Estimates in the modulation spaces for the Dirac equation with potential

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Abstract
In the present paper we obtain estimates in the modulation spaces for the solutions to the Dirac equation with quadratic and sub-quadratic potentials. We derive a representation for the Dirac operator that permits to solve approximately the perturbed Dirac equation and to obtain the desired estimates for the solution.

1 Introduction.
In this paper, we aim to obtain estimates in the modulation spaces for the solutions to the Cauchy problem for the Dirac equation
\[
\begin{cases}
i \partial_t \psi(t, x) = H_0 \psi(t, x) + V(t, x) \psi(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
\psi(0, x) = \psi_0(x), & x \in \mathbb{R}^3,
\end{cases}
\]
where \(i = \sqrt{-1}\), \(\psi(t, x) = (\psi_1(t, x), \psi_2(t, x), \psi_3(t, x), \psi_4(t, x))^T \in \mathbb{C}^4\) is a four-spinor field,
\[
H_0 = -i \alpha \cdot \nabla + m \beta,
\]
is the free Dirac operator with \(m\) - the mass of the particle, \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\), and \(\alpha_j, j = 1, 2, 3, 4\), are \(4 \times 4\) Hermitian matrices that satisfy the relation:
\[
\alpha_j \alpha_k + \alpha_k \alpha_j = 2 \delta_{jk}, \quad 1 \leq j, k \leq 4,
\]
where \(\delta_{jk}\) denotes the Kronecker symbol. The standard choice of \(\alpha_j\) is ([23]):
\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad 1 \leq j \leq 3, \quad \alpha_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \beta,
\]
\((I_n\) is the \(n \times n\) unit matrix) and
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
are the Pauli matrices. The potential \(V(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R}^3)\) is a \((4 \times 4)\)-matrix valued function which entries \(V_{jk}(t, x)\), \(1 \leq j, k \leq 4\), for all multi-indices \(\alpha\) with \(|\alpha| \geq 2\) or \(|\alpha| \geq 1\), satisfy the estimates
\[
|\partial_x^\alpha V_{jk}(t, x)| \leq C_\alpha, \quad 1 \leq j, k \leq 4.
\]
The usual framework for equation (1.1) is a \(L^2\) based space, as for instance, the Sobolev spaces \(H^s\). The question arises if it is possible to remove the \(L^2\) constraint and consider equation (1.1) in functional spaces which are not \(L^2\) based. In the

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case when $H_0 = -\Delta$ several approaches were used in order to answer this question. For example, the local well-posedness of the NLS equation was studied on Zhidkov spaces in [14]. Different spaces of infinite mass were introduced in [32] and [12] to study the well-posedness problem for the NLS equation. We also mention the papers [15] and [26] that consider other frameworks which are not $L^2$ based.

In general, for a Fourier multiplier $e^{itH}$, the main drawback in working in Lebesgue spaces $L^p$ is that $e^{itH}$ may be unbounded. This means that the initial properties are not preserved by the time evolution. In the case of unimodular Fourier multipliers $e^{it\xi^2}$ with general symbols $e^{i|\xi|^\alpha}$, where $\alpha \in [0, 2]$, the modulation spaces $M^{p,q}_{\alpha}$ (see Section 2.3 for the definition of these spaces) have resulted to be an alternative for the study of $e^{itH}$. It was shown in [3] that such multipliers are bounded on all modulation spaces, even if they are unbounded on usual $L^p$-spaces. There exist a large literature concerning the modulation spaces and their applications to the Schrödinger equation or other equations, such as the wave equation or the Klein-Gordon equation. For example, we mention the works [2]-[11], [16]-[21], [25], [28]-[31], and the references cited therein. As far as we know, there are no papers concerning the Dirac equation in the framework of the modulation spaces. We pretend to fill this gap by proving some estimates in modulation spaces for the solutions of equation (1.1).

Let us first recall some known results. In the case of the free Schrödinger operator $H_0 = -\frac{1}{2}\Delta$ and $V \equiv 0$, estimates on the modulation spaces for the solutions to the corresponding Cauchy problem were obtained in [3], [29] and [30]. More precisely, the following was proved. Consider the Schrödinger equation

$$
\begin{align*}
\left\{ \begin{array}{l}
 i\partial_t u (t, x) &= -\frac{1}{2}\Delta u (t, x) + V (t, x) u (t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
 u (0, x) &= u_0 (x), \quad x \in \mathbb{R}^N,
\end{array} \right.
\end{align*}
$$

\tag{1.4}

where $u (t, x)$ is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, $V (t, x)$ is a real-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $u_0 (x)$ is a complex-valued function of $x \in \mathbb{R}^N$.

**Theorem A.** (See [3].) Let $1 \leq p, q \leq \infty$. Suppose that $V (t, x) \equiv 0$. Then, there exists a positive constant $C$ such that

$$
\| u (t, \cdot) \|_{M^{p,q}} \leq C (1 + |t|)^{N/2} \| u_0 \|_{M^{p,q}}, \quad u_0 \in S (\mathbb{R}^N),
$$

for all $t \in \mathbb{R}$, where $u (t, x)$ is the solution of (1.4).

ii) (See [29].) Let $2 \leq p < \infty$, $1 \leq q < \infty$, $1/p + 1/p' = 1$. Suppose that $V (t, x) \equiv 0$. Then, there exists positive constants $C$ and $C'$ such that

$$
\| u (t, \cdot) \|_{M^{p,q}} \leq C (1 + |t|)^{-N(1/2 - 1/p)} \| u_0 \|_{M^{p',q}}, \quad u_0 \in S (\mathbb{R}^N),
$$

and

$$
\| u (t, \cdot) \|_{M^{p,q}} \leq C' (1 + |t|)^{N(1/2 - 1/p)} \| u_0 \|_{M^{p',q}}, \quad u_0 \in S (\mathbb{R}^N),
$$

for all $t \in \mathbb{R}$, where $u (t, x)$ is the solution of (1.4).

A new representation for the Schrödinger operator via the wave packet transform was derived in [16]-[17] and used to study equation (1.4) in the context of the modulation spaces. In particular, the results of Theorem A were proved in [17] by using this representation. As it was showed in [18] and [19], the approach of [16]-[17] may be applied to study equation (1.4) with quadratic and sub-quadratic potentials to estimate its solution in the modulation spaces. The following results are due to [18] and [19]. (Below we emphasize the dependence of the modulation spaces $M^{p,q}_{\phi}$ on the window $\phi$, see Section 2.3 for more details.)

**Theorem B.** (See [18].) Let $1 \leq p \leq \infty$ and $\phi_0 \in S (\mathbb{R}^N) \setminus \{0\}$. Suppose that $V (t, x) \equiv \pm \frac{1}{2} |x|^2$. Then,

$$
\| u (t, \cdot) \|_{M^{p,q}_{\phi_0}} = \| u_0 \|_{M^{p,q}_{\phi_0}}, \quad u_0 \in S (\mathbb{R}^N),
$$

for all $t \in \mathbb{R}$, where $u (t, x)$ and $\phi (t, x)$ are the solutions of (1.4) with $u (0, x) = u_0 (x)$ and $\phi (t, x) = \phi_0 (x)$, respectively.

