OPTIMALITY AND DUALITY FOR NONDIFFERENTIABLE FRACTIONAL PROGRAMMING WITH GENERALIZED INVEXITY

GWI SOO KIM* AND MOON HEE KIM**

Abstract. We establish necessary and sufficient optimality conditions for a class of generalized nondifferentiable fractional optimization programming problems. Moreover, we prove the weak and strong duality theorems under \((V, \rho)\)-invexity assumption.

1. Introduction and preliminaries

Many authors have introduced various concepts of generalized convexity and have obtained optimality and duality results for optimization programming problem ([1]-[4], [6]-[12]). Many practical problems encountered in economics, engineering design, and management science, and so forth can be described by nonsmooth functions. The theory of nonsmooth optimization using locally Lipschitz functions was introduced by Clarke [5].

We consider the following generalized nondifferentiable fractional optimization problem (GFP):

\[
\text{(GFP) Minimize } \max \left\{ \frac{f_i(x)}{g_i(x)} \mid i = 1, \cdots, p \right\} \\
\text{subject to } h_j(x) \leq 0, \ j = 1, \cdots, m,
\]

where \(f := (f_1, \cdots, f_p) : \mathbb{R}^n \rightarrow \mathbb{R}^p\), \(g := (g_1, \cdots, g_p) : \mathbb{R}^n \rightarrow \mathbb{R}^p\) and \(h := (h_1, \cdots, h_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m\) are locally Lipschitz function. We assume that \(f_i(x) \geq 0\) and \(g_i(x) > 0, \ i = 1, \cdots, p\). Let \(X_0 := \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, \ j = 1, \cdots, m\}\).

Received April 27, 2016; Accepted July 15, 2016.

2010 Mathematics Subject Classification: Primary 90C25, 90C29; Secondary 90C46.

Key words and phrases: fractional optimization problem, optimality theorem, duality.

Correspondence should be addressed to Moon Hee Kim, mooni@tu.ac.kr.
0, \ j = 1, \cdots, m} be the feasible set of (GFP). Let \( J = \{ 1, 2, \cdots, m \} \) and \( J(x_0) = \{ j \in J \mid h_j(x_0) = 0 \} \).

We consider the following fractional optimization problem (FP):

\[
\begin{align*}
\text{(FP)} \quad & \quad \text{Minimize} \quad \max \left\{ \frac{f_i(x) + s(x)C_i}{g_i(x)} \mid i = 1, \cdots, p \right\} \\
& \quad \text{subject to} \quad h_j(x) \leq 0, \quad j = 1, \cdots, m,
\end{align*}
\]

where \( f := (f_1, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p \), \( g := (g_1, \cdots, g_p) : \mathbb{R}^n \to \mathbb{R}^p \) and \( h := (h_1, \cdots, h_m) : \mathbb{R}^n \to \mathbb{R}^m \) are continuously differentiable function. For each \( i = 1, \cdots, p \), \( C_i \) is compact convex set of \( \mathbb{R}^n \) and \( s(x)C_i := \max\{ \langle x, y_i \rangle \mid y_i \in C_i \} \).

Recently, Kim and Kim [7] consider the nondifferentiable fractional optimization problem (FP), in which each component of the objective function contains a term involving the support function of a compact convex set. They established necessary and sufficient optimality conditions for fractional optimization problem (FP). And they formulated a Mond-Weir type dual problem for (FP) and showed that the weak and strong duality.

In this paper, we apply the approach of Kim and Kim[7] to the generalized nondifferentiable fractional optimization problem (GFP), we establish necessary and sufficient optimality conditions for a nondifferentiable fractional optimization programming involving locally Lipschitz functions. Moreover, we prove the weak and strong duality theorems under \((V, \rho)\)-invexity assumption.

Now we give some notations for our results in this section:

Let a function \( f : \mathbb{R}^n \to \mathbb{R} \) be given. We shall suppose that \( f \) is locally Lipschitz, that is, for each \( x \in \mathbb{R}^n \), there exist an open neighborhood \( U \) and a constant \( L > 0 \) such that for all \( y \) and \( z \) in \( U \),

\[
|f(y) - f(z)| \leq L\|y - z\|.
\]

Let \( g : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \) be a convex function. The subdifferential of \( g \) at \( a \in \text{dom}g \) is defined by

\[
\partial g(a) := \{ v \in \mathbb{R}^n \mid g(x) \geq g(a) + \langle v, x - a \rangle \forall x \in \text{dom}g \},
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product on \( \mathbb{R}^n \) and \( \text{dom}g := \{ x \in \mathbb{R}^n : g(x) < +\infty \} \).

