Decentralized Policy Gradient for Nash Equilibria Learning of General-sum Stochastic Games

Yan Chen

School of Mathematical Sciences
East China Normal University
No. 5005, South Lianhua Road, Shanghai 200241, China

Tao Li

School of Mathematical Sciences
East China Normal University
No. 5005, South Lianhua Road, Shanghai 200241, China

Abstract

We study Nash equilibria learning of a general-sum stochastic game with an unknown transition probability density function. Agents take actions at the current environment state and their joint action influences the transition of the environment state and their immediate rewards. Each agent only observes the environment state and its own immediate reward and is unknown about the actions or immediate rewards of others. We introduce the concepts of weighted asymptotic Nash equilibrium with probability 1 and in probability. For the case with exact pseudo gradients, we design a two-loop algorithm by the equivalence of Nash equilibrium and variational inequality problems. In the outer loop, we sequentially update a constructed strongly monotone variational inequality by updating a proximal parameter while employing a single-call extra-gradient algorithm in the inner loop for solving the constructed variational inequality. We show that if the associated Minty variational inequality has a solution, then the designed algorithm converges to the $k^{1/2}$-weighted asymptotic Nash equilibrium. Further, for the case with unknown pseudo gradients, we propose a decentralized algorithm, where the G(PO)MDP gradient estimator of the pseudo gradient is provided by Monte-Carlo simulations. The convergence to the $k^{4}$-weighted asymptotic Nash equilibrium in probability is achieved.

Keywords: Stochastic Game, Policy Gradient, Nash Equilibrium, Multi-agent Reinforcement Learning, Variational Inequality

1. Introduction

In a Markov decision process, an agent aims at finding a policy to maximize its own expectation of cumulative discounted immediate rewards. At any given environment state, an agent chooses a policy and takes an action. Then, the agent gains an immediate reward and the action causes the environment a transition to the next state. In a multi-agent scenario, the decision-making of an agent is affected not only by the environment state, but also by the behaviors of other agents. Game theory studies how multiple agents make decisions when they interact directly and dynamically, and how these decisions reach an equilibrium. Motivated by which, Shapley came up with the framework of stochastic games (Shapley, 1953), also known as Markov games (Littman, 1994), which allows people to...
extend the Markov decision process to the case with multiple agents. In a stochastic game, each agent take its action independently and both state transitions of the environment and immediate rewards depend on agents’ actions. Quantitative classical algorithms, such as iterated algorithms (Shapley, 1953; Hoffman and Karp, 1966) and mathematical programming (Filar et al., 1991; Filar and Vrieze, 2012), have been proposed for finding Nash equilibria of general-sum stochastic games when agents know the transition probability density function of the environment state. For the case with the unknown state transition probability density function, Littman (1994) introduced multi-agent reinforcement learning as a framework and applied Q-Learning to a simple two-player zero-sum stochastic game. Since then, multi-agent reinforcement learning algorithms, like Q-Learning (Watkins and Dayan, 1992) and Actor-Critic (Konda and Tsitsiklis, 1999), have been widely used. These algorithms allow agents to act while learning if the transition probability density of the environment state is unknown. Hu and Wellman (1998) proposed a Q-Learning algorithm of two-player general-sum stochastic games, where they constructed a bimatrix game based on the current estimation of action-value function and computed the Nash equilibrium for the bimatrix game in each iteration. It is shown that the Q-Learning algorithm is convergent if every bimatrix game arising from the learning process has a global optimum point or a saddle point and agents update the estimations of action-value functions according to values at this point. Hu and Wellman (2003) extended the result to a multi-agent context and designed an improved algorithm: Nash Q-Learning (NashQ). Littman (2001) presented Friend or Foe Q-Learning (FFQ), where agents assume its opponent as either a friend or foe, which can be considered as an extension of NashQ. They showed that FFQ learns the action-value function under a Nash equilibrium if there exists an adversarial equilibrium or coordination equilibrium of the stochastic game. Greenwald et al. (2003) generalized NashQ and FFQ and introduced Correlated Q-Learning. They demonstrated the convergence by simulations but didn’t give the theoretical proof. Prasad et al. (2015) designed an actor-critic algorithm for Nash equilibria of general-sum stochastic games from the view of dynamic programming. The critic updates the state-value functions and the actor performs gradient descent on policies. They established that the algorithm converges to a Nash equilibrium asymptotically. Perolat et al. (2018) built a stochastic approximation for a fictitious play process using an Actor-Critic algorithm. They proved the convergence of the method towards a Nash equilibrium for both cases with two-player zero-sum and cooperative (that is, when players receive the same immediate rewards) stochastic games.

All algorithms mentioned above generated by Q-Learning and Actor-Critic are based on the estimations of action-value functions. They learn the estimations of action-value functions and then choose their actions according to the estimations. It’s impossible to design policies without the estimations of action-value functions for these algorithms (Sutton et al., 1999). In particular, these algorithms generated by Q-Learning encounter the computation difficulty of Nash equilibria of stage games which makes the algorithms implementation more difficult. At the same time, these algorithms need huge tables to store the estimations of action-value functions. So they can’t deal with large action spaces or a continuum of action spaces. To this end, Sutton et al. (1999) introduced policy gradient methods. Firstly, policies of each agent are parameterized and then policy parameters are updated according to the gradient of state-value functions with respect to parameters. Policy gradient methods learn parameterized policies directly and action selections no longer
depend directly on action-value functions. At this point, action-value functions can still be used to learn the parameters of policies but are unnecessary for action selections. By virtue of this advantage, policy gradient methods provide a practical way to handle the stochastic games where the action space is massive or even a continuum. At the same time, policy gradient methods search directly in the parameter space. As a result, they enjoy better theoretical convergence guarantees (Yang et al., 2018; Zhang et al., 2020; Agarwal et al., 2020). Commonly used classes of parameterized policies include direct parameterization, $\alpha$-greedy direct parameterization, Gaussian parameterization, softmax parameterization, log-linear parameterization and so on. In recent years, with the great success of the theoretical research on policy gradient methods for a Markov decision process, policy gradient methods with different classes of parameterized policies are also applied to zero-sum and general-sum stochastic games. For a two-agent zero-sum stochastic game, Daskalakis et al. (2020) focused on $\alpha$-greedy direct parameterization and used the REINFORCE gradient estimator of the policy gradient. They showed that if agents descend and ascend along the gradients of total reward functions with respect to their parameters respectively, their policies converge to a min-max equilibrium of the game, as long as their learning rates follow a two-timescale rule. Zhao et al. (2022) considered softmax parameterization. They found a minimax equilibrium for the matrix game constructed by the estimations of state-value functions and then performed natural policy gradient descent to update the estimations in each iteration. They proved the algorithm can find a near-optimal policy. Wei et al. (2021) used direct parameterization and updated the policies of two agents by running an optimistic gradient descent or ascent algorithm with a critic that slowly learns the state-value function of each state. They showed that the algorithm converges to the set of Nash equilibria if the induced discounted Markov chain under any stationary policies is irreducible. For general-sum stochastic games, Leonardos et al. (2021) defined the notion of Markov potential games by transplanting potential games into the setting of Markov games. They took $\alpha$-greedy direct parameterization and replaced the actual gradients of total reward functions with respect to parameters with REINFORCE gradient estimators. The convergence to the $\epsilon$-Nash equilibrium of the gradient ascent algorithm of parameters is given. Zhang et al. (2021) considered direct parameterization and showed that Nash equilibria and first-order Nash equilibria of general-sum stochastic games are equivalent. They gave the rate of converging to strict Nash equilibria if agents perform gradient descent for the case with known gradients of state-value functions with respect to policy parameters. Also, they gave global convergence rates for both exact gradients and gradients estimated by samples of Markov potential games. Mao and Başar (2022) proposed a decentralized algorithm in which each agent independently uses an optimistic V-learning and performs a mirror descent for policy updating with direct parameterization. Their algorithm converges to a coarse correlated equilibrium, a solution concept that generalizes Nash equilibrium by allowing possible correlated equilibria among the agents’ policies.

It is worth noting that the results of Daskalakis et al. (2020), Leonardos et al. (2021) and Zhang et al. (2021) depend on the fact that one can find Nash equilibria by the first-order necessary optimality conditions of total reward functions with respect to the parameter of each player if the total reward functions satisfy the gradient domination theorem. Inspired by the above research, Daskalakis et al. (2020), Leonardos et al. (2021) and Zhang et al. (2021) find Nash equilibria of zero-sum stochastic games and Markov
potential games with unknown transition probability density functions. While for general-sum stochastic games, Zhang et al. (2021) is restricted to the case where the state and action spaces are finite, the gradients of total reward functions with respect to parameters are known to each agent, and policies are with direct parameterization. In this paper, we study learning Nash equilibria of a general-sum stochastic game with an unknown transition probability density function. The joint actions of agents influence the state transition of the environment and their immediate rewards. Each agent only observes states and their immediate rewards and is unaware of the actions or rewards of other agents. Compared with Daskalakis et al. (2020), Leonardos et al. (2021) and Zhang et al. (2021), we study the general-sum stochastic game in which the state and action spaces are compact and convex and the transition probability density function is unknown to all agents. Focusing on the equivalence between Nash equilibrium and variational inequality problems, we propose algorithms for learning Nash equilibria. We introduce the concepts of weighted asymptotic Nash equilibrium with probability 1 and in probability and illuminate the connection between these concepts. It is shown that the algorithms converge to the weighted asymptotic Nash equilibrium for the case of exact gradients and the weighted asymptotic Nash equilibrium in probability for the case with unknown gradients respectively. Leonardos et al. (2021) and Zhang et al. (2021) updated policy parameters by gradient ascent and they showed the convergence of the algorithm using non-convex optimization by the existing of potential functions in Markov potential games. While for the general-sum stochastic games, there are no longer potential functions, thus, one can not ensure the convergence if updating policy parameters by gradient ascent. Different from Leonardos et al. (2021) and Zhang et al. (2021), we consider the variational inequality problem which is equivalent to the Nash equilibrium problem and design a two-loop algorithm. For the case with exact pseudo gradients, we design a two-loop algorithm in which we sequentially update a constructed strongly monotone variational inequality in the outer loop by updating a proximal parameter and employ a single-call extra-gradient algorithm in the inner loop for solving the constructed variational inequality. As a consequence, it is possible to employ the variational inequality to establish the convergence to the $k^{\frac{1}{2}}$-weighted asymptotic Nash equilibrium of our algorithm if the related Minty variational inequality has a solution. For the case with unknown pseudo gradients, Daskalakis et al. (2020) assumes that agents negotiate learning rates at the beginning of a zero-sum stochastic game, which leads to an incompletely decentralized algorithm. While it’s unnecessary to negotiate learning rates in advance, which cuts down the communication cost and therefore our algorithm is completely decentralized. Agents estimate pseudo gradients by interacting with the environment in the scenario of unknown pseudo gradients. The G(PO)MDP gradient estimator with a single trajectory has a high variance. Therefore, different from Daskalakis et al. (2020), we make use of the average of multiple trajectories. When agents interact with the environment, although the unbiased G(PO)MDP estimator of the pseudo gradient samples from an infinite time horizon, the Monte Carlo simulations are not feasible to sample in an infinite time horizon, so we adopt the G(PO)MDP estimator of a finite time horizon (Chen et al., 2021; Lu et al., 2021). The errors between the estimated pseudo gradient and the real pseudo gradient are analysed and we establish the convergence to the $k^{\frac{1}{4}}$-weighted asymptotic Nash equilibrium in probability.
The remainder of this paper is organized as follows. In Section II, the stochastic game problem is formulated. In Section III, we present the equivalence between Nash equilibrium and variational inequality problems and the existence of Nash equilibrium. In Section IV, we propose the algorithm for learning Nash equilibria and analyse the convergence of the algorithm for the case with exact pseudo gradients. In Section V, for the case with unknown pseudo gradients, the algorithm for learning Nash equilibria is given by introducing the G(PO)MDP gradient estimator and the convergence of the algorithm is showed. In Section VI, numerical examples are given to illustrate our algorithms. In Section VII, conclusion and future research topics are given.

The following notations will be used throughout this paper: For a given vector $x = (x_i)_i$, $\|x\| = \sqrt{\sum_i x_i^2}$ denotes its Euclidean norm and $|x| = \sum_i |x_i|$ denotes its $L^1$-norm. $E[X]$ denotes the expectation of stochastic variable $X$. $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors in Euclidian space. $0$ denotes vector 0. $\delta(\cdot)$ denotes Dirac delta function.

2. Problem Formulation

A stochastic game is an extension of a Markov decision process of a single agent. A stochastic game is denoted by a tuple $\Gamma = (\mathcal{N}, \mathcal{S}, \times_{i=1}^N \mathcal{A}_i, \rho, (r_i)_{i=1}^N, \gamma)$, where

(i) $\mathcal{N} = \{1, 2, \ldots, N\}$ is the set of agents and its cardinality is $N$;
(ii) $\mathcal{S}$ is the environment state space which is observed by all agents and is compact in $\mathbb{R}^q$, where $q$ is a positive integer;
(iii) $\times_{i=1}^N \mathcal{A}_i$ is the joint action space, where $\mathcal{A}_i$ is the action space of agent $i$, which is compact in $\mathbb{R}^{m_i}$, where $m_i$ is a positive integer;
(iv) $\rho_{t,t+1}(s' | s, a)$ is the conditional transition probability density function of the environment state from $s(t) = s$ to $s(t+1) = s'$ if agents take a joint action $a(t) = a = (a_i)_{i=1}^N$ at time $t$, that is,

$$\rho_{t,t+1}(s' | s, a) \geq 0, \forall s, s' \in \mathcal{S}, \forall a \in \mathcal{A}, \int_\mathcal{S} \rho_{t,t+1}(s' | s, a) \, ds' = 1,$$

and, especially, the conditional transition probability density function degenerates to a Dirac delta function if the state space is discrete, and so without loss of generality, we only focus on a continuum of state space;
(v) $r_i(s, a)$ is the immediate reward function of agent $i$ if the environment state is $s$ and the joint action $a$ is chosen by the agents;
(vi) $\gamma \in (0, 1)$ is the discount factor that describes the influence of the rewards obtained in the future on the agents’ policy (Prasad et al., 2015).

For any given state $s$, a policy of agent $i$ is defined as a conditional probability density function over $\mathcal{A}_i$. The admissible policy set of agent $i$ is defined as

$$\Pi_i \triangleq \left\{ \pi_i(\cdot | s), \forall s \in \mathcal{S} \left| \int_{\mathcal{A}_i} \pi_i(a_i' | s) \, da_i' = 1, \pi_i(a_i | s) \geq 0, \forall a_i \in \mathcal{A}_i \right. \right\}.$$
We assume that the policies of all agents are stationary, that is, the policies are independent of time. The set of joint policies of all agents is denoted by $\Pi = \times_{i=1}^{N} \Pi_i$. A joint policy is denoted by $\pi (a \mid s) = \prod_{i=1}^{N} \pi_i (a_i \mid s)$, $a = (a_i)_{i=1}^{N} \in \mathcal{A}$, $s \in \mathcal{S}$.

Let $\rho^\pi$ denote the Markov kernel of the Markov chain induced by the policy $\pi$, that is, for any state $s(t) = s$ at time $t$ and any state $s(t+1) = s'$ at time $t+1$, it follows that $\rho^\pi_{t,t+1} (s' \mid s) = \int_{\mathcal{A}} \rho_{t,t+1} (s' \mid s, a) \pi (a \mid s) da$. We denote the probability density function of the initial state $s(t)$ by $\rho_t$.

We consider more generalized policy parameterization with stochastic parameters. Let stochastic parameter $\theta_i$ be a stochastic variable on some probability space $(\Omega, \mathcal{F}, P)$ taking values in $\Theta_i \subseteq \mathbb{R}^{d_i}$, where $d_i$ is a positive integer. Let $\theta = (\theta_i)_{i=1}^{N} \in \Theta = \times_{i=1}^{N} \Theta_i \subseteq \mathbb{R}^{\sum_{i=1}^{N} d_i}$.

Let $\pi_\theta (a \mid s) = \prod_{i=1}^{N} \pi_{\theta_i} (a_i \mid s)$ denote the joint policy.

The state-value function $V^\pi_\theta (s, t)$ of agent $i$ is defined as the conditional expectation of the discounted sum of immediate rewards starting from the initial state $s(t) = s$ at time $t$ by choosing actions according to the policy $\pi_\theta$, that is,

$$V^\pi_\theta (s, t) = \mathbb{E} \left[ \sum_{l=t}^{\infty} \gamma^{l-t} r_i (s(l), a(l)) \mid s(t) = s, \theta \right]. \tag{1}$$

The total reward function $J^\pi_\theta (t)$ of agent $i$ is defined as the conditional expectation of the discounted sum of immediate rewards starting from the initial state $s(t)$ at time $t$ with the probability density function $\rho_t$ by choosing actions according to the policy $\pi_\theta$, that is,

$$J^\pi_\theta (t) = \mathbb{E} \left[ \sum_{l=t}^{\infty} \gamma^{l-t} r_i (s(l), a(l)) \mid \theta \right]. \tag{2}$$

It is known that $J^\pi_\theta (t)$ is a Borel measurable function of $\theta$. For notational convenience, we denote $V^\pi_\theta (s, t)$ by $V_\theta (s, t)$, $J^\pi_\theta (t)$ by $J_\theta (t)$ and $\rho^\pi_\theta$ by $\rho^\theta$.

At each time $t = t, t+1, \ldots$, agents take an action $a(l) \in \mathcal{A}$ according to the policy $\pi_\theta$ given the current state $s(l) \in \mathcal{S}$ observed by all agents. Then, agent $i$ gains an immediate reward $r_i (s(l), a(l))$, $i = 1, 2, \ldots, N$, and the state transitions to the next state $s(l+1) \in \mathcal{S}$. Each agent aims at maximizing its total reward function.

In our model, we assume that agent $i$ observes its own immediate reward and is completely unknown about the rewards and actions of other agents. The transition probability density function is unknown to all agents.

We study learning Nash equilibria of the general-sum stochastic game with parameterized policies. The Nash equilibrium of the game is defined as follows.

**Definition 1** (Nash, 1951) (Nash equilibrium) For the game $\Gamma$, if there exists $\theta^* = (\theta^*_i)_{i=1}^{N} \in \Theta$ such that

$$\sup_{i \in \mathcal{N}} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta^*_i)} (t) - J_i^{(\theta^*_i, \theta^*_i)} (t) \right) \leq 0,$$

then $(\pi^*_i)_{i=1}^{N}$ is called a Nash equilibrium.

**Definition 2** (Daskalakis et al., 2006) ($\epsilon$-Nash equilibrium) For the game $\Gamma$ and for any given $\epsilon > 0$, if there exists $\theta^* = (\theta^*_i)_{i=1}^{N} \in \Theta$ such that

$$\sup_{i \in \mathcal{N}} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta^*_i)} (t) - J_i^{(\theta^*_i, \theta^*_i)} (t) \right) \leq \epsilon,$$

it is said that $(\pi^*_i)_{i=1}^{N}$ is a $\epsilon$-Nash equilibrium.
then \((\pi^{\theta_i}_i)_{i=1}^N\) is called an \(\epsilon\)-Nash equilibrium.

The iterative output of the algorithm to learn a Nash equilibrium is often a sequence of random variables. Then we introduce the following concepts of weighted asymptotic Nash equilibrium with probability 1 and in probability.

**Definition 3** For the game \(\Gamma\) and for a given random parameter sequence \(\{\theta_k = (\theta_{i,k})_{i=1}^N, k \geq 1\}\), if there exists a positive sequence \(\{\gamma_k, k \geq 1\}\) and a nonnegative random sequence \(\{\epsilon_k, k \geq 1\}\), satisfying \(\sum_{k=1}^{\infty} \gamma_k = \infty\) and

\[
\sup_{i \in N} \frac{\sum_{k=1}^{K} \gamma_k \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i,\theta_{-i},k)}(t) - J_i^{(\theta_{i,k},\theta_{-i,k})}(t) \right)}{\sum_{k=1}^{K} \gamma_k} \leq \epsilon_K, \ K = 1, 2, \ldots, \text{a.s.,}
\]

for any given countable set \(\tilde{\Theta}_i \subseteq \Theta_i\),

then the sequence of policies \((\pi_{i,k})_{i=1}^N\) \(\infty_{k=1}\) is called a \(\gamma_k\)-weighted \(\epsilon_k\)-Nash equilibrium with probability 1.

**Definition 4** For the game \(\Gamma\) and for a given random parameter sequence \(\{\theta_k = (\theta_{i,k})_{i=1}^N, k \geq 1\}\), if there exists a positive sequence \(\{\gamma_k, k \geq 1\}\), satisfying \(\sum_{k=1}^{\infty} \gamma_k = \infty\) and

\[
\limsup_{K \to \infty} \frac{\sum_{k=1}^{K} \gamma_k \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i,\theta_{-i},k)}(t) - J_i^{(\theta_{i,k},\theta_{-i,k})}(t) \right)}{\sum_{k=1}^{K} \gamma_k} = 0 \text{ a.s.,}
\]

for any given countable set \(\tilde{\Theta}_i \subseteq \Theta_i\),

then the sequence of policies \((\pi_{i,k})_{i=1}^N\) \(\infty_{k=1}\) is called a \(\gamma_k\)-weighted asymptotic Nash equilibrium with probability 1.