**Theorem C.** (See [19].) i) Let $1 \leq p \leq \infty$, $\phi_0 \in S (\mathbb{R}^N) \setminus \{0\}$ and $T > 0$. Set $\phi (t, x) := e^{it\frac{1}{2} \phi_0}$. Suppose that $V (t, x)$ is a real-valued function satisfying (1.3) for $|\alpha| \geq 2$. Then, there exists a positive constant $C_T$ such that

$$
\| u (t, \cdot) \|_{M^{p,q}_{\phi_0}} \leq C_T \| u_0 \|_{M^{p,q}_{\phi_0}}, \quad u_0 \in S (\mathbb{R}^N),
$$

(1.5)

uniformly for $t \in [-T, T]$, where $u (t, x)$ is the solution of (1.4) in $C (\mathbb{R}; L^2 (\mathbb{R}^N))$.

ii) Let $1 \leq p, q \leq \infty$, $\phi_0 \in S (\mathbb{R}^N) \setminus \{0\}$ and $T > 0$. Set $\phi (t, x) := e^{it\frac{1}{2} \phi_0}$. Suppose that $V (t, x)$ is a real-valued function satisfying (1.3) for $|\alpha| \geq 1$. Then, there exists a positive constant $C_T$ such that

$$
\| u (t, \cdot) \|_{M^{p,q}_{\phi_0}} \leq C_T \| u_0 \|_{M^{p,q}_{\phi_0}}, \quad u_0 \in S (\mathbb{R}^N),
$$

(1.6)

uniformly for $t \in [-T, T]$, where $u (t, x)$ is the solution of (1.4) in $C (\mathbb{R}; L^2 (\mathbb{R}^N))$.

We observe that the approach of [16]-[18] was also used in [31] to prove a result similar to Theorem C for more general Schrödinger-type equations with time-dependent quadratic or sub-quadratic potentials.
Main results.

We now present our main results. First, we consider the free case, similar to Theorem A.

**Theorem 1.1** i) Let $1 \leq p, q \leq \infty$, and $T > 0$. Suppose that $V(t, x) = 0$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. Then, the solution $\psi(t, x)$ of (1.1) satisfies

$$\|\psi(t, \cdot)\|_{M^{p,q}} \leq C_T \|\psi_0\|_{M^{p,q}}, \quad \psi_0 \in \mathcal{S} (\mathbb{R}^N),$$

uniformly for $t \in [-T, T]$.

ii) Let $0 < q < \infty$. Suppose that $V(t, x) = 0$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. Then

$$\|\psi(t, \cdot)\|_{M^{0,p,2q}} \leq C (T)^{-\theta(3/2-3/p)} \|\psi_0\|_{M^{p,q}}, \quad \psi_0 \in \mathcal{S} (\mathbb{R}^N),$$

where $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\theta \in [0, 1]$ and $2\sigma = 5\theta \left( \frac{1}{2} - \frac{1}{p} \right)$.

Now, we present the estimates analogous to Theorem C for the equation (1.1) with quadratic and sub-quadratic potentials.

**Theorem 1.2** i) Let $1 \leq p \leq \infty$, and $T > 0$. Suppose that the potential in problem (1.1) decomposes as $V(t, x) = Q(t, x)I_1 + V_2(t, x)$, where $Q(t, x) \in C^\infty (\mathbb{R}^3 \times \mathbb{R}^3)$ is a real-valued function satisfying (1.3) for $|\alpha| \geq 2$ and $V_2(t, x)$ is a $(4 \times 4)$ -matrix-valued function whose components verify (1.3) for $|\alpha| \geq 0$. Then,

$$\|\psi(t, \cdot)\|_{M^{p,q}} \leq C_T \|\psi_0\|_{M^{p,q}}, \quad \psi_0 \in \mathcal{S} (\mathbb{R}^N),$$

uniformly for $t \in [-T, T]$.

ii) Let $1 \leq p, q \leq \infty$, and $T > 0$. Suppose that $V(t, x) = V_1(t, x) + V_2(t, x)$, where $V_1(t, x) = \{(V_1)_{jk}(t, x)\}$, $1 \leq j, k \leq 4$, is a matrix-valued function with $(V_1)_{jk}(t, x) \in C^\infty (\mathbb{R} \times \mathbb{R}^3)$ which entries satisfy (1.3) for $|\alpha| \geq 1$ and $V_2(t, x)$ is a $(4 \times 4)$ -matrix-valued function whose components verify (1.3) for $|\alpha| \geq 0$. Then,

$$\|\psi(t, \cdot)\|_{M^{p,q}} \leq C_T \|\psi_0\|_{M^{p,q}}, \quad \psi_0 \in \mathcal{S} (\mathbb{R}^N),$$

uniformly for $t \in [-T, T]$.

**Remark 1.3** Here are some comments on Theorems 1.1 and 1.2.

1. For definiteness, we choose the space dimension to be $N = 3$. Theorems 1.1 and 1.2 remain valid for the Dirac equation in any dimension.

2. We note that we prove Theorem 1.2 for more general equations (3.6). See Theorems 3.2 and 3.3.

3. We expect that in general it is not possible to replace the $M^{p,p}$-norm in (1.9) by the $M^{p,q}$-norm. In the case of the Schrödinger equations it is known that (1.5) might be false if $p \neq q$. Indeed, let $u(t, x)$ solves (1.4) with $V(t, x) = \frac{1}{2} |x|^2$. Then (see [16])

$$\|u\left(\frac{\pi}{2}, \cdot\right)\|_{M^{p,q}, \phi \left(\frac{\pi}{2}, \cdot\right)} = \left\|W_{\phi_0} u_0 (\xi, x)\right\|_{L^p_x} \left\|L^q_{\phi} \right\|_{L^q_x}.$$

However, $\left\|W_{\phi_0} u_0 (\xi, x)\right\|_{L^p_x} \left\|L^q_{\phi} \right\|_{L^q_x}$ it is not true in general.

4. We observe that in Theorem 1.2 the potential $V_2(t, x)$ is a general $(4 \times 4)$ -matrix-valued function, not necessarily Hermitian.

5. In the second part of Theorem 1.2 the potential $V_1(t, x)$ is a general Hermitian matrix. Therefore, it includes the case of the electromagnetic potential

$$V_1(t, x) = \begin{pmatrix} Q_+ (t, x) & \alpha \cdot A (t, x) \\ \alpha \cdot A (t, x) & Q_- (t, x) \end{pmatrix},$$

where $A(t, x) = (A_1(t, x), A_2(t, x), A_3(t, x))$, and $Q_\pm, A_j \in C^\infty (\mathbb{R} \times \mathbb{R}^3)$, $j = 1, 2, 3$, satisfy (1.3) for $|\alpha| \geq 1$.

**Comments on the proof.** Our proof is based on the strategy developed in the papers [16]-[18]. As a first step, we need to derive a representation for the Dirac operator that permits to transform the Dirac equation (1.1) into a system of ordinary differential equations as the transform obtained in [17] for the Schrödinger equation. Since the free Dirac equation is a system of coupled equations, it seems impossible to obtain such a representation for the Dirac operator. Fortunately for us, it is enough to obtain approximate representation that, to the main order, transform the original equation (1.1) into a
new one that can be solved "approximately" (see (3.13), (3.17) and (3.18) below). Then, using the integral equation (3.18) for the solution to this transformed equation, we are able to obtain the desired estimates for the solution of (1.1). Since we use an approximate representation, the window \( \phi \) in (1.9) and (1.10) is fixed, while in (1.5) and (1.6) \( \phi(t,x) \) is asked to solve the free Schrödinger equation. We observe that in order to obtain the integral representation (3.18), we restrict ourselves in (1.9) to diagonal quadratic potentials \( Q(t, x) I_4 \).

