Cosmological implications and structure formation from a time varying vacuum

Spyros Basilakos

Academy of Athens, Research Center for Astronomy & Applied Mathematics, Soronou Efessiou 4, 11-527, Athens, Greece

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ABSTRACT

We study the dynamics of the Friedmann–Lemaitre–Robertson–Walker (FLRW) flat cosmological models in which the vacuum energy varies with time, \( \Lambda(t) \). In this model, we find that the main cosmological functions such as the scale factor of the universe and the Hubble flow are defined in terms of exponential functions. Applying a joint likelihood analysis of the recent Type Ia supernovae data, the cosmic microwave background shift parameter and the baryonic acoustic oscillations traced by the Sloan Digital Sky Survey (SDSS) galaxies, we place tight constraints on the main cosmological parameters of the \( \Lambda(t) \) scenario. Also, we compare the \( \Lambda(t) \) model with the traditional \( \Lambda \) cosmology and we find that the former model provides a Hubble expansion which compares well with that of the \( \Lambda \) cosmology. However, the \( \Lambda(t) \) scenario predicts stronger small scale dynamics, which implies a faster growth rate of perturbations with respect to the usual \( \Lambda \) cosmology, despite the fact that they share the same equation of state parameter. In this framework, we find that galaxy clusters in the \( \Lambda(t) \) model appear to form earlier than in the \( \Lambda \) model.

Key words: cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

The detailed analysis of the available high quality cosmological observations (Riess et al. 1998; Perlmutter et al. 1999; Efstathiou et al. 2002; Basilakos & Plionis 2005; Tegmark et al. 2006; Davis et al. 2007; Kowalski et al. 2008; Komatsu et al. 2009) has converged during the last decade towards a cosmic expansion history that involves a spatial flat geometry and a recent accelerating expansion of the universe. This expansion has been attributed to an energy component (dark energy) with negative pressure which dominates the universe at late times and causes the observed accelerating expansion. The simplest type of dark energy corresponds to the cosmological constant (see for review Peebles & Ratra 2003). The nature of the dark energy is still a mystery and indeed it is one of the most fundamental current problems in physics and cosmology.

In the literature, there are many theoretical speculations regarding the physics of the above exotic dark energy. The simplest approach is to consider a real scalar field \( \phi \) which rolls down the potential energy \( V(\phi) \) and therefore could mimic the dark energy (Ratra & Peebles 1988; Weinberg 1989; Turner & White 1997; Caldwell, Dave & Steinhardt 1998; Padmanabhan 2003). Alternatively, Ozer & Taha (1987) proposed a different scenario in which a time varying \( \Lambda \) parameter could be a possible candidate for the dark energy (see also Bertolami 1986; Freese et al. 1987; Peebles & Ratra 1988; Carvalho, Lima & Waga 1992; Overduin & Cooperstock 1998; Bertolami & Martins 2000; Alcaniz & Maia 2003; Opher & Pellison 2004; Bauer 2005; Barrow & Clifton 2006; Montenegro & Carneiro 2007 and references therein). In this cosmological model, the dark energy equation of state parameter \( w \equiv P_{DE}/\rho_{DE} \), is strictly equal to \(-1\), but the vacuum energy density (or \( \Lambda \)) varies with time. It is interesting to mention here that the renormalization group (RG) in quantum field theory (Shapiro & Solé 2000; Babić et al. 2002) provides a time varying vacuum, in which the \( \Lambda \) component evolves as \( H^2(t) \) [see Grande, Solá & Stefancic 2006], where \( H \) is the Hubble parameter. On the other hand, based on the holographic principle (Bousso 2002; Padmanabhan 2005), one can prove that \( \Lambda \sim H^2 \).

However, in the \( \Lambda(t) \) cosmological model there is a coupling between the time-dependent vacuum and matter (Carneiro et al. 2008). In particular, using the combination of the conservation of the total energy with the variation of the vacuum energy one can prove that the \( \Lambda(t) \) model provides either a process of a particle production or the mass of the dark matter particles increases. The latter general properties can be explained within the framework of the interacting dark energy models (Alcaniz & Lima 2005 and references therein). We would like to stress here that most of the recent papers in dark energy studies are based on the assumption that the dark energy evolves independently of the dark matter. Of course, the unknown nature of both dark matter and dark energy implies that at the moment we cannot exclude the possibility of interactions in the dark sector. The confirmation of such a possibility...
would be of paramount importance because interactions between dark matter and dark energy could provide possible solutions to the cosmological coincidence problem. In general, several papers have been published in this area (e.g. Zimdahl, Pavón & Chimento 2001; Amendola et al. 2003; Cai & Wang 2005; Binder & Kremer 2006; Das, Corasaniti & Khoury 2006; Olivares, Atrio-Barandela & Pavón 2008 and references therein) proposing that the dark energy and dark matter could be coupled.

The aim of the present work is to investigate the observational consequences of the overall dynamics by using the $\Lambda(t)$ cosmological model. Due to the absence of a physically well-motivated functional form for the $\Lambda(t)$ parameter, we consider a power series form in $H$ up to a second order. Doing so, we include the effects of the de-Sitter space–time. The plan of the paper is as follows.

The basic theoretical elements of the problem are presented in Section 2 by solving analytically [for a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) geometry] the basic cosmological equations. In Section 3, we place constraints on the main parameters of our model by performing a joint likelihood analysis utilizing the Union08 Type Ia supernovae (SNIa) data (Kowalski et al. 2008), the shift parameter of the cosmic microwave background (CMB; Komatsu et al. 2009) and the observed baryonic acoustic oscillations (BAOs; Eisenstein et al. 2005; Padmanabhan et al. 2007). Section 4 outlines the comparison between the time varying vacuum model with the traditional $\Lambda$ cosmology. Also, in Section 4, we solve analytically the time evolution equation of the mass density contrast for the $\Lambda(t)$ model while in Section 5 we present theoretical predictions regarding the formation of the galaxy clusters. In Section 6, we draw our conclusions. Finally, in the appendix, we have treated analytically, the basic cosmological equations considering that the time varying $\Lambda(t)$ parameter can be expressed with the aid of a power series expansion in $H$ up to a third order. Note that throughout the paper we use $H_0 = 70.5\, \text{km}\, \text{s}^{-1}\, \text{Mpc}^{-1}$ (Freedman et al. 2001; Komatsu et al. 2009).

