Optimization-Based Block Coordinate Gradient Coding for Mitigating Partial Stragglers in Distributed Learning

Qi Wang, Ying Cui, Member, IEEE, Chenglin Li, Member, IEEE, Junni Zou, Member, IEEE, and Hongkai Xiong, Senior Member, IEEE

Abstract—Gradient coding schemes effectively mitigate full stragglers in distributed learning by introducing identical redundancy in coded local partial derivatives corresponding to all model parameters. However, they are no longer effective for partial stragglers as they cannot utilize incomplete computation results from partial stragglers. This paper aims to design a new gradient coding scheme for mitigating partial stragglers in distributed learning. Specifically, we consider a distributed system consisting of one master and N workers, characterized by a general partial straggler model and focuses on solving a general large-scale machine learning problem with L model parameters using gradient coding. First, we propose a coordinate gradient coding scheme with L coding parameters representing L possibly different diversities for the L coordinates, which generates most gradient coding schemes. Then, we consider the minimization of the expected overall runtime and the maximization of the completion probability with respect to the L coding parameters for coordinates, which are challenging discrete optimization problems. To reduce computational complexity, we first transform each to an equivalent but much simpler discrete problem with N ≪ L variables representing the partition of the L coordinates into N blocks, each with identical redundancy. This represents a coordinate gradient coding scheme for L coding parameters for blocks. Then, we adopt continuous relaxation to further reduce computational complexity. For the resulting minimization of expected overall runtime, we develop an iterative algorithm of computational complexity O(N^2) to obtain an optimal solution and derive two closed-form approximate solutions both with computational complexity O(N). For the result maximization of the completion probability, we develop an iterative algorithm of computational complexity O(N^2) to obtain a stationary point and derive a closed-form approximate solution with computational complexity O(N) at a large threshold. Finally, numerical results show that the proposed solutions significantly outperform existing coded computation schemes and their extensions.

Index Terms—Big Data, coded computation, distributed learning, gradient coding, stochastic optimization.

I. INTRODUCTION

Due to the explosion in the numbers of samples and features of modern datasets, it is generally impossible to train a model by solving a large-scale machine learning problem on a single node. This challenge naturally leads to distributed learning in a master-worker distributed computation system. Recently, distributed learning (or federated learning) has been actively investigated [2], [3], [4], [5], [6]. Due to various factors such as insufficient power, contention of shared resources, imbalanced work allocation and network congestions [7], [8], some processing nodes may be slower than others or even fail from time to time. These nodes, referred to as stragglers, can significantly affect the overall computation efficiency. Generally speaking, there are two commonly used straggler models. One is the full (persistent) straggler model where stragglers are unavailable permanently [9], [10], [11], [12], [13]. The other is the partial (non-persistent) straggler model where stragglers are slow but can conduct a certain amount of work [9], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28]. The partial straggler model is more general than the full straggler model, as the former with each worker’s computing time following a Bernoulli distribution degenerates to the latter.

Recently, several coded distributed computation techniques, including coded computation [9], [10], [11], [12], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [29] and approximate coded computation [12], [13], [27], [28], [30], have been proposed to mitigate the effect of stragglers in training the model via gradient descent (or stochastic gradient descent) algorithms [31]. The common idea is to enable robust collaborative computation of a gradient (or stochastic gradient) for reducing the overall computation time in the presence of stragglers. Note that approximate coded computation schemes achieve faster computation speeds with higher accuracy losses.
than coded computation schemes. In both coded computation and approximate coded computation, matrix multiplications (called coded matrix multiplication) [9], [10], [14], [15], [16], [17], [18], [19], [20], [21], [27], [28], [29] and calculations of gradients in general forms (called gradient coding) [11], [12], [13], [22], [23], [24], [25], [26], [30] are investigated. Specifically, in [9], [10], [14], [15], [16], [20], [27], [28], [29], coded submatrix-submatrix products corresponding to all data blocks have identical redundancy, and in [11], [12], [13], [22], [23], [24], coded local partial derivatives corresponding to all coordinates have identical redundancy. Note that with identical redundancy, the computing capabilities of partial stragglers cannot be effectively utilized since incomplete computation results from stragglers are wasted. To avoid wasting incomplete computation results, in [17], [18], [19], [21], [25], [26], diverse redundancies are introduced in coded computation schemes, and incomplete computation results such as coded submatrix-submatrix products corresponding to some data blocks [9], [10], [14], [15], [16], [20], [27], [28] and coded local partial derivatives corresponding to some coordinates [11], [12], [13], [22], [23], [24] are utilized. Note that in [17], [21], [25], [26] coding parameters determining the amount of redundancies are artificially fixed, which may limit the performances of coded computation schemes. To address the limitation, in [18], [19], the optimization of coding parameters for coded matrix multiplication is formulated, and an efficient solution for an approximated problem is obtained. Nevertheless, the optimization of coding parameters for gradient coding, which is significantly different from that of coded matrix multiplication, still remains open.

To shed some light, this paper investigates optimization-based gradient coding schemes, which are applicable to a broader range of applications. Specifically, we consider a distributed computation system consisting of one master and \( N \) workers, characterized by a general partial straggler model with the computing times of workers independent and identically distributed (i.i.d.) according to an arbitrary distribution, and focus on solving a general large-scale machine learning problem with \( L \) model parameters using gradient descent methods. Our considered general partial straggler model includes the one under the shifted exponential distribution [9], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28] as special cases. Besides, our results also apply to mini-batch stochastic gradient descent methods. Our detailed contributions are summarized below.

- We propose a coordinate gradient coding scheme with \( L \) coding parameters, one for each coordinate, to maximally diversify the redundancies in coded local partial derivatives corresponding to all \( L \) coordinates. It is worth noting that it systematically generates existing gradient coding schemes [11], [22] by allowing introduced redundancies for all coordinates to be different.

- We formulate the minimization of the expected overall runtime with respect to the \( L \) coding parameters for coordinates. The problem is a challenging stochastic optimization problem with a large number of \( (L) \) discrete variables. First, we transform the original problem with \( L \) variables to an equivalent but much simpler problem with \( N \ll L \) variables representing \( L \) possibly different diversities for the \( L \) coordinates and focus on solving the continuous relaxation of the equivalent problem with a negligible approximation error at \( N \ll L \). Then, we develop an iterative algorithm of computational complexity \( O(N^2) \) to obtain a stationary point of the relaxed non-convex problem using stochastic projected subgradient method. Besides, we obtain a closed-form approximate solution of the relaxed problem with computational complexity \( O(N) \) at a large threshold.

- Numerical results show that the proposed solutions significantly outperform the optimized version of the gradient coding scheme in [11] and two extensions of the coded matrix multiplication scheme in [18]. Numerical results also show the impacts of the system parameters on the performances of the proposed solutions and the close-to-optimal performances of the approximate solutions.

To the best of our knowledge, this is the first work that optimizes redundancies for gradient coding to effectively utilize computing capabilities of partial stragglers. A preliminary version of this paper appeared as [1]. In this paper, we additionally consider the minimization of the completion probability consisting of theoretical analysis and numerical results.

**Notation:** Throughout this paper, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}_+ \) denotes the set of positive real numbers, \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{N}_+ \) denotes the set of positive integers, and \( [N] \) denotes set \( \{1, \ldots, N\} \), for any \( N \in \mathbb{N} \). We also use calligraphic capitalized letters, e.g., \( \mathcal{K} \), to denote sets. \( |\mathcal{K}| \) denotes the cardinality of set \( \mathcal{K} \). For random quantities, we use upper case italic letters, e.g., \( T \), for scalars, upper case non-italic bold letters, e.g., \( \mathbf{T} \), for vectors. For deterministic quantities, we use lower case italic letters, e.g., \( t \), for scalars, lower case bold letters, e.g., \( \mathbf{t} \), for vectors, \( x_i \) denotes the \( i \)-th coordinate of \( x \). \( I(\cdot) \) denotes the indicator function. \( 1_m \) denotes the \( m \times 1 \) identity vector and \( 1_{m \times n} \) denotes the \( m \times n \) identity matrix.

\[
(k_0, k_1, \ldots, k_{n-1}) \triangleq \begin{pmatrix} k_0 & k_1 & \cdots & k_{n-1} \\
1 & 1 & \cdots & 1 \
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

denotes the multinomial coefficient, where \( \sum_{i=0}^{n-1} k_i = n \).

## II. SYSTEM SETUP

We consider a master-worker distributed computation system which consists of one master and \( N \) workers all with computation and communication capabilities [9], [11], [12], [13], [14], [17], [18], [19], [20], [21], [22], [23], [25], [30], [36], [37], [38].
Let \([N] \triangleq \{1, \ldots, N\}\) denote the set of worker indices. We assume that the master and each worker are connected by a fast communication link, and hence we omit the communication time, as in [18], [19], [20], [21]. We consider a general partial straggler model for the workers. Specifically, at any instant, the CPU cycle times of the \(N\) workers, denoted by \(T_n, n \in [N]\), are i.i.d. random variables. The values of \(T_n, n \in [N]\) at each instant are not known to the master, but the common distribution is known to the master. Let \(f_T(\cdot)\) and \(F_T(\cdot)\) denote the cumulative distribution function (CDF) and probability density function (PDF) of \(T_n, n \in [N]\), respectively. We shall see that most theoretical results in this paper do not require any assumption on the distribution of \(T_n, n \in [N]\).

Remark 1: The adopted straggler model is more general than those in [9], [11], [12], [13], [30], [36], [38], [39]. Specifically, when \(T_n, n \in [N]\) follow a Bernoulli distribution, the adopted straggler model degenerates to the full straggler model in [9], [11], [13], [30], [36], [38], [39]; when \(T_n \in [T_{lb}, T_{ub}], n \in [N]\), with \(T_{lb}, T_{ub} > 0\) and \(T_{ub} = \alpha T_{lb}\) for some constant \(\alpha > 1\), the adopted straggler model degenerates to the \(\alpha\)-partial straggler model in [12].

