A Fourier view on the $R$-transform and related asymptotics of spherical integrals

Alice Guionnet, Mylène Maïda

Abstract

We estimate the asymptotics of spherical integrals of real symmetric or Hermitian matrices when the rank of one matrix is much smaller than its dimension. We show that it is given in terms of the $R$-transform of the spectral measure of the full rank matrix and give a new proof of the fact that the $R$-transform is additive under free convolution. These asymptotics also extend to the case where one matrix has rank one but complex eigenvalue, a result related with the analyticity of the corresponding spherical integrals.

Keywords: Large deviations, random matrices, non-commutative measure, $R$-transform.

MSC: 60F10, 15A52, 46L50.

1 Introduction

1.1 General framework and statement of the results

In this article, we consider the spherical integrals

$$I_N^{(β)}(D_N, E_N) := \int \exp\{N \text{tr}(UD_NU^*E_N)\} dm_N^{(β)}(U),$$

where $m_N^{(β)}$ denote the Haar measure on the orthogonal group $O_N$ when $β = 1$ and on the unitary group $U_N$ when $β = 2$, and $D_N, E_N$ are $N \times N$ matrices that we can assume diagonal without loss of generality. Such integrals are often called, in the physics literature, Itzykson-Zuber or Harish-Chandra integrals. We do not consider the case $β = 4$ mostly to lighten the notations.

The interest for these objects goes back in particular to the work of Harish-Chandra ([14], [15]) who intended to define a notion of Fourier transform on Lie algebras. They have been then extensively studied in the framework of so-called matrix models that are related to the problem of enumerating maps (after [16], it has been developed in physics for example in [27], [19] or [21], in mathematics in [6] or [11]; a very nice introduction to these links is provided in [28]). The asymptotics of the spherical integrals needed to solve matrix models were investigated in [13]. More precisely, when $D_N, E_N$ have $N$ distinct real eigenvalues $(θ_i(D_N), λ_i(E_N))_{1 ≤ i ≤ N}$ and the spectral measures...
\[ \hat{\mu}_{DN}^N = \frac{1}{N} \sum \delta_{\theta_i(D_N)} \text{ and } \hat{\mu}_{EN}^N = \frac{1}{N} \sum \delta_{\lambda_i(E_N)} \] converge respectively to \( \mu_D \) and \( \mu_E \), it is proved in Theorem 1.1 of [13] that

\[ \lim_{N \to \infty} \frac{1}{N^2} \log I_N^{(\beta)}(D_N, E_N) = I^{(\beta)}(\mu_D, \mu_E) \]

exists under some technical assumptions and a (complicated) formula for this limit is given.

In this paper, we investigate different asymptotics of the spherical integrals, namely the case where one of the matrix, say \( D_N \), has rank much smaller than \( N \).

Such asymptotics were also already used in physics (see [20], where they consider replicated spin glasses, the number of replica being there the rank of \( D_N \)) or stated for instance in [6], section 1, as a formal limit (the spherical integral being seen as a serie in \( \theta \) when \( D_N = \text{diag}(\theta, 0, \cdots, 0) \) whose coefficients are converging as \( N \) goes to infinity). However, to our knowledge, there is no rigorous derivation of this limit available in the literature. We here study this problem by use of large deviations techniques. The proofs are however rather different from those of [13] ; they rely on large deviations for Gaussian variables and not on their Brownian motion interpretation and stochastic analysis as in [13].

Before stating our results, we now introduce some notations and make a few remarks. Let \( D_N = \text{diag}(\theta, 0, \cdots, 0) \) have rank one so that

\[ I_N^{(\beta)}(D_N, E_N) = I_N^{(\beta)}(\theta, E_N) = \int e^{\theta N(UE_NU^*)} 1_{\mathbb{R}}(z) \, d\mu_{E,N}(\lambda). \]

Note that in general, in the case \( \beta = 1 \), we will omit the superscript \( (\beta) \) in all these notations.

We make the following hypothesis :

**Hypothesis 1.1**

1. \( \hat{\mu}_{EN}^N \) converges weakly towards a compactly supported measure \( \mu_E \).

2. \( \lambda_{\min}(E_N) := \min_{1 \leq i \leq N} \lambda_i(E_N) \) and \( \lambda_{\max}(E_N) := \max_{1 \leq i \leq N} \lambda_i(E_N) \) converge respectively to \( \lambda_{\min} \) and \( \lambda_{\max} \) which are finite.

Note that under Hypothesis 1.1, the support of \( \mu_E \), which we shall denote \( \text{supp}(\mu_E) \), is included into \( [\lambda_{\min}, \lambda_{\max}] \).

Let us denote, for a probability measure \( \mu_E \), its Hilbert transform by \( H_{\mu_E} \):

\[ H_{\mu_E} : I_E := \mathbb{R} \setminus \text{supp}(\mu_E) \longrightarrow \mathbb{R} \]

\[ z \longmapsto \int \frac{1}{z - \lambda} d\mu_E(\lambda). \]

It is easily seen (c.f subsection 1.2 for details) that \( H_{\mu_E} : I_E \to H_{\mu_E}(I_E) \) is invertible, with inverse denoted \( K_{\mu_E} \). For \( z \in H_{\mu_E}(I_E) \), we set \( R_{\mu_E}(z) = K_{\mu_E}(z) - z^{-1} \) to be the so-called \( R \)-transform of \( \mu_E \). In the case of the spectral measure \( \hat{\mu}_{EN}^N \) of \( E_N \), we denote by \( H_{EN} \) its Hilbert transform given by

\[ H_{EN}(x) = \frac{1}{N} \text{tr}(x - E_N)^{-1} = \frac{1}{N} \sum_{i=1}^N (x - \lambda_i(E_N))^{-1}. \]

The central result of this paper can be stated as follows :
Theorem 1.2 Let $\beta = 1$ or 2. If we assume that Hypothesis 1.1 is satisfied and that there is $\epsilon > 0$ such that
\[ \|E_N\|_{\infty} := \max\{\lambda_{\max}(E_N), |\lambda_{\min}(E_N)|\} = O\left(N^{\frac{1}{2}-\epsilon}\right), \] (4)
then for $\theta$ small enough so that there exists $\eta > 0$ so that
\[ \frac{2\theta}{\beta} \in \bigcup_{N_0 \geq 0} \bigcap_{N \geq N_0} H_{E_N}([\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c), \] (5)
\[ I_N^{(\beta)}(\theta) := \lim_{N \to \infty} \frac{1}{N} \log I_N^{(\beta)}(\theta, E_N) = \frac{\beta}{2} \int_0^{2\theta} R_{\mu_E}(v)dv. \] (6)
Under Hypothesis 1.1.2, (4) is obviously satisfied and (5) is equivalent to
\[ \frac{2\theta}{\beta} \in H_{\mu_E}([\lambda_{\min}, \lambda_{\max}]^c). \]
This result is proved in section 2 and appears in a way as a by-product of Lemma 2.1. It raises several remarks and generalisations that we shall investigate in this paper.

Note that in Theorems 1.3, 1.4 and 1.5 hereafter we consider the case $\beta = 1$, which requires simpler notations but every statement could be extended to the case $\beta = 2$. The main difference to extend these theorems to the case $\beta = 2$ is that, following Fact 1.8, it requires to deal with twice as much Gaussian variables, and hence to consider covariance matrices with twice bigger dimension (the difficulty lying then in showing that these matrices are positive definite).

The first question we can ask is how to precise the convergence (6). Indeed, in the full rank asymptotics, in particular in the framework of [13], the second order term has not yet been rigorously derived. In our case, if $d$ is the Dudley distance between measures (which is compatible with the weak topology) given by
\[ d(\mu, \nu) = \sup \left\{ \int f d\mu - \int f d\nu ; |f(x)| \text{ and } \left| \frac{f(x) - f(y)}{x - y} \right| \leq 1, \forall x \neq y \right\}, \] (7)
we have

Theorem 1.3 Assume Hypothesis 1.2 and
\[ d(\hat{\mu}_N, \mu_E) = o(\sqrt{N}^{-1}). \]
Let $\theta$ be such that $2\theta \in H_{\mu_E}([\lambda_{\min}, \lambda_{\max}]^c)$.

• If $\mu_E$ is not a Dirac measure at a single point, then, with $v = R_{\mu_E}(2\theta)$,
\[ \lim_{N \to \infty} e^{-N(\theta v - \frac{1}{2\theta} \sum_{i=1}^N \log(1 - 2\theta \lambda_i(E_N) + 2\theta v))} I_N(\theta, E_N) = \frac{\sqrt{Z - 4\theta^2}}{\theta \sqrt{Z}}, \]
with $Z := \int_0^{\infty} \frac{1}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda)$. 

\[ \]
• If $\mu_E = \delta_e$ for some $e \in \mathbb{R}$,
  \[
  \lim_{N \to \infty} e^{-N\delta_e} I_N(\theta, E_N) = 1.
  \]

This theorem gives the second order term for the convergence given in Theorem 1.2 above. Indeed, with $2\theta \in H_{\mu_E}([\lambda_{\text{min}}, \lambda_{\text{max}}])$, under Hypothesis 1.1.2, there exists (c.f. [14] for details) $\eta(\theta) > 0$ so that for $N$ large enough

\[
1 - 2\theta \lambda_i(E_N) + 2\theta v > \eta(\theta).
\]

Therefore, there exists a finite constant $C(\theta) \leq (\eta(\theta)^{-1} + |\log(\eta(\theta))|)$ such that for $N$ sufficiently large

\[
\left| \frac{1}{2N} \sum_{i=1}^{N} \log(1 - 2\theta \lambda_i(E_N) + 2\theta v) - \frac{1}{2} \int \log(1 - 2\theta \lambda + 2\theta v) d\mu_E(\lambda) \right| \leq C(\theta) d \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(E_N)}, \mu_E \right),
\]

where $d$ is the Dudley distance.

Moreover, with $v = R_{\mu_E}(2\theta)$, it is easy to see that

\[
\theta v - \frac{1}{2} \int \log(1 - 2\theta \lambda + 2\theta v) d\mu_E(\lambda) = \frac{1}{2} \int_0^{2\theta} R_{\mu_E}(u) du,
\]

showing how Theorem 1.3 relates with Theorem 1.2.

Another remark is that Theorem 1.2 can be seen as giving an interpretation of the primitive of the $R$-transform $R_{\mu_E}$ as a Laplace transform of $(UE_NU^*)_{11}$ for large $N$ and for compactly supported probability measures $\mu_E$.

A natural question is to wonder whether it can be extended to the case where $\theta$ is complex, to get an analogy with the Fourier transform that seems to have originally motivated Harish-Chandra. In the case of the different asymptotics studied in [13], this question is open: in physics, formal analytic extensions of the formula obtained for Hermitian matrices to any matrices are commonly used, but S. Zelditch [26] found that such an extension could be false by exhibiting counter-examples. In the context of the asymptotics we consider here, we shall however see that this extension is valid for $|\theta|$ small enough. Note that, as far as $\mu_E$ is compactly supported, $R_{\mu_E}$ can be extended analytically at least in a complex neighborhood of the origin (see Proposition 1.13 for further details).

**Theorem 1.4** Take $\beta = 1$ and assume that $(E_N)_{N \in \mathbb{N}}$ is a uniformly bounded sequence of matrices satisfying Hypothesis 1.1.1 where $\mu_E$ is not a Dirac mass.

Assume furthermore that $d(\mu_{E_N}^N, \mu_E) = o(\sqrt{N}^{-1})$, where $d$ is the Dudley distance defined by (7).

Then, there exists an $r > 0$ such that, for any $\theta \in \mathbb{C}$, such that $|\theta| \leq r$,

\[
\lim_{N \to \infty} \frac{1}{N} \log I_N(\theta, E_N) = \theta v(\theta) - \frac{1}{2} \int \log(1 + 2\theta v(\theta) - 2\theta \lambda) d\mu_E(\lambda),
\]
where $\log(\cdot)$ is the main branch of the logarithm in $\mathbb{C}$ and $\nu(\theta) = R_{\mu_E}(2\theta)$. More precisely, we prove that for $\theta$ in a small complex neighborhood of the origin,

$$
\lim_{N \to \infty} e^{-N(\theta v - \frac{1}{2N} \sum_{i=1}^N \log(1-2\theta \lambda_i(E_N)+2\theta v))} I_N(\theta, E_N) = \frac{\sqrt{Z - 4\theta^2}}{\theta \sqrt{Z}},
$$

with $Z := \int \frac{1}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda)$.

It is not hard to see that the above convergence is uniform in a small complex neighborhood of the origin. Consequently, there exists $\theta_0 > 0$, $N_0 \in \mathbb{N}$, such that for $|\theta| \leq \theta_0$, for all $N \geq N_0$, $f_N(\theta) := \frac{1}{N} \log I_N(\theta, E_N)$, is bounded from above and below. Moreover, under Hypothesis 1.1 the $f_N$’s are holomorphic and uniformly bounded. Therefore, by Cauchy’s formula

$$
\partial^{(n)} f_N|_{z=0} = -\frac{1}{2\pi i} \int_{|z| = \theta_0/2} \frac{f_N(z)}{z^{n+1}} dz
$$

insures with dominated convergence theorem’s that for all $n \in \mathbb{N}^*$,

$$
\lim_{N \to \infty} \partial^{(n)} f_N|_{z=0} = \partial^{(n)} f|_{z=0} = 2^{k-1} \theta^{n-1} R_{\mu_E}|_{z=0}
$$

with $f(\theta) = \theta v(\theta) - \frac{1}{2} \int \log(1+2\theta v(\theta)-2\theta\lambda)d\mu_E(\lambda)$). Hence, we give a new proof of B. Collins’ result [4] (here in the orthogonal setting rather than in the unitary one) and validate the strategy, commonly used in physics, of computing $f$ to calculate $\lim_{N \to \infty} \partial^{(n)} f_N|_{z=0}$.

Note that the case $\mu_E = \delta_\varepsilon$ is trivial if we assume additionally Hypothesis 1.1 with $\lambda_{\min}$ and $\lambda_{\max}$ the edges of the support of $\mu_E$ since then $\max_{1 \leq i \leq N} |\lambda_i - \varepsilon|$ goes to zero with $N$ which entails

$$
\lim_{N \to \infty} \frac{1}{N} \log I_N(\theta, E_N) = \theta \varepsilon
$$

for all $\theta$ in $\mathbb{C}$.

The proof of Theorem 1.4 will be more involved than the real case treated in sections 2 and 3 and the difficulty lies of course in the fact that the integral is now oscillatory, forcing us to control more precisely the deviations in order to make sure that the term of order one in the large $N$ expansion does not vanish. This is the object of section 4.

Once the view of spherical integrals as Fourier transforms has been justified by the extension to the complex plane, a second natural question is to wonder whether we can use it to see that the $R$-transform is additive under free convolution. Let us make some reminder about free probability: in this set up, the notion of freeness replaces the standard notion of independence and the R-transform is analogous to the logarithm of the Fourier transform of a measure. Now, it is well known that the log-Laplace (or Fourier) transform is additive under convolution i.e. for any probability measures $\mu$, $\nu$ on $\mathbb{R}$ (say compactly supported to simplify), any $\lambda \in \mathbb{R}$, (or $\mathbb{C}$)

$$
\log \int e^{\lambda x} d\nu * \mu(x) = \log \int e^{\lambda x} d\mu(x) + \log \int e^{\lambda x} d\nu(x).
$$

Moreover, this property, if it holds for $\lambda$’s in a neighbourhood of the origin, characterizes uniquely the convolution. Similarly, if we denote $\mu \boxplus \nu$ the free convolution of two compactly supported probability measures on $\mathbb{R}$, it is uniquely described by the fact that

$$
R_{\mu \boxplus \nu}(\lambda) = R_{\mu}(\lambda) + R_{\nu}(\lambda)
$$
for sufficiently small \( \lambda \)'s. Theorem \(^2\) provides an interpretation of this result. Indeed, Voiculescu \(^2\) proved that if \( A_N, B_N \) are two diagonal matrices with spectral measures converging towards \( \mu_A \) and \( \mu_B \) respectively, with uniformly bounded spectral radius, then the spectral measure of \( A_N + U_B N U^* \) converges, if \( U \) follows \( m_N^{(2)} \), towards \( \mu_A \bigoplus \mu_B \). This result extends naturally to the case where \( U \) follows \( m_N^{(1)} \) (see \(^\[7\] \) Theorem 5.2 for instance). Therefore, it is natural to expect the following result:

**Theorem 1.5** Let \( \beta = 1, (A_N, B_N)_{N \in \mathbb{N}} \) be a sequence of uniformly bounded real diagonal matrices and \( V_N \) following \( m_N^{(1)} \).

1. Then

\[
\lim_{N \to \infty} \left( \frac{1}{N} \log I_N(\theta, A_N + V_N B_N V_N^*) - \int \frac{1}{N} \log I_N(\theta, A_N + V_N B_N V_N^*) dm_N^{(1)}(V_N) \right) = 0 \text{ a.s.} \quad (8)
\]

2. If additionally the spectral measures of \( A_N \) and \( B_N \) converge respectively to \( \mu_A \) and \( \mu_B \) fast enough (i.e. such that \( d(\tilde{\mu}_A, \mu_A) + d(\tilde{\mu}_B, \mu_B) = o(\sqrt{N^{-1}}) \)) and \( \mu_A \) and \( \mu_B \) are not Dirac masses at a point, then, for any \( \theta \) small enough,

\[
\lim_{N \to \infty} \frac{1}{N} \log I_N(\theta, A_N + V_N B_N V_N^*) = \lim_{N \to \infty} \frac{1}{N} \log I_N(\theta, A_N) + \lim_{N \to \infty} \frac{1}{N} \log I_N(\theta, B_N) \text{ a.s.} \quad (9)
\]

Then the additivity of the \( R \)-transform (cf. Corollary \(^6\)) is a direct consequence of this result together with the continuity of the spherical integrals with respect to the empirical measure of the full rank matrix (which will be shown in Lemma \(^2\)).

Note that the case where \( \mu_A \) or \( \mu_B \) are Dirac masses is trivial if we assume that the edges of the spectrum of \( A_N \) or \( B_N \) converge towards this point. The general case could be handled as well but, since it has no motivation for the \( R \)-transform (for which we can always assume that the above condition holds, see Corollary \(^6\)), we shall not detail it. Section \(^8\) will be devoted to the proof of this theorem which decomposes mainly in two steps: to get the first point, we establish a result of concentration under \( m_N^{(1)} \) that will give us \(^9\); then to prove the second point once we have the first one it is enough to consider the expectation of \( \frac{1}{N} \log I_N(\theta, A_N + V_N B_N V_N^*) \) and if one assumes that

\[
\lim_{N \to \infty} \frac{1}{N} \int \left( \log \int e^{\theta N(V_N A_N U^* + U V_N B_N V_N^*)11} dm_N^{(1)}(U) \right) dm_N^{(1)}(V) = \lim_{N \to \infty} \frac{1}{N} \log \int \int e^{\theta N(V_N A_N U^* + U V_N B_N V_N^*)11} dm_N^{(1)}(U) dm_N^{(1)}(V) \quad (10)
\]

the equality \(^9\) follows from the observation that the right hand side equals \( N^{-1} \log I_N(\theta, A_N) + N^{-1} \log I_N(\theta, B_N) \).

Note that equation \(^10\) is rather typical to what should be expected for disordered particle systems in the high temperature regime and indeed our proof follows some very smart ideas of Talagrand that he developed in the context of Sherrington-Kirkpatrick model of spin glasses at high temperature (see \(^23\)). This proof is however rather technical because the required control on the \( L^2 \) norm of the partition function is based on the study of second order corrections of replicated...
Theorem 1.7 Let $\beta = 1$ or 2. Assume $\hat{\mu}^{N}_{E}$ satisfy Hypothesis 1.1. If we let $H_{\min} : = \lim_{z \to \lambda_{\min}} H_{\mu_E}(z)$ and $H_{\max} : = \lim_{z \to \lambda_{\max}} H_{\mu_E}(z)$, then

$$\lim_{N \to \infty} \frac{1}{N} \log I^{(3)}_{N} (\theta, E_N) = I^{(3)}_{\mu_E}(\theta) = \theta v(\theta) - \frac{\beta}{2} \int \log \left(1 + \frac{2}{\beta} \theta v(\theta) - 2 \frac{\theta}{\beta} \lambda\right) d\mu_{E}(\lambda)$$

with

$$v(\theta) = \begin{cases} \frac{R_{\mu_E} (\beta \theta)}{\beta} & \text{if } H_{\min} \leq \frac{2 \theta}{\beta} \leq H_{\max} \\ \lambda_{\max} - \frac{\beta}{2 \theta} & \text{if } \frac{2 \theta}{\beta} > H_{\max} \\ \lambda_{\min} - \frac{\beta}{2 \theta} & \text{if } \frac{2 \theta}{\beta} < H_{\min}. \end{cases}$$

Note here that the values of $\lambda_{\min}$ and $\lambda_{\max}$ do affect the value of the limit of spherical integrals in the asymptotics we consider here, contrarily to what happens in the full rank asymptotics considered in [13].

