Strong Convergence Rates in Averaging Principle for Slow-Fast McKean-Vlasov SPDEs

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\textbf{Abstract.} In this paper, we aim to study the asymptotic behaviour for a class of McKean-Vlasov stochastic partial differential equations with slow and fast time-scales. Using the variational approach and classical Khasminskii time discretization, we show that the slow component strongly converges to the solution of the associated averaged equation. In particular, the corresponding convergence rates are also obtained. The main results can be applied to demonstrate the averaging principle for various McKean-Vlasov nonlinear SPDEs such as stochastic porous media type equation, stochastic $p$-Laplace type equation and also some McKean-Vlasov stochastic differential equations.

\textbf{Keywords:} SPDE; Distribution dependence; Averaging principle; Convergence rate; Porous media equation; $p$-Laplace equation.

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\section{Introduction}

The McKean-Vlasov SDEs, also called mean-field SDEs or distribution dependent SDEs, have attracted much attention in recent years, which was initiated by McKean \cite{37}. Roughly speaking, these are SDEs where their coefficients also depend on the distribution of solutions. This type of models can be used to characterize the limiting behaviors of $N$-interacting particle systems of mean-field type while $N$ goes to infinity (also called propagation of chaos), one can see \cite{38} for more background on this topic. The main motivation for studying the McKean-Vlasov SDEs is due to its wide applications since the evolution of stochastic systems often rely on both the microcosmic position and the macrocosmic distribution of the particles. Furthermore, the McKean-Vlasov SDEs also have some intrinsic link with the nonlinear Fokker-Planck-Kolmogorov equations (cf. \cite{11, 29}). More precisely, the corresponding distribution density (denoted by $\rho_t$) of solutions to McKean-Vlasov SDEs solves the following PDE

$$\partial_t \rho_t = L^* \rho_t, \quad t \geq 0,$$

where $L$ is a second order differential operator and $L^*$ denotes its adjoint operator.

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McKean-Vlasov S(P)DEs have been extensively investigated in recent years. For instance, Wang [48] proved the strong and weak existence and uniqueness of solutions to McKean-Vlasov monotone SDEs, and also studied the corresponding exponential ergodicity and Harnack type inequality under some strongly dissipative conditions, which are applicable to e.g. the homogeneous Landau equations. After that, Zhang [50] investigated the weak solutions of McKean-Vlasov SDEs with singular coefficients, which can be used to characterize the existence of weak solutions to 2D Navier-Stokes equations with measure as initial vorticity. Recently, the authors [28] used the generalized variational framework to study the existence of unique strong solution for a class of distribution dependent stochastic porous media equation. Barbu and R"ockner [4] also used the nonlinear Fokker-Planck equations to investigate some McKean-Vlasov SDEs. We refer the interested reader to [5, 11, 26, 30, 41] and references therein for more recent results on this topic.

In this paper, we will consider the following slow-fast McKean-Vlasov stochastic partial differential equations

\[
\begin{align*}
    dX_t^\varepsilon &= \left[ A_1(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) + f(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon) \right] dt + B_1(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})dW_1^t, \\
    dY_t^\varepsilon &= \frac{1}{\varepsilon} A_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}} B_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon)dW_2^t, \\
    X_0^\varepsilon &= x, Y_0^\varepsilon = y,
\end{align*}
\]  

(1.1)

where \( \{W_i^t\}_{t \in [0,T]}, i = 1, 2, \) are independent cylindrical Wiener processes defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\), \( \mathcal{L}_{X_t^\varepsilon} \) denotes the law of \( X_t^\varepsilon \), \( \varepsilon \) is a small and positive parameter describing the ratio of time-scale between processes \( X_t^\varepsilon \) and \( Y_t^\varepsilon \). With this time-scale, the variable \( X_t^\varepsilon \) is referred to as the slow component and \( Y_t^\varepsilon \) is referred to as the fast component. Multiscale systems are very common in many fields of sciences, like material sciences, fluids dynamics, climate dynamics, etc. For example, dynamics of chemical reaction networks often take place on notably different time-scales, from the order of nanoseconds to the order of several days, the reader can see [6, 16, 25, 39] and the references therein for more precise background and applications.

One natural question is what will happen to the solution of the system (1.1) as \( \varepsilon \to 0 \)? This question arises naturally from both physical and mathematical standpoints. Averaging principle is a powerful tool for some qualitative analysis of stochastic dynamical systems with different time-scales. The averaging principle for stochastic dynamical systems with fast and slow time-scales can be viewed as a law of large numbers, in the cases where a slow component is driven by an equation with coefficients depending on a fast component, which is an ergodic stochastic process: when the separation of time-scales goes to infinity, the slow component converges to the solution of an averaged equation whose coefficients have been averaged out with respect to some invariant probability distribution for the fast component.

Apart from the above motivations, the averaging principle itself is also theoretically interesting, which has been studied a lot in the literatures. The averaging principle for dynamical systems with different time-scales was first studied by Bogoliubov and Mitropolsky [8] for the deterministic systems, afterwards Khasminskii [31] developed the averaging principles for stochastic dynamical systems, see e.g. [20, 24, 34] for further generalizations on different types of SDEs. Recently, the averaging principles for SPDEs have also been intensively investigated in the literature. For example, Dong et al. [15] studied the strong and weak averaging principle for stochastic Burgers equations, Bréhier [9, 10] gave the strong and weak orders in averaging for stochastic evolution equation of parabolic type with slow and fast time-scales. The averaging principle for the nonautonomous slow-fast systems of stochastic reaction-diffusion equations was considered in [13]. Moreover, Liu et al. [36] also established...
the strong averaging principle for a class of SPDEs with locally monotone coefficients. For
more results on this subject, we refer to [1, 2, 12, 18, 19, 21, 22, 40, 44, 46, 47, 49] and the
references therein.

However, to the best of our knowledge, there is no result concerning the averaging prin-
ciple for McKean-Vlasov type SPDEs in the literature so far. Recently, based on the techniques
of time discretization and Poisson equation, Röckner et al. [42] established the strong conver-
gence rates of averaging principle for McKean-Vlasov SDEs with global Lipschitz coefficients.
Bezemek and Spiliopoulos [7] also studied the large deviations principle for interacting par-
ticle systems of diffusion type in multiscale environments. Note that the above results are
for the finite dimensional SDE case. In this paper, we aim to study the strong averaging
principle for a class of McKean-Vlasov (nonlinear) SPDEs with slow and fast time-scales.
More precisely, under some appropriate assumptions, we shall prove that

\[ E \left( \sup_{t \in [0,T]} \| X^\varepsilon_t - \bar{X}_t \|_{H_1}^2 \right) \leq C \varepsilon^{-1/3} \to 0, \quad \text{as } \varepsilon \to 0, \quad (1.2) \]

where \( \bar{X}_t \) is the solution of the averaged equation (see equation (2.5) below). In particular,
the corresponding convergence rate of (1.2) is also derived, which is very important in some
applications. For instance, the rate of convergence is crucial for the analysis of numerical
schemes used to approximate the slow component \( X^\varepsilon \).

In the distribution-independent case, the convergence rates for two-time-scale SDEs have
been studied in some works, see e.g. [20, 33, 44, 43] and the references therein. Note that
there are only few results concerning the strong convergence rates for SPDEs in the literature.
Fu et al. [18] established the convergence rate of order 1/4 for a class of stochastic hyperbolic-
parabolic equations. Dong et al. [15] also studied the strong convergence of stochastic Burgers
equations with some Logarithmic convergence order. An important development concerning
strong convergence rate for SPDEs was established by Bréhier [10] with the convergence
rate of order 1/2, which is the optimal order of strong convergence in general. However,
most papers in the literature investigated strong convergence rate using the mild solution
approach, which is only applicable to some semilinear SPDEs. In this paper, we establish
the convergence rate of order 1/6 for a class of McKean-Vlasov quasilinear SPDEs. In [10],
to obtain the optimal convergence order, some fairly strong conditions such as the regularity
of second and higher order derivatives of the coefficients and more regular initial value are
assumed. The convergence rate obtained here might not be optimal, since we only assume
the coefficients satisfy some monotonicity and coercivity conditions, which is in general much
weaker than the assumptions in [10]. As examples, our main results are applicable to some
McKean-Vlasov quasilinear SPDEs such as distribution dependent stochastic porous media
type equations, stochastic \( p \)-Laplace type equations, which are also new in the distribution-
independent case.

It should be mentioned that this is the first averaging principle result for two-time-scale
McKean-Vlasov (nonlinear) SPDEs. In addition, We also remark that there are some merits
to analyze nonlinear operators (even linear operators) on a Gelfand triple replacing a single
space, which helps us to deal with the McKean-Vlasov type SPDEs with nonlinear terms (cf.
e.g. [17, 32]). Since the well-posedness of the two-time-scale McKean-Vlasov SPDEs (1.1)
is not covered by the classical theory of monotone SPDEs [35] and the McKean-Vlasov
case (26, 28), based on the technique of Galerkin type approximation and monotonic-
ity arguments, we first prove the existence and uniqueness of variational solutions for the
two-time-scale McKean-Vlasov SPDEs. Then we aim to investigate the strong averaging
principle for this type of models. The proof here is mainly inspired by the well-known time
results of McKean-Vlasov SDEs in [42], in order to cover some infinite dimensional nonlinear dynamical systems under random influences. We need to point out that compared with the discretization method, which was first developed by Khasminskii in [31] for finite dimensional nonlinear SPDE models, we now consider the system in two Gelfand triples, thus we have to derive some apriori estimates of solutions involving different spaces and overcome some non-trivial difficulties caused by the nonlinear terms, which is quite different to the finite dimensional case.

The remainder of this manuscript is organized as follows. In section 2, we construct the variational framework for a class of McKean-Vlasov SPDEs and give the main results of the present paper. In section 3, we show the existence and uniqueness of solutions to the system (1.1). In section 4, we devote to proving the averaging principle for the system (1.1), and in section 5 some concrete McKean-Vlasov SPDE models are given to illustrate the applications of the main results.

2 Main Results

Let us denote by \((U_i,\langle \cdot,\cdot \rangle_{U_i})\) and \((H_i,\langle \cdot,\cdot \rangle_{H_i})\), \(i = 1, 2\), some separable Hilbert spaces, and \(H_i^*\) the dual space of \(H_i\). Let \(V_i\), \(i = 1, 2\), denote the reflexive Banach spaces such that the embedding \(V_i \subset H_i\) is continuous and dense. We identify \(H_i\) with its dual space according to the Riesz isomorphism, which gives the following Gelfand triples

\[
V_i \subset H_i(\cong H_i^*) \subset V_i^*.
\]

The dualization between spaces \(V_i\) and \(V_i^*\) is denoted by \(V_i^*(\cdot,\cdot)_{V_i}\). It is obvious that

\[
V_i^*(\cdot,\cdot)_{V_i}|_{H_i \times V_i} = \langle \cdot,\cdot \rangle_{H_i},\quad i = 1, 2.
\]

Let \(L_2(U_i, H_i)\) be the space of all Hilbert-Schmidt operators from \(U_i\) to \(H_i\).

Denote by \(\mathcal{P}(H_1)\) the space of all probability measures on \(H_1\) equipped with the weak topology. Now we define

\[
\mathcal{P}_2(H_1) := \left\{ \mu \in \mathcal{P}(H_1) : \mu(\|\cdot\|_{H_1}^2) := \int_{H_1} \|\xi\|_{H_1}^2 \mu(d\xi) < \infty \right\}.
\]

Then \(\mathcal{P}_2(H_1)\) is a Polish space under the following \(L^2\)-Wasserstein metric

\[
\mathcal{W}_2,H_1(\mu,\nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left( \int_{H_1 \times H_1} \|\xi - \eta\|_{H_i}^2 \pi(d\xi, d\eta) \right)^{\frac{1}{2}}, \quad \mu, \nu \in \mathcal{P}_2(H_1),
\]

here \(\mathcal{C}(\mu,\nu)\) stands for the set of all couplings for the measures \(\mu\) and \(\nu\), i.e., \(\pi \in \mathcal{C}(\mu,\nu)\) is a probability measure on \(H_1 \times H_1\) such that \(\pi(\cdot \times H_1) = \mu\) and \(\pi(H_1 \times \cdot) = \nu\).

For some measurable maps

\[
A_1 : V_1 \times \mathcal{P}_2(H_1) \to V_1^*, \quad f : H_1 \times \mathcal{P}_2(H_1) \times H_2 \to H_1, \quad B_1 : V_1 \times \mathcal{P}_2(H_1) \to L_2(U_1, H_1),
\]

and

\[
A_2 : H_1 \times \mathcal{P}_2(H_1) \times V_2 \to V_2^*, \quad B_2 : H_1 \times \mathcal{P}_2(H_1) \times V_2 \to L_2(U_2, H_2),
\]

we consider the following two-time-scale McKean-Vlasov SPDEs

\[
\begin{aligned}
&dX_t^\varepsilon = \left[ A_1(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) + f(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon) \right] dt + B_1(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})dW_t^1, \\
&dY_t^\varepsilon = \frac{1}{\varepsilon} A_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}} B_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon)dW_t^2, \\
&X_0^\varepsilon = x, \quad Y_0^\varepsilon = y,
\end{aligned}
\]  

(2.1)
where \( \{ W_i^j \}_{i \in [0,T]}, i = 1,2, \) are \( U_i \)-valued independent cylindrical Wiener process defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_{\geq 0}, \mathbb{P})\), initial values \( x, y \) belong to \( H_1, H_2 \) respectively.

We first assume that the coefficients in (2.1) satisfy the following two hypothesises.

**Hypothesis 2.1** There are some constants \( \alpha \geq 2, \theta > 0 \) and \( c_1 > 0 \) such that for all \( u, v \in V_1, u_1, u_2 \in H_1, v_1, v_2 \in H_2 \) and \( \mu, \nu \in \mathcal{P}_2(H_1) \) we have

(A1) (Demicontinuity) The map

\[
V_1 \times \mathcal{P}_2(H_1) \ni (u, \mu) \mapsto v_1 \langle A_1(t, u, \mu), v \rangle_{V_1}
\]

is continuous.

(A2) (Monotonicity and Lipschitz)

\[
v_1 \langle A_1(u, \mu) - A_1(v, \nu), u - v \rangle_{V_1} \leq c_1(\|u - v\|^2_{H_1} + \mathbb{W}_{2,H_1}(\mu, \nu)^2).
\]

Moreover,

\[
\| f(u_1, \mu, v_1) - f(u_2, \nu, v_2) \|_{H_1}
\leq c_1(\|u_1 - u_2\|_{H_1} + \|v_1 - v_2\|_{H_2} + \mathbb{W}_{2,H_1}(\mu, \nu))
\]

and

\[
\| B_1(u, \mu) - B_1(v, \nu) \|_{L_2(V_1, H_1)} \leq c_1(\|u - v\|_{H_1} + \mathbb{W}_{2,H_1}(\mu, \nu)).
\]

(A3) (Coercivity)

\[
2v_1 \langle A_1(u, \mu), u \rangle_{V_1} + \| B_1(u, \mu) \|^2_{L_2(V_1, H_1)} \leq -\theta \|u\|^2_{V_1} + c_1(1 + \|u\|^2_{H_1} + \mu(\| \cdot \|^2_{H_1})).
\]

(A4) (Growth)

\[
\| A_1(u, \mu) \|_{V_1} \leq c_1(1 + \|u\|^\alpha_{V_1} + \mu(\| \cdot \|^2_{H_1})).
\]

**Hypothesis 2.2** There are some constants \( \beta > 1, \eta, \kappa > 0, L_{B_2}, c_2 > 0 \) such that for all \( u, u_1, u_2 \in H_1, v, v_1, v_2, w \in V_2 \) and \( \mu, \nu \in \mathcal{P}_2(H_1) \) we have

(H1) (Demicontinuity) The map

\[
H_1 \times \mathcal{P}_2(H_1) \times V_2 \ni (u, \mu, v) \mapsto v_2 \langle A_2(u, \mu, v), w \rangle_{V_2}
\]

is continuous.

