The loop homology algebra of discrete torsion

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Abstract

We show that Lupercio-Uribe-Xicoténcatl’s orbifold loop product and coproduct can be described by a group cohomology class in some cases. By computing this cohomology class, we show that the orbifold loop product is a non-trivial orbifold invariant.

1 Introduction

Let $M$ be a closed oriented manifold. In their seminal work [5], Chas-Sullivan show that there exist a graded commutative associative product structure on the free loop space homology:

\[ H_p(LM; \mathbb{Z}) \otimes H_q(LM; \mathbb{Z}) \to H_{p+q-\dim M}(LM; \mathbb{Z}). \]  

(1.1)

This product is called the \textit{loop product} and plays a central role in many consider- able works on rich algebraic structures on $H_*(LM; \mathbb{Z})$, so called \textit{String topology}. These study have uncovered a deep relationship with many areas of mathematics including Riemannian geometry, low dimensional topology, and mathematical physics. To develop string topology for orbifolds is thus of great worth and is one of the significant problem in algebraic topology. Let $G$ be a finite group acting smoothly on $M$. It is well known that the homotopy type of the Borel construction $M_G := M \times_G EG$ is an orbifold invariant of the global quotient orbifold $[M/G]$. In [10] Lupercio-Uribe-Xicoténcatl show that there exist a graded commutative associative product structure, named the \textit{orbifold loop product}:

\[ H_p(LM_G; k) \otimes H_q(LM_G; k) \to H_{p+q-\dim M}(LM_G; k), \]  

(1.2)

for all coefficient fields $k$ satisfying that its characteristic is coprime to the order of $G$. They also constructed some rich algebraic structures analogous to string topology, and they coined \textit{Orbifold string topology} for the study of these structures. When the action of $G$ on $M$ is free, it is known that $M_G$ is homotopy equivalent to the naive quotient $M/G$. Then the all algebraic structures under consideration coincide with the counterpart in ordinary string topology. Similar to the ordinary case, the orbifold loop product plays a central role in orbifold string topology. Furthermore, it is shown in [10] that the orbifold loop product is an orbifold invariant of $[M/G]$. Hence we can expect the orbifold loop homology $H_*(LM_G; k)$ to be a strong tool to study orbifolds as well as the other cohomology theories for
orbifolds like Chen-Ruan cohomology and stringy theory which are in progress by many authors ([1]). However, it is hard to check whether the orbifold loop product is indeed a non-trivial invariant. In the same paper, Lupercio-Uribe-Xicoténcatl computes this product structure for lens spaces. In [2] the author himself refined their method and computed the product structure for a wide class of orbifolds. Their computation show that the orbifold loop product has no information on the action of $G$ on $M$ for the orbifolds they considered, because the loop homology $H_*(LM_G; k)$ turns out to split as an algebra into the tensor product of the ordinary loop homology $H_*(LM; k)$ with the center of group ring $Z(k[G])$.

In this article, we describe the algebra structure of $H_*(LM_G; k)$ as an algebra over the ordinary loop homology ring $H_*(LM; k)$ for some kinds of orbifolds including those which is under consideration in [2]. Then it turns out that the orbifold loop product is a non-trivial orbifold invariant. We also determine the coalgebra structure of $H_*(LM_G; k)$ for these orbifolds, which is defined in [8]. In [8], they show that this coalgebra structure is compatible with the orbifold loop product and these forms a nearly Frobenius algebra, namely a two dimensional topological quantum field theory without the counit. In our case, the resulting nearly Frobenius algebra is a variation of the algebra of discrete torsion which is studied pure algebraically by R. Kaufmann in [9].

Now we state our main results. The first part reveal the linear structure of $H_*(LM_G; k)$ under some conditions on the action of $G$. We assume that the action of $G$ on $M$ is homotopically trivial in the meaning that each $g \in G$ is homotopic to the identity map $id_M$. We identify $G$ with its image in aut$_1 M$, the identity component of the self-mapping space of $M$. If $g \in G$ is homotopic to the identity map by a homotopy $H_g : X \times [0, 1] \rightarrow M$ with $H(m, 0) = x$ 
$H(m, 1) = g(m)$,
we have a homotopy equivalence map $\tau_{H_g} : P_g M \rightarrow LM$ defined as $\tau_{H_g}(\sigma)(t) = \begin{cases} \sigma(2t) & (0 \leq t \leq 1/2) \\ H_g(\sigma(0), 2 - 2t) & (1/2 \leq t \leq 1) \end{cases}$. Its homotopy inverse map $\eta_{H_g}$ is defined as $\eta_{H_g}(l)(t) = \begin{cases} l(2t) & (0 \leq t \leq 1/2) \\ H_g(l(1), 2t - 1) & (1/2 \leq t \leq 1) \end{cases}$. We set $\tau_{H_g^0} := \bigsqcup_{g \in G} \tau_{H_g}$. We remark that a homotopy $H_g : M \times [0, 1] \rightarrow M$ can be seen as a path in aut$_1 M$ which connects the identity map and $g$. We denote the order of a group $K$ by $|K|$, and the subgroup of torsion elements in $K$ by $K^{tor}$. We use the notation $Z(A)$ for the center of an algebra $A$. The following is stated as Theorem 3.2 in this article.

**Theorem.** If there exists a sub H-space $S$ of aut$_1 M$ containing $G$ and satisfying

(A) $S$ is path-connected,
then there is a path family \( \theta_G = \{ \theta_g : [0, 1] \to S \}_{g \in G} \) with \( \theta_g(0) = id \) and \( \theta_g(1) = g \), such that the associated homotopy equivalence

\[
\tau_{\theta_G} : P_G M \simeq LM \times G,
\]

is \( G \)-equivariant up to homotopy. Here, the action of \( G \) on the RHS above is defined by \((l, g) \cdot h = (lh, h^{-1}gh)\).

