Quantum Speed Limits for Time Evolution of a System Subspace

S. Albeverio\textsuperscript{a} * and A. K. Motovilov\textsuperscript{b, c}, **

\textsuperscript{a} Institut für Angewandte Mathematik and HCM, Universität Bonn, Bonn, 53115 Germany
\textsuperscript{b} Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, 141980 Russia
\textsuperscript{c} Dubna State University, Dubna, 141980 Russia

* e-mail: albeverio@uni-bonn.de
** e-mail: motovilv@theor.jinr.ru

Received March 11, 2021; revised July 8, 2021; accepted July 18, 2021

Abstract—One of the fundamental physical limits on the speed of time evolution of a quantum state is known in the form of the celebrated Mandelstam–Tamm inequality. This inequality gives an answer to the question on how fast an isolated quantum system can evolve from its initial state to an orthogonal one. In its turn, the Fleming bound is an extension of the Mandelstam–Tamm inequality that gives an optimal speed bound for the evolution between non-orthogonal initial and final states. In the present work, we are concerned not with a single state but with a whole (possibly infinite-dimensional) subspace of the system states that are subject to the Schrödinger evolution. By using the concept of maximal angle between subspaces we derive an optimal estimate on the speed of such a subspace evolution that may be viewed as a natural generalization of the Fleming bound.

Keywords: Mandelstam–Tamm inequality, Fleming bound, quantum speed limit, subspace evolution

DOI: 10.1134/S1063779622020058

1. INTRODUCTION

By a quantum speed limit one understands a lower bound on the time required for a quantum system to evolve between two (given) distinguishable states. Literature on quantum speed limits is enormous and in this short article we make no attempt to present a more or less complete survey of all the results on the subject. Instead, we simply refer the reader to the recent comprehensive review articles [1] and [2].

To introduce the main known quantum speed limits, we consider an isolated quantum system with a Hamiltonian $H$, which is supposed to be a time-independent self-adjoint operator on the complex Hilbert space $\mathcal{H}$. The unit (i.e. norm one) vectors of $\mathcal{H}$ represent possible pure states of the system. To be more precise, a pure state $\psi$ of a quantum system is a class of equivalence of vectors on a unit sphere in $\mathcal{H}$: the vectors $\phi, \psi \in \mathcal{H}$ represent the same state if there is $\alpha \in [0, 2\pi)$ such that $\psi = e^{i\alpha} \phi$. In an obvious way, one may identify the state $\psi$ with a one-dimensional subspace $\mathcal{H}_\psi$ which is a linear span of an arbitrarily chosen vector $\psi$ in $\mathcal{H}$, $\mathcal{H}_\psi = \{ \phi = \lambda \psi | \lambda \in \mathbb{C} \}$.

In the following, we will assume, for shortness, that the measurement units are chosen in such a way that $\hbar = 1$. It is supposed that the evolution of a state vector $\psi(t) \in \mathcal{H}$, $t \in \mathbb{R}$, is governed by the Schrödinger equation

\begin{equation}
\label{eq:Schrodinger}
\tag{1}
 i \frac{d}{dt} \psi = H \psi,
\end{equation}

\begin{equation}
\label{eq:initial}
\tag{2}
\psi(t)|_{t = 0} = \psi_0,
\end{equation}

where the vector $\psi_0$ is taken in the domain $\text{Dom}(H)$ of $H$. This vector represents an initial state of the system.

Let $t_0 = 0$. Then the solution to (1), (2) is given by

\begin{equation}
\label{eq:solution}
\tag{3}
\psi(t) = U(t) \psi_0,
\end{equation}

where

\begin{equation}
\label{eq:unitary}
\tag{4}
U(t) = e^{-iHt}, \quad t \in \mathbb{R},
\end{equation}

form a strongly continuous unitary group. Studies of quantum speed limits originate from the very basic question: How fast can the isolated system with the Hamiltonian $H$ evolve to a state orthogonal to its initial state $\psi_\theta$?

The importance of this question is obvious in many respects. Probably, the most recent motivation comes from quantum information theory and quantum computing (see, e.g., [1, 2]).

Known answers to the above basic question have been given in the form of lower bounds for the so-called orthogonalization time $T_1$, that is, for the time
needed for the system to evolve from $\psi_0$ to a state $\psi(T_{\perp})$ such that $\langle \psi_0, \psi(T_{\perp}) \rangle = 0$.

