An Improved Bushell-Okrasinski Type Inequality for Sugeno Integrals

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Abstract

Recently, Roman-Flores et al. (2008) proposed a Bushell-Okrasinski type inequality for fuzzy integrals. In this paper, we improve the result of Roman-Flores et al. by finding the optimal constant $H_s$ for which the following Bushell-Okrasinski type inequality for fuzzy integrals

$$H_s \left( \int_0^1 (1-t)^{s-1} g(t) \, d\mu \right)^s \geq \left( \int_0^1 g(t) \, d\mu \right)^s$$

holds, where $s \geq 1$, $g : [0, 1] \to [0, \infty)$ is a non-increasing function. The case of nondecreasing function is also treated. The result of Roman-Flores et al. is a special case of our result.

Keywords: Fuzzy measure, Sugeno integral, Bushell-Okrasinski type inequality

1. Introduction and Preliminaries

A number of studies have examined the Sugeno integral since its introduction in 1974 [1]. Ralescu and Adams [2] generalized a range of fuzzy measures and gave several equivalent definitions of fuzzy integrals. Wang and Klir [3] provided an overview of fuzzy measure theory.

Caballero and Sadarangani [4–6] proved a Hermite-Hadamard type inequality, a Cauchy type inequality, and Fritz Carlson’s inequality for fuzzy integrals. Roman-Flores et al. [7–10] presented several new types of inequalities for Sugeno integrals, including a Prekopa-Leindler type inequality, a Jensen type inequality, a Young type inequality, and some convolution type inequalities. Flores-Franulic et al. [11, 12] presented Chebyshev’s inequality and Stolarsky’s inequality for fuzzy integrals. Ouyang and Fang [13] generalized their main results to prove some optimal upper bounds for the Sugeno integral of the monotone function in [9]. Ouyang et al. [14] generalized a Chebyshev type inequality for the fuzzy integral of monotone functions based on an arbitrary fuzzy measure. Hong [15] extended previous research by presenting a Hardy-type inequality for Sugeno integrals, Hong [16] proposed a Liapunov type inequality for Sugeno integrals, and Hong [17] proposed a Berward and Favard type inequalities for fuzzy integrals. Hong et al. [19] considered Steffensen’s integral inequality for Sugeno integral. Recently, Roman-Flores et al. [10] showed a Bushell-Okrasinski type inequality for fuzzy integrals, but this inequality is not optimal. In this paper, we improve the results of Roman-Flores et al. [10] for $s \geq 1$. Specifically, we find an optimal constant for which a Bushell-Okrasinski type inequality for Sugeno integrals holds for non-increasing functions.

Definition 1. Let $\Sigma$ be a $\sigma$-algebra of subsets of $\mathbb{R}$ and let $\mu : \Sigma \to [0, \infty]$ be a non-negative, extended real-valued set function. We say that $\mu$ is a fuzzy measure if and only if
(a) \( \mu(\emptyset) = 0 \).

(b) \( E, F \in \Sigma \) and \( E \subseteq F \) imply \( \mu(E) \leq \mu(F) \) (monotonicity).

(c) \( \{E_p\} \subseteq \Sigma \) and \( E_1 \subseteq E_2 \subseteq \cdots \) imply \( \lim_{p \to \infty} \mu(E_p) = \mu \left( \bigcup_{p=1}^{\infty} E_p \right) \) (continuity form below).

(d) \( \{E_p\} \subseteq \Sigma \), \( E_1 \supseteq E_2 \supseteq \cdots \), and \( \mu(E) < \infty \) imply \( \lim_{p \to \infty} \mu(E_p) = \mu \left( \bigcap_{p=1}^{\infty} E_p \right) \) (continuity form above).

