TEST FUNCTIONS, KERNELS, REALIZATIONS AND INTERPOLATION

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ABSTRACT. Jim Agler revolutionized the area of Pick interpolation with his realization theorem for what is now called the Agler-Schur class for the unit ball in $\mathbb{C}^d$. We discuss an extension of these results to algebras of functions arising from test functions and the dual notion of a family of reproducing kernels, as well as the related interpolation theorem. When working with test functions, one ideally wants to use as small a collection as possible. Nevertheless, in some situations infinite sets of test functions are unavoidable. When this is the case, certain topological considerations come to the fore. We illustrate this with examples, including the multiplier algebra of an annulus and the infinite polydisk.

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1. INTRODUCTION

Let $\mathbb{D}^d$ denote the $d$-polydisk in $\mathbb{C}^d$, $\mathbb{D}^d = \{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_j| < 1, j = 1, 2, \ldots, d \}$. The Agler-Schur class $S_d$ consists of those functions $f : \mathbb{D}^d \to \mathbb{C}$ for which there exist positive (that is, positive semidefinite) kernels $\Gamma_j : \mathbb{D}^d \times \mathbb{D}^d \to \mathbb{C}$ such that

$$1 - f(z)f(w)^* = \sum_{j=1}^{d} \Gamma_j(z, w)(1 - z_j w_j^*).$$

(1.1)

Here $\zeta^*$ denotes the conjugate of the complex number $\zeta$.

A $\mathbb{D}^d$ unitary colligation is a pair $\Sigma = (U, E)$, where $E = \bigoplus_1^d E_j$ is an auxiliary Hilbert space and

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \bigoplus E \to \bigoplus E$$

is unitary. With respect to the decomposition of $E$, let $Z = \bigoplus z_j I_{E_j}$, the $d \times d$ block diagonal operator matrix with diagonal entries $z_j I_{E_j}$. The transfer function $W_{\Sigma} : \mathbb{D}^d \to \mathbb{C}$ of the colligation $\Sigma$ is defined as

$$W_{\Sigma} = D + CZ(I - AZ)^{-1}B.$$ 

The primum mobile for contemporary work on multivariable realization with applications to interpolation theory is the following result of Agler, stated for what is now called the Agler-Schur class functions on the polydisk. See [2], [4], [5], [11]. This theorem has been generalized in many directions ([11] [8], [7], [20], [9], to give a few instances). An operator $C$ on a Hilbert space $H$ is a strict contraction provided $\|C\| < 1$.

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Theorem 1.1 (\cite{2}). Suppose $f: \mathbb{D}^d \rightarrow \mathbb{C}$. The following are equivalent.

(i) If $k$ is a positive kernel on $\mathbb{D}^d$ and if

$$(z, w) \mapsto (1 - z_j w_j^*) k(z, w)$$

is a positive kernel on $\mathbb{D}^d$ for each $j = 1, 2, \ldots, d$, then the kernel

$$(z, w) \mapsto (1 - f(z)f(w^*) k(z, w)$$

is also positive;

(ii) $f \in \mathcal{S}_d$;

(iii) There is a unitary colligation $\Sigma$ so that $f = W_{\Sigma}$; and

(iv) For each tuple $T = (T_1, \ldots, T_d)$ of commuting strict contractions on a Hilbert space $H$,

$$\|f(T_1, \ldots, T_d)\| \leq 1.$$ 

In newer versions $Z$ is replaced by some matrix function on a domain in $\mathbb{C}^d$, the domain being determined by those values at which the norm of the function is less than 1 (\cite{7}, \cite{9}). As an example, choosing $Z = (z_1, z_2, \ldots, z_d)$ on the unit ball $B = \{z \in \mathbb{C}^d : \|z\| < 1\}$ in $\mathbb{C}^d$ gives rise to the Agler-Schur class of functions associated with the row contractions and the Nevanlinna-Pick kernel $k(z, w) = (1 - \langle z, w \rangle)^{-1}$. Of course, choosing $Z$ to be the diagonal matrix with diagonal entries $z_j$ leads to the Agler-Schur class of Theorem 1.1.

In the paper \cite{12} Agler-Schur classes and Agler-Pick interpolation were considered in the very general setting of algebras of functions over semigroupoids. With hopes of mollifying those who might otherwise be put off by the algebraic formalism of parts of \cite{12}, we drop the semigroupoid structure (or rather, work with a trivial case). The realization and interpolation theorems we present have proofs which in outline follow those found in \cite{12}. However some simplification is achieved in the present setting, and as a novelty we include a von Neumann type inequality along the lines of part (iv) of Theorem 1.1. Furthermore, we show that strictly contractive functions can be replaced by a certain class $\mathcal{P}$ of representations of the algebra generated by the test functions. Representations in this family allow for nice approximations in terms of what we call “simple representations” — essentially representations involving only finitely many test functions. Regarding the interpolation theorem, the realization formula for the interpolating function in the Agler-Pick interpolation theorem has certain additional structure which we highlight.

As Jim Agler first discovered (\cite{3}, see also \cite{5}), realization and interpolation problems in function algebras which might not be multiplier algebras for some reproducing kernel Hilbert space can be effectively handled by means of so-called test functions (in the realization theorem for the polydisk given above, these are the coordinate functions). The test functions are used to delineate the unit ball in the algebra by means of a duality with a family of reproducing kernels. The function algebra is then the intersection of the multiplier algebras of the reproducing kernel Hilbert spaces.

The work of Ambrozie \cite{6} highlighted a second duality at play in the Agler realization theorem, and in particular in the definition of the Agler-Schur class. A similar duality is also evident in the work of Agler and McCarthy \cite{5}. One can view the test functions as points in some abstract space. Then in \cite{12}, the sum is replaced by a single kernel $\Gamma$ multiplied by $1 - E(z)E(w)^*$, where $E(z)$ is the evaluation functional at the point $z$. The principle difference between Ambrozie’s approach and that of Agler and McCarthy then comes down to whether one views the kernel $\Gamma$ as an element of a predual (Agler and McCarthy), or as an element of a dual space, along with the introduction of a more general notion of unitary colligation (Ambrozie). There is little to distinguish these when there
are only finitely many test functions. But when there are infinitely many test functions (something looked at by both), the two approaches are quite distinct. We feel that the dual approach offers certain advantages, which hopefully will be made apparent in what follows.

Our initial motivation was an interest in variants of Theorem 13.8 from [5] (see also [3]) which covers Agler-Pick interpolation on multiply connected domains in \( \mathbb{C} \) (see also [1], [13], [17], [16], [18], and [19]), and at the suggestion of John McCarthy [15], versions of the Agler algebra for the infinite polydisk. While the examples we consider are replete with analytic structure, a noteworthy feature of the general theory is that no such structure need be imposed, as will be clear from the axioms for test functions given below.

Function algebras on multiply connected domains and the infinite polydisk are examples where infinite families of test functions are required. We look at these in some detail, concentrating on the annulus as the multiply connected domain since all of the salient features of more general domains are already evident in this example. In these two cases, the emphasis will be somewhat different. The infinite polydisk has a natural choice of test functions. However this collection is not compact, the consequence being that there are functions in the Agler-Schur class that are not simply represented by some infinite version of (1.1), despite the fact that our definition of the Agler-Schur class naturally reduces to the original version for the finite polydisk. In terms of transfer function representations for functions in the Agler-Schur class, this is manifested in the need for the inclusion of representations in the colligation — or rather a broader class of representations, since in fact the decomposition of \( \mathcal{E} \) used in the construction of \( Z \) in the transfer function is essentially a representation of a rather simple form.

The situation is reversed for the annulus in that we are handed a function algebra (the bounded analytic functions on the annulus), and the first step then is to find a good set of test functions. Roughly speaking, such a set should be as small as possible. The collection of test functions we construct in this example is compact, and we show that it is minimal in the sense that there is no closed subset of this collection which is also a collection of test functions for this algebra.

## 2. Preliminaries and Main Results

This section contains a discussion of the ingredients that go into the statement and proof of our generalization of Theorem 2.2 and ends with a statement of the realization formula and Agler-Pick interpolation theorem.

### 2.1. Test Functions and Evaluations

For a finite subset \( F \) of \( X \), let \( P(F) \) denote all complex-valued functions on \( X \). By declaring the indicator functions of points in \( X \) to be an orthonormal basis, \( P(F) \) can be identified with the Hilbert space \( \mathbb{C}^F \). For a collection \( \Psi \) of functions on \( X \), let \( \Psi|_F = \{ \psi|_F : \psi \in \Psi \} \subset P(F) \).

A collection \( \Psi \) of functions on \( X \) is a collection of test functions provided,

(i) For each \( x \in X \),

\[
\sup \{|\psi(x)| : \psi \in \Psi \} < 1; \quad \text{and}
\]

(ii) for each finite set \( F \), the unital algebra generated by \( \Psi|_F \) is all of \( P(F) \).

The second hypothesis, while not essential, simplifies the exposition. It implies, among other things, that the functions in \( \Psi \) separate the points of \( X \). As we see shortly, the first hypothesis allows us to use the test functions to define a Banach algebra norm on the algebra of functions over \( X \) with addition and multiplication defined pointwise.

The function algebras we will be working with may be multiplier algebras for \( H^2(k) \) for some reproducing kernel \( k \) (this is what happens in the classical setting when one studies Nevanlinna-Pick...
interpolation), though this is but a special version of what we wish to consider here. Rather, test functions will allow us to manage in a broader context by means of a familiar dual construction. To this end, we introduce the algebra of bounded continuous functions over $\Psi$ with pointwise algebra operations, denoted by $C_b(\Psi)$.

Let $\Psi$ be a collection of test functions. This is a subspace of $B(X, \overline{D})$, the collection of bounded functions from $X$ into the closed unit disk $\overline{D}$ (equivalently, $\overline{D}^X$), which we endow with the topology of pointwise convergence. By Tychonov’s theorem, this is a compact Hausdorff space. Now $\overline{D}$ is a Tychonov space in the usual metric topology (that is, points are closed and for any closed set and point disjoint from it, there is a continuous function separating the two). Consequently, $\overline{D}^X$ is Tychonov, as is any subspace. In particular, $\Psi$ is a Tychonov space.

