$k$-Component $q$-deformed charge coherent states and their nonclassical properties

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Abstract

$k$-Component $q$-deformed charge coherent states are constructed, their (over)completeness proved and their generation explored. The $q$-deformed charge coherent states and the even (odd) $q$-deformed charge coherent states are the two special cases of them as $k$ becomes 1 and 2, respectively. A $D$-algebra realization of the SU$_q$(1,1) generators is given in terms of them. Their nonclassical properties are studied and it is shown that for $k \geq 3$, they exhibit two-mode $q$-antibunching, but neither SU$_q$(1,1) squeezing, nor one- or two-mode $q$-squeezing.

Keywords: $k$-Component charge coherent states; $q$-Deformation; Completeness relations; Squeezing

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1. Introduction

The coherent states introduced by Schrödinger [1] and Glauber [2] are the eigenstates of the boson annihilation operator, and have widespread applications in the fields of physics [3–7]. However, in all the cases the quanta involved are uncharged. In 1976, Bhaumik et al. [4,8,9] constructed the boson coherent states which, carrying definite charge, are the eigenstates of both the pair boson annihilation operator and the charge operator. This kind of states are the so-called charge coherent states. Based on this work, the charge coherent states for SU(2) [10], SU(3) [11], and arbitrary compact Lie groups [12] were also put forward.

The concept of charge coherent states has proved to be very useful in many areas, such as elementary particle physics [9,13–17], quantum field theory [12,18,19], nuclear physics [20], thermodynamics [21–23], quantum mechanics [24], and quantum optics [25–27]. Moreover, some schemes for generating charge coherent states in quantum optics were proposed [25,26,28,29].

As is well known, the even and odd coherent states [30], which are the two orthonormalized eigenstates of the square of the boson annihilation operator, play an important role in quantum optics [31–33]. An extension of the even and odd coherent states is to define the $k$-component coherent states [34,35], which are the $k$ ($k \geq 1$) orthonormalized eigenstates of the $k$th power of the boson annihilation operator. The coherent states and the even (odd) ones are the two special cases of the $k$-component coherent states as $k$ becomes 1 and 2, respectively. Inspired by the above idea, in Ref. [36] one of the authors (X.-M.L.) has generalized the charge coherent states to the even and odd charge coherent states, defined as the two orthonormalized eigenstates
of both the square of the pair boson annihilation operator and the charge operator; in Ref. [37] he has further extended the even and odd charge coherent states to the $k$-component charge coherent states, defined as the $k$ orthonormalized eigenstates of both the $k$th power of the pair boson annihilation operator and the charge operator. The charge coherent states and the even (odd) ones are the two special cases of the $k$-component charge coherent states as $k$ becomes 1 and 2, respectively.

On the other hand, quantum groups [38,39], introduced as a mathematical description of deformed Lie algebras, have given the possibility of generalizing the notion of coherent states to the case of $q$-deformations [40–44]. A $q$-deformed harmonic oscillator [40,45] was defined in terms of $q$-boson annihilation and creation operators, the latter satisfying the quantum Heisenberg-Weyl algebra [40,45,46], which plays an important role in quantum groups. The $q$-deformed coherent states introduced by Biedenharn [40] are the eigenstates of the $q$-boson annihilation operator. Such states have been well studied [41,42,47,48], and widely applied to quantum optics and mathematical physics [44,49–53]. Furthermore, the $q$-deformed charge coherent states [54,55] were constructed as the eigenstates of both the pair $q$-boson annihilation operator and the charge operator.

A natural extension of the $q$-deformed coherent states is provided by the even and odd $q$-deformed coherent states [56], which are the two orthonormalized eigenstates of the square of the $q$-boson annihilation operator. In a previous Letter [57], motivated by the above idea, the authors (X.-M.L. and C.Q.) have generalized the $q$-deformed charge coherent states to the even and odd $q$-deformed charge coherent states, defined as the two orthonormalized eigenstates of both the square of the pair $q$-boson annihilation operator and the charge operator. A further extension of the even and odd
$q$-deformed coherent states is given by the $k$-component $q$-deformed coherent states [58,59], which are the $k$ orthonormalized eigenstates of the $k$th power of the $q$-boson annihilation operator. The $q$-deformed coherent states and the even (odd) ones are the two special cases of the $k$-component $q$-deformed coherent states as $k$ becomes 1 and 2, respectively. In a parallel way, it is very desirable to generalize the even and odd $q$-deformed charge coherent states to the $k$-component $q$-deformed charge coherent states, defined as the $k$ orthonormalized eigenstates of both the $k$th power of the pair $q$-boson annihilation operator and the charge operator. The $q$-deformed charge coherent states and the even (odd) ones are the two special cases of the $k$-component $q$-deformed charge coherent states as $k$ becomes 1 and 2, respectively.

This paper is arranged as follows. In Section 2, the $k$-component $q$-deformed charge coherent states are constructed. Their completeness is proved in Section 3. Section 4 is devoted to generating them. In Section 5, they are used to provide a $D$-algebra realization of the $SU_q(1, 1)$ generators. Their nonclassical properties, such as $SU_q(1, 1)$ squeezing, single- or two-mode $q$-squeezing, and two-mode $q$-antibunching, are studied in Section 6. Section 7 contains a summary of the results.

2. $k$-Component $q$-deformed charge coherent states

Two mutually commuting $q$-deformed harmonic oscillators are defined in terms of two pairs of independent $q$-boson annihilation and creation operators $a_i, a_i^+$ ($i = 1, 2$), together with corresponding number operators $N_i$, satisfying the quantum Heisenberg-Weyl algebra

\[ a_i a_i^+ - q a_i^+ a_i = q^{-N_i}, \]

\[ [N_i, a_i^+] = a_i^+, \quad [N_i, a_i] = -a_i, \]

\[ [N_i, N_j] = 0, \quad [a_i, a_j] = [a_i, a_j^+] = [a_i^+, a_j^+] = 0. \]
where $q$ is a positive real deformation parameter. The operators $a_i$, $a_i^+$, and $N_i$ act in the Fock space with basis $|n\rangle_i$ ($n = 0, 1, 2, \ldots$), such that

$$a_i|0\rangle_i = 0, \quad |n\rangle_i = \frac{(a_i^+)^n}{\sqrt{[n]!}}|0\rangle_i,$$

where

$$[n]! \equiv [n]_q! = [n]_q[n-1]_q \ldots [1]_q, \quad [0]! = 1,$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \equiv [n].$$

Their action on the basis states is given by

$$a_i|n\rangle_i = \sqrt{|n|}|n-1\rangle_i, \quad a_i^+|n\rangle_i = \sqrt{|n+1|}|n+1\rangle_i, \quad N_i|n\rangle_i = n|n\rangle_i.$$  

Note that $[n]$ is invariant under $q \leftrightarrow 1/q$. In the following, $[n]$ will refer to the $q$-deformed $n$ defined by (5) corresponding to the base $q$. If the base is different, then it will be indicated explicitly.