From the above theorem, we can deduce the following which is stated as Theorem 3.2.

**Theorem.** Let \( k \) be a field whose characteristic is coprime to \(|G|\). The homotopy equivalence \( \tau_{\theta_G} \) above induces a homology group isomorphism

\[
\Theta_G : H_*(LM_G; k) \cong H_*(LM; k) \otimes Z(k[G]).
\]

Next we describe the algebra and the coalgebra structures on (1.4). Let \([M] \) be the fundamental class of \( M \). For every field \( k \), we have a map \( \varphi_k : \pi_1 S \to H_{dim M}(LM; k) \) defined by \( \varphi_k([l]) = l_*[M] \), here we consider \( l \in \pi_1 S \) as a map \( M \to LM \). We set \( P_k := \varphi_k^{-1}(c, [M]) \), where \( c \) denotes the map assigning constant loops \( M \to LM \). We can easily see that \( P_k \) is a subgroup of \( \pi_1 S \). The following is stated as Proposition 4.1 and Proposition 4.2 in this article.

**Theorem.** The linear isomorphism \( \Theta_G : H_*(LM_G; k) \cong H_*(LM; k) \otimes Z(k[G]) \) is an algebra isomorphism, where the product structure of the right hand side is described by a cocycle \( c_{\theta_G} \in Z^2(G; \pi_1 S/P_k) \) as

\[
(x \otimes g) \circ c_{\theta_G}(y \otimes h) = x \circ y \circ c_{\theta_G}(g, h) \otimes gh.
\]

**Theorem.** The homology class \([c_{\theta_G}] \) in \( H^2(G; \pi_1 S/P_k) \) does not depend on the choice of the path family \( \theta_G \). Furthermore, if two cocycles \( c \) and \( c' \) are cohomologous each other, then the corresponding algebras \( H_*(LM; k) \otimes c Z(k[G]) \) and \( H_*(LM; k) \otimes c' Z(k[G]) \) are isomorphic.

The similar statements for the coproduct structure also holds. We have the followings which is stated as Proposition 4.3 and Proposition 4.4 in this article.

**Theorem.** The linear isomorphism \( \Theta_G : H_*(LM_G; k) \cong H_*(LM; k) \otimes Z(k[G]) \) is a coalgebra isomorphism, where the coproduct structure of the right hand side is described by the cocycle \( c_{\theta_G} \in Z^2(G; \pi_1 S/P_k) \) as

\[
\delta_{c_{\theta_G}}(x \otimes g) = \sum_{k \in G} \delta(xc_{\theta_G}^{-1}(gk, k^{-1})) \otimes (gk \otimes k^{-1}).
\]
Theorem. If two cocycles \( c \) and \( c' \) are cohomologous each other, then the corresponding coalgebras \( H_*(LM; k) \otimes Z(k[G]) \) and \( H_*(LM; k) \otimes Z(k[G]) \) are isomorphic.

This article is organized as follows. We briefly review the orbifold string topology in section 2. In section 3, we study the certain path space defined by Lupercio-Uribe as loop groupoid in [10], and reveal the linear structure of homology of \( LM_G \). In section 4, we study the loop product and coproduct by chacing the diagram of loop groupoids, and determine the algebra and coalgebra structure on the homology studied in Section 3. Finally, in Section 5, we compute the loop homology algebra for some global quotient orbifolds and see that orbifold loop product can be non-trivial. Hence the orbifold string topology turns out to be pivotal for studying orbifolds.

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2 Preliminaries on orbifold string topology

In this section, we briefly review about the fundamental notions in the theory of orbifold string topology. We recommend the reader to refer [10] and [8] for the detail.

String topology

Let \( M \) be a closed oriented manifold. We use the notation \( LM := C^0(S^1, M) \). In [5], Chas-Sullivan constructed an associative graded commutative product on the homology of \( LM \), which is called loop product. In [3], Cohen-Godin constructed the coproduct structure and showed that these makes \( H_*(LM; \mathbb{Z}) \) a 2d-TQFT without counit. These structures have many deep interaction with the other area of mathematics.

Orbifold loop product

In [10], Lupercio-Uribe-Xicoténcatl constructed an analogous product for orbifolds so that it coincides with the original one when the target orbifold is just a
manifold. We explain the construction of this product. Let $G$ be a finite group acting smoothly on a $M$. Let $P_gM$ be the space $\{\sigma : [0,1] \to M \mid \sigma(1) = \sigma(0)g\}$, and we set $P_GM := \bigsqcup_{g \in G} P_gM$. There is a $G$ action on $P_GM$ defined as $\varphi \cdot h := \varphi h \in P_{h^{-1}gh}M$ for every $\varphi \in P_gM$ and $h \in G$. These spaces are defined in [10] as the loop groupoid. In [10], they prove the weak homotopy equivalence $L(M \times_G EG) \simeq P_GM \times_G EG$, whence we have an isomorphism $H_\ast(L(M \times_G EG)) \simeq H_\ast(P_GM \times_G EG)$ by Whitehead theorem. When the characteristic of the coefficient field $k$ is coprime to the group order $|G|$, the homology group $H_\ast(P_GM \times_G EG; k)$ can be viewed as a subspace of $H_\ast(P_GM; k)$ via the covering transfer map. We fix such a field $k$. We construct the orbifold loop product $\circ : H_\ast(L(X \times_G EG); k) \otimes H_\ast(L(X \times_G EG); k) \to H_{\ast - \dim M}(L(X \times_G EG); k)$ as follows. For every $g, h \in G$, we consider the following pullback diagram

