Simultaneous State and Unknown Input Set-Valued Observers for Nonlinear Dynamical Systems

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Abstract

In this paper, we propose fixed-order set-valued observers for nonlinear bounded-error dynamical systems with unknown input signals that simultaneously find bounded sets of states and unknown inputs that include the true states and inputs. Sufficient conditions in the form of Linear Matrix Inequalities (LMIs) for the stability of the proposed observers are derived for general nonlinear systems and furthermore, less restrictive sufficient conditions are provided for three classes of nonlinear systems: (I) Linear Parameter-Varying (LPV), (II) Lipschitz continuous, and (III) Decremental Quadratic Constrained (DQC) systems. This includes a new DQC property that is at least as general as the incremental quadratic constrained property for nonlinear systems. In addition, we design the optimal $H_\infty$ observer among those that satisfy the stability conditions, using semi-definite programs with additional LMIs constraints. Furthermore, sufficient conditions are provided for the upper bounds of the estimation errors to converge to steady state values and finally, the effectiveness of the proposed set-valued observers is demonstrated through illustrative examples, where we compare the performance of our observers with some existing observers.

Key words: Set-Valued Estimation; Nonlinear Systems; State and Input Estimation; Resilient Estimation.

1 Introduction

1.1 Motivation

Cyber-physical systems (CPS), e.g., power grids, autonomous vehicles, medical devices, etc., are systems in which computational and communication components are deeply intertwined and interacting with each other in several ways to control physical entities. Such safety-critical systems, if jeopardized or malfunctioning, can cause serious detriment to their operators, controlled physical components and the people utilizing them. A need for CPS security and for new designs of resilient estimation and control has been accentuated by recent incidents of attacks on CPS, e.g., the Iranian nuclear plant, the Ukrainian power grid and the Maroochy water service [5,13,32,34,43]. Specifically, false data injection attack is one of the most serious types of attacks on CPS, where malicious and/or strategic attackers inject counterfeit data signals into the sensor measurements and actuator signals to cause damage, steal energy etc. Given the strategic nature of these false data injection signals, they are not well-modeled by a zero-mean, Gaussian white noise nor by signals with known bounds. Hence, traditional Kalman filtering and unknown input observers do not apply. Nevertheless, reliable estimates of states and unknown inputs are indispensable and useful for the sake of attack identification, resilient control, etc. Similar state and input estimation problems can be found across a wide range of disciplines, from input estimation in physiological systems [11], to fault detection and diagnosis [29], to the estimation of mean areal precipitation [21].

1.2 Literature Review

Much of the research focus has been on simultaneous input and state estimation for stochastic systems with unknown inputs, assuming that the noise signals are Gaussian and white, via minimum variance unbiased (MVU) estimation approaches (e.g., [15,16,39,41]), modified double-model adaptive estimation methods (e.g., [23]), or robust regularized least square approaches as in [1]. However, such Kalman filtering inspired approaches are not applicable for set-membership estimation problems in bounded-error settings, as is considered in this paper, where set-valued uncertainties are considered and sets of states and unknown inputs that are compatible with measurements are desired (cf. [37] for a comprehensive discussion).
In the context of attack-resilient estimation, numerous approaches were proposed for deterministic systems (e.g., [10,14,28,33]), stochastic systems (e.g., [20,40,42]) and bounded-error systems [25,27,38], against false data injection attacks. However, these approaches mainly yield point estimates, i.e., the most likely or best single estimate, as opposed to set-valued estimates. On the other hand, the work in [27] only computes error bounds for the initial state and [25] assumes zero initial states and does not consider any optimality criteria.

In addition, unknown input observer designs for different classes of discrete-time nonlinear systems are relatively scarce. The method proposed in [35] leverages discrete-time sliding mode observers for calculating state and scarce. The method proposed in [35] leverages discrete-time sliding mode observers for calculating state and unknown input point estimates, assuming that the unknown inputs have known bounds and evolve as known functions of states, which may not be directly applicable when considering adversaries in the system. The authors in [22] proposed an LMI-based state estimation approach for globally Lipschitz nonlinear discrete-time systems, but did not consider unknown input reconstruction. An LMI-based approach was also used in [17] for simultaneous estimation of state and unknown input for a class of continuous-time dynamic systems with Lipschitz nonlinearities, but the authors did not address optimality or stability properties for their observer, as well as only considered point estimates.

The work in [2] designed an asymptotic observer to calculate point estimates for a class of continuous-time systems whose nonlinear terms satisfy an incremental quadratic inequality property. Similar work was done for the same class of discrete-time nonlinear systems in [7]. However, none of them addressed unknown input estimation. Moreover, the restrictive assumption of bounded unknown inputs is needed in order to obtain convergent estimates. Considering bounded unknown inputs, but with unknown bounds, the work in [8] applied second-order series expansions to construct observer for state estimation in nonlinear discrete-time systems. The authors also provided sufficient conditions for stability and optimality of the designed estimator. However, their method does not compute unknown input estimates. On the other hand, in a recent and interesting work in [6], the authors designed an observer for reconstruction of unknown exogenous inputs in nonlinear continuous-time systems with unknown and potentially unbounded inputs, providing sufficient LMI conditions for \( L_\infty \)-stability of the observer. However, their observer does not simultaneously estimate the state, the unknown input estimates are point estimates and the optimality of their approach was not analyzed.

The author in [37] and references therein discussed the advantages of set-valued observers (when compared to point estimators) in terms of providing hard accuracy bounds, which are important to guarantee safety [4]. In addition, the use of fixed-order set-valued methods can help decrease the complexity of optimal observers [24], which grows with time. Hence, a fixed-order set-valued observer for linear time-invariant discrete time systems with bounded errors, was presented in [37], that simultaneously finds bounded sets of compatible states and unknown inputs that are optimal in the minimum \( H_\infty \)-norm sense, i.e., with minimum average power amplification. In our preliminary work [18], we extended the approach in [37] to linear parameter-varying systems, while in [19], we generalized the method to switched linear systems with unknown modes and sparse unknown inputs (attacks). In this paper, we aim to further design novel set-valued observers for broader classes of nonlinear systems.

### 1.3 Contribution

The goal of this paper is to bridge the gap between set-valued state estimation without unknown inputs and point-valued state and unknown input estimation for a broad range of nonlinear dynamical systems. In particular, we propose fixed-order set-valued observers for nonlinear discrete-time bounded-error systems that simultaneously find bounded sets of states and unknown inputs that contain the true state and unknown input, are compatible/consistent with measurement outputs and are optimal in the minimum \( H_\infty \)-norm sense, i.e., with minimum average power amplification.

First, we introduce a novel class of nonlinear vector fields, Decremental Quadratic Constraint (DQC) systems, and show that they include a broad range of nonlinearities. We also derive some results on the relationship between DQC functions with some other classes of nonlinearities, such as incremental quadratic constraint, Lipschitz continuous and linear parameter-varying (LPV) functions.

Then, we present our three-step recursive set-valued observer for nonlinear discrete-time systems. In particular, we derive sufficient conditions for the stability of the observer in the form of LMIs for general nonlinear systems, as well as less restrictive sufficient LMI conditions for stability of the observer for three classes of nonlinearities: (I) DQC, (II) Lipschitz continuous and (III) LPV systems. Furthermore, we design \( H_\infty \) observers, using additional LMIs for each of the aforementioned classes of systems. Finally, we derive sufficient conditions for convergence of estimation errors for each class of functions.

Note that we consider completely unknown inputs (different from noise signals) without imposing any assumptions on them (such as being norm bounded, with limited energy or being included in some known set). Considering resilient estimation in cyber-physical systems, our set-valued observers are applicable for achieving attack-resiliency against false data injection attacks on both actuator and sensor signals. It is worth mentioning that
in our preliminary work [18], we designed set-valued $H_\infty$ observers for the special case of LPV systems.

## 2 Preliminary Material

### 2.1 Notation

$\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space and $\mathbb{N}$ nonnegative integers, while $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the sets of non-negative and positive real numbers. For a vector $v \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{p \times q}$, $\|v\| \triangleq \sqrt{v^T v}$ and $\|M\|$ denote their (induced) 2-norm. Moreover, the transpose, inverse, Moore-Penrose pseudoinverse and rank of $M$ are given by $M^T$, $M^{-1}$, $M^+$ and $\text{rk}(M)$. For symmetric matrices $S$ and $S'$, $S \succ 0$, $S \succeq 0$, $S \prec 0$, and $S \preceq 0$ mean that $S$ is positive semi-definite, positive definite, positive semi-negative and positive negative, respectively. Moreover, $S \succeq S'$ and $S \preceq S'$ mean $S - S'$ is positive semi-definite and positive semi-negative, respectively.

### 2.2 Structural Properties

Here, we briefly introduce the structural properties that we will consider for our different classes of systems, so that we will be able to refer to them later when needed.

**Definition 2.1 (Strong Detectability [37])** The following bounded-error Linear Time Invariant (LTI) system:

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k + Gd_k + w_k, \\
y_k &= Cx_k + Du_k + Hd_k + v_k, \\
\end{align*}
\]

\(i.e., the tuple (A, G, C, H), is strongly detectable if \(y_k = 0 \forall k \geq 0\) implies \(x_k \to 0\) as \(k \to \infty\), for all initial states and input sequences \(\{d_i\}_{i \in \mathbb{N}}\), where A, B, G, C, D, H are known constant matrices with appropriate dimensions, and \(x_k, u_k, y_k, d_k, w_k\) and \(v_k\) are system state, known input, output, unknown input, bounded norm process noise and measurement noise signals, respectively.

**Remark 2.2** Several necessary and sufficient rank conditions are provided in [37, Theorem 1] to check the strong detectability of system (1), i.e., $(A, G, C, H)$, including $\text{rk}R_S(z) \triangleq \text{rk}\left[ zI - A - G \\ C H \right] = n+p$ for all $z \in \mathbb{C}$, $|z| \geq 1$.

It is worth mentioning that all the aforementioned conditions are equivalent to the system being minimum-phase (i.e., the invariant zeros of $R_S(z)$ are stable). Moreover, strong detectability implies that the pair $(A, C)$ is detectable, and if $l = p$, then strong detectability implies that the pair $(A, G)$ is stabilizable (cf. [37, Theorem 1] for more details).

**Definition 2.3 (Lipschitz Vector Fields)** A vector field $f(\cdot) : D_f \to \mathbb{R}^m$ is globally $L_f$-Lipschitz continuous on $D_f \subseteq \mathbb{R}^n$, if there exists $L_f \in \mathbb{R}_{++}$, such that $\|f(x_1) - f(x_2)\| \leq L_f \|x_1 - x_2\|, \forall x_1, x_2 \in D_f$.

**Definition 2.4 (LPV Functions)** A vector field $f(\cdot) : \mathbb{R}^p \to \mathbb{R}^q$ is Linear Parameter-Varying (LPV), if at each time step $k$, $f(x_k)$ can be decomposed into a convex combination of linear functions with known coefficients, i.e., $\forall k \geq 0, \exists \mathcal{N} \in \mathbb{N}$ such that $\forall i \in \{1, \ldots, N\}$, there exist known $\lambda_{i,k} \in [0,1]$ and $A^i \in \mathbb{R}^{p \times q}$ such that $\sum_{i=1}^{N} \lambda_{i,k} = 1$ and $f(x_k) = \sum_{i=1}^{N} \lambda_{i,k} A^i x_k$. Each linear function $A^i x$ is called a constituent function of the original nonlinear function.