The $q$-boson operators $a_i$ and $a_i^+$ can be constructed from the conventional boson annihilation and creation operators $b_i$, $b_i^+$ in the following way [60]:

$$a_i = \sqrt{\frac{[N_i + 1]}{N_i + 1}}b_i, \quad a_i^+ = b_i^+ \sqrt{\frac{[N_i + 1]}{N_i + 1}},$$

where $N_i = b_i^+b_i$. It is worth noticing that $[N_i] = a_i^+a_i$.

The operators $a_1(a_1^+)$ and $a_2(a_2^+)$ are assigned the “charge” quanta 1 and $-1$, respectively. Thus the charge operator is given by

$$Q = N_1 - N_2.$$  

In view of the fact that

$$[Q, (a_1a_2)^k] = 0,$$

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where $k$ is a positive integer ($k = 1, 2, 3, ...$), we may seek the $k$-component $q$-deformed charge coherent states, which are the $k$ orthonormalized eigenstates of both the $k$th power $(a_1 a_2)^k$ of the pair $q$-boson annihilation operator $a_1 a_2$ and the charge operator $Q$.

Let $|m, n\rangle = |m\rangle_1 |n\rangle_2$ denote the basis states of two-mode Fock space, where $|m\rangle_1$ and $|n\rangle_2$ are the eigenstates of $N_1$ and $N_2$ corresponding to the eigenvalues $m$ and $n$, respectively. They satisfy the completeness relation

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m, n\rangle \langle m, n| = I.$$ \hspace{1cm} (10)

We now consider the following states:

$$|\xi, q, k\rangle_j = N_{kj}^j \sum_{p=\text{max}(0,-q/k)}^{\infty} \frac{\xi^{kp+j+\min(0,q)}}{\{[kp+j]![kp+j+q]!\}^{1/2}} |kp+j+q,kp+j\rangle$$

$$= \begin{cases} N_{kj}^j \sum_{n=0}^{\infty} \frac{\xi^{kn+j}}{\{[kn+j]![kn+j+q]!\}^{1/2}} |kn+j+q,kn+j\rangle, & q \geq 0, \\ N_{kj}^j \sum_{n=0}^{\infty} \frac{\xi^{kn+j}}{\{[kn+j]![kn+j-q]!\}^{1/2}} |kn+j,kn+j-q\rangle, & q \leq 0, \end{cases}$$ \hspace{1cm} (11)

where $j = 0, 1, \ldots, k-1$, $\xi$ is a complex number, $q$ is a fixed integer and $N_{kj}^j$ are normalization factors given by

$$N_{kj}^j = N_{kj}^j (|\xi|^2) = \left( \sum_{n=0}^{\infty} \frac{(|\xi|^2)^{kn+j}}{[kn+j]![kn+j+q]!} \right)^{-1/2}.$$ \hspace{1cm} (12)

As can be verified by explicit calculations, these states satisfy the relations

$$(a_1 a_2)^k |\xi, q, k\rangle_j = \xi^k |\xi, q, k\rangle_j, \quad Q |\xi, q, k\rangle_j = q |\xi, q, k\rangle_j, \quad j \langle \xi, q, k |\xi, q, k\rangle_j = \delta_{jj'}.$$ \hspace{1cm} (13)

It indicates that $|\xi, q, k\rangle_j$ ($j = 0, 1, \ldots, k-1$) in (11) are exactly the $k$ orthonormalized eigenstates of both the operator $(a_1 a_2)^k$ and $Q$ corresponding to the eigenvalues $\xi^k$ and $q$, respectively. Obviously, $q$ is the charge number which the states $|\xi, q, k\rangle_j$ carry.
Therefore, the \( k \) states of (11) are just what we want, that is to say, they are the \( k \)-component \( q \)-deformed charge coherent states. In the limit \( q \rightarrow 1 \), they reduce to the usual \( k \)-component charge coherent states constructed by the author (X.-M.L.) [37].

According to (11), for \( k = 1 \), we obtain

\[
|\xi, q, 1\rangle_0 = N_q \sum_{p=\text{max}(0,-q)} \xi^{p+\min(0,q)} |p + q, p\rangle \\
= \begin{cases} \\
N_q \sum_{n=0}^{\infty} \frac{\xi^{n}}{[n+q]!} |n + q, n\rangle, & q \geq 0, \\
N_q \sum_{n=0}^{\infty} \frac{\xi^{n}}{[n-q]!} |n, n-q\rangle, & q \leq 0,
\end{cases}
\]

\[
\equiv |\xi, q\rangle, \quad (14)
\]

where

\[
N_q \equiv N_q^{0}(|\xi|^{2}) = \left( \sum_{n=0}^{\infty} \frac{(|\xi|^{2})^{n}}{[n]![n+q]!} \right)^{-1/2}. \quad (15)
\]

It is evident that \( |\xi, q, 1\rangle_0 (\equiv |\xi, q\rangle) \) are exactly the so called \( q \)-deformed charge coherent states given in Ref. [54].

