SOME REMARKS ON PRODUCING HOPF ALGEBRAS

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We report some observations concerning two well-known approaches to construction of quantum groups. Thus, starting from a bialgebra of inhomogeneous type and imposing quadratic, cubic or quartic commutation relations on a subset of its generators we come, in each case, to a $q$-deformed universal enveloping algebra of a certain simple Lie algebra. An interesting correlation between the order of initial commutation relations and the Cartan matrix of the resulting algebra is observed. Another example demonstrates that the bialgebra structure of $sl_q(2)$ can be completely determined by requiring the $q$-oscillator algebra to be its covariant comodule, in analogy with Manin’s approach to define $SL_q(2)$ as a symmetry algebra of the bosonic and fermionic quantum planes.

A lot of recipes are known to produce new examples of quantum groups (quasitriangular Hopf algebras). The aim of the present note is to demonstrate some interesting results which can be obtained within the two of these approaches. Although the Hopf algebras thus obtained proved to be not new, the way of producing them looks instructive and fruitful.

Our first example deals with the Hopf algebras generated by a constant matrix solution $R$ of the Yang-Baxter equations. The appropriate technique has been elaborated in [1, 2]. Here we actually use its modification described in [3, 4]. Let us consider a bialgebra of inhomogeneous type [5] with generators $\{1, t^i_j, u^i_j, E_i, F_i\}$ which form matrices $T, U$, a row $E$ and a column $F$, respectively. Its multiplication relations are

\begin{align}
R_{12} T_1 T_2 &= T_2 T_1 R_{12}, & E_1 T_2 &= T_2 E_1 R_{12}, \\
R_{12} U_1 U_2 &= U_2 U_1 R_{12}, & F_2 U_1 &= R_{12} U_1 F_2, \\
R_{12} U_1 T_2 &= T_2 U_1 R_{12}, & T_2 F_1 &= R_{12} F_1 T_2, \\
E F - F E &= T - U, & U_1 E_2 &= E_2 U_1 R_{12},
\end{align}

the coalgebra structure is

\begin{align}
\Delta(T) &= T \otimes T, & \Delta(E) &= E \otimes T + 1 \otimes E, \\
\Delta(U) &= U \otimes U, & \Delta(F) &= F \otimes 1 + U \otimes F,
\end{align}

and the duality conditions look like

\begin{align}
< U, T >= R, & \quad < 1, T >=< U, 1 >=< F, E >= 1.
\end{align}

In [4] it is shown how an antipode can be introduced into this bialgebra thus making it a Hopf algebra.
One sees that commutation relations between the $E_i$-generators, as well as between $F_i$, are missing in (1). In [3] a method is proposed how to add such relations to (1) without destroying the bialgebra (the Hopf algebra structure also survives). The method consists in imposing upon $E$ the relations of the $N$-th order

$$E_1 \ldots E_N \omega_{1\ldots N} = 0 \quad (4)$$

or, in more explicit form,

$$E_{i_1} \ldots E_{i_N} \omega_{i_1\ldots i_N} = 0,$$

for every solution $\omega$ of the following system of equations:

$$\left[ \begin{array}{c} N \\ n \end{array} \right]_B \omega_{1\ldots N} = 0 \quad (n = 1, 2, \ldots, N - 1), \quad (5)$$

where $B \equiv \tilde{R} \equiv \sigma R$ and $\left[ \begin{array}{c} N \\ n \end{array} \right]_B$ are generalized binomial coefficients [3], [4], which are defined as follows. Let some $x$ and $y$ obey

$$x_1y_2 = y_1x_2B_{12}. \quad (6)$$

Then

$$(x_1 + y_1) \ldots (x_N + y_N) = \sum_{n=0}^{N} y_1 \ldots y_n x_{n+1} \ldots x_N \left[ \begin{array}{c} N \\ n \end{array} \right]_B. \quad (7)$$

Generalized binomial coefficients enter (5) due to the second relation in the first line of (1) which entails

$$(1 \otimes E_1)(E_2 \otimes T_2) = (E_1 \otimes T_1)(1 \otimes E_2)B_{12}. \quad$$

Taking then the coproduct of (4) and identifying $\Delta(E)$ with $y + x$ we can use (5),(7) and come to the condition (5).

It can be easily shown that $\theta = \omega^t$ (i.e. $\theta_{i\ldots j} = \omega^{i\ldots j}$) serves to define the commutation rules for $F^i$:

$$\theta_{1\ldots N} F_1 \ldots F_N = 0. \quad (8)$$

This completes the definition of our bialgebra (actually, a Hopf algebra).

