Extrinsic eigenvalues upper bounds for submanifolds in weighted manifolds

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Abstract
We prove Reilly-type upper bounds for divergence-type operators of the second order as well as for Steklov problems on submanifolds of Riemannian manifolds of bounded sectional curvature endowed with a weighted measure.

Keywords Submanifolds · Reilly-type upper bounds · Eigenvalues estimates · Divergence-type operators · Steklov problems

Mathematics Subject Classification 53C24 · 53C42 · 58J50

1 Introduction
Let \((M^n, g)\) be an \(n\)-dimensional compact, connected, oriented manifold without boundary, and consider an isometric immersion \(X : M^n \to \mathbb{R}^{n+1}\) in the Euclidean space. The spectrum of the Laplacian of \((M, g)\) is an increasing sequence of real numbers

\[0 = \lambda_0(\Delta) < \lambda_1(\Delta) \leq \lambda_2(\Delta) \leq \cdots \leq \lambda_k(\Delta) \leq \cdots \to +\infty.\]
The eigenvalue 0 (corresponding to constant functions) is simple and \( \lambda_1(\Delta) \) is the first positive eigenvalue. In [12], Reilly proved the following well-known upper bound for \( \lambda_1(\Delta) \)

\[
\lambda_1(\Delta) \leq \frac{n}{V(M)} \int_M H^2 \, dv_g,
\]

(1)

where \( H \) is the mean curvature of the immersion. He also proved an analogous inequality involving the higher-order mean curvatures. Namely, for \( r \in \{1, \ldots, n\} \)

\[
\lambda_1(\Delta) \left( \int_M H_{r-1} \, dv_g \right)^2 \leq V(M) \int_M H^2 \, dv_g,
\]

(2)

where \( H_r \) is the \( r \)-th mean curvature, defined by the \( r \)-th symmetric polynomial of the principal curvatures. Moreover, Reilly studied the equality cases and proved that equality in (1) or (2) is attained if and only if \( X(M) \) is a geodesic sphere.

In the case of higher codimension, Reilly also proved that

\[
\lambda_1(\Delta) \leq \frac{n}{V(M)} \int_M \|\mathbf{H}\|^2 \, dv_g,
\]

(3)

where \( \mathbf{H} \) is here the mean curvature vector, with equality if and only if \( M \) is minimally immersed in a geodesic sphere.

The Reilly inequality can be easily extended to submanifolds of the sphere \( S^n \) using the canonical embedding of \( S^n \) into \( \mathbb{R}^{n+1} \):

\[
\lambda_1(\Delta) \leq \frac{n}{V(M)} \int_M (\|\mathbf{H}\|^2 + 1) \, dv_g.
\]

(4)

Moreover, El Soufi and Ilias [7] proved an analogue for submanifold of the hyperbolic space as

\[
\lambda_1(\Delta) \leq \frac{n}{V(M)} \int_M (\|\mathbf{H}\|^2 - 1) \, dv_g.
\]

(5)

In case the ambient space has non-constant sectional curvature, Heintze [10] proved the following weaker inequality

\[
\lambda_1(\Delta) \leq n(\|\mathbf{H}\|^2 + \delta),
\]

(6)

where the ambient sectional curvature is bounded above by \( \delta \).

On the other hand, more recently, in [14], the second author prove the following general inequality

\[
\lambda_1(L_T, f) \left( \int_M \text{tr} (S) \mu_f \right)^2 \leq \left( \int_M \text{tr} (T) \mu_f \right) \int_M (\|H_s\|^2 + \|S \nabla f\|^2) \, \mu_f,
\]

(7)

where \( \mu_f = e^{-f} \, dv_g \) is the weighted measure of \((M, g)\) endowed with the density \( e^{-f} \), \( T, S \) are two symmetric, free-divergence \((1,1)\)-tensors with \( T \) positive definite, and \( L_T, f \) is the second-order differential operator defined for any smooth function \( u \) on \( M \) by

\[
L_T, f = -\text{div}(T \nabla u) + (\nabla f, T \nabla u).
\]

When \( f = 0 \) and the tensor \( S \) and \( T \) are associated with higher order mean curvatures \( H_s \) and \( H_r \), we recover the inequality of Alias and Malacarné [2] and, in particular, Reilly’s inequality (2) if \( r = 0 \).

The first result of this paper gives upper bounds for the first eigenvalue of the operator \( L_T, f \) for submanifolds of Riemannian manifolds with sectional curvature bounded by above
which generalizes inequality (7) in the non-constant curvature case. Namely, we prove the following.

**Theorem 1.1** Let \((\tilde{M}^{n+p}, \tilde{g}, \tilde{\mu}_f)\) be a weighted Riemannian manifold with sectional curvature \(\text{sect}_{\tilde{M}} \leq \delta\) and \(\tilde{\mu}_f = e^{-f}dv_{\tilde{g}}\). Let \((M, g)\) be a closed Riemannian manifold isometrically immersed into \((M^{n+p}, \tilde{g})\) by \(X\). We endow \(M\) with the weighted measure \(\mu_f = e^{-f}dv_{\tilde{g}}\). Let \(T\) be a positive definite \((1, 1)\)-tensor on \(M\) and denote by \(\lambda_1\) the first positive eigenvalue of the operator \(L_{T,f}\).

1. If \(\delta \leq 0\), then
   \[
   \lambda_1 \leq \sup_M \left[ \delta \text{tr} (T) + \sup_M \left( \frac{\|H_T - T\nabla f\|}{\text{tr}(S)} \right) \|H_S - S\nabla f\| \right].
   \]

2. If \(\delta > 0\) and \(X(M)\) is contained in a geodesic ball of radius \(\frac{\pi}{4\sqrt{\delta}}\),
   \[
   \lambda_1 \leq \frac{\int_M \text{tr} (T) \mu_f}{V_f(M)} \left( \delta + \frac{\int_M \|H_S - S\nabla f\|^2 \mu_f}{V_f(M) \inf (\text{tr}(S)^2)} \right).
   \]

The second eigenvalue problem that we consider in this paper is the Steklov problem associated with the operator \(L_{T,f}\) on a submanifold \(\Omega\) with non-empty boundary \(\partial\Omega = M\) of a Riemannian manifold with sectional curvature bounded by above. We can consider the following generalized weighted Steklov problem

\[
\begin{cases}
    L_{T,f}u = 0 \text{ on } \Omega, \\
    \frac{\partial u}{\partial v_T} = \sigma u \text{ on } M = \partial\Omega,
\end{cases}
\]

where \(\frac{\partial u}{\partial v_T} = \langle T(\nabla u), v\rangle\). In the case where \(f\) is constant, the operator \(L_{T,f}\) is of particular interest for the study of \(r\)-stability when \(T = T_r\) is the tensor associated with \(r\)-th mean curvature (see [1] for instance). More precisely, from [3], we know that this problem (8) has a discrete nonnegative spectrum and we denote by \(\sigma_1\) its first eigenvalue. In [13], the second author has obtained upper bounds for this problem for domains of a manifold lying in a Euclidean space. Namely, he has proved

\[
\sigma_1 \left( \int_M \text{tr} (S) \tilde{\mu}_f \right)^2 \leq \left( \int_\Omega \text{tr} (T) \mu_f \right) \int_M (\|H_S\|^2 + \|S\nabla f\|^2) \tilde{\mu}_f,
\]

where \(\mu_f = e^{-f}dv_{\tilde{g}}\) and \(\tilde{\mu}_f = e^{-f}dv_{\tilde{g}}\) are, respectively, the weighted measures of \((\Omega, g)\) and \((M, \tilde{g})\), endowed with the density \(e^{-f}\) and where \(T\) and \(S\) are two symmetric, free-divergence \((1, 1)\)-tensors with \(T\) positive definite. Note that without density and for \(T = \text{Id}\), this inequality has been proven by Ilias and Makhouli [11].

