By transforming a 1D second-order linear oscillator into a 2D first-order polar motion differential equation, it can be shown that the finite smoothness (i.e., the presence of jump in finite order derivatives) of the applied Newtonian forcing constitutes the sufficient and necessary condition for instantaneous excitation of free eigen-mode. This condition can be met by forcing functions originated from turbulent and multiphase fluid motions. Sub-macroscopic transition time associated with astatic elastic deformation limits the physical smoothness of the applied forcing for the Earth’s polar motion. Eigen-modes can also be excited by an infinitely smooth forcing that has a finite domain of non-zero values. The eigen-period serves as a macroscopic timescale to characterize the inertia of a linear oscillator. If a zero mean irregular forcing of finite smoothness exhibits a high degree of randomness and the timescale is much shorter than the eigen-period, then for negligible damping the eigen-waveform will increase in proportion to the square root of time, while the waveform distortion is statistically a constant. As a result, the pattern of distinctive eigen-oscillation will dominate the forced solution for longer enough duration.

1. Introduction

Setting a 1D linear oscillator in motion by a continuous force is a solved problem in 18th century’s physics. Often the forced motion is accompanied by a free eigen-mode of the oscillator. It sounds naïve to ask how the eigen-mode is excited, as the excitation of eigen-mode by impulsive forces, by prescribed initial conditions and by resonance are well formulated in textbooks and easy to implement in engineering and household devices.

Yet if the device is so huge like the Earth for instance, the question becomes much less naïve, as i) it is difficult to find significant impulsive forces in a fluid-dominant environment that can interrupt the continuity of momentum or angular momentum of planetary motion; ii) the initial condition is at the infinite history and most likely faded away; iii) there is no natural sources of resonance operating at the eigen-frequency. A prominent eigen-mode of a gigantic linear oscillator that persists in spite of the three shortages is the Chandler Wobble: a supposedly free circular eigen-wobble by the Earth’s rotational axis around the axis of spheroidal symmetry at a period of \( \approx 433 \) days (e.g., refs. [1–6]). Related but a little more complicated than a 1D oscillator is the phenomenon of the Earth’s background noise known as the hum, that is, irregular but persistent oscillations detected by superconducting gravimeters in a very quiet environment.[7]

A subtle and rather amusing problem associated with the Fourier spectrum method for eigen-mode excitations in the context of planetary and geosciences can be outlined as follows. In the complex domain, the general solution of the forced motion of a linear oscillator is in the form of an exponential eigen-mode multiplied by its complex amplitude (see Sections 3 and 4). The complex amplitude is exactly the Fourier transform of the applied forcing, terminated by causality at the present time \( t \) and taking value at the eigen-frequency. For a fixed \( t \), a forcing that delivers non-zero solution must have resonance spectral power at the eigen-frequency, since it is the solution by itself. As the time progresses from \( t = -\infty \), however, the eigen-mode may or may not rise in the forced solution at all! Apparently, something has not been accounted for, if one just looks at the Fourier spectrum for answers of the eigen-mode excitation. Often people follow the standard procedure in spectral analysis to make Fourier decomposition of the available forcing data for the entire timespan, and then, seek resonance solution in a narrow band around the eigen-frequency. For an arbitrary epoch \( t_0 \) within the timespan, this approach violates the principle of causality by injecting future information at \( t > t_0 \) back into the past. The artificial eigen-frequency band so derived could not explain the time varying amplitude of the eigen-mode exhibited in the time series of observation (e.g., ref. [3–5]), nor could it provide the mechanism of how the eigen-mode is excited before any epoch within the timespan.

In this paper, we explore the mechanism of eigen-mode excitation by going deeper into Newton’s second law behind the three textbook ingredients and away from the difficulties associated with the spectral method. The key point is that the sluggish
inertia of a linear oscillator manifests in any order derivative of the displacement beyond the second-order acceleration for the very reason that the mathematical structure of the linear governing equation is differential invariant to any order.

It is well known from textbooks (e.g., ref. [8]) that the inertia of a linear oscillator gives rise to free eigen-mode to prevent jumps in velocity and displacement. The differential invariance of governing equation to any, say, the Nth order implies that the inertia of a linear oscillator gives rise to free eigen-mode to prevent jumps in the \((N - n)\)th \((n = 1, 2, \ldots, N)\) order derivatives of the displacement.

Since there is no standard terminology for derivatives of the conventional acceleration, we call them high order accelerations or any order accelerations.

We will show that the sufficient and necessary condition for instantaneously excited eigen-mode (IEEM) is the presence of jumps in any order acceleration, or equivalently, in any order derivative of the applied Newtonian forcing.

The physical significance of this development lies in the revelation that the free eigen-mode can be excited without interrupting the continuity of the applied forcing and its derivatives up to any finite order. Jumps in any order derivatives of the applied Newtonian forcing constitutes a weaker requirement for IEEM than jumps and delta impulses in the Newtonian forcing itself, and, thus, is suitable to the forcing generated from turbulence and multiphase fluid dynamics, such as the dynamics of air–sea interface where the fluid motions are continuous but continuous to a finite order of differentiations. Visually, a forcing with jumps in greater-than-one th order of its time derivatives is very smooth. The inertia of a linear oscillator including the Earth’s polar motion is more sensitive than human eyes to those invisible jumps as manifested by instantaneous excitation of the eigen-modes.

Jumps in time derivatives of the applied forcing are mathematical abstraction of limited timescales a macro-physical system can resolve under the influence of microscopic fluctuations. The Earth’s polar motional response to the applied forcing experiences two levels of microscopic fluctuations: the sub-macroscopic astatic elastic deformation that can stagnate the polar motion; and the sub-sub-macroscopic molecular fluctuation that affect the creeping of elastic deformation. We use the continuum theory of elasticity to estimate the sub-macroscopic transition time that limits the physical smoothness of the applied forcing in the Earth’s polar motional response.

In contrast to IEEM, the eigen-mode can also be produced continuously by a forcing \(f(t)\) of local support (i.e., \(f(t) \neq 0\) only in a finite domain). Macroscopic timescales are crucial under both regimes of IEEM and continuous excitation for boosting robust eigen-signals. The condition for significant eigen-signal is inline with an educated guess, namely, the macroscopic timescales of variation of the applied forcing should be shorter than the oscillator’s eigen-period. As a scaling factor, the macroscopic timescale also plays an important role in numerical solutions in terms of controlling the spurious eigen-signals. Demonstrations of this fact are given in numerical case studies comparable to the Earth’s polar motion.

Successive excitations of eigen-mode incur phase interference and may not result in a distinctive pattern of eigen-oscillation. For practical purpose, we give a brief account of how a recognizable pattern of eigen-oscillation emerges under a continuous applied forcing with a very short timescale and high degree of randomness.

The paper is organized as follows. In section 2, we used textbook examples to highlight the differential jumps in high-order acceleration. The 1D linear oscillator and the polar motion problems are united in Section 3. The next two sections are devoted to the sufficient and necessary condition for IEEM. The physical smoothness of the forcing is discussed in Section 6. We examine in Section 7 the influence of timescales on the robustness of eigen-signals excited by both the continuous excitation and IEEM. Section 8 is taken for the pattern recognition problem. Finally, concluding remarks are in Section 9.

Throughout this paper, we use the word “forcing” as a combined term of the applied force and torque. It may also serve as a reminder that forces and torques in natural sciences are often difficult to specify, and have to be expressed indirectly through conservation laws or other relations.

2. Smoothness in High-Order Acceleration of a 1D Oscillator

Consider the displacement \(x\) of a 1D linear oscillator with a unit mass, starting from rest at \(t = 0\) driven by the force \(f(t) = \sin \omega t\)

\[
\frac{d^2x}{dt^2} + \sigma^2 x = H(t) \sin \omega t, \quad x(0) = \frac{dx(0)}{dt} = 0
\] (1)

where the eigen-frequency \(\sigma\) and the forcing frequency \(\omega\) are both non-zero real numbers, \(\sigma \neq \omega\), and \(H(t)\) the Heaviside step function. By a standard textbook procedure, the general solution of (1) is a sum of a homogeneous eigen-solution with two integral constants \(A, B\), and a particular forced solution, that is, the “steady-state” solution as called in many textbooks (e.g., ref. [8])

\[
x(t) = (A \cos \sigma t + B \sin \sigma t) + H(t) \frac{\sin \omega t}{(\sigma^2 - \omega^2)}
\] (2)

Using the initial conditions in (1) to determine the integral constants we complete the solution

\[
x(t) = -H(t) \frac{\omega \sin \sigma t}{\sigma(\sigma^2 - \omega^2)} + H(t) \frac{\sin \omega t}{(\sigma^2 - \omega^2)}
\] (3)

This procedure may mislead beginners by giving the impression that the IEEM term \(\sin \sigma t\) is due to the initial condition rather than to the forcing. In fact, the initial condition does relate to the forcing (see Section 3).