As in [9], [11], [12], [13], [14], [17], [18], [19], [20], [21], [22], [23], [25], [30], [36], [37], [38], [39], we focus on the following distributed computation scenario. The master holds a data set of \(M\) samples, denoted by \(D \triangleq \{y_i : i \in [M]\}\), and aims to train a model. The model is parameterized by an \(L\)-dimensional vector \(\theta \in \mathbb{R}^L\). Notice that the model size \(L\) is usually much larger than the number of workers \(N\). For a given \(\theta \in \mathbb{R}^L\), define the loss incurred by \(y_i\) as \(\ell(\theta; y_i)\). We assume that \(\ell(\cdot)\) is differentiable but not necessarily convex. Then, the risk function \(\hat{\ell} : \mathbb{R}^L \rightarrow \mathbb{R}\) of the model parameter \(\theta \in \mathbb{R}^L\) is defined as

\[
\hat{\ell}(\theta; D) \triangleq \sum_{y \in D} \ell(\theta; y).
\]

The master aims to minimize the risk function with respect to the model parameter \(\theta\) using commonly used gradient descent methods.2

To handle a massive amount of training data, the master implements a gradient descent method with the help of all workers. Specifically, the master partitions the whole data set and sends to each worker some particular subsets so that the master and \(N\) workers can collaboratively compute the gradient \(\nabla \hat{\ell}(\theta; D) \triangleq \sum_{y \in D} \nabla \ell(\theta; y)\) in each iteration. In contrast with the case in [11], [22], [23] where each worker starts to send the computation results after completing all \(L\) subtasks, we investigate the case where each worker sequentially computes \(L\) subtasks and sends the computation result of each subtask to the master once its computation is completed. Based on the computation results of the \(l\)-th subtasks of all workers, the master can compute the \(l\)-th partial derivative \(\frac{\partial \hat{\ell}(\theta; D)}{\partial \theta_{l}}\), i.e., the \(l\)-th coordinate of the gradient \(\nabla \hat{\ell}(\theta; D)\).3 We aim to design a coordinate gradient coding scheme for the considered case.

III. COORDINATE GRADIENT CODING

We propose a coordinate gradient coding scheme parameterized by the coding parameters \(s \triangleq (s_l)_{l \in [L]}\) for the \(L\) coordinates, which satisfy

\[
s_l \in \{0, 1, \ldots, N-1\}, \quad l \in [L].
\] (1)

Here, \(s_l\) represents the redundancy in the computation of a coded local partial derivative corresponding to the \(l\)-th coordinate. It also represents the number of stragglers the master can tolerate when recovering the \(l\)-th partial derivative \(\frac{\partial \hat{\ell}(\theta; D)}{\partial \theta_{l}}\). In other words, the master only needs to receive the coded partial derivatives corresponding to the \(l\)-th coordinate from arbitrary \(N - s_l\) workers to recover \(\frac{\partial \hat{\ell}(\theta; D)}{\partial \theta_{l}}\). The proposed scheme operates in two phases, as illustrated below. It generates the gradient coding scheme in [11], [22] by allowing \(s_l, l \in [L]\) to be different. Later, we shall see that the proposed coordinate gradient coding scheme with optimal coding parameters \(s\) turns to a block coordinate gradient coding scheme, which can be implemented more easily.

Sample Allocation Phase: First, the master partitions dataset \(D\) into \(N\) subsets of size \(\frac{M}{N}\), denoted by \(D_l, i \in [N]\) ([11], [12], [22], [23]). Then, the master allocates the same number of data subsets to each worker. Specifically, for all \(n \in [N]\), the master allocates the \(\max_{l \in [L]} s_l + 1\) subsets, \(D_l, i \in [N] \triangleq \{j \oplus (n-1) : j \in [\max_{l \in [L]} s_l + 1]\}\), to worker \(n\), where the operator \(\oplus\) is defined as: \(a_1 \oplus a_2 \triangleq \begin{cases} a_1 + a_2, & \text{if } a_1 + a_2 \leq N \\ a_1 + a_2 - N, & \text{if } a_1 + a_2 > N \end{cases}\) for all \(a_1, a_2 \in [N]\).

Collaborative Training Phase: In each iteration of a gradient descent method, the master first sends the latest \(\theta\) to all workers. Then, each worker \(n \in [N]\) sequentially computes and sends coded local partial derivatives corresponding to coordinates \(1, 2, \ldots, L\) to the master. Specifically, for all \(l \in [L]\), worker \(n \in [N]\) computes local partial derivatives \(\frac{\partial \ell(\theta; D_i)}{\partial \theta_{l}}, i \in [N] \triangleq \{j \oplus (n-1) : j \in [s_l + 1]\}\) and the coded local partial derivative \(b_{l,n}g_l\) and sends \(b_{l,n}g_l\) to the master, where \(b_{l,n}\) denotes the \(n\)-th row of encoding matrix \(B_{l}\) generated according to [11] with \(s_l\) in [6] being \(s_l\) (please refer to Algorithm 3 in Appendix N for details), and \(g_l \triangleq \left(\frac{\partial \ell(\theta; D_1)}{\partial \theta_{l}}, \frac{\partial \ell(\theta; D_2)}{\partial \theta_{l}}, \ldots, \frac{\partial \ell(\theta; D_N)}{\partial \theta_{l}}\right)^T\).

Next, the master sequentially receives coded local partial derivatives \(b_{l,1}g_l, b_{l,2}g_l, \ldots, b_{l,N}g_l\) corresponding to coordinates \(1, 2, \ldots, L\) from each worker and recovers partial derivatives \(\frac{\partial \ell(\theta; D_1)}{\partial \theta_{l}}, \frac{\partial \ell(\theta; D_2)}{\partial \theta_{l}}, \ldots, \frac{\partial \ell(\theta; D_N)}{\partial \theta_{l}}\). Let \(T(1), T(2), \ldots, T(N)\) be \(T_n, n \in [N]\) arranged in the increasing order, so that \(T(n)\) is the \(n\)-th smallest one. Specifically, for all \(l \in [L]\), the master computes the \(l\)-th partial derivative \(\frac{\partial \hat{\ell}(\theta; D)}{\partial \theta_{l}}\) via the coded local partial derivatives from the \(N - s_l\) fastest workers with CPU cycle times \(T(n), n \in [N - s_l]\), where \(a_{l,j}\) denotes the \(j\)-th row of decoding matrix \(A_l\) generated according to [6] with \(s_l\) in [6] being \(s_l\) (please refer to Algorithm 4 in Appendix N for details), \(\text{supp}(a_{l,j})\) denotes the support of \(a_{l,j}, F_l \triangleq \{i_1, i_2, \ldots, i_{N-s_l}\}\) denotes the indices of the \(N - s_l\) fastest workers, \(B_{l,F_l}\) denotes the sub-matrix of \(B_{l}\) with rows indexed by \(F_l\), and \(j \in [s_l]\) is chosen such that \(\text{supp}(a_{l,j})B_{l,F_l} = 1_{N_l}^T\). Note that the orders for computing, sending, and receiving the coded local partial derivatives are

2 We present the results based on gradient descent methods for ease of exposition. Note that our results also apply to mini-batch stochastic gradient descent methods [9], [11], [12], [13], [14], [17], [18], [19], [20], [21], [22], [23], [25], [30], [36], [37], [38], [39], where the master randomly selects a mini-batch of samples and notifies the workers the index of the mini-batch.

3 We choose a coordinate as the basic computing and communication unit for ease of exposition. Note that our results also apply to the case where a block of coordinates (which can associate with one layer of a neural network) is viewed as the basic unit.

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1, \ldots, L. Once the master has recovered \( \frac{\partial \hat{h}(\theta; D)}{\partial \theta_l} \), \( l \in [L] \), it readily obtains the gradient \( \nabla \hat{h}(\theta; D) \) \([11]\).

Let \( b \) denote the maximum of the numbers of CPU cycles for computing \( \frac{\partial \hat{h}(\theta; D)}{\partial \theta_l} \), \( l \in [L] \). \(^4\) We omit the computation loads for encoding at each worker and decoding at the master, as they are usually much smaller than the computation load for calculating the partial derivatives in practice. Thus, for all \( n \in [N] \) and \( l \in [L] \), the completion time for computing the coded local partial derivative corresponding to the \( l \)-th coordinate at worker \( n \) is \( \frac{M}{N} b T_{\max} \sum_{i=1}^l (s_i + 1) \). For all \( l \in [L] \), the completion time for recovering \( \frac{\partial \hat{h}(\theta; D)}{\partial \theta_l} \) at the master is \( \frac{M}{N} b T_{(N-s_l)} \sum_{i=1}^l (s_i + 1) \). Therefore, the overall runtime for the master and workers to collaboratively compute the gradient \( \nabla \hat{h}(\theta; D) \) is

\[
\tau(s, T) = \frac{M}{N} b \max_{l \in [L]} T_{(N-s_l)} \sum_{i=1}^l (s_i + 1),
\]

where \( T = (T_n)_{n \in [N]} \). Note that \( \tau(s, T) \) is a function of the parameters \( s \) and random vector \( T \) and hence is also random.

Remark 2 (Motivation of Diversifying Redundancy): Each worker sequentially computes and sends coded local partial derivatives (corresponding to coordinates \( 1, 2, \ldots, L \)) to the master. Note that for any \( s = (s_l)_{l \in [L]} \) and any \( n \in [N] \), the length of the interval between the times receiving the coded local partial derivatives from the fastest worker and from the \( n \)-th fastest worker, \( \frac{M}{N} b (T_{(n)} - T_{(1)}) \sum_{i=1}^l (s_i + 1) \), increases with \( l \). To reduce the waiting time for collecting additional coded local partial derivatives for decoding, we intended to reduce the number of required coded local partial derivatives for recovering the \( l \)-th partial derivative \( \frac{\partial \hat{h}(\theta; D)}{\partial \theta_l} \), \( N - s_l \), when \( l \) increases. That is, intuitively, letting \( s_l \) (representing the redundancy in the computation of a coded local partial derivative corresponding to the \( l \)-th coordinate) increase with \( l \) helps reduce the completion time for computing the coded local partial derivative corresponding to the \( l \)-th coordinate, thereby reducing the overall runtime. This reveals the ineffectiveness of posing identical redundancy (i.e., identical \( s_l \), \( l \in [L] \)) and motivates diverse (more specifically, increase) redundancies in the computation of coded local partial derivatives for \( L \) coordinates (i.e., different \( s_l \), \( l \in [L] \)). A motivating example is illustrated in Fig. 1. Specifically, in Fig. 1(a), the computation results of the coded local partial derivatives corresponding to the 3rd and 4th coordinates from worker 1 and from worker 2 are not utilized; in Fig. 1(b), the computation result of the coded local partial derivative corresponding to the 1st coordinate from worker 3 is not utilized; in Fig. 1(c), the computation results of the coded local partial derivatives corresponding to all coordinates from all workers are utilized. Fig. 1 shows that the overall runtime of the proposed coordinate gradient coding scheme with coding parameters \( s = (1, 1, 2, 2) \) (shown in Fig. 1(c)) is shorter than those of the original gradient coding scheme with coding parameter \( s = 1 \) (shown in Fig. 1(a)) and \( s = 2 \) (shown in Fig. 1(b)).

