PECULIARITY OF STRING THEORY ON ORBIFOLDS
IN THE PRESENCE OF
AN ANTSYMMETRIC BACKGROUND FIELD†

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ABSTRACT

We study string theory on orbifolds in the presence of an antisymmetric constant background field and discuss some of new aspects of the theory. It is shown that the term with the antisymmetric field has a topological nature like a Chern-Simons term or a Wess-Zumino term. Due to this property, the theory exhibits various anomalous behavior: Zero mode variables obey nontrivial quantization conditions. Coordinate transformations which define orbifolds are modified at the quantum level. Wavefunctions of twisted strings acquire phase factors when they move around non-contractible loops on orbifolds. Zero mode eigenvalues are shifted from naively expected values, in favor of modular invariance.

1. Introduction

Geometrical notion of string theory is far from obvious. For instance in toroidal compactification we can obtain enhanced gauge symmetries by tuning the moduli of the torus\(^1,2\) and there exists a striking isomorphism between the large radius and small radius tori\(^3\). Another surprising example is that strings on some orbifolds\(^4\) can equivalently be described as strings on tori\(^5−8\) although orbifolds are geometrically quite different from tori. An antisymmetric background field has first been introduced by Narain, Sarmadi and Witten\(^9\) to explain Narain torus compactification\(^10\) in the conventional canonical approach. Orbifold models based on Narain torus compactification have been discussed by many authors and various “realistic” models have been proposed\(^11−14\). Nevertheless, one interesting topological nature of the antisymmetric field has not been discussed before. Our aim of this paper is to discuss some of new aspects of string theory on orbifolds in the presence of the antisymmetric background field. As we will explain below, orbifold models considered before are trivial in a topological point of view. Our concern of this paper is topologically nontrivial orbifolds. Since fermionic degrees of freedom play no important role in our discussion, we will restrict our considerations to bosonic strings.

† To appear in the proceedings of the International Workshop on “String Theory, Quantum Gravity and the Unification of the Fundamental Interactions”, Rome, Italy on September 21-26, 1992.
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The action from which we shall start is

\[ S[X] = \int d\tau \int_0^\pi d\sigma \frac{1}{2\pi} \{ \eta^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^I + B^{IJ} \varepsilon^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^J \}, \quad (1.1) \]

where \( B^{IJ} \) (\( I, J = 1, \cdots, D \)) is the antisymmetric constant background field and \( X^I(\tau, \sigma) \) (\( I, \cdots, D \)) describes a string coordinate on a \( D \)-dimensional torus \( T^D \) which is defined by identifying a point \( X^I \) to \( X^I + \pi w^I \) for all \( w^I \in \Lambda \), where \( \Lambda \) is a \( D \)-dimensional lattice. An orbifold \(^4\) is obtained by dividing a torus by the action of a discrete symmetry group \( G \) of the torus. Any element \( g \) of \( G \) can be represented (for symmetric orbifolds) by\(^4,^{15}\)

\[ g = (U, v), \quad (1.2) \]

where \( U \) denotes a rotation and \( v \) is a translation. To simplify the discussion we will set the shift vector \( v = 0 \) throughout this paper. The shift vector will play no essential role in our discussion. In section 2 we clarify a topological nature of the second term of the action (1.1): Consider a transformation,

\[ g: \quad X^I \rightarrow U^{IJ} X^J. \quad (1.3) \]

It is shown in the next section that a Euclidean action \( S_E[X] \) in the path integral formalism\(^{16}\) is not invariant under the transformation (1.3) unless the rotation matrix \( U^{IJ} \) commutes with \( B^{IJ} \) but in general

\[ S_E[X] - S_E[U X] = i 2\pi n \quad \text{with} \quad n \in \mathbb{Z}. \quad (1.4) \]

This means that the transformation (1.3) is not a symmetry of the Euclidean action \( S_E[X] \) unless \([U, B] = 0\) but is a symmetry of the theory at the quantum level. Thus, the second term of the action (1.1) can be regarded as a topological term like a Chern-Simons term or a Wess-Zumino term. Much attention of previous works has been paid to the case

\[ [U, B] = 0, \quad (1.5) \]

or simply \( B^{IJ} = 0 \), i.e., to topologically “trivial” orbifold models. The generalization to the case

\[ [U, B] \neq 0, \quad (1.6) \]

is not, however, trivial, as expected from the result (1.4). A naive construction of orbifold models with Eq.(1.6) would give modular non-invariant partition functions and also destroy the duality of amplitudes\(^{15,17,18}\).

Due to the property (1.4), the orbifold models with Eq.(1.6) exhibit various anomalous behavior. In section 3 we discuss the duality of amplitudes and clarify cocycle properties of vertex operators. Then we find that zero modes of strings should obey nontrivial quantization conditions, which are the origin of the anomalous behavior of the theory in an operator formalism point of view.