It is straightforward to find from (3)

\[
x(0) = \frac{dx(0)}{dt} = \frac{d^2x(0)}{dt^2} = 0, \quad \frac{d^3x(0)}{dt^3} = \omega \neq 0
\] (4)

That the third-order acceleration is discontinuous at the onset \(t = 0\) is because the derivative of the applied forcing is discontinuous at \(t = 0\). Since the forcing in (1) is sinusoidal, the discontinuity at \(t = 0\) may occur in accelerations of 5th, 7th ... order. We shall prove below that each of the discontinuities excites the same IEEM but with different coefficients.
Before getting into general analysis, let us pursue a little further by intuition. To have acceleration smooth to higher order, one shall set a smoother forcing accordingly. For instance, the forcing \( f(t) = H(t) \sin^2 \omega t \) is continuous and has continuous first-order derivative \( f'(0) = 0 \), but not the second-order \( f''(0) \neq 0 \). Let us look at another onset problem

\[
\frac{d^2x}{dt^2} + \sigma^2 x = H(t) \sin^2 \omega t, \quad x(0) = \frac{dx(0)}{dt} = 0
\]

Its solution becomes

\[
x(t) = H(t) \frac{2\omega^2 \cos \omega t - 1}{\sigma^2 \omega^2 - 4\omega^4} + H(t) \frac{\sin^2 \omega t}{\sigma^2 - 4\omega^2}
\]

Indeed, we have smooth accelerations up to the third order

\[
x(0) = \frac{dx(0)}{dt} = \frac{d^2x(0)}{dt^2} = \frac{d^3x(0)}{dt^3} = 0, \quad \frac{d^4x(0)}{dt^4} = 2\omega^2 \neq 0
\]

It is easy to show by repeatedly differentiating the equation \( \ddot{x} + \sigma^2 x = H(t)f(t) \) that for a general forcing \( f(t) \) continuous in \( n \)th order derivatives, \( f^{(n)}(0) = 0 \), until \( n = N \), the smoothness of the oscillator’s acceleration is up to \( (N+2) \)th order. For a specific forcing, we can use ramp functions to make it have continuous derivatives up to any order \( N \) (known as \( C^N \) function) at the onset. For example

\[
f(t) = H(t) \left(1 - e^{-\omega t}\right)^{N-1} \sin \omega t, \quad a > 0
\]

There have been established smoothing techniques in the theory of distribution (generalized function) to ramp a given forcing function differentiable up to infinite order at any boundaries of compact support (e.g., ref. [9]). A related question we have to address, then, is whether such ramped forcing excites eigen-mode.

### 3.1 Oscillator and 2D Polar Motion Unified

To dig down to the fundamentals of the eigen-mode excitation, we should rid the homogeneous eigen-solution at least for the onset problems. This can be achieved by transforming the second-order oscillator \( \ddot{x} + \sigma^2 x = f(t) \) into a first-order system

\[
\frac{dx}{dt} + \sigma y = 0, \quad \frac{dy}{dt} - \sigma x = -\frac{1}{\sigma} f(t)
\]

Using the complex notation \( m = x + iy \), we obtain

\[
\frac{dm}{dt} - i\sigma m = Ef(t), \quad E = \frac{1}{i\sigma}
\]

We call the variable \( m(t) \) complex velocity or simply the velocity. The solution for the oscillator \( x(t) \) is found in the real part of the complex velocity. If one does not specify the forcing \( f(t) \) as a real valued function, Equation (10) is identical to the linearized governing equation for the Earth’s polar motion (e.g., ref. [2]). Thus, the two problems can be combined at this point. The factor \( 1/(i\sigma) \) on the right hand side of (10) is used for tracking the 1D oscillator problems. Later on, we will use a different coefficient \( E \) for the 2D polar motion problem. Incidentally, by admitting negative eigen-frequency \(-\sigma\), Equation (10) also governs the cyclotron motion of free charged particles in the magnetic field (e.g., ref. [10]), and the main results below do apply to \(-\sigma\).

The general solution of Equation (10) is again a combination of a homogeneous eigen-solution and a particular forced solution.

\[
m(t) = \Lambda e^{i\omega t} + E \int_0^t e^{i\omega(t-\tau)} f(\tau) d\tau
\]

where \( \Lambda \) is the integral constant for the eigen-solution. The integral in (11) can be interpreted in three different ways: i) It is the complex version of the Green function solution for an oscillator (e.g., [11]); ii) the convolution between the eigen-mode and the forcing; iii) the eigen-mode \( e^{i\omega t} \) multiplied by its complex amplitude, which is the Fourier spectrum of the forcing time series at the eigen-frequency \( \sigma \). The Fourier transform of the forcing terminates at the present time \( t \) by causality; that is, the future motion cannot affect that in the past. The third interpretation is especially interesting for our purpose as it hints at the rise of eigen-mode in the solution, whereas it is not always the case at all.

For onset problems starting from rest at \( t = 0 \), we have

\[
m(t) = \Lambda e^{i\omega t} + E \int_0^t e^{i\omega(t-\tau)} f(\tau) d\tau
\]

Notice that the homogeneous initial condition in Equation (1) is equivalent to \( m(0) = 0 \) after transforming into (10). For a bounded Newtonian forcing \( f(t) \) that does not contain the impulsive delta function, the definite integral on the right hand side of (12) tends to zero as \( t \to 0 \). Consequently, \( \Lambda = 0 \), and the homogeneous eigen-solution \( e^{i\omega t} \) is eliminated.

\[
m(t) = E \int_0^t e^{i\omega(t-\tau)} f(\tau) d\tau
\]

This forced particular solution relates the eigen-mode, if exits, directly to the forcing \( f(t) \). For a quick check of its correctness, we take the integrand \( f(t) = \sin \omega t \) and \( E = 1/i\sigma \) in (13) to find the identical solution as (3):

\[
x(t) = \text{Re} \left[ \frac{1}{i\sigma} \int_0^t e^{i\omega(t-\tau)} \sin \omega \tau d\tau \right]
\]

\[
= -H(t) \frac{\omega \sin \sigma t}{\sigma(\sigma^2 - \omega^2)} + H(t) \frac{\sin \omega t}{(\sigma^2 - \omega^2)}
\]

It is evident from Equation (14) that the forced particular solution can excite the IEEM directly. This revelation is especially important for the study of polar motion where there have long been efforts in search of mechanisms of transporting energy from the forced particular solution to the independent eigen-mode \( e^{i\omega t} \). We now see that the independent eigen-mode \( e^{i\omega t} \) is not needed at all for the rise of the eigen-mode.

To compare the solution (14) with the one that extends the onset of motion back to the infinite history \( t \to \infty \), we should let the memory for the infinite history fade away. This can be done by setting a small damping to the eigen-frequency:

\[
\sigma \leftrightarrow \sigma \left(1 + \frac{i}{2Q}\right), \quad Q \gg 1
\]
The solution now contains no eigen-mode
\[
x(t) = \text{Re} \left[ \frac{1}{\omega^2} \int_{-\infty}^{t} e^{i\omega(t-\tau)} \sin \omega \tau d\tau \right] = \frac{\sin \omega t}{(\sigma^2 - \omega^2)}
\] (16)

A comparison between the solutions (14) and (16) suggests that the eigen-mode is excited by some form of interruption in the forcing, even though the forcing is continuous. We shall prove next that the IEEM will rise when the forcing is smooth to any finite order differentiation.

