ON NEW GENERAL INTEGRAL INEQUALITIES FOR $h$--CONVEX FUNCTIONS

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Abstract. In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are $h$--convex and we point out the results for some special classes of functions. Some applications to special means of real numbers are also given.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following inequality

\begin{equation}
  f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},
\end{equation}

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [2]-[7],[11]-[14],[16],[22], the results of the generalization, improvement and extension of the famous integral inequality (1.1).

In the paper [22] a large class of non-negative functions, the so-called $h$--convex functions is considered. This class contains several well-known classes of functions such as non-negative convex functions, $s$--convex in the second sense, Godunova–Levin functions and $P$--functions. Let us recall definitions of these special classes of functions.

Definition 1. $f : I \rightarrow R$ is a Godunova–Levin function or that $f$ belongs to the class $Q(I)$ if $f$ is non-negative and for all $x, y \in I$ and $\alpha \in (0, 1)$ we have

\begin{equation}
  f(\alpha x + (1 - \alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1 - \alpha}.
\end{equation}

The class $Q(I)$ was firstly described in [8] by Godunova and Levin. Some further properties of it are given in [7] [14] [15]. Among others, it is noted that non-negative monotone and non-negative convex functions belong to this class of functions.

In 1978, Breckner introduced $s$-convex functions as a generalization of convex functions as follows [4]:

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Definition 2. Let \( s \in (0, 1] \) be a fixed real number. A function \( f : [0, \infty) \to [0, \infty) \) is said to be \( s \)-convex (in the second sense) if \( f \) belongs to the class \( K^2_s \), if
\[
  f(ax + (1 - \alpha)y) \leq \alpha^s f(x) + (1 - \alpha)^s f(y)
\]
for all \( x, y \in [0, \infty) \) and \( \alpha \in [0, 1] \).

Of course, \( s \)-convexity means just convexity when \( s = 1 \). In [7] Dragomir et al. defined the concept of \( P \)-function as the following:

Definition 3. We say that \( f : I \to \mathbb{R} \) is a \( P \)-function, or that \( f \) belongs to the class \( P(I) \), if \( f \) is a non-negative function and for all \( x, y \in I \), \( \alpha \in [0, 1] \), we have
\[
  f(ax + (1 - \alpha)y) \leq f(x) + f(y).
\]

Let \( I \) and \( J \) be intervals in \( \mathbb{R} \), \( (0, 1) \subseteq J \) and \( h \) and \( f \) be real non-negative functions defined on \( J \) and \( I \), respectively. In [22], Varošanec defined the concept of \( h \)-convexity as follows:

Definition 4. Let \( h : J \to \mathbb{R} \) be a non-negative function, \( h \neq 0 \). We say that \( f : I \to \mathbb{R} \) is an \( h \)-convex function or that \( f \) belongs to the class \( SX(h, I) \), if \( f \) is non-negative function and for all \( x, y \in I \) and \( \alpha \in (0, 1) \) we have
\[
  f(ax + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).
\]

If inequality (1.2) is reversed, then \( f \) is said to be \( h \)-concave, i.e. \( f \in SV(h, I) \). The notion of \( h \)-convexity unifies and generalizes the known classes of functions, \( s \)-convex functions, Godunova-Levin functions and \( P \)-functions, which are obtained by putting in (1.2), \( h(t) = t \), \( h(t) = t^s \), \( h(t) = \frac{1}{t} \), and \( h(t) = 1 \), respectively.

In [5], Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for \( s \)-convex functions in the second sense.

Theorem 1. Suppose that \( f : [0, \infty) \to [0, \infty) \) is an \( s \)-convex function in the second sense, where \( s \in (0, 1) \), and let \( a, b \in [0, \infty) \), \( a < b \). If \( f \in L[a, b] \) then the following inequalities hold
\[
  2^{s - 1} f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s + 1}.
\]

The constant \( k = \frac{1}{s + 1} \) is the best possible in the second inequality in (1.3).

Theorem 2. Let \( f \in Q(I) \), \( a, b \in I \) with \( a < b \) and \( f \in L[a, b] \). Then
\[
  f\left(\frac{a + b}{2}\right) \leq \frac{4}{b - a} \int_a^b f(x)dx.
\]

Theorem 3. Let \( f \in P(I) \), \( a, b \in I \) with \( a < b \) and \( f \in L[a, b] \). Then
\[
  f\left(\frac{a + b}{2}\right) \leq \frac{2}{b - a} \int_a^b f(x)dx \leq 2[f(a) + f(b)].
\]

In [8], Dragomir et. al. proved two inequalities of Hadamard type for classes of Godunova-Levin functions and \( P \)-functions.

In [18], Sarikaya et. al. established a new Hadamard-type inequality for \( h \)-convex functions.
Theorem 4. Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$ and $f \in L([a, b])$. Then

\begin{equation}
\frac{1}{2h}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq [f(a) + f(b)] \int_0^1 h(t)dt.
\end{equation}

The following inequality is well known in the literature as Simpson’s inequality.

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a, b)$ and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^2.$$ 

In recent years many authors have studied error estimations for Simpson’s inequality; for refinements, counterparts, generalizations and new Simpson’s type inequalities, see [1, 17, 19, 20]. In [10], Iscan obtained a new generalization of some integral inequalities for differentiable convex mapping which are connected Simpson’s, midpoint and trapezoid inequalities, and he used the following lemma to prove this.

Lemma 1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^c$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. Then the following equality holds:

$$\lambda(\alpha f(a) + (1 - \alpha)f(b)) + (1 - \lambda)f(\alpha a + (1 - \alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx$$

$$= (b-a) \left[ \int_0^{1-\alpha} (t - \alpha\lambda) f' \left( tb + (1-t)a \right) dt \right] + \int_{1-\alpha}^1 (t-1 + \lambda(1-\alpha)) f' \left( tb + (1-t)a \right) dt \right].$$

The main inequality in [10], pointed out, is as follows.

