Pseudo-classical theory for directed transport at quantum resonance

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Abstract. Recent studies have demonstrated that a directed current arises in kicked atom systems at quantum resonance (so-called ‘resonance ratchets’). Here, we demonstrate that this effect can be explained using a pseudo-classical model by taking classical initial conditions analogous to the initial quantum state. A corollary of our result is that a current is also expected to arise in the \textit{actual} standard classical limit of the kicked atoms, demonstrating that the phenomenon can arise even in the absence of quantum interference. We show that in the standard classical limit, the momentum current undergoes far less saturation due to quasi-momentum spread than for the quantum resonance case. Additionally, we demonstrate that a phase-independent analytical scaling law exists for the mean momentum as a function of a single combined parameter and show that it predicts an unexpected current inversion regime.

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1. Introduction

Dynamical systems that display directed motion in the absence of unbalanced forces are of considerable interest across scientific fields. On the one hand, physicists hope to take advantage of such effects to realize novel transport solutions at the atomic scale [1] and such studies can also improve our understanding of basic thermodynamic phenomena [2, 3]. On the other hand, biologists have an interest in studying mechanisms for movement in situations where thermal fluctuations are large compared with the forces available to propel an organism. The study of Brownian and biological motors [4] is indeed one of the driving forces behind the study of directed diffusion in general.

In the last few years, there have been a number of studies of systems where fluctuations provided by Hamiltonian chaos (rather than noise) drive a current in a system with broken symmetries [5]. The coherent nature of these noiseless ratchet realizations allows quantum effects such as current saturation and current reversal to be easily observed. Only a few realizations of such Hamiltonian ratchets have been performed experimentally [6]. However, a closely related system, which uses atoms in an initial quantum superposition of two motional states exposed to a sinusoidal potential pulsed at the quantum resonance (QR) [7] (i.e. at the Talbot time [8]) has produced readily controllable momentum currents in two separate experiments [9, 10]. In these studies, and in related studies at QR, using an asymmetric potential [11], the strictly quantum nature of the initial state along with the necessity of the wave-mechanical Talbot effect has justifiably led to the claim that this ratchet-like directed motion is fundamentally quantum-mechanical.

Although we do not aim to undermine the above claim, it is nonetheless clear from previous studies of the QR phenomenon that a pseudo-classical theory suffices to explain near-resonant dynamics to an excellent approximation [12]–[16]. The so-called $\epsilon$-classical standard map ($\epsilon$SM) and the associated scaling law for the quantum resonant peaks of the atom optics kicked rotor provide an elegant theoretical structure for understanding quantum dynamics for small detuning from resonance $\epsilon$. Since the QR ratchets of [9, 10] rely on QR to produce a momentum current, one might expect that a pseudo-classical description of the phenomenon is also possible.

In the following, we will show that this is indeed the case. In fact, there is only one hurdle to overcome before the existing $\epsilon$-classical theory can be adapted to the case of resonant ratchets—the identification of appropriate classical initial conditions that correspond to the initial quantum superposition state required to produce directed diffusion. Once this has been done, the phenomena documented in [9, 10] can readily be explained using the pseudo-classical theory of [13].

Along with the $\epsilon$-classical formulation of the resonance ratchet problem come a number of interesting revelations about the system. Firstly, we find that a momentum current also exists in the semi-classical limit of vanishing kicking period. Furthermore, in this limit, the momentum current is much less sensitive to the initial quasi-momentum spread, and so is more useful for real transport applications. Additionally, we demonstrate that a scaling law exists for the momentum current as a function of a combined parameter of the kick strength, detuning from QR, and the kick number. This scaling law reveals that current reversals should occur in the resonant ratchet, not just for particular parameter values but for certain parameter families.

4 Strictly, since no ratchet potential is applied, the system does not constitute a ratchet in the normal sense. Henceforth, however, we will use ‘ratchet’ without quotation marks with the understanding that it refers to the behaviour observed in [9, 10].
Such a hitherto unnoticed phenomenon should certainly be observable in experiments at finite detuning from exact QR. Furthermore, the scaling law for the current may be expressed in a form which is independent of the quantum phase $\phi$, raising interesting questions about the role of quantum interference in the dynamics of the atoms.