**Definition 2.5 ($\delta$-QC Vector Fields [2])** A symmetric matrix $M \in \mathbb{R}^{(n_q + n_r) \times (n_q + n_r)}$ is an incremental multiplier matrix ($\delta M$) for $f(\cdot)$, if the following incremental quadratic constraint (\(\delta\)-QC) is satisfied for all $q_1, q_2 \in \mathbb{R}^{n_q}$:

$$
\left[ (\Delta f)^T (\Delta q)^T \right] M \left[ (\Delta f)^T (\Delta q)^T \right]^T \geq 0,
$$

where $\Delta q \triangleq q_2 - q_1$ and $\Delta f \triangleq f(q_2) - f(q_1)$.

Next, we introduce a new class of systems we call decremental quadratic constrained (DQC) that is at least as general as $\delta$-QC and includes a broad range of nonlinearities.

**Definition 2.6 (DQC Functions)** A vector field $f(\cdot) : \mathbb{R}^p \to \mathbb{R}^q$ is $(M, \gamma)$-Decremental Quadratic Constrained (DQC), if there exists symmetric matrix $M \in \mathbb{R}^{(p+q) \times (p+q)}$ and $\gamma \in \mathbb{R}_+$ such that

$$
\left[ (\Delta f)^T (\Delta x)^T \right] M \left[ (\Delta f)^T (\Delta x)^T \right]^T \leq \gamma,
$$

for all $x_1, x_2 \in \mathbb{R}^p$, where $\Delta x \triangleq x_2 - x_1$ and $\Delta f \triangleq f(x_2) - f(x_1)$. We call $M$ a decremental multiplier matrix for function $f(\cdot)$.

First of all, we show that a vector field may satisfy DQC property with different pairs of $(M, \gamma)$’s. For clarity, all proofs are provided in the Appendix.

**Proposition 2.7** Suppose $f(\cdot)$ is $(M, \gamma)$-DQC. Then it is also $(\kappa M, \kappa \gamma)$-DQC, $(\nu M, \gamma)$-DQC, $(M, \rho \gamma)$-DQC and $(M', \gamma')$-DQC for every $\kappa \geq 0$, $0 \leq \nu \leq 1$, $\rho \geq \gamma$ and $M' \preceq M$.

Moreover, we next show that the DQC property includes Lipschitz continuity and is at least as general as the incremental quadratic constrained (\(\delta\)-QC) property (cf. Definition 2.5), which recently has received considerable attention in nonlinear system state and input estimation (e.g., in [2,6,7]). Consequently, the class of DQC functions is a generalization of several types of nonlinearities (cf. Corollary 2.10).
Proposition 2.8 Every globally $L_f$-Lipschitz continuous function is $\delta$-QC with multiplier matrix

$$M = \begin{bmatrix} -I & 0 \\ 0 & L_f^2 \end{bmatrix}. $$

Proposition 2.9 Every nonlinearity which is $\delta$-QC with multiplier matrix $M$ is $(-M, \gamma)$-DQC for any $\gamma \geq 0$.

Corollary 2.10 Lipschitz nonlinearities, incrementally sector bounded nonlinearities and nonlinearities with matrix parameterizations [2], etc., which are $\delta$-QC (cf. [2,6,7] for definitions, demonstrations and more detailed examples), are also DQC.

Next, we provide some instances of nonlinear DQC vector fields, that to our best knowledge, have not been shown to be $\delta$-QC.

Example 1 Consider any monotonically increasing vector-filed $f(.) : \mathbb{R}^n \to \mathbb{R}^n$, which is not necessarily globally Lipschitz. By monotonically increasing, we mean that $\Delta f^T \Delta x \geq 0$, for all $x_1, x_2 \in \mathbb{D}_f$, where $\Delta f$ and $\Delta x$ are defined in Definition 2.6. As simple examples, the reader can consider $g(x) = x^5$ with $\mathbb{D}_g = \mathbb{R}$ or $h(x) = \tan(x)$ with $\mathbb{D}_h = (\mathbb{R}, \mathbb{R})$. It can be easily validated that such functions are $(M, \gamma)$-DQC with $M = \begin{bmatrix} 0_{n \times n} & -I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{bmatrix}$ and any $\gamma \geq 0$. Similarly, any monotonically decreasing vector field is $(-M, \gamma)$-DQC.

Example 2 Now, consider $f(x) = x^2$ with $\mathbb{D}_f = [-\bar{x}, \bar{x}] \subset \mathbb{R}$, $\bar{x} \geq 0.5$, which is not a monotone function. Let $M_0 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. It can be verified that

$$\left(\Delta f \right)^T \left(\Delta x \right) M_0 \left(\Delta f \right)^T \left(\Delta x \right)^T = \Delta f - \Delta x \| \Delta f - \Delta x \|^2 = \| \Delta x \|^2 - \Delta x \|^2 \leq 2\bar{x}(2\bar{x} + 1) = 9, $$

for $x_1, x_2 \in \mathbb{D}_f$. Hence, $f(x) = x^2$ for all $x \in [-\bar{x}, \bar{x}] \subset \mathbb{R}$ with $\bar{x} \geq 0.5$ is $(M_0, 9)$-DQC.

Furthermore, considering a specific structure for the decremental multiplier matrix $M$, we introduce a new class of functions that is a subset of the DQC class.

Definition 2.11 (DQC* Functions) A vector field $f(.)$ is a DQC* function, if it is $(M, \gamma)$-DQC for some known $M \in \mathbb{R}^{n \times n}$ and $\gamma \geq 0$, and there exists a known $A \in \mathbb{R}^{n \times n}$, such that

$$\begin{bmatrix} I_{n \times n} & -A \\ -A^T & A^T A \end{bmatrix} \leq M. $$

Now we present some results that establish the relationships between the aforementioned classes of nonlinearities.

Proposition 2.12 Suppose $f(.)$ is globally $L_f$-Lipschitz continuous and the state space, $\mathcal{X}$, is bounded, i.e., there exists $r \in \mathbb{R}_+$ such that for all $x \in \mathcal{X}$, $\|x\| \leq r$. Then, $f(.)$ is a DQC* function with $A = I_{n \times n}$.

Proposition 2.13 Suppose vector field $f(.)$ can be decomposed as the sum of an affine and a bounded nonlinear function $g(.)$, i.e., $f(x) = Ax + h(x)$, where $A \in \mathbb{R}^{n \times n}$, $h \in \mathbb{R}^n$ and $\|g(x)\| \leq r \in \mathbb{R}_+$ for all $x \in \mathcal{D}_a$. Then, $f(.)$ is a DQC* function with $A = A$, $M = \begin{bmatrix} I_{n \times n} & -A \\ -A^T & A^T A \end{bmatrix}$ and any $\gamma \geq (2r)^2$. Note that some DQC systems are also DQC*. The following Proposition 2.14 helps with finding such an $A$ for some specific structures of $M$.

Proposition 2.14 Suppose $f(.) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a $(M, \gamma)$-DQC vector field, with $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, where $M_{11}, M_{12}, M_{22} \in \mathbb{R}^{n \times n}$, $M_{11} - I_{n \times n} \geq 0$ and $M_{22} - M_{12}^T M_{12} \geq 0$. Then, $f(.)$ is a DQC* function with $A = -M_{12}$.

The reader can verify that such sufficient conditions in Proposition 2.14 hold for the function in Example 2.

Proposition 2.15 Every LPV function $f(.)$ with constituent matrices $A_i$, $\forall i \in 1 \ldots N$, is $\|A^m\|$-globally Lipschitz continuous, where $\|A^m\| = \max_{i \in 1 \ldots N} \|A^i\|$.

Corollary 2.16 As a direct corollary of Propositions 2.12 and 2.15, any bounded domain LPV function is a DQC* function.

Figure 1 summarizes all the above results on the relationships between several classes of nonlinearities.

3 Problem Statement

In this section, we describe the system, vector field and unknown input signal assumptions as well as formally state the observer design problem.

System Assumptions. Consider the nonlinear discrete-time bounded-error system

$$x_{k+1} = f(x_k) + Bu_k + Gd_k + Ww_k, $$

$$y_k = Cx_k + Du_k + Hd_k + v_k, $$

(3)
where $x_k \in \mathbb{R}^n$ is the state vector at time $k \in \mathbb{N}$, $u_k \in \mathbb{R}^m$ is a known input vector, $d_k \in \mathbb{R}^l$ is an unknown input vector, and $y_k \in \mathbb{R}^l$ is the measurement vector. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^l$ are assumed to be bounded, with $\|w_k\| \leq \eta_w$ and $\|v_k\| \leq \eta_v$ (thus, they are $\ell_\infty$ sequences). We also assume an estimate $\hat{x}_0$ of the initial state $x_0$ is available, where $\|\hat{x}_0 - x_0\| \leq \delta_0$. The vector field $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and matrices $B, C, D, G, W$ and $H$ are known and of appropriate dimensions, where $G$ and $H$ are matrices that encode the locations through which the unknown input or attack signal can affect the system dynamics and measurements. Note that no assumption is made on $H$ to be either the zero matrix (no direct feedthrough), or to have full column rank when there is direct feedthrough. Without loss of generality, we assume that $\text{rk}(G^\top H^\top) = p$, $n \geq l \geq 1$, $l \geq p \geq 0$ and $m \geq 0$.

**Vector Field Assumptions.** Here, we formally state the classes of nonlinear systems, related to the assumptions about the nonlinear vector field $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n = \begin{bmatrix} f_1(\cdot) & \cdots & f_l(\cdot) & \cdots & f_n(\cdot) \end{bmatrix}^\top$, $\forall j \in \{1, \ldots, n\}$, that we consider in this paper.

**Class 0** Nonlinear systems without any additional assumptions.

For this general case of Class 0 systems, we expect to derive conservative sufficient conditions for stability and optimality of the designed observers. However, to enable the computation of upper bounds for the estimation errors, we need some assumptions on the variations of the vector field in terms of state variations.

**Class I** Globally $L_1$-Lipschitz continuous systems.

**Class II** $DQC^*$ systems, with some known $M \in \mathbb{R}^{2n \times 2n}$, $\gamma \geq 0$, and $A \in \mathbb{R}^{n \times n}$.

**Class III** LPV systems with constituent matrices $A^i \in \mathbb{R}^{n \times n}, \forall i \in \{1, \ldots, N\}$.

For Class III of systems, the system dynamics is governed by an LPV system with known parameters at run-time. We call each tuple $(A^i, C, A, H), \forall i \in \{1 \ldots N\}$, an LTI constituent of system (3).

**Unknown Input (or Attack) Signal Assumptions.** The unknown inputs $d_k$ are not constrained to be a signal of any type (random or strategic) nor to follow any model, thus no prior ‘useful’ knowledge of the dynamics of $d_k$ is available (independent of $\{d_\ell\} \ni \forall \ell, \{w_\ell\}$ and $\{v_\ell\} \forall \ell$). We also do not assume that $d_k$ is bounded or has known bounds and thus, $d_k$ is suitable for representing adversarial attack signals.