The second result of the present paper gives a generalization of this estimate when the manifold with boundary \((M, g)\) is immersed into an ambient Riemannian manifold of sectional curvature bounded by above. Namely, we prove the following.

**Theorem 1.2** Let \((\tilde{M}^{n+p}, \tilde{g}, \tilde{\mu}_f)\) be a weighted Riemannian manifold with sectional curvature \(\text{sect}_{\tilde{M}} \leq \delta\) and \(\tilde{\mu}_f = e^{-f}dv_{\tilde{g}}\). Let \((\Omega, g)\) be a compact Riemannian manifold with non-empty boundary \(M\) isometrically immersed into \((\tilde{M}^{n+p}, \tilde{g})\) by \(X\). We endow \(\Omega\) and \(M\), respectively, with the weighted measure \(\mu_f = e^{-f}dv_{\tilde{g}}\) and \(\tilde{\mu}_f = e^{-f}dv_{\tilde{g}}\), where \(\tilde{g}\) is the
induced metric on $M$. Let $T$, $S$ be a symmetric, divergence-free and positive definite $(1, 1)$-tensors on $\Omega$ and $M$, respectively, and denote by $\sigma_1$ the first eigenvalue of the Steklov problem (8).

1) If $\delta \leq 0$ and $X(\Omega)$ is contained in the geodesic ball $B(p, R)$ of radius $R$, where $p$ is the center of mass of $M$ for the measure $\tilde{\mu}$, then

$$
\sigma_1 \leq \sup_{\Omega} \left[ \delta \text{tr}(T) + \sup_{\Omega} \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr}(T)} \right) \|H_T - T(\nabla f)\| \right] 
\times \left[ \delta + \frac{\sup_M \|H_S - S(\nabla f)\|^2}{\inf_M (\text{tr}(S))^2} \right] \left( \frac{V_f(\Omega)}{V_f(M)} \right)^2 \frac{\delta^2}{\delta (R)}.
$$

2) If $\delta > 0$ and $X(\Omega)$ is contained in a geodesic ball of radius $\frac{\pi}{4\sqrt{\delta}}$, then

$$
\sigma_1 \leq \int_{\Omega} \frac{\text{tr}(T) \mu_f}{V_f(M)} \left( \delta + \frac{\int_M \|H_S - S(\nabla f)\|^2 \, \mu_f}{V_f(M) \inf (\text{tr}(S))^2} \right).
$$

Finally, we will consider the so-called eigenvalue problem for Wentzell boundary conditions

$$
\begin{align*}
\Delta u & = 0 \quad \text{in } \Omega, \\
-b\Delta u - \frac{\partial u}{\partial \nu} & = \alpha u \quad \text{on } M,
\end{align*}
$$

where $b$ is a given positive constant, $\Omega$ is a submanifold with non-empty boundary $\partial \Omega = M$ of a Riemannian manifold $M$ with sectional curvature bounded by above, and $\Delta$, $\tilde{\Delta}$ denote the Laplacians on $\Omega$ and $M$, respectively. It is clear that if $b = 0$, then we recover the classical Steklov problem. The spectrum of this problem is an increasing sequence (see [5]) with 0 as first eigenvalue which is simple and the corresponding eigenfunctions are the constant ones. We denote by $\alpha_1$ the first positive eigenvalue. In [14], the second author proved the following estimate when $\Omega$ is a submanifold of the Euclidean space $\mathbb{R}^n$

$$
\alpha_1 \left( \int_{\partial M} \text{tr}(S) \, dv_g \right)^2 \leq \left( n V(M) + b(n - 1) V(\partial M) \right) \left( \int_{\partial M} \|H_S\|^2 \, dv_g \right).
$$

In the following theorem, we obtain a comparable estimate when the ambient space is of bounded sectional curvature. Namely, we prove the following.

**Theorem 1.3** Let $(\tilde{M}^{n+p}, \tilde{g})$ be a Riemannian manifold with sectional curvature $\text{sect}_{\tilde{M}} \leq \delta$. Let $(\Omega, g)$ be a compact Riemannian manifold with non-empty boundary $M$ isometrically immersed into $(\tilde{M}^{n+p}, \tilde{g})$ by $X$. We denote by $\tilde{g}$ the induced metric on $M$. Let $S$ be a symmetric, divergence-free and positive definite $(1, 1)$-tensor on $M$ and denote by $\alpha_1$ the first eigenvalue of the Steklov–Wentzell problem (9).

1) If $\delta \leq 0$ and $X(\Omega)$ is contained in the geodesic ball $B(p, R)$ of radius $R$, where $p$ is the center of mass of $\tilde{M}$, then

$$
\alpha_1 \leq \left[ n \frac{V(\Omega)}{V(M)} + b(n - 1) - \delta s_\delta^2(R) \left( \frac{V(\Omega)}{V(M)} + b \right) \right] \left( \delta + \frac{\sup_M \|H_S\|^2}{\inf_M (\text{tr}(S))^2} \right).
$$
(2) If \( \delta > 0 \) and so \( X(\Omega) \) is contained in a geodesic ball of radius \( \frac{\pi}{4\sqrt{\delta}} \), then

\[
\alpha_1 \leq \left( n \frac{V(\Omega)}{V(M)} + b(n - 1) \right) \left( \delta + \frac{\int_M \|H_S\|^2 dv_{\tilde{g}}}{V(M) \inf (\text{tr} (S)^2)} \right).
\]

2 Preliminaries

Let \((\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)\) be a weighted Riemannian manifold with sectional curvature \( \text{sect}_{\bar{M}} \leq \delta \) and weighted measure \( \bar{\mu}_f = e^{-f} dv_{\bar{g}} \). Let \( p \) a fixed point in \( \bar{M} \), we denote by \( r(x) \) the geodesic distance between \( x \) and \( p \). Moreover, we define the vector field \( X \) by \( X(x) := s_\delta(r(x))(\bar{\nabla} r)(x) \), \( s_\delta \) is the function defined by

\[
s_\delta(r) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}r) & \text{if } \delta > 0 \\ r & \text{if } \delta = 0 \\ \frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) & \text{if } \delta < 0. \end{cases}
\]