As a consequence of Theorem 1.6 we can see that there are two phase transitions at $H_{\max} \beta / 2$ and $H_{\min} \beta / 2$ which are of second order in general (the second derivatives of $I^{(3)}_{\mu_E}(\theta)$ being discontinuous at these points, except when $\lambda_{\max}^{H_{\mu_E} \beta}(\lambda_{\max}) = 1$ (or similar equation with $\lambda_{\min}$ instead of $\lambda_{\max}$), in which case the transition is of order 3). These transitions can in fact be characterized by the asymptotic behaviour of $(U E_N U^*)_{11}$ under the Gibbs measure

$$d\mu^{\beta, \theta}_{N} (U) = \frac{1}{I^{(3)}_{N} (\theta, E_N)} e^{N \theta (U E_N U^*)_{11}} d\mu^{(\beta)}_{N} (U).$$

For $\theta \in \left[\frac{H_{\min} \beta}{2}, \frac{H_{\max} \beta}{2}\right]^{c}$, $(U E_N U^*)_{11}$ saturates and converges $\mu^{\beta, \theta}_{N}$ almost surely towards $\lambda_{\max} - \frac{\beta}{2 \theta}$ (resp. $\lambda_{\min} - \frac{\beta}{2 \theta}$). Hence, up to a small component of norm of order $\theta^{-1}$, with high probability, the first column vector $U_1$ of $U$ will align on the eigenvector corresponding to either the smallest or the largest eigenvalue of $E_N$, whereas for smaller $\theta$’s, $U_1$ will prefer to charge all the eigenspaces of $E_N$.

Another natural question is to wonder what happens when $D_N$ has not rank one but rank negligible compared to $N$. It is not very hard to see that in the case where all the eigenvalues of $D_N$ are small enough (namely when they all lie inside $H_{\mu_E} ([\lambda_{\min}, \lambda_{\max}]^{c})$), we find that the spherical integral approximately factorizes into a product of integrals of rank one. More precisely,

Theorem 1.7 Let $\beta = 1$ or 2. Let $D_N = \text{diag}(\theta_1^{N}, \ldots, \theta_{M(N)}^{N}, 0, \ldots, 0)$ with $M(N)$ which is $o(N^{\frac{4}{7} - \varepsilon})$ for some $\varepsilon > 0$. Assume that $\hat{\mu}^{N}_{E}$ fulfills Hypothesis 1.1, that $\|E_N\|_{\infty} = o(N^{\frac{4}{7} - \varepsilon})$ for some $\varepsilon > 0$ and that there exists $N_0 \in \mathbb{N}$ and $\eta > 0$ such that, for all $N \geq N_0$ and $i$ from 1 to $M(N)$, $\frac{2 \eta}{\beta} \in H_{E_N} ([\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^{c})$.

Then, if $\frac{1}{M(N)} \sum_{i=1}^{M(N)} \delta_{\theta_i^{N}}$ converges weakly to $\mu_{D}$,

$$I^{(\beta)}_{\mu_E}(D) : = \lim_{N \to \infty} \frac{1}{NM(N)} \log I^{(\beta)}_{N} (D_N, E_N)$$
exists and is given by

\[ I^{(b)}_{\mu E}(D) = \lim_{N \to \infty} \frac{1}{M(N)} \sum_{i=1}^{M(N)} I^{(b)}_{\mu E}(\theta_i^N) = \int I^{(b)}_{\mu E}(\theta) d\mu_D(\theta). \]  

(11)

This will be shown at the end of section 2, the proof being very similar to the case of rank one. It relies mainly on Fact 1.8 hereafter and comes from the fact that in such asymptotics the \(M(N)\) first column vectors of an orthogonal or unitary matrix distributed according to the Haar measure behave approximately like independent vectors uniformly distributed on the sphere. This can be compared with the very old result of E. Borel \[5\] which says that one entry of an orthogonal matrix distributed according to the Haar measure behaves like a Gaussian variable. That kind of considerations finds continuation for example in a recent work of A. D’Aristotile, P. Diaconis and C. M. Newman \[8\] where they consider a number of element of the orthogonal group going to infinity not too fast with \(N\). In the same direction, one can also mention the recent work of T. Jiang \[17\] where he shows that the entries of the first \(O(N/\log N)\) columns of an Haar distributed unitary matrix can be simultaneously approximated by independent standard normal variables.

Recently, P. Śniady could prove by different techniques that the asymptotics we are talking about extend to \(M(N) = o(N)\).

Of course we would like to generalize also the full asymptotics we’ve got in Theorem 1.6 to the set up of finite rank i.e. in particular consider the case where some (a \(o(N)\) number) of the eigenvalues of \(E_N\) could converge away from the support. It seems to involve not only the deviations of \(\lambda_{\text{max}}\) but those of the first \(M\) ones when the rank is \(M\). As it becomes rather complicate and as the proof is already rather involved in rank one, we postpone this issue to further research.

To finish this introduction, we also want to mention that the results we’ve just presented give (maybe) less obvious relations between the \(R\)-transform and Schur functions or vicious walkers. Indeed, if \(s_{\lambda}\) denotes the Schur function associated with a Young tableau \(\lambda\) (cf. \[22\] for more details), then, it can be checked (cf. \[12\] for instance) that

\[ s_{\lambda}(M) = I^{(2)}_{N} \left( \log M, \frac{l}{N} \right) \Delta \left( \frac{l}{N} \right) \frac{\Delta(\log M)}{\Delta(M)} \]

with \(l_i = \lambda_i + N - i, 1 \leq i \leq N\) and \(\Delta(M) = \prod_{i<j}(M_i - M_j)\) when \(M = \text{diag}(M_1, \ldots, M_N)\). Thus, our results also give the asymptotics of Schur functions when \(N^{-1}\delta_{N^{-1}(\lambda_i+N-i)}\) converges towards some compactly supported probability measure \(\mu\). For instance, Theorem 1.2 implies that for \(\theta\) small enough

\[ \lim_{N \to \infty} \frac{1}{N} \log \left( \prod_{j>i} (N^{-1}(\lambda_j - j - \lambda_i + i))^{-1} s_{\lambda}(e^\theta, 1, \ldots, 1) \right) = \int_0^\theta R_\mu(u)du + \log(\theta(e^\theta - 1)^{-1}). \]

Such asymptotics should be more directly related with the combinatorics of the symmetric group and more precisely with non-crossing partitions which play a key role in free convolution.

On the other hand, it is also known that spherical integrals are related with the density kernel of vicious walkers, that is Brownian motions conditionned to avoid each others, either by using the fact that the eigenvalues of the Hermitian Brownian motion are described by such vicious walkers (more commonly named in this context Dyson’s Brownian motions) or by applying directly the result of Karlin-McGregor \[18\]. Hence, the study of the asymptotics of spherical integrals we are
considering allows to estimate this density kernel when $N - 1$ vicious walkers start at the origin, the last one starting at $\theta$ and at time one reach $(x_1, \ldots, x_N)$ whose empirical distribution approximates a given compactly supported probability measure.

### 1.2 Preliminary properties and notations

Before going into the proofs themselves, we gather here somematerial and notations that will be useful throughout the paper.

#### 1.2.1 Gaussian representation of Haar measure

In the different cases we will develop, the first step will be always the same: we will represent the column vectors of unitary or orthogonal matrices distributed according to Haar measure via Gaussian vectors. To be more precise, we recall the following fact:

**Fact 1.8** Let $k \leq N$ be fixed.

- **Orthogonal case.**
  Let $U = (u_{ij})_{1 \leq i,j \leq N}$ be a random orthogonal matrix distributed according to $m_N^{(1)}$, the Haar measure on $O_N$. Denote by $(u^{(i)})_{1 \leq i \leq N}$ the column vectors of $U$.
  Let $(g^{(1)}, \ldots, g^{(k)})$ be $k$ independent standard Gaussian vectors in $\mathbb{R}^N$ and let $(\tilde{g}^{(1)}, \ldots, \tilde{g}^{(k)})$ the vectors obtained from $(g^{(1)}, \ldots, g^{(k)})$ by the standard Schmidt orthogonalisation procedure.
  Then it is well known that
  $$(u^{(1)}, \ldots, u^{(k)}) \sim \left( \frac{\tilde{g}^{(1)}}{\|\tilde{g}^{(1)}\|}, \ldots, \frac{\tilde{g}^{(k)}}{\|\tilde{g}^{(k)}\|} \right),$$
  where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$ and the equality $\sim$ means that the two $k \times N$-matrices have the same law.

- **Unitary case.**
  With the same notations, let $U$ be distributed according to $m_N^{(2)}$, the Haar measure on $U_N$. Let $(g^{(1),R}, \ldots, g^{(k),R}, g^{(1),i}, \ldots, g^{(k),i})$ be $2k$ independent standard Gaussian vectors in $\mathbb{R}^N$ and let $(\tilde{G}^{(1)}, \ldots, \tilde{G}^{(k)})$ be the $k$ vectors obtained from $(g^{(1),R} + ig^{(1),i}, \ldots, g^{(k),R} + ig^{(k),i})$ by the standard Schmidt orthogonalisation procedure with respect to the usual scalar product in $\mathbb{C}^N$.
  Then we get that
  $$(u^{(1)}, \ldots, u^{(k)}) \sim \left( \frac{\tilde{G}^{(1)}}{\|\tilde{G}^{(1)}\|}, \ldots, \frac{\tilde{G}^{(k)}}{\|\tilde{G}^{(k)}\|} \right),$$
  where $\|\cdot\|$ denotes the usual norm in $\mathbb{C}^N$.

Note that heuristically, the above representation in terms of Gaussian vectors allows us to understand why the limit in the finite rank case behaves as a sum of functions of each of the eigenvalues of $D_N$. Indeed, in high dimension, we know that a bunch of $k$ (independent of the dimension) Gaussian vectors are almost orthogonal one from another so that the orthogonalisation procedure let them almost independent.
1.2.2 Some properties of the Hilbert and the R-transforms of a compactly supported probability measure on $\mathbb{R}$

Let $\lambda_{\min}(E)$ and $\lambda_{\max}(E)$ be the edges of the support of $\mu_E$. For all $\lambda_{\min} \leq \lambda_{\min}(E)$ and $\lambda_{\max} \geq \lambda_{\max}(E)$, let us denote by $H_{\mu_E}^{\min} := \lim_{z \uparrow \lambda_{\min}} H_{\mu_E}(z)$ and $H_{\mu_E}^{\max} := \lim_{z \downarrow \lambda_{\max}} H_{\mu_E}(z)$, where $H_{\mu_E}$ was defined in (3).

We sum up the properties of $H_{\mu_E}$ that will be useful for us in the following

Property 1.9:
1. $H_{\mu_E}$ is decreasing and positive on $\{z > \lambda_{\max}\}$ and decreasing and negative on $\{z < \lambda_{\min}\}$.
2. Therefore, $H_{\mu_E}^{\min}$ exists in $\mathbb{R}^* \cup \{-\infty\}$ and $H_{\mu_E}^{\max}$ exists in $\mathbb{R}^* \cup \{+\infty\}$.
3. $H_{\mu_E}$ is bijective from $I = \mathbb{R}\setminus[\lambda_{\min}, \lambda_{\max}]$ onto its image $I' = [H_{\mu_E}^{\min}, H_{\mu_E}^{\max}] \setminus \{0\}$.
4. $H_{\mu_E}$ is analytic on $I$ and its derivative never cancels on $I$.

The third point of the property above allows the following

Definition 1.10:
1. $K_{\mu_E}$ is defined on $I'$ as the functional inverse of $H_{\mu_E}$.
2. $I'$ does not contain 0 so that, on $I'$, we can define $R_{\mu_E}$ given by $R_{\mu_E}(\gamma) = K_{\mu_E}(\gamma) - \frac{1}{\gamma}$ for any $\gamma \in I'$.

We will need to consider the inverse $Q_{\mu_E}$ of $R_{\mu_E}$. To define it properly, we have to look more carefully at the properties of $R_{\mu_E}$. We have :

Property 1.11:
1. $K_{\mu_E}$ and $R_{\mu_E}$ are analytic (and in particular continuously differentiable) on $I'$.
2. $R_{\mu_E}$ is increasing and its derivative never cancels.
3. $\lim_{\gamma \to 0^{-}} R_{\mu_E}(\gamma) = \lim_{\gamma \to 0^{+}} R_{\mu_E}(\gamma) = m := \int_{\lambda_{\min}}^{\lambda_{\max}} d\mu_E(\lambda)$.
4. $R_{\mu_E}$ is bijective from $I'$ onto its image $I'' := \left[\lambda_{\min} - \frac{1}{H_{\mu_E}^{\min}}, \lambda_{\max} - \frac{1}{H_{\mu_E}^{\max}}\right] \setminus \{m\}$ so that we can define its inverse $Q_{\mu_E}$ from $I''$ to $I'$. Moreover, $Q_{\mu_E}$ is differentiable on $I''$.

The proof of these properties is easy and left to the reader.

The following property deals with the behaviour of these functions on the complex plane. A proof of it can be found for example in [24]. We first extend the definition of the Hilbert transform, that we denote again $H_{\mu_E}$ by

$$H_{\mu_E} : \mathbb{C} \setminus \text{supp}(\mu_E) \rightarrow \mathbb{C} \quad \frac{1}{z} \mapsto \int \frac{1}{z - \lambda} d\mu_E(\lambda).$$
Property 1.12:

1. There exists a neighbourhood $\mathcal{A}$ of $\infty$ such that $H_{\mu_E}$ is bijective from $\mathcal{A}$ into $H_{\mu_E}(\mathcal{A})$, which is a neighbourhood of 0.

2. We denote by $K_{\mu_E}^{(c)}$ its functional inverse on $H_{\mu_E}(\mathcal{A})$ and $R_{\mu_E}^{(c)}$ is given by $R_{\mu_E}^{(c)}(\gamma) = K_{\mu_E}^{(c)}(\gamma) - \frac{1}{\gamma}$ for any $\gamma \in H_{\mu_E}(\mathcal{A})$ (that does not contain 0).

3. $R_{\mu_E}^{(c)}$ is analytic and coincides with $R_{\mu_E}$ on $I' \cap H_{\mu_E}(\mathcal{A})$. Therefore, we denote it again $R_{\mu_E}$.

Note that throughout the paper, we will denote $\lambda_i := \lambda_i(E_N)$, $\theta_i := \theta_i(D_N)$ (and even $\theta$ will denote $\theta_i(D_N)$ in the case of rank one) and denote in short $H_{E_N}(x) = \frac{1}{N} \text{tr}(x - E_N)^{-1}$.

We now state the following property, which will be useful in the proof of Theorem 1.4:

**Proposition 1.13** If $(E_N)_{N \in \mathbb{N}}$ is uniformly bounded and satisfying Hypothesis 1.1.1, there exists $r > 0$ such that, for any $\theta \in \mathbb{C}$ such that $|\theta| \leq r$, there is a solution of

$$H_{E_N} \left( \frac{1}{2\theta} + v_N(\theta) \right) = 2\theta,$$

such that $v_N(\theta) \xrightarrow{N \to \infty} R_{\mu_E}(2\theta)$.

**Proof of Proposition 1.13**: Let $\mathcal{A}_N$ be a neighbourhood of $\infty$ on which $H_{E_N}$ is invertible ($\mathcal{A}_N$ can be given as $\{ z/|z| > R_N \}$, for some $R_N$). For any $\eta > 0$, we denote by $\mathcal{A}_N^\eta := \{ x \in \mathcal{A}_N / d(x, \mathcal{A}_N^\eta) \geq \eta \}$. Let $\theta$ be such that there exists $\eta > 0$ such that $2\theta \in \bigcup_{N \geq 0} \bigcap_{N \geq N_0} H_{E_N}(\mathcal{A}_N^\eta)$, we take $v_N(\theta)$ the unique solution in $\mathcal{A}_N^\eta - (2\theta)^{-1}$ of

$$H_{E_N} \left( \frac{1}{2\theta} + v_N(\theta) \right) = 2\theta.$$

Since, for all $\lambda \in \bigcup_{N \geq 0} \bigcap_{N \geq N_0} \text{supp}(\tilde{\mu}_{E_N}^N)$, the application $z \mapsto (z - \lambda)^{-1}$ is continuous bounded on $\bigcup_{N \geq 0} \bigcap_{N \geq N_0} \mathcal{A}_N^\eta$ under Hypothesis 1.1.1, $v_N(\theta)$ converges to $R_{\mu_E}(2\theta)$.

Furthermore, the fact that $(E_N)_{N \in \mathbb{N}}$ is uniformly bounded ensures that we can choose the $\mathcal{A}_N$’s such that there exists $r > 0$ such that $\bigcup_{N \geq 0} \bigcap_{N \geq N_0} H_{E_N}(\mathcal{A}_N^\eta) \supset \{ \theta / |\theta| \leq r \}. \hfill \square$

## 2 Proof of Theorems 1.2, 1.7 and related results

Before going into more details, let us state and prove a lemma which deals with the continuity of $I_N$ and its limit. We state here a trivial continuity in the finite rank matrix but also a weaker continuity result in the spectral measure of the diverging rank matrix, on which the proof of Theorem 1.2 is based.

**Lemma 2.1**

1. For any $N \in \mathbb{N}$, any sequence of matrices $(E_N)_{N \in \mathbb{N}}$ with spectral radius $\|E_N\|_{\infty}$ uniformly bounded by $\|E\|_{\infty}$, any Hermitian matrices $(D_N, \tilde{D}_N)_{N \in \mathbb{N}}$,

$$\left| \frac{1}{N} \log I_N^{(\beta)}(D_N, E_N) - \frac{1}{N} \log I_N^{(\beta)}(\tilde{D}_N, E_N) \right| \leq \|E\|_{\infty} \text{tr}|D_N - \tilde{D}_N|$$
2. Let \(D_N = \text{diag}(\theta, 0, \ldots, 0)\). Assume that there is a positive \(\eta\) and a finite integer \(N_0\) such that for \(N \geq N_0\), \(\frac{2\theta}{\beta} \in H_{E_N}(\lfloor \lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta \rfloor^c)\). We let \(v_N\) be the unique solution in \(-\beta(2\theta)^{-1} + [\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c\) of the equation
\[
\frac{\beta}{2\theta} H_{E_N} \left( \frac{\beta}{2\theta} + v_N \right) = 1. \tag{13}
\]

Then, \(v_N \in [\lambda_{\min}(E_N), \lambda_{\max}(E_N)]\) and for any \(\zeta \in (0, \frac{1}{2})\), there exists a finite constant \(C(\eta, \zeta)\) depending only on \(\eta\) and \(\zeta\) such that for all \(N \geq N_0\)
\[
\left| \frac{1}{N} \log I_N^{(\beta)}(\theta, E_N) - \theta v_N + \frac{\beta}{2N} \sum_{i=1}^{N} \log \left( 1 + \frac{2\theta}{\beta} v_N - \frac{2\theta}{\beta} \lambda_i \right) \right| \leq C(\eta, \zeta) N^{-\frac{1}{2} + \zeta} \|E_N\|_\infty.
\]

3. Let \(D_N = \text{diag}(\theta, 0, \ldots, 0)\). Let \(E_N, \tilde{E}_N\) be two matrices such that
\[
d(\hat{\mu}_N^{E_N}, \hat{\mu}_N^{\tilde{E}_N}) \leq \delta,
\]
where \(d\) is the Dudley distance on \(P(\mathbb{R})\) and so that both \(E_N\) and \(\tilde{E}_N\) satisfy (1). Let \(\eta > 0\). Assume that there exists \(N_0 < \infty\) so that for \(N \geq N_0\), \(\frac{2\theta}{\beta} \in H_{E_N}(\lfloor \lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta \rfloor^c) \cap H_{\tilde{E}_N}(\lfloor \lambda_{\min}(\tilde{E}_N) - \eta, \lambda_{\max}(\tilde{E}_N) + \eta \rfloor^c)\). Then, there exists a function \(g(\delta, \eta)\) (independent of \(N\)) going to zero with \(\delta\) for any \(\eta\) and such that for all \(N \geq N_0\)
\[
\left| \frac{1}{N} \log I_N^{(\beta)}(D_N, E_N) - \frac{1}{N} \log I_N^{(\beta)}(D_N, \tilde{E}_N) \right| \leq g(\delta, \eta)
\]

Note that the third point is analogous to the continuity statement obtained in the case where \(D_N\) has also rank \(N\) in [13], Lemma 5.1. However, let us mention again that there is an important difference here which lies in the fact that the smallest and largest eigenvalues play quite an important role. In fact, it can be seen (see Theorem [13]) that if we let one eigenvalue be much larger than the support of the limiting spectral distribution, then the limit of the spherical integral will change dramatically. However, Lemma [2] shows that this limit will not depend on these escaping eigenvalues provided \(|\theta|\) is smaller than some critical value \(\theta_0(\lambda_{\min}, \lambda_{\max}) = \min(|H_{\min}/\beta^2|, |H_{\max}/\beta^2|)\).

