the inequalities in Theorem 2.6 is naturally considered in two cases correspondingly. We will see that the inequalities in Theorem 2.6 for even times iteration path are our main difficult to prove that the brake orbit found in Section 3 has minimal period, though we believe this kind brake orbit has minimal period, we can only prove that it has minimal period or it is 2-times iteration of a brake orbit with minimal period.

§3 APPLICATIONS TO NONLINEAR HAMILTONIAN SYSTEMS

We now apply Theorem 2.6 to the brake orbit problem of autonomous Hamiltonian system

$$\begin{cases} -Jx = Bx + H'(x), & x \in \mathbb{R}^{2n}, \\ x(\tau/2 + t) = Nx(\tau/2 - t), \\ x(\tau + t) = x(t), & t \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $H(Nx) = H(x)$ and $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ is a $2n \times 2n$ symmetric semi-positive definite matrix whose operator norm is denoted by $\|B\|$, $B_1$ and $B_2$ are $n \times n$ symmetric matrices. A solution $(\tau, x)$ of the problem (3.1) is a brake orbit of the Hamiltonian system, and $\tau$ is the brake period of $x$. To find a brake orbit of the Hamiltonian system in (3.1), it is sufficient to solve the following problem

$$\begin{cases} -Jx(t) = Bx + H'(x(t)), & x \in \mathbb{R}^{2n}, t \in [0, \tau/2], \\ x(0) \in L_0, x(\tau/2) \in L_0. \end{cases} \tag{3.2}$$
Any solution $x$ of problem (3.2) can be extended to a brake orbit $(\tau, x)$ with the mirror symmetry of $L_0$ by $x(\tau/2 + t) = Nx(\tau/2 - t), \ t \in [0, \tau/2]$ and $x(\tau + t) = x(t), \ t \in \mathbb{R}$.

**Theorem 3.1.** Suppose the Hamiltonian function $H$ satisfies the conditions:

- $(H1)$ $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ satisfying $H(Nx) = H(x), \ \forall x \in \mathbb{R}^{2n}$.
- $(H2)$ there are constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu H(x) \leq H'(x) \cdot x, \ \forall |x| \geq r_0.$$  

- $(H3)$ $H(x) = o(|x|^2)$ at $x = 0$.
- $(H4)$ $H(x) \geq 0, \ \forall x \in \mathbb{R}^{2n}.$

Then for every $0 < \tau < \frac{2\pi}{\|B\|}$, the system (3.1) possesses a non-constant brake orbit $(\tau, x)$ satisfying

$$i_{L_0}(x, \tau/2) \leq 1. \quad (3.3)$$

Moreover, if $x$ further satisfies the following condition:

- $(HX) \ H''(x(t)) \geq 0 \ \forall t \in \mathbb{R}$ and $\int_0^{\tau/2} H''(x(t)) \ dt > 0.$

Then the minimal brake period of $x$ is $\tau$ or $\tau/2$.

**Proof.** We divide the proof into two steps.

**Step 1.** Show that there exists a brake orbit $(\tau, x)$ satisfying (3.3) for $0 < \tau < \frac{2\pi}{\|B\|}$.

Fix $\tau \in (0, \frac{2\pi}{\|B\|})$. Without loss generality, we suppose $\tau = 2$, then $\tau < \frac{2\pi}{\|B\|}$ implies $\|B\| < \pi$. By conditions (H1)-(H4), we can find a non-constant $\tau$-periodic solution $x$ of (3.2) via the saddle point theorem such that (3.3) holds. For reader's convenience, we sketch the proof here and refer the reader to Theorem 3.5 of [15] for the case of periodic solution. We note that the main ideas here are the same as that in the periodic case. We refer the paper [11] for some details.

In fact, following P. Rabinowitz' pioneering work [24], let $K > 0$ and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(t) = 1$ if $t \leq K$, $\chi(t) = 0$ if $t \geq K + 1$, and $\chi'(t) < 0$ if $y \in (K, K + 1)$. The number $K$ will be determined later. Set

$$\bar{H}_K(z) = \frac{1}{2}(Bz, z) + H_K(z),$$

with

$$H_K(z) = \chi(|z|)H(z) + (1 - \chi(|z|))R_K|z|^4,$$

where the constant $R_K$ satisfies

$$R_K \geq \max_{K \leq |z| \leq K + 1} \frac{H(z)}{|z|^4}.$$  

We set $L^2 = L^2([0, 1], \mathbb{R}^{2n})$ and define a Hilbert space $E := \mathcal{W}_{L_0} = W^{1/2, 2}_{L_0}([0, 1], \mathbb{R}^{2n})$ with $L_0$ boundary conditions by

$$\mathcal{W}_{L_0} = \{ z \in L^2 \mid z(t) = \sum_{k \in \mathbb{Z}} \exp(k\pi t J)a_k, \ a_k \in L_0, \ |z|^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)|a_k|^2 < \infty \}.$$
We denote its inner product by $\langle \cdot , \cdot \rangle$. By the well-known Sobolev embedding theorem, for any $s \in [1, +\infty)$, there is a constant $C_s > 0$ such that

$$\|z\|_{L^s} \leq C_s \|z\|, \ \forall z \in \mathcal{W}_{L_0}.$$ 

Define a functional $f_K$ on $E$ by

$$f_K(z) = \int_0^1 \left( \frac{1}{2} \dot{z} \cdot Jz - \dot{H}_K(z) \right) dt, \ \forall z \in E. \quad (3.4)$$

