HEURISTICS FOR $\ell$-TORSION IN VERONESE SYZYGIES

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Abstract. To what extent should we expect the syzygies of Veronese embeddings of projective space to depend on the characteristic of the field? As computation of syzygies is impossible for large degree Veronese embeddings, we instead develop an heuristic approach based on random flag complexes. We prove that the corresponding Stanley–Reisner ideals have Betti numbers which almost always depend on the characteristic, and we use this to conjecture that the syzygies of the $d$-uple embedding of projective $r$-space with $r \geq 7$ should depend on the characteristic for almost all $d$.

1. Introduction

Imagine $\mathbb{P}^{10}$ embedded into a larger projective space by the $d$-uple Veronese embedding, where $d$ is some large integer like $d = 100$ or $d = 100000$. What should we expect about the syzygies? Such questions were raised by Ein and Lazarsfeld in [17] and later in [15]. While they focused on quantitative behaviors that are independent of the ground field, we ask: To what extent should we expect the syzygies to depend on the characteristic, if at all? Given the impossibility of computing data for large $d$, how can we make a reasonable conjecture?

The central idea in this paper is the development of an heuristic—based on a random flag complex construction—for modelling the syzygies of Veronese embeddings of projective space. The resulting conjectures propose that, when it comes to dependence on the characteristic of the ground field, pathologies are the norm. Let us make this more precise.

For any integers $r, d \geq 1$ and any field $k$, we may consider the $d$-uple embedding of $\mathbb{P}^r_k$ into $\mathbb{P}^{(r+d)-1}$; the image is given by an ideal $I \subset S$, where $S$ is a polynomial ring in $r+d$ variables over $k$. We denote the algebraic Betti numbers of the image by $\beta_{i,j}(\mathbb{P}^r_k; d) := \dim_k \text{Tor}^S_i(S/I, k)_j$. These encode the number of degree $j$ generators for the $i$th syzygies, and a major open question is to describe the Betti table $\beta(\mathbb{P}^r_k; d)$, which is the collection of all these Betti numbers [2, 6, 7, 9, 16, 17, 20–22, 26–28, 31].

Since each individual Betti number is invariant under flat extensions, the Betti table is determined by the integers $r, d$ and the characteristic of $k$. For a prime $\ell$, we say that $\beta(\mathbb{P}^r; d)$ has $\ell$-torsion if $\beta(\mathbb{P}^r_{\overline{k}}; d) \neq \beta(\mathbb{P}^r_{\overline{Q}}; d)$, and we say that $\beta(\mathbb{P}^r; d)$ depends on the characteristic if this occurs for some $\ell$. There are two known cases.

- For $r = 1$ and any $d$, the Betti numbers in $\beta(\mathbb{P}^r; d)$ do not depend on the characteristic, as any rational normal curve is resolved by an Eagon-Northcott complex.
- If $r \geq 7$, Andersen’s thesis [2] shows that $\beta_{5,7}(\mathbb{P}^r; 2)$ has 5-torsion.

Very little else seems to be known or even conjectured about the dependence of Veronese syzygies on the characteristic, including no known examples of $\ell$-torsion for $\ell \neq 5$. 

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1This is equivalent to a certain integral Tor group having $\ell$-torsion: see Remark 2.1.
One key challenge in this area is the difficulty of generating good data. For instance, the syzygies of \( \mathbb{P}^2 \) under the 5-uple embedding were only recently computed \([7,9]\). For larger values of \( d \) and \( r \), computation is essentially impossible: in the case of \( \mathbb{P}^{10} \) and \( d = 100 \), the computation would involve \( \approx 4.68 \times 10^{15} \) variables.

Heuristics can provide an alternate route for generating conjectures, especially when computation is infeasible. (Such an approach is quite common for predicting properties of how the prime numbers are distributed, for instance.) In this paper, we use an heuristic model to motivate conjectures about \( \ell \)-torsion in \( \beta(\mathbb{P}^r; d) \). For instance, we are led to conjecture that dependence on the characteristic should be commonplace as \( d \to \infty \).

**Conjecture 1.1.** Let \( r \geq 7 \). For any \( d \gg 0 \), the Betti table of \( \mathbb{P}^r \) under the d-uple embedding depends on the characteristic.

This conjecture is based upon corresponding properties of the following model for Veronese syzygies. We let \( \Delta \sim \Delta(n, p) \) denote a random flag complex on \( n \) vertices with attaching probability \( p \). (See [22] for details.) For a given field \( k \), we let \( I_\Delta \) be the corresponding Stanley–Reisner ideal in \( S = k[x_1, \ldots, x_n] \). Ein and Lazarsfeld showed that if \( d \gg 0 \), then almost all of the Betti numbers in rows \( 1, \ldots, r \) of \( \beta(\mathbb{P}^r; d) \) are nonzero (see for instance [18, Theorem 1.1]).

Theorem 1.3 of [18] gives that a similar result holds for \( I_\Delta \) as long as \( n^{-1/(r-1)} \ll p \ll n^{-1/r} \) and \( n \gg 0 \). Thus, if \( p \) is in the specified range, then the Betti table \( \beta(S/I_\Delta) \) as \( n \to \infty \) satisfies similar nonvanishing properties\(^2\) as \( \beta(\mathbb{P}^r; d) \) as \( d \to \infty \); in this sense, the Betti tables \( \beta(S/I_\Delta) \) determined by \( \Delta(n, p) \) can act as a random model for Veronese syzygies.

To predict how \( \beta(\mathbb{P}^r; d) \) depends on the characteristic, we will therefore consider the corresponding questions for \( \beta(S/I_\Delta) \) for various fields \( k \). As with Veronese syzygies, we say that the Betti table of the Stanley–Reisner ideal of \( \Delta \) has \( \ell \)-torsion if this Betti table is different when defined over a field of characteristic \( \ell \) than it is over \( \mathbb{Q} \), and we say that this Betti table depends on the characteristic if this occurs for some \( \ell \). We prove:

**Theorem 1.2.** Let \( r \geq 7 \), and let \( \Delta \sim \Delta(n, p) \) be a random flag complex with \( n^{-1/(r-1)} \ll p \ll n^{-1/r} \). With high probability as \( n \to \infty \), the Betti table of the Stanley–Reisner ideal of \( \Delta \) depends on the characteristic.

In other words, if \( p \) is in the range where the Betti table of the Stanley–Reisner ideal of \( \Delta \) behaves like \( \mathbb{P}^r \)—in the sense of [18, Theorem 1.3]—then this Betti table will almost always depend on the characteristic for \( n \gg 0 \). This theorem is the basis of Conjecture 1.1. Since our \( r \geq 7 \) hypothesis in Conjecture 1.1 is based upon properties of the \( \Delta(n, p) \) model, the fact that this hypothesis lines up with Andersen’s example appears to be a coincidence; see Remarks 1.3 and 7.4 for more details. Note also that, based on [2], we might even find \( \ell \)-torsion in \( \beta(\mathbb{P}^r; d) \) for small values of \( d \) as well; however, Theorem 1.2 is asymptotic in nature, which motivates the \( d \gg 0 \) hypothesis in Conjecture 1.1.

In fact, we prove the sharper result:

**Theorem 1.3.** Let \( m \geq 2 \), and let \( \Delta \sim \Delta(n, p) \) be a random flag complex with \( n^{-1/6} \ll p \leq 1 - \epsilon \) for some \( \epsilon > 0 \). With high probability as \( n \to \infty \), the Betti table of the Stanley–Reisner ideal of \( \Delta \) has \( \ell \)-torsion for every \( \ell \) dividing \( m \). In particular, this holds for \( \Delta \sim \Delta(n, p) \) where \( n^{-1/(r-1)} \ll p \ll n^{-1/r} \) for any \( r \geq 7 \).

The proof of Theorem 1.3 (which implies Theorem 1.2) proceeds as follows. By Hochster’s formula [8, Theorem 5.5.1], it suffices to show that some induced subcomplex of \( \Delta \) has
$m$-torsion in its homology. So, for each $m$, we construct a flag complex $X_m$ with a small number of vertices and with $m$-torsion in $H_1(X_m)$. This complex is derived from Newman’s construction of a two-dimensional simplicial complex $X$ where $H_1(X)$ has $m$-torsion [25 §3], though we modify his work to ensure that $X_m$ is a flag complex and to lower the maximal vertex degree. We then apply Bollobás’s theorem on subgraphs of a random graph [3 Theorem 8]—or rather a minor variant of that result for induced subgraphs—to prove that $X_m$ appears as an induced subcomplex of $\Delta$ with high probability as $n \to \infty$, yielding Theorem [1.3].

Theorem [1.3] fits into an emerging literature on random monomial ideals. Our current work seems to be the first application of random monomial ideal methods to generate new conjectures outside of the world of monomial ideals. Random monomial ideals first appeared in the work of De Loera-Petrović-Silverstein-Stasi-Wilburne [14], which outlined an array of frameworks for studying random monomial ideals, including the model used in this paper, as well as models related to other types of random simplicial complexes such as [12] [24]; they also proved threshold results for dimension and other invariants of these ideals. In [13], similar methods are applied to study the average behavior of Betti tables of random monomial ideals and to compare these with certain resolutions of generic monomial ideals. Recent work of Banerjee and Yogeshwaran analyzes homological properties of the edge ideals of Erdős-Rényi random graphs [3]. The forthcoming [30] looks more closely at threshold phenomena in the phase transitions of the random models from [14]. There is also the previously referenced [18], which uses random monomial methods to demonstrate some asymptotic syzygy phenomena observed/conjectured in [15] [17].