The simultaneous input and state set-valued observer design problem is twofold and can be stated as follows:

**Problem 1** Given the nonlinear discrete-time bounded-error system with unknown inputs (3),

1) Design stable observers that simultaneously find bounded sets of compatible states and unknown inputs for the four classes of nonlinear systems.

2) Among the observers that satisfy 1), find the optimal observer in the minimum $H_\infty$-norm sense, i.e., with minimum average power amplification.

**4 Fixed-Order Simultaneous Input and State Set-Valued Observer Framework**

In this paper, we propose recursive set-valued observers that consist of three steps, similar to the framework in our previous works [18,19,37]. The three steps are (1) an unknown input estimation step that uses the current measurement and the set of compatible states to estimate the set of compatible unknown inputs, (2) a time update step which propagates the compatible set of states based on the system dynamics, and (3) a measurement update step that uses the current measurement to update the set of compatible states. To sum up, our target is to design a three-step recursive set-valued observer of the form:

**Unknown Input Estimation**: $\hat{D}_{k-1} = F_d(\hat{X}_{k-1}, u_k)$,

**Time Update**: $\hat{X}_k^* = F^*_{x}(\hat{X}_{k-1}, \hat{D}_{k-1}, u_k)$,

**Measurement Update**: $\hat{X}_k = F_x(\hat{X}_k^*, u_k, y_k)$,

where $F_d, F^*_x$ and $F_x$ are to-be-designed set mappings, while $\hat{D}_{k-1}, \hat{X}_k^*$ and $\hat{X}_k$ are the sets of compatible unknown inputs at time $k - 1$, propagated, and updated states at time $k$, correspondingly. It is important to note that $d_{2,k}$ cannot be estimated from $y_k$ since it does not affect $z_{1,k}$ and $z_{2,k}$. Thus, the only estimate we can obtain in light of (7) is a (one-step) delayed estimate of...
\( \hat{D}_{k-1} \). The reader may refer to [39] for a complete discussion on when a delay is absent or when we can expect further delays. Similar to [4,9,37], since the complexity of optimal observers increases with time, we only consider fixed-order recursive filters. In particular, we choose set-valued estimates of the form:

\[
\begin{align*}
\hat{D}_{k-1} &= \{d \in \mathbb{R}^n : \|d_{k-1} - \hat{d}_{k-1}\| \leq \delta_{k-1}^d\}, \\
\hat{X}_{k}^{*} &= \{x \in \mathbb{R}^n : \|x_k - \hat{x}_{k|k}\| \leq \delta_{k}^x\}
\end{align*}
\]

In other words, we restrict the estimation errors to balls of norm \( \delta \). In this setting, the observer design problem is equivalent to finding the centroids \( \hat{d}_{k-1}, \hat{x}_{k|k}^{*} \) and \( \hat{x}_{k|k} \) as well as the radii \( \delta_{k-1}^d, \delta_{k}^x \) and \( \delta_{k}^x \) of the sets \( \hat{D}_{k-1} \), \( \hat{X}_{k}^{*} \) and \( \hat{X}_{k} \), respectively. In addition, we limit our attention to observers for the centroids \( \hat{d}_{k-1}, \hat{x}_{k|k}^{*} \) and \( \hat{x}_{k|k} \) that belong to the class of three-step recursive filters given in [16] and [41], with \( \hat{x}_{0|0} \equiv \hat{x}_0 \).

4.1 System Transformation

To aid the observer design, we first carry out a transformation to decouple the output equation into two components, one with a full rank direct feedthrough matrix and the other without direct feedthrough. Note that this similarity transformation is similar to the one in [18,19,37] and is not the same as the one for stochastic systems in [41].

Let \( p_H \equiv \text{rk}(H) \). Using singular value decomposition, we rewrite the direct feedthrough matrix \( H \) as \( H = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \), where \( \Sigma \in \mathbb{R}^{p_H \times p_H} \) is a diagonal matrix of full rank, \( U_1 \in \mathbb{R}^{l \times p_H}, U_2 \in \mathbb{R}^{l \times (l-p_H)}, V_1 \in \mathbb{R}^{p_H \times p_H} \) and \( V_2 \in \mathbb{R}^{p \times (p-H)} \), while \( U \equiv \begin{bmatrix} U_1 & U_2 \end{bmatrix} \) and \( V \equiv \begin{bmatrix} V_1 & V_2 \end{bmatrix} \) are unitary matrices. When there is no direct feedthrough, \( \Sigma, U_1 \) and \( V_1 \) are empty matrices and \( U_2 \) and \( V_2 \) are arbitrary unitary matrices.

Then, we decouple the unknown input into two orthogonal components:

\[
d_{1,k} = V_1^T d_k, \quad d_{2,k} = V_2^T d_k.
\]

Considering that \( V \) is unitary,

\[
d_k = V_1 d_{1,k} + V_2 d_{2,k},
\]

and we can represent the system (3) as:

\[
x_{k+1} = f(x_k) + B u_k + G_1 d_{1,k} + G_2 d_{2,k},
\]

\[
y_k = C x_k + D u_k + H_1 d_{1,k},
\]

where \( G_1 \equiv G V_1, G_2 \equiv G V_2 \) and \( H_1 \equiv H V_1 \). Next, the output \( y_k \) is decoupled using a nonsingular transformation \( T = \begin{bmatrix} T_1^T & T_2^T \end{bmatrix} \), \( \hat{U} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \) to obtain \( z_{1,k} \in \mathbb{R}^{p_H} \) and \( z_{2,k} \in \mathbb{R}^{l-p_H} \) given by

\[
\begin{align*}
z_{1,k} &\equiv T_1 y_k = U_1^T y_k = C_1 x_k + \Sigma d_{1,k} + D_1 u_k + v_{1,k}, \\
z_{2,k} &\equiv T_2 y_k = U_2^T y_k = C_2 x_k + D_2 u_k + v_{2,k},
\end{align*}
\]

where \( C_1 \equiv U_1^T C, C_2 \equiv U_2^T C, D_1 \equiv U_1^T D, D_2 \equiv U_2^T D, v_{1,k} \equiv U_1^T v_k \) and \( v_{2,k} \equiv U_2^T v_k \). This transformation is also chosen such that \( \| v_{1,k}^T v_{2,k} \| = \| U^T v_k \| = \| v_k \| \).

4.2 Observer Structure

Using the above transformation, we propose the following three-step recursive observer structure to compute the state and input estimate sets:

**Unknown Input Estimation (UIE):**

\[
\begin{align*}
\hat{d}_{1,k} &= M_1(z_{1,k} - C_1 \hat{x}_{k|k} - D_1 u_k), \\
\hat{d}_{2,k-1} &= M_2(z_{2,k} - C_2 \hat{x}_{k|k-1} - D_2 u_k), \\
\hat{d}_{k-1} &= V_1 \hat{d}_{1,k-1} + V_2 \hat{d}_{2,k-1}.
\end{align*}
\]

**Time Update (TU):**

\[
\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}) + B u_{k-1} + G_1 \hat{d}_{1,k-1},
\]

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + G_2 \hat{d}_{2,k-1}.
\]

**Measurement Update (MU):**

\[
\hat{x}_{k|k} = \hat{x}_{k|k}^* + L(y_k - C \hat{x}_{k|k} - D u_k) = \hat{x}_{k|k}^* + \tilde{L}(z_{2,k} - C_2 \hat{x}_{k|k} - D_2 u_k),
\]

where \( L \in \mathbb{R}^{n \times l}, \tilde{L} \equiv L U_2 \in \mathbb{R}^{n \times (l-p_H)} \), \( M_1 \in \mathbb{R}^{p_H \times p_H} \) and \( M_2 \in \mathbb{R}^{(p-H) \times (l-p_H)} \) are observer gain matrices that are designed according to Theorem 5.8, where the main result in is obtained by minimizing the “volume” of the set of compatible states and unknown inputs, quantified by the radii \( \delta_{k-1}^x, \delta_{k}^x \) and \( \delta_{k}^x \). Note also that we applied \( L = L U_2 U_2^T = \tilde{L} U_2^T \).
5 Observer Design and Analysis

In this section we derive LMI conditions for designing observers that are stable (section 5.1) and optimal in the $\mathcal{H}_\infty$ sense (section 5.2). Moreover, we derive the resulting radii of the state and input estimates (section 5.3). To do so, first, we will derive our observer error dynamics through the following Lemma 5.1. For conciseness, all proofs are provided in the Appendix.

Lemma 5.1 Consider system (3) and the observer (8)-(13). Suppose $\text{rk}(C_2G_2) = p - p_H$. Then, designing observer matrix gains as $M_1 = \Sigma^{-1}$, $M_2 = (C_2G_2)^\dagger$, $LU_1 = 0$ and $L = LU_2U_2^\dagger = \hat{L}U_2^\dagger$ yields $M_1\Sigma = I$ and $M_2C_2G_2 = I$ and lead to the following difference equation for the state estimation error dynamics:

$$\hat{x}_{k+1|k+1} = (I - \hat{L}C_2)\Phi(\Delta \hat{x}_k - \Psi \hat{x}_k) + \mathcal{W}(\hat{L})w_k,$$

(14)

where

$$\Delta \hat{x}_k \triangleq f(x_k) - f(\hat{x}_k), \quad \Phi \triangleq I - G_2M_2C_2,$$

$$\mathcal{W}_k \triangleq \left[\frac{(\sqrt{\mathcal{V}})w_k^T}{(\sqrt{\mathcal{V}})\mathcal{W}_k^T} \right]^{T},$$

$$R \triangleq \left[-\sqrt{2}\Phi G_1 M_1 T_1 - \Phi W - \sqrt{2}G_2 M_2 T_2 \right],$$

$$Q \triangleq \left[0_{(p-H)xI} 0_{(p-H)xN} - \sqrt{2}T_2 \right],$$

$$\Psi \triangleq G_1 M_1 C_1, \quad \mathcal{W}(\hat{L}) \triangleq (I - \hat{L}C_2)R + \hat{L}Q.$$

Note that $\mathcal{W}_k$ is chosen such that $\lim_{k \to \infty} \frac{1}{k+1} \sum_{i=0}^{k} \mathcal{W}_i^T \mathcal{W}_i = \lim_{k \to \infty} \frac{1}{k+1} \sum_{i=0}^{k} \mathcal{W}_i^T \mathcal{W}_i$. The results (14) shows that we successfully decoupled/canceled out $d_k$ from the error dynamics, otherwise there would be a potentially unbounded and unknown term in the error dynamics.

5.1 Stable Observer Design

We first study the stability of the observer in the sense of Lyapunov. For the sake of clarity, we first formally define the notion of stability.

Definition 5.2 [Lyapunov Stability] A simultaneous state and input set-valued observer is Lyapunov stable, if its estimation error norm sequences $\{\|\hat{x}_{k|k}\| \triangleq \|x_k - \hat{x}_{k|k}\|, \|d_{k-1|k-1}\| \triangleq \|d_{k-1} - d_{k-1}\|\}_{k=1}^\infty$ are bounded.

Now, we are ready to provide our first set of main results on sufficient conditions for bounded-error stability of the observer (8)-(13), by supposing for the moment that there is no exogenous bounded noise $w_k$ and $v_k$.