The rest of the paper is organized as follows. In Section 2, we introduce some notation, we recall some definitions and properties of the vector-valued modulation spaces and we present some known results for the Dirac operator. Section 3 is dedicated to the proof of our main results and it is divided in two parts: in the first part we prove Theorem 1.1, whilst in the second part, we present the proof of Theorem 1.2.

2 Preliminaries.

2.1 Notation.

Let \( N, m \geq 1 \) be entire. For \( 1 \leq p < \infty \) we denote by \( L^p(\mathbb{R}^N; \mathbb{C}^m) \) the Lebesgue spaces of \( \mathbb{C}^m \)-vector valued functions. Also, we introduce the weighted \( L^2(\mathbb{R}^N; \mathbb{C}^m) \) spaces for \( s \in \mathbb{R} \), \( L^2_s := \{ f : \langle x \rangle^s f(x) \in L^2(\mathbb{R}^N; \mathbb{C}^m) \} \), \( \| f \|_{L^2(\mathbb{R}^N; \mathbb{C}^m)} := \| \langle x \rangle^s f(x) \|_{L^2(\mathbb{R}^N; \mathbb{C}^m)} \), where \( \langle x \rangle = \left( 1 + |x|^2 \right)^{1/2} \). For \( 1 \leq p, q \leq \infty \), \( r, s \in \mathbb{R} \), \( f \in L^p_r(\mathbb{R}^{2N}; \mathbb{C}^m) \), then

\[
\| f \|_{L^p_r(\mathbb{R}^{2N}; \mathbb{C}^m)} = \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |f(x, \xi)|^p \langle \xi \rangle^{ps} dx \right) dx \right)^{1/p} < \infty,
\]

with the standard modification when \( p \) or \( q \) are equal to infinity. We denote by \( \langle \cdot, \cdot \rangle \) the \( L^2 \) scalar product. The Fourier transform \( \mathcal{F} \) is given by

\[
\hat{f}(\xi) = (\mathcal{F}f)(\xi) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} f(x) \, dx
\]

\((\xi \cdot x = \sum_{j=1}^N x_j \xi_j)\) and the inverse Fourier transform \( \mathcal{F}^{-1} \) is defined by

\[
\hat{f}^{-1}(\xi) = (\mathcal{F}^{-1}f)(x) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{i\xi \cdot x} f(x) \, d\xi.
\]

Finally, we denote by \( C > 0 \) constants that may be different in each occasion.

2.2 The wave packet transform of vector-valued tempered distributions.

We now present the definition of the wave packet transform of short-time Fourier transform and recall some results of [27]. The wave packet transform of \( f \in L^p(\mathbb{R}^N; \mathbb{C}^m) \) with respect to a window function \( \phi \in L^p(\mathbb{R}^N) \) is defined by

\[
W_\phi f(x, \xi) := \int_{\mathbb{R}^N} \overline{\phi(x - y)} f(y) e^{-i\xi \cdot y} dy.
\]  

(2.1)

Similarly to the scalar case, the wave packet transform for a vector valued distribution \( f \in \mathcal{S}'(\mathbb{R}^N; \mathbb{C}^m) \) with respect to the window \( \phi \in \mathcal{S}(\mathbb{R}^N) \) is defined by the right-hand side of (2.1), where the "integral" in this case means distribution action. The following result holds (see Lemma 2.1 of [27]).

**Proposition 2.1** Let \( f \in \mathcal{S}'(\mathbb{R}^N; \mathbb{C}^m) \) and \( \phi \in \mathcal{S}(\mathbb{R}^N) \). Then, \( W_\phi f \in C^\infty(\mathbb{R}^{2N}; \mathbb{C}^m) \) and there exist an integer \( M > 0 \) and \( C > 0 \) such that

\[
|W_\phi f(x, \xi)|_{\mathbb{C}^m} \leq C (1 + |x| + |\xi|)^M, \quad (x, \xi) \in \mathbb{R}^{2N}.
\]

For a strongly measurable function \( F : \mathbb{R}^{2N} \to \mathbb{C}^m \) and \( \psi \in \mathcal{S}(\mathbb{R}^N) \) we define the map

\[
\langle W_\psi^* F, \phi \rangle := \iint_{\mathbb{R}^{2N}} F(y, \xi) \langle \psi(x - y), \phi(y) \rangle e^{i\xi \cdot y} dy \, d\xi,
\]

with \( d\xi = (2\pi)^{-n} \, d\xi \), and denote,

\[
W_\psi^* F = \iint_{\mathbb{R}^{2N}} F(y, \xi) \psi(x - y) e^{i\xi \cdot y} dy \, d\xi.
\]  

(2.2)
The next result shows that the wave packet transform is invertible on $S'(\mathbb{R}^N;\mathbb{C}^m)$ and the adjoint operator $W_\phi^*$ is defined by (2.2) (see Proposition 2.5 of [27]).

**Proposition 2.2** Let $\phi, \psi \in S(\mathbb{R}^N)$ be such that $\langle \psi, \phi \rangle \neq 0$. Then,

$$f = \langle \psi, \phi \rangle^{-1} W_\phi^* W_\phi f, \ f \in S'.$$

### 2.3 Modulation spaces.

Let us now define the modulation spaces and recall some properties of these spaces. Let $1 \leq p, q \leq \infty$, $r, s \in \mathbb{R}$. For $f \in S'(\mathbb{R}^N;\mathbb{C}^m)$ and $W_\phi f \in L^p_{r,s}(\mathbb{R}^N;\mathbb{C}^m)$, with $\phi \in S(\mathbb{R}^N) \setminus \{0\}$, the modulation space norm is defined by (see [27], [24])

$$\|f\|_{M_{r,s}^p(\mathbb{R}^N;\mathbb{C}^m)} := \|W_\phi f\|_{L^p_{r,s}(\mathbb{R}^N;\mathbb{C}^m)}, \ \phi \in S(\mathbb{R}^N) \setminus \{0\}.$$

At first, this definition is dependent on the window $\phi \in S(\mathbb{R}^N) \setminus \{0\}$. Nevertheless, it is not: it follows from Proposition 3.2 of [27] that if $\phi, \psi \in S(\mathbb{R}^N) \setminus \{0\}$, then the norm $\|\|_{(M_{r,s}^p)_{\phi}(\mathbb{R}^N;\mathbb{C}^m)}$ associated to $\phi$ and the norm $\|\|_{(M_{r,s}^p)_{\psi}(\mathbb{R}^N;\mathbb{C}^m)}$ corresponding to $\psi$ are equivalent. For $1 \leq p, q \leq \infty$, $r, s \in \mathbb{R}$, the modulation space $M_{r,s}^p(\mathbb{R}^N;\mathbb{C}^m) \subset S'(\mathbb{R}^N;\mathbb{C}^m)$, is defined as the set of all $f \in S'(\mathbb{R}^N;\mathbb{C}^m)$ such that $\|f\|_{M_{r,s}^p(\mathbb{R}^N;\mathbb{C}^m)} < \infty$. We write $M^p(\mathbb{R}^N;\mathbb{C}^m) := M_{0,0}^p(\mathbb{R}^N;\mathbb{C}^m)$ and $M^{p^*}(\mathbb{R}^N;\mathbb{C}^m) := M^p(\mathbb{R}^N;\mathbb{C}^m)$. The following result holds (see Proposition 3.3 of [27]):