The above construction has been illustrated in [3] for the case of diagonal $R$. Here we choose to consider the simplest non-diagonal $R$-matrix

$$R = q^\nu \left( \begin{array}{cccc} q & 1 & \ldots & 1 \\ 1 & q - q^{-1} & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & \ldots & q \end{array} \right). \quad (9)$$

Dots denote zeros, and a factor $q^\nu$ is needed to achieve such a normalization of the $R$- or $B$-matrix for which the solutions of (5) may exist. Below we give several (not all) explicit solutions for $\omega$ at $N = 2, 3, 4$ with $\nu$ equal to 1, 0, $-1/3$, respectively.
Taking into account the explicit form of generalized binomial coefficients results in the following equations for $\omega$ (as usual, $B_i \equiv B_{i+1}$):

$N = 2, \nu = 1$:

$$ (1 + B) \omega = 0; \quad (10) $$

$N = 3, \nu = 0$:

$$ \begin{cases} (1 + B_1 + B_1 B_2) \omega = 0, \\ (1 + B_2 + B_2 B_1) \omega = 0; \end{cases} \quad (11) $$

$N = 4, \nu = -1/3$:

$$ \begin{cases} (1 + B_1 + B_1 B_2 + B_1 B_2 B_3) \omega = 0, \\ (1 + B_2 + B_2 B_1 + B_2 B_3 + B_2 B_1 B_3 + B_2 B_1 B_3 B_2) \omega = 0, \\ (1 + B_3 + B_3 B_2 + B_3 B_2 B_1) \omega = 0. \end{cases} \quad (12) $$

For $N = 2$ a solution (tensor $\omega^{ij}$) depends on a single parameter $\rho$:

$$ \omega^{11} = \omega^{22} = 0, \quad \omega^{21} = \rho, \quad \omega^{12} = -q\rho. $$

For $N = 3$ it is 2-parametric,

$$ \omega^{111} = \omega^{222} = 0, \quad \omega^{211} = \rho, \quad \omega^{112} = q\rho, \quad \omega^{221} = \sigma, $$

$$ \omega^{121} = -(1 + q)\rho, \quad \omega^{122} = q\sigma, \quad \omega^{212} = -(1 + q)\sigma, $$

and for $N = 4$ it has 3 parameters:

$$ \begin{align*}
\omega^{1111} & = \omega^{2222} = 0, \quad \omega^{2221} = \rho, \quad \omega^{1222} = -q\rho, \quad \omega^{2111} = \sigma, \quad \omega^{1112} = -q\sigma, \\
\omega^{2212} & = -q^{-1/3}(1 + q^{2/3} + q^{4/3})\rho, \quad \omega^{2122} = (1 + q^{2/3} + q^{4/3})\rho, \\
\omega^{1211} & = -q^{-1/3}(1 + q^{2/3} + q^{4/3})\sigma, \quad \omega^{1121} = (1 + q^{2/3} + q^{4/3})\sigma, \\
\omega^{2211} & = -q^{1/3}(1 + q^{2/3} + q^{4/3})\varphi, \quad \omega^{1212} = -q^{-1/3}(1 + q^{2/3} + q^{4/3})\varphi, \\
\omega^{2121} & = -(q + 2q^{1/3} + q^{5/3})\varphi, \quad \omega^{1212} = (q^{-1} + q^{-1/3} + 2q^{1/3})\varphi.
\end{align*} $$

These solutions produce the following commutation relations.

$N = 2$:

$$ E_2 E_1 - q E_1 E_2 = 0. \quad (13) $$

$N = 3$:

$$ \begin{align*}
E_2 E_1 E_1 - (1 + q) E_1 E_2 E_1 + q E_1 E_1 E_2 & = 0, \\
E_2 E_2 E_1 - (1 + q) E_2 E_1 E_2 + q E_1 E_2 E_2 & = 0.
\end{align*} \quad (14) $$

$N = 4$:

$$ \begin{align*}
E_2 E_1 E_1 E_1 + (1 + q^{2/3} + q^{4/3})(E_1 E_2 E_1 E_1 - q^{-1/3} E_1 E_2 E_1 E_1) - q E_1 E_1 E_2 E_2 & = 0, \\
(q^{1/3} + q^{-1/3})(E_2 E_2 E_1 E_1 - E_1 E_2 E_1 E_2) + (1 + q^{-1})(E_1 E_2 E_2 E_1 + E_2 E_1 E_1 E_2) & = 0, \\
(2 + q^{-2/3} + q^{-4/3}) E_1 E_2 E_1 E_2 - (2 + q^{2/3} + q^{4/3}) E_1 E_1 E_2 E_1 & = 0, \\
E_2 E_2 E_1 E_1 + (1 + q^{2/3} + q^{4/3})(E_2 E_2 E_1 E_2 - q^{-1/3} E_2 E_2 E_1 E_2) - q E_1 E_2 E_2 E_2 & = 0.
\end{align*} \quad (15) $$
The corresponding equations for $F^i$ are the same, with lower indices replaced by upper ones.

