ON PARTIAL DIFFERENTIAL ENCODINGS OF BOOLEAN FUNCTIONS

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Abstract. We introduce partial differential encodings of Boolean functions as a way of measuring the complexity of Boolean functions. These encodings enable us to derive from group actions non-trivial bounds on the Chow-Rank of polynomials used to specify partial differential encodings of Boolean functions. We also introduce variants of partial differential encodings called partial differential programs. We show that such programs optimally describe important families of polynomials including determinants and permanents. Partial differential programs also enable us to quantitatively contrast these two families of polynomials. Finally we derive from polynomial constructions inspired by partial differential programs which exhibit an unconditional exponential separation between high order hypergraph isomorphism instances and their sub-isomorphism counterparts.

1. Introduction

In an epoch-making book titled “An Investigation of the Laws of Thought”, George Boole [Boo54] laid the foundations for the Boolean algebra. This algebra serves as the first of two pillars of the computing revolution. Interestingly, George Boole also initiated the branch of mathematics known as invariant theory [Wol08]. There is a recognition [GIM+19, CMQ16, Aar16, Gro20] that a rich interplay relates these seemingly separate branches both pioneered by Boole. Invariant theory emphasizes consequences which stem from symmetries or lack thereof. The importance of symmetries in the analysis of Boolean functions was well known to pioneers of the field, such as Shannon, Pólya and Redfield [Sha49, Pol40, Pol37, Red27]. We investigate in the present work, partial differential incarnations of Turing machines. Turing machines were introduced by Alan Turing [Tur36] as a mathematical model of computation. Turing machines are the second pillar of the computing revolution. The use of differential operators in invariant theory is also very old. Their origin can be traced back to the work of early pioneers of invariant theory. Most notably to the work of George Boole, Arthur Cayley and James Joseph Sylvester [Cay89, Syl52] who instigated the use of differential operators to construct invariants of group actions. The framework is also known as Cayley’s Ω process. In complexity theory, differential operators were investigated in the context of arithmetic complexity by Baur and Strassen [BS83]. More recently, Cornelius Brand and Kevin Pratt [BP20] were able to match the runtime of the fastest known deterministic algorithm for detecting subgraphs of bounded path-width using a method of partial derivatives. We refer the reader to excellent recent surveys on partial differential methods in arithmetic complexity [CKW11, SY10]. The importance of investigating partial differential operators is reinforced by the central role they play in physics and machine learning. The present work formally ties, aspects of low depth arithmetic circuit complexity to Boolean De Morgan circuit complexity. Recent depth reduction results [VSS81, AV08, Raz13, GKKS16, Hya79] motivate our focus on low depth arithmetic circuits. In the present work, we introduce partial differential encodings of Boolean functions and their relaxations. These encodings enable us to determine the fraction of optimal encodings. Our main result is a general method for deriving Chow-Rank bounds of polynomial from group actions. Our method is a discrete analog of representation theory methods which devise bound on the border rank from Lie group actions [Lan17, Gro15]. We also introduce variants of partial differential encodings called partial differential programs. We show that such programs optimally describe important families of polynomials including determinants and permanents. Partial differential programs also enable us to quantitatively contrast these two families of polynomials. Finally we derive from polynomial constructions inspired by...
partial differential programs which exhibit an unconditional exponential separation between high order hypergraph isomorphism instances and their sub-isomorphism counterparts.

2. Partial Differential Encodings.

Recall the “needles in a haystack” conundrum [Sha49]. The conundrum roughly translates into the observation that there are at most $s^3$ Boolean circuits (expressed in the De Morgan basis) of size $s$ for as many Boolean functions among the $2^{2^n}$ possible Boolean functions on $n$ bits. Consequently, most circuits have size at most $O(2^{2n})$. Unfortunately, the argument does not bound to the size of circuit encodings of specific Boolean functions. We circumvent this drawback by considering an algebraic variant of the conundrum. The algebraic variant is based upon Boole’s correspondence

\[
\begin{align*}
\text{True} & \rightarrow 1, \\
\text{False} & \rightarrow 0, \\
\neg x_i & \rightarrow 1 - x_i, \\
x_i \land x_j & \rightarrow x_i x_j, \\
x_i \lor x_j & \rightarrow x_i + x_j - x_i x_j, \\

(2.1)
\end{align*}
\]

and conversely

\[
\begin{align*}
1 & \rightarrow \text{True}, \\
0 & \rightarrow \text{False}, \\
1 - x_i & \rightarrow \neg x_i, \\
x_i x_j & \rightarrow x_i \land x_j, \\
(x_i + x_j) \mod 2 & \rightarrow (x_i \lor x_j) \land \neg (x_i \land x_j) := (x_i \lor x_j).

(2.2)
\end{align*}
\]

To reflect the fact that the variables \( \{x_i : i \in \mathbb{Z}_n\} \) are Boolean, algebraic expression are often taken modulo the binary algebraic relations

\[
\begin{align*}
\left\{ (x_i)^2 \equiv x_i \bigg| i \in \mathbb{Z}_n \right\}.

(2.3)
\end{align*}
\]

**Proposition 1.** An arbitrary Boolean function

\[
F : \{0, 1\}^n \rightarrow \{0, 1\}
\]

admits a canonical depth-3 $\sum \Pi \sum$ arithmetic formula expression over $\mathbb{Q}$ as well as a canonical depth-2 $\Pi \sum$ arithmetic formula over $\mathbb{C}$ prescribed modulo relations described in Eq. (2.3).

\footnote{For notational convenience let $\mathbb{Z}_n := \{0, n\} \cap \mathbb{Z}$.}
Proof. By Lagrange’s interpolation theorem \( F (x) \) admits a unique multilinear interpolant given by

\[
F (x) = \sum_{b \in \{0,1\}^{n+1}} \prod_{i \in \mathbb{Z}_n} \left( \prod_{d \in \{0,1\} \setminus \{b_i\}} \frac{x_i - d_i}{b_i - d_i} \right) = \sum_{b \in \{0,1\}^{n+1}} \prod_{i \in \mathbb{Z}_n} \frac{x_i - (1 - b_i)}{2b_i - 1}.
\]

Thereby expressing the desired depth–3 \( \sum \prod \sum \) formula. Furthermore, Lagrange’s interpolation construction yields the congruence identity

\[
\sum_{b \in \{0,1\}^{n+1}} \prod_{i \in \mathbb{Z}_n} \frac{x_i - (1 - b_i)}{2b_i - 1} \equiv \sum_{b \in \{0,1\}^{n+1}} \prod_{d \in \{0,1\}^{n+1} \setminus \{b\}} \prod_{j \in \mathbb{Z}_n} \frac{\sum_{j \in \mathbb{Z}_n} 2^j x_j - \sum_{k \in \mathbb{Z}_n} 2^k b_k}{\sum_{j \in \mathbb{Z}_n} 2^j b_j - \sum_{k \in \mathbb{Z}_n} 2^k d_k} \mod \{ (x_i)^2 - x_i \}_{i \in \mathbb{Z}_n}
\]

By the fundamental theorem of algebra there exist \( \{r_i : 0 \leq i < 2^n\} \subset \mathbb{C} \) such that

\[
r_0 \prod_{0 < i < 2^n} \left( r_i + \sum_{j \in \mathbb{Z}_n} 2^j x_j \right) = \sum_{b \in \{0,1\}^{n+1}} \prod_{d \in \{0,1\}^{n+1} \setminus \{b\}} \prod_{j \in \mathbb{Z}_n} \frac{\sum_{j \in \mathbb{Z}_n} 2^j x_j - \sum_{k \in \mathbb{Z}_n} 2^k b_k}{\sum_{j \in \mathbb{Z}_n} 2^j b_j - \sum_{k \in \mathbb{Z}_n} 2^k d_k} \mod \{ (x_i)^2 - x_i \}_{i \in \mathbb{Z}_n}.
\]

Consequently

\[
F (x) \equiv r_0 \prod_{0 < i < 2^n} \left( r_i + \sum_{j \in \mathbb{Z}_n} 2^j x_j \right) \mod \{ (x_i)^2 - x_i \}_{i \in \mathbb{Z}_n}.
\]

Thereby expressing the desired depth–2 \( \prod \sum \) formula expressing the Boolean function \( F \) modulo relations described in Eq. (2.3). \( \square \)

Arithmetic formulas derived in the proof of Prop. (1) feature expressions of the form

\[
\sum_{0 \leq u < v < d} \prod_{0 \leq u < v < d} \left( B[u, v, 0] + \sum_{w \in \mathbb{Z}_n} B[u, v, w + 1] x_w \right).
\]

We say that the hypermatrix \( B \in \mathbb{C}^{d \times (n+1)} \) underlies the corresponding depth–3 \( \sum \prod \sum \) arithmetic formulas.

2.1. Partial Differential Encoding of Boolean functions and their relaxations. For notational convenience, let \( \mathbb{Z}_n \) denote the set formed by the first \( n \) consecutive non-negative integers i.e.

\[
\mathbb{Z}_n := \{0, n\} \cap \mathbb{Z}.
\]

For simplicity take \( n \) to be a perfect square. Edges of the complete graph on \( \sqrt{n} \) vertices allowing for loop edges are associated with members of \( \mathbb{Z}_n \) as prescribed by the following identification:

the edge \((i,j) \in \mathbb{Z}_{\sqrt{n}} \times \mathbb{Z}_{\sqrt{n}}\) is associated with the integer \(i\sqrt{n} + j \in \mathbb{Z}_n\).

