HARMONIC MAPS AND TWISTORIAL STRUCTURES

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This paper is dedicated to the memory of Ştefan Papadima.

Abstract. We introduce the notion of Riemannian twistorial structure and we show that it provides new natural constructions of harmonic maps.

Introduction

Perhaps that the main apparently embarrassing problem in differential geometry is the lack of an obvious notion of morphism. We argue that this is not an well posed problem, that is, the definition of a ‘linear $G$-structure’, for some Lie group $G$, as a faithful representation of $G$ is too raw: essentially, we have to consider only the representations preserving a holomorphic embedding of a compact complex manifold $Y$ into some complex Grassmannian of the representation’s vector space $U$ (cf. [20]). Note that, this is not too restrictive as, assuming the complexified representation irreducible, we can, for example, take $Y$ to be the closed $G^\mathbb{C}$-orbit in the projectivisation of the complexification of $U^*$, setting which proved to be crucial in the classification of irreducible torsion free affine holonomies [14], [15]. Consequently, the ‘$G$-structures’ should be given by suitable twistorial structures and the ‘morphisms’ be just the corresponding twistorial maps of [22], [11].

One of the main tasks of this paper is to construct the suitable definition of ‘Riemannian twistorial structure’. This is based on the ‘Euclidean (twistorial) $Y$-structures’, that is, embeddings of $Y$ into the generalized Grassmannian of coisotropic subspaces (that is, orthogonal complements of isotropic subspaces) of the corresponding Euclidean space. As is usual in twistor theory, examples abound (see Section 1, below) with the more or less obvious particular cases of Hermitian (where $Y$ is a two points space) and metric quaternionic ($Y$ the Riemann sphere [19]) geometries. Furthermore, as coisotropy means isotropic orthogonal complement, there is always an almost CR manifold underlying the construction of a Riemannian twistorial structure and this, by following an idea of [17] (see [23]), can be shown to provide natural constructions of harmonic maps (see Section 2).

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This brings us to the other main task of this paper, namely, to find new natural con-
structions of harmonic maps. On one hand, this is in continuation of our investigation
of the interplay of the properties of a map of being twistorial and a harmonic mor-
phism (\cite{22}, \cite{10}, \cite{11}) and, on the other, it provides the adequate level of generality
for previously known twistorial constructions of harmonic maps (\cite{5}, \cite{6}). We achieve
this just because, in our setting, any Riemannian symmetric space admits a nontrivial
Riemannian twistorial structure invariant under the isometry group (Theorem 3.11)
leading, for example, to the natural twistorial constructions of Example 3.13, based on
the new twistorial structure of Example 3.10(3).

But there is also a feature of twistorial structures which seems to have not been
observed before: the canonical projection from the ‘total space’ onto the twistor space,
endowed with natural Riemannian metrics, usually, pulls back holomorphic functions
to harmonic functions and, consequently (see \cite{9}, \cite{1}, \cite{4} for more information on such
maps), if the twistor space is Kähler (as it is the case of the main examples provided
by generalized flag manifolds) this projection is a harmonic map. This gives Theorem
3.4 and Corollary 3.5 with the abundant sources of examples provided by Examples 3.9
and 3.10.

1. Euclidean twistorial structures

The following definition specializes a notion of \cite{20} (cf. \cite{19}).

**Definition 1.1.** Let $U$ be an Euclidean space, $\dim U = n$, let $k \in \mathbb{N}$, $n/2 \leq k \leq n$,
and let $Y$ be a compact complex manifold endowed with a conjugation; that is, an
involutive antiholomorphic diffeomorphism.

An Euclidean (twistorial) $Y$-structure on $U$ is a a holomorphic embedding $\sigma$ of $Y$
into $\text{Gr}_k(U^\mathbb{C})$ such that:

1. $\sigma$ intertwines the conjugations;
2. $\sigma(y) \subseteq U^\mathbb{C}$ is coisotropic, for any $y \in Y$;

We say that $\sigma$ is full/maximal if, further, the following condition is satisfied:

3. The tautological exact sequence of holomorphic vector bundles over $Y$

\begin{equation}
0 \rightarrow \bigsqcup_{y \in Y} \sigma(y) \rightarrow Y \times U^\mathbb{C} \rightarrow U \rightarrow 0
\end{equation}

induces a linear map from $U^\mathbb{C}$ to $H^0(U)$ which is injective/bijective.

A Hermitian $Y$-structure on $U$ is an Euclidean $Y$-structure on $U$ with $k = n/2$.

If $\sigma_j$ are Euclidean $Y_j$-structures on $U_j$, $(j = 1, 2)$, a morphism from $(U_1, \sigma_1)$ to
$(U_2, \sigma_2)$ is a pair $(A, \alpha)$, where $A : U_1 \rightarrow U_2$ is linear, $\alpha : Y_1 \rightarrow Y_2$ is holomorphic and
intertwines the conjugations, and $A(\sigma_1(y)) \subseteq \sigma_2(\alpha(y))$, for any $y \in Y_1$.

If $\sigma$ is an Euclidean $Y$-structure on $U$, an automorphism of $(U, \sigma)$ is an isomorphism
$(A, \alpha) : (U, \sigma) \rightarrow (U, \sigma)$ such that $A \in \text{SO}(U)$.
Remark 1.2. If $U$ is endowed with an Euclidean $Y$-structure $\sigma$ then $Y$ is Kähler. Also, any automorphism $(A, \alpha)$ of $(U, \sigma)$ preserves the tautological exact sequence $\mathbf{1.1}$, and $\alpha$ is an isometry of $Y$.

Example 1.3. 1) In Definition $\mathbf{1.1}$, if $k = n$ (and $Y$ is formed of just one point) then we obtain the trivial Euclidean structure of $U$.

2) Any Euclidean $Y$-structure is the product of a trivial Euclidean structure and a full Euclidean $Y$-structure. Indeed, if $\sigma$ is an Euclidean $Y$-structure on $U$ then $\bigcap_{y \in Y} \sigma(y) = V^C$ for some vector subspace $V$ of $U$ and $\sigma(y) = V^C \oplus \tau(y)$, for any $y \in Y$, where $\tau$ is a full Euclidean $Y$-structure on $V^\perp(\subseteq U)$.

3) Let $n = \dim U$ be even and $Y$ be formed of just two points. Then any Hermitian $Y$-structure on $U$ is given by an orthogonal complex structure on $U$.

4) Let $U = \mathbb{H}^m$ and let $S^2$ be the sphere of unit imaginary quaternions. By associating to any $q \in S^2$ the $-i$ the eigenspace of the linear complex structure given by quaternionic multiplication to the left with $q$, we obtain a maximal Hermitian $S^2$-structure whose automorphism group is $\text{Sp}(1) \cdot \text{Sp}(m)$ embedded into $(\text{Sp}(1) \cdot \text{Sp}(m)) \times SO(3)$ as the graph of the obvious Lie groups morphism from $\text{Sp}(1) \cdot \text{Sp}(m)$ to $SO(3)$.