There is also a great deal of literature on the study of $\ell$-torsion arising in random constructions. The most relevant such study is perhaps the recent work by Kahle-Lutz-Newman on $\ell$-torsion in the homology of random simplicial complexes [23], which conjectures the existence of bursts of torsion homology at specific thresholds. For comparison, those authors are interested in $\ell$-torsion in the global homology of a complex like $\Delta(n,p)$, whereas, due to Hochster’s formula, we analyze the simpler question of finding $\ell$-torsion in the homology of any induced subgraph of $\Delta(n,p)$.

Remark 1.4. We note that the bound $r \geq 7$ in Theorem [1.3] is not necessarily sharp. In fact, we undertake a detailed investigation of the 2-torsion of the Betti table of the Stanley–Reisner ideal of $\Delta$ in §5, which yields a bound of $r \geq 4$. See Remarks [4.3] and [7.4] for further discussion on restrictions on $r$ in both Conjecture [1.1] and Theorem [1.3].

Theorem [1.3] also leads us to a stronger conjecture on Veronese $\ell$-torsion:

**Conjecture 1.5.** Let $r \geq 7$. As $d \to \infty$, the number of primes $\ell$ such that $\beta(P^r;d)$ has $\ell$-torsion will be unbounded.

Regarding Conjectures [1.1] and [1.5] it is worth emphasizing the total lack of direct evidence. As noted above, [2] appears to provide the only known instance of $\ell$-torsion for any Veronese embedding. These conjectures are based primarily upon the heuristic model and, to a lesser extent, upon the nonvanishing results of [16] [17], both of which rely on an inductive structure where pathologies in $\beta(P^r;d)$ tend to propagate as $d \to \infty$, and both of which show that the asymptotic behavior of syzygies exhibits a strong uniformity.

However, we do not expect our random flag complex model to be a perfect predictor of all properties of Veronese syzygies. In fact, the results in [18] imply that while the Betti tables associated to $\Delta$ have similar overarching nonvanishing properties as Veronese
embeddings, these Betti tables do not demonstrate more nuanced properties such as Green’s $N_r$-property [21]: our model will not give correct predictions about these properties. This is why Conjectures 1.1 and 1.5 echo certain qualitative aspects of Theorem 1.3 as opposed to more specific and quantitative predictions about $\ell$-torsion in $\beta(P^r; d)$.

In a rather different direction, an alternate heuristic model for Veronese syzygies is considered in [15]. That model is based on Boij-Söderberg theory and is used to generate quantitative conjectures about the entries of $\beta(P^r_k; d)$ for $d \gg 0$. However, since this model does not take into account the characteristic of the field, it cannot be used to generate conjectures such as those above. See also the results of [11], which provided a combinatorial parallel of the asymptotic results of [17].

Remark 1.6. Ein and Lazarsfeld’s asymptotic nonvanishing results are more-or-less uniform for any smooth variety of dimension $r$ [17, Theorem A], and these were even expanded to integral varieties by Zhou [32]. In this paper, we restrict attention to $P^n$ for concreteness, but we would expect that Conjecture 1.1 would likely apply to the $d$-uple embeddings of any $r$-dimensional integral variety which is flat over $\mathbb{Z}$, including products of projective spaces, toric varieties, hypersurfaces, Grassmanians, and more.

This paper is organized as follows. In §2, we review notation and background, including on Betti numbers, Hochster’s formula, and random flag complexes. §3 contains our main construction in which we construct an explicit flag complex $X_m$ with $m$-torsion in homology; see Theorem 3.1. In §4, we apply a minor variant of Bollobás’s Theorem on subgraphs of a random graph to show that, with high probability, $X_m$ appears as an induced subgraph of $\Delta(n, p)$ for any $\frac{n}{2} \leq p \leq 1$ and $m \geq 2$. In §6, we then combine this result with Hochster’s formula to prove Theorem 1.3. §5 is a bit separate from the main results as we analyze the case of 2-torsion more closely. Finally, §7 returns to the geometric setting where we use our results to produce heuristics and conjectures about $\ell$-torsion in Veronese syzygies.

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2. BACKGROUND AND NOTATION

2.1. Betti tables for Veronese embeddings. For a given $r, d \geq 1$ and field $k$, we have the $d$-uple Veronese embedding $P^r_k \rightarrow P^{r+d}_k$. The image is determined by a homogeneous ideal $I \subset S$ where $S$ is a polynomial ring with coefficients in $k$ and $(r+d)$ variables. The homogeneous coordinate ring $S/I$ of the image is a graded $S$-module. We can thus take a minimal free resolution $F_0 \leftarrow F_1 \leftarrow \cdots$ of $S/I$, where each $F_i$ is a graded free $S$-module, $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(S/I)}$. This provides one way to define the algebraic Betti numbers; an alternate definition is $\beta_{i,j}(S/I) := \dim_k \text{Tor}_i^S(S/I, k)_j$. To emphasize the dependence on $r$ and $d$ (and to avoid referencing the ambient ring $S$ and the homogeneous coordinate ring $S/I$, both of which change with $d$), we will denote these Betti numbers by $\beta_{i,j}(P^r_k; d)$ instead of the more standard $\beta_{i,j}(S/I)$. Further, we write $\beta(P^r_k; d)$ for the Betti table of this embedding, which is the collection of all $\beta_{i,j}(P^r_k; d)$. 


2.2. Torsion in Betti tables. Throughout this paper we will analyze graded algebras, all of which have the following form: there is an ideal \( I \) in a polynomial ring \( T \) with coefficients in \( \mathbb{Z} \), where \( T/I \) is flat over \( \mathbb{Z} \), and we are interested in specializations \( (T/I) \otimes_{\mathbb{Z}} k \) to various different fields \( k \). We consider such graded algebras that arise in two ways: as the coordinate rings of Veronese embeddings of projective space and as the Stanley–Reisner rings of simplicial complexes. The central questions of this paper are concerned with when the Betti numbers of such algebras depend on the choice of the characteristic of \( k \).

First, we consider the Veronese embeddings. For any positive integers \( r \) and \( d \), we can embed \( \mathbb{P}_\mathbb{Z}^d \to \mathbb{P}_{\mathbb{Z}}^d(r^d-1) \) via the \( d \)-uple Veronese embedding. If \( T \) is the polynomial ring for the larger projective space, then there is an ideal \( I \subset T \) defining the image of this map. Since \( T/I \) is flat over \( \mathbb{Z} \), the coordinate ring of the Veronese embedding over a field \( k \) is given by \( (T/I) \otimes_{\mathbb{Z}} k \). As noted in the previous subsection (with \( S = T \otimes_{\mathbb{Z}} k \)), the algebraic Betti numbers are defined as

\[
\beta_{i,j}(\mathbb{P}_r^d; d) := \dim_k \Tor_{i}^{S}(T/I \otimes_{\mathbb{Z}} k, k)_j.
\]

Since field extensions are flat, algebraic Betti numbers are invariant under field extensions, and thus, \( \beta(\mathbb{P}_r^d; d) \) only depends on \( r, d \) and the characteristic of \( k \). Moreover, by semi-continuity, we have an inequality \( \beta_{i,j}(\mathbb{P}_r^d; d) \leq \beta_{i,j}(\mathbb{P}_r^d; d) \) for any prime \( \ell \) (with equality for all but finitely many \( \ell \)). As noted in the introduction, we will say that \( \beta(\mathbb{P}_r^d; d) \) has \( \ell \)-torsion if this inequality is strict for some \( i, j \), and we will say that \( \beta(\mathbb{P}_r^d; d) \) depends on the characteristic if this inequality is strict for some \( i, j \) and some \( \ell \).

Remark 2.1. Let \( I \) be an ideal in \( T = \mathbb{Z}[x_1, \ldots, x_n] \) which is flat over \( \mathbb{Z} \). Let \( S' = T \otimes_{\mathbb{Z}} \mathbb{F}_\ell = \mathbb{F}_\ell[x_1, \ldots, x_n] \) and \( I' = IS' \). By a standard argument, it follows that

\[
\dim_{\mathbb{F}_\ell} \Tor_{i}^{S'}(T/I', \mathbb{F}_\ell)_j = \dim_{\mathbb{F}_\ell}(\Tor_{i}^{S}(T/I, \mathbb{Z})_j \otimes_{\mathbb{Z}} \mathbb{F}_\ell) + \dim_{\mathbb{F}_\ell}(\Tor_{i}^{S}(\Tor_{i+1}^{T}(T/I, \mathbb{Z})_j, \mathbb{F}_\ell)).
\]

In particular, the Betti table of such an ideal has \( \ell \)-torsion in the sense of the introduction if and only if one of the \( \Tor_{i+1}^{T}(T/I, \mathbb{Z})_j \) has \( \ell \)-torsion as an abelian group.

We next consider notation for monomial ideals since Stanley–Reisner ideals of simplicial complexes are monomial ideals. Let \( J \) be a monomial ideal in \( T = \mathbb{Z}[x_1, \ldots, x_n] \). For a field \( k \), the algebraic Betti numbers of \( (T/J) \otimes_{\mathbb{Z}} k \) are given by

\[
\beta_{i,j}((T/J) \otimes_{\mathbb{Z}} k) := \dim_k \Tor_{i}^{T}(T/J \otimes_{\mathbb{Z}} k, k)_j.
\]

As in the Veronese case, these only depend on the characteristic of the field, and we have the same inequality \( \beta_{i,j}((T/J) \otimes_{\mathbb{Z}} \mathbb{Q}) \leq \beta_{i,j}((T/J) \otimes_{\mathbb{Z}} \mathbb{F}_\ell) \). As in the introduction, we say that \( \beta(T/J) \) has \( \ell \)-torsion if this inequality is strict for some \( i, j \), and we say that \( \beta(T/J) \) depends on the characteristic if it has \( \ell \)-torsion for some \( \ell \).