4. The Sufficient and Necessary Condition for IEEM

We assume the applied forcing \( H(t) \) differentiable to any order in the time domain \((0, t)\). For an easy confirmation of differential invariance, we differentiate the governing Equation (10) to the \( n \)th order to have the equation for the \( n \)th order velocity \( m^{(n)}(t) \), that is, the \((n - 1)\)th order acceleration

\[
\frac{dm^{(n)}(t)}{dt} - i\sigma m^{(n)}(t) = E H(t) f^{(n)}(t)
\] (17)

Solution of this equation under the homogeneous initial condition \( m^{(n)}(0) = 0 \) is

\[
m^{(n)}(t) = E e^{i\sigma t} \int_{0}^{t} e^{-i\sigma \tau} f^{(n)}(\tau) d\tau
\] (18)

To expose the origin of IEEM, let us perform integration by parts successively to \((N - 1)\)th order on solution (13)

\[
m(t) = E \sum_{n=0}^{N-1} \frac{e^{i\sigma t} f^{(n)}(0) - f^{(n)}(t)}{(i\sigma)^{n+1}} +
\]

\[
E \frac{1}{(i\sigma)^{N+1}} \int_{0}^{t} e^{-i\sigma \tau} f^{(N)}(\tau) d\tau
\] (19)

There are three important revelations from this expression. First, the IEEM \( \propto e^{i\sigma t} \) arises at each of the differential jumps across the onset \( f^{(n)}(0) \neq 0 \). Second, the integral term in (19) is in essence the solution for the \( N\)th-order acceleration \( m^{(N)}(t) \) with the homogeneous initial condition \( m^{(N)}(0) = 0 \) (see Equation (18)). This leads to the third revelation: If there are more IEEM terms in the form of \( B_n e^{i\sigma t} \) other than already excited by differential jumps \( f^{(n)}(0) \neq 0 \), they must be contained exclusively in the \( N \)th order velocity \( m^{(N)}(t) \). Here, \( B_n \) denotes constants distinguished by the index \( n \) that relates to higher-order derivatives of the applied forcing. The same procedure may continue if the new forcing function \( f^{(N)}(t) \) has further differential jumps at \( t = 0 \).

Conversely, suppose \( N < \infty \) is the highest order of differential jumps in the applied forcing

\[
f^{(N)}(0) \neq 0, \quad f^{(N+1)}(0) = 0, \quad n = 1, 2, \ldots
\] (20)

We claim that there are no IEEM terms, \( B_n e^{i\sigma t} \), in the solution of \( m^{(N+1)}(t) \) under the homogeneous initial condition \( m^{(N+1)}(0) = 0 \). In other words, all of the IEEM terms are excited by the jumps \( f^{(n)}(0) \neq 0, \quad n \leq N \).

To see this is true, we deduce, based on the prescribed initial condition \( m^{(N+1)}(0) = 0 \) together with the relation (20) and Equation (17) that

\[
m^{(N+1)}(0) = 0, \quad n = 1, 2, \ldots
\] (21)

Suppose the solution of \( m^{(N+1)}(t) \) contains an IEEM term \( B_{N+1} e^{i\sigma t} \), we can write

\[
m^{(N+1)}(t) = B_{N+1} e^{i\sigma t} + q(t)
\] (22)

where \( q(t) \) is a function to be determined. Differentiating Equation (22) successively and taking the values at \( t = 0 \), we have from (21) for \( n = 1, 2, \ldots \)

\[
m^{(N+1)}(0) = (i\sigma)^{n-1} B_{N+1} + \frac{d^{n-1} q(0)}{dt^{n-1}} = 0
\] (23)

It follows by the Taylor series that

\[
q(t) = -B_{N+1} \left( 1 + \frac{i\sigma t}{1!} + \frac{(i\sigma t)^2}{2!} + \cdots \right) = -B_{N+1} e^{i\sigma t}
\] (24)

Thus, the function \( q(t) \) has to contain another term \( -B_{N+1} e^{i\sigma t} \) to cancel out the IEEM all together. Two extremes are worth noting. When \( N \to \infty \), the IEEM will arise in solutions of every order velocity \( m^{(n)}(t), \quad n = 1, 2, \ldots \); when \( N = 0 \), the forcing \( H(t) f(t) \) is a smooth \( C^\infty \) and no IEEM can exist.

We analyze the onset behavior to escape from the initial condition. The essential of the excitation mechanism is the presence of differential jump in any order derivative of the applied Newtonian forcing, regardless of what had happened before and will happen after the jump. Therefore, we have established the following sufficient and necessary condition for the rise of IEEM. For an oscillator governed by Equation (10) with an arbitrary applied forcing \( f(t) \), the eigen-mode is excited instantaneously at those points in the time domain \((t \to \infty, \tau)\) where discontinuity occurs in any order differentiation of the applied Newtonian forcing, or equivalently in any order acceleration of the displacement.

Notice, the sufficient and necessary condition applies to accelerations of negative order, namely, the velocity and displacement. Without prescribed initial conditions, the IEEM generated by discontinuities in velocity requires delta impulses in the applied forcing, and the IEEM generated by discontinuities in displacement requires singular derivatives of delta impulses in the applied forcing. Furthermore, the sufficient and necessary condition also applies to resonant IEEM, although there is no need for integration by parts in (19) to deal with the resonance. The onset of resonant forcing \( f(t) \propto H(t) e^{i\sigma t} \) can never be \( C^\infty \).

5. Examples

To confirm the theory and to expose the process of IEEM, we present three examples where the maximum order of differential jumps is \( N = \) finite, \( N \to \infty \), and \( N = 0 \), respectively. For simplicity, we continue to proceed with Heaviside type of onset problems as proxies of differential jumps in general forcing functions. Throughout this section, we choose the coefficient \( E = 1/i\sigma \).
5.1. Differential Jumps to Finite Order $N = \text{Finite}$

Consider the forcing $f(t) = H(t)\sigma N^t$. This function has continuous derivatives at $t = 0$ for all orders except at the order $N$:

$$f^{(n)}(0) = 0, \quad (n = 0, 1 \cdots N - 1, N + 1 \cdots), \quad f^{(N)}(0) = N^t$$  \hspace{1em} (25)

According to the sufficient and necessary condition, we shall expect to have only one IEEM excited by the only jump $f^{(N)}(0) = N^t$. The equation for the $N$th order velocity is

$$\frac{d^m(N)}{dt} - \sigma m(N)(t) = \frac{1}{i\sigma} H(t)N^t, \quad m(N)(0) = 0$$  \hspace{1em} (26)

Since $f^{(N+1)}(0) = 0, \quad n = 1, 2 \cdots$, the solution by integration by parts is simply

$$m(N)(t) = H(t)\frac{N^t}{(i\sigma)^{N+1}}[e^{i\sigma t} - 1]$$  \hspace{1em} (27)

The first and the only IEEM is thus excited by the $N$th order jump $f^{(N)}(0) = N^t$. One can perform definite integrations from 0 to $t$ for $N$ times to arrive at the final solution

$$m(t) = H(t)\frac{N^t}{(i\sigma)^{N+1}}e^{i\sigma t} - H(t)\sum_{n=0}^{N-1} \frac{N(N-1) \cdots (n+1)}{(i\sigma)^{N+1-n}} t^n$$  \hspace{1em} (28)

5.2. Differential Jumps to Infinite Order $N = \infty$

The next example is the familiar problem $f(t) = H(t)\sin \omega t$. This function has successive jumps at the onset $t = 0$ in every other order of derivatives

$$f^{(2n)}(0) = 0, \quad f^{(2n+1)}(0) = (-1)^n \omega^{2n+1}, \quad n = 0, 1 \cdots$$  \hspace{1em} (29)

According to the sufficient and necessary condition, we shall expect to have the first eigen-mode excited by the first jump $f^{(1)}(0) = \omega$, and more IEEM at each one of the subsequent jumps at order $2n + 1$ till $n$ goes to infinity.

We start from the equation for the first derivative, $n = 1$, where differential jumps in the applied Newtonian forcing occurs for the first time

$$\frac{d^{(1)}(t)}{dt} - i\sigma m^{(1)}(t) = H(t)\frac{\omega}{i\sigma} \cos \omega t, \quad m^{(1)}(0) = 0$$  \hspace{1em} (30)

Taking integration by parts on its solution (18), we have

$$m^{(1)}(t) = \frac{\omega}{i\sigma} e^{i\sigma t} \int_0^t e^{-i\sigma t} \cos \omega \tau \, d\tau -$$

$$= \frac{\omega}{(i\sigma)^2} (e^{i\sigma t} - \cos \omega t) -$$

$$= \frac{\omega^2}{(i\sigma)^3} \left( e^{i\sigma t} - 2\cos \omega t + \sin \omega \tau \right)$$  \hspace{1em} (31)

The first eigen-term $e^{i\sigma t}$ is excited by the first jump, $f^{(1)}(0) = \omega$. Take integration by parts once more on the integral term in (31) to obtain

$$m^{(1)}(t) = \frac{\omega}{i\sigma} e^{i\sigma t} \int_0^t e^{-i\sigma t} \cos \omega \tau \, d\tau$$

$$= \frac{\omega}{(i\sigma)^2} \left( e^{i\sigma t} - \cos \omega t + \frac{\omega \sin \omega t}{i\sigma} \right)$$  \hspace{1em} (32)

$$= \frac{\omega^2}{(i\sigma)^3} \left( e^{i\sigma t} - 2\cos \omega t + \sin \omega \tau \right)$$