Theorem 5. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^c$ such that $f' \in L[a, b]$, where $a, b \in I^c$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is convex on $[a, b]$,
where

\[ q \geq 1, \text{ then the following inequality holds:} \]

\[
(1.5) \quad \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x) dx \geq
\]

\[
(b-a) \left\{ \gamma_2 \left( \frac{1}{4} (\mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q)^\frac{1}{q} \right) + v_2 \left( \eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q \right)^\frac{1}{q} \right\}, \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha)
\]

\[
(b-a) \left\{ \gamma_2 \left( \frac{1}{4} (\mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q)^\frac{1}{q} \right) + v_2 \left( \eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q \right)^\frac{1}{q} \right\}, \quad \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha
\]

\[
(b-a) \left\{ \gamma_1 \left( \frac{1}{4} (\mu_3 |f'(b)|^q + \mu_4 |f'(a)|^q)^\frac{1}{q} \right) + v_1 \left( \eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q \right)^\frac{1}{q} \right\}, \quad 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\]

where

\[ \gamma_1 = (1 - \alpha) \left[ \alpha \lambda - \frac{(1 - \alpha)}{2} \right] \]

\[ \gamma_2 = (\alpha \lambda)^2 - \gamma_1 \]

\[ v_1 = \frac{1 - (1 - \alpha)^2}{2} - \alpha [1 - \lambda (1 - \alpha)] \]

\[ v_2 = \frac{1 + (1 - \alpha)^2}{2} - (\lambda + 1) [1 - \lambda (1 - \alpha)] \]

\[ \mu_1 = \frac{(\alpha \lambda)^3 + (1 - \alpha)^3}{3} - \frac{\alpha \lambda (1 - \alpha)^2}{2} \]

\[ \mu_2 = \frac{1 + \alpha^3 + (1 - \alpha \lambda)^3}{3} - \frac{(1 - \alpha \lambda)}{2} (1 + \alpha^2) \]

\[ \mu_3 = \frac{\alpha \lambda (1 - \alpha)^2}{2} - \frac{(1 - \alpha)^3}{3} \]

\[ \mu_4 = \frac{(\alpha \lambda - 1) (1 - \alpha^2)}{2} + \frac{1 - \alpha^3}{3} \]

\[ \eta_1 = \frac{1 - (1 - \alpha)^3}{3} - \frac{[1 - \lambda (1 - \alpha)]}{2} \alpha (2 - \alpha) \]

\[ \eta_2 = \frac{\lambda (1 - \alpha) \alpha^2}{2} - \frac{\alpha^3}{3} \]

\[ \eta_3 = \frac{[1 - \lambda (1 - \alpha)]^3}{3} - \frac{[1 - \lambda (1 - \alpha)]}{2} (1 + (1 - \alpha)^2) + \frac{1 + (1 - \alpha)^3}{3} \]

\[ \eta_4 = \frac{[\lambda (1 - \alpha)]^3}{3} - \frac{\lambda (1 - \alpha) \alpha^2}{2} + \frac{\alpha^3}{3} \]

In [2] Alomari et al. obtained the following inequalities of the left-hand side of Hermite-Hadamard’s inequality for s-convex mappings.
Theorem 6. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^s \), such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q, \ q \geq 1 \), is \( s \)-convex on \([a,b]\), for some fixed \( s \in (0,1] \), then the following inequality holds:

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left( \frac{2}{(s+1)(s+2)} \right)^{\frac{1}{p}} \left\{ (2^{1-s} + 1) |f'(b)|^q + 2^{1-s} |f'(a)|^q \right\}^{\frac{1}{q}}
\]

(1.6)

\[
+ \left\{ (2^{1-s} + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q \right\}^{\frac{1}{q}}.
\]

Theorem 7. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^s \), such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^p \), \( p > 1 \), is \( s \)-convex on \([a,b]\), for some fixed \( s \in (0,1] \), then the following inequality holds:

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right) \left( \frac{1}{s+1} \right)^{\frac{1}{q}}
\]

\[
\times \left\{ (2^{1-s} + s + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q \right\}^{\frac{1}{p}}
\]

(1.7)

\[
+ \left\{ (2^{1-s} + s + 1) |f'(b)|^q + 2^{1-s} |f'(a)|^q \right\}^{\frac{1}{q}},
\]

where \( p \) is the conjugate of \( q \), \( q = \frac{p}{p-1} \).

In [19], Sarikaya et al. obtained a new upper bound for the right-hand side of Simpson’s inequality for \( s \)-convex mapping as follows:

Theorem 8. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^s \), such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \), \( q > 1 \), is \( s \)-convex on \([a,b]\), for some fixed \( s \in (0,1] \) and \( q > 1 \), then the following inequality holds:

(1.8)

\[
\left| \frac{1}{6} f(a) + 4 f\left( \frac{a+b}{2} \right) + f(b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{12} \left( \frac{1 + 2^{p+1}}{3(p+1)} \right)^{\frac{1}{q}}
\]

\[
\times \left\{ \left( \frac{|f'\left( \frac{a+b}{2} \right)|^q + |f'(a)|^q}{s+1} \right)^{\frac{1}{p}} + \left( \frac{|f'\left( \frac{a+b}{2} \right)|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{p}} \right\},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

In [12], Kirmaci et al. proved the following trapezoid inequality:

Theorem 9. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^s \), such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \), \( q > 1 \), is \( s \)-convex on \([a,b]\), for some fixed \( s \in (0,1] \) and \( q > 1 \), then

(1.9)

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left( \frac{q-1}{2(2q-1)} \right)^{\frac{1}{q}} \left( \frac{1}{s+1} \right)^{\frac{1}{q}}
\]

\[
\times \left\{ \left( \left| f'\left( \frac{a+b}{2} \right) \right|^q + |f'(a)|^q \right)^{\frac{1}{q}} + \left( \left| f'\left( \frac{a+b}{2} \right) \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).
2. Main results

The following theorems give a new result of integral inequalities for $h$–convex functions. In the sequel of the paper $I$ and $J$ are intervals in $\mathbb{R}$, $(0,1) \subset J$ and $h$ and $f$ are real non-negative functions defined on $J$ and $I$, respectively and $h \in L[0,1]$, $h \neq 0$.