*A priori* the set $X$ is not assumed to have a topology, though in fact it inherits one as a subspace of $C_b(\Psi)$. Since $\Psi$ separates the points of $X$, there is an injective mapping $E : X \rightarrow C_b(\Psi)$, where $E(x) = e_x$ is the evaluation mapping $e_x(\psi) = \psi(x)$. Hence we can view $X$ as a subset of the unit ball of $C_b(\Psi)$. The same arguments applied above to $\Psi$ now show that with this topology, $X$ is Tychonov. In particular, if a net $\{x_\alpha\}$ converges to $x$ in $X$, then $\psi(x_\alpha)$ converges to $\psi(x)$. Hence $\Psi$ is a subset of the unit ball of $C_b(X)$.

By the same token, for a collection of test functions $\Psi$, the evaluation mapping $F : \Psi \rightarrow C_b(\Psi)^*$ can be defined by $F(\psi) = f_\psi$, where $f_\psi(g) = g(\psi)$ for all $g \in C_b(\Psi)$. The mapping $F$ is continuous in the weak-* topology. Since $\Psi$ is Tychonov, the map $\Psi \rightarrow F(\Psi)$ is a homeomorphism and $F(\Psi)$ is a subset of the closed unit ball of $C_b(\Psi)^*$, which is compact. The Stone-Čech compactification $\beta\Psi$ of $\Psi$ is then the weak-* closure of $F(\Psi)$. Since $\beta\Psi$ is a compact function space, it is pointwise closed, and so contains the image of the pointwise closure $\overline{\Psi}$ of $\Psi$. But then $F$ extends to a homeomorphism of $\overline{\Psi}$ to $F(\Psi)$. Therefore we can identify $\beta\Psi$ with $\overline{\Psi}$.

By the way, none of the above depends on the assumption that the test functions are complex valued, with the exception of the conclusion that $C_b(\Psi)$ is an algebra generated by $\{E(x) : x \in X\}$. We could, for example, take our test functions to have values in a Hilbert $C^*$-module $\mathcal{M}$ which we may concretely view as a norm closed subalgebra of $B(\mathcal{H}, \mathcal{K})$, the bounded operators between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. This is a corner of a $C^*$-subalgebra $\mathcal{C}$ of $B(\mathcal{H} \oplus \mathcal{K})$ with the property that each element of $\mathcal{M}$ is the corner of some element of $\mathcal{C}$ with the same norm. Since this added generality might obscure the main thrust of the paper, we restrict our attention to the simpler setting of scalar valued test functions.

### 2.2. Kernel/test function duality and the Agler-Schur class.

Let $\mathcal{B}$ denote a $C^*$-algebra with Banach space dual $\mathcal{B}^*$. A positive (that is, positive semidefinite) kernel on a subset $Y$ of $X$ with values in $\mathcal{B}^*$ is a function $\Gamma : Y \times Y \rightarrow \mathcal{B}^*$ such that for any finite subset $F$ of $Y$ and function $f : F \rightarrow \mathcal{B}$

$$\sum_{a, b \in F} \Gamma(a, b)(f(b)^* f(a)) \geq 0.$$ 

Naturally the restriction of a positive kernel on $X$ to a subset $F$ is still positive. We use $M(F, \mathcal{B}^*)^+$ to denote the collection of positive definite kernels on $F \subseteq X$ with values in $\mathcal{B}^*$. In the case that $\mathcal{B} = \mathbb{C}$ the modifier “with values in” is dropped.

Given a collection $\Psi$ of test functions, write $\mathcal{K}_\Psi$ for the collection of positive kernels $k$ on $X$ such that for each $\psi \in \Psi$ the kernel,

$$X \times X \ni (x, y) \mapsto (1 - \psi(x)\psi(y)^*)k(x, y)$$

is a probability measure. For any $\psi, \psi' \in \Psi$ the kernel (almost surely)

$$x, y \in X \mapsto \psi(x)\psi'(y)^*$$

is positive, as is the kernel (almost surely)

$$x, y \in X \mapsto \psi(x)\psi(y)^*.$$
is positive. This is nonempty, since it contains the kernel which is identically 0. Condition (i) in the definition of a collection of test functions implies that the so-called Toeplitz kernel $s$ with $s(x, x) = 1$ for all $x$ and $s(x, y) = 0$ if $y \neq x$ is also in $\mathcal{K}_\Psi$.

Say that a function $\varphi : X \to \mathbb{C}$ is in $H^\infty(\mathcal{K}_\Psi)$ if there is a real number $C$ so that, for each $k \in \mathcal{K}_\Psi$, the kernel

$$X \times X \ni (x, y) \mapsto (C^2 - \varphi(x)\varphi(y)^*)k(x, y)$$  \hspace{1cm} (2.1)

is positive, in which case write $C_\varphi$ for the infimum over all such $C$ (so $C_\varphi$ is independent of $k$). Then

$$\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} = C_\varphi$$

defines a Banach algebra norm on $H^\infty(\mathcal{K}_\Psi)$. Since the Toeplitz kernel $s$ is in $\mathcal{K}_\Psi$, norm convergent sequences in $H^\infty(\mathcal{K}_\Psi)$ converge pointwise, and since positivity of the kernels in (2.1) is verified on finite sets, completeness is easily checked. By definition, the test functions are in the unit ball of $H^\infty(\mathcal{K}_\Psi)$. Indeed, if $\mathcal{K}_\Psi$ consists solely of those kernels which are conjugate equivalent to the Toeplitz kernel (that is, each $k \in \mathcal{K}_\Psi$ has the form $k(x, y) = c(x)s(x, y)c(y)^*$ for some function $c$) then we then have $H^\infty(\mathcal{K}_\Psi) = C_b(X)$.

Replacing $\Psi$ by its pointwise closure (equivalently, $\beta\Psi$) adds nothing new. We end up with the same collection of kernels (that is, $\mathcal{K}_\beta\Psi = \mathcal{K}_\Psi$), and hence the same space of functions $H^\infty(\mathcal{K}_\Psi)$ with the same topology.

A function $f : X \to \mathbb{C}$ is said to be in the Agler-Schur class if there exists a positive kernel $\Gamma : X \times X \to C_b(\Psi)^*$ so that for all $x, y \in X$,

$$1 - f(x)f(y)^* = \Gamma(x, y)(1 - E(x)E(y)^*).$$

In the case that $X = \mathbb{D}^d$ and $\Psi$ consists of the coordinate functions, we recover the original definition of the Agler-Schur class from the Introduction.

2.3. $C_b(\Psi)$-unitary colligations, transfer functions, and the class $\mathbb{F}$. For a collection of test functions $\Psi$, following [3], define a $C_b(\Psi)$-unitary colligation $\Sigma$ to be a tuple $\Sigma = (U, E, \rho)$, $E$ a Hilbert space, $U$ unitary on $E \oplus \mathbb{C}$, and

$$\rho : C_b(\Psi) \to B(E)$$

a unital $*$-representation. Writing $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the transfer function associated to $\Sigma$ is defined as

$$W_\Sigma(x) = D + CZ(x)(I - AZ(x))^{-1}B,$$  \hspace{1cm} (2.2)

where $Z(x) = \rho(E(x))$. As observed earlier, by assumption (i) for test functions, $\|E(x)\| < 1$ for all $x \in X$, and since $\rho$ is unital, it is contractive. Hence $\|Z(x)\| < 1$ for all $x \in X$ and the definition of $W_\Sigma$ makes sense. Additionally, as the next lemma indicates, $W_\Sigma$ is contractive.

Lemma 2.1. Let $\Sigma = (U, E, \rho)$ be a $C_b(\Psi)$-unitary colligation with associated transfer function $W_\Sigma$. Then $\|W_\Sigma\| \leq 1$.

Proof. This is a standard calculation. Using the relations between the elements of $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

implied by the assumption that it is unitary, we find that for any $k \in \mathcal{K}_\Psi$,

$$(1 - W_\Sigma(x)W_\Sigma(y)^*)k(x, y) = k(x, y) - W_\Sigma(x)k(x, y)W_\Sigma(y)^*$$

$$= C(I - Z(x)A)^{-1}(k(x, y) - Z(x)k(x, y)Z(y)^*)(I - Z(y)A)^{-1}C^*.$$

Given a finite set $F$, the matrix

$$(k(x, y) - E(x)k(x, y)E(y)^*)_{x,y \in F}$$
is positive, since its value at a test function $\psi$ is
\[(k(x, y) - \psi(x)k(x, y)\psi(y)^*)_{x, y \in F}.
\]
Since $\rho$ is contractive, the result follows. \hfill \Box

A unital representation $\pi : H^\infty(K_\Psi) \to B(H)$ is weakly continuous if whenever $\varphi_\alpha$ is a bounded net from $H^\infty(K_\Psi)$ which converges pointwise to a $\varphi \in H^\infty(K_\Psi)$, then $\pi(\varphi_\alpha)$ converges in the weak operator topology to $\pi(\varphi)$.

We say that $\pi$ is in the class $\mathbb{F}$ if $\pi$ is a representation of $H^\infty(K_\Psi)$ on a Hilbert space $H$ such that
(i) $\pi$ is weakly continuous; and
(ii) $\pi$ is contractive on test functions (that is, $\|\pi(\psi)\| \leq 1$ for each $\psi \in \Psi$).

An example of such a representation is one for which $\|\pi(\psi)\| < 1$ for each $\psi \in \Psi$. However the class $\mathbb{F}$ includes somewhat more, since for example, it also contains the identity representation, $\pi(\psi) = \psi$ for all $\psi \in \Psi$. The main fact about this class is that these representations are automatically contractive and approximately respect transfer function representations.

### 2.4. Abstract realization and Agler-Pick interpolation.

The following theorem is the analogue of Theorem 1.1 and in fact contains that theorem as a special case when $X = \mathbb{D}^d$ and $\Psi$ consists of the $d$ coordinate functions. More or less a corollary of this is the Agler-Pick interpolation theorem which follows it. With the exception of (iv) in Theorem 2.2 and the concrete form of the space $E$ for the unitary colligation in Theorem 2.3, they are in fact special cases of results to be found in [12]. Note that a corollary of Theorem 2.2 is that the representations in $\mathbb{F}$ are contractive (recall that a priori only their behavior on test functions is prescribed).

