GENERALIZED EQUIVARIANT MODEL STRUCTURES ON $\text{Cat}^I$

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ABSTRACT. Let $I$ be a small category, $\mathcal{C}$ be the category $\text{Cat}$, $\text{Ac}$ or $\text{Pos}$ of small categories, acyclic categories, or posets, respectively. Let $\emptyset$ be a locally small class of objects in $\text{Set}^I$ such that $\text{colim}_I O = \ast$ for every $O \in \emptyset$. We prove that $\mathcal{C}^I$ admits the $\emptyset$-equivariant model structure in the sense of Farjoun, and that it is Quillen equivalent to the $\emptyset$-equivariant model structure on $\text{sSet}^I$. This generalizes previous results of Bohmann-Mazur-Osorno-Ozornova-Ponto-Yarnall and of May-Stephan-Zakharevich when $I = G$ is a discrete group and $\emptyset$ is the set of orbits of $G$.

CONTENTS

1. Introduction 1
2. Acknowledgements 3
3. Preliminaries 3
3.1. The Thomason model structure 3
3.2. The $\emptyset$-equivariant model structure 4
3.3. Chorny’s theorem 5
4. Comparison between $\emptyset$-equivariant model structures 5
5. The $\emptyset$-equivariant model structure on $\text{Cat}^I$ 7
6. The $\emptyset$-equivariant model structures on $\text{Ac}^I$ and $\text{Pos}^I$ 12
7. Applications 13
7.1. Equivariant diagrams 13
7.2. The collection of all $I$-orbits 14
Appendix A. Properness 14
References 15

1. INTRODUCTION

In [Tho80], Thomason defined a model structure on $\text{Cat}$, the category of small categories, which is Quillen equivalent to the standard model structure on $\text{sSet}$. The Thomason model structure has been shown to transfer to other categories. Raptis [Rap10] showed that $\text{Pos}$, the category of posets, admits a model structure that is Quillen equivalent to the Thomason model structure on $\text{Cat}$. Recently Bruckner [Bru15] showed that $\text{Ac}$, the category of small acyclic categories, admits a model structure that is Quillen equivalent to the Thomason model structure on $\text{Cat}$. The following diagram shows the relevant Quillen equivalences.

\[
\begin{array}{ccc}
\text{sSet} & \overset{\text{Ex}^I N}{\underset{i_A}{\longrightarrow}} & \text{Cat} & \overset{p_A}{\longrightarrow} & \text{Ac} & \overset{p_P}{\longrightarrow} & \text{Pos}
\end{array}
\]
In the diagram, \( N : \text{Cat} \to \text{sSet} \) is the nerve functor, \( c \) is its left adjoint, \( \text{Sd} : \text{sSet} \to \text{sSet} \) is the barycentric subdivision, and \( \text{Ex} \) is its right adjoint. The functors \( i_A \) and \( i_P \) are the natural inclusions, and \( p_A \) and \( p_P \) respectively are their left adjoints.

Let \( G \) be a discrete group. The above Quillen equivalences have been generalized to \( G \)-spaces. In [BMO+13] Bohmann-Mazur-Osorno-Ozornova-Ponto-Yarnall showed that the Quillen equivalence between \( \text{sSet} \) and \( \text{Cat} \) can be lifted to a Quillen equivalence between \( \text{sSet}^G \) and \( \text{Cat}^G \), each equipped with the fixed point model structure. May-Stephan-Zakharevich [MSZ16] further showed that the Quillen equivalence between \( \text{Cat}^G \) and \( \text{Pos}^G \) can also be lifted. The following diagram shows the relevant Quillen equivalences.

\[
\begin{array}{cccc}
\text{sSet}^G & \xrightarrow{\text{Sd}^G} & \text{Cat}^G & \xrightarrow{\text{Ex}^G} \\
\xleftarrow{\text{N}} & & & \xleftarrow{i_A^G} \\
\xrightarrow{\text{p}_A^G} & & & \xrightarrow{\text{p}_P^G} \\
\end{array}
\]

Our main result is a generalization of results of [BMO+13] and of [MSZ16] to diagram categories indexed by arbitrary small categories. The \( G \)-fixed point model structures are replaced with the \( O \)-equivariant model structures, where a morphism \( X \to Y \) is a weak equivalence (resp. fibration) if and only if for every \( O \in \mathcal{O} \), the induced map \( \text{Hom}(O, X) \to \text{Hom}(O, Y) \) is a weak equivalence (resp. fibration) in \( \text{sSet} \). (The full definition is given in section 3.2). We also need to specify \( \text{sSet} \)-enriched structures on \( \text{Cat}^I \), \( \text{Ac}^I \) and \( \text{Pos}^I \), which are different from the usual \( \text{sSet} \)-enriched structures and are discussed later.

**Theorem 1.1.** Let \( I \) be a small category, \( \mathcal{C} \) be the category \( \text{Cat} \), \( \text{Ac} \) or \( \text{Pos} \). Let \( \mathcal{O} \) be a locally small class of diagrams in \( \text{Set}^I \) such that \( \text{colim}_I O = \ast \) for every \( O \in \mathcal{O} \). Then \( \mathcal{C}^I \) admits the \( \mathcal{O} \)-equivariant model structure, and this model structure is Quillen equivalent to \( \text{sSet}^I \) equipped with the \( \mathcal{O} \)-equivariant structure.

When \( I = G \) is a discrete group and \( \mathcal{O} = \mathcal{O}_G \) is the category of \( G \)-orbits, the theorem reduces to [BMO+13], Theorem A and B, and [MSZ16], Theorem 1.1 and 1.2.

The proofs of [BMO+13] and [MSZ16] are based on a theorem of Stephan [Ste13], which says that for suitable categories \( \mathcal{C} \), the category \( \mathcal{C}^G \) admits the fixed point model structure, and this model structure is Quillen equivalent to \( \mathcal{C}^{\mathcal{O}_G} \) equipped with the projective model structure. This can be seen as a generalization of Elmendorf’s theorem [Elm83], which says that \( \text{sSet}^G \) equipped with the fixed point model structure is Quillen equivalent to \( \text{sSet}^{\mathcal{O}_G^G} \) equipped with the projective model structure.

In our proof of Theorem 1.1 we use a generalization of Elmendorf’s theorem in another direction. Dwyer and Kan [DK84] proved that, for a bicomplete \( \text{sSet} \)-enriched category \( M \) and a small full subcategory of orbits \( \mathcal{O} \subseteq M \) satisfying certain axioms, \( M \) admits the \( \mathcal{O} \)-equivariant model structure, and this model structure is Quillen equivalent to \( \text{sSet}^{\mathcal{O}_G} \) with the projective model structure.

Farjoun [Far87] generalized the \( \mathcal{O} \)-equivariant model structure to cases where \( \mathcal{O} \) can be a proper class rather than a set. Farjoun also applied this theory to \( M = \text{sSet}^I \), where \( I \) is a small category. In this case, Farjoun showed that \( \mathcal{O}_I \), the class of all diagrams whose colimits over \( I \) are points, is a collection of orbits.

