Generalized Vaidya Spacetime in Lovelock Gravity and Thermodynamics on Apparent Horizon

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We present a kind of generalized Vaidya solutions in a generic Lovelock gravity. This solution generalizes the simple case in Gauss-Bonnet gravity reported recently by some authors. We study the thermodynamics of apparent horizon in this generalized Vaidya spacetime. Treating those terms except for the Einstein tensor as an effective energy-momentum tensor in the gravitational field equations, and using the unified first law in Einstein gravity theory, we obtain an entropy expression for the apparent horizon. We also obtain an energy expression of this spacetime, which coincides with the generalized Misner-Sharp energy proposed by Maeda and Nozawa in Lovelock gravity.
I. INTRODUCTION

The Lovelock gravity [1] is a natural generalization of general relativity (Einstein gravity theory) in higher dimensional spacetimes. Its action is a sum of some dimensionally extended Euler densities, where there are no more than second order derivatives with respect to metric in equations of motion. Also it is known that the Lovelock gravity is free of ghost. Over the past years, due to the development of some theories in higher dimensional (>4) spacetimes such as string theories, brane world scenarios etc, the Lovelock gravity has attracted a lot of attention, and some new features and properties have been revealed.

The action of Lovelock gravity can be written as

\[ \mathcal{L} = \sum_{i=0}^{p} c_i \mathcal{L}_i, \]  

(1.1)

where \( p \leq \lfloor (n-1)/2 \rfloor \) (\( \lfloor N \rfloor \) denotes the integer part of the number \( N \)), \( c_i \) are arbitrary constants with dimension of \([\text{Length}]^{2i-2}\), \( n \) is the spacetime dimension and \( \mathcal{L}_i \) are the Euler densities

\[ \mathcal{L}_i = \frac{1}{2i} \sqrt{-g} \delta_{c_1 \cdots c_i | a_1 b_1 \cdots a_i | d_1} R_{c_1 d_1 | a_1 b_1} \cdots R_{c_i d_i | a_i b_i}, \]

(1.2)

where the generalized delta function is totally antisymmetric in both sets of indices. From the Lagrangian (1.1), one can get the equations of motion, \( \mathcal{G}_{ab} = 0 \), where

\[ \mathcal{G}_b^a = \sum_{i=0}^{p} \frac{1}{2i+1} c_i \delta_{b c_1 \cdots c_i | a_1 b_1 \cdots a_i | d_1} R_{a_1 b_1 \cdots a_i b_i} \cdots R_{c_i d_i | a_i b_i}. \]

(1.3)

After introducing matter into the theory, we can get the equations of motion including the energy-momentum tensor of matter

\[ \mathcal{G}_{ab} = 8\pi G T_{ab}. \]

(1.4)

From (1.3), it can be shown that the equations of motion do not have more than second order derivatives with respect to metric and that the Lovelock theory has the same degrees of freedom as the ordinary Einstein gravity theory. Just because of this fact, the Lovelock gravity is free of ghost when expanded on a flat spacetime, avoiding any problem with unitarity [2]. Note that, \( \mathcal{L}_0 \) denotes the unity, and \( \mathcal{L}_1 \) gives us the usual curvature scalar term, while \( \mathcal{L}_2 \) is just the Gauss-Bonnet term. Usually in order for the Einstein gravity to be recovered in the low energy limit, the constant \( c_0 \) should be identified as the cosmological
constant up to a constant and $c_1$ should be positive (for simplicity one may take $c_1 = 1$). 
For example, in $n = 4$, we have 
\[ \frac{1}{2} c_0 g_{ab} + c_1 (R_{ab} - \frac{1}{2} R g_{ab}) = 8\pi G T_{ab}. \] 

(1.5) 
This is just the Einstein equation with the cosmological constant $\Lambda = -c_0/2$ if we set $c_1$ to be one. 
When $n = 5$ and $p = 2$, after expanding the Kroneker-Delta, one gets the equations of motion for Gauss-Bonnet gravity in the usual manner.

In the literature on the Lovelock gravity, the most extensively studied theory is the so-called Einstein-Gauss-Bonnet (EGB) gravity. The EGB gravity is a special case of Lovelock's theory of gravitation, whose Lagrangian just contains the first three terms in (1.1). The Gauss-Bonnet term naturally appears in the low energy effective action of heterotic string theory. Spherically symmetric black hole solutions in the Gauss-Bonnet gravity have been found and discussed in [4, 5, 6], and topological nontrivial black holes have been studied in [7]. Rotating Gauss-Bonnet black holes have been discussed in [8]. Some other extensions such as including the perturbative AdS black hole solutions in gravity theories with second order curvature corrections could be seen in [9]. In addition, the references in [10] have investigated the holographic properties associated with the Gauss-Bonnet theory. And the papers in [11, 12, 13, 14] gave some exact solutions for Vaidya-like solution in the Einstein-Gauss-Bonnet gravity.

For a generic case, although the Lagrangian (1.1) looks complicated, some exact black hole solutions have been found and their associated thermodynamics was investigated in [15, 16, 17, 18, 19, 20]. In the so-called third-order Lovelock gravity, that is, containing the first four terms in (1.1), some exact solutions have been found in [21]. Furthermore, it is also known that those higher derivative terms in the Lagrangian (1.1) with positive coefficients arise as higher order corrections in superstring theories, and their cosmological implication has been studied [22].

In this paper, we are mainly interested in dynamical black hole solutions in Lovelock gravity by generalizing those discussions in [11, 12, 13, 14]. The organization of the paper is as follows. In Sec. II, we present a kind of generalized Vaidya spacetime in the general Lovelock gravity (1.1). In Sec. III, we study thermodynamics of apparent horizon of the generalized Vaidya spacetime, and give corresponding entropy expression associated with the apparent horizon. In addition, we propose a way to obtain the generalized Misner-Sharp
energy in the Lovelock gravity. Section IV. is devoted to conclusion and discussion.