**Theorem 2.2.** Suppose $\Psi$ is a collection of test functions. The following are equivalent:

(i) $\varphi \in H^\infty(K_\Psi)$ and $\|\varphi\|_{H^\infty(K_\Psi)} \leq 1$;

(ii) For each finite set $F \subset X$ there exists a positive kernel $\Gamma : F \times F \to C_b(\Psi)^*$ so that for all $x, y \in F$
\[1 - \varphi(x)\varphi(y)^* = \Gamma(x, y)(1 - E(x)E(y)^*);
\]

(iii) There exists a positive kernel $\Gamma : X \times X \to C_b(\Psi)^*$ so that for all $x, y \in X$
\[1 - \varphi(x)\varphi(y)^* = \Gamma(x, y)(1 - E(x)E(y)^*);
\]

(iv) There is a colligation $\Sigma$ so that $\varphi = W_\Sigma$;

(ivS) For every representation $\pi$ of $H^\infty(K_\Psi)$ such that $\|\pi(\psi)\| < 1$ for all $\psi \in \Psi, \|\pi(\varphi)\| \leq 1$; and

(ivF) For every representation of $H^\infty(K_\Psi)$ in $\mathbb{F}$, $\|\pi(\varphi)\| \leq 1$.

The Agler-Pick interpolation theorem corresponding to Theorem 2.2 is the following.

**Theorem 2.3.** Suppose $\Psi$ is a collection of test functions. Let $F$, a finite subset of $X$, and $\xi : F \to \mathbb{D}$ be given.

(i) There exists $\varphi \in H^\infty(K_\Psi)$ so that $\|\varphi\|_{H^\infty(K_\Psi)} \leq 1$ and $\varphi|_F = \xi$;

(ii) for each $k \in K_\Psi$, the kernel
\[F \times F \ni (x, y) \mapsto (1 - \xi(x)\xi(y)^*)k(x, y)
\]
is positive;

(iii) there exists a positive kernel $\Gamma : F \times F \to C_b(\Psi)^*$ so that for all $x, y \in F$
\[1 - \xi(x)\xi(y)^* = \Gamma(x, y)(1 - E(x)E(y)^*).
\]

(2.3)
Moreover, in this case we also have $\Psi$ is compact (say replacing $\Psi$ by its closure) there is a bounded positive measure $\mu$ on $\Psi$ so that the interpolant $\varphi$ has a transfer function realization of the form

$$\varphi(x) = D + CE(x)(I - AE(x))^{-1}B$$

for a unitary

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}: \mathbb{C}^n \otimes L^2(\mu) \oplus \mathbb{C} \rightarrow \mathbb{C}^n \otimes L^2(\mu) \oplus \mathbb{C},$$

where the $E(x)$ is interpreted as the multiplication operator on $\mathbb{C}^n \otimes L^2(\mu)$ given by $E(x)h \otimes f = h \otimes E(x)f$.

**Remark 2.4.** (In the last theorem the representation $\rho$ in the definition of $\Psi$-unitary colligation is simply a multiple of the representation of $C(\Psi)$ as multiplication operators on $L^2(\mu)$.

Note that in Theorem 2.2 it is certainly not the case that all positive $\Gamma$ give rise to a $\varphi$. Indeed, Theorem 2.3 says a $\Gamma$ corresponding to an Agler-Pick interpolation problem can be chosen with additional structure. This will be highlighted in our discussion of the annulus.

As a consequence of the realization theorem, we also get the following.

**Proposition 2.5.** The Agler-Schur class is closed in the topology of pointwise convergence. In particular, it contains the closure of the test functions.

### 2.5. Organization

The remainder of the paper is organized as follows. Section 3 collects results needed to prove Theorem 2.2 which is then given in Section 4.1. Section 4.2 contains the proof of Theorem 2.3 the basic Agler-Pick interpolation result companion to Theorem 2.2. Examples are found in Section 5. These include the annulus algebra and the Agler algebra of the infinite polydisk. The case of the annulus $\mathbb{A}$ is covered in the greatest detail, and it is shown that modulo natural equivalences, the set of scalar-valued inner functions defined on $\mathbb{A}$ with precisely two zeros in $\mathbb{A}$ and a particular normalization is a minimal collection of test functions for $H^\infty(\mathbb{A})$. Further examples show that a certain amount of care is needed in working in the context presented here, especially when the collection of test functions is not finite.

### 3. Ingredients

This section collects results preliminary to the proofs of Theorems 2.2 and 2.3.

#### 3.1. Simple representations and the class $F$.

When dealing with an infinite collection of test functions, it is often useful to approximate using a finite subset. This is the idea behind simple representations, which are central to our proof that (iii) implies (iv) in Theorem 2.2 where we approximate the function $x \mapsto \rho(E(x))$ appearing in the transfer function realization.

To be more precise, given a collection of test functions $\Psi$ with $\psi_j \in \Psi$, $j = 1, \ldots, N$, and orthogonal projections $P_j$ such that $\sum_{j=1}^N P_j = I$, define a simple representation $\rho: C_b(\Psi) \rightarrow B(\mathcal{H})$ to have the form

$$\rho(f) = \sum_{j=1}^N P_j f(\psi_j).$$

Clearly $\rho$ is unital.
Set \( Z(x) = \sum_{j=1}^{N} P_j \psi_j(x) = \rho(E(x)) \) and suppose \( \pi \) is a representation of \( H^\infty(K_\psi) \) on \( B(\mathcal{H}) \). Then it is natural to define
\[
\pi(Z) = \sum_{j=1}^{N} P_j \otimes \pi(\psi_j).
\]

Lemma 3.1. If
(i) \( \Sigma = (U, \mathcal{E}, \rho) \) is a \( C_b(\Psi) \)-unitary colligation;
(ii) the representation \( \rho \) is simple;
(iii) \( \pi \in \mathbb{F} \); and
(iv) \( \varphi = D + CZ(I - AZ)^{-1}B \), where \( Z = \rho(E) \),
then \( \pi(\varphi) \) is a contraction.

Proof. Let 0 < \( r < 1 \) and define
\[
\varphi_r = D + CrZ(I - rAZ)^{-1}B.
\]

Fix \( x \in X \). Since \( E(x) \) is a strict contraction and \( \rho \) is a contractive representation, \( Z = \rho(E(x)) \) is also a strict contraction. It follows that pointwise, \( \sum_{1}^{M}(rAZ(x))^n \) converges in norm to \( (I - rAZ(x))^{-1} \). Consequently,
\[
\varphi_r^M = D + CZ \sum_{1}^{M}(AZ)^n B,
\]
converges pointwise with \( M \) to \( \varphi_r \) and so the sequence \( \pi(\varphi_r^M) \) converges weakly to \( \pi(\varphi_r) \).

Let \( A_j = AP_j \) and for an \( n \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with each \( \alpha_j \in \{1, 2, \ldots, N\} \), let \( A_\alpha = A_{\alpha_1}A_{\alpha_2} \cdots A_{\alpha_n} \). Define \( \psi_\alpha \) similarly. Let \( |\alpha| = n \). By expanding \( \varphi_r^M \) we have
\[
\varphi_M = D + \sum_j \sum_{|\alpha| \leq M} r^{|\alpha|+1}CP_jA_\alpha B\psi_j\psi_\alpha.
\]

Thus
\[
\pi(\varphi_r^M) = D \otimes I + \sum_j \sum_{|\alpha| \leq M} r^{|\alpha|+1}CP_jA_\alpha B\pi(\psi_j)\pi(\psi_\alpha).
\]
\[
= D \otimes I + (C \otimes I)r\pi(Z)\sum_{n=1}^{M}((rA \otimes I)\pi(Z))^n(B \otimes I),
\]
according to the definition of \( \pi(Z) \) in (3.1). The right side converges in norm with \( M \) to
\[
D \otimes I + (C \otimes I)r\pi(Z)((I \otimes I) - r(A \otimes I)\pi(Z))^{-1}(B \otimes I),
\]
giving a transfer function representation for \( \pi(\varphi_r) \). The proof that \( \pi(\varphi) \) is a contraction proceeds along the lines of that given for Lemma 2.1 and makes use of the assumption (built into the definition of \( \pi \)) that each \( \pi(\psi_j) \) is a contraction.

To complete the proof, note that \( \varphi_r \) converges (with \( r \)) pointwise boundedly to \( \varphi \) and thus \( \pi(\varphi_r) \) converges WOT to \( \pi(\varphi) \). Since each \( \pi(\varphi_r) \) is contractive, so is \( \pi(\varphi) \). \( \square \)

Proposition 3.2. Suppose \( \pi \in \mathbb{F} \). If
(i) \( \Sigma = (U, \mathcal{E}, \rho) \) is a \( C_b(\Psi) \)-unitary colligation; and
(ii) \( \varphi = D + CZ(I - AZ)^{-1}B \), where \( Z = \rho(E) \),
then there exists a net of simple representations \( \rho_\alpha : C_b(\Psi) \to B(\mathcal{E}) \) such that the net
\[
\varphi_\alpha = D + CZ_\alpha(I - AZ_\alpha)^{-1}B,
\]
converges pointwise to \( \varphi \), where \( Z_\alpha = \rho_\alpha(E) \). Consequently, \( \pi(\varphi_\alpha) \) converges weakly to \( \pi(\varphi) \), and so \( \pi(\varphi) \) is a contraction.

**Proof.** Consider the collection \( \mathcal{F} \) consisting of ordered pairs \((F, \epsilon)\) where \( F \) is a finite subset of \( X \) and \( \epsilon > 0 \) ordered by \((F, \epsilon) \leq (G, \delta)\) if \( F \subset G \) and \( \delta \leq \epsilon \). With this order \( \mathcal{F} \) is a directed set.