**Proposition 2.3** For any $1 \leq p, q < \infty$, and $r, s \in \mathbb{R}$, the set $S(\mathbb{R}^N;\mathbb{C}^m)$ is dense in $M_{r,s}^p(\mathbb{R}^N;\mathbb{C}^m)$. If $p_1 \leq p_2$, $q_1 \leq q_2$, $r_1 \geq r_2$, $s_1 \geq s_2$, $M_{r_1,s_1}^{p_1,q_1}(\mathbb{R}^N;\mathbb{C}^m) \hookrightarrow M_{r_2,s_2}^{p_2,q_2}(\mathbb{R}^N;\mathbb{C}^m)$. Moreover, $M_0^p(\mathbb{R}^N;\mathbb{C}^m) = L_p^2(\mathbb{R}^N;\mathbb{C}^m)$ and $M_0^{p^*}(\mathbb{R}^N;\mathbb{C}^m) = \mathcal{F} L_p^2(\mathbb{R}^N;\mathbb{C}^m)$ with equivalent norms.

As in the case of the modulation spaces for scalar distributions (see [13]), the complex interpolation theory for the modulation spaces $M_{r,s}^p(\mathbb{R}^N;\mathbb{C}^m)$ stands as follows:

**Proposition 2.4** Let $0 < \theta < 1$ and $1 \leq p_i, q_i < \infty, \ i = 1, 2$. Set $1/p = (1-\theta)/p_1 + \theta/p_2, 1/q = (1-\theta)/q_1 + \theta/q_2$, then $(M_{r_1,s_1}^{p_1,q_1}, M_{r_2,s_2}^{p_2,q_2})[\theta] = M_{r,s}^{p,q}$.

We conclude this Subsection by presenting the following result (see Corollary 2.3 of [24]).

**Proposition 2.5** For $s \in \mathbb{R}$, $(D)^s$ is a continuous bijective map from $M_{0,s}^{p,q}$ to $M_{0,s}^{q',p'}$, with continuous inverse.

### 2.4 Dirac equation.

The free Dirac operator $H_0$ defined by (1.2) is a self-adjoint operator on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ with domain $D(H_0) = \mathcal{H}^1(\mathbb{R}^3;\mathbb{C}^4)$, the Sobolev space of order 1 ([1]). We can diagonalize $H_0$ by the Fourier transform $\mathcal{F}$. Actually, $H_0 \mathcal{F}^*$ acts as multiplication by the matrix $h_0(\xi) = \alpha \cdot \xi + m\beta$. This matrix has two eigenvalues $E = \pm \sqrt{\xi^2 + m^2}$ and each eigenspace $X\pm(\xi)$ is a two-dimensional subspace of $\mathbb{C}^4$. The orthogonal projections onto these eigenspaces are given by (see Section 2 of [22])

$$P_\pm(\xi) := \frac{1}{2} \left( I_4 \pm (\xi^2 + m^2)^{-1/2} (\alpha \cdot \xi + m\beta) \right).$$

(2.3)

Note that

$$P_\pm(\xi) \mathcal{F} = (\mathcal{F} P_\pm)(\xi),$$

where

$$P_\pm := \frac{1}{2} \left( I_4 \pm \frac{H_0}{|H_0|} \right),$$

(2.4)

are the projections on positive and negative energies of the Dirac operator $H_0$. 

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5
3 Proof of Theorems 1.1 and 1.2.

3.1 Proof of Theorem 1.1.

Estimate (1.7) follows from (1.10) for $V(t,x) \equiv 0$. In order to prove the second assertion, we multiply the both sides of equation (1.1) with $V(t,x) \equiv 0$ by $P_\pm$, defined in (2.4). Then, we obtain two equations

$$
\begin{align*}
\left\{ \begin{array}{l}
i\partial_t \psi_\pm(t,x) = \pm \left( \sqrt{m^2 - \Delta} \right) \psi_\pm(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
\psi_\pm(0,x) = (\psi_0)_{\pm}(x), \quad x \in \mathbb{R}^N,
\end{array} \right.
\end{align*}
$$

(3.1)

where $\psi_\pm(t,x) := P_\pm \psi(t,x)$. We need an estimate for the Klein-Gordon semigroup $e^{it\omega t/2}, \omega = (m^2 - \Delta)$ in terms of the modulation spaces. For $2 \le p < \infty, 0 < q < \infty, \phi \in \mathcal{S}(\mathbb{R}^N) \setminus \{0\}$ and $f \in \mathcal{M}^{p,q}$, we have the following inequality (see Proposition 4.2 of [29])

$$
\left\| e^{it\omega t/2} f \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} \le C \left\langle t \right\rangle^{-N\theta[1/2-1/p]} \left\| f \right\|_{\mathcal{M}^{p,q}}, \quad \frac{1}{p} + \frac{1}{p'} = 1,
$$

(3.2)

where $\theta \in [0,1]$ and $2\sigma = (N + 2) \theta \left( \frac{1}{2} - \frac{1}{p} \right)$. Using (3.2) in (3.1) we estimate

$$
\left\| \psi_\pm(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} \le C \left\langle t \right\rangle^{-N\theta[1/2-1/p]} \left\| (\psi_0)_\pm \right\|_{\mathcal{M}^{p,q}}.
$$

(3.3)

Since $P_+ + P_- = I$, we have

$$
\left\| \psi(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} \le \left\| \psi_+(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} + \left\| \psi_-(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}}.
$$

(3.4)

Moreover, as $\alpha_j^2 = I, j = 1, 2, 3, 4,$

$$
\left\| H_0 \psi(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} \le \left\| \psi(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma+1}} + m \left\| \psi(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} \le (1 + m) \left\| \psi(t,\cdot) \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}}.
$$

Then, using Lemma 2.5 we deduce

$$
\left\| (\psi_0)_\pm \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} = \left\| P_\pm \psi_0 \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} \le \frac{1}{2} \left( \left\| \psi_0 \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} + \left\| \frac{H_0}{(D)} \psi_0 \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}} \right) \le \left( 1 + \frac{m}{2} \right) \left\| \psi_0 \right\|_{\mathcal{M}^{p,q}_{0,-2\sigma}}.
$$

(3.5)

Therefore, by (3.3), (3.4) and (3.5) we attain (1.8). Theorem 1.1 is proved.

3.2 Proof of Theorem 1.2.

Instead of proving Theorem 1.2 for equation (1.1) directly, we consider a more general system

$$
\begin{align*}
\left\{ \begin{array}{l}
i\partial_t u(t,x) = a(D) u(t,x) + V(t,x) u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
u(0,x) = u_0(x), \quad x \in \mathbb{R}^N,
\end{array} \right.
\end{align*}
$$

(3.6)

where $a(D) = \mathcal{F}^{-1} a(\xi) \mathcal{F}^{-1}, a(\xi)$ is an Hermitian $(m \times m)$-matrix valued symbol such that

$$
b(\eta;\xi) = a(\xi - \eta) - a(\xi)
$$

(3.7)

satisfies

$$
\left| \partial^k_\eta b(\eta;\xi) \right| \le C_a(\eta)^k, \quad \text{for all } \eta, \xi \in \mathbb{R}^N,
$$

(3.8)

for some $k \ge 0$, and $V(t,x)$ is a $(m \times m)$-matrix valued function. Since $\mathcal{F}^{-1} H_0 \mathcal{F} = a \cdot \xi + m\beta$, the case of the equation (1.1) is included in (3.6). We prove the following elementary estimate that is involved in the proof of the main results.