Before going into the proof of Lemma [2], let us show that Theorem [1.2] is a direct consequence of its second point.

**Proof of Theorem [1.2]:** Since we assumed that, for \(N\) large enough, \(2\theta \beta^{-1} \in H_{E_N}(\lfloor \lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta \rfloor^c)\), we can find a \(v_N\) satisfying (13). Note that \(v_N\) is unique by strict monotonicity of \(H_{E_N}\) on \(]-\infty, \lambda_{\min}(E_N) - \eta]\), where it is negative, and on \(]\lambda_{\max}(E_N) + \eta, \infty]\), where it is positive. Therefore,
\[
(2\theta)^{-1} + v_N \in [\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c
\]
enforces that
\[
1 - \frac{2\theta}{\beta} \lambda_i + \frac{2\theta}{\beta} v_N > \frac{2|\theta|}{\beta} \eta \tag{14}
\]
so that, because of the uniform continuity of \(H_{E_N}\) on \([\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c\), as \(\hat{\mu}_N^{E_N}\) converges to \(\mu_{E}\), \(v_N\) converges to \(v\) the solution of \(H_{\mu_E}(v) = 2\theta\beta^{-1}\) and
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \log \left( 1 + \frac{2\theta}{\beta} v_N - \frac{2\theta}{\beta} \lambda_i \right) = \int \log \left( 1 + \frac{2\theta}{\beta} v - \frac{2\theta}{\beta} \lambda \right) d\mu_E(\lambda).
\]
Furthermore, the computation of the derivative of
\[ \theta \mapsto \theta v - \frac{\beta}{2} \int \log \left( 1 + \frac{2\theta}{\beta} v - \frac{2\theta}{\beta} \lambda \right) d\mu_E(\lambda), \]
with this particular \( v = R_{\mu_E}(2\theta\beta^{-1}) \) allows us to get the explicit expression
\[ \theta v - \frac{\beta}{2} \int \log \left( 1 + \frac{2\theta}{\beta} v - \frac{2\theta}{\beta} \lambda \right) d\mu_E(\lambda) = \frac{\beta}{2} \int_0^{\frac{2\theta}{\beta}} R_{\mu_E}(u)du. \]

Therefore, Hypothesis (4) together with Lemma 2.1.2 finishes the proof of (6).

Now the last point is to check that under Hypothesis 1.1, the assumption of Lemma 2.1.2 is equivalent to \( 2\theta/\beta \in H_{\mu_E}([\lambda_{\min}, \lambda_{\max}]^c) \).

Let us first observe that \( H_{\mu_E}([\lambda_{\min}, \lambda_{\max}]^c) = \bigcup_{\eta > 0} H_{\mu_E}([\lambda_{\min} - \eta, \lambda_{\max} + \eta]^c) \) and that, under Hypothesis 1.1,
\[ H_{\mu_E}([\lambda_{\min} - 2\eta, \lambda_{\max} + 2\eta]^c) \subseteq \bigcup_{N_0 \geq 0} \bigcap_{N \geq N_0} H_{E_N}([\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c), \]
since, for any \( \lambda \in \bigcup_{N_0 \geq 0} \bigcap_{N \geq N_0} \text{supp}(\hat{\mu}_{E_N}^N) \), the application \( z \mapsto (z - \lambda)^{-1} \) is continuous bounded on \( [\lambda_{\min} - 2\eta, \lambda_{\max} + 2\eta]^c \). Therefore, \( \frac{2\theta}{\beta} \in H_{\mu_E}([\lambda_{\min}, \lambda_{\max}]^c) \) implies the assumption of Lemma 2.1.2.

Conversely, we get by the same arguments that
\[ \bigcup_{N_0 \geq 0} \bigcap_{N \geq N_0} H_{E_N}([\lambda_{\min}(E_N) - 2\eta, \lambda_{\max}(E_N) + 2\eta]^c) \subseteq H_{\mu_E}([\lambda_{\min} - \eta, \lambda_{\max} + \eta]^c), \]
what completes the proof.

### 2.1 Proof of Lemma 2.1

- The first point is trivial since the matrix \( U \) is unitary or orthogonal and hence bounded.
- Let us consider the second point. We now stick to the case \( \beta = 1 \) and will summarize at the end of the proof the changes to perform for the case \( \beta = 2 \). We can assume that the \( \{\lambda_1(E_N), \ldots, \lambda_N(E_N)\} \) is not reduced to a single point \( \{e\} \) since otherwise the result is straightforward. We write in short \( I_N(\theta, E_N) = I_N^{(1)}(D_N, E_N) \). The ideas of the proof are very close to usual large deviations techniques, and in fact in some sense simpler because strong concentration arguments are available for free (cf. 13). Following Fact 1.8 we can write, with \( (\lambda_1, \ldots, \lambda_N) \) the eigenvalues of \( E_N \),
\[ I_N(\theta, E_N) = \mathbb{E} \left[ \exp \left\{ N\theta \frac{\sum_{i=1}^N \lambda_i g_i^2}{\sum_{i=1}^N g_i^2} \right\} \right] \]
where the \( g_i \)'s are i.i.d standard Gaussian variables. Now, writing the Gaussian vector \((g_1, \ldots, g_N)\) in its polar decomposition, we realize of course that the spherical integral does not depend on its radius \( r = \| g \| \) which follows the law
\[ \rho_N(dr) := Z_N^{-1} r^{N-1} e^{-\frac{1}{2} r^2} dr, \]
with $Z_N$ the appropriate normalizing constant.

The idea of the proof is now that $r$ will of course concentrate around $\sqrt{N}$ so that we are reduced to study the numerator and to make the adequate change of variable so that it concentrates around $v_N$. For $\kappa < 1/2$, there exists a finite constant $C(\kappa)$ such that

$$\rho_N \left( \left| \frac{r^2}{N} - 1 \right| \geq N^{-\kappa} \right) \leq C(\kappa) e^{-\frac{1}{4} N^{1-2\kappa}}. \quad (15)$$

Such an estimate can be readily obtained by applying standard precise Laplace method to the law $\tilde{\rho}_N$ of $(N-2)^{-1}r^2$ which is given by

$$\tilde{\rho}_N(dx) = \tilde{Z}_N^{-1} 1_{x \geq 0} e^{-\frac{N-2}{2}f(x)} dx$$

with $f(x) = x - \log x$. Indeed, $f$ achieves its minimal value at $x = 1$ so that for any $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that $\tilde{Z}_N \tilde{\rho}_N(\{ |x-1| > \epsilon \}) \leq e^{-c(\epsilon)N}$. Now, $\sigma_\epsilon = \inf\{ f''(x), |x-1| \leq \epsilon \} > 0$ so that Taylor expansion results with

$$\tilde{Z}_N \tilde{\rho}_N(\{ |x-1| \geq N^{-\kappa} \}) \leq e^{-c(\epsilon)N} + \int_{\theta \geq N^{-\kappa}} e^{-\frac{N-2}{2} \sigma_\epsilon y^2} dy \leq e^{-\frac{\sigma_\epsilon}{4} N^{1-2\kappa}}$$

where the last inequality holds for $N$ large enough. A lower bound on $\tilde{Z}_N$ is obtained similarly by considering $\tilde{\sigma}_\epsilon = \sup\{ f''(x), |x-1| \leq \epsilon \} > 0$ showing that $\tilde{Z}_N \geq \tilde{c}(\epsilon) \sqrt{N}^{-1}$. We conclude by noticing that $\sigma_\epsilon$ goes to one as $\epsilon$ goes to zero. Note that such a result can also be seen as a direct consequence of moderate deviations (cf. section 3.7 in [9]).

From this, if we introduce the event $A_N(\kappa) := \{ \frac{\|g\|_2^2}{N} - 1 \leq N^{-\kappa} \}$, it is not hard to see that for any $\kappa < \frac{1}{2}$ and for $N$ large enough (such that $1 - C(\kappa) e^{-\frac{1}{4} N^{1-2\kappa}} > 0$), we have

$$1 \leq \frac{I_N(\theta, E_N)}{\mathbb{E} \left[ 1_{A_N(\kappa)} \exp \left\{ N \theta^2 \sum_{i=1}^{N} \lambda_i g_i^2 \right\} \right]} \leq \delta(\kappa, N)$$

where $\delta(\kappa, N) = \frac{1}{1 - C(\kappa) e^{-\frac{1}{4} N^{1-2\kappa}}}$. Therefore,

$$I_N(\theta, E_N) \leq \delta(\kappa, N) \mathbb{E} \left[ 1_{A_N(\kappa)} \exp \left\{ N \theta^2 \sum_{i=1}^{N} \lambda_i g_i^2 \right\} \right] \leq \delta(\kappa, N) e^{N\theta v + N^{1-\kappa} \theta (\|E_N\|_\infty + |v|)} \mathbb{E} \left[ 1_{A_N(\kappa)} \exp \left\{ \theta \sum_{i=1}^{N} \lambda_i g_i^2 - v \theta \sum_{i=1}^{N} g_i^2 \right\} \right]$$

for any $v \in \mathbb{R}$. Now,

$$\mathbb{E} \left[ 1_{A_N(\kappa)} \exp \left\{ \theta \sum_{i=1}^{N} \lambda_i g_i^2 - v \theta \sum_{i=1}^{N} g_i^2 \right\} \right] = \prod_{i=1}^{N} \left[ \sqrt{1 + 2\theta v - 2\theta \lambda_i} \right]^{-1} P_N(A_N(\kappa)) \quad (17)$$

with $P_N$ the probability measure on $\mathbb{R}^N$ given by

$$P_N(dg_1, \ldots, dg_N) = \frac{1}{\sqrt{2\pi}} \prod_{i=1}^{N} \left[ \sqrt{1 + 2\theta v - 2\theta \lambda_i} \right]^{-1} e^{-\frac{1}{2} (1 + 2\theta v - 2\theta \lambda_i) g_i^2} dg_i$$
which is well defined provided we choose \( v \) so that
\[
1 + 2\theta v - 2\theta \lambda_i > 0 \quad \forall i \text{ from } 1 \text{ to } N.
\] (18)

Thus, for any such \( v \)'s, we get from (16) and (17), that for any \( \kappa = \frac{1}{2} - \zeta \) with \( \zeta > 0 \) and \( N \) large enough, since \( P_N(A_N(\kappa)) \leq 1 \),
\[
I_N(\theta, E_N) \leq \delta(\kappa, N) \prod_{i=1}^{N} \left( \sqrt{1 + 2\theta v - 2\theta \lambda_i} \right)^{-1} e^{N\theta v + N^1-\kappa\theta v\|E_N\|\infty}. 
\] (19)

We similarly obtain the lower bound
\[
I_N(\theta, E_N) \geq e^{N\theta v - N^1-\kappa\theta v\|E_N\|\infty} \prod_{i=1}^{N} \left( \sqrt{1 + 2\theta v - 2\theta \lambda_i} \right)^{-1} P_N(A_N(\kappa))
\]

Now, we show that we can choose \( v \) wisely so that for \( N \geq N(\kappa) \),
\[
P_N(A_N(\kappa)) = P_N(\|A_N\|^2 - 1 \leq N^{-\kappa}) \geq \frac{1}{2}. 
\] (20)

This will finish to prove, with this choice of \( v \), that
\[
I_N(\theta, E_N) \geq \frac{1}{2} e^{N\theta v - N^1-\kappa\theta (\|E_N\|\infty + |v|)} \prod_{i=1}^{N} \left( \sqrt{1 + 2\theta v - 2\theta \lambda_i} \right)^{-1} P_N(A_N(\kappa))
\] (21)

yielding the desired lower bound.

We know that \( P_N \) is a product measure under which \( \tilde{g}_i = \sqrt{1 + 2\theta v - 2\theta \lambda_i} g_i \) are i.i.d standard Gaussian variables. Let us now choose \( v = v_N \) in \( -(2\theta)^{-1} + [\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c \) satisfying
\[
E_{P_N} \left[ \frac{1}{N} \|g\|^2 \right] = \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{\tilde{g}_i^2}{1 + 2\theta v_N - 2\theta \lambda_i} \right] = \frac{1}{2\theta} H_{E_N} ((2\theta)^{-1} + v_N) = 1. 
\] (22)

We recall from (14) that \( 1 - 2\theta \lambda_i + 2\theta v_N > 2\theta \eta > 0 \) so that all our computations are validated by this final choice.

With this choice of \( v_N \), we have
\[
E_{P_N} \left[ \left( \frac{1}{N} \|g\|^2 - 1 \right)^2 \right] = \frac{2}{N^2} \sum_{i=1}^{N} \frac{1}{(1 + 2\theta v_N - 2\theta \lambda_i)^2} \leq \frac{2}{N\theta^2\eta^2}
\]

so that by Chebychev’s inequality
\[
P_N(\|A_N\|^2 - 1 \leq N^{-\kappa}) \leq \frac{2}{\eta^2\theta^2} N^{2\kappa - 1},
\]
which is smaller than $2^{-1}$ for sufficiently large $N$ since $2\kappa < 1$, resulting with (20).

Finally, since by definition

$$\frac{1}{N} \sum_{i=1}^{N} \frac{1}{1 - 2\theta \lambda_i + 2\theta v_N} = 1$$

with $(\lambda_i)_{1 \leq i \leq N}$ which do not all take the same value, there exists $i$ and $j$ so that

$$-2\theta \lambda_i + 2\theta v_N > 0, \quad -2\theta \lambda_j + 2\theta v_N < 0$$

so that $v_N \in [\lambda_{\min}(E_N), \lambda_{\max}(E_N)]$. Thus, (21) together with (19) give the second point of the lemma for $\beta = 1$.

In the case where $\beta = 2$, the $g_i^2$ have to be replaced everywhere by $g_i^2 + \tilde{g}_i^2$ with independent Gaussian variables $(g_i, \tilde{g}_i)_{1 \leq i \leq N}$. This time, we can concentrate

$$\frac{1}{N} \|g\|^2 = \frac{1}{N} \sum_{i=1}^{N} g_i^2 + \frac{1}{N} \sum_{i=1}^{N} \tilde{g}_i^2$$

around 2. Everything then follows by dividing $\theta$ by two and noticing that we will get the same Gaussian integrals squared.

• The last point is an easy consequence of the second since, for any $\lambda \in \bigcup_{N_0 \geq 0} \bigcap_{N \geq N_0} (\text{supp}(\hat{\mu}_{E_N}^N) \cap \text{supp}(\hat{\mu}_{E_N}^N))$, the application $z \mapsto (z - \lambda)^{-1}$ is continuous bounded (with norm depending on $\eta$) on $\bigcup_{N_0 \geq 0} \bigcap_{N \geq N_0} [\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c$.

2.2 Generalisation of the method to the multi-dimensional case

In the sequel, we want to apply the strategy we used above to show Theorem 1.7, that is to say study the behaviour of the spherical integrals as the rank of $D_N$ remains negligible compared to $\sqrt{N}$. In this case and if all the eigenvalues of $D_N$ are small enough, we show that it behaves like a product, namely that we have the equality (11). To lighten the notations, we let $\theta_i := \theta_i^N$, for all $i \leq M(N)$.

We will rely again on Fact 1.8 and write in the case $\beta = 1$,

$$I_N(D_N, E_N) = \mathbb{E} \left[ \exp \left\{ N \sum_{m=1}^{M} \sum_{i=1}^{N} \lambda_i \left( \frac{\hat{\mu}_{g_i}^{(m)}}{\sqrt{\sum_{i=1}^{N} (\hat{g}_i^{(m)})^2}} \right)^2 \right\} \right],$$

where the expectation is taken under the standard Gaussian measure and the vectors $(\hat{g}^{(1)}, \ldots, \hat{g}^{(M)})$ are obtained from the Gaussian vectors $(g^{(1)}, \ldots, g^{(M)})$ by a standard Schmidt orthogonalisation procedure.

This means that there exists a lower triangular matrix $A = (A_{ij})_{1 \leq i, j \leq M}$ such that for any integer $m$ between 1 and $M$,

$$\hat{g}^{(m)} = g^{(m)} + \sum_{j=1}^{m-1} A_{mj} g^{(j)}$$
and the $A_{ij}$’s are solutions of the following system : for all $p$ from 1 to $m - 1$,

$$\langle g^{(m)}, g^{(p)} \rangle + \sum_{j=1}^{m-1} A_{mj} \langle g^{(j)}, g^{(p)} \rangle = 0,$$

(24)

with $\langle . , . \rangle$ the usual scalar product in $\mathbb{R}^N$.

Therefore, if we denote, for $i$ and $j$ between 1 and $M$, with $i \leq j$,

$$X^{ij}_N := \frac{1}{N} \langle g^{(i)}, g^{(j)} \rangle$$

and

$$Y^{ij}_N := \frac{1}{N} \sum_{l=1}^{N} \lambda_l g^{(i)}_{l} g^{(j)}_{l},$$

then, for each $m$ from 1 to $M$, there exists a rational function $F_m : \mathbb{R}^{m(m+1)} \to \mathbb{R}$ such that

$$\sum_{i=1}^{N} \lambda_l (g^{(m)}_{l})^2 = F_m((X^{ij}_N, Y^{ij}_N)_{1 \leq i \leq j \leq m})$$

(25)

and a rational function $G_m : \mathbb{R}^{\frac{m(m+1)}{2}} \to \mathbb{R}$ such that

$$\frac{1}{N} \sum_{i=1}^{N} (g^{(m)}_{i})^2 = G_m((X^{ij}_N)_{1 \leq i \leq j \leq m}).$$

(26)

We now adopt the following system of coordinates in $\mathbb{R}^{MN}$ : $r_1, \alpha^{(1)}_1, \ldots, \alpha^{(1)}_{N-1}$ are the polar coordinates of $g^{(1)}$, $r_2 := \|g^{(2)}\|$, $\beta_2$ is the angle between $g^{(1)}$ and $g^{(2)}$, $\alpha^{(2)}_1, \ldots, \alpha^{(2)}_{N-2}$ are the angles needed to spot $g^{(2)}$ on the cone of angle $\beta_2$ around $g^{(1)}$, then $r_3 := \|g^{(3)}\|$, $\beta_3$ the angle between $g^{(3)}$ and $g^{(i)}$ ($i = 1, 2$) and $\alpha^{(3)}_1, \ldots, \alpha^{(3)}_{N-3}$ the angles needed to spot $g^{(3)}$ on the intersection of the two cones...etc...

Then observe that $F_m((X^{ij}_N, Y^{ij}_N)_{1 \leq i \leq j \leq m})$ depends only on the $\alpha$’s (because the $\frac{g^{(i)}_{i}}{\|g^{(i)}\|}$ do) whereas $G_m((X^{ij}_N)_{1 \leq i \leq j \leq m})$ depends on the $r$’s and the $\beta$’s. Therefore, if we consider the event

$$B_N(\kappa) := \left\{ \forall i, \ |X^{ii}_N - 1| \leq N^{-\kappa}, \ \forall i \neq j \ |X^{ij}_N| \leq N^{-\kappa} \right\},$$

then, as in the case of rank one, we can write that

$$I_N(D_N, E_N) \leq \mathbb{E} \left[ 1_{B_N(\kappa)} e^{\gamma_{\theta m} F_m(X^{ij}_N, Y^{ij}_N)} + P(B_N(\kappa)^c) I_N(D_N, E_N) \right].$$

(27)

Now we claim that, for $N$ large enough, for any $\kappa > 0$, there exists an $\alpha > 0$ such that

$$P(B_N(\kappa)^c) \leq C'(\kappa) e^{-\alpha N^{1-2\kappa}}.$$ 

(28)

Indeed, as in $\cite{[L]}$,

$$P(B_N(\kappa)^c) \leq \sum_{i=1}^{M} P \left( |X^{ii}_N - 1| > N^{-k} \right) + \sum_{i,j=1}^{M} P \left( |X^{ij}_N| > N^{-k} \right)$$

$$\leq c_1(\kappa) M e^{-\frac{1}{4} N^{1-2\kappa}} + c_2(\kappa) M^2 e^{-\frac{1}{8} N^{1-2\kappa}},$$
what gives immediately (28).