In section 4 we see that the (classical) transformation should be modified at the quantum level and that a vertex operator \( V(k_L, k_R; z) \) transforms as

\[ g: \quad V(k_L, k_R; z) \rightarrow \rho V(U^T k_L, U^T k_R; z). \quad (1.7) \]
The phase factor $\rho$ can be regarded as a kind of a quantum effect and plays an important role in extracting physical states. It is pointed out that in an algebraic point of view the phase $\rho$ has a connection with automorphisms of algebras.

In section 5 we find an Aharonov-Bohm like effect in our system. Strings on orbifolds in the presence of the antisymmetric background field are very similar to electrons in the presence of an infinitely long solenoid: The space is not simply-connected. The antisymmetric background field plays the same role as an external gauge field. If an electron moves around the solenoid, a wavefunction of the electron in general acquires a phase. The same thing happens to twisted strings. A wavefunction of a twisted string is not, in general, periodic with respect to a torus shift but

$$
\Psi(x^I + \pi w^I) = e^{i\pi w \cdot v} \Psi(x^I),
$$

(1.8)

for $w^I \in \Lambda$ such that $w^I = U^{IJ} w^J$. The constant shift vector $v^I$ depends on the commutator $[U, B]$. We also discuss a physical implication of Eq.(1.8) and see how a naively expected spectrum is modified.

Section 6 is devoted to conclusions.

2. Topological Nature of a $B^{IJ}$-Term

In this section, we shall clarify a topological nature of the second term of the action (1.1) from a path integral point of view.

In the Euclidean path integral formalism, the one-loop vacuum amplitude of the closed bosonic string theory on a torus is given by the functional integral

$$
\int \frac{[dg_{\alpha\beta}][dX^I]}{\mathcal{V}_{ol}} \exp\{-S[X, g]\},
$$

(2.1)

where $g_{\alpha\beta}$ is a Euclidean metric of the two dimensional world sheet with the topology of a torus and $\mathcal{V}_{ol}$ is a volume of local gauge symmetry groups. The Euclidean action is given by

$$
S[X, g] = \int_0^1 d^2 \sigma \frac{1}{2\pi} \{\sqrt{g} g^{\alpha\beta} \partial_{\alpha} X^I \partial_{\beta} X^I - i B^{IJ} \varepsilon^{\alpha\beta} \partial_{\alpha} X^I \partial_{\beta} X^J\}.
$$

(2.2)

It should be emphasized that the imaginary number $i$ appears in the second term of the Euclidean action (2.2) due to the antisymmetric property of $\varepsilon^{\alpha\beta}$. Since strings propagate on a torus defined by identifying a point $X^I$ with $X^I + \pi w^I$ for all $w^I \in \Lambda$, where $\Lambda$ is a $D$-dimensional lattice, the string coordinate $X^I(\sigma^1, \sigma^2)$ obeys the following boundary conditions:

$$
X^I(\sigma^1 + 1, \sigma^2) = X^I(\sigma^1, \sigma^2) + \pi w^I,
$$

$$
X^I(\sigma^1, \sigma^2 + 1) = X^I(\sigma^1, \sigma^2) + \pi w'^I,
$$

(2.3)

for some $w^I, w'^I \in \Lambda$. 

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Let us consider a transformation,

$$g : X^I \rightarrow U^{IJ} X^J \quad \text{with} \quad U^T U = 1,$$  \hspace{1cm} (2.4)

where $U^{IJ}$ is an orthogonal matrix. The first term of the action (2.2) is trivially invariant under the transformation (2.4) but the second term is not invariant if $U^{IJ}$ does not commute with $B^{IJ}$. Since $B^{IJ}$ is a constant antisymmetric field, the second term of the action (2.2) can be written as a total divergence, i.e.,

$$B^{IJ} \varepsilon^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^J = \partial_\alpha (B^{IJ} \varepsilon^{\alpha\beta} X^I \partial_\beta X^J).$$  \hspace{1cm} (2.5)

It turns out that the difference between $S[X,g]$ and $S[UX,g]$ is given by

$$S[X,g] - S[UX,g] = i\pi \omega^I (B - U^T BU)^{IJ} w^J.$$  \hspace{1cm} (2.6)

From this relation, we come to an important conclusion that the transformation (2.4) is a quantum symmetry of the theory in a path integral point of view if

$$(B - U^T BU)^{IJ} w^J \in 2\Lambda^* \quad \text{for all} \quad w^I \in \Lambda,$$  \hspace{1cm} (2.7)

where $\Lambda^*$ is the dual lattice of $\Lambda$, although the Euclidean action (2.2) itself is not invariant under the transformation (2.4) if $[U,B] \neq 0$. Then, the second term of the action (2.2) can be regarded as a topological term like a Chern-Simons term or a Wess-Zumino term. In the following, we will show that the condition (2.7) is nothing but a consistency condition for constructing string theory on orbifolds.

