We introduce a new non-zero-sum game of optimal stopping with asymmetric information. Given a stochastic process modelling the value of an asset, one player has full access to the information and observes the process completely, while the other player can access it only periodically at independent Poisson arrival times. The first one to stop receives a reward, different for each player, while the other one gets nothing. We study how each player balances the maximisation of gains against the maximisation of the likelihood of stopping before the opponent. In such a setup, driven by a Lévy process with positive jumps, we not only prove the existence, but also explicitly construct a Nash equilibrium with values of the game written in terms of the scale function. Numerical illustrations with put-option payoffs are also provided to study the behaviour of the players’ strategies as well as the quantification of the value of information.

AMS 2020 Subject Classifications: 60G51, 60G40, 91A15, 91A27, 90B50.

Keywords: optimal stopping, Lévy processes, non-zero-sum game, asymmetric information, periodic observations

1. INTRODUCTION

We consider a game of timing between two players who are after the same underlying asset, whose value is evolving stochastically according to a spectrally positive Lévy process \(X\), that allows for positive jumps. The players have asymmetric information on \(X\), based on which they can choose when to stop the game; the first one to stop receives an associated reward, while the other one gets nothing. More precisely, the fully informed player \(C\) (a representative “expert”) can observe the process \(X\) continuously (without delay) thanks to the information provided to them. On the contrary, the partially informed player \(P\) (a representative “non-expert”) observes the process \(X\) only periodically at some random times, given by the jump times of an independent Poisson process with rate \(\lambda > 0\). Player \(C\) is aware that competition exists by a partially informed player, knows that the opponent’s rate of observations is \(\lambda > 0\), but cannot know the actual (random) times of the opponent’s observations (only finds out after player \(P\) stops). Likewise, player \(P\) is also aware that the fully informed player exists. Even though they are both after the same asset, the additional information provided to player \(C\) (as opposed to player \(P\)) yields additional fees for player \(C\). Thus, for each value of \(X\), the net (immediate) reward of player \(C\) is strictly lower than that of player \(P\), which creates a trade-off for the aforementioned asymmetry in information. The question we aim at answering in this paper is “What is the optimal strategy for each player that balances the maximisation of gains (resp., minimisation of costs) against the simultaneous likelihood maximisation of stopping before the opponent?”. Given that each player has their own optimisation criterion based on their individual reward and information, the aforementioned question results in a competition which is formulated as a non-zero-sum game (NZSG) of optimal stopping under asymmetric information.
Many real-world situations present themselves as games where the participants have asymmetric information about the random environment affecting the games’ outcomes. Our motivation in the present paper comes from the fields of finance and operational research. A classical example comes from spot market platforms, where companies offer their excess capacity of supply in commodities, raw materials, physical goods, container shipments, cargo, etc. The non-negotiable (exogenous) price of the asset is affected solely by the market dynamics and can experience both “continuous” changes as well as positive jumps, which are driven for instance by sudden increases of demand or news. The players in such a scenario would be big corporations with departments of spot buying (experts continuously monitoring the price) and smaller companies with resource constraints (non-experts monitoring the price periodically) aiming at purchasing the limited supply of the asset that is on sale. In general, the spot market is an increasingly important operational tool for purchasing supply against the risk of higher than anticipated demand for goods (see, e.g. [40] and references therein). Another class of examples comes from the online hospitality and travel industry, where travel agencies or other platforms compete with consumers at purchasing limited special offers or last remaining rooms and tickets available. All aforementioned scenarios can be formulated as NZSGs of optimal stopping under the asymmetric information setting introduced in this paper.

The main contributions of the paper are the following:

1. we prove the existence of a Nash equilibrium for this class of NZSGs of optimal stopping with asymmetric information and concave reward functions, which is the class of rewards associated with risk-averse utility maximisation theory;
2. we develop a methodology that not only achieves the crucial theoretical existence of a Nash equilibrium, but also provides its explicit construction, which is of fundamental importance to applications;
3. this explicit nature of our results allows us to also (i) find a unique Nash equilibrium that is Pareto-superior to any other in case there is more than one, and present (ii) a straightforward numerical in-depth analysis of a case study with put-option payoffs, (iii) comparative statics with respect to the rate λ of player P’s observations, as well as (iv) a numerical study of the value of information, or in other words, the “fair consultant fee” that player C has to pay for accessing their advantageous additional information.

To the best of our knowledge, this is the first paper studying a NZSG of optimal stopping with asymmetric information, driven by continuous and periodic observations. Moreover, our method for constructing a Nash equilibrium allows us to go beyond the main focus of existing papers, which (even in zero-sum optimal stopping games with asymmetric information) is the proof of existence of a value for the game. Finally, we offer an alternative case study on the value of information in optimal decision timing under continuous versus periodic observations (see e.g. [2] and [3] for other considerations).

The literature on NZSGs of optimal stopping is mainly concerned with the existence of a Nash equilibrium. In a continuous time model, the first such existence result was proved via the study of a system of quasi-variational inequalities (QVIs) for diffusion processes [6], and afterwards, in terms of Dirichlet forms for symmetric Markov processes [32], in a probabilistic way for special (non-symmetric) right-processes [8], via the martingale approach for a class of non-cyclic games [23] and for (even non-Markovian) processes with positive jumps [22] (see also [19] for an earlier non-Markovian case of supermartingales). Another adequate solution concept, the weaker requirement of existence of an ε–equilibrium, was firstly proved in non-randomised [31] and more recently in randomised stopping times [27]. Special existence results are also obtained in various application-driven problems, such as game options in incomplete markets with utility-based arguments [24]. Beyond existence results, examples where such equilibria have been constructed are very limited and they are always under a symmetric full information setting, such as the convertible bonds pricing with corporate taxes and default risk in a geometric Brownian motion model [10], and the determination of sufficient conditions for the optimality of threshold-type stopping strategies under regular linear diffusion models for absorbing [4], or natural and not non-singular state-space boundaries [12].
In this paper, we expand the literature on NZSGs of optimal stopping both by proving the existence of a Nash equilibrium and by explicitly constructing the optimal threshold strategies, in a novel setting where (i) the underlying dynamics is given by a Lévy model with positive jumps and (ii) players have asymmetric information.

The asymmetry in information has attracted the attention of recent literature on optimal stopping games, though has focused only on zero-sum formulations – we extend this to a non-zero-sum formulation. A wide spectrum of information asymmetry forms has been considered. In this paper, the partially informed player gets to observe the underlying process only at discrete random times (not known to opponent), while the fully informed player observes it continuously. The asymmetry is therefore both in the content and the size of the two players’ filtrations. Differently to our paper, stopping strategies belonging to a strictly larger filtration than the opponent were used in [18] assuming that one player has additional information about the existence of a random variable and in [28] of a random time horizon, while a different filtration content was considered in [17] for two players with heterogeneous beliefs about the drift of a diffusion, who agree to disagree about it. In all aforementioned settings (including ours), the partially informed player cannot access the additional information or simply has no knowledge of its existence, thus the actions of opponents do not offer any additional insights. Therefore, proving the existence of a Nash equilibrium under such an asymmetry falls outside the standard theory of optimal stopping.

For completeness, we point out that in other considerations of asymmetric information (again only in zero-sum games), players are assumed to have partial knowledge about the rewards’ structure [21] (see also [20] for a simpler version), or about a Bernoulli random variable affecting the process’s drift [11] (see also [13] for a similar problem where both players have partial information), and partially informed players can learn about the unobservable rewards/drift from the opponent’s actions and their observations of the underlying processes. Such considerations lead to completely different formulations, where problems can actually be transformed to ones with full information using filtering techniques and can result in the use of randomised strategies by the initially fully informed players.

A random periodicity in observations, similar to the one of the underlying Lévy process \( X \) by the partially informed player \( P \) in our game, has been considered in one-player optimal stopping settings with various applications in the literature. The vanilla put/call option with exercise times restricted to be Poisson arrival times has been studied by [14] in a Brownian motion model and [36] in a Lévy model, while an endogenous bankruptcy (Leland-Toft model) in the Poisson observation setting was studied by [34]. On the other hand, to the best of our knowledge, optimal stopping games featuring periodic observation times have not been studied before.

NZSGs of optimal stopping in continuous time under full information have been studied via different methodologies, including QVIs [6], QVIs in Dirichlet forms [32], stochastic processes theory [19], backward induction [33], potential theory of Ray–Markov processes [8], and Snell envelope theory [22] (see also subgame-perfect equilibrium methods [37]). In this paper, we consider the case of a NZSG with asymmetric information and an underlying Lévy process \( X \). Either of these features sets our framework outside the well-developed standard theory (see, e.g. references above). Drawing on the theory of optimal stopping for diffusion processes is also non-feasible as it has limitations when dealing with jumps. It is also worth mentioning that, in zero-sum games of optimal stopping, the characterisation of threshold equilibria is given by the optimisation of a single expected reward, while in our NZSG we deal with the joint optimisation of two coupled expected rewards, whose complexity amplifies due to the information asymmetry, i.e. filtrations according to which each player is optimising.

In order to tackle our problem and the aforementioned complexity, we develop in this paper a more “direct” approach for proving the existence and eventually for explicitly constructing a Nash equilibrium. To be more precise, we firstly use a probabilistic approach focused on the values associated with “threshold strategies”, the cutting-edge fluctuation theory of Lévy processes, their scale functions and first-order conditions. Using this approach, we show the existence and explicitly construct a Nash equilibrium in the class of threshold strategies, for a wide class of underlying Lévy processes with positive jumps. More precisely, player \( C \) stops (i.e. buys the asset) at the first
time its price decreases to (or is below) a threshold $a^*$, while player $P$ stops at the first observation time when the asset price is below a threshold $l^*$ such that $l^* > a^*$. Note that, contrary to the former, the latter strategy is not a pure threshold one, since the price may hit or even cross below $l^*$ and then recover above it, before an observation time occurs, resulting in player $P$ not stopping in this case (see Figure 1 for an illustration of the information flow for both players and their stopping strategies). Notice that, even under the simplest and nonetheless important example of a Brownian motion model, classical optimal stopping methods would fail because the process at player $P$’s exercise time is random, as illustrated also in Figure 1. The power of our direct approach is also reflected by the fact that it does not rely on (or a priori assume) neither the continuous nor the smooth-fit condition. We however prove a posteriori that the value function of player $C$ (resp., player $P$) is always smooth at the optimal stopping threshold $a^*$ (resp., $l^*$) in both unbounded and bounded variation Lévy models.

We further prove that the values of these threshold strategies are sufficiently regular (see Proposition 4.13) to apply Itô’s formula, a necessary requirement for developing a verification theorem with a system of coupled variational inequalities (VIs), to potentially prove that their associated optimality is the strongest one possible – amongst all possible stopping strategies. We indeed develop the VI in this paper for such an upgrade of “optimality” for player $P$’s threshold strategy. This generalises the standard VIs [6] to our case of asymmetric information, a result appearing here for the first time to the best of our knowledge; the verification lemma associated to this VI (esp. its proof) is also non-standard (Lemma 5.4). However, the above methodology has limitations when used to upgrade the “optimality” of player $C$’s threshold strategy under the generality of our problem formulation in Lévy models and reward functions (additional assumptions seem to be required on both model and rewards, that are also hard to verify). Hence, we take a different route for player $C$, based on the use of an average problem approach (see, e.g., [41, 38, 30]), and we combine it with the aforementioned VI. In all, our proposed amalgamated methodology allows for the construction of a Nash equilibrium and verification of its optimality over all possible

**Figure 1.** Illustration of player $C$ and player $P$’s stopping strategies. The solid black trajectory shows the path of $X$ and the piecewise horizontal blue lines show player $P$’s most recent information on $X$; observation times are shown by dotted vertical lines. Given some $l^* > a^*$, player $P$ stops at the first observation time of $X$ below $l^*$ (indicated by red circles) and player $C$ stops at the classical hitting time below $a^*$ (indicated by green squares). **Case 1** shows the scenario when player $P$ gets to observe the process when $X$ is in $(a^*, l^*)$ (shown by the red strip) and thus exercises first. **Case 2** shows the scenario when player $P$ does not get to observe $X$ before it enters $(-\infty, a^*)$ (shown by the blue strip) and thus player $C$ exercises first.
stopping of the same asset, where the first one to stop receives an associated reward. To be more precise, player $C$ (resp., $P$) aims at maximising a discounted reward function $f_C : \mathbb{R} \to \mathbb{R}$ (resp., $f_P : \mathbb{R} \to \mathbb{R}$) by stopping the game before player $P$ (resp., $C$), otherwise receives nothing. Both players discount their future gains with a constant discount rate $q > 0$. Even though they are both after the same asset, the additional information provided to player $C$ (as opposed to player $P$) yields an additional fee for player $C$, if and when successfully stopping before player $P$.

A pair of stopping times $(\tau, \sigma)$ in this game consists of $\tau \in \mathcal{T}_C$ and $\sigma \in \mathcal{T}_P$, where $\mathcal{T}_C$ is the set of $\mathcal{F}$-stopping times and $\mathcal{T}_P := \{T^{(M)} : M \text{ is a } \mathcal{G}\text{-stopping time}\}$. This means that while player $C$ can stop in the “usual” way, player $P$ can stop only at the Poisson observation times. Each player aims at maximising their expected discounted reward functions given by

$$V_C(\tau, \sigma; x) := \mathbb{E}_x\left[e^{-q\tau} f_C(X_\tau) 1_{\{\tau < \sigma\}}\right]$$

and

$$V_P(\tau, \sigma; x) := \mathbb{E}_x\left[e^{-q\sigma} f_P(X_\sigma) 1_{\{\sigma < \tau\}}\right].$$

The rest of the paper is organised as follows. In Section 2, we provide a completely new mathematical formulation of the game between the two players with continuous and periodic observations, which can be understood for a general class of stochastic processes $X$ (not only Lévy models). In Section 3, we review the fluctuation theory of Lévy processes and the scale function. In particular, in Section 3.2, we also develop new identities required for expressing the expected rewards under threshold strategies, which can also be applied to study other optimal stopping problems and games under such information asymmetry; these could be natural directions for future research. In Section 4, we obtain a Nash equilibrium in a weak sense, by restricting the set of strategies to those of threshold-type. In Section 5, we strengthen the optimality and show that the one obtained in Section 4 is actually a Nash equilibrium in the strong sense (with strategy sets given by general sets of stopping times). We conclude the paper with numerical results on strategy optimality, comparative statics and the value of information in Section 6. Several long or technical proofs are deferred to the Appendices A–C.
The main aim of this paper is therefore to obtain, for each \( x \in \mathbb{R} \), a Nash equilibrium \((\tau^*, \sigma^*) \in T_c \times T_p\) such that
\[
V_c(\tau^*, \sigma^*; x) \geq V_c(\tau, \sigma^*; x), \quad \forall \tau \in T_c,
\]
\[
V_p(\tau^*, \sigma^*; x) \geq V_p(\tau^*, \sigma; x), \quad \forall \sigma \in T_p.
\]

3. Fluctuation identities

Throughout this paper, we focus on the case when \( X = \{X_t : t \geq 0\} \) is a spectrally positive Lévy process whose Laplace exponent is given by
\[
\psi(s) := \log \mathbb{E} \left[ e^{-sX_1} \right] = \gamma s + \frac{1}{2} \nu^2 s^2 + \int_{(0,\infty)} (e^{-sz} - 1 + sz 1_{(0<z<1)}) \Pi(dz), \quad s \geq 0,
\]
where \( \Pi \) is a Lévy measure on \((0, \infty)\) that satisfies the integrability condition \( \int_{(0,\infty)} (1 \wedge z^2) \Pi(dz) < \infty \). Note that, the process \( X \) has paths of bounded variation if and only if \( \nu = 0 \) and \( \int_{(0,1)} z \Pi(dz) < \infty \); in this case, we write (3.1) as
\[
\psi(s) = \mu s + \int_{(0,\infty)} (e^{-sz} - 1) \Pi(dz), \quad s \geq 0, \quad \text{where} \quad \mu := \gamma + \int_{(0,1)} z \Pi(dz).
\]
We exclude the case in which \( X \) is a subordinator (i.e., \( X \) has monotone paths a.s.). This implies that \( \mu > 0 \) when \( X \) is of bounded variation.

In the sequel, we will prove that, for a large class of reward functions \( f_p \) and \( f_c \) satisfying only certain mild assumptions, a pair of threshold strategies leads to the Nash equilibrium of our NZSG. In particular, player \( C \)'s optimal strategy will be to stop at the first down-crossing time of some level, while player \( P \)'s optimal strategy will be to stop at the first Poisson time at which the process is below some other level. To this end, we further denote, for \( b \in \mathbb{R} \), the random times
\[
\tau_b^- := \inf\{t > 0 : X_t < b\} \in T_c \quad \text{and} \quad T_b^- := \inf\{T^{(n)} : X_{T^{(n)}} < b\} \in T_p,
\]
where we recall \((T^{(n)})_{n \in \mathbb{N}}\) are the jump times of an independent Poisson process with rate \( \lambda \). The objective of this section is to thus derive the expressions, for \( x \in \mathbb{R} \) and \( a \leq l \), of the values of these threshold strategies
\[
v_c(x; a, l) := \mathbb{E}_x \left[ e^{-q\tau_a^-} f_c(X_{\tau_a^-}) 1_{\{\tau_a^- < T_l^-\}} \right], \quad \text{and} \quad v_p(x; a, l) := \mathbb{E}_x \left[ e^{-qT_l^-} f_p(X_{T_l^-}) 1_{\{T_l^- < \tau_a^-\}} \right]
\]
in terms of the scale function of \( X \). Note that \( T_l^- \neq \tau_a^- \) a.s. thanks to the independence between \( X \) and \( N \).