The first of these bounds is the celebrated Mandelstam–Tamm inequality of 1945 discovered in [3]:

$$T_{\perp} \geq \frac{2\pi}{\Delta E},$$  \hfill (5)

where $\Delta E$ is the energy spread (dispersion) in the initial state $\psi_0$,

$$\Delta E = \sqrt{\langle H\psi_0, \psi_0 \rangle - \langle H\psi_0, \psi_0 \rangle^2}, \quad \psi_0 \in \text{Dom}(H).$$  \hfill (6)

Another lower bound for the orthogonalization time, the Margolus–Levitin inequality [4] has been found more than half a century later, in 1998. This bound has the following form:

$$T_{\parallel} \geq \frac{2\pi}{\delta E},$$  \hfill (7)

where

$$\delta E = \langle H\psi_0, \psi_0 \rangle - \min(\text{spec}(H))$$  \hfill (8)

is nothing but the average energy for the state $\psi_0$ measured relative to the lower edge of the spectrum $\text{spec}(H)$ of the Hamiltonian $H$.

Both the bounds (5) and (7) have been proven to be sharp (see, e.g., [1, p. 7] and [2, p. 3923]).

It is worth to remark that the inequalities (5) and (7) recall the uncertainty relation for energy and time but are very different from this relation in the essence since both (5) and (7) are related not to the standard deviation in the measurement of $t$ but to the well-founded time for a given state to evolve into an orthogonal state.

There is also a lower bound for intermediate time moments, namely, the Fleming bound (1973) derived in [5]:

$$T_\theta \geq \frac{\theta}{\Delta E},$$  \hfill (9)

where $T_\theta$ stands for the time moment at which the acute angle

$$\angle(\psi_0, \psi(t)) := \arccos\left|\langle \psi_0, \psi(t) \rangle\right|$$

between the states represented by the vectors $\psi_0$ and $\psi(t)$ reaches a certain value $\theta \in (0, \pi/2]$.

Obviously, the Mandelstam–Tamm bound represents a particular case of the Fleming bound for $\theta = \pi/2$. In fact, through the years, the Mandelstam–Tamm inequality (5) has been rediscovered several times by various researchers (for the corresponding discussion, see, e.g., [1, p. 5]). Also, there are generalizations of this bound to the evolution of mixed states. Furthermore, there are more detailed evolution speed estimates for particular classes of quantum-mechanical evolutionary problems (see [1, 2]). The existence of Mandelstam–Tamm-type bounds for the orthogonalization time is discussed even for certain non-self-adjoint (so-called pseudo-Hermitian and, in particular, PT-symmetric) Hamiltonians (see [6, 7] and references therein).

In the present work, we are concerned with a possibly infinite-dimensional subspace of the system states that evolve accordingly to the non-stationary Schrödinger equation. Notice that by a subspace of the Hilbert space $\mathcal{F}$ we always understand a closed linear subset in $\mathcal{F}$. By using the concept of maximal angle between subspaces we then derive several estimates on the speed of such a subspace evolution. The estimate (26) we attain in Theorem 5 below may be viewed as a natural extension of the Fleming bound (9).

2. BOUNDS FOR THE SPEED OF THE SUBSPACE EVOLUTION

We are concerned not with a single state but with a whole (possibly infinite-dimensional) subspace spanned by the system states that are subject to the Schrödinger evolution. That is, we consider a subspace $\mathcal{F}_0 \subset \mathcal{F}$ every element $\psi_0$ of which is subject to the Schrödinger evolution (1), (2), i.e.,

$$i\frac{d}{dt}\psi = H\psi,$$  \hfill (10)

$$\psi(t)\rvert_{t=0} = \psi_0, \quad \psi_0 \in \mathcal{F}_0.$$  \hfill (11)

For simplicity (in fact, mainly for avoiding discussion of the domains), the Hamiltonian $H$ is assumed to be a bounded operator.

Given $t \geq 0$, by $\mathcal{P}(t)$ we denote the subspace of $\mathcal{F}$ spanned by the values $\psi(t)$ of the vector-valued functions that solve (10), (11) for various $\psi_0 \in \mathcal{F}_0$. So that we deal with a path $\mathcal{P}(t)$, $t \geq 0$, in the set of all subspaces of the Hilbert space $\mathcal{F}$. Or (and this is the same) with the path

$$P(t), \quad t \geq 0, \quad \text{Ran}(P(t)) = \mathcal{P}(t),$$  \hfill (12)

of the orthogonal projections $P(t)$ in $\mathcal{F}$ onto the respective subspaces $\mathcal{P}(t)$.