In this paper, we consider two performance metrics, namely, the expected overall runtime, \( \mathbb{E}[\tau(s, T)] \) (as considered in \([11, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 37, 38]\), and the completion probability by time threshold \( t \), \( P(s, t) \leq \Pr[\tau(s, T) \leq t] \) (as considered in \([9, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]\)). By \([18]\), we have

\[
P(s, t) = \sum_{k_0, k_1, \ldots, k_L}^{N} \left( \prod_{l=0}^{L} (F_l(s, t) - F_{k_l}(s, t)) \right)^{k_1},
\]

where \( k \equiv (k_l)_{l=0}^{L-1} \), \( K(s, N, L) \equiv \{ k \in \mathbb{N}^{L+1} : \sum_{l=0}^{L-1} k_l \leq s_l, l \in [L], \sum_{l=0}^{L-1} s_l = N \} \), and \( F_l(s, t) \equiv \Pr\left[ \frac{M}{N} b T_{s_l} \sum_{i=1}^l (s_i + 1) < t \right] = F_T \left( \frac{t}{M b \sum_{i=1}^L (s_i + 1)} \right), l \in [L] \). For ease of exposition, we let \( F_0(s, t) = 1 \) and \( F_{L+1}(s, t) = 0 \). In Sections IV and V, we will minimize \( \mathbb{E}[\tau(s, T)] \) and maximize \( P(s, t) \), respectively, by optimizing the coding parameters \( s \) for the \( L \) coordinates under the constraints in (1).

IV. EXPECTED OVERALL RUNTIME MINIMIZATION

In this section, we first formulate the minimization of the expected overall runtime with respect to the coding parameters for coordinates. Then, we obtain an optimal solution of the continuous relaxation of the original problem. Finally, we obtain two low-complexity closed-form approximate solutions of the relaxed problem.

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\(^4\) For tractability, we use the maximum, \( b \), when optimizing the coding parameters in this paper. The proposed optimization framework can be extended to consider the exact numbers of CPU cycles for computing \( \frac{\partial \hat{h}(\theta; D)}{\partial \theta_l} \), \( l \in [L] \) in optimizing the coding parameters.
A. Problem Formulation

We would like to minimize \( \mathbb{E}[\tau(s, T)] \) by optimizing the coding parameters \( s \) for the \( L \) coordinates under the constraints in (1).

**Problem 1 (Expected Overall Runtime Minimization):**

\[
\tau_{\text{avg}} \triangleq \min_s \mathbb{E}[\tau(s, T)]
\]

s.t. (1).

In general, the objective function \( \mathbb{E}[\tau(s, T)] \) does not have an analytical expression, and the number of variables (model size \( L \)) is usually quite large.\(^5\) Thus, Problem 1 is a challenging stochastic optimization problem. First, we characterize the monotonicity of an optimal solution of Problem 1.

**Lemma 1 (Monotonicity of Optimal Solution of Problem 1):**

An optimal solution \( s^* \triangleq (s^*_l)_{l \in [L]} \) of Problem 1 satisfies \( s^*_1 \leq s^*_2 \leq \cdots \leq s^*_L \).

**Proof:** Please refer to Appendix A.

Lemma 1 verifies the intuition in Remark 2. Lemma 1, together with the constraints in (1), indicates that we can optimally partition the \( L \) coordinates into \( N \) blocks, each with identical redundancy, for tolerating 0, 1, \ldots, \( N-1 \) stragglers, respectively. That is, the proposed coordinate gradient coding scheme with \( s^* \) becomes a block coordinate gradient coding scheme with at most \( N \) blocks, each with one coding parameter. It can be implemented with lighter overhead. Specifically, each worker can send multiple coded local partial derivatives corresponding to one block together in one transmission once their computations are completed without influencing the overall runtime.

Next, based on Lemma 1 and the constraints in (1), we transform Problem 1 to an equivalent problem with \( N \) variables, which optimizes the partition of the \( L \) coordinates into \( N \) blocks.

**Problem 2 (Equivalent Problem of Problem 1):**

\[
\hat{\tau}_{\text{avg}} \triangleq \min_x \mathbb{E}[\hat{\tau}(x, T)]
\]

s.t. (4).

where \( x \triangleq (x_n)_{n=0,1,\ldots,N-1} \) and

\[
\hat{\tau}(x, T) \triangleq \frac{M}{N} \max_{n=0,1,\ldots,N-1} T_{(N-n)} \sum_{i=0}^{n}(i+1)x_i.
\]

**Theorem 1 (Equivalence between Problem 1 and Problem 2):** An optimal solution of Problem 1, denoted by \( s^* = (s^*_l)_{l \in [L]} \), and an optimal solution of Problem 2, denoted by \( x^* = (x^*_n)_{n=0,1,\ldots,N-1} \) satisfy

\[
x^*_n = \sum_{l \in [L]} I(s^*_l = n), \quad n = 0, 1, \ldots, N-1,
\]

(7)

\[
s^*_l = \min_{i: \sum_{n=0}^{i} x^*_n \geq l} \left\{ i \right\}, \quad l \in [L].
\]

(8)

Furthermore, their optimal values, \( \tau^*_{\text{avg}} \) and \( \hat{\tau}^*_{\text{avg}} \), satisfy \( \tau^*_{\text{avg}} = \hat{\tau}^*_{\text{avg}} \).

**Proof:** Please refer to Appendix B.

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\(^5\)For example, AlexNet has 60 million parameters [40]; VGG-16 has 138 million parameters [41]; GoogleNet has 13 million parameters [42]; ResNet-152 has 60 million parameters [43].
Algorithm 1: Algorithm for Obtaining An Optimal Solution of Problem 3.

1: initialization: Set $i = 1$ and choose any feasible point $x^{(0)}$ of Problem 3.

2: repeat
3: Generate $S$ random samples of $\mathbf{T}$, i.e., $\mathbf{T}^{1}, \ldots, \mathbf{T}^{S}$, according to the distribution of $T_n, n \in [N]$.
4: Obtain a nonbiased subgradient $\hat{g}(x^{(i-1)})$ according to (10).
5: Update $x^{(i)}$ according to (11).
6: Compute $\lambda$ by solving the equation in (14) with bisection search, and compute $x^{(i)}$ according to (13).
7: Set $i = i + 1$.
8: until Some convergence criterion is met.

Next, we project $x^{(i)}$ on the feasible set of Problem 3 which is convex. The projection is given by the solution of the following problem.

Problem 4 (Projection of $x^{(i)}$):

$$x^{(i)} = \arg\min_{x} \|x^{(i)} - x\|^2_2$$

s.t. (4), (9).

Problem 4 is a convex quadratic programming. By the Karush-Kuhn-Tucker (KKT) conditions [44, Section 5.5.3], we obtain the optimal solution of it.

Lemma 2 (Optimal Solution of Problem 4): The optimal solution $x^{(i)} \triangleq (x^{n(i)})_{n=0,1,\ldots,N-1}$ of Problem 4 is given by

$$x^{n(i)} = \max \left\{ \hat{x}^{n(i)} + \lambda, 0 \right\}, \quad n = 0, 1, \ldots, N - 1,$$

where $\lambda$ satisfies

$$\sum_{n=0}^{N-1} \max \left\{ \hat{x}^{n(i)} + \lambda, 0 \right\} = L.$$  \hspace{1cm} (14)

Proof: Please refer to Appendix C.

Note that $\lambda$ can be obtained by bisection search. The details of the algorithm are summarized in Algorithm 1. We can easily verify that the computational complexities of Steps 3, 4, 5, 6, and 7 in Algorithm 1 are $O(N)$, $O(N^2)$, $O(N)$, and $O(N)$, respectively. Thus, the overall computational complexity of Algorithm 1 is $O(N^2)$.

C. Approximate Solutions

In the following, we obtain two closed-form approximate solutions of Problem 3 which are more computationally efficient than the optimal solution given by Lemma 2.

1) Approximate Solution Based on Deterministic CPU Cycle Times: We approximate the objective function of Problem 3 by replacing the random vector $\mathbf{T}$ with the deterministic vector $\mathbf{t} \triangleq (t^n)_{n \in [N]}$, where $t_n \triangleq \mathbb{E}[T_n]$ (which can be numerically computed for an arbitrary form of $F_T(\cdot)$ and analytically computed for some special forms of $F_T(\cdot)$).

Problem 5 (Approximation of Problem 3 at $\mathbf{t}$):

$$x^{(E,t)} = \arg\min_{x} \bar{\tau}(x, t)$$

s.t. (4), (9).

Note that Problem 5 is a challenging convex problem with a non-differentiable objective function due to the point-wise maximum in $\bar{\tau}(x, \mathbf{T})$. By contradiction and construction, we obtain the optimal solution of Problem 5 given as follows.

Theorem 2 (Closed-form Optimal Solution of Problem 5):

$$x^{(E,t)} = \frac{1}{t_n} z^{(E,t)} , \quad z^{(E,t)} = \frac{1}{n+1} \left( \frac{1}{t_{N-n}} - \frac{1}{t_{N+1-n}} \right) z^{(E,t)}, \quad n \in [N - 1],$$

where $z^{(E,t)} \triangleq \frac{\sum_{n=1}^{N} \mathbb{E}[T_{N+1-n}]}{N+1} + \frac{1}{N+1}$.

Proof: Please refer to Appendix D.

$x^{(E,t)}$ can be interpreted as an optimal solution of Problem 3 for a master-worker distributed computation system where $N$ workers have deterministic CPU cycle times $t$, and can be treated as an approximate solution of Problem 3.

Finally, we characterize the computational complexity and suboptimality of $x^{(E,t)}$ under the assumption that the distribution of $T_n, n \in [N]$ is a shifted-exponential distribution, i.e.,

$$F_T(t) = 1 - e^{-\mu(t-t_0)}, \quad t \geq t_0,$$

where $\mu$ is the rate parameter and $t_0 > 0$ is the shift parameter.

Note that shifted-exponential distributions are widely considered in modeling stragglers in distributed computation systems [14], [18], [19], [22], [23]. By Rényi representation theorem of exponential order statistics [47, Corollary 2.26], we have

$$t_n = \mathbb{E}[T_n] = \frac{1}{\mu} (H_N - H_{N-n}) + t_0, \quad n \in [N],$$

where $H_n \triangleq \sum_{i=1}^{n} \frac{1}{i}$ is the $n$-th harmonic number. The computational complexity for calculating $t$ according to (16) is $O(N)$. Given $t$, the computational complexity for calculating $z^{(E,t)}$ is $O(N)$. Thus, the computational complexity for calculating $x^{(E,t)}$ is $O(N)$. The suboptimality of $x^{(E,t)}$ is given as follows.

Theorem 3 (Sub-optimality of $x^{(E,t)}$):

$$\mathbb{E}[\hat{\tau}(x^{(E,t)}, \mathbf{T})] = O(\log^2(N)),$$

where $\hat{\tau}_{\text{avg-ct}}$ denotes the optimal value of Problem 3.

Proof: Please refer to Appendix E.