However, it is not easy to prove an Elmendorf’s theorem when \( \mathcal{O} \) is a proper class, because in this case, \( \text{sSet}^{\mathcal{O}_G^G} \), “the category of functors from \( \mathcal{O}_G^G \) to \( \text{sSet} \),
is not well-defined. This issue is partially resolved by Chorny and Dwyer [CD09], and eventually resolved by Chorny [Cho14], by replacing \( \text{sSet}^{\text{op}} \) with \( \mathcal{P}(\mathcal{M}) \), the category of small functors from \( \mathcal{M}^{\text{op}} \) to \( \text{sSet} \), equipped with the \( \mathcal{O} \)-relative model structure (recalled in section 3.3). Chorny’s theorem says that \( \mathcal{M} \) equipped with the \( \mathcal{O} \)-equivariant model structure is Quillen equivalent to \( \mathcal{P}(\mathcal{M}) \) equipped with the \( \mathcal{O} \)-relative model structure. Chorny’s theorem is an essential ingredient of our proof of Theorem 1.1.

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3. Preliminaries

In this section we review the necessary definitions and results.

3.1. The Thomason model structure. Thomason [Tho80] defined a model structure on \( \text{Cat} \) in which a morphism \( f \) in \( \text{Cat} \) is a weak equivalence (resp. fibration) if and only if \( \text{Ex}^2Nf \) is a weak equivalence (resp. fibration) in \( \text{sSet} \) equipped with the standard model structure, where \( N : \text{Cat} \to \text{sSet} \) is the nerve functor, and \( \text{Ex} : \text{sSet} \to \text{sSet} \) is the right adjoint of the barycentric subdivision functor \( \text{Sd} : \text{sSet} \to \text{sSet} \).

Let \( I_{\text{sSet}} \) be the set of generating cofibrations \( \partial\Delta[n] \to \Delta[n] \), and \( J_{\text{sSet}} \) be the set of generating trivial cofibrations \( \Lambda^k[n] \to \Delta[n] \) for the standard model structure on \( \text{sSet} \). Then \( I_{\text{Cat}} = c\text{Sd}^2I_{\text{sSet}} \) and \( J_{\text{Cat}} = c\text{Sd}^2J_{\text{sSet}} \) are sets of generating cofibrations and trivial cofibrations for the Thomason model structure on \( \text{Cat} \), where \( c : \text{sSet} \to \text{Cat} \) is the left adjoint to \( N \) and \( \text{Sd} : \text{sSet} \to \text{sSet} \) is the barycentric subdivision functor, which is left adjoint to \( \text{Ex} \).

The notion of Dwyer maps is important for the Thomason model structure.

**Definition 3.1** ([Tho80], Definition 4.1). Let \( i : A \to B \) be a monomorphism in \( \text{Cat} \).

We say \( i \) is a **sieve** if for every \( a \in \text{ob}(A) \) and morphism \( f : b \to i(a) \) in \( B \), there exists morphism \( f' : b' \to a \) in \( A \) such that \( i(f') = f \).

We say \( i \) is a **cosieve** if for every \( a \in \text{ob}(A) \) and morphism \( g : i(a) \to b \) in \( B \), there exists morphism \( f' : a \to b' \) in \( A \) such that \( i(f') = f \).

We say \( i \) is a **Dwyer map** if it is a sieve and factorizes as \( A \xrightarrow{i} W \xrightarrow{j} B \) such that \( f \) is a monomorphism, \( j : W \to B \) is a cosieve and there is a right adjoint \( r : W \to A \) to \( f \).

The category \( \text{Ac} \) of small acyclic categories and the category \( \text{Pos} \) of posets are reflective subcategories of \( \text{Cat} \). Both of them admits the Thomason model structure, by Raptis [Rap10] for \( \text{Pos} \) and Bruckner [Bru15] for \( \text{Ac} \). In the Thomason model structures on \( \text{Ac} \) and \( \text{Pos} \), a morphism is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) as a morphism in \( \text{Cat} \). All morphisms in \( I_{\text{Cat}} \) and \( J_{\text{Cat}} \) are Dwyer maps between posets. For \( I = \text{Ac} \) and \( \text{Pos} \), write \( I_{\text{e}} = I_{\text{Cat}} \) and \( J_{\text{e}} = J_{\text{Cat}} \). Then \( I_{\text{e}} \) is the set of generating cofibrations...
and $\mathcal{C}_{\mathcal{C}}$ is the set of generating trivial cofibrations for the Thomason model structure on $\mathcal{C}$.

### 3.2. The $\mathcal{O}$-equivariant model structure.

Let $\mathcal{M}$ be a bicomplete $\text{sSet}$-enriched category. Here “bicomplete” means that the underlying category of $\mathcal{M}$ is bicomplete, and $\mathcal{M}$ is powered and copowered (i.e. cotensored and tensored) over $\text{sSet}$. Let $\otimes$ denote the copower structure on $\mathcal{M}$. Let $\text{Hom}(-, -)$ denote the $\text{sSet}$-enriched hom. Let $\mathcal{O}$ be a class of objects of $\mathcal{M}$.

**Definition 3.2.** We say $\mathcal{M}$ admits the $\mathcal{O}$-equivariant model structure if there is a model structure on $\mathcal{M}$ such that a morphism $X \to Y$ in $\mathcal{M}$ is a weak equivalence (resp. fibration) if and only if for every $O \in \mathcal{O}$, the induced map $\text{Hom}(O, X) \to \text{Hom}(O, Y)$ is a weak equivalence (resp. fibration) in $\text{sSet}$ with the standard model structure.

**Definition 3.3** ([Far87], Definition 1.1). A class $\mathcal{O}$ of objects of $\mathcal{M}$ is called locally small if for every object $M \in \mathcal{M}$ there exists a set of objects $\mathcal{O}' \subseteq \mathcal{O}$ such that every morphism from an object in $\mathcal{O}$ to $M$ factors through an object in $\mathcal{O}'$.

**Remark 3.4.** In particular, if $\mathcal{O}$ is a set, then $\mathcal{O}$ is locally small.

Dwyer and Kan [DK84] defined a collection of orbits when $\mathcal{O}$ is a set. Farjoun [Far87] generalized the definition to the case when $\mathcal{O}$ can be a proper class.

**Definition 3.5** ([DK84], [Far87]). A locally small class $\mathcal{O}$ of objects in $\mathcal{M}$ is called a collection of orbits, if the following axioms hold.

**Q1.** Let

\[
\begin{array}{ccc}
O \otimes K & \longrightarrow & X_a \\
\downarrow & & \downarrow \\
O \otimes L & \longrightarrow & X_{a+1}
\end{array}
\]

be a pushout in $\mathcal{M}$ where $O \in \mathcal{O}$ and $K \to L \in \text{I}_{\text{sSet}}$. Then for $O' \in \mathcal{O}$, the diagram

\[
\begin{array}{ccc}
\text{Hom}(O', O \otimes K) & \longrightarrow & \text{Hom}(O', X_a) \\
\downarrow & & \downarrow \\
\text{Hom}(O', O \otimes L) & \longrightarrow & \text{Hom}(O', X_{a+1})
\end{array}
\]

is a homotopy pushout in $\text{sSet}$.

**Q2.** Let $O \in \mathcal{O}$ and $X_0 \to \cdots \to X_a \to X_{a+1} \to \cdots$ be a continuous transfinite sequence in $\mathcal{M}$ where each map $X_a \to X_{a+1}$ is as in Q1. Then the natural map

\[\text{colim}_a \text{Hom}(O, X_a) \to \text{Hom}(O, \text{colim}_a X_a)\]

is a weak equivalence in $\text{sSet}$.

**Q3.** There exists a limit ordinal $c$ such that if the sequence in Q2 is indexed by ordinals $< c$, then the natural map is an isomorphism.

The following proposition is proved by Dwyer and Kan [DK84] when $\mathcal{O}$ is a set, and by Farjoun [Far87] when $\mathcal{O}$ can be a proper class.

