1. INTRODUCTION

Reconsidering chiral gauge theories on the lattice we generalize the basic structure which has been introduced in the overlap formalism of Narayanan and Neuberger [1] and in the formulation of Lüscher [2]. Only requiring that the Dirac operator $D$ allows a decomposition into Weyl operators we still extend the large class of operators describing massless fermions on the lattice which we have found recently [3]. In addition to Ginsparg-Wilson (GW) fermions [4] this class includes the ones proposed by Fujikawa [5] and in the overlap formalism of Narayanan and Neuberger [1] and in the formulation of Lüscher [2].

Noting that the operators $D$ are functions of a basic unitary and $\gamma_5$-Hermitian operator $V$ we introduce chiral projections of form [7]

$$P_{\pm} = \frac{1}{2}(1 \pm \gamma_5 G), \quad \bar{P}_{\pm} = \frac{1}{2}(1 \pm \bar{G} \gamma_5).$$

with functions $G(V)$ and $\bar{G}(V)$ satisfying

$$G^{-1} = G^\dagger = \gamma_5 G \gamma_5, \quad \bar{G}^{-1} = \bar{G}^\dagger = \gamma_5 \bar{G} \gamma_5.$$

Requiring the decomposition

$$D = P_+ DP_- + P_- DP_+$$

then leads to the relations

$$\bar{P}_+ DP_+ = DP_+ = \bar{P}_- D$$

and gives the general condition

$$D + D^\dagger \bar{G} G = 0.$$

The forms considered in Refs. [2] and [8] in the GW case correspond to the special choices $G = V, \bar{G} = 1$ and $G = ((1 - s)1 + sV)/N, \bar{G} = (s1 + (1 - s)V)/N$ with a real parameter $s$ and normalization factor $N$, respectively.

2. GENERAL RELATIONS

We start from the spectral representation of $V$

$$V = P_1^{(+)} + P_1^{(-)} - P_2^{(+)} - P_2^{(-)} + \sum_{k \ (0 < \varphi_k < \pi)} (e^{i\varphi_k} P_k^{(I)} + e^{-i\varphi_k} P_k^{(II)}),$$

with the orthogonal projections satisfy $\gamma_5 P_j^{(\pm)} \gamma_5 = \pm P_j^{(\pm)}, \gamma_5 P_k^{(I)} = P_k^{(II)} \gamma_5$, which implies that one has

$$N_+(1) - N_-(1) = \frac{1}{2}\text{Tr}(\gamma_5 V)$$

for $N_{\pm}(1) = \text{Tr}(P_{1j}^{(\pm)}), N_{\pm}(-1) = \text{Tr}(P_{2j}^{(\pm)}), \text{and}$

$$\text{Tr}(\gamma_5 P_k^{(I)}) = \text{Tr}(\gamma_5 P_k^{(II)}) = 0,$$

$$\text{Tr} P_j^{(I)} = \text{Tr} P_j^{(II)}.$$

The representation of $D = F(V)$ then becomes

$$D = f(1)(P_1^{(+)} + P_1^{(-)} + f(-1)(P_2^{(+)} + P_2^{(-)})$$

$$+ \sum_{k \ (0 < \varphi_k < \pi)} (f(e^{i\varphi_k}) P_k^{(I)} + f(e^{-i\varphi_k}) P_k^{(II)}),$$

with the conditions on the spectral functions

$$f(1) = 0, \ f(-1) \neq 0, \ f^*(v) = f(v^*),$$

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corresponding to masslessness, allowance for a non-zero index, \( \gamma_5 \)-Hermiticity, respectively. The index of \( D \) then gets \( I = N_+ (1) - N_- (1) \), which thus is given by \( \bar{S} \) and subject to \( \bar{S} \).

For \( G \) and \( \bar{G} \) analogous representations to \( (11) \) hold with spectral functions \( g \) and \( \bar{g} \), respectively. The latter satisfy \( |g|^2 = 1 \), \( g^* (v) = g (v^*) \), \( |\bar{g}|^2 = 1 \), \( \bar{g}^* (v) = \bar{g} (v^*) \). In terms of spectral functions the general condition \( \bar{g} \) now reads

\[
f + f^* \bar{g} g = 0.
\]

Because of \( f (-1) \neq 0 \) this implies

\[
\bar{g} (-1) = -g (-1),
\]

with the important consequence that \( G \) and \( \bar{G} \) must be generally different.

To have \( I = \bar{N} - N \) for the index, where \( \bar{N} = \text{Tr} \bar{P}_+ \), \( N = \text{Tr} P_- \), one needs \( \bar{g} (1) = g (1) = 1 \). Then for the choices \( g (-1) = -\bar{g} (-1) = \pm 1 \) one gets \( N = \frac{1}{2} \text{Tr} \mathbf{1} + I \), \( \bar{N} = \frac{1}{2} \text{Tr} \mathbf{1} \) and \( N = \frac{1}{2} \text{Tr} \mathbf{1} \), \( \bar{N} = \frac{1}{2} \text{Tr} \mathbf{1} + I \), respectively.