**Theorem 10.** Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^0$ such that $f' \in L[a,b]$, where $a, b \in I^0$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is $h$–convex on $[a,b]$, $q \geq 1$, then the following inequality holds:

\[
\lambda (\alpha f(a)+(1-\alpha)f(b))+(1-\lambda) f(\alpha a+(1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \leq \left\{(b-a) \left[ \frac{1}{2} \alpha^2 \lambda + \frac{1}{2} B^2 \right] \right\}
\]

where

\[
\gamma_1 = (1-\alpha) \left[ \alpha \lambda - \left( \frac{1-\alpha}{2} \right) \right], \quad \gamma_2 = (\alpha \lambda)^2 - \gamma_1,
\]

\[
v_1 = \frac{1-(1-\alpha)^2}{2} - \alpha [1-\lambda (1-\alpha)],
\]

\[
v_2 = \frac{1+(1-\alpha)^2}{2} - (\lambda + 1) (1-\alpha) [1-\lambda (1-\alpha)],
\]

\[
A = |f'(b)|^q \int_0^{1-\alpha} |t-\alpha | h(t) dt + |f'(a)|^q \int_0^{1-\alpha} |t-\alpha | h(1-t) dt,
\]

\[
B = |f'(b)|^q \int_{1-\alpha}^{1} |t-1 + \lambda (1-\alpha)| h(t) dt + |f'(a)|^q \int_{1-\alpha}^{1} |t-1 + \lambda (1-\alpha)| h(1-t) dt
\]

**Proof.** Suppose that $q \geq 1$. From Lemma 1 and using the well known power mean inequality, we have

\[
\left| \lambda (\alpha f(a)+(1-\alpha)f(b))+(1-\lambda) f(\alpha a+(1-\alpha)b) \right| - \frac{1}{b-a} \int_a^b f(x)dx \leq (b-a) \left[ \int_0^{1-\alpha} |t-\alpha | |f'(tb+(1-t)a)| dt + \int_{1-\alpha}^{1} |t-1 + \lambda (1-\alpha)| |f'(tb+(1-t)a)| dt \right]
\]

\[
\leq (b-a) \left\{ \left( \int_0^{1-\alpha} |t-\alpha | dt \right)^{1-\frac{1}{q}} \left( \int_0^{1-\alpha} |t-\alpha | |f'(tb+(1-t)a)|^q dt \right)^{\frac{1}{q}} \right\}
\]
\[
\int_0^1 |t - \alpha \lambda| |f'(tb + (1-t)a)|^q \, dt, \quad I_2 = \int_{1-\alpha}^1 |t - 1 + \lambda (1-\alpha)| |f'(tb + (1-t)a)|^q \, dt
\]

Since \(|f'|^q\) is \(h\)-convex on \([a, b]\),

\[
I_1 \leq |f'(b)|^q \int_0^{1-\alpha} |t - \alpha \lambda| h(t) \, dt + |f'(a)|^q \int_0^{1-\alpha} |t - \alpha \lambda| h(1-t) \, dt.
\]

Similarly

\[
I_2 \leq |f'(b)|^q \int_{1-\alpha}^1 |t - 1 + \lambda (1-\alpha)| h(t) \, dt + |f'(a)|^q \int_{1-\alpha}^1 |t - 1 + \lambda (1-\alpha)| h(1-t) \, dt.
\]

Additionally, by simple computation

\[
\int_0^{1-\alpha} |t - \alpha \lambda| \, dt = \begin{cases} 
\gamma_2, & \alpha \lambda \leq 1 - \alpha \\
\gamma_1, & \alpha \lambda \geq 1 - \alpha
\end{cases},
\]

\[
\gamma_1 = (1-\alpha) \left[ \alpha \lambda - \frac{(1-\alpha)}{2} \right], \quad \gamma_2 = (\alpha \lambda)^2 - \gamma_1,
\]

\[
\int_{1-\alpha}^1 |t - 1 + \lambda (1-\alpha)| \, dt = \begin{cases} 
v_1, & 1 - \lambda (1-\alpha) \leq 1 - \alpha \\
v_2, & 1 - \lambda (1-\alpha) \geq 1 - \alpha
\end{cases},
\]

\[
v_1 = \frac{1 - (1-\alpha)^2}{2} - \alpha \left[ 1 - \lambda (1-\alpha) \right],
\]

\[
v_2 = \frac{1 + (1-\alpha)^2}{2} - (\lambda + 1) (1-\alpha) \left[ 1 - \lambda (1-\alpha) \right].
\]

Thus, using (2.5), (2.8) in (2.4), we obtain the inequality (2.1). This completes the proof. \(\square\)
Corollary 1. Under the assumptions of Theorem 10 with \( q = 1 \), we have

\[
\left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x)\,dx \right|
\]

\[
\leq (b - a) \left\{ \left| f'(b) \right| \left[ \int_0^{1-\alpha} |t - \alpha \lambda| h(t)\,dt + \int_{1-\alpha}^1 |t - 1 + \lambda (1 - \alpha)| h(t)\,dt \right] \\
|f'(a)| \left[ \int_0^{1-\alpha} |t - \alpha \lambda| h(1 - t)\,dt + \int_{1-\alpha}^1 |t - 1 + \lambda (1 - \alpha)| h(1 - t)\,dt \right] \right\}.
\]