Theorem 5.3 (Observer Stability) Consider system (3) and the observer (8)-(13). Suppose there is no bounded noise $w_k$ and $v_k$ and all the conditions in Lemma 5.1 hold. Then, the observer error dynamics is Lyapunov stable, if there exist matrices $0 < \bar{P} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{n \times (l-p_H)}$ and $0 < \Gamma \in \mathbb{R}^{(l-p_H) \times (l-p_H)}$, such that the following LMIs hold:

$$\Pi \triangleq q \begin{bmatrix} I - \Gamma & 0 & 0 \\ 0 & Y^T & P \end{bmatrix} \geq 0, \quad \Upsilon \triangleq \Lambda^T \Xi \geq 0, \quad \forall i \in \{1 \ldots N\},$$

(15)

where $q \in \{0, 1\}$, $\Theta$, $\Lambda^i$, $\Xi$ and $N$ are defined for different cases as follows:

(0) If $f(.)$ is a Class 0 function, then $\bar{P} \triangleq 1$, $N \triangleq 1$ and

$$\Theta \triangleq \Phi^T (F - C_2^T C_2) \Phi,$$

$$\Lambda^i \triangleq \Phi^T (P - C_2^T Y^T - Y C_2) \Phi,$$

$$\Xi \triangleq P + \Psi^T \Phi^T F \Phi \Psi - \Phi^T C_2^T C_2 \Phi.$$

(1) If $f(.)$ is a Class I function, then $\bar{P} \triangleq 1$, $N \triangleq 1$ and

$$\Theta \triangleq \Phi^T (F - C_2^T C_2) \Phi + I,$$

$$\Lambda^i \triangleq \Phi^T (P - C_2^T Y^T - Y C_2) \Phi,$$

$$\Xi \triangleq P + \Psi^T \Phi^T F \Phi \Psi - \Phi^T C_2^T C_2 \Phi - L_2^2 I.$$

(II) If $f(.)$ is a Class II function, then $\bar{P} \triangleq 1$, $N \triangleq 1$ and

$$\Theta \triangleq \Phi^T (F - C_2^T C_2) \Phi,$$

$$\Lambda^i \triangleq \Phi^T (P - C_2^T Y^T - Y C_2) \Phi (\Psi - A),$$

$$\Xi \triangleq P + (\Psi - A)^T \Phi^T F \Phi (\Psi - A) - \Phi^T C_2^T C_2 \Phi.$$

(III) If $f(.)$ is a Class III function, then $\bar{P} \triangleq 0$, $N$ is the number of constituent LTI systems and $\forall i \in \{1 \ldots N\}$:

$$\Theta \triangleq P, \Xi \triangleq P, \Lambda^i \triangleq (\Lambda^i - \Psi) \Phi^T (P - C_2^T Y^T),$$

(19)

with $F \triangleq Y C_2 + C_2^T Y^T - P - C_2^T \Gamma C_2$. Moreover, the corresponding observer gain can be obtained as $\hat{L} = P^{-1}Y$.

It is worth mentioning that for Class I functions, i.e., when the nonlinear vector field $f(.)$ is globally Lipschitz continuous, stability of the observer can also be demonstrated using more succinct LMIs in the following Lemma 5.4.

Lemma 5.4 (Alternative LMIs (Class I)) Consider system (3) and the observer (8)-(13). Suppose all the conditions in Lemma 5.1 hold, $f(.)$ is a Class I function and there is no bounded noise $w_k$ and $v_k$. Then,
the observer error dynamics is Lyapunov stable with the observer gain \( \hat{L} = P^{-1}Y \), if \( \exists 0 < P \in \mathbb{R}^{n \times n} \) and \( Y \in \mathbb{R}^{n \times (l-p\mu)} \) such that

\[
\begin{bmatrix}
  I & (P - YC_2) & 0 \\
  (P - YC_2) & P & 0 \\
  0 & 0 & \Delta
\end{bmatrix} \succeq 0,
\]

where \( \Delta \triangleq P - 2L_2^2\lambda_{\max}(\Phi^T\Phi)I - 2\Psi^T\Phi^T\Psi, \) with \( \Phi \) and \( \Psi \) defined in Lemma 5.1.

Moreover, if \( f(.) \) is a Class III function, then we can provide necessary and/or sufficient conditions for the existence of stable observers. The necessary conditions are conveniently testable. They are also beneficial in the sense that if they are not satisfied, the designer knows a priori that there does not exist any \( \mathcal{H}_\infty \)-observer for such modified systems with unknown inputs/attacks. The conditions are formally derived in the following Lemma 5.5.

**Lemma 5.5 (Existence of Stable Observers)**

Suppose \( f(.) \) is a Class III function and all the conditions in Lemma 5.1 hold. Then, there exists a stable observer for the system \( (3) \), with any sequence \( \{\lambda_{i,k}\}_{k=0}^\infty \) for all \( i = 1,2,\ldots,N \) that satisfies \( 0 \leq \lambda_{i,k} \leq 1, \sum_{i=1}^N \lambda_{i,k} = 1, \forall k, \) if \( (A_k,C_2) \) be uniformly detectable for each \( k \), and only if all constituent LTI systems \( (A^i,G,C,H) \), \( \forall i \in \{1\ldots N\} \), are strongly detectable (cf. Definition 2.1), where \( \overline{\lambda}_{i,k} \triangleq \Phi(\sum_{i=1}^N \lambda_{i,k}A^i - \Psi), \) with \( \Phi \) and \( \Psi \) defined in Lemma 5.1.

**Corollary 5.6** There exists a stable simultaneous state and input set-calculated observer for the LTI system \( (1) \), through \( (8)-(13) \), if and only if the tuple \( (A,G,C,H) \) is strongly detectable and only if \( \text{rk}(C_2G_2) = p - p\mu \). Moreover, the observer gain matrices can be designed as \( M_1 = \Sigma^{-1}, M_2 = (C_2G_2)^\dagger \) and \( L = LU_2^\dagger \) and \( \hat{L} = P^{-1}Y \), where \( P > 0 \) and \( Y \) solve the following feasibility program with LMI constraints:

\[
\begin{align*}
\text{Find} \quad & (P > 0, Y) \\
\text{s.t.} \quad & \begin{bmatrix}
  P & A \\
  A^T & P
\end{bmatrix} \succeq 0,
\end{align*}
\]

with \( \Lambda \triangleq (A - \Psi)^T\Phi^T(P - C_2^TY^T) \) and \( \Phi \) and \( \Psi \) defined in Lemma 5.1.

\[\uparrow\] The readers are referred to [3, Section 2] for the concise definition of uniform detectability. A spectral test can be found in [30].

### 5.2 \( \mathcal{H}_\infty \) Observer Design

The goal of this section is to provide additional sufficient conditions to guarantee optimality of the observers in the \( \mathcal{H}_\infty \) sense. We first define our considered notion of optimality via the following Definition 5.7.

**Definition 5.7 (\( \mathcal{H}_\infty \)-Observer)** Let \( T_{\hat{x},w,v} \) denote the transfer function matrix that maps the noise signals \( \hat{w}_k \triangleq [w_k^T \quad v_k^T]^T \) to the updated state estimation error \( \hat{x}_{k|k} \triangleq \hat{x}_k - \hat{x}_{k|k} \). For a given \( \eta \in \mathbb{R}_+, \) the observer performance satisfies \( \mathcal{H}_\infty \) norm bounded by \( \eta \), if \( \|T_{\hat{x},w,v}\|_\infty \leq \eta, \) i.e., the maximum average signal power amplification is upper-bounded by \( \eta^2 \):

\[
\lim_{k \to \infty} \frac{1}{k+1} \sum_{i=0}^k \hat{x}_{i|i}^T \hat{x}_{i|i} \leq \frac{\|T_{\hat{x},w,v}\|_\infty^2}{\eta^2}.
\]

Now we present our second set of main results, on designing stable and optimal observers in the minimum \( \mathcal{H}_\infty \) sense.

**Theorem 5.8 (\( \mathcal{H}_\infty \)-Observer Design)** Consider system \( (3) \), the observer \( (8)-(13) \), and given \( \eta > 0 \). Suppose all the conditions in Theorem 5.3 hold, consider \( \Phi, \Psi, Q \) and \( R \) defined in Lemma 5.1 and let \( \Omega \triangleq C_2R - Q \). Then, with the gain \( \hat{L} = P^{-1}Y \), we obtain a stable observer with \( \mathcal{H}_\infty \) norm bounded by \( \eta \), if (15) holds and

\[
\begin{bmatrix}
  N_{11} & * & * \\
  N_{21} & N_{22} & * \\
  N_{31} & N_{32} & N_{33} & * \\
  N_{41} & N_{42} & N_{43} & N_{44}
\end{bmatrix} \succeq 0, \forall i \in \{1 \ldots N\},
\]

where \( N_{11}, N_{21}, N_{22}, N_{31}, N_{32}, N_{33}, N_{41}, N_{42}, N_{43}, N_{44} \) and \( N \) are defined for different cases as follows:

(0) If \( f(.) \) is a Class 0 function, then \( N \triangleq 1 \) and

\[
\begin{align*}
N_{11} & \triangleq \eta^2 I + R^TY\Omega + \Omega^TY^TR - R^TPR - \Omega^T(\Gamma + 2I)\Omega \\
N_{21} & \triangleq \Psi^T\Phi^T(PR - Y\Omega - C_2^TY^T), \\
N_{22} & \triangleq -I - \Psi^T\Phi^TC_2^TC_2\Phi\Psi, \\
N_{31} & \triangleq Y\Omega + C_2^TY^TR - PR, \\
N_{32} & \triangleq -\Phi^TC_2^TC_2\Phi,
\end{align*}
\]

and \( N_{32}, N_{41}, N_{42}, N_{43} \) are zero matrices with appropriate dimensions.
(I) If \( f(.) \) is a Class I function, then \( N \triangleq 1 \),
\[
N_{11} \triangleq \eta^2 I + R^\top Y \Omega + \Omega^\top Y^\top R - R^\top P R - \Omega^\top (\Gamma+2I) \Omega,
\]
\[
N_{21} \triangleq -I - (A - \Psi)^\top \Phi^\top (P R - Y \Omega - C_2^\top Y^\top R),
\]
\[
N_{22} \triangleq I - \Psi^\top C_2^\top C_2 \Phi - L_f^2 \lambda_{max}(\Phi^\top C_2^\top C_2 \Phi) I,
\]
\[
N_{31} \triangleq \Omega + C_2^\top Y^\top R - P R, N_{33} \triangleq 0, N_{44} \triangleq I,
\]
and \( N_{32}, N_{41}, N_{42}, N_{43} \) are zero matrices with appropriate dimensions.

(II) If \( f(.) \) is a Class II function, then \( N \triangleq 1 \),
\[
N_{11} \triangleq \eta^2 I + R^\top Y \Omega + \Omega^\top Y^\top R - R^\top P R - \Omega^\top (\Gamma+2I) \Omega,
\]
\[
N_{21} \triangleq -(A - \Psi)^\top \Phi^\top (P R - Y \Omega - C_2^\top Y^\top R),
\]
\[
N_{22} \triangleq I - (A - \Psi)^\top \Phi^\top C_2^\top C_2 \Phi (A - \Psi), N_{44} \triangleq I,
\]
\[
N_{31} \triangleq \Omega + C_2^\top Y^\top R - P R, N_{33} \triangleq -\Phi^\top C_2^\top C_2 \Phi,
\]
and \( N_{32}, N_{41}, N_{42}, N_{43} \) are zero matrices with appropriate dimensions.