Moreover, any maximal Hermitian $S^2$-structure is obtained this way.

5) Let $U = \mathbb{R}^3 \times \mathbb{R}^m$ and let $S^2$ be embedded as the (isotropic) conic in the projectivisation of the complexification of the Euclidean space $\mathbb{R}^3$. By defining $\sigma(\ell) = \ell^\perp \otimes \mathbb{C}^m$, for any $\ell \in S^2(\subseteq \mathbb{C}P^2)$, we obtain a maximal Euclidean $S^2$-structure whose automorphism group is $SO(3) \times SO(m)$ (see $\mathbf{13}$ for more details on these structures).

More generally, if $U$ is endowed with a linear $Y$-structure and $V$ is endowed with a (trivial) Euclidean structure then $U \otimes V$ is endowed with an Euclidean $Y$-structure whose automorphism group is covered by the direct product of the automorphism group of $U$ and $SO(V)$.

6) Let $G$ be a complex semisimple Lie group and $P \subseteq G$ a parabolic subgroup such that $G$ acts effectively on $Y = G/P$. Let $G_\mathbb{R}$ be a compact real form of $G$ and denote $\mathfrak{g}, \mathfrak{g}_\mathbb{R}, \mathfrak{p}$ the Lie algebras of $G$, $G_\mathbb{R}$, $P$, respectively.

As $\mathfrak{p}$ is its own normalizer in $\mathfrak{g}$ and $P$ is connected, we have a holomorphic embedding $\sigma : Y \to \text{Gr}_k(\mathfrak{g})$ given by $\sigma(aP) = (\text{Ad}a)(\mathfrak{p})$, for any $a \in G$. Consequently, $\sigma$ is an Euclidean $Y$-structure on $\mathfrak{g}_\mathbb{R}$, where $\mathfrak{g}_\mathbb{R}$ is endowed with the opposite of its Killing form. Moreover, this is maximal as the corresponding tautological exact sequence $\mathbf{1.1}$ is the canonical exact sequence $\mathbf{3}$ associated to the holomorphic principal bundle $(G, Y, P)$, and by $\mathbf{26}$ Proposition II, we have $\mathfrak{g} = H^0(T^{1,0}Y)$.

Furthermore, the automorphism group of $\sigma$ is a semidirect product $H \rtimes G_\mathbb{R}$, where the complexification of the Lie algebra of $H$ is $H^0(\text{End}(T^{1,0}Y))$.

Definition 1.4. Let $\rho : E \to U$ be an orthogonal projection between Euclidean spaces. Suppose that $E$ is endowed with an Euclidean $Y$-structure $\sigma : Y \to \text{Gr}(E^C)$ such that $\sigma(y) \cap (\ker \rho)^C = \{0\}$, for any $y \in Y$. 

Then $Y \to \text{Gr}(U^C)$, $y \mapsto \rho^C(\sigma(y))$, $(y \in Y)$, is the Euclidean $Y$-structure on $U$ induced by $\rho$ and $\sigma$.

**Lemma 1.5.** Let $E$ be endowed with an Euclidean $Y$-structure $\sigma$ and let $\rho : E \to U$ be an orthogonal projection which induces a $Y$-structure on $U$.

Then, for any $y \in Y$, we have $\rho(\sigma(y)) = (\sigma(y))^\perp \cap U^C$. Consequently, for any $y \in Y$, the orthogonal projection of $\rho(\sigma(y))^\perp$ onto $(\rho(\sigma(y)) \cap \rho(\sigma(y))^\perp)^\perp$ is equal to $\rho(\sigma(y))^\perp$.

**Proof.** Let $y \in Y$ and denote $C = \sigma(y)$. As $U$ is the orthogonal complement in $E$ of $\ker \rho$, the orthogonal complement in $U^C$ of $\rho(C)$ is equal to the orthogonal complement in $E^C$ of $\rho(C) + (\ker \rho)^C$. Thus, we have to prove that $(\rho(C) + (\ker \rho)^C)^\perp = C^\perp \cap U^C$; equivalently, $\rho(C) + (\ker \rho)^C = C + (\ker \rho)^C$ which holds because both sides are equal to $\rho^{-1}(\rho(C))$.

As $\rho(C)^\perp = C^\perp \cap U^C$, we have $\rho(C)^\perp \subseteq \rho(C^\perp)$ and the last statement follows. \hfill $\square$

**Remark 1.6.** Any maximal (quaternionic-like) Euclidean $S^2$-structure (in particular, the one given by Example [13,5]) is induced by a Hermitian $S^2$-structure (as given by Example [13,4]); moreover, maximality is a necessary condition for this to hold (see [19]).

But, not all maximal Euclidean $Y$-structures are induced by Hermitian $Y$-structures. For an example, let $Y$ be the hyperquadric of isotropic directions in the complexification of an Euclidean space $U$. If $\dim U \geq 4$ then $\sigma : Y \to P(U^C)$, $\sigma(y) = y^+$, $(y \in Y)$, is a maximal Euclidean $Y$-structure which cannot be induced by a Hermitian one.

Indeed, let $0 \to E \to Y \times U^C \to U \to 0$ be the corresponding tautological exact sequence of holomorphic vector bundles. From this exact sequence we deduce that $H^0(E) = H^1(E) = \{0\}$, and from its dual, because $H^0(U^*) = H^1(U^*) = \{0\}$ (by the Kodaira vanishing theorem and Serre duality), we deduce $U^C = H^0(E^*)$.

Now, if $\sigma$ would be induced by a Hermitian $Y$-structure on some Euclidean space $E$ then its tautological exact sequence would be $0 \to E \to Y \times E^C \to E^* \to 0$ which would imply $E^C = H^0(E^*) = U^C$, a contradiction.

More generally, by taking $Y$ to be any generalized flag manifold, $\dim_C Y \geq 2$, endowed with suitable very ample line bundles, we obtain maximal Euclidean $Y$-structures not induced by Hermitian $Y$-structures.

Similarly, it can be proved that the Euclidean $Y$-structure of Example [13,6] is induced by a Hermitian $Y$-structure if and only if $Y$ is a product of Riemann spheres.

Finally, let $E$ be endowed with an Euclidean $Y$-structure $\sigma$. Then an orthogonal projection $\rho : E \to U$ induces a maximal Hermitian $Y$-structure on $U$ if and only if it is an isometry.

