AUTOCORRELATIONS OF BINARY SEQUENCES AND RUN STRUCTURE

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Abstract. We analyze the connection between the autocorrelation of a binary sequence and its run structure given by the run length encoding. We show that both the periodic and the aperiodic autocorrelation of a binary sequence can be formulated in terms of the run structure. The run structure is given by the consecutive runs of the sequence. Let \( C = (C_0, C_1, \cdots, C_n) \) denote the autocorrelation vector of a binary sequence and \( \Delta \) the difference operator. We prove that the \( k \)th component of \( \Delta^2(C) \) can be directly calculated by using the consecutive runs of total length \( k \). In particular this shows that the \( k \)th autocorrelation is already determined by all consecutive runs of total length \( l < k \). In the aperiodic case we show how the run vector \( R \) can be efficiently calculated and give a characterization of skew-symmetric sequences in terms of their run length encoding.

1. Introduction

Let \( n \) be a positive integer and and let \( a = (a_1, a_2, \cdots, a_n) \) be a (finite) sequence of real numbers. The length \( n \) of the sequence \( a \) will be denoted by \( |a| \). \( a \) is called a binary sequence if \( a_i \in \{-1, 1\} \) for all \( i = 1, \cdots, n \). In the following we analyze in detail the connection between the autocorrelation of a binary sequence and its run structure given by the run length encoding. Binary sequences with suitable autocorrelation properties play an important part in a wide range of different engineering applications. For example they are used in signal processing in order to detect signals in a noisy background. The autocorrelation measures the similarity between the original sequence and its translate. In many applications it is of interest to collectively minimize the absolute values of the off-peak autocorrelations. For a survey on this topic we refer to [1, 2, 3]. Depending on the type of application there are two types of autocorrelations commonly used: the aperiodic and the periodic autocorrelation.

For \( k = 0, 1, \cdots, n - 1 \) the \( k \)th aperiodic autocorrelation is given by

\[
C_k(a) := \sum_{i=1}^{n-k} a_i a_{i+k}.
\]

In the periodic case put \( a_{n+i} := a_i \) for \( i \geq 1 \); for \( k = 0, 1, \cdots, n - 1 \) the \( k \)th periodic autocorrelation is then defined by
In the following we will additionally put $C_n(a) := 0$ and $\tilde{C}_n(a) := n$. Note that for a binary sequence the peak autocorrelation equals the length of the sequence: $C_0(a) = \tilde{C}_0(a) = n$. Note further that for $k = 1, \cdots, n$ we have $\tilde{C}_k(a) = C_{n-k}(a)$ and

\begin{equation}
\tilde{C}_k(a) = C_k(a) + C_{n-k}(a).
\end{equation}

In the following we analyze in detail the connection between the autocorrelations of a binary sequence and its run structure. A run is defined as a substring of maximal length where all elements have the same value. Runs as well as autocorrelation values were used in [4] in order to measure apparent randomness in a binary sequence (according to [5] these randomness postulates first appeared in [6]).

We prove that both the periodic and the aperiodic autocorrelation of a binary sequence can be formulated in terms of the run structure. As we will see the run structure is given by the consecutive runs of $a$. Consecutive runs of $a$ with total length $k$ determine $R_k$, the $k$th element of $R$; we call $R$ the run vector. If $C := (C_0(a), C_1(a), \cdots, C_n(a))$ denotes the autocorrelation vector of $a$ and $\triangle$ the forward difference operator, then we will prove that $\triangle^2(C) = -2R$. In particular this shows that the consecutive runs of total length up to $k - 1$ determine the $k$th autocorrelation.

For the periodic case this was established in [7]. The objective of this paper is twofold: to simplify the proof in [7] and to find a similar relationship for the aperiodic autocorrelation. We start with the latter; in the first part of this paper the aperiodic case is considered and it is shown how the aperiodic autocorrelation of a binary sequence can be formulated in terms of the run structure. In the aperiodic case we show how the run vector can be efficiently calculated based on a simple algorithm and we derive a further practical formula for calculating the elements of the run vector. Furthermore, we give a characterization of skew-symmetric sequences in terms of their run length encoding. Finally, we consider the periodic case. The presented proof of the aperiodic case can also be applied to the periodic case with minor modifications, resulting in a new simplified and more direct proof. This paper, however, follows a different and even shorter approach: the results for the periodic case are directly derived from the aperiodic case by using (1.3).

After the completion of this paper the author became aware that a different run correlation technique for the aperiodic case was published in [8]. There a tabular arrangement for the correlation calculation was developed by using a set of sequential relations which relate the aperiodic autocorrelation to the run structure. This is essentially the result of Theorem I which we prove in the first part of this paper. In [9] these results were used for a search strategy in order to construct binary sequences with specified aperiodic autocorrelation values by explicitly eliminating large subsets of binary sequences and thus reducing the search space.
2. Preliminaries

In the following \( a \) will always be a fixed binary sequence of length \( n \). Furthermore, in the aperiodic case we will always put \( a_0 := 0 \) and \( a_{n+1} := 0 \) in order to circumvent boundary problems.

A substring of \( a \) always represents a non-empty contiguous part \( (a_i, a_{i+1}, \ldots, a_{j-1}) \) of \( a \) with \( 1 \leq i < j \leq n + 1 \); quite similar to the concept of an half-open interval it will be denoted by \( a(i,j) \). In the following we will also always distinguish between the substrings \( a(i,j) \) and \( a(i',j') \) whenever \( (i,j) \neq (i',j') \). Hence \( a \) has \( \frac{n(n+1)}{2} \) different substrings. Note that by our definition a substring is always non-empty and that the length \( |a(i,j)| \) of the substring \( a(i,j) \) is given by \( |a(i,j)| = j - i \). If \( 1 < i \) and \( j < n + 1 \), then \( a(i,j) \) is called an inner substring, otherwise \( a(i,j) \) is called an outer substring.

2.1. Runs, Run Blocks and Run Length Encoding. A run of \( a \) is a substring of \( a \) with maximal length where all elements have the same value. Thus if \( a(i,j) \) is a run of \( a \), then \( a_{i-1} = a_i = a_{i+1} = \cdots = a_{j-1} = a_j \). In the following \( \gamma \) will always denote the total number of runs of the sequence \( a \). Consecutive runs form what we will call a run block. Thus for \( 1 \leq i < j \leq n + 1 \) a substring \( a(i,j) \) is a run block of \( a \), if and only if \( a_{i-1} = a_i \) and \( a_{j-1} = a_j \). Let for example \( a \) be the binary sequence of length 13 given by \( a = (++++++−−−−−++) \); here in the following the symbol ‘+’ stands for 1 and the symbol ‘−’ for -1. In this case we have \( \gamma = 3 \) since \( a \) has three runs, namely \( a(1,8), a(8,11) \) and \( a(11,14) \). Furthermore, \( a \) has a total of six run blocks: there are the three run blocks \( a(1,8), a(8,11) \) and \( a(11,14) \) consisting of just a single run, there are the two run blocks \( a(1,11) \) and \( a(8,14) \) consisting of two consecutive runs and there is the run block \( a(1,14) \) which is \( a \) itself consisting of three consecutive runs.

For an inner run block \( a(i,j) \) (i.e. a run block which is an inner substring such as, for example \( a(8,11) \) of the previous example) we have \( a_{i-1} = -a_i \) and \( a_{j-1} = -a_j \). Note since \( a_0 = a_{n+1} = 0 \) the sequence \( a \) itself is represented by the run block \( a(1,n+1) \) and that each run block of \( a \) can be uniquely divided in runs of \( a \). In particular, the sequence \( a \) can be uniquely divided in \( \gamma \) runs \( a(ik,ik+1) \) with \( 1 = i_1 < i_2 < \cdots < i_{\gamma+1} = n + 1 \); \( a(ik,ik+1) \) is then called the \( k \)th run of \( a \). If we put \( r_k := |a(ik,ik+1)| = ik+1 - ik \), then the sequence \( r = (r_1, r_2, \ldots, r_\gamma) \) is called the run length encoding of \( a \). Note that beside \( a \) only the binary sequence \((-a_1, -a_2, \ldots, -a_n)\) has the same run length encoding as \( a \).