$$
P_gM \times_M P_hM \xrightarrow{\Delta} P_gM \times P_hM \xrightarrow{\varepsilon \times \varepsilon_0} M \xrightarrow{\Delta} M \times M,
$$

where $\Delta$ denotes the diagonal embedding, and $\varepsilon_0$ denotes the evaluation map with $\varepsilon_0(\varphi) = \varphi(t)$. The map $\Delta$ is a codimension $d$ embedding and we have an umkehr map $\Delta^\prime : H_\ast(P_gM \times P_hM) \to H_{\ast - \dim M}(P_gM \times_M P_hM)$ by composing the Pontrjagin-Thom map and the Thom isomorphism. We define the homomorphisms $\bullet_{g,h}$ by the sequence

$$
\bullet_{g,h} : H_p(P_gM; k) \otimes H_q(P_hM; k) \xrightarrow{\gamma} H_{p+q}(P_gM \times P_hM; k) \xrightarrow{\Delta^\prime} H_{p+q - \dim M}(P_gM \times_M P_hM; k).
$$

Here, $\gamma$ denotes the concatenation map $P_gM \times_M P_hM \to P_{gh}M$ defined by $\gamma(\sigma_g, \sigma_h) = \sigma_g \ast \sigma_h = \begin{cases} 
\sigma_g(2t) & (0 \leq t \leq 1/2) \\
\sigma_h(2t-1) & (1/2 \leq t \leq 1)
\end{cases}$. By taking summation over $g, h \in G$, we obtain the homomorphism

$$
\bullet := \oplus_{g,h \in G} \bullet_{g,h} : H_p(P_GM; k) \otimes H_q(P_GM; k) \to H_{p+q - \dim M}(P_GM; k).
$$

The orbifold loop product

$$
\circ : H_p(P_GM \times_G EG; k) \otimes H_q(P_GM \times_G EG; k) \to H_{p+q - \dim M}(P_GM \times_G EG; k),
$$

is defined as the restriction homomorphism of $\bullet$. It is shown that the obtained orbifold loop product is an orbifold invariant of the global quotient orbifold $[M/G]$. 5
Orbifold loop coproduct

Similar to the product, we define the orbifold loop coproduct $\delta$ as follows. We set $\mathcal{P}_{gh} M := \{ \sigma \in \mathcal{P}_{gh} M \mid \sigma(1/2) = \sigma(0) g \}$. For every $g, h \in G$, we consider the following pullback diagram

$$
P_{gh} M \xrightarrow{\tilde{\Delta}} P_{gh} M \xrightarrow{\Delta} \mathcal{P}_{gh} M \xrightarrow{(\epsilon_1, \epsilon_0)} M \xrightarrow{\Delta} M \times M.
$$

We define the homomorphims $\flat_{g,h}$ by the sequence

$$
\begin{align*}
H_*(P_{gh} M; k) &\xrightarrow{\tilde{\lambda}} H_{*-\dim M}(P_{gh} M; k) \\
&\xrightarrow{\lambda} H_{*-\dim M}(P_g M \times M P_h M; k) \\
&\equiv \bigoplus_{p+q=-\dim M} H_p(P_g M; k) \otimes H_q(P_h M; k),
\end{align*}
$$

where $\lambda$ is defined by $\lambda(\sigma)(s, t) = (\sigma(\frac{s}{2}), \sigma(\frac{s}{2} + \frac{t}{2}))$, and $i$ denotes the inclusion map. By taking summation over $g, h \in G$, we obtain the homomorphism $b$ defined by

$$
b |_{H_*(P_g M; k)} := \bigoplus_{l=g,h} b_{g,h} : H_*(P_l M; k) \rightarrow \bigoplus_{p+q=-\dim M} H_p(P_G M; k) \otimes H_q(P_G M; k).
$$

When the coefficient field $k$ has the characteristic coprime to $|G|$, the orbifold loop coproduct $\delta$ is defined to be the restriction of $b$ to the $G$-invariant part $H_*(P_G M; k)^G$.

TQFT structure

It is shown in [8] that the orbifold loop product and coproduct defined as above makes the homology $H_*(L(M \times_G EG); k)$ a $(1 + 1)$-dimensional topological quantum field theory with positive boundaries, i.e. they enjoy the associativity, the coassociativity, the Frobenius relation, and the other compatibility conditions except for the existence of the counit. See [8] for the proof.

3 The linear structure

Let $G$ be a finite group acting on a topological space $X$. If $g \in G$ is homotopic to the identity map by a homotopy $H_g : X \times [0, 1] \rightarrow X$ with

$$
\begin{align*}
H(x, 0) &= x \\
H(x, 1) &= g(x)
\end{align*}
$$

we have a homotopy equivalence map $\tau_{H_g} : P_g X \rightarrow LX$ defined as $\tau_{H_g}(\sigma)(t) =$...
can be viewed as a subspace of \( \text{aut} \) defined by \( \left\{ l(2t) \right\}(0 \leq t \leq 1/2) \). Its homotopy inverse map \( \eta_{H_g} \) is defined as

\[
\eta_{H_g}(l)(t) = \begin{cases} l(2t) & (0 \leq t \leq 1/2) \\ H_g(l(1), 2t - 1) & (1/2 \leq t \leq 1) \end{cases}
\]

We show that we can choose a family of paths \( \tau_{H_g} := \bigcup_{g \in G} \tau_{H_g} \). We remark that a homotopy \( H_g : X \times [0, 1] \to X \) can be seen as a path in \( \text{aut}_1(X) \) which connects the identity map and \( g \).

