Boundary asymptotics of the relative Bergman kernel metric for curves

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Abstract
We study the behaviors of the relative Bergman kernel metrics on holomorphic families of degenerating hyperelliptic Riemann surfaces and their Jacobian varieties. Near a node or cusp, we obtain precise asymptotic formulas with explicit coefficients. In general the Bergman kernels on a given cuspidal family do not always converge to that on the regular part of the limiting surface, which is different from the nodal case. It turns out that information on both the singularity and the complex structure contributes to various asymptotic behaviors of the Bergman kernel. Our method involves the classical Taylor expansion for Abelian differentials and period matrices.

Mathematics Subject Classification
Primary 32A25; Secondary 32G20 · 32G15 · 14H45 · 32Q28 · 14D06

1 Introduction

On an $n$-dimensional complex manifold $X$, the Bergman kernel is a reproducing kernel of the space of square integrable holomorphic top forms. For the canonical bundle $\mathcal{K}$, which is the $n$th exterior power of the cotangent bundle on $X$, the Bergman kernel is defined as

$$\kappa := \sum e_j \otimes \overline{e_j},$$

where $\{e_j\}$ is a complete orthonormal basis of $H^0_{\mathbb{L}_2}(X, \mathcal{K})$. Thus, the Bergman kernel is a canonical volume-form determined only by the complex structure. As the complex structure deforms, the log-plurisubharmonic variation results of the Bergman kernels on pseudoconvex domains were obtained by Maitani and Yamaguchi [49] and later by Berndtsson [1, 2]. More generally, further important developments on general Stein manifolds and complex projective algebraic manifolds [2, 7, 17, 48, 61, 70, 72] state certain positivity properties of the direct images of the relative canonical bundles, and turn out to have intimate connections with
the space of Kähler metrics \([3, 18, 19, 71]\), the invariance of pligrgenera \([8, 46, 59, 65, 66]\), and the sharp Ohsawa-Takegoshi theorem \([6, 9, 11, 15, 34–36, 53, 58, 64]\). Conversely, the Ohsawa-Takegoshi theorem with optimal constant was applied in \([11, 34]\) to prove the variation results of the Bergman kernels.

Consider the family \(X \xrightarrow{\pi} \Delta\), where \(X\) is a complex \((n+1)\)-dimensional Stein manifold fibred over the unit disc \(\Delta \subset \mathbb{C}\), and \(\pi\) is a surjective holomorphic map with connected fibers. Moreover, \(\pi\) is a holomorphic submersion over \(\Delta^* := \Delta \setminus \{0\}\). For \(\lambda \in \Delta^*\), the fibres \(X_\lambda = \pi^{-1}(\lambda)\) are \(n\)-dimensional Stein manifolds, whose Bergman kernels on the diagonal are denoted by \(\kappa_{X_\lambda}\). Let \((z^1, \ldots, z^n, \lambda)\) be a coordinate of \(X_\lambda \times \Delta^*\), which is a local trivialization of the fibration, and write \(\kappa_{X_\lambda} := k_\lambda(z)(dz^1 \wedge \cdots \wedge dz^n) \otimes (dz^1 \wedge \cdots \wedge dz^n)\) locally. Then, the above mentioned positivity results imply that \(\psi := \log k_\lambda(z)\) is subharmonic with respect to \(\lambda \in \Delta^*\), i.e.,

\[
\frac{\partial^2 \psi}{\partial \lambda \partial \overline{\lambda}} \geq 0,
\]

if the Bergman kernel is not identically zero. Here as \(\lambda\) varies, \(e^\psi\) induces on \(\mathcal{K}_{X/\Delta}\) the so-called relative Bergman kernel metric, which is represented by different local functions, given different local trivializations (see [7]). When the fibration \(\pi\) has a singular fiber \(X_0\), our goal is to quantitatively characterize at degeneration the asymptotic of \(\psi\), as \(\lambda \to 0\).

In fact, the study of the asymptotic analysis of period integrals – and, in particular, the Abelian differentials on algebraic curves – is a very classical subject and has been pursued by many experts in the field of Hodge theory \([13, 40, 41, 73, 77]\), especially the nilpotent orbit theorem \([33, 62]\) in the variations of Hodge structures. In \([32]\), the (pluri)subharmonicity in the base direction of the above \(\psi\) was described as “a pleasant surprise” by Griffiths. It is known classically that the earlier works of Griffiths \([31, 32]\), Fujita \([30]\), as well as other important results in algebraic geometry including \([42, 44]\) played a decisive role in understanding the behavior of \(\psi\). For more background on the \(L^2\)-analysis in several complex variables, and its interaction with Hodge theory, see \([5, 14, 54–57, 60]\).

In the theory of Riemann surfaces and their moduli space, the degenerations of analytic differentials have also been extensively studied via a general method called the pinching coordinate (see \([29, 50, 75]\) and the references therein). The spaces of degenerating Riemann surfaces correspond to paths in the moduli space leading to the boundary points, and are obtained from compact surfaces by shrinking finitely many closed loops to points, called nodes. Near nodal singularities of general curves, Wentworth \([74]\) obtained precise estimates for the Arakelov metric with applications to arithmetic geometry and string theory (see also \([16, 39]\)); Habermann and Jost \([37, 38]\) studied the behavior of the Bergman kernels and their induced \(L^2\) metrics on Teichmüller spaces, with a strong motivation in minimal surface theory. Specifically, the result in \([37]\) shows that the Bergman kernels on a degenerating family of compact Riemann surfaces converge to that on the normalization of the limiting nodal curve, with the second term having asymptotic growth \((-\log |\lambda|)^{-1}\), as \(\lambda \to 0\).

On the other hand, there exist worse singularities such as cusps, with which the pinching coordinate method cannot deal. Boucksom and Jonsson \([10]\) studied the asymptotics of volume forms on degenerating compact manifolds and established a measure-theoretic version of the Kontsevich-Soibelman conjecture, which deals with the limiting behavior of a family of Calabi-Yau manifolds approaching a “cusp” in the moduli space boundary. In \([20–23]\), the author obtained quantitative results for the Bergman kernels on Legendre and other degenerating families of elliptic curves, by using elliptic functions as well as a method based on the Taylor series expansion of Abelian differentials and period matrices (see \([12]\)).
In this paper, for the fibers $X_\lambda$ being algebraic curves or their Jacobian varieties, we determine the precise asymptotic behaviors of $\psi$ at degeneration, using the classical Taylor expansion method. Our first motivation is to write down explicitly the asymptotic coefficients, which involve the complex structure information and reflect the geometry of the base varieties and their singularities, for various families of hyperelliptic curves degenerating to singular ones with nodes or cusps. Our second motivation is to investigate whether similar convergence results hold true for the Bergman kernel on cuspidal degenerating families of curves, in comparison with [10, 37, 38, 63, 74]. It turns out that this is not always the case as seen in Theorem 2.2. However, in Theorem 2.3 we find that the Bergman kernels on some cuspidal families of curves indeed converge at degeneration. As a last motivation, we apply the results on higher genus curves to the study of their Jacobians, and generalize the results in [20, 21, 23] toward higher dimensions.

The organization of this paper is as follows. In Sect. 2, we state our main results on the degeneration of the Bergman kernel. In Sect. 3, we study the Bergman kernel on the normalization of general algebraic curves. In Sects. 4, 5, we work on nodal curves. In Sects. 6, 7 and 8, 9, we work on cuspidal curves of types I and II, respectively. Our results on the Jacobian varieties are proved in Sect. 10.

## 2 Main results

Throughout this paper, a smooth algebraic curve means a Riemann surface, and we consider the complex analytic families of hyperelliptic curves $X_\lambda := \{(x, y) \in \mathbb{C}^2 \mid y^2 = h_\lambda(x) P(x)\}$ of genus $g \geq 2$, parameterized by $\lambda \in \Delta^*$. Here, $P$ denotes a polynomial of degree $2g - 2$ with complex roots $a_j$ such that $1 < |a_1| < |a_2| < \cdots < |a_{2g-2}|$. For each $\lambda \in \Delta^*$, $h_\lambda$ is a degree 3 polynomial with distinct roots of absolute values less than or equal to 1. It is well known that on the smooth curve $X_\lambda$, there exist globally defined Abelian differentials

$$\omega_i := \frac{x^{i-1}dx}{y}, \quad i = 1, \ldots, g. \tag{2.1}$$

By its classical construction, the hyperelliptic curve $X_\lambda$ can be realized as a 2-sheeted ramified covering of the Riemann sphere $\mathbb{P}^1$ (see [12]). Take a homology basis $\delta_i$ and $\gamma_j$ of $H_1(X_\lambda, \mathbb{Z})$ such that their intersection numbers are $\delta_i \cdot \gamma_j = 0 = \gamma_i \cdot \gamma_j$, and $\delta_i \cdot \gamma_j = \delta_{i,j} = -\gamma_j \cdot \delta_i$ (here $\delta_{i,j}$ is the Kronecker $\delta$). Then, for the above $\omega_i \in H^0(X_\lambda, \mathcal{K})$, we set

$$A_{i,j} := \int_{\delta_j} \omega_i, \quad B_{i,j} := \int_{\delta_j} \gamma_i. \tag{2.2}$$

By the Hodge-Riemann bilinear relation, the matrix $(A_{ij})_{1 \leq i, j \leq g}$ is invertible and define the normalized period matrix $Z := A^{-1}B$. Then, $Z$ is symmetric and has a positive definite imaginary part, i.e., $\text{Im} Z > 0$. With respect to the chosen homology basis, we call the matrices $A$, $B$ and $Z$, the $A$-period matrix, $B$-period matrix, and normalized period matrix, respectively. Moreover, we use the symbol “$\sim$” to denote that the ratio of its both sides tends to 1, as $\lambda \to 0$; in the case of matrices, it refers to the entrywise ratio.

### Nodal cases

In affine coordinates, consider a family of hyperelliptic curves

$$X_\lambda := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x - \lambda)(x - 1) P(x)\}. \tag{2.3}$$

As $\lambda \to 0$, $X_\lambda$ degenerates to a singular curve $X_0$ with a non-separating node. The normalization of $X_0$ is a smooth curve $Y := \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x - 1) P(x)\}$, whose $A$-period
matrix and normalized period matrix are denoted by $A_0$ and $Z_0$, respectively. By the $L^2$ removable singularity theorem, $Y$ and the regular part of $X_0$ have the same Bergman kernel, denoted by $\kappa_0$. We write $\kappa_{X_0} = k_0(z)dz \otimes \overline{dz}$ and $\kappa_0 = k_0(z)dz \otimes \overline{dz}$ in the local coordinate $z := \sqrt{x}$ near $(0, 0)$; see their precise formulas in (3.2) and (3.8).

**Theorem 2.1** (Nodal cases) For $X_\lambda$ defined by (2.3), as $\lambda \to 0$, it holds that

(i) $\kappa_{X_\lambda} \to \kappa_0$;

(ii) for $0 \neq |z|$ small,

$$\psi - \log k_0(z) \sim -\frac{\pi}{\log |\lambda|} \frac{1 - 2 \text{Re} \sum_{i=1}^{g-1} ((\text{Im } Z_0)^{-1} \text{Im}(A_0^{-1} \star))_i z^{2i}}{\sum_{i,j=1}^{g-1} ((\text{Im } Z_0)^{-1})_{i,j} (z^i \overline{z}^j)^2},$$

where $\star$ is a column vector with $g - 1$ rows whose entries all equal $-2i\sqrt{P(0)}^{-1}$.

Part (i) is essentially due to Habermann and Jost [37], who in fact have used the pinching coordinate method to study general (possibly non-hyperelliptic) algebraic curves degenerating to singular ones with separating or non-separating nodes. To write down the explicit asymptotic coefficients in Part (ii), we rely on the study of genus two curves in our Theorem 4.1.

