A 2-Categorical Study of Graded and Indexed Monads

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Abstract

In the study of computational effects, it is important to consider the notion of computational effects with parameters. The need of such a notion arises when, for example, statically estimating the range of effects caused by a program, or studying the ways in which effects with local scopes are derived from effects with only the global scope. Extending the classical observation that computational effects can be modeled by monads, these computational effects with parameters are modeled by various mathematical structures including graded monads and indexed monads, which are two different generalizations of ordinary monads. The former has been employed in the semantics of effect systems, whereas the latter in the study of the relationship between the local state monads and the global state monads, each exemplifying the two situations mentioned above. However, despite their importance, the mathematical theory of graded and indexed monads is far less developed than that of ordinary monads.

Here we develop the mathematical theory of graded and indexed monads from a 2-categorical viewpoint. We first introduce four 2-categories and observe that in two of them graded monads are in fact monads in the 2-categorical sense, and similarly indexed monads are monads in the 2-categorical sense in the other two. We then construct explicitly the Eilenberg–Moore and the Kleisli objects of graded monads, and the Eilenberg–Moore objects of indexed monads in the sense of Street in appropriate 2-categories among these four. The corresponding results for graded and indexed comonads also follow.

We expect that the current work will provide a theoretical foundation to a unified study of computational effects with parameters, or dually (using the comonad variants), of computational resources with parameters, arising for example in Bounded Linear Logic.
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Chapter 1

Introduction

1.1 Background and related work

Roughly speaking, there are two lines of research on which the current work is based, one computational and one mathematical. From a computational point of view, this work can be considered as a contribution to a theoretical study of computational effects, originating in Moggi’s work in the late 1980’s; this perspective is explained in Section 1.1.2. The mathematical aspect of the thesis relies on the 2-categorical theory of monads developed by Street in the 1970’s, as recalled in Section 1.1.3.

1.1.1 Monads

Let us begin with a brief discussion on the basic theory of monads, which forms the common basis for both of the two lines of research mentioned above.

The notion of monad originates in pure mathematics, and has been one of the main subjects of research in category theory. Monads are so fundamental that any reasonable introduction to category theory contains some account of them; see for example [24]. Among the earliest important constructions in the theory of monads are those of Eilenberg–Moore [6] and Kleisli [21] obtained in mid 1960’s, each building a category \( \mathcal{D} \) out of a monad \( T \) on a category \( \mathcal{C} \), together with an adjunction

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\perp} & \mathcal{D} \\
F & \downarrow & \quad U
\end{array}
\]

that generates \( T \) in the sense that \( T = U \circ F \) (and similarly for the other components).

What we achieve in this thesis is, intuitively, the generalization of these constructions of Eilenberg–Moore and Kleisli from ordinary monads to generalized notions of monad called graded or indexed monads.

1.1.2 Computational effects and monads

Examples of computational effects include those behaviors of programs involving global state, nondeterministic branch, or input/output. Moggi [29] brought breakthrough to the theory of computational effects by advocating that these diverse kinds of computational effects can be uniformly modeled by monads. For example, corresponding to the notion of global state there is a monad \( T \) (say, on \( \text{Set} \)) called the global state monad,
CHAPTER 1. INTRODUCTION

defined as $TX := S \Rightarrow (S \times X)$ where $S$ is the set of states. Similarly, there are monads for nondeterministic branch, input/output, etc. What is especially interesting in Moggi’s approach is that, not only mathematical concepts that model these computational effects possess the suitable monad structures, the theory of monads developed in pure mathematics can be fruitfully applied to the study of computational effects. Indeed, Moggi [29, 30] constructed categorical models of his calculus for computational effects via the Kleisli construction.

Nowadays, computational effects with parameters are becoming increasingly important. For example, there are type systems called effect systems [24], which statically estimate the range of effects caused by a program. Effect systems achieve this estimation by introducing, instead of a single effect (e.g., global state), a family of effects parametrized by ranges (e.g., global state together with a set of registers specified, with the intuition being that it contains those registers that are manipulated in an execution of the program). On the other hand, there is an active line of research on the nature of local state [32, 35, 34, 25] (which, in addition to manipulating registers, is able to increase or decrease the number of registers in use), especially on its relation to global state. One attracting view [25] is that local state is obtained by “gluing” a family of global states with different numbers of registers, and there again the idea of parameters shows itself.

Corresponding to this parametrization on the side of computational effects, various notions of monads with parameters have been employed in the study of those computational effects with parameters. The notion of graded monad has been used to give semantics of effect systems [18], whereas the notion of indexed monad arose and has been applied in the study of local state [35, 34, 25].

This thesis, by developing the fundamental mathematical theory of graded and indexed monads, aims to lay the foundation of the unified study of computational effects with parameters, just as mathematical theory of monads has been the basis of the unified study of computational effects initiated by Moggi.

1.1.3 Street’s 2-categorical study of monads

In 1972, Street [38] presented a highly abstract 2-categorical framework that collects main results in the theory of monads obtained until then (including the Eilenberg–Moore and Kleisli constructions), and reconstructs them inside an arbitrary 2-category (subject to certain completeness and cocompleteness requirements). For the basic 2-categorical notions, see [20]. In particular, he defined the Eilenberg–Moore and Kleisli objects of a monad in a 2-category, indicating how the Eilenberg–Moore and Kleisli constructions should be generalized to 2-categories other than $\mathcal{Cat}$ (the 2-category of categories, functors, and natural transformations).

This framework of Street is particularly suited for our study of graded and indexed monads, since it turns out that the framework is general enough to include theories of graded and indexed monads as particular instances. Indeed, the main theorems of this thesis (Theorems 4.20, 4.41, and 4.61) amount to state that certain concrete constructions defined in the thesis produce Eilenberg–Moore or Kleisli objects in the sense of Street, providing a compelling justification for our constructions.
1.2 Chapter overview

We begin with the definition of graded and indexed monads in Chapter 2, together with examples illustrating them: the graded state monad (Section 2.1.2) and the indexed state monad (Section 2.2.2), each extending the usual global state monad in different directions. The dual notions of graded and indexed comonad are also introduced.

Then, in Chapter 3, we introduce four 2-categories by “enlarging” the familiar 2-category $Cat$ in various ways. These 2-categories are now “large” enough to incorporate the notions of graded and indexed monads as mere monads (in the 2-categorical sense) inside them.

The above observation has a key consequence that it now makes perfect sense to speculate on Eilenberg–Moore and Kleisli objects of graded and indexed monads in the sense of Street, by working inside these “larger” 2-categories instead of $Cat$. We show that indeed there do exist the following constructions that produce the Eilenberg–Moore or Kleisli objects:

- The Eilenberg–Moore construction for graded monads.
- The Kleisli construction for graded monads.
- The Eilenberg–Moore construction for indexed monads.

The definition, and establishment of the relevant universal property of them are the main tasks of Chapter 4, which are also the main contributions of the current thesis. We also present the corresponding constructions for graded and indexed comonads. The quest for the following construction is left as future work:

- The Kleisli construction for indexed monads.

We then present two applications of these constructions in Chapter 5. In Section 5.1 we apply the first two constructions, the Eilenberg–Moore and Kleisli constructions for graded monads, to the study of lax monoidal actions. We show how they give solutions to the problem of decomposing lax monoidal actions into strict monoidal actions and adjunctions, a situation which generalizes the one that constitute the very origin of the classical Eilenberg–Moore and Kleisli constructions for ordinary monads [6, 21]. The second application, presented in Section 5.2, employs the Eilenberg–Moore construction for indexed monads. The story begins with the realization that the categories identical to our Eilenberg–Moore categories of indexed monads has already appeared in a recent paper [25] by Maillard and Melliès. They did not introduce these categories as Eilenberg–Moore categories, but established a nice connection to the categories of models of indexed Lawvere theories defined by Power [34, 35]. We show how the perspective provided by our construction enables one to understand more conceptually this result of Maillard and Melliès.

Finally in Chapter 6 we conclude the thesis and indicate possible directions for future work.

1.3 Notation

For the various compositions inside a 2-category, we adopt the following notation: we denote composition of 1-cells by $\circ$, vertical composition of 2-cells by $\cdot$, and horizontal composition of 2-cells by $\ast$. We often abbreviate whiskerings $id_g \ast \alpha$ and $\beta \ast id_f$ by $g \ast \alpha$.
and $\beta * f$ respectively. We also omit various composition symbols and write compositions simply by concatenation when confusion is unlikely.

Given a 2-category $\mathcal{K}$, by $\mathcal{K}^{\text{op}(1)}$, $\mathcal{K}^{\text{op}(2)}$, and $\mathcal{K}^{\text{op}(1,2)}$, we mean the 2-categories obtained from $\mathcal{K}$ by reversing, respectively, 1-cells but not 2-cells, 2-cells but not 1-cells, and both 1-cells and 2-cells. When $\mathcal{K}$ is actually a category, we also write $\mathcal{K}^{\text{op}(1)}$ as $\mathcal{K}^{\text{op}}$.

We denote an adjunction of type

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
\downarrow \\
B
\end{array}
\]

by $L \dashv R : B \to A$. 

Chapter 2
Adding parameters to monads

In this chapter, we introduce two different ways to generalize the notion of monad on a category so as to incorporate the intuitive idea of parameters into it. These two ways give rise to the notions of graded monad and indexed monad respectively. The transition from monads to graded or indexed monads is motivated by the various ways to add parameters to computational effects. We illustrate this point of view by two typical examples: the graded state monad and the indexed state monad.

One can also argue that the notions of graded and indexed monad are mathematically natural and are in some sense derived from a single general principle; see Appendix B for the discussion on this unified view.

2.1 Graded monads

A graded monad takes its parameters from a monoidal category.

2.1.1 The definition

Definition 2.1. Let $\mathcal{M} = (\mathcal{M}, \otimes, I)$ be a strict monoidal category and $\mathcal{C}$ be a category. An $\mathcal{M}$-graded monad on $\mathcal{C}$ is a (lax) monoidal functor

\[ T : (\mathcal{M}, \otimes, I) \rightarrow ([\mathcal{C}, \mathcal{C}], \circ, \text{id}_\mathcal{C}). \]

To spell out the definition, an $\mathcal{M}$-graded monad on $\mathcal{C}$ consists of the following data:

- A functor $T_m : \mathcal{C} \rightarrow \mathcal{C}$ for each object $m$ of $\mathcal{M}$.
- A natural transformation $T_u : T_m \Rightarrow T_{m'}$ for each morphism $u : m \rightarrow m'$ of $\mathcal{M}$.
- A natural transformation $\eta^{(T)} = \eta : \text{id}_\mathcal{C} \Rightarrow T_I$.
- A natural transformation $\mu^{(T)}_{m,n} = \mu_{m,n} : T_m \circ T_n \Rightarrow T_{m \otimes n}$ for each pair of objects $m, n$ of $\mathcal{M}$.

These data are subject to the following axioms:

(GM1) $\text{id}_{T_m} = T_{\text{id}_m}$ for each object $m$ of $\mathcal{M}$.

(GM2) $T_{u'} \cdot T_u = T_{u' \circ u}$ for each composable pair of morphisms $u, u'$ of $\mathcal{M}$.
CHAPTER 2. ADDING PARAMETERS TO MONADS

\( T_m \circ T_n \overset{\mu_{m,n}}{\Rightarrow} T_{m \otimes n} \)  

(GM3) \( T_u \circ T_v \) commutes for each pair of morphisms \( u : m \rightarrow m', v : n \rightarrow n' \) of \( M \).

\( T_{m'} \circ T_{n'} \overset{\mu_{m',n'}}{\Rightarrow} T_{m' \otimes n'} \)

\( T_{m} \overset{\eta_{m}}{\Rightarrow} T_I \circ T_m \)

(GM4) commutes for each object \( m \) of \( M \).

\( \eta_{m} \overset{T_I}{\Rightarrow} \mu_{I,m} \overset{id_{T_m}}{\Rightarrow} T_m \)

\( T_I \circ T_m \overset{T_I \mu_{m,n}}{\Rightarrow} T_I \circ T_{m \otimes n} \)

(GM6) commutes for each triple of objects \( I, m, n \) of \( M \).

\( \mu_{I,m,n} \overset{T_I \circ T_m \circ T_n \circ T_I \circ T_m \circ T_n \circ T_I \circ T_m \circ T_n \circ T_I \circ T_m \circ T_n}{\Rightarrow} T_I \circ T_{m \otimes n} \)

Graded monads generalize ordinary monads (monads in \( \mathcal{C} \)), in the following sense.

**Proposition 2.2.** Let 1 be the terminal monoidal category and \( \mathcal{C} \) a category. A 1-graded monad on \( \mathcal{C} \) is nothing but a monad on \( \mathcal{C} \).

That monoidal functors from 1 to \([\mathcal{C}, \mathcal{C}]\) is the same as monads on \( \mathcal{C} \) has been known for a while; see for example [3].

Graded monads, which were previously also known as parametric monads [28, 27, 18], arose in Melliès’ study [28] of Tensorial Logic, a refinement of Linear Logic. This notion was used by Katsumata [18] in the study of semantics of effect systems [23, 17], which are type systems to statically estimate the range of effects caused by a program (e.g., the set of registers that may be manipulated during an execution of the program). The intuition underlying Katsumata’s work is that graded monads model computational effects with parameters, extending Moggi’s classical observation [29, 30] that monads model computational effects.

On the other hand, the dual notion of graded comonad, which will be introduced below, have appeared in the study of computational resources with parameters, as seen in e.g., the work of Petricek, Orchard and Mycroft [31]. Graded comonads also arise in the study of higher-order model checking; see e.g., Grellois–Melliès [12] and Tsukada–Ong [39].

2.1.2 An example: the graded state monad

One of the motivating examples of graded monads is the graded state monad [7], which is an \( \text{Inj} \)-graded monad on \( \text{Set} \). Let us see its definition in detail. We first define the monoidal category \( \text{Inj} = (\text{Inj}, +, 0) \).
Definition 2.3. Define the strict monoidal category \((\text{Inj}, +, 0)\) as follows:

- An object of \(\text{Inj}\) is a natural number \(m = \{0, \ldots, m - 1\}\).
- A morphism \(u\) of \(\text{Inj}\) from \(m\) to \(m'\) is an injective function between sets
  \[
  u : \{0, \ldots, m - 1\} \rightarrow \{0, \ldots, m' - 1\}.
  \]
- The monoidal product \(+\) on \(\text{Inj}\) is the restriction of the binary coproduct on \(\text{Set}\).

Observe that the standard choice of \(+\) indeed makes \((\text{Inj}, +, 0)\) a strict monoidal category. \(\blacksquare\)

Definition 2.4. Let \(V\) be a set, interpreted as the set of values that can be stored in one register. The **graded state monad** with the set of values \(V\) is the \(\text{Inj}\)-graded monad \(T\) on \(\text{Set}\) defined as follows:

- For a natural number \(m\), define the functor \(T_m : \text{Set} \rightarrow \text{Set}\) by
  \[
  T_m \quad X \quad \mapsto \quad V^m \Rightarrow (V^m \times X) \cong (V^m \Rightarrow V^m) \times (V^m \Rightarrow X),
  \]
  where \(\Rightarrow\) denotes the exponential in \(\text{Set}\).
- For an injective function \(u : m \rightarrow m'\), define the natural transformation \(T_u : T_m \Rightarrow T_{m'}\) by setting its \(X\)-component,
  \[
  T_{u,X} \quad : \quad (V^m \Rightarrow V^m) \times (V^m \Rightarrow X) \rightarrow (V^{m'} \Rightarrow V^{m'}) \times (V^{m'} \Rightarrow X),
  \]
  as \((\tau, \xi) \mapsto (u \cdot \tau, \xi \circ V^l)\), where given a function \(\tau : V^m \rightarrow V^m\), the function
  \(u \cdot \tau : V^{m'} \rightarrow V^{m'}\) is defined as
  \[
  (u \cdot \tau)(val_{k'})_{k' \in m'} := \begin{cases}
  (\tau(val_{u(k)})_{k \in m})_l & \text{if } l' = u(l) \text{ for some } l \in m; \\
  val_{l'} & \text{otherwise}.
  \end{cases}
  \]
- Define the natural transformation \(\eta^{(T)} : \text{id}_{\text{Set}} \Rightarrow T_0\) by setting its \(X\)-component,
  \[
  \eta^{(T)}_X : X \rightarrow (V^0 \Rightarrow V^0) \times (V^0 \Rightarrow X),
  \]
  to be the canonical bijection.
- For a pair of natural numbers \(m, n\), define the natural transformation \(\mu^{(T)}_{m,n} : T_m \circ T_n \Rightarrow T_{m+n}\), by setting its \(X\)-component,
  \[
  \mu^{(T)}_{m,n,X} : (V^m \Rightarrow V^m) \times (V^{m+n} \Rightarrow V^n) \times (V^{m+n} \Rightarrow X) \rightarrow (V^{m+n} \Rightarrow V^{m+n}) \times (V^{m+n} \Rightarrow X),
  \]
  as \((\tau, \sigma, \xi) \mapsto (\tau \circ \sigma, \xi)\), where given functions \(\tau : V^m \rightarrow V^m\) and \(\sigma : V^{m+n} \rightarrow V^n\),
  the function \(\tau \circ \sigma : V^{m+n} \rightarrow V^{m+n}\) is defined as
  \[
  (\tau \circ \sigma)(val_k, wal_{k'})_{k \in m, k' \in n}) := \begin{cases}
  (\tau(val_k)_{k \in m})_l & \text{if } l \in m; \\
  (\sigma(val_k, wal_{k'})_{k \in m, k' \in n})_l & \text{if } l \in n. \quad \blacksquare
  \end{cases}
  \]
2.1.3 Graded comonads

The dual notion of graded comonad is of course

**Definition 2.5.** Let \( M = (\mathbb{M}, \otimes, I) \) be a strict monoidal category and \( C \) be a category. An \( M \)-graded comonad on \( C \) is an oplax monoidal functor

\[
S : (\mathbb{M}, \otimes, I) \rightarrow ([C, C], \circ, \text{id}_C). 
\]

We write down the explicit description of an \( M \)-graded comonad on \( C \) for the sake of concreteness; it consists of the following data:

- A functor \( S_m : C \rightarrow C \) for each object \( m \) of \( M \).
- A natural transformation \( S_u : S_m \Rightarrow S_{m'} \) for each morphism \( u : m \rightarrow m' \) of \( M \).
- A natural transformation \( \varepsilon : S_I \Rightarrow \text{id}_C \).
- A natural transformation \( \delta^{(S)}_{m,n} : S_{m \otimes n} \Rightarrow S_m \circ S_n \) for each pair of objects \( m, n \) of \( M \).

These data are subject to the following axioms:

**GC1** \( \text{id}_{S_m} = S_{\text{id}_m} \) for each object \( m \) of \( M \).

**GC2** \( S_{u'} \circ S_u = S_{u' \circ u} \) for each composable pair of morphisms \( u, u' \) of \( M \).

**GC3** \( S_{m \otimes n} \Rightarrow S_m \circ S_n \)

**GC4** \( \text{id}_{S_m} \Rightarrow S_m \Rightarrow \varepsilon \ast S_m \)

**GC5** \( \text{id}_{S_m} \Rightarrow S_m \Rightarrow S_m \ast \varepsilon \)

**GC6** \( S_{l \otimes m \otimes n} \Rightarrow S_{l \otimes m} \circ S_n \)

It should be clear that \( (\mathbb{M}, \otimes, I) \)-graded comonads on \( C \) are nothing but \( (\mathbb{M}^{\text{op}}, \otimes, I) \)-graded monads on \( C^{\text{op}} \). Note that we reversed the orientations of morphisms of \( M \) and \( \mathbb{C} \), but not the orientation of the monoidal product of \( M \).
2.2 Indexed monads

An *indexed monad* takes its parameters from a *category*.

### 2.2.1 The definition

We begin with a preliminary definition.

**Definition 2.6.** Let \( C \) be a category. Define the category \( \mathbf{Mnd}(C) \) of monads on \( C \) as follows:

- An object of \( \mathbf{Mnd}(C) \) is a monad \( T = (T, \eta, \mu) \) on \( C \).
- A morphism of \( \mathbf{Mnd}(C) \) from \( (T, \eta, \mu) \) to \( (T', \eta', \mu') \) is a natural transformation \( \tau: T' \rightarrow T \) commuting with the monad structures, i.e., such that the diagrams

\[
\begin{array}{ccc}
id_C & \xrightarrow{id_{id_{C}}} & id_C \\
\eta' & \downarrow{\eta} & \eta \\
T' & \tau & T \\
\end{array} \quad \begin{array}{ccc}
T' \circ T' & \xrightarrow{\tau \circ \tau} & T \circ T \\
\mu' & \downarrow{\mu} & \mu \\
T' & \tau & T \\
\end{array}
\]

commute. 

The direction of morphisms of \( \mathbf{Mnd}(C) \) follows that of Street [38]. Note that this direction makes the ordinary Eilenberg–Moore construction a covariant functor

\[
\mathbb{C}^{(-)} : \mathbf{Mnd}(C) \rightarrow \mathbf{Cat},
\]

with \( \mathbb{C}^{(-)}: C^T \rightarrow C^{T'} \) given by \( \left( \begin{array}{c} T_c \\ c \end{array} \right) \mapsto \left( \begin{array}{c} T'_c \\ c \circ \tau_c \end{array} \right) \).

**Definition 2.7.** Let \( \mathbb{B} \) and \( C \) be categories. A \( \mathbb{B} \)-indexed monad on \( C \) is a functor

\[
\mathcal{F} : \mathbb{B}^{\text{op}} \rightarrow \mathbf{Mnd}(C).
\]

Explicitly, a \( \mathbb{B} \)-indexed monad on \( C \) consists of the following data:

- A functor \( \mathcal{F}_b: C \rightarrow C \) for each object \( b \) of \( \mathbb{B} \).
- A natural transformation \( \mathcal{F}_u: \mathcal{F}_b \Rightarrow \mathcal{F}_{b'} \) for each morphism \( u: b \rightarrow b' \) of \( \mathbb{B} \).
- A natural transformation \( \eta^{(\mathcal{F})}_b = \eta_b: \text{id}_C \Rightarrow \mathcal{F}_b \) for each object \( b \) of \( \mathbb{B} \).
- A natural transformation \( \mu^{(\mathcal{F})}_b = \mu_b: \mathcal{F}_b \circ \mathcal{F}_b \Rightarrow \mathcal{F}_b \) for each object \( b \) of \( \mathbb{B} \).

These data are subject to the following axioms:

**\( \text{(IM1)} \)** \( \text{id}_{\mathcal{F}_b} = \mathcal{F}_{\text{id}_b} \) for each object \( b \) of \( \mathbb{B} \).

**\( \text{(IM2)} \)** \( \mathcal{F}_{u'} \circ \mathcal{F}_u = \mathcal{F}_{u' \circ u} \) for each composable pair of morphisms \( u, u' \) of \( \mathbb{B} \).

\[
\begin{array}{ccc}
id_C & \xrightarrow{id_{id_{C}}} & id_C \\
\eta_b & \downarrow{id_{\mathcal{F}_b}} & \eta_{b'} \\
\mathcal{F}_b & \mathcal{F}_u & \mathcal{F}_{b'}
\end{array}
\]

\( \text{(IM3)} \) commutes for each morphism \( u: b \rightarrow b' \) of \( \mathbb{B} \).
Indexed monads also provide a generalization of ordinary monads:

**Proposition 2.8.** Let 1 be the terminal category and \( C \) a category. A 1-indexed monad on \( C \) is nothing but a monad on \( C \).

Indexed monads have also arisen in the study of computational effects recently. They appeared, in the form of indexed Lawvere theory, in the work of Power \([35, 34]\) on the local state monad. Maillard and Melliès \([25]\) shed new light on Power’s work using indexed monads in their sense; our notion of \( B \)-indexed monad on \( C \) is a restriction of their notion of \( B \)-indexed monad (without further qualification). The primary example of indexed monads used in this series of study is the indexed state monad which we recall shortly, and can be intuitively thought of as a family of the state monads indexed by (parametrized by) the number of registers; so indexed monads also model the idea of computational effects with parameters. We review in more detail Power’s and Maillard and Melliès’ work in Section 5.2 where we relate their results to ours.

### 2.2.2 An example: the indexed state monad

The indexed state monad \([35, 34, 25]\) is an \( \text{Inj} \)-indexed monad on \( \text{Set} \). Note that this time we regard \( \text{Inj} \) as a mere category rather than a strict monoidal category (\( \text{Inj}, +, 0 \)).

**Definition 2.9.** Let \( V \) be a set, interpreted again as the set of values that can be stored in one register. The **indexed state monad** with the set of values \( V \) is the \( \text{Inj} \)-indexed monad \( \mathcal{T} \) defined as follows:

- For a natural number \( m \), define the functor \( \mathcal{T}_m : \text{Set} \to \text{Set} \) by \( \mathcal{T}_m = T_m \).
- For an injective function \( u : m \to m' \), define the natural transformation \( \mathcal{T}_u : \mathcal{T}_m \Rightarrow \mathcal{T}_{m'} \) by \( \mathcal{T}_u = T_u \).
For a natural number \( m \), define the natural transformation \( \eta_m(\mathcal{F}) : \text{id}_{\text{Set}} \Rightarrow \mathcal{F}_m \) by setting its \( X \)-component,

\[
\eta_{m,X}^{(\mathcal{F})} : X \rightarrow (V^m \Rightarrow V^m) \times (V^m \Rightarrow X),
\]
as \( x \mapsto (\text{id}_{V^m}, \text{const}_x = (- \mapsto x)) \).

For a natural number \( m \), define the natural transformation \( \mu_m(\mathcal{F}) : \mathcal{F}_m \circ \mathcal{F}_m \Rightarrow \mathcal{F}_m \) by setting its \( X \)-component,

\[
\mu_{m,X}^{(\mathcal{F})} : (V^m \Rightarrow V^m) \times (V^m \Rightarrow V^m) \times (V^m \Rightarrow (V^m \Rightarrow X)) \rightarrow (V^m \Rightarrow V^m) \times (V^m \Rightarrow X),
\]
by \( (\tau, \sigma, \xi) \mapsto (\tau \triangleright \sigma, \tau \triangleright \xi) \), where

- given functions \( \tau : V^m \rightarrow V^m \) and \( \sigma : V^m \rightarrow (V^m \Rightarrow V^m) \), the function \( \tau \triangleright \sigma : V^m \rightarrow V^m \) is defined as
  \[
  (\tau \triangleright \sigma)(v) := \sigma(v)(\tau v);
  \]
- given functions \( \tau : V^m \rightarrow V^m \) and \( \xi : V^m \rightarrow (V^m \Rightarrow X) \), the function \( \tau \triangleright \xi : V^m \rightarrow X \) is defined as
  \[
  (\tau \triangleright \xi)(v) := \xi(v)(\tau v). \quad \Box
  \]

Power \cite{34, 35} derives the indexed Lawvere theory equivalent to the indexed state monad abstractly from the global state monad, using the universality of \((\text{Inj}, +, 0)\) as the free monoidal category with the initial unit, and the tensor product operation naturally defined on Lawvere theories. Now the reader might noticed the obvious similarity between the indexed state monad and the graded state monad in Section 2.1.2. Indeed, one can also derive the graded state monad from the indexed state monad abstractly; the requirement is again that \((\text{Inj}, +, 0)\) has the initial unit.

**Proposition 2.10.** Let \( \mathcal{M} = (\mathcal{M}, \otimes, I) \) be a strict monoidal category such that the monoidal unit \( I \) is the initial object of \( \mathcal{M} \), and \( \mathcal{C} \) be a category. Then every \( \mathcal{M} \)-indexed monad \( \mathcal{F} = (\mathcal{F}, \eta(\mathcal{F}), \mu(\mathcal{F})) \) on \( \mathcal{C} \) naturally induces an \((\mathcal{M}, \otimes, I)\)-graded monad \( T = (T, \eta(T), \mu(T)) \) on \( \mathcal{C} \).

**Proof.** First define \( T_m := \mathcal{F}_m \) and \( T_n := \mathcal{F}_n \) for each object \( m \) and morphism \( u \) of \( \mathcal{M} \).

Then define \( \eta(T) : \text{id}_{\mathcal{C}} \Rightarrow T_I \) by \( \eta(T) := \eta_{I}^{(\mathcal{F})} \).