We also define

\[
c_\delta(r) = \begin{cases} \cos(\sqrt{\delta}r) & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \\ \cosh(\sqrt{|\delta|}r) & \text{if } \delta < 0. \end{cases}
\]

Hence, we have

\[c_\delta^2 + \delta s_\delta^2 = 1, \quad s_\delta' = c_\delta \quad \text{and} \quad c_\delta' = -\delta s_\delta.\]

In addition, let \((M^n, g)\) be a closed Riemannian manifold isometrically immersed into \((\bar{M}^{n+p}, \bar{g})\) by \( \phi \). If \( \delta > 0 \), then we assume that \( \phi(M) \) is contained in a geodesic ball of radius \( \frac{\pi}{2\sqrt{\delta}} \). We endow \( M \) with the weighted measure \( \mu_f = e^{-f} dv_g \). We can define on \( M \) a divergence associated with the volume form \( \mu_f = e^{-f} dv_g \) by

\[
\text{div}_f Y = \text{div} Y - \langle \nabla f, Y \rangle
\]

or, equivalently,

\[
d(\iota_Y \mu_f) = \text{div}_f(Y) \mu_f,
\]

where \( \nabla \) is the gradient on \( \Sigma \), that is, the projection on \( T \Sigma \) of the gradient \( \bar{\nabla} \) on \( \bar{M} \). We call it the \( f \)-divergence. We recall briefly some basic facts about the \( f \)-divergence. In the case where \( \Sigma \) is closed, we first have the weighted version of the divergence theorem:

\[
\int_\Sigma \text{div}_f Y \mu_f = 0, \quad \text{(10)}
\]

for any vector field \( Y \) on \( \Sigma \). From this, we deduce easily the integration by parts formula

\[
\int_\Sigma u \text{div}_f Y \mu_f = -\int_\Sigma \langle \nabla u, Y \rangle \mu_f, \quad \text{(11)}
\]

for any smooth function \( u \) and any vector field \( Y \) on \( \Sigma \). First, we prove the following elementary lemma which generalize in non-constant curvature the classical Hsiung-Minkowski formula (see [8, 13] for instance).
Lemma 2.1 Let $T$ be a symmetric divergence-free positive $(1,1)$-tensor on $M$. Then the following hold

1. $\text{div}_f(TX^\top) \geq \text{tr}(T)c_\delta + \langle X, H_T - T(\nabla f) \rangle$.
2. $\int_M \text{tr}(T)c_\delta \mu_f \leq -\int_M \langle X, H_T - T(\nabla f) \rangle \mu_f$.
3. $\delta \int_M \langle TX^\top, X^\top \rangle \mu_f \geq \int_M \text{tr}(T)c_\delta^2 \mu_f - \int_M \|H_T - T(\nabla f)\|s_\delta c_\delta \mu_f$.

Proof The proof is a straightforward consequence of the analogue non-weighted result proven by Grosjean. Namely, in [8], the author has shown that

$$\text{div}_M(TX^\top) \geq \text{tr}(T)c_\delta(r) + \langle X, H_T \rangle.$$

Hence, from the definition of the $f$-divergence, we have

$$\text{div}_f(TX^\top) = \text{div}(TX^\top) - \langle \nabla f, TX^\top \rangle \geq \text{tr}(T)c_\delta(r) + \langle X, H_T - T(\nabla f) \rangle,$$

and this proves part (1). For the second part, we integrate the last inequality with respect to the measure $\mu_f$ and we get immediately

$$\int_M \text{tr}(T)c_\delta \mu_f \leq -\int_M \langle X, H_T - T(\nabla f) \rangle \mu_f,$$

since

$$\int_M \text{div}_f(TX^\top) \mu_f = 0.$$

Finally, for the last part, if $\delta = 0$, then $c_\delta = 1$ and we get directly the conclusion from the second one by using

$$|\langle X, H_T - T(\nabla f) \rangle| \leq ||X|| \cdot ||H_T - T(\nabla f)|| = s_\delta ||H_T - T(\nabla f)||.$$

If $\delta \neq 0$, since $X^\top = s_\delta(r)\nabla r = -\delta \nabla c_\delta(r)$, then we have

$$\delta \int_M \langle TX^\top, X^\top \rangle \mu_f = \frac{1}{\delta} \int_M \langle T(\nabla c_\delta, \nabla c_\delta) \rangle \mu_f$$

$$= -\frac{1}{\delta} \int_M \text{div}_f(T\nabla c_\delta)c_\delta \mu_f$$

$$= \int_M \text{div}_f(X^\top)c_\delta \mu_f$$

$$\geq \int_M \text{tr}(T)c_\delta^2 \mu_f - \int_M \|H_T - T(\nabla f)\|s_\delta c_\delta \mu_f,$$

where we have used the first part of the lemma and the well-known Cauchy–Schwarz inequality.

\[\square\]

3 Two key lemma

In this section we will prove two basic key lemma that will be used throughout the paper.
Lemma 3.1 Let \((\tilde{M}^{n+p}, \tilde{g}, \tilde{\mu}_f)\) be a weighted Riemannian manifold with sectional curvature \(\text{sect}_\tilde{M} \leq \delta \leq 0\), and \(\tilde{\mu}_f = e^{-f} d\tilde{v}_g\). Let \((M, g)\) be a closed Riemannian manifold isometrically immersed into \((\tilde{M}^{n+p}, \tilde{g})\), and we endow \(M\) with the weighted measure \(\mu_f = e^{-f} d\nu_g\).

Let \(S\) be a symmetric, divergence-free and positive definite \((1, 1)\)-tensor on \(M\). Then, we have

\[
\frac{\int_M \|X\|^2 \mu_f}{V_f(M)} \geq \frac{1}{\delta + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\text{tr}(S))^2}}.
\]

**Proof** We have

\[
\int_M \left(\text{tr}(S) - \delta \langle SX^T, X^T\rangle\right) \mu_f = \int_M \left(\text{tr}(S) - \text{div}(SX^T) c_3(r)\right) \mu_f
\]

\[
\leq \int_M \left(\delta \text{tr}(S) s_3^2(r) - \langle H_S - S(\nabla f), X\rangle c_3(r)\right) \mu_f
\]

\[
\leq \int_M \left(\delta \text{tr}(S) s_3^2(r) + \frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\text{tr}(S))} \int_M \text{tr}(S) s_3(r) c_3(r)\right) \mu_f
\]

\[
\leq - \frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\text{tr}(S))} \int_M c_3(r) s_3(r) \langle S\nabla M r, \nabla M r \rangle + s_3^2(r) \langle H_S - S(\nabla f), \nabla M r \rangle \mu_f
\]

\[
+ \delta \inf(\text{tr}(S)) \int_M s_3^2(r) \mu_f
\]

\[
\leq \left(\delta \inf(\text{tr}(S)) + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\text{tr}(S))} - \frac{H_S - S(\nabla f)\|_\infty}{\inf(\text{tr}(S))}\right) \int_M c_3(r) s_3(r) \langle S\nabla M r, \nabla M r \rangle \mu_f.
\]