Depth–3 arithmetic formulas expressing Boolean functions suggest alternative partial differential encoding of Boolean functions.
Definition 2. A Partial Differential Encoding (or PDE for short) of a Boolean function
\[ F : \{0,1\}^{n \times 1} \rightarrow \{0,1\} \]
is one of two encodings of the Boolean function \( F \). The first is of the form:
\[
F(1_T) = \left( \prod_{i \in T} \frac{\partial}{\partial a_i} \right) \sum_{0 \leq u < \rho} \prod_{0 \leq v < d} \left( B[u,v,0] + \sum_{0 \leq i,j < \sqrt{n}} B[u,v,1+i\sqrt{n}+j] a_{i,j} \right)_{A=0,v<\sqrt{n}}^m,
\]
for all \( T \subseteq \mathbb{Z}_n \) and where \( 1_T \) denotes the indicator vector of the edge subset \( T \). Note that such a PDE is specified via a multilinear polynomial in the \( n \) variables \( \{a_{0,0}, \cdots, a_{\sqrt{n}-1,\sqrt{n}-1}\} \). In particular when \( m = 1 \) the said multilinear polynomial is
\[
\sum_{0 \leq u < \rho} \prod_{0 \leq v < d} \left( B[u,v,0] + \sum_{0 \leq i,j < \sqrt{n}} B[u,v,1+i\sqrt{n}+j] a_{i,j} \right) = \sum_{b \in \{0,1\}^{n \times 1}} \prod_{0 \leq i,j < \sqrt{n}} (a_{i,j})^{b_i \sqrt{n}+j}.
\]
In its second form, a PDE of \( F \) is specified by in polynomial in the \( \sqrt{n} \) variables \( \{x_0, \cdots, x_{\sqrt{n}-1}\} \) not necessarily multilinear as follows
\[
F(1_T) = \left( \prod_{0 \leq i,j < \sqrt{n}} \frac{\partial}{\partial x_i} \right)^{j} \sum_{0 \leq u < \rho} \prod_{0 \leq v < d} \left( H[u,v,0] + \sum_{0 \leq w < \sqrt{n}} H[u,v,1+w] x_w \right)_{x=0,\sqrt{n}}^m.
\]
In particular when \( m = 1 \) the polynomial used to specify the PDE is
\[
\sum_{0 \leq u < \rho} \prod_{0 \leq v < d} \left( H[u,v,0] + \sum_{0 \leq w < \sqrt{n}} H[u,v,1+w] x_w \right) = \sum_{b \in \{0,1\}^{n \times 1}} \prod_{0 \leq i,j < \sqrt{n}} (x_i)^{b_i \sqrt{n}+j},
\]
In both forms, the positive integer \( m \) is called the exponent parameter of the PDE. We see that hypermatrices \( B \in \mathbb{C}^{\rho \times d \times (1+n)} \) and \( H \in \mathbb{C}^{\rho \times d \times (1+\sqrt{n})} \) completely specify the PDE. Similarly, a PDE relaxation is encodings of one of the form
\[
\left( \prod_{i \in T} \frac{\partial}{\partial a_i} \right) \sum_{0 \leq u < \rho} \prod_{0 \leq v < d} \left( B[u,v,0] + \sum_{0 \leq i,j < \sqrt{n}} B[u,v,1+i\sqrt{n}+j] a_{i,j} \right)_{A=0} \begin{cases} \neq 0 & \text{if } F(1_T) = 1 \\ 0 & \text{otherwise} \end{cases},
\]
or alternatively as
\[
\prod_{i \in T} \left( \frac{\partial}{\partial x_i} \right)^{j} \sum_{0 \leq u < \rho} \prod_{0 \leq v < d} \left( H[u,v,0] + \sum_{0 \leq w < \sqrt{n}} H[u,v,1+w] x_w \right)_{x=0} \begin{cases} \neq 0 & \text{if } F(1_T) = 1 \\ 0 & \text{otherwise} \end{cases}.
\]
More generally, PDEs and their relaxations can be defined for Boolean function on \( m \)–uniform hypergraphs. In that setting a PDE relaxation is expressed as
\[
\left( \prod_{\text{lex}(i_0, \cdots, i_{m-1}) \in T} \frac{\partial}{\partial a_{i_0, \cdots, i_{m-1}}} \right) \sum_{0 \leq u < \rho} \prod_{0 \leq v < d} \left( B[u,v,0] + \sum_{0 \leq i_0, \cdots, i_{m-1} < n} B[u,v,1+\text{lex}(i_0, \cdots, i_{m-1})] a_{i_0, \cdots, i_{m-1}} \right)_{A=0} \begin{cases} \neq 0 & \text{if } F(1_T) = 1 \\ 0 & \text{otherwise} \end{cases}.
\]
\[
\text{is } \begin{cases} 
\neq 0 & \text{if } F(\mathbf{1}_T) = 1 \\
0 & \text{otherwise}
\end{cases},
\]

where
\[
\operatorname{lex}(i_0, \cdots, i_{m-1}) = \sum_{0 \leq k < m} i_k n^{\frac{k}{m}}, \quad \forall (i_0, \cdots, i_{m-1}) \in (\mathbb{Z}/m)^m.
\]

PDEs exemplify our sought variant of the “needles in a haystack” conundrum. Consider Boolean functions specified in terms of a given arbitrary subset \( S \subseteq \mathbb{Z}_n \) such that
\[
(2.7) \quad F_{\subseteq S}(\mathbf{1}_T) = \begin{cases} 
1 & \text{if } T \subseteq S \\
0 & \text{otherwise}
\end{cases}, \quad F_{\supseteq S}(\mathbf{1}_T) = \begin{cases} 
1 & \text{if } T \supseteq S \\
0 & \text{otherwise}
\end{cases}, \quad F_{=} S(\mathbf{1}_T) = \begin{cases} 
1 & \text{if } T = S \\
0 & \text{otherwise}
\end{cases}.
\]

In other words these Boolean functions test whether or not the input graph whose edges make up the subset \( T \subset \mathbb{Z}_n \) is a subgraph respectively supergraph or equal to of some fixed given graph whose edge make up the subset \( S \subseteq \mathbb{Z}_n \). PDEs of \( F_{\subseteq S}, F_{\supseteq S} \) and \( F_{=} S \) are given by
\[
\forall \mathbf{1}_T \in \{0, 1\}^{n \times 1}, \quad F_{\subseteq S}(\mathbf{1}_T) = \left( \prod_{i \sqrt{n} + j \in T} \frac{\partial}{\partial a_{i,j}} \right) \left. P_{\subseteq S}(\mathbf{A}) \right|_{A=0, \sqrt{n} \times \sqrt{n}}^m,
\]
\[
\forall \mathbf{1}_T \in \{0, 1\}^{n \times 1}, \quad F_{\supseteq S}(\mathbf{1}_T) = \left( \prod_{i \sqrt{n} + j \in T} \frac{\partial}{\partial a_{i,j}} \right) \left. P_{\supseteq S}(\mathbf{A}) \right|_{A=0, \sqrt{n} \times \sqrt{n}}^m,
\]
and
\[
\forall \mathbf{1}_T \in \{0, 1\}^{n \times 1}, \quad F_{=} S(\mathbf{1}_T) = \left( \prod_{i \sqrt{n} + j \in T} \frac{\partial}{\partial a_{i,j}} \right) \left. P_{=} S(\mathbf{A}) \right|_{A=0, \sqrt{n} \times \sqrt{n}}^m,
\]

where
\[
P_{\subseteq S}(\mathbf{A}) \in \left\{ \sum_{R \subseteq S} \omega_R \prod_{i \sqrt{n} + j \in R} a_{i,j} : (\omega_R)^m = 1 \right\},
\]
\[
P_{\supseteq S}(\mathbf{A}) \in \left\{ \sum_{R \supseteq S} \omega_R \prod_{i \sqrt{n} + j \in R} a_{i,j} : (\omega_R)^m = 1 \right\},
\]
and
\[
P_{=} S(\mathbf{A}) \in \left\{ \omega_S \prod_{i \sqrt{n} + j \in S} a_{i,j} : (\omega_S)^m = 1 \right\}.
\]

**Example 3.** For instance take \( n = 4 \), and for simplicity take the exponent parameter to be \( m = 1 \). In that setting the edges of the complete graph allowing for loop edges on 2 vertices are identified with members of
\[ \text{lex}(0,1) = 1 \]
\[ \text{lex}(1,0) = 2 \]

**Figure 2.1.** Graph \( T \) and its edges.

\[ Z_4 = \{ \text{lex}(0,0) = 0, \text{lex}(0,1) = 1, \text{lex}(1,0) = 2, \text{lex}(1,1) = 3 \}. \]

Further let the chosen subset of edges which make our chosen graph be given by \( S = \{ 0, 1, 2 \} \) then

\[ P_{\subseteq S}(A) = a_{00}a_{01}a_{10} + a_{00}a_{01} + a_{00}a_{10} + a_{01} + a_{10} + 1, \]
\[ P_{\supseteq S}(A) = a_{00}a_{01}a_{10}a_{11} + a_{00}a_{01}a_{10}, \]
\[ P_{=} S(A) = a_{00}a_{01}a_{10}. \]

In particular given \( T = \{ 1, 2 \} \),

**Example 4.** the corresponding indicator vector is

\[ 1_T = ( 0 \ 1 \ 1 \ 0 )^T. \]

We see that

\[ F_{\subseteq S}(1_T) = \left( \left( \prod_{2i+j \in T} \frac{\partial}{\partial a_{i,j}} \right) P_{\subseteq S}(A) \bigg|_{A=0_{2 \times 2}} \right) = 1, \]
\[ F_{\supseteq S}(1_T) = \left( \left( \prod_{2i+j \in T} \frac{\partial}{\partial a_{i,j}} \right) P_{\supseteq S}(A) \bigg|_{A=0_{2 \times 2}} \right) = 0, \]
\[ F_{=} S(1_T) = \left( \left( \prod_{2i+j \in T} \frac{\partial}{\partial a_{i,j}} \right) P_{=} S(A) \bigg|_{A=0_{2 \times 2}} \right) = 0. \]

For a fixed exponent parameter \( m \), there are exactly \( m^{(2^{\lvert S \rvert})} \) distinct choices for \( m \)-th roots of unity which make up non-vanishing coefficients of \( P_{\subseteq S}(A) \) and \( m^{(2^n-\lvert S \rvert)} \) distinct choices for \( m \)-th roots of unity which make up non-vanishing coefficients of \( P_{\supseteq S}(A) \). On the one hand, such PDEs of \( F_{\subseteq S} \) make up the “haystack”. On the other hand, the “needles” embedded in this haystack are **optimal** PDEs. A PDE of \( F_{\subseteq S} \) is optimal if the hypermatrix which underlies depth–3 \( \sum \prod \sum \) arithmetic formula used to specify the PDE is such that product of dimensions \( \rho \cdot d \) is the minimum possible. Let \( B \in \mathbb{C}^{\rho \times d \times (1+n)} \) underly the depth–3 \( \sum \prod \sum \) arithmetic formula expressing \( P_{\subseteq S} \), such that \( \rho \) is the smallest possible integer, then recall that \( \rho \) is the Chow-rank (over \( \mathbb{C} \)) of the polynomial \( P_{\subseteq S} \). For instance, recall that for a multilinear polynomial of total degree two in the \( n \) variables \( a_{0,0}, \cdots , a_{i,j}, \cdots , a_{n,n} \leq n \)
yield optimal PDEs and uses to specify PDEs for \( P(\sum_{\partial a_n} \omega_n \langle 1, \ldots, 1 \rangle) \) where \( M \in \mathbb{C}^{n \times n} \),

given
\[
\begin{pmatrix}
a_{0,0} \\
\vdots \\
a_{i,j} \\
\vdots \\
a_{\sqrt{n}, \sqrt{n}}
\end{pmatrix}^T \begin{pmatrix}
(M \circ (1_{n \times n} - I_n))
\end{pmatrix} \begin{pmatrix}
a_{0,0} \\
\vdots \\
a_{i,j} \\
\vdots \\
a_{\sqrt{n}, \sqrt{n}}
\end{pmatrix}
\]
where \( \circ \) denotes the entry-wise product.