2. $\epsilon$-classical map and initial conditions

Here we will consider the well-known atom optics quantum kicked rotor (AOQKR) system [17]. Atoms are subject to pulses from an off-resonant optical standing wave. Assuming spontaneous emission can be neglected, atomic motion is governed by the following Hamiltonian [18] in dimensionless units [12]–[14]:

$$H = \frac{\tau p^2}{2} + k \cos (\Theta) \sum_{t=1}^{N} \delta(t' - t),$$

where $\Theta = 2k_l x_{\text{phys}}$ (the ‘phys’ subscript denotes physical units as opposed to dimensionless units) is the scaled position and $p = p_{\text{phys}}/2p_{\text{rec}}$, $p_{\text{rec}} = \hbar k_l$, its conjugate momentum, with $2k_l$ being the wavenumber of the standing wave. $k = V_0 \Delta / \hbar$ is the kick strength (for an optical standing wave potential depth of $V_0$ pulsed on for time $\Delta$). We also transform the kicking period $T_{\text{phys}}$ into the dimensionless quantity $\tau = 8\omega_{\text{rec}} T_{\text{phys}}$, where $\omega_{\text{rec}} = p_{\text{rec}}^2/(2M\hbar)$ for an atomic mass $M$, is the recoil frequency of the atoms in the field. Note that when the kicking period $T_{\text{phys}}$ is an integer multiple of the QR time, then $\tau$ is an integer multiple of $2\pi$. We also use a dimensionless continuous time variable $t'$ and the integer kick counter $t$.

Classically, the system is a paradigm of chaos studies; the Hamiltonian may be replaced by a discrete map that gives discrete time atomic position $x_t$ and momentum $p_t$ after each $\delta$-kick. For sufficiently large $k$, the system exhibits global chaos [19]. However, if the system is quantized, it is well known that two uniquely quantum behaviours can appear. Best studied is dynamical localization where atomic diffusion is halted after a certain time due to quantum interference. Also unique to the quantum system is QR (which concerns us here), which occurs is dynamical localization where atomic diffusion is halted after a certain time due to quantum interference. Also unique to the quantum system is QR (which concerns us here), which occurs is dynamical localization where atomic diffusion is halted after a certain time due to quantum interference. Also unique to the quantum system is QR (which concerns us here), which occurs is dynamical localization where atomic diffusion is halted after a certain time due to quantum interference.

Starting from the quantum evolution operator over a single kick $U = \exp (-i k \cos(\Theta) ) \exp (-i \tau p^2/2)$, it was shown in [13] that introducing a fictitious Planck constant $\epsilon = 2\pi l - \tau$ allowed the near resonant dynamics to be approximated by a pseudoclassical map (‘pseudo-classical’ due to the presence of a variable quantity in the role of Planck’s constant). Using appropriate definitions, the $\epsilon$SM may be written in the same form as the usual classical map viz. [14]

$$J_{t+1} = J_t + \tilde{k} \cos (\theta_t), \quad \theta_{t+1} = \theta_t + J_t,$$

where $J_t$ is a scaled momentum variable defined by $J_t = \epsilon p_t + l \pi + \tau \beta$ with $\beta$ the non-integer part of the momentum or quasi-momentum and $\theta_t = \Theta_t + \pi (1 - \text{sign}(\epsilon))/2$ is the rescaled position. The effective Planck constant $\epsilon$ enters in the definition $\tilde{k} = \epsilon |k|$.

The $\epsilon$SM reproduces the dynamics of the quantum kicked rotor near QR (i.e. when $\tau$ is a multiple of $2\pi$) very well as was demonstrated theoretically in [12, 13] and experimentally in [14]. However, if we seek to apply it to the phenomenon of directed transport at QR as investigated experimentally in [9, 10], we run into a problem: for the derivation of the

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scaling results in [13]–[16], the pseudo-classical formalism reviewed above averages over an initial momentum distribution covering the entire Brillouin zone (i.e. $0 \leq \beta \leq 1$) and uses the pendulum Hamiltonian as an approximation to the true kicked rotor Hamiltonian (see e.g. [13] for details). By symmetry of those initial conditions and of the pendulum dynamics we know that there can be no net momentum gain. Nonetheless, we will show that it is possible to adapt the usual $\epsilon$-classical method to dynamics to describe the resonance ratchet effect by choosing classical initial conditions which are analogous (but not necessarily equivalent) to the quantum case.