Theorem 3 indicates that for any model size $L$, the expected overall runtime of $x^{(E,t)}$ and the minimum expected overall runtime have a sub-linear multiplicative gap in $N$. As $\log^2(N) = 5.3$ at $N = 10$, the analytical upper bound on the gap is not large at a small or moderate $N$. Later in Section VI, we shall see that the actual gap is very small even at $N = 50$.

2) Approximate Solution Based on Deterministic CPU Frequencies: We approximate the objective function of Problem 3 by replacing random vector $\mathbf{T}$ with deterministic vector $\mathbf{t}' \triangleq (t'_n)_{n \in [N]}$, where $t'_n \triangleq 1/\mathbb{E}[1/T_n], n \in [N]$ (which can be numerically computed for an arbitrary form of $F_T(\cdot)$ and analytically computed for some special forms of $F_T(\cdot)$).

Problem 6 (Approximation of Problem 3 at $\mathbf{t}'$):

$$x^{(E,t')} = \arg\min_{x} \bar{\tau}(x, \mathbf{t}')$$

s.t. (4), (9).
Problem 6 has the same structure as Problem 5 and can be solved with the same method.

**Theorem 4 (Closed-form Optimal Solution of Problem 6):**
\[
x_0^{(E,f)} = \frac{1}{t_n} z^{(E,f)}, \quad x_n^{(E,f)} = \frac{N}{n+1} \left( \frac{1}{t_{n+1}} - \frac{1}{t_n} \right) z^{(E,f)}, \quad n \in [N-1],
\]
where \( z^{(E,f)} \) is given in \([10] \).

**Proof:** The proof is similar to that of Theorem 2 and is omitted due to page limitation.

Let \( t_n = \frac{1}{t_n}, \quad n \in [N] \) denote the CPU frequencies of the \( N \) workers. Thus, \( x^{(E,f)} \) can be interpreted as an optimal solution of Problem 3 for a distributed computation system where \( N \) workers have deterministic CPU frequencies \( t' \), and can be treated as an approximate solution of Problem 3. Given \( t' \), the computational complexity of calculating \( x^{(E,f)} \) is \( \mathcal{O}(N) \).

Finally, we characterize the computational complexity and suboptimality of \( x^{(E,f)} \) under a shifted-exponential distribution as in Subsec. IV-C1. We first derive the expression of \( t' \) in the following Lemma.

**Lemma 3 (Parameters of Problem 6 under a shifted-exponential distribution):** We have \( t_n' \) given in (17) as shown at the bottom of this page, where \( E_i(x) \triangleq \int_{-\infty}^{x} \frac{1}{t} dt \) is the exponential integral.

**Proof:** Please refer to Appendix F.

The computational complexity for calculating \( t' \) according to (17) is \( \mathcal{O}(N^2) \). Given \( t' \), the computational complexity for calculating \( z^{(E,f)} \) is \( \mathcal{O}(N) \). Thus, the computational complexity for calculating \( x^{(E,f)} \) is \( \mathcal{O}(N^2) \). The suboptimality of \( x^{(E,f)} \) is given as follows.

**Theorem 5 (Sub-optimality of \( x^{(E,f)} \)):**
\[
\nu(t_n') = \frac{E_i(x^{(E,f)})}{\log(N)} = \mathcal{O}(\log(N)),
\]
where \( \hat{t}_{\log} \) denotes the optimal value of Problem 3.

**Proof:** Please refer to Appendix G.

Similarly, Theorem 5 indicates that for any model size \( L \), the expected overall runtime of \( x^{(E,f)} \) and the minimum expected overall runtime have a sub-linear multiplicative gap in \( N \). In summary, the multiplicative gap for \( x^{(E,f)} \) is smaller than that for \( x^{(E,t)} \), but the computational complexity for calculating \( x^{(E,f)} \) is higher than that for \( x^{(E,t)} \).

V. COMPLETION PROBABILITY MAXIMIZATION

In this section, we first formulate the maximization of the completion probability with respect to the coding parameters for coordinates. Then, we obtain a stationary point of the continuous relaxation of the original problem. Finally, we obtain a low-complexity closed-form approximate solution of the relaxed problem at large threshold \( t \) which is close to the optimal one.

A. Problem Formulation

We would like to maximize \( P(s,t) \) by optimizing the coding parameters \( s \) for the \( L \) coordinates under the constraints in (1).

**Problem 7 (Completion Probability Maximization):**
\[
P^*(t) \triangleq \max_s P(s,t) \\
\text{s.t. (1)},
\]

where \( P(s,t) \) is given by (3).

The objective function \( P(s,t) \) is non-convex and has a large number of summands for large \( N \) or \( L \). In addition, the number of variables (model size \( L \)) is usually quite large. Hence, Problem 7 is a challenging non-convex problem. First, we characterize the monotonicity of an optimal solution of Problem 7.

**Lemma 4 (Monotonicity of Optimal Solution of Problem 7):**

The optimal solution \( s^* \triangleq (s_i')_{i \in [L]} \) of Problem 7 satisfies \( s_1^* \leq s_2^* \leq \cdots \leq s_L^* \).

**Proof:** Please refer to Appendix H.

Notice that Lemma 4 resembles Lemma 1. Thus, similar conclusions can be drawn. Next, based on Lemma 4, we transform Problem 7 to an equivalent problem with \( N \) variables, which optimizes the partition of the \( L \) coordinates into \( N \) blocks.

**Problem 8 (Equivalent Problem of Problem 7):**
\[
\hat{P}^*(t) \triangleq \max_{x} \hat{P}(x,t) \\
\text{s.t. (4), (5)},
\]

where \( \hat{P}(x,t) \) is given by
\[
\hat{P}(x,t) \triangleq \sum_{k \in \mathcal{K}(N)} g(x,t,k).
\]

with \( \hat{k} \triangleq \hat{k}_i \in (0 \cup [N-1]) \),
\[
\mathcal{K}(N) \triangleq \left\{ \hat{k} \in [N^N] : \sum_{i=0}^{N-1} \hat{k}_i = n \in [N], \sum_{i=0}^{N-1} \hat{k}_i = N \right\}.
\]

and \( g(x,t,k) \) is given in (18) as shown at the bottom of this page.

**Theorem 6 (Equivalence between Problem 7 and Problem 8):**

An optimal solution of Problem 7, denoted by \( s^* = (s_i')_{i \in [L]} \), and an optimal solution of Problem 8, denoted by \( x^* = (x_n')_{n \in [N-1]} \), satisfy (7) and (8), and their optimal values, \( P^*(t) \) and \( \hat{P}^*(t) \), satisfy \( P^*(t) = \hat{P}^*(t) \).

**Proof:** Please refer to Appendix I.

Theorem 6 resembles Theorem 1, and hence similar conclusions can be drawn. We can solve Problem 8 whose number of variables, \( N \), is usually much smaller than the number of variables of Problem 7, \( L \). By relaxing the integer constraints in (5), we obtain a more tractable problem compared with Problem 7.

**Problem 9 (Relaxed Continuous Problem of Problem 8):**
\[
\hat{P}^*_c(t) \triangleq \max_{x} \hat{P}(x,t) \\
\text{s.t. (4), (9)},
\]

As illustrated in Section IV-A, the rounding method [44, pp. 386] can be applied to obtain a good approximate solution of
Problem 7 from an optimal solution of Problem 9. Thus, in the following section, we focus on solving the relaxed problem in Problem 9.

**B. Stationary Point**

First, we calculate the number of summands in \( \hat{P}(x, t) \).

**Lemma 5:** The number of summands in \( \hat{P}(x, t) \) is given by

\[
|\hat{K}(N)| = \frac{2}{N+1} \binom{2N-1}{N-1} \geq 2^{N-1},
\]

where the equality holds if and only if \( N = 1 \).

**Proof:** Please refer to Appendix J.

As the number of summands in \( \hat{P}(x, t) \) is prohibitively large when \( N \) is large, directly tackling Problem 8 is not computationally efficient. We hence solve its equivalent stochastic version to reduce the computation time.

**Problem 10 (Equivalent Stochastic Problem of Problem 9):**

\[
\begin{align*}
\max_{x} & \quad \mathbb{E}[g(x, t, \xi)] \\
\text{s.t.} & \quad (4), (9),
\end{align*}
\]

where random vector \( \xi \triangleq (\xi_n)_{n=0,1,...,N-1} \) follows the uniform probability distribution on \( \hat{K}(N) \). It is obvious that \( \hat{P}(x, t) = |\hat{K}(N)| \mathbb{E}[g(x, t, \xi)] \), implying that Problem 9 and Problem 10 are equivalent. Problem 10 is a stochastic non-convex problem with deterministic convex constraints. We obtain a stationary point of Problem 10 by using SSCA [48]. The idea is to iteratively solve a sequence of convex approximations of Problem 10, each of which is obtained by replacing the objective function with a convex surrogate function based on a randomly generated realization of \( \xi \). Specifically, in the \( i \)-th iteration, generate a realization of \( \xi \), denoted by \( \hat{\xi}^{(i)} \), and approximate Problem 10 with the following convex problem.

**Problem 11 (Approximate Convex Problem of Problem 10 in the \( i \)-th Iteration):**

\[
\hat{x}^{(i)} \triangleq \underset{x}{\text{argmax}} \quad f^{(i)}(x, x^{(i-1)}, t, \hat{\xi}^{(i)})
\]

\[
\text{s.t.} \quad (4), (9),
\]

where \( f^{(i)}(x, x^{(i-1)}, t, \xi^{(i)}) \triangleq h^{(i)}(x - x^{(i-1)}) - \tau \| x - x^{(i-1)} \|^2 \) is a concave surrogate function of \( \mathbb{E}[g(x, t, \xi)] \) around \( x^{(i-1)} \), with \( h^{(i)} = (1 - \sigma^{(i)})h^{(i-1)} + \sigma^{(i)}\nabla g(x^{(i)}, t, \xi^{(i)}) \in \mathbb{R}^N \) as an approximation of \( \nabla \mathbb{E}[g(x, \xi)] \). Here, \( \{\sigma^{(i)}\} \) is a positive diminishing stepsize sequence satisfying (25), and \( h^{(0)} = 0 \). Problem 11 is a convex quadratic programming. By the KKT conditions [44, Section 5.5.3], we obtain the optimal solution of Problem 11.