An orbifold \(^4\) is obtained by dividing a torus by the action of a suitable discrete group $G$. In the construction of an orbifold model, we start with a $D$-dimensional toroidally compactified closed bosonic string theory which is specified by a $(D+2)$-dimensional Lorentzian even self-dual lattice $\Gamma^{D,D}$, on which the left- and right-moving momentum $(p^I_L, p^I_R)$ $(I = 1, \cdots, D)$ lies. In general any group element $g$ of $G$ (for symmetric orbifolds) can be represented by

$$g = (U, v),$$  \hspace{1cm} (2.8)

where $U$ is a rotation and $v$ is a translation. As mentioned in the introduction, we will set the shift vector $v = 0$ for simplicity. To define an orbifold consistently, $G$ must be a discrete symmetry group of the torus. This means that any element $g$ of $G$ must be an automorphism of the lattice $\Gamma^{D,D}$, i.e.,

$$g : (p^I_L, p^I_R) \rightarrow (U^{IJ} p^J_L, U^{IJ} p^J_R) \in \Gamma^{D,D},$$  \hspace{1cm} (2.9)

for all $(p^I_L, p^I_R) \in \Gamma^{D,D}$. The left- and the right- moving momenta, $p^I_L$ and $p^I_R$, are related to the center of mass momentum $p^I$ and the winding number $w^I$ of a string as follows\(^9\):

$$p^I_L = \frac{1}{2} p^I + \frac{1}{2} (1 - B)^{IJ} w^J,$$

$$p^I_R = \frac{1}{2} p^I - \frac{1}{2} (1 + B)^{IJ} w^J.$$  \hspace{1cm} (2.10)
The winding number \( w^I \), by definition, lies on the lattice \( \Lambda \), i.e.,
\[
w^I \in \Lambda.
\] (2.11)

Since a wavefunction \( \Psi(x^I) \) of a string on a torus must be periodic, i.e., \( \Psi(x^I + \pi w^I) = \Psi(x^I) \) for all \( w^I \in \Lambda \), the allowed momentum is
\[
p^I \in 2\Lambda^*.
\] (2.12)

It follows from Eqs. (2.9) and (2.10) that \( g \) acts on \( w^I \) and \( p^I \) as follows:
\[
g : \begin{align*}
w^I & \rightarrow U^{IJ}w^J, \\
p^I & \rightarrow U^{IJ}p^J - [U,B]^{IJ}w^J.
\end{align*}
\] (2.13)

For the action of \( g \) on \( w^I \) to be well-defined, it must be an automorphism of \( \Lambda \), i.e.,
\[
U^{IJ}w^J \in \Lambda \text{ for all } w^I \in \Lambda.
\] (2.14)

For the action of \( g \) on \( p^I \) to be well-defined, we further require the following condition:
\[
[U,B]^{IJ}w^J \in 2\Lambda^*,
\]
or equivalently
\[
(B - U^T B U)^{IJ}w^J \in 2\Lambda^* \text{ for all } w^I \in \Lambda.
\] (2.15)

This is just the condition (2.7), as announced before.

In the following sections, we shall construct the operator formalism of string theory on orbifolds in the presence of the antisymmetric constant background field and reveal various anomalous behavior of the theory. We will see that the necessity of nontrivial cocycle operators in vertex operators is the origin of the anomalous behavior in the operator formalism.

Before closing this section, it may be instructive to present some examples of \( B^{IJ} \) and \( U^{IJ} \), which satisfy the condition (2.15). Let us consider the following \( (D + D) \)-dimensional Lorentzian even self-dual lattice, which has been introduced by Englert and Neveu\(^{20}\),
\[
\Gamma^{D,D} = \{ (p^I_L,p^I_R) \mid p^I_L,p^I_R \in \Lambda_W(\mathcal{G}), \ p^I_L - p^I_R \in \Lambda_R(\mathcal{G}) \},
\] (2.16)