2.3. Graphs and simplicial complexes. For a simplicial complex \( X \), we write \( V(X) \), \( E(X) \), and \( F(X) \) for the set of vertices, edges, and (2-dimensional) faces of \( X \), respectively. We use \( |*| \) to denote the number of elements in these sets. The degree of a vertex \( v \) (denoted \( \deg(v) \)) is the number of edges in \( X \) containing \( v \). We write \( \max\deg(X) \) for the maximum degree of any vertex of \( X \), and we write \( \avg\deg(X) \) for the average degree of a vertex in \( X \).

For a pair of graphs \( H, G \), we write \( H \subset G \) if \( H \) is a subgraph of \( G \). We write \( H \subset^\ind G \) if \( H \) is an induced subgraph of \( G \), that is, if the vertices of \( H \) are a subset of the vertices of \( G \) and the edges of \( H \) are precisely the edges connecting those vertices within \( G \) (see Figure 1). We use similar definitions and notations for a simplicial complex \( \Delta' \) to be a subcomplex (or
Figure 1. In the graphs show above, $H$ is a subgraph of $G$, but it is not the induced subgraph on the vertex set \{1, 2, 3\} since $H$ is missing the diagonal edge connecting vertices 1 and 3.

an induced subcomplex) of another complex $\Delta$. If $\alpha \subset V(\Delta)$, then we let $\Delta|_{\alpha}$ denote the induced subcomplex of $\Delta$ on $\alpha$.

The following definitions, adapted from [5] and [10], will be used in sections 4, 5, and 6.

**Definition 2.2.** The **essential density** of a graph $G$ is

$$m(G) := \max \left\{ \frac{|E(H)|}{|V(H)|} : H \subset G, |V(H)| > 0 \right\},$$

and $G$ is **strictly balanced** if $m(H) < m(G)$ for all subgraphs $H \subset G$.

From a simplicial complex $\Delta$ on $n$ vertices, there is a corresponding Stanley–Reisner ideal $I_{\Delta} \subset S = k[x_1, \ldots, x_n]$. Since these $I_{\Delta}$ are squarefree monomial ideals, Hochster’s Formula [8, Theorem 5.5.1] relates the Betti table of $S/I_{\Delta}$ to topological properties of $\Delta$, providing our key tool for studying $\beta(S/I_{\Delta})$ for various fields $k$. An immediate consequence of Hochster’s formula is the following fact, which characterizes when these Betti tables are different over a field of characteristic $\ell$ than over $\mathbb{Q}$.

**Fact 2.3.** For a simplicial complex $\Delta$, the Betti table of the Stanley–Reisner ideal $I_{\Delta}$ has $\ell$-torsion if and only if there exists a subset $\alpha \subset V(\Delta)$ such that $\Delta|_{\alpha}$ has $\ell$-torsion in one of its homology groups.

**2.4. Monomial ideals from random flag complexes.** Our monomial ideals are Stanley–Reisner ideals associated to random flag complexes. Recall that a flag complex is a simplicial complex obtained from a graph by adjoining a $k$-simplex to every $(k+1)$-clique in the graph. In particular, a flag complex is entirely determined by its underlying graph, and the process of obtaining a flag complex from its underlying graph is called taking the clique complex. We write $\Delta \sim \Delta(n, p)$ to denote the flag complex which is the clique complex of an Erdős-Rényi random graph $G(n, p)$ on $n$ vertices, where each edge is attached with probability $p$. If $\alpha \subset V(\Delta)$, then we note that $\Delta|_{\alpha}$ is also flag. The properties of random flag complexes have been analyzed extensively, with [24] providing an overview. As discussed in the introduction, the syzygies of Stanley–Reisner ideals of random flag complexes were first studied in [18].

**2.5. Probability.** We use the notation $P[\ast]$ for the probability of an event. For a random variable $X$, we use $E(X)$ for the expected value of $X$ and $\text{Var}(X)$ for the variance of $X$.

For functions $f(x)$ and $g(x)$, we write $f \ll g$ if $\lim_{x \to \infty} f/g \to 0$. We use $f \in O(g)$ if there is a constant $N$ where $|f(x)| \leq N|g(x)|$ for all sufficiently large values of $x$, and we use $f \in \Omega(g)$ if there is a constant $N'$ where $|f(x)| \geq N'|g(x)|$ for all sufficiently large values of $x$. 
3. Constructing a flag complex with \( m \)-torsion homology

The goal of this section is to prove the following result:

**Theorem 3.1.** For every \( m \geq 2 \), there exists a two-dimensional flag complex \( X_m \) such that the torsion subgroup of \( H_1(X_m) \) is isomorphic to \( \mathbb{Z}/m\mathbb{Z} \) and \( \text{maxdeg}(X_m) \leq 12 \).

This result is the foundation of our proof of Theorem 1.3 as we will show that this specific complex \( X_m \) appears as an induced subcomplex of \( \Delta(n, p) \) with high probability under the hypotheses of that theorem.

Here is an overview of our proof of Theorem 3.1, which is largely based on ideas from [25]. Given an integer \( m \geq 2 \), we write its binary expansion as \( m = 2^{n_1} + \cdots + 2^{n_k} \) with \( 0 \leq n_1 < \cdots < n_k \). Note that \( k \) is the Hamming weight of \( m \) and \( n_k = \lfloor \log_2(m) \rfloor \). With this setup, the “repeated squares presentation” of \( \mathbb{Z}/m\mathbb{Z} \) is given by

\[
\mathbb{Z}/m\mathbb{Z} = \langle \gamma_0, \gamma_1, \ldots, \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \ldots, 2\gamma_{n_k-1} = \gamma_{n_k}, \gamma_{n_k} + \cdots + \gamma_{n_k} = 0 \rangle.
\]

We will construct a two-dimensional flag complex \( X_m \) such that the torsion subgroup of \( H_1(X_m) \) has this presentation. To do so, we follow Newman’s “telescope and sphere” construction in [25], where \( Y_1 \) is the telescope satisfying

\[
H_1(Y_1) \cong \langle \gamma_0, \gamma_1, \ldots, \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \ldots, 2\gamma_{n_k-1} = \gamma_{n_k} \rangle,
\]

\( Y_2 \) is the sphere satisfying

\[
H_1(Y_2) \cong \langle \tau_1, \ldots, \tau_k \mid \tau_1 + \cdots + \tau_k = 0 \rangle,
\]

and \( X_m \) is created by gluing \( Y_1 \) and \( Y_2 \) together to yield a complex with the desired \( H_1 \)-group. Because we want our construction to be a flag complex with \( \text{maxdeg}(X_m) \leq 12 \), we cannot simply quote Newman’s results. Instead, we must alter the triangulations to ensure that \( Y_1, Y_2, \) and \( X_m \) are flag complexes. Then, we must further alter the construction to reduce \( \text{maxdeg}(X_m) \). However, each of our constructions is homeomorphic to each of Newman’s constructions.

**Notation 3.2.** Throughout the remainder of this section we assume that \( m \geq 2 \) is given. We write \( m = 2^{n_1} + \cdots + 2^{n_k} \) with \( 0 \leq n_1 < \cdots < n_k \). To simplify notation, we also denote \( X_m \) by \( X \) for the remainder of this section.

3.1. **The telescope construction.** The telescope \( Y_1 \) that we construct will be homeomorphic to the \( Y_1 \) that Newman constructs in [25] Proof of Lemma 3.1] for the \( d = 2 \) case. We start with building blocks which are punctured projective planes; in contrast with [25], our blocks are triangulated so that each is a flag complex. Explicitly, for each \( i = 0, \ldots, (n_k - 1) \), we produce a building block which is a triangulated projective plane with a square face removed, with vertices, edges, and faces as illustrated in Figure 2. Our building blocks differ from Newman’s in order to ensure that \( Y_1 \) and the final simplicial complex \( X \) are flag complexes; for instance, we need to add extra vertices \( v'_{8i}, \ldots, v'_{8i+7} \).

We construct \( Y_1 \) by identifying edges and vertices of these \( n_k \) building blocks as labeled. The underlying vertex set is \( V(Y_1) = \{ v_0, v_1, v_2, \ldots, v_{4n_k+3}, v'_0, v'_1, \ldots, v'_{8n_k-1} \} \), so we have \( |V(Y_1)| = (4n_k + 4) + 8n_k = 12n_k + 4 \). Since each building block has 44 edges, 4 of which are glued to the next building block, and 28 faces, a similar computation yields \( |E(Y_1)| = 40n_k + 4 \) and \( |F(Y_1)| = 28n_k \). In addition, observe that the vertices of highest degree are those in the squares in the “middle” of the telescope, such as vertex \( v_4 \) when \( n_k \geq 2 \). In this case, \( v_4 \) is
adjacent to \( v_5, v_7, v'_0, v'_1, v'_2, v'_5, v'_{15}, v'_{11}, \) and \( v'_{12}, \) so \( \deg(v_4) = 9. \) By the symmetry of \( Y_1, \) we have that \( \maxdeg(Y_1) = 9 \) when \( n_k \geq 2, \) and \( \maxdeg(Y_1) = 6 \) when \( n_k = 1 \) (when \( m = 2, 3 \)).