**Definition 5** For the game \(\Gamma\) and for a given random parameter sequence \(\{\theta_k = (\theta_{i,k})_{i=1}^N, k \geq 1\}\), if there exists a positive sequence \(\{\gamma_k, k \geq 1\}\), satisfying \(\sum_{k=1}^{\infty} \gamma_k = \infty\), and for any \(\delta \in (0, 1]\), \(\epsilon > 0\), there exists \(K_0 > 0\) such that

\[
P \left\{ \sup_{i \in N} \frac{\sum_{k=1}^{K} \gamma_k \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i,\theta_{-i},k)}(t) - J_i^{(\theta_{i,k},\theta_{-i,k})}(t) \right)}{\sum_{k=1}^{K} \gamma_k} \leq \epsilon \right\} \geq 1 - \delta, \ K \geq K_0,
\]

for any given countable set \(\tilde{\Theta}_i \subseteq \Theta_i\),

then the sequence of policies \((\pi_{i,k})_{i=1}^N\) \(\infty_{k=1}\) is called a \(\gamma_k\)-weighted asymptotic Nash equilibrium in probability.

**Remark 6** Since \(\Theta_i\) is an uncountable set, \(\sup_{\theta_i \in \Theta_i} J_i^{(\theta_i,\theta_{-i},k)}(t)\) may be not a random variable anymore. To avoid making complex separability assumptions on \(J_i^{(\theta_i,\theta_{-i},k)}(t)\), we only focus on the supremum of \(J_i^{(\theta_i,\theta_{-i},k)}(t)\) over any countable subset of \(\Theta_i\) in Definitions 3-5. Especially, if the random parameter sequence \(\{\theta_{i,k}, k \geq 1\}\) is deterministic, then \(\tilde{\Theta}_i\) can be replaced by \(\Theta_i\) in the above definitions.
It’s easy to prove the following theorem which implies the connection of the above three definitions.

**Theorem 7** For the game \( \Gamma \), if there exists a random sequence \( \{\epsilon_k, k \geq 1\} \) such that \( (\{\pi_{\theta_i,k}\}_{i=1}^{N})_{k=1}^{\infty} \) is a \( \gamma_k \)-weighted \( \epsilon_k \)-Nash equilibrium with probability 1 and \( \{\epsilon_k, k \geq 1\} \) tends to zero with probability 1 (in probability), then \( (\{\pi_{\theta_i,k}\}_{i=1}^{N})_{k=1}^{\infty} \) is a \( \gamma_k \)-weighted asymptotic Nash equilibrium with probability 1 (in probability).

The following theorem illustrates that if a sequence of policies is a \( \gamma_k \)-weighted \( \epsilon_k \)-Nash equilibrium of the game \( \Gamma \) with probability 1, then a random subsequence of the original sequence is an \( \epsilon_k \)-Nash equilibrium in expectation.

**Theorem 8** For the game \( \Gamma \), if \( (\{\pi_{\theta_i,k}\}_{i=1}^{N})_{k=1}^{\infty} \) is a \( \gamma_k \)-weighted \( \epsilon_k \)-Nash equilibrium with probability 1 and for any \( K = 1, 2, \ldots \), there exist random variables \( \tau_K \in \{1, \ldots, K\} \) such that \( P(\tau_k = k) = \frac{\gamma_k}{\sum_{k=1}^{\infty} \gamma_k} \) and \( \{\tau_k, k \geq 1\} \) is independent of \( \{\theta_k = (\theta_{i,k})_{i=1}^{N}, k \geq 1\} \), then

\[
\sup_{i \in \mathbb{N}} \mathbb{E}\left[ \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{\tau_k, k - 1} K)}(t) - J_i^{(\theta_i, \tau_k K)}(t) \middle| \theta_k = 1, 2, \ldots, K \right] \leq \epsilon_K,
\]

where \( \Theta_i \) is any countable subset of \( \Theta_i \).

**Proof** See Appendix B.

### 3. Existence of Nash Equilibrium

In this section, we will show that the policy parameterization satisfies a gradient domination theorem and will prove the equivalence between Nash equilibrium and variational inequality problems. Then the existence of Nash equilibrium is established.

We start with the following assumptions on policy parameters and immediate rewards.

**Assumption 1** The conditional probability density function of the initial state for any given \( \theta \in \Theta \) is independent of \( \theta \), that is, \( \rho_t(s \mid \theta) = \rho_t(s) \).

**Assumption 2** There exists \( U_R > 0 \) such that

\[
\sup_{i \in \mathcal{N}} \sup_{(s,a) \in \mathcal{S} \times \mathcal{A}} |r_i(s,a)| \leq U_R.
\]

\( \Theta_i \subseteq \mathbb{R}^{d_i} \) is a nonempty compact convex set. By the compactness of \( \Theta_i \), we may assume there exists \( D_i > 0 \) such that \( \|\theta_i\| \leq \sqrt{2}D_i \). \( \pi_{\theta_i}(a_i \mid s) \) is concave and continuously differentiable with respect to \( \theta_i \in \Theta_i \).

In particular, direct parameterization, \( \alpha \)-greedy direct parameterization and Gaussian parameterization under some conditions satisfy Assumption 2.

**Assumption 3** For any \( i \in \mathcal{N} \), the policy \( \pi_{\theta_i} \) satisfies the following conditions:

\[
\nabla_{\theta_i} \log \pi_{\theta_i}(a_i \mid s) \text{ exists}, \forall s \in \mathcal{S}, \forall a_i \in \mathcal{A}_i, \text{ and there exist } L_\Theta > 0 \text{ and } B_\Theta > 0 \text{ such that}
\]

\[
\left\| \nabla_{\theta_i} \log \pi_{\theta_i}(a_i \mid s) - \nabla_{\theta_i} \log \pi_{\theta_i}(a_i \mid s) \right\| \leq L_\Theta \left\| \theta_i^1 - \theta_i^2 \right\|, \forall \theta_i^1, \theta_i^2 \in \Theta_i, \forall s \in \mathcal{S}, \forall a_i \in \mathcal{A}_i;
\]

\[
\sup_{\theta_i \in \Theta_i, s \in \mathcal{S}, a_i \in \mathcal{A}_i} \left\| \nabla_{\theta_i} \log \pi_{\theta_i}(a_i \mid s) \right\| \leq B_\Theta.
\]
Remark 9 Some commonly used parameterized policies such as Gaussian policy under some conditions satisfy Assumptions 2.3. For Gaussian policy,
\[
\pi_{\theta_i}(a_i \mid s) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(a_i - \phi_i^T(s)\theta_i)^2}{2\sigma^2} \right),
\]
where \(\phi_i^T(s)\theta_i\) is the mean of actions, \(\sigma^2\) is the variance, and \(\phi_i(s) \in \mathbb{R}^{d_i}\) is the feature vector to approximate the mean action at the state \(s\), if the following conditions are satisfied: (i) \(A\) is bounded; (ii) \(\sup_{s \in S} \|\phi_i(s)\| < \infty\); (iii) \(\sup_{i \in N, s \in S, a_i \in A_i, \theta_i \in \Theta_i} \|\phi_i^T(s)\theta_i - a_i\| \leq \sigma\); (iv) \(\Theta_i\) is compact and convex in \(\mathbb{R}^{d_i}\), then Assumptions 2.3 hold.

The action-value function \(Q_i^\theta(s, a, t)\) of agent \(i\) is defined as the conditional expectation of the discounted sum of immediate rewards starting from the initial state \(s(t) = s\) and the joint action \(a(t) = a\) at time \(t\) by choosing actions according to the policy \(\pi_\theta\), that is,
\[
Q_i^\theta(s, a, t) = r_i(s, a, t) + \gamma \sum_{l=t+1}^{\infty} \gamma^{l-t} r_i(s(l), a(l)) \bigg| s(t) = s, a(t) = a, \theta \bigg] .
\]
(3)

We denote \(Q_i^\theta(s, a, t)\) by \(Q_i^\theta(s, a, t)\).

The relationship between the state-value function and the action-value function is given as
\[
V_i^\theta(s, t) = \int_A Q_i^\theta(s, a, t) \pi_\theta(a \mid s) da ,
\]
(4)
\[
Q_i^\theta(s, a, t) = r_i(s, a, t) + \gamma \int_S V_i^\theta(s', t+1) \rho_{t+1}(s' \mid s, a) ds' .
\]
(5)

(1) and (3) demonstrate that although \(V_i^\theta(s, t)\) and \(Q_i^\theta(s, a, t)\) are marked out the initial time \(t\), \(V_i^\theta(s, t)\) and \(Q_i^\theta(s, a, t)\) are dependent on the initial state rather than the initial time.

By Assumptions 1-3, (1), (2), (4) and (5), we have the following proposition.

Proposition 10 If Assumptions 1-3 hold, then
\[
\nabla_{\theta_i} V_i^\theta(s, t) = \frac{1}{1-\gamma} \int_{S \times A} d_{\rho_i}^\theta(s') \nabla_{\theta_i} \pi_{\theta_i}(a_i \mid s') \pi_{\theta_i}(a_{-i} \mid s') Q_i^\theta(s', a, t+1) ds' da ,
\]
where
\[
\rho_{t,l}^\theta(s' \mid s) = \begin{cases} \delta(s' - s), & l = t, \\ \int_S \rho_{t-1,l}^\theta(s' \mid s'') \rho_{t-1,l}^\theta(s'' \mid s) ds'', & l \geq t + 1, \end{cases}
\]
\[
\rho_{t,l}^\theta(s' \mid s) = \rho_{t,l}^\theta(s(t) = s' \mid s(t) = s) \text{ and } d_{\rho_i}^\theta(s') = (1 - \gamma) \int_S \sum_{l=t}^{\infty} \gamma^{l-t} \rho_{t,l}^\theta(s' \mid s) \rho_{l}(s) ds
\]
called the probability density function of the discounted state distribution induced by \(\pi_\theta\).

Proof We will prove by induction that
\[
\nabla_{\theta_i} V_i^\theta(s, t) = \int_{S \times A} \sum_{l=t}^{t+k} \gamma^{l-t} \rho_{t,l}^\theta(s' \mid s) \nabla_{\theta_i} \log \pi_{\theta_i}(a_i \mid s') \pi_{\theta}(a \mid s') Q_i^\theta(s', a, l) ds' da
\]
Let $k = 0$, by Assumptions 2-3, taking the derivative with respect to $\theta_i$ on both sides of (4) and combining with (5) give

$$
\nabla_{\theta_i} V_i^\theta(s, t) = \nabla_{\theta_i} \left( \int_A Q_i^\theta(s, a, t) \pi_\theta(a) \, da \right)
= \int_A Q_i^\theta(s, a, t) \nabla_{\theta_i} \pi_\theta(a) \, da + \int_A \nabla_{\theta_i} Q_i^\theta(s, a, t) \pi_\theta(a) \, da
= \int_A Q_i^\theta(s, a, t) \nabla_{\theta_i} \pi_\theta(a_i | s) \pi_{\theta_{-i}}(a_{-i} | s) \, da
+ \int_A \nabla_{\theta_i} (r_i(s, a, t) + \gamma \int_S \rho_{t, t+1}(s', s, a) V_i^\theta(s', t + 1) \, ds') \pi_\theta(a | s) \, da
= \int_A Q_i^\theta(s, a, t) \nabla_{\theta_i} \log \pi_\theta(a_i | s) \pi_\theta(a | s) \, da
+ \gamma \int_S \rho_{t, t+1}^\theta(s' | s) \nabla_{\theta_i} V_i^\theta(s', t + 1) \, ds'.
$$

Thus, (6) is true for $k = 0$. Let $m$ be any positive integer and suppose (6) is true for $k = m$, that is,

$$
\nabla_{\theta_i} V_i^\theta(s, t) = \int_{S \times A} \sum_{l=t}^{t+m} \gamma^{l-t} \rho_{l,t}^\theta(s' | s) \nabla_{\theta_i} \pi_\theta(a_i | s') \pi_\theta(a | s') Q_i^\theta(s', a, l) \, ds' \, da
+ \gamma^{m+1} \int_S \rho_{t, t+m+1}^\theta(s' | s) \nabla_{\theta_i} V_i^\theta(s', t + m + 1) \, ds'.
$$

For the term $\nabla_{\theta_i} V_i^\theta(s', t + m + 1)$, by Assumptions 2-3, similar to the proof of (7), we have

$$
\nabla_{\theta_i} V_i^\theta(s', t + m + 1) = \int_A Q_i^\theta(s', a, t + m + 1) \nabla_{\theta_i} \log \pi_\theta(a_i | s') \pi_\theta(a | s') \, da
+ \gamma \int_S \rho_{t+m+1, t+m+2}^\theta(s'' | s') \nabla_{\theta_i} V_i^\theta(s'', t + m + 2) \, ds'.'
$$

By (8) and the above inequality, we have

$$
\nabla_{\theta_i} V_i^\theta(s, t)
= \int_{S \times A} \sum_{l=t}^{t+m} \gamma^{l-t} \rho_{l,t}^\theta(s' | s) \nabla_{\theta_i} \log \pi_\theta(a_i | s') \pi_\theta(a | s') Q_i^\theta(s', a, l) \, ds' \, da
+ \gamma^{m+1} \left( \int_{S \times A} \rho_{t, t+m+1}^\theta(s' | s) Q_i^\theta(s', a, t + m + 1) \nabla_{\theta_i} \log \pi_\theta(a_i | s') \pi_\theta(a | s') \, dads' \right)
+ \gamma^{m+2} \int_S \rho_{t, t+m+2}^\theta(s' | s) \nabla_{\theta_i} V_i^\theta(s', t + m + 2) \, ds'
= \int_{S \times A} \sum_{l=t}^{t+m+1} \gamma^{l-t} \rho_{l,t}^\theta(s' | s) \nabla_{\theta_i} \log \pi_\theta(a_i | s') \pi_\theta(a | s') Q_i^\theta(s', a, l) \, ds' \, da
$$
\[ + \gamma^{m+2} \left( \int_S \rho_{t,m+2}^\theta (s' \mid s) \nabla_{\theta_i} V_i^\theta (s', t + m + 2) ds' \right). \]

Thus, (6) holds for \( k = m + 1 \). By the principle of induction, (6) is true for all \( k \in \mathbb{N} \).

From (1), (3) and Assumption 2, it follows that
\[ \sup_{s \in S, a \in A} |Q_i^\theta (s', a, l)| \leq \frac{U_R}{1 - \gamma}, \]
which together with Assumption 3 and (7) implies
\[
\sup_{s \in S} \left| \nabla_{\theta_i} V_i^\theta (s, t) \right| \leq \sup_{s \in S} \left| \int_A Q_i^\theta (s, a, t) \nabla_{\theta_i} \log \pi_{\theta_i} (a_i \mid s) \pi_\theta (a \mid s) da \right|
+ \gamma \sup_{s \in S} \left| \int_S \rho_{t+1}^\theta (s') \nabla_{\theta_i} V_i^\theta (s', t + 1) ds' \right|
\leq \frac{U_R B_\theta}{1 - \gamma} + \gamma \sup_{s \in S} \left| \nabla_{\theta_i} V_i^\theta (s, t + 1) \right|.
\]

This together with the fact that \( V_i^\theta (s, t) \) are dependent on the initial state rather than the initial time gives
\[ \sup_{s \in S} \left| \nabla_{\theta_i} V_i^\theta (s, t) \right| \leq \frac{U_R B_\theta}{(1 - \gamma)^2}. \quad (9) \]

By Assumptions 2-3, we have
\[
\sum_{l=t}^{\infty} \int_{S \times A} \gamma^{l-t} \rho_{t,l}^\theta (s' \mid s) \left| \nabla_{\theta_i} \log \pi_{\theta_i} (a_i \mid s') \pi_\theta (a \mid s') \right| Q_i^\theta (s', a, l) ds'da
\leq \frac{B_\theta U_R}{1 - \gamma} \sum_{l=t}^{\infty} \int_{S \times A} \rho_{t,l}^\theta (s' \mid s) \pi_\theta (a \mid s') ds'da
\leq \frac{B_\theta U_R}{(1 - \gamma)^2}.
\]

Then by the above inequality, (9) and the Dominated Convergence Theorem, letting \( k \) tends to \( \infty \) on both sides of (6) gives
\[
\nabla_{\theta_i} V_i^\theta (s, t) = \int_{S \times A} \sum_{l=t}^{\infty} \gamma^{l-t} \rho_{t,l}^\theta (s' \mid s) \nabla_{\theta_i} \log \pi_{\theta_i} (a_i \mid s') \pi_\theta (a \mid s') Q_i^\theta (s', a, l) ds'da.
\]

By Assumption 1, it is known that the conditional probability density function of \( s(t) \) for a given \( \theta \) satisfies \( \rho_t (s \mid \theta) = \rho_t (s) \), which together with (1), (2), the above equality and the property of conditional expectation gives
\[
\nabla_{\theta_i} J_i^\theta (t) = \nabla_{\theta_i} \mathbb{E} \left[ \sum_{l=t}^{\infty} \gamma^{l-t} \rho_i (s(l), a(l)) \bigg| \theta \right]
= \nabla_{\theta_i} \mathbb{E} \left[ \mathbb{E} \left[ \sum_{l=t}^{\infty} \gamma^{l-t} \rho_i (s(l), a(l)) \bigg| s(t) = s, \theta \right] \bigg| \theta \right]
= \nabla_{\theta_i} \int_S \mathbb{E} \left[ \sum_{l=t}^{\infty} \gamma^{l-t} \rho_i (s(l), a(l)) \bigg| s(t) = s, \rho_t (s \mid \theta) ds.
\]
By Assumption 1, Assumption 4 and Lemma 30, we have

\[
M \leq \sup_{s,t} |\nabla_{\theta_i} \log \pi_{\theta_i}(a_i \mid s') \pi_{\theta}(a \mid s') Q^\theta_{i,t}(s', a, l) d\rho_t(s) ds
\]

To give the gradient domination theorem, we need the following assumption widely used in literatures Zhang et al. (2020), Daskalakis et al. (2020) and Zhang et al. (2021).

**Assumption 4** The induced discounted Markov chain by \( N \) agents satisfy \( d^0_{\rho_t}(s') > 0, \forall s' \in S, \forall \theta \in \Theta \).

We define the pseudo gradient mapping \( F : \Theta \to \mathbb{R}^{\sum_{i=1}^N d_i} \) as \( F(\theta) = (F_i(\theta))_{i=1}^N = (-\nabla_{\theta_i} J_i^0(t))_{i=1}^N, \forall \theta \in \Theta \). Then we have the following gradient domination theorem by Assumptions 1-4.

**Lemma 11** (Gradient domination theorem) If Assumptions 1-4 hold, then for any \( \theta = (\theta_i, \theta_{-i}) \in \Theta \), we have

\[
\sup_{\theta_i' \in \Theta_i} J_i^{(\theta_i', \theta_{-i})}(t) - J_i^{(\theta_i, \theta_{-i})}(t) \leq M_1 \sup_{\theta_i \in \Theta_i} \langle F_i(\theta), \bar{\theta}_i - \theta_i \rangle, \forall i = 1, 2, \ldots, N, \quad (10)
\]

where \( M_1 = \sup_{\theta, \theta' \in \Theta} \| \frac{\partial^i}{\partial \theta_i^2} \|_\infty, \| \frac{\partial^i}{\partial \theta_i} \|_\infty = \sup_{s' \in S} d^0_{\rho_t}(s') \) and \( \bar{\theta}^i = (\theta_i', \theta_{-i}) \in \Theta \).