**Proposition 3.1.** Let \( G \) be a finite group acting on \( X \) homotopically trivially, hence \( G \) can be viewed as a subspace of \( \text{aut}_1X \). We denote the order of a group \( K \) by \( |K| \), and the subgroup of torsion elements in \( K \) by \( K^{\text{tor}} \). If there exists a sub \( H \)-space \( S \) of \( \text{aut}_1X \) containing \( G \) and satisfying

(A) \( S \) is path-connected,

(B) \( |G| \) and \( |(\pi_1S)^{\text{tor}}| \) are coprime,

then there is a path family \( \theta_G = \{ \theta_g : [0, 1] \to S \}_{g \in G} \) with \( \begin{cases} \theta_g(0) = \text{id} \\ \theta_g(1) = g \end{cases} \), such that the associated homotopy equivalence

\[
\tau_{\theta_G} : P_{G}X \approx LX \times G,
\]

is \( G \)-equivariant up to homotopy. Here, the action of \( G \) on the RHS above is defined by \((l, g) \cdot h = (lh, h^{-1}gh)\).

**Proof.** We show that we can choose a family of paths \( \{ \theta_g : [0, 1] \to S \}_{g \in G} \) such that \( \begin{cases} \theta_g(0) = \text{id} \\ \theta_g(1) = g \end{cases} \), and \( h^{-1}\theta_g h \) is homotopic to \( \theta_{h^{-1} \theta_g h} \) relative to endpoints for each \( g, h \in G \). Then the map \( \tau_{\theta_G} \) will be the desired one. To prove this, it is sufficient to show that \( h^{-1}\theta_g h \) is homotopic to \( \theta_g \) relative to endpoints for each \( g \in G \) and \( h \in C(g) \), the center of \( g \). Let \( \Gamma_h : S \to S \) be the map defined by conjugation by \( h \in G \). If we take the identity map as the base point of \( S \), then \( \Gamma_h \) is a based map and induces an identity on \( \pi_1S \) by the assumption (A). On the other hand, we can easily see that \( \Gamma_h(\theta_g * (h^{-1}\theta_g h)^{-1}) = (h^{-1}\theta_g h) * (h^{-2}\theta_g h^2) \). Here we denote by \((h^{-1}\theta_g h)^{-1}\) the inverse trajectory of the path \( h^{-1}\theta_g h \). Hence we obtain

\[
2[\theta_g * (h^{-1}\theta_g h)^{-1}] = [\theta_g * (h^{-2}\theta_g h^2)] = [\theta_g * (h^{-1}\theta_g h)^{-1} + (h^{-1}\theta_g h) * (h^{-2}\theta_g h^2)] \]

in \( \pi_1S \). By repeating this, we obtain \( n[\theta_g * (h^{-1}\theta_g h)^{-1}] = [\theta_g * (h^{-n}\theta_g h^n)^{-1}] \) for every positive integer \( n \). Since \( G \) is a finite group, we have \( N[\theta_g * (h^{-1}\theta_g h)^{-1}] = \)

\[
\ldots
\]

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\[ [\theta_g * (\theta_g)^{-1}] = 0 \] for some \( N \), from which we deduce \([\theta_g * (h^{-1}\theta_g h)^{-1}] = 0 \) because of the assumption \([B]\). Hence we obtain \( h^{-1}\theta_g h \approx \theta_g \) relative to endpoints. \( \square \)

The following can be deduced from Proposition \([3.1]\) and an easy application of the covering transfer map.

**Theorem 3.2.** Let \( k \) be a field whose characteristic is coprime to \( |G| \). Under the assumption of Proposition \([3.1]\), the homotopy equivalence \( \tau_{\theta_G} \) in Proposition \([3.1]\) induces a homotopy group isomorphism

\[
H_*(LM_G; k) \cong H_*(LM; k) \otimes Z(k[G]),
\]

where \( Z(\cdot) \) denotes the center of an algebra.

**Proof.** Since the space \( LM_G \) is homotopy equivalent to the Borel construction \( P_G M \times_G EG \), we consider the homology \( H_*(P_G M \times_G EG; k) \), which is isomorphic to the \( G \)-invariant space \( H_*(P_G M; k)^G \) by the covering transfer. Then the homotopy equivalence \( P_G M \cong LM \times G \) obtained in Proposition \([3.1]\) induces an isomorphism \( H_*(P_G M; k)^G \cong H_*(LM \times G; k)^G \) since it is \( G \)-equivariant up to homotopy. Since \( G \) acts on \( LM \) homotopically trivially, we obtain \( H_*(LM \times G; k)^G \cong H_*(LM; k) \otimes H_*(G; k)^G \) by the Künneth formula. Finally, we can deduce \( H_*(G; k)^G = Z(k[G]) \) by a calculation. \( \square \)

## 4 The loop product and the coproduct structure

In this section, we consider a closed oriented manifold \( M \), and assume that \( G \) acts smoothly on \( M \). We further assume that the action of \( G \) is homotopically trivial, and there exists a path-connected sub \( H \)-space \( S \) of \( \text{aut}_1 M \) containing \( G \) such that \( |G| \) and \( |(\pi_1 S)^{tor}| \) are coprime. Let \( [M] \) be the fundamental class of \( M \). For every field \( k \), we have a map \( \varphi_k : \pi_1 S \rightarrow H_{dim M}(LM; k) \) defined by \( \varphi_k[I] = l_*[M] \), here we consider \( l \in \pi_1 S \) as a map \( M \rightarrow LM \). We set \( P_k := \varphi_k^{-1}(c_*[M]) \), where \( c \) denotes the map assigning constant loops \( M \rightarrow LM \). We can easily see that \( P_k \) is a subgroup of \( \pi_1 S \).