To compute \( H_1(Y_1) \), we simply apply the identical argument from \([25]\). We order the vertices in the natural way, where \( v_j > v_k \) if \( j > k, \) similarly for the \( v'_i, \) and where \( v'_i > v_j \) for all \( \ell, j. \) We let these vertex orderings induce orientations on the edges and faces of \( Y_1. \) For each \( i = 0, \ldots, n_k, \) denote by \( \gamma_i \) the 1-cycle of \( Y_1 \) represented by \([v_{4i}, v_{4i+1}] + [v_{4i+1}, v_{4i+2}] + [v_{4i+2}, v_{4i+3}] - [v_{4i}, v_{4i+3}]\). Then \( 2\gamma_i - \gamma_{i+1} \) is a 1-boundary of \( Y_1 \) for each \( i = 0, \ldots, (n_k-1) \), and, as in Newman’s construction, we have that \( H_1(Y_1) \) can be presented as \( \langle \gamma_0, \gamma_1, \ldots, \gamma_{n_k} | 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \ldots, 2\gamma_{n_k-1} = \gamma_{n_k} \rangle. \)

3.2. The sphere construction. The sphere part \( Y_2 \) is a flag triangulation of the sphere \( S^2 \) that has \( k \) square holes such that the squares are all vertex disjoint and nonadjacent. Our \( Y_2 \) will be homeomorphic to the \( Y_2 \) that Newman constructs in \([25]\) for the \( d = 2 \) case, but our construction involves a few different steps. First, we will show that for any integer \( k \geq 1, \) there exists a flag triangulation \( T_i \) of \( S^2 \) (here \( i = \lfloor \frac{k}{4} \rfloor \)) with at least \( k \) faces such that \( \maxdeg(T_i) \leq 6. \) Then, we will insert square holes on \( k \) of the faces of \( T_i, \) while subdividing the edges, and call the resulting flag complex \( \widetilde{T_i}. \) Finally, we describe a process to replace each vertex of degree 14 in \( \widetilde{T_i} \) with two degree 9 vertices so that the resulting complex, \( Y_2, \) has \( \maxdeg(Y_2) \leq 12. \) Throughout these constructions, we will have four cases corresponding to the value of \( k \mod 4, \) and we carefully keep track of the degrees of each vertex in \( T_i, \widetilde{T_i}, \) and \( Y_2 \) for each case.

3.2.1. \( T_i \) and flag bistellar 0-moves. We begin by constructing an infinite sequence \( T_0, T_1, T_2, \ldots \) of flag triangulations of \( S^2 \) such that \( \maxdeg(T_i) \leq 6 \) for all \( i. \) To do so, we adapt the bistellar 0-moves used in \([25]\) Lemma 5.6. Let \( T_0 \) be the 3-simplex boundary on the vertex set \( \{w_0, w_1, w_2, w_3\} \). Note that each vertex of \( T_0 \) has degree 3. We will construct the remaining \( T_i \) inductively. To build \( T_1, \) first remove the face \([w_1, w_2, w_3]\) and edge \([w_1, w_3]\). Then, add two new vertices \( w_4 \) and \( w_5 \) as well as new edges \([w_0, w_4], [w_1, w_4], [w_3, w_4], [w_1, w_5], [w_2, w_5], [w_3, w_5], \) and \([w_4, w_5]. \) Taking the clique complex will then give \( T_1. \) See Figure 3.
degree 4 \( w_i \) vertices. Each \( T_{i+1} \) for \( i \geq 0 \) will be obtained from \( T_i \) by performing a flag bistellar 0-move on the face \( \{w_{2i+1}, w_{2i+2}, w_{2i+3}\} \) of \( T_i \). Explicitly, to construct \( T_{i+1} \), remove the face \( \{w_{2i+1}, w_{2i+2}, w_{2i+3}\} \) and the edge \( \{w_{2i+1}, w_{2i+3}\} \). Then, add new vertices \( w_{2i+4} \) and \( w_{2i+5} \) and new edges \( \{w_{2i}, w_{2i+4}\}, \{w_{2i+1}, w_{2i+4}\}, \{w_{2i+3}, w_{2i+4}\}, \{w_{2i+1}, w_{2i+5}\}, \{w_{2i+2}, w_{2i+5}\}, \{w_{2i+3}, w_{2i+5}\} \), and take the clique complex to get \( T_{i+1} \). Note that each flag bistellar 0-move adds 2 vertices, 6 edges, and 4 faces. Since \( |V(T_0)| = 4, |E(T_0)| = 6, \) and \( |F(T_0)| = 4 \), this means that \( |V(T_i)| = 2i + 4, |E(T_i)| = 6i + 6, \) and \( |F(T_i)| = 4i + 4 \).

Further, Table 1 summarizes the degrees of the vertices in each \( T_i \). To compute the degrees of vertices in \( T_i \) for \( i \geq 3 \), observe that when the new vertices \( w_{2i+2} \) and \( w_{2i+3} \) are added, they have degree 4 in \( T_i \). For each of the next two iterations of the flag bistellar-0 move, the degree of these vertices increases by one, resulting in degree 6 in \( T_{i+2} \). In the remaining triangulations \( T_j \) with \( j \geq i + 3 \), these vertices are not affected. Therefore, max\( \deg(T_i) \leq 6 \) for each \( i \).

From this infinite sequence of flag triangulations of \( S^2 \) with bounded degree, we are interested in the particular \( T_i \) with \( i = \left\lfloor \frac{k-1}{2} \right\rfloor \) to use in our construction of \( Y_2 \), where \( k \) is the Hamming weight of \( m \) as in Notation 3.2. Note that this \( T_i \) has vertex set \( \{w_0, \ldots, w_{2i+3}\} \) and has \( 4\left\lfloor \frac{k-1}{2} \right\rfloor + 4 \) faces. Let \( \delta \) be the integer \( 0 \leq \delta \leq 3 \) where \( \delta \equiv -k \mod 4 \). Then \( T_i \) has exactly \( k + \delta \) faces.

3.2.2. Constructing \( \tilde{T}_i \). Next, we insert square holes in the first \( k \) faces of \( T_i \) and subdivide the remaining faces in such a way that the squares will be vertex disjoint and nonadjacent.

![Figure 3. The first few flag triangulations of \( S^2 \) using flag bistellar 0-moves.](image)
Call this complex $\tilde{\mathbf{v}}$ and consider the case when $\delta = 0$ and all $k$ faces of $T_i$ have a square removed from them. Table 4 (left), and subdivided triangulation on remaining faces (right).

First, we will insert square holes in $k$ of the faces of $T_i$, making sure to triangulate the resulting faces and take the clique complex so that our simplicial complex remains flag. Let $[w_r, w_s, w_t]$ with $r < s < t$ be the $j$th of these $k$ faces with respect to a fixed ordering of the faces (where $j$ ranges from 1 to $k$). We remove this face and subdivide the edges by adding new vertices $w_{r,s}, w_{r,t},$ and $w_{s,t}$ and new edges $[w_r, w'_{r,s}], [w_s, w'_{r,s}], [w_t, w'_{r,t}], [w_r, w'_{s,t}], [w_s, w'_{s,t}],$ and $[w_t, w'_{s,t}]$. Then, we add vertices $u_{4j-4}, u_{4j-3}, u_{4j-2},$ and $u_{4j-1}$ to form a square inside the original face with indices increasing counterclockwise. Moreover, we add edges $\begin{bmatrix} w_r, u_{4j-4} \end{bmatrix}, \begin{bmatrix} r, u_{4j-1} \end{bmatrix}, \begin{bmatrix} u_{4j-4}, u_{4j-3} \end{bmatrix}, \begin{bmatrix} u_{4j-3}, w_s \end{bmatrix}, \begin{bmatrix} w_s, u_{4j-3} \end{bmatrix}$ $\begin{bmatrix} u_{4j-3}, w'_{s,t} \end{bmatrix}, \begin{bmatrix} w_t, u_{4j-2} \end{bmatrix}, \begin{bmatrix} u_{4j-2}, w'_{s,t} \end{bmatrix}, \begin{bmatrix} u_{4j-1}, w'_r \end{bmatrix}$. After applying this process, we take the clique complex. The result of this operation on face $[w_r, w_s, w_t]$ is depicted in Figure 4 (left).

The remaining $\delta$ faces of $T_i$ will simply be subdivided and triangulated before taking the clique complex. Explicitly, this means that after removing the face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ and its edges, we add vertices $w'_{2i+1,2i+2}, w'_{2i+1,2i+3},$ and $w'_{2i+2,2i+3}$ and edges $\begin{bmatrix} w_{2i+1}, w'_{2i+1,2i+2} \end{bmatrix}, \begin{bmatrix} w_{2i+1}, w'_{2i+1,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+2}, w'_{2i+1,2i+2} \end{bmatrix}, \begin{bmatrix} w_{2i+2}, w'_{2i+1,2i+3} \end{bmatrix}$ $\begin{bmatrix} w_{2i+3}, w'_{2i+2,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+2}, w'_{2i+2,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+3}, w'_{2i+1,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+3}, w'_{2i+2,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+2}, w'_{2i+2,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+3}, w'_{2i+1,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+2}, w'_{2i+2,2i+3} \end{bmatrix}, \begin{bmatrix} w_{2i+3}, w'_{2i+1,2i+3} \end{bmatrix}$. This subdivision of face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ is shown in Figure 4 (right). We do similarly for the faces $[w_{2i-1}, w_{2i+2}, w_{2i+3}]$ and $[w_{2i}, w_{2i+1}, w_{2i+3}]$, if necessary. The clique complex of this construction is a flag complex which is homeomorphic to $S^2$ with $k$ distinct points removed. Call this complex $\tilde{T}_i$.