### 2. Harmonic maps and CR twistor spaces

Let $M$ be a (smooth, oriented) manifold, and let $(Y,C)$ be an almost CR manifold; that is, $C \subseteq T^C Y$ such that $C \cap \overline{C} = 0$. Then $(Y,C)$ is an *almost CR twistor space* (cf.
Moreover, the metric on \( N \) and let \( \phi \). We may suppose that the image of \( \pi : Y \to M \) such that:

1. For any fibre \( Y_x = \pi^{-1}(x) \), \( x \in M \), the intersection \( \mathcal{C} \cap T^C Y_x \) is the anti-holomorphic tangent bundle of a complex structure on \( Y_x \);
2. For any \( x \in M \), the map \( Y_x \to \text{Gr}(T^C_x M) \), \( y \mapsto d\pi(C_y) \), \( y \in Y_x \), is a holomorphic embedding.

Suppose, now, that \( M \) is endowed with a Riemannian metric. Then the almost CR twistor space \( (Y, \mathcal{C}) \) is Riemannian if \( Y \) is endowed with a Riemannian metric such that the following conditions are, also, satisfied:

3. \( \pi \) is a Riemannian submersion with totally geodesic fibres;
4. \( \mathcal{C} \) is isotropic;
5. For any \( x \in M \), the Riemannian metric induced from \( Y \) on \( Y_x \) is the same with the Kähler metric induced from \( \text{Gr}(T^C_x M) \);
6. Under the obvious embedding \( Y \subseteq \text{Gr}(T^C M) \), the horizontal distribution \( \mathcal{H} = (\ker d\pi)^\perp \) on \( Y \) is induced by the Levi-Civita connection of \( M \).

**Remark 2.1.** Let \( x_0 \in M \) and let \( G \) be the automorphism group of the Euclidean \( Y_{x_0} \)-structure given by \( Y_{x_0} \to \text{Gr}(T^C_{x_0} M) \), \( y \mapsto d\pi(C_y)^\perp \), \( y \in Y_{x_0} \).

Then \( G \) contains the holonomy group of \( M \) at \( x_0 \) and, consequently, we may retrieve \( (Y, \mathcal{C}, \pi) \) from the \( Y_{x_0} \)-structure on \( T_{x_0} M \) and the holonomy bundle of the orthonormal frame bundle of \( M \).

Let \( (Y, \mathcal{C}) \) be a Riemannian almost CR twistor space given by \( \pi : Y \to M \) and let \( \mathcal{H} = (\ker d\pi)^\perp \). Then \( \tilde{\mathcal{C}} = (\mathcal{C} \cap \mathcal{H} \cap \mathcal{C}) \oplus (\ker d\pi)^{1,0} \) is an isotropic almost CR structure on \( Y \). Essentially, the following result was obtained in [23, §5] (cf. [7]).

**Theorem 2.2.** Let \( M \) be a Riemannian manifold endowed with a Riemannian almost CR twistor space \( (Y, \mathcal{C}) \) given by \( \pi : Y \to M \). Let \( (N, J) \) be an almost complex manifold and let \( \varphi : N \to Y \) be an immersion such that:

1. \( d\varphi(TN) \cap (\ker d\pi) = 0 \);
2. \( \varphi : (N, J) \to (Y, \tilde{\mathcal{C}}) \) is a CR map (that is, \( d\varphi(T^{0,1} N) \subseteq \tilde{\mathcal{C}} \)).

Then \( \pi \circ \varphi : (N, J) \to M \) is a \((1,1)\)-geodesic immersion and, in particular, minimal. Moreover, the metric on \( N \) pulled back by \( \pi \circ \varphi \) from \( M \) is \((1,2)\)-symplectic, with respect to \( J \).

**Proof.** We may suppose that the image of \( \varphi \) is the image of a section of \( Y \) over \( N (\subseteq M) \).

It is, also, convenient (see [23]) to associate to \( \mathcal{C} \) the almost \( f \)-structure \( \mathcal{F} \) on \( Y \) (that is, \( \mathcal{F} \) is a section of \( \text{End}(TY) \) such that \( \mathcal{F}^2 + \mathcal{F} = 0 \)) which is skew-adjoint and \( \ker(\mathcal{F} + i) = \mathcal{C} \).

Accordingly, \( Y \) is a bundle of skew-adjoint linear \( f \)-structures on \( M \) and therefore the section of \( Y \) giving \( \varphi \) corresponds to a section \( F \) of \( \text{End}(TM|_N) \). Furthermore, standard arguments show that condition (ii) is equivalent to the following:

1. \( (i) \) \( TN \subseteq \ker(F^2 + 1) \) and \( F|_N = J \);
2. \( (ii) \) \( \nabla_{Fx} F = (\nabla_X F) F \), for any \( X \in TN \), where \( \nabla \) is the Levi-Civita connection.
of $M$.

Consequently, for any sections $X$ and $Y$ of $T^{1,0}M$, $\nabla_X Y$ is a section of $\ker(F - i)$; equivalently, $(N, J)$ endowed with the metric induces from $M$ is $(1,2)$-symplectic, and $b(\overline{X}, Y) \in \ker(F - i)$, where $b$ is the second fundamental form of $N \subseteq M$. Together with the fact that $b$ is symmetric we deduce that $b(\overline{X}, Y) = 0$, for any $X, Y \in T^{1,0}N$. \hfill $\square$

**Remark 2.3.** In (ii) of Theorem 2.2, if we replace $\tilde{C}$ by $C$ then similarly we obtain (essentially, [23, §5]) that $(N, J)$ is integrable.

Essentially, the following result was obtained in [23, §5].

**Corollary 2.4.** Let $M$ be a Riemannian manifold endowed with a Riemannian almost CR twistor space $(Y, C)$ given by $\pi : Y \to M$. Let $(N, J)$ be an almost complex manifold and let $\varphi : N \to Y$ be an immersion such that:

(i) $\varphi(N)$ is orthogonal to the fibres of $\pi$;

(ii) $\varphi : (N, J) \to (Y, C)$ is a CR map.

Then the immersion $\pi \circ \varphi : (N, J) \to M$ is $(1,1)$-geodesic and, in particular, minimal. Moreover, the metric on $N$ pulled back by $\pi \circ \varphi$ from $M$ is Kähler, with respect to $J$, and, thus, $\varphi$ is pluriharmonic.

**Proof.** This is an immediate consequence of Theorem 2.2 and Remark 2.3. \hfill $\square$

In certain cases, the converse of Theorem 2.2 (and, similarly, of Remark 2.3) also holds. For example, let $M$ be a Riemannian manifold, let $k \in \mathbb{N}$, $2k \leq \dim M$, and let $Y$ be the bundle of isotropic subspaces of dimension $k$ on $T^C M$. Then [17] (see Remark 2.4; cf. [8], [24]) $Y$ is endowed with a canonical almost CR structure $\mathcal{C}$ which if $k = 1$ is integrable [24] (cf. [8]), and if $k \geq 2$ then $\mathcal{C}$ is integrable if and only if $M$ is conformally flat [17]. Obviously, $(Y, \mathcal{C})$ is a Riemannian almost CR twistor space.