**Cuspidal cases** Since in [29, 37–39, 50, 74, 75] only the nodal degeneration was considered, we continue investigating whether similar convergence results hold true for the Bergman kernel on cuspidal degenerating families of curves, and find that in general this is not the case. For the singular curve $X_0 := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3 P(x)\}$ with an ordinary cusp at $(0, 0)$, its normalization is the smooth curve $Y := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x P(x)\}$, whose $A$-period matrix and normalized period matrix are denoted by $A_0$ and $Z_0$, respectively. Then, $Y$ and the regular part of $X_0$ have the same Bergman kernel, denoted by $\kappa_0$. In the following two cases, we explore different families of hyperelliptic curves $X_\lambda$ that give rise to $X_0$, as $\lambda \to 0$.

Similarly, we write $\kappa_{X_\lambda} = k_\lambda(z)dz \otimes \overline{dz}$ and $\kappa_0 = k_0(z)dz \otimes \overline{dz}$ in the local coordinate $z := \sqrt{x}$ near $(0, 0)$; see their precise formulas in Sect. 3.

**Theorem 2.2** (Cuspidal cases, I) For the hyperelliptic curves

$$X_\lambda := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x^2 - \lambda) P(x)\},$$

as $\lambda \to 0$, for $0 \neq |z|$ small, it holds that

(i) $k_\lambda(z) \to k_0(z) + 4|z^4 P(z^2)|^{-1}$, i.e., $\kappa_{X_\lambda} \not\to \kappa_0$;

(ii) $\psi - \log \left(k_0(z) + \frac{4}{|z^4 P(z^2)|}\right) \sim \frac{-2 \text{Re} \sum_{i=1}^{g-1} ((\text{Im } Z_0)^{-1} \text{Im}(A_0^{-1} (\Diamond - \sqrt{-1} \diamond))))_i z^{2i}}{1 + \sum_{i,j=1}^{g-1} ((\text{Im } Z_0)^{-1})_{i,j} (z^i \overline{z}^j)^2},$

where $\Diamond$ and $\diamond$ are column vectors and with $g - 1$ rows involving $\lambda$.

**Theorem 2.3** (Cuspidal cases, II) For the hyperelliptic curves

$$X_\lambda := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x - \lambda)(x - \lambda^2) P(x)\},$$

as $\lambda \to 0$, it holds that

$\lambda$ See their definitions in Lemmata 7.1 and 7.2.
(i) \( \kappa_{X_\lambda} \rightarrow \kappa_0 \);
(ii) for \( 0 \neq |z| \) small,
\[
\psi - \log k_0(z) \sim \frac{4 \pi}{k_0(z)|z^4 P(z^2)|} \cdot \frac{1}{- \log |\lambda|}.
\]

Remarks.

(a) In Theorem 2.2, (ii), the right hand side in the expansion of \( \psi \) is harmonic in \( \lambda \) of order \( O(\lambda^{1/4}) \), which is different from Theorems 2.1 and 2.3 where the growth order \((- \log |\lambda|)^{-1} \) appears. The harmonicity in the Bergman kernel variation detects the triviality of holomorphic fibrations, with applications to the Suita conjecture (see [4, 6, 25–27, 68]).

(b) The proofs of Theorems 2.2 and 2.3 rely on the proofs of Theorems 6.1 and 8.1, respectively, for genus-two curves with more explicit asymptotic coefficients when \( P = (x - a)(x - b) \). In particular, Theorem 8.1 says that the right hand side in Theorem 2.3, (ii), reduces to
\[
\pi \cdot \text{Im} \tau \cdot \frac{1}{- \log |\lambda| \cdot |z|^4},
\]
where \( \tau \) is the period (scalar) of the elliptic curve \( (x, y) \in \mathbb{C}^2 \mid y^2 = x(x - a)(x - b) \).

(c) In Theorems 2.1 and 2.3, the coefficients of the first and second terms in the expansions of \( \psi \) depend only on the information away from the singularity. The boundary behavior of canonically defined plurisubharmonic functions often contains significant information, in general. Recall that (see [45]) the Lelong number for a plurisubharmonic function \( u \) in a neighbourhood of \( 0 \in \mathbb{C}^n \) is defined as
\[
\lim_{r \to 0^+} \sup_{|z| = r} \frac{u(z)}{\log r},
\]
which is the supreme of all numbers \( v \geq 0 \) such that \( u(z) \leq v \log |z| + O(1) \) near 0. For a holomorphic family of Riemann surfaces \( X_\lambda \) parameterized by \( \lambda \in \Delta^* \), each fiberwise Bergman kernel is locally written as \( \kappa_{X_\lambda} := k_\lambda(z)dz \otimes d\bar{z} \), in some coordinate \( z \) for some function \( k_\lambda \). By (1.2), the potential \( \psi := \log k_\lambda(z) \) is subharmonic with respect to \( \lambda \in \Delta^* \). Moreover, the boundedness of \( \psi \) near \( \lambda = 0 \) proved in Theorems 2.1, 2.2 and 2.3 yields the following corollary.

**Corollary 2.4** For each family of hyperelliptic curves defined by (2.3), (2.4) and (2.5), as \( \lambda \to 0 \), \( \psi := \log k_\lambda(z) \) extends to a subharmonic function in \( \lambda \in \Delta \), where the local coordinate is \( z := \sqrt{x} \neq 0 \). Moreover, \( \psi \) has Lelong number zero at \( \lambda = 0 \).

Thus, in our studies of curves, the subharmonic function \( \psi \) tends to be small, not big, near the singular fibers (cf. [7]).

**Jacobian varieties** For a curve \( X_\lambda \) of genus \( g \geq 2 \), its Jacobian variety \( \text{Jac}(X_\lambda) \) is a \( g \)-dimensional complex torus whose Bergman kernel by the definition (1.1) is written as \( \mu_\lambda(dw^1 \wedge \cdots \wedge dw^g) \otimes (dw^1 \wedge \cdots \wedge dw^g) \), under the coordinate \( (w^1, \ldots, w^g) \) induced from \( \mathbb{C}^g \). For each family of hyperelliptic curves defined by (2.3), (2.4), or (2.5), we determine the asymptotic behaviors of the Bergman kernels on the naturally associated family of Jacobians \( \text{Jac}(X_\lambda) \) as follows.
Theorem 2.5 (Jacobian varieties) As $\lambda \to 0$, it holds that
\[
\log \mu_\lambda = \begin{cases}
- \log(-\log |\lambda|) + \log \det(\text{Im } Z_0) + O\left( (\log |\lambda|)^{-1} \right), & \text{for } X_\lambda \text{ in (2.3) or (2.5)}; \\
- \log \det(\text{Im } Z_0) + O(\lambda^{1/2}), & \text{for } X_\lambda \text{ in (2.4)}.
\end{cases}
\]

It is worth mentioning that as the curves $X_\lambda$ in (2.4) degenerate to a cuspidal one, the family of their Jacobian varieties $\text{Jac}(X_\lambda)$ has a smooth fiber at $\lambda = 0$. However, for two other families of curves $X_\lambda$ in (2.3) and (2.5), their Jacobian varieties $\text{Jac}(X_\lambda)$ indeed degenerate, as $\lambda \to 0$. Theorem 2.5 generalizes the results in [20, 21, 23] on elliptic curves toward higher dimensional Abelian varieties. One may further compare our results with recent works [28, 43, 67, 69, 76]. An announcement of this paper for curves appeared in [24].

3 Bergman kernel on hyperelliptic curves

The Bergman kernel, defined in (1.1), is independent of the choices of the orthonormal basis for the Hilbert space of $L^2$ holomorphic top-forms. We consider the complex analytic families of hyperelliptic curves $X_\lambda := \{(x, y) \in C^2 \mid y^2 = h_\lambda(x) P(x)\}$ of genus $g \geq 2$, parameterized by $\lambda \in \Delta^*$. It is known (cf. [52, 74]) that the Bergman kernel $\kappa_{X_\lambda}$ on the curve $X_\lambda$ can be written as
\[
\kappa_{X_\lambda} := \sum_{i,j=1}^{g} ((\text{Im } Z)^{-1})_{i,j} \omega_i \otimes \bar{\omega}_j,
\]
where $\omega_i$ is given in (2.1) and $Z$ is the normalized Riemann period matrix defined by (2.2). To get the above alternative definition of $\kappa_{X_\lambda}$, it suffices to observe that by the Hodge-Riemann bilinear relation
\[
\frac{\sqrt{-1}}{2} \int_{X_\lambda} \omega_i \wedge \bar{\omega}_j = (\text{Im } Z)_{ij},
\]
which is independent of the choice of the homology basis. Since each $\omega_i$ is a linear combination of $\{e_j\}_{j=1}^{g}$, a complete orthonormal basis of $H^0_{L^2}(X, K)$, the Bergman kernel defined in (1.1) can be alternatively written as above in view of (3.1).

For our convenience, we fix once and for all a canonical homology basis $\delta_i$ and $\gamma_j$ of $H_1(X_\lambda, \mathbb{Z})$ for each family of hyperelliptic curves defined by (2.3), (2.4) and (2.5) in the following way. On $X_\lambda$ there is a canonical involution induced by $y \mapsto -y$, which we denote by $\sigma: X_\lambda \to X_\lambda$, and the fixed points of $\sigma$ are the Weierstraß points. For concreteness, we first deal with the family of curves defined by (2.3), and the Weierstraß points correspond to the points $x = 0, x = \lambda, x = 1, x = a_1, \ldots, x = a_{2g-2}$ and the point $\infty$. We denote them by $W_1, \ldots, W_{2g+2}$, where $W_{2g+2} = \infty$. We let $\delta_1$ be a cycle lying in a single sheet of $X_\lambda$ such that it contains $0, \lambda, \in \mathbb{C}$. Let $\delta_2$ be a cycle lying in a single sheet of $X_\lambda$ such that it contains $0, \lambda, 1, a_1 \in \mathbb{C}$. Similarly, for each $j = 3, \ldots, g$, let $\delta_j$ be a cycle lying in a single sheet of $X_\lambda$ such that it contains $0, \lambda, 1, a_1, \ldots, a_{2j-3} \in \mathbb{C}$. On the other hand, let $\gamma_1$ be a cycle passing from one sheet to the other as we pass the branch cut such that it contains only two Weierstraß points $W_2$ and $W_3$. Thus, $\gamma_1$ is made of two pieces, one in one sheet and one in the other sheet. In general, for each $j = 2, \ldots, g$, let $\gamma_j$ be a cycle passing from one sheet to the other as we pass the branch cut such that it contains only two Weierstraß points $W_{2j}$ and $W_{2j+1}$. Next, for the family of curves defined by (2.4), the Weierstraß points correspond to the points $x = -\sqrt{\lambda}, x = 0, x = \sqrt{\lambda}, x = a_1, \ldots, x = a_{2g-2}$ and the point.
Asymptotics of the relative Bergman kernel

A nodal Riemann surface

Nodal family

\( \lambda \)

Cuspidal family, Case I

degenerates to a singular curve

\( X \)

degenerates to a node. For the family of hyperelliptic curves in (2.3), as \( \lambda \to 0 \), \( X \) degenerates to a singular curve \( X_0 \) with a non-separating node. In the above \( z \)-coordinate, and the Bergman kernel \( \kappa_{X_\lambda} \) can be written as

\[
\sum_{i,j=1}^{g} (\text{Im } Z)^{-1}_{i,j}(z^j \bar{z}^i)^2 \frac{4dz \otimes d\bar{z}}{|z^4(z^2 - \lambda)(z^2 - 1) P(z^2)|} =: k_\lambda(z) dz \otimes d\bar{z}. \quad (3.2)
\]

Particularly, if \( P = (x - a)(x - b) \), then (3.2) reduces to

\[
4 \frac{(\text{Im } Z)^{-1}_{1,1} + (\text{Im } Z)^{-1}_{2,1} z^2 + (\text{Im } Z)^{-1}_{1,2} z^2 + (\text{Im } Z)^{-1}_{2,2} z^4}{|z^4(z^2 - a)(z^2 - b)(z^2 - \lambda)|} dz \otimes d\bar{z}. \quad (3.3)
\]

The pinching coordinate could be used for general (possibly non-hyperelliptic) algebraic curves degenerating to a nodal curve, which represents a boundary point of the Deligne-Mumford compactification of moduli space (see [29, 50, 74, 75]). For a compact Riemann surface of genus \( g \geq 2 \), its non-hyperellipticity is characterized by the non-vanishing of the Gaussian curvature of its Bergman kernel (see [47]). However, in this paper we do not use the effective pinching coordinate method, which works for both non-separating and separating nodal cases, since we need to deal with the cuspidal curves as well.

