APPENDIX TO ‘ROTH’S THEOREM ON PROGRESSIONS
REVISITED’ BY J. BOURGAIN

TOM SANDERS

1. Introduction

In this appendix we flesh out the details of two results stated in the paper. The
first is a refinement of Freiman’s theorem, a result which has an extensive literature
(see, for example, [Bil99, Cha02, Fre73, Ruz96, Nat96] and [TV06]); familiarity with
Cha02 and Ruz96 will be assumed, although it is not logically necessary.

Theorem 1.1. Suppose that \(A \subset \mathbb{Z}\) is a finite set with \(|A + A| \leq K|A|\). Then \(A\) is
contained in a multidimensional arithmetic progression \(P\) with
\[
\dim P = O(K^{7/4} \log^3 K) \quad \text{and} \quad |P| \leq \exp(O(K^{7/4} \log^3 K))|A|.
\]

The previous best estimates are due to Chang [Cha02] who showed the above
result (up to logarithmic factors) with \(2\) in place of \(7/4\). Note that one cannot
hope to improve the dimension bound past \(\lfloor K - 1 \rfloor\), or the exponent of \(K\) in the
size bound below \(1\); at the end of [Cha02] Chang (using arguments of Bilu [Bil99])
actually shows how to bootstrap the dimension bound to \(\lfloor K - 1 \rfloor\) for a small cost in
the size bound. See the notes [Gre05] of Green for an exposition of this argument.

The second result we shall show is an improvement of a theorem of Konyagin
and Laba from [KL06]. For \(\alpha \in \mathbb{R}\) and \(A \subset \mathbb{R}\) we write \(\alpha.A := \{\alpha a : a \in A\}\).

Theorem 1.2. Suppose that \(A \subset \mathbb{R}\) is a finite set and \(\alpha \in \mathbb{R}\) is transcendental.
Then
\[
|A + \alpha.A| \gg \frac{(\log |A|)^{4/3}}{(\log \log |A|)^{8/3}}|A|.
\]

In [KL06] the above result was shown with a \(1\) in place of \(4/3\) – again, up
to factors of \(\log \log |A|\) – and it was observed that for any transcendental \(\alpha\) and
positive integer \(N\) there is a simple construction of a set \(A\) with \(|A| = N\) and
\(|A + \alpha.A| = \exp(O(\sqrt{\log |A|}))|A|\).

The improvements in both Theorem 1.1 and Theorem 1.2 stem from the following
result, the proof of which is the content of this appendix. It will be appropriate for
us to consider Bohr sets in \(\mathbb{Z}/N\mathbb{Z}\) rather than the generalizations presented in the
paper: if \(\Gamma\) is a set of characters on \(\mathbb{Z}/N\mathbb{Z}\) and \(\delta \in (0, 1]\), then we write
\[
\mathcal{B}(\Gamma, \delta) := \{x \in \mathbb{Z}/N\mathbb{Z} : \|\gamma(x)\| \leq \delta \text{ for all } \gamma \in \Gamma\}.
\]

Theorem 1.3. Suppose that \(A, B \subset \mathbb{Z}/N\mathbb{Z}\) have \(|A + B| \leq K|B|\) and \(A\) has density
\(\alpha\). Then \(2A - 2B\) contains \(\mathcal{B}(\Gamma, \delta) \cap \mathcal{B}(\Lambda, \epsilon)\), where
\[
|\Gamma| = O(K^{1/2} \log \alpha^{-1}) \quad \text{and} \quad \log \delta^{-1} = O(K^{1/4} \log \alpha^{-1} \log(K \alpha^{-1}))
\]
and
\[
|\Lambda| = O(K^{3/4} \log \alpha^{-1}) \quad \text{and} \quad \log \epsilon^{-1} = O(\log(K \log \alpha^{-1})).
\]
The appendix now splits into five sections. In §2 we recall the basic facts about Bohr sets and dissociativity which we shall need; in §3 we detail the key new density increment developed in the paper, before completing the proof of Theorem 1.3 in §4. Finally, in §5 we prove Theorem 1.1 and in §6 we prove Theorem 1.2.

2. Bohr sets and dissociativity

We say that a Bohr set $B(\Gamma, \delta)$ is regular if

$$1 - 2^4|\Gamma||\eta| \leq \frac{|B(\Gamma, (1 + \eta)\delta)|}{|B(\Gamma, \delta)|} \leq 1 + 2^4|\Gamma||\eta|$$

whenever $|\Gamma||\eta| \leq 2^{-4}$.

Typically Bohr sets are regular, a fact implicit in the proof of the following proposition, which may be found, for example, as Lemma 4.25 for [TV06].

**Proposition 2.1.** Suppose that $B(\Gamma, \delta)$ is a Bohr set. Then there is a $\delta'$ with $\delta \leq \delta' < 2\delta$ such that $B(\Gamma, \delta')$ is regular.

There is a natural candidate for ‘approximate Haar measure’ on $B(\Gamma, \delta)$: we write $\beta_{\Gamma, \delta}$ for the unique uniform probability measure on $B(\Gamma, \delta)$. Having identified such a measure there are various possible formulations of the ‘approximate annihilator’ of a Bohr set and the following lemma helps us pass between them.