For $m \in \mathbb{N}$, define $E^0 = L_0$,

$$E_m = \{ z \in E \mid z(t) = \sum_{k=-m}^{m} \exp(k \pi t) a_k, \ a_k \in L_0 \},$$

$$E^\pm = \{ z \in E \mid z(t) = \sum_{\pm k > 0} \exp(k \pi t) a_k, \ a_k \in L_0 \},$$

and $E^+_m = E_m \cap E^+$, $E^-_m = E_m \cap E^-$. We have $E_m = E^-_m \oplus E^0 \oplus E^+_m$. Let $P_m$ be the projection $P_m : E \to E_m$. Then $\{E_m, P_m\}_{m \in \mathbb{N}}$ form a Galerkin approximation scheme of the operator $-Jd/dt$ on $E$. Denote by $f_{K,m} = f_K|_{E_m}$. Set $Q_m = \{ r_0 \leq r \leq r_1 \} \oplus \{ B_{r_1}(0) \cap (E^-_m \oplus E^0_m) \}$ with some $c \in \partial B_1(0) \cap E^+_m$. Then for large $r_1 > 0$ and small $\rho > 0$, $\partial Q_m$ and $B_{\rho}(0) \cap E^+_m$ form a topological (in fact homologically) link (cf. P84 of [2]). By the condition $\|B\| < \pi$, we obtain a constant $\beta = \beta(K) > 0$ such that

I) $f_{K,m}(z) \geq \beta > 0$, $\forall z \in \partial B_{2\rho}(0) \cap E^+_m$,

II) $f_{K,m}(z) \leq 0$, $\forall z \in \partial Q_m$.

In fact, by (H2), for any $\varepsilon > 0$, there is a $\delta > 0$ such that $H_K(z) \leq \varepsilon |z|^2$ if $|z| \leq \delta$. Since $\dot{H}_K(z)|z|^{-4}$ is uniformly bounded as $|z| \to +\infty$, there is an $M_1 = M_1(K)$ such that $\dot{H}_K(z) \leq M_1|z|^4$ for $|z| \geq \delta$. Hence

$$\dot{H}_K(z) \leq \varepsilon |z|^2 + M_1 |z|^4, \ \forall z \in \mathbb{R}^{2n}.$$ 

For $z \in \partial B_{\rho}(0) \cap E^+_m$, we have

$$\int_0^1 H_K(t, z) dt \leq \varepsilon \|z\|_{L^2}^2 + M_1 \|z\|_{L^4}^4 \leq (\varepsilon C_2^2 + M_1 C_4^4 \|z\|^2) \|z\|^2.$$ 

So we have

$$f_{K,m}(z) = \frac{1}{2} \langle Az, z \rangle - \frac{1}{2} \langle Bz, z \rangle - \int_0^1 H_K(z(t)) dt$$

$$\geq \frac{\pi}{2} \|z\|^2 - \frac{\|B\|}{2} \|z\|^2 - (\varepsilon C_2^2 + M_1 C_4^4 \|z\|^2) \|z\|^2$$

$$= \frac{\pi}{2} \rho^2 - \frac{\|B\|}{2} \rho^2 - (\varepsilon C_2^2 + M_1 C_4^4 \rho^2) \rho^2.$$
Since $\|B\| < \pi$, we can choose constants $\rho = \rho(K) > 0$ and $\beta = \beta(K) > 0$, which are sufficiently small and independent of $m$, such that for $z \in \partial B_\rho(0) \cap E_m^+$,\[f_{K,m}(z) \geq \beta > 0.\]

Hence (I) holds.

Let $e \in E_m^+ \cap \partial B_1$ and $z = z^- + z^0 \in E_m^- \oplus E^0$. We have
\[
f_{K,m}(z + re) = \frac{1}{2} \langle Az^-, z^- \rangle + \frac{1}{2} r^2 \langle Ae, e \rangle - \frac{1}{2} \langle B(z + re), z + re \rangle - \int_0^1 \dot{H}_K(z + re) dt \]
\[\leq -\frac{\pi}{2} \|z^-\|^2 + \frac{\pi}{2} r^2 - \int_0^1 \dot{H}_K(z + re) dt,
\]
If $r = 0$, from condition (H4), there holds
\[f_{K,m}(z + re) \leq -\frac{\pi}{2} \|z^-\|^2 \leq 0.
\]
If $r = r_1$ or $\|z\| = r_1$, then from (H2), we have
\[H_K(z) \geq b_1 |z|^\mu - b_2,
\]
where $b_1 > 0$, $b_2$ are two constants independent of $K$ and $m$. Then there holds
\[
\int_0^1 \dot{H}_K(z + re) dt \geq b_1 \int_0^1 |z + re|^\mu dt - b_2
\]
\[\geq b_3 \left( \int_0^1 |z + re|^2 dt \right)^{\frac{\mu}{2}} - b_4
\]
\[\geq b_5 (\|z^0\|^\mu + r^\mu) - b_4,
\]
where $b_3$, $b_4$ are constants and $b_5 > 0$ independent of $K$ and $m$. Thus there holds
\[f_{K,m}(z + re) \leq -\frac{\pi}{2} \|z^-\|^2 + \frac{\pi}{2} r^2 - b_5 (\|z^0\|^\mu + r^\mu) + b_4,
\]
So we can choose large enough $r_1$ independent of $K$ and $m$ such that
\[\varphi_m(z + re) \leq 0, \quad \text{on} \ \partial Q_m.
\]
Then (II) holds.