Essentially, the following result was obtained in [23, §5] (see, also, [25], for the case $k = 1$).

**Corollary 2.5.** Let $M$ be a Riemannian manifold, let $k \in \mathbb{N}$, $2 \leq 2k \leq \dim M$, and let $Y$ be the bundle of isotropic subspaces of dimension $k$ on $T^C M$ endowed with its canonical almost CR structure $\mathcal{C}$.

Let $(N, J)$ be a $(1,2)$-symplectic almost Hermitian manifold, $\dim N = 2k$, and let $\varphi : N \to M$ be an isometric immersion. Then the following assertions are equivalent:

(i) $\varphi$ is a $(1,1)$-geodesic map;

(ii) $(N, J) \to (Y, \mathcal{C})$, $x \mapsto d\varphi(T^{0,1}_x N)$, $(x \in N)$, is a CR map.

**Proof.** This is an immediate consequence of Theorem 2.2. \hfill $\square$

**Example 2.6.** Let $k, n \in \mathbb{N}$, $2 \leq 2k \leq n$, let $(N, J)$ be a $(1,2)$-symplectic almost Hermitian manifold, $\dim N = 2k$, and let $\varphi : N \to \mathbb{R}^n$ be an isometric immersion. Denote by $\text{Gr}^0_k(n, \mathbb{C})$ the space of isotropic complex vector subspaces of dimension $k$ of $\mathbb{C}^n$. 
Then \( \varphi \) is \((1,1)\)-geodesic (and, in particular, harmonic) if an only if it induces an antiholomorphic map from \((N,J)\) to \(\text{Gr}_k^0(n,\mathbb{C})\) (the case \(k = 1\) is classical, see [25] and the references therein). This quickly gives the Weierstrass representation [2] of pluriharmonic immersions from Kähler manifolds to Euclidean spaces.

Some of the results of this section admit more general formulations which, however, require the following definitions.

**Definition 2.7.** Let \((M,g)\) and \((N,h)\) be Riemannian manifolds, \((N,h)\) equipped with a skew-adjoint almost \(f\)-structure \(F\). We say that a map \(\varphi : (N,h,F) \to (M,g)\) is

i) pseudo-harmonic if

\[
\text{trace}\left( \nabla d\varphi \bigg|_{H \times H} \right) = 0,
\]

where \(H = \ker(F^2 + \text{Id})\).

ii) pseudo-\((1,1)\)-geodesic if

\[
\nabla d\varphi(Z,Z) = 0,
\]

for all \(Z \in \ker(F-i)\).

**Definition 2.8.** Let \((N,h)\) be a Riemannian manifold, equipped with a skew-adjoint almost \(f\)-structure \(F\). We say that \((N,h,F)\) is CR-cosymplectic if the restriction of \(\nabla^N J\) to \(\ker(F^2 + i) \otimes \ker(F - i)\) is trace-free (with respect to the metric \(h\)), i.e.

\[
\sum_i \nabla Z_i \in \ker(F - i),
\]

where \(\{Z_i\}\) is an orthonormal basis of \(\ker(F - i)\).

ii) \((N,h,F)\) is CR-\((1,2)\)-symplectic if \(\nabla^N J\) maps \(\ker(F^2 + i) \otimes \ker(F - i)\) into \(\ker(F - i)\).

**Corollary 2.9.** Let \(M\) be a Riemannian manifold endowed with a Riemannian almost CR twistor space \((Y,\mathcal{C})\) given by \(\pi : Y \to M\).

Let \(N\) be a Riemannian manifold endowed with a skew-adjoint almost \(f\)-structure \(J\) such that \((N,h,J)\) is a CR-cosymplectic Riemannian manifold.

If \(\varphi : (N,\ker(J+i)) \to (Y,\mathcal{C})\) is a CR map then \(\pi \circ \varphi\) is pseudo-harmonic.

**Proof.** Similarly to the proof of Theorem 2.2, the map \(\varphi\) corresponds to a section \(F\) of \((\pi \circ \varphi)^*(Y) \subseteq \text{End}((\pi \circ \varphi)^*(TM))\). Also, as \(\varphi : (N,\ker(J+i)) \to (Y,\mathcal{C})\) is a CR map, we have \(((\pi \circ \varphi)^*(\nabla^M)) F = (((\pi \circ \varphi)^*(\nabla^M)) X F)\), for any \(X \in \ker(J^2 + 1)\), where \(\nabla^M\) is the Levi-Civita connection of \(M\).

Thus, \(((\pi \circ \varphi)^*(\nabla^M)) (d(\pi \circ \varphi)(Y))\) is a section of \(\ker(F - i)\) for any sections \(X\) and \(Y\) of \(\ker(J - i)\), where \(d(\pi \circ \varphi)\) is seen as a section of \(\text{Hom}(TN,(\pi \circ \varphi)^*(TM))\).

Consequently, the trace of \(\nabla d(\pi \circ \varphi)\) restricted to \(\ker(J^2 + 1)\) takes values in \(\ker(F - i)\) and the proof quickly follows. \(\square\)
Corollary 2.9 gives [23, Theorem 5.6] if $J$ is an almost Hermitian structure.

**Corollary 2.10.** Let $(M, g)$ be a Riemannian manifold, $k \in \mathbb{N}$, $2 \leq 2k \leq \dim M$, and $Y$ the bundle of isotropic subspaces of dimension $k$ on $T^C M$ endowed with its canonical CR structure $\tilde{\mathcal{C}}$. Let $(N, h)$ be a Riemannian manifold endowed with a skew-adjoint almost $f$-structure $F$ of rank $2k$ and such that $(N, h, F)$ is CR-(1,2)-symplectic. Let $\varphi : N \to M$ be an immersion such that $(\varphi^* h)^{(2,0)} = 0$. Then the following assertions are equivalent:

i) $\varphi$ is pseudo-(1,1)-geodesic.

ii) $\tilde{\varphi} : (N, F) \to (Y, \tilde{\mathcal{C}})$, $x \mapsto d\varphi \left( \ker (F + i) \right)$ is a CR map.

**Corollary 2.11.** Let $(M, g)$ be an oriented Riemannian $(2n + 2)$-manifold and $Y$ the bundle of maximal isotropic subspaces on $T^C M$ endowed with its canonical CR structure $\tilde{\mathcal{C}}$. Let $(N, h, F)$ be a Riemannian manifold endowed with a skew-adjoint almost $f$-structure $F$ of rank $2n$ such that $(N, h, F)$ is CR-(1,2)-symplectic. Let $\varphi : N \to M$ be an immersion such that $(\varphi^* h)^{(2,0)} = 0$. Then the following assertions are equivalent:

i) $\varphi$ is pseudo-(1,1)-geodesic

ii) $\tilde{\varphi} : (N, F) \to (Y, \tilde{\mathcal{C}})$ is a CR map.

