HOMOGENEOUS DISTRIBUTIONS ON FINITE DIMENSIONAL VECTOR SPACES

HU AJIAN XUE

Abstract. Let $V$ be a finite dimensional vector space over a local field $F$. Let $\chi : F^\times \to \mathbb{C}^\times$ be an arbitrary character of $F^\times$. We determine the structure of the natural representation of $\mathrm{GL}(V)$ on the space $S^*(V)^\chi$ of $\chi$-invariant distributions on $V$.

1. Introduction

Let $V$ be a vector space over a local field $F$, of finite dimension $n \geq 1$. For every $g \in \mathrm{GL}(V)$ and every distribution $\eta$ on $V$, define

$$ (1) \quad g.\eta := \text{the push-forward of } \eta \text{ through the map } g : V \to V. $$

For each character $\chi : F^\times \to \mathbb{C}^\times$, a distribution $\eta$ on $V$ is said to be $\chi$-invariant if

$$ a.\eta = \chi(a) \eta, \quad \text{for all } a \in F^\times. $$

Here and as usual, $F^\times$ is identified with the center of $\mathrm{GL}(V)$. By [AGS, Theorem 4.0.2], we know that every $\chi$-invariant distribution on $V$ is tempered when $F$ is archimedean. By convention, every distribution on $V$ is defined to be tempered when $F$ is non-archimedean. The goal of this paper is to understand the space

$$ (2) \quad S^*(V)^\chi := \{ \eta \in S^*(V) \mid a.\eta = \chi(a) \eta \} $$

of $\chi$-invariant tempered distributions on $V$, as a representation of $\mathrm{GL}(V)$ under the action (1). Here and as usual, $S(V)$ denotes the space of Schwartz or Schwartz-Bruhat functions on $V$, when $F$ is respectively archimedean or non-archimedean; and $S^*(V)$ denotes the space of all (continuous in the archimedean case) linear functionals on $S(V)$. It is a fundamental fact in Tate’s thesis that the space (2) is one dimensional when $n = 1$ (see [Wei, Section 1]). Thus we will focus on the case when $n \geq 2$.

Dualizing the action (1), we have a representation of $\mathrm{GL}(V)$ on $S(V)$ by

$$ (g.f)(x) := f(g^{-1}x), \quad \text{for all } g \in \mathrm{GL}(V), f \in S(V), x \in V. $$

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Define the normalized \((F^\times, \chi)\)-coinvariant space
\[
\mathcal{S}_\chi(V) := \left( \mathcal{S}(V) \otimes |\det|_F^{-\frac{1}{2}} \right)_{F^\times, \chi},
\]
where \(|\det|_F\) denotes the positive character
\[
\text{GL}(V) \to \mathbb{C}^\times, \quad g \mapsto |\det(g)|_F,
\]
\(|\cdot|_F\) denotes the normalized absolute value on \(F\), and for a smooth representation \(U\) of \(\text{GL}(V)\), \(U_{F^\times, \chi}\) denotes the maximal (Hausdorff in the archimedean case) quotient of \(U\) on which \(F^\times\) acts through the character \(\chi\). Here and as usual, we do not distinguish a one dimensional representation with its corresponding character. Then we have
\[
\mathcal{S}^\ast(V) \cong \left( \mathcal{S}_{\chi^{-1}}(V) \otimes |\det|_F^{-\frac{1}{2}} \right)_{F^\times, \chi},
\]
as representations of \(\text{GL}(V)\). Here and henceforth, we use a superscript \(^\ast\) to indicate the dual space in various contexts. Thus we only need to study the representation (3).

As before, write \(\mathcal{S}(V \setminus \{0\})\) for the space of Schwartz or Schwartz-Bruhat functions on \(V \setminus \{0\}\), when \(F\) is respectively archimedean or non-archimedean (see [AG] for the definition of Schwartz functions in the archimedean case). Similar to (3), define
\[
\mathcal{S}_\chi(V \setminus \{0\}) := \left( \mathcal{S}(V \setminus \{0\}) \otimes |\det|_F^{-\frac{1}{2}} \right)_{F^\times, \chi}.
\]
Then the embedding \(\mathcal{S}(V \setminus \{0\}) \hookrightarrow \mathcal{S}(V)\) induces a homomorphism
\[
j_\chi : \mathcal{S}_\chi(V \setminus \{0\}) \to \mathcal{S}_\chi(V)
\]
of representations of \(\text{GL}(V)\).

