Nonlinear sigma model approach for phase disorder transitions
and the pseudogap phase in chiral Gross-Neveu, Nambu–Jona-Lasinio models
and strong-coupling superconductors

Egor Babaev
Institute for Theoretical Physics, Uppsala University Box 803, S-75108 Uppsala, Sweden

We briefly review the nonlinear sigma model approach for the subject of increasing interest:
“two-step” phase transitions in the Gross-Neveu and the modified Nambu–Jona-Lasinio models
at low \( N \) and condensation from pseudogap phase in strong-coupling superconductors. Recent
success in describing “Bose-type” superconductors that possess two characteristic temperatures and
a pseudogap above \( T_c \) is the development approximately comparable with the BCS theory. One can
expect that it should have influence on high-energy physics, similar to impact of the BCS theory on
this subject. Although first generalizations of this concept to particle physics were made recently,
these results were not systematized. In this review we summarize this development and discuss
similarities and differences of the appearance of the pseudogap phase in superconductors and the
Gross-Neveu and Nambu–Jona-Lasinio - like models. We discuss its possible relevance for chiral
phase transition in QCD and color superconductors. This paper is organized in three parts: in the
first section we briefly review the separation of temperatures of pair formation and pair condensation
in strong - coupling and low carrier density superconductors (i.e. the formation of the pseudogap
phase ). Second part is a review of nonlinear sigma model approach to an analogous phenomenon in
the Chiral Gross-Neveu model at small \( N \). In the third section we discuss the modified Nambu–Jona-
Lasinio model where the chiral phase transition is accompanied by a formation of a phase analogous
to the pseudogap phase.

I. INTRODUCTION

Many concepts of particle physics have a close relation to superconductivity, for example the Nambu–Jona-Lasinio
model \([1]-[3]\) was proposed in analogy to the BCS theory of superconductivity and is considered as a low-energy
effective theory of QCD. Recently a substantial progress has been made in the theory of superconductivity in systems
with strong attraction and low carrier density. That is, it has been observed that away from the limits of infinitesimally
weak coupling strength or very high carrier density, BCS-like mean-field theories are qualitatively wrong and these
systems possess along with superconductive phase an additional phase where there exist Cooper pairs but no symmetry
is broken due to phase fluctuations (the pseudogap phase ). What may be regarded as an indication of the importance
of this concept to particle physics is that recently the formation of the pseudogap phase due to dynamic quantum
fluctuations at low \( N \) was found in the chiral Gross-Neveu model in \( 2 + \epsilon \) dimensions \([4]\).

Separation of the temperatures of the pair formation and of the onset of phase coherence (pair condensation) in
strong-coupling superconductors, in fact, has been known already for many years (Crossover from BCS supercon-
ductivity to Bose-Einstein Condensation (BEC) of tightly bound fermion pairs) \([5,6]\). Intensive theoretical study of
these phenomena in the recent years (see for example \([7]-[21]\)), was sparked by experimental results on underdoped
(low carrier density) cuprates that display “gap-like” feature above critical temperature \( T_c \) that disappears only at a
substantially higher temperature \( T^* \). There is experimental evidence that this phenomenon in high-\( T_c \) superconduc-
tors may be connected with precritical pairing fluctuations above \( T_c \). At present, this crossover has been studied by
variety of methods and in many different models.

Because of intimate relationship of many problems in particle physics to superconductivity it seems natural to guess
that the pseudogap may become a fruitful concept in high energy physics too.

Below we review these phenomena in superconductors and discuss its possible implications for QCD.

II. PSEUDOGAP PHASE IN STRONG-COUPLING AND LOW CARRIER DENSITY THEORIES OF
SUPERCONDUCTIVITY

*email: egor@teorfys.uu.se http://www.teorfys.uu.se/PEOPLE/egor/
A. Perturbative results

The BCS theory describes metallic superconductors perfectly. However, it failed to describe even qualitatively superconductivity in underdoped High-$T_c$ compounds. One of the most exotic properties of the latter materials is the existence of a pseudogap in the spectrum of the normal state well above critical temperature that from an experimental point of view, manifests itself as a significant suppression of low frequency spectral weight, thus being in contrast to the exactly zero spectral weight in the case of the superconductive gap. Moreover, spectroscopy experiments show that a superconductive gap evolves smoothly in magnitude and wave vector dependence to a pseudogap in normal state. Besides that, NMR and tunneling experiments indicate the existence of incoherent Cooper pairs well above $T_c$. In principle it is easy to guess what is hidden behind these circumstances, and why BCS theory is incapable of describing it. Let us imagine for a moment that we are able to bind electrons in Cooper pairs infinitely tightly - obviously this implies that the characteristic temperature of thermal pair decomposition will also be infinitely high, but this does not imply that the long-range order will survive at infinitely high temperatures. As first observed in [6], long-range order will be destroyed in a similar way, as say, in superfluid $^4$He, i.e., tightly bound Cooper pairs, at a certain temperature will acquire a nonzero momentum and thus we will have gas of tightly bound Cooper pairs but no macroscopic occupation of the zero momentum level $\mathbf{q} = 0$ and with it no long-range order. Thus phase diagram of a strong-coupling superconductor has three regions:

- The superconductive phase where there are condensed fermion pairs.
- The pseudogap phase where there exist fermion pairs but there is no condensate and thus there is no symmetry breakdown and no superconductivity.
- The normal phase with thermally decomposed Cooper pairs.

Of course, the existence of bound pairs above the critical temperature will result in deviations from Fermi-liquid behavior that make the pseudogap phase a very interesting object of study. In order to describe superconductivity in such a system the theory should incorporate pairs with nonzero momentum. Thus, the BCS scenario is invalid for description of spontaneous symmetry breakdown in a system with strong attractive interaction or low carrier density (see [6], [7] and references therein). So in principle in a strong-coupling superconductor onset of long range order has nothing to do with pair formation transition. The existence of the paired fermions is only necessary but not sufficient condition for symmetry breakdown. The BCS limit is a rather exotic case of infinitesimally weak coupling strength and high carrier density when the disappearance of superconductivity can approximately be described as a pair-breaking transition. The strong-coupling limit is another exotic case where the temperatures of pair decomposition and symmetry breakdown can be arbitrarily separated. There is nothing surprising in it: formally, in the case of Bose condensation of $^4$He we can also introduce a characteristic temperature of thermal decomposition of the $^4$He atom; however this does not mean that this temperature is somehow related to the temperature of the Bose condensation of the gas of $^4$He atoms. A schematic phase diagram of a superconductor is in shown in Fig 1.
Let us show how one can obtain a pseudogap phase starting from the BCS Hamiltonian. This was first done in the pioneering work by Nozieres and Schmitt-Rink and in the functional integral formalism for a system with \( \delta \)-function attraction by Sa de Melo, Randeria and Engelbrecht. In this subsection we briefly reproduce a part of the transparent article and in the following section we will show how qualitatively the same result can be obtained within nonlinear sigma model (3D XY-model) approach proposed by the author. In the following sections we discuss analogous nonlinear-sigma model approach to the similar phenomena in the chiral GN and NJL models.