**Example 2.12.** Any orientable 3-dimensional manifold admits a foliation by Riemann surfaces, hence a Levi flat CR structure with $(\nabla F)_H = 0$, where $H = \ker (F^2 + \text{Id})$. Therefore, any 3-dimensional manifold can be equipped with a CR-(1,2)-symplectic $f$-structure.

In particular, by the previous corollary, Eells-Salamon’s bijective correspondence between $\varphi$ pseudo-harmonic and $\tilde{\varphi}$ CR map, extends to isometric immersions $\varphi : (N^3, h) \to (M^4, g)$.

**Example 2.13.** An almost contact manifold $(N, h, F)$ is called nearly cosymplectic if $\nabla F$ is skew-symmetric. This condition implies that the associated $f$-structure is CR-(1,2)-symplectic. If $N$ is a hypersurface of a nearly Kahler manifold $M$, then it inherits an almost contact structure which, under some conditions, is nearly cosymplectic. The inclusion map $\varphi$ will then satisfy the conditions of the last corollary.

### 3. Riemannian Twistorial Structures

Let $E$ be a Riemannian vector bundle over a manifold $M$. A subbundle $Y \subseteq \text{Gr}(E^C)$ such that $Y_x \subseteq \text{Gr}(E^C_x)$ is an Euclidean $Y_x$-structure on $E_x$, for any $x \in M$, is called a Riemannian $Y$-structure on $E$. Suppose, further, that $M$ is Riemannian and there is a morphism of vector bundles $\rho : E \to TM$ which is an orthogonal projection at each point. If $\rho_x$ and $\sigma_x$ induce a $Y_x$-structure on $T_x M$, for any $x \in M$, where $\sigma : Y \to \text{Gr}(E^C)$ is the inclusion, then we say that $\rho$ and $\sigma$ induce a Riemannian $Y$-structure on $M$. 
Let $E$ be, further, endowed with a compatible connection $\nabla$; that is, $\nabla$ preserves the Riemannian structure of $E$ and induces a connection $\mathcal{H} \subseteq TY$ on $Y$. Then we can construct an almost co-CR structure $\mathcal{C}$ on $Y$ (that is, $\mathcal{C} \subseteq T^\mathcal{C}Y$ is a complex distribution such that $\mathcal{C} + \mathcal{C}^\perp = T^\mathcal{C}Y$), as follows. Firstly, let $\mathcal{B} \subseteq \mathcal{H}^\mathcal{C}$ be such that $d\pi(B_y) = \rho(\sigma(y))$, for any $y \in Y$, where $\pi : Y \rightarrow M$ is the projection. Then $\mathcal{C} = \mathcal{B} \oplus (\ker d\pi)^{0,1}$ is an almost co-CR structure on $Y$.

Let $T = d\nabla_\iota$ be the $\iota$-torsion of $\nabla$, where $\iota$ is the inclusion of $TM$ into $E$ as the orthogonal complement of $\ker\rho$. If $(X,Y,Z) \mapsto \langle T(X,Y),Z \rangle$ is a section of $\Lambda^3T^*M$, where $\langle \cdot,\cdot \rangle$ is the metric on $E$, then $T$ is called totally antisymmetric.

The following definition specializes notions of [22], [11]; compare, also, [23], [10], [13].

**Definition 3.1.** If the $\iota$-torsion of $\nabla$ is totally antisymmetric we say that $(E,\rho,\sigma,\nabla)$ is a Riemannian almost twistorial structure (on $M$). If, also, $\mathcal{C}$ is integrable we say that $(E,\rho,\sigma,\nabla)$ is a Riemannian twistorial structure.

Note that, similarly to [18], if the automorphism group of the typical fibre of $Y$ acts tranzitively on it, the integrability of $\mathcal{C}$ is determined by the curvature form and the $\rho$-torsion of $\nabla$.

Let $M$ be endowed with a Riemannian twistorial structure $(E,\rho,\sigma,\nabla)$. Suppose that there exists a complex manifold $Z$ endowed with a conjugation and a surjective submersion $\varphi : Y \rightarrow Z$ intertwining the conjugations and such that $\mathcal{C} = (d\varphi)^{-1}(T^{0,1}Z)$ (we take $\varphi$ such that $(\ker d\varphi)^\mathcal{C} = \mathcal{C} \cap \mathcal{C}^\perp$). We call $Z$ the twistor space of $(E,\rho,\sigma,\nabla)$.

**Remark 3.2.** Let $M$ be an oriented Riemannian manifold, $\dim M = n$. Let $Y^0$ be a compact complex manifold endowed with a conjugation and let $G \subseteq \text{SO}(n)$ be the automorphism group of an Euclidean $Y^0$-structure on $\mathbb{R}^n$ (endowed with its canonical metric). If $G$ contains the holonomy group of $M$ then we can build $\pi : Y \rightarrow M$ and an embedding $\sigma : Y \rightarrow \text{Gr}(T^\mathcal{C}M)$ such that $(TM,\text{Id}_{TM},\sigma,\nabla)$ is a Riemannian almost twistorial structure, where $\nabla$ is the Levi-Civita connection of $M$.

Moreover, assuming integrability, the twistor space can be defined at least by restricting to any convex open subset of $M$.

**Remark 3.3.** a) Let $(E,\rho,\sigma,\nabla)$ be a Riemannian almost twistorial structure on $M$. Endow $Y$ with the Riemannian metric with respect to which $\mathcal{H} = (\ker d\pi)^\perp$, $\pi$ is a Riemannian submersion and on its fibres the metrics are the same with the ones induced from $\text{Gr}(E^\mathcal{C})$. Then $\mathcal{C}$ is coisotropic.

b) Conversely, let $M$ be a Riemannian manifold and let $Z$ be a complex manifold endowed with a conjugation. Let $\pi$ and $\varphi$ be submersions from a Riemannian manifold $Y$ onto $M$ and $Z$, respectively, such that:

1. $\pi$ is a proper Riemannian submersion with totally geodesic fibres;
2. $\mathcal{C} = (d\varphi)^{-1}(T^{0,1}Z)$ is a coisotropic distribution on $Y$;
3. For any $x \in M$, the map $Y_x \rightarrow \text{Gr}(T_x^\mathcal{C}M)$, $y \mapsto d\pi(C_y)$, $(y \in Y_x)$, is an
Euclidean $Y_z$-structure on $T_xM$, where $Y_x = \pi^{-1}(x)$:

(4) For any $x \in M$, the inclusion map $Y_x \to Y$ is an isometric embedding, where $Y_x$ is endowed with the Kähler metric induced from $\text{Gr}(T^C_xM)$;

(5) For any $x \in M$, the restriction of $\varphi$ to $Y_x$ is a holomorphic embedding intertwining the conjugations.