**Lemma 2.2.** Suppose that $B(\Gamma, \delta)$ is a regular Bohr set and $\kappa > 0$ is a parameter. Then

$$\{\gamma : |\beta_{\Gamma, \delta}(\gamma)| \geq \kappa\} \subset \{\gamma : |1 - \gamma(x)| \leq 2^5|\Gamma|\kappa^{-1}\delta^{-1} \text{ for all } x \in B(\Gamma, \delta')\}.$$

Dissociativity is an important concept and for us and we shall require a local analogue, but first we need some notation. If $\Lambda$ is a set of characters and $m : \Lambda \rightarrow \{-1, 0, 1\}$ then we write

$$m.\Lambda := \sum_{\lambda \in \Lambda} m_\lambda \lambda$$

and $\langle \Lambda \rangle := \{m.\Lambda : m \in \{-1, 0, 1\}^\Lambda\}$.

If $S$ is a symmetric neighborhood of the trivial character then we say that a set of characters $\Lambda$ is $S$-dissociated if

$$m \in \{-1, 0, 1\}^\Lambda$$

and $m.\Lambda \in S$ implies that $m \equiv 0$.

The usual notion of dissociativity corresponds to taking $S$ equal to the set containing just the trivial character, and typically for us $S$ will be a set of the form

$$\{\gamma : |\beta_{\Gamma, \delta}(\gamma)| \geq \kappa\};$$

the following lemma is the tool by which we make use of this notion.

**Lemma 2.3.** Suppose that $B(\Gamma, \delta)$ is a regular Bohr set, $\mathcal{L}$ is a set of characters and $\Lambda$ is a maximal $S := \{\gamma : |\beta_{\Gamma, \delta}(\gamma)| \geq \kappa\}$-dissociated subset of characters. Then $\mathcal{L}$ is contained in

$$\{\gamma : |1 - \gamma(x)| \leq 2^5|\Gamma|\kappa^{-1}\delta^{-1} + 2^4|\Lambda|\epsilon \text{ for all } x \in B(\Gamma, \delta') \cap B(\Lambda, \epsilon)\}.$$

**Proof.** We begin by proving that $\mathcal{L} \subset \langle \Lambda \rangle + S$. Suppose (for a contradiction) that there is some character $\gamma \in \mathcal{L} \setminus \langle \Lambda \rangle + S$. Let $\Lambda' := \Lambda \cup \{\gamma\}$ which is a strict superset of $\Lambda$. We shall show that $\Lambda'$ is dissociated contradicting the maximality of $\Lambda$. Suppose that $m \in \{-1, 0, 1\}^{\Lambda'}$ is such that $m.\Lambda' \in S$. We have three cases.

(i) $m_\gamma = 0$ in which case $m|_{\Lambda} \Lambda \in S$ and so $m|_{\Lambda} \equiv 0$ by $S$-dissociativity of $\Lambda$. It follows that $m \equiv 0$. 

The usual notion of dissociativity corresponds to taking $S$ equal to the set containing just the trivial character, and typically for us $S$ will be a set of the form

$$\{\gamma : |\beta_{\Gamma, \delta}(\gamma)| \geq \kappa\};$$

the following lemma is the tool by which we make use of this notion.
Proposition 3.1. Suppose that $B(\Gamma, \delta)$ is a regular Bohr set, $A \subset B(\Gamma, \delta)$ has relative density $\alpha$ and $\rho \in (0, 1]$ is a parameter. Suppose, further, that there is a Bohr set $B(\Gamma', \delta')$ with

$$2 \left( \frac{\rho \alpha}{2(1 + |\Gamma|)} \right)^{2^6} \delta \leq \delta' \leq 4 \left( \frac{\rho \alpha}{2(1 + |\Gamma|)} \right)^{2^6} \delta,$$

and a $\{\gamma : |\beta_{\Gamma, \delta}(\gamma)| \geq 1/3\}$-dissociated set $\Lambda$ of at least $2^7 \rho^{-1}(1 + \log \alpha^{-1})$ characters such that for each $\lambda \in \Lambda$ we have $|1_A d\beta_{\Gamma, \delta}(\lambda)| \geq \rho \alpha$. Then there is a regular Bohr set $B(\Gamma'', \delta'')$ with

$$\delta'' \geq \left( \frac{\rho \alpha}{2(1 + |\Gamma|)} \right)^{2^6} \delta$$

and $|\Gamma''| \leq |\Gamma| + \rho|\Lambda|/2^4(1 + \log \alpha^{-1})$

such that

$$\|1_A \ast \beta_{\Gamma'', \delta''}\|_{\infty} \geq \alpha \left( 1 + \frac{\rho^2|\Lambda|}{2^4(1 + \log \alpha^{-1})} \right).$$

This proposition is proved using Proposition (*) which essentially introduces Riesz products. We shall now formalize some appropriate notation and definitions to deal with them. Suppose that $\Lambda$ is a symmetric set of characters. $\omega : \Lambda \rightarrow D := \ldots$
\{z \in \mathbb{C} : |z| \leq 1\} is hermitian if \(\omega(\lambda) = \overline{\omega(-\lambda)}\) for all \(\lambda \in \Lambda\). Given a hermitian \(\omega : \Lambda \to D\) we define the product
\[
p_\omega := \prod_{\{\lambda, -\lambda\} \subset \Lambda} \left(1 + \frac{\omega(\lambda)\lambda + \overline{\omega(\lambda)}\lambda}{2}\right)
\]
and call it a Riesz product.