It is easy to see that the representation \(\mathcal{S}_\chi(V \setminus \{0\})\) is isomorphic to a degenerate principal series. More precisely, fix an arbitrary nonzero vector \(v_0 \in V\), and write \(P(v_0) = F^\times \times P^\circ(v_0)\) for the maximal parabolic subgroup of \(\text{GL}(V)\) stabilizing \(Fv_0\), where \(P^\circ(v_0)\) denotes the stabilizer of \(v_0\) in \(\text{GL}(V)\). Then the linear map
\[
\mathcal{S}(V \setminus \{0\}) \otimes |\det|_F^{-\frac{1}{2}} \to C^\infty(\text{GL}(V)),
\]
\[
\phi \otimes 1 \mapsto \left( g \mapsto \int_{F^\times} \phi((ga)^{-1}v_0) \cdot |\det(ga)|_F^{-\frac{1}{2}} \cdot \chi^{-1}(a) \, d^\times a \right)
\]
induces a \(\text{GL}(V)\)-intertwining isomorphism
\[
\mathcal{S}_\chi(V \setminus \{0\}) \cong \text{Ind}_{P(v_0)}^{\text{GL}(V)} \chi \otimes 1.
\]
Here \(d^\times a\) is a Haar measure on the multiplicative group \(F^\times\). And “Ind” indicates the normalized smooth induction, on which \(\text{GL}(V)\) acts by right translation. While “1” stands for the trivial character of \(P^\circ(v_0)\). The structure of this degenerate principal series is well-known (see Lemma 4.1).
Denote by $C_F$ the submonoid of $\text{Hom}(F^x, C^x)$ generated by characters of the form $\iota|_{F^x} : F^x \to C^x$, where $\iota : F \hookrightarrow C$ is a continuous field embedding. Explicitly, denote by $N$ the set of non-negative integers. If $F = \mathbb{R}$, let $\iota$ be the natural imbedding of $\mathbb{R}$ into $C$. If $F = \mathbb{C}$, let $\iota_1, \iota_2$ be the identity map and complex conjugate respectively. Then

$$C_F = \begin{cases} \{(\iota|_{F^x})^r | r \in \mathbb{N}\}, & \text{if } F = \mathbb{R} ; \\ \{(\iota_1|_{C^x})^r (\iota_2|_{C^x})^s | r, s \in \mathbb{N}\}, & \text{if } F = \mathbb{C}; \\ \{1\}, & \text{if } F \text{ is non-archimedean}. \end{cases}$$

In Section 2 we will define an irreducible finite dimensional representation $\sigma_{V, \chi}$ of $\text{GL}(V)$ for each $\chi \in C_F$. Note that

$$C^+(n) := \{ | \cdot |_{F^x}^\frac{n}{2} C_F \} \text{ and } C^-(n) := \{ \chi^{-1} | \chi \in C^+(n) \}$$

are disjoint subsets of $\text{Hom}(F^x, C^x)$.

Now the main result of this paper is formulated as follows.

**Theorem 1.1.** Let $V$ be an $n$-dimensional $(n \geq 2)$ vector space over a local field $F$, and let $\chi$ be a character of $F^x$. Define the homomorphism $j_\chi : S_\chi(V \setminus \{0\}) \to S_\chi(V)$ as $[\text{6}]$. Then we have the following:

(a) If $\chi \notin C^+(n) \cup C^-(n)$, then $j_\chi$ is an isomorphism of irreducible representations.

(b) If $\chi \in C^+(n)$, then $j_\chi$ is an isomorphism and $S_\chi(V)$ has a unique irreducible subrepresentation, and the corresponding quotient representation is isomorphic to $\sigma_{V, \chi} |_{\cdot |_{F^x}^\frac{n}{2} C_F} \otimes |\det|_{F^x}^\frac{1}{2}$.