(III) If \( f(.) \) is a Class III function, then \( N \) is the number of constituent LTI systems,
\[
N_{11} \triangleq N_{22} \triangleq P, N_{32} \triangleq ((P - Y C_2) R + Y Q)^\top,
\]
\[
N_{21} \triangleq (P - Y C_2) \Phi (A^\top - \Psi), N_{44} \triangleq \eta^2 I,
\]
\[
N_{41} \triangleq I, N_{33} \triangleq \eta^2 I,
\]
and \( N_{31}, N_{42} \) and \( N_{43} \) are zero matrices with appropriate dimensions.

Finally, the minimum \( H_\infty \) bound can be found by solving the following semi-definite program with LMI constraints:
\[
(\eta^*)^2 = \min_{\eta > 0, \eta^2 > 0} \eta^2 \text{ s.t. (15), (22),}
\]
where \( \eta^2 \) is a decision variable. Solving this Semi-Definite Program (SDP), we derive \( \|T_{z,w,v}\|_\infty \leq \eta^* \). This bound is obtained by applying the observer gain \( \bar{L}^* = P^{**-1} Y^* \), where \( (P^*, Y^*, \Gamma^*) \) solves the above SDP.

5.3 Radii of Estimates and Convergence of Errors

In this section, we are interested in computing closed form expressions for the estimation radii and sufficient conditions for their convergence, as well as their steady state values (if they exist). Notice that considering the general case of Class 0 functions, i.e. without imposing any additional assumption on \( f(.) \), to the best of our knowledge, there is no guarantee that any closed form expressions for the radii can be found using (14), since there is no means to relate the state error \( \tilde{x}_{v,k} \) to the function increment \( \Delta f_k \), whereas when \( f(.) \) belongs to either of the classes I, II or III, it is possible to relate function variations to the estimation errors and to find closed form expressions for the radii (cf. Theorem 5.9).

It is worth mentioning that for linear time-invariant systems, strong detectability of the system is a sufficient condition for the convergence of the radii \( \delta_k^c \) and \( \delta_k^d \) to steady state [37], but it is less clear for general nonlinear systems. Notice that if \( f(.) \) is a Class III function, i.e., in the LPV case, even strong detectability of all constituent LTI systems does not guarantee that the radii converge. The reason is that the convergence hinges on the stability of the product of time-varying matrices (cf. proof of Theorem 5.9), which is not guaranteed even if all the multiplicands are stable. In the following, we discuss some sufficient conditions for the convergence of the radii to steady state, where first we characterize the resulting radii \( \delta_k^c \) and \( \delta_k^d \) when using our proposed observer.

**Theorem 5.9 (Radii of Estimates)** Consider system (3) along with the observer (8)–(13). Suppose the conditions of Theorem 5.8 hold. Let \( \bar{R} \triangleq - (\bar{C} \bar{G}_1 M_1 T_1 + \bar{C}_2 M_2 T_2 + \bar{L} T_2), \alpha \triangleq \|V_2 M_2 C_2 \| \eta_w + \| (V_2 M_2 G_1 - V_1) M_1 T_1 \| \eta_w \) and \( \eta \triangleq \| \bar{R} \| \eta_w + \| \bar{C} \bar{G}_1 \| \eta_w \), with \( \bar{C} \) and \( \bar{G}_1 \) defined in Lemma 5.1. Then, the radii \( \delta_k^c \) and \( \delta_k^d \) can be obtained as:
\[
\delta_k^c = \delta_k^d = \eta \beta + \alpha \sum_{i=1}^{k} \theta i^{-1},
\]
\[
\delta_k^d = \delta_k^d - \eta \beta + \alpha \sum_{i=1}^{k} \theta i^{-1},
\]
where \( \theta, \beta, \alpha \) and \( \alpha \) are defined for the different function classes as follows:

(I) If \( f(.) \) is a Class I function, then
\[
\theta \triangleq \|L_f + \| \bar{C} \bar{G}_1 \| \| (I - \bar{L} C_2) \|),
\]
\[
\bar{\eta} \triangleq \bar{T},
\]
\[
\beta \triangleq \|V_1 M_1 C_1 - V_2 M_2 C_2 \| + L_f \|V_2 M_2 C_2 \|,
\]
\[
\bar{\alpha} \triangleq \alpha.
\]

(II) If \( f(.) \) is a Class II function, then
\[
\theta \triangleq \| (I - \bar{L} C_2) \| \| (A - \Psi) \|),
\]
\[
\bar{\eta} \triangleq \bar{T} + \| (I - \bar{L} C_2) \| \gamma,
\]
\[
\beta \triangleq \|V_1 M_1 C_1 + V_2 M_2 C_2 (A - \Psi) \|,
\]
\[
\bar{\alpha} \triangleq \alpha + \|V_2 M_2 C_2 \| \gamma.
\]

(III) If \( f(.) \) is a Class III function, then
\[
\theta \triangleq \max_{i \in \{1,2,\ldots,N\}} \| A_{c,i} \|,
\]
\[
\bar{\eta} \triangleq \bar{T},
\]
\[
\beta \triangleq \max_{i \in \{1,2,\ldots,N\}} \|V_1 M_1 C_1 + V_2 M_2 C_2 A_{c,i} \|,
\]
\[
\bar{\alpha} \triangleq \alpha.
\]
with \( A_{e,i} \triangleq (I - \hat{L}C_2)\Phi(A^i - \Psi) \), for all \( i \in \{1, \ldots, N\} \).

Furthermore, the radii \( \delta^e_k \) and \( \delta^d_{k-1} \), are convergent if \( \theta < 1 \) and if so, the steady state radii are given by:

\[
\lim_{k \to \infty} \delta^e_k = \frac{\eta}{1 - \theta}, \quad \lim_{k \to \infty} \delta^d_k = \frac{\eta \beta}{1 - \theta} + \overline{\epsilon}.
\]

The resulting fixed-order set-valued observer is summarized in Algorithm 1.

**Corollary 5.10** If \( f(.) \) is a Class III function and the conditions of Theorem 5.8 hold, then, the radii \( \delta^e_k \) and \( \delta^d_{k-1} \), computed in (27) and (28), are convergent if \( \|A_{e,i}\| < 1 \) for all \( i \in \{1, 2, \ldots, N\} \), where \( A_{e,i} \triangleq (I - \hat{L}C_2)\Phi(A^i - \Psi) \), with \( \Phi \) and \( \Psi \) defined in Lemma 5.1.

**Remark 5.11** Alternatively, we can trade off between “optimality” of the observer and “convergence” of the radii (i.e., the steady state values). We can find \( \eta \) (e.g., by line search) that satisfies the following feasibility problem:

\[
\begin{align*}
\text{Find } \{P, Y, \Gamma\} & \quad \text{s.t. } (22), \\
\end{align*}
\]

as well as the sufficient condition in Theorem 5.9, i.e., \( \theta < 1 \). Although the designed observer may not be optimum in the minimum \( H_\infty \) sense when using this alternative method, we can guarantee the steady state convergence of the radii instead.

6 Simulation Results and Comparison with Benchmark Observers

Two simulation examples are considered in this section to demonstrate the performance of the proposed observer. In the first example, where the dynamic system belongs to classes I and II, we consider simultaneous input and state estimation problem and design observers for each class to study their performances. Our second example is a benchmark dynamical Lipschitz continuous (i.e., Class I) system, where we compare the results of our observer with two other existing observers in the literature, [7,8]. We consider two different scenarios, one with a bounded unknown input, and the other with an unbounded unknown input. The results show that in the unbounded input scenario, when applying the observers in [7,8], the estimation errors diverge, while as expected from our theoretical results, the errors converge to steady state values when applying our proposed observer.

**Algorithm 1** Simultaneous Input and State Observer

1: Initialize:
   \( M_1 = \Sigma^{-1}; M_2 = (C_2G_2)^T; \Phi = I - G_2M_2C_2; \)

2: Compute \( \hat{L} \) via Theorem 5.8 and \( \theta, \overline{\eta}, \overline{\epsilon}, \overline{\zeta} \) via Theorem 5.9;

3: \( \hat{x}_{t_{0}} = \hat{x}_0 = \text{centroid}(\hat{X}_0) \);

4: \( \delta^e_{k-1} = M_1(z_{1,t} - C_1\hat{x}_{t_{0}} - D_1u_{0}); \)

5: \( \delta^d_{k-1} = \min\{\|x - \hat{x}_{t_{0}}\| \leq \delta, \forall x \in \hat{X}_0\}; \)

6: for \( k = 1 \) to \( K \) do

   \( \triangleright \) Estimation of \( d_{2,k-1} \) and \( d_{k-1} \)

8: \( \hat{x}^{*}_{k} = \hat{x}_{1,k-1} + C_2\hat{d}_{2,k-1}; \)

9: \( \hat{x}^{*}_{k} = \hat{x}^{*}_{k} + \hat{L}(\hat{x}_{2,k} - C_2\hat{x}^{*}_{k}) - D_2u_k); \)

10: \( \delta^d_{k} = \beta(\delta^d_{k-1} + \sum_{i=1}^{k} \theta^{i-1}) + \overline{\epsilon}; \)

11: \( \hat{x}_{k} = \{x \in \mathbb{R}^l : \|x - \hat{x}_{k}\| \leq \delta^d_{k}\}; \)

12: \( \hat{d}_{k} = M_1(z_{1,k} - C_1\hat{x}_{k} - D_1u_k); \)

13: end for

6.1 Single-Link Flexible-Joint Robotic System

We consider a single-link manipulator with flexible joints [1,31], where the system has 4 states. We slightly modify the dynamical system described in [1], by ignoring the dynamics for the unknown inputs (different from the existing bounded disturbances) to make them completely unknown input signals. We also consider bounded-norm disturbances (instead of stochastic noise signals in [1,31]). So, we have the dynamical system (3) with \( n = 4, f(x) = Ax + \left[0 \ 2.16T_s \ 0 \ -3.33T_s\sin(x_3)\right]^T \),

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-48.6T_s & 1 - 1.25T_s & 48.6T_s & 0 \\
0 & 0 & 1 & T_s \\
19.5T_s & 0 & -19.5T_s & 1
\end{bmatrix}, \quad p = m = 1,
\]

\( l = 2, B = 0_{4 \times 1}, G = T_s \begin{bmatrix}5 & 5 & 2 & 1 \end{bmatrix}^T, \quad C = \begin{bmatrix}1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}^T, \)