Of course, in the ratchet experiments an initial quantum superposition of momentum states (and thus position states) was created, which has no direct classical analogy. Thus, our strategy is to choose initial conditions that best correspond to the quantum resonant ratchet initial conditions by considering the momentum and spatial distributions of atoms placed into the $|\psi_i\rangle = 1/\sqrt{2}(|p = 0\rangle + |p = 2\bar{h}k\rangle)$ initial state. We will show that by matching the quantum and classical distributions (both in momentum and position space) the $\epsilon$SM may be used to describe the directed transport behaviour seen in [9, 10].

Firstly, we consider the initial momentum of the atoms. The momentum probability distribution corresponding to the initial momentum superposition state $\psi_i$ is

$$P(p_0) = \frac{1}{2}(\delta_{p_0,0} + \delta_{p_0,1}).$$

We will take this to be the momentum distribution for the classical atoms as we continue below.

What about the position space distribution of atoms? If the momentum space wavefunction is $\psi_i$, then the position space distribution is non-trivially the Fourier transform of $\psi_i$. We note that for classical atoms, no such relation between the momentum and position space distributions for the atoms is actually required, given the absence of the uncertainty principle in classical physics. Writing the momentum eigenstates as $\delta$ functions in momentum space, we see that the appropriate Fourier pair is $\langle p | \psi_i \rangle = \delta(p) + \delta(p+1) \Rightarrow \frac{1}{2\pi}(1 + \exp(i\theta))$. We can also take into account the possibility that the initial spatial distribution has a phase difference compared to the periodic potential by adding a phase term $\phi$ inside the complex exponential. This mimics the role of quantum phase in the pseudo-classical system. This leads to a position space probability distribution of

$$P(\theta) = |\psi(\theta)|^2 = \frac{1}{2\pi}(1 + \cos(\theta + \phi)).$$

3. $\epsilon$-classical momentum current

We now show that if these probability distributions are used, the $\epsilon$-classical map predicts the current found in the resonance ratchet system. We proceed by iterating the standard map in the usual way to find $J_i$ dependent on particular initial conditions $J_0$ and $\theta_0$ governed by the definitions of the $J$ and $\theta$ variables along with the probability distributions given by equations (3) and (4). This gives

$$J_N = J_0 + \tilde{k} \sum_{t=0}^{N-1} \cos(\theta_t).$$

The mean momentum is found by averaging over initial conditions. At perfect QR, the atoms have the same value of $\theta$ modulo $2\pi$ after each iteration, we can replace $\theta_t$ with $\theta_0$. Thus, we
Figure 1. Simulations of the $\epsilon$SM (discrete points) compared with the predictions of equation (8). The data sets are for $\phi = 0$ (crosses, dashed line), $\phi = \pi/2$ (circles, dotted line) and $\phi = \pi$ (squares, dash-dotted line).

have at a particular discrete time $t$

$$\langle J_{t,\text{res}} \rangle = \langle J_0 \rangle + \tilde{k} t \langle \cos (\theta_0) \rangle.$$  

Finally, to find the physical mean momentum at resonance from the $\epsilon$SM, we calculate the quantity [13]

$$\langle p_{t,\text{res}} \rangle = \frac{|J - J_0|}{|\epsilon|} = \frac{\tilde{k} t}{|\epsilon|} \langle \cos (\theta_0) \rangle.$$  

(7)

For kicked rotor experiments with cold thermal atoms (i.e. atoms which still have a broad momentum spread compared with the width of the Brillouin zone of the periodic kick potential), equation (7) is trivially zero because atoms are distributed uniformly over the standing wave to a good approximation. However, in the resonant-ratchet experiments, the position space probability distribution is given by equation (4). In this case, we can show that $\langle \cos (\theta_0) \rangle = \cos (\phi)/2$. Thus, we find that

$$\langle p_{t,\text{res}} \rangle = \frac{kt}{2} \cos (\phi).$$  

(8)

It is immediately apparent that equation (8) will reproduce the $\phi$ dependent momentum current seen in [9]. This is demonstrated in figure 1, where $\epsilon$-classical simulation results are seen to be in good agreement with the predictions of equation (8).