Now define $\Omega = \{ \Phi \in C(Q_m, E_m) \mid \Phi(x) = x \ \text{for} \ x \in \partial Q_m \}$, and set
\[c_{K,m} = \inf_{\Phi \in \Omega} \sup_{x \in \Phi(Q_m)} f_{K,m}(x).
\]

It is well known that $f_K$ satisfies the usual (P.S)* condition on $E$, i.e. a sequence $\{x_m\}$ with $x_m \in E_m$ possesses a convergent subsequence in $E$, provided $f'_K(x_m) \to 0$ as $m \to \infty$ and $|f_{K,m}(x_m)| \leq b$ for some $b > 0$ and all $m \in \mathbb{N}$.
(see [11] for a proof). Thus by the saddle point theorem (cf. [25]), we see that
c_{K,m} \geq \beta > 0 is a critical value of \( f_{K,m} \), we denote the corresponding critical point
by \( x_{K,m} \). The Morse index of \( x_{K,m} \) satisfies

\[
m^-(x_{K,m}) \leq \dim Q_m = mn + n + 1.
\]

By taking \( m \to +\infty \), we obtain a critical point \( x_K \) such that \( x_{K,m} \to x_K \), \( m \to +\infty \)
and \( m^-_d(x_K) \leq \dim Q_m = 1 + n + mn \), \( 0 < c_K \equiv f_K(x_K) \leq M_1 \), where the \( d \)-Morse
index \( m^-_d(x_K) \) is defined to the total number of the eigenvalues of \( f''_K \) belonging to
\((-\infty, d]\) for \( d > 0 \) small enough, and \( M_1 \) is a constant independent of \( K \). Moreover,
by the Galerkin approximation method, Theorem 2.1 of [14], we have the \( d \)-Morse
index satisfying

\[
m^-_d(x_K) = mn + n + i_{L_0}(x_K, 1) \leq 1 + n + mn.
\]

Thus we have

\[
i_{L_0}(x_K, 1) \leq 1.
\]

Now the similar arguments as in the section 6 of [25] yields a constant \( M_2 \) in-
dependent of \( K \) such that \( \| x_K \|_\infty \leq M_2 \). Choose \( K > M_2 \). Then \( x \equiv x_K \) is
a non-constant solution of the problem (3.2) satisfying (3.3). By extending the
domain with mirror symmetry of \( L_0 \), we obtain a 2-periodic brake orbit \((2, x)\) of
problem (3.1).

**Step 2.** Estimate the brake period of \((2, x)\).

Denote the minimal period of the brake orbit \( x \) by \( 2/k \) for some \( k \in \mathbb{N} \), i.e.,
\((x, 1/k)\) is a solution of the problem (3.2). By the condition (HX) and \( B \) being
semi-positive definite, using (9.17) of [4], we have that \( i_1(x, 2/k) \geq n \) for every \( 2/k-
periodic solution \((x, 2/k)\) (see also (4.2) in the appendix below), and by Theorem
5.2 of [13] we see that \( i_{L_0}(x, 1/k) \geq 0 \) for the \( L_0 \)-solution \((x, 1/k)\) (see also (4.3)
in the appendix below). Together with (3.21) of [16] (see also (4.14) in the appendix below),
we obtain

\[
i_{L_0}(x, 1/k) \geq i_{L_0}(x, 1/k) \geq 0, \quad i_1(x, 2/k) \geq n.
\]

Since the system (3.1) is autonomous, we have

\[
\nu_1(x, 2/k) \geq 1.
\]

Therefore, by Theorem 2.6, (3.3) and (3.5)-(3.6), we obtain \( k = 1, 2, 3, 4 \).
If \( k = 3 \), by (3.3) and (3.5)-(3.6), and by using Theorem 2.6 again we find
the left equality of (2.7) holds for \( k = 3 \) and \( i_{L_0}(x, 1/3) = 0, i_1(x, 2/3) = n \), and
\( \nu_1(x, 2/3) = 1 \).

The left side hand equality in the inequality (2.7) holds if and only if \( I_{2p} \circ N_1(1, -1)^{q} \circ K \in \Omega_0(\gamma(2/3)) \) for some non-negative integers \( p \) and \( q \) satisfying
\( p + q \leq n \) and some \( K \in \operatorname{Sp}(2(n-p-q)) \) satisfying \( \sigma(K) \subset U \setminus \mathbb{R} \). If \( r = n-p-q > 0 \), then by List 12 in P198 of [21] (see also the list after Definition 4.4
in the appendix below), we have \( R(\theta_{r_i}) \circ \cdots \circ R(\theta_{r}) \in \Omega_0(K) \) for some \( \theta_j \in (0, \pi) \). 
In this case, all eigenvalues of \( K \) on \( U^+ \) (on \( U^- \)) are located on the arc between
1 and \( \exp(2\pi \sqrt{-1}/k) \) (and \( \exp(-2\pi \sqrt{-1}/k) \)) on \( U^+ \) (in \( U^- \)) and are all Krein negative (positive) definite. We remind that \( \gamma(t) \) is the fundamental solution of the linearized system at \( (2/3, x) \). By the condition \( \nu_1(x, 2/3) = 1 \), we have \( p = 0, q = 1 \).