To pass between the notion of dissociaitivity defined in the previous section and the ‘Riesz product condition’ towards the end of Proposition (*) we use the following technical lemma.

**Lemma 3.2.** Suppose that \(B(\Gamma, \delta)\) is a regular Bohr set, \(\Lambda\) is a symmetric \(\{\gamma : |\hat{\beta}_{\Gamma, \delta}(\gamma)| \geq 1/3\}\)-dissociated set of characters and \(\omega : \Lambda \to D\) is hermitian. Then
\[
\int p_\omega d\beta_{\Gamma, \delta + \delta''} \leq 1 + 2^{7}(||\Gamma|||\Lambda|\delta'\delta^{-1})^{1/2}
\]
for all \(\delta'' \leq \delta'\).

**Proof.** We need to introduce some smoothed measures. Let \(L\) be an integer to be optimized later and write \(\tilde{\beta}\) for the measure \(\beta_{\Gamma, \delta} + \delta' + 2L\delta'\) where \(\beta_{\Gamma, \delta}\) occurs \(2L\) times. Now \(p_\omega \geq 0\) and \(\tilde{\beta}\) is uniform on \(B(\Gamma, \delta + \delta'')\) so
\[
\int p_\omega d\beta_{\Gamma, \delta + \delta''} \leq \int p_\omega d\tilde{\beta} \times \frac{|B(\Gamma, \delta + \delta'' + 2L\delta')|}{|B(\Gamma, \delta + \delta'')|} \leq (1 + 2^{8}||\Gamma|||L\delta'\delta^{-1}) \int p_\omega d\tilde{\beta};
\]
the last inequality by regularity and the fact that \(\delta'' \leq \delta'\). Now Plancherel’s theorem tells us that
\[
\int p_\omega d\tilde{\beta} \leq \sum_{\gamma \in (\Lambda)} |\hat{\beta}_{\Gamma, \delta}(\gamma)|^{2L} \leq 1 + 3^{-L} \leq 1 + \frac{L}{2}
\]
if \(L \geq ||\Lambda||\). \(L\) can now be optimized with ease. \(\square\)

The content of the next proof is simply the observation that a Riesz product on \(\Lambda\) is roughly constant on a small enough Bohr set on the characters \(\Lambda\).

**Proof of Proposition 3.1.** For each \(\lambda \in \Lambda\) let \(\omega(\lambda)\) be a complex number such that
\[
\omega(\lambda)1\lambda \hat{\beta}_{\Gamma, \delta}(\lambda) = |1\lambda \hat{\beta}_{\Gamma, \delta}(\lambda)|.
\]
Note that \(\omega\) is hermitian since \(1\lambda \hat{\beta}_{\Gamma, \delta}\) is real. We let \(\Phi\) be the set \(\{(\omega(\lambda)\lambda + \omega(\lambda)\lambda)/2 : \lambda \in \Lambda\}\) so that \(|\Lambda| \geq |\Phi| \geq |\Lambda|/2\). From the definition of \(\omega\) we see that \(\langle 1\lambda, \phi \rangle \geq \rho a\) for all \(\phi \in \Phi\) and since
\[
\delta' \leq \delta/2^{14}(1 + |\Lambda|)(1 + |\Gamma|),
\]
Lemma 3.2 applies and we have
\[
\int \prod_{\phi \in \Phi} (1 + a(\phi)\phi) d\beta_{\Gamma, \delta} \leq 2
\]
for all \(a : \Phi \to [-1, 1]\).
We apply Proposition (*) (with constants which come out of the proof) to conclude that there is a set \( \Phi' \subset \Phi \) with
\[
|\Phi'| \leq \frac{\rho|\Lambda|}{2^5(1 + \log \alpha^{-1})}
\]
such that
\[
\int 1_A \prod_{\phi \in \Phi'} (1 + \phi)d\beta_{\Gamma,\delta} \geq \alpha \left( 1 + \frac{\rho^2|\Lambda|}{2^8(1 + \log \alpha^{-1})} \right).
\]

Let \( \Lambda' \) be the subset of \( \Lambda \) such that \( \Phi' = \{(\omega(\lambda)\lambda + \omega(\lambda)\bar{\lambda})/2 : \lambda \in \Lambda'\} \) and \( \omega' := \omega|_{\Lambda'} \), so that \( p_{\omega'} = \prod_{\phi \in \Phi'} (1 + \phi) \). We take \( \Gamma' := \Gamma \cup \Lambda' \) and it follows that
\[
|\Gamma'| \leq |\Gamma| + \frac{\rho|\Lambda|}{2^4(1 + \log \alpha^{-1})}.
\]