Figure 2 shows the phase space of the resonance ratchet system for three different kick numbers for an initial phase $\phi = 0$ with initial (classical) distributions of momentum and position given by equations (3) and (4), respectively. The figures show that the phase space accumulates an imbalance in positive and negative momentum trajectories as time increases. Because the case considered here is not exactly on resonance, after a finite time ($\sim400$ kicks for the parameters in figure 2) the phase space becomes symmetrically filled and the current disappears. We discuss the off-resonant behaviour of the system further in section 5.
The $\epsilon$-classical phase space shown for a sinusoidal initial position distribution and $\epsilon = 10^{-4}$ for a phase of 0 for total number of kicks (a) $N = 25$, (b) $N = 100$ and (c) $N = 250$ of the potential relative to the spatial density.

4. The effect of quasi-momentum

We now look at the more general case where the atoms have an initial quasi-momentum. Such an analysis is required to explain the results of the Summy group [10]. We start with the standard result from $\epsilon$-classical theory that the momentum at QR is given by

$$\langle p_{t, \text{res}} \rangle = \lim_{\epsilon \to 0} \frac{\langle J_t - J_0 \rangle}{\epsilon} = k \left\{ \frac{\sin (t J_0/2)}{\sin (J_0/2)} \left[ \sin [\theta_0 + (t - 1)J_0/2] \right] \right\},$$

where the average is taken at fixed time $t$ over initial conditions in momentum and position as given by equations (3) and (4). The reader may refer to [13], where a similar limit was calculated.

We proceed by computing the averages of the terms $\sin (\theta_0) \cos (\theta_0)$ over the distribution $P(\theta_0) = 1/(2\pi)[1 + \cos(\theta_0 + \phi)]$ for $\theta_0 \in [0, 2\pi]$. We find that

$$\langle \cos (\theta_0) \rangle = \frac{1}{2} \cos (\phi), \quad \langle \sin (\theta_0) \rangle = -\frac{1}{2} \sin (\phi).$$

Using equation (10) and trigonometric identities, and noting that by definition of $J$, $J_0 \to 0$ in the $\epsilon \to 0$ limit which we are considering here, we can average equation (9) over the initial conditions to give

$$\langle p_{t, \text{res}} \rangle = \frac{k \sin [(\pi l + \tau \beta)t/2]}{2 \sin [(\pi l + \tau \beta)/2]} \sin [(t - 1)(\pi l + \tau \beta)/2 - \phi].$$

This formula is essentially the same as equation (1) from [10], where it was derived using a purely quantum analysis. In figure 3, we show the variation of the momentum current with $\beta$ for two different QRs.

The above analysis shows that an $\epsilon$-classical treatment correctly models the behaviour of the quantum resonant ratchet. This is despite the fact that the behaviour of an atomic wavepacket subject to pulses from a standing wave at QR is fundamentally quantum in nature. The success of the $\epsilon$-classical approach implies that the salient aspects of the quantum behaviour can be
Figure 3. The momentum current as a function of $\beta$ as given by equation (11) for QRs with (a) $l = 1$ and (b) $l = 2$. In each case, the solid line shows the results for $\phi = -\pi/2$ and the dashed line shows the results for $\phi = \pi/2$. The case in (a) demonstrates exactly the behaviour seen in [10].

captured in $\epsilon$-classical correlations between the atomic position distribution and the driving field. We note that, as pointed out in [10], the actual classical mechanics for a quantum resonant pulse period do not exhibit a momentum current precisely because the mixing caused by classical chaos destroys the sinusoidal position distribution necessary for the ratchet effect to occur.