Cuspidal family, Case I

For the family of hyperelliptic curves in (2.4), as \( \lambda \to 0 \), \( X \) degenerates to a singular curve \( X_0 \) with a cusp. Similarly, in the above \( z \)-coordinate,

\[
\omega_i = \frac{2z^{2(i-1)} dz}{\sqrt{(z^4 - \lambda) P(z^2)}}, \quad i = 1, \ldots, g,
\]

and the Bergman kernel \( \kappa_{X_\lambda} \) can be written as

\[
\kappa_{X_\lambda} = \sum_{i,j=1}^{g} (\text{Im } Z)^{-1}_{i,j}(z^j \bar{z}^i)^2 \frac{4dz \otimes d\bar{z}}{|z^4(z^2 - \lambda) P(z^2)|} =: k_\lambda(z) dz \otimes d\bar{z}. \quad (3.4)
\]
Particularly, if $P = (x - a)(x - b)$, then (3.4) reduces to

$$4 \frac{(\Im Z)^{-1})_{1,1} + ((\Im Z)^{-1})_{1,2}z^2 + ((\Im Z)^{-1})_{2,1}z^2 + ((\Im Z)^{-1})_{2,2}|z|^4}{|z^2 - a)(z^2 - b)(z^4 - \lambda)|} dz \otimes d\bar{z}. \quad (3.5)$$

**Cuspidal family, Case II** For the family of hyperelliptic curves in (2.5), as $\lambda \to 0$, $X_\lambda$ degenerates to a singular curve $X_0$ with a cusp. Similarly, in the above $z$-coordinate,

$$\omega_i = \frac{2z^{2(i-1)}dz}{\sqrt{(z^2 - \lambda)(z^2 - \lambda^2)}P(z^2)}, \quad i = 1, \ldots, g,$$

and the Bergman kernel $\kappa_{X_\lambda}$ can be written as

$$\kappa_{X_\lambda} = \sum_{i,j=1}^{g} ((\Im Z)^{-1})_{i,j}(z^i\bar{z}^j)^2 \frac{4dz \otimes d\bar{z}}{|z^4(z^2 - \lambda)(z^2 - \lambda^2)P(z^2)|} =: k_{\lambda}(z)dz \otimes d\bar{z}. \quad (3.6)$$

Particularly, if $P = (x - a)(x - b)$, then (3.6) reduces to

$$4 \frac{(\Im Z)^{-1})_{1,1} + ((\Im Z)^{-1})_{1,2}z^2 + ((\Im Z)^{-1})_{2,1}z^2 + ((\Im Z)^{-1})_{2,2}|z|^4}{|z^2 - a)(z^2 - b)(z^4 - \lambda^2)|} dz \otimes d\bar{z}. \quad (3.7)$$

**Normalization of singular curves** For the nodal singular curve $X_0$ in (2.3), we consider its normalization $Y := \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x - 1)P(x)\}$, which is a smooth curve of genus $g - 1$. The global Abelian differentials $\omega_i$ can be expressed in the above $z$-coordinate as

$$\omega_i = \frac{x^{(i-1)}dx}{\sqrt{(x - 1)P(x)}} = \frac{2z^{2i-1}dz}{\sqrt{(z^2 - 1)P(z^2)}}, \quad i = 1, \ldots, g - 1.$$

Let $Z_0$ be the normalized period matrix of $Y$. By the $L^2$ removable singularity theorem, $Y$ and the regular part of $X_0$ have the same Bergman kernel which can be written as $\kappa_0 = k_0(z)dz \otimes d\bar{z}$, where

$$k_0(z) = \frac{4}{|z^4(z^2 - 1)P(z^2)|} \sum_{i,j=1}^{g-1} ((\Im Z_0)^{-1})_{i,j}(z^i\bar{z}^j)^2. \quad (3.8)$$

For the cuspidal singular curve $X_0 := \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3P(x)\}$, its normalization is $Y := \{(x,y) \in \mathbb{C}^2 \mid y^2 = xP(x)\}$. Similarly, in the above $z$-coordinate, the Abelian differentials $\omega_i$ can be expressed as

$$\omega_i = \frac{x^{(i-1)}dx}{\sqrt{xP(x)}} = \frac{2z^{2i-2}dz}{\sqrt{P(z^2)}}, \quad i = 1, \ldots, g - 1,$$

and the Bergman kernel on $Y$ can be written as $\kappa_0 = k_0(z)dz \otimes d\bar{z}$, where

$$k_0(z) = \frac{4}{|z^4P(z^2)|} \sum_{i,j=1}^{g-1} ((\Im Z_0)^{-1})_{i,j}(z^i\bar{z}^j)^2. \quad (3.9)$$
4 Non-separating node: genus-two curves

In this section, we consider a family of genus two curves

$$X_{\lambda} := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x - \lambda)(x - 1)(x - a)(x - b)\},$$

where $a, b, \lambda$ are distinct complex numbers satisfying $0 < |\lambda| < 1 < |a| < |b|$. As $\lambda \to 0$, $X_{\lambda}$ degenerates to a singular curve $X_0$ with a non-separating node. The normalization of $X_0$ is an elliptic curve $\{(x, y) \in \mathbb{C}^2 \mid y^2 = (x - 1)(x - a)(x - b)\}$, whose period is $c$ given in (4.6). Let

$$c_1 := \int_1^a \frac{\sqrt{ab} \, dx}{\sqrt{(x - 1)(x - a)(x - b)}}$$

be a constant depending on $a, b$. The Bergman kernels on $X_{\lambda}$ and on the normalization of $X_0$ are denoted by $\kappa_{X_{\lambda}}$ and $\kappa_0$, respectively. In the local coordinate $z := \sqrt{x}$ near $(0, 0)$, we write $\kappa_{X_{\lambda}} = k_\lambda(z)dz \otimes d\bar{z}$, and $\kappa_0 = k_0(z)dz \otimes d\bar{z}$. Then, our result on the asymptotic behaviour of the Bergman kernel $\kappa_{X_{\lambda}}$ with precise coefficients is stated as follows.

**Theorem 4.1** For $X_{\lambda}$ defined by (4.1), as $\lambda \to 0$, it holds that

(i) $\kappa_{X_{\lambda}} \to \kappa_0$;

(ii) for small $|z| \neq 0$,

$$\log k_\lambda(z) - \log k_0(z) \sim \frac{\pi}{-\log |\lambda|} \frac{\text{Im } c + (z^2 + \bar{z}^2) \text{Re } c_1^{-1}}{|z|^4}.$$

In fact, the results in Sect. 5 on general hyperelliptic curves largely rely on our proofs in this section. To prove Theorem 4.1, we need the following two lemmata by analyzing the asymptotics of the $A$-period matrix and $B$-period matrix on $X_{\lambda}$.

**Lemma 4.2** Under the same assumptions as in Theorem 4.1, as $\lambda \to 0$, it holds that

$$A = \begin{pmatrix} \frac{-2\pi}{\sqrt{ab}} c_2 & \frac{c_2}{\sqrt{ab}} \\ 0 & -2c_1 \end{pmatrix} + O(\lambda),$$

where $c_2 := \int_{\delta_2} \frac{dx}{x\sqrt{(x - 1)(x - a)(x - b)}}$ and $\lim_{\lambda \to 0} \frac{O(\lambda)}{\lambda}$ is a finite matrix.

**Proof** To estimate the four entries of $A$ one by one, we use the choice of cycles $\delta_1, \delta_2$ specified in Sect. 3. Firstly, $A_{1,1} = \int_{\delta_1} \omega_1$, where $\delta_1$ only contains 0 and $\lambda$. By the substitutions $t = \lambda^{-1}$ and $s = x^{-1}$, we get the dual cycle $\tilde{\delta}_1$ which contains $\{\infty, t\}$ so that $-\tilde{\delta}_1$ contains $\{0, 1, a^{-1}, b^{-1}\}$, for $|s| \in (1, |t|)$. As $\lambda \to 0$,

$$A_{11} = \int_{\delta_1} \frac{-s^{-2} \, ds}{\sqrt{s^{-1}(s^{-1} - t^{-1})(s^{-1} - 1)(s^{-1} - a)(s^{-1} - b)}}$$

$$= -\int_{-\tilde{\delta}_1} \frac{-\sqrt{s} \, ds}{\sqrt{ab}(s - 1)(s - a^{-1})(s - b^{-1})(s - t)}$$

$$= \int_{-\tilde{\delta}_1} \frac{\sqrt{s} \, ds}{-\sqrt{ab}(s - 1)(s - a^{-1})(s - b^{-1})(s - t)} \left(1 + s \frac{2}{t^2} + O\left(\frac{s^2}{t^2}\right)\right)$$

$$= \int_{-\tilde{\delta}_1} \frac{ds}{-s\sqrt{-ab}} \left(1 + s \frac{2}{t^2} + O\left(\frac{s^2}{t^2}\right)\right) \left(1 + O(s^{-1})\right) \left(1 + O(s^{-1})\right) \left(1 + O(s^{-1})\right)$$
\[= \int_{-\delta_1}^{\delta_1} \frac{ds}{s \sqrt{-ab}} \left( 1 + \frac{s}{2t} + O\left(\frac{s^2}{t^2}\right) \right) \left( 1 + O(s^{-1}) \right)\]
\[= \frac{1}{-\sqrt{-ab}} \int_{-\delta_1}^{\delta_1} \left( 1 + O(t^{-1}) \right) \frac{ds}{s} = \frac{2\pi}{-\sqrt{-ab}} \left( 1 + O(\lambda) \right).\]

Notice that we have used the Maclaurin expansion
\[\frac{1}{\sqrt{s-a}} = \begin{cases} \sqrt{a}^{-1} \left( 1 + O\left(\frac{sa}{s-a}\right) \right), & |s| < |a|; \\ \sqrt{s}^{-1} \left( 1 + O\left(\frac{s}{s-a}\right) \right), & |s| > |a|. \end{cases} \tag{4.2}\]

Secondly, look at \(A_{21} = \int_{\delta_1}^{\omega_2} \omega \) and similarly it holds that
\[A_{21} = \int_{-\delta_1}^{\delta_1} ds \cdot \sqrt{s \cdot a} \left( 1 + O\left(\frac{s^2}{t^2}\right) \right) \left( 1 + O(s^{-1}) \right)\]
\[= \frac{1}{-\sqrt{-ab}} \int_{-\delta_1}^{\delta_1} \left( \frac{1}{2t} + O(t^{-2}) \right) \frac{ds}{s} = \frac{2\pi}{-\sqrt{-ab}} \left( \lambda^2 + O(\lambda^2) \right).\]

Thirdly, since \(\delta_2\) contains \(\{0, \lambda, 1, a\}\), as \(\lambda \to 0\), it holds that
\[A_{12} = \int_{\delta_2}^{\delta_2} dx \cdot \sqrt{(x-a)(x-b)} \left( 1 + \frac{\lambda}{2x} + O\left(\frac{\lambda^2}{x^2}\right) \right)\]
\[= \int_{\delta_2}^{\delta_2} dx \cdot \sqrt{(x-a)(x-b)} \left( 1 + O(\lambda) \right).\]

Lastly,
\[A_{22} = \int_{\delta_2}^{\delta_2} dx \cdot \sqrt{(x-a)(x-b)} \left( 1 + O(\lambda) \right) = -2 \int_{1}^{a} dx \cdot \sqrt{(x-a)(x-b)} \left( 1 + O(\lambda) \right).\]