By Lemma 4.3 in the appendix below, there are paths \( \alpha \in \mathcal{P}_{2/3}(2), \beta \in \mathcal{P}_{2/3}(2n-2) \) such that \( \gamma \sim \alpha \circ \beta, \alpha(2/3) = N_1(1, -1), \beta(\tau) = K. \)

By the locations of the end point matrix \( \alpha(2/3) \) and \( \beta(2/3) \), there are two integers \( k_1, k_2 \) such that (see the proof of Theorem 4.3 in [15], specially (4.18) and (4.19) there).

\[
i_1(\alpha, 2/3) = 2k_1, \; i_1(\beta, 2/3) = 2k_2 + n - 1.
\]

From this result, we see that if \( n = 1 \), then \( N_1(1, -1) \in \Omega_0(\gamma(2/3)) \), and \( i_1(x, 2/3) \) must be even, so \( i_1(x, 2/3) = n = 1 \) is impossible. If \( n > 1 \), we have \( n - 1 > 0 \) and

\[
i_1(x, 2/3) = 2(k_1 + k_2) + n - 1.
\]

But \( i_1(x, 2/3) = n \), so \( k_1 + k_2 = \frac{1}{2} \). It is also impossible.

If \( k = 4 \), the solution \( (1/2, x) \) itself is a brake orbit. Thus \( i_1(x, 1/2) \) and \( i_1(x, 1) \) are well defined and by Theorem 2.6, we have that the left hand side equality in (2.8) holds for \( k = 4 \) and

\[
i_1(x, 1/2) = n, \; \nu_1(x, 1/2) = 1, \; i_{L_0}(x, 1/4) = i_{L_0}^{L_0}(x, 1/4) = 0.
\]

By the same arguments as above, we still get \( i_1(x, 1/2) = 2(k_1 + k_2) + n - 1 \). This is also impossible.

\[\blacksquare\]

Remark 3.2. If \( B = 0 \), the results of Theorem 3.1 hold for every \( \tau > 0 \). The following condition is more accessible than (HX) but it implies the condition (HX).

(H6) \( H''(x) \geq 0 \) for all \( x \in \mathbb{R}^{2n} \), the set \( D = \{ x \in \mathbb{R}^{2n} | H''(x) \neq 0, \; 0 \in \sigma(H''(x)) \} \) is hereditarily disconnected, i.e. every connected component of \( D \) contains only one point.

Similarly, we consider the brake orbit minimal periodic problem for the following autonomous second order Hamiltonian system

\[
\begin{aligned}
\dot{x} + V'(x) & = 0, \quad x \in \mathbb{R}^n, \\
x(0) & = x(\tau/2) = 0 \\
x(\tau/2 + t) & = -x(\tau/2 - t), \; x(\tau + t) = x(t).
\end{aligned}
\] (3.7)

A solution \( (\tau, x) \) of (3.7) is a kind of brake orbit for the second order Hamiltonian system.

In this paper, we consider the following conditions on \( V \):

(V1) \( V \in C^2(\mathbb{R}^n, \mathbb{R}). \)

(V2) There exist constants \( \mu > 2 \) and \( r_0 > 0 \) such that

\[
0 < \mu V(x) \leq V'(x) \cdot x, \quad \forall |x| \geq r_0.
\]

(V3) \( V(x) \geq V(0) = 0 \; \forall x \in \mathbb{R}^n. \)

(V4) \( V(x) = o(|x|^2) \), at \( x = 0. \)

(V5) \( V(-x) = V(x), \; \forall x \in \mathbb{R}^n. \)

(V6) \( V''(x) > 0, \; \forall x \in \mathbb{R}. \)
Theorem 3.3. Suppose $V$ satisfies the conditions (V1)-(V6). Then for every $\tau > 0$, the problem (3.7) possesses a non-constant solution $(\tau, x)$ such that the minimal period of $x$ is $\tau$ or $\tau/2$.

Proof. Without loss generality, we suppose $\tau = 2$. We define a Hilbert space $W$ which is a subspace of $W^{1,2}([0, 1], \mathbb{R}^n)$ by

$$W = \{ x \in W^{1,2}([0, 1], \mathbb{R}^n) | x(t) = \sum_{k=1}^{\infty} \sin k \pi t \cdot a_k, a_k \in \mathbb{R}^n \}.$$ 

The inner product of $W$ is still the $W^{1,2}$ inner product.

We consider the following functional

$$\psi(x) = \int_0^1 \left( \frac{1}{2} |\dot{x}|^2 - V(x) \right) dt, \quad \forall x \in W. \tag{3.8}$$

A critical point $x$ of $\psi$ is a solution of the problem (3.7) by extending the domain to $\mathbb{R}$ via $x(1 + t) = -x(1 - t)$ and $x(2 + t) = x(t)$. The condition (V3) implies $\psi(0) = 0$. The condition (V4) implies $\psi(\partial B_\rho(0)) \geq \alpha_0$ with $\partial B_\rho(0) = \{ x \in W | \|x\| = \rho \}$ for some small $\rho > 0$ and $\alpha_0 > 0$. In fact, there exists a constant $c_1 > 0$ such that

$$\int_0^1 |\dot{x}|^2 dt \geq c_1 \|x\|_{W}^2. \tag{3.9}$$

If $\|x\|_W \to 0$, then $\|x\|_\infty \to 0$. So by condition (V4), for any $0 < \varepsilon < \frac{c_1}{2}$, there exists small $\rho > 0$ such that

$$\int_0^1 V(x(t)) dt \leq \varepsilon \|x\|_2^2 \leq \varepsilon \|x\|_{W}^2, \quad \|x\|_W = \rho.$$ 

Thus we have

$$\psi(x) = \int_0^1 \left( \frac{1}{2} |\dot{x}|^2 - V(x) \right) dt \geq \left( \frac{c_1}{2} - \varepsilon \right) \rho^2 := \alpha_0 > 0.$$