Corollary 2. Under the assumptions of Theorem 10 with \( I \subseteq [0, \infty) \), \( h(t) = t^s \), \( s \in (0, 1] \), we have

\[
(2.9) \left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x)\,dx \right|
\]

\[
\leq \left\{ \begin{array}{l}
(b - a) \left[ \gamma_2 \left( \mu_1^s |f'(b)|^q + \mu_2^s |f'(a)|^q \right) \right]^{1/4} \\
+ v_2 \left[ \frac{1}{4} \left( \nu_1^s |f'(b)|^q + \nu_2^s |f'(a)|^q \right) \right]^{1/4}, \\
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
(b - a) \left[ \gamma_2 \left( \mu_1^s |f'(b)|^q + \mu_2^s |f'(a)|^q \right) \right]^{1/4} \\
+ v_2 \left[ \frac{1}{4} \left( \nu_1^s |f'(b)|^q + \nu_2^s |f'(a)|^q \right) \right]^{1/4}, \\
\end{array} \right.
\]

\[
\leq \left\{ \begin{array}{l}
(b - a) \left[ \gamma_2 \left( \mu_1^s |f'(b)|^q + \mu_2^s |f'(a)|^q \right) \right]^{1/4} \\
+ v_2 \left[ \frac{1}{4} \left( \nu_1^s |f'(b)|^q + \nu_2^s |f'(a)|^q \right) \right]^{1/4}, \\
\end{array} \right.
\]

where \( \gamma_1, \gamma_2, \nu_1 \) and \( \nu_2 \) are defined as in (2.2) - (2.3) and

\[
\mu_1^s = \frac{(\alpha \lambda)^{s+2}}{(s + 1)(s + 2)} - \frac{2}{s + 1} - \frac{(1 - \alpha)^{s+1}}{s + 1} + \frac{(1 - \alpha)^{s+2}}{s + 2},
\]

\[
\mu_2^s = \frac{(1 - \alpha \lambda)^{s+2}}{(s + 1)(s + 2)} + \frac{2}{s + 1} - \frac{(1 - \alpha) (1 + a^{s+1})}{s + 1} + \frac{1 + \alpha^{s+2}}{s + 2},
\]

\[
\mu_3^s = \frac{(\alpha \lambda) (1 - a^{s+1})}{s + 1} - \frac{(1 - \alpha a^{s+1})}{s + 2},
\]

\[
\mu_4^s = \frac{(\alpha \lambda - 1) (1 - \alpha a^{s+1})}{s + 1} + \frac{1 - \alpha^{s+2}}{s + 2}.
\]
Remark 2. In Corollary 2, if we take
\[ \eta_1 = \frac{1 - (1 - \alpha)^{s+2}}{s + 2} - \frac{1 - \lambda (1 - \alpha)}{s + 1} \left[ 1 - (1 - \alpha)^{s+1} \right], \]
\[ \eta_2 = \frac{\lambda (1 - \alpha) \alpha^{s+1}}{s + 1} - \frac{\alpha^{s+2}}{s + 2}, \]
\[ \eta_3 = \frac{2 [1 - \lambda (1 - \alpha)^{s+2}]}{(s + 1)(s + 2)} - \frac{1 + (1 - \alpha)^{s+1}}{s + 1} \left[ 1 - \lambda (1 - \alpha) \right] + \frac{1 + (1 - \alpha)^{s+2}}{s + 2}, \]
\[ \eta_4 = \left[ \lambda (1 - \alpha)^{s+2} \right] - \frac{2}{(s + 1)(s + 2)} - \lambda (1 - \alpha) \frac{\alpha^{s+1}}{s + 1} + \frac{\alpha^{s+2}}{s + 2}. \]

Corollary 3. Let the assumptions of Theorem 10 hold. Then for \( h(t) = t \) the inequality \( (2.7) \) reduced to the inequality \( (1.5) \).

Corollary 4. Under the assumptions of Theorem 10 with \( h(t) = 1 \), we have
\[ \begin{align*}
\lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x)dx &= \\
\leq (b - a) \left( \left| f'(b) \right|^q + \left| f'(a) \right|^q \right)^{\frac{1}{q}} \times \left\{ \begin{array}{ll}
\gamma_2 + \nu_2 & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
\gamma_2 + \nu_1 & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
\gamma_1 + \nu_2 & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{array} \right.,
\end{align*} \]
where \( \gamma_1, \gamma_2, \nu_1 \) and \( \nu_2 \) are defined as in \( (2.2) \) - \( (2.3) \).

Remark 1. In Corollary 2, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{3} \), then we have the following Simpson type inequality
\[ (2.10) \quad \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{2} \left( \frac{5}{36} \right)^{\frac{1}{q}}, \]
which is the same of the inequality in \( \text{[19]} \) Theorem 10.

Remark 2. In Corollary 2, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), then we have following midpoint inequality
\[ (2.11) \quad \left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{8} \left( \frac{2}{(s + 1)(s + 2)} \right)^{\frac{1}{q}} \times \left\{ \begin{array}{ll}
\frac{2^{1 - s} (s + 1)}{2} \left| f'(b) \right|^q + \frac{2^{1 - s} (2s + 2 - s - 3)}{2} \left| f'(a) \right|^q \left( \frac{1}{q} \right) \right.
\end{array} \right. \]
\[ + \left. \frac{2^{1 - s} s}{2} \left( \frac{2s + 2 - s - 3}{2} \right) \right\}. \]
We note that the obtained midpoint inequality (2.11) is better than the inequality (1.6). Because \( \frac{s+1}{2} \leq 1 \) and \( \frac{2s+2-s-1}{2} \leq \frac{2^{s+1}+1}{2^{s+1}+1} \).

