Abstract

In this paper, we obtain some sufficient conditions for the $D$-completion of a $T_0$ space to be the well-filterification of this space, the well-filterification of a $T_0$ space to be the sobrification of this space and the $D$-completion of a $T_0$ space to be the sobrification, respectively. Moreover, we give an example to show that a tapered closed set may be neither the closure of a directed set nor the closed $KF$-set, respectively. Because the tapered closed set is a closed $WD$-set, the example also gives a negative answer to a problem proposed by Xu. Meantime, a new direct characterization of the $D$-completion of a $T_0$ space is given by using the notion of pre-$c$-compact elements.

Keywords: $D$-completion, Well-filterification, Sobrification, Join continuous

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1. Introduction

$D$-completion, well-filterification and sobrification of a $T_0$ space play a fundamental role in non-Hausdorff topological spaces. We know that the $D$-completion of a $T_0$ space is contained in the well-filterification of this space, and the well-filterification is contained in the sobrification. But the converses of them are not necessarily true.

In [6], Keimel and Lawson verified that the $D$-completion of a $T_0$ space is the sobrification of this space if the space is a $c$-space or $qc$-space. Lawson, Wu and Xi gave the sufficient condition that the space $X$ is core-compact for a well-filtered space $X$ to be sober in [8]. Xu and Shen proved that every first countable well-filtered space is sober in [14]. Xi and Lawson obtained the result that every monotone convergence space $X$ with the property that $\downarrow(K \cap A)$ is a closed subset of $X$ for any closed $KF$ set $A$ and compact saturated set $K$ is well-filtered in [12]. Therefore, naturally, there are some questions in the following:

1. Whether the well-filterification of a first countable $T_0$ space $X$ coincides with the sobrification of this space;
2. Whether the well-filterification of a second countable $T_0$ space agrees with the sobrification;
3. Whether the $D$-completion of a $T_0$ space $X$ with the property that $\downarrow(K \cap A)$ is a closed subset of $X$ for any closed $KF$ set $A$ and compact saturated set $K$ coincides with the well-
filterification.

In [10], Miao and Li proposed the concept of join-continuous poset and showed that every core-
compact join continuous dcpo is sober. They also obtain the result that a monotone convergence
space $X$ with the property that $\downarrow(A \cap W)$ is a closed subset of $X$ for any irreducible closed subset $A$ of $X$ and upper set $W$ is a sober space. Naturally, some questions are raised below.

(4) Whether the $D$-completion of a locally compact $T_0$ space $X$ with the property that $\downarrow(K \cap A)$ is a closed subset of $X$ for any closed $KF$ set $A$ and compact saturated set $K$ agrees with the sobrification;

(5) Whether the $D$-completion of a core-compact join-continuous poset $P$ coincides with the sobrification;

(6) Whether the $D$-completion of a $T_0$ space $X$ with the property that $\downarrow(A \cap W)$ is a closed subset of $X$ for any irreducible closed subset $A$ of $X$ and upper set $W$ coincides with the sobrification.

In [9], Liu, Li and Wu give a counterexample to show that $WD(X)$ may not agree with $KF(X)$ for any $T_0$ space $X$, which solved the open problem proposed by Xu in [15]. Zhang and Li provided a direct characterization of the $D$-completion of a $T_0$ space using the tapered closed subsets ([16]). We know that every directed set is a tapered set. So, naturally there is a problem in the following.

**Problem 1.1.** Let $X$ be a $T_0$ space and $A$ a tapered closed subset of $X$. Does there exist a directed subset $D$ of $X$ such that $A = cl(D)$?

In [17], Zhao and Fan gave the $D$-completion of a poset. Keimel and Lawson presented a standard $D$-completion of a $T_0$ topological space in [6]. However, both of their work do not give the concrete elements of the $D$-completion of a poset or a $T_0$ space like the sobrification of the spaces. In [16], Zhang and Li provided a direct description of the elements of the $D$-completion of a $T_0$ space by continuous functions. Shall we just characterize the $D$-completion of a $T_0$ space by the elements of the space itself?

The purpose of this paper is to investigate the above questions. We give some sufficient conditions for the $D$-completion of a $T_0$ space to be the well-filterification, the well-filterification of a $T_0$ space to be the sobrification and the $D$-completion of a $T_0$ space to be the sobrification, respectively. For Problem 1.1, we propose a counterexample to reveal that a tapered closed set may not be the closure of a directed set and the closed $KF$-set, respectively. Meantime, a new direct characterization of the $D$-completion of a $T_0$ space is given by using the notion of pre-c-compact elements.

2. Preliminaries

Without further references, the posets mentioned here are all endowed with the Scott topology.

Let $X$ be a topological space. We denote the set of all closed sets of $X$ by $\Gamma(X)$ and all open sets by $O(X)$. If $P$ is a poset, $\sigma(P)$ denotes the set of all Scott open sets and $\Gamma(P)$ the set of all Scott closed sets. $A$ is called $d$-closed if $D \subseteq A$ implies that $\sup D \in A$ for any directed subset of $A$. $cl_d(A)$ represents $A$ is $d$-closed. Given a topological space $(X, \tau)$, we define $x \leq y$ iff $x \in cl(y)$. Hence, $X$ with its specialization order is a poset. For any set $A \subseteq X$, we denote the closure of $A$ by $cl(A)$. The irreducible sets of $X$ are denoted by $IRR(X)$ and the irreducible closed sets by $IRR(X)$, all upper sets of $X$ by $up(X)$.
Definition 2.1. ([3])

(i) Let \( P \) be a poset. A subset \( D \) of \( P \) is directed provided it is nonempty and every finite subset of \( D \) has an upper bound in \( D \). We denote that \( D \) is a directed subset of \( P \) by \( D \subseteq^\uparrow P \).

(ii) A poset \( P \) is a dcpo if every directed subset \( D \) has the supremum.

Definition 2.2. ([II])

(i) A topological space \( X \) is locally compact if for every \( x \in X \) and any open neighborhood \( U \) of \( x \), there is a compact saturated subset \( Q \) of \( X \) such that \( x \in \text{int}(Q) \) and \( Q \subseteq U \).

A set \( K \) of a topological space is called saturated if it is the intersection of its open neighborhood \((K = \uparrow K \text{ in its specialization order})\). If \( A \) is any subset of \( X \), the intersection sat\( A \) of all its open neighborhood is a saturated set called its saturation.

(ii) A topological space \( X \) is core-compact if \( \mathcal{O}(X) \) is a continuous lattice.

(iii) A topological space \( X \) is said to be first countable if each point has a countable neighbourhood basis (local base). That is, for each point \( x \in X \) there exists a sequence \( N_1, N_2, \ldots \) of the neighbourhoods of \( x \) such that for any neighbourhood \( N \) of \( x \), there exists an integer \( i \) with \( N_i \) contained in \( N \).

(iv) A topological space is said to be second countable if its topology has a countable base.

Definition 2.3. ([II]) A topological space \( X \) is sober if it is \( T_0 \) and every irreducible closed subset of \( X \) is the closure of a (unique) point.