II. GENERALIZED VAIDYA SOLUTION IN LOVELOCK THEORY

In the four-dimensional general relativity, the Vaidya spacetime is a typical dynamical one. In this spacetime there exists a pure radiation matter. The metric of the spacetime can be written as

\[ ds^2 = -f(r,v)dv^2 + 2dvdr + r^2d\Omega_2^2. \]  \hspace{1cm} (2.1)

where \( f = 1 - 2m(v)/r \), and \( d\Omega_2^2 \) is the line element of a two-dimensional unit sphere. In this spacetime, the apparent horizon is given by \( f = 0 \), or \( r = 2m(v) \). The energy-momentum tensor for the radiation matter in the spacetime is given by

\[ T_{ab} = \mu l_a l_b, \]

where \( l_a = (1,0,0,0) \) in coordinates \((v,r,\theta,\phi)\). The quantity \( \mu \) is the energy density of the radiation matter. Now, in an \( n \)-dimensional spacetime, assume a similar form of spherically symmetric metric

\[ ds^2 = -f(r,v)dv^2 + 2dvdr + r^2d\Omega_n^2. \]  \hspace{1cm} (2.2)

The energy-momentum tensor of radiation matter has a similar form as that in four dimensions. For this metric, it is not hard to calculate the nonvanishing components of Riemann tensor given by

\[ R^{vr}_{\phantom{vr}vr} = \frac{-f''}{2}, \quad R^{vi}_{\phantom{vi}vj} = \frac{-f'}{2r}\delta^i_j, \quad R^{ri}_{\phantom{ri}rj} = \frac{-f'}{2r}\delta^i_j, \]

\[ R^{ij}_{\phantom{ij}kl} = \frac{1 - f}{r^2}\delta^{ij}_{kl}. \]  \hspace{1cm} (2.3)

Here a prime/overdot denotes the derivative with respect to \( r/v \). Substituting these results into the equations of motion \((1.4)\), and using identities

\[ \delta^{a_1\cdots a_m}_{b_1\cdots b_m} \delta^{b_m}_{a_m} = [n - (m - 1)]\delta^{a_1\cdots a_{m-1}}_{b_1\cdots b_{m-1}} \]

and

\[ \delta^{a_1\cdots a_{m-1}a_m}_{b_1\cdots b_{m-1}b_m} \delta^{b_m}_{a_m} = 2[n - (m - 1)][n - (m - 2)]\delta^{a_1\cdots a_{m-2}}_{b_1\cdots b_{m-2}}, \]

we find the equations of motion

\[ G^v_v = \sum_i^p c_i \frac{(n-2)!}{(n-2i-1)!} \left[ i \left( \frac{-f'}{2r} \right) \left( \frac{1 - f}{r^2} \right)^{i-1} + \frac{1}{2} (n-2i-1) \left( \frac{1 - f}{r^2} \right)^i \right] = 0, \]  \hspace{1cm} (2.4)
\[ \mathcal{G}_r = \sum_i^p c_i \frac{(n-2)!}{(n-2i-1)!} \left[ i \left( -\frac{f}{2r} \right) \left( 1 - \frac{f}{r^2} \right)^{i-1} \right] = 8\pi G \mu , \quad (2.5) \]

and
\[ \mathcal{G}_k^j = 0 = \delta_k^j \sum_i^p c_i \frac{1}{2} \left\{ \frac{(n-3)!}{(n-2i-3)!} \left( 1 - \frac{f}{r^2} \right)^i \right. \\
+ 4i \frac{(n-3)!}{(n-2i-2)!} \left( -\frac{f'}{2r} \right) \left( 1 - \frac{f}{r^2} \right)^{i-1} \\
+ 2i \frac{(n-3)!}{(n-2i-1)!} \left( -\frac{f''}{2} \right) \left( 1 - \frac{f}{r^2} \right)^{i-1} \\
+ 4i(i-1) \frac{(n-3)!}{(n-2i-1)!} \left( -\frac{f'}{2r} \right)^2 \left( 1 - \frac{f}{r^2} \right)^{i-2} \right\} . \quad (2.6) \]

Other components of \( \mathcal{G}_a^a \) are given by \( \mathcal{G}_r^r = \mathcal{G}_v^v, \mathcal{G}_r^v = 0 \). The components \( \mathcal{G}_i^j \) are not independent, because they are linearly expressed in terms of \( \partial_v \mathcal{G}_v^v \) and \( \mathcal{G}_v^v \), as will be shown below. Defining a new function \( F(v, r) \)
\[ F(v, r) = \frac{1 - f(v, r)}{r^2} , \quad (2.7) \]
we can put the equations (2.4) and (2.6) into the forms
\[ \sum_i^p c_i \frac{(n-2)!}{2(n-2i-1)!} \frac{1}{r^{n-2}} \left[ r^{n-1} F^i \right]' = 0 , \quad (2.8) \]
and
\[ \mathcal{G}_k^j = \delta_k^j \sum_i^p c_i \frac{(n-3)!}{2(n-2i-1)!} \frac{1}{r^{n-3}} \left[ r^{n-1} F^i \right]'' = 0 . \quad (2.9) \]

From these two equations, it is easy to show \( \mathcal{G}_j^j = \delta_j^i \left[ r^i \partial_v \mathcal{G}_v^v / (n-2) + \mathcal{G}_v^v \right] \), so \( \mathcal{G}_j^j = 0 \) do not yield independent equations. Integrating the equation (2.8) leads to an order-\( p \) algebraic equation for \( F(v, r) \) or \( f(v, r) \)
\[ \sum_i^p c_i \frac{(n-2)!}{(n-2i-1)!} F^i = \frac{16\pi G m(v)}{\Omega_{n-2} r^{|n-1|}} , \quad (2.10) \]
where \( m(v) \) is an arbitrary function of \( v \), which appears as an integration constant. (Certainly, to ensure some energy condition, this mass function should be positive.) The coefficient \( 16\pi G / \Omega_{n-2} \) is chosen such that \( m \) can be interpreted as the mass of the solution when \( m \) is a constant. Using
\[ \frac{\partial}{\partial v} F^i = i F^{i-1} \hat{F} = i \left( -\frac{f}{r^2} \right) \left( 1 - \frac{f}{r^2} \right)^{i-1} , \quad (2.11) \]
in the equation (2.5), we have
\[
8\pi G \mu = \sum_i c_i \frac{(n-2)!r}{2(n-2i-1)!} \partial_v F^i .
\] (2.12)