Lemma 4.3 Under the same assumptions as in Theorem 4.1, as \(\lambda \to 0\), it holds that
\[B \sim \left( \frac{-2 \log \lambda}{\sqrt{-ab}} d_1, \frac{2}{\sqrt{-ab}} d_2 \right).\]

where \(d_1 := -2 \int_{a}^{b} \frac{dx}{x \sqrt{(x-a)(x-b)}}\) and \(d_2 := -2 \int_{a}^{b} \frac{dx}{\sqrt{(x-a)(x-b)}}.\)

Proof As \(t \to \infty\), we make use of the following computations (cf. [12]).
\[\int_{1}^{t} \frac{ds}{s \sqrt{s-t}} = \frac{2}{\sqrt{t}} \sqrt{-1} \log \left( \sqrt{\frac{t}{s}} + \sqrt{\frac{t}{s} - 1} \right) \left|_{1}^{t} \right. \sim \frac{\sqrt{-1}}{\sqrt{t}} \log t.\tag{4.3}\]
\[\int_{1}^{t} \frac{ds}{s^2 \sqrt{s-t}} = \frac{2}{\sqrt{t}} \sqrt{-1} \left. \frac{1}{2t} \int_{1}^{t} \frac{ds}{s \sqrt{s-t}} \right|_{1}^{t} = \frac{\sqrt{-1}}{\sqrt{t}} + \frac{\sqrt{-1}}{2t \sqrt{t}} \log t \sim -\frac{\sqrt{-1}}{\sqrt{t}}.\tag{4.4}\]
In particular, (4.4) yields the boundedness of
\[
\int_1^t \frac{\sqrt{t}}{s \sqrt{s - t}} \text{O}(s^{-1}) ds.
\]
More generally, for any integer \( \alpha \geq 1 \), as \( t \to \infty \), it holds that
\[
\int_1^t \frac{ds}{s^{\alpha+1} \sqrt{s - t}} = \frac{\sqrt{s - t}}{\alpha t s^\alpha} \left| \int_1^t \frac{ds}{s^\alpha \sqrt{s - t}} \right| + \frac{2\alpha - 1}{2\alpha t} \int_1^t \frac{ds}{s^{\alpha+1} \sqrt{s - t}} \sim -\frac{\sqrt{1}}{\alpha \sqrt{t}}. \tag{4.5}
\]

Similar to the proof of Lemma 4.2, we use the choice of cycles \( \gamma_1, \gamma_2 \) specified in Sect. 3 to estimate the four entries of \( B \) one by one. Firstly, by (4.3) and (4.4), for \( |s| \in (1, |t|) \),
\[
B_{11} = -2 \int_1^1 \frac{dx}{\sqrt{x(x - \lambda)(x - 1)}(x - a)(x - b)}
\]
\[
= -2 \int_1^1 \frac{1}{\sqrt{ab(s - 1)(s - a^{-1})(s - b^{-1})(s - t)}}
\]
\[
= -2 \int_1^t \frac{1}{\sqrt{s - t}ab \sqrt{s}} (1 + \text{O}(s^{-1})) \left( 1 + \text{O}(s^{-1}) \right) \left( 1 + \text{O}(s^{-1}) \right)
\]
\[
= -2 \frac{2\sqrt{t}}{\sqrt{ab}} \int_1^t \frac{ds}{s^{\alpha+1} \sqrt{s - t}} \left( 1 + \text{O}(s^{-1}) \right)
\]
\[
\sim -\frac{2 \log \lambda}{\sqrt{-ab}},
\]
as \( \lambda \to 0 \). Secondly, by (4.4),
\[
B_{21} = -2 \int_1^t \frac{dx}{\sqrt{x(x - 1)}(x - a)(x - b)}
\]
\[
= -2 \int_1^t \frac{1}{\sqrt{s - t}ab \sqrt{s}} \left( 1 + \text{O}(s^{-1}) \right) \sim -\frac{2}{\sqrt{-ab}}.
\]
Finally, similar to \( A_{12} \),
\[
B_{12} \sim -2 \int_a^b \frac{dx}{\sqrt{(x - 1)(x - a)(x - b)}},
\]
and
\[
B_{22} \sim -2 \int_a^b \frac{dx}{\sqrt{(x - 1)(x - a)(x - b)}}.
\]

Combining Lemmata 4.2 and 4.3 with (3.3), we get the asymptotics of the Bergman kernels.

**Proof of Theorem 4.1** Notice that the period is defined as
\[
c := \frac{\int_a^b \sqrt{(x - 1)(x - a)(x - b)} \frac{dx}{\sqrt{(x - 1)(x - a)(x - b)}}}{\int_1^a \sqrt{(x - 1)(x - a)(x - b)} \frac{dx}{\sqrt{(x - 1)(x - a)(x - b)}}} = \tau \left( \frac{1 - b}{1 - a} \right), \tag{4.6}
\]
where \( \tau(\cdot) \) is the inverse of the elliptic modular lambda function. On the normalization of the nodal curve \( X_0 \), by (3.8), in the local coordinate \( z = \sqrt{x} \), the Bergman kernel is exactly
\[
k_0 = k_0(z) dz \otimes \overline{dz},
\]
where
\[
k_0(z) = \frac{4|z|^2}{(\text{Im } c)|(z^2 - 1)(z^2 - a)(z^2 - b)|}. \tag{4.7}
\]
By Lemma 4.2, as \( \lambda \to 0 \),
\[
A^{-1} \sim \left( \begin{array}{cc}
\frac{-2\pi}{\sqrt{ab}} & c_2 \\
O(\lambda) & \frac{-2c_1}{\sqrt{ab}}
\end{array} \right)^{-1} = \left( \begin{array}{cc}
\frac{-\sqrt{ab}}{2\pi} & \frac{\sqrt{abc_2}}{2\pi} \\
O(\lambda) & \frac{\sqrt{ab}}{2\pi}
\end{array} \right).
\]

Let \( Z = A^{-1}B \) denote the normalized period matrix of \( X_\lambda \). Then, it follows that
\[
Z \sim \left( \begin{array}{c}
\frac{-\sqrt{-1}}{\pi} \log \lambda \\
\frac{-\sqrt{-1}}{\pi} \log \lambda
\end{array} \right), \quad \text{Im} \ Z \sim \left( \begin{array}{c}
\frac{-\log |\lambda|}{\pi} - \Re\{c_1^{-1}\} \\
-\Re\{c_1^{-1}\} \quad \text{Im} \ c
\end{array} \right).
\]

So, as \( \lambda \to 0 \), it holds that
\[
(\text{Im} \ Z)^{-1} \sim \left( \begin{array}{c}
\frac{-\log |\lambda|}{\pi} - \Re\{c_1^{-1}\} \pi \\
\frac{-\log |\lambda|}{\pi} \quad \text{Im} \ c
\end{array} \right)\cdot \left( \begin{array}{c}
\frac{-\log |\lambda|}{\pi} - \Re\{c_1^{-1}\} \pi \\
\frac{-\log |\lambda|}{\pi} \quad \text{Im} \ c
\end{array} \right)^{-1}.
\]

By (3.3) and (4.7), \( \kappa_{X_\lambda} \to \kappa_0 \). Moreover, in the local coordinate \( z = \sqrt{x} \) near \((0, 0)\),
\[
k_\lambda(z) - k_0(z) \sim \frac{4\pi}{|z^2 - 1|} \frac{1 + \Re\{c_1^{-1}\}}{\text{Im} \ c} \left( z^2 + z^2 \right),
\]
which yields the conclusion. \( \square \)

Since the moduli space of genus-two curves is 3 dimensional, one may consider the more general family \( X_{\lambda, a, b} := \{ y^2 = x(x - 1)(x - \lambda)(x - a)(x - b) \} \), parameterized by three distinct complex numbers \( \lambda, a, b \in \mathbb{C} \setminus [0, 1] \). As \( \lambda, a \) or \( b \) tends to 0, 1 or \( \infty \), or towards one another, \( X_{\lambda, a, b} \) will become singular. In our setting, we fix the other two parameters \( a \) and \( b \), and move \( \lambda \) only, so the precise asymptotic coefficients we have obtained depend on both \( a \) and \( b \).

## 5 Non-separating node: hyperelliptic and general curves

This section is devoted to the proof of Theorem 2.1, as a generalization of Theorem 4.1 towards the hyperelliptic case. When \( \lambda \in \mathbb{C} \setminus [0, 1, a_1, \ldots, a_{2g-2}] \), \( X_\lambda \) defined in (2.3) has genus \( g \). To prove Theorem 2.1, we use the choice of cycles \( \delta_j, \gamma_j \) specified in Sect. 3, and need to analyze on \( X_\lambda \) the asymptotics of its \( A \)-period matrix and \( B \)-period matrix, denoted by \( A \) and \( B \), respectively. Meanwhile, for the normalization \( Y := \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x - 1) P(x)\} \) of genus \( g - 1 \), denote its \( A \)-period matrix and \( B \)-period matrix by \( A_0 \) and \( B_0 \), respectively.

**Lemma 5.1** Under the same assumptions as in Theorem 2.1, as \( \lambda \to 0 \), it holds that
\[
A \sim \left( \begin{array}{c}
\frac{2\pi}{\sqrt{P(0)}} \quad O(1) \\
* \quad A_0
\end{array} \right),
\]
where \(* = (O(\lambda), 0, \ldots, 0)^T\) is a column vector with \( g - 1 \) rows.

**Proof** Firstly, \( A_{11} = \int_{\delta_1} \omega_1 \), where \( \delta_1 \) only contains \( 0 \) and \( \lambda \). By the substitutions \( t = \lambda^{-1} \) and \( s = x^{-1} \), we get the dual cycle \( \delta_1 \) which contains \( \{\infty, t\} \) so that \( -\delta_1 \) contains \( \{0, 1, a_1^{-1}, \ldots, a_{2g-2}^{-1}\} \) for \(|s| \in (1, |t|)\). As \( \lambda \to 0 \),
\[\square\]
\[ A_{11} = \int_{-\delta_1} ds \frac{1}{\sqrt{-1s\sqrt{P(0)}}} \left( 1 + O\left( \frac{s^i}{t} \right) \right) \left( 1 + O(s^{-1}) \right) \]
\[ \sim \int_{-\delta_1} ds \frac{1}{\sqrt{-1s\sqrt{P(0)}}} \left( 1 + O(s^{-1}) \right) \]
\[ = \int_{-\delta_1} ds \frac{1}{\sqrt{-1s\sqrt{P(0)}}} \]
\[ = \frac{2\pi}{\sqrt{P(0)}}. \]

\[ A_{21} = \int_{-\delta_1} ds \frac{1}{\sqrt{-1s^2\sqrt{P(0)}}} \left( 1 + O\left( \frac{s^i}{t} \right) \right) \left( 1 + O(s^{-1}) \right) \]
\[ = \int_{-\delta_1} ds \frac{1}{\sqrt{-1\sqrt{P(0)}}} \left( s^{-i} + O\left( \frac{1}{ts^{i-1}} \right) \right) \left( 1 + O(s^{-1}) \right) \]
\[ = \int_{-\delta_1} ds \frac{1}{\sqrt{-1\sqrt{P(0)}}} \left( + O \left( \frac{1}{ts} \right) \right) \]
\[ = O(\lambda). \]

For \( 3 \leq i \leq g \),

\[ A_{i1} = \int_{-\delta_1} ds \frac{1}{\sqrt{-1s^i\sqrt{P(0)}}} \left( 1 + O\left( \frac{s^i}{t} \right) \right) \left( 1 + O(s^{-1}) \right) \]
\[ = \int_{-\delta_1} ds \frac{1}{\sqrt{-1s^i\sqrt{P(0)}}} \left( s^{-i} + O\left( \frac{1}{ts^{i-1}} \right) \right) \left( 1 + O(s^{-1}) \right) \]
\[ = 0. \]