**Proposition 4.1.** The linear isomorphism \( \tau_{\theta_G} : H_*(LM_G; k) \cong H_*(LM; k) \otimes Z(k[G]) \) in Proposition \([3.1]\) is an algebra isomorphism, where the product structure of the right hand side is described by a cocycle \( c \in Z^2(G; \pi_1 S/P_k) \) as

\[
(x \otimes g) \circ_c (y \otimes h) = x \circ y \circ c(g, h) \otimes gh.
\]

**Proof.** We define a 2-cocycle \( c \in Z^2(G; \pi_1 S/P_k) \) by \( c(g, h) := [\theta_h * \theta_g h * \theta_g^{-1}] \in \pi_1 S/P_k \). It is easily checked that \( dc = 0 \). By taking the homology of the following
homotopy commutative diagram

\[ \begin{array}{c}
LM \xrightarrow{\gamma_{(g,h)}} LM \xrightarrow{\gamma} LM \times M LM \\
\xrightarrow{\tau_{gh}} P_{gh}M \xleftarrow{\gamma_{(g,h)}} P_gM \times M P_hM.
\end{array} \quad (4.2) \]

we obtain \((x \otimes g) \circ (y \otimes h) = x \circ y \circ c(g,h) \otimes gh. \)

The following proposition implies that the ambiguity of the choice of the path family \(\theta_0\) so far does not effect when considering the isomorphism class of algebra. We denote the algebra described by a cocycle \(c\) as in Proposition 4.1 by \(H_*(LM;k) \otimes_{\mathcal{C}} Z(k[G]).\)

**Proposition 4.2.** If \(\theta_0\) and \(\theta_0'\) are path families satisfying the assumption of Proposition 3.1 then the corresponding 2-cocycles \(c\) and \(c'\) are cohomologous each other. Furthermore, if two cocycles \(c\) and \(c'\) are cohomologous each other, then the corresponding algebras \(H_*(LM;k) \otimes_{\mathcal{C}} Z(k[G])\) and \(H_*(LM;k) \otimes_{\mathcal{C}} Z(k[G])\) are isomorphic.

**Proof.** By setting \(\xi(g) = \theta_0 \ast \theta_0'^{-1}\), we obtain \(c = cd\xi\), which shows the former part. If two cocycles \(c, c' \in Z^2(G; \pi_1 S/P_k)\) are cohomologous, there exists a function \(\xi : G \to \pi_1 S/P_k\) such that \(c = c'd\xi\). Then the map \(F_\xi : H_*(LM;k) \otimes_{\mathcal{C}} Z(k[G]) \to H_*(LM;k) \otimes_{\mathcal{C}} Z(k[G])\) defined by \(F_\xi(x \otimes g) = \xi(g)x \otimes g\) is a well-defined algebra homomorphism because \(F_\xi((x \otimes g) \circ_c (y \otimes h)) = F_\xi(xy_c'(g,h)d\xi(g,h) \otimes gh = xyc'(g,h)\xi(g)\xi(h) \otimes gh = (x\xi(g) \otimes g) \circ_{c'} (y\xi(h) \otimes h) = F_\xi(x \otimes g) \circ_{c'} F_\xi(y \otimes h).\) This map is apparently an isomorphism. \(\square\)

Next we consider the coproduct structure. The similar arguments work as in the case of the products.

**Proposition 4.3.** The linear isomorphism \(\tau_{\theta_0} : H_*(LMG;k) \cong H_*(LM;k) \otimes Z(k[G])\) in Proposition 3.1 is a coalgebra isomorphism, where the coproduct structure of the right hand side is described by a cocycle \(c \in Z^2(G; \pi_1 S/P_k)\) as

\[ \delta_*(x \otimes g) = \sum_{k \in G} \delta((xc^{-1}(gk,k^{-1})) \otimes (gk \otimes k^{-1})]. \quad (4.3) \]

**Proof.** Since \(\tau_{\theta_0h} \simeq (\circ c(g,h)) \circ (\tau_{\theta_0} \ast \theta_h)\), this follows from the following commuta-
Proposition 4.4. If two cocycles \( c \) and \( c' \) are cohomologous each other, then the corresponding coalgebras \( H_*(LM;k) \otimes \chi Z(k[G]) \) and \( H_*(LM;k) \otimes \epsilon Z(k[G]) \) are isomorphic.

To prove this, we use the following lemma.

Lemma 4.5. For every \( x, y, z \in H_*(LM;\mathbb{Z}) \), \( \delta(x) \) is of the form \( \delta_1(x) \otimes \delta_2(x) \), and we have \( \delta(x \circ y \circ z) = \delta_1(x) \circ y \otimes \delta_2(x) \circ z \).