Now place some total order \( < \) on \( \Phi' \). Then
\[
p_{\omega'}(x + y) - p_{\omega'}(x) = \sum_{\phi' \in \Phi'} \prod_{\phi < \phi'} (1 + \phi(x + y))(\phi'(x + y) - \phi'(x)) \prod_{\phi' < \phi} (1 + \phi(x)).
\]
It follows that
\[
|p_{\omega'}(x + y) - p_{\omega'}(x)| \leq \sum_{\phi' \in \Phi'} \prod_{\phi < \phi'} (1 + \phi(x + y))|\lambda(y) - 1| \prod_{\phi' < \phi} (1 + \phi(x)),
\]
and hence if \( y \in B(\Gamma', \delta'') \) we conclude that
\[
|p_{\omega'}(x + y) - p_{\omega'}(x)| \leq 2^{4}\delta'' \sum_{\phi' \in \Phi'} \prod_{\phi < \phi'} (1 + \phi(x + y)) \prod_{\phi' < \phi} (1 + \phi(x)).
\]
If we define
\[
\omega_{\phi'}(\lambda) := \begin{cases} 
\omega(\lambda)\lambda(y) & \text{if } (\omega(\lambda)\lambda + \omega(\lambda)\bar{\lambda})/2 < \phi' \\
0 & \text{if } (\omega(\lambda)\lambda + \omega(\lambda)\bar{\lambda})/2 = \phi' \\
\omega(\lambda) & \text{if } (\omega(\lambda)\lambda + \omega(\lambda)\bar{\lambda})/2 > \phi'
\end{cases}
\]
then this last expression can be written as
\[
|p_{\omega'}(x + y) - p_{\omega'}(x)| \leq 2^{4}\delta'' \sum_{\phi' \in \Phi'} p_{\omega_{\phi'}}(x).
\]
Hence, by Lemma 3.2 we have
\[
\int p_{\omega_{\phi'}} d\beta_{\Gamma,\delta} \leq 2,
\]
whence
\[
\int 1_A|p_{\omega'} * \beta_{\Gamma',\delta''} - p_{\omega'}|d\beta_{\Gamma,\delta} \leq 2^{5}\delta''|\Phi'|.
\]

Pick \( \delta'' \) satisfying the lower bound of the proposition and regular for \( \Gamma' \) by Proposition 2.1, such that
\[
|\int 1_A p_{\omega'} * \beta_{\Gamma',\delta''} d\beta_{\Gamma,\delta} - \int 1_A p_{\omega'} d\beta_{\Gamma,\delta}| \leq 2^{5}|\Lambda|\delta'' \leq \frac{\alpha \rho^2}{210(1 + \log \alpha^{-1})},
\]
where the last inequality is by choice of \( \delta'' \). Consequently
\[
\int 1_A p_{\omega'} * \beta_{\Gamma',\delta''} d\beta_{\Gamma,\delta} \geq \alpha \left( 1 + \frac{\rho^2|\Lambda|}{2^8(1 + \log \alpha^{-1})} \right).
\]
But
\[
\int 1_{A^*} \ast \beta_{\Gamma', \delta'} d\beta_{\Gamma, \delta} \leq \frac{|B(\Gamma, \delta + \delta')|}{|B(\Gamma, \delta)|} \times \int 1_A \ast \beta_{\Gamma', \delta'} p_{\omega'} d\beta_{\Gamma, \delta + \delta'} \\
\leq (1 + 2^4|\Gamma|\delta''^{-1}) ||1_A \ast \beta_{\Gamma', \delta'}||_\infty \\
\times (1 + 2^7(|\Gamma||\Lambda'|\delta''^{-1})^{1/2}).
\]
It follows that
\[
||1_A \ast \beta_{\Gamma', \delta'}||_\infty \geq 1 - 2^{10}(|\Lambda|\delta''^{-1})^{1/2} \int 1_{A^*} \ast \beta_{\Gamma', \delta'} d\beta_{\Gamma, \delta},
\]
from which we retrieve the result. □

4. The Proof of Theorem 1.3

The proof is iterative with the following lemma as the driving ingredient.

Lemma 4.1 (Iteration lemma). Suppose that $B(\Gamma, \delta)$ is a regular Bohr set and $A \subset \mathbb{Z}/N\mathbb{Z}$ has relative density $\alpha$ in $B(\Gamma, \delta)$. Suppose, further, that there is a set $B \subset \mathbb{Z}/N\mathbb{Z}$ such that $|A + B| \leq K|B|$. Then at least one of the following conclusions is true.

(i) $2A - 2B$ contains $B(\Gamma, \delta'') \cap B(\Lambda, \epsilon)$ where $\Lambda$ is a set of size $O(K^{3/4} \log \alpha^{-1})$ and $\delta''^{-1} = (2\alpha^{-1}K(1 + |\Gamma|))^{O(1)} \delta^{-1}$ and $\epsilon^{-1} = O(K \log \alpha^{-1})$.