According to (11), for \( k = 2 \), we obtain

\[
|\xi, q, 2\rangle_j = N_q^{j} \sum_{p=\text{max}(0,-q)} \xi^{2p+j+\min(0,q)} |2p + j + q, 2p + j\rangle \\
= \begin{cases} \\
N_q^{j} \sum_{n=0}^{\infty} \frac{\xi^{2n+j}}{[2n+j]![2n+j+q]!} |2n + j + q, 2n + j\rangle, & q \geq 0, \\
N_q^{j} \sum_{n=0}^{\infty} \frac{\xi^{2n+j}}{[2n+j]![2n+j-q]!} |2n + j, 2n + j - q\rangle, & q \leq 0,
\end{cases}
\]

\[
\equiv |\xi, q\rangle, \quad (16)
\]

where \( j = 0, 1 \). It is evident that \( |\xi, q, 2\rangle_0 (|\xi, q, 2\rangle_1) \) are exactly the so called even (odd) \( q \)-deformed charge coherent states obtained by the authors (X.-M.L. and C.Q.) [57].

From (11), it follows that

\[
j(\xi, q, k|\xi', q', k')_j = N_{kq}^{j}(|\xi|^{2}) N_{kq}^{j}(|\xi'|^{2}) \left[ N_{kq}^{j}(\xi^{*}\xi') \right]^{-2} \delta_{gg'j} \delta_{jj'}.
\]

\[7\]
This further shows that, for the same value of $k$, the states $|\xi, q, k\rangle_j$ are orthogonal to one another with respect to both the subscript $j$ and the charge number $q$. However, they are nonorthogonal with respect to the parameter $\xi$.

For the mean values of the operators $N_1$ and $N_2$, there exists the relation

$$j\langle \xi, q, k|N_1|\xi, q, k\rangle_j = q + j\langle \xi, q, k|N_2|\xi, q, k\rangle_j.$$  \hspace{1cm} (18)

In terms of the $k$-component $q$-deformed charge coherent states, the $q$-deformed charge coherent states can be expanded as

$$|\xi, q\rangle = N_q\left[\sum_{j=0}^{k-1} (N_{kj}^q)^{-1}|\xi, q, k\rangle_j\right],$$  \hspace{1cm} (19)

where the normalization factors are such that

$$N_q^{-2} = \sum_{j=0}^{k-1} (N_{kj}^q)^{-2}.$$  \hspace{1cm} (20)

3. Completeness of $k$-component $q$-deformed charge coherent states

Let us begin with some $q$-deformed formulas which are useful in the proof of completeness of the $k$-component $q$-deformed charge coherent states. The $q$-deformed Bessel function of (integer) order $\nu$ may be defined by [61]

$$J_\nu(q, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]!(\nu + k)!} \left(\frac{x}{\sqrt{q}[2]_{\sqrt{q}}}\right)^{\nu+2k},$$  \hspace{1cm} (21)

where $[n]_{\sqrt{q}}$ is defined as in Eq. (5) except for replacing $q$ by $\sqrt{q}$. An integral representation of the $q$-deformed modified Bessel function of order $\nu$ is given by [62]

$$K_\nu(q, x) = \frac{1}{[2]_{\sqrt{q}}} \left(\frac{x}{[2]_{\sqrt{q}}}\right)^\nu \int_0^\infty dq t^{\nu+1} e_q(-t)e_q \left(-\frac{x^2}{([2]_{\sqrt{q}})^2 t}\right),$$  \hspace{1cm} (22)
where $d_q t$ is a standard $q$-integration [47,63,64], and $e_q(x)$ is a $q$-exponential function [47]

$$e_q(x) = \begin{cases} \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, & \text{for } x > -\zeta, \\ 0, & \text{otherwise}, \end{cases}$$

(23)

with $-\zeta$ being the largest zero of $e_q(x)$. Then, it follows that [62]

$$\int_0^\infty d\sqrt{q}u u^{2p+\nu+1}K_\nu(q, [2]\sqrt{q}u) = \frac{[\nu + p]![p]!}{([2]\sqrt{q})^2}.$$ \hspace{1cm} (24)

We now prove that the $k$-component $q$-deformed charge coherent states form an (over)complete set, that is to say

$$\sum_{q=\infty}^\infty \int \frac{d^2 \xi}{\pi} \phi_q(\xi) N^2_q \left[ \sum_{j=0}^{k-1} (N^2_q)^{-2} |\xi, q, k\rangle_j \langle \xi, q, k| \right] = \sum_{q=\infty}^\infty I_q = I, \hspace{1cm} (25)$$

where

$$d^2_q \xi = |\xi| d\sqrt{q} |\xi| d\theta, \hspace{1cm} \xi = |\xi| e^{i\theta},$$

(26)

and

$$\phi_q(\xi) = \frac{([2]\sqrt{q})^2}{2} (-i)^2 J_q(q, i\sqrt{q}[2]\sqrt{q}|\xi|) K_q(q, [2]\sqrt{q}|\xi|).$$ \hspace{1cm} (27)

Note that the integral over $\theta$ is a standard integration while that over $|\xi|$ is a $q$-integration.

In fact, for $q \geq 0$, we have

$$I_q = \int \frac{d^2 \xi}{\pi} \phi_q(\xi) N^2_q \sum_{k=0}^{k-1} \sum_{n,m} \frac{\xi^{k+n+j} \xi^{k+m+j} |kn + j + q, kn + j\rangle \langle km + j + q, km + j|}{(|kn + j|)[kn + j + q]!(km + j)!(km + j + q)!}^{1/2} \hspace{1cm} (28)$$

$$= \int_0^\infty \frac{d\sqrt{q}|\xi| ([2]\sqrt{q})^2 |\xi|^{q+1} K_q(q, [2]\sqrt{q}|\xi|) \sum_{j=0}^{k-1} \sum_{n,m} |\xi|^{k(n+m)+2j} \int_{-\pi}^\pi d\theta e^{ik(n-m)\theta} \hspace{1cm} (29)$$

$$\times \frac{|kn + j + q, kn + j\rangle \langle km + j + q, km + j|}{(|kn + j|)[kn + j + q]!(km + j)!(km + j + q)!}^{1/2} \hspace{1cm} (30)$$

$$= \int_0^\infty \frac{d\sqrt{q}|\xi| ([2]\sqrt{q})^2 |\xi|^{q+1} K_q(q, [2]\sqrt{q}|\xi|) \sum_{j=0}^{k-1} \sum_{n=0}^\infty \frac{|\xi|^{2n+j} |kn + j + q, kn + j\rangle \langle kn + j + q, kn + j|}{|kn + j|}[kn + j + q]! \hspace{1cm} (31)$$

$$\times \frac{[kn + j]!(kn + j + q)!}{[km + j]!(km + j + q)!}}$$

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\[ I_q = \sum_{n=0}^{\infty} |n+q, n\rangle \langle n+q, n| \]  

(28)

Similarly, for \( q \leq 0 \), we get

\[ I_q = \sum_{n=0}^{\infty} |n, n-q\rangle \langle n, n-q| \].