The Hamiltonian of the BCS model is:

\[
H = \sum_{\sigma} \int d^D x \psi_\sigma^\dagger(x) \left( -\frac{\nabla^2}{2m} - \mu \right) \psi_\sigma(x) + g \int d^D x \psi_\uparrow^\dagger(x) \psi_\downarrow^\dagger(x) \psi_\downarrow(x) \psi_\uparrow(x),
\]

where \( \psi_\sigma(x) \) is the Fermi field operator, \( \sigma = \uparrow, \downarrow \) denotes the spin components, \( m \) is the fermionic mass, and \( g < 0 \) the strength of an attractive potential \( g \delta(x-x') \).

The mean-field equations for the gap parameter \( \Delta \) and the chemical potential \( \mu \) can be obtained with a standard variation procedure:

\[
-\frac{1}{g} = \frac{1}{V} \sum_k \frac{1}{E_k} \tanh \frac{E_k}{2T},
\]

\[
n = \frac{1}{V} \sum_k \left( 1 - \frac{\xi_k}{E_k} \tanh \frac{E_k}{2T} \right),
\]

where the sum runs over all wave vectors \( k \), \( N \) is the total number of fermions, \( V \) the volume of the system, and

\[
E_k = \sqrt{\xi_k^2 + \Delta^2} \quad \text{with} \quad \xi_k = \frac{k^2}{2m} - \mu
\]

are the energies of single-particle excitations.

The \( \delta \)-function potential produces divergence and requires regularization. A BCS superconductor possesses a natural cutoff supplied by the Debye frequency \( \omega_D \). For the crossover problem to be treated here this is no longer a useful quantity, since in the strong-coupling limit all fermions participate in the interaction, not only those in a thin shell of width \( \omega_D \) around the Fermi surface. To be applicable in this regime, we renormalize the gap equation in three dimensions with the help of the \( s \)-wave scattering length \( a_s \), for which the low-energy limit of the two-body scattering process gives an equally divergent expression:

\[
\frac{m}{4\pi a_s} = \frac{1}{g} + \frac{1}{V} \sum_k \frac{m}{k^2}.
\]

Eliminating \( g \) from (5) we obtain a renormalized gap equation

\[
-\frac{m}{4\pi a_s} = \frac{1}{V} \sum_k \left[ \frac{1}{2E_k} \tanh \frac{E_k}{2T} - \frac{m}{k^2} \right],
\]
in which $1/k_Fa_s$ plays the role of a dimensionless coupling constant which monotonically increases from $-\infty$ to $\infty$ as the bare coupling constant $g$ runs from small (BCS limit) to large values (BEC limit). This equation is to be solved simultaneously with (3). These mean-field equations were analyzed e.g. in Ref. 9.

In the BCS limit, the chemical potential $\mu$ does not differ much from the Fermi energy $\epsilon_F$, whereas with increasing interaction strength, the distribution function $n_k$ broadens and $\mu$ decreases, and in the BEC limit we have tightly bound pairs and nondegenerate fermions with a large negative chemical potential $|\mu|$ that in the strong-coupling regime lies above the temperature of the onset of phase coherence (6, 9). Obviously bound pairs and nondegenerate fermions with a large negative chemical potential $\mu$ (6) we have from (6) the critical temperature in the BCS limit ($\mu \gg T_c$) $T_c^{BCS} = 8e^{-2}e^\gamma\pi^{-1}\epsilon_F \exp(-\pi/(2k_Fa_s))$ where $\gamma = -\Gamma'(1)/\Gamma(1) = 0.577\ldots$, from (6) we have that chemical potential in this case is $\mu = \epsilon_F$. In the strong coupling limit, from Eqs. (6), (8) we obtain that in the BEC limit $\mu = -E_b/2$, where $E_b = 1/ma_s^2$ is the binding energy of the bound pairs. In the BEC limit, the mean-field eq. (4) gives that the "gap" sets in at $T^* \approx E_b/2\log(E_b/\epsilon_F)^{3/2}$. A simple "chemical" equilibrium estimate ($\mu_b = 2\mu_F$) yields for the temperature of pair dissociation: $T_{dissociation} \approx E_b/\log(E_b/\epsilon_F)^{3/2}$ which shows that at strong couplings $T^*$ is indeed related to pair formation (8) (which in the strong-coupling regime lies above the temperature of the onset of phase coherence (3)). Obviously $T^*$ is a monotonous function of coupling strength. If we take into the account gaussian fluctuations we can see that in the strong coupling regime, the temperature $T^*$, obtained with the above estimate is not related in any respect to the critical temperature of the onset of phase coherence. Expression for the thermodynamic potential with gaussian corrections reads (4):

$$\Omega = \Omega_0 - T \sum_{q,iq} \ln \Gamma(q,iq),$$

where

$$\Gamma^{-1}(q,iq) = \sum_k \left\{ \frac{1-n_k-n_{k+q}}{iq-\xi_k-\xi_{k+q}} + \frac{m}{k^2} \right\} - \frac{m}{4\pi a_s}.$$ (8)

Where $n_k$ is Fermi occupation and $iq = iT2\pi n$. It is convenient following to (4) to rewrite $\Omega$ in terms of a phase shift according to definition: $\Gamma(q,\omega \pm i0) = |\Gamma(q,\omega)| \exp(\pm i\delta(q,\omega))$. After inclusion of Gaussian correction the number equation $N = -\partial\Omega/\partial\mu$ reads:

$$n = n_0(\mu, T) + \sum_q \int_{-\infty}^{\infty} \frac{d\omega}{\pi} n_B(\omega) \frac{\partial\delta}{\partial\mu}(q,\omega)$$ (9)

Where $n_0$ is density of "free" fermions defined in (8) and $n_B(\omega) = 1/(\exp(\omega/T) - 1)$ is the Bose function. In order to study behavior of $T_c$ one should solve a set of the number and gap equations. In the BCS limit $T_c$ is not affected substantially by Gaussian corrections, thus the superconductive transition can be described by mean-field theory and correspondingly $T_c \approx T^*$. In the opposite limit, numerical solution (8, 9) show that the temperature of the superconductive phase transition tends to a constant value that does not depend on the coupling strength and is equal to the temperature of condensation of the ideal Bose gas of particles of mass $2m$ and density $n/2$, where $m$ and $n$ are the mass and density of electrons correspondingly:

$$T_c = \left[ \frac{n}{2\zeta(3/2)} \right]^{2/3} \frac{\pi}{m} = 0.218\epsilon_F.$$ (10)

Where $\epsilon_F$ was used simply as a dimensional constant, namely the Fermi energy of the gas of free fermions with the density $n$ and mass $m$ (obviously at very strong coupling strength, when all fermions are paired there is no Fermi surface).