**Lemma 3.1** Suppose that $b$ satisfies (3.8). Then, for any $\phi \in \mathcal{S}(\mathbb{R}^N)$ the estimate

$$
\left| \partial^\beta_\xi (\mathcal{F}^{-1}_{\eta \rightarrow z} (b(\eta,\xi) \phi(\eta))) (z) \right| \le C_\beta (z)^{-2n},
$$

(3.9)

for any $|\beta| \ge 0$ and $n \in \mathbb{N}$ such that $2n > N$.

**Proof.** Note that

$$
\partial^\beta_\xi (\mathcal{F}^{-1}_{\eta \rightarrow z} (b(\eta,\xi) \phi(\eta))) (z) = \int_{\mathbb{R}^N} e^{iz \cdot \eta} (i\eta)^\beta b(\eta,\xi) \phi(\eta) d\eta.
$$

(3.10)

Using the equality

$$
(1 - \Delta_\eta)^n e^{iz \cdot \eta} = \langle z \rangle^{2n} e^{iz \cdot \eta}
$$

(3.11)

in the right-hand side of (3.10) and integrating by parts we obtain the estimate (3.9). ■

The first assertion of Theorem 1.2 is consequence of the following result.
Theorem 3.2 Let \( 1 \leq p \leq \infty \) and \( T > 0 \). Set \( m \in \mathbb{N} \). Let \( a ( \xi ) = \{ a_{jk} ( \xi ) \} \) be an Hermitian \((m \times m)\)-matrix valued function such that \( a_{jk} ( \xi ) \in C^\infty (\mathbb{R}^N) \) and that satisfies the estimate (3.8). Suppose that \( V (t, x) = Q (t, x) I_m + V_2 (t, x) \), where \( Q (t, x) \in C^\infty (\mathbb{R}^N \times \mathbb{R}^N) \) is a real-valued function satisfying (1.3) for \( \alpha \geq 2 \) and \( V_2 (t, x) \in C^\infty (\mathbb{R}^N \times \mathbb{R}^N) \) is a \((m \times m)\)-matrix valued function that verifies (1.3) for \( \alpha \geq 0 \). Then, the solution \( u (t, x) \) of (3.6)

\[
\| u (t, \cdot) \|_{M^p} \leq C_T \| u_0 \|_{M^p},
\]

uniformly for \( t \in [-T, T] \).

Proof. We consider the case \( t \in [0, T] \). Note that

\[
W_\phi (a (D) u) (x, \xi) = \int_{\mathbb{R}^N} e^{-iy \cdot \xi} \overline{\phi (x-y)} \left( \int_{\mathbb{R}^N} e^{iy \cdot \eta} a (\eta) \overline{u (\eta)} \, d\eta \right) dy
= \int_{\mathbb{R}^N} e^{-iy \cdot \xi} \overline{\phi (x-y)} \left( \int_{\mathbb{R}^N} e^{i(\xi - \eta) \cdot \eta} a (\xi - \eta) \overline{u (\eta)} \, d\eta \right) dy
= a (\xi) W_\phi u (x, \xi) + R_0 u (x, \xi),
\]

where

\[
R_0 u (x, \xi) = \int_{\mathbb{R}^N} S (x-y, \xi) e^{-ix \cdot y} u (y) \, dy
\]

and

\[
S (z, \xi) := (F_{\eta \to z}^{-1} ((a (\xi - \eta) - a (\xi)) (F_{\phi}^{-1} (\eta)))) (z).
\]

Expanding the potential \( Q \) in Taylor's series, we have

\[
Q (t, y) = Q (t, x) + (y - x) \cdot (\nabla Q) (t, x) + \sum_{j,k=1}^{N} (y_j - x_j) (y_k - x_k) Q_{jk} (t, y, x),
\]

with

\[
Q_{jk} (t, y, x) := \int_{0}^{1} (\partial_{x_j} \partial_{x_k} Q (t, x + \theta (y - x))) (1 - \theta) \, d\theta.
\]

Then,

\[
W_\phi (Qu) (t, x, \xi) = (Q (t, x) + i (\nabla Q) (t, x) \cdot \nabla \xi - (x \cdot \nabla Q) (t, x)) W_\phi u (t, x, \xi) + Ru (t, x, \xi),
\]

where

\[
Ru (t, x, \xi) = \int_{\mathbb{R}^N} e^{-iy \cdot \xi} \overline{\phi (x-y)} \sum_{j,k=1}^{N} (y_j - x_j) (y_k - x_k) Q_{jk} (y, x) u (y) \, dy.
\]

Thus, by (3.13) and (3.15) we transform equation (3.6) into

\[
\begin{cases}
(i \partial_t - i (\nabla Q) (t, x) \cdot \nabla \xi - Q (t, x) + (x \cdot \nabla Q) (t, x) - a (\xi)) W_\phi u (t, x, \xi) \\
= Ru (t, x, \xi) + \tilde{R} u (t, x, \xi) + R_0 u (x, \xi),
\end{cases}
\]

\[
W_\phi (u) (0, x, \xi) = W_\phi (u_0) (x, \xi).
\]

with

\[
\tilde{R} u (x, \xi) := \int_{\mathbb{R}^N} e^{-iy \cdot \xi} \overline{\phi (x-y)} V_2 (t, y) u (y) \, dy
\]

We solve problem (3.17) by the method of characteristics. The solution to (3.17) is given by

\[
W_\phi u (t, x, \xi) = e^{-i \int_{0}^{t} h (s, t, x, \xi) \, ds} \left( W_\phi (u_0) (x, g (0; t, x, \xi)) - i \int_{0}^{t} e^{i \int_{0}^{\tau} h (s, t, x, \xi) \, ds} \left( Ru + \tilde{R} u + R_0 u \right) (\tau, x, g (\tau; t, x, \xi)) \, d\tau \right),
\]

where

\[
h (s; t, x, \xi) := a (g (s; t, x, \xi)) + Q (s, x) - (x \cdot \nabla Q) (s, x)
\]

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and \( g(s; t, x, \xi) = \xi - \int_0^s (\nabla V)(\tau, x) \, d\tau \). Taking the \( L^p\)-norm with respect to \( x \) and \( \xi \) on the both side of (3.18) we obtain

\[
||u(t, \cdot)||_{L^p} = ||W \phi u(t, x, \xi)||_{L^p_{x, \xi}} \leq ||I_1||_{L^p_{x, \xi}} + \int_0^t \left( ||I_2||_{L^p_{x, \xi}} + ||I_3||_{L^p_{x, \xi}} \right) \, d\tau,
\]

(3.19)

where

\[
I_1 := W \phi (u_0)(x, g(0; t, x, \xi)),
\]

(3.20)

\[
I_2 := R u(\tau, x, g(\tau; t, x, \xi)) \]

\[
I_3 := R \phi (\tau, x, g(\tau; t, x, \xi)) \]

and

\[
I_4 := R_0 u(\tau, x, g(\tau; t, x, \xi)).
\]

We begin by estimating \( I_1 \). Let us consider the change of variables \( \Xi = g(0; t, x, \xi) \). Since \( \det J(g(s; t, x, \xi)) = 1 \), for all \( s, t, x \) and \( \xi \), (\( J \) denotes the Jacobian matrix) the implicit function theorem imply

\[
\left| \frac{\partial (\xi)}{\partial (\Xi)} \right| = 1.
\]