(ii) There is a regular Bohr set $B(\Gamma', \delta''')$ with $|\Gamma'| \leq |\Gamma| + O(K^{1/4})$ and $\delta''''^{-1} = (2\alpha^{-1}K(1 + |\Gamma|))^{O(1)} \delta^{-1}$, and such that

\[
||1_A \ast \beta_{\Gamma', \delta'''}||_\infty \geq 1 + 2^{-7}K^{-1/4}.
\]

Proof. Apply Proposition 2.3 to pick $\delta'$ regular for $\Gamma$ such that

\[
2 \left( \frac{\alpha}{4\sqrt{K(1 + |\Gamma|)}} \right) \frac{\delta}{2^{6}} \leq \delta' \leq 4 \left( \frac{\alpha}{4\sqrt{K(1 + |\Gamma|)}} \right) \frac{\delta}{2^{6}}.
\]

Let $\Lambda$ be a maximal $\{\gamma : |\beta_{\Gamma, \delta'}(\gamma)| \geq 1/3\}$-dissociated subset of

\[
\mathcal{L} := \{\gamma : |1_{A^*} d\beta_{\Gamma, \delta}(\gamma)| \geq \alpha/2\sqrt{K}\}.
\]

If $|\Lambda| \leq 2^{8}K^{3/4}(1 + \log \alpha^{-1})$ then apply Lemma 2.3 to see that $\mathcal{L}$ is contained in

\[
\{\gamma : |1 - \gamma(x)| \leq 1/4 \text{ for all } x \in B(\Gamma, \delta'') \cap B(\Lambda, \epsilon)\},
\]

where

\[
\delta'' = \frac{\delta'}{2^{10}(1 + |\Gamma|)} \text{ and } \epsilon = \frac{1}{215K(1 + \log \alpha^{-1})}.
\]

Write $|B| = \beta N$. By the Cauchy-Schwarz inequality we have

\[
|\beta^2\alpha^2| = \left| \mathbb{E}_{1_B \ast (1_A d\beta_{\Gamma, \delta})} \right|^2 \leq \mathbb{E}(1_B \ast (1_A d\beta_{\Gamma, \delta}))^2|B|\beta.
\]

It follows that if we write $f := 1_B \ast (1_A d\beta_{\Gamma, \delta}) \ast 1_B \ast (1_A d\beta_{\Gamma, \delta})$ then $f(0) \geq \alpha^2 \beta/K$.

By the inversion formula we have

\[
f(x) = \sum_{\gamma} |1_B(\gamma)|^2 |1_A d\beta_{\Gamma, \delta}(\gamma)|^2 \gamma(x),
\]
whence, by Parseval’s theorem, we have
\begin{align*}
|f(0) - f(x)| &\leq \sum_{\gamma \in \mathcal{L}} |\widehat{1_B}(\gamma)|^2 |1_A \widehat{\beta_{\Gamma',\delta'}}(\gamma)|^2 |\gamma(x) - 1| \\
&\quad + 2 \sum_{\gamma \not\in \mathcal{L}} |\widehat{1_B}(\gamma)|^2 |1_A \widehat{\beta_{\Gamma',\delta'}}(\gamma)|^2 \\
&\leq \sup_{\gamma \in \mathcal{L}} |1 - \gamma(x)| \sum_{\gamma} |\widehat{1_B}(\gamma)|^2 |1_A \widehat{\beta_{\Gamma',\delta'}}(\gamma)|^2 \\
&\quad + \beta \alpha^2 / 2K \\
&\leq \left( \sup_{\gamma \in \mathcal{L}} |1 - \gamma(x)| + 1/2 \right) f(0) \leq 3f(0)/4
\end{align*}
if \(x \in B(\Gamma, \delta''') \cap B(\Lambda, \epsilon)\). It follows that we are in case (i).

In the other case we discard (if necessary) just enough elements of \(\Lambda\) to ensure that the inequality \(|\Lambda| \leq 2^9 K^{3/4}(1 + \log \alpha^{-1})\) holds and then apply Proposition 3.3 with parameter \(\rho = 1/2\sqrt{K}\). It follows that there is a regular Bohr set \(B(\Gamma', \delta''')\) with
\[\delta''' \geq (\alpha/2K(1 + |\Gamma|))^{27} \delta\] and \(|\Gamma'| \leq |\Gamma| + 2^4 K^{1/4}\),
and
\[\|1_A * \beta_{\Gamma',\delta'''}\|_\infty \geq \alpha(1 + 2^{-7} K^{-1/4}).\]
It follows that we are in case (ii). \(\square\)

Iterating this to yield Theorem 1.3 is a simple exercise.

**Proof of Theorem 1.3**. We construct a sequence of regular Bohr sets \(B(\Gamma_k, \delta_k)\) iteratively initializing with \(\Gamma_0\) as the set containing the trivial character and \(\delta_0 = 1\) which has \(B(\Gamma_0, \delta_0)\) regular for trivial reasons. Write \(\alpha_k = \|1_A * \beta_{\Gamma_k,\delta_k}\|_\infty\) so that \(\alpha_0 = \alpha\) and let \(x_k\) be such that \(1_A * \beta_{\Gamma_k,\delta_k}(x_k) = \alpha_k\). We apply Lemma 4.1 repeatedly to the sets \(x_k - A\) and the Bohr sets \(B(\Gamma_k, \delta_k)\). If after \(k\) steps of the iteration we have never found ourselves in the first case of Lemma 4.1 then
\[\alpha_k \geq \alpha(1 + 2^{-7} K^{-1/4})^k, |\Gamma_k| = O(K^{1/4}k)\text{ and }\delta_k^{-1} = (2\alpha^{-1} K^{3/4}k)^{O(k)}\]
Since \(\alpha_k \leq 1\) the first of these ensures that \(k = O(K^{1/4}(\log \alpha^{-1})\) and so there is some \(k\) of size \(O(K^{1/4} \log \alpha^{-1})\) for which we end up in the first case of Lemma 4.1 and at that stage we have
\[|\Gamma_k| = O(K^{1/2} \log \alpha^{-1})\]
and
\[\delta_k^{-1} = \exp(O(K^{1/4} \log \alpha^{-1} \log \alpha^{-1} K)).\]
The result follows. \(\square\)