## 2 COSMOLOGY WITH A TIME DEPENDENT VACUUM

In the framework of a spatially flat FLRW geometry, the basic equations which governs the global dynamics of the universe are

$$\rho_m + \rho_\Lambda = 3H^2$$

(1)

and

$$\frac{d(\rho_m + \rho_\Lambda)}{dt} + 3H(\rho_m + P_m + \rho_\Lambda + P_\Lambda) = 0,$$

(2)

where $\rho_m$ and $\rho_\Lambda$ are the matter density and vacuum density, respectively, while $P_m = 0$ and $P_\Lambda$ is the corresponding vacuum pressure. Note that for simplicity we use geometrical units ($8\pi G = c = 1$) in which $\rho_\Lambda = \Lambda$. In order to study the above system of differential equations, we need to define explicitly the functional form of the $\Lambda$ component. Within the framework of the $\Lambda(t)$ model, it is interesting to note that the equation of state takes the usual form of $P_\Lambda = -\rho_\Lambda(t) = -\Lambda(t)$ [Ozer & Taha 1987; Peebles & Ratra 1988].

On the other hand, introducing in the global dynamics the above idea in a form of the time-dependent vacuum, it is possible to explain the physical properties of the dark energy. Considering now equation (2), we have the following useful formula (see also Carneiro et al. 2008):

$$\rho_m + 3H\rho_m = -\Lambda$$

(3)

and indeed, using equation (1), we obtain

$$2H + 3H^2 = \Lambda$$

(4)

or

$$\int_{s}^{t} \frac{dy}{y - 3y^2} = \int_{0}^{t} \frac{dt}{2 - \frac{t}{2}},$$

(5)

where the over-dot denotes derivatives with respect to time. Of course, the traditional $\Lambda$ cosmology can be described by the above integration (equation 5) using a constant vacuum term $\Lambda = \text{const}$ (for more details see Section 3.5).

Now, from equation (3), it becomes evident that in this cosmological scenario there is a coupling between the time-dependent vacuum and matter. Actually, the idea for possible interactions in the dark sector is not really new in this kind of studies. It has been shown that the coupling between dark matter and dark energy could provide possible solutions to the cosmological coincidence problem (e.g. Zimdahl et al. 2001; Amendola et al. 2003; Cai & Wang 2005; Binder & Kremer 2006; Das et al. 2006; Olivares et al. 2008 and references therein). In this context, one of the most important issues and unknowns is the precise functional form of the equation of state parameter $w(a)$, where $a$ is the scale factor. The usual procedure is to derive a $w(a)$ approximate functional form, by using a Taylor expansion around the present epoch (e.g. Chevallier & Polarski 2001; Linder 2003), which then provides approximate solutions of the global density evolution. However, the current approach is somewhat different in the sense that we do not ‘design’ the equation of state parameter such that to produce the desired (accelerated) cosmic evolution. Rather, we investigate whether a generalized vacuum component with $w(t) = -1$ and $P_\Lambda = -\rho_\Lambda(t)$ (Ozer & Taha 1987; Peebles & Ratra 1988) in the expanding Universe allows for a late accelerated phase of the Universe and under which circumstances such a solution provides a viable alternative to the dark energy.

Although, we do not have a fundamental theory to model the time-dependent $\Lambda(t)$ function, we can parametrize the latter using a phenomenological approach. Indeed, in a series of recent papers, authors (see e.g. Ray, Mukhopadhyay & Meng 2007; Sil & Som 2008 and references therein) have investigated the global dynamical properties of the universe considering that the vacuum energy density decreases linearly either with the energy density or the square of the energy density. Alternatively, Carneiro et al. (2008) proposed a different pattern in which the vacuum term is proportional to the Hubble parameter, $\Lambda(a) \propto H(a)$. However, this model fails to fit the current CMB data (see also Section 3.4). In this context, attempts to provide a theoretical explanation for the $\Lambda(t)$ have been presented in the literature (see Grande et al. 2006 and references therein). These authors found that a time-dependent vacuum could arise from the RG in quantum field theory. The corresponding solution for a running $\Lambda(t)$ is found to be $\Lambda(t) \sim c_1 H^2(t)$ [where $c_1$ is a constant] and it can mimic the quintessence or phantom behaviour and transit smoothly between the two. It is worth noting that at late enough times the above solution asymptotically reaches the de-Sitter regime $\Lambda \sim H^2$, as far as the global dynamics is concerned.

In this paper, we parametrize the functional form of $\Lambda(t)$ by applying a power series expansion in $H$ up to the second order (see the appendix for a third order expansion which interestingly predict models with late acceleration but without initial singularities):

$$\Lambda(t) = n_1 H + n_2 H^2.$$

(6)