DEFINITION 1.1. A vector function \( f : \mathbb{R}^n \to \mathbb{R}^p \) is said to be \((V, \rho)\)-invex at \( u \in \mathbb{R}^n \) with respect to the function \( \eta \) and \( \theta_i : \mathbb{R}^n \to \mathbb{R}^n \) if there exists \( \alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\} \) and \( \rho_i \in \mathbb{R}, i = 1, \ldots, p \) such that for any
\[\xi_i \in \partial f_i(u), \ i = 1, \ldots, p \text{ and any } x \in \mathbb{R}^n, \ \text{and for all } i = 1, \ldots, p,\]
\[\alpha_i(x,u)[f_i(x) - f_i(u)] \geq \xi_i^T \eta(x,u) + \rho_i \|\theta_i(x,u)\|^2.\]

**Lemma 1.2.** [5] Let \( f \) and \( g \) be Lipschitz near \( x \) and suppose that \( g(x) \neq 0 \). Then \( \frac{f}{g} \) is Lipschitz near \( x \), and one has
\[\partial \left( \frac{f}{g} \right)(x) \subset g(x)\partial f(x) - f(x)\partial g(x) \quad \{g(x)\}^2.\]
If in addition \( f(x) \geq 0 \), \( g(x) > 0 \) and if \( f \) and \(-g\) are regular at \( x \), then equality holds and \( \frac{f}{g} \) is regular at \( x \).

**Theorem 1.3.** Assume that \( f \) and \( g \) are vector-valued differentiable functions defined on \( \mathbb{R}^n \) and \( f(x) \geq 0 \), \( g(x) > 0 \) for all \( x \in \mathbb{R}^n \). If \( f \) and \( -g \) are regular and \((V,\rho)\)-invex at \( x_0 \), then \( \frac{f}{g} \) is \((V,\rho)\)-invex at \( x_0 \), where
\[\bar{\alpha}_i(x,x_0) = \frac{g_i(x)}{g_i(x_0)} \alpha_i(x,x_0) \quad \bar{\theta}_i(x,x_0) = \left( \frac{1}{g_i(x_0)} \right)^{\frac{1}{2}} \theta_i(x,x_0).\]

**Proof.** Let \( x, x_0 \in X_0 \). Then, by the \((V,\rho)\)-invexity of \( f \) and \(-g\), there exists \( \alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\} \) and \( \rho_i \in \mathbb{R}, \ i = 1, \ldots, p \) such that for any \( \xi_i \in \partial f_i(x_0), \ \zeta_i \in \partial g_i(x_0), \ i = 1, \ldots, p \) and \( x \in \mathbb{R}^n \), and for all \( i = 1, \ldots, p \),
\[\alpha_i(x,x_0)[f_i(x) - f_i(x_0)] \geq \xi_i^T \eta(x,x_0) + \rho_i \|\theta_i(x,x_0)\|^2,\]
\[\alpha_i(x,x_0)[g_i(x) - g_i(x_0)] \geq \zeta_i^T \eta(x,x_0) + \rho_i \|\theta_i(x,x_0)\|^2.\]
So, we have for any \( \xi_i \in \partial f_i(x_0), \ \zeta_i \in \partial g_i(x_0), \ i = 1, \ldots, p \) and \( x \in \mathbb{R}^n \), and for all \( i = 1, \ldots, p \),
\[\alpha_i(x,x_0) \left( \frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)} \right) = \alpha_i(x,x_0) \left( \frac{f_i(x) - f_i(x_0)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x)} \right) \frac{g_i(x)}{g_i(x)} \frac{g_i(x) - g_i(x_0)}{g_i(x_0)} \]
\[\geq \xi_i^T \eta(x,x_0) + \rho_i \|\theta_i(x,x_0)\|^2 \frac{f_i(x_0)}{g_i(x_0)} - \frac{f_i(x_0)}{g_i(x_0)} \left( \zeta_i^T \eta(x,x_0) + \rho_i \|\theta_i(x,x_0)\|^2 \right).\]
Since \( g_i(x) > 0, \ i = 1, \ldots, p \) for all \( x \in X_0 \), we have for any \( \xi_i \in \partial f_i(x_0), \ \zeta_i \in \partial g_i(x_0), \ i = 1, \ldots, p \) and \( x \in \mathbb{R}^n \), and for all \( i = 1, \ldots, p \),
Thus, from Lemma 1.2, for any \( \omega_i \in \partial \left( \frac{f_i}{g_i} \right)(x_0), \xi_i \in \partial f_i(x_0), \zeta_i \in \partial g_i(x_0), i = 1, \ldots, p \) and for all \( i = 1, \ldots, p, \)