There is no new eigen-term excited by the second round of integration by parts because the forcing $f^{(3)}(t) \sin \omega t$ under the integral sign in (31) has no jump at $t = 0$. The two integrals in (32) are identical, meaning that additional IEEM are excited by higher-order differential jumps repeatedly in exactly the same way as by the first jump; the only difference is in the coefficients at each jump. We can write from Equations (31) and (32)

$$m^{(1)}(t) = \frac{1}{i\sigma} \left( e^{i\sigma t} - \cos \omega t + \frac{\omega \sin \omega t}{i\sigma} \right) \times$$

$$\left[ \frac{\omega}{i\sigma} \left( \frac{\omega}{i\sigma} \right)^3 + \left( \frac{\omega}{i\sigma} \right)^5 + \cdots \right]$$  \hspace{1em} (33)

The first power term in the square parenthesis is the IEEM excited by the first jump, $f^{(1)}(0) = \omega$, the second power term is the IEEM by the second jump, $f^{(3)}(0) = -\omega^3$, and so on to infinity. Suppose $|\omega| < |\sigma|$, then, after summing up the infinite power series, we have

$$m^{(1)}(t) = -\left( e^{i\sigma t} - \cos \omega t + \frac{\omega \sin \omega t}{i\sigma} \right) \frac{\omega}{\sigma^2 - \omega^2}$$  \hspace{1em} (34)

For $|\omega| > |\sigma|$, we can use Euler’s complex formula for $\sin \omega t$ and perform integration by parts on $\exp(\pm i\omega t)$ instead of on $e^{i\sigma t}$. The result is obtained by flopping the positions of $i\sigma$ and $\omega$ in the power series

$$m^{(1)}(t) = \frac{1}{i\sigma} \left( e^{i\sigma t} - \cos \omega t + \frac{\omega \sin \omega t}{i\sigma} \right) \times$$

$$\left[ \frac{i\sigma}{\omega} \left( \frac{i\sigma}{\omega} \right)^3 + \left( \frac{i\sigma}{\omega} \right)^5 + \cdots \right]$$  \hspace{1em} (35)

After summing up the infinite power series, we arrive at the same Equation (34). The physical solution for the first derivative $x^{(1)}(t)$ is in the real part of (34)

$$x^{(1)}(t) = -H(t)\frac{\omega}{\sigma^2 - \omega^2} \cos \sigma t - \omega \sin \omega t$$  \hspace{1em} (36)

By integration from 0 to $t$ on (36), we recover the desired solution $x(t)$ identical to solution (3)

$$x(t) = -H(t)\frac{\omega}{\sigma^2 - \omega^2} + \sin \omega t$$  \hspace{1em} (37)

Incidentally, the expressions (33) and (35) provide the details of how the inertia of the oscillator, characterized by the
eigen-frequency $\sigma$, drags the forced motion. When $|\omega| < |\sigma|$, the eigen-frequency stays in the denominator in (33) to prevent the infinite number of excitations from blowing up the displacement, whereas for $|\omega| > |\sigma|$, it flips to the numerator in (35) for the same purpose. Only when $|\omega| = |\sigma|$, the inertial drag and the applied forcing get balanced in any order of derivatives and the infinite summation in (33) and (35) blows up. Notice, the free eigen-mode also prevents unbounded increase of amplitude at the resonance frequency. This can be seen by taking the limit $\omega \to \sigma$ in solution (37) to have $\lim_{\omega \to \sigma} x(t) = 0$. The true unbounded resonance occurs only when the free eigen-oscillation damps out. In this case, we have to remove the eigen-mode $e^{i\omega t}$ from (33) and (35) to obtain

$$m^{(1)}(t) = \frac{1}{\sigma} \{ \cos \sigma t + i \sin \sigma t \} [1 + 1 + 1 + \cdots \}$$

(38)

5.3. No Differential Jump $N = 0$

For comparison, we use a ramp function $R(t)$ to smooth the forcing $H(t) \sin \omega t$. A solid example of the ramp often used in mathematics is $R(t) = e^{-1/(at^2)}$ (e.g., ref. [9])

$$f(t) = H(t) e^{-1/(at^2)} \sin \omega t, \quad a > 0$$

(39)

This is a $C^\infty(-\infty, t]$ function with $f^{(n)}(0) = 0$ ($n = 0, 1, \ldots, \infty$). The drawback of this forcing lies in the lack of closed-form solution (13). So, we choose another ramp function smooth to $N$th order

$$f(t) = \left(1 - e^{-t/N}\right)^N H(t) \sin \omega t$$

(40)

The solution under a $C^\infty$ forcing is achieved by taking the limit $N \to \infty$ after solving $m(t)$ for each $N$. Let us decompose the forcing with binomial

$$f(t) = \sum_{n=0}^{N} f_n(t)$$

(41)

$$f_n(t) = (-1)^n \left(\begin{array}{c} N \\ n \end{array}\right) e^{-nt/N} H(t) \sin \omega t$$

The equation of motion and the initial condition under each $f_n(t)$ is

$$\frac{d^2 m_n(t)}{dt^2} - i\sigma m_n(t) = \frac{1}{i\sigma} f_n(t), \quad m_n(0) = 0$$

(42)

The desired solution $m(t)$ is the sum of $m_n(t)$.

The solution for each $m_n(t)$ can be expressed by a sum of an eigen-mode $m_n^e(t)$ and a forced-mode $m_n^f(t)$

$$m_n^e(t) = (-1)^n \left(\begin{array}{c} N \\ n \end{array}\right) e^{nt/N} \left[ \frac{-\omega}{(\sigma - in/N)^2 - \omega^2} \right]$$

$$m_n^f(t) = (-1)^n \left(\begin{array}{c} N \\ n \end{array}\right) e^{nt/N} \left[ \frac{i(\sigma - in/N) \sin \omega t + \omega \cos \omega t}{(\sigma - in/N)^2 - \omega^2} \right]$$

(43)

Taking the summation and the limit $N \to \infty$, we have, after some limiting operations

$$m^e(t) = \lim_{N \to \infty} \frac{-\omega e^{it}}{i(\sigma^2 - \omega^2)} \sum_{n=0}^{N} (-1)^n \left(\begin{array}{c} N \\ n \end{array}\right)$$

$$= \frac{-\omega e^{it}}{i(\sigma^2 - \omega^2)} \lim_{N \to \infty} \left(1 - 1\right)^N = 0$$

(44)

$$m^f(t) = \lim_{N \to \infty} \frac{i(\sigma \sin \omega t + \omega \cos \omega t)}{i(\sigma^2 - \omega^2)} \sum_{n=0}^{N} (-1)^n \left(\begin{array}{c} N \\ n \end{array}\right) e^{-nt/N}$$

$$= \frac{i(\sigma \sin \omega t + \omega \cos \omega t)}{i(\sigma^2 - \omega^2)} \lim_{N \to \infty} \left[ 1 - e^{-t/N} \right]^N$$

(45)

The solution with $N \to \infty$ is simply

$$x(t) = \text{Re} \left\{ m^e(t) + m^f(t) \right\} = H(t) \frac{\sin \omega t}{(\sigma^2 - \omega^2)}$$

(46)

Comparing this solution with (3) and (16), we see that a $C^\infty$ ramp of infinitesimal timescale near the onset can eliminate the eigen-mode as effective as viscous damping during the entire history.

For a general treatment, we can extend the lower integral limit for a $C^\infty$ forcing in the integral solution (13)

$$\int_0^t e^{-i\sigma \tau} R(t) f(t) d\tau = \int_0^t e^{-i\sigma \tau} H(t) R(t) f(t) d\tau$$

(47)

$$= \int_0^t e^{-i\sigma \tau} R(t) f(t) dt$$

The far right of Equation (47) is an indefinite integral. The designation from the definite integral to an indefinite integral in (47) reflects an easy-to-prove mathematical property as follows:

If the integrant $f(t)$ is a $C^\infty$ function with $\frac{d^nf(t)}{dt^n} = 0$, $n = 0, 1, \ldots, \infty$, for $t \leq t_0$, then, the definite integration is independent of the lower integral limit, provided that the lower limit $\leq t_0$. This independence physically eliminates the excitation of eigen-mode at the onset. If the $C^\infty$ ramp function $R(t)$ rapidly approaches to a Heaviside step function, for example, the ramp in (39), the indefinite integral can be written

$$\int_0^t e^{-i\sigma \tau} R(t) f(t) dt \approx \int_0^t e^{-i\sigma \tau} f(t) dt$$

(48)

Substitution of $f(t) = \sin \omega t$ in (48) yields the solution (46). Notice, for a general initial value problem where $m(t_0) \neq 0$, the $C^\infty$ ramp only prevents the IEEM at the initial time $t_0$; it cannot affect pre-existing eigen-motions.

