ON FUZZY SEMIHYPERRINGS

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Abstract. In this article we introduce the study of fuzzy semihyperrings and fuzzy \( R \)-semihypermodules, where \( R \) is a semihyperring and \( R \)-semihypermodules are representations of \( R \). In particular, semihyperrings all of whose hyperideals are idempotent, called fully idempotent semihyperrings, are investigated in a fuzzy context. It is proved, among other results, that a semihyperring \( R \) is fully idempotent if and only if the lattices of fuzzy hyperideals of \( R \) is distributive under the sum and product of fuzzy hyperideals. It is also shown that the set of proper fuzzy prime hyperideals of a fully idempotent semihyperring \( R \) admits the structure of a topological space, called the fuzzy prime spectrum of \( R \).

1. Introduction

The concept of hyperstructure was first introduced by Marty [15] at the eighth Congress of Scandinavian Mathematicians in 1934, when he defined hypergroups and started to analyze their properties. Now, the theory of algebraic hyperstructures has become a well-established branch in algebraic theory and it has extensive applications in many branches of mathematics and applied science. Later on, people have developed the semi-hypergroups, which are the simplest algebraic hyperstructures having closure and associative properties. A comprehensive review of the theory of hyperstructures can be found in [7, 8, 20]. The canonical hypergroups are a special type of hypergroup which Mittas [17], is the first one that studied them extensively. The theory of hypermodules which their additive structure is just a canonical hypergroup, several authors have studied that, for example, Massouros [16], Corsini [6], Davvaz [9, 10], Zhan et al. [22], Ameri [5] and Zahedi and Ameri [26].

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The theory of fuzzy sets, introduced by Zadeh [23] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods and tools. In [1, 2, 4, 13, 14, 19, 24, 25], some applications of this theory in algebraic structures and hyperstructures can be seen.

In this paper, however, we pursue an algebraic approach to investigate the concept of fuzzy semihyperring and related notion in order to set the ground for future work. In section 1, we provide basic definitions and establish some preliminary results. In Section 2, we investigate fully idempotent semihyperrings, that is, semihyperrings all of whose hyperideals are idempotent. It is proved that such semihyperrings are characterized by the property that each proper fuzzy hyperideal is the intersection of fuzzy prime hyperideals containing it. In section 3, we construct the fuzzy prime spectrum of fully idempotent semihyperrings in a manner analogous to the construction of the prime spectrum in classical semiring theory.

2. Preliminaries

A hyperstructure is a non-empty set say $H$ together with a mapping "$\circ$": $H \times H \to P^*(H)$, where $P^*(H)$ is the set of all the non-empty subsets of $H$ and "$\circ$" is called hyperoperation. If $x \in H$ and $A, B \in P^*(H)$, then by $A \circ B$, $A \circ x$, and $x \circ B$, we mean $A \circ B = \bigcup\{a \circ b : a \in A$ and $b \in B\}$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$, respectively [7, 8]. A hypergroupoid is a set $H$ with a binary hyperoperation $\circ$ and a commutative hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$ is called a semi-hypergroup. A hypergroup is a semi-hypergroup, such that, for all $x \in H$, we have $x \circ H = H \circ x$, which is called reproduction axiom. If $H$ is a hypergroup and $K$ is a nonempty subset of $H$. Then $K$ is a subhypergroup of $H$ if $K$ itself a hypergroup under hyperoperation, defined in $H$. Hence it is clear that a subset $K$ of $H$ is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on $H$.

A set $H$ together a hyperoperation $\circ$ is called a polygroup, if the following conditions are satisfied:

1. $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$.
2. There exist a unique element, $e \in H$ such that $e \circ x = x \circ e = x$, for all $x \in H$.
3. For all $x \in H$, there exist a unique element, say $x' \in H$ such that $e \in x \circ x' \cap x' \circ x$ (where $x' = x^{-1}$).
4. For all $x, y, z \in H$, $z \in x \circ y \Rightarrow x \in z \circ y^{-1} \Rightarrow y \in x^{-1} \circ z$.

A non-empty subset $K$ of a polygroup $(H, \circ)$ is called a subpolygroup if $(K, \circ)$ is itself a polygroup. In this case we write $K \triangleleft_p H$.

A commutative polygroup is called canonical hypergroup.

Definition 2.1. A semihyperring is an algebraic hypersystem $(R, \oplus, \cdot)$ consisting of a non-empty set $R$ together with one hyperoperation "$\oplus$" and one binary operation "$\cdot$" on $R$, such that $(R, \oplus)$ is a commutative semihypergroup and $(R, \cdot)$ is a semigroup. For all $x, y, z \in R$, the binary operation of multiplication is distributive over hyperoperation from both sides that is, $x.(y \oplus z) = x.y \oplus x.z$ and $(x \oplus y).z = x.z \oplus y.z$. An element $0 \in R$, is an absorbing element, such that $0 \oplus x = x = x \oplus 0$, and $0.x = 0 = x.0$ for all $x \in R$ [18]. By a subsemihyperring of $R$, we mean a non-empty subset $S$ of $R$ such that for all $x, y \in S$, we have $x.y \in S$ and $x \oplus y \subseteq S$. 

In[1, 2, 4, 13, 14, 19, 24, 25], some applications of this theory in algebraic structures and hyperstructures can be seen.
A semihyperring \((R,\oplus,\cdot)\) is called a hyperring if \((R,\oplus)\) is a canonical hypergroup and \((R,\cdot)\) is a semigroup.

**Example 2.2.** Let \(X\) be a non-empty finite set and \(\tau\) is a topology on \(X\). We define the hyperoperation of the addition and the multiplication on \(\tau\) as;

For any \(A, B \in \tau\), \(A \oplus B = A \cup B\) and \(A.B = A \cap B\). Then \((\tau,\oplus,\cdot)\) is a semihyperring with absorbing element and additive identity \(\Phi\) and multiplicative identity \(X\).

**Example 2.3.** Let us consider a set \(S = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in W\}\). Where \(W\) is a set of whole numbers. We define the hyperoperation of addition and multiplication as;

For \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) and \(B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\) be taken from \(S\).

\[
A \oplus B = \left[\begin{array}{ll}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & + & \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}
\end{array}\right] = \left[\begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix}\right] \subseteq S
\]

\[
A.B = \left[\begin{array}{ll}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & \cdot & \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}
\end{array}\right] = \left[\begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}\right] \in S
\]

Then \((S,\oplus,\cdot)\) is a semihyperring with hyperidentity as a null matrix and multiplicative identity as an identity matrix.