Thus,

\[
||I_1||_{L^p_{x, \xi}} = \left( \int_{\mathbb{R}^{2N}} |W \phi (u_0)(x, \Xi)|^p \left| \frac{\partial (\xi)}{\partial (\Xi)} \right| \, dx \, d\Xi \right)^{1/p} = ||u_0||_{L^p}.
\]

(3.22)

Next, we consider \( I_2 \). By the inversion formula (2.2), from (3.16) we deduce

\[
R u(t, x, \xi) = \frac{1}{||\phi||_{L^2}} \sum_{j,k=1}^{N} \int \int \phi_{jk}(y-x) Q_{jk}(t, y, x) \phi(y-z) W \phi u(t, z, \eta) e^{ip(y-\xi)} \, dz \, d\eta dy,
\]

(3.23)

where \( \phi_{jk}(x) := x_j x_k \phi(x) \). Then,

\[
I_2 = R u(\tau, x, g(\tau; t, x, \xi)) = \frac{1}{||\phi||_{L^2}} \sum_{j,k=1}^{N} \int \int \phi_{jk}(y-x) \phi(y-z) Q_{jk}(\tau, y, x) W \phi u(\tau, z, \eta) e^{ip(y-\xi)} \, dz \, d\eta dy.
\]

(3.24)

Let us consider \( n \in \mathbb{N} \) such that \( 2n > N \). Using (3.11) in the right-hand side of (3.24) and integrating by parts we have

\[
||I_2||_{L^p_{x, \xi}} \leq \frac{1}{||\phi||_{L^2}} \sum_{j,k=1}^{N} \int \int_{\mathbb{R}^{2N}} \left| (1 - \Delta y)^n (\phi_{jk}(y-x) Q_{jk}(\tau, y, x) \phi(y-z)) \right| \left| \frac{W \phi u(\tau, z, \eta)}{(\eta - g(\tau; t, x, \xi))^{2n}} \right| \, dz \, d\eta dy.
\]

\[
\leq \frac{1}{||\phi||_{L^2}} \sum_{j,k=1}^{N} \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2n} \left| \int \int_{\mathbb{R}^{2N}} \partial_{y}^{\beta_1} \partial_{\xi}^{\beta_2} Q_{jk}(\tau, y, x) \partial_{y}^{\beta_3} \phi(y-z) \frac{W \phi u(\tau, z, \eta)}{(\eta - g(\tau; t, x, \xi))^{2n}} \, dz \, d\eta dy \right|.
\]

Since \( |\partial_{y}^{\beta} Q_{jk}(\tau, y, x)| \leq C_{\beta_2}, C_{\beta_2} > 0 \), we estimate

\[
||I_2||_{L^p_{x, \xi}} \leq \frac{1}{||\phi||_{L^2}} \sum_{j,k=1}^{N} \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2n} \left| \int \int_{\mathbb{R}^{2N}} C_{\beta_2} |\partial_{y}^{\beta_1} \phi_{jk}(y-x)| |\partial_{y}^{\beta_3} \phi(y-z)\frac{W \phi u(\tau, z, \eta)}{(\eta - g(\tau; t, x, \xi))^{2n}} \, dz \, d\eta dy \right|.
\]

(3.25)
Making the change of variables $\Xi = g(\tau; t, x, \xi)$ in the integral on the right-hand side of (3.25) and using (3.21) we obtain

\[
\frac{1}{p} \left\| I_2 \right\| \leq \frac{1}{p} \left\| \phi \right\| L^2 \sum_{j,k=1}^{N} \left( \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^{2n}} C_{\beta_2} \left| \partial^{\beta_1} \phi_{jk} (y-x) \right| \left| \partial^{\beta_2} \phi (y-z) \right| \left| W_{\phi u} (\tau, z, \eta) \right| \frac{d\eta d\eta}{(\eta - \Xi)^{2n}} \right)^p dx d\Xi \right)^{1/p}.
\]

(3.26)

Then, by Young’s inequality it follows

\[
\left\| I_2 \right\| \leq \frac{1}{p} \left\| \phi \right\| L^2 \sum_{j,k=1}^{N} \left( \int_{\mathbb{R}^{2n}} C_{\beta_2} \left\| \partial^{\beta_1} \phi_{jk} \right\| L^1 \left\| \partial^{\beta_2} \phi \right\| L^1 \left( \frac{\tau}{\eta - \Xi} \right)^{2n} \left\| u (\tau, \cdot) \right\| M^p \right)^{1/p} \leq C_T \left\| u (\tau, \cdot) \right\| M^p,
\]

(3.27)

uniformly for $t \in [0,T]$ and $\tau \in [0,t]$. Since $V_2$ satisfies (1.3) with $|\alpha| \geq 0$, similarly to (3.27) we prove that

\[
\left\| I_3 \right\| \leq C_T \left\| u (\tau, \cdot) \right\| M^p,
\]

(3.28)

uniformly for $t \in [0,T]$ and $\tau \in [0,t]$.

At last, we estimate $I_4$. Again, by (2.2), from (3.14) we have

\[
R_0 u (x, \xi) = \frac{1}{2} \left\| \phi \right\| L^2 \left( \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} S(x-y, \xi, \eta) \phi(y-z) W_{\phi u}(t, z, \eta) e^{i \eta (\tau - \xi)} dy d\eta \right).
\]

and then,

\[
\left\| I_4 \right\| \leq \frac{1}{p} \left\| \phi \right\| L^2 \left( \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} S(x-y, \xi, \eta) \phi(y-z) W_{\phi u}(t, z, \eta) e^{i \eta (\tau - \xi)} dy d\eta \right)^{1/p}.
\]

(3.29)

Using (3.11) in the right-hand side of (3.29) and integrating by parts we get

\[
\left\| I_4 \right\| \leq \frac{1}{p} \left\| \phi \right\| L^2 \left( \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left| \partial^{\beta_1} S(x-y, \xi, \eta) \right| \left| \partial^{\beta_2} \phi(y-z) \right| \left| W_{\phi u}(t, z, \eta) \right| \frac{d\eta d\eta}{(\eta - \Xi)^{2n}} \right)^{1/p}.
\]

Then, by Young’s inequality we deduce

\[
\left\| I_4 \right\| \leq \frac{1}{p} \left\| \phi \right\| L^2 \left( \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left| \partial^{\beta_1} S(x-y, \xi, \eta) \right| \left| \partial^{\beta_2} \phi(y-z) \right| \left| W_{\phi u}(t, z, \eta) \right| \frac{d\eta d\eta}{(\eta - \Xi)^{2n}} \right)^{1/p}.
\]

(3.30)

uniformly for $t \in [0,T]$ and $\tau \in [0,t]$.

Finally, using (3.22), (3.27) and (3.30) in (3.19) we arrive to

\[
\left\| u (t, \cdot) \right\| M^p \leq \left\| u_0 \right\| M^p + C \int_0^t \left\| u (\tau, \cdot) \right\| M^p d\tau.
\]

(3.31)

Applying Gronwall’s lemma to (3.31), we attain (3.12). The case $t \in [-T,0]$ can be considered similarly.