The system of the gap and number equations can be solved analytically in the strong-coupling limit. First as it was pointed out in (9) one can make the following approximation: to retain Gaussian corrections only in the number equation and solve it together with mean-field "gap" equation. Near $T_c$ in the strong -coupling regime one finds that $\mu(T_c) = -E_b/2$, where $E_b$ is the energy required to break a pair. One can observe that in this limit $\Gamma(q,z)$ has an isolated pole

\footnote{As first discussed in 1960s, even in BCS superconductors there is a narrow region of precritical pairing fluctuations. This gives rise, e.g., to the so-called paraconductivity effect. In particle physics, this phenomenon was pointed out by Hatsuda and Kunihiro (9).}
on the real axis for each \( q \), representing a two-body bound state with momentum \( q \). Since formally in this limit we can make energy required to break a pair arbitrarily large, this pole is widely separated from the branch cut representing the continuum of two-particle excitations. The low energy physics at temperatures much lower that temperature of thermal pair decomposition is governed by this pole and one can write \( \Gamma(q, iq_m) \approx R(q)/[iq_m - \omega_b(q) + 2\mu] \), where \( \omega_b(q) \approx -E_b + |q|^2/4m \). The partition function then may be written in the following form:

\[
Z = Z_0 \int d\tilde{\phi} d\phi \exp \left\{ \sum_{q, iq} \tilde{\phi}_q (iq_0 - \omega_b(q) + 2\mu)\phi_q \right\}.
\]  

(11)

Correspondingly the strong-coupling number equation reads:

\[
n = n_0 + \sum_q n_B [\omega_b(q) - 2\mu].
\]

(12)

From which neglecting \( n_0 \) follows the result (10) \[6,9\].

B. Nonperturbative nonlinear sigma-model (NLSM) approach to the BCS-BEC crossover in superconductors

The above discussed crossover in the BCS model was recently studied in details perturbatively in a variety of approximations (see for example \[10,11\]). Qualitatively, essential features of this crossover can be reproduced in another simple model system - with the help of deriving of an effective non-linear sigma model (i.e. 3D XY-model) \[7\]. Moreover in the same framework of nonlinear sigma model one can study as well analogous crossover in 2D superconductor \[13,14,19,7\] that cannot be addressed with discussed in the previous section perturbative method due to absence of the long-range order in 2D. In two dimensions \( T_c \) is identified with a temperature of the Kosterlitz-Thouless transition \( T_{KT} \) in 2D XY-model. In the strong-coupling (or low carrier density) regime \( T_{KT} \) lies significantly lower than the temperature of the pair formation \[13,14,19,7\].

As we have shown in \[7\], many essential features known from numerical study of strong-coupling and low carrier density superconductors are reproduced with very good accuracy within 3DXY-model approach. In particular, in the framework of NLSM approach there is no artificial maximum in \( T_c \) in the regime of intermediate couplings (which appears in the approximation discussed in the previous section).

1. XY-model approach to 3D superconductors

Let us now reproduce results of previous subsection in the framework of NLSM approach proposed by the author \[7\].

It is very transparent to study the properties of the BCS-BEC crossover to employ ”modulus-phase” variables. Following to Witten \[24\] we can write a partition function as a functional integral over modulus and phase of the Hubbard-Stratonovich field \( \Delta \exp(i\varphi) \):

\[
Z(\mu,T) = \int \mathcal{D}\Delta \mathcal{D}\varphi \exp \left\{ -\beta\Omega(T,\Delta(x),\varphi(x)) \right\}.
\]

(13)

Assuming that phase fluctuations do not affect local modulus of the complex Hubbard -Stratonovich field we can write thermodynamic potential as a sum of “potential” and “kinetic” (gradient) terms:

\[
\Omega(\Delta(x),\varphi(x)) \approx \Omega_{\text{grad}}(T,\Delta,\varphi(x)) + \Omega_{\text{pot}}(T,\Delta) = \int d^3x \frac{J(T,\Delta)}{2} (\nabla \varphi)^2 + \Omega_{\text{pot}}(T,\Delta).
\]

(14)

The above estimate by \[10\] when we retained corrections only in the number equation renders correct limiting result \[10\], however as it was observed by the authors of \[10\] in the intermediate coupling regime it gives an artificial maximum in \( T_c \) as a function of coupling strength, this artifact is removed in higher approximations \[10\].

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Obviously in the above expression the effective potential \( \Omega_{\text{pot}}(T,\Delta) \) coincides with the ordinary mean-field effective potential. The gradient term \( \Omega_{\text{grad}}(T,\Delta,\partial \varphi(x)) \) coincides with the Hamiltonian of 3D XY model with a stiffness \( J(\Delta, T) \).

Let us reproduce low-temperature expression for the phase stiffness in strong-coupling regime from \([7]\):

\[
J = \frac{n}{4m} - \frac{3\sqrt{2} \pi m}{16\pi^2} T^{3/2} \exp \left[ -\frac{\sqrt{\mu^2 + \Delta^2}}{T} \right]. \tag{15}
\]

Where \( n \) and \( m \) are density and mass of fermions. We can see that thermal corrections to the first term in this regime are exponentially suppressed and the r.h.s. tends in this limit quickly to

\[
J_{\text{BE}} = \frac{n}{4m}. \tag{16}
\]

The form of this expression is not surprising - we see that at sufficiently strong coupling strength all fermions are bound to pairs and stiffness becomes equal of the low-temperature phase stiffness of the Bose gas of density \( n/2 \) and boson mass \( 2m \). Obviously all information about internal structure of composite bosons is evaporated from this expression in this approximation since at low temperature in this regime there are no thermal pair decomposition effects \([1]\). Thus we see that low-temperature expression for the stiffness of the phase fluctuations reaches a plateau value with increasing coupling strength, whereas temperature of the thermal pair decomposition is a monotonously growing function of the coupling strength.

In principle knowledge of lowest gradient term governing gaussian fluctuations is not sufficient for the study of the position of the phase decoherence transition in a system with preformed pairs. In continuum, 3D XY model is a free field theory and there is no phase transition. Phase transition of 3D XY model was studied in great details on a lattice, so we can consider a lattice model of BCS-BEC crossover and we can verify aposteriory to what extend lattice field theory and there is no phase transition. Phase transition of 3D XY model was studied in great details on a

In \([7]\) we employed a finite-temperature generalization of the gradient expansion at \( T = 0 \) discussed in \([\textbf{25}]\).

Critical temperature of the phase transition \( T_c^{3D\text{XY}} \) of 3D XY model can be obtained with a simple mean-field estimate \([\textbf{26}]\):

\[
T_c^{3D\text{XY}} \approx 3Ja. \tag{17}
\]

Where \( a = n^{-1/3}_{\text{pair}} \) is the lattice spacing. In contrast to the ordinary 3DXY-model, in order to find temperature of the phase transition we should study a system of the equations \([\textbf{6}]\), \([\textbf{14}]\), \([\textbf{15}]\) and \([\textbf{17}]\). System of these equations can be solved however analytically in the strong coupling limit, the result is \([\textbf{14}]\):

\[
T_c = \frac{3}{2m} \left( \frac{n}{2} \right)^{2/3} - \frac{1}{n^{1/3}} \frac{1}{2\pi^2} \frac{1}{n^{3/2}} T^{3/2} m^{3/2} \exp \left( -\frac{\sqrt{\mu^2 + \Delta^2}}{T_c} \right). \tag{18}
\]

With increasing coupling strength, this quickly tends from below to the value (compare with \([\textbf{15}]\)):

\[
T_c = \frac{3n^{2/3}}{2^{5/3}m} = \frac{3}{(6\pi^2)^{2/3}} \epsilon_F \approx 0.2 \epsilon_F. \tag{19}
\]

Where constant \( \epsilon_F \) is Fermi energy of free fermi gas of density \( n \) and fermion mass \( m \). We should observe that in nonperturbative NLSM approach \( T_c \) approaches plateau value \([\textbf{14}]\), that depends only on mass \( 2m \) and density \( n/2 \)

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3In \([\textbf{1}]\) we employed a finite-temperature generalization of the gradient expansion at \( T = 0 \) discussed in \([\textbf{25}]\).