Chow-rank \( \left\{ \begin{pmatrix}
a_{0,0} \\
\vdots \\
a_{i,j} \\
\vdots \\
a_{\sqrt{n}, \sqrt{n}}
\end{pmatrix}^T (M \circ (1_{n \times n} - I_n)) \begin{pmatrix}
a_{0,0} \\
\vdots \\
a_{i,j} \\
\vdots \\
a_{\sqrt{n}, \sqrt{n}}
\end{pmatrix} = \inf_{H \in \mathbb{C}^{n \times n}} \text{tensor-rank} \{ H + M \circ (1_{n \times n} - I_n) \} \}
\]

A similar definition extends to higher total degree multilinear polynomials. Reading directly from the expanded forms on the left hand side of equalities

\[
\sum_{R \subseteq S} \omega_R \prod_{i, j \in R} a_{i,j} = \sum_{0 \leq u < \rho} \prod_{0 \leq v < \delta} \left( B[u, v, 0] + \sum_{0 \leq i, j} B[u, v, 1 + i \sqrt{n} + j] a_{i,j} \right),
\]

\[
\sum_{R \subseteq S} \omega_R \prod_{i, j \in R} a_{i,j} = \sum_{0 \leq u < \rho} \prod_{0 \leq v < \delta'} \left( B'[u, v, 0] + \sum_{0 \leq i, j} B'[u, v, 1 + i \sqrt{n} + j] a_{i,j} \right),
\]
yields respective Chow-rank and degree bounds \( \rho \leq 2^{|S|}, \delta \geq |S| \) and \( \rho' \leq 2^{|n|-|S|}, \delta' \geq n \). Multilinear polynomials

(2.8) \( P_{\subseteq S} (A) = \prod_{i \sqrt{n} + j \in S} (1 + a_{i,j}) \) and \( P_{\supseteq S} (A) = \prod_{i \sqrt{n} + j \in S} a_{i,j} \prod_{i \sqrt{n} + j \subseteq S} (1 + a_{i,j}) \),
yield optimal PDEs

\[
F_{\subseteq S} (1_T) = \left( \left. \prod_{i \sqrt{n} + j \in T} \frac{\partial}{\partial a_{i,j}} \right|_{A=0, \pi_{\sqrt{n}}} \right)^m P_{\subseteq S} (A),
\]

and

\[
F_{\supseteq S} (1_T) = \left( \left. \prod_{i \sqrt{n} + j \in T} \frac{\partial}{\partial a_{i,j}} \right|_{A=0, \pi_{\sqrt{n}}} \right)^m P_{\supseteq S} (A).
\]

These PDEs are optimal in the sense that both the total degree and the Chow-rank of polynomials \( P_{\subseteq S} \) and \( P_{\supseteq S} \) used to specify PDEs for \( F_{\subseteq S} \) and \( F_{\supseteq S} \) are as small possible.

**Proposition 5.** Optimal choices for \( P_{\subseteq S} \) and \( P_{\supseteq S} \) are

\[
P_{\subseteq S} (A) \in \left\{ \omega_S \prod_{i \sqrt{n} + j \in S} (1 + \omega_{i,j} a_{i,j}) : (\omega_{i,j})_m^m = 1 \right\} \subseteq \left\{ \sum_{R \subseteq S} \omega_R \prod_{i \sqrt{n} + j \in R} a_{i,j} : (\omega_R)_R^m = 1 \right\},
\]
and

\[ P_{2S}(A) \in \left\{ \left( \begin{array}{cc} \omega_S & \prod_{i,v+j \in S} a_{i,j} \\ \prod_{i,v+j \in S} (1 + \omega_{i,j} a_{i,j}) & \prod_{i,v+j \in S} (1 + \omega_{i,j} a_{i,j}) \end{array} \right) \mid \omega_S = 1, \prod_{i,v+j \in S} (1 + \omega_{i,j} a_{i,j}) \right\} \supset \left\{ \sum_{R \supset S} \omega_R \prod_{i,v+j \in R} a_{i,j} : (\omega_R)^m = 1 \right\}. \]

In which case the sparse and thin hypermatrices which underly the respective depth-3 \( \sum \prod \) arithmetic formulas are of size \( 1 \times |S| \times (1 + n) \) and \( 1 \times n \times (1 + n) \).

**Proof.** Prime factors in the factorization of the integer count for the number of non-vanishing monomial terms in the expanded form of \( P_{\leq S}(x) \) and \( P_{\geq S}(x) \) yield lower bounds for the number of non-vanishing terms which make up each linear form. There are \( 2^{|S|} \) non-vanishing monomial terms in the expanded form of \( P_{\leq S}(x) \) and \( 2^{(n-|S|)} \) non-vanishing terms in the expanded form of \( P_{\geq S}(x) \). The chosen expression for \( P_{\leq S}(x) \) and \( P_{\geq S}(x) \) have Chow rank one. Consequently the Chow decomposition upper-bound matches the factorization lower bound. Thus completing the proof.

The fractions of optimal PDEs for \( F_{\leq S} \) and \( F_{\geq S} \) are respectively \( m(|S| - 2^{|S|}) \) and \( m(n - |S|) - 2^{(n-|S|)} \). Optimal PDEs devised for Boolean functions \( F_{\leq S} \) and \( F_{\geq S} \), epitomize their membership into the complexity class \( \text{P}/\text{Poly} \). Namely the class of Boolean functions which admit efficient PDEs (i.e. PDEs whose underlying hypermatrices are upper bounded in size by some polynomial in \( n \)). We conclude this section by describing some PDEs as well as some PDE relaxations realizing some important families of Boolean functions.

**Example 6.** Let

\[ F_{\text{func}} : \{0, 1\}^{\sqrt{n} \times \sqrt{n}} \to \{0, 1\}, \]

The Boolean function \( F_{\text{func}} \) takes as input the adjacency matrix \( M \in \{0, 1\}^{\sqrt{n} \times \sqrt{n}} \) of a directed graph \( G \) (allowing for loop edges) and tests whether or not every vertex in the input graph has out-degree equal to one.

\[ F_{\text{func}}(M) = \begin{cases} 1 & \text{if } G \text{ is a functional directed graph} \\ 0 & \text{otherwise} \end{cases}, \]

where \( M \in \{0, 1\}^{\sqrt{n} \times \sqrt{n}} \) denotes the adjacency matrix of the input \( \sqrt{n} \)-vertex graph \( G \). The entries of \( M \) are such that

\[ M[u, v] = \begin{cases} 1 & \text{if } (u, v) \in E(G), \quad 0 \leq u, v < \sqrt{n} \\ 0 & \text{otherwise} \end{cases}. \]

PDEs of \( F_{\text{func}} \) with exponent parameter \( m \) are of the form

\[ F_{\text{func}}(M) = \left( \prod_{0 \leq i, j < \sqrt{n}} \partial^{M[i,j]} \right)^m P_{\text{func}}(A) \bigg|_{A=0_{\sqrt{n} \times \sqrt{n}}}, \]

where

\[ P_{\text{func}}(A) \in \left\{ \sum_{f \in \mathbb{Z} \sqrt{n}} \omega_f \prod_{i \in \mathbb{Z} n} a_{i,f(i)} : (\omega_f)^m = 1 \right\}. \]
The integer factorization lower-bound argument used to prove Prop. (3) can be applied to $P_{\text{funt}}$. The Boolean function $F_{\text{funt}}$ also lies in the class $\mathbf{P}/\mathbf{Poly}$ since $P_{\text{funt}}$ can be taken such that

$$P_{\text{funt}}(A) \in \left\{ \prod_{i \in \mathbb{Z}_n} \sum_{j \in \mathbb{Z}_n} \omega_{i,j} a_{i,j} : \frac{(\omega_{i,j})^m = 1}{0 \leq i, j < \sqrt{n}} \right\} \subset \left\{ \sum_{f \in \mathbb{Z}_n^m} \omega_f \prod_{i \in \mathbb{Z}_n} A[i, f(i)] : \frac{(\omega_f)^m = 1}{f \in (\mathbb{Z}_n^\sqrt{m})} \right\}.$$ 

When $\sqrt{n}$ is prime the count factorization lower-bound of $\sqrt{n}$ non-vanishing terms per linear functional matches the number of terms in the irreducible factors of the Chow-rank one decomposition. Finally, for a fixed exponent parameter $m$, the fraction of optimal PDEs is $m(n-n^{\sqrt{m}})$. Over the transformation monoid of functions whose domain and codomain are both $\mathbb{Z}_n$, we describe additional families of Boolean functions

$$F_{\mathbb{Z}_n^\sqrt{m}} : (\mathbb{Z}_n^\sqrt{m}) \to \{0, 1\}$$ 

defined such that

$$F_{(\mathbb{Z}_n^\sqrt{m})}^\mathbb{Z}_n^\sqrt{m} = \begin{cases} 1 & \text{if } \exists g \in (\mathbb{Z}_n^\sqrt{m}) \text{ s.t. } f = g \circ h, \\ 0 & \text{otherwise} \end{cases}, \forall f \in (\mathbb{Z}_n^\sqrt{m}).$$

In other words, the Boolean function tests whether or not the input function $f \in (\mathbb{Z}_n^\sqrt{m})$ lies in the $h$-right coset of the transformation monoid $(\mathbb{Z}_n^\sqrt{m})^\mathbb{Z}_n^\sqrt{m}$. The Boolean function $F_{(\mathbb{Z}_n^\sqrt{m})}^\mathbb{Z}_n^\sqrt{m}$ admits a PDE relaxation with exponent parameter $m = 1$ given by

$$F_{(\mathbb{Z}_n^\sqrt{m})}^\mathbb{Z}_n^\sqrt{m} = \left( \prod_{i \in \mathbb{Z}_n^\sqrt{m}} \frac{\partial}{\partial h_i(f(i))} \right) \left|_{A=0} \right. \left. P_{(\mathbb{Z}_n^\sqrt{m})}^\mathbb{Z}_n^\sqrt{m} = \prod_{k \in \mathbb{Z}_n^\sqrt{m}} a_{k,j} \right),$$

expressed in terms of the multivariate polynomial

$$P_{(\mathbb{Z}_n^\sqrt{m})}^\mathbb{Z}_n^\sqrt{m}(A) = \prod_{i \in \mathbb{Z}_n^\sqrt{m}} \left( \sum_{j \in \mathbb{Z}_n^\sqrt{m}} \prod_{k \in \mathbb{Z}_n^\sqrt{m}} a_{k,j} \right).$$

3. PDEs of cardinality variants of $F_{\leq S}$ and $F_{\geq S}$ and Group Orbitals.

We discuss symmetric variants of Boolean functions $F_{\leq S}$, $F_{\geq S}$ and $F_{= S}$ in $\{0, 1\}^{(0,1)^n}$ defined such that

$$F_{\leq |S|}(1_T) = \begin{cases} 1 & \text{if } |T| \leq |S| \\ 0 & \text{otherwise} \end{cases}, \quad F_{\geq |S|}(1_T) = \begin{cases} 1 & \text{if } |T| \geq |S| \\ 0 & \text{otherwise} \end{cases} \quad \text{and}\quad F_{= |S|}(1_T) = \begin{cases} 1 & \text{if } |T| = |S| \\ 0 & \text{otherwise} \end{cases}.$$ 

Prior to describing PDE constructions for the Boolean functions $F_{\leq |S|}$, $F_{\geq |S|}$ and $F_{= |S|}$ we start by defining important notions used to devise various polynomial constructions.

**Definition 7.** Given a multivariate polynomial $P \in \mathbb{C}[y_0, \cdots, y_{m-1}]$ the canonical representative of the congruence class

$$P(y_0, \cdots, y_{m-1}) \mod \left( \prod_{i \in \mathbb{Z}_n} (y_i)^{f(i)} - \prod_{j \in \mathbb{Z}_n} (x_j)^{g(j)} \right)$$

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where $f \in \mathbb{Z}_d^n$ and $g \in \mathbb{Z}_d^n$ is the multivariate polynomial in $\mathbb{C} [x_0, \cdots, x_{m-1}, y_0, \cdots, y_{n-1}]$ obtained by replacing in the expanded form of $P$ every occurrence of the monomial $\prod_{i \in \mathbb{Z}_n} (y_i)^{f(i)}$ by the monomial $\prod_{j \in \mathbb{Z}_m} (x_j)^{g(j)}$.