**Lemma 6 (Optimal Solution of Problem 11):** The optimal solution \( \hat{x}^{(i)} \triangleq (\hat{x}^{(i)}_n)_{n=0,1,...,N-1} \) of Problem 11 is given by

\[
x^{(i)}_n = \max \left\{ x^{(i-1)}_n + \frac{1}{2\tau} \left( \sigma^{(i)} \frac{\partial}{\partial x_n} g(x^{(i-1)}, \xi^{(i)}) \\
- (1 - \sigma^{(i)})h^{(i-1)}_n \right) + \frac{1}{\tau} \lambda, 0 \right\}, \quad n = 0, 1, \ldots, N - 1,
\]

where \( \lambda \) satisfies

\[
\sum_{n=0}^{N-1} \max \left\{ x^{(i-1)}_n + \frac{1}{2\tau} \left( \sigma^{(i)} \frac{\partial}{\partial x_n} g(x^{(i-1)}, \xi^{(i)}) \\
- (1 - \sigma^{(i)})h^{(i-1)}_n \right) + \frac{1}{\tau} \lambda, 0 \right\} = L.
\]

**Proof:** Please refer to Appendix K.

Then, based on Theorem 7, we generate \( \xi^{(i)} \) by generating \( \xi_0, \ldots, \xi_{N-1} \) successively according to (25) and (27). From (25) and (27), we know that the time complexity and space complexity of the proposed method for generating \( \xi^{(i)} \) are \( O(N^2) \) and \( O(N) \), respectively. Obviously, both the time complexity and space complexity of the new method are much lower than those of the naive method mentioned above.

Finally, we summarize the details of the algorithm for obtaining a stationary point of Problem 10 in Algorithm 2. Here, \( I \) denotes the total number of iterations, irrelevant to the problem size [48]. The computational complexity of Steps 4 in Algorithm 2 is \( O(N) \), and the computational complexities of Steps 6 and 7 in Algorithm 2 are \( O(N) \). Thus, the computational complexity of generating a discrete random variable with \( N \) outcomes in the sample space is \( O(\log N) \) [49].
Algorithm 2: Algorithm for Obtaining a Stationary Point of Problem 10.

1: **initialization:** Choose any feasible point $x^{(0)}$ of Problem 10.
2: for $i = 1, 2, \ldots, I$ do
3:     for $n = 0, 1, \ldots, N - 1$ do
4:         Generate $\xi_n^{(i)}$ for given $\xi_0, \ldots, \xi_{n-1}$, according to (27).
5:     end for
6:     Compute $\lambda$ by solving the equation in (23) with bisection search, and compute $x^{(i)}$ according to (22).
7:     Update $x^{(i)}$ according to (24).
8: end for

complexity of Algorithm 2 is $O(N^2)$. It has been shown in [48] that every limit point of the sequence $\{x^{(i)}\}$ generated by Algorithm 2 is a stationary point, denoted by $x^{(CDF,\text{st})}$, of Problem 10 almost surely.

C. Approximate Solution at Large Threshold

In this part, assuming that the distribution of $T_n, n \in [N]$ follows the shifted-exponential distribution given in (15), we obtain a low-complexity approximate solution of Problem 9 at large $t$. First, we obtain an asymptotic approximation of $1 - \hat{P}(x, t)$ at large $t$.

**Lemma 7 (Asymptotic Failure Probability at Large t):** $1 - \hat{P}(x, t) \xrightarrow{t \to \infty} E_{\text{lg}}(x, t)$, where

$$E_{\text{lg}}(x, t) \triangleq \left( \sum_{n \in N_2(x)} \left( \frac{N}{n + 2} \right) e^{\mu t(n+2)} \right) e^{-\frac{N}{n \in N_1(x)} \sum_{i=0}^{n+1} \frac{1}{i+1}}.$$

Here, $N_2(x) \triangleq \{ n \in [N-1] : x_n \neq 0 \}$ and $N_2(x) \triangleq \{ n \in [N] : x_n \neq 0 \}$ (i.e., $N_2(x) \triangleq \{ n \in N_1(x) : \sum_{n=0}^{n+1} \frac{1}{i+1} \}$).

**Proof:** Please refer to Appendix L.

By Lemma 7, Problem 9 is asymptotically equivalent to the following problem, as $t \to \infty$:

$$\min_{x} E_{\text{lg}}(x, t) \quad \text{s.t.} \quad (4), (9).$$

Problem 12 is still a quite challenging non-convex problem. For tractability, we approximate $E_{\text{lg}}(x, t)$ with an upper bound, denoted as $E_{\text{ub}}(x, t)$, which is given by (26) as shown at the bottom of this page. It is clear that $\hat{P}_{\text{lg}}(x, t) \leq E_{\text{ub}}(x, t)$, as $\mathcal{N}_2(x) \subseteq \mathcal{N}_1(x) \subseteq \{0, 1, \ldots, N - 1\}$. Therefore, we can approximate Problem 12 with the following problem.

**Problem 13 (Approximate Problem of Problem 12):**

$$x^{(CDF,\text{ub})} \triangleq \arg\min_{x} E_{\text{ub}}^{\text{opt}}(x, t) \quad \text{s.t.} \quad (4), (9).$$

Then, by contradiction and construction, we obtain an optimal solution of Problem 13.

**Theorem 8 (Closed-form Optimal Solution of Problem 13):** An optimal solution of Problem 13 is given by

$$x^{(CDF,\text{ub})}_n = \frac{L}{H_N}, \quad x^{(CDF,\text{ub})}_n = \frac{1}{n+1}, \quad n \in [N-1],$$

where $H_n \triangleq \sum_{i=1}^{n} \frac{1}{i}$ is the $n$-th harmonic number.

**Proof:** Please refer to Appendix M.

From Theorem 8, we can see that $x^{(CDF,\text{ub})}_n$ decreases with $n$, i.e., blocks with less redundancy include more coordinates. Obviously, the computational complexity for calculating $x^{(CDF,\text{ub})}$ according to (29) is $O(N)$.

VI. Numerical Results

In this section, we numerically evaluate the performance of the integer-valued approximations (obtained using the rounding method in [44, pp. 386]) of the proposed solutions. Let $\hat{x}^{(E, \text{opt})}$, $\hat{x}^{(E, t)}$, and $\hat{x}^{(E, t)}$ denote the integer-valued approximations of the proposed solutions $x^{(E, \text{opt})}$, $x^{(E, t)}$, and $x^{(E, t)}$ in Section IV.

Let $x^{(CDF,\text{st})}$ and $x^{(CDF,\text{ub})}$ denote the integer-valued approximations of the proposed solutions $x^{(CDF,\text{st})}$ and $x^{(CDF,\text{ub})}$ in Section V. We consider the subsequent four baseline schemes. Single-block coordinate gradient coding (BCCG) corresponds to the integer-valued solution obtained by solving Problem 2 (or Problem 8) with extra constraints $||x|| = 1$ and rounding the optimal solution using the rounding method in [44, pp. 386]. Notice that single-BCCG can be viewed as an optimized version of the gradient coding scheme for full stragglers in [11]. Tandon et al.’s gradient coding corresponds to the optimal gradient coding scheme in [11] for $\alpha$-partial stragglers with $\alpha = \frac{2 \cdot \Pr[T_n < t] \cdot \Pr[T_n > t]}{2 \cdot \Pr[T_n < t] + \Pr[T_n > t]} = 6$, where $t$ satisfies $\Pr[T_n < t] = 0.5$. Ferdinard et al.’s coded computation ($r = L$) and ($r = L/2$) correspond to the optimal coding scheme in [18], with the optimized coding parameter at the number of layers $r = L$ and $r = L/2$, respectively. We set $M = 50$ and $b = 1$.\footnote{Source code for the simulations is available at https://github.com/ CuiYing123456/Block_Gradient_Coding.\footnote{This paper focuses on the overall runtime for the master and workers to collaboratively compute the gradient in each iteration of a gradient descent methods for solving a machine learning problem. Thus, we can evaluate the overall runtime of one iteration without a specific machine learning dataset.}}

A. Shifted-Exponential Distribution

In this part, $T_n, n \in [N]$ follow a shifted-exponential distribution (with parameters $\mu$ and $t_0$) given in (15) as in [14, 18], [19, 22, 23].

First, we consider expected overall runtime minimization. Fig. 3(a), (b), and (c) illustrate the expected runtime versus the
number of workers $N$, the rate parameter $\mu$, and the number of model parameters $L$, respectively. From Fig. 3(a), we see that the expected overall runtime of each scheme decreases with $N$, due to the increase of the overall computation resource with $N$. From Fig. 3(b), we see that the expected overall runtime of each scheme decreases with $\mu$ due to the decrease of $E[T_n] = \frac{1}{\mu} + t_0$ with $\mu$. From Fig. 3(c), we see that the expected overall runtime of each scheme increases with $L$, due to the increase of computation load with $L$. Furthermore, from Fig. 3, we can draw the following conclusions. The proposed solutions significantly outperform the four baseline schemes. For instance, the proposed solutions can achieve reductions of 44% in the expected overall runtime over the best baseline scheme at $\mu = 10^{-2.6}$ in Fig. 3(b). The gains over single BCGC and Tandon et al.’s gradient coding are due to the diverse redundancy introduced across partial derivatives. The gains over Ferdinand et al.’s coded computation schemes at $r = L$ and $r = L/2$ indicate that an optimal coded computation scheme for calculating matrix-vector multiplications is no longer effective for calculating a general gradient. The proposed closed-form approximate solutions are quite close to the proposed optimal solution and hence have significant practical values. Besides, note that $\hat{x}^{(E,N)}$ slightly outperforms $\hat{x}^{(E,1)}$ which can be drawn from Theorem 3 and Theorem 5.

Next, we consider completion probability maximization. Fig. 4(a), (b), and (c) illustrate the completion probability versus the number of workers $N$, the rate parameter $\mu$, and the number of model parameters $L$, respectively. From Fig. 4, we see that the completion probability of each scheme increases with $N$ and $\mu$, and decreases with $L$. The explanations are similar to those for the expected overall runtime minimization shown in Fig. 3. Fig. 5 illustrates the completion probability versus the threshold $t$. From Fig. 5, we see that the completion probability of each scheme increases with $t$ as expected. Furthermore, from Figs. 4 and 5, we can draw the following conclusions. The proposed solutions significantly outperform the four baseline schemes. For instance, in Fig. 4(a), the proposed solutions can achieve reductions of 15% in the completion probability over the best baseline scheme at $N = 10$. Besides, the proposed closed-form approximate solution achieves similar completion probability to the proposed stationary point with much lower computational complexity and their gap reduces with $t$, indicating the significant practical value of the proposed approximate solution.