4In the ordinary attractive Hubbard model considered in \([\textbf{1}]\) the critical temperature is a nonmonotonous function of the coupling strength: \( T_c \) decreases in the strong-coupling limit due to in that model composite bosons move via virtual ionization.

5We should stress that estimation of critical temperature of the phase transition of the effective 3D XY model with mean-field methods has nothing to do with BCS mean field approximation since derivation of the phase stiffness coefficient \([\textbf{14}]\) required studying the gaussian fluctuations in the BCS model \([\textbf{3}]\), and thus this may be regarded as the approximation of the same level as considered in the previous section.
of composite bosons, from below. This is in agreement with numerical study in higher approximations \[10\], whereas in the approach presented in previous subsection \(T_c\) has an artificial maximum at the intermediate coupling strength thus approaching the limiting value from above. Another circumstance that we discuss below is that NLSM approach gives also qualitatively correct results in the opposite limit of weak-coupling strength \[7\]. In the weak-coupling limit near \(T_c\), the stiffness coefficient may be derived with the help of Gorkov’s well-known method:

\[
J_{\text{BCS}} = \frac{7}{48\pi^2} \zeta(3) \frac{p_F^3}{m} \frac{\Delta^2}{T^*},
\]

This is precisely the coefficient of the gradient term in the Ginzburg-Landau expansion. In the weak-coupling limit the two temperatures of the onset of pairing correlations and the onset of phase coherence \(T^*\) and \(T_c\) merge according to the formula \[7\]:

\[
T_c = T^* - \frac{(2\pi^2)^{2/3}}{2} \frac{T^{*5/2}}{\epsilon_F^{3/2}} \rightarrow T^*,
\]

With it one can see \[7\] that in the weak-coupling limit, the temperature of the phase transition of the effective XY-model tends from below to the characteristic temperature of the disappearance of the effective potential and merges with it for infinitesimally weak coupling strength. Thus we arrive at some sort of aposteriori verification of BCS behavior in this limit in the model of hard-core composite bosons on the lattice (i.e. in this limit if nonzero modulus of the complex gap function \(\Delta e^{i\varphi(x)}\) appears at some temperature, at the same temperature phase coherence is established and continuous symmetry is broken). In the weak and moderate coupling strength limits the disappearance of superconductivity is a competition of two processes - pairbreaking which is thermal excitation of individual particles and decoherence process which is thermal excitation of collective modes.

Let us now summarize the results that follow from the NLSM consideration. In this model in strong-coupling or low carrier density regimes system possesses three phases:

1. **Superconductive phase** \((T < T_{\text{c}^{3DXY}})\).

2. **Pseudogap phase** \((T_{\text{c}^{3DXY}} < T < T^*)\) - the phase where there exists a local gap modulus \(\Delta\) that signalizes existence of the tightly bound (but noncondenced) fermion pairs but phase is random so average of the complex gap is zero \(<|\Delta| \exp(i\varphi)| = 0\). So in this phase there is no superconductive gap and with it no symmetry breakdown.

3. **Normal phase** \((T > T^*)\) the phase with thermally decomposed Cooper pairs.

2. **XY-model approach to 2D superconductors**

In two dimension there is no proper long-range order and superconductive phase transition is associated with a Kosterlitz-Thouless transition. In order to study this transition it is sufficient to extract lowest gradient term that determines temperature of the phase transition according to the formula \[13\], \[14\], \[19\], \[7\]

\[
T_{\text{KT}} = \frac{\pi}{2} J(\mu, T_{\text{KT}}, \Delta(\mu, T_{\text{KT}})).
\]

This equation just like in the discussed above 3D case should be solved self-consistently with equations for the gap modulus and chemical potential \[4\], \[19\]. The result in the strong-coupling limit is \[7\]:

\[
T_{\text{KT}} \sim \frac{\pi}{8} \frac{n}{m} \left\{ 1 - \frac{1}{8} \exp \left[\frac{2\mu}{\epsilon_F} - 4\right] \right\}.
\]

Thus for increasing coupling strength, the phase-decoherence temperature \(T_{\text{KT}}\) tends very quickly towards a constant value \[13\], \[14\], \[19\], \[7\] corresponding to KT transition in system of bosons with density \(n/2\) and mass \(2m\) (whereas

\[6\] In principle there is no KT phase transition in a charged system due to Meissner effect, however coupling to electromagnetic field is always neglected in discussion of 2D superconductors, due to experimentally in-plane penetration length in high-\(T_c\) materials is much larger than coherence length.
characteristic temperature of the thermal pair decomposition continues to grow monotonously with the growing coupling strength):

$$T_{KT} = \frac{\pi n}{8 m}. \tag{24}$$

So, this phenomenon in 2D is qualitatively similar to the above discussed 3D case.

There is no superconductivity in the pseudogap phase however it exhibits rich exotic non-Fermi-liquid behavior due to local pairing correlations that makes it as interesting an object of theoretical and experimental study as the superconductive phase itself. In particular, along with specific heat, optical conductivity and tunneling experiments there are following circumstances observed in the pseudogap phase: In experiments on YBCO a significant suppression of in-plane conductivity $\sigma_{\text{ab}}(\omega)$ was observed at frequencies below $500 \text{ cm}^{-1}$ beginning at temperatures much above $T_c$. Experiments on underdoped samples revealed deviations from the linear resistivity law. In particular $\sigma_{\text{ab}}(\omega = 0; T)$ increases slightly with decreasing $T$ below a certain temperature. NMR and neutrons observations show that below temperatures $T^*$ much higher than $T_c$, spin susceptibility starts decreasing.

In conclusion, let us once more emphasis essential features of this phenomenon in superconductivity:

- Away from a very special limit of infinitesimally weak coupling strength and high carrier density, superconductors are characterized by two temperatures $T_c$ and $T^*(>> T_c)$. $T^*$ is the characteristic temperature below which pair correlations become important (or in the regime of strong interaction it is the characteristic temperature of the formation of real bound pairs). $T_c$ corresponds to the onset of phase coherence in a system of preformed fermion pairs. The region of non-Fermi liquid behavior between $T_c$ and $T^*$ calls the pseudogap phase, however the term “pseudogap”, originated in early experimental papers, may seem somewhat misleading since, even though a substantial depletion of low-frequency spectral weight is observed in this region experimentally - there is no superconductive gap in the spectrum.

- One should note that there is no proper phase transition at $T^*$, which is simply a characteristic temperature of thermal decomposition of certain fraction of noncondensed Cooper pairs. Even though the position of this temperature may be reasonable estimated with mean-field methods, second-order phase transition at $T^*$ is certainly an artifact of the discussed above approximation. Experiments on specific heat indicate however certain features at this characteristic temperature.