\( W = I, D = 0_{2 \times 1}, T_s = 0.01, H = T_s \begin{bmatrix}1.1 & 2 \end{bmatrix}^T \)

and \( \eta_{u} = \eta_{v} = 0.1 \). The unknown input signal is depicted in Figure 2. Vector field \( f(.) \) is a Class I function with \( L_f = 3.33T_s[\text{diag}(0.0,0.1)]\| = 3.33T_s[1] \), \( f(.) \) is a Class II function as well, with \( A = A \) and \( \gamma = 0.56 \) (cf. Lemma 2.13). Considering Theorem 5.8, cases (I) and (II) of the sufficient conditions are satisfied. Solving the corresponding SDPs
returns $P_{L}^\star = \begin{bmatrix} 1.3347 & -1.4121 & -0.2308 & -0.1154 \\ -1.4121 & 2.4642 & -0.1435 & 0.0717 \\ -0.2308 & -0.1435 & 1.1048 & -0.0364 \\ -0.1154 & -0.0717 & 0.0364 & 1.1594 \end{bmatrix}$, 
\( (Y_{L}^\star)^\top = \begin{bmatrix} 0.3237 & 0.2261 & 0.0265 & 0.0132 \end{bmatrix}^\top \), $\Gamma_{L} = 0.6391$, $\eta_{L} = 1.0560$ and consequently $(\hat{L}_{L}^\star)^\top = \begin{bmatrix} 1.1360 & 0.7694 & 0.3672 & 0.1836 \end{bmatrix}^\top$ for case (I), and

$P_{DQC}^\star = \begin{bmatrix} 2.8641 & -2.9917 & -1.0025 & -0.5012 \\ -2.9917 & 4.9445 & -0.7205 & -0.3602 \\ -1.0025 & -0.7205 & 4.5920 & -0.1786 \\ -0.5012 & -0.3602 & -0.1786 & 4.8599 \end{bmatrix}$, 
\( (Y_{DQC}^\star)^\top = \begin{bmatrix} -0.3234 & 1.0288 & 0.0453 & 0.0227 \end{bmatrix}^\top \), $\Gamma_{DQC} = 1.1473$, $\eta_{DQC} = 0.9641$ and $(\hat{L}_{DQC}^\star)^\top = \begin{bmatrix} 0.8205 & 0.7619 & 0.3147 & 0.1573 \end{bmatrix}^\top$ for case (II). We observe from Figure 2 that our proposed observer, i.e., Algorithm 1 is able to find set-valued estimates of the states and unknown inputs, for Lipschitz continuous (Class I) and DQC* (Class II) functions. The actual estimation errors are also within the predicted upper bounds (cf. Figure 3), which converge to steady-state values as established in Theorem 5.9. Furthermore, Figures 2 and 3 show that for this specific example system, estimation errors and their radii are tighter when applying the obtained observer gains for Class I (i.e., Lipschitz) functions, when compared to applying the ones corresponding to the Class II (i.e., DQC*) functions.

6.2 Comparison with Benchmark Observers

In this section, we illustrate the effectiveness of our Simultaneous Input and State Set-Valued Observer (SISO), by comparing its performance with two benchmark observers in [7] and [8] (we call them Chen-Hu and Chak-Stan-Shre observers, respectively). The designed estimator in [7] calculates both (point) state and unknown input estimates, while the observer in [8], only obtains (point) state estimates. For comparison, we apply the three observers on a benchmark dynamical system in [7], which is in the form of (3) with $n = 2$, $m = l = p = 1$, $f(x) = \begin{bmatrix} -0.42x_1 + x_2 \\ -0.6x_1 - 1.25\tanh(x_1) \end{bmatrix}^\top$, $G = \begin{bmatrix} 1 & -0.65 \end{bmatrix}^\top$, $B = D = H = 0_{1 \times 1}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $W = I$, $\eta_w = 0.2$ and $\eta_u = 0.1$. The vector field $\tilde{f}(\cdot)$ is Lipschitz continuous (i.e., Class I) with $L_f = 1.1711$. We consider two scenarios for the unknown input. In the first, we consider a random signal with bounded norm, i.e., $\|d_k\| \leq 0.2$ for the unknown input $d_k$, while $d_k$ in the second scenario is a ramp that becomes unbounded when time increases. As is demonstrated in Figures 4 and 5, in the first scenario, i.e., bounded unknown inputs, the set estimates of our approach (i.e., SISO estimates) converge to steady values and the point estimates of the two benchmark approaches [7,8] are
within the predicted upper bounds and exhibit a convergent behavior. In this scenario, the two benchmark approaches result in slightly better performance than SISO, since they are benefiting from the additional assumption of bounded input.

More interestingly, considering the second scenario, i.e., with unbounded unknown inputs, Figure 6 demonstrates that our set-valued estimates still converge, i.e., our observer remains stable, with $P^* = \begin{bmatrix} 1.9543 & 1.2561 \\ 1.2561 & 5.1084 \end{bmatrix}$, $Y^* = \begin{bmatrix} -0.1196 & 0.3887 \\ -0.1307 & 0.1082 \end{bmatrix}$, $\Gamma^* = 0.6360$ and $\eta^* = 1.993$, while the estimates of the two benchmark approaches exceed the boundaries of the compatible sets of states and inputs after some time steps of our approach and display a divergent behavior (cf. Figure 7).

![Fig. 6. Actual states $x_1$, $x_2$, and their estimates, as well as unknown input $d$ and its estimates in the unbounded unknown input scenario.](image)

![Fig. 7. Estimation errors in the unbounded unknown input scenario.](image)

7 Conclusion and Future Work

We presented fixed-order set-valued $\mathcal{H}_\infty$-observers for nonlinear bounded-error discrete-time dynamic systems with unknown inputs. Sufficient Linear Matrix Inequalities for Lyapunov stability of the designed observer were derived for different classes of nonlinear systems, including general nonlinear systems, Lipschitz continuous systems, Decremental Quadratic Constrained systems and Linear Parameter-Varying systems. Moreover, we derived additional LMI conditions and corresponding tractable semi-definite programs for obtaining the minimum $\mathcal{H}_\infty$ norm for the transfer function that maps the noise signal to the state error of the stable observers.

In addition, we derived sufficient conditions for the convergence of the radii of the set-valued state and input estimates and derived their steady state values. Finally, using two illustrative examples, we demonstrated the effectiveness of our proposed design, as well as its advantages over two existing benchmark observers. For future work, we plan to generalize this framework to hybrid and switched nonlinear systems and consider other forms of CPS attacks.
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A Appendix

In this Appendix, we provide proofs for our propositions, lemmas and theorems. First, for the sake of reader’s convenience, we restate a lemma from [36] that we will frequently use in deriving some of our results.

Lemma A.1 [36, Lemma 2.2] Let $D$, $S$ and $F$ be real matrices of appropriate dimensions and $F^T F \leq I$. Then, for any scalar $\epsilon > 0$ and $x, y \in \mathbb{R}^n$,

\[ 2x^T DFSy \leq \epsilon^{-1}x^T DD^T x + \epsilon y^T S^T Sy. \]

### A.1 Proof of Proposition 2.7

The results follow from the facts that an inequality in $\mathbb{R}$ is preserved by multiplying the both sides by a non-negative number, or by multiplying the left hand side by a non-negative number that is not greater than 1, or by increasing the right hand side, as well as $A \preceq B \implies x^T (A - B)x \preceq 0$. \(\square\)

### A.2 Proof of Proposition 2.8

Considering $M = \begin{bmatrix} -I & 0 \\ 0 & L_f^2 \end{bmatrix}$, we have $[(\Delta f)^T (\Delta q)^T] M = (\Delta f)^T \Delta f + \lambda L_f^2 (\Delta q)^T \Delta q \geq 0$, where the inequality is implied by the Lipschitz continuity of $f(\cdot)$. \(\square\)

### A.3 Proof of Proposition 2.9

By definition, $f$ is $\delta$-QC with multiplier matrix $M$ means that $[(\Delta f)^T (\Delta q)^T] M [(\Delta f)^T (\Delta q)^T]^T \geq 0$. Then, it follows in a straightforward manner that $[(\Delta f)^T (\Delta q)^T] (-M) [(\Delta f)^T (\Delta q)^T]^T \leq \gamma$ for every $\gamma \geq 0$. \(\square\)

### A.4 Proof of Proposition 2.12

We observe that $[(\Delta f)^T (\Delta x)^T] M [(\Delta f)^T (\Delta x)^T]^T = (\Delta f)^T (\Delta f) \leq L_f^2 \|x\|^2 \leq L_f^2 (2r)^2 = 4r^2 L_f^2$, where the second and third inequalities hold by Lipschitz continuity of $f(\cdot)$ and boundedness of the state space, respectively. \(\square\)

### A.5 Proof of Lemma 2.13

First, notice that $\Delta f = A \Delta x + \Delta g$. Given this and $\|g(x)\| \leq r$, we can conclude that $[(\Delta f)^T (\Delta x)^T] M (\Delta f)^T (\Delta f) + (\Delta x)^T A^T A (\Delta x) = (\Delta f - A \Delta x)^T A f - A \Delta x) = (\Delta g)^T (\Delta g) \leq (2r)^2$. \(\square\)

### A.6 Proof of Proposition 2.14

By construction, we have $M - \begin{bmatrix} I_{n \times n} & -A \\ -A^T & A^T A \end{bmatrix} = \begin{bmatrix} M_{11} - I & 0 \\ 0 & M_{22} - M_{12} M_{12} \end{bmatrix} \succeq 0$, since both submatrices on the diagonal are positive semi-definite by assumption. \(\square\)

### A.7 Proof of Proposition 2.15

The global Lipschitz continuity of LPV systems can be shown as follows:

\[ \Delta f_k \triangleq \|f(x_1 - x_2)\| = \|\sum_{i=1}^{N} \lambda_{i,k} A_i \Delta x_k\| \leq \sum_{i=1}^{N} \lambda_{i,k} \|A_i\| \|\Delta x_k\| \leq \|A^m\| \|\Delta x_k\|, \]

with $\|A^m\| = \max_{i=1,\ldots,N} \|A_i\|$, where the first and second inequalities hold by sub-multiplicative inequality for norms and positivity of $\lambda_{i,k}$, the third inequality holds by the facts that $0 \leq \lambda_{i,k} \leq 1$ and $\sum_{i=1}^{N} \lambda_{i,k} = 1$. \(\square\)
A.8 Proof of Lemma 5.1

Aiming to derive the governing equation for the evolution of the state errors, from (7) and (8), we obtain

$$\dot{d}_{1,k} = M_1(C_1 \dot{x}_{k|k} + \Sigma d_{1,k} + v_{1,k}).$$  \hspace{1cm} (A.1)

Moreover, from (3), (7) and (9)–(12), we have

$$\dot{d}_{2,k-1} = M_2[C_2(\Delta f(x_{k-1}) + G_1 \dot{d}_{1,k-1} + G_2 d_{2,k-1} + w_{k-1}) + v_{2,k}],$$  \hspace{1cm} (A.2)

and by plugging $M_1 = \Sigma^{-1}$ into (A.1), we obtain

$$\dot{d}_{1,k} = d_{1,k} - \dot{d}_{1,k} = -M_1(C_1 \dot{x}_{k|k} + v_{1,k}),$$  \hspace{1cm} (A.3)

where $\Delta f(x_k) \triangleq f(x_k) - f(\dot{x}_k)$. Then, by setting $M_2 = (C_2 G_2)^\top$ in (A.2) and using (A.3), we have

$$\dot{d}_{2,k-1} = -M_2(C_2(\Delta f(x_{k-1}) - G_1 M_1 C_1 \dot{x}_{k-1|k-1}) + w_{k-1} + v_{2,k}).$$

Furthermore, it follows from (3), (11) and (12) that

$$\tilde{x}_{k|k} = (I - LC_2)\tilde{x}_{k|k} - \tilde{w}_k,$$

where $\tilde{w}_k \triangleq v_{2,k}$, is a Class II function.

$$\tilde{x}_{k|k} = \Phi[\Delta f(x_{k-1}) - G_1 M_1 C_1 \dot{x}_{k-1|k-1}] + \tilde{w}_k,$$  \hspace{1cm} (A.6)\hspace{1cm} (A.7)

Now we consider each of the four cases, separately.

\begin{itemize}
  \item Case (0): Applying Lemma A.1 to (A.11), we obtain
    $$\Delta V_k^{\text{wn}} \leq -\Delta f_k^\top \Theta \Delta f_k + \tilde{x}_{k|k}^\top \Xi \tilde{x}_{k|k} + 2\Delta f_k^\top \Lambda^i \tilde{x}_{k|k} \hspace{1cm} (A.12)$$

    with $\Theta$, $\Xi$, and $\Lambda^i$ defined in (16) and $\Psi_i$ defined in (15).