Hence, we get

\[
\int_M \text{tr}(S) \mu_f \leq \left(\delta \inf(\text{tr}(S)) + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\text{tr}(S))} \right) \int_M s_3^2(r) \mu_f
\]

\[
- \frac{H_S - S(\nabla f)\|_\infty}{\inf(\text{tr}(S))} \int_M c_3(r) s_3(r) \langle S\nabla M r, \nabla M r \rangle \mu_f
\]

\[
+ \delta \int_M s_3^2(r) \langle S\nabla M r, \nabla M r \rangle \mu_f
\]

\[
\leq \left(\delta \inf(\text{tr}(S)) + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\text{tr}(S))}\right) \int_M s_3^2(r) \mu_f
\]

\[
+ \int_M \left(\delta s_3^2(r) - c_3(r) s_3(r) \frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\text{tr}(S))}\right) \langle S\nabla M r, \nabla M r \rangle \mu_f.
\]

Since \(\delta \leq 0\), the second term of the right-hand side is nonpositive, and thus we get
\[\inf(\tr(S)) V_f(M) \leq \int_M \tr(S) \mu_f \]
\[
\leq \left( \delta \inf(\tr(S)) + \frac{\|H_S - S(\nabla f)\|_2}{\inf(\tr(S))} \right) \int_M s_\beta^2(r) \mu_f ,
\]
which gives immediately the result since \(\|X\| = s_\beta(r)\).

**Lemma 3.2** Let \((\tilde{M}^{n+p}, \tilde{g}, \tilde{\mu}_f)\) be a weighted Riemannian manifold with sectional curvature \(\sect_{\tilde{M}} \leq \delta\), with \(\delta > 0\), and \(\tilde{\mu}_f = e^{-f} \, dv_{\tilde{g}}\). Let \((M, g)\) be a closed Riemannian manifold isometrically immersed into \((\tilde{M}^{n+p}, \tilde{g})\) by \(X\) so that \(X(M)\) is contained in a geodesic ball of radius \(\frac{\pi}{2\sqrt{\delta}}\). We endow \(M\) with the weighted measure \(\mu_f = e^{-f} \, dv_{\tilde{g}}\). Let \(S\) be a symmetric, divergence-free and positive definite \((1, 1)\)-tensor on \(M\). Then, we have

\[
1 - \left( \frac{\int_M c_\beta(r) \mu_f}{V_f(M)} \right)^2 \geq \frac{1}{1 + \frac{\int_M \|H_S - S(\nabla f)\|_2^2 \mu_f}{\delta \inf(\tr(S))^2 V_f(M)}}.
\]

**Proof** For a sake of compactness, we will write

\[
\alpha = \frac{\int_M c_\beta(r) \mu_f}{V_f(M)} \quad \text{and} \quad \beta = 1 + \frac{\int_M \|H_S - S(\nabla f)\|_2^2 \mu_f}{\delta \inf(\tr(S))^2 V_f(M)}.
\]

We thus have to show that \((1 - \alpha^2) \beta \geq 1\). We have

\[
(1 - \alpha^2) \beta = \beta - \left( \frac{\int_M c_\beta(r) \mu_f}{V_f(M)} \right)^2 - \left( \frac{\int_M c_\beta(r) \mu_f}{V_f(M)} \right)^2 \frac{\int_M \|H_S - S(\nabla f)\|_2^2 \mu_f}{\delta \inf(\tr(S))^2 V_f(M)}
\]
\[
\geq \beta - \left( \frac{\int_M \tr(S) c_\beta(r) \mu_f}{\inf(\tr(S)) V_f(M)} \right)^2 - \left( \frac{\int_M c_\beta(r) \mu_f}{V_f(M)} \right)^2 \frac{\int_M \|H_S - S(\nabla f)\|_2^2 \mu_f}{\delta \inf(\tr(S))^2 V_f(M)}
\]
\[
\geq \beta - \left( \frac{\int_M s_\beta(r) \|H_S - S(\nabla f)\|_2 \mu_f}{\inf(\tr(S)) V_f(M)} \right)^2 - \left( \frac{\int_M c_\beta(r) \mu_f}{V_f(M)} \right)^2 \frac{\int_M \|H_S - S(\nabla f)\|_2^2 \mu_f}{\delta \inf(\tr(S))^2 V_f(M)}
\]
\[
\geq \beta - \frac{\int_M s_\beta^2(r) \mu_f}{\inf(\tr(S))^2 V_f(M)^2}
\]
\[
- \left( \frac{\int_M c_\beta(r) \mu_f}{V_f(M)} \right)^2 \frac{\int_M \|H_S - S(\nabla f)\|_2^2 \mu_f}{\delta \inf(\tr(S))^2 V_f(M)}
\]
\[
\geq \beta - \frac{\int_M \|H_S - S(\nabla f)\|_2^2 \mu_f}{\delta \inf(\tr(S))^2 V_f(M)^2} \left( \int_M (s_\beta^2(r) + \delta^2 c_\beta^2(r) \mu_f) \right)
\]
\[ \beta - \int_M \frac{\|HS - S(\nabla f)\|^2 \mu_f}{\delta \inf(\text{tr}(S))^2 V_f(M)} = 1, \]

and this concludes the proof. \qed

4 Proofs of the Theorems

**Proof of Theorem 1.1** Case \( \delta \leq 0 \). Let \( p \in \overline{M} \) be a fixed point and let \( \{x_1, \cdots, x_N\} \) be the normal coordinates of \( \overline{M} \) centered at \( p \). For any \( x \in \overline{M} \), \( r(x) \) is the geodesic distance between \( p \) and \( x \) over \( \overline{M} \). We want to use as test functions the functions

\[ \frac{s_\delta(r)}{r} x_i, \]

for \( 1 \leq i \leq N \). For this purpose, we will choose \( p \) as the center of mass of \( M \) with respect for the measure \( \mu_f \), that is, \( p \) is the only point in \( M \) so that

\[ \int_M \frac{s_\delta(r)}{r} x_i \mu_f = 0, \]

for any \( i \in \{1, \ldots, N\} \). Note at that point that we assume that \( M \) is contained in a ball of radius \( \frac{\pi}{4\sqrt{\delta}} \) when \( \delta \) is positive allows us to ensure that \( M \) is contained in a ball of radius \( \frac{\pi}{4\sqrt{\delta}} \) centered at \( p \). This holds also for Theorems 1.2 and 1.3. These functions are candidates for test functions since they are \( L^2(\mu_f) \)-orthogonal to the constant functions which are the eigenfunctions for the first eigenvalue \( \lambda_0 = 0 \). Thus, we have

\[ \lambda_1 \int_M \sum_{i=1}^N \frac{s_\delta^2(r)}{r^2} x_i^2 \mu_f \leq \int_M \sum_{i=1}^N \left( T \nabla \left( \frac{s_\delta(r)}{r} x_i \right), \nabla \left( \frac{s_\delta(r)}{r} x_i \right) \right) \mu_f. \]  