However, the fact that the $\epsilon$-classical description is successful at QR suggests that there should also be a current in the semi-classical limit $\epsilon \equiv \tau \to 0$. This was established for the resonance in mean energy in [15], and its origin lies in the dependence of $J$ on the product $\tau \beta$, which for this case tends to zero, too, as $\epsilon \to 0$.

Nonetheless, there is a very important difference between the $\epsilon$-classical map for $l = 0$, $\epsilon \to 0$ (the semi-classical limit) and $l > 0$, $\epsilon \to 0$ which we will now comment on. The general behaviour of the resonance ratchet in the presence of an initial momentum spread, as has already been noted [10], is saturation of the momentum current at a finite kick number, rather than the unbounded momentum growth that is predicted in the absence of other limiting factors. This saturation, of course, limits the usefulness of the ratchet effect in all realistic experimental situations where a quasi-momentum spread is inevitable.

However, the effect of quasi-momentum on the system is not independent of the kicking period. Indeed, the longer the kicking period is, the more time there is for the system to resolve the quasi-momentum differences between atoms. For this reason, as has been noted elsewhere [15], the QR peaks in energy for the kicked rotor are narrower and taller at the $l = 0$ case than for $l > 0$. For similar reasons, when $l = 0$, the momentum current becomes insensitive to the initial momentum spread, in principle preventing saturation of the momentum current, cf, the above argument $\tau \beta = \epsilon \beta \to 0$.

Figure 4 shows the effects of an initial quasi-momentum spread for $l = 0$ and 2, with $\epsilon = 0$ in both cases. In the upper and lower plots of figure 4, solid lines show $\epsilon$-classical
Figure 4. Simulation results for $\epsilon$-classical simulations (lines) and quantum simulations (discrete points) showing the effect of initial momentum spread. The upper plot shows the results as a function of kick number when $l = 2$ for uniform quasi-momentum distributions of width $0\hbar k$ (solid line), $0.01 \times \hbar k$ (dashed line, crosses), $0.1 \times 2\hbar k$ (dash-dotted line, circles) and $2\hbar k$ (dotted line, squares). The lower plot shows simulations for the same values of quasi-momentum spread with the same symbols, but this time near the true classical limit (i.e. $l = 0$ in the definition of $J$, see around equation (2), $\tau \beta = \epsilon \beta \rightarrow 0$).

Simulation results whereas discrete points are from quantum simulations. We see that as the quasi-momentum spread is increased from $0.01 \times 2\hbar k$, to $2\hbar k$, the momentum current saturates at a low kick number in the $l > 0$ case. However, in the $l = 0$ case (the semi-classical limit), $\langle p \rangle$ does not saturate.

It should not come as a surprise that in the classical limit the effect of quasi-momentum, which is an inherently quantum-mechanical quantity, becomes negligible. However, this fact is very helpful because it allows us to eliminate the principle problem with the QR ratchet—saturation due to quasi-momentum spread—while retaining the benefits of a momentum current. Of course, the flip-side to this benefit is that the control over the momentum current afforded by altering the quasi-momentum is not present in the $l = 0$ limit. This can be seen immediately by inspecting equation (11), since the quantity $\tau \beta$, which gives the variation of $\langle p \rangle$ with $\beta$ is always approximately 0 in the $l = 0$ limit, for which $\tau = \epsilon \rightarrow 0$.

5. Scaling law for momentum current

So far in the study of directional momentum transport in these resonant systems, the effect of detuning from exact QR has not been considered. In general, it is not a trivial matter to calculate a closed form solution for off-resonant dynamics (representative examples of which are shown in figure 5), and the scaling functions for the atomic energy found in [13, 14] are thus considered to be important results.

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Before considering the possibility of a scaling law for the momentum current, we will inspect the time behaviour of the resonance ratchet as the kicking time diverges from exact resonance.

In [14] it was shown that a one parameter scaling law exists for the mean energy of the quantum kicked rotor near QR. In particular, the scaled energy is given by

$$E_{k \phi t} = 1 - \Phi(x) + \frac{\pi}{4x} G(x),$$

(12)

where $x = \sqrt{k|\epsilon|t}$ and $\Phi(x)$ and $G(x)$ are closed form functions derived by considering the topological changes in the $\epsilon$-classical phase space as $\epsilon$ increases.