6. Physical Smoothness of the Applied Forcing: A Case Study with the Earth’s Polar Motion

Forcing functions in natural sciences, such as the Earth’s polar motion, are obtained based on a combined effort of observation

[Image 48x737 to 137x758]
6.1. Sub-Macroscopic Transition Time versus Jumps

We have seen in solution (46) and the general treatment (48) that the IEEM at any epoch can be eliminated by $C^\infty$ ramps in however short timescales. This mathematical achievement means little to physical realities, as no macroscopic physical system could react to the applied forcing in unlimited small timescales. Differential jumps in the Newtonian forcing are mathematical abstraction of variations in too short timescales for the system to resolve.

From a statistical physics point of view, a macroscopic physical system undergoes microscopic molecular fluctuations. In response to the applied forcing, the molecules adjust their motions by exchanging energy in transition from one equilibrium state to another, and this process takes time. If its timescale of variation is shorter than the transition time, then the applied Newtonian forcing should be treated as a jump function. Treatise of the transition time at the molecular level has to be based on quantum mechanics or particle dynamics through the Langevin equation or the Fokker–Planck equation (e.g., refs. [18, 19]). Transition times associated with molecular fluctuations are usually negligibly small by macroscopic physical standard, and hence not so often considered in classical mechanics.

For rotational motions, including the polar motion of the planetary Earth, the macroscopic system is the whole solid Earth as a rigid body. The immediate sub-macroscopic impact on the Earth’s rotational response to the applied forcing is the astatic elastic deformation in the Earth’s interior, which stagnates the rotational response to the applied forcing by absorbing the rotational energy into astatic vibrations. Molecular fluctuations affect the creeping of local elastic deformation at the sub-sub-macroscopic level. To put the sub-macroscopic and sub-sub-macroscopic levels in perspective, we note that the Earth mass is $\approx 6 \times 10^{24}$ kg. By taking 10 kg for each sub-macroscopic parcel, the Earth has $\approx 10^{23}$ polar motional sub-macroscopic parcels. The sub-sub-macroscopic molecular particles for each sub-macroscopic parcel is in the order of magnitude of the Avogadro’s number, which is also $\approx 10^{23}$. Thus, the sub-macroscopic transition time for the Earth’s polar motional response to the applied forcing can be estimated based on the continuum theory of elasticity. Note, since the restoring force of the equatorial bulge plays a key role in polar motion, static elastic deformation induced by polar motion also affects the Earth's polar motion itself by reducing the equatorial bulge. This equilibrium static elastic deformation is reached only after the transition time and, hence, should be included in the regular rotational response.

The Earth undergoes background vibrations known as the “hum”.[7] For a perfect elastic medium of finite volume, those elastic waves eventually settle into repeated pattern of oscillations by reflections and refractions. If the variation of the applied forcing $f(t)$ that drives the polar motion is slow enough not to perturb the pattern of elastic oscillations, the forcing $f(t)$ is resolvable in the Earth’s polar motional response. Conversely, if the forcing’s variation is too rapid within a time window as to be able to disturb the pattern of elastic oscillations, it can be treated as a jump. By this reasoning, the transition time of elastic adjustment by the sub-macroscopic material parcels to the Earth’s polar motion should be comparable to a round trip travel time of the seismic waves through the diameter of the Earth $d_E = 12742$ km (one cycle of the repeated pattern). The speed of seismic waves is $v \approx 8$ km sec$^{-1}$, and the transition time is $\approx 2d_E/v = 53$ min. This value is close to the period of the grand normal mode of the Earth’s astatic elastic free oscillation of $54$ min. (e.g., ref. [20]). Physically, a significant change in the applied forcing within the time window of $54$ min is capable of exciting the grand normal mode. The physical mechanism is similar to what we will consider in Section 7.

We can assert at this point that significant changes in the applied Newtonian forcing $f(t)$ within a time window of $50$ min can be treated on average as a jump in $f(t)$ for the Earth’s polar motion. The differential invariance of the linear oscillator equation (Equation (17)) allows us to assert further that significant changes within the time window of $50$ min in the $n$th order derivative of the applied forcing $f^{(n)}(t)$ can be treated on average as the $n$th order differential jump in $f^{(n)}(t)$ for the Earth’s polar motion.

The $54$ min transition time window for effective differential jumps in the applied forcing $f^{(n)}(t)$, ($n = 0, 1, 2, \ldots$) suggests that the polar motional eigen-mode, that is, the Chandler Wobble can be excited virtually by all kinds of fluid motions. It has been established that short-period oceanic waves can induce vibrations in the solid Earth ranging from low-frequency tremors[21] to detectable real earthquakes.[22] These waves are certainly capable of exciting the Chandler Wobble. Practically, however, those short-period localized waves are most likely averaged out in global integrations unless they are sizable storms. The main source for the eigen-mode excitation in the Earth’s polar motion case are large scale fluid motions in the atmosphere and ocean. Averaged time variations of those large scale motions maybe slow enough to be treated as continuous, but their time derivatives may not necessarily be due to their turbulent nature (e.g., refs. [23, 24]).

6.2. Physical Smoothness for Turbulent Flows on Earth

Globally, the turbulent flows in the atmospheric boundary layer and the top mixing-layer of the ocean are inhomogeneous because of the longitudinal (diurnal) and latitudinal differences in the driving power of solar radiation.[24] For this reason, regional turbulence eddies cannot sum to zero by global integration in formation of the polar motional forcing $f(t)$ (refs. [14, 25] for detail).

Time derivatives for turbulent flows are exceedingly difficult to estimate. The so-called observed time derivatives of turbulent transport quantities are actually inferences based on G. I. Taylor’s hypothesis.[24,26] The essence of Taylor’s hypothesis is to convert time derivatives into manageable spatial derivatives by assuming that the turbulent patterns are carried downstream by a laminar main flow. The situation is getting worse for estimating the time
derivatives of the applied forcing $f^n(t)$ for the Earth’s polar motion, as there is no spatial variable in the forcing function $f(t)$.

For a qualitative illustration of finite smoothness of the turbulent flows, we consider the diffusion of an effluent puff in a locally homogeneous turbulent atmospheric flow, known as the Batchelor problem ([24,27]). The motion of a single puff is hardly strong enough to drive the Earth’s polar motion. Nonetheless, its qualitatively manageable evolution may shed some light on the temporal variability of synoptic turbulent flows. Let $D(t)$ be the approximate diameter of an effluent puff with $L \ll D \ll \eta$, where $L$ is the size of the turbulent carrying air mass and $\eta$ the size of molecular diffusion eddies.

By recognizing the puff as an energy-containing turbulent eddy at the basic level of its energy cascade, Batchelor ([27]) is able to employ Kolmogorov's [28] scaling law to derive the rate $D(t)$ and to show that away from the initial state the evolution of the puff can be expressed as $D(t) \approx \epsilon^{1/4} t^{1/2}$, where $\epsilon$ is the turbulent energy dissipation rate that can be considered constant. A qualitative estimate of the time derivatives can be written $D^{(n)}(t) \propto t^{(5-2n)/2}$. Suppose the puff is dissipated in 1 h duration from the epoch $t = 1$ h to $t = 2$ h, which can be considered continuous according to the 54 min transition time window for the Earth’s polar motion. The normalized derivatives are plotted in Figure 1, where we see clearly that significant reduction ($\approx 1/e$) in the 5th order derivative $D^{(5)}(t)$ (redline) takes place within 0.25 h (15 min). By the 54 min transition time window, this $D^{(5)}(t)$ can be regarded as having a jump at the epoch $t = 1$ h. The physical smoothness of this simple model is thus limited to $C^4$.

### 6.3. Physical Smoothness for Interface Dynamics on Earth

The qualitative $C^4$ physical smoothness is likely an overestimate for the realistic physical smoothness of the ocean-atmosphere forcing on the Earth’s polar motion, as it has not taken into account the interface dynamics. Consider rainfall as a simplest example of interface dynamics. Rainfalls increase the Earth’s moment of inertia by adding mass to the surface. Suppose the raindrops start hitting the ground at $t = 0$. Let $M(t)$ be the mass accumulation per unit area on the surface, $u$ the constant speed of raindrops, and $\rho(t)$ the raindrop number per unit volume near the surface, then suppose the mass of each raindrop is unit

$$M(t) = \int_0^t u \rho(t) dt$$  \hspace{1cm} (49)

Variation of the number density $\rho(t)$ looks similar to a diffusion that starts from $\rho(0) = 0$ and quickly saturates to a constant (Figure 2), but the Fick’s law does not apply to rainfall, since the raindrops do not collide with each other to bounce into other directions. To a good approximation, the onset increase of $\rho(t)$ is linear in time: $\rho(t) = H(t) Kt$, where $K$ is a proper constant. Now we have

$$M(0) = 0, \quad \frac{dM(0)}{dt} = 0, \quad \frac{d^2M(0)}{dt^2} = uK \neq 0$$ \hspace{1cm} (50)

This is a $C^1$ function. Pressure variations at the air–sea interface are more complicated than rainfalls, but the breach of smoothness at the second-order $C^2$ can be seen by the fact that stresses from winds, including the relative motion during propagation of tidal ocean waves, break the continuous water mass into pieces to crash the ocean surface. This process takes place continuously in time with the tidal motions.