Definition 2.4. ([III]) We shall say that a space \( X \) is well-filtered if for each filter basis \( \mathcal{C} \) of compact saturated sets and each open set \( U \) with \( \bigcap \mathcal{C} \subseteq U \), there is a \( K \in \mathcal{C} \) with \( K \subseteq U \).

For a topological space \( X \), the compact saturated subsets of \( X \) are denoted by \( Q(X) \). We write \( \mathcal{F} \subseteq_{fl} Q(X) \) represents that \( \mathcal{F} \) is a filtered subfamily of \( Q(X) \) and \( F \subseteq_{fin} X \) represents that \( F \) is a finite subset of \( X \).

Definition 2.5. ([III]) Let \( X \) be a topological space. A nonempty subset \( A \in KF(X) \) if and only if there exists \( \mathcal{F} \subseteq_{fl} Q(X) \) such that \( cl(A) \) is a minimal closed set that intersects all members of \( \mathcal{F} \). The set of all closed \( KF \)-subsets of \( X \) is denoted by \( KF(X) \).

Definition 2.6. ([14]) A \( T_0 \) space \( X \) is called \( \omega \)-well-filtered, if for any countable filtered family \( \{K_i : i < \omega\} \subseteq Q(X) \) and \( U \in O(X) \), it satisfies

\[
\bigcap_{i<\omega} K_i \subseteq U \Rightarrow \exists i_0 < \omega, K_{i_0} \subseteq U
\]

Definition 2.7. ([14]) Let \( X \) be a \( T_0 \) space. A nonempty subset \( A \in KF_\omega(X) \) if and only if there exists a countable filtered family \( \mathcal{K} \subseteq Q(X) \) such that \( cl(A) \) is a minimal closed set that intersects all members of \( \mathcal{K} \). The set of all closed \( KF_\omega \)-subsets of \( X \) is denoted by \( KF_\omega(X) \).

Definition 2.8. ([14]) A subset \( A \) of a \( T_0 \) space \( X \) is called a \( \omega \)-well-filtered determined set, \( WD_\omega \) set for short, if for any continuous mapping \( f : X \rightarrow Y \) to a \( \omega \)-well-filtered space \( Y \), there exists a unique \( y_A \in Y \) such that \( cl(f(A)) = cl(\{y_A\}) \). The set of all closed \( \omega \)-well-filtered determined subsets of \( X \) is denoted by \( WD_\omega(X) \).

Definition 2.9. ([6]) A \( T_0 \) space is a monotone convergence space if and only if the closure of every directed sets (in the specialization order) is the closure of a unique point.
Lemma 2.13. (4) If a topological space $X$ is a $T_0$ space $X$ is a monotone convergence space $Y$ with a topological embedding $j : X \to Y$ and $\text{cl}_d(j(X)) = Y$.

Definition 2.10. (5) A $D$-completion of a $T_0$ space $X$ is a monotone convergence space $Y$ with a topological embedding $j : X \to Y$ and $\text{cl}_d(j(X)) = Y$.

Definition 2.11. (10) A subset $A$ of a $T_0$ space $X$ is called a well-filtered determined set, $WD$ set for short, if for any continuous mapping $f : X \to Y$ to a well-filtered space $Y$, there exists a unique $y_A \in Y$ such that $\text{cl}(f(A)) = \text{cl}(\{y_A\})$. The set of all closed well-filtered determined subsets of $X$ is denoted by $\text{WD}(X)$.

Lemma 2.13. (4) If a topological space $X$ is second countable, then $X$ is first countable.

Lemma 2.14. (10) The topological space of all tapered closed subsets of a $T_0$ space $X$ is the standard $D$-completion of $X$.

The following construction is due to Ershov [2]. Let topological spaces $X$ and $Y_x$, and $x \in X$, be given. Let

$$Z = \bigcup_{x \in X} Y_x \times \{x\}$$

$$\tau = \{U \subseteq Z \mid (U)_x \in \tau(Y_x) \text{ for any } x \in X \text{ and } (U)_X \in \tau(X)\},$$

where $U_x = \{y \in Y_x \mid (y, x) \in U\}$ for any $x \in X$ and $(U)_X = \{x \in X \mid (U)_x \neq \emptyset\}$.

In this paper, the space $Z = (Z, \tau)$ is denoted by $\sum_X Y_x$. For any subset $A \subseteq Z$, put $(A)_x = \{y \in Y_x \mid (y, x) \in A\}$, and $(A)_X = \{x \in X \mid (A)_x \neq \emptyset\}$.

Lemma 2.15. [2] Let $X$ be a $T_0$ space, $Y_x$ an irreducible $T_0$ space for any $x \in X$, and $Z = \sum_X Y_x$. For all $(y_0, x_0), (y_1, x_1) \in Z$, we have $(y_0, x_0) \leq (y_1, x_1)$ if and only if the following two alternatives hold:

1. $x_0 = x_1$, $y_0 \leq x_0 \leq y_1$,
2. $x_0 < x_1$ and $y_1 = \top_{x_1}$ is the greatest element in $Y_{x_1}$ with respect to the specialization order.

Lemma 2.16. [2] Let $\mathbb{N} = (\mathbb{N}, \tau_{cof})$, where $\tau_{cof}$ denotes the cofinite topology, and $X_n$ an irreducible $T_0$ space for every $n \in \mathbb{N}$, such that there are at most finitely many $X_n$’s that have a greatest element under the specialization order $\leq_{X_n}$. Let $Z = \sum_{\mathbb{N}} X_n$. Then $A$ is a $KF$-set of $Z$ iff there exists a unique $n \in \mathbb{N}$ such that $A \subseteq X_n \times \{n\}$ and $\{y \in X_n \mid (y, x) \in A\}$ is a $KF$-set of $X_n$.

Proposition 2.17. ([2]) For a monotone convergence space $X$, $A$ is a monotone convergence subspace iff $A$ is a sub-decpo of $X$ with the specialization order.

Theorem 2.18. [13] Every locally compact $T_0$ space is a Rudin space, that is, $\text{IRR}(X) = \text{KF}(X)$.

Theorem 2.19. [13] Let $L$ be a dcpo. Then the following statements are equivalent:

1. $\sigma(L)$ is a continuous lattice;
2. For every dcpo or complete lattice $S$, one has $\sigma(S \times L) = \sigma(S) \times \sigma(L)$.

Theorem 2.20. [13] Let $X$ be a $T_0$ space. Then $\text{WD}(X)$ with the lower Vietoris topology is the well-filtered reflection of $X$.
3. Sufficient conditions

**Lemma 3.1.** Let $X$ be a $T_0$ space. Then the following statements are equivalent:

1. If $A \in \text{KF}(X)$, then $\downarrow (A \cap K) \in \Gamma(X)$ for any $K \in Q(X)$;
2. $A$ is a directed closed subset of $X$.

**Proof.** (1) $\Rightarrow$ (2) It suffices to prove that $A$ is directed. Suppose $A \in \text{KF}(X)$. Then there exists $\{K_i\}_{i \in I} \subseteq \text{flt} \, Q(X)$ such that $A$ is a minimal closed set that intersects all members of $\{K_i\}_{i \in I}$ by Definition 2.5.