The spectral functions have been used in Ref. \[3\] to construct various concrete examples of Dirac operators. The respective methods have been seen in Ref. \[7\] to extend to the more general class of operators satisfying \[5\].

3. CORRELATION FUNCTIONS

General fermionic correlation functions can be written as \[12\]

\[
\langle \psi_{\sigma_{i_{1}} \cdots \sigma_{i_{n}}} \cdots \bar{\psi}_{\tau_{l_{1}} \cdots \tau_{l_{m}}} \rangle_{\mathcal{L}}
\]

\[
= \frac{1}{r!} \sum_{\sigma_{i_{1}}, \cdots, \sigma_{i_{r}} \cdots, \tau_{l_{1}}, \cdots, \tau_{l_{m}}} \sum_{\mathcal{L}} \mathcal{Y}_{\sigma_{i_{1}} \cdots \sigma_{i_{r}}} \mathcal{Y}_{\tau_{l_{1}} \cdots \tau_{l_{m}}} D_{\sigma_{i_{1}}} \cdots D_{\sigma_{i_{r}}}
\]

with alternating multilinear forms \( \mathcal{Y}_{\sigma_{i_{1}} \cdots \sigma_{i_{r}}} \) and \( \bar{\mathcal{Y}}_{\tau_{l_{1}} \cdots \tau_{l_{m}}} \), which are explicitly represented by

\[
\mathcal{Y}_{\sigma_{i_{1}} \cdots \sigma_{i_{r}}} = \sum_{j_{1}, \ldots, j_{N}} \epsilon_{j_{1}, \ldots, j_{N}} u_{\sigma_{i_{1}} j_{1}} \cdots u_{\sigma_{i_{r}} j_{N}}
\]

and an analogous expression for \( \bar{\mathcal{Y}}_{\tau_{l_{1}} \cdots \tau_{l_{m}}} \). The bases in such expressions satisfy

\[
P_- = uu^\dagger, u^\dagger u = 1_w, \bar{P}_+ = \bar{u} \bar{u}^\dagger, \bar{u}^\dagger \bar{u} = 1_{\bar{w}},
\]

where \( 1_w \) and \( 1_{\bar{w}} \) are the identity operators in the spaces of the Weyl degrees of freedom. Comparing with vector theory it is seen that instead of \( \epsilon_{\sigma_{i_{1}} \cdots \sigma_{i_{r}}} \) and \( \epsilon_{\bar{\sigma}_{l_{1}} \cdots \bar{\sigma}_{l_{m}}} \) with \( K = \text{Tr} \mathbf{1} \) there, one has \( \mathcal{Y}_{\sigma_{i_{1}} \cdots \sigma_{i_{r}}} \) and \( \bar{\mathcal{Y}}_{\bar{\sigma}_{l_{1}} \cdots \bar{\sigma}_{l_{m}}} \) here.

By \( (17) \) the bases are fixed up to unitary transformations, \( \bar{u} \mathcal{S} = u \mathcal{S}, \bar{u} \mathcal{S} = u \mathcal{S} \). In addition requiring unimodularity, \( \text{det}_{w} \mathcal{S} = 1 \), \( \text{det}_{w} \mathcal{S} = 1 \), the correlation functions \( (15) \) get invariant. While without this additional restriction the transformations \( \mathcal{S} \) connect all bases of the subspace on which \( P_- \) projects, the unimodular \( \mathcal{S} \) connect only subsets thereof. The total set of bases \( u(\mathcal{S}) \) thus decomposes into subsets. Because the formulation of the theory has to be restricted to one of such subsets (which are not equivalent) there is the question to which one of them. Analogous considerations apply to the bases \( \bar{u}(\mathcal{S}) \).

4. GAUGE TRANSFORMATIONS

Considering the gauge transformation \( P' = \mathcal{T} P \mathcal{T}^\dagger \) for \( G \neq \mathbf{1} \) with \( [\mathcal{T}, P_-] \neq 0 \), given a solution \( u \) satisfying \( (17) \) then \( u' = \mathcal{T} u \mathcal{S} \) is a solution of the transformed conditions \( P'_- = u' u'^\dagger \) and \( u'^\dagger u' = 1_w \). With the restriction to unimodular \( \mathcal{S} \) this constitutes a mapping between the respective subset and the transformed subset of bases. With analogous considerations for \( \bar{u} \) it becomes obvious that for \( G \neq \mathbf{1}, \mathcal{G} \neq \mathbf{1} \) the correlation functions \( (15) \) transform gauge-covariantly,

\[
\langle \psi_{\sigma_{i_{1}} \cdots \sigma_{i_{n}}} \cdots \bar{\psi}_{\tau_{l_{1}} \cdots \tau_{l_{m}}} \rangle_{\mathcal{L}}
\]

\[
= \sum_{\sigma_{i_{1}}, \cdots, \sigma_{i_{r}} \cdots, \tau_{l_{1}}, \cdots, \tau_{l_{m}}} \mathcal{T}_{\sigma_{i_{1}}} \cdots \mathcal{T}_{\sigma_{i_{r}}} \mathcal{S}_{\tau_{l_{1}}} \cdots \mathcal{S}_{\tau_{l_{m}}} \langle \psi_{\sigma_{i_{1}} \cdots \sigma_{i_{n}}} \cdots \bar{\psi}_{\tau_{l_{1}} \cdots \tau_{l_{m}}} \rangle_{\mathcal{L}}^{\prime}
\]