**Proposition 3.6** ([Far87], Proposition 1.3). Let $\mathcal{O} \subseteq \mathcal{M}$ be a collection of orbits. Then $\mathcal{M}$ admits an $\mathcal{O}$-equivariant model structure.
3.3. Chorny’s theorem. Let $\mathcal{M}$ be a bicomplete $s\text{Set}$-enriched category and $\mathcal{O}$ be a collection of orbits in $\mathcal{M}$. Chorny’s theorem compares the $\mathcal{O}$-equivariant model structure on $\mathcal{M}$ with the $\mathcal{O}$-relative model structure on $\mathcal{P}(\mathcal{M})$, and is used in our proof of Theorem 1.1.

Recall the definition of $\mathcal{P}(\mathcal{M})$.

**Definition 3.7.** A functor $\mathcal{M}^{\text{op}} \to s\text{Set}$ is *small* if it is the left Kan extension of a functor $\mathcal{J}^{\text{op}} \to s\text{Set}$, where $\mathcal{J}$ is a small full subcategory of $\mathcal{M}$. We denote by $\mathcal{P}(\mathcal{M})$ the category of small functors from $\mathcal{M}^{\text{op}} \to s\text{Set}$.

As stated in [DL07], small functors can be alternatively characterized as small weighted colimits of representable functors.

Chorny [Cho14] defined the $\mathcal{O}$-relative model structure on $\mathcal{P}(\mathcal{M})$, denoted $\mathcal{P}(\mathcal{M}, \mathcal{O})$.

A morphism $X \to Y$ in $\mathcal{P}(\mathcal{M}, \mathcal{O})$ is a weak equivalence (resp. fibration) if for every $O \in \mathcal{O}$, the induced map $X(O) \to Y(O)$ is a weak equivalence (resp. fibration) in $s\text{Set}$ with the standard model structure. As discussed in [Cho14], Proposition 2.8, the $\mathcal{O}$-relative model structure on $\mathcal{P}(\mathcal{M})$ is the same as the $\{\text{Hom}(\quad, O) : O \in \mathcal{O}\}$-equivariant model structure on $\mathcal{P}(\mathcal{M})$.

Now we can state Chorny’s theorem.

**Theorem 3.8 (Cho14, Theorem 3.1).** Let $\mathcal{M}$ be a bicomplete $s\text{Set}$-enriched category and $\mathcal{O}$ be a collection of orbits in $\mathcal{M}$. Then there is a Quillen equivalence

$$Z : \mathcal{P}(\mathcal{M}, \mathcal{O}) \rightleftarrows \mathcal{M} : Y$$

where $Y$ sends an object $M \in \mathcal{M}$ to the functor $\text{Hom}(M, \quad)$ corepresented by $M$, and $Z = \text{Id}_\mathcal{M} \otimes \mathcal{M}$ is the coend with the identity functor.

4. Comparison between $\mathcal{O}$-equivariant model structures

In this section, we prove a comparison result between $\mathcal{O}$-equivariant model structures on different $s\text{Set}$-enriched categories.

**Theorem 4.1.** Let $\mathcal{M}$, $\mathcal{N}$ be two simplicial categories with an adjunction

$$L : \mathcal{M} \rightleftarrows \mathcal{N} : R.$$

Let $\mathcal{O}_\mathcal{M}$ be a collection of orbits in $\mathcal{M}$ and $\mathcal{O}_\mathcal{N}$ be a collections of orbits in $\mathcal{N}$, in the sense of Definition 3.5. Assume that $L\mathcal{O}_\mathcal{M} \subseteq \mathcal{O}_\mathcal{N}$ and $R\mathcal{O}_\mathcal{N} \subseteq \mathcal{O}_\mathcal{M}$, and that $(L, R)$ restricts to an equivalence of categories between $\mathcal{O}_\mathcal{M}$ and $\mathcal{O}_\mathcal{N}$. Then $(L, R)$ defines a Quillen equivalence between $\mathcal{M}$ with the $\mathcal{O}_\mathcal{M}$-equivariant model structure and $\mathcal{N}$ with the $\mathcal{O}_\mathcal{N}$-equivariant model structure.

We prove the theorem in several steps. Recall that $\mathcal{P}(\mathcal{M})$ is the category of small functors from $\mathcal{M}^{\text{op}}$ to $s\text{Set}$.

**Lemma 4.2.** In the setting of Theorem 4.1, we have an adjunction

$$R^* : \mathcal{P}(\mathcal{M}) \rightleftarrows \mathcal{P}(\mathcal{N}) : L^*.$$

where $L^*$ and $R^*$ are restrictions along $L$ and $R$, respectively.

**Proof.** It is not even obvious that $L^*$ and $R^*$ are well-defined functors. By definition, every object in $\mathcal{P}(\mathcal{M})$ is the left Kan extension along the inclusion functor of a small full subcategory of $\mathcal{M}$. So we have a well-defined pushforward functor $L_* : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ given by left Kan extension along $L$. We would like to apply [DL07], Proposition 3.3 to show that $L^*$ is well defined and is the right adjoint of
Let $L$. So we need to verify the condition that $N(L-, X) : M^{op} \to \mathcal{V}$ is small for every object $X \in N$. In fact, since $L$ is a left adjoint, $N(L-, X) \simeq M(-, RX)$ is a representable functor, and is small.

Now we have an adjunction

$$L : \mathcal{P}(M) \rightleftarrows \mathcal{P}(N) : L^*.$$  

We would like to prove that $L$ and $R^*$ are naturally equivalent. To show this, consider an object $Lan_i F$ in $\mathcal{P}(M)$, where $i : \beta \to M$ is the inclusion of a small full subcategory, and $F \in \mathcal{P}(\beta)$. We have

$$L_i Lan_i F = \text{colim}_{\lambda \in \beta} F(\lambda) \simeq \text{colim}_{\alpha \in M} F(\alpha) = R^* Lan_i F.$$  

Recall that the model category $\mathcal{P}(M, \mathcal{O}_M)$ is the category $\mathcal{P}(M)$ equipped with the $\mathcal{O}_M$-relative model structure where a morphism is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) evaluated at every object in $\mathcal{O}_M$.

**Lemma 4.3.** In the setting of Theorem 4.1, we have a Quillen equivalence

$$R^* : \mathcal{P}(M, \mathcal{O}_M) \rightleftarrows \mathcal{P}(N, \mathcal{O}_N) : L^*.$$  

**Proof.** By Lemma 4.2, $(R^*, L^*)$ is a pair of adjoint functors.

We first prove that $(R^*, L^*)$ is a Quillen adjunction, i.e.
$L^*$ preserves fibrations and trivial fibrations. A morphism $\eta$ in $\mathcal{P}(\mathcal{N}, \mathcal{O}_N)$ is a fibration (resp. trivial fibration) if and only if $\eta(O)$ is a fibration (resp. trivial fibration) for every $O \in \mathcal{O}_N$. Its restriction $L^*\eta$ in $\mathcal{P}(\mathcal{M}, \mathcal{O}_M)$ is a fibration (resp. trivial fibration) if and only if $L^*\eta(O') = \eta(LO')$ is a fibration (resp. trivial fibration) for every $O' \in \mathcal{O}_M$. We have $LO_M \subseteq \mathcal{O}_N$, so if $\eta$ is a fibration (resp. trivial fibration), then $L^*\eta$ is a fibration (resp. trivial fibration).