Claim 1: $A = \downarrow (A \cap K_j)$ for any $j \in I$.

For any $i \in I$, since $\{K_i\}_{i \in I} \subseteq \text{flt} \, Q(X)$, there exists $r \in I$ such that $K_r \subseteq K_i \cap K_j$. This implies that $\emptyset \neq A \cap K_r \subseteq A \cap K_j \cap K_i \subseteq \downarrow (A \cap K_j) \cap K_i$. Then we have $A = \downarrow (A \cap K_j)$ by the minimality of $A$.

Claim 2: $\uparrow x \cap K_i \cap A \neq \emptyset$ for any $x \in A$ and $i \in I$.

$x \in A = \downarrow (A \cap K_i)$ by Claim 1. Thus $\uparrow x \cap A \cap K_i \neq \emptyset$.

Claim 3: $A$ is directed.

Let $x, y \in A$. Then we have $\downarrow (\uparrow x \cap A) \cap K_i \neq \emptyset$ by Claim 2. It follows that $\downarrow (\uparrow x \cap A) = A$ by the minimality of $A$. Note that $y \in A = \downarrow (\uparrow x \cap A)$, thus $\uparrow y \cap \uparrow x \cap A \neq \emptyset$.

(2) $\Rightarrow$ (1) It remains to prove that $\downarrow (A \cap K) \in \Gamma(X)$.

Claim 1: $\downarrow (\uparrow x \cap A) \in \Gamma(X)$ for any $x \in X$.

If $\uparrow x \cap A = \emptyset$, then we have $\downarrow (\uparrow x \cap A) = \emptyset \in \Gamma(X)$. Else $\uparrow x \cap A \neq \emptyset$. Then $x \in A$ because $A$ is a lower set. Obviously, $\downarrow (\uparrow x \cap A) \subseteq A$. Conversely, $\uparrow a \cap \uparrow x \cap A \neq \emptyset$ for any $a \in A$ since $A$ is directed. So we have $\downarrow (\uparrow x \cap A) = A \in \Gamma(X)$.

Claim 2: $\downarrow (K \cap A) \in \Gamma(X)$.

If $K \cap A = \emptyset$, then $\downarrow (K \cap A) = \emptyset \in \Gamma(X)$. Else $K \cap A \neq \emptyset$. Pick $k \in K \cap A$. This means that $A \subseteq \downarrow (k \cap A) \subseteq \downarrow (K \cap A) \subseteq A$ by Claim 1. Hence, $\downarrow (K \cap A) = A \in \Gamma(X)$.

**Theorem 3.2.** Let $X$ be a locally compact $T_0$ space with the property $\downarrow (K \cap A) \in \Gamma(X)$ for any $A \in \text{KF}(X)$ and $K \in Q(X)$. Then the $D$-completion of $X$ coincides with the soberification of $X$.

**Proof.** Let $A \in \text{IRR}(X)$. It suffices to prove that $A$ is directed. By Theorem 2.18, we have $\text{IRR}(X) = \text{KF}(X)$. This implies that $A$ is directed by Lemma 3.1.

Theorem 3.2 gives a positive answer for the problem (4) in the introduction.

**Theorem 3.3.** Let $X$ be a $T_0$ space with the property $\downarrow (K \cap A) \in \Gamma(X)$ for any $A \in \text{KF}(X)$ and $K \in Q(X)$. If $\text{KF}(X)$ endowed with the lower Vietoris topology is the well-filterification of $X$. Then the $D$-completion of $X$ agrees with the well-filterification of $X$.

**Proof.** By Lemma 2.14, we know that all tapered closed subsets of $X$ is the $D$-completion of $X$. Note that directed closed subsets of $X$ must be tapered. We need to prove that $A$ is a directed subset of $X$ for any tapered closed subset $A$ of $X$. It suffices to prove that $D = \{\text{cl}(D) \mid D \subseteq \uparrow X\}$ is a subdepo of $\Gamma(X)$. Let $\{\text{cl}(D_i)\}_{i \in I}$ be any directed subset of $\Gamma(X)$ contained in $D$. Note that $\sup_{i \in I} \text{cl}(D_i) = \text{cl}(\bigcup_{i \in I} \text{cl}(D_i))$. We need to verify that $\sup_{i \in I} \text{cl}(D_i) \in D$. Since $D \subseteq \text{KF}(X)$, we have $\text{cl}(D_i)$ is directed for any $i \in I$ by Lemma 3.1. It suffices to show that $\bigcup_{i \in I} \text{cl}(D_i)$ is a directed subset of $X$. Now let $x, y \in \bigcup_{i \in I} \text{cl}(D_i)$. Then there exists $\{i_x, i_y\} \subseteq I$ such that $x \in \text{cl}(D_{i_x})$ and $y \in \text{cl}(D_{i_y})$. Hence, $\text{cl}(D_{i_x}) \cap \text{cl}(D_{i_y})$ is a directed subset of $X$. By Theorem 2.16, we have $\text{cl}(D_{i_x}) \cap \text{cl}(D_{i_y}) = \text{cl}(D_{i})$ for some $i \in I$. Then $x, y \in \text{cl}(D_i)$, thus $\text{cl}(D_{i_x}) \cap \text{cl}(D_{i_y}) = \text{cl}(D_i)$ is directed, so $\bigcup_{i \in I} \text{cl}(D_i)$ is directed.
\( cl(D_{i_2}), y \in cl(D_{i_3}) \). It follows that there exists \( i \in I \) such that \( \{ x, y \} \subseteq cl(D_{i_2}) \cup cl(D_{i_3}) \subseteq cl(D_i) \) by the directionality of \((cl(D_i))_{i \in I}\). Thus there exists \( z \in cl(D_i) \subseteq \bigcup_{i \in I} cl(D_i) \) such that \( z \) is an upper bound of \( x, y \) because \( cl(D_i) \) is a directed subset of \( X \). So \( \sup_{i \in I} cl(D_i) \subseteq D \). By Lemma 3.1, we have \( \mathbf{KF}(X) \subseteq D \). Hence, the \( D \)-completion of \( X \) agrees with the well-filterification of \( X \). \[ \square \]

We know that a monotone convergence space with the property \( \downarrow (K \cap A) \in \Gamma(X) \) for any \( A \in \mathbf{KF}(X) \) and \( K \in Q(X) \) is a well-filtered space by [12]. So there is a problem below:

**Problem 3.4.** Without the condition that \( \mathbf{KF}(X) \) endowed with the lower Vietoris topology is the well-filterification of \( X \), whether the statement in Theorem 3.3 holds for any \( T_0 \) space \( X \) with the property \( \downarrow (K \cap A) \in \Gamma(X) \) for any \( A \in \mathbf{KF}(X) \) and \( K \in Q(X) \).

Note that Problem 3.4 actually is the problem (3) in the introduction. For this problem, we give a counterexample as follows.