Let for example \( a \) be the binary sequence of length 13 given by \( a = (++++++−−−−−++) \), then \( \gamma = 2 \) and the run length encoding \( r \) of \( a \) is given by \( r = (6, 7) \). In this case \( a \) has two runs and three blocks. If \( a = (++++++−−−−−++) \), then the run length encoding of \( a \) is given by \( r = (7, 3, 3) \) and as already noted \( a \) has three runs and six run blocks. The binary sequence of \( a = (++++++−−−−−+++) \) has four runs and its run length encoding is given by \( r = (3, 6, 3, 3) \).

As we will see later the weight of a run block will be used in order to calculate the autocorrelation. The weight \( w \) of a substring \( a(i,j) \) is defined by

\[
(2.1) \quad w(a(i,j)) := \begin{cases} 
2a_i \cdot a_{j-1} & \text{if } a(i,j) \text{ is an inner run block} \\
3a_i \cdot a_{j-1} & \text{if } a(i,j) \text{ is an outer run block} \\
0 & \text{otherwise.}
\end{cases}
\]
Note that for a substring $b$ we have $w(b) = 0$ unless $b$ is a run block. In this case $|w(b)| = 2$ if $b$ is an inner and $|w(b)| = 1$ if $b$ is an outer run block. We remark further, that if a run block $b = a(i,j)$ consists of $m$ consecutive runs, then $a_i \cdot a_{j-1} = -(-1)^m$ and thus $w(b) > 0$ if $m$ is odd and $w(b) < 0$ if $m$ is even.

2.2. The Aperiodic Run Structure. In the following $r$ will always denote the run length encoding of $a$ with $r_k = i_{k+1} - i_k$ and $\gamma$ as defined above. Let us define the aperiodic run structure $R$ of $a$ as the set of all substrings of $r$.

Next we want to show that there is a one-to-one correspondence between the run blocks of $a$ and the substrings of the run length encoding $r$. If $b$ is a run block of $a$ of length $k$, then we have $b = a(i_p,i_q)$ for some $1 \leq p < q \leq \gamma + 1$ and the run block $b = a(i_p,i_q)$ corresponds to the substring $r(p,q)$. The run length encoding gives us therefore a mapping $\Phi$ from the set $B$ of all run blocks of $a$ to the aperiodic run structure $R$ of $a$ defined by $\Phi(a(i_p,i_q)) = r(p,q)$. Note that $\Phi$ is bijective and that $\Phi$ maps an inner run block of $a$ to an inner substring of $r$. Furthermore for $k = 1, \cdots ,n$ let $B_k$ denote the set of all run blocks of $a$ with length $k$ and let $R_k$ denote the set of all substrings $r(p,q)$ of $r$ whose sum $\sum_{j=p}^{q-1} r_j$ is equal to $k$; obviously $B_k \subseteq B$ and $R_k \subseteq R$. If $b = a(i_p,i_q)$ is a run block of $a$ with length $k$, then we have

$$i_q - i_p = \sum_{j=p}^{q-1} r_j = k$$

and hence

$$\Phi(B_k) = R_k.$$ 

Thus each element $u \in R_k$ (i.e. each substring $u$ of $r$ whose sum is equal to $k$) corresponds uniquely to a run block of length $k$ consisting of $|u|$ consecutive runs, and vice versa.

For a substring $u$ of $r$ let

$$\alpha(u) := \begin{cases} 2 & \text{if } u \text{ is an inner substring of } r \\ 1 & \text{otherwise.} \end{cases}$$

If the run block $a(i_p,i_q)$ consists of $m$ consecutive runs, then $a_{i_p} \cdot a_{i_q-1} = -(-1)^m$ as already noted. Since $\Phi(a(i_p,i_q)) = r(p,q)$ we have $m = |r(p,q)|$ and it follows from (2.4) that $w(a(i_p,i_q)) = -\alpha(r(p,q)) \cdot (-1)^{|r(p,q)|}$. Since the mapping $\Phi$ is bijective, (2.3) together with the last remark of the previous subsection gives us that

$$\sum_{u \in R_k} \alpha(u) \cdot (-1)^{|u|} = - \sum_{b \in B_k} w(b).$$

3. The Main Result for the Aperiodic Case

In this section we want to analyze the connection between the aperiodic autocorrelations $C_k(a)$ of the binary sequence $a$ and its run structure given by the run length encoding $r = (r_1, r_2, \cdots , r_n)$. Note that $\sum_{j=1}^{\gamma} r_j = |a| = n = C_0(a)$. Moreover, we have

$$C_1(a) = n + 1 - 2\gamma.$$
since for each \( j = 1, 2, \cdots, \gamma - 1 \) the \( j \)th run of \( a \) contributes \( r_j - 2 \) to the sum \( C_1(a) \) whereas the last run contributes \( r_{\gamma} - 1 \) to the sum \( C_1(a) \). Hence we have
\[
C_1(a) = \sum_{j=1}^{\gamma-1} (r_j - 2) + (r_{\gamma} - 1) = 1 + \sum_{j=1}^{\gamma} (r_j - 2) = 1 + n - 2\gamma.
\]
For \( k = 1, 2, \cdots, n - 1 \) put
\[
R_k := \sum_{u \in R_k} \alpha(u) \cdot (-1)^{|u|}
\]
and \( R(a) := (R_1, R_2, \cdots, R_{n-1}) \); we call \( R(a) \) the run vector of \( a \). Thus by (2.5) we have
\[
R_k = - \sum_{b \in B_k} w(b).
\]
Therefore, in order to calculate \( R_k \) all consecutive runs of \( a \) with a total length of \( k \) (i.e. all run blocks of \( a \) with length \( k \)) have to be considered. Each run block \( b \) of length \( k \) contributes the (negative) weight \(-w(b)\) as defined in (2.1) to the sum in (3.3). As already noted \(|w(b)|\) equals 1 if \( b \) is an outer run block, \(|w(b)|\) equals 2 if \( b \) is an inner run block and the sign of \( w(b) \) depends only on the number of runs: we have \( w(b) > 0 \) if the run block \( b \) consists of an odd number of consecutive runs and \( w(b) < 0 \) if \( b \) consists of an even number of consecutive runs.

Let us give an example. Consider the run vector \( R(a) \) for the three sequences of the previous example. If \( r = (6, 7) \), then apart from \( a \) itself there are only two run blocks: they both consist of a single run, their length is 6 resp. 7 and both are outer run blocks. Thus we have \( R_k = \emptyset \) for \( k \neq 6, 7 \) and \( R_6 = \{r(1, 2)\} \), \( R_7 = \{r(2, 3)\} \); hence by (3.2) \( R(a) = (0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0) \).

For \( r = (7, 3, 3) \) there are three run blocks consisting of a single run; their length is 7, 3 and 3, their weight -1, -2 and -1 and they correspond to the substrings \( r(1, 2) \), \( r(2, 3) \) and \( r(3, 4) \). In addition there are two (outer) run blocks consisting of two consecutive runs. Their length is 10 resp. 6, both have weight 1 and they correspond to the substrings \( r(1, 3) \) and \( r(2, 4) \). Altogether, we have \( R_3 = \{r(2, 3), r(3, 4)\} \), \( R_6 = \{r(2, 4)\} \), \( R_7 = \{r(1, 2)\} \), \( R_{10} = \{r(1, 3)\} \) and \( R_k = \emptyset \) for the remaining cases \( k = 1, 2, 4, 5, 8, 9, 11, 12 \); this shows that \( R(a) = (0, 0, -3, 0, 1, -1, 0, 0, 1, 0, 0) \).

