K-THEORY AND 0-CYCLES ON SCHEMES

RAHUL GUPTA, AMALENDU KRISHNA

Abstract. We prove Bloch’s formula for 0-cycles on affine schemes over algebraically closed fields. We prove this formula also for projective schemes over algebraically closed fields which are regular in codimension one. Several applications, including Bloch’s formula for 0-cycles with modulus, are derived.

Contents

1. Introduction 1
2. Review of 0-cycles and Milnor $K$-sheaves 6
3. The Bloch-Quillen map for 0-cycles 12
4. Proof of Theorem 1.1 19
5. Bloch’s formula for 0-cycles with modulus 23
6. The question of Kerz-Saito 27
7. Euler class groups of affine algebras 30
8. The kernel of the cycle class map 35
References 37

1. Introduction

The principal aim of this paper is to study the Chow group of 0-cycles on singular schemes and the Chow group of 0-cycles with modulus on smooth schemes. We prove Bloch’s formula for these groups and show that the canonical cycle class map from them to the appropriate $K$-theory and relative $K$-theory groups have torsion kernels of bounded exponents. These results directly extend the analogous classical results about the 0-cycle groups on smooth schemes to the setting of 0-cycles on singular schemes and 0-cycles with modulus on smooth schemes. We derive several outstanding consequences of these results. This section provides the background of these problems, a summary of main results and applications, their statements and outline of proofs.

1.1. Bloch’s formula. The Bloch-Quillen formula in the theory of algebraic cycles provides a description for the Chow group of 0-cycles on a smooth quasi-projective scheme over a field in terms of the Zariski cohomology of the Quillen $K$-theory sheaves. This yields a direct connection between the Chow groups and algebraic $K$-theory of smooth quasi-projective schemes. This formula for curves is classical and the case of surfaces was derived by Bloch [9]. The general case of the formula for all smooth quasi-projective schemes was established by Quillen [39]. Kato [15] showed that this formula also holds if one replaces the Quillen $K$-theory sheaves by the Milnor $K$-theory sheaves and the Zariski cohomology by the Nisnevich cohomology. In conclusion, for a smooth quasi-projective scheme $X$ of dimension $d \geq 0$ over a field, one knows that

$$\text{CH}^d(X) \simeq H^{d}_{\text{zar}}(X, K^M_{d,X}) \simeq H^{d}_{\text{nis}}(X, K^M_{d,X}) \simeq H^{d}_{\text{zar}}(X, K_{d,X}).$$

2010 Mathematics Subject Classification. Primary 14C25; Secondary 19E08, 19E15.

Key words and phrases. Algebraic $K$-theory, algebraic cycles, singular varieties.
However, no complete generalization of this formula has been found for singular schemes despite the fact that there is a well established theory of 0-cycles on such schemes after the work of Levine and Weibel [33] in 1980’s. This formula was shown for singular curves by Levine and Weibel [33]. Collino [10] proved the Bloch-Quillen formula for a scheme which is almost non-singular (meaning that it has only one singular point). For a quasi-projective surface whose singular locus is affine, a Bloch-Quillen formula of the type (1.1) was proven by Pedrini and Weibel [38]. Levine [31] proved this formula for all singular surfaces over algebraically closed fields (see [8] for details). These are the only cases of singular schemes for which any of the isomorphisms in (1.1) is presently known.

On the contrary, Levine and Srinivas showed [44, §3.2] that the formula of Quillen for the Chow group of 0-cycles can not be generalized to singular schemes, even with the rational coefficients. They showed that if \( X \) is the boundary of the 4-simplex in \( \mathbb{A}^4_{\mathbb{C}} \), given by the equation \( f(x,y,z,w) = xyzw(1 - x - y - z - w) = 0 \), then \( \text{CH}^3(X) \cong K_3^M(\mathbb{C})_{\mathbb{Q}} \), whereas \( H^3_{\text{zar}}(X, K_{3,X})_{\mathbb{Q}} \cong K_3(\mathbb{C})_{\mathbb{Q}} \).

This example shows that Quillen’s generalization of Bloch’s formula can not be extended to higher dimensional affine schemes. Kato’s generalization of Bloch’s formula to singular schemes however remains an open question. Our first main result answers this question as follows. For any Noetherian scheme \( X \), let us denote its Nisnevich (resp. Zariski) site by \( X_{\text{nis}} \) (resp. \( X_{\text{zar}} \)). For a separated and reduced scheme \( X \) of finite type over field, we let \( \text{CH}^0_{\text{LW}}(X) \) denote the Levine-Weibel Chow group of 0-cycles. Let \( K^M_{i,X} \) denote the Nisnevich (or Zariski) sheaves of Milnor \( K \)-groups on \( X \) (see §2).

**Theorem 1.1.** Let \( k \) be an infinite perfect field and let \( X \) be a reduced quasi-projective scheme of pure dimension \( d \geq 0 \) over \( k \). Then there is a canonical surjective map
\[
\rho_X : \text{CH}^0_{\text{LW}}(X) \twoheadrightarrow H^d_{\text{nis}}(X, K^M_{d,X}).
\]
This is an isomorphism if \( k \) is algebraically closed and either of the following holds.

1. \( X \) is affine.
2. \( X \) is projective and regular in codimension one.

In our proof of the isomorphism, the assumption that \( k \) is algebraically closed plays a crucial role. Without this assumption, we can prove Theorem 1.1 for affine surfaces and for a modified version of the Chow group (see §2.1).

**Theorem 1.2.** Let \( X \) be a reduced affine surface over a perfect field \( k \). Then there is a canonical isomorphism
\[
\rho_X : \text{CH}^0(X) \cong H^2_{\text{nis}}(X, K^M_{2,X}).
\]

### 1.2. Bloch’s formula for Chow group with modulus.

Just as Bloch’s higher Chow groups are the motivic cohomology which describe algebraic \( K \)-theory of smooth schemes, the higher Chow groups with modulus [7] are supposed to describe the relative algebraic \( K \)-theory of the pair \((X,D)\), where \( X \) is a smooth scheme and \( D \subset X \) is an effective Cartier divisor. One of the primary goals of the program of connecting cycles with modulus and relative \( K \)-theory is the proof of Bloch-Quillen-Kato type formula for the Chow groups with modulus. As an application of Theorems 1.1 and 1.2, we solve this problem for 0-cycles with modulus as follows.

**Theorem 1.3.** Let \( k \) be an algebraically closed field and let \( X \) be a smooth quasi-projective scheme of dimension \( d \geq 1 \) over \( k \). Let \( D \subset X \) be an effective Cartier divisor. Then there is a canonical isomorphism
\[
\rho_{X|D} : \text{CH}^0(X|D) \cong H^d_{\text{nis}}(X, K^M_{d,(X,D)}).
\]
in the following cases.

1. $X$ is affine.
2. $X$ is projective and $D$ is integral.

If $d \leq 2$, then part (2) of the above theorem holds without any condition on $D$ and this was shown in [6]. If $k$ is not necessarily algebraically closed, we can prove the following.

**Theorem 1.4.** Let $X$ be a smooth affine surface over a perfect field $k$ and let $D \subset X$ be an effective Cartier divisor. Then there is a canonical isomorphism

$$
\rho_{X|D} : \text{CH}_0(X|D) \xrightarrow{\cong} H^2_{\text{nis}}(X, K^M_{2}(X,D)).
$$

1.3. **The cycle class map.** For a Noetherian scheme $X$, there is a cycle class map $\lambda_X : \text{CH}^{LW}_0(X) \to K_0(X)$ (see [33], Proposition 2.1). Let $F^d K_0(X)$ be its image. If $X$ is a smooth scheme and $D \subset X$ is an effective Cartier divisor, there is a cycle class map $\lambda_{X|D} : \text{CH}_0(X|D) \to K_0(X,D)$ (see [6] and § 5.1). As an application of the Chern class maps, Grothendieck [16, § 4.3] proved that for a smooth quasi-projective scheme $X$ of dimension $d \geq 1$ over a field, the kernel of the cycle class map $\lambda_X$ is a torsion group of exponent $(d - 1)!$.

The second principal aim of this text is to generalize this result to 0-cycles on singular schemes and 0-cycles with modulus on smooth schemes as follows. When $k = \overline{k}$, part (1) of the following theorem gives an independent proof of an old unpublished result of Levine (see [30], Corollary 5.4) for affine schemes. When $k$ is not algebraically closed, this result is completely new.

**Theorem 1.5.** Let $A$ be a geometrically reduced affine algebra of dimension $d \geq 1$ over a field $k$. Assume that either $A$ is algebraically closed or $(d - 1)! \in k^\times$. Let $X = \text{Spec} (A)$ and let $D \subset X$ be an effective Cartier divisor. Then the following hold.

1. If $k$ is an infinite field, then the kernel of the cycle class map $\lambda_A : \text{CH}^{LW}_0(A) \to K_0(A)$ is a torsion group of exponent $(d - 1)!$.
2. For arbitrary fields, the kernel of the cycle class map $\lambda_A : \text{CH}_0(A) \to K_0(A)$ is a torsion group of exponent $(d - 1)!$, where as before $\text{CH}_0(A)$ is a modified version of the Chow group.
3. If $k$ is perfect and $X$ is smooth, then the kernel of the cycle class map $\lambda_{X|D} : \text{CH}_0(X|D) \to K_0(X,D)$ is a torsion group of exponent $(d - 1)!$.

We now give several applications of the above results.

1.4. **Motivic cohomology with modulus.** Let $X$ be a smooth quasi-projective scheme of pure dimension $d$ over a field $k$ and let $D \subset X$ be an effective Cartier divisor. Let $H^i_M(X|D, \mathbb{Z}(r))$ denote the motivic cohomology of the presheaf of cycle complexes with modulus. There exists a canonical map $\text{CH}^i(X|D, 2r - i) \to H^i_M(X|D, \mathbb{Z}(r))$ (see § 2.3). It is known that this natural map is an isomorphism if $D = \emptyset$. For an arbitrary effective divisor $D$, one does not know much about the nature of this map. As an application of Bloch’s formula for 0-cycles with modulus, we show the following.

**Theorem 1.6.** Let $k$ be an algebraically closed field. Let $X$ be a smooth quasi-projective scheme of pure dimension $d \geq 1$ over $k$ and let $D \subset X$ be an effective Cartier divisor such that $D_{\text{red}}$ is a simple normal crossing divisor. Then the map $\text{can}_M : \text{CH}^d(X|D) \to H^d_M(X|D, \mathbb{Z}(d))$ of (2.1) is surjective in the following cases.

1. $d \leq 2$.
2. $X$ is affine.
(3) \( \text{char}(k) = 0 \) and \( X \) is projective.
(4) \( \text{char}(k) > 0, \) \( X \) is projective and \( D \) is reduced.

Moreover, it is an isomorphism if in addition, \( D \) is connected and smooth.

If \( k \) is a finite field and \( D_{\text{red}} \) is a simple normal crossing divisor, then Rülling and Saito \cite{44} proved that the map \( \text{can}_{M} \) induces an isomorphism of the pro-abelian groups \( \text{can}_{M} : \left\{ \lim \right\}_n \text{CH}^d(X|D) \rightarrow \left\{ \lim \right\}_n H^d_{LW}(X|D,\mathbb{Z}(d)) \). In the following theorem, we are able to prove this isomorphism for affine schemes and for quasi-projective surfaces over algebraically closed fields.

**Theorem 1.7.** Let \( k \) be an algebraically closed field. Let \( X \) be a smooth quasi-projective scheme of pure dimension \( d \geq 1 \) over \( k \) and let \( D \subset X \) be a simple normal crossing divisor. Then the map of pro-abelian groups \( \text{can}_{M} : \left\{ \lim \right\}_n \text{CH}^d(X|nD) \rightarrow \left\{ \lim \right\}_n H^d_{LW}(X|nD,\mathbb{Z}(d)) \) is an isomorphism in the following cases.

1. \( X \) is affine.
2. \( X \) is a quasi-projective scheme and \( d = 2 \).

1.5. **The strong Bloch-Srinivas conjecture.** Let \( X \) be a reduced affine or projective scheme of dimension \( d \geq 1 \) over an algebraically closed field \( k \). Assume that \( X \) is regular in codimension one and there exists a resolution of singularities \( \pi : \bar{X} \rightarrow X \). Let \( E_0 \subset \bar{X} \) be the reduced exceptional divisor. It is known that there exists a surjective pull-back map \( \pi^* : \text{CH}^d_{LW}(X) \rightarrow \text{CH}^d_0(\bar{X}) \). It is not hard to see that this map has a factorization \( \text{CH}^d_0(LW)(X) \xrightarrow{\pi_n^*} \text{CH}^d_0(\bar{X}|nE_0) \rightarrow \text{CH}^d_0(\bar{X}) \) for every \( n \geq 1 \). As an application of our proof of Theorem 1.1 we can prove the following result about the maps \( \pi_n^* \).

**Theorem 1.8.** The map \( \pi_n^* : \text{CH}^d_{LW}(X) \rightarrow \text{CH}^d_0(\bar{X}|nE_0) \) is an isomorphism for all \( n \gg 0 \). If \( \text{char}(k) > 0, \) then \( \pi_n^* \) is an isomorphism for every \( n \geq 1 \).

The first part of Theorem 1.8 was conjectured (in a different but equivalent form) by Bloch and Srinivas \cite{43} for normal surfaces. Its proof was given in \cite{28}. This conjecture allows us to estimate the kernel of the map \( \text{CH}^d_{LW}(X) \rightarrow \text{CH}^d_0(\bar{X}) \).

1.6. **Question of Kerz and Saito.** Let \( X \) be a smooth projective scheme of dimension \( d \geq 1 \) over a perfect field \( k \) of positive characteristic and let \( U \subset X \) be an open subset whose complement is supported on a divisor. Then a question posed by Kerz and Saito \cite{22} asks if there is an isomorphism

\[ \lim \left\{ \lim \left( \text{CH}^d(X|D) \xrightarrow{\pi} H^d_{\text{HS}}(X,K^M_{d,(X,D)}) \right) \right\}, \]

where the limits are taken over effective divisors on \( X \) with support outside \( U \).

This question was answered positively by Kerz and Saito if \( k \) is a finite field, using an earlier result of Kato and Saito \cite{19}. As an application of Theorem 1.8 we prove the following stronger version of this question whenever \( k \) is algebraically closed and \( X \setminus U \) can be contracted to a smaller dimensional scheme without changing \( U \).

**Theorem 1.9.** Let \( Y \) be a reduced projective scheme of pure dimension \( d \geq 1 \) over an algebraically closed field \( k \) of positive characteristic. Assume that \( Y \) is regular in codimension one and there exists a resolution of singularities \( \pi : X \rightarrow Y \). Let \( E_0 \subset X \) be the reduced exceptional
divisor. Then for any effective divisor $D \subset X$ with support $E_0$, there is a commutative diagram

\[
\begin{array}{ccc}
\text{CH}_0(X|D) & \xrightarrow{\rho_X|D} & H^d_{\text{nis}}(X, K^M_{d,(X,D)}) \\
\downarrow & & \downarrow \\
\text{CH}_0(X|E_0) & \xrightarrow{\rho_X|E_0} & H^d_{\text{nis}}(X, K^M_{d,(X,E_0)})
\end{array}
\]

in which all arrows are isomorphisms.

We warn the reader that this result does not follow from Theorem 1.3. Rather, it gives new cases of Bloch’s formula for Chow groups with modulus.

1.7. Schlichting’s theorem. In [42], Schlichting gave a necessary and sufficient condition for vector bundles of top rank on affine schemes to admit nowhere vanishing sections. By combining Theorem 1.1 with [25, Theorem 1.2], we recover Schlichting’s theorem over algebraically closed fields.

**Corollary 1.10.** Let $A$ be a reduced affine algebra of pure dimension $d \geq 1$ over an algebraically closed field and let $X = \text{Spec}(A)$. Let $P$ be a projective $A$-module of rank $d$. Then $P$ admits an Euler class $e(P) \in H_{\text{zar}}^d(X, K^M_{d,X})$. Furthermore, $P$ splits off a free summand of positive rank if and only if $e(P)$ dies in $H^d_{\text{nis}}(X, K^M_{d,X})$.

1.8. Euler class group and Chow group. The Euler and weak Euler class groups of a commutative Noetherian ring $A$ were introduced by Bhatwadekar and R. Sridharan [5] in order to study the question of existence of nowhere vanishing sections of projective modules of rank $d = \dim(A)$. If $A$ is a smooth affine algebra over an infinite perfect field, it was conjectured by Bhatwadekar and R. Sridharan (see [41, Remark 3.13]) that the weak Euler class group $E_0(A)$ coincides with the Chow group of 0-cycles $\text{CH}_0(A)$. This was proven by Bhatwadekar (unpublished) if $\dim(A) \leq 2$ and by Asok and Fasel [2] in general. As part of the proof of Theorem 1.2, we establish the following partial generalization of these results to non-smooth algebras.

**Theorem 1.11.** Let $A$ be a 2-dimensional geometrically reduced affine algebra over an infinite field. Then there is an isomorphism

\[
E_0(A) \xrightarrow{\cong} \text{CH}_0^{\text{LW}}(A).
\]

1.9. Outline of proofs. We now briefly outline the strategy of our proofs. Bloch’s formula for 0-cycles on singular schemes has three main steps: the construction of the Bloch-Quillen map, showing its surjectivity and, its injectivity.

The construction of the Bloch-Quillen map from the Chow group to the cohomology of the Milnor $K$-theory sheaves is a major obstacle. In the smooth case, this directly follows from the Gersten complex. But this method breaks down in the singular case. Another possibility is to construct this map for surfaces and then reduce the general case to the surface case. But this too breaks down due to the lack of a push-forward map on the top cohomology of the Milnor $K$-theory sheaf.

Instead, what we observe here is that in any dimension, the Cousin complex still gives a presentation of the desired cohomology group. The heart of the proof then is to compare some part of the Gersten complex with Cousin complex of Milnor $K$-theory sheaves to kill the rational equivalence on the group of 0-cycles. The key role here is played by our Proposition 3.4.

We complete the proof of Theorem 1.1 in § 4. The main ingredients for the surjectivity are some results of Kato and Saito [19]. The injectivity is shown using the Roitman torsion
theorems of \cite{25} and \cite{28}. Apart from these, we also need to use a technique of Levine \cite{32} to study the relation between the $K$-theory of a normal projective scheme and its Albanese variety.

In §5 we prove Bloch’s formula for the Chow group with modulus. In order to do so, we generalize Theorem 1.1 to certain kind of projective schemes which are not regular in codimension one. The crucial ingredient here is the Roitman torsion theorem of \cite{26}. We derive Bloch’s formula in the modulus setting using this and a decomposition theorem for the Chow group of 0-cycles from \cite{6}. In this section, we also prove Theorem 1.6 and Theorem 1.7. The main idea here is to use a theorem of Rülling and Saito \cite{41} which compares motivic cohomology with modulus with the cohomology of a relative Milnor $K$-group. Then the theorems then follow from Bloch’s formula with modulus and the comparisons between Rülling-Saito relative Milnor $K$-groups and Kato-Saito relative Milnor $K$-groups.

The strong Bloch-Srinivas conjecture is proven in §6 using Theorem 1.1, the recent pro-descent theorem of Kerz, Strunk and Tamme \cite{23} and some results on the $K$-theory in positive characteristic from \cite{25}. A question of Kerz-Saito is answered in a special case as an application of our proof of the strong version of the Bloch-Srinivas conjecture.

In §7, we present the proof of Theorem 1.5. In order to do this, we use the theory of Euler class groups of commutative Noetherian rings. For singular rings, these groups are difficult to study directly. To circumvent this, we introduce a modified version of the Euler class group. We then show that this modified version coincides with the classical one for singular affine schemes. This uses a hard result of Van der Kallen \cite{48}, a theorem of Das-Zinna \cite{12} and the Bertini theorem of Murthy and Swan.

Using this isomorphism, the Bertini theorems of Murthy and Swan, the cancellation theorem of Suslin and some results of Bhatwadekar-R. Sridharan on the Euler classes of projective modules, we complete the proof of Theorem 1.5. Using these Euler class groups, we derive Bloch’s formula for affine surfaces over arbitrary perfect fields in §8.

1.10. Notations. The following notations will be followed in this text. The word scheme will mean a separated Noetherian scheme of finite Krull dimension and the word ring will mean a commutative Noetherian ring. For a scheme $X$, the normalization of $X_{\text{red}}$ will be denoted by $X^N$. We shall denote the Nisnevich (resp. Zariski) site of $X$ by $X_{\text{nis}}$ (resp. $X_{\text{zar}}$). For a point $x \in X$, we shall denote the scheme $\text{Spec}(O_{X,x})$ by $X_x$. We let $X^o_x = X_x \setminus \{x\}$ and $\eta_x = \text{Spec}(k(x))$. For a closed subscheme $Z \subset X$, we shall let $|Z|$ denote the support of $Z$.

Throughout this text, we shall fix a perfect field $k$ and let $\text{Sch}_k$ denote the category of separated schemes of finite type over $k$. We shall let $\text{Sm}_k$ denote the category of those schemes in $\text{Sch}_k$ which are smooth over $k$. For $X,Y \in \text{Sch}_k$, we shall denote $X \times \text{Spec}(k) Y$ simply by $X \times_k Y$.

For abelian groups $A$ and $B$, we shall write $A \otimes \mathbb{Z} B$ in short as $A \otimes B$. For a prime $p$, we shall let $A(p)$ denote the $p$-primary torsion subgroup of $A$.

2. Review of 0-cycles and Milnor $K$-sheaves

In this section, we recall the definitions of various 0-cycle groups and relations between them. We also recall the definition of the Milnor $K$-theory sheaves which is one of our main objects of study in order to prove the Bloch-Quillen type formula for Chow groups on singular varieties and Chow groups with modulus. We also prove some other preliminary results that will be used in the proofs of the main results.

