A SIMPLY CONNECTED SURFACE OF GENERAL TYPE
WITH $p_g = 1$, $q = 0$, AND $K^2 = 8$

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Abstract. We construct a new family of simply connected minimal complex surfaces with $p_g = 1$, $q = 0$, and $K^2 = 8$ using a $\mathbb{Q}$-Gorenstein smoothing theory.

1. Introduction

This paper is an addendum to the authors' work [5], in which we constructed a family of minimal complex surfaces of general type with $p_g = 1$, $q = 0$, and $1 \leq K^2 \leq 2$ and simply connected surfaces with $p_g = 1$, $q = 0$, and $3 \leq K^2 \leq 6$ using a $\mathbb{Q}$-Gorenstein smoothing theory. We extend the results to the $K^2 = 8$ case in this paper. The main result of this paper is the following theorem.

Theorem 1.1. There exists a simply connected minimal complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 8$.

We briefly sketch the proof. At first we blow up a K3 surface $Y$ in a suitable set of points so that we obtain a surface with some special disjoint linear chains of rational curves which can be contracted to singularities class $T$ on a singular surface $X$ with $H^2(T_X) \neq 0$. In order to prove the existence of a global $\mathbb{Q}$-Gorenstein smoothing of $X$, we apply the cyclic covering trick developed in Y. Lee and J. Park [2]. The cyclic covering trick says that, if a cyclic covering $\pi : V \to W$ of singular surfaces satisfies certain conditions and the base $W$ has a $\mathbb{Q}$-Gorenstein smoothing, then the cover $V$ has also a $\mathbb{Q}$-Gorenstein smoothing.

The main ingredient of this paper is that we construct an unramified double covering $\pi : \overline{X} \to X$ to a singular surface $X$ constructed in a recent paper [4] of the first author. It is a main result of H. Park [4] that the singular surface $X$ has a global $\mathbb{Q}$-Gorenstein smoothing and a general fiber $X_t$ of the smoothing of $X$ is a surface of general type with $p_g = 0$, $K^2 = 4$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. We show that the double covering $\overline{X} \to X$ satisfies all the conditions of the cyclic covering trick; hence, there is a global $\mathbb{Q}$-Gorenstein smoothing of $\overline{X}$. Then it is not difficult to show that a general fiber $\overline{X}_t$ of the smoothing of $\overline{X}$ is the desired surface.

2. Construction

According to Kondo [1], there is an Enriques surface $Y$ with an elliptic fibration over $\mathbb{P}^1$ which has an $I_0$-singular fiber, a nodal singular fiber $F$, and two bisections $S_1$ and $S_2$; Figure 1. Again by Kondo [1], there is an unbranched double covering...
\( \pi : Y \to Y \) of \( Y \) where \( Y \) is an elliptic \( K3 \) surface which has two \( I_9 \)-singular fiber, two nodal singular fiber \( F_1 \) and \( F_2 \), and four sections \( S_1, \ldots, S_4 \) such that \( \pi(F_1) = \pi(F_2) = F, \pi(S_1) = \pi(S_3) = S_1 \), and \( \pi(S_2) = \pi(S_4) = S_2 \); Figure 2

![Figure 1: An Enriques surface](image1)

![Figure 2: A K3 surface](image2)

We blow up the \( K3 \) surface \( Y \) totally 30 times at the marked points \( \bullet \) and \( \bigcirc \). We then get a surface \( \mathcal{Z} = Y \times_{\mathbb{P}^2} 30 \mathbb{P}^2 \); Figure 3. There exist four disjoint linear
chains of \(\mathbb{CP}^1\)'s in \(\mathbb{Z}\):

\[
\begin{align*}
C_{19,6} & : -2 -2 -9 -2 -2 -2 -4 \\
C_{19,6} & : 0 -2 -2 -2 -2 -2 -4 \\
C_{73,50} & : 0 -2 -7 -2 -3 -2 -4 \\
C_{73,50} & : 0 -2 -7 -2 -3 -2 -4
\end{align*}
\]

Figure 3: A surface \(\mathbb{Z} = Y \# 30 \mathbb{CP}^2\)

We contract these four chains of \(\mathbb{CP}^1\)'s from the surface \(\mathbb{Z}\) so that it produces a normal projective surface \(\overline{X}\) with four singular points of class \(T\). It is not difficult to show that \(H^2(X, T_X) \neq 0\).