Obviously, equation (6) can be seen as a combination of the above ansatzs namely $H(t)$ (Carneiro et al. 2008) and $H^2(t)$ (quantum field theory; Grande et al. 2006), respectively. It is now routine to integrate equation (5) and obtain the Hubble function predicted by
the current $\Lambda(t)$ model,
\[
H(t) = \frac{n_1}{\beta} e^{\beta t/2} e^{-\beta t/2 - 1},
\]
where the range of $\beta (= 3 - n_2)$ values for which the above integration is valid is $\beta \in (0, +\infty)$ (or $n_2 < 3$). Of course, if we consider different patterns for the vacuum density then we can obtain different solutions for the Hubble parameter. Using now the definition of the Hubble parameter $H \equiv \dot{a}/a$, the scale factor of the universe $a(t)$, evolves with time as
\[
a(t) = a_1 \left(e^{\beta t/2} - 1\right)^{2/\beta},
\]
where $a_1$ is the constant of integration. As expected, at late enough times the above solution reduces to the de-Sitter case. Note that for $\beta \rightarrow 3$ and at early times the $\Lambda(t)$ model tends to the Einstein de-Sitter case. Now from equations (7) and (8), we can easily write the corresponding Hubble flow as a function of the scale factor
\[
H(a) = \frac{n_1}{\beta} \left[1 + \left(\frac{a}{a_1}\right)^{-\beta/2}\right].
\]
Evaluating equation (9) at the present time ($a \equiv 1$), we obtain
\[
n_1 = \frac{\beta H_0}{1 + a_1^{2/\beta}}.
\]
where $H_0$ is the Hubble constant. From equations (9) and (10), using the usual unit-less $\Omega$-parametrization, we have after some algebra that
\[
E(a) = \frac{H(a)}{H_0} = \left(1 - \Omega_m + \Omega_m a^{-\beta/2}\right),
\]
while the corresponding matter density parameter is $\Omega_m(a) = \Omega_m a^{-\beta/2}/E(a)$. The normalized scale factor of the universe becomes
\[
a(t) = (\frac{\Omega_m}{1 - \Omega_m})^{2/\beta} \left[e^{(1-\Omega_m)H_0/2} - 1\right]^{2/\beta},
\]
or
\[
t(a) = \frac{2}{\beta(1 - \Omega_m)} H_0^{-1} \ln \left[\frac{a^{\beta/2} E(a)}{\Omega_m}\right],
\]
where $a_1^{2/\beta} = \Omega_m/(1 - \Omega_m)$. It is interesting to point here that the current age of the universe [at $a = 1$, $E(1) = 1$] is
\[
t_0 = \frac{2}{\beta} H_0^{-1} \ln \Omega_m.\]
We now investigate the circumstances under which an inflection point exists and therefore have an acceleration phase of the scale factor. This crucial period in the cosmic history corresponds to $\ddot{a}(t) = 0$. Differentiating twice equation (12), we then have
\[
a_t = \frac{(\beta - 2)\Omega_m}{2(1 - \Omega_m)}^{2/\beta} t_1 = \frac{2}{\beta(1 - \Omega_m)} H_0^{-1} \ln \left(\frac{\beta}{2}\right),
\]
which implies that the condition for which an inflection point is present in the evolution of the scale factor is $\beta > 2$.

3 COSMOLOGICAL CONSTRAINTS

3.1 The likelihood from the CMB shift parameter
A very accurate and deep geometrical probe of dark energy is the angular scale of the sound horizon at the last scattering surface as encoded in the location of the first peak of the CMB temperature perturbation spectrum. This probe is described by the so-called CMB shift parameter (cf. Bond, Efstathiou & Tegmark 1997; Trotta 2004; Nesseris & Perivolaropoulos 2007) which is a normalized quantity and it is defined as
\[
R = \sqrt{\Omega_m} \int_{z_0}^{1} \frac{dz}{E(z)} = \sqrt{\Omega_m} \int_{z_0}^{z_0} \frac{dz}{E(z)}.
\]
One of the merits of using the shift parameter in cosmological studies is that its dependence on the Hubble constant is negligible (for details see Melchiorri et al. 2003; Nesseris & Perivolaropoulos 2007 and references therein). The shift parameter measured from the Wilkinson Microwave Anisotropy Probe (WMAP) 5-yr data (Komatsu et al. 2009) is $R = 1.71 \pm 0.019$ at $z_0 = 1090$ [or $a_0 = (1 + z_0)^{-1} \approx 9.17 \times 10^{-4}$ and $E(z) = H(z)/H_0$] is the normalized Hubble flow. Therefore, the corresponding $\chi^2_{\text{cmb}}$ function is simply written
\[
\chi^2_{\text{cmb}}(p) = \frac{[R(p) - 1.71]^2}{0.019^2},
\]
where $p$ is a vector containing the cosmological parameters that we want to fit. Note that we sample the unknown parameters as follows: $\Omega_m \in [0.1, 1]$ and $\beta \in [2, 5]$ in steps of 0.01. In Fig. 1(a), we present the $1\sigma$, $2\sigma$ and $3\sigma$ confidence levels in the $(\Omega_m, \beta)$ plane. It is evident that the $\beta$ parameter is tightly constrained ($\beta \approx 3.58$) while the matter density parameter is not and all the values in the interval $0.1 \leq \Omega_m \leq 1$ are acceptable (see Table 1). However, following the WMAP 5-yr results (Komatsu et al. 2009) of the full temperature perturbation spectrum $\Delta T/T$, we can use an additional constrain which is $\Omega_m h^2 = 0.1326 \pm 0.0063$. Thus, for $h \approx 0.71$ (Freedman et al. 2001; Komatsu et al. 2009), we find $0.24 \leq \Omega_m \leq 0.29$ ($2\sigma$ limits).
Table 1. Results of the likelihood function analysis. The 1st column indicates the data used (the last two rows corresponds to the inflection points). Errors of the fitted parameters represent 1σ uncertainties. Finally, the current age of the universe $t_0$ has units of Gyr (for $H_0 = 70.5$ km s$^{-1}$ Mpc$^{-1}$).