Finally, we consider the example \( r = (3, 6, 3, 3) \). In order to calculate for instance \( R_6 \) we have to consider all consecutive runs of \( a \) which have a total length (i.e. run block length) of 6. There are exactly two run blocks of \( a \) with length 6. The first one consists of a single run, has weight -2 and corresponds to the inner substrings \( r(2, 3) \); the second one consists of two consecutive runs, has weight 1 and corresponds to the outer substring \( r(3, 5) \). This shows that \( R_6 = \{r(2, 3), r(3, 5)\} \) and \( R_6 = -1 \).

Similarly, we have \( R_3 = \{r(1, 2), r(3, 4), r(4, 5)\} \), \( R_9 = \{r(1, 3), r(2, 4)\} \), \( R_{12} = \{r(1, 4), r(2, 5)\} \) and \( R_k = \emptyset \) if \( k \) is not a multiple of 3. Hence it follows that \( R(a) = (0, 0, -4, 0, 0, -1, 0, 0, 3, 0, 0, -2, 0, 0) \).

The next theorem (cf. [3]) is the main result for the aperiodic case. It shows that for a binary sequence \( a \) the aperiodic autocorrelations and and the run vector \( R(a) \) are closely related.

**Theorem 1.** Let \( k = 1, \cdots, n - 1 \); then
\[
C_{k+1}(a) - 2C_k(a) + C_{k-1}(a) = -2R_k.
\]
**Proof.** Let \( \delta := (\delta_1, \delta_2, \cdots, \delta_{n+1}) \) be the sequence defined by \( \delta_i := a_i - a_{i-1} \) for all \( i = 1, 2, \cdots, n + 1 \). Since \( a(i, j) \) is a run block of \( a \) if and only if \( a_{i-1} \neq a_i \) and
\[ a_{j-1} \neq a_j \] it follows that

\[ a(i, j) \] is a block of a \( \iff \delta_i \cdot \delta_j \neq 0 \)

Now let \( 1 \leq i < j \leq n + 1 \) with \( j - i < n \). If \( a(i, j) \) is a run block, then
\[
\delta_i \delta_j = (a_i - a_{i-1})(a_j - a_{j-1})
\]
\[
= a_i a_j - a_i a_{j-1} - a_{i-1} a_j + a_{i-1} a_{j-1}
\]
\[
= -2(a_i a_{j-1} + a_{i-1} a_j)
\]
\[
= -2w(a(i, j))
\]

Hence by (2.4)
\[
2R_k = 2 \sum_{u \in \mathbb{R}_k} a(u) \cdot (-1)^{|u|} = -2 \sum_{b \in \mathbb{B}_k} w(b)
\]
\[
= -2 \sum_{i=1}^{n+1-k} w(a(i, i + k))
\]
\[
= \sum_{i=1}^{n+1-k} \delta_i \delta_{i+k} = C_k(\delta).
\]

Now Theorem 1 follows directly from the following lemma. \( \Box \)

**Lemma 2.** Let \( \delta = (\delta_1, \delta_2, \cdots, \delta_{n+1}) \) be the sequence defined by \( \delta_i := a_i - a_{i-1} \) for all \( i = 1, 2, \cdots, n+1 \); then for \( k = 1, 2, \cdots, n-1 \)
\[
C_{k+1}(a) - 2C_k(a) + C_{k-1}(a) = -C_k(\delta).
\]

**Proof.** Let \( 1 \leq k \leq n - 1 \); then
\[
C_k(\delta)
\]
\[
= \sum_{i=1}^{n+1-k} \delta_i \delta_{i+k} = \sum_{i=1}^{n+1-k} (a_i - a_{i-1})(a_{i+k} - a_{i+k-1})
\]
\[
= \sum_{i=1}^{n+1-k} a_i a_{i+k} - a_i a_{i+k-1} - a_{i-1} a_{i+k} + a_{i-1} a_{i+k-1}
\]
\[
= (C_k(a) + a_{n+1-k} a_{n+1}) - C_{k-1}(a)
\]
\[
- (a_0 a_{k+1} + C_k(a) + a_{n-k} a_{n+1})
\]
\[
+ (a_0 a_k + C_k(a))
\]
\[
= -C_{k+1}(a) + 2C_k(a) - C_{k-1}(a).
\]

We have \( C_0(a) = n \) and \( C_1(a) = n + 1 - 2\gamma \) by (3.1); Theorem 1 gives us that
\( C_2(a) = 2C_1(a) - n - 2R_1 \) and thus \( C_2(a) = n + 2 - 4\gamma - 2R_1 \). Furthermore Theorem 1 shows that \( C_{k+1}(a) = 2C_k(a) - C_k(a) - 2R_k \) for \( k = 1, 2, \cdots, n-1 \).

Let us denote by \( C(a) := (C_0(a), C_1(a), \cdots, C_n(a)) \) the aperiodic autocorrelation vector of \( a \). We can now use Theorem 1 in order to compute \( C(a) \) for the three sequences of the previous examples. As we have seen, if \( r = (6, 7) \) then we have \( R(a) = (0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0) \). It follows that \( C_0(a) = n = 13, C_1(a) = n + 1 - 2\gamma = 13 + 1 - 2 \cdot 2 = 10, C_2(a) = 2C_1(a) - n - 2R_1 = \)
20 - 13 = 7, C_3(a) = 2C_2(a) - C_1(a) - 2R_2 = 4 and so on, which gives us
C(a) = (13, 10, 7, 4, 1, -2, -5, -6, -5, -4, -3, -2, -1, 0). Furthermore, by a simple
calculation we get C(a) = (13, 8, 3, -2, -1, 0, 1, 0, 1, 2, 3, 2, 1, 0) if r = (7, 3, 3) and
C(a) = (15, 8, 1, -6, -5, -4, -3, 0, 3, 6, 3, 0, -3, -2, -1, 0) if r = (3, 6, 3, 3).

Theorem 1 can be reformulated by using the difference operator. The (forward)
difference operator \( \triangle(b) \) of a sequence \( b = (b_1, b_2, \ldots, b_m) \) with \( m \geq 2 \) is given by
\[
\triangle(b) := (b_2 - b_1, b_3 - b_2, \ldots, b_m - b_{m-1}).
\]
For \( m \geq 3 \) we have \( \triangle^2(b) := \triangle(\triangle(b)) = (b_3 - 2b_2 + b_1, b_4 - 2b_3 + b_2, \ldots, b_m - 2b_{m-1} + b_{m-2}) \).

**Corollary 3.** \( \triangle^2(C(a)) = -2R(a) \) for \( n \geq 3 \).

**Proof.** This is just a reformulation of Theorem 1. \( \square \)

**Remark.** If we put \( \bar{\pi} := (0, a_1, a_2, \ldots, a_n, 0) \), then Lemma 2 shows that for \( n \geq 3 \)
\[
\triangle^2(C(a)) = -(C_1(\triangle(\bar{\pi})), C_2(\triangle(\bar{\pi})), \ldots, C_{n-1}(\triangle(\bar{\pi}))).
\]
A more explicit relationship between the aperiodic autocorrelation and the run
vector gives the next result. Note that by (3.1) \( C_1(a) - C_0(a) = 1 - 2\gamma \).