Now, as far as \( \kappa < \frac{1}{2} \), (27) together with (28) give

\[
1 \leq \frac{I_N(D_N, E_N)}{\mathbb{E} \left[ 1_{B_N(\kappa)} e^{\text{NF}_m(X_{ij}^{(1)}, Y_{ij}^{(1)})} \right]} \leq 1 + \epsilon(N, k),
\]

with \( \epsilon(N, k) \) going to zero.

We now want to expand \( F_M \) on \( B_N(\kappa) \) as we did in the previous subsection.

As the \( A_{ij} \)'s satisfy the linear system (24), we can write the Cramer’s formulas corresponding to it and get

\[
A_{ij} = \frac{\det(R_{kl}^{N})_{1 \leq k, l \leq i-1}}{\det(X_{kl}^{N})_{1 \leq k, l \leq i-1}},
\]

where

\[
R_{kl}^{N} = \begin{cases} X_{kl}^{N}, & \text{if } l \neq j \\ -X_{ki}^{N}, & \text{if } l = j. \end{cases}
\]

Now, we look at the denominator and can show that

\[
\det(X_{kl}^{N})_{1 \leq k, l \leq i-1} \geq 1 - \sum_{s=1}^{i-1} (MN^{-\kappa})^s \geq \frac{1}{2},
\]

where the last inequality holds for \( N \) large enough as far as \( M = o(N^\kappa) \).

We now go to the numerator : expanding over the \( j \)th column, we get this time that

\[
\det(R_{kl}^{N})_{1 \leq k, l \leq i-1} \leq N^{-\kappa} + (M - 1)N^{-2\kappa} \sum_{s=1}^{i-1} (MN^{-\kappa})^s \leq cN^{-\kappa},
\]

where again the last equality holds as far as \( M = o(N^\kappa) \) and \( c \) is a fixed constant.

From the two last inequalities, we have that, on \( B_N(\kappa) \), \( \sup_{i<j} |A_{ij}| \leq c'N^{-\kappa} \).

From that we can easily deduce that, for any \( m \) less than \( M \), we have

\[
\frac{1}{N} \left\| \tilde{g}(m) - g(m) \right\|^2 \leq \frac{1}{N} \sum_{i,j=1}^{m-1} |A_{mj}A_{mi}| \langle g(i), g(j) \rangle^2 \leq c''N^{-2\kappa}(M^2N^{-2\kappa} + M) \leq c_2N^{-\kappa}.
\]

From these estimations and (28), for any \( v_j^N \), we get the following upper bound :

\[
I_N(D_N, E_N) \leq (1 + \epsilon(\kappa, N)) \exp \left\{ N \sum_{j=1}^{M} \lambda_j v_j^N \right\}
\]

\[
\mathbb{E} \left[ 1_{B_N(\kappa)} \prod_{j=1}^{M} \exp \left\{ N \lambda_j v_j^N + \frac{N}{2} \sum_{i=1}^{N} (\tilde{g}_i^{(j)})^2 - v_j^N \frac{1}{N} \sum_{i=1}^{N} (\tilde{g}_i^{(j)})^2 \right\} \right]
\]

\[
\frac{1}{1 + \frac{N}{\left\| g^{(j)} \right\|^2} + \left( \frac{1}{N} \left\| g^{(j)} \right\|^2 - 1 \right)}
\]

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\[ \leq (1 + \epsilon(\kappa, N)) \exp \left\{ N \sum_{j=1}^{M} \theta_j v_j^N \right\} \]

\[ \mathbb{E} \left[ 1_{B_N(\kappa)} \prod_{j=1}^{M} \exp \left\{ \left( \theta_j \sum_{i=1}^{N} \lambda_i (\hat{g}_i^{(j)})^2 - v_j^N \theta_j \sum_{i=1}^{N} (\hat{g}_i^{(j)})^2 \right) \right\} \left[ 1 + c_4 N^{-\kappa} \right] \right] \]

\[ \leq (1 + \epsilon(\kappa, N)) e^{N \sum_{j=1}^{M} \theta_j v_j^N} e^{C \sup |\theta_j| \left( ||E_N||_{\infty} + \sup |v_j^N| \right) M N^{1-\kappa}} \]

\[ \mathbb{E} \left[ \prod_{j=1}^{M} \exp \left\{ \theta_j \sum_{i=1}^{N} \lambda_i (g_i^{(j)})^2 - v_j^N \sum_{i=1}^{N} (g_i^{(j)})^2 \right\} \right]. \]

where \( C \) is again a fixed constant.

From the hypotheses of Theorem 1.7, we know that there exists an \( N \) such that \( 2 \theta_j \in H_{E_N}(\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta^\nu) \), from which we can easily deduce that \( |2 \theta_j| \leq \eta^{-1} \). Moreover, as in the proof of Lemma 2.12, \( |v_j^N| \leq ||E_N||_{\infty} \) is uniformly bounded. Therefore, we get

\[ \limsup_{N \to \infty} \frac{1}{NM(N)} \log I_N(D_N, E_N) \leq \int I_{\mu_E}(\theta) d\mu_D(\theta). \]

We also get a similar lower bound and conclude similarly to the preceding subsection by considering the shifted probability measure \( P_{\theta_1, \ldots, \theta_M}^N = \otimes_{j=1}^{M} P_{\theta_j}^N \) where

\[ P_{\theta_j}^N(dg_1, \ldots, dg_N) = \frac{1}{\sqrt{2\pi}} \prod_{i=1}^{N} \sqrt{1 + 2 \theta_j v_j^N - 2 \theta_j \lambda_i} e^{-\frac{1}{2} \left( 1 + 2 \theta_j v_j^N - 2 \theta_j \lambda_i \right) \lambda_i^2} dg_i. \]

This concludes the proof of Theorem 1.7.

### 3 Central limit theorem in the case of rank one

Under the hypotheses of Theorem 1.2, \( v_N \) (defined by (13)) is converging to \( v = R_{\mu_E} \left( \frac{2\theta}{\sqrt{2}} \right) \) and we established that the spherical integral is converging to \( \theta v - \frac{\beta}{2} \int \log \left( 1 + \frac{2\theta}{\sqrt{2}} v - \frac{2\theta}{\sqrt{2}} \lambda \right) d\mu_E(\lambda) \). In the case where the fluctuations of the eigenvalues do not interfere, we can get sharper estimates, given, in the case \( \beta = 1 \), by Theorem 1.3. This section is devoted to its proof, namely the study of the behaviour of \( e^{-N \left( \theta R_{\mu_E} (2\theta) - \frac{1}{2N} \sum \log(1 + 2R_{\mu_E} (2\theta) - 2\delta \lambda) \right)} I_N(\theta, E_N) \).

**Proof of Theorem 1.3**

- We first treat the non degenerate case \( \mu_E \neq \delta_e \).

Let us first make an important remark : the hypothesis that \( d(\mu_{E_N}, \mu_E) = o(\sqrt{N}^{-1}) \) has the two following consequences :

\[ |v - v_N| = o(\sqrt{N}^{-1}) \quad (29) \]

and \( \lim_{N \to \infty} \sqrt{N}(H_{E_N} - H_{\mu_E}(K_{\mu_E}(2\theta))) = 0. \quad (30) \)
Indeed, since $2\theta \in H_{\mu_E}([\lambda_{\min}, \lambda_{\max}]^c)$, there is an $\eta > 0$, such that, for $N$ large enough, $2\theta \in H_{E_N}([\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c)$. Therefore, as for any $\lambda$ which is in $\text{supp}(\mu_{E_N}^N)$ for $N$ large enough, $z \mapsto (z - \lambda)^{-1}$ is uniformly bounded and Lipschitz on $\bigcap_{N \geq N_0} [\lambda_{\min}(E_N) - \eta, \lambda_{\max}(E_N) + \eta]^c$, we get directly \cite{29}, and also \cite{30} as we know that $K_{\mu_E}(2\theta) \in [\lambda_{\min}, \lambda_{\max}]^c$.

For $v = R_{\mu_E}(2\theta)$, we set
\[
\gamma_N = \left( \frac{1}{N} \sum_{i=1}^{N} g_i^2 - 1 \right) \quad \text{and} \quad \hat{\gamma}_N = \left( \frac{1}{N} \sum_{i=1}^{N} \lambda_i g_i^2 - v \right).
\]
Let us also define for $\epsilon > 0$
\[
I_N^\epsilon(\theta, E_N) := \int_{|\gamma_N| \leq \epsilon, |\hat{\gamma}_N| \leq \epsilon} \exp \left\{ \theta N \frac{\gamma_N + v}{\gamma_N + 1} \right\} \prod_{i=1}^{N} dP(g_i),
\]
with $P$ the standard Gaussian probability measure on $\mathbb{R}$. We claim that, for any $\zeta > 0$, for $N$ large enough,
\[
|I_N(\theta, E_N) - I_N^\epsilon(\theta, E_N)| \leq e^{-N^{1-\zeta}} I_N(\theta, E_N).
\] (31)

Indeed, consider
\[
\mu_N^\theta(dg) = \frac{1}{I_N(\theta, E_N)} \exp \left\{ \theta N \frac{\gamma_N + v}{\gamma_N + 1} \right\} \prod_{i=1}^{N} dP(g_i). 
\]
(31) is equivalent to
\[
\mu_N^\theta(|\gamma_N| \geq \epsilon) \leq \frac{1}{2} e^{-N^{1-\zeta}} \quad \text{and} \quad \mu_N^\theta(|\hat{\gamma}_N| \geq \epsilon) \leq \frac{1}{2} e^{-N^{1-\zeta}}
\] (32)
The first inequality is trivial since by \cite{15}, for $\kappa < \frac{1}{2}$,
\[
\mu_N^\theta(|\gamma_N| \geq N^{-\kappa}) = \rho_N \left( \frac{r^2}{N} - 1 \right) \geq N^{-\kappa} \leq e^{-\frac{1}{2} N^{1-2\kappa}}.
\]
To show the second point, following the proof of Lemma \cite{24}, we find a finite constant $C(\kappa)$ so that
\[
\mu_N^\theta(|\hat{\gamma}_N| \geq \epsilon) \leq C(\kappa) e^{C(\kappa)N^{1-\kappa - |\theta| ||E_N||}} P_N(|\hat{\gamma}_N| \geq \epsilon)
\]
where under $P_N$ the $g_i$ are independent centered Gaussian variable with covariance $(1 - 2\theta \lambda_i + 2\theta v_N)^{-1}$. Hence
\[
P_N(|\hat{\gamma}_N| \geq \epsilon) = P_N(\left| \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i}{1 - 2\theta \lambda_i + 2\theta v_N} g_i^2 - v \right| \geq \epsilon).
\]
Let us denote $\bar{E}_N = \phi_{v_N}(E_N)$ with $\phi_v(x) = x(1 - 2\theta x + 2\theta v)^{-1}$. Then, the spectral measure of $\bar{E}_N$ converges towards $\mu_{\bar{E}} := \phi_\theta \mu_E$ since $v_N$ converges towards $v$ (see \cite{29}). Moreover $\lambda_{\min}(\bar{E}_N)$ and $\lambda_{\max}(\bar{E}_N)$ converge. Hence, we can apply Lemma \cite{5.3} to obtain a large deviation principle for the law of $\frac{1}{N} \sum_{i=1}^{N} \lambda_i(\bar{E}_N) g_i^2$ under $P_N$ with good rate function $J$. One checks that $J$ has a unique minimizer which is
\[ z_0 = R_{\mu}(0) = \int \frac{\lambda}{1 - 2\theta \lambda + 2\theta v} \, d\mu_E(\lambda) = v. \]

As a consequence, for \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \) so that for \( N \) large enough

\[
P^{\otimes N} \left( \left| \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i}{1 - 2\theta \lambda_i + 2\theta v} \tilde{g}_i^2 - v \right| > \epsilon \right) \leq e^{-\delta(\epsilon)N}.
\]

This completes the proof of (32).

We now deal with \( I_N(\theta, E_N) \). We use the expansion \( \frac{1}{1 + \gamma_N} = 1 - \gamma_N + \frac{\gamma_N^2}{1 + \gamma_N} \) to get that

\[
I_N(\theta, E_N) = e^{\theta v} \int \left[ \exp \left\{ -\theta N \gamma_N \frac{\nu N - v\gamma_N}{\gamma_N + 1} \right\} e^{(\theta N \nu - v\gamma_N) N} \right] \prod_{i=1}^{N} dP(g_i).
\]

We note that

\[
\exp \{ \theta N (\nu N - v\gamma_N) \} \prod_{i=1}^{N} dP(g_i) = \prod_{i=1}^{N} \left[ \sqrt{1 + 2\theta v - 2\theta \lambda_i} \right]^{-1} \prod_{i=1}^{N} dP_i(g_i)
\]

with \( P_i \) the centered Gaussian probability measure

\[
dP_i(x) = \sqrt{(2\pi)^{-1}(1 + 2\theta v - 2\theta \lambda_i)} \exp \left\{ -\frac{1}{2}(1 + 2\theta v - 2\theta \lambda_i) x^2 \right\} dx.
\]

We have that

\[
1 + 2\theta v - 2\theta \lambda_i = 2\theta (K_{\mu_E}(2\theta) - \lambda_i)
\]

and we know that \( K_{\mu_E}(2\theta) \in [\lambda_{\min}, \lambda_{\max}]^c \). Further, arguing as in (14), we find, for any given \( \theta > 0 \), a constant \( \eta_{\theta} > 0 \) such that

\[
\inf_{1 \leq i \leq N} \left( 1 + 2\theta v - 2\theta \lambda_i \right) > \eta_{\theta}
\]

insuring that the \( P_i \) are well defined. Therefore,

\[
I_N(\theta, E_N) = e^{\theta v} \int \left[ \exp \left\{ -\theta N \gamma_N \frac{\nu N - v\gamma_N}{\gamma_N + 1} \right\} \right] \prod_{i=1}^{N} dP_i(g_i)
\]

(34)

Now, under \( \prod_{i=1}^{N} dP_i(g_i) \), \( (\sqrt{N}\gamma_N, \sqrt{N}\tilde{\gamma}_N) \) converges in law towards a centered two-dimensional Gaussian variables \( (\Gamma_1, \Gamma_2) \) as soon as their covariances converge. We investigate this convergence.

Hereafter, we shall write \( g_i = (1 + 2\theta(v - \lambda_i))^{-\frac{1}{2}} \tilde{g}_i \) with standard independent Gaussian variables \( \tilde{g}_i \). Then,

\[
\mathbb{E}(\langle \sqrt{N}\gamma_N \rangle^2) = N \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \frac{\tilde{g}_i^2 - 1}{1 + 2\theta v - 2\theta \lambda_i} + \frac{1}{2\theta} (H_{E_N} - H_{\mu_E})(K_{\mu_E}(2\theta)) \right)^2 \right]
\]

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where we used that
\[ 2\theta = H_{\mu_E}(K_{\mu_E}(2\theta)) = \int \frac{1}{K_{\mu_E}(2\theta) - \lambda} d\mu_E(\lambda), \]
and (33). Equation (30) implies
\[ \lim_{N \to \infty} \mathbb{E}((\sqrt{N}\gamma_N)^2) = \lim_{N \to \infty} N \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} \frac{\theta_i^2 - 1}{1 + 2\theta - 2\lambda_i} \right)^2 \right] \]
\[ = \lim_{N \to \infty} \frac{2}{N} \sum_{i=1}^{N} \frac{1}{(1 + 2\theta - 2\lambda_i)^2} \]
\[ = \frac{1}{2\theta^2} \int \frac{1}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) := \frac{Z}{2\theta^2}, \]
where the above convergence holds since $K_{\mu_E}(2\theta)$ lies outside $[\lambda_{\text{min}}, \lambda_{\text{max}}]$ and therefore outside the support of $\mu_E$.

Similar computations give that under the same hypotheses,
\[ \lim_{N \to \infty} \mathbb{E}((\sqrt{N}\gamma_N)^2) = \frac{1}{2\theta^2} \int \frac{\lambda^2}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) \]
and that
\[ \lim_{N \to \infty} \mathbb{E}(\sqrt{N}\gamma_N \sqrt{N}\gamma_N) = \frac{1}{2\theta^2} \int \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda). \]

Therefore, provided that the Gaussian integral is well defined, we find that
\[ I_N(\theta, E_N) = e^{N\theta v - \frac{N}{2} \int \log\left(2\theta(K_{\mu_E}(2\theta) - \lambda)\right) d\mu_E(\lambda)} \int e^{-(\theta x - yz)} d\Gamma(x, y) (1 + o(1)), \]
with $\Gamma$ a centered Gaussian measure on $\mathbb{R}^2$ with covariance matrix
\[ R = \frac{1}{2\theta^2} \left[ \begin{array}{cc} \int \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) & \int \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) \\ \int \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) & \int \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) \end{array} \right], \]
where we used the notation $K_{\mu_E} := K_{\mu_E}(2\theta)$.

Following the ideas [4] as outlined in appendix 7, we know that there is one step needed to justify this derivation, namely to check that the Gaussian integration in (36) is non-degenerate. If we set $D := 4\theta^4 \det R$, then, using the relation (35), one finds that $D = Z - 4\theta^2$, and that the Gaussian integral in (36) equals
\[ \frac{\theta^2}{\pi \sqrt{D}} \int \exp \left( -\frac{1}{2} \sum_{i,j=1}^{2} K_{i,j} x_i x_j \right) dx_1 dx_2, \]
where the matrix $K$ equals $\theta \left[ \begin{array}{cc} -2v & 1 \\ 1 & 0 \end{array} \right] + R^{-1}$, that is
\[ K = 2\theta^2 \left[ \begin{array}{cc} \frac{\lambda^2}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) - \frac{(K_{\mu_E}(2\theta) - \lambda)^D}{\theta} & \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) + \frac{D}{2\theta^2} \\ \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) + \frac{D}{2\theta^2} & \frac{\lambda}{(K_{\mu_E}(2\theta) - \lambda)^2} d\mu_E(\lambda) \end{array} \right]. \]
Our task is to verify that \( K \) is positive definite. It is enough to check that \( K_{11} > 0 \) and \( \det K > 0 \).

Re-expressing \( K_{11} \), one finds that

\[
K_{11} = \frac{2\theta^2}{D} \left( 1 - 4\theta K_{\mu E} + K_{\mu E}^2 Z - \frac{1}{\theta} (Z - 4\theta^2) \left( K_{\mu E} - \frac{1}{2\theta} \right) \right)
\]

\[
= \frac{2\theta^2}{D} \left( \left( K_{\mu E} - \frac{1}{2\theta} \right)^2 Z + \frac{Z}{4\theta^2} - 1 \right)
\]

But Schwarz’s inequality applied to (35) yields that \( Z > \frac{4}{\theta^2} \) as soon as \( \mu_E \) is not degenerate, implying that

\[
K_{11} > \left( K_{\mu E} - \frac{1}{2\theta} \right)^2 Z \geq 0,
\]

as needed. Turning to the evaluation of the determinant, note that

\[
\det K = \frac{4\theta^4}{D^2} Z \left( \frac{Z}{4\theta^2} - 1 \right) > 0,
\]

where the last inequality is again due to (35).

- Let us finally consider the case \( \mu_E = \delta_e \). In this case, \( H_{\mu E}(x) = (x - e)^{-1} \) and \( K_{\mu E}(x) = x^{-1} + e, \) \( v = e \) (note also that \( Z \) in Theorem 1.3 is equal to \( 4\theta^2 \)). We can follow the previous proof but then

\[
\lim_{N \to \infty} E[(\sqrt{N}(\gamma_N - v\gamma_N))^2] = 0.
\]

From here, we argue again using appendix 7 that

\[
\lim_{N \to \infty} E[|\gamma_N| \leq \epsilon, |\gamma_N - v\gamma_N| \leq \epsilon] e^{-\theta(1+\gamma_N)^{-1}\sqrt{N}\gamma_N\sqrt{N}(\gamma_N - v\gamma_N)} = 1
\]

which completes the proof of Theorem 1.3.