**Remark 3.1.** Notice that, for all threshold strategies with choices of \( l \leq a \) and all \( x \in \mathbb{R} \) in (3.3), we have \( \tau_a^- < T_l^- \) a.s., hence
\[
v_c(x; a, l) = v_c^\phi(x; a) := \mathbb{E}_x \left[ e^{-q\tau_a^-} f_c(X_{\tau_a^-}) 1_{\{\tau_a^- < \infty\}} \right] \quad \text{and} \quad v_p(x; a, l) = 0
\]
which boils down to a one-player maximisation problem for player \( C \), while player \( P \) does not participate in the game under such a choice of \( l \leq a \).

3.1. Scale functions. We denote by \( W^{(q)} : \mathbb{R} \to [0, \infty) \) the \( q \)-scale function corresponding to \( X \) for \( q > 0 \). It takes value zero on the negative half-line, while on the positive half-line it is the unique continuous and strictly increasing function defined by
\[
\int_0^\infty e^{-\theta x} W^{(q)}(x) \, dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q), \quad \text{where} \quad \Phi(q) := \sup\{s \geq 0 : \psi(s) = q\}.
\]

**Remark 3.2.** Some known properties of the scale function \( W^{(q)} \) are summarised as follows:
(i) The scale function $W^{(q)}$ is differentiable a.e. In particular, if $X$ is of unbounded variation or the Lévy measure $\Pi$ is atomless, it is known that $W^{(q)}$ is $C^1(\mathbb{R}\setminus\{0\})$; see, e.g., [9, Theorem 3].

(ii) As in Lemma 3.1 of [25], we have

$$W^{(q)}(0) = \begin{cases} 0 & \text{if } X \text{ is of unbounded variation}, \\ \mu^{-1} & \text{if } X \text{ is of bounded variation}. \end{cases}$$

We also define for $r > 0$,

$$Z^{(r)}(x; \theta) := e^{\theta x} \left( 1 + (r - \psi(\theta)) \int_0^x e^{-\theta u} W^{(r)}(u) \, du \right), \quad x \in \mathbb{R}, \, \theta \geq 0,$$

which further yields

$$Z^{(q+\lambda)}(x; \Phi(q)) = e^{\Phi(q)x} \left( 1 + \lambda \int_0^x e^{-\Phi(q)u} W^{(q+\lambda)}(u) \, du \right), \quad x \in \mathbb{R},$$

$$Z^{(q+\lambda)'}(x; \Phi(q)) = \Phi(q) Z^{(q+\lambda)}(x; \Phi(q)) + \lambda W^{(q+\lambda)}(x), \quad x \in \mathbb{R}\setminus\{0\}.$$

The following related result is proved in Appendix A.1.

**Lemma 3.3.** The mapping $u \mapsto W^{(q+\lambda)}(u)/Z^{(q+\lambda)}(u; \Phi(q))$ is increasing on $(0, \infty)$.

The scale functions are closely linked with several known fluctuation identities that will be used for the derivation of (3.3). By Theorem 3.12 in [26], we have

$$\mathbb{E}_x \left[ e^{-q\tau^+_0} ; \tau^-_0 < \infty \right] = e^{-\Phi(q)x}, \quad x \geq 0.$$

Let $\tau^+_b := \inf \{ t > 0 : X_t > b \}$ for $b \in \mathbb{R}$. By identity (8.11) of [26], we have

$$\mathbb{E}_x \left[ e^{-q\tau^+_0} - \theta(X^+_0 - b) ; \tau^-_0 > \tau^+_b \right] = Z^{(q)}(b - x; \theta) - \frac{Z^{(q)}(b; \theta)}{W^{(q)}(b)} W^{(q)}(b - x), \quad x, b, \theta \geq 0.$$

Using Theorem 2.7.(i) in [25], for any locally bounded measurable function $f$, constants $b > a$ and $y \geq a$, the resolvents are given by

$$\mathbb{E}_x \left[ \int_0^{\tau^-_0 + \tau^+_b} e^{-qs} f(X_s) \, ds \right] = \int_0^{b-a} f(b - u) \left\{ \frac{W^{(q)}(b - x)}{W^{(q)}(b)} W^{(q)}(b - a - u) - W^{(q)}(b - x - u) \right\} \, du.$$

### 3.2. Computation of $v_c(x; a, l)$ and $v_p(x; a, l)$ in (3.3)

We firstly fix $\lambda > 0$ and define

$$W^{(q+\lambda)}(x) := W^{(q)}(x + b) + \lambda \int_0^x W^{(q+\lambda)}(x - u) W^{(q)}(a + u) \, du \quad \forall x, b \in \mathbb{R},$$

where, in particular,

$$W^{(q+\lambda)}(x) = W^{(q)}(x), \quad \forall x \leq 0 \quad \text{and} \quad b \in \mathbb{R}.$$

The proof of the following result is given in Appendix A.2.

**Lemma 3.4.** For $b \geq l \geq a$ and any locally bounded measurable function $f_c$ on $\mathbb{R}$, we have

$$\mathbb{E}_x \left[ e^{-q\tau^-_a} f_c(X_{\tau^-_a}) 1_{\{\tau^-_a < \tau^+_0 \land \tau^+_b \}} \right] = \begin{cases} f_c(a) \frac{W^{(q+\lambda)}(l-x)}{W^{(q+\lambda)}(l-a)}, & x > a, \\ f_c(x), & x \leq a. \end{cases}$$
By taking the limit as $b \uparrow \infty$ in (3.12), we obtain the desired expression for $v_c(x; a, l)$ defined in (3.3). The proof of the following is deferred to Appendix A.3.

**Proposition 3.5.** For $l \geq a$ and any locally bounded measurable function $f_c$ on $\mathbb{R}$, the function $v_c(x; a, l)$ from (3.3) is given by

$$v_c(x; a, l) = \begin{cases} f_c(a) \frac{Z(q+\lambda)(l-x; \Phi(q))}{Z(q+\lambda)(l-a; \Phi(q))}, & \text{for } x > a, \\ f_c(x), & \text{for } x \leq a. \end{cases}$$

By (3.4) together with Proposition 3.5, for $x \geq l$, we particularly have

$$v_c(x; a, l) = f_c(a) \frac{e^{\Phi(q)(l-x)}}{Z(q+\lambda)(l-a; \Phi(q))} = e^{\Phi(q)(l-x)} v_c(l; a, l),$$

where

$$v_c(l; a, l) = \frac{f_c(a)}{Z(q+\lambda)(l-a; \Phi(q))}.$$ 

Consider now $f_p$ to be any locally bounded measurable function on $\mathbb{R}$. Then, we also define

$$\Gamma(x; l) := \int_0^{l-x} f_p(l-u) W^{(q+\lambda)}(l-x-u) \, du = \int_0^{l-x} f_p(u+x) W^{(q+\lambda)}(u) \, du, \quad \forall \ x, l \in \mathbb{R}. $$

The proof of the following result is given in Appendix A.4.

**Lemma 3.6.** For $b \geq l \geq a$ and any locally bounded measurable function $f_p$ on $\mathbb{R}$, we have

$$\mathbb{E}_x \left[ e^{-q T_{l-}^-} f_p(X_{T_{l-}^-}) 1_{\{T_{l-}^- < T_{l}^- \land \tau^+_b\}} \right] = \begin{cases} \lambda \left( \frac{\gamma^{(q,\lambda)}(l-x)}{\gamma^{(q,\lambda)}(l-a)} \Gamma(a; l) - \Gamma(x; l) \right), & \text{for } x > a, \\ 0, & \text{for } x \leq a. \end{cases}$$

By taking the limit as $b \uparrow \infty$ in (3.16), we obtain the desired expression for $v_p(x; a, l)$ defined in (3.3). The proof of the following is deferred to Appendix A.5.

**Proposition 3.7.** For $l \geq a$ and any locally bounded measurable function $f_p$ on $\mathbb{R}$, the function $v_p(x; a, l)$ from (3.3) is given by

$$v_p(x; a, l) = \begin{cases} \lambda \left( \frac{Z(q+\lambda)(l-x; \Phi(q))}{Z(q+\lambda)(l-a; \Phi(q))} \Gamma(a; l) - \Gamma(x; l) \right), & \text{for } x > a, \\ 0, & \text{for } x \leq a. \end{cases}$$

By Proposition 3.7, for $x \geq l$, we particularly have

$$v_p(x; a, l) = \lambda \frac{e^{\Phi(q)(l-x)}}{Z(q+\lambda)(l-a; \Phi(q))} \Gamma(a; l) = e^{\Phi(q)(l-x)} v_p(l; a, l),$$

where

$$v_p(l; a, l) = \frac{\lambda}{Z(q+\lambda)(l-a; \Phi(q))} \Gamma(a; l).$$
4. Optimality over threshold strategies

In this section, we consider a version of the game where admissible strategies are restricted to be of threshold-type (this will be strengthened in Section 5). Hence, the value functions of the two players take the form of (3.3) and the objective is to find a Nash equilibrium \((a^*, l^*) \in \mathbb{R}^2\) satisfying simultaneously the following two equations:

\[
v_c(x; a^*, l^*) = \max_{a \in \mathbb{R}} v_c(x; a, l^*),
\]

\[
v_p(x; a^*, l^*) = \max_{l \in \mathbb{R}} v_p(x; a^*, l).
\]

(4.1)

Although the barriers \((a, l)\) are allowed to depend on \(x\), the values of \((a^*, l^*)\) that we will obtain are invariant of \(x\).

For the rest of the paper, we make the following assumption on the reward functions.

**Assumption 4.1.** The reward functions \(f_c(\cdot)\) and \(f_p(\cdot)\) satisfy the following properties:

(i) We have \(f_c(\cdot) < f_p(\cdot)\) on \(\mathbb{R}\).

(ii) For \(i \in \{c, p\}\), the function \(f_i(\cdot)\) is strictly decreasing, continuously differentiable and concave on \(\mathbb{R}\) and admits a constant \(\overline{x}_i \in \mathbb{R}\) such that

\[
\begin{align*}
&f_i(\overline{x}_i) > 0, \quad x < \overline{x}_i, \\
&f_i(\overline{x}_i) < 0, \quad x > \overline{x}_i.
\end{align*}
\]

(4.2)

It is worth noting that Assumption 4.1.(i) reflects the additional costs bared by player \(C\) for the additional information provided, if and when successfully stopping before player \(P\). One important framework satisfying Assumption 4.1.(ii) is the perpetual American option pricing driven by an exponential Lévy process (see Section 6). Apart from this, Assumption 4.1.(ii) is a natural condition widely applicable. The decreasing reward functions reflect the game’s “optimal purchasing” nature, while the class of concave reward functions associates the present setting with the widely-used risk-averse utility maximisation theory, or even risk-neutral given that linear functions also satisfy this assumption.

**Remark 4.1.** By considering the dual process \(-X\), the results in this paper hold also for the case driven by a spectrally negative Lévy process and strictly increasing, continuously differentiable and concave \(f_i\), for \(i \in \{c, p\}\).

In view of Assumption 4.1, we automatically get that

\[
\overline{x}_c < \overline{x}_p.
\]

The values \(\overline{x}_c\) and \(\overline{x}_p\) will act as upper bounds for the optimal barriers for players \(C\) and \(P\); this is intuitive because it is obviously suboptimal to stop when the reward is negative.

### 4.1. Benchmark case: Single-player setting

We begin our analysis with the consideration of the special case when \(\lambda = 0\), i.e. player \(P\) can never stop, as a benchmark. This involves only player \(C\) whose expected reward under a threshold strategy \(\tau_a^-\) is derived by (3.6) and is given by (cf. Remark 3.1)

\[
v_c^0(x; a) := \mathbb{E}_x \left[ e^{-\eta_{\tau_a^-}} f_c(X_{\tau_a^-}) 1_{\{\tau_a^- < \infty\}} \right] = \begin{cases} f_c(x) & \text{for } x \leq a, \\
e^{\Phi(q)(a-x)} f_c(a) & \text{for } x > a. \end{cases}
\]

Straightforward differentiation gives

\[
\frac{\partial}{\partial a} v_c^0(x; a) = \begin{cases} 0 & \text{for } x < a, \\
e^{\Phi(q)(a-x)} h_c^0(a) & \text{for } x > a, \end{cases}
\]

(4.4)

where we define

\[
h_c^0(x) := \Phi(q) f_c(x) + f_c'(x), \quad x \in \mathbb{R}.
\]

(4.5)
By Assumption 4.1, the function \( h_c^0(\cdot) \) is continuous, monotonically decreasing and in particular satisfies \( h_c^0(\tau_c) = f'_{c}(\tau_c) < 0 \). Hence, there exists

\[
(4.6) \quad \alpha \in [\alpha, \tau_c) \quad \text{such that for } x \in \mathbb{R}, \quad h_c^0(x) \begin{cases} > 0, & x < \alpha, \\ < 0, & x > \alpha. \end{cases}
\]

Using Theorem 2.2 of [30], we can conclude that \( \tau_c^- \) is in fact the maximiser over all stopping times when \( \alpha > -\infty \). Indeed, in light of Remark 4.1, the assumptions imposed in [30] (where they consider the spectrally negative case) are satisfied by the properties of \( h_c^0 \) defined in (4.5). Hence, we have

\[
(4.7) \quad v_c^0(x; \alpha) = \max_{a \in \mathbb{R}} v_c^0(x; a) = \sup_{\tau \in T_c} E_x \left[ e^{-q\tau} f_c(X_\tau) 1_{\{\tau < \infty\}} \right], \quad x \in \mathbb{R}.
\]

Instead, when \( \alpha = -\infty \), an optimal stopping time does not exist.

Identity (4.7) also provides the solution to the degenerate case discussed in Remark 3.1.

### 4.2. Preliminary results

In view of Section 4.1, we make the following standing assumption in the rest of the paper, which essentially rules out the case when player \( C \) should optimally never stop (when there is no opponent), as this is not interesting in terms of applications.

**Assumption 4.2.** We assume that \( \alpha \) defined in (4.6) satisfies \( \alpha > -\infty \).

Analogous to \( h_c^0(\cdot) \) as in (4.5), we define the following continuous functions, for all \( x \in \mathbb{R} \),

\[
(4.8) \quad h_c(x) := (\Phi(q) + \lambda W^{(q+\lambda)}(0)) f_c(x) + f'_{c}(x) = h_c^0(x) + \lambda W^{(q+\lambda)}(0) f_c(x), \\
\quad h_p(x) := \Phi(q) f_p(x) + f'_{p}(x).
\]

For \( i \in \{c, p\} \), thanks to Assumption 4.1 and \( h_i(\tau_i) = f'_i(\tau_i) < 0 \), there exist

\[
(4.9) \quad \tau_i \in [\alpha, \tau_i) \quad \text{such that for } x \in \mathbb{R}, \quad h_i(x) \begin{cases} > 0, & x < \tau_i, \\ < 0, & x > \tau_i. \end{cases}
\]

**Remark 4.2.** It is straightforward to see from (4.8) that the function \( h_c(\cdot) \) coincides with \( h_c^0(\cdot) \) and \( \tau_c^- = \alpha \) if and only if \( W^{(q+\lambda)}(0) = 0 \) or if and only if \( X \) is of unbounded variation (see Remark 3.2.(ii)).

While \( \tau_p^- \) may be equal to \( -\infty \), Assumption 4.2 guarantees the finiteness of \( \tau_c^- \) as shown in the following result. This will be important in showing the existence of a Nash equilibrium, since \( \tau_c^- \) will act as a lower bound for \((a^*, l^*)\) (see Lemma 4.6 below). The proof is given in Appendix B.1.

**Lemma 4.3.** Recall \( \alpha \) and \( \tau_c^- \) defined in (4.6) and (4.9), respectively. We have \( -\infty < \alpha \leq \tau_c^- \).

Given the above result in Lemma 4.3 and the observations in Remark 3.1, we aim in the following result at restricting our focus on a strict subset of \( \mathbb{R}^2 \) for the selection of \((a^*, l^*)\) leading to the maximisation of (4.1). The proof is given in Appendix B.2.

**Lemma 4.4.** For \( x \in \mathbb{R} \), the problem in (4.1) satisfies:

(i) For any \( l \in \mathbb{R} \), we have \( \max_{a \in \mathbb{R}} v_c(x; a, l) = \max_{a \in (\alpha, \tau_c^-)} v_c(x; a, l) \);

(ii) If \( l \geq \alpha \) in (i), then we have \( \max_{a \in \mathbb{R}} v_c(x; a, l) = \max_{a \in (\alpha, \tau_c^-)} v_c(x; a, l) \);

(ii) For any \( a \in \mathbb{R} \), we have \( \max_{l \in \mathbb{R}} v_p(x; a, l) = \max_{l \in [a, \tau_p^\infty]} v_p(x; a, l) \).
Given that a Nash equilibrium \((a^*, l^*) \in \mathbb{R}^2\) must satisfy both equations in (4.1) simultaneously, the latter system can be written in light of Lemma 4.4.(i)–(ii) in the form of strategies:

\[
v_c(x; a^*, l^*) = \max_{a \in (-\infty, x \wedge l^* \wedge x]} v_c(x; a, l^*),
\]

\[
v_p(x; a^*, l^*) = \max_{l \in [a^*, \bar{p}]} v_p(x; a^*, l).
\]

4.3. First-order conditions. In this section we will characterise the candidate (optimal) thresholds \((a^*, l^*)\) by means of using the first-order conditions for the candidate value functions in (3.3), given by (3.13) and (3.17).