If \( f \) is a non-negative real-valued function defined on \( \mathbb{R} \), then we denote by \( F \) the support of \( E \), and can be found in [3]:

\[
\begin{align*}
\text{if } f & \text{ is the support of } \mu, \text{ for } \alpha > 0, \text{ and } F_0 = \{ x \in X | f(x) > 0 \} = \text{supp}(f) \text{ is the support of } f \text{.} \\
\text{We note that } \alpha \leq \beta & \Rightarrow \{ \gamma \geq \beta \} \subseteq \{ \gamma \geq \alpha \}. \\
\end{align*}
\]

If \( \mu \) is a fuzzy measure on \( A \subseteq \mathbb{R} \), then we define the following:

\[
\mathcal{F}^\mu(A) = \{ f : A \to [0, \infty) | f \text{ is } \mu \text{-measurable} \}.
\]

**Definition 2.** Let \( \mu \) be a fuzzy measure on \( (\mathbb{R}, \Sigma) \). If \( f \in \mathcal{F}^\mu(\mathbb{R}) \) and \( A \in \Sigma \), then the Sugeno integral (or the fuzzy integral) of \( f \) on \( A \), with respect to the fuzzy measure \( \mu \), is defined as

\[
(S) \int_A f \, d\mu = \sup_{\alpha \in [0, \infty)} [ \alpha \land \mu(A \cap F_\alpha) ].
\]

In particular, if \( A = X \) then

\[
(S) \int_X f \, d\mu = (S) \int f \, d\mu = \sup_{\alpha \in [0, \infty)} [ \alpha \land \mu(F_\alpha) ].
\]

The following properties of the Sugeno integral are well known and can be found in [3]:

**Proposition 1** [3]. If \( \mu \) is a fuzzy measure on \( \mathbb{R} \) and \( f, g \in \mathcal{F}^\mu(\mathbb{R}) \), then

\[
\begin{align*}
(i) & \quad (S) \int_A f \, d\mu \leq \mu(A) ; \\
(ii) & \quad (S) \int_A K \, d\mu = K \land \mu(A) \text{ for any constant } K \in [0, \infty) ; \\
(iii) & \quad (S) \int_A f \, d\mu \leq (S) \int_A g \, d\mu, \text{ if } f \leq g \text{ on } A ; \\
(iv) & \quad \mu(A \cap \{ f \geq \alpha \}) \geq \alpha \Rightarrow (S) \int_A f \, d\mu \geq \alpha ; \\
(v) & \quad \mu(A \cap \{ f \geq \alpha \}) \leq \alpha \Rightarrow (S) \int_A f \, d\mu \leq \alpha ; \\
\end{align*}
\]

\[
\begin{align*}
(iii) & \quad (S) \int_A f \, d\mu < \alpha \Leftrightarrow \text{there exists } \gamma < \alpha \text{ such that } \\
& \quad \mu(A \cap \{ f \geq \gamma \}) < \alpha ; \\
\end{align*}
\]

\[
\begin{align*}
(ii) & \quad (S) \int_A f \, d\mu > \alpha \Leftrightarrow \text{there exists } \gamma > \alpha \text{ such that } \\
& \quad \mu(A \cap \{ f \geq \gamma \}) > \alpha .
\end{align*}
\]

**Note 1.** Let \( F(\alpha) = \mu(A \cap \{ f \geq \alpha \}) \). Then by Proposition 1, (iv), (v),

\[
F(\alpha) = \alpha \Leftrightarrow \int_0^1 f(x) \, d\mu = \alpha .
\]

**Theorem 1** [13]. Let \( f : [0, \infty) \to [0, \infty) \) be continuous and non-increasing or non-decreasing functions and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Let \( (S) \int_0^\alpha f(x) \, d\mu = p \). If \( 0 < p < a \), then \( f(p) = p \) and \( f(a - p) = p \), respectively.