**Corollary 4.** Let \( k = 0, 1, \ldots, n \); then
\[
C_k(a) = n + (1 - 2\gamma)k - 2 \sum_{j=1}^{k-1} (k - j)R_j.
\]

**Proof.** Since \( C_0(a) = n \) and by (3.1) \( C_1(a) = n + 1 - 2\gamma \) the statement is true
for \( k = 0, 1 \). For \( k \geq 2 \) it follows directly from \( \triangle^2(C(a)) = -2R(a) \) and the next
Lemma. \( \square \)

**Lemma 5.** For \( m \geq 3 \) let \( b = (b_1, b_2, \ldots, b_m) \) be a sequence of real numbers and
let \( \epsilon_j \) denote the \( j \)th component of \( \triangle^2(b) \), i.e. \( \triangle^2(b) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{m-2}) \); then for
\( k = 0, 1, \ldots, m - 1 \)
\[
b_{k+1} = b_1 + (b_2 - b_1)k + \sum_{j=1}^{k-1} (k - j)\epsilon_j.
\]

**Proof.** This can be easily proved by induction on \( k \). \( \square \)

Next we want to show that based on the symmetry of the operator \( \triangle^2 \) a
complementary formulation of Corollary 4 can be derived. If \( b \) and \( \triangle^2(b) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{m-2}) \) are defined as in Lemma 5 then we also have for \( k = 0, 1, \ldots, m - 1 \)
\[
b_{k+1} = b_m + (b_{m-1} - b_m)(m - k - 1) + \sum_{j=k+1}^{m-2} (j - k)\epsilon_j.
\]

The proof is similar to the proof of Lemma 5. By applying this to \( \triangle^2(C(a)) =
-2R(a) \) and noting that \( C_{n-1}(a) = (-1)^{n+1} \) and \( C_n(a) = 0 \) it follows as in the
proof of Corollary 4 that for \( k = 0, 1, \ldots, n \)
\[
C_k(a) = (-1)^{n+1}(n - k) - 2 \sum_{j=k+1}^{n-1} (j - k)R_j.
\]
Similarly, from \( \epsilon_k = b_{k+2} - 2b_{k+1} + b_k \) it easily follows that \( \sum_{j=1}^{m-2} \epsilon_j = b_1 - b_2 + b_m - b_{m-1} \). Applying this to \( \triangle^2(C(a)) = -2R(a) \) gives us that \( 2 \sum_{k=1}^{n-1} R_k = C_0(a) - C_1(a) + C_n(a) - C_{n-1}(a) = 1 - 2\gamma + (-1)^{\gamma+1} \) and hence

\[
\sum_{k=1}^{n-1} R_k = \begin{cases} -\gamma & \text{if } \gamma \text{ even} \\ 1 - \gamma & \text{if } \gamma \text{ odd}. \end{cases}
\]

4. Applying the Results

In this section we will first present an example where the run vector \( R(a) \) is used in order to establish a relationship between the correlations of certain related sequences. Next we show how the run vector \( R(a) \) can be efficiently calculated based on a simple algorithm. After that we develop an alternative formula which shows how \( R_k \) can be expressed in terms of the sums \( r_1 + r_2 + \cdots + r_j \). The rest of this section discusses skew-symmetric sequences and their run length encoding. For a skew-symmetric sequence \( a \) we have \( C_k(a) = 0 \) whenever \( k \) is odd; if \( k > 0 \) is even, then we will show that \( C_k(a) = R_k(a) \). Moreover, we give a characterization of skew-symmetric sequences in terms of their run length encoding.

4.1. A First Example. For \( m > 1 \) we consider the sequence \( b = (b_1, b_2, \cdots, b_{nm}) \) obtained by repeating each element of \( a \) exactly \( m \) times so that the run length encoding \( r(b) \) of \( b \) is given by \( (mr_1, mr_2, \cdots, mr_\gamma) \). For example \( m = 2 \) gives us \( b = (a_1, a_1, a_2, a_2, \cdots, a_n, a_n) \). We will show that for \( 0 \leq k < n \) and \( 0 \leq s < m \) the aperiodic autocorrelations of \( b \) are given by

\[
C_{km+s}(b) = (m-s)C_k(a) + sC_{k+1}(a)
\]

where we have put as before \( C_0(a) := 0 \).

For \( 0 \leq k < n \) and \( 0 \leq s < m \) put \( D_{km+s} := (m-s)C_k(a) + sC_{k+1}(a) \), \( D_{nm} = 0 \) and \( D := (D_0, D_1, \cdots, D_{nm}) \). In order to prove (4.1) using Lemma 5 and Corollary 4 it is sufficient to show that

\[
C_0(b) = D_0, \quad C_1(b) = D_1 \quad \text{and} \quad \triangle^2(D) = -2R(b).
\]

We have \( C_0(b) = mn = mC_0(a) = D_0 \) and by (4.1) \( C_1(b) = mn + 1 - 2\gamma = (m-1)n + s_1 + 2\gamma = (m-1)C_0(a) + C_1(a) = D_1 \). Since \( r(b) = (mr_1, mr_2, \cdots, mr_\gamma) \) it is not difficult to see that for \( 1 \leq j < nm \)

\[
R_j(b) = \begin{cases} R_{\gamma}(a) & \text{if } j = 0 \mod m \\ 0 & \text{otherwise}. \end{cases}
\]

Now let \( \triangle^2(D) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_{nm-1}) \). For \( 0 \leq k < n \) we have \( D_{km+s+1} = D_{km+s} = C_{k+1}(a) - C_k(a) \) for \( 0 \leq s < m - 1 \) and also for \( s = m - 1 \). Since \( \triangle^2(D) := \triangle(\triangle(D)) \) we have \( \epsilon_j = 0 \) unless \( j \) is a multiple of \( m \); in this case we have \( \epsilon_{i,m} = C_{i+1}(a) - 2C_i(a) + C_{i-1}(a) = -2R_i(a) = -2R_{i,m}(b) \). Hence \( \triangle^2(D) = -2R(b) \).

4.2. Calculation of the Run Vector. The results of the previous section show how the run structure and the aperiodic autocorrelations are related. The main result which says that \( \triangle^2(C(a)) = -2R(a) \) can be used in order to calculate the autocorrelation vector \( C(a) \). However, the presented form, in particular (3.2), is not very well suited for practical purposes. However, \( R(a) \) can be efficiently computed based on the following simple algorithm, which can be easily derived from (3.2).
As a precondition we will assume that the run length encoding $r$ and its length $\gamma$ are known and that each component of the array $R$ is initialized to zero. After the final step of the algorithm the array $R$ contains the computed run vector $R(a)$.

The algorithm computes $R(a)$ in two steps. The first step considers only the outer substrings of the run length encoding $r = (r(j))$ and uses the fact that $|r(1, j)| = \gamma - |r(j, \gamma + 1)|$:

\[
\hat{\gamma} \leftarrow (-1)^\gamma \\
\alpha \leftarrow -1 \\
s \leftarrow 0
\]

for $j = 1$ to $\gamma - 1$

\[
s \leftarrow s + r_j \\
R_s \leftarrow R_s + \alpha \\
R_{n-s} \leftarrow R_{n-s} + \hat{\gamma} \cdot \alpha \\
\alpha \leftarrow -\alpha
\]

end for

The second and final step of the algorithm takes into account all inner substrings of $r$:

for $i = 2$ to $\gamma - 1$

\[
\alpha \leftarrow -2 \\
s \leftarrow 0
\]

for $j = i$ to $\gamma - 1$

\[
s \leftarrow s + r_j \\
R_s \leftarrow R_s + \alpha \\
\alpha \leftarrow -\alpha
\]

end for

end for

In the first step of the algorithm there are $\gamma - 1$ iterations and we have a total of $\frac{(\gamma - 1)(\gamma - 2)}{2}$ iterations in the second step. In the second step by enrolling the inner loop into two separate loops any multiplication can be avoided; the same is true for the loop in the first step. Thus the computation of $R(a)$ requires $4(\gamma - 1) + (\gamma - 1)(\gamma - 2) = (\gamma - 1)(\gamma + 2)$ additions and no multiplications.

Let us compare this with the direct calculation of the autocorrelations $C_1(a), C_2(a), \cdots, C_n(a)$ as defined in (1.1) which requires $\frac{n(n-1)}{2}$ multiplications and $\frac{n(n-1)(n-2)}{2}$ additions. Now, at this point let us assume that $\gamma \approx \frac{n}{2}$; we will come back to this assumption below. If $\gamma \approx \frac{n}{2}$ then the above algorithm for the computations of $R(a)$ requires approximately $\frac{n^2}{2}$ additions and no multiplication. Even if we do not distinguish between the cost of an addition and a multiplication, then the above algorithms should be approximately four times faster than the direct calculation using (1.1), a very similar result can be found in [8].