To look more closely at the structure of the resulting algebra let us arrange its generators as follows:

$$T = \left( \begin{array}{cc} a & 0 \\ b & c \end{array} \right), \quad U = \left( \begin{array}{cc} x & y \\ 0 & z \end{array} \right), \quad E = (d e), \quad F = \left( \begin{array}{c} f \\ g \end{array} \right)$$

(zeros are to make the duality relations (3) nondegenerate). The following commutation relations are of special interest for us:

$$N = 2, 3, 4:$$

$$eb - q^\nu be = q^\nu (q - q^{-1}) cd, \quad db - q^{1+\nu} bd = 0$$

(17)

(these stem from (1)) and, in addition, $N = 2$:

$$ed - qde = 0,$$

(18)

$N = 3$:

$$eed - (1 + q) ede + qdee = 0,$$

(19)

$N = 4$:

$$eed + (1 + q^{2/3} + q^{4/3}) (edere - q^{-1/3} eede) - qdee = 0,$$

(20)

which are the last equations in (13)-(15). Remarkably, all the other relations in (14), (15) prove to be dependent on (17), (19) and (20).

Moreover, it is not difficult to show that our Hopf algebras for $N = 2, 3, 4$ are isomorphic to $U_q sl(2), U_q sp(2)$ and $U_q g_2$, respectively. In other words, the $N$-th power in the relations (4) leads to the $q$-deformed universal enveloping algebra $U_q g$ with $g$ defined by the Cartan matrix

$$\left( \begin{array}{cc} 2 & -1 \\ -N + 1 & 2 \end{array} \right).$$

(21)

The isomorphism is given by the following relations (we omit normalization factors which can be easily restored in each case):

$$\bar{c}b \sim X_1^+, \quad a \sim k_1^2 k_2^2, \quad y\bar{z} \sim X_1^-, \quad \bar{x} \sim k_1'^2 k_2'^2$$

$$e \sim X_2^+, \quad c \sim k_2^2, \quad g \sim X_2^-, \quad \bar{z} \sim k_2'^2$$

(22)

where $\bar{c}, \bar{z}, \bar{x}$ are inverses of $c, x, z$, and $\{k_i, X_i^\pm\}$ the standard Drinfeld-Jimbo generators of $U_q g$. The second pair $k_1', k_2'$ of Cartan generators is to be identified with $k_1, k_2$. Second equations in (17), as well as (13)-(21), are associated (by means of the first equations in (17)) with $q$-deformed Serre relations produced by the Cartan matrix (21). An intriguing question whether this correspondence will remain for $N > 4$ is now under investigation.

The second topic of the present note concerns quite another procedure for constructing Hopf algebras, advocated by Manin [7] in the case of $q$-plane. Recall [2] that
the noncommutative algebras of the coordinate functions on the quantum planes, both
bosonic and fermionic, are respected by coaction of $SL_q(2)$. Moreover, the require-
ment of such covariance unambiguously fixes the bialgebra structure of $SL_q(2)$. Let
us see now that, quite analogously, the bialgebra structure of $sl_q(2)$ is unambiguously
determined if we require the $q$-oscillator algebra to be its covariant comodule.

The idea of this possibility has been prompted by Kulish’s paper [8] where the
$sl_q(2)$-comodule structure of the $q$-oscillator has been discovered. Let us invert the
problem and consider the $q$-oscillator-type algebra
\[ AB - q^2 BA = 1 \]  
(23)
as a would-be covariant comodule under the coaction of some bialgebra. Namely, let
us require that the commutation relations (23) are respected by the coaction
\[
A \rightarrow z \otimes A + x \otimes 1,
B \rightarrow z \otimes B + y \otimes 1
\]  
(24)
of some bialgebra $A$ with generators $x, y, z$. Then the (coassociative) coalgebra struc-
ture of $A$ is unambiguously fixed by
\[
\Delta(z) = z \otimes z,
\Delta(x) = x \otimes 1 + z \otimes x,
\Delta(y) = y \otimes 1 + z \otimes y,
\]  
(25)
and its algebra structure by
\[
xz = q^2 zx,
yz = q^2 yz,
q^2 yx - xy = z^2 - 1.
\]  
(26)
Using the substitution
\[
z = q^{-H}, \quad x = \sqrt{q - q^{-1} q^{-H/2} X^+}, \quad y = \sqrt{q - q^{-1} X^{-} q^{-H/2}},
\]  
(27)
we come to a conclusion that $A$ is precisely $sl_q(2)$. Really, equations (25),(26) imply
\[
\Delta(X^\pm) = X^\pm \otimes q^{H/2} + q^{-H/2} \otimes X^\pm, \quad \Delta(H) = H \otimes 1 + 1 \otimes H,
\]
\[
[H, X^\pm] = \pm 2 X^\pm, \quad [X^+, X^-] = q^{H} - q^{-H}.
\]
This nonstandard approach to deriving $sl_q(2)$ may shed some light on the role of $q$-
oscillator in the general context of quantum groups.

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