### B. Distribution Based-on Real-World Platform

In this part, $T_n, n \in [N]$ follow the distribution for an Amazon Elastic Compute Cloud (Amazon EC2) cluster, reported in [14, Fig. 7]. Specifically, we first approximate the complementary CDF for the Amazon EC2 cluster [14, Fig. 7] with a complementary CDF of a shifted-exponential distribution (with parameters $\mu$ and $t_0$) given in (15) using moment matching. Consequently, we have $t_0 = 0$, $\mu = \frac{1}{E[T_n]} = 5 \times 10^{-2}$. Then, we obtain the proposed solutions and baseline solutions under the approximate shifted-exponential distribution. Finally, we evaluate the expected overall runtime and completion probability using the samples generated from the complementary CDF for the Amazon EC2 cluster [14, Fig. 7]. Fig. 6(a) and (b) show the corresponding expected overall runtime and completion probability versus the number of workers $N$, respectively. Similar observations can be made. Thus, we can conclude that the proposed solution framework relying on a shifted-exponential distribution have significant practical values.
VII. CONCLUSION

In this paper, we proposed a coordinate gradient coding scheme with L coding parameters, representing L possibly different diversities for the L coordinates, which generated most gradient coding schemes. We considered the minimization of the expected overall runtime and the maximization of the completion probability with respect to the L coding parameters for coordinates. We transformed the coordinate gradient coding scheme to an equivalent but more easily implemented block coordinate gradient coding scheme with \( N \ll L \) coding parameters for blocks and adopted continuous relaxation. For the resulting minimization of expected overall runtime, we derived an optimal solution and two closed-form approximate solutions. For the resultant maximization of the completion probability, we derived a stationary point and a closed-form approximate solution. Numerical results demonstrated that the proposed solutions significantly outperformed existing coded computation schemes and their extensions, and the approximate solutions had close-to-optimal performances. To the best of our knowledge, this is the first work that optimizes the redundancies for gradient coding to effectively utilize the computing capabilities of partial stragglers.

APPENDIX A
PROOF OF LEMMA 1

We prove Lemma 1 by contradiction. First, suppose that for the optimal solution \( s^* \), there exists \( m \in [L - 1] \) such that \( s^*_m > s^*_{m+1} \). Construct a feasible solution \( \hat{s} \triangleq (\hat{s}_l)_{l \leq |L|} \), where

\[
\hat{s}_l = s^*_l, \quad l \neq m,
\]

\[
\hat{s}_m = s^*_m + 1, \quad l = m.
\]

Then, for all \( l \in [m - 1] \), we have

\[
\frac{M}{N} b T_{(N - \hat{s}_l)} \sum_{i=1}^{l} (\hat{s}_i + 1) \leq \frac{M}{N} b T_{(N - s_l)} \sum_{i=1}^{l} (s^*_i + 1) \leq \tau(s^*, T).
\]

For all \( l \in [N]\{m\} \), we have

\[
\frac{M}{N} b T_{(N - \hat{s}_l)} \sum_{i=1}^{l} (\hat{s}_i + 1) \leq \frac{M}{N} b T_{(N - s_l)} \sum_{i=1}^{l} (s^*_i + 1) \leq \tau(s^*, T).
\]

For \( l = m \), we have

\[
\frac{M}{N} b T_{(N - \hat{s}_m)} \sum_{i=1}^{m} (\hat{s}_i + 1) \leq \frac{M}{N} b T_{(N - s_m)} \sum_{i=1}^{m} (s^*_i + 1) \leq \tau(s^*, T).
\]

\[
\frac{M}{N} b T_{(N - s^*_m + 1)} \sum_{i=1}^{m+1} (s^*_i + 1) \leq \tau(s^*, T).
\]

Here, (a), (b), and (c) are due to (30) and (31). By (2), (32), (33) and (34), we have \( \tau(\hat{s}, T) \leq \tau(s^*, T), T \in \mathbb{R}_+^N \). Besides, \( \tau(\hat{s}, T) < \tau(s^*, T), T \in S \triangleq \{ T : T_n = t, n \in [N], t \in \mathbb{R}_+ \} \). As the Lebesgue measure is strictly positive on the non-empty open set \( S, \mathbb{E}[\tau(\hat{s}, T)] < \mathbb{E}[\tau(s^*, T)] \). This indicates that \( s^*_m \) is not an optimal solution, which contradicts with the assumption. Therefore, we complete the proof of Lemma 1.

APPENDIX B
PROOF OF THEOREM 1

Based on Lemma 1, we first equivalently transform Problem 1 to the following problem.

\[
\min \ E[\tau(s, T)] \quad \text{s.t. } (1),
\]

\[
s_i \leq s_{i+1}, \quad i \in [L - 1].
\]

Next, we equivalently transform the problem in (35) to Problem 2. Introduce auxiliary variables \( x \triangleq (x_n)_{n=0,1,\ldots,N-1} \) with \( x_n = \sum_{i \in [l]} I(s_i = n) \in \mathbb{N}, \quad n = 0, 1, \ldots, N - 1 \), together with (36) implying \( s_i = \min\{ i : \sum_{i=n}^{N-1} x_i \geq l \}, \quad l \in [L] \). For notation simplicity, treat \( \sum_{i=0}^{n-1} x_{i} \triangleq 0 \) for ease of illustration and define \( S_n \triangleq \{ \sum_{i=0}^{n-1} x_{i} + 1, \sum_{i=0}^{n-1} x_{i} + 2, \ldots, \sum_{i=0}^{n-1} x_{i} \}, \quad n = 0, 1, \ldots, N - 1 \). Thus, we have

\[
\sum_{i=0}^{n-1} x_i + 1 = \sum_{i=0}^{n} (i + 1) x_i.
\]

Besides, we have

\[
\tau(s, T) = \frac{M}{N} b \max_{l \in [L]} T_{(N - s_l)} \sum_{i=0}^{l} (s_i + 1) = \frac{M}{N} b \max_{l \in [L]} \max_{i \in S_n} T_{(N - s_l)} \left( \sum_{i \in S_n} (s_i + 1) + \sum_{i \in S_n} (s_i + 1) \right).
\]

\[
\frac{M}{N} b \max_{n=0,\ldots,N-1} \max_{l \in [L]} T_{(N - s_n)} \left( \sum_{i \in S_n} (s_i + 1) + \sum_{i \in S_n} (s_i + 1) \right).
\]

where (a) is due to \( [L] = \cup_{n=0}^{N-1} S_n \) and \( [l] = \cup_{n=0}^{n-1} S_n \cup \{ S_n \cap [l] \} \) (by the definition of \( S_n \)), (b) is due to \( s_l = n, l \in S_n \) (by the expression of \( s_l \) and the definition of \( S_n \)), and (c) is due to (37). By \( x_n = \sum_{i \in [l]} I(s_i = n) \in \mathbb{N}, \quad n = 0, 1, \ldots, N - 1 \), we can obtain (4) and (5). Thus, the problem in (35) is equivalent to Problem 2, implying (7) and (8) hold and \( \tau^* = \hat{\tau}^* \). Therefore, we can show Theorem 1.
APPENDIX C
Proof of Lemma 2
Let $\lambda^{(i)} = (\lambda^{(i)}_n)_{n=0,1,\ldots,N-1}$ denote the Lagrange multipliers corresponding to the inequality constraints in (9), and let $\nu$ denote the Lagrange multiplier corresponding to the equality constraint in (4). The KKT conditions are given by
\begin{align}
\sum_{n=0}^{N-1} x^{(i)}_n &= L, \\
x^{(i)}_n &\geq 0, \quad n = 0, 1, \ldots, N - 1, \\
\lambda^{(i)}_n &\geq 0, \quad n = 0, 1, \ldots, N - 1, \\
\lambda^{(i)}_n x^{(i)}_n &= 0, \quad n = 0, 1, \ldots, N - 1, \\
2(x^{(i)}_n - \hat{x}^{(i)}_n) - \lambda^{(i)}_n + \nu^{(i)} &= 0, \quad n = 0, 1, \ldots, N - 1. 
\end{align}

By the KKT conditions, we can obtain (50)
\begin{align}
x^{(i)}_n &= \max \left\{ \hat{x}^{(i)}_n - \frac{1}{2} \nu^{(i)}, 0 \right\}, \quad n = 0, 1, \ldots, N - 1,
\end{align}
where $\nu^{(i)}$ satisfies
\begin{align}
\sum_{n=0}^{N-1} \max \left\{ \hat{x}^{(i)}_n - \frac{1}{2} \nu^{(i)}, 0 \right\} &= L. \tag{44}
\end{align}

It can be verified that (44) has a unique solution which is readily determined by bisection search with initial lower bound and upper bound on $\nu^{(i)}$ given by $\frac{2}{\int_{0}^{x^{(i)}_n - \frac{1}{2} \nu^{(i)}} dx}$ and $\sum_{n=0}^{N-1} x^{(i)}_n - L$, respectively [44, pp. 146]. Note that Problem 4 is a convex problem with differentiable objective and constraint functions, and it satisfies Slater’s condition. Thus, the KKT conditions provide necessary and sufficient conditions for optimality. By (43), (44), and replacing $\frac{1}{2} \nu^{(i)}$ with $\nu$ for simplicity, we can show Lemma 2.

APPENDIX D
Proof of Theorem 2
First, we prove $\hat{\tau}(x, t) \geq \frac{L}{N} b_2(z^{(E)}_t)$ for all $x$ satisfying (4) and (9) by contradiction and construction. Recall $\hat{\tau}(x, t) = \frac{M}{N} \max_{n=0,1,\ldots,N-1} (x_n - x_{n+1})^T x_n$. Suppose $\hat{\tau}(x, t) < \frac{L}{N} b_2(z^{(E)}_t)$. Define $\beta_i \triangleq \frac{1}{i+1} \tau^{(i+1)}$, $i \in [N]$. We have $L = \sum_{i=0}^{N-1} \beta_i \frac{1}{i+1} \tau^{(i+1)}$. By $\sum_{i=0}^{N-1} \beta_i = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \frac{1}{j+1} \tau^{(j)} = \frac{1}{N} \sum_{j=0}^{N-1} (j+1) \tau^{(j)}$, and $\sum_{j=0}^{N-1} j \tau^{(j)}$, which leads to a contradiction, where (a) is due to (4), (b) is due to the expansion of $\sum_{i=0}^{N-1} x_i$ in terms of $\sum_{i=0}^{N-1} x_i$, (c) is due to the assumption, and (d) is due to the definition of $\hat{\tau}(x, t)$ in Theorem 2 and $\beta_i$, $i \in [N]$. Thus, we can show $\hat{\tau}(x, t) \geq \frac{L}{N} b_2(z^{(E)}_t)$. Next, it is obvious that $x^{(E)}_t$ is a feasible point and achieves the minimum, i.e., $\hat{\tau}(x^{(E)}_t, t) = \hat{\tau}(x^{(E)}_t)$. By $\hat{\tau}(x^{(E)}_t) \geq \hat{\tau}(x^{(E)}_t, t)$, we know that $x^{(E)}_t$ is an optimal solution of Problem 5. Therefore, we complete the proof of Theorem 2.