4 Extension of the results to the complex plane

In this section, we would like to extend the results of section 2 to the case where \( \theta \) is complex, that is to show Theorem 1.4.

As in the real case, we first would like to write that

\[
I_N(\theta, E_N) = \prod_{i=1}^{N} \sqrt{\zeta_i} \int \exp \left\{ \theta N \sum_{i=1}^{N} \frac{\lambda_i \zeta_i g_i^2}{2} - \frac{1}{2} \sum_{i=1}^{N} \zeta_i g_i^2 \right\} \prod_{i=1}^{N} dg_i,
\]

with \( \zeta_i = \frac{1}{1 + 2\theta v - 2\theta \lambda_i} \), for \( v \) such that \( \Re(\zeta_i) > 0, \) \( \forall i \) with \( 1 \leq i \leq N. \)

This is a direct consequence of the following lemma

**Lemma 4.1** For any function \( f : \mathbb{C}^N \to \mathbb{C} \) which is invariant by \( x \mapsto -x \), analytic outside 0 and bounded on \( \{ z = x + iy \in \mathbb{C} / |y| < x \}^N \) and for any \( (\zeta_1, \ldots, \zeta_N) \) such that \( \Re(\zeta_i) > 0 \) for any \( i \) from
1 to $N$, we have that

$$J_N := \int f(g_1, \ldots, g_N)e^{-\frac{1}{2} \sum_{i=1}^{N} \zeta_i^2} \prod_{i=1}^{N} dg_i$$

$$= \prod_{i=1}^{N} \sqrt{\zeta_i} \int f(\sqrt{\zeta_1}g_1, \ldots, \sqrt{\zeta_N}g_N)e^{-\frac{1}{2} \sum_{i=1}^{N} \zeta_i^2} \prod_{i=1}^{N} dg_i,$$

with $\sqrt{\cdot}$ is the principal branch of the square root in $\mathbb{C}$.

**Proof of Lemma 4.1**

We denote by $r_j$ the modulus of $\zeta_j$ and $\alpha_j$ its phase ($\zeta_j = r_j e^{\alpha_j}$).

As $f$ is bounded on $\mathbb{R}^N$, dominated convergence gives that

$$J_N = \lim_{R \to \infty, \epsilon \to 0} \int_{[-R, R] \setminus [-\epsilon, \epsilon]^N} f(g_1, \ldots, g_N)e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i.$$

Thanks to invariance of $f$ by $x \mapsto -x$, we also have that

$$J_N = \lim_{R \to \infty, \epsilon \to 0} 2^N \int_{[\epsilon, R]} f(g_1, \ldots, g_N)e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i.$$

For each $j$ from 1 to $N$ and $R \in \mathbb{R}^+$, we define the following segments in $\mathbb{C}$:

$$C_{R, \epsilon}^j := \left\{ r e^{i \alpha_j^ \frac{\alpha_j}{2}}; \epsilon \leq r \leq R \right\},$$

and the following arc of circles

$$D_\epsilon^j := \left\{ e^{i \alpha}; 0 \leq \alpha \leq \frac{\alpha_j}{2} \right\} \quad \text{and} \quad D_R^j := \left\{ Re^{i \alpha}; 0 \leq \alpha \leq \frac{\alpha_j}{2} \right\},$$

so that, for each $j$, $[\epsilon, R]$ run from $\epsilon$ to $R$ followed by $D_R^j$ run counterclockwise, followed by $C_{R, \epsilon}^j$ run from $Re^{i \alpha_j^ \frac{\alpha_j}{2}}$ to $e^{i \alpha_j^ \frac{\alpha_j}{2}}$ followed by $D_\epsilon^j$ run clockwise form a closed path.

Therefore, if we let

$$f_{1,2,\ldots,N}^x : \mathbb{C} \to \mathbb{C}$$

$$x \mapsto f(x, x_2, \ldots, x_N),$$

then for any $(x_2, \ldots, x_N) \in \mathbb{C}^{N-1}$, $x \mapsto f_{1,2,\ldots,N}^x(x)e^{-\frac{1}{2}x^2}$ is analytic inside the contour $[\epsilon, R] \cup D_R^j \cup C_{R, \epsilon}^j \cup D_\epsilon^j$, so that Cauchy’s theorem implies

$$\int_{[\epsilon, R]} f_{1,2,\ldots,N}^x(g_1)e^{-\frac{1}{2}g_1^2}dg_1 = \int_{C_{R, \epsilon}^j} f_{1,2,\ldots,N}^x(g_1)e^{-\frac{1}{2}g_1^2}dg_1$$

$$= \int_{D_R^j} f_{1,2,\ldots,N}^x(g_1)e^{-\frac{1}{2}g_1^2}dg_1 + \int_{D_\epsilon^j} f_{1,2,\ldots,N}^x(g_1)e^{-\frac{1}{2}g_1^2}dg_1.$$
If we denote by
\[ J_{N,R}^1 = \int_{[\varepsilon,R]^{N-1}} e^{-\frac{1}{2} \sum_{i=2}^{N} g_i^2} \int_{D_R^1} f_{g_1,\ldots,g_N}^2 (g_1) e^{-\frac{1}{2} g_1^2} dg_1 \ldots dg_N, \]
we have that
\[ |J_{N,R}^1| = \int_{[\varepsilon,R]^{N-1}} \int_0^\alpha \int_0^{\frac{\pi}{2}} f(g_1,\ldots,g_N) e^{-\frac{1}{2} \sum_{i=2}^{N} g_i^2} R e^{-\frac{1}{2} R^2 \cos(2\alpha_1)} du_1 dg_2 \ldots dg_N, \]
\[ \leq \|f\|_\infty \sqrt{2\pi N} \alpha_1^\frac{1}{2} R e^{-\frac{1}{4} R^2 \cos(\alpha_1)}. \]
As \( \cos(\alpha_1) > 0 \), we have that for any \( \varepsilon \), \( \lim_{R \to \infty} |J_{N,R}^1| = 0. \)
In the same way, if we let
\[ L_{N,\varepsilon}^1 = \int_{[\varepsilon,R]^{N-1}} e^{-\frac{1}{2} \sum_{i=2}^{N} g_i^2} \int_{D^1_\varepsilon} f(g_1,\ldots,g_N) e^{-\frac{1}{2} g_1^2} dg_1 \ldots dg_N, \]
then we have that
\[ |L_{N,\varepsilon}^1| \leq \|f\|_\infty \sqrt{2\pi N} \varepsilon \alpha_1^\frac{1}{2}, \]
so that \( \lim_{\varepsilon \to 0} |L_{N,\varepsilon}^1| = 0. \)
By doing the same computation for each variable, we get that
\[ \lim_{R \to \infty, \varepsilon \to 0} \int_{[\varepsilon,R]^{N}} \int_0^{r\varepsilon} f(g_1,\ldots,g_N) e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i = \lim_{R \to \infty, \varepsilon \to 0} \int_{\Pi_{i=1}^{N}} f(g_1,\ldots,g_N) e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i. \]
The last step is to make the change of variable in \( \mathbb{R} \) which consist in letting \( \tilde{g}_j = \sqrt{r_j} g_j \) to get the result announced in the lemma [4.1] and therefore the formula (38).

We now go back to the proof of Theorem 1.4 and proceed as in section 2. We let
\[ \gamma_N := \frac{1}{N} \sum_{i=1}^{N} \zeta_i g_i^2 - 1 \quad \text{and} \quad \hat{\gamma}_N := \frac{1}{N} \sum_{i=1}^{N} \lambda_i \zeta_i g_i^2 - v(\theta), \]
with \( v(\theta) = R_{\mu_R}(2\theta) \), which, for \( |\theta| \) small enough, is well defined and such that \( \Re \zeta_i > 0 \), by virtue of Property 1.12 and Proposition 1.13.
Therefore, we find that
\[ I_N(\theta,E_N) = \prod_{i=1}^{N} \sqrt{\zeta_i} e^{N\theta\nu_{\gamma_N}} \int \exp \left\{ N\theta \frac{\gamma_N(v(\theta) - \hat{\gamma}_N)}{1 + \gamma_N} \right\} e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i, \] (39)
which is almost similar to what we got in (34) except that in the complex plane this is not so easy to “localize” the integral around 0 as we did before.
Our goal is now to show that
\[ \lim_{N \to \infty} \int \exp \left\{ N\theta \frac{\gamma_N(v(\theta) - \hat{\gamma}_N)}{1 + \gamma_N} \right\} e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i \]
exists and is not null.

Denote $\gamma_N = u_1^N + iu_2^N - 1$ and $\tilde{\gamma}_N = v_1^N + iv_2^N - v(\theta)$, and let

$$X^N + X_0 := (u_1^N, u_2^N, v_1^N, v_2^N) = \left( \int \zeta_1(\lambda)x^2 d\tilde{\mu}^N(x, \lambda), \ldots, \int \zeta_4(\lambda)x^2 d\tilde{\mu}^N(x, \lambda) \right)$$

with $d\tilde{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i, \theta_i}$,

$$\zeta_1(\lambda) = \Re((1 + 2v(\theta)\theta - 2\theta\lambda)^{-1}), \quad \zeta_2(\lambda) = \Im((1 + 2v(\theta)\theta - 2\theta\lambda)^{-1}),$$

$$\zeta_3(\lambda) = \Re(\lambda(1 + 2v(\theta)\theta - 2\theta\lambda)^{-1}), \quad \zeta_4(\lambda) = \Im(\lambda(1 + 2v(\theta)\theta - 2\theta\lambda)^{-1})$$

and $X_0 = (1, 0, \Re(v(\theta)), \Im(v(\theta)))$.

Then, we easily see as in [2] (cf Lemma 4.1 therein) that the law of $X^N$ under $\prod_{i=1}^{N} \sqrt{2\pi}^{-1} e^{-\frac{1}{2}g_i^2} dg_i$ satisfies a large deviation principle on $\mathbb{R}^4$ with rate function

$$\Lambda^*(X) = \sup_{Y \in \mathbb{R}^4} \left\{ \langle Y, X + X_0 \rangle + \frac{1}{2} \int \log \left( 1 - 2\zeta(\lambda), Y \right) d\mu_E(\lambda) \right\},$$

with $\langle \cdot, \cdot \rangle$ the usual scalar product on $\mathbb{R}^4$.

We denote

$$F(X^N) := \theta \frac{\gamma_N(v_{\gamma_N} - \tilde{\gamma}_N)}{1 + \gamma_N} = F_1(X^N) + iF_2(X^N)$$

with $F_1$ and $F_2$ respectively the real and imaginary part of $F$. With these notations, our problem boils down to show that $E[e^{NF(X^N)}]$ converges towards a non-zero limit. Following [1], we know that it is enough for us to check that

1. there is a vector $X^*$ so that $F(X^*) = 0$ and

$$\lim_{M \to \infty} \lim_{N \to \infty} \left( \frac{1}{N} \log E[e^{NF_1(X^N)}] - \frac{1}{N} \log E[1_{|X^N-X^*| \leq \frac{M}{\sqrt{N}}}] e^{NF_1(X^N)} \right) = 0.$$

To prove this, the main part of the work will be to show that

a) $X^*$ is the unique minimizer of $\Lambda^* - F_1$ (This indeed entails that the expectation can be localized in a small ball around $X^*$), and then we will check that

b) $X^*$ is a not degenerate minimizer i.e the Hessian of $\Lambda^* - F_1$ is positive definite at $X^*$ (As shown in appendix [7] this will allow us to take this small ball of radius of order $\sqrt{N^{-1}}$).

2. $X^*$ is also a critical point of $F_2$. This second point allows to see that there is no fast oscillations which reduces the first order of the integral.

Once these two points are checked, it is not hard to see that

$$E[e^{NF(X^N)}] = E[e^{ND^2F[X^*]|X^N-X^*,X^N-X}](1 + o(1)) = \det(D^2(\Lambda^* - F)[X^*])^{-\frac{1}{2}}(1 + o(1)).$$

This formula extends analytically the result of Theorem 4.4. In our case, $F$ depends linearly on $\theta$ and $X^*$ is the origin, from which it is easy to see that the convergence, if it holds for some
complex $\theta \neq 0$, will hold in a neighborhood of the origin since non degeneracy and uniqueness of the minimizer questions will continuously depend on $\theta$. Moreover, it is not hard to see that the convergence will actually hold uniformly in such a neighborhood of the origin (again because error terms will depend continuously on $\theta$).

**Proof of the first point:** To prove a), let us notice that by our choice of $v(\theta)$ (see Proposition 4.1.3), $\Lambda^*$ is minimum at the origin and that the differential of $F_1$ at the origin is null. Hence, the origin is a critical point of $F_1 - \Lambda^*$ (where this function is null) and we shall now prove that it is the unique one when $|\theta|$ is small enough.

For that, we adopt the strategy used in [2] and consider the joint deviations of the law of $(X^N, \hat{\mu}^N)$. A slight generalization of Lemma 4.1 therein shows that it satisfies a large deviations principle on $\mathbb{R}^4 \times \mathcal{P}(\mathbb{R})$ with good rate function

$$J(X, \mu) = I(\mu | \mu_E \otimes P) + \tau \left( X + X_0 - \int \zeta(\lambda) x^2 d\mu(\lambda, x) \right),$$

with $I(.)$ the usual relative entropy, $P$ a standard Gaussian measure and

$$\tau(X) = \sup_{\alpha \in D_0} \{ \langle \alpha, X \rangle \},$$

where $D_0 = \{ \alpha \in \mathbb{R}^4 : 1 - 2\langle \alpha, \zeta(\lambda) \rangle \geq 0 \mu_E \text{ a.s. } \}$. From that and the contraction principle we have that

$$I(X) := \Lambda^*(X) - F_1(X)$$

$$= \inf_{\mu \in \mathcal{P}(\mathbb{R})} \sup_{\alpha \in D_0} \left\{ I(\mu | \mu_E \otimes P) + \langle X + X_0 - \int \zeta(\lambda) x^2 d\mu(\lambda, x), \alpha \rangle - F_1(X) \right\}. \quad (40)$$

If we set

$$\mu^\alpha(dx, d\lambda) = \frac{1}{Z_\alpha} e^{-\frac{1}{2}(1 - 2\langle \zeta(\lambda), \alpha \rangle) x^2} dx d\mu_E(\lambda)$$

then

$$I(\mu | \mu^\alpha) = I(\mu | \mu_E \otimes P) - \langle \alpha, \int \zeta(\lambda) x^2 d\mu(\lambda, x) \rangle - \frac{1}{2} \int \log(1 - 2\langle \zeta(\lambda), \alpha \rangle) d\mu_E(\lambda).$$

Thus,

$$I(X) = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \sup_{\alpha} \left\{ I(\mu | \mu^\alpha) + \langle X + X_0, \alpha \rangle \right. \right.$$

$$\left. \left. \quad + \frac{1}{2} \int \log(1 - 2\langle \zeta(\lambda), \alpha \rangle) d\mu_E(\lambda) - F_1(X) \right\}.$$

Observe that the supremum in $\Lambda^*(X)$ is achieved at some $Y^X$ since $Y \mapsto - \int \log(1 - 2\langle \zeta(\lambda), Y \rangle) d\mu_E(\lambda)$ is lower semicontinuous and $\{ Y \in \mathbb{R}^4 : 1 - 2\langle \zeta(\lambda), Y \rangle \geq 0 \mu_E \text{ a.s. } \}$ is compact when $\mu_E$ is not a Dirac mass. Indeed, from the definition of $v(\theta)$, we find that $\mu_E(\zeta(\lambda) > 0) > 0$ as well as $\mu_E(\zeta_i(\lambda) < 0) > 0$ for $1 \leq i \leq 4$ from which the compactness follows. Moreover $Y^X$ satisfies

$$(X + X_0)_i = \int \frac{\zeta_i(\lambda)}{1 - 2 < \zeta(\lambda), Y^X>} d\mu_E(\lambda), \quad 1 \leq i \leq 4. \quad (41)$$
Consequently,

\[ \Lambda^*(X) - F_1(X) = I(X) \geq \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left\{ I(\mu | \mu^Y) + \Lambda^*(X) - F_1(X) \right\}. \]

Since \( I(\mu | \mu^Y) \geq 0 \), we deduce that the infimum in \( \mu \) is taken at \( \mu = \mu^Y \). We also check that \( \int \zeta(\lambda)x^2 d\mu^Y(\lambda, x) = X + X_0 \) due to (41). Hence, going back to (40), we find that \( I(X) = \mathcal{I}(\mu^Y) \) with

\[ \mathcal{I}(\mu) = I(\mu | \mu_E \otimes P) - F_1 \left( \int \zeta(\lambda)x^2 d\mu(x, \lambda) - X_0 \right). \]

We next show that \( \mathcal{I} \) has a unique minimizer for \( \theta \) small enough, and this minimizer satisfies \( \int \zeta(\lambda)x^2 d\mu(x, \lambda) = X_0 \). If the infimum is actually reached at a point \( \mu^* \) such that \( F_1 \) is regular enough at the vicinity of \( \int \zeta(\lambda)x^2 d\mu^*(x, \lambda) - X_0 \) then this saddle point satisfies the equation

\[ d\mu(x, \lambda) = \frac{1}{Z_\mu} e^{DF_1(\int \zeta(\lambda)x^2 d\mu(x, \lambda) - X_0)\zeta(\lambda)x^2 - \frac{1}{2} x^2 d\mu_E(\lambda)}. \quad (42) \]

Before going on the proof, let us justify that it is indeed the case. Note first that as \( \theta \) goes to zero, \( v(\theta) \) goes to \( m = \int \lambda d\mu_E(\lambda) \) and \( \Re[(1 + 2\theta - 2\theta)^{-1}] \) is bounded below by say \( 2^{-1} \). Consequently, \( \Re[\gamma_N + 1 \geq 2^{-1} \frac{1}{N} \sum_{i=1}^N \gamma_i^2] \). The rate function for the deviations of the latest is \( x - \log x - 1 \) which goes to infinity as \( x \) goes to zero as \( \log x \). Therefore, for \( \theta \) small enough,

\[ \Lambda^*(X) \geq \log(2X_1)^{-1} \]

Since \( F_1(X) \) is locally bounded, we deduce that the infimum has to be taken on \( X_1 \geq \epsilon \) for some fixed \( \epsilon > 0 \). In particular, \( F_1 \) is \( C^\infty \) on this set and equation (42) is well defined.

We now want to use this saddlepoint equation to show uniqueness. Suppose that there are two minimizers \( \mu \) and \( \nu \) satisfying (42). Then

\[ \Delta := \left| \int \zeta(\lambda)x^2 d\mu(x, \lambda) - \int \zeta(\lambda)x^2 d\nu(x, \lambda) \right| \leq 4C|\theta| \sup_i \int |\zeta_i(\lambda)|x^2(\mu(x, \lambda) + d\nu(x, \lambda)) \Delta, \]

as we have that \( y \to DF_1(y)[x] \) is Lipschitz, with Lipschitz norm of order \( C|\theta||x| \). We have now to show that for \( \theta \) small enough, these covariances are uniformly bounded. This can be done using some arguments very similar to the ones we gave above to justify that the critical points are such that \( X_1 \geq \epsilon \). We let it to the reader. For \( \theta \) small enough, we obtain a contraction so that \( \Delta = 0 \), which entails also \( \mu = \nu \). It is easy to check that \( \mu \) such that \( \int \zeta(\lambda)x^2 d\mu(x, \lambda) = X_0 \) is always a solution to (42), and hence the unique one when \( \theta \) is small enough. Observe now that by (42), this minimizer is of the form \( \mu^* = \mu^* = \mu^Y \), so that \( X^* = \int \zeta(\lambda)x^2 d\mu^*(x, \lambda) - X_0 = 0 \) minimizes indeed \( I \) and is actually its unique minimizer.

This concludes the proof of point a), which was the hard part of the work.

As we announced at the beginning and following [1], we now have to show b), that is to say to check that this minimizer is non-degenerate. To see that, remark that the second order derivative
of $F_1$ at the origin is simply

$$D^2F_1[0](U, V) = \Re(\theta(U(vU - V))) \leq C|\theta|(|U|^2 + |V|^2) = C|\theta| \left( \sum_{i=1}^{4} X_i^2 \right)$$

(43)

with $U = X_1 + iX_2, V = X_3 + iX_4$.