Finally, we have to define \( \mu_{m,n}^{(T)} : T_m \circ T_n \Rightarrow T_{m \otimes n} \). To this purpose, first observe that there are morphisms

\[
\begin{align*}
in_{m,n} & : m = m \otimes I \xrightarrow{\text{id}_m \otimes_{n} i_n} m \otimes n \quad \text{and} \quad \text{inr}_{m,n} : n = I \otimes n \xrightarrow{i_m \otimes_{n} \text{id}_n} m \otimes n
\end{align*}
\]
of \( \mathcal{M} \). Now define

\[
\mu_{m,n}^{(T)} := \mathcal{F}_m \circ \mathcal{F}_n \xrightarrow{\mathcal{F}_{\text{in}_{m,n}} \ast \mathcal{F}_{\text{in}_{n,m}}} \mathcal{F}_{m \otimes n} \circ \mathcal{F}_{m \otimes n} \xrightarrow{\mu_{m \otimes n}^{(\mathcal{F})}} \mathcal{F}_{m \otimes n}.
\]

This gives an abstract explanation of how our leading examples of “monads with parameters”, the graded and indexed state monads, are obtained from the global state monad.
2.2.3 Indexed comonads

**Definition 2.11.** Let \( \mathcal{C} \) be a category. Define the category \( \text{Comnd}(\mathcal{C}) \) of comonads on \( \mathcal{C} \) as follows:

- An object of \( \text{Comnd}(\mathcal{C}) \) is a comonad \( S = (S, \varepsilon, \delta) \) on \( \mathcal{C} \).
- A morphism of \( \text{Comnd}(\mathcal{C}) \) from \( (S, \varepsilon, \delta) \) to \( (S', \varepsilon', \delta') \) is a natural transformation \( \sigma : S \to S' \) commuting with the comonad structures, i.e., such that the diagrams

  \[
  \begin{array}{ccc}
  S & \xrightarrow{\sigma} & S' \\
  \downarrow{\varepsilon} & & \downarrow{\varepsilon'} \\
  \text{id}_C & > & \text{id}_C \\
  \end{array}
  \quad \text{and} \quad
  \begin{array}{ccc}
  S & \xrightarrow{\sigma} & S' \\
  \downarrow{\delta} & & \downarrow{\delta'} \\
  S \circ S & > & S' \circ S' \\
  \end{array}
  \]

  
  commute.

This definition makes the co-Eilenberg–Moore construction for comonads a covariant functor

\[
\mathcal{C}(-) : \text{Comnd}(\mathcal{C}) \to \mathcal{Cat},
\]

with \( C : \mathcal{C} \to \mathcal{C}' \) given by \( \left( \begin{array}{c} c \\ S \end{array} \right) \mapsto \left( \begin{array}{c} c \\ S \circ \sigma \end{array} \right) \).

**Definition 2.12.** Let \( \mathcal{B} \) and \( \mathcal{C} \) be categories. A \( \mathcal{B} \)-indexed comonad on \( \mathcal{C} \) is a functor

\[
\mathcal{J} : \mathcal{B} \to \text{Comnd}(\mathcal{C}).
\]

A \( \mathcal{B} \)-indexed comonad on \( \mathcal{C} \) consists of the following data:

- A functor \( \mathcal{J}_b : \mathcal{C} \to \mathcal{C} \) for each object \( b \) of \( \mathcal{B} \).
- A natural transformation \( \varepsilon\mathcal{J}_b : \mathcal{J}_b \Rightarrow \text{id}_\mathcal{C} \) for each morphism \( u : b \to b' \) of \( \mathcal{B} \).
- A natural transformation \( \delta\mathcal{J}_b : \mathcal{J}_b \Rightarrow \mathcal{J}_b \circ \mathcal{J}_b \) for each object \( b \) of \( \mathcal{B} \).

These data are subject to the following axioms:

- **(IC1)** \( \text{id}_{\mathcal{J}_b} = \mathcal{J}_{\text{id}_b} \) for each object \( b \) of \( \mathcal{B} \).
- **(IC2)** \( \mathcal{J}_{u' \circ u} = \mathcal{J}_u \circ \mathcal{J}_{u'} \) for each composable pair of morphisms \( u, u' \) of \( \mathcal{B} \).
- **(IC3)** \( \varepsilon\mathcal{J}_b \downarrow \mathcal{J}_u \Rightarrow \varepsilon\mathcal{J}_{b'} \) commutes for each morphism \( u : b \to b' \) of \( \mathcal{B} \).
- **(IC4)** \( \delta\mathcal{J}_b \circ \mathcal{J}_u \Rightarrow \mathcal{J}_{u} \circ \mathcal{J}_{b'} \) commutes for each morphism \( u : b \to b' \) of \( \mathcal{B} \).
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\[
\begin{align*}
\mathcal{J}_b & \xrightarrow{\delta_b} \mathcal{J}_b \circ \mathcal{J}_b \\
\mathcal{J}_b \circ \mathcal{J}_b & \xrightarrow{\delta_b \circ \delta_b} \mathcal{J}_b \circ \mathcal{J}_b \circ \mathcal{J}_b
\end{align*}
\]

(IC5) \(\mathcal{J}_b \xrightarrow{\epsilon_b \circ \mathcal{J}_b} \mathcal{J}_b\) commutes for each object \(b\) of \(\mathcal{B}\).

(IC6) \(\mathcal{J}_b \xrightarrow{\epsilon_b \circ \mathcal{J}_b} \mathcal{J}_b\) commutes for each object \(b\) of \(\mathcal{B}\).

(IC7) \(\mathcal{J}_b \xrightarrow{\epsilon_b \circ \mathcal{J}_b} \mathcal{J}_b\) commutes for each object \(b\) of \(\mathcal{B}\).

Observe that \(\mathcal{B}\)-indexed comonads on \(\mathcal{C}\) are equivalent to \(\mathcal{B}^{\text{op}}\)-indexed monads on \(\mathcal{C}^{\text{op}}\).

Notes

For the reason why we have decided to change the terminology from parametric monads to graded monads, see Section 2 of [7]. The graded state monad, which appears in [7], was introduced by Paul-André Melliès. I learned the abstract construction of the graded state monad from the indexed state monad (Proposition 2.10) from the anonymous reviewer of the paper [7].
Chapter 3

The four 2-categories

In this chapter, we introduce four 2-categories $E^{++}$, $E^{+-}$, $E^{-+}$ and $E^{--}$. These 2-categories enable us to regard graded and indexed monads as mere monads in the 2-categorical sense, thus paving the way to a mathematical theory of graded and indexed monads inside the celebrated abstract framework of Street [38]. More precisely, the relationship of the notions of graded and indexed (co)monad and the 2-categories $E^{++}$, $E^{+-}$, $E^{-+}$ and $E^{--}$ is summarized in the following table:

|                      | $E^{++}$ | $E^{+-}$ | $E^{-+}$ | $E^{--}$ |
|----------------------|----------|----------|----------|----------|
| Graded monads        | ✓        |          | ✓        | ✓        |
| Indexed monads       | ✓        | ✓        |          |          |
| Graded comonads      | ✓        |          | ✓        |          |
| Indexed comonads     | ✓        |          |          | ✓        |

The ✓ mark indicates that the row notion can be seen as mere (co)monads in the column 2-category.

3.1 The formal theory of monads

In his seminal and influential paper [38], Street developed an abstract theory of monads relative to an arbitrary 2-category $\mathcal{K}$, so that the usual theory of monads is regained by instantiating $\mathcal{K}$ by $\text{Cat}$, the 2-category of categories, functors, and natural transformations. Since our principal justification of the constructions presented in Chapter 4 will be that they produce Eilenberg–Moore or Kleisli objects, which are among the key notions introduced in [38], we begin this chapter with a quick review of Street’s work.

**Definition 3.1.** Let $\mathcal{H}$ be a 2-category and $K$ a 0-cell of $\mathcal{H}$. A monad $T$ in $\mathcal{H}$ on $K$ is given by a 1-cell $T: K \to K$, and 2-cells $\eta: \text{id}_K \Rightarrow T$ and $\mu: T \circ T \Rightarrow T$ of $\mathcal{H}$, satisfying the usual axioms $\mu.\eta T = \text{id}_T = \mu.T\eta$ and $\mu.T\mu = \mu.\mu T$. ■

We fix a monad $T$ in $\mathcal{H}$ on $K$. For the definition of Eilenberg–Moore and Kleisli objects, we adopt the following representable one, which appears for instance in [37]:

**Definition 3.2.** The Eilenberg–Moore object of $T$ is a 0-cell $K^T$ such that there is a family of isomorphisms of categories

$$\mathcal{H}(X, K^T) \cong \mathcal{H}(X, K)^{\mathcal{H}(X, T)}$$

2-natural in $X \in \mathcal{H}$. Here the category on the right hand side is the (usual) Eilenberg–Moore category of the monad $\mathcal{H}(X, T)$ on the category $\mathcal{H}(X, K)$. ■
Definition 3.3. The Kleisli object of $T$ is a 0-cell $K_T$ such that there is a family of isomorphisms of categories

$$\mathcal{K}(K_T, X) \cong \mathcal{K}(K, X)^{\mathcal{K}(T, X)}$$

2-natural in $X \in \mathcal{K}$. Here the category on the right hand side is the (usual) Eilenberg–Moore category of the monad $\mathcal{K}(T, X)$ on the category $\mathcal{K}(K, X)$. ■

Noting that a monad in $\mathcal{K}$ is the same thing as a monad in $\mathcal{K}^{op(1)}$, the Kleisli object of $T$ in $\mathcal{K}$ can be equivalently defined as the Eilenberg–Moore object of $T$ in $\mathcal{K}^{op(1)}$.

Now a remarkable point is that from this simple and abstract definition, one can reconstruct a fair amount of the well-known property of Eilenberg–Moore or Kleisli categories, including the existence of adjunctions which generate the monads, and the existence and uniqueness of comparison 1-cells. The interested reader should consult [38] for an ingenious 2-categorical manipulation achieving this reconstruction.

3.2 Enlarging $\mathbf{Cat}$

The four 2-categories $\mathcal{E}^{++}$, $\mathcal{E}^{+-}$, $\mathcal{E}^{-+}$ and $\mathcal{E}^{--}$ we now introduce are obtained via suitable lax comma constructions applied to the 3-category $2\mathbf{Cat}$ of 2-categories, 2-functors, 2-natural transformations and modifications. The intuitive idea is that these four 2-categories are obtained by “enlarging” the familiar 2-category $\mathbf{Cat}$; actually, they “contain” an arbitrary 2-category, in the sense that for any 2-category $\mathcal{K}$ (subject to a certain size condition) there are canonical inclusion 2-functors $\mathcal{K} \hookrightarrow \mathcal{E}^{++}$, $\mathcal{K} \hookrightarrow \mathcal{E}^{+-}$, $\mathcal{K} \hookrightarrow \mathcal{E}^{-+}$ and $\mathcal{K} \hookrightarrow \mathcal{E}^{--}$.

In the following definitions, we denote the terminal 2-category (the 2-category consisting of a single 0-cell, a single 1-cell and a single 2-cell) by 1.

3.2.1 The 2-category $\mathcal{E}^{++}$

Definition 3.4. We define the 2-category $\mathcal{E}^{++}$ by the following data.

- A 0-cell of $\mathcal{E}^{++}$ is a 2-functor $A: 1 \to \mathcal{A}$ where $\mathcal{A}$ is a 2-category; equivalently, it is a pair $(\mathcal{A}, A)$ where $A$ is a 0-cell of $\mathcal{A}$.

- A 1-cell of $\mathcal{E}^{++}$ from $(\mathcal{A}, A)$ to $(\mathcal{B}, B)$ is a diagram in $2\mathbf{Cat}$

$$
\begin{array}{ccc}
A & \xrightarrow{F} & \mathcal{B} \\
\downarrow^{f} & & \downarrow^{B} \\
\mathcal{A} & \xrightarrow{1} & \mathcal{B}
\end{array}
$$

where $F$ is a 2-functor and $f$ a 2-natural transformation; equivalently, it is a pair $(F, f)$ where $f: FA \to B$ is a 1-cell of $\mathcal{B}$. 
A 2-cell of $\mathcal{E}^{++}$ from $(F, f)$ to $(F', f')$ is a diagram in $2\text{-}\mathsf{Cat}$

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \Theta & & \downarrow \Theta \\
F & \xrightarrow{\alpha} & F'
\end{array}
\]

where $\Theta$ is a 2-natural transformation and $\alpha$ a modification; equivalently, it is a pair $(\Theta, \alpha)$ where $\alpha$ is a 2-cell of $B$ of the following type:

\[
\begin{array}{ccc}
FA & \xrightarrow{f} & B \\
\downarrow \Theta_A & & \downarrow \Theta_A \\
F'A & \xrightarrow{\alpha} & F'
\end{array}
\]

The compositions in $\mathcal{E}^{++}$ are defined abstractly by obvious pasting diagrams in $2\text{-}\mathsf{Cat}$. We also provide a concrete description of the compositions in Appendix [A].

The first projection of the data defines a 2-functor $p^{++} : \mathcal{E}^{++} \to 2\text{-}\mathsf{Cat}_2$, where $2\text{-}\mathsf{Cat}_2$ is the 2-category of 2-categories, 2-functors, and 2-natural transformations. We take a fibrational viewpoint [2, 15, 14] and say a notion $X$ is above $I$ if $p^{++}(X) = I$. Observe that the fiber over the object $\mathcal{K}$ of $2\text{-}\mathsf{Cat}_2$ is $\mathcal{K}$ itself; this is why one can claim that $\mathcal{E}^{++}$ “contains” an arbitrary 2-category.

### 3.2.2 The 2-category $\mathcal{E}^{+-}$

**Definition 3.5.** We define the 2-category $\mathcal{E}^{+-}$ by the following data.

- A 0-cell of $\mathcal{E}^{+-}$ is a 2-functor $A : 1 \to \mathcal{A}$ where $\mathcal{A}$ is a 2-category; equivalently, it is a pair $(\mathcal{A}, A)$ where $A$ is a 0-cell of $\mathcal{A}$.

- A 1-cell of $\mathcal{E}^{+-}$ from $(\mathcal{A}, A)$ to $(\mathcal{B}, B)$ is a diagram in $2\text{-}\mathsf{Cat}$

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow F & & \downarrow F \\
\mathcal{A} & \xrightarrow{\alpha} & \mathcal{B}
\end{array}
\]

where $F$ is a 2-functor and $f$ a 2-natural transformation; equivalently, it is a pair $(F, f)$ where $f : FA \to B$ is a 1-cell of $\mathcal{B}$. 
A 2-cell of $\mathcal{E}^{+-}$ from $(F, f)$ to $(F', f')$ is a diagram in $2$-$\text{Cat}$

$$\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
F & \xrightarrow{\Theta} & F'
\end{array}
\end{array} \quad \Rightarrow 
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{f'} & B \\
\downarrow & & \downarrow \\
F' & \xrightarrow{\Theta} & F
\end{array}
\end{array}$$

where $\Theta$ is a 2-natural transformation and $\alpha$ a modification; equivalently, it is a pair $(\Theta, \alpha)$ where $\alpha$ is a 2-cell of $\mathcal{B}$ of the following type:

$$
\begin{array}{c}
\Theta_A \\
\xrightarrow{\alpha} \\
\xrightarrow{f'}
\end{array}
\begin{array}{c}
F A \\
\xrightarrow{\alpha} \\
\xrightarrow{f'}
\end{array}
\begin{array}{c}
F' A \\
\xrightarrow{\Theta} \\
\xrightarrow{\alpha}
\end{array}
\begin{array}{c}
B
\end{array}
\begin{array}{c}
\xrightarrow{\alpha}
\end{array}
\begin{array}{c}
\xrightarrow{f'}
\end{array}
\begin{array}{c}
B
\end{array}
$$

For this 2-category, we may define by the first projection the projection 2-functor $p^{+-}: \mathcal{E}^{+-} \rightarrow 2$-$\text{Cat}^{op(2)}$.

### 3.2.3 The 2-category $\mathcal{E}^{--}$

**Definition 3.6.** We define the 2-category $\mathcal{E}^{--}$ by the following data.

- A 0-cell of $\mathcal{E}^{--}$ is a 2-functor $A: 1 \rightarrow \mathcal{A}$ where $\mathcal{A}$ is a 2-category; equivalently, it is a pair $(\mathcal{A}, A)$ where $A$ is a 0-cell of $\mathcal{A}$.
- A 1-cell of $\mathcal{E}^{--}$ from $(\mathcal{A}, A)$ to $(\mathcal{B}, B)$ is a diagram in $2$-$\text{Cat}$

$$
\begin{array}{c}
A \\
\xrightarrow{f} \\
\xrightarrow{F}
\end{array}
\begin{array}{c}
\xrightarrow{1} \\
\xrightarrow{B}
\end{array}
\begin{array}{c}
\xrightarrow{F}
\end{array}
\begin{array}{c}
\xrightarrow{B}
\end{array}
$$

where $F$ is a 2-functor and $f$ a 2-natural transformation; equivalently, it is a pair $(F, f)$ where $f: A \rightarrow FB$ is a 1-cell of $\mathcal{A}$.
- A 2-cell of $\mathcal{E}^{--}$ from $(F, f)$ to $(F', f')$ is a diagram in $2$-$\text{Cat}$

$$
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
F & \xrightarrow{\Theta} & F'
\end{array}
\end{array} \quad \Rightarrow 
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{f'} & B \\
\downarrow & & \downarrow \\
F' & \xrightarrow{\Theta} & F
\end{array}
\end{array}$$
where $\Theta$ is a 2-natural transformation and $\alpha$ a modification; equivalently, it is a pair $(\Theta, \alpha)$ where $\alpha$ is a 2-cell of $\mathscr{A}$ of the following type:

![Diagram](image)

The first projection defines a 2-functor $p^+: \mathfrak{E}^- \rightarrow \mathcal{2-Cat}_2^{op(1)}$.

### 3.2.4 The 2-category $\mathfrak{E}^{--}$

**Definition 3.7.** We define the 2-category $\mathfrak{E}^{--}$ by the following data.

- A 0-cell of $\mathfrak{E}^{--}$ is a 2-functor $A: 1 \rightarrow \mathscr{A}$ where $\mathscr{A}$ is a 2-category; equivalently, it is a pair $(\mathscr{A}, A)$ where $A$ is a 0-cell of $\mathscr{A}$.

- A 1-cell of $\mathfrak{E}^{--}$ from $(\mathscr{A}, A)$ to $(\mathscr{B}, B)$ is a diagram in $\mathcal{2-Cat}$

![Diagram](image)

where $F$ is a 2-functor and $f$ a 2-natural transformation; equivalently, it is a pair $(F, f)$ where $f: A \rightarrow FB$ is a 1-cell of $\mathscr{A}$.

- A 2-cell of $\mathfrak{E}^{--}$ from $(F, f)$ to $(F', f')$ is a diagram in $\mathcal{2-Cat}$

![Diagram](image)

where $\Theta$ is a 2-natural transformation and $\alpha$ a modification; equivalently, it is a pair $(\Theta, \alpha)$ where $\alpha$ is a 2-cell of $\mathscr{A}$ of the following type:

![Diagram](image)

The first projection defines a 2-functor $p^{--}: \mathfrak{E}^{--} \rightarrow \mathcal{2-Cat}_2^{op(1,2)}$. 
3.3 Reducing monads with parameters to mere monads

Let us proceed to see how the notions of graded and indexed monad can be seen as mere monads in the 2-categorical sense of Definition 3.1 inside the appropriate 2-categories among $\mathcal{E}^{++}$, $\mathcal{E}^{-+}$, $\mathcal{E}^{-}$ and $\mathcal{E}^{--}$; this also helps digress and motivate the definitions of these 2-categories, which might look a little bit complicated at first sight.

3.3.1 Graded monads as monads in $\mathcal{E}^{++}$

The first reduction for graded monads takes place in the 2-category $\mathcal{E}^{++}$, and will later be employed to perform the Eilenberg–Moore construction for graded monads.

Let us fix a strict monoidal category $\mathbb{M} = (\mathbb{M}, \otimes, I)$. First we need a preliminary definition.

**Definition 3.8.** There is a 2-monad $\mathbb{M} \times (-) = (\mathbb{M} \times (-), H, M)$ on $\mathsf{Cat}$ defined as follows.

- The 2-functor $\mathbb{M} \times (-) : \mathsf{Cat} \to \mathsf{Cat}$ is the product category construction.
- The 2-natural transformation $H : \mathbb{M} \times (-) \Rightarrow \mathbb{M} \times (-)$ has as its $X$-component the functor $H^X : X \to \mathbb{M} \times X$ given by $x \mapsto (I, x)$.
- The 2-natural transformation $M : \mathbb{M} \times \mathbb{M} \times (-) \Rightarrow \mathbb{M} \times (-)$ has as its $X$-component the functor $M^X : \mathbb{M} \times \mathbb{M} \times X \to \mathbb{M} \times X$ given by $(m, n, x) \mapsto (m \otimes n, x)$.

See Appendix B.1 for an abstract way to understand this 2-monad, as well as other 2-(co)monads introduced subsequently.

Now we have the following result:

**Proposition 3.9.** Let $\mathcal{C}$ be a category. Then an $\mathbb{M}$-graded monad on $\mathcal{C}$ is the same thing as a monad in $\mathcal{E}^{++}$ on $(\mathsf{Cat}, \mathcal{C})$ above the 2-monad $\mathbb{M} \times (-) : \mathsf{Cat} \to \mathsf{Cat}$.

**Proof.** Let us spell out what a monad in $\mathcal{E}^{++}$ on $(\mathsf{Cat}, \mathcal{C})$ above $\mathbb{M} \times (-)$, actually is. Such a thing consists of the following data:

- A 1-cell $(\mathbb{M} \times (-), \mathbf{T}) : (\mathsf{Cat}, \mathcal{C}) \to (\mathsf{Cat}, \mathcal{C})$ of $\mathcal{E}^{++}$, where $\mathbf{T}$ is a functor of type $\mathbf{T} : \mathbb{M} \times \mathcal{C} \to \mathcal{C}$.

- A 2-cell $(H^\mathcal{C}, \eta) : (id_{\mathsf{Cat}}, id_{\mathcal{C}}) \Rightarrow (\mathbb{M} \times (-), \mathbf{T})$ of $\mathcal{E}^{++}$, where $\eta$ is a natural transformation of type

  $\begin{array}{ccc}
  \mathcal{C} & \xymatrix{\ar[r]^{\eta} & \Rightarrow} & \mathcal{C} \\
  \ar[u]^{H^\mathcal{C}} & \mathbb{M} \times \mathcal{C} & \ar[l]_{id_{\mathbb{M}}} \\
  \mathbb{M} \times \mathcal{C} & \ar[u]_{\mathbf{T}} & \mathcal{C}
  \end{array}$

- A 2-cell $(M^\mathcal{C}, \mu) : (\mathbb{M} \times (-), \mathbf{T}) \circ (\mathbb{M} \times (-), \mathbf{T}) \Rightarrow (\mathbb{M} \times (-), \mathbf{T})$ of $\mathcal{E}^{++}$, where $\mu$ is a natural transformation of type

  $\begin{array}{ccc}
  \mathbb{M} \times \mathbb{M} \times \mathcal{C} & \xymatrix{\ar[r]^{M \times \mathbf{T}} & \Rightarrow} & \mathbb{M} \times \mathcal{C} \\
  \ar[r]_{M^\mathcal{C}} & \mathbb{M} \times \mathcal{C} & \ar[l]_{\mathbf{T}} \\
  \mathbb{M} \times \mathcal{C} & \ar[u]_{\mathbf{T}} & \mathcal{C}
  \end{array}$
These data satisfy the following axioms:

$$
\begin{align*}
M \times C & \xrightarrow{T} C \\
M \times M \times C & \xrightarrow{\mu} M \times C \xrightarrow{T} C = M \times C \xrightarrow{T} C & (3.1) \\
M \times M \times C & \xrightarrow{\eta} M \times C \xrightarrow{T} C = M \times C \xrightarrow{T} C & (3.2) \\
M \times M \times M \times C & \xrightarrow{\mu} M \times M \times C \xrightarrow{T} M \times C = M \times C \xrightarrow{T} C & (3.3)
\end{align*}
$$

To see the equivalence of this notion and that of $M$-graded monad on $C$, first note that by adjointness the functor $T: M \times C \to C$ is equivalent to its transpose, also denoted by $T: M \to [C, C]$. Now it is routine to see that in the definition of graded monads, giving the data $T_m$ and $T_u$ satisfying (GM1) and (GM2) is equivalent to giving $T$, and giving $\mu_{m,n}$ satisfying (GM3) is equivalent to giving $\mu$. Finally observe that (GM4), (GM5) and (GM6) are equivalent to (3.1), (3.2) and (3.3) respectively. □

### 3.3.2 Graded monads as monads in $\mathcal{E}^{\rightarrow}$

In this section we observe that there is another way to understand graded monads as monads in the 2-categorical sense, by working inside the 2-category $\mathcal{E}^{\rightarrow}$; this will be used for the Kleisli construction for graded monads.

Again we fix a strict monoidal category $M = (\mathcal{M}, \otimes, I)$.

**Definition 3.10.** There is a 2-comonad $[M, -] = ([M, -], H_M, M_M)$ on $\mathcal{C}at$ defined as follows.
• The 2-functor $[M, -] : \mathcal{C} \to \mathcal{C}$ is the functor category construction.

• The 2-natural transformation $H_M : [M, -] \Rightarrow \text{id}_{\mathcal{C}}$ has as its $X$-component the functor $H_{M,X} = H_X : [M, X] \to X$ given by $X \mapsto X(I)$.

• The 2-natural transformation $M_M : [M, -] \Rightarrow [M, [M, -]]$ has as its $X$-component the functor $M_{M,X} = M_X : [M, X] \to [M, [M, X]]$ given by $X \mapsto (m \mapsto X(- \otimes m))$.

**Proposition 3.11.** Let $C$ be a category. Then an $M$-graded monad on $C$ is the same thing as a monad in $\mathcal{E}^{-\Rightarrow}$ on $(\mathcal{C} \mathcal{C}, C)$ above the 2-comonad $[M, -] : \mathcal{C} \to \mathcal{C}$.

**Proof.** Indeed, the latter notion is given by the following data:

- A 1-cell $([M, -, T]) : (\mathcal{C} \mathcal{C}, C) \to (\mathcal{C} \mathcal{C}, C)$ of $\mathcal{E}^{-\Rightarrow}$, where $T$ is a functor of type $T : C \to [M, C]$.

- A 2-cell $(H_M, \eta) : (\text{id}_{\mathcal{C}}, \text{id}_{C}) \Rightarrow ([M, -], T)$ of $\mathcal{E}^{-\Rightarrow}$, where $\eta$ is a natural transformation of type $\eta : C \to [M, [M, C]]$.

- A 2-cell $(M_M, \mu) : ([M, -], T) \circ ([M, -], T) \Rightarrow ([M, -], T)$ of $\mathcal{E}^{-\Rightarrow}$, where $\mu$ is a natural transformation of type $\mu : C \to [M, C]$.

These data satisfy the following axioms:
3.3. REDUCING MONADS WITH PARAMETERS TO MERE MONADS

\[ [M, C] \xrightarrow{[M, T]} [M, [M, C]] \xrightarrow{[M, [M, T]]} [M, [M, [M, C]]] \]

\[ \xrightarrow{\mu} \]

\[ C \xrightarrow{T} [M, C] \xrightarrow{[M, T]} [M, [M, C]] \xrightarrow{\mu} [M, M, C] \]

\[ \xrightarrow{\nabla} \]

\[ [M, [M, C]] \xrightarrow{[M, [M, T]]} [M, [M, [M, C]]] \]

\[ \xrightarrow{\mu} \]

\[ = \]

\[ C \xrightarrow{T} [M, C] \xrightarrow{[M, T]} [M, [M, C]] \xrightarrow{\mu} [M, M, C] \]

The axioms correspond respectively to (GM4), (GM5) and (GM6). □

3.3.3 Indexed monads as monads in \( \mathcal{E}^{+\cdot} \)

Indexed monads also admit similar reductions to mere monads in the 2-categorical sense. The first such reduction is achieved in the 2-category \( \mathcal{E}^{+\cdot} \), and will provide a basis for the Eilenberg–Moore construction for indexed monads.

Let us fix a category \( \mathcal{B} \).

Definition 3.12. There is a 2-comonad \( \mathcal{B} \times (-) = (\mathcal{B} \times (-), H^\mathcal{B}, M^\mathcal{B}) \) on \( \mathcal{C}at \) defined as follows.

- The 2-functor \( \mathcal{B} \times (-) \) is the product category construction.
- The 2-natural transformation \( H^\mathcal{B} : \mathcal{B} \times (-) \Rightarrow Id_{\mathcal{C}at} \) has as its \( X \)-component the functor \( H^\mathcal{B}_X = H_X : \mathcal{B} \times X \to X \) given by \( (b, x) \mapsto x \).
- The 2-natural transformation \( M^\mathcal{B} : \mathcal{B} \times (-) \Rightarrow \mathcal{B} \times \mathcal{B} \times (-) \) has as its \( X \)-component the functor \( M^\mathcal{B}_X = M_X : \mathcal{B} \times X \to \mathcal{B} \times \mathcal{B} \times X \) given by \( (b, x) \mapsto (b, b, x) \). ■

Proposition 3.13. Let \( \mathcal{C} \) be a category. Then a \( \mathcal{B} \)-indexed monad on \( \mathcal{C} \) is the same thing as a monad in \( \mathcal{E}^{+\cdot} \) on \( \mathcal{C}at \) above the 2-comonad \( \mathcal{B} \times (-) : \mathcal{C}at \to \mathcal{C}at \).