Let’s consider the degrees of the vertices of $\tilde{T}_i$. We have that $\deg(w'_{m,n}) = 6$ for all $m, n$ and $\deg(u_\ell) \in \{4, 5\}$ for all $\ell$, where the “top” $u_\ell$ have degree 4 and the “bottom” $u_\ell$ have degree 5. To determine the degrees of the $w_j$ vertices, we need to consider their degrees in $T_i$ and how their degrees increase during the subdivision and square face removal processes. As we are interested in bounding the maximum degree of the vertices of $\tilde{T}_i$, we need only consider the case when $\delta = 0$ and all $k$ faces of $T_i$ have a square removed from them. Table
Table 2. Degrees of the vertices in $\tilde{T}_i$ when $k \equiv 0 \mod 4$.

| $T_i$  | Degree | Vertices          |
|-------|--------|-------------------|
| $\tilde{T}_0$ (k = 4) | 6      | $w_2, w_3$         |
|       | 7      | $w_1$             |
|       | 9      | $w_0$             |
| $\tilde{T}_1$ (k = 8) | 8      | $w_4, w_5$         |
|       | 9      | $w_2, w_3$         |
|       | 10     | $w_1$             |
|       | 12     | $w_0$             |
| $\tilde{T}_2$ (k = 12) | 8      | $w_6, w_7$         |
|       | 10     | $w_1$             |
|       | 11     | $w_4, w_5$         |
|       | 12     | $w_0, w_2, w_3$    |
| $\tilde{T}_i$ (k = 3i + 4) | 8      | $w_{2i+2}, w_{2i+3}$ |
|       | 10     | $w_1$             |
|       | 11     | $w_{2i}, w_{2i+1}$ |
|       | 12     | $w_0, w_2, w_3$    |
|       | 14     | $w_4, \ldots, w_{2i-1}$ |

2 gives the degrees of each of the $w_j$ vertices in $\tilde{T}_i$ when $\delta = 0$.

To verify the degrees of the $w_j$ in $\tilde{T}_i$ when $i \geq 3$, we consider how the degrees of the vertices change as $i$ increases. Between $\tilde{T}_{i-1}$ and $\tilde{T}_i$ (with $\delta = 0$ for both), the only vertices that change degree are $w_{2i-2}, w_{2i-1}, w_{2i}, w_{2i+1}$, each of which increase degree by 3. This is because they each get one new edge from the $T$ flag bistellar 0-move and two new edges from the square removal triangulation process (since each vertex is the smallest indexed and hence the “top” vertex of one new triangular face). Further, the new vertices $w_{2i+2}, w_{2i+3}$ in $\tilde{T}_i$ have degree 8, and they increase degree by 3 in the next two iterations, resulting in degree 14 in $\tilde{T}_{i+2}$ and all future iterations.

The above argument shows that regardless of $m$ and $k$, $\text{maxdeg}(\tilde{T}_i) \leq 14$, where $i = \left\lfloor \frac{k-1}{4} \right\rfloor$. Furthermore, the only vertices that could have degree 14 are $w_4, \ldots, w_{2i-1}$, each of which is separated from the others by a $w_{m,n}'$ vertex, which only has degree 6. We want to know exactly which vertices in $\tilde{T}_i$ have degree 14, for all possible $k$ with $i \geq 3$, because we plan to alter these vertices to decrease $\text{maxdeg}(\tilde{T}_i)$. Note that as $\delta$ increases from 0 to 3, the degree of each $w_j$ vertex is nonincreasing. When $k = 4i + 4$ and $\delta = 0$, the above table gives that $w_4, \ldots, w_{2i-1}$ have degree 14. When $k = 4i + 3$ and $\delta = 1$, the face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ is subdivided instead of having a square removed, but this does not change the degrees of $w_4, \ldots, w_{2i-1}$, so these all still have degree 14. When $k = 4i + 2$ and $\delta = 2$, the faces $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ and $[w_{2i-1}, w_{2i+2}, w_{2i+3}]$ are subdivided. Therefore, $w_{2i-1}$ has two fewer edges than in the previous case since $w_{2i-1}$ is the smallest indexed vertex in $[w_{2i-1}, w_{2i+2}, w_{2i+3}]$ and so would have two “top” $u$ adjacent to it if this face had a square removed from it. So, in this case, $w_4, \ldots, w_{2i-2}$ have degree 14 and $w_0, w_2, w_3, w_{2i-1}$ have degree 12 in $\tilde{T}_i$. Finally, if $k = 4i + 1$ and $\delta = 3$, then additionally the face $[w_{2i}, w_{2i+1}, w_{2i+3}]$ is subdivided, which means that the degree 12 and 14 vertices are the same as in the previous cases.
3.2.3. **Replacing degree 14 vertices to construct** $Y_2$. Having identified the vertices of $\widetilde{T_i}$ of the highest degree, we now describe a process by which we will replace each vertex of degree 14 by two vertices of degree 9 in order to ensure that $\text{maxdeg}(\widetilde{T_i}) \leq 12$ for all $k$ and $i$. The resulting flag complex, given by taking the clique complex of this construction, will be the final $Y_2$, and it will be homeomorphic to $\widetilde{T_i}$. The process is summarized by Figure 5 and described in detail in the following paragraphs.

Suppose $w_j$ is a vertex of degree 14 in $\widetilde{T_i}$. Locally, on a small neighborhood of $w_j$, $\widetilde{T_i}$ is homeomorphic to a 2-manifold. Since $\text{deg}(w_j) = 14$, $w_j$ is surrounded by six triangular faces coming from $T_i$, all of which have had a square removed. By our construction, two of these squares (which are in adjacent triangular faces) have both of their “top” $u_{\ell}$ vertices connected to $w_j$, but the other four squares just have a single edge connecting one of their “bottom” $u_{\ell}$ vertices to $w_j$. So, $w_j$ has six $w_{m,n}'$ neighbors and eight $u_{\ell}$ neighbors, which form a 14-sided polygon with $w_j$ as its “star” point. Choose two $w_{m,n}'$ vertices which are across from each other in this 14-sided polygon, say $w_{a,b}'$ and $w_{c,d}'$. Next, we will remove $w_j$ and all of the 14 faces that it is contained in. Then, we add vertices $w_{j_1}$ and $w_{j_2}$ in place of $w_j$ and add edges in such a way that $\text{deg}(w_{j_1}) = \text{deg}(w_{j_2}) = 9$, there are edges $[w_{j_1}, w_{j_2}], [w_{j_1}, w_{a,b}'], [w_{j_1}, w_{c,d}'], [w_{j_2}, w_{a,b}']$, and $[w_{j_2}, w_{c,d}']$, and the 14-sided polygon is triangulated with 16 triangles. This process only changes the degree of $w_{a,b}'$ and $w_{c,d}'$, each of which now have degree 7. Therefore, the maximum degree of $w_{j_1}, w_{j_2}$, and the 14 vertices in the polygon is 9 (since $\text{deg}(u_{\ell}) \in \{4, 5\}$ and $\text{deg}(w_{m,n}') = 6$). To illustrate this construction, we consider the case when $k = 20$. Then $i = 4$, $\delta = 0$, and $\text{deg}(w_7) = 14$ in $\widetilde{T_4}$. Figure 5 depicts this process when $w_{a,b}' = w_{3,7}'$ and $w_{c,d}' = w_{7,11}'$.

After repeating the above process for each degree 14 vertex in $\widetilde{T_i}$, we take the clique complex and call the resulting flag complex $Y_2$. Observe that this process increases the number of vertices by 1, the number of edges by 3, and the number of faces by 2 each time a degree 14 vertex in $\widetilde{T_i}$ is replaced. Also, note that $\text{maxdeg}(Y_2) \leq 12$ for all $m$.

Now, we give the $w_j, w_{m,n}'$, and $u_{\ell}$ vertices their natural orderings and say that $w_{m,n}' > w_j$ and $w_{m,n}' > u_{\ell}$ for all $\ell, m, n$, and $j$, and then let these vertex orderings induce orientations on the edges and faces of $Y_2$ (as shown in Figure 5). Counting the vertices, edges, and
faces of $Y_2$ we have that if $0 \leq k \leq 12$, then there were no degree 14 vertices to remove, so $|V(Y_2)| = 6k + 2\delta + 2$, $|E(Y_2)| = 17k + 6\delta$, and $|F(Y_2)| = 10k + 4\delta$. If $k \geq 13$, then $i \geq 3$ and at least one degree 14 vertex was removed to construct $Y_2$ from $\widetilde{T}_i$. Table 3 gives the number of vertices, edges, and faces of $Y_2$ for all values of $k \geq 13$.

| $k$  | $\delta$ | $|V(Y_2)|$ | $|E(Y_2)|$ | $|F(Y_2)|$ |
|------|----------|------------|------------|------------|
| $4i + 4$ | 0 | $\frac{13}{2}k - 4$ | $\frac{37}{2}k - 18$ | $11k - 12$ |
| $4i + 3$ | 1 | $\frac{13}{2}k - \frac{3}{2}$ | $\frac{37}{2}k - \frac{21}{2}$ | $11k - 7$ |
| $4i + 2$ | 2 | $\frac{13}{2}k$ | $\frac{37}{2}k - 6$ | $11k - 4$ |
| $4i + 1$ | 3 | $\frac{13}{2}k + \frac{5}{2}$ | $\frac{37}{2}k + \frac{3}{2}$ | $11k + 1$ |

Table 3. Number of vertices, edges, and faces in $Y_2$ when $k \geq 13$.