It is well known, and this is easily verified by inspection, that the projection path $P(t)$ is the (unique) solution to the Cauchy problem

$$i\frac{d}{dt}P = [P, H],$$  \hfill (13)

$$P(t)\rvert_{t=0} = P_0,$$  \hfill (14)

where $[P, H] := PH - HP$ denotes the commutator of $P = P(t)$ and $H$, and $\text{Ran}(P_0) = \mathcal{F}_0$. The solution to (13), (14) is explicitly given by

$$P(t) = U(t)P_0 U(t)^*, = e^{-iHt}P_0 e^{iHt}.$$  \hfill (15)
We further notice that the set of all orthogonal projections in the Hilbert space $\mathcal{H}$ (and hence the set of all subspaces of $\mathcal{H}$) is a metric space with distance given by the standard operator norm,

$$
\rho(Q_1, Q_2) := \|Q_1 - Q_2\|, \quad \rho(Q_1, Q_2) := \rho(Q_1, Q_2),
$$

where $Q_1$, $Q_2$ are arbitrary orthogonal projections and $\mathcal{O}_1$, $\mathcal{O}_2$, their ranges.

It is, however, much less known that there is another natural metric on the set of all orthogonal projections in $\mathcal{H}$ (and hence on the set of all the subspaces of $\mathcal{H}$). The corresponding distance is defined by

$$
\vartheta(\mathcal{O}_1, \mathcal{O}_2) := \vartheta(Q_1, Q_2) := \arcsin(\|Q_1 - Q_2\|). \quad (16)
$$

That (16) is a metric has been proven in 1993 by Lawrence Brown [8]. An alternative proof may be found in [9].

The quantity $\vartheta(\mathcal{O}_1, \mathcal{O}_2)$ is called the maximal angle between the subspaces $\mathcal{O}_1$ and $\mathcal{O}_2$.

**Remark 1.** The concept of maximal angle between subspaces can be traced back to Krein, Krasnoselsky, and Milman [10]. Assuming that $(\mathcal{O}_1, \mathcal{O}_2)$ is an ordered pair of subspaces with $\mathcal{O}_1 \neq \{0\}$, they applied the notion of the (relative) maximal angle between the subspaces $\mathcal{O}_1$ and $\mathcal{O}_2$ to the number $\varphi(\mathcal{O}_1, \mathcal{O}_2) \in [0, \pi/2]$ such that

$$
\sin \varphi(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \text{dist}(x, \mathcal{O}_2). \quad (17)
$$

If both $\mathcal{O}_1 \neq \{0\}$ and $\mathcal{O}_2 \neq \{0\}$ then

$$
\vartheta(\mathcal{O}_1, \mathcal{O}_2) = \max(\varphi(\mathcal{O}_1, \mathcal{O}_2), \varphi(\mathcal{O}_2, \mathcal{O}_1)). \quad (18)
$$

Unlike $\varphi(\mathcal{O}_1, \mathcal{O}_2)$, the maximal angle $\vartheta(\mathcal{O}_1, \mathcal{O}_2)$ is always symmetric with respect to the interchange of the arguments $\mathcal{O}_1$ and $\mathcal{O}_2$. Furthermore,

$$
\varphi(\mathcal{O}_2, \mathcal{O}_1) = \varphi(\mathcal{O}_1, \mathcal{O}_2) = \vartheta(\mathcal{O}_1, \mathcal{O}_2)
$$

whenever $\|Q_1 - Q_2\| < 1$.

To give a quantum-mechanical interpretation of the maximal angle between subspaces we follow the concept of a subspace-state of a quantum system. Namely, given a subspace $\mathcal{O} \subset \mathcal{H}$, one says that the system is in the $\mathcal{O}$-state if it is in a pure state described by a (non-specified) normalized vector $x \in \mathcal{O}$. Clearly, by (17) and (18) the quantity $\cos^2 \vartheta(\mathcal{O}_1, \mathcal{O}_2)$ may be understood as a minimum probability for a quantum system which is in a $\mathcal{O}_1$-state to be found also in a $\mathcal{O}_2$-state.

Now assume that $Q(t), t \geq 0$, is an arbitrary piecewise smooth path in the set of orthogonal projections on the Hilbert space $\mathcal{H}$. By using the triangle inequality for subspaces one verifies (see, e.g., [11], Theorem 1) that

$$
\vartheta(\mathcal{O}_0, \mathcal{O}_t) \leq \int_0^t \|Q(\tau)\| d\tau, \quad (19)
$$

where $Q(t) = \frac{dQ(t)}{dt}$ and $\mathcal{O}_t := \text{Ran}(Q(t)), t \geq 0$.