**Example 3.5.** Let \( X = (\mathbb{N}, \tau_{cof}) \), \( Y_n = (\mathbb{N}, \sigma(\mathbb{N})) \) for any \( n \in X \), \( Z = \sum X Y_n \), where \( \mathbb{N} \) is the set of natural numbers. Then \( \downarrow (K \cap A) \in \Gamma(Z) \) for any \( A \in \mathbf{KF}(Z) \) and \( K \in Q(Z) \), but the \( D \)-completion \( Z^d \) of \( Z \) does not agree with the well-filterification \( Z^w \) of \( Z \).

**Proof.** By Lemma 2.16 and Lemma 2.15, we have that \( A \) is a directed subset of \( Z \) for any \( A \in \mathbf{KF}(Z) \). This implies that \( \downarrow (K \cap A) \in \Gamma(Z) \) for any \( A \in \mathbf{KF}(Z) \) and \( K \in Q(Z) \) because of Lemma 3.1. Let \( N = \{ Y_n \times \{ n \} \mid n \in X \} \). Note that \( Y_n \times \{ n \} \subseteq \uparrow Z \) for any \( n \in X \). It follows that \( N \subseteq \downarrow Z \).

Claim 1: \( A \) is compact in \( Z^w \) for any \( A \subseteq N \).

Let \( \{ U_i \}_{i \in I} \subseteq O(Z) \) with \( A \subseteq \bigcup_{i \in I} \uparrow U_i \). Pick \( Y_{n_0} \times \{ n_0 \} \in A \subseteq \bigcup_{i \in I} \uparrow U_i \). So there exists \( i_{n_0} \in I \) such that \( Y_{n_0} \times \{ n_0 \} \in \uparrow U_{i_{n_0}} \). (\( U_{i_{n_0}} \)) is open in \( Z \). Let \( B = X \setminus (U_{i_{n_0}}), C = \{ n \in X \mid Y_n \times \{ n \} \subseteq A, Y_n \times \{ n \} \cap U_{i_{n_0}} = \emptyset \} \). Note that \( B \) is finite and \( C \subseteq B \). Then there exists finitely many members of \( \{ \uparrow U_i \}_{i \in I} \) to cover \( A \).

Let \( A_n = N \setminus \{ Y_i \times \{ i \} \mid i \in \{ 1, 2, \cdots, n \} \} \). Thus \( \{ \uparrow Z \wedge A_n \}_{n \in N} \subseteq \uparrow Z \) by Claim 1.

Since \( Z^w \) is well-filtered, we have \( Z \subseteq \bigcap_{n \in N} \uparrow Z \wedge A_n \neq \emptyset \). Choose \( B \in \bigcap_{n \in N} \uparrow Z \wedge A_n \), i.e., \( B \in Z^w \).

Claim 2: \( (B)_X \) is infinite.

Suppose not, if \( (B)_X \) is finite. Let \( n_1 = \max (B)_X \). Since \( B \in \bigcap_{n \in X} \uparrow Z \wedge A_n \subseteq \uparrow Z \wedge A_{n_1} \), we have \( \downarrow Z \wedge B \cap A_{n_1} \neq \emptyset \). Then there exists \( n_2 > n_1 \) such that \( Y_{n_2} \times \{ n_2 \} \in \downarrow Z \wedge B \cap A_{n_1} \), which contradicts \( n_2 \notin (B)_X \).

By Lemma 2.15, \( B \) is not directed. This implies that \( B \) is not a tapered closed subset of \( Z \) by the proof of Theorem 3.3. Thus \( B \in Z^w \setminus Z^d \), that is, the \( D \)-completion \( Z^d \) of \( Z \) does not agree with the well-filterification \( Z^w \) of \( Z \). \[ \square \]

**Lemma 3.6.** Let \( L \) be a poset. Then the following statements are equivalent:

1. \( \sigma(L) \) is a continuous lattice;
2. For every dcpo or complete lattice \( S \), one has \( \sigma(S \times L) = \sigma(S) \times \sigma(L) \).

**Proof.** The proof is similar to Theorem 2.19. \[ \square \]
Theorem 3.7. Let $L$ be a core-compact and join continuous poset. Then the $D$-completion of $L$ coincides with the soberification of $L$.

Proof. Let $A$ be an irreducible closed subset of $X$. Then it suffices to prove that $A$ is directed. Define $F : (L \times L, \sigma(L \times L)) \rightarrow up(L)$ by $F(x, y) = \uparrow x \cap \uparrow y$ for any $(x, y) \in L \times L$.

Claim 1: $F$ is continuous.

Let $U \in \sigma(L)$. We need to prove that $F^{-1}(\Box U)$ is Scott open in $L \times L$. It suffices to prove that $F^{-1}(\Box U)$ is Scott open when $F^{-1}(\Box U) \neq \emptyset$. Obviously, $F^{-1}(\Box U) = \uparrow(F^{-1}(\Box U))$.

Now let $\{(x_i, y_i)\}_{i \in I}$ be any directed subset of $L \times L$ such that $\sup_{i \in I} (x_i, y_i)$ exists in $L \times L$ with $\sup_{i \in I} (x_i, y_i) = (\sup_{i \in I} x_i, \sup_{i \in I} y_i) \in F^{-1}(\Box U)$. This implies that $\uparrow \sup_{i \in I} x_i \cap \uparrow \sup_{i \in I} y_i \in \Box U$. So we have $f_{\sup_{i \in I} x_i}(\sup_{i \in I} y_i) \in \Box U$. It follows that $\sup_{i \in I} y_i \in f_{\sup_{i \in I} x_i}(\Box U)$ is Scott open in $L$ because $L$ is join continuous. This implies that there exists $i_0 \in I$ such that $y_{i_0} \in f_{\sup_{i \in I} x_i}(\Box U)$. Thus $\uparrow y_{i_0} \cap \uparrow \sup_{i \in I} x_i \in \Box U$. Then we have $\sup_{i \in I} x_i \in f_{y_{i_0}}^{-1}(\Box U)$. By the join continuity of $L$, there exists $i_1 \in I$ such that $x_{i_1} \in f_{y_{i_0}}^{-1}(\Box U)$. Thus there exists $i \in I$ such that $(x_i, y_i)$ is an upper bound of $\{(x_{i_0}, y_{i_0}), (x_{i_1}, y_{i_1})\}$ because $\{(x_i, y_i)\}_{i \in I}$ is directed. Therefore, $(x_i, y_i) \in F^{-1}(\Box U)$. Hence, $F$ is continuous.

Claim 2: $A$ is a directed subset of $X$.