APPENDIX E
Proof of Theorem 3
Recall $\hat{\tau}^{\ast}_{\text{avg}-ct}$, $x^{(E, \text{opt})}_t$, and $x^{(E, t)}_t$ denote the optimal value of Problem 3, an optimal solution of Problem 3, and an optimal solution of Problem 5, respectively. First, we have
\begin{align}
\hat{\tau}^{\ast}_{\text{avg}-ct} &\geq \hat{\tau}(x^{(E, \text{opt})}_t, t) \geq \hat{\tau}(x^{(E, t)}_t, t),
\end{align}
where (a) is due to Jensen’s inequality, and (b) is due to the optimality of $x^{(E, t)}_t$. Then, we have $\frac{\hat{\tau}^{\ast}_{\text{avg}-ct} - \hat{\tau}(x^{(E, t)}_t, t)}{\hat{\tau}^{\ast}_{\text{avg}-ct}} \leq \sum_{n=0}^{N-1} \frac{1}{N} \int_{T(n)}^{T(n+1)} \frac{dx}{x^{(E, t)}_t} \left[ \sup_{x^{(E, t)}_t} \frac{\partial}{\partial x} \frac{1}{x^{(E, t)}_t} \right] dx,$
\begin{align}
\frac{\hat{\tau}^{\ast}_{\text{avg}-ct} - \hat{\tau}(x^{(E, t)}_t, t)}{\hat{\tau}^{\ast}_{\text{avg}-ct}} &\leq \sum_{n=0}^{N-1} \frac{1}{N} \int_{T(n)}^{T(n+1)} \frac{dx}{x^{(E, t)}_t} \left[ \sup_{x^{(E, t)}_t} \frac{\partial}{\partial x} \frac{1}{x^{(E, t)}_t} \right] dx.
\end{align}

APPENDIX F
Proof of Lemma 3
For notation simplicity, denote $I(p, q) \triangleq \int_{0}^{\infty} e^{-s\log(x+\mu)} dx$, $p, q \in \mathbb{N}_+$. Before proving Lemma 3, we first show the following lemma.

Lemma 8: $I(p, q) = \sum_{n=0}^{q-1} (-1)^{n} \frac{e^{\mu p+q}}{p+q}$.

Proof: Denote $F(n) \triangleq e^{-\mu x} E_1(-\mu x)$, $n \in \mathbb{N}_+$, where $E_1(x)$ is the exponential integral. We prove Lemma 8 by induction on $q$ for any fixed $p \in \mathbb{N}_+$. For $q = 1$, we have $I(p, 1) = e^{-\mu} \int_{0}^{\infty} e^{-\mu x} dx = f(p)$, for $p \in \mathbb{N}_+$. Suppose that for a given $t \geq 2$, Lemma 8 holds for $q = t$. For $q = t + 1$, we have $I(p, t+1) = I(p, t) - I(p + 1, t) = \sum_{i=0}^{q-1} (-1)^{i+1} i! f(p - i) - \sum_{i=0}^{q-1} (-1)^{i+1} i! f(p + 1 + i) = \sum_{i=0}^{q-1} (-1)^{i+1} i! f(p + 1 + i)$, where (a) is due to $I(p, q) = I(p + 1, q) + I(p, q + 1)$, and (b) is due to the assumption. Thus, Lemma 8 also holds for $q = t + 1$. Therefore, by induction, we can show Lemma 8.

Now, we prove Lemma 3. We have $\frac{1}{N} \frac{\hat{\tau}^{\ast}_{\text{avg}-ct}}{\hat{\tau}(x^{(E, \text{opt})}_t, t)} \left[ \sup_{x^{(E, \text{opt})}_t} \frac{\partial}{\partial x} \hat{\tau}(x^{(E, \text{opt})}_t, t) \right] \leq \frac{\hat{\tau}^{\ast}_{\text{avg}-ct}}{\hat{\tau}(x^{(E, \text{opt})}_t, t)} \left[ \sup_{x^{(E, \text{opt})}_t} \frac{\partial}{\partial x} \hat{\tau}(x^{(E, \text{opt})}_t, t) \right].$

APPENDIX G
Proof of Theorem 5
First, by Cauchy-Schwarz inequality, we have $\mathbb{E}[T(n)] \leq \mathbb{E}[\frac{1}{T(n)}] \geq 1$, implying $\mathbb{E}[\frac{1}{T(n)}] \leq \mathbb{E}[\frac{1}{T(n)}].$
Then, we have
\[
\frac{\mathbb{E}[\tilde{r}(x^{(E,0)}, T)]}{\tau_{N}} \leq \frac{\mathbb{E}[\tilde{r}(x^{(E,0)}, T)]}{\mathbb{E}[r(x^{(E,0)}, T)_{k}]} \leq \frac{1}{\mathbb{E}[\tau_{N}]} + \frac{1}{\mathbb{E}[\tau_{N}]} + \frac{1}{\mathbb{E}[\tau_{N}]} \mathbb{E} \left[ \max_{n \in [N]} T_{n} \mathbb{E} \left[ \frac{1}{\tau_{n}} \right] \right] \leq \mathbb{E} \left[ \max_{n \in [N]} T_{n} \mathbb{E} \left[ \frac{1}{\tau_{n}} \right] \right] \leq \mathbb{E} \left[ \frac{1}{\tau_{N}} \right]
\]
where \(a\) is due to (45), \(b\) is due to the definition of \(\tilde{r}()\) in (6) and Theorem 4, \(c\) is due to (48) since \(\mathbb{E}[\tau_{n}] \leq \mathbb{E}[\tau_{N}]\). Therefore, we complete the proof of Theorem 5.

**APPENDIX I**

**PROOF OF THEOREM 6**

Based on Lemma 4, we first equivalently transform Problem 7 to the following problem.
\[
\max_{s} P(s, t)\quad \text{s.t. (1), (36).}
\]

Next, we equivalently transform the problem in (52) to Problem 8. Introduce auxiliary variables \(x \triangleq (x_{n})_{n=0,1,\ldots,N-1}\) with \(x_{n} = \sum_{l \in [L]} I(s_{l} = n) \in \mathbb{N},\ n = 0, 1, \ldots, N - 1,\) together with (36) implying \(s_{l} = \min \{i : \sum_{n=0}^{i} x_{n} \geq l,\ l \in [L]\). For notation simplicity, treat \(k_{-1} \triangleq 0\) for ease of illustration, and define \(x = (k_{l})_{l=0}^{N-1} = (\sum_{j=0}^{l} x_{j})_{l=0}^{N-1} \in \mathbb{N}^{N},\ n = 0, 1, \ldots, N - 1,\)
\(k_{N}(x) \triangleq k_{L} \in \mathbb{N},\ \mathcal{U}(x, k) \triangleq \{k \in \mathbb{N}^{N} : [k]_{1} = [k]_{L}\} \subseteq \mathbb{N}^{N+1}\). Recall \(\mathcal{K}(s, N, L) \triangleq \{k \in \mathbb{N}^{N+1} : \sum_{l=1}^{L} k_{l} \leq s_{l},\ l \in [L]\},\ \sum_{l=0}^{L} k_{l} = N\). Thus, we have
\[
\mathcal{K}(s, N, L) = \bigcup_{\mathcal{K} \subseteq \mathcal{N}} \left( \mathcal{U}(x_{0}, k_{-1} - 1) \times \mathcal{U}(x_{1}, k_{0}) \times \cdots \right) \times \mathcal{U}(x_{N-1}, k_{N-2} - 1) \times \mathcal{U}(1, k_{N-1}).
\]

Besides, we have
\[
\begin{align*}
P(s, t) &= \sum_{k \in \mathcal{K}(s, N, L)} \left( \sum_{k_{0}, k_{1}, \ldots, k_{L}} \prod_{l=0}^{L} (F_{l}(s, t) - F_{l+1}(s, t))^{k_{l}} \right) \\
&= \sum_{k \in \mathcal{K}(N)} \sum_{k_{0}, k_{1}, \ldots, k_{L}} \frac{1}{k_{0}! k_{1}! \cdots k_{L}!} \prod_{l=0}^{L} (F_{l}(s, t) - F_{l+1}(s, t))^{k_{l}} \\
&= \sum_{k \in \mathcal{K}(N)} \sum_{k_{0}, k_{1}, \ldots, k_{L}} \frac{1}{k_{0}! k_{1}! \cdots k_{L}!} \prod_{l=0}^{L} (F_{l}(s, t) - F_{l+1}(s, t))^{k_{l}} \\
&= \sum_{k \in \mathcal{K}(N)} \sum_{k_{0}, k_{1}, \ldots, k_{L}} \frac{1}{k_{0}! k_{1}! \cdots k_{L}!} \prod_{l=0}^{L} (F_{l}(s, t) - F_{l+1}(s, t))^{k_{l}}
\end{align*}
\]

Finally, we have
\[
\begin{align*}
P(\hat{s}, t) - P(s^{*}, t) &= \Pr[\tau(\hat{s}, T) \leq t] - \Pr[\tau(s^{*}, T) \leq t] \\
&\geq \Pr[\tau(T) \in \mathcal{K}(\hat{s}) \cup \mathcal{K}(s^{*})] \\
&\geq \Pr[\tau(T) \in \mathcal{K}(\hat{s})] \\
&= \sum_{k \in \mathcal{K}(N)} \sum_{k_{0}, k_{1}, \ldots, k_{L}} \frac{1}{k_{0}! k_{1}! \cdots k_{L}!} \prod_{l=0}^{L} (F_{l}(s, t) - F_{l+1}(s, t))^{k_{l}}
\end{align*}
\]

This indicates that \(s^{*}\) is not an optimal solution, which contradicts with the assumption. Therefore, we complete the proof of Lemma 4.
Appendix J
Proof of Lemma 5

First, we show

\[
f(m, n) \triangleq \sum_{i=0}^{m+1} \frac{i + 2}{n + i + 1} \left( 2n + i - 1 \right) - m + 2 \left( 2n + m + 1 \right)
\]

\[= 0, n \in \mathbb{N}_+, m \in \mathbb{N}, \tag{54}\]

by induction on \(m\) for any fixed \(n \in \mathbb{N}_+\). For \(m = 0\), \(f(m, n) = \frac{2}{n+1} + \frac{3}{n+2} - \frac{2}{n+2} = 0\) \((a)\), where \((a)\) is due to \(\binom{n}{0} = \frac{n!}{m!(n-m)!}\). Suppose that for a given \(t \geq 1\), \(f(m, n) = 0\) holds for \(m = t\). Then, for \(m = t + 1\), \(f(m, n) = \sum_{i=0}^{t+2} \frac{i + 2}{n + i + 1} \left( 2n + i - 1 \right) = \frac{t+2}{n+2} \left( 2n + t + 1 \right) + \frac{t+4}{n+3} \left( 2n + t + 1 \right) = 0\), \((b)\), where \((b)\) is due to the assumption, and \((c)\) is due to \(\binom{n}{m} = \frac{n!}{m!(n-m)!}\). Thus, \(f(m, n) = 0\) also holds for \(m = t + 1\).