Given \( \alpha = (F, \epsilon) \in \mathcal{F} \), by the compactness of \( \beta \Psi \) there exists a finite collection \( \mathcal{U} = \{U_1, \ldots, U_m\} \) of nonempty open sets which covers \( \beta \Psi \) with the property that for any two \( \psi', \psi'' \in U_j \) and any \( x \in F \),
\[
|\psi'(x) - \psi''(x)| < \epsilon.
\]

We construct a partition \( \Delta_\alpha = \{\Delta_1, \ldots, \Delta_m\} \) of \( \beta \Psi \) from \( \mathcal{U} \) in the usual way. Let
\[
\Delta_m = U_m \setminus (\cup_{j=1}^{m-1} U_j).
\]
Choose points \( \psi_j^\alpha \in U_j \cap \Psi \). While \( \psi_j^\alpha \) need not be in \( \Delta_j \) it is the case that if \( \psi \in \Delta_j \), then (3.2) holds with \( \psi' = \psi_j \) and \( \psi'' = \psi \). Since, for \( x \in F \),
\[
\left\| \sum_j \psi_j^\alpha(x)Q(\Delta_j) - \int \psi(x)dQ(\psi) \right\| \leq \sum_j \left\| \int (\psi_j^\alpha(x) - \psi(x))dQ(\psi) \right\| \leq \epsilon
\]

Let \( Q \) denote spectral measure associated to \( \rho \) so that
\[
\rho(f) = \int_{\beta \Psi} f \ dQ.
\]
Define
\[
\rho_\alpha(f) = \sum_{j=1}^m f(\psi_j)Q(\Delta_j)
\]
and let \( Z_\alpha(x) = \rho_\alpha^\prime(E(x)) \). It follows by (3.3) that for \( \alpha = (F, \epsilon) \), \( \|Z_\alpha(x) - Z(x)\| \leq \epsilon \) for \( x \in F \) and this remains true if \( \alpha \) is replaced by any \( \beta \geq \alpha \). Since \( \|E(x)\| < 1 \), we have \( 0 < \delta = \sup_{x \in F} (1 - \|E(x)\|)/2 \), and so for \( r = 1 - \delta/2 \) and \( \epsilon < \delta/2 \), it follows that \( \|Z_\alpha(x)\| < r = 1 - \epsilon \).

By Lemma 2.1 for any \( \alpha \),
\[
\varphi_\alpha = D + CZ_\alpha(I - AZ_\alpha)^{-1}B.
\]
is a contraction. Note that
\[
(AZ_\alpha)^n - (AZ)^n
= A(Z_\alpha - Z)AZ_\alpha \cdots AZ_\alpha + AZ_\alpha(Z_\alpha - Z)AZ_\alpha \cdots AZ_\alpha + \cdots + AZ_\alpha \cdots AZ_\alpha Z_\alpha - Z
\]
and so
\[
\|(AZ_\alpha)^n - (AZ)^n\| \leq n\|A\|r^{n-1}.
\]
Hence for suitably chosen \( \alpha \),
\[
\|Z_\alpha(I - AZ_\alpha)^{-1} - Z(I - AZ)^{-1}\|
= \|(Z_\alpha - Z)(I - AZ_\alpha)^{-1} + Z[(I - AZ_\alpha)^{-1} - Z(I - AZ)^{-1}]\|
\leq \epsilon \left[ \frac{1}{1-r} + \frac{r^2}{(1-r)^2} \right],
\]
Thus the bounded net \( \varphi_\alpha \) converges pointwise to \( \varphi \).
As constructed each \( \varphi_\alpha \) has a simple transfer function representation, and so by Lemma 3.1, \( \pi(\varphi_\alpha) \) is a contraction. Since the net \( \varphi_\alpha \) is bounded and converges pointwise to \( \varphi \), then net \( \pi(\varphi_\alpha) \) converges in the weak operator topology to \( \pi(\varphi) \). Hence \( \pi(\varphi) \) is a contraction.

3.2. **Factorization.** The engine powering the lurking isometry argument in the proof of (iiX) implies (iii) of Theorem 2.2 is the factorization in the following proposition. A similar result may be found in [6]. A detailed proof, which we hint at, is given in [12]. See also the book [5] Theorem 2.53 proof 1.

**Proposition 3.3.** If \( \Gamma : X \times X \to C_b(\Psi)^* \) is positive, then there exists a Hilbert space \( \mathcal{E} \) and a function \( L : X \to B(C_b(\Psi), \mathcal{E}) \) such that

\[
\Gamma(x,y)(fg^*) = \langle L(x)f, L(y)g \rangle
\]

for all \( f, g \in C_b(\Psi) \).

Further, there exists a unital \( * \)-representation \( \rho : C_b(\Psi) \to B(\mathcal{E}) \) such that \( L(x)ab = \rho(a)L(x)b \) for all \( x \in X, a, b \in C_b(\Psi) \).

**Sketch of the proof.** Let \( V \) denote the vector space with basis \( X \). On the vector space \( V \otimes C_b(\Psi) \) introduce the positive semidefinite sesquilinear form induced by

\[
\langle x \otimes f, y \otimes g \rangle = \Gamma(x,y)(g^*f),
\]

where \( x, y \in X \) and \( f, g \in C_b(\Psi) \). This is positive semidefinite by the hypothesis that \( \Gamma \) is positive semidefinite. Mod out by the kernel and complete to get the Hilbert space \( \mathcal{E} \). Define \( L \) by \( L(x)a = x \otimes a \) and verify that this is a bounded operator.

The \( * \)-representation is induced by the left regular representation of \( C_b(\Psi) \), \( \rho : C_b(\Psi) \to B(\mathcal{E}) \) with \( \rho(a)(x \otimes f) = x \otimes af \). Then check that \( \rho \) is a contractive unital representation of \( C_b(\Psi) \) satisfying \( L(x)ab = \rho(a)L(x)b \) for all \( x \in X, a, b \in C_b(\Psi) \). □

3.3. **A closed cone.** The proof of (i) implies (ii) in Theorem 2.2 is based on a cone separation argument which, in order to work, requires that the cone be closed and have nonempty interior. We present some of the background material here.

Given a finite subset \( F \subset X \), let \( C_b(\Psi)_{\mathcal{F}}^+ \) denote the collection of positive kernels \( \Gamma : F \times F \to C_b(\Psi)^* \). If \( \Gamma \in C_b(\Psi)_{\mathcal{F}}^+ \) and \( x \in F \), then \( \Gamma(x,x) \) is a positive linear functional on the unital \( C^* \)-algebra \( C_b(\Psi) \) and therefore, \( \|\Gamma\| = \Gamma(1) \).

Details of the proof outlined below can be found in [12].

**Lemma 3.4.** Let \( \Psi \) be a set of test functions. If for each \( x \in X, \|E(x)\|_{\infty} < 1 \), then

\[
\mathcal{C}_F = \{ (\Gamma(x,y)(I - E(x)E(y)^*))_{x,y \in F} : \Gamma \in C_b(\Psi)_{\mathcal{F}}^+ \}
\]

is a closed cone of \( |F| \times |F| \) matrices (where \( |F| \) is the cardinality of \( F \)).

**Sketch of the proof.** Suppose \( M = (\Gamma(x,y)(I - E(x)E(y)^*)) \in \mathcal{C}_F \). Since \( \|E(x)\| < 1, 1 - E(x)E(x)^* > \epsilon I \) for some \( \epsilon > 0 \). Hence \( \frac{1}{\epsilon} M(x,x) \geq \Gamma(x,x)11 = \|\Gamma(x,x)\| \), and so \( \|\Gamma(x,x)\| \lesssim \frac{1}{\epsilon} \|M\| \). Finiteness of \( F \) means that there is in fact a single \( \epsilon \) which will do for all \( x \in F \), while positivity of \( \Gamma \) implies that for \( g \in C_b(\Psi) \),

\[
2|\Gamma(x,y)g| \leq \Gamma(x,x)1 + \Gamma gg^* \leq \|\Gamma(x,x)\| + \|\Gamma(y,y)\|\|g\|\|g^*\|.
\]

Consequently, \( \|\Gamma(x,y)\| \lesssim \frac{1}{\epsilon} \|M\| \) for all \( x, y \in F \).

Now suppose \( M_j \in \mathcal{C}_F \) is a Cauchy sequence. For each \( j \) there exists \( \Gamma_j \in C_b(\Psi)_{\mathcal{F}}^+ \) so that

\[
M_j = (\Gamma_j(x,y)(I - E(x)E(y)^*))_{x,y \in F}.
\]
Since the $M_j$’s are uniformly bounded, $\Gamma_j(x, y)$ is uniformly bounded for all $x, y$ and $j$. Thus there is a subsequence $\Gamma_{j_i}$ such that $\Gamma_{j_i}(x, y)$ converges weak*- to $\Gamma(x, y)$. Likewise, $\Gamma_j(x, y)E(x)E(y)^*$ converges weak*- to $\Gamma(x, y)E(x)E(y)^*$. Hence $M = \lim_j M_j = (\Gamma(x, y)(1 - E(x)E(y)^*))_{x, y \in F}$. Positivity of $\Gamma$ is a consequence of the positivity of the $\Gamma_j$’s. We conclude that $C_F$ is closed. \hfill \Box

The next lemma gives an example of a positive kernel in $C_b(\Psi)_F$ which will be particularly useful in showing that the cone $C_F$ in (3.4) has nonempty interior in the subsequent lemma.

**Lemma 3.5.** Let $\Psi$ be a set of test functions for which $\|E(x)\| < 1$ for all $x$. For each $\psi \in \Psi$ the function $\Gamma_\psi : X \times X \to C_b(\Psi)^*$ given by

$$\Gamma_\psi(x, y)(f) = \frac{f(\psi)}{1 - \psi(x)\psi(y)^*}$$

is a positive kernel. Here $f \in C_b(\Psi)$.

**Proof.** Note that $|\psi(x)| \leq \|E(x)\| < 1$ so that the formula makes sense and moreover,

$$S(x, y) = \frac{1}{1 - \psi(x)\psi(y)^*}$$

defines a positive kernel on $X$. With $y = x$,

$$|\Gamma_\psi(x, x)(f)| = \frac{|f(\psi)|}{|1 - |\psi(x)||^2} \leq \|f\| \frac{1}{|1 - |\psi(x)||^2}$$

so that $\Gamma_\psi(x, x)$ is indeed in $C_b(\Psi)^*$.

For a finite set $F \subset X$ and function $f : F \to C_b(\Psi)$

$$\sum_{x, y \in F} \Gamma_\psi(x, y)(f(x)f(y)^*) = \sum_{x, y \in F} f(x)(\psi)f(y)(\psi)^*S(x, y) \geq 0,$$

since $S$ is a positive kernel on $X$. It follows that each $\Gamma_\psi(x, y) \in C_b(\Psi)^*$ and $\Gamma_\psi$ is a positive kernel on $X$. \hfill \Box

**Lemma 3.6.** Let $\Psi$ be a set of test functions, $F \subset X$ a finite set, and $C_F$ the cone in (3.4). Then $C_F$ contains all positive $|F| \times |F|$ matrices, and hence has nonempty interior.