3.2.4. Homology of $Y_2$. Since $Y_2$ is an oriented flag triangulation of $S^2$ with $k$ square holes, each of which are vertex disjoint and nonadjacent, our $Y_2$ is homeomorphic to Newman’s $Y_2$ in the $d = 2$ case of [25, Lemma 5.7], and we can apply the same argument to compute the homology of $Y_2$. We denote the 1-cycles that are the boundaries of the $k$ square holes by $\tau_1, \ldots, \tau_k$. Explicitly, for $j = 1, \ldots, k$, we define

$$\tau_j := [u_{4j-4}, u_{4j-3}] + [u_{4j-3}, u_{4j-2}] + [u_{4j-2}, u_{4j-1}] - [u_{4j-4}, u_{4j-1}],$$

Then, by our construction, each $\tau_j$ is a positively-oriented 1-cycle in $H_1(Y_2)$, and exactly as in [25, Proof of Lemma 5.7], we have that $H_1(Y_2) = \langle \tau_1, \ldots, \tau_k | \tau_1 + \cdots + \tau_k = 0 \rangle$.

3.3. Construction of $X$ and proof of Theorem 3.1. Now we attach $Y_1$ and $Y_2$ together to form the two-dimensional flag complex $X$ such that the torsion subgroup of $H_1(X)$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. This part essentially follows [25, §3], though we must confirm that the resulting complex is flag and satisfies the desired bound of vertex degree.

Proof of Theorem 3.1. For a given $m$, let $Y_1$ and $Y_2$ be the complexes constructed in the previous subsections. Let $S$ denote the subcomplex of $Y_2$ induced by the $4k$ vertices $u_0, \ldots, u_{4k-1}$. Since the square holes in $Y_2$ are vertex-disjoint and have no edges between any two of them, $S$ is a disjoint union of $k$ square boundaries. Let $f : S \to Y_1$ be the simplicial map defined, for $j = 1, \ldots, k$, by

$$u_{4j-4} \mapsto v_{4n_j}, \quad u_{4j-3} \mapsto v_{4n_j+1}, \quad u_{4j-2} \mapsto v_{4n_j+2}, \quad u_{4j-1} \mapsto v_{4n_j+3}.$$

Following [25, §3], let $X = Y_1 \sqcup_f Y_2$ and observe that this is a simplicial complex by the same argument as Newman gives. In addition, $X$ is a flag complex because $Y_1$ and $Y_2$ are flag, and we subdivided the edges of $Y_1$ and $Y_2$ to avoid the possibility that $X$ might contain a 3-cycle which doesn’t have a face. Furthermore, in $X$ the squares $\tau_j$ and $\gamma_{n_j}$ are identified by $f$ for $j = 1, \ldots, k$, and, as in [25],

$$H_1(X) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/m\mathbb{Z},$$

where $\mathbb{Z}/m\mathbb{Z}$ has the repeated squares representation given by

$$\langle \gamma_0, \gamma_1, \ldots, \gamma_n | 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \ldots, 2\gamma_{n-1} = \gamma_n, \gamma_{n} + \cdots + \gamma_n = 0 \rangle.$$
Finally, using our counts for the number of vertices, edges, and faces of $Y_1$ and $Y_2$ and with δ defined as above, we have

$$|V(X)| = 2k + 12n_k + 6 + 2\delta, \quad |E(X)| = 13k + 40n_k + 4 + 6\delta, \quad |F(X)| = 10k + 28n_k + 4\delta.$$  

If $k \geq 13$, then Table 4 gives the number of vertices, edges, and faces in $X$ (where $i = \lfloor \frac{k - 1}{4} \rfloor$).

| $k$ | $\delta$ | $|V(X)|$ | $|E(X)|$ | $|F(X)|$ |
|-----|---------|--------|--------|--------|
| $4i + 4$ | 0 | $\frac{5}{2}k + 12n_k$ | $\frac{29}{2}k + 40n_k - 14$ | $11k + 28n_k - 12$ |
| $4i + 3$ | 1 | $\frac{5}{3}k + 12n_k + \frac{5}{3}$ | $\frac{29}{2}k + 40n_k - \frac{13}{2}$ | $11k + 28n_k - 7$ |
| $4i + 2$ | 2 | $\frac{5}{3}k + 12n_k + 4$ | $\frac{29}{2}k + 40n_k - 2$ | $11k + 28n_k - 4$ |
| $4i + 1$ | 3 | $\frac{5}{3}k + 12n_k + \frac{13}{2}$ | $\frac{29}{2}k + 40n_k + \frac{11}{2}$ | $11k + 28n_k + 1$ |

Table 4. Number of vertices, edges, and faces in $X$ when $k \geq 13$.

Additionally, recall that $\maxdeg(Y_1) \leq 9$ and $\maxdeg(Y_2) \leq 12$. Since in $X$ we are only identifying the squares of $Y_2$ with $k$ of the squares of $Y_1$, to find the maximum degree of any vertex of $X$, we need only check the degrees of the identified vertices. In $Y_1$, we know that $\deg(v_j) \leq 9$ for each $j$, and in $Y_2$, we know that $\deg(u_\ell) \in \{4, 5\}$ for each $\ell$. Let $v_j$ and $u_\ell$ be vertices that are identified in $X$. Since two of their adjacent edges in the squares are identified as well, in $X$ we see that $\deg(v_j) = \deg(u_\ell) \leq 12$. Thus, $\maxdeg(X) \leq 12$. □

We also note the following corollary:

**Corollary 3.3.** For every finite abelian group $G$ there is a two-dimensional flag complex $X$ such that the torsion subgroup of $H_1(X)$ is isomorphic to $G$ and $\maxdeg(X) \leq 12$.

**Proof.** Let $G = \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z}$ with $m_1|m_2|\cdots|m_r$ be an arbitrary finite abelian group. By Theorem 3.1, there exist two-dimensional flag complexes $X_{m_i}$ such that the torsion subgroup of $H_1(X_{m_i})$ is isomorphic to $\mathbb{Z}/m_i\mathbb{Z}$ and $\maxdeg(X_{m_i}) \leq 12$. If $X$ is the disjoint union of all the $X_{m_i}$, then $X$ satisfies the hypotheses of the corollary. □

4. Appearance of subcomplexes in $\Delta(n, p)$

The goal of this section is to show, for attaching probabilities $p$ in an appropriate range, the flag complex $X_m$ from Theorem 3.1 will appear with high probability as an induced subcomplex of $\Delta(n, p)$. See §2 for the relevant definitions and notation used throughout this section. Here is our main result:

**Proposition 4.1.** Let $m \geq 2$, and let $X_m$ be as in Theorem 3.1. If $\Delta \sim \Delta(n, p)$ is a random flag complex with $n^{-1/6} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$, then $\Pr \left[ X_m \text{ind} \Delta(n, p) \right] \to 1$ as $n \to \infty$.

Our proof of this result will rely on Bollobás’s theorem on the appearance of subgraphs of a random graph, which we state here for reference.
Theorem 4.2 (Bollobás [5]). Let $G'$ be a fixed graph, let $m(G')$ be the essential density of $G'$ defined in Definition 2.2, and let $G(n, p)$ be the Erdős-Rényi random graph on $n$ vertices with attaching probability $p$. As $n \to \infty$, we have

$$
P \left[ G' \subset G(n, p) \right] \to \begin{cases} 
0 & \text{if } p \leq n^{-1/m(G')} \\
1 & \text{if } p \geq n^{-1/m(G')} . 
\end{cases}
$$

Since any flag complex is determined by its underlying graph, we can almost apply this to prove Proposition 4.1. However, Proposition 4.1 (and our eventual application of it via Hochster’s formula to Theorem 1.3) requires $X_m$ to appear as an induced subcomplex, whereas Bollobás’s result is for not necessarily induced subgraphs. The following proposition, which is likely known to experts, shows that so long as $p$ is bounded away from 1, this distinction is immaterial in the limit.

Proposition 4.3. Let $G'$ be a fixed graph, let $m(G')$ be the essential density of $G'$ defined in Definition 2.2, and let $G(n, p)$ be the Erdős-Rényi random graph on $n$ vertices with attaching probability $p$. Suppose $p = p(n) \leq 1 - \epsilon$ for some constant $\epsilon > 0$. Then as $n \to \infty$, we have

$$
P \left[ G' \text{ ind} \subset G(n, p) \right] \to \begin{cases} 
0 & \text{if } p \leq n^{-1/m(G')} \\
1 & \text{if } p \geq n^{-1/m(G')} . 
\end{cases}
$$

Proof. Since an induced subgraph is a subgraph, if $P[G' \subset G(n, p)] \to 0$, then

$$
P \left[ G' \text{ ind} \subset G(n, p) \right] \to 0 .$$

Thus, the first half of the threshold is a direct consequence of Theorem 4.2 and all that needs to be shown is the second half of the threshold.

So, suppose that $p \geq n^{-1/m(G')}$. We will mirror the proof of Bollobás’s theorem from [19, Theorem 5.3] (originally due to [29]), which relies on the second moment method. Let $\Lambda(G', n)$ be the set containing all of the possible ways that $G'$ can appear as an induced subgraph of $G(n, p)$. Thus, an element $H \in \Lambda(G', n)$ corresponds to a subset of the $n$ vertices and specified edges among those vertices such that the resulting graph is a copy of $G'$. We want to count the number of times $G'$ appears as an induced subgraph of $G(n, p)$. For each $H \in \Lambda(G', n)$, we let $1_H$ be the corresponding indicator random variable, where $1_H = 1$ occurs in the event that restricting $G(n, p)$ to the vertices of $H$ is precisely the copy of $G'$ indicated by $H$. Note that the random variables $1_H$ are not independent, as two distinct elements from $\Lambda(G', n)$ might have overlapping vertex sets. If we let $N_{G'}$ be the random variable for the number of copies of $G'$ appearing as induced subgraphs in $G(n, p)$, then we have $N_{G'} = \sum_{H \in \Lambda(G', n)} 1_H$.