Comparing equations (2.10) and (2.12), we obtain
\[
\mu = \frac{\dot{m}(v)}{\Omega_{n-2} r^{n-2}} .
\] (2.13)

Thus, we obtain a kind of radiating Vaidya spacetime in a generic Lovelock gravity by solving equation (2.10). When the dimension of spacetime is five, that is, \( n = 5 \), the solution reduces to the one reported by some authors in references [11, 12, 13, 14], while in \( n = 4 \), the solution is just the familiar Vaidya solution of general relativity.

Now we further generalize the Vaidya spacetime in Lovelock gravity to more general case. Note that for the metric (2.2) we have \( \mathcal{G}_r^r = \mathcal{G}_v^v \), so the energy-momentum tensor of matter has to satisfy \( T_r^r = T_v^v \). Certainly, the matter of pure radiation discussed above satisfies the constraint. In fact, they are \( T_r^r = T_v^v = 0 \). If we further assume the spherical part of the energy-momentum tensor has the form \( T_i^i = \sigma T_r^r = \sigma T_v^v \) (where \( \sigma \) is a constant, and the repeat index \( i \) does not sum), then from the equation \( \nabla_a T^a_b = 0 \) or the explicit expressions of \( \mathcal{G}_b^a \) in equations (2.4), (2.5) and (2.6), we can find
\[
\partial_v T^v_v + \partial_r T^r_v + \frac{n-2}{r} T^r_v = 0 ,
\] (2.14)

\[
\partial_r T^r_v + \frac{(n-2)(1-\sigma)}{r} T^r_v = 0 .
\] (2.15)

As a result, for the pure radiation matter with \( T_r^r = T_v^v = 0 \), one can find that \( T^r_v \) has to be proportional to \( 1/r^{n-2} \). This is consistent with the equation (2.13). Next, in the case with \( T_r^r = T_v^v \neq 0 \), the equation (2.15) tells us that \( T_r^r \) and \( T_v^v \) have the form
\[
T_r^r = T_v^v = C(v) r^{-(n-2)(1-\sigma)} ,
\] (2.16)

where \( C(v) \) is a function of \( v \). The off-diagonal part of the energy-momentum tensor \( T_b^a \), i.e., the component \( T_v^r \) has to satisfy the equation (2.14). Now the equations of motion \( \mathcal{G}_v^v = \mathcal{G}_v^r = 8\pi G C(v) r^{-(n-2)(1-\sigma)} \) modify the equation (2.8) to
\[
\sum_i c_i \frac{(n-2)!}{2(n-2i-1)!} [r^{n-1} F^i]^{'} = 8\pi G C(v) r^{(n-2)\sigma} .
\] (2.17)
Integrating this equation, we have
\[ \sum_i c_i \frac{(n-2)!}{(n-2i-1)!} F^i = 16\pi G \left( \frac{m(v)}{\Omega_{n-2} r^{n-1}} + \frac{C(v)\Theta(r)}{r^{n-1}} \right), \] (2.18)

where \( m(v) \) is an arbitrary function, appearing as an integration constant again. Here,
\[ \Theta(r) = \int dr r^{(n-2)\sigma}, \]
for \( \sigma = -1/(n-2) \) and
\[ \Theta(r) = \frac{r^{(n-2)\sigma+1}}{(n-2)\sigma + 1}, \] (2.20)
otherwise. Certainly, to ensure some energy condition for the energy-momentum tensor, the parameter \( \sigma \) and function \( m(v) \) and \( C(v) \) should satisfy certain consistency relations. For example, \( C(v) \leq 0 \) and \(-1 \leq \sigma \leq 0\). These relations have been discussed in [13]. One can see from the relation (2.11) that \( T^v_v \) is given by the partial derivative of the equation (2.18) with respect to \( v \). Thus we have
\[ T^v_v = \tilde{\mu} = \frac{\dot{m}(v)}{\Omega_{n-2} r^{n-2}} + \frac{\dot{C}(v)\Theta(r)}{r^{n-2}}. \] (2.21)
This is consistent with the equation (2.14). So the energy-momentum tensor of matter in this case can be written as
\[ T_{ab} = \tilde{\mu} l_a l_b - P(l_a n_b + n_a l_b) + \sigma P q_{ab}, \] (2.22)
where \( n_a \) is a null vector which satisfies \( l_a n^a = -1 \). In coordinates \( \{v, r, \cdots\} \), we have \( l_a = (1, 0, 0, \cdots) \), and \( n_a = (f/2, -1, 0, \cdots) \). The tensor \( q_{ab} \) is a projection operator which is given by \( q_{ab} = g_{ab} + l_a n_b + l_b n_a \). So the metric (2.2) can be put into the form \( g_{ab} = h_{ab} + q_{ab} \), where
\[ h_{ab} = -l_a n_b - l_b n_a \] (2.23)
is the metric of the two-dimensional spacetime transverse to the \( (n-2) \)-dimensional sphere. Certainly, in coordinates \( \{v, r, \cdots\} \), the line element of \( h_{ab} \) can be expressed as \(-f(v, r)dv^2 + 2dv dr\). The quantity \( P \) is the radial pressure with the form \( P = -C(v) r^{-(n-2)(1-\sigma)}. \)