Then we prove that $(R^*, L^*)$ is a Quillen equivalence, i.e. when $M \in \mathcal{P}(M, \mathcal{O}_M)$ is a cofibrant object and $N \in \mathcal{P}(N, \mathcal{O}_N)$ is a fibrant object, a morphism $M \to L^*N$ is a weak equivalence if and only if its adjoint $R^*M \to N$ is a weak equivalence. The morphism $M \to L^*N$ is a weak equivalence if and only if $M(O) \to L^*N(O) = N(LO)$ is a weak equivalence for every object $O \in \mathcal{O}_M$. On the other hand, $R^*M \to N$ is a weak equivalence if and only if $M(RO') \simeq R^*M(RO') \to N(O')$ is a weak equivalence for every object $O' \in \mathcal{O}_N$. Therefore if for every $O \in \mathcal{O}_M$ we can find $O' \in \mathcal{O}_N$ such that $RO' \simeq O$, and for every $O' \in \mathcal{O}_N$ we can find $O \in \mathcal{O}_M$ such that $LO \simeq O'$, the above two conditions coincide. This is the case because $(L, R)$ defines an equivalence of categories between $\mathcal{O}_M$ and $\mathcal{O}_N$.  

**Proof of Theorem 4.1.** By Chorny’s theorem, we have the following diagram.

```
M --------> N
|                 |                 |
| L \(\rightarrow\) \(\leftarrow\) R |
|                 |                 |
Z \(\rightarrow\) Y \(\leftarrow\) Z

\mathcal{P}(M, \mathcal{O}_M) --------> \mathcal{P}(N, \mathcal{O}_N)
```

The vertical $(Z, Y)$ pairs are Quillen equivalences by Theorem 3.8, and the pair $(R^*, L^*)$ is a Quillen equivalence by Lemma 4.3. To show that $(L, R)$ is a Quillen equivalence, we apply the 2-out-of-3 property of Quillen equivalences. So we need to show that $LZ \simeq ZR^*$, and $YR \simeq L^*Y$.  

□
Consider any object \( M \in P(M, \Omega_M) \). To show that \( LZM \simeq ZR^*M \), we only need to show that they represent the same functor in \( N \). For any \( n \in N \), we have

\[
N(LZM, n) \simeq M(ZM, Rn) \simeq P(M, \Omega_M)(M, YRn)
\]

and

\[
N(ZR^*M, n) \simeq P(N, \Omega_N)(R^*M, Yn) \simeq P(M, \Omega_M)(M, L^*Yn).
\]

It remains to prove that \( YR \simeq L^*Y \). For any \( m \in M \), we have

\[
(YRn)(m) = M(m, Rn) \simeq N(Lm, n) = (L^*Yn)(m).
\]

\[\square\]

5. The \( \mathcal{O} \)-equivariant model structure on \( \text{Cat}^I \)

In this section, we prove Theorem 1.1 for \( \text{Cat}^I \).

Let \( I \) be a small category and \( \mathcal{O} \) be a locally small class of objects in \( \text{Set}^I \) such that \( \text{colim}_I O = * \) for every \( O \in \mathcal{O} \). Clearly \( \text{Set}^I \) embeds naturally in \( \text{sSet}^I, \text{Cat}^I, \text{Ac}^I \) and \( \text{Pos}^I \). Farjoun [Far87] proved that \( \mathcal{O} \) is a collection of orbits in \( \text{sSet}^I \). Therefore \( \text{sSet}^I \) admits the \( \mathcal{O} \)-equivariant model structure.

In order to construct the \( \mathcal{O} \)-equivariant model structure on \( \text{Cat}^I \), we need \( \text{Cat}^I \) to be \( \text{sSet} \)-enriched. We know that \( \text{Cat} \) is Cartesian closed, i.e. enriched over itself. Let \( \mathcal{H}om \) denote the internal hom of \( \text{Cat} \). The nerve functor \( N : \text{Cat} \to \text{sSet} \) gives the usual \( \text{sSet} \)-enriched structure on \( \text{Cat} \), but it is not suitable for our purpose. We consider the \( \text{sSet} \)-enriched on \( \text{Cat} \) given by the strict monoidal functor \( \text{Ex}^2N : \text{Cat} \to \text{sSet} \). Let \( \text{Hom} \) denote \( \text{Ex}^2N\mathcal{H}om \).

Now we show that \( \text{Cat} \) is a bicomplete \( \text{sSet} \)-enriched category. It is well-known that underlying category of \( \text{Cat} \) is bicomplete, so we only need to show that the \( \text{sSet} \)-enriched structure is powered and copowered. For \( C \in \mathcal{C} \) and \( X \in \text{sSet} \), the copower \( C \otimes X \) is \( C \times \text{Sd}^2X \) and the power \( [X, C] \) is \( \mathcal{H}om(\text{Sd}^2X, C) \). The correctness is easily verified. Therefore \( \text{Cat} \) is a bicmplete \( \text{sSet} \)-enriched category.

Slightly abusing notation, we use \( \mathcal{H}om \) (resp. \( \text{Hom} \)) to denote the hom functor in \( \text{Cat}^I \) induced from \( \mathcal{H}om \) (resp. \( \text{Hom} \)) in \( \text{Cat} \).

**Theorem 5.1** (Theorem 1.1 for \( \text{Cat}^I \)). Let \( I \) be a small category and \( \mathcal{O} \) be a locally small class of objects in \( \text{Set}^I \) such that \( \text{colim}_I O = * \) for every \( O \in \mathcal{O} \). Then \( \text{Cat}^I \) admits the \( \mathcal{O} \)-equivariant model structure, and there is a Quillen equivalence

\[
c\text{Sd}^2 : \text{sSet}^I \rightleftarrows \text{Cat}^I : \text{Ex}^2N
\]

where both sides are equipped with the \( \mathcal{O} \)-equivariant model structures.

We prove that \( \mathcal{O} \) is a collection of orbits in \( \text{Cat}^I \) by first proving analogous orbit axioms Q1-Q3 with \( \mathcal{H}om \) replaced with \( \mathcal{H}om \).

**Proposition 5.2** (Analogue of Q1). Let

\[
\begin{array}{ccc}
O \times K & \longrightarrow & X_a \\
\downarrow & & \downarrow \\
O \times L & \longrightarrow & X_{a+1}
\end{array}
\]
be a pushout in $\mathbf{Cat}^I$ where $O \in \mathcal{O}$ and $K \hookrightarrow L \in \mathcal{J}_{\mathbf{Cat}}$. Then
\[
\begin{array}{ccc}
\mathcal{H} \text{Hom}(O', O \times K) & \longrightarrow & \mathcal{H} \text{Hom}(O', O \times L) \\
\downarrow & & \downarrow \\
\mathcal{H} \text{Hom}(O', O \times K) & \longrightarrow & \mathcal{H} \text{Hom}(O', O \times L)
\end{array}
\]
is a pushout in $\mathbf{Cat}$ for $O' \in \mathcal{O}$.

The proof is divided into several steps.

**Lemma 5.3.** Let $K \to L$ be a Dwyer map between posets and $O \in \mathbf{Cat}^I$ be a diagram. Then the natural map $O \times K \to O \times L$ is pointwise a Dwyer map between posets.