**Example 2.4.** On four element semihyperring \((R,\oplus,\cdot)\) defined by the following two tables:

| \(\oplus\) | 0   | a   | b   | c   | 0   | a   | b   | c   |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|
| 0         | \{0\} | \{0\} | \{0\} | \{0\} | 0   | 0   | 0   | 0   |
| a         | \{0\} | \{a,b\} | \{b\} | \{c\} | a   | a   | a   | a   |
| b         | \{0\} | \{b\} | \{0,b\} | \{c\} | b   | b   | b   | b   |
| c         | \{0\} | \{c\} | \{c\} | \{0,c\} | c   | c   | c   | c   |

By routine calculation \(\{0\}, \{0,b\}, \{0,c\}, \{0,a,b\},\{0,b,c\}, R\), are subsemihyperrings of \(R\).

**Definition 2.5.** By a left (right) hyperideal of \(R\), we mean a subsemihyperring \(I\) of \(R\) such that for all \(r \in R\) and \(x \in I\), we have \(r.x \in I\) (\(x.r \in I\)).

By a hyperideal , we mean a subsemihyperring of \(R\) which is both a left and a right hyperideal of \(R\). A hyperideal generated by a non-empty subset \(A\) of a semihyperring \(R\), will be denoted by \(\langle A \rangle\), which is intersection of all hyperideals of \(R\), which contains \(A\). If \(I\) and \(J\) be two hyperideals of a semihyperring \(R\), then the sum and product of two hyperideals are also a hyperideal and defined as respectively:

\[
I \oplus J = \bigcup_{a_i \in I, b_j \in J} (a_i \oplus b_j)
\]

and

\[
IJ = \left\{ \sum_{finite} a_i.b_j; a_i \in I, b_j \in J \right\}
\]

**Example 2.6.** On four element semihyperring \((R,\oplus,\cdot)\) defined by the example 2.4

These all \(\{0,b\}, \{0,c\}, \{0,a,b\}, \{0,b,c\}\), are right hyperideals of semihyperring \(R\).
**Definition 2.7.** A non-empty set $M$, which is commutative semihypergroup with respect to addition, with an absorbing element 0 is called a right, $R$-semihypermodule $M_R$, if $R$ is a semihyperring and there is a function $\alpha : M \times R \rightarrow P^*(M)$, where $P^*(M) = P(M) \setminus \{0\}$, such that if $\alpha(m, x)$ is denoted by $mx$ and $mx \subseteq M$, for all $x \in R$ and $m \in M$. Then the following conditions hold, for all $x, y \in R$ and $m_1, m_2, m \in M$:

(i) $(m_1 \oplus m_2)x = m_1x \oplus m_2x$
(ii) $m(x \oplus y) = mx \oplus my$
(iii) $m(xy) = (mx)y$
(iv) $0x = 0 = 0.$

Similarly, we can define a left $R$-semihypermodules $_RM$. A semihyperring $R$ is a right semihypermodules over itself which will be denoted by $R_R$. A non-empty subset $N$ of a right $R$-semihypermodule $M$ is called a subsemihypermodule of $M$, if $(N, \oplus)$ is a subsemihypergroup of $(M, \oplus)$ and $RN \subseteq P^*(N)$. Also note that, the right (left) subsemihypermodules $R_R$ are right (left) hyperideals of $R$.

**Definition 2.8.** Every hyperideal of a semihyperring $R$ is a semihypermodule of $R$.

Therefore by example [24], semihyperring $R$, $\{0, b\}$, $\{0, c\}$, $\{0, a, b\}$ and $\{0, b, c\}$, are right $R$-semihypermodules of semihyperring $R$.

**Definition 2.9.** If $X$ is a universe and $A \subseteq X$, then characteristic function of $A$ is a function $\chi_A : X \rightarrow \{0, 1\}$, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

A fuzzy subset $\lambda$ of $X$ is a function $\lambda : X \rightarrow [0,1]$, for all $x \in X$, where $\lambda$ is fuzzy subset of $X$ such that for each $x \in X$, $0 \leq \lambda(x) \leq 1$. For any two fuzzy subsets $\lambda$ and $\mu$ of $X$, $\lambda \leq \mu$ if and only if $\lambda(x) \leq \mu(x)$, for all $x \in X$. The symbols $\lambda \land \mu$, and $\lambda \lor \mu$ will mean the following fuzzy subsets of $X$, for all $x \in X$.

$$(\lambda \land \mu)(x) = \min\{\lambda(x), \mu(x)\}$$

$$(\lambda \lor \mu)(x) = \max\{\lambda(x), \mu(x)\}$$

More generally, if $\{\lambda_i : i \in \Lambda\}$ is a family of fuzzy subsets of $X$, then $\land_{i \in \Lambda} \lambda_i$ and $\lor_{i \in \Lambda} \lambda_i$ are defined by

$$(\land_{i \in \Lambda} \lambda_i)(x) = \min_{i \in \Lambda} \lambda_i(x)$$

$$(\lor_{i \in \Lambda} \lambda_i)(x) = \max_{i \in \Lambda} \lambda_i(x)$$

and will called the intersection and union of the family $\{\lambda_i : i \in \Lambda\}$ of fuzzy subsets of $X$.

Let $\lambda$ be a fuzzy subset of $X$ and $t \in (0,1]$. Then the set $U^\lambda_t = \{x \in X : \lambda(x) \geq t\}$ is called the level subset of $X$.

**Definition 2.10.** Let $R$ be a semihyperring and $\mu$ a fuzzy set in $R$. Then, $\mu$ is said to be a fuzzy hyperideal of $R$ if for all $r, x, y \in R$ the following axioms hold:
(i) \( \inf_{x \in x \oplus y} \mu(z) \geq \mu(x) \wedge \mu(y) \), for all \( x, y \in R \).

(ii) \( \mu(xy) \geq \mu(x) \) and \( \mu(yx) \geq \mu(x) \) for all \( x, y \in R \).

**Theorem 2.11.** Let \( \mu \) be a fuzzy set in a semihyperring \( R \). Then, \( \mu \) is a fuzzy hyperideal of \( R \) if and only if for every \( t \in (0, 1) \), the level subset

\[
\mu_t = \{ x \in R | \mu(x) \geq t \} \neq \emptyset
\]

is an hyperideal of \( R \).