To this end, for any fixed \(l \in \mathbb{R}\), we define for \(a \in (-\infty, l]\) the function

\[
I(a; l) := f'_c(a) + (\Phi(q) Z^{(q+\lambda)}(l - a; \Phi(q)) + \lambda W^{(q+\lambda)}(l - a)) v_c(l; a, l)
\]

\[
= f'_c(a) + \Phi(q) f_c(a) + \lambda W^{(q+\lambda)}(l - a) v_c(l; a, l),
\]

where the second equality holds by (3.14). In particular, for \(a = l \in \mathbb{R}\), we have from (3.14) that \(v_c(l; l, l) = f_c(l)\), hence by (4.8) we get

\[
I(l; l) = h_c(l).
\]

Below, we compute the partial derivatives of (3.13) and (3.17) with respect to the threshold under control of each player; the proof is given in Appendix B.3.

**Lemma 4.5.** Consider \(v_c\) and \(v_p\) in (3.13) and (3.17), respectively. Then, we have:

(i) For \(a < l \land x\),

\[
\frac{\partial}{\partial a} v_c(x; a, l) = \frac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))} I(a; l).
\]

(ii) For \(x \geq a\) and \(l > a\) such that \(l \neq x\),

\[
\frac{\partial}{\partial l} v_p(x; a, l) = \lambda \left( \frac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))} W^{(q+\lambda)}(l - a) - W^{(q+\lambda)}(l - x) \right) \left( f_p(l) - v_p(l; a, l) \right).
\]

Using an appropriate modification of the identity in (3.8) for \(l \geq a\), we observe that

\[
\frac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))} W^{(q+\lambda)}(l - a) - W^{(q+\lambda)}(l - x) > 0, \quad \forall x \geq a,
\]

and hence we conclude from Lemma 4.5.(ii) that

\[
\text{sgn}\left( \frac{\partial}{\partial l} v_p(x; a, l) \right) = \text{sgn}\left( f_p(l) - v_p(l; a, l) \right), \quad \forall x \geq a, \ l > a.
\]

We can therefore extract from Lemma 4.5 the following two necessary conditions for the optimality of threshold strategies:

1. The first-order condition \(\frac{\partial}{\partial a} v_c(x; a, l) = 0\) for \(x \geq a\), required for the optimality (best response to any given \(l\)) of the candidate threshold \(a\), implies that the following condition should hold:

\[
C_a : I(a; l) = 0.
\]

2. The first-order condition \(\frac{\partial}{\partial l} v_p(x; a, l) = 0\) for \(x \geq a\), required for the optimality (best response to any given \(a\)) of the candidate threshold \(l\), implies that the following condition should hold:

\[
C_l : f_p(l) = v_p(l; a, l).
\]

Overall, the candidate (equilibrium) threshold pair \((a^*, l^*)\) should satisfy both conditions (4.14) and (4.15).
4.4. Existence and uniqueness of the best response threshold strategies $a^*$ and $l^*$. Now, we will check conditions for the existence and uniqueness of the candidate (optimal) thresholds $a^*$ and $l^*$ as best responses to arbitrary choices of thresholds by the opponent player.

4.4.1. Best response for Player C. Recall that, in order for the candidate (equilibrium) threshold pair $(a^*, l^*)$ to satisfy condition $C_a$ given by (4.14), we must have $I(a^*; l^*) = 0$. Hence, we aim at proving the existence of a unique solution to the equation $I(\cdot; l) = 0$ for an appropriate range of values of $l$. We begin with the most general case of $l \in \mathbb{R}$, by focusing only on player $C$ (cf. Lemma 4.4.(i)–(ii)), even though for optimality we will later only require $l^* \leq x_p$ (cf. (4.10) for $v_p$). In view of (4.10) for $v_c$, we therefore search for solutions to $I(\cdot; l) = 0$ on $(-\infty, x_c \land l]$. Refer to Figure 2.(i) in Section 6 for sample plots of the function $a \mapsto I(a; l)$.

**Lemma 4.6.** Recall that $x_c > -\infty$ (cf. Lemma 4.3) and the function $I(\cdot; l)$ defined in (4.11).

- (I) Suppose that $l < x_c$. Then, there does not exist $a \leq l$ such that $I(a; l) = 0$.
- (II) Suppose that $l \geq x_c$. Then, there exists on $(-\infty, x_c \land l]$, a unique root $\bar{a}(l)$ such that $I(\bar{a}(l); l) = 0$, i.e. satisfying (4.14). In addition, we have
  - (i) $\bar{a}(l) \in [x_c, x_c \land l]$ and $h_c(\bar{a}(l)) \leq 0$. In particular, $\bar{a}(x_c) = x_c$.
  - (ii) $I(\bar{a}(l); l) > 0$ for all $a < \bar{a}(l)$ and $I(\bar{a}(l); l) < 0$ for all $a > \bar{a}(l)$.
  - (iii) for all $x \in \mathbb{R}$, that $v_c(x; \bar{a}(l), l) = \max_{a \in (-\infty, x \land l \land x_c]} v_c(x; a, l)$ which is further upgraded to

$$
v_c(x; \bar{a}(l), l) = \max_{a \in \mathbb{R}} v_c(x; a, l).
$$

**Proof.** Suppose $a \leq x_c$ and $a < l$. Then, we have by (4.9) that $h_c(a) \geq 0$, or equivalently by (4.8) that $f_c'(a) + \Phi(q) f_c(a) \geq -\lambda W^{(q+\lambda)}(0) f_c(a)$. Taking this into account together with the definition (4.11) of $I(\cdot; l)$ and the equation (3.14), we get

$$
I(a; l) \geq \lambda \left( W^{(q+\lambda)}(l - a) v_c(l; a, l) - W^{(q+\lambda)}(0) f_c(a) \right)
$$

$$
= \lambda f_c(a) \left( \frac{W^{(q+\lambda)}(l - a)}{Z^{(q+\lambda)}(l - a; \Phi(q))} - W^{(q+\lambda)}(0) \right) > 0,
$$

where the latter inequality follows from Lemma 3.3, the fact that $Z^{(q+\lambda)}(0; \Phi(q)) = 1$ and because $f_c(a) > 0$ thanks to $a \leq x_c < x_c$ and (4.2).

**Part (I).** For $l < x_c$, it is straightforward to see from the above result that $I(a; l) > 0$ for all $a < l < x_c$.

**Part (II). Step 1: Existence.** For $l = x_c$, (4.12) gives (recalling $x_c > -\infty$ by Lemma 4.3)

$$
I(x_c; x_c) = h_c(x_c) = 0,
$$

thus existence is straightforward. For $l > x_c$, we have from (4.16) that $I(x_c; l) > 0$. On the other hand, it follows from (4.12) and (4.9) with $l > x_c$, that

$$
I(l; l) = h_c(l) < 0,
$$

and when $l > x_c$, we also have from (4.11) and (3.14) that

$$
I(x_c; l) = f_c'(x_c) < 0.
$$

Now thanks to the continuity of $I(\cdot; l)$, there must exist at least one $a$ such that $I(a; l) = 0$ on $[x_c, l \land x_c]$.

**Step 2: Uniqueness.** By differentiating (4.11) and using Lemma 4.5.(i), we obtain that

$$
\frac{\partial}{\partial a} I(a+; l) = f_c''(a+) + \Phi(q) f_c'(a) - \lambda W^{(q+\lambda)}((l - a) -) v_c(l; a, l) + \lambda \frac{W^{(q+\lambda)}(l - a)}{Z^{(q+\lambda)}(l - a; \Phi(q))} I(a; l).
$$
We now define, for all $a \in (-\infty, l \wedge \bar{x}_c]$, the function
\begin{equation}
(4.19) \quad I(a; l) := \exp \left\{ \lambda \int_a^l \frac{W(q+\lambda)(l-u)}{Z(q+\lambda)(l-u; \Phi(q))} \, du \right\} I(a; l) .
\end{equation}

The first derivative of $I(\cdot; l)$ can be obtained by using (4.18), namely, we have for all $a \leq l \wedge \bar{x}_c$
that
\[
\frac{\partial}{\partial a} I(a; l) = \exp \left\{ \lambda \int_a^l \frac{W(q+\lambda)(l-u)}{Z(q+\lambda)(l-u; \Phi(q))} \, du \right\} \left( f''_c(a) + \Phi(q) f'_c(a) - \lambda W(q+\lambda)'(l-a) v_c(l; a, l) \right)
\]
which is negative due to Assumption 4.1 and the positivity of $W(q+\lambda)'(l-a)$ and $v_c(l; a, l)$ thanks to $a < \pi_c$.
Hence, for any fixed $l \in \mathbb{R}$, the mapping $a \mapsto I(a; l)$ is decreasing on $(-\infty, l \wedge \bar{x}_c)$, which yields $I(\cdot; l) = 0$ has at most one solution on $(-\infty, l \wedge \bar{x}_c)$. In view of its definition in (4.19), it is straightforward to see that also $I(\cdot; l) = 0$ has at most one solution on $(-\infty, l \wedge \bar{x}_c)$. Hence, the solution to the equation $I(\bar{a}(l); l) = 0$, established in Step 1 on $[\bar{x}_c, \bar{x}_c \wedge l]$ is unique and denoted by $\bar{a}(l)$.

Step 3: Proof of part (i). This follows from the inequalities obtained in step 1.

Step 4: Proof of part (ii). This follows directly from the results in steps 1 and 2.

Step 5: Proof of part (iii). Lemma 4.5(i) and the above results imply that $\bar{a}(l)$ satisfies
\[
v_c(x; \bar{a}(l), l) = \max_{a \in (-\infty, x \wedge l \wedge \bar{x}_c]} v_c(x; a, l).
\]

Since $l \geq x_c \geq a$, Lemma 4.4(i)* gives $v_c(x; \bar{a}(l), l) = \max_{a \in \mathbb{R}} v_c(x; a, l)$, which completes the proof. \qed

Below we present the continuity and monotonicity of the threshold $\bar{a}(l)$ with respect to the arbitrarily chosen – until this stage of analysis – threshold $l \in [\bar{x}_c, \infty)$, according to Lemma 4.6.(II).

**Lemma 4.7.** The function $l \mapsto \bar{a}(l)$, which is defined in Lemma 4.6.(II), is continuous and strictly increasing in $l$ on $(\bar{x}_c, \infty)$.

**Proof.** To show the continuity, we argue by contradiction, assuming that it fails to be continuous at some $l^\dagger$, so that there exist two sequences $(l_n^\dagger)$ and $(l^\dagger_n)$ converging to $l^\dagger$ with $\lim_{n \to \infty} \bar{a}(l_n^\dagger) \neq \lim_{n \to \infty} \bar{a}(l^\dagger_n)$. Then, noting that $(a, l) \mapsto I(a; l)$ is continuous, we have
\[
I(\lim_{n \to \infty} \bar{a}(l_n^\dagger); l^\dagger) = \lim_{n \to \infty} I(\bar{a}(l_n^\dagger); l_n^\dagger) = 0 = \lim_{n \to \infty} I(\bar{a}(l^\dagger_n); l_n^\dagger) = I(\lim_{n \to \infty} \bar{a}(l^\dagger_n); l^\dagger).
\]
This contradicts with the uniqueness of the root $\bar{a}(l^\dagger)$ of $I(\cdot; l^\dagger) = 0$ as established in Lemma 4.6.

Now to show the monotonicity, we combine the definition of $I(a; l)$ in (4.11) with the expression in (3.14) and we calculate the partial derivative of $I(a; l)$ with respect to $l$, for all $a < \bar{x}_c$ (so that $f_c(a) > 0$), given by
\begin{equation}
(4.20) \quad \frac{\partial}{\partial l} I(a; l) = \lambda \frac{\partial}{\partial l} \left( W(q+\lambda)(l-a) v_c(l; a, l) \right) = \lambda f_c(a) \frac{\partial}{\partial l} \left( \frac{W(q+\lambda)(l-a)}{Z(q+\lambda)(l-a; \Phi(q))} \right) > 0,
\end{equation}
for a.e. $l > a$; the latter inequality follows from Lemma 3.3. Then, we argue again by contradiction, assuming that there exists $l^\dagger \in (\bar{x}_c, \infty)$ and $\delta > 0$, so that $\bar{a}(l^\dagger) \geq \bar{a}(l^\dagger + \delta)$. Since $I(\bar{a}(l^\dagger); l^\dagger) = I(\bar{a}(l^\dagger + \delta); l^\dagger + \delta) = 0$, we have
\[
I(\bar{a}(l^\dagger + \delta); l^\dagger + \delta) - I(\bar{a}(l^\dagger); l^\dagger + \delta) = I(\bar{a}(l^\dagger); l^\dagger) - I(\bar{a}(l^\dagger); l^\dagger + \delta) < 0,
\]
where the last inequality holds by (4.20) (and because $\bar{a}(l^\dagger) < \bar{x}_c$ by Lemma 4.6.(II).(i)). Hence, again by $I(\bar{a}(l^\dagger + \delta); l^\dagger + \delta) = 0$, we get
\[
I(\bar{a}(l^\dagger); l^\dagger + \delta) > 0.
\]
Now the assumption $\bar{a}(l^\dagger) \geq \bar{a}(l^\dagger + \delta)$ contradicts with the fact that $I(a; l^\dagger + \delta) \leq 0$ for $a \geq \bar{a}(l^\dagger + \delta)$ as in Lemma 4.6.(II).(ii), which completes the proof. \qed
4.4.2. Best response for Player $P$. For any fixed $a \in \mathbb{R}$, we now define for $l \in [a, \infty)$ the function

\begin{equation}
J(l; a) := Z^{(q+\lambda)}(l - a; \Phi(q)) \left( f_p(l) - v_p(l; a, l) \right) = f_p(l) Z^{(q+\lambda)}(l - a; \Phi(q)) - \lambda \Gamma(a; l),
\end{equation}

where the second equality holds by (3.19). In particular, taking into account (3.4) and (3.15), we get

\begin{equation}
J(a; a) = f_p(a), \quad \forall a \in \mathbb{R}.
\end{equation}

Then, by Proposition 3.7 and the expressions in (3.19) and (4.21), we get, for $a \leq x$, that

\begin{equation}
v_p(x; a, l) = Z^{(q+\lambda)}(l - x; \Phi(q)) v_p(l; a, l) - \lambda \Gamma(x; l)
= Z^{(q+\lambda)}(l - x; \Phi(q)) \left( v_p(l; a, l) - f_p(l) \right) + J(l; x).
\end{equation}

Recall that, in order for the candidate equilibrium threshold pair $(a^*, l^*)$ to satisfy condition $C_l$ given by (4.15), we must have $J(l^*; a^*) = 0$. Hence, we aim at proving the existence of a unique solution to the equation $J(\cdot; a) = 0$ for an appropriate range of values of $a$. We begin with the most general case of $a \in (-\infty, \pi_p)$, by focusing only on player $P$ (cf. Lemma 4.4(ii)), even though for optimality we will later only require $a^* \leq \pi_c$ (cf. (4.10) for $v_c$).

In view of (4.10) for $v_p$, we therefore search for solutions to $J(\cdot; a) = 0$ on $[a, \pi_p]$. Refer to Figure 2.(ii) in Section 6 for sample plots of the function $l \mapsto J(l; a)$.

**Lemma 4.8.** Fix $a < \pi_p$ and recall the function $J(\cdot; a)$ defined in (4.21).

(i) There exists on $[a, \pi_p]$, a unique root $\tilde{l}(a)$ such that $J(\tilde{l}(a); a) = 0$. In addition, this satisfies

\begin{equation}
\tilde{l}(a) \in (\pi_p \lor a, \pi_p) \quad \text{and} \quad h_p(\tilde{l}(a)) < 0, \quad \text{where } h_p \text{ is defined in (4.8)}.
\end{equation}

(ii) We have $J(l; a) > 0$, for all $l < \tilde{l}(a)$, and $J(l; a) < 0$, for all $l > \tilde{l}(a)$.

(iii) For $x \in \mathbb{R}$, we have $v_p(x; a, \tilde{l}(a)) = \max_{l \in \mathbb{R}} v_p(x; a, l)$ which is further upgraded to

\begin{equation}
v_p(x; a, \tilde{l}(a)) = \max_{l \in \mathbb{R}} v_p(x; a, l).
\end{equation}

**Proof.** We first prove parts (i) and (ii) together in the first two steps and part (iii) in the third step.

**Step 1.** Thanks to $a < \pi_p$, (3.15) and (4.2), we have

\begin{equation}
J(a; a) = f_p(a) > 0 \quad \text{and} \quad J(\pi_p; a) = -\lambda \Gamma(a; \pi_p) < 0.
\end{equation}

This together with the continuity of $l \mapsto J(l; a)$ shows that there exists at least one $l \in (a, \pi_p)$ such that $J(l; a) = 0$.

