The properties of the $\sigma$ and $\kappa$ resonances in a new unitarization approach

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Abstract. A new unitarization approach is discussed and applied to study the elastic $\pi K$ scattering process. The existence of the light $\kappa$ resonance is firmly established if the scattering length in the $I=1/2$ channel does not deviate too much from its value obtained from chiral perturbation theory, and a precise determination of the mass and width of the $\kappa$ resonance requires a precise determination of the scattering length parameter.

In history the $\sigma$ particle was firstly proposed by Gell-Mann and Levy in association with the linear $\sigma$ model. Non-linear realization of chiral symmetry was later discovered [1] and since then there existed argument that the $\sigma$ meson is unnecessary for chiral symmetry and even in contradiction to experiments. However there are also people, many of them are among the audience, insist on the existence of $\sigma$ resonance which results in the return of $\sigma$ in the Review of Particle Properties (named as $f_0(600)$) after disappearing for about 30 years. [2] The postulated $\kappa$ resonance also has a rather long history [3] and the status is even more intriguing despite the recent experimental results from the E791 and the BES Collaborations. [4] The dynamics related to the $\sigma$ and $\kappa$ resonance is of highly non-perturbative, strong interaction nature. Since it is always not easy to separate a distant pole from the background contributions, conclusions found in the literature are very often model dependent.

However, in Ref. [5, 6], a model independent dispersive analysis has been proposed and it is demonstrated that the non-linear realization of chiral symmetry, or chiral perturbation theory ($\chi$PT) actually needs the $\sigma$ resonance to accommodate for the experimental data. The crucial point to attack the problem is to consider the dispersion relations for the following two quantities,

$$F(s) = \frac{1}{2i\rho(s)}(S(s) - \frac{1}{\overline{S(s)}}), \quad \tilde{F}(s) = \frac{1}{2}(S(s) + \frac{1}{S(s)}),$$

where $\rho(s)$ is the kinematic factor. The functions $\rho(s)F(s)$ and $\tilde{F}(s)$ are respectively the analytic continuation of the imaginary and real part of the partial wave $S$-matrix defined in the elastic region and they satisfy the following dispersion relations:

$$F(s) = \alpha - \sum_j \frac{1}{S'(z^R_j)} - \frac{1}{2i\rho(z^I_j)} + \frac{1}{\pi} \int_L \frac{\operatorname{Im} F(s')}{s' - s} ds' + \frac{1}{\pi} \int_R \frac{\operatorname{Im} F(s')}{s' - s} ds',$$
\[ \tilde{F}(s) = \tilde{\alpha} + \sum_j \frac{1}{2S'(z_j^2)(s - z_j^2)} + \frac{1}{\pi} \int_L \frac{\text{Im} \tilde{F}(s')}{s' - s} ds' + \frac{1}{\pi} \int_R \frac{\text{Im} \tilde{F}(s')}{s' - s} ds'. \quad (2) \]

In the above expressions the subscript \(L\) represents dynamical cuts rather than the physical right hand cuts. \(R\) represents right hand cuts starting from the second physical threshold. The left hand cut is in general rather complicated but for equal mass scatterings like \(\pi\pi\) scatterings the situation is much simplified: \(L = (-\infty, 0]\) and \(R = [4m_K^2, \infty)\). One subtraction to the dispersion integrals is understood. In Eq. (2) \(\tilde{\alpha}\) and \(\tilde{\alpha}\) are subtraction constants and \(z_j\) denotes pole positions on the second sheet. When \(z_j\) is real it represents a virtual state pole, when \(z_j\) is complex it must appear in one conjugate pair together with \(z_j^\ast\), representing a resonance. The experimental curve of the function \(\tilde{F}\) is convex, yet chiral perturbation theory predicts a negative and concave left hand integral contribution. This fact unambiguously establishes the existence of the \(\sigma\) resonance, if the chiral prediction to the cut integral is qualitatively correct. For more details we refer to Ref. [5].