Proof. The latter notion is given by the following data:

- A 1-cell \( (\mathcal{B} \times (-), \mathcal{F}) : (\mathcal{C}at, \mathcal{C}) \to (\mathcal{C}at, \mathcal{C}) \) of \( \mathcal{E}^{+\cdot} \), where \( \mathcal{F} \) is a functor of type

\[ \mathcal{F} : \mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{C}. \]

- A 2-cell \( (H^\mathcal{B}, \eta) : (Id_{\mathcal{C}at}, Id_{\mathcal{C}}) \Rightarrow (\mathcal{B} \times (-), \mathcal{F}) \) of \( \mathcal{E}^{+\cdot} \), where \( \eta \) is a natural transformation of type

\[ \xrightarrow{H^\mathcal{B}} \]

\[ \xrightarrow{\eta} \]

\[ \mathcal{B} \times \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C}. \]
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- A 2-cell \((M^B, \mu): (B \times (-), \mathcal{F}) \circ (B \times (-), \mathcal{F}) \Rightarrow (B \times (-), \mathcal{F})\) of \(\mathcal{E}^{+-}\), where \(\mu\) is a natural transformation of type

\[
\begin{align*}
&\xymatrix{ & B \times B \times C \ar[r]^-{\mathcal{F}} & B \times C \ar[r]^-{\mathcal{F}} & C \\
& B \times C \ar[r]_-{\mathcal{F}} & B \times C \ar[r]^-{\mathcal{F}} & C \\
& B \times C \ar[u]^-{M_C} & B \times C \ar[u]_-{\mu} & C \ar[u]_-{\mathcal{F}} \ar[u]_-{\mu}
\end{align*}
\]

These data satisfy the following axioms:

\[
\begin{align*}
&\xymatrix{ & B \times C \ar[r]^-{\mathcal{F}} & C \ar[dl]_-{H_{B \times C}} & \\
& B \times B \times C \ar[r]^-{\mathcal{F}} & B \times C \ar[r]^-{\mathcal{F}} & C \ar[u]_-{\eta} & \\
& B \times C \ar[r]_-{\mathcal{F}} & C \ar[dl]_-{H_C} & \\
& B \times B \times C \ar[r]_-{\mathcal{F}} & B \times C \ar[r]_-{\mathcal{F}} & C \ar[u]_-{\eta} \ar[u]_-{\mathcal{F}} \ar[u]_-{\mu} \ar[u]_-{\mu} & \\
& B \times C \ar[u]_-{\mathcal{F}} \ar[u]_-{\mathcal{F}} & C \ar[u]_-{\mathcal{F}} \ar[u]_-{\mathcal{F}} \ar[u]_-{\mathcal{F}}}
\end{align*}
\]

The axioms correspond respectively to (IM5), (IM6) and (IM7).

\[\square\]

3.3.4 Indexed monads as monads in \(\mathcal{E}^{+-}\)

We fix a category \(B\).
Definition 3.14. There is a 2-monad $\mathcal{B} = ([\mathcal{B}, -], H_\mathcal{B}, M_\mathcal{B})$ on $\mathcal{C}$ defined as follows.

- The 2-functor $[\mathcal{B}, -]: \mathcal{C} \to \mathcal{C}$ is the functor category construction.
- The 2-natural transformation $H_\mathcal{B}: \text{id}_{\mathcal{C}} \Rightarrow [\mathcal{B}, -]$ has as its $X$-component the functor $H_\mathcal{B}, X = H_X: \mathcal{B} \to [\mathcal{B}, X]$ given by $x \mapsto (b \mapsto x)$.
- The 2-natural transformation $M_\mathcal{B}: [\mathcal{B}, [\mathcal{B}, -]] \Rightarrow [\mathcal{B}, -]$ has as its $X$-component the functor $M_\mathcal{B}, X = M_X: [\mathcal{B}, [\mathcal{B}, X]] \to [\mathcal{B}, X]$ given by $\Xi \mapsto (b \mapsto (\Xi b)b)$.

Proposition 3.15. Let $\mathcal{C}$ be a category. Then a $\mathcal{B}$-indexed monad on $\mathcal{C}$ is the same thing as a monad in $\mathcal{E}^{-+}$ on $(\mathcal{C}, \mathcal{C})$ above the 2-monad $[\mathcal{B}, -]: \mathcal{C} \to \mathcal{C}$.

Proof. The latter notion is given by the following data:

- A 1-cell $([\mathcal{B}, -], F): (\mathcal{C}, \mathcal{C}) \to ([\mathcal{B}, \mathcal{C}], \mathcal{C})$ of $\mathcal{E}^{-+}$, where $F$ is a functor of type $F: \mathcal{C} \to [\mathcal{B}, \mathcal{C}]$.
- A 2-cell $(H_\mathcal{B}, \eta): (\text{id}_{\mathcal{C}}, \text{id}_\mathcal{C}) \Rightarrow ([\mathcal{B}, -], F)$ of $\mathcal{E}^{-+}$, where $\eta$ is a natural transformation of type

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\eta} & [\mathcal{B}, \mathcal{C}] \\
\downarrow{H_\mathcal{B}} & & \downarrow{H_{[\mathcal{B}, \mathcal{C}]}} \\
\mathcal{C} & \xrightarrow{F} & [\mathcal{B}, \mathcal{C}] \\
\end{array}
\]

- A 2-cell $(M_\mathcal{B}, \mu): ([\mathcal{B}, -], F) \circ ([\mathcal{B}, -], F) \Rightarrow ([\mathcal{B}, -], F)$ of $\mathcal{E}^{-+}$, where $\mu$ is a natural transformation of type

\[
\begin{array}{ccc}
[\mathcal{B}, \mathcal{C}] & \xrightarrow{\mu} & [\mathcal{B}, [\mathcal{B}, \mathcal{C}]] \\
\downarrow{M_\mathcal{B}} & & \downarrow{M_{[\mathcal{B}, \mathcal{C}]}} \\
[\mathcal{B}, \mathcal{C}] & \xrightarrow{F} & [\mathcal{B}, \mathcal{C}] \\
\end{array}
\]

These data satisfy the following axioms:
The axioms correspond respectively to (IM5), (IM6) and (IM7).

3.3.5 Graded and indexed comonads as comonads

Here we list the analogous results for graded and indexed comonads. Let $\mathbb{M} = (\mathbb{M}, \otimes, I)$ be a strict monoidal category and $\mathbb{B}$ a category.

**Proposition 3.16.** Let $\mathcal{C}$ be a category. Then an $\mathbb{M}$-graded comonad on $\mathcal{C}$ is the same thing as a comonad in $\mathcal{E}^{+\mathbb{B}}$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-monad $\mathbb{M} \times (-) : \mathcal{C}at \to \mathcal{C}at$.

**Proof.** The latter notion is given by the following data:

- A 1-cell $(\mathbb{M} \times (-), S) : (\mathcal{C}at, \mathcal{C}) \to (\mathcal{C}at, \mathcal{C})$ of $\mathcal{E}^{+\mathbb{B}}$, where $S$ is a functor of type

  $$S : \mathbb{M} \times \mathcal{C} \to \mathcal{C}.$$

- A 2-cell $(H^{\mathbb{M}}, \varepsilon) : (\mathbb{M} \times (-), S) \Rightarrow (\text{id}_{\mathcal{C}at}, \text{id}_{\mathcal{C}})$ of $\mathcal{E}^{+\mathbb{B}}$, where $\varepsilon$ is a natural transformation of type

  $$\varepsilon : \mathbb{M} \times \mathcal{C} \to \mathcal{C}.$$

- A 2-cell $(M^{\mathbb{M}}, \delta) : (\mathbb{M} \times (-), S) \Rightarrow (\mathbb{M} \times (-), S) \circ (\mathbb{M} \times (-), S)$ of $\mathcal{E}^{+\mathbb{B}}$, where $\delta$ is a natural transformation of type

  $$\delta : \mathbb{M} \times \mathbb{M} \times \mathcal{C} \to \mathbb{M} \times \mathcal{C}.$$

They satisfy axioms corresponding to (GC4), (GC5) and (GC6). \qed
Proposition 3.17. Let $\mathcal{C}$ be a category. Then an $\mathcal{M}$-graded comonad on $\mathcal{C}$ is the same thing as a comonad in $\mathcal{E}^{--}$ on $(\mathcal{Cat}, \mathcal{C})$ above the 2-comonad $[\mathcal{M}, -]: \mathcal{Cat} \to \mathcal{Cat}$.

Proof. The latter notion is given by the following data:

- A 1-cell $([\mathcal{M}, -], S): (\mathcal{Cat}, \mathcal{C}) \to (\mathcal{Cat}, \mathcal{C})$ of $\mathcal{E}^{--}$, where $S$ is a functor of type

$$S : \mathcal{C} \to [\mathcal{M}, \mathcal{C}].$$

- A 2-cell $(H_M, \varepsilon): ([\mathcal{M}, -], S) \Rightarrow (\text{id}_{\mathcal{Cat}}, \text{id}_\mathcal{C})$ of $\mathcal{E}^{--}$, where $\varepsilon$ is a natural transformation of type

$$\varepsilon : \mathcal{C} \to [\mathcal{M}, \mathcal{C}].$$

- A 2-cell $(M_M, \delta): ([\mathcal{M}, -], S) \Rightarrow ([\mathcal{M}, -], S) \circ ([\mathcal{M}, -], S)$ of $\mathcal{E}^{--}$, where $\delta$ is a natural transformation of type

They satisfy axioms corresponding to (GC4), (GC5) and (GC6). $\square$

Proposition 3.18. Let $\mathcal{C}$ be a category. Then a $\mathcal{B}$-indexed comonad on $\mathcal{C}$ is the same thing as a comonad in $\mathcal{E}^{++}$ on $(\mathcal{Cat}, \mathcal{C})$ above the 2-comonad $\mathcal{B} \times (-): \mathcal{Cat} \to \mathcal{Cat}$.

Proof. The latter notion is given by the following data:

- A 1-cell $(\mathcal{B} \times (-), \mathcal{I}): (\mathcal{Cat}, \mathcal{C}) \to (\mathcal{Cat}, \mathcal{C})$ of $\mathcal{E}^{++}$, where $\mathcal{I}$ is a functor of type

$$\mathcal{I} : \mathcal{B} \times \mathcal{C} \to \mathcal{C}.$$

- A 2-cell $(H_B, \varepsilon): (\mathcal{B} \times (-), \mathcal{I}) \Rightarrow (\text{id}_{\mathcal{Cat}}, \text{id}_\mathcal{C})$ of $\mathcal{E}^{++}$, where $\varepsilon$ is a natural transformation of type

- A 2-cell $(M_B, \delta): (\mathcal{B} \times (-), \mathcal{I}) \Rightarrow (\mathcal{B} \times (-), \mathcal{I}) \circ (\mathcal{B} \times (-), \mathcal{I})$ of $\mathcal{E}^{++}$, where $\delta$ is a natural transformation of type
They satisfy axioms corresponding to (IC5), (IC6) and (IC7).

**Proposition 3.19.** Let $\mathcal{C}$ be a category. Then an $\mathbb{B}$-graded comonad on $\mathcal{C}$ is the same thing as a comonad in $\mathcal{E}^{--}$ on $(\mathcal{Cat}, \mathcal{C})$ above the 2-monad $[\mathbb{B}, -] : \mathcal{Cat} \to \mathcal{Cat}$.

**Proof.** The latter notion is given by the following data:

- A 1-cell $([\mathbb{B}, -], \mathcal{J}) : (\mathcal{Cat}, \mathcal{C}) \to (\mathcal{Cat}, \mathcal{C})$ of $\mathcal{E}^{--}$, where $\mathcal{J}$ is a functor of type

$$\mathcal{J} : \mathcal{C} \to [\mathbb{B}, \mathcal{C}].$$

- A 2-cell $\varepsilon : ([\mathbb{B}, -], \mathcal{J}) \Rightarrow (\mathcal{C}, [\mathbb{B}, -], \mathcal{J})$ of $\mathcal{E}^{--}$, where $\varepsilon$ is a natural transformation of type

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathcal{J}} & [\mathbb{B}, \mathcal{C}] \\
\downarrow{id_C} & & \downarrow{\mathcal{J}} \\
\mathcal{C} & \xrightarrow{\varepsilon} & [\mathbb{B}, \mathcal{C}]
\end{array}$$

- A 2-cell $\delta : ([\mathbb{B}, -], \mathcal{J}) \Rightarrow ([\mathbb{B}, -], \mathcal{J}) \circ ([\mathbb{B}, -], \mathcal{J})$ of $\mathcal{E}^{--}$, where $\delta$ is a natural transformation of type

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathcal{J}} & [\mathbb{B}, \mathcal{C}] \\
\downarrow{[\mathbb{B}, \mathcal{J}]} & & \downarrow{[\mathbb{B}, \mathcal{J}]} \\
[\mathbb{B}, \mathcal{C}] & \xrightarrow{\delta} & [\mathbb{B}, \mathcal{C}]
\end{array}$$

They satisfy axioms corresponding to (IC5), (IC6) and (IC7).

**Notes**

The definitions of the 2-categories $\mathcal{E}^{++}$, $\mathcal{E}^{+-}$, $\mathcal{E}^{-+}$ and $\mathcal{E}^{--}$ have occurred to me after I learned from Paul-André Melliès his key observation that, by “enlarging” $\mathcal{Cat}$ in a certain way, one can regard graded monads as mere monads in the 2-categorical sense; indeed, his 2-category was a full sub 2-category of $\mathcal{E}^{++}$. When actually writing down the definition of the 2-category $\mathcal{E}^{++}$ for the first time, Kenji Maillard helped me by telling me the (perhaps folklore) view of the Grothendieck construction as a certain comma construction.

The reduction of graded monads to mere monads are presented in [7].

The observation that the notion of indexed monad can also be reduced to monads using these 2-categories seems to be new here.
Chapter 4

The main constructions

In this chapter, we describe the Eilenberg–Moore and the Kleisli constructions for graded and indexed monads (except for the Kleisli construction for indexed monads). These constructions are natural yet nontrivial generalization of the classical Eilenberg–Moore and Kleisli constructions. Moreover, our constructions satisfy the relevant 2-dimensional universal properties in naturally arising 2-categories, i.e., they produce Eilenberg–Moore and Kleisli objects respectively in appropriate 2-categories introduced in Chapter 3; we regard this fact as the major justification of the definitions given below. More precisely, the relationship of the 2-categories \( E^{++}, E^{+-}, E^{-+}, E^{--} \), graded and indexed (co)monads, and the generalized (co)Eilenberg–Moore and (co)Kleisli constructions for them is summarized in the following table, which is a refinement of the table appearing at the beginning of the previous chapter:

| Graded monads | \( E^{++} \) | \( E^{+-} \) | \( E^{-+} \) | \( E^{--} \) |
|---------------|-------------|-------------|-------------|-------------|
| Indexed monads| EM          | EM          | Kle         |
| Graded comonads| coEM        | coEM        | coKl        |
| Indexed comonads| coEM        | coEM        | coKl        |

So far, we have not been able to identify the (co)Kleisli constructions for indexed (co)monads; the * mark indicates the conjectural status of the construction.

As we have already mentioned in Section 3.1 the 2-dimensional universality of Eilenberg–Moore and Kleisli objects is powerful enough to reconstruct some of the main development of the classical Eilenberg–Moore and Kleisli construction abstractly. However, in the current chapter we have chosen to start by following more closely the style of the classical theory, and construct adjunctions that generate monads and comparison maps explicitly. The discussion on 2-categorical properties of our constructions, which seems to be largely of interest only to the experts, is placed after that and the reader can harmlessly skip these parts.

We conclude this chapter by briefly indicating the suitable co-Eilenberg–Moore and co-Kleisli constructions for graded and indexed comonads, again except for the conjectural co-Kleisli construction for indexed comonads which is left for future work.
4.1 The Eilenberg–Moore construction for graded monads

Let $\mathcal{M} = (\mathcal{M}, \otimes, I)$ be a strict monoidal category, $\mathcal{C}$ a category, and $T$ an $\mathcal{M}$-graded monad on $\mathcal{C}$. Recall from Section 3.3.1 that $T$ may be seen as a monad in $\mathcal{E}^{++}$; the Eilenberg–Moore adjunction for $T$ lives in $\mathcal{E}^{++}$, and lies above the Eilenberg–Moore adjunction for the 2-monad $\mathcal{M} \times (-)$ on $\text{Cat}$. See the picture below for an illustration.

We will define the data appearing in the picture, one by one. We write the functor part of $T$ as $*: \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ as well and use the infix notation for it.

4.1.1 The Eilenberg–Moore category

Extending the classical construction of the Eilenberg–Moore category of an ordinary monad as the category of algebras, the Eilenberg–Moore category of a graded monad is given as the category of graded algebras.

**Definition 4.1.** Define the category $\mathcal{C}^T$ as follows:

- An object of $\mathcal{C}^T$ is a **graded $T$-algebra**, i.e., a pair $(A, h)$ where $A: \mathcal{M} \to \mathcal{C}$ is a functor and $h$ is a natural transformation of type

$$
\mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M} \times \mathcal{C} \xrightarrow{M \times A} \mathcal{M} \times \mathcal{M} \xrightarrow{h} \mathcal{M} \times \mathcal{C} \xrightarrow{T} \mathcal{C}
$$

So the component of $h$ at $(m, n) \in \mathcal{M} \times \mathcal{M}$ is of type

$$
h_{m,n} : m \ast A_n \longrightarrow A_{m \otimes n}.
$$

These data are subject to the following axioms:
4.1. THE EILENBERG–MOORE CONSTRUCTION FOR GRADED MONADS

A morphism of \( C^T \) from \((A,h)\) to \((A',h')\) is a **homomorphism** of graded \( T \)-algebras between them, i.e., a natural transformation \( \varphi: A \Rightarrow A' \) making the diagram

\[
\begin{array}{ccc}
A_n & \xrightarrow{m*\varphi_n} & A'_n \\
\downarrow{h_{m,n}} & & \downarrow{h'_{m,n}} \\
A_{m\otimes n} & \xrightarrow{\varphi_{m\otimes n}} & A'_{m\otimes n}
\end{array}
\]

commute for each pair of objects \( m, n \) of \( M \).

Let us introduce a convenient notation for graded \( T \)-algebras. We write a graded \( T \)-algebra \((A,h)\) as \(((A_n)_{n\in M}, (h_{m,n})_{m,n\in M})\), and use this notation to indicate definitions in what follows. In principle we need to check the relevant functoriality or naturality to validate such definitions, but these are all completely routine and left to the interested reader. Similarly, we denote a homomorphism \( \varphi: (A,h) \Rightarrow (A',h') \) by \((\varphi_n)_{n\in M}\).

The category \( C^T \) becomes an object of the 2-category \( \text{CAlg}^M \times (-) \) by the following functor \( \otimes \):

**Definition 4.2.** Define the functor

\[ \otimes : M \times C^T \rightarrow C^T \]

as follows:

- Given objects \( p \) and \((A,h)\) of \( M \) and \( C^T \) respectively, we define the graded \( T \)-algebra \( p \otimes (A,h) \) by the precomposition of \((-) \otimes p: M \rightarrow M\):

\[
p \otimes (A,h) := \left( (A_{n \otimes p})_{n \in M}, (h_{m,n \otimes p})_{m,n \in M} \right).
\]

- Given morphisms \( u: p \rightarrow p' \) and \( \varphi: (A,h) \rightarrow (A',h') \) of \( M \) and \( C^T \) respectively, we define the homomorphism \( u \otimes \varphi: p \otimes (A,h) \rightarrow p' \otimes (A',h') \) by setting the component \((u \otimes \varphi)_n: A_{n \otimes p} \rightarrow A'_{n \otimes p'}\) at \( n \in M \) to be either of the following two equivalent composites:

\[
\begin{array}{ccc}
A_n & \xrightarrow{\varphi_n \otimes p} & A'_{n \otimes p} \\
\downarrow{A \otimes u} & & \downarrow{A' \otimes u} \\
A_n \otimes p' & \xrightarrow{\varphi_n \otimes p'} & A'_{n \otimes p'}
\end{array}
\]
That this functor $\otimes: \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ gives a $\mathcal{C}at^\mathcal{M} \times (-)$-algebra structure, also known as a strict left action of $(\mathcal{M}, \otimes, I)$, is easily verified; a key step is the following:

\[
p \otimes (q \otimes (A, h)) = p \otimes ((A_n \otimes q)_{n \in \mathcal{M}}, (h_{m,n} \otimes q)_{m,n \in \mathcal{M}}) = (p \otimes q) \otimes (A, h).
\]

### 4.1.2 The Eilenberg–Moore adjunction

#### The left adjoint

We define the 1-cell \((F^\mathcal{M}, f^T): (\mathcal{C}at, \mathcal{C}) \to (\mathcal{C}at^\mathcal{M} \times (-), \mathcal{M} \times \mathcal{C}^T)\) of \(\mathcal{E}^{++}\) as follows:

**Definition 4.3.** The 2-functor \(F^\mathcal{M}: \mathcal{C}at \to \mathcal{C}at^\mathcal{M} \times \mathcal{C}\) is the free 2-functor \(X \mapsto \) where \(M_X = M_X\) is the one defined in Definition 3.8. ■

**Definition 4.4.** The 1-cell \(f^T: (\mathcal{M} \times \mathcal{M} \times \mathcal{C}, M \times \mathcal{C}) \to (\mathcal{C}at^\mathcal{M} \times (-), \mathcal{M} \times \mathcal{C}^T)\) of \(\mathcal{C}at^\mathcal{M} \times (-)\) is the functor \(f^T: \mathcal{M} \times \mathcal{C} \to \mathcal{C}^T\) defined as

\[
f^T(p, c) := ((n \otimes p) \ast c)_{n \in \mathcal{M}};
\]

\[
(\mu_{m,n \otimes p, c}: m \ast ((n \otimes p) \ast c) \to (m \otimes n \otimes p) \ast c)_{m,n \in \mathcal{M}}
\]
on an object \((p, c)\) and

\[
f^T(u, f) := ((n \otimes u) \ast f: (n \otimes p) \ast c \to (n \otimes p') \ast c')_{n \in \mathcal{M}}
\]
on a morphism \((u, f): (p, c) \to (p', c').\) ■

#### The right adjoint

We define the 1-cell

\[
(U^\mathcal{M}, u^T): (\mathcal{C}at^\mathcal{M} \times (-), \mathcal{M} \times \mathcal{C}^T) \to (\mathcal{C}at, \mathcal{C})
\]
of \(\mathcal{E}^{++}\) as follows:

**Definition 4.5.** The 2-functor \(U^\mathcal{M}: \mathcal{C}at^\mathcal{M} \times (-) \to \mathcal{C}at\) is the forgetful 2-functor \(\binom{\mathcal{M} \times A}{A}\) defined as

\[
U^\mathcal{M}(A, h) := A_I
\]

for an object \((A, h)\) and \(U^\mathcal{M}(\varphi) := \varphi_I\) for a morphism \(\varphi.\) ■

**Definition 4.6.** The functor \(u^T: \mathcal{C}^T \to \mathcal{C}\) is given by the evaluation at the monoidal unit \(I \in \mathcal{M}: u^T(A, h) := A_I\) for an object \((A, h)\) and \(u^T(\varphi) := \varphi_I\) for a morphism \(\varphi.\) ■
4.1. THE EILENBERG–MOORE CONSTRUCTION FOR GRADED MONADS

The unit

We define the 2-cell

\[
\begin{array}{c}
\text{(Cat, } \mathbb{C}) \\
\downarrow (H^M, \eta_T) \\
\text{(Cat, } \mathbb{C})
\end{array}
\xrightarrow{(id_{\text{Cat}}, id_{\mathbb{C}})}
\begin{array}{c}
\text{(Cat, } \mathbb{C}) \\
\downarrow (U^M, u^T) \\
\text{(Cat, } \mathbb{C})
\end{array}
\]

of \( \mathcal{E}^{++} \) as follows:

**Definition 4.7.** The 2-natural transformation \( H^M : \text{id}_{\text{Cat}} \Rightarrow U^M \circ F^M \) is the one defined in Definition 3.8.

**Definition 4.8.** The natural transformation

\[
\begin{array}{c}
\mathbb{C} \\
\downarrow H_c \\
M \times \mathbb{C}
\end{array}
\xrightarrow{id_{\mathbb{C}}} 
\begin{array}{c}
\mathbb{C} \\
\downarrow u^T \\
M \times \mathbb{C}
\end{array}
\]

has components \( \eta^T_c : c \to I \ast c \) given by the data of the graded monad \( T \).

The counit

We define the 2-cell

\[
\begin{array}{c}
\text{(Cat, } \mathbb{C}) \\
\downarrow (U^M, u^T) \\
\text{(Cat, } \mathbb{C})
\end{array}
\xrightarrow{(id_{\text{Cat}}, id_{\mathbb{C}})}
\begin{array}{c}
\text{(Cat, } \mathbb{C}) \\
\downarrow (E^M, \varepsilon^T) \\
\text{(Cat, } \mathbb{C})
\end{array}
\]

of \( \mathcal{E}^{++} \) as follows:

**Definition 4.9.** The 2-natural transformation \( E^M : E^M \circ U^M \Rightarrow \text{id}_{\text{Cat}^{M \times (-)}} \) has components \( E^M_\alpha : (M \times M \times \mathbb{A}) \to (M \times \mathbb{A}) \) given by \( E^M_\alpha = E_\alpha := \alpha : M \times \mathbb{A} \to \mathbb{A} \).
Definition 4.10. The 2-cell

\[
\begin{array}{ccc}
(M \times M \times C_T & \xrightarrow{\mu} & M \times C_T \\
\downarrow \mu & & \downarrow \mu \\
M \times C_T & \xrightarrow{\varepsilon_T} & C_T
\end{array}
\]

of \(\mathbb{Cat}^{M \times (-)}\) is the natural transformation with its component at \((p, (A, h)) \in M \times C_T\) of type

\[
\varepsilon_T^{p,(A,h)} : \left( ((n \otimes p) \ast A_I)_{n \in M}, \ (\mu_{m,n \otimes p,A_I})_{m,n \in M} \right) \rightarrow \left( (A_{n \otimes p})_{n \in M}, \ (h_{m,n \otimes p})_{m,n \in M} \right),
\]

itself with the component at \(n \in M\) given by

\[
\varepsilon_T^{p,(A,h),n} : h_{n \otimes p,I} : (n \otimes p) \ast A_I \rightarrow (A_{n \otimes p}).
\]

Observe that \(\varepsilon_T^{p,(A,h)}\) is indeed a homomorphism of graded \(T\)-algebras, i.e., the diagram

\[
\begin{array}{ccc}
m \ast (n \otimes p) \ast A_I & \xrightarrow{m \ast h_{n \otimes p,I}} & m \ast A_{n \otimes p} \\
\downarrow \mu_{m,n \otimes p,A_I} & & \downarrow h_{m,n \otimes p} \\
(m \otimes n \otimes p) \ast A_I & \xrightarrow{h_{m \otimes n \otimes p,I}} & A_{m \otimes n \otimes p}
\end{array}
\]

commutes, thanks to one of the axioms of graded \(T\)-algebras.

4.1.3 Comparison maps

Suppose we have an adjunction \((L, l) \dashv (R, r) : (\mathcal{D}, D) \rightarrow (\mathbb{Cat}, C)\) in \(\mathcal{E}^{++}\) with unit \((H, \eta)\): \((id_{\mathbb{Cat}}, id_C) \Rightarrow (R, r) \circ (L, l)\) and counit \((E, \varepsilon) : (L, l) \circ (R, r) \Rightarrow (id_{\mathcal{D}}, id_D)\), which gives a resolution of the monad \((M \times (-), T)\), i.e., such that the following equations hold:

\[
\begin{array}{ccc}
\mathbb{Cat} & \xrightarrow{M \times (-)} & \mathbb{Cat} \\
\downarrow M \times (-) & & \downarrow M \times (-) \\
\mathbb{Cat} & \xrightarrow{M \times (-)} & \mathbb{Cat}
\end{array}
\]

(4.1)

\[
\begin{array}{ccc}
M \times C & \xrightarrow{T} & C \\
\downarrow H^M & & \downarrow H \\
C & \xrightarrow{\eta^T} & C
\end{array}
\]

(4.2)

\[
\begin{array}{ccc}
\mathbb{Cat} & \xrightarrow{L} & \mathbb{Cat} \\
\downarrow R & & \downarrow L \\
\mathbb{Cat} & \xrightarrow{id_{\mathcal{D}}} & \mathbb{Cat}
\end{array}
\]

(4.3)

\[
\begin{array}{ccc}
\mathbb{Cat} & \xrightarrow{L} & \mathbb{Cat} \\
\downarrow R & & \downarrow L \\
\mathbb{Cat} & \xrightarrow{id_{\mathcal{D}}} & \mathbb{Cat}
\end{array}
\]

(4.4)
4.1. The Eilenberg–Moore Construction for Graded Monads

First note that equations (4.1), (4.3) and (4.5) imply the existence of the comparison 2-functor $K$ by a classical result of 2-monad theory (or rather enriched monad theory).