where \( \Lambda_R(\mathcal{G}) \) (\( \Lambda_W(\mathcal{G}) \)) is the root (weight) lattice of a simply-laced Lie algebra \( \mathcal{G} \) with rank \( D \) and the squared length of the root vectors is normalized to two. In this normalization, the weight lattice \( \Lambda_W(\mathcal{G}) \) is just the dual lattice of \( \Lambda_R(\mathcal{G}) \). It should be emphasized that the lattice (2.16) plays a crucial role of gauge symmetries in closed string theory\(^{1,2}\). The lattice (2.16) can be obtained through the relations (2.10) by choosing \( \Lambda \) and \( B^{IJ} \) as follows:
\[
\Lambda = \Lambda_R(\mathcal{G}),
\] (2.17)
and
\[ \alpha_i^I B^{IJ} \alpha_j^J = \alpha_i^I \alpha_j^J \mod 2, \quad (2.18) \]
where \( \alpha_i \) is a simple root of \( G \) which is normalized to \( (\alpha_i)^2 = 2 \). If we choose the rotation matrix \( U \) to be an automorphism of the root lattice \( \Lambda_R(G) \), i.e.,
\[ U^{IJ} w^J \in \Lambda_R(G) \quad \text{for all} \quad w^I \in \Lambda_R(G), \quad (2.19) \]
then the transformation (2.9) is an automorphism of the lattice (2.16) and the matrix \( U \) always satisfies the condition (2.15), i.e.,
\[ (B - U^T B U)^{IJ} w^J \in 2\Lambda_W(G) \quad \text{for all} \quad w^I \in \Lambda_R(G). \quad (2.20) \]

3. Cocycle Properties of Vertex Operators

In this section we shall investigate cocycle properties of vertex operators and show that zero modes of strings should obey nontrivial quantization conditions.

Let us consider a vertex operator \( V(k_L^I, k_R^I, z) \) which describes the emission of a state with the momentum \( (k_L^I, k_R^I) \in \Gamma^{D,D} \). The vertex operator will be of the form,
\[ V(k_L, k_R; z) = f_{k_L, k_R}(z) : e^{ik_L \cdot X_L(z) + ik_R \cdot X_R(\bar{z})} C_{k_L, k_R}, \quad (3.1) \]
where \( X_L^I(z) \) (\( X_R^I(\bar{z}) \)) are the left- (right-) moving string coordinate and \( f_{k_L, k_R}(z) \) is a normalization factor, which will in general depend on \( k_L^I, k_R^I, z \) and \( \bar{z} \) in twisted sectors. The \( C_{k_L, k_R} \) is, if necessary, some extra (cocycle) operator. The product of two vertex operators
\[ V(k_L, k_R; z)V(k_L', k_R'; z'), \quad (3.2) \]
is well-defined if \( |z| > |z'| \). The different ordering of the two vertex operators corresponds to the different “time”-ordering. To obtain scattering amplitudes, we must sum over all possible “time”-ordering for the emission of states. We must then establish that each contribution is independent of the order of the vertex operators to enlarge the regions of integrations over \( z \) variables\(^{21}\). Thus the product (3.2), with respect to \( z \) and \( z' \), has to be analytically continued to the region \( |z'| > |z| \) and to be identical to
\[ V(k_L', k_R'; z')V(k_L, k_R; z), \quad (3.3) \]
for \( |z'| > |z| \). In terms of zero modes, the above statement can be translated into the following condition:
\[ V_0(k_L, k_R)V_0(k_L', k_R') = \eta V_0(k_L', k_R')V_0(k_L, k_R), \quad (3.4) \]
where
\[ V_0(k_L, k_R) = e^{ik_L \cdot \hat{x}_L + ik_R \cdot \hat{x}_R} C_{k_L, k_R}, \quad (3.5) \]
The \( \hat{x}_L^I \) (\( \hat{x}_R^I \)) denotes the left-(right-) moving “center of mass” coordinate. The phase factor \( \eta \) is required to compensate a phase appearing in reversing the order
of the oscillator modes of the vertex operators. In the untwisted sector, the phase $\eta$ is given by\textsuperscript{1,2}
\[
\eta = (-1)^{k_L \cdot k'_L - k_R \cdot k'_R}. \tag{3.6}
\]

In the $U$-twisted sector with $U^N = 1$, the phase $\eta$ is given by\textsuperscript{22}
\[
\eta = \exp\{i\pi k^I_L (1 + \sum_{\ell=1}^N \frac{\ell}{N} (U^{\ell} - U^{\ell')))^{IJ} k'^J_L - i\pi k^I_R (1 + \sum_{\ell=1}^N \frac{\ell}{N} (U^{\ell} - U^{\ell')))^{IJ} k'^J_R}\}.
\tag{3.7}
\]

In the untwisted sector, Frenkel and Kač\textsuperscript{1,2} have shown that the cocycle operator can be constructed without introducing any more degrees of freedom. However, instead of introducing the cocycle operator $C_{k_L, k_R}$ it is possible to achieve the relation (3.4). In ref.23, it has been shown that multiplying vertex operators by cocycle operators is equivalent to modifying commutation relations for zero modes and that the following commutation relations \textsuperscript{†} with $C_{k_L, k_R} = 1$:
\[
[\hat{x}^I_L, \hat{x}^J_L] = i\pi B^{IJ},
[\hat{x}^I_R, \hat{x}^J_R] = i\pi B^{IJ},
[\hat{x}^I_L, \hat{x}^J_R] = i\pi (1 - B)^{IJ}, \tag{3.8}
\]

lead to the relation (3.4) with the correct phase (3.6). A geometrical meaning of the above commutation relations has also been discussed in ref.23.