(H2) (Monotonicity and Lipschitz)

\[
v_2 \langle A_2(u_1, \mu, v_1) - A_2(u_2, \nu, v_2), v_1 - v_2 \rangle_{V_2}
\leq -\kappa \|v_1 - v_2\|^2_{H_2} + c_2(\|u_1 - u_2\|^2_{H_1} + \mathbb{W}_{2,H_1}(\mu, \nu)^2)
\]

and

\[
\| B_2(u_1, \mu, v_1) - B_2(u_2, \nu, v_2) \|_{L_2(V_1, H_2)} \leq L_{B_2} \|v_1 - v_2\|_{H_2} + c_2(\|u_1 - u_2\|_{H_1} + \mathbb{W}_{2,H_1}(\mu, \nu)).
\]
\((H3) \ (Coercivity)\)

\[
2\nu^2 (A_2(u, \mu, v), v)_{V_2} + \|B_2(u, \mu, v)\|_{L_2(V_2,H_2)}^2 \\
\leq c_2 (1 + \|v\|_{H_2}^2 + \|u\|_{H_1}^2 + \mu(\|\cdot\|_{H_1}^2)) - \eta \|v\|^2_{V_2}.
\]

\((H4) \ (Growth)\)

\[
\|A_2(u, \mu, v)\|_{V_2}^2 \leq c_2 (1 + \|v\|_{V_2}^2 + \|u\|_{H_1}^2 + \mu(\|\cdot\|_{H_1}^2)).
\]

**Remark 2.1**

(i) Note that the assumptions for the slow component of system (2.1) in Hypothesis [2.2] extends the classical variational framework to the distribution dependent case, which are applicable to various McKean-Vlasov quasilinear and semilinear SPDEs, such as distribution dependent stochastic porous media type equations and stochastic p-Laplace type equations.

(ii) The strictly monotone condition (2.3) is used to guarantee the existence and uniqueness of invariant probability measure and the associated exponential ergodicity for the frozen equation (see Eq. (4.2) below) of the fast component of system (2.1). A typical example satisfying hypothesis 2.1 will be presented in section 5.

The definition of variational solution to system (2.1) is given as follows.

**Definition 2.1** For any \(\varepsilon > 0\), we call a continuous \(H_1 \times H_2\)-valued \((\mathcal{F}_t)_{t\geq0}\)-adapted process

\((X^\varepsilon_t, Y^\varepsilon_t)_{t\in[0,T]}\)

is a solution of the system (2.1), if for its \(dt \times \mathbb{P}\)-equivalent class \((\hat{X}^\varepsilon_t, \hat{Y}^\varepsilon_t)_{t\in[0,T]}\) satisfying

\[
\hat{X}^\varepsilon \in L^\alpha([0,T] \times \Omega, dt \times \mathbb{P}; V_1) \cap L^2([0,T] \times \Omega, dt \times \mathbb{P}; H_1),
\]

\[
\hat{Y}^\varepsilon \in L^\beta([0,T] \times \Omega, dt \times \mathbb{P}; V_2) \cap L^2([0,T] \times \Omega, dt \times \mathbb{P}; H_2),
\]

where \(\alpha, \beta\) is the same as defined in (A3) and (H3), respectively, and \(\mathbb{P}\)-a.s.,

\[
\begin{align*}
\begin{cases}
\ dX^\varepsilon_t = x + \int_0^t [A_1(\hat{X}^\varepsilon_s, \mathcal{L}_{X^\varepsilon_s}) + f(\hat{X}^\varepsilon_s, \mathcal{L}_{X^\varepsilon_s}, \hat{Y}^\varepsilon_s)] \, ds + \int_0^t B_1(\hat{X}^\varepsilon_s, \mathcal{L}_{X^\varepsilon_s}) \, dW^1_s,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\ dY^\varepsilon_t = y + \frac{1}{\varepsilon} \int_0^t A_2(\hat{X}^\varepsilon_s, \mathcal{L}_{X^\varepsilon_s}, \hat{Y}^\varepsilon_s) \, ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t B_2(\hat{X}^\varepsilon_s, \mathcal{L}_{X^\varepsilon_s}, \hat{Y}^\varepsilon_s) \, dW^2_s,
\end{cases}
\end{align*}
\]

here \((\hat{X}^\varepsilon, \hat{Y}^\varepsilon)\) is an \(V_1 \times V_2\)-valued progressively measurable \(dt \times \mathbb{P}\)-version of \((\hat{X}^\varepsilon, \hat{Y}^\varepsilon)\).

The first result is about the existence and uniqueness of solutions to system (2.1).

**Theorem 2.1** Suppose that the assumptions (A1)-(A4) and (H1)-(H4) hold. For each \(\varepsilon > 0\) and initial values \(x \in H_1\), \(y \in H_2\), system (2.1) has a unique solution \((X^\varepsilon_t, Y^\varepsilon_t)_{t\in[0,T]}\) in the sense of Definition 2.1.

The next main result of this paper is the strong averaging principle for the system (2.1).

**Theorem 2.2** Suppose that the assumptions (A1)-(A4) and (H1)-(H4) hold. If \(\kappa > 2L^2_{B_2}\), then for any initial values \(x \in H_1\), \(y \in H_2\) and \(T > 0\), we have

\[
\mathbb{E} \left( \sup_{t\in[0,T]} \|X^\varepsilon_t - \hat{X}_t\|_{H_1}^2 \right) \leq C_T (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2) \varepsilon^{1/3} \to 0, \quad \text{as } \varepsilon \to 0,
\]

as } \varepsilon \to 0. \quad (2.4)
where \( C_T \) is a constant only depending on \( T \), \( \bar{X}_t \) is the solution of the following averaged equation
\[
\begin{align*}
\left\{ \begin{array}{l}
    d\bar{X}_t = \left[ A_1(\bar{X}_t, L_{X_t}) + f(\bar{X}_t, L_{X_t}) \right] dt + B_1(\bar{X}_t, L_{X_t})dW^1_t, \\
    \bar{X}_0 = x.
\end{array} \right.
\end{align*}
\]
(2.5)

Here the nonlinear coefficient \( \bar{f}(x, \mu) := \int_{H_2} f(x, \mu, y)\nu^{\mu, y}(dy) \) is the average of \( f \) with \( \nu^{\mu, y} \) being the unique invariant distribution of the frozen equation below with respect to any fixed \( x \in H_1 \) and \( \mu \in \mathcal{P}_2(H_1) \),
\[
\left\{ \begin{array}{l}
    dY_t = A_2(x, \mu, Y_t)dt + B_2(x, \mu, Y_t)d\tilde{W}^2_t, \\
    Y_0 = y,
\end{array} \right.
\]

where \( \tilde{W}^2_t \) is an \( U_2 \)-valued cylindrical Wiener process defined on another probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\).

Throughout the paper, \( C, C_T \) denote some positive constants which may change from line to line, and \( C_T \) is used to stress that the constant only depends on \( T \).

3 Proof of Existence and uniqueness

In this section, we will use the technique of Galerkin type approximation to get the existence and uniqueness of strong solutions to system (2.1).

Without loss of the generality, we assume \( \varepsilon = 1 \) in system (2.1) and consider the following equation
\[
\left\{ \begin{array}{l}
    dX^1_t = \left[ A_1(X^1_t, L_{X^1_t}) + f(X^1_t, L_{X^1_t}, Y^2_t) \right] dt + B_1(X^1_t, L_{X^1_t})dW^1_t, \\
    dY^2_t = A_2(X^1_t, L_{X^1_t}, Y^2_t)dt + B_2(X^1_t, L_{X^1_t}, Y^2_t)d\tilde{W}^2_t, \\
    X^1_0 = x, Y^2_0 = y.
\end{array} \right.
\]
(3.1)

Choosing \( \{e_1, e_2, \cdots \} \subset V_1 \) as an orthonormal basis (ONB) on \( H_1 \) and \( \{l_1, l_2, \cdots \} \subset V_2 \) as an ONB on \( H_2 \). Define the following maps
\[
\Pi^n_1 : V_1^* \rightarrow H_1^n := \text{span}\{e_1, e_2, \cdots , e_n\}, \quad \Pi^n_2 : V_2^* \rightarrow H_2^n := \text{span}\{l_1, l_2, \cdots , l_n\}, \quad n \geq 1,
\]
respectively by
\[
\Pi^n_1 x := \sum_{i=1}^{n} \langle x, e_i \rangle V e_i, \quad x \in V_1^*,
\]
and
\[
\Pi^n_2 y := \sum_{i=1}^{n} \langle y, l_i \rangle V_2 l_i, \quad y \in V_2^*.
\]

It is easy to see that if we restrict \( \Pi^n_1 \) to \( H_1 \), denoted by \( \Pi^n_1|_{H_1} \), then it is an orthogonal projection onto \( H_1^n \) on \( H_1 \). Denote by \( \{g_1, g_2, \cdots \} \) and \( \{j_1, j_2, \cdots \} \) the ONBs on \( U_1 \) and \( U_2 \), respectively. Let
\[
W^{1,n}_t := \Pi^n_1 W^1_t = \sum_{i=1}^{n} \langle W^1_t, g_i \rangle U_1 g_i, \quad n \geq 1,
\]
and
\[
W^{2,n}_t := \Pi^n_2 W^2_t = \sum_{i=1}^{n} \langle W^2_t, j_i \rangle U_2 j_i, \quad n \geq 1,
\]
respectively. Denote by \( \bar{A}_n \) and \( \bar{B}_n \) the \( n \)-dimensional truncation of \( A \) and \( B \), respectively. It is easy to see

where \( \Pi_n^1 \) is an orthonormal projection onto \( U_1^n := \text{span}\{g_1, g_2, \ldots, g_n\} \) on \( U_1 \), and analogously for \( \Pi_n^2 \).

For any \( n \geq 1 \), we consider the following finite dimensional equation

\[
\begin{aligned}
\frac{dX_t^{1,n}}{dt} &= \Pi_n^1 \left[ A_1(X_t^{1,n}, \mathcal{L}_X^{1,n}) + f(X_t^{1,n}, \mathcal{L}_X^{1,n}, Y_t^{2,n}) \right] dt + \Pi_n^1 B_1(X_t^{1,n}, \mathcal{L}_X^{1,n}) dW_t^{1,n}, \\
\frac{dY_t^{2,n}}{dt} &= \Pi_n^2 A_2(X_t^{1,n}, \mathcal{L}_X^{1,n}, Y_t^{2,n}) dt + \Pi_n^2 B_2(X_t^{1,n}, \mathcal{L}_X^{1,n}, Y_t^{2,n}) dW_t^{2,n}, \\
X_0^{1,n} &= x^n, \quad Y_0^{2,n} = y^n,
\end{aligned}
\]

(3.2)

here we denote \( X_1^{1,n} := \Pi_1^n X, Y_2^{2,n} := \Pi_2^n Y, x^n := \Pi_1^n x \) and \( y^n := \Pi_2^n y \).

We now introduce the following product spaces. Let \( \mathcal{H} := H_1 \times H_2 \) be the product Hilbert space. For any \( (\phi_1, \phi_2), \varphi := (\varphi_1, \varphi_2) \in \mathcal{H} \), we denote the scalar product and the induced norm by

\[
(\phi, \varphi)_\mathcal{H} = (\phi_1, \varphi_1)_{H_1} + (\varphi_2, \varphi_2)_{H_2}, \quad \|\phi\|_\mathcal{H} = \sqrt{(\phi, \phi)_\mathcal{H}} = \sqrt{\|\phi_1\|_{H_1}^2 + \|\phi_2\|_{H_2}^2}.
\]

Similarly, we also define \( \mathcal{U} := U_1 \times U_2 \) and \( \mathcal{V} := V_1 \times V_2 \). Then \( \mathcal{V} \) is a reflexive Banach space with the norm,

\[
\|\psi\|_\mathcal{V} = \sqrt{\langle\psi, \psi\rangle_\mathcal{V}} = \sqrt{\|\psi_1\|_{V_1}^2 + \|\psi_2\|_{V_2}^2}, \quad \text{for any} \ \psi = (\psi_1, \psi_2) \in \mathcal{V}.
\]

We rewrite the systems \((3.1)\) and \((3.2)\) for \( \Theta = (X^1, Y^2) \) and \( \Theta^n = (X_1^{1,n}, Y_2^{2,n}) \), respectively, as

\[
\begin{aligned}
d\Theta_t &= A(\Theta_t, \mathcal{L}_\Theta) dt + B(\Theta_t, \mathcal{L}_\Theta) dW_t, \quad \Theta_0 = (x, y), \\
\frac{d\Theta_t^n}{dt} &= \Pi_n A(\Theta^n_t, \mathcal{L}_{\Theta^n}) dt + \Pi_n B(\Theta^n_t, \mathcal{L}_{\Theta^n}) dW^n_t, \quad \Theta^n_0 = (x^n, y^n),
\end{aligned}
\]

(3.3)\hspace{1cm}(3.4)

where \( \mathcal{L}_\Theta \in C([0, T]; \mathcal{P}_2(\mathcal{H})) \) with its marginal distribution \( \mathcal{L}_{X_1} \in C([0, T]; \mathcal{P}_2(H_1)) \), analogously for \( \mathcal{L}_{\Theta^n} \), \( \Pi_n := \text{diag}(\Pi_1^n, \Pi_2^n) \) and

\[
\begin{aligned}
A(\Theta_t, \mathcal{L}_\Theta) &= (A_1(X_t^1, \mathcal{L}_{X_1}) + f(X_t^1, \mathcal{L}_{X_1}, Y_t^2), A_2(X_t^1, \mathcal{L}_{X_1}, Y_t^2)), \\
A(\Theta^n_t, \mathcal{L}_{\Theta^n}) &= (A_1(X_t^{1,n}, \mathcal{L}_{X_1^{1,n}}) + f(X_t^{1,n}, \mathcal{L}_{X_1^{1,n}}, Y_t^{2,n}), A_2(X_t^{1,n}, \mathcal{L}_{X_1^{1,n}}, Y_t^{2,n})), \\
B(\Theta_t, \mathcal{L}_\Theta) &= \text{diag} \left( B_1(X_t^1, \mathcal{L}_{X_1}), B_2(X_t^1, \mathcal{L}_{X_1}, Y_t^2) \right), \\
B(\Theta^n_t, \mathcal{L}_{\Theta^n}) &= \text{diag} \left( B_1(X_t^{1,n}, \mathcal{L}_{X_1^{1,n}}), B_2(X_t^{1,n}, \mathcal{L}_{X_1^{1,n}}, Y_t^{2,n}) \right),
\end{aligned}
\]

and \( W_t := (W_t^1, W_t^2), W^n_t := (W_t^{1,n}, W_t^{2,n}) \). Let \( L_2(\mathcal{U}, \mathcal{H}) \) denotes the space of Hilbert-Schmidt operators from \( \mathcal{U} \) to \( \mathcal{H} \), with the norm:

\[
\|S\|_{L_2(\mathcal{U}, \mathcal{H})} := \sqrt{\|S_1\|_{L_2(U_1, H_1)}^2 + \|S_2\|_{L_2(U_2, H_2)}^2}, \quad S = (S_1, S_2),
\]

where \( S_i \in L_2(U_i, H_i), \ i = 1, 2 \). Let \( \mathcal{V}^* \) be the dual space of \( \mathcal{V} \), it is obvious that \( \mathcal{V}^* = V_1^* \times V_2^* \), and we consider the following Gelfand triple

\[
\mathcal{V} \subset \mathcal{H} \cong \mathcal{H}^* \subset \mathcal{V}^*.
\]

It is easy to see that the following mappings

\[
A : \mathcal{V} \times \mathcal{P}_2(\mathcal{H}) \rightarrow \mathcal{V}^*, \quad B : \mathcal{V} \times \mathcal{P}_2(\mathcal{H}) \rightarrow \mathcal{V}^*
\]
are well defined.

To complete the proof, we first verify the new coefficients in equation (3.3) satisfy the monotonicity condition similar to (2.24). Indeed, for any \( w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in \mathcal{V} \), and \( \vartheta_1 = (\mu_1, \nu_1), \vartheta_2 = (\mu_2, \nu_2) \in \mathcal{P}(\mathcal{H}) \), by conditions (A2) and (H2), we have

\[
\nu_i(A(w_1, \vartheta_1) - A(w_2, \vartheta_2), w_1 - w_2)_Y \\
= \nu_i(A_1(u_1, \mu_1) - A_1(u_2, \mu_2), u_1 - u_2)_Y + \nu_i(A_2(u_1, \mu_1, \nu_1) - A_2(u_2, \mu_2, \nu_2), v_1 - v_2)_Y \\
\leq C(\|u_1 - u_2\|_{\mathcal{H}}^2 + \|v_1 - v_2\|_{\mathcal{H}}^2 + \mathbb{W}_{2, \mathcal{H}}(\mu_1, \mu_2)^2) \\
\leq C(\|w_1 - w_2\|_{\mathcal{H}}^2 + \mathbb{W}_{2, \mathcal{H}}(\vartheta_1, \vartheta_2)^2) \\
\tag{3.5}
\]

and

\[
\|B(w_1, \vartheta_1) - B(w_2, \vartheta_2)\|_{L_2(\mathcal{H})}^2 \\
= \|B_1(u_1, \mu_1) - B_1(u_2, \mu_2)\|_{L_2(\Omega, \mathcal{H})}^2 + \|B_2(u_1, \mu_1, \nu_1) - B_2(u_2, \mu_2, \nu_2)\|_{L_2(\Omega, \mathcal{H})}^2 \\
\leq C(\|w_1 - w_2\|_{\mathcal{H}}^2 + \mathbb{W}_{2, \mathcal{H}}(\vartheta_1, \vartheta_2)^2). \\
\tag{3.6}
\]

Following from [11, Lemma 2.2] or [29, Theorem 3.3], by (3.5), (3.6), (A1), (H1), (A4) and (H4), system (3.2) has a unique continuous solution \((X^{1,n}, Y^{2,n})\). We now define the following spaces equipped with the associated norms

\[
J_i := L^2([0, T] \times \Omega, dt \times \mathbb{P}; L_2(U_i, H_i)), \quad i = 1, 2, \\
K_1 := L^\alpha([0, T] \times \Omega, dt \times \mathbb{P}; V_1), \quad K_2 := L^\beta([0, T] \times \Omega, dt \times \mathbb{P}; V_2), \\
K_1^* := L^\alpha([0, T] \times \Omega, dt \times \mathbb{P}; V_1^*), \quad K_2^* := L^\beta([0, T] \times \Omega, dt \times \mathbb{P}; V_2^*). 
\]

In order to prove the existence of solutions, we first give the following apriori estimates.