**Step 2.** By taking the partial derivative of (3.15), for all $a < l$, which gives $\frac{\partial}{\partial l} \Gamma(a; l) = f_p(l) W^{(q+\lambda)}(l - a)$ and using (3.5), we obtain

\begin{equation}
\frac{\partial}{\partial l} J(l; a) = f_p(l) Z^{(q+\lambda)}(l - a; \Phi(q)) + f_p(l) Z^{(q+\lambda)\prime}(l - a; \Phi(q)) - \lambda \frac{\partial}{\partial l} \Gamma(a; l) = h_p(l) Z^{(q+\lambda)}(l - a; \Phi(q)).
\end{equation}

Using (4.9) and the facts that $Z^{(q+\lambda)}(\cdot; \Phi(q))$ is uniformly positive and $h_p(\cdot)$ is continuous, we have

\begin{equation}
\frac{\partial}{\partial l} J(l; a) \begin{cases} > 0, & l < \pi_p, \\ < 0, & l > \pi_p. \end{cases}
\end{equation}

Combining this (when $\pi_p = -\infty, J(\cdot; a)$ is monotonically decreasing on $(-\infty, \pi_p]$) with (4.25), the solution in step 1 is unique and we denote it by $\tilde{l}(a)$. We also obtain the claims in (4.24) – thus completing part (i) – as well as the claim in part (ii).

**Step 3.** Combining part (ii) with (4.13), we conclude that $\tilde{l}(a)$ is indeed the maximiser over $[a, \pi_p]$. Thus, Lemma 4.4(ii) completes the proof. \qed
For the threshold pair \((a, \tilde{l}(a))\), we get in light of condition \(C_l\) from (4.15) and the expression (4.23) of \(v_p\), that
\[
v_p(x; a, \tilde{l}(a)) = J(\tilde{l}(a); x), \quad x \geq a,
\]
where in particular, recalling once again the condition \(C_l\), it is straightforward to confirm that \(v_p(a; a, \tilde{l}(a)) = J(\tilde{l}(a); a) = 0\). This expression will be useful both in the proof of the forthcoming result as well as later for strengthening the results in Section 5.

Similar to Lemma 4.7, we now present the continuity and monotonicity of the threshold \(\tilde{l}(a)\) with respect to the arbitrarily chosen – until this stage of analysis – threshold \(a \in (-\infty, \bar{x}_p)\), according to Lemma 4.8.

**Lemma 4.9.** The function \(a \mapsto \tilde{l}(a)\), which is defined in Lemma 4.8.(i), is continuous and strictly increasing in \(a\) on \((-\infty, \bar{x}_p)\).

**Proof.** To show the continuity, we argue by contradiction, assuming that it fails to be continuous at some \(a^+ < \bar{x}_p\), so that there exist two sequences \((a_n^-)\) and \((a_n^+)\) converging to \(a^+\) with \(\lim_{n \to \infty} \tilde{l}(a_n^-) \neq \lim_{n \to \infty} \tilde{l}(a_n^+)\). Then, noting that \((l, a) \mapsto J(l; a)\) is continuous, we have
\[
J(\lim_{n \to \infty} \tilde{l}(a_n^-); a^+) = \lim_{n \to \infty} J(\tilde{l}(a_n^-); a_n^-) = 0 = \lim_{n \to \infty} J(\tilde{l}(a_n^+); a_n^+) = J(\lim_{n \to \infty} \tilde{l}(a_n^+); a^+).
\]
This contradicts with the uniqueness of the root \(\tilde{l}(a)\) of \(J(\cdot; a^+) = 0\) established in Lemma 4.8.

To show the monotonicity, we again argue by contradiction, assuming that there is \(a^+ \in (-\infty, \bar{x}_p)\) and sufficiently small \(\delta > 0\), so that \(a^+ + \delta < \bar{x}_p\) and \(\tilde{l}(a^+ + \delta) \leq \tilde{l}(a^+)\). Since \(J(\tilde{l}(a^+); a^+) = J(\tilde{l}(a^+ + \delta); a^+ + \delta) = 0\), we have
\[
J(\tilde{l}(a^+); a^+) - J(\tilde{l}(a^+); a^+ + \delta) = J(\tilde{l}(a^+ + \delta); a^+ + \delta) - J(\tilde{l}(a^+); a^+ + \delta) \geq 0
\]
where the last inequality holds by (4.26) (recall that \(\tilde{l}(a^+ + \delta) > \bar{x}_p\) for all \(a \in [a^+, a^+ + \delta]\) by (4.24)) and by our assumption that \(\tilde{l}(a^+ + \delta) \leq \tilde{l}(a^+)\). Using once again the fact that \(J(\tilde{l}(a^+); a^+) = 0\), we obtain
\[
J(\tilde{l}(a^+); a^+ + \delta) \leq 0
\]
This contradicts with the fact that (4.27) is strictly positive given that \(\tilde{l}(a^+) < \bar{x}_p\). This completes the proof. □

4.5. **Construction of Nash equilibrium.** In the previous section, we established that, for any threshold \(l\) chosen by player \(P\) from an appropriate domain, player \(C\) chooses a unique best response \(\tilde{a}(l)\) such that (4.14) holds (cf. Lemma 4.6), i.e. \(I(\tilde{a}(l); l) = 0\). Moreover, for any threshold \(a\) chosen by player \(C\) from an appropriate domain, player \(P\) chooses a unique best response \(\tilde{l}(a)\) such that (4.15) holds (cf. Lemma 4.8), i.e. \(J(\tilde{l}(a); a) = 0\). The uniqueness of \(\tilde{a}(\cdot)\) and \(\tilde{l}(\cdot)\) and their continuity (cf. Lemmata 4.7 and 4.9) will be used in the proofs of Propositions 4.10 and 4.11. Sample plots of the functions \(a \mapsto I(a; \tilde{l}(a))\) and \(l \mapsto J(l; \tilde{a}(l))\) are shown in Figure 3 in Section 6.

In this subsection, we aim at analysing a fixed point \((a^*, l^*)\) satisfying
\[
l^* = \tilde{l}(a^*) \quad \text{and} \quad a^* = \tilde{a}(l^*).
\]
Namely, we shall prove that the associated stopping times to these threshold strategies are the best responses to each other. This condition can be shown to be equivalent to the equilibrium relation (4.1).

**Proposition 4.10.** A pair of barriers \((a^*, l^*)\) satisfies (4.28) if and only if it satisfies (4.1).

**Proof.** We prove the sufficiency and necessity separately in the following two steps.

**Step 1.** Suppose \((a^*, l^*)\) satisfies (4.28). Then, by Lemmata 4.6.(II).(iii) and 4.8.(iii), \((a^*, l^*)\) solves also (4.1).

**Step 2.** Suppose \((a^*, l^*)\) satisfies (4.1). Since the partial derivatives of \(v_c\) and \(v_p\) with respect to \(a\) and \(l\), respectively, were shown to be continuous in Lemma 4.5, \((a^*, l^*)\) must satisfy the first-order conditions in (4.14) and (4.15), which are equivalent to \(J(l^*; a^*) = I(a^*; l^*) = 0\). Now, by Lemma 4.6.(II), the equality \(I(a^*; l^*) = 0\)
requires that \( l^* \geq \bar{x}_c \) and by the uniqueness of the root \( \bar{a}(\cdot) \) we must have \( a^* = \bar{a}(l^*) \in [\bar{x}_c, \bar{x}_c \wedge l^*] \). Furthermore, given that \( a^* < \bar{x}_c < \bar{x}_p \), Lemma 4.8 implies that the equality \( J(l^*; a^*) = 0 \) guarantees (again by the uniqueness) that \( l^* = \bar{l}(a^*) \). Hence \((a^*, l^*)\) must satisfy the relationships in (4.28).

\(\square\)

We now show the root \( l^* \) of \( J(\cdot; \bar{a} (\cdot)) = 0 \) exists and together with the corresponding best response \( a^* \) form a Nash equilibrium in the sense of (4.28) (or equivalently (4.1), in light of Proposition 4.10).

**Proposition 4.11.** Fix \( x \in \mathbb{R} \).

(i) There exists a root \( l^* \) to the equation \( J(\cdot; \bar{a} (\cdot)) = 0 \), such that \( l^* \in (\bar{x}_c, \bar{x}_p) \).

(ii) For any root \( l^* \) in (i) and \( a^* := \bar{a}(l^*) \in (\bar{x}_c, \bar{x}_c \wedge l^*) \), the pair \((a^*, l^*)\) satisfies (4.28), hence (4.1) as well.

**Proof.** We prove the two parts separately.

**Proof of part (i).** By Lemma 4.6.(II),(i) and (4.22) and because \( \bar{x}_c < \bar{x}_p \) due to (4.3) and (4.9), we get

\[
J(\bar{x}_c; \bar{a}(\bar{x}_c)) = J(\bar{x}_c; \bar{x}_c) = f_p(\bar{x}_c) > 0.
\]

Furthermore, since \( f_p(\bar{x}_p) = 0 \) and \( \bar{x}_p \geq \bar{x}_c > \bar{a}(\bar{x}_p) \) by (4.3) and Lemma 4.6.(II),(i), (4.21) also yields that

\[
J(\bar{x}_p; \bar{a}(\bar{x}_p)) = -\lambda \Gamma(\bar{a}(\bar{x}_p); \bar{x}_p) = -\lambda \int_{\bar{a}(\bar{x}_p)}^{\bar{x}_p} f_p(u) W(u + \lambda)(u - \bar{a}(\bar{x}_p)) \, du < 0.
\]

Using (4.29)–(4.30) together with the fact that the function \( l \mapsto J(l; \bar{a}(l)) \) is continuous on \((\bar{x}_c, \bar{x}_p)\) (due to the continuity of \( l \mapsto \bar{a}(l) \) from Lemma 4.7), we complete the proof of this part.

**Proof of part (ii).** Using \( l^* > \bar{x}_c \) from part (i) to define \( a^* := \bar{a}(l^*) \), we conclude from Lemma 4.6 that \( a^* \in (\bar{x}_c, \bar{x}_c \wedge l^*) \). Thanks to the uniqueness of \( \bar{l}(\cdot) \) from Lemma 4.8, we have \( l^* = \bar{l}(a^*) \) if and only if \( J(l^*; a^*) = 0 \). This is indeed true, due to the ways \( l^* \) was derived and \( a^* := \bar{a}(l^*) \) was defined, which imply \( J(l^*; \bar{a}(l^*)) = 0 \). \(\square\)

Notice that the equivalence of (4.1) and (4.28) implies that \( J(l^*; \bar{a}(l^*)) = 0 \) (and \( a^* = \bar{a}(l^*) \)) is a necessary condition for \((a^*, l^*)\) to be a Nash equilibrium and hence the ones obtained in Proposition 4.11 are the only Nash equilibria. In particular, if \( l^* \) satisfying Proposition 4.11.(i) is unique, then the Nash equilibrium in the sense of (4.1) (i.e. the pair of \((a^*, l^*)\) satisfying the two equalities in (4.1)) is unique. However, there may be multiple \( l^* \) satisfying Proposition 4.11.(i). In such a scenario, we show in the following proposition, using also the monotonicity of \( \bar{a} (\cdot) \) and \( \bar{l} (\cdot) \) (cf. Lemmata 4.7 and 4.9), that by choosing the smallest (threshold) root in Proposition 4.11.(i), we can construct the unique Nash equilibrium that is Pareto-superior to any other Nash equilibrium. If we can assume that both players are rational and intelligent enough, we can discard other (Pareto-dominated) equilibria, because all agents are strictly better-off if they switch to these Pareto-superior equilibrium strategies.

To this end, we denote by \( l^*_{\text{min}} \) the minimum root in Proposition 4.11.(i), defined by

\[
l^*_{\text{min}} := \min\{ l \in (\bar{x}_c, \bar{x}_p) : J(l; \bar{a}(l)) = 0 \}.
\]

Then, by Proposition 4.11.(ii) with \( a^*_{\text{min}} := \bar{a}(l^*_{\text{min}}) \), we conclude that \((a^*_{\text{min}}, l^*_{\text{min}})\) is a Nash equilibrium (in the sense of (4.28) and (4.1)).

**Proposition 4.12.** The Nash equilibrium \((a^*_{\text{min}}, l^*_{\text{min}})\) is Pareto-superior to any other Nash equilibrium \((a^*, l^*)\) satisfying (4.28) (or equivalently (4.1)). In other words,

\[
v_i(x; a^*, l^*) \leq v_i(x; a^*_{\text{min}}, l^*_{\text{min}}), \quad \text{for both } i \in \{c, p\} \text{ and all } x \in \mathbb{R}.
\]

In particular, if \( x > a^*_{\text{min}} \), then the above inequality is strict.
Proof. We prove the desired claims in the following three steps.

Step 1. Suppose there exists \((a^*, l^*)\) satisfying (4.28), i.e., \(l^*_o = \overline{a}(l^*_o)\) and \(a^*_o = \overline{a}(l^*_o)\), different from \((a^*_\min, l^*_\min)\). This implies that \(J(l^*_o, \overline{a}(l^*_o)) = 0\), while \(a^*_o = \overline{a}(l^*_o)\) implies \(I(a^*_o; l^*_o) = 0\), hence we know from Lemma 4.6.(II).(i) that \(l^*_o > \overline{a}_c\). Due to Proposition 4.10 and the equivalence of (4.1) and (4.10), we must also have \(l^*_o < \overline{p}_p\). Combining these with the definition of \(l^*_\min\) in (3.41) and the fact that \(\overline{a}(\cdot)\) is unique due to Lemma 4.6.(II), we conclude that \(l^*_\min < l^*_o\). Using this together with the monotonicity of the function \(\overline{a}(\cdot)\) in Lemma 4.7, we get that

\[
a^*_o = \overline{a}(l^*_o) > \overline{a}(l^*_\min) = a^*_\min.
\]

Step 2. It is straightforward to see that \(l^*_\min < l^*_o\) (thus \(T^+_0 \leq T^*_\min\)) a.s. implies that \(\tau^+_a < T^*_\min\) a.s., which implies for \(x \in \mathbb{R}\) (recalling that \(a^*_o < \overline{p}_p\) from Lemma 4.6.(II), thus \(f_c(X_{l^*_o}^{-}) > 0\) a.s. by (4.2)), that

\[
v_c(x; a^*_o, l^*_o) = \mathbb{E}_x \left[ e^{-q^T_0} f_c(X_{l^*_o}^{-}) 1\{\tau^+_a < T^*_\min\} \right] \leq \mathbb{E}_x \left[ e^{-q^T_0} f_c(X_{l^*_o}^{-}) 1\{\tau^+_a < T^*_\min\} \right] \leq v_c(x; a^*_\min, l^*_\min),
\]

where the last inequality holds due to \(a^*_\min\) being the best response to \(l^*_\min\) and is strict when \(x > a^*_\min\).

Step 3. Similarly, \(a^*_\min < a^*_o\) (thus \(\tau^+_a \leq \tau^+_{a^*_\min}\)) a.s. implies that \(T^*_0^+ \leq \{\tau^+_a < T^*_\min\}\) a.s., hence for \(x \in \mathbb{R}\) (recalling that \(l^*_o < \overline{p}_p\) from Lemma 4.8.(i), thus \(f_p(X_{l^*_o}^+) > 0\) a.s. by (4.2)), we obtain

\[
v_p(x; a^*_o, l^*_o) = \mathbb{E}_x \left[ e^{-q^T_0} f_p(X_{l^*_o}^{-}) 1\{\tau^+_a < T^*_\min\} \right] \leq \mathbb{E}_x \left[ e^{-q^T_0} f_p(X_{l^*_o}^{-}) 1\{\tau^+_a < T^*_\min\} \right] \leq v_p(x; a^*_\min, l^*_\min),
\]

where the last inequality holds due to \(l^*_\min\) being the best response to \(a^*_\min\) and is strict when \(x > a^*_\min\). \(\square\)

4.6. Properties of \(v_c(\cdot; a^*, l^*)\) and \(v_p(\cdot; a^*, l^*)\). Before concluding this section, we obtain some properties of \(v_c(\cdot; a^*, l^*)\) and \(v_p(\cdot; a^*, l^*)\) for \((a^*, l^*)\) satisfying (4.28).

Observe that, by (4.15), (4.21) and due to \(J(l^*_o; a^*) = 0\), we obtain

\[
\lambda \Gamma(a^*; l^*) = f_p(l^*) = v_p(l^*; a^*, l^*).
\]

Substituting this in (3.17) for \(v_p(x; a^*, l^*)\), we get

\[
v_p(x; a^*, l^*) = Z^{(a+\lambda)}(l^* - x; \Phi(q)) v_p(l^*; a^*, l^*) - \lambda \Gamma(x; l^*) , \quad \text{for } x \geq a^*.
\]

This alternative expression will be used both for proving the next result (see Appendix B.4) as well as in Section 5.

Proposition 4.13 (Smoothness and convexity). Recall \(v_c(\cdot; a^*, l^*)\) and \(v_p(\cdot; a^*, l^*)\) defined in (3.3) and satisfying (4.1).