**Remark 3.** In Corollary 2, if we take \( \alpha = \frac{1}{2} \), and \( \lambda = 1 \), then we get the following trapezoid inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{8} \left( \frac{2^{1-s}}{(s+1)(s+2)} \right)^{\frac{1}{q}} \times \left\{ (|f'(b)|^q + |f'(a)|^q (2^{s+1} + 1))^{\frac{1}{q}} + (|f'(a)|^q + |f'(b)|^q (2^{s+1} + 1))^{\frac{1}{q}} \right\}
\]

Using Lemma 1, we shall give another result for convex functions as follows.

**Theorem 11.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \) and \( \alpha, \lambda \in [0,1] \). If \( |f'|^q \) is \( h \)-convex on \( [a,b] \), \( q > 1 \), then the following inequality holds:

(2.12)

\[
\lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x)\,dx \leq (b-a)
\]

\[
\times \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \int_0^b 1 \right)^{\frac{1}{q}} \left\{ \begin{array}{ll}
\frac{1}{\epsilon_1} C_1^{\frac{1}{q}} + \frac{1}{\epsilon_2} D_1^{\frac{1}{q}}, & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \\
\frac{1}{\epsilon_1} C_2^{\frac{1}{q}} + \frac{1}{\epsilon_2} D_2^{\frac{1}{q}}, & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
\frac{1}{\epsilon_2} C_3^{\frac{1}{q}} + \frac{1}{\epsilon_2} D_3^{\frac{1}{q}}, & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{array} \right.
\]

where

(2.13)

\[
C = (1 - \alpha) \left[ |f'((1 - \alpha) b + \alpha a)|^q + |f'(a)|^q \right],
\]

\[
D = \alpha \left[ |f'((1 - \alpha) b + \alpha a)|^q + |f'(b)|^q \right],
\]

\[
\epsilon_1 = (\alpha \lambda)^{p+1} + (1 - \alpha - \alpha \lambda)^{p+1}, \quad \epsilon_2 = (\alpha \lambda)^{p+1} - (\alpha \lambda - 1 + \alpha)^{p+1},
\]

\[
\epsilon_3 = [\lambda (1 - \alpha)]^{p+1} + [\alpha - \lambda (1 - \alpha)]^{p+1}, \quad \epsilon_4 = [\lambda (1 - \alpha)]^{p+1} - [\lambda (1 - \alpha) - \alpha]^{p+1},
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** From Lemma 1 and by Hölder’s integral inequality, we have

\[
\left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x)\,dx \right|
\]

\[
\leq (b-a) \left[ \int_0^1 t - \alpha \lambda \int |f'(tb + (1 - t)a)|\,dt + \int_0^1 |t - 1 + \lambda (1 - \alpha)| \int |f'(tb + (1 - t)a)|\,dt \right]
\]

\[
\leq (b-a) \left( \int_0^1 t - \alpha \lambda |p|\,dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tb + (1 - t)a)|^q\,dt \right)^{\frac{1}{q}}
\]
The inequality (2.16) holds for $\alpha = 1$ too. Similarly, for $\alpha \in (0, 1]$ by the inequality (1.4), we get
\[
\int_0^1 \frac{1}{(1-\alpha)(b-a)} \int_a^b |f'(x)|^q \, dx \geq (1-\alpha) \left[ |f'((1-\alpha)b+\alpha a)|^q + |f'(a)|^q \right] \int_0^1 h(t) \, dt.
\]
(2.15)

The inequality (2.16) holds for $\alpha = 0$ too. By simple computation
\[
\int_0^1 |t-\alpha| \lambda \, dt = \begin{cases} \frac{(\alpha \lambda)^{p+1}+(1-\alpha-\alpha \lambda)^{p+1}}{(\alpha \lambda)^{p+1}-(\alpha \lambda-1+\alpha)^{p+1}}, & \alpha \lambda \leq 1-\alpha, \\ \frac{(\alpha \lambda)^{p+1}+(1-\alpha-\alpha \lambda)^{p+1}}{(\alpha \lambda)^{p+1}-(\alpha \lambda-1+\alpha)^{p+1}}, & \alpha \lambda \geq 1-\alpha, \end{cases}
\]
and
\[
\int_0^1 |t-1+\lambda(1-\alpha)| \lambda \, dt = \begin{cases} \frac{[\lambda(1-\alpha)]^{p+1}+[\alpha-\lambda(1-\alpha)]^{p+1}}{[\lambda(1-\alpha)]^{p+1}+[\lambda(1-\alpha)-\alpha]^{p+1}}, & 1-\alpha \leq 1-\lambda(1-\alpha), \\ \frac{[\lambda(1-\alpha)]^{p+1}+[\alpha-\lambda(1-\alpha)]^{p+1}}{[\lambda(1-\alpha)]^{p+1}+[\lambda(1-\alpha)-\alpha]^{p+1}}, & 1-\alpha \geq 1-\lambda(1-\alpha), \end{cases}
\]
thus, using (2.15)-(2.18) in (2.14), we obtain the inequality (2.12). This completes the proof.

\[\square\]