Definition 4.11. The 2-functor $K : \mathcal{D} \to \mathcal{C}^\mathcal{M \times (-)}$ is the comparison 2-functor $D' \mapsto \left( \mathcal{M \times RD'} \downarrow \mathcal{R}_{RD'} \right)$.

This 2-functor $K$ becomes the 2-functor part of the comparison map $(K, k)$ under construction. So now it remains to construct an appropriate 1-cell $k$ of $\mathcal{C}^\mathcal{M \times (-)}$.

Definition 4.12. The 1-cell $k : \left( \mathcal{M \times RD} \downarrow \mathcal{R}_{RD} \right) \to \left( \mathcal{M \times C} \downarrow \mathcal{C}^\mathcal{T} \right) \downarrow \mathcal{C}$ of $\mathcal{C}^\mathcal{M \times (-)}$ is the functor $k : RD \to \mathcal{C}^\mathcal{T}$ defined as

$$k(d) := \left( (rRE_D(n,d))_{n \in \mathcal{M}}, (rR_{\mathcal{M \times RD}(n,d)} m \in \mathcal{M} \} \right)$$

on an object $d$. The types of the structure maps indeed match:

$$m * rRE_D(n,d) = rRl(m, rRE_D(n,d)) \quad \text{by (4.2)}$$

$$= rRlRLr(m, RE_D(n,d))$$

$$\xrightarrow{rR_{\mathcal{M \times RD}(n,d)}} rRE_D(m, RE_D(n,d))$$

$$= rRE_DRLRE_D(m, n, d)$$

$$= rRE_DRE_{LRD}(m, n, d)$$

$$= rRE_D(m \otimes n, d) \quad \text{by (4.5)}.$$

$k$ is defined as

$$k(w) := \left( rRE_D(n, w) : rRE_D(n,d) \to rRE_D(n,d') \right)_{n \in \mathcal{M}}$$

on a morphism $w : d \to d'$. ■

Proposition 4.13. The 1-cell $(K, k)$ of $\mathcal{E}^{++}$ satisfies the equations $(K, k) \circ (L, l) = (F^\mathcal{M}, f^\mathcal{T})$ and $(R, r) = (U^\mathcal{M}, u^\mathcal{T}) \circ (K, k)$. Moreover, it is the unique such.

We omit a proof since it follows from the 2-dimensional universality discussed below.
4.1.4 The 2-dimensional universality

Statement of the theorem

We will show that there is a family of isomorphisms of categories

\[ E^{++} \left( (\mathcal{X}, X), \left( \text{Cat}^{M \times (-)}, \downarrow_{\otimes} \mathbb{C}^T \right) \right) \]

\[ \cong E^{++} \left( (\mathcal{X}, X), (\text{Cat}, \mathbb{C}) \right) \]

2-natural in \((\mathcal{X}, X) \in E^{++}\); cf. Definition [3.2]. More precisely, we claim that the data

\[ \left( \text{Cat}^{M \times (-)}, \downarrow_{\otimes} \mathbb{C}^T \right) \]

provides the universal left \(T\)-module, in the sense that every left \(T\)-module

\[ (\mathcal{X}, X) \xrightarrow{(G, g)} (\text{Cat}, \mathbb{C}) \]

i.e., an object of the category \(E^{++}((\mathcal{X}, X), (\text{Cat}, \mathbb{C}))\), factors uniquely as

\[ (\mathcal{X}, X) \xrightarrow{(G, g)} \left( \text{Cat}^{M \times (-)}, \downarrow_{\otimes} \mathbb{C}^T \right) \]

and similarly every morphism of left \(T\)-modules

\[ (\mathcal{X}, X) \xrightarrow{(G, g)} (\text{Cat}, \mathbb{C}) \]
i.e., a morphism of the category \( \mathcal{E}^{++}((\mathcal{X}, X), (\mathcal{C}, \mathcal{C}))) \) factors uniquely as

\[
\begin{array}{c}
(\mathcal{X}, X) \\
\downarrow (\Omega, \varnothing) \\
(\mathcal{C}, \mathcal{C})
\end{array}
\xrightarrow{(G, g)}
\begin{array}{c}
\mathcal{C}^\mathcal{M} \\
\downarrow (u, u^T) \\
\mathcal{C}^\mathcal{M} \mathcal{T}
\end{array}
\xrightarrow{(\tilde{\Omega}, \tilde{\omega})}
\begin{array}{c}
\mathcal{C}^\mathcal{M} \\
\downarrow (\mathcal{M} \times (-), \mathcal{T}) \\
\mathcal{C}^\mathcal{M} \mathcal{T}
\end{array}
\xrightarrow{(\tilde{\Omega}', \tilde{\omega}')} \\
\mathcal{C}^\mathcal{M} \mathcal{T}
\end{array}
\]

The 1-dimensional aspect

Let us first verify the unique factorization for left \( \mathcal{T} \)-modules. Suppose we have a left \( \mathcal{T} \)-modules, i.e., a piece of data

\[
(\mathcal{X}, X) \\
\downarrow (\Omega, \varnothing) \\
(\mathcal{C}, \mathcal{C})
\xrightarrow{(G, g)}
\begin{array}{c}
\mathcal{C}^\mathcal{M} \\
\downarrow (\mathcal{M} \times (-), \mathcal{T}) \\
\mathcal{C}^\mathcal{M} \mathcal{T}
\end{array}
\xrightarrow{(\tilde{\Omega}, \tilde{\omega})}
\begin{array}{c}
\mathcal{C}^\mathcal{M} \\
\downarrow (\mathcal{M} \times (-), \mathcal{T}) \\
\mathcal{C}^\mathcal{M} \mathcal{T}
\end{array}
\xrightarrow{(\tilde{\Omega}', \tilde{\omega}')} \\
\mathcal{C}^\mathcal{M} \mathcal{T}
\]

in \( \mathcal{E}^{++} \) satisfying

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \Gamma \\
\mathcal{C}
\end{array}
\xrightarrow{G}
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{M} \times (-) \\
\mathcal{C}
\end{array}
\xrightarrow{\text{id}_{\mathcal{C}}} \\
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{M} \times (-) \\
\mathcal{C}
\end{array}
\xrightarrow{H_{\mathcal{M}}} \\
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{M} \times (-) \\
\mathcal{C}
\end{array}
\xrightarrow{H_{\mathcal{C}}} \\
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{M} \times (-) \\
\mathcal{C}
\end{array}
\xrightarrow{\text{id}_{\mathcal{C}}} \\
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{M} \times (-) \\
\mathcal{C}
\end{array}
\xrightarrow{\text{id}_{\mathcal{C}}}
\]

(4.7)

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \Gamma \\
\mathcal{C}
\end{array}
\xrightarrow{G}
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{M} \times (-) \\
\mathcal{C}
\end{array}
\xrightarrow{\text{id}_{\mathcal{C}}}
\]

(4.8)

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \Gamma \\
\mathcal{C}
\end{array}
\xrightarrow{G}
\begin{array}{c}
\mathcal{C} \\
\downarrow \mathcal{M} \times (-) \\
\mathcal{C}
\end{array}
\xrightarrow{\text{id}_{\mathcal{C}}}
\]

(4.9)
First note that equations (4.7) and (4.9) imply the unique factorization

\[ M \times G X \xrightarrow{M \times M \times g} M \times M \times C \rightarrow M \times C \]

since the Eilenberg–Moore 2-category $\mathcal{C}at^M_{\times(-)}$ is the Eilenberg–Moore object in $2\mathcal{C}at$. Concretely, the 2-functor $\tilde{G}$ is defined as follows:

**Definition 4.14.** The 2-functor $\tilde{G} : \mathcal{X} \rightarrow \mathcal{C}at^M_{\times(-)}$ is the mediating 2-functor $X' \mapsto (M \times G X \xrightarrow{M \times M \times g} M \times M \times C)\cdot(\Gamma X \xrightarrow{\gamma} G X)$.

This 2-functor is the only possible choice for the 2-functor part of the desired factorization; thus it remains to construct a 1-cell $\tilde{g} : (M \times G X \xrightarrow{M \times \Gamma X} M \times C \xrightarrow{\mu} M \times T)\cdot(\Gamma X \xrightarrow{\gamma} G X)$ which satisfies

\[ (4.11) \]

\[ (4.12) \]
and show its uniqueness.

**Definition 4.15.** The 1-cell $\tilde{g} : \left( \mathcal{M} \times \mathcal{G}_X \right) \to \left( \mathcal{M} \times \mathcal{C}_T^\circ \right)$ of $\mathcal{Cat}^{\mathcal{M} \times (-)}$ is a functor of type $\tilde{g} : \mathcal{G}_X \to \mathcal{C}_T$. As a functor, it is defined as

$$\tilde{g}(x) := \left( (g \Gamma_X(n, x))_{n \in \mathcal{M}}, (\gamma_m, \Gamma_X(n, x))_{m, n \in \mathcal{M}} \right)$$

on an object $x$. Let us observe that the structure maps are well-typed:

$$m \ast g \Gamma_X(n, x) = T(\mathcal{M} \times g)(m, \Gamma_X(n, x)) = g \Gamma_X(m, \Gamma_X(n, x))$$

by (4.9)

$$= g \Gamma_X(M \times \Gamma_X)(n, u, z) = g \Gamma_X(M \times \Gamma_X)(n, u, z) \oplus n, x) \text{ by (4.9)}$$

$$= p \oplus (\tilde{g} \Gamma_X(n, x))_{n \in \mathcal{M}}, (\gamma_m, \Gamma_X(n, x))_{m, n \in \mathcal{M}}$$

on objects and

$$\tilde{g} \Gamma_X(u, z) = (g \Gamma_X(n, \Gamma_X(u, z)))_{n \in \mathcal{M}}$$

$$= (g \Gamma_X(M \times \Gamma_X)(n, u, z))_{n \in \mathcal{M}}$$

$$= (g \Gamma_X(n \oplus u, z))_{n \in \mathcal{M}}$$

$$= u \oplus (\tilde{g} \Gamma_X(n, z))_{n \in \mathcal{M}}$$

$$= u \oplus \tilde{g} z$$

**Proposition 4.16.** The functor $\tilde{g}$ defined above is indeed a 1-cell of $\mathcal{Cat}^{\mathcal{M} \times (-)}$ which satisfies (4.11) and (4.12). Moreover, it is the unique such.

**Proof.** First, $\tilde{g}$ is a 1-cell of $\mathcal{Cat}^{\mathcal{M} \times (-)}$, i.e., the diagram

$$\begin{array}{ccc}
\mathcal{M} \times \mathcal{G}_X & \xrightarrow{\tilde{g}} & \mathcal{M} \times \mathcal{C}_T^\circ \\
\Gamma_X & \searrow & \downarrow \\
\mathcal{G}_X & \xrightarrow{\tilde{g}} & \mathcal{C}_T^\circ
\end{array}$$

(4.13)

commutes, because,

$$\tilde{g} \Gamma_X(p, x) = ((g \Gamma_X(n, \Gamma_X(p, x)))_{n \in \mathcal{M}}, (\gamma_m, \Gamma_X(n, \Gamma_X(p, x)))_{m, n \in \mathcal{M}}$$

$$= ((g \Gamma_X(M \times \Gamma_X)(n, p, x))_{n \in \mathcal{M}}, (\gamma_m, \Gamma_X(M \times \Gamma_X)(n, p, x))_{m, n \in \mathcal{M}})$$

$$= ((g \Gamma_X M \mathcal{G}_X(n, p, x))_{n \in \mathcal{M}}, (\gamma_m, \Gamma_X M \mathcal{G}_X(n, p, x))_{m, n \in \mathcal{M}}) \text{ by (4.9)}$$

$$= p \oplus ((g \Gamma_X(n, x))_{n \in \mathcal{M}}, (\gamma_m, \Gamma_X(n, x))_{m, n \in \mathcal{M}})$$

$$= p \oplus \tilde{g} x$$

on objects and
on morphisms. \( \tilde{g} \) satisfies (4.11) since,
\[
gx = g_{\Gamma X} H_{GX} x
\]
\[
= g_{\Gamma X}(I, x)
\]
\[
= u^T((g_{\Gamma X}(n, x))_{n \in M}, (\gamma_m, \Gamma_X(n, x))_{m \in M})
\]
\[
= u^T \tilde{g}x
\]
on objects and
\[
gz = g_{\Gamma X} H_{GX} z
\]
\[
= g_{\Gamma X}(I, z)
\]
\[
= u^T(g_{\Gamma X}(n, z))_{n \in M}
\]
\[
= u^T \tilde{g}z
\]
on morphisms.
Finally, \( \tilde{g} \) satisfies (4.12) since,
\[
\gamma_{m,x} = \gamma_m, \Gamma_X(I, x)
\]
\[
= \varepsilon_m, \tilde{g}x, I
\]
\[
= u^T \varepsilon_m, \tilde{g}x
\]
on objects.
For the uniqueness, suppose that a functor \( \hat{g}: GX \to C_T \) with \( \hat{g}x = ((A^x_n)_{n \in M}, (h^x_{m,n})_{m,n \in M}) \) on objects and \( \hat{g}z = (\varphi^z_n)_{n \in M} \) on morphisms also satisfy the conditions. First, equation (4.11) forces \( A^x_I = gx \) and \( \varphi^z_I = gz \), whereas equation (4.12) says that \( h^x_{m,I} = \gamma_{m,x} \). Now the requirement that \( \hat{g} \) is a 1-cell of \( \mathcal{C}_{\text{at}}^{M \times (-)} \) determines everything else. By chasing diagram (4.11) (with \( \tilde{g} \) replaced by \( \hat{g} \)) starting from the object \((n, x)\) and evaluating at \( I \) we may conclude \( A^x_n = \Gamma_{X}(n, x) \) and \( h^x_{m,n} = \gamma_{m,\Gamma_{X}(n, x)} \); whereas chasing it starting from the morphism \((\text{id}_n, z)\) and evaluating at \( I \) enables us to conclude \( \varphi^z_n = g_{\Gamma X}(n, z) \).
\( \square \)

**The 2-dimensional aspect**

Let us proceed to the unique factorization for morphisms of left \( T \)-modules. Suppose we have a morphism of left \( T \)-modules, i.e., a 2-cell
\[
(G, g)
\]
\[
\xrightarrow{(G, \gamma)}
\]
\[
\xleftarrow{(\Omega, \omega)}
\]
\[
\xrightarrow{(\mathcal{C}_{\text{at}}, C)}
\]
of \( \mathcal{E}^{++} \) satisfying
\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{C}_{\text{at}} \\
\downarrow \Gamma' & & \downarrow \mathcal{M} \times (-) \\
\mathcal{C}_{\text{at}} & \xrightarrow{G'} & \mathcal{E}
\end{array}
\]
\[
= \begin{array}{ccc}
\mathcal{C}_{\text{at}} & \xrightarrow{G} & \mathcal{E} \\
\downarrow \Gamma & & \downarrow \mathcal{M} \times (-) \\
\mathcal{E} & \xrightarrow{G'} & \mathcal{C}_{\text{at}}
\end{array}
\] (4.14)
4.1. THE EILENBERG–MOORE CONSTRUCTION FOR GRADED MONADS

Equation (4.14) implies the unique factorization

\[
\begin{aligned}
\mathcal{B} \xrightarrow{\Omega} \mathcal{Cat} &= \mathcal{B} \xrightarrow{\hat{\Omega}} \mathcal{Cat}^{\mathcal{M} \times (-)} \xrightarrow{U^\mathcal{M}} \mathcal{Cat}
\end{aligned}
\]

where the 2-natural transformation \(\hat{\Omega}\) is defined as follows:

**Definition 4.17.** The 2-natural transformation \(\hat{\Omega}: \hat{G} \Rightarrow \hat{G}'\) has, as components, 1-cells \(\hat{\Omega}_X: \left(\mathcal{M} \times G X \downarrow \Gamma X \right) \rightarrow \left(\mathcal{M} \times G' X \downarrow \Gamma' X \right)\) of \(\mathcal{Cat}^{\mathcal{M} \times (-)}\) given by \(\hat{\Omega}_X := \Omega_X\) as functors.

Next we construct a 2-cell

\[
\begin{aligned}
\left(\mathcal{M} \times G X \downarrow \Gamma X \right) \xrightarrow{\hat{g}} \left(\mathcal{M} \times \mathcal{C} \downarrow \mathcal{C} \right) = \left(\mathcal{M} \times G' X \downarrow \Gamma' X \right) \xrightarrow{\hat{g}'} \left(\mathcal{M} \times \mathcal{C} \downarrow \mathcal{C} \right)
\end{aligned}
\]

of \(\mathcal{Cat}^{\mathcal{M} \times (-)}\) satisfying

\[
\begin{aligned}
GX \xrightarrow{g} \mathcal{C} &= \mathcal{C} \xrightarrow{u^T} \mathcal{C}
\end{aligned}
\]
Definition 4.18. The 2-cell \( \tilde{\omega} \) of \( \mathcal{C}a^m \times (-) \) is the natural transformation consisting of components at \( x \in GX \)

\[
\tilde{\omega}_x : \left( (g \Gamma_X(n,x))_{n \in M}, (\gamma_{m, \Gamma_X(n,x)})_{m, n \in M} \right) \rightarrow \left( (g' \Gamma_X(n, \Omega_X x))_{n \in M}, (\gamma'_{m, \Gamma_X(n, \Omega_X x)})_{m, n \in M} \right)
\]
defined as

\[
\tilde{\omega}_{x,n} := \omega_{\Gamma_X(n,x)} : g \Gamma_X(n,x) \rightarrow g' \Omega_X \Gamma_X(n,x) \quad (4.14)
\]

Proposition 4.19. The natural transformation \( \tilde{\omega} \) defined above is indeed a 2-cell of \( \mathcal{C}a^m \times (-) \) which satisfies (4.16). Moreover, it is the unique such.

Proof. \( \tilde{\omega} \) is a 2-cell of \( \mathcal{C}a^m \times (-) \), i.e.,

\[
\begin{array}{ccc}
\mathbb{M} \times GX & \xrightarrow{\gamma_X} & \mathbb{M} \times \mathbb{C}T \\
\xrightarrow{\tilde{\omega}} & & \xleftarrow{\mathbb{M} \times \tilde{\omega}} \\
\Gamma_X & \xleftarrow{\tilde{\omega}'} & \mathbb{M} \times G'X \end{array}
\]

holds, since

\[
\tilde{\omega}_{X(n,x), m} = \omega_{\Gamma_X(m, \Gamma_X(n,x))} = \omega_{\Gamma_X(M \times \Gamma_X)(m, n, x)} = \omega_{\Gamma_X M_G X(m, n, x)} = \omega_{\Gamma_X(m \otimes n, x)} = \tilde{\omega}_{x, m \otimes n} = (n \otimes \tilde{\omega}_x)_m
\]
on objects.

\( \tilde{\omega} \) satisfies (4.16) since,

\[
\omega_x = \omega_{\Gamma_X H_{G,X_x}} = \omega_{\Gamma_X(I, x)} = u^T(\omega_{\Gamma_X(n, x)}), n \in M
\]
on objects.

For the uniqueness, suppose a natural transformation \( \tilde{\omega} : \tilde{\omega}' \circ \Omega_X \Rightarrow g \) with components \( \tilde{\omega}_x = (\varphi^x_n)_{n \in M} \) also satisfies the conditions. Equation (4.16) implies that \( \varphi^x_I = \omega_x \), and putting the object \((n, x)\) into equation (4.17) and evaluating the resulting morphism at \( I \) determines that \( \varphi^x_n = \omega_{\Gamma_X(n, x)} \).

Finally we may conclude:

Theorem 4.20. The object \( \left( \mathcal{C}a^m \times (-), \mathbb{M} \times \mathbb{C}T \right) \) of \( \mathcal{E}^+ \) is the Eilenberg–Moore object of the graded monad \( T \), considered as a monad \( \left( \mathbb{M} \times (-), T \right) \) in \( \mathcal{E}^+ \) on \( (\mathcal{C}a^m, \mathbb{C}) \).
4.2 The Kleisli construction for graded monads

As in the previous section, suppose we have a strict monoidal category \( M = (M, \otimes, I) \), a category \( C \), and an \( M \)-graded monad \( T \) on \( C \). We continue to write the functor part of \( T \) as \( * : M \times C \to C \), by identifying the adjoint transposes. Following the observation in Section 3.3.2 this time, the Kleisli adjunction for \( T \) lives in \( \mathcal{E}^{--} \) and lies above the co-Eilenberg–Moore adjunction for the 2-comonad \( [M, -] \) on \( \mathcal{C}at \), as in the picture below:

\[
\begin{array}{cccc}
\mathcal{C}at & \mathcal{C}at^{[M, -]} & \mathcal{C}at^T & \mathcal{C}at \\
\mathcal{C}at & \mathcal{C}at^{[M, -]} & \mathcal{C}at^T & \mathcal{C}at
\end{array}
\]

4.2.1 The Kleisli category

**Definition 4.21.** The category \( \mathcal{C}at_T \) is defined as follows:

- An object of \( \mathcal{C}at_T \) is a pair \((m, c)\) where \( m \) and \( c \) are objects of \( M \) and \( C \) respectively.
- The set of morphisms from \((m, c)\) to \((m', c')\) is defined by the coend formula

\[
\mathcal{C}at_T((m, c), (m', c')) := \int_{n \in M} M(m \otimes n, m') \times \mathcal{C}(c, n* c').
\]

Explicitly, a morphism \((m, c) \to (m', c')\) is an equivalence class \([n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n* c']\) of tuples consisting of an object \( n \in M \) and morphisms \( v, f \), where the equivalence relation is generated by

\[
(n, m \otimes n \xrightarrow{m \otimes w} m \otimes n' \xrightarrow{v} m', c \xrightarrow{f} n* c') \sim (n', m \otimes n' \xrightarrow{v} m', c \xrightarrow{f} n* c' \xrightarrow{w*c'} n'* c')
\]

for each morphism \( w: n \to n' \) of \( M \).
- The identity morphism on \((m, c)\) is given by \([I, m \otimes I \xrightarrow{id_m} m, c \xrightarrow{\eta_c} I*c]\).
• For two composable morphisms

\[
[n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n \ast c'] : (m, c) \rightarrow (m', c'),
\]
\[
[n', m' \otimes n' \xrightarrow{v'} m'', c' \xrightarrow{f'} n' \ast c''] : (m', c') \rightarrow (m'', c''),
\]
their composite is given by

\[
[n \otimes n', m \otimes n \otimes n' \xrightarrow{v \otimes v'} m' \otimes n' \otimes n'' \xrightarrow{\mu_{n,n',n''}} (n \otimes n') \ast c''] : (m, c) \rightarrow (m'', c'').
\]

We need to check well-definedness to validate such definitions as that of compositions in \(C_T\). All these are established through straightforward calculation.

**Definition 4.22.** Define the functor

\[\otimes : \mathbb{C}_T \rightarrow \mathbb{M},\mathbb{C}_T\]

as follows; we will use the infix notation \(l \otimes (m, c)\) to denote the value of the functor \(\otimes(m, c) : \mathbb{M} \rightarrow \mathbb{C}_T\) applied to \(l \in \mathbb{M}\) and similarly for morphisms.

- Given objects \(l\) and \((m, c)\) of \(\mathbb{M}\) and \(\mathbb{C}_T\) respectively, we define \(l \otimes (m, c) := (l \otimes m, c)\).

- Given morphisms \(u : l \rightarrow l'\) and \([n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n \ast c'] : (m, c) \rightarrow (m', c')\) of \(\mathbb{M}\) and \(\mathbb{C}_T\) respectively, we define

\[
u \otimes [n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n \ast c'] := [n, l \otimes m \otimes n \xrightarrow{u \otimes v} l' \otimes m', c \xrightarrow{f} n \ast c'] : (l \otimes m, c) \rightarrow (l' \otimes m', c').
\]

### 4.2.2 The Kleisli adjunction

**The left adjoint**

We define the 1-cell

\[
(F_M, f_T) : (\mathcal{C}^{\text{op}}, \mathbb{C}) \rightarrow (\mathcal{C}^{\text{op}}[\mathbb{M}, -], \mathbb{C}_T[\mathbb{M}, \mathbb{C}_T])
\]

of \(\mathcal{C}^{\text{op}}\) as follows:

**Definition 4.23.** The 2-functor \(F_M : \mathcal{C}^{\text{op}}[\mathbb{M}, -] \rightarrow \mathcal{C}^{\text{op}}\) is the forgetful 2-functor \(\mathbb{A}[\mathbb{M}, \mathbb{A}] \rightarrow \mathbb{A}\).

**Definition 4.24.** The functor \(f_T : \mathbb{C} \rightarrow \mathbb{C}_T\) is defined as \(f_T(c) := (I, c)\) on an object \(c\) and

\[
f_T(f) := [I, I \otimes I \xrightarrow{id} I, c \xrightarrow{f} c' \xrightarrow{\eta c} I \ast c'] : (I, c) \rightarrow (I, c')
\]
on a morphism \(f : c \rightarrow c'\).
4.2. THE KLEISLI CONSTRUCTION FOR GRADED MONADS

The right adjoint

We define the 1-cell

\[(U_M, u_T) : \left( \mathcal{C}^{\mathbb{N}, -}, \underbrace{\mathcal{C}}_{\text{Graded Monads}} \right) \rightarrow (\mathcal{C})\]

of \(\mathcal{E}^{\mathbb{N}}\) as follows:

**Definition 4.25.** The 2-functor \(U_M : \mathcal{C} \rightarrow \mathcal{C}^{\mathbb{N}, -}\) is the cofree 2-functor \(X \mapsto \underbrace{\mathcal{C}}_{\text{Graded Monads}}\), where \(M_M, X = M_X\) is the one defined in Definition 3.10.

**Definition 4.26.** The 1-cell \(u_T : (\mathcal{C}^{\mathbb{N}, -}) \rightarrow (\mathcal{C})\) of \(\mathcal{E}^{\mathbb{N}}\) is the functor \(u_T : \mathcal{C} \rightarrow [M, C]\) defined as

\[u_T(m, c) := ((- \otimes m)c : M \rightarrow C)\]

on an object \((m, c)\) and

\[\left( u_T(n, m \otimes n \rightarrow m', c \xrightarrow{f} n \ast c') \right)_l := \left( (l \otimes m)c \xrightarrow{\mu_{l \otimes m, n \ast c'}} (l \otimes m \ast n \ast c') \xrightarrow{\mu_{l \otimes m, n \ast c'}} (l \otimes m) \ast n \ast c' \right)\]

on a morphism \([n, m \otimes n \rightarrow m', c \xrightarrow{f} n \ast c'] : (m, c) \rightarrow (m', c')\).

The unit

We define the 2-cell

\[\eta_T : (\mathcal{C}) \rightarrow (\mathcal{C})\]

of \(\mathcal{E}^{\mathbb{N}}\) as follows:

**Definition 4.27.** The 2-natural transformation \(H_M : F_M \circ U_M \Rightarrow \text{id}_{\mathcal{C}}\) is the one defined in Definition 3.10.

**Definition 4.28.** The natural transformation

\[\eta_T : C \rightarrow [M, C]\]

has components \(\eta_{T,c} : c \rightarrow (I \ast c)\) given by the data of the graded monad.
CHAPTER 4. THE MAIN CONSTRUCTIONS

The counit

We define the 2-cell

\[
\begin{align*}
(Cat[M, -], & \circ) \rightarrow (\mathcal{C}at, \circ) \rightarrow (C_T[M, -], \circ) \rightarrow (\mathcal{C}at[M, -], \circ) \\
(U_M, u_T) & \rightarrow (Cat, \circ) \rightarrow (E_M, \varepsilon_T) \rightarrow (C_T[M, -], \circ) \\
(id_{\mathcal{C}at[M, -], \circ}) & \rightarrow (\mathcal{C}at, \circ) \rightarrow (\mathcal{C}at[M, -], \circ) \\
\end{align*}
\]

of \(\mathcal{C}at[M, -]\) as follows:

**Definition 4.29.** The 2-natural transformation \(E_M: id_{\mathcal{C}at[M, -]} \Rightarrow U_M \circ F_M\) has components \(E_{M, \alpha}: [A, B] \rightarrow [M, A] \rightarrow [M, B]\) given by \(E_{M, \alpha} = \alpha: [B, A] \rightarrow A\).