**Proof.** The proof is straight forward by considering the definition. \( \Box \)

**Definition 2.12.** Let \( M \) be a right (left) \( R \)-semihypermodule. A function \( \mu : M \to [0, 1] \), is called a fuzzy subsemihypermodule of \( M_R \) (\( R_M \)), if the following conditions hold for all \( m_1, m_2, m \in M \):

(i) \( \mu(0_M) = 1 \)

(ii) \( \inf_{m' \in m_1 \oplus m_2} \mu(m') \geq \mu(m_1) \wedge \mu(m_2) \), for all \( m_1, m_2 \in M \),

(iii) \( \mu(mr) \geq \mu(m) \), \( \mu(mr) \geq \mu(m) \) for all \( r \in R \) and \( m \in M \).

Also note that, fuzzy subsemihypermodules of \( R_R \) (\( R_L \)) are called fuzzy hyperideals of \( R \).

Generalizing the notion of a fuzzy hypermodule \( [1, 13, 18] \), we formulate the following definition.

**Definition 2.13.** Let \( \lambda \) be a fuzzy subsemihypermodule of a right semihypermodule \( M_R \) and \( \mu \) a fuzzy hyperideal of \( R \).

Then the fuzzy subset \( \lambda \mu \) of \( M \) is defined by

\[
(\lambda \mu)(x) = \bigvee_{x \in \Sigma_{j=1}^{p} y_j z_j} \left[ \left( \bigwedge_{i=1}^{q} \mu(y_j) \wedge \mu(z_i) \right) \right]
\]

where \( x \in M, y_j \in M, z_i \in R \) and \( p \in N \).

**Proposition 2.14.** If \( \lambda \) is a fuzzy subsemihypermodule of \( M_R \) and \( \mu \) a fuzzy hyperideal of \( R \).

Then the fuzzy subset \( \lambda \mu \) is a fuzzy subsemihypermodule of \( M \).

**Proof.** We have

(i) \( (\lambda \mu)(0_M) = \bigvee_{0 \in \Sigma_{j=1}^{p} y_j z_j} \left[ \left( \bigwedge_{i=1}^{q} \mu(y_j) \wedge \mu(z_i) \right) \right] \geq \lambda(0_M) \wedge \mu(0) = 1 \).

Thus \( (\lambda \mu)(0_M) = 1 \).

For (ii)

\[
(\lambda \mu)(m) = \bigvee_{m' \in \Sigma_{j=1}^{p} y_j' z_j'} \left[ \left( \bigwedge_{i=1}^{q} \mu(y_j') \wedge \mu(z_i') \right) \right],
\]

and

\[
(\lambda \mu)(m') = \bigvee_{m' \in \Sigma_{k=1}^{r} y_k' z_k'} \left[ \left( \bigwedge_{i=1}^{q} \mu(y_k') \wedge \mu(z_i') \right) \right],
\]

where \( m, m' \in M \).

Thus

\[
(\lambda \mu)(m) \wedge (\lambda \mu)(m') = \bigvee_{m' \in \Sigma_{j=1}^{p} y_j' z_j'} \left[ \left( \bigwedge_{i=1}^{q} \mu(y_j') \wedge (\mu(z_i')) \right) \right] \wedge \bigvee_{m' \in \Sigma_{k=1}^{r} y_k' z_k'} \left[ \left( \bigwedge_{i=1}^{q} \mu(y_k') \wedge (\mu(z_i')) \right) \right]
\]

(using the infinite meet distributive law)

\[
= \bigvee_{m' \in \Sigma_{j=1}^{p} y_j' z_j'} \bigvee_{m' \in \Sigma_{k=1}^{r} y_k' z_k'} \left[ \left( \bigwedge_{i=1}^{q} \mu(y_j') \wedge (\mu(z_i')) \right) \right] \wedge \bigvee_{1 \leq k \leq r} \left[ \left( \bigwedge_{1 \leq k \leq r} (\mu(y_k') \wedge (\mu(z_k'))) \right) \right]
\]
Corollary 2.15. If $\lambda$ and $\mu$ are fuzzy hyperideals of $R$, then $\lambda \mu$ is a fuzzy hyperideal of $R$, called the product of $\lambda$ and $\mu$.

Remark 2.16. If $\lambda$ and $\mu$ are fuzzy hyperideals of $R$, then $\lambda \land \mu$ is clearly a fuzzy hyperideal of $R$. In general, $\lambda \land \mu \neq \lambda \mu$.

Definition 2.17. If $\lambda$ and $\mu$ are fuzzy hyperideals of $R$. The fuzzy subset $\lambda \oplus \mu$ of $R$ is defined by

$$(\lambda \oplus \mu)(x) = \bigvee_{z,y \in z} [\lambda(y) \land \mu(z)],$$

for $x \in R$.

Proposition 2.18. For fuzzy hyperideals $\lambda$ and $\mu$ of $R$, $\lambda \oplus \mu$ is a fuzzy hyperideal of $R$ (called the sum of $\lambda$ and $\mu$).

Proof. For any $x, x' \in R$

$$(\lambda \oplus \mu)(x) \land (\lambda \oplus \mu)(x') = \bigvee_{x' \in y \land z} (\lambda(y) \land \mu(z)) \land \bigvee_{x' \in y' \land z'} (\lambda(y') \land \mu(z'))$$

$$= \bigvee_{x \in y \land z} \bigvee_{x' \in y' \land z'} (\lambda(y) \land \mu(z)) \land \bigvee_{x' \in y' \land z'} (\lambda(y') \land \mu(z'))$$

$$= \bigvee_{x \in y \land z} \bigvee_{x' \in y' \land z'} (\lambda(y) \land \lambda(y')) \land (\mu(z) \land \mu(z'))$$

$$\leq \bigvee_{x \in y \land z} \inf_{z' \in z \land z' \land z} (\lambda(y) \land \lambda(y')) \land \inf_{z' \in z \land z' \land z} (\mu(z) \land \mu(z'))$$

$$\leq \inf_{x \in y \land z} (\lambda \oplus \mu)(x').$$

Again

$$(\lambda \oplus \mu)(x) = \bigvee_{x \in y \land z} (\lambda(y) \land \mu(z))$$

$$\leq \bigvee_{x \in y \land z} (\lambda(ya) \land \mu(za))$$

(where $a$ is any element of $R$)

$$\leq \bigvee_{x \in y \land z} (\lambda(y') \land \mu(z'))$$

$$= \inf_{b \in xa} (\lambda \oplus \mu)(b)$$

Hence $\lambda \oplus \mu$ is a fuzzy hyperideal of $R$. 

\[\blacksquare\]

3. Fully idempotent semihyperrings

A semihyperring $R$ is called fully idempotent if each hyperideal of $R$ is idempotent (a hyperideal $I$ is idempotent if $I^2 = I$), and a semihyperring $R$ is said to be regular if for each $x \in R$, there exist $a, b \in R$ such that $x = axa$.