2.1. Levine-Weibel Chow group of singular schemes. We recall the definition of the cohomological Chow group of 0-cycles for singular schemes from \cite{6} and \cite{33}. Let $X$ be a reduced quasi-projective scheme of dimension $d \geq 1$ over $k$. Let $X_{\text{sing}}$ and $X_{\text{reg}}$ respectively denote the loci of the singular and the regular points of $X$. Given a nowhere dense closed
subset $Y \subset X$ such that $X_{\text{sing}} \subseteq Y$ and no component of $X$ is contained in $Y$, we let $Z_0(X,Y)$ denote the free abelian group on the closed points of $X \setminus Y$. We write $Z_0(X,X_{\text{sing}})$ in short as $Z_0(X)$.

**Definition 2.1.** Let $C$ be a pure dimension one reduced scheme in $\text{Sch}_k$. We shall say that a pair $(C,Z)$ is a good curve relative to $X$ if there exists a finite morphism $\nu : C \to X$ and a closed proper subset $Z \subseteq C$ such that the following hold.

1. No component of $C$ is contained in $Z$.
2. $\nu^{-1}(X_{\text{sing}}) \cup C_{\text{sing}} \subseteq Z$.
3. $\nu$ is local complete intersection at every point $x \in C$ such that $\nu(x) \in X_{\text{sing}}$.

Let $(C,Z)$ be a good curve relative to $X$ and let $\{\eta_1,\ldots,\eta_r\}$ be the set of generic points of $C$. Let $O_{C,Z}$ denote the semilocal ring of $C$ at $S = Z \cup \{\eta_1,\ldots,\eta_r\}$. Let $k(C)$ denote the ring of total quotients of $C$ and write $O_{C,Z}$ for the group of units in $O_{C,Z}$. Notice that $O_{C,Z}$ coincides with $k(C)$ if $|Z| = \emptyset$. As $C$ is Cohen-Macaulay, $O_{C,Z}$ is the subgroup of $k(C)^*$ consisting of those $f$ which are regular and invertible in the local rings $O_{C,x}$ for every $x \in Z$.

Given any $f \in O_{C,Z}^{\times} \to k(C)^*$, we denote by $\text{div}_C(f)$ (or $\text{div}(f)$ in short) the divisor of zeros and poles of $f$ on $C$, which is defined as follows. If $C_1,\ldots,C_r$ are the irreducible components of $C$, and $f_i$ is the factor of $f$ in $k(C_i)$, we set $\text{div}(f)$ to be the 0-cycle $\sum_{i=1}^r \text{div}(f_i)$, where $\text{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the sense of [14]. As $f$ is an invertible regular function on $C$ along $Z$, $\text{div}(f) \in Z_0(C,Z)$.

By definition, given any good curve $(C,Z)$ relative to $X$, we have a push-forward map $Z_0(C,Z) \xrightarrow{\nu_*} Z_0(X)$. We shall write $R_0(C,Z,X)$ for the subgroup of $Z_0(X)$ generated by the set $\{\nu_*(\text{div}(f))f \in O_{C,Z}^{\times}\}$. Let $R_0(X)$ denote the subgroup of $Z_0(X)$ generated by the image of the map $R_0(C,Z,X) \to Z_0(X)$, where $(C,Z)$ runs through all good curves relative to $X$. We let $CH_0(X) = \frac{Z_0(X)}{R_0(X)}$.

If we let $R_{0,\text{LW}}^0(X)$ denote the subgroup of $Z_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $\nu : C \to X$ is a closed immersion, then the resulting quotient group $Z_0(X)/R_{0,\text{LW}}^0(X)$ is denoted by $CH_{0,\text{LW}}^0(X)$. Such curves on $X$ are called the Cartier curves. There is a canonical surjection $CH_{0,\text{LW}}^0(X) \to CH_0(X)$. The Chow group $CH_{0,\text{LW}}^0(X)$ was discovered by Levine and Weibel [33] in an attempt to describe the Grothendieck group of a singular scheme in terms of algebraic cycles. The modified version $CH_0(X)$ was introduced in [9].

We shall use the following moving lemma type result from [13] Lemma 1.3, Corollary 1.4] in the proof of Theorems [14] and [1,8].

**Lemma 2.2.** Let $X$ be a reduced quasi-projective scheme over an infinite perfect field $k$. Let $Y$ be a nowhere dense closed subscheme of $X$ containing $X_{\text{sing}}$ such that the codimension of $Y$ in $X$ is at least two. Let $R_{0,\text{LW}}^0(X,Y) \subset R_{0,\text{LW}}^0(X)$ denote the subgroup generated by $(f)_C$ where $C$ is an integral curve such that $C \cap Y = \emptyset$ and $f \in k(C)^*$. Then the map

$$\frac{Z_0(X,Y)}{R_{0,\text{LW}}^0(X,Y)} \to \frac{Z_0(X)}{R_{0,\text{LW}}^0(X)}$$

is an isomorphism.

2.2. Higher Chow groups with modulus. For $n \geq 1$, let $\square^n$ denote the scheme $\mathbb{A}^n_k = (\mathbb{P}^1_k \setminus \{\infty\})^n$. Let $(y_1,\ldots,y_n)$ denote the coordinate of a point on $\square^n$. We shall denote the scheme $(\mathbb{P}^1_k)^n$ by $\mathbb{P}^n$. For $1 \leq i \leq n$, let $F_{n,i}^\infty$ denote the closed subscheme of $\mathbb{P}^n$ given by the equation $\{y_i = \infty\}$. We shall denote the divisor $\sum_{i=1}^n F_{n,i}^\infty$ by $F_{n}^\infty$. 
Let \( X \) be a smooth quasi-projective scheme of dimension \( d \geq 0 \) over \( k \) and let \( D \subseteq X \) be an effective Cartier divisor. For \( r \in \mathbb{Z} \) and \( n \geq 0 \), let \( z_r(X|D, n) \) be the free abelian group on integral closed subschemes \( V \) of \( X \times \square^n \) of dimension \( r + n \) satisfying the following conditions:

1. (Face condition) For each face \( F \) of \( \square^n \), \( V \) intersects \( X \times F \) properly:
   \[
   \dim_k(V \cap (X \times F)) \leq r + \dim_k(F),
   \]
   and

2. (Modulus condition) \( V \) is a cycle with modulus \( D \) relative to \( F_n^\infty \):
   \[
   \nu^*(D \times \square^n) \leq \nu^*(X \times F_n^\infty),
   \]
where \( \overline{V} \) is the closure of \( V \) in \( X \times \square^n \) and \( \nu : \overline{V}^N \to \overline{V} \to X \times \square^n \) is the composite map from the normalization of \( \overline{V} \). We let \( z_r(X|D, n)_{\text{degn}} \) denote the subgroup of \( z_r(X|D, n) \) generated by cycles which are pull-back of some cycles under various projections \( X \times \square^n \to X \times \square^m \) with \( m < n \).

**Definition 2.3.** The cycle complex with modulus \( (z_r(X|D, \bullet), d) \) of \( X \) in dimension \( r \) and with modulus \( D \) is the non-degenerate complex associated to the cubical abelian group \( n \mapsto z_r(X|D, n) \), i.e.,
\[
z_r(X|D, n) := \frac{z_r(X|D, n)}{z_r(X|D, n)_{\text{degn}}}.
\]

The homology \( \text{CH}_r(X|D, n) = H_n(z_r(X|D, \bullet)) \) is called a higher Chow group of \( X \) with modulus \( D \). Sometimes, we also write it as the Chow group of the modulus pair \( (X, D) \). If \( X \) has pure dimension \( d \), we write \( \text{CH}^r(X|D, n) = \text{CH}^r(X|D, n) \). We shall often write \( \text{CH}^r(X|D, 0) \) as \( \text{CH}^r(X|D) \). We refer to [27] for further details on this definition. The reader should note that \( \text{CH}^r(X|D, n) \) coincides with the usual higher Chow group of Bloch \( \text{CH}_r(X, n) \) if \( D = \emptyset \).

### 2.3. Motivic cohomology with modulus

Let \( X \) be a smooth quasi-projective scheme of pure dimension \( d \geq 0 \) over \( k \) and let \( D \subseteq X \) be an effective Cartier divisor. For an étale map \( V \to X \), we let \( D_V \) denote the pull-back of \( D \) to \( V \). Then the presheaves \( z^r(-|D, n) \) on the site \( X_{\text{et}} \) defined by \( (V \to X) \mapsto z^r(V|D_V, n) \) are sheaves for the étale topology and therefore for the Nisnevich topology.

If \( X \) is smooth and \( D \) is an effective Cartier divisor on \( X \), then the \( r \)-th motivic complex of the pair \( (X, D) \) is defined to be the complex of the Nisnevich sheaves on \( X \):
\[
\mathbb{Z}(r)_{X|D} = z^r(-|D, 2r - \bullet).
\]

and the motivic cohomology of the pair \( (X, D) \) is defined to be:
\[
H^i_M(X|D, \mathbb{Z}(r)) := \mathbb{H}^i_{\text{nis}}(X, \mathbb{Z}(r)_{X|D}).
\]

Note that, for \( 0 \leq r \) and \( 0 \leq i \leq 2r \), there exists a natural map
\[
(2.1) \quad \text{CH}^r(X|D, 2r - i) \to H^i_M(X|D, \mathbb{Z}(r)).
\]

Indeed, considering \( z^r(X|D, 2r - \bullet) \) as a complex of constant Nisnevich sheaves on \( X \), we have a natural map \( z^r(X|D, 2r - \bullet) \to \mathbb{Z}(r)_{X|D, \text{nis}} \). Taking the hypercohomology of these complexes, we get \( \text{CH}^r(X|D, 2r - i) = \mathbb{H}^i_{\text{nis}}(z^r(X|D, 2r - \bullet)) \to H^i_M(X|D, \mathbb{Z}(r)) \).

### 2.4. The double and its Chow group

Let \( X \) be a smooth quasi-projective scheme of dimension \( d \) over \( k \) and let \( D \subseteq X \) be an effective Cartier divisor. Recall from [9] § 2.1 that the double of \( X \) along \( D \) is a quasi-projective scheme \( S(X, D) = X \cup_D X \) so that
\[
(2.2) \quad \xymatrix{ D \ar[r]^i & X \ar[d]^{\iota_+} \ar[l]_{\iota_-} \ar[r]_{\iota_+} & S(X, D) \ar[l]_{\iota_-} }
\]
is a co-Cartesian square in \( \text{Sch}_k \). In particular, the identity map of \( X \) induces a finite map \( \forall : S(X, D) \to X \) such that \( b^* \circ \imath_\pm = \text{id}_X \) and \( \pi = \imath_+ \cup \imath_- : X \cup X \to S(X, D) \) is the normalization map. We let \( X_\pm = \imath_\pm(X) \subset S(X, D) \) denote the two irreducible components of \( S(X, D) \). We shall often write \( S(X, D) \) as \( S_X \) when the divisor \( D \) is understood. \( S_X \) is a reduced quasi-projective scheme whose singular locus is \( D_{\text{red}} \subset S_X \). It is projective whenever \( X \) is so. It follows from [26 Lemma 2.2] that \( S_X \) is also a Cartesian square.

It is clear that the map \( Z_0(S_X, D) \xrightarrow{(\iota^*\,\iota^-)} Z_0(X_+, D) \oplus Z_0(X_-, D) \) is an isomorphism. Notice also that there are push-forward inclusion maps \( \ker : Z_0(X, D) \to Z_0(S_X, D) \) such that \( \iota^*_+ \circ \iota^+ \equiv \text{id} \) and \( \iota^*_- \circ \iota^- \equiv 0 \). The fundamental result that connects the 0-cycles with modulus on \( X \) and 0-cycles on \( S_X \) is the following.

**Theorem 2.4.** ([5 Theorem 1.9]) Let \( X \) be a smooth quasi-projective scheme over \( k \) and let \( D \subset X \) be an effective Cartier divisor. Then there is a split short exact sequence

\[
0 \to \text{CH}_0(X|D) \xrightarrow{p_*} \text{CH}_0(S_X) \xrightarrow{\iota_*} \text{CH}_0(X) \to 0.
\]

2.5. **Milnor K-theory sheaves.** Let \( A \) be a ring. Let \( T(A^\times) \) denote the \( \mathbb{Z} \)-tensor algebra over the group of units in \( A \). Recall that the Milnor \( K \)-groups \( K_i^M(A) \) of \( A \) is the \( i \)-th graded piece of the quotient of \( T(A^\times) \) by the homogenous ideal generated by \( a \otimes (1 - a) \in A^\times \otimes A^\times \) with \( a, 1 - a \in A^\times \). Given an ideal \( I \subset A \), we let \( K_i^M(A, I) = \text{Ker}(K_i^M(A) \to K_i^M(A/I)) \).

For \( a_1, \ldots, a_i \in A^\times \), we let \( \{a_1, \ldots, a_i\} \) denote the image of \( a_1 \otimes \cdots \otimes a_i \) in \( K_i^M(A) \). We shall frequently use the following description of \( K_i^M(A, I) \) for local rings from [19 Lemma 2.5].

**Lemma 2.5.** ([19 Lemma 1.3.1]) Let \( A \) be a finite product of local rings and let \( I \subset A \) be an ideal. Then \( K_i^M(A, I) \) coincides with the subgroup of \( K_i^M(A) \) generated by elements of the form \( \{a_1, \ldots, a_i\} \) such that \( a_j \in \text{Ker}(A^\times \to (A/I)^\times) \) for some \( j \).

We shall need the following local result later in the proof of Theorem 2.3. Let

\[
\begin{array}{ccc}
R & \xrightarrow{\psi_1} & A_1 \\
\downarrow \psi_2 & & \downarrow \phi_1 \\
A_2 & \xrightarrow{\phi_2} & B
\end{array}
\]

be a Cartesian square of rings.

**Lemma 2.6.** Associated to the Milnor square \((2.3)\), the restriction map \( \text{Ker}(K_i^M(R) \to K_i^M(A_2)) \to \text{Ker}(K_i^M(A_1) \to K_i^M(B)) \) is surjective if \( A_1 \) and \( A_2 \) are local rings.

**Proof.** We let \( J_i = \text{Ker}(\phi_i) \) and \( I_i = \text{Ker}(\psi_i) \) for \( i = 1, 2 \). We need to show that the map of relative Milnor \( K \)-groups \( K_i^M(R, I_2) \to K_i^M(A_1, J_1) \) is surjective. If \( q \leq 1 \), then it follows from [36 Theorem 6.2, Lemma 4.1] that this map is actually an isomorphism. So we assume \( q \geq 2 \).

It is easy to check that \( R \) is a local ring. It follows from Lemma 2.5 that \( K_i^M(A_1, J_1) \) is generated by the Milnor symbols \( \{b_1, \ldots, b_q\} \) such that \( b_j \in (1 + J_1)^\times \) for some \( 1 \leq j \leq q \). A similar presentation holds for \( K_i^M(R, I_2) \). We choose such a symbol \( \{b_1, \ldots, b_q\} \) \( K_i^M(A_1, J_1) \).

Suppose that \( b_j \in (1 + J_1)^\times \) for some \( 1 \leq j \leq q \). Since the map \( R^\times \to A_1^\times \) is surjective, we can find \( b'_i \in R^\times \) such that \( \psi_1(b'_i) = b_i \) for \( 1 \leq i \neq j \leq q \). Furthermore, we have isomorphism \( (1 + I_2)^\times = K_1^M(R, I_2) \xrightarrow{\psi_2} K_i^M(A_1, J_1) = (1 + J_1)^\times \) by \( q = 1 \) case. So we can choose \( b'_j \in (1 + I_2)^\times \) such that \( \psi_1(b'_j) = b_j \). It is now immediate that \( \{b'_1, \ldots, b'_q\} \) \( K_i^M(R, I_2) \) and \( \psi_1(\{b'_1, \ldots, b'_q\}) = \{b_1, \ldots, b_q\} \). This finishes the proof.

**Definition 2.7.** For a scheme \( X \) and closed immersion \( i : Y \hookrightarrow X \), we let \( K_i^M(X, Y) \) denote the Zariski (resp. Nisnevich) sheaf on \( X_{\text{zar}} \) (resp. \( X_{\text{nis}} \) associated to the presheaf \( U \mapsto \text{Ker}(K_i^M(\Gamma(O_U)) \to K_i^M(\Gamma(O_{Y \times X \times U}))) \).
Since the Zariski or the Nisnevich cohomology of the push-forward sheaf $\iota_*(\mathcal{K}_{r,Y}^M)$ coincides with that of $\mathcal{K}_{r,Y}^M$, we shall not distinguish between these two sheaves in the sequel. It follows immediately from the above definition that there is a short exact sequence of Zariski (or Nisnevich) sheaves

\[(2.4) \quad 0 \to \mathcal{K}_{r,(X,Y)}^M \to \mathcal{K}_{r,X}^M \to \mathcal{K}_{r,Y}^M \to 0.\]

2.6. Rülling-Saito relative Milnor $K$-theory. Let $k$ be a field. Let $X$ be a smooth scheme over $k$ and let $D$ be an effective Cartier divisor on $X$. Then Rülling and Saito \cite{Rullingsaito} defined a variant of relative Milnor $K$-groups of the pair $(X,D)$ as follows.

For a smooth scheme $X$, we let $\hat{\mathcal{K}}_{r,X}^M$ denote the kernel of the map of Zariski sheaves

\[\bigcup_{x \in X} (i_x)_*(K^M_i(x)) \to \bigcup_{x \in X} (i_x)_*(K^M_{i-1}(x)).\]

Note that, Kerz \cite{Kerz2007} gave a description of these sheaves. Moreover, there exists a natural surjective map $\mathcal{K}_{r,X}^M \to \hat{\mathcal{K}}_{r,X}^M$ which is an isomorphism at the generic points of $X$. By \cite{Kerz2005}, it also follows that this map is an isomorphism if $k$ is an infinite field.

Let $j : V \to X$ denote the compliment $X \setminus D$. Then the Zariski sheaf $\mathcal{K}_{r,X|D}^M$ is defined to be the image of the natural map

\[(2.5) \quad \text{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_D^\times) \otimes_k j_* \hat{\mathcal{K}}_{r-1,V}^M \to j_* \hat{\mathcal{K}}_{r,V}^M, \quad a \otimes \{b_1, \ldots, b_{r-1}\} \mapsto \{a, b_1, \ldots, b_{r-1}\}.\]

In particular, $\mathcal{K}_{r,X|D}^M = 0$ for $r \leq 0$ and $\mathcal{K}_{r,X,D}^M = \text{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_D^\times)$. Observe that by definition $\mathcal{K}_{r,X|D}^M$ is the subsheaf of the Zariski sheaf $\bigcup_{x \in X} (i_x)_*(K^M_i(x))$. We let $\mathcal{K}_{r,X|D,nis}^M$ denote the Nisnevich sheaf associated to the presheaf on the small Nisnevich site of $X$

\[(2.6) \quad X_{nis} \to \text{(abelian groups)}, \quad (v : V \to X) \mapsto H^0(V, \mathcal{K}_{r,V|n, nis}^M).\]

**Lemma 2.8.** Let $k$ be an infinite filed and let $(X,D)$ be as above. Then there exists an injective map of sheaves $\mathcal{K}_{r,(X,D),nis}^M \to \mathcal{K}_{r,X|D,nis}^M$ and it is an isomorphism if $D$ is smooth. Moreover, if the support of $D$ has a simple normal crossing, then the map induces an isomorphism of pro-sheaves

\[
\lim_n \mathcal{K}_{r,(X,nD),nis}^M \to \lim_n \mathcal{K}_{r,X|n,D,nis}^M.
\]

**Proof.** By the definition of the Nisnevich sheaves $\mathcal{K}_{r,(X,D),nis}^M$ and $\mathcal{K}_{r,X|D,nis}^M$ (Definition 2.7 and (2.6)), it suffices to show that the claims of Lemma 2.8 hold for the Zariski sheaves $\mathcal{K}_{r,(X,D)}^M$ and $\mathcal{K}_{r,X|D}^M$. Since $k$ is infinite, we have $\mathcal{K}_{r,V}^M = \hat{\mathcal{K}}_{r,V}^M$. For the existence of the injective map, consider the following diagram:

\[
\begin{array}{ccc}
\text{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_D^\times) & \otimes_k & \mathcal{K}_{r-1,X}^M \\
\downarrow \text{id} \otimes j^* & & \downarrow j^* \\
\text{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_D^\times) & \otimes_k & j_* \hat{\mathcal{K}}_{r-1,V}^M \\
& & \downarrow (i_x)_* (K^M_i(x)) \\
& & \bigcup_{x \in X} (i_x)_*(K^M_i(x)).
\end{array}
\]

It follows easily that the square and the triangle in (2.7) commute. By Lemma 2.6 and \cite{Kerz2005} Lemma 2.2, the image of the top horizontal arrow in (2.7) is the sheaf $\mathcal{K}_{r,(X,D)}^M$ while by definition, the image of the bottom left horizontal arrow is the sheaf $\mathcal{K}_{r,X|D}^M$. Therefore, there exists an inclusion of the Zariski sheaves $\mathcal{K}_{r,(X,D)}^M \to \mathcal{K}_{r,X|D}^M$. 