Our goal is to show that $P[N_{G'} \geq 1] \to 1$, or equivalently that $P[N_{G'} = 0] \to 0$. Since $N_{G'}$ is non-negative, the second moment method as seen in [1, Theorem 4.3.1] states that $P[N_{G'} = 0] \leq \frac{\text{Var}(N_{G'})}{\text{E}[N_{G'}]^2}$, so it suffices to show that $\frac{\text{Var}(N_{G'})}{\text{E}[N_{G'}]^2} \to 0$. To start, we will bound the expected value. To simplify notation throughout the following computation, we let
If we can restrict to the case where they share at least two vertices, which gives

\[ E[N_{G'}] = \sum_{H \in \Lambda(G',n)} E[1_H] \]
\[ = \sum_{H \in \Lambda(G',n)} p^{e}(1 - p)^{e} \]
\[ = \Omega(n^v) \cdot p^{e}(1 - p)^{e}. \]

Now let us repeat this with the variance instead.

\[ \text{Var}(N_{G'}) = \sum_{H, H' \in \Lambda(G',n)} E[1_H 1_{H'}] - E[1_H]E[1_{H'}] \]
\[ = \sum_{H, H' \in \Lambda(G',n)} P[1_H = 1 \text{ and } 1_{H'} = 1] - P[1_H = 1]P[1_{H'} = 1] \]
\[ = \sum_{H, H' \in \Lambda(G',n)} P[1_H = 1] (P[1_{H'} = 1 | 1_H = 1] - P[1_{H'} = 1]) \]
\[ = p^{e}(1 - p)^{e} \sum_{H, H' \in \Lambda(G',n)} P[1_{H'} = 1 | 1_H = 1] - P[1_{H'} = 1]. \]

If \( H \) and \( H' \) don’t share at least two vertices, \( 1_H \) and \( 1_{H'} \) are independent of each other, so we can restrict to the case where they share at least two vertices, which gives

\[ = p^{e}(1 - p)^{e} \sum_{i=2}^{v} \sum_{H, H' \in \Lambda(G',n)} P[1_{H'} = 1 | 1_H = 1] - P[1_{H'} = 1]. \]

We now come to the key observation, which is also at the heart of the proof in [19] Theorem 5.3: \( P[1_{H'} = 1 | 1_H = 1] \) is maximized if those edges and non-edges in \( H \) are exactly those that are required by \( H' \). Thus, by applying the fact that any subgraph of \( G' \) with \( i \) vertices, has at most \( i \cdot m(G') \) edges and at most \( \binom{i}{2} \) non-edges we get the following bound for \( H, H' \in \Lambda(G',n) \) sharing \( i \) vertices:

\[ P[1_{H'} = 1 | 1_H = 1] \leq P[1_{H'} = 1] \cdot p^{-i \cdot m(G')} (1 - p)^{-\binom{i}{2}}. \]

From here, it is a standard computation. Substituting this back into the previous equation and simplifying, we get

\[ \text{Var}(N_{G'}) \leq p^{e}(1 - p)^{e} \sum_{i=2}^{v} \sum_{H, H' \in \Lambda(G',n)} P[1_{H'} = 1] \left( p^{-i \cdot m(G')} (1 - p)^{-\binom{i}{2}} - 1 \right) \]
\[ \leq \left( p^{e}(1 - p)^{e} \right)^{2} \sum_{i=2}^{v} O \left( n^{2v-i} \left( p^{-i \cdot m(G')} (1 - p)^{-\binom{i}{2}} - 1 \right) \right). \]

And since \( p \) is bounded away from 1 and \( 1 - p \) is bounded away from 0, we get

\[ \leq \left( p^{e}(1 - p)^{e} \right)^{2} \sum_{i=2}^{v} O \left( n^{2v-i} p^{-i \cdot m(G')} \right). \]
Finally, applying the second moment method gives

\[ P[N_{G'} \leq 0] \leq \frac{\text{Var}(N_{G'})}{\mathbb{E}[N_{G'}]^2} = \sum_{i=2}^{v} \frac{O\left(n^{2v-i}p^{-i}m(G')\right)}{\Omega(n^{2v})} = \sum_{i=2}^{v} O\left(n^{-i}p^{-i}m(G')\right). \]

Since \( p \gg n^{-1/m(G')} \), we conclude that \( np^{m(G')} \to \infty \), and therefore, \( P[N_{G'} = 0] \to 0 \). It follows that \( P\left[G'_{\text{ind}} \subset G(n,p)\right] \to 1 \). \( \square \)

We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. Recall that \( X_m \) is the complex from Theorem 3.1, and let \( H_m \) be its underlying graph. Moreover, the underlying graph of \( \Delta(n, p) \) is the Erdős-Rényi random graph \( G(n, p) \). Since a flag complex is uniquely determined by its 1-skeleton, it suffices to show that \( P\left[H_m^{\text{ind}} \subset G(n, p)\right] \to 1 \).

Since \( \text{maxdeg}(H_m) \leq 12 \), every subgraph has average degree at most 12. Thus, the essential density \( m(H_m) \) satisfies \( m(H_m) \leq 6 \). Since \( p \gg n^{-1/6} \), we have \( p \gg n^{-1/m(H_m)} \). Applying Proposition 4.3 gives \( P\left[H_m^{\text{ind}} \subset G(n, p)\right] \to 1 \); thus, \( P\left[X_m^{\text{ind}} \subset \Delta(n, p)\right] \to 1 \). \( \square \)

Remark 4.4. Explicitly computing the essential density \( m(H_m) \) seems difficult in general, and our chosen bound \( m(H_m) \leq 6 \), which is determined by the fact that \( 6 = \frac{1}{2} \text{maxdeg}(X_m) \), is likely too coarse. It would be interesting to see a sharper result on \( m(H_m) \), as this could potentially provide an heuristic for decreasing the bound on \( r \) in Conjecture 1.1. Might it even be the case that \( m(H_m) \) is half the average degree, \( \frac{1}{2} \text{avg}(H_m) \)?

In any case, \( \frac{1}{2} \text{avg}(H_m) \) at least provides a lower bound on \( m(H_m) \). Due to the detailed nature of the constructions in [31], we can estimate this value. Let \( k \geq 13 \) and \( m \gg 0 \). By Table 4, \( n_k = \lfloor \log_2(m) \rfloor \) will be much larger than \( \delta \), and so the number of vertices will be approximately \( \frac{3}{2}k + 12n_k \) and the number of edges will be approximately \( \frac{29}{7}k + 40n_k \). The smallest ratio of edges to vertices can be is when \( n_k \gg k \), in which case the ratio will be approximately \( \frac{3}{3} \). A similar computation holds for \( k \leq 12 \) and for \( m \gg 0 \). We can conclude that \( m(H_m) \geq 3\frac{1}{3} - \epsilon \), where \( \epsilon \) is a positive constant that goes to 0 as \( m \to \infty \). \( \square \)

5. A detailed analysis of 2-torsion

The goal of this section is to provide a more detailed analysis of what happens in the case of 2-torsion. We use a known flag triangulation of \( \mathbb{R}P^2 \) that minimizes the number of vertices and where we can easily compute its essential density to produce induced subcomplexes of \( \Delta(n, p) \) with 2-torsion.

In [4], the authors find two (nonisomorphic) minimal flag triangulations of \( \mathbb{R}P^2 \), each of which have 11 vertices and 30 edges and differ by a single bistellar 0-move. One of these flag triangulations is depicted in Figure 6.

For the remainder of this section, let \( G \) denote the underlying graph of this flag triangulation of \( \mathbb{R}P^2 \), which we denote by \( \Delta(G) \) as it is the clique complex of \( G \). To understand the probability that this particular triangulation of \( \mathbb{R}P^2 \) appears as an induced subcomplex of \( \Delta(n, p) \), we need to compute the essential density \( m(G) \).
Lemma 5.1. For the graph $G$ underlying the flag triangulation of $\mathbb{R}P^2$ exhibited in Figure 6, the essential density $m(G)$ is $30/11$.

Proof. This amounts to an exhaustive computation, which is summarized in Table 5. In particular, Table 5 identifies the maximal number of edges that a subgraph $H \subset G$ on $|V(H)|$ vertices can have, for each $|V(H)| \leq 11$. One can see from the table that $m(G)$ is maximized by the entire graph, and thus $m(G) = |E(G)|/|V(G)| = 30/11$.

Lemma 5.1 shows that the graph $G$ is strongly balanced in the sense of Definition 2.2. While we expect the essential density of our complexes $X_m$ to be lower than the coarse bound of $1/2 \maxdeg(X_m)$ (see Remark 4.4), we note that in the case of the graph $G$, this difference is not very large. In fact, we have $1/2 \maxdeg(G) = 3$ and $m(G) = 30/11 \approx 2.72$.

Combining Lemma 5.1 and Theorem 4.2 we obtain an analogue of Proposition 4.1.

Proposition 5.2. if $\Delta \sim \Delta(n, p)$ is a random flag complex with $n^{-11/30} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$, then $P\left[ \Delta(G) \cap \Delta(n, p) \right] \to 1$ as $n \to \infty$.