In what follows we discuss possible implication of these results to QCD that may posses a phase analogous to the pseudogap phase in strong-coupling superconductors. The simplest model related to particle physics that displays pseudogap behavior of dynamical origin is Chiral Gross-Neveu model at low $N$ that is discussed in the next section in $2 + \epsilon$ dimensions.

III. PSEUDOGAP PHASE IN CHIRAL GROSS-NEVEU MODEL IN $2 + \epsilon$ DIMENSIONS AT LOW $N$

Let us now discuss a phenomenon similar to pseudogap in a simple field-theoretic model - chiral version of the Gross-Neveu model [27], whose Lagrange density is

$$\mathcal{L} = \bar{\psi}_a i \gamma_\mu \psi_a + \frac{g_0}{2N} \left[ (\bar{\psi}_a \psi_a)^2 + (\bar{\psi}_a i \gamma_5 \psi_a)^2 \right]. \tag{25}$$

Where index $a$ runs from 1 to $N$. Appearance of the pseudogap phase in this model has quite similar roots with the above discussed phenomenon in strong-coupling superconductors. In superconductors we observed appearance of the pseudogap phase on the phase diagram in the region away from the limits of infinitesimally weak coupling strength or extremely high carrier density - i.e. in the regime when BCS mean-field treatment is no longer valid. Chiral Gross-Neveu model can be treated in the limit of infinite number of field components $N$ in a framework of mean-field approach quite similar to BCS theory. Transparently in the mean-field approximation one can find only one phase transition at certain value of the coupling strength, similar to BCS phase transition. However, at low $N$, system start to perform dynamic chiral fluctuations which as we have shown in give rise to a second, phase disorder, transition. So at low $N$ the model possesses two transitions at two characteristic values of renormalized coupling constant. Let us reproduce this result.
One can write the collective field action for this model as:

$$A_{\text{coll}}[\sigma] = N \left\{ -\frac{1}{2g_0}(\sigma^2 + \pi^2) - i\text{Tr} \log [i\bar{\phi} - \sigma(x) - i\gamma_5\pi] \right\}. \tag{26}$$

This expression is invariant under the continuous set of chiral O(2) transformations which rotate \(\sigma\) and \(\pi\) fields into each other. This model is equivalent to another one:

$$\mathcal{L} = \bar{\psi}_a i\bar{\phi} \psi_a + \frac{g_0}{2N} \left( \bar{\psi}_a C \psi_a^T \right) \left( \psi_b^T C \psi_b \right). \tag{27}$$

Here \(C\) is the matrix of charge conjugation which is defined by

$$C\gamma^\mu C^{-1} = -\gamma^\mu^T. \tag{28}$$

In two dimensions, we choose the \(\gamma\)-matrices as \(\gamma^0 = \sigma^1, \gamma^1 = -i\sigma^2\), and \(C = \gamma^1\). The second model goes over into the first by replacing \(\psi \rightarrow \frac{1}{2}(1 - \gamma_5)\psi + \frac{1}{2}(1 + \gamma_5)\bar{\psi}\), where superscript T denotes transposition. In the Lagrange density \(\mathcal{L}\) we introduce a complex collective field by adding a term \((N/2g_0) \left| \Delta - \frac{g_0}{N} \psi_b^T C \psi_b \right|^2\), leading to the partition function

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\Delta \mathcal{D}\Delta^\dagger \exp \left\{ i \int d^Dx \left[ \bar{\psi}_a i\bar{\phi} \psi_a + \frac{1}{2} \left( \Delta^\dagger \psi_a^T C \psi_a + \text{c.c.} \right) + \bar{\eta} \psi + \bar{\psi} \eta - \frac{N}{2g_0} |\Delta|^2 \right]\right\}. \tag{29}$$

The relation with the previous collective fields \(\sigma\) and \(\pi\) is \(\Delta = \sigma + i\pi\).

Following to BCS procedure we can fix phase of the order parameter, then for a constant \(\Delta\), the effective action gives rise to an effective potential that in \(2 + \epsilon\) dimensions reads:

$$\frac{1}{N} v(\Delta) = \frac{\mu^\epsilon}{2} \left[ \frac{\Delta^2}{g_0 \mu^\epsilon} - b_\epsilon \left( \frac{\Delta}{\mu} \right)^{2+\epsilon} \right], \tag{30}$$

where \(\mu\) is an arbitrary mass scale, and the constant \(b_\epsilon\) stands for

$$b_\epsilon = \frac{2}{D} 2^{\epsilon/2} S_D \Gamma(D/2) \Gamma(1 - D/2) = \frac{2}{D} \frac{1}{(2\pi)^{D/2}} \Gamma(1 - D/2), \tag{31}$$

which has an \(\epsilon\)-expansion \(b_\epsilon \sim -\left[ 1 - (\epsilon/2) \log(2\pi e^{-\gamma}) \right] / \epsilon \mu + \mathcal{O}(\epsilon)\). A renormalized coupling constant \(g\) may be introduced by the equation

$$\frac{1}{g_0 \mu^\epsilon} - b_\epsilon \equiv \frac{1}{g}, \tag{32}$$

so that

$$\frac{1}{N} v(\Delta) = \frac{\mu^\epsilon}{2} \left\{ \frac{\Delta^2}{g} + b_\epsilon \Delta^2 \left[ 1 - \left( \frac{\Delta}{\mu} \right)^{\epsilon} \right] \right\}. \tag{33}$$

Extremizing this we obtain either \(\Delta_0 = 0\) or a nonzero \(\Delta_0\) that solves the gap equation

$$1 = g_0 \text{tr}(1) \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + \Delta_0^2}, \tag{34}$$

in the following form:

$$1 - \frac{g^*}{g} = \frac{D}{2} \left( \frac{\Delta_0}{\mu} \right)^\epsilon, \tag{35}$$

where \(g^* = -1/b_\epsilon \approx \pi\epsilon\).

In the limit \(N \to \infty\) this result is exact. On the other hand in the opposite limit of low \(N\) the system starts to perform fluctuations around saddle-point solution and in order to describe this system properly one should go beyond mean-field approximation and study propagator of the \(\theta\)-field - where \(\theta\) is the phase of the order parameter.
Let us consider first the case $\epsilon = 0$ where the collective field theory consists of complex field $\Delta$ with O(2)-symmetry $\Delta = |\Delta| e^{i\theta}$. Such a system possesses macroscopic excitations of the form of vortices and antivortices that attract each other by a logarithmic Coulomb potential. It is known [28] that in such field theory involving a pure phase field $\theta(x)$, with a Lagrange density

$$L = \frac{\beta}{2} |\partial \theta(x)|^2,$$

(36)

where $\beta$ is the stiffness of the $\theta$-fluctuations, there is a Kostrelitz-Thouless transition when the stiffness falls below $\beta_{KT} = 2/\pi$ [28].

Let us return to Gross-Neveu model. Performing an expansion around saddle point solution one can find propagator of the $\theta$-field when $\epsilon = 0$ [4,24]:

$$G_{\theta \theta} \approx i \frac{4\pi}{N} q^2 + \text{regular terms} \quad (37)$$

Comparing this with the propagator for the model Lagrange density (36)

$$G_{\theta \theta} = \frac{1}{\beta} i \frac{q^2}{q^2} \quad (38)$$

we identify the stiffness $\beta = N/4\pi$. The pair version of the chiral Gross-Neveu model has therefore a vortex-antivortex pair breaking transition if $N$ falls below the critical value $N_c = 8$ [4].