The following statement is the main tool that we use below in establishing quantum speed limits for evolution of subspaces. It represents itself the first of such limits.

**Theorem 2.** Assume that $P(t), t \geq 0$, is the path (15) where $P_0$ is an orthogonal projection in $\mathcal{H}$. Then the following inequality holds

$$
\vartheta(\Psi_0, \Psi(t)) \leq V_{H, P} t,
$$

where $\Psi_0 = \text{Ran}(P_0), \Psi(t) = \text{Ran}(P(t)), t \geq 0$, and

$$
V_{H, P} := \|P_0 H P_0^+\| = \|P_0^+ H P_0\|. \quad (20)
$$

**Proof.** In the case under consideration the operator norm of the commutator of $P(t)$ and $H$ does not depend on $t$, thus,

$$
\|\{P(t), H\}\| = \|P_0 H P_0^+\| = \|P_0^+ H P_0\|, \quad (21)
$$

where $\Psi_0 = \text{Ran}(P_0)$, $\Psi(t) = \text{Ran}(P(t)), t \geq 0$, and

$$
\{P_0, H\} = P_0 H P_0^+ - P_0^+ H P_0. \quad (22)
$$

From (23) it immediately follows that

$$
\|\{P_0, H\}\| = \|P_0 H P_0^+\| = \|P_0^+ H P_0\|. \quad (24)
$$

Thus, in order to conclude with (20) it only remains to combine (13) with (22) and (23) and then to apply to $P(t)$ the inequality (19).

The proof is complete.

It is worth to notice that by (20), (21) only the off-diagonal entries $P_0 H P_0^+$ and $P_0^+ H P_0$ of $H$ contribute into the variation of the subspace $\Psi_0$. If the Hamiltonian $H$ is block diagonal with respect to the orthogonal decomposition $\mathcal{H} = \Psi_0 \oplus \Psi_0^\perp$, more precisely,

$$
\{P_0, H\} = P_0 H P_0^+ - P_0^+ H P_0.
$$

**Corollary 3.** Under the hypothesis of Theorem 2, assume that $T_0$ is a time moment for which the maximal angle between the initial subspace $\Psi_0$ and a subspace in the path $\Psi(t), t \geq 0$, reaches the value of $\theta, 0 < \theta \leq \pi/2$, that is,

$$
\vartheta(\Psi_0, \Psi(T_0)) = \theta.
$$

(24)
Then
\[ T_0 \geq \frac{\theta}{V_{H,P_0}} \]  \hspace{1cm} (25)

**Example 4.** Let the Hamiltonian \( H \) correspond to a two-level system with non-degenerate bound states \( e_1 \) and \( e_2 \), that is, \( \|e_1\| = \|e_2\| = 1 \), \( \langle e_1, e_2 \rangle = 0 \), and
\[ H = E_1\langle e_1, e_1\rangle e_1 + E_2\langle e_2, e_2\rangle e_2, \]
where the binding energies \( E_1 \) and \( E_2 \) are different, \( E_1 \neq E_2 \). Assume that \( P_0 \), \( \langle e \rangle \), is projection on the one-dimensional subspace spanned by the vector 
\[ e = \frac{1}{\sqrt{2}}(e_1 + e_2). \]

Notice that Example 4 is employed in many papers on quantum speed limits (see, e.g., \([1, 2, 6, 7]\)). In particular, this example proves tightness of both the Mandelstam–Tamm and Margolus–Levitin inequalities (see, e.g., \([1]\), Section 2.4). One easily verifies that this example also works well for the bound (25) turning this bound into equality. That is, the bound (25) is optimal.

**Theorem 5.** Assume the hypothesis of Theorem 2. Let \( \theta \) and \( T_0 \) be the same as in Corollary 3. Then the following inequality holds:
\[ T_0 \geq \frac{\theta}{\Delta E_{\Psi_0}}, \]  \hspace{1cm} (26)
where
\[ \Delta E_{\Psi_0} = \sup_{\psi \in \Psi_0} \left( \|H^2\psi\| - \langle H\psi, \psi \rangle \right)^{1/2}. \]  \hspace{1cm} (27)

Skipping the proof, we only notice that (26) is proven by Theorem 2 by taking into account that \( V_{H,P_0} \leq \Delta E_{\Psi_0} \).