**Proof.** Assume $i : K \to L$ factors as $K \xrightarrow{f} W \xrightarrow{j} L$, where $f$ is a monomorphism, $j$ is a cosieve, and $f$ admits a right adjoint $r$. We prove that $A \times i : A \times K \hookrightarrow A \times L$ is a Dwyer map between posets for any set $A$. It is clear that $A \times i$ is a sieve. Consider the sequence $A \times K \xrightarrow{A \times f} A \times W \xrightarrow{A \times j} A \times L$. Clearly $A \times j$ is a cosieve. It remains to prove that $A \times r : A \times W \to A \times K$ is the right adjoint of $A \times f$. For any $(a, x) \in A \times K$ and $(b, y) \in A \times W$, we have
\[
\begin{align*}
(A \times W)((A \times f)(a, x), (b, y)) &= (A \times W)((a, fx), (b, y)) \\
&= \begin{cases} 
W(fx, y) & \text{if } a = b, \\
\emptyset & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
K(x, ry) & \text{if } a = b, \\
\emptyset & \text{otherwise}
\end{cases} \\
&= (A \times K)((a, x), (b, ry)) \\
&= (A \times K)((a, x), (A \times r)(b, y)).
\end{align*}
\]
\[\square\]

**Lemma 5.4.** Let $O \in \mathcal{O}$, and $D$ be a diagram in $\mathbf{Cat}^I$. Let $K$ be a poset. Then $\mathcal{H} \text{Hom}(O, D \times K) = \mathcal{H} \text{Hom}(O, D) \times K$.

**Proof.** We have a natural monomorphism $\mathcal{H} \text{Hom}(O, D) \times K \to \mathcal{H} \text{Hom}(O, D \times K)$ where a pair $(f_0, k) \in \mathcal{H} \text{Hom}(O, D) \times K$ is sent to the morphism $x \mapsto (f_0(x), k)$. Let $f \in \mathcal{H} \text{Hom}(O, D \times K)$. We prove that for all $i \in I$ and $a \in O(i)$, the second components of $f(a) \in D(i) \times K$ are the same. If this holds, then $f$ is in the image of the natural monomorphism and the lemma follows.

Consider a morphism $g : i \to j$ in $I$. For all $x \in O(i)$, we have $f(x) \in D(i) \times K$ and the second components of $f(g(x)) \in D(j) \times K$ are the same because the maps $D(i) \times K \to D(j) \times K$ in the diagram $D \times K$ preserve the second component.

By assumption, $\text{colim}_I O = *$. So any two elements in $\bigsqcup_{i \in I} O(i)$ are equivalent under the equivalence relation generated by $x \sim g(x)$ for every object $i, j \in I$, morphism $g : i \to j$, and $x \in O(i)$. Therefore the images of them under the map $f$ have the same second components. \[\square\]

**Lemma 5.5.** Let $K, L, O$ be as in Lemma 5.3 and $O' \in \mathcal{O}$. Then the natural map $\mathcal{H} \text{Hom}(O', O \times K) \to \mathcal{H} \text{Hom}(O', O \times L)$ is a Dwyer map between posets.
Proof. By Lemma 5.4, we only need to prove that $\mathcal{H}om(O', O) \times K \rightarrow \mathcal{H}om(O', O) \times L$ is a Dwyer map between posets. This follows from Lemma 5.3. □

Before proving Proposition 5.2, we need one lemma from [BMO+13], which gives an explicit description of pushouts in $\text{Cat}$ along Dwyer maps between posets.

**Lemma 5.6** ([BMO+13], Lemma 2.5). Let $i : A \to B$ be a Dwyer map between posets with cosieve $W$ and retraction $r$, and $F : A \to C$ be a functor. Let $D$ be the pushout of $i$ and $F$. Then $\text{ob}(D) = \text{ob}(C) \sqcup (\text{ob}(B) \setminus \text{ob}(A))$ and for $d, d' \in \text{ob}(D)$,

$$D(d, d') = \begin{cases} B(d, d') & \text{if } d, d' \in \text{ob}(B) \setminus \text{ob}(A), \\ C(d, d') & \text{if } d, d' \in \text{ob}(C), \\ \emptyset & \text{if } d \in \text{ob}(B) \setminus \text{ob}(A) \text{ and } d' \in \text{ob}(C), \\ C(d, F(r(d'))) & \text{if } d \in \text{ob}(C) \text{ and } d' \in \text{ob}(B) \setminus \text{ob}(A). \end{cases}$$

**Proof of Proposition 5.2.** Let $D$ be the following pushout.

$$\xymatrix{ \mathcal{H}om(O', O \times K) \ar[d] \ar[r] & \mathcal{H}om(O', X_a) \ar[d] \\ \mathcal{H}om(O', O \times L) \ar[r] & D }$$

We prove that $D = \mathcal{H}om(O', X_{a+1})$. By Lemma 5.6,

$$\text{ob}(D) = \text{ob}(\mathcal{H}om(O', X_a)) \sqcup (\text{ob}(\mathcal{H}om(O', O \times L)) \setminus \text{ob}(\mathcal{H}om(O', O \times K))).$$

By Lemma 5.4, we have $\mathcal{H}om(O', O \times L) = \mathcal{H}om(O', O) \times L$ and $\mathcal{H}om(O', O \times K) = \mathcal{H}om(O', O) \times K$. So

$$\text{ob}(D) = \text{ob}(\mathcal{H}om(O', X_a)) \sqcup \text{ob}(\mathcal{H}om(O', O) \times (L \setminus K)).$$

Now let us consider $X_{a+1}$. By Lemma 5.6, for each $i$,

$$\text{ob}(X_{a+1}(i)) = \text{ob}(X_a(i)) \sqcup \text{ob}(O(i) \times (L \setminus K)).$$

Clearly there is a monomorphism $D \to \mathcal{H}om(O', X_{a+1})$. We prove that this is an isomorphism.

Let $f \in \mathcal{H}om(O', X_{a+1})$. For any objects $i, j \in I$, morphism $g : i \to j$, and $x \in O'(i)$, we have $f(x) \in X_a(i)$ if and only if $f(g(x)) \in X_a(j)$. By assumption, $\text{colim}_I O' = *$. So either $f(x) \in X_a(i)$ for all $i \in I$ and $x \in O'(i)$, or $f(g(x)) \in O(i) \times (L \setminus K)$ for all such $i$ and $x$. So

$$\text{ob}(\mathcal{H}om(O', X_{a+1})) = \text{ob}(\mathcal{H}om(O', X_a)) \sqcup \text{ob}(\mathcal{H}om(O', O) \times (L \setminus K)) = \text{ob}(D).$$

So the natural map $D \to \mathcal{H}om(O', X_{a+1})$ is an isomorphism on objects. It is easy to see that the map is also an isomorphism on morphisms. □

**Proposition 5.7** (Analogue of Q2 and Q3). Let $O \in 0$ and $X_1 \to \cdots \to X_a \to X_{a+1} \to \cdots$ be a continuous transfinite sequence in $\text{Cat}^I$ where each map $X_a \to X_{a+1}$ is as in Proposition 5.2. The natural map

$$\text{colim}_a \mathcal{H}om(O, X_a) \to \mathcal{H}om(O, \text{colim}_a X_a)$$

is an isomorphism.
Proof. By Lemma 5.6, each $X_a \to X_{a+1}$ is a monomorphism. So colim$_a X_a$ can be understood as an infinite union of $X_a$, and colim$_a Hom(O, X_a)$ can be understood as an infinite union of $Hom(O, X_a)$.

Let $f \in Hom(O, \text{colim}_a X_a)$. We prove that there exists some index $b$ such that $f$ factors as $O \to X_b \to \text{colim}_a X_a$. If this holds then $f \in \text{colim}_a Hom(O, X_a)$ and the proposition follows.