In twisted sectors, the relation (3.4) with the phase (3.7) may require new degrees of freedom, which are called fixed points (fixed lines, fixed surfaces, · · ·) by physicists. The explicit realization has been obtained in ref.24 in the case of $[U, B] = 0$ and $det(1 - U) \neq 0$. In ref.24, the quantization of zero modes of twisted strings has been clarified from a geometrical point of view and the nontrivial commutation relations of the zero modes with $C_{k_L, k_R} = 1$ have been shown to naturally satisfy the relation (3.4) with the correct phase (3.7). For $[U, B] \neq 0$ and/or $det(1 - U) = 0$ we need to modify the results given in ref.24. It turns out that the following commutation relations with $C_{k_L, k_R} = 1$ satisfy the relation (3.4) with the phase (3.7):
\[
[\hat{x}^I_L, \hat{x}^J_L] = i\pi (B - \sum_{\ell=1}^N \frac{\ell}{N} (U^{\ell} - U^{\ell')))^{IJ},
[\hat{x}^I_R, \hat{x}^J_R] = i\pi (B + \sum_{\ell=1}^N \frac{\ell}{N} (U^{\ell} - U^{\ell')))^{IJ},
[\hat{x}^I_L, \hat{x}^J_R] = i\pi (1 - B)^{IJ}, \tag{3.9}
\]

These commutation relations reduce to those given in ref. 24 for $[U, B] = 0$ and $det(1 - U) \neq 0$ and also reduce to the relations (3.8) for $U = 1$. We will see in the

\textsuperscript{†} The normalization of $\hat{x}^I_L$ and $\hat{x}^I_R$ is different from that in ref.23 by factor two.
following sections that the above nontrivial commutation relations are the origin of anomalous features of the theory.

4. Anomalous Transformations

In this section, we shall show that the action of \( g \) on the string coordinate becomes anomalous at the quantum level if \([U, B] \neq 0\).

The relevant operators in string theory are the momentum operators, \( P^I_L(z) \equiv i\partial_z X^I_L(z) \) and \( P^I_R(\bar{z}) \equiv i\partial_{\bar{z}} X^I_R(\bar{z}) \), and the vertex operator \( V(k_L, k_R; z) \). Other operators can be obtained from the operator products of these operators. For example the energy-momentum tensors of the left- and the right- movers are given by

\[
T(z) = \lim_{w \to z} \frac{1}{2} \left( P^I_L(w) P^I_L(z) - \frac{D}{(w - z)^2} \right),
\]

\[
\bar{T}(\bar{z}) = \lim_{\bar{w} \to \bar{z}} \frac{1}{2} \left( P^I_R(\bar{w}) P^I_R(\bar{z}) - \frac{D}{(\bar{w} - \bar{z})^2} \right).
\]

Under the action of \( g \), the momentum operators transform as

\[
g: \quad (P^I_L(z), P^I_R(\bar{z})) \rightarrow (U^{IJ} P^I_L(z), U^{IJ} P^I_R(\bar{z})),
\]

which leaves the energy-momentum tensors invariant, as it should be. One might expect that under the action of \( g \) the left- and the right- moving string coordinates, \( X^I_L(z) \) and \( X^I_R(\bar{z}) \), transform in a similar way to \( P^I_L(z) \) and \( P^I_R(\bar{z}) \). However, it is not the case if \( U^{IJ} \) does not commute with \( B^{IJ} \). Suppose that under the action of \( g \), \( \hat{x}^I_L \) and \( \hat{x}^I_R \) would transform as

\[
g: \quad \hat{x}^I_L \rightarrow U^{IJ} \hat{x}^J_L,
\]

\[
\hat{x}^I_R \rightarrow U^{IJ} \hat{x}^J_R.
\]

It is easy to see that the transformations (4.3) is inconsistent with the commutation relations (3.8) or (3.9) unless \( U^{IJ} \) commutes with \( B^{IJ} \). In ref.15, it has been shown that in the untwisted sector the correct action of \( g \) on \( \hat{x}^I_L \) and \( \hat{x}^I_R \) is given by \(^\dag\)

\[
g: \quad \hat{x}^I_L \rightarrow U^{IJ} \hat{x}^J_L + \pi U^{IJ}(\frac{1}{2}(B - U^T BU) - C_U)^{JK} \hat{w}^K,
\]

\[
\hat{x}^I_R \rightarrow U^{IJ} \hat{x}^J_R - \pi U^{IJ}(\frac{1}{2}(B - U^T BU) - C_U)^{JK} \hat{w}^K,
\]

where \( \hat{w}^k \) is the winding number operator, which satisfies the following commutation relations\(^\ddagger\):

\[
[\hat{x}^I_L, \hat{w}^J] = i\delta^{IJ},
\]

\[
[\hat{x}^I_R, \hat{w}^J] = -i\delta^{IJ}.
\]

\(^\dag\) Set \( U_L = U_R = U \) and assume the form (6-13) in the first reference of ref.15.