**Proof** By Assumption 1, Assumption 4 and Lemma 30, we have

\[
J_i^{(\theta_i', \theta_{-i})}(t) - J_i^{(\theta_i, \theta_{-i})}(t)
\]

\[
= \frac{1}{1 - \gamma} \int_{S \times A_i} d_{\rho_t}^i(s') \pi_{\theta_i'}(a_i \mid s') \left( \int_{A_{-i}} \pi_{\theta_{-i}}(a_{-i} \mid s') A_i^0(s', a, t + 1) da_{-i} \right) da_i ds'
\]

\[
\leq \frac{1}{1 - \gamma} \sup_{\bar{\theta}_i \in \Theta_i} \int_{S \times A_i} d_{\rho_t}^i(s') d_{\rho_t}^i(s') \pi_{\bar{\theta}_i}(a_i \mid s') \left( \int_{A_{-i}} \pi_{\theta_{-i}}(a_{-i} \mid s') \right) da_i \times A_i^0(s', a, t + 1) da_{-i}
\]

\[
\leq \frac{1}{1 - \gamma} \sup_{\bar{\theta}_i \in \Theta_i} \int_{S \times A_i} d_{\rho_t}^i(s') d_{\rho_t}^i(s') \pi_{\bar{\theta}_i}(a_i \mid s') \left( \int_{A_{-i}} \pi_{\theta_{-i}}(a_{-i} \mid s') \right) da_i \times A_i^0(s', a, t + 1) da_{-i}
\]

\[
\leq M_1 \sup_{\theta_i \in \Theta_i} \langle F_i(\theta), \bar{\theta}_i - \theta_i \rangle, \forall i = 1, 2, \ldots, N.
\]
\[
\begin{align*}
&\leq \left\| \frac{d\tilde{\theta}_t}{d\theta_t} \right\|_{\infty} \frac{1}{1 - \gamma} \sup_{\hat{\theta}_i \in \Theta_i} \int_{S \times A_i} d_{\rho_t}^\theta (s') \pi_{\hat{\theta}_i} (a_i | s') \left( \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \right) \\
&\quad \times A_t^\theta(s', a, t + 1) da_{-i}) da_i ds' \\
&= \left\| \frac{d\tilde{\theta}_t}{d\theta_t} \right\|_{\infty} \sup_{\hat{\theta}_i \in \Theta_i} \int_{S \times A_i} (\pi_{\hat{\theta}_i} (a_i | s') - \pi_{\theta_i} (a_i | s')) \left( \frac{1}{1 - \gamma} d_{\rho_t}^\theta (s') \right) \\
&\quad \times \left( \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') A_t^\theta(s', a_i, a_{-i}, t + 1) da_{-i} \right) da_i ds' \\
&= \left\| \frac{d\tilde{\theta}_t}{d\theta_t} \right\|_{\infty} \sup_{\hat{\theta}_i \in \Theta_i} \int_{S \times A_i} (\pi_{\hat{\theta}_i} (a_i | s') - \pi_{\theta_i} (a_i | s')) \left( \frac{1}{1 - \gamma} d_{\rho_t}^\theta (s') \right) \\
&\quad \times \left( \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') Q_t^\theta(s', a_i, a_{-i}, t + 1) da_{-i} \right) da_i ds',}
\end{align*}
\]
where \( A_t^\theta(s', a, t + 1) = Q_t^\theta(s', a_i, a_{-i}, t + 1) - V_t^\theta(s', t + 1), \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') da_{-i} = 0 \) is used in the third equality and the last equality is by \( \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') da_{-i} da_i \).

Noting that \( \pi_{\theta_i} (a_i | s') \) is concave with respect to \( \theta_i \), we know that \( \pi_{\hat{\theta}_i} (a_i | s') - \pi_{\theta_i} (a_i | s') \leq (\hat{\theta}_i - \theta_i)^T \nabla_{\theta_i} \pi_{\theta_i} (a_i | s') \). This together with Assumption 4, Proposition 10 and (11) yields

\[
\sup_{\hat{\theta}_i' \in \Theta_i} J_{t+1}^{(\hat{\theta}_i', \theta_{-i})} (t) - J_t^{(\theta_i, \theta_{-i})} (t) \\
\leq \sup_{\hat{\theta}_i' \in \Theta_i} \left\| \frac{d\tilde{\theta}_t}{d\theta_t} \right\|_{\infty} \sup_{\hat{\theta}_i' \in \Theta_i} \int_{S \times A_i} (\hat{\theta}_i - \theta_i)^T \nabla_{\theta_i} \pi_{\theta_i} (a_i | s') \left( \frac{1}{1 - \gamma} d_{\rho_t}^\theta (s') \right) \\
&\quad \times \left( \int_{A_{-i}} \pi_{\theta_{-i}} (a_{-i} | s') Q_t^\theta(s', a_i, a_{-i}, t + 1) da_{-i} \right) da_i ds' \\
= \sup_{\hat{\theta}_i' \in \Theta_i} \left\| \frac{d\tilde{\theta}_t}{d\theta_t} \right\|_{\infty} \sup_{\hat{\theta}_i' \in \Theta_i} (\hat{\theta}_i - \theta_i)^T \nabla_{\theta_i} J_t^{\theta_i} (t) \\
\leq \sup_{\hat{\theta}_i' \in \Theta_i} \left\| \frac{d\tilde{\theta}_t}{d\theta_t} \right\|_{\infty} \sup_{\hat{\theta}_i' \in \Theta_i} (\hat{\theta}_i - \theta_i)^T \nabla_{\theta_i} J_t^{\theta_i} (t) = M_1 \sup_{\hat{\theta}_i' \in \Theta_i} (F_t(\theta), \theta_i - \hat{\theta}_i),
\]

that is, (10) follows.

We assume the parameterized policies are concave with respect to parameters, so the gradient domination theorems for direct parameterization (Zhang et al., 2021) and \( \alpha \)-greedy direct parameterization (Zhang et al., 2020; Daskalakis et al., 2020) are special cases of our result.
Lemma 11 enables agents to approximate their best response to other agents’ policies if they update policies in the algorithm by controlling the upper bound of the right side of (10) or using the first-order necessary optimality conditions of the total reward functions with respect to the parameter of each player, which makes it possible to learn a Nash equilibrium (Daskalakis et al., 2020; Zhang et al., 2021).

Inspired by Lemma 11, we can find a Nash equilibrium by means of the first-order necessary optimality conditions of the total reward functions with respect to the parameter of each player. Hence we give the equivalence between Nash equilibrium and variational inequality problems.

Definition 12 (Kinderlehrer and Stampacchia, 1980) For a given subset $K$ in $\mathbb{R}^n$ and a mapping $G : K \rightarrow \mathbb{R}^n$, a variational inequality problem is to find a vector $x^* \in K$ such that

$$
\langle G(x^*), x^* - x \rangle \leq 0, \forall x \in K,
$$

or $x^*$ is called a solution of SVI($G, K$).

Definition 13 (Minty, 1962) For a given subset $K$ in $\mathbb{R}^n$ and a mapping $G : K \rightarrow \mathbb{R}^n$, a Minty variational inequality problem is to find a vector $x^* \in K$ such that

$$
\langle G(x), x^* - x \rangle \geq 0, \forall x \in K,
$$

or $x^*$ is called a solution of MVI($G, K$).

Definition 14 (Zhang et al., 2021) (First-order Nash equilibrium) For the game $\Gamma$, if there exists $\theta^* = (\theta^*_i)_{i=1}^N \in \Theta$ which is a solution of SVI($F, \Theta$), that is,

$$
\sup_{i \in N} \left( \sup_{\theta_i \in \Theta_i} \langle F_i(\theta^*), \theta^*_i - \theta_i \rangle \right) \leq 0,
$$

then $(\pi_{\theta^*_i})_{i=1}^N$ is called a first-order Nash equilibrium.

Definition 15 ($\epsilon$-first-order Nash equilibrium) For the game $\Gamma$ and for any given $\epsilon > 0$, if there exists $\theta^* = (\theta^*_i)_{i=1}^N \in \Theta$ such that

$$
\sup_{i \in N} \left( \sup_{\theta_i \in \Theta_i} \langle F_i(\theta^*), \theta^*_i - \theta_i \rangle \right) \leq \epsilon,
$$

then $(\pi_{\theta^*_i})_{i=1}^N$ is called an $\epsilon$-first-order Nash equilibrium.

Lemma 16 For the game $\Gamma$, if Assumptions 1-4 hold, then $(\pi_{\theta^*_i})_{i=1}^N$ is a first-order Nash equilibrium if and only if it is a Nash equilibrium; $(\pi_{\theta^*_i})_{i=1}^N$ is an $\epsilon$-first-order Nash equilibrium if and only if it is a $M_1\epsilon$-Nash equilibrium, where $M_1$ is given by Lemma 11.

By Definition 14 and Lemma 16, we can characterize the Nash equilibrium problem in terms of SVI($F, \Theta$).

Theorem 17 If Assumptions 1-4 hold, then there exists a solution of SVI($F, \Theta$) and the game $\Gamma$ has a Nash equilibrium.

Proof See Appendix B.
4. Learning Nash Equilibria with Exact Pseudo Gradients

In this section, we assume all agents access to the exact pseudo gradient $F(\theta)$. Before we design the algorithm, we will prove that $F(\theta)$ is Lipschitz continuous with respect to $\theta$ at first.

**Lemma 18** If Assumptions 1-3 hold, then $F(\theta)$ is $L$-Lipschitz continuous with respect to $\theta \in \Theta$, where the Lipschitz constant

$$L = \sqrt{\frac{2(U_R)^2(L_\theta)^2}{(1-\gamma)^6} + \frac{2(1+\gamma)^2(U_R)^2N(B_\theta)^4}{(1-\gamma)^6}}.$$ 

**Proof** See Appendix B. 

Then, we propose Algorithm 1 for learning Nash equilibrium with exact pseudo gradients.

---

**Algorithm 1** Algorithm for Exact Pseudo gradients

1: Input: Lipschitz constant $L$, integer $K \geq 1$, weight $\gamma_k$, initial values $\theta_1 = (\theta_{i,1})_{i=1}^N \in \Theta,$
2: $\beta \in (0, \frac{1}{L})$, $\bar{\eta} = \frac{1}{2\sqrt{L^2 + \frac{1}{\beta^2}}},$
3: for $k = 1, \ldots, K$ do
4: Input: initial values $\theta^1 = (\theta^1_i)_{i=1}^N = z^1 \in \Theta$, integer $H_k$, stepsize $\eta$ satisfying
5: $\eta \in \left(0, \min \left\{ \frac{1}{(\frac{1}{L} - \delta)} + [\sqrt{(\frac{1}{L} - \delta)^2 + 32(L^2 + \frac{1}{\beta^2})} - (\frac{1}{L} - \delta)] + 2(L^2 + \frac{1}{\beta^2}) \right\} \right).$
6: Let $F_k(\theta) = (F_{i,k}(\theta))_{i=1}^N = F(\theta) + \frac{1}{\beta}(\theta - \theta_k) = \left(F_i(\theta) + \frac{1}{\beta}(\theta_i - \theta_i,k)) \right)_{i=1}^N.$
7: for $h = 1, \ldots, H_k$ do
8: for $i = 1, \ldots, N$ do
9: $\theta^{h+1}_i = \text{arg min}_{\theta_i \in \Theta_i} \{ \langle 2\eta F_{i,k}(\theta^h), \theta_i \rangle + \| \theta_i - z^h_i \|^2 \}. $
10: $z^{h+1}_i = \text{arg min}_{\theta_i \in \Theta_i} \{ \langle 2\eta F_{i,k}(\theta^{h+1}), \theta_i \rangle + \| \theta_i - z^h_i \|^2 \}. $
11: end for
12: end for
13: $z^{H_k+1}_i = (z^{H_k+1}_i)_{i=1}^N.$
14: $\theta_{k+1} = \text{arg min}_{\theta \in \Theta} \{ \langle 2\eta F_k(z^{H_k+1}), \theta \rangle + \| \theta - z^{H_k+1} \|^2 \}. $
15: end for
16: Randomly choose $\tau_K$ satisfying $P(\tau_K = k) = \frac{\gamma_k}{\sum_{k=1}^{K} \gamma_k}$, $k = 1, \ldots, K.$
17: Output: $\theta_{\tau_K}.$

---

In Algorithm 1, similar to Liu et al. (2021) and Koshal et al. (2010, 2013), by adding a strongly monotone term $\frac{1}{\beta}(\theta - \theta_k)$ to $F$, we construct SVI($F_k, \Theta$) in the outer loop and provide SVI($F_k, \Theta$) is $\left(\frac{1}{\beta} - L\right)$-strongly monotone if $\frac{1}{\beta} > L$ by Lemma 28. We update
SVI($F_k, \Theta$) by updating the proximal parameter $\theta_k$. In the inner loop, we employ a single-call extra-gradient algorithm for solving the constructed strongly monotone variational inequality while Liu et al. (2021) adopted an extra-gradient algorithm with two calls of pseudo gradients. We aim at alleviating the cost of pseudo gradients per iteration. Hsieh et al. (2019) also used a single-call extra-gradient algorithm to approximate the solution of a strongly monotone variational inequality. They measured the performance of the average of the inner loop output by the dual gap function (Definition 26) of SVI($F_k, \Theta$) while Liu et al. (2021) measured the difference between the output of the outer loop and the performance of the output of the inner loop by the prime gap function (Definition 26) of SVI($F_k, \Theta$). From Definition 26, we know our result is not a corollary of Hsieh et al. (2019). For the convergence result of the outer-loop iteration, we measure the performance of the outer loop by the prime gap function of SVI($F, \Theta$) while Liu et al. (2021) measured the difference between the output of the outer loop and the real solution of SVI($F_k, \Theta$).

For the convergence of our algorithm, we make the following assumption which has been adopted in recent works on variational inequalities (Liu et al., 2021; Song et al., 2020).

**Assumption 5** $MVI(F, \Theta)$ has a solution.

Below we will give the lemma and the theorem in this section.

**Lemma 19** If Assumptions 1-5 hold and we choose $\gamma_k = \frac{1}{k^2}$, $\gamma_0 = 0$ and $H_k = k$ in Algorithm 1, then $((\pi_{\theta_i,k})_{k=1}^N)^\infty_{k=1}$ given by Algorithm 1 satisfies

$$\sum_{k=1}^K k^\frac{1}{2} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \Theta_i, \theta_{-i, k})}(t) - J_i^{(\theta_i, \Theta_i, \theta_{-i, k})}(t) \right) \leq L(K),$$

and is a $k^\frac{1}{2}$-weighted $L(k)$-Nash equilibrium of the game $\Gamma$, where $L(K) = 2\sqrt{3}D_M\left(4L \sqrt{\frac{2D}{\beta} + \frac{\sqrt{N}B_{\alpha}U_{\beta}}{1-\gamma^2}} \frac{1}{K^2} + 2\sqrt{6}(1 + \sqrt{2}) \frac{1}{2} D \left( \frac{1}{2\beta - L} + 2\beta \right) \frac{1}{4} \left( 1 + 2L^2 + \frac{2}{\beta \gamma^2} \right) \left( \frac{L^2 + \frac{1}{\beta \gamma^2}}{\eta \left( \frac{1}{\beta - L} \right)} \right) \frac{1}{4} \right) (4LD + 2D \frac{1}{\beta} + \frac{\sqrt{N}B_{\alpha}U_{\beta}}{1-\gamma^2}) M_k^2$. $M_1$ is given by Lemma 11, $\epsilon$ is Euler’s number and $D = \sqrt{\sum_{i=1}^N D_i^2}$.

**Proof** By Lemma 31 (ii), we have

$$\frac{\|z_i^{h+1} - \theta_i\|^2}{2} \leq \frac{\|z_i^h - \theta_i\|^2}{2} - \left( \eta F_{i,k}(\theta_i^{h+1}), \theta_i^{h+1} - \theta_i \right) + \frac{\|\eta F_{i,k}(\theta_i^{h+1}) - \eta F_{i,k}(\theta_i)\|^2}{2} - \frac{\|\theta_i^{h+1} - z_i^h\|^2}{2}, \forall i \in N.$$

Taking summation for both sides of the above inequality from $i = 1$ to $N$ and rearranging the above inequality lead to

$$\left( \eta F_{k}(\theta_i^{h+1}), \theta_i^{h+1} - \theta \right) \leq \frac{\|z^h - \theta\|^2}{2} - \frac{\|z_i^{h+1} - \theta\|^2}{2} + \frac{\|\eta F_{k}(\theta_i^{h+1}) - \eta F_{k}(\theta_i)\|^2}{2} - \frac{\|\theta_i^{h+1} - z_i^h\|^2}{2}.$$  

(13)
For the third term on the right side of the above inequality, by Assumptions 1-3 and Lemma 27, we have

\[
\frac{\| \eta F_k(\theta^{h+1}) - \eta F_k(\theta^h) \|^2}{2} \leq \eta^2 (L^2 + \frac{1}{\beta^2}) \| \theta^{h+1} - \theta^h \|^2.
\]

This together with (13) gives

\[
\langle \eta F_k(\theta^{h+1}), \theta^{h+1} - \theta \rangle \leq \frac{\| z^h - \theta \|^2}{2} - \frac{\| z^{h+1} - \theta \|^2}{2} + \eta^2 (L^2 + \frac{1}{\beta^2}) \| \theta^{h+1} - \theta^h \|^2
\]

\[
- \| \theta^{h+1} - z^h \|^2 \frac{2}{2}.
\]

(14)

For the term \( \| \theta^{h+1} - \theta^h \|^2 \) in the above inequality, by \( C_2 \) inequality, we have

\[
\| \theta^{h+1} - \theta^h \|^2 \leq 2 \| \theta^{h+1} - z^h \|^2 + 2 \| \theta^h - z^h \|^2.
\]

(15)

For the first term on the right side of (15), by the non-expansion property of the proximal mapping in Lemma 31 (iii), Assumptions 1-3 and Lemma 27, we have

\[
\| \theta^h - z^h \|^2 = \sum_{i=1}^{N} \| \theta^h_i - z^h_i \|^2 \leq \sum_{i=1}^{N} \| F_i(\theta^{h-1}) - F_i(\theta^h) \|^2
\]

\[
= \eta^2 \left\| F(\theta^{h-1}) - F(\theta^h) \right\|^2 \leq \eta^2 \left( 2L^2 + \frac{2}{\beta^2} \right) \| \theta^{h-1} - \theta^h \|^2.
\]

This together with (15) gives

\[
\| \theta^{h+1} - \theta^h \|^2 \leq 2 \| \theta^{h+1} - z^h \|^2 + 2\eta^2 \left( 2L^2 + \frac{2}{\beta^2} \right) \| \theta^{h-1} - \theta^h \|^2.
\]

By \( \| \theta^{h+1} - \theta^h \|^2 = 2 \| \theta^{h+1} - \theta^h \|^2 - \| \theta^{h+1} - \theta^h \|^2 \) and the above inequality, we have

\[
\| \theta^{h+1} - \theta^h \|^2 \leq 4 \| \theta^{h+1} - z^h \|^2 + 4\eta^2 \left( 2L^2 + \frac{2}{\beta^2} \right) \| \theta^{h-1} - \theta^h \|^2 - \| \theta^{h+1} - \theta^h \|^2.
\]

Combining (14) with the above inequality gives

\[
\langle \eta F_k(\theta^{h+1}), \theta^{h+1} - \theta \rangle \leq \frac{\| z^h - \theta \|^2}{2} - \frac{\| z^{h+1} - \theta \|^2}{2} + \left( 4\eta^2 \left( L^2 + \frac{1}{\beta^2} \right) - \frac{1}{2} \right) \| \theta^{h+1} - z^h \|^2
\]

\[
+ 8\eta^4 \left( L^2 + \frac{1}{\beta^2} \right) \| \theta^{h-1} - \theta^h \|^2 - \eta^2 \left( L^2 + \frac{1}{\beta^2} \right) \| \theta^{h+1} - \theta^h \|^2.
\]

According to Assumptions 1-3 and Lemma 28, we know that \( F_k(\theta) \) is \( \left( \frac{1}{\beta} - L \right) \)-strongly monotone and then SVI(\( F_k, \Theta \)) has a unique solution \( \text{(Kinderlehrer and Stampacchia, 1980)} \), which is denoted by \( \overline{\theta}_k \). Substituting \( \overline{\theta}_k \) for \( \theta \) in the above inequality leads to

\[
\langle \eta F_k(\theta^{h+1}), \theta^{h+1} - \overline{\theta}_k \rangle
\]
\[
\leq \frac{\|z^h - \theta_k\|^2}{2} + \frac{\|z^{h+1} - \theta_k\|^2}{2} + \left(4\eta^2 \left(L^2 + \frac{1}{\beta^2}\right) - \frac{1}{2}\right) \|\theta^{h+1} - z^h\|^2 + 8\eta^4 \left(L^2 + \frac{1}{\beta^2}\right)^2 \\
\times \|\theta^{h-1} - \theta_k\|^2 - \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \|\theta^{h+1} - \theta_k\|^2.
\]

Noting that \(\theta_k\) solves SVI\((F_k, \Theta)\), by \(C_2\) inequality and Lemma 28, we have
\[
\eta \left(\frac{1}{\beta} - L\right) \left\|\frac{z^h - \theta_k}{2}\right\|^2 - \eta \left(\frac{1}{\beta} - L\right) \left\|z^h - \theta^{h+1}\right\|^2 \leq \eta \left(\frac{1}{\beta} - L\right) \left\|\theta^{h+1} - \theta_k\right\|^2 \\
\leq \eta \left<F_k(\theta^{h+1}) - F_k(\theta_k), \theta^{h+1} - \theta_k\right> \\
\leq \left<\eta F_k(\theta^{h+1}), \theta^{h+1} - \theta_k\right>.
\]

Taking summation for both sides of (16) and the above inequality gives
\[
\frac{\|z^{h+1} - \theta_k\|^2}{2} \leq \left(1 - \eta \left(\frac{1}{\beta} - L\right)\right) \frac{\|z^h - \theta_k\|^2}{2} + 8\eta^4 \left(L^2 + \frac{1}{\beta^2}\right)^2 \|\theta^{h-1} - \theta_k\|^2 - \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \|\theta^{h+1} - \theta_k\|^2 \\
+ \left[4\eta^2 \left(L^2 + \frac{1}{\beta^2}\right) + \eta \left(\frac{1}{\beta} - L\right) - \frac{1}{2}\right] \|\theta^{h+1} - z^h\|^2.
\]