For \( \mathcal{G} = \mathbf{1} \) with \( [\mathcal{T}, \bar{P}_+] = 0 \) and \( \bar{P}_+ = \bar{P}_+ \), given a gauge-field independent solution \( \bar{u}_c \) of \( (17) \), also \( \bar{u} = \bar{u}_c \mathcal{S} \) with \( \mathcal{S} \) is a solution, representing the respective invariant subset of bases. This solution can be rewritten as \( \bar{u} = \mathcal{T} \bar{u}_c \mathcal{S} \mathcal{T}^\dagger \mathcal{S} \) with \( \mathcal{T} = e^\mathbf{S} \) and \( \mathcal{B}_{n,n} = i \delta_{n,n} \sum_{B_n} b_n T^\ell \). Thus the additional condition \( \text{Tr} \mathcal{T}^\ell = 0 \) on the generators \( T^\ell \) is needed to get rid of that factor.
5. CP TRANSFORMATIONS

With the charge conjugation matrix $C$, $P_{u'n} = \delta_{u'n}$, $U_{4n}^{CP} = U_{4n}^{*}$, $U_{kn}^{CP} = U_{k,n-k}^{*}$ for $k = 1, 2, 3$, where $\bar{n} = (-\bar{n}, n_4)$, we have

$$O(U^{CP}) = W(O(U))^T W^\dagger, \quad W = P_{\gamma 4} C^\dagger$$

for $V$, $D$, $G$, $\bar{G}$, while for the projections $P_{-}^{CP}(U^{CP})$, $P_{+}(U)$, $P_{+}^{CP}(U^{CP})$, $P_{-}(U)$, we get

$$P_{-}^{CP} = WP_{+}^T W^\dagger, \quad P_{+}^{CP} = WP_{+}^T W^\dagger,$$

$$P_{-} = \frac{1}{2}(1 - \gamma_5 \bar{G}), \quad P_{+} = \frac{1}{2}(1 + \gamma_5 \bar{G}).$$

It is seen that (21) differs from (1) by an interchange of $G$ and $\bar{G}$. Since in [3] only the product enters, the interchanged choice is associated to the same situation of continuum theory.

Given solutions $u$ and $\bar{u}$ of (14), then $u^{CP} = \bar{W} u \tilde{S}$ and $\bar{u}^{CP} = W u^* S$ are the solutions of the CP transformed conditions. With unimodular $S$ and $\tilde{S}$ this leads to the transformations

$$\langle \psi_{\sigma_1}^{CP}, \ldots, \psi_{\sigma_L}^{CP} \rangle \rightarrow \psi_{\sigma_1}^{CP}, \ldots, \psi_{\sigma_L}^{CP}$$

$$= \sum_{\sigma_1, \ldots, \sigma_L} \sum_{\sigma_1, \ldots, \sigma_L} \tilde{W}^\dagger_{\sigma_1, \sigma_1} \ldots \tilde{W}^\dagger_{\sigma_L, \sigma_L} \psi_{\sigma_1} \ldots \psi_{\sigma_L} L,$$

$$\langle \bar{\psi}_{\bar{\sigma}_1}^{C}, \ldots, \bar{\psi}_{\bar{\sigma}_L}^{C} \rangle \rightarrow \bar{\psi}_{\bar{\sigma}_1}^{C}, \ldots, \bar{\psi}_{\bar{\sigma}_L}^{C},$$

where the operators $\bar{P}_-$, $\bar{D}$, $\bar{P}_+$ are the ones restricted to the subspace without zero modes.

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REFERENCES

1. R. Narayanan, H. Neuberger, Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B442 (1994) 574; Nucl. Phys. B443 (1995) 305.
2. M. Lüscher, Nucl. Phys. B549 (1999) 295; Nucl. Phys. B568 (2000) 162.
3. W. Kerler, Nucl. Phys. B646 (2002) 201.
4. P.H. Ginsparg, K.G. Wilson, Phys. Rev. D25 (1982) 2649.
5. K. Fujikawa, Nucl. Phys. B589 (2000) 487.
6. K. Fujikawa, M. Ishibashi, H. Suzuki Phys. Lett. B538 (2002) 197; JHEP 0204 (2002) 046.
7. W. Kerler, hep-lat/0307011 to appear in Nucl. Phys. B.
8. P. Hasenfratz, Nucl. Phys. (Proc. Suppl.) 106 (2002) 159.