Secondly, since \( \delta_2 \) contains only \( 0, \lambda, 1 \) and \( a_1 \) (but \( a_2, \ldots, a_{2g-2} \)), it holds that

\[ A_{12} = \int_{\delta_2} \frac{dx}{\sqrt{x(x-\lambda)(x-1)P(x)}} \]
\[ = \int_{\delta_2} \frac{dx}{x\sqrt{(x-1)P(x)}} \left( 1 + O\left( \frac{\lambda}{x} \right) \right) \]
\[ \sim \int_{\delta_2} \frac{dx}{x\sqrt{(x-1)P(x)}}. \]

In general, for \( A_{i2} \), there is an extra \( x^{i-1} \) in the original integrand above, so

\[ A_{i2} \sim \int_{\delta_2} \frac{x^{i-2}dx}{\sqrt{(x-1)P(x)}}. \]

Thirdly, \( \delta_j \) contains only \( 0, \lambda^2, \lambda, a_1, \ldots, a_{2j-3} \) (but \( a_{2j-2}, \ldots, a_{2g-2} \)). In general, for \( j \geq 2 \), \( A_{ij} \) is asymptotic to the same integrand along \( \delta_j \) instead of along \( \delta_2 \), i.e.,

\[ A_{ij} \sim \int_{\delta_j} \frac{x^{i-2}dx}{\sqrt{(x-1)P(x)}}, \]

which is exactly the same as the corresponding entry of \( A_0 \) when \( i \geq 2 \).
Lemma 5.2 \textit{Under the same assumptions as in Theorem 2.1, as } \lambda \to 0, \text{ it holds that}

\[ B \sim \begin{pmatrix} -\frac{2\sqrt{-1}}{\sqrt{P(0)}} \log \lambda \, O(1) \\ \star \end{pmatrix}, \]

where \( \star \) is a column vector whose entries are all \( \frac{2\sqrt{-1}}{\sqrt{P(0)}} \) with \( g - 1 \) rows.

\textbf{Proof} Firstly, by (4.3) and (4.4), for \( |s| \in (1, |t|) \),

\[
B_{11} = -2 \int_\lambda^1 \frac{dx}{\sqrt{x(x-\lambda)(x-1) \cdot P(x)}} = -2 \int_\lambda^1 \frac{dx}{\sqrt{x(x-\lambda)}} \sqrt{-P(0)} (1 + O(x)) = 2 \sqrt{t} \int_1^t \frac{ds}{s\sqrt{s-t}} (1 + O(s^{-1})) \sim \frac{-2\sqrt{-1}}{\sqrt{P(0)}} \sqrt{-1 \log \lambda},
\]

where the last equality holds due to the substitutions \( t = \lambda^{-1} \) and \( s = x^{-1} \). Thus, for \( 2 \leq i \leq g \), by (4.5),

\[
B_{ij} \sim -2 \int_{a_{ij}^{-2}}^{a_{ij}^{-3}} \frac{x^{i-2}dx}{\sqrt{(x-1)P(x)}},
\]

which is exactly the same as the corresponding entry of \( B_0 \) when \( i \geq 2 \).

Now we will give a proof of Theorem 2.1.

\textbf{Proof of Theorem 2.1} By Lemma 5.1 and the block matrix inversion, we know that

\[
A^{-1} \sim \begin{pmatrix} \frac{\sqrt{P(0)}}{2\pi} & O(1) \\ O(\lambda) & A_0^{-1} \end{pmatrix},
\]

where both \( (O(1))^T \) and \( \lim_{\lambda \to 0} O(\lambda) \) are finite column vectors with \( g - 1 \) rows. Therefore, as \( \lambda \to 0 \),

\[
Z = A^{-1} B \sim \begin{pmatrix} -\frac{\sqrt{-1}}{\pi} \log \lambda \, O(1) \\ A_0^{-1} \star \end{pmatrix}, \quad \text{Im} \, Z \sim \begin{pmatrix} \frac{\log |\lambda|}{-\pi} & O(1) \\ \text{Im}(A_0^{-1} \star) & \text{Im} \, Z_0 \end{pmatrix},
\]

and

\[
(\text{Im} \, Z)^{-1} \sim \begin{pmatrix} \frac{-\pi}{\log |\lambda|} & O((\log |\lambda|)^{-1}) \\ (\text{Im} \, Z_0)^{-1} \text{Im}(A_0^{-1} \star) \frac{\pi}{\log |\lambda|} & (\text{Im} \, Z_0)^{-1} \end{pmatrix}.
\]
In fact, since $Z$ is symmetric, the off-diagonal block matrices in each matrix above concerning $Z$ are the transpose of each other. By (3.2) and (3.8), as $\lambda \to 0$, $\kappa_{X_{\lambda}} \to \kappa_0$. Moreover, in the local coordinate $z = \sqrt{x}$ near $(0, 0)$, it holds that

$$k_{\lambda}(z) - k_0(z) \sim \frac{4\pi}{|z^2 - \lambda)(z^2 - 1)|} \left(1 - 2 \text{Re} \sum_{i=1}^{g-1} \left((\text{Im } Z_0)^{-1} \text{Im}(A_0^{-1} \ast)\right)_i z^{2i}ight) \left|1 - 2 \text{Re } \frac{\lambda}{4} \sum_{i=1}^{|z| |\lambda|} \left(\text{Im } \tau \right) + 1\right)$$

which yields the conclusion.

If we ignore the precise asymptotic coefficients, then the leading term growth in Theorem 2.1 corresponds to [37, Proposition 3.2]. The degeneration of nodal non-hyperelliptic curves was also treated in [37].

**6 Cusp I: genus-two curves**

In this section, we consider a family of genus two curves

$$X_{\lambda} := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x^2 - \lambda)(x - a)(x - b)\}, \quad (6.1)$$

where $a, b, \lambda$ are distinct complex numbers satisfying $0 < |\lambda| < |a| < |b|$. As $\lambda \to 0$, $X_{\lambda}$ degenerates to a singular curve $X_0$ with an ordinary cusp. The normalization of $X_0$ is an elliptic curve $\{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x - a)(x - b)\}$, whose period is $\tau$ given in (6.2). Let

$$c_4 := -2 \int_0^1 \frac{(x - 1)dx}{\sqrt{x(x - 1)(x - 2)}}$$

and let

$$c_3 := \int_0^a \frac{\sqrt{ab}dx}{\sqrt{x(x - a)(x - b)}},$$

be a constant depending on $a, b$. The Bergman kernels on $X_{\lambda}$ and on the normalization of $X_0$ are denoted by $\kappa_{X_{\lambda}}$ and $\kappa_0$, respectively. In the local coordinate $z := \sqrt{x}$ near $(0, 0)$, we write $\kappa_{X_{\lambda}} = k_{\lambda}(z)dz \otimes d\bar{z}$, and $\kappa_0 = k_0(z)dz \otimes d\bar{z}$. Then, our result on the asymptotic behaviour of the Bergman kernel $\kappa_{X_{\lambda}}$ with precise coefficients is stated as follows.

**Theorem 6.1** For $X_{\lambda}$ defined by (6.1), as $\lambda \to 0$, for small $|z| \neq 0$, it holds that

(i) $k_{\lambda}(z) \to k_0(z) \left(\frac{\text{Im } \tau}{|z^4|} + 1\right)$, i.e., $\kappa_{X_{\lambda}} \not\to \kappa_0$;

(ii) $\log k_{\lambda}(z) - \log k_0(z) - \log \left(\frac{\text{Im } \tau}{|z^4|} + 1\right) \sim \frac{\text{Re} \left\{\lambda^{1/4} c_4 c_3 \right\} \left(z^2 + \bar{z}^2\right)}{|z|^4 + \text{Im } \tau},$ 

where $\tau$ is the period (scalar) of the elliptic curve $\{y^2 = x(x - a)(x - b)\}$.

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2 The author is grateful to Professor Z. Huang for bringing attention the paper [37] during the 2016 Tsinghua Sanya International Mathematics Forum, where a preliminary version of this work was presented.
To prove Theorem 6.1, we need the following two lemmata by analyzing the asymptotics of the \(A\)-period matrix and \(B\)-period matrix on \(X_\lambda\). Our choice of cycles \(\delta_j, \gamma_j\) is specified in Sect. 3.

**Lemma 6.2** Under the same assumptions as in Theorem 6.1, as \(\lambda \to 0\), it holds that

\[
A \sim \begin{pmatrix}
c_5 \lambda^{-1/4} & c_6 \\
c_4 \lambda^{1/4} - 2c_3/\sqrt{ab}
\end{pmatrix},
\]

where \(c_5 := -2 \int_0^1 \frac{du}{\sqrt{u(u-1)(u-2)}}\) and \(c_6\) depends on \(a, b\).

**Proof** We estimate the four entries one by one. Firstly, let \(\delta_1\) contain only \(-\sqrt{\lambda}\) and 0. By the Cauchy Integral Theorem and Taylor series expansion, we know that

\[
A_{11} = -2 \int_{-\sqrt{\lambda}}^0 \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}}
= -2 \int_{-\sqrt{\lambda}}^0 \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}} (1 + O(x))
= -2\lambda^{-1/4} \int_0^1 \frac{du}{\sqrt{u(u-1)(u-2)}} (1 + O\left(\sqrt{\lambda}(u-1)\right))
\sim -2\lambda^{-1/4} \int_0^1 \frac{du}{\sqrt{u(u-1)(u-2)}} =: \frac{c_5}{-\sqrt{ab}} \lambda^{-1/4},
\]

where the third equality holds due to the substitution \(x = (u-1)\sqrt{\lambda}\). Secondly,

\[
A_{21} = -2\lambda^{1/4} \int_0^1 \frac{(u-1)du}{\sqrt{u(u-1)(u-2)}} (1 + O\left(\sqrt{\lambda}(u-1)\right))
\sim -2\lambda^{1/4} \int_0^1 \frac{(u-1)du}{\sqrt{u(u-1)(u-2)}} =: \frac{c_4}{-\sqrt{ab}} \lambda^{1/4}.
\]

Thirdly, let \(\delta_2\) contain only \(-\sqrt{\lambda}, 0, \sqrt{\lambda}\) and \(a\) (but \(b\)). Then,

\[
A_{12} = \int_{\delta_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{x}} \left(1 + O\left(\sqrt{\lambda}/x\right)\right) \frac{1}{\sqrt{x}} \left(1 + O\left(\sqrt{\lambda}/x\right)\right)
\sim \int_{\delta_2} \frac{dx}{x\sqrt{x(x-a)(x-b)}} =: c_6.
\]

Lastly,

\[
A_{22} = \int_{\delta_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \left(1 + O\left(\sqrt{\lambda}/x\right)\right)
\sim \int_{\delta_2} \frac{dx}{\sqrt{x(x-a)(x-b)}}
= -2 \int_a^b \frac{dx}{\sqrt{x(x-a)(x-b)}} = -2c_3/\sqrt{ab}.
\]
Lemma 6.3 Under the same assumptions as in Lemma 6.2, as \( \lambda \to 0 \), it holds that

\[
B \sim \frac{1}{\sqrt{-ab}} \left( c_5 \lambda^{-1/4} d_3 \sqrt{-ab} - c_4 \lambda^{1/4} d_4 \sqrt{-ab} \right),
\]

where \( d_3 := -2 \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} \) and \( d_4 := -2 \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} \).

Proof Again, we estimate all the four entries one by one. Firstly, let \( \gamma_1 \) contain only 0 and \( \sqrt{\lambda} \). By Cauchy Integral Theorem, we can get that

\[
B_{11} = -2 \int_{0}^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x^2 - \lambda)(x-a)(x-b)}} = -2 \int_{0}^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}} \left( 1 + O(x) \right)
\]

\[
= \frac{2\lambda^{-1/4}}{\sqrt{ab}} \int_{1}^{0} \frac{du}{\sqrt{-u(u-1)(u-2)}} \left( 1 + O((u-1) \cdot \sqrt{\lambda}) \right)
\]

\[
\sim \frac{2\lambda^{-1/4}}{\sqrt{ab}} \int_{0}^{1} \frac{\sqrt{-1}du}{\sqrt{u(u-1)(u-2)}} =: \frac{c_5}{\sqrt{-ab}} \lambda^{-1/4},
\]

where the third equality holds due to the substitution \( x = (1-u)\sqrt{\lambda} \).