**Proof.** Let $\Gamma_\psi$ denote the positive kernel from Lemma 3.5. Then

$$[1] = \Gamma_\psi(x, y)(I - E(x)E(y)^*) \in C_F,$$

where $[1]$ is the matrix with all entries equal to 1. For $P$ be a positive $|F| \times |F|$ matrix, $\tilde{\Gamma}$ defined by $\tilde{\Gamma}(x, y) = P(x, y)\Gamma_\psi(x, y)$ is a positive kernel, and so $P(x, y) = \tilde{\Gamma}(x, y)(I - E(x)E(y)^*)$. \hfill \Box

**Lemma 3.7.** The cone $C_F$ in (3.4) is closed under conjugation by diagonal matrices; i.e., if $M = (M(x, y)) \in C_F$ and $c : F \to \mathbb{C}$, then $cMc^* = (c(x)M(x, y)c(y)^*) \in C_F$.

**Proof.** Simply note that if $\Gamma : F \times F \to C_b(\Psi)^*$ is positive, then so is $c\Gamma c^*$ defined by $(c\Gamma c^*)(x, y) = c(x)c(y)^*\Gamma(x, y)$. \hfill \Box

The next proposition connects the closed unit ball of of $H^\infty(K_\psi)$ with the cone $C_F$. Further details of the proof sketched below can be found in Lemmas 5.5 and 3.4 of [12].
Proposition 3.8. Let $F \subset X$ be finite and $\varphi \in H^\infty(\mathcal{K}_\Psi)$. If $M_\varphi$ defined by
\[
M_\varphi(x,y) = 1 - \varphi(x)\varphi(y)^* , \quad x,y \in F,
\]
is not in $C_F$, then there is a kernel $k \in \mathcal{K}_\Psi$ such that the matrix
\[
(1 - \varphi(x)\varphi(y)^*)k(x,y)_{x,y \in F}
\]
is not positive. That is, $\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} > 1$.

Sketch of the proof. Use a version of the Hahn-Banach theorem to find a linear functional $\lambda \neq 0$ on the selfadjoint $|F| \times |F|$ matrices such that $\lambda(M) \geq 0$ for all $M \in C_F$ but $\lambda(M_\varphi) < 0$. We can find such a $\lambda$ since $C_F$ is closed and has nonempty interior by Lemmas 3.3 and 3.6.

For $f,g \in P(F)$ (viewed as vectors in $\mathbb{C}^F$), define $\langle f, g \rangle = \lambda(fg^*)$. Since $C_F$ contains all positive $|F| \times |F|$ matrices, this is positive. Mod out by the kernel and call the resulting space $\mathcal{H}$. Let $q$ be the quotient map. Show that $\lambda(M_\varphi) < 0$ implies $\lambda([1]) > 0$, and hence $q(\delta_F) \neq 0$, where $\delta_F$ is the function in $P(F)$ which is identically 1.

Let $\mu$ be a representation of $P(F)$ given by $\mu(g)q(f) = q(fg)$, where the product $fg$ is defined pointwise. Verify that $\mu$ is contractive on test functions and that $\mu([1] - \varphi|F\varphi^*|F) < 0$.

What is more, $\delta_F$ is a cyclic vector for $\mu$. Hence if $\xi_x \in P(F)$ is defined to be 1 at $x$ and zero elsewhere, then $\{\xi_x = \mu(\xi_x)\delta_F\}_{x \in F}$ is a basis for $\mathcal{H}$. Let $\{k_x\}$ be the dual basis, $k(x,y) = \langle k_x, k_y \rangle$. Then for $c \in \mathbb{C},$
\[
\langle \mu(c\xi_x)^*k_a, \ell_b \rangle = c^* \langle k_a, \mu(\xi_x)\ell_b \rangle = c^* \langle k_a, \mu(\xi_x)\mu(\xi_b)\delta_F \rangle = \begin{cases} c^* & \text{if } x = b = a \\ 0 & \text{otherwise} \end{cases} = \langle c^*k_a, \ell_b \rangle.
\]
So for $f \in P(F)$, $\mu(f)^*k_a = f(a)^*k_a$. If $f$ is the test function $\psi$, this yields that the matrix
\[
((1 - \psi(x)\psi(y)^*)k(x,y))_{x,y \in F}
\]
is positive, while with $f = \varphi$, it is strictly negative. Extend $k$ to all of $X \times X$ by setting $k(x,y) = 0$ if either $x$ or $y$ are not in $F$. We then have $((1 - f(x)f(y)^*)k(x,y))_{x,y \in X}$ is positive when $f$ is a test function (so that $k \in \mathcal{K}_\Psi$), but not positive for $f = \varphi$. \hfill \Box

3.4. A compact set. The proof of (iiF) implies (iiX) in Theorem 2.2 uses Kurosh’s theorem (5), Theorem 2.56), the application of which requires that certain sets be compact.

Fix $\varphi : X \to \mathbb{C}$ and a collection of test functions $\Psi$. For $F \subset X$, let
\[
\Phi_F = \{ \Gamma \in C_0(\Psi)_F^+ : 1 - \varphi(x)\varphi(y)^* = \Gamma(x,y)(1 - E(x)E(y)^*) \text{ for } x,y \in F \}.
\]
The set $\Phi_F$ is naturally identified with a subset of the product of $C_0(\Psi)^*$ with itself $|F|^2$ times.

Lemma 3.9. If for each $x \in X$, $\|E(x)\| < 1$, then the set $\Phi_F$ is compact.

Proof. Let $\Gamma_\alpha$ be a net in $\Phi_F$. Arguing as in the proof of Lemma 3.4, we find each $\Gamma_\alpha(x,x)$ is a bounded net and thus each $\Gamma_\alpha(x,y)$ is also a bounded net. By weak-$*$ compactness of the unit ball in $C_0(\Psi)^*$ there exists a $\Gamma$ and subnet $\Gamma_\beta$ of $\Gamma_\alpha$ so that for each $x, y \in F$, $\Gamma_\beta(x,y)$ converges to $\Gamma(x,y)$. \hfill \Box

4. Proofs

We are now set to prove the theorems stated in Subsection 2.4.
4.1. Proof of Theorem 2.2

4.1.1. Proof of (i) implies (iiF). Let \( \varphi \in H^\infty(K\Psi) \). If we suppose (iiF) does not hold, then by Proposition 3.8, \( \|\varphi\|_{H^\infty(K\Psi)} > 1 \).

4.1.2. Proof of (iiF) implies (iiX). The proof here uses Kurosh’s Theorem and in much the same way as in [5].

The hypothesis is that for every finite subset \( F \subset X \), \( \Phi_F \), as defined in subsection 3.4, is not empty, and so by Lemma 3.9, \( \Phi_F \) is compact. For finite set \( F \subset G \), define \( \pi_G^F : \Phi_F \to \Phi_G \) by

\[
\pi_G^F(\Gamma)_F = \Gamma_{|F \times F}.
\]

Thus, with \( F \) equal to the collection of all finite subsets of \( X \) partially ordered by inclusion, the triple \( (\Phi_F, \pi_G^F, F) \) is an inverse limit of nonempty compact spaces. Consequently, by Kurosh’s Theorem, for each \( F \) there is a \( \Gamma_F \in \Phi_F \) so that whenever \( F, G \in F \) and \( F \subset G \),

\[
\pi_G^F(\Gamma_G) = \Gamma_F.
\]

Define \( \Gamma : X \times X \to C_b(\Psi) \) by

\[
\Gamma(x,y) = \Gamma_F(x,y)
\]

where \( F \in F \) and \( x,y \in F \). This is well defined by the relation in equation (4.1). If \( F \) is any finite set and \( f : F \to C_b(\Psi) \) is any function, then

\[
\sum_{x,y \in F} \Gamma(x,y)(f(x)f(y)^*) = \sum_{x,y \in F} \Gamma_F(x,y)(f(x)f(y)^*) \geq 0
\]

since \( \Gamma_F \in C_b(\Psi)_F \). Hence \( \Gamma \) is positive.

4.1.3. Proof of (iiX) implies (iii). Let \( \Gamma \) denote the positive kernel of the hypothesis of (iiX). Apply Lemma 3.3 to find \( E, L : X \to B(C_b(\Psi), E) \), and \( \rho : C_b(\Psi) \to B(E) \) as in the conclusion of the lemma.

Rewrite condition (iiX) as

\[
1 + \langle Z(x)L(x)1, Z(x)L(x)1 \rangle = \varphi(x)\varphi(y)^* + \langle L(x)1, L(x)1 \rangle,
\]

where we use Proposition 3.3 to express \( L(x)E(x) = Z(x)L(x)1 \) with \( Z(x) = \rho(E(x)) \). From here the remainder of the proof is the standard lurking isometry argument.

Let \( \mathcal{E}_d \) denote finite linear combinations of

\[
\begin{pmatrix}
Z(x)L(x)1 \\
1
\end{pmatrix} \in \mathcal{E}
\]

and let \( \mathcal{E}_r \) denote finite linear combinations of

\[
\begin{pmatrix}
L(x)1 \\
\varphi(x)
\end{pmatrix} \in \mathcal{E}.
\]

Define \( V : \mathcal{E}_d \to \mathcal{E}_r \) by

\[
V\begin{pmatrix}
Z(x)L(x)1 \\
1
\end{pmatrix} = \begin{pmatrix}
L(x)1 \\
\varphi(x)
\end{pmatrix},
\]

extend by linearity and show that \( V \) is a well defined isometry on \( \mathcal{E}_d \), and hence on \( \mathcal{E}_d \). This further extends to a unitary operator

\[
U = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} : \mathcal{H} \to \mathcal{H},
\]

where

\[
\begin{pmatrix}
A \\
C
\end{pmatrix} : \mathcal{H} \to \mathcal{H},
\]

and

\[
\begin{pmatrix}
B \\
D
\end{pmatrix} : \mathcal{H} \to \mathcal{H}.
\]
with $U$ restricted to $\mathcal{E}_d$ equal to $V$; that is $U\gamma = V\gamma$ for $\gamma \in \mathcal{E}_d$.

This gives the system of equations
\[
AZ(x)L(x)1 + B = L(x)1 \\
CZ(x)L(x)1 + D = \varphi(x),
\]
which, when solved for $\varphi$, yields
\[
\varphi(x) = D + CZ(x)(I - AZ(x))^{-1}B,
\]

as desired.