Generally, it is not easy to solve equation (2.18) when \( p \) is greater than one. However, for some special cases, one can explicitly get analytic solutions. Now, we give some simple examples.
(1). Gauss-Bonnet theory: The simple radiating Vaidya solution in the Gauss-Bonnet gravity without a cosmological constant has been found in the references \[11, 12, 13, 14\]. Here we can directly solve the second order algebraic equation (2.18), and give the solution

\[ ds^2_{GB} = -f(v, r) dv^2 + 2 dvdr + r^2 d\Omega_{n-2}^2, \]  

(2.24)

where \( f(v, r) \) satisfies the equation (2.18), which in this case becomes

\[ F + \alpha(n-3)(n-4) F^2 = \frac{16\pi G}{n-2} \left( \frac{m(v)}{\Omega_{n-2} r^{n-1}} + \frac{\mathcal{C}(v) \Theta(r)}{r^{n-1}} \right). \]  

(2.25)

Here, we have set \( c_0 = 0, c_1 = 1 \) and \( c_2 = \alpha \). This parameter \( \alpha \), called the Gauss-Bonnet coefficient, has the dimension of length squared. Since it is an algebraic equation of order two, the solutions have two branches in general. Only one branch has the general relativity limit and is given by

\[ f(v, r) = 1 + \frac{r^2}{2(n-3)(n-4)\alpha} \left[ 1 - \sqrt{1 + \frac{64\pi(n-3)(n-4)\alpha}{n-2} \left( \frac{m(v)}{\Omega_{n-2} r^{n-1}} + \frac{\mathcal{C}(v) \Theta(r)}{r^{n-1}} \right)} \right], \]  

(2.26)

where, for simplicity, we have set \( G = 1 \). From \( f(v, r) = 0 \), we can obtain the trapping horizon, which is also its apparent horizon in this case. The radius of apparent horizon satisfies

\[ r_A^{n-3} + \alpha(n-3)(n-4) r_A^{n-5} = \frac{16\pi}{n-2} \left( \frac{m(v)}{\Omega_{n-2}} + \frac{\mathcal{C}(v) \Theta(r_A)}{r^{n-1}} \right). \]  

(2.27)

Thus, in general, the radius of the apparent horizon is a function of \( v \). This solution is the same as the one in \[13\] if a cosmological constant is included.

(2). Dimensionally continued Lovelock gravity: An interesting case is to choose some special values for the coefficients \( c_i, i = 0, \cdots, p \). In \[15, 16, 17, 18\] a set of special coefficients has been chosen so that the equation (2.18) has a simple expression. In odd dimensions the action is the Chern-Simons form for the AdS group, while in even dimensions it is called Born-Infeld theories constructed with the Lorentz part of the AdS curvature tensor. In the odd-dimensional case, say \( n = 2p + 1 \), i.e., Chern-Simons theory, we can choose

\[ c_i = (n - 2i - 1)! \binom{p}{i} \ell^{-n+2i}, \]  

(2.28)

where the parameter \( \ell \) is a length scale. Note that \( c_i \) differ from the coefficients \( \alpha_i \) in the references \[15, 16, 17, 18\] only by a factor of \( (n - 2i)! \). At the same time we choose \( 1/16\pi G \) as

\[ \frac{\Omega_{n-2}}{16\pi G} = \frac{\ell}{(n-2)!}. \]  

(2.29)
Then the equation (2.18) gives the solution

\[ f(v, r) = 1 - \left[ m(v) + \Omega_{n-2}C(v)\Theta(r) \right]^{\frac{1}{p}} + \frac{r^2}{\ell^2}, \]  
(2.30)

For the even-dimensional case, say \( n = 2p + 2 \), we set \( c_i, i = 1, \ldots, p \), to be

\[ c_i = (n - 2i - 1)! \binom{p}{i} \ell^{-n+2i}, \]  
(2.31)

and the gravity coupling constant

\[ \frac{\Omega_{n-2}}{16\pi G} = \frac{\ell^2}{(n-2)!}, \]  
(2.32)

Then the equation (2.18) gives the solution

\[ f(v, r) = 1 - \left[ \frac{m(v) + \Omega_{n-2}C(v)\Theta(r)}{r} \right]^{\frac{1}{p}} + \frac{r^2}{\ell^2}. \]  
(2.33)

The solutions (2.30) and (2.33) reduce to the static cases if \( m(v) \) and \( C(v) \) do not depend on coordinate \( v \). In these cases, the solutions have a unique AdS vacuum [15, 16, 17, 18]. These solutions (2.30) and (2.33) with vanishing \( C(v) \) have already been found by M. Nozawa and H. Maeda in [45].

(3). Pure Lovelock gravity: In this theory [23, 24, 25], only two of coefficients \( c_i \) are non-vanishing: one is \( c_0 \) and the other is \( c_k \) with \( 1 \leq k \leq p \). We can normalize them as \( c_0/(n-1)(n-2) = -1/\ell^2 \) and \( c_k(n-1)!/(n-2k-1)! = \alpha^{2k-2} \), where \( \ell \) and \( \alpha \) are two length scales. Then the equation (2.18) becomes

\[ \alpha^{2k-2}F^k = \frac{1}{\ell^2} + \frac{16\pi G}{n-2} \left( \frac{m(v)}{\Omega_{n-2}r^{n-1}} + \frac{C(v)\Theta(r)}{r^{n-1}} \right). \]  
(2.34)

Thus, in the case of even dimensions, if the right hand of (2.34) is non-negative, we have

\[ f(v, r) = 1 \pm \frac{r^2}{\alpha^2} \left[ \frac{\alpha^2}{\ell^2} + \frac{16\pi G\alpha^2}{n-2} \left( \frac{m(v)}{\Omega_{n-2}r^{n-1}} + \frac{C(v)\Theta(r)}{r^{n-1}} \right) \right]^{\frac{1}{p}}, \]  
(2.35)

while, in the other case of odd dimensions,

\[ f(v, r) = 1 + \frac{r^2}{\alpha^2} \left[ \frac{\alpha^2}{\ell^2} + \frac{16\pi G\alpha^2}{n-2} \left( \frac{m(v)}{\Omega_{n-2}r^{n-1}} + \frac{C(v)\Theta(r)}{r^{n-1}} \right) \right]^{\frac{1}{p}}. \]  
(2.36)

Thus, for \( k = 1 \), one gets a generalized Vaidya black hole with a cosmological constant. When \( m(v) \) and \( C(v) \) are two constants, these solutions reduce to the static solutions.