On the other side, observe that, as $d(\hat{\mu}_{E_N}^N, \mu_E) = o(\sqrt{N}^{-1})$, the covariance matrix of $\sqrt{N}(u_1^N, (\Im(\theta))^{-1}u_2^N, v_1^N, (\Im(\theta))^{-1}v_2^N)$ converges as $N$ goes to infinity towards a $4 \times 4$ matrix $K(\theta)$ which is positive definite. Now, remark that $v(\theta) = R_{\mu_E}(2\theta)$ implies that $\Re(\theta(\Im(\theta))^{-1}\Im(v(\theta)))$ converges as $|\theta|$ goes to zero, from which we argue that $K(0)$ is positive definite and bounded. By continuity in $\theta$ of $K(\theta)$ we deduce that $K(\theta) \leq C I$ for some $C > 0$ and $\theta$ small enough. and the limiting covariances $\sqrt{N}(u_1^N, u_2^N, v_1^N, v_2^N)$ (which are also given by the second order derivatives of $\Lambda^*$) converges towards a matrix $K'(\theta)$ such that

$$D^2\Lambda^*[0](X, X) = \langle X, K'(\theta)^{-1}X \rangle \geq C^{-1}(X_1^2 + X_2^2 + (\Im(\theta))^{-2}X_3^2 + (\Im(\theta))^{-2}X_4^2)$$

and hence, this together with (43) gives that, for $|\theta|$ small enough, $\frac{1}{T}D^2\Lambda^*[0] - D^2F_1[0] \geq 0$.

- **Proof of the second point:** To get Theorem 1.4, the last step is now to establish the second point, namely to check that 0 is also a critical point for $F_2$, which is straightforward computation since $F$ behaves in the neighborhood of the origin as a sum of monomials of degree 2 in $X$.

## 5 Full asymptotics in the real rank one case

The goal of this section is to establish the convergence and to find an explicit expression for $I_{\mu_E}(\theta) := \lim_{N \to \infty} \frac{1}{N} \log I_N(\theta, E_N)$ as far as $E_N$ satisfies Hypothesis 1.1 but $\theta$ do not necessarily satisfy the hypotheses of Theorem 1.2. This corresponds to show Theorem 1.6 (we again restrict to the case $\beta = 1$ to avoid heavy notations).

We recall that

$$I_N(\theta, E_N) = \mathbb{E} \left[ \exp \left( N\theta \sum_{i=1}^{N} \frac{\lambda_i g_i^2}{\sum_{i=1}^{N} g_i^2} \right) \right],$$

therefore one main step of the proof will be to get a large deviation principle for $z_N := \sum_{i=1}^{N} \lambda_i g_i^2 / \sum_{i=1}^{N} g_i^2$.

### 5.1 Large deviation bounds for $z_N$

We denote by $u_N := \frac{1}{N} \sum_{i=1}^{N} g_i^2$ and $v_N := \frac{1}{N} \sum_{i=1}^{N} \lambda_i g_i^2$. We intend to get the following result

**Proposition 5.1** If the empirical measure $\hat{\mu}_{E_N}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ satisfies Hypothesis 1.7, the law $\hat{\pi}_{\lambda}$ of $(u_N^N, v_N^N)$ under the standard $N$-dimensional Gaussian measure satisfies a large deviation principle in the scale $N$ with good rate function

$$T(\alpha) = \begin{cases} \frac{1}{T}h_{\alpha}(K_{\mu_E}(Q_{\mu_E}(\alpha))) & \text{if } \alpha \in [\alpha_{\min}, \alpha_{\max}], \\ \frac{1}{T}\max_\alpha & \text{if } \alpha \in [\alpha_{\max}, \lambda_{\max}], \\ \frac{1}{T}\min_\alpha & \text{if } \alpha \in [\lambda_{\min}, \alpha_{\min}], \\ +\infty & \text{if } \alpha \notin [\lambda_{\min}, \lambda_{\max}] \end{cases}$$

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Lemma 5.3
We denote by

\[ h_\alpha(\kappa) = \int \log \left( \frac{\kappa - \lambda}{\kappa - \alpha} \right) d\mu_\lambda(\lambda), \]

where we recall that \( H_{\text{max}} = \lim_{z \to \lambda_{\text{max}}} \int \frac{1}{z - \lambda} d\mu_\lambda(\lambda) \) and \( H_{\text{min}} = \lim_{z \to \lambda_{\text{min}}} \int \frac{1}{z - \lambda} d\mu_\lambda(\lambda) \); we denote also, for \( \kappa \in [\lambda_{\text{min}}, \lambda_{\text{max}}], \)

\[ h_\alpha^{\text{min}} = \lim_{\kappa \to \lambda_{\text{min}}} h_\alpha(\kappa) \quad \text{and} \quad h_\alpha^{\text{max}} = \lim_{\kappa \to \lambda_{\text{max}}} h_\alpha(\kappa). \]

Finally, the functions \( K_{\mu_{\lambda}} \) and \( Q_{\mu_{\lambda}} \) were defined respectively in Definition 1.10 and Property 1.11.

Note that \( H_{\text{max}} \) and \( H_{\text{min}} \) can be infinite (respectively \( +\infty \) and \( -\infty \)); in this case, we adopt the convention that \( \frac{1}{\infty} = 0 \).

The proof of Proposition 5.1 decomposes mainly in four steps, expressed in the following four lemmata:

**Lemma 5.2** For any \( \alpha \in [\lambda_{\text{min}}, \lambda_{\text{max}}], \)

\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \hat{\pi}_N \left( |v_N - \alpha u_N| < \sqrt{\epsilon} \right) \\
\leq \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \hat{\pi}_N \left( \left| \frac{v_N}{u_N} - \alpha \right| < \epsilon \right) \\
\leq \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \hat{\pi}_N \left( \left| \frac{v_N}{u_N} - \alpha \right| < \epsilon \right) \\
\leq \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \hat{\pi}_N \left( |v_N - \alpha u_N| < \sqrt{\epsilon} \right)
\]

**Lemma 5.3** We denote by \( v_N(\gamma) := N^{-1} \sum_{i=1}^{N} \gamma_i g_i^2 \) and we assume that the \( \gamma_i \)'s are such that

1. \( \gamma_{\text{max}} := \max_{1 \leq i \leq N} \gamma_i \) (resp. \( \gamma_{\text{min}} := \min_{1 \leq i \leq N} \gamma_i \)) converges towards \( \gamma_{\text{max}} < \infty \) (resp. \( \gamma_{\text{min}} > -\infty \)).

2. The empirical measure \( N^{-1} \sum_{i=1}^{N} \delta_{\gamma_i} \) converges to a compactly supported measure \( \mu \); we denote by \( \gamma^+ \) and \( \gamma^- \) the edges of the support of \( \mu \).

Then, the law of \( v_N(\gamma) \) satisfies a large deviation principle in the scale \( N \) with rate function

\[
J_{\mu, \gamma_{\text{min}}, \gamma_{\text{max}}}(x) = \begin{cases} 
L(x) & \text{if } x \in [x_1, x_2] \\
L(x_1) + \frac{1}{2\gamma_{\text{min}}}(x - x_1) & \text{if } x < x_1 \\
L(x_2) + \frac{1}{2\gamma_{\text{max}}}(x - x_2) & \text{if } x > x_2 
\end{cases}
\]

with

\[ L(x) = \sup \left\{ ux + \frac{1}{2} \int \log(1 - 2\lambda u)d\mu_\lambda(\lambda) \right\} \]

where the supremum is taken over \( u \) such that \( 1 - 2\lambda u > 0 \) for every \( \lambda \in [\gamma_{\text{min}}, \gamma_{\text{max}}], \)

\[ x_1 = \begin{cases} 
\gamma_{\text{min}}(\gamma_{\text{min}} H_{\text{min}}^\gamma - 1) & \text{if } \gamma_{\text{min}} < 0 \\
-\infty & \text{otherwise},
\end{cases} \]

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whereas
\[ x_2 = \begin{cases} 
\gamma_{\text{max}}(\gamma_{\text{max}}H_{\text{max}}^\gamma - 1), & \text{if } \gamma_{\text{max}} > 0 \\
\infty & \text{otherwise,}
\end{cases} \]

with the obvious notations \( H_{\text{max}}^\gamma = \lim_{z \to \gamma_{\text{max}}} H_\mu(z) \) and \( H_{\text{min}}^\gamma = \lim_{z \to \gamma_{\text{min}}} H_\mu(z) \).

**Lemma 5.4** If we denote \( \gamma_i^\alpha := \lambda_i - \alpha \), \( \mu^\alpha \) the weak limit of the empirical measure \( \frac{1}{N} \sum_{i=1}^{N} \delta_{\gamma_i^\alpha} \) (note that \( \mu^\alpha \) is just \( \tau_{-\alpha} \mu \), where \( \tau_{-\alpha} \) is the shift given by \( \tau_{-\alpha}(x) = x - \alpha \)), \( \gamma_{\text{max}}^\alpha \) and \( \gamma_{\text{min}}^\alpha \) are respectively the limits of \( \max \gamma_i^\alpha \) and \( \min \gamma_i^\alpha \), then
\[ J_{\mu^\alpha, \gamma_{\text{max}}^\alpha, \gamma_{\text{min}}^\alpha}(0) = T(\alpha), \]

with \( T \) as defined in Proposition 5.1.

**Lemma 5.5** \( T \) is a good rate function.

Then, Proposition 5.1 follows easily from these lemmata. Indeed, by definition of \( u_N \) and \( v_N \), we have that, for all \( \epsilon > 0 \) and \( N \) large enough \( z_N \in [\lambda_{\text{min}} - \epsilon, \lambda_{\text{max}} + \epsilon] \) so that,
\[ \limsup_{N \to \infty} \frac{1}{N} \log \hat{\pi}_N \left( z_N \in [\lambda_{\text{min}} - \epsilon, \lambda_{\text{max}} + \epsilon] \right) = -\infty. \]

Thus, from Theorem 4.1.11 in [9], it is enough to consider small balls ie to show that, for any \( \alpha \in [\lambda_{\text{min}}, \lambda_{\text{max}}] \),
\[ \limsup_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \hat{\pi}_N \left( |z_N - \alpha| \leq \epsilon \right) \leq -T(\alpha), \]

and
\[ \liminf_{\epsilon \to 0} \liminf_{N \to \infty} \frac{1}{N} \log \hat{\pi}_N \left( |z_N - \alpha| < \epsilon \right) \geq -T(\alpha). \]

Now, if \( \gamma_i^\alpha = \lambda_i - \alpha \) and the \( \lambda_i \)'s satisfy Hypothesis 1.1, \( v_N(\gamma^\alpha) := \frac{1}{N} \sum (\lambda_i - \alpha)g_i^2 = v_N - \alpha u_N \) satisfy the hypotheses (1) and (2) of Lemma 5.3. Therefore it satisfies a large deviation principle with rate function \( J_{\mu^\alpha, \gamma_{\text{max}}^\alpha, \gamma_{\text{min}}^\alpha} \). In particular this gives that in Lemma 5.2 the rightmost and leftmost members coincide, so that
\[ \lim \liminf \frac{1}{N} \log \hat{\pi}_N \left( |v_N - \alpha u_N| < \sqrt{\epsilon} \right) \]
\[ = \lim \limsup \frac{1}{N} \log \hat{\pi}_N \left( |v_N - \alpha u_N| < \sqrt{\epsilon} \right) = -J_{\mu^\alpha, \gamma_{\text{max}}^\alpha, \gamma_{\text{min}}^\alpha}(0) = -T(\alpha) \]

where the last equality comes from Lemma 5.4.

The study of the function \( T \), that will give Lemma 5.5 allows to conclude the proof.

### 5.2 Proofs of the lemmata

**Proof of Lemma 5.2**

For any \( \alpha \in \mathbb{R} \) and \( \epsilon > 0 \), we have
\[ \hat{\pi}_N \left( |v_N - \alpha u_N| < \sqrt{\epsilon} \right) - \hat{\pi}_N \left( |u_N| \geq \sqrt{\epsilon}^{-1} \right) \leq \hat{\pi}_N \left( \frac{|v_N - \alpha|}{|u_N|} < \epsilon \right) \]
\[ \leq \hat{\pi}_N \left( |v_N - \alpha u_N| < \sqrt{\epsilon} \right) + \hat{\pi}_N \left( |u_N| \geq \sqrt{\epsilon}^{-1} \right). \]
Now, by Chebychev’s inequality,
\[ \hat{\pi}_N \left( |u_N| \geq \sqrt{\epsilon^{-1}} \right) \leq e^{-\frac{1}{2} \hat{\pi}_N \left( e^{1/2 u_N} \right)} \leq 2^N e^{-\frac{1}{4} \sqrt{\epsilon}} \hat{\pi}_N \left( e^{1/4 u_N} \right) \leq 2 \sqrt{N} e^{-\frac{1}{4} \sqrt{\epsilon N}}, \]
so that
\[ \lim \limsup_{\epsilon \to 0} \frac{1}{N} \log \hat{\pi}_N \left( |u_N| \geq \sqrt{\epsilon^{-1}} \right) = -\infty, \]
what gives immediately Lemma 5.2.

Lemma 5.3 is proved in [3], Theorem 1; we omit it here.

Proof of Lemma 5.4:
Our goal is to identify \( T(\alpha) = J^{\alpha}_{\gamma_{\text{min}}, \gamma_{\text{max}}}(0) \). As we said above, it is enough to restrict to \( \alpha \in [\lambda_{\text{min}}, \lambda_{\text{max}}] \).
We have of course \( \gamma_{\text{min}} = \lambda_{\text{min}} - \alpha \) and \( \gamma_{\text{max}} = \lambda_{\text{max}} - \alpha \) and it is easy to check that
\[ H_{\text{max}}^\alpha := \lim_{z \downarrow \lambda_{\text{max}} - \alpha} \int \frac{1}{z - \lambda} d\mu(\lambda) = H_{\text{max}} \]
(and respectively for \( H_{\text{min}} \)).
Therefore, if we denote by \( x_1^\alpha \) and \( x_2^\alpha \) the bounds corresponding to \( \mu^\alpha \), we have that:
\[ x_1^\alpha = (\lambda_{\text{min}} - \alpha)((\lambda_{\text{min}} - \alpha)H_{\text{min}} - 1) \]
as the inequality \( \gamma_{\text{min}} = \lambda_{\text{min}} - \alpha < 0 \) is always satisfied for the \( \alpha \)'s we are interested in) and similarly \( x_2^\alpha = (\lambda_{\text{max}} - \alpha)((\lambda_{\text{max}} - \alpha)H_{\text{max}} - 1) \). We now have to determine the sign of \( x_1^\alpha \) and \( x_2^\alpha \) with respect to \( \alpha \). It is easy to check that
\begin{itemize}
  \item \( x_1^\alpha \leq 0 \) and \( x_2^\alpha \geq 0 \) if \( \alpha \in \left[ \alpha_{\text{min}} := \lambda_{\text{min}} - \frac{1}{H_{\text{min}}}, \alpha_{\text{max}} := \lambda_{\text{max}} - \frac{1}{H_{\text{max}}} \right] \)
  \item \( x_1^\alpha \leq 0 \) and \( x_2^\alpha \leq 0 \) if \( \alpha \in [\alpha_{\text{max}}, \lambda_{\text{max}}] \)
  \item \( x_1^\alpha \geq 0 \) and \( x_2^\alpha \geq 0 \) if \( \alpha \in [\alpha_{\text{min}}, \lambda_{\text{min}}] \)
\end{itemize}

Therefore, we deduce
\[ J^{\alpha}_{\gamma_{\text{min}}, \gamma_{\text{max}}}(0) = \begin{cases} 
  L^\alpha(0) & \text{if } \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}] \\
  L^\alpha(x_2^\alpha) - \frac{1}{2} H_{\text{max}}(\alpha_{\text{max}} - \alpha) & \text{if } \alpha_{\text{max}} \leq \alpha \leq \lambda_{\text{max}} \\
  L^\alpha(x_1^\alpha) - \frac{1}{2} H_{\text{min}}(\alpha_{\text{min}} - \alpha) & \text{if } \lambda_{\text{min}} \leq \alpha \leq \alpha_{\text{min}} 
\end{cases} \]
where we recall that
\[ L^\alpha(x) = \sup \left\{ ux + \frac{1}{2} \int \log(1 + 2\alpha u - 2\lambda u) d\mu(\lambda) \right\}, \]
with the supremum on \( u \) such that \( 1 + 2\alpha u - 2\lambda u > 0 \) for all \( \lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}] \).

We now get interested in the expression of \( L^\alpha \) on \( [x_1^\alpha, x_2^\alpha] \).
Obviously, the supremum is not reached at \( u = 0 \).
For \( u \neq 0 \), we denote \( \kappa := \alpha + \frac{1}{2u} \), then we have that \( 1 + 2\alpha u - 2\lambda u = \frac{\kappa - \lambda}{\kappa - \alpha} \). Moreover, if for all
\[ \lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}], \quad 1 + 2\alpha u - 2\lambda u > 0 \text{ then } (\kappa > \lambda_{\text{max}} \text{ and } u > 0) \text{ or } (\kappa < \lambda_{\text{min}} \text{ and } u < 0) \text{ and conversely, so that} \\
\]
\[ L^\alpha(x) = \frac{1}{2} \sup_{\kappa \in [\lambda_{\text{min}}, \lambda_{\text{max}}]} \left\{ \frac{x}{\kappa - \alpha} + h_\alpha(\kappa) \right\}, \]

with the notations of Proposition \[\text{Proposition} \]

- If \( \alpha \in I'' := [\alpha_{\text{min}}, \alpha_{\text{max}}] \),

\[ J_{\mu, \gamma_{\alpha}^{\min}, \gamma_{\alpha}^{\max}}(0) = L^\alpha(0) = \frac{1}{2} \sup_{\kappa \in [\lambda_{\text{min}}, \lambda_{\text{max}}]} h_\alpha(\kappa). \]

We now want to check that in this case, the supremum of \( h_\alpha \) is reached at \( \kappa_0 = K_{\mu_E}(Q_{\mu_E}(\alpha)) \).

The first point is to show that in this case, there is a unique \( \kappa_0 \) where \( h'_\alpha \) cancels. Indeed:

\[ h'_\alpha(\kappa_0) = 0 \iff H_{\mu_E}(\kappa_0) = \frac{1}{\kappa_0 - \alpha} \iff \kappa_0 = K_{\mu_E}(Q_{\mu_E}(\alpha)) \]

We now check that the maximum of \( h_\alpha \) is reached at \( \kappa_0 \);

- if \( \kappa_0 > \lambda_{\text{max}} \), \( h_\alpha \) is decreasing from 0 to \( h_{\alpha_{\text{min}}} \) on \( ]-\infty, \lambda_{\text{min}}[ \), it is increasing from \( h_{\alpha_{\text{max}}} \) to \( h_\alpha(\kappa_0) \) on \([\lambda_{\text{max}}, \kappa_0] \) and then decreasing from \( h_\alpha(\kappa_0) \) to 0 on \([\kappa_0, +\infty[ \),

- if \( \kappa_0 < \lambda_{\text{min}} \), \( h_\alpha \) is increasing from 0 to \( h_\alpha(\kappa_0) \) on \( ]-\infty, \kappa_0[ \) then decreasing from \( h_\alpha(\kappa_0) \) to \( h_{\alpha_{\text{min}}} \) on \([\kappa_0, \lambda_{\text{min}}[ \), it is increasing from \( h_{\alpha_{\text{max}}} \) to 0 on \([\lambda_{\text{max}}, +\infty[ \).

We treat in details the proof of the first point, when \( \kappa_0 > \lambda_{\text{max}} \), the other one being very similar. We recall from Property \[\text{Property} \] that \( I'' \) is the image of \( R_{\mu_E} \).

If \( \kappa_0 > \lambda_{\text{max}} \), \( h'_\alpha \) does not cancel on \( ]-\infty, \lambda_{\text{min}}[ \). It is negative since, when \( \alpha \in I'' \), \( \lambda_{\text{min}} - \frac{1}{H_{\mu_E}} \) and so \( \lim_{\kappa_0 \to \lambda_{\text{min}}} h'_\alpha(\kappa) < 0 \). On the other side, we want to find the sign of \( h'_\alpha \) on \([\lambda_{\text{max}}, +\infty[ \) knowing that it cancels at \( \kappa_0 \). As above, we show that \( \lim_{\kappa_0 \to \lambda_{\text{max}}} h'_\alpha(\kappa) > 0 \) and we deduce from that and the continuity of \( h'_\alpha \), that it is positive till \( \kappa_0 \). Furthermore, \( h_\alpha \) is also twice differentiable at \( \kappa_0 \) and

\[ h''_\alpha(\kappa_0) = -\int \frac{1}{(\kappa_0 - \lambda)^2} d\mu_E(\lambda) + \left( \frac{1}{\kappa_0 - \alpha} \right)^2 \\
< -\left( \int \frac{1}{\kappa_0 - \lambda} d\mu_E(\lambda) \right)^2 + (H_{\mu_E}(\kappa_0))^2 < 0, \]

where we used Cauchy-Schwarz inequality and the definition of \( \kappa_0 \). Therefore \( h'_\alpha \) is negative for \( \kappa > \kappa_0 \) and the fact that \( \lim_{\kappa_0 \to +\infty} h_\alpha(\kappa) = 0 \) concludes the proof of the first point.