Consider any $i \in I$, an index $b$ and an object $x \in X_b(i) \subseteq \text{colim}_a X_a(i)$. By the explicit description of $X_a \to X_{a+1}$, we see that

1. If $g : i \to j$ is a map in $I$, then $f(g(x)) \in X_b(j)$.
2. If $g : j \to i$ is a map in $I$ and $y \in \text{colim}_a X_a(j)$ is an object such that $g(y) = x$, then $f(y) \in X_b(j)$.

Therefore for any objects $i, j \in I$, morphism $g : i \to j$ and $x \in O(i)$, we have that $f(x) \in X_b(i)$ if and only if $f(g(x)) \in X_b(j)$. By assumption, colim$_I O = \ast$. So there exists some index $b$ such that for all $i \in I$ and $x \in O(i)$, we have $f(x) \in X_b(i)$. □

Now we transfer our analogous orbit axioms (Proposition 5.2 and 5.7) to the actual orbit axioms.

**Proposition 5.8 (Q1).** In the setting of Theorem 5.1, the class $\emptyset \subseteq \text{Cat}'$ satisfies Q1.

**Proof.** Let $O, O' \in \emptyset$ and $K \hookrightarrow L \in \mathcal{J}_{\text{Set}}$. Let $K' \hookrightarrow L' = cSd^2(K \hookrightarrow L) \in \mathcal{J}_{\text{Cat}}$. By definition, $O \otimes K = O \times K'$ and $O \otimes L = O \times L'$. Let

$$
\begin{array}{ccc}
O \otimes K & \longrightarrow & X_a \\
\uparrow & & \uparrow \\
O \otimes L & \longrightarrow & X_{a+1}
\end{array}
$$

be a pushout in $\text{Cat}'$. By Proposition 5.2,

$$
\begin{array}{ccc}
Hom(O', O \otimes K) & \longrightarrow & Hom(O', X_a) \\
\uparrow & & \uparrow \\
Hom(O', O \otimes L) & \longrightarrow & Hom(O', X_{a+1})
\end{array}
$$

is a pushout in $\text{Cat}$.

In fact, by the proof of Proposition 5.2, this is a pushout along a Dwyer map between posets. By [Tho80], Proposition 4.3, the natural map

$$
D = N\text{Hom}(O', X_a) \cup N\text{Hom}(O', O \otimes K) \to N\text{Hom}(O', X_{a+1})
$$

is a weak equivalence.
There is a natural transformation $\eta : \text{Id} \to \text{Ex}$ which is objectwise a weak equivalence ([GJ09], Theorem III.4.6). So we have a commutative cube

$$
\begin{array}{c}
\text{Hom}(O', O \otimes K) \\
\text{Hom}(O', O \otimes L)
\end{array}
\begin{array}{c}
\text{Hom}(O', X_a) \\
\text{Hom}(O', X_{a+1})
\end{array}
\begin{array}{c}
\text{NHom}(O', O \otimes K) \\
\text{NHom}(O', O \otimes L)
\end{array}
\begin{array}{c}
\text{NHom}(O', X_a) \\
D
\end{array}
$$

where the back square is a homotopy pushout (because it is a pushout and the map $\text{NHom}(O', O \otimes K) \to \text{NHom}(O', O \otimes L)$ is a cofibration), and the four arrows from the back square to the front square are weak equivalences. Hence the front square is a homotopy pushout.

**Proposition 5.9** (Q2 and Q3). In the setting of Theorem 5.1, the class $\mathcal{O} \subseteq \text{Cat}^I$ satisfies Q3, thus also satisfies Q2.

**Proof.** The nerve functor $N$ commutes with filtered colimits ([Lac]). So we only need to prove that the functor $\text{Ex}$ commutes with filtered colimits. This is true because $(\text{Ex}^-)_n$ is corepresented by $\text{Sd}\Delta^n$, which is a compact object.

**Proposition 5.10.** In the setting of Theorem 5.1, the class $\mathcal{O}$ is locally small in $\text{Cat}^I$.

**Proof.** The inclusion functor $\text{Set}^I \to \text{Cat}^I$ is the left adjoint of the forgetful functor $\text{Cat}^I \to \text{Set}^I$ that forgets the morphisms. So the propositions follows from that $\mathcal{O}$ is locally small in $\text{Set}^I$.

Now we can construct the $\mathcal{O}$-equivariant model structure on $\text{Cat}^I$.

**Proof of Theorem 5.1.** By Proposition 5.8, 5.9, and 5.10, the class $\mathcal{O}$ is locally small in $\text{Cat}^I$ and satisfies Q1-Q3, thus is a collection of orbits. The existence of the model structure follows from Farjoun’s Proposition 3.6.

The functors $c, \text{Sd}, \text{Ex}, N$ all preserve sets. So $c\text{Sd}^2$ and $\text{Ex}^2 N$ preserve sets, i.e. they restrict to equivalences of categories between $\mathcal{O}$ considered as a full subcategory of $\text{Cat}^I$ and $\mathcal{O}$ considered as a full subcategory of $\text{sSet}^I$. The Quillen equivalence follows from Theorem 4.1.

**Remark 5.11.** The $\mathcal{O}$-equivariant model structure on $\text{Cat}^I$ can also be described as follows. A morphism $X \to Y$ in $\text{Cat}^I$ is a weak equivalence (resp. fibration) if and only if for all $O \in \mathcal{O}$, $\text{Hom}(O, X) \to \text{Hom}(O, Y)$ is a weak equivalence (resp. fibration) in $\text{Cat}$ with the Thomason model structure.

Note, on the other hand, that our proof does not use the Thomason model structure directly. If we take $I = 1$ and $\mathcal{O} = \{\ast\}$ in Theorem 5.1, then the $\mathcal{O}$-equivariant model structure on $\text{Cat}^I = I$ reduces to the Thomason model structure.
6. The $\mathcal{O}$-equivariant model structures on $\text{Ac}^I$ and $\text{Pos}^I$

In this section we prove Theorem 1.1 for $\text{Ac}^I$ and $\text{Pos}^I$.

Recall that $\text{Ac}$ is the category of the small acyclic categories and $\text{Pos}$ is the category of posets. Let $\mathcal{C}$ denote the category $\text{Ac}$ or $\text{Pos}$. The category $\text{Pos}$ is a full subcategory of $\text{Ac}$, which is in turn a full subcategory of $\text{Cat}$. So we can see $\text{Ex}^2N$ as a functor from $\mathcal{C}$ to $\text{sSet}$, and define $\text{Hom}$ and $\text{Hom}$ in $\mathcal{C}$ using the corresponding hom functors in $\text{Cat}^I$. By [Tho80], Lemma 5.6, the functor $cSd^2$ takes values in posets. Therefore we can see $cSd^2$ as a functor from $\text{sSet}$ to $\mathcal{C}$.

The functor $\text{Hom}$ gives $\mathcal{C}$ an $\text{sSet}$-enriched structure. It is well known that the underlying category of $\mathcal{C}$ is bicomplete. The power and copower structure of $\mathcal{C}$ as an $\text{sSet}$-enriched category is similar to that of $\text{Cat}$. For $C \in \mathcal{C}$ and $X \in \text{sSet}$, the copower $C \otimes X$ is $C \times cSd^2X$ and the power $[X,C]$ is $\text{Hom}(cSd^2X,C)$. Therefore $\mathcal{C}$ is a bicomplete $\text{sSet}$-enriched category.