Note that the congruence relation relating $P$ to its canonical representative is an immediate consequence of the binomial theorem. For instance, given a polynomial interpolation of a Boolean function

$$F : \{0, 1\}^{n \times 1} \to \{0, 1\}$$

given by

$$F(y_0, \cdots, y_{n-1}) = \sum_{b \in \{0, 1\}^{n \times 1}} \prod_{i \in \mathbb{Z}_n} \left( \frac{y_i - (1 - b_i)}{2b_i - 1} \right),$$

s.t. $F(b) = 1$

The multilinear polynomial used to specify the corresponding PDE with exponent parameter $m = 1$ is the canonical representative of the congruence class

$$F(y_0, \cdots, y_{n-1}) \mod \left\{ \prod_{i \in \mathbb{Z}_n} \left( \frac{y_i - (1 - b_i)}{2b_i - 1} \right) - \prod_{i \in \mathbb{Z}_n} (x_i)^{b_i} : b \in \{0, 1\}^{n \times 1}, \text{s.t. } F(b) = 1 \right\}.$$

**Definition 8.** Let $G$ denote an arbitrary subgroup of the symmetric group $S_n$, let

$$\text{lex}_G : G \to \mathbb{Z}_{|G|}$$

denote an arbitrary bijective/lexicographic map. Let $Z$ denote a symbolic $n \times |G|$ matrix. The $n \times 1$ $G$-orbital vector depicts orbits of the action of $G$ on the vertex set of a complete directed graph allowing for loop edges

$$O_{Z,G} [i \sqrt{n} + j] = \prod_{\sigma \in G} Z [\sigma (i \sqrt{n} + j), \text{lex}_G (\sigma)], \forall (i, j) \in \mathbb{Z}_{\sqrt{n}} \times \mathbb{Z}_{\sqrt{n}}.$$

We illustrate an orbital construction used to devise PDEs for Boolean functions $F_{\leq |S|}$, $F_{\geq |S|}$ and $F_{= |S|}$ from PDEs for Boolean functions $F_{\leq S}$, $F_{\geq S}$ and $F_{= S}$. We take the group to be the whole symmetric group. As a result, each entry of the orbital vector $O_Z$ depicts the action of the symmetric group $S_n \subseteq \mathbb{Z}_n^\times$ on the corresponding edge. In particular, each entry of $O_Z$ is a monomial in the entries of a symbolic $n \times (n!)$ matrix $Z$ such that

$$O_Z [i \sqrt{n} + j] = \prod_{\sigma \in S_n} Z [\sigma (i \sqrt{n} + j), \text{lex}_{S_n} (\sigma)], \forall (i, j) \in \mathbb{Z}_{\sqrt{n}} \times \mathbb{Z}_{\sqrt{n}},$$

where

$$\text{lex}_{S_n} (\sigma) = \sum_{k \in \mathbb{Z}_n} (n - 1 - k)! \{ |(\sigma (i) > \sigma (k)) : 0 \leq i < k < n \}, \text{for all } \sigma \in S_n.$$  

For this choice $\text{lex}_{S_n}(\text{id}) = 0$ and $\text{lex}_{S_n}(n - 1 - \text{id}) = n! - 1$. Similarly, let

$$\text{lex}_{\mathcal{Q}(\mathbb{Z}_n)} : \mathcal{Q}(\mathbb{Z}_n) \to (\mathbb{Z}_{2^n}),$$

map bijectively members of the power set $\mathcal{Q}(\mathbb{Z}_n)$ to $\mathbb{Z}_{2^n}$ as follows

$$\text{lex}_{\mathcal{Q}(\mathbb{Z}_n)} (R) = \sum_{j \in R} 2^j, \text{ for all } R \subseteq \mathbb{Z}_n.$$  

For simplicity we take the exponent parameter $m = 1$. Let the canonical representative of $P_{\leq S} (O_Z)$ modulo binomial relations

$$P_{\leq S} (O_Z) \mod \left\{ \prod_{i \in R} Z [i, \text{lex}_{S_n} (\sigma)] - \prod_{j \in R} Y [j, \text{lex}_{\mathcal{Q}(\mathbb{Z}_n)} (R), \sigma \in S_n] : R \subseteq \mathbb{Z}_n, \text{ for all } R \subseteq \mathbb{Z}_n \right\},$$

(3.3)
denote the unique polynomial depending upon entries of \( Y \) and crucially not depend upon any entry of \( Z \). Note that the order with which we perform the reduction modulo prescribed binomial relations matters. The canonical representative of the congruence class in Eq. (3.3) is obtained by reducing \( P \subseteq S (O_Z) \) modulo relations taken in decreasing order of magnitude of the cardinality of the set parameter \( R \).

**Proposition 9.** The canonical representative of the congruence class

\[
P_{\subseteq S} (O_Z) \mod \left\{ \prod_{i \in R} Z[i, \text{lex}_{S_n} (\sigma)] - \prod_{j \in R} Y[j, \text{lex}_{\varphi(Z_n)} (R)] : R \subseteq Z_n, |R| \leq |S| \right\},
\]

is the orbit list generating polynomial

\[
\sum_{0 \leq t \leq |S|} \binom{|S|}{t} \prod_{R \subseteq Z_n} \left( \prod_{|R| = t} Y[j, \text{lex}_{\varphi(Z_n)} (R)] \right)^{(n-|R|)! |R|!}.
\]

**Proof.** The canonical representative is devised by substituting into the expanded form of \( P_{\subseteq S} (O_Z) \) each monomial occurrence of the form

\[
\prod_{i \in R} Z[i, \text{lex}_{S_n} (\sigma)], \quad \text{for all } R \subseteq Z_n, \sigma \in S_n,
\]

with the corresponding monomial

\[
\prod_{j \in R} Y[j, \text{lex}_{\varphi(Z_n)} (R)].
\]

The canonical representative is thus given by

\[
\sum_{R \subseteq Z_n} \prod_{|R| \leq |S|} \left( \prod_{|T| = |R|} Y[j, \text{lex}_{\varphi(Z_n)} (T)] \right)^{(n-|T|)! |T|!}.
\]

\[
= \sum_{0 \leq t \leq |S|} \binom{|S|}{t} \prod_{R \subseteq Z_n} \left( \prod_{|R| = t} Y[j, \text{lex}_{\varphi(Z_n)} (R)] \right)^{(n-|R|)! |R|!}.
\]

Similarly, the canonical representative of the congruence

\[
P_{\geq S} (O_Z) \mod \left\{ \prod_{i \in R} Z[i, \text{lex}_{S_n} (\sigma)] - \prod_{j \in R} Y[j, \text{lex}_{\varphi(Z_n)} (R)] : R \subseteq Z_n, |R| \geq |S|, \sigma \in S_n \right\},
\]
is the polynomial
\[
\sum_{R \subseteq \mathbb{Z}_n} \prod_{T \subseteq \mathbb{Z}_n} \left( \prod_{j \in R} Y_{j, \text{lex} \mathcal{Q}(\mathbb{Z}_n)}(T) \right)^{(n-\vert T\vert)! \vert T\vert!},
\]
\[
= \sum_{\vert S\vert \leq t \leq n} \frac{(n-\vert S\vert)}{(n-\vert T\vert)! \vert T\vert!} \prod_{R \subseteq \mathbb{Z}_n} \left( \prod_{j \in R} Y_{j, \text{lex} \mathcal{Q}(\mathbb{Z}_n)}(R) \right)^{(n-\vert R\vert)! \vert R\vert!},
\]
and the canonical representative of the congruence class
\[
P_{=S}(O_{\mathbb{Z}}) \mod \left\{ \prod_{i \in R} Z_{i, \text{lex} \mathcal{S}_n}(\sigma) - \prod_{j \in R} Y_{j, \text{lex} \mathcal{Q}(\mathbb{Z}_n)}(R) : \begin{array}{l} R \subseteq \mathbb{Z}_n, \\ \vert R\vert \geq \vert S\vert, \\ \sigma \in \mathcal{S}_n \end{array} \right\},
\]
is the polynomial
\[
\prod_{T \subseteq \mathbb{Z}_n} \left( \prod_{j \in R} Y_{j, \text{lex} \mathcal{Q}(\mathbb{Z}_n)}(T) \right)^{(n-\vert S\vert)! \vert S\vert!},
\]
Let \( P_{\leq \vert S\vert} \) and \( P_{\geq \vert S\vert} \)
\[
(3.4) \quad P_{\leq \vert S\vert} = \left( P_{\leq \vert S\vert}(O_{\mathbb{Z}}) \mod \left\{ \prod_{i \in R} Z_{i, \text{lex} \mathcal{S}_n}(\sigma) - \prod_{j \in R} Y_{j, \text{lex} \mathcal{Q}(\mathbb{Z}_n)}(R) : \begin{array}{l} R \subseteq \mathbb{Z}_n, \\ \vert R\vert \leq \vert S\vert, \\ \sigma \in \mathcal{S}_n \end{array} \right\} \right)[0, 1],
\]
and
\[
(3.5) \quad P_{\geq \vert S\vert} = \left( P_{\geq \vert S\vert}(O_{\mathbb{Z}}) \mod \left\{ \prod_{i \in R} Z_{i, \text{lex} \mathcal{S}_n}(\sigma) - \prod_{j \in R} Y_{j, \text{lex} \mathcal{Q}(\mathbb{Z}_n)}(R) : \begin{array}{l} R \subseteq \mathbb{Z}_n, \\ \vert R\vert \geq \vert S\vert, \\ \sigma \in \mathcal{S}_n \end{array} \right\} \right)[0, 1].
\]
The respective representative of the congruence classes are polynomials in the class which depend only upon entries of \( A \) and do not depend upon entries of \( \mathbb{Z} \).

**Proposition 10.** Polynomials \( P_{\leq \vert S\vert} \) and \( P_{\geq \vert S\vert} \) are used to specify PDEs
\[
(3.6) \quad F_{\leq \vert S\vert}(1_{T}) = \left( \prod_{i \in \mathbb{Z}_n} \frac{\partial}{\partial a_{ij}} \right) \left. P_{\leq \vert S\vert}(A) \right|_{A=0,v_{\pi}x_{\pi}}^{m} \text{ and } F_{\geq \vert S\vert}(1_{T}) = \left( \prod_{i \in \mathbb{Z}_n} \frac{\partial}{\partial a_{ij}} \right) \left. P_{\geq \vert S\vert}(A) \right|_{A=0,v_{\pi}x_{\pi}}^{m}.
\]

**Proof.** Similarly to the argument used to prove Prop. (3), the canonical representative for the first of these congruence classes is obtained by successively replacing into the expanded form of \( P_{\leq \vert S\vert}(O_{\mathbb{Z}}) \) every occurrence of monomials of the form
\[
\prod_{i \in R} Z_{i, \text{lex} \mathcal{S}_n}(\sigma), \quad \forall \begin{array}{l} R \subseteq \mathbb{Z}_n, \\ \sigma \in \mathcal{S}_n \end{array},
\]
with the corresponding upper triangular $2 \times 2$ matrix
\[
\begin{pmatrix}
1 & \prod_{i,j \in R} a_{i,j} / (n - |R|)! |R|! \left( \binom{|R|}{i} \right) \\
0 & 1
\end{pmatrix}
\]
followed taking the $[0, 1]$ entry of the $2 \times 2$ matrix resulting from the said substitutions. Similarly, the canonical representatives for the second of the two congruence classes is obtained by successively replacing into the expanded form of $P_{\geq S} (OZ)$ every occurrence of monomials in the entries of $Z$ given by
\[
\prod_{i \in R} Z[i, \text{lex}_{S_n} (\sigma)] , \quad \forall \ R \subseteq \mathbb{Z}_{n^k} , \quad \sigma \in S_{n^k} ,
\]
with the corresponding upper triangular $2 \times 2$ matrix
\[
\begin{pmatrix}
1 & \prod_{i,j \in R} a_{i,j} / (n - |R|)! |R|! \left( \binom{|R|}{i} \right) \\
0 & 1
\end{pmatrix}
\]
followed by taking the $[0, 1]$ entry of the $2 \times 2$ matrix resulting from the said substitutions. \[\square\]