Finally, we got that if \( \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}] \),

\[ J_{\mu, \gamma_{\alpha}^{\min}, \gamma_{\alpha}^{\max}}(0) = \frac{1}{2} h_\alpha(K_{\mu_E}(Q_{\mu_E}(\alpha))) \]

- If \( \alpha > \alpha_{\text{max}} \), our starting point is

\[ J_{\mu, \gamma_{\alpha}^{\min}, \gamma_{\alpha}^{\max}}(0) = \frac{1}{2} \sup_{\kappa \in [\lambda_{\text{min}}, \lambda_{\text{max}}]} \left\{ \alpha^2 \left( \frac{1}{\kappa - \alpha} - \frac{1}{\lambda_{\text{max}} - \alpha} \right) + h_\alpha(\kappa) \right\} \]
Using arguments as above, we show that the function
\[ g_\alpha(k) = \frac{x_2^\alpha}{k - \alpha} + h_\alpha(k) \]
on \([\lambda_{\text{min}}, \lambda_{\text{max}}]^c\) takes its supremum as \(k\) goes to \(\lambda_{\text{max}}\) by showing that its derivative is negative on \([\lambda_{\text{min}}, \lambda_{\text{max}}]^c\). Hence, \(J_{\mu} G_\alpha(0) = \frac{1}{2} \lambda_{\text{max}}^\alpha\).

- The case \(\alpha < \alpha_{\text{min}}\) is treated similarly, which concludes the proof of Lemma 5.6.

The proof of Lemma 5.5 is easy: \(T\) is in fact continuous on \([\lambda_{\text{min}}, \lambda_{\text{max}}]\). Indeed, it is continuous on each interval \([\lambda_{\text{min}}, \alpha_{\text{min}}[\[, \alpha_{\text{min}}, \alpha_{\text{max}}]\[, \lambda_{\text{max}}, \alpha_{\text{max}}]\[, \lambda_{\text{max}}, \lambda_{\text{max}}]\). So that it is enough to check that \(K_E G_\mu(\alpha) \xrightarrow{\alpha \to \alpha_{\text{max}}} \lambda_{\text{max}}\) (see Property 1.11) so that \(T(\alpha) \xrightarrow{\alpha \to \alpha_{\text{max}}} \frac{1}{2} \lambda_{\text{max}}^\alpha\); and similarly at \(\alpha_{\text{min}}\).

### 5.3 Proof of Theorem 1.6

By Varadhan’s lemma, we have

\[ \lim_{N \to \infty} \frac{1}{N} \log I_N(\theta, E_N) = \sup_{\alpha} \{ \theta \alpha - T(\alpha) \}. \]

Lemma 5.6 therefore gives the existence of the limit, the last step to conclude the proof of Theorem 1.6 is to check that it coincides with the function \(I_{\mu_E}\) introduced in Theorem 1.6.

We denote by
\[ G(\theta) := \sup_{\alpha \in I''} \left[ \theta \alpha - \frac{1}{2} h_\alpha(\lambda_{\text{max}}) \right], \]
\[ G_1(\theta) := \sup_{\alpha \in I_1} \left[ \theta \alpha - \frac{1}{2} h_{\lambda_{\text{max}}} \right], \quad G_2(\theta) := \sup_{\alpha \in I_2} \left[ \theta \alpha - \frac{1}{2} h_{\alpha_{\text{min}}} \right], \]
where we recall that \(I'' = [\alpha_{\text{min}}, \alpha_{\text{max}}]\) and we denote by \(I_1 = [\alpha_{\text{max}}, \lambda_{\text{max}}]\) and \(I_2 = [\lambda_{\text{min}}, \alpha_{\text{min}}]\).

The main part of the work for this last step will rely on proving

**Lemma 5.7** With the notations introduced above, we have\(^1\)
\[ G(\theta) = \begin{cases} \frac{1}{2} \int_0^\theta R_{\mu_E}(u)du, & \text{if } 2\theta \in I' \cup \{0\} = [H_{\text{min}}, H_{\text{max}}], \\ \theta \alpha_{\text{min}} - \frac{1}{2} \int \log(H_{\text{min}}(\lambda_{\text{min}} - \lambda))d\mu_E(\lambda)^\#, & \text{if } 2\theta \leq H_{\text{min}}, \\ \theta \alpha_{\text{max}} - \frac{1}{2} \int \log(H_{\text{max}}(\lambda_{\text{max}} - \lambda))d\mu_E(\lambda)^\ast, & \text{if } 2\theta \geq H_{\text{max}}, \\ \end{cases} \]
\[ G_1(\theta) = \begin{cases} \theta \left( \lambda_{\text{max}} - \frac{1}{2\theta} \right) - \frac{1}{2} \int \log(2\theta(\lambda_{\text{max}} - \lambda))d\mu_E(\lambda)^\ast, & \text{if } 2\theta > H_{\text{max}}, \\ \theta \alpha_{\text{max}} - \frac{1}{2} \int \log(H_{\text{max}}(\lambda_{\text{max}} - \lambda))d\mu_E(\lambda)^\ast, & \text{if } 2\theta < H_{\text{max}}, \\ \end{cases} \]

\(^1\# = -\infty\) if \(H_{\text{min}} = -\infty\) and otherwise these expressions are well defined in virtue of the fact that \(\int \frac{1}{\lambda} d\mu(\lambda) < +\infty \Rightarrow - \int \log \lambda d\mu(\lambda) < +\infty\),\n\* = -\infty\) if \(H_{\text{max}} = +\infty\) and otherwise these expressions are well defined for the same reason.

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\begin{align*}
G_2(\theta) &= \begin{cases} 
\theta \left( \lambda_{\min} - \frac{1}{2\theta} \right) - \frac{1}{2} \int \log(2\theta(\lambda_{\min} - \lambda))d\mu_E(\lambda) \chi, & \text{if } 2\theta < H_{\min} \\
\theta \left( \lambda_{\min} - \frac{1}{H_{\min}} \right) - \frac{1}{2} \int \log(H_{\min}(\lambda_{\min} - \lambda))d\mu_E(\lambda) \chi, & \text{if } 2\theta > H_{\min}.
\end{cases}
\end{align*}

**Proof of Lemma 5.7**:  
- We first study $G$.
- This is finding the supremum of $j_\theta(\alpha) := \theta \alpha - \frac{1}{2} h_\alpha (K_{\mu_E} (Q_{\mu_E}(\alpha)))$ on $I''$. From Definition 1.10 and Property 1.11, we have that $j_\theta$ is differentiable on $I''$ and an easy computation gives

$$j_\theta'(\alpha) = \frac{1}{2} (2\theta - Q_{\mu_E}(\alpha)).$$

- If $2\theta \in I'$, $j_\theta$ is maximized at $\alpha_0 = R_{\mu_E}(2\theta)$ and so, if $2\theta \in [H_{\min}, H_{\max}] \setminus \{0\}$,

$$G(\theta) = \frac{1}{2} \left( 2\theta R_{\mu_E}(2\theta) - \log(2\theta) - \int \log(K_{\mu_E}(2\theta) - \lambda)d\mu_E(\lambda) \right)$$

$$= \frac{1}{2} \int_{0}^{2\theta} R_{\mu_E}(u)du.$$ 

- If $H_{\min} > -\infty$ and $2\theta < H_{\min}$, the equation $j_\theta'(\alpha_0) = 0$ has no solution and actually $j_\theta'$ is negativeso that the supremum is reached at the left boundary $\alpha_{\min}$ of $I''$ and is equal to

$$\theta \alpha_{\min} - \frac{1}{2} \int \log(H_{\min}(\lambda_{\min} - \lambda))d\mu_E(\lambda).$$

- If $H_{\max} < +\infty$, a similar treatment in the case $2\theta > H_{\max}$ concludes the proof for $G$.

- The formulas for $G_1$ and $G_2$ are derived similarly.

By virtue of Lemmata 5.6 and 5.7, to finish the proof of Theorem 1.6, we have now

1. to compare $G|_{I'}$, $G_1|_{I'}$ and $G_2|_{I'}$ to get $I_{\mu,E}|_{I'}$.
   - Since $\lim_{\alpha \uparrow \lambda_{\max}} j_\theta(\alpha) = G_1(\theta)$ and $\lim_{\alpha \downarrow \lambda_{\min}} j_\theta(\alpha) = G_2(\theta)$ whereas $G(\theta) = \sup_{\alpha \in I'} [j_\theta(\alpha)]$, we get that $I_{\mu,E}|_{I'} = G|_{I'}$.

2. if $H_{\max} < +\infty$, to compare $G|_{\{2\theta > H_{\max}\}}$, $G_1|_{\{2\theta > H_{\max}\}}$ and $G_2|_{\{2\theta > H_{\min}\}}$ to get $I_{\mu,E}|_{\{2\theta > H_{\max}\}}$.
   - By studying the function $x \mapsto -\frac{\theta}{x} - \frac{1}{2} \log x$, which reaches its maximum at $\theta$, we can easily deduce that $G_1(\theta)$ is decreasing.
   - Moreover $G_1|_{\{2\theta > H_{\max}\}}$ and $G_2|_{\{2\theta > H_{\max}\}}$ are the limits of $j_\theta$ respectively at $\alpha_{\max}$ and $\alpha_{\min}$ and we know that in the case $2\theta > H_{\max}$, $j_\theta$ is increasing. This gives $G_2|_{\{2\theta > H_{\max}\}} < G_1|_{\{2\theta > H_{\max}\}}$.
   - In this case we conclude that the maximum is given by $G_1(\{2\theta > H_{\max}\})$.

3. Arguing similarly, we can see that in the case where $2\theta < H_{\min}$ the maximum is given by $G_2|_{\{2\theta < H_{\min}\}}$.

To conclude the proof of Theorem 1.6 we use the continuity of $I_{\mu,E}$ with respect to $\theta$ given by the first point of Lemma 2.1 to specify its value at $\lambda_{\min}$, $\alpha_{\min}$, $\alpha_{\max}$ and $\lambda_{\max}$.
6 Asymptotic independence and free convolution

In this section, we want to prove Theorem 1.5, that is to say concentration and decorrelation properties for the spherical integrals.

We recall first that as an immediate Corollary of Theorem 1.5, we get that

**Corollary 6.1**

For \( \theta \) sufficiently small

\[
R_{\mu_B \boxplus \mu_A}(\theta) = R_{\mu_A}(\theta) + R_{\mu_B}(\theta),
\]

where \( \boxplus \) denotes the free convolution of measures.

**Proof.** In fact, being given \( \mu_A, \mu_B \), we take \( \lambda_1(A) \) (resp. \( \lambda_1(B) \)) to be the lower edge of the support of \( \mu_A \) (resp. \( \mu_B \)) and then set for \( i \geq 2 \)

\[
\lambda_i(A) = \inf \left\{ x \geq \lambda_{i-1}(A) : \mu_A([\lambda_1(A), x]) \geq \frac{i}{N} \right\},
\]

\[
\lambda_i(B) = \inf \left\{ x \geq \lambda_{i-1}(A) : \mu_B([\lambda_1(B), x]) \geq \frac{i}{N} \right\}.
\]

It is easily seen that with this choice, \( A_N = \text{diag}(\lambda_i(A)) \) and \( B_N = \text{diag}(\lambda_i(B)) \) satisfy Hypothesis \[1.1\]. Since \( \mu_A \) and \( \mu_B \) are compactly supported, \( A_N \) and \( B_N \) have uniformly bounded spectral radius and so does \( A_N + UB_NU^* \). Hence, for \( \theta \) small enough, \( A_N, B_N \) and \( A_N + UB_NU^* \) satisfy the hypotheses of Theorem \[1.2\] (recall that \( A_N \) and \( UB_NU^* \) are asymptotically free (c.f Theorem 5.2 in \[7\]) so that \( \hat{\mu}_N^{A_N + UB_NU^*} \) converges towards \( \mu_B \boxplus \mu_A \)). Moreover, we can check that \( d(\hat{\mu}_N^{A_N + UB_NU^*}, \mu_A) \leq 2\|A_N\|_{\infty} N^{-1} \) and similarly for \( \mu_B \) so that \( d(\hat{\mu}_N^{A_N}, \mu_A) + d(\hat{\mu}_N^{B_N}, \mu_B) = o(\sqrt{N^{-1}}) \).

Thus, combining Theorem \[1.5\] and Theorem \[1.2\] imply

\[
\int_0^{2\theta} R_{\mu_B \boxplus \mu_A}(v)dv = \int_0^{2\theta} R_{\mu_A}(v)dv + \int_0^{2\theta} R_{\mu_B}(v)dv.
\]

Differentiating with respect to \( \theta \) gives Corollary \[6.1\].

Since the \( R \)-transform is analytic in a neighbourhood of the origin, this entails the famous additivity property of the \( R \)-transform. So, Theorem \[1.5\] provides a new proof of this property, independent of cumulant techniques.

As announced in the introduction, the first step will be to use a result of concentration for orthogonal matrices.

### 6.1 Concentration of measure for orthogonal matrices

In this section, we prove the first point of Theorem \[1.5\] that relies on the following lemma, which is a direct consequence of a theorem due to Gromov \[10\].

**Lemma 6.2** [Gromov, \[10\], p. 128] Let \( M_N^{(1)} \) denote the Haar measure on the special orthogonal group \( SO(N) \). There exists a positive constant \( c > 0 \), independent of \( N \), such that for any function
$F : SO(N) \to \mathbb{R}$ so that there is a real $\|F\|_L$ such that, for any $U, U' \in SO(N)$

$$|F(U) - F(U')| \leq \|F\|_L \left( \sum_{i,j=1}^{N} |u_{ij} - u'_{ij}|^2 \right)^{\frac{1}{2}},$$

then, for any $\varepsilon > 0$,

$$M_N^{(1)} \left( \left| F(U) - \int F(U) dM_N^{(1)}(U) \right| \geq \varepsilon \right) \leq e^{-cN\|F\|_L^2\varepsilon^2}.$$

**Proof of lemma 6.2**: In [10], the author prove such a lemma using the fact that the Ricci curvature of $SO(N)$ is of order\(^2 N\), and their result holds when $F$ is Lipschitz with respect to the standard bivariant metric which measures the length of the geodesic in $SO(N)$ between two elements $U, U' \in SO(N)$. This distance is of course greater than the length of the geodesic in the whole space of matrices, given by the Euclidean distance, so that Lemma 6.2 is a direct consequence of [10].

To prove Theorem 1.5.1, we now apply our result with $F(U_N) = \frac{1}{N} \log I_N(\theta, A_N + U_N BU_N^*)$. To get (8), we have to check that this $F$ satisfies the hypotheses of Lemma 6.2 i.e. that $F$ is Lipschitz.

We have, for any matrices $W, \tilde{W}$ in $M_N := \{ W \in \mathcal{M}_N(\mathbb{C}); WW^* \leq 1 \}$,

$$\left| \frac{1}{N} \log I_N(\theta, A_N + WB_N W^*) - \frac{1}{N} \log I_N(\theta, A_N + \tilde{W}B_N \tilde{W}^*) \right|$$

$$\leq 2\theta \|B\|_\infty \sup_{|v|=1} \langle v, |W - \tilde{W}|v \rangle \leq 2\theta \|B\|_\infty \left( \sum_{i,j=1}^{N} |w_{ij} - \tilde{w}_{ij}|^2 \right)^{\frac{1}{2}}.$$

Moreover, if $T$ is for example the transformation changing the first column vector $U_1$ of the matrix $U$ into $-U_1$, $O(N) = SO(N) \sqcup T(SO(N))$. Note that

$$F(TU) = \frac{1}{N} \log I_N(\theta, T^* A_N T + U_N B_N (U_N)^*).$$

Now, if we set $E_N = A_N + U_N BU_N^*$ and $E'_N = T^* A_N T + U_N B_N U_N^*$, we easily see that

$$d(||\mu_{E_N}||, ||\mu_{E'_N}||) \leq \frac{1}{N} \text{tr}|E'_N - E_N| \leq \frac{2\|A\|_\infty}{N}.$$

Hence, Lemma 2.13 implies that

$$\delta_N = \sup_{U \in SO(N)} |F(U) - F(TU)| \to 0 \text{ as } N \to \infty$$

Since

$$\int_{O(N)} F(U)dm_N^{(1)}(U) = \frac{1}{2} \int_{SO(N)} F(U)dM_N^{(1)}(U) + \frac{1}{2} \int_{SO(N)} F(TU)dM_N^{(1)}(U),$$

in [13] it is reported that the Ricci curvature is given by $N/4$ whereas J.C Sikorav and Y. Ollivier reported to us that it is in fact $(N - 2)/2$. 

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we deduce that
\[ \left| \int_{O(N)} F(U) dm_N(U) - \int_{SO(N)} F(U) dM_N(U) \right| \leq \delta_N. \]
Thus, Lemma 6.2 implies that for \( \epsilon > 0 \)
\[ M_N^{(1)} \left( \left| F(U) - \int_{O(N)} F(U) dm_N(U) \right| \geq \epsilon + \delta_N \right) \leq e^{-cN\|F\|_2^2\epsilon^2} \] (44)
and similarly for \( F(TU) \) so that
\[ m_N^{(1)} \left( \left| F(U) - \int_{O(N)} F(U) dm_N(U) \right| \geq \epsilon + \delta_N \right) \leq e^{-cN\|F\|_2^2\epsilon^2}, \]
what gives Theorem 1.5.1.

6.2 Exchanging integration with the logarithm

We are now seeking to establish the second point of Theorem 1.5. By Jensen's inequality,
\[ \mathbb{E}[\log I_N(\theta, A_N + V_N B_N(V_N)^*)] \leq \log \mathbb{E}[I_N(\theta, A_N + V_N B_N(V_N)^*)] \]
so that we only need here to prove the converse inequality.

The whole idea to get it is contained in the following

**Lemma 6.3** For any uniformly bounded sequence of matrices \((A_N, B_N)_{N \in \mathbb{N}}\) and \( \theta \) small enough, there exists a finite constant \( C(A, B, \theta) \) such that for \( N \) large enough
\[ \mathbb{E}[I_N(\theta, A_N + V_N B_N(V_N)^*)^2] \leq C(\theta, A, B) \]

Let us conclude the **proof of Theorem 1.5.2** before proving this lemma. Hereafter, \( \epsilon > 0 \) is fixed. We introduce the event
\[ \mathcal{A} = \left\{ I_N(\theta, A_N + V_N B_N(V_N)^*) \geq \frac{1}{2} \mathbb{E}[I_N(\theta, A_N + V_N B_N(V_N)^*)] \right\} \]
Following [23], we have, if \( I_N := I_N(\theta, A_N + V_N B_N(V_N)^*) \) that
\[ \mathbb{E}[I_N] = \mathbb{E}[I_N 1_{\mathcal{A}^c}] + \mathbb{E}[I_N 1_{\mathcal{A}}] \leq \frac{1}{2} \mathbb{E}[I_N] + \mathbb{E}[I_N^2] \mathbb{P}^\frac{1}{2}(\mathcal{A}) \]
so that
\[ \frac{1}{4C(A, B, \theta)} \leq \mathbb{P}(\mathcal{A}). \]

Furthermore, let
\[ t = \frac{1}{N} \log \mathbb{E} \left[ \frac{1}{2} I_N(\theta, A_N + V_N B_N(V_N)^*) \right] - \frac{1}{N} \mathbb{E}[\log I_N(\theta, A_N + V_N B_N(V_N)^*)] \]
We can assume that $t \geq \delta_N$ (since $\delta_N$ being given in (44)) since otherwise we are done. We then get by (44) that for any $t \geq \delta_N$ and $N$ large enough,

$$\mathbb{P}(A) \leq \frac{1}{N} \log I_N(\theta, A_N + UB_NU^*) - m_N^{(1)} \left( \frac{1}{N} \log I_N(\theta, A_N + UB_NU^*) \right) \geq t \right) \leq e^{-cN(t-\delta_N)^2}$$

with $c' = c(2|\theta||B||_{\infty})^{-2}$. As a consequence,

$$\frac{1}{4C(A, B, \theta)} \leq e^{-cN(t-\delta_N)^2}, \quad \text{so that} \quad t \leq \delta_N + \left( \frac{1}{c'N} \log(4C(A, B, \theta)) \right)^{\frac{1}{2}}.$$

Hence, since $\delta_N$ goes to zero with $N$,

$$\lim_{N \to \infty} \left( \frac{1}{N} \log \mathbb{E} \left[ \frac{1}{2} I_N(\theta, A_N + V_NB_N(V_N)^*) \right] - \frac{1}{N} \mathbb{E}[\log I_N(\theta, A_N + V_NB_N(V_N)^*)] \right) = 0$$

which completes the proof of Theorem 1.5.2.