| Sample        | $\Omega_m$  | $\beta$     | $t_0$ | $a_1$ | $t_1/t_0$ |
|---------------|-------------|-------------|-------|-------|-----------|
| CMB           | uncons. ($\Omega_m = 0.13$) | 3.58$^{+0.04}_{-0.32}$ | 18.2  | 0.30  | 0.29      |
| SNIa          | 0.20$^{+0.12}_{-0.01}$ | uncons. ($\beta = 4.6$) | 12.4  | 0.59  | 0.50      |
| BAO           | 0.28$^{+0.11}_{-0.04}$ | uncons. ($\beta = 3.50$) | 16.0  | 0.36  | 0.33      |
| SNIa-BAO      | 0.28$^{+0.03}_{-0.03}$ | 3.50$^{+0.30}_{-0.36}$ | 14.0  | 0.49  | 0.44      |
| CMB-BAO       | 0.29$^{+0.03}_{-0.03}$ | 3.44$^{+0.02}_{-0.02}$ | 14.1  | 0.49  | 0.44      |
| SNIa-CMB      | 0.29$^{+0.01}_{-0.03}$ | 3.44$^{+0.02}_{-0.02}$ | 14.1  | 0.49  | 0.44      |
| ALL           | 0.29$^{+0.01}_{-0.02}$ | 3.44$^{+0.02}_{-0.02}$ | 14.1  | 0.49  | 0.44      |

3.2 The likelihood from the SNIa

We now use the publicly available Union08 sample of 307 supernovae of Kowalski et al. (2008) in order to constrain $\Omega_m$. In this case, the likelihood function can be written as

$$\chi^2_{\text{SNIa}}(p) = \sum_{i=1}^{307} \frac{\left[\mu^b(a_i, p) - \mu^{ob}(a_i)\right]^2}{\sigma_i^2},$$

where $a_i = (1 + z_i)^{-1}$ is the observed scale factor of the universe, $z_i$ is the observed redshift, $\mu$ is the distance modulus $\mu = m - M = 5 \log d_L + 25$ and $d_L(a, p)$ is the luminosity distance

$$d_L(a, p) = \frac{c}{H_0} \int_x^1 \frac{da}{E(x)},$$

where $c$ is the speed of light (≡ h) here. Fig. 1(a) also shows the 1σ, 2σ and 3σ confidence levels in the $(\Omega_m, \beta)$ plane. Although, the $\beta$ parameter is not constrained by this analysis the matter density parameter has an upper limit of $\Omega_m \leq 0.29$ within the 1σ uncertainty (see Table 1).

3.3 The likelihood from BAOs

In this section, we utilize the so-called BAOs. BAOs are produced by pressure (acoustic) waves in the photon-baryon plasma in the early universe, generated by dark matter overdensities. Evidence of this excess has been found in the clustering properties of the luminous Sloan Digital Sky Survey (SDSS) red galaxies (Eisenstein et al. 2005; Padmanabhan et al. 2007) and it can provide a ‘standard ruler’ with which we can put constraints on the cosmological models.

1 http://supernova.lbl.gov/Union. This catalogue includes the following components in the error budget of the distance moduli: (a) $\sigma_{\text{sys}}$, (see Kowalski et al. 2008) obtained due to lensing, Milky way dust extinction and host galaxy peculiar velocities (b) the systematic error $\sigma_{\text{sys}}$, and (c) the uncertainty which is related with the light-curve fitting. For the latter uncertainty, Kowalski et al. take into account the stretch $s$ and colour $c$ corrections via $\mu_g = m_g - M + \alpha (s - 1) - bc$ for a specific cosmological model ($\Omega_m, w$) = (0.29, -0.97). We would like to caution the reader that we do not minimize $\chi^2$ over the parameters $a$ and $b$, which implies that in the case of the $\Lambda(t)$ model, we may not be treating the third component of the error budget properly. However, we would like to stress that according to Kowalski et al. (2008) the corresponding constants $a$ and $b$ are rather insensitive to the assumed cosmological parameters (see their Section 5.1). Thus, had we included the proper light-curve uncertainty in our fit we would have obtained a larger solution space [see Fig. 1(a)].

particular, we use the following estimator:

$$A(p) = \sqrt{\Omega_m} \left( \int_a^{1} \frac{da}{a^2 E(a)} \right)^{2/3}$$

measured from the SDSS data to be $A = 0.469 \pm 0.017$, where $z_i = 0.35$ [or $a_i = (1 + z_i)^{-1} \approx 0.75$]. In this case, the $\chi^2_{\text{BAO}}$ function is given

$$\chi^2_{\text{BAO}}(p).$$

It is evident [see Fig. 1(a)], that the matter density parameter is constrained ($\Omega_m \approx 0.28$) by this analysis, while the $\beta$ parameter is not visible (see also Table 1).

3.4 The joint likelihoods

We can combine the above probes by using a joint likelihood analysis,

$$L_\text{tot}(p) = L_{\text{BAO}} \times L_{\text{cmb}} \times L_{\text{SNIa}}$$

or

$$\chi^2_{\text{tot}}(p) = \chi^2_{\text{BAO}} + \chi^2_{\text{cmb}} + \chi^2_{\text{SNIa}},$$

in order to put even further constraints on the parameter space used. Note, that we define the likelihood estimator as $L_j \propto \exp(-\chi^2_j/2)$. The resulting best-fitting parameters are presented in the second two rows of Table 1. The overall likelihood function peaks at $\Omega_m = 0.29^{+0.01}_{-0.02}$, $\beta = 3.44 \pm 0.02$ and the corresponding $\chi^2_{\text{tot}}$ ($\Omega_m, \beta$) is 310.2 (d.o.f. = 307). In this cosmological scenario, the current age of the universe is found to be $t_0 \approx 14.1$ Gyr and the inflection point is located at $(a_1, t_1) \approx (0.49, 0.44t_0)$. In Figs 1(b)–(d), we present, for various observational pairs, the corresponding likelihood contours (see also Table 1).