When $|S|$ is not a fixed constant independent of $n$ say $|S| = O(\sqrt{n})$, then the PDE construction above do not certify membership of $F_{\leq |S|}$ and $F_{\geq |S|}$ into the complexity class $P \slash Poly$. In the setting where $|S|$ depends on $n$, Ben Or [NW96] devises an optimal Chow decompositions for $P_{\leq |S|} (x)$ and $P_{\geq |S|} (x)$ via Cramer’s rule as follows
\[
P_{\leq |S|} (A) = \sum_{0 \leq i < |S|} \prod_{0 \leq u < v < n} \det V_i \exp \left\{ \frac{2 \pi u \sqrt{-1}}{n} \right\} - \exp \left\{ \frac{2 \pi u \sqrt{-1}}{n} \right\}.
\]
and
\[
P_{\geq |S|} (A) = \sum_{|S| \leq i \leq n} \prod_{0 \leq u < v < n} \det V_i \exp \left\{ \frac{2 \pi u \sqrt{-1}}{n} \right\} - \exp \left\{ \frac{2 \pi u \sqrt{-1}}{n} \right\}.
\]
The $n \times n$ matrix $V_i$ has entries given by
\[
V_k [u, v] = \left\{ \begin{array}{ll}
1 + \exp \left\{ \frac{2 \pi u \sqrt{-1}}{n} \right\} a_{i,j} & \text{if } k = v \\
\exp \left\{ \frac{2 \pi u \sqrt{-1}}{n} \right\} & \text{otherwise}
\end{array} \right.
\]
Such expansions describe depth–3 $\sum \prod \sum$ arithmetic formula whose underlying hypermatrix is of size $n \times n \times (n + 1)$.

**Example 11.** Let us illustrate the orbital construction in the case $n = 4$, and $S = \{0, 1, 3\}$. It follows from the setup that
\[
P_{\leq S} (A) = (1 + a_{00}) (1 + a_{01}) (1 + a_{11}) .
\]
\[
\Rightarrow P_{\leq S} (OZ) = \left( 1 + \prod_{\sigma \in \mathcal{S}_4} Z[\sigma (2 \cdot 0 + 0), \text{lex} (\sigma)] \right) \left( 1 + \prod_{\sigma \in \mathcal{S}_4} Z[\sigma (2 \cdot 0 + 1), \text{lex} (\sigma)] \right) \left( 1 + \prod_{\sigma \in \mathcal{S}_4} Z[\sigma (2 \cdot 1 + 1), \text{lex} (\sigma)] \right).
\]
Hence
\[
\left( P_{\leq S} (OZ) \mod \left\{ \prod_{i \in R} Z[i, \text{lex}_{S_n} (\sigma)] - \prod_{j \in R} Y[j, \text{lex}_{\mathcal{O}(Z_n)} (R)] : \ R \subseteq \mathbb{Z}_{n^k} , \ |R| \leq |S| \right\} \right) = \left\{ \frac{R \subseteq \mathbb{Z}_{n^k}}{\sigma \in S_{n^k}} \right\}
\]
For a concrete example, consider a PDE for the Boolean function specified by the truth table:

\[ \begin{array}{ccc}
    x_0 & x_1 & F(x) \\
    0 & 0 & 1 \\
    0 & 1 & 0 \\
    1 & 0 & 1 \\
    1 & 1 & 1 \\
\end{array} \]

This expands to:

\[ F(1_T) = \left( \frac{\partial}{\partial x_0} \right)^{1_T[0]} \left( \frac{\partial}{\partial x_1} \right)^{1_T[1]} P_F(x) \bigg|_{x=0_{2,1}}. \]

The multilinear polynomial \( P_F(x) \) used to specify a PDE for \( F \) with exponent parameter equal to one is given by:

\[ P_F(x) = 1 + x_0 + x_0 x_1. \]

Trivially, a \( 3 \times 2 \times 3 \) hypermatrix underlies the depth–3 \( \sum \sum \) arithmetic formula which expresses the expanded form of \( P_F(x) \). However, hypermatrices which underly optimal depth–3 \( \sum \sum \) arithmetic formula for \( P_F(x) \) are
of size $2 \times 2 \times 3$ as seen from the equality

$$
P_F(x) = (B[0,0,0] + B[0,0,1]x_0 + B[0,0,2]x_1) \times (B[0,1,0] + B[0,1,1]x_0 + B[0,1,2]x_1) + (B[1,0,0] + B[1,0,1]x_0 + B[1,0,2]x_1) \times (B[1,1,0] + B[1,1,1]x_0 + B[1,1,2]x_1)$$

where for instance we take non-zero entries of $B \in \{0,1\}^{2 \times 2 \times 3}$ to be

$$
B[0,0,0] = B[0,1,0] = B[1,0,1] = B[1,1,0] = B[1,1,2] = 1.
$$

Hence taking

$$
P_F(x) \in \{\mu + u_0 x_0 (1 + u_1 x_1) : \mu^m = (u_0)^m = (u_1)^m = 1\}.
$$

yields optimal PDEs for $F$ with exponent parameter $m$ of the form

$$
F(1_T) = \left(\frac{\partial}{\partial x_0} 1_T[0] \quad \frac{\partial}{\partial x_1} 1_T[1] \quad \mu + u_0 x_0 (1 + u_1 x_1)\right)_{x=0_{2 \times 1}}^m,
$$

Alternatively, we prescribe $P_F(x)$ up to congruence modulo the Boolean relations

$$(x_0)^2 \equiv x_0 \text{ and } (x_1)^2 \equiv x_1.$$ 

In which case a PDP for $F$ is specified by a polynomial $Q_F(x)$ member of the congruence class of $P_F(x)$. We write

$$
F(1_T) = \left(\frac{\partial}{\partial x_0} 1_T[0] \quad \frac{\partial}{\partial x_1} 1_T[1] \quad Q_F(x) \mod \left\{\begin{array}{l}
(x_0)^2 - x_0 \\
(x_1)^2 - x_1
\end{array}\right\}\right)_{x=0_{2 \times 1}}.
$$

Choices for a hypermatrix $B'$ which underlies optimal depth-3 $\sum \prod \sum$ arithmetic formula for $Q_F(x)$ are of size $1 \times 1 \times 3$ as seen from the expression

$$
Q_F(x) = (B'[0,0,0] + B'[0,0,1]x_0 + B'[0,0,2]x_1) (B'[0,1,0] + B'[0,1,1]x_0 + B'[0,1,2]x_1),
$$

where for instance an optimal PDP for $F$ is completely determined by the taking $B'$ such that

$$
B'[0,0,0] = B'[0,0,1] = 1, B'[0,1,0] = B'[0,1,1] = 1, B'[0,1,2] = -\frac{1}{2}.
$$

Our example illustrates an instance in where PDPs are smaller than optimal PDEs for the same Boolean function. Reduction in size is achieved at the expense of introducing some non-trivial algebraic relations. By definition, PDEs form a proper subset of PDPs. There are finitely many PDEs for any given Boolean function (assuming a fixed exponent parameter $m$). By contrast there are infinitely many PDPs for a given Boolean functions (assuming a fixed exponent parameter $m$).

### 4.1. Orbital Chow-rank bound.

We describe here the simplest illustration of a general method for devising Chow rank bounds from group actions. As a concrete illustration for the orbital bound argument, we derive bounds on the size of a hypermatrix $H$ which underlies an optimal Chow-decompositions over $C$ for $Q_{\leq |S|}$, $Q_{\geq |S|}$ and $Q_{|= |S|}$ ($x$) used to specify PDPs for Boolean functions

$$
F_{\leq |S|}(1_T) = \begin{cases} 1 & \text{if } |T| \leq |S| \\ 0 & \text{otherwise} \end{cases}, \quad F_{\geq |S|}(1_T) = \begin{cases} 1 & \text{if } |T| \geq |S| \\ 0 & \text{otherwise} \end{cases}.
$$
PDPs for such Boolean \( F \) substituting each entry of Theorem 12.

A hypermatrix \( F \) or each one of the congruence identities we have that for all \( \forall = 1 \)

Expanding the right hand side yields

\[
F_{\leq |S|} (1_T) = \left\{ \begin{array}{ll}
1 & \text{if } |T| = |S| \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
F_{\leq |S|} (1_T) = \left( \prod_{i \in T} \frac{\partial}{\partial x_i} \right) Q_{\leq |S|} (x_0, \ldots, x_{n-1}) \mod \left\{ \left( \frac{(x_i)^2 - x_i}{i \in \mathbb{Z}_n} \right) \right\}_{x=0_{n \times 1}}^m.
\]

\[
F_{\geq |S|} (1_T) = \left( \prod_{i \in T} \frac{\partial}{\partial x_i} \right) Q_{\geq |S|} (x_0, \ldots, x_{n-1}) \mod \left\{ \left( \frac{(x_i)^2 - x_i}{i \in \mathbb{Z}_n} \right) \right\}_{x=0_{n \times 1}}^m.
\]

**Theorem 12.** A hypermatrix \( H \in \mathbb{C}^{p \times d \times (1+n)} \) can be chosen to underly a depth-3 \( \sum \prod \sum \) arithmetic formula for \( Q_{\leq |S|}, Q_{\geq |S|} \) and \( Q_{= |S|} \) respectively used to specify PDPs for Boolean functions \( F_{\leq |S|}, F_{\geq |S|} \) and \( F_{=} |S| \) such that \( \rho = 1 \).

**Proof.** For each Boolean function \( F_{\leq |S|}, F_{\geq |S|} \) and \( F_{=} |S| \), \( H \) is determined by congruence identities of the form

\[
\sum R \leq |S| \in R \prod x_i \equiv \sum 0 \leq u < \rho 0 \leq v < d \left( H[u, v, 0] + \sum_{w \in \mathbb{Z}_n} H[u, v, 1 + w] x_w \right) \mod \left\{ \left( \frac{(x_i)^2 - x_i}{i \in \mathbb{Z}_n} \right) \right\},
\]

\[
\sum R \geq |S| \in R \prod x_i \equiv \sum 0 \leq u < \rho 0 \leq v < d \left( H[u, v, 0] + \sum_{w \in \mathbb{Z}_n} H[u, v, 1 + w] x_w \right) \mod \left\{ \left( \frac{(x_i)^2 - x_i}{i \in \mathbb{Z}_n} \right) \right\},
\]

\[
\sum |R|= |S| \in R \prod x_i \equiv \sum 0 \leq u < \rho 0 \leq v < d \left( H[u, v, 0] + \sum_{w \in \mathbb{Z}_n} H[u, v, 1 + w] x_w \right) \mod \left\{ \left( \frac{(x_i)^2 - x_i}{i \in \mathbb{Z}_n} \right) \right\}.
\]

Expanding the right hand side yields

\[
\sum |R| \leq |S| \in R \prod x_i = \sum R \leq |S| \in R \prod x_i, \quad \sum |R| \geq |S| \in R \prod x_i = \sum R \geq |S| \in R \prod x_i, \quad \sum |R|= |S| \in R \prod x_i = \sum R \in \mathbb{Z}_n \prod x_i.
\]