(c) If $\chi \in C^-(n)$, then both $S_\chi(V)$ and $S_\chi(V \setminus \{0\})$ have length 2 and have a unique irreducible subrepresentation. The unique irreducible subrepresentation of $S_\chi(V)$ is isomorphic to

$$\ker(j_\chi) \cong \text{coker}(j_\chi) \cong \left( \sigma_{V, \chi} |_{\cdot |_{F^x}^\frac{n}{2} C_F} \otimes |\det|_{F^x}^\frac{1}{2} \right)^*.$$ 

Note that $S(V)$ carries an action of $\text{GL}_1(F) \times \text{GL}(V)$ which is defined by

$$((g_1, g_2), f)(x) := f(g_2^{-1} x g_1), \quad (g_1, g_2) \in \text{GL}_1(F) \times \text{GL}(V), f \in S(V), x \in V.$$ 

Denote by $\Theta(\chi)$ the full theta lift of the representation $\chi$ of $\text{GL}_1(F)$ to $\text{GL}(V)$. Then we have

$$\left( S(V) \otimes (|\cdot |_{F^x}^\frac{n}{2} \otimes |\det|_{F^x}^\frac{1}{2}) \right)_{\text{GL}_1(F), \chi} \cong \chi \otimes \Theta(\chi)$$

as representations of $\text{GL}_1(F) \times \text{GL}(V)$. It follows that $\Theta(\chi) \cong S_{\chi^{-1}}(V)$ in our setting. It is a fundamental fact that the full theta lift always has a unique irreducible quotient whenever it is nonzero (see [How, Wald, Min, GT, GaS]). Howe also expects that in many cases, the full theta lift also has a unique irreducible
subrepresentation, and the irreducible subrepresentation is “large” and the irreducible quotient representation is “small”. Theorem 1.1 provides some evidences for these expectations.

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2. Distributions supported at 0

We continue with the notation of the Introduction. For every \( \chi \in C_F \), we shall define a representation \( \sigma_{V, \chi} \) of \( GL(V) \) in what follows. If \( F \) is non-archimedean, we define \( \sigma_{V,1} \) to be the one dimensional trivial representation for the unique member 1 \( \in C_F \). If \( F \cong \mathbb{R} \), let \( \chi = (t_1|_{F^\times})^r \cdot (t_2|_{F^\times})^s \in C_F \), \( (r, s \in \mathbb{N}) \) (as is defined in the Introduction), we define

\[
\sigma_{V, \chi} := \text{Sym}^r(V \otimes_{F^\times} \mathbb{C}) \otimes \text{Sym}^s(V \otimes_{F^\times} \mathbb{C}).
\]

Here and henceforth, Sym\(^r\) indicates the \( r \)-th symmetric power. If \( F \cong \mathbb{C} \), let \( \chi = (t_1|_{F^\times})^r \cdot (t_2|_{F^\times})^s \in C_F \), \( (r, s \in \mathbb{N}) \), we define

\[
\sigma_{V, \chi} := \text{Sym}^r(V \otimes_{F^\times} \mathbb{C}) \otimes \text{Sym}^s(V \otimes_{F^\times} \mathbb{C}).
\]

In all cases, \( \sigma_{V, \chi} \) is an irreducible finite dimensional representation of \( GL(V) \) of central character \( \chi \).

Let \( S^*(V, \{0\}) \) denotes the space of tempered distributions on \( V \) whose support is contained in \( \{0\} \). For each character \( \chi : F^\times \to \mathbb{C}^\times \), put

\[
S^*(V, \{0\})^\chi := S^*(V, \{0\}) \cap S^*(V)^\chi,
\]

which is a representation of \( GL(V) \).