At least for large $n$ this remains true, even if we consider the additional cost in order to compute $C(a)$ and the run length encoding $r(a)$ of $a$: if $-2R(a)$ is already computed then we need less than $2n$ additions in order to calculate the autocorrelation vector $C(a)$ using $\Delta^2(C(a)) = -2R(a)$ and not more than $n + \gamma$ additions are necessary in order to calculate $r$.

Let us come back to the assumption that $\gamma \approx \frac{n}{2}$. Note that this is true for most binary sequences if $n$ is large. If, however, $\gamma > \frac{n+1}{2}$, then consider the binary sequence $\bar{a}$ of length $n$ one gets, when inverting every second element of $a$, i.e.
\[ \bar{a}_i := a_i \text{ if } i \text{ is odd and } \bar{a}_i := -a_i \text{ if } i \text{ is even.} \] Then \( C_k(\bar{a}) = C_k(a) \) if \( k \) is even and \( C_k(\bar{a}) = -C_k(a) \) if \( k \) is odd. By induction it is not difficult to show that \( \gamma + \bar{\gamma} = n + 1 \) where \( \bar{\gamma} \) denotes the length of the run length encoding of \( \bar{a} \). Thus if \( \gamma > \frac{n+1}{2} \), then we have \( \bar{\gamma} \leq \frac{2}{2} \) and \( \bar{a} \) has up to every second sign the same aperiodic autocorrelation vector. So in the above algorithm \( \bar{a} \) instead of \( a \) can be used in this case in order to compute the autocorrelation vector \( C(a) \).

4.3. A More Practical Formula for \( R_k \). In this subsection we develop a formula which shows how \( R_k \) can be expressed in terms of the sums \( r_1 + r_2 + \cdots + r_j \); this gives rise to a different algorithm for calculating \( R_k \) which is for example particularly useful in a branch-and-bound like exhaustive search where only parts of the sequence \( a \) are known.

For \( j = 1, 2, \cdots, \gamma \) let
\[ s_j := r_1 + r_2 + \cdots + r_j \] and
\[ t_j := r_{\gamma} + r_{\gamma-1} + \cdots + r_{\gamma-j+1}. \]
Note that then \( 1 \leq s_1 < s_2 < \cdots < s_\gamma = n, 1 \leq t_1 < t_2 < \cdots < t_\gamma = n \) and that for \( j = 1, 2, \cdots, \gamma - 1 \)
\[ s_j + t_{\gamma-j} = n. \]
Furthermore, let
\[ S := \{s_1, s_2, \cdots, s_\gamma-1\} \] and
\[ T := \{t_1, t_2, \cdots, t_{\gamma-1}\}. \]

The functions \( f_S, f_T : \mathbb{Z} \rightarrow \{-1, 0, 1\} \) defined by
\[ f_S(k) := \begin{cases} (-1)^j & \text{if } k \in S \text{ with } k = s_j \\ 0 & \text{otherwise} \end{cases} \]
\[ f_T(k) := \begin{cases} (-1)^j & \text{if } k \in T \text{ with } k = t_j \\ 0 & \text{otherwise} \end{cases} \]
will play an important role in the following. Note that by (4.4) we have for all \( k \in \mathbb{Z} \)
\[ f_S(k) = (-1)^\gamma f_T(n - k). \]

The next theorem shows how \( R_k \) can be expressed in terms of \( s_1, s_2, \cdots, s_\gamma-1 \).

**Theorem 6.** Let \( k = 1, 2, \cdots, n - 1; \) then
\[ R_k = f_S(k) + (-1)^\gamma f_S(n - k) + 2 \sum_{j=1}^{\gamma-1} (-1)^j f_S(s_j - k) \]

**Proof.** As in the proof of Theorem 4 let \( \delta := (\delta_1, \delta_2, \cdots, \delta_{n+1}) \) be the sequence defined by \( \delta_i := a_i - a_{i-1} \) for all \( i = 1, 2, \cdots, n + 1 \). Then \( \delta_1 = a_1, \delta_{n+1} = (-1)^\gamma a_1 \).
and for $i = 1, 2, \cdots, n-1$ we have $2f_S(i) = a_1\delta_{i+1}$. Let $1 \leq k < n$. Hence by Theorem 4 and Lemma 2
\[
R_k = \frac{1}{2}C_k(\delta) = \frac{1}{2} \sum_{i=1}^{n+1-k} \delta_i \delta_{i+k} = \frac{1}{2}(\delta_1 \delta_{k+1} + \delta_{n+1-k} \delta_{n+1} + \sum_{i=2}^{n-k} \delta_i \delta_{i+k}) = f_S(k) + (-1)^k f_S(n-k) + 2 \sum_{i=1}^{n-k-1} f_S(i) f_S(i+k).
\]

Since $f_S(i) = 0$ for $i \geq n$, Theorem 4 follows directly from the following Lemma. □

**Lemma 7.** Let $k = 1, 2, \cdots, n-1$; then
\[
\sum_{j=1}^{\gamma-1} (-1)^j \cdot f_S(s_j - k) = \sum_{j=1}^{\gamma-1} (-1)^j \cdot f_S(s_j + k) = \sum_{i=1}^{n-1} f_S(i + k) \cdot f_S(i).
\]

**Proof.** We have
\[
\sum_{j=1}^{\gamma-1} (-1)^j \cdot f_S(s_j - k) = \sum_{j=1}^{\gamma-1} f_S(s_j) \cdot f_S(s_j - k) = \sum_{i=1}^{n-1} f_S(i) \cdot f_S(i - k) = \sum_{i=1}^{n-1} f_S(i + k) \cdot f_S(i) = \sum_{j=1}^{\gamma-1} f_S(s_j + k) \cdot f_S(s_j) = \sum_{j=1}^{\gamma-1} f_S(s_j + k) \cdot (-1)^j.
\]

\[\square\]

$R_k$ can be expressed by Theorem 6 as the sum of $\gamma + 1$ terms. Each of these terms can be easily computed; for example by means of a pre-calculated array which holds the values $f_S(i)$ for $i = 1, 2, \cdots, n-1$. As before we may assume that $\gamma \approx \frac{n}{2}$ and that $n$ is large. But then, on average (taken over all binary sequences of length $n$ and all $1 \leq k < n$) only about a quarter of these terms are not zero, since on average we have $|\{1 \leq j < \gamma : s_j - k > 0 \text{ and } f_S(s_j - k) \neq 0\}| \approx \frac{\gamma-k}{4}$.

If only the first and the last part of the binary sequence $a$ are known, then we will see that a reformulation of Theorem 6 is useful.

**Corollary 8.** Let $k = 1, 2, \cdots, n-1$; then
\[
(-1)^k R_{n-k} = f_S(k) + f_T(k) + 2 \sum_{j=1}^{\gamma-1} (-1)^j f_T(k - s_j).
\]

**Proof.** The result follows directly from Theorem 6, Lemma 7 and (4.9). □

Now let $1 \leq k < m \leq n$ and let us assume that the first and the last $m$ elements of $a$ are known. Then all $s_j \in S$ with $s_j < m$ and all $t_j \in T$ with $t_j < m$ can...
be easily determined; the same is true for \( f_S(i) \) and \( f_T(i) \) for all \( i < m \). Hence Corollary 8 can be applied in order to calculate \( R_{n-k} \). In particular, this can be applied in a branch-and-bound like exhaustive search (as for example in [10]) in order to find binary sequences with low autocorrelations.