Proof. According to [7], we have a formula \( \delta(x \circ y \circ z) = \chi(M)[c_0] \circ x \otimes [c_0] \circ y \circ z \), where \([c_0]\) denotes the constant loop at the base point, and \( \chi(M) \) denotes the Euler characteristic of \( M \). Hence we have \( \delta_1(x) \circ y \otimes \delta_2(x) \circ z = \chi(M)x \circ y \otimes z = \delta(x \circ y \circ z) \). \( \square \)

Proof. If two cocycles \( c, c' \in Z^2(G;\pi_1S/P_k) \) are cohomologous, then there exists a function \( \xi : G \to \pi_1S/P_k \) such that \( c = c'd\xi \). Then the map \( F_\xi : H_*(LM;k) \otimes \epsilon Z(k[G]) \to H_*(LM;k) \otimes \epsilon Z(k[G]) \) defined by \( F_\xi(x \otimes g) = \xi(g)x \otimes g \) is a well-defined coalgebra homomorphism because we have

\[
\delta_\epsilon \circ F_\xi(x \otimes g) = \delta_\epsilon(\xi(g) \otimes g) = \sum_{k \in G} \delta(x \xi(g)c^{-1}(gk,k^{-1})) \otimes (gk \otimes k^{-1}) = \\
\sum_{k \in G} \delta(x \xi(gk)\xi(k^{-1})c^{-1}(gk,k^{-1})) \otimes (gk \otimes k^{-1}) = \\
\sum_{k \in G} (\delta_1(xc^{-1}(gk,k^{-1}))\xi(gk) \otimes gk) \otimes (\delta_2(xc^{-1}(gk,k^{-1}))\xi(k^{-1}) \otimes k^{-1}) = F_\xi \otimes F_\xi(\delta_\epsilon(x \otimes g)).
\]

(4.5)

Here, we use Lemma 4.5 for the fourth equality. This map is apparently an isomorphism. \( \square \)
5 Orbifold loop product is a non-trivial orbifold invariant.

In [10] and [2] they and the author himself compute the orbifold loop product for some global quotient orbifolds including lens spaces. However, it results in that the orbifold loop product computed by them has no information on the action of groups on manifolds. This is because the loop homology $H_*(LMG; k)$ splits into the tensor product $H_*(LM; k) \otimes \mathbb{Z}(k[G])$ as an algebra for the orbifolds $[M/G]$ under their consideration. Here, we use the notation $H_* := H_{*+\dim M}$. In this section we compute the loop homology algebra structure by computing the cohomology class $c \in H^2(G; \pi_1 S/P_k)$ for some global quotient orbifolds. Consequently, we show that some of them have non-trivial information of the group action. Although we only consider the algebra structure in this section, we can also determine the coalgebra structure in the similar way.

The quotiented circle $[S^1/C_n]$

We first consider the orbifold $[S^1/C_n]$, where $C_n$ denotes the cyclic group of order $n$ acting on $S^1 \subset \mathbb{C}$ by clockwise rotation. It is known by [4] that the loop homology of the circle is isomorphic to $\Lambda(a) \otimes \mathbb{Z}[t, t^{-1}]$ as algebra, where $\Lambda$ denotes the exterior algebra and we set $\deg a = \deg t = 0$. The tensor product is always taken over a field $k$ in the following arguments.

**Proposition 5.1.** Let $C_n = \langle g \rangle$ be the cyclic group in the above, and $k$ be a field whose characteristic is coprime to $n$. Then we have an algebra isomorphism

$$H_*(L[S^1/C_n]; k) \cong \mathbb{H}_*(LS^1; k) \otimes k[C_n],$$

where the product structure on the RHS is defined by

$$(x \otimes g^i) \circ (y \otimes g^j) = \begin{cases} xy \otimes g^{i+j} & (i + j < n) \\ xy^i \otimes g^{i+j} & (i + j \geq n) \end{cases},$$

for every $x, y \in \mathbb{H}_*(LC^p; k)$ and $0 \leq i, j < n$. Furthermore, this algebra is not isomorphic to the tensor product of algebras $\mathbb{H}_*(LS^1; k) \otimes k[C_n]$.

**Proof.** The isomorphism as vector spaces follows from Proposition [3.1] and Theorem [3.2] since $\pi_1 S^1 = \mathbb{Z}$. It is easy to check that the homomorphism $\varphi_k : \pi_1 S^1 \to H_1(LS^1; k)$ sends the generator of $\pi_1 S^1$ to $1 \otimes t \in H_1(LS^1; k)$. Hence $\varphi_k$ is injective and $\pi_1 S^1/P_k = \mathbb{Z} = \langle t \rangle$. As in the proof of Proposition [4.1] we define a cocycle
Proposition 5.3. Let \( \Omega \) be the cocycle \( \theta \) cohomologous. Thus we can deduce the non-triviality of this product by Proposition 4.1.

Remark 5.2. The cocycle \( \omega \) above becomes a generator of \( H^2(C_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \), hence is not null-cohomologous. Thus we can deduce the non-triviality of this product by Proposition 4.2.

The quotiented projective space \([\mathbb{C}P^l/G]\)

Next we consider the orbifold \([\mathbb{C}P^l/G]\), where \( G \) denotes the finite subgroup of the unitary group \( U(l+1) \) acting on the complex projective space \( \mathbb{C}P^l \) in the obvious manner. It is known by [4] that \( H_2(l\mathbb{C}P^l; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/(l+1)\mathbb{Z} = E_{2,0}^\infty \oplus E_{0,2}^\infty \), where \( E_{2,0}^\infty \) and \( E_{0,2}^\infty \) denote the infinity term of the homology Serre spectral sequence for the fibration \( \Omega \mathbb{C}P^l \to l\mathbb{C}P^l \to \mathbb{C}P^l \). By the standard argument, we deduce that the map \( i \) and \( p \) induces a surjection \( i_\ast : H_2(\Omega \mathbb{C}P^l; \mathbb{Z}) \to E_{0,2}^\infty \) and an isomorphism \( p_\ast : E_{2,0}^\infty \to H_2(\mathbb{C}P^l; \mathbb{Z}) \), and we have a homotopy equivalence \( \Omega \mathbb{C}P^l \cong S^1 \times \Omega S^{2l+1} \) by which we obtain \( H_2(\mathbb{C}P^l; \mathbb{Z}) \cong H_2(\Omega S^{2l+1}; \mathbb{Z}) \cong \mathbb{Z} \).