**Lemma 3.1** Suppose that (A3) and (H3) hold. Then there exists a constant \( C_T > 0 \), which is independent of \( n \), such that for all \( n \geq 1 \)

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \|X_t^{1,n}\|_{H_1}^2 \right] + \mathbb{E}\left[ \sup_{t \in [0, T]} \|Y_t^{2,n}\|_{H_2}^2 \right] + \|X^{1,n}\|_{K_1} + \|Y^{2,n}\|_{K_2} \leq C_T(1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2). 
\]

**Proof** By Itô’s formula for finite dimensional case and (H3), we have

\[
d\|Y_t^{2,n}\|_{H_2}^2 = \left[ 2V_t(\Pi^n_2 A_2(X_t^{1,n}, \mathcal{L}^{1,n}, Y_t^{2,n}), Y_t^{2,n})_{V_2} + \|\Pi^n_2 B_2(X_t^{1,n}, \mathcal{L}^{1,n}, Y_t^{2,n})\|_{L_2(U_2, \mathcal{H})}^2 \right] dt \\
+ 2\langle \Pi^n_2 B_2(X_t^{1,n}, \mathcal{L}^{1,n}, Y_t^{2,n}) dW_t^{2,n}, Y_t^{2,n} \rangle_{H_2} \\
\leq \left[ -\eta \|Y_t^{2,n}\|_{V_2}^2 + c_2(1 + \|X_t^{1,n}\|_{H_1}^2 + \mathcal{L}^{1,n}([\|\cdot\|_{H_1}^2 + \|Y_t^{2,n}\|_{H_2}^2]) dt + dM_t^n, 
\right.
\]

where we denote \( dM^n_t := 2\langle \Pi^n_2 B_2(X_t^{1,n}, \mathcal{L}^{1,n}, Y_t^{2,n}) dW_t^{2,n}, Y_t^{2,n} \rangle_{H_2} \).

We set the following stopping time

\[
\tau^n_R := \inf \left\{ t \in [0, T] : \|X_t^{1,n}\|_{H_1} + \|Y_t^{2,n}\|_{H_2} > R \right\}, \quad R > 0.
\]
Then, using Burkholder-Davis-Gundy’s inequality, we have

$$
E \left[ \sup_{t \in [0,T \wedge \tau^R]} \left\| Y_{t}^{2,n} \right\|_{H_2}^2 \right] + \eta E \int_0^{T \wedge \tau^R} \left\| Y_{t}^{2,n} \right\|_{V_2}^\beta dt 
$$

$$
= \left\| y^n \right\|_{H_2}^2 + CT + CE \int_0^{T \wedge \tau^R} \left\| Y_{t}^{2,n} \right\|_{H_2}^2 dt
$$

$$
+ CE \int_0^{T \wedge \tau^R} \left( \left\| X_{t}^{1,n} \right\|_{H_2}^2 + \mathcal{L}_{X_{t}^{1,n}}(\left\| \cdot \right\|_{H_2}) \right) dt + E \left[ \sup_{t \in [0,T \wedge \tau^R]} | M_t^n | \right] 
$$

$$
\leq \left\| y \right\|_{H_2}^2 + \frac{1}{2} E \left[ \sup_{t \in [0,T \wedge \tau^R]} \left\| Y_{t}^{2,n} \right\|_{H_2}^2 \right] + CT + CE \int_0^{T \wedge \tau^R} \left\| Y_{t}^{2,n} \right\|_{H_2}^2 dt
$$

$$
+ CE \int_0^{T \wedge \tau^R} \left( \left\| X_{t}^{1,n} \right\|_{H_2}^2 + \mathcal{L}_{X_{t}^{1,n}}(\left\| \cdot \right\|_{H_2}) \right) dt,
$$

which implies

$$
E \left[ \sup_{t \in [0,T \wedge \tau^R]} \left\| Y_{t}^{2,n} \right\|_{H_2}^2 \right] + 2\eta E \int_0^{T \wedge \tau^R} \left\| Y_{t}^{2,n} \right\|_{V_2}^\beta dt
$$

$$
\leq C \left\| y \right\|_{H_2}^2 + CT + C \int_0^T E \sup_{s \in [0,t \wedge \tau^R]} \left\| Y_{s}^{2,n} \right\|_{H_2}^2 dt
$$

$$
+ CE \int_0^{T \wedge \tau^R} \left( \left\| X_{t}^{1,n} \right\|_{H_2}^2 + \mathcal{L}_{X_{t}^{1,n}}(\left\| \cdot \right\|_{H_2}) \right) dt.
$$

Applying Gronwall’s inequality, we obtain

$$
E \left[ \sup_{t \in [0,T \wedge \tau^R]} \left\| Y_{t}^{2,n} \right\|_{H_2}^2 \right] + 2\eta E \int_0^{T \wedge \tau^R} \left\| Y_{t}^{2,n} \right\|_{V_2}^\beta dt
$$

$$
\leq CT \left\| y \right\|_{H_2}^2 + CT + CTE \int_0^{T \wedge \tau^R} \left( \left\| X_{t}^{1,n} \right\|_{H_2}^2 + \mathcal{L}_{X_{t}^{1,n}}(\left\| \cdot \right\|_{H_2}) \right) dt.
$$

Similarly, applying Itô’s formula to $\left\| X_{t}^{1,n} \right\|_{H_2}$ and using (A3), we have

$$
d\left\| X_{t}^{1,n} \right\|_{H_2}^2 = \left[ 2\Pi_t (\Pi_t A_1(X_{t}^{1,n}, \mathcal{L}_{X_{t}^{1,n}}), X_{t}^{1,n})_{V_1} + 2\Pi_t f(X_{t}^{1,n}, \mathcal{L}_{X_{t}^{1,n}}, Y_{t}^{2,n}), X_{t}^{1,n})_{H_1} + \|\Pi_t B_1(X_{t}^{1,n}, \mathcal{L}_{X_{t}^{1,n}})\|_{L_2(U_1, H_1)} \right] dt + dN_t^n
$$

$$
\leq \left[ -\theta \left\| X_{t}^{1,n} \right\|_{V_1}^\alpha + C(1 + \left\| X_{t}^{1,n} \right\|_{H_2}^2 + \mathcal{L}_{X_{t}^{1,n}}(\left\| \cdot \right\|_{H_2})) + C \left\| Y_{t}^{2,n} \right\|_{H_2} \right] dt + dN_t^n,
$$

here we denote $dN_t^n := 2\langle \Pi_t B_1(X_{t}^{1,n}, \mathcal{L}_{X_{t}^{1,n}})dW_{t}^{1,n}, X_{t}^{1,n} \rangle_{H_1}$.

By Burkholder-Davis-Gundy’s inequality we infer that

$$
E \left[ \sup_{t \in [0,T \wedge \tau^R]} \left\| X_{t}^{1,n} \right\|_{H_2}^2 \right] + \theta E \int_0^{T \wedge \tau^R} \left\| X_{t}^{1,n} \right\|_{V_1}^\alpha dt
$$

$$
\leq \left\| x \right\|_{H_1}^2 + CT + CE \int_0^{T \wedge \tau^R} \left( \left\| X_{t}^{1,n} \right\|_{H_2}^2 + \mathcal{L}_{X_{t}^{1,n}}(\left\| \cdot \right\|_{H_2}) \right) dt
$$

$$
+ CE \int_0^{T \wedge \tau^R} \left\| Y_{t}^{2,n} \right\|_{H_2}^2 dt + E \left[ \sup_{t \in [0,T \wedge \tau^R]} | N_t^n | \right]
$$

$$
\leq CT(1 + \left\| x \right\|_{H_2}^2 + \left\| y \right\|_{H_2}^2) + CT \int_0^T E \left\| X_{t}^{1,n} \right\|_{H_2}^2 dt + \frac{1}{2} E \left[ \sup_{t \in [0,T \wedge \tau^R]} \left\| X_{t}^{1,n} \right\|_{H_2}^2 \right].
where we used (3.8) in the last step.

Rearranging the above inequality and taking $R \to \infty$, then by the monotone convergence theorem we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|X_t^{1,n}\|_{H_1}^2 \right] + 2\theta \mathbb{E} \int_0^T \|X_t^{1,n}\|_{V_1} dt \\
\leq C_T (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2) + C_T \int_0^T \mathbb{E} \|X_t^{1,n}\|_{H_1}^2 dt.
\]

Thus applying Gronwall’s lemma gives that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|X_t^{1,n}\|_{H_1}^2 \right] + 2\theta \mathbb{E} \int_0^T \|X_t^{1,n}\|_{V_1} dt \leq C_T (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2).
\]

Recalling (3.7) and following the same procedure as (3.9), it is easy to get that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|Y_t^{2,n}\|_{H_2}^2 \right] + 2\eta \mathbb{E} \int_0^T \|Y_t^{2,n}\|_{\dot{V}_2} dt \leq C_T (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2),
\]

which completes the proof.

Combining (A3), (A4), (H3), (H4) with Lemma 3.1 it is easy to obtain the following estimates.

**Lemma 3.2** Suppose (A3), (A4), (H3) and (H4). There exists a constant $C_T > 0$ which is independent of $n$ such that

\[
\|A_1(X_t^{1,n}, \mathcal{L}_{X_t^{1,n}})\|_{K_1} + \|f(X_t^{1,n}, \mathcal{L}_{X_t^{1,n}}, Y_t^{2,n})\|_{L^2([0,T] \times \Omega; H_1)} + \|B_1(X_t^{1,n}, \mathcal{L}_{X_t^{1,n}})\|_{J_1} \leq C_T (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2),
\]

and

\[
\|A_2(X_t^{1,n}, \mathcal{L}_{X_t^{1,n}}, Y_t^{2,n})\|_{K_2} + \|B_1(X_t^{1,n}, \mathcal{L}_{X_t^{1,n}}, Y_t^{2,n})\|_{J_2} \leq C_T (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2),
\]

for all $n \geq 1$.

**Proof of Theorem 2.1** Due to the reflexivity of $J_i, K_i, K_i^*$ and $L^2([0,T] \times \Omega, dt \times \mathbb{P}; H_i)$, $i = 1, 2$, there exist common subsequences $n_k$ such that for $k \to \infty$,

(i) $X_t^{1,n_k} \to \tilde{X}$ weakly in $K_1$ and weakly in $L^2([0,T] \times \Omega, dt \times \mathbb{P}; H_1)$,

(ii) $Y_t^{2,n_k} \to \tilde{Y}$ weakly in $K_2$ and weakly in $L^2([0,T] \times \Omega, dt \times \mathbb{P}; H_2)$,

(iii) $A_1(X_t^{1,n_k}, \mathcal{L}_{X_t^{1,n_k}}) \to \tilde{F}^1$ weakly in $K_1^*$,

(iv) $A_2(X_t^{1,n_k}, \mathcal{L}_{X_t^{1,n_k}}, Y_t^{2,n_k}) \to \tilde{F}^2$ weakly in $K_2^*$,

(v) $f(X_t^{1,n_k}, \mathcal{L}_{X_t^{1,n_k}}, Y_t^{2,n_k}) \to \tilde{f}$ weakly in $L^2([0,T] \times \Omega, dt \times \mathbb{P}; H_1)$,

(vi) $B_1(X_t^{1,n_k}, \mathcal{L}_{X_t^{1,n_k}}) \to \tilde{Z}^1$ weakly in $J_1$,

(vii) $B_2(X_t^{1,n_k}, \mathcal{L}_{X_t^{1,n_k}}, Y_t^{2,n_k}) \to \tilde{Z}^2$ weakly in $J_2$.

Due to $\alpha \geq 2$, it is obvious that

\[
f(X_t^{1,n_k}, \mathcal{L}_{X_t^{1,n_k}}, Y_t^{2,n_k}) \to \tilde{f} \text{ weakly in } K_1^*.
\]

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Since the bounded linear operator between two Banach space is weakly continuous, it leads to \( \int_0^T \Pi_{t,s} B_1(X_{t,s}^{X_1}, \mathcal{L}_{X_1}^{X_1}) dW_{s}^{Y_2} \) weakly in \( \mathcal{M}_T^2(H_1) \) (the space of all continuous square integrable martingales from \([0, T] \times \Omega \) to \( H_1 \)), and analogously for \( \int_0^T \Pi_{t,s} B_2(X_{t,s}^{X_2}, \mathcal{L}_{X_2}^{X_2}, Y_{t,s}^{Y_2}) dW_{s}^{Y_2} \).

In addition, the approximants are progressively measurable, it follows that all of the above limits are progressively measurable.

Note that \( V_i, i = 1, 2 \), are separable, by the definition of \( (X_1^{X_1}, Y_2^{X_2}) \) that for any \( \tilde{\varepsilon} \in V_1 \) and \( \tilde{\varepsilon} \in V_2, dt \times \mathbb{P}\text{-a.e.} \)

\[
\begin{align*}
\left\{ \begin{array}{l}
\psi_t (\bar{X}_t, \tilde{\varepsilon})_{V_1} = \psi_t (x, \tilde{\varepsilon})_{V_1} + \int_0^t \psi_t (\bar{F}_s^1, \tilde{\varepsilon})_{V_1} ds + \int_0^t \langle \bar{f}_s, \tilde{\varepsilon} \rangle_{H_1} ds + \int_0^t \langle \bar{Z}_s^1 dW_s^1, \tilde{\varepsilon} \rangle_{H_1}, \\
\psi_t (\bar{Y}_t, \tilde{\varepsilon})_{V_2} = \psi_t (y, \tilde{\varepsilon})_{V_2} + \int_0^t \psi_t (\bar{F}_s^2, \tilde{\varepsilon})_{V_2} ds + \int_0^t \langle \bar{Z}_s^2 dW_s^2, \tilde{\varepsilon} \rangle_{H_2},
\end{array} \right.
\end{align*}
\]

Let us define

\[
\begin{align*}
X_t^1 &= x + \int_0^t \bar{F}_s^1 ds + \int_0^t \bar{f}_s ds + \int_0^t \bar{Z}_s^1 dW_s^1, \ t \in [0, T], \\
Y_t^2 &= y + \int_0^t \bar{F}_s^2 ds + \int_0^t \bar{Z}_s^2 dW_s^2, \ t \in [0, T],
\end{align*}
\]

then it gives that \( X_1 = \bar{X}, \ Y_2 = \bar{Y}, \ dt \times \mathbb{P}\text{-a.e.} \) According to Lemma \( \text{[3.1]} \) and \( \text{[3.5]} \) Theorem \( 4.2.5 \) that \( (X_1, Y_2) \) is a continuous \( H_1 \times H_2 \)-valued \( (\mathcal{F}_t) \)-adapted process. Thus, it suffices to prove that \( dt \times \mathbb{P}\text{-a.e.} \)

\[
\bar{F}^1 = A_1(X_1, \mathcal{L}_{X_1}), \quad \bar{f} = f(X_1, \mathcal{L}_{X_1}, Y_2), \quad \bar{F}^2 = A_2(X_1, \mathcal{L}_{X_1}, Y_2), \quad \text{(3.10)}
\]

and

\[
\bar{Z}^1 = B_1(X_1, \mathcal{L}_{X_1}), \quad \bar{Z}^2 = B_2(X_1, \mathcal{L}_{X_1}, Y_2), \quad \text{(3.11)}
\]

which implies the existence of solutions to system \( \text{[3.1]} \).