Consider now the model in $2 + \epsilon$ dimensions where pairs form when renormalized coupling constant becomes larger than the critical value $g = g^* \approx \pi\epsilon$. In this case the expression for the stiffness of phase fluctuations reads [4]

$$\beta = \frac{N}{4\pi} \left( 1 - \frac{g^*}{g} \right). \quad (39)$$

What implies a KT transition in the neighborhood of two dimensions [4] at:

$$N_c \approx 8 \left( 1 - \frac{g^*}{g} \right)^{-1} \quad (40)$$

The resulting phase diagram is shown in Fig. 2.

![Phase Diagram](image)

**FIG. 2.** The two transition lines in the $N - g$-plane of the chiral Gross-Neveu model in $2 + \epsilon$ dimensions. In order to stress difference between the local gap (i.e. "pseudogap") and the order parameter analogous to the "superconductive" gap we denote by $M$ *modulus* of the order parameter ($M = |\Delta_0|$). In this model $M$ plays the role of the "quark" mass. For $\epsilon = 0$, the vertical transition line coincides with the $N$-axis, and the solid hyperbola degenerates into a horizontal line at $N_c = 8$. In the limit $N \to \infty$ generation of the quark mass happens simultaneously with "phase ordering" transition.

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[7] There is a misleading statement about 3D case in [4].
In the chiral formulation of the same model in $2+\epsilon$ dimensions, the “pseudogap” phase has chiral symmetry in spite of a nonzero spontaneously generated “quark mass” $M = |\Delta_0| \neq 0$. This phase is directly related to the pseudogap phase of a strong-coupling superconductor - where there are Cooper pairs but there is no symmetry breakdown due to violent phase fluctuations. The reason why this is possible is that the “quark mass” depends only on $|\Delta_0|$, thus allowing for arbitrary phase fluctuations preserving chiral symmetry. It is very easy to see that the solid hyperbola in Fig. 3 is not simply the proper (albeit approximate) continuation of the vertical line for smaller $N$. There are two simple arguments. One is formal: For infinitesimal $\epsilon$ the first transition lies precisely at $g = g^* = \pi \epsilon$ for all $N$, so that the horizontal transition line is clearly distinguished from it (stiffness of the phase fluctuation in the regime $g^* / g \to 0$, just like in the case of superconductor, reaches plateau value that does not depend on the coupling strength and $\epsilon$).

The other argument is physical and also has a clear analogy in the corresponding phenomena in superconductivity. If $N$ is lowered at some very large $g$, the binding energy of the pairs increases with $1 / N$ [in two dimensions, the binding energy is $4 M \sin^2(\pi / 2(N - 1))$]. It is then impossible that the phase fluctuations on the horizontal branch of the transition line, which are low-energy excitations, unbind the strongly bound pairs. Accuracy of the “BCS” scenario in the limit $N \to \infty$ is clearly seen from the form of phase stiffness which has a factor $N$ that “freezes” the phase fluctuations in this limit and thus all the physics is essentially governed by the size of the gap modulus.

The $2+1$-dimensional Chiral Gross-Neveu model \[29\] also exhibits an analogous behavior at finite temperature \[30\] where a similar effect is governed by thermal fluctuations. At finite $N$ the temperature of KT transition deviates from mean-field temperature of the gap modulus formation, however in $D=2+1$ the phase diagram is substantially different from the phase diagram of the same model in $D = 2 + \epsilon$ at $T = 0$ (see detailed discussion in \[30\]).

**IV. CHIRAL FLUCTUATIONS IN THE THE NJL MODEL AT ZERO TEMPERATURE**

Recently an attempt was made \[31\] to generalize to the NJL model the nonlinear-sigma approach for description of chiral fluctuations proposed in \[14\]. The authors \[31\] claimed that at $N_c = 3$ the NJL model does not display spontaneous symmetry breakdown due to chiral fluctuations. We show below that the NLSM approach does not allow to prove that chiral symmetry is always restored by fluctuations in the NJL model at $N_c = 3$. Below we also discuss differences from the chiral GN model, where the NLSM approach allows one to reach a similar conclusion at low $N$.

The Lagrangian of the NJL model reads \[1\]

$$
\mathcal{L} = \bar{\psi} i \partial \! / \! \psi + \frac{g_0}{2 N_c} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} \lambda_1 \gamma_5 \psi)^2 \right].
$$

(41)

The three $2 \times 2$-dimensional matrices $\lambda_a / 2$, generate the fundamental representation of flavor $SU(2)$, and are normalized by $\text{tr}(\lambda_a \lambda_b) = 2 \delta_{ab}$. One can introduce Hubbard - Stratonovich fields $\sigma$ and $\pi_a$:

$$
\mathcal{L} = \bar{\psi} \left( i \partial \! / \! - \sigma - i \gamma_5 \lambda_a \pi_a \right) \psi - \frac{N_c}{2 g_0} \left( \sigma^2 + \pi_a^2 \right).
$$

(42)

After integrating out quark fields, following a standard mean-field variation procedure one can choose the pseudoscalar solution $\pi_a$ to vanish and the scalar solution $\sigma \equiv M$ to be given by a gap equation:

$$
\frac{1}{g_0} = i (\text{tr} f 1)(\text{tr} \gamma_1) \int \frac{d^D p}{(2 \pi)^D} \frac{1}{p^2 - M^2}.
$$

(43)

The momentum integral is regularized by means of a cutoff $\Lambda$. The constituent quark mass $M$ in the limit $N_c \to \infty$ is analogous to the superconductive gap in the BCS limit of the theory of superconductivity.

At finite $N_c$ one can study fluctuations around the saddle point solution. The quadratic terms of expansion around the saddle point are:

\[\text{It should be emphasized that the existence of the pseudogap phase in 2+1 dimensions due to thermal fluctuations in the Chiral Gross-Neveu model can not be rigorously proven in contrast to 2+\epsilon-dimensional case discussed in this section (when there are two small parameters $\epsilon$ and $1/N$). The 2+1 dimensional problem lacks a small parameter which would allow to estimate accurately the position of $T^*$ at very small $N$, however it can be argued that rough low-$N$ estimates show the appearance of thermal-fluctuations-induced pseudogap phase in this model, especially pronounced at $N \leq 4$.}[/sup]
\[ A_0[\sigma', \pi'] = \frac{1}{2} \int d^4q \left[ \left( \frac{\pi_a'(q)}{\sigma'(q)} \right)^T \left( \begin{array}{cc} G_{\sigma}^{-1} & 0 \\ 0 & G_{\pi}^{-1} \end{array} \right) \left( \begin{array}{c} \pi'_a(-q) \\ \sigma'(-q) \end{array} \right) \right], \]  

(44)

where \((\sigma', \pi'_a) \equiv (\sigma - M, \pi_a)\) and \(G_{\sigma,\pi}^{-1}\) are the inverse bosonic propagators. Implementing a momentum cutoff \(\Lambda\), we can write \(G_{\sigma,\pi}^{-1}\) for small \(qE\) as:

\[ G_{\sigma,\pi}^{-1} \approx -\frac{N_c}{(2\pi)^2} \left[ \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - \frac{\Lambda^2}{\Lambda^2 + M^2} \right] q^2 \equiv -Z(M/\Lambda)q^2; \quad G_{\sigma,\pi}^{-1} \approx -Z(M/\Lambda)(q^2 + 4M^2). \]  

(45)

In analogy to 3D \(XY\)-model approach to strong-coupling superconductivity \([7]\), the authors of \([31]\) introduced a unit vector field \(n_i \equiv (n_0, n_a) \equiv (\sigma, \pi_a)/\rho\) and set up an effective nonlinear sigma-model

\[ A_0[n_i] = \frac{\beta}{2} \int d^4x [\partial n_i(x)]^2. \]  

(46)

The prefactor \(\beta = M^2Z(M/\Lambda)\), that follows from Eqs. \([14]\) and \([15]\) plays the role of the stiffness of the unit field fluctuations.