We go back to the proof of Lemma 6.3. Observe first that

$$L_N(\theta, A, B) := \mathbb{E}[I_N(\theta, A_N + V_NB_N(V_N)^*)^2]$$

$$= \int e^{\theta N((UAU^*)_{11} + (UA^*U^*)_{11} + (VNUB(UV_N^*)_{11} + (\tilde{U}V_NB(\tilde{U}V_N)^*)_{11})} dm_N^{(1) \otimes 3}(U, \tilde{U}, V_N)$$

$$= \int e^{\theta N((UAU^*)_{11} + (UA^*U^*)_{11} + (VBU^*)_{11} + (\tilde{U}U^*VBU^*U^*)_{11})} dm_N^{(1) \otimes 3}(V, U, \tilde{U})$$

where we used that $m_N^{(1)}$ is invariant by the action of the orthogonal group. We shall now prove that $L_N(\theta, A, B)$ factorizes. The proof requires sharp estimates of spherical integrals. We already got the kind of estimates we need in section 3. The ideas here will be very similar although the calculations will be more involved.

To rewrite $L_N(\theta, A, B)$ in a more proper way, the key observation is that, if we consider the column vector $W := (V^*U\tilde{U}^*)$ then $\langle V_1, W \rangle = \langle U_1, \tilde{U}_1 \rangle$ so that we have the decomposition

$$W = \langle U_1, \tilde{U}_1 \rangle V_1 + (1 - ||U_1, \tilde{U}_1||^2)^{\frac{1}{2}} V_2$$

with $(V_1, V_2)$ orthogonal and distributed uniformly on the sphere. Therefore,

$$L_N(\theta, A, B) = \mathbb{E} \left[ \exp\{N\theta(F_1^N + F_2^N + F_3^N + F_4^N + F_5^N)\} \right]$$

with

$$F_1^N = \langle U, AU \rangle$$
$$F_2^N = \langle \tilde{U}, A\tilde{U} \rangle$$
$$F_3^N = (1 + \langle U, \tilde{U} \rangle^2) \langle V_1, BV_1 \rangle$$
$$F_4^N = 2(1 - ||U, \tilde{U}||^2)^{\frac{1}{2}} \langle U, \tilde{U} \rangle \langle V_1, BV_2 \rangle$$
$$F_5^N = (1 - \langle U, \tilde{U} \rangle^2) \langle V_2, BV_2 \rangle$$
where \( U, \tilde{U} \) are two independent vectors following the uniform law on the sphere of radius \( \sqrt{N} \) in \( \mathbb{R}^N \) and \( V_1, V_2 \) are the two first column vectors of a matrix \( V \) following \( m_N^{(1)} \), \( U, \tilde{U} \) and \( V \) being independent.

We now adopt the same strategy as in section 2 to show that the \( F_i \)'s will become asymptotically independent (or negligible). More precisely, we use again Fact 1.3 and recall that we can write
\[
U = \frac{g^{(1)}}{\|g^{(1)}\|}, \quad \tilde{U} = \frac{g^{(2)}}{\|g^{(2)}\|}, \quad V_1 = \frac{g^{(3)}}{\|g^{(3)}\|} \quad \text{and} \quad V_2 = \frac{G}{\|G\|} \quad \text{with} \quad G = g^{(4)} - \frac{\langle g^{(3)}, g^{(4)} \rangle}{\|g^{(4)}\|^2} g^{(3)}
\]
where \( g^{(1)}, g^{(2)}, g^{(3)} \) and \( g^{(4)} \) are i.i.d standard Gaussian vectors. We now set for \( i = 1, 2, 3, 4 \), with \( \lambda^{(i)}_j \) the eigenvalues of \( A \) for \( i = 1 \) or \( 2 \) and of \( B \) for \( i = 3 \) or \( 4 \), we have
\[
\tilde{U}_i^N = \frac{1}{N} \sum_{j=1}^{N} (g^{(i)}_j)^2 - 1, \quad \text{and} \quad \hat{V}_i^N = \frac{1}{N} \sum_{j=1}^{N} \lambda^{(i)}_j (g^{(i)}_j)^2 - v_i
\]
Moreover, we let for \( i = 1 \) or \( 2 \),
\[
\hat{W}_i^N = \frac{1}{N} \sum_{j=1}^{N} \lambda^{(i)}_j g^{(2i-1)}_j g^{(2i)}_j \quad \text{and} \quad \hat{Z}_i^N = \frac{1}{N} \sum_{j=1}^{N} g^{(2i-1)}_j g^{(2i)}_j.
\]
Under the Gaussian measure, all these quantities are going to zero almost surely and we can localize \( L_N \) as we made it in section 2 that is to say restrict the integration to the event \( A'_N := \{ \hat{U}_i^N, \hat{V}_i^N, \hat{W}_i^N, \hat{Z}_i^N \text{ are } o(N^{-\frac{1}{2}+\kappa}) \} \), for any \( \kappa > 0 \). We then express the \( F_i \)'s as function of these variables and on \( A'_N \) we expand them till \( o(N^{-1}) \). For example, on \( A'_N \),
\[
F_1 = \frac{\hat{V}_i^N + v_1}{\hat{U}_i^N + 1} = v_1 + (\hat{V}_i^N - v_1 \hat{U}_1^N) - \hat{U}_1^N (\hat{V}_i^N - v_1 \hat{U}_1^N) + o(N^{-1})
\]
and all the calculations go the same way so that we get that the full second order in \( \sum_i F_i \) is
\[
\Xi^N = -\sum_{i=1}^{4} \hat{U}_i^N (\hat{V}_i^N - v_1 \hat{U}_i^N) + 2(\hat{Z}_i^N - \hat{Z}_2^N)\hat{W}_2^N - 2v_2 \hat{Z}_2^N \hat{Z}_1^N + 2v_2 (\hat{Z}_2^N)^2
\]
Now, as before, we consider the shifted probability measure \( P_N \) (which contains all the first order term above) under which \( (\tilde{g}^{(i)})_{i=1,...,4} \) defined by \( \tilde{g}^{(i)}_j = \sqrt{1 + 2\theta v_i - 2\theta \lambda^{(i)}_j g^{(i)}_j} \) are i.i.d standard Gaussian vectors.

Under \( P_N \), the \( (\hat{U}_i^N, \hat{V}_i^N)_{1 \leq i \leq 4} \) are still independent with the same law than for the one dimensional case. Moreover, we see that for \( i = 1, 2, 3, 4 \), \( j = 1, 2 \),
\[
\lim_{N \to \infty} N\mathbb{E}[\hat{U}_i^N \hat{Z}_j^N] = 0, \quad \lim_{N \to \infty} N\mathbb{E}[\hat{U}_i^N \hat{W}_j^N] = 0.
\]
Similarly, \( (\hat{Z}_i^N, \hat{W}_i^N)_{i=1,2} \) are asymptotically uncorrelated. Moreover, with \( \mu_1 = \mu_A \) and \( \mu_2 = \mu_B \),
\[
\lim_{N \to \infty} N\mathbb{E}[\hat{W}_i^N \hat{Z}_i^N] = \int \frac{x}{(1 + 2\theta(v_i - x))^2} d\mu_i(x)
\]
\[
\lim_{N \to \infty} N\mathbb{E}[(\hat{W}_i^N)^2] = \int \frac{x^2}{(1 + 2\theta(v_i - x))^2} d\mu_i(x)
\]
\[
\lim_{N \to \infty} N\mathbb{E}[(\hat{Z}_i^N)^2] = \int \frac{1}{(1 + 2\theta(v_i - x))^2} d\mu_i(x).
\]
Thus, with \(G_i^N = \theta v_i - \frac{1}{2N} \sum_{j=1}^{N} \log(1 - 2\theta \lambda_j^{(i)} + 2\theta v_i)\) and if the Gaussian integral is well defined, we have

\[
L_N(\theta, A, B) = \frac{e^{2NG_1^N + 2NG_2^N}}{\det(K_A) \det(K_B)} \int \exp\left\{2\theta(\hat{z}_1 - \hat{z}_2)\hat{w}_2 - 2\nu_2 \theta \hat{z}_2 \hat{z}_1 + 2\nu_2 \theta \hat{z}_2^2 \right\} \prod_{i=1,2} dP_i(\hat{w}_i, \hat{z}_i)(1 + o(1))
\]

with \(P_i\) the law of two Gaussian variables with covariance matrix

\[
R_i = \frac{1}{2} \left( \begin{array}{cc} \int \frac{1}{(1+2\theta v_i - x)^2} d\mu_i(x) & \int \frac{x}{(1+2\theta v_i - x)^2} d\mu_i(x) \\ \int \frac{x}{(1+2\theta v_i - x)^2} d\mu_i(x) & \int \frac{1}{(1+2\theta v_i - x)^2} d\mu_i(x) \end{array} \right)
\]

and \(K_A\) and \(K_B\) as defined in (37) if we replace \(\mu_E\) therein respectively by \(\mu_A\) or \(\mu_B\).

We now integrate on the variables \((\hat{z}_2, \hat{w}_2)\) so that the Gaussian computation gives

\[
L_N(\theta, A, B) = \frac{e^{2NG_1^N + 2NG_2^N}}{\det(K_A) \det(K_B)} \int \exp\{\theta^2(\epsilon, K_B^{-1} \epsilon) \hat{z}_1^2 \} dP_i(\hat{z}_1, \hat{w}_1)(1 + o(1))
\]

with \(\epsilon = (-\nu_2, 1)\). To show that the remaining integral is finite it is enough to check that

\[-2\theta^2(\epsilon, K_B^{-1} \epsilon) + \text{var} \hat{z}_1 \geq 0,
\]

at least for \(\theta\) small enough. But we can check that \(\theta^2(\epsilon, K_B^{-1} \epsilon) \approx \theta^2 \sigma_2\), with \(\sigma_2 = \int x^2 d\mu_B(x)\) whereas the variance of \(\hat{z}_1\) is of order 1.

This finishes to prove that for sufficiently small \(\theta\)'s there exists a finite constant \(C(\theta, A, B)\) such that

\[
L_N(\theta, A, B) = \frac{e^{NG_1^N}}{\det(K_A)}(1 + o(1)) \quad \text{and} \quad L_N(\theta, B) = \frac{e^{NG_2^N}}{\det(K_B)}(1 + o(1))
\]

Since on the other hand we have seen in section 3 that

\[
I_N(\theta, A) = \frac{e^{NG_1^N}}{\det(K_A)}(1 + o(1)) \quad \text{and} \quad I_N(\theta, B) = \frac{e^{NG_2^N}}{\det(K_B)}(1 + o(1)),
\]

we have proved Lemma 6.3.

7 Appendix

In this Appendix, we clarify the derivation of the central limit theorem of Theorems 1.3 and 1.4 and Lemma 6.3. We follow the ideas of [4], where only sums of i.i.d entries \(N^{-1} \sum_{i=1}^{N} x_i\) were considered rather than ponderated sums \(N^{-1} \sum_{i=1}^{N} \lambda_i x_i\). We consider the case of Theorem 1.4 which is the most complicated;

\[
I_N(\theta, E_N) = \prod_{i=1}^{N} \sqrt{\xi_i} e^{N\theta v} \int \exp \left\{ N\theta \frac{g_i^N (\nu \gamma_N - \hat{\gamma}_N)}{1 + \gamma_N} \right\} e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i,
\]

(45)
where we recall that \( \zeta_i := (1 + 2\theta v - 2\theta \lambda_i) \), \( \gamma_N := \frac{1}{N} \sum_{i=1}^{N} \zeta_i g_i^2 - 1 \) and \( \hat{\gamma}_N = \frac{1}{N} \sum_{i=1}^{N} \lambda_i \zeta_i g_i^2 - v \). We denote
\[
J_N(\theta, E_N) = \sqrt{2\pi} N \int \exp \left\{ N \theta \frac{\gamma_N (v \gamma_N - \hat{\gamma}_N)}{1 + \gamma_N} \right\} e^{-\frac{1}{2} \sum_{i=1}^{N} g_i^2} \prod_{i=1}^{N} dg_i.
\]
The idea is the following:

- The first step is to derive a large deviation principle for \((\gamma_N, \hat{\gamma}_N)\) under the Gibbs measure

\[
\mu_N^0(dg) = J_N(\theta, E_N)^{-1} \prod_{i=1}^{N} P(dg_i).
\]

As we showed that the unique minimizer is zero, it entitles us to write
\[
J_N(\theta, E_N) = (1 + \delta(\epsilon, \epsilon', N)) J_N^{\epsilon,\epsilon'}(\theta, E_N)
\]
with
\[
J_N^{\epsilon,\epsilon'}(\theta, E_N) = \int_{|\gamma_N| \leq \epsilon, |\hat{\gamma}_N| \leq \epsilon'} \exp \left\{ N \theta \frac{\gamma_N (v \gamma_N - \hat{\gamma}_N)}{1 + \gamma_N} \right\} \prod_{i=1}^{N} P(dg_i)
\]
where \(\delta(\epsilon, \epsilon', N)\) goes to zero as \(N\) goes to infinity for any \(\epsilon, \epsilon' > 0\).

- Let us assume that we can take above \(\epsilon = M/\sqrt{N}, \epsilon' = M'/\sqrt{N}\) with \(\delta(M \sqrt{N}^{-1}, M' \sqrt{N}^{-1}, N)\) going to zero as \(N\) and then \(M, M'\) go to infinity. On the set \(|\gamma_N| \leq N^{-\frac{1}{2}} M, |\gamma_N| \leq N^{-\frac{1}{2}} M'\),

\[
f(\sqrt{N} \gamma_N, \sqrt{N} \hat{\gamma}_N) = N \theta \frac{\gamma_N (v \gamma_N - \hat{\gamma}_N)}{1 + \gamma_N} = N \theta \gamma_N (v \gamma_N - \hat{\gamma}_N) + O((M + M')^3 N^{-\frac{1}{2}})
\]

and \(f(\sqrt{N} \gamma_N, \sqrt{N} \hat{\gamma}_N)\) is uniformly bounded. Further, the law of \((N^{\frac{1}{2}} \gamma_N, N^{\frac{1}{2}} \hat{\gamma}_N)\) converges under \(P^{\otimes N}\) towards a two-dimensionnal complex Gaussian process with covariance matrix \(K'(\theta)\). Hence, we can apply dominated convergence theorem to see that

\[
\lim_{N \to \infty} \int_{|\gamma_N| \leq N^{-\frac{1}{2}} M, |\hat{\gamma}_N| \leq N^{-\frac{1}{2}} M'} \exp \left\{ N \theta \frac{\gamma_N (v \gamma_N - \hat{\gamma}_N)}{1 + \gamma_N} \right\} \prod_{i=1}^{N} P(dg_i)
\]

\[
= (2\pi)^{-\frac{1}{2}} \det(K'(\theta))^{-\frac{1}{2}} \int \int_{|x| \leq M, |y| \leq M'} e^{\theta x(x-y) - \frac{1}{2} <x,y, K'(\theta)^{-1}(x,y)> } dx dy.
\]

In the proof of Theorem 1.3, we established that the bilinear form \(x, y \to \theta x(x-y) - \frac{1}{2} <x,y, K'(\theta)^{-1}(x,y)>\) is strictly negative for \(|\theta|\) small enough, therefore we can now let \(M, M'\) going to infinity to obtain a limit.

- To see that we can take \(\epsilon = M/\sqrt{N}, \epsilon' = M'/\sqrt{N}\), we can simplify the argument by recalling that the spherical integral does not depend on \(\gamma_N\). Therefore,

\[
(1 - P^{\otimes N}(\epsilon \geq |\gamma_N| \geq M \sqrt{N}^{-1})) J_N^{\epsilon,\epsilon'}(\theta, E_N) = J_N^M + \frac{1}{2} \epsilon'(\theta, E_N)
\]

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But, \( \sqrt{N} \gamma_N = G_N^1 + iG_N^2 \) has, under \( P^{\otimes N} \), sub-Gaussian exponential moments since
\[
\mathbb{E}[e^{aG_N^i}] = \prod_{i=1}^{N} \left[ (1 - 2a\sqrt{N}^{-1} \zeta_j(\lambda_i))^{-\frac{1}{2}} e^{-a\sqrt{N}^{-1} \zeta_j(\lambda_i)} \right] \leq e^{ca^2}
\]
for some finite constant \( c \) which only depends on a uniform bound on the \( \zeta_j(\lambda_i) \), where we recall that \( \zeta_j(\lambda_i) = \Re \zeta_i \) if \( j = 1 \) and \( \zeta_j(\lambda_i) = \Im \zeta_i \) if \( j = 2 \). By Chebychev’s inequality, we therefore conclude that for \( M \) big enough,
\[
P^{\otimes N}(|\gamma_N| \geq M\sqrt{N}^{-1}) \leq e^{-\frac{c}{8}M^2}.
\]
Finally let us consider
\[
J_N^{M,M',\epsilon'} = \int_{|\gamma_N| \leq M\sqrt{N}^{-1}, M'\sqrt{N}^{-1} \leq |x,y| \leq \epsilon'} \exp \left\{ N\theta \frac{\gamma_N(x,y) - \hat{\gamma}_N}{1 + \gamma_N} \right\} \prod_{i=1}^{N} P(dg_i).
\]
Clearly, we find a finite constant \( C \) (depending on \( \theta \) and \( \epsilon' \)) such that
\[
|J_N^{M,M',\epsilon'}| \leq e^{CM^2} \int_{|\gamma_N| \leq M\sqrt{N}^{-1}, M'\sqrt{N}^{-1} \leq |x,y| \leq \epsilon'} \exp \left\{ CM|\sqrt{N}\hat{\gamma}_N| \right\} dP^{\otimes N}(g).
\]
Again, \( \sqrt{N}\hat{\gamma}_N \) has sub-Gaussian tail so that we find \( C' > 0 \) so that
\[
|J_N^{M,M',\epsilon'}| \leq e^{(C + \frac{C^2}{2})M^2 - C'(M')^2}.
\]
Now, by the previous point, we know that
\[
I(\theta, \mu_E) = \lim_{M,M' \to \infty} \lim_{N \to 0} \int_{[|\gamma_N| \leq N^{-\frac{1}{2}}, M|\hat{\gamma}_N| \leq N^{-\frac{1}{2}} M']} e^{\left\{ N\theta \frac{\gamma_N(x,y) - \hat{\gamma}_N}{1 + \gamma_N} \right\}} \prod_{i=1}^{N} P(dg_i)
\]
exists and moreover goes to one as \( \theta \) goes to zero. Hence, for \(|\theta|\) small enough, this term dominates \( J_N^{M,M',\epsilon'} \) for \( N, M, M' \) large enough ( \( M' \gg M \) ) and we conclude that
\[
\lim_{N \to \infty} J_N(\theta, E_N) = \lim_{M,M' \to \infty} \lim_{N \to 0} J_N^{M,\frac{1}{2},M',\frac{1}{2}} = I(\theta, \mu_E).
\]
Of course, this strategy only requires non-degeneracy of the minimum and \( I(\theta, \mu_E) \neq 0 \). In the setting of Theorem \[3\] this is verified on the whole interval \( 2\theta \in H_{\mu_E}(\left[\lambda_{\text{min}}, \lambda_{\text{max}}\right]) \). In Lemma \[6.3\] we can also apply it by noting that \( L_N(\theta, A, B) \) does not depend on \( ||g^{(1)}||, ||g^{(2)}||, ||g^{(3)}||, ||G|| \) to localize these quantities and proceed.

Acknowledgments: We are very grateful to O. Zeitouni for helpful discussion at the beginning of this work, which in particular allowed us to obtain the second order correction in the full high temperature region. We would like also to thank P. Śniady for many useful comments during this research. We thank Y. Ollivier for pointing out \[10\] and showing us how Lemma \[5.2\] could be deduced, which simplified a lot the argument. Finally, we are also very grateful to an anonymous referee whose careful reading and useful comments helped us to improve the coherence of the paper.
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