\(^\ddagger\)
The $C^{IJ}_U$ is a symmetric matrix defined through the relation,

$$w^I C^{IJ}_U w'^J = \frac{1}{2} w^I (B - U^T BU)^{IJ} w'^J \mod 2,$$

for all $w^I, w'^I \in \Lambda$. The existence of a symmetric matrix $C_U$ is guaranteed by the fact that

$$\frac{1}{2} w^I (B - U^T BU)^{IJ} w'^J \in \mathbb{Z} \quad \text{for all} \quad w^I, w'^I \in \Lambda.$$

In ref.15, the transformation (4.4) has been derived from the requirement that the action of $g$ on vertex operators has to preserve the duality property of amplitudes. It is important to verify that the transformation (4.4) is consistent with the commutation relations (3.8). We can further show that the transformation (4.4) is still correct in the $U$-twisted sector and is consistent with the commutation relations (3.9).

It seems that the transformation (4.4) loses its geometrical meaning unless $U^{IJ}$ commutes with $B^{IJ}$. However, the transformation (4.4) is geometrically still well-defined because the anomalous terms in Eq.(4.4) can be regarded as a torus shift, i.e.,

$$g : \ (\hat{x}^I_L, \hat{x}^J_R) \rightarrow (U^{IJ} \hat{x}^I_L, U^{IJ} \hat{x}^J_R) + \text{torus shift}. \quad (4.8)$$

To see this, let us consider the action of $g$ on $k_L \cdot \hat{x}_L + k_R \cdot \hat{x}_R$ for $(k^I_L, k^I_R) \in \Gamma^{D,D}$.

$$g : \ k_L \cdot \hat{x}_L + k_R \cdot \hat{x}_R \rightarrow k_L \cdot U \hat{x}_L + k_R \cdot U \hat{x}_R + \pi (k_L - k_R) \cdot U \left( \frac{1}{2} (B - U^T BU) - C_U \right) \hat{w}. \quad (4.9)$$

The last term of the right hand side may be regarded as a trivial shift because

$$(k_L - k_R) \cdot U \left( \frac{1}{2} (B - U^T BU) - C_U \right) \hat{w} = 0 \mod 2, \quad (4.10)$$

for all $(k^I_L, k^I_R) \in \Gamma^{D,D}$ and $\hat{w} \in \Lambda$. (Recall that $k^I_L - k^I_R \in \Lambda$.)

What role do the anomalous terms in Eq.(4.4) play? To answer this question, let us consider the action of $g$ on a vertex operator. The result is

$$g : \ V(k_L, k_R; \alpha) \rightarrow e^{-i \frac{\pi}{2} (k_L - k_R) \cdot U C_U U^T (k_L - k_R)} V(U^T k_L, U^T k_R; \alpha). \quad (4.11)$$

It should be noted that to derive the relation (4.11) we cannot use the relation (4.10) directly in the exponent because $\hat{w}^I$ is not a c-number but a q-number. We can use the relation (4.10) only after separating the terms depending on $\hat{w}^I$ from $V(U^T k_L, U^T k_R; \alpha)$ by use of the Hausdorff formula. The phase factor appearing in the transformation (4.11) plays an important role in extracting physical states, which must be $g$-invariant. The following example may be helpful to understand a role of the phase in Eq.(4.11): Let us consider the (left-moving) momentum operator $H^I(z) \equiv P^I_L(z) \ (I = 1, 2)$ and the vertex operators $E^\alpha(z) \equiv V(\alpha, 0; z)$, where $\alpha$'s are root vectors of $SU(3)$ with $\alpha^2 = 2$. They will form level one $SU(3)$ Kač-Moody
algebra\(^1,2\). Let \(\alpha_i\) \((i = 1, 2)\) be a simple root of \(SU(3)\). Consider the following transformation:

\[ g : \quad \alpha_1 \leftrightarrow \alpha_2. \quad (4.12) \]