We recall that Grosjean proved in [8, Lemma 2.1] that

\[ \sum_{i=1}^N \left( T \nabla \left( \frac{s_\delta(r)}{r} x_i \right), \nabla \left( \frac{s_\delta(r)}{r} x_i \right) \right) \leq \text{tr}(T) - \delta \left( TX^T, X^T \right). \]  

Moreover, by Lemma 2.1, item (3), we have

\[ \delta \int_M \langle TX^T, X^T \rangle \mu_f \geq \int_M \text{tr}(T)c_\delta^2(r) \mu_f - \int_M \|H_T - T(\nabla f)\|s_\delta(r)c_\delta(r) \mu_f \]

which together with (12) and (13) gives

\[ \lambda_1 \int_M s_\delta^2(r) \mu_f \leq \int_M \left( \text{tr}(T) - c_\delta^2(r) \text{tr}(T) + \|H_T - T(\nabla f)\|s_\delta(r)c_\delta(r) \right) \mu_f \]

\[ \leq \delta \int_M s_\delta^2(r) \text{tr}(T) \mu_f + \int_M \|H_T - T(\nabla f)\|s_\delta(r)c_\delta(r) \mu_f, \]

where we have used \( c_\delta^2 + \delta s_\delta^2 = 1 \). Hence, we obtain
\[ \lambda_1 \int_M s_\delta^2(r) \mu_f \leq \delta \int_M s_\delta^2(r) \text{tr}(T) \mu_f \]

\[ + \sup_M \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr}(S)} \right) \int_M \text{tr}(S)s_\delta(r)c_\delta(r) \mu_f. \] 

(14)

Now, we claim the following.

**Lemma 4.1** We have

\[ \int_M \text{tr}(S)s_\delta(r)c_\delta(r) \mu_f \leq \int_M \|H_S - S(\nabla f)\|s_\delta^2(r) \mu_f. \]

Reporting this into (14), we get

\[ \lambda_1 \int_M s_\delta^2(r) \mu_f \leq \int_M \left[ \delta \text{tr}(T) + \sup_M \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr}(S)} \right) \|H_S - S(\nabla f)\| \right] s_\delta^2(r) \mu_f. \]

which gives immediately the desired estimate

\[ \lambda_1 \leq \sup_M \left[ \delta \text{tr}(T) + \sup_M \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr}(S)} \right) \|H_S - S(\nabla f)\| \right]. \]

This concludes the proof for the case \( \delta < 0 \), up to the proof of the lemma that we give now.

**Proof of Lemma 4.1** Multiplying the first part of Lemma 2.1 for the tensor \( S \) by \( s_\delta(r) \), we get

\[ \text{div}_f (SX^\top)s_\delta(r) \geq \text{tr}(S)c_\delta(r)s_\delta(r) - \langle X, H_S - S(\nabla f) \rangle, \]

and thus, by the Cauchy-Schwarz inequality, we have

\[ \text{div}_f (SX^\top)s_\delta(r) \geq \text{tr}(S)c_\delta(r)s_\delta(r) - \|H_S - S(\nabla f)\|s_\delta(r). \]

Now, integrating this relation and using the integration by parts in the formula (11), we get

\[ \int_M \text{tr}(S)c_\delta(r)s_\delta(r) \mu_f - \int_M \|H_S - S(\nabla f)\|s_\delta(r) \mu_f \leq - \int_M \langle \nabla s_\delta(r), SX^\top \rangle \mu_f \]

\[ \leq - \int_M c_\delta(r)s_\delta(r)\langle \nabla r, S\nabla \rangle \mu_f \leq 0, \]

since \( S \) is positive. This concludes the proof of Lemma 4.1.

\[ \square \]

Case \( \delta > 0 \). Like in the case \( \delta < 0 \), we use \( \frac{s_\delta(r)}{\delta} x_i, 1 \leq i \leq N \), as test functions. Using (13) again, we get

\[ \lambda_1 \int_M s_\delta^2(r) \mu_f \leq \int_M \left( \text{tr}(T) - \frac{\|H_T - T(\nabla f)\|}{\|S\|} \right) \mu_f. \]

(15)

On the other, we use another test function in this case, namely \( c_\delta(r) - \overline{c_\delta} \) where for more convenience, we have denoted by \( \overline{c_\delta} \) the mean value of \( c_\delta(r) \), that is \( \overline{c_\delta} = \frac{1}{V_f(M)} \int_M c_\delta(r) \mu_f. \)

Here again, this function is \( L^2(\mu_f) \)-orthogonal to the constant functions, so it is a candidate for being a test function. Hence, we have
\[
\lambda_1 \int_M (c_\delta(r) - \overline{c}_\delta)^2 \mu_f \leq \int_M \langle T \nabla (c_\delta(r) - \overline{c}_\delta), \nabla (c_\delta(r) - \overline{c}_\delta) \rangle \mu_f \\
\leq \int_M \langle T \nabla c_\delta(r), \nabla c_\delta(r) \rangle \mu_f \\
\leq \delta^2 \int_M s_\delta^2(r) \langle T \nabla r, \nabla r \rangle \mu_f \\
\leq \delta^2 \int_M \langle TX^\top, X^\top \rangle \mu_f.
\]

From this, we deduce immediately that
\[
\lambda_1 \int_M c_\delta^2(r) \mu_f \leq \delta^2 \int_M \langle TX^\top, X^\top \rangle + \lambda_1 V_f(M) \left( \int_M c_\delta(r) \mu_f \right)^2.
\] (16)

Now, using the fact that \(c_\delta^2 + \delta s_\delta^2 = 1\), (16) plus \(\delta\) times (15) gives
\[
\lambda_1 V_f(M) \leq \delta \int_M \text{tr} \, (T) \mu_f + \left( \int_M c_\delta(r) \mu_f \right)^2.
\]

and thus
\[
\lambda_1 V_f(M) \left(1 - \left( \frac{\int_M c_\delta(r) \mu_f}{V_f(M)} \right)^2\right) \leq \delta \int_M \text{tr} \, (T) \mu_f.
\]

Now, we conclude by using Lemma 3.2 to get the desired upper bound
\[
\lambda_1 \leq \frac{\int_M \text{tr} \, (T) \mu_f}{V_f(M)} \left( \delta + \frac{\int_M \|H_S - S(\nabla f)\|^2 \mu_f}{V_f(M) \inf \left( \text{tr} \, (S)^2 \right)} \right),
\]

and this concludes the proof of Theorem 1.1.