Now let us observe that from the arguments given in \([31]\) it does not follow that the NJL model necessarily remains in a chirally symmetric phase at \(N_c = 3\). At first, in contrast to the \((2 + \epsilon)\)-dimensional case, discussed in \([1]\), one can not make unfortunately, any similar calculations in a closed form in 3 + 1-dimensions because this theory is not renormalizable. It was already observed in \([22]\)-\([24]\) that the cutoff of meson loops cannot be set equal to the cutoff for quark loops and thus the \(1/N_c\) corrected theory \([34]\) possesses two independent parameters that may be adjusted at will. We present another argument of a different nature rooted in the nonuniversality of the critical stiffness of a NLSM in four dimensions, which does not allow one to reach the conclusion of \([31]\) in the framework of the NLSM approach. Our observation also applies to the NLSM description of precritical fluctuations in general systems. It also allows us to show that the additional cutoff discussed below can not be related to the inverse coherence length of the radial fluctuations in the effective potential as suggested in \([31]\).

The authors of \([31]\) by deriving \(G_{\sigma,\pi}\) have essentially extracted two characteristics from the initial system: the stiffness of the phase fluctuations in the degenerate minimum of the effective potential and the mass of the radial fluctuation. However, knowledge of these characteristics does not allow one in principle to judge if directional fluctuations will destroy long range order or the system will possess a BCS-like phase transition. The reason is that the critical stiffness of the nonlinear sigma model is not an universal quantity in 3 + 1-dimensions. So in principle knowledge of the stiffness of NJL model is not sufficient for finding the position of the phase transition in the effective nonlinear sigma model. The situation is just like that in a Heisenberg magnet, where the critical temperature depends on the stiffness along with lattice spacing and lattice structure. Thus if one is given only a stiffness coefficient one can not determine the temperature of the phase transition. The situation is in contrast to the 2D case where the position of a KT-transition can be deduced from the stiffness coefficient \([28]\).

Let us recall a procedure for expressing the critical stiffness of the O(4)-nonlinear sigma model via an additional parameter: one can relax the constraint \(n_i^2 = 1\) and introduce an extra integration over the lagrange multiplier \(\lambda\), rewriting Eq. \([44]\) as: \((\beta/2) \int d^4x \left\{[\partial n_i(x)]^2 + \lambda \left[ n_i^2(x) - 1 \right] \right\}\). Integrating out the \(n_i(x)\)-fields, yields:

\[ A_0[\lambda] = -\frac{\beta}{2} \int d^4x \frac{\lambda(x)}{2} + \frac{N_n}{2} \text{Tr} \ln \left[-\partial^2 + \lambda(x)\right], \]  

(47)

where \(N_n\) is the number of components of \(n_i(x)\) and \(\text{Tr}\) denotes the functional trace. This yields a gap equation:

\[ \beta = N_n \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \lambda}. \]  

(48)

The model has a phase transition at a critical stiffness that depends on an unspecified additional cutoff parameter that should be applied to the gap equation:

\[ \beta^\sigma = N_n \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2}. \]  

(49)

For example, in the case of magnets the additional cutoff needed in Eq. \([49]\) is naturally related to the lattice spacing. In \([32]\) a criterion was proposed that states that one can relate the inverse coherence length extracted from radial
fluctuations in an effective potential of an initial theory to the cutoff in the integral \(49\) so that all the parameters in a theory would be expressed from quantities derived from an initial model, and thus this modified model possesses a universal critical stiffness. However, unfortunately there is no reason for relating the cutoff needed in Eq. \(49\) to the coherence length of the modulus fluctuations and moreover we show that this procedure leads in general to unphysical consequences. It should also be observed that it does not make the theory consistent anyway because of the following circumstance: The authors of \(31\) suggested that, “since pions in the symmetry broken phase are composite they are not “defined” over the length scales much shorter than the inverse binding energy of the pair wave function which is equal to \(2M^2\).” Thus the authors of \(31\) performed integral in \(49\) up to cutoff \(4M^2\). However, since, as suggested in \(31\), the pion fields are not “defined” over the length scales much shorter than the inverse binding energy this may only serve as an estimate for “upper boundary” of what would be the universal critical stiffness value. So, unfortunately one cannot make the conclusion of absence of symmetry breakdown in such a modified theory by observing that stiffness derived from the initial model is smaller than the maximal possible value of would be universal critical stiffness. It was also supposed in \(31\) that the relation of the coherence length to the cutoff in the equation \(49\) yields a universal criterion for judging the nature of symmetry breakdown in general physical systems. There is a simple counterexample: in the case of a strong-coupling superconductor, the effective nonlinear sigma model that describes fluctuations in a degenerate valley of the effective potential is a 3D XY-model. In the continuous case it is a free field theory and has no phase transition at all. The phase transition appears only in the lattice theory and, of course its temperature depends on the lattice spacing. As we discussed above with increasing coupling strength the low-temperature phase stiffness of the effective 3D XY model tends to a plateau value \(J = n/4m\), where \(n\) and \(m\) are the density and the mass of fermions \(\delta\). Thus the temperature of the phase transition of the effective 3D XY-model is

\[
T_{c}^{3DXY} \propto \frac{m}{m-a},
\]

(50)

where \(a\) is the lattice spacing. To be careful one should remark that accurate analysis shows that a strong coupling superconductor possesses two characteristic length scales: size of the Cooper pairs which tends to zero with increasing coupling strength, and a coherence length that tends to infinity with increasing coupling strength as the system evolves towards a weakly nonideal gas of true composite bosons \(\delta\). First if one relates the constant \(a\) in \(50\) to the size of the Cooper pairs following to the arguments of \(31\) one will come to an incorrect conclusion of the absence of superconductivity in strong-coupling superconductors, in a similar way as the authors of \(31\) came to the conclusion of the nonexistence of symmetry breakdown in the NJL model. This is in a direct contradiction with behavior of the strong coupling superconductors discussed above. Second, if one attempts to relate \(a\) in \(50\) to the second length scale of the theory, namely, the true coherence length, which tends to infinity with increasing coupling strength, then one will also come to a qualitatively incorrect conclusion \(31\).