### 6.4. A Numerical Demonstration

A function $f(t)$ with jumps in greater-than-one-th order derivatives is quite smooth in visual appearance. It is remarkable that a linear oscillator can expose those high-order invisible jumps by exciting the IEEM. Here, we present a numerical example by taking $E = i\Omega$, and calculate comparatively in Figure 3 $C^0$ and a $C^2$ IEEM. The physical setting is artificial but consistent in order of magnitude to the real situation of the Earth’s polar motion. The governing equation is

$$\frac{dm}{dt} - i\sigma m = i\Omega f(t)$$ \hspace{1cm} (51)
where $\Omega$ is the frequency of 1-day period. The forcing comprises two sinusoidal wave trains at the frequency $\omega$ of 365.25-day (1 year) period. The first wave train is a $C^\infty$ function extending back to the infinite history $t = -\infty$, and the second superimposes on the first at the epoch $t_0$ with the smoothness of $C^0$ and $C^2$, respectively (Figure 3a).

\[
\begin{align*}
C^0 : f(t) & = W \sin \omega t + H(t - t_0) W_1 \sin \omega(t - t_0) \\
C^2 : f(t) & = W' \sin \omega t + H(t - t_0) W'_1 \sin^3 \omega(t - t_0)
\end{align*}
\]  

(52)

The arbitrary constants are set $W, W', W_1, W'_1 \approx 10^{-9}$, a number in line with the polar motional perturbation induced by the atmospheric angular momentum variation. The eigen-frequency is taken of a 433-day (1.2 year) period. The first wave train is a steady sinusoidal wave train at the forcing frequency because the damping is compensated by the applied forcing. Addition of the second wave train at $t_0$ breaks the $C^\infty$ smoothness of the forcing down to $C^0$ and $C^2$. The IEEM arises at $t_0$ as evidenced in the near 6-year beat between the eigen-frequency $Re(\sigma)$ and the forcing frequency $\omega$. Damping as exhibited in Figure 1b after $t_0$ is due to the free eigen-mode alone $\propto \exp[i(1 + i/2Q)]$. A blowup plot at the vicinity of $t_0$ to visualize the smoothness of the forcing (the $C^2$ curve in the blowup rotates $45^\circ$ for better display). The bumpy deflection of the $C^0$ point at $t_0$ is clearly seen in the blowup, whereas the transition of the $C^2$ forcing through the $t_0$ point is very smooth by visual inspection. Though the rise of the IEEM is instant at $t_0$ (Figure 3c), the total solution $m(t)$ is, as a general rule, one order smoother than the corresponding forcing. This means that the restoring forcing $i\omega m$ is, as a general rule, one order smoother than the applied Newtonian forcing $f(t)$.

Some people have difficulty accepting IEEM under the perception that a jumpy IEEM is unrealistic to achieve on oversized oscillators such as the Earth. This notion confuses IEEM with the instantaneous jump in the complete solution. The truth is on the contrary: The IEEM arises by the inertia to avoid instantaneous jump in the complete solution. This point is well demonstrated in Figure 3, where we can see that the complete solutions are continuous despite the jumpy IEEM shown in Figure 3c.

7. Macroscopic Timescale in Continuous Excitation and IEEM

We have seen the sub-macroscopic transition timescales that can be approximated as differential jumps. Here, we consider another timescale at the macroscopic level that is crucial for producing robust eigen-signals. In connection with the macroscopic timescale, let us first examine the so-called continuous excitation by a $C^\infty$ forcing.

7.1. Continuous Excitation

Suppose the forcing $f(t)$ is a $C^\infty$ function with a local support $(0, T)$

\[
f(t) \neq 0, \quad t \in (0, T); \quad f(t) = 0, \quad t \not\in (0, T)
\]  

(53)

We know now that no IEEM can be excited by a $C^\infty$ function. However, the solution (13) can be written

\[
m(t) = E e^{i\sigma(t-T)} \left[ e^{i\omega T} \int_0^T e^{-i\sigma t} f(t) dt \right], \quad t > T
\]  

(54)

For $t > T$, the integral in Equation (54) stays constant and the solution $m(t)$ becomes a pure eigen-mode $\propto \exp[i(\sigma(t - T)]$. We interpret it as being exited continuously during the time interval $(0, T)$. This mathematical identity has no guarantee of robust eigen-signal, as the integral term in (54) could be exceedingly small.

Denote by $F(\sigma, T)$ the Fourier spectrum of the applied Newtonian forcing $f(t)$ up to the time $T$. As mentioned earlier, the integral part of the solution (54) is just the Fourier transform $F(\sigma, T)$ of the applied forcing at the eigen-frequency. Thus, the sufficient
and necessary condition for producing significant eigen-signal by a $C^\infty$ applied forcing of local support (53) is to have significant power in the Fourier spectrum $F(\sigma, T)$. To be more specific, we consider the $C^\infty$ tapering or apodization functions of timescale $T$. Apparently, this timescale $T$ is at the macroscopic level where the sub-macroscopic structures are aggregated as a rigid unit.

Tapering functions are normally used in signal processing to ramp the abrupt termination of time series (e.g., ref. [29]). Here, we take them as the building blocks of aperiodic $C^\infty$ forcing in the semi-infinite time domain ($-\infty, t$). One can approximate an aperiodic $C^\infty$ function by a set of various tapering functions with the aid of $C^\infty$ ramps to connect them. The impulsive delta function can also be realized by the limit of a tapering function with shrinking time scale $T$ and increasing spikiness at the same time. Roughly, the Fourier transform $F(\omega, T)$ of a tapering function $f(t)$ is also a tapering function centered at the zero frequency $\omega = 0$. The bandwidth $\Xi$ of effective frequency of $F(\omega, T)$ follows the Heisenberg–Gabor limit to be the reciprocal of the timescale: $\Xi \approx 1 / 2T$ (e.g., ref. [10]). Thus, the only way to increase the power in $F(\sigma, T)$ is to broaden the bandwidth $\Xi$ by shrinking the timescale of variation $T$.

As an example, consider the Gaussian forcing

$$f(t) = \frac{W}{\nu \sqrt{2\pi}} \exp \left( -\frac{t^2}{2\nu^2} \right)$$

(55)

This is a tapering function in the time domain ($-\infty, \infty$). The standard deviation $\nu$ characterizes the time scale of variation $T$ by the relation $\nu = T / 6$ (e.g., Figure 4a). Although it does not have a local support, the function converges to zero exponentially for $|t| > T$. Similar to calculations in Figure 3, we choose the amplitude $W \approx 10^{-3}$, the eigen-frequency $\sigma$ of the 433-day period, with no damping. The solution of Equation (51) under the Gaussian forcing is

$$m(t) = i\Omega \frac{W}{\nu \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma \tau} \exp \left( -\frac{\tau^2}{2\nu^2} \right) d\tau$$

(56)

For sufficiently large time $t \gg 6\nu = T$, we can use the approximation

$$m(t) \approx i\Omega \frac{W}{\nu \sqrt{2\pi}} e^{i\sigma t} \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2\nu^2}} d\tau = i\Omega W \exp \left( -\frac{\sigma^2 \nu^2}{2} \right) e^{i\sigma t}$$

(57)

The last result is a pure eigen-mode with its amplitude critically dependent on the Fourier frequency bandwidth $1/\nu$. For $\sigma^2 \nu^2 \gg 1$, the eigen-signal is practically zero at time $t \gg T$. The condition for significant eigen-signal is $\sigma^2 \nu^2 \leq 1$. Writing with the eigen-period, $\sigma = 2\pi / T^\nu$, and the time scale, $T$, the condition becomes $T^\nu = \frac{1}{\sigma} \leq \frac{T}{6}$, or $T \leq T^\nu$

(58)

Numerical calculation of solution (52) is presented in Figure 4 for standard deviations $\nu$ ranging from 0.4 year to 0.8 year. Significant eigen-signal already emerges for $\nu = 0.4$ y, that is $T \approx 2T^\nu$, while there is no visible eigen-signal when $\nu$ is only twice longer, $\nu = 0.8$ y.