(29)

Consequently, we derive

\[
\sum_{q=-\infty}^{\infty} I_q = \sum_{n=0}^{\infty} \left( \sum_{q=-\infty}^{-1} |n, n-q\rangle \langle n, n-q| + \sum_{q=0}^{\infty} |n+q, n\rangle \langle n+q, n| \right) 
\]

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m, n\rangle \langle m, n| = I. \]

(30)

Hence, the \( k \)-component \( q \)-deformed charge coherent states are qualified to make up an (over)complete representation. It should be mentioned that \( I_q \) represents the resolution of unity in the subspace where \( Q = q \).

In the two special cases of \( k = 1 \) and \( k = 2 \), the above demonstration gives the proof of completeness of the \( q \)-deformed charge coherent states \[65\] and the even (odd) \( q \)-deformed charge coherent states \[57\], respectively.

### 4. Generation of \( k \)-component \( q \)-deformed charge coherent states

The \( k \)-component \( q \)-deformed coherent states, defined as the \( k \) orthonormalized eigenstates of the \( k \)th power of the \( q \)-boson annihilation operator, can be expanded in the single-mode Fock space as \[58,59\]

\[ |\xi, k\rangle_j = N_k^j \sum_{n=0}^{\infty} \frac{\xi^{kn+j}}{\sqrt{[kn+j]!}} |kn+j\rangle, \]  

(31)
where \( j = 0, 1, \ldots, k-1 \) and

\[
N_j^k \equiv N_j^k(|\xi|^2) = \left\{ \sum_{n=0}^{\infty} \frac{(|\xi|^2)^{kn+j}}{[kn+j]!} \right\}^{-1/2}.
\] (32)

As a special case, for \( k = 1 \), \(|\xi, 1\rangle_0\) are exactly the \( q \)-deformed coherent states, i.e.,

\[
|\xi, 1\rangle_0 = e^{-1/2q(|\xi|^2)} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} |n\rangle \equiv |\xi\rangle.
\] (33)

The \( k \)-component \( q \)-deformed charge coherent states can also be obtained from the states (31) and (33) according to the following expression

\[
|\xi, q, k\rangle_j = \begin{cases} 
N_j^k e^{1/2q(|\xi|^2)} [N_j^k(|\xi_1|^2)]^{-1} \xi_1^{-q} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{i\alpha} |e^{-i\alpha} \xi_1 \rangle \otimes |e^{i\alpha} \xi_2 \rangle, & q \geq 0, \\
N_j^k e^{1/2q(|\xi|^2)} [N_j^k(|\xi|^2)]^{-1} \xi_1^{+q} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha} |e^{i\alpha} \xi_2 \rangle \otimes |e^{-i\alpha} \xi_1 \rangle, & q \leq 0,
\end{cases}
\] (34)

where \( \xi = \xi_1 \xi_2 \). Such a representation is very useful since the properties of \( q \)-deformed coherent states and \( k \)-component \( q \)-deformed coherent states can now be employed in a study of the properties of \( k \)-component \( q \)-deformed charge coherent states. The expression for the latter given in (34) has a very simple group-theoretical interpretation: in (34) one suitably averages over the U(1)-group (caused by the charge operator \( Q \)) action on the product of \( q \)-deformed coherent states and \( k \)-component \( q \)-deformed coherent states, which then projects out the \( Q = q \) charge subspace contribution.

It is easy to see that in the limit \( q \to 1 \), the above discussion gives back the corresponding results for the usual \( k \)-component charge coherent states obtained in Ref. [37], and that in the two special cases of \( k = 1 \) and \( k = 2 \), it gives the corresponding results for the \( q \)-deformed charge coherent states [54] and the even (odd) \( q \)-deformed charge coherent states [57], respectively.
5. **D-algebra realization of SU\(_q(1, 1)\) generators**

As is well known, the coherent state D-algebra [6,66] is a mapping of quantum observables onto a differential form that acts on the parameter space of coherent states, and has a beautiful application in the reformulation of the entire laser theory in terms of C-number differential equations [67]. We shall construct the D-algebra realization of the \(q\)-deformed SU\(_q(1, 1)\) generators corresponding to the unnormalized \(k\)-component \(q\)-deformed charge coherent states, defined by

\[
||\xi, q, k\rangle_j = (N^{-1}_{kg})|\xi, q, k\rangle_j.
\]

Let \(||\xi\rangle\) denote a column vector composed of \(||\xi\rangle_j (j = 0, 1, \ldots, k - 1)\), i.e.,

\[
||\xi\rangle = \begin{bmatrix}
||\xi\rangle_0 \\
||\xi\rangle_1 \\
\vdots \\
||\xi\rangle_{k-1}
\end{bmatrix}.
\]

The action of the operators \(a_i, a_i^+, N_i\) on this column vector can be written in the matrix form:

\[
\begin{align*}
P_{\text{Positive}} & \quad \text{Negative Q} \\
N_1||\xi\rangle = (\xi \frac{d}{d\xi} + q) ||\xi\rangle, & \quad N_1||\xi\rangle = \xi \frac{d}{d\xi} ||\xi\rangle, \\
N_2||\xi\rangle = \xi \frac{d}{d\xi} ||\xi\rangle, & \quad N_2||\xi\rangle = \left(\xi \frac{d}{d\xi} - q\right) ||\xi\rangle,
\end{align*}
\]

where \(d/d\xi\) is a standard differential operator, whereas \(d/dq\xi\) is a \(q\)-differential one.
\[
\frac{d}{dq}\xi f(q\xi) = \frac{f(q\xi) - f(q^{-1}\xi)}{q\xi - q^{-1}\xi};
\]  
(38)

and

\[
M = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}, \quad N = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]  
(39)

Obviously, \(N\) is both the inverse and the transpose of \(M\).