Now, assume that the support of \(D\) has simple normal crossings. Then [41, Proposition 2.8] gives a description of \(K^M_{r(X,D),x}\) for \(x \in D\). It follows from this description that \(K^M_{r,X}\) is a subsheaf of Zariski sheaf \(K^M_{r,X}\), and for \(m \geq 2\), we have

\[
K^M_{r(X,mD)} \hookrightarrow K^M_{r,X|mD} \hookrightarrow K^M_{r(X,(m-1)D)} \hookrightarrow K^M_{r,X}.
\]

Therefore, we have an isomorphism of pro-sheaves 

\[
\lim_{\rightarrow} K^M_{r(X,nD),\text{nis}} \cong \lim_{\rightarrow} K^M_{r(X,nD),\text{nis}}
\]

Now, let \(D\) be smooth. Let \(x \in D\) and let \(\eta \in X\) be the generic point of the component of \(X\) which contains \(x\). Since \(D\) is smooth, it is defined at \(x\) by an irreducible element \(t \in \mathcal{O}_{X,x}\). Then by [41, Proposition 2.8], \(K^M_{r,X,D,\eta}\) is the subgroup of \(K^M_r(k(\eta))\) generated by the elements of the form \(\{1+at, u_2, \ldots, u_r\}\) and \(\{1+b, 1+u_1t, u_3, \ldots, u_r\} = \{1+u_1t, 1+b, u_3, \ldots, u_r\}\), where \(a \in \mathcal{O}_{X,x}\) and \(1+b, u_1 \in \mathcal{O}^X_{X,x}\). Therefore, by Lemma 2.5, the natural inclusion \(K^M_{r(X,D)} \hookrightarrow K^M_{r,X,D}\) is an isomorphism. This completes the proof of the lemma. \(\square\)

### 2.7. Gersten and Cousin complexes

Let \(k\) be a perfect field. Let \(X\) be an equi-dimensional scheme of dimension \(d\) which is a localization of a reduced quasi-projective scheme over \(k\). For any point \(x \in X\), let \(K^i_M(x) = K^i_M(k(x))\). Let \(X^{(q)}\) be the set of codimension \(q\) points on \(X\). For any \(x \in X^{(q)}\) and \(y \in X^{(q+1)}\), let \(Z = \{x\}\). We let \(\partial^M_{x,y} : K^i_M(x) \to K^i_{M-1}(y)\) be the map

\[
\partial^M_{x,y} = \begin{cases} 0 & \text{if } y \notin Z \\ \sum_{z \mid y} N_{k(z)/k(y)} \circ \partial_z & \text{otherwise}, \end{cases}
\]

where \(z\) runs through the closed points in \(Z^N\) over \(y\) and \(\partial_z : K^i_M(k(x)) = K^i_M(k(Z^N)) \to K^i_{M-1}(k(z))\) is the classical boundary map on the quotient field of a dvr defined in [3].

Recall from [18, Proposition 1] (see also [40, Lemma 3.3] for a generalization) that there is a Gersten complex of Zariski sheaves

\[
0 \to K^M_{i,X} \to \bigcup_{x \in X^{(0)}} (i^{-1}x)_* (K^M_i(x)) \to \bigcup_{x \in X^{(1)}} (i^{-1}x)_* (K^M_i(x)) \to \ldots
\]

\[
\ldots \to \bigcup_{x \in X^{(d-1)}} (i^{-1}x)_* (K^M_{i-d+1}(x)) \xrightarrow{\partial^M_{x,y}} \bigcup_{x \in X^{(d)}} (i^{-1}x)_* (K^M_{i-d}(x)),
\]

where \(\epsilon\) is the usual restriction map to the generic points. The other boundary maps consist of the sums of homomorphisms \(\partial^M_{x,y}\) for \(x \in X^{q}, y \in X^{q+1}\).

For a Zariski sheaf \(\mathcal{F}\) on \(X\) and a point \(x \in X\) (not necessarily closed), recall that \(H^q_x(X_{zar}, \mathcal{F})\) is defined as the colimit \(\lim_{\rightarrow} H^q_x(U, \mathcal{F}|_U)\), where the limit is over all open neighborhoods of \(x\) in \(X\). The Nisnevich cohomology \(H^q_x(X_{nis}, \mathcal{F})\) is defined in an analogous way.

Recall also that for any Zariski sheaf of abelian groups \(\mathcal{F}\) on \(X\), the filtration by codimension of support (coniveau filtration) of the Zariski cohomology with support gives rise to the Cousin complex of Zariski cohomology sheaves

\[
C^*_{\mathcal{F}} : \bigcup_{x \in X^{(0)}} (i^{-1}x)_* H^q_x(X, \mathcal{F}) \to \bigcup_{x \in X^{(1)}} (i^{-1}x)_* H^q_{x+1}(X, \mathcal{F}) \to \ldots
\]

\[
\ldots \to \bigcup_{x \in X^{(d-1)}} (i^{-1}x)_* H^q_{x+d-1}(X, \mathcal{F}) \xrightarrow{\partial^S_{x,y}} \bigcup_{x \in X^{(d)}} (i^{-1}x)_* H^q_{x+d}(X, \mathcal{F})
\]

with a map \(\epsilon : H^q(\mathcal{F}) \to C^*_{\mathcal{F}}\), where \(H^q(\mathcal{F})\) is the Zariski sheaf on \(X\) associated to the presheaf \(U \mapsto H^q_{zar}(U, \mathcal{F})\). Let \(f_S : \bigcup_{x \in X^{(d)}} H^d_x(X, \mathcal{F}) \to H^d_{zar}(X, \mathcal{F})\) denote the sum of the ‘forget support’ maps \(f_{S,x} : H^d_x(X, \mathcal{F}) \to H^d_{zar}(X, \mathcal{F})\).
For \( x \in X^{(q)} \) and \( y \in X^{(q+1)} \), let \( \partial^{S}_{x,y} : H^{n}_{x}(X, F) \to H^{n+1}_{y}(X, F) \) be the composite map
\[
(2.11) \quad H^{n}_{x}(X, F) \to \bigcup_{x \in X^{(q)}} H^{n}_{x}(X, F) \xrightarrow{\partial^{S}} \bigcup_{y \in X^{(q+1)}} H^{n+1}_{y}(X, F) \to H^{n+1}_{y}(X, F).
\]

**Lemma 2.9.** The Cousin complex induces an exact sequence of Zariski cohomology
\[
(2.12) \quad \bigcup_{x \in X^{(d)}} H^{d-1}_{x}(X, F) \xrightarrow{\partial^{S}} \bigcup_{x \in X^{(d)}} H^{d}_{x}(X, F) \xrightarrow{f^{q}} H^{d}_{\text{zar}}(X, F) \to 0.
\]

**Proof.** The complex \((2.10)\) gives rise to a spectral sequence \( E_{1}^{pq} = \bigcup_{x \in X^{(q)}} H^{p+q}_{x}(X, F) \Rightarrow H^{p+q}_{\text{zar}}(X, F) \). The lemma is now an easy consequence of the fact that \( H^{i}_{x}(X, F) \cong H^{i}_{x}(X, F) \) for every \( x \in X^{(q)} \) and \( i \geq 0 \) by excision, and \( \dim(X_{x} \setminus \{x\}) = q - 1 \). \( \square \)

2.8. **Algebraic K-theory.** Given a scheme \( X \), we let \( K(X) \) denote the Bass-Thomason-Trobaugh non-connective K-theory spectrum of the biWaldhausen category of perfect complexes on \( X \). This coincides with the K-theory spectrum of the exact category of locally free sheaves if \( X \) is regular. We let \( K_{i}(X) \) denote the stable homotopy groups of the spectrum \( K(X) \) for \( i \in \mathbb{Z} \). Given a map \( f : Y \to X \) of schemes, we let \( K(X, Y) \) denote the homotopy fiber of the map of spectra \( f^{*} : K(X) \to K(Y) \). If \( f \) is an open immersion, we write \( K(X, Y) \) as \( K^{X \setminus Y}(X) \). We let \( K_{i}(X, Y) \) denote the stable homotopy groups of the spectrum \( K(X, Y) \) for \( i \in \mathbb{Z} \) and we denote by \( \mathcal{K}_{i}(X, Y) \) the image of the natural map \( K_{i}(X, Y) \to K_{i}(X) \). Let \( \mathcal{K}_{i,X} \) denote the Zariski (or Nisnevich) sheaf on \( X \) associated to the presheaf \( U \mapsto K_{i}(U) \). The sheaves \( \mathcal{K}_{i}(X, Y) \) and \( \mathcal{K}_{i}(X, Y) \) are defined similarly.

For \( X = \text{Spec}(A) \) and an ideal \( I \subset A \) with \( Y = \text{Spec}(A/I) \), we shall use the identifications \( K(X) \cong K(A) \) and \( K(X, Y) \cong K(A, I) \). The ring structure on \( K_{*}(A) \) and the natural map \( K^{M}_{1}(A) = A^{\times} \to K_{1}(A) \) define the maps of presheaves \( \mathcal{K}^{M}_{i,X} \to \mathcal{K}^{M}_{i,X} \) and \( \mathcal{K}^{M}_{i}(X, Y) \to \mathcal{K}^{M}_{i}(X, Y) \). Let \( f : Y \to X \) be a closed immersion such that \( \dim(Y) < \dim(X) = d \). Since the kernel of the surjective map \( \mathcal{K}_{i}(X, Y) \to \mathcal{K}_{i}(X, Y) \) is supported on \( Y \), the induced map \( H^{d}_{\text{nis}}(X, \mathcal{K}_{i}(X, Y)) \to H^{d}_{\text{nis}}(X, \mathcal{K}_{i}(X, Y)) \) is an isomorphism. Therefore, there is a natural map \( H^{d}_{\text{nis}}(X, \mathcal{K}^{M}_{i}(X, Y)) \to H^{d}_{\text{nis}}(X, \mathcal{K}^{M}_{i}(X, Y)) \).

3. **The Bloch-Quillen map for 0-cycles.**

The Bloch-Quillen-Kato formula for smooth schemes is immediately proven using the Gersten resolution for the Milnor and Quillen K-theory sheaves. But it is not hard to see that the Gersten complex in its current form can not give an acyclic resolution for the Milnor or Quillen K-theory sheaves on singular schemes. This poses a great difficulty in proving analogues of the Bloch-Quillen-Kato formula for singular schemes.

Due to the lack of the Gersten resolution, the construction of a Bloch-Quillen-Kato type map from the Chow group to the cohomology of the Milnor K-theory sheaf becomes the first major obstacle in proving the Bloch-Quillen-Kato formula for singular schemes. The goal of this section is to construct this map. The idea we use is to look at the Cousin complex instead of the Gersten complex. It is not hard to see that this complex does give an expression of the top cohomology of the Milnor K-theory sheaf in terms of the cohomology with supports. The problem then boils down to unraveling the appropriate boundary maps in the Cousin complex. In the rest of this section, we show how it is achieved.

3.1. **The map \( p_{X} \) on the group of 0-cycles.** Let \( k \) be a perfect field and let \( X \) be a reduced quasi-projective scheme of pure dimension \( d \geq 0 \) over \( k \). Let \( x \in X_{\text{reg}} \) be a closed point. We have the ‘forget support’ map \( f_{S,x} : H^{d}_{x}(X, \mathcal{K}^{M}_{d,X}) \to H^{d}_{\text{zar}}(X, \mathcal{K}^{M}_{d,X}) \) between the
Zariski cohomology groups. By [18, Theorem 2], there is, for every pair of integers \( n, q \geq 0 \) and \( x \in X^{(q)} \), a canonical isomorphism

\[
(3.1) \quad \rho_X : K^M_{n-q}(k(x)) \xrightarrow{=} H^q_x(X_{\text{zar}}, K^M_{n,X}).
\]

In particular, for \( x \in X^{(d)} \), we have \( Z \cong K^M_0(k(x)) \) \( \xrightarrow{\rho_x} H^d_x(X, K^M_{d,X}) \). We let \( \rho^\text{zar}_X([x]) \) denote the image of \( 1 \in K^M_0(k(x)) \) in \( H^d_{\text{zar}}(X, K^M_{d,X}) \) under the forget support map. Extending this linearly, we obtain a map \( \rho^\text{zar}_X : Z_0(X) \rightarrow H^d_{\text{zar}}(X, K^M_{d,X}) \), which we shall call 'the Zariski Bloch-Quillen map'. Composing this with the canonical map \( H^d_{\text{zar}}(X, K^M_{d,X}) \rightarrow H^d_{\text{nis}}(X, K^M_{d,X}) \), we obtain our main object of study: the (Nisnevich) Bloch-Quillen map

\[
(3.2) \quad \rho_X : Z_0(X) \rightarrow H^d_{\text{nis}}(X, K^M_{d,X}).
\]

Since \( x \) is a regular point of \( X \), the excision property of the cohomology with support tells us that the map \( H^d_x(X, K_{d,X}) \rightarrow H^d_x(X_{\text{reg}}, K_{d,X_{\text{reg}}}) \) is an isomorphism. By Gersten resolution for the Quillen \( K \)-theory sheaf \( K_{d,X_{\text{reg}}} \), we have an isomorphism \( Z \cong K_0(k(x)) \) \( \xrightarrow{\rho_x} H^d_x(X, K_{d,X}) \). As before, this gives a Bloch-Quillen map

\[
\rho'_X : Z_0(X) \rightarrow H^d_{\text{nis}}(X, K_{d,X}).
\]

**Lemma 3.1.** With the notations as above, the diagram

\[
(3.3) \quad \begin{array}{ccc}
Z_0(X) & \xrightarrow{\rho_X} & H^d_{\text{nis}}(X, K^M_{d,X}) \\
\downarrow{\rho'_X} \quad & & \downarrow{\kappa} \\
H^d_{\text{nis}}(X, K_{d,X}) \\
\end{array}
\]

is commutative, where the arrow going down on the right is induced by the canonical map from the Milnor to Quillen \( K \)-theory sheaves.

**Proof.** By definitions of \( \rho_X \) and \( \rho'_X \), it suffices to show more generally that for \( x \in X_{\text{reg}} \cap X^{(q)} \) and \( n \geq 0 \), there is a commutative diagram

\[
(3.4) \quad \begin{array}{ccc}
K^M_{n-q}(k(x)) & \xrightarrow{\rho_x} & H^0_x(X, K^M_{n,X}) \\
\downarrow{\kappa} \quad & & \downarrow{\kappa} \\
K_{n-q}(k(x)) & \xrightarrow{\rho'_x} & H^0(x, K_{n,X})
\end{array}
\]

where first row is the isomorphism of \( (3.1) \) and the bottom row is an isomorphism by Gersten resolution for the Zariski sheaf \( K_{n,X_{\text{reg}}} \) by Quillen [39].

We prove that \( (3.4) \) commutes by induction on \( q \). If \( q = 0 \) and we let \( \eta_x = \text{Spec}(k(x)) \), then the terms on the right are \( H^0(\eta_x, K^M_{n,\eta_x}) \) and \( H^0(\eta_x, K_{n,\eta_x}) \). Moreover, \( \rho_x \) and \( \rho'_x \) are defined (in [18] and [39]) to be the canonical isomorphisms \( K^M_n(k(x)) \xrightarrow{=} H^0(\eta_x, K^M_{n,\eta_x}) \) and \( K_n(k(x)) \xrightarrow{=} H^0(\eta_x, K_{n,\eta_x}) \). The diagram then clearly commutes.
We now assume \( q \geq 1 \). We let \( T = (X_x)^{(q-1)} \) and consider the following diagram

\[
\begin{array}{c}
\bigcup_{y \in T} H^{q-1}_y(X, K^M_{n,X}) \\
\downarrow \rho_x \\
\bigcup_{y \in T} K^{M}_{n-q+1}(k(y)) \\
\downarrow \partial^M_x \\
K_{n-q}(k(x))
\end{array}
\]

We want to show that the right face of the above cube commutes. Since the map \( \partial^M_x : \bigcup_{y \in T} K^{M}_{n-q+1}(k(y)) \to K_{n-q}(k(x)) \) is surjective by [18 Theorem 1], it suffices to show that all other faces of (3.5) commute. The left face commutes by induction hypothesis. The top face commutes by the naturality of the theory of supports. The commutativity of the back face is part of Kato’s definition of \( \rho_x \) (see [18 § 4]). The bottom face commutes because of the well known fact that the canonical map from the Milnor \( K \)-theory to the Quillen \( K \)-theory commutes with the boundary maps in the Gersten complexes on the regular scheme \( X_x \). Finally, the exactness of the Gersten complex for \( \mathcal{K}_{n,X} \) allows us to use this resolution to compute the boundary map in the long exact sequence for support cohomology. It follows that under the isomorphism \( \rho'_x \), given by the resolution, the front face commutes.

3.2. Compatibility with the Bloch-Quillen map to \( K \)-theory. Before we prove that \( \rho_X \) kills the 0-cycles which are rationally equivalent to zero, we explain how it is compatible with the cycle class map \( \lambda_X : Z_0(X) \to K_0(X) \). Recall that any regular closed point \( x \in X \) has the property that the inclusion map \( \text{Spec}(k(x)) \to X \) is a regular embedding. In particular, there is a push-forward map \( \iota_{x,*} : K_0(k(x)) \to K_0(X) \) and \( \lambda_X([x]) \) is the image of 1 \( \in K_0(k(x)) \) under this map. It is shown in [33 Proposition 2.1] that \( \lambda_X \) factors through the rational equivalence to give a cycle class map \( \lambda_X : \text{CH}_0^{\text{LW}}(X) \to K_0(X) \). It is further shown in [6 Lemma 3.13] that it factors through the modified Chow group. Hence we have the maps

\[
\lambda_X : \text{CH}_0^{\text{LW}}(X) \to \text{CH}_0(X) \to K_0(X).
\]

The Nisnevich descent spectral sequence of Thomason and Trobaugh [47] gives rise to a natural map \( \kappa_X : H^d_{\text{nis}}(X, \mathcal{K}^M_{d,X}) \to H^d_{\text{nis}}(X, \mathcal{K}_{d,X}) \to K_0(X) \).

**Lemma 3.2.** There is a commutative diagram

\[
\begin{array}{c}
Z_0(X) \\
\downarrow \lambda_X \\
\bigoplus \mathcal{K}^M_{d,X} \\
\downarrow \kappa_X \\
K_0(X)
\end{array}
\]
Proof. By Lemma 3.3, it suffices to show that (3.7) commutes if we replace $H^d_{\text{nis}}(X, \mathcal{K}^M_{d,X})$ by $H^d_{\text{nis}}(X, \mathcal{K}_{d,X})$. Furthermore, we have a diagram

$$Z_0(X) \xrightarrow{\rho'_X} H^d_{\text{zar}}(X, \mathcal{K}_{d,X}) \xrightarrow{E} H^d_{\text{nis}}(X, \mathcal{K}_{d,X})$$

in which the triangle on the right commutes. We can therefore work with the Zariski cohomology. Note here that the map $H^d_{(x)}(X_{\text{zar}}, \mathcal{K}^M_{d,X}) \rightarrow H^d_{(x)}(X_{\text{nis}}, \mathcal{K}^M_{d,X})$ is an isomorphism for $x \in X_{\text{reg}}$.

We fix a closed point $x \in X_{\text{reg}}$ and let $S = \text{Spec}(k(x))$. We consider the diagram

$$Z(x) \xrightarrow{\zeta} H^0(S, \mathcal{K}_{0,S}) \xrightarrow{\rho'_X} H^d_x(X_{\text{reg}}, \mathcal{K}^M_{d,X_{\text{reg}}}) \xrightarrow{\zeta} H^d_x(X_{\text{zar}}, \mathcal{K}_{d,X}) \xrightarrow{\ddagger} H^d_{\text{zar}}(X, \mathcal{K}_{d,X})$$

By the definition of $\rho'_X$, the image of $x$ in $Z(x)$ maps under the composition of the top row of the diagram to $\rho'_X(x) \in H^d_{\text{zar}}(X, \mathcal{K}_{d,X})$. The composition of the bottom row sends $x$ to the element $\lambda_X([x]) \in K_0(X)$. It suffices therefore to show that all squares in (3.9) commute. The middle square commutes by the naturality of the Zariski descent spectral sequence of Thomason-Trobaugh for pull back along open immersion while the right square in (3.9) commutes by [47, Corollaries 10.5, 10.10]. We are left to show that the left square in the diagram commutes. But this is a direct consequence of the comparison between the Thomason-Trobaugh and Quillen spectral sequences for the $K$-theory of the regular scheme $X_{\text{reg}}$ with support.

Indeed, the vertical arrows in the left square in (3.9) are the edge maps of the Thomason-Trobaugh spectral sequences for $K_0(S)$ and $K^S_0(X_{\text{reg}})$. Equivalently, these are the edge maps of the Brown-Gersten hypercohomology spectral sequences for $K_0(S)$ and $K^S_0(X_{\text{reg}})$ [see, for example, [47, Proof of Theorem 10.3]]. On the other hand, it follows from [15, Corollary 74] that the Brown-Gersten hypercohomology spectral sequences for $K_0(S)$ and $K^S_0(X_{\text{reg}})$ coincide with the corresponding Quillen spectral sequences from $E_2$-page onwards. So we can identify the two vertical arrows of the left square in (3.9) with the edge maps of the Quillen spectral sequences for $K$-theory with support. We are now done because the top horizontal arrow in this square is induced by the push-forward map on the Quillen spectral sequences (see [15, § 2.5.4, Theorem 65]) and the bottom horizontal arrow is the push-forward map on the limits of these spectral sequences. \[\square\]

3.3. The boundary maps in Gersten and Cousin complexes. We shall now prove a general result which will be the key step in the proof of the factorization of the Bloch-Quillen map through the rational equivalence. We begin with the following elementary but useful observation from commutative algebra.

Lemma 3.3. If $A$ is a reduced ring, then all its associated primes are minimal.