**Theorem 2.1.** The singular surface \(\overline{X}\) has a global \(\mathbb{Q}\)-Gorenstein smoothing. A general fiber \(\overline{X}_t\) of the smoothing of \(\overline{X}\) is a simply connected minimal complex surface of general type with \(p_g = 1\), \(q = 0\), and \(K^2 = 8\).

In order to prove Theorem 2.1, we apply the following proposition.

**Proposition 2.2** (Y. Lee and J. Park [2]). Let \(V\) be a normal projective surface with singularities of class \(T\). Assume that a cyclic group \(G\) acts on \(X\) such that

1. \(W = V/G\) is a normal projective surface with singularities of class \(T\),
2. \(p_g(W) = q(W) = 0\),
3. \(W\) has a \(\mathbb{Q}\)-Gorenstein smoothing,
4. the map \(\sigma : V \to W\) induced by a cyclic covering is flat, and the branch locus \(D\) (resp. the ramification locus) of the map \(\sigma : V \to W\) is a nonsingular curve lying outside the singular locus of \(W\) (resp. of \(V\)), and
5. \(H^1(W, O_W(D)) = 0\).
Then there exists a $\mathbb{Q}$-Gorenstein smoothing of $V$ that is compatible with a $\mathbb{Q}$-Gorenstein smoothing of $W$. Furthermore the cyclic covering extends to the $\mathbb{Q}$-Gorenstein smoothing.

We now construct an unramified double covering from the singular surface $X$ to another singular surface. We begin with the Enriques surface $Y$ in Figure 1. We blow up totally 15 times at the marked points • and ◯; Figure 4. We then get a surface $Z = Y^{\#15}\mathbb{CP}^2$; Figure 5. There exist two disjoint linear chains of $\mathbb{CP}^1$'s in $Z$:

\[
C_{19,6} : \begin{array}{cccccccccc}
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
\end{array}
\]

\[
C_{73,50} : \begin{array}{cccccccccc}
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
\end{array}
\]

Figure 4: An Enriques surface $Y$ with marked points

We contract the two chains of $\mathbb{CP}^1$'s from the surface $Z$ so that it produces a normal projective surface $X$ with two singular points class $T$. It is clear that there is an unbranched double covering $\pi : X \rightarrow X$. The singular surface $X$ satisfies the third condition of Proposition 2.2.

Proposition 2.3 (H. Park [4]). The singular surface $X$ has a global $\mathbb{Q}$-Gorenstein smoothing. A general fiber $X_t$ of the smoothing of $X$ is a minimal complex surface of general type with $p_g = 0$, $K^2 = 4$, and $\pi_1(X_t) = \mathbb{Z}/2\mathbb{Z}$.

Proof of Theorem 2.1. It is easy to show that the covering $\pi : X \rightarrow X$ satisfies all conditions of Proposition 2.2. Therefore the singular surface $X$ has a global $\mathbb{Q}$-Gorenstein smoothing. Let $X_t$ be a general fiber of the smoothing of $X$. Since $p_g(X) = 1$, $q(X) = 0$, and $K^2_X = 8$, by applying general results of complex surface theory and $\mathbb{Q}$-Gorenstein smoothing theory, one may conclude that a general fiber $X_t$ is a complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 8$. Furthermore, it is not difficult to show that a general fiber $X_t$ is minimal by using a similar technique in [3, 6, 7].
Claim. A general fiber $\overline{X_t}$ is simply connected: By Proposition 2.2, there is an induced unbranched double covering $\overline{X_t} \to X_t$; hence, a general fiber $\overline{X_t}$ is simply connected because $\pi(X_t) = \mathbb{Z}/2\mathbb{Z}$ by Proposition 2.3.

References

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