Finally, it is worth noting that Carneiro et al. (2008) considered a different ansatz in order to parametrize the time dependence of the vacuum energy. Their assumption is based on the fact that $\Lambda(t)$ is proportional to the Hubble parameter (in our formulation $n_2 = 0$). They found that this model fits the observational data (BAO+SNIa+CMB) at 2σ level for $\Omega_m \approx 0.43$. In our case, if we marginalize over $\beta = 3$ (or $n_2 = 0$), then the joint likelihood analysis provides a best-fitting value of $\Omega_m \approx 0.35$, but the fit is rather poor $\chi^2_{\text{tot}}(\Omega_m) \approx 383$ (d.o.f. = 308). We investigate a bit

2 Likelihoods are normalized to their maximum values. Note that the errors of the fitted parameters represent 1σ uncertainties.
further this result and we reveal that the poor joint fit is due to the fact that the best-fitting value provided by the likelihood analysis of CMB shift parameter is found to be more than 3σ away, Ω_m ≃ 0.80 (see solid line in Fig. 2), from the SNIa/BAO solution Ω_m ≃ 0.32 (see dashed line in Fig. 2). This implies that the functional form E(a) = 1 − Ω_m + Ω_mα^{−3/2} fails to fit the CMB data. We thus argue that the Λ(t) ∝ H(t) relation produces a discrepancy between the SNIa/BAO and CMB shift parameter which may lead to misleading cosmological results. We further confirm the latter result, by using a Bayesian statistics (see for example Davis et al. 2007), in which the corresponding estimator is defined as BIC = χ^2 + k ln N (where k is the number of parameters and N is the number of data points used in the fit) BIC = Bayesian Information Criterion. The next step is to estimate the relative deviation between the two models ∆BIC = BIC_{Λ}(a) − BIC_{Λ+1}(a). In general, a difference in BIC of ∆BIC > 6 is considered strong evidence against that model which occurs the larger ∆BIC. In our case, we find ∆BIC ≃ 69 which implies a strong evidence against the Λ(t) ∝ H(t) model.

### 3.5 The standard Λ cosmology

In this section, we wish to remind the reader of some basic elements of the concordance Λ cosmology in order to appreciate the differences with the Λ(t) cosmology. In the case of Λ = const, it is straightforward to integrate equation (5). Therefore, the Hubble function predicted by the Λ model is

$$H(t) = \frac{\Lambda}{3} \coth \left( \frac{3}{2} \sqrt{\frac{\Lambda}{3}} t \right)$$

where Λ = 3H_0^2 (1 − Ω_m). Then the normalized Hubble function is written as

$$E_Λ(a) = \frac{H(a)}{H_0} = [1 − Ω_m + Ω_m a^{−3}]^{1/2}$$

while Ω_m(a) = Ω_m a^{−3} E_Λ^2(a). To this end, the scale factor of the universe is given by

$$a_Λ(t) = \left( \frac{Ω_m}{1 − Ω_m} \right)^{1/3} \sinh^{2/3} \left( \frac{\sqrt{3} H_0 \sqrt{1 − Ω_m} t}{2} \right)$$

or

$$t_Λ = \frac{2}{3 \sqrt{1 − Ω_m} H_0^{-1}} \ln \left[ \sqrt{1 − Ω_m + E_Λ(a)} \left( a^{−3/2} \sqrt{Ω_m} \right) \right]$$

Comparing the Λ model with the observational data, we find that the best-fitting value is Ω_m = 0.28 ± 0.02 with χ^2_{min}(Ω_m) ≃ 308.5 (d.o.f. = 308) in a good agreement with the 5-yr WMAP data (Komatsu et al. 2009). Note that Davis et al. (2007) using the Union08–SNIa+BAO+CMB and a Bayesian statistics found Ω_m = 0.27 ± 0.04, while Kowalski et al. (2008) utilizing the Union08–SNIa+BAO+CMB obtained Ω_m = 0.274^{+0.016+0.013}_{−0.016−0.012} (for w ∼ −1). Obviously, our results coincide within 1σ errors.

The current age of the universe is given by

$$t_Λ = \frac{2}{3 \sqrt{1 − Ω_m} H_0^{-1}} \ln \left( \frac{\sqrt{1 − Ω_m + 1}}{\sqrt{Ω_m}} \right)$$

while the inflection point takes place at

$$t_Λ = \frac{2}{3 \sqrt{1 − Ω_m} H_0^{-1}} \ln \left( \frac{\sqrt{3} + 1}{2} \right)$$

Therefore, we estimate t_Λ ≃ 13.9 Gyr, t_Λ ≃ 0.52Ω_m and a_Λ ≃ 0.58. Finally, using the previously described Bayesian statistics, we find that ∆BIC = BIC_{Λ+1}(a) − BIC_{Λ} ≃ 5. This comparison implies a preference for the usual Λ cosmology.

### 4 Comparison between different types of vacuum

In this section, we investigate in more detail the correspondence of the Λ(t) model with the traditional Λ cosmology (see previous sections) in order to show the extent to which they compare.

#### 4.1 Compare the cosmic evolution

Knowing now the parameter space (Ω_m, β), we present the evolution of the Λ(t) scale factor seen in the upper panel of Fig. 3 as the solid line. It can be seen that it closely resembles the corresponding scale factor of the Λ cosmology (dashed line). We have checked the cosmic phases of the Λ(t) scenario against the concordance cosmology by utilizing the deceleration parameter, q(a) = −(1 − a d ln H/da). The evolution of the deceleration parameter is presented in the bottom panel of Fig. 3, while in the insert figure we plot the relative deviation of the deceleration parameter, ∆(q − q_Λ), between the two vacuum models. We find the following phases: (a) at early enough times a < 0.21 the deceleration parameters are both positive with q > q_Λ, which means that the cosmic expansion in the Λ(t) model is more rapid decelerated than in the Λ case, (b) between 0.21 < a < 0.49 the deceleration parameters remain positive but q < q_Λ, (c) then for 0.49 < a < 0.58 the traditional Λ model remains in the decelerated regime q_0 > 0, but the Λ(t) is starting to accelerate q < 0 and (d) for 0.58 < a < 0.80, the deceleration parameters are both negative and as long as q < q_Λ, the Λ(t) model predicts a much more acceleration than in the Λ model (the opposite situation seems to hold prior to the present epoch 0.80 < a ≤ 1). In a special case, where ∆(q − q_Λ) = 0 [q = q_Λ, either at a ≃ 0.21 or a ≃ 0.80], the two vacuum models predict exactly the same expansion of the universe. From Fig. 3, it becomes clear that the Λ(t) model reaches a maximum deviation from the Λ cosmology prior to a ∼ 0.1 (z ∼ 9) and a ∼ 0.45 (z ∼ 1.2). Therefore, in order to investigate whether the expansion of the observed universe follows one of the above possibilities, we need a robust extragalactic distance indicator at redshifts z > 1.2. Finally, the deceleration parameters at the present time are q_0 ≃ −0.50 and q_0 ≃ −0.57.
4.2 Compare the linear growth factor