For dynamical black holes, it is difficult to study their thermodynamical properties. In four-dimensional Einstein gravity, Hayward has proposed a method to discuss this issue [30, 31, 32, 33]. In the next section, we will discuss the thermodynamics of the apparent horizon of the new solutions in this section.
III. ENTROPY AND ENERGY OF THE APPARENT HORIZON

In the early 1970s, it was found that four laws of black hole mechanics in general relativity are very analogous to four laws of thermodynamics. Due to Hawking’s discovery that black hole radiates thermal radiation, it turns out that it is not just an analog, but they are indeed identical with each other. Then thermodynamics of black hole has been established soundly (for a review, see [26]). In black hole thermodynamics, the temperature and entropy of a black hole are given by $T_{EH} = \kappa/2\pi$ and $S_{EH} = A/4$, where $\kappa$ and $A$ are the surface gravity and area of event horizon of the black hole, respectively. In higher derivative theory of gravity, such as Gauss-Bonnet gravity, it turns out that the area formula for black hole entropy no longer holds. In fact, black hole entropy gets corrections from these higher derivative terms. For a diffeomorphism invariant theory, Wald [27] showed that the entropy of black hole is a kind of Noether charge, and obtained a formula for black hole entropy, now called Wald formula, associated with the event horizon of black hole. For more details, see the review paper by Wald [26].

In the literature, most of discussions on thermodynamics of black holes have been focused on stationary black holes. Thermodynamics of a black hole is associated with the event horizon of the black hole, which is the boundary of the past of future infinity. Therefore the event horizon is a null hypersurface and depends on some global structures of the spacetime, so it is difficult to study the event horizon of a dynamical (time-dependent) spacetime. In general relativity, there exists another kind of horizon, named apparent horizon, which is not heavily dependent of global structure of spacetime. In the standard definition of apparent horizon, one first slices the spacetime, and then finds the boundary of trapped region in each slice. This boundary (two-dimensional surface) is called the apparent horizon [28]. Also the three-dimensional hypersurface of the union of all these two-dimensional surfaces is usually called the apparent horizon. Over the past years, several generalizations of horizon have been proposed, such as the trapping horizon by Hayward, the isolated horizon and the dynamical horizon by Ashtekar et al. The relations and differences among those horizons have been discussed in a recent review [29]. For a dynamical black hole, the outer trapping horizon and the dynamical horizon are not null hypersurfaces but spacelike hypersurfaces of the spacetime [29]. Therefore, some well-known results associated with the event horizon, such as Wald entropy formula, may not be applicable to those horizons.
In this section, we will discuss the entropy and the energy associated with the apparent horizon of the generalized Vaidya solution in Lovelock gravity theory \((2.18)\). Before going on, let us first give a brief review on the work of Hayward \([30, 31, 32, 33]\). Although his work focuses on four-dimensional Einstein gravity, it can be straightforwardly generalized to higher dimensional Einstein gravity \([34, 35]\). These discussions reveal the deep relation between equations of motion and thermodynamics of the spacetimes. For relevant discussions, see also \([36, 37, 38, 39, 40, 41]\). For an \(n\)-dimensional spherically symmetric spacetime \(((M, g_{ab})\), we can write the metric in the double null form

\[
ds^2 = h_{ab}dx^a dx^b + r^2(x)d\Omega_{n-2}^2,
\]

where \(\{x^a\}\) are coordinates of the two-dimensional spacetime \((M, h_{ab})\) which is transverse to the \((n-2)\)-dimensional sphere, \(r(x)\) is the radius of the sphere. Similarly, one can also divide the energy-momentum tensor of matter into two parts. One part denoted by \(T_{ab}\) (do not be confused with the total energy-momentum tensor) corresponds to the two-dimensional spacetime \(h_{ab}\). From this energy-momentum tensor, one can define two important physical quantities: the work density \(W = -1/2h_{ab}T_{ab}\), which corresponds to the work term in the first law, and the energy supply \(\Psi_a = T^b_a \partial_b r + W \partial_a r\). By using these two quantities and the Misner-Sharp energy inside the sphere with radius \(r\),

\[
E = \frac{(n-2)}{16\pi} \Omega_{n-2} r^{n-3} \left(1 - h_{ab} \partial_a r \partial_b r\right),
\]

one can put some components of the Einstein equations into the so-called unified first law

\[
dE = A \Psi + W dV,
\]

where \(A = \Omega_{n-2} r^{n-2}\) and \(V = \Omega_{n-2} r^{n-1}/(n-1)\) are the area and the volume of an \((n-2)\)-sphere with radius \(r\). For a vector \(\xi\) tangent to the trapping horizon of the spacetime, Hayward showed that on trapping horizon, one has \([32]\)

\[
A\Psi_a \xi^a = \frac{\kappa}{8\pi} \mathcal{L}_\xi A = \frac{\kappa}{2\pi} \mathcal{L}_\xi S = T \delta S,
\]

where \(S = A/4\), \(T = \kappa/2\pi\), and \(\kappa\) is the surface gravity defined by \(\kappa = D_a D^a r/2\). Here, \(D_a\) is the covariant derivative associated with metric \(h_{ab}\). The \(\delta\) operator in the right hand side of the equation \((3.4)\) should be understood as follows: take a Lie derivative with respect to \(\xi\), and then evaluate it on the apparent horizon. Note that the left hand side of the
equation (3.4) represents the amount of energy crossing the trapping horizon. Therefore the equation (3.4) may be understood as the Clausius relation for mechanics of dynamic black holes: \( \delta Q = T \delta S \) with \( \delta Q = A \Psi_a \xi^a \). By projecting the unified first law onto the trapping horizon, the first law of thermodynamics is given by

\[
\delta E = T \delta S + W \delta V .
\]