The condition (V2) implies that there exists an element $x_0 \in W$ with $\|x_0\| > \rho$, such that $\psi(x_0) < 0$. In fact, we take an element $e \in W$ with $\|e\| = 1$ and by (V3) we assume $\int_0^1 V(e(t)) dt > 0$. Consider $x = \lambda e$ for $\lambda > 0$. Condition (V2) implies that there is a constant $c_2 > 0$ such that $V(\lambda e) \geq \lambda^\mu V(e) - c_2$ for $\lambda$ large enough, and there holds

$$\psi(\lambda e) \leq \lambda^2 \int_0^1 \frac{1}{2} |\dot{e}|^2 dt - \lambda^\mu \int_0^1 V(e(t)) dt + c_2 < 0.$$ 

Then we take $x_0 = \lambda e$ for large $\lambda$ such that the above inequalities holds.

We define

$$\Gamma = \{ h \in C([0, 1], W) | h(0) = 0, \ h(1) = x_0 \}$$
and

\[ c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} \psi(h(s)). \]

By using the Mountain pass theorem (cf. Theorem 2.2 of [25]), from the conditions (V2)-(V4) it is well known that there exists a critical point \( x \in W \) of \( \psi \) with critical value \( c > 0 \) which is a Mountain pass point such that its Morse index satisfying \( m^-(x, 1) \leq 1 \). If we set \( y = \dot{x} \) and \( z = (x, y) \in \mathbb{R}^{2n} \), the problem (3.7) can be transformed into the following problem

\[
\begin{cases}
\dot{z} = -JH'(z), \\
z(0) \in L_0, \quad z(1) \in L_0
\end{cases}
\]

with \( H(z) = H(x, y) = \frac{1}{2}|y|^2 + V(x) \). We note that (V5) implies \( H(Nz) = H(z) \), so \((2, z)\) is a brake orbit with brake period 2. We remind that in this case the complex structure is \(-J\), but it does not cause any difficult to apply the index theory. By Theorem 5.1 of [13], the Morse index \( m^-(x, 1) \) of \( x \) is just the \( L_0 \)-index \( i_{L_0}(z, 1) \) of \((1, z)\). i.e., there holds(see also Lemma 4.6 in the appendix below)

\[ m^-(x, 1) = i_{L_0}(z, 1), \quad m^0(x, 1) = \nu_{L_0}(z, 1). \]

We can suppose the minimal period of \( x \) is \( 2/k \) for \( k \in \mathbb{N} \). But \( i_{L_0}(z, 1/k) = m^-(x, 1/k) \geq 0 \), and from the convexity condition (V6), we have \( i_1(z, 2/k) \geq n \). With the same arguments as in the proof of Theorem 3.1, we get \( k \in \{1, 2\} \). \( \blacksquare \)

We note that the functional \( \psi \) is even, there may be infinite many solutions \((\tau, x)\) satisfying Theorem 3.3. We also note that Theorem 3.3 is not a special case of Theorem 3.1, since the Hamiltonian function \( H(x, y) = \frac{1}{2}|y|^2 + V(x) \) is quadratic in the variables \( y \), in this case \( B = \left( \begin{array}{cc} 0 & 0 \\ 0 & I_n \end{array} \right) \) with \( \|B\| = 1 \). Thus when applying Theorem 3.1 to this case, we can only get the result of Theorem 3.3 for \( 0 < \tau < 2\pi \).

We now consider the following problem

\[
\begin{cases}
\ddot{x} + V'(x) = 0, & x \in \mathbb{R}^n, \\
\dot{x}(0) = \dot{x}(\tau/2) = 0, \\
x(\tau/2 + t) = x(\tau/2 - t), \quad x(\tau + t) = x(t).
\end{cases}
\]

A solution of (3.10) is also a kind of brake orbit for the second order Hamiltonian system.

By set \( y = \dot{x}, \quad z = (y, x) \) and \( H(z) = H(y, x) = \frac{1}{2}|y|^2 + V(x) \), the problem (3.10) can be transformed into the following \( L_0 \)-boundary value problem

\[
\begin{cases}
\dot{z} = JH'(z) \\
z(0) \in L_0, \quad z(\tau/2) \in L_0.
\end{cases}
\]

In this case the condition \( H(Nz) = H(z) \) is satisfied automatically. Set \( B = \left( \begin{array}{cc} I_n & 0 \\ 0 & 0 \end{array} \right) \), then \( \|B\| = 1 \). The following result is a direct consequence of Theorem 3.1.
Corollary 3.4. Suppose $V$ satisfies the conditions (V1)-(V4) and (V6). Then for every $0 < \tau < 2\pi$, the problem (3.10) possesses a non-constant solution $(\tau, x)$ such that $x$ has minimal period $\tau$ or $\tau/2$.

We note that if we directly solve the problem (3.10) by the same way as in the proof of Theorem 3.3, the formation of the functional is still $\psi$ as defined in (3.8), but the domain should be

$$W_1 = \{ x \in W^{1,2}([0,1], \mathbb{R}^n) | x(t) = \sum_{k=0}^{\infty} \cos k\pi t \cdot a_k, \ a_k \in \mathbb{R}^n \}.$$ 

In this time, it is not able to apply the Mountain pass theorem to get a critical point directly due to the fact $\mathbb{R}^n \subset W_1$, so the inequality (3.9) is not true.