Suppose that $Y$ is induced through a vector bundles morphism $\rho : E \to TM$ from a Riemannian vector bundle $E$ endowed with a compatible connection $\nabla$ with totally antisymmetric torsion, and $Y \subseteq \text{Gr}(E^C)$ and $\mathcal{H} = (\ker d\pi)^\perp$ associated to $\nabla$. Then $(E, \rho, \sigma, \nabla)$ is a Riemannian twistorial structure, with twistor space $Z$, where $\sigma : Y \to \text{Gr}(E^C)$ is the inclusion. For brevity, we say that $(E, \rho, \sigma, \nabla)$ is given by $\varphi$ and $\pi$.

Let $M$ be endowed with a Riemannian twistorial structure given by $\varphi : Y \to Z$ and $\pi : Y \to M$. Let $\mathcal{H} = (\ker d\pi)^\perp$ and $\mathcal{K} = (\ker d\varphi)^\perp$ be the horizontal distributions on $Y$ determined by $\pi$ and $\varphi$, respectively. Let $J$ be the orthogonal complex structure on $\mathcal{H}$ with respect to which $d\varphi|_\mathcal{H}$ is complex linear at each point. Note that, $J$ restricts to give the complex structures of the fibres of $\pi$.

Because $\mathcal{C}$ is coisotropic and $(\ker d\varphi)^C = \mathcal{C} \cap \overline{\mathcal{C}}$ we have $\mathcal{H} = \mathcal{C}^\perp \oplus \overline{\mathcal{C}^\perp}$. Together with $\mathcal{C}^\perp \supseteq (\ker d\pi)^{0,1}$, this implies $\ker d\pi \subseteq \mathcal{H}$; equivalently, $\ker d\varphi \subseteq \mathcal{H}$. We shall denote by $P_{\mathcal{H} \cap \mathcal{K}}$ the orthogonal projection onto $\mathcal{H} \cap \mathcal{K}$.

**Theorem 3.4.** Let $M$ be endowed with a Riemannian twistorial structure given by $\varphi : Y \to Z$ and $\pi : Y \to M$. If the fibres of $\varphi$ are minimal then the following assertions are equivalent:

(i) $\varphi$ pulls back holomorphic functions on $Z$ to harmonic functions on $Y$;

(ii) $\text{trace}_\omega (\mathcal{H}^\perp + P_{\mathcal{H} \cap \mathcal{K}} T) = 0$, where $\mathcal{H}^\perp$ is the integrability tensor of $\mathcal{H}$, $\omega$ is the Kähler form of $(\mathcal{H} \cap \mathcal{K}, J|_{\mathcal{H} \cap \mathcal{K}})$, and $T$ is the section of $\mathcal{H} \otimes \Lambda^2 \mathcal{K}^*$ corresponding to the torsion of the induced connection on $M$.

Consequently, if $Z$ is Kähler, the fibres of $\varphi$ are minimal and (ii) holds then $\varphi$ is a harmonic map.

**Proof.** Let $(E, \rho, \sigma, \nabla)$ be the given Riemannian twistorial structure. Note that, $\rho \nabla$ is related to the Levi-Civita connection $\nabla^M$ of $M$ by $\rho \nabla = \nabla^M + \frac{1}{2} T$, where $T$, also, denotes the torsion of $\rho \nabla$.

Let $(y_t)$ be a curve on $Y$ horizontal with respect to $\pi$; that is, $\dot{y}_t \in \mathcal{H}_{y_t}$, for any $t$. Because $\mathcal{H}$ is induced by $\nabla$, for any section $s$ of the pull back by $(\pi(y_t))_t$ of $E$ such that $s_t \in \sigma(y_t)$, for any $t$, we have $\nabla_{d\pi(y_t)} s \in \sigma(y_t)$, for any $t$. Thus, if $s_t \in \sigma(y_t) \cap T^C_{\pi(y_t)} M$, for any $t$, then $(\rho \nabla)_{d\pi(y_t)} s \in \rho(\sigma(y_t))$, for any $t$. Together with Lemma 1.5, we deduce that if $s_t \in \rho(\sigma(y_t))^\perp$, for any $t$, then $(\rho \nabla)_{d\pi(y_t)} s \in \rho(\sigma(y_t)^\perp)$, for any $t$.

As the fibres of $\varphi$ are minimal, with notations similar to [21, Appendix A], assertion (i) is equivalent to $\text{div}_{\mathcal{K}} J = 0$.

Note that, at each $y \in Y$, the $-i$ eigenspace of $J$ restricted to $\mathcal{H}_y \cap \mathcal{K}_y$ is the
horizontal lift, with respect to $\pi$, of $\rho(\sigma(y))_\perp$. Consequently, for any sections $X_1$ and $X_2$ of $\mathcal{H}$ such that $X_2$ is a section of $\ker(J + i)$, the orthogonal projection onto $\mathcal{H} \cap \mathcal{K}$ of $\pi^*(\rho\nabla)_{X_1}X_2$ is a section of $\ker(J + i)$.

Together with the fact that $\pi$ is a Riemannian submersion and its fibres are Kähler manifolds, a straightforward calculation gives $\text{div}_\pi J = -\frac{1}{2} \text{trace}_\omega(I_{\mathcal{H}} + P_{\mathcal{H} \cap \mathcal{K}}T)$.

The last statement follows from [9] (or [21, Appendix A]). □

**Corollary 3.5.** Let $M$ be endowed with a Riemannian twistorial structure given by $\varphi : Y \to Z$ and $\pi : Y \to M$, and such that $E = TM$. Then the following assertions are equivalent:

(i) $\varphi$ pulls back holomorphic functions on $Z$ to harmonic functions on $Y$;

(ii) $\text{trace}_\omega(I_{\mathcal{H}} + P_{\mathcal{H} \cap \mathcal{K}}T) = 0$, where $I_{\mathcal{H}}$ is the integrability tensor of $\mathcal{H}$, $\omega$ is the Kähler form of $(\mathcal{H} \cap \mathcal{K}, J|_{\mathcal{H} \cap \mathcal{K}})$ and $T$ is the section of $\mathcal{H} \otimes \Lambda^2 \mathcal{H}^*$ corresponding to the torsion of the connection.

Consequently, if $Z$ is Kähler and (ii) holds then $\varphi$ is a harmonic map.