The \(q\)-deformed SU\(_q\)(1,1) algebra consists of three generators \(K_0, K_+,\) and \(K_-\), satisfying the commutation relations

\[
[K_+, K_-] = -[2K_0], \quad [K_0, K_\pm] = \pm K_\pm,
\]  
(40)

and is realized in terms of the two-mode \(q\)-boson operators as

\[
K_- = a_1 a_2, \quad K_+ = a_1^+ a_2^+, \quad K_0 = \frac{1}{2} (N_1 + N_2 + 1).
\]  
(41)

Actually, the \(k\)-component \(q\)-deformed charge coherent states are also the \(k\) orthonormalized eigenstates of the \(k\)th power of \(K_-\).

The \(D\)-algebra of the SU\(_q\)(1,1) generators \(A\) may be defined for the action on the ket coherent states (36) or for that on the corresponding bras as

\[
A||q\rangle = D^k(A)||q\rangle,
\]  
(42)

\[
\langle q||A = D^b(A)\langle q||,
\]  
(43)

respectively. Using (37) and (41), we get for the former

\[
D^k(K_-) = \xi M,
\]  
(44)

\[
D^k(K_+) = \xi^{-|q|} \frac{d}{d\xi} \xi^{|q|+1} \frac{d}{d\xi} N,
\]  
(45)

\[
D^k(K_0) = \frac{1}{2} \left(2\xi \frac{d}{d\xi} + |q| + 1\right) I,
\]  
(46)
while the latter can be obtained from the adjoint relation

\[ D^b(A) = [D^k(A^+)]^*. \]  

Thus, the \( D \)-algebra of the SU\(_q(1,1)\) generators corresponding to the unnormalized \( k \)-component \( q \)-deformed charge coherent states has been realized in a \( q \)-differential-operator matrix form.

From (36), (39), (42) and (44), we clearly see that by the successive actions of the operator \( a_1a_2 \), each component of the \( k \)-component \( q \)-deformed charge coherent states, apart from normalization, can be transformed into another in this way: \( |\xi, \underline{q}, k\rangle_0 \rightarrow |\xi, \underline{q}, k\rangle_{k-1} \rightarrow |\xi, \underline{q}, k\rangle_{k-2} \rightarrow \cdots \rightarrow |\xi, \underline{q}, k\rangle_1 \rightarrow |\xi, \underline{q}, k\rangle_0 \). Actually, \( a_1a_2 \) plays the role of a rotating operator among these \( k \)-component states.

It is easy to check that in the limit \( q \rightarrow 1 \), the above discussion gives back the corresponding results for the usual \( k \)-component charge coherent states obtained in Ref. [37], and that in the two special cases of \( k = 1 \) and \( k = 2 \), it gives the corresponding results for the \( q \)-deformed charge coherent states [65] and the even (odd) \( q \)-deformed charge coherent states [57], respectively.

6. Nonclassical properties of \( k \)-component \( q \)-deformed charge coherent states

In Ref. [57], the authors (X.-M.L. and C.Q.) have examined the even and odd \( q \)-deformed charge coherent states for some nonclassical properties, such as SU\(_q(1,1)\) squeezing, single- or two-mode \( q \)-squeezing, and two-mode \( q \)-antibunching. In this section, we will study the nonclassical properties of the \( k \)-component \( q \)-deformed charge coherent states with \( k \geq 3 \).
6.1. SU\(_q(1,1)\) squeezing

In analogy with the definition of SU(1,1) squeezing \cite{68}, we have introduced SU\(_q(1,1)\) squeezing \cite{57} in terms of the Hermitian \(q\)-deformed quadrature operators

\[
X_1 = \frac{K_+ + K_-}{2}, \quad X_2 = \frac{i(K_+ - K_-)}{2},
\]

(48)

which satisfy the commutation relation

\[
[X_1, X_2] = \frac{i}{2}[2K_0]
\]

(49)

and the uncertainty relation

\[
\langle(\Delta X_1)^2\rangle \langle(\Delta X_2)^2\rangle \geq \frac{1}{16}|\langle[2K_0]\rangle|^2.
\]

(50)

A state is said to be SU\(_q(1,1)\) squeezed if

\[
\langle(\Delta X_i)^2\rangle < \frac{1}{4}|\langle[2K_0]\rangle| \quad (i = 1 \text{ or } 2).
\]

(51)

Let us now calculate the fluctuations (variances) of \(X_1\) and \(X_2\) with respect to the \(k\)-component \(q\)-deformed charge coherent states. Using (41) – (44) and (47), we get

\[
o\langle\xi, q, k|K_+ K_-|\xi, q, k\rangle_0 = |\xi|^2 (N_{kq}^0)^2 / (N_{kq}^{k-1})^2,
\]

(52)

\[
m\langle\xi, q, k|K_+ K_-|\xi, q, k\rangle_m = |\xi|^2 (N_{kq}^m)^2 / (N_{kq}^{m-1})^2, \quad m = 1, 2, \ldots, k - 1.
\]

(53)

In the meantime, for the states \(|\xi, q, k\rangle_j \quad (k \geq 3)\), it always follows that

\[
j\langle\xi, q, k|K_-|\xi, q, k\rangle_j = j\langle\xi, q, k|K_-^2|\xi, q, k\rangle_j = 0, \quad j = 0, 1, \ldots, k - 1.
\]

(54)

Therefore, for \(|\xi, q, k\rangle_0\) and \(|\xi, q, k\rangle_m \quad (m = 1, 2, \ldots, k - 1)\), the fluctuations are given by

\[
o\langle\xi, q, k|(\Delta X_1)^2|\xi, q, k\rangle_0 = o\langle\xi, q, k|(\Delta X_2)^2|\xi, q, k\rangle_0
\]
\[ m\langle \xi, q, k | (\Delta X_1)^2 | \xi, q, k \rangle_m = m\langle \xi, q, k | (\Delta X_2)^2 | \xi, q, k \rangle_m = \frac{1}{4} m\langle \xi, q, k | [2K_0] | \xi, q, k \rangle_m + \frac{1}{2} |\xi|^2 (N_{kq}^0)^2 / (N_{kq}^{k-1})^2. \tag{55} \]

Consequently, for \( k \geq 3 \), we find

\[ j\langle \xi, q, k | (\Delta X_1)^2 | \xi, q, k \rangle_j = j\langle \xi, q, k | (\Delta X_2)^2 | \xi, q, k \rangle_j \geq \frac{1}{4} j\langle \xi, q, k | [2K_0] | \xi, q, k \rangle_j, \quad j = 0, 1, \ldots, k - 1. \tag{56} \]

The inequalities in (57) say that there is no \( \text{SU}_{q}(1, 1) \) squeezing in the \( k \)-component \( q \)-deformed charge coherent states with \( k \geq 3 \). However, there is such squeezing in the even and odd \( q \)-deformed charge coherent states [57].