§4 Appendix. Some properties for the indeces

4.1. Some properties of Maslov-type index. For a symplectic path $\gamma \in \mathcal{P}(2n)$, its Maslov-type index is a pair of integers $(i_1(\gamma), \nu_1(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\}$ (cf. [20],[21]). If $\gamma \in \mathcal{P}(2n)$ is the fundamental solution of a linear Hamiltonian system

$$\dot{x} = JB(t)x$$

with continuous symmetric matrix function $B(t)$, its Maslov-type index usually denoted by $(i_1(B), \nu_1(B))$. The following result was proved in [12].

Lemma 4.1. If $B_1(t) - B_2(t) > 0$ is a positive definite matrix function, then there holds

$$i_1(B_1) \geq i_1(B_2) + \nu_1(B_2). \quad (4.1)$$

(4.1) also holds under the following condition

$$B(t) = B_1(t) - B_2(t) \geq 0, \ \int_0^1 B(t)dt > 0.$$ 

As a direct consequence, if the continuous symmetric matrix function satisfying $B(t) \geq 0$ and $\int_0^1 B(t)dt > 0$, then there holds

$$i_1(B) \geq n. \quad (4.2)$$

Definition 4.2. ([20],[21]) Two symplectic paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}(2n)$ are homotopic on $[0,1]$, denoted by $\gamma_0 \sim \gamma_1$, if there exists a map $\delta \in C([0,1] \times [0,1], \text{Sp}(2n))$ such that $\delta(0, \cdot) = \gamma_0(\cdot)$, $\delta(1, \cdot) = \gamma_1(\cdot)$, $\delta(s, \cdot) = I_{2n}$, and $\nu_1(\delta(s, 1))$ is constant for $0 \leq s \leq 1$.

We note that for two paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}(2n)$ with the same end points $\gamma_0(1) = \gamma_1(1)$, $\gamma_0 \sim \gamma_1$ with fixed end points if and only if $i_1(\gamma_0) = i_1(\gamma_1)$. By choosing suitable zigzag standard paths $\alpha_{n,k}$ in $\mathcal{P}^*(2n)$ with $i_1(\alpha_{n,k}) = k$ and $\alpha_{n,k}(1) = M_n^\pm$ if $(-1)^k = \pm 1$ as in [22], and by the definition of the Maslov-type index, we have the following result.
Lemma 4.3. For a symplectic path $\gamma \in \mathcal{P}(2n)$ with $\gamma(1) = M_1 \circ M_2$, $M_j \in \text{Sp}(2n)$, $j = 1, 2$, $n_1 + n_2 = n$, there exists two symplectic paths $\gamma_j \in \mathcal{P}(2n_j)$ such that $\gamma \sim \gamma_1 \circ \gamma_2$ and $\gamma_j(1) = M_j$.

The index function $(i_\omega(\gamma), \nu_\omega(\gamma))$ was defined for $\omega \in \mathbb{U} := \{ z \in \mathbb{C} | |z| = 1 \}$ in [22] by Y.Long.

Definition 4.4. ([22]) For any $M \in \text{Sp}(2n)$ and $\omega \in \mathbb{U}$, choosing $\gamma \in \mathcal{P}(2n)$ with $\gamma(1) = M$, the splitting numbers of $M$ are defined by

\[ S_M^+(\omega) = \lim_{\epsilon \to 0^+} i_{\exp(\pm \epsilon \sqrt{-1})\omega}(\gamma) - i_\omega(\gamma). \]

The following list for the splitting number comes from [21].

1. $(S_M^+(1), S_M^-(1)) = (1, 1)$ for $M = N_1(1, b)$ with $b = 1$ or 0.
2. $(S_M^+(1), S_M^-(1)) = (0, 0)$ for $M = N_1(1, -1)$.
3. $(S_M^+(1), S_M^-(1)) = (1, 1)$ for $M = N_1(-1, b)$ with $b = 0$ or 0.
4. $(S_M^+(1), S_M^-(1)) = (0, 0)$ for $M = N_1(-1, 1)$.
5. $(S_M^+(e^{-\sqrt{-1}b}), S_M^-(e^{-\sqrt{-1}b})) = (0, 1)$ for $M = R(\theta)$ with $\theta \in (0, \pi) \cup (\pi, 2\pi)$.
6. $(S_M^+(\omega), S_M^-(\omega)) = (1, 1)$ for $M = N_2(\omega, b)$ being non-trivial with $\omega = e^{\sqrt{-1}\theta} \in \mathbb{U} \setminus \mathbb{R}$.
7. $(S_M^+(\omega), S_M^-(\omega)) = (0, 0)$ for $M = N_2(\omega, b)$ being trivial with $\omega = e^{\sqrt{-1}\theta} \in \mathbb{U} \setminus \mathbb{R}$.
8. $(S_M^+(\omega), S_M^-(\omega)) = (0, 0)$ for any $\omega \in \mathbb{U}$ and $M \in \text{Sp}(2n)$ satisfying $\sigma(M) \cap \mathbb{U} = \emptyset$.

4.2. Some properties of the $L_0$-index. For a symplectic path $\gamma \in \mathcal{P}(2n)$, the so called $L_0$-index $(i_{L_0}(\gamma), \nu_{L_0}(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, n\}$ was first defined in [13]. We have a brief introduction of this index theory in the section 2 of this paper. The following result was proved in [13].