Lemma 3.1. A semihyperring $R$ is regular if and only if for any right hyperideal $I$ and for any left hyperideal $L$ of $R$, we have $IL = I \cap L$.

Concerning these semihyperrings, we prove the following characterization theorem.
Theorem 3.2. The following conditions for a semihyperring \( R \), are equivalent:

1. \( R \) is fully idempotent,
2. Fuzzy hyperideal of \( R \) is idempotent.
3. For each pair of fuzzy hyperideals \( \lambda \) and \( \mu \) of \( R \), \( \lambda \land \mu = \lambda \mu \).

If \( R \) is assumed to be commutative (that is, \( xy = yx \) for all \( x, y \in R \)), then the above conditions are equivalent to:

4. \( R \) is regular.

Proof. (1)\( \Rightarrow \) (2). Let \( \delta \) be a fuzzy hyperideal of \( R \). For any \( x \in R \),

\[
\delta^2(x) = (\delta \delta)(x)
\]

\[
= \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \land \left( \delta(y_i) \land \delta(z_i) \right) \right]
\]

\[
\leq \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \land \left( \delta(y_i z_i) \land \delta(y_i z_i) \right) \right]
\]

\[
= \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \delta(y_i z_i) \land \delta(y_i z_i) \right]
\]

\[
\leq \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \delta(x) \land \delta(x) = \delta(x).
\]

Since each hyperideal of \( R \) is idempotent, therefore, \( (x) = (x)^2 \), for each \( x \in R \).

Since \( x \in (x) \) it follows that \( x \in (x)^2 = R_{x R R x} \). Hence, \( x = \sum_{i=1}^q a_i x^a_i b_i x^b_i \)

where \( a_i, a^i, b_i, b^i \in R \) and \( q \in N \). Now, \( \delta(x) = \delta(x) \land \delta(x) \leq \delta(a_i x^a_i) \land \delta(b_i x^b_i) \)

\((1 \leq i \leq q)\).

Therefore,

\[
\delta(x) \leq \bigvee_{1 \leq i \leq q} \left[ \delta(a_i x^a_i) \land \delta(b_i x^b_i) \right]
\]

\[
\leq \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \delta(y_i z_i) \land \delta(y_i z_i) \right]
\]

\[
\leq \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \delta(x) \land \delta(x) = \delta(x).
\]

Thus \( \delta^2 = \delta \).

(2)\( \Rightarrow \) (1). Let \( I \) be an hyperideal of \( R \). Thus \( \delta_I \), the characteristic function of \( I \), is a fuzzy hyperideal of \( R \). Hence \( \delta_I^2 = \delta_I \). Therefore, \( \delta_I \delta_I = \delta_I \), hence \( \delta_I^2 = \delta_I \). It follows that \( I^2 = I \). Hence (2)\( \Leftrightarrow \) (1).

(1)\( \Rightarrow \) (3). Let \( \lambda \) and \( \mu \) be any pair of fuzzy hyperideals of \( R \). For any \( x \in R \),

\[
(\lambda \mu)(x) = \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \lambda(y_i) \land \mu(z_i) \right]
\]

\[
\leq \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \lambda(y_i z_i) \land \mu(y_i z_i) \right]
\]

\[
\leq \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \lambda(y_i z_i) \land \mu(y_i z_i) \right]
\]

\[
= \lambda(x) \land \mu(x).
\]

Again, since \( R \) is fully idempotent, \( x = x^2 \), for any \( x \in R \). Hence, as argued in the first part of the proof of this theorem, we have

\[
(\lambda \land \mu)(x) = \lambda(x) \land \mu(x)
\]

\[
\leq \bigvee_{x \in \Sigma_{i=1}^p y_i, \sum_1^i \leq p} \left[ \lambda(y_i) \land \mu(z_i) \right]
\]

\[
= (\lambda \mu)(x).
\]

Thus \( \lambda \land \mu = \lambda \mu \).

(3)\( \Rightarrow \) (1). Let \( \lambda \) and \( \mu \) be any pair of fuzzy hyperideals of \( R \). We have, \( \lambda \land \mu = \lambda \mu \). take \( \mu = \lambda \).
Thus \( \lambda \land \lambda = \lambda^2 \), that is, \( \lambda = \lambda^2 \), where \( \lambda \) is any fuzzy hyperideal of \( R \). Hence, (3) \( \Rightarrow \) (2). Since we already proved that (1) and (2) are equivalent, hence it follows that (3) \( \Rightarrow \) (1) and (1) \( \Rightarrow \) (3). This establishes (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3). Finally, if \( R \) is commutative then it is easy to verify that (1) \( \Leftrightarrow \) (4).

Next, we prove another characterization theorem for fully idempotent semihyperrings.

**Theorem 3.3.** The following conditions for a semihyperring \( R \), are equivalent:

1. \( R \) is fully idempotent,
2. The set of all fuzzy hyperideals of \( R \) (ordered by \( \leq \)) forms a distributive lattice \( FI_R \) under the sum and intersection of fuzzy hyperideals with \( \lambda \land \mu = \lambda \mu \), for each pair of fuzzy hyperideals \( \lambda, \mu \) of \( R \).

**Proof.** (1) \( \Rightarrow \) (2). The set \( FI_R \) of all fuzzy hyperideals of \( R \) (ordered by \( \leq \)) is clearly a lattice under the sum and intersection of fuzzy hyperideals. Moreover, since \( R \) is a fully idempotent semihyperring, it follows from above Theorem that \( \lambda \land \mu = \lambda \mu \), for each pair of fuzzy hyperideals \( \lambda, \mu \) of \( R \). We now show that \( FI_R \) is a distributive lattice, that is, for fuzzy hyperideals \( \lambda, \delta \) and \( \eta \) of \( R \), we have

\[
[(\lambda \land \delta) \lor \eta] = [(\lambda \lor \eta) \land (\delta \lor \eta)],
\]

For any \( x \in R \),

\[
[(\lambda \land \delta) \lor \eta](x) = \bigvee_{y \in y \lor z} [(\lambda \land \delta)(y) \land \eta(z)]
\]

\[
= \bigvee_{y \in y \lor z} [\lambda(y) \land \delta(y) \land \eta(z)]
\]

\[
= \bigvee_{y \in y \lor z} [\lambda(y) \land \delta(y) \land \eta(z) \land \eta(z)]
\]

\[
= \bigvee_{y \in y \lor z} [\lambda(y) \land \eta(z) \land \delta(y) \land \eta(z)]
\]

\[
= \bigvee_{y \in y \lor z} [(\lambda \lor \eta)(x) \land (\delta \lor \eta)(x)]
\]

because, for \( x \in y \lor z \), \( \lambda(y) \lor \eta(z) \leq (\lambda \lor \eta)(x) \) and, similarly, \( \delta(y) \land \eta(z) \leq (\delta \lor \eta)(x) \).