4.1.4. Proof of (iii) implies (ivF). This is a direct consequence of Proposition 3.2.

4.1.5. Proof that (ivF) is equivalent to (ivS). This is trivial in one direction. In the other, it follows from the proof of Lemma 3.1.

4.1.6. Proof of (ivF) implies (i). Take $\pi$ to be the identity representation in Proposition 3.2.

4.2. Agler-Pick Interpolation: Proof of Theorem 2.3. It turns out that in the Agler-Pick interpolation setting more can be said about the transfer function realization of the interpolant. Suppose $\mu$ is a (positive) measure on $\Psi$. The functions $E(x)$ determine multiplication operators on $L^2(\mu)$ by the formula $(E(x)f)(\psi) = \psi(x)f(\psi)$. Abusing notation, for a positive integer $n$, let $E(x)$ also denote the operator $I_n \otimes E(x)$ on $\mathbb{C}^n \otimes L^2(\mu)$, or more precisely, the representation $\rho(E(x)) = I_n \otimes E(x)$.

Proof of Theorem 2.3. If $\varphi$ exists, the implication (i) implies (iiF) of Theorem 2.2 applied to $F$ establishes the existence of $\Gamma$.

Conversely, suppose a positive $\Gamma$ satisfying equation (2.3) exists. View $\Gamma$ as an $n \times n$ matrix
\[
\Gamma = (\Gamma(x_\ell, x_j))_{j,\ell}
\]
with entries from $C_b(\Psi)^*$. The converse can be proved using the factorization from Proposition 3.3. However the proof of the last part about the measure $\mu$ requires a somewhat more concrete factorization of $\Gamma$.

Choose a positive measure $\mu$ on $\Psi$ so that each $\Gamma(x_\ell, x_j)$ is absolutely continuous with respect to $\mu$. We can without loss of generality assume that the measure is defined on the closure of $\Psi$ (if this is not already closed), and hence we may assume that the measure $\mu$ is bounded. By Radon-Nikodym, there exist $L^\infty(\mu)$ functions $F_{\ell,j}$ so that $\Gamma(x_\ell, x_j) = F_{\ell,j} d\mu$. In particular, the matrix-valued function $F$ can be identified with an element of the $C^*$-algebra of $n \times n$ matrices with entries from $L^\infty(\mu)$. The fact that $\Gamma$ is positive implies that $F$ is (almost everywhere $\mu$) pointwise positive. Consequently, there exists vectors $H$ from $\mathbb{C}^n \otimes L^\infty(\mu)$ so that $F_{\ell,j} = H(x_\ell)H(x_j)^*$. This gives the factorization,
\[
\Gamma = HH^* d\mu.
\]

Observe
\[
\Gamma(x_\ell, x_j)(1 - E(x_\ell)E(x_j)^*)
= \int H(x_\ell)H(x_j)^* d\mu - \int H(x_\ell)E(x_\ell)^*E(x_j)H(x_j)^* d\mu
= \langle H(x_\ell), H(x_j) \rangle - \langle E(x_\ell)H(x_j), E(x_\ell)H(x_\ell) \rangle.
\]
Thus, equation (2.3) becomes,
\[
1 + \langle E(x_\ell)H(x_j), E(x_\ell)H(x_\ell) \rangle = \xi(x_j)\xi(x_\ell)^* + \langle H(x_\ell), H(x_j) \rangle.
\] (4.2)
A lurking isometry argument as in the proof of (iiX) implies (iii) for Theorem 2.2 allows us to define a unitary operator

$$U = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) : \mathbb{C}^n \otimes L^2(\mu) \oplus \mathbb{C}^n \otimes L^2(\mu) \rightarrow \mathbb{C}$$

with

$$U \left( \begin{array}{c} E(x_j)H(x_j) \\ 1 \end{array} \right) = \left( \begin{array}{c} H(x_j) \\ \xi(x_j) \end{array} \right),$$

which can then be solved to give

$$\xi(x_j) = D + CE(x_j)(I - AE(x_j))^{-1}B.$$

Define

$$\varphi(x) = D + CE(x)(I - AE(x))^{-1}B$$

for $$x \in X$$. Then $$\varphi$$ extends $$\xi$$ and the implication (iii) implies (i) of Theorem 2.2 completes the proof. \qed

4.3. **Proof of Proposition 2.5** Let $$\Psi$$ be a collection of test functions, $$K_{\Psi}$$ and $$H^\infty(K_{\Psi})$$ as above, and suppose $$\varphi_\alpha$$ is a net in the Agler-Schur class of $$H^\infty(K_{\Psi})$$. Then for all $$\alpha$$, $$\|\varphi_\alpha\| \leq 1$$, and so $$\|\varphi\| \leq 1$$. Fix $$F \subset X$$ finite. Then there is a $$\Gamma_{F,\alpha} \geq 0$$ such that

$$1 - \varphi_\alpha(x)\varphi_\alpha(y)^* = \Gamma_{F,\alpha}(x, y)(1 - E(x)E(y)^*), \quad x, y \in F.$$ 

So the matrix $$M_\alpha = (1 - \varphi_\alpha(x)\varphi_\alpha(y)^*) \in C_F$$. Since by Lemma 3.4 $$C_F$$ is closed, arguing as at the end of the proof of that lemma, we have a $$\Gamma_F \geq 0$$ such that

$$1 - \varphi(x)\varphi(y)^* = \Gamma_F(x, y)(1 - E(x)E(y)^*), \quad x, y \in F.$$ 

Applying (iiF) implies (iiX) of Theorem 2.2 it follows that $$\varphi$$ is in the Agler-Schur class of $$H^\infty(K_{\Psi})$$. Finally, since the test functions are a subset of the Agler-Schur class, the last statement is obvious.

5. **Examples**

In this section we concentrate on two main examples where an infinite collection of test functions is required; the annulus and the infinite polydisk. We then close with a few further examples illustrating the necessity of various parts of our definitions of test functions.

5.1. **The annulus.** Fix $$q \in (0, 1)$$ and write $$\mathbb{A} = \mathbb{A}_q$$ for the annulus

$$\mathbb{A} = \{ z \in \mathbb{C} : q < |z| < 1 \}.$$ 

Let $$H^\infty(\mathbb{A})$$ denote the bounded analytic functions on $$\mathbb{A}$$. There is a collection of functions $$\vartheta_t$$ naturally parameterized by $$t$$ in the unit circle $$\mathbb{T}$$ with the property that each $$\vartheta_t$$ is unimodular on the boundary of $$\mathbb{A}$$ (and so extending analytically across the boundary) and has precisely two zeros in $$\mathbb{A}$$. Moreover, any function with these properties is, up to pre-composition with an automorphism of $$\mathbb{A}$$ and post-composition with an automorphism of $$\mathbb{D}$$, one of these $$\vartheta_t$$. We begin by constructing $$\vartheta_t$$ and showing that $$\Theta = \{ \vartheta_t : t \in \mathbb{T} \}$$ is indeed a family of test functions for $$H^\infty(\mathbb{A})$$.

Let $$B_0 = \{|z| = 1\}$$ and $$B_1 = \{|z| = q\}$$ denote the boundary components of the boundary $$B$$ of $$\mathbb{A}$$. For normalization, fix a base point $$b \in \mathbb{A}$$ such that $$|b| \neq \sqrt{q}$$. Using Green’s functions (or otherwise), for each point $$\alpha \in B$$ there exists a unique positive harmonic function $$h_\alpha$$ whose boundary
values come from the measure on $B$ with point mass at $\alpha$. If $h$ is any positive harmonic function on $\mathbb{A}$ there is a (positive) measure $\mu$ on $B$ so that

$$h(z) = \int_B h_\gamma \, d\mu(\gamma) = \int_{B_0} h_\alpha \, d\mu(\alpha) + \int_{B_1} h_\beta \, d\mu(\beta).$$

(5.1)

The harmonic function $h$ is the real part of an analytic function $f$ if and only if $\mu(B_0) = \mu(B_1)$. In particular, given $\alpha \in B_0$ and $\beta \in B_1$, the function $h_\alpha + h_\beta$ is the real part of an analytic function $f_{\alpha,\beta}$ which we may normalize by requiring the imaginary part of $f_{\alpha,\beta}(b) = 0$. Since both $h_\alpha$ and $h_\beta$ are nonnegative on the boundary they are both positive inside the annulus and so $f(b) > 0$. Then because the boundary values for the $h_\alpha$'s are point masses, (5.1) can be re-expressed as

$$h(z) = \frac{2}{\mu(B)} \Re \int_{B_0} \int_{B_1} f_{\alpha,\beta} \, d\mu(\beta) \, d\mu(\alpha),$$

and when $\mu(B_0) = \mu(B_1)$, this will be the real part of an analytic function $f$ with $f(b) > 0$. Given $\alpha \in B_0$ and $\beta \in B_1$, let

$$\psi_{\alpha,\beta} = \frac{f_{\alpha,\beta} - f_{\alpha,\beta}(b)}{f_{\alpha,\beta} + f_{\alpha,\beta}(b)}.$$ 

(5.2)

Then

$$f_{\alpha,\beta} = f_{\alpha,\beta}(b) \frac{\psi_{\alpha,\beta} + 1}{\psi_{\alpha,\beta} - 1}.$$ 

Note that $\psi_{\alpha,\beta}$ is unimodular on $B$, takes the value 0 at $b$, and in fact extends to an analytic function on a region containing $\mathbb{A}$. Further, $\psi_{\alpha,\beta}$ takes the value 1 on $B$ precisely at those points where $f_{\alpha,\beta} = \infty$; namely $\alpha$ and $\beta$. Thus, $\psi_{\alpha,\beta}$ is two to one, and by the Maximum Modulus Principle has two zeros in $\mathbb{A}$. Since the product of the moduli of the zeros is $q$, the second zero is also on the circle $\{|z| = \frac{q}{\sqrt{q}}\}$.

The assumption that $b \neq \sqrt{q}$ thus ensures that the zeros are distinct.