This is clearly an automorphism of the root lattice of \(SU(3)\). Then one might expect that under the transformation \((4.12)\) the generators of \(SU(3)\) Kac-Moody algebra would transform as follows:

\[ \begin{align*}
\alpha_1 \cdot H(z) & \longrightarrow \alpha_2 \cdot H(z), \\
E^{\pm \alpha_1}(z) & \longrightarrow E^{\pm \alpha_2}(z),
\end{align*} \quad (4.13) \]

and \(E^{\pm(\alpha_1+\alpha_2)}(z)\) would be left invariant. If it were true, we would have a strange result: The invariant generators, \((\alpha_1+\alpha_2)\cdot H(z), E^{\pm \alpha_1}(z) + E^{\pm \alpha_2}(z)\) and \(E^{\pm(\alpha_1+\alpha_2)}(z)\) could form a subalgebra of \(SU(3)\). This is not, however, acceptable because five generators must form an algebra with rank more than 3 while rank of \(SU(3)\) is 2. The key to resolve this inconsistency is the phase in Eq.\((4.11)\): We may take an Englert-Neveu lattice \((2.16)\) with \(\Lambda^R(SU(3))\) in order for \(E^\alpha(z)\) to be well-defined\(^1,2\). This means that we take a nontrivial antisymmetric field \(B^{IJ}\) through the relation \((2.18)\). It turns out that \(B^{IJ}\) does not commute with the transformation \((4.12)\). Then, our formalism tells us that the transformations \((4.13)\) are still true but \(E^{\pm(\alpha_1+\alpha_2)}(z)\) must transform as

\[ g : \quad E^{\pm(\alpha_1+\alpha_2)}(z) \longrightarrow -E^{\pm(\alpha_1+\alpha_2)}(z). \quad (4.14) \]

Therefore, \(E^{\pm(\alpha_1+\alpha_2)}(z)\) are not invariant under the action of \(g\) although \(\alpha_1 + \alpha_2\) is. In an algebraic point of view the phase in Eq.\((4.11)\) has a connection with automorphisms of algebras rather than automorphisms of lattices.

5. Aharonov-Bohm like Effect

Strings on orbifolds in the presence of the antisymmetric background field may be in a similar situation to electrons in the presence of an infinitely long solenoid. Both underlying spaces are not simply-connected and possess non contractible loops. The antisymmetric background field \(B^{IJ}\) may play a similar role to an external gauge field \(A^\mu\). The gauge field \(A^\mu\) is not gauge invariant and \(B^{IJ}\) is not invariant under the transformation \(B^{IJ} \rightarrow (U^T B U)^{IJ}\) if \(U^{IJ}\) does not commute with \(B^{IJ}\). The relation \((2.6)\) means that \(B^{IJ}\) cannot be defined as a single-valued “function” for twisted strings with \([U, B] \neq 0\). This fact suggests that if a twisted string moves around a non-contractible loop on the orbifold the wavefunction may acquire a nontrivial phase. In fact we can show that

\[ \Psi(x^I + \pi w^I) = e^{-i \frac{\pi}{2} w^I C_{IJ}^I w^J} \Psi(x^I), \quad (5.1) \]

where

\[ w^I \in \Lambda_U = \{ w^I \in \Lambda \mid w^I = U^{IJ} w^J \}. \quad (5.2) \]
What is a physical implication of the relation (5.1)? To see this, we first note that the left hand side of Eq.(5.1) can be expressed as
\[ \Psi(x^I + \pi w^I) = e^{-i\pi w^I \hat{p}^I} \Psi(x^I), \] (5.3)
where \( \hat{p}^I \) is the canonical conjugate momentum restricted to the \( U \)-invariant subspace, i.e., \( \hat{p}^I = U^{IJ} \hat{p}_J^I \). Introduce a vector \( v^I \) with \( v^I = U^{IJ} v_J^I \) through the relation
\[ w^I v^I = \frac{1}{2} w^I C_{U}^{IJ} w^J \mod 2, \] (5.4)
for all \( w^I \in \Lambda_U \). Comparing Eq.(5.1) with (5.3), we conclude that
\[ \hat{p}^I \in v^I + \Lambda_U^*, \] (5.5)
where \( \Lambda_U^* \) is the dual lattice of \( \Lambda_U \). Thus, allowed eigenvalues of \( \hat{p}^I \) are different from naively expected values by \( v^I \). This result exactly agrees with the one expected from the argument of modular invariance in ref.15. Further we can show that Eq.(5.5) is consistent with the single-valuedness of \( g \)-invariant operators with respect to \( z \).

Before deriving Eq.(5.1) or (5.5), it may be instructive to recall quantum mechanics on a circle with a radius \( L \), where a point \( x \) is identified with \( x + 2\pi nL \) for all \( n \in \mathbb{Z} \). It is important to understand that the coordinate operator \( \hat{x} \) itself is not well-defined on the circle while the canonical momentum operator \( \hat{p} \) is well-defined. However, the following operator
\[ e^{ik\hat{x}} \quad \text{with} \quad k = \frac{m}{L} \quad (m \in \mathbb{Z}), \] (5.6)
is well-defined on the circle because it is consistent with the torus identification. Physically, the operator (5.6) plays a role of a momentum shift by \( k = \frac{m}{L} \). It turns out that an operator \( e^{i2\pi L \hat{p}} \) commutes with all operators, i.e., \( \hat{p} \) and \( e^{ik\hat{x}} \), and hence it must be a \( c \)-number. Indeed we have
\[ e^{i2\pi L \hat{p}} = 1. \] (5.7)
This leads to the well-known result,
\[ \hat{p} = \frac{m}{L} \quad \text{with} \quad m \in \mathbb{Z}. \] (5.8)