Again,

\[
[(\lambda \lor \eta) \land (\delta \lor \eta)](x)
\]

\[
= [(\lambda \lor \eta)(\delta \lor \eta)](x)
\]

\[
= \bigvee_{x \in x \lor y} \bigwedge_{1 \leq i \leq p} [(\lambda \lor \eta)(y_i) \land (\delta \lor \eta)(z_i)]
\]

\[
= \bigvee_{x \in x \lor y} \bigwedge_{1 \leq i \leq p} \bigg[ \bigvee_{y_i \in y_i \lor z_i} (\lambda(y_i) \land \eta(s_i)) \land \bigvee_{z_i \in z_i \lor u_i} (\delta(t_i) \land \eta(u_i)) \bigg]
\]

\[
= \bigvee_{x \in x \lor y} \bigwedge_{1 \leq i \leq p} \bigg[ \bigvee_{y_i \in y_i \lor z_i} (\lambda(y_i) \land \eta(s_i) \land (\delta(t_i) \land \eta(u_i))) \bigg]
\]

using infinite meet distributive law

\[
= \bigvee_{x \in x \lor y} \bigwedge_{1 \leq i \leq p} \bigg[ \bigvee_{y_i \in y_i \lor z_i} (\lambda(y_i) \land \eta(s_i) \land (\delta(t_i) \land \eta(u_i))) \bigg]
\]

\[
\leq \bigvee_{x \in x \lor y} \bigwedge_{1 \leq i \leq p} \bigg[ \bigvee_{y_i \in y_i \lor z_i} (\lambda(y_i) \land \eta(s_i) \land (\delta(t_i) \land \eta(u_i))) \bigg]
\]

\[
\leq \bigvee_{x \in x \lor y} \bigwedge_{1 \leq i \leq p} \bigg[ \bigvee_{y_i \in y_i \lor z_i} (\lambda(y_i) \land \eta(s_i) \land (\delta(t_i) \land \eta(u_i))) \bigg]
\]

\[
\leq \bigvee_{x \in x \lor y} \bigwedge_{1 \leq i \leq p} \bigg[ (\lambda \land \delta)(y_i, z_i) \bigg]
\]

\[
\leq \bigvee_{x \in x \lor y} [(\lambda \land \delta) \land \eta](x)
\]

We claim that

\[
\begin{align*}
\text{Lemma 3.8.} & \quad \text{Let } A \text{ be a hyperideal of } R \text{, then } A \subseteq I \text{ implies that either } A \subseteq I \text{ or } B \subseteq I. \\
\text{Definition 3.5.} & \quad \text{A hyperideal } I \text{ of a semihyperring } R \text{ is called an irreducible if for all hyperideals } A, B \text{ of } R, A \cap B = I, \text{ implies } A = I \text{ or } B = I. \\
\text{Definition 3.6.} & \quad \text{A fuzzy hyperideal } \eta \text{ of a semihyperring } R \text{ is called a fuzzy prime hyperideal of } R \text{ if for fuzzy hyperideals } \lambda, \mu \text{ of } R, \lambda \leq \mu \Rightarrow \lambda = \eta \text{ or } \mu = \eta. \\
\text{Theorem 3.7.} & \quad \text{Let } R \text{ be a fully idempotent semihyperring. For a fuzzy hyperideal } \eta \text{ of } R, \text{ the following conditions are equivalent:}
\end{align*}
\]

\[
\begin{align*}
(1) & \quad \eta \text{ is a fuzzy prime hyperideal,} \\
(2) & \quad \eta \text{ is a fuzzy irreducible hyperideal.}
\end{align*}
\]

Proof. (1) Assume that \( \eta \) is a fuzzy prime hyperideal. We show that \( \eta \) is fuzzy irreducible, that is, for fuzzy hyperideals \( \lambda, \mu \) of \( R, \lambda \leq \mu = \eta \Rightarrow \lambda = \eta \text{ or } \mu = \eta. \) Since \( R \) is a fully idempotent semihyperring, the set of fuzzy hyperideals of \( R \) (ordered by \( \leq \)) is a distributive lattice under the sum and intersection of fuzzy hyperideals by Definition 3.4.

This implies that \( \eta = g.l.b. \{ \lambda, \mu \} \), since \( \eta = \lambda \wedge \mu. \) Thus it follows that \( \lambda \leq \eta \text{ and } \eta \leq \mu. \) On the other hand, as \( \lambda \leq R \text{ is fully idempotent, } \) it follows from Definition 3.3 that \( \lambda \wedge \mu = \lambda \mu. \) Hence \( \eta = \lambda \wedge \mu = \lambda \mu. \) Since \( \eta \) is a fuzzy prime hyperideal, by the above definition, either \( \lambda \leq \eta \text{ or } \mu \leq \eta. \) As already noted, \( \eta \leq \lambda \text{ and } \eta \leq \mu; \) so it follows that either \( \lambda = \eta \text{ or } \mu = \eta. \) Hence \( \eta \) is a fuzzy irreducible hyperideal.

(2) Conversely, assume that \( \eta \) is a fuzzy irreducible hyperideal. We show that \( \eta \) is a fuzzy prime hyperideal. Suppose there exist fuzzy hyperideals \( \lambda \) and \( \mu \) such that \( \lambda \mu \leq \eta. \) Since \( R \) is assumed to be a fully idempotent semihyperring, it follows from Definition 3.3 that the set of fuzzy hyperideals of \( R \) (ordered by \( \leq \)) is a distributive lattice with respect to the sum and intersection of fuzzy hyperideals. Hence the inequality \( \lambda \wedge \mu \leq \eta \Rightarrow (\lambda \wedge \mu) \vee \eta = \eta, \) and using the distributivity of this lattice, we have \( \eta = (\lambda \wedge \mu) \vee (\mu \wedge \eta) = (\lambda \wedge \eta) \vee (\mu \wedge \eta) = (\lambda \wedge \eta) \wedge (\lambda \wedge \mu). \) Since \( \eta \) is fuzzy irreducible, it follows that either \( \lambda \wedge \eta = \eta \text{ or } \mu \wedge \eta = \eta. \) This implies that either \( \lambda \leq \eta \text{ or } \mu \leq \eta. \) Hence \( \eta \) is a fuzzy prime hyperideal.