As an example let \( r = (5, 2, 2, 1, 2, \cdots, 5, 3, 1, 4) \) and assume that only the first 12 and the last 12 elements of \( a \) are known; let us assume further that \( \gamma \) is even. Thus we know that \((s_1, s_2, s_3, s_4) = (5, 7, 9, 10), (t_1, t_2, t_3) = (4, 5, 8), s_5 \geq 12 \) and \( t_4 \geq 12 \). Thus \( f_S(j) \) has for \( j = 1, 2, \cdots, 11 \) the following values:

\[
0 \ 0 \ 0 \ -1 \ 1 \ 0 \ -1 \ 1 \ 0
\]

and \( f_T(j) \) has for \( j = 1, 2, \cdots, 11 \) the following values:

\[
0 \ 0 \ 0 \ -1 \ 1 \ 0 \ -1 \ 0 \ 0 \ 0
\]

For \( k < 12 \) let \( S_k := \{ k - s > 0 : s \in S \} \) and \( T_k := S_k \cap T \) then

\[
\begin{align*}
S_{11} &= \{ 6, 4, 2, 1 \} & T_{11} &= \{ 4 \} \\
S_{10} &= \{ 5, 3, 1 \} & T_{10} &= \{ 5 \} \\
S_9 &= \{ 4, 2 \} & T_9 &= \{ 4 \} \\
S_8 &= \{ 3, 1 \} \\
S_7 &= \{ 2 \} \\
S_6 &= \{ 1 \}
\end{align*}
\]

\( S_k = \emptyset \) for \( k < 6 \) and \( T_k = \emptyset \) for \( k < 9 \). Thus for \( g(j) := f_S(j) \) and \( f_T(j) \) we have \( g(4) = g(8) = g(9) = -1 \) and \( g(7) = g(10) = 1 \); all other values of \( g(j) \) for \( j < 12 \) are zero. Note that for \( 1 \leq j < \gamma \) we have \( f_T(k - s_j) \neq 0 \) if and only if \( k - s_j \in T_k \); hence for \( R_{n-k}^{(in)} := 2 \sum_{j=1}^{\gamma-1} (-1)^j f_T(k - s_j) \) it follows that \( R_{n-k}^{(in)} = 0 \) for \( i = 1, 2, \cdots, 8 \) and that \( R_{n-9}^{(in)} = -2 f_T(4) = 2 \), \( R_{n-10}^{(in)} = -2 f_T(5) = -2 \) and \( R_{n-11}^{(in)} = 2 f_T(4) = -2 \). Therefore, \( R_{n-k} = g(k) + R_{n-k}^{(in)} \) has by Corollary 8 for \( k = 1, 2, \cdots, 11 \) the following values:

\[
0 \ 0 \ 0 \ -1 \ 1 \ 0 \ 0 \ 1 \ -1 \ 1 \ -1 \ -2
\]

4.4. Skew-Symmetric Sequences and Run Length Encoding. An odd length binary sequence \( a \) of length \( 2m - 1 \) is called skew-symmetric if for \( i = 1, 2, \cdots, m - 1 \)

\[
a_{m-i} = (-1)^i a_{m+i}.
\]

Skew-symmetric sequences are of particular interest in different areas. For example, consider the merit factor \( F(a) \) which is defined by

\[
F(a) := \frac{n^2}{2 \sum_{k=1}^{n-1} C_k^2(a)}.
\]

In many applications it is of interest to collectively minimize the absolute values of the autocorrelations; the merit factor can be used as a possible measure, see [11] for a survey on this topic. Let \( F_n \) be the highest merit factor possible for all binary sequences of length \( n \). We say that a binary sequence \( a \) has an optimal merit factor if \( F(a) = F_n \) (where as always \( n \) denotes the length of \( a \)). Many of the known odd length binary sequences with an optimal merit factor are skew-symmetric and it is even conjectured in [11] that a restriction to skew-symmetric sequences does not change the asymptotic behavior of \( F_n \). Furthermore, all odd length Barker sequences are skew-symmetric; a Barker sequence of length \( n \) is a binary sequence with \( |C_k(a)| \leq 1 \) for all \( k = 1, 2, \cdots, n - 1 \). Since for a binary sequence \( a \) we have
If then \( \gamma > \frac{a}{\gamma} \), we can construct a set \( I \) and only if its run length encoding is skew-symmetric and let \( I \) be the set of all \( b \) such that \( \gamma > a \). Furthermore, by Theorem 1 we have \( C_{k+1}(a) - 2C_k(a) + C_{k-1}(a) = -2R_k \) for \( k = 1, 2, \cdots, n - 1 \). Therefore, \( \hat{R}_k(a) = C_k(a) \) if \( k > 0 \) is even and \( \hat{R}_k(a) = \frac{a}{\gamma}(C_{k-1}(a) + C_{k+1}(a)) \) if \( k \) is odd. In particular, for an odd length Barker sequence \( a \) we have \( \hat{R}_k(a) = (-1)^k \gamma + 1 \) for \( k = 2, 3, \cdots, n - 1 \) and \( \hat{R}_k(a) = 1 - \gamma \) if \( \gamma \) is odd and \( \hat{R}_k(a) = 1 - \gamma \) if \( \gamma \) is even.

We will now want to describe skew-symmetric sequences in terms of their run length encoding. The run length encoding \( r \) of \( a \) is called skew-symmetric if \( a \) itself is skew-symmetric. Note that this is well defined. We call the run length encoding \( r \) balanced if

\[
\hat{S} \cup T = \{1, 2, \cdots, n - 1\} \quad \text{and} \quad \hat{S} \cap T = \emptyset.
\]

Note that if \( r \) is balanced, then \( n = 2\gamma - 1 \). Furthermore, \( r \) is balanced if and only if we have for each \( k = 1, 2, \cdots, n - 1 \) and \( r \in \hat{S} \) or \( n - r \in \hat{S} \). Hence if \( r \) is balanced, then \( f_k(1) \neq 0 \) \( \iff \) \( f_k(0) \neq 0 \) \( \iff \) \( f_k(n - k) = 0 \) for \( k = 1, 2, \cdots, n - 1 \).

We call a run length encoding \( r = (r_1, r_2, \cdots, r_s) \) reducible if either \( r_1 = 1 \) and \( r_s > 1 \) or \( r_1 > 1 \) and \( r_s = 1 \). If \( r \) is reducible, then the reduced run length encoding \( \hat{r} \) is given by \( \hat{r} = (r_2, r_3, \cdots, r_{s-1}) \) if \( r_1 = 1 \) and \( r_s - 1 \). Moreover, \( \hat{r} = (r_2, r_3, \cdots, r_{s-1}) \) if \( r_1 > 1 \). Given \( r \) we will in the following always denote the so defined reduced run length encoding by \( \hat{r} \).

Let \( L \) denote the set of all run length encodings of binary sequences of length \( n \geq 1 \). Furthermore, let \( L_s \) be the set of all \( r \in L \) which are skew-symmetric and let \( L_b \) be the set of all \( r \in L \) which are balanced. Now let \( r \) be a run length encoding with \( \gamma > 1 \); then it quite easily follows that

\[
(4.12) \quad \hat{S} \cup T = \{1, 2, \cdots, n - 1\} \quad \text{and} \quad \hat{S} \cap T = \emptyset.
\]

We will show that \( L_s = L_b \), i.e. that a binary sequence is skew-symmetric if and only if its run length encoding is balanced. In particular, we will inductively construct a set \( \hat{I} \) such that \( L_s = \hat{I} \) and \( L_b = \hat{I} \). Let \( \hat{I}^{(1)} = \{1\} \), i.e. \( \hat{I}^{(1)} \) is the one-element set containing the run length encoding of the two binary sequences of length \( 1 \). For \( k \geq 1 \) let \( \hat{I}^{(k+1)} \) be the set of all reducible \( r \in L \) with \( \hat{r} \in \hat{I}^{(k)} \) and finally let \( \hat{I} := \bigcup_{k \geq 1} \hat{I}^{(k)} \). It is easy to see that the elements of \( \hat{I} \) can be interpreted as the nodes of an infinite binary tree: \( r = (1) \) is the root, and if \( r \in \hat{I} \) with \( \gamma > 1 \), then \( \hat{r} \) is its parent. Moreover, \( \hat{I}^{(k)} \) presents the set of nodes having depth \( k - 1 \). If \( r \in \hat{I} \) then it easily follows by induction that \( r \in \hat{I}^{(\gamma)} \). Furthermore, we have for \( \gamma > 1 \) that \( r \in \hat{I} \) if and only if \( \hat{r} \in \hat{I} \).