**Corollary 5.** Under the assumptions of Theorem [11] with $I \subseteq [0, \infty)$, $h(t) = t^s$, $s \in (0, 1]$, we have
\[
\lambda (\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(aa + (1-\alpha) b) - \frac{1}{b-a} \int_a^b f(x) \, dx \leq (b-a)
\]
(2.19)
\[
\times \left( \frac{1}{p+1} \right) \left( \frac{1}{s+1} \right)^{\frac{1}{2}} \left\{ \begin{array}{ll}
\epsilon_1 C^\frac{1}{2} + \frac{\epsilon_2}{2} D^\frac{1}{2}, & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1-\alpha), \\
\epsilon_1 C^\frac{1}{2} + \frac{\epsilon_3}{2} D^\frac{1}{2}, & \alpha \lambda \leq 1 - \lambda (1-\alpha) \leq 1 - \alpha, \\
\epsilon_2 C^\frac{1}{2} + \frac{\epsilon_4}{2} D^\frac{1}{2}, & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1-\alpha),
\end{array} \right.
\]
where $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, $\epsilon_4$, $C$ and $D$ are defined as in (2.13).
Remark 4. In Corollary 5, if we take

\[ \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(aa + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x) dx \leq (b-a) \]

where \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, C \) and \( D \) are defined as in (2.13).

Corollary 7. Under the assumptions of Theorem 11 with \( h(t) = 1 \), we have

\[ \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(aa + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x) dx \leq (b-a) \]

\[ \times \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left\{ \begin{array}{l}
\varepsilon_1^\frac{1}{q} C^\frac{1}{q} + \varepsilon_3^\frac{1}{q} D^\frac{1}{q}, \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
\varepsilon_1^\frac{1}{q} C^\frac{1}{q} + \varepsilon_4^\frac{1}{q} D^\frac{1}{q}, \quad \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha , \\
\varepsilon_2^\frac{1}{q} C^\frac{1}{q} + \varepsilon_4^\frac{1}{q} D^\frac{1}{q}, \quad 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha) 
\end{array} \right. \]

Remark 4. In Corollary 5, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{3} \), then we have the following Simpson type inequality

\[ (2.20) \quad \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{12} \left( \frac{1 + 2p+1}{3 (p+1)} \right) \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f'(a) \right|^q \right\}^{\frac{1}{q}} + \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f'(b) \right|^q \right\}^{\frac{1}{q}}, \]

which is the same of the inequality (1.8).

Remark 5. In Corollary 5, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), then we have the following midpoint inequality

\[ \left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f'(a) \right|^q \right\}^{\frac{1}{q}} + \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f'(b) \right|^q \right\}^{\frac{1}{q}}. \]

We note that by inequality

\[ 2^{s-1} \left| f' \left( \frac{a+b}{2} \right) \right|^q \leq \left| f'(a) \right|^q + \left| f'(b) \right|^q \]
We proceed similarly as in the proof Theorem 11. Since

\[ f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \left( \frac{b - a}{4} \right) \left( \frac{1}{p + 1} \right) \left( \frac{1}{s + 1} \right)^{\frac{1}{s}} \times \left[ \left( \int (2^{1-s} + s + 1)^{\frac{q}{s}} \right) \left( \int f''(a)^q + 2^{1-s} |f'(b)|^q \right)^{\frac{1}{s}} \right]^{\frac{1}{s}}, \]

which is the same of the inequality (1.7).

**Remark 6.** In Corollary [6] if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), then we have the following trapezoid inequality

\[ \left( \frac{f(a) + f(b)}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{4} \left( \frac{1}{p + 1} \right)^{\frac{1}{s}} \times \left\{ \left( \left( \frac{f''(a) + f''(b)}{2} \right)^{\frac{q}{s}} + \left( \frac{f'(a)}{s + 1} \right)^{\frac{q}{s}} \right)^{\frac{1}{s}} \right\}. \]

We note that the obtained midpoint inequality (2.21) is better than the inequality (1.7).

**Theorem 12.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) such that \( f' \in L[a, b] \), where \( a, b \in I^0 \) with \( a < b \) and \( \alpha, \lambda \in [0, 1] \). If \( |f'|^q \) is \( h \)-concave on \([a, b], q > 1\), then the following inequality holds:

\[ \left( \lambda \alpha f(a) + (1 - \lambda f(b)) + (1 - \lambda) f(\alpha a + (1 - \lambda) b) - \frac{1}{b - a} \int_a^b f(x) \, dx \right) \leq (b - a) \left( \frac{1}{2h (\frac{1}{2})} \right)^{\frac{1}{s}} \left( \frac{1}{p + 1} \right)^{\frac{1}{s}} \left\{ \frac{1}{\frac{3}{2} E^+ + \frac{1}{2} F^+} \right\}^{\frac{1}{s}}, \]

where

\[ E = (1 - \alpha) \left( \frac{f'}{\left( \frac{(1 - \alpha)b + (1 + \alpha)a}{2} \right)^q} \right), \]

\[ F = \alpha \left( \frac{f'}{\left( \frac{(2 - \alpha)b + \alpha a}{2} \right)^q} \right), \]

and \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) are defined as in (2.13).