**Definition 4.30.** The 2-cell \((C_T[M, -], \circ) \rightarrow (\mathcal{C}at, \circ) \rightarrow (E_M, \varepsilon_T) \rightarrow (C_T[M, -], \circ) \rightarrow (id_{\mathcal{C}at[M, -], \circ})\) of \(\mathcal{C}at[M, -]\) is the natural transformation with the component at \((m, c) \in C_T\) being \(\varepsilon_{(m, c)}: (I, ((-) \otimes m)*c) \Rightarrow ((-) \otimes m, c)\), itself with the component at \(l \in M\) given by \(\varepsilon_{(m, c), l} = [l \otimes m, l \otimes m \otimes m \Rightarrow l \otimes m, (l \otimes m)*c \otimes c \Rightarrow (l \otimes m)*c] = (I, (l \otimes m)*c) \rightarrow (l \otimes m, c)\).

4.2.3 Comparison maps

Suppose we have an adjunction \((L, l) \dashv (R, r): (\mathcal{D}, D) \rightarrow (\mathcal{C}at, \circ)\) in \(\mathcal{E}\) with unit \((H, \eta): (id_{\mathcal{C}at}, id_{\mathcal{C}at}) \Rightarrow (R, r) \circ (L, l)\) and counit \((E, \varepsilon): (L, l) \circ (R, r) \Rightarrow (id_{\mathcal{D}}, id_{\mathcal{D}})\), which gives a resolution of the monad \([M, -], T\), i.e., such that the following equations hold:

\[
\begin{align*}
\mathcal{C}at[M, -] & \Rightarrow \mathcal{C}at \Rightarrow \mathcal{C}at \Rightarrow \mathcal{C}at \Rightarrow \mathcal{C}at \\
\mathcal{C} \Rightarrow [M, \mathcal{C}] & \Rightarrow \mathcal{C} \Rightarrow LF \Rightarrow LR \Rightarrow LRC \\
H_M & = H \Rightarrow \eta_T = \eta
\end{align*}
\]
4.2. THE KLEISLI CONSTRUCTION FOR GRADED MONADS

Equations (4.18), (4.20) and (4.22) imply the existence of the comparison 2-functor $K$:

**Definition 4.31.** The 2-functor $K : D \to \mathcal{C}at^{[M,-]}$ is the comparison 2-functor $D' \mapsto \left( L_D' \downarrow \mathcal{C}at \right)_{[M,LD]}$. \hfill $\blacksquare$

This provides the 2-functor part of the comparison map under construction. The remaining piece of data is given as follows:

**Definition 4.32.** The 1-cell $k : \left( C_T \downarrow \mathcal{C}at \right)_{[M,CT]} \to \left( L_D \downarrow \mathcal{C}at \right)_{[M,L_D]}$ of $\mathcal{C}at^{[M,-]}$ is the functor $k : C_T \to LD$ defined as

$$k(m, c) := (LE_Dc)m$$

on an object $(m, c)$ and

$$k(n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n*c') := ((LE_Dc')v \circ (LE_DLE_c,n)m \circ (LE_df)m$$

on a morphism $[n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n*c'] : (m, c) \to (m', c')$. Let us check that the type of $k[n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n*c']$ is indeed the right one:

$$\begin{align*}
(LE_Dc)m & \xrightarrow{(LE_Dlf)m} (LE_Dl(n*c'))m \\
& \xrightarrow{(LE_DLE_c)n} (LE_D((LRlRl)c)n)m \quad \text{by (4.19)} \\
& \xrightarrow{(LE_DLE_c)n} (LE_D((LE_Dl)c)n)m \\
& \xrightarrow{(LE_RlDl)c'n} (M_{LD}c'n)m \quad \text{by (4.22)} \\
& \xrightarrow{(LE_Dl)c'm} (LE_Dl)c'm'.
\end{align*}$$

**Proposition 4.33.** The 1-cell $(K, k)$ of $\mathcal{C}at^{[M,-]}$ satisfies the equations $(K,k) \circ (F_M, f_T) = (L,l)$ and $(U_M, u_T) = (R,r) \circ (K,k)$. Moreover, it is the unique such.
4.2.4 The 2-dimensional universality

Statement of the theorem

We will show that there is a family of isomorphisms of categories

$$
\mathcal{E}^{--} \left( \left( \mathbf{Cat}^{[M,-]}, \mathbf{C}_T \right) \downarrow \mathcal{M}, \mathcal{C}_T \right), (\mathcal{X}, X) \cong \mathcal{E}^{--}((\mathbf{Cat}, \mathcal{C}), (\mathcal{X}, X))^{\mathcal{E}^{--}(([M,-],T), (\mathcal{X},X))}
$$

2-natural in $$(\mathcal{X}, X) \in \mathcal{E}^{--}$$; cf. Definition 3.3. We claim that the data

provides the universal right $T$-module, in the sense that every right $T$-module

$$
(G, g) : \mathcal{E}^{--}((\mathbf{Cat}, \mathbf{C}), (\mathcal{X}, X)) \rightarrow (\mathcal{X}, X)
$$

i.e., an object of the category $\mathcal{E}^{--}((\mathbf{Cat}, \mathcal{C}), (\mathcal{X}, X))^{\mathcal{E}^{--}(([M,-],T), (\mathcal{X},X))}$, factors uniquely as

and similarly every morphism of right $T$-modules

$$
(G, g)
$$
i.e., a morphism of the category $\mathcal{E}^{--}((\mathcal{C}at, \mathcal{C}), (\mathcal{B}, X))^{--}((\mathcal{M}, T), (\mathcal{B}, X))$, factors uniquely as

$$
\begin{array}{c}
\xymatrix{
(\mathcal{C}at, \mathcal{C}) & (\mathcal{C}at)^{\mathcal{M}, -} & (\mathcal{B}, X) \\
(\mathcal{C}at, \mathcal{C}) \ar[r]^-{(F_M, F_T)} & (\mathcal{C}at)^{\mathcal{M}, -} \ar[r]^-{C_T} & (\mathcal{B}, X) \\
(\mathcal{C}at, \mathcal{C}) \ar[r]^-{(G, g)} & (\mathcal{B}, X) \ar[r]_-{(G', g')} & (\mathcal{B}, X)
}
\end{array}
$$

The 1-dimensional aspect

Suppose we have a right $T$-module, i.e., a piece of data

$$
\begin{array}{c}
\xymatrix{
(\mathcal{C}at, \mathcal{C}) & (\mathcal{B}, X) \\
(\mathcal{C}at, \mathcal{C}) \ar[r]^-{(G, g)} & (\mathcal{B}, X)
}
\end{array}
$$

in $\mathcal{E}^{--}$ satisfying

\begin{align}
\xymatrix{
\mathcal{C} & \mathcal{C} \ar[l]^-{\text{id}_{\mathcal{C}}} & \mathcal{C} \ar[l]^-{\text{id}_{\mathcal{C}}} \\
[M, \mathcal{C}] & [M, \mathcal{C}] \ar[l]^-{H_M} & [M, \mathcal{C}] \ar[l]^-{H_M} \\
[M, GX] & [M, GX] \ar[l]^-{\gamma} & [M, GX] \ar[l]^-{\gamma}
}
\end{align}

\begin{align}
\xymatrix{
\mathcal{C} & \mathcal{C} \\
[M, \mathcal{C}] & [M, \mathcal{C}] \ar[l]^-{\text{id}_{\mathcal{C}}} & [M, \mathcal{C}] \ar[l]^-{\text{id}_{\mathcal{C}}} \\
[M, \mathcal{C}] & [M, \mathcal{C}] \ar[l]^-{H_M} & [M, \mathcal{C}] \ar[l]^-{H_M} \\
[M, \mathcal{C}] & [M, \mathcal{C}] \ar[l]^-{\gamma} & [M, \mathcal{C}] \ar[l]^-{\gamma}
}
\end{align}

\begin{align}
\xymatrix{
\mathcal{C} & \mathcal{C} \\
[M, \mathcal{C}] & [M, \mathcal{C}] \ar[l]^-{\text{id}_{\mathcal{C}}} & [M, \mathcal{C}] \ar[l]^-{\text{id}_{\mathcal{C}}} \\
[M, \mathcal{C}] & [M, \mathcal{C}] \ar[l]^-{H_M} & [M, \mathcal{C}] \ar[l]^-{H_M} \\
[M, \mathcal{C}] & [M, \mathcal{C}] \ar[l]^-{\gamma} & [M, \mathcal{C}] \ar[l]^-{\gamma}
}
\end{align}
Equations (4.24) and (4.26) imply the unique factorization

$$\begin{array}{c}
\text{Cat} \\ [M, -]
\end{array} \quad \xleftarrow{G} \quad [\mathcal{X}, \Gamma X'] = \quad \begin{array}{c}
\text{Cat} \\ [M, -]
\end{array} \quad \xleftarrow{\mathcal{G}} \quad \mathcal{X}'
$$

since the co-Eilenberg–Moore 2-category $\text{Cat}^{[M, -]}$ is the co-Eilenberg–Moore object in 2-$\text{Cat}$. 

Definition 4.34. The 2-functor $\tilde{G}: \mathcal{X} \to \text{Cat}^{[M, -]}$ is the mediating 2-functor $X' \mapsto \left( \begin{array}{c} GX' \\ [M, GCX'] \\ [M, GX'] \end{array} \right)$.

Now it remains to construct a 1-cell $\tilde{g}: \left( \begin{array}{c} C_T \\ [M, C_T] \end{array} \right) \to \left( \begin{array}{c} GX \\ \Gamma X \\ [M, GX] \end{array} \right)$ of $\text{Cat}^{[M, -]}$ which satisfies

$$\begin{array}{c}
\text{Cat} \\ [M, C]
\end{array} \quad \xrightarrow{g} \quad [M, GX]
$$

$$\begin{array}{c}
\text{Cat} \\ [M, C]
\end{array} \quad \xrightarrow{fr} \quad C_T \quad \xrightarrow{\tilde{g}} \quad GX
$$
and show its uniqueness.

**Definition 4.35.** The 1-cell $\tilde{g} : \left( \frac{C_T}{\mathcal{M}, C_T} \right) \to \left( \frac{GX}{\mathcal{M}, GX} \right)$ is the functor $\tilde{g} : C_T \to GX$ defined as

$$\tilde{g}(m, c) := (\Gamma_X gc)m$$
on an object $(m, c)$ and

$$\tilde{g}[n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n \ast c'] := ((\Gamma_X gc') \circ (\Gamma_X \eta_{c', n})_m \circ (\Gamma_X \varphi)_m)(n).$$

on a morphism $[n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n \ast c'] : (m, c) \to (m', c')$. To check the type of $\tilde{g}[n, v, f]$, observe

$$\begin{align*}
\frac{(\Gamma_X gc)m \xrightarrow{(\Gamma_X \varphi)_m}}{(\Gamma_X \eta_{c', n})_m \circ (\Gamma_X \varphi)_m} & \quad \frac{\Gamma_X(([\mathcal{M}, g][Tc'])n)m}{(\Gamma_X((\Gamma_X gc')n))m} \\
& \quad \frac{(M_{GX} \Gamma_X gc')n}{(\Gamma_X gc')(m \otimes n)} \\
& \quad \frac{(\Gamma_X gc')v}{(\Gamma_X gc')m'}.
\end{align*}$$

In order to show the uniqueness of factorization, we need the following calculational result.

**Lemma 4.36.** Every morphism $[n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n \ast c'] : (m, c) \to (m', c')$ of $C_T$ can be decomposed as

$$[n, m \otimes n \xrightarrow{v} m', c \xrightarrow{f} n \ast c'] = (m, c) \xrightarrow{m \otimes f_T(f)} (m, n \ast c') \xrightarrow{\varphi \circ f_T(c')} (m', c').$$

**Proof.** As equivalence classes, the morphisms on the right hand side are

$$m \circ f_T(f) = m \circ [I, I \xrightarrow{id} I, c \xrightarrow{f} n \ast c' \xrightarrow{\eta_{c', I}} I \ast n \ast c'] = [I, m \xrightarrow{id} m, c \xrightarrow{f} n \ast c' \xrightarrow{\eta_{c', I}} I \ast n \ast c'],$$

$$m \circ \varphi_{T,(I, c'),n} = m \circ [n, n \xrightarrow{id} n, n \ast c' \xrightarrow{id} n \ast c'] = [n, m \otimes n \xrightarrow{id} m \otimes n, n \ast c' \xrightarrow{id} n \ast c'],$$

$$v \circ f_T(c') = v \circ [I, I \xrightarrow{id} I, c' \xrightarrow{\eta_{c'}} I \ast c'] = [I, m \otimes n \xrightarrow{v} m', c' \xrightarrow{\eta_{c'}} I \ast c'].$$

The composite of the first two morphisms is

$$(m \circ \varphi_{T,(I, c'),n}) \circ (m \circ f_T(f)) = [n, m \otimes n \xrightarrow{id} m \otimes n, c \xrightarrow{f} n \ast c' \xrightarrow{\eta_{c', I}} I \ast n \ast c' \xrightarrow{\mu_{I, n \ast c'}} n \ast c'] = [n, m \otimes n \xrightarrow{id} m \otimes n, c \xrightarrow{f} n \ast c'].$$
so finally the composite of the three morphisms is
\[
(v \circ f_T(c')) \circ (m \circ \epsilon_{T,(I,c'),n}) \circ (m \circ f_T(f))
\]
\[
= [n, m \circ n \to m', \ c \xrightarrow{f} n \circ e' \xrightarrow{n \circ h'} n \circ I \circ e' \xrightarrow{\mu_{n,e'}} n \circ e']
\]
\[
= [n, m \circ n \to m', \ c \xrightarrow{f} n \circ e'].
\]

**Proposition 4.37.** The functor \( \tilde{g} \) defined above is indeed a 1-cell of \( \mathcal{C}at^{[M, -]} \) which satisfies (4.28) and (4.29). Moreover, it is the unique such.

**Proof.** First observe that thanks to Lemma 4.36, the uniqueness is obvious. Indeed, equation (4.28) determines the value of \( \tilde{g} \) at \( f_T(f) \) and \( f_T(c') \), and equation (4.29) determines the value of \( \tilde{g} \) at \( \epsilon_{T,(I,e'),n} = \epsilon_{T,f_T(c'),n} \). Finally, the requirement that \( \tilde{g} \) is a 1-cell of \( \mathcal{C}at^{[M, -]} \) enforces the equation \( \tilde{g}(m \circ -) = (\Gamma_X \tilde{g}(\_))m \).

That \( \tilde{g} \) indeed satisfies the conditions is straightforward to check. \( \tilde{g} \) is a 1-cell of \( \mathcal{C}at^{[M, -]} \) because, on objects
\[
\tilde{g}([I \circ (m, c)] = \tilde{g}(l \circ m, c)
\]
\[
= (\Gamma_X gc)(l \circ m)
\]
\[
= ((M_{GX} \Gamma_X gc)m)l
\]
\[
= (\Gamma_X ((M_{GX} gc)m))l
\]
\[
= (\Gamma_X \tilde{g}(m, c))l,
\]
and on morphisms
\[
\tilde{g}(u \circ [n, m \circ n \to m', c \xrightarrow{f} n \circ e'])
\]
\[
= \tilde{g}[n, l \circ m \circ n \xrightarrow{u \circ v} l' \circ m', c \xrightarrow{f} n \circ e']
\]
\[
= (\Gamma_X gc')(u \circ v) \circ (\Gamma_X \gamma_{e',n}) \circ (\Gamma_X uf) \circ (\Gamma_X I m)
\]
\[
= ((M_{GX} \Gamma_X gc')v) \circ (M_{GX} \Gamma_X \gamma_{e',n}) \circ (M_{GX} \Gamma_X uf) \circ (M_{GX} \Gamma_X I m)
\]
\[
= ((M_{GX} \Gamma_X gc')v) \circ (M_{GX} \Gamma_X \gamma_{e',n}) \circ (M_{GX} \Gamma_X uf) \circ (M_{GX} \Gamma_X I m)
\]
\[
= (\Gamma_X ((M_{GX} gc')v) \circ (\Gamma_X \gamma_{e',n}) \circ (\Gamma_X uf) \circ (\Gamma_X I m))u
\]
\[
= (\Gamma_X \tilde{g}(n, v, f))u
\]
\[
\tilde{g} \text{ satisfies (4.28) since}
\]
\[
gc = H_{GX} \Gamma_X gc
\]
\[
= (\Gamma_X gc)I
\]
\[
= \tilde{g}(I, c)
\]
\[
= \tilde{g}f_T(c)
\]
on objects and
\[
gf = \gamma_{e',I} \circ \eta_{e'} \circ \gf
\]
\[
= (H_{GX} \Gamma_X \gamma_{e',I}) \circ (H_{GX} \Gamma_X g(\eta_{e'} \circ f))
\]
\[
= (\Gamma_X \gamma_{e',I}) \circ (\Gamma_X g(\eta_{e'} \circ f)) \circ (\Gamma_X I c')
\]
\[
= \tilde{g}[I, I \xrightarrow{id} I, c \xrightarrow{f} c' \xrightarrow{\eta_{c'}} I \circ e']
\]
\[
= \tilde{g}f_T(f)
on morphisms.

Finally, \( \tilde{g} \) satisfies (4.29) since
\[
\gamma_{c,n} = H_{G X} \Gamma X \gamma_{c,n} \\
= (\Gamma X \gamma_{c,n}) I \\
= \tilde{g} [n, n \xrightarrow{id} n, n \xrightarrow{\ast c} n \xrightarrow{\ast c}] \\
= \tilde{g} \varepsilon \mathcal{T}_{(I, c), n} \\
= ([M, \tilde{g}] \varepsilon \mathcal{T}_{f (c)})_n
\]
holds. \( \square \)

The 2-dimensional aspect

Suppose we have a morphism of right \( T \)-modules, i.e., a 2-cell
\[
\begin{array}{c}
\text{Cat} \xrightarrow{(G, g)} (\mathcal{X}, X) \\
\text{Cat} \xrightarrow{(G', g')} (\mathcal{X}', X')
\end{array}
\]
of \( \mathcal{E} \) satisfying
\[
\xymatrix{ 
\text{Cat} \ar[rr]^{[M, G X]} & & [M, G' X] \\
\text{Cat} \ar[u]^{[M, \Gamma X]} \ar[rrru]_{[M, \gamma X]} & & [M, G' X] \ar[u]_{[M, \Omega X]} \ar[rrru] \\
C \ar[r]^T & [M, \text{Cat}] & [M, G' X] \\
\ar[rrru]_{g'} & & [M, G' X] \\
\ar[u]_{[M, g]} & & [M, G' X] \\
\end{array}
\]
(4.30)

Equation (4.30) implies the unique factorization
\[
\xymatrix{ 
\text{Cat} \ar[r]^{G} & (\mathcal{X}, X) \\
\text{Cat} \ar[r]^{F_M} \ar[u]^{\tilde{G}} & \text{Cat}^{[M, -]} \ar[u]^{\tilde{\Omega}} \\
C \ar[u]^{\Omega} \ar[r]^T & \mathcal{X} \ar[u]_{\Gamma X} \\
\ar[u]_{g} & & \mathcal{X} \ar[u]_{G' X} \ar[r]_{\Omega X} & GX \\
\ar[ru]_{g'} & & \mathcal{X} \ar[u]_{G' X} \\
\end{array}
\]
(4.31)
where the 2-natural transformation \( \tilde{\Omega} \) is defined as follows:

**Definition 4.38.** The 2-natural transformation \( \tilde{\Omega} : \tilde{G} \Rightarrow \tilde{G} \) has, as components, 1-cells \( \tilde{\Omega}_X : (G'X, \Gamma'X) \rightarrow (GX, \Gamma X) \) of \( \mathcal{C}at^{[M,-]} \) given by \( \tilde{\Omega}_X := \Omega_X \) as functors. ■

Next we proceed to construct a 2-cell of \( \mathcal{C}at^{[M,-]} \) satisfying

\[
\begin{align*}
\tilde{\omega} & : \left( \begin{array}{c}
\mathcal{C}T \\
[M, \mathcal{C}T]
\end{array} \right) \rightarrow \left( \begin{array}{c}
GX \\
[M, GX]
\end{array} \right) \\
\tilde{\omega} & : \left( \begin{array}{c}
G'X \\
[M, G'X]
\end{array} \right)
\end{align*}
\]

of \( \mathcal{C}at^{[M,-]} \) at \((m,c) \in \mathcal{C}T\).

**Definition 4.39.** The 2-cell \( \tilde{\omega} \) of \( \mathcal{C}at^{[M,-]} \) has components given by

\[
\tilde{\omega}_{(m,c)} : (\Gamma_Xg)c m \xrightarrow{(\Gamma_X\omega)c m} (\Gamma_X\Omega_Xg'c)m = (4.30)
\]

at \((m,c) \in \mathcal{C}T\). ■

**Proposition 4.40.** The natural transformation \( \tilde{\omega} \) defined above is indeed a 2-cell of \( \mathcal{C}at^{[M,-]} \) which satisfies (4.32). Moreover, it is the unique such.

**Proof.** Let us first check that \( \tilde{\omega} \) satisfies the conditions. \( \tilde{\omega} \) is a 2-cell of \( \mathcal{C}at^{[M,-]} \), i.e., \( \tilde{\omega}_{(m,c)} = (\Gamma_X\tilde{\omega}_{(m,c)})_t \) holds, since

\[
\begin{align*}
\tilde{\omega}_{(m,c)} & = \tilde{\omega}_{(\otimes m,c)} \\
& = (\Gamma_X\omega_c)_{\otimes m} \\
& = (M_GX\Gamma_X\omega_c)_{m,l} \\
& = (\Gamma_X(\Gamma_X\omega_c)m)l \\
& = (\Gamma_X\tilde{\omega}_{(m,c)})_t. \quad \text{by (4.26)}
\end{align*}
\]

\( \tilde{\omega} \) satisfies (4.32) since

\[
\begin{align*}
\omega_c & = H_{GX}\Gamma_X\omega_c \\
& = (\Gamma_X\omega_c)_I \\
& = \tilde{\omega}_{(I,c)} \\
& = \tilde{\omega}_{f_T(c)}.
\end{align*}
\]

For uniqueness, observe that equation (4.32) determines the \((I,c)\)-component of \( \tilde{\omega} \), and the requirement that \( \tilde{\omega} \) is a 2-cell of \( \mathcal{C}at^{[M,-]} \) implies \( \tilde{\omega}_{(m,c)} = (\Gamma_X\tilde{\omega}_{(I,c)})_m \), thus determining all the components. □
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**Theorem 4.41.** The object \( \mathbb{C} \mathsf{at}^{[\mathbb{M},-]} \downarrow \mathcal{T} \) of \( \mathcal{E} \) is the Kleisli object of the graded monad \( T \), considered as a monad \( ([\mathbb{M},-], T) \) in \( \mathcal{E} \) on \( \mathbb{C} \mathsf{at} \).

### 4.3 The Eilenberg–Moore construction for indexed monads

Let \( \mathbb{B} \) and \( \mathbb{C} \) be categories, and \( \mathcal{F} \) a \( \mathbb{B} \)-indexed monad on \( \mathbb{C} \). The Eilenberg–Moore adjunction for \( \mathcal{F} \), which lives in \( \mathcal{E}^{+} \) (cf. Section 3.3.3), lies above the co-Eilenberg–Moore adjunction for the 2-comonad \( \mathbb{B} \times (-) \) on \( \mathbb{C} \mathsf{at} \).

**Definition 4.42.** Define the category \( \mathbb{C} \mathcal{F} \) as follows:

- An object of \( \mathbb{C} \mathcal{F} \) is a pair \( (b, \mathcal{T} b c, \chi) \), or more concisely \( (b, \chi) \), where \( b \) is an object of \( \mathbb{B} \) and \( \mathcal{T} b c, \chi \) is a \( \mathcal{T} b \)-algebra (an object of \( \mathbb{C} \mathcal{T} b \)).

- A morphism from \( (b, \mathcal{T} b c, \chi) \) to \( (b', \mathcal{T} b' c', \chi' \circ \mathcal{T} u c) \) is a pair \( (u, h) \) where \( u: b \to b' \) is a morphism of \( \mathbb{B} \) and \( h: \mathcal{T} b c, \chi \to \mathcal{T} b' c', \chi' \circ \mathcal{T} u c \) is a homomorphism of \( \mathcal{T} b \)-algebras.
i.e., a morphism $h: c \to c'$ in $\mathbb{C}$ which makes the diagram

\[
\begin{array}{ccc}
\mathcal{B}_c & \xrightarrow{\mathcal{B}_h} & \mathcal{B}_{c'} \\
\chi \downarrow & & \downarrow \chi' \\
\tau_c & \longleftarrow^h & \tau_{c'}
\end{array}
\]

commute.

- The identity morphism on $(b, \frac{\mathcal{B}_c}{\chi})$ is given by $(\text{id}_b, \text{id}_\chi)$.

- For two composable morphisms
  \[
  \left( \begin{array}{c}
  b \xrightarrow{u} b', \left( \frac{\mathcal{B}_c}{\chi} \right) h \\
  b' \xrightarrow{u'} b'', \left( \frac{\mathcal{B}_{c'}}{\chi'} \right) h'
  \end{array} \right) : (b, \chi) \to (b', \chi'),
  \quad
  \left( \begin{array}{c}
  \left( \frac{\mathcal{B}_{c'}}{\chi'} \right) h' \\
  \left( \frac{\mathcal{B}_{c''}}{\chi''} \right) h''
  \end{array} \right)
  : (b', \chi') \to (b'', \chi''),
  \]
  their composite is given by

\[
\left( \begin{array}{c}
  b \xrightarrow{u} b', \left( \frac{\mathcal{B}_c}{\chi} \right) h \\
  b' \xrightarrow{u'} b'', \left( \frac{\mathcal{B}_{c'}}{\chi'} \right) h'
  \end{array} \right)
  \xrightarrow{\left( \begin{array}{c}
  \left( \frac{\mathcal{B}_{c'}}{\chi'} \right) h' \\
  \left( \frac{\mathcal{B}_{c''}}{\chi''} \right) h''
  \end{array} \right)}
  \left( \begin{array}{c}
  b, \chi \\
  b'', \chi''
  \end{array} \right).
\]

**Definition 4.43.** Define the functor $\pi: \mathbb{C}^\mathcal{F} \to \mathbb{B} \times \mathbb{C}^\mathcal{F}$ by $(b, \chi) \mapsto (b, (b, \chi))$ and $(u, h) \mapsto (u, (u, h))$.

### 4.3.2 The Eilenberg–Moore adjunction

The left adjoint

We define the 1-cell

\[
(F^\mathcal{B}, f^\mathcal{F}) : (\mathcal{C}^\mathcal{B}, \mathcal{C}) \to (\mathcal{C}^\mathcal{B} \times (-), \mathcal{C}^\mathcal{F})
\]

of $\mathcal{C}^+$ as follows:

**Definition 4.44.** The 2-functor $F^\mathcal{B}: \mathcal{C}^\mathcal{B} \to \mathcal{C}^\mathcal{B} \times (-)$ is the cofree 2-functor $\mathcal{X} \mapsto \left( \frac{\mathcal{B} \times \mathcal{X}}{\mathcal{B} \times \mathcal{B} \times \mathcal{X}} \right)$ where $M^\mathcal{B}_\mathcal{X} = M_\mathcal{X}$ is the one defined in Definition 3.12.

**Definition 4.45.** The 1-cell $f^\mathcal{F}: \mathbb{B} \times \mathcal{C} \to \mathbb{C}^\mathcal{F}$ defined as

\[
  f^\mathcal{F}(b, c) := \left( b, \frac{\mathcal{B}_c}{\mathcal{F}_c} \mathcal{B}_b \mathcal{C}_c \right)
\]
on an object $(b, c)$ and

\[
  f^\mathcal{F}(u, f) := \left( b \xrightarrow{u} b', \left( \mathcal{B}_b \mathcal{B}_c \right) \mathcal{F}_u \mathcal{C}_c \mathcal{F}_b \mathcal{F}_c \xrightarrow{\mathcal{F}_u \mathcal{C}_c \mathcal{F}_b \mathcal{F}_c} \left( \mathcal{B}_b \mathcal{B}_c \mathcal{C}_c \mathcal{F}_b \mathcal{F}_c \right) \mathcal{F}_u \mathcal{C}_c \mathcal{F}_b \mathcal{F}_c \mathcal{F}_u \mathcal{C}_c \mathcal{F}_b \mathcal{F}_c \right)
\]
on a morphism $(u, f): (b, c) \to (b', c')$.  