For each one of the congruence identities we have that for all \( \forall R \subseteq \mathbb{Z}_n \), the polynomial in the entries of the unknown matrix \( H \) given by \( K_R (H) \) is given by

\[
K_R (H) = \left( \frac{\partial}{\partial x_i} \right)_{d_i \leq d \leq \rho} \sum 0 \leq u < \rho 0 \leq v < d \left( H[u, v, 0] + \sum_{w \in \mathbb{Z}_n} H[u, v, 1 + w] x_w \right)_{x=0_{n \times 1}}^m,
\]

substituting each entry of \( x \) for the corresponding entry of the orbital vector \( O \) yields constraints

\[
\sum |R| \leq |S| \in R \prod O[i] = \sum R \leq |S| \in R \prod O[i],
\]

\[
\sum |R| \geq |S| \in R \prod O[i] = \sum R \geq |S| \in R \prod O[i],
\]

\[
\sum |R|= |S| \in R \prod O[i] = \sum R \in \mathbb{Z}_n \prod O[i].
\]
\[
\sum_{|R|=|S|} \prod_{i \in R} O_Z[i] = \sum_{R \subseteq \mathbb{Z}_n} K_R(H) \prod_{i \in R} O_Z[i].
\]

Equating corresponding coefficients on both sides of the equal sign in the respective canonical representatives of

\[
\sum_{|R| \leq |S|} \prod_{i \in R} O_Z[i] \mod \left\{ \prod_{i \in R} Z[i, \text{lex}(\sigma)] - \frac{\prod_{j \in R} Y[j, \text{lex}(R)]}{\binom{n}{|R|}} : |R| \leq |S|, \sigma \in S_n \right\},
\]

\[
\sum_{|R| \geq |S|} \prod_{i \in R} O_Z[i] \mod \left\{ \prod_{i \in R} Z[i, \text{lex}(\sigma)] - \frac{\prod_{j \in R} Y[j, \text{lex}(R)]}{\binom{n}{|R|}} : |R| \geq |S|, \sigma \in S_n \right\},
\]

\[
\sum_{|R|=|S|} \prod_{i \in R} O_Z[i] \mod \left\{ \prod_{i \in R} Z[i, \text{lex}(\sigma)] - \frac{\prod_{j \in R} Y[j, \text{lex}(R)]}{\binom{n}{|R|}} : |R| = |S|, \sigma \in S_n \right\},
\]
yields respectively

\[
\sum_{0 \leq t \leq |S|} \prod_{R \subseteq \mathbb{Z}_n} \left( \prod_{i \in R} Y[i, \text{lex}(R)] \right), \sum_{|S| \leq t \leq n} \prod_{R \subseteq \mathbb{Z}_n} \left( \prod_{i \in R} Y[i, \text{lex}(R)] \right), \prod_{R \subseteq \mathbb{Z}_n} \left( \prod_{i \in R} Y[i, \text{lex}(R)] \right).
\]

with the corresponding coefficients in the canonical representative of the congruence class

\[
\sum_{R \subseteq \mathbb{Z}_n} K_R(H) \prod_{i \in R} O_Z[i] \mod \left\{ \prod_{i \in R} Z[i, \text{lex}(\sigma)] - \frac{\prod_{j \in R} Y[j, \text{lex}(R)]}{\binom{n}{|R|}} : |R| \leq |S|, \sigma \in S_n \right\},
\]
yields for each of the three constraints a different systems of \((n+1)\) equations in the \(\rho \cdot d \cdot (1+n)\) unknown entries for \(H \in \mathbb{C}^{\rho \times d \times (1+n)}\) respectively of the form

\[
\left\{ 1 = \sum_{R \subseteq \mathbb{Z}_n} K_R(H) : 0 \leq t \leq |S| \right\} \cup \left\{ 0 = \sum_{R \subseteq \mathbb{Z}_n} K_R(H) : |S| < t \leq n \right\}
\]

\[
\left\{ 1 = \sum_{R \subseteq \mathbb{Z}_n} K_R(H) : |S| \leq t \leq n \right\} \cup \left\{ 0 = \sum_{R \subseteq \mathbb{Z}_n} K_R(H) : 0 \leq t < |S| \right\}
\]

\[
\left\{ 1 = \sum_{R \subseteq \mathbb{Z}_n} K_R(H) : |S| \leq t \leq n \right\} \cup \left\{ 0 = \sum_{R \subseteq \mathbb{Z}_n} K_R(H) : 0 \leq t < |S| \right\}
\]
Proposition 13. For all integer $t$ the sparsest set of generators (for the same polynomial ring) given by independent constraints $\leq$ permutations of their input variables. So too are polynomials ring of symmetric polynomials given by Newton–Girard identities. These identities relate the densest (in their monomial support) set of generators for the can express them in a way that the fundamental theorem of symmetric polynomials tells us. Recall the well known claim holds. $\square$

We now proceed to devise the PDPs for $F_{\leq |S|}, F_{\geq |S|}$ specified in terms of Chow rank one polynomials $Q_{\leq |S|}$ and $Q_{\geq |S|}$ as suggested by Thrm. [12]. Note that Boolean functions $F_{\leq |S|}, F_{\geq |S|}$ are both symmetric with respect to permutations of their input variables. So too are polynomials $P_{\leq |S|}$ and $P_{\geq |S|}$ used to specify their PDEs. So we can express them in a way that the fundamental theorem of symmetric polynomials tells us. Recall the well known Newton–Girard identities. These identities relate the densest (in their monomial support) set of generators for the ring of symmetric polynomials given by

$$e_t(x) = \sum_{R \subseteq \mathbb{Z}_k \atop |R| = t} \prod_{j \in R} x_j, \quad \forall t \in \mathbb{Z}_{n+1} \setminus \{0\},$$

to the sparsest set of generators (for the same polynomial ring) given by

$$p_t(x) = \sum_{0 \leq i < n} (x_i)^t, \quad \forall t \in \mathbb{Z}_{n+1} \setminus \{0\}$$

**Proposition 13.** For all integer $0 < t \leq n$, we have

$$e_t(x) = (-1)^t \sum_{m_1 + \cdots + m_t = t \atop m_1 \geq 0, \ldots, m_t \geq 0} \prod_{1 \leq i \leq t} \frac{(-p_i(x))^{m_i}}{m_i! \cdot i^{m_i}}.$$ 

**Proof.** Consider the polynomial

$$\frac{1}{(n-1)!} \sum_{\sigma \in S_n} \prod_{\gamma \in S_n} (x_{\sigma(0)} - y_{\gamma(1)})^{-1 m_{\gamma}} = \sum_{i \in \mathbb{Z}_n} \prod_{j \in \mathbb{Z}_n} (x_i - y_j)$$

$$\implies 0 = \sum_{0 \leq t \leq n} p_t(x) e_{n-t}(x).$$

Solving via back-substitution the resulting triangular system of linear equations in the unknowns $e_t(x)$, for all integer $0 < t \leq n$ yields the Newton-Girard identity

$$e_t(x) = (-1)^t \sum_{m_1 + \cdots + m_t = t \atop m_1 \geq 0, \ldots, m_t \geq 0} \prod_{1 \leq i \leq t} \frac{(-p_i(x))^{m_i}}{m_i! \cdot i^{m_i}}. \quad \square$$

Using the Newton–Girard identity and exploiting the binary algebraic relations in Eq. [23], we eliminate cross terms from multilinear polynomials $F_{\leq |S|}$ and $P_{\geq |S|}$. 

$$\left\{ 1 = \sum_{R \subseteq \mathbb{Z}_n \atop |R| = |S|} K_R(H) : |R| = |S| \right\} \cup \left\{ 0 = \sum_{R \subseteq \mathbb{Z}_n \atop |R| = t} K_R(H) : 0 \leq t \neq |S| \leq n \right\}.$$
Theorem 14. Boolean functions $F_{\leq |S|}$ and $F_{\geq |S|}$ admit PDPs respectively expressed in terms of polynomials

$$Q_{\leq |S|} (x) \in \left\{ \sum_{R \subseteq \mathbb{Z}_n, |R| \leq |S|} \omega_R \prod_{j \in R} x_j : (\omega_R)^m = 1 \right\}$$

and

$$Q_{\geq |S|} (x) \in \left\{ \sum_{R \subseteq \mathbb{Z}_n, |R| \geq |S|} \omega_R \prod_{j \in R} x_j : (\omega_R)^m = 1 \right\},$$

which can both be chosen to have Chow-rank 1.

Proof. We describe an elimination procedure which exploits congruence identities

$$p_t (x) \equiv p_t (x) \mod \left\{ (x_i)^2 - x_i \atop i \in \mathbb{Z}_n \right\}, \quad \forall t \in \mathbb{Z}_{n+1} \setminus \{1, 0\}.$$ 

to reduce the number of terms. By Prop. (13) taken modulo binary algebraic relation described in Eq. (2.3), we have

$$e_t (x) \equiv (-1)^t \sum_{\left\{ m_1 + 2m_2 + \cdots + tm_t = t \atop m_1 \geq 0, \ldots, m_t \geq 0 \right\}} \prod_{1 \leq i \leq t} \frac{(-1) (x_i)^{m_i}}{m_i ! ^{m_i}} \mod \left\{ (x_i)^2 - x_i \atop i \in \mathbb{Z}_n \right\}.$$ 

It follows that within respective congruence classes

$$\left( P_{\leq |S|} (x) \mod \left\{ (x_i)^2 - x_i \atop i \in \mathbb{Z}_n \right\} \right) \text{ and } \left( P_{\geq |S|} (x) \mod \left\{ (x_i)^2 - x_i \atop i \in \mathbb{Z}_n \right\} \right)$$

lies univariate polynomials in the linear functional $\sum_{i \in \mathbb{Z}_n} x_i$ of degree $|S|$ and $n$ respectively. By the Fundamental Theorem of Algebra, there exists

$$\{ \alpha, \beta \} \cup \left\{ \alpha_i, \beta_j : 0 \leq i < |S|, \quad 0 \leq j < n \right\} \subset \mathbb{C},$$

with which we express optimal PDPs for $F_{\leq |S|}$, $F_{\geq |S|}$ and $F_{= |S|}$ as follows

$$F_{\leq |S|} (1_T) = \left( \alpha \prod_{i \in \mathbb{Z}_n} \left( \frac{\partial}{\partial x_i} \right)^{1_T[i]} \prod_{0 \leq j < |S|} \left( \alpha_j + \sum_{i \in \mathbb{Z}_n} x_i \right) \mod \left\{ (x_i)^2 - x_i \atop i \in \mathbb{Z}_n \right\} \right) \bigg| \begin{array}{c} x = 0_{n \times 1} \end{array}.$$

The hypermatrix underlying the polynomial used to specify the PDP above is of size $1 \times |S| \times (1 + n)$.

$$F_{\geq |S|} (1_T) = \left( \beta \prod_{i \in \mathbb{Z}_n} \left( \frac{\partial}{\partial x_i} \right)^{1_T[i]} \prod_{0 \leq j < n} \left( \beta_j + \sum_{i \in \mathbb{Z}_n} x_i \right) \mod \left\{ (x_i)^2 - x_i \atop i \in \mathbb{Z}_n \right\} \right) \bigg| \begin{array}{c} x = 0_{n \times 1} \end{array}.$$

The hypermatrix underlying the depth-3 $\sum \prod \sum$ arithmetic formula used to specify the PDP is of size $1 \times n \times (1 + n)$. \qed
We see that Ben Or \cite{BW96} constructions yield optimal PDEs for Boolean functions \( F_{\leq|S|}, F_{\geq|S|} \) specified respectively via polynomials \( P_{\leq|S|}(x) \) and \( P_{\geq|S|}(x) \) whose Chow rank are at most \( O(n) \) however Thrm. \cite{Wig19} establishes that PDPs for the same Boolean function are specified respectively via polynomials \( Q_{\leq|S|}(x) \) and \( Q_{\geq|S|}(x) \) whose Chow rank is equal to one. We further remark that the distinction between PDEs and PDPs is akin to the distinctions between time complexity \cite{Wig19} and Kolmogorov complexity \cite{For01}.