**Proof.** We proceed similarly as in the proof Theorem [11] Since \( |f'|^q \) is \( h \)-concave on \([a, b], \) for \( \alpha \in [0, 1] \) by the inequality (1.4), we get

\[ \int_0^{1-\alpha} |f'(tb + (1-t)a)|^d \, dt = (1 - \alpha) \left[ \frac{1}{(1 - \alpha)(b - a)} \int_0^{(1 - \alpha)b + \alpha a} f'(x)^d \, dx \right] \]

\[ \leq \frac{(1 - \alpha)}{2h (\frac{1}{2})} \left[ f' \left( \frac{(1 - \alpha)b + (1 + \alpha)a}{2} \right)^q \right]. \]

\[ (2.23) \]
The inequality (2.23) holds for $\alpha = 1$ too. Similarly, for $\alpha \in (0, 1]$ by the inequality (1.4), we have

$$
\int_{1-a}^{1} |f'(tb + (1-t)a)|^q \, dt = \alpha \left[ \frac{1}{\alpha(b-a)} \int_{(1-a)b+aa}^{b} |f'(x)|^q \, dx \right]
$$

(2.24)

$$
\leq \frac{\alpha}{2(b \left( \frac{1}{4} \right)^2)} \left| f' \left( \frac{(2-\alpha)b + \alpha a}{2} \right) \right|^q.
$$

The inequality (2.24) holds for $\alpha = 0$ too. Thus, using (2.17), (2.18), (2.23) and (2.24) in (2.14), we obtain the inequality (2.22). This completes the proof. \qed

**Corollary 8.** Under the assumptions of Theorem 12 with $I \subseteq [0, \infty)$, $h(t) = t^s$, $s \in (0, 1]$, we have

$$
\left| \lambda (\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(aa + (1-\alpha) b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq (b-a)
$$

$$
\times 2^{\frac{p-1}{p}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \begin{array}{ll}
\varepsilon_1^\frac{1}{p} E^\frac{1}{p} + \varepsilon_3^\frac{1}{p} F^\frac{1}{p} & , \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1-\alpha) \\
\varepsilon_1^\frac{1}{p} E^\frac{1}{p} + \varepsilon_4^\frac{1}{p} F^\frac{1}{p} & , \quad \alpha \lambda \leq 1 - \lambda (1-\alpha) \leq 1 - \alpha \\
\varepsilon_2^\frac{1}{p} E^\frac{1}{p} + \varepsilon_3^\frac{1}{p} F^\frac{1}{p} & , \quad 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1-\alpha)
\end{array} \right.
$$

where $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\varepsilon_4$, $E$ and $F$ are defined as in Theorem 12.

**Corollary 9.** Under the assumptions of Theorem 12 with $h(t) = t$, we have

$$
\left| \lambda (\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(aa + (1-\alpha) b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq (b-a)
$$

$$
\times \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \begin{array}{ll}
\varepsilon_1^\frac{1}{p} E^\frac{1}{p} + \varepsilon_3^\frac{1}{p} F^\frac{1}{p} & , \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1-\alpha) \\
\varepsilon_1^\frac{1}{p} E^\frac{1}{p} + \varepsilon_4^\frac{1}{p} F^\frac{1}{p} & , \quad \alpha \lambda \leq 1 - \lambda (1-\alpha) \leq 1 - \alpha \\
\varepsilon_2^\frac{1}{p} E^\frac{1}{p} + \varepsilon_3^\frac{1}{p} F^\frac{1}{p} & , \quad 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1-\alpha)
\end{array} \right.
$$

where $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\varepsilon_4$, $E$ and $F$ are defined as in Theorem 12.

**Corollary 10.** Under the assumptions of Theorem 12 with $h(t) = 1$, we have

$$
\left| \lambda (\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(aa + (1-\alpha) b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq (b-a)
$$

$$
\times 2^{\frac{p-1}{p}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \begin{array}{ll}
\varepsilon_1^\frac{1}{p} E^\frac{1}{p} + \varepsilon_3^\frac{1}{p} F^\frac{1}{p} & , \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1-\alpha) \\
\varepsilon_1^\frac{1}{p} E^\frac{1}{p} + \varepsilon_4^\frac{1}{p} F^\frac{1}{p} & , \quad \alpha \lambda \leq 1 - \lambda (1-\alpha) \leq 1 - \alpha \\
\varepsilon_2^\frac{1}{p} E^\frac{1}{p} + \varepsilon_3^\frac{1}{p} F^\frac{1}{p} & , \quad 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1-\alpha)
\end{array} \right.
$$
where \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, E \) and \( F \) are defined as in Theorem 12.

**Corollary 11.** Under the assumptions of Theorem 12 with \( h(t) = \frac{1}{t}, t \in (0,1) \), we have

\[
\left| \lambda (f' \alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq (b - a)
\]

\[\times 4^{\frac{1}{p}} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \cdot \left\{ \begin{array}{ll}
\varepsilon_4^\frac{1}{p} E_{\frac{1}{p}} + \varepsilon_5^\frac{1}{p} F_{\frac{1}{p}} & , \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
\varepsilon_1^\frac{1}{q} E_{\frac{1}{q}} + \varepsilon_3^\frac{1}{q} F_{\frac{1}{q}} & , \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha ,
\end{array} \right. \]

where \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, E \) and \( F \) are defined as in Theorem 12.

**Remark 7.** In Corollary 8, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), then we have the following trapezoid inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \\
\leq \frac{b - a}{4} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \times \left( \frac{1}{2} \right)^{\frac{1 - s}{p}} \left[ |f' \left( \frac{3b + a}{4} \right)| + |f' \left( \frac{3a + b}{4} \right)| \right]
\]

which is the same of the inequality in [16, Theorem 8 (i)].

**Remark 8.** In Corollary 8, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), then we have the following midpoint inequality

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) dx \right| \\
\leq \frac{b - a}{4} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \times \left( \frac{1}{2} \right)^{\frac{1 - s}{p}} \left[ |f' \left( \frac{3b + a}{4} \right)| + |f' \left( \frac{3a + b}{4} \right)| \right]
\]

which is the same of the inequality in [16, Theorem 8 (ii)].