**Theorem 6.1** (Theorem 1.1 for $\text{Ac}^I$ and $\text{Pos}^I$). Let $\mathcal{C}$ be $\text{Ac}$ or $\text{Pos}$. Let $I$ be a small category and $\mathcal{O}$ be a locally small class of objects in $\text{Set}^I$ such that $\text{colim}_I O = *$ for every $O \in \mathcal{O}$. Then $\mathcal{C}^I$ admits the $\mathcal{O}$-equivariant model structure and there is a Quillen equivalence

$$cSd^2 : \text{sSet}^I \rightleftarrows \mathcal{C}^I : \text{Ex}^2N$$

where both sides are equipped with the $\mathcal{O}$-equivariant model structures.

We prove that $\mathcal{O}$ satisfies the orbit axioms Q1-Q3.

**Proposition 6.2** (Q1). In the setting of Theorem 6.1, the class $\mathcal{O} \subseteq \mathcal{C}^I$ satisfies Q1.

**Proof.** By [Tho80], Lemma 5.6, the inclusion $\text{Pos} \rightarrow \text{Cat}$ preserves pushouts along Dwyer maps between posets. By [Bru15], Proposition 4.5, the inclusion $\text{Ac} \rightarrow \text{Cat}$ preserves pushouts whose leg is a Dwyer map between posets. So the relevant pushouts in $\mathcal{C}$ can be performed in $\text{Cat}$. □

**Proposition 6.3** (Q2 and Q3). In the setting of Theorem 6.1, the class $\mathcal{O} \subseteq \mathcal{C}^I$ satisfies Q3, thus also satisfies Q2.

**Proof.** By [Tho80], Lemma 5.6, the inclusion $\text{Pos} \rightarrow \text{Cat}$ preserves filtered colimits. By [Bru15], Proposition 4.1, the inclusion $\text{Ac} \rightarrow \text{Cat}$ preserves filtered colimits. So the relevant colimits in $\mathcal{C}$ can be performed in $\text{Cat}$. □

**Proposition 6.4.** In the setting of Theorem 6.1, the class $\mathcal{O}$ is locally small in $\mathcal{C}^I$.

**Proof.** Same as the proof of Proposition 5.10. □

**Proof of Theorem 6.1.** By Proposition 6.2, 6.3, and 6.4, the class $\mathcal{O}$ is a collection of orbits in $\mathcal{C}^I$. The existence of the model structure follows from Proposition 3.6. The proof of the Quillen equivalence follows from Theorem 4.1. □

**Remark 6.5.** Similar to Remark 5.11, we can also describe the $\mathcal{O}$-equivariant model structure on $\mathcal{C}^I$ using the Thomason model structure. A morphism $X \rightarrow Y$ in $\mathcal{C}^I$ is a weak equivalence (resp. fibration) if and only if for all $O \in \mathcal{O}$, the induced map $\text{Hom}(O,X) \rightarrow \text{Hom}(O,Y)$ is a weak equivalence (resp. fibration) in $\mathcal{C}$ (or $\text{Cat}$) equipped with the Thomason model structure.
7. Applications

In this section we discuss some applications of our theorem.

7.1. Equivariant diagrams. Let $G$ be a discrete group acting on a small category $I$. Define $G \rtimes_a I$ to be the category whose objects are $\text{ob}(I)$, and a morphism $i \to j$ is a pair $(g, \alpha : gi \to j)$ where $g \in G$ and $\alpha \in \text{mor}(I)$. The category of $G$-diagrams in a category $\mathcal{C}$ is defined to be $\mathcal{C}^{G \rtimes_a I}$.

When $\mathcal{C}$ admits cellular fixed-point functors in the sense of Guillou and May [GM11], Dotto and Moi [DM16] defined a model structure on $\mathcal{C}^{G \rtimes_a I}$, called the “$G$-projective” model structure. For every $i \in I$, let $G_i \subseteq G$ be the group of stabilizers of $i$. This group acts on $i$ via the morphism $G_i \hookrightarrow G \rtimes_a I$ defined by $* \mapsto i$ and $g \mapsto (g, \text{id})$. A morphism $X \to Y$ in $\mathcal{C}^{G \rtimes a I}$ is a weak equivalence (resp. fibration), if and only if for every $i \in I$ and every subgroup $H$ of $G_i$, the induced map $X(i)^H \to Y(i)^H$ is a weak equivalence (resp. fibration) in $\mathcal{C}$.

We give an alternative proof of the existence of the model structure using orbits.

**Proposition 7.1.** Let $\mathcal{C}$ be $\text{sSet}$ equipped with the standard model structure, or $\text{Cat}$, $\text{Ac}$ or $\text{Pos}$ equipped with the Thomason model structure. Then $\mathcal{C}^{G \rtimes a I}$ admits the “$G$-projective” model structure. In fact, this model structure is the $\emptyset$-equivariant model structure for a certain set $\emptyset$ of objects in $\text{Set}^{G \rtimes a I}$ such that $\text{colim}_{G \rtimes a I} O = *$ for every $O \in \emptyset$.

**Proof.** The existence of the “$G$-projective” model structure follows from the second claim. So we only need to find an appropriate $\emptyset$.

Fix $k \in I$ and a subgroup $H \subseteq G_k$. Consider the diagram $O_{k,H}$ in $\text{Set}^{G \rtimes a I}$, where

1. an object $i$ is mapped to $(G \rtimes_a I)(k,i)/H$, where $h \in H$ acts on the morphism set $(G \rtimes_a I)(k,i)$ by sending a morphism $(g, \alpha : gk \to i)$ to $(gh, \alpha : ghk = gk \to i)$;
2. a morphism $(g, \alpha : gi \to j)$ is mapped to the map that sends the orbit of $(g', \alpha' : g'k \to i)$ to the orbit of $(gg', \alpha \circ (ga') : gg'k \to gi \to j)$.

In short, $O_{k,H} = (G \rtimes_a I)(k,-)/H$. We know that $\text{colim}_{G \rtimes a I}(G \rtimes a I)(k,-) = *$, so $\text{colim}_{G \rtimes a I} O_{k,H} = *$.

Let $\emptyset$ be the set of $O_{k,H}$, for all $k \in I$ and subgroup $H \subseteq G_k$. Then $\emptyset$ is locally small (because it is a set) and consists of diagrams whose colimits over $G \rtimes a I$ are one-element sets. So we have an $\emptyset$-equivariant model structure on $\mathcal{C}^I$ by Theorem 1.1.

To prove that the $\emptyset$-equivariant model structure is the “$G$-projective” model structure, we only need to show that $O_{k,H}$ corepresents the functor $(-)(k)^H$ in $\mathcal{C}^{G \rtimes a I}$. For any $X \in \mathcal{C}^{G \rtimes a I}$, a map from $(G \rtimes a I)(k,-)$ to $X$ is uniquely determined by the image of the identity in $(G \rtimes a I)(k,-)$ in $X(k)$, i.e. an object in $X(k)$. So a map from $(G \rtimes a I)(k,-)/H$ to $X$ is uniquely determined by an object in $X(k)$ that is invariant under $H$ action. This proves that the $\emptyset$-skeltons of $\text{Hom}((G \rtimes a I)(k,-), X)$ and $X(k)^H$ agree. The proof for higher skeletons is similar. $\square$

**Remark 7.2.** It is well known that $\text{sSet}$ equipped with the standard model structure is “nice”. It has been verified in [BMO+13] and in [MSZ16] that $\text{Cat}$ and $\text{Pos}$ equipped with the Thomason model structures are “nice”. So the “$G$-projective” model structure is known to exist in these cases.
Corollary 7.3. In the setting of Proposition 7.1, the “G-projective” model structures on $\text{sSet}^{G \times I}$, $\text{Cat}^{G \times I}$, $\text{Ac}^{G \times I}$ and $\text{Pos}^{G \times I}$ are Quillen equivalent.