5. PDEs/PDPs over the transformation monoid \( \mathbb{Z}^n_n \).

We emphasize salient features of PDEs/PDPs by focusing on Boolean functions whose domain are members of the transformation monoid \( \mathbb{Z}^n_n \) in other words functions whose domain and codomain is \( \mathbb{Z}_n \). Given an arbitrary \( T \subseteq \mathbb{Z}^n_n \), let \( F_T : \mathbb{Z}^n_n \to \{0, 1\} \) be such that

\[
F_T(f) = \begin{cases} 
1 & \text{if } f \in T \\
0 & \text{otherwise}
\end{cases} \quad \text{for all } f \in \mathbb{Z}^n_n.
\]

The Boolean function \( F_T \) therefore tests for membership of an input function \( f \in \mathbb{Z}^n_n \) into the subset \( T \). PDEs of \( F_T \) with exponent parameter \( m \) are of the form

\[
F_T(f) = \left( \frac{\partial^n P_T(A)}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \right)^m, \quad \text{for all } f \in \mathbb{Z}^n_n,
\]

where

\[
P_T(A) \in \left\{ \sum_{g \in T} \omega_g \prod_{i \in \mathbb{Z}_n} a_{i,g(i)} : (\omega_g)^m = 1, \quad g \in T \right\}.
\]

Note when expressing such PDEs, evaluations of each entries of \( A \) at 0 are no longer needed. Furthermore it is easy to see that for all \( S, T \subseteq \mathbb{Z}^n_n \),

\[
\neg F_T(f) = \left( \frac{\partial^n P_T(A)}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \right)^m, \quad \text{for all } f \in \mathbb{Z}^n_n,
\]

where \( \mathcal{T} := \mathbb{Z}^n_n \setminus T \).

\[
F_S(f) \lor F_T(f) = \left( \frac{\partial^n P_{S\cup T}(A)}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \right)^m, \quad \text{for all } f \in \mathbb{Z}^n_n,
\]

and

\[
F_S(f) \land F_T(f) = \left( \frac{\partial^n P_{S\cap T}(A)}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \right)^m, \quad \text{for all } f \in \mathbb{Z}^n_n.
\]

There are \( m^{|T|} \) distinct choices for \( P_T(A) \) with Chow-rank trivially upper-bounded by \( |T| \).

Example 15. Take

\[
T = \left\{ f \in \mathbb{Z}^n_n : \begin{cases} 
f(0) = 0 \\
f(i) < i, \forall i \in \mathbb{Z}_n \setminus \{0\}
\end{cases} \right\}.
\]
the an optimal PDE of $F_T$ with exponent parameter $m = 1$ is given by

$$ F_T (f) = \left( \frac{\partial^n P_T (A)}{\prod_{i \in \mathbb{Z}_n} \partial a_i, f (i)} \right)^m, \quad \text{for all } f \in \mathbb{Z}_n^m, $$

where

$$ P_T (A) = A [0, 0] \prod_{i \in \mathbb{Z}_n \setminus \{0\}} \sum_{i < j < n} A [i, j] = \sum_{f \in \mathbb{Z}_n^m} \prod_{i \in \mathbb{Z}_n} A [i, f (i)]. $$

Consider for example the Boolean function

$$ F_{S_n} : (\mathbb{Z}_n)^n \rightarrow \{0, 1\} $$

defined such that

$$ F_{S_n} (f) = \begin{cases} 1 & \text{if } f \in S_n \\ 0 & \text{otherwise} \end{cases}, \quad \text{for all } f \in \mathbb{Z}_n^n. $$

The corresponding PDE with exponent parameter $m$ specified via a multilinear polynomial is given by

$$ F_{S_n} (f) = \left( \left. \left( \prod_{i \in \mathbb{Z}_n} \frac{\partial}{\partial a_i, f (i)} \right) \right|_{A = \mathbf{0}_{n \times n}} P_{S_n} (A) \right)^m, $$

where

$$ P_{S_n} (A) \in \left\{ \sum_{\sigma \in S_n} \omega_\sigma \prod_{i \in \mathbb{Z}_n} a_{i, \sigma (i)} : \left( \omega_\sigma \right)^m = 1 \quad \sigma \in S_n \right\}. $$

Alternatively an optimal PDE with exponent parameter $m = 2$ is devised for $F_{S_n}$ by specifying it using non-multilinear polynomial as follows

$$ F_{S_n} (f) = \left( \prod_{i \in \mathbb{Z}_n} \left( \frac{\partial}{\partial \sqrt{f (i)!} \partial x_i} \right) p_{S_n} (x_0, \ldots, x_{n-1}) \right|_{x = \mathbf{0}_{n \times 1}}^2, $$

where

$$ p_{S_n} (x_0, \ldots, x_{n-1}) \in \left\{ \prod_{0 \leq i < j < n} (\omega_0 x_i - \omega_u x_u) : (\omega_0)^2 = 1 \quad u \in \mathbb{Z}_n \right\}. $$

**Proposition 16.** Let $OZ$ denote the orbital vector $n \times 1$ vector with entries

$$ OZ [i] = \prod_{\gamma \in S_n} \mathbb{Z} [\gamma (i), \text{lex} (\gamma)], \quad \forall i \in \mathbb{Z}_n. $$

The $[0, 1]$ entry of the canonical representative of the congruence class

$$ \prod_{i \in \mathbb{Z}_n} (OZ [i])^i \mod \left\{ \prod_{i \in \mathbb{Z}_n} (\mathbb{Z} [\gamma (i), \text{lex} (\gamma)])^i \right\} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \prod_{i \in \mathbb{Z}_n} (x_{\gamma (i)})^i \right\} \gamma \in S_n \} \} $
yields the polynomial \( p_{S_n}(x_0, \ldots, x_{n-1}) \) used to specify the PDE

\[
F_{S_n}(f) = \left( \prod_{i \in S_n} \frac{\partial}{\partial x_i} \right)^{f(i)} p_{S_n}(x_0, \ldots, x_{n-1}) \bigg|_{x=0_{n \times 1}}^m,
\]

with exponent parameter \( m = 1 \).

**Proof.** When the exponent parameter \( m = 1 \),

\[
p_{S_n}(x_0, \ldots, x_{n-1}) = \text{Per} \{ \text{Vandermonde} (x) \}.
\]

Recall that

\[
(Vandermonde (x)) [i, j] = (x_i)^j, \forall 0 \leq i, j < n.
\]

We see that

\[
\prod_{i \in \mathbb{Z}} (O_{\mathbb{Z}} [i])^i \mod \left\{ \prod_{i \in \mathbb{Z}} (\mathbb{Z} [\gamma (i), \text{lex} (\gamma)])^i - \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \prod_{i \in \mathbb{Z}} (x_{\gamma (i)})^i \right\} \equiv \text{Per} \{ \text{Vandermonde} (x) \}.
\]

From which the desired claim follows. \(\square\)

**Proposition 17.** An optimal PDE for the Boolean function \( F_{S_n} \) with exponent parameter \( m = 2 \) is

\[
F_{S_n}(f) = \left( \prod_{i \in \mathbb{Z}} \frac{\partial}{\partial x_i} \right)^{f(i)} p_{\text{inv}}(x_0, \ldots, x_{n-1}) \bigg|_{x=0_{n \times 1}}^m,
\]

where

\[
p_{S_n} \in \{ \det (\text{Vandermonde} (\text{diag} (s) x)) : I_n = \text{diag} (s)^m \} \subseteq \left\{ \sum_{\sigma \in S_n} \omega_\sigma \prod_{i \in \mathbb{Z}} (x_{\sigma (i)})^i : (\omega_\sigma)^m = 1 \right\}.
\]

**Proof.** The proof immediately follows from the Chow-rank one decomposition of the well known determinant of the Vandermonde matrix given by

\[
\prod_{0 \leq i < j < n} (x_j - x_i) = \det (\text{Vandermonde} (x)).
\]

\(\square\)

**Definition 18.** The subset \( T \) is a normal subset of \( \mathbb{Z}_{2n}^n \) if

\[
T = \sigma T \sigma^{-1} := \left\{ \sigma f \sigma^{-1} : f \in T \right\}, \quad \text{for all } \sigma \in S_n.
\]

For instance \( \mathbb{Z}_{2}^n, S_n, S_n \setminus \mathbb{Z}_{2}^n \) and \( \{ f \in \mathbb{Z}_{2}^n : 1 = |f^{(n-1)} (\mathbb{Z}_n)| \} \) all form normal subsets of the transformation monoid \( \mathbb{Z}_{2n}^n \). Consider linear transformations prescribed with respect to the standard Euclidean basis over the finite field with \( p \) elements \( \mathbb{F}_p \) where \( p \) is prime. With respect to the standard Euclidean basis, linear transformations are represented by matrices in \( \mathbb{F}_p^{[\log_p (n)] \times [\log_p (n)]} \). We abuse of notation slightly and view such linear transformations as members of the transformation monoid \( \mathbb{F}_p^{[\log_p (n)] \times 1} \). Assume for simplicity throughout the
subsequent discussion that \(n\) is a power of \(p\) and let the canonical embedding of the vector space \(\mathbb{F}_p^{\log_p(n) \times 1}\) into \(\mathbb{Z}_n\) be prescribed by the map

\[
\eta : \mathbb{F}_p^{\log_p(n) \times 1} \rightarrow \mathbb{Z}_n, \quad \text{such that} \quad \eta(b) = \sum_{0 \leq i < \log_n} b_i p^i, \quad \text{for all} \quad b \in (\mathbb{F}_p)^{\log_p(n) \times 1}.
\]

Via this embedding, the monoid of linear transformations from \(\mathbb{F}_p^{\log_p(n) \times 1}\) to \(\mathbb{F}_p^{\log_p(n) \times 1}\) is isomorphic to a sub-monoid of \(\mathbb{Z}_n^{\log_p(n)}\) of order \(n^{\log_p(n)}\). We call this particular monoid the monoid of endomorphism and is denoted for notational convenience \(\text{End}_{\log_p(n)}(\mathbb{F}_p)\). The largest group in \(\text{End}_{\log_p(n)}(\mathbb{F}_p)\) is isomorphic to \(\text{GL}_{\log_p(n)}(\mathbb{F}_p)\) thus isomorphic to a subgroup of the permutation group \(S_n \subseteq \mathbb{Z}_n\) of order

\[
\prod_{0 \leq k < \log_p(n)} (n - p^k).
\]

We abuse notion and identify \(\text{GL}_{\log_p(n)}(\mathbb{F}_2)\) with its isomorphic image in \(S_n \subseteq \mathbb{Z}_n\). Consider the \(\log_p(n) \times \log_p(n)\) orbital matrix \(O_{\mathbb{Z}}\) depicts orbits of the action of the Abelian group \(\mathbb{F}_p^{\log_p(n) \times 1}\)

\[
O_{\mathbb{Z}}[i,j] = \prod_{b \in \mathbb{F}_p^{\log_p(n)}} (\mathbb{Z}[i, \eta(b)])^{b_j}, \quad \forall (i,j) \in \mathbb{Z}_{\log_p(n)} \times \mathbb{Z}_{\log_p(n)}.
\]