Lemma 4.5. Suppose $\gamma \in \mathcal{P}(2n)$ is the fundamental solution of the following linear Hamiltonian system

\[ \dot{x}(t) = JB(t)x(t), \quad x(t) \in \mathbb{R}^{2n}, \]

where $B(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}$ is symmetric with $n \times n$ blocks $S_{jk}$. If $S_{22}(t) > 0$ (positive definite), there holds

\[ i_{L_0}(\gamma) = i_{L_0}(B) \geq 0. \tag{4.3} \]

(4.3) is also true if $S_{22}(t) \geq 0$ and $\int_0^1 S_{22}(t) dt > 0$.

We consider the following problem

\[ \begin{cases} 
[P(t)x'(t) - Q(t)x(t)]' + Q^T(t)x'(t) + R(t)x = 0, \\
x(0) = x(1) = 0, 
\end{cases} \tag{4.4} \]

where $P$ and $R$ are symmetric $n \times n$ matrix functions, we suppose $-P > 0$ (positive definite). For simplicity, We assume $P, Q$ are smooth and $R$ is continuous. The equations in (4.4) was studied by M.Morse. We turn it into a first order equations
with Lagrangian boundary condition by setting \( z(t) = (x(t), y(t))^T \in \mathbb{R}^{2n} \) with \( y = P(t)x'(t) - Q(t)x(t) \):

\[
\begin{aligned}
    \dot{z} &= JB(t)z \\
    z(0), \ z(1) &\in L_0,
\end{aligned}
\]  

(4.5)

where \( B = B(t) \) is defined by

\[
    B(t) = \begin{pmatrix} -R(t) - Q^T(t)P^{-1}(t)Q(t) & -Q^T(t)P^{-1}(t) \\ -P^{-1}(t)Q(t) & -P^{-1}(t) \end{pmatrix}.
\]

We take the space \( W = W^{1,2}_0([0,1], \mathbb{R}^n) \), the subspace of \( W^{1,2}([0,1], \mathbb{R}^n) \) with the elements \( x \) satisfying \( x(0) = x(1) = 0 \). Define the following functional on \( W \)

\[
    \varphi(x) = -\frac{1}{2} \int_0^1 \langle P^{-1}(t)(P(t)x'(t) - Q(t)x(t)), P(t)x'(t) - Q(t)x(t) \rangle \\
    - \langle (R(t) + Q^T(t)P^{-1}(t)Q(t))x(t), x(t) \rangle \, dt.
\]

The critical point of \( \varphi \) is a solution of the problem (4.4), and so we get a solution of the problem (4.5). Denote the Morse index of the functional \( \varphi \) at \( x = 0 \) by \( m^{L_0}(B) \), which is the total multiplicity of the negative eigenvalues of the Hessian of \( \varphi \) at \( x = 0 \), and the nullity by \( n^{L_0}(B) \). The following result was proved in [13].

**Lemma 4.6.** There holds

\[ i_{L_0}(B) = m^{L_0}(B), \quad \nu_{L_0}(B) = n^{L_0}(B). \]

Let \( E \) be a separable Hilbert space, and \( Q = A - B : E \to E \) be a bounded self-adjoint linear operators with \( B : E \to E \) a compact self-adjoint operator. \( N = \ker Q \) and \( \dim N < +\infty \). \( Q|_{N^\perp} \) is invertible. \( P : E \to N \) the orthogonal projection. Set \( d = \frac{1}{4}\| (Q|_{N^\perp})^{-1} \|^{-1} \). \( \Gamma = \{ P_k | k = 1, 2, \ldots \} \) be the Galerkin approximation sequence of \( A \):

1. \( E_k := P_k E \) is finite dimensional for all \( k \in \mathbb{N} \),
2. \( P_k \to I \) strongly as \( k \to +\infty \)
3. \( P_k A = AP_k \).

For an operator \( S \), we denote by \( M^*(S) \) the eigenspaces of \( S \) with eigenvalues belonging to \((0, +\infty)\), \( \{0\} \) and \((-\infty, 0) \) with \( * = +, 0 \) and \( * = - \), respectively. We denote by \( m^*(S) = \dim M^*(S) \). Similarly, we denote by \( M^d_*(S) \) the \( d \)-eigenspaces of \( S \) with eigenvalues belonging to \((d, +\infty)\), \((-d, d) \) and \((-\infty, -d) \) with \( * = +, 0 \) and \( * = - \), respectively. We denote by \( m^*_d(S) = \dim M^*_d(S) \).

**Lemma 4.7.** (Lemma 3.3 in [16]) Let \( B \) be a linear symmetric compact operator. Then the difference of the \( d \)-Morse indices

\[
m_d^-(P_m(A - B)P_m) - m_d^-(P_mAP_m)
\]

eventually becomes a constant independent of \( m \), where \( d > 0 \) is determined by the operators \( A \) and \( A - B \). Moreover \( m_d^0(P_m(A - B)P_m) \) eventually becomes a constant independent of \( m \) and for large \( m \), there holds

\[
m_d^0(P_m(A - B)P_m) = m_0^0(A - B).
\]
Definition 4.8. ([16]) For the operators $A$ and $B$ in Lemma 4.7, $\Gamma$ is an Galerkin approximation sequence w.r.t $A$, we define the relative index by

$$I(A, A - B) = m_d(P_m(A - B)P_m) - m_d(P_mA_P_m), m \geq m^*, \quad (4.8)$$

where $m^* > 0$ is a constant large enough such that the difference in (4.6) becomes a constant independent of $m \geq m^*$.