Secondly,

\[
B_{21} = \frac{2\lambda^{-1/4}}{\sqrt{ab}} \int_{1}^{0} \frac{(-u+1)\sqrt{\lambda}du}{\sqrt{-u(u-1)(u-2)}} \left( 1 + O((u-1) \cdot \sqrt{\lambda}) \right)
\]

\[
\sim \frac{2\lambda^{1/4}}{\sqrt{ab}} \int_{0}^{1} \frac{(u-1)du}{\sqrt{-u(u-1)(u-2)}} =: \frac{\sqrt{-1}\lambda^{1/4}}{\sqrt{ab}} c_4.
\]

Thirdly, let \( \gamma_2 \) contain only \( a \) and \( b \). Then, it holds that

\[
B_{12} = \int_{\gamma_2} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}}
\]

\[
= -2 \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{\lambda}} \left( 1 + O \left( \frac{\sqrt{\lambda}}{2x} \right) \right) \frac{1}{\sqrt{\lambda}} \left( 1 + O \left( \frac{\sqrt{\lambda}}{2x} \right) \right)
\]

\[
\sim -2 \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} =: d_3.
\]

Lastly,

\[
B_{22} = \int_{\gamma_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \left( 1 + O \left( \frac{\sqrt{\lambda}}{2x} \right) \right) \sim -2 \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} =: d_4.
\]

Combining Lemmata 6.2 and 6.3 with (3.5), we get the asymptotics of the Bergman kernels.

Proof of Theorem 6.1 Notice that the period is defined as

\[
\tau := \begin{cases} 
\int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} \\
\int_{0}^{\sqrt{\lambda}} \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} \int_{0}^{\sqrt{\lambda}} \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}}
\end{cases} = \tau \left( \frac{b}{a} \right),
\]
where $\tau(\cdot)$ is the inverse of the elliptic modular lambda function. On the regular part of the cuspidal curve $X_0$, by (3.9), in the local coordinate $z = \sqrt{x}$, the Bergman kernel is exactly $K_0 = k_0(z)dz \otimes d\bar{z}$, where

$$k_0(z) = \frac{4}{\text{Im } \tau \cdot |(z^2 - a)(z^2 - b)|}.$$  \hfill (6.3)

By Lemma 6.2, as $\lambda \to 0$,

$$A^{-1} \sim \left(\begin{array}{cc}
\frac{c_5}{\sqrt{ab}} \lambda^{-1/4} & c_6 \\
\frac{c_4}{\sqrt{ab}} \lambda^{-1/4} & c_3
\end{array}\right)^{-1} \sim \frac{ab \lambda^{1/4}}{2c_3c_5} \left(\begin{array}{cc}
\frac{-2c_3}{\sqrt{ab}} & -c_6 \\
\frac{c_5}{\sqrt{ab}} \lambda^{-1/4} & \frac{c_5}{\sqrt{ab}} \lambda^{-1/4}
\end{array}\right).$$

Let $Z = A^{-1}B$ denote the normalized period matrix of $X_\lambda$. Then, it follows that

$$Z \sim \frac{ab \lambda^{1/4}}{2c_3c_5} \left(\begin{array}{cc}
\frac{-2c_3}{\sqrt{ab}} & -c_6 \\
\frac{c_5}{\sqrt{ab}} \lambda^{-1/4} & \frac{c_5}{\sqrt{ab}} \lambda^{-1/4}
\end{array}\right) \frac{1}{\sqrt{-ab}} \left(\begin{array}{cc}
\frac{c_5 \lambda^{-1/4}}{d_3 \sqrt{-ab}} & d_3 \sqrt{-ab} \\
d_3 \sqrt{-ab} & -c_4 \lambda^{-1/4} \frac{d_4 \lambda^{-1/4}}{d_3 \sqrt{-ab}}
\end{array}\right) \sim \left(\begin{array}{cc}
\sqrt{-1} & \sqrt{-1} \lambda^{1/4} \\
\sqrt{-1} \lambda^{1/4} & \frac{c_4}{c_3}
\end{array}\right) \left(\begin{array}{cc}
d_3 \sqrt{-ab} & c_5^{-1} \\
\tau & 1
\end{array}\right),$$

$$\text{Im } Z \sim \left(\begin{array}{cc}
1 & \text{O}(\lambda^{1/4}) \\
\text{Re } \left\lfloor \frac{1}{\lambda^{1/4} \frac{c_4}{c_3}} \right\rfloor & \text{Im } \tau
\end{array}\right).$$

So, as $\lambda \to 0$,

$$(\text{Im } Z)^{-1} \sim \left(\begin{array}{cc}
1 & -\text{O}(\lambda^{-1/4}) \\
(\text{Im } \tau)^{-1} \text{Re } \left\lfloor \frac{1}{\lambda^{1/4} \frac{c_4}{c_3}} \right\rfloor & (\text{Im } \tau)^{-1}
\end{array}\right).$$

By (3.7) and (6.3),

$$k_\lambda(z) \sim \frac{1}{\text{Im } \tau} \text{Re } \left\lfloor \frac{1}{\lambda^{1/4} \frac{c_4}{c_3}} \right\rfloor \frac{|z^2 + z^2|^4}{|z^2 - a)(z^2 - b)|} \to \frac{\text{Im } \tau}{|z^4|} + 1 \rightarrow k_0(z),$$

which means that $k_{X_\lambda} \to k_0$. Moreover, in the local coordinate $z = \sqrt{x}$ near $(0, 0)$,

$$k_\lambda(z) = \left(\frac{\text{Im } \tau}{|z|^4} + 1\right) k_0(z) \sim \frac{4(\text{Im } \tau)^{-1} \text{Re } \left\lfloor \frac{1}{\lambda^{1/4} \frac{c_4}{c_3}} \right\rfloor |z^2 + z^2|^4}{|z^4 - a)(z^2 - b)|},$$

which yields the conclusion. \hfill $\square$

### 7 Cusp I: hyperelliptic curves

This section is devoted to the proof of Theorem 2.2. When $\lambda \in \mathbb{C} \setminus \{0, a_1, \ldots, a_{2g-2}\}$, $X_\lambda$ defined in (2.4) has genus $g$. To prove Theorem 2.2, we use the choice of cycles $\delta_j, \gamma_j$ specified in Sect. 3, and need to analyze on $X_\lambda$ the asymptotics of its $A$-period matrix and $B$-period matrix, denoted by $A$ and $B$, respectively. Meanwhile, for the normalization $Y := \{(x, y) \in \mathbb{C}^2 \mid y^2 = xP(x)\}$ of genus $g - 1$, denote its $A$-period matrix and $B$-period matrix by $A_0$ and $B_0$, respectively.
Lemma 7.1 Under the same assumptions as in Theorem 2.2, as $\lambda \to 0$, it holds that
\[
A \sim \begin{pmatrix}
  c_5 \lambda^{-\frac{1}{4}} \frac{1}{\sqrt{P(0)}} & O(1) \\
  \diamond & A_0
\end{pmatrix},
\]
where $\diamond$ is a column vector with $g-1$ rows whose entries are given by (7.1) below.

Proof We will estimate all the $g \times g$ elements one by one. Firstly, as $\lambda \to 0$, it holds that
\[
A_{11} = -2 \int_{0}^{1} \frac{dx}{\sqrt{x(x^2 - \lambda)} P(x)}
\]
\[
= -2 \int_{0}^{1} \frac{dx}{\sqrt{x(x^2 - \lambda)}} \frac{1}{\sqrt{P(0)}} (1 + O(x))
\]
\[
= -2\lambda^{-\frac{1}{4}} \int_{0}^{1} \frac{du}{\sqrt{u(u-1)(u-2)}} \frac{1}{\sqrt{P(0)}} (1 + O((u-1)\sqrt{\lambda}))
\]
\[
\sim -2\lambda^{-\frac{1}{4}} \int_{0}^{1} \frac{du}{\sqrt{u(u-1)(u-2)}} \frac{1}{\sqrt{P(0)}} =: \lambda^{-\frac{1}{4}} c_5 \frac{1}{\sqrt{P(0)}}.
\]
where the third equality holds due to the substitution $x = (u-1)\sqrt{\lambda}$. In general, for $a_{i1}$, $2 \leq i \leq g$, there is an extra $x^{i-1}$ in the original integrand and thus an extra $\sqrt{\lambda}^{-i}(u-1)^{i-1}$ in the numerator of the above last expression, so
\[
A_{i1} \sim -2 \int_{0}^{1} \frac{(u-1)^{i-1} du}{\sqrt{u(u-1)(u-2)}} \frac{1}{\sqrt{x^{i-2} P(0)}}.
\]
(7.1)

Secondly, let $\delta_2$ contain only $-\sqrt{\lambda}$, $0$, $\sqrt{\lambda}$ and $a_1$ (but $a_2, \ldots, a_{2g-2}$). Then,
\[
A_{12} = \int_{\delta_2} \frac{dx}{\sqrt{x(x^2 - \lambda)} P(x)}
\]
\[
= \int_{\delta_2} \frac{dx}{x \sqrt{P(x)}} \left(1 + O\left(\frac{\lambda}{x^2}\right)\right)
\]
\[
\sim \int_{\delta_2} \frac{dx}{x \sqrt{P(x)}}.
\]

For general $a_{i2}$, there is an extra $x^{i-1}$ in the original integrand, so
\[
A_{i2} \sim \int_{\delta_2} \frac{x^{i-2} dx}{\sqrt{x} P(x)}
\]

In general, for $j \geq 2$, let $\delta_j$ contain only $-\sqrt{\lambda}$, $0$, $\sqrt{\lambda}$, $a_1, \ldots, a_{2j-3}$ (but $a_{2j-2}, \ldots, a_{2g-2}$). The entry $A_{ij}$ is asymptotic to the same integrand along $\delta_j$ instead of along $\delta_2$, i.e.,
\[
A_{ij} \sim \int_{\delta_j} \frac{x^{i-2} dx}{\sqrt{x} P(x)},
\]
which is exactly the same as the corresponding entry of $A_0$ when $i \geq 2$.

Lemma 7.2 Under the same assumptions as in Theorem 2.2, as $\lambda \to 0$, it holds that
\[
B \sim \begin{pmatrix}
  c_5 \lambda^{-\frac{1}{4}} \frac{1}{\sqrt{P(0)}} & O(1) \\
  \diamond & B_0
\end{pmatrix},
\]
where ♦ is a column vector whose entries are \( B_{i1} = (-1)^{i-1} \sqrt{-1} A_{i1} \) for \( A_{i1} \) in (7.1) and \( 2 \leq i \leq g \).

**Proof** For the first column of \( B \), we make the change of coordinates (similar to the proof of Lemma 6.3) by setting \( x = (-u + 1) \sqrt{x} \), and get for \( 1 \leq i \leq g \) that

\[
B_{i1} \sim (-1)^{i-1} \sqrt{-1} a_{i1}.
\]

For Column \( j, 2 \leq j \leq g \), we use Taylor expansion of \((x^2 - \lambda)^{-1/2}\) and get that

\[
B_{ij} \sim -2 \int_{a_{j-2}}^{a_{j-1}} x^{i-2} dx.
\]

Now we will give a proof of Theorem 2.2.