Proof. The proof is nearly identical to that of Proposition 4.1 so we omit the details.

Question 5.3. It would be interesting to know whether $p \ll n^{-11/30}$ is a sharp threshold for the appearance of 2-torsion in the homology of $\Delta(n, p)$. A closely related question is whether there exists a flag complex $X$ with 2-torsion homology and a smaller essential density.

6. Torsion in the Betti tables associated to $\Delta$

We now prove Theorem 1.3. The hard work was done in the previous sections.

Proof of Theorem 1.3. Assume $n^{-1/6} \ll p \leq 1 - \epsilon$ and let $\Delta \sim \Delta(n, p)$. Let $X_m$ be as in Theorem 3.1. By Proposition 4.1, $\Delta$ contains $X_m$ as an induced subcomplex, with high probability. Since $H_1(X_m)$ has $m$-torsion, Hochster’s Formula (see Fact 2.3) gives that the Betti table of the Stanley–Reisner ideal of $\Delta$ has $\ell$-torsion for every $\ell$ dividing $m$.

We can also apply the more detailed study of 2-torsion from §5 to obtain a result on the appearance of 2-torsion in the Betti tables of random flag complexes.
Proposition 6.1. Let \( r \geq 4 \), and let \( \Delta \sim \Delta(n, p) \) be a random flag complex with \( n^{-1/(r-1)} \ll p \ll n^{-1/r} \). With high probability as \( n \to \infty \), the Betti table of the Stanley–Reisner ideal of \( \Delta \) has 2-torsion.

Proof. The proof is the same as the proof of Theorem 1.3, but utilizing Proposition 5.2 in place of Proposition 4.1 since \( r \geq 4 \) and \( n^{-1/(r-1)} \ll p \) gives \( n^{-11/30} \ll p \). \( \square \)

Note that the bound on \( r \) for the appearance of 2-torsion in Proposition 6.1 is lower than in Theorem 1.3. This is due to our ability to sharply compute the essential density in this case; in contrast, for Theorem 1.3 we work with a bound on the essential density. See Question 7.3 and Remark 7.4 for more on the possibility of lowering the bound on \( r \) in Theorem 1.3. It would be interesting to understand a precise threshold on the attaching probability \( p \) such that the Betti table of the Stanley–Reisner ideal of \( \Delta \) does not depend on the characteristic. A related question is posed in Question 7.3.

We also note that our constructions are based entirely on torsion in the \( H_1 \)-groups, and thus we obtain Betti tables where the entries in the second row of the Betti table (that is the row of entries of the form \( \beta_{i,i+2} \)) depend on the characteristic. Since Newman’s work also produces small simplicial complexes where the \( H_r \)-groups have torsion, for any \( i \geq 1 \) [25, Theorem 1], one could likely apply the methods of [33] to produce thresholds for where the other rows of the Betti table would depend on the characteristic, and it might be interesting to explore the resulting thresholds.

| \(|V(H)|\) | max\(|\{E(H)\}|\) | \(V(H)\) | max \(\frac{|E(H)|}{|V(H)|}\) |
|------|------------------|--------|------------------|
| 1    | 0                | \(\{v_1\}\) | 0                |
| 2    | 1                | \(\{v_1, v_2\}\) | \(\frac{1}{2}\) |
| 3    | 3                | \(\{v_1, v_2, v_6\}\) | 1                |
| 4    | 5                | \(\{v_1, v_2, v_5, v_6\}\) | \(\frac{5}{4}\) |
| 5    | 7                | \(\{v_1, v_2, v_4, v_5, v_6\}\) | \(\frac{7}{5}\) |
| 6    | 10               | \(\{v_1, v_4, v_7, v_8, v_9, v_{11}\}\) | \(\frac{5}{3}\) |
| 7    | 13               | \(\{v_1, v_2, v_4, v_7, v_8, v_9, v_{11}\}\) | \(\frac{13}{7}\) |
| 8    | 17               | \(\{v_1, v_2, v_4, v_6, v_7, v_8, v_9, v_{11}\}\) | \(\frac{17}{8}\) |
| 9    | 21               | \(\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{11}\}\) | \(\frac{7}{3}\) |
| 10   | 25               | \(\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{11}\}\) | \(\frac{5}{2}\) |
| 11   | 30               | \(\{v_1, \ldots, v_{11}\}\) | \(\frac{30}{11}\) |

Table 5. With \( G \) as the underlying graph of the complex in Figure 6, this table computes the maximal number of edges of subgraphs \( H \subset G \) with varying number of vertices.
Finally, we return to the question of $\ell$-torsion in Veronese syzygies. Since there is very little computational evidence either in favor or in opposition to Conjecture 1.1, we base the conjecture upon an heuristic model.

As noted in the introduction, one of the central results of [18] is that for $\Delta \sim \Delta(n, p)$ with $n^{-1/(r-1)} \ll p \ll n^{-1/r}$ and $S = k[x_1, \ldots, x_n]$, the Betti table $\beta(S/I_\Delta)$ as $n \to \infty$ will exhibit the known nonvanishing properties of the Betti table of the Veronese embeddings $\beta(\mathbb{P}^r_k; d)$ as $d \to \infty$. Based on this connection, we use Theorem 1.3 as an heuristic for understanding the behavior of $\beta(\mathbb{P}^r_k; d)$, in particular, when these Betti tables depend on the characteristic.

For Conjecture 1.1, we set $r \geq 7$ and use the framework of Theorem 1.3. With these hypotheses, as $n \to \infty$, the Betti table associated to $\Delta$ will depend on the characteristic with high probability. We thus conjecture a corresponding statement for $\beta(\mathbb{P}^r_k; d)$ with $r \geq 7$ and $d \to \infty$. While we conjecture that this dependence on characteristic should be quite widespread, the only known examples of such behavior come from [2]. It would thus be very interesting to produce any new examples (or non-examples!) of torsion in Veronese syzygies. For instance:

**Question 7.1.** Can one find any new examples of Veronese embeddings whose Betti tables depend on the characteristic? For a given $\ell$, can one produce a Betti table with $\ell$-torsion? Can one find some $\beta(\mathbb{P}^r_k; d)$ which has $\ell$-torsion for two (or more) distinct primes?

We find it especially surprising that there are no known examples of 2-torsion. Conjecture 1.5 represents one way to sharpen Conjecture 1.1. In particular, since Theorem 1.3 shows that, with $r \geq 7$ and within the given framework, $m$-torsion appears with high probability as $n \to \infty$ in the Betti table of the Stanley–Reisner ideal of $\Delta$, we conjecture that $m$-torsion should appear frequently in the Betti tables of the $d$-uple Veronese embeddings for $\mathbb{P}^r$ as $d \to \infty$.

There are many follow-up questions one might ask, and we assemble some of these below.

**Question 7.2.** What is the minimal value of $r$ such that $\beta(\mathbb{P}^r_k; d)$ depends on the characteristic for some $d$? (It is known that $1 < r \leq 6$.)

To develop an heuristic for this question, along the lines of this paper, one would need to consider the following question, which seeks to sharpen Theorem 1.3.

**Question 7.3.** Let $m \geq 2$. For a random flag complex $\Delta \sim \Delta(n, p)$, what is the threshold on $p$ such that the Betti table of the Stanley–Reisner ideal of $\Delta$ has $m$-torsion with high probability as $n \to \infty$?

**Remark 7.4.** We know of two natural ways that one could improve the bound on $r$ in Theorem 1.3. First, one could perform a more detailed study of the essential density $m(H_m)$, as that value is surely lower than our chosen bound $\frac{1}{2} \maxdeg(X_m)$. Second, one could aim to produce flag complexes $X'_m$ with torsion homology (not necessarily in $H_1$) which have a lower essential density than $X_m$. Of course, following the heuristic at the heart of this paper, any such improvement of the bound on $r$ in Theorem 1.3 would suggest a corresponding improvement of the bound on $r$ in Conjectures 1.1 and 1.5.

In a different direction, one might ask about how large $n$ needs to be before we expect to see that the Betti table associated to $\Delta$ has $\ell$-torsion.
Question 7.5. Fix a prime $\ell$ and integer $r \geq 7$. Let $\Delta \sim \Delta(n, p)$ be a random flag complex with $n^{-1/(r-1)} \ll p \ll n^{-1/r}$. For a constant $0 < \epsilon < 1$, approximately how large does $n$ need to be to guarantee that

$$
P \left[ \text{Betti table associated to } \Delta \text{ has } \ell\text{-torsion} \right] \geq 1 - \epsilon?
$$

It would be interesting to even answer this question for 2-torsion, where the concrete constructions from §5 make the question seemingly more tractable. The corresponding question for Veronese embeddings would be the following:

Question 7.6. Fix a prime $\ell$ and integer $r \geq 7$. Can one provide lower/upper bounds on the minimal value of $d$ such that $\beta(\mathbb{P}^r; d)$ has $\ell$-torsion?

We could turn to even more quantitative questions related to Conjecture 1.5 as well.

Question 7.7. Fix a prime $\ell$ and an integer $r \geq 7$. Can one describe the set of $d \in \mathbb{Z}$ such that $\beta(\mathbb{P}^r; d)$ has $\ell$-torsion? Can one bound or estimate the density of that set?

Question 7.8. Can one estimate or bound the growth rate of the number of primes $\ell$ such that $\beta(\mathbb{P}^r; d)$ has $\ell$-torsion as $d \to \infty$?

Even a compelling heuristic for these last two questions could be quite interesting.

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