Proposition 5.3. Let \( k \) be a field whose characteristic is coprime to \( |G| \). As vector spaces, we have an isomorphism

\[
H_s(L[\mathbb{C}P^l/G]; k) \cong H_s(l\mathbb{C}P^l; k) \otimes k[G].
\]  

Proof. This immediately follows from Proposition 4.1 Theorem 3.2 and the fact \( \pi_1 U(l+1) \cong \mathbb{Z} \).

Let \( \text{const} : H_2(l\mathbb{C}P^l) \to H_2(l\mathbb{C}P^l) \) be the map assigning constant loops. Let \( [\mathbb{C}P^l] \) be the fundamental class of \( \mathbb{C}P^l \). We use the abbreviation \([\mathbb{C}P^l] = \text{const}([\mathbb{C}P^l]) \in H_2(l\mathbb{C}P^l)\). This is the unit element of \( H_s(l\mathbb{C}P^l; k) \). We fix a generator of second term of \( H_2(l\mathbb{C}P^l; k) \cong k \otimes k \otimes \mathbb{Z}/(l+1)\mathbb{Z} \), denoted by \( \varepsilon_k \). Hence \([\mathbb{C}P^l] \) and \( \varepsilon_k \) are additive generators of \( H_2(l\mathbb{C}P^l; k) \). For a reason of the degree, we have \( \varepsilon_k^2 = 0 \). Hence the set \( \cup_k := \{ [\mathbb{C}P^l] + i\varepsilon_k | 0 \leq i \leq l \} \) forms the cyclic group of order \( (\text{ch } k, l+1) \) by the loop product. We denote the characteristic of a field \( k \) by \( \text{ch } k \) here and in the below.

Proposition 5.4. Let \( C_n = \langle g \rangle \) be a finite cyclic subgroup of \( U(l+1) \), and \( k \) be a field with \( (\text{ch } k, n) = 1 \) and \( (\text{ch } k, l+1) > 1 \). Then we have an algebra isomorphism

\[
\mathbb{H}_s(L[\mathbb{C}P^l/C_n]; k) \cong \mathbb{H}_s(l\mathbb{C}P^l; k) \otimes k[C_n],
\]  

\( 12 \)
where the product structure on the RHS is defined by

\[(x \otimes g^i) \circ (y \otimes g^j) = \begin{cases} xy \otimes g^{i+j} & (i + j < n) \\ xy([C\mathbb{P}^l] + \varepsilon_k) \otimes g^{i+j} & (i + j \geq n) \end{cases}, \quad (5.5)\]

for every \(x, y \in \mathbb{H}_*(LC\mathbb{P}^l; k)\) and \(0 \leq i, j < n\). Furthermore, this algebra is not isomorphic to the tensor product of algebras \(\mathbb{H}_*(LC\mathbb{P}^l; k) \otimes k[C_n]\).

**Proof.** The additive isomorphism follows by the previous proposition. To prove the algebra part, we use the following lemma which we prove later.

**Lemma 5.5.** The image of the homomorphism \(\varphi_k : \pi_1 U(l + 1) \to H_2(LC\mathbb{P}^l; k)\) is \(\mathbb{U}_k\). Hence \(\pi_1 U(l + 1)/P_k \cong \mathbb{U}_k\).

We can assume that the 1-torus in \(U(l + 1)\) containing \(C_n\) hits \([C\mathbb{P}^l] + \varepsilon_k\) by \(\varphi_k\) as an element of \(\pi_1 U(l + 1)\). Hence the cocycle \(c \in Z^2(C_n; \pi_1 U(l + 1)/P_k)\) defined by \(c(g^i, g^j) := [\theta_{g^i} \ast \theta_{g^j} g^i \ast \theta_{g^j}^{-1}]\), where \(\theta_g\) denotes the clockwise shortest geodesic in \(U(l + 1)\) (hence in the 1-torus above) connecting 1 and \(g^i, g^j\), is actually of the form \(c(g^i, g^j) = \begin{cases} [C\mathbb{P}^l] & i + j < n \\ [C\mathbb{P}^l] + \varepsilon_k & i + j \geq n \end{cases}\) for \(0 \leq i, j < n\). Then the isomorphism (5.4) follows from Proposition 4.1. The projection \(\mathbb{Z} \to \mathbb{Z}/(\text{ch } k, l + 1)\mathbb{Z}\) induces a surjection \(H^2(C_n; \mathbb{Z}) \to H^2(C_n; \mathbb{Z}/(\text{ch } k, l + 1)\mathbb{Z})\), which is shown by considering the exact sequence of group cohomology and from the fact \(H^\text{odd}(C_n; \mathbb{Z}) = 0\). Hence the cocycle above become the generator of \(H^2(C_n; \mathbb{Z}/(\text{ch } k, l + 1)\mathbb{Z})\). Thus we can deduce the non-triviality of this product by Proposition 4.2. \(\square\)