From \(\eta < \frac{-\left(\frac{1}{\beta} - L\right) + \left((\frac{1}{\beta} - L)^2 + 8(L^2 + \frac{1}{\beta^2})\right)^{1/2}}{8}\), we know that \(4\eta^2 \left(L^2 + \frac{1}{\beta^2}\right) + \eta \left(\frac{1}{\beta} - L\right) - \frac{1}{2} \leq 0\), which together with the above inequality gives
\[
\frac{\|z^{h+1} - \theta_k\|^2}{2} \leq \left(1 - \eta \left(\frac{1}{\beta} - L\right)\right) \frac{\|z^h - \theta_k\|^2}{2} + 8\eta^4 \left(L^2 + \frac{1}{\beta^2}\right)^2 \|\theta^{h-1} - \theta_k\|^2 \\
- \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \|\theta^{h+1} - \theta_k\|^2.
\]

Then, from \(\eta < \frac{-\left(\frac{1}{\beta} - L\right) + \left((\frac{1}{\beta} - L)^2 + 32(L^2 + \frac{1}{\beta^2})\right)^{1/2}}{16}\), we know that \(8\eta^4 \left(L^2 + \frac{1}{\beta^2}\right)^2 \leq \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \left(1 - \eta \left(\frac{1}{\beta} - L\right)\right)\). This together with the above inequality gives
\[
\frac{\|z^{h+1} - \theta_k\|^2}{2} + \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \|\theta^{h+1} - \theta_k\|^2 \leq l_1 \left(\frac{\|z^h - \theta_k\|^2}{2} + \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \|\theta^h - \theta_k\|^2 \right) \\
\times \|\theta^h - \theta^{h-1}\|^2,
\]
where \(l_1 = 1 - \eta \left(\frac{1}{\beta} - L\right)\). From \(0 < \eta < \frac{1}{\beta - L}\), it follows that \(l_1 \in (0, 1)\). By Assumption 2 and the above inequality, we have
\[
\frac{\|z^{H_k+1} - \theta_k\|^2}{2} + \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \|\theta^{H_k+1} - \theta^{H_k}\|^2 \\
\leq l_1^{H_k} \left(\frac{\|z^1 - \theta_k\|^2}{2} + \eta^2 \left(L^2 + \frac{1}{\beta^2}\right) \|\theta^1 - \theta^0\|^2 \right) \leq l_1^{H_k} \left(1 + 2 \left(L^2 + \frac{1}{\beta^2}\right)\right) 2D^2,
\]

18
By Assumptions 1-3, Lemma 28 and Lemma 32, we have

\[ \langle F_k(\theta_{k+1}), \theta_{k+1} - \theta \rangle \leq 2D \left( 2L^2 + \frac{2}{\beta^2} \right)^{\frac{1}{2}} \left( 2 + \sqrt{2} \right) \| z^{H_k+1} - \overline{\theta}_k \|. \]

Substituting (18) into the above inequality gives

\[ \langle F_k(\theta_{k+1}), \theta_{k+1} - \theta \rangle \leq C_1 l_1^{\frac{H_k}{2}}, \quad \forall \theta \in \Theta, \quad (19) \]

where \( C_1 = 8(1 + \sqrt{2})D^2 \left( L^2 + \frac{1}{\beta^2} \right)^{\frac{1}{2}} \left( 1 + 2 \left( L^2 + \frac{1}{\beta^2} \right) \right)^{\frac{1}{2}} \). Noting that \( \overline{\theta}_k \) is the solution of SVI(\( F_k, \Theta \)), by (19), we have

\[ \langle F_k(\theta_{k+1}) - F_k(\overline{\theta}_k), \theta_{k+1} - \overline{\theta}_k \rangle = \langle F_k(\theta_{k+1}), \theta_{k+1} - \overline{\theta}_k \rangle + \langle -F_k(\overline{\theta}_k), \theta_{k+1} - \overline{\theta}_k \rangle \leq C_1 l_1^{\frac{H_k}{2}}. \quad (20) \]

Combining Assumptions 1-3 and Lemma 28 with the above inequality implies

\[ \left( \frac{1}{\beta} - L \right) \| \theta_{k+1} - \overline{\theta}_k \|^2 \leq \langle F_k(\theta_{k+1}) - F_k(\overline{\theta}_k), \theta_{k+1} - \overline{\theta}_k \rangle \leq C_1 l_1^{\frac{H_k}{2}}. \quad (21) \]

By Assumption 5, we know that MVI(\( F, \Theta \)) has a solution, which is denoted by \( \theta_* \). This together with \( F(\theta_{k+1}) = F_k(\theta_{k+1}) - \frac{1}{\beta} (\theta_{k+1} - \theta_k) \) gives

\[ \langle \beta F_k(\theta_{k+1}) - (\theta_{k+1} - \theta_k), \theta_{k+1} - \theta_* \rangle \geq 0. \]

Then, combining (19) with the above inequality leads to

\[ -\beta \left( C_1 l_1^{\frac{H_k}{2}} \right) \leq \langle \beta F_k(\theta_{k+1}), \theta_* - \theta_{k+1} \rangle \leq \langle \theta_{k+1} - \theta_k, \theta_* - \theta_{k+1} \rangle, \]

that is,

\[ -\beta \left( C_1 l_1^{\frac{H_k}{2}} \right) \leq \langle \theta_{k+1} - \theta_k, \theta_* - \theta_{k+1} \rangle. \]

This together with \( \frac{1}{2} \| \theta_k - \overline{\theta}_k \|^2 + 2 \| \overline{\theta}_k - \theta_{k+1} \|^2 \geq -2 \langle \theta_k - \overline{\theta}_k, \overline{\theta}_k - \theta_{k+1} \rangle \) gives

\[ \| \theta_k - \theta_* \|^2 = \| \theta_k - \theta_{k+1} + \theta_{k+1} - \theta_* \|^2 = \| \theta_k - \theta_{k+1} \|^2 + \| \theta_{k+1} - \theta_* \|^2 + 2 \langle \theta_k - \theta_{k+1}, \theta_{k+1} - \theta_* \rangle \geq \| \theta_k - \theta_{k+1} \|^2 + \| \theta_{k+1} - \theta_* \|^2 - 2\beta \left( C_1 l_1^{\frac{H_k}{2}} \right) \]

\[ = \| \theta_k - \overline{\theta}_k \|^2 + \| \overline{\theta}_k - \theta_{k+1} \|^2 + 2 \langle \theta_k - \overline{\theta}_k, \overline{\theta}_k - \theta_{k+1} \rangle + \| \theta_{k+1} - \theta_* \|^2 - 2\beta \left( C_1 l_1^{\frac{H_k}{2}} \right) \]

19
Dividing both sides of the above inequality by \( \frac{1}{2} \), we have

\[
\sum_{k=1}^{K} \gamma_k \left( \theta_k - \theta_0 \right) \leq \sum_{k=1}^{K} \gamma_k \left( \| \theta_k - \theta_0 \|^2 - \| \theta_{k+1} - \theta_0 \|^2 \right) + \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=1}^{K} \gamma_k \left( \frac{H_k}{C_1 \ell_1^2} \right)
\]

Then we have

\[
\frac{1}{2} \| \theta_k - \theta_0 \|^2 \leq \| \theta_k - \theta_0 \|^2 - \| \theta_{k+1} - \theta_0 \|^2 + \| \theta_k - \theta_{k+1} \|^2 + 2\beta C_1 \ell_1^2 \frac{H_k}{\gamma_k}.
\]

Multiply both sides of the above inequality by \( \gamma_k \) and take summation for both sides from \( i = 1 \) to \( N \). Then, by (21), we have

\[
\frac{1}{2} \sum_{k=1}^{K} \gamma_k \| \theta_k - \theta_0 \|^2 \leq \sum_{k=1}^{K} \gamma_k \left( \| \theta_k - \theta_0 \|^2 - \| \theta_{k+1} - \theta_0 \|^2 \right) + \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=1}^{K} \gamma_k \left( \frac{H_k}{C_1 \ell_1^2} \right)
\]

\[
= \sum_{k=1}^{K} \gamma_k \left( \| \theta_k - \theta_0 \|^2 - \| \theta_{k+1} - \theta_0 \|^2 \right) + \sum_{k=1}^{K} \left( \gamma_k - \gamma_k - 2 \right) \| \theta_k - \theta_0 \|^2
\]

\[
= \gamma_0 \| \theta_0 - \theta_0 \|^2 - K \| \theta_0 - \theta_0 \|^2 + (\gamma_K - \gamma_0) D^2 + \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=0}^{K-1} \gamma_k \left( \frac{H_k}{C_1 \ell_1^2} \right)
\]

\[
\leq 4\gamma D^2 + \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=1}^{K} \gamma_k \left( \frac{H_k}{C_1 \ell_1^2} \right).
\]

Dividing both sides of the above inequality by \( \sum_{k=1}^{K} \gamma_k \), we have

\[
\frac{\sum_{k=1}^{K} \gamma_k \| \theta_k - \theta_0 \|^2}{\sum_{k=1}^{K} \gamma_k} \leq 8 \sum_{k=1}^{K} \gamma_k D^2 + \frac{2 \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=1}^{K} \gamma_k C_1 \ell_1^2}{\sum_{k=1}^{K} \gamma_k} \tag{22}
\]

From \( l_1 = 1 - \eta \left( \frac{1}{\beta - L} \right) \in (0, 1) \), we know that \( l_1^{H_k} < \frac{1}{e^{H_k} \eta \left( \frac{1}{\beta - L} \right)} \leq \frac{1}{H_k^2} \left( \eta \left( \frac{1}{\beta - L} \right) \right)^2 \).

This together with the above inequality implies

\[
\frac{\sum_{k=1}^{K} \gamma_k \| \theta_k - \theta_0 \|^2}{\sum_{k=1}^{K} \gamma_k} \leq 8 \sqrt{\gamma D^2 + \frac{2 \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=1}^{K} \gamma_k C_1 \ell_1^2}{\sum_{k=1}^{K} \gamma_k}} \tag{23}
\]

Combining \( H_k = k, \gamma_k = k^\frac{1}{2} \) and \( \sum_{k=1}^{K} k^{\frac{1}{2}} \geq \frac{2}{3} K^{\frac{3}{2}} \) with the above inequality, we have

\[
\frac{\sum_{k=1}^{K} k^{\frac{1}{2}} \| \theta_k - \theta_0 \|^2}{\sum_{k=1}^{K} k^{\frac{1}{2}}} \leq 12 D^2 \frac{1}{K} + 3 \left( \frac{1}{\beta - L} + 2\beta \right) \frac{C_1}{\left( \eta \left( \frac{1}{\beta - L} \right) \right)^{\frac{1}{2}} K^{\frac{3}{2}}}. \tag{23}
\]
Applying Lyapunov inequality into the above inequality leads to
\[
\sum_{k=1}^{K} k^{\frac{1}{2}} \left\| \theta_k - \overline{\theta}_k \right\| \leq \left( \frac{\sum_{k=1}^{K} k^{\frac{1}{2}} \left\| \theta_k - \overline{\theta}_k \right\|^2}{\sum_{k=1}^{K} k^{\frac{1}{2}}} \right)^{\frac{1}{2}}
\leq 2\sqrt{3}D \frac{1}{K^{\frac{1}{2}}} + \left( 3 \left( \frac{1}{\beta} - L \right) + 2\beta \right) \left( \frac{C_1}{\epsilon \eta \left( \frac{1}{\beta} - L \right)} \right)^{\frac{1}{2}} \frac{1}{K^{\frac{1}{4}}}. \tag{24}
\]
By Assumptions 1-3, Lemma 18 and Lemma 29, we have
\[
\langle F(\theta_k), \theta_k - \theta \rangle = \langle F(\theta_k) - F(\overline{\theta}_k), \theta_k - \overline{\theta}_k \rangle + \langle F(\theta_k) - F(\overline{\theta}_k), \overline{\theta}_k - \theta \rangle
\leq 2LD \left\| \theta_k - \overline{\theta}_k \right\| + 2LD \left\| \theta_k - \overline{\theta}_k \right\| + \left\| F(\overline{\theta}_k) \right\| \left\| \theta_k - \overline{\theta}_k \right\| + \frac{2D}{\beta} \left\| \theta_k - \overline{\theta}_k \right\|.
\tag{25}
\]
By Proposition 10, we have
\[
\left\| F(\theta) \right\|^2 = \sum_{n=1}^{N} \left\| \nabla_{\theta_i} J^\theta(t) \right\|^2 \leq \sum_{n=1}^{N} \left[ \int_{S \times A} \int_{S} \sum_{l=t}^{\infty} \gamma^{l-t} \rho^\theta(s(l) = s' \mid s(t) = s) \nabla_{\theta_i} \pi^\theta(a_i \mid s') \pi_{\theta_{-i}}(a_{-i} \mid s') Q^\theta_i(s', a, l) \rho_t(s) ds ds' da \right]^2.
\]
From (1), (3) and Assumption 2, it follows that \( |Q^\theta_i(s', a, l)| \leq \frac{U_R}{\gamma} \), which together with Assumption 1, Assumption 3 and the above inequality implies
\[
\left\| F(\overline{\theta}_k) \right\| \leq \left( \sum_{n=1}^{N} \sum_{l=t}^{\infty} \gamma^{l-t} B_\theta \frac{U_R}{1 - \gamma} \int_{S \times A} \int_{S} \rho^\theta(s(l) = s' \mid s(t) = s) \rho_t(s) ds ds' da \right)^{\frac{1}{2}} \leq \frac{\sqrt{\bar{N}} B_\theta U_R}{(1 - \gamma)^{\frac{1}{2}}}. \tag{26}
\]
By (25) and the above inequality, we have
\[
\langle F(\theta_k), \theta_k - \theta \rangle \leq \left( 4LD + \frac{2D}{\beta} + \frac{\sqrt{\bar{N}} B_\theta U_R}{(1 - \gamma)^{\frac{1}{2}}} \right) \left\| \theta_k - \overline{\theta}_k \right\|. \tag{27}
\]
Then, combining (24) with the above inequality gives
\[
\frac{\sum_{k=1}^{K} k^{\frac{1}{2}} \left( \sup_{\theta \in \Theta} \langle F(\theta_k), \theta_k - \theta \rangle \right)}{\sum_{k=1}^{K} k^{\frac{1}{2}}} \leq \frac{\sum_{k=1}^{K} k^{\frac{1}{2}} \left( 4LD + \frac{2D}{\beta} + \frac{\sqrt{\bar{N}} B_\theta U_R}{(1 - \gamma)^{\frac{1}{2}}} \right) \left\| \theta_k - \overline{\theta}_k \right\|}{\sum_{k=1}^{K} k^{\frac{1}{2}}}
\leq 2\sqrt{3}D \left( 4LD + \frac{2D}{\beta} + \frac{\sqrt{\bar{N}} B_\theta U_R}{(1 - \gamma)^{\frac{1}{2}}} \right) \frac{1}{K^{\frac{1}{2}}} + C_2 \frac{1}{K^{\frac{1}{4}}},
\]
where \( C_2 = \left( 4LD + \frac{2D}{\beta} + \sqrt{NB\theta U_R} \right) \left( 3 \left( \frac{1}{\gamma - L} + 2\beta \right) \frac{C_1}{(\epsilon_0(\gamma - L))^{\frac{1}{2}}} \right)^{\frac{1}{2}}. \) Note that
\[
\sup_{\theta_i \in \Theta_i} \langle F_i(\theta_i), \theta_i - \theta \rangle \leq \sup_{\theta \in \Theta} \langle F(\theta), \theta - \theta \rangle.
\]
This together with the above inequality gives
\[
\sum_{k=1}^{K} k^{\frac{1}{2}} \left( \sup_{\theta_i \in \Theta_i} \langle F_i(\theta_i), \theta_i - \theta \rangle \right) \leq 2\sqrt{3}D \left( 4LD + \frac{2D}{\beta} + \frac{\sqrt{NB\theta U_R}}{(1 - \gamma)^2} \right) \frac{1}{K^{\frac{1}{2}}} + C_2 \frac{1}{K^{\frac{1}{2}}},
\]
Then, from Assumptions 1-4, Lemma 11 and the above inequality, it follows that
\[
\sum_{k=1}^{K} k^{\frac{1}{2}} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k), \theta_{-i}, k})(t) \right) \leq \sup_{i \in \mathcal{N}} \sum_{k=1}^{K} k^{\frac{1}{2}} M_i \left( \sup_{\theta_i \in \Theta_i} \langle F_i(\theta_i), \theta_i - \theta_i \rangle \right) \leq \sum_{k=1}^{K} k^{\frac{1}{2}} \leq 2\sqrt{3}DM_i \left( 4LD + \frac{2D}{\beta} + \frac{\sqrt{NB\theta U_R}}{(1 - \gamma)^2} \right) \frac{1}{K^{\frac{1}{2}}} + C_2 M_i \frac{1}{K^{\frac{1}{2}}} = L(K).
\]
By the above inequality, for any countable set \( \widetilde{\Theta}_i \subseteq \Theta_i \), we have
\[
\sum_{k=1}^{K} k^{\frac{1}{2}} \left( \sup_{\theta_i \in \widetilde{\Theta}_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k), \theta_{-i}, k})(t) \right) \leq \sum_{k=1}^{K} k^{\frac{1}{2}} \leq \frac{2\sqrt{3}DM_i}{K^{\frac{1}{2}}} \left( 4LD + \frac{2D}{\beta} + \frac{\sqrt{NB\theta U_R}}{(1 - \gamma)^2} \right) \frac{1}{K^{\frac{1}{2}}} + C_2 M_i \frac{1}{K^{\frac{1}{2}}} \leq L(K),
\]
that is, (12) holds. Then from Definition 3, we know \( (\pi_{\theta_i, k})_{i=1}^{\infty} \) is a \( k^{\frac{1}{2}} \)-weighted \( L(k)-Nash \) equilibrium of the game \( \Gamma \). \( \square \)

**Theorem 20** If Assumptions 1-5 hold and we choose \( \gamma_k = k^{\frac{1}{2}}, \gamma_0 = 0 \) and \( H_k = k \), then \( (\pi_{\theta_i, k})_{i=1}^{\infty} \) given by Algorithm 1 is a \( k^{\frac{1}{2}} \)-weighted asymptotic Nash equilibrium of the game \( \Gamma \) and \( \lim_{K \to \infty} \sup_{i \in \mathcal{N}} E \left[ \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k), \theta_{-i}, k})(t) \right] \theta_k, k = 1, 2, ..., K = 0 \), where \( \widetilde{\Theta}_i \) is any countable subset of \( \Theta_i \).

**Proof** If Assumptions 1-5 hold, by Lemma 19, we have
\[
\sum_{k=1}^{K} k^{\frac{1}{2}} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k), \theta_{-i}, k})(t) \right) \leq L(K).
\]
Noting that \( \lim_{k \to \infty} L(k) = 0 \), by Theorem 7, we know that \((\pi_{\theta_{1,k}})_{i=1}^{N})_{k=1}^{\infty}\) is a \(k^2\)-weighted asymptotic Nash equilibrium of the game \(\Gamma\). Then, by Theorem 8, it follows that
\[
\lim_{K \to \infty} \sup_{i \in \mathcal{N}} \left[ \sup_{\theta_{i} \in \Theta_{i}} J_{i}^{\theta_{i}, \theta_{1:k}, \tau_{K}}(t) - J_{i}^{\theta_{i}, \tau_{K}, \theta_{1:k}, \tau_{K}}(t) \right] \theta_{k}, k = 1, 2, \ldots, K = 0.
\]

5. Learning Nash Equilibria with Unknown Pseudo Gradients

In this section, we design Algorithm 2 for learning Nash equilibria with unknown pseudo gradients. We employ Monte-Carlo simulations for the estimations of pseudo gradients. Similar to Algorithm 1, Algorithm 2 is also a two-loop algorithm. In the inner loop, we construct \(\tilde{F}_{k}(\theta) = \left(\tilde{F}_{i}(\theta) + \frac{1}{\beta}(\theta_{i} - \theta_{i,k})\right)_{i=1}^{N} = \left(-\nabla_{\theta_{i}}^{T,K_{1}} J_{i}^{\theta}(t) + \frac{1}{\beta}(\theta_{i} - \theta_{i,k})\right)_{i=1}^{N}\) and SVI(\(\tilde{F}_{k}, \Theta\)), where \(-\nabla_{\theta_{i}}^{T,K_{1}} J_{i}^{\theta}(t)\) is the estimation of \(F_{i}(\theta)\) with \(K_{1}\) trajectories and a finite time horizon of length \(T\). Then we update the proximal parameter in each iteration. In the inner loop, we provide a single-call extra-gradient algorithm for the constructed variational inequality.