**Remark 9.** In Corollary 8, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), then we have the following trapezoid inequality

\[
(2.25) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \\
\leq \frac{b - a}{4} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left[ |f' \left( \frac{3b + a}{4} \right)| + |f' \left( \frac{3a + b}{4} \right)| \right]
\]

which is the same of the inequality in [12, Theorem 2].
Remark 10. In Corollary \[7 \] if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), then we have the following trapezoid inequality

\[
(2.26) \quad \left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left[ \left| f' \left( \frac{3b + a}{4} \right) \right| + \left| f' \left( \frac{3a + b}{4} \right) \right| \right]
\]

which is the same of the inequality in \[2 \] Theorem 2.5.

Remark 11. In Corollary \[7 \] since \( |f'|^q \), \( q > 1 \), is concave on \([a, b]\), using the power mean inequality, we have

\[
|f' (\lambda x + (1 - \lambda) y)|^q \geq \lambda |f' (x)|^q + (1 - \lambda) |f' (y)|^q \\
\geq (\lambda |f' (x)| + (1 - \lambda) |f' (y)|)^q,
\]

\( \forall x, y \in [a, b] \) and \( \lambda \in [0, 1] \). Hence

\[
|f' (\lambda x + (1 - \lambda) y)| \geq \lambda |f' (x)| + (1 - \lambda) |f' (y)|
\]

so \( |f'| \) is also concave. Then by the inequality (1.1), we have

\[
(2.27) \quad \left| f' \left( \frac{3b + a}{4} \right) \right| + \left| f' \left( \frac{3a + b}{4} \right) \right| \leq 2 \left| f' \left( \frac{a + b}{2} \right) \right|.
\]

Thus, using the inequality (2.27) in (2.20) and (2.20) we get

\[
\left| f \left( \frac{a + f(b)}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{2} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left| f' \left( \frac{a + b}{2} \right) \right|,
\]

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{2} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left| f' \left( \frac{a + b}{2} \right) \right|.
\]

3. Some applications for special means

Let us recall the following special means of arbitrary real numbers \( a, b \) with \( a \neq b \) and \( \alpha \in [0, 1] \):

1. The weighted arithmetic mean

\[
A_{\alpha} (a, b) := \alpha a + (1 - \alpha)b, \ a, b \in \mathbb{R}.
\]

2. The unweighted arithmetic mean

\[
A (a, b) := \frac{a + b}{2}, \ a, b \in \mathbb{R}.
\]

3. The Logarithmic mean

\[
L (a, b) := \frac{b - a}{\ln |b| - \ln |a|}, \ |a| \neq |b|, \ ab \neq 0.
\]

4. Then \( p \)–Logarithmic mean

\[
L_p (a, b) := \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, \ p \in \mathbb{R} \setminus \{-1, 0\}, \ a, b > 0.
\]
Proposition 2. Let \( t \) have the following inequality:

\[ \varepsilon \in \mathbb{R} \]

Proof. Corollary 2.

Proposition 1. Let \( a, b \in \mathbb{R} \) with \( 0 < a < b \), \( q \geq 1 \) and \( s \in \left( 0, \frac{1}{q} \right) \) we have the following inequality:

\[ |\lambda A_\alpha \left( a^{s+1}, b^{s+1} \right) + (1 - \lambda) A_\alpha^{s+1} (a, b) - L_\alpha^{s+1} (a, b)\] \]

\[ \leq \left\{ \begin{align*}
& (b - a) (s + 1) \left\{ \frac{1}{2} \left( \mu_1^2 b^q + \mu_2^2 a^q \right) \right\}, \\
& + v_2 \left( \eta_3^2 a^q + \eta_4^2 a^q \right) \right\}, \\
& \alpha \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha)
\end{align*} \]

where \( \gamma_1, \gamma_2, v_1, v_2, \mu_1^2, \mu_2^2, \mu_3^2, \mu_4^2, \eta_1^2, \eta_2^2, \eta_3^2, \eta_4^2 \) numbers are defined as in Corollary 2.

Proof. The assertion follows from applied the inequality (2.10) to the function \( f(t) = t^{s+1}, t \in [a, b] \) and \( s \in \left( 0, \frac{1}{q} \right) \), which implies that \( f'(t) = (s + 1)t^s \), \( t \in [a, b] \) and \( |f'(t)|^q = (s + 1)^q t^{qs} \), \( t \in [a, b] \) is a \( s \)-convex function in the second sense since \( qs \in (0, 1) \) and \( (s + 1)^q > 0 \).

Proposition 2. Let \( a, b \in \mathbb{R} \) with \( 0 < a < b \), \( p, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( s \in \left( 0, \frac{1}{q} \right) \) we have the following inequality:

\[ |\lambda A_\alpha \left( a^{s+1}, b^{s+1} \right) + (1 - \lambda) A_\alpha^{s+1} (a, b) - L_\alpha^{s+1} (a, b)\] \]

\[ \leq (b - a) \left( \frac{1}{p + 1} \right) \left( s + 1 \right)^{1 - \frac{1}{q}} \]

\[ \times \left\{ \begin{align*}
& (1 - \alpha) \frac{1}{2} \varepsilon_1^2 \theta_1 + a^s q \varepsilon_2^2 \theta_2, \\
& \alpha \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha)
\end{align*} \]

where \( \theta_1 = A_\alpha (a, b) + b^s q \), \( \theta_2 = A_\alpha (a, b) + b^s q \), \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) numbers are defined as in Corollary 2.

Proof. The assertion follows from applied the inequality (2.10) to the function \( f(t) = t^{s+1}, t \in [a, b] \) and \( s \in \left( 0, \frac{1}{q} \right) \), which implies that \( f'(t) = (s + 1)t^s \), \( t \in [a, b] \) and \( |f'(t)|^q = (s + 1)^q t^{qs} \), \( t \in [a, b] \) is a \( s \)-convex function in the second sense since \( qs \in (0, 1) \) and \( (s + 1)^q > 0 \).
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