From Eq. (2) one obtains the generalized unitarity condition which holds on the entire complex \(s\) plane [6]:

\[ \tilde{F}^2 + (\rho F)^2 = 1. \quad (3) \]

This equation is used to obtain solutions of the simplest \(S\) matrices. Here ‘simplest’ means those solutions of unitary \(S\) matrices contain no cut integrals as appeared in Eq. (2) and contain minimal set of poles, i.e., one or two. The one pole solution represents a virtual/bound state whereas the two pole (on the second sheet) solution represents a resonance. The solution representing a resonance located at \(z_0\) (having positive imaginary part) and \(z_0^\ast\) for un-equal mass scatterings is the following:

\[ S(s) = \frac{M^2(z_0) - s + i\rho(s)sG[z_0]}{M^2(z_0) - s - i\rho(s)sG[z_0]}, \quad (4) \]

where

\[ M^2(z_0) = \text{Re}[z_0] + \frac{\text{Im}[z_0]\text{Im}[z_0\rho(z_0)]}{\text{Re}[z_0\rho(z_0)]}, \quad G[z_0] = \frac{\text{Im}[z_0]}{\text{Re}[z_0\rho(z_0)]}. \quad (5) \]

The Eq. (4) is very interesting as it reveals the remarkable difference between a narrow resonance located far above the threshold and a light and broad resonance. In fact, \(s = M^2(z_0)\) is the place where the resonance contribution to the phase shift passes \(\pi/2\). However, a light and broad resonance corresponds to a very large \(M(z_0)^2\). When \(\text{Re}[z_0] \leq (s_L + s_R)/2\) \((s_L = (m_K - m_\pi)^2, s_R = (m_K + m_\pi)^2)\), the phase shift never reaches \(\pi/2\)! See fig. 11 for more illustrations.

It is worthwhile to make a pedagogical analysis to a widely used parameterization form found in the literature:

\[ S = \frac{M^2 - s + i\rho(s)g}{M^2 - s - i\rho(s)g}. \quad (6) \]

For a sufficiently large \(M^2\) and small \(g\) and for equal mass scatterings, such an \(S\) matrix contains a resonance and a virtual state [7]. However, the latter is not predicted by \(\chi PT\)
and violates the validity of chiral expansion at the pole position and therefore should be abandoned. For unequal mass scattering the $S$ matrix Eq. (6) contains two resonance poles. The situation is depicted in fig. 2. The virtual state pole in $\pi\pi$ scattering and the resonance pole in $\pi K$ scattering have a common origin: they are both generated from $s = 0$ due to the kinematical singularity of $\rho(s)$ at $s = 0$ and should therefore be removed on the same footing. Moreover, the physical pole contribution and the spurious pole contribution to the scattering length are additive and are both positive. The accompanied spurious pole can have a larger contribution than the resonance contribution itself if the resonance is light and broad! It is therefore incorrect to use Eq. (6) to discuss a light and broad resonance.

Since a unitary matrix divided by any unitary matrix is still unitary. If we single out all poles of an $S$ matrix, we have without any loss of generality the following form:

$$S^{\text{phy.}} = \prod S^{\text{poles}} \cdot S^{\text{cut}}. \quad (7)$$
where \( S^{\text{cut}} \) no longer contains any pole and it can be parameterized as:

\[
S^{\text{cut}} = e^{2i\rho f(s)},
\]

\[
f(s) = f_0 + \frac{s - s_0}{\pi} \int_L \frac{\text{Im}_L f(s')}{(s' - s_0)(s' - s)} + \frac{s - s_0}{\pi} \int_R \frac{\text{Im}_R f(s')}{(s' - s_0)(s' - s)}. \tag{8}
\]

It can be demonstrated that the discontinuity of \( f \) obeys the following simple relations:

\[
disc f = \text{disc}\left\{ \frac{1}{2i\rho(s)} \log \left[ S^{\text{phy}}(s) \right] \right\}. \tag{9}
\]

The equation (9) is useful when estimating the background contributions. For example we can simply approximate \( S^{\text{phy}} \) in Eq. (9) by \( S^{\chi \text{PT}} \) on \( L \),

\[
disc f_L = \text{disc}\left\{ \frac{1}{2i\rho(s)} \log \left[ S^{\chi \text{PT}}(s) \right] \right\}, \tag{10}
\]

to estimate the background contributions.