**Proof of Theorem 1.2** Case \(\delta \leq 0\). Like in the proof of Theorem 1.1, we will consider \(p \in \overline{M}\) as the center of mass of \(\Omega, \{x_1, \ldots, x_N\}\) the normal coordinates of \(\overline{M}\) centered at \(p\) and by \(r(x)\) the geodesic distance (on \(\overline{M}\)) between \(x\) and \(p\), for any \(x \in \overline{M}\). By the choice of \(p\), we have
\[
\int_M \frac{s_\delta(r)}{r} x_i \tilde{\mu}_f = 0,
\]
for any \(i \in \{1, \ldots, N\}\), and we can use the functions \(\frac{s_\delta(r)}{r} x_i\) as test functions in the variational characterization of \(\sigma_1\). Thus, we have
\[
\sigma_1 \int_M \sum_{i=1}^N \frac{s_\delta^2(r)}{r^2} x_i^2 \tilde{\mu}_f \leq \int_{\Omega} \sum_{i=1}^N \left( T \nabla \left( \frac{s_\delta(r)}{r} x_i \right), \nabla \left( \frac{s_\delta(r)}{r} x_i \right) \right) \mu_f \\
\leq \int_{\Omega} \left( \text{tr} \, (T) - \delta \langle TX^\top, X^\top \rangle \right) \mu_f.
\] (17)

where we have used inequality (13) to get the second line. Here, \(X = \frac{s_\delta(r)}{r} \nabla r\) and \(X^\top = \frac{s_\delta(r)}{r} \nabla r\) is the tangent component of \(X\) to \(\Omega\). On the other hand, by item (3) of Lemma 2.1 applied to \(\overline{M}\), we have
\[
\delta \int_{\Omega} \langle TX^T, X^T \rangle \mu_f \geq \int_{\Omega} \text{tr} \,(T) c_\delta^2 \mu_f - \int_{\Omega} \|H_T - T(\nabla f)\| s_\delta c_\delta \mu_f.
\]

Reporting into (17), we have
\[
\sigma_1 \int_M s_\delta^2(r) \tilde{\mu}_f \leq \int_{\Omega} \left( \text{tr} \,(T) - c_\delta^2(r) \text{tr} \,(T) + \|H_T - T(\nabla f)\| s_\delta(r) c_\delta(r) \right) \mu_f
\]
\[
\leq \delta \int_{\Omega} s_\delta^2(r) \text{tr} \,(T) \mu_f + \int_{\Omega} \|H_T - T(\nabla f)\| s_\delta(r) c_\delta(r) \mu_f,
\]
\[
\leq \delta \int_{\Omega} s_\delta^2(r) \text{tr} \,(T) \mu_f + \sup _{\Omega} \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr} \,(T)} \right) \int_{\Omega} \text{tr} \,(T) s_\delta(r) c_\delta(r) \mu_f.
\]

From Lemma 4.1 applied on \(\Omega\) for the tensor \(T\), we have
\[
\int_{\Omega} \text{tr} \,(T) s_\delta(r) c_\delta(r) \mu_f \leq \int_{\Omega} \|H_T - T(\nabla f)\| s_\delta^2(r) \mu_f,
\]
which yields to
\[
\sigma_1 \int_M s_\delta^2(r) \tilde{\mu}_f \leq \int_{\Omega} s_\delta^2(r) \left[ \delta \text{tr} \,(T) + \sup _{\Omega} \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr} \,(T)} \right) \|H_T - T(\nabla f)\| \right] \mu_f.
\]

Moreover, from the assumption that \(\Omega\) is contained in the ball of radius \(B(p, R)\), we get that \(s_\delta(r) \leq s_\delta(R)\) and thus
\[
\sigma_1 \int_M s_\delta^2(r) \tilde{\mu}_f \leq s_\delta^2(R) V_f(\Omega) \sup _{\Omega} \left[ \delta \text{tr} \,(T) + \sup _{\Omega} \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr} \,(T)} \right) \|H_T - T(\nabla f)\| \right].
\]

Finally, by Lemma 3.2, we have
\[
\int_M s_\delta^2(r) \tilde{\mu}_f \geq \frac{V_f(M)}{\inf _M (\text{tr} \,(S))^2} \frac{1}{\delta + \frac{\sup _M \|H_S - S(\nabla f)\|^2}{\inf _M (\text{tr} \,(S))^2}},
\]
which give the desired estimate when \(\delta \leq 0\):
\[
\sigma_1 \leq \sup _{\Omega} \left[ \delta \text{tr} \,(T) + \sup _{\Omega} \left( \frac{\|H_T - T(\nabla f)\|}{\text{tr} \,(T)} \right) \|H_T - T(\nabla f)\| \right] \times \left[ \delta + \frac{\sup _M \|H_S - S(\nabla f)\|^2}{\inf _M (\text{tr} \,(S))^2} \right] \frac{V_f(\Omega)}{V_f(M)} s_\delta^2(R).
\]

Case \(\delta > 0\). Like in the case \(\delta \leq 0\), we have
\[
\sigma_1 \int_M s_\delta^2(r) \tilde{\mu}_f \leq \int_{\Omega} \left( \text{tr} \,(T) - \delta \left( TX^T, X^T \right) \right) \mu_f
\]
(18)

by using \(\frac{s_\delta(r)}{r} x_i, 1 \leq i \leq N\) as test functions. In addition, we also use another test function, \(c_\delta(r) - \tilde{c}_\delta\), with \(\tilde{c}_\delta = \frac{1}{V_f(M)} \int_M c_\delta(r) \tilde{\mu}_f\). By a computation analogue to the proof of Theorem 1.1, we have

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\[
\sigma_1 \int_M (c_\delta(r) - \tilde{c}_\delta)^2 \tilde{\mu}_f \leq \int_{\Omega} \langle T \nabla (c_\delta(r) - \tilde{c}_\delta), \nabla (c_\delta(r) - \tilde{c}_\delta) \rangle \mu_f \\
\leq \int_{\Omega} \langle T \nabla c_\delta(r), \nabla c_\delta(r) \rangle \tilde{\mu}_f \\
\leq \delta^2 \int_{\Omega} s_\delta^2(r) \langle T r, \nabla r \rangle \mu_f \\
\leq \delta^2 \int_{\Omega} \langle TX^\top, X^\top \rangle \mu_f.
\]

From this, we deduce
\[
\sigma_1 \int_M c_\delta^2(r) \tilde{\mu}_f \leq \delta^2 \int_{\Omega} \langle TX^\top, X^\top \rangle \mu_f + \frac{\sigma_1}{V_f(M)} \left( \int_M c_\delta(r) \tilde{\mu}_f \right)^2. \tag{19}
\]