Proof. Suppose that there is a strict inclusion of associated primes $p \subset q$. Let $\overline{A} = A/p$ and let $\overline{q}$ be the image of $q$ in $\overline{A}$. We can write $q = \text{ann}(a)$ for some $a \in A$. Since $A$ is reduced, it follows that $a \notin q$ and in particular, $\overline{a} \neq 0$. On the other hand, $p \subset q$ implies that there exists $0 \neq \overline{b} \in \overline{q}$. Since $\overline{ab} = 0$, we reach a contradiction as $\overline{A}$ is an integral domain. \[\square\]
Let $k$ be any field. Let $X$ be a reduced quasi-projective scheme of pure dimension $d \geq 2$ over $k$. Let $n \geq 0$ be an integer. For a Zariski sheaf $\mathcal{F}$ on $X$, let

$$
\partial^S : \bigcup_{x \in X^{(q)}} (i_x)_* H^q_x(X, \mathcal{F}) \to \bigcup_{y \in X^{(q+1)}} (i_y)_* H^{n+1}_y(X, \mathcal{F})
$$

be the boundary map of the Cousin complex (2.10).

Since $\partial^S x \cdot = 0$ if $y \notin \overline{x}$, as follows from the construction of (2.10), we have a commutative diagram (where $y \in z$ means $y \in \{z\}$ and $\partial^S_y = \sum_{y \in z \in X^{(q)}} \partial^S_{z,y}$)

We now restrict to the case where $\mathcal{F}$ is a Milnor $K$-theory sheaf. Let $Y \subset X$ be a reduced closed subscheme and let $y \in Y^{(1)}$. For a generic point $x$ of $Y$, let $\phi_{x,y} : K^n_M(\mathcal{O}_{Y,y}) \to K^n_M(k(x))$ be zero if $y \notin \overline{x}$ and otherwise, we let it be the composition $K^n_M(\mathcal{O}_{Y,y}) \to \bigcup_{y \in z \in Y^{(q)}} K^n_M(k(z)) \to K^n_M(k(x)))$ along the composition $\mathcal{O}_{Y,y} \to \prod_{y \in z \in Y^{(q)}} k(z) \to k(x)$.

We set

$$
\Phi_{Y,y} = \bigcup_{x \in Y^{(q)}} \phi_{x,y} : K^n_M(\mathcal{O}_{Y,y}) \to \bigcup_{x \in Y^{(q)}} K^n_M(k(x)).
$$

Let $k$ be an infinite field. In this case, Kerz [20] has shown that the Milnor $K$-theory sheaf on $X_{\text{reg}}$ has a Gersten resolution. In fact, it is easy to verify that the Gersten complex of Kerz coincides with the one defined earlier by Kato [18]. This implies in particular that for a point $x \in X_{\text{reg}}$, there is a canonical isomorphism

$$
\psi_x : K^n_{n-m}(k(x)) \cong H^m_x(X, \kappa_{n,X}).
$$

Moreover, the isomorphism $\psi_x$ is same as the map $\rho_x$ in (3.11). We shall use this identification throughout this text. The key step in the proof of the factorization of the Bloch-Quillen map through the rational equivalence is provided by the following.

**Proposition 3.4.** Let $X$ be a reduced quasi-projective scheme of pure dimension $d \geq 2$ over an infinite field and let $n \geq 0$ be an integer. Let $Y_{d-1} \subset Y_{d-2} \subset \cdots \subset Y_1 \subset Y_0 = X$ be a sequence of reduced closed subschemes such that the following hold.

1. $Y_i$ has pure codimension one in $Y_{i-1}$.
2. For each $1 \leq i \leq d-1$, there exists a line bundle $\mathcal{L}_i$ on $Y_{i-1}$ with a section $s_i \in \Gamma(Y_{i-1}, \mathcal{L}_i)$ such that $Y_i$ is the zero-locus of $s_i$.
3. For each $1 \leq i \leq d-1$, the subset $Y_i \cap X_{\text{sing}}$ is nowhere dense in $Y_i$.

Then for each $0 \leq i \leq d-1$ and $y \in Y_i^{(1)}$, the composition map

$$
K^n_{n-i}(\mathcal{O}_{Y_i,y}) \xrightarrow{\Phi_{Y_i,y}} \bigcup_{x \in Y_i^{(1)}} K^n_{n-i}(k(x)) \xrightarrow{\psi_x} \bigcup_{x \in Y_i^{(1)}} H^i_x(X, \kappa_{n,X}) \xrightarrow{\partial^S_y} H^{i+1}_y(X, \kappa_{n,X})
$$

is zero.

**Proof.** We shall prove the proposition by induction on $i$. Before we do this, let us note that the isomorphism in the middle of (3.14) is by (3.13) and our assumption (3). We let $\Phi_{Y_i,y}$ denote the composition of the middle isomorphism in (3.14) with $\Phi_{Y_i,y}$.
STEP 1. We let \( i = 0 \) and fix a point \( y \in X^{(1)} \). The long exact sequence for the cohomology with support gives us an exact sequence (where \( j_{y}: X^{\circ}_{y} \to X_{y} \))

\[
H^{0}(X_{y}, K_{n,X_{y}}^{M}) \xrightarrow{j_{y}^{*}} H^{0}(X^{\circ}_{y}, K_{n,X_{y}}^{M}) \xrightarrow{\partial_{y}^{S}} H^{1}_{y}(X_{y}, K_{n,X_{y}}^{M}).
\]

We consider the diagram

\[
\begin{array}{c}
\Phi_{X,y} \quad \Phi_{X,y} \quad \Phi_{X,y} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
K_{n}^{M}(O_{X,y}) \quad K_{n}^{M}(O_{X,y}) \quad K_{n}^{M}(O_{X,y}) \\
\downarrow \quad \downarrow \quad \downarrow \\
H^{0}(\eta_{x}, K_{n,\eta_{x}}^{M}) \quad H^{0}(\eta_{x}, K_{n,\eta_{x}}^{M}) \quad H^{1}_{y}(X, K_{n,X}^{M}) \\
\downarrow \quad \downarrow \quad \downarrow \\
K_{n}^{M}(O_{X,y}) \quad K_{n}^{M}(O_{X,y}) \quad K_{n}^{M}(O_{X,y}) \\
\downarrow \quad \downarrow \quad \downarrow \\
\Phi_{X,y} \quad \Phi_{X,y} \quad \Phi_{X,y} \\
\end{array}
\]

We need to show that the composite arrow on the bottom row is zero. But this follows because the top composite arrow is zero and all the squares evidently commute. We just have to observe that \( \phi_{x,y} \) is simply the restriction of \( j_{y}^{*} \) to \( \eta_{x} \), by definition. This proves the base case \( i = 0 \).

STEP 2. Before we prove the proposition for \( i > 0 \), we claim that for every \( i > 0 \), the following hold.

1. The closed subscheme \( Y_{i} \subset Y_{i-1} \) is a Cartier divisor.
2. \( \mathcal{O}_{Y_{i-1},x} \) is a discrete valuation ring for every \( x \in Y_{i-1}^{(0)} \).
3. Every irreducible component of \( Y_{i} \) is contained in exactly one irreducible component of \( Y_{i-1} \).

We let \( w \in Y \) be a closed point and let \( a_{i} \) be the image of \( s_{i} \in \Gamma(Y_{i-1}, \mathcal{L}_{i}) \) under the restriction map \( \Gamma(Y_{i-1}, \mathcal{L}_{i}) \to \mathcal{O}_{Y_{i-1},w} \cong \Gamma(\mathcal{O}_{Y_{i-1},w}, \mathcal{L}_{i}|_{\mathcal{O}_{Y_{i-1},w}}) \). Since \( Y_{i-1} \) is reduced, it follows from the assumption (1) of the proposition and Lemma 3.3 that \( a_{i} \) is a non-zero divisor in \( \mathcal{O}_{Y_{i-1},w} \) and \( \mathcal{O}_{Y_{i-1},w} = \mathcal{O}_{Y_{i-1},w}/(a_{i}) \). This proves (1). If we let \( w \in \{ x \} \) for any \( x \in Y_{i}^{(0)} \subseteq Y_{i-1}^{(0)} \) and let \( p \) be the minimal prime of \( (a_{i}) \) defining \( x \), then the assumption that \( Y_{i} \) is reduced implies that \( \mathcal{O}_{Y_{i-1},p} \) is an 1-dimensional local ring whose maximal ideal \( p\mathcal{O}_{Y_{i-1},p} \) is generated by the image of \( a_{i} \) under the localization \( \mathcal{O}_{Y_{i-1},w} \to \mathcal{O}_{Y_{i-1},p} \). We conclude from [35, Theorem 11.2] that \( \mathcal{O}_{Y_{i-1},p} \) is a discrete valuation ring. This proves (2) and (3) is immediate from (2) as any intersection of two or more components of \( Y_{i-1} \) is part of its singular locus. This proves the claim.

STEP 3. We now assume \( i > 0 \). We fix a point \( y \in Y_{i}^{(1)} \) and an element \( \alpha \in K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},y}) \). Since the map \( K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},y}) \to K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},y}) \) is surjective, we can choose a lift \( \tilde{\alpha} \) of \( \alpha \) in \( K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},y}) \). Let \( a_{i} \) be the image of \( s_{i} \in \Gamma(Y_{i-1}, \mathcal{L}_{i}) \) in \( \mathcal{O}_{Y_{i-1},y} \) under the restriction map so that \( \mathcal{O}_{Y_{i-1},y} = \mathcal{O}_{Y_{i-1},y}/(a_{i}) \).

For any \( x \in Y_{i-1} \) such that \( y \in \{ x \} \), we let \( \tilde{\alpha}_{x} \) be the image of \( \tilde{\alpha} \) under the restriction map \( K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},y}) \to K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},x}) \). We let \( \pi_{x} \) be the image of \( \tilde{\alpha} \) under the composition \( K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},y}) \to K_{n-i}^{M}(\mathcal{O}_{Y_{i-1},x}) \to K_{n-i}^{M}(k(x)) \).

Let \( \{ p_{1}, \ldots, p_{m} \} \) be the set of minimal primes of \( (a_{i}) \) in \( \mathcal{O}_{Y_{i-1},y} \). These are the generic points of \( Y_{i} \) containing \( y \). If \( q \subset \mathcal{O}_{Y_{i-1},y} \) is a height one prime ideal such that \( q \notin \{ p_{1}, \ldots, p_{m} \} \), then we
must have $a_i \notin q$. It follows that any $x \in Y^{(1)}_i - Y^{(0)}_i$ such that $y \in \{x\}$, we have $a_i \in \mathcal{O}_{Y_i,x}$. In particular, there is an element $\beta_x = a_i \cdot \overline{\alpha}_x \in K^M_{n+i-1}(\mathcal{O}_{Y_i,x})$.

For $y \in \{z\}$ with $z \in Y^{(1)}_i$, let $a_{i,z}$ be the non-zero (as $a_i$ is a non-zero divisor by STEP 2) image of $a_i$ in $k(z)$. Let $\beta_z = a_{i,z} \cdot \overline{\alpha}_z$ if $y \in \{z\}$ and zero otherwise (note that $\overline{\alpha}_z = \overline{\alpha}_x$). Set

$$\beta = (\beta_z)_{z \in Y^{(0)}_i} \in \bigcup_{z \in Y^{(0)}_i} K^M_{n+i-1}(k(z)).$$

**STEP 4.** We claim that for any $x \in Y^{(1)}_i - Y^{(0)}_i$ such that $y \in \{x\}$, one has $\partial_x^S(\beta) = 0$ under the map $\bigcup_{z \in Y^{(0)}_i} K^M_{n+i-1}(k(z)) \to \bigcup_{z \in Y^{(0)}_i} H^{i-1}_z(X, \mathcal{K}_{n,X}) \to H^i_x(X, \mathcal{K}_{n,X})$.

We know by the induction hypothesis that the composition

$$K^M_{n+i-1}(\mathcal{O}_{Y_i,x}) \xrightarrow{\Phi_{Y_i,x}} \bigcup_{z \in Y^{(0)}_i} K^M_{n+i-1}(k(z)) \xrightarrow{\partial_x^S} \bigcup_{z \in Y^{(0)}_i} H^{i-1}_z(X, \mathcal{K}_{n,X}) \to H^i_x(X, \mathcal{K}_{n,X})$$

is zero.

In particular, we get $\partial_x^S(\Phi_{Y_i,x}(\beta_x)) = 0$. It suffices therefore to show that $\partial_x^S(\beta') = 0$ if we write $\beta' = \beta - \Phi_{Y_i,x}(\beta_x)$. Using (3.11), we only need to show that $\beta'_x = 0$ if $x \in \{z\}$. To prove this, we note that if $z \in Y^{(1)}_i$ is such that $x \in \{z\}$, then there is a factorization $\mathcal{O}_{Y_i,y} \to \mathcal{O}_{Y_i,x} \to \mathcal{O}_{Y_i,z} = k(z)$. It follows from the above construction in this case that $\beta_z = \phi_{x,z}(\beta_x)$. Equivalently, $\beta'_z = 0$. This proves the claim.

**STEP 5.** In this step, we shall study what happens to $\partial_x^S(\beta)$ when $x \in Y^{(1)}_i - Y^{(0)}_i$ and $y \in \{x\}$. We now recall from STEP 2 that if $x \in Y^{(1)}_i - Y^{(0)}_i$ is such that $y \in \{x\}$, then $\mathcal{O}_{Y_i,x}$ is a discrete valuation ring. It follows (see [3]) that for any $z \in Y^{(0)}_i$ with $x \in \{z\}$ and $y \in \{x\}$, one has $\partial_x^S(\beta_z) = \partial_x^S(a_{i,z} \cdot \overline{\alpha}_z) = \overline{\alpha}_x$ under the boundary map $\partial^S_x : K^M_{n+i-1}(k(z)) \to K^M_{n-i}(k(x))$. Using the commutative diagram

$$K^M_{n-1}(\mathcal{O}_{Y_i,y}) \xrightarrow{\phi_{x,y}} K^M_{n-1}(\mathcal{O}_{Y_i,x})$$

\begin{tikzcd}
K^M_{n-1}(\mathcal{O}_{Y_i,y}) \arrow[r, equal] \arrow[d, equal] & K^M_{n-1}(\mathcal{O}_{Y_i,x}) \arrow[d, \phi_{x,y}] \\
K^M_{n-1}(\mathcal{O}_{Y_i,y}) & K^M_{n-1}(k(x))
\end{tikzcd}

we get $\partial^M_{x,y}(\beta_z) = \phi_{x,y}(\alpha)$.

Let $z \in Y^{(0)}_i$ be the unique point such that $x \in \{z\}$ by STEP 2. Since $k$ is infinite and $x \in X_{\text{reg}}$ by assumption (3) of the proposition, it follows from the Gersten resolution of $\mathcal{K}^M_{n,X_{\text{reg}}}$ by Kerz [20] that there is a commutative diagram

$$K^M_{n+i-1}(k(z)) \xrightarrow{\partial^M_{z,x}} K^M_{n-i}(k(x))$$

\begin{tikzcd}
K^M_{n+i-1}(k(z)) \arrow[r, equal] \arrow[d, equal] & K^M_{n-i}(k(x)) \arrow[d, equal] \\
H^{i-1}_z(X, \mathcal{K}_{n,X}) & H^i_x(X, \mathcal{K}_{n,X}).
\end{tikzcd}

If we identify the top and the bottom rows of (3.19) (see (3.13)), and combine this with (3.18), we see that for any $x \in Y^{(1)}_i$ such that $y \in \{x\}$, one has

$$\partial_x^S(\beta) = \partial^M_{z,x}(\beta_z) = \phi_{x,y}(\alpha) \in H^i_x(X, \mathcal{K}_{n,X}).$$

We note here that the first equality uses the uniqueness of $z \in Y^{(0)}_i$ such that $x \in \{z\}$. 


STEP 6. In the final step, we consider the commutative diagram

\[
\begin{array}{c}
\beta \in \bigcup_{z \in Y_{i-1}^{(0)}} H_{i-1}^{z}(X, K_{n,X}^{M}) \xrightarrow{\partial^{S}} \bigcup_{x \in Y_{i-1}^{(1)}} H_{i}^{x}(X, K_{n,X}^{M}) \xrightarrow{\partial^{S}_{y}} H_{i+1}^{y}(X, K_{n,X}^{M}) \\
\Phi_{Y_{i},y}(\alpha) \in \bigcup_{x \in Y_{i}^{(0)}} H_{i}^{x}(X, K_{n,X}^{M}) \xrightarrow{\partial^{S}_{y}} H_{i+1}^{y}(X, K_{n,X}^{M}).
\end{array}
\]

The top row of (3.21) is a complex, as one can immediately see from the Cousin complex \([2,10]\). We need to show that \(\partial^{S}_{y}(\Phi_{Y_{i},y}(\alpha)) = 0\). Equivalently, we need to show that \(\partial^{S}_{y} \circ \iota(\Phi_{Y_{i},y}(\alpha)) = 0\).

To show this last statement, let us write \(\alpha' = \iota(\Phi_{Y_{i},y}(\alpha))\). It suffices to show that \(\alpha' = \partial^{S}(\beta)\) for every \(x \in Y_{i-1}^{(1)}\) such that \(y \in \{x\}\). Suppose first that \(x \in Y_{i-1}^{(1)} \setminus Y_{i}^{(0)}\). In this case, \(\alpha'\) is anyway zero and \(\partial^{S}(\beta) = \partial^{S}_{y}(\beta) = 0\) by STEP 4. If \(x \in Y_{i}^{(0)}\), then \(\alpha' = \partial^{S}_{y}(\beta)\) by (3.20) in STEP 5. This completes the proof.

Remark 3.5. If we take \(n = \dim(X)\), then Proposition 3.4 and its proof remain valid over finite fields as well in view of \([18\) Theorem 2] and \([19\ 2.7.1]\).

4. Proof of Theorem 1.1

We shall prove Theorem 1.1 in this section. We begin by showing that \(\rho^{zar}\) factors through the rational equivalence classes.

4.1. Factorization of \(\rho^{zar}\) through rational equivalence. Let \(k\) be an infinite perfect field and let \(X\) be a reduced quasi-projective scheme of pure dimension \(d \geq 0\) over \(k\). Recall from §3.1 that the Bloch-Quillen map \(\rho^{zar}_{X} : Z_{0}(X) \to H^{d}_{zar}(X, K_{d,X}^{M})\) takes a regular closed point \(x \in X_{reg}\) to the image of \([x] \in K_{0}(k(x))\) under the forget support map \(K_{0}(k(x)) \cong H^{d}_{x}(X_{zar}, K_{d,X}^{M}) \to H^{d}_{zar}(X, K_{d,X}^{M})\).

Theorem 4.1. The Zariski Bloch-Quillen map induces a homomorphism

\[
\rho^{zar}_{X} : CH^{LW}_{0}(X) \to H^{d}_{zar}(X, K_{d,X}^{M}).
\]

In particular, the Nisnevich Bloch-Quillen map induces a surjective group homomorphism

\[
\rho_{X} : CH^{LW}_{0}(X) \to H^{d}_{nis}(X, K_{d,X}^{M}).
\]

Proof. For \(d = 1\), the theorem follows from \([33\ Proposition 1.4]\). We can therefore assume that \(d \geq 2\). We need to show that the map \(\rho^{zar}_{X} : Z_{0}(X) \to H^{d}_{zar}(X, K_{d,X}^{M})\) kills \(R^{LW}_{0}(X)\). It follows from \([32\ Lemmas 1.3, 1.4]\) that \(R^{LW}_{0}(X)\) is generated by \(\text{div}(f)\), where \(C \subset X\) is a Cartier curve and \(f \in O_{C,C \cap X_{sing}}\) such that the following hold.

1. There is a sequence of reduced closed subschemes \(C = Y_{d-1} \subset Y_{d-2} \subset \cdots \subset Y_{1} \subset Y_{0} = X\).
2. For each \(1 \leq i \leq d-1\), there is a line bundle \(\mathcal{L}_{i}\) on \(Y_{i-1}\) with a section \(s_{i} \in \Gamma(Y_{i-1}, \mathcal{L}_{i})\) such that \(Y_{i}\) is the zero-locus of \(s_{i}\).
3. \(Y_{i}\) has pure codimension one in \(Y_{i-1}\).
4. For each \(1 \leq i \leq d-1\), the subset \(Y_{i} \cap X_{sing}\) is nowhere dense in \(Y_{i}\).
By Lemma 2.9 it suffices to show the commutativity of the diagram

\[
\begin{array}{ccc}
\bigcup_{x \in X^{(d-1)}} H^d_x(X, \mathcal{K}^M_{d,X}) & \xrightarrow{\delta^S} & \bigcup_{x \in X^{(d)}} H^d_x(X, \mathcal{K}^M_{d,X}) \\
\bigcup_{x \in C^{(0)}} H^d_x(X, \mathcal{K}^M_{d,X}) & \equiv & \bigcup_{x \in X_{\text{reg}}} \mathcal{K}^M_1(k(x)) \\
\bigcup_{x \in C^{(0)}} K^M_1(k(x)) & \xrightarrow{\equiv} & \bigcup_{x \in X^{(d)}} H^d_x(X, \mathcal{K}^M_{d,X}) \\
\mathcal{O}^X_{C,C \cap X_{\text{sing}}} & \xrightarrow{\text{div}} & \mathcal{O}_0(X).
\end{array}
\]

We let \( \theta_C \) denote the composite of all vertical arrows on the left in (4.2). We fix a point \( y \in X^{(d)} \). It is clear that \( (\delta^S \circ \theta_C)_y = 0 = (\text{div})_y \) whenever \( y \notin C \). So we can assume that \( y \in C^{(1)} \).