It is easy to verify that the \( q \)-deformed charge coherent states satisfy the equality in (50) and that \( \langle (\Delta X_1)^2 \rangle = \langle (\Delta X_2)^2 \rangle \). This point has been observed in Ref. [57]. Therefore, the \( q \)-deformed charge coherent states are not \( \text{SU}_{q}(1, 1) \) squeezed.

### 6.2. Single-mode \( q \)-squeezing

In analogy with the definition of single-mode squeezing [27], we have introduced single-mode \( q \)-squeezing [57] in terms of the Hermitian \( q \)-deformed quadrature operators for the individual modes

\[ Y_1 = \frac{a_1^+ + a_1}{2}, \quad Y_2 = \frac{i(a_1^+ - a_1)}{2}, \]
\[ Z_1 = \frac{a_2^+ + a_2}{2}, \quad Z_2 = \frac{i(a_2^+ - a_2)}{2}, \tag{58} \]

which satisfy the commutation relations

\[ [Y_1, Y_2] = \frac{i}{2} [a_1, a_1^+], \quad [Z_1, Z_2] = \frac{i}{2} [a_2, a_2^+]. \tag{59} \]
and the uncertainty relations

\[
\langle (\Delta Y_1)^2 \rangle \langle (\Delta Y_2)^2 \rangle \geq \frac{1}{16} |\langle [a_1, a_1^+] \rangle|^2, \quad \langle (\Delta Z_1)^2 \rangle \langle (\Delta Z_2)^2 \rangle \geq \frac{1}{16} |\langle [a_2, a_2^+] \rangle|^2.
\]  

(60)

A state is said to be single-mode \(q\)-squeezed if

\[
\langle (\Delta Y_i)^2 \rangle < \frac{1}{4} |\langle [a_i, a_i^+] \rangle|, \quad \langle (\Delta Z_i)^2 \rangle < \frac{1}{4} |\langle [a_i, a_i^+] \rangle| \quad (i = 1 \text{ or } 2).
\]  

(61)

For the states \(|\xi, q, k\rangle_j\) \((k \geq 1)\), it always follows that

\[
j\langle a_{1\downarrow} \rangle_j = j\langle a_{2\downarrow} \rangle_j = j\langle a_{1\uparrow}^2 \rangle_j = j\langle a_{2\uparrow}^2 \rangle_j = 0, \quad j = 0, 1, \ldots, k - 1.
\]  

(62)

Thus, the fluctuations are given by

\[
\begin{align*}
j\langle \xi, q, k | (\Delta Y_1)^2 | \xi, q, k \rangle_j &= j\langle \xi, q, k | (\Delta Y_2)^2 | \xi, q, k \rangle_j \\
&= \frac{1}{4} \left\{ j\langle \xi, q, k | [a_1, a_1^+] \rangle_j |\xi, q, k\rangle_j + 2 j\langle \xi, q, k | a_{1\downarrow} a_{2\uparrow} \rangle_j |\xi, q, k\rangle_j \right\} \\
&> \frac{1}{4} j\langle \xi, q, k | [a_1, a_1^+] \rangle_j |\xi, q, k\rangle_j, \quad (63)
\end{align*}
\]

\[
\begin{align*}
j\langle \xi, q, k | (\Delta Z_1)^2 | \xi, q, k \rangle_j &= j\langle \xi, q, k | (\Delta Z_2)^2 | \xi, q, k \rangle_j \\
&= \frac{1}{4} \left\{ j\langle \xi, q, k | [a_2, a_2^+] \rangle_j |\xi, q, k\rangle_j + 2 j\langle \xi, q, k | a_{2\downarrow} a_{2\uparrow} \rangle_j |\xi, q, k\rangle_j \right\} \\
&> \frac{1}{4} j\langle \xi, q, k | [a_2, a_2^+] \rangle_j |\xi, q, k\rangle_j.
\end{align*}
\]  

(64)

This shows that there is no single-mode \(q\)-squeezing in the \(k\)-component \(q\)-deformed charge coherent states with \(k \geq 1\). As two special cases, there is no such \(q\)-squeezing in the \(q\)-deformed charge coherent states [65] and the even (odd) \(q\)-deformed charge coherent states [57] as \(k\) becomes 1 and 2, respectively.

6.3. Two-mode \(q\)-squeezing

In analogy with the definition of two-mode squeezing [69], we have introduced two-mode \(q\)-squeezing [57] in terms of the Hermitian \(q\)-deformed quadrature operators for
the two modes

\[ W_1 = \frac{Y_1 + Z_1}{\sqrt{2}} = \frac{1}{\sqrt{8}}(a_1^+ + a_2^+ + a_1 + a_2), \quad W_2 = \frac{Y_2 + Z_2}{\sqrt{2}} = \frac{i}{\sqrt{8}}(a_1^+ + a_2^+ - a_1 - a_2), \]  

(65)

which satisfy the commutation relation

\[ [W_1, W_2] = \frac{1}{4} i \left\{ [a_1, a_1^+] + [a_2, a_2^+] \right\} \]  

(66)

and the uncertainty relation

\[ \langle (\Delta W_1)^2 \rangle \langle (\Delta W_2)^2 \rangle \geq \frac{1}{64} \left[ \langle [a_1, a_1^+] \rangle + \langle [a_2, a_2^+] \rangle \right]^2. \]  