The next proposition shows that the skew-symmetric sequences are exactly the binary sequences which have a balanced run length encoding.

**Proposition 9.** \( L_s = L_b \).

**Proof.** First we will show that \( L_s \subseteq \hat{I} \). Assume that this is not the case and choose \( r \in L_s \) with \( \gamma > 1 \) such that \( r \notin \hat{I} \). Then \( \gamma > 1 \) and by (4.13) \( r \) is reducible and \( \hat{r} \in L_s \). Since \( r \notin \hat{I} \) we also have \( \hat{r} \notin \hat{I} \) which contradicts the minimality of \( \gamma \).

Next we want to show that \( \hat{I} \subseteq L_s \). Similar as before, assume that this is not the case and choose \( r \in \hat{I} \) with \( \gamma > 1 \) such that \( r \notin L_s \). Then \( \gamma > 1 \), \( r \) is reducible and \( \hat{r} \in \hat{I} \). By (4.13) \( \hat{r} \) is not skew-symmetric which contradicts the minimality of \( \gamma \).
of $\gamma$. Therefore, we have $L_S = I$. Using (1.14) similar arguments show that also $L_S = I$ which completes the proof. \hfill $\square$

5. The Periodic Case

The situation in the periodic case is quite similar to the aperiodic one. In order to formulate and prove the relationship between the periodic autocorrelations of binary sequences and their run structure we have to adapt some of the previous definitions to the periodic case. In the following we will always assume that $a$ is a binary sequence of length $n$ which is not constant, i.e. $a_i \neq a_j$ for some $1 \leq i < j \leq n$. This is no loss of generality as we will see at the end of this section. We could prove the following results in a very similar way as we have done in the aperiodic case. However, by using (1.3) we can directly transfer the aperiodic results of the previous section to the periodic case.

In the following we will in addition always assume that $a_1 \neq a_n$. Again this is no loss of generality since the periodic autocorrelations are shift-invariant, i.e. for $b := (a_n, a_1, a_2, \cdots, a_{n-1})$ we have $\tilde{C}_k(b) = \tilde{C}_k(a)$ for all $k = 0, 1, \cdots, n$. The main reason for the additional assumption $a_1 \neq a_n$ is that we can in this case adopt the definition of the the run length encoding $r = (r_1, r_2, \cdots, r_\gamma)$ of $a$ without any modification. Note that $\gamma$ is always even if $a_1 \neq a_n$.

5.1. Preliminaries for the Periodic Case. As in the periodic case substrings of the run length encoding $r = (r_1, r_2, \cdots, r_\gamma)$ of $a$ play an important role. For $1 \leq i \leq \gamma$ we will in the following always put $r_{\gamma+1} := r_i$. Similar to the concept of half-open interval on a circle we have to adapt the definition of a substring to the periodic case. In order to distinguish this new definition from the previous one we will refer to $p$-substrings. As we will see, $p$-substrings $r(i, j)$ of $r$ will be only defined if and only if $1 \leq i, j \leq \gamma$ and $i \neq j$.

For $1 \leq i < j \leq \gamma$ the $p$-substring $r(i, j)$ is the same as in the periodic case; thus representing $(r_i, r_{i+1}, \cdots, r_{j-1})$. For $1 \leq j < i \leq \gamma$ the $p$-substring $r(i, j)$ will represent the non-empty contiguous part $(r_i, r_{i+1}, \cdots, r_{\gamma+j-1})$ of the periodic extension of $r$. Unless otherwise stated we will in the following always assume that $1 \leq i, j \leq \gamma$ and $i \neq j$. As before $|r(i, j)|$ denotes the length of the $p$-substring, hence

$$|r(i, j)| = \begin{cases} j - i & \text{if } 1 \leq i < j \leq \gamma \\ \gamma + j - i & \text{if } 1 \leq j < i \leq \gamma. \end{cases}$$

In particular we have $1 \leq |r(i, j)| < \gamma$. Note that the $p$-substrings $r(i, j)$ and $r(j, i)$ are complementary: they have no element in common and their concatenation represents all elements of $r$; in particular we have

$$|r(i, j)| + |r(j, i)| = \gamma. \quad (5.1)$$

5.2. The Main Result for the Periodic Case. The periodic run structure $R$ of $a$ will be defined as the set of all $p$-substrings of $r$. Similar to the periodic case we will analyze the connection between the periodic autocorrelations $\tilde{C}_k(a)$ of a binary sequence $a$ of length $n$ and their periodic run structure given by the run length encoding $r = (r_1, r_2, \cdots, r_\gamma)$ of $a$. The case $k = 1$ is easy; similar to the aperiodic case we have

$$\tilde{C}_1(a) = n - 2\gamma \quad (5.2)$$
since for each \( j = 1, 2, \ldots, \gamma \) the \( j \)th run of \( a \) contributes \( r_j - 2 \) to the sum \( \tilde{C}_1(a) \). Thus \( \tilde{C}_1(a) = \sum_{j=1}^{\gamma} (r_j - 2) = n - 2\gamma \).

The sum \( S(r(i,j)) \) for a p-substring \( r(i,j) \) will be defined as

\[
S(r(i,j)) = \begin{cases} 
\sum_{m=i}^{r} r_m & \text{if } 1 \leq i < j \leq \gamma \\
\sum_{m=i}^{\gamma+j-1} r_m & \text{if } 1 \leq j < i \leq \gamma.
\end{cases}
\]

Let \( 1 \leq k < n \); we will denote by \( \tilde{R}_k \) the set of all p-substrings \( r(i,j) \) of \( r \) with \( S(r(i,j)) = k \) (where as always \( 1 \leq i, j \leq \gamma \) and \( i \neq j \)) and put

\[
(5.3) \quad \tilde{R}_k := \sum_{u \in \tilde{R}_k} (-1)^{|u|}.
\]

Similar as in the aperiodic case a p-substring of \( r \) corresponds in a one-to-one way to consecutive runs of \( a \). Note however, that consecutive has in the periodic case a slightly different meaning due to the periodic extension of \( a \). Similar to the aperiodic case \( \tilde{R}_k \) can be interpreted as the sum of weights of those consecutive runs of \( a \) which (total) length equals \( k \); the weight is in the periodic case either 1 or -1 depending on whether the number of runs is even or odd. In particular, \( -\tilde{R}_1 \) equals the number of runs of length 1.

Since \( S(r(i,j)) + S(r(j,i)) = n \), we have \( r(i,j) \in \tilde{R}_k \) if and only if \( r(j,i) \in \tilde{R}_{n-k} \). In particular, this shows that \( |\tilde{R}_k| = |\tilde{R}_{n-k}| \). Moreover, since \( \gamma \) is even, it follows from (5.1) that

\[
(5.4) \quad (-1)^{|r(i,j)|} = (-1)^{|r(j,i)|}
\]

and hence \( \tilde{R}_k = \tilde{R}_{n-k} \).