**2. Bushell-Okrasinski Type Inequality**

The classical Bushell-Okrasinski inequality provides the following inequality [19]:

\[
\int_0^x (x - t)^{s-1} g(t)^s \, d\mu \leq \left( \int_0^x g(t) \, d\mu \right)^s, \quad 0 \leq x \leq b,
\]

where \( 1 \leq s, g : [0, 1] \to [0, \infty) \) is a continuous and increasing function. After changing the variable \( t = sx \), Malamud [20] analyzed the Bushell-Okrasinski inequality in the following new form:

\[
S \int_0^1 (1 - t)^s g(t)^s \, d\mu \leq \left( \int_0^1 g(t) \, d\mu \right)^s. \tag{1}
\]

Recently, Roman-Flores et al. [10] showed a Bushell-Okrasinski inequality derived from (1) for Sugeno integrals as follows:

**Theorem 2** (Roman-Flores et al. [10]). Let \( g : [0, 1] \to [0, \infty) \) be a continuous decreasing function and let \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then

\[
S \int_0^1 (1 - t)^s g(t)^s \, d\mu \geq \left( \int_0^1 g(t) \, d\mu \right)^s
\]

holds for all \( 2 \leq s \).

We now improve the above result.

**Theorem 3** (Fuzzy Bushell-Okrasinski inequality). Let \( g : [0, 1] \to [0, \infty) \) be a continuous non-increasing function and let \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then

\[
\left( \int_0^1 (1 - t)^s g(t)^s \, d\mu \right)^{-1} \left( \int_0^1 (1 - t)^s g(t)^s \, d\mu \right) \geq \left( \int_0^1 g(t) \, d\mu \right)^s.
\]
We consider

\[ \left( S \int_0^1 g(t) d\mu \right)^{1-s} \]

holds for all \( 1 \leq s \).

**Proof.** Let

\[ H_s = \sup \left\{ \frac{\left( S \int_0^1 g(t) d\mu \right)^{1-s}}{\left( S \int_0^1 (1-t)^{s-1} g(t) d\mu \right)^{1-s}} \mid g \in G \right\}, \]

where \( G \) is the set of functions which are non-increasing on \([0, 1]\) and let

\[ H_s(\alpha) = \sup \left\{ \frac{\alpha^s}{\left( S \int_0^1 (1-t)^{s-1} g(t) d\mu \right)^{1-s}} \mid g \in G_\alpha \right\}, \]

where \( G_\alpha = \left\{ g \in G \mid \left( S \int_0^1 g(t) d\mu \right) = \alpha \right\} \) for \( \alpha \in [0, 1] \). Then

\[ H_s = \sup_{0 < \alpha \leq 1} H_s(\alpha). \]

We consider \( H_s(\alpha) \). Let

\[ g_0(x) = \begin{cases} \alpha, & \text{if } x \in [0, \alpha), \\ 0, & \text{if } x \in [\alpha, 1]. \end{cases} \]

Because \( \mu\{g_0 \geq \alpha\} = \alpha \), by Note 1

\[ \left( S \int_0^1 g_0(x) d\mu \right) = \alpha. \]

Then it is easy to check that \( g_0 = \inf G_\alpha \). Thus, we have

\[ H_s(\alpha) = \frac{\alpha^s}{\left( S \int_0^1 (1-t)^{s-1} g_0(t) d\mu \right)^{1-s}}. \]

Now, let

\[ \left( S \int_0^1 (1-t)^{s-1} g_0(t) d\mu \right) = t_0. \]

Because \( g_0 \) is continuous and decreasing on \([0, \alpha]\) and the left limit of \( g_0 \) at \( \alpha \) is less than \( \alpha \), by Theorem 1,

\[ t_0 = (1-t_0)^{s-1} \alpha^s. \]

To find \( H_s \) we now consider the following optimization problem:

\[ H_s = \sup_{0 < \alpha \leq 1} H_s(\alpha) = \text{Maximize} \frac{\alpha^s}{t}, \quad (2) \]

where

\[ t = (1-t)^{s-1} \alpha^s, \quad 0 < \alpha \leq 1. \]

We first show that \( H_s(\alpha) = \frac{\alpha^s}{t} \) is an increasing function of \( \alpha \). We have

\[ H_s'(\alpha) = \frac{\alpha^{s-1}}{t^2} \left( st - \alpha \frac{dt}{d\alpha} \right). \]