(67)

A state is said to be two-mode \( q \)-squeezed if

\[ \langle (\Delta W_i)^2 \rangle < \frac{1}{8} \langle [a_1, a_1^+] \rangle + \langle [a_2, a_2^+] \rangle \quad (i = 1 \text{ or } 2). \]  

(68)

For the states \( |\xi, q, k\rangle_j \) \((k \geq 2)\), the fluctuations are given by

\[
\langle \xi, q, k | (\Delta W_1)^2 | \xi, q, k \rangle_j = j \langle \xi, q, k | (\Delta W_2)^2 | \xi, q, k \rangle_j = \frac{1}{2} \left\{ j \langle \xi, q, k | (\Delta Y_1)^2 | \xi, q, k \rangle_j + j \langle \xi, q, k | (\Delta Z_1)^2 | \xi, q, k \rangle_j \right\} \\
= \frac{1}{2} \left\{ j \langle \xi, q, k | (\Delta Y_2)^2 | \xi, q, k \rangle_j + j \langle \xi, q, k | (\Delta Z_2)^2 | \xi, q, k \rangle_j \right\} \\
= \frac{1}{8} \left\{ j \langle \xi, q, k | [a_1, a_1^+] \rangle \xi, q, k \rangle_j + j \langle \xi, q, k | [a_2, a_2^+] \rangle \xi, q, k \rangle_j \\
+ 2 j \langle \xi, q, k | a_1^+ a_1 | \xi, q, k \rangle_j + 2 j \langle \xi, q, k | a_2^+ a_2 | \xi, q, k \rangle_j \right\} \geq \frac{1}{8} \left\{ j \langle \xi, q, k | [a_1, a_1^+] \rangle \xi, q, k \rangle_j + j \langle \xi, q, k | [a_2, a_2^+] \rangle \xi, q, k \rangle_j \right\}. \]  

(69)

This shows that there is no two-mode \( q \)-squeezing in the \( k \)-component \( q \)-deformed charge coherent states with \( k \geq 2 \). As a special case, there is no such \( q \)-squeezing in the even and odd \( q \)-deformed charge coherent states [57] as \( k \) becomes 2. However, there is such \( q \)-squeezing in the \( q \)-deformed charge coherent states [65].
6.4. Two-mode \( q \)-antibunching

In analogy with the definition of two-mode antibunching [36], we have introduced a two-mode \( q \)-correlation function as [57]

\[
g^{(2)}(0) = \frac{\langle (a_1^+ a_2^+ a_1 a_2)^2 \rangle}{\langle a_1^+ a_2^+ a_1 a_2 \rangle^2} = \frac{\langle : (|N_1| |N_2|) : \rangle}{\langle |N_1| |N_2| \rangle^2} = \frac{\langle K_+^2 K_-^2 \rangle}{\langle K_+ K_- \rangle^2},
\]

where \( a_i \) and \( a_i^+ \) represent the annihilation and creation operators of \( q \)-deformed photons of a deformed light field and \( : \) denotes normal ordering. We call \( g^{(2)}(0) \) the two-mode \( q \)-correlation degree. Physically, \( g^{(2)}(0) \) is a measure of \( q \)-deformed two-photon correlations in the \( q \)-deformed two-mode field and is related to the \( q \)-deformed two-photon number distributions. A state is said to be two-mode \( q \)-antibunched if

\[
g^{(2)}(0) < 1.
\]

Let us now study the two-mode \( q \)-antibunching effect for the \( k \)-component \( q \)-deformed charge coherent states with \( k \geq 3 \). First, for \( k \geq 3 \), one can easily obtain the following relations:

\[
0 \langle \xi, q, k | K_+^2 K_-^2 | \xi, q, k \rangle_0 = |\xi|^4 (N_{kq}^0)^2 / (N_{kq}^{k-2})^2,
\]

\[
1 \langle \xi, q, k | K_+^2 K_-^2 | \xi, q, k \rangle_1 = |\xi|^4 (N_{kq}^1)^2 / (N_{kq}^{k-1})^2,
\]

\[
l \langle \xi, q, k | K_+^2 K_-^2 | \xi, q, k \rangle_l = |\xi|^4 (N_{kq}^l)^2 / (N_{kq}^{l-2})^2, \quad l = 2, 3, \ldots, k - 1.
\]

According to (52), (53) and (72) – (74), the two-mode \( q \)-correlation degrees of the \( k \)-component \( q \)-deformed charge coherent states can be obtained as follows:

\[
g^{(2)}_0(0) = \frac{0 \langle \xi, q, k | K_+^2 K_-^2 | \xi, q, k \rangle_0}{0 \langle \xi, q, k | K_+ K_- | \xi, q, k \rangle_0^2} = \frac{(N_{kq}^{k-1})^4}{(N_{kq}^0)^2 (N_{kq}^{k-2})^2},
\]

\[
g^{(2)}_1(0) = \frac{1 \langle \xi, q, k | K_+^2 K_-^2 | \xi, q, k \rangle_1}{1 \langle \xi, q, k | K_+ K_- | \xi, q, k \rangle_1^2} = \frac{(N_{kq}^0)^4}{(N_{kq}^1)^2 (N_{kq}^{k-1})^2},
\]

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\[ g_l^{(2)}(0) = \frac{\langle \xi, q, k | K_+^2 K_-^2 | \xi, q, k \rangle_l}{\langle \xi, q, k | K_+ K_- | \xi, q, k \rangle_l^2} = \frac{(N_{kq}^{l-1})^4}{(N_{kq}^{l-2})^2(N_{kq}^l)^2}, \quad l = 2, 3, \ldots, k - 1. \tag{77} \]

Evidently, the following relation exists:

\[ \prod_{j=0}^{k-1} g_j^{(2)}(0) = 1. \tag{78} \]

We shall prove that for \( k \geq 3 \), the \( k \)-component \( q \)-deformed charge coherent states show two-mode \( q \)-antibunching.