By Lemma 3.6 and Claim 1, we have $F : \sigma(L) \times \sigma(L) \rightarrow up(L)$ is continuous. Let $\{x, y\} \subseteq A$. We need to prove that $\uparrow x \cap \uparrow y \cap A \neq \emptyset$. Suppose not, if $\uparrow x \cap \uparrow y \cap A = \emptyset$. It follows that $\uparrow x \cap \uparrow y \subseteq \Box X \setminus A$, that is, $F((x, y)) \in \Box X \setminus A$. This implies that there exists $\{U, V\} \subseteq \sigma(L)$ such that $(x, y) \in U \times V \subseteq F^{-1}(\Box X \setminus A)$ by the continuity of $F$. Notice that $x \in A \cap U$, $y \in A \cap V$. We have $A \cap U \cap V \neq \emptyset$ since $A$ is irreducible. Pick $z \in A \cap U \cap V$. Hence $(z, z) \in U \times V \subseteq F^{-1}(\Box X \setminus A)$. This means that $z \in \uparrow z \cap \uparrow z \subseteq (X \setminus A)$, which contradicts $z \in A$. So $A$ is directed. 

For the problem (5) mentioned in the introduction, we give a positive answer by Theorem 3.7.

Corollary 3.8. Let $L$ be a core-compact poset with the property that $\downarrow (K \cap A) \in \Gamma(L)$ for any $A \in \text{IRR}(L)$ and $K \in Q(L)$. Then the $D$-completion of $L$ agrees with the soberification of $L$.

Proof. The proof is similar to Theorem 3.7.

In contrast to Theorems 3.2 and Corollary 3.8, we raise the following question.

Problem 3.9. Whether the statement in Corollary 3.8 holds for arbitrary core-compact topological spaces $X$ with the property that $\downarrow (K \cap A) \in \Gamma(X)$ for any $A \in \text{IRR}(X)$ and $K \in Q(X)$.

Proposition 3.10. Let $X$ be a second countable space. Then the well-filterification of $X$ coincides with the soberification of $X$.

Proof. It suffices to prove that the well-filterification of $X$ is second countable by Theorem 4.1 in [14]. Let $B$ be a countable basis of $X$, $\mathcal{U} = \{\uparrow U \mid U \in B\}$. Since $\mathcal{U}$ is countable, we only need to prove that $\mathcal{U}$ is a basis of the well-filterification of $X$. Let $A \in \text{WD}(X), V \in \mathcal{O}(X), A \in \uparrow V$. Then $A \cap V \neq \emptyset$. Pick $x \in A \cap V$. This implies that there exists $U \in \mathcal{B}$ such that $x \in U \subseteq V$. It follows that $A \in \uparrow U \subseteq \uparrow V$.
We give a positive answer for the problem (2) in the introduction by Theorem 3.10.

Lemma 3.11. Let \( X \) be a \( T_0 \) space, \( X^* \) the soberification of \( X \). For each ordinal \( \beta \), define
\[
(1) \ X_{\beta + 1} = \{ x \in X^* \mid \exists F \in \mathbf{KF}(X_\beta), cl_{X^*}(F) = \downarrow_{X^*} x \}; \\
(2) \ X_\beta = \bigcup _{\gamma < \beta} X_\gamma \text{ for a limit ordinal } \beta.
\]

Then there exists an ordinal \( \alpha_X \) such that \( X_{\alpha_X} = X_{\alpha_X + 1} \) and the \( \omega \)-well-filtered reflection is \( X_{\alpha_X} \) endowed with the lower Vietoris topology.

Proof. The proof is similar to Proposition 3.8 in [14]. \( \square \)

Theorem 3.12. Let \( X \) be a first countable space. Then the well-filterification of \( X \) agrees with the soberification of \( X \).

Proof. By Theorem 4.1 in [14], we only need to prove that the \( \omega \)-well-filterification of \( X \) is first countable. There exists an ordinal \( \alpha_X \) such that \( X_{\alpha_X} = X_{\alpha_X + 1} \) and the \( \omega \)-well-filtered reflection is \( X_{\alpha_X} \) endowed with the lower Vietoris topology by Lemma 3.11. We want to prove that \( X_\alpha \) endowed with the lower Vietoris topology is first countable for any ordinal \( \alpha \leq \alpha_X \). We use induction on \( \alpha \). For \( \alpha = 0 \), \( X_0 = \{ \downarrow x \mid x \in X \} \). Then \( X \) and \( X_0 \) are homeomorphic. This implies that \( X_0 \) is first countable. Let \( \alpha \) be such that \( \alpha + 1 \leq \alpha_X \), and let \( X_\alpha \) be first countable.

Claim 1: \( X_{\alpha + 1} \) is first countable.

Let \( A \in X_{\alpha + 1} \). Then there exists \( F \in \mathbf{KF}(X_\alpha) \) such that \( cl_{X^*}(F) = \downarrow_{X^*} A \). Since \( F \in \mathbf{KF}(X_\alpha) \), there exists \( (K_n)_{n \in \mathbb{N}} \subseteq Q(X_\alpha) \) such that \( F \) is a minimal closed set that intersects all members of \( \{K_n \mid n \in \mathbb{N}\} \). Pick \( A_n \in F \cap K_n \) for any \( n \in \mathbb{N} \). It follows that \( F = cl_{X_\alpha}(A) \) from the minimality of \( F \), where \( A = \{ A_n \mid n \in \mathbb{N} \} \). We want to prove that \( cl_{X^*}(F) = cl_{X^*}(A) \). Since \( cl_{X^*}(F) \cap X_\alpha = F = cl_{X_\alpha}(A) = cl_{X_\alpha}(A) \cap X_\alpha \), we have \( F \subseteq cl_{X^*}(A) \). So \( cl_{X^*}(F) \subseteq cl_{X^*}(A) \).

Conversely, note that \( A_n \in K_n \subseteq X_\alpha \) for any \( n \in \mathbb{N} \). Hence, \( A \subseteq cl_{X^*}(A) \cap X_\alpha = cl_{X^*}(F) \cap X_\alpha \). This implies that \( A \subseteq cl_{X^*}(F) \). Therefore, \( cl_{X^*}(F) = cl_{X^*}(A) \). Because \( X_\alpha \) is first countable, there exists a countable neighborhood basis \( B_n = \{ \downarrow U \cap X_\alpha \mid \downarrow U \cap X_\alpha \in B_n \} \) of \( A_n \) for any \( n \in \mathbb{N} \). Let \( B_A = \{ \downarrow U \cap X_{\alpha + 1} \mid \downarrow U \cap X_\alpha \in \bigcup_{n \in \mathbb{N}} B_n \} \). Note that \( B_A \) is countable. We only need to prove that \( B_A \) is a countable neighborhood basis of \( A \) in \( X_{\alpha + 1} \). Let \( V \in O(X) \) with \( A \in \downarrow V \cap X_{\alpha + 1} \). It follows that \( A \cap \downarrow V \neq \emptyset \) since \( \downarrow_{X^*} A = cl_{X^*}(F) = cl_{X^*}(A) \). Choose \( A_n \in A \cap \downarrow V \cap X_\alpha \). Then there exists \( \downarrow U \cap X_\alpha \in B_n \) such that \( A_n \in \downarrow U \cap X_\alpha \subseteq \downarrow V \cap X_\alpha \). We need to prove that \( U \subseteq V \). Suppose not, if \( U \not\subseteq V \), then there exists \( u \in U \cap (X \setminus V) \). This implies that \( \downarrow u \in \downarrow V \cap X_\alpha \subseteq \downarrow V \cap X_{\alpha + 1} \).