**Algorithm 2** Algorithm for Unknown Pseudo gradients

1. Input: Lipschitz constant \(L\), integer \(K \geq 1\), weight \(\gamma_{k}\), initial values \(\theta_{1} = (\theta_{1,1})_{i=1}^{N} \in \Theta\), \(\beta \in (0, \frac{1}{L}), \bar{\eta} = \frac{1}{2L^2\beta^2}\), number of trajectories \(K_{1}\), time horizon \(T\).
2. for \(k = 1, \ldots, K\) do
3. Input: \(l_{2} \in \left[\max\left\{\frac{1}{\beta} - L, 6\left(L^2 + \frac{1}{\beta^2}\right)\right\}, \infty\right), l_{1} = \min\left\{\frac{1}{2\sqrt{L}}, \frac{1}{\beta}\right\}\), stepsize \(\eta_{h}\), initial values \(\theta^{1} = (\theta_{1}^{1})_{i=1}^{N} = z_{1}^{1} \in \Theta\), integer \(H_{k}\).
4. Let \(\hat{F}_{k}(\theta) = (\hat{F}_{i,k}(\theta))_{i=1}^{N} = \hat{F}(\theta) + \frac{1}{\beta}(\theta - \theta_{k}) = \left(\hat{F}_{i}(\theta) + \frac{1}{\beta}(\theta_{i} - \theta_{i,k})\right)_{i=1}^{N} \).
5. for \(h = 1, \ldots, H_{k}\) do
6. for \(i = 1, \ldots, N\) do
7. Sample \(\nabla_{\theta_{i}}^{T,K_{1}} J_{i}^{\theta_{i}}(t)\) by Monte-Carlo simulations.
8. \(\theta_{i}^{h+1} = \arg\min_{\theta_{i} \in \Theta_{i}} \left\{2\eta_{h} \hat{F}_{i,k}(\theta_{i}) + ||\theta_{i} - z_{i}^{h}||^{2}\right\} \).
9. Sample \(\nabla_{\theta_{i}}^{T,K_{1}} J_{i}^{\theta_{i}^{h+1}}(t)\) by Monte-Carlo simulations.
10. \(z_{i}^{h+1} = \arg\min_{\theta_{i} \in \Theta_{i}} \left\{2\eta_{h} \hat{F}_{i,k}(\theta_{i}) + ||\theta_{i} - z_{i}^{h}||^{2}\right\} \).
11. end for
12. end for
13. \(z_{H_{k}+1} = \left(z_{i}^{H_{k}+1}\right)_{i=1}^{N} \).
14. Sample \(\nabla_{\theta_{i}}^{T,K_{1}} J_{i}^{z_{H_{k}+1}}(t)\) by Monte-Carlo simulations.
15. \(\theta_{k+1} = \arg\min_{\theta \in \Theta} \left\{2\eta_{H_{k}} \hat{F}_{k}(\theta_{H_{k}+1}) + ||\theta - z_{H_{k}+1}||^{2}\right\} \).
16. end for
17. Randomly choose \(\tau_{K}\) satisfying \(P(\tau_{K} = k) = \frac{\gamma_{k}}{\sum_{k=1}^{K} \gamma_{k}}\), \(k = 1, \ldots, K\).
18. Output: \(\theta_{\tau_{K}}\).
19. end for
20. Output: \(\theta_{\tau_{K}}\).
5.1 Error Analysis of Pseudo Gradient Estimation

We will give the difference between the estimated pseudo gradient and the real pseudo gradient at first.

By Assumptions 1-3, for agent 1, \( \nabla_{\theta_1} J^\theta_1(t) \) can be written as

\[
\nabla_{\theta_1} J^\theta_1(t) = \mathbb{E}_{T \sim \rho_\theta} \left[ \sum_{l=1}^{\infty} \left( \sum_{\tau=t}^{l} \nabla_{\theta_1} \log \pi_{\theta_1}(a_1(\tau, w) \mid s(\tau, w)) \right) \gamma^{l-t} r_1(s(l, a(l, w)), a(l, w)) \right],
\]

where \( T \) is the trajectory of agent 1 for the given \( \pi_\theta \), that is, \( T = (s(t, w), a(t, w), \ldots, s(l, w), a(l, w), \ldots) \), \( w \) is the sample path and \( \rho_\theta \) is the probability distribution density function of the trajectory \( T \) (Baxter and Bartlett, 2001). If the transition probability density function is unknown, it’s intractable to compute the expectation of (28). Thus we employ the stochastic estimator of (28). The G(PO)MDP gradient estimator of \( \nabla_{\theta_1} J^\theta_1(t) \) is

\[
\hat{\nabla}_{\theta_1} J^\theta_1(t) = \mathbb{E}_{T \sim \rho_\theta} \left[ \sum_{l=1}^{\infty} \left( \sum_{\tau=t}^{l} \nabla_{\theta_1} \log \pi_{\theta_1}(a_1(\tau, w) \mid s(\tau, w)) \right) \gamma^{l-t} r_1(s(l, w), a(l, w)) \right].
\]

Notice that sampling from a single trajectory \( (s(t, w), a(t, w), \ldots, s(l, w), a(l, w), \ldots) \) in (29) may cause a high variance of the G(PO)MDP gradient estimator and sampling from an infinite horizon is not tractable in (29). To this end, Chen et al. (2021) proposed the stochastic estimator with \( K_1 \) trajectories and a finite time horizon of length \( T \) which can be expressed as

\[
\hat{\nabla}_{\theta_1}^{T,K_1} J^\theta_1(t) = \frac{1}{K_1} \sum_{w=1}^{K_1} \sum_{l=1}^{T} \left( \sum_{\tau=t}^{l} \nabla_{\theta_1} \log \pi_{\theta_1}(a_1(\tau, w) \mid s(\tau, w)) \right) \gamma^{l-t} r_1(s(l, a(l, w))).
\]

Denote \( \hat{F}(\theta) = \left( \hat{F}_i(\theta) \right)_{i=1}^{N} = \left( -\hat{\nabla}_{\theta_i}^{T,K_1} J^\theta_i(t) \right)_{i=1}^{N} \) and

\[
\hat{\nabla}_{\theta_1}^{T} J^\theta_1(t) = \mathbb{E}_{T \sim \rho_\theta} \left[ \sum_{l=1}^{T} \left( \sum_{\tau=t}^{l} \nabla_{\theta_1} \log \pi_{\theta_1}(a_1(\tau) \mid s(\tau)) \right) \gamma^{l-t} r_1(s(l), a(l)) \right],
\]

where \( T \) is the trajectory according to the policy \( \pi_\theta \). Denote \( F(\theta, T) = (F_i(\theta, T))_{i=1}^{N} = ( -\nabla_{\theta_i} J^\theta_i(t) )_{i=1}^{N}, \forall \theta \in \Theta. \)

**Lemma 21** If Assumptions 1-3 hold, then

\[
P \left( \| \hat{F}(\theta) - F(\theta) \|^2 \leq M(T, K_1, \delta) \right) \geq 1 - \frac{\delta}{4K_1}, \forall \theta \in \Theta, \forall \delta \in (0, 1],
\]

where \( M(T, K_1, \delta) = 2N(B_\Theta U_R)^2 \left( \frac{T+1}{\gamma} + \frac{\gamma}{(1-\gamma)^2} \right)^2 \gamma^{T+1} + 16 \log \left( \frac{8K_1}{\delta} \right) \frac{NB_\Theta^2 U_R^2 \gamma^2}{(1-\gamma)^2 K_1}. \)

**Proof** By C2 inequality, we have

\[
\| \hat{F}(\theta) - F(\theta) \|^2 = \| \hat{F}(\theta) - F(\theta, T) + F(\theta, T) - F(\theta) \|^2 \< 2 \| \hat{F}(\theta) - F(\theta, T) \|^2 + 2 \| F(\theta, T) - F(\theta) \|^2.
\]

(30)
For the second term in the above inequality, from Assumptions 1-3 and Lemma 6 in Chen et al. (2021), it follows that

\[
2 \| F(\theta, T) - F(\theta) \|^2 = 2 \sum_{i=1}^{N} \left\| -\nabla_{\theta_i}^{T} J_{i}^{\theta}(t) + \nabla_{\theta_i} J_{i}^{\theta}(t) \right\|^2 \leq \sigma_T,
\]

where \( \sigma_T = 2N(B\theta U_{R})^2 \left[ \left( \frac{T+1}{1-\gamma} + \frac{\gamma}{(1-\gamma)^2} \right) \gamma^{T+1} \right]^2 \). For the first term in (30), denote

\[
\hat{F}(\theta, T, w) = \left( \hat{F}_i(\theta, T, w) \right)_{i=1}^{N}
\]

\[
= \left( \sum_{l=1}^{T+t} \left( \sum_{\tau=t}^{l} \nabla_{\theta_i} \log \pi_{\theta_i}(a_{i}(\tau, w) \mid s(\tau, w)) \gamma^{l-t} r_{i}(s(l, w), a(l, w)) \right) \right)_{i=1}^{N},
\]

and then \( \tilde{F}(\theta) = \sum_{K=1}^{K_1} \frac{\hat{F}(\theta, T, w)}{K} \). Noting that \( \hat{F}_i(\theta, T, w) \) is an unbiased estimator of \( F_i(\theta, T) \) (Liu et al., 2020), we know that \( \hat{F}(\theta, T, w) \) is the unbiased estimator of \( F(\theta, T) \). By Assumptions 1-3 and Lemma 5 in Chen et al. (2021), we have \( \| \hat{F}_i(\theta, T, w) - F_i(\theta, T) \|^2 \leq \left( \frac{2B\theta U_{R} \gamma}{(1-\gamma)^2} \right)^2 \), and then

\[
\| \hat{F}(\theta, T, w) - F(\theta, T) \|^2 = \sum_{i=1}^{N} \| \hat{F}_i(\theta, T, w) - F_i(\theta, T) \|^2 \leq N \left( \frac{2B\theta U_{R} \gamma}{(1-\gamma)^2} \right)^2 \frac{1}{K_1}.
\]

Hence, by Lemma 33, it follows that

\[
P \left\{ \| \hat{F}(\theta) - F(\theta) \|^2 \leq 8 \log \left( \frac{8K_1}{\delta} \right) N B\theta U_{R}^2 \gamma^2 \right\} \geq 1 - \frac{\delta}{4K_1}, \quad \forall \delta \in (0, 1].
\]

Then combining (30) and (31) with the above inequality gives

\[
P \left\{ \| \hat{F}(\theta) - F(\theta) \|^2 \leq M(T, K_1, \delta) \right\} \geq 1 - \frac{\delta}{4K_1}, \quad \forall \delta \in (0, 1].
\]

\[\square\]

5.2 Convergence Analysis of Algorithm 2

In this section, we establish the convergence of Algorithm 2

**Lemma 22** If Assumptions 1-5 hold, and we choose \( \eta_h = \frac{\eta_k}{k^{\frac{1}{4}}} \), \( \gamma_k = k^{\frac{1}{4}} \) and \( H_k = k \) in Algorithm 2, then \( \left( (\pi_{\theta_i,k})_{i=1}^{N} \right)_{k=1}^{\infty} \) given by Algorithm 2 satisfies

\[
P \left\{ \sup_{i \in N} \sum_{k=1}^{K} \frac{k^{\frac{1}{4}}}{k} \left( \sup_{\theta_i \in \Theta_i} J_{i}^{(\theta_i, \theta_{-i,k})}(t) - J_{i}^{\theta_k}(t) \right) \leq L(K, K_1, T, \delta) + (\phi(K))^{\frac{1}{4}} \right\} \geq 1 - \frac{K}{K_1}, \quad \forall \delta \in (0, 1],
\]

for any given countable set \( \widetilde{\Theta}_i \subseteq \Theta_i, \)
where $L(K, K_1, T, \delta) = \sqrt{\frac{M}{2}} \left(4LD + \frac{2D}{\beta} + \frac{\sqrt{NBaUR}}{(1-\gamma)^2}\right) + \sqrt{2M_1 \left(\frac{1}{3-L} + 2\beta\right)^2} \left(4LD + \frac{2D}{\beta} + \frac{\sqrt{NBaUR}}{(1-\gamma)^2}\right)
\left[\frac{\sqrt{2M}}{2\left(L^2 + \frac{1}{\beta}\right)}\right] + \sqrt{\frac{30}{11} \left(\frac{1}{3-L} + 2\beta\right)^2} \left(4LD + \frac{2D}{\beta} + \frac{\sqrt{NBaUR}}{(1-\gamma)^2}\right) \left(2D \left(2L^2 + \frac{2}{\beta}\right) \left(2 + \sqrt{2}\right) \left(\frac{1}{3-L}\right)^2 \left(\phi(K)\right)^{\frac{3}{2}} = o \left(\frac{1}{K^2}\right)\right).

M_1 is given by Lemma 11, $M = M(T, K_1, \delta) = 2N (B_\theta U_R)^2 \left[\left(T + \frac{1}{1-\gamma} + \frac{\gamma}{(1-\gamma)^2}\right) \gamma^{T+1}\right] + 16 \log \left(\frac{8K_1}{\delta}\right) \frac{NB_\theta^2 U_R^2}{(1-\gamma)^4 K_1}$ and $D = \sqrt{\sum_{i=1}^{N} \frac{D_i^2}{T}}$.

**Proof** By Lemma 31 (ii), we have

$$\left\|h_i^{h+1} - \theta_i\right\|^2 \leq \left\|z_i^{h} - \theta_i\right\|^2 - \left\langle \eta h_i \hat{F}_i, (\theta_i^{h+1}, \theta_i^{h+1} - \theta_i) \right\rangle + \frac{\left\|\eta h_i \hat{F}_i (\theta_i^{h+1}) - \eta h_i \hat{F}_i (\theta_i)\right\|^2}{2}
- \frac{\left\|\theta_i^{h+1} - z_i^{h}\right\|^2}{2}, \forall i \in \mathbb{N}.$$

Taking summation for both sides of the above inequality from $i = 1$ to $N$ and rearranging the above inequality lead to

$$\left\langle \eta h_i F_i (\theta_i^{h+1}), \theta_i^{h+1} - \theta_i\right\rangle \leq \left\|z_i^{h} - \theta_i\right\|^2 - \left\|z_i^{h+1} - \theta_i\right\|^2 + \frac{\left\|\eta h_i \hat{F}_i (\theta_i^{h+1}) - \eta h_i \hat{F}_i (\theta_i)\right\|^2}{2}
+ \left\langle -\eta h_i \hat{F}_i (\theta_i^{h+1}) + \eta h_i F_i (\theta_i^{h+1}), \theta_i^{h+1} - \theta_i\right\rangle - \frac{\left\|\theta_i^{h+1} - z_i^{h}\right\|^2}{2}.
(32)$$

For the fourth term on the right side of (32), by Assumption 2, we have

$$\left\langle -\eta h_i \hat{F}_i (\theta_i^{h+1}) + \eta h_i F_i (\theta_i^{h+1}), \theta_i^{h+1} - \theta_i\right\rangle \leq \eta h_i \left\|\hat{F}_i (\theta_i^{h+1}) - F_i (\theta_i^{h+1})\right\| \left\|\theta_i^{h+1} - \theta_i\right\|
\leq 2\eta h_i D \left\|\hat{F}_i (\theta_i^{h+1}) - F_i (\theta_i^{h+1})\right\|.
(33)$$

For the third term on the right side of (32), by Cauchy-Schwartz inequality, Assumptions 1-3 and Lemma 27, we have

$$\left\|\eta h_i \hat{F}_i (\theta_i^{h+1}) - \eta h_i \hat{F}_i (\theta_i)\right\|^2 = \eta_i^2 \left\| \hat{F}_i (\theta_i^{h+1}) - F_i (\theta_i^{h+1}) + F_i (\theta_i^{h+1}) - F_i (\theta_i) + F_i (\theta_i) - \hat{F}_i (\theta_i) \right\|^2
\leq \frac{3\eta_i^2}{2} \left( \left\| \hat{F}_i (\theta_i^{h+1}) - F_i (\theta_i^{h+1}) \right\|^2 + \left\| F_i (\theta_i^{h+1}) - F_i (\theta_i) \right\|^2 + \left\| F_i (\theta_i) - \hat{F}_i (\theta_i) \right\|^2 \right)$$
Lemma 27, we have proximal mapping in Lemma 31 (iii), Cauchy-Schwartz inequality, Assumptions 1-3 and for the term 

Applying $C$ into the term $\|\theta^{h+1} - \theta^h\|^2$ in the above inequality leads to

$$\|\theta^{h+1} - \theta^h\|^2 \leq 2\|\theta^{h+1} - z^h\|^2 + 2\|\theta^h - z^h\|^2.$$  \hspace{1cm} (35)

For the term $\|\theta^h - z^h\|^2$ in the above inequality, by the non-expansion property of the proximal mapping in Lemma 31 (iii), Cauchy-Schwartz inequality, Assumptions 1-3 and Lemma 27, we have

$$\|\theta^h - z^h\|^2 \\
= \sum_{i=1}^{N} \|\theta^h - z^h\|^2 \\
\leq \sum_{i=1}^{N} \| \eta_{h-1} F_{ik}(\theta^{h-1}) + \eta_{h-1} \hat{F}_{ik}(\theta^h) \|^2 \\
= \sum_{i=1}^{N} \left| \eta_{h-1} \hat{F}_{ik}(\theta^{h-1}) + \eta_{h-1} F_{ik}(\theta^{h-1}) - \eta_{h-1} F_{ik}(\theta^{h-1}) + \eta_{h-1} F_{ik}(\theta^h) \\
- \eta_{h-1} F_{ik}(\theta^h) + \eta_{h-1} \hat{F}_{ik}(\theta^h) \right|^2 \\
\leq 3\eta_{h-1}^2 \left\{ \sum_{i=1}^{N} \| \hat{F}_{ik}(\theta^{h-1}) + F_{ik}(\theta^{h-1}) \|^2 + \sum_{i=1}^{N} \| F_{ik}(\theta^{h-1}) - F_{ik}(\theta^h) \|^2 \right\} \\
= 3\eta_{h-1}^2 \left\{ \| \hat{F}_{k}(\theta^{h-1}) + F_{k}(\theta^{h-1}) \|^2 + \| F_{k}(\theta^{h-1}) - F_{k}(\theta^h) \|^2 \right\} \\
\leq 3\eta_{h-1}^2 \left\{ \| \hat{F}_{k}(\theta^{h-1}) + F_{k}(\theta^{h-1}) \|^2 + \| F_{k}(\theta^h) - \hat{F}_{k}(\theta^h) \|^2 \right\} \\
+ 3\eta_{h-1}^2 \left( 2L^2 + \frac{2}{\beta^2} \right) \| \theta^{h-1} - \theta^h \|^2.$$

Substituting the above inequality into (35) gives

$$\|\theta^{h+1} - \theta^h\|^2 \leq 6\eta_{h-1}^2 \left\{ \| \hat{F}_{k}(\theta^{h-1}) + F_{k}(\theta^{h-1}) \|^2 + \| F_{k}(\theta^h) - \hat{F}_{k}(\theta^h) \|^2 \right\} \\
+ 6\eta_{h-1}^2 \left( 2L^2 + \frac{2}{\beta^2} \right) \| \theta^{h-1} - \theta^h \|^2 + 2\|\theta^{h+1} - z^h\|^2.$$
By (32), (33), (34) and the above inequality, we have

$$\langle \eta_h F_k(\theta^{h+1}), \theta^{h+1} - \theta \rangle \leq \frac{\|z^h - \theta\|^2}{2} - \frac{\|z^{h+1} - \theta\|^2}{2} + 2\eta_h D \left\| - \hat{F}_k(\theta^{h+1}) + F_k(\theta^{h+1}) \right\|
\quad + \frac{3\eta_h^2}{2} \left( \left\| \hat{F}_k(\theta^{h+1}) - F_k(\theta^{h+1}) \right\|^2 + \left\| F_k(\theta^h) - \hat{F}_k(\theta^h) \right\|^2 \right)
\quad + 9 \left( 2L^2 + \frac{2}{\beta^2} \right) \eta_h^2 \eta_{h-1} \left\{ \left\| - \hat{F}_k(\theta^{h-1}) + F_k(\theta^{h-1}) \right\|^2 + \left\| F_k(\theta^h) - \hat{F}_k(\theta^h) \right\|^2 \right\}
\quad + \left\| F_k(\theta^h) - \hat{F}_k(\theta^h) \right\|^2
\quad + \left( 3\eta_h^2 \left( 2L^2 + \frac{2}{\beta^2} \right) - \frac{1}{2} \right) \left\| \theta^{h+1} - z^h \right\|^2.
$$