From (12) and (75), it follows that

\[ g_0^{(2)}(0) = \frac{f(x)}{x^k \phi(x)}, \]

where

\[ f(x) = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{1}{[kn]![kn+|q|]![km-kn+k-2]![km-kn+k-2+|q|]!} \right\} x^{km}, \]

\[ \phi(x) = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{1}{[kn+k-1]![kn+k-1+|q|]![km-kn+k-1]![km-kn+k-1+|q|]!} \right\} x^{km} \]

and \( x = |\xi|^2 \). For \( k \geq 3 \), we have

\[ \sum_{n=0}^{m} \frac{1}{[kn]![kn+|q|]![km-kn+k-2]![km-kn+k-2+|q|]!} > \sum_{n=0}^{m} \frac{1}{[kn+k-1]![kn+k-1+|q|]![km-kn+k-1]![km-kn+k-1+|q|]!}, \]

and thus \( f(x) > \phi(x) \) when \( x > 0 \). Hence, \( g_0^{(2)}(0) > 1 \) when \( 0 < x \leq 1 \). However, when \( x > 1 \), the following inequality

\[ \frac{f(x)}{x^k \phi(x)} < 1, \text{ i.e., } x^k > \frac{f(x)}{\phi(x)} \]

may have real roots. Consequently, in the region of \( x > 1 \), for arbitrary fixed values of \( q \) and \( q \), there surely exists some range of \( x \) values such that

\[ g_0^{(2)}(0) = \frac{f(x)}{x^k \phi(x)} < 1. \]
To make the above statement clear, we plot $g^{(2)}_0(0)$ against $x$ for various $k$, $q$ and $q$ in Fig. 1.

From (12) and (76), it follows that

$$g^{(2)}_1(0) = \frac{x^k f_1(x)}{\varphi_1(x)},$$

where

$$f_1(x) = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{1}{[kn+1][kn+1+|q|][bm-1]!} \right\} x^{km},$$

$$\varphi_1(x) = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{1}{[kn][kn+|q|][bm-1]!} \right\} x^{km}.$$

Apparently,

$$\sum_{n=0}^{m} \frac{1}{[kn+1][kn+1+|q|][bm-1]!} < \sum_{n=0}^{m} \frac{1}{kn[km-kn+|q|][bm-1]},$$

so that $f_1(x) < \varphi_1(x)$. Therefore, $g^{(2)}_1(0) < x^k$, namely, $g^{(2)}_1(0) < 1$ as $x \leq 1$.

From (12) and (77), it follows that

$$g^{(2)}_l(0) = \frac{f_2(x)}{\varphi_2(x)},$$

where

$$f_2(x) = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{1}{[kn+l-2][kn+l-2+|q|][kn+l]!} \right\} x^{km},$$

$$\varphi_2(x) = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{1}{[kn+l-1][kn+l-1+|q|][kn+l-1]} \right\} x^{km}.$$

Obviously,

$$f_2(x) < \frac{1}{[l-2][l-2+|q|][l]+|q|} \sum_{m=0}^{\infty} (m+1) x^{km},$$

$$\varphi_2(x) > \frac{1}{\{km+l-1\}[km+l-1+|q|]} \frac{1}{\{l-1\}[l-1+|q|]}. $$

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Thus, we obtain

\[
g^{(2)}_l(0) < \frac{[(l-1)][l-1+|q|]}{[l-2][l+|q|][l+|q|]} \sum_{m=0}^{\infty} (m+1)x^{km}.
\]

For \(x < 1\), it reads

\[
\sum_{m=0}^{\infty} (m+1)x^{km} = \frac{1}{(1-x^k)^2}.
\]

Therefore, as \(x < 1\), we get

\[
g^{(2)}_l(0) < \frac{[(l-1)][l-1+|q|]}{[l][l+|q|]} \frac{1}{(1-x^k)^2}.
\]

As a result, if \(x^k \leq 1 - \{[(l-1)][l-1+|q|]/[l][l+|q|]\}^{1/2}\), then

\[
g^{(2)}_l(0) < 1, \quad l = 2, 3, \ldots, k-1.
\]

From the above discussion, we see that for \(k \geq 3\), the two-mode \(q\)-correlation degrees \(g^{(2)}_l(0)\) \((j = 0, 1, \ldots, k-1)\) can be less than 1 over some particular range of \(x\) values. This indicates that two-mode \(q\)-antibunching exists for the \(k\)-component \(q\)-deformed charge coherent states with \(k \geq 3\). The same situation occurs for the even and odd \(q\)-deformed charge coherent states [57]. However, for the \(q\)-deformed charge coherent states we have \(g^{(2)}(0) = 1\) so that no two-mode \(q\)-antibunching exists.

It can be shown that in the limit \(q \rightarrow 1\), the nonclassical properties of the usual \(k\)-component charge coherent states, studied in Ref. [37], are retrieved as expected.

7. Summary

Let us sum up the results obtained in the present paper:

(1) The \(k\)-component \(q\)-deformed charge coherent states, defined as the \(k\) \((k \geq 1)\) orthonormalized eigenstates of both the \(k\)th power of the pair \(q\)-boson annihilation operations.
operator and the charge operator, have been constructed and their (over)completeness proved. Such $q$-deformed states become the usual $k$-component charge coherent states in the limit $q \to 1$. They become the $q$-deformed charge coherent states and the even (odd) $q$-deformed charge coherent states in the two special cases of $k = 1$ and $k = 2$, respectively.

(2) The $k$-component $q$-deformed charge coherent states have been shown to be generated by a suitable average over the $U(1)$-group (caused by the charge operator) action on the product of $q$-deformed coherent states and $k$-component $q$-deformed coherent states.

(3) The $D$-algebra of the SU$_q$(1,1) generators corresponding to the $k$-component $q$-deformed charge coherent states has been realized in a $q$-differential-operator matrix form.

(4) For $k \geq 3$, the $k$-component $q$-deformed charge coherent states have been shown to exhibit two-mode $q$-antibunching, but neither SU$_q$(1,1) squeezing, nor single- or two-mode $q$-squeezing.

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Figure caption

Fig. 1. $g (\equiv g_0^{(2)}(0))$ against $x$ for $k = 3, 4, 5$, with (a) $q = \pm 2, q = 0.9$ and (b) $q = \pm 3, q = 0.8$. 
(a)