We now demonstrate that such a law also exists in the case of the momentum current. We proceed according to [13]. First, we employ the scaled variable $J' = J/(\sqrt{k|\epsilon|})$ and apply the pendulum approximation to the kicked rotor Hamiltonian [19] (in the scaled variables $J'$ and $\theta$). The motion is then described—in continuous time—by $H' \approx (J')^2/2 + |\epsilon|k \cos(\theta)$. Expressing the original momentum $p$ in terms of those scaled variables, we have

$$\langle p \rangle = \frac{1}{|\epsilon|} \sqrt{k|\epsilon|} \langle J' - J'_0 \rangle.$$

(13)

Under the pendulum approximation, we can determine $J' = J'(\theta_0, J'_0, x)$ using solutions for the pendulum trajectories at arbitrary initial conditions. Because the definition of $J$ multiplies the physical momentum $p$ by $|\epsilon|$, for small $\epsilon$, the initial conditions for $J'_0$ when $p = 0$ or 1 are essentially the same. Therefore, we take the initial momentum $J'$ to be zero and merely integrate over the (non-uniform) initial position distribution:

$$\langle J' - J'_0 \rangle = \int_{-\pi}^{\pi} d\theta_0 P(\theta_0) (J'(\theta_0, J'_0, x) - J'_0).$$

(14)
Figure 6. Scaled mean momentum $\langle p \rangle / (-kt \sin(\phi))$ plotted against the scaling variable $x = \sqrt{k|\epsilon|/t}$. Discrete points show $\epsilon$-classical (crosses) and quantum (all other symbols) simulation results for a variety of experimentally relevant parameters as indicated in the legend. The solid line shows the analytical scaling function given by equation (17). The same data are shown in the inset on logarithmic axes, where excellent agreement is found over at least two orders of magnitude on both axes.

Taking $P(\theta_0)$ as given by equation (4) and expanding the cosine term, we find that

$$\langle J' - J'_0 \rangle = \int_{-\pi}^{\pi} d\theta_0 (J'(\theta_0, J'_0 = 0, x)) + \cos(\phi) \int_{-\pi}^{\pi} d\theta_0 \cos \theta_0 (J'(\theta_0, J'_0 = 0, x))$$

$$- \sin(\phi) \int_{-\pi}^{\pi} d\theta_0 \sin \theta_0 (J'(\theta_0, J'_0 = 0, x)).$$

(15)

Firstly, we note that by symmetry of the initial conditions, the first term on the rhs of equation (15) vanishes. Furthermore, because the solutions for $J'$ are odd, the term in $\cos(\phi)$ also vanishes and we are left with just the sine term in $\phi$. If we define $F(x) \equiv \int_{-\pi}^{\pi} d\theta_0 \sin \theta_0 (J'(\theta_0, J'_0 = 0, x))$, then substituting into equation (13) gives

$$\langle p \rangle = -\frac{\sqrt{k}}{\sqrt{|\epsilon|}} \sin(\phi) F(x).$$

(16)

Finally, dividing by $-kt \sin(\phi)$ gives the following new scaling law:

$$\frac{\langle p_{t,\epsilon} \rangle}{-kt \sin(\phi)} \approx R(x) \equiv \frac{F(x)}{x}.$$ 

(17)

We tested this scaling law by performing quantum simulations for a wide variety of all parameters of the system ($\phi$, $\epsilon$, $t$ and $k$), and compared the results with scaled energies from an $\epsilon$-classical simulation for a single parameter set. The results are shown in figure 6. We see very...
good agreement between quantum simulations (done over a broad range of parameters) and the scaled $\epsilon$-classical simulation (for a single parameter set with varying $\epsilon$). For values of $k > 10$, there seems to be a breakdown in scaling, which probably results from the large spread of the wavepackets suppressing the current. However, in general, it is clear that a single parameter scaling law is valid for the momentum current over a broad parameter range, as long as time is neither too short (where our argument based on a continuous time evolution and on average momentum values fails) nor too large (where the $\epsilon$-classical approximation would fail).