    Finally, (A.12) and (15) imply that $\Delta V_k^{\text{wn}} \leq 0$.

  \item Case (I): Adding and subtracting $\Delta f_k^\top \Phi \Delta f_k$ from the right hand side of (A.11), as well as from the Lipschitz continuity of $f(.)$, we have
    $$\Delta V_k^{\text{wn}} \leq \Delta f_k^\top \Phi (S + C_2^2 \tilde{L}) \Phi \Delta f_k + \tilde{x}_{k|k}^\top (\Psi \Phi + P) \tilde{x}_{k|k} - 2\Delta f_k^\top \Phi S \Phi \tilde{x}_{k|k},$$  \hspace{1cm} (A.13)\hspace{1cm} (A.14)

    Now, applying Lemma A.1 to (A.13) results in (A.12) with $\Theta$, $\Xi$, and $\Lambda^i$ defined in (18) and $\Psi_i$ defined in (15), which implies that $\Delta V_k^{\text{wn}} \leq 0$.

  \item Case (II): To prove this, we first derive the following lemma.
  \end{itemize}

**Lemma A.2.** Suppose $f(.)$ is a Class II function. Then, at each time step $k$, $\Delta f_k$ can be decomposed into a linear function of $\tilde{x}_{k|k}$ and a bounded norm uncertain nonlinear term, i.e., $\Delta f_k = A \tilde{x}_{k|k} + s_k$, where $\|s_k\| \leq \gamma$. 

for each case. First, notice that for cases (0)–(II) and considering $Y = PL$, $\Pi \geq 0 \iff I - \Gamma \geq 0$ and

$$\begin{bmatrix}
\Gamma & Y \\
Y^\top & P
\end{bmatrix} \succeq 0,$$

which by pre- and post-multiplication by

$$\begin{bmatrix}
I & 0 \\
0 & P^{-1}
\end{bmatrix},$$

is equivalent to $I - \Gamma \geq 0$ and

$$0 \preceq \tilde{L}^\top P \tilde{L} \succeq I.$$  \hspace{1cm} (A.10)
**Proof.** Define \( s_k \triangleq \Delta f(x_k) - A\hat{x}_{k|k} \). Then, notice that
\[
\|s_k\|^2 = s_k^\top s_k = (\Delta f(x_k) - A\hat{x}_{k|k})^\top (\Delta f(x_k) - A\hat{x}_{k|k})
\]
\[
= \Delta f(x_k)^\top \Delta f_k - 2\hat{x}_{k|k}^\top A^\top \Delta f(x_k) + \hat{x}_{k|k}^\top A^\top A\hat{x}_{k|k}
\]
\[
= \left[(\Delta f(x_k))^\top \hat{x}_{k|k}\right] M \left[(\Delta f(x_k))^\top \hat{x}_{k|k}\right]^\top \leq \gamma^2,
\]
where the last inequality holds since \( f(.) \) is a DQC* function.
\[ \square \]

Now, from Lemma A.2 and (A.8), we have
\[
\hat{x}_{k+1|k+1} = (I - \hat{L}C_2)\Phi(s_k - (\Psi - A)e_k).
\]
Comparing this with (A.8), the rest of the proof is similar to the one for case (0), with the only difference being the use of \( \Delta f_k \) and \( \Psi \) in the place of \( s_k \) and \( \Psi - A \), respectively.

- **Case (III):** By \( f(.) \) being an LPV function as well as (A.8), we obtain
\[
\hat{x}_{k+1|k+1} = (I - \hat{L}C_2)\Phi \hat{A}_k \hat{x}_k,
\]  
(A.14)

where \( \hat{A}_k \triangleq \sum_{i=1}^{N} \lambda_i k(A_i^\top - \Psi) \). Then, the result follows directly from applying [26, Lemma 1].
\[ \square \]

A.10 Proof of Lemma 5.4

We define a similar candidate Lyapunov function \( V^{wn}_k \) as in the proof of Theorem 5.3, and we show that (20) implies \( \Delta V^{wn}_k \leq 0 \). First, notice that (20) is equivalent to
\[
\Delta \geq 0 \text{ and } \begin{bmatrix} I & (I - \hat{L}C_2)^\top P \\ P(I - \hat{L}C_2) \end{bmatrix} \succeq 0,
\]
by pre- and post-multiplication by \( \begin{bmatrix} P(-\frac{1}{2}) & 0 \\ 0 & P^{-1} \end{bmatrix} \), in turn, equivalent to
\[
\begin{bmatrix} P^{-1} & P(-\frac{1}{2}) (I - \hat{L}C_2)^\top \\ (I - \hat{L}C_2)P(-\frac{1}{2}) & P^{-1} \end{bmatrix} \succeq 0,
\]
and \( \Delta \geq 0 \). Applying Schur complement, we obtain equivalently that \( \Delta \geq 0 \) and \( P^{-1} - P(-\frac{1}{2})(I - \hat{L}C_2)^\top P(I - \hat{L}C_2)(P^{-\frac{1}{2}}) \succeq 0 \). Pre- and post-multiplication by \( P^\frac{1}{2} \) returns, equivalently,
\[
\Delta \triangleq P - 2L_2^2 \lambda_{\text{max}}(\Phi^\top \Phi) I - 2\Psi^\top \Phi^\top \Phi \Psi > 0,
\]  
(A.15)

\[
(I - \hat{L}C_2)^\top P(I - \hat{L}C_2) < I.
\]  
(A.16)

Finally, by (A.9), (A.16), Lemma A.1, Lipschitz continuity of \( f(.) \) and (A.15), we obtain
\[
\Delta V^{wn}_k \leq \Delta f_k^\top (2\Phi^\top \Phi) \Delta f_k + \hat{x}_{k|k}^\top (2\Psi^\top \Phi^\top \Phi \Psi - P) \hat{x}_{k|k}
\]
\[
\leq 2\lambda_{\text{max}}(\Phi^\top \Phi) \Delta f_k^\top \Delta f_k + \hat{x}_{k|k}^\top (2\Psi^\top \Phi^\top \Phi \Psi - P) \hat{x}_{k|k}
\]
\[
\leq \hat{x}_{k|k}^\top (2L_2^2 \lambda_{\text{max}}(\Phi^\top \Phi) I + 2\Psi^\top \Phi^\top \Phi \Psi - P) \hat{x}_{k|k} \leq 0.
\]
\[ \square \]

A.11 Proof of Lemma 5.5

To show that uniform detectability is sufficient for existence of an observer, notice that for a Class III function \( f(.) \), (14) can be written as
\[
\hat{x}_{k|k} = (I - \hat{L}C_2)\bar{A}_{k-1} \hat{x}_{k-1|k-1} + (I - \hat{L}C_2)\bar{w}_{k-1} - \hat{L}\bar{v}_{k-1},
\]  
(A.17)

where
\[
\bar{w}_{k-1} \triangleq -(I - G_2 M_2 C_2)(G_1 M_1 v_{1,k-1} - w_{k-1}),
\]
\[
\bar{A}_k \triangleq \Phi \sum_{i=1}^{N} \lambda_{i,k} A_i^\top - \Psi, \quad \bar{v}_{k-1} \triangleq v_{2,k}.
\]

Now, consider the following linear time-varying system without unknown inputs:
\[
x_{k+1} = \bar{A}_k x_k + \bar{w}_k, y_k = C_2 x_k + \bar{v}_k.
\]  
(A.18)

Systems (A.17) and (A.18) are equivalent from the viewpoint of estimation, since the estimation error equations for both problems are the same, hence they both have the same objective. Therefore, the pair \((\bar{A}_k, C_2)\) needs to be uniformly detectable such that the observer is stable [3, Section 5].

Moreover, as for the necessity of the strong detectability of the constituent LTI systems, suppose for contradiction, that there exists a stable observer for system (3) with any sequence \( \{\lambda_{i,k}\}_{k=0}^\infty \) for all \( i \in \{1, 2, \ldots, N\} \) that satisfies \( 0 \leq \lambda_{i,k} \leq 1, \sum_{i=1}^{N} \lambda_{i,k} = 1, \forall k \), but one of the constituent linear time-invariant systems (e.g., \( (A^j, G, C, H) \)) is not strongly detectable. Since the observer exists for any sequence of \( \lambda_{i,k} \), that means that an observer also exists when \( \lambda_{j,k} = 1 \) and \( \lambda_{i,k} = 0 \), \( \forall i \neq j \) for all \( k \). However, we know from [37] that strong detectability is necessary for the stability of the linear time-invariant system \( (A^j, G, C, H) \), which is a contradiction. Hence, the proof is complete.
\[ \square \]

A.12 Proof of Theorem 5.8

We use a similar approach as in the proof of Theorem 5.3 for Class 0, I and II systems and a different approach for Class III systems. First, for Class 0, I and II systems, consider the error dynamics with bounded noise signals (14) and the candidate Lyapunov function \( V^n_k \triangleq x_{k|k}^\top P x_{k|k} \). Observe that
\[
\Delta V^n_k \triangleq V^n_{k+1} - V^n_k = \Delta V^{wn}_k + \Delta r_k,
\]  
(A.19)
where $V_k^{wn}$ is the Lyapunov function for the error dynamics without noise signals, defined in (A.9), and
\[
\Delta r_k \triangleq 2(\Delta f_k^T - \tilde{x}_{k|k}^T (\Phi \Psi) (I - \tilde{L}C_2)^T PW(\tilde{L}) \tilde{m}_k \\
+ \tilde{m}_k^T W(\tilde{L})^T PW(\tilde{L}) \tilde{m}_k),
\] (A.20)
with $\Phi, \Psi, \tilde{m}_k$ and $W(\tilde{L})$ defined in Lemma 5.1. We will show for each of the cases (0), (I) and (II) that
\[
\Delta V_k^{wn} \leq 0 \quad \text{follows from Theorem 5.3},
\] (A.21)
and
\[
\Delta V_k^{wn} \leq \eta^2 \tilde{w}_k^T \tilde{w}_k - \tilde{x}_{k|k}^T \tilde{x}_{k|k}.
\] (A.22)
Summing up both sides of (A.22) from zero to infinity, returns $V_k^{wn} - V_0^{wn} \leq \eta^2 \sum_{k=0}^{\infty} \tilde{w}_k^T \tilde{w}_k - \sum_{k=0}^{\infty} \tilde{x}_{k|k}^T \tilde{x}_{k|k} = \eta^2 \sum_{k=0}^{\infty} \tilde{w}_k^T \tilde{w}_k - \sum_{k=0}^{\infty} \tilde{x}_{k|k}^T \tilde{x}_{k|k}$, where at each time step $k$, $\tilde{w}_k^T = \begin{bmatrix} \tilde{w}_k^T & \tilde{v}_k^T \end{bmatrix}$. Then, it follows from setting the initial conditions to zero that $\sum_{k=0}^{\infty} \tilde{x}_{k|k}^T \tilde{x}_{k|k} \leq \eta^2 \sum_{k=0}^{\infty} \tilde{w}_k^T \tilde{w}_k$.