**Remark 6.** Example 4 shows that the bound (26) is sharp. (In the case of a one-dimensional subspace \( \Psi_0 \) this bound simply turns into the Fleming bound for the speed of a state evolution).

The next statement is easy to prove and it is rather well known. We present it here only for convenience of the reader.

**Lemma 7.** For the maximal energy dispersion \( \Delta E_{\Psi_0} \) on the subspace \( \Psi_0 \), defined by (27), one always has the (optimal) bound
\[ \Delta E_{\Psi_0} \leq \frac{E_{\max}(H) - E_{\min}(H)}{2}, \]  \hspace{1cm} (28)
where
\[ E_{\min}(H) = \min(\text{spec}(H)) \]
and \( E_{\max}(H) = \max(\text{spec}(H)) \) are respectively the upper and lower bounds of the spectrum of the Hermitian Hamiltonian \( H \).

With Lemma 7 one easily obtains the following corollary to Theorem 5.

**Corollary 8.** Assume that \( \Omega \) is a non-negative number and let \( \mathcal{B}_d(\mathcal{S}) \) be the set of all bounded self-adjoint operators \( H \) on the Hilbert space \( \mathcal{S} \) (with \( \dim \mathcal{S} \geq 2 \)) such that
\[ E_{\max}(H) - E_{\min}(H) = \Omega. \]
Then
\[ \inf_{H \in \mathcal{B}_d(\mathcal{S})} T_0(H) = \frac{2\theta}{\Omega}, \]  \hspace{1cm} (29)
where \( T_0(H) \) is a time moment for which the maximal angle between the initial subspace \( \Psi_0 \) and a subspace in the path \( \Psi(t) \), \( t \geq 0 \), given by (15) reaches the value of \( \theta \leq \frac{\pi}{2} \).

The bound (29) represents a generalization to subspaces of the optimal passage time estimate established for the quantum brachistochrone problem (see, e.g. \([6, \text{Eq. (12.17)}]\)). The latter estimate is nothing but the equality in the Fleming bound (9) with \( \Delta E \) replaced by \( \frac{1}{2} \Omega \) where the quantity \( \Omega \) is introduced in Corollary 8.

**FUNDING**
This study was supported by the Heisenberg–Landau Program.

**CONFLICT OF INTEREST**
The authors declare that they have no conflicts of interest.

**REFERENCES**

1. S. Deffner and S. Campbell, “Quantum speed limits: From Heisenberg’s uncertainty principle to optimal quantum control,” J. Phys. A: Math. Gen. 50, 453001 (2017).
2. M. R. Frey, “Quantum speed limits—primer, perspectives, and potential future directions,” Quantum Inf. Process. 15, 3919–3950 (2016).
3. L. I. Mandelstam and I. E. Tamm, “The uncertainty relation between energy and time in nonrelativistic quantum mechanics,” J. Phys. (USSR) 9, 249–254 (1945).
4. N. Margolus and L. B. Levitin, “The maximum speed of dynamical evolution,” Physica D 120, 188–195 (1998).
5. G. N. Fleming, “A unitary bound on the evolution of nonstationary states,” Nuovo Cimento A 16, 232–240 (1973).
6. C. M. Bender and D. C. Brody, “Optimal time evolution for Hermitian and non-Hermitian Hamiltonians,” Lect. Notes Phys. 789, 341–361 (2009).
7. W. H. Wang, Z. L. Chen, Y. Song, and Y. J. Fan, “Optimal time evolution for pseudo-Hermitian Hamiltonians,” Theor. Math. Phys. 204, 1020–1032 (2020).
8. L. G. Brown, “The rectifiable metric on the set of closed subspaces of Hilbert space,” Trans. Am. Math. Soc. 227, 279–289 (1993).
9. S. Albeverio and A. K. Motovilov, “Sharpening the norm bound in the subspace perturbation theory,” Compl. Anal. Oper. Theory 7, 1389–1416 (2013).
10. M. G. Krein, M. A. Krasnoselsky, and D. P. Milman, “On defect numbers of linear operators in Banach space and some geometric problems”, Sbornik Trudov Inst. Mat. Akad. Nauk Ukr. SSR., No. 11, 97–112 (1948).
11. K. A. Makarov and A. Seelmann, “The length metric on the set of orthogonal projections and new estimates in the subspace perturbation problem,” J. Reine Angew. Math. 708, 1–15 (2015).