Let us first assume that \( y \in C \cap X_{\text{sing}} \). We then have a commutative diagram

\[
\begin{array}{ccc}
\bigcup_{x \in C^{(0)}} K^M_1(k(x)) & \xrightarrow{=} & \bigcup_{x \in C^{(0)}} H^d_x(X, \mathcal{K}^M_{d,X}) \\
\bigcup_{x \in C^{(0)}} K^M_1(k(x)) & \xrightarrow{=} & \bigcup_{x \in X^{(d)}} H^d_x(X, \mathcal{K}^M_{d,X}) \\
\mathcal{O}^X_{C,C \cap X_{\text{sing}}} & \xrightarrow{\Phi_{C,y}} & \mathcal{O}^X_{C,y} \\
\mathcal{O}^X_{C,y} & \xrightarrow{\Phi_{C,y}} & \mathcal{O}^X_{C,C \cap X_{\text{sing}}} \\
\mathcal{O}^X_{C,C \cap X_{\text{sing}}} & \xrightarrow{\text{div}} & \mathcal{O}_0(X).
\end{array}
\]

On the other hand, Proposition 3.3 says that the composite of all bottom horizontal arrows in (4.3) is zero (note that \( \mathcal{O}^X_{C,y} \cong K^M_1(\mathcal{O}^X_{C,y}) \)). It follows that \( \delta^S_y \circ \theta_C = (\delta^S \circ \theta_C)_y = 0 \). Since the map \( \text{div} \) has support only on \( X_{\text{reg}} \), we also have \( (\text{div})_y = 0 \) so that we get \( (\delta^S \circ \theta_C)_y = (\text{div})_y \).

Suppose now that \( y \in C^{(1)} \cap X_{\text{reg}} \). In this case, we have a diagram

\[
\begin{array}{ccc}
\bigcup_{x \in C^{(0)}} K^M_1(k(x)) & \xrightarrow{=} & \bigcup_{x \in C^{(0)}} H^d_x(X, \mathcal{K}^M_{d,X}) \\
\bigcup_{x \in C^{(0)}} K^M_1(k(x)) & \xrightarrow{=} & \bigcup_{x \in X^{(d)}} H^d_x(X, \mathcal{K}^M_{d,X}) \\
\mathcal{O}^X_{C,C \cap X_{\text{sing}}} & \xrightarrow{\text{div}} & \mathcal{O}_0(X) \\
\mathcal{O}_0(X) & \xrightarrow{\delta^M} & \mathcal{O}_0(X) \\
\mathcal{O}_0(X) & \xrightarrow{\delta^S} & \mathcal{O}_0(X).
\end{array}
\]

Since \( X_{\text{reg}} \) is regular, it is well known that \( \delta^M \) coincides with the divisor map. In particular, the left square commutes. The right square commutes by the Gersten resolution of \( \mathcal{K}^M_{d,X} \) on \( X_{\text{reg}} \). But this implies that \( (\delta^S \circ \theta_C)_y = (\text{div})_y \). We have thus shown that (4.2) commutes. This shows that \( \rho^M_{\text{zar}} \) kills \( R^0_{LM}(X) \).

To show that \( \rho^M_X : CH^M_0(X) \to H^d_{\text{nis}}(X, \mathcal{K}^M_{d,X}) \) is surjective, it suffices to show that the map \( \rho^M_X : \mathcal{O}_0(X) = \bigcup_{x \in X^{(d)}} \mathcal{K}^M_0(k(x)) \to H^d_{\text{nis}}(X, \mathcal{K}^M_{d,X}) \) is surjective. But this follows from [19].

Theorem 2.5] since \( k \) is perfect and hence \( U := X_{\text{reg}} \) is nice in the sense of [19, Definition 2.2]. Moreover, \( U \) is dense in \( X \). The proof of the theorem is now complete. \( \square \)

4.2. **Theorem 1.1 for affine schemes.** As a consequence of Theorem 4.1 we can now prove Theorem 1.1 for affine schemes as follows.
Theorem 4.2. Let $k$ be an algebraically closed field and let $X$ be a reduced affine scheme of pure dimension $d \geq 0$ over $k$. Then the map

$$\rho_X : \text{CH}^d_{\text{LW}}(X) \to H^d_{\text{mis}}(X, \mathcal{K}^M_{d,X})$$

is an isomorphism.

Proof. In view of Theorem 4.1 we only need to show that $\rho_X$ is injective. Using Lemma 3.2 it suffices to show that the Bloch-Quillen map $\lambda_X : \text{CH}^d_{\text{LW}}(X) \to K_0(X)$ is injective. For $d \leq 1$, this follows from [33, Theorem 2.3]. For $d \geq 2$, this is [35, Corollary 7.6].

4.3. Theorem 1.1 for projective schemes. We shall now prove Theorem 1.1 for projective schemes over an algebraically closed field which are regular in codimension one. We fix an algebraically closed field $k$ and a reduced projective scheme $X$ of dimension $d \geq 1$ over $k$ which is regular in codimension one. Note that if $d = 1$, this means that $X$ is regular. Let $\pi : X^N \to X$ denote the normalization morphism and let $Y = \pi^{-1}(X_{\text{sing}})$. This clearly induces the pull-back map $\pi^* : Z_0(X) \to Z_0(X^N, Y) \subset Z_0(X^N)$. We begin with the following reduction.

Lemma 4.3. The map $\pi^* : Z_0(X) \to Z_0(X^N)$ induces an isomorphism $\pi^* : \text{CH}^d_{\text{LW}}(X) \cong \text{CH}^d_{\text{LW}}(X^N)$.

Proof. There is nothing to prove when $d = 1$ so we assume $d \geq 2$. By Lemma 2.2 it suffices to show that the map $\pi^* : \text{CH}^d_{\text{LW}}(X) \to \text{CH}^d_{\text{LW}}(X^N, Y) := Z_0(X^N, Y) / \mathcal{R}^d_{\text{LW}}(X^N, Y)$ is an isomorphism.

The map $\pi^* : Z_0(X) \to Z_0(X^N, Y)$ is just the identity map. Furthermore, any element of $\mathcal{R}^d_{\text{LW}}(X^N, Y)$ is of the form $\text{div}(f)$, where $C \subset X_{\text{reg}}$ is an integral curve and $f \in k(C)^*$. But this uniquely defines an element of $\mathcal{R}^d_{\text{LW}}(X^N, Y)$. Conversely, any element of $\mathcal{R}^d_{\text{LW}}(X^N, Y)$ is of the form $\text{div}(f)$, where $C \subset X^N \setminus Y$ is an integral curve and $f \in k(C)^*$. But $\pi(C)$ and $\pi_*(f)$ then uniquely define an element of $\mathcal{R}^d_{\text{LW}}(X)$. This proves the desired bijection $\pi^* : \text{CH}^d_{\text{LW}}(X) \cong \text{CH}^d_{\text{LW}}(X^N, Y)$. \hfill $\square$

The key step for proving Theorem 1.1 for projective schemes is the following result of independent interest. This result was proven by Levine (see [32, Theorem 3.2]) modulo $p$-torsion if char$(k) = p > 0$. We shall follow Levine’s outline in making his result unconditional.

Theorem 4.4. Let $X$ be as above. Then the cycle class map $\lambda_X : \text{CH}^d_{\text{LW}}(X) \to K_0(X)$ is injective.

Proof. We consider the commutative diagram

$$\begin{align*}
\text{CH}^d_{\text{LW}}(X) & \xrightarrow{\lambda_X} K_0(X) \\
\pi^* & \downarrow \quad \pi^* \\
\text{CH}^d_{\text{LW}}(X^N) & \cong \Lambda^N_{\text{LW}}(X^N) 
\end{align*}$$

(4.5)

It follows from Lemma 4.3 that the left vertical arrow is an isomorphism. This shows that we can assume that $X$ is normal. In particular, we can assume that $X$ is integral. Levine has shown that the map $\text{CH}^d_{\text{LW}}(X)_0 \to K_0(X)_0$ is injective (see [30, Corollary 5.4] and [32, Corollary 2.7]). So the heart of the proof is to show that the map $\lambda_X : \text{CH}^d_{\text{LW}}(X)_{\text{tor}} \to K_0(X)$ is injective. Let $\text{CH}^d_{\text{LW}}(X)_0$ denote the kernel of the degree map $\deg : \text{CH}^d_{\text{LW}}(X) \to \mathbb{Z}$. It is clear that $\text{CH}^d_{\text{LW}}(X)_{\text{tor}} \subset \text{CH}^d_{\text{LW}}(X)_0$.

Recall from [28] that the normal projective variety $X$ admits an albanese variety $A := \text{Alb}(X)$ in the sense of [29, Chap. II, § 3] and an albanese rational map $u : X \to A$ which is regular on $X_{\text{reg}}$. If we fix a closed point $P \in X_{\text{reg}}$, then $u$ defines a surjective group
homomorphism \( \tau_X : CH_0^{LW}(X)_0 \to A(k) \) such that \( u(x) = \tau_X([x] - [P]) \). We shall make no distinction between \( A \) and \( A(k) \) in the rest of the proof as long as the context makes it clear whether we are talking about the variety \( A \) or the group \( A(k) \).

Let \( X^* \subset X \times A \) be the closure of the graph of \( u \) with projections \( p : X^* \to X \) and \( q : X^* \to A \). Since \( A \) is regular and \( q \) is projective (because \( X \) is projective), there is a push-forward map \( q_* : K_0(X^*) \to K_0(A) \) (see [17, 3.16.5]). We let \( u_t : K_0(X) \to K_0(A) \) denote the composite map \( q_\circ p^* \). Note that \( u_t \) is defined on higher- \( K \)-groups as well, but we do not need this general version.

The morphism \( p \) is an isomorphism over \( X_{\text{reg}} \) and \( q \) agrees with \( u \) under this isomorphism. If \( x \in X_{\text{reg}} \) is a closed point and \( y = p^{-1}(x) \), then we have in \( K_0(A) \):

\[
(4.6) \quad u_t \circ \lambda_X([x]) = q_\circ p^* \circ \lambda_X([x]) = q_\circ \lambda_X*([y]) = \sum_i \{ R^iq_*(k(y)) \} = [k(u(x))] = \lambda_A([u(x)]).
\]

If we identify \( Z_0(X) \) with \( Z_0(X^*, p^{-1}(X_{\text{sing}})) \subset Z_0(X^*) \), then it follows from (4.6) that for a 0-cycle \( \alpha \in Z_0(X) \), one has

\[
(4.7) \quad u_t \circ \lambda_X(\alpha) = \lambda_A \circ u_t(\alpha) \in K_0(A).
\]

Let \( \mathcal{P} \) be the Poincaré line bundle on \( A \times \hat{A} \), where \( \hat{A} = \text{Pic}^0(A) \) is the dual abelian variety to \( A \). Then \( \mathcal{P} \) defines a map \( \sim \mathcal{P} : K_0(A) \to K_0(\hat{A}) \) such that \( \sim \mathcal{P}(\beta) = (p_{\hat{A}})_* (p_{\hat{A}}^*(\beta) \otimes [\mathcal{P}]) \) for \( \beta \in K_0(A) \). If \( x \in A \) is a closed point, then \( \sim \mathcal{P}(\lambda_A([x])) = (p_{\hat{A}})_* (\mathcal{O}_{\{x\} \times \hat{A}} \otimes [\mathcal{P}]) \). Since the map \( \{x\} \times \hat{A} \to \hat{A} \) is an isomorphism under \( p_{\hat{A}} \), we see that \( \sim \mathcal{P} \) restricts to \( \sim \mathcal{P} : F_0K_0(A) \to \text{Pic}^0(\hat{A}) \), where \( F_0K_0(A) \) is the subgroup of \( K_0(A) \) generated by the classes of closed points. As \( (p_{\hat{A}})_* (\mathcal{O}_{\{x\} \times \hat{A}} \otimes [\mathcal{P}]) \) is identified with the pull-back of \( \mathcal{P} \) under the embedding \( \hat{A} \to A \times \hat{A} \), given by \( y \mapsto (x,y) \), we see furthermore that \( \sim \mathcal{P}(\lambda_A([x])) = x \) under the isomorphism \( \text{Pic}^0(\hat{A}) \cong A \) via \( \mathcal{P} \). We thus get a homomorphism

\[
(4.8) \quad \sim \mathcal{P} : F_0K_0(A) \to A
\]
such that \( \sim \mathcal{P} \circ \lambda_A(\alpha) = \alpha \).

Combining the construction of (4.7) with (4.8), one gets a commutative diagram

\[
(4.9) \quad X_{\text{reg}} \xrightarrow{u} CH_0^{LW}(X)_0 \xrightarrow{\tau_X} A \\
\hspace{1cm} \lambda_X \downarrow \quad \sim \mathcal{P} \downarrow \quad \lambda_X
\]

\[
F_0K_0(X) \xrightarrow{u_t} F_0K_0(A).
\]

Since the map \( \tau_X : CH_0^{LW}(X)_{\text{tor}} \to A_{\text{tor}} \) is injective by [28, Theorem 1.6] as \( X \) is normal and projective over \( k \), it follows from (4.9) that the map \( \lambda_X : CH_0^{LW}(X)_{\text{tor}} \to F_0K_0(X)_{\text{tor}} \to K_0(X)_{\text{tor}} \) is also injective.

A combination of Theorems 1.1, 1.4 and Lemma 3.2 yields the following result and brings us to the end of the proof of Theorem 1.1.

**Corollary 4.5.** Let \( X \) be a reduced projective scheme of pure dimension \( d \geq 1 \) over an algebraically closed field. Assume that \( X \) is regular in codimension one. Then the Bloch-Quillen map

\[
\rho_X : CH_0^{LW}(X) \to H_{\text{nis}}^d(X, \mathcal{K}_d^M)
\]
is an isomorphism.

5. Bloch’s formula for 0-cycles with modulus

Our goal in this section is to prove Theorem 1.3 which provides Bloch’s formula for the Chow group of 0-cycles with modulus. We shall do this using the double construction of §2.4 and Theorem 2.4. We fix an algebraically closed field $k$ and a smooth quasi-projective scheme $X$ of dimension $d \geq 1$ over $k$. We fix an effective Cartier divisor $D \subset X$. Recall that the double of $X$ along $D$ is the scheme $S_X = X \cup D$. There is a fold map $\nabla : S_X \to X$ and inclusions as irreducible components $\iota_{\pm} : X \to S_X$ such that $\nabla \circ \iota_{\pm}$ is identity.

5.1. Bloch’s formula for $S_X$. Our aim is to derive Theorem 1.3 from Bloch’s formula for the singular scheme $S_X$. If $X$ is affine, this already follows from Theorem 1.1. However, this is not the case when $X$ is projective. The reason is that $S_X$ is not regular in codimension one. We shall now extend Theorem 1.1 to the case of projective schemes of the type $S_X$ under some condition on $D$.

**Theorem 5.1.** Let $D \subset X$ be an inclusion of a divisor as above. Assume that $X$ is projective and $D$ is integral. Then the Bloch-Quillen map

$$\rho_{S_X} : \text{CH}_0^{\text{LW}}(S_X) \to H^d_{\text{nis}}(X, \mathbb{Z}_d, S_X)$$

is an isomorphism.

**Proof.** By Theorem 1.1 and Lemma 3.2, we only have to show that the cycle class map $\lambda_{S_X} : \text{CH}_0^{\text{LW}}(S_X) \to K_0(S_X)$ is injective. Since $\lambda_{S_X}$ is injective with $\mathbb{Q}$-coefficients (see [30, Corollary 5.4] and [32, Corollary 2.7]), the theorem is reduced to showing that the map $\text{CH}_0^{\text{LW}}(S_X)_{\text{tor}} \to K_0(S_X)$ is injective. We can assume $d \geq 2$ by [33, Proposition 1.4]. We can also assume that $X$ is connected.

We have a commutative diagram

$$
\begin{array}{ccc}
\text{CH}_0^{\text{LW}}(S_X) & \xrightarrow{\lambda_{S_X}} & K_0(S_X) \\
\pi^* & & | \pi^* \\
\text{CH}_0^{\text{LW}}(S_X^N) & \xrightarrow{\lambda_{S_X}^N} & K_0(S_X^N).
\end{array}
$$

Since $S_X^N = X_+ \cup X_-$ is smooth and projective, the map $\lambda_{S_X}^N$ is injective by [32, Theorem 3.2]. It suffices therefore to show that the map $\pi^* : \text{CH}_0^{\text{LW}}(S_X)_{\text{tor}} \to \text{CH}_0^{\text{LW}}(S_X^N)$ is injective.

Let $A^d(S_X)$ denote the Albanese variety of $S_X$ and let $\tau_{S_X} : \text{CH}_0^{\text{LW}}(S_X)_{0} \to A^d(S_X)$ denote the universal regular homomorphism (see [13, Theorem 1]). In general, $A^d(S_X)$ is a connected commutative algebraic group whose abelian variety quotient is the Albanese variety of $S_X^N$ as in [29]. However, under our assumption that $D$ is reduced, it is shown in [26, Theorem 6.5] that $A^d(S_X)$ is a semi-abelian variety and there exists a commutative diagram

$$
\begin{array}{ccc}
\text{CH}_0^{\text{LW}}(S_X)_{0} & \xrightarrow{\pi^*} & \text{CH}_0^{\text{LW}}(S_X^N)_{0} \\
\tau_{S_X} & & | \tau_{S_X} \\
1 \to T & \xrightarrow{\pi^*} & A^d(S_X) \xrightarrow{\pi^*} A^d(S_X^N) & \to 1,
\end{array}
$$

where the bottom sequence is exact and $T \cong \mathbb{G}_m$ is a torus over $k$. Furthermore, the vertical arrows in (5.2) are isomorphisms on the torsion subgroups. We have therefore reduced the theorem to showing that the map $\pi^* : A^d(S_X)_{\text{tor}} \to A^d(S_X^N)_{\text{tor}}$ is injective. We shall in
that $\Lambda^{1}$ canonical maps $\Lambda^{1}$ D

Dposite map and let $\Lambda^{T}(5.3) \Lambda^{1}$

Kleiman [1, Theorem 7] implies that we can find a smooth connected curve

Since $X$ is connected and smooth of dimension $d \geq 2$, the Bertini theorem of Altman and Kleiman [1] Theorem 7] implies that we can find a smooth connected curve $C \subset X$ which contains $S$. It is then clear that $C \not\approx D$ and the intersection number $(D \cdot C)$ is positive. In particular, $D$ is not numerically equivalent to zero on $X$. This completes the proof of the theorem.

5.2. Bloch-Quillen map with modulus. Let $(X,D)$ be as before. Then for $x \in X^{(d)} \setminus D$, we have $\mathbb{Z} \cong K^{0}_{0}(k(x)) \cong H^{d}_{x}(V_{\text{nis}}, K^{M}_{d,V}) \cong H^{d}_{x}(X_{\text{nis}}, K^{M}_{d,(X,D)})$, where $V = X \setminus D$. Therefore, we have a group homomorphism

$$
(5.4) \rho_{X|D} : z_{0}(X|D) \to H^{d}_{\text{nis}}(X, K^{M}_{d,(X,D)})
$$
such that for $x \in X^{(d)} \setminus D$, the element $\rho_{X|D}(x)$ is the image of 1 in $H^{d}_{\text{nis}}(X, K^{M}_{d,(X,D)})$ under the forget support map. The aim of this section is to list the pairs $(X,D)$ for which this homomorphism factors through the Chow group with modulus $\text{CH}_{0}(X|D)$. The essential idea is to use Theorem [2.4] and to use it we need to know that when is the natural map $\text{can}_{X} : \text{CH}_{0}^{\text{LW}}(S_{X}) \to \text{CH}_{0}(S_{X})$ an isomorphism. The following lemma provides all known cases in which the map $\text{can}_{X}$ is indeed an isomorphism.

**Lemma 5.2.** Let $(k,X,D)$ be as before. Then the canonical map $\text{can}_{X} : \text{CH}_{0}^{\text{LW}}(S_{X}) \to \text{CH}_{0}(S_{X})$ is isomorphism in the following cases.

1. $d \leq 2$.
2. $X$ is affine.
(3) \( \text{char}(k) = 0 \) and \( X \) is projective.
(4) \( \text{char}(k) > 0 \), \( X \) is projective and \( D \) is reduced.

**Proof.** The assertions (1)-(3) follow from [6, Theorem 3.17] while [26, Theorem 6.6] yields (4). \( \square \)

**Proposition 5.3.** Let \((k,X,D)\) be as in Lemma 5.2. Then the map in (5.4) induces a surjective group homomorphism \( \rho_{X|D} : \text{CH}_0(X|D) \to H^d_{\text{nis}}(X,K^M_{d,(X,D)}) \).

**Proof.** By Lemma 5.2, the canonical map \( \text{CH}^0_{\text{nis}}(S_X) \to \text{CH}_0(S_X) \) is an isomorphism. Combining this with Theorem 2.4 and noting that the composite map \( X \to S_X \to X \) is identity, we get a commutative diagram of split exact sequences

\[
\begin{array}{cccccc}
0 & \to & \text{CH}_0(X|D) & \xrightarrow{\rho_{X|D}} & \text{CH}^0_{\text{nis}}(S_X) & \xrightarrow{\iota^*} & \text{CH}_0(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^d_{\text{nis}}(S_X,K^M_{d,(S_X,X)}) & \xrightarrow{\rho_{X|D}} & H^d_{\text{nis}}(S_X,M^M_{d,S_X}) & \xrightarrow{\iota^*} & H^d_{\text{nis}}(X,K^M_{d,X}) & \to & 0.
\end{array}
\]

This yields a canonical homomorphism \( \tilde{\rho}_{X|D} : \text{CH}_0(X|D) \to H^d_{\text{nis}}(S_X,K^M_{d,(S_X,X)}) \). Moreover, it follows from Lemma 2.0 that the map of Nisnevich sheaves \( \tau^*_+ : K^M_{d,(S_X,X)} \to \tau^*_+(K^M_{d,(X,D)}) \) is surjective. Furthermore, its kernel is clearly supported on \( D \). It then follows from the bound on the Nisnevich cohomological dimension that the map \( \iota^*_+ : H^d_{\text{nis}}(S_X,K^M_{d,(S_X,X)}) \to H^d_{\text{nis}}(S_X,\rho_{X|D}(K^M_{d,(X,D)})) \) is an isomorphism. The factorization of the map \( \rho_{X|D} \) now follows because \( H^d_{\text{nis}}(S_X,\rho_{X|D}(K^M_{d,(X,D)})) \cong H^d_{\text{nis}}(X,K^M_{d,(X,D)}) \) and the composite map \( \tilde{\rho}_{X|D} \) agrees with the map in (5.4).