From (i)-(vi) above, it is easy to get the corresponding convergence for \( \Theta^{n_k} A(\Theta^{n_k}, \mathcal{L}_{\Theta^{n_k}}) \) and \( B(\Theta^{n_k}, \mathcal{L}_{\Theta^{n_k}}) \), respectively. For instance,

\[
A(\Theta^{n_k}, \mathcal{L}_{\Theta^{n_k}}) \to \bar{F} := (\bar{F}^1 + \bar{f}, \bar{F}^2) \text{ weakly in } K_1 \times K_2, \text{ as } k \to \infty.
\]

By \( \text{[3.10]} \) and \( \text{[3.11]} \), it suffices to show that \( dt \times \mathbb{P}\text{-a.e.} \)

\[
A(\Theta, \mathcal{L}_\Theta) = \bar{F} \text{ and } B(\Theta, \mathcal{L}_\Theta) = \bar{Z} := \text{diag}(\bar{Z}^1, \bar{Z}^2).
\]

Combining \( \text{[3.5]}, \text{[3.6]} \) with \( \text{(A1)} \) and \( \text{(H1)}, \text{[3.12]} \) follows from the monotonicity arguments, we include the details here for completeness.

Take any non-negative \( \psi \in L^\infty([0, T], dt; \mathbb{R}) \), by Hölder’s inequality, we have

\[
\mathbb{E} \left[ \int_0^T \psi_t \| \Theta_t \|_{H_1}^2 dt \right] = \lim_{k \to \infty} \mathbb{E} \left[ \int_0^T \langle \psi_t \Theta_t, \Theta_t^{n_k} \rangle_{H_1} dt \right]
\leq \left[ \mathbb{E} \left( \int_0^T \psi_t \| \Theta_t \|_{H_1}^2 dt \right) \right]^{1/2} \liminf_{k \to \infty} \left[ \mathbb{E} \left( \int_0^T \psi_t \| \Theta_t^{n_k} \|_{H_1}^2 dt \right) \right]^{1/2},
\]

which implies the following lower semi-continuity

\[
\mathbb{E} \left[ \int_0^T \psi_t \| \Theta_t \|_{H_1}^2 dt \right] \leq \liminf_{k \to \infty} \mathbb{E} \left[ \int_0^T \psi_t \| \Theta_t^{n_k} \|_{H_1}^2 dt \right]. \quad \text{(3.13)}
\]
For any $\phi := (\phi_1, \phi_2) \in K_1 \times K_2$ and $\lambda \geq 0$, Itô’s formula yields that
\[
\mathbb{E} \left[ e^{-\lambda t} \| \Theta_t^n \|_{\mathcal{H}}^2 - \| \Theta_0^n \|_{\mathcal{H}}^2 \right] 
\leq \mathbb{E} \left[ \int_0^t e^{-\lambda s} \left( 2 \nu_s \langle A(\Theta_s^n, \mathcal{L}_{\Theta_s^n}), \Theta_s^n \rangle \nu + \| B(\Theta_s^n, \mathcal{L}_{\Theta_s^n}) \|_{L_2(\mathcal{H}, \mathcal{H})}^2 - \lambda \| \Theta_s^n \|_{\mathcal{H}}^2 \right) ds \right]
\leq \mathbb{E} \left[ \int_0^t e^{-\lambda s} \left( 2 \nu_s \langle A(\Theta_s^n, \mathcal{L}_{\Theta_s^n}) - A(\phi_s, \mathcal{L}_{\phi_s}), \Theta_s^n \rangle \nu \right.ight.
\left. + \| B(\Theta_s^n, \mathcal{L}_{\Theta_s^n}) - B(\phi_s, \mathcal{L}_{\phi_s}) \|_{L_2(\mathcal{H}, \mathcal{H})}^2 - \left. \lambda \| \Theta_s^n - \phi_s \|_{\mathcal{H}}^2 \right) ds \right]
\leq \mathbb{E} \left\{ \int_0^t e^{-\lambda s} \left[ c \left( \| \Theta_s^n - \phi_s \|_{\mathcal{H}}^2 + \mathbb{W}_{2,H}(\mathcal{L}_{\Theta_s^n}, \mathcal{L}_{\phi_s})^2 \right) - \lambda \| \Theta_s^n - \phi_s \|_{\mathcal{H}}^2 \right] ds \right\} = 0.
\]

Inserting (3.15) into (3.14), by the lower semi-continuity (3.13) we get
\[
\mathbb{E} \left[ \int_0^T \psi_t \left( e^{-\lambda t} \| \Theta_t \|_{\mathcal{H}}^2 - \| \Theta_0 \|_{\mathcal{H}}^2 \right) dt \right] 
\leq \liminf_{k \to \infty} \mathbb{E} \left[ \int_0^T \psi_t \left( e^{-\lambda t} \| \Theta_t^n \|_{\mathcal{H}}^2 - \| \Theta_0^n \|_{\mathcal{H}}^2 \right) dt \right]
\leq \mathbb{E} \left\{ \int_0^T \psi_t \left[ \int_0^t e^{-\lambda s} \left( 2 \nu_s \langle A(\phi_s, \mathcal{L}_{\phi_s}), \Theta_s \rangle \nu + 2 \nu_s \langle F_s - A(\phi_s, \mathcal{L}_{\phi_s}), \phi_s \rangle \nu \right.ight.
\left. + 2 \nu_s \langle B(\phi_s, \mathcal{L}_{\phi_s}), L_{\mathcal{H}} \rangle - \left. \| B(\phi_s, \mathcal{L}_{\phi_s}) \|_{L_2(\mathcal{H}, \mathcal{H})}^2 \right. 
\right. \left. \left. - 2 \lambda \left( \Theta_s, \phi_s \right) \mathcal{H} + \lambda \| \phi_s \|_{\mathcal{H}}^2 \right) ds \right] dt \right\}.
\]

Applying Itô’s formula and the product rule,
\[
\mathbb{E} \left[ e^{-\lambda t} \| \Theta_t \|_{\mathcal{H}}^2 - \| \Theta_0 \|_{\mathcal{H}}^2 \right] 
= \mathbb{E} \left[ \int_0^t e^{-\lambda s} \left( 2 \nu_s \langle A(\Theta_s, \mathcal{L}_{\Theta_s}), \Theta_s \rangle \nu + \| B(\Theta_s, \mathcal{L}_{\Theta_s}) \|_{L_2(\mathcal{H}, \mathcal{H})}^2 - \lambda \| \Theta_s \|_{\mathcal{H}}^2 \right) ds \right].
\]

Inserting (3.17) into (3.16) and rearranging it implies
\[
\mathbb{E} \left\{ \int_0^T \psi_t \left[ \int_0^t e^{-\lambda s} \left( 2 \nu_s \langle F_s - A(\phi_s, \mathcal{L}_{\phi_s}), \Theta_s - \phi_s \rangle \nu + \| Z_s - B(\phi_s, \mathcal{L}_{\phi_s}) \|_{L_2(\mathcal{H}, \mathcal{H})}^2 \right) ds \right] dt \right\} \leq 0.
\]
First, taking $\phi = \Theta$ implies that $B(\Theta, \mathcal{L}_\Theta) = \bar{Z}$. Next, letting $\phi = \Theta - \eta \tilde{\phi} v$ for any $\eta > 0$, $v \in V$ and $\tilde{\phi} \in L^\infty([0, T] \times \Omega, dt \times \mathbb{P}; \mathbb{R})$. It follows that
\[
\mathbb{W}_{2,H}(\mathcal{L}_\Theta, \mathcal{L}_\Theta)^2 \leq \mathbb{E}\|\eta \tilde{\phi} v\|^2_H \leq \eta \|\tilde{\phi}\|^2_\infty \|v\|^2_H \downarrow 0, \text{ as } \eta \downarrow 0.
\]
Then taking $\eta \to 0$ by dominated convergence theorem we have
\[
\mathbb{E}\left\{ \int_0^T \psi_t \left( \int_0^t e^{-\lambda s} v' \langle \tilde{F}_s - A(\Theta, \mathcal{L}_\Theta), \tilde{\phi}_s v \rangle_V \right) dt \right\} \leq 0.
\]
The converse follows by taking $\tilde{\phi} = -\tilde{\phi}$, which concludes $A(\Theta, \mathcal{L}_\Theta) = \bar{F}$.

The uniqueness of solutions to systems (3.1) follows from the Itô’s formula, (3.5) and (3.6) directly. Hence we complete the proof of Theorem 2.1. □

4 Proof of Averaging principle

In this section, we aim to prove that the slow component of system (2.1) strongly converges to the solution of the corresponding averaged equation, which is mainly based on the technique of Khasminskii time discretization. In particular, the corresponding convergence rate is also derived.

4.1 Some apriori estimates for system (2.1)

We first give some uniform bounds with respect to $\varepsilon \in (0, 1)$ for the solutions $(X^\varepsilon, Y^\varepsilon)$ of system (2.1).

Lemma 4.1 For any $T > 0$, there exists a constant $C_T > 0$ such that,
\[
\sup_{\varepsilon \in (0, 1)} \mathbb{E} \left( \sup_{t \in [0, T]} \|X^\varepsilon_t\|^4_{H_1} \right) + \sup_{\varepsilon \in (0, 1)} \mathbb{E} \left( \int_0^T \|X^\varepsilon_t\|^4_{V_1} dt \right) \leq C_T \left( 1 + \|x\|_{H_1}^4 + \|y\|_{H_2}^4 \right) \quad (4.1)
\]
and
\[
\sup_{\varepsilon \in (0, 1)} \sup_{t \in [0, T]} \mathbb{E} \|Y^\varepsilon_t\|^4_{H_2} \leq C_T \left( 1 + \|x\|_{H_1}^4 + \|y\|_{H_2}^4 \right). \quad (4.2)
\]

Proof Applying Itô’s formula for $\|Y^\varepsilon_t\|^4_{H_2}$, we have
\[
\|Y^\varepsilon_t\|^4_{H_2} = \|y\|^4_{H_2} + \frac{4}{\varepsilon} \int_0^t \|Y^\varepsilon_s\|^2_{H_2} v_2 \langle A_2(X^\varepsilon_s, \mathcal{L}X^\varepsilon_s, Y^\varepsilon_s), Y^\varepsilon_s \rangle_{V_2} ds
\]
\[
+ \frac{4}{\varepsilon} \int_0^t \|B_2(X^\varepsilon_s, \mathcal{L}X^\varepsilon_s, Y^\varepsilon_s) Y^\varepsilon_s\|^2_{U_2} ds + \frac{2}{\varepsilon} \int_0^t \|Y^\varepsilon_s\|^2_{H_2} \|B_2(X^\varepsilon_s, \mathcal{L}X^\varepsilon_s, Y^\varepsilon_s)\|^2_{L_2(U_2, H_2)} ds
\]
\[
+ \frac{4}{\varepsilon} \int_0^t \|Y^\varepsilon_s\|^2_{H_2} (B_2(X^\varepsilon_s, \mathcal{L}X^\varepsilon_s, Y^\varepsilon_s)) dW^2_s, Y^\varepsilon_s)_{H_2}.
\]

Following the same calculations as in the proof of [35, Lemma 4.3.8], by Hypothesis 2.2 there is a constant $\lambda \in (0, \kappa)$ such that for any $u \in H_1, \mu \in \mathcal{P}_2(H_1)$ and $v \in V_2$,
\[
2V_2 \langle A_2(u, \mu, v), v \rangle_{V_2} + \|B_2(u, \mu, v)\|^2_{L_2(U_2, H_2)} \leq -\lambda \|v\|_{H_2}^2 + C \left( 1 + \|u\|_{H_1}^2 + \mu(\|\cdot\|_{H_1}^2) \right). \quad (4.3)
\]
Taking expectation and differentiating with respect to $t$, by (1.3) we deduce that

$$
\frac{d}{dt} \mathbb{E} \|Y_t^\varepsilon\|^4_{H^2} = \frac{4}{\varepsilon} \mathbb{E} \left( \|Y_t^\varepsilon\|^2_{H^2} \mathbb{V}_2 \left( A_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon), Y_t^\varepsilon \right) \right) + \frac{4}{\varepsilon} \mathbb{E} \left( \|B_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon)\|^2 \right) + \frac{2}{\varepsilon} \mathbb{E} \left( \|Y_t^\varepsilon\|^2_{H^2} \mathbb{V}_2 \left( B_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon) \right) \right) \\
\leq \frac{2}{\varepsilon} \mathbb{E} \left[ \|Y_t^\varepsilon\|^2_{H^2} \left( 2V_t^\varepsilon \left( A_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon), Y_t^\varepsilon \right) \right) + 2 \mathbb{E} \left( \|B_2(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon)\|^2 \right) \right] \\
\leq \frac{2}{\varepsilon} \mathbb{E} \left[ \|Y_t^\varepsilon\|^2_{H^2} \left( -\lambda \|Y_t^\varepsilon\|^2_{H^2} + C \|X_t^\varepsilon\|^2_{H^1} + C \mathcal{L}_{X_t^\varepsilon} (\| \cdot \|^2_{H^1} + C) \right) \right] \\
\leq \frac{2\lambda_0}{\varepsilon} \mathbb{E} \|Y_t^\varepsilon\|^4_{H^2} + \frac{C}{\varepsilon} \mathbb{E} \|X_t^\varepsilon\|^4_{H^1} + \frac{C}{\varepsilon},
$$

where $\lambda_0 \in (0, \lambda)$ and we used the fact that $\mathcal{L}_{X_t^\varepsilon} (\| \cdot \|^2_{H^1}) = \mathbb{E} \|X_t^\varepsilon\|^2_{H^1}$. Hence, by the comparison theorem, it is easy to see that

$$
\mathbb{E} \|Y_t^\varepsilon\|^4_{H^2} \leq \|y\|^4_{H^2} e^{-\frac{2\lambda_0}{\varepsilon} t} + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\lambda_0}{\varepsilon} (t-s)} \left( 1 + \mathbb{E} \|X_s^\varepsilon\|^4_{H^1} \right) ds. \tag{4.4}
$$

On the other hand, using Itô’s formula again, we also have

$$
\|X_t^\varepsilon\|^4_{H^1} = \|x\|^4_{H^1} + 4 \int_0^t \|X_s^\varepsilon\|^2_{H^1} V_1 (A_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) + f(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon), \mathcal{L}_{X_s^\varepsilon}) ds + 4 \int_0^t \|B_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon})\|^2_{H^1} \|X_s^\varepsilon\|^2_{H^1} ds + 2 \int_0^t \|X_s^\varepsilon\|^2_{H^1} \|B_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon})\|^2_{H^1} ds \\
+ 4 \int_0^t \|X_s^\varepsilon\|^2_{H^1} \|B_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) dW_s^1, X_s^\varepsilon \|_{H^1} ds.
$$

Then by Burkholder-Davis-Gundy’s inequality, (1.4) and Hypothesis 2.1 it holds that

$$
\mathbb{E} \left( \sup_{t \in [0,T]} \|X_t^\varepsilon\|^4_{H^1} \right) + 4 \mathbb{E} \left( \int_0^T \|X_t^\varepsilon\|^2_{H^1} \|X_t^\varepsilon\|^2_{V_1} dt \right) \\
\leq \|x\|^4_{H^1} + C_T + C \int_0^T \mathbb{E} \|X_t^\varepsilon\|^4_{H^1} dt + C \int_0^T \mathbb{E} \|Y_t^\varepsilon\|^4_{H^2} dt \\
\leq C_T \left( 1 + \|x\|^4_{H^1} + \|y\|^4_{H^2} \right) + C \int_0^T \mathbb{E} \|X_t^\varepsilon\|^4_{H^1} dt \\
+ \frac{C}{\varepsilon} \int_0^T \int_0^t e^{-\frac{2\lambda_0}{\varepsilon} (t-s)} \left( 1 + \mathbb{E} \|X_s^\varepsilon\|^4_{H^1} \right) ds dt \\
\leq C_T \left( 1 + \|x\|^4_{H^1} + \|y\|^4_{H^2} \right) + C \int_0^T \mathbb{E} \|X_t^\varepsilon\|^4_{H^1} dt.
$$

Hence, applying Gronwall’s inequality, we get

$$
\mathbb{E} \left( \sup_{t \in [0,T]} \|X_t^\varepsilon\|^4_{H^1} \right) + \mathbb{E} \left( \int_0^T \|X_t^\varepsilon\|^2_{H^1} \|X_t^\varepsilon\|^2_{V_1} dt \right) \leq C_T \left( 1 + \|x\|^4_{H^1} + \|y\|^4_{H^2} \right), \tag{4.5}
$$

which also gives

$$
\mathbb{E} \|Y_t^\varepsilon\|^4_{H^2} \leq C_T \left( 1 + \|x\|^4_{H^1} + \|y\|^4_{H^2} \right).
Moreover, applying Itô's formula to \( \|X_t^\varepsilon\|_{H_1}^2, \|Y_t^\varepsilon\|_{H_2}^2 \) and following the same procedure as \([45]\), it is obvious that

\[
\mathbb{E} \left( \int_0^T \|X_t^\varepsilon\|_{V_1}^\alpha dt \right) \leq C_T \left( 1 + \|x\|_{H_1}^4 + \|y\|_{H_2}^4 \right).
\]

The proof is complete. \( \square \)

The following Lemma is an estimate of the integral of the time increment of \( X_t^\varepsilon \), which is weaker than the Hölder continuity of time (see e.g. \([15, 18, 19]\) but strong enough for our purpose, and the advantage is it only needs initial value \( x \in H_1, y \in H_2 \).