Similar to the periodic case we put \( \tilde{R}(a) := (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_{n-1}) \). Let us compute \( \tilde{R}(a) \) for the following two sequences of the previous examples. For \( a = (+ + + + + + + + + +) \) we have \( n = 13 \), \( \gamma = 2 \) and \( r = (6,7) \). Furthermore, we have \( \tilde{R}_k = \emptyset \) for \( 1 \leq k \leq 12 \) unless \( k = 6 \) or \( k = 7 \); in these cases we have \( \tilde{R}_6 = \{r(1,2)\} \) and \( \tilde{R}_7 = \{r(2,1)\} \). Hence \( \tilde{R}(a) = (0,0,0,0,0,0,-1,-1,0,0,0,0,0) \) by (5.3). For \( a = (+ + + + + + + + + + + + + + + + + + + +) \) we have \( n = 15 \), \( \gamma = 4 \) and \( r = (3,6,3,3) \). In order to calculate for instance \( \tilde{R}_6 \) we have by (5.3) to consider all consecutive runs of \( a \) which have a total length of 6. In this case these consecutive runs correspond to the p-substrings \( r(2,3), r(3,1) \) and \( r(4,2) \); the p-substrings \( r(2,3) \) consists of a single run, whereas the p-substrings \( r(3,1) \) and \( r(4,2) \) each consists of two consecutive runs. This shows that \( \tilde{R}_6 = \{r(2,3), r(3,1), r(4,2)\} \) and \( \tilde{R}_6 = 1 \).

Similarly we have \( \tilde{R}_3 = \{r(1,2), r(3,4), r(4,1)\}, \tilde{R}_9 = \{r(1,3), r(2,4), r(3,2)\}, \tilde{R}_{12} = \{r(1,4), r(2,1), r(4,3)\} \) and \( \tilde{R}_k = \emptyset \) if \( k \) is not a multiple of 3; hence \( \tilde{R}(a) = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \).

**Lemma 10.** \( 2 \tilde{R}_k = \tilde{R}_k + \tilde{R}_{n-k} \) for \( k = 1, 2, \ldots, n-1 \).

**Proof.** Let \( 1 \leq k < n \). We can write \( \tilde{R}_k \) as the (disjoint) union \( \tilde{R}_k = \tilde{R}_k^{(in)} \cup \tilde{R}_k^{(out)} \) where \( \tilde{R}_k^{(in)} \) denotes the set of all inner substrings in \( \tilde{R}_k \) and \( \tilde{R}_k^{(out)} \) denotes the set of all outer substrings in \( \tilde{R}_k \). Similarly we can write \( \tilde{R}_k \) as the (disjoint) union

\[
\tilde{R}_k = \tilde{R}_k^{(in)} \cup \tilde{R}_k^{(out)} \cup \tilde{R}_k^{(amid)}
\]

where \( \tilde{R}_k^{(in)} := \{r(i,j) \in \tilde{R}_k : 1 < i < j \leq \gamma\} \), \( \tilde{R}_k^{(out)} := \{r(i,j) \in \tilde{R}_k : i = 1 \text{ or } j = 1\} \) and \( \tilde{R}_k^{(amid)} := \{r(i,j) \in \tilde{R}_k : 1 < j < i \leq \gamma\} \).
If \( u \in \tilde{R}_k^{(in)} \) then \( u \) is an inner substring; moreover, we have \( u \in \tilde{R}_k^{(in)} \) if and only if \( u \in R_k^{(in)} \) and thus

\[
2 \sum_{u \in \tilde{R}_k^{(in)}} (-1)^{|u|} = \sum_{u \in R_k^{(in)}} \alpha(u) \cdot (-1)^{|u|}.
\]

If \( r(i, j) \in \tilde{R}_k^{(amid)} \) then \( r(j, i) \) is an inner substring; furthermore we have \( r(i, j) \in \tilde{R}_k^{(amid)} \) if and only if \( r(j, i) \in R_{n-k}^{(in)} \) and thus by (5.4)

\[
2 \sum_{u \in \tilde{R}_k^{(amid)}} (-1)^{|u|} = \sum_{u \in R_{n-k}^{(in)}} \alpha(u) \cdot (-1)^{|u|}.
\]

For \( r(i, j) \in \tilde{R}_k^{(out)} \) we consider the two possible cases \( i = 1 \) and \( j = 1 \) separately:

\[
r(1, j) \in \tilde{R}_k^{(out)} \Leftrightarrow r(1, j) \in R_k^{(out)} \Leftrightarrow r(j, \gamma + 1) \in R_{n-k}^{(out)}
\]

\[
r(i, 1) \in \tilde{R}_k^{(out)} \Leftrightarrow r(i, \gamma + 1) \in R_k^{(out)} \Leftrightarrow r(1, i) \in R_{n-k}^{(out)}.
\]

Hence by (5.4)

\[
\sum_{u \in \tilde{R}_k^{(out)}} (-1)^{|u|} = \sum_{u \in R_k^{(out)}} \alpha(u) \cdot (-1)^{|u|} = \sum_{u \in R_{n-k}^{(out)}} \alpha(u) \cdot (-1)^{|u|}.
\]

Summing up we obtain

\[
2 \sum_{u \in \tilde{R}_k} (-1)^{|u|} = \sum_{u \in R_k} \alpha(u) \cdot (-1)^{|u|} + \sum_{u \in R_{n-k}} \alpha(u) \cdot (-1)^{|u|}.
\]

\[\Box\]

Since \( \gamma \) is even it easily follows from (3.7) and Lemma 11 that \( \sum_{k=1}^{n-1} R_k = -\gamma \).

The next theorem (cf. [7]) which is the main result of this section shows that for a binary sequence \( a \) the periodic autocorrelations and \( \check{R}(a) \) are closely related.

**Theorem 11.** Let \( k = 1, 2, \ldots, n-1 \); then

\[
\check{C}_{k+1}(a) - 2\check{C}_k(a) + \check{C}_{k-1}(a) = -4\check{R}_k.
\]

**Proof.** Let \( 1 \leq k < n \). By Theorem 11 we have \( C_{k+1}(a) - 2C_k(a) + C_{k-1}(a) = -2R_k \)
and \( C_{(n-k)+1}(a) - 2C_{n-k}(a) + C_{(n-k)-1}(a) = -2R_{n-k} \). Hence the theorem follows
by (1.3) and Lemma 10. \[\Box\]

By (5.2) we have \( \check{C}_1(a) = n - 2\gamma \) and Theorem 11 gives us that \( \check{C}_2(a) = 2\check{C}_1(a) - n - 4\check{R}_1 \)
and therefore \( \check{C}_2(a) = n - 4\gamma - 4\check{R}_1 \).

Let us denote by \( \check{C}(a) := (\check{C}_0(a), \check{C}_1(a), \ldots, \check{C}_n(a)) \) the periodic autocorrelation
vector of \( a \).

**Corollary 12.** \( \Delta^2(\check{C}(a)) = -4\check{R}(a) \) for \( n \geq 3 \).

**Proof.** This is just a reformulation of Theorem 11. \[\Box\]

**Corollary 13.** Let \( k = 0, 1, \ldots, n \); then

\[
\check{C}_k(a) = n - 2\gamma k - 4\sum_{j=1}^{k-1} (k-j)\check{R}_j.
\]
Proof. Since $\tilde{C}_0(a) = n$ and by (5.2) $\tilde{C}_1(a) = n - 2\gamma$ the statement is true for $k = 0, 1$. For $k \geq 2$ it follows directly from $\Delta^2(\tilde{C}(a)) = 2\tilde{R}(a)$ and Lemma [5]. \hfill \Box

Finally, as in the aperiodic case a complementary formulation for $\tilde{C}_k(a)$ in Corollary [13] can be easily derived; since $\tilde{C}_n(a) = n$ and $\tilde{C}_{n-1}(a) = \tilde{C}_1(a) = n - 2\gamma$ equation (5.5) gives us that for $k = 0, 1, \ldots, n$

$$\tilde{C}_k(a) = n - 2\gamma(n - k) - 4 \sum_{j=k+1}^{n-1} (j - k)\tilde{R}_j.$$  

Remark. In this section we have for technical reasons assumed that the binary sequence $a$ is not constant. This however is no loss of generality since Theorem [11] and the following results are also true if $a$ is a constant binary sequence. This follows immediately since for a constant binary sequence $a$ of length $n$ we have $\tilde{C}_k(a) = n$ and also $\tilde{R}_k = 0$ for $k = 1, 2, \ldots, n - 1$.

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