Thus \( u \in V \), which contradicts \( u \in X \setminus V \). So \( A \in \downarrow U \cap X_{\alpha + 1} \subseteq \downarrow V \cap X_{\alpha + 1} \).

Suppose now that \( \alpha \leq \alpha_X \) is a limit ordinal and \( X_\beta \) is first countable for any \( \beta < \alpha \). By definition of \( X_\alpha \), we have \( X_\alpha = \bigcup_{\beta < \alpha} X_\alpha \).

Claim 2: \( X_\alpha \) is first countable.

For any \( A \in X_\alpha \), there exists \( \beta < \alpha \) such that \( A \in X_\beta \). We have that there exists a countable neighborhood basis \( B = \{ \downarrow U \cap X_\beta \mid \downarrow U \cap X_\beta \in B \} \) of \( A \) in \( X_\beta \) since \( X_\beta \) is first countable. It suffices to prove that \( B_A = \{ \downarrow U \cap X_\alpha \mid \downarrow U \cap X_\beta \in B \} \) is a countable neighborhood basis of \( A \) in \( X_\alpha \). It is obvious that \( B_A \) is countable. Now let \( V \in O(X) \) with \( A \in X_\alpha \cap \downarrow V \). Then \( A \in \downarrow V \cap X_\beta \). It follows that there exists \( \downarrow U \cap X_\beta \in B \) such that \( A \in \downarrow U \cap X_\beta \subseteq \downarrow V \cap X_\beta \). Similar to the proof of Claim 1, we have \( U \subseteq V \). So \( A \in \downarrow U \cap X_\alpha \subseteq \downarrow V \cap X_\alpha \). This proves that \( X_\alpha \) endowed with the lower Vietoris topology is first countable for any ordinal \( \alpha \leq \alpha_X \). Hence, the \( \omega \)-well-filtered reflection \( X_{\alpha_X} \) of \( X \) is first countable.

\( \square \)
We give a positive answer by Theorem 3.12 to the problem (1) in the introduction.

**Theorem 3.13.** Let $X$ be a $T_0$ space with the property $\downarrow (A \cap W) \in \Gamma (X)$ for any $A \in \text{IRR}(X)$ and $W \in \text{up}(X)$. Then the $D$-completion of $X$ coincides with the soberification of $X$.

**Proof.** Let $A \in \text{IRR}(X)$. It suffices to prove that $A$ is directed. For any $x, y \in A$, we need to prove that $\uparrow x \cap \uparrow y \cap A \neq \emptyset$. Suppose not, $\uparrow x \cap \uparrow y \cap A = \emptyset$. Let $B = \downarrow (\uparrow x \cap A)$. Then $\downarrow (X \setminus B) \in \Gamma (X)$. It follows that $A = (B \cup (X \setminus B)) \cap A = (B \cap A) \cup ((X \setminus B) \cap A) = B \cup ((X \setminus B) \cap A) \subseteq B \cup \downarrow ((X \setminus B) \cap A)$. Thus $A \subseteq B$ or $A \subseteq \downarrow ((X \setminus B) \cap A)$. But $y \in A \cap (X \setminus B)$ and $\uparrow x \cap A \cap (X \setminus B) = \emptyset$ implies that $A \not\subseteq B$ and $A \not\subseteq \downarrow ((X \setminus B) \cap A)$, which contradicts $A \subseteq B$ or $A \subseteq \downarrow ((X \setminus B) \cap A)$. □

For the problem (6) in the introduction, we give a positive answer by Theorem 3.13.

4. A counterexample

In the following, we construct an example to give a negative answer to Problem 1.1. It also can answer Xu’s problem ([15]).

![Figure 1: A tapered non-directed-determined poset.](image)

**Example 4.1.** Let $L = \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \infty)$, where $\mathbb{N}$ is the set of natural numbers. We define an order $\leq$ on $L$ as follows:

$(n_1, i_1, j_1) \leq (n_2, i_2, j_2)$ if and only if:

- $n_1 = n_2, i_1 = i_2, j_1 \leq j_2$;
- $n_1 = n_2, i_2 = j_2 = j_1, i_1 \leq i_2$;
- $n_2 = n_1 + 1, j_1 \leq i_2, j_2 = \infty$.

$L$ can be easily depicted as in Figure 1. Then $L$ is tapered closed in $(L, \sigma (L))$, but $L \notin \text{KF}(L)$ and $L \notin \{ \text{cl}(D) \mid D \subseteq \uparrow L \}$.

**Proof.** Claim 1: $L$ is tapered closed.
Let $A_n = \{(n, i, j) \mid (i, j) \in \mathbb{N} \times (\mathbb{N} \cup \infty)\}$. Note that $\bigcup_{i=1}^n A_i = cl(A_n \setminus \max A_n)$ for any $n \in \mathbb{N}$ and $A_n \setminus \max A_n \subseteq L$. Hence, $\bigcup_{i=1}^n A_i$ is tapered closed. Let $B_n = \bigcup_{i=1}^n A_i$. It follows that $\{B_n\}_{n \in \mathbb{N}} \subseteq \Gamma(L)$. Then we have $L = \sup_{n \in \mathbb{N}} B_n$ is tapered closed.

Claim 2: $L \notin \{cl(D) \mid D \subseteq L\}$.

Suppose not, there exists a directed subset $D$ of $L$ such that $cl(D) = L$. We need to prove that $\langle D \rangle = \{n \in \mathbb{N} \mid A_n \cap D \neq \emptyset\}$ is infinite. Assume $\langle D \rangle$ is finite. Let $n_0 = \max(\langle D \rangle)$. Then we have $D \subseteq \bigcup_{i \in \langle D \rangle} A_i \subseteq \bigcup_{i=1}^{n_0} A_i = B_{n_0}$. So $L = cl(D) \subseteq B_{n_0}$, which contradicts $B_{n_0} \nsubseteq L$. Thus $\langle D \rangle$ is infinite. Let $n_1 \in \langle D \rangle$. Pick $(n_1, i_1, j_1) \in D \cap A_{n_1}$. By the infiniteness of $\langle D \rangle$, there exists $m \in \langle D \rangle$ such that $m \geq n_1 + 2$. Choose $(m, i_m, j_m) \in D \cap A_m$. Obviously, $\nuparrow(n_1, i_1, j_1) \subseteq A_{n_1} \cup A_{n_1+1}$ and $\nuparrow(m, i_m, j_m) \subseteq A_m \cup A_{m+1}$. $(A_{n_1} \cup A_{n_1+1}) \cap (A_m \cup A_{m+1}) = \emptyset$. It follows that $\nuparrow(n_1, i_1, j_1) \cap \nuparrow(m, i_m, j_m) = \emptyset$, which contradicts that $D$ is directed.

Claim 3: $(K)_N = \{n \in \mathbb{N} \mid K \cap A_n \neq \emptyset\}$ is finite for any $K \subseteq Q(L)$.