\[
\alpha_i(x, x_0) \left( \frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)} \right) \\
\geq \frac{g_i(x_0)}{g_i(x)} \left[ \xi_i^T \eta(x, x_0) + \rho_i \| \theta_i(x, x_0) \|^2 \right] \\
- \frac{g_i(x_0)}{g_i(x)} \left( f_i(x_0) \xi_i^T \eta(x, x_0) \right) + \rho_i \left( \frac{f_i(x_0)}{(g_i(x_0))^2} \right)^{\frac{1}{2}} \theta_i(x, x_0) \|^2 \right). 
\]

Since \( 1 + \left( \frac{f_i(x_0)}{g_i(x_0)} \right)^{\frac{1}{2}} \geq 1, i = 1, \ldots, p, \) we have for any \( \omega_i \in \partial \left( \frac{f_i}{g_i} \right)(x_0), \)

\[
\alpha_i(x, x_0) \left( \frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)} \right) \geq \frac{g_i(x_0)}{g_i(x)} \left[ \omega_i^T \eta(x, x_0) + \rho_i \left( \frac{1}{(g_i(x_0))^2} \right)^{\frac{1}{2}} \theta_i(x, x_0) \|^2 \right]. 
\]
Thus, the function $\frac{f}{g}$ is $(V, \rho)$-invex at $x_0$, where

$$\bar{\alpha}_i(x, x_0) = \frac{g_i(x)}{g_i(x_0)} \alpha_i(x, x_0), \quad \bar{\theta}_i(x, x_0) = \frac{1}{(g_i(x_0))^{\frac{1}{2}}} \theta_i(x, x_0).$$

\[\square\]

2. Optimality theorems

Now, we establish the Kuhn-Tucker necessary and sufficient conditions for a solution of (GFP).

**Theorem 2.1. (Kuhn-Tucker Necessary Optimality Theorem)**

Assume that $f$ and $-g$ are regular. If $x_0$ is a solution of (GFP), and assume that $0 \notin \text{co}\{\partial h_j(x_0) \mid j \in J(x_0)\}$, then there exist $\lambda_i \geq 0$, $i \in I(x_0) := \{i \mid \max \left\{ \frac{f_i(x_0)}{g_i(x_0)} \mid i = 1, \ldots, p \right\} = \frac{f_i(x_0)}{g_i(x_0)} \}$, $\sum_{i \in I(x_0)} \lambda_i = 1$ and $\mu_j \geq 0$, $j = 1, \ldots, m$ such that

$$0 \in \sum_{i \in I(x_0)} \lambda_i \partial \left( \frac{f_i}{g_i} \right)(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)$$

and

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

**Proof.** Let $\phi_i(x) = \frac{f_i(x)}{g_i(x)}$, $i = 1, \ldots, p$. Let $x_0$ be a solution of (GFP) and let $I(x_0) = \{i \mid \max \{\phi_i(x_0) \mid i = 1, \ldots, p\} = \phi_i(x_0)\}$. Then by Proposition 2.3.12 in [5] and Corollary 5.1.8 in [11], there exist $\mu_j \geq 0$, $j = 1, \ldots, m$,

$$0 \in \text{co}\{\partial \phi_i(x_0) \mid i \in I(x_0)\} + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)$$

and

$$\mu_j h_j(x_0) = 0,$$

where $\text{co}A$ is the convex hull of the set $A$. By Lemma 1.2,

$$\partial \phi_i(x_0) = \frac{g_i(x_0) \partial f_i(x_0) - \partial g_i(x_0) f_i(x_0)}{(g_i(x_0))^2}$$

$$= \partial \left( \frac{f_i}{g_i} \right)(x_0).$$
and hence from (2.1), there exist \( \lambda_i \geq 0, \ i \in I(x_0) \), \( \sum_{i \in I(x_0)} \lambda_i = 1 \) and \( \mu_j \geq 0, \ j = 1, \ldots, m \) such that

\[
0 \in \sum_{i \in I(x_0)} \lambda_i \partial \left( \frac{f_i}{g_i} \right)(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)
\]

and

\[
\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.
\]