From $t_2 \geq 3 \left( 2L^2 + \frac{2}{\beta^2} \right)$, $t_1 = \min \left\{ \frac{1}{2t_2}, \frac{1}{4t_2} \right\}$ and $\eta_h = \frac{t_1}{h^2}$, we know that $9 \left( 2L^2 + \frac{2}{\beta^2} \right) \eta_{h-1}^2 \leq \frac{3}{4}, 9 \left( 2L^2 + \frac{2}{\beta^2} \right)^2 \eta_{h-1}^2 \leq \frac{1}{16}$. This together with the above inequality gives

$$\langle \eta_h F_k(\theta^{h+1}), \theta^{h+1} - \theta \rangle \leq \frac{\|z^h - \theta\|^2}{2} - \frac{\|z^{h+1} - \theta\|^2}{2} + 2\eta_h D \left\| - \hat{F}_k(\theta^{h+1}) + F_k(\theta^{h+1}) \right\|
\quad + \frac{3\eta_h^2}{2} \left( \left\| \hat{F}_k(\theta^{h+1}) - F_k(\theta^{h+1}) \right\|^2 + \left\| F_k(\theta^h) - \hat{F}_k(\theta^h) \right\|^2 \right)
\quad + \frac{3}{4} \eta_h^2 \left\{ \left\| - \hat{F}_k(\theta^{h-1}) + F_k(\theta^{h-1}) \right\|^2 + \left\| F_k(\theta^h) - \hat{F}_k(\theta^h) \right\|^2 \right\}
\quad + \left( 3\eta_h^2 \left( 2L^2 + \frac{2}{\beta^2} \right) - \frac{1}{2} \right) \left\| \theta^{h+1} - z^h \right\|^2.
$$

According to Assumptions 1-3 and Lemma 28, $F_k(\theta)$ is $\left( \frac{1}{\beta} - L \right)$-strongly monotone and then SVI($F_k, \Theta$) has a unique solution (Kinderlehrer and Stampacchia, 1980), which is denoted by $\bar{\theta}_k$. Substituting $\bar{\theta}_k$ for $\theta$ in the above inequality implies

$$\langle \eta_h F_k(\theta^{h+1}), \theta^{h+1} - \bar{\theta}_k \rangle \leq \frac{\|z^h - \bar{\theta}_k\|^2}{2} - \frac{\|z^{h+1} - \bar{\theta}_k\|^2}{2} + 2\eta_h D \left\| - \hat{F}_k(\theta^{h+1}) + F_k(\theta^{h+1}) \right\|
\quad + \frac{3\eta_h^2}{2} \left( \left\| \hat{F}_k(\theta^{h+1}) - F_k(\theta^{h+1}) \right\|^2 + \left\| F_k(\theta^h) - \hat{F}_k(\theta^h) \right\|^2 \right)
\quad + \frac{3}{4} \eta_h^2 \left\{ \left\| - \hat{F}_k(\theta^{h-1}) + F_k(\theta^{h-1}) \right\|^2 + \left\| F_k(\theta^h) - \hat{F}_k(\theta^h) \right\|^2 \right\}
\quad + \left( 3\eta_h^2 \left( 2L^2 + \frac{2}{\beta^2} \right) - \frac{1}{2} \right) \left\| \theta^{h+1} - z^h \right\|^2.
$$

(36)

Similar to the proof of (17) in Lemma 19, we have

$$\eta_h \left( \frac{1}{\beta} - L \right) \frac{\|z^h - \bar{\theta}_k\|^2}{2} - \eta_h \left( \frac{1}{\beta} - L \right) \left\| z^h - \theta^{h+1} \right\|^2 \leq \langle \eta_h F_k(\theta^{h+1}), \theta^{h+1} - \bar{\theta}_k \rangle.
$$
Taking summation for both sides of (36) and the above inequality gives

\[
\frac{\|z^{h+1} - \overline{\theta}_k\|^2}{2} \leq \left(1 - \eta_h \left(\frac{1}{\beta} - L\right)\right) \frac{\|z^h - \overline{\theta}_k\|^2}{2} + 2\eta_h D \left\|\hat{F}_k(\theta^{h+1}) + F_k(\theta^{h+1})\right\|
+ \frac{3\eta_h^2}{2} \left\|\hat{F}_k(\theta^{h+1}) - F_k(\theta^{h+1})\right\|^2 + \left\|F_k(\theta^h) - \hat{F}_k(\theta^h)\right\|^2
+ \frac{3}{4} \eta_h^2 \left\|F_k(\theta^{h-1}) + F_k(\theta^{h-1})\right\|^2 + \left\|F_k(\theta^h) - \hat{F}_k(\theta^h)\right\|^2
+ \frac{1}{16} \eta_h^2 \left\|\theta^{h-1} - \theta^h\right\|^2 + \left[3\eta_h^2 \left(2L^2 + \frac{2}{\beta^2}\right) + \eta_h \left(\frac{1}{\beta} - L\right) - \frac{1}{2}\right] \left\|\theta^{h+1} - z^h\right\|^2.
\]

From \(l_2 \geq \max \left\{3 \left(2L^2 + \frac{2}{\beta^2}\right), \left(\frac{1}{\beta} - L\right)\right\}\), \(l_1 = \min \left\{\frac{1}{2\sqrt{\beta}}, \frac{1}{\beta - L}\right\}\) and \(\eta_h = \frac{l_1}{h^4}\), we know that \(3\eta_h^2 \left(2L^2 + \frac{2}{\beta^2}\right) + \eta_h \left(\frac{1}{\beta} - L\right) \leq l_2^2 \frac{1}{h^2} - l_2 \frac{1}{h} - l_2 \frac{l_1}{h^2} + l_2 - l_2 \frac{l_1}{h^2} \leq l_2^2 + l_2 - l_2 \frac{l_1}{h^2} \leq \frac{1}{2}\), that is, \(3\eta_h^2 \left(2L^2 + \frac{2}{\beta^2}\right) + \eta_h \left(\frac{1}{\beta} - L\right) - \frac{1}{2} < 0\). This together with the above inequality leads to

\[
\frac{\|z^{h+1} - \overline{\theta}_k\|^2}{2} \leq \left(1 - \eta_h \left(\frac{1}{\beta} - L\right)\right) \frac{\|z^h - \overline{\theta}_k\|^2}{2} + 2\eta_h D \left\|\hat{F}_k(\theta^{h+1}) + F_k(\theta^{h+1})\right\|
+ \frac{3\eta_h^2}{2} \left\|\hat{F}_k(\theta^{h+1}) - F_k(\theta^{h+1})\right\|^2 + \left\|F_k(\theta^h) - \hat{F}_k(\theta^h)\right\|^2
+ \frac{3}{4} \eta_h^2 \left\|F_k(\theta^{h-1}) + F_k(\theta^{h-1})\right\|^2 + \left\|F_k(\theta^h) - \hat{F}_k(\theta^h)\right\|^2
+ \frac{1}{16} \eta_h^2 \left\|\theta^{h-1} - \theta^h\right\|^2.
\]

Denote \(A_k(\theta) = \left\{ w \in \Omega : \|\hat{F}_k(\theta) - F_k(\theta)\|^2 \leq M(T, K_1, \delta) \right\}\) and \(A_k = A_k(\theta^{H_k-1}) \cap A_k(\theta^{H_k}) \cap A_k(z^{H_k+1})\). By Assumptions 1 and Lemma 21, we know that \(P\{A_k(\theta)\} \geq 1 - \frac{\epsilon}{\pi T}\), for any \(\theta \in \Theta\). Hence, we have \(P\{A_k\} \geq 1 - \frac{\epsilon}{\pi T}\). Then, by (37), we obtain

\[
\frac{\|z^{H_k+1} - \overline{\theta}_k\|^2}{2} \leq \left(1 - \eta_{H_k} \left(\frac{1}{\beta} - L\right)\right) \frac{\|z^{H_k} - \overline{\theta}_k\|^2}{2} + 2\eta_{H_k} D\sqrt{M} + \frac{9\eta_{H_k}^2}{2} M + \frac{D^2}{4} \eta_{H_k}^2 \forall w \in A_k.
\]

Combining \(\eta_{H_k} \left(\frac{1}{\beta} - L\right) \geq 0\) and the above inequality with Lemma 34 gives

\[
\|z^{H_k+1} - \overline{\theta}_k\| \leq \left(\frac{4D\sqrt{M}}{\frac{1}{\beta} - L}\right)^{\frac{1}{2}} + \left(\frac{l_1 \left(9M + \frac{D^2}{2}\right)}{\left(\frac{1}{\beta} - L\right)}\right)^{\frac{1}{2}} \frac{1}{H_k^2} + (\phi(H_k))^\frac{1}{2}, \forall w \in A_k, \quad (38)
\]
where \( \phi(H_k) = o\left(\frac{1}{H_k^2}\right) \). Denote \( \tilde{\theta}_{k+1} = \arg \min_{\theta \in \Theta} \left\{ \langle 2\tilde{\eta}F_k(z^{H_k+1}), \theta \rangle + \| \theta - z^{H_k+1} \|^2 \right\} \), where \( \tilde{\eta} = \frac{1}{2\sqrt{L^2 + \beta^2}} \). According to the non-expansion property of the proximal mapping in Lemma 31 (iii), we have

\[
\| \theta_{k+1} - \tilde{\theta}_{k+1} \|^2 \leq \tilde{\eta}^2 \| F_k(z^{H_k+1}) - \hat{F}_k(z^{H_k+1}) \|^2. \tag{39}
\]

By Assumptions 1-3, Lemma 28 and Lemma 32, we have

\[
\langle F_k(\theta_{k+1}), \theta_{k+1} - \theta \rangle = \langle F_k(\theta_{k+1}) - F_k(\tilde{\theta}_{k+1}) + F_k(\tilde{\theta}_{k+1}), \theta_{k+1} - \tilde{\theta}_{k+1} + \tilde{\theta}_{k+1} - \theta \rangle \\
= \langle F_k(\theta_{k+1}) - F_k(\tilde{\theta}_{k+1}), \theta_{k+1} - \tilde{\theta}_{k+1} \rangle + \langle F_k(\tilde{\theta}_{k+1}), \theta_{k+1} - \tilde{\theta}_{k+1} \rangle \\
+ \langle F_k(\theta_{k+1}) - F_k(\tilde{\theta}_{k+1}), \tilde{\theta}_{k+1} - \theta \rangle + \langle F_k(\tilde{\theta}_{k+1}), \tilde{\theta}_{k+1} - \theta \rangle \\
\leq 2 \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} D \| \theta_{k+1} - \tilde{\theta}_{k+1} \| + 2 \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} D \| \theta_{k+1} - \tilde{\theta}_{k+1} \| \\
+ \| F_k(\tilde{\theta}_{k+1}) \| \| \theta_{k+1} - \tilde{\theta}_{k+1} \| + 2D \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} (2 + \sqrt{2}) \| z^{H_k+1} - \tilde{\theta}_k \|. \tag{40}
\]

Similar to the proof of (26) in Lemma 19, by Assumptions 1-3, we know that \( \| F(\tilde{\theta}_{k+1}) \| \leq \sqrt{NB_k U_k} \), then \( \| F_k(\tilde{\theta}_{k+1}) \| \leq \left( 2 \| F(\tilde{\theta}_{k+1}) \|^2 + \frac{2}{\beta^2} \| \tilde{\theta}_{k+1} - \theta_k \| \right)^{1/2} \leq \left( \frac{2NB_k^2 U_k^2}{(1-\gamma)^2} + \frac{8D^2}{\beta^2} \right)^{1/2} \).

This together with (38), (39) and (40) gives

\[
\langle F_k(\theta_{k+1}), \theta_{k+1} - \theta \rangle \leq 4 \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} D \| \theta_{k+1} - \tilde{\theta}_{k+1} \| + \left( \frac{2NB_k^2 U_k^2}{(1-\gamma)^2} + \frac{8D^2}{\beta^2} \right)^{1/2} \\
\times \| \theta_{k+1} - \tilde{\theta}_{k+1} \| + 2D \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} (2 + \sqrt{2}) \| z^{H_k+1} - \tilde{\theta}_k \| \\
\leq C_3 \left( F_k(z^{H_k+1}) - \hat{F}_k(z^{H_k+1}) \right) + C_4 + C_5 \frac{1}{H_k} + C_6 (\phi(H_k))^{1/2}, \tag{41}
\]

where \( C_3 = \tilde{\eta} \left( 4 \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} D + \left( \frac{2NB_k^2 U_k^2}{(1-\gamma)^2} + \frac{8D^2}{\beta^2} \right)^{1/2} \right) \), \( C_4 = 2D \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} (2 + \sqrt{2}) \left( \frac{4D\sqrt{M}}{\beta - L} \right)^{1/2} \), \( C_5 = 2D \sqrt{2L^2 + \frac{2}{\beta^2} (2 + \sqrt{2}) \left( \frac{1}{\beta - L} \right)^{1/2}} \) and \( C_6 = 2D \left( 2L^2 + \frac{2}{\beta^2} \right) \frac{1}{2} (2 + \sqrt{2}) \). From \( A_k \subseteq A_k(z^{H_k+1}) \) and (41), it follows that

\[
\langle F_k(\theta_{k+1}), \theta_{k+1} - \theta \rangle \leq C_3 \sqrt{M} + C_4 + C_5 \frac{1}{H_k} + C_6 (\phi(H_k))^{1/2}, \forall \ w \in A_k.
\]

Recall that \( P\{A_k\} \geq 1 - \frac{\delta}{K_1} \). This together with the above inequality gives

\[
P \left\{ \langle F_k(\theta_{k+1}), \theta_{k+1} - \theta \rangle \leq C_3 \sqrt{M} + C_4 + C_5 \frac{1}{H_k} + C_6 (\phi(H_k))^{1/2} \right\} \geq P\{A_k\} \geq 1 - \frac{\delta}{K_1}.
\]
Noting that \( P \{ A_k \} \geq 1 - \frac{\delta}{K_1} \), we have
\[
P \left\{ \bigcap_{k=1}^{K} A_k \bigcap H_k = k \right\} = 1 - P \left\{ \bigcup_{k=1}^{K} A_k \right\} \geq 1 - \frac{K}{K_1} \delta. \quad (42)
\]
Noting that \( \gamma_k = \frac{1}{k^2} \) and \( H_k = k \), we denote \( \tilde{A} = \left\{ w \in \Omega : \sum_{k=1}^{K} \frac{1}{k^2} \| \theta_k - \bar{\theta}_k \|^2 \leq 2 \frac{K_2}{K} D^2 + \right. \]
\[
\left. \frac{2 \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=1}^{K} \frac{1}{k^{2}} \right) \right\}, \text{ where } P_k = C_3 \sqrt{M} + C_4 + C_5 \frac{1}{H_k} + C_6 (\phi(k))^{\frac{1}{2}}. \text{ By Assumption 5,}
\]
similar to the proof of (20)-(22) in Lemma 19, for any \( w \in \bigcap_{k=1}^{K} A_k \), we obtain that \( w \in \tilde{A} \). Hence, \( \bigcap_{k=1}^{K} A_k \subseteq \tilde{A} \). This together with (42) gives
\[
P \left\{ \tilde{A} \right\} \geq P \left\{ \bigcap_{k=1}^{K} A_k \right\} \geq 1 - \frac{K}{K_1} \delta. \quad (43)
\]
Noting that \( \sum_{k=1}^{K} \frac{1}{k^2} \geq \frac{1}{2} K_2^2 \) and \( \sum_{k=1}^{K} \frac{1}{k^{2}} \leq \frac{12}{11} K_1^{\frac{1}{2}} \), we have
\[
\frac{\sum_{k=1}^{K} \frac{1}{k^2} \| \theta_k - \bar{\theta}_k \|^2}{\sum_{k=1}^{K} \frac{1}{k^{2}}} \leq \frac{2 K_2^2 D^2}{\sum_{k=1}^{K} \frac{1}{k^{2}}} + \frac{2 \left( \frac{1}{\beta - L} + 2\beta \right) \sum_{k=1}^{K} \frac{1}{k^{2}} P_k}{\sum_{k=1}^{K} \frac{1}{k^{2}}} 
\leq \frac{5 D}{2} \frac{1}{K} + 2 \left( \frac{1}{\beta - L} + 2\beta \right) \left( C_3 \sqrt{M} + C_4 \right) 
+ \frac{30}{11} \left( \frac{1}{\beta - L} + 2\beta \right) C_5 \frac{1}{K^{\frac{1}{2}}} + (\phi(K))^{\frac{1}{2}}, \forall w \in \tilde{A}.
\]
Then, by Lyapunov inequality and the above inequality, we have
\[
\frac{\sum_{k=1}^{K} \frac{1}{k^2} \| \theta_k - \bar{\theta}_k \|^2}{\sum_{k=1}^{K} \frac{1}{k^{2}}} \leq \sqrt{\frac{\sum_{k=1}^{K} \frac{1}{k^2} \| \theta_k - \bar{\theta}_k \|^2}{\sum_{k=1}^{K} \frac{1}{k^{2}}}} \leq \sqrt{\frac{10}{2} D \frac{1}{K^2} + \sqrt{2} \left( \frac{1}{\beta - L} + 2\beta \right)^{\frac{1}{2}} \left( C_3 \sqrt{M} + C_4 \right)^{\frac{1}{2}} 
+ \sqrt{\frac{30}{11} \left( \frac{1}{\beta - L} + 2\beta \right) C_5 \frac{1}{K^{\frac{1}{2}}} + (\phi(K))^{\frac{1}{2}}}, \forall w \in \tilde{A}. \quad (44)
\]
By Assumptions 1-3, similar to the proof of (27) in Lemma 19, we have
\[
\langle F(\theta_k), \theta_k - \theta \rangle \leq \left( 4 L D + \frac{2 D}{\beta} + \frac{\sqrt{N} B \phi U_R}{(1 - \gamma)^2} \right) \| \theta_k - \bar{\theta}_k \|. 
\]
Then combining (44) with the above inequality gives
\[
\frac{\sum_{k=1}^{K} \frac{1}{k^2} \left( \sup_{\theta \in \Theta} \langle F(\theta_k), \theta_k - \theta \rangle \right)}{\sum_{k=1}^{K} \frac{1}{k^{2}}} \leq \left( 4 L D + \frac{2 D}{\beta} + \frac{\sqrt{N} B \phi U_R}{(1 - \gamma)^2} \right) \frac{\sum_{k=1}^{K} \frac{1}{k^2} \| \theta_k - \bar{\theta}_k \|}{\sum_{k=1}^{K} \frac{1}{k^{2}}}
\]
\[ L_1 \frac{1}{K^2} + L_2 + L_3 \frac{1}{K} + (\phi(K))^{\frac{1}{2}}, \forall w \in \bar{A}, \]  

(45)

where \( L_1 = \sqrt{\frac{\gamma}{2}} D \left( 4LD + \frac{2D}{\beta} + \sqrt{NB_uU_u} \right) \), \( L_2 = \sqrt{2} \left( \frac{1}{\gamma} - L \right)^{\frac{1}{2}} \left( 4LD + \frac{2D}{\beta} \right) \)

\[ + \sqrt{\frac{NB_uU_u}{(1-\gamma)^2}} \right) \left( C_3 \sqrt{M} + C_4 \right)^{\frac{1}{2}} \text{ and } L_3 = \sqrt{\frac{\gamma}{11}} \left( 4LD + \frac{2D}{\beta} + \sqrt{NB_uU_u} \right) \left( \frac{1}{\gamma} - L \right)^{\frac{1}{2}} \frac{2C_5}{\beta}. \]

By Assumptions 1-4 and Lemma 11, we have

\[ \sum_{k=1}^{K} k^\frac{1}{2} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k}, \theta_{-i,k})}(t) \right) \]

\[ \leq \sup_{i \in N} \frac{\sum_{k=1}^{K} k^\frac{1}{2} \sum_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k}, \theta_{-i,k})}(t) \right)}{\sum_{k=1}^{K} k^\frac{1}{2}} \]

Noting that \( \sup_{\theta_i \in \Theta_i} \langle F_i(\theta_k), \theta_{i,k} - \theta_i \rangle \leq \sup_{\theta \in \Theta} \langle F(\theta_k), \theta_k - \theta \rangle \), for any countable set \( \widetilde{\Theta}_i \subseteq \Theta_i \), it follows that \( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k}, \theta_{-i,k})}(t) \leq \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k}, \theta_{-i,k})}(t) \). This together with the above inequality leads to

\[ \sum_{k=1}^{K} k^\frac{1}{2} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k}, \theta_{-i,k})}(t) \right) \]

\[ \leq \sup_{i \in N} \frac{\sum_{k=1}^{K} k^\frac{1}{2} \sum_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k}, \theta_{-i,k})}(t) \right)}{\sum_{k=1}^{K} k^\frac{1}{2}} \]

\[ \leq \sup_{i \in N} \frac{\sum_{k=1}^{K} k^\frac{1}{2} \sum_{\theta_i \in \Theta_i} \langle F(\theta_k), \theta_k - \theta \rangle \right)}{\sum_{k=1}^{K} k^\frac{1}{2}} \]

Hence, by (45) and the above inequality, we have

\[ \sum_{k=1}^{K} k^\frac{1}{2} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k})}(t) \right) \leq L(K, K_1, T, \delta) + (\phi(K))^{\frac{1}{2}}, \forall w \in \bar{A}. \]

Then, from (43) and the above inequality, we have

\[ P \left\{ \sum_{k=1}^{K} k^\frac{1}{2} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{i,k})}(t) - J_i^{(\theta_i, \theta_{i,k})}(t) \right) \leq L(K, K_1, T, \delta) + (\phi(K))^{\frac{1}{2}} \right\} \geq 1 - \frac{K}{K_1}. \]
that is, Lemma 22 holds.