**Proof of Theorem 2.2** By Lemma 7.1 and the block matrix inversion, we know that

\[
A^{-1} \sim \begin{pmatrix} \lambda^{-1} \sqrt{P(0)} & O(\lambda^{1/4}) \\ (-A_0^{-1} \diamond \lambda^{-1} \sqrt{P(0)}) & A_0^{-1} \end{pmatrix},
\]

where \( \lim_{\lambda \to 0} (O(\lambda^{1/2}))' \frac{\lambda^{1/2}}{\lambda^2} \) is a finite column vectors with \( g - 1 \) rows. Therefore, as \( \lambda \to 0 \),

\[
Z = A^{-1} B \sim \begin{pmatrix} \sqrt{-1} & O(\lambda^{1/2}) \\ A_0^{-1}(-\lambda \sqrt{-1} \diamond) A_0^{-1} B_0 \end{pmatrix}, \quad \text{Im } Z \sim \begin{pmatrix} 1 & O(\lambda^{1/4}) \\ \text{Im}(A_0^{-1}(-\lambda \sqrt{-1} \diamond)) & \text{Im } Z_0 \end{pmatrix},
\]

and

\[
(\text{Im } Z)^{-1} \sim \begin{pmatrix} 1 & O(\lambda^{1/4}) \\ (\text{Im } Z_0)^{-1} \text{Im}(A_0^{-1}(-\lambda \sqrt{-1} \diamond)) (\text{Im } Z_0)^{-1} \end{pmatrix}.
\]

By (3.4) and (3.9), as \( \lambda \to 0 \), in the local coordinate \( z = \sqrt{x} \) near \((0, 0)\), it holds that

\[
k_\lambda(z) \to 1 + \sum_{i,j=1}^{g-1} (\text{Im } Z_0)^{-1}_{i,j} (z^i z^j)^2 = k_0(z) + \frac{4}{|z^4 P(z^2)|},
\]

so \( \kappa_{X_\lambda} \neq \kappa_0 \). Moreover,

\[
k_\lambda(z) - k_0(z) - \frac{4}{|z^4 P(z^2)|} \sim \frac{-8 \text{Re} \sum_{i=1}^{g-1} (\text{Im } Z_0)^{-1} \text{Im}(A_0^{-1}(-\lambda \sqrt{-1} \diamond))_i z^{2i}}{|z^4 P(z^2)|},
\]

which yields the conclusion.

Since \( Z \) is symmetric, \( A_0^{-1}(-\lambda \sqrt{-1} \diamond) = O(\lambda^{1/4}) \), and both the leading and subleading terms in the expansion of \( \kappa_{X_\lambda} \) are harmonic with respect to \( \lambda \).

**8 Cusp II: genus-two curves**

In this section, we consider a family of genus two curves

\[
X_\lambda := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x - \lambda)(x - 1)(x - a)(x - b)\}, \quad (8.1)
\]
where \(a, b, \lambda\) are distinct complex numbers satisfying \(0 < |\lambda| < |a| < |b|\). As \(\lambda \to 0\), \(X_\lambda\) degenerates to a singular curve \(X_0\) with an ordinary cusp. The normalization of \(X_0\) is an elliptic curve \(\{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x - a)(x - b)\}\), whose period is \(\tau\) given in (6.2). The Bergman kernels on \(X_\lambda\) and on the normalization of \(X_0\) are denoted by \(\kappa_{X_\lambda}\) and \(\kappa_0\), respectively. In the local coordinate \(z := \sqrt{x}\) near \((0,0)\), we write \(\kappa_{X_\lambda} = k_\lambda(z)dz \otimes d\bar{z}\), and \(\kappa_0 = k_0(z)dz \otimes d\bar{z}\). Then, our result on the asymptotic behaviour of the Bergman kernel \(\kappa_{X_\lambda}\) with precise coefficients is stated as follows.

**Theorem 8.1** For \(X_\lambda\) defined by (8.1), as \(\lambda \to 0\), it holds that

\(\begin{align*}
(i) \quad &\kappa_{X_\lambda} \to \kappa_0; \\
(ii) \quad &\text{for small } |z| \neq 0,
\end{align*}\)

\[
\log k_\lambda(z) - \log k_0(z) \sim \frac{\pi \cdot \text{Im} \; \tau}{-\log |\lambda| \cdot |z|^4},
\]

To prove Theorem 8.1, we need the following two lemmata by analyzing the asymptotics of the \(A\)-period matrix and \(B\)-period matrix on \(X_\lambda\). Our choice of cycles \(\delta_j, \gamma_j\) is specified in Sect. 3.

**Lemma 8.2** Under the same assumptions as in Theorem 8.1, as \(\lambda \to 0\), it holds that

\[
A \sim \begin{pmatrix} -\frac{2\pi}{\sqrt{ab}} & c_6 \\ \sqrt{ab}/\sqrt{\lambda} & -\frac{2c_3}{\sqrt{ab}} \end{pmatrix},
\]

where \(c_7 := \frac{2}{\sqrt{ab}} \int_0^1 \sqrt{v^{-1} - 1} \; dv\) and \(c_3, c_6\) are the same as in Lemma 6.2.

**Proof** Letting \(x = \lambda^2(1 - v)\), we consider the integrals

\[
\begin{align*}
\int_0^{\lambda^2} \frac{dx}{\sqrt{x(x - \lambda)(x - \lambda^2)}} = & \int_0^1 \frac{-\sqrt{-1} \lambda^{-1} \; dv}{\sqrt{v(v - 1)(v - 1 + \lambda^{-1})}} \\
& \sim -\frac{\lambda^{-1}}{2\sqrt{-1}} \int_C \frac{dv}{v\lambda^{-1/2}} = -\frac{-\pi}{\sqrt{\lambda}},
\end{align*}
\]

where \(C\) is a large cycle containing 0, 1, and

\[
\begin{align*}
\int_0^{\lambda^2} \frac{xdx}{\sqrt{x(x - \lambda)(x - \lambda^2)}} = & -\lambda \int_0^1 \frac{(v - 1) \; dv}{\sqrt{v(v - 1)(v - 1 + \lambda^{-1})}} \\
& \sim -\frac{-\lambda^{3/2}}{-\lambda} \int_0^1 \sqrt{\frac{v - 1}{v}} \; dv.
\end{align*}
\]

We estimate the four entries one by one. Firstly, let \(\delta_1\) only contain 0 and \(\lambda^2\), and similar to the proof of Lemma 6.2 we know that

\[
A_{11} = -2 \int_0^{\lambda^2} \frac{dx}{\sqrt{x(x - \lambda)(x - \lambda^2)}} \frac{1}{\sqrt{-a}} \frac{1}{\sqrt{-b}} \left(1 + \frac{x}{2a} + O(x^2)\right) \left(1 - \frac{x}{2b} + O(x^2)\right)
\]

\[
\sim -2 \int_0^{\lambda^2} \frac{dx}{\sqrt{x(x - \lambda)(x - \lambda^2)}} \\
\sim -2\frac{\sqrt{ab}}{\sqrt{\lambda^3}}.
\]

Secondly,

\[
A_{21} \sim \frac{2}{\sqrt{ab}} \int_0^{\lambda^2} \frac{xdx}{\sqrt{x(x - \lambda)(x - \lambda^2)}} \sim -2\frac{\lambda^{3/2}}{\sqrt{ab}} \int_0^1 \sqrt{\frac{v - 1}{v}} \; dv =: c_7\lambda^{3/2}.
\]
Thirdly, let $\delta_2$ contain only $0, \lambda, \lambda^2$ and $a$. Then, it holds that
\[
A_{12} = \int_{\delta_2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}}
= \int_{\delta_2} \frac{dx}{x \sqrt{x(x-a)(x-b)}} \left(1 + O\left(\frac{\lambda}{x}\right)\right) \left(1 + O\left(\frac{\lambda^2}{x}\right)\right)
\sim \int_{\delta_2} \frac{dx}{x \sqrt{x(x-a)(x-b)}} =: c_6,
\]
Lastly,
\[
A_{22} \sim \int_{\delta_2} \frac{dx}{x \sqrt{x(x-a)(x-b)}} = \int_{0}^{a} \frac{dx}{x \sqrt{x(x-a)(x-b)}}.
\]

**Lemma 8.3** Under the same assumptions as in Theorem 8.1, as $\lambda \to 0$, it holds that
\[
B \sim \left(\frac{2\sqrt{-1} \log \lambda}{\sqrt{ab \sqrt{-\lambda}}} d_3\right),
\]
where $d_3, d_4$ are the same as in Lemma 6.3.

**Proof** By [23] or (4.3), it is known that, as $\lambda \to 0$,
\[
\int_{\lambda^2}^{\lambda} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} = \frac{1}{\sqrt{\lambda}} \int_{\lambda}^{1} \frac{du}{\sqrt{u(u-1)(u-\lambda)}} \sim \frac{\sqrt{-1} \log \lambda}{\sqrt{\lambda}}.
\]
Firstly, let $\gamma_1$ contain only $\lambda$ and $\lambda^2$, and we get that
\[
B_{11} = -2 \int_{\lambda^2}^{\lambda} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}}
= -2 \int_{-\sqrt{ab}}^{\sqrt{ab}} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \left(1 + O(x)\right)
\sim 2 \sqrt{ab} \int_{\lambda^2}^{\lambda} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}}
\sim 2 \sqrt{-1} \log \lambda.
\]
Secondly, by (4.4) and the substitutions $t = \lambda^{-1}$ and $x = \lambda s^{-1}$, as $\lambda \to 0$,
\[
B_{21} \sim \frac{2}{\sqrt{ab}} \int_{\lambda^2}^{\lambda} \frac{x dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} = \frac{2}{\sqrt{ab}} \int_{1}^{t} \frac{ds}{s^2 \sqrt{s-t}} \sim \frac{2}{\sqrt{ab}} \sqrt{-\lambda}.
\]
Thirdly,
\[
B_{12} = -2 \int_{a}^{b} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}}
= -2 \int_{a}^{b} \frac{dx \left(1 + O\left(\lambda x^{-1}\right)\right) \left(1 + O\left(\lambda^2 x^{-1}\right)\right)}{x \sqrt{x(x-a)(x-b)}}
\sim -2 \int_{a}^{b} \frac{dx}{x \sqrt{x(x-a)(x-b)}} := d_3.
\]
Lastly,

\[ B_{22} \sim -2 \int_{a}^{b} \frac{dx}{\sqrt{x(x-a)(x-b)}} := d_4. \]

**Proof of Theorem 8.1** On the regular part of the cuspidal curve \( X_0 \), (6.3) gives the formula for the Bergman kernel \( \kappa_0 = k_0(z)|dz|^2 \) in the local coordinate \( z = \sqrt{x} \). By Lemma 8.2, as \( \lambda \to 0 \),

\[ A^{-1} \sim \begin{pmatrix} -2\pi \sqrt{ab} \lambda^3/2 & c_6 \sqrt{\lambda} \\ \frac{-2c_3}{\sqrt{ab}} \lambda^{3/2} & -2 \pi \sqrt{ab} \end{pmatrix}^{-1} \sim \frac{ab \sqrt{\lambda}}{4\pi c_3} \begin{pmatrix} -2c_3 & -c_6 \\ \frac{-2c_3}{\sqrt{ab}} \lambda^{3/2} & -2 \pi \sqrt{ab} \lambda \end{pmatrix}. \]

Let \( Z = A^{-1}B \) denote the normalized period matrix of \( X_\lambda \). Then, it follows that

\[ Z \sim \begin{pmatrix} -\frac{1}{\sqrt{\lambda}} \log \lambda \pi \sqrt{ab} \lambda^{3/2} & -2c_3 \frac{\sqrt{d_3}}{d_4} - c_6 d_4 \\ \frac{-2c_3}{\sqrt{ab}} \lambda^{3/2} & -2 \pi \sqrt{ab} \lambda \end{pmatrix}, \quad \text{Im} \ Z \sim \begin{pmatrix} -\frac{1}{\sqrt{\lambda}} \log \lambda \pi \sqrt{ab} \lambda^{3/2} & O(\lambda^{1/2}) \\ -\frac{1}{\sqrt{\lambda}} \log \lambda \pi \sqrt{ab} \lambda^{3/2} & \text{Im} \tau \end{pmatrix}. \]

So, as \( \lambda \to 0 \),

\[ (\text{Im} \ Z)^{-1} \sim \begin{pmatrix} -\frac{1}{\sqrt{\lambda}} \log \lambda \pi \sqrt{ab} \lambda^{3/2} & O(\lambda^{1/2}) \\ -\frac{1}{\sqrt{\lambda}} \log \lambda \pi \sqrt{ab} \lambda^{3/2} & \text{Im} \tau \end{pmatrix}^{-1}. \]

By (3.7) and (6.3), \( \kappa_{X_\lambda} \to \kappa_0 \). Moreover, in the local coordinate \( z = \sqrt{x} \) near \((0, 0)\),

\[ k_\lambda(z) - k_0(z) \sim \frac{4\pi}{|z^2 - a)(z^2 - b)(z^2 - c)|} \cdot \frac{1 + O(\lambda^{1/2})}{-\log |\lambda|}, \]

which yields the conclusion.