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The right adjoint

We define the 1-cell
\[(U^B, u^\mathcal{F}) : (\mathcal{C}at^B \times (-), \frac{\mathcal{C}at \times \mathcal{F}}{B \times \mathcal{F}}) \to (\mathcal{C}at, \mathcal{C})\]
of \(\mathcal{E}^{+-}\) as follows:

**Definition 4.46.** The 2-functor \(U^B : \mathcal{C}at^B \times (-) \to \mathcal{C}at\) is the forgetful 2-functor \(\left( \frac{A}{B \times A} \right) \mapsto A\).

**Definition 4.47.** The functor \(u^\mathcal{F} : \mathcal{C} \to \mathcal{C}\) is also given as the forgetful functor \((b, \frac{c}{\chi}) \mapsto c\) and \((u, h) \mapsto h\).

The unit

We define the 2-cell

\[\xymatrix{(\mathcal{C}at, \mathcal{C}) \ar[r]^{(id_{\mathcal{C}at}, id_{\mathcal{C}})} \ar[r]_{(H^B, \eta^\mathcal{F})} & (\mathcal{C}at, \mathcal{C}) \ar[r]^{(U^B, u^\mathcal{F})} & (\mathcal{C}at, \mathcal{C})}\]
of \(\mathcal{E}^{+-}\) as follows:

**Definition 4.48.** The 2-natural transformation \(H^B : U^B \circ F^B \Rightarrow id_{\mathcal{C}at}\) has components \(H^B_X = H_X\) defined in Definition 3.12.

**Definition 4.49.** The natural transformation

\[\xymatrix{\mathcal{B} \times \mathcal{C} \ar[r]^{f^\mathcal{F}} \ar[r]_{H_\mathcal{C}} & \mathcal{C} \ar[r]^{id_{\mathcal{C}}} & \mathcal{C} \ar[r]_{\eta^\mathcal{F}} & \mathcal{C} \ar[r]^{u^\mathcal{F}} & \mathcal{C}}\]

has components \(\eta^\mathcal{F}_{b,c} : c \to \mathcal{F}b\mathcal{C}\) given by the data of the indexed monad.
The counit

We define the 2-cell

\[
\begin{align*}
&\exists (U^\mathbb{B}, u^\mathcal{T}) \oplus \exists (E^\mathbb{B}, \varepsilon^\mathcal{T}) \oplus \exists (\mathcal{B} \times (-), \mathcal{C}^\mathcal{T}) \\
&\quad \downarrow \mathcal{B} \times \mathcal{C}^\mathcal{T} \oplus \downarrow \mathcal{B} \times \mathcal{C}^\mathcal{T} \\
&\quad \exists (\mathcal{B} \times (-), \mathcal{C}^\mathcal{T}) \oplus \exists \mathcal{B} \times (-) \oplus \exists \mathcal{C}^\mathcal{T}
\end{align*}
\]

of \( \mathcal{C}^\mathcal{T} \) as follows:

**Definition 4.50.** The 2-natural transformation \( E^\mathbb{B} \colon \text{id}_{\mathcal{B} \times (-)} \Rightarrow F^\mathbb{B} \circ U^\mathbb{B} \) has components \( E^\mathbb{B}_\alpha \colon (A \mathcal{B} \times A \mathcal{B} \times A) \rightarrow (B \mathcal{B} \times A \mathcal{B} \times A) \) given by \( E^\mathbb{B}_\alpha := \alpha : A \rightarrow B \).

**Definition 4.51.** The 2-cell \( (\mathcal{B} \mathcal{C}^\mathcal{T}) \oplus \exists \mathcal{B} \times (-) \oplus \exists \mathcal{C}^\mathcal{T} \) of \( \mathcal{C}^\mathcal{T} \) is the natural transformation with the component at \( (b, \chi) \in \mathcal{C}^\mathcal{T} \), \( \varepsilon^\mathcal{T}_{(b, \chi)} : (b, \chi) \rightarrow (\chi, \rho_{bc}) \), given by \( \varepsilon^\mathcal{T}_{(b, \chi)} := (\text{id}_b, \chi) \).

### 4.3.3 Comparison maps

Suppose we have an adjunction \( (L, l) \vdash (R, r) \colon (\mathcal{D}, D) \rightarrow (\mathcal{Cat}, \mathcal{C}) \) in \( \mathcal{C}^{+} \) with unit \( (H, \eta) : (\text{id}_{\mathcal{Cat}}, \text{id}_{\mathcal{C}}) \Rightarrow (R, r) \circ (L, l) \) and counit \( (E, \varepsilon) : (L, l) \circ (R, r) \Rightarrow (\text{id}_{\mathcal{D}}, \text{id}_{\mathcal{D}}) \), which gives a resolution of the monad \( \mathcal{B} \times (-) \), i.e., such that the following equations hold:

\[
\begin{align*}
\mathcal{Cat}^{\mathcal{B} \times (-)} \mathcal{Cat} &= \mathcal{Cat} \xrightarrow{L} \mathcal{D} \xrightarrow{R} \mathcal{Cat} \\
\mathcal{B} \mathcal{C}^{\mathcal{T}} \mathcal{C}^{\mathcal{T}} &= RLC \xrightarrow{RL} RD \xrightarrow{R} \mathcal{C} \\
H^\mathbb{B} &= H \\
\eta^\mathcal{T} &= \eta \\
\mathcal{B} \mathcal{C}^{\mathcal{T}} \mathcal{B} \mathcal{C}^{\mathcal{T}} &= \mathcal{Cat} \xrightarrow{L} \mathcal{D} \xrightarrow{E} \mathcal{Cat} \xrightarrow{id_{\mathcal{D}}} \mathcal{D} \xrightarrow{R} \mathcal{Cat}
\end{align*}
\]
The equations (4.33), (4.35) and (4.37) imply the existence of the comparison 2-functor $K$:

**Definition 4.52.** The 2-functor $K: \mathcal{D} \to \mathcal{C} \text{at}^B \times (-)$ is the comparison 2-functor $D' \mapsto (R_{B \times D'}, R_{B \times D'})$.

Before constructing the 1-cell $k$, we introduce a notation to describe the structure maps of coalgebras for the 2-comonad $B \times (-)$. Given an object $(A, B \times A)$ of $\mathcal{C} \text{at}^B \times (-)$, we will write $\alpha = (\alpha_0(-), \alpha_1(-))$. Note that by one of the axioms for $B \times (-)$-coalgebras saying that $A \mapsto B \times A \alpha$ commutes, $\alpha_1 = \text{id}_A$ holds; thus the only meaningful data of $\alpha$ is actually $\alpha_0: A \to B$, which is subject to no axioms. In fact, there is an isomorphism of 2-categories $\mathcal{C} \text{at}^B \times (-) \cong \mathcal{C} \text{at}/B$ given by $\left( \frac{A}{B}, \frac{\alpha}{A \times B} \right) \mapsto \left( \frac{A}{B}, \frac{\text{id}_A}{B} \right)$.

**Definition 4.53.** The 1-cell $k: \left( \frac{R_{B \times D'}}{B \times D'} \right) \to \left( \frac{T_{B \times \mathcal{C} \times (-)}}{B \times \mathcal{C} \times (-)} \right)$ of $\mathcal{C} \text{at}^B \times (-)$ is defined as

\[
k(d) := \left( \frac{RE_{D,0}d, \mathcal{T}_{RE_{D,0}d}}{rd} \right)
\]
on an object $d$. To check the type of the structure map, observe

\[
\mathcal{T}_{RE_{D,0}d} = rRl(RE_{D,0}d, rd)
\]
\[
= rRlRLl(RE_{D,0}d, d)
\]
\[
= rRlRLlRE_{D}d
\]
\[
\xrightarrow{rRe_d} rd.
\]
k is defined as

\[
k(w) := \left( \frac{RE_{D,0}d, \mathcal{T}_{RE_{D,0}d}}{rd} \right)
\]
on an object $w$. To check the type of the structure map, observe

\[
\mathcal{T}_{RE_{D,0}w} = rRl(RE_{D,0}w, rd)
\]
\[
= rRlRLl(RE_{D,0}w, d)
\]
\[
= rRlRLlRE_{D}d
\]
\[
\xrightarrow{rRe_d} rd.
\]
on a morphism \( w : d \to d' \).

**Proposition 4.54.** The 1-cell \((K,k)\) of \(\mathcal{E}^{+-}\) satisfies the equations \((K,k) \circ (L,l) = (F^B,f^\mathcal{F})\) and \((R,r) = (U^B,u^\mathcal{F}) \circ (K,k)\). Moreover, it is the unique such.

### 4.3.4 The 2-dimensional universality

**Statement of the theorem**

We will show that there is a family of isomorphisms of categories

\[
\mathcal{E}^{+-}(\mathcal{X},X), \text{Cat}^{\mathcal{B} \times (-), \mathcal{C}^{\mathcal{F}}}_{\mathcal{B} \times \mathcal{C}^{\mathcal{F}}}) \cong \mathcal{E}^{+-}(\mathcal{X},X), \text{Cat}^{\mathcal{B} \times (-), \mathcal{C}^{\mathcal{F}}}_{\mathcal{B} \times \mathcal{C}^{\mathcal{F}}})
\]

2-natural in \((\mathcal{X},X) \in \mathcal{E}^{+-}\), by showing that the data

\[
\begin{align*}
&\begin{array}{c}
\text{Cat}^{\mathcal{B} \times (-), \mathcal{C}^{\mathcal{F}}}_{\mathcal{B} \times \mathcal{C}^{\mathcal{F}}} \\
\downarrow \text{id}_{\text{Cat}^{\mathcal{B} \times (-), \mathcal{C}^{\mathcal{F}}}}
\end{array} \\
\rightarrow &\begin{array}{c}
\text{Cat}^{\mathcal{B} \times (-), \mathcal{C}^{\mathcal{F}}}_{\mathcal{B} \times \mathcal{C}^{\mathcal{F}}} \\
\downarrow \text{id}_{\text{Cat}^{\mathcal{B} \times (-), \mathcal{C}^{\mathcal{F}}}}
\end{array}
\end{align*}
\]

provides the universal left \(\mathcal{F}\)-module.

**The 1-dimensional aspect**

Suppose we have a left \(\mathcal{F}\)-module, i.e., a diagram

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \Gamma \\
\mathcal{B} \times (-) \\
\downarrow \text{id}_{\mathcal{B} \times (-)}
\end{array} \rightarrow \begin{array}{c}
\text{Cat} \\
\downarrow \text{id}\text{Cat}
\end{array} \\
\rightarrow \begin{array}{c}
\mathcal{C} \\
\downarrow \text{id}\text{C}
\end{array}
\]

in \(\mathcal{E}^{+-}\) satisfying

\[
\begin{align*}
\mathcal{X} & \xrightarrow{G} \text{Cat} \\
\mathcal{B} \times (-) & \xrightarrow{H^\mathcal{B}} \text{Cat} \\
\text{id}_{\text{Cat}} & = \mathcal{X} \xrightarrow{G} \text{Cat} \\
\end{align*}
\] (4.39)

\[
\begin{align*}
\mathcal{X} & \xrightarrow{g} \mathcal{C} \\
\mathcal{B} \times \mathcal{X} & \xrightarrow{B \times g} \mathcal{B} \times \mathcal{C} \\
\mathcal{C} & = \mathcal{X} \xrightarrow{g} \mathcal{C} \\
\end{align*}
\] (4.40)
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\[ \mathcal{B} \xrightarrow{G} \text{Cat} \quad \xrightarrow{\Gamma \times (\_ \, \_)} \quad \text{Cat} \]

\[ \mathcal{B} \xrightarrow{G} \text{Cat} \quad \xrightarrow{\Gamma} \quad \text{Cat} \]

Equations (4.41) and (4.42) imply the unique factorization

\[ \mathcal{B} \times \mathcal{B} \times \mathcal{G} \xrightarrow{\mathcal{B} \times \mathcal{B} \times g} \mathcal{B} \times \mathcal{B} \times \mathcal{C} \xrightarrow{\mathcal{B} \times \mathcal{S}} \mathcal{B} \times \mathcal{C} \]

\[ \Gamma_X \]

\[ \mathcal{B} \times \mathcal{G} \xrightarrow{\mathcal{B} \times \mathcal{S}} \mathcal{C} \]

since the co-Eilenberg–Moore 2-category \( \mathcal{C} \) is the co-Eilenberg–Moore object in \( 2 \)-\( \mathcal{C} \).

**Definition 4.55.** The 2-functor \( \tilde{G} : \mathcal{B} \rightarrow \mathcal{C} \) is the mediating 2-functor \( \mathcal{X}' \mapsto \mathcal{G} \).

It remains to construct a 1-cell \( \tilde{g} : \left( \mathcal{G} \xrightarrow{\Gamma X} \mathcal{B} \right) \rightarrow \left( \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C} \right) \) of \( \mathcal{C} \) which satisfies

\[ \mathcal{B} \xrightarrow{\mathcal{B} \times (\_ \, \_)} \mathcal{C} \]
and show its uniqueness. To describe the definition of this functor concisely, let us write $\Gamma_X x = (\Gamma_X, x) = (\Gamma x, x) \in \mathbb{B} \times GX$ in what follows.

**Definition 4.56.** The 1-cell $\tilde{g}: (\Gamma X, \mathbb{B} \times GX) \to (\mathcal{C}, \mathbb{B} \times \mathcal{C})$ of $\mathcal{C}at^{\mathbb{B} \times (-)}$ is the functor $\tilde{g}: GX \to \mathcal{C}$ defined as

$$
\tilde{g}(x) := \left( \Gamma x, \frac{\mathcal{F}_{x}^{gx}}{g x} \right)
$$

on an object $x$ and

$$
\tilde{g}(z) := \left( \Gamma x \xrightarrow{\gamma_z} \Gamma x', \frac{\mathcal{F}_{x}^{gx}}{g x} \xrightarrow{\gamma_z} \frac{\mathcal{F}_{x}^{gx'}}{g x'} \xrightarrow{\gamma_{z'}} \frac{\mathcal{F}_{x}^{gx'} x'}{g x' x'} \xrightarrow{\gamma_{z'} x} \frac{\mathcal{F}_{x}^{gx'} x'}{g x' x'} \right)
$$

on a morphism $z: x \to x'$.

**Proposition 4.57.** The functor $\tilde{g}$ defined above is indeed a 1-cell of $\mathcal{C}at^{\mathbb{B} \times (-)}$ which satisfies (4.43) and (4.44). Moreover, it is the unique such.

**Proof.** First, $\tilde{g}$ is a 1-cell of $\mathcal{C}at^{\mathbb{B} \times (-)}$ because

$$
(id_{\mathbb{B}} \times \tilde{g}) \Gamma_X x = (\Gamma x, \tilde{g} x) = (\Gamma x, (\Gamma x, \gamma_x)) = \pi \tilde{g} x
$$

on objects and

$$
(id_{\mathbb{B}} \times \tilde{g}) \Gamma_X z = (\Gamma z, \tilde{g} z) = (\Gamma z, (\Gamma z, g z)) = \pi \tilde{g} z
$$

on morphisms.

$\tilde{g}$ satisfies (4.43) since

$$
g x = u^{\mathcal{F}} \left( \Gamma x, \frac{\mathcal{F}_{x}^{gx}}{g x} \right) = u^{\mathcal{F}} \tilde{g} x
$$
on objects and
\[ gz = u^\mathcal{T} \left( \Gamma_x \xrightarrow{\Gamma_x} \Gamma_{x'}, \left( \frac{\mathcal{R}_{x,gx}}{\mathcal{R}_{x,gx}'} \right) \right) \frac{g_{\gamma}}{g_{\gamma'}} \]
\[ = u^\mathcal{T} \bar{g}z \]
on morphisms.
Finally, \( \bar{g} \) satisfies (4.44) since
\[ \gamma_x = u^\mathcal{T} (\text{id}_{\Gamma_x}, \gamma_x) = u^\mathcal{T} \varepsilon(\Gamma_x, \gamma_x) = u^\mathcal{T} \varepsilon\mathcal{T}_{\gamma_x}. \]

Let us now move on to the proof of uniqueness. The requirement that \( \bar{g} \) is a 1-cell of \( \mathcal{C}at^{B \times (-)} \) determines the first components of \( \bar{g}x \). (4.43) determines the underlying object of the algebra part of \( \bar{g}x \). Finally, (4.44) forces the structure map of the algebra part of \( \bar{g}x \) to be \( \gamma_x \), thus completely specifies the definition of \( \bar{g} \).
\[ \square \]

The 2-dimensional aspect

Suppose we have a morphism of left \( \mathcal{T} \)-modules, i.e., a 2-cell
\[ (\mathcal{X}, X) \xrightarrow{(G, g)} (\mathcal{C}at, \mathcal{C}) \]
of \( \mathcal{G}^{+} \) satisfying
\[ (4.45) \]
\[ (4.46) \]
Equation (4.45) implies the unique factorization

\[
\begin{array}{ccc}
\mathcal{X}^G & \overset{\Omega}{\longrightarrow} & \mathcal{C} \\
\downarrow G^g & & \downarrow u^B \\
\mathcal{X}^{G'} & \overset{\tilde{\Omega}}{\longrightarrow} & \mathcal{C}^{B \times (-)}
\end{array}
\]

where the 2-natural transformation \( \tilde{\Omega} \) is defined as follows:

**Definition 4.58.** The 2-natural transformation \( \tilde{\Omega} : \tilde{G} \Rightarrow \tilde{G} \) has, as components, 1-cells \( \tilde{\Omega}_X : (G^X)_{B \times G^X} \to (G^X)_{B \times G^X} \) of \( \mathcal{C}^{B \times (-)} \) given by \( \tilde{\Omega}_X := \Omega_X \) as functors. ■

Next we proceed to construct a 2-cell

\[
\begin{array}{c}
\tilde{\omega} : \begin{array}{c}
(G^X)_{B \times G^X} \\
\tilde{\Omega}_X
\end{array} \longrightarrow \begin{array}{c}
(G^X)_{B \times G^X} \\
\tilde{\omega}
\end{array}
\end{array}
\]

of \( \mathcal{C}^{B \times (-)} \) satisfying

\[
\begin{array}{ccc}
\mathcal{X}^{G'} & \overset{\omega}{\longrightarrow} & \mathcal{C} \\
\downarrow g' & & \downarrow u^\mathcal{C} \\
\mathcal{X}^{G} & \overset{\tilde{\omega}}{\longrightarrow} & \mathcal{C}^{B \times (-)}
\end{array}
\]

**Definition 4.59.** To define the 2-cell \( \tilde{\omega} \) of \( \mathcal{C}^{B \times (-)} \), with components

\[
\tilde{\omega}_x : \left( \Omega_{X,x} \right)_{\Gamma_{\mathcal{C}X}} \to \left( \Gamma'_{X,x} \right)_{\Gamma'_{\mathcal{C}X}},
\]

at \( x \in G'X \), first observe that by equation (4.45), \( \Omega_{X,x} = \Gamma'_{X,x} \). Now we define \( \tilde{\omega}_x \) to be

\[
\tilde{\omega}_x := \left( \Omega_{X,x} \xrightarrow{id} \Gamma'_{X,x}, \left( \Omega_{X,x} \xrightarrow{g\Omega_{X,x}} \Gamma'_{X,x} \right) \xrightarrow{\omega_x} \left( \Gamma'_{X,x} \xrightarrow{g'\omega_x} \Gamma'_{X,x} \right) \right).
\]

■

**Proposition 4.60.** The natural transformation \( \tilde{\omega} \) defined above is indeed a 2-cell of \( \mathcal{C}^{B \times (-)} \) which satisfies (4.47). Moreover, it is the unique such.

**Proof.** First, \( \tilde{\omega} \) is a 2-cell of \( \mathcal{C}^{B \times (-)} \), i.e., \( (\text{id}_B \times \tilde{\omega})_{\Gamma_{X,x}} = \pi \tilde{\omega}_x \) holds, since

\[
\begin{array}{l}
\left( \text{id}_B \times \tilde{\omega} \right)_{\Gamma_{X,x}} = \left( \Gamma'_{X,x} \xrightarrow{id} \Gamma'_{X,x}, \tilde{\omega}_x \right) \\
= \left( \Omega_{X,x} \xrightarrow{id} \Gamma'_{X,x}, \tilde{\omega}_x \right) \\
= \pi \tilde{\omega}_x.
\end{array}
\]
That $\tilde{\omega}$ satisfies (4.47) is also easy to see:

$$\omega_x = u^\mathcal{F} \left( \Gamma \Omega_x x \xrightarrow{\text{id}} \Gamma' x, \left( g \Gamma \Omega_x x \xrightarrow{\text{id}} g \Gamma' x \right) \omega_x \xrightarrow{\tilde{\omega}} \left( g' \Gamma \Omega_x x \xrightarrow{\text{id}} g' \Gamma' x \right) \right)$$

$$= u^\mathcal{F} \tilde{\omega}_x.$$ 

For the uniqueness, observe that the requirement that $\tilde{\omega}$ is a 2-cell of $\mathcal{C}at^{B \times (-)}$ forces the first component of $\tilde{\omega}_x$ to be the identity, and the requirement that $\tilde{\omega}$ satisfies (4.47) determines the second component.

**Theorem 4.61.** The object $\left( \mathcal{C}at^{B \times (-)}, \mathcal{C}at^\mathcal{F} \right)$ of $\mathcal{E}^+-\mathcal{E}$ is the Eilenberg–Moore object of the indexed monad $\mathcal{T}$, considered as a monad $(B \times (-), \mathcal{T})$ in $\mathcal{E}^+-\mathcal{E}$ on $(\mathcal{C}at, \mathcal{C})$.

### 4.4 Constructions for graded and indexed comonads

The constructions introduced so far duality to those for graded and indexed *comonads* rather straightforwardly. We briefly describe how the co-Eilenberg–Moore and co-Kleisli categories look like.

#### 4.4.1 Co-Eilenberg–Moore categories of graded comonads

Let us fix a strict monoidal category $M = (\mathcal{M}, \otimes, I)$, a category $\mathcal{C}$, and an $\mathcal{M}$-graded comonad $S$ on $\mathcal{C}$. We write the functor part of $S$ also as $\ast: \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$ and use the infix notation. As observed in Section 3.3.5, graded comonads can be seen as comonads in the 2-category $\mathcal{E}^+-\mathcal{E}$; the co-Eilenberg–Moore construction is performed inside $\mathcal{E}^+-\mathcal{E}$.

**Definition 4.62.** Define the category $\mathcal{C}^S$ as follows:

- An object of $\mathcal{C}^S$ is a *graded $S$-coalgebra*, i.e., a pair $(A, h)$ where $A: \mathcal{M} \rightarrow \mathcal{C}$ is a functor and $h$ is a natural transformation of type

$$\mathcal{M} \times \mathcal{M} \xrightarrow{\mathcal{M} \times A} \mathcal{M} \times \mathcal{C}$$

$$\otimes \xrightarrow{h} S \xrightarrow{A} \mathcal{C}$$

So the component of $h$ at $(m, n) \in \mathcal{M} \times \mathcal{M}$ is of type

$$h_{m,n} : A_{m \otimes n} \rightarrow m \ast A_n.$$ 

These data are subject to the following axioms:

- $A_n \xrightarrow{h_{I,n}} I \ast A_n$
- $A_n \xrightarrow{\text{id}_{A_n}} A_n \xrightarrow{\varepsilon A_n} A_n$ commutes for each object $n$ of $\mathcal{M}$.
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\[ A \otimes m \otimes n \xrightarrow{h_{l,m,n}} l \ast A \otimes m \otimes n \]
\[ (l \otimes m) \ast A \otimes n \delta_{l,m,A} \xrightarrow{(l \ast m) \ast A} l \ast m \ast A \]
commutes for each triple of objects \( l, m, n \) of \( M \).

- A morphism of \( C^S \) from \((A, h)\) to \((A', h')\) is a homomorphism of graded \( S \)-coalgebras between them, i.e., a natural transformation \( \varphi: A \Rightarrow A' \) making the diagram

\[
\begin{array}{ccc}
A \otimes n & \xrightarrow{\varphi \otimes n} & A' \otimes n \\
\downarrow h_{m,n} & & \downarrow h'_{m,n} \\
m \ast A & \xrightarrow{m \ast \varphi_n} & m \ast A'
\end{array}
\]
commute for each pair of objects \( m, n \) of \( M \).

**Definition 4.63.** Define the functor \( \otimes : M \times C^S \rightarrow C^S \) as follows:

- Given objects \( p \) and \((A, h)\) of \( M \) and \( C^S \) respectively, we define the graded \( S \)-algebra \( p \otimes (A, h) \) by the precomposition of \((-) \otimes p: M \rightarrow M\): \( p \otimes (A, h) := ((A_n \otimes p)_{n \in M}, (h_{m,n} \otimes p)_{m, n \in M}) \).

- Given morphisms \( u: p \rightarrow p' \) and \( \varphi: (A, h) \rightarrow (A', h') \) of \( M \) and \( C^S \) respectively, we define the homomorphism \( u \otimes \varphi: p \otimes (A, h) \rightarrow p' \otimes (A', h') \) by setting the component \((u \otimes \varphi)_n: A_n \otimes p \rightarrow A'_n \otimes p'\) to be either of the following two equivalent composites:

\[
\begin{array}{ccc}
A_n \otimes p & \xrightarrow{\varphi_n \otimes p} & A'_n \otimes p \\
\downarrow A_n \otimes u & & \downarrow A'_n \otimes u \\
A_n \otimes p' & \xrightarrow{\varphi_n \otimes p'} & A'_n \otimes p'
\end{array}
\]

**Theorem 4.64.** The object \( (\text{Cat} \overset{(-)}{\times} (M, \otimes, I), \otimes) \) of \( \mathcal{E}^{++} \) is the co-Eilenberg–Moore object of the graded comonad \( S \), considered as a comonad \((M \times (-), S)\) in \( \mathcal{E}^{+-} \) on \( (\text{Cat}, C) \).

Recall that a \((M, \otimes, I)\)-graded comonad on \( C \) is the same thing as a \((M^{\text{op}}, \otimes, I)\)-graded monad on \( C^{\text{op}} \). Let \( S \) denote the \((M^{\text{op}}, \otimes, I)\)-graded monad on \( C^{\text{op}} \) corresponding to \( S \). Now, Eilenberg–Moore categories for graded monads and co-Eilenberg–Moore categories for graded comonads are related to each other in the following way:

\[ (C^{\text{op}})^S \cong (C^S)^{\text{op}}. \]

Actually, one can say more: the canonical \( M^{\text{op}} \times (-) \)-algebra structure \( \otimes: M^{\text{op}} \times (C^{\text{op}})^S \rightarrow (C^{\text{op}})^S \) on \( (C^{\text{op}})^S \) corresponds to the canonical \( M \times (-) \)-algebra structure \( \otimes: M \times C^S \rightarrow C^S \) on \( C^S \) via this dualization: \( \otimes \cong \otimes^{\text{op}} \).
4.4.2 Co-Kleisli categories of graded comonads

Again we fix a strict monoidal category \( \mathcal{M} = (\mathcal{M}, \otimes, I) \), a category \( \mathcal{C} \), and an \( \mathcal{M} \)-graded comonad \( \mathcal{S} \) on \( \mathcal{C} \); we continue to write the functor part of \( \mathcal{S} \) also as \( *: \mathcal{M} \times \mathcal{C} \to \mathcal{C} \) and use the infix notation. For the co-Kleisli construction for graded comonads, we use another observation in Section 3.3.5 that graded comonads can also be seen as comonads in the 2-category \( \mathcal{E}^{++} \).

**Definition 4.65.** Define the category \( \mathcal{C}_\mathcal{S} \) as follows:

- An object of \( \mathcal{C}_\mathcal{S} \) is a pair \((m, c)\) where \( m \) and \( c \) are objects of \( \mathcal{M} \) and \( \mathcal{C} \) respectively.
- The set of morphisms from \((m, c)\) to \((m', c')\) is defined by the coend formula

\[
\mathcal{C}_\mathcal{S}((m, c), (m', c')) := \int_{n \in \mathcal{M}} \mathcal{M}(m, m' \otimes n) \times \mathcal{C}(n* c, c').
\]

Explicitly, a morphism \((m, c) \to (m', c')\) is an equivalence class \([n, m \overset{v}{\to} m' \otimes n, n* c \overset{f}{\to} c']\) of tuples consisting of an object \( n \in \mathcal{M} \) and morphisms \( v, f \), where the equivalence relation is generated by

\[
(n, m \overset{v}{\to} m' \otimes n, n* c \overset{f}{\to} c') \sim (n', m \overset{v}{\to} m' \otimes n, m' \otimes n', n' * c \overset{f}{\to} c')
\]

for each morphism \( w: n \to n' \) of \( \mathcal{M} \).