The condition $T \leq T^\nu$ for robust eigen-signal stands as a general rule applicable to any $C^\infty$ tapering forcing functions. This is because of the generality of the Heisenberg–Gabor limit. On physical grounds, the eigen-period $T^\nu$ characterizes the timescale of inertial response to an applied forcing. When $T < T^\nu$, the injection of kinetic energy by the applied forcing is faster than absorption by the inertial restoring force, and the sluggish inertial restoring force could not catch up with variations of the applied forcing. After the applied forcing is gone, the inertia maintains the acquired energy by moving under its own inertial restoring force $iam$. Conversely, When $T > T^\nu$, the injection of kinetic energy is relatively slow, and there is enough time for the inertia to adjust its motion in response to variations in the applied forcing. As a result, there is a weak eigen-signal left after the applied forcing vanishes.

The maximum amplitude for the eigen-mode is reached at the limit $\nu \rightarrow 0$. The forcing (55) at the limit is an impulsive delta function, and the continuously excited eigen-mode becomes an IEEM. Put it in another way, the IEEM is just an extreme case of the continuously excited eigen-mode with the time scale $T \rightarrow 0$. Figure 4. Continuous excitation of the eigen-mode by the Gaussian forcing (red) with various timescales of forcing variation. Delayed response before the peak of the forcing is clearly seen in the plots. Eigen signals appear after the forcing effectively reduces to zero. The robustness of the eigen-signals depends on the timescale $T = 6\nu$. $T = \frac{1}{\sigma}$ or $T = \frac{1}{\nu \sqrt{2\pi}}$, or $T \leq T^\nu$ (58)
7.2. Macroscopic Timescales for Robust IEEM

The macroscopic timescale of variation also affects the amplitude of IEEM in a similar way as it determines the amplitude of continuous excitation. A good analogy can be drawn from daily experience: A walking man holding a glass of water can never prevent triggering the IEEM oscillation of the water in the cup, but he could control the IEEM’s amplitude by walking slowly. A similar phenomenon with a linear oscillator can be quantified by scaling the applied forcing \( f(t) \rightarrow f(t/T) \). Substituting the scaled forcing into solution (19), we can write the total of the IEEM part of the solution \( m^E(t) \)

\[
m^E(t) = \mathcal{E} e^{\sigma t} \left[ f(0) + \frac{f^{(1)}(0)}{T^2 (i\sigma)} + \frac{f^{(2)}(0)}{T^4 (i\sigma)^2} \cdots \right] =
\]

\[E e^{\sigma t} \left[ f(0) + \frac{f^{(1)}(0)}{2\pi i} \left( \frac{T^2}{T} \right) + \frac{f^{(2)}(0)}{(2\pi i)^2} \left( \frac{T^4}{T^4} \right)^2 + \cdots \right]
\]

Mathematically, the instantaneous excitation of the eigen-mode does not depend upon the macroscopic timescale \( T \), but the amplitude of IEEM does, since the macroscopic timescale of variation \( T \) determines the strengths of the differential jumps.

The right-hand side of Equation (59) suggests that the condition \( T \leq T^\sigma \) for robust eigen-signal to appear remains valid for IEEM. For a given scaled forcing \( f(t/T) \) the \( n \)th power of the ratio between the two timescales, \( (T^\sigma/T^n)^n \), \( n = 1, 2, \ldots \), gives the physical insight of how a robust IEEM signal arises in the \( n \)th-order energy injection by the applied forcing is higher than the \( n \)th order rate of energy absorption by the sluggish inertia, and the \( n \)th order derivative of the inertial restoring force could not catch up with the variation of the \( n \)th order derivative of the applied forcing. Conversely, the eigen-mode is diminished when \( T > T^\sigma \) because the \( n \)th order energy injection is slow, and there is enough time for readjustment by the \( n \)th order derivative of the inertial restoring force to the \( n \)th order variation of the applied forcing.

For \( f(0) = 0 \), the amplitude of the IEEM reduces to zero when the timescale \( T \rightarrow \infty \).

As an example, consider the following scaled \( C^1 \) forcing in the governing Equation (51)

\[f(t) = H(t) \left[ 1 - \exp \left( -\frac{t}{T} \right) \right]^2.\]

Starting from \( n = 2 \), this applied forcing has differential jump at \( t = 0 \) in every order derivative, \( f^{(n)}(0) \neq 0 \). The solution of Equation (51) under this forcing is \( m(t) = m^E(t) + m^f(t) \)

\[
m^E(t) = i\Omega e^{\sigma t} \left[ \frac{1}{i\sigma} - \frac{2}{i\sigma^2 + 1/T} + \frac{1}{i\sigma^2 + 2/T} \right]
\]

\[
m^f(t) = -i\Omega \left[ \frac{1}{i\sigma} - \frac{2e^{-i/T}}{i\sigma + 1/T} + \frac{e^{-2i/T}}{i\sigma + 2/T} \right]
\]

One can verify that the coefficient expressed in the square parenthesis of \( m^E(t) \) is a result of summing up the infinite IEEM by Equation (59). Indeed, the amplitude of the eigen-mode \( m^E(t) \) diminishes to zero monotonically as \( T \rightarrow \infty \).

Figure 5. IEEM excitation of the eigen-mode by exponential forcing (red) with different timescales of forcing variation. Though the eigen-mode is excited instantly, significant eigen-signals appear only after the forcing reaches its asymptotic values. The robustness of the eigen-signals depends on the largeness of the ratio of timescales \( T^\sigma/T^n \), where \( T^\sigma = 1.2 \) year.

Depicted in Figure 5 are the numerical time series of the solution (61) for different timescales \( T \). Comparing the eigen-signals in Figures 4 and 5, one can hardly identify which is continuous excitation and which is IEEM. This is because the total signal \( m(t) \) in Figure 5 has to evolve starting from zero. Robust eigen-signals emerge only after the forced signals \( m^f(t) \) reach their asymptotic values. Thus, the IEEM signals in Figure 5 manifest as if they are produced by continuous excitation. The boost of eigen-mode for \( T \leq T^\sigma = 1.2 \) y is clearly seen in Figure 5c,d.

7.3. About the Spurious Eigen-Signals Induced by Numerical Discretization

Aside from characterizing the sluggish inertia of the linear oscillator, the macroscopic timescale \( T^\sigma \) also has an important implication in numerical simulations of the IEEM. In fact, none-dimensionalization in a numerical scheme is equivalent to scaling the time variable, like we did in Equation (59). Discretized forcing \( f(t) \) would induce some spurious IEEM by artificial differential jumps, as no discretization could faithfully restore the
ideal $C^\infty$ smoothness. The strength of the $n$th order artificial differential jump are in general proportional to the power of the integration subinterval $\alpha (\Delta t)^n$. Our result in Figure 5 suggests the use of scaling time $T \geq T^*$ and as smaller $\Delta t$ as possible to control the spurious eigen-signals. Although spurious eigen-signals can never be eliminated in numerical calculations, it can be kept small as indicated in Figure 5a.

Numerical testing cases have already been conducted in Figures 3, 4, and 5, where we chose the scaling time $T = T^* = 433$ days and the subinterval $\Delta t = 1$ day during numerical calculations. It is readily shown in those Figures that only the physical eigen-signals are distinctively on display; spurious eigen-signals are simply invisible. It is even difficult to visualize the difference between the numerical result and full analytical solutions. This is why we skipped these comparisons.

8. Distortion of Eigen-Waveform

As the last point, we digress from the excitation mechanism to consider pattern formation of the eigen-mode under a $C^2$ irregu-ular forcing with short macroscopic timescale of variation. After all, the eigen-modes in many real applications are better identified through visual recognition in graphic displays (e.g., refs. [2, 3, 5]). Intuitively, successive irregular excitations should be respon- sible for time variation of the eigen-waveforms on a linear oscillator. In addition to confirming the intuition, we try to quantify the distortion of the eigen-waveforms under a forcing with a high degree of randomness.

Generally speaking, if the forcing $f(t)$ is strictly periodic in the time domain $(-\infty, t)$ with its period $T \ll T^*$, there will be no distinctive pattern of eigen-oscillation shown in the solution $m(t)$, even though the eigen-mode is excited periodically. The main reason is the regularly repeated phase interferences that kills the pattern of eigen-oscillation before a full eigen-wavelet can be developed.

For an aperiodic forcing in $(-\infty, t)$ with its macroscopic characteristic timescale of variation $T \ll T^*$, there will be a distinctive pattern of eigen-oscillation, provided that i) the irregularity exhibits a high degree of randomness, ii) the duration is longer enough, and iii) the damping is small.