In the framework of a time varying vacuum, the corresponding time evolution equation for the mass density contrast, in a pressureless fluid is given by (Arcuri & Waga 1994; see also Borges et al. 2008)

\[ D + (2H + Q)D - \left[ \frac{\rho_m}{2} - 2HQ - Q \right] D = 0, \]  

(28)

where \( \rho_m = 3H^2 - \Lambda \) (see equation 1) and \( Q(t) = -\Lambda/\rho_m \). It becomes clear, that the interacting vacuum energy affects the growth factor via the function \( Q(t) \). Obviously, in the case of a constant \( \Lambda [Q(t) = 0] \), the above equation reduces to the usual time evolution equation for the mass density contrast (see Peebles 1993). In this context, the growing solution as a function of redshift is given by

\[ D_\Lambda (z) = \frac{5\Omega_m E_\Lambda (z)}{2} \int_{z}^{\infty} \frac{(1 + x) E_m^2 (x)}{x} dx. \]

(29)

We now proceed in an attempt to analytically solve equation (28). To do so, we change variables from \( t \) to a new one following the transformation

\[ y = \exp(n t/2) \quad 0 < y < 1. \]

(30)

Doing so, equation (28) can be written

\[ \beta^2 y (y - 1)^2 D' + 2\beta (y - 1)(5y - \beta) D' - 2(6 - \beta)(\beta - 2y) D = 0, \]

(31)

where \( \beta \) prime denotes derivatives with respect to \( y \). We find that equation (31) has a decaying solution for \( \beta < 8 \) of the form \( D_1 (y) = (y - 1)^{(6 - \beta)/\beta} \). The second independent solution (growing mode) of equation (31) can be found easily from the following expression

\[ D(y) = D_1 (y) \int_{y}^{1} \frac{(a - 1)^{2/\beta} da}{u^2}. \]

(32)

Inserting equations (10) and (13) into equation (30), the \( y \) variable is related with the scale factor as

\[ y = \frac{a^{6/\beta}(1 - \Omega_m) + \Omega_m}{\Omega_m}. \]

(33)

In the redshift regime \( [a = (1 + z)^{-1}] \), the combination of the above two equations leads to the following growing mode

\[ D(z) = C(\Omega_m)(1 + z)^{(6 - \beta)/2} \int_{z}^{\infty} \frac{(x + 1)^{(6 - 4)/2} dx}{E^2(x)}, \]

(34)

where

\[ C(\Omega_m) = \frac{\beta}{2} \Omega_m^{\beta} \left( \frac{1 - \Omega_m}{\Omega_m} \right)^{\left(2 - 4/\beta\right)}. \]

(35)

In the upper panel of Fig. 4, we present the growth factor evolution, derived by integrating equations (29) and (34), for the two vacuum models. Note that the growth factors are normalized to unity at the present time. Despite the fact that the global cosmological behaviour of the \( \Lambda(t) \) vacuum model is in a good agreement with the usual \( \Lambda \) cosmology (as it seen in Fig. 3), the two vacuum cosmological models trace differently the evolution of the matter fluctuation field. In particular, close to the present epoch \( (z < 0.3) \) the \( \Lambda(t) \) growth factor reaches a plateau, which means that the matter fluctuations are effectively frozen. It is obvious that the growth factor in the \( \Lambda(t) \) model is much greater than that of the concordance \( \Lambda \) cosmology. Indeed, assuming that clusters of galaxies have formed prior to the epoch of \( z \sim 1.4 (a \sim 0.42) \), in which the most distant cluster has been found (Mullis et al. 2005; Stanford et al. 2006), the deviation \( (1 - D/D_\Lambda) \) per cent, of the growth factor \( D(a) \) for the
Table 2. Cosmological data of the growth rate of clustering (see Nesseris & Perivolaropoulos 2008). The correspondence of the columns is as follows: redshift, observed growth rate and references.

| $z$   | $f_{\text{obs}}$ | Refs.             |
|-------|------------------|-------------------|
| 0.15  | 0.51 ± 0.11      | Verde et al. (2002); Hawkins et al. (2003) |
| 0.35  | 0.70 ± 0.18      | Tegmark et al. (2006) |
| 0.55  | 0.75 ± 0.18      | Ross et al. (2007) |
| 1.40  | 0.90 ± 0.24      | da Angela et al. (2008) |
| 3.00  | 1.46 ± 0.29      | McDonald et al. (2005) |

$\Lambda(t)$ scenario with respect to the $\Lambda$ solution $D_{\Lambda}(a)$ is $-51$ per cent while prior to the inflection point ($a_{i} \sim 0.5$) we find $-43$ per cent. We conclude that the behaviour of the growth factor is sensitive to the different types of vacuum with $D(z) > D_{\Lambda}(z)$ and it is expected that this difference will affect also the predictions related with the formation of the cosmic structures (see Section 5).