(I) Regarding the function \(v_c(\cdot; a^*, l^*)\), we have the following:

(i) \(v_c(\cdot; a^*, l^*)\) is continuous on \(\mathbb{R}\) and \(C^2\) (resp., \(C^1\)) on \((a^*, \infty)\setminus\{l^*\}\) when \(X\) is of unbounded (resp., bounded) variation.

(ii) \(v_c(\cdot; a^*, l^*)\) is continuously differentiable at \(a^*\).

(iii) \(v_c(\cdot; a^*, l^*)\) is continuously differentiable at \(l^*\), only when \(X\) is of unbounded variation.

(iv) \(v_c(\cdot; a^*, l^*)\) is decreasing and convex on \((a^*, \infty)\).

(II) Regarding the function \(v_p(\cdot; a^*, l^*)\), we have the following:

(i) \(v_p(\cdot; a^*, l^*)\) is continuous on \(\mathbb{R}\) and twice continuously differentiable on \(\mathbb{R}\setminus\{a^*, l^*\}\).

(ii) \(v_p(\cdot; a^*, l^*)\) is continuously differentiable at \(l^*\).

(iii) \(v_p(\cdot; a^*, l^*)\) is twice continuously differentiable at \(l^*\), only when \(X\) is of unbounded variation.
5. Optimality over all stopping times

In Proposition 4.11, we showed the existence of the solutions \((a^*, l^*)\) to (4.14) and (4.15) and that \((a^*, l^*)\) is a Nash equilibrium in the sense of (4.1) where strategies are restricted to be of threshold-type. In Proposition 4.12, we showed that by choosing \((a_{\text{min}}^*, l_{\text{min}}^*)\) as in (4.31), we can construct a Nash equilibrium that is Pareto-superior to other \((a^*, l^*)\) satisfying (4.28) or equivalently (4.1) (cf. Proposition 4.10).

In this section, we strengthen the results by considering a larger set of admissible strategies. For \(x \in \mathbb{R}\), we aim at showing the following two propositions. Note that, the results of this section hold for any selection of \((a^*, l^*)\) as long as (4.28) or equivalently (4.1) is satisfied.

**Proposition 5.1.** With \((a^*, l^*)\) obtained in Proposition 4.11, we have

\[
(5.1) \quad v_p(x; a^*, l^*) = \sup_{\sigma \in \mathcal{T}_p} V_p(\tau_{a^*}, \sigma; x), \quad x \in \mathbb{R}.
\]

**Proposition 5.2.** With \((a^*, l^*)\) obtained in Proposition 4.11, we have

\[
(5.2) \quad v_c(x; a^*, l^*) = \sup_{\tau \in \mathcal{T}_c} V_c(\tau, T_{l^*}^-; x), \quad x \in \mathbb{R}.
\]

By Propositions 5.1 and 5.2, we immediately get that the pair of strategies \((\tau_{a^*}, T_{l^*}^-)\) is a Nash equilibrium as in Section 2, when the strategy sets of both players are unrestricted (most general ones possible); this is formally stated in the following.

**Theorem 5.3.** With \((a^*, l^*)\) obtained in Proposition 4.11, we have for all \(x \in \mathbb{R}\),

\[
\begin{align*}
V_c(\tau_{a^*}, T_{l^*}^-; x) & \geq V_c(\tau, T_{l^*}^-; x), \quad \forall \tau \in \mathcal{T}_c, \\
V_p(\tau_{a^*}, T_{l^*}^-; x) & \geq V_p(\tau_{a^*}, \sigma; x), \quad \forall \sigma \in \mathcal{T}_p.
\end{align*}
\]

The rest of this section is devoted to the proofs of Propositions 5.1 and 5.2.

### 5.1. Proof of Proposition 5.1.

Below, we prove (5.1) via the use of verification arguments. To this end, we define the infinitesimal generator \(\mathcal{L}\) acting on \(v_p(\cdot; a^*, l^*)\) as follows:

\[
\mathcal{L}v_p(x; a^*, l^*) := -\gamma v_p'(x; a^*, l^*) + \frac{\nu^2}{2} v_p''(x; a^*, l^*) + \int_{(0, \infty)} [v_p(x + z; a^*, l^*) - v_p(x; a^*, l^*) - v_p'(x; a^*, l^*)] \Pi(dz).
\]

By the smoothness obtained in Proposition 4.13.(II), and because \(v_p(x; a^*, l^*)\) is bounded in \(x\) by \(\max_{a^*_y \leq x \leq l^*} f_p(y) = f_p(a^*) \leq f_p(x) \leq f_p(x_c) < \infty\), due to the monotonicity of \(f_p\) in Assumption 4.1.(ii), the admissible interval for \(a^* = \tilde{a}(l^*)\) in Lemma 4.6.(II),(i), and the finiteness of \(x_c\) in Lemma 4.3, we conclude that \(\mathcal{L}v_p(x; a^*, l^*)\) is defined for all \(x \in \mathbb{R}\setminus \{a^*\}\).

We first obtain sufficient conditions for the optimality (verification lemma) in the optimal stopping problem (5.1). The proof of the following result is deferred to Appendix C.

**Lemma 5.4** (Verification lemma for (5.1)). Suppose that

(i) \((\mathcal{L} - q)v_p(x; a^*, l^*) = 0\), for \(x \geq l^*\),

(ii) \((\mathcal{L} - q)v_p(x; a^*, l^*) - \lambda(v_p(x; a^*, l^*) - f_p(x)) = 0\), for \(x \in (a^*, l^*)\),

(iii) \(v_p(x; a^*, l^*) \geq f_p(x)\), for \(x \geq l^*\),

(iv) \(v_p(x; a^*, l^*) \leq f_p(x)\), for \(x \in [a^*, l^*]\),

(v) \(v_p(x; a^*, l^*) = 0\), for \(x \leq a^*\).
Then, (5.1) holds true.

The remainder of the proof requires the verification of the five conditions assumed in Lemma 5.4 to eventually conclude the desired optimality.

Part (i). This follows from the expression (3.18) of \( v_p \) and the direct computation of

\[
(\mathcal{L} - q)e^{\Phi(q)(l^* - x)} = 0.
\]

Part (ii). By (4.20) in [35] and Lemma 4.5 in [15], respectively, we get

\[
(\mathcal{L} - (q + \lambda))Z^{(q+\lambda)}(l^* - x; \Phi(q)) = 0 \quad \text{and} \quad (\mathcal{L} - (q + \lambda))\Gamma(x; l^*) = f_p(x).
\]

Applying these equations to the expression (3.17) of \( v_p \), we obtain the desired \((\mathcal{L}-(q+\lambda))v_p(x; a^*, l^*) = -\lambda f_p(x)\).

Parts (iii) and (iv). Using the expression (4.32) of \( v_p \), we define

\[
R_p(x) := v_p(x; a^*, l^*) - f_p(x) = Z^{(q+\lambda)}(l^* - x; \Phi(q)) f_p(l^*) - \lambda \Gamma(x; l^*) - f_p(x), \quad x \geq a^*.
\]

In particular, we have by (3.17)

\[
R_p(a^*) = v_p(a^*, a^*, l^*) - f_p(a^*) = -f_p(a^*) < 0,
\]

and by condition \( C_1 \) in (4.15), we have

\[
R_p(l^*) = 0.
\]

First, suppose that \( x \geq l^* \) aiming for the proof of (iii). By (3.18), we get the simplified expression

\[
R_p(x) = v_p(x; a^*, l^*) - f_p(x) = e^{\Phi(q)(l^*-x)} v_p(l^*; a^*, l^*) - f_p(x), \quad x \geq l^*.
\]

By calculating its first and second derivatives, we get

\[
R'_p(x) = -\Phi(q) e^{\Phi(q)(l^*-x)} v_p(l^*; a^*, l^*) - f'_p(x),
\]

\[
R''_p(x) = \Phi^2(q) e^{\Phi(q)(l^*-x)} v_p(l^*; a^*, l^*) - f''_p(x) > 0,
\]

where the last inequality follows from Assumption 4.1(ii) and because \( v_p(l^*; a^*, l^*) > 0 \). We can thus immediately see that \( R_p(\cdot) \) is convex on \((l^*, \infty)\), while the inequality (4.24) for the equilibrium threshold pair \((l, a) = (l^*, a^*)\) and the definition (4.8) of \( h_p \), imply that

\[
R'_p(l^*) = -\Phi(q) v_p(l^*; a^*, l^*) - f'_p(l^*) = -\Phi(q) f_p(l^*) - f'_p(l^*) = -h_p(l^*) > 0.
\]

Therefore, combining the above inequality with the convexity of \( R_p(\cdot) \) we get that \( R'_p(x) > 0 \) for all \( x > l^* \). By this and (5.4), we have that \( R_p(x) \geq 0 \) for all \( x \geq l^* \). In all, the definition (5.5) yields \( v_p(x; a^*, l^*) \geq f_p(x) \) for all \( x \geq l^* \), proving that part (iii) indeed holds true.

Suppose now that \( a^* \leq x < l^* \) aiming for the proof of (iv). Differentiating (5.3) and using (4.8), we get

\[
R'_p(x) = -\Phi(q) Z^{(q+\lambda)}(l^* - x; \Phi(q)) f_p(l^*) - \lambda \int_{l^*-x}^{l^*} f'_p(u + x) W^{(q+\lambda)}(u) \, du - f'_p(x)
\]

\[
= -\Phi(q) R_p(x) - \lambda \int_{l^*-x}^{l^*} h_p(u + x) W^{(q+\lambda)}(u) \, du - h_p(x).
\]

Then, we define \( \overline{R}_p(x) := e^{\Phi(q)x} R_p(x) \) and observe that

\[
\overline{R}_p(x) = e^{\Phi(q)x} \left[ -\lambda \int_{l^*-x}^{l^*} h_p(u + x) W^{(q+\lambda)}(u) \, du - h_p(x) \right].
\]
In view of (4.9), we have for all \( x \in (x_p, l^*) \), which is a well-defined interval (cf. Lemma 4.8), that \( h_p(x) < 0 \). Therefore, \( \overline{R}_p(x) > 0 \) for all \( x \in (x_p, l^*) \). Combining this monotonicity with the fact that \( \overline{R}_p(l^*) = 0 \) (see (5.4) and the definition of \( \overline{R}_p \)), we conclude that
\[
\overline{R}_p(x) < 0, \quad \text{for all } x \in (x_p, l^*).
\]

For \( x \leq x_p \), using the probabilistic expression of \( v_p \) in the definition (3.3), we observe that
\[
v_p(x; a^*, l^*) = \mathbb{E}_x \left[ e^{-qT_{l^*}} f_p(X_{T_{l^*}}) 1_{\{T_{l^*} < \tau_{a^*}\}} \right] \leq \mathbb{E}_x \left[ e^{-qT_{l^*}} f_p(X_{T_{l^*}}) 1_{\{T_{l^*} < \infty\}} \right] \\
\leq \sup_{\tau \in \mathcal{T}_c} \mathbb{E}_x \left[ e^{-q\tau} f_p(X_\tau) 1_{\{\tau < \infty\}} \right] = \mathbb{E}_x \left[ e^{-q\tau_y} f_p(X_{\tau_y}) 1_{\{\tau_y < \infty\}} \right],
\]
where the last optimality holds similarly to Section 4.1 (by using \( f_p \) instead of \( f_c \) and \( h_p \) from (4.8) instead of \( h_c^0 \) in (4.5) and using (4.9)). Hence,
\[
v_p(x; a^*, l^*) \leq \mathbb{E}_x \left[ e^{-q\tau_y} f_p(X_{\tau_y}) 1_{\{\tau_y < \infty\}} \right] = f_p(x), \quad \text{for all } x \in [a^* \vee x_p, x_p].
\]

**Part (v).** This follows directly from the definition of \( v_p(x; a^*, l^*) \) in (3.3).

### 5.2. Proof of Proposition 5.2.

To upgrade the optimality of threshold-type strategies over all stopping times for player \( C \), we will not rely on an analytical verification theorem, as we performed in Section 5.1 for player \( P \). In fact, by attempting to prove optimality in the latter way for the general setting of reward functions \( f_c(\cdot) \) and Lévy models considered in this paper, one observes that further assumptions should be imposed, that are also hard to verify. Instead, we employ here a different methodology that is based on the use of an *average problem approach* (see e.g. [30], [38] and [41]) to prove the optimality of threshold type strategies over all stopping times. In this way, we successfully maintain the original general setting of our paper without relying on further assumptions.

To this end, we denote the right hand side of (5.2) by \( u(x) \). We thus aim at proving that
\[
u(x) = v_c(x; a^*, l^*).
\]

Using the definition (3.2) of \( T_{l^*}^- \) and the independence of the Poisson process \( N \) and Lévy process \( X \), we can rewrite the current optimal stopping problem with random time-horizon \( T_{l^*}^- \), in the form of
\[
u(x) = \sup_{\tau \in \mathcal{T}_c} \mathbb{E}_x \left[ e^{-AQX_{\tau}} f_c(X_\tau) 1_{\{\tau < \infty\}} \right],
\]
where the latter is a perpetual optimal stopping problem with stochastic discounting given by the occupation time
\[
A_t^X := qt + \lambda \int_0^t 1_{\{X_u < l^*\}} \, du, \quad \forall t \geq 0,
\]
(see [30] and [38] for other optimal stopping problems with occupation time discounting under one-dimensional Lévy models, and [39] under two-dimensional ones). It is easy to see that \( A_t^X \) is a continuous additive functional.

In order to use the results of [30], where they consider the spectrally negative case, we define the process \( Y_t := -X_t \), for all \( t \geq 0 \), which is a spectrally negative Lévy process starting from \( Y_0 = -x \). We then define
\[
\hat{f}_c(y) := f_c(-y) \quad \text{for all } y \in \mathbb{R},
\]
and we observe from Assumption 4.1.(ii) that \( \hat{f}_c(\cdot) \) is a strictly increasing, continuously differentiable and concave function on \( \mathbb{R} \), such that \( \hat{f}_c(y) > 0 \) if and only if \( y > -\pi_c \). Hence, the optimal stopping problem (5.7) can be further rewritten in terms of the process \( Y \), with \( \overline{E}_y := \mathbb{E}_{-y} \) and \( x = -y \), taking the form
\[
u(x) = \sup_{\tau \in \mathcal{T}_c} \overline{E}_y \left[ e^{-AQ^{-Y}_{\tau}} f_c(\tau_{-Y}) 1_{\{\tau < \infty\}} \right] = \sup_{\tau \in \mathcal{T}_c} \overline{E}_y \left[ e^{-AQ^{Y} f_c(Y_\tau)} 1_{\{\tau < \infty\}} \right] := \hat{u}(y),
\]}
where we notice that
\[ A^+_t = \hat{A}^+_t := qt + \lambda \int_0^t 1_{\{Y_u > -t^*\}} \, du, \quad \forall \, t \geq 0. \]

Then, we define the left-inverse \( \zeta \) of \( \hat{A}^Y \) at an independent exponential time e with unit mean by
\[ \zeta \equiv (\hat{A}^Y)^{-1}(e) := \inf\{ t > 0 : \hat{A}^Y_t > e \} \]
and the running maximum process of \( Y \) by \( \overline{Y}_t := \sup_{0 \leq u \leq t} Y_u \). Using similar arguments to [38, Section 4.1], we obtain for \( y \leq -a \), that
\[ \overline{P}_y(\overline{Y}_\zeta > -a) = \overline{P}_y \left[ \exp(-\hat{A}^Y_{\zeta-a}(e)) 1_{\{\zeta-a < \infty\}} \right] = \overline{E}_y \left[ \exp(-A^X_{\zeta-a}) 1_{\{\zeta-a < \infty\}} \right] = \overline{E}_y \left[ e^{q\tau^-} 1_{\{\tau^- < \infty\}} \right] = \frac{Z(q+\lambda)(I^* + y; \Phi(q))}{Z(q+\lambda)(I^* - a; \Phi(q))}, \]
where the second equality follows from the definition of \( Y \), which yields
\[ \hat{\tau}^-_a := \inf\{ t > 0 : Y_t > -a \} = \inf\{ t > 0 : X_t < a \} = \tau_a^-, \]
while the last equality follows from (3.13) for \( f_c(\cdot) \equiv 1 \) in Proposition 3.5. Therefore, using (3.5), we can define the “hazard rate” function \( \Lambda(\cdot) \) on \( \mathbb{R} \) such that
\[ \Lambda(z) := -\frac{1}{\overline{P}_y(\overline{Y}_\zeta > z)} \overline{P}_y(\overline{Y}_\zeta > z) \frac{\partial}{\partial z} \left[ \overline{P}_y(\overline{Y}_\zeta > z) \right] = \Phi(q) + \lambda \frac{W(q+\lambda)(I^* + z)}{Z(q+\lambda)(I^* + z; \Phi(q))}. \]
which holds for all \( y \leq z \). It is clear that \( \int_{-\infty}^{\infty} \Lambda(z) \, dz = \infty \) and hence satisfies [30, Assumption 2.2]. Now define the function \( \tilde{h}(\cdot) \) on \( \mathbb{R} \), which is given by (cf. (5.8))
\[ \tilde{h}(y) := \tilde{f}_c(y) - \frac{\tilde{f}_c'(y)}{\Lambda(y)} = f_c(-y) + \frac{f_c'(-y)}{\Lambda(y)}. \]
Due to the definition (4.11) of \( I(\cdot; l^*) \) together with (3.14) and the expression (5.11) of \( \Lambda \), we further have
\[ \tilde{h}(y) = (\Lambda(y))^{-1} \left( f_c(-y) \Phi(q) + \lambda \frac{W(q+\lambda)(I^* + y)}{Z(q+\lambda)(I^* + y; \Phi(q))} \right) + f_c'(-y) = I(-y; l^*) \Lambda(y), \]
which implies via the positivity of \( \Lambda(\cdot) \) on \( \mathbb{R} \) and the results in Lemma 4.6.(II).(ii) for \( a^* = \alpha(l^*) \) that
\[ \tilde{h}(y) \begin{cases} > 0, & \text{for } -y < a^* \Leftrightarrow y > -a^*, \\ < 0, & \text{for } -y \geq a^* \Leftrightarrow y \leq -a^*. \end{cases} \]
Finally, by taking the first derivative in (5.12), we obtain by straightforward calculations that
\[ \tilde{h}'(y) = -f_c'(-y) - \frac{f_c''(-y)}{\Lambda(y)} - \frac{f_c'(-y)\Lambda'(y)}{\Lambda^2(y)} > 0, \quad \text{for a.e. } y \in \mathbb{R}. \]
The positivity is due to Assumption 4.1.(ii) and the fact that the combination of the definition (5.11) of \( \Lambda(\cdot) \) together with Lemma 3.3 implies that
\[ y \mapsto \Lambda(y) = \Phi(q) + \lambda \frac{W(q+\lambda)(I^* + y)}{Z(q+\lambda)(I^* + y; \Phi(q))} \text{ is increasing on } \mathbb{R}. \]
This shows that [30, Assumption 2.3] holds true (with \( x^* \) replaced with \( -a^* \)).