- The identity morphism on \((m, c)\) is given by \([I, m \overset{id_m}{\to} m \otimes I, I* c \overset{\varepsilon}{\to} c]\).
- For two composable morphisms

\[
[n, m \overset{v}{\to} m' \otimes n, n* c \overset{f}{\to} c'] : (m, c) \to (m', c'),
\]

\[
[n', m' \overset{v'}{\to} m'' \otimes n', n' * c' \overset{f'}{\to} c''] : (m', c') \to (m'', c''),
\]

their composite is given by

\[
[n' \otimes n, m \overset{v}{\to} m' \otimes n, n' * c \overset{f}{\to} c'] : (m, c) \to (m'', c'').
\]

\[
\left( n' \otimes n \right) \overset{\delta_{n', n} \cdot c}{\to} n' * n * c \overset{n' * f}{\to} n' * c' \overset{f'}{\to} c' \right)
\]

\[
: (m, c) \to (m'', c'').
\]

Note that the first component of the composite morphism defined above is \( n' \otimes n \), rather than \( n \otimes n' \).

**Definition 4.66.** Define the functor

\[
\bigodot: \mathcal{C}_\mathcal{S} \to [\mathcal{M}, \mathcal{C}_\mathcal{S}]
\]

as follows: note that we will use the infix notation \( l \bigodot (m, c) \) to denote the value of the functor \( \bigodot(m, c): \mathcal{M} \to \mathcal{C}_\mathcal{S} \) applied to \( l \in \mathcal{M} \) and similarly for morphisms.

- Given objects \( l \) and \((m, c)\) of \( \mathcal{M} \) and \( \mathcal{C}_\mathcal{S} \) respectively, we define \( l \bigodot (m, c) := (l \otimes m, c). \)
• Given morphisms $u: l \to l'$ and $[n, m \xrightarrow{v} m' \otimes n, n \star c \xrightarrow{f} c'] : (m, c) \to (m', c')$ of $\mathcal{M}$ and $\mathcal{C}_S$ respectively, we define

$$u \odot [n, m \xrightarrow{v} m' \otimes n, n \star c \xrightarrow{f} c'] := [n, l \otimes m \xrightarrow{u \odot v} l' \otimes m' \otimes n, n \star c \xrightarrow{f} c'] : (l \otimes m, c) \to (l' \otimes m', c').$$



**Theorem 4.67.** The object $(\mathcal{C}at^{[\mathcal{M}, -]}, \mathcal{C}_S \downarrow \odot \mathcal{M, C}_S)$ of $\mathcal{E}^+ \mathcal{M}$ is the co-Kleisli object of the graded comonad $S$, considered as a comonad $([\mathcal{M}, -], S)$ in $\mathcal{E}^+ \mathcal{M}$ on $(\mathcal{C}at, \mathcal{C})$.

The relation to the Kleisli construction for graded monads is again given by $(\mathcal{C}^{op})_S \cong (\mathcal{C}_S)^{op}$ and $\odot \cong \circ^{op}$, where $\odot : (\mathcal{C}^{op})_S \to [\mathcal{M}^{op}, (\mathcal{C}^{op})_S]$ and $\circ : \mathcal{C}_S \to [\mathcal{M}, \mathcal{C}_S]$.

### 4.4.3 Co-Eilenberg–Moore categories of indexed comonads

Let us fix categories $\mathcal{B}, \mathcal{C}$, and a $\mathcal{B}$-indexed comonad $\mathcal{J}$ on $\mathcal{C}$. Based on the observation in Section 3.3.5 that indexed comonads can be considered as comonads in the 2-category $\mathcal{E}^{++}$, we construct the co-Eilenberg–Moore category of $\mathcal{J}$ in $\mathcal{E}^{++}$.

**Definition 4.68.** Define the category $\mathcal{C}_\mathcal{J}$ as follows:

- An object of $\mathcal{C}_\mathcal{J}$ is a pair $(b, \chi_b)$, or more concisely $(b, \chi)$, where $b$ is an object of $\mathcal{B}$ and $(b, \chi_b)$ is a $\mathcal{J}_b$-coalgebra (an object of $\mathcal{C}_\mathcal{J}$).

- A morphism from $(b, \chi_b)$ to $(b', \chi_{b'})$ is a pair $(u, h)$ where $u : b \to b'$ is a morphism of $\mathcal{B}$ and $h : (b, \chi_b) \to (b', \chi_{b'})$ is a homomorphism of $\mathcal{J}_b$-coalgebras, i.e., a morphism $h : c \to c'$ in $\mathcal{C}$ which makes the diagram

$$
\begin{array}{ccc}
\mathcal{J}_b c & \xrightarrow{h} & \mathcal{J}_b c' \\
\downarrow \chi & & \downarrow \chi' \\
\mathcal{J}_{u,c} & \xrightarrow{\chi_{u,c} \circ \chi} & \mathcal{J}_{u,c}'
\end{array}
$$

commute.

- The identity morphism on $(b, \chi_b)$ is given by $(id_b, id_{\chi})$.

- For two composable morphisms

$$
\left( b \xrightarrow{u'} b', \left(c, \chi_{u,c} \circ \chi\right) \xrightarrow{h} \left(c', \chi'_{u',c'}\right) \right) : (b, \chi) \to (b', \chi'),
$$

$$
\left( b' \xrightarrow{u''} b'', \left(c', \chi'_{u',c'} \circ \chi'\right) \xrightarrow{h'} \left(c'', \chi''_{u'',c''}\right) \right) : (b', \chi') \to (b'', \chi''),
$$

where $u = u'' \circ u'$, $b = b''$, $c = c''$, $\chi = \chi''_{u'',c''} \circ \chi'_{u',c'} \circ \chi$, and $h = h' \circ h'' \circ (id_b, id_{\chi})$. 


their composite is given by
\[
\begin{align*}
(b, u) &\mapsto b' \xrightarrow{u'} b''', \\
\left( \frac{c}{\mathcal{A}^{\text{op}}} \circ u \circ c \chi \right) &\mapsto \left( \frac{c'}{\mathcal{A}^{\text{op}}} \circ u' \circ c' \chi' \right) \\
\left( \frac{c''}{\mathcal{A}^{\text{op}}} \chi'' \right) &\mapsto \left( \frac{c'''}{\mathcal{A}^{\text{op}}} \chi''' \right)
\end{align*}
\]
\[\vdash (b, \chi) \rightarrow (b'', \chi').\]

**Definition 4.69.** Define the functor \( \pi: C^\mathcal{A} \rightarrow \mathbb{B} \times C^\mathcal{A} \) by \((b, \chi) \mapsto (b, (b, \chi))\) and \((u, h) \mapsto (u, (u, h))\).

**Theorem 4.70.** The object \( \left( \mathcal{C}^\mathcal{A}, \frac{C^\mathcal{A}}{\mathcal{B} \times C^\mathcal{A}} \right) \) of \( \mathcal{E}^{++} \) is the co-Eilenberg–Moore object of the indexed comonad \( \mathcal{I} \), considered as a comonad \((\mathbb{B} \times (-), \mathcal{I})\) in \( \mathcal{E}^{++} \) on \((\mathcal{C}^\mathcal{A}, C)\).

A \( \mathbb{B} \)-indexed comonad on \( C \) corresponds to a \( \mathbb{B}^{\text{op}} \)-indexed monad on \( C^{\text{op}} \); let \( \mathcal{I} \) be the \( \mathbb{B}^{\text{op}} \)-indexed monad on \( C^{\text{op}} \) corresponding to \( \mathcal{I} \). The relationship of the Eilenberg–Moore construction for indexed monads and the co-Eilenberg–Moore construction for indexed comonads is given by \( (C^{\text{op}})^{\mathcal{I}} \cong (C^\mathcal{A})^{\text{op}} \) and \( \pi \cong \pi^{\text{op}} \), where \( \pi: (C^{\text{op}})^{\mathcal{I}} \rightarrow \mathbb{B}^{\text{op}} \times (C^{\text{op}})^{\mathcal{I}} \) and \( \mathcal{I}: C^\mathcal{A} \rightarrow \mathbb{B} \times C^\mathcal{A} \).

Recall the isomorphism of 2-categories \( \mathcal{C} \mathcal{A}^{\mathbb{B} \times (-)} \cong \mathcal{C} \mathcal{A}/\mathbb{B} \). In contrast to the phenomenon that Eilenberg–Moore categories for \( \mathbb{B} \)-indexed monads give rise to Grothendieck fibrations over \( \mathbb{B} \) under this isomorphism, co-Eilenberg–Moore categories for \( \mathbb{B} \)-indexed comonads give rise to Grothendieck opfibrations over \( \mathbb{B} \).

**Notes**

The definition of the Eilenberg–Moore category of a graded monad has been suggested to me independently by Shin-ya Katsumata and by Paul-André Melliès. I learned the definition of the Kleisli category of a graded monad, together with Lemma 4.36, from Shin-ya Katsumata. After learning these definition, I formulated and proved the 2-dimensional universality of the Eilenberg–Moore and Kleisli categories of a graded monad (Theorems 4.20 and 4.41). The contents of Sections 4.1 and 4.2 are included in the paper [7].

I defined the Eilenberg–Moore category of an indexed monad, and proved its universality (Theorem 4.61). The construction turned out to be essentially the same as the one that appears in [25] (but not as the Eilenberg–Moore category).
Chapter 5

Applications of the constructions

In this chapter, we present two applications of the constructions presented in the previous chapter. The first one is discussed in Section 5.1 and is an application of the Eilenberg–Moore and Kleisli constructions for graded monads; we shall show that they can be understood as giving ways to decompose lax monoidal actions into strict monoidal actions and adjunctions. Then, in Section 5.2, we see how the other construction, the Eilenberg–Moore construction for indexed monads, sheds new light to the previous work of Power [35, 34] and Maillard–Melliès [25], by revealing the related notions which have been implicit in their work.

5.1 Decomposition of lax monoidal actions

Let us fix a strict monoidal category \( M = (M, \otimes, I) \) throughout this section. For \( C \) a category, an \( M \)-graded monad on \( C \) is equivalent to the notion known as lax action of \( M \) on \( C \), in the sense of lax algebras for 2-monads [5]. On the other hand, objects of the 2-category \( \text{Cat}^{M \times (-)} \cong \text{Cat}^{[M, -]} \) are naturally thought of as categories equipped with strict actions of \( M \). In this section, we will explain that the Eilenberg–Moore and Kleisli constructions for graded monads developed in Sections 4.1 and 4.2 can also be understood as a result relating these different types of actions of a monoidal category; in a certain sense, these constructions show that we can always reduce the general notion of lax action to the more restrictive notion of strict action.

5.1.1 Lax and strict actions

Let us first fix the definitions of strict and lax actions.

**Definition 5.1.** Let \( A \) be a category. A strict \( M \)-action on \( A \) is a functor

\[
\otimes : M \times A \rightarrow A
\]

satisfying the equalities

\[
a = I \otimes a, \quad m \otimes n \otimes a = (m \otimes n) \otimes a.
\]

A category \( A \) equipped with a strict \( M \)-action \( \otimes \) on it is a strict \( M \)-category \( \left( M \times A \atop I \otimes A \right) \).
**Definition 5.2.** Let $C$ be a category. A **lax $M$-action on $A$** is a functor

$$ * : M \times C \to C,$$

together with a family of morphisms

$$c \xrightarrow{\eta_c} I \ast c, \quad m \ast n \ast c \xrightarrow{\mu_{m,n,c}} (m \otimes n) \ast c$$

satisfying the suitable coherence axioms corresponding to those for graded monads. ■

Therefore the notion of lax action can be obtained by relaxing that of strict action, by systematically replacing equalities with morphisms, which in turn are subject to new coherence axioms; cf. [1]. Now, as a general phenomenon in the 2-monad theory, we have the following proposition.

**Proposition 5.3.** Let $\left( \begin{array}{c} M \\ \otimes \end{array} \right)$ be a strict $M$-category, $C$ a category, and $\triangleright$ an adjunction between the category $C$ and the underlying category $A$ of $\left( \begin{array}{c} M \\ \otimes \end{array} \right)$. Then, the composite functor

$$M \times C \xrightarrow{M \times l} M \times A \xrightarrow{\otimes} A \xrightarrow{r} C$$

naturally has a structure of lax $M$-action on $C$.

This leads us to the notion of **resolution** of a lax monoidal action.

### 5.1.2 Resolutions

**Definition 5.4.** For a lax $M$-action $* = (\ast, \eta, \mu)$ on a category $C$, define the category $\text{Res}(*)$ as follows.

- An object of $\text{Res}(*)$ is a **resolution** of $\ast$, which is given by the following data:
  - A strict $M$-category $\left( \begin{array}{c} M \\ \otimes \end{array} \right)$.
  - An adjunction

$$\begin{array}{ccc}
A & \overset{l}{\leftarrow} & C \\
\downarrow{t} & & \downarrow{r} \\
A & \overset{r}{\leftarrow} & C
\end{array}$$

These data must satisfy the condition that, by the procedure of Proposition 5.3, they yield $(\ast, \eta, \mu)$. 
Suppose we have two resolutions of $\ast$:

\[
\rho = \left( \begin{array}{c}
M \times A \\
\downarrow \Theta \\
A
\end{array}, \; l \dashv r : A \to C \right), \\
\rho' = \left( \begin{array}{c}
M \times A' \\
\downarrow \Theta' \\
A'
\end{array}, \; l' \dashv r' : A' \to C \right).
\]

A morphism of $\text{Res}(\ast)$ from $\rho$ to $\rho'$ is a morphism of strict $M$-actions

\[
k : \left( \begin{array}{c}
M \times A \\
\downarrow \Theta \\
A
\end{array} \right) \longrightarrow \left( \begin{array}{c}
M \times A' \\
\downarrow \Theta' \\
A'
\end{array} \right),
\]

i.e., a functor $k : A \to A'$ making the diagram commute, satisfying the following equations:

\[
k \circ l = l', \\
r = r' \circ k.
\]

A natural question to ask at this point is: given an arbitrary lax $M$-action $\ast$, does there exist a resolution of $\ast$? This problem generalizes the one of finding an adjunction that generates an arbitrary monad (replace $M$ by 1), of which there are two solutions obtained way back in 1960’s, one by Eilenberg–Moore [6] and one by Kleisli [21]. Actually, our constructions of Eilenberg–Moore and Kleisli categories of graded monads, which generalize constructions in [6] and [21] respectively, provide answers to this generalized problem as well.

### 5.1.3 The fibrational correspondence of adjunctions

In order to solve the problem of finding resolutions of the lax $M$-action $\ast$ via the Eilenberg–Moore and Kleisli constructions for graded monads, we need the following 2-fibrational property [14] of the 2-functor $p^{++} : \mathcal{D}^{++} \to 2\text{-}\mathcal{C}at_2$ concerning the correspondence of adjunctions in the total 2-category and those in a fiber 2-category:

**Proposition 5.5.** Let $\mathcal{C}$ and $\mathcal{A}$ be 2-categories, $L \dashv R : \mathcal{A} \to \mathcal{C}$ be a 2-adjunction (adjunction in 2-$\mathcal{C}at$), and $C$ and $A$ be objects of $\mathcal{C}$ and $\mathcal{A}$ respectively. Then there is a bijective correspondence between the following two notions:
• Adjunctions in $\mathcal{E}^{++}$ between $(\mathcal{C}, C)$ and $(\mathcal{A}, A)$ above $L \dashv R$:

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$(\mathcal{C}, C)$};
  \node (b) at (3,0) {$(\mathcal{A}, A)$};
  \node (c) at (1.5,1.5) {$(\mathcal{C}, C)$};
  \node (d) at (1.5,-1.5) {$(\mathcal{A}, A)$};
  \draw[->] (a) to node {(L,l')} (b);
  \draw[->] (b) to node {(R,r)} (a);
  \draw[->] (c) to node {\text{Adjunctions in $\mathcal{C}$ between $C$ and RA:}} (d);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\mathcal{C}$};
  \node (b) at (3,0) {$\mathcal{A}$};
  \node (c) at (1.5,1.5) {$(\mathcal{C}, C)$};
  \node (d) at (1.5,-1.5) {$(\mathcal{A}, A)$};
  \draw[->] (a) to node {L} (b);
  \draw[->] (b) to node {R} (a);
\end{tikzpicture}
\end{center}

• Adjunctions in $\mathcal{C}$ between $C$ and RA:

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$(\mathcal{C}, RA)$};
  \node (b) at (3,0) {$(\mathcal{C}, C)$};
  \node (c) at (1.5,1.5) {$\mathcal{C}$};
  \node (d) at (1.5,-1.5) {$\mathcal{C}$};
  \draw[->] (a) to node {(id$_\mathcal{C}$, l)} (b);
  \draw[->] (b) to node {(id$_\mathcal{C}$, r)} (a);
  \draw[->] (c) to node {\text{2-Cat}} (d);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\mathcal{C}$};
  \node (b) at (3,0) {$\mathcal{A}$};
  \node (c) at (1.5,1.5) {$\mathcal{C}$};
  \node (d) at (1.5,-1.5) {$\mathcal{C}$};
  \draw[->] (a) to node {L} (b);
  \draw[->] (b) to node {R} (a);
\end{tikzpicture}
\end{center}

\textbf{Proof.} Indeed, the former notion is given by the following data:

• A 1-cell $l'$: $LC \to A$ of $\mathcal{A}$.

• A 1-cell $r$: $RA \to C$ of $\mathcal{C}$.

• A 2-cell $\eta$ of $\mathcal{C}$ of the following type:

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$C$};
  \node (b) at (3,0) {$C$};
  \node (c) at (1.5,1.5) {$C$};
  \node (d) at (1.5,-1.5) {$C$};
  \draw[->] (a) to node {$\text{id}_C$} (b);
  \draw[->] (b) to node {$\eta$} (a);
  \draw[->] (c) to node {$H_C$} (d);
  \draw[->] (d) to node {$Rl'$} (c);
\end{tikzpicture}
\end{center}
5.1. DECOMPOSITION OF LAX MONOIDAL ACTIONS

• A 2-cell \( \varepsilon' \) of \( \mathcal{A} \) of the following type:

\[
\begin{array}{ccc}
LRA & \xrightarrow{Lr} & LC \\
\downarrow \varepsilon' & & \downarrow l' \\
A & \xrightarrow{id_A} & A
\end{array}
\]

Here, \( H \) and \( E \) are the unit and counit of the 2-adjunction \( L \dashv R \) respectively.

On the other hand, the latter notion is given by the following data:

• A 1-cell \( l: C \to RA \) of \( \mathcal{C} \).

• A 1-cell \( r: RA \to C \) of \( \mathcal{C} \).

• A 2-cell \( \eta \) of \( \mathcal{C} \) of the following type:

\[
\begin{array}{ccc}
C & \xrightarrow{id_C} & C \\
\downarrow l & \downarrow \eta & \downarrow r \\
RA & \xrightarrow{} & C
\end{array}
\]

• A 2-cell \( \varepsilon \) of \( \mathcal{C} \) of the following type:

\[
\begin{array}{ccc}
RA & \xrightarrow{\varepsilon} & RA \\
\downarrow id_{RA} & & \downarrow id_{RA} \\
RA & \xrightarrow{} & RA
\end{array}
\]

Now the correspondence should be clear; \( l' \) and \( \varepsilon' \) correspond respectively to \( l \) and \( \varepsilon \) under the 2-adjunction \( L \dashv R \). That this correspondence preserves and reflects the triangular identity is straightforward to check. \( \square \)

We also have an analogous result for \( p^{--}: \mathcal{C}^{op(1,2)} \to 2\text{-}\mathcal{Cat}_{2}^{op(1,2)} \):

**Proposition 5.6.** Let \( \mathcal{C} \) and \( \mathcal{A} \) be 2-categories, \( L \dashv R: \mathcal{C} \to \mathcal{A} \) be a 2-adjunction (adjunction in 2-\( \mathcal{Cat} \)), and \( C \) and \( A \) be objects of \( \mathcal{C} \) and \( \mathcal{A} \) respectively. Then there is a bijective correspondence between the following two notions:
• Adjunctions in $\mathcal{E}^{--}$ between $(\mathcal{C}, C)$ and $(\mathcal{A}, A)$ above $L \dashv R$

\[
\begin{array}{c}
\mathcal{C}
\end{array}
\begin{array}{c}
\perp
\end{array}
\begin{array}{c}
\mathcal{A}
\end{array}
\begin{array}{c}
\mathcal{C}
\end{array}
\begin{array}{c}
\perp
\end{array}
\begin{array}{c}
\mathcal{A}
\end{array}
\begin{array}{c}
\mathcal{E}^{--}
\end{array}
\]

• Adjunctions in $\mathcal{E}$ between $C$ and $LA$:

\[
\begin{array}{c}
\mathcal{C}
\end{array}
\begin{array}{c}
\perp
\end{array}
\begin{array}{c}
\mathcal{A}
\end{array}
\begin{array}{c}
\mathcal{C}
\end{array}
\begin{array}{c}
\perp
\end{array}
\begin{array}{c}
\mathcal{A}
\end{array}
\begin{array}{c}
\mathcal{E}^{--}
\end{array}
\]

5.1.4 Existence of the terminal and initial resolutions

Now we are ready to connect the Eilenberg–Moore and Kleisli constructions for graded monads to the notion of resolutions of a lax action. Let us fix an $M$-graded monad $\mathbf{T}$, or equivalently a lax $M$-action $\ast$, on a category $\mathcal{C}$.

We begin with the case of the Eilenberg–Moore construction. Applying Proposition 5.5 to the free-forgetful 2-adjunction

\[F^M \vdash U^M : \mathcal{Cat}^{M \times (-)} \longrightarrow \mathcal{Cat},\]

and objects $\mathcal{C}$ and $\left(\frac{M \times \mathcal{C}^T}{\downarrow \otimes \mathcal{C}^T}\right)$ of $\mathcal{Cat}$ and $\mathcal{Cat}^{M \times (-)}$ respectively, we obtain from the Eilenberg–Moore adjunction for $\mathbf{T}$

\[
\begin{array}{c}
\mathcal{Cat}, \mathcal{C}
\end{array}
\begin{array}{c}
\perp
\end{array}
\begin{array}{c}
\mathcal{Cat}^{M \times (-)}, \frac{M \times \mathcal{C}^T}{\downarrow \otimes \mathcal{C}^T}
\end{array}
\begin{array}{c}
\mathcal{Cat}, \mathcal{C}
\end{array}
\begin{array}{c}
\perp
\end{array}
\begin{array}{c}
\mathcal{Cat}^{M \times (-)}, \frac{M \times \mathcal{C}^T}{\downarrow \otimes \mathcal{C}^T}
\end{array}
\begin{array}{c}
\mathcal{E}^{--}
\end{array}
\]
5.1. DECOMPOSITION OF LAX MONOIDAL ACTIONS

the following resolution:

**Definition 5.7.** The Eilenberg–Moore resolution for the lax $M$-action $*$ is given by the strict $M$-category $(\mathcal{M} \times \mathcal{C}_T) \downarrow \mathcal{C}_T$ and the adjunction

\[
f_T \circ H_{\mathcal{M}}^\mathcal{C}_T \quad \dashv \quad \mathcal{C}_T \quad \dashv \quad \mathcal{C}_T \quad \downarrow \quad u_T
\]

between categories $\mathcal{C}$ and $\mathcal{C}_T$. ■

Similarly, using Proposition 5.6 to the forgetful-cofree 2-adjunction

\[
F_M \dashv U_M : \mathcal{Cat} \rightarrow \mathcal{Cat}^{[M,-]},
\]

and objects $\mathcal{C}$ and $(\mathcal{C}_T \downarrow [M, \mathcal{C}_T])$ of $\mathcal{Cat}$ and $\mathcal{Cat}^{[M,-]}$ respectively, it follows that the Kleisli adjunction for $T$

\[
(F_M, f_T) \quad \dashv \quad \left(\mathcal{Cat}, \mathcal{C}_T \downarrow [M, \mathcal{C}_T]\right)
\]

gives rise to:

**Definition 5.8.** The Kleisli resolution for the lax $M$-action $*$ is given by the strict $M$-category $(\mathcal{C}_T \downarrow [M, \mathcal{C}_T])$ and the adjunction

\[
f_T \quad \dashv \quad \mathcal{C}_T
\]

between categories $\mathcal{C}$ and $\mathcal{C}_T$. ■

Moreover, as an easy corollary of the comparison theorems (Propositions 4.13 and 4.33), we may conclude:

**Theorem 5.9.** The category $\text{Res}(*)$ has both the terminal and initial objects, given respectively by the Eilenberg–Moore and Kleisli resolutions.
5.2 A construction of Maillard and Melliès

Maillard and Melliès [25] introduced indexed monads, which are actually somewhat more general than what we call indexed monads here. They also introduced a construction which produces a 2-fibration over a 2-category $\mathcal{B}$ for each $\mathcal{B}$-indexed monad in their sense. Interestingly, our construction of the Eilenberg–Moore categories of indexed monads turns out to constitute particular instances of their construction. In this chapter, we see how the result of [25] connecting their construction to the notion of model of an indexed Lawvere theory introduced by Power [34, 35], can be understood in the light of notions related to the Eilenberg–Moore construction for indexed monads.

5.2.1 Indexed Lawvere theories and their models

In his investigation of the relationship between the global state monad and the local state monad [32], Power [35, 34] introduced the notion of indexed Lawvere theory, whose definition is recalled below.

**Definition 5.10.** Let $\mathcal{B}$ be a category. A $\mathcal{B}$-indexed Lawvere theory is a functor

$$ \mathcal{L} : \mathcal{B} \to \text{Law}. $$

Here, Law is the category consisting of Lawvere theories and maps of them; see [16] for the detailed definition. Thanks to the well-known correspondence of (ordinary) Lawvere theories and finitary monads on Set [33], i.e., the inclusion functor $\iota : \text{Law} \to \text{Mnd}(\text{Set})^\text{op}$, it turns out that every $\mathcal{B}$-indexed Lawvere theory defines a $\mathcal{B}$-indexed monad on Set (in our sense) by postcomposing $\iota$, as observed in [25].

Power also defined models of an indexed Lawvere theory, generalizing the classical notion of model of a Lawvere theory.

**Definition 5.11.** Let $\mathcal{B}$ be a category, $\mathcal{L}$ a $\mathcal{B}$-indexed Lawvere theory, and $\mathcal{C}$ a category with finite products. Define the category $\text{Mod}(\mathcal{L}, \mathcal{C})$ as follows:

- An object of $\text{Mod}(\mathcal{L}, \mathcal{C})$ is a model $M$ of $\mathcal{L}$ in $\mathcal{C}$. It consists of the following data:
  - For each object $b$ of $\mathcal{B}$, a model $M_b$ of the Lawvere theory $\mathcal{L}_b$ in $\mathcal{C}$, i.e., a finite product preserving functor $M_b : \mathcal{L}_b \to \mathcal{C}$.
  - For each morphism $u : b \to b'$ of $\mathcal{B}$, a natural transformation of type

$$
\begin{array}{ccc}
\mathcal{L}_b & \xrightarrow{M_b} & \mathcal{C} \\
\downarrow M_u & & \downarrow M_u \\
\mathcal{L}_{b'} & \xrightarrow{M_{b'}} & \mathcal{C}
\end{array}
$$

These data are subject to the following axioms:

- $M_{\text{id}_b} = \text{id}_{M_b}$ for each object $b$ of $\mathcal{B}$.
5.2. A CONSTRUCTION OF MAILLARD AND MELLIÈS

5.2.2 The base change adjoint triple

Recall from Section 4.3 that given a $B$-indexed monad $T$ on the category $C$, one may find its Eilenberg–Moore object by considering $T$ as a monad in the 2-category $\mathcal{C}^{\mathcal{B}}$, and is given as $(\mathcal{C}^{\mathcal{B} \times (-)}, \mathcal{C}^{\mathcal{B} \times (-)}, \Pi) \in \mathcal{C}^{\mathcal{B} \times (-)}$. As remarked in Section 4.3, the 2-category $\mathcal{C}^{\mathcal{B} \times (-)}$ is isomorphic to the slice 2-category $\mathcal{C}^{B/\mathcal{B}}$, by the 2-functor $(\mathcal{A}, \mathcal{A}) \mapsto \mathcal{A}$. Let us now remember the base change adjoint triple

\[
\mathcal{C}^{B/\mathcal{B}} \xleftarrow{\Pi} \mathcal{C} \xrightarrow{\Pi} \mathcal{C}^{B/\mathcal{B}}
\]

connecting the 2-categories $\mathcal{C}$ and $\mathcal{C}^{B/\mathcal{B}}$. This notion allows us to nicely describe the relation between the category of models and the Eilenberg–Moore category; before recalling the precise definition of the adjoint triple, we state our main theorem in the current section:

**Theorem 5.12.** Let $B$ be a category and $L$ a $B$-indexed Lawvere theory; note that $L$ determines a $B$-indexed monad $\iota L$ on $\mathbf{Set}$. There is an equivalence of categories between $\text{Mod}(\mathcal{L}, \mathbf{Set})$ and $\prod_B \left( \frac{\mathbf{Set}}{B} \right)$. 