5. **Improving Freiman’s theorem: the proof of Theorem 1.1**

Ruzsa’s proof of Freiman’s theorem in [Ruz96] naturally splits into four stages: finding a good model; Bogoliouboff’s argument; determining the structure of Bohr sets; and, Chang’s pullback and covering argument. The improvement of this work arises from replacing Bogoliouboff’s argument by the more sophisticated Theorem 1.3

To ‘find a good model’ we use the following proposition due to Ruzsa which essentially appears as Theorem 8.9 in [Nat96] for example.
Proposition 5.1. Suppose that $A$ is a finite set of integers with $|A + A| \leq K|A|$. Then there is an integer $N$ with $N = K^{O(1)}|A|$ and a set $A' \subset A$ with $|A'| \geq |A|/8$ such that $A'$ is Freiman 8-isomorphic to a subset of $\mathbb{Z}/N\mathbb{Z}$.

We have already dealt with ‘Bogolioúboff’s argument’, so we turn to determining the structure of Bohr sets. It was a key insight of Ruzsa in [Ruz96] that Bohr sets contain large multidimensional progressions. Fortunately the same is true for intersections of Bohr sets. The following is a straightforward generalization of [TV06, Proposition 4.23].

Proposition 5.2. Suppose that $\Gamma$ is a set of characters on $\mathbb{Z}/N\mathbb{Z}$ and $(\delta_\gamma)_{\gamma \in \Gamma} \in (0,1]^\Gamma$. Then $B(\Gamma, (\delta_\gamma)_{\gamma \in \Gamma})$ contains a symmetric multidimensional progression $P$ of dimension $|\Gamma|$ and size at least $\prod_{\gamma \in \Gamma} (\delta_\gamma/|\Gamma|)N$.

Finally, ‘Chang’s pullback and covering argument’ is the following result which converts a large progression contained in $2A - 2A$ into a progression containing $A$.

Proposition 5.3. Suppose that $A$ is a finite set of integers with $|A + A| \leq K|A|$ and $2A - 2A$ contains a multidimensional progression of dimension $d$ and size $\eta|A|$. Then $A$ is contained in a multidimensional progression of size at most $\exp(O(d + K\log Kn^{-1}))/|A|$ and dimension at most $O(d + K\log Kn^{-1})$.

With these ingredients we can now prove Theorem 1.1.

Proof of Theorem 1.1. We apply Proposition 5.1 to get an integer $N$ with $N = K^{O(1)}|A|$ and a set $A' \subset A$ with $|A'| \geq |A|/8$ such that $A'$ is Freiman 8-isomorphic to a subset $A''$ of $\mathbb{Z}/N\mathbb{Z}$. Note that $A''$ has $|A'' + A''| \leq K|A| \leq 8K|A''|$ and density $K^{-O(1)}$. Apply Theorem 1.3 to get that $2A'' - 2A''$ contains $B(\Gamma, \delta) \cap B(\Lambda, \epsilon)$ where $|\Gamma| = O(K^{1/2} \log K)$ and $\log \delta^{-1} = O(K^{1/4} \log^2 K)$, and $|\Lambda| = O(K^{3/4} \log K)$ and $\log \epsilon^{-1} = O(\log K)$.

Proposition 5.2 then ensures that $2A'' - 2A''$ contains a multidimensional progression of dimension $O(K^{3/4} \log K)$ and size at least $\exp(-O(K^{3/4} \log^3 K))N$. Since $A''$ is Freiman 8-isomorphic to a subset of $A$ we have that $2A'' - 2A''$ is Freiman 2-isomorphic to a subset of $2A - 2A$ which thus contains a multidimensional progression of dimension $O(K^{3/4} \log K)$ and size at least $\exp(-O(K^{3/4} \log^3 K))|A|$. The result follows from Proposition 5.3.

It is worth remarking that for the purpose of improving the bounds in Freiman’s theorem for general abelian groups (the current best such appearing in the paper [GR07] of Green and Ruzsa) it would be desirable to pay closer attention to the $\alpha$-dependencies in Theorem 1.3. These contribute logarithmic terms in $\mathbb{Z}/N\mathbb{Z}$, but polynomial terms when we do not have a modelling lemma of the strength of Proposition 5.1, as is the case in general.
6. IMPROVING THE KONYAGIN-LABA THEOREM: THE PROOF OF THEOREM 1.2

We require two preliminary results.