(3.5)

In the generalized Vaidya spacetime found in the previous section, the apparent horizon is just a kind of trapping horizon, so we do not emphasize the difference between these two concepts in the present paper. For the generalized Vaidya spacetime (2.2), the apparent horizon is given by \( f(v, r) = 0 \). From the definition of surface gravity, it is easy to find \( \kappa = f'(v, r_A)/2 \). We now discuss thermodynamics on apparent horizon/trapping horizon for the dynamical solutions in Lovelock gravity theory. Following [35], we may move all terms of \( G_{ab} \) except for the Einstein tensor into the right hand side of the field equations \( G_{ab} = -8\pi T_{ab} \) and rewrite them in the standard form for Einstein gravity

\[
G_{ab} = 8\pi T_{ab} = 8\pi \left( T_{ab}^{(m)} + T_{ab}^{(e)} \right).
\]

(3.6)

Here, \( T_{ab}^{(m)} \) is the energy-momentum tensor of matter (2.22). The effective energy-momentum tensor \( T_{ab}^{(e)} \) has the form

\[
8\pi T_{ab}^{(e)} = H_{ab} = G_{ab} - G_{ab},
\]

(3.7)

where \( G_{ab} \) is the Einstein tensor of the spacetime. In this way, we can go along the line of Einstein gravity, although we are discussing a gravity theory beyond the Einstein gravity. Thus, the work term and the energy supply defined on the two-dimensional spacetime \((M^2, h_{ab})\) are, respectively,

\[
W = -\frac{1}{2} h^{ab} T_{ab} = W^{(m)} + W^{(e)},
\]

\[
\Psi_a = T_a^b \partial_b r + W \partial_a r = \Psi_a^{(m)} + \Psi_a^{(e)},
\]

(3.8)

where \( W^{(m)} \) and \( \Psi_a^{(m)} \) are defined by using \( T_{ab}^{(m)} \), while \( W^{(e)} \) and \( \Psi_a^{(e)} \) by \( T_{ab}^{(e)} \), and the contraction is taken over the two-dimensional spacetime \( h_{ab} \). It is easy to find in our case,

\[
W^{(m)} = -P = C(v) r^{-(n-2)(1-\sigma)},
\]

(3.9)

so the energy supply for the matter is given by

\[
\Psi_a^{(m)} = \tilde{\mu} l_a.
\]

(3.10)
Now, because we are treating an effective Einstein gravity (i.e. view the non-Einstein term $H_{ab}$ as an energy-momentum tensor of higher curvature terms), the Misner-Sharp energy (3.2) is applicable. For the generalized Vaidya spacetime, we can replace the term $h^{ab} \partial_a r \partial_b r$ in equation (3.2) by $f(v, r)$. It is easy to see that equations (3.8) and (3.2) satisfy the unified first law (3.3).

On the apparent horizon, we have the Clausius-like equation

$$A \Psi_a \xi^a = A \Psi^{(m)}_a \xi^a + A \Psi^{(e)}_a \xi^a = \frac{\kappa}{8\pi} \delta A. \quad (3.11)$$

where $\xi$ is a vector field tangent to the apparent horizon. This vector can be determined as follows. Since $f = 0$ on the $(n-2)$-dimensional marginal surface, the Lie derivative of $f$ with respect to $\xi$ should vanish, so $\mathcal{L}_\xi f = 0$ on this surface. This means that on the apparent horizon we have

$$\xi^v \partial_v f + \xi^r \partial_r f = 0. \quad (3.12)$$

Noting that $\kappa = f'(v, r_A)/2$, we have $\xi^r/\xi^v = -\dot{f}/2\kappa$. In the case of $\ddot{f} = \partial_v f = 0$, we have $\xi^r = 0$. Therefore on the apparent horizon, it is reasonable to set $\xi_a$ to be

$$\xi_a = \left( \frac{\dot{f}}{2\kappa} \right) l_a + n_a. \quad (3.13)$$

Here, the normalization of $\xi$ is not important and will not play a key role in the following discussion. It is easy to find $\xi^a \xi_a = -\dot{f}/\kappa$. So the generator or the tangent vector of the apparent horizon is not a null vector unless $\dot{f} = 0$, i.e., the static case. On the other hand, for the static cases, one has $\xi^a = n^a = (\partial/\partial v)^a$ on the apparent horizon, as expected.

Note that the heat flow $\delta Q$ is determined by the pure matter energy-momentum tensor or pure matter energy-supply. This point has been emphasized in [35]. Thus, on the apparent horizon we can rewrite the equation (3.11) as

$$\delta Q \equiv A \xi^a \Psi^{(m)}_a = \frac{\kappa}{8\pi} \delta A - A \xi^a \Psi^{(e)}_a. \quad (3.13)$$

An interesting question is whether the right hand side of the equation (3.13) can be cast into a form $T\delta S$ of the Clausius relation as in the Einstein gravity. The answer is affirmative, since the right hand side of the (3.13) indeed can be written in a form of $T\delta S$, where $S$ will be given below. One can get this entropy expression by directly substituting the expression of $H_{ab}$ in equation (3.7) into the equation (3.13). In fact, the authors in [35]
obtain an entropy expression of apparent horizon in Lovelock gravity, but in a setting of FRW universe. Here let us stress that it is not always possible to rewrite the right hand side of the equation (3.13) in a form $T \delta S$, scalar-tensor theory and $f(R)$ gravitational theory being the counterexamples [35].