**Proof of Lemma 5.5.** Let \(F\) be the space \(\{f : C\mathbb{P}^l \to C\mathbb{P}^l \mid f|_{C\mathbb{P}^{l-1}} = \iota\}\), where \(\iota : C\mathbb{P}^{l-1} \to C\mathbb{P}^l\) is the natural inclusion. It is known by Sasao [6] that \(F\) is homotopy equivalent to \(\Omega^2 C\mathbb{P}^l\). We define the map \(a_* : \pi_1 \text{aut}_1(C\mathbb{P}^l) \to [C\mathbb{P}^l, LC\mathbb{P}^l]\) by adjoint of maps. We also define a map \(\pi_1 \text{Map}_*(C\mathbb{P}^l, C\mathbb{P}^l) \to [C\mathbb{P}^l, LC\mathbb{P}^l]\) by adjoint, and also denote it by \(a_*\). Since the homomorphism \(\pi_1 \text{Map}_*(C\mathbb{P}^l, C\mathbb{P}^l) \to \pi_1 \text{aut}_1(C\mathbb{P}^l)\) induced from the inclusion is surjective, the images of both \(a_*\) coincides. We have a map \(p : C\mathbb{P}^l \to C\mathbb{P}^l \vee S^2\) defined by “pinching” the 2l-cell of \(C\mathbb{P}^l\). For \(g \in \Omega^2 LC\mathbb{P}^l\), we can define a map \((c \vee g) \circ p : C\mathbb{P}^l \to LC\mathbb{P}^l\), where \(c : C\mathbb{P}^l \to C\mathbb{P}^l\) is the map assigning constant loops. Then we have the following
Two dimensional spherical orbifolds

We consider spherical orbifolds $[S^2/G]$ for $G$ being a finite subgroup of $SO(3)$ acting on $S^2$ by isometry. Since $H_2(LS^2;\mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_2(LS^2; k) \cong \begin{cases} k & (\text{ch } k, 2) = 1 \\ k \oplus k & (\text{ch } k, 2) > 1 \end{cases}$. In [2], we show that $\mathbb{H}_*(L[S^2/G]; k)$ splits into $\mathbb{H}_*(LS^2; k) \otimes Z(k[G])$ as an algebra when $(\text{ch } k, 2) = 1$ and $(\text{ch } k, |G|) = 1$. By using our new method, we show the same splitting formula in the other case, namely when $(\text{ch } k, 2) > 1$ and $(\text{ch } k, |G|) = 1$.

**Proposition 5.6.** For every two dimensional spherical orbifolds $[S^2/G]$, its loop homology $\mathbb{H}_*(L[S^2/G]; k)$ splits into $\mathbb{H}_*(LS^2; k) \otimes Z(k[G])$ whenever $(\text{ch } k, |G|) = 1$. 

\begin{align}
H_2(\Omega C P^l) &\xrightarrow{i_*} H_2(L C P^l) \xrightarrow{+[C P^l]} H_2(L C P^l) \\
\pi_2 \Omega C P^l &\xrightarrow{i_*} \pi_2 L C P^l \xrightarrow{(\cdot^2)\circ p} [C P^l, L C P^l] \\
\pi_1 \Omega C P^l &\xrightarrow{=} \pi_1 \text{Map}_\ast(C P^l, C P^l) \xrightarrow{\pi_1 \text{aut}_1(C P^l)} \\
\pi_1 F &\xrightarrow{=} \pi_1 \text{Map}_\ast(C P^l, C P^l) \xrightarrow{\pi_1 \text{aut}_1(C P^l)},
\end{align}

where $\mathcal{H}$ denotes the Hurewicz map, the map $[C P^l, L C P^l] \to H_2(L C P^l)$ is defined by $f \mapsto f \cdot [C P^l]$, and the homomorphism $\pi_1 F \to \pi_1 \text{Map}_\ast(C P^l, C P^l)$ is induced from the inclusion. We note that $F$ is the fiber of the fibration $\text{Map}_\ast(C P^l, C P^l) \to \text{Map}_\ast(C P^{l-1}, C P^l)$. Since we have $\pi_1 \text{Map}_\ast(C P^{l-1}, C P^l) = [C P^{l-1}, \Omega C P^l] = [C P^{l-1}, S^1 \times \Omega S^{2l+1}] = [S^1 \wedge C P^{l-1}, S^{2l+1}] = \ast$, the homomorphism $\pi_1 F \to \pi_1 \text{Map}_\ast(C P^l, C P^l)$ is surjective. Furthermore, the following commutative diagram shows that the Hurewicz map $\pi_2 \Omega C P^l \to H_2(\Omega C P^l)$ is an isomorphism. Thus the statement follows from the fact that the homomorphism $i_* : H_2(\Omega C P^l; \mathbb{Z}) \to E^\infty_{0,2}$ is surjective.

\begin{align}
H_2(\Omega C P^l) &\xrightarrow{\mathcal{H}} H_2(S^1 \times \Omega S^{2l+1}) \xrightarrow{pr_*} H_2(\Omega S^{2l+1}) \\
\pi_2 \Omega C P^l &\xrightarrow{=} \pi_2(S^1 \times \Omega S^{2l+1}) \xrightarrow{pr_*} \pi_2 \Omega S^{2l+1}.
\end{align}
Proof. We only have to consider the case that \( \text{ch} \, k \) is even and \(|G|\) is odd as mentioned above. Since \(|\pi_1\text{SO}(3)| = 2\), it splits as vector a space by Proposition 3.1. Furthermore, the group cohomology \( H^2(G; A) \) vanishes whenever \(|A|\) is even, hence it splits as an algebra by Proposition 4.1 and Proposition 4.2. □

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