**Lemma 6.1.** Suppose that \( A \subset \mathbb{R}, \alpha \in \mathbb{R} \setminus \{0\} \) and \(|A + \alpha A| \leq K|A|\). Then

\[
|2A - 2A| + \alpha(2A - 2A) + \cdots + \alpha^{l-1}(2A - 2A) | \leq K^{O(l)}|A|.
\]

**Proof.** Write \( A' := 2A - 2A \) and \( A'' := \alpha(2A - 2A) \). By the Plünnecke-Ruzsa estimates ([TV06, Corollary 6.27]) we have \( |kB'| \leq K^{4k}|A| \) for all \( k \). Since \( \alpha \neq 0 \) we have that \( |kB'| = |kA'| \), whence \( |kA'| \leq K^{4k}|A| \) and, in particular, \( |A' - A'| \leq K^8|A| \) and \( |3A' - 3A'| \leq K^{24}|A| \). By this second inequality and Ruzsa’s covering lemma ([TV06, Lemma 2.14]) there is a set \( S \) with \(|S| \leq K^{24}\) such that \( 3A' - 2A' \subset A' - A' + S \). Again, by Ruzsa’s covering lemma and the fact that \(|A + \alpha A| \leq K|A|\) there is also a set \( T \) with \(|T| \leq K \) and \( \alpha A \subset A - A + T \), whence \( B' \subset A' - A' + 2T - 2T \). Put \( T' := 2T - 2T \) and note that \(|T'| \leq K^4\).

Now, write \( B_l := A' + \alpha A' + \cdots + \alpha^{l-1}A' \) and define a sequence of sets \( T_l \) recursively by letting \( T_1 \) be some set containing precisely one element of \( A' \) and \( T_{l+1} := S + T' - T' + \alpha T_l \). We shall show by induction that \( B_l \subset A' - A' + T_l \). For \( l = 1 \) this is immediate. Assume that we have shown the result for some \( l \). Then

\[
B_{l+1} = A' + \alpha B_l \subset A' + \alpha A' - \alpha A' + \alpha T_l \subset 3A' - 2A' + T' + T' + \alpha T_l = A' - A' + T_{l+1}.
\]

The claim follows. It remains to note that \(|B_l| \leq |A' - A'||T_l| \leq K^8|A'||T_l| \leq K^{32l-24}|A'|\) as required. \(\square\)

Note that a direct application of the Plünnecke-Ruzsa estimates gives only a bound of the form \( K^{O(l)}|A| \).

The following is a straightforward modification of Proposition 5.1.

**Proposition 6.2.** Suppose that \( A \) and \( B \) are finite sets of integers with \(|A + B| \leq K \min\{|A|, |B|\}\) and \( k \geq 2 \) is a positive integer. Then there is an integer \( N \) with \( N = K^{O(k)} \min\{|A|, |B|\} \) a subset \( A \) of size at least \(|A|/k\), a subset \( B \), and a Freiman \( k \)-isomorphism mapping these to \( A'' \) and \( B'' \) in \( \mathbb{Z}/N\mathbb{Z} \) respectively. Furthermore, \(|A'' + B''| \leq kK \min\{|A''|, |B''|\} \).

**Proof.** There is clearly a Freiman \( k \)-isomorphism which maps \( A \) and \( B \) onto \( A' \) and \( B' \), respectively, some subsets of \( \mathbb{Z}/p\mathbb{Z} \) for a sufficiently large prime \( p \). Since \( k \geq 2 \) we have \(|A' + B'| = |A + B|\) and consequently we shall assume that \( A \) and \( B \) are subsets of \( \mathbb{Z}/p\mathbb{Z} \).

Suppose that \( q \in \mathbb{Z}/p\mathbb{Z}^* \) and define \( \phi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}; x \mapsto qx \) and \( \phi' : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z} \) to be the map which takes \( x + p\mathbb{Z} \) to its least non-negative member. The range of \( \phi' \) is partitioned by the \( k \)-sets

\[
I_j := \left\{ x \in \mathbb{Z} : \frac{j-1}{k}p \leq x < \frac{j}{k}p - 1 \right\} \text{ for } j \in [k],
\]

and, moreover, \( \phi'|_{\phi^{-1}(I_j)} \) is clearly a Freiman \( k \)-isomorphism for each \( j \in [k] \).

By the pigeon-hole principle there is some \( j = j(q) \) such that

\[
A(q) := (\phi' \circ \phi)^{-1}(I_j \cap A \text{ has } |A(q)| \geq |A|/k);
\]
similarly there is some \( x = x(q) \in \mathbb{Z}/p\mathbb{Z} \) such that
\[
B(q) := (\phi' \circ \phi)^{-1}(I_{(q)}) \cap (x + B) \text{ has } |B(q)| \geq |B|/k.
\]

Put \( C(q) = A(q) \cup B(q) \), and note that \( \phi' \circ \phi|_{C(q)} \) is a Frei\v{m}an \( k \)-isomorphism.