Now from [30, Theorem 2.2] (see also [38, Section 4.1–4.2] for a similar result), which states under [30, Assumptions 2.2 and 2.3] that the root of \( \tilde{h}(\cdot) = 0 \) (in our case \(-a^*) \) gives the optimal strategy. In other words, the optimal stopping time for \( \tilde{u} \) in (5.9) is given by \( \tilde{\tau}^+_{-a^*} \). This yields in view of (5.10) that the optimal stopping time for \( u \) is given by \( \tau_{a^*} \) and consequently (5.6) holds true, which completes the proof.
6. Numerical Results

In this section, we confirm the analytical results focusing on a case study with put-type payoffs for both players. Suppose that \( f_i(x) = K_i - e^{\alpha x} \), for \( i \in \{c, p\} \), for some fixed \( K_p > K_c \). Then, Assumption 4.1 is satisfied with \( \pi_t = \log K_i \), while Assumption 4.2 holds true with \( a = \log \left( \frac{\Phi(q)K_c}{1+\Phi(q)} \right) \). We also have in view of (4.9) that

\[
\tilde{x}_p = \log \frac{K_p \Phi(q)}{1 + \Phi(q)} = \pi_p + \log \frac{\Phi(q)}{1 + \Phi(q)},
\]

\[
\tilde{x}_c = \log \frac{(\Phi(q) + \lambda W^{q+\lambda}) K_c}{1 + \Phi(q) + \lambda W^{q+\lambda}} = \pi_c + \log \frac{\Phi(q) + \lambda W^{q+\lambda}}{1 + \Phi(q) + \lambda W^{q+\lambda}}.
\]

As an underlying asset price \( e^X \), we consider the case of \( X \) that is of the form

\[ X_t = X_0 - \mu t + \nu B_t + \sum_{n=1}^{M_t} Z_n, \quad 0 \leq t < \infty, \]

where \( \mu > 0 \) and \( \nu \geq 0 \) are constants, \( B = (B_t : t \geq 0) \) is a standard Brownian motion, \( M = (M_t : t \geq 0) \) is a Poisson process with arrival rate \( \alpha \), and \( Z = (Z_n : n = 1, 2, \ldots) \) is an i.i.d. sequence of exponential random variables with parameter \( \beta \). The processes \( B, M, \) and \( Z \) are assumed mutually independent. In this model, the scale functions admit explicit expressions that can be found, e.g., in [16, 25].

In all forthcoming analyses, we use the parameter values \( \nu = 0.2, \alpha = 1, \beta = 2, q = 0.05 \) and \( \mu = 0.31333 \) so that \( (e^{-\nu t + X_t} : t \geq 0) \) becomes a martingale. In particular, we set \( K_p = 60 \) and \( K_c = 50 \), and unless stated otherwise, we also set \( \lambda = 1 \).

6.1. Optimality. As discussed in Section 4.3, the optimal barriers corresponding to the Nash equilibrium \((a^*, l^*)\) are those satisfying simultaneously the conditions \( C_a \) and \( C_l \), given by (4.14)–(4.15) and expressed as the roots of \( a \mapsto I(a; l) \) in (4.11) and \( l \mapsto J(l; a) \) in (4.21), respectively.

In Figure 2.(i), we plot \( a \mapsto I(a; l) \) for five values of \( l \) equally spaced between \( \tilde{x}_c \) and \( \pi_p \) (cf. Proposition 4.11). It is observed that \( I(\cdot; l) \) starts from positive values and ends at non-positive values – admitting a unique root \( \tilde{a}(l) \) between \( \tilde{x}_c \) and \( \pi_c \wedge l \). This confirms Lemma 4.6.(II).(i)–(ii). In Figure 2.(ii), we plot \( l \mapsto J(l; a) \) for five values of \( a \) equally spaced between \( \pi_c \) and \( \tilde{x}_c \) (cf. Proposition 4.11). It is observed that \( J(\cdot; a) \) starts from positive values and ends at negative values – admitting a unique root \( \tilde{l}(a) \) which is between \( \pi_p \) and \( \pi_p \). This confirms Lemma 4.8.(i)–(ii).

The roots \( \tilde{a}(l) \) and \( \tilde{l}(a) \) can be computed via classical bisection. In Proposition 4.11, we showed the existence of \( l^* \) such that \( J(l^*; \tilde{a}(l^*)) = 0 \) and hence that \( (\tilde{a}(l^*), l^*) \) becomes a Nash equilibrium. In fact, as shown in Figure 3.(ii), the mapping \( l \mapsto J(l; \tilde{a}(l)) \) is monotone and hence the value \( l^* \) (and therefore also \( a^* := \tilde{a}(l^*) \)) is unique in this case study. For completeness, we also plot \( a \mapsto I(a; \tilde{l}(a)) \) in Figure 3.(i). It is also confirmed to be monotone and the unique root \( a^* \) such that \( I(a^*; \tilde{l}(a^*)) = 0 \) leads to the unique Nash equilibrium \( (a^*, \tilde{l}(a^*)) \equiv (\tilde{a}(l^*), l^*). \)

With \( a^* \) and \( l^* \) obtained by the above procedures, we then compute the value functions for both players \( C \) and \( P \) and confirm their optimality. To this end, we compare them with those for different choices of \( a \) and \( l \) while the opponent’s strategy remains fixed at \( l^* \) and \( a^* \), respectively. In Figure 4, we illustrate the results, showing that the value function indeed dominates those with wrong barrier selections for both players. For player \( P \), we also plot the reward function \( e^x \mapsto K_p - e^{\alpha x} \), to confirm that the optimal barrier \( l^* \) (in the plot, \( l^* \)) is the only barrier at which the value function and reward function coincide, confirming also the optimality condition \( C_l \) in (4.15).

6.2. Sensitivity with respect to \( \lambda \). We shall now analyse how the equilibrium strategies of both players change with respect to the observation rate \( \lambda \) of player \( P \). In Figure 5.(i)–(ii), we plot the equilibrium value function of
each player for \( \lambda \) ranging from 0.1 to 500. We see that \( v_c(x; a^*, l^*) \) is monotonically decreasing in \( \lambda \) for all asset values \( e^x \), whereas \( v_p(x; a^*, l^*) \) seems monotonically increasing for large values of \( e^x \), but the monotonicity is non-conclusive when the asset value \( e^x \) is low, due to the involvement of player \( C \).

To complete the picture, we plot in Figure 5(iii) the barriers \( a^* \) and \( l^* \) as functions of \( \lambda \). As \( \lambda \) increases, the threshold \( a^* \) seems to converge to some value close to, but slightly smaller than \( \pi_c = \log K_c = 50 \), so that it yields a positive reward at stopping, while \( l^* \) converges to some value larger than \( \pi_c \). When \( \lambda \) is very large, the observations of player \( P \) are almost as frequent as those of player \( C \). However, player \( C \)'s advantageous right to always stop first when \( a^* = l^* \), no matter how large \( \lambda \) is, leads player \( P \) to select \( l^* \) strictly larger than \( a^* \) (otherwise Player \( P \)'s reward will always be zero). Moreover, if player \( P \) chooses \( l < \pi_c \) and has a very high observation.

\[
F I G U R E \ 2. \ (i) \ Plot \ of \ a \mapsto I(a; l) \ on \ [x_c - 0.5, l \wedge \pi_c] \ for \ l = x_c, \ldots, \pi_p. \ The \ roots \ of \ I(\cdot; l) = 0 \ are \ indicated \ by \ circles \ and \ the \ vertical \ dotted \ lines \ correspond \ to \ a = x_c, \pi_c. \ (ii) \ Plot \ of \ l \mapsto J(l; a) \ on \ [a, \pi_p] \ for \ a = x_c, \ldots, \pi_c. \ The \ roots \ of \ J(\cdot; a) = 0 \ are \ indicated \ by \ circles \ and \ the \ vertical \ dotted \ lines \ correspond \ to \ l = \pi_p, \pi_c.
\]

\[
F I G U R E \ 3. \ (i) \ Plot \ of \ a \mapsto I(a; \tilde{l}(a)) \ on \ [x_c, \pi_c]. \ The \ root \ of \ I(\cdot; \tilde{l}(\cdot)) = 0 \ is \ indicated \ by \ the \ circle. \ (ii) \ Plot \ of \ l \mapsto J(l; \tilde{a}(l)) \ on \ [x_c, \pi_p]. \ The \ root \ of \ J(\cdot; \tilde{a}(\cdot)) = 0 \ is \ indicated \ by \ the \ circle.
\]
For $\lambda$ player $P$ to increase frequency, then player $C$ will try to increase $a$ to $l$ to stop before player $P$ – in response to this, player $P$ needs to increase $l$. In this way, the selection of barriers $l < \overline{\tau}_e$ cannot be part of an equilibrium strategy, justifying the aforementioned asymptotic behaviour.

6.3. Value of information. We finally define the value of additional information to be the amount $\delta := f_p(x) - f_c(x) = K_p - K_c$ that player $C$ should pay as fees to an expert or to obtain the information continuously, such that $v_c(x; a^*, l^*) = v_p(x; a^*, l^*)$ (i.e. the value functions of both players coincide). In our numerical results, the difference $v_c(x; a^*, l^*) - v_p(x; a^*, l^*)$ is monotone in $K_c$, hence we obtain via the bisection method the unique zero that leads to the desired $\delta$ value. In order to analyse how this $\delta$ changes with respect to the starting value $e^x$ and player $P$'s observation rate $\lambda$, we plot $\delta$ in Figure 6 as a function of $e^x$ (when $\lambda = 1$) and $\lambda$ (when $e^x = K_p = 60$). It is observed that the value of information decreases both in $e^x$ and $\lambda$, however it does not seem to converge to zero as $\lambda \to \infty$, because player $C$ still has the right to stop before player $P$ no matter how large $\lambda$ is.

APPENDIX A. PROOFS OF FLUCTUATION IDENTITIES

We first obtain the following two results, which will be used in the proofs of Lemmata 3.4 and 3.6.

Lemma A.1. For $b > 0$, $q \geq 0$ and $0 \leq x \leq a$, we have

$$
\mathbb{E}_{-x} \left[ e^{-(q+\lambda)\tau_0^+} W(q)(b - X_{\tau_0^+}); \tau_0^+ < \tau_a^- \right] = \mathcal{W}_b^{(q,\lambda)}(x) - \frac{W(q+\lambda)(x)}{W(q+\lambda)(a)} \mathcal{W}_b^{(q,\lambda)}(a).
$$

Proof. This is a direct consequence of Lemma 2.1 in [29] (see also [29, Lemma 2.2]) and the spatial homogeneity of Lévy processes.

Lemma A.2. For $c \geq 0$ and $b > 0$, we have

$$
\lambda \int_0^b W(q)(u) W(q+\lambda)(b + c - u) du = W(q+\lambda)(b + c) - \mathcal{W}_b^{(q,\lambda)}(c).
$$

Proof. This completes the proof.

\[\square\]
Figure 5. The value functions (i) $e^x \mapsto v_c(x; a^*, l^*)$ and (ii) $e^x \mapsto v_p(x; a^*, l^*)$, for $\lambda = 0.1, 0.5, 1, 2, 3, 4, 5, 10, 20, 50, 100, 200, 300, 500$. (iii) The barriers $l^*$ and $a^*$ as functions of $\lambda$.

Figure 6. Value of additional information $\delta := K_p - K_c$, such that $v_c(x; a^*, l^*) = v_p(x; a^*, l^*)$, as a function of (i) the initial asset value $e^x$ and (ii) the observation rate $\lambda$. 
Proof. By identity (6) in [29], we have for \( p, q \geq 0 \) and \( x \in \mathbb{R} \)
\[
W^{(q)}(x) - W^{(p)}(x) = (q - p) \int_0^x W^{(p)}(x - y)W^{(q)}(y)\,dy.
\] (A.1)

Using the equation (A.1) and the definition (3.10) of \( \mathcal{W}_b^{(q,\lambda)}(\cdot) \), we get that
\[
\lambda \int_0^b W^{(q)}(u)W^{(q,\lambda)}(b + c - u)\,du
= \lambda \int_0^{b+c} W^{(q)}(u)W^{(q,\lambda)}(b + c - u)\,du - \lambda \int_b^{b+c} W^{(q)}(u)W^{(q,\lambda)}(b + c - u)\,du
= W^{(q,\lambda)}(b + c) - W^{(q)}(b + c) - \lambda \int_0^c W^{(q)}(u + b)W^{(q,\lambda)}(c - u)\,du = W^{(q,\lambda)}(b + c) - \mathcal{W}_b^{(q,\lambda)}(c)
\]
which completes the proof. \( \square \)

A.1. **Proof of Lemma 3.3.** By the probabilistic expression (3.8), for \( 0 < x < y \), we have
\[
Z^{(q,\lambda)}(x; \Phi(q)) - Z^{(q,\lambda)}(y; \Phi(q))W^{(q,\lambda)}(x) > 0.
\]
Hence,
\[
\frac{W^{(q,\lambda)}(x)}{Z^{(q,\lambda)}(x; \Phi(q))} < \frac{W^{(q,\lambda)}(y)}{Z^{(q,\lambda)}(y; \Phi(q))}, \quad \text{for all} \quad 0 < x < y.
\]

A.2. **Proof of Lemma 3.4.** Throughout this proof, we write \( \bar{g}(x) := \mathbb{E}_x \left[ e^{-q\tau_a} f_c(X_{\tau_a}) \mathbbm{1}_{\{\tau_a^- < T_1^- \wedge \tau_b^+ \}} \right] \) for \( x \in \mathbb{R} \).

It is straightforward to see by the definitions of \( \bar{g}(x) \) and \( \tau_a^- \), that \( \bar{g}(x) = f_c(x) \) for all \( x < a \). The remainder of the proof is devoted to the case of \( x \geq a \). We prove this result in the following steps.