**Proof.** As $\pi$ is a Riemannian submersion mapping the fibres of $\varphi$ to totally geodesic immersions to $M$, the fibres of $\varphi$ are totally geodesic, and the proof follows from Theorem 3.4. □

**Remark 3.6.** Conditions (ii) of Theorem 3.4 and Corollary 3.5 are automatically satisfied if $I_{\mathcal{H}}(X, X) = T(X, X) = 0$, for any $X \in \mathcal{C}_\perp \cap \mathcal{H}^\perp$.

**Proposition 3.7.** Let $M$ be endowed with a Riemannian twistorial structure given by $\varphi : Y \to Z$ and $\pi : Y \to M$, and such that $E = TM$ and $\nabla$ is the Levi-Civita connection of $M$.

Then, for any complex submanifold $N$ of $Z$ whose preimage by $\varphi$ is horizontal with respect to $\pi$, the immersion $\pi|_{\varphi^{-1}(N)} : \varphi^{-1}(N) \to M$ is minimal.

**Proof.** This similar to the proof of Theorem 2.2 and will be omitted. □

**Remark 3.8.** With the same notations as in Proposition 3.7, similarly to Corollary 2.4, we obtain that $\varphi|_{\varphi^{-1}(N)}$ (has geodesic fibres and) is $\text{PHH}$, in the sense of [1] (see [4]).

**Example 3.9.** Let $U$ be endowed with an Euclidean $Y$-structure $\sigma$. Then on endowing $U$ with the trivial flat connection we obtain a Riemannian twistorial structure whose twistor space is the holomorphic vector bundle $\mathcal{U}$ of the tautological exact sequence (1.1). Note that, $\pi$ is the canonical projection from $Y \times U$ to $U$ and $\varphi$ is the restriction to $Y \times U$ of the canonical projection from $Y \times U^\perp$ to $\mathcal{U}$.

Furthermore, if $\sigma$ is a Hermitian $Y$-structure then the radial projection from $U \setminus \{0\}$ onto the unit sphere $S \subseteq U$ induces a Riemannian twistorial structure on $S$, with $\nabla$ the restriction to $S$ of the trivial connection on $U$. The corresponding twistor space is the complex projectivisation $\mathcal{PU}$ and, note that, the radial projection $U \setminus \{0\} \to S$ is the twistorial map (22), corresponding to the holomorphic submersion $U \setminus \{0\} \to \mathcal{PU}$. 

Example 3.10. Let $G$ be a semisimple complex Lie group and let $P \subseteq G$ be a parabolic subgroup. Let $H \subseteq G$ be a closed complex Lie subgroup which is semisimple, contains no normal subgroup of $G$, and $G/H$ is reductive with respect to $m \subseteq g$, where $g$ is the Lie algebra of $G$.

Suppose that $P$ and $(H, m)$ satisfy the following compatibility conditions:

(i) $H \cap P$ is a parabolic subgroup of $H$;

(ii) $m$ is nondegenerate with respect to the Killing form of $g$;

(iii) $p = h \cap p + m \cap p$, where $p$ and $h$ are the Lie algebras of $P$ and $H$, respectively;

(iv) $p^\perp = h \cap p^\perp + m \cap p^\perp$ and $m \cap p^\perp$ is the orthogonal complement of $m \cap p$ in $m$ (in particular, $m \cap p$ is coisotropic in $m$);

(v) $H \cap P = \{ a \in H \mid (\text{Ad} a)(m \cap p) = m \cap p \}$ (note that, ‘$\subseteq$’ is automatically satisfied).

We may choose a compact real form $G^\mathbb{R}$ of $G$ such that $H^\mathbb{R} = G^\mathbb{R} \cap H$ is a compact real form of $H$, and we assume that $m^\mathbb{R} = m \cap g^\mathbb{R}$ is complementary to $h^\mathbb{R}$ in $g^\mathbb{R}$, where $g^\mathbb{R}$ and $h^\mathbb{R}$ are the Lie algebras of $G^\mathbb{R}$ and $H^\mathbb{R}$, respectively.

Endow $M = G^\mathbb{R}/H^\mathbb{R}$ with the canonical connection $\nabla$ determined by $m^\mathbb{R}$ and with the $G^\mathbb{R}$-invariant Riemannian metric given by the restriction to $m^\mathbb{R}$ of the opposite of the Killing form of $g^\mathbb{R}$. Then the torsion of $\nabla$ is totally antisymmetric. Moreover, $(M, \nabla)$ satisfy (ii) of Corollary 3.5 (this follows from the fact that $p^\perp$ is nilpotent), with $Y = G^\mathbb{R}/(H^\mathbb{R} \cap P)$, $Z = G/P$, and $\pi : Y \to M$ and $\varphi : Y \to Z$ the canonical projections (but, note that, the Riemannian metric on $Y$ is not necessarily the quotient metric). Furthermore, the Riemannian twistorial structure on $M$ is of Hermitian type if and only if $h$ contains the reductive part of $p$; that is, $h \supseteq p \cap \overline{p}$ (equivalently, $m \cap p = m \cap p^\perp$).

A significant particular case is when $m = h^\perp$ in which case (ii), (iii) and (iv), above, are automatically satisfied. This contains the case when $\nabla$ is torsion free; equivalently, up to Riemannian covering spaces, $M$ is a Riemannian symmetric space.

Thus, the examples of twistor spaces of [5] and [6] fit into our setting. Also, all of the examples of [13] (see in [19] the cases with splitting normal bundle exact sequence for the twistor spheres) whose twistor spaces are generalized flag manifolds, also, provide concrete examples of Riemannian twistorial structures.

Moreover, similarly to [5] and [6], if $m = h^\perp$ and $\nabla$ is torsion free, we may construct a $G$-invariant holomorphic distribution $\mathcal{F}$ on $Z$ whose preimage by $d\varphi$ is horizontal with respect to $\pi$ and, thus, by Corollary 2.4 and Proposition 3.7, provides a construction of minimal submanifolds of $M$. Note that, geometrically, $\mathcal{F}$ can be constructed as follows. Firstly, embed $Z$ into a complex projective space through the ‘generalized Plücker embedding’. Then $\mathcal{F}$ is generated by the tangent spaces to the complex projective lines contained in $Z$ whose preimages by $\varphi$ are horizontal with respect to $\pi$ (note that, unless $G = H$, the distribution $\mathcal{F}$ is nonzero).

Among the concrete examples (satisfying (ii) of Corollary 3.5) not previously considered are the following:
1a) Take $M = G^\mathbb{R}$ with its Riemannian symmetric space structure, $Z = G/P \times G/P$ and $Y = G^\mathbb{R} \times G/P$.

1b) Also, with $\nabla$ the (−)-connection we obtain another example of Riemannian twistorial structure on $G^\mathbb{R}$ with the same twistor space but with a different Riemannian structure on $Y$.