**Theorem 23** If Assumptions 1-5 hold, and we choose \( \eta_h = \frac{l_1}{h^2}, \gamma_k = k^{\frac{1}{2}}, H_k = k \) and \( K_1 = K \) in Algorithm 2, then \( \left( (\pi_{\theta, i, k})_{i=1}^{N} \right)_{k=1}^{\infty} \) given by Algorithm 2 is a \( k^{\frac{1}{4}} \)-weighted asymptotic Nash equilibrium of the game \( \Gamma \) in probability.

**Proof** If Assumptions 1-5 hold, then by Lemma 22, for any countable set \( \tilde{\Theta}_i \subseteq \Theta_i \), we have

\[
P\left\{ \sup_{i \in \mathcal{N}} \frac{\sum_{k=1}^{K} k^{\frac{1}{4}} \left( \sup_{\theta_{i} \in \tilde{\Theta}_i} J_{i}^{(\theta_{i}, \theta_{-i, k})}(t) - J_{i}^{(\theta_{i})}(t) \right)}{\sum_{k=1}^{K} k^{\frac{1}{4}}} \leq R_1(T, K) + R_2(K) \right\} \geq 1 - \delta,
\]

where \( R_1(T, K) = \sqrt{2}M_1 \left( \frac{1}{\beta - L} + 2\beta \right)^{\frac{1}{2}} \left( 4LD + \frac{\sqrt{N}BaUR}{(1-\gamma)^2} + \frac{2D}{\beta} \right) \left( \frac{M_1}{\sqrt{K}} + \sqrt{\frac{10}{11} \left( \frac{1}{\beta - L} + 2\beta \right)^{\frac{1}{2}} \left( 4LD + \frac{2D}{\beta} + \frac{\sqrt{N}BaUR}{(1-\gamma)^2} \right)} \right)^{\frac{1}{2}} \), \( R_2(K) = \frac{\sqrt{10D}}{2} \left( \left( \frac{1}{\beta - L} + 2\beta \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \phi(K) \right)^{\frac{1}{2}}. \) From \( M = M(T, K, \delta) = 2N(B \alpha U_R)^2 \left( \frac{T+1}{T+\gamma} + \frac{\gamma}{(T+1)^2} \right)^{-1} + 16 \log \left( \frac{8K}{\delta} \right) \frac{N^2 B^2 \delta^2 \alpha^2}{(1-\gamma)^4 \gamma^2 K} \) and \( \gamma \in (0, 1), \) we have \( \lim_{T \to \infty, K \to \infty} R_1(T, K) = 0. \) Therefore, for any \( \epsilon > 0, \) there exist \( \tilde{T} > 0 \) and \( K_2 > 0 \) such that \( R_1(T, K) \leq \frac{\epsilon}{2} \) if \( T \geq \tilde{T} \) and \( K \geq K_2. \) Recalling that \( (\phi(K))^{\frac{1}{2}} = o \left( \frac{1}{K^{\frac{1}{2}}} \right) \), we have \( \lim_{K \to \infty} R_2(K) = 0. \) Therefore, there exists \( K_3 > 0 \) such that \( R_2(K) \leq \frac{\epsilon}{2} \) if \( K \geq K_3. \) To sum up, for any \( \delta \in (0, 1] \) and \( \epsilon > 0, \) there exist \( \tilde{T} \) and \( \tilde{K} = \max \{ K_2, K_3 \} \) such that \( R_1(T, K) + R_2(K) \leq \epsilon \) if \( T \geq \tilde{T} \) and \( K \geq \tilde{K}. \) This together with (46) gives

\[
P\left\{ \sup_{i \in \mathcal{N}} \frac{\sum_{k=1}^{K} k^{\frac{1}{4}} \left( \sup_{\theta_{i} \in \tilde{\Theta}_i} J_{i}^{(\theta_{i}, \theta_{-i, k})}(t) - J_{i}^{(\theta_{i}, \theta_{-i, k})}(t) \right)}{\sum_{k=1}^{K} k^{\frac{1}{4}}} \leq \epsilon \right\} \geq 1 - \delta.
\]

Hence, from Definition 5, we know \( \left( (\pi_{\theta, i, k})_{i=1}^{N} \right)_{k=1}^{\infty} \) is a \( k^{\frac{1}{4}} \)-weighted asymptotic Nash equilibrium of the game \( \Gamma \) in probability. □
Remark 24 For the stochastic game with the finite state and action space, if we consider
direct parameterization, that is, \( \pi_\theta(a_i | s) = \theta_{s,a_i} \), where \( \theta_{s,a_i} \geq 0 \) and \( \sum_{a_i \in \mathcal{A}_i} \theta_{s,a_i} = 1 \),
then the direct parameterization doesn’t meet with the conditions in Assumption 3. However,
there exists a similar conclusion to Lemma 18 in Lemma 7 in Zhang et al. (2021). To be
more exact, constant \( L \) changes into \( \frac{2\sup_{A} \sum_{i=1}^{n} |\mathcal{A}_i|}{1-\gamma} \), where \( |\mathcal{A}_i| \) is the number of actions of
agent \( i \). If we consider \( \alpha \)-greedy direct parameterization, the parameterization doesn’t meet
with the conditions in Assumption 3 neither. we can still come to the same conclusion as
Lemma 18 by means of the proof of Lemma 7 in Zhang et al. (2021).

For the case with finite state and action space, direct parameterization satisfies gradient
dominant theorem from Lemma 3 in Zhang et al. (2021). Since \( \alpha \)-greedy direct parameter-
ization satisfies Assumption 2, then gradient domination theorem holds for \( \alpha \)-greedy direct
parameterization by Lemma 11. If Assumption 4 holds, the Nash equilibrium problem can be
equivalent to SVI(\( F, \Theta \)) under the two classes of parameterization. Furthermore, we have
illustrated that the two classes of parameterization satisfies Lemma 18. So, Algorithm 1 can
be used in the context of the two kinds of parameterization. Thus, Theorem 20 holds for
the two kinds of parameterization. Similarly, Algorithm 2 can be used under \( \alpha \)-greedy direct
parameterization.

6. Numerical Example

Consider a two-person stochastic game, where the set of agents, the state space and the
action space are given by \( \mathcal{N} = \{1,2\}, \mathcal{S} = \{1,2\}, \mathcal{A}_1 = \mathcal{A}_2 = \{1,2\} \). We assume
that the immediate rewards of agents are independent of the state and are given in Table 1.
The discount factor \( \gamma \) is taken as 0.9. We consider \( \alpha \)-greedy direct parameterization,
that is, \( \pi_\theta(a_i | s) = \theta_{s,a_i} \), where \( \theta_{s,a_i} \geq 0 \) and \( \sum_{a_i \in \mathcal{A}_i} \theta_{s,a_i} = 1 \),
then the direct parameterization doesn’t meet with the conditions in Assumption 3. However,
there exists a similar conclusion to Lemma 18 in Lemma 7 in Zhang et al. (2021). To be
more exact, constant \( L \) changes into \( \frac{2\sup_{A} \sum_{i=1}^{n} |\mathcal{A}_i|}{1-\gamma} \), where \( |\mathcal{A}_i| \) is the number of actions of
agent \( i \). If we consider \( \alpha \)-greedy direct parameterization, the parameterization doesn’t meet
with the conditions in Assumption 3 neither. we can still come to the same conclusion as
Lemma 18 by means of the proof of Lemma 7 in Zhang et al. (2021).

For the case with finite state and action space, direct parameterization satisfies gradient
dominant theorem from Lemma 3 in Zhang et al. (2021). Since \( \alpha \)-greedy direct parameter-
ization satisfies Assumption 2, then gradient domination theorem holds for \( \alpha \)-greedy direct
parameterization by Lemma 11. If Assumption 4 holds, the Nash equilibrium problem can be
equivalent to SVI(\( F, \Theta \)) under the two classes of parameterization. Furthermore, we have
illustrated that the two classes of parameterization satisfies Lemma 18. So, Algorithm 1 can
be used in the context of the two kinds of parameterization. Thus, Theorem 20 holds for
the two kinds of parameterization. Similarly, Algorithm 2 can be used under \( \alpha \)-greedy direct
parameterization.

Consider a two-person stochastic game, where the set of agents, the state space and the
action space are given by \( \mathcal{N} = \{1,2\}, \mathcal{S} = \{1,2\}, \mathcal{A}_1 = \mathcal{A}_2 = \{1,2\} \). We assume
that the immediate rewards of agents are independent of the state and are given in Table 1.
The discount factor \( \gamma \) is taken as 0.9. We consider \( \alpha \)-greedy direct parameterization,
that is, \( \pi_\theta(a_i | s) = \theta_{s,a_i} \), where \( \theta_{s,a_i} \geq 0 \) and \( \sum_{a_i \in \mathcal{A}_i} \theta_{s,a_i} = 1 \),
then the direct parameterization doesn’t meet with the conditions in Assumption 3. However,
there exists a similar conclusion to Lemma 18 in Lemma 7 in Zhang et al. (2021). To be
more exact, constant \( L \) changes into \( \frac{2\sup_{A} \sum_{i=1}^{n} |\mathcal{A}_i|}{1-\gamma} \), where \( |\mathcal{A}_i| \) is the number of actions of
agent \( i \). If we consider \( \alpha \)-greedy direct parameterization, the parameterization doesn’t meet
with the conditions in Assumption 3 neither. we can still come to the same conclusion as
Lemma 18 by means of the proof of Lemma 7 in Zhang et al. (2021).

For the case with finite state and action space, direct parameterization satisfies gradient
dominant theorem from Lemma 3 in Zhang et al. (2021). Since \( \alpha \)-greedy direct parameter-
ization satisfies Assumption 2, then gradient domination theorem holds for \( \alpha \)-greedy direct
parameterization by Lemma 11. If Assumption 4 holds, the Nash equilibrium problem can be
equivalent to SVI(\( F, \Theta \)) under the two classes of parameterization. Furthermore, we have
illustrated that the two classes of parameterization satisfies Lemma 18. So, Algorithm 1 can
be used in the context of the two kinds of parameterization. Thus, Theorem 20 holds for
the two kinds of parameterization. Similarly, Algorithm 2 can be used under \( \alpha \)-greedy direct
parameterization.

Consider a two-person stochastic game, where the set of agents, the state space and the
action space are given by \( \mathcal{N} = \{1,2\}, \mathcal{S} = \{1,2\}, \mathcal{A}_1 = \mathcal{A}_2 = \{1,2\} \). We assume
that the immediate rewards of agents are independent of the state and are given in Table 1.
The discount factor \( \gamma \) is taken as 0.9. We consider \( \alpha \)-greedy direct parameterization,
that is, \( \pi_\theta(a_i | s) = \theta_{s,a_i} \), where \( \theta_{s,a_i} \geq 0 \) and \( \sum_{a_i \in \mathcal{A}_i} \theta_{s,a_i} = 1 \),
then the direct parameterization doesn’t meet with the conditions in Assumption 3. However,
there exists a similar conclusion to Lemma 18 in Lemma 7 in Zhang et al. (2021). To be
more exact, constant \( L \) changes into \( \frac{2\sup_{A} \sum_{i=1}^{n} |\mathcal{A}_i|}{1-\gamma} \), where \( |\mathcal{A}_i| \) is the number of actions of
agent \( i \). If we consider \( \alpha \)-greedy direct parameterization, the parameterization doesn’t meet
with the conditions in Assumption 3 neither. we can still come to the same conclusion as
Lemma 18 by means of the proof of Lemma 7 in Zhang et al. (2021).

For the case with finite state and action space, direct parameterization satisfies gradient
dominant theorem from Lemma 3 in Zhang et al. (2021). Since \( \alpha \)-greedy direct parameter-
ization satisfies Assumption 2, then gradient domination theorem holds for \( \alpha \)-greedy direct
parameterization by Lemma 11. If Assumption 4 holds, the Nash equilibrium problem can be
equivalent to SVI(\( F, \Theta \)) under the two classes of parameterization. Furthermore, we have
illustrated that the two classes of parameterization satisfies Lemma 18. So, Algorithm 1 can
be used in the context of the two kinds of parameterization. Thus, Theorem 20 holds for
the two kinds of parameterization. Similarly, Algorithm 2 can be used under \( \alpha \)-greedy direct
parameterization.
Decentralized Policy Gradient Nash Equilibria Learning of Stochastic Games

Table 1: Reward Table

|   | $a_2 = 1$ | $a_2 = 2$ |
|---|-----------|-----------|
| $a_1 = 1$ | (3, 3)    | (4, 0)    |
| $a_1 = 2$ | (0, 4)    | (2, 2)    |

Figure 1: Curve of $\epsilon_{1,K}$ with respect to $K$ for the case with the exact pseudo gradients.

Figure 2: Curve of $\epsilon_{2,K}$ with respect to $K$ for the case with the unknown pseudo gradients.

For the case with the exact pseudo gradients (Algorithm 1), Figure 1 shows the curve of $\epsilon_{1,K}$ with respect to $K$. It can be seen that $\epsilon_{1,K}$ vanishes to 0 as $K$ increases, which implies $((\pi_{\theta_{1,k}})_{i=1}^{N})_{k=1}^{\infty}$ is a $k^{\frac{1}{2}}$-weighted asymptotic Nash equilibrium of the game, that is, Theorem 20 follows. For the case with the unknown pseudo gradients (Algorithm 2), the sampling time horizon of the G(PO)MDP estimator and the number of trajectories are taken as $T = 20$ and $K_1 = K + 1$. Figure 2 shows $\epsilon_{2,K}$ vanishes to 0 as $K$ increases. It can be seen from Figure 2 that $((\pi_{\theta_{1,k}})_{i=1}^{N})_{k=1}^{\infty}$ is a $k^{\frac{1}{4}}$-weighted asymptotic Nash equilibrium of the game with probability one, which is a better result comparing from Theorem 23 where $((\pi_{\theta_{1,k}})_{i=1}^{N})_{k=1}^{\infty}$ is a $k^{\frac{1}{4}}$-weighted asymptotic Nash equilibrium of the game in probability. Comparing Figure 1 with Figure 2, it can be seen that Algorithm 2 have a slower convergence rate than Algorithm 1.

7. Conclusion

The general-sum stochastic game with an unknown transition probability density function is investigated in this paper. Each agent only observes the environment state and its own reward and is unknown about the transition probability density function of the environment state and the others’ actions and rewards. We define the concepts of weighted asymptotic Nash equilibrium for a given sequence with probability 1 and in probability under policy parameterization and prove the equivalence between Nash equilibrium and variational inequality problems. We have proposed two-loop algorithms for the solution of the variational inequality problem for the cases with exact and unknown pseudo gradients, respectively. In the outer loop, we sequentially update the constructed strongly monotone variational inequality and we employ a single-call extra-gradient algorithm for solving the constructed strongly monotone variational inequality in the inner loop. It is shown that the algorithm is
convergent to the $k^{\frac{1}{2}}$-weighted asymptotic Nash equilibrium in the context of exact pseudo gradients. Further, in the context of unknown pseudo gradients, a decentralized algorithm is proposed by leveraging the G(PO)MDP gradient estimator of the pseudo gradient. Also, we provide the convergence guarantee to the $k^{\frac{1}{4}}$-weighted asymptotic Nash equilibrium in probability.

It is worth noting that our assumptions about parameterization do not hold for softmax parameterization. So, how to design an algorithm for learning Nash equilibria under softmax parameterization is a future direction. It is also worth considering how to make better-verified assumptions than Assumption 5 and choose appropriate parameterization so that pseudo gradients have better properties. In addition, we would consider how to design an algorithm to estimate the pseudo gradient with a lower computation complexity in future.

Appendix A Supplement Definitions and Lemmas

**Definition 25** (Facchinei and Pang, 2003) Mapping $G(x) : K \rightarrow \mathbb{R}^n$ is

(i) monotone: if

$$
\langle G(x) - G(x'), x - x' \rangle \geq 0, \quad \forall \ x, \ x' \in K;
$$

(ii) strongly monotone: if

$$
\langle G(x) - G(x'), x - x' \rangle > 0, \quad \forall \ x, \ x' \in K;
$$

(iii) $\mu$-strongly monotone: if there exists a constant $\mu > 0$, such that

$$
\langle G(x) - G(x'), x - x' \rangle \geq \mu \|x - x'\|^2, \quad \forall \ x, \ x' \in K.
$$

**Definition 26** (Facchinei and Pang, 2003) For SVI $(G, K)$, its prime gap function is $G_{\text{gap}}(x) \equiv \sup_{y \in K} G^T(x)(x - y), \ x \in K$ and its dual gap function is $G_{\text{dual}}(x) \equiv \sup_{y \in K} G^T(y)(x - y), \ x \in K$.

**Lemma 27** If Assumptions 1-3 hold, then for any $\theta \in \Theta$, $F_k(\theta)$ is $\sqrt{2L^2 + \frac{2}{\gamma^2}}$-Lipschitz continuous with respect to $\theta$, where $L$ is given by Lemma 18.

**Proof** If Assumptions 1 3 hold, then by Lemma 18, it follows that

$$
\|F_k(\theta) - F_k(\theta')\|^2 = \left\|F(\theta) - F(\theta') + \frac{1}{\gamma}(\theta - \theta')\right\|^2
\leq 2 \|F(\theta) - F(\theta')\|^2 + 2 \left\|\frac{1}{\gamma}(\theta - \theta')\right\|^2
\leq 2L^2 \|\theta - \theta'\|^2 + 2 \left\|\frac{1}{\gamma^2}\right\|^2 \|\theta - \theta'\|^2
\leq \left(2L^2 + \frac{2}{\gamma^2}\right) \|\theta - \theta'\|^2, \ \forall \ \theta, \ \theta' \in \Theta.
$$
Lemma 28  If Assumptions 1-3 hold, then $F_k(\theta)$ is $\left(\frac{1}{\beta} - L\right)$-strongly monotone with respect to $\theta \in \Theta$, where $L$ is given by Lemma 18.

Proof  If Assumptions 1-3 hold, by Lemma 18, it follows that

$$
\langle F_k(\theta) - F_k(\theta'), \theta - \theta' \rangle = \langle F(\theta) + \frac{1}{\beta}(\theta - \theta_k) - F(\theta') - \frac{1}{\beta}(\theta' - \theta_k), \theta - \theta' \rangle \\
\geq \langle F(\theta) - F(\theta'), \theta - \theta' \rangle + \frac{1}{\beta}\|\theta - \theta'\|^2 \\
\geq -L\|\theta - \theta'\|^2 + \frac{1}{\beta}\|\theta - \theta'\|^2 \\
\geq \left(\frac{1}{\beta} - L\right)\|\theta - \theta'\|^2, \forall \theta, \theta' \in \Theta.
$$

By Definition 25 (iii), the lemma is true. ■

Lemma 29  If Assumptions 1-3 hold and denote $\hat{\theta}$ as the solution of SVI($F_k, \Theta$), then

$$
\sup_{\theta \in \Theta} \left\langle F(\hat{\theta}), \hat{\theta} - \theta \right\rangle \leq \frac{2D}{\beta} \|\theta_k - \hat{\theta}\|.
$$

In particular, if $\theta = (\theta_i, \hat{\theta}_{-i})$, then

$$
\sup_{\hat{\theta} \in \Theta_i} \left\langle F_i(\hat{\theta}), \hat{\theta}_i - \theta_i \right\rangle \leq \frac{2D}{\beta} \|\theta_k - \hat{\theta}\|.
$$

Proof  Noting that $\hat{\theta}$ is the solution of SVI($F_k, \Theta$), we have $\sup_{\theta \in \Theta} \langle F_k(\hat{\theta}), \hat{\theta} - \theta \rangle = \sup_{\theta \in \Theta} \langle F_k(\hat{\theta}) + \frac{1}{\beta}(\hat{\theta} - \theta_k), \hat{\theta} - \theta \rangle \leq 0$, which together with Assumption 2 gives $\sup_{\theta \in \Theta} \langle F(\hat{\theta}), \hat{\theta} - \theta \rangle \leq \frac{1}{\beta} \langle \hat{\theta} - \theta_k, \hat{\theta} - \theta \rangle \leq \frac{2D}{\beta} \|\theta_k - \hat{\theta}\|$.
Lemma 33 (Pinelis, 1994) (Concentration inequality) If $X_1, X_2, ..., X_N \in \mathbb{R}^d$ denote a vector-valued martingale difference sequence satisfying $\|X_n\| \leq V$ and $E[X_n|X_1, ..., X_{n-1}] = 0, \forall n \in \{1, ..., N\}$, then for any $\delta \in (0, 1]$, we have

$$P\left\{ \sup_{n=1}^{N} \left\| \sum_{i=1}^{N} X_n \right\|^2 > 2 \log(2/\delta) V^2 N \right\} \leq \delta.$$ 

Lemma 34 (Chung, 1954) If $\{u_k, k \geq 1\}$ is a real sequence satisfying

$$u_{k+1} \leq \left(1 - \frac{c}{k^s}\right) u_k + \frac{d}{k^t}, \quad 0 < s < 1, \quad s < t, \quad c > 0, \quad d > 0,$$

then

$$u_k \leq \frac{d}{c} \frac{1}{k^{t-s}} + \phi(k),$$

where $\phi(k) = o\left(\frac{1}{k^{t-s}}\right)$.