Note that the present scaling law only exists for very small spread in the initial quasi-momentum. In the case of a uniform quasi-momentum distribution, there is no momentum current because of the symmetric filling of the pseudo-classical phase space.

The most intriguing feature of the scaling function is the fact that the scaled current (and thus the unscaled current as well) becomes negative for certain $x$. This so-called current inversion has also been found in other quantum ratchets $[1, 5]$, but to our knowledge this is the first prediction of current inversion due to parameter variation for the QR ratchet system (aside from the obvious dependence of the current on $\phi$ and $\beta$). Indeed, studies of the system up until now have focused on exactly resonant kicking. This precludes the possibility of finding the current inversion phenomenon, which requires nonzero $x$, and hence nonzero $\epsilon$, or in other words off-resonant kicking. It should be a fairly simple matter to adapt current experiments to look for the scaling and current inversion effects.

Lastly, we note that we can use equation (8) to simplify the scaling function even further. Note that $\sin \phi = \pm \sqrt{1 - \cos^2 \phi} = \pm \sqrt{1 - (p_{t,\text{res}}/kt)^2}$, we may write the scaling function as

$$\frac{\text{sign}(\langle p_{t,\text{res}} \rangle \langle p_{t,\epsilon} \rangle)}{\sqrt{k^2 t^2 - \langle p_{t,\text{res}} \rangle^2}} \approx R(x) \equiv \frac{F(x)}{x}. \quad (18)$$

In equation (18) there is no dependence on the phase $\phi$. Thus to scale experimental measurements for comparison with the scaling function, it is sufficient to know simply the resonant momentum current and the kick strength $k$. The exact quantum phase plays no part in determining the unique dynamics of the system.

Equation (18) underlines an intriguing feature of the work presented here. Despite the explanation of the directed transport in $[9]$ in terms of interference of matter waves spreading from two initial wavepackets, the dynamics of the atomic ensemble may be predicted to a good approximation by a theory that takes only classical probabilities into account. We know that, in general, it is certainly not possible to replace quantum probability distributions by their classical counterparts since to do so ignores the interference terms that are responsible for uniquely quantum behaviour. However, in the present case, it is apparent that merely taking account of the spatial correlation between the atom density and the optical field intensity allows all the important features of the dynamics to be reproduced. Whether such an approach can be generalized to other systems where quantum interference is of importance is an interesting question for further research.

6. Conclusion

In the present paper, we have shown that a pseudo-classical approach may be used to describe the dynamics of QR ratchets so far studied using purely quantum methods. We first demonstrated that the pseudo-classical dynamics reproduced the main results of $[9, 10]$, including the existence of a momentum current dependent on the quantum phase $\phi$ and the variation of the current as a function of the atomic quasi-momentum $\beta$.  

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We then demonstrated a number of surprising new facts about this system, which were illuminated by our pseudo- or \( \epsilon \)-classical approach. Firstly, it follows as a simple corollary that the ratchet effect at QR also occurs in the limit as \( \tau \to 0 \). Although this does not change the fact that the ratchet effect seen when \( \tau \) equals the Talbot time (i.e. equals the quantum resonant value) is purely quantum, it does show that an equivalent effect exists even for purely classical atoms.

Secondly, we demonstrated that the current suppressing effect of a nonzero quasi-momentum spread is absent in the semi-classical limit. Surprisingly, this means that the use of a vanishing kicking period is actually the best strategy to accelerate atoms using the methods developed in [9, 10]. However, in this case, we also lose the possibilities for quantum control offered by the dependence of the momentum current on the quasi-momentum.

Finally, we demonstrated the existence of a scaling law for the momentum current as a function of a combined variable \( x = \sqrt{\epsilon |k| t} \). We tested the scaling law for a broad range of parameters and found good agreement between simulations and the one-parameter law so long as the kicking strength was not too large. Of particular interest was the current inversion predicted for certain \( x \). This inversion has not yet been observed experimentally, but the advantage of the scaling law formulation is that it predicts a family of experimental parameters \( \epsilon, k \) and \( t \) for which inversion should occur, allowing experimentalists freedom to find the most accessible regime in which to observe the effect.

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