We claim that $\Theta' = \{\psi_{\alpha,\beta}\}$ is a collection of test functions for $H^\infty(\mathbb{A})$; that is, that the unit ball of $H^\infty(\mathbb{A})$ is the same as the unit ball of $H^\infty(\Theta')$. One direction is nearly automatic. Since $\Theta'$ is a subset of the unit ball of $H^\infty(\mathbb{A})$ it follows that the Szeg"{o} kernel $s$ for $\mathbb{A}$ is in $K_{K_{\Theta'}}$. Thus, if $\varphi$ is in the unit ball of $H^\infty(K_{\Theta'})$, then

$$((1 - \varphi(x)\varphi(y)^*)s(x,y)) \geq 0.$$ 

Hence $\varphi$ is in the unit ball of $H^\infty(\mathbb{A})$. (In general if $\Psi$ is contained in $\Psi'$, then the unit ball of $H^\infty(K_{\Psi'})$ is contained in the unit ball of $H^\infty(K_{\Psi'})$.)

To prove the converse inclusion, suppose $\xi : \mathbb{A} \to \mathbb{D}$ is analytic and $\xi(b)$ is real. There exists a $\mu$ so that

$$\frac{1 + \xi}{1 - \xi} = \int_{B_0} \int_{B_1} f_{\alpha,\beta} \, d\mu(\beta) \, d\mu(\alpha)$$

$$= \int_{B_0} \int_{B_1} f_{\alpha,\beta}(b) \frac{\psi_{\alpha,\beta} + 1}{\psi_{\alpha,\beta} - 1} \, d\mu(\beta) \, d\mu(\alpha).$$

For $z, w \in \mathbb{A},$

$$\frac{1 + \xi}{1 - \xi}(z) + \frac{1 + \xi}{1 - \xi}(w)^* = \frac{1}{1 - \xi(z)(1 - \xi(z)\xi(w)^*)}$$

$$= \int_{B_0} \int_{B_1} f_{\alpha,\beta}(b) \frac{1 - \psi_{\alpha,\beta}(z)\psi_{\alpha,\beta}(w)^*}{(1 - \psi_{\alpha,\beta}(z))(1 - \psi_{\alpha,\beta}(w)^*)} \, d\mu(\beta) \, d\mu(\alpha).$$
Thus, there exist functions $H_{\alpha,\beta}(z, w)$, analytic in $z$, conjugate analytic in $w$ and continuous in $\alpha, \beta$ for fixed $z, w$ so that

$$1 - \xi(z)\xi^*(w) = \int_{B_0} \int_{B_1} H_{\alpha,\beta}(z, w)(1 - \psi_{\alpha,\beta}(z)\psi_{\alpha,\beta}(w)^*)d\mu(\beta)d\mu(\alpha).$$

and the claim is proved.

There is some redundancy in our choice of test functions. Given $t \in T$, let $\vartheta_t$ denote the function analytic in $A$, unimodular on $B$ with zeros at $b$ and $q_t b$ and with $\vartheta_t(1) = 1$. The collection $\Theta = \{\vartheta_t : t \in T\}$ is uniformly continuous in $t$ and $z$. For each $\alpha, \beta$, there exist $t, \gamma$ in $T$ so that $\psi_{\alpha,\beta} = \gamma \vartheta_t$. Thus, $\Theta$ is a totally bounded collection of test functions for $H^\infty(\mathbb{A})$. (As an alternate, use the parameterization of the unimodular functions with precisely two zeros in terms of theta functions [14]).

**Lemma 5.1.** The collection $\Theta$ is a collection of test functions for $H^\infty(\mathbb{A})$ and is compact in the norm topology of $H^\infty(\mathbb{A})$.

Similar results for triply connected domains may be found in [13]. See also a comment in [18].

The realization theorem, Theorem 2.2, now reads as follows.

**Proposition 5.2.** Suppose $\varphi : \mathbb{A} \to \mathbb{C}$. The following are equivalent.

(i) $\varphi \in H^\infty(\mathbb{A})$ with norm less than or equal to one;

(ii) There is a positive kernel $\Gamma : \mathbb{A} \times \mathbb{A} \to C(T)^*$ so that

$$1 - \varphi(z)\varphi(w)^* = \Gamma(z, w)(1 - E(z)E(w)^*)$$

where $E(z)(\vartheta_t) = \vartheta_t(z)$; and

(iii) there exists an auxiliary Hilbert space $E$ and an analytic function $\Phi : \mathbb{A} \to B(E)$ whose values $\{\Phi(z)\}$ are commuting normal contraction operators and a unitary

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : E \oplus C \to E \oplus C$$

so that $\varphi$ has the unitary colligation transfer function realization

$$\varphi(z) = D + C\Phi(z)(I - A\Phi(z))^{-1}B.$$ 

We were able to reduce our original collection of test functions for the annulus to $\Theta$. It is reasonable to wonder if it is possible to throw out even more. The next proposition shows that the answer is “no”. It resembles a result of [10] which says that in a sense all of the Sarason/Abrahamsen reproducing kernels for the annulus are needed for Nevanlinna-Pick interpolation on the annulus.

**Proposition 5.3.** No proper closed subset of $\Theta$ is a set of test functions for $H^\infty(\mathbb{A})$.

Note that a dense subset of the test functions will also be a set of test functions, though in this case there is no real advantage to taking such a set. The situation will be quite different in the case of the infinite polydisk, as we shall see.

**Proof of Proposition 5.3.** Suppose $C$ is a proper subset of $\Theta$, and that $\vartheta_0 = \vartheta_{t_0}$ is not in $C$.

Let $k : \mathbb{A} \times \mathbb{A} \to \mathbb{C}$ denote the Szegö kernel for the annulus with respect to harmonic measure $\omega$ for the base point $b \in \mathbb{A}$ (recall that we assume $|b| \neq \sqrt{q}$). Let $X$ denote the Schottky double of $\mathbb{A}$ and write $Jz$ for the twin of $z$ in the double. According to Fay [14] (see also [10]), for each $a \in \mathbb{A}$ the kernel $k_a = k(\cdot, a)$ is meromorphic on $X$ with exactly two poles with the exception $k_b = 1$ (so in
particular, \(k(b, z) = k(z, b) = 1\) for all \(z\). Moreover, there exists a point \(P\) in the complement of the closure of \(\mathbb{A}\) so that \(k_0\) has poles at \(P\) (independent of \(a\)) and \(J_a\).

The kernels
\[
\Delta_t(z, w) = (1 - \vartheta_t(z)\vartheta_t(w)^*)k(z, w)
\]
are positive and have rank two. To see this, observe that \(M_t\), the operator of multiplication by \(\vartheta_t\) on \(H^2(k)\), is an isometry, so \(1 - M_tM_t^*\) is the projection onto \(\ker M_t^*\). Furthermore, if \(b\) and \(a_t\) are the two zeros of \(\vartheta_t\) (distinct since \(|b| \neq \sqrt{a}\)), then the identity \(M_t^*k_w = \vartheta_t(w)^*k_w\) implies that \(\ker M_t^* = \mathcal{K}_t = \text{span} \{k_b, k_{a_t}\}\). If we choose \(f_t = k_b = 1\) and \(g_t = k_{a_t} - k_b = k_{a_t} - 1\), then
\[
\Delta_t = f_tf_t^* + g_tg_t^* = 1 + g_tg_t^*.
\]
It is useful to remark for later use that \(g_t\) has the same poles as \(k_{a_t}\); namely \(P\) and \(Ja_t\).

Choose three distinct points \(z_1, z_2, z_3 \in \mathbb{A}\) and consider the Agler-Pick interpolation problem of finding a \(\varphi \in H^\infty(\mathbb{A})\) so that \(\varphi(z_t) = \vartheta_0(z_t)\). The fact that the \(3 \times 3\) matrix
\[
(k(z_t, z_m)(1 - \vartheta_0(z_t)\vartheta_0(z_m)^*))^3_{j,m=1}
\]
has rank two implies that this interpolation problem has a unique solution, namely \(\varphi = \vartheta_0\). On the other hand, there is a bounded positive measure \(\mu\) on \(C\) so that \(\varphi\) has a realization of the form in Theorem 2.3 with \(n = 3\). The usual computations convert that realization to
\[
1 - \vartheta_0(z)\vartheta_0(w)^* = \int_C \sum_{\nu=1}^3 h_{\nu}(z, \vartheta)h_{\nu}(w, \vartheta)^*(1 - \vartheta(z)\vartheta(w)^*)\,d\mu(\vartheta)
\]
for functions \(h_\nu(z, \cdot) \in L^2(\mu)\). In particular, multiplying through by \(k(z, w)\) gives
\[
\Delta_0(z, w) = \int_C \sum_{\nu=1}^3 h_{\nu}(z, \vartheta)\Delta_0(z, w)h_{\nu}(w, \vartheta)^*\,d\mu(\vartheta)
\]
where \(\Delta_0(z, w) = (1 - \vartheta(z)\vartheta(w)^*)k(z, w)\).

Fix \(z\). Since \(\Delta_0(z, z) \geq 0 \mu\text{-a.s.},\) given \(\delta > 0\), there is a set \(C' \subset C\) and a constant \(c_\delta > 0\) such that \(\mu(C - C') < \delta\) and for all \(z \in \mathbb{A}\),
\[
\Delta_0(z, z) \geq c_\delta h_{\nu}(z, \vartheta)\Delta_0(z, w)h_{\nu}(w, \vartheta)^*, \quad \vartheta \in C'.
\]
Then using the factorization of the \(\Delta\)’s given above, by Douglas’ lemma there are constants \(c_k, k = 1, 2, 3, 4,\) such that for fixed \(\vartheta \in C'\),
\[
h_{\nu}(\cdot, \vartheta) = c_1 + c_2g_{a_0},
\]
\[
h_{\nu}(\cdot, \vartheta)g_{a_0} = c_3 + c_4g_{a_0}.
\]
Since the kernels extend meromorphically to \(X\), the same is true for \(h_{\nu}(\cdot, \vartheta)\) by the first equation. That equation also implies that either \(h_{\nu}(\cdot, \vartheta)\) is constant or that it has the same poles as \(g_{a_0}\); that is, simple poles at \(P\) and \(a_0\). If \(h_{\nu}(\cdot, \vartheta)\) is not constant, then the left side of the second equation has a double pole at \(P\), while the right only has a single pole. Hence \(h_{\nu}(\cdot, \vartheta)\) must be constant. If it is a nonzero constant, the second equation would imply that the poles of \(g_{a_1}\) and \(g_{a_0}\) agree, and in particular, that \(a_1 = a_0\), contradicting the assumption that \(\vartheta_0 \notin C\). Hence \(h_{\nu}(\cdot, \vartheta) = 0\).