Appendix B Proofs of Theorem 8, Theorem 17 and Lemma 18

Proof of Theorem 8: For any countable set $\widehat{\Theta}_i \subseteq \Theta_i$, by the property of conditional expectation, we have

$$E\left[ \sup_{\theta_i \in \widehat{\Theta}_i} J_{i}^{(\theta_i, \theta - i, \gamma_K)}(t) - J_{i}^{(\theta_i, \gamma_K, \theta - i, \gamma_K)}(t) \bigg| \theta_K, k = 1, ..., K \right]$$
Combining the above inequality with (46) gives

\[
E \left\{ \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, \tau_K)}(t) - J_i^{(\theta_i, \tau_K, \theta_{-i}, \tau_K)}(t) \mid \tau_{K}, \theta_k, k = 1, \ldots, K \right\} = \sum_{k=1}^{K} \frac{\gamma_k}{\sum_{k=1}^{K} \gamma_k} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, \tau)}(t) - J_i^{(\theta_i, \tau, \theta_{-i}, \tau)}(t) \right) P\{\tau_K = k\}.
\]

Noting that \(\tau_K\) is independent of \(\theta_k\), it follows that

\[
P\{\tau_K = k \mid \theta_k, k = 1, \ldots, K\} = P\{\tau_K = k\}.
\]

It’s easy to see that

\[
E \left\{ \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k, \theta_{-i}, k)}(t) \mid \tau_K = k, \theta_k, k = 1, \ldots, K \right\}.
\]

From the definition of total reward function, we have

\[
E \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k, \theta_{-i}, k)}(t) = \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k, \theta_{-i}, k)}(t).
\]

This together with (47) gives

\[
E \left\{ \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, k)}(t) - J_i^{(\theta_i, k, \theta_{-i}, k)}(t) \mid \tau_K = k, \theta_k, k = 1, \ldots, K \right\}
\]

Combining the above inequality with (46) gives

\[
E \left\{ \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, \tau_K)}(t) - J_i^{(\theta_i, \tau_K, \theta_{-i}, \tau_K)}(t) \mid \theta_k, k = 1, \ldots, K \right\} = \sum_{k=1}^{K} \frac{\gamma_k}{\sum_{k=1}^{K} \gamma_k} \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i}, \tau)}(t) - J_i^{(\theta_i, \tau, \theta_{-i}, \tau)}(t) \right) P\{\tau_K = k\}.
\]
Noting that \( \left( (\pi_{\theta_i})^N_{i=1} \right)_{k=1}^\infty \) is a \( \gamma_k \)-weighted \( \epsilon_k \)-Nash equilibrium of the game \( \Gamma \), it follows that

\[
\sup_{i \in N} \mathbb{E} \left[ \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i} = k)}(t) - J_i^{(\theta_k, \theta_{-i} = k)}(t) \bigg| \theta_k, k = 1, ..., K \right] = \sup_{i \in N} \frac{\sum_{k=1}^K \gamma_k \left( \sup_{\theta_i \in \Theta_i} J_i^{(\theta_i, \theta_{-i} = k)}(t) - J_i^{(\theta_k, \theta_{-i} = k)}(t) \right)}{\sum_{k=1}^K \gamma_k} \leq \epsilon_K,
\]

then the theorem holds.

\[\boxdot\]

**Proof of Theorem 17:** By Assumption 2 and Lemma 3.1 in Hartman and Stampacchia (1966), we know that SVI\((F, \Theta)\) has a solution. By Assumption 1 and Assumption 4, Lemma 16 implies the equivalence between the Nash equilibrium problem and SVI\((F, \Theta)\). Therefore, the game has a Nash equilibrium.

\[\boxdot\]

**Proof of Lemma 18:** For agent \( i \), by Assumptions 1-2, Lemma 3.2 in Zhang et al. (2020) and Proposition 10, we have

\[
\nabla_{\theta_i} V_i^\theta_l(s, t) = \sum_{l=1}^N \sum_{t=0}^\infty \gamma^{l+\tau-t} \int r_i(s(l + \tau), a(l + \tau)) \nabla_{\theta_i} \log \pi_{\theta_i}(a_i(l) \mid s(l)) \rho_{\theta_i,t:l+\tau}ds_{l+1}...ds_{l+\tau}da_{l+1}...da_{l+\tau},
\]

where \( \rho_{\theta_i,t:l+\tau} = \left[ \prod_{u=t}^{l+\tau-1} \rho(s(u + 1) \mid s(u), a(u)) \right] \left[ \prod_{u=t}^{l+\tau} \pi_{\theta_i}(a(u) \mid s(u)) \right] \) is the probability density function of the trajectory \( (s(t), a(t), ..., s(l + \tau), a(l + \tau)) \) and the integral is over all trajectories. Then, it follows that

\[
\left\| \nabla_{\theta_i} V_i^\theta_1(s, t) - \nabla_{\theta_i} V_i^\theta_2(s, t) \right\| \\
\leq \sum_{l=1}^\infty \sum_{t=0}^\infty \gamma^{l+\tau-t} \left( \int \left\| \nabla_{\theta_i} \log \pi_{\theta_i}^1(a_i(l) \mid s(l)) - \nabla_{\theta_i} \log \pi_{\theta_i}^2(a_i(l) \mid s(l)) \right\| \right. \\
\times r_i(s(l + \tau), a(l + \tau)) \rho_{\theta_i,t:l+\tau} + \int r_i(s(l + \tau), a(l + \tau)) \nabla_{\theta_i} \log \pi_{\theta_i}^2(a_i(l) \mid s(l)) \\
\left. \times (\rho_{\theta_i,t:l+\tau} - \rho_{\theta_i,t:l+\tau}^2) \right) \left| ds_{l+1}...ds_{l+\tau}da_{l+1}...da_{l+\tau} \right| \\
\leq \sum_{l=1}^\infty \sum_{t=0}^\infty \gamma^{l+\tau-t} \left( \int \left\| \nabla_{\theta_i} \log \pi_{\theta_i}^1(a_i(l) \mid s(l)) - \nabla_{\theta_i} \log \pi_{\theta_i}^2(a_i(l) \mid s(l)) \right\| \right. \\
\times |r_i(s(l + \tau), a(l + \tau))| \rho_{\theta_i,t:l+\tau}ds_{l+1}...ds_{l+\tau}da_{l+1}...da_{l+\tau} + \int |r_i(s(l + \tau), a(l + \tau))| \\
\times \left\| \nabla_{\theta_i} \log \pi_{\theta_i}^2(a_i(l) \mid s(l)) \right\| \left| \rho_{\theta_i,t:l+\tau} - \rho_{\theta_i,t:l+\tau}^2 \right| ds_{l+1}...ds_{l+\tau}da_{l+1}...da_{l+\tau}. \quad (48)
\]
Denote
\[
I_1 = \int \left| r_i (s(l + \tau), a(l + \tau)) \right| \left\| \nabla_{\theta_1} \log \pi_{\theta_1} (a_i(l) \mid s(l)) - \nabla_{\theta_2} \log \pi_{\theta_2} (a_i(l) \mid s(l)) \right\|
\times \rho_{\theta_1, t:t+\tau} ds_{t+1} \ldots ds_{t+\tau} da_{t+1} \ldots da_{t+\tau},
\]
\[
I_2 = \int \left| r_i (s(l + \tau), a(l + \tau)) \right| \left\| \nabla_{\theta_1} \log \pi_{\theta_1} (a_i(l) \mid s(l)) \right\|
\times \left| \rho_{\theta_1, t:t+\tau} - \rho_{\theta_2, t:t+\tau} \right| ds_{t+1} \ldots ds_{t+\tau} da_{t+1} \ldots da_{t+\tau}.
\]
(49)

For the term $I_1$, by Assumption 2, we have
\[
I_1 \leq UR_L \| \theta_1^1 - \theta_2^1 \|.
\]
(51)

For the term $I_2$, denote $U_{t+\tau} = \{ u : u = t, ..., l + \tau \}$. From the definitions of $\rho_{\theta_1, t:t+\tau}$ and $\rho_{\theta_2, t:t+\tau}$, we have
\[
\rho_{\theta_1, t:t+\tau} - \rho_{\theta_2, t:t+\tau} = \prod_{u=t}^{t+\tau-1} \rho (s(u + 1) \mid s(u), a(u))
\times \left[ \prod_{u \in U_{t+\tau}} \pi_{\theta_1} (a(u) \mid s(u)) - \prod_{u \in U_{t+\tau}} \pi_{\theta_2} (a(u) \mid s(u)) \right].
\]
(52)

From Assumption 3 and Taylor expansion of $\prod_{u \in U_{t+\tau}} \pi_{\theta} (a(u) \mid s(u))$ near $\theta = \theta_1$, there exists some $\hat{\theta} = \lambda \theta_1 + (1 - \lambda) \theta_2$ and $\lambda \in [0, 1]$, such that
\[
\left| \prod_{u \in U_{t+\tau}} \pi_{\theta_1} (a(u) \mid s(u)) - \prod_{u \in U_{t+\tau}} \pi_{\theta_2} (a(u) \mid s(u)) \right|
= (\theta_1 - \theta_2)^T \left[ \sum_{m \in U_{t+\tau}} \nabla \log \pi_{\hat{\theta}} (a(m) \mid s(m)) \prod_{u \in U_{t+\tau}, u \neq m} \pi_{\hat{\theta}} (a(u) \mid s(u)) \right]
= (\theta_1 - \theta_2)^T \left[ \sum_{m \in U_{t+\tau}} \nabla \log \pi_{\hat{\theta}} (a(m) \mid s(m)) \prod_{u \in U_{t+\tau}} \pi_{\hat{\theta}} (a(u) \mid s(u)) \right]
\leq \| \theta_1 - \theta_2 \| \sum_{m \in U_{t+\tau}} \| \nabla \log \pi_{\hat{\theta}} (a(m) \mid s(m)) \| \prod_{u \in U_{t+\tau}} \pi_{\hat{\theta}} (a(u) \mid s(u))
\leq \| \theta_1 - \theta_2 \| (l + \tau - t + 1)B_\Theta \sqrt{N} \prod_{u \in U_{t+\tau}} \pi_{\hat{\theta}} (a(u) \mid s(u)),
\]
where $\nabla \log \pi_{\hat{\theta}} (a(u) \mid s(u)) = (\nabla_{\theta_i} \log \pi_{\theta_i} (a_i(u) \mid s(u)))_{i=1}^N$. From (50), (52), the above inequality and Assumption 3, it follows that
\[
I_2 \leq \| \theta_1 - \theta_2 \| UR \sqrt{N} B_\Theta^2 \int \left[ \prod_{u=t}^{t+\tau-1} \rho (s(u + 1) \mid s(u), a(u)) \right] (l + \tau - t + 1)
\]
41
By (48), (51) and the above inequality, we have

\[ H \leq \frac{1}{2}|\theta_1 - \theta_2|^2 U_R \sqrt{N} B_\theta^2 (l + \tau - t + 1). \]

Hence, we have

\[ F(\theta_1) - F(\theta_2) \leq L \| \theta_1 - \theta_2 \|. \]

This together with Assumption 2 and Proposition 10 gives

\[ \| \nabla_{\theta_i} V_i^{\theta_1} (s, t) - \nabla_{\theta_i} V_i^{\theta_2} (s, t) \| \leq \sum_{l=1}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{l+\tau-t} (I_1 + I_2) \]
\[ \leq \sum_{l=1}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{l+\tau-t} \left( U_R L_\theta \| \theta_1 - \theta_2 \| + (l + \tau - t + 1) U_R \sqrt{N} B_\theta^2 \| \theta_1 - \theta_2 \| \right) \]
\[ \leq \frac{1}{(1 - \gamma)^2} U_R L_\theta \| \theta_1 - \theta_2 \| + \frac{1 + \gamma}{(1 - \gamma)^3} U_R \sqrt{N} B_\theta^2 \| \theta_1 - \theta_2 \|. \]

Hence, we have

\[ \sum_{i=1}^N \| \nabla_{\theta_i} J_i^{\theta_1} (t) - \nabla_{\theta_i} J_i^{\theta_2} (t) \| \]
\[ \leq 2 \sum_{i=1}^N \left( \frac{U_R \cdot L_\theta}{(1 - \gamma)^2} \right)^2 \| \theta_1 - \theta_2 \| \]
\[ \leq 2 \left( \frac{U_R L_\theta}{(1 - \gamma)^2} \right)^2 \| \theta_1 - \theta_2 \| \]
\[ \leq 2 \left( \frac{U_R L_\theta}{(1 - \gamma)^2} \right)^2 \| \theta_1 - \theta_2 \| \]
\[ \leq \left[ 2 \left( \frac{U_R L_\theta}{(1 - \gamma)^2} \right)^2 + 2 \left( \frac{1 + \gamma}{(1 - \gamma)^3} U_R \sqrt{N} B_\theta^2 \right)^2 \| \theta_1 - \theta_2 \|^2 \right]. \]

that is,

\[ \| F(\theta_1) - F(\theta_2) \| \leq L \| \theta_1 - \theta_2 \|. \]

References

A. Agarwal, S. M. Kakade, J. D. Lee, and G. Mahajan. Optimality and approximation with policy gradient methods in Markov decision processes. In Proceedings of the 33rd Conference on Learning Theory, pages 64–66, Graz, Austria, 2020.
J. Baxter and P. L. Bartlett. Infinite-horizon policy-gradient estimation. *Journal of Artificial Intelligence Research*, 15:319–350, 2001.

T. Y. Chen, K. Zhang, G. B. Giannakis, and T. Başar. Communication efficient policy gradient methods for distributed reinforcement learning. *IEEE Transactions on Control of Network Systems*, forthcoming, 2021.

K. L. Chung. On a stochastic approximation method. *Annals of Mathematical Statistics*, 25(3):463–483, 1954.

C. Daskalakis, A. Mehta, and C. Papadimitriou. A note on approximate Nash equilibria. In *Proceedings of the 2nd International Workshop on Internet and Network Economics*, pages 297–306, Patras, Greece, 2006.

C. Daskalakis, D. J. Foster, and N. Golowich. Independent policy gradient methods for competitive reinforcement learning. *Advances in Neural Information Processing Systems*, 33:5527–5540, 2020.

F. Facchinei and J. S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. I*. Springer-Verlag, New York, USA, 2003.

J. A. Filar and K. Vrieze. *Competitive Markov Decision Processes*. Springer Science and Business Media, Berlin, German, 2012.

J. A. Filar, T. A. Schultz, F. Thuijsman, and O. J. Vrieze. Nonlinear programming and stationary equilibria in stochastic games. *Mathematical Programming*, 50(1):227–237, 1991.

A. Greenwald, K. Hall, and R. Serrano. Correlated Q-learning. In *Proceedings of the 20th International Conference on International Conference on Machine Learning*, volume 20, pages 242–249, Washington, USA, 2003.

P. Hartman and G. Stampacchia. On some nonlinear elliptic differential-functional equations. *Acta Mathematica*, 115:271–310, 1966.

A. Hoffman and R. Karp. On nonterminating stochastic games. *Management Science*, 12 (5):359–370, 1966.

Y. G. Hsieh, J. Malick F. Iutzeler, and P. Mertikopoulos. On the convergence of single-call stochastic extra-gradient methods. In *Proceedings of the 33rd International Conference on Neural Information Processing Systems*, pages 6938–6948, Virtual, 2019.

J. Hu and M. P. Wellman. Multiagent reinforcement learning: theoretical framework and an algorithm. In *Proceedings of the 15th International Conference on Machine Learning*, volume 98, pages 242–250, Wisconson, USA, 1998.

J. Hu and M. P. Wellman. Nash Q-learning for general-sum stochastic games. *Journal of Machine Learning Research*, 4:1039–1069, 2003.

D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications*. Academic Press, New York, USA, 1980.
V. Konda and J. Tsitsiklis. Actor-Critic Algorithms. Advances in Neural Information Processing Systems, pages 1008–1014, 1999.

J. Koshal, A. Nedić, and U. V. Shanbhag. Single timescale regularized stochastic approximation schemes for monotone Nash games under uncertainty. In Proceedings of the 49th IEEE Conference on Decision and Control, page 231–236, Georgia, USA, 2010.

J. Koshal, A. Nedić, and U. V. Shanbhag. Regularized iterative stochastic approximation methods for stochastic variational inequality problems. IEEE Transactions on Automatic Control, 58(3):594–609, 2013.

S. Leonardos, W. Overman, I. Panageas, and G. Piliouras. Global convergence of multi-agent policy gradient in Markov potential games. arXiv preprint arXiv:2106.01969, 2021.

M. L. Littman. Markov games as a framework for multi-agent reinforcement learning. In Proceedings of the 11th International Conference on Machine Learning, pages 157–163, New Jersey, USA, 1994.

M. L. Littman. Friend-or-Foe Q-learning in general-sum games. In Proceedings of the 18th International Conference on Machine Learning, volume 1, pages 322–328, Massachusetts, USA, 2001.

M. Liu, H. Rafique, Q. Lin, and T. Yang. First-order convergence theory for weakly-convex-weakly-concave min-max problems. Journal of Machine Learning Research, 22(169):1–34, 2021.

Y. Liu, K. Zhang, T. Başar, and W. Yin. An improved analysis of (variance-reduced) policy gradient and natural policy gradient methods. Advances in Neural Information Processing Systems, 33:7624–7636, 2020.

S. Lu, K. Zhang, T. Chen, T. Başar, and L. Horesh. Decentralized policy gradient descent ascent for safe multi-agent reinforcement learning. In Proceedings of the 35th AAAI Conference on Artificial Intelligence, volume 35, pages 8767–8775, Virtual, 2021.

W. C. Mao and T. Başar. Provably efficient reinforcement learning in decentralized general-sum Markov games. Dynamic Games and Applications, pages 1–22, 2022.

G. J. Minty. Monotone (nonlinear) operators in Hilbert space. Duke Mathematical Journal, 29(3):341–346, 1962.

J. F. Nash. Non-cooperative games. Annals of Mathematics, 54(2):286–295, 1951.

J. Perolat, B. Piot, and O. Pietquin. Actor-Critic fictitious play in simultaneous move multistage games. In Proceedings of the 21st International Conference on Artificial Intelligence and Statistics, pages 919–928, Playa Blanca, Lanzarote, 2018.

I. Pinelis. Optimum bounds for the distributions of martingales in Banach spaces. The Annals of Probability, 22(4):1679–1706, 1994.
H. L. Prasad, L. A. Prashanth, and S. Bhatnagar. Two-timescale algorithms for learning Nash equilibria in general-sum stochastic games. In Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems, pages 1371–1379, Istanbul, Turkey, 2015.

L. S. Shapley. Stochastic games. Proceedings of the National Academy of Sciences of the United States of America, pages 1095–1100, 1953.

C. Song, Z. Zhou, Y. Jiang Y. Zhou, and Y. Ma. Optimistic dual extrapolation for coherent non-monotone variational inequalities. Advances in Neural Information Processing Systems, 33:14303–14314, 2020.

R. S. Sutton, S. P. Singh D. A. McAllester, and Y. Mansour. Policy gradient methods for reinforcement learning with function approximation. Advances in Neural Information Processing Systems, pages 1057–1063, 1999.

C. J. Watkins and P. Dayan. Q-learning. Machine Learning, 8(3):279–292, 1992.

C. Y. Wei, C. W. Lee, M. X. Zhang, and H. P. Luo. Last-iterate convergence of decentralized optimistic gradient descent/ascent in infinite-horizon competitive Markov games. In Proceedings of the 34th Conference on Learning Theory, volume 134, pages 4259–4299, Colorado, USA, 2021.

Z. Yang, M. Hong K. Zhang, and T. Başar. A finite sample analysis of the Actor-Critic algorithm. In Proceedings of the 57th IEEE Conference on Decision and Control, Florida, USA, 2018.

K. Zhang, A. Koppel, H. Zhu, and T. Başar. Global convergence of policy gradient methods to (almost) locally optimal policies. SIAM Journal on Control and Optimization, 58(6): 3586–3612, 2020.

R. Zhang, Z. Ren, and N. Li. Gradient play in stochastic games: stationary points, convergence, and sample complexity. arXiv preprint arXiv: 2106.00198, 2021.

Y. Zhao, Y. Tian, J. D. Lee, and S. S. Du. Provably efficient policy gradient methods for two-player zero-sum Markov games. In Proceedings of the 25th International Conference on Artificial Intelligence and Statistics, volume 151, pages 2736–2761, Virtual, 2022.