Suppose not, $(K)_N$ is infinite. For any $n \in (K)_N$, pick $(n, i_n, j_n) \in K \cap A_n$. Then $(n, i_n, \infty) \in K \cap A_n$. Let $F \subseteq_{fin}(K)_N, B_F = \{\{n \in (K)_N \mid F \downarrow(n, i_n, \infty)\} \mid n \in (K)_N\}$. We need to verify that $B_F \in \Gamma(L)$. Now let $D \subseteq_{fin} B_F$. If $D$ is finite, then $\sup D \in D \subseteq B_F$. Else, there exists a unique $n \in (K)_N$ such that $D \subseteq \downarrow(n, i_n, \infty)$. This implies that $\sup D \in \downarrow(n, i_n, \infty) \subseteq B_F$. $\downarrow B_F = B_F$ is obvious. Note that $B_F \cap K \neq \emptyset$ for any $F \subseteq_{fin}(K)_N$. So we have $K \cap \bigcap_{F \subseteq_{fin}(K)_N} B_F \neq \emptyset$. Therefore, there exists $(n, i, j) \in K \cap \bigcap_{F \subseteq_{fin}(K)_N} B_F$. Since $(K)_N$ is infinite, we have $m \in (K)_N$ such that $m \geq n + 2$.

We now distinguish two cases:

Case 1, $n + 1 \in (K)_N$: Then $(n, i, j) \in \bigcap_{F \subseteq_{fin}(K)_N} B_F \subseteq B_{\{n, n+1\} \cap A_n}$, which contradicts $B_{\{n, n+1\} \cap A_n} = \emptyset$.

Case 2, $n + 1 \notin (K)_N$: Then $(n, i, j) \in \bigcap_{F \subseteq_{fin}(K)_N} B_F \subseteq B_{\{n\} \cap A_n}$, which contradicts $B_{\{n\} \cap A_n} = \emptyset$.

Claim 4: $L \notin \text{KF}(L)$.

Suppose not, $L \in \text{KF}(L)$, then there exists $(K_i)_{i \in I} \subseteq_{filt} Q(L)$ such that $L$ is a minimal closed set that intersects all members of $(K_i)_{i \in I}$ by Definition 2.5. Pick $i_0 \in I$. $(K_i)_{i \in I}$ is finite by Claim 3. Now let $m_0 = \max(K_{i_0})$. This implies that $K_{i_0} \subseteq B_{m_0}$. We need to prove that $B_{m_0} \cap K_i \neq \emptyset$ for any $i \in I$. For any $i \in I$, there exists $j \in I$ such that $K_j \subseteq K_i \cap K_{i_0}$ because $(K_i)_{i \in I}$ is filter. It follows that $K_j = B_{m_0} \cap K_j \subseteq B_{m_0} \cap K_i$. So $L = B_{m_0}$ by the minimality of $L$, which contradicts $L \neq B_{m_0}$.

Example 4.2. Let $J$ be Johnstone’s Example. Then $J \in \text{KF}(J)$, but $J$ is not a tapered closed set.

Remark 4.3. By Example 4.1 and 4.2 we know that there is no relation between the tapered closed subsets and the closed $\text{KF}$-subsets of a $T_0$ space $X$.

Figure 2 shows certain relations among some kinds of subsets of a $T_0$ space.

5. A direct characterization for $D$-completion

Let $P$ be a poset, $\mathcal{H} = \{D \subseteq \Gamma(P) \mid \emptyset \mid \forall A \in D, \forall a \in A, \downarrow a \in D, cl_d(D) = D\}$ and $A, B \in \Gamma(P)$. We say that $A$ is beneath $B$ denoted by $A \prec B$, if for every $D \in \mathcal{H}$, the relation $B \subseteq \sup D$.
always implies that \( A \in \mathcal{D} \).

Notice that the relation which we define above is not an auxiliary relation because the elements of \( \mathcal{H} \) may not be lower sets.

**Definition 5.1.** Let \( P \) be a poset. An element \( A \) of \( \Gamma(P) \) is called pre-C-compact if \( A \prec A \). We use \( K(\Gamma(P)) \) to denote the set of all the pre-C-compact elements of \( P \).

Ho and Zhao introduce \( C \)-continuous posets and \( C \)-compact elements in [19]. They proved that the \( C \)-compact elements of a poset \( P \) are co-primes. They find that the \( C \)-compact elements of \( \Gamma(P) \) are more than the elements of the \( D \)-completion of the posets. Now we want to find a way to reduce the \( C \)-compact elements of \( P \) such that they are equal.

**Definition 5.2.** [19] Let \( P \) be a poset and \( x, y \in P \). We say that \( x \) is beneath \( y \), denoted by \( x \prec y \), if for every nonempty Scott closed set \( C \subseteq P \) for which \( \text{sup} C \) exists, the relation \( y \leq \text{sup} C \) always implies that \( x \in C \). An element \( x \) of a poset \( P \) is called \( C \)-compact if \( x \prec x \).

The pre-C-compact elements are \( C \)-compact by Definition 5.2. The following example reveals that \( C \)-compact elements may not be pre-C-compact.

**Example 5.3.** Let \( J = \mathbb{N} \times (\mathbb{N} \cup \{\infty\}) \), that is Johnstone's example. The partial order is \((m,n) \leq (a,b)\) iff either \( m = a \) and \( n \leq b \) or \( b = \infty \) and \( n \leq a \). That \( J \) is \( C \)-compact have been proved in Remark 5.3 of [19]. We now show that \( J \) is not pre-C-compact. Let \( \mathcal{D} = \{\downarrow x \mid x \in J\} \). Note that \( \mathcal{D} \in \mathcal{H} \) and \( \text{sup} \mathcal{D} = J \) but \( J \notin \mathcal{D} \).

The following proposition reveals some order-theoretic properties of the lattice \((\Gamma(P), \subseteq)\) for an arbitrary poset \( P \).

**Proposition 5.4.** Let \( P \) be a poset. Then for any \( \mathcal{D} \in \mathcal{H} \), \( \text{sup}_{\Gamma(P)} \mathcal{D} = \bigcup \mathcal{D} \).

**Proof.** Note that each member of \( \mathcal{H} \) is a Scott-closed subset of \( P \). So to prove the equation, it suffices to show that \( \bigcup \mathcal{D} \in \Gamma(P) \). Obviously, \( \bigcup \mathcal{D} \) is a lower set. Now Let \( D \) be any directed subset of \( P \) contained in \( \bigcup \mathcal{D} \) such that \( \text{sup} D \) exists in \( P \). We want to prove that \( \text{sup} D \in C \) for some \( C \in \mathcal{D} \). First note that \( C = \{\downarrow d \mid d \in D\} \) is a directed subset of \( \Gamma(P) \). Moreover, \( C \subseteq \mathcal{D} \) by the definition of \( \mathcal{D} \). Since \( \mathcal{D} \) is a \( d \)-closed set, so \( \text{sup} C \in \mathcal{D} \). But \( \text{sup} C \) is precisely \( \downarrow \text{sup} D \). Hence \( \text{sup} D \in C \) for some \( C \in \mathcal{D} \). \( \square \)
Proposition 5.5. Let $P$ be a poset. If $A \in \mathcal{K}(\Gamma(P))$, then $A$ is an irreducible closed set.