### 9 Cusp II: hyperelliptic curves

This section is devoted to the proof of Theorem 2.3. When \( \lambda \in \mathbb{C} \setminus \{0, 1, a_1, \ldots, a_{2g-2}\} \), \( X_\lambda \) defined in (2.5) has genus \( g \). To prove Theorem 2.3, we use the choice of cycles \( \delta_j, \gamma_j \) specified in Sect. 3, and need to analyze on \( X_\lambda \) the asymptotics of its \( A \)-period matrix and \( B \)-period matrix, denoted by \( A \) and \( B \), respectively. Meanwhile, for the normalization \( Y := \{(x, y) \in \mathbb{C}^2 | y^2 = x \ P(x) \} \) of genus \( g-1 \), denote its \( A \)-period matrix and \( B \)-period matrix by \( A_0 \) and \( B_0 \), respectively.

**Lemma 9.1** Under the same assumptions as in Theorem 2.3, as \( \lambda \to 0 \), it holds that

\[ A \sim \begin{pmatrix} \frac{2\pi}{\sqrt{\lambda} P(0)} O(1) \\ *\** A_0 \end{pmatrix}, \]

where ** is a column vector with \( g-1 \) rows whose entries are at most \( O(\lambda^{3/2}) \).
Proof} Firstly, as \( \lambda \to 0 \),

\[
A_{11} = -2 \int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \frac{1}{\sqrt{P(0)}} (1 + O(x))
\]

\[
\sim -2 \int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \frac{1}{\sqrt{P(0)}}
\]

\[
\sim \frac{2\pi}{\sqrt{\lambda P(0)}}.
\]

\[
A_{21} \sim -\frac{2}{\sqrt{P(0)}} \int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \sim -\frac{2}{\sqrt{P(0)}} \frac{\lambda^{3/2}}{\sqrt{-1}} \int_0^1 \sqrt{\frac{v-1}{v}} dv.
\]

Here we change the variable by letting \( x = \lambda^2(1-v) \). In general, for \( 2 \leq i \leq g \),

\[
A_{i1} \sim -\frac{2}{\sqrt{P(0)}} \frac{\lambda^{3/2}}{\sqrt{-1}} \int_0^1 \lambda^{2i-4} (1-v)^{i-2} \sqrt{\frac{v-1}{v}} dv = O(\lambda^{2i-2.5}).
\]

Secondly, let \( \delta_2 \) contain only \( 0, \lambda^2, \lambda \) and \( a_1 \) (but \( a_2, \ldots, a_{2g-2} \)). Then,

\[
A_{12} = \int_{\delta_2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} P(x)
\]

\[
= \int_{\delta_2} \frac{dx}{x\sqrt{xP(x)}} \left( 1 + O\left( \frac{\lambda}{x^2} \right) \right)
\]

\[
\sim \int_{\delta_2} \frac{dx}{x\sqrt{xP(x)}}.
\]

In general, for \( A_{12} \), there is an extra \( x^{i-1} \) in the original integrand above, so

\[
A_{i2} \sim \int_{\delta_2} \frac{x^{i-2}dx}{\sqrt{xP(x)}}.
\]

Thirdly, for \( j \geq 2 \), let \( \delta_j \) contain only \( 0, \lambda^2, \lambda, a_1, \ldots, a_{2j-3} \) (but \( a_{2j-2}, \ldots, a_{2g-2} \)). In general, the entry \( A_{ij} \) is asymptotic to the same integrand along \( \delta_j \) instead of along \( \delta_2 \), i.e.,

\[
A_{ij} \sim \int_{\delta_j} \frac{x^{i-2}dx}{\sqrt{xP(x)}}.
\]

which is exactly the same as the corresponding entry of \( A_0 \) when \( i \geq 2 \).

Lemma 9.2 Under the same assumptions as in Theorem 2.3, as \( \lambda \to 0 \), it holds that

\[
B \sim \begin{pmatrix} -\frac{2}{\sqrt{P(0)}} \frac{\sqrt{-1} \log \lambda}{\sqrt{\lambda}} & O(1) \end{pmatrix},
\]

where \( \heartsuit \) is a column vector whose entries are \( -\frac{2\lambda^{i-2}}{\sqrt{P(0)}} \), for \( 2 \leq i \leq g \).
Proof

\[
B_{11} = -2 \int_{\lambda^2}^{\lambda} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \cdot \frac{1}{\sqrt{P(x)} (1 + O(x))}
\]

\[
\sim -2 \int_{\lambda^2}^{\lambda} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \cdot \frac{1}{\sqrt{P(0)}}
\]

\[
\sim -2 \frac{\sqrt{-1 \log \lambda}}{\sqrt{\lambda}}.
\]

Thus, for \(2 \leq i \leq g\), by (4.5),

\[
B_{ij} = -2 \int_{\lambda^2}^{\lambda} \frac{x^{i-1}dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \cdot \frac{1}{\sqrt{P(0)}} (1 + O(x))
\]

\[
\sim -2 \int_{\lambda^2}^{\lambda} \frac{x^{i-1}dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \cdot \frac{1}{\sqrt{P(0)}}
\]

\[
= \frac{2\lambda^{i-2}}{\sqrt{P(0)}} \int_{a_j-3}^{a_j-2} ds \frac{s^{i-2}}{s^2 - \lambda} \frac{1}{\sqrt{s^2 - \lambda}}
\]

\[
\sim \frac{2\lambda^{i-2}}{\sqrt{P(0)}} \frac{1}{\sqrt{\lambda}},
\]

where the last equality holds due to the substitutions \(t = \lambda^{-1}\) and \(x = \lambda s^{-1}\). For Column \(j\), \(2 \leq j \leq g\), we use Taylor expansion of \(\sqrt{(x-\lambda)(x-\lambda^2)^{-1}}\) to get that

\[
B_{ij} \sim -2 \int_{a_j-3}^{a_j-2} \frac{x^{i-2}dx}{\sqrt{x^2 P(x)}},
\]

which is exactly the same as the corresponding entry of \(B_0\) when \(i \geq 2\).

Now we will give a proof of Theorem 2.3.

Proof of Theorem 2.3  By Lemma 9.1 and the block matrix inversion, we know that

\[
A^{-1} \sim \begin{pmatrix}
\frac{\sqrt{\lambda P(0)}}{2\pi} & O(\frac{1}{\lambda^2}) \\
O(\frac{1}{\lambda^2}) & A_0^{-1}
\end{pmatrix},
\]

where both \(\lim_{\lambda \to 0} \frac{(O(\lambda^2))^T}{\lambda^2} \) and \(\lim_{\lambda \to 0} \frac{O(\lambda^2)}{\lambda^2} \) are finite column vectors with \(g-1\) rows. Therefore, as \(\lambda \to 0\),

\[
Z = A^{-1} B \sim \begin{pmatrix}
-\sqrt{\frac{1}{\pi}} \log \lambda & O(\lambda^\frac{1}{2}) \\
O(\lambda^\frac{1}{2}) & A_0^{-1} B_0
\end{pmatrix}, \quad \text{Im } Z \sim \begin{pmatrix}
\log |\lambda| & O(\lambda^\frac{1}{2}) \\
\text{Im}(A_0^{-1} B_0) & \text{Im } Z_0
\end{pmatrix},
\]

and

\[
(\text{Im } Z)^{-1} \sim \begin{pmatrix}
-\frac{\log |\lambda|}{\pi} & O((\log |\lambda|)^{-1} \frac{1}{\lambda^\frac{1}{2}}) \\
(\text{Im } Z_0)^{-1} \text{Im}(A_0^{-1} B_0) & \frac{\pi}{\log |\lambda|} (\text{Im } Z_0)^{-1}
\end{pmatrix}.
\]
In fact, since $Z$ is symmetric, the off-diagonal block matrices in each matrix above concerning $Z$ are the transpose of each other. On the regular part of the cuspidal curve $X_0$, the formula for the Bergman kernel $\kappa_0 = k_0(z)|dz|^2$ is given in (3.9). This together with (3.6) will imply that $\kappa_{X_\lambda} \to \kappa_0$, as $\lambda \to 0$. Moreover, it holds that

$$k_\lambda(z) - k_0(z) \sim \frac{4\pi}{(z^2 - \lambda)(z^2 - \lambda^2)P(z^2)} \left( 1 - 2 \text{Re} \sum_{i=1}^{g-1} \left( (\text{Im} Z_0)^{-1} \text{Im}(A_0^{-1} \mathcal{O}) \right) \right) z^{2i},$$

which yields that

$$\psi - \log k_0(z) \sim \frac{\pi}{-\log |\lambda|} \frac{1}{\sum_{i,j=1}^{g-1} (\text{Im} Z_0)^{-1} i, j (z^i z^j)^2}.$$ 

## 10 Jacobian varieties

We will give a proof of Theorem 2.5 by using the results obtained in the proofs of Theorems 2.1, 2.2, 2.3. Let $X_\lambda$ be a compact curve of genus $g \geq 2$, and let $Z$ be its period matrix with respect to some chosen homology basis. The Jacobian variety of $X_\lambda$, which is denoted by $\text{Jac}(X_\lambda)$, is then identified with the $g$-dimensional complex torus $C^g/\mathbb{Z}^g + \mathbb{Z}^g$. It is well known that the Abel-Jacobi (period) map $X_\lambda \to \text{Jac}(X_\lambda)$ is a holomorphic embedding, and the Bergman kernel on a smooth algebraic curve is the pull back of the Euclidean metric from the Jacobian variety via this map.

**Proof of Theorem 2.5** By definition (1.1), the Bergman kernel on $\text{Jac}(X_\lambda)$ can be written as $\mu_\lambda(\text{d}w^1 \wedge \cdots \wedge \text{d}w^8) \wedge (\text{d}\overline{w}^1 \wedge \cdots \wedge \text{d}\overline{w}^8)$, under the coordinate $(w^1, \ldots, w^8)$ induced from $\mathbb{C}^8$, where $\mu_\lambda = (\text{det}(\text{Im} Z))^\lambda$. After adding a one-point compactification at $\infty$, one may assume that the curve $X_\lambda$ is compact. For $X_\lambda$ defined in (2.3) or (2.5), as $\lambda \to 0$, it holds that

$$\text{det}(\text{Im} Z) = \frac{\log |\lambda|}{-\pi} \text{det}(\text{Im} Z_0) + O(1) \to +\infty,$$

which yields the first part of the conclusion in Theorem 2.5. For $X_\lambda$ defined in (2.4), more careful analysis in the proof of Theorem 2.2 shows that as $\lambda \to 0$,

$$\text{Im} Z = \begin{pmatrix} 1 + O(\lambda^{1/2}) & O(\lambda^{1/4}) \\ O(\lambda^{1/4}) & \text{Im} Z_0 + O(\lambda^{1/2}) \end{pmatrix}, \quad \text{det}(\text{Im} Z) = \det \text{Im} Z_0 + O(\lambda^{1/2}) < +\infty,$$

which yields the second part of the conclusion.

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