Now, using the fact that \(c_\delta^2 + \delta s_\delta^2 = 1\), (19) plus \(\delta\) times (18) gives
\[
\sigma_1 V_f(M) \leq \delta \int_{\Omega} \text{tr} \, (T) \mu_f + \left( \int_M c_\delta(r) \tilde{\mu}_f \right)^2
\]
and so
\[
\sigma_1 V_f(M) \left( 1 - \left( \frac{\int_M c_\delta(r) \tilde{\mu}_f}{V_f(M)} \right)^2 \right) \leq \delta \int_{\Omega} \text{tr} \, (T) \mu_f.
\]

Finally, since we have
\[
1 - \left( \frac{\int_M c_\delta(r) \tilde{\mu}_f}{V_f(M)} \right)^2 \geq \frac{1}{1 + \left( \frac{\int_M \| H_S - S(\tilde{\nabla} f) \|^2 \tilde{\mu}_f}{\delta \inf(\text{tr} \, (S))^2 V_f(M)} \right)^2},
\]
by Lemma 3.2, we get
\[
\sigma_1 \leq \frac{\int_{\Omega} \text{tr} \, (T) \mu_f}{V_f(M)} \left( \delta + \frac{\int_M \| H_S - S(\tilde{\nabla} f) \|^2 \tilde{\mu}_f}{V_f(M) \inf(\text{tr} \, (S))^2} \right),
\]
and this concludes the proof. \(\square\)

**Proof of Theorem 1.3** Here again we consider differently the two cases. **Case** \(\delta \leq 0\). First, we recall the variational characterization of \(\alpha_1\) (see [5])
\[
\alpha_1 = \inf \left\{ \frac{\int_{\Omega} \| \nabla u \|^2 \, dv_{\tilde{g}} + b \int_M \| \nabla u \|^2 \, dv_{\tilde{g}}}{\int_M u^2 \, dv_{\tilde{g}}} \mid \int_M u \, dv_{\tilde{g}} = 0 \right\}. \tag{20}
\]

As in the proof of Theorem 1.2, we use \(s_\delta(r) r^{-\gamma} x_i\) as test functions, where \(r\) is the geodesic distance to the center of mass \(p\) of \(\Omega\) and \(\{x_1, \ldots, x_N\}\) the normal coordinates of \(\tilde{M}\) centered at \(p\). Hence, we have

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\[ \alpha_1 \int_M \sum_{i=1}^N \frac{s_\delta(r)}{r^2} x_i^2 \, dv_{\bar{g}} \leq \int_\Omega \sum_{i=1}^N \left\| \nabla \left( \frac{s_\delta(r)}{r} x_i \right) \right\| dv_{\bar{g}} + b \int_M \sum_{i=1}^N \left\| \nabla \left( \frac{s_\delta(r)}{r} x_i \right) \right\| dv_{\bar{g}} \]

that is,

\[ \alpha_1 \int_M s_\delta(r)^2 \, dv_{\bar{g}} \leq \int_\Omega \left( n - \delta \|X^\top\|^2 \right) \, dv_{\bar{g}} + b \int_M \left( (n - 1) - \delta \|X^\top\|^2 \right) \, dv_{\bar{g}}. \]  

(21)

where we use inequality (13) twice, once on \( \Omega \) for the first term and once on \( M \) for the second term. Moreover we have, \( \|X^\top\| \leq \|X\| = s_\delta(r) \) and since \( \delta \) is nonpositive, \( s_\delta \) is an increasing function. So, by the assumption that \( \Omega \) is contained in the ball \( B(p, R) \), we have

\[ \alpha_1 \int_M s_\delta(r)^2 \, dv_{\bar{g}} \leq \left( nV(\Omega) + b(n - 1)V(M) \right) \left( V(\Omega) + bV(M) \right) - \delta s_\delta(R)^2 \left( V(\Omega) + bV(M) \right). \]

We conclude by applying Lemma 3.2, which says for \( f = 0 \),

\[ \int_M s_\delta(r)^2 \, dv_{\bar{g}} \leq \frac{1}{\delta + \frac{\sup_M \|H_S\|^2}{\inf_M (\tr(S))^2}}, \]

to obtain the desired estimate when \( \delta \leq 0 \), that is,

\[ \alpha_1 \leq \left[ n \frac{V(\Omega)}{V(M)} + b(n - 1) - \delta s_\delta^2(R) \left( \frac{V(\Omega)}{V(M)} + b \right) \right] \left( \delta + \frac{\sup_M \|H_S\|^2}{\inf_M (\tr(S))^2} \right). \]

Case \( \delta > 0 \). Like in the case \( \delta \leq 0 \), using the functions \( \frac{s_\delta(r)}{r} x_i \) as test functions in the variational characterization of \( \alpha_1 \), we get

\[ \alpha_1 \int_M s_\delta^2(r) \, dv_{\bar{g}} \leq \int_\Omega \left( n - \delta \|X^\top\|^2 \right) \, dv_{\bar{g}} + b \int_M \left( n - 1 - \delta \|X^\top\|^2 \right) \, dv_{\bar{g}}. \]

(23)

Moreover, we use another test function, \( c_\delta(r) - \bar{c}_\delta \) with \( \bar{c}_\delta = \frac{1}{V(M)} \int_M c_\delta(r) \, dv_{\bar{g}} \).

\[ \alpha_1 \int_M (c_\delta(r) - \bar{c}_\delta)^2 \, dv_{\bar{g}} \leq \int_\Omega \left( \nabla (c_\delta(r) - \bar{c}_\delta), \nabla (c_\delta(r) - \bar{c}_\delta) \right) \, dv_{\bar{g}} \]

\[ + b \int_M \left( \nabla (c_\delta(r) - \bar{c}_\delta), \nabla (c_\delta(r) - \bar{c}_\delta) \right) \, dv_{\bar{g}} \]

\[ \leq \int_\Omega \nabla c_\delta(r), \nabla c_\delta(r) \right) \, dv_{\bar{g}} + b \int_M \left( \bar{V} c_\delta(r), \bar{V} c_\delta(r) \right) \, dv_{\bar{g}} \]

\[ \leq \delta^2 \int_\Omega s_\delta^2(r) \, \langle \nabla r, \nabla r \rangle \, dv_{\bar{g}} + b \delta^2 \int_M s_\delta^2(r) \, \langle \bar{V} r, \bar{V} r \rangle \, dv_{\bar{g}} \]

\[ \leq \delta^2 \int_\Omega \|X^\top\|^2 \, dv_{\bar{g}} + b \delta^2 \int_M \|X^\top\|^2 \, dv_{\bar{g}}. \]
where \(X^\top = s_\delta (r) \nabla r\) is the tangent component of \(X\) to \(\Omega\) and \(X^\top = s_\delta (r) \tilde{\nabla} r\) is the part of \(X\) tangent to \(M\). From this, we get

\[
\alpha_1 \int_M c_\delta (r)^2 dv_{\tilde{g}} \leq \delta^2 \int_\Omega \|X^\top\|^2 dv_\bar{g} + b \delta^2 \int_M \|X^\top\|^2 dv_{\bar{g}} + \frac{\alpha_1}{V(M)} \left( \int_M c_\delta (r) dv_{\bar{g}} \right)^2. 
\]