Therefore, by induction, we can show (54).

Next, we show

\[
\hat{K}_m(N) = \frac{m + 2}{N + m + 1} \left( 2N + m - 1 \right), \quad m, n \in \mathbb{N}_+, \tag{55}\]

where \(\hat{K}_m(N) \triangleq \{ k \in \mathbb{N} : \sum_{i=0}^{m+1} \hat{k}_i \leq n + m, n \in \mathbb{N}_+ \}, \sum_{i=0}^{m+1} \hat{k}_i = N + m\), by induction on \(N\) for any fixed \(m \in \mathbb{N}_+\). For \(N = 1\), \(\hat{K}_m(N) = 1 = \frac{m + 2}{N + m + 1} \left( 2N + m - 1 \right)\). Suppose that for a given \(n \geq 1\), (55) holds for \(N = n + 1\). Then, for \(N = n + 1\), \(\hat{K}_m(n + 1) = \left( \frac{m + 2}{n + m + 1} \right) \sum_{i=0}^{m+1} \hat{k}_i \left( n + 1 \right) = \frac{m + 2}{n + m + 1} \left( 2N + m + 1 \right)\), \((d)\), where \((d)\) is due to \(\hat{K}_m(n + 1) = \sum_{i=0}^{m+1} \hat{k}_i \left( n + 1 \right)\), \((e)\) is due to the assumption, and \((f)\) is due to (54). Thus, (55) also holds for \(N = n + 1\). Therefore, by induction, we can show (55).

Finally, we have \(|\hat{K}_m(N)| = |\hat{K}_m(N)| \triangleq \frac{2}{N + 1} \left( 2N - 1 \right) = \prod_{n=2}^{\frac{n+n+1}{2}} \geq \prod_{n=2}^{2} = 2 \left( 2N - 1 \right), \quad where \((g)\) is due to (55).

Therefore, we complete the proof of Lemma 5.

Appendix K
Proof of Theorem 7

First, by \(\Pr[\xi = k] = \frac{1}{|\{\sigma\}|}\), we obtain (25) and (27). We have \(\Pr[\xi = k] = \frac{1}{|\{\sigma\}|} \sum_{n=0}^{N-n} \hat{k}_n (N-n=) = \frac{|\{\sigma\}|}{N-n} \hat{k}_n (N-n=) = \frac{2}{N-n} \left( 2N - 1 \right)\), \((a)\) is due to the definition of conditional probability, \((b)\) is due to \(\mathcal{K}(N) \cap \{ \xi \in \mathbb{N}^N : \hat{k}_0 = \hat{k}_1 = \ldots = \hat{k}_{n-1} = \hat{k}_n = 0 \} = \mathcal{K}(N) \cap \{ \xi \in \mathbb{N}^N : \hat{k}_0 = \hat{k}_1 = \ldots = \hat{k}_{n-1} = \hat{k}_n = 0 \} = \mathcal{K}(N) \cap \{ \xi \in \mathbb{N}^N : \hat{k}_0 = \hat{k}_1 = \ldots = \hat{k}_{n-1} = \hat{k}_n = 0 \} = \mathcal{K}(N) \cap \{ \xi \in \mathbb{N}^N : \hat{k}_0 = \hat{k}_1 = \ldots = \hat{k}_{n-1} = \hat{k}_n = 0 \}\), \((c)\) is due to (55). Next, by (25) and (27), we obtain \(\Pr[\xi = k] = \frac{1}{|\{\sigma\}|}\). We have

\[
\Pr[\xi = k] = \left( \prod_{n=0}^{N-n} \frac{1}{|\{\sigma\}|} \right) = \frac{1}{|\{\sigma\}|}, \quad where \((d)\) is due to the chain rule of probability and \((e)\) is due to (25), (27), and (55). Therefore, we complete the proof of Theorem 7.

Appendix L
Proof of Lemma 7

For notation simplicity, define \(\Omega(N) \triangleq \{ \hat{k} \in \mathbb{N}^N : \sum_{n=0}^{N-1} \hat{k}_n = N \} = \mathcal{T}_2(N) \triangleq \{ \hat{k} \in \mathbb{N}^N : \hat{k}_0 = 0, n \in \{0, 1, \ldots, N-2\} \}\). Recall \(\hat{K}(N)\) given in (20) and \(\mathcal{T}_1(N) \triangleq \{ \hat{k} \in \mathbb{N}^N : \hat{k}_0 = 0, n \in \{0, 1, \ldots, N-2\} \}\). Before proving Lemma 7, we first show the following lemma.

Lemma 9: The optimal value of \(\min_{\hat{k} \in \mathcal{T}_2(N)} f_{IP}(\hat{k}, x)\), where \(f_{IP}(\hat{k}, x) \triangleq \sum_{n \in \mathcal{T}_1(N)} \frac{k_n}{1+i(S+n)}, \quad and \mathcal{T}_2(N) \triangleq \frac{\Omega(N) \setminus \mathcal{T}_1(N)}{\setminus \mathcal{T}_1(N)}\), denoted by \(f_{IP}(x)\), is \(f_{IP}(x) = \min_{\hat{k} \in \mathcal{T}_2(N)} f_{IP}(\hat{k}, x)\).

Proof: For notation simplicity, let \(\mathcal{T}_2(N) = \{ \hat{k} \in \mathbb{N}^N : \hat{k}_0 = 0, n \in \mathcal{T}_1(N) \}\), \(\mathcal{S}_N \triangleq \{ \hat{k} \in \mathcal{T}_2(N) : \hat{k}_0 = 0, n \in \mathcal{T}_1(N) \}\), \(\mathcal{S}_N = \{ \hat{k} \in \mathcal{T}_2(N) : \hat{k}_0 = 0, n \in \mathcal{T}_1(N) \}\), and \(\mathcal{S}_N \setminus \mathcal{T}_1(N) = \{ \hat{k} \in \mathcal{T}_2(N) : \hat{k}_0 = 0, n \in \mathcal{T}_1(N) \}\). We decompose the problem in Lemma 9 (i.e., \(\min_{\hat{k} \in \mathcal{T}_2(N)} f_{IP}(\hat{k}, x)\)) into \(\mathcal{T}_1(N)\) subproblems, i.e., \(f_{IP}(x) = \min_{\hat{k} \in \mathcal{T}_2(N)} f_{IP}(\hat{k}, x)\), where \(f_{IP}(\hat{k}, x) \triangleq \min_{\hat{k} \in \mathcal{T}_2(N)} f_{IP}(\hat{k}, x), m \in \{\mathcal{S}_N\}\). Then, we analyze \(f_{IP}(x) \triangleq \min_{\hat{k} \in \mathcal{T}_2(N)} f_{IP}(\hat{k}, x)\).
(i) We analyze $f_1^*$:

$$f_1^* = \min_{\mathbf{k} \in T(N)} \min_{\mathbf{N}} f_{IP}(\mathbf{k}, \mathbf{x}) = \min_{\mathbf{N}} \min_{\mathbf{k} \in T(N)} f_{IP}(\mathbf{k}, \mathbf{x})$$

$$\overset{(a)}{=} \min_{\mathbf{k} \in T(N)} \min_{\mathbf{N}} f_{IP}(\mathbf{k}, \mathbf{x}) = \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i$$

$$\overset{(b)}{=} \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i \overset{(c)}{=} \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i$$

$$\overset{(d)}{=} \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i \overset{(e)}{=} \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i$$

$$\overset{(f)}{=} \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i$$

$$\overset{(g)}{=} \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i$$

$$\overset{(h)}{=} \min_{\mathbf{N} \in \mathcal{N}(x)} \sum_{i=0}^{n-1}(i+1)x_i$$

where $\mathcal{N} \triangleq \{a_1, \ldots, a_m\}$ with $0 \leq a_1 < a_2 < \cdots < a_m \leq N - 2$.

For $k \in \mathcal{N}(x)$, we have $\hat{K}_n \in \mathcal{N}(x)$.

$\mathcal{N}(x)$ is due to $
\mathcal{N}_1(N) = \mathcal{N}_2(N) \cap \{k \in \mathcal{N}(x) : \hat{k}_n \neq 0, \hat{k}_i = 0, i \in \{0, \ldots, N - 2\} \} \overset{(a)}{=} \mathcal{N}(x)$.

Therefore, we complete the proof of Lemma 7.

APPENDIX M

PROOF OF THEOREM 8

First, it is obvious that Problem 13 is equivalent to the following problem.

$$\min_{\mathbf{x}} \max_{n=0,1,\ldots,N-1} \sum_{i=0}^{n-1}(i+1)x_i \overset{(a)}{=} \min_{\mathbf{x}} \max_{n=0,1,\ldots,N-1} \sum_{i=0}^{n-1}(i+1)x_i$$

s.t. $(4), (9)$.

Then, by Theorem 2 and by letting $t_n = \frac{1}{N-n+1}$, $n \in [N]$, we obtain an optimal solution of the above problem given by (29).

Therefore, we complete the proof of Theorem 8.

APPENDIX N

ENCODING AND DECODING MATRIX FOR COMPUTING $\frac{\partial \hat{H}(\theta,D)}{\partial \theta}$

WHERE $l \in [L]$

Algorithm 3: Algorithm for Obtaining Encoding Matrix $B_l$

1: input: Number of workers $N$ and number of full stragglers $s_l$.
2: output: Encoding matrix $B_l \in \mathbb{R}^{N \times N}$.
3: \textbf{H} = randn($s_l, N$);
4: $H(\cdot,N) = -\text{sum}(H(\cdot, 1 : N - 1), 2)$;
5: $B_l = \text{zeros}(N)$;
6: \textbf{i} = 0 : 1 : N
7: \textbf{j} = mod(\textbf{i} - 1 : s_l + i - 1, N) + 1;
8: $B_l(i, j) = [1; -H(i, j(2 : s_l + 1)))H(i, j(1))];$
9: \textbf{end for}

Algorithm 4: Algorithm for Obtaining Decoding Matrix $A_l$

1: input: Number of workers $N$, number of full stragglers $s_l$, and encoding matrix $B_l$.
2: output: Decoding matrix $A_l \in \mathbb{R}^{(N \times N)}$.
3: \textbf{f} = nchoosek(N, s_l);
4: $A_l = \text{zeros}($\textbf{f}$, N)$;
5: \textit{for} $l = 1 : N$\textit{do}
6: $x = \text{zeros}(1, N)$;
7: $y = \text{ones}(1, k)B_l(1, i)$;
8: $x(i) = y$;
9: $A_l = [A_l; x]$;
10: \textbf{end for}