Proof. Suppose $A$ is pre-$C$-compact and $B, C \in \Gamma(P)$, $A \subseteq B \cup C$. Let $D = \downarrow\{B, C\}$. Then $D \in \mathcal{H}$ and $\text{sup}D = B \cup C$. Hence, $A \subseteq D$, so $A \subseteq B$ or $A \subseteq C$.

Lemma 5.6. Let $P$ be a poset and $D$ be a directed subset of $P$. Then $\text{cl}(D) \in \mathcal{K}(\Gamma(P))$.

Proof. Let $D \in \mathcal{H}$. Suppose $\text{cl}(D) \subseteq \text{sup}D$. Then by Proposition 5.4 $D \subseteq \text{sup}D$ and $\downarrow\{d \mid d \in D\}$ is a directed subset of $D$. Hence, $\text{cl}(D) = \text{sup}\{d \mid d \in D\} \in D$.

Corollary 5.7. Let $P$ be a poset. Then for each $x \in P$, $\downarrow x < \downarrow x$ holds.

Theorem 5.8. For any poset $P$, $\mathcal{K}(\Gamma(P))$ is a dcpo with the inclusion order.

Proof. Let $(A_i)_{i \in I}$ be a directed subset in $\mathcal{K}(\Gamma(P))$. It suffices to show that $\text{sup}_{i \in I} A_i < \text{sup}_{i \in I} A_i$. Suppose $D \in \mathcal{H}$ with $\text{sup}_{i \in I} A_i \subseteq \text{sup}D$. Then $A_i \subseteq \text{sup}D$ for all $i \in I$. Since $A_i \in \mathcal{K}(\Gamma(P))$, it follows that $A_i < A_i$ and so $A_i \in D$. Because $D$ is $d$-closed, this implies that $\text{sup}_{i \in I} A_i \in D$.

We write $\hat{P} = \mathcal{K}(\Gamma(P))$ with the relative topology of $\Gamma(P)$.

Proposition 5.9. Let $P$ be a poset. Then $\hat{P}$ is a monotone convergence space.

Proof. It is easy to prove by Proposition 2.17.

Lemma 5.10. Let $P$ be a poset. If $A \subseteq P$ is tapered and closed, then $A \in \mathcal{K}(\Gamma(P))$.

Proof. Assume $P^\vee$ is the set of all tapered closed subsets and $A \in P^\vee$, then $(P^\vee, j)$ is the $D$-completion of $P$ by Lemma 2.14. Define $f : P \rightarrow \mathcal{K}(\Gamma(P))$ by $f(x) = \downarrow x$ for all $x \in P$. Then, immediately, $f$ is continuous. As $(P^\vee, j)$ is the $D$-completion of $P$, it follows that there exists a unique continuous map $f : P^\vee \rightarrow \mathcal{K}(\Gamma(P))$ s.t. $f = f \circ j$. Thus $\text{sup}f(A) = f(A)$ by Theorem 3.10 (2) of [19]. We need to prove $A \in \mathcal{K}(\Gamma(P))$. Suppose not, $A \notin \mathcal{K}(\Gamma(P))$. Since $\downarrow a \leq A$ for any $a \in A$ and $\hat{f}$ is continuous, we have $\downarrow a = \hat{f}(\downarrow a) \leq \hat{f}(A) = \text{sup}f(A)$. Hence, $A \not\subseteq \text{sup}f(A)$, i.e. $\text{sup}f(A) \cap A^\circ \neq \emptyset$. So we have $\text{sup}f(A) \in \hat{f}(A^\circ)$. Because $\hat{f}$ is continuous, this implies $\hat{f}^{-1}(\hat{f}(A^\circ))$ is open in $P^\vee$ and $A \in \hat{f}^{-1}(\hat{f}(A^\circ))$. Hence, there exists $U \in \sigma(P)$ such that $A \in \hat{f}(U) \subseteq \hat{f}^{-1}(\hat{f}(A^\circ))$. This means that $A \cap U \neq \emptyset$, then there exists $a \in A \cap U$ such that $\downarrow a \in \hat{f}(U) \subseteq \hat{f}^{-1}(\hat{f}(A^\circ))$, that is $\hat{f}(\downarrow a) = \downarrow a \in \hat{f}(A^\circ)$ which contradicts $a \in A$. Thus we have shown the Lemma.

Theorem 5.11. Let $P$ be a poset. Then $\mathcal{K}(\Gamma(P)) = \text{cl}_d(\eta(P))$ where $\eta : P \rightarrow \Gamma(P)$ is defined by $\eta(x) = \downarrow x$ for all $x \in P$, i.e. $(\hat{P}, \eta)$ is the $D$-completion of $P$.

Proof. For any $x \in P$, $\eta(x) = \downarrow x \in \mathcal{K}(\Gamma(P))$ by Corollary 5.7. Then $\eta(P) \subseteq \mathcal{K}(\Gamma(P))$. We have shown that $\mathcal{K}(\Gamma(P))$ is a subdcpo of $\Gamma(P)$, so $\text{cl}_d(\eta(P)) \subseteq \mathcal{K}(\Gamma(P))$. Conversely, note that $\text{cl}_d(\eta(P)) \in \mathcal{H}$ and $\text{sup} \text{cl}_d(\eta(P)) = P$. If $A \in \mathcal{K}(\Gamma(P))$, then $A \subseteq \text{sup} \text{cl}_d(\eta(P))$. This implies $A \in \text{cl}_d(\eta(P))$ since $A \prec A$.

From the proof of Theorem 5.11 we have the following:

Corollary 5.12. Let $P$ be a poset. Then the pre-$C$-compact elements are exactly the tapered closed sets.
Definition 5.13. Let $(X, \tau)$ be a topological space. $X$ is called determined by its directed subsets if $\tau = \{ U \subseteq X \mid cl(D) \cap U \neq \emptyset \text{ implies } D \cap U \neq \emptyset \}$ for all directed subsets with respect to the order of specialization.

The following two theorems have the similar proof to that of posets.

**Theorem 5.14.** Let $X$ be a topological space. Suppose $X$ is determined by its directed subsets, then we can get $(K(\Gamma(X)), \eta)$ is the $D$-completion of $X$, where $\eta : X \rightarrow \Gamma(X)$ is defined by $\eta(x) = \downarrow x$ for all $x \in X$.

**Theorem 5.15.** Suppose $X$ is a topological space. Define $H = \{ D \subseteq \Gamma(X) \mid \forall A \in D, \forall a \in A, \downarrow a \in D, cl_d(D) = D, cl(\bigcup D) = \bigcup D \}$. Then $(K(\Gamma(X)), \eta)$ is the $D$-completion of $X$.

**Corollary 5.16.** A topological space $X$ is a monotone convergence space if and only if for any $A \in K(\Gamma(X))$, there exists a unique point $x$ in $X$, such that $A = \downarrow x$.

6. Reference

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