4.3 Compare the growth rate of clustering

We further compare the two vacuum cosmological scenarios by utilizing the well-known indicator of clustering, namely the growth rate $f(a) \equiv d \ln D/d \ln a$ (Peebles 1993). The corresponding parametrization of the growth rate of clustering can be achieved by introducing a growth index $\gamma$ (see Wang & Steinhardt 1998) defined by

$$f(a) = \Omega_{m}^{\gamma}(a).$$  

(36)

In order to quantify the growth index, we perform a standard $\chi^{2}$ minimization procedure (described previously) between the measured growth rate of the 2dF (Two Degree Field) and SDSS catalogues [see Table 2; Nesseris & Perivolaropoulos (2008)] with those expected in our spatially flat cosmological models

$$\chi^{2}(\gamma) = \sum_{i=1}^{5} \left( f_{\text{obs}}(z_{i}) - f_{\text{model}}(z_{i}, \gamma) \right)^{2} / \sigma_{i},$$  

(37)

where $\sigma_{i}$ is the observed growth rate uncertainty. In the bottom panel of Fig. 4, we present the measured $f_{\text{obs}}(z)$ (filled symbols) with the estimated growth rate function $f(z) = \Omega_{m}^{\gamma}(z)$ for the considered cosmological models. Note that for the $\Lambda(t)$ model (solid line), we use $(\Omega_{m}, \beta) = (0.29, 3.44)$ and for the $\Lambda$ case (dashed line) we impose $\Omega_{m} = 0.28$. Also, in the insert panel of Fig. 4, we plot the variation of $\Delta \chi^{2} = \chi^{2}(\gamma) - \chi^{2}_{\text{min}}(\gamma)$ around the best $\gamma$ fit. We find that the growth index is $\gamma = 0.50^{+0.14}_{-0.13}$ (in the case of the $\Lambda(t)$, which is somewhat less (but still within $1\sigma$ errors) than the $\Lambda$ growth index, $\gamma_{\Lambda} = 0.62^{+0.15}_{-0.13}$).

5 THE FORMATION OF GALAXY CLUSTERS

In this section, we attempt to briefly investigate the cluster formation processes by generalizing the basic equations which govern the behaviour of the matter perturbations within the framework of a $\Lambda(t)$ flat cosmology. Also we compare our predictions with those found for the traditional $\Lambda$ cosmology. This can help us to understand better the theoretical expectations of the $\Lambda(t)$ cosmological scenario as well as the variants from the $\Lambda$ model.

The concept of estimating the fractional rate of cluster formation has been brought up by different authors (cf. Peebles 1984; Weinberg 1987; Martel & Wasserman 1990; Richstone, Loeb & Turner 1992). The above authors introduced a methodology which computes the rate at which mass joins virialized structures, which grow from small initial perturbations in the universe. In particular, the basic tool is the so-called Press–Schechter formalism which considers the fraction of mass in the universe contained in gravitationally bound structures (such as galaxy clusters) with matter fluctuations greater than a critical value $\delta_c$. Assuming that the density contrast is normally distributed with zero mean and variance $\sigma^2(\delta)$ we have

$$dP(\delta) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{\delta^2}{2\sigma^2(\delta)} \right] d\delta,$$  

(38)

where $\delta_c$ is the linearly extrapolated density threshold above which structures collapse, i.e., $\delta_c = 1.686$. Note, that it has been shown that $\delta_c$ depends only weakly on $\Omega_m$ (Eke, Cole & Frenk 1996). In this kind of studies, it is common to parameterize the rms mass fluctuation amplitude at 8 h$^{-1}$ Mpc which can be expressed as a function of redshift as $\sigma(M, z) = \sigma_s(z) = D(z)\sigma_s$. The current cosmological models are normalized by the analysis of the WMAP 5-yr data $\sigma_s = 0.80$ (Komatsu et al. 2009). The integration of equation (38) provides the fraction of the universe, on some specific mass scale, that has already collapsed producing cosmic structures (galaxy clusters) at redshift $z$ and is given by (see also Richstone et al. 1992)

$$P(z) = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{\delta_c}{\sqrt{2}\sigma(z)} \right) \right].$$  

(39)

Obviously the above generic of form equation (39) depends on the choice of the background cosmology. The next step is to normalize the probability to give the number of clusters which have already collapsed by the epoch $z$ (cumulative distribution), divided by the number of clusters which have collapsed at the present epoch ($z = 0$), $F(z) = P(z) / P(0)$. In Fig. 5, we present in a logarithmic scale the behaviour of normalized cluster formation rate as a function of redshift for the two cosmological models. In particular, for the traditional $\Lambda$ cosmology we find the known behaviour in which galaxy clusters appear to be formed at high redshifts $z \sim 2$ (see e.g. Basilakos 2003 and references therein), while the same general picture seems to hold for the $\Lambda(t)$ model. However, in the latter case, we find the following differences: (i) clusters appear to form earlier ($z \sim 3.5$) with respect to the $\Lambda$ model and (ii) prior to $z \sim 0.4$ the cluster formation has terminated due to the fact that the matter fluctuation field effectively freezes (see Section 3.4). It is worth noting that the different formation rates between the two vacuum models are due to the fact that the evolution of the corresponding growth factors is different (see the upper panel of Fig. 4). Finally, for

\[\text{Figure 5. Theoretical predictions of the fractional rate of cluster formation as a function of redshift. Note that the solid and dashed line corresponds to the different formation rates between the two vacuum types.}\]
a higher $\sigma_8$ value ($\sigma_8 = 0.95$), the corresponding cluster formation rate moves to higher redshifts [see Fig. 5: $\Lambda(t)$-open triangles and $\Lambda$-open points]. The opposite situation is true for $\sigma_8 < 0.80$.