**Step 1: Computation of \( \bar{g}(x) \) in terms of \( \bar{g}(l) \).** On one hand, for \( x \geq l \), using the strong Markov property, spatial homogeneity of Lévy processes and (3.7), we obtain
\[
\bar{g}(x) = \mathbb{E}_x \left[ e^{-q\tau_a^-} \bar{g}(X_{\tau_a^-}); \tau_a^- < \tau_b^+ \right] = \bar{g}(l) \frac{W^{(q)}(b - x)}{W^{(q)}(b - l)}.
\] (A.2)

On the other hand, for \( x \in [a, l) \), using again the strong Markov property, we obtain
\[
\bar{g}(x) = \mathbb{E}_x \left[ e^{-q\tau_a} f_c(X_{\tau_a}); \tau_a^- < T(1) \wedge \tau_b^+ \right] + \mathbb{E}_x \left[ e^{-q\tau_a^-} \bar{g}(X_{\tau_a^-}); \tau_a^- < \tau_b^+ \right] = \mathbb{E}_x \left[ e^{-(q+\lambda)\tau_a} f_c(X_{\tau_a}); \tau_a^- < \tau_b^+ \right] = f_c(a) \frac{W^{(q+\lambda)}(l - x)}{W^{(q+\lambda)}(l - a)}.
\] (A.3)
Step 2: Computation of \( \bar{g}(l) \). We note that we can write

\[
\bar{g}(l) = \mathbb{E}_l \left[ e^{-q \tau_a^-} f_c( X_{\tau_a^-}^-) ; \tau_a^- < T^{(1)} \land \tau_b^+ \right] + \mathbb{E}_l \left[ e^{-q T^{(1)}} \bar{g}( X_{T^{(1)}}) 1 \{ X_{T^{(1)}} \geq l \} ; T^{(1)} < \tau_a^- \land \tau_b^+ \right].
\]

The first term on the right-hand side of the above equation can be expressed as

\[
\mathbb{E}_l \left[ e^{-q \tau_a^-} f_c( X_{\tau_a^-}^-) ; \tau_a^- < T^{(1)} \land \tau_b^+ \right] = f_c(a) \mathbb{E}_l \left[ e^{-(q+\lambda) \tau_a^-} ; \tau_a^- < \tau_b^+ \right] = f_c(a) \frac{W(q+\lambda)(b-l)}{W(q+\lambda)(b-a)}.
\]

Using (3.9), the second term can be calculated by

\[
\mathbb{E}_l \left[ e^{-q T^{(1)}} \bar{g}( X_{T^{(1)}}) 1 \{ X_{T^{(1)}} \geq l \} ; T^{(1)} < \tau_a^- \land \tau_b^+ \right] = \lambda \mathbb{E}_l \left[ \int_0^{\tau_a^- \land \tau_b^+} e^{-(q+\lambda)s} \bar{g}(X_s) 1 \{ X_s \geq l \} \, ds \right]
\]

\[
= \lambda \int_0^{b-l} \bar{g}(b-u) \left( \frac{W(q+\lambda)(b-l)}{W(q+\lambda)(b-a)} W(q+\lambda)(b-a-u) - W(q+\lambda)(b-l-u) \right) \, du
\]

\[
= \lambda \frac{\bar{g}(l)}{W(q)(b-l)} \int_0^{b-l} W(q)(u) \left( \frac{W(q+\lambda)(b-l)}{W(q+\lambda)(b-a)} W(q+\lambda)(b-a-u) - W(q+\lambda)(b-l-u) \right) \, du,
\]

where the last equality follows from (A.2). Next, using Lemma A.2 and (A.1) with (3.11) we further get

\[
\begin{align*}
\mathbb{E}_l \left[ e^{-q T^{(1)}} \bar{g}( X_{T^{(1)}}) 1 \{ X_{T^{(1)}} \geq l \} ; T^{(1)} < \tau_a^- \land \tau_b^+ \right] &= \frac{\bar{g}(l)}{W(q)(b-l)} \left( \frac{W(q+\lambda)(b-l)}{W(q+\lambda)(b-a)} W(q+\lambda)(b-a-l) - W(q+\lambda)(b-l) + W(q)(b-l) \right) \\
&= \frac{\bar{g}(l)}{W(q)(b-l)} \frac{\mathcal{C}(q,\lambda)(l-a) W(q+\lambda)(b-l)}{W(q)(b-l) W(q+\lambda)(b-a)}.
\end{align*}
\]

Hence, putting all the pieces together, we get

\[
\bar{g}(l) = f_c(a) \frac{W(q+\lambda)(b-l)}{W(q+\lambda)(b-a)} + \bar{g}(l) - \bar{g}(l) \frac{\mathcal{C}(q,\lambda)(l-a) W(q+\lambda)(b-l)}{W(q)(b-l) W(q+\lambda)(b-a)}.
\]

Therefore, by solving for \( \bar{g}(l) \), we finally obtain

\[
\bar{g}(l) = f_c(a) \frac{W(q)(b-l)}{\mathcal{C}(q,\lambda)(l-a)}.
\]

Step 3: Computation of \( \bar{g}(x) \). Substituting (A.6) in (A.4) we obtain the result.

A.3. Proof of Proposition 3.5. First, we note that by using Exercise 8.5.(i) in [26] and recalling the definition (3.10) of \( \mathcal{W}_{b-l}^{(q,\lambda)}(\cdot) \), we have for \( x \in \mathbb{R} \), the limit

\[
\lim_{b \to \infty} \frac{\mathcal{W}_{b-l}^{(q,\lambda)}(x)}{W(q)(b-l)} = \lim_{b \to \infty} \left( \frac{W(q)(x+b-l)}{W(q)(b-l)} + \lambda \int_0^x W(q+\lambda)(x-u) \frac{W(q)(u+b-l)}{W(q)(b-l)} \, du \right)
\]

\[
= e^{\Phi(q) x} + \lambda \int_0^x e^{\Phi(q) u} W(q+\lambda)(x-u) \, du = Z^{(q+\lambda)}(x; \Phi(q)) ,
\]

where the last equality follows from (3.4). Then, by taking \( b \to \infty \) in (12.12) we get the desired result.
A.4. Proof of Lemma 3.6. Throughout this proof, we write \( g(x) := \mathbb{E}_x \left[ e^{-qT_{1}} f_p(X_{T_{1}}) 1_{\{T_{1}<\tau_{a} \wedge \tau_{b}^{+}\}} \right] \) for \( x \in \mathbb{R} \). It is straightforward to see by the definitions of \( g(x) \) and \( \tau_{a}^{-} \), that \( g(x) = 0 \) for all \( x \leq a \). Hence, the remainder of the proof is devoted to the case of \( x > a \). We prove this result in the following steps.

Step 1: Computation of \( g(x) \) in terms of \( g(l) \). On one hand, for \( x > l \), using the strong Markov property, spatial homogeneity of Lévy processes and (3.7), we obtain

\[
(A.8) \quad g(x) = \mathbb{E}_{x-l} \left[ e^{-q\tau_{0}^{-}} g(X_{\tau_{0}^{-}} + l); \tau_{0}^{-} < \tau_{b}^{-} \right] = g(l) \frac{W_{(q)}(b - x)}{W_{(q)}(b - l)}.
\]

On the other hand, for \( x \in [a, l) \), using again the strong Markov property, we obtain

\[
(A.9) \quad g(x) = \mathbb{E}_x \left[ e^{-qT_{(1)}} f_p(X_{T_{(1)}}); T_{(1)} < \tau_{a}^{-} \wedge \tau_{l}^{+} \right] + \mathbb{E}_x \left[ e^{-q\tau_{0}^{+}} g(X_{\tau_{0}^{+}}); \tau_{0}^{+} < T_{(1)} \wedge \tau_{a}^{-} \right].
\]

For the first term on the right-hand side of (A.9), using the spatial homogeneity of Lévy processes and (3.9),

\[
(A.10) \quad \mathbb{E}_x \left[ e^{-qT_{(1)}} f_p(X_{T_{(1)}}); T_{(1)} < \tau_{a}^{-} \wedge \tau_{l}^{+} \right] = \lambda \mathbb{E}_x \left[ \int_0^{\tau_{a}^{-} \wedge \tau_{l}^{+}} e^{-(q+\lambda)s} f_p(X_s) ds \right]
\]

\[
= \lambda \int_0^{t-a} f_p(l - u) \left( \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} W_{(q+\lambda)}(l - a - u) - W_{(q+\lambda)}(l - x - u) \right) du
\]

\[
= \lambda \left( \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \Gamma(a; l) - \Gamma(x; l) \right),
\]

where the last equality follows from (3.15).

Now, for the second term on the right-hand side of (A.9), we use (A.8) together with Lemma A.1 to obtain

\[
\mathbb{E}_x \left[ e^{-q\tau_{0}^{+}} g(X_{\tau_{0}^{+}}); \tau_{0}^{+} < T_{(1)} \wedge \tau_{a}^{-} \right] = \mathbb{E}_{x-l} \left[ e^{-q\tau_{0}^{+}} g(X_{\tau_{0}^{+}} + l); \tau_{0}^{+} < T_{(1)} \wedge \tau_{a}^{-} \right]
\]

\[
= \mathbb{E}_{x-l} \left[ e^{-q\tau_{0}^{+}} g(l) \frac{W_{(q)}(b - X_{\tau_{0}^{+}} + l)}{W_{(q)}(b - l)}; \tau_{0}^{+} < T_{(1)} \wedge \tau_{a}^{-} \right]
\]

\[
= \lambda \frac{g(l)}{W_{(q)}(b - l)} \left( \mathbb{E}_{b-l} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} - \mathbb{E}_{b-l} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \right).
\]

Therefore, by putting all the pieces together, we obtain for \( x \in [a, l) \), that

\[
(A.11) \quad g(x) = \lambda \left( \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \Gamma(a; l) - \Gamma(x; l) \right) + g(l) \left( \mathbb{E}_{b-l} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} - \mathbb{E}_{b-l} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \right).
\]

Step 2: Computation of \( g(l) \). We note that

\[
g(l) = \mathbb{E}_l \left[ e^{-qT_{(1)}} f_p(X_{T_{(1)}} 1_{\{X_{T_{(1)}} < l\}}); T_{(1)} < \tau_{a}^{-} \wedge \tau_{b}^{+} \right] + \mathbb{E}_l \left[ e^{-qT_{(1)}} g(X_{T_{(1)}} 1_{\{X_{T_{(1)}} \geq l\}}); T_{(1)} < \tau_{a}^{-} \wedge \tau_{b}^{+} \right].
\]

Modifying (A.10) (with \( f_p(\cdot) \) instead of \( f_p(\cdot) \), and using (3.15), the first expectation becomes

\[
\mathbb{E}_l \left[ e^{-qT_{(1)}} f_p(X_{T_{(1)}} 1_{\{X_{T_{(1)}} < l\}}); T_{(1)} < \tau_{a}^{-} \wedge \tau_{b}^{+} \right] = \lambda \mathbb{E}_l \left[ \int_0^{\tau_{a}^{-} \wedge \tau_{b}^{+}} e^{-(q+\lambda)s} f_p(X_s) 1_{\{X_{s} < l\}} ds \right]
\]

\[
= \lambda \frac{W_{(q+\lambda)}(b - l)}{W_{(q+\lambda)}(b - a)} \int_{b-l}^{b-a} f_p(b - u) W_{(q+\lambda)}(b - a - u) du = \lambda \frac{W_{(q+\lambda)}(b - l)}{W_{(q+\lambda)}(b - a)} \Gamma(a; l),
\]

while for the final expectation, we simply replace \( \bar{g} \) with \( g \) in (A.5) to obtain

\[
\mathbb{E}_l \left[ e^{-qT_{(1)}} g(X_{T_{(1)}} 1_{\{X_{T_{(1)}} \geq l\}}); T_{(1)} < \tau_{a}^{-} \wedge \tau_{b}^{+} \right] = g(l) - g(l) \frac{\mathbb{E}_{b-l} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} - \mathbb{E}_{b-l} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \frac{W_{(q+\lambda)}(l - x)}{W_{(q+\lambda)}(l - a)} \right).
\]
Hence, putting all the pieces together in the original equation, we get
\[ g(l) = \lambda \frac{W^{(q+\lambda)}(b-l)}{W^{(q+\lambda)}(b-a)} \Gamma(a;l) + g(l) - \frac{\gamma^{(q,\lambda)}_{b-l} (l-a)}{W^{(q+\lambda)}(b-l) W^{(q+\lambda)}(b-a)}. \]

Therefore, by solving for \( g(l) \), we finally obtain
\[ g(l) = \lambda \frac{W^{(q)}(b-l)}{\gamma^{(q,\lambda)}_{b-l} (l-a)} \Gamma(a;l). \]  
(A.12)

**Step 3: Computation of \( g(x) \).** By substituting (A.12) in (A.11) from the previous two steps, we obtain the result.

### A.5. Proof of Proposition 3.7.

Taking \( b \to \infty \) in (3.16) together with an application of equation (A.7) leads to the desired result.

### Appendix B. Proofs of Some Technical Results in Section 4

#### B.1. Proof of Lemma 4.3.

By Remark 4.2, \( a = \underline{x} \), when \( X \) is of unbounded variation. It thus remains to prove the claim only for the bounded variation case, i.e. when \( W^{(q+\lambda)}(0) > 0 \) (see Remark 3.2(ii)). By Assumption 4.1 and (4.6), we have \( a < \overline{x} \) and \( f_c(a) > 0 \). Hence, by (4.8) we have \( h_c(a) = \lambda W^{(q+\lambda)}(0)f_c(a) > 0 \), implying in view of (4.9) that \( a < \underline{x} \).

#### B.2. Proof of Lemma 4.4.

We prove each part separately.

**Proof of (i).** We first show that \( \max_{a \geq x} v_c(x; a, l) \leq v_c(x; x, l) \), which is straightforward to see since \( v_c(x; a, l) = v_c(x; x, l) = f_c(x) \) for all \( a \geq x \). Then, it remains to show that \( \max_{a \leq \overline{x}} v_c(x; a, l) \leq v_c(x; \overline{x}, l) \). For \( a > \overline{x} \), we have \( f_c(a) < 0 \) due to (4.2), hence it is clear that \( v_c(x; a, l) \leq v_c(x; \overline{x}, l) \).

**Proof of (i).** We have from Remark 3.1 that
\[
\max_{a \in \mathbb{R}} v_c(x; a, l) = \max_{a \leq l} v_c(x; a, l) \land \max_{a > l} v_c(x; a, l) = \max_{a \leq l} v_c(x; a, l) \lor \max_{a > l} v_c^0(x; a).
\]

Due to the assumption \( l \geq a \) and given that \( a \mapsto v_c^0(x; a) \) is decreasing on \([l, \infty) \subseteq [a, \infty) \) (see (4.4)–(4.6)), we get
\[
\max_{a \in \mathbb{R}} v_c(x; a, l) = \max_{a \leq l} v_c(x; a, l) \lor v_c^0(x; l) = \max_{a \leq l} v_c(x; a, l) \lor v_c(x; l, l) = \max_{a \leq l} v_c(x; a, l).
\]

Combining this together with (i), we complete the proof.

**Proof of (ii).** It suffices to show that \( \max_{l \in [a, \overline{x}]} v_p(x; a, l) \leq v_p(x; a, \overline{x}) \). Indeed, on one hand, for \( l < a \), we have by Remark 3.1, that \( v_p(x; a, l) = 0 \leq v_p(x; a, \overline{x}) \). On the other hand, for \( l > \overline{x} \), we have
\[
v_p(x; a, l) - v_p(x; a, \overline{x}) = \mathbb{E}_x \left[ e^{-qT_{l}} f_p(X_{T_{l}}^{-}) 1_{\{T_{l} < T_{a}, T_{a} < T_{\overline{x}}\}} \right] - \mathbb{E}_x \left[ e^{-qT_{\overline{x}}} f_p(X_{T_{\overline{x}}^{-}}) 1_{\{T_{\overline{x}} < T_{a}, T_{a} < T_{\overline{x}}\}} \right] \leq 0
\]
where the last inequality holds because, on \( \{T_{l} < T_{\overline{x}}\} \) we have \( X_{T_{l}}^{-} > \overline{x} \), thus \( f_p(X_{T_{l}}^{-}) < 0 \) while \( f_p(X_{T_{\overline{x}}}^{-}) \geq 0 \) due to (4.2). This completes the proof.
B.3. **Proof of Lemma 4.5.** We prove the two parts separately.

**Proof of part (i).** By (3.5), we have

\[
\frac{\partial}{\partial a} v_c(x; a, l) = f'_c(a) Z^{(q+\lambda)}(l - x; \Phi(q)) + f_c(a) Z^{(q+\lambda)}(l - x; \Phi(q)) \left( \frac{\Phi(q) Z^{(q+\lambda)}(l - a; \Phi(q)) + \lambda W^{(q+\lambda)}(l - a)}{(Z^{(q+\lambda)}(l - a; \Phi(q)))^2} \right)
\]

\[
= Z^{(q+\lambda)}(l - x; \Phi(q)) \left( f'_c(a) + f_c(a) \frac{\Phi(q) Z^{(q+\lambda)}(l - a; \Phi(q)) + \lambda W^{(q+\lambda)}(l - a)}{Z^{(q+\lambda)}(l - a; \Phi(q))} \right)
\]

where the last equality holds by (3.14).

**Proof of part (ii).** Using (3.15) we have \( \frac{\partial}{\partial l} \Gamma(x; l) = f_p(l) W^{(q+\lambda)}(l - x) \) and by (3.5)

\[
\frac{\partial}{\partial l} \frac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))} = \frac{\Phi(q) Z^{(q+\lambda)}(l - x; \Phi(q)) + \lambda W^{(q+\lambda)}(l - x)}{Z^{(q+\lambda)}(l - a; \Phi(q))}
\]

\[
- \frac{Z^{(q+\lambda)}(l - x; \Phi(q)) \Phi(q) Z^{(q+\lambda)}(l - a; \Phi(q)) + \lambda W^{(q+\lambda)}(l - a)}{(Z^{(q+\lambda)}(l - a; \Phi(q)))^2}
\]

\[
= \frac{\lambda}{Z^{(q+\lambda)}(l - a; \Phi(q))} \left( W^{(q+\lambda)}(l - x) - \frac{Z^{(q+\lambda)}(l - x; \Phi(q)) W^{(q+\lambda)}(l - a)}{Z^{(q+\lambda)}(l - a; \Phi(q))} \right) \Gamma(a; l)
\]

Hence, differentiating (3.17), we get

\[
\lambda^{-1} \frac{\partial}{\partial l} v_p(x; a, l) = \frac{\lambda}{Z^{(q+\lambda)}(l - a; \Phi(q))} \left( W^{(q+\lambda)}(l - x) - \frac{Z^{(q+\lambda)}(l - x; \Phi(q)) W^{(q+\lambda)}(l - a)}{Z^{(q+\lambda)}(l - a; \Phi(q))} \right) \Gamma(a; l)
\]

\[
+ \frac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))} f_p(l) W^{(q+\lambda)}(l - a) - f_p(l) W^{(q+\lambda)}(l - x)
\]

\[
= \left( \frac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))} W^{(q+\lambda)}(l - a) - W^{(q+\lambda)}(l - x) \right) \left( f_p(l) - v_p(l; a, l) \right)
\]

where the last equality holds by (3.19).