Lemma 3.8. Let \( R \) be a fully idempotent semihyperring. If \( \lambda \) is a fuzzy hyperideal of \( R \) with \( \lambda(a) = \alpha, \) where \( a \) is any element of \( R \) and \( \alpha \in [0, 1], \) then there exists a fuzzy prime hyperideal \( \eta \) of \( R \) such that \( \lambda \leq \eta \text{ and } \eta(a) = \alpha. \)

Proof. Let \( X = \{ \mu : \mu \text{ is a fuzzy hyperideal of } R, \mu(a) = \alpha, \text{ and } \lambda \leq \mu \}. \) Then \( X \neq \emptyset, \) since \( \lambda \in X. \) Let \( \tau \) be a totally ordered subset of \( X, \) say \( \tau = \{ \lambda_i : i \in I \}. \)

We claim that \( \bigvee_{i \in I} \lambda_i \) is a fuzzy hyperideal of \( R. \) Clearly, \( \bigvee_{i \in I} \lambda_i(x) = 1. \) Also, for any \( x, r \in R, \) we have
\[\forall \lambda_i(x) = \forall \lambda_i(x) \leq [\forall (\lambda_i(x))] = \forall \lambda_i(x).\]

Similarly, \(\forall \lambda_i(x) \leq \forall \lambda_i(x)\). Finally, we show that \(\inf_{x,y} \forall \lambda_i(x) \geq \forall \lambda_i(x) \land \forall \lambda_i(x)\), for any \(x, y \in R\). Consider

\[
\forall \lambda_i(x) \land \forall \lambda_i(y) = \forall \lambda_i(x) \land \forall \lambda_j(y)
\]

\[
= [\forall \lambda_i(x)] \land \forall \lambda_j(y)
\]

\[
= \forall \lambda_i(x) \land \forall \lambda_j(y)
\]

\[
\leq \forall \lambda_i(x) \land \lambda_j(y)]
\]

Thus \(\forall \lambda_i(x)\) is a fuzzy hyperideal of \(R\). Clearly \(\forall \lambda_i(\alpha) = \forall \lambda_i(\alpha) = \alpha\). Thus \(\forall \lambda_i(\alpha)\) is the l.u.b of \(\tau\).

Hence, by Zorn’s lemma, there exists a fuzzy hyperideal \(\eta\) of \(R\) which is maximal with respect to the property that \(\lambda \leq \eta\) and \(\eta(\alpha) = \alpha\). We now show that \(\eta\) is a fuzzy irreducible hyperideal of \(R\). Suppose \(\eta = \delta_1 \land \delta_2\), where \(\delta_1\) and \(\delta_2\) are fuzzy hyperideals of \(R\). Since \(R\) is assumed to be a fully idempotent semihyperring, so by Theorem 3.9, the set of fuzzy hyperideals of \(R\) (ordered by \(\leq\)) is a distributive lattice under the sum and intersection of fuzzy hyperideals. Hence \(\eta = \delta_1 \land \delta_2 = g.l.b.\{\delta_1, \delta_2\}\). This implies that \(\eta \leq \delta_2\) and \(\eta \leq \delta_2\). We claim that either \(\eta = \delta_1\) or \(\eta = \delta_2\). Suppose, on the contrary, \(\eta \neq \delta_1\) and \(\eta \neq \delta_2\), it follows that \(\delta_1(a) \neq \alpha\) and \(\delta_2(a) \neq \alpha\). Hence \(\alpha = \eta(\alpha) = (\delta_1 \land \delta_2)(\alpha) = \{\delta_1(a) \land \delta_2(a)\} \neq \alpha\), which is contradiction. Hence either \(\eta = \delta_1\) or \(\eta = \delta_2\). This proves that \(\eta\) is a fuzzy irreducible hyperideal. Hence by Theorem 3.8, \(\eta\) is a fuzzy prime hyperideal.

Now, we have to prove the main characterization theorem for fully idempotent semihyperrings.

**Theorem 3.9.** The following conditions for a semihyperring \(R\) are equivalent:

1. \(R\) is fully idempotent,
2. The lattice of all fuzzy hyperideals of \(R\) (ordered by \(\leq\)) is a distributive lattice under the sum and intersection of fuzzy hyperideals with \(\lambda \land \mu = \lambda \mu\), for each pair of fuzzy hyperideals \(\lambda, \mu\) of \(R\).
3. Each fuzzy hyperideal is the intersection of those fuzzy prime hyperideals of \(R\) which contain it. If, in addition, \(R\) is assumed to be commutative, then the above conditions are equivalent to:
4. \(R\) is regular.

**Proof.** (1)\(\Rightarrow\)(2). This is Theorem 3.3.

(2)\(\Rightarrow\)(3). Let \(\lambda\) be a fuzzy hyperideal of \(R\) and let \(\{\lambda_s : s \in \Omega\}\) be the family of all fuzzy prime hyperideals of \(R\) which contain \(\lambda\). Obviously, \(\lambda \leq \bigwedge_{s \in \Omega} \lambda_s\). We now prove that \(\bigwedge_{s \in \Omega} \lambda_s \leq \lambda\). Let \(\alpha\) be any element of \(R\). By Lemma 3.8, there
exists a fuzzy prime hyperideal $λ_t$ (say) such that $λ ≤ λ_t$ and $λ(a) = λ_t(a)$. Thus
$λ_t ∈ \{λ_s : s ∈ Ω\}$. Hence $∧_{s ∈ Ω} λ_s ≤ λ_t$, so $∧_{s ∈ Ω} λ_s(a) ≤ λ_t(a) = λ(a)$. This implies
that $∧_{s ∈ Ω} λ_s ≤ λ$, so $∧_{s ∈ Ω} λ_s = λ$.

(3)⇒(1). Let $λ$ be any fuzzy hyperideal of $R$. Then $λ^2$ is also a fuzzy hyperideal
of $R$. Hence, according to statement (3), $λ^2$ can be written as $λ^2 = ∧_{s ∈ Ω} λ_s$, where
$\{λ_s : s ∈ Ω\}$ is the family of all fuzzy prime hyperideals of $R$ which contains $λ^2$.
Now $λ^2 ≤ λ$ is always true. Hence, $λ^2 = λ$. Therefore, $R$ is fully idempotent.
Finally, if $R$ is assumed to be commutative, then as noted in Theorem 3.2 (1)⇔(4).
This completes the proof of the theorem. □

At the end of this section, we prove the following fuzzy theoretic characterization
of regular semihyperring. First we recall the following definition.