Now it is at the stage to use the newly established unitarization scheme to study the \( \pi K \) scatterings. In here \( R \) in principle starts from \((m_K + m_\eta)^2 \) to \(+\infty\) but this cut is rather weak in practice and therefore it is appropriate to take \( R = [(m_K + m_\eta)^2, +\infty) \). We fit the LASS data \([9]\) up to 1430MeV (about 20 MeV below the \( K\eta' \) threshold). There are totally six parameters \( a_0^{1/2}, a_0^{3/2}, \) two pole parameters for \( \kappa \) and two for \( K^* \) (1430). It is found appropriate to truncate the left hand integral at \( \Lambda_{L,I}^2 = 1/2 \simeq \Lambda_{L,I}^2 = 3/2 \simeq 1.5\text{GeV}^2 \) (since the once subtracted dispersion integral is convergent the numerical output is actually not very sensitive to the truncation) and we obtain:

\[
\chi^2_{d.o.f.} = 38.35/(60-6) ;
\]
\[
M_\kappa = 594 \pm 79\text{MeV} , \quad \Gamma_\kappa = 724 \pm 332\text{MeV} ;
\]
\[
a_0^{1/2} = 0.284 \pm 0.089 , \quad a_0^{3/2} = -0.129 \pm 0.006 ;
\]
\[
M_{K^*} = 1456 \pm 8\text{MeV} , \quad \Gamma_{K^*} = 217 \pm 31\text{MeV}. \tag{11}
\]

The fit quality of the above results are rather good though the width of the \( \kappa \) resonance is quite flexible. However the fit prefers a somewhat larger scattering length in the I=1/2 channel and much larger (in magnitude) scattering length parameter in the I=3/2 channel comparing with the \( \chi \text{PT} \) predictions to the \( \pi K \) scattering lengths \([10]\): \( a_0^{1/2} = 0.18 \pm 0.02, a_0^{3/2} = -0.05 \pm 0.02 \). It is necessary to further clarify the issue whether the \( \kappa \) resonance exist. According to Pennington, a resonance exist if without it the total \( \chi^2 \) of the fit increase significantly. If we freeze the \( \kappa \) resonance, we get, \( \chi^2_{d.o.f.} = 63.67/(60-4), \quad a_0^{1/2} = 0.446 \pm 0.006 \) and other outputs are similar to Eq. (12). Comparing with the results in Eq. (12) the \( \chi^2_{d.o.f.} \) given in this way is increased by a factor of 1.7. If this is still not enough to support the existence of \( \kappa \), the value of \( a_0^{1/2} \) given by freezing the \( \kappa \) degree of freedom is too large comparing with the \( \chi \text{PT} \) value: the fit value of \( a_0^{3/2} \)
is about $4 \sigma$ away from $\chi$PT result whereas $a_0^{1/2}$ is $14 \sigma$ away! The conclusion is that if $a_0^{1/2}$ does not deviate much from its value predicted by $\chi$PT then $\kappa$ resonance must exist. On the contrary, it is estimated that if $a_0^{1/2}$ becomes larger than roughly 0.35 then the existence of the $\kappa$ resonance becomes doubtful.

It is sometimes found in the literature the discussions on the constraint of Adler zero on the scattering amplitude. In our scheme, it is possible to embed this constraint into our parameterization form. In fact, the Adler zero automatically emerges in our approach if the subtraction constant $f_0$ is limited within certain range. This is because all the resonance (and virtual state) $S$ matrices are real and less than 1 when $s_L < s < s_R$. On the contrary, the cut integrals contribute a factor larger than 1, therefore the $T$ matrix zero is obtainable in the right place when $f_0$ is confined to a certain range, which in turn put some constraints on the magnitude of the scattering length parameter itself. For example, within the range $0 < a_0^{1/2} < 0.26$ there exists a $T$ matrix zero in the region $s_L < s < s_R$, and when $a_0^{1/2} \simeq 0.20$ the zero locates in the place close to the one loop $\chi$PT prediction.

We make further fit by confining $a_0^{1/2}$ in the region $0.18 \pm 0.02$ (other conditions are the same as the fit described in above) and the results follow:

$$\chi^2_{d.o.f.} = 38.96/(60 - 6);$$
$$M_\kappa = 646 \pm 7MeV, \quad \Gamma_\kappa = 540 \pm 42MeV;$$
$$a_0^{1/2} = 0.2, \quad a_0^{3/2} = -0.128 \pm 0.006;$$
$$M_{K^*} = 1450 \pm 5MeV, \quad \Gamma_{K^*} = 232 \pm 25MeV.$$  \(12\)

The Adler zero position is now at $s_A \simeq 0.245$ GeV. If in this fit we further freeze the $\kappa$ degrees of freedom we would obtain a total $\chi^2 \sim 750$. This clearly demonstrates the existence of the $\kappa$ resonance. For more details of our analysis we refer to Ref. \[8\].

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