Proof. It follows from Proposition 7.1 and Theorem 1.1. □

Remark 7.4. Take $\mathcal{C} = \text{Cat}$, and $I = 1$. Let $G$ act on $I$ trivially. Corollary 7.3 gives the main result of [BMO+13], which says that $\text{Cat}^G$ admits the fixed point model structure and this model structure is Quillen equivalent to $\text{sSet}^I$ equipped with the fixed point structure.

Remark 7.5. Take $\mathcal{C} = \text{Pos}$, and $I = 1$. Let $G$ act on $I$ trivially. Corollary 7.3 gives the main result of [MSZ16], which says that $\text{Pos}^G$ admits the fixed point model structure and this model structure is Quillen equivalent to $\text{sSet}^I$ equipped with the fixed point structure.

7.2. The collection of all $I$-orbits. Finally, we give an example where $\mathcal{O}$ is a proper class rather than a set.

Proposition 7.6. Let $I$ be a small category, $\mathcal{C}$ be the category $\text{Cat}$, $\text{Ac}$ or $\text{Pos}$, and $\mathcal{O}$ be the class of all diagrams in $\text{Set}^I$ whose colimits are one-element sets. Then $\mathcal{C}^I$ admits the $\mathcal{O}$-equivariant model structure, and this model structure is Quillen equivalent to $\text{sSet}^I$ equipped with the $\mathcal{O}$-equivariant model structure.

Proof. To apply Theorem 1.1, we only need to verify that $\mathcal{O}$ is locally small in $\text{Set}^I$. This is well-known by [Far87]. □

Appendix A. Properness

In this appendix we prove that the $\mathcal{O}$-equivariant model structures on $\text{sSet}^I$, $\text{Cat}^I$, $\text{Ac}^I$, and $\text{Pos}^I$ are proper.

Proposition A.1. Let $\mathcal{M}$ be a bicomplete $\text{sSet}$-enriched category and $\mathcal{O}$ be a collection of orbits in $\mathcal{M}$. Then the $\mathcal{O}$-equivariant model structure on $\mathcal{M}$ is right proper.

Proof. We prove that weak equivalences are preserved by pullbacks along fibrations. Let $D = B \times_A C$ be a pullback in $\mathcal{M}$ where the morphism $B \rightarrow A$ is a weak equivalence and the morphism $C \rightarrow A$ is a fibration. Then for every $O \in \mathcal{O}$, the induced map $\text{Hom}(O, B) \rightarrow \text{Hom}(O, A)$ is a weak equivalence and $\text{Hom}(O, C) \rightarrow \text{Hom}(O, A)$ is a fibration. The functor $\text{Hom}(O, -)$ preserves limits, so $\text{Hom}(O, D) = \text{Hom}(O, B) \times_{\text{Hom}(O, A)} \text{Hom}(O, C)$. Because $\text{sSet}$ with the standard model structure is right proper, the map $\text{Hom}(O, D) \rightarrow \text{Hom}(O, C)$ is a weak equivalence. Hence $D \rightarrow C$ is a weak equivalence. □

Proposition A.2. Let $I$ be a small category, $\mathcal{C}$ be the category $\text{sSet}$, $\text{Cat}$, $\text{Ac}$ or $\text{Pos}$. Let $\mathcal{O}$ be a locally small class of objects in $\text{Set}^I$ such that $\text{colim}_I O = *$ for every $O \in \mathcal{O}$. Then the $\mathcal{O}$-equivariant model structure on $\mathcal{C}^I$ is left proper.

Proof. When $\mathcal{C} = \text{sSet}$, we equip $\mathcal{C}$ with the standard model structure; in other cases, we equip $\mathcal{C}$ with the Thomason model structure. Then $\mathcal{C}$ is proper. (The case $\mathcal{C} = \text{sSet}$ is well-known; the case $\mathcal{C} = \text{Cat}$ is by [Cis99]; the case $\mathcal{C} = \text{Pos}$ is by [Rap10]; the case $\mathcal{C} = \text{Ac}$ is by [Bru15].)

Let $\text{Hom}$ denote the internal hom of $\mathcal{C}$. Then in the $\mathcal{O}$-equivariant model structure on $\mathcal{C}^I$, a map $X \rightarrow Y$ is a weak equivalence (resp. fibration) if and only if for every $O \in \mathcal{O}$, the induced map $\text{Hom}(O, X) \rightarrow \text{Hom}(O, Y)$ is a weak equivalence
(resp. fibration) in $\mathcal{C}$. (The case $\mathcal{C} = \mathbf{sSet}$ is by definition; the case $\mathcal{C} = \mathbf{Cat}$ is by Remark 5.11; the case $\mathcal{C} = \mathbf{Ac}$ or $\mathbf{Pos}$ is by Remark 6.5.)

Now we prove that for every $O \in \mathcal{O}$, the functor $\mathcal{H}om(O, -) : \mathcal{E}^I \to \mathcal{C}$ preserves cofibrations. Farjoun’s proof [Far87] of the existence of the $\mathcal{O}$-equivariant model structure says that the class $\mathcal{J}_\mathcal{C}^\mathcal{O} = \{O \otimes K \to O \otimes L : K \to L \in \mathcal{J}_{\mathbf{sSet}}, O \in \mathcal{O}\}$ is a class of generating cofibrations in the $\mathcal{O}$-equivariant model structure. Let $\mathcal{J}_\mathcal{C}^\mathcal{O} = \mathcal{J}_\mathcal{Ac}^\mathcal{O} = \mathcal{J}_\mathcal{Pos}^\mathcal{O} = c\mathbf{Sd}^2\mathcal{J}_{\mathbf{sSet}}$. Then $\mathcal{J}_\mathcal{C}^\mathcal{O}$ can be written as $\{O \times K \to O \times L : K \to L \in \mathcal{J}_\mathcal{C}, O \in \mathcal{O}\}$.

By Lemma 5.4, for every $O \times K \to O \times L \in \mathcal{J}_\mathcal{C}^\mathcal{O}$ and every $O' \in \mathcal{O}$, the map $\mathcal{H}om(O', O \times K) \to \mathcal{H}om(O', O \times L)$ is by definition, the case $\mathcal{C}$ is left proper, the proofs hold for $\mathbf{sSet}$, $\mathbf{Ac}$ and $\mathbf{Pos}$.) So $\mathcal{H}om(O, -)$ preserves cofibrations. By [BMO+13], Lemma 3.1, the functor $\mathcal{H}om(O, -)$ preserves pushouts along cofibrations.

Now let $D = B \cup_A C$ be a pushout in $\mathcal{C}^I$ where the morphism $A \to B$ is a weak equivalence and the morphism $A \to C$ is a cofibration. We prove that the map $C \to D$ is a weak equivalence. For every $O \in \mathcal{O}$, the induced map $\mathcal{H}om(O, A) \to \mathcal{H}om(O, B)$ is a weak equivalence and $\mathcal{H}om(O, A) \to \mathcal{H}om(O, C)$ is a cofibration. The functor $\mathcal{H}om(O, -)$ preserves pushouts of cofibrations, so $\mathcal{H}om(O, D) = \mathcal{H}om(O, B) \cup_{\mathcal{H}om(O, A)} \mathcal{H}om(O, C)$. Because $\mathcal{C}$ is left proper, the natural map $\mathcal{H}om(O, C) \to \mathcal{H}om(O, D)$ is a weak equivalence. Hence $C \to D$ is a weak equivalence.

□

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