Finally, let \( N := |k(A + B) - k(A + B)| \) and \( \phi' : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z} \) be the usual quotient map. For every \( q \), \( \psi := \phi' \circ \phi|_{C(q)} \) is a Frei\v{m}an \( k \)-homomorphism; it is also a Frei\v{m}an \( k \)-isomorphism: put
\[
I(q) := \left\{ q \left( \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} a'_i \right) : a_1, \ldots, a_k, a'_1, \ldots, a'_k \in C(q) \right\},
\]
and
\[
M(N, p) := \{ kN + p\mathbb{Z} : 0 \leq k \leq |p/N| \}.
\]

\( \psi \) is a Frei\v{m}an \( k \)-isomorphism for some \( q \) if \( I(q) \cap M(N, p) = \{ p\mathbb{Z} \} \). For each \( q \) there are at most \(|kC(q) - kC(q)| - 1 \leq N - 1\) non-zero elements in \( I(q) \), whence if \( q \) is chosen uniformly at random from \( \mathbb{Z}/p\mathbb{Z}^* \), we have
\[
E_q|I(q) \cap M(N, p)| = 1 + \frac{|M(N, p)| - 1}{p - 1} \leq 1 + \frac{|p/N| - 1}{p - 1} \frac{N}{2} < 2.
\]

We conclude that there is some \( q \) such that \( |I(q) \cap M(N, p)| < 2 \), and so \( I(q) \cap M(N, p) = \{ p\mathbb{Z} \} \), and \( \psi \) is a Frei\v{m}an \( k \)-isomorphism.

To finish the proof we put \( A'' := \psi(A(q)) \) and \( B'' := \psi(B(q)) \), and since \( \psi \) is a Frei\v{m}an \( 2 \)-isomorphism,
\[
|A'' + B''| = |A(q) + B(q)| \leq |A + (x(q) + B)| \leq K \min\{|A|, |B|\} \leq kK \min\{|A''|, |B''|\}.
\]

The bound on \( N \) follows from the Plünnecke-Ruzsa estimates ([TV06, Corollary 6.27]).

The following argument is due to J. Bourgain.

**Proof of Theorem 1.3** Since \( A \cup \alpha A \) is finite it generates a finite dimensional \( \mathbb{Z} \)-module in \( \mathbb{R} \), whence \( A \cup \alpha A \) is Frei\v{m}an isomorphic of all orders to a subset \( A' \cup B' \) of \( \mathbb{Z}^d \). (Here, of course, \( A \) is isomorphic to \( A' \) and \( \alpha A \) to \( B' \).)

Let \( L \) be the the largest coefficient occurring in any element of \( A' \cup B' \) (we may assume, by a translation, that they are all positive). The map
\[
(x_1, \ldots, x_d) \mapsto x_1 + 17Lx_2 + (17L)^2x_3 + \cdots + (17L)^{d-1}x_d
\]
is a Frei\v{m}an \( 8 \)-isomorphism of \( A' \cup B' \) into \( \mathbb{Z} \) from which it follows that \(|A' + B'| \leq K \min\{|A'|, |B'|\} \) since \( |A'| = |B'| \). By Proposition 6.2 there is an integer \( N \) with \( N = K^{O(1)}|A| \), a subset of \( A' \) of size at least \(|A|/8 \), a subset of \( B' \) and a Frei\v{m}an 8-isomorphism mapping these sets to \( A'' \) and \( B'' \) in \( \mathbb{Z}/N\mathbb{Z} \) respectively such that \(|A'' + B''| \leq 8K \min\{|A''|, |B''|\} \).

It follows from Theorem 1.3 that \( 2A'' - 2B'' \) contains \( B(\Gamma, \delta) \cap B(A, \epsilon) \) where
\[
|\Gamma| = O(K^{1/2} \log K) \text{ and } \log \delta^{-1} = O(K^{1/4} \log K)
\]
and
\[
|A| = O(K^{3/4} \log K) \text{ and } \log \epsilon^{-1} = O(\log K).
\]
(Essentially) by Lemma 2.0(i) of the paper we have that
\[ |B(\Gamma, \eta\delta) \cap B(\Lambda, \eta\epsilon)| \geq (\eta\delta)^{|\Gamma|}(\eta\epsilon)^{|\Lambda|}N. \]

If \( \eta > \max\{\delta^{-1}N^{-1/2}|\Gamma|, \epsilon^{-1}N^{-1/2}|\Lambda|\} \) then this intersection has size greater than 1 and hence contains a non-zero element say \( d \). Moreover by the triangle inequality, for any \( k \) with \( |k| \leq \min\{\delta N^{1/2}|\Gamma|, \epsilon N^{1/2}|\Lambda|\} \) we have \( kd \in B(\Gamma, \eta\delta) \cap B(\Lambda, \eta\epsilon) \).

Thus \( 2A'' - 2B'' \) contains an arithmetic progression of length
\[ L \geq \min\{\delta N^{1/2}|\Gamma|, \epsilon N^{1/2}|\Lambda|\} \gg K^{-O(1)}|A|^{1/CK^{3/4}}\log K \]
for some absolute \( C > 0 \), and hence \( 2A - 2(\alpha.A) \) contains an arithmetic progression \( P \) of length \( L \).

By Lemma 6.1 with parameter \( l + 1 \) and some manipulations we get that
\[ |(2A - 2(\alpha.A)) + \alpha(2A - 2(\alpha.A)) + \cdots + \alpha^{l-1}(2A - 2(\alpha.A))| \leq K^{O(l)}|A|, \]
and since \( 2A - 2(\alpha.A) \) contains \( P \) we have (by the transcendence of \( \alpha \)) that the left hand side is at least \( |P|^l \). Taking \( l = \lceil 2CK^{3/4} \log K \rceil \) and inserting the lower bound for \( |P| \), the result follows. \( \square \)

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