The second assertion of Theorem 1.2 follows from:
Theorem 3.3 Let \( 1 \leq p, q \leq \infty \) and \( T > 0 \). Set \( m \in \mathbb{N} \). Let \( a_1(\xi) = \{a_{jk}(\xi)\} \) be an Hermitian \((m \times m)\)-matrix valued function such that \( a_{jk}(\xi) \in C^\infty(\mathbb{R}^N) \) and the estimate (3.8) is valid. Suppose that \( \mathbf{V}(t, x) = \mathbf{V}_1(t, x) + \mathbf{V}_2(t, x) \), where \( \mathbf{V}_1(t, x) = \{(\mathbf{V}_1)_{jk}(t, x)\} \) is a matrix-valued function with \( Q_{jk}(t, x) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N) \) that satisfies (1.3) for \( |\alpha| \geq 1 \). Moreover, let \( \mathbf{V}_2(t, x) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N) \) be a matrix-valued function that verifies (1.3) for \( |\alpha| \geq 0 \). Then, the solution \( u(t, x) \) of (3.6)

\[
\|u(t, \cdot)\|_{M^p,q} \leq C_T \|u_0\|_{M^p,q},
\]

uniformly for \( t \in [-T, T] \).

Proof. We consider the case \( t \in [0, T] \). We expand the potential \( \mathbf{V}_1 \) as

\[
\mathbf{V}_1(t, x) = \mathbf{V}_1(t, x) + \sum_{k=1}^{N} (y_k - x_k) \mathbf{V}_k(t, x),
\]

where

\[
\mathbf{V}_k(t, y, x) := \int_{0}^{1} \partial_{\theta} \mathbf{V}_1(t, x + \theta(y - x)) \, d\theta.
\]

Then,

\[
W_{\phi}(qu)(t, x, \xi) = \mathbf{V}_1(t, x) W_{\phi}u(t, x, \xi) + R_1u(t, x, \xi),
\]

where

\[
R_1u(t, x, \xi) = \sum_{k=1}^{N} \int_{\mathbb{R}^N} e^{-i\alpha(\xi)} \tilde{\phi}_k(y) \mathbf{V}_k(t, y, x) u(t, y) \, dy,
\]

and \( \tilde{\phi}_k : y \mapsto y_k \phi(y) \). By (3.13) and (3.33) we transform equation (3.6) into

\[
\left\{ (i \partial_t - \mathbf{V}_1(t, x) - a(\xi)) W_{\phi}u(t, x, \xi) = R_1u(t, x, \xi) + \tilde{R}u(t, x, \xi) + R_0u(t, x, \xi), W_{\phi}(u)(0, x, \xi) = W_{\phi}(u_0)(x, \xi).
\]

Thus, the solution of (3.34) is given by

\[
W_{\phi}u(t, x, \xi) = e^{-ita(\xi)} - i \int_{0}^{t} \mathbf{V}_1(s, x) ds \left( W_{\phi}(u_0)(x, \xi) - i \int_{0}^{t} e^{-ia(\xi)} - i \int_{0}^{t} \mathbf{V}_1(s, x) ds \left( R_1u + \tilde{R}u + R_0u \right)(\tau, x, \xi) d\tau \right).
\]

Therefore,

\[
|W_{\phi}u(t, x, \xi)| \leq \left| \tilde{I}_1 \right| + \int_{0}^{t} \left( \left| \tilde{I}_2 \right| + \left| \tilde{I}_3 \right| + \left| \tilde{I}_4 \right| \right) d\tau,
\]

where \( \tilde{I}_1 := W_{\phi}(u_0)(x, \xi) \), \( \tilde{I}_2 := R_1u(t, x, \xi) \), \( \tilde{I}_3 := \tilde{R}u(t, x, \xi) \) and \( \tilde{I}_4 := R_0u(t, x, \xi) \). Let us estimate the norms \( \left\| \tilde{I}_j \right\|_{M_{\phi}^{1,\infty}} \) and \( \left\| \tilde{I}_j \right\|_{M_{\phi}^{p,q}} \), \( 1 \leq p \leq \infty \), for \( j = 2, 3, 4 \). By the inversion formula for the wave-packet transform we have

\[
R_1u(t, x, \xi) = \frac{1}{\|\phi\|_{L^2}} \sum_{k=1}^{N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{\phi}_k(y - x) \mathbf{V}_k(t, y, x) \phi(y - z) W_{\phi}u(t, z, \eta) e^{i(y - \xi)} d\eta dy.
\]

Taking into account (3.11), integrating by parts in the right-hand side of (3.36) and using (1.3) we estimate

\[
\left| \tilde{I}_2 \right| \leq \frac{1}{\|\phi\|_{L^2}} \sum_{j, k=1}^{N} \sum_{|\beta_1| + |\beta_2| + |\beta_3| + |\beta_4| \leq 2n} C_{\beta_2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \partial_{\eta}^{\beta_2} \tilde{\phi}_k(y) \right| \left| \partial_{y}^{\beta_4} \phi(y - z) \right| \left| W_{\phi}u(t, z, \eta) \right| \, d\eta dy.
\]

Then, by Young’s inequality

\[
\left\| \tilde{I}_2 \right\|_{L_{\phi}^{p,q}} \leq C_T \left\| u(t, \cdot) \right\|_{M_{\phi}^{\infty,1}}^2,
\]

where

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \partial_{\eta}^{\beta_2} \tilde{\phi}_k(y) \right| \left| \partial_{y}^{\beta_4} \phi(y - z) \right| \left| W_{\phi}u(t, z, \eta) \right| \, d\eta dy \right)^{\frac{1}{p}} \leq C_{\beta_2} \left\| \partial_{\eta}^{\beta_2} \tilde{\phi}_k \right\|_{L^1} \left\| \partial_{y}^{\beta_4} \phi \right\|_{L^1}.
\]
then by Young’s inequality and
\[ \| I_2 \|_{L^p_T} \leq C_T \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}}, \]  
uniformly for \( t \in [0, T] \) and \( \tau \in [0, t] \). Since \( V_2 \) satisfies (1.3) with \( |\alpha| \geq 0 \), similarly to (3.38)-(3.40) we prove that
\[ \left\| \int_{\tau}^{t} \right\|_{L^p_T} \leq C_T \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}}, \]  
and
\[ \left\| I_3 \right\|_{L^p_T} \leq C_T \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}}, \]  
and
\[ \left\| I_3 \right\|_{L^p_T} \leq C_T \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}}, \]  
uniformly for \( t \in [0, T] \) and \( \tau \in [0, t] \). Next, using (3.11) and integrating by parts in the right-hand side of (3.23) we have

\[ \left| \int_{\tau}^{t} \right| \leq \frac{1}{\| \phi \|_{L^2_{|\beta_1|+|\beta_2|\leq 2n}}} \iint S(x-y, \xi) \left| \partial_{y}^2 \phi(y-z) \right| \frac{|W_\phi u(t, z, \eta)|}{(\eta-\xi)^{2n}} d\eta d\eta. \]

Moreover, using Lemma 3.1 we get
\[ \left| \int_{\tau}^{t} \right| \leq C \| \phi \|_{L^2_{|\beta_1|+|\beta_2|\leq 2n}} \iint (x-y)^{-2n} \left| \partial_{y}^2 \phi(y-z) \right| \frac{|W_\phi u(t, z, \eta)|}{(\eta-\xi)^{2n}} d\eta d\eta. \]