Denote by \( \delta_0 \) the Dirac distribution. Set \( V = F^n \). If \( F = \mathbb{R} \), for \( I = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n \), let \( \partial^I \) be the differential operator which takes \( k_i \)-th derivative for the \( i \)-th variable for all \( 1 \leq i \leq n \). If \( F = \mathbb{C} \), define \( \partial^{I_1} \partial^{I_2} \) for \( I_1, I_2 \in \mathbb{N}^n \) similarly. By [SS Theorem 1.7], we have

\[
S^*(V, \{0\}) = \begin{cases} 
\text{span}\{\partial^I \delta_0 | I \in \mathbb{N}^n\}, & \text{if } F = \mathbb{R}; \\
\text{span}\{\partial^{I_1} \partial^{I_2} \delta_0 | I_1, I_2 \in \mathbb{N}^n\}, & \text{if } F = \mathbb{C}; \\
\text{span}\{\delta_0\}, & \text{if } F \text{ is non-archimedean}.
\end{cases}
\]

Then it is easy to deduce the following

Lemma 2.1. If \( \chi \in C_F \), then \( S^*(V, \{0\})^\chi \cong \sigma_{V, \chi} \); otherwise it is zero. Moreover,

\[
S^*(V, \{0\}) = \bigoplus_{\chi \in C_F} S^*(V, \{0\})^\chi.
\]
3. Fourier transform

Denote by $V^*$ the dual space of $V$. The Fourier transform yields a linear isomorphism

$$ F : \mathcal{S}(V) \otimes |\text{det}|_{F}^{-\frac{1}{2}} \rightarrow \mathcal{S}(V^*) \otimes |\text{det}|_{F}^{-\frac{1}{2}}, \quad \phi \otimes 1 \mapsto \hat{\phi} \otimes 1, $$

where

$$ \hat{\phi}(\lambda) := \int_{V} \phi(x) \psi(\lambda(x)) \, dx. $$

Here $d\, x$ is a fixed Haar measure on $V$, and $\psi$ is a fixed non-trivial unitary character on $F$. It is routine to check that

$$ F(g.\eta) = g^{-t}(F(\eta)), \quad \text{for all } g \in \text{GL}(V), \eta \in \mathcal{S}(V) \otimes |\text{det}|_{F}^{-\frac{1}{2}}. $$

Here $g^{-t} \in \text{GL}(V^*)$ denotes the inverse transpose of $g$. Consequently, $F$ induces a linear isomorphism

$$ (8) \quad \mathcal{S}_\chi(V) \cong \mathcal{S}_{\chi^{-1}}(V^*) $$

which is intertwining with respect to the isomorphism $\text{GL}(V) \rightarrow \text{GL}(V^*), g \mapsto g^{-t}$.

**Lemma 3.1.** If $n \geq 2$, then the space $\mathcal{S}_\chi(V)$ is infinite dimensional.

**Proof.** Since $C^+(n)$ and $C^-(n)$ are disjoint, by using the isomorphism (8), it suffices to prove the lemma in the case when $\chi \notin C^-(n)$.

As a reformulation of (8), we have

$$ (9) \quad \mathcal{S}^*(V) \chi^{-1} \cdot |\cdot|_{F}^{-\frac{1}{2}} \cong (\mathcal{S}_\chi(V))^* \otimes |\text{det}|_{F}^{-\frac{1}{2}}, $$

Note that $\chi \notin C^-(n)$ if and only if $\chi^{-1} \cdot |\cdot|_{F}^{-\frac{1}{2}} \notin \text{Hom}_{\text{alg}}(F^\chi, \mathbb{C}^\chi)$. Thus we only need to show that $\mathcal{S}^*(V) \chi$ is infinite dimensional when $\chi \notin \text{Hom}_{\text{alg}}(F^\chi, \mathbb{C}^\chi)$.

For each one dimensional subspace $L$ of $V$, let $\eta_L$ denote a nonzero distribution in the one dimensional space $\mathcal{S}^*(L)^\chi$. Assuming $\chi \notin \text{Hom}_{\text{alg}}(F^\chi, \mathbb{C}^\chi)$, we know from Lemma 2.1 that the support of $\eta_L$ equals $L$. Then the infinite family

$$ \{ \text{the push-forward of } \eta_L \text{ through the embedding } L \hookrightarrow V \}, $$

where $L$ runs over all one dimensional subspace of $V$, is linearly independent in $\mathcal{S}^*(V)^\chi$. This shows that the space $\mathcal{S}^*(V)^\chi$ is infinite dimensional.

\[ \square \]

4. **Proof of theorem 1.1**

In this section, assume that $n \geq 2$. Recall the infinite dimensional representation $\mathcal{S}_\chi(V \setminus \{0\}) \cong \text{Ind}_{\text{GL}(V)}^{\text{GL}(V^*)} \chi \otimes 1$ from the Introduction. Combining [HL Section 2.4 and 3.4] and [GoS Theorem 1.1], we get the following lemma.
Lemma 4.1. (a) If $\chi \notin C^+(n) \cup C^-(n)$, then the representation $\text{Ind}_{P(v_0)} GL(V) \chi \otimes 1$ is irreducible; otherwise, it has length 2 and has a unique irreducible subrepresentation.