**Lemma 4.2** For any \( T > 0 \), there exists a constant \( C_T > 0 \) such that for any \( \varepsilon \in (0, 1) \) and \( \delta > 0 \) small enough,

\[
\mathbb{E} \left[ \int_0^T \|X_t^\varepsilon - X_{\delta t}^\varepsilon\|_{H_1}^2 dt \right] \leq C_T \delta \left( 1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2 \right),
\]

where \( t(\delta) := [\frac{\delta}{\varepsilon}] \delta \) and \([s]\) denotes the integer part of \( s \).

**Proof** Using \((4.1)\), it is easy to get that

\[
\mathbb{E} \left[ \int_0^T \|X_t^\varepsilon - X_{\delta t}^\varepsilon\|_{H_1}^2 dt \right]
= \mathbb{E} \left[ \int_0^\delta \|X_t^\varepsilon - x\|_{H_1}^2 dt \right] + \mathbb{E} \left[ \int_\delta^T \|X_t^\varepsilon - X_{\delta t}^\varepsilon\|_{H_1}^2 dt \right]
\leq C \left( 1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2 \right) \delta
+ 2\mathbb{E} \left( \int_\delta^T \|X_t^\varepsilon - X_{\delta t}^\varepsilon\|_{H_1}^2 dt \right) + 2\mathbb{E} \left( \int_\delta^T \|X_{\delta t}^\varepsilon - X_{t-\delta}^\varepsilon\|_{H_1}^2 dt \right).
\]

It follows from Itô’s formula that

\[
\|X_t^\varepsilon - X_{t-\delta}^\varepsilon\|_{H_1}^2 = 2 \int_{t-\delta}^t \langle A_1(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon), X_s^\varepsilon - X_{t-\delta}^\varepsilon \rangle_{V_1} ds
+ 2 \int_{t-\delta}^t \langle f(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon), X_s^\varepsilon - X_{t-\delta}^\varepsilon \rangle_{V_1} ds
+ \int_{t-\delta}^t \|B_1(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon)\|_{L_2(V_1, H_1)}^2 ds
+ 2 \int_{t-\delta}^t \langle B_1(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon) dW_s^1, X_s^\varepsilon - X_{t-\delta}^\varepsilon \rangle_{H_1} ds
:= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\]

For the first term \( I_1(t) \), by condition \((A4)\), there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \left( \int_\delta^T I_1(t) dt \right)
\leq C \mathbb{E} \left( \int_\delta^T \int_{t-\delta}^t \|A_1(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon)\|_{V_1} \|X_s^\varepsilon - X_{t-\delta}^\varepsilon\|_{V_1} ds dt \right)
\leq C \left[ \mathbb{E} \int_\delta^T \int_{t-\delta}^t \|A_1(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon)\|_{V_1}^{\alpha-1} ds \right]^{\alpha/(\alpha-1)} \mathbb{E} \left[ \int_\delta^T \int_{t-\delta}^t \|X_s^\varepsilon - X_{t-\delta}^\varepsilon\|_{V_1} ds dt \right]^{1/\alpha}
\leq C \left[ \delta \mathbb{E} \int_0^T (1 + \|X_s^\varepsilon\|_{V_1}^\alpha + \mathcal{L}X_s^\varepsilon(\|\cdot\|_{H_1}^2)) ds \right]^{(\alpha-1)/\alpha} \left[ \delta \mathbb{E} \int_0^T \|X_s^\varepsilon\|_{V_1}^\alpha ds \right]^{1/\alpha}
\leq C_T \delta \left( 1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2 \right),
\]

(4.9)
where we use Fubini’s theorem and (4.1) in the third and fourth inequalities respectively.

For $I_2(t)$ and $I_3(t)$, by condition $(A2)$, (4.1), and (4.2), we get

\[
\mathbb{E} \left( \int_0^T I_2(t) dt \right) 
\leq C \mathbb{E} \left[ \int_0^T \int_{t-\delta}^t \left( 1 + \|X_s^\varepsilon\|_{H_1} + \|Y_s^\varepsilon\|_{H_2} + (\mathcal{L}X_s^\varepsilon(\|\cdot\|_{H_1}))^{1/2} \right) \left( \|X_s^\varepsilon\|_{H_1} + \|X_{t-\delta}^\varepsilon\|_{H_1} \right) ds dt \right]
\]

\[
\leq C T \delta \mathbb{E} \left[ \sup_{s \in [0,T]} (1 + \|X_s^\varepsilon\|_{H_1}^2) \right] + C T \delta^{1/2} \left[ \mathbb{E} \left( \sup_{s \in [0,T]} \|X_s^\varepsilon\|_{H_1}^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( \int_0^T \int_{t-\delta}^t \|Y_s^\varepsilon\|_{H_2}^2 ds dt \right) \right]^{1/2}
\]

\[
\leq C T \delta \mathbb{E} \left[ \sup_{s \in [0,T]} (1 + \|X_s^\varepsilon\|_{H_1}^2) \right] + C T \delta \int_0^T \mathbb{E} \|Y_s^\varepsilon\|_{H_2}^2 ds
\]

\[
\leq C T \delta \left( 1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2 \right)
\]  

(4.10)

and

\[
\mathbb{E} \left( \int_0^T I_3(t) dt \right) \leq C \mathbb{E} \left[ \int_0^T \int_{t-\delta}^t \left( 1 + \|X_s^\varepsilon\|_{H_1}^2 + \mathcal{L}X_s^\varepsilon(\|\cdot\|_{H_1}^2) \right) ds dt \right]
\]

\[
\leq C T \delta \mathbb{E} \left[ \sup_{s \in [0,T]} (1 + \|X_s^\varepsilon\|_{H_1}^2 + \mathbb{E} \|X_s^\varepsilon\|_{H_1}^2) \right]
\]

\[
\leq C T \delta \left( 1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2 \right).
\]  

(4.11)

For $I_4(t)$, due to Lemma 4.1, it is easy to see that

\[
\mathbb{E} \left( \int_0^T I_4(t) dt \right) = \int_0^T \mathbb{E} \left[ \int_{t-\delta}^t \langle B_1(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon) dW_s^1, X_s^\varepsilon - X_{t-\delta}^\varepsilon \rangle_{H_1} \right] dt
\]

\[
= 0.
\]  

(4.12)

Combining estimates (4.3)–(4.12), we get that

\[
\mathbb{E} \left( \int_0^T \|X_s^\varepsilon - X_{t-\delta}^\varepsilon\|_{H_1}^2 dt \right) \leq C T \delta (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2).
\]  

(4.13)

By a similar argument as above, we can also get

\[
\mathbb{E} \left( \int_0^T \|X_s^\varepsilon - X_{t-\delta}^\varepsilon\|_{H_2}^2 dt \right) \leq C T \delta (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2).
\]  

(4.14)

Hence, (4.7), (4.13) and (4.14) implies (4.6) holds. The proof is complete. \hfill \qed

4.2 Estimates of auxiliary process

Inspired by the time discretization method developed in [31], we divide $[0, T]$ into intervals of size $\delta$, where $\delta$ is a fixed positive number depending on $\varepsilon$ and will be chosen later. Then,
we construct an auxiliary process $\hat{Y}_t^\varepsilon \in H_2$, with $\hat{Y}_0^\varepsilon = Y_0^\varepsilon = y$, and for any $k \in \mathbb{N}$ and $t \in [k\delta, \min((k+1)\delta, T)],$

$$\hat{Y}_t^\varepsilon = \hat{Y}_0^\varepsilon + \frac{1}{\varepsilon} \int_{k\delta}^t A_2(X_{k\delta}^\varepsilon, \mathcal{L}X_{k\delta}^\varepsilon, \hat{Y}_s) ds + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^t B_2(X_{k\delta}^\varepsilon, \mathcal{L}X_{k\delta}^\varepsilon, \hat{Y}_s) dW_s^2, \quad (4.15)$$

which is equivalent to

$$d\hat{Y}_t^\varepsilon = \frac{1}{\varepsilon} \left[A_2 \left(X_{t(\delta)}^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon \right) \right] dt + \frac{1}{\sqrt{\varepsilon}} B_2 \left(X_{t(\delta)}^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon \right) dW_t^2, \quad \hat{Y}_0^\varepsilon = y.$$

By the construction of $\hat{Y}_t^\varepsilon$, we can obtain the following estimates which will be used below.

**Lemma 4.3** For any $T > 0$, there exists a constant $C_T > 0$ such that,

$$\sup_{\varepsilon \in (0, 1)} \sup_{t \in [0, T]} \mathbb{E} \|\hat{Y}_t^\varepsilon\|_{H_2}^2 \leq C_T (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2) \quad (4.16)$$

and

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E} \left( \int_{0}^{T} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^2_{H_2} dt \right) \leq C_T \delta \left(1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2\right) \quad (4.17).$$

**Proof** Since the proof of (4.16) is similar to Lemma 4.1, we omit it here. Next, we will prove (4.17). It is easy to see that $Y_t^\varepsilon - \hat{Y}_t^\varepsilon$ satisfies the following equation

$$\left\{ d(Y_t^\varepsilon - \hat{Y}_t^\varepsilon) = \frac{1}{\varepsilon} \left[A_2 \left(X_{t(\delta)}^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon, Y_t^\varepsilon \right) - A_2 \left(X_{t(\delta)}^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon \right) \right] dt \\
+ \frac{1}{\sqrt{\varepsilon}} \left[B_2 \left(X_{t(\delta)}^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon, Y_t^\varepsilon \right) - B_2 \left(X_{t(\delta)}^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon \right) \right] dW_t^2, \quad (4.18)\right.$$

Thus, applying Itô’s formula and taking expectation, we get

$$\mathbb{E} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^2_{H_2} = \mathbb{E} \int_{0}^{t} \|V_2 \left(A_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon \right) - A_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon \right), Y_s^\varepsilon - \hat{Y}_s^\varepsilon \right) \|^2_{V_2} ds \\
+ \frac{1}{\varepsilon} \mathbb{E} \int_{0}^{t} \left\| \left[B_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon \right) - B_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon \right) \right] \|^2_{L_2(U_2, H_2)} ds.$$ 

Then by condition (H2), we have

$$\frac{d}{dt} \mathbb{E} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^2_{H_2}$$

$$= \frac{2}{\varepsilon} \mathbb{E} \int_{0}^{t} \left\| A_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon \right) - A_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon \right), Y_s^\varepsilon - \hat{Y}_s^\varepsilon \right\|_{V_2}^2 + \frac{1}{\varepsilon} \mathbb{E} \left\| B_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon \right) - B_2 \left(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon \right) \right\|^2_{L_2(U_2, H_2)}$$

$$\leq -\frac{2k}{\varepsilon} \mathbb{E} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|^2_{H_2} + \frac{c_2}{\varepsilon} \mathbb{E} \left[ \|X_t^\varepsilon - X_{t(\delta)}^\varepsilon\|^2_{H_1} + \mathbb{W}_{2, H_1}(\mathcal{L}X_t^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon) \right]$$

$$+ \frac{1}{\varepsilon} \left[ L_{B_2} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\|_{H_2} + c_2 \|X_t^\varepsilon - X_{t(\delta)}^\varepsilon\|_{H_1} + c_2 \mathbb{W}_{2, H_1}(\mathcal{L}X_t^\varepsilon, \mathcal{L}X_{t(\delta)}^\varepsilon) \right]^2. \quad (4.19)$$
Note that
\[ \mathcal{W}_{2,H_1}(\mathcal{L}X_t^\varepsilon, \mathcal{L}X_t^{\varepsilon(\delta)})^2 \leq \mathbb{E}||X_t^\varepsilon - X_t^{\varepsilon(\delta)}||^2_{H_1}. \]  
(4.20)

Due to \(2\kappa > L_B^2\), then according to (4.19) and (4.20) there exists \(\theta > 0\) such that
\[ \frac{d}{dt} \mathbb{E} ||Y_t^\varepsilon - \hat{Y}_t^\varepsilon||^2_{H_2} \leq -\frac{\theta}{\varepsilon} \mathbb{E} ||Y_t^\varepsilon - \hat{Y}_t^\varepsilon||^2_{H_2} + \frac{C}{\varepsilon} \mathbb{E} ||X_t^\varepsilon - X_t^{\varepsilon(\delta)}||^2_{H_1}. \]

Therefore, by the comparison theorem we have
\[ \mathbb{E} ||Y_t^\varepsilon - \hat{Y}_t^\varepsilon||^2_{H_2} \leq \frac{C}{\varepsilon} \int_0^t e^{-\frac{\theta(t-s)}{\varepsilon}} \mathbb{E} ||X_s^\varepsilon - X_s^{\varepsilon(\delta)}||^2_{H_1} ds. \]

Using Fubini’s theorem, we can get that for any \(T > 0\),
\[ \mathbb{E} \left( \int_0^T ||Y_t^\varepsilon - \hat{Y}_t^\varepsilon||^2_{H_2} dt \right) \leq \frac{C}{\varepsilon} \int_0^T \int_0^t e^{-\frac{\theta(t-s)}{\varepsilon}} \mathbb{E} ||X_s^\varepsilon - X_s^{\varepsilon(\delta)}||^2_{H_1} ds dt \]
\[ = \frac{C}{\varepsilon} \mathbb{E} \left[ \int_0^T ||X_s^\varepsilon - X_s^{\varepsilon(\delta)}||^2_{H_1} \left( \int_s^T e^{-\frac{\theta(t-s)}{\varepsilon}} dt \right) ds \right] \]
\[ \leq C \mathbb{E} \left( \int_0^T ||X_s^\varepsilon - X_s^{\varepsilon(\delta)}||^2_{H_1} ds \right). \]

It follows from Lemma 4.2 that
\[ \mathbb{E} \left( \int_0^T ||Y_t^\varepsilon - \hat{Y}_t^\varepsilon||^2_{H_2} dt \right) \leq C_T \delta (1 + ||x||^2_{H_1} + ||y||^2_{H_2}). \]

The proof is complete. \(\square\)

### 4.3 The frozen and averaged equations

In this subsection, we first introduce the frozen equation associated with the fast equation for a fixed slow component \(x \in H_1\) and \(\mu \in \mathcal{P}_2(H_1)\), i.e.,
\[ \begin{cases} dY_t = [A_2(x, \mu, Y_t)]dt + B_2(x, \mu, Y_t)d\tilde{W}_t^2, \\ Y_0 = y \in H_2, \end{cases} \]
(4.21)

where \(\tilde{W}_t^2\) is a cylindrical Wiener process in a separable Hilbert space \(U_2\) on another probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) with natural filtration \((\tilde{\mathcal{F}}_t)_{t \geq 0}\).

Since \(x\) and \(\mu\) are fixed in equation (4.21), following from [35, Theorem 4.2.4] under Hypothesis 2.2 there is a unique solution denoted by \(Y_t^{x,\mu,y}\) to equation (4.21), which is a homogeneous Markov process. Let \(P_t^{x,\mu}\) be the transition semigroup of \(Y_t^{x,\mu,y}\), that is, for any bounded measurable function \(\varphi\) on \(H_2\),
\[ P_t^{x,\mu} \varphi(y) = \tilde{\mathbb{E}}[\varphi(Y_t^{x,\mu,y})], \quad y \in H_2, \quad t > 0, \]
where \(\tilde{\mathbb{E}}\) is the expectation on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\). Then by [35, Theorem 4.3.9], \(P_t^{x,\mu}\) has a unique invariant measure \(\nu^{x,\mu}\). Moreover, we have the following two propositions.
Proposition 4.1 There exists a constant $C > 0$ such that for any $x, x_1, x_2 \in H_1, y \in H_2$ and $\mu, \nu \in \mathcal{P}_2(H_1)$,

$$
\sup_{t \in [0, \infty)} \tilde{E}\|Y^{x,\mu,y}_t\|_{H_2}^2 \leq C \left( 1 + \|x\|_{H_1} + \|y\|_{H_2} + \mu(\| \cdot \|^2_{H_1}) \right),
$$

(4.22)

$$
\sup_{t \in [0, \infty)} \tilde{E}\|Y^{x_1,\mu,y}_t - Y^{x_2,\nu,y}_t\|_{H_2}^2 \leq C \left( \|x_1 - x_2\|_{H_1} + \mathbb{W}_{2,H_1}(\mu, \nu)^2 \right).
$$

(4.23)

**Proof** By Itô’s formula, we have

$$
\|Y^{x,\mu,y}_t\|_{H_2}^2 = \|y\|_{H_2}^2 + 2 \int_0^t \langle A_2(x, \mu, Y^{x,\mu,y}_s), Y^{x,\mu,y}_s \rangle ds + \int_0^t \|B_2(x, \mu, Y^{x,\mu,y}_s)\|_{L_2(U_2,H_2)}^2 ds
$$

$$
+ 2 \int_0^t \langle B_2(x, \mu, Y^{x,\mu,y}_s) d\tilde{W}_s, Y^{x,\mu,y}_s \rangle_{H_2}.
$$

(4.24)

Taking expectation on both sides of (4.24), by (4.3) we obtain

$$
\frac{d}{dt} \tilde{E}\|Y^{x,\mu,y}_t\|_{H_2}^2 = \tilde{E}\left( 2\langle A_2(x, \mu, Y^{x,\mu,y}_t), Y^{x,\mu,y}_t \rangle_{V_2} + \|B_2(x, \mu, Y^{x,\mu,y}_t)\|_{L_2(U_2,H_2)}^2 \right)
$$

$$
\leq -\lambda \|Y^{x,\mu,y}_t\|_{H_2}^2 + C \left( 1 + \|x\|_{H_1} + \mu(\| \cdot \|^2_{H_1}) \right).
$$

Hence, applying the comparison theorem yields

$$
\tilde{E}\|Y^{x,\mu,y}_t\|_{H_2}^2 \leq \|y\|_{H_2}^2 e^{-\lambda t} + C \int_0^t e^{-\lambda(t-s)} \left( 1 + \|x\|^2_{H_1} + \mu(\| \cdot \|^2_{H_1}) \right) ds
$$

$$
\leq \|y\|_{H_2}^2 e^{-\lambda t} + C \left( 1 + \|x\|^2_{H_1} + \mu(\| \cdot \|^2_{H_1}) \right),
$$

(4.25)

which gives (4.22).