**Corollary 2.2.** Let \( f := (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p \), \( g := (g_1, \ldots, g_p) : \mathbb{R}^n \to \mathbb{R}^p \) and \( h := (h_1, \ldots, h_m) : \mathbb{R}^n \to \mathbb{R}^m \) are continuously differentiable. If \( x_0 \) is a solution of (GFP), and assume that \( 0 / \in \text{co} \{ \nabla h_j(x_0) \mid j \in J(x_0) \} \), then there exist \( \lambda_i \geq 0, i \in I(x_0) := \{ i \mid \max \left\{ \frac{f_i(x_0)}{g_i(x_0)} \mid i = 1, \ldots, p \right\} = \frac{f_i(x_0)}{g_i(x_0)} \}, \sum_{i \in I(x_0)} \lambda_i = 1 \) and \( \mu_j \geq 0, j = 1, \ldots, m \) such that

\[
\sum_{i \in I(x_0)} \lambda_i \nabla \left( \frac{f_i(x_0)}{g_i(x_0)} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,
\]

\[
\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.
\]

**Theorem 2.3. (Kuhn-Tucker Sufficient Optimality Theorem)** Assume that \( f \) and \( -g \) are regular. Let \( x_0 \) be a feasible solution of (GFP). Suppose that there exist \( \lambda_i \geq 0, i \in I(x_0), \sum_{i \in I(x_0)} \lambda_i = 1 \) and \( \mu_j \geq 0, j = 1, \ldots, m \) such that

\[
0 \in \sum_{i \in I(x_0)} \lambda_i \partial \left( \frac{f_i}{g_i} \right)(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)
\]

and \( \sum_{j=1}^{m} \mu_j h_j(x_0) = 0. \) If \( f(\cdot) \) and \( -g(\cdot) \) are \((V, \rho)\)-invex at \( x_0 \), and \( h \) is \( \eta \)-invex at \( x_0 \) with respect to the same \( \eta \), and \( \sum_{i \in I(x_0)} \lambda_i \rho_i \| \theta_i(x, x_0) \|^2 \geq 0 \), then \( x_0 \) is a solution of (GFP).

**Proof.** Suppose that \( x_0 \) is not a solution of (GFP). Then there exist a feasible solution \( x \) of (GFP) such that

\[
\max_{1 \leq i \leq p} \frac{f_i(x_0)}{g_i(x_0)} > \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}.
\]
Then
\[ \frac{f_i(x_0)}{g_i(x_0)} > \frac{f_i(x)}{g_i(x)}, \text{ for all } i \in I(x_0), \]
and hence \( \bar{\alpha}_i(x, x_0) > 0, \)
\[ \bar{\alpha}_i(x, x_0) \left[ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)} \right] < 0. \]
Since \( f(\cdot) \) and \( -g(\cdot) \) are \((V, \rho)-\text{invex and regular at } x_0, \) by Theorem 1.3, we have for any \( w_i \in \partial \left( \frac{f_i(x)}{g_i(x)} \right)(x_0), i \in I(x_0) \)
\[ w_i \eta(x, x_0) + \rho_i \| \bar{\theta}(x, x_0) \|^2 < 0. \]
Hence, there exist \( \lambda_i \geq 0, i \in I(x_0), \sum_{i \in I(x_0)} \lambda_i = 1 \) such that
\[ \sum_{i \in I(x_0)} \lambda_i w_i \eta(x, x_0) + \sum_{i \in I(x_0)} \lambda_i \rho_i \| \bar{\theta}(x, x_0) \|^2 < 0. \]
Since \( \sum_{i \in I(x_0)} \lambda_i \rho_i \| \bar{\theta}(x, x_0) \|^2 \geq 0, \)
\[ \sum_{i \in I(x_0)} \lambda_i w_i \eta(x, x_0) < 0, \]
and so, it follows from (2.2) that there exist \( \nu_j \in \partial h_j(x_0), j = 1, \ldots, m \) such that
\[ \sum_{j=1}^{m} \mu_j \nu_j \eta(x, x_0) > 0. \]
Then, by the \( \eta \)-invexity of \( h, \) we have
\[ \sum_{j=1}^{m} \mu_j h_j(x) > \sum_{j=1}^{m} \mu_j h_j(x_0). \]
Since \( \sum_{j=1}^{m} \mu_j h_j(x_0) = 0, \) we have \( \sum_{j=1}^{m} \mu_j h_j(x) > 0, \) which is a contradiction since \( \mu_j \geq 0, j = 1, \ldots, m \) and \( x \) is a feasible solution of (GFP). Consequently, \( x_0 \) is a solution of (GFP). \( \square \)
3. Duality theorems