Definition 3.10. Let $λ$ and $μ$ be fuzzy subsets of a semihyperring $R$. Then the
fuzzy subset $λ ◦ μ$ is defined by $(λ ◦ μ)(x) = \lor_{x,y,z} [(λ(y) ∧ μ(z))]$, for all $x ∈ R$.

Theorem 3.11. The following conditions for a semihyperring $R$ are equivalent:

1. $R$ is regular,
2. For any right hyperideal of $R$ and any left hyperideal $L$ of $R$, $R ∩ L = RL$,
3. For any fuzzy right hyperideal $λ$ and any fuzzy left hyperideal $μ$ of $R$,
   $λ ∧ μ = λ ◦ μ$.

Proof: For (1)⇒(2), we refer to Golan [4, proposition 5.27, p.63]. So we have to
prove only (1)⇒(3). Suppose that $R$ is regular. Let $δ$ be any fuzzy right hyperideal
and $μ$ any fuzzy left hyperideal of $R$. We show that $λ ∧ μ = λ ◦ μ$. Let $x ∈ R$. Then
$(λ ◦ μ)(x) = \lor_{x,y,z} [λ(y) ∧ μ(z)]$
$≤ \lor_{x,y,z} [λ(yz) ∧ μ(yz)] = \lor_{x,y,z} [λ(x) ∧ μ(x)]$
$= \lor_{x,y,z} [λ(x) ∧ μ(x)] = (λ ∧ μ)(x)$.

Thus $(λ ◦ μ) ≤ (λ ∧ μ)$. This does not depend upon the hypothesis. We now show
that $(λ ∧ μ) ≤ (λ ◦ μ)$. Let $x ∈ R$. Since $R$ is von Neumann regular, there exists
$a ∈ R$ such that $x = xa$. Thus
$(λ ∧ μ)(x) = (λ(x) ∧ μ(x)) ≤ (λ(xa) ∧ μ(x)) ≤ \lor_{x,y,z} (λ(y) ∧ μ(z)) = (λ ◦ μ)(x)$.

Hence $λ ◦ μ = λ ∧ μ$. Conversely, assume that $λ ∧ μ = λ ◦ μ$ for any fuzzy right
hyperideal $λ$ and any left hyperideal $μ$ of $R$. We show that $R$ is regular. Let $x ∈ R$.
$xR$ and $Rx$ are the principal right and left hyperideals of $R$, respectively, which
are generated by $x$. Thus, if $δ_{xR}$ and $δ_{Rx}$ denote, respectively, the characteristic
functions of $xR$ and $Rx$, then clearly $δ_{xR}$ and $δ_{Rx}$ are fuzzy right and left hyperideals
of $R$. Hence, by the assumption $δ_{xR} ∧ δ_{Rx} = δ_{xR} ◦ δ_{Rx}$. This implies that $xR ∩ Rx = xRRx$.
Thus $x ∈ xR ∩ Rx = xRRx ⊆ xRx$. Hence, there exists $x ∈ R$ such that
$x = xa$, thus showing that $R$ is regular. □