Following the similar calculations as above, by (H2) it is obvious that (4.23) holds. $\square$

Proposition 4.2 There exist $C > 0$ and $\rho > 0$ such that for any $x \in H_1, y \in H_2$ and $\mu \in \mathcal{P}_2(H_1)$,

$$
\left\| \tilde{E}f(x, \mu, Y^{x,\mu,y}_t) - \tilde{f}(x, \mu) \right\|_{H_1} \leq C \left( 1 + \|x\|_{H_1} + \|y\|_{H_2} + \mu(\| \cdot \|^2_{H_1})^{1/2} \right) e^{-\frac{\rho}{4} t},
$$

(4.26)

where $\tilde{f}(x, \mu) = \int_{H_2} f(x, \mu, z) \nu^{x,\mu}(dz)$.

**Proof** We denote by $Y^{x,\mu,y'}_t$ the solution of Eq. (4.21) with initial value $Y_0 = y'$. Using Itô’s formula and (H2), similar to (4.25), there exists a constant $\rho > 0$ such that

$$
\tilde{E}\|Y^{x,\mu,y}_t - Y^{x,\mu,y'}_t\|^2_{H_2} \leq \|y - y'\|^2_{H_2} e^{-\rho t},
$$

(4.27)

for any $y, y' \in H_2$.

Then by the invariance of $\nu^{x,\mu}$ and (4.25), we have

$$
\int_{H_2} \|y'\|^2_{H_2} \nu^{x,\mu}(dy') = \int_{H_2} \tilde{E}\|Y^{x,\mu,y'}_t\|^2_{H_2} \nu^{x,\mu}(dy')
$$

$$
\leq e^{-\lambda t} \int_{H_2} \|y'\|^2_{H_2} \nu^{x,\mu}(dy') + C \left( 1 + \|x\|^2_{H_1} + \mu(\| \cdot \|^2_{H_1}) \right).
$$

Take $t = t_0$ such that $e^{-\lambda t_0} < 1$, we have

$$
\int_{H_2} \|y'\|^2_{H_2} \nu^{x,\mu}(dy') \leq C \left( 1 + \|x\|^2_{H_1} + \mu(\| \cdot \|^2_{H_1}) \right).
$$

(4.28)
Then, using the invariance of $\nu^{x,\mu}$, (4.27) and (4.28), we have

$$
\left\| \mathbb{E} f(x, \mu, Y_t^{x,\mu,y}) - \bar{f}(x, \mu) \right\|_{H_1} = \left\| \mathbb{E} f(x, \mu, Y_t^{x,\mu,y}) - \int_{H_2} f(x, \mu, y') \nu^{x,\mu}(dy') \right\|_{H_1}
$$

$$
= \left\| \int_{H_2} \left[ \mathbb{E} f(x, \mu, Y_t^{x,\mu,y}) - \mathbb{E} f(x, \mu, Y_t^{x,\mu,y'}) \right] \nu^{x,\mu}(dy') \right\|_{H_1}
$$

$$
\leq C \int_{H_2} \mathbb{E} \left\| Y_t^{x,\mu,y} - Y_t^{x,\mu,y'} \right\|_{H_2} \nu^{x,\mu}(dy')
$$

$$
\leq C e^{-\frac{t}{2}} \int_{H_2} \|y - y'\|_{H_2} \nu^{x,\mu}(dy')
$$

$$
\leq C e^{-\frac{t}{2}} (1 + \|x\|_{H_1} + \|y\|_{H_2} + (\mu(\cdot) \cdot \|\cdot\|_{H_1}^2)^{1/2}),
$$

which concludes the proof of Proposition 4.2. \qed

Next, we consider the corresponding averaged equation, i.e.,

$$
d\bar{X}_t = A_1(\bar{X}_t, \mathcal{L}X_t)dt + \bar{f}(X_t, \mathcal{L}X_t)dt + B_1(\bar{X}_t, \mathcal{L}X_t)dW_t^1,
$$

(4.29)

with

$$
\bar{f}(x, \mu) = \int_{H_2} f(x, \mu, z) \nu^{x,\mu}(dz),
$$

with $\nu^{x,\mu}$ being the unique invariant measure of equation (4.21).

**Remark 4.1** In terms of exponential ergodicity (4.26) and Lipschitz of $f$, one can easily check that $\bar{f}$ is also Lipschitz continuous, i.e.

$$
\|\bar{f}(x_1, \mu) - \bar{f}(x_2, \nu)\|_{H_1} \leq C (\|x_1 - x_2\|_{H_1} + \mathbb{W}_2(\mu, \nu)), x_1, x_2 \in H_1, \mu, \nu \in \mathcal{P}_2(H_1).
$$

(4.30)

**Proof** For any $x_1, x_2 \in H_1$ and $\mu, \nu \in \mathcal{P}_2(H_1)$, by (4.23) and (4.24),

$$
\left\| \bar{f}(x_1, \mu) - \bar{f}(x_2, \nu) \right\|_{H_1}^2
$$

$$
\leq C \left\| \bar{f}(x_1, \mu) - \mathbb{E} f(x_1, \mu, Y_t^{x_1,\mu,y}) \right\|_{H_1}^2 + C \left\| \bar{f}(x_2, \nu) - \mathbb{E} f(x_2, \nu, Y_t^{x_2,\mu,y}) \right\|_{H_1}^2
$$

$$
+ C \left\| \mathbb{E} f(x_1, \mu, Y_t^{x_1,\mu,y}) - \mathbb{E} f(x_2, \nu, Y_t^{x_2,\mu,y}) \right\|_{H_1}^2
$$

$$
\leq C \left(\|x_1\|_{H_1}^2 + \|x_2\|_{H_1}^2 + \|y\|_{H_2}^2 + \|\mu(\cdot)\|_{H_1}^2 + \nu(\|\cdot\|_{H_1}^2) \right) e^{-\rho t}
$$

$$
+ C \left(\|x_1 - x_2\|_{H_1}^2 + \mathbb{W}_2(\mu, \nu)^2 \right).
$$

(4.31)

Taking $t \to \infty$ for both sides of (4.31) leads to the desired estimate (4.30). \qed

Thus, similar to Theorem 2.1 for any $x \in H_1$, Eq. (4.29) has a unique solution $\bar{X}_t$. Moreover, using an argument similar to that in Lemma 4.1 and Lemma 4.2 we also have the following estimates.

**Lemma 4.4** For any $T > 0$, there exists a constant $C_T > 0$ such that

$$
\mathbb{E} \left( \sup_{t \in [0,T]} \|\bar{X}_t\|_{H_1}^2 \right) \leq C_T (1 + \|x\|_{H_1}^2)
$$

and

$$
\mathbb{E} \left[ \int_0^T \|\bar{X}_t - \bar{X}_t(\cdot)\|_{H_1}^2 dt \right] \leq C_T \delta \left(1 + \|x\|_{H_1}^2\right).
$$

(4.32)

Now we are in the position to finish the proof of the second main result.
4.4 Proof of Theorem 2.2

The proof of Theorem 2.2 will be divided into the following three steps.

Step 1. It is easy to see that \( X^ε_t - \bar{X}_t \) satisfies the following equation

\[
\begin{align*}
\begin{cases}
    d(X^ε_t - \bar{X}_t) &= \left[ A_t \left( X^ε_t, \mathcal{L}_{X^ε_t} \right) - A_t \left( \bar{X}_t, \mathcal{L}_{\bar{X}_t} \right) + f \left( X^ε_t, \mathcal{L}_{X^ε_t}, Y^ε_t \right) - \bar{f} \left( \bar{X}_t, \mathcal{L}_{\bar{X}_t} \right) \right] dt \\
    &\quad + \left[ B_t \left( X^ε_t, \mathcal{L}_{X^ε_t} \right) - B_t \left( \bar{X}_t, \mathcal{L}_{\bar{X}_t} \right) \right] dW^1_t,
\end{cases}
\end{align*}
\]

\( X^ε_0 - \bar{X}_0 = 0 \).

Thus, applying Itô’s formula yields

\[
\begin{align*}
\| X^ε_t - \bar{X}_t \|^2_{H_1} &= 2 \int_0^t V_s \left( A_s \left( X^ε_s, \mathcal{L}_{X^ε_s} \right) - A_s \left( \bar{X}_s, \mathcal{L}_{\bar{X}_s} \right), X^ε_s - \bar{X}_s \right) ds \\
    &\quad + 2 \int_0^t \left\langle f \left( X^ε_s, \mathcal{L}_{X^ε_s}, Y^ε_s \right) - \bar{f} \left( \bar{X}_s, \mathcal{L}_{\bar{X}_s} \right), X^ε_s - \bar{X}_s \right\rangle_{H_1} ds \\
    &\quad + \int_0^t \| B_s \left( X^ε_s, \mathcal{L}_{X^ε_s} \right) - B_s \left( \bar{X}_s, \mathcal{L}_{\bar{X}_s} \right) \|^2_{L_2(U_1, H_1)} ds \\
    &\quad + 2 \int_0^t \left\langle \left[ B_s \left( X^ε_s, \mathcal{L}_{X^ε_s} \right) - B_s \left( \bar{X}_s, \mathcal{L}_{\bar{X}_s} \right) \right] dW^1_s, X^ε_s - \bar{X}_s \right\rangle_{H_1} \\
    &:= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{align*}
\]

By condition (A2), we have

\[
\mathbb{E} \left( \sup_{t \in [0,T]} (I_1(t) + I_3(t)) \right) \leq C \mathbb{E} \int_0^T \| X^ε_t - \bar{X}_t \|^2_{H_1} + \mathbb{W}_{2,H_1}(\mathcal{L}_{X^ε_t}, \mathcal{L}_{\bar{X}_t})^2 dt \\
\leq C \mathbb{E} \int_0^T \| X^ε_t - \bar{X}_t \|^2_{H_1} dt. \tag{4.34}
\]

Then by Burkholder-Davis-Gundy’s inequality, condition (A2), it holds that

\[
\begin{align*}
\mathbb{E} \left( \sup_{t \in [0,T]} I_4(t) \right) &\leq C \mathbb{E} \left[ \int_0^T \| B_t \left( X^ε_t, \mathcal{L}_{X^ε_t} \right) - B_t \left( \bar{X}_t, \mathcal{L}_{\bar{X}_t} \right) \|_{L_2(U_1, H_1)}^2 \| X^ε_t - \bar{X}_t \|^2_{H_1} dt \right]^{1/2} \\
&\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \| X^ε_t - \bar{X}_t \|^2_{H_1} \int_0^T \| B_t \left( X^ε_t, \mathcal{L}_{X^ε_t} \right) - B_t \left( \bar{X}_t, \mathcal{L}_{\bar{X}_t} \right) \|^2_{L_2(U_1, H_1)} dt \right]^{1/2} \\
&\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} \| X^ε_t - \bar{X}_t \|^2_{H_1} \right] + C \mathbb{E} \int_0^T \| X^ε_t - \bar{X}_t \|^2_{H_1} + \mathbb{W}_{2,H_1}(\mathcal{L}_{X^ε_t}, \mathcal{L}_{\bar{X}_t})^2 dt \\
&\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} \| X^ε_t - \bar{X}_t \|^2_{H_1} \right] + C \mathbb{E} \int_0^T \| X^ε_t - \bar{X}_t \|^2_{H_1} dt. \tag{4.35}
\end{align*}
\]
As for $I_2(t)$, we first rewrite it as

$$I_2(t) = 2 \int_0^t \langle f \left( X_s^e, \mathcal{L}X_s^e, Y_s^e \right) - f \left( X_{s(\delta)}^e, \mathcal{L}X_{s(\delta)}^e, \dot{Y}_{s(\delta)}^e \right), X_s^e - X_{s(\delta)} \rangle_{H_1} ds$$

$$+ 2 \int_0^t \langle f \left( X_s^e, \mathcal{L}X_s^e, \dot{Y}_s^e \right) - \dot{f} \left( X_{s(\delta)}^e, \mathcal{L}X_{s(\delta)}^e \right), X_s^e - X_{s(\delta)} \rangle_{H_1} ds$$

$$+ 2 \int_0^t \langle \dot{f} \left( X_s^e, \mathcal{L}X_s^e \right) - \dot{f} \left( X_{s(\delta)}^e, \mathcal{L}X_{s(\delta)}^e \right), X_s^e - X_{s(\delta)} \rangle_{H_1} ds$$

$$:= I_{21}(t) + I_{22}(t) + I_{23}(t) + I_{24}(t). \quad (4.36)$$

According to (A2), (4.16), (4.17), and (4.30), it is easy to see that

$$E \left( \sup_{t \in [0,T]} (I_{21}(t) + I_{23}(t)) \right) \leq C \int_0^T \left( \| X_t^e - X_{i(\delta)}(t) \|_{H_1} + \mathcal{W}_{2,H_1}(LX_t^e, LX_{i(\delta)}^e) + \| Y_t^e - \dot{Y}_{i(\delta)}^e \|_{H_2} \right) \| X_t^e - X_i \|_{H_1} dt$$

$$\leq C \int_0^T \| X_t^e - X_i(\delta) \|_{H_1}^2 dt + C \int_0^T \| X_t^e - X_{i(\delta)}^e \|_{H_1}^2 dt$$

$$\leq C \int_0^T \| X_t^e - X_i \|_{H_1}^2 dt + C \int_0^T \| X_t^e - X_{i(\delta)} \|_{H_1}^2 dt$$

and

$$E \left( \sup_{t \in [0,T]} I_{23}(t) \right) \leq C \int_0^T \left( \| X_t^e - X_{i(\delta)}(t) \|_{H_1} + \mathcal{W}_{2,H_1}(LX_t^e, LX_{i(\delta)}^e) \right) \| X_t^e - X_i \|_{H_1} dt$$

$$\leq C \int_0^T \| X_t^e - X_i(\delta) \|_{H_1}^2 dt. \quad (4.38)$$

As for $I_{22}(t)$, we rewrite it as

$$I_{22}(t) = 2 \int_0^t \langle f \left( X_s^e, \mathcal{L}X_s^e, \dot{Y}_s^e \right) - f \left( X_{s(\delta)}^e, \mathcal{L}X_{s(\delta)}^e \right), X_s^e - X_{s(\delta)} \rangle_{H_1} ds$$

$$+ 2 \int_0^t \langle f \left( X_s^e, \mathcal{L}X_s^e, \dot{Y}_s^e \right) - f \left( X_{s(\delta)}^e, \mathcal{L}X_{s(\delta)}^e \right), X_s^e - X_{s(\delta)} \rangle_{H_1} ds$$

$$+ 2 \int_0^t \langle f \left( X_s^e, \mathcal{L}X_s^e \right) - f \left( X_{s(\delta)}^e, \mathcal{L}X_{s(\delta)}^e \right), X_s(\delta) - X_{s(\delta)} \rangle_{H_1} ds$$

$$:= J_1(t) + J_2(t) + J_3(t). \quad (4.39)$$

By Lemma 4.1 and Lemma 4.2, we obtain

$$E \left( \sup_{t \in [0,T]} J_1(t) \right) \leq C \int_0^T \| f(X_{i(\delta)}^e, \mathcal{L}X_{i(\delta)}^e, \dot{Y}_{i(\delta)}^e) - f(X_{i(\delta)}^e, \mathcal{L}X_{i(\delta)}^e) \|_{H_1} \| X_i^e - X_{i(\delta)} \|_{H_1} dt$$

$$\leq C \left[ \int_0^T \left( 1 + \| X_t^e \|_{H_1}^2 + \mathcal{W}_{2,H_1}(LX_t^e, LX_{i(\delta)}^e) + \| Y_t^e \|_{H_2}^2 \right) dt \right]^{1/2} \left[ \int_0^T \| X_t^e - X_{i(\delta)} \|_{H_1}^2 dt \right]^{1/2}$$

$$\leq C \delta^{1/2}(1 + \| x \|_{H_1}^2 + \| y \|_{H_2}^2). \quad (4.40)$$
Similarly, by Lemma 4.4 we can also get
\[ \mathbb{E} \left( \sup_{t \in [0,T]} J_2(t) \right) \leq C_T \delta^{1/2} (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2). \] (4.41)