Now, we propose the following Mond-Weir type dual problem (DGFP):

\((DGFP)\) Maximize \(\max \left\{ \frac{f_i(u)}{g_i(u)} \mid i = 1, \ldots, p \right\}\)

subject to \(0 \in \sum_{i \in I(u)} \lambda_i \partial \left( \frac{f_i}{g_i} \right)(u) + \sum_{j=1}^{m} \mu_j \partial h_j(u)\)

\(\sum_{j=1}^{m} \mu_j h_j(u) = 0,\)
\(\lambda_i \geq 0, i \in I(u), \sum_{i \in I(u)} \lambda_i = 1, \mu_j \geq 0, j = 1, \ldots, m.\)

Now we show that the following weak duality theorem holds between (GFP) and (DGFP).

**Theorem 3.1. (Weak Duality)** Assume that \(f\) and \(-g\) are regular. Let \(x\) be any feasible for (GFP) and let \((u, \lambda, \mu)\) be any feasible for (DGFP). Assume that \(f(\cdot)\) and \(-g(\cdot)\) are \((V, \rho)\)-invex at \(u\), and let \(h\) is \(\eta\)-invex at \(u\) with respect to the same \(\eta\), and \(\sum_{i \in I(u)} \lambda_i \rho_i \|\theta_i(x, u)\|^2 > 0\). Then the following holds:

\[\max \left\{ \frac{f_i(x)}{g_i(x)} \mid i = 1, \ldots, p \right\} \geq \max \left\{ \frac{f_i(u)}{g_i(u)} \mid i = 1, \ldots, p \right\}.\]

**Proof.** Let \(x\) be any feasible for (GFP) and let \((u, \lambda, \mu)\) be any feasible for (DGFP). Then we have

\[\sum_{j=1}^{m} \mu_j h_j(x) \leq 0 \leq \sum_{j=1}^{m} \mu_j h_j(u).\]

By the \(\eta\)-invexity of \(h_j(u), j = 1, \ldots, m\), there exists \(\nu_j^* \in \partial h_j(u), j = 1, \ldots, m\) such that

\[\sum_{j=1}^{m} \mu_j \nu_j^* \eta(x, u) \leq 0.\]

Using (3.1), we have there exists \(w_i^* \in \partial \left( \frac{f_i}{g_i} \right)(u), i \in I(u),\)

\[\sum_{i \in I(u)} \lambda_i \ w_i^* \eta(x, u) \geq 0.\]

Now suppose that
max \left\{ \frac{f_i(x)}{g_i(x)} \mid i = 1, \ldots, p \right\} < \max \left\{ \frac{f_i(u)}{g_i(u)} \mid i = 1, \ldots, p \right\}.

Then

\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}, \text{ for all } i \in I(u).

By Theorem 1.3, we have there exists \( w_i^* \in \partial \left\{ \frac{f_i}{g_i} \right\}(u), \ i \in I(u) \) such that

\[ 0 > \bar{\alpha}_i(x, u) \left[ \frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)} \right] \geq w_i^* \eta(x, u) + \rho_i \| \bar{\theta}_i(x, u) \|^2. \]

By using \( \lambda_i \geq 0, \ i \in I(u) \), we have,

\[ \sum_{i \in I(u)} \lambda_i w_i^* \eta(x, u) + \sum_{i \in I(u)} \lambda_i \rho_i \| \bar{\theta}_i(x, u) \|^2 < 0. \]

Since \( \sum_{i \in I(u)} \lambda_i \rho_i \| \bar{\theta}_i(x, u) \|^2 \geq 0 \), we have

\[ \sum_{i \in I(u)} \lambda_i w_i^* \eta(x, u) < 0, \]

which contradicts (3.2). Hence the result holds.

Now we give a strong duality theorem which holds between (GFP) and (DGFP).

**Theorem 3.2. (Strong Duality)** If \( \bar{x} \) is a solution of (GFP) and suppose that \( 0 \notin \text{co}\{ \partial h_j(x_0) \mid j \in J(x_0) \} \). Then there exist \( \bar{\lambda} \in \mathbb{R}^p \) and \( \bar{\mu} \in \mathbb{R}^m \) such that \( (\bar{x}, \bar{\lambda}, \bar{\mu}) \) is feasible for (DGFP). Moreover if the weak duality holds, then \( (\bar{x}, \bar{\lambda}, \bar{\mu}) \) is a solution of (DGFP).