Thus, it remains to show that (A.21) holds for each case (0)–(II). Plugging the expression for $W(\tilde{L})$ from Lemma 5.1 into (A.20), we obtain
\[
\Delta r_k = \tilde{x}_{k|k}^T \tilde{x}_{k|k} + 2(\Delta f_k - \Psi \tilde{x}_{k|k})^T \Phi \Psi \Phi^T (PR - \Omega) - C_2 \tilde{L}^T P \tilde{L} \Omega \tilde{m}_k + \tilde{m}_k^T (R^T PR - R^T \Omega - \Omega^T Y R + \Omega^T \Gamma \Omega - \eta^2 I) \tilde{m}_k,
\]
which by (A.10) and Lemma A.1, implies that:
\[
\Delta r_k \leq \tilde{m}_k^T (R^T PR - R^T \Omega) \Omega \tilde{m}_k + \tilde{x}_{k|k}^T (I + \Psi \Phi \Phi^T C_2 \tilde{C}_2 \Psi) \tilde{x}_{k|k} + 2 \tilde{f}_k^T \Phi (PR - \Omega - C_2 \tilde{L}^T PR) \tilde{m}_k - 2 \tilde{x}_{k|k}^T \Psi \Phi (PR - \Omega - C_2 \tilde{L}^T PR) \tilde{m}_k \leq \Delta f_k^T (\Phi C_2^T \tilde{C}_2 \Phi) \Delta f_k.
\] (A.23)
\[
\Delta r_k \leq \Delta f_k^T (\Phi C_2^T \tilde{C}_2 \Phi) \Delta f_k \leq \lambda_{max}(\Phi^T C_2^T \tilde{C}_2 \Phi) \Delta f_k^T \Delta f_k \leq \tilde{x}_{k|k}^T (L^2 \tilde{L} A_{max}(\Phi^T C_2^T \tilde{C}_2 \Phi) I) \tilde{x}_{k|k},
\] (A.24)
where the second inequality is implied by Lipschitz continuity of $f(.)$. Then, it can be concluded from (A.23) and (A.24) that $\Delta r_k \leq -\zeta^T \psi \leq 0$, where $\zeta \triangleq \begin{bmatrix} \tilde{m}_k^T & \tilde{x}_{k|k}^T & \Delta f_k^T \end{bmatrix}^T$ and $\psi$ is the matrix in (22) with its elements defined in (24).

- **Case (II):** By Lemma A.2 and (A.20) we have
\[
\Delta r_k = \tilde{m}_k^T W(\tilde{L})^T PW(\tilde{L}) \tilde{m}_k + 2(\tilde{x}_{k|k}^T (A - \Psi) \Psi + s_k^T) \Phi^T (I - \tilde{L}C_2)^T PW(\tilde{L}) \tilde{m}_k,
\] (A.25)
where $s_k = \Delta f_k - \Psi \tilde{x}_{k|k}$. Comparing (A.25) with (A.20), the rest of the proof is similar to the one for case (0), by replacing $\Delta f_k$ and $\Psi$ with $s_k$ and $\Psi - A$, respectively, which results in $\Delta r_k \leq -\zeta^T \psi \leq 0$, where $\zeta \triangleq \begin{bmatrix} \tilde{m}_k^T & \tilde{x}_{k|k}^T & \Delta f_k^T \end{bmatrix}^T$ and $\psi$ is the matrix in (22) with its elements defined in (25).

- **Case (III):** For this case, we consider a different approach compared to the previous cases. By $f(.)$ being LPV and (14), we can define a system with $\tilde{x}_{k|k}$ as its state and $\tilde{z}_{k|k} = \tilde{x}_{k|k}$ as the output:
\[
\tilde{x}_{k|k} = (I - \tilde{L}C_2) \tilde{A}_{k-1} \tilde{x}_{k-1|k-1} + [(I - \tilde{L}C_2) R + \tilde{L}Q] \tilde{w}_{k-1},
\] (A.26)
\[
\tilde{z}_{k|k} = \tilde{x}_{k|k},
\]
where $\tilde{A}_{k} \triangleq \Phi \sum_{i=1}^{N} \lambda_{i,k} (A^i - \Psi)$, each $A^i$ is a constituent matrix of $f(.)$ and $\Phi$ and $\Psi$ are defined in Lemma 5.1. Now, by [12, Lemma 3], system (A.26) has an $H_\infty$ performance bounded by $\eta$, if there exists a symmetric positive definite matrix $S$ such that:
\[
S \triangleq 
\begin{bmatrix}
S & (I - \tilde{L}C_2) \tilde{A}S & (I - \tilde{L}C_2) R + \tilde{L}Q & 0 \\
* & S & 0 & S \\
* & * & \eta I & 0 \\
* & * & * & \eta I
\end{bmatrix} > 0,
\] (A.27)
where $\tilde{A} \triangleq \Phi (A^i - \Psi)$. Notice that the referenced lemma requires the existence of a bounded matrix sequence, which in our case is a sequence of time-invariant matrices ($S$ is the same for each $k$), that is obviously bounded. By plugging $P = S^{-1} > 0$, defining $P = P^T \triangleq
\begin{bmatrix}
0 & P & 0 & 0 \\
P & 0 & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
$ and applying some
similarity transformations, we obtain
\[
PSP = \begin{bmatrix} P \overline{\overline{A}}^T (I - C_2^T \bar{L}^T) P & 0 & I \\ \ast & P & (I - \bar{L} C_2) R + \bar{L} Q \\ \ast & \ast & I \\ \ast & \ast & \ast \end{bmatrix} \eta I \geq 0, \forall i \in \{1, 2, \ldots, N\}.
\]

Setting \( Y \triangleq P \bar{L} \) completes the proof. \( \square \)

\subsection*{A.13 Proof of Theorem 5.9}

(I) If \( f(.) \) is a Class I function, then, the result in (27) with \( \theta \) and \( \overline{\overline{\eta}} \) defined in (29), directly follows from Lipschitz continuity of \( f(.) \), as well as applying triangle and sub-multiplicative inequalities for norms on (14). Moreover, the result in (28) with \( \beta \) and \( \overline{\overline{\beta}} \) defined in (29), is obtained by triangle and sub-multiplicative inequalities, (5), (A.3) and (A.4).

(II) If \( f(.) \) is a Class II function, by Lemma A.2, (14), (A.3), (A.4) and triangle and sub-multiplicative inequalities, we obtain the results in (27) and (28) with \( \theta, \overline{\overline{\eta}}, \beta \) and \( \overline{\overline{\beta}} \) defined in (30).

(III) If \( f(.) \) is a Class III function, we first need to find closed form expressions for the state and input estimation errors through the following lemma.

\textbf{Lemma A.3} The state and input estimation errors are
\[
\hat{x}_{k|k} = \sum_{i=1}^{k-2} A_{e,k-j}(\Psi \hat{w}_{k-j} - \bar{L}\hat{v}_{k-i}) + \sum_{i=1}^{k-1} A_{e,k-j}\hat{x}_{0|0},
\]
\[
\hat{d}_{k-1} = -\sum_{i=1}^{N} \lambda_{i,k-1}(V_1 M_1 C_1 + V_2 M_2 C_2 A_{e,i})\hat{x}_{k-1|k-1} + \sum_{i=1}^{N} \lambda_{i,k-1}(\bar{V}_1 M_1 - \bar{V}_2 M_2 C_2 A_{e,i})\hat{x}_{k-1|k-1} + \bar{V}_2 M_2 C_2 w_{k-1} - V_2 M_2 T_2 v_k.
\]

\textbf{Proof.} Starting from (A.17) and applying simple induction return the results for the state errors. Then, the expression for the input errors follows from (A.3), (A.4) and (5).

Now, we are ready to prove Theorem 5.9 for LPV functions. First, we define
\[
B_{e,k} \triangleq \prod_{j=0}^{k-1} A_{e,k-j}, \quad C_{e,k}^i \triangleq \prod_{j=0}^{k-2} A_{e,k-j}, \quad \hat{t}_k \triangleq \Psi \hat{w}_k - \bar{L}\hat{v}_k,
\]
for \( 1 \leq i \leq k \). Then, from Lemma A.3, we have
\[
\| \hat{x}_{k|k} \| \leq \| B_{e,k} \| \| \hat{x}_{0|0} \| + \sum_{i=1}^{k} C_{e,k}^i \| \hat{t}_{k-i} \|,
\]
by triangle inequality and submultiplicativity of norms. Moreover, by similar reasoning, we find
\[
\| B_{e,k} \| \leq \prod_{j=0}^{k-1} \sum_{i=1}^{N} \lambda_{i,k-j} \| \Psi \Phi(A^i - G_1 M_1 C_1) \| \leq \theta^k,
\]
\[
\| \prod_{i=1}^{k} C_{e,k}^i \| \| \hat{t}_{k-i} \| \leq \sum_{i=1}^{k} \| C_{e,k}^i \| \| \hat{t}_{k-i} \|,
\]
\[
\| C_{e,k}^i \| \leq \prod_{j=0}^{k-2} \sum_{i=1}^{N} \lambda_{i,k-j} A_{e,i} \| \leq \theta^{-1}.
\]
Moreover, from (A.28),
\[
\| \hat{t}_{k-i} \| = \| \Psi \Phi \hat{w}_{k-i} \| \leq \overline{\overline{\eta}},
\]
with \( \overline{\overline{\rho}} \triangleq -\Psi \Phi G_1 M_1 T_1 + \Psi G_2 M_2 T_2 + \bar{L}T_2 \). Then, from (A.29)–(A.31), we obtain (27) with \( \theta \) and \( \overline{\overline{\eta}} \) defined in (31). Furthermore, the result in (28) with \( \beta \) and \( \overline{\overline{\beta}} \) defined in (31), follows from applying Lemma A.3, as well as triangle inequality, the facts that \( 0 \leq \lambda_{i,k} \leq 1 \), \( \sum_{i=1}^{N} \lambda_{i,k} = 1 \) and sub-multiplicativity of matrix norms.

Finally, the steady state values are obtained by taking the limit from both sides of (27) and (28), assuming \( \theta < 1 \). \( \square \)

\subsection*{A.14 Proof of Corollary 5.10}

Clearly \( \| A_{e,i} \| < 1 \) implies that \( \theta < 1 \), which is a sufficient condition for the convergence of errors by Theorem 5.9. \( \square \)