Thus, combining (4.33)-(4.41) yields
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \| X^\varepsilon_t - \tilde{X}_t \|_{H_1}^2 \right] \leq 2 \mathbb{E} \left( \sup_{t \in [0,T]} J_2(t) \right) + C_T \delta^{1/2} (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2) + C \mathbb{E} \int_0^T \| X^\varepsilon_t - \tilde{X}_t \|_{H_1}^2 \, dt. \] (4.42)

**Step 2.** In this step, we will use the time discretization technique to deal with \( J_2(t) \).

Note that
\[
|J_2(t)| = 2 \sum_{k=0}^{[t/\delta]-1} \int_{k \delta}^{(k+1) \delta} \left\langle f \left( X^\varepsilon_{s(\delta)}, \mathcal{L} X^\varepsilon_{s(\delta)}, \mathcal{Y}^\varepsilon_{s(\delta)} \right), X^\varepsilon_{s(\delta)} - \tilde{X}_{s(\delta)} \right\rangle_{H_1} \, ds \\
+ \int_{t(\delta)}^t \left\langle f \left( X^\varepsilon_{s(\delta)}, \mathcal{L} X^\varepsilon_{s(\delta)}, \mathcal{Y}^\varepsilon_{s(\delta)} \right), X^\varepsilon_{s(\delta)} - \tilde{X}_{s(\delta)} \right\rangle_{H_1} \, ds \\
\leq 2 \sum_{k=0}^{[t/\delta]-1} \left| \int_{k \delta}^{(k+1) \delta} \left\langle f \left( X^\varepsilon_{s(\delta)}, \mathcal{L} X^\varepsilon_{s(\delta)}, \mathcal{Y}^\varepsilon_{s(\delta)} \right), X^\varepsilon_{s(\delta)} - \tilde{X}_{s(\delta)} \right\rangle_{H_1} \, ds \right| \\
+ 2 \left| \int_{t(\delta)}^t \left\langle f \left( X^\varepsilon_{s(\delta)}, \mathcal{L} X^\varepsilon_{s(\delta)}, \mathcal{Y}^\varepsilon_{s(\delta)} \right), X^\varepsilon_{s(\delta)} - \tilde{X}_{s(\delta)} \right\rangle_{H_1} \, ds \right| \\
:= J_{21}(t) + J_{22}(t). \] (4.43)

By Lemma 4.1 and Lemma 4.2 it is easy to prove that
\[
\mathbb{E} \left( \sup_{t \in [0,T]} J_{22}(t) \right) \\
\leq C \left[ \mathbb{E} \sup_{t \in [0,T]} \int_{t(\delta)}^t \left( 1 + \| X^\varepsilon_{s(\delta)} \|_{H_1}^2 + \| \mathcal{L} X^\varepsilon_{s(\delta)} \|_{H_1}^2 + \| \mathcal{Y}^\varepsilon_{s(\delta)} \|_{H_2}^2 \right) \, ds \right]^{1/2} \\
\times \left[ \mathbb{E} \sup_{t \in [0,T]} \int_{t(\delta)}^t \left( \| X^\varepsilon_{s(\delta)} - \tilde{X}_{s(\delta)} \|_{H_1}^2 \right) \, ds \right]^{1/2} \\
\leq C \delta^{1/2} \left[ \mathbb{E} \int_0^T \left( 1 + \| X^\varepsilon_t \|_{H_1}^2 + \| \mathcal{L} X^\varepsilon_t \|_{H_1}^2 + \| \mathcal{Y}^\varepsilon_t \|_{H_2}^2 \right) \, dt \right]^{1/2} \left[ \mathbb{E} \sup_{t \in [0,T]} \| X^\varepsilon_t - \tilde{X}_t \|_{H_1}^2 \right]^{1/2} \\
\leq C_T \delta^{1/2} (1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2). \] (4.44)
As for the term $J_{21}(t)$, we can control it as follows.

\[
\mathbb{E} \left( \sup_{t \in [0,T]} J_{21}(t) \right) \leq 2 \mathbb{E} \sum_{k=0}^\frac{\lfloor T/\delta \rfloor - 1}{(k+1)\delta} \left| \int_{k\delta}^{(k+1)\delta} \left\langle f \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}, \tilde{Y}_s^\varepsilon \right) - \tilde{f} \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta} \right), X_{k\delta}^\varepsilon - \tilde{X}_{k\delta} \right\rangle_{H_1} ds \right| \leq C_T \max_{0 \leq k \leq \lfloor T/\delta \rfloor - 1} \frac{\delta}{\varepsilon^2} \mathbb{E} \left[ \int_{k\delta}^{(k+1)\delta} \left| \sum_{k=0}^\frac{\lfloor T/\delta \rfloor - 1}{(k+1)\delta} \left( f \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}, \tilde{Y}_s^\varepsilon \right) - \tilde{f} \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta} \right) \right) ds \right| \right]^{1/2}
\]

\[
\leq \frac{C_T}{\delta^2} \max_{0 \leq k \leq \lfloor T/\delta \rfloor - 1} \left[ \mathbb{E} \left[ \int_{0}^{\delta} \left\| f \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}, \tilde{Y}_s^\varepsilon \right) - \tilde{f} \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta} \right) \right\|^2_{H_1} ds \right] \right]^{1/2} + \frac{1}{4} \left( \sup_{t \in [0,T]} \mathbb{E} \left[ \| X_t^\varepsilon - \tilde{X}_t \|^2_{H_1} \right] \right),
\]

where for any $0 \leq r \leq s \leq \frac{\delta}{\varepsilon}$,

\[
\Phi_k(s, r) := \mathbb{E} \left[ \left\langle f \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}, \tilde{Y}_s^\varepsilon \right) - \tilde{f} \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta} \right), f \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}, \tilde{Y}_r^\varepsilon \right) - \tilde{f} \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta} \right) \right\rangle_{H_1} \right].
\]

For any $s > 0$, $\mu \in \mathcal{P}_2(H_1)$, and any $\mathcal{F}_s$-measurable $H_1$-valued random variable $X$ and $H_2$-valued random variable $Y$, we consider the following equation

\[
d\tilde{Y}_t = \frac{1}{\varepsilon} A_2(X, \mu, \tilde{Y}_t) dt + \frac{1}{\sqrt{\varepsilon}} B_2(X, \mu, \tilde{Y}_t) dW_t^2, \quad t \geq s, \quad \tilde{Y}_s = Y.
\]

Then, by Theorem 4.2.4, it is easy to see that Eq. (4.46) has a unique solution denoted by $\tilde{Y}_t^\varepsilon,s,X,\mu,Y$. Following the construction of $\tilde{Y}_t^\varepsilon$ in (4.15), for any $k \in \mathbb{N}$, it is easy to check that

\[
\tilde{Y}_t^\varepsilon = Y_t^\varepsilon,k\varepsilon,X_{k\varepsilon}^\varepsilon,\mathcal{L}_{k\varepsilon},Y_{k\varepsilon}^\varepsilon, \quad t \in [k\delta, (k+1)\delta].
\]

Thus, we have

\[
\Phi_k(s, r) = \mathbb{E} \left[ \left\langle f \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}, \tilde{Y}_r^\varepsilon, k\varepsilon,X_{k\delta}^\varepsilon,\mathcal{L}_{k\delta},Y_{k\delta}^\varepsilon \right) - \tilde{f} \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta} \right), f \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}, \tilde{Y}_s^\varepsilon, k\varepsilon,X_{k\delta}^\varepsilon,\mathcal{L}_{k\delta},Y_{k\delta}^\varepsilon \right) - \tilde{f} \left( X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta} \right) \right\rangle_{H_1} \right].
\]
Note that for any fixed \( x \in H_1 \) and \( y \in H_2 \), \( \hat{Y}^{w_{s_{e+k\delta}},x,\mu,y}_{s_{e+k\delta}} \) is independent of \( \mathcal{F}_{k\delta} \), and \( X_{s_{k\delta}}^e, \hat{Y}_{k\delta}^e \) are \( \mathcal{F}_{k\delta} \)-measurable, thus we have

\[
\Phi_k(s,r) = \mathbb{E}\left\{ \mathbb{E}\left[ \left( f(X_{s_{k\delta}}^e, \mathcal{L}_{X_{k\delta}}^e, Y_{s_{k\delta}}^{e,\delta, x,\mu,y} - \bar{f}(X_{k\delta}, \mathcal{L}_{X_{k\delta}}) \right) \left( X_{s_{k\delta}}^e, \mathcal{L}_{X_{k\delta}}^e, \hat{Y}_{k\delta}^e \right) - \bar{f}(X_{k\delta}, \mathcal{L}_{X_{k\delta}}) \right) _H \right\} \}
\]

Recall the definition of the process \( \{\hat{Y}^{e,\delta,x,\mu,y}_{s_{e+k\delta}}\}_{s \geq 0} \), it is easy to see that

\[
\hat{Y}^{e,\delta,x,\mu,y}_{s_{e+k\delta}} = y + \frac{1}{\varepsilon} \int_{s_{k\delta}}^{s_{e+k\delta}} A_2(x, \mu, \hat{Y}_{s_{e+k\delta}}^{e,\delta,x,\mu,y}) \, dr + \frac{1}{\varepsilon} \int_{s_{k\delta}}^{s_{e+k\delta}} B_2(x, \mu, \hat{Y}_{s_{e+k\delta}}^{e,\delta,x,\mu,y}) \, dW^2_r
\]

\[
= y + \frac{1}{\varepsilon} \int_{0}^{s_{k\delta}} A_2(x, \mu, \hat{Y}_{s_{k\delta}}^{e,\delta,x,\mu,y}) \, dr + \frac{1}{\varepsilon} \int_{0}^{s_{k\delta}} B_2(x, \mu, \hat{Y}_{s_{k\delta}}^{e,\delta,x,\mu,y}) \, dW^2_r
\]

\[
= y + \int_{0}^{s_{k\delta}} A_2(x, \mu, \hat{Y}_{s_{k\delta}}^{e,\delta,x,\mu,y}) \, dr + \int_{0}^{s_{k\delta}} B_2(x, \mu, \hat{Y}_{s_{k\delta}}^{e,\delta,x,\mu,y}) \, dW^2_r, \quad (4.47)
\]

where \( \left\{ W^2_{r_{2,k\delta}} := W^2_{r+k\delta} - W^2_{r} \right\}_{r \geq 0} \) and \( \left\{ \hat{W}^2_{r_{2,k\delta}} := \frac{1}{\varepsilon} W^2_{r_{2,k\delta}} \right\}_{r \geq 0} \).

Note that the solution of the frozen equation satisfies

\[
Y^{e,x,\mu,y}_{s_{k\delta}} = y + \int_{0}^{s_{k\delta}} A_2(x, \mu, Y^{e,x,\mu,y}_{s_{k\delta}}) \, dr + \int_{0}^{s_{k\delta}} B_2(x, \mu, Y^{e,x,\mu,y}_{s_{k\delta}}) \, dW^2_r. \quad (4.48)
\]

Then, the uniqueness of the solution of (4.47) and (4.48) implies that the distribution of \( \{Y^{e,\delta,x,\mu,y}_{s_{e+k\delta}}\}_{0 \leq s \leq \frac{4}{\varepsilon}} \) coincides with the distribution of \( \{Y^{e,x,\mu,y}_{s_{e+k\delta}}\}_{0 \leq s \leq \frac{4}{\varepsilon}} \). Thus, using Markov and time-homogenous properties of process \( Y^{e,x,\mu,y} \), we have

\[
\Phi_k(s,r) = \mathbb{E}\left\{ \mathbb{E}\left[ \left( f(x, \mathcal{L}_{X_{k\delta}}^e, Y_{s_{k\delta}}^{e,\delta,x,\mu,y} - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) \left( x, \mathcal{L}_{X_{k\delta}}^e, \hat{Y}_{k\delta}^e \right) - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) _H \right\} \}
\]

\[
= \mathbb{E}\left\{ \mathbb{E}\left[ \left( f(x, \mathcal{L}_{X_{k\delta}}^e, Y_{s_{k\delta}}^{e,\delta,x,\mu,y} - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) \left( x, \mathcal{L}_{X_{k\delta}}^e, \hat{Y}_{k\delta}^e \right) - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) _H \right\} \}
\]

\[
= \mathbb{E}\left\{ \mathbb{E}\left[ \left( f(x, \mathcal{L}_{X_{k\delta}}^e, Y_{s_{k\delta}}^{e,\delta,x,\mu,y} - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) \left( x, \mathcal{L}_{X_{k\delta}}^e, \hat{Y}_{k\delta}^e \right) - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) _H \right\} \}
\]

\[
= \mathbb{E}\left\{ \mathbb{E}\left[ \left( f(x, \mathcal{L}_{X_{k\delta}}^e, Y_{s_{k\delta}}^{e,\delta,x,\mu,y} - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) \left( x, \mathcal{L}_{X_{k\delta}}^e, \hat{Y}_{k\delta}^e \right) - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) _H \right\} \}
\]

\[
= \mathbb{E}\left\{ \mathbb{E}\left[ \left( f(x, \mathcal{L}_{X_{k\delta}}^e, Y_{s_{k\delta}}^{e,\delta,x,\mu,y} - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) \left( x, \mathcal{L}_{X_{k\delta}}^e, \hat{Y}_{k\delta}^e \right) - \bar{f}(x, \mathcal{L}_{X_{k\delta}}) \right) _H \right\} \}
\]
Therefore, according to Proposition \ref{prop1} and \ref{prop2} we arrive

\[
\Phi_{k}(s, r) \leq C_{T}\mathbb{E}\left\{ \mathbb{E}\left[ 1 + \|X_{k,s}^{\varepsilon}\|^2_{H_1} + \mathcal{L}_{X_{k,s}^{\varepsilon}}(\|\cdot\|^2_{H_1}) + \|Y_{r} X_{k,s}^{\varepsilon} \mathcal{L}_{X_{k,s}^{\varepsilon}} Y_{k,s}^{\varepsilon}\|^2_{H_2} \right] e^{-\frac{(s-r)^{2}}{2}} \right\}
\]

\[
\leq C_{T}\mathbb{E}\left\{ 1 + \|X_{k,s}^{\varepsilon}\|^2_{H_1} + \mathcal{L}_{X_{k,s}^{\varepsilon}}(\|\cdot\|^2_{H_1}) + \|Y_{k,s}^{\varepsilon}\|^2_{H_2} \right\} e^{-\frac{(s-r)^{2}}{2}}
\]

\[
\leq C_{T}(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2})e^{-\frac{(s-r)^{2}}{2}}.
\]

By (4.45) and (4.49), we deduce that

\[
\mathbb{E}\left( \sup_{t \in [0,T]} J_{21}(t) \right) \leq C_{T}(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2}) \frac{\varepsilon^2}{\delta^2} \left[ \int_{0}^{T} \int_{r}^{T} e^{-\frac{(s-r)^{2}}{2}} dsdr \right]
\]

\[
+ \frac{1}{4} \mathbb{E}\left[ \sup_{t \in [0,T]} \|X_{t}^{\varepsilon} - \bar{X}_{t}\|^2_{H_1} \right]
\]

\[
= C_{T}(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2}) \frac{\varepsilon^2}{\delta^2} \left( \frac{2\delta}{\rho\varepsilon} - \frac{4}{\rho^2} + \frac{4}{\rho^2} e^{\frac{-\rho^2}{\rho^2}} \right)
\]

\[
+ \frac{1}{4} \mathbb{E}\left[ \sup_{t \in [0,T]} \|X_{t}^{\varepsilon} - \bar{X}_{t}\|^2_{H_1} \right]
\]

\[
\leq C_{T}(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2}) \left( \frac{\varepsilon^2}{\delta^2} + \frac{\varepsilon}{\delta} + \delta^{1/2} \right) + \frac{1}{4} \mathbb{E}\left[ \sup_{t \in [0,T]} \|X_{t}^{\varepsilon} - \bar{X}_{t}\|^2_{H_1} \right].
\] (4.50)