In string theory on orbifolds, \( \hat{p} \) and \( e^{ik\hat{x}} \) will be replaced by the momentum operators \( P^I_L(z), \hat{P}^I_R(\bar{z}) \) and the vertex operator \( V(k_L, k_R; z) \), respectively, and an analog of the identity (5.7) in the \( U \)-twisted sector is
\[ \xi_k e^{-ik_L \cdot \hat{U} \hat{x}_L - ik_R \cdot \hat{U} \hat{x}_R} e^{ik_L \cdot \hat{x}_L + ik_R \cdot \hat{x}_R} e^{i2\pi(k_L \cdot P_U \hat{p}_L - k_R \cdot P_U \hat{p}_R)} = 1, \] (5.9)
where
\[
\xi_k = \exp\{i\pi (k_L \cdot \mathcal{P} U k_L - k_R \cdot \mathcal{P} U k_R) + \frac{i\pi}{2} (k_L - k_R) \cdot UC_U U^T (k_L - k_R)\}. \quad (5.10)
\]

The $\mathcal{P} U$ is a projection matrix defined by $(U^N = 1)$
\[
\mathcal{P} U = \frac{1}{N} \sum_{\ell=1}^N U^\ell. \quad (5.11)
\]

It is not difficult to show that the left hand side of Eq.(5.9) commutes with all the operators by use of the commutation relations (3.9). A physical meaning of the identity (5.9) is obvious for $[U, B] = 0$. The identity (5.9) then reduces to
\[
e^{i\pi k \cdot \hat{\omega}} e^{i\pi w \cdot U \hat{p}} = 1, \quad (5.12)
\]
where
\[
w^I = k^I_L - k^I_R \in \Lambda, \quad k^I = (1 + B)^{IJ} k^J_L + (1 - B)^{IJ} k^J_R \in 2\Lambda^*. \quad (5.13)
\]
The $\hat{p}^I$ is the canonical momentum conjugate to $\hat{x}^I$ in the $U$-twisted sector. The identity (5.12) implies that
\[
\hat{w}^I \in \Lambda, \quad \hat{p}^I \in 2\Lambda^*. \quad (5.14)
\]
These results are consistent with the identification $x^I \sim x^I + \pi w^I$ for all $w^I \in \Lambda$. For $[U, B] \neq 0$, the first relation of (5.14) still holds while the second one does not. In fact, putting $k^I = 0$ and $w^I \in \Lambda_U$ in Eq.(5.9), we have
\[
\exp\{i\pi w \cdot \hat{p} - i\frac{\pi}{2} w \cdot C_U w\} = 1. \quad (5.15)
\]
This is equivalent to Eq. (5.1) or (5.5).

6. Conclusions

In this paper, we have discussed some of new aspects of string theory on orbifolds in the presence of the antisymmetric constant background field $B^{IJ}$ both in a path integral point of view and in an operator formalism point of view, although we have omitted some of the details. This work is, however, far from complete. We have shown that the transformation (2.4) is a quantum symmetry of string theory on tori. To clarify the full topological nature of the $B^{IJ}$-term, it is necessary to construct the path integral formalism of string theory on orbifolds in the presence of the antisymmetric background field and also beyond one-loop. For topologically nontrivial cases, i.e., $[U, B] \neq 0$, we have not yet found a Euclidean action of string

\[\dagger\] For the precise definition of $\hat{p}^I$ in the $U$-twisted sector, see Eq.(3.15) in ref.17.
theory on orbifolds, which respects the duality and modular invariance of amplitudes \(^{17,25}\). In the operator formalism, we also have to construct Hilbert spaces of twisted sectors and to verify modular invariance of amplitudes. Subtle part for constructing twisted Hilbert spaces is the zero modes, which obey the nontrivial algebras (3.9). A necessary condition for modular invariance is left-right level matching conditions \(^{26,27}\). In ref.15, the level matching conditions have been proved to be satisfied. The level matching conditions may be necessary and sufficient for modular invariance as discussed in ref. 26 and 27 although the discussions may not directly be applicable to our case. The details of this paper and the subjects addressed above will be reported elsewhere.

Acknowledgements

I would like to thank K. Ito, H. Ooguri, J. Petersen, F. Ruiz and M. Tabuse for useful discussions and also would like to acknowledge the hospitality for the Niels Bohr Institute where this work was done.

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