From the equation \( t = (1-t)^{s-1} \alpha^s \), we now have

\[ sa^{s-1} = \frac{(1-t)^{s-1} + t(s-1)(1-t)^{s-2}}{(1-t)^{2s-2}} \frac{dt}{d\alpha} \]

which implies that

\[ \frac{dt}{d\alpha} = \frac{1}{1+t(s-1)(1-t)^{-1}} st \leq st. \]

Then \( H_s'(\alpha) \geq 0 \), that is, \( H_s \) is nondecreasing and thus

\[ H_s = H_s(1) = \frac{1}{\left( S \int_0^1 (1-t)^{s-1} d\mu \right)^{1-s}}, \]

which completes the proof. \( \square \)

**Note 2.** As shown in the proof of Theorem 3, the continuity assumption of \( g \) is not needed.

**Note 3.** If \( g(t) = 1 \) in the inequality of Theorem 3, then the equality holds. Therefore, we see that the inequality in Theorem 3 is optimal.

**Lemma 1.** Let \( (S \int_0^1 (1-t)^{s-1} d\mu = x^* \), \( s \geq 2 \). Then

\[ \frac{1}{s} \leq \frac{2(s-1)^{s-1}}{(s-1)^{s-1} + s^{s-2}} < x^*. \]

**Proof.** Suppose that \( f(t) = (1-t)^{s-1} - t \). Then by Theorem 1, \( f(x^*) = 0 \). We note that

\[ f'(t) = -(s-1)(1-t)^{s-2} - 1 < 0, \quad f'(0) = -s, \]

and

\[ f''(t) = (s-1)(s-2)(1-t)^{s-3} \geq 0. \]

Because \( f \) is decreasing and convex, we have

\[ \frac{1}{s} = -\frac{f(0)}{f'(0)} < x^*, \]
and similarly, we have
\[
\frac{1}{s} \leq \frac{1}{s} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} < x^s
\]
and
\[
\frac{1}{s} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} = \frac{1}{s} - \frac{(1 - \frac{1}{2})^{s-1} - \frac{1}{2}}{1 - (s - 1)(1 - \frac{1}{2})^{s-2}}
= \frac{1}{s} \left(\frac{2(s - 1)^{s-1}}{(s - 1)^{s-1} + s^{s-2}}\right),
\]
which completes the proof.

The following result of Roman-Flores et al. [10] is a special case of our results.

**Corollary 1** (Roman-Flores et al. [?]). Let \( g : [0, 1] \to [0, \infty) \) be a non-increasing function and let \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then
\[
s \left(\int_0^1 (1 - t)^{s-1} g(t) d\mu\right) \geq \left(\int_0^1 g(t) d\mu\right)^s
\]
holds for all \( 2 \leq s \).

**Corollary 2.** Let \( g : [0, 1] \to [0, \infty) \) be a non-increasing function and that \( \mu \) is the Lebesgue measure on \( \mathbb{R} \). Then
\[
s \left(\int_0^1 (1 - t)^{s-1} t^{s-2} g(t) d\mu\right) \geq \left(\int_0^1 t^{s-1} g(t) d\mu\right)^s
\]
holds for all \( 2 \leq s \).

**Example 1.** We compare our result with that of Roman-Flores et al. [10] for \( s = 2, 3, 4, 5, 10, 20 \). That is,
\[
\begin{array}{|c|c|}
\hline
s & \left(\int_0^1 (1 - t)^{s-1} d\mu\right)^{-1} \\
\hline
2 & 2.000 \\
3 & 2.618 \\
4 & 3.148 \\
5 & 3.630 \\
10 & 5.692 \\
20 & 9.110 \\
\hline
\end{array}
\]
Our results indicates much lower than that of Roman-Flores et al. [10].

The case of a non-increasing function is similar.

**Theorem 4.** Let \( g : [0, 1] \to [0, \infty) \) be a non-decreasing function and let \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then
\[
\left(\int_0^1 t^{s-1} d\mu\right)^{-1} \left(\int_0^1 t^{s-1} g(t) d\mu\right) \geq \left(\int_0^1 g(t) d\mu\right)^s
\]
holds for all \( 1 \geq s \).

**Conflict of Interest**

No potential conflict of interest relevant to this article was reported.

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