In order to find the entropy expression on the apparent horizon, here we use a little different method from the one in [35]. By using the explicit form of $\Psi_a^{(m)}$ in equation (3.10), we have

$$\delta Q = A \Psi_a^{(m)} \xi^a = A \tilde{\mu} l_a \xi^a = - A \tilde{\mu} .$$

(3.14)

\[ \text{¿From equations (2.11), (2.18) and (2.21), it is not hard to find} \]

$$- 2 A \tilde{\mu} \frac{\dot{f}}{f} = \frac{A}{8 \pi} \sum_i c_i \frac{i(n-2)!}{(n-2i-1)!} r^{1-2i} .$$

(3.15)

Hence we have on the apparent horizon,

$$\delta Q = - A \tilde{\mu} = \frac{\kappa}{2\pi} \left( \frac{\dot{f}}{2\kappa} \right) \left( \frac{A}{4} \sum_i c_i \frac{i(n-2)!}{(n-2i-1)!} r^{1-2i} \right) = \frac{\kappa}{2\pi} L \xi S = \frac{\kappa}{2\pi} \delta S ,$$

(3.16)

where all the calculations are done on the apparent horizon. This is very similar to the Clausius relation $\delta Q = T \delta S$ if we define the temperature by $T = \kappa/2\pi$ and the entropy of apparent horizon $S$ by

$$S = \frac{A}{4} \sum_i c_i \frac{i(n-2)!}{(n-2i)!} r_A^{2-2i} .$$

(3.17)

The entropy (3.17) is the same as that in the static cases if one replaces $r_A$ with the event horizon radius $r_+$ of static black holes in Lovelock gravity [19].

Now, we have shown that $A \Psi^{(m)}$ term can be written as $T \delta S$ term in the first law. Can the equations of motion in Lovelock gravity be rewritten as the unified first law (3.3) in Einstein gravity theory, where the left-hand side is completely determined by spacetime geometry, while the right hand side is determined by matter in spacetime? We can rewrite (3.3) as

$$dE - A \Psi^{(e)} - W^{(e)} dV = A \Psi^{(m)} + W^{(m)} dV .$$

(3.18)

Note that the left hand side of the equation is totally determined by geometry because $\Psi^{(e)}$ and $W^{(e)}$ are defined by geometric quantities as well as $E$. This implies that the left hand side of the equation (3.18) can give us an energy form like the Misner-Sharp energy in Einstein gravity. Indeed it is not hard to show that the left hand side of the equation can
be cast into a form of \( dE_L \), where the function \( E_L \) is given by

\[
E_L = \frac{\Omega_{n-2}}{16\pi} \sum_i \frac{(n-2)!c_i}{(n-2i-1)!} r^{n-2i-1} (1 - f(v, r))^i. \tag{3.19}
\]

Thus in the Vaidya-like spacetime for Lovelock gravity theory we arrive at a generalized unified first law

\[
dE_L = A\Psi^{(m)} + W^{(m)} dV. \tag{3.20}
\]

After projecting on the apparent horizon, we obtain the first law of apparent horizon

\[
\delta E_L = T\delta S - P\delta V. \tag{3.21}
\]

where we have used \( W^{(m)} = -P \). If we write the function \( f(v, r) \) back into the form \( h^{ab}\partial_a r \partial_b r \), the energy \( E_L \) is nothing but the generalized Minser-Sharp energy suggested by H. Maeda and M. Nozawa in recent papers [43, 44], where the generalized Misner-Sharp energy is shown to be a quasilocal conserved charge associated with a locally conserved current constructed from the generalized Kodama vector. Here let us stress that our procedure to find the entropy and the energy expressions is a direct generalization of the method proposed by Hayward to Lovelock gravity theory. The two methods, though different, lead to the same result. It would be interesting to use our method to study the unified first law in other gravity theories and to get the corresponding Miner-Sharp energy.

In some sense, Lovelock gravity theory is special since this theory is ghost-free, and the equations of motion are in fact second order for metric, as in the case of Einstein gravity. These properties may help us separating matter from geometry effectively. For other theories without such properties, such as \( f(R) \) gravity, the relation between equations of motion and thermodynamics is complicated. Recently, Eling, Guedens and Jacobson have pointed out that equations of motion for \( f(R) \) gravity correspond to a non-equilibrium thermodynamics of spacetime. To get correct equations of motion, an entropy production term has to be added to the Clausius relation [37]. However, to what extent can the spacetime be described by equilibrium thermodynamics is still an open question.

In summary, the definition of heat variation \( \delta Q \) is important in our procedure. The above discussion shows that \( \delta Q \) should be defined by pure matter energy supply or pure matter energy-momentum tensor, through which we can find a reasonable entropy associated with the apparent horizon. With this entropy and the unified first law in Einstein gravity
theory, we can obtain a proper energy form, i.e. generalized Misner-Sharp energy, completely determined by spacetime geometry. Thus we can establish the generalized unified first law for Lovelock gravity theory.

The generalized unified first law (3.21) of the apparent horizon is a version of physical process for the first law of black hole thermodynamics, i.e., an active version of the first law. In this version of the first law, one considers a spacetime and matter energy-momentum tensor, and calculates the variation of thermodynamical quantities which are induced by the matter (it enters into horizon). The first law then establishes a relation among the variations of those thermodynamic quantities. In the passive version or phase space version of first law, on the other hand, one compares the thermodynamical quantities of two spacetimes differing from each other by a small amount of variables, and then gives variations of these quantities, which lead to the first law. This physical process of the first law can shed some light on the passive version of the first law. For the solution (2.18), here we give some suggestion on the passive version of the first law. By using equations (2.18) and the definitions of the apparent horizon and the surface gravity, the surface gravity can be written as a function of the radius of the apparent horizon

$$\kappa = \frac{1}{2r_A} \sum_i \tilde{c}_i r_A^{-2i+2} \left[ \sum_i (n - 2i - 1) \tilde{c}_i r_A^{-2i+2} + 16\pi \mathcal{C}(v) r_A^{(n-2)\sigma-n+4} \right], \quad (3.22)$$