Then, by Young’s inequality
\[ \left\| \int_{\tau}^{t} \right\|_{L^p_T} \leq C_T \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}}, \]  
and
\[ \left\| \int_{\tau}^{t} \right\|_{L^p_T} \leq C_T \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}}, \]  
and
\[ \left\| \int_{\tau}^{t} \right\|_{L^p_T} \leq C_T \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}}, \]  
uniformly for \( t \in [0, T] \) and \( \tau \in [0, t] \). Using estimates (3.38)-(3.43), (3.44)-(3.46) in (3.35) we get
\[ \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} \leq C \| u_0 \|_{M_{\phi}^{1,\infty}} + C_0 \int_{0}^{t} \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} d\tau, \]
\[ \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} \leq C \| u_0 \|_{M_{\phi}^{1,\infty}} + C_0 \int_{0}^{t} \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} d\tau, \]
and
\[ \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} \leq C \| u_0 \|_{M_{\phi}^{1,\infty}} + C_0 \int_{0}^{t} \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} d\tau. \]

Then, Gronwall’s lemma yields
\[ \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} \leq C \| u_0 \|_{M_{\phi}^{1,\infty}}, \]  
\[ \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} \leq C \| u_0 \|_{M_{\phi}^{1,\infty}}, \]  
and
\[ \| u(\tau, \cdot) \|_{M_{\phi}^{1,\infty}} \leq C \| u_0 \|_{M_{\phi}^{1,\infty}}. \]  

Estimate (3.32) follows from (3.47)-(3.49) by the complex interpolation theorem for modulation spaces (see Proposition 2.4). The case \( t \in [0, T] \) can be considered similarly. \( \square \)
References

[1] R. A. Adams and J.J.F. Fournier, Sobolev Spaces, 2nd edition, Academic Press, New York, (2003).

[2] Á. Bényi and K. Okoudjou, Local well-posedness of nonlinear dispersive equations on modulation spaces, Bull. Lond. Math. Soc. 41 (2009) 549–558.

[3] Á. Bényi, K. Gröchenig, K. Okoudjou and L.G.Rogers, Unimodular Fourier multipliers for modulation spaces, J. Funct. Anal. 246 (2007) 366–384.

[4] D. Bhimani and P.K.Ratnakumar, Functions operating on modulation spaces and nonlinear dispersive equations. J. Funct. Anal. 270 2 (2016) 621–648.

[5] J. Chen and D. Fan, Estimates for wave and Klein-Gordon equations on modulation spaces. Sci. China Math. 55 10 (2012), 2109–2123.

[6] J.Chen, W.Guo and G. Zhao, Klein-Gordon equations on modulation spaces. Abstr. Appl. Anal. (2014), 15 pp.

[7] E. Cordero and F.Nicola, Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation, J. Funct. Anal. 254 (2008) 506–534.

[8] E.Cordero and F.Nicola, Strichartz estimates in Wiener amalgam spaces for the Schrödinger equation, Math.Nachr. 281 (2008) 25–41.

[9] E.Cordero, K.Gröchenig, F.Nicola and L.Rodino, Wiener algebras of Fourier integral operators, J. Math. Pures Appl. 99 (2013) 219–233.

[10] E.Cordero, K.Gröchenig, F.Nicola and L.Rodino, Generalized metaplectic operators and the Schrödinger equation with a potential in the Sjöstrand class, J. Math. Phys. 55 8 (2014) 17 pp.

[11] E.Cordero and F.Nicola, On the Schrödinger equation with potential in modulation spaces, J. Pseudo-Differ. Oper. Appl. 5 3 (2014), 319–341.

[12] Correia, S; Local Cauchy theory for the nonlinear Schrödinger equation in spaces of infinite mass. Rev. Mat. Complut. 31 2 (2018), 449–465.

[13] H.G.Feichtinger, Modulation spaces on locally compact abelian groups, in: M.Krishna, R.Radha, S.Thangavelu (Eds.), Wavelets and Their Applications, Allied Publishers, Chennai, India, NewDelhi, (2003), pp.99–140, updated version of a technical report, University of Vienna, (1983).

[14] Gallo, C., Schrödinger group on Zhidkov spaces. Adv. Differ. Equ. 9 5–6 , 509–538 (2004)

[15] Gérard, P., The Cauchy problem for the Gross–Pitaevskii equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 23 5, 765–779 (2006)

[16] K. Kato, M. Kobayashi and S. Ito, Remark on wave front sets of solutions to Schrödinger equation of a free particle and anharmonic oscillator, SUT J. Math. 47 (2011) 175–183.

[17] K. Kato, M. Kobayashi and S. Ito, Representation of Schrödinger operator of a free particle via short time Fourier transform and its applications, Tohoku Math. J. 64 (2012) 223–231.

[18] K. Kato, M. Kobayashi and S. Ito, Remarks on Wiener Amalgam space type estimates for Schrödinger equation, in: RIMS Kōkyūroku Bessatsu ,B33, Res. Inst. Math. Sci. (RIMS), Kyoto, 2012, pp.41–48.

[19] K. Kato, M. Kobayashi and S. Ito, Estimates on modulation spaces for Schrödinger evolution operators with quadratic and sub-quadratic potentials, J. Funct. Anal. 266 (2014) 733–753.

[20] M.Kobayashi and M.Sugimoto, The inclusion relation between Sobolev and modulation spaces, J. Funct. Anal. 260 (2011) 3189–3208.

[21] A.Miyachi, F.Nicola, S.Rivetti, A.Tabacco and N.Tomita, Estimates for unimodular Fourier multipliers on modulation spaces, Proc. Amer. Math. Soc. 137 (2009) 3869–3883.

[22] Naumkin I P and Weder R, High-energy and smoothness asymptotic expansion of the scattering amplitude for the Dirac equation and application, Math. Meth. Appl. Sci. 38 (2015) 2427-2465.
[23] B. Thaller, The Dirac equation, Texts and Monographs in Physics Berlin: Springer-Verlag, (1992)

[24] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, II, Ann. Global Anal. Geom. 26 (2004) 73–106.

[25] N. Tomita, Unimodular Fourier multipliers on modulation spaces $M_{p,q}$ for $0 < p < 1$. Harmonic analysis and nonlinear partial differential equations, in: RIMS Kôkyûroku Bessatsu, B18, Res.Inst.Math.Sci. (RIMS), Kyoto, 2010, pp.125–131.

[26] Vargas, A., Vega, L., Global wellposedness for 1D non-linear Schrödinger equation for data with an infinite $L^2$ norm. J. Math. Pures Appl. (9) 80 10, 1029–1044 (2001)

[27] P. Wahlberg, Vector-valued Modulation Spaces and Localization Operators with Operator-valued Symbols, Integr. equ. oper. theory 59 (2007) 99-128.

[28] B.Wang and C.Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, J. Differential Equations 239 (2007) 213–250.

[29] B.Wang and H.Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, J. Differential Equations 232 (2007) 36–73.

[30] B. Wang, L. Zhao and B.Guo, Isometric decomposition operators, function spaces $E^A_{p,q}$ and applications to nonlinear evolution equations, J. Funct. Anal. 233 (2006) 1–39.

[31] W.Wei, Modulation space estimates for Schrödinger type equations with time-dependent potentials, Czechoslovak Math. J. 64 139 2 (2014), 539–566.

[32] Zhou, Y.: Cauchy problem of nonlinear Schrödinger equation with initial data in Sobolev space $W^{s,p}$ for $p < 2$. Trans. Am. Math. Soc. 362 9, 4683–4694 (2010)