Thus, in general, the nonlinear sigma model approach for precritical fluctuations possesses an additional fitting parameter, which is the cutoff in the gap equation \(49\), which can not be related to the inverse coherence length extracted from radial fluctuations in an effective potential. Thus within the NLSM approach one can not prove if the NJL model displays necessarily the directional fluctuations-driven restoration of chiral symmetry at low \(N_{c}\).

V. CHIRAL FLUCTUATIONS AT FINITE TEMPERATURE AND A MODIFIED NJL MODEL WITH A PSEUDOGAP

This section is based on the paper \(38\). The authors of \(31\) employed NLSM arguments in an attempt to show that the NJL model cannot serve the study of the chiral symmetry breakdown. We have shown in the above that this conclusion appears to be incorrect since the critical stiffness in 3+1-dimensions is not a universal quantity and one has an additional fitting parameter. This is an inherent feature of the discussed NLSM approach in 3+1 dimensions (compare with the cutoffs discussions in nonrenormalizable models in a different approach \(32\), \(33\), and also \(37\)). The above circumstance allows one to fix the critical stiffness from phenomenological considerations. However, we argue below that, what is missed in \(31\) is that, in principle, the low-\(N_{c}\) fluctuation instabilities, when properly treated, have a clear physical meaning. Moreover, we argue that one can employ a NLSM for describing the chiral fluctuations (e.g. at finite temperature), provided that special care is taken of the additional cutoff parameter. Indeed, it was already discussed in the literature that at finite temperatures the chiral phase transition should be accompanied by developed fluctuations (\(33\) and references therein). We argue that this process at low \(N_{c}\) should give rise to a phase analogous to the pseudogap phase that may be conveniently described within a nonlinear sigma model approach. There are indeed other ways to describe these phenomena, however the NLSM approach seems to be especially convenient. The
description of the two-step chiral phase transition and appearance of the intermediate phase requires one to study the system at the next-to-mean-field level. Unfortunately, the NJL model is not renormalizable and does not allow one to make any conclusions about the importance of fluctuations in a closed form [44]. On the other hand, a pseudogap phase is a general feature of Fermi systems with composite bosons. The NLSM construction discussed below, because of its nonperturbative nature, can not be regarded as a regular approximation but may be considered as a tractable modification of the NJL model that has a pseudogap. One can also find an additional motivation for employing these arguments in the fact that the NLSM approach allows one to prove the existence of the phase analogous to pseudogap in the chiral GN model [4,30] which is the closest relative of the NJL model. Also the NLSM approach works well for the description of precritical fluctuations in superconductors [4] - where essentially the same results have been obtained with different methods and in different models. We stress that these phenomena are a general feature of any Fermi system with attraction. Also, to a certain extent similar crossovers are known in a large variety of condensed matter systems. In particular, besides superconductors we might mention the excitonic condensate in semiconductors, Josephson junction arrays, itinerant and local-momentum theories of magnetism and ferroelectrics.

Let us now consider the chiral fluctuations in the NJL model at finite temperature. Then, following standard dimensional reduction arguments [see e.g. [39]], the chiral fluctuations should be described by a $3\text{-dimensional}$ classical $O(4)$-nonlinear sigma model with stiffness $J_T$ is the stiffness of thermal fluctuations in the degenerate valley of the effective potential. The temperature - dependent quark mass $M$ that enters this expression is given by a standard mean-field gap equation which also should be regularized with the cutoff $\Lambda_T$:

$$J_T/T = N_c \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + \lambda}$$

(51)

The temperature of the phase transition of the three dimensional classical $O(4)$ sigma model with stiffness $J_T$ is expressed via the additional parameter $\Lambda_T$ needed in (51) as:

$$T_c = \frac{\pi^2 J_T}{2 N_c}$$

(52)

The stiffness of thermal fluctuations $J_T$ can be readily extracted from the NJL model. At finite temperature the inverse bosonic propagator of the collective field $\pi$ for small $q$ can be written as:

$$G_\pi^{-1} = -2^{D/2}N_c \int \frac{d^3p}{(2\pi)^3} \sum_n \left[ \frac{T}{(p^2 + M^2 + \omega_n^2)^2} \right] q^2 =$$

$$= -2^{D/2}N_c \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{8 (p^2 + M^2)^{3/2}} \tanh \left( \frac{\sqrt{p^2 + M^2}}{2T} \right) \right] q^2 =$$

$$- K(T, \Lambda_T, M, N_c) q^2,$$

(53)

where $\Lambda_T$ is a momentum cutoff. The propagator (53) gives the gradient term that allows one to set up an effective classical $3D O(4)$-nonlinear sigma model:

$$E = \frac{J_T(T, \Lambda_T, M, N_c)}{2} \int d^3x [\partial n_i(x)]^2,$$

(54)

where

$$J_T(T, \Lambda_T, M, N_c) = K(T, \Lambda_T, M, N_c) M^2(T, \Lambda_T)$$

(55)

is the stiffness of the thermal fluctuations in the degenerate valley of the effective potential. The temperature dependent quark mass $M$ that enters this expression is given by a standard mean-field gap equation which also should be regularized with the cutoff $\Lambda_T$:

$$\frac{1}{g_0} = 2 \times 2^{D/2} \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{T}{p^2 + M^2 + \omega_n^2}.$$
(this roughly corresponds to thermal pair breaking in a superconductor) and the process of thermal excitations of the directional fluctuations in the degenerate minimum of the effective potential. The “BCS” limit corresponds to the situation where $T^*$ merges with $T_c$ and it is easily seen that this scenario always holds true at $N_c \to \infty$. That is, at infinite $N$ the mean-field theory is always accurate just as BCS theory works well in weak-coupling superconductors. In the framework of this NLSM construction, at low $N_c$ the scenario of the phase transition depends on the choice of $M(0), \Lambda_T$, and $\tilde{\Lambda}_T$, which should be fixed from phenomenological considerations.

VI. CONCLUSION

The precursor pairing fluctuations is a general feature of any Fermi system with composite bosons and it is the dominant region of a phase diagram of strong-coupling and low-carrier-density superconductors. At the moment it is a subject of increasing interest in different branches of physics. In the first part of this paper we briefly outlined the nonlinear sigma model approach to this phenomenon in superconductors in two and three dimensions. In the second part we discussed similar phenomena in the chiral Gross-Neveu and Nambu–Jona-Lasinio models. This discussion should have relevance for hot QCD and color superconductors.

We also note that in some sense similar phenomena are known in a large variety of condensed matter systems, in particular, besides superconductors we can mention itinerant and local-momentum theories of magnetism, exitonic condensate in semiconductors, ferroelectrics and Josephson junction arrays [10].

We would like to stress that the main purpose of this paper is to summarize present discussions of precursor fluctuations in the Gross-Neveu and Nambu–Jona-Lasinio models. We illustrated the discussion with a few examples from superconductivity, outlining occurrence of similar phenomena in several arbitrarily chosen models of superconductors with pseudogaps. Thus this paper can not be regarded as a review of this phenomenon in superconductivity which evolved recently to a very large branch of condensed matter physics. Thus our references to the papers on superconductivity are by definition incomplete, for more complete set of references reader may consult corresponding reviews on superconductivity [e.g. the review [13]].

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