To substantiate the latter claim, we consider a zero-mean noise-like irregular forcing with very short timescale $T \approx 1$ day, as compared to the eigen-period $T^* \approx 433$ days (Figure 6a). We call the forcing an irregular function instead of stochastic process because it is a continuous function smooth to $C^2$ ensured by the technique of spline interpolation. Such a function can serve as a proxy of the de-trend irregular variation of the global atmospheric angular momentum observed by concerted efforts of meteorologists all over the world (e.g., refs. [14, 25]). Globally, the irregular variation of the atmospheric angular momentum is persistent without preferred direction. Thus, the statistical behavior of the spiky forcing model in Figure 6a can be constrained by a Gaussian white noise.

According to the theoretical development in previous sections, each of the spikes shown in Figure 6a excites the IEEM. The phase interferences among the successively excited eigen-mode and their intermingling with the forced-mode constantly change the waveform of the solution.

Figure 6. a) Highly irregular $C^2$ continuous forcing function $f(t)$. The 1-day timescale of variation is so short compared to the 433-day eigen-period that the statistical behavior of the forcing time series is like a Gaussian white noise with a standard deviation $\nu = 0.25/2$. b) The $m_2(t)$ component of the solution with the random-walk like asymptotic behavior is shown by blue line. The green lines mark the range of significant waveform distortion $v^2_m = \nu^2 T^* \Omega/\sqrt{\nu^2}$.

We can formally separate the eigen-oscillation and the wave- form distortion by partitioning the solution of equation (51) at a fixed epoch $\xi$

$$m(t) = i\Omega e^{-i\xi} \int_{-\infty}^{t} e^{-i\nu T^* d\tau} f(\tau) d\tau$$

$$= e^{-i\nu T^* \xi} \left[ i\Omega e^{-i\xi} \int_{-\infty}^{\xi} e^{-i\nu T^* d\tau} f(\tau) d\tau \right]$$

$$+ i\Omega \int_{\xi}^{t} e^{-i\nu T^* d\tau} f(\tau) d\tau$$

The first term on the far right of (62) is a pure eigen-mode starting from the epoch $\xi$. Denote this formal eigen-mode by $m^\nu(t - \xi)$. During the interval of one eigen-period $0 < t - \xi < T^*$, the non-zero second term on the far right of (62) continues to distort the waveform of the eigen-mode $m^\nu(t - \xi)$. At the end of one eigen-cycle $t - \xi = T^*$, a new formal eigen-mode starts from the epoch $T^* + \xi$ and the process of formal-eigenmode-distorsion repeats again and again. This simple reinterpretation of the solution $m(t)$ allows us to estimate the magnitude of distortion by averaging the variance of the second term on the far right of (62) over one eigen-period. Let this quantity be $v^2$, we have

$$v^2 = \frac{1}{T^*} \int_{\xi}^{\xi + T^*} \left( m(t) - m^\nu(t - \xi) \right)^2 dt$$

where the brackets represents the mathematical expectation by treating the irregular forcing as if it is a Gaussian white noise. The variance of the applied forcing $v^2$ can be computed directly...
from the forcing data (Figure 6a) based on the ergodic hypothesis (e.g., ref. [19]). Substitution of (62) into (63) yields

\[
v^2_\lambda = \frac{\Omega^2}{T^2} \int_\xi^{\xi+\tau'} \left[ \int_\xi^{t'} e^{-i(t'-\tau')} \langle f(\tau)f(\tau') \rangle d\tau d\tau' \right] dt
\]

Making use of the white noise covariance in the form of delta function

\[
\langle f(\tau)f(\tau') \rangle = v^2 \delta(\tau - \tau')
\]

we find

\[
v^2_\lambda = \frac{\Omega^2 v^2}{T^2} \int_\xi^{\xi+\tau'} \int_\xi^{t'} dt = \frac{\Omega^2 v^2 T^\nu}{2^\nu} \langle t \rangle
\]

where \( \langle t \rangle \) is the dimensional unit of the time variable. By taking \( \langle t \rangle = T^\nu \), we have \( v_\lambda = v_\Omega T^\nu \sqrt{2/\nu} \). Equation (66) states that the distortion of waveform is statistically at a constant level \( v_\lambda \), independent of the epoch \( \xi \), and hence, independent of any particular eigen-wavelet in the time series \( m(t) \). If the amplitude of the solution \( |m(t)| \) is smaller than \( v_\lambda \), then distortion of the waveform as a distinctive eigen-oscillation will be more severe. The larger the amplitude \( |m(t)| \), the smaller the relative distortion of the eigen-waveform. In other words, the larger the amplitude \( |m(t)| \), the more robust the eigen-signal appears.

As for the amplitude, when the randomness of the forcing function \( f(t) \) is white noise like, the solution \( m(t) \) can be viewed as a Weiner random walk with the standard deviation \( \approx \sqrt{t} \). This means that without damping, the amplitude \( |m(t)| \) will drift to infinity by \( \approx \sqrt{t} \). Thus, for longer enough duration, a random like forcing with \( T < T^\nu \) will produce distinctive pattern of eigen-oscillation in the time series of the forced solution \( m(t) \).

Figure 6b shows the numerical solution \( m(t) \) under the random like forcing in Figure 6a. For effective control of spurious eigen-signals, we choose the eigen-period \( T^\nu = 433 \) days as the scaling time and the integration subinterval \( \Delta t = 0.1 \) day. The trend of amplitude increase \( \approx \sqrt{t} \) is seen with increasingly robust pattern of eigen-oscillation. On closer inspection of Figure 6b, one can notice some apparent changes in the eigen-frequency among those severely distorted eigen-wavelets. This frequency alteration together with the waveform distortion is a diagnostic feature of successive and irregular excitation of the eigen-mode.

9. Concluding Remark

Eigen-mode excitation is a physical problem and should be treated as such. In this regard, the high-order accelerations considered in this paper do have physical meaning. Newton’s second law defines the force according to the dimension of acceleration \( |M| |L| / |S|^2 \). Higher-order accelerations in the dimensions \( |M| |L| / |S|^n \), \( n > 0 \) represent the rate of change in acceleration to any order, or equivalently, the rate of change in force to any order. The any order rate of change in the Newtonian forcing characterizes the way in which the forcing is applied either by controlled process or by natural cause. Newton’s second law did not explicitly address the consequences of finite \( C^N (N < \infty) \) smoothness in application of the forces. Indeed, there are no consequenses worth noting in many mechanical problems. As an example, consider the free-fall \( z(t) \) of a unit point mass starting from rest \( t = 0 \) by the constant gravitational force, \( g \)

\[
z(t) = \frac{1}{2} g t^2, \quad z(0) = z^{(1)}(0) = 0, \quad z^{(2)}(0) = g \neq 0
\]

This expression says that the gravitational force \( g \) is applied instantly at the onset \( t = 0 \). Using a \( C^\infty \) ramp to smooth out the application of \( g \) at \( t = 0 \) makes no difference in subsequent motion of the free-fall. In contrast, a \( C^\infty \) ramp makes a fundamental difference in terms of IEEM for a linear oscillator including the Earth’s polar motion, the cyclotron motion of free charged particles in the magnetic field, and similar problems.

A \( C^N (N \geq 1) \) function is deceivingly smooth by visual inspection. The sluggish inertia of a linear oscillator is more sensitive than human eyes as it can expose the hidden jumps in any order derivatives of the applied forcing by exciting the IEEM. The sensitivity even constitutes the foundation for the familiar phenomenon of unbounded resonance: One can deduce from the example in Section 5.2 that had the inertial sensitivity to differential jumps of the applied forcing limited to a finite order, the amplitude of resonance would have been bounded.

The Earth’s polar motional response to the applied forcing experiences two levels of microscopic fluctuations. The sub-macroscopic astatic elastic deformation that can stagnate the polar motion and the sub-sub-macroscopic molecular fluctuation that affects the creeping of elastic deformation. The sub-macroscopic transition time associated with the astatic elastic deformation is \( \approx 54 \) min. This sub-macroscopic timescale limits the physical smoothness of the applied forcing to a finite order \( C^N \).

The eigen-period of a linear oscillator makes the macroscopic timescale that characterizes the inertia of a linear oscillator. For boosting robust eigen-signals under both excitation regimes of IEEM and continuous excitation, the timescale of variation in the applied forcing has to be shorter than the eigen-period.

Multiple excitations of the eigen-mode may not necessarily result in distinctive pattern of eigen-oscillation, but if the irregularity of a \( C^N (N \neq \infty) \) continuous forcing has high degree of randomness that is statistically close to a white noise, and the timescale is much shorter than the eigen-period, then, for longer enough duration and small damping the pattern of distinctive eigen-oscillation will appear in the time series of the forced solution. The greater the amplitude, the more robust the eigen-oscillation.

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Conflict of Interest

The authors declare no conflict of interest.
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