(b) If $\chi \in C^+(n)$, then the irreducible quotient representation of $\text{Ind}_{P(v_0)} GL(V) \chi \otimes 1$ is isomorphic to $\sigma_{v_0} \otimes \det |a|^{\frac{1}{2}}$.

(c) If $\chi \in C^-(n)$, then the irreducible subrepresentation of $\text{Ind}_{P(v_0)} GL(V) \chi \otimes 1$ is isomorphic to $\left( \sigma_{v_0} \otimes \det |a|^{\frac{1}{2}} \right)^*$.

Proof. Let $F = \mathbb{R}$. Set $G = GL(V)$ and $P = P(v_0)$. We may assume that $v_0 = (1, 0, \ldots, 0)^t$, thus $P$ has the form $P = \begin{pmatrix} \mathbb{R}^* & \text{GL}_{n-1}(\mathbb{R}) \\ 0 & \mathbb{R}^* \end{pmatrix}$, and $P^o = P^o(v_0)$ is the subgroup of $P$ with the first column being $(1, 0, \ldots, 0)^t$. The modular character $\Delta_P$ of $P$ satisfies

$$\Delta_P(p) = |\det(\text{Ad}_p)| = |a|^{n-1}|\det g|^{-1} = |a|^{n}|\det p|^{-1},$$

where $p = \begin{pmatrix} a & x \\ 0 & g \end{pmatrix} \in P$. Denote by $u\text{Ind}_P^G(\chi \otimes 1)$ the non-normalized induction and observe that

$$u\text{Ind}_P^G((\chi \otimes 1) \otimes \Delta_P^*) = \text{Ind}_P^G(\chi \otimes 1).$$

For any representation $\pi$ of $P$ and any character $\chi'$ of $G$, there is a natural isomorphism

$$\text{Ind}_P^G(\pi \otimes \chi'|_F) \cong \chi' \otimes u\text{Ind}_P^G \pi. \quad (10)$$

It follows that

$$\text{Ind}_P^G(\chi \otimes 1) \cong |\det|^{-\frac{1}{2}} \otimes u\text{Ind}_P^G(\chi \cdot |\cdot|^{\frac{1}{2}} \otimes 1) \quad (11)$$

by setting $\pi = \chi \otimes 1$ and $\chi' = |\det|^{-\frac{1}{2}}$.

Let $Q = \begin{pmatrix} \text{GL}_{n-1}(\mathbb{R}) & \mathbb{R}^* \\ 0 & \mathbb{R}^* \end{pmatrix}$ be another parabolic subgroup of $G$, and let $Q^o$ be the subgroup of $Q$ with the last row being $(0, \ldots, 0, 1)$. Then $Q = Q^o \times \mathbb{R}^*$. Set $w = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. We have an isomorphism

$$\tau : u\text{Ind}_P^G(\chi \otimes 1) \overset{\cong}{\rightarrow} u\text{Ind}_Q^G(1 \otimes \chi^{-1}), \quad f \mapsto (g \mapsto f(wg^{-t}w)), \quad (12)$$

which is intertwining with respect to the isomorphism $G \rightarrow G, g \mapsto wg^{-t}w$.

By [III, Theorem 3.41, 3.42, 3.43 and 3.44, applied with $k = 1$], the representation $u\text{Ind}_Q^G(1 \otimes \chi)$ is irreducible if $\chi \notin |\cdot|^{-\frac{1}{2}}C^+(n) \cup |\cdot|^{-\frac{1}{2}}C^-(n)$; otherwise, it has
length 2 and has a unique irreducible subrepresentation. Hence the assertion (a) follows from the isomorphisms (11) and (12).