(24)

Hence summing (24) and \(\delta\) times (23), using the fact that \(c_\delta^2 + \delta s_\delta^2 = 1\), gives

\[
\alpha_1 V(M) \leq \delta \left( n V(\Omega) + b (n - 1) V(M) \right) + \frac{\alpha_1}{V(M)} \left( \int_M c_\delta (r) dv_{\bar{g}} \right)^2,
\]

and so

\[
\alpha_1 V(M) \left( 1 - \left( \frac{1}{V(M)} \int_M c_\delta (r) dv_{\bar{g}} \right)^2 \right) \leq \delta \left( n V(\Omega) + b (n - 1) V(M) \right).
\]

Moreover, from we have Lemma 3.2 (with \(f = 0\)), we have

\[
1 - \left( \frac{\int_M c_\delta (r) dv_{\bar{g}}}{V(M)} \right)^2 \geq \frac{1}{1 + \frac{\int_M \|H_S\|^2 dv_{\bar{g}}}{\delta \inf (\text{tr} (S))^2 V(M)}}
\]

which gives

\[
\alpha_1 V(M) \leq \delta \left( n V(\Omega) + b (n - 1) V(M) \right) \left( \delta + \frac{\int_M \|H_S\|^2 dv_{\bar{g}}}{V(M) \inf (\text{tr} (S))^2} \right)
\]

and finally the desired estimate

\[
\alpha_1 \leq \left( n \frac{V(\Omega)}{V(M)} + b (n - 1) \right) \left( \delta + \frac{\int_M \|H_S\|^2 dv_{\bar{g}}}{V(M) \inf (\text{tr} (S))^2} \right).
\]

This concludes the proof. \(\square\)

5 A remark about the case \(\delta > 0\)

The aim of this section is to compare the estimates in both case \(\delta > 0\) and \(\delta \leq 0\). Indeed, in the estimates of Theorems 1.2, the radius \(R\) of a ball containing \(\Omega\) appears when \(\delta \leq 0\) but not for the estimates for \(\delta > 0\). It turns out that when \(\delta > 0\), we can bound the radius \(R\) from above in terms of \(H_T\) and \(\text{tr} T\) and so obtain upper bounds comparable to those obtained for \(\delta \leq 0\). First, we have the following

Proposition 5.1 Let \((\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)\) be a weighted Riemannian manifold with sectional curvature \(\text{sect}_{\bar{g}} \leq \delta\), with \(\delta > 0\), and \(\bar{\mu}_f = e^{-f} dv_{\bar{g}}\). Let \((\Omega, g)\) be a compact Riemannian manifold with non-empty boundary \(M\) isometrically immersed into \((\bar{M}^{n+p}, \bar{g})\) by \(X\). We
endow $\Omega$ with the weighted measure $\mu_f = e^{-f} \, dv_g$. Let $T$ be a symmetric, divergence-free and positive definite $(1, 1)$-tensor on $\Omega$. If $\Omega$ is contained a ball of radius $R$, then

$$s_\delta^2(R) \left( \frac{\|H_T - T \nabla f\|_\infty^{2}}{\inf_{\Omega}(\text{tr}(T))^2} + \delta \right) \geq 1.$$  

**Proof** From the second part of Lemma 2.1 applied on $\Omega$, we have

$$\int_{\Omega} c_\delta(r) \text{tr}(T) \mu_f \leq - \int_{\Omega} (X, H_T - T \nabla f) \mu_f.$$

We recall that $X = s_\delta(r) \nabla r$, which gives

$$\int_{\Omega} c_\delta(r) \text{tr}(T) \mu_f \leq \int_{\Omega} \|H_T - T \nabla f\| s_\delta(r) \mu_f.$$

We are in the case where $\delta > 0$, so $c_\delta$ and $s_\delta$ are, respectively, decreasing and increasing on $[0, \frac{\pi}{2\sqrt{\delta}}]$. Moreover, $\Omega$ is contained in a ball of radius $R < \frac{\pi}{4\sqrt{\delta}}$ which implies that $\Omega$ is contained in a ball of radius $\frac{\pi}{2\sqrt{\delta}}$ centered at $p$. Hence, $r < R$ and so $cd(r) \geq c_\delta(R)$ and $s_\delta(r) \leq s_\delta(R)$ on $\Omega$. Thus, we get

$$c_\delta(R) \inf_{\Omega}(\text{tr}(T)) \leq s_\delta(R) \sup_{\Omega} \|H_T - T \nabla f\|.$$

We deduce easily from this and the fact that $c_\delta^2 + \delta s_\delta^2 = 1$ that

$$s_\delta^2(R) \left( \frac{\|H_T - T \nabla f\|_\infty^{2}}{\inf_{\Omega}(\text{tr}(T))^2} + \delta \right) \geq 1,$$

which concludes the proof of the proposition.  

Now, using the above proposition together with the estimate of Theorem 1.2 in the case $\delta > 0$, we get the following estimate

$$\sigma_1 \leq \left( \frac{\|H_T - T \nabla f\|_\infty^{2}}{\inf_{\Omega}(\text{tr}(T))^2} + \delta \right) \left( \delta + \frac{\int_{M} \|H_S - S(\nabla f)\|^2 \tilde{\mu}_f}{V_f(M) \inf_{\Omega}(\text{tr}(S))^2} \right) \frac{\int_{\Omega} \text{tr}(T) \mu_f}{V_f(M)} s_\delta^2(R)$$

$$\leq \sup_{\Omega}(\text{tr}(T)) \left( \frac{\|H_T - T \nabla f\|_\infty^{2}}{\inf_{\Omega}(\text{tr}(T))^2} + \delta \right) \left( \delta + \frac{\int_{M} \|H_S - S(\nabla f)\|^2 \tilde{\mu}_f}{V_f(M) \inf_{\Omega}(\text{tr}(S))^2} \right) \frac{V_f(\Omega)}{V_f(M)} s_\delta^2(R)$$

which is comparable to the estimate for the case $\delta \leq 0$:

$$\sigma_1 \leq \sup_{\Omega} \left[ \delta \text{tr}(T) + \sup_{\Omega} \left( \frac{\|H_T - T (\nabla f)\|}{\text{tr}(T)} \right) \|H_T - T (\nabla f)\| \right]$$

$$\times \left[ \delta + \sup_{M} \frac{\|H_S - S(\nabla f)\|^2}{\inf_{M}(\text{tr}(S))^2} \right] \frac{V_f(\Omega)}{V_f(M)} s_\delta^2(R).$$

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
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