B.4. **Proof of Proposition 4.13.** We prove the two sets of properties separately.

**Proof of part (I).** The continuity is clear by (3.13). Fix \( x \in (a^*, \infty) \setminus \{l^*\} \). By (3.13), (3.5) and (3.14),

\[
v'_c(x; a^*, l^*) = -f_c(a^*) \frac{\Phi(q) Z^{(q+\lambda)}(l^* - x; \Phi(q)) + \lambda W^{(q+\lambda)}(l^* - x)}{Z^{(q+\lambda)}(l^* - a^*; \Phi(q))}
\]

\[
= -\left( \Phi(q) Z^{(q+\lambda)}(l^* - x; \Phi(q)) + \lambda W^{(q+\lambda)}(l^* - x) \right) v_c(l^*; a^*, l^*)
\]

Differentiating this further and using (B.1), we get

\[
v''(x; a^*, l^*) = (\Phi(q) Z^{(q+\lambda)}(l^* - x; \Phi(q)) + \lambda W^{(q+\lambda)}(l^* - x) + \lambda W^{(q+\lambda)'(l^* - x)} - \lambda W^{(q+\lambda)'(l^* - x) - x}) v_c(l^*; a^*, l^*)
\]

\[
= -\Phi(q) v'_c(x; a^*, l^*) + \lambda W^{(q+\lambda)'(l^* - x)} v_c(l^*; a^*, l^*).
\]

By (B.1) and (B.2) together with Remark 3.2, we prove part (i).

To prove part (ii), we use the definition (4.11) and the fact that \( I(a^*; l^*) = 0 \), to conclude from (B.1) that

\[
v'_c(a^*; a^*, l^*) = -\left( \Phi(q) Z^{(q+\lambda)}(l^* - a^*; \Phi(q)) + \lambda W^{(q+\lambda)}(l^* - a^*) \right) v_c(l^*; a^*, l^*)
\]

\[
= f'_c(a^*) - I(a^*; l^*) = f'_c(a^*).
\]

This coincides with \( v'_c(a^*; a^*, l^*) = f'_c(a^*) \) (see (3.13)), hence \( v_c(\cdot; a^*, l^*) \) is continuously differentiable at \( a^* \).
Part (iii) holds true by combining (B.1) with Remark 3.2.

Using once again (B.1) with the positivity of the scale function, yielding \( v'_c(x; a^*, l^*) \leq 0 \) for \( x \in (a^*, \infty) \setminus \{l^*\} \), together with the continuity of \( v_c(\cdot; a^*, l^*) \) at \( l^* \) from part (i), we conclude the monotonicity in part (iv). Then, (B.2), the monotonicity and positivity of \( v_c(x; a^*, l^*) \) imply that \( v''_c(x; a^*, l^*) > 0 \) for all \( x \) except for \( l^* \) and the discontinuity points of \( W^{(q+\lambda)'}(\cdot) \). However, in the unbounded variation case, \( v'_c(\cdot; a^*, l^*) \) is continuous at \( l^* \) from part (iii), while in the bounded variation case, we have from (B.1) that

\[
v'_c(l^* -; a^*, l^*) = v'_c(l^* +; a^*, l^*) - \lambda W^{(q+\lambda)}(0) v_c(l^*; a^*, l^*) < v'_c(l^* +; a^*, l^*).
\]

This shows (in view of \( v_c \) begin decreasing) the desired convexity on \( (a^*, \infty) \) in part (iv).

**Proof of part (II).** The continuity is clear by (4.32). Fix \( x \in (a^*, \infty) \setminus \{l^*\} \). By (4.32), (3.5) and (4.15),

\[
v'_p(x; a^*, l^*) = -\left(\Phi(q)Z^{(q+\lambda)}(l^* - x; \Phi(q)) + \lambda W^{(q+\lambda)}(l^* - x)\right)v_p(l^*; a^*, l^*) + \lambda f_p(l^*)W^{(q+\lambda)}(l^* - x) - \lambda \int_0^{l^* - x} f'_p(u + x)W^{(q+\lambda)}(u) \, du
\]

(B.3)

while further differentiation yields

\[
v''_p(x; a^*, l^*) = \Phi(q)\left(\Phi(q)Z^{(q+\lambda)}(l^* - x; \Phi(q)) + \lambda W^{(q+\lambda)}(l^* - x)\right)v_p(l^*; a^*, l^*) + \lambda f'_p(l^*)W^{(q+\lambda)}(l^* - x) - \lambda \int_0^{l^* - x} f''_p(u + x)W^{(q+\lambda)}(u) \, du.
\]

(B.4)

Thus, the \( C^2 \) property in part (i) follows by using also Assumption 4.1.(ii), while parts (ii)–(iii) follow from the continuity and smoothness of the scale function in Remark 3.2 applied to (B.3)–(B.4).

**APPENDIX C. PROOF OF LEMMA 5.4 (VERIFICATION FOR PLAYER \( P \))**

Throughout this proof, we define \( w_p(x) := v_p(x; a^*, l^*) \), for all \( x \in \mathbb{R} \), \( e_\lambda \) to be an exponential random variable independent of \( X \) and \( T(0):= 0 \).

The proof for \( x < a^* \) is straightforward, since \( V_p(\tau_{a^*}; x; \sigma; x) = 0 \) for all \( \sigma \in \mathcal{T}_p \) and therefore \( \sup_{\sigma \in \mathcal{T}_p} V_p(\tau_{a^*}; x; \sigma; x) = 0 = w_p(x) \) by condition (v).

In the rest of the proof, we assume \( x \geq a^* \) and fix \( \varepsilon > 0 \) and \( m > 0 \). In view of the smoothness of \( w_p \) on \([a^* + \varepsilon, \infty)\) from Proposition 4.13.(II), it follows by Itô’s formula for all \( n \in \mathbb{N} \) and \( t \geq 0 \), that

\[
e^{-\lambda(t)}w_p(x_{t \wedge \tau_{a^*}^{\infty} \wedge \tau_m^{\infty}}) - w_p(x) = \int_0^{t \wedge \tau_{a^*}^{\infty} \wedge \tau_m^{\infty}} e^{-\lambda(s)} \mathcal{L}(\mathcal{L} = (q + \lambda)) w_p(X_s) \, ds + M_{t \wedge \tau_{a^*}^{\infty} \wedge \tau_m^{\infty}},
\]

where \( (M_t)_{t \geq 0} \) is a zero-mean local martingale with respect to the filtration \( \mathcal{F} \) and is defined by

\[
M_t := \int_0^t \nu e^{-qs} w'_p(X_s) \, dB_s + \lim_{\varepsilon \downarrow 0} \int_{[0,t]} \int_{(\varepsilon,1]} e^{-qs} w'_p(X_{s-}) y(N(ds \times dy) - \Pi(dy)\,ds)
+ \int_{[0,t]} \int_{(0,\infty)} e^{-qs} (w_p(X_{s-} + y) - w_p(X_{s-}) - w'_p(X_{s-}) y 1_{\{y \in (0,1]\}})(N(ds \times dy) - \Pi(dy)\,ds).
\]

Next, we aim at deriving a probabilistic expression of \( w_p \), which will involve the function \( \overline{w}_p \) defined by

\[
\overline{w}_p(x) := \max\{f_p(x), w_p(x)\} = f_p(x) 1_{\{x \leq l^*\}} + w_p(x) 1_{\{x > l^*\}}, \quad x \in [a^*, \infty),
\]

where the latter equality holds true due to conditions (iii) and (iv).
Lemma C.1. For $x \geq a^*$, we have

$$
w_p(x) = \mathbb{E}_x \left[ \int_0^{\tau_a^*} e^{-(q+\lambda)s} \overline{w}_p(X_s) \, ds \right] = \mathbb{E}_x \left[ e^{-qe^\lambda} \overline{w}_p(X_{e^\lambda}) 1_{\{e^\lambda < \tau_a^*\}} \right].$$

Proof. For $x > a^*$ (where $Lw_p(x)$ is well-defined by Proposition 4.13), we have

$$(L - q)w_p(x) + \lambda \max \{ f_p(x) - w_p(x), 0 \} = (L - (q + \lambda))w_p(x) + \lambda \overline{w}_p(x) = 0.$$ 

The last equality holds for $x \geq l^*$ by conditions (i) and (iii), and for $a^* < x < l^*$ by conditions (ii) and (iv). Substituting this back in (C.1), we obtain

$$e^{-(q+\lambda)(t\wedge\tau_{a^*+\epsilon}\wedge\tau_{a^*}^m)} w_p(X_{t\wedge\tau_{a^*+\epsilon}\wedge\tau_{a^*}^m}) = w_p(x) - \lambda \int_0^{t\wedge\tau_{a^*+\epsilon}\wedge\tau_{a^*}^m} e^{-(q+\lambda)s} \overline{w}_p(X_s) \, ds + M_{t\wedge\tau_{a^*+\epsilon}\wedge\tau_{a^*}^m}.$$ 

Therefore, by rearranging the terms and taking expectations, the optional sampling theorem gives

$$w_p(x) = \mathbb{E}_x \left[ e^{-(q+\lambda)(t\wedge\tau_{a^*+\epsilon}\wedge\tau_{a^*}^m)} w_p(X_{t\wedge\tau_{a^*+\epsilon}\wedge\tau_{a^*}^m}) \right] + \mathbb{E}_x \left[ \lambda \int_0^{t\wedge\tau_{a^*+\epsilon}\wedge\tau_{a^*}^m} e^{-(q+\lambda)s} \overline{w}_p(X_s) \, ds \right].$$ 

Since $\overline{w}_p$ is bounded, the dominated convergence theorem then gives, upon taking $m \to \infty$, that

$$w_p(x) = \mathbb{E}_x \left[ e^{-(q+\lambda)(t\wedge\tau_{a^*+\epsilon})} w_p(X_{t\wedge\tau_{a^*+\epsilon}}) \right] + \mathbb{E}_x \left[ \lambda \int_0^{t\wedge\tau_{a^*+\epsilon}} e^{-(q+\lambda)s} \overline{w}_p(X_s) \, ds \right].$$ 

Moreover, the boundedness and non-negativity of $w_p$, imply again by the dominated convergence theorem that

$$\lim_{t \to \infty} \mathbb{E}_x \left[ e^{-(q+\lambda)(t\wedge\tau_{a^*+\epsilon})} w_p(X_{t\wedge\tau_{a^*+\epsilon}}) \right] = \mathbb{E}_x \left[ e^{-(q+\lambda)\tau_{a^*}^*} w_p(X_{\tau_{a^*}^*}) 1_{\{\tau_{a^*}^* < \infty\}} \right] \leq \mathbb{E}_x \left[ e^{-(q+\lambda)\tau_{a^*}^*} \max_{a^* \leq y \leq a^*+\epsilon} w_p(y) \right],$$ 

where the latter inequality holds because $X_{\tau_{a^*}^*} \leq a^* + \epsilon$ a.s. and by the condition (v). Since $w_p$ is continuous on $[a^*, \infty)$ by Proposition 4.13.(II), and $w_p(a^*) = 0$ by condition (v), we have $\max_{a^* \leq y \leq a^*+\epsilon} w_p(y) \to 0$ as $\epsilon \downarrow 0$, hence

$$\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} \mathbb{E}_x \left[ e^{-(q+\lambda)(t\wedge\tau_{a^*+\epsilon})} w_p(X_{t\wedge\tau_{a^*+\epsilon}}) \right] = 0.$$ 

As $X$ is a spectrally positive Lévy process, $(\tau_{-b})_{b \geq 0}$ is a $\mathbb{P}$-subordinator with potential killing (see, e.g., the proof of Lemma VII.23 of [7]), hence $\tau_{a^*}$ at any time $a^*$ is continuous $\mathbb{P}_x$-a.s.; this implies $\tau_{a^*+\epsilon} \to \tau_{a^*}$ as $\epsilon \downarrow 0$ on $\{\tau_{a^*}^* < \infty\}$, Therefore, by the monotone convergence theorem, we obtain

$$\mathbb{E}_x \left[ \int_0^{t\wedge\tau_{a^*+\epsilon}} e^{-(q+\lambda)s} \overline{w}_p(X_s) \, ds \right] \overset{t \to \infty, \epsilon \downarrow 0}{\longrightarrow} \mathbb{E}_x \left[ \int_0^{\tau_{a^*}^*} e^{-(q+\lambda)s} \overline{w}_p(X_s) \, ds \right].$$ 

Finally, taking the limit as $t \to \infty$ and $\epsilon \downarrow 0$ in (C.4), we complete the proof of the first equality in (C.3). The second equality in (C.3) then holds true due to the definition of $e_\lambda$. \hfill $\Box$

Then, by Lemma C.1 and the definition (C.2) of $\overline{w}_p(\cdot)$, we have the inequality

$$\overline{w}_p(x) \geq w_p(x) = \mathbb{E}_x \left[ e^{-qe^\lambda} \overline{w}_p(X_{e^\lambda}) 1_{\{e^\lambda < \tau_{a^*}^*\}} \right].$$

To proceed further, we also define by $(\mathcal{H}_t)_{t \geq 0}$ the filtration generated by the two-dimensional process $(X, N)$ and let $\overline{\mathbb{G}} = (\overline{\mathcal{G}}_n)_{n \geq 0}$ with $\overline{\mathcal{G}}_n := \mathcal{H}_{T(n)}$, $n \geq 0$. Notice that player $P$’s filtration $\mathbb{G}$ satisfies $\mathbb{G}_n \subset \overline{\mathcal{G}}_n$ for all $n \geq 0$. 

Now, fix \( n \geq 0 \) and observe that the event \( \{ T^{(n)} < \tau^{-}_a \} \) is \( \mathcal{G}_n \)-measurable, hence by the strong Markov property of \((X, N)\), we get
\[
\mathbb{E}_x \left[ e^{-qT^{(n+1)}} \mathbb{E}_p \left( X_{T^{(n+1)}} \mathbf{1}_{\{ T^{(n+1)} < \tau^{-}_a \}} \bigg| \mathcal{G}_n \right) \right] = \mathbb{E}_x \left[ e^{-qT^{(n)}} \mathbb{E}_p \left( X_{T^{(n)}} \mathbf{1}_{\{ T^{(n)} < \tau^{-}_a \}} \bigg| \mathcal{G}_n \right) \right]
= \mathbb{E}_x \left[ e^{-qT^{(n)}} \mathbb{E}_p \left( X_{T^{(n)}} \mathbf{1}_{\{ T^{(n)} < \tau^{-}_a \}} \bigg| \mathcal{G}_n \right) \right]
\]
where the last inequality holds by (C.5). This shows that \( \{ e^{-qT^{(n)}} \mathbb{E}_p \left( X_{T^{(n)}} \mathbf{1}_{\{ T^{(n)} < \tau^{-}_a \}} \bigg| \mathcal{G}_n \right) \}_{n \in \mathbb{N}} \) is a \( \mathcal{G} \)-supermartingale.

Since it is always suboptimal to stop when the reward is negative, without loss of generality, let \( \sigma \in \mathcal{T}_p \) be such that
\[
f_p(X_\sigma) \geq 0 \quad \text{a.s.}
\]
Combining the aforementioned property and the definition (C.2) of \( \mathbb{E}_p(\cdot) \) together with Lemma C.1, we can conclude that for any stopping time \( \sigma \in \mathcal{T}_p \) satisfying (C.6), we have by Fatou’s lemma followed by optional sampling theorem (noting that \( \sigma \) is a \( \mathcal{G} \)-stopping time as \( \mathcal{G}_n \subset \mathcal{G}_n \)) that
\[
V_p(\tau^{-}_a, \sigma; x) \equiv \mathbb{E}_x \left[ e^{-q\sigma} f_p(X_\sigma) \mathbf{1}_{\{ \sigma < \tau^{-}_a \}} \right] \leq \liminf_{N \to \infty} \mathbb{E}_x \left[ e^{-q(\sigma \wedge T(N))} f_p(X_{\sigma \wedge T(N)}) \mathbf{1}_{\{ \sigma \wedge T(N) < \tau^{-}_a \}} \right]
\leq \liminf_{N \to \infty} \mathbb{E}_x \left[ e^{-q(\sigma \wedge T(N))} \mathbb{E}_p(X_{\sigma \wedge T(N)}) \mathbf{1}_{\{ \sigma \wedge T(N) < \tau^{-}_a \}} \right] \leq \mathbb{E}_x \left[ e^{-q(\sigma \wedge T(1))} \mathbb{E}_p(X_{\sigma \wedge T(1)}) \mathbf{1}_{\{ \sigma \wedge T(1) < \tau^{-}_a \}} \right] = w_p(x),
\]
where the last equality holds because \( \sigma \geq T^{(1)} \) a.s. together with (C.3). The arbitrariness of \( \sigma \in \mathcal{T}_p \) then completes the proof.

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