For $\omega = e^{\sqrt{-1} \theta}$, we define a Hilbert space $E^\omega = E^\omega_{L_0}$ consisting of those $x(t)$ in $L^2([0, 1], \mathbb{C}^n)$ such that $e^{-\theta t J} x(t)$ has Fourier series

$$e^{-\theta t J} x(t) = \sum_{j \in \mathbb{Z}} e^{j \pi t J} \left( \begin{array}{c} 0 \\ a_j \end{array} \right), \quad a_j \in \mathbb{C}^n$$

and

$$\|x\|^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)|a_k|^2 < \infty.$$ 

For $x \in E^\omega$, we can write

$$x(t) = e^{\theta t J} \sum_{j \in \mathbb{Z}} e^{j \pi t J} \left( \begin{array}{c} 0 \\ a_j \end{array} \right) = \sum_{j \in \mathbb{Z}} e^{(\theta + j \pi) t \sqrt{-1}} \left( \begin{array}{c} \sqrt{-1} a_j/2 \\ a_j/2 \end{array} \right) + e^{-(\theta + j \pi) t \sqrt{-1}} \left( \begin{array}{c} -\sqrt{-1} a_j/2 \\ a_j/2 \end{array} \right).$$

So we can write

$$x(t) = \xi(t) + N\xi(-t), \quad \xi(t) = \sum_{j \in \mathbb{Z}} e^{(\theta + j \pi) t \sqrt{-1}} \left( \begin{array}{c} \sqrt{-1} a_j/2 \\ a_j/2 \end{array} \right). \quad (4.9)$$

For $\omega = 1$, i.e., $\theta = 0$, we define two self-adjoint operators $A_1, B_1 \in \mathcal{L}(E^1)$ by extending the bilinear forms

$$\langle A_1 x, y \rangle = \int_0^1 (-J \dot{x}(t), y(t)) dt, \quad \langle B_1 x, y \rangle = \int_0^1 (B(t)x, y) dt$$

on $E^1$, here $(\cdot, \cdot)$ is the Hermitian inner product in $\mathbb{C}^n$. Then $B$ is compact. For $\omega = e^{\sqrt{-1} \theta}$, $\theta \in [0, \pi)$, we define two self-adjoint operators $A^\omega, B^\omega \in \mathcal{L}(E^\omega)$ by extending the bilinear forms

$$\langle A^\omega x, y \rangle = \int_0^1 (-J \dot{x}(t), y(t)) dt,$$

$$\langle B^\omega x, y \rangle = \int_0^1 (B(t)x(t), y(t)) dt$$

on $E^\omega$, where we have written $x(t) = \xi(t) + N\xi(-t)$, $y(t) = \eta(t) + N\eta(-t)$ as in (3.10). Then $B^\omega$ is also compact.

By Theorem 2.1 of [14], we have the following formula

$$I(A_1, A_1 - B_1) = i L_0(B) + n. \quad (4.10)$$
Definition 4.9. ([16]) We define the index function
\[ i_{\omega}^{L_0}(B) := I(A^\omega, A^\omega - B^\omega), \quad \nu^{L_0}_\omega(B) := \mu^0(A^\omega - B^\omega), \quad \omega = e^{\sqrt{-1}\theta}, \quad \theta \in (0, \pi). \]

Lemma 4.10. ([16]) The index function \( i_{\omega}^{L_0}(B) \) is locally constant. For \( \omega_0 = e^{\sqrt{-1}\theta_0}, \quad \theta_0 \in (0, \pi) \) is a point of discontinuity of \( i_{\omega}^{L_0}(B) \), then \( \nu^{L_0}_{\omega_0}(B) > 0 \) and so \( \dim(\gamma(1) L_0 \cap e^{0+} L_0) > 0 \). Moreover there hold
\[
\begin{aligned}
| i^{L_0}_{\omega_0^+}(B) - i^{L_0}_{\omega_0^-}(B) | &\leq \nu^{L_0}_{\omega_0}(B), \quad | i^{L_0}_{\omega_0^+}(B) - i^{L_0}_{\omega_0^-}(B) | \leq \nu^{L_0}_{\omega_0}(B), \\
| i^{L_0}_{\omega_0^-}(B) - i^{L_0}_{\omega_0^+}(B) | &\leq \nu^{L_0}_{\omega_0}(B), \quad | i_0(B) + n - i^{L_0}_{\omega_0^+}(B) | \leq \nu^{L_0}_0(B),
\end{aligned}
\]
(4.11)
where \( i^{L_0}_{\omega_0^+}(B), i^{L_0}_{\omega_0^-}(B) \) are the right and left limit respectively of the index function \( i^{L_0}_\omega(B) \) at \( \omega_0 = e^{\sqrt{-1}\theta_0} \) as a function of \( \theta \).

By (4.10), Definition 4.9 and Lemma 4.10, we see that for any \( \omega_0 = e^{\sqrt{-1}\theta_0}, \quad \theta_0 \in (0, \pi) \), there holds
\[ i^{L_0}_{\omega_0^-}(B) \geq i_0(B) + n - \sum_{\omega = e^{\sqrt{-1}\theta}, \quad 0 \leq \theta \leq \theta_0} \nu^{L_0}_\omega(B). \]  
(4.12)
We note that
\[ \sum_{\omega = e^{\sqrt{-1}\theta}, \quad 0 \leq \theta \leq \theta_0} \nu^{L_0}_\omega(B) \leq n. \]  
(4.13)
So we have
\[ i_0(B) \leq i^{L_0}_{\omega_0^-}(B) \leq i^{L_0}_0(B) + n. \]  
(4.14)

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