where $r_A = r_A(v)$ and the coefficients $\tilde{c}_i$ are defined by

$$\tilde{c}_i = c_i \frac{(n - 2)!}{(n - 2i - 1)!}. \quad (3.23)$$

The energy inside the apparent horizon is given by

$$M_L = \frac{\Omega_{n-2}}{16\pi} \sum_i \tilde{c}_i r_A^{n-2i-1}, \quad (3.24)$$

and the pressure on the apparent horizon by

$$P = -C(v) r_A^{-(n-2)(1-\sigma)}. \quad (3.25)$$

Then it is easy to check that the following relation holds

$$dM_L = TdS - PdV. \quad (3.26)$$

where the entropy is given by (3.17), while $dr_A$ denotes a variation of the apparent horizon, i.e., from $r_A$ to $r_A + dr_A$. This is a variation of the state parameter $r_A$ of the dynamical
black hole. It should be noted that the variation operator “d” is not an exterior derivative “d” of the spacetime manifold, but a difference of two nearby points in the solution space of the theory. Equation (3.26) is just the passive version of the first law. Another point we wish to stress is that the physical meaning of energy $M_L$ is ambiguous. It is not an ADM mass of the spacetime (although it is an ADM mass in the static limit), but a generalized Misner-Sharp energy inside the apparent horizon. By using equation (2.18), the energy $M_L$ can be expressed as

$$M_L = m(v) + \Omega_{n-2}C(v)\Theta(r_A).$$

(3.27)

Thus, we see that the generalized Misner-sharp energy inside the apparent horizon for the solution (2.18) gets the physical meaning as a generalized Bondi mass if the matter is pure radiation. However, for other type of matters, such as the term leading to the nonvanishing $C(v)$, the Bondi mass should be modified. Further, it turns out difficult to define the Bondi mass at null infinity in odd dimensions. This has been pointed out in references [46, 47].

For the static case, $r_A = r_+$, equations (3.22) and (3.26) reduce to the first law of black hole thermodynamics in Lovelock gravity theory [19].

IV. CONCLUSION AND DISCUSSION

In this paper, we have obtained a kind of generalized Vaidya solutions (2.18) in a generic Lovelock gravity. Explicit forms of these solutions are given for some special cases, such as Gauss-Bonnet gravity, dimensional continued gravity and pure Lovelock gravity. We have also investigated the unified first law on the apparent horizon of this spacetime, and given the entropy (3.17) associated with the apparent horizon and the energy (3.19) for $(n-2)$-dimensional sphere with radius $r$. In our procedure, we treated all terms of Lovelock gravity except for the Einstein tensor in equations of motion as an effective energy-momentum tensor in Einstein gravity theory, to which we could apply the unified first law proposed by Hayward for Einstein gravity theory. Defining the heat flux by pure matter energy-momentum tensor, we obtained an entropy expression (3.17) associated with the apparent horizon. Substituting this entropy into the unified first law leads to an energy form (3.19) for Lovelock gravity, which is the same as the one suggested by H. Maeda and M. Nozawa [43, 44]. This method can be further extended to other theories of gravity, for example, brane world theory.

Here a comment not discussed in section III is in order. We did not discuss whether the
The apparent horizon (or trapping horizon) is “outer” and “inner” [32]. In general, for higher dimensional cases, the parameter $p$ in (1.1) is more than one, so the equation (2.18) is a higher order algebraic equation of $f(v, r)$ and may have multi-horizons. We have assumed in section III that the apparent horizon is an outer apparent horizon (trapping horizon) of black hole spacetime with positive surface gravity $1/2f'(v, r)|_{r=r_A} > 0$. In fact, the result of this paper also holds for the cosmological event horizon (inner apparent horizon) with negative surface gravity $1/2f'(v, r)|_{r=r_A} < 0$. In this case the Hawking temperature of the apparent horizon is $T = |\kappa/2\pi|$.

Note that, while investigating the thermodynamical properties of the new solution, we just focussed on the mechanics of the dynamical black hole. To establish the thermodynamics of apparent horizon, one has to show that there exists Hawking radiation with the temperature $T = \kappa/2\pi$ as in the case of static black holes. In fact, this may be proved by following some recent works on Hawking radiation and entropy of dynamical spacetime [48, 49, 50, 51, 52] (for a recent review, see [53]).

In a dynamical spacetime the apparent horizon is not necessarily the same as the event horizon. For the solution (2.18), the apparent horizon is simply given by an algebraic equation $f(v, r_A(v)) = 0$, from which the event horizon is very different. The event horizon requires the global information of the spacetime. However, if we assume the event horizon is a null hypersurface of the spacetime, it is not hard to find that the event horizon of solution (2.18) is given by $f(v, r_+(v)) = 2dr_+(v)/dv$. Therefore, in general, it is not easy to give an explicit expression for the location of event horizon because we have to solve a differential equation instead of an algebraic equation. For some simple cases, such as the four-dimensional Vaidya spacetime and the five-dimensional Vaidya spacetime with the Gauss-Bonnet term, we may give the explicit formula of the location of the event horizon for some simple mass functions $m(v)$. Taking the four-dimensional Vaidya spacetime in the Einstein gravity as an example, the event horizon is given by $r_+ = 2m(v)/(1 - 2\dot{r}_+)$. Some authors have calculated the associated temperature and entropy by using brick wall model and gravity anomalies method [54, 55]:

$$T = \frac{1 - 2\dot{r}_+}{8\pi m(v)} = \frac{1}{4\pi r_+}, \quad S = \frac{A}{4} = \pi r_+^2.$$  \hspace{1cm} (4.1)

At this stage a serious problem may arise. For a dynamical spacetime such as the Vaidya solution, which horizon, the apparent horizon or the event horizon or both of them, is indeed


associated with Hawking radiation and the entropy? This issue has been commented in [56] (references therein). But it is fair to say that this is still an open question. Obviously, this is a quite interesting issue worth further studying.

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