**Step 3.** Now, we are in the position to complete the proof. Combining (4.42)-(4.44) and (4.50) yields

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|X_{t}^{\varepsilon} - \bar{X}_{t}\|^2_{H_1} \right] \leq C_{T}(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2}) \left( \frac{\varepsilon^2}{\delta^2} + \frac{\varepsilon}{\delta} + \delta^{1/2} \right)
\]

\[
+ C \mathbb{E}\int_{0}^{T} \|X_{t}^{\varepsilon} - \bar{X}_{t}\|^2_{H_1} dt.
\]

Using the Gronwall’s inequality yields

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|X_{t}^{\varepsilon} - \bar{X}_{t}\|^2_{H_1} \right] \leq C_{T}(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2}) \left( \frac{\varepsilon^2}{\delta^2} + \frac{\varepsilon}{\delta} + \delta^{1/2} \right). \quad (4.51)
\]

Then, by taking \( \delta = \varepsilon^{2/3} \) in (4.51) we deduce that

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|X_{t}^{\varepsilon} - \bar{X}_{t}\|^2_{H_1} \right] \leq C_{T}(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2}) \varepsilon^{1/3}.
\]

The proof is complete. \( \square \)

5 Application to examples

In this section, we shall apply the main results in Theorem 2.1 and 2.2 to various two-time-scale McKean-Vlasov SPDE models, which also generalize some existing works in the literature from classical SPDEs to distribution dependent case.
Throughout this section, we assume $\Lambda \subset \mathbb{R}^d$ as a bounded domain with smooth boundary $\partial \Lambda$. Let $C_0^\infty(\Lambda, \mathbb{R}^d)$ be the space of all smooth functions from $\Lambda$ to $\mathbb{R}^d$ with compact support. For any $r \geq 1$, let $L^r(\Lambda, \mathbb{R}^d)$ be the vector valued $L^r$-space with the norm $\| \cdot \|_{L^r}$. For any integer $m > 0$, we denote by $W^{m,r}_0(\Lambda, \mathbb{R}^d)$ the classical Sobolev space (with Dirichlet boundary condition) from domain $\Lambda$ to $\mathbb{R}^d$ equipped with the (equivalent) norm

$$
\|u\|_{W^{m,r}} = \left( \sum_{|\alpha| = m} \int_{\Lambda} |D^\alpha u|^r dx \right)^{\frac{1}{r}}.
$$

5.1 Slow-fast McKean-Vlasov stochastic porous media equation

The first example is the McKean-Vlasov stochastic porous media type equation, which is the dynamics of gas flow in a porous medium (cf. e.g. [3, 14, 23, 45]). More precisely, we consider the following slow-fast McKean-Vlasov stochastic evolution equations

$$
\begin{cases}
    dX^\varepsilon_t = \left[ \Delta \Psi(X^\varepsilon_t, L^\varepsilon_t) + f(X^\varepsilon_t, L^\varepsilon_t, Y^\varepsilon_t) \right] dt + B_1(X^\varepsilon_t, L^\varepsilon_t) dW^1_t, \\
    dY^\varepsilon_t = \frac{1}{\varepsilon} \left[ \Delta Y^\varepsilon_t + g(X^\varepsilon_t, L^\varepsilon_t, Y^\varepsilon_t) \right] dt + \frac{1}{\sqrt{\varepsilon}} B_2(X^\varepsilon_t, L^\varepsilon_t, Y^\varepsilon_t) dW^2_t, \\
    X^\varepsilon_0 = x, Y^\varepsilon_0 = y.
\end{cases}
$$

where $\Delta$ denotes the Laplace operator, and $\Psi, f, g, B_1, B_2$ satisfy some assumptions below.

For any $r \geq 2$, we set the following Gelfand triple for the slow equation

$$
V_1 := L^r(\Lambda) \subset H_1 := (W_0^{1,2}(\Lambda))^* \subset V_1^*,
$$

and the following Gelfand triple for the fast equation

$$
V_2 := W_0^{1,2}(\Lambda) \subset H_2 := L^2(\Lambda) \subset V_2^*.
$$

We recall the following useful lemma (see e.g. [35, Lemma 4.1.13]).

**Lemma 5.1** The map

$$
\Delta : W_0^{1,2}(\Lambda) \to (L^r(\Lambda))^*
$$

could be extend to a linear isometry

$$
\Delta : L^{\frac{d}{d-1}}(\Lambda) \to (L^r(\Lambda))^*.
$$

Furthermore, for any $u \in L^{\frac{d}{d-1}}(\Lambda)$, $v \in L^r(\Lambda)$ we have

$$
V_1 \langle -\Delta u, v \rangle_{V_1} = L^{\frac{d}{d-1}} \langle u, v \rangle_{L^r} = \int_{\Lambda} u(\xi)v(\xi) d\xi.
$$

We first formulate the assumptions on $\Psi$. Suppose the map

$$
\Psi : V_1 \times \mathcal{P}_2(H_1) \to L^{\frac{d}{d-1}}(\Lambda)
$$

is measurable, and satisfies the following hypothesis.

**Hypothesis 5.1** For all $u, v \in V_1$ and $\mu, \nu \in \mathcal{P}_2(H_1)$,
(Ψ1) The map
\[ V_1 \times \mathscr{P}_2(H_1) \ni (u, \mu) \mapsto \int_{\Lambda} \Psi(u, \mu)(\xi)v(\xi) d\xi \]
is continuous.

(Ψ2) There are some constants \(C, \theta > 0\) such that
\[ \int_{\Lambda} \Psi(u, \mu)(\xi)v(\xi) d\xi \geq -C(1 + \|u\|_{H_1}^2 + \mu(\|v\|_{H_1})) + \theta \|u\|_{V_1}^r. \]

(Ψ3)
\[ \int_{\Lambda} (\Psi(u, \mu)(\xi) - \Psi(v, \nu)(\xi))(u(\xi) - v(\xi)) d\xi \geq 0. \]

(Ψ4) There is a constant \(C > 0\),
\[ \|\Psi(u, \mu)\|_{L^\infty_{\mathbb{R}^+}} \leq C(1 + \|u\|_{V_1}^r + \mu(\|v\|_{H_1}^2)). \]

After the preparations above, we now define map \(A_1 : V_1 \times \mathscr{P}_2(H_1) \to V_1^*\) by
\[ A_1(u, \mu) := \Delta \Psi(u, \mu). \]

The Lemma 5.1 ensures that the map \(A_1\) is well-defined and takes value in \(V_1^*\). Moreover, it is easy to check that the conditions (Ψ1)-(Ψ4) imply (A1)-(A4). In order to prove the main result, we further assume that the measurable maps
\[ f : H_1 \times \mathscr{P}_2(H_1) \times H_2 \to H_1, \quad B_1 : V_1 \times \mathscr{P}_2(H_1) \to L_2(U_1, H_1), \]
and
\[ g : H_1 \times \mathscr{P}_2(H_1) \times V_2 \to V_2^*, \quad B_2 : H_1 \times \mathscr{P}_2(H_1) \times V_2 \to L_2(U_2, H_2) \]
are Lipschitz continuous. More precisely, there are some positive constants \(L_g, L_{B_2}\) and \(C\) such that for all \(u_1, u_2 \in H_1, v_1, v_2 \in H_2\) and \(\mu_1, \mu_2 \in \mathscr{P}_2(H_1)\),
\[ \|f(u_1, \mu_1, v_1) - f(u_2, \mu_2, v_2)\|_{H_1} \leq C\left(\|u_1 - u_2\|_{H_1} + \|v_1 - v_2\|_{H_2} + \mathbb{W}_{2, H_1}(\mu_1, \mu_2)\right), \quad \text{(5.2)} \]
\[ \|B_1(u_1, \mu_1) - B_1(u_2, \mu_2)\|_{L_2(U_1, H_1)} \leq C\left(\|u_1 - u_2\|_{H_1} + \mathbb{W}_{2, H_1}(\mu_1, \mu_2)\), \quad \text{(5.3)} \]
\[ \|g(u_1, \mu_1, v_1) - g(u_2, \mu_2, v_2)\|_{H_1} \leq L_g\|v_1 - v_2\|_{H_2} + C\left(\|u_1 - u_2\|_{H_1} + \mathbb{W}_{2, H_1}(\mu_1, \mu_2)\right), \quad \text{(5.4)} \]
\[ \|B_2(u_1, \mu_1, v_1) - B_2(u_2, \mu_2, v_2)\|_{L_2(U_2, H_2)} \leq L_{B_2}\|v_1 - v_2\|_{H_2} + C\left(\|u_1 - u_2\|_{H_1} + \mathbb{W}_{2, H_1}(\mu_1, \mu_2)\). \quad \text{(5.5)} \]

Furthermore, we also assume that the smallest eigenvalue \(\lambda_1\) of map \(-\Delta\) satisfies
\[ \lambda_1 - L_g - L_{B_2}^2 > 0. \quad \text{(5.6)} \]

Hence, according to Theorem 2.1 and 2.2, we have the following result for the slow-fast distribution dependent stochastic porous media equation.
Theorem 5.1 Assume that (5.2)-(5.4) hold and $\Psi$ fulfills the conditions ($\Psi$1)-($\Psi$4) above. Then for any initial values $x \in H_1$, $y \in H_2$ and $T > 0$, system (5.7) has a unique solution $(X_t^\varepsilon, Y_t^\varepsilon)_{t \in [0, T]}$ such that

$$
\mathbb{E}\left( \sup_{t \in [0, T]} \|X_t^\varepsilon - \bar{X}_t\|^2_{H_1} \right) \leq C_T(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2})\varepsilon^{1/3} \to 0, \text{ as } \varepsilon \to 0,
$$

where $C_T$ is a constant only depending on $T$, and $\bar{X}_t$ is the solution of the corresponding averaged equation.

Remark 5.1 (i) In [27, 28], the authors have established the well-posedness and large deviation principle for McKean-Vlasov stochastic porous media equations. To the best of our knowledge, there is no result on the averaging principle in the literature obtained for two-time-scale McKean-Vlasov SPDE such as stochastic porous media equations here and stochastic $p$-Laplace equations below.

(ii) In [30], the authors have established the averaging principle result for classical (i.e. distribution independent) stochastic quasilinear SPDEs with slow and fast time-scales. In comparison to [30], we not only extend the corresponding averaging principle result to the distribution dependent case, but also explicitly obtain the strong convergence rate for the system in this work.

5.2 Slow-fast McKean-Vlasov stochastic $p$-Laplace equations

Now we apply our main results to establish the averaging principle for following slow-fast McKean-Vlasov stochastic $p$-Laplace equations

$$
\begin{align*}
\begin{cases}
    dX_t^\varepsilon &= \left[\text{div}(|\nabla X_t^\varepsilon|^{p-2}\nabla X_t^\varepsilon) + f(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon)\right] dt + B_1(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon)dW_t^1, \\
    dY_t^\varepsilon &= \frac{1}{\varepsilon}[\Delta Y_t^\varepsilon + g(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon)] dt + \frac{1}{\sqrt{\varepsilon}}B_2(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon)dW_t^2, \\
    X_0^\varepsilon &= x, Y_0^\varepsilon = y.
\end{cases}
\end{align*}
$$

(5.7)

For any $p \geq 2$, we set the following Gelfand triple for the slow equation

$$
V_1 := W_0^{1,p}(\Lambda) \subset H_1 := L^2(\Lambda) \subset V_1^*,
$$

and the following Gelfand triple for the fast equation

$$
V_2 := W_0^{1,2}(\Lambda) \subset H_2 := L^2(\Lambda) \subset V_2^*.
$$

Denote $\mathcal{A}_1(u) := \text{div}(|\nabla u|^{p-2}\nabla u)$, which is called $p$-Laplacian operator. It is well-known that the operator $\mathcal{A}_1$ satisfies (A1)-(A4), interested readers can refer to e.g. [35, Example 4.1.9] for the detailed proof. Thus, according to Theorem 2.1 and 2.2, we have the following result for the slow-fast distribution dependent stochastic $p$-Laplace equations.

Theorem 5.2 Assume that (5.2)-(5.4) hold, then for any initial values $x \in H_1$, $y \in H_2$ and $T > 0$, system (5.7) has a unique solution $(X_t^\varepsilon, Y_t^\varepsilon)_{t \in [0, T]}$ such that

$$
\mathbb{E}\left( \sup_{t \in [0, T]} \|X_t^\varepsilon - \bar{X}_t\|^2_{H_1} \right) \leq C_T(1 + \|x\|^2_{H_1} + \|y\|^2_{H_2})\varepsilon^{1/3} \to 0, \text{ as } \varepsilon \to 0,
$$

where $C_T$ is a constant only depending on $T$, and $\bar{X}_t$ is the solution of the corresponding averaged equation.
Remark 5.2 In particular, if we take \( p = 2 \), \( \tilde{A} \) reduces to the classical Laplace operator. Therefore, our result above also covers some slow-fast distribution dependent semilinear SPDEs.

5.3 Slow-fast McKean-Vlasov SDEs

Besides the above McKean-Vlasov SPDEs, our main results are also applicable to McKean-Vlasov SDE models. For instance, we consider \( V_i = H_i = \mathbb{R}^d \) (\( i = 1, 2 \)) with the Euclidean norm |·| and inner product \( \langle \cdot, \cdot \rangle \),

\[
\begin{aligned}
&dX_t^\varepsilon = b_1(X_t^\varepsilon, X_t^\varepsilon, Y_t^\varepsilon)dt + \sigma_1(X_t^\varepsilon, X_t^\varepsilon)dW_t^1, \\
&dY_t^\varepsilon = \frac{1}{\varepsilon}b_2(X_t^\varepsilon, X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}\sigma_2(X_t^\varepsilon, X_t^\varepsilon, Y_t^\varepsilon)dW_t^2, \\
&X_0^\varepsilon = x, Y_0^\varepsilon = y.
\end{aligned}
\]  

(5.8)

Suppose the coefficients \( b_i : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \rightarrow \mathbb{R}^d \), \( \sigma_1 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times d} \), \( \sigma_2 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d} \) are measurable and satisfy the following conditions (here \( \mathbb{R}^{d \times d} \) denotes the set of real \( d \times d \) matrices).

Hypothesis 5.2 For all \( u, v, u_1, u_2, v_1, v_2, w \in \mathbb{R}^d \) and \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \).

(C1) There exists some constants \( C, \kappa > 0 \) such that

\[ |b_1(u_1, \mu, v_1) - b_1(u_2, \nu, v_2)| \leq C(|u_1 - u_2| + |v_1 - v_2| + \mathcal{W}_{2, \mathbb{R}^d}(\mu, \nu)). \]

Moreover,

\[ \langle b_2(u_1, \mu, v_1) - b_2(u_2, \nu, v_2), v_1 - v_2 \rangle \leq -\kappa|v_1 - v_2|^2 + C(|u_1 - u_2|^2 + \mathcal{W}_{2, \mathbb{R}^d}(\mu, \nu)^2). \]

(C2) There are some constants \( L_{B_2}, C > 0 \) such that

\[ \|\sigma_1(u, \mu) - \sigma_1(v, \nu)\| \leq C(|u - v| + \mathcal{W}_{2, \mathbb{R}^d}(\mu, \nu)), \]

and

\[ \|\sigma_2(u_1, \mu, v_1) - \sigma_2(u_2, \nu, v_2)\| \leq L_{\sigma_2}|u_1 - u_2| + C(|v_1 - v_2| + \mathcal{W}_{2, \mathbb{R}^d}(\mu, \nu)), \]

where \( \| \cdot \| \) denotes the matrix norm.

By Theorem 2.1 and 2.2, we can derive the averaging principle for the slow-fast McKean-Vlasov SDEs.

Theorem 5.3 Assume that Hypothesis 5.2 hold and \( \kappa > 2L_{\sigma_2}^2 \), then for any initial values \( x, y \in \mathbb{R}^d \) and \( T > 0 \), system (5.8) has a unique solution \( (X_t^\varepsilon, Y_t^\varepsilon)_{t \in [0, T]} \) such that

\[ \mathbb{E} \left( \sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t|^2 \right) \leq C_T(1 + \|x\|_{H_1}^2 + \|y\|_{H_2}^2)\varepsilon^1/3 \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \]

where \( C_T \) is a constant only depending on \( T \), and \( \bar{X}_t \) is the solution of the corresponding averaged equation.
Remark 5.3 Using the techniques of time discretization and Poisson equation, Röckner et al. \cite{42} established the optimal strong convergence rate $1/2$ of averaging principle for two-time-scale McKean-Vlasov SDEs under some fairly strong conditions, such as the regularity of first-order and second-order partial derivatives of the coefficients. The convergence rate obtained here is not optimal, since we only assume the coefficients satisfy some monotonicity and Lipschitz conditions, which is in general much weaker than the assumptions in \cite{42}. Moreover, our main results are not only covering this type of models, but also applicable to various two-time-scale McKean-Vlasov (nonlinear) SPDEs.

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