One of the most striking results on the classical correspondence between Lawvere theories $L$ and finitary monads $T_L = \iota(L)$ on $\mathbf{Set}$ says that there is an equivalence of categories between $\text{Mod}(L, \mathbf{Set})$ and $\mathbf{Set} T_L$. Now we claim that this intimate relation between the category of models of $L$ in $\mathbf{Set}$ and the Eilenberg–Moore category of $T_L$ generalizes to the indexed setting, in a somewhat nontrivial manner; this is exactly what we intend to show in the current section.
Definition 5.13. The 2-functor $\prod_B : \text{Cat}/B \to \text{Cat}$ is, up to the isomorphism $\text{Cat}^B \cong \text{Cat}/B$, the forgetful 2-functor $U^B : \text{Cat}^{B \times (-)} \to \text{Cat}$ defined in Definition 4.46. $\blacksquare$

Definition 5.14. The 2-functor $B^* : \text{Cat} \to \text{Cat}/B$ is, up to the isomorphism $\text{Cat}^B \cong \text{Cat}/B$, the cofree 2-functor $F^B : \text{Cat} \to \text{Cat}$ defined in Definition 4.44. $\blacksquare$

So the 2-adjunction $\prod_B \dashv B^* : \text{Cat} \to \text{Cat}/B$ is, essentially, the co-Eilenberg–Moore 2-adjunction $U^B \dashv F^B : \text{Cat} \to \text{Cat}$ for the 2-comonad $B \times (-)$ on which we heavily relied when performing the Eilenberg–Moore construction for indexed monads. The following construction is new.

Definition 5.15. We define the 2-functor $\prod_B : \text{Cat}/B \to \text{Cat}$ in the following way.

- Given an object $\left(\frac{E}{p}_B\right)$ of $\text{Cat}/B$, define the category $\prod_B \left(\frac{E}{p}_B\right)$ as follows:
  - Its object is a section of $p$, i.e., a functor $s : B \to E$ satisfying $p \circ s = \text{id}_B$.
  - Its morphism from $s$ to $s'$ is a natural transformation $\psi : s \Rightarrow s' : B \to E$ satisfying $p^* \psi = \text{id}_B$.

- Given a morphism $f : \left(\frac{E}{p}_B\right) \to \left(\frac{E'}{p'}_B\right)$ of $\text{Cat}/B$, i.e., a functor $f : E \to E'$ making the diagram
  \[
  \begin{array}{ccc}
  E & \xrightarrow{f} & E' \\
  p \downarrow & & \downarrow p' \\
  B & & B
  \end{array}
  \]
  commute, define the functor $\prod_B f : \prod_B \left(\frac{E}{p}_B\right) \to \prod_B \left(\frac{E'}{p'}_B\right)$ as follows:
  - It sends a section $s$ of $p$ to the section $f \circ s : B \to E'$ of $p'$; observe that $p' \circ f \circ s = p \circ s = \text{id}_B$.
  - It sends a morphism $\varphi : s \to s'$ of sections of $p$ to $f^* \varphi : f \circ s \to f \circ s'$.

- Given a 2-cell $\alpha : f \Rightarrow f' : \left(\frac{E}{p}_B\right) \to \left(\frac{E'}{p'}_B\right)$ of $\text{Cat}/B$, i.e., a natural transformation $\alpha : f \Rightarrow f' : E \to E'$ satisfying the equation
  \[
  \begin{array}{ccc}
  E & \xrightarrow{f} & E' \\
  p \downarrow & & \downarrow p' \\
  B & & B
  \end{array}
  \]
  define the natural transformation $\prod_B \alpha : \prod_B f \Rightarrow \prod_B f' : \prod_B \left(\frac{E}{p}_B\right) \to \prod_B \left(\frac{E'}{p'}_B\right)$ as follows:
5.2. A CONSTRUCTION OF MAILLARD AND MELLIÈS

- Its s-component \((\prod_B \alpha)_s : f \circ s \to f' \circ s\) for a section \(s\) of \(p\) is given by 
  \((\prod_B \alpha)_s := \alpha \ast s\). ■

Or more concisely,

\[
\prod_B := \text{Cat}/B \left( \left( \B_{\text{id}_B} \right), - \right) : \text{Cat}/B \rightarrow \text{Cat}.
\]

5.2.3 Models as sections

Let us finally prove Theorem 5.12, assuming the classical equivalence between the category of models of a Lawvere theory in \(\text{Set}\) and the Eilenberg–Moore category of the corresponding finitary monad on \(\text{Set}\). Below we give a more concrete description of the category \(\prod_B \left( \text{Set}^{\mathcal{L}_B} \right)\). To avoid too heavy notation we abbreviate the finitary monad \(\mathcal{T}_{\mathcal{L}_B}\) on \(\text{Set}\) corresponding to the Lawvere theory \(\mathcal{L}_B\) as \(\mathcal{T}_B\); similarly for the relevant monad morphisms.

- An object of \(\prod_B \left( \text{Set}^{\mathcal{L}_B} \right)\) consists of the following data:
  - For each object \(b\) of \(\mathcal{B}\), a \(\mathcal{T}_B\)-algebra \((T_b c_b, \chi_b)\).
  - For each morphism \(u : b \rightarrow b'\) of \(\mathcal{B}\), a morphism of \(\mathcal{T}_B\)-algebras
    \[
    h_u : \left( \begin{array}{c} T_b c_b \\ \downarrow \chi_b \end{array} \right) \rightarrow \left( \begin{array}{c} T_{b'} c_{b'} \\ \downarrow \chi_{b'} \end{array} \right) \circ \mathcal{T}_B u \circ \mathcal{T}_B c_b
    \]

  These data are subject to the following functoriality axioms:

  - \(h_{\text{id}_b} = \text{id}_{\chi_b}\) for each object \(b\) of \(\mathcal{B}\).
  - \(h_{u' \circ u} = h_{u'} \circ h_u\) for each pair of composable morphisms \(u, u'\) of \(\mathcal{B}\).

- A morphism \(\psi : ((\chi_b), (h_u)) \rightarrow ((\chi'_b), (h'_u))\) of \(\prod_B \left( \text{Set}^{\mathcal{L}_B} \right)\) is a family of morphisms of \(\mathcal{T}_B\)-algebras
  \[
  \psi_b : \left( \begin{array}{c} T_b c_b \\ \downarrow \chi_b \end{array} \right) \rightarrow \left( \begin{array}{c} T_b c_b' \\ \downarrow \chi_{b'} \end{array} \right)
  \]

for each \(b \in \mathcal{B}\), such that the following naturality square

\[
\begin{array}{ccc}
T_b c_b \\
\downarrow \chi_b & \searrow \psi_b \\
T_b c_b' & \circ \mathcal{T}_B u \circ \mathcal{T}_B c_b \\
\downarrow h_u & & \downarrow h'_u \\
T_{b'} c_{b'} \\
\downarrow \chi_{b'} & \searrow \psi_{b'} \\
T_{b'} c_{b'}' & \circ \mathcal{T}_B u \circ \mathcal{T}_B c_{b'}
\end{array}
\]

commutes for each morphism \(u : b \rightarrow b'\) of \(\mathcal{B}\).

Now it only remains to apply the classical equivalence of models of a Lawvere theory in \(\text{Set}\) and algebras of the corresponding finitary monad on \(\text{Set}\), before we reach Definition 5.11.
Notes

The idea of decomposing a lax action into a strict action together with an adjunction is presented in the paper [7]. Among the authors of [7], this material has been mainly developed by Shin-ya Katsumata, and indeed it was in this context that he arrived at the definitions of Eilenberg–Moore and Kleisli categories of graded monads. (These are then informed to me, as mentioned in Notes for Chapter 4.) Then I noticed that thanks to the 2-fibrational property of $p^{++} : \mathcal{E}^{++} \to 2\text{-Cat}_2$ and $p^{--} : \mathcal{E}^{--} \to 2\text{-Cat}_2^{\text{op}(1,2)}$ concerning the correspondence of adjunctions (Section 5.1.3), decompositions (called resolutions here) can be seen as instances of the more familiar 2-categorical situation of adjunctions that generate monads. The adjoint correspondence theorem (Propositions 5.5 and 5.6) seem not to have been stated explicitly in the literature as far as I am aware, but the similar result (for a particular 2-fibration) has been shown as the main theorem in the paper [14] by Hermida. I conjecture that this property is possessed more generally by an arbitrary 2-fibration.

The main conceptual novelty in Section 5.2 is the observation brought by our Eilenberg–Moore construction for indexed monads that, the functor $\left( \begin{array}{c} C \times \pi_0 \\ B \end{array} \right)$ naturally lives in the 2-category $\mathcal{C}at/B$ (rather than other alternatives such as $\mathcal{F}ib(B)$). This enables us to identify the dependent product 2-functor $\prod_B : \mathcal{C}at/B \to \mathcal{C}at$ as the right construction of the “category of sections”. Note that this actually corrects a subtle mistake in the paper [25] concerning the definition of morphisms of the category of sections; Maillard and Melliès defined their category of sections as a full subcategory of the functor category, but it is indeed our more restricted definition that establishes an equivalence with the category of models of an indexed Lawvere theory as defined by Power [34, 35].
Chapter 6

Conclusions and future work

6.1 Conclusions

In this thesis we initiated a unified mathematical study on graded and indexed monads. After providing in Chapter 3 the novel 2-categorical understanding of these notions (see also Appendix B), we defined in Chapter 4 the following constructions and established their 2-dimensional universality:

- The Eilenberg–Moore construction for graded monads.
- The Kleisli construction for graded monads.
- The Eilenberg–Moore construction for indexed monads.

These constructions are then applied in Chapter 5 to two situations. The first (Section 5.1) deals with lax actions of monoidal categories, and we showed that our Eilenberg–Moore and Kleisli constructions for graded monads provide canonical resolutions of a lax action, canonical in the sense that they are the terminal and initial ones respectively. In doing so we encountered a theorem on the correspondence of adjunctions (Section 5.1.3), whose nature seems to be 2-fibrational.

As the second application (Section 5.2) we reconstructed a proof of a beautiful theorem in [25] providing the view that models of an indexed Lawvere theory [34, 35] can be seen as sections. We employed the novel perspective brought to us by our Eilenberg–Moore construction for indexed monads, and gave a more conceptual construction of the category of sections using the rightmost 2-functor \( \prod \) constituting the famous base change adjoint triple (Section 5.2.2). Not only our formulation now enables one to state more clearly in what sense the theorem generalizes the well-known close relationship between the category of models (in \( \text{Set} \)) of a Lawvere theory and the Eilenberg–Moore category of the corresponding monad, by providing clearer understanding of the situation we were able to point out a missing condition in [25] that should have been posed on morphisms of sections (see Notes for Chapter 5).

6.2 Directions for further research

6.2.1 The Kleisli construction for indexed monads

One thing that is obviously missing from the current thesis is the Kleisli construction for indexed monads. Naturally we believe that the Kleisli construction should take
place in the 2-category $\mathcal{E}^{-+}$, using the observation in Section 3.3.4. We conjecture the existence of a suitable construction completing the following picture:

![Diagram](image)

All the attempts to define the object $\left(\mathcal{B}, \mathcal{C}_\mathcal{F}\right)$ of $\mathcal{C}at^{[\mathcal{B}, \mathcal{C}_\mathcal{F}]}$ so far have failed. One possible approach for this problem would be to understand the other three constructions (Eilenberg–Moore and Kleisli for graded monads, and Eilenberg–Moore for indexed monads) much more abstractly so that it is immediate how one can obtain the Kleisli construction for indexed monads. The observation presented in Appendix B might be valuable for this strategy.

### 6.2.2 A 3-categorical study of graded and indexed monads

The category $2\text{-}\mathcal{C}at$ of 2-categories, which we have employed when constructing the 2-categories $\mathcal{E}^{++}$, $\mathcal{E}^{+-}$, $\mathcal{E}^{-+}$ and $\mathcal{E}^{--}$, is inherently a 3-category. In fact, it may well be more natural to consider $\mathcal{E}^{++}$, $\mathcal{E}^{+-}$, $\mathcal{E}^{-+}$ and $\mathcal{E}^{--}$ as 3-categories as well; indeed, there are fairly natural definitions of 3-cells of them. They seem to arise as sub-3-categories of appropriate functor 3-categories.

In this thesis we have confined ourselves to dealing only with strict monoidal categories when considering graded monads. Although this covers a large class of graded monads currently employed, there do exist natural examples of graded monads graded by non-strict monoidal categories [28]. If one wishes to take an arbitrary monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ as the parameter category of graded monads, it seems inevitable to manipulate the pseudomonad $\mathcal{M} \times (-)$ and the pseudocomonad $[\mathcal{M}, -]$ on $\mathcal{C}at$. Just like monads live in 2-categories, pseudomonads live in 3-categories (or perhaps better: Gray-categories), and there is a work by Lack [22] which may naturally be thought of as the 3-dimensional version of Street’s formal theory of monads [38]. In particular, Lack’s work includes an abstract definition of the object of pseudoalgebras of a pseudomonad in a Gray-category as a certain 3-dimensional limit, providing the 3-dimensional analogue of the Eilenberg–Moore object of a monad.

Therefore there are evidences which support the claim that our work would be done more properly in the setting of 3-categories or Gray-categories [11, 13]. As the world
of 3-dimensional category theory still seems to remain rather unexplored, the upgraded 3-categorical study of graded and indexed monads could bring important contribution to pure category theory, too.

### 6.2.3 Categorical semantics of Bounded Linear Logic

As mentioned at the end of Section 2.1.1, the notion of graded comonad has been employed in the study of computational resources with parameters. Perhaps the most celebrated logical system dealing with parametrized computational resources is *Bounded Linear Logic* [10], which replaces the of course modality ! of Linear Logic [9] by a family of modalities !x parametrized by resource polynomials x. Bounded Linear Logic has recently been generalized [8] so as to be able to take an arbitrary semiring as the collection of parameters.

On the other hand, there has been a line of research seeking for the appropriate categorical semantics for Linear Logic, such as [36, 4], to name just two; see [26] for a nice survey. The biggest challenge was the identification of a suitable categorical structure modeling the modality !, and the consensus reached is that ! should be modeled as a certain comonad. Mathematical results on the co-Eilenberg–Moore and co-Kleisli categories for comonads have been useful in clarifying the relationship between the various proposed semantics.

We expect that our co-Eilenberg–Moore and co-Kleisli constructions for graded comonads can be fruitfully employed in the study of categorical semantics of Bounded Linear Logic, on which it looks that not much work has been done.

### 6.2.4 Syntactical development

From the viewpoint of the theory of computational effects, the development presented in the current thesis remains entirely semantical. The recent theoretical study of computational effects has been greatly benefited by the syntactical approach, in which the emphasis is placed on Lawvere theories rather than monads, as is conspicuous for example in the seminal paper [32] by Plotkin and Power.

We believe that the development of suitable mathematical theories of graded Lawvere theories and indexed Lawvere theories would be an important contribution to the theory of computational effects with parameters. We do not know yet what a graded Lawvere theory means, and although Power [34, 35] has already defined the notion of indexed Lawvere theory which possesses a nice relationship to our notion of indexed monad (Section 5.2), Power himself makes it clear in his papers that his definition of indexed Lawvere theory is not a definitive one; so this direction of research could bring us to a whole new world.

**Notes**

I had a valuable discussion on possible definitions of the Kleisli category of an indexed monad with Kazuyuki Asada and Takeshi Tsukada.

The possibility of a 3-categorical approach has been in a sense evident as soon as I defined δ++, but it was a series of enlightening discussions with John Power which brought me much clearer insight into this.

The research theme on categorical semantics of Bounded Linear Logic was suggested to me by Ichiro Hasuo.
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Appendix A

Compositions in $\mathcal{E}^{++}$

We describe in detail how 1-cells and 2-cells in the 2-category $\mathcal{E}^{++}$ are composed; compositions in the other three 2-categories $\mathcal{E}^{+-}$, $\mathcal{E}^{-+}$ and $\mathcal{E}^{--}$ are completely similar.

A.1 Compositions of 1-cells

Suppose we have the following diagram in $\mathcal{E}^{++}$:

$$(A', A) \xrightarrow{(F, f)} (B, B) \xrightarrow{(G, g)} (C, C)$$

Recall that $f: FA \to B$ can be thought of as a 1-cell in $\mathcal{B}$ and $g: GB \to C$ as a 1-cell in $\mathcal{C}$. We define

$$(G, g) \circ (F, f) := (GF, g \circ Gf)$$

where the second component is the 1-cell $GFA \xrightarrow{Gf} GB \xrightarrow{g} C$ in $\mathcal{C}$.

A.2 Vertical compositions of 2-cells

Suppose we have the following diagram in $\mathcal{E}^{++}$:

We regard $\alpha$ and $\alpha'$ respectively as:

$$(F, f) \xrightarrow{(\Theta, \alpha)} (F', f') \xrightarrow{(\Theta', \alpha') (F'', f'') \in \mathcal{B}$$

$$(F', f') \xrightarrow{(\Theta', \alpha') (F'', f'') \in \mathcal{B}$$

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Now define
\[(\Theta', \alpha').(\Theta, \alpha) := (\Theta', \Theta, (\alpha' \ast \Theta_A).\alpha)\]
with the second component being the 2-cell

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$FA$};
  \node (B) at (2,0) {$B$};
  \node (C) at (0,-2) {$F'A$};
  \node (D) at (2,-2) {$F''A$};
  \draw[->] (A) to node[auto] {$f$} (B);
  \draw[->] (A) to node[auto,swap] {$\Theta_A$} (C);
  \draw[->] (B) to node[auto] {$\alpha$} (D);
  \draw[->] (C) to node[auto,swap] {$\alpha'$} (D);
\end{tikzpicture}
\end{array}
\]

in \(\mathcal{B}\).

### A.3 Whiskerings

Before describing the somewhat complicated horizontal compositions of 2-cells in \(\mathcal{E}^{++}\), we begin with the simpler situations of whiskerings.

Suppose we have the following diagram in \(\mathcal{E}^{++}\):

\[
\big((\mathcal{A}, A), (\mathcal{B}, B), (\mathcal{C}, C)\big)
\]

We define
\[(G, g) \ast (\Theta, \alpha) := (G \ast \Theta, g \ast G\alpha)\]
with the second component

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$GFA$};
  \node (B) at (2,0) {$GB$};
  \node (C) at (2,-2) {$C$};
  \node (D) at (0,-2) {$GF'A$};
  \draw[->] (A) to node[auto] {$GF\alpha$} (B);
  \draw[->] (A) to node[auto,swap] {$G\Theta_A$} (D);
  \draw[->] (B) to node[auto] {$g$} (C);
  \draw[->] (C) to node[auto,swap] {$g'$} (D);
\end{tikzpicture}
\end{array}
\]

Suppose we have the following diagram in \(\mathcal{E}^{++}\):

\[
\big((\mathcal{A}, A), (\mathcal{B}, B), (\mathcal{C}, C)\big)
\]

Define
\[(\Xi, \beta) \ast (F, f) := (\Xi \ast F, \beta \ast Gf)\]
A.4 Horizontal compositions of 2-cells

Suppose we have the following diagram in $\mathcal{E}^{++}$:

\[
\begin{array}{ccc}
(F,f) & \longrightarrow & (G,g) \\
(\alpha', A) & \downarrow & (\Xi, \beta) \\
(F', f') & \longrightarrow & (G', g') \\
(\Theta, A) & \downarrow & (\Theta, \alpha) \\
(\Theta, B) & \longrightarrow & (\Theta, C) \\
(G,B) & \downarrow & (G,C) \\
GFA & \longrightarrow & GF'B \\
G'FA & \longrightarrow & G'F'B \\
GFA & \longrightarrow & GF'B \\
G'FA & \longrightarrow & G'F'B \\
\end{array}
\]

We should have, as in any 2-category, the following identity

\[
(\Xi, \beta)*(\Theta, \alpha) = ((\Xi, \beta)*(F', f')) . ((G, g)*(\Theta, \alpha)).
\]

Therefore we can take this as the definition:

\[
(\Xi, \beta)*(\Theta, \alpha) := (\Xi*(\Theta, \beta*G\alpha)
\]

with the second component depicted as follows:

\[
\begin{array}{ccc}
GFA & \stackrel{Gf}{\longrightarrow} & GB \\
G\Theta_A & \downarrow & G\beta_B \\
GF'A & \stackrel{Gf'}{\longrightarrow} & G'B \\
G\Theta_A & \downarrow & G\beta_B \\
\Xi_{F,A} & \stackrel{\Xi_{F,A}}{\longrightarrow} & G'\alpha_B \\
G'F'A & \stackrel{G'f'}{\longrightarrow} & G'F'B \\
\Xi_{F,A} & \stackrel{\Xi_{F,A}}{\longrightarrow} & G'\alpha_B \\
G'F'A & \stackrel{G'f'}{\longrightarrow} & G'F'B \\
\end{array}
\]

Or alternatively, we can start from the following identity that also holds in any 2-category

\[
(\Xi, \beta)*(\Theta, \alpha) = ((G', g')*(\Theta, \alpha)) . ((\Xi, \beta)*(F, f))
\]

and define

\[
(\Xi, \beta)*(\Theta, \alpha) := (\Xi*(\Theta, \beta*G\alpha)).(\beta*GF)
\]

with the second component
That these two definitions of vertical compositions of 2-cells coincide is an immediate consequence of the 2-naturality of $\Xi$. 
Appendix B

An abstract view of graded and indexed monads

B.1 Enriched (co)monads via (co)powers

In developing the mathematical theory of graded and indexed monads, we heavily employed the following 2-monads and 2-comonads on $\mathbf{Cat}$:

- The 2-monad $M \times (-)$ where $(M, \otimes, I)$ is a strict monoidal category.
- The 2-comonad $[M, -]$ where $(M, \otimes, I)$ is a strict monoidal category.
- The 2-comonad $B \times (-)$ where $B$ is a category.
- The 2-monad $[B, -]$ where $B$ is a category.

In fact there is a general construction in enriched category theory [19] of which these are instances; the construction induces enriched monads and comonads on a suitably complete and cocomplete enriched category from monoids and comonoids in the enriching category, through powers and copowers.

Suppose that $\mathcal{V} = (\mathcal{V}, \bullet, [-,-], 1)$ is a symmetric monoidal closed category and $\mathcal{A}$ a $\mathcal{V}$-category.

**Definition B.1.** Given objects $V$ and $A$ of $\mathcal{V}$ and $\mathcal{A}$ respectively, the power of $A$ by $V$ is the object $V \downarrow A$ of $\mathcal{A}$ satisfying

$$\mathcal{A}(X, V \downarrow A) \cong [V, \mathcal{A}(X, A)]$$

$\mathcal{V}$-natural in $X \in \mathcal{A}$. A $\mathcal{V}$-category is said to be **powered** if it has all powers.

**Definition B.2.** Given objects $V$ and $A$ of $\mathcal{V}$ and $\mathcal{A}$ respectively, the copower of $A$ by $V$ is the object $V \otimes A$ of $\mathcal{A}$ satisfying

$$\mathcal{A}(V \otimes A, X) \cong [V, \mathcal{A}(A, X)]$$

$\mathcal{V}$-natural in $X \in \mathcal{A}$. A $\mathcal{V}$-category is said to be **copowered** if it has all copowers.

Suppose a $\mathcal{V}$-category $\mathcal{A}$ is both powered and copowered. Now the general construction alluded above is the following:

- The functor $M \otimes (-) : \mathcal{A} \to \mathcal{A}$ becomes a $\mathcal{V}$-monad on $\mathcal{A}$ when $(M, M \bullet M \overset{m}{\to} M, 1 \overset{u}{\to} M)$ is a monoid in $\mathcal{V}$. 
• The functor $M \otimes (-) : \mathcal{A} \to \mathcal{A}$ becomes a $\mathcal{V}$-comonad on $\mathcal{A}$ when $(M, M \cdot M \xrightarrow{m} M, 1 \xrightarrow{u} M)$ is a monoid in $\mathcal{V}$.

• The functor $B \otimes (-) : \mathcal{A} \to \mathcal{A}$ becomes a $\mathcal{V}$-comonad on $\mathcal{A}$ when $(B, B \xrightarrow{d} B \cdot B, B \xrightarrow{e} 1)$ is a comonoid in $\mathcal{V}$.

• The functor $B \otimes (-) : \mathcal{A} \to \mathcal{A}$ becomes a $\mathcal{V}$-monad on $\mathcal{A}$ when $(B, B \xrightarrow{d} B \cdot B, B \xrightarrow{e} 1)$ is a comonoid in $\mathcal{V}$.

Note that 2-(co)monads are nothing but $\mathcal{C}at$-(co)monads, and strict monoidal categories are nothing but monoids in $(\mathcal{C}at, \times, 1)$. Also, as in any Cartesian monoidal category, a comonoid in $(\mathcal{C}at, \times, 1)$ is the same thing as an object of $\mathcal{C}at$; every category $\mathcal{B}$ admits a unique comonoid structure given by $(\mathcal{B}, \mathcal{B} \xrightarrow{\Delta} \mathcal{B} \times \mathcal{B}, \mathcal{B} \xrightarrow{1} 1)$. Combining these observations with the fact that powers and copowers in $\mathcal{C}at$ as a $\mathcal{C}at$-category are given by exponentials and Cartesian product respectively, we now obtain a unified abstract explanation of the 2-(co)monads at the beginning of this section.

### B.2 Generalizing monads in $\mathcal{C}at$ via lifting

Now that we have understood abstractly the four kinds of 2-(co)monads used in studying graded and indexed monads, let us proceed to look again the relationship of graded and indexed monads and these 2-(co)monads. Assuming the constructions of 2-categories $\mathcal{E}^{++}, \mathcal{E}^{+-}, \mathcal{E}^{-+}$ and $\mathcal{E}^{--}$, a priori there are eight notions of generalized monad on $\mathcal{C} \in \mathcal{C}at$ obtained by lifting these 2-(co)monads:

1. Monads in $\mathcal{E}^{++}$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-monad $M \times (-)$:
   \[
   T : M \times \mathcal{C} \to \mathcal{C}.
   \]

2. Monads in $\mathcal{E}^{++}$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-monad $[\mathcal{B}, -]$:
   \[
   T : [\mathcal{B}, \mathcal{C}] \to \mathcal{C}.
   \]

3. Monads in $\mathcal{E}^{+-}$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-comonad $[\mathcal{M}, -]$:
   \[
   T : [\mathcal{M}, \mathcal{C}] \to \mathcal{C}.
   \]

4. Monads in $\mathcal{E}^{+-}$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-comonad $\mathcal{B} \times (-)$:
   \[
   T : \mathcal{B} \times \mathcal{C} \to \mathcal{C}.
   \]

5. Monads in $\mathcal{E}^{-+}$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-monad $\mathcal{M} \times (-)$:
   \[
   T : \mathcal{C} \to \mathcal{M} \times \mathcal{C}.
   \]

6. Monads in $\mathcal{E}^{-+}$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-monad $[\mathcal{B}, -]$:
   \[
   T : \mathcal{C} \to [\mathcal{B}, \mathcal{C}].
   \]
7. Monads in $\mathcal{E}^-$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-comonad $[M, -]$: 

$$T : \mathcal{C} \rightarrow [M, \mathcal{C}].$$

8. Monads in $\mathcal{E}^-$ on $(\mathcal{C}at, \mathcal{C})$ above the 2-comonad $\mathbb{B} \times (-)$:

$$T : \mathcal{C} \rightarrow \mathbb{B} \times \mathcal{C}.$$

Actually, it turns out that the notions 1 and 7 coincide and are that of graded monad, and the notions 4 and 6 coincide and are that of indexed monad; this coincidence is because of the general adjointness

$$V \otimes (-) \dashv V \pitchfork (-) \quad (B.1)$$

between copowers and powers.

Therefore one can characterize the notions of graded and indexed monads exactly as those notions of generalized monad on a category listed above, for which there are two different ways of thinking about them thanks to the fundamental adjointness $[B.1]$. Note that indeed we used this dual view to construct (or at least try to construct) both Eilenberg–Moore and Kleisli categories.

**Notes**

The materials contained in this chapter have occurred to my mind through discussions with John Power. In particular I learned the construction of enriched (co)monads from (co)monoids from him.