**Proof.** By Theorem 2.1, there exist \( \bar{\lambda}_i \geq 0, i \in I(\bar{x}) := \{ i \mid \max \left\{ \frac{f_i(\bar{x})}{g_i(\bar{x})} \mid i = 1, \ldots, p \right\} = \frac{f_i(\bar{x})}{g_i(\bar{x})} \}, \sum_{i \in I(\bar{x})} \bar{\lambda}_i = 1 \text{ and } \bar{\mu}_j \geq 0, j = 1, \ldots, m \) such that

\[ 0 \in \sum_{i \in I(\bar{x})} \bar{\lambda}_i \partial \left( \frac{f_i}{g_i} \right)(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j \partial h_j(\bar{x}) \]

and \( \sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}) = 0. \)
Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible for (DGFP). On the other hand, by weak duality (Theorem 3.1),

$$\max \left\{ \frac{f_i(\bar{x})}{g_i(\bar{x})} \mid i = 1, \cdots, p \right\} \geq \max \left\{ \frac{f_i(u)}{g_i(u)} \mid i = 1, \cdots, p \right\}$$

for any (DGFP) feasible solution $(u, \lambda, \mu)$. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a solution of (DGFP).

\section{4. Conclusions}

This paper is concerned with optimality conditions and duality theorems for fractional optimization problems involving locally Lipschitz functions. Using Clarke’s generalized subdifferential, we gave necessary and sufficient optimality theorems for the problems. The sufficient optimality conditions were verified under generalized invexity conditions on involved functions. The Mond-Weir dual problems were formulated, and then duality theorems were established, that is, weak and strong duality theorems for the nondifferentiable fractional optimization problems.

\begin{thebibliography}{9}

[1] R. P. Agarwal, I. Ahmad, and S. Al-Homidan, \textit{Optimality and duality for nondifferentiable multiobjective programming problems involving generalized $d-\rho-(n, \theta)$ Type I invex functions}, Journal of Nonlinear and Convex Analysis 13 (2012), no. 4, 733-744.

[2] I. Ahmad, S. K. Gupta, and A. Jayswal, \textit{On sufficiency and duality for non-smooth multiobjective programming problems involving generalized $V-r$-invex functions}, Nonlinear Analysis: Theory, Methods & Applications 74 (17) (2011), 5920-5928.

[3] R. P. Agarwal, I. Ahmad, Z. Husain, and A. Jayswal, \textit{Optimality and duality in nonsmooth multiobjective optimization involving $V$-type I invex functions}, Journal of Inequalities and Applications 21 (2010).

[4] I. Ahmad and S. Sharma, \textit{Optimality conditions and duality in nonsmooth multiobjective optimization}, Journal of Nonlinear and Convex Analysis 8 (2007), 417-430.

[5] F. H. Clarke, \textit{Optimization and Nonsmooth Analysis}, A Wiley-Interscience Publication, John Wiley & Sons, 1983.

[6] D. S. Kim, S. J. Kim, and M. H. Kim, \textit{Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems}, Journal of Optimization Theory and Applications 129 (2006), no. 1, 131-146.

[7] M. H. Kim and G. S Kim, \textit{On optimality and duality for generalized nondifferentiable fractional optimization problems}, Communications of the Korean Mathematical Society 25 (2010), 139-147.

\end{thebibliography}
[8] H. Kuk, G. M. Lee, and D. S. Kim, Nonsmooth multiobjective programs with 
\((V,\rho)\)-invexity, Indian Journal of Pure and Applied Mathematics 29 (1998), 405-412.

[9] H. Kuk, G. M. Lee, and T. Tanino, Optimality and duality for nonsmooth multiobjective fractional programming with generalized invexity, Journal of Mathematical Analysis and Applications 262 (2001), 365-375.

[10] Z. Liang, H. Huang, and P. M. Pardalos, Optimality conditions and duality for a class of nonlinear fractional programming problems, Journal of Optimization Theory and Applications 110 (2001), 611-619.

[11] M. M. Mäkelä and P. Neittaanmäki, Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control, World Scientific Publishing Co. Pte. Ltd. 1992.

[12] Z. Y. Peng and S. S. Chang, Some properties of semi-G-preinvex functions, Taiwan Journal of Mathematics 17 (2013), no. 3, 873-884.

* Department of Applied Mathematics
Pukyong National University
Busan 48513, Republic of Korea
E-mail: gwisoo1103@hanmail.net

** Department of Refrigeration Engineering
Tongmyong University
Busan 608-711, Republic of Korea
E-mail: mooni@tu.ac.kr