To show that \( \rho_{X|D} : \text{CH}_0(X|D) \to H^d_{\text{nis}}(X,K^M_{d,(X,D)}) \) is surjective, it suffices to show that the map \( \rho_{X|D} : \mathcal{O}(X|D) = \bigcup_{x \in V^{(o)}} K^M_0(k(x)) \to H^d_{\text{nis}}(X,K^M_{d,(X,D)}) \) is surjective. This follows from [19, Theorem 2.5] since \( k \) is perfect and hence \( V = X \setminus D \) is nice in the sense of [19, Definition 2.2]. The proof of the proposition is now complete. \( \square \)

### 5.3. Proof of Theorem 1.3

Now Theorem 1.3 follows from Proposition 5.3, Theorem 1.1 and Theorem 5.1. Indeed, the two solid arrows in (5.5) are isomorphisms by Theorem 1.1 (if \( X \) is affine) or by Theorem 5.1 (if \( X \) is projective and \( D \) is integral). Therefore, if \( (X,D) \) is as in Theorem 1.3, then the natural map \( \rho_{X|D} : \text{CH}_0(X|D) \to H^d_{\text{nis}}(X,K^M_{d,(X,D)}) \) is an isomorphism. This completes the proof. \( \square \)

### 5.4. Proofs of Theorems 1.6 and 1.7

Let \( k \) be a field and let \( X \) be a smooth quasi-projective scheme of pure dimension \( d \) over \( k \). Let \( D \) be an effective Cartier divisor on \( X \). In this section, we shall assume that \( D_{\text{red}} \) is a simple normal crossing divisor, i.e., if \( D_1, \ldots, D_s \) are the irreducible components of \( D \), then the intersections \( c_{ij} \cap D_i \) are smooth over \( k \) and have codimension \( r \), where \( I \subset \{1, \ldots, s\} \) and \( r \) is the cardinality of \( I \).

As an application of Theorem 1.1, Theorem 1.3, Proposition 5.3 and [11, Theorem 3.8], we shall now prove Theorems 1.6 and 1.7.

Let \( (X,D) \) be as in Theorem 1.6. Then by Proposition 5.3, the Bloch-Quillen map in (5.4) induces a surjective homomorphism \( \rho_{X|D} : \text{CH}^d(X|nD) \to H^d_{\text{nis}}(X,K^M_{d,(X,D)}) \). Consider the diagram:
Theorem 1.6 (assuming the diagram (5.6) commutes).

Cartier divisor on \(X\) (5.6). Now assume that \(D\) is connected and smooth. It then follows from Theorem 1.3 that the top horizontal arrow in (5.6) is an isomorphism. Therefore, it suffices to show that the right vertical arrow in (5.6) is surjective. By [41, Theorem 3.8], the bottom horizontal arrow in (5.6) is an isomorphism. This completes the proof of Theorem 1.7 (assuming (5.6) commutes).

Since the cokernel of the inclusion \(\mathcal{K}^M_{d,X,D} \to \mathcal{K}^M_{d,X|D}\) is supported on \(D\), the right vertical arrow in (5.6) is surjective. By [11] Theorem 3.8, the bottom horizontal arrow in (5.6) is an isomorphism. Therefore, it suffices to show that the right vertical arrow in (5.6) is an isomorphism. As before, by [41, Theorem 3.8], the bottom horizontal arrow in (5.6) is an isomorphism. Therefore, it suffices to show that the right vertical arrow in (5.6) induces an isomorphism of pro-abelian groups. This follows from Lemma 2.8. We now assume that the diagram (5.6) commutes and complete the proofs of Theorems 1.6 and 1.7.

Now, assume that \(X\) is either affine or a quasi-projective surface. Let \(D\) be an effective Cartier divisor on \(X\) such that \(D_{\text{red}}\) is a simple normal crossing divisor. Then by Theorem 1.1 (if \(X\) is affine) or by [6] Theorem 1.8 (if \(X\) is a quasi-projective surface), the top horizontal arrow in (5.6) is an isomorphism. Now assume that \(D\) is connected and smooth. It then follows from Theorem 1.3 that the top horizontal arrow in (5.6) is an isomorphism. Therefore, it suffices to show that the right vertical arrow in (5.6) is an isomorphism. As before, by [11] Theorem 3.8, the bottom horizontal arrow in (5.6) is an isomorphism. Therefore, it suffices to show that the right vertical arrow in (5.6) is an isomorphism. This follows from Lemma 2.8 where we proved that the inclusion \(\mathcal{K}^M_{d,X,D} \to \mathcal{K}^M_{d,X|D}\) is an isomorphism. This completes the proof of Theorem 1.7 (assuming (5.6) commutes).

Now, we prove that the diagram (5.6) commutes. To see this, let \(z^\sigma(X|D,2r-\bullet)_c\) denote the complex of constant Nisnevich sheaves on \(X\) defined by the complex \(z^\sigma(X|D,2r-\bullet)\). Recall that the map \(\text{can}_M\) is induced by the morphism of complex of Nisnevich sheaves \(z^\sigma(X|D,2r-\bullet)_c \to z^\sigma(-|D,2r-\bullet)\). Given \(x \in X \setminus D\), consider the following diagram.

(5.7) \[
\begin{array}{ccc}
\text{CH}^d(X|D) & \to & H^d_{\text{nis}}(X,\mathcal{K}^M_{d,X,D}) \\
& \searrow & \downarrow \\
& H^d_{\text{nis}}(X,\mathcal{K}^M_{d,X,D}) & \to & H^d_{\text{nis}}(X,\mathcal{K}^M_{d,X|D}) \\
\mathbb{H}^2d(X,\mathbb{Z}(d)_{X|D}) & \to & H^d_{\text{nis}}(X,\mathcal{K}^M_{d,X|D}) \\
& \searrow & \downarrow \\
\mathbb{H}^2d(X,\mathbb{Z}(d)_{X|D}) & \to & H^d_{\text{nis}}(X,\mathcal{K}^M_{d,X|D}) \\
& \swarrow & \downarrow \\
& & & H^d_{\text{nis}}(X,\mathcal{K}^M_{d,X|D}).
\end{array}
\]

We have to show that the front face in the diagram (5.7) commutes. Since \(\text{CH}^d(X|D)\) is generated by the classes of the closed points \(x \in X^{(d)} \setminus D\), it suffices to show that all other faces in (5.7) commute. The top face commutes by the definition of the Bloch-Quillen map \(\lambda_{X|D}\) (see (5.4)). Note that, \(\mathbb{H}^2d(X,\mathbb{Z}(d)_{X|D},2d-\bullet)_{\text{nis}} = \mathbb{Z}(x)\) and \(\mathbb{H}^2d(X,\mathbb{Z}(d)_{X|D},2d-\bullet)_{\text{nis}} = \text{CH}^d(X|D)\). The right face, the left face and the bottom face of (5.7) commute by the naturality of the ‘forget support’ map on cohomologies. Therefore, it suffices to show that the
back face of \([5.7]\) commutes. By excision, we can assume that \(X\) is smooth. But for smooth scheme, the commutativity of the back face is well known. This completes the proof of the Theorems \([1.6]\) and \([1.7]\).

6. The question of Kerz-Saito

We now prove Theorem \([1.8]\) as an application of Theorem \([1.1]\). We shall then use Theorem \([1.8]\) and its proof to give a proof of Theorem \([1.9]\). Let \(X\) be a reduced affine or projective scheme of pure dimension \(d \geq 1\) over an algebraically closed field \(k\) and let \(\pi : \tilde{X} \to X\) be a resolution of singularities. Let \(E_0 \subset \tilde{X}\) be the reduced exceptional divisor. Assume that \(X\) is regular in codimension one and let \(U = X_{\text{reg}} = \tilde{X} \setminus E_0\). We let \(S = X_{\text{sing}}\) with the reduced induced closed subscheme structure and we let \(E\) denote the scheme theoretic inverse image \(\pi^{-1}(S)\). Note that \(E\) is supported on \(E_0\) and there exists \(m \geq 1\) such that we have

\[
E_0 \cong E \cong mE_0.
\]

Let \(D\) be a divisor on \(X\) supported on \(E_0\). Given a closed point \(x \in U\), the composite map of \(K\)-theory spectra \(K(k(x)) \to K(\tilde{X}) \to K(D)\) is null-homotopic. This yields a map \(u_x : K(k(x)) \to K(\tilde{X}, D)\). Letting \(\lambda_{\tilde{X}|D}(\{x\}) = u_x(1) \in K_0(\tilde{X}, D)\), we get a cycle class map \(\lambda_{\tilde{X}|D} : Z_0(\tilde{X}, D) \to K_0(\tilde{X}, D)\). By the same reason, we also have a map \(\lambda_{X|nS} : Z_0(X) \to K_0(X, nS)\).

It follows from \([6, \text{Theorem 12.4}]\) that \(\lambda_{\tilde{X}|D}\) factors through \(\lambda_{\tilde{X}|D} : CH_0(\tilde{X}|D) \to K_0(\tilde{X}, D)\). We let \(F_0K_0(\tilde{X}, D)\) be the image of this cycle class map. Using Lemma \([2.2]\) and Definition \([2.3]\) it also follows easily that the map \(\pi^* : Z_0(X) \to Z_0(\tilde{X}, D)\) factors through the rational equivalence classes. We also need the following refinement of the cycle class map \(\lambda_X : CH_0^{LW}(X) \to K_0(X)\).

**Lemma 6.1.** The map \(\lambda_{X|nS} : Z_0(X) \to K_0(X, nS)\) factors through the rational equivalence classes.

**Proof.** We let \(C \subset X\) be an integral curve such that \(C \cap S = \emptyset\) and let \(f \in k(C)^\times\). By Lemma \([2.2]\), it suffices to show that \(\lambda_{X|nS}(\text{div}(f)) = 0\). Let \(C_N \to C\) be the normalization map and let \(\nu : C_N \to C \to X\) denote the composite map. It is then clear that \(\text{div}(f) = \nu_*(\text{div}(f))\), where \(f \in k(C)^\times = k(C_N)^\times\).

Since \(C \cap S = \emptyset\), the finite map \(\nu : C_N \to X\) has finite tor-dimension and the resulting push-forward map \(\nu_* : K(C_N) \to K(X)\) factors through \(K(C_N) \to K(X, nS) \to K(X)\) just as above. We thus have a commutative diagram

\[
\begin{array}{ccc}
Z_0(C_N) & \xrightarrow{\lambda_{C_N}} & K_0(C_N) \\
\downarrow \nu_* & & \downarrow \nu_* \\
Z_0(X) & \xrightarrow{\lambda_{X|nS}} & K_0(X, nS).
\end{array}
\]

We are now done since \(\lambda_{C_N}(\text{div}(f)) = 0\). \(\Box\)

6.1. **Proof of Theorem \([1.8]\)** We shall now prove Theorem \([1.8]\) Using Lemma \([6.1]\) and the construction of various other maps before it, we obtain a commutative diagram for every \(n \geq 1\):
Theorem 4.4 (if \(LW\)

It follows that all arrows in the left square and in the middle square of (6.3) are isomorphisms. By \([25, \text{Theorem 1.6}]\), it follows that the composite map \(CH_{0}^LW(X) \xrightarrow{\pi} CH_{0}(\tilde{X}|nE) \rightarrow CH_{0}(\tilde{X}|nE_{0}) \rightarrow CH_{0}(\tilde{X})\).

The map \(\lambda_{X}\) on the left is an isomorphism by \([25, \text{Corollary 7.6}]\) (if \(X\) is affine) and Theorem 4.4 (if \(X\) is projective). It follows that all arrows in the triangle on the left are isomorphisms. By \([23, \text{Theorem A}]\), the canonical homomorphism of pro-abelian groups \(\varprojlim_{n} K_{0}(X,nS) \rightarrow \varprojlim_{n} K_{0}(\tilde{X},nE)\) is an isomorphism. In particular, its restriction \(\varprojlim_{n} F_{0}K_{0}(X,nS) \rightarrow \varprojlim_{n} F_{0}K_{0}(\tilde{X},nE)\) is an isomorphism too. By \([6.1]\), it follows that the map of pro-abelian groups \(\varprojlim_{n} F_{0}K_{0}(\tilde{X},nE) \rightarrow \varprojlim_{n} F_{0}K_{0}(\tilde{X},nE_{0})\) is an isomorphism. As \(\lambda_{X|nS}\) is an isomorphism for all \(n \geq 1\), we get an isomorphism of pro-abelian groups \(CH_{0}^LW(X) \xrightarrow{\pi} \varprojlim_{n} F_{0}K_{0}(\tilde{X},nE_{0})\). Since \(CH_{0}^LW(X)\) is a constant pro-abelian group, an elementary calculation shows that we must have \(CH_{0}^LW(X) \cong \varprojlim_{n} F_{0}K_{0}(\tilde{X},nE_{0})\) for all \(n \gg 1\). It follows then that all arrows in the left square and in the middle square of (6.3) are isomorphisms for all \(n \gg 1\). This proves the standard version of the Bloch-Srinivas conjecture (part (1) of Theorem 1.8).

We assume now that \(\text{char}(k) = p > 0\) and prove the strong version, namely, that \(CH_{0}^LW(X) \cong CH_{0}(\tilde{X}|E_{0}) \xrightarrow{\pi} F_{0}K_{0}(\tilde{X},E_{0})\).

Using the homotopy fiber sequence of spectra

\[K(\tilde{X},nE_{0}) \rightarrow K(\tilde{X},E_{0}) \rightarrow K(nE_{0},E_{0})\]

and \([25, \text{Lemma 3.5}]\), it follows that the kernel of the map \(F_{0}K_{0}(\tilde{X},nE_{0}) \rightarrow F_{0}K_{0}(\tilde{X},E_{0})\) is a \(p\)-primary torsion group of bounded exponent.

We choose \(n \gg 1\) such that \(CH_{0}^LW(X) \cong CH_{0}(\tilde{X}|nE_{0}) \cong F_{0}K_{0}(\tilde{X},nE_{0})\). It follows then that the kernel of the composite map \(CH_{0}^LW(X) \rightarrow CH_{0}(\tilde{X}|E_{0}) \rightarrow F_{0}K_{0}(\tilde{X},E_{0})\) is a \(p\)-primary torsion group of bounded exponent.

If \(X\) is affine, this kernel must be zero by \([25, \text{Theorem 1.1}]\). If \(X\) is projective, we have a commutative diagram

\[
\begin{array}{ccc}
CH_{0}^LW(X)_{\text{tor}} & \longrightarrow & CH_{0}(\tilde{X}|E_{0})_{\text{tor}} \\
\downarrow \cong & & \downarrow \cong \\
F_{0}K_{0}(X)_{\text{tor}} & \longrightarrow & F_{0}K_{0}(\tilde{X},E_{0})_{\text{tor}} \\
\downarrow \cong & & \downarrow \cong \\
CH_{0}^LW(X^{N})_{\text{tor}} & \xrightarrow{\pi} & F_{0}K_{0}(X^{N})_{\text{tor}} \\
\downarrow \tau_{XN} \cong & & \downarrow A^{d}(X^{N})_{\text{tor}} \cong \\
& & A^{d}(\tilde{X})_{\text{tor}}.
\end{array}
\]

All arrows in the left triangle are isomorphisms by Lemma 4.3 and Theorem 4.4. By the same reason, the vertical arrow on the top right is an isomorphism. It follows from this diagram and \([28, \text{Theorem 1.6}]\) that the composite map \(CH_{0}^LW(X)_{\text{tor}} \rightarrow F_{0}K_{0}(\tilde{X},E_{0})_{\text{tor}} \rightarrow F_{0}K_{0}(\tilde{X})_{\text{tor}} \rightarrow A^{d}(\tilde{X})_{\text{tor}}\) is an isomorphism. In particular, the map \(CH_{0}^LW(X)_{\text{tor}} \rightarrow F_{0}K_{0}(\tilde{X},E_{0})_{\text{tor}}\) is injective. It follows that \(\text{Ker}(CH_{0}^LW(X) \rightarrow F_{0}K_{0}(\tilde{X},E_{0}))\) must be zero. The proof of Theorem 1.8 is now complete.

\[\square\]
6.2. Proof of Theorem 1.9. We now prove Theorem 1.9. We shall follow the notations of the statement of Theorem 1.9 in its proof. Recall from Theorem 1.9 that $Y$ is a reduced projective scheme of pure dimension $d$ over an algebraically closed field $k$ of positive characteristic. Our assumption is that $Y$ is regular in codimension one and $\pi : X \to Y$ is a resolution of singularities with the reduced exceptional divisor $E_0 \subset X$. Let $S \subset Y$ be the singular locus with the reduced subscheme structure. Moreover, let $E$ denote the scheme theoretic inverse image $\pi^{-1}(S)$. Note that these notations are little different from the ones in Theorem 1.8.

We fix an integer $n \geq 1$ and consider the commutative diagram (6.5)

$$
\begin{array}{ccc}
\text{CH}_0^{LW}(Y) & \xrightarrow{\lambda_{Y|nS}} & H^d_{\text{nis}}(Y, K^M_{d|Y,(Y,nS)}) \\
\rho_Y & \approx & \approx \\
H^d_{\text{nis}}(Y, K^M_{d|Y}) & \xrightarrow{\approx} & F_0K_0(Y).
\end{array}
$$

The left vertical arrow on the bottom square is an isomorphism as $\dim(S) \leq d-2$. We have shown in the proof of Theorem 1.9 that all solid arrows in (6.5) are isomorphisms. It follows that the map $\rho_{Y|nS} : Z_0(Y) \to H^d_{\text{nis}}(Y, K^M_{d|Y,(Y,nS)})$ factors through the Chow group $\text{CH}_0^{LW}(Y)$ so that (6.5) is commutative and all maps are isomorphisms.

We next consider the commutative diagram (6.6)

$$
\begin{array}{ccc}
Z_0(Y) & \xrightarrow{\theta_{X|nE}} & CH_0^{LW}(Y) \\
\xrightarrow{\approx} & \xrightarrow{\approx} & \xrightarrow{\approx} \\
\pi^* & \theta_{Y|nS} & H^d_{\text{nis}}(Y, K^M_{d|Y,(Y,nS)}) \\
\pi^* & \approx & \approx \\
Z_0(X|nE) & \xrightarrow{\theta_{X|nE}} & CH_0(X|nE) \\
\pi^* & \approx & \approx \\
F_0K_0(X|nE) & \approx & F_0(X, E),
\end{array}
$$

where $\theta_{X|nE}$ is the composition of the edge map in the Thomason-Trobaugh spectral sequence with natural map $H^d_{\text{nis}}(X, K^M_{d|X,(X,nE)}) \to H^d_{\text{nis}}(X, K^M_{d|X,(X,nE)})$ of §2.8.

We have shown in the proof of Theorem 1.8 that all solid arrows in (6.6) (except possibly the middle vertical arrow) are isomorphisms. A simple diagram chase shows that the dotted arrow $\rho_{X|nE}$ is in fact a solid arrow. We now consider the commutative diagram (6.7)

$$
\begin{array}{ccc}
\text{CH}_0^{LW}(Y) & \xrightarrow{\pi^*} & CH_0(X|nE) \\
\xrightarrow{\approx} & \xrightarrow{\approx} & \xrightarrow{\approx} \\
\pi^* & \theta_{X|nE} & H^d_{\text{nis}}(X, K^M_{d|X,(X,nE)}) \\
\pi^* & \approx & \approx \\
\text{CH}_0(X|nE_0) & \xrightarrow{\approx} & F_0K_0(X, nE_0),
\end{array}
$$

where vertical arrows exist as $E_0 \subset E$. By Theorem 1.8 it follows that the left vertical arrow in (6.7) is an isomorphism. A simple diagram chase shows that the dotted arrow in $\rho_{X|nE_0}$ is in fact a solid arrow. Since we proved in Theorem 1.8 that the composite map $\text{CH}_0^{LW}(Y) \to F_0K_0(X, E_0)$ is an isomorphism, it follows the map $\rho_{X|nE_0}$ is injective. On the other hand, the composite map $Z_0(X|nE_0) \to CH_0(X|nE_0)$ is surjective by [19, Theorem 2.5]. We conclude that all arrows in (6.7) are isomorphisms. In particular, $\rho_{X|nE_0}$ is an isomorphism for every $n \geq 1$.