If \( \chi = (\iota|\mathbb{R}|\times(\mathbb{R} \times \mathbb{R})) \), for \( I = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n \) which satisfies \(|I| := k_1 + k_2 + \cdots + k_n = r\), we define

\[
 f_I \begin{pmatrix}
  * & * & \cdots & * \\
  \vdots & \vdots & \ddots & \vdots \\
  * & * & \cdots & * \\
  a_1 & a_2 & \cdots & a_n
\end{pmatrix} := a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}.
\]

It is easy to check that \( f_I \in \mathsf{Ind}_P^G(1 \otimes \chi) \), and \( W := \text{span}\{f_I | I \in \mathbb{N}^n, |I| = r\} \) is a (finite dimensional) subrepresentation of \( \mathsf{Ind}_P^G(1 \otimes \chi) \) which is isomorphic to \( \sigma_{V,\chi} \).

Applying the isomorphisms (11) and (12), we get the Part (c).

Note that the representations \( \mathsf{Ind}_P^G(\chi \otimes 1) \) and \( \mathsf{Ind}_P^G(\chi^{-1} \otimes 1) \) are contragredient ([Wal Lemma 5.2.4]). Thus (c) implies (b), and we complete the proof for \( F = \mathbb{R} \).

The proofs for the cases that \( F = \mathbb{C} \) and \( F \) is non-archimedean are analogous, and we omit the details here.

\[\square\]

The short exact sequence

\[0 \to \mathcal{S}(V \setminus \{0\}) \to \mathcal{S}(V) \to (\mathcal{S}^*(V, \{0\}))^* \to 0\]

induces an exact sequence

\[\mathcal{S}_\chi(V \setminus \{0\}) \xrightarrow{j_\chi} \mathcal{S}_\chi(V) \to \left(\mathcal{S}^*(V, \{0\})^{\chi^{-1} \cdot |F|^{\frac{1}{2}}} \right)^* \otimes |\det|^{-\frac{1}{2}} \to 0.\]

Here we have used the natural isomorphism

\[
 \left(\mathcal{S}^*(V, \{0\})^* \otimes |\det|^{-\frac{1}{2}} \right)_{F^*,\chi} \cong \left(\mathcal{S}^*(V, \{0\})^{\chi^{-1} \cdot |F|^{\frac{1}{2}}} \right)^* \otimes |\det|^{-\frac{1}{2}}
\]

of representations of \( \text{GL}(V) \).

If \( \chi \notin C^-/n \), then Lemma 2.1 and the exactness of (13) imply that \( j_\chi \) is surjective. We argue according to the three cases of Theorem 1.1.

**Case a:** \( \chi \notin C^+(n) \cup C^-(n) \).

In this case, \( \mathcal{S}_\chi(V \setminus \{0\}) \) is irreducible by Part (a) of Lemma 4.1. By Lemma 3.1, \( \mathcal{S}_\chi(V \setminus \{0\}) \) is nonzero. Therefore the surjective homomorphism \( j_\chi \) is an isomorphism. This proves Part (a) of Theorem 1.1.

**Case b:** \( \chi \in C^+(n) \).

In this case, by Lemma 5.1, \( \ker(j_\chi) \) is a subrepresentation of \( \mathcal{S}_\chi(V \setminus \{0\}) \) of infinite codimension. Then Parts (a) and (b) of Lemma 4.1 implies that \( \ker(j_\chi) = \{0\} \), which further implies Part (b) of Theorem 1.1.

**Case c:** \( \chi \in C^-(n) \).
In this case, applying Part (b) of Theorem 1.1 to $V^*$, and using the isomorphism (8), we know that $S_\chi(V)$ has length 2 and has a unique irreducible subrepresentation. The exact sequence (13) and Lemma 2.1 imply that
\[ \text{coker}(j_\chi) \cong \left( \sigma_{V,\chi^{-1} \cdot | \cdot |^{-n/2} \otimes |\text{det}|^{1/2}} \right)^*. \]
Thus the image of $j_\chi$ is irreducible, and hence $\ker(j_\chi)$ is irreducible as $S_\chi(V \setminus \{0\})$ has length 2. Applying Part (c) of Lemma 4.1, this proves Part (c) of Theorem 1.1.

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ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA
E-mail address: xuehuajian@126.com