4. Fuzzy prime spectrum of a fully
idempotent semihyperring

In this section $R$ will denote a fully idempotent semihyperring, $FI_R$ will denote
the lattice of fuzzy hyperideals of $R$, and $FP_R$ the set of all proper fuzzy prime
hyperideals of $R$. For any fuzzy hyperideal $λ$ of $R$, we define $O_λ = \{μ ∈ FP_R : λ ⊈$
\[ \begin{array}{l}
\text{Theorem 4.1.} \quad \text{The set } \tau(FP_R) \text{ forms a topology on the set } FP_R. \text{ The assignment } \\
\lambda \mapsto O_{\lambda} \text{ is an isomorphism between the lattice } FI_R \text{ of fuzzy hyperideals of } R \text{ and } \\
\text{the lattice of open subsets of } FP_R. \\
\text{Proof.} \quad \text{First we show that } \tau(FP_R) \text{ forms a topology on the set } FP_R. \text{ Note that } \\
O_{\varphi} = \{ \mu \in FP_R : \varphi \not\leq \mu \} = \varphi, \text{ where } \varphi \text{ is the usual empty set and } \varphi \text{ is the fuzzy } \\
\text{zero hyperideal of } R \text{ defined by } \varphi(a) = 0 \text{ for all } a \in R. \text{ This follows since } \varphi \text{ is } \\
\text{contained in every fuzzy (prime) hyperideal of } R. \text{ Thus } O_{\varphi} \text{ is the empty subset of } \\
\tau(FP_R). \text{ On the other hand, we have } O_A = \{ \mu \in FP_R : A \not\leq \mu \} = FP_R. \text{ This is } \\
true, \text{ since } FP_R \text{ is the set of proper fuzzy prime hyperideals of } R. \text{ So } O_A = FP_R \\
is an element of } \tau(FP_R). \text{ Now, let } O_{\delta_1}, O_{\delta_2} \in FP_R \text{ with } \delta_1 \text{ and } \delta_2 \in FI_R. \text{ Then } \\
O_{\delta_1} \cap O_{\delta_2} = \{ \mu \in FP_R : \delta_1 \leq \mu \} \cap \{ \mu \in FP_R : \delta_2 \leq \mu \} = O_{\delta_1 \wedge \delta_2}. \\
\text{Let us now consider an arbitrary family } \\
\{ \eta_i \}_{i \in I} \text{ of fuzzy hyperideals of } R. \text{ Since } \\
\bigcup_{i \in I} \eta_i = \bigcup_{i \in I} \{ \mu \in FP_R : \eta_i \not\leq \mu \} = \{ \mu \in FP_R : \exists k \in I \text{ so that } \eta_k \not\leq \mu \}. \\
\text{Note that } \\
(\sum_{i \in I} \eta_i)(x) = \bigvee_{x \in x_1 \oplus x_2 \oplus x_3 \oplus \ldots} (\eta_1(x_1) \wedge \eta_2(x_2) \wedge \eta_3(x_3) \wedge \ldots) \\
\text{where only a finite number of the } x'_i \text{'s are not } 0. \text{ Thus, since } \eta_i(0) = 1, \text{ therefore, } \\
\text{we are considering the infimum of a finite number of terms, because the } 1 \text{'s are effectively not being considered.} \\
\text{Now, if for some } k \in I, \eta_k \not\leq \mu, \text{ then there exists } x \in R \text{ such that } \eta_k > \mu. \\
\text{Consider the particular factorization of } x \text{ for which } x_k = x \text{ and } x_i = 0 \text{ for all } i \neq k. \text{ We see that } \\
\eta_k(x) \text{ is an element of the set whose supremum is defined to be } (\sum_{i \in I} \eta_i)(x). \text{ Thus, } (\sum_{i \in I} \eta_i)(x) \geq \eta_k(x) > \mu(x). \text{ Thus } (\sum_{i \in I} \eta_i)(x) > \mu(x). \text{ Hence, we } \\
have \sum_{i \in I} \eta_i \not\leq \mu. \\
\text{Hence, } \eta_k \not\leq \mu \text{ for some } k \in I \text{ implies that } \sum_{i \in I} \eta_i \not\leq \mu. \\
\text{Conversely, suppose that } \sum_{i \in I} \eta_i \not\leq \mu. \text{ Therefore, there exists an element } x \in R \text{ such that } \\
(\bigcup_{i \in I} \eta_i)(x) > \mu(x). \text{ This means that } \\
\bigvee_{x \in x_1 \oplus x_2 \oplus x_3 \oplus \ldots} (\eta_1(x_1) \wedge \eta_2(x_2) \wedge \eta_3(x_3) \wedge \ldots) > \mu(x). \\
\text{Now, if all the elements of the set, whose supremum we are taking, are individually } \\
less than or equal to } \mu(x), \text{ then we have } (\sum_{i \in I} \eta_i)(x) = \bigvee_{x \in x_1 \oplus x_2 \oplus x_3 \oplus \ldots} (\eta_1(x_1) \wedge \\
\eta_2(x_2) \wedge \eta_3(x_3) \wedge \ldots) \leq \mu(x) \\
\text{which does not agree with what we have assumed. Thus, there is at least one } \\
element of the set (whose supremum we are taking), \text{ say, } \eta_1(x'_1) \wedge \eta_2(x'_2) \wedge \eta_3(x'_3) \wedge \ldots \text{which is greater than } \mu(x)(x \in x'_1 \oplus x'_2 \oplus x'_3 \oplus \ldots \text{being the corresponding breakup of } x, \text{ where only a finite number of the } x'_i \text{'s are nonzero). Thus } \\
\eta_1(x'_1) \wedge \eta_2(x'_2) \wedge \eta_3(x'_3) \wedge \ldots > \mu(x'_1) \wedge \mu(x'_2) \wedge \mu(x'_3) \wedge \ldots \\
\text{That is, } \eta_1(x'_1) \wedge \eta_2(x'_2) \wedge \eta_3(x'_3) \wedge \ldots > \mu(x'_1) \wedge \mu(x'_2) \wedge \mu(x'_3) \wedge \ldots.
\end{array} \]
That is, \( \eta_1(x'_1) \land \eta_2(x'_2) \land \eta_3(x'_3) \land \ldots > \mu(x'_p) \)
where \( \mu(x'_p) = \mu(x'_1) \land \mu(x'_2) \land \mu(x'_3) \land \ldots (p \in I) \).
Hence \( \eta_1(x'_1) > \mu(x'_p) \). It follows that \( \eta_p \not< \mu \) for some \( p \in N. \) Hence, \( \sum_{i \in I} \eta_i \not< \mu \Rightarrow \eta_p \not< \mu \) for some \( p \in N. \) Hence, the two statements, that is,
(i) \( \sum_{i \in I} \eta_i \not< \mu \), and
(ii) \( \eta_p \not< \mu \) for some \( p \in I \) are equivalent. Hence
\[ \bigcup_{i \in I} O_{\eta_i} = \bigvee_{i \in I} \{ \mu \in FP_R : \eta_i \not< \mu \} \]
\[ = \bigcup_{i \in I} \{ \mu \in FP_R : \sum_{i \in I} \eta_i \not< \mu \} \]
\[ = O_{\sum_{i \in I} \eta_i} \]
because, \( \sum_{i \in I} \eta_i \) is also a fuzzy hyperideal of \( R. \) Thus, \( \bigcup_{i \in I} O_{\eta_i} \in \tau(FP_R) \). Hence it follows that \( \tau(FP_R) \) forms a topology on the set \( FP_R. \) Let \( \phi : FI_R \rightarrow FP_R \) be the mapping defined by \( \lambda \rightarrow O_{\lambda}. \) It follows from the above that the prescription \( \phi(\lambda) = O_\lambda \) preserves finite intersection and arbitrary union. Thus \( \phi \) is a lattice homomorphism. To conclude the proof, we must show that \( \phi \) is bijective. In fact, we need to prove the equivalence \( \delta_1 = \delta_2, \) if and only if \( O_{\delta_1} = O_{\delta_2}, \) for \( \delta_1, \delta_2 \) in \( L_A. \) Suppose that \( O_{\delta_1} = O_{\delta_2}. \) If \( \delta_1 \neq \delta_2, \) then there exists \( x \in R \) such that \( \delta_1(x) \neq \delta_2(x) \). Thus, either \( \delta_1(x) > \delta_2(x) \) or \( \delta_2(x) > \delta_1(x) \). Suppose that \( \delta_1(x) > \delta_2(x) \). Using Lemma 3.3 there exists a fuzzy prime hyperideal \( \mu \) of \( R \) such that \( \delta_2 \leq \mu \) and \( \delta_2(x) = \mu(x) \). Hence, \( \delta_1 \not< \mu, \) because \( \delta_1(x) > \delta_2(x) = \mu(x) \). Therefore, \( \delta_1(x) > \mu(x) \). Thus, \( \mu \in O_{\delta_1}. \) Our assumption is that \( O_{\delta_1} = O_{\delta_2}. \) Hence, we have \( \mu \in O_{\delta_2}. \) Hence \( \delta_2 \not< \mu. \) This is a contradiction. If, in the beginning, we had assumed that \( \delta_2(x) > \delta_1(x) \) then, again, by similar reasoning, we get a contradiction. Thus, \( O_{\delta_1} = O_{\delta_2} \) implies that \( \delta_1 = \delta_2. \) Conversely, if \( \delta_1 = \delta_2, \) then, by definition, it is obvious that \( O_{\delta_1} = O_{\delta_2}. \) Thus, we have proved that \( \delta_1 = \delta_2 \) if and only if \( O_{\delta_1} = O_{\delta_2} \) for \( \delta_1 \) and \( \delta_2 \) in \( L_A. \) This completes the proof of the theorem.
The set \( FP_R \) will be called the fuzzy prime spectrum of \( R \) and the topology \( \tau(FP_R) \) constructed in the above theorem will be called the fuzzy spectral topology on \( FP_R. \) The associated topological space will be called the fuzzy spectral space of \( R. \)

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