To finish the proof of Theorem 1.9, we let $D \subset X$ be any effective Cartier divisor with support $E_0$. We can then find two inclusions $E_0 \subset D \subset nE_0$ for some $n \gg 0$. This gives rise to
a commutative diagram

\[
\begin{array}{cccccc}
\text{CH}_0^{\text{LW}}(Y) & \xrightarrow{\cong} & \text{CH}_0(X;nE_0) & \xrightarrow{\cong} & \text{CH}_0(X|D) & \xrightarrow{\cong} & \text{CH}_0(X|E_0) \\
\cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong \\
H^d_{\text{nis}}(Y;\mathcal{K}_{d,Y}^M) & \xrightarrow{\cong} & H^d_{\text{nis}}(X;\mathcal{K}_{d,(X,nE_0)}^M) & \xrightarrow{\cong} & H^d_{\text{nis}}(X;\mathcal{K}_{d,(X,D)}^M) & \xrightarrow{\cong} & H^d_{\text{nis}}(X;\mathcal{K}_{d,(X,E_0)}^M).
\end{array}
\]

A diagram chase shows that all solid arrows in (6.8) are isomorphisms. This implies that the vertical dotted arrow is in fact a solid arrow and is an isomorphism. In other words, the Bloch-Quillen map \( Z_0(X|D) \to H^d_{\text{nis}}(Y;\mathcal{K}_{d,Y}^M) \) induces an isomorphism \( \rho_{X|D} : \text{CH}_0(X|D) \xrightarrow{\cong} H^d_{\text{nis}}(X;\mathcal{K}_{d,(X,D)}^M) \). We have thus proven Theorem \[1.9\] \[ \square \]

7. Euler class groups of affine algebras

In order to prove Theorems \[1.2\], \[1.4\] and \[1.5\] we shall use the theory of Euler class groups of affine algebras. The Euler class group of a \( k \)-algebra \( A \) has an advantage that any class in this group is the class of a nice enough ideal \( J \subset A \) which has a class \([A/J]\) in \( K_0(A) \) as well. If this class dies in \( K_0(A) \), then there are some commutative algebra results which allow us to conclude that the class of \( J \) is zero in the Euler class group as well. So the key to proving a result like Theorem \[1.5\] is to connect the Levine-Weibel Chow group with these Euler class groups.

Unfortunately, the Euler class group has cycles which are supported on the singular locus of \( \text{Spec}(A) \), and hence, it is very hard to directly connect this group with the Chow group. To circumvent this problem, we introduce a new version of the Euler class group. This new version is closely related to the Chow group. The key result of this section is that this new version is canonically isomorphic to the original one. This will be used in the next section to finish the proofs of Theorems \[1.2\], \[1.4\] and \[1.5\].

Throughout this section, we fix a field \( k \) and all rings we consider will be geometrically reduced equi-dimensional affine algebras over \( k \).

7.1. The Euler class groups. We recall the definitions of the Euler class groups from \[5\]. Let \( A \) be an affine \( k \)-algebra of dimension \( d \geq 2 \). Let \( G(A) \) be the free abelian group on the pairs \((n, \omega_n)\), where \( n \subset A \) is an \( m \)-primary ideal for a maximal ideal \( m \subset A \) of height \( d \) and \( \omega_n : (A/n)^d \to n/n^2 \) is an \( A \)-linear surjection.

Given an ideal \( J \subset A \) of height \( d \) with the irredundant primary decomposition \( J = n_1 \cap \cdots \cap n_r \) and a surjection \( \omega_J : (A/J)^d \to J/J^2 \), the Chinese remainder theorem yields surjections \( \omega_{n_i} : (A/n_i)^d \to n_i/n_i^2 \). In particular, the ideal \( J \) and the map \( \omega_J \) together define a unique class \((J, \omega_J) = \sum_{i=1}^r (n_i, \omega_{n_i}) \in G(A)\). Let \( H(A) \subset G(A) \) be the subgroup generated by the classes \((J, \omega_J)\) as above such that there is a commutative diagram of \( A \)-modules:

\[
\begin{array}{cccccc}
A^d & \xrightarrow{\omega_J} & J \\
\downarrow & \downarrow & \downarrow \\
(A/J)^d & \xrightarrow{\omega_J} & J/J^2.
\end{array}
\]

The Euler class group of \( A \) is defined to be the group \( E(A) = G(A)/H(A) \).

The weak version of the Euler class group is defined as follows. Let \( G_0(A) \) denote the free abelian group on the set of ideals \( n \subset A \) such that \( n \) is an \( m \)-primary ideal for some maximal ideal of height \( d \) in \( A \) and there is a surjective \( A \)-linear map \( \omega_n : (A/n)^d \to n/n^2 \).
Given an ideal \( J \subset A \) of height \( d \) with the irredundant primary decomposition \( J = n_1 \cap \cdots \cap n_r \) and a surjection \( \omega_J : (A/J)^d \twoheadrightarrow J/J^2 \), the Chinese remainder theorem yields surjections \( \omega_{n_i} : (A/n_i)^d \twoheadrightarrow n_i/n_i^2 \). In particular, the ideal \( J \) defines a unique class \((J) = \sum_{i=1}^{r} n_i \in G_0(A)\).

Let \( H_0(A) \subset G_0(A) \) be the subgroup generated by the classes \((J)\) such that \( J \) is generated by \( d \) elements. The weak Euler class group is defined as \( E_0(A) = G_0(A)/H_0(A) \).

It is clear that there is a canonical forget orientation map \( \psi_A : E(A) \to E_0(A) \).

7.2. The Segre exact sequence. Given a positive integer \( n \), let \( Um_n(A) \) denote the set of unimodular rows of length \( n \) over \( A \). Recall here that a row \( \underline{a} := [a_1, \ldots, a_n] \in M_{1,n}(A) \) is called unimodular, if the ideal \((a_1, \ldots, a_n)\) is \( A \). If \( B \in M_{n,1} \) is such that \( \underline{a}B = \underline{a}M^{-1}B \) for any \( M \in GL_n(A) \). Setting \( B' = M^{-1}B \), we get \( (\underline{a}B)B' = 1 \). Using this, one can easily show that \( GL_n(A) \) acts on \( Um_n(A) \). We let \( WS_n(A) = Um_n(A)/E_n(A) \) be the quotient for the action of the elementary matrices \( E_n(A) \) on \( Um_n(A) \). It was shown by van der Kallen [45] that \( WS_n(A) \) is an abelian group. For any \( \underline{a} \in Um_n(A) \), let \([\underline{a}]\) denote its equivalence class in \( WS_n(A) \). We now quote the following independent results of Das-Zinna [12] and van der Kallen [49]. When \( d \geq 2 \) is even and \( Q \subset A \), this was earlier proven by Bhatwadekar-R. Sridharan [5 Theorem 7.6].

**Theorem 7.1.** There is an exact sequence

\[
WS_{d+1}(A) \xrightarrow{\phi_A} E(A) \xrightarrow{\psi_A} E_0(A) \to 0.
\]

7.3. The modified Euler class groups. We now introduce the modified Euler class groups. We shall say that an ideal \( J \subset A \) is regular, if it is reduced (i.e., \( J = \sqrt{J} \)) and the localization \( A_p \) is a regular local ring for every minimal prime \( p \) of \( J \). For any finitely generated \( A \)-module \( M \), let \( \mu(M) \) denote the smallest positive integer \( m \) such that \( M \) is generated by \( m \) elements.

(1) Let \( G^s(A) \) denote the free abelian group on the set of pairs \((m, \omega_m)\), where \( m \subset A \) is a regular maximal ideal of height \( d \) and \( \omega_m : (A/m)^d \twoheadrightarrow m/m^2 \) is an isomorphism.

Given a regular ideal \( J \subset A \) of height \( d \) with the primary decomposition \( J = m_1 \cap \cdots \cap m_r \) and an isomorphism \( \omega_J : (A/J)^d \xrightarrow{\cong} J/J^2 \), the Chinese remainder theorem yields isomorphisms \( \omega_{m_i} : (A/m_i)^d \xrightarrow{\cong} m_i/m_i^2 \). In particular, the ideal \( J \) and the map \( \omega_J \) together define a unique class \((J, \omega_J) = \sum_{i=1}^{r} (m_i, \omega_{m_i}) \in G^s(A)\).

Let \( H^s(A) \subset G^s(A) \) be the subgroup generated by the classes \((J, \omega_J)\) as above such that there is a commutative diagram of \( A \)-modules:

\[
\begin{array}{ccc}
A^d & \xrightarrow{\omega_J} & J \\
\downarrow & & \downarrow \\
(A/J)^d & \xrightarrow{\omega_J} & J/J^2.
\end{array}
\]

We let \( E^s(A) = G^s(A)/H^s(A) \).

(2) Let \( G^s_0(A) \) denote the free abelian group on the set of regular maximal ideals \( m \subset A \) of height \( d \). Given a regular ideal \( J \subset A \) of height \( d \) with the primary decomposition \( J = m_1 \cap \cdots \cap m_r \), we let \((J) = \sum_{i=1}^{r} m_i \in G^s_0(A)\). Let \( H^s_0(A) \subset G^s_0(A) \) be the subgroup generated by the classes \((J)\) as above such that \( \mu(J) = d \). We let \( E^s_0(A) = G^s_0(A)/H^s_0(A) \).
We now consider the following commutative diagram of short exact sequences.

\[
\begin{array}{c}
0 & 0 & 0 \\
0 & F^s(A) & H^s(A) & H_0^s(A) & 0 \\
0 & T^s(A) & G^s(A) & G_0^s(A) & 0 \\
0 & L^s(A) & E^s(A) & E_0^s(A) & 0 \\
& 0 & 0 & 0 & \\
\end{array}
\]

The only thing one needs to observe to get this diagram is that the map \( H^s(A) \to H_0^s(A) \) is surjective (by above definitions). It is easy to see that \( T^s(A) \) is generated by classes \((m, \omega_m) - (m, \omega'_m)\), where \( \omega_m : (A/m)^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2 \) and \( \omega'_m : (A/m)^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2 \) are two isomorphisms. It follows from (7.4) that the same holds for \( L^s(A) \) as well. A similar commutative diagram exists if we remove the superscript ‘s’ everywhere.

**Lemma 7.2.** Let \( k \) be an infinite field and let \( A \) be a geometrically reduced affine \( k \)-algebra. Then \( L^s(A) \subset E^s(A) \) is generated by elements of the type \((J, \omega_J)\), where \( J \) is a regular ideal of height \( d \) with \( \mu(J) = d \).

**Proof.** Let \( \tilde{L}^s(A) \) denote the subgroup of \( E^s(A) \) generated by elements \((J, \omega_J)\), where \( J \) is a regular ideal of height \( d \) with \( \mu(J) = d \). It is clear that \( \tilde{L}^s(A) \subseteq L^s(A) \). To prove the reverse inclusion, it suffices to show using the above description of \( L^s(A) \) that an element of the type \((m, \omega_m) - (m, \omega'_m)\) lies in \( \tilde{L}^s(A) \). The proof of this is a direct translation of [3] Lemma 3.3] and goes as follows.

If \((m, \omega'_m) = 0 \) in \( E^s(A) \), then it follows from [3] Theorem 4.2] that \((m, \omega_m) \in \tilde{L}^s(A) \). So we can assume that \((m, \omega'_m) \neq 0 \) in \( E^s(A) \). In this case, we can apply the Murthy-Swan Bertini theorem (see the proof of Lemma [7.3 below) to find a regular ideal \( I \) of height \( d \) which is co-maximal with \( m \) such that there is a surjection \( \tau : A^d \twoheadrightarrow J = m \cap I \) and \( \omega_m = \tau|_{A/m} \). If we let \( \omega_J = \tau|_{A/I} \), then we get \((m, \omega'_m) + (I, \omega_I) = (J, \tau|_{A/I}) = 0 \) in \( E^s(A) \).

On the other hand, since \( J = mI \) and \( m + I = A \), it follows that \( \omega_m \) and \( \omega_I \) induce a surjection \( \omega_J : (A/I)^d \twoheadrightarrow J/J^2 \) and hence \((m, \omega_m) + (I, \omega_I) = (J, \omega_J) \) in \( E^s(A) \). We conclude that \((m, \omega_m) - (m, \omega'_m) = (J, \omega_J) - (J, \tau|_{A/I}) = (J, \omega_J) \in L^s(A) \). This proves the lemma.

### 7.4. Connection between the classical and modified Euler class groups

We shall assume in this subsection that \( k \) is an infinite field. There is an obvious commutative diagram of the Euler class groups

\[
\begin{array}{ccc}
E^s(A) & \xrightarrow{\psi^s_A} & E_0^s(A) \\
\gamma_A \downarrow & & \downarrow \gamma^0_A \\
E(A) & \xrightarrow{\psi_A} & E_0(A)
\end{array}
\]

The goal of this section is to prove that the vertical arrows are isomorphisms. We begin with the easy part of this goal.
Lemma 7.3. Let \( A \) be a geometrically reduced affine algebra of dimension \( d \geq 2 \) over \( k \). Then there is a canonical isomorphism

\[
\gamma_A : E^s(A) \overset{\sim}{\to} E(A).
\]

Proof. We prove the surjectivity of \( \gamma_A \) using the Bertini theorems of Murthy [37, Theorem 2.3] and Swan [16, Theorem 1.3]. Let \( J \) be an ideal of \( A \) of height \( d \) with a surjection \( \omega_J : (A/J)^d \to J/J^2 \). Let \( \{m_1, \ldots, m_r\} \) be a set of smooth maximal ideals of \( A \). A special case of the Murthy-Swan Bertini theorem says that there exists an ideal \( I \subset A \) (called a residual of \( J \)) such that the following hold (see [37, Corollary 2.6 and Remarks 2.8, 3.2]).

1. There exists a surjection \( \alpha : A^d \to IJ \).
2. \( I + J = I + m_i = A \) for \( 1 \leq i \leq r \).
3. \( I \) is a regular ideal of \( A \) of height \( d \).
4. \( \alpha|_{A/J} = \omega_J \).

It follows from (1), (2), and (4) that \( (I, \alpha|_{A/I}) + (J, \omega_J) = (IJ, \alpha|_{A/J}) = 0 \) in \( E(A) \) and (3) says that \( (I, \alpha|_{A/J}) \in E^s(A) \). This shows that \( \gamma_A \) is surjective.

To show that \( \gamma_A \) is injective, let \( \alpha \in E^s(A) \) be such that \( \gamma_A(\alpha) = 0 \). By repeatedly applying the above Bertini theorem again, we can write \( \alpha = (J, \omega_J) \), where \( J \) is a regular ideal of height \( d \) in \( A \). We now apply [5, Theorem 4.2] to conclude that \( \omega_J \) lifts to a surjection \( \overline{\omega_J} : A^d \to J \). In particular, \( \alpha = (J, \omega_J) = 0 \) in \( E^s(A) \). This shows that \( \gamma_A \) is injective. \( \square \)

Using Theorem 7.1 and Lemma 7.3, we can now prove our main comparison result.

Proposition 7.4. Let \( A \) be a geometrically reduced affine algebra of dimension \( d \geq 2 \) over \( k \). Then there is a canonical isomorphism

\[
\gamma_A^0 : E^0_0(A) \overset{\sim}{\to} E_0(A).
\]

Proof. We have a commutative diagram of short exact sequences

\[
\begin{array}{ccc}
0 & \to & L^s(A) \to E^s(A) \to E^s_0(A) \to 0 \\
& & \downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
0 & \to & L(A) \to E(A) \to E_0(A) \to 0
\end{array}
\]

(7.6)

Using Lemma 7.3, it suffices to show that the left vertical arrow in this diagram is surjective. Using Theorem 7.1, it suffices to show that if \( \underline{a} = [a_1, \ldots, a_{d+1}] \) is a unimodular row, then \( \phi_A(\underline{a}) \in L^s(A) \). Note that we can identify \( L^s(A) \) with its image in \( L(A) \).

Let \( \{e_1, \ldots, e_d\} \) be the standard basis of the free \( A \)-module \( A^d \). Let \( \alpha : A^d \to A \) be given by \( \alpha(e_i) = a_i \) for \( 1 \leq i \leq d \). For a projective \( A \)-module \( P \) and an \( A \)-linear map \( f : P^* \to A \), let \( Z(f) \) denote the closed subscheme of \( \text{Spec} \( A \) \) where the induced map \( f^* : \text{Spec} \( A \) \to \text{Spec} \( \text{Sym}(P^*) \) \) vanishes.

We fix a surjective \( k \)-algebra homomorphism \( u : k[X_1, \ldots, X_n] \to A \) and let \( x_i = u(X_i) \) for \( 1 \leq i \leq n \). Let \( X = \text{Spec} \( A \) \) and let \( I \subset A \) be the reduced ideal such that \( \text{Spec} \( A/I \) = X_{\text{sing}}. \)

Let \( \overline{I} : k[X_1, \ldots, X_n] \to A \to A/I \) be the composite map. For any \( A \)-module \( M \) and an element \( m \in M \), let \( \overline{m} \) denote its image under the map \( M \to M/I \). For \( 1 \leq i \leq n \) and \( 1 \leq j \leq d \), we let \( t_{ij} = x_i e_j \in A^d \).

In this case, Swan’s Bertini theorem (see the proof of [16, Theorem 1.4]) says that there exists a dense open subset \( U_1 \subset A_k^{n(1+d)} \) such that for every \( (\lambda_i, \gamma_{ij}) \in U_1(k) \), the following hold.

1. \( Z(\alpha+a_{d+1}(\Sigma_i \lambda_i e_i + \Sigma_{i,j} \gamma_{ij} t_{ij})) \) is a reduced closed subscheme of \( X \) of pure codimension \( \geq d \).
2. \( Z(\alpha + a_{d+1}(\Sigma_i \lambda_i e_i + \Sigma_{i,j} \gamma_{ij} t_{ij})) \cap X_{\text{reg}} \) is regular.
Similarly, by applying the Bertini theorem to the composite embedding $X_{\text{sing}} \hookrightarrow X \hookrightarrow \mathbb{A}^n_k$, we get a dense open subset $U_2 \subset \mathbb{A}^{n+d}_k$ such that for every $((\lambda_i), (\gamma_{ij})) \in U_2(k)$, the following holds.

(3) $Z((\alpha + \overline{\alpha}_{d+1}(\sum_i \lambda_i e_i + \sum_{i,j} \gamma_{ij} e_{ij}))$ is a closed subscheme of $X_{\text{sing}}$ of pure codimension $d$ (or is empty).

Since $A$ is geometrically reduced and hence $\dim(X_{\text{sing}}) \leq d - 1$, it follows from (1), (2) and (3) that for a general set of elements $\{b_1, \ldots, b_d\}$ in $A$, the ideal $J = (a_1 + b_1 a_{d+1}, \ldots, a_d + b_d a_{d+1})$ has the property that either $J = A$ or it is a regular ideal of height $d$ in $A$.

If $J = A$, we have $\phi_A(\{a\}) = 0$. If $J$ is a regular ideal of height $d$, then it is easy to check that the equivalence class of the unimodular row $\overline{a'} := [a'_1, \ldots, a'_{d+1}]$ in $W_{S_{d+1}}(A)$ is same as that of $a$. Moreover, it follows from [11 § 7, p. 214] when $d = 2$ and from [12 Remark 3.7] when $d \geq 3$ that $\phi_A(\{a\}) = (J, \overline{a_{d+1}} \omega_J)$. Since $(J, \overline{a_{d+1}} \omega_J) \in G^s(A)$ and since $\mu(J) = d$, it follows from Lemma 7.2 that $(J, \overline{a_{d+1}} \omega_J) \in L^s(A)$. This finishes the proof of the proposition. □

### 7.5. Euler class group and K-theory

Let $k$ be a field and let $A$ be an equi-dimensional geometrically reduced affine algebra of dimension $d \geq 2$ over $k$. One can check from the definition that a generator of $E_0(A)$ may not be a local complete intersection ideal in $A$ in general. So there is no evident cycle class map $E_0(A) \to K_0(A)$. One immediate advantage of $E_0^s(A)$ is that each of its generator is a local complete intersection ideal. Using this idea and Proposition 7.4, one can construct a cycle class map $\text{cyc}_A : E_0^s(A) \to K_0(A)$ as follows.

If $m \in A$ is a regular maximal ideal, then $A/m$ admits a class $[A/m] \in K_0(A)$. Extending it linearly, one gets a map $G_0^s(A) \to K_0(A)$. If $J \subset A$ is a regular ideal of height $d$ such that $\mu(J) = d$, then one knows that it must be a complete intersection ideal (see [17 Theorems 135, 125]). Using the Koszul resolution of $A/J$, it easily follows that $\text{cyc}_A((J)) = [A/J] = 0$ in $K_0(A)$. We therefore get a map

(7.7) $\text{cyc}_A : E_0^s(A) \to K_0(A)$.

Using Proposition 7.4 we can now prove our main result of this section:

**Theorem 7.5.** Let $k$ be an infinite field and let $A$ be a geometrically reduced affine algebra of dimension $d \geq 2$ over $k$. Assume that one of the following holds.

1. $k$ is algebraically closed.
2. $(d - 1)! \in k^*$.

Then $\ker(\text{cyc}_A)$ is a torsion group of exponent $(d - 1)!$.

**Proof.** Let $\alpha \in E_0^s(A)$ be such that $\text{cyc}_A(\alpha) = 0$. By repeatedly applying the Murthy-Swan Bertini theorem, as in the proof of Lemma 7.3, we can assume that $\alpha = (I)$, where $I \subset A$ is a regular ideal of height $d$. Our assumption then says that $[A/I] = 0$ in $K_0(A)$. Since $I$ supported on the Cohen-Macaulay locus of $A$, the proof of [31 Lemma 1.2] shows that there exists an $A$-regular sequence $(f_1, \ldots, f_d)$ such that $I = (f_1, \ldots, f_d) + I^2$. Let $J = (f_1, \ldots, f_{d-1}) + I^{(d-1)!}$. It follows from [11 Lemma 4.1] that $(J) = (d - 1)! (I)$ in $E_0(A)$. If we can show that $(J) = 0$ in $E_0(A)$, then it will follow that $(d - 1)! (I) = 0$ in $E_0(A)$. We can then conclude from Proposition 7.4 that $(d - 1)! (I) = 0$ in $E_0^s(A)$. We have therefore reduced the problem to showing that $(J) = 0 \in E_0(A)$.