6 CONCLUSIONS

In this paper, we study analytically and numerically the large and small scale dynamical properties of the FLRW flat cosmologies in which the ‘vacuum’ energy is a function of the cosmic time $\Lambda(t)$. Assuming that the vacuum component can be expressed as a power series $\Lambda = n_1 H + (3 - \beta)H^2$, we find that the time evolution of the basic cosmological functions are described in terms of exponential functions which can accommodate a late-time accelerated expansion, equivalent to the standard $\Lambda$ model. Performing, a joint likelihood analysis using the current observational data (SNIa, CMB anisotropies, equivalent to the standard $\Lambda$ model), we put tight constraints on the main cosmological parameters of the $\Lambda(t)$ model. In particular, we find $\Omega_m \approx 0.29, \beta \approx 3.44$ and the age of the universe is $t_0 \approx 14.1$ Gyr ($H = H_0/100 \approx 0.705$). Also, we compare the $\Lambda(t)$ scenario with the traditional $\Lambda$ cosmology. We find that the behaviour of the global expansion in the $\Lambda(t)$ model compares well with that of the usual $\Lambda$ cosmology. However, there are differences especially when we consider the small-scale dynamics. Indeed, we reveal that the $\Lambda(t)$ cosmological model has two important differences over the considered $\Lambda$ cosmology.

(i) The amplitude and the shape of the linear growth of perturbations are different with respect to the $\Lambda$ solution. As an example, prior to the inflection point the $\Lambda(t)$ growth factor increases by a factor of $\approx 43$ per cent. In this context, the growth index of clustering ($\gamma \approx 0.50$) is somewhat different with that of the $\Lambda$ model ($\gamma_\Lambda \approx 0.62$).

(ii) The large-scale structures (such as galaxy clusters) form earlier ($z \approx 3.5$) with respect to those produced in the framework of the concordance $\Lambda$ model ($z \approx 2$).

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APPENDIX

In this appendix, we treat analytically, as much as possible, the problem of the time varying $\Lambda(t)$ parameter with the aid of a power series in $H$ up to a third order: $\Lambda = n_1 H + n_2 H^2 + n_3 H^3$ ($n_1 \neq 0$). The time evolution equation for the Hubble flow is
obtained by equation (5) as

$$\int_{y(\rho_2)}^{\rho_1} \frac{dy}{y(y^2 - \beta y + n_1)} = \frac{t}{2},$$  \hspace{1cm} (A1)$$

where \( \beta = 3 - n_2 \). In particular, the discriminant \( D = \beta^2 - 4n_1n_3 \)
characterizes the solutions of equation (A1) as

(i) Case 1: \( D > 0 (\beta^2 - 4n_1n_2 > 0) \): the corresponding general
solution of equation (A1) is written

$$\ln \left\{ \frac{H_1(\rho_1, \beta)}{(H_1 - \rho_1)^{1/2}} \right\} = \frac{n_3 t}{2},$$  \hspace{1cm} (A2)$$

where \( \rho_{1,2} = \frac{\beta \pm \sqrt{D}}{2n_3} \neq 0 \). As an example for \( \beta = 0 \) (or \( n_2 = 3, \rho_1 = -\rho_2 \)) we obtain

$$H(t) = \frac{\rho_2}{\sqrt{1 - e^{\beta \rho_2 t}}},$$  \hspace{1cm} (A3)$$

and

$$a(t) = a_1 \left( \frac{1 + \sqrt{1 - e^{\beta \rho_2 t}}}{1 - \sqrt{1 - e^{\beta \rho_2 t}}} \right)^{-1/(n_3 \rho_2)}.$$

(A4)

where \( n_3 < 0 \) and \( \rho_2 > 0 \). We would like to point out that as long
as the cosmic time takes large values \( t \gg 1 \), the \( \Lambda(t) \) model has
the de-Sitter feature due to \( a(t) \sim e^{\beta t} \).
On the other hand, it is very
interesting the fact that this model has no initial singularity. Indeed,
for \( t \rightarrow 0 \), we get \( a(t) \rightarrow a_1 \).

Now, if we consider \( \beta \neq 0 \) then the situation becomes complicated
(see equation A2) but for the special case of \( \rho_1 = 2\rho_2 \), we can derive
the following analytical solutions:

$$H(t) = \rho_2 + \frac{\rho_2}{\sqrt{1 - e^{\beta \rho_2 t}}}$$  \hspace{1cm} (A5)$$

and

$$a(t) = a_1 e^{\beta t} \left( \frac{1 + \sqrt{1 - e^{\beta \rho_2 t}}}{1 - \sqrt{1 - e^{\beta \rho_2 t}}} \right)^{-1/(n_3 \rho_2)}.$$

(A6)

where \( n_3 < 0, \rho_2 > 0 \) and \( \beta < 0 \). Again, the \( \Lambda(t) \) model asymptotically
reaches the de-Sitter regime \( a(t) \sim e^{\beta t} \), while for \( t \rightarrow 0 \),
we again find no singularity \( a(t) \rightarrow a_1 \).

(ii) Case 2: \( D = 0 (\beta^2 = 4n_1n_2) \): in this case, the integration
of equation (A1) leads to the solution of

$$\ln \left( \frac{H}{H - \rho} \right) - \frac{\rho}{H - \rho} = \frac{\rho^2 n_3 t}{2},$$  \hspace{1cm} (A7)$$

where \( \rho = \beta/2n_3 \neq 0 \). Now if \( \beta = 0 \) (\( \rho = 0 \)), which implies that
\( n_1 = 0 \), then the solution of equation (A1) is given by

$$H(t) = \sqrt{\frac{1}{n_3 t}}, \hspace{0.5cm} n_3 < 0$$

(A8)

and

$$a(t) = a_1 e^{\beta/\sqrt{2n_3}}.$$

(A9)

(iii) Case 3: \( D < 0 (\beta^2 - 4n_1n_2 < 0) \): in this case, the integration
of equation (A1) leads to the solution of

$$\ln \left( \frac{n_3 H^2}{n_3 H^2 - \beta H + n_1} \right) + \frac{2\beta}{\sqrt{-D}} \left[ \tan^{-1} \frac{G(H)}{2} \right] = n_1 t,$$

(A10)

where \( G(H) = (2n_3 H - \beta)/\sqrt{-D} \).

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