Taking \(\delta\) going to 0, we see that the subset of \(C'\) on which \(h_{\nu}(\cdot, \vartheta)\) is nonzero has \(\mu\) measure zero, yielding a contradiction.

\[\square\]
5.2. The infinite polydisk. By the infinite polydisk $\mathbb{D}^\infty$, we mean the open unit ball of $C_b(\mathbb{N})$. Thus,

$$\mathbb{D}^\infty = \{ z : \mathbb{N} \to \mathbb{D} : \text{sup}\{ |z(n)| : n < 1 \} \}.$$

Let $e_n$ denote the function $e_n : \mathbb{D}^\infty \to \mathbb{C}$ given by $e_n(z) = z(n)$. The set of test functions $\Psi = \{ e_n : n \in \mathbb{N} \}$ is topologized by the inclusion $\Psi \subset B(\mathbb{D}^\infty, \mathbb{D})$. The spaces $\Psi$ and $\mathbb{N}$ are homeomorphic and hence $\beta\Psi$ is identified with $\beta\mathbb{N}$.

A $\chi \in \beta\mathbb{N}\backslash\mathbb{N}$ determines a function $\varphi_\chi : X \to \mathbb{D}$ given by

$$\varphi_\chi(z) = z(\chi),$$

where we have identified $z \in X$ with its unique extension to a continuous function $z : \beta\mathbb{N} \to \mathbb{C}$. This identification follows from the general discussion of $\Psi$ and $\beta\Psi$ in subsection 2.7. Further, $\varphi_\chi$ is in the unit ball of $H^\infty(\Psi)$.

Theorem 2.2 now implies that there is a positive kernel with entries in $C(\mathbb{D}^\infty)^*$ such that

$$1 - \varphi(z)\varphi(w)^* = (z, w)(1 - E(z)E(w)^*) \geq 0.$$

In this case there is a clear choice for $\Gamma$; namely, $\Gamma(z, w) = \gamma$, where $\gamma(e) = e(\chi)$ for $e \in C(\overline{\mathbb{D}^\infty})$.

**Proposition 5.4.** There do not exist positive kernels $\Gamma_n : X \times X \to \mathbb{C}$ such that

$$1 - \varphi_\chi(z)\varphi_\chi(w)^* = \sum_n \Gamma_n(z, w)(1 - z(n)w(n)^*).$$

Similarly, if $C$ is any closed subset of $\beta\mathbb{N}$ with $\chi \notin C$, then there does not exist a positive $\Gamma : X \times X \to C(C)^*$ such that

$$1 - \varphi_\chi(z)\varphi_\chi(w)^* = (z, w)(I - E(z)E(w)^*).$$

**Proof.** For the first part observe that if $z(n)$ converges to $L$ as $n \to \infty$, then $\chi(z) = L$ for any $\chi \in \beta\mathbb{N}\backslash\mathbb{N}$.

Choose $z(n) = \frac{1}{2}\left(1 - \frac{1}{n+1}\right)$ we have $\frac{1}{2} = z(\chi) > z(n) \geq 0$ for all $n$. Let $0$ denote the zero sequence. Suppose that the first representation in the proposition holds. Then

$$\frac{1}{2} = \sum_n \Gamma_n(z, z) = \frac{1}{2} \sum_n \Gamma_n(z, z),$$

and so $\sum_n \Gamma_n(z, z) < 1. $ Obviously $\sum_n \Gamma_n(z, 0) = \sum_n \Gamma_n(0, z) = 1.$ Also, for each $n$,

$$\begin{pmatrix}
\Gamma_n(z, z) & \Gamma_n(z, 0) \\
\Gamma_n(0, z) & \Gamma_n(0, 0)
\end{pmatrix} \geq 0,$$

so

$$\sum_n \begin{pmatrix}
\Gamma_n(z, z) & \Gamma_n(z, 0) \\
\Gamma_n(0, z) & \Gamma_n(0, 0)
\end{pmatrix} = \begin{pmatrix}
\sum_n \Gamma_n(z, z) & 1 \\
1 & 1
\end{pmatrix} \geq 0,$$

and thus $\sum_n \Gamma_n(z, z) \geq 1$, a contradiction.

The second part of the Proposition is proved similarly, in this situation choosing a function $z$ with $z(C) = 0, z(\chi) = \sqrt{\frac{1}{2}},$ and $0 \leq z \leq \sqrt{\frac{1}{2}}$ (such a function exists since $\beta\mathbb{N}$ is Tychonov).

By the way, if we define $P_n$ as the projection of $f \in \mathbb{D}^\infty$ onto its first $n$ components, then despite the fact that $P_nf$ converges pointwise to $f$, $\varphi(P_nf) = 0$, and so obviously does not converge to $\varphi(f)$ in general. However this does not contradict Proposition 5.2 since this only says that there is some net of simple representations converging to $\varphi$, and obviously this is not one!
5.3. Further Examples. We end with a few examples illustrating some of the pitfalls into which an unwary applicant of the results presented can fall.

The following example shows that it is sometimes important and natural to use the compactification of $\Psi$, and illustrates more simply the phenomena observed with the infinite polydisk.

5.3.1. Example 1. Choose $X$ equal the unit disk $\mathbb{D}$ and let, for $n = 1, 2, \ldots$, $\psi_n(z) = \sqrt{\frac{1}{n}} z$. The collection $\Psi = \{\psi_n : n\}$ is a set of test functions for $H^\infty(\mathbb{D})$ and the function $\xi(z) = z$ is in $H^\infty(\mathbb{D})$ with $||\xi|| = 1$.

Lemma 5.5. There do not exist positive kernels $\Gamma_n$ so that

$$1 - zw^* = \sum_{n \in \mathbb{N}} \Gamma_n(z, w)(1 - \psi_n(z)\psi_n(w)^*). \quad (5.3)$$

Proof. Suppose (5.3) holds for some positive kernels $\Gamma_j$. Divide through by $1 - zw^*$ to obtain,

$$1 = \sum \Gamma_n(z, w) + \frac{1}{n} \frac{\Gamma_n(z, w)}{1 - zw^*}.$$

Note that the left side is the rank one positive matrix $[1]$ consisting of all 1’s, and also each term on the right side is positive. Hence each term on the right side is a nonnegative constant multiple of $[1]$, which is clearly a contradiction. □

Interestingly, if we had used the function $\sqrt{\frac{1}{n}} z$ instead of $z$, then there is an obvious choice for the $\Gamma_k$’s; namely, $\Gamma_n = [1]$ and all others equal to 0. Furthermore, for a finite set $F \subset X$, the matrices

$$(1 - (1 - \frac{1}{n}) zw^*)_{z,w \in F},$$

converge to

$$(1 - zw^*)_{z,w \in F},$$

and so by the proof of Lemma 3.4 there must be a $\Gamma$ such that $1 - zw = \Gamma(z, w)(1 - E(z)E(w)^*)$ (the proof of the lemma makes no use, either explicit or implicit, of the compactness of the set of test functions).

The kernel $\Gamma$ has entries which are continuous functions over $C_b(\Psi)$, and this includes the point evaluations which we tried to use above. However there are point evaluations we have not considered — the ones coming from points in the Stone-Čech compactification $\beta\Psi$ of $\Psi$. In this case, this agrees with the one point compactification where we add the function $\psi_\infty(z) = z$. The functions $E(z)$ extend uniquely to $\Psi$, and if the positive linear functional $\gamma \in C_b(\Psi_0)^*$ is defined by $\gamma(e) = e(\psi_\infty)$, then the choice $\Gamma(x, y) = \gamma$ for all $x, y$ gets us out of our quandary.

In this example, it is clear that there would be no harm (in fact, it would be to our advantage) to include the functions in the Stone-Čech compactification of the set of test functions, particularly since it does not much effect the size of the set of test functions. This is in stark contrast to the case of the infinite polydisk, where the compactification increases the set size from being countable to at least having cardinality of $2^\mathbb{N}$.

5.3.2. Example 2. Let $X = \{x_1, x_2\}$ denote a two point set and define $\psi(x_1) = 0$ and $\psi(x_2) = 1$. The set $\Psi = \{\psi\}$ is a set of test functions for $C(X)$. However, the function $\tilde{\psi} = 1 - \psi$ is in the unit ball of $C(X)$, but there does not exist a positive $\Gamma$ such that

$$1 - \tilde{\psi}(x)\tilde{\psi}(y)^* = \Gamma(x, y)(1 - \psi(x)\psi(y)^*). \quad (5.4)$$

The remainder of this subsection is devoted to these assertions.
Suppose $k \in K_\Psi$; that is, $k$ is a positive kernel and the $2 \times 2$ matrix
\[
\begin{pmatrix}
(1 - \psi(x_1)\psi^*(x_1))k(x_1, x_1) & (1 - \psi(x_1)\psi^*(x_2))k(x_1, x_2) \\
(1 - \psi(x_2)\psi^*(x_1))k(x_2, x_1) & (1 - \psi(x_2)\psi^*(x_2))k(x_2, x_2)
\end{pmatrix}
= \begin{pmatrix}
k(x_1, x_1) & k(x_1, x_2) \\
k(x_2, x_1) & 0
\end{pmatrix}
\]
is positive. It follows that $k(x_1, x_1)$ and $k(x_2, x_2)$ are nonnegative and $k(x_1, x_2) = 0 = k(x_2, x_1)$. Now for $\varphi : X \to \mathbb{C}$ it is readily verified that $(1 - \varphi(x)\varphi^*(y))k(x, y)$ is positive for all such $k$ if and only if $|\varphi(x_j)| \leq 1$ for $j = 1, 2$. Hence $H^\infty(K_\Psi) = C(X)$ and $\tilde{\psi}$ is contractive.

Since the $(x_2, x_2)$ entry of the left hand side of equation (5.4) is 1, but the same entry on the right hand side of this equation is 0, no such $\Gamma$ exists. So what went wrong? To begin with, $\psi$ violates condition (i) for a test function. However this is not so serious in this case. More to the point, it also violates (ii), since if we choose the set $F = \{x_2\}$, $\Psi|F$ does not generate $P(F)$. We could fix this either by taking a quotient or by adding another test function. A natural choice is $\tilde{\psi}$; sadly this still violates condition (i).

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