Now, it follows from [37 Theorem 2.2] that there exists a projective $A$-module $P$ of rank $d$ and a surjection $P \twoheadrightarrow J$ such that $[P] - [A^d] = -[A/I]$ in $K_0(A)$. It follows from our hypothesis on $I$ that $[P] = [A^d] \in K_0(A)$ so that $P$ is stably free. If $k$ is algebraically closed, it follows from [45 Theorem 6] that $P$ is free. But this implies that $\mu(J) = d$ so that $(J) = 0$ in $E_0(A)$.
Suppose now that \((d - 1)! \in k^\times\). At any rate, it follows from the cancellation theorem of Bass (see \[45\] Theorem 1) that \(P \oplus A \cong A^{d+1}\) so that \(P\) is the kernel of a surjection \(A^{d+1} \to A\). In this case, it is shown on \[5\] Page 214] that there exists an ideal \(J\) of height \(d\) with \(\mu(J') = d\) and a surjection \(P \to J'\). Furthermore, under the assumption that \((d - 1)! \in k^\times\), it is shown in \[5\] § 4] that the weak Euler class \(e_0(P)\) of \(P\) is well defined in \(E_0(A)\) and \((J) = e_0(P) = (J')\). We are now done because \((J') = 0 = E_0(A)\). This finishes the proof. \(\Box\)

Combining Proposition \([7,4]\) and Theorem \([7,5]\) we obtain the following. When \(\text{char}(k) = 0\) and \(A\) is Cohen-Macaulay, this was earlier proven independently by Bhatwadekar (unpublished) and Das-Mandal \([11\] Corollary 4.2]\).

**Corollary 7.6.** Let \(A\) be a geometrically reduced affine algebra of dimension \(d \geq 2\) over an infinite field \(k\). Then there is a cycle class map \(E_0(A) \to K_0(A)\) whose kernel is torsion of exponent \((d - 1)!\) if either \(k = \overline{k}\) or \((d - 1)! \in k^\times\).

## 8. The kernel of the cycle class map

We shall now prove Theorems \([1,2]\) \([1,4]\) \([1,5]\) and \([1,11]\) with the help of the results of § 7. In order to do so, we need to recall the cycle class map for 0-cycles in \([3,6]\) in the modulus setting.

### 8.1. The cycle class map with modulus

Let \(X\) be a smooth quasi-projective scheme of dimension \(d \geq 1\) over a perfect field \(k\) and let \(D \subset X\) be an effective Cartier divisor. Recall from \([6\] Theorem 12.4\] that there is a cycle class map with modulus

\[
\lambda_{X|D} : \text{CH}_0(X|D) \to K_0(X, D).
\]

This was constructed as the composition of the left arrows in the following commutative diagram of short exact sequences.

\[
\begin{array}{ccccccccc}
0 & \to & \text{CH}_0(X|D) & \xrightarrow{p_{+,*}} & \text{CH}_0(S_X) & \xrightarrow{\iota_*} & \text{CH}_0(X) & \to & 0 \\
\downarrow{\lambda_{X|D}} & & \downarrow{\lambda_{S_X}} & & \downarrow{\lambda_X} & & \downarrow{\lambda_X} & & \\
0 & \to & K_0(S_X, X_-) & \xrightarrow{p_{+,*}} & K_0(S_X) & \xrightarrow{\iota_*} & K_0(X) & \to & 0 \\
\downarrow{\phi_0} & & \downarrow{\iota_*} & & \downarrow{\iota_*} & & \downarrow{\iota_*} & & \\
K_0(X, D) & \xrightarrow{\iota_*} & K_0(X) & \xrightarrow{\iota_*} & K_0(D),
\end{array}
\]

where \(\lambda_{S_X}\) and \(\lambda_X\) are as in \([5,6]\).

Observe that, given \(x \in z_0(X|D)\), the composition of the maps of spectra \(K(k(x)) \to K(X) \to K(D)\) is null homotopic and hence it defines a map \(K_0(k(x)) \to K_0(X, D)\). Extending linearly, we have a homomorphism \(z_0(X|D) \to K_0(X, D)\). Moreover, this homomorphism factors through \(\phi_0 : K_0(S_X, X_+) \to K_0(X, D)\) and, on cycles, it is same as \(\lambda_{X|D}\). To see this, let \(p_{+,x} : \text{Spec}(k(x)) \to X \times S_X\) be the composition of the inclusions \(\iota_x : \text{Spec}(k(x)) \to X\) and \(\iota_+ : X \hookrightarrow S_X\). We then have the commutative diagram

\[
\begin{array}{ccccccccc}
K(k(x)) & \xrightarrow{\iota_{+,x}} & K^{(x)}(X_+ \setminus D) & \xrightarrow{\bar{\iota}_*} & K^{(x)}(S_X) & \to & K(S_X, X_-) & \to & K(k(x)) \\
\downarrow{\cong} & & \downarrow{\iota_{+,x}^*} & & \downarrow{\iota_*^*} & & \downarrow{\phi_0} & & \\
K(k(x)) & \xrightarrow{\iota_{+,x}} & K^{(x)}(X \setminus D) & \xrightarrow{\bar{\iota}_*} & K^{(x)}(X) & \to & K(X, D)
\end{array}
\]
such that the composition of the arrows in the top and bottom rows give the maps $\tilde{\lambda}_{X|D}$ and $\lambda_{X|D}$, respectively.

8.2. Proof of Theorem 1.5. We fix a field $k$. We also fix a (equi-dimensional) geometrically reduced affine algebra $A$ of dimension $d \geq 2$ over $k$ and let $X = \text{Spec}(A)$. We shall interchangeably use the notations $\text{CH}_0^{LW}(X)$ and $\text{CH}_0^{LW}(A)$. We begin with the following connection between the Euler class group and the Chow group of $A$.

It is clear from the definition of $E_0^s(A)$ in §7.3 that there are canonical maps $G_0^s(A) \xrightarrow{\sim} \mathcal{Z}_0(A) \rightarrow \text{CH}_0^{LW}(A)$ which sends a regular maximal ideal $m$ to the cycle class $[x] \in \mathcal{Z}_0(A)$, where $x = \text{Spec}(A/m) \in X_{\text{reg}}$. In order to show that the composite map factors through $E_0^s(A)$, we need to show that if $J \in A$ is a regular ideal of height $d$ such that $\mu(J) = d$, then the image of $(J)$ in $\mathcal{Z}_0(A)$ lies in $\mathcal{R}_0^{LW}(A)$. By [33 Lemma 2.2], it suffices to show that $J$ is a complete intersection ideal. But this follows directly from [17 Theorems 125, 135] because a regular ideal is always a local complete intersection. We have therefore constructed a canonical surjective map

$$ (8.4) \quad cyc_A : E_0^s(A) \rightarrow \text{CH}_0^{LW}(A) $$

and it is immediate from (7.4) that there is a commutative diagram

$$ (8.5) \quad E_0(A) \xrightarrow{\gamma_0^A} E_0^s(A) \xrightarrow{\cyc_A} \text{CH}_0^{LW}(A) \rightarrow \text{CH}(A) \rightarrow K_0(A). $$

As a combination of Theorem 7.5 and (8.5), we immediately get the following result about the cycle class map for the 0-cycles. When $k = \overline{k}$, this gives an independent proof of an old unpublished result of Levine (see [30 Corollary 5.4]). When $k$ is not algebraically closed, this result is completely new.

**Theorem 8.1.** Let $A$ be geometrically reduced affine algebra of dimension $d \geq 2$ over an field $k$. Assume that either $k$ is algebraically closed or $(d - 1)! \in k^*$. Let $X = \text{Spec}(A)$ and let $D \subset X$ be an effective Cartier divisor. Then the following hold.

1. If $k$ is an infinite field, then the kernel of the cycle class map $\lambda_A : \text{CH}_0^{LW}(A) \rightarrow K_0(A)$ is a torsion group of exponent $(d - 1)!$.

2. For arbitrary fields, the kernel of the cycle class map $\lambda_A : \text{CH}_0(A) \rightarrow K_0(A)$ is a torsion group of exponent $(d - 1)!$.

3. If $k$ is perfect and $X$ is smooth, then the kernel of the cycle class map $\lambda_{X|D} : \text{CH}_0(X|D) \rightarrow K_0(X, D)$ as in (8.1) is a torsion group of exponent $(d - 1)!$.

**Proof.** When $k$ is infinite, Theorem 7.5 and (8.5) together prove (1) and (2). We now assume $k$ is finite and prove (2). We denote the map $\text{CH}_0(A) \rightarrow K_0(A)$ by $\lambda_A$. Let $\alpha \in \text{CH}_0(A)$ be such that $\lambda_A(\alpha) = 0$. Let $\beta = (d - 1)\alpha \in \text{CH}_0(A)$. We choose two distinct primes $\ell_1$ and $\ell_2$ different from char($k$) and let $k_i$ denote the pro-$\ell_i$ extension of $k$ for $i = 1, 2$.

It follows from the case of infinite fields, the compatibility of the cycle class map with respect to field extensions and [6 Proposition 6.1] that $\beta_{k_i} = 0$ for $i = 1, 2$. Note that each $k_i$ is a limit of finite separable extensions of the perfect field $k$ and hence the hypotheses of [6 Proposition 6.1] are satisfied. Another application of [6 Proposition 6.1] shows that we can find two finite extensions $k'_1$ and $k'_2$ of $k$ of relatively prime degrees such that $\beta_{k'_i} = 0$ for $i = 1, 2$. We conclude by applying [6 Proposition 6.1] once again that $(d - 1)\alpha = \beta = 0$.

Now assume that $k$ is perfect and $X = \text{Spec}(A)$ is smooth. Then by (2) it follows that the kernel of the cycle class map $\lambda_{S_X} : \text{CH}_0(S_X) \rightarrow K_0(S_X)$ is of exponent $(d - 1)!$. It follows
from \([3.6]\) and \([8.2]\) that the same is true of the kernel of the map \(\lambda_{X|D} : \text{CH}_0(X|D) \to K_0(S(X, X_+))\). The assertion (3) then follows from \([36, \text{Lemma 4.1}]\) which yields that the natural map \(\phi_0 : K_0(S(X, X_-)) \to K_0(X, D)\) is an isomorphism.

\[\square\]

8.3. Bloch’s formula for affine surfaces over arbitrary field. As a corollary, we can prove Theorems 1.2, 1.4 and 1.11 as follows. The part (2) of the theorem below was conjectured by Bhatwadekar and R. Sridharan (see \([4, \text{Remark 3.13}]\)). Assuming that \(A\) is regular, this conjecture was proven by Bhatwadekar (unpublished) in dimension two and, by Asok and Fasel \([2]\) in general.

**Theorem 8.2.** Let \(k\) be a field and let \(A\) be a geometrically reduced affine algebra of dimension two over \(k\). Let \(X = \text{Spec}(A)\) and let \(D \subset X\) be an effective Cartier divisor. Then the following hold.

1. The map \(\text{CH}_0(X) \to K_0(X)\) is injective.
2. If \(k\) is finite, then there are isomorphisms \(E_0(A) \xrightarrow{\gamma} \text{CH}_0^{\text{LW}}(A) \xrightarrow{\xi} \text{CH}_0(A)\).
3. If \(k\) is a perfect field, then the map \(\rho_X : \mathcal{Z}_0(X) \to H^2_{\text{nis}}(X, \mathcal{K}^M_{2,X})\) induces an isomorphism \(\rho_X : \text{CH}_0(X) \xrightarrow{\xi} H^2_{\text{nis}}(X, \mathcal{K}^M_{2,X})\).
4. If \(k\) is a perfect field and \(X\) is smooth, there is an isomorphism \(\rho_X|D : \text{CH}_0(X|D) \xrightarrow{\xi} H^2_{\text{nis}}(X, \mathcal{K}^M_{2,(X,D)})\).

**Proof.** When \(k\) is finite, it follows from Proposition \([7.4]\), Theorem \([7.5]\) and \([8.5]\) that the maps

\[
E_0(A) \xleftarrow{\gamma_0} E'_0(A) \xrightarrow{\xi_0} \text{CH}_0^{\text{LW}}(A) \xrightarrow{\xi} \text{CH}_0(A)
\]

are all isomorphisms and the map \(\text{CH}_0(A) \to K_0(A)\) is injective. This proves (2) while the assertion (1) follows from Theorem \([8.1]\).

We now prove (3). By \([21, \text{Proposition 14}]\), the natural map of sheaves \(\mathcal{K}^M_{2,X} \to \mathcal{K}_{2,X}\) is surjective. Since they are generically same, it follows that \(H^2_{\text{nis}}(X, \mathcal{K}^M_{2,X}) \to H^2_{\text{nis}}(X, \mathcal{K}_{2,X})\) is an isomorphism. The existence of \(\rho_X : \text{CH}_0(X) \to H^2_{\text{nis}}(X, \mathcal{K}^M_{2,X})\) follows immediately from Lemma \([3.2]\) and \([24, \text{Lemma 2.1}]\), which shows that the map \(\kappa_X : H^2_{\text{zar}}(X, \mathcal{K}_{2,X}) \xrightarrow{\xi} H^2_{\text{nis}}(X, \mathcal{K}_{2,X}) \to K_0(X)\) is injective. It follows from (1) and Lemma \([3.2]\) that \(\rho_X\) is injective and its surjectivity follows from \([19, \text{Theorem 2.5}]\) (see the proof of Theorem \([8.1]\)). This proves (3). The assertion (4) follows from (3) and Theorem \([2.4]\) exactly as we proved Theorem \([1.3]\) in \(\S\) \([5.3]\).

**Remark 8.3.** We warn the reader that the affineness of \(X\) is an essential condition in Theorem \([8.2]\). One should not expect Bloch’s formula \(\text{CH}_0(X|D) \xrightarrow{\xi} H^2_{\text{nis}}(X, \mathcal{K}^M_{2,(X,D)})\) when \(X\) is a smooth projective surface over a finite field. The reason for this is that a result of Kerz and Saito \([22]\) says that there is an isomorphism \(\text{CH}_0(X|D)_0 \xrightarrow{\lambda_0} \pi_1(X, D)_0\), where \(\pi_1(X, D)\) is a quotient of the étale fundamental group of \(\pi_1(X \setminus D)\) which characterizes abelian étale overs of \(X \setminus D\) which have ramification along \(|D|\) bounded by the divisor \(D\).

On the other hand, a result of Kato and Saito \([19]\) implies that there is a surjection \(\pi_1(X, D) \to H^2_{\text{nis}}(X, \mathcal{K}^M_{2,(X,D)})\) which is not expected to be an isomorphism in general. This suggests that the relative Milnor \(K\)-theory needs to be suitably re-defined in order to solve this anomaly.

**References**

[1] A. Altman, S. Kleiman, *Bertini theorems for hypersurface sections containing a subscheme*, Comm. Alg., 7, (1979), no. 8, 775–790.

[2] A. Asok, J. Fasel, *Euler class groups and motivic stable cohomotopy*, [arXiv:1001.05723](http://arxiv.org/abs/1001.05723) (2016).
[3] H. Bass, J. Tate, The Milnor ring of a global field, Algebraic K-theory II, Lect. Notes Math., 342, (1972), 349–446.
[4] S. M. Bhatwadekar, R. Sridharan, Zero cycles and the Euler class groups of smooth real affine varieties, Invent. Math., 136, (1999), 287–322.
[5] S. M. Bhatwadekar, R. Sridharan, The Euler class group of a Noetherian ring, Comp. Math., 122, (2000), 183–222.
[6] F. Binda, A. Krishna, Zero cycles with modulus and zero cycles on singular varieties, Comp. Math., 154, (2018), 120–187.
[7] F. Binda, S. Saito, Relative cycles with moduli and regulator maps, J. Math. Inst. Jussieu, (to appear), (2017), arXiv:1412.0385
[8] J. Biswas, V. Srinivas, The Chow ring of a singular surface, Proc. Indian Acad. Sci., 108, (1998), 227–249.
[9] S. Bloch, $K_2$ and algebraic cycles, Ann. Math., 99, (1974), 349–379.
[10] A. Collino, Quillen’s K-theory and algebraic cycles on almost non-singular varieties, Ill. J. Math., 25, (1981), 654–666.
[11] M. K. Das, S. Mandal, A Riemann-Roch theorem, J. Algebra 301, (2006), 148–164.
[12] M. K. Das, Md. A. Zinna, Strong Euler class of a stably free module of odd rank, J. Algebra, 432, (2015), 185–204.
[13] H. Esnault, V. Srinivas, E. Viehweg, The Universal regular quotient of Chow group of points on projective varieties, Invent. Math., 135, (1999), 595–664.
[14] W. Fulton, Intersection theory, Second Edition, Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Folge. A Series of Modern Surveys in Mathematics, 2, Springer-Verlag, Berlin, (1998).
[15] H. Gillet, K-theory and intersection theory, in Handbook on K-theory, Vol. 1 , Springer Verlag, (2005), 235–293.
[16] A. Grothendieck, La théorie des classes de Chern, Bulletin de la Société 86, (1958), 137–154.
[17] I. Kaplansky, Commutative Rings, Boston, (1970).
[18] K. Kato, Milnor K-theory and the Chow group of zero cycles, Applications of algebraic K-theory to algebraic geometry and number theory, Contemporary Mathematics, 55, Amer. Math. Soc, Providence, RI, (1986), 241–253.
[19] K. Kato, S. Saito, Global class field theory of arithmetic schemes, Applications of algebraic K-theory to algebraic geometry and number theory, Contemporary Mathematics, 55, Amer. Math. Soc, Providence, RI, (1986), 255–331.
[20] M. Kerz, The Gersten conjecture for Milnor K-theory, Invent. Math., 175, (2009), 1–33.
[21] M. Kerz, Milnor K-theory of local rings with finite residue fields, J. Algebraic Geom., 19, (2010), 173–191.
[22] M. Kerz, S. Saito, Chow group of 0-cycles with modulus and higher dimensional class field theory, Duke Math. J., 165, (2016), no. 15, 2811–2897.
[23] M. Kerz, F. Strunk, G. Tamme, Algebraic K-theory and descent for blow-ups, Invent. Math. (to appear), (2017), arXiv:1611.08466.
[24] A. Krishna, On 0-cycles with modulus, Algebra & Number Theory, 9, (2015) no. 10, 2397–2415.
[25] A. Krishna, Murthy’s conjecture on 0-cycles, arXiv:1511.04221v1, (2015).
[26] A. Krishna, Torsion in the 0-cycle group with modulus, Algebra & Number Theory, to appear (2018), arXiv:1607.01493v2.
[27] A. Krishna, J. Park, A module structure and a vanishing theorem for cycles with modulus, Math. Res. Lett., 24, (2017), no. 4, 1147–1176.
[28] A. Krishna, V. Srinivas, Zero-cycles and K-theory on normal surfaces, Ann. of Math., 156, (2002), 155–195.
[29] S. Lang, Abelian Varieties, Interscience Tracts in Pure and Applied Mathematics, no. 7, Interscience Publishers, (1959).
[30] M. Levine, A geometric theory of the Chow ring of a singular variety, Unpublished manuscript.
[31] M. Levine, Bloch’s formula for singular surfaces, Topology, 24, (1985), 165–174.
[32] M. Levine, Zero-cycles and K-theory on singular varieties, Proc. Symp. Pure Math., 46, Amer. Math. Soc, Providence, (1987), 451–462.
[33] M. Levine, C. Weibel, Zero cycles and complete intersections on singular varieties, J. Reine Angew. Math., 359, (1985), 106–120.
[34] S. Mandal, Complete intersection K-theory and Chern classes, Math. Z. 227, (1998), 423–454.
[35] H. Matsumura, Commutative ring theory, Cambridge studies in advanced mathematics, Cambridge university press, 8, (1997).
[36] J. Milnor, Introduction To Algebraic K-Theory, Annals of Math Studies, 72, Princeton, (1971).
[37] M. P. Murthy, Zero cycles and projective modules, Ann. Math., 140, (1994), 405–434.
[38] C. Pedrini, C. Weibel, *Divisibility in the Chow group of zero-cycles on a singular surface*, Asterisque, 226, (1994), 371–409.

[39] D. Quillen, *Higher Algebraic K-theory*, LNM Series, 341, Springer-Verlag, Berlin, (1973), 85–147.

[40] M. Rost, *Chow groups with coefficients*, Doc. Math., 1, (1996), 319–393.

[41] K. Rülling, S. Saito, *Higher Chow groups with modulus and relative Milnor K-theory*, Trans. Amer. Math. Soc., to appear, doi:10.1090/tran/7018. Google Scholar.

[42] M. Schlichting, *Euler class groups, and the homology of elementary and special linear groups*, Adv. Math., 320, (2107), 1–81.

[43] V. Srinivas, *Zero cycles on a singular surface II*, J. Reine angew. Math., 362, (1985), 4–27.

[44] V. Srinivas, *Algebraic cycles on singular varieties*, Proceedings of the International Congress of Mathematicians, 2, Hindustan Book Agency, New Delhi, (2010), 603–623.

[45] A. Suslin, *A cancellation theorem for projective modules over algebras*, Soviet Math. Dokl., 18, (5) (1977), 323–338.

[46] R. G. Swan, *A cancellation theorem for projective modules in the metastable range*, Invent. Math., 27, (1974), 23–43.

[47] R. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., Birkhäuser, Boston, MA, 88, 1990, 247–435.

[48] W. van der Kallen, *A module structure on certain orbit set of unimodular rows*, J. Pure Appl. Algebra, 57 (1989), no. 3, 281–316.

[49] W. van der Kallen, *Extrapolating an Euler class*, J. Algebra, 434, (2015), 65–71.

School of Mathematics, Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Colaba, Mumbai, India

E-mail address: amal@math.tifr.res.in

E-mail address: rahul@math.tifr.res.in