Note that, if $G^\mathbb{R}/H^\mathbb{R}$ is a symmetric space with twistor space $G/P$, as above, then the usual totally geodesic embedding of $G^\mathbb{R}/H^\mathbb{R}$ into $G^\mathbb{R}$, endowed with the twistorial structure given by (1a), above, is twistorial. This twistorial map corresponds to the diagonal (holomorphic) embedding of $G/P$ into $G/P \times G/P$. This way, we have the following:

2) Let $k, p, q \in \mathbb{N} \setminus \{0\}$, $k \leq p/2$, $M = \text{Gr}^+_{p,q}(p + q, \mathbb{R})$ with its Riemannian symmetric space structure, $Z$ the Grassmannian of isotropic $k$-dimensional subspaces of $\mathbb{C}^{p+q}$ and $Y = \{(u, v) \in Z \times M \mid u \subseteq v^\mathbb{C}\}$. Note that, this Riemannian twistorial structure is of Hermitian type if and only if $p = 2k$, case considered in [5] and [6]; also, the case $k = 1$, $p = 3$ was considered in [13] (see [19]).

3) Let $k \in \mathbb{N} \setminus \{0\}$ and $p_1, \ldots, p_k, n \in \mathbb{N}$ such that $0 < p_1 < \cdots < p_k < n$. The flag manifold $Z = F_{p_1, \ldots, p_k}(n, \mathbb{C})$, $M = \text{Gr}_{p_1-p_2+p_3-\cdots}(n, \mathbb{C})$, with $Y$ formed of those pairs $((\alpha_1, \ldots, \alpha_k), \beta) \in Z \times M$ such that $(\alpha_1 \cap \beta, \alpha_3 \cap \beta, \ldots) \in F_{p_1-p_3-p_4-\cdots}(n, \mathbb{C})$ and $(\alpha_2 \cap \beta^+, \alpha_4 \cap \beta^+, \ldots) \in F_{p_2-p_4-p_5-\cdots}(n, \mathbb{C})$. The case $k \leq 2$ is covered by [5] and [6], up to the twistoriality of the embeddings of $M$ into $\text{SU}(n)$.

Note that, also, [13, Example 4.7] (see [19, Example 4.6]) provides an example of a Riemannian reductive homogeneous space which is not a symmetric space and which is endowed with a Riemannian twistorial structure, as above.

**Theorem 3.11.** Any Riemannian symmetric space admits a nontrivial Riemannian twistorial structure invariant under the isometry group.

**Proof.** This follows from Examples 3.9 and 3.10. □

**Remark 3.12.** Let $G$ be a semisimple complex Lie group, $P \subseteq G$ a parabolic subgroup and $G^\mathbb{R}$ a compact real form of $G$. Let $\varphi : Y \to Z$ and $\pi : Y \to G^\mathbb{R}$ be the submersions giving the corresponding Riemannian twistorial on $G^\mathbb{R}$ as a symmetric space, where $Y = G^\mathbb{R} \times G/P$ and $Z = G/P \times G/P$.

Let $a : N \to G^\mathbb{R}$ be an immersion. In view of Corollary 2.4 and Proposition 3.7 it is useful to understand when $Y$ admits a section over $a$ which is horizontal with respect to the connection induced on $Y$ by the the Levi-Civita connection of $G^\mathbb{R}$. A straightforward argument shows that, locally, this condition holds if and only if there exists a map $b : N \to G^\mathbb{R}$ such that $b^*\theta + \text{Ad}(b^{-1})A$ takes values in $\mathfrak{p}$, where $A = \frac{1}{2}a^*\theta$, with $\theta$ the canonical (left invariant) one-form on $G^\mathbb{R}$ and $\mathfrak{p}$ the Lie algebra of $P$.

Conversely, let $A$ be a $\mathfrak{g}^\mathbb{R}$-valued one-form on $N$, where $\mathfrak{g}^\mathbb{R}$ is the Lie algebra of $G^\mathbb{R}$.

Suppose that the following conditions are satisfied:

(i) $A_x : T_xN \to \mathfrak{g}^\mathbb{R}$ is injective, for any $x \in N$;

(ii) $dA + 2[A, A] = 0$;
(iii) Locally, there exists $b : N \to G^\mathbb{R}$ such that $b^* \theta + \text{Ad}(b^{-1})A$ takes values in $\mathfrak{p}$.

Then, locally, we can integrate $a^* \theta = 2A$ to obtain immersions $a : N \to G^\mathbb{R}$ admitting horizontal lifts to $Y$. Finally, if the CR structure on $Y$ induces a complex structure on the image of such a lift, from Corollary 11, we deduce that $a$ is pluriharmonic and the corresponding statement for Proposition 5.7 also holds. In fact, locally, $\pi|_{\varphi^{-1}(N)}$ is minimal as in Proposition 5.7 (with ‘$\varphi(N)$’ replacing ‘$N$’) which shows that the latter approach is more general if the CR twistor space is associated to a Riemannian twistorial structure.

The simplest case is when $P/G$ is a Hermitian symmetric space. Then the totally geodesic embeddings of $P/G$ into $G^\mathbb{R}$ are twistorial pluriharmonic and their restrictions to complex submanifolds of $P/G$ give, as is well known, pluriharmonic immersions of uniton number one, in the sense of [16] (cf. [27]), with respect to any unitary representation of $G^\mathbb{R}$.

Example 3.13. Let $k \in \mathbb{N} \setminus \{0\}$, $p_1, \ldots, p_k, n \in \mathbb{N}$ such that $0 < p_1 < \cdots < p_k < n$. Let $Z = F_{p_1, \ldots, p_k}(n, \mathbb{C})$ be the twistor space of Example 3.10(3) and $\psi = (\psi_1, \ldots, \psi_k)$ a holomorphic immersion to $Z$ such that if $k \geq 2$ then $d(\Gamma(\psi_j)) \subseteq \Gamma(\psi_{j+1})$, for $j = 1, \ldots, k - 1$, where $\psi$ also denotes the corresponding flag of holomorphic vector bundles, $\Gamma$ is the functor giving the sheaves of sections, and $d$ is the covariant derivation of the trivial flat connection on $Z \times \mathbb{C}^n$. Denote by $\Pi_\alpha$ the orthogonal projection onto the subspace $\alpha$ of $\mathbb{C}^n$ and let $\varphi_j = \Pi_{\psi_j} - \Pi_{\psi_j}^\perp$, $j = 1, \ldots, k$. Then $\varphi = \varphi_1 \cdots \varphi_k$ is a twistorial pluriharmonic immersion as in Corollary 2.4 (the case $k \leq 2$ is well known; compare [28] and the references therein), and $\psi$ provides an example for Proposition 5.7. Moreover, to deduce that $\varphi$ is pluriharmonic it is sufficient to assume $\psi$ holomorphic.

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