Using the orbital matrix we devise a listing of \(\text{End}_{\log_p(n)}\) as follows

**Proposition 19.** The [0,1] entry of the 2 \times 2 canonical representative of the congruence class

\[
\prod_{0 \leq i,j < \log_p(n)} \sum_{k \in F_p} (O_{\mathbb{Z}}[i,j])^k \mod \prod_{0 \leq u < \log_p(n)} (\mathbb{Z}[u, \eta(b)])^{\sum_{0 \leq v < \log_p(n)} M[u,v]b_v} - \left( \prod_{i \in \mathbb{Z}_n} A[i, \eta(M^{-1}M[i]) \right) : M \in \mathbb{F}_p^{\log_p(n) \times \log_p(n)}
\]

yields the listing

\[
F_{\text{End}_{\log_p(n)}}(A) = \sum_{f \in \text{End}_{\log_p(n)}(\mathbb{F}_p)} \prod_{i \in \mathbb{Z}_n} A[i, f(i)].
\]

**Proof.** The proof stems from the matrix identity

\[
\prod_{0 \leq i,j < \log_p(n)} \sum_{0 \leq k < p} (X[i,j])^k = \sum_{M \in \mathbb{F}_p^{\log_p(n) \times \log_p(n)}} \prod_{0 \leq i,j < \log_p(n)} (X[i,j])^{M[i,j]}.
\]

Substituting into the polynomial identity the orbital matrix \(O_{\mathbb{Z}}\) for the matrix \(X\) yields the equality

\[
\prod_{0 \leq i,j < n} \sum_{0 \leq k < p} \left( \prod_{b \in \mathbb{F}_p^{\log_p(n)}} \tilde{z}_{i,j}(b) \right) = \sum_{M \in \mathbb{F}_p^{\log_p(n) \times \log_p(n)}} \prod_{0 \leq i,j < \log_p(n)} \left( \prod_{b \in \mathbb{F}_p^{\log_p(n)}} \left( \tilde{z}_{i,j}(b) \right)^{M[i,j]b_j} \right).
\]
The canonical representative for the first of these congruence classes is obtained by successively replacing into the expanded form of $P_{\leq S} (O_2)$ every occurrence of monomials of the form

$$0 \leq u, v < \log_p (n)$$

$$b \in \mathbb{F}_p^{\log_p (n)}$$

with the $2 \times 2$ upper triangular matrix

$$
\begin{pmatrix}
1 & \prod_{i \in \mathbb{Z}_m} A \left[ i, \eta \left( M^{-1} (i) \right) \right] \\
0 & 1
\end{pmatrix},
$$

followed by taking the $[0, 1]$ entry of the $2 \times 2$ matrix resulting from the said substitutions.

The ensuing PDE with exponent parameter $m = 1$ is

$$F_{\text{End}_{\log_p (n)} (f)} = \left( \left( \prod_{i \in \mathbb{Z}_m} \frac{\partial}{\partial a_i, f (i)} \right) P_{\text{End}_{\log_p (n)} (A)} \right)_{A=0}$$. 

For simplicity take $p = 2$ and let $\log$ denote the logarithm base 2. The canonical embedding of the vector space $(\mathbb{F}_2)^{\log \times 1}$ into $\mathbb{Z}_n$ is prescribed by the map

$$\eta : (\mathbb{F}_2)^{\log \times 1} \rightarrow \mathbb{Z}_n, \text{ such that } \eta (b) = \sum_{0 \leq i < \log} b_i 2^i, \text{ for all } b \in (\mathbb{F}_2)^{\log \times 1}$$.

Via this embedding, the monoid of linear transformations from $(\mathbb{F}_2)^{\log \times 1}$ to $(\mathbb{F}_2)^{\log \times 1}$ is isomorphic to a submonoid of $(\mathbb{Z}_n)^{\log n}$ of order $n^{\log n}$. We call this particular monoid the monoid of endomorphism and is denoted for notational convenience $\text{End}_{\log} (\mathbb{F}_2)$.

Boolean functions of fundamental importance are Boolean function which test

$$f \in \text{End}_{\log} (\mathbb{F}_2) \subset \mathbb{Z}_n$$

subject to the invariance

$$\sigma T \gamma = T, \text{ for all } (\sigma, \gamma) \in \text{GL}_{\log} (\mathbb{F}_2) \times \text{GL}_{\log} (\mathbb{F}_2).$$

Their importance stem from their invariance to coordinate change. PDPs of $F_T$ are of the form

$$F_T (f) = \left( \prod_{i \in \mathbb{Z}_m} \frac{\partial^n}{\partial a_i, f (i)} \right) \left( Q_T (A) \mod \left\{ \begin{array}{c} (a_{ij})^2 - a_{ij} \\ 0 \leq i, j < n \end{array} \right\} \right)^m, \text{ for all } f \in \mathbb{Z}_n^\log.$$

Let the rank of $f \in \text{End}_{\log} (\mathbb{F}_2)$ denote the rank of the corresponding $\log \times \log$ matrix.

**Theorem 20.** Let $S \subset \mathbb{Z}_{1+\log}$ and $T \subset \text{End}_{\log} (\mathbb{F}_2) \subset \mathbb{Z}_n^\log$ be such that

$$T = \{ f \in \text{End}_{\log} (\mathbb{F}_2) : \text{Rank} (f) \in S \}$$

then there exist a PDP for $F_T$ specified via a polynomial $Q_T (A)$ up to binary algebraic relations of Chow rank at most

$$\left[ \frac{1+ \log}{1+ \log} \right] = 1$$

whereby

$$\left( Q_T (A) \mod \left\{ \begin{array}{c} (a_{ij})^2 - a_{ij} \\ 0 \leq i, j < n \end{array} \right\} \right) \equiv P_T (A) \in \left\{ \sum_{g \in T} \omega_g \prod_{i \in \mathbb{Z}_n} a_{i, g (i)} : \ (\omega_g)^m = 1, \ g \in T \right\}$$.
Proof. For simplicity we take the exponent parameter \( m = 1 \). Consider the symbolic listing of \( \text{End}_{\text{ign}}(\mathbb{F}_2) \) given by
\[
\sum_{\mathbf{M} \in (\mathbb{F}_2)^{\log n} \times \text{GL}_n} \prod_{i \in \mathbb{Z}_n} \mathbf{A} \left[ i, \eta \left( \mathbf{M}_{\eta^{-1}}(i) \right) \right] = \sum_{f \in \text{End}_{\text{ign}}(\mathbb{F}_2)} \prod_{i \in \mathbb{Z}_n} a_{i, f(i)}.
\]
Given the prescribed invariance
\[
\sigma T \gamma = T, \quad \text{for all } (\sigma, \gamma) \in \text{GL}_{\text{ign}}(\mathbb{F}_2) \times \text{GL}_{\text{ign}}(\mathbb{F}_2),
\]

We consider the \( n \times n \) orbital matrix associated with the corresponding group action as follows
\[
\mathcal{O}_Z[i, j] = \prod_{(\sigma, \gamma) \in \text{GL}_{\text{ign}}(\mathbb{F}_2) \times \text{GL}_{\text{ign}}(\mathbb{F}_2)} Z[\sigma^{(-1)}(i), \gamma(j), \text{lex}_{\text{GL}}(\sigma, \gamma)], \quad \text{for all } 0 \leq i, j < n.
\]
The lexicographic map above is an arbitrary bijection from \( \text{GL}_{\text{ign}}(\mathbb{F}_2) \times \text{GL}_{\text{ign}}(\mathbb{F}_2) \) to \( \mathbb{Z}^{(\text{GL}_{\text{ign}}(\mathbb{F}_2))^2} \). Let \( \text{lex}_{\text{End}} \) denote an arbitrary bijection from \( \text{End}_{\text{ign}}(\mathbb{F}_2) \) to \( \mathbb{Z}^{n \times n} \) then the corresponding orbit list generating polynomial is
\[
\sum_{f \in \text{End}_{\text{ign}}(\mathbb{F}_2)} \prod_{i \in \mathbb{Z}_n} \mathcal{O}_Z[i, f(i)] \mod \left\{ \prod_{i \in \mathbb{Z}_n} Z[i, f(i), \text{lex}_{\text{GL}}(\sigma, \gamma)] - \prod_{j \in \mathbb{Z}_n} Y[j, f(j), \text{lex}_{\text{End}}(f)] \right\},
\]
and is given by
\[
\sum_{0 \leq i \leq \log n} |(\text{GL}_{\text{ign}}(\mathbb{F}_2) \times \text{GL}_{\text{ign}}(\mathbb{F}_2))/\mathcal{A}(f)| \prod_{f \in \text{End}_{\text{ign}}(\mathbb{F}_2)} \left( \prod_{i \in \mathbb{Z}_n} Y[i, f(i), \text{lex}(f)] \right)^{|\mathcal{A}(f)|}
\]
where \( \mathcal{A}(f) \) denotes the \( f \)-Left-Right invariant subgroup of \( \text{GL}_{\text{ign}}(\mathbb{F}_2) \times \text{GL}_{\text{ign}}(\mathbb{F}_2) \) in other words
\[
\mathcal{A}(f) := \{(\sigma, \gamma) \in \text{GL}_{\text{ign}}(\mathbb{F}_2) \times \text{GL}_{\text{ign}}(\mathbb{F}_2) : \sigma f \gamma = f\}.
\]

We bound the Chow-rank of \( Q_T \) via the orbital argument. Consider the equality
\[
\sum_{f \in \text{End}_{\text{ign}}(\mathbb{F}_2)} \prod_{i \in \mathbb{Z}_n} a_{i, f(i)} = \sum_{0 \leq u < \rho \leq 0 \leq v < d} \left( \sum_{0 \leq i, j < n} H[u, v, 0] + \sum_{0 \leq i, j < n} H[u, v, 1 + n i + j] a_{i,j} \right)
\]
Substituting on both side of the equal sign above entries of \( \mathbf{A} \) by corresponding entries of the orbital matrix yields
\[
\sum_{f \in \text{End}_{\text{ign}}(\mathbb{F}_2)} \prod_{i \in \mathbb{Z}_n} \mathcal{O}_Z[i, f(i)] \equiv \sum_{f \in \text{End}_{\text{ign}}(\mathbb{F}_2)} K_f(\mathbf{H}) \prod_{i \in \mathbb{Z}_n} \mathcal{O}_Z[i, f(i)] \mod \left\{ (a_{i,j})^2 - a_{i,j} \right\},
\]
where the polynomial \( K_f(\mathbf{H}) \) is given by
\[
K_f(\mathbf{H}) = \sum_{\{1 \leq d_k \leq \rho : k \in \mathbb{Z}_n\}} \prod_{\{0 \leq u < \rho, 0 \leq v < d\}} \left( H[u, v, 0] + \sum_{0 \leq i, j < n} H[u, v, 1 + n i + j] a_{i,j} \right)
\]
with the corresponding coefficients in the canonical representative of the congruence class
\[
\sum_{f \in \text{End}_{\text{ign}}(\mathbb{F}_2)} \prod_{i \in \mathbb{Z}_n} \mathcal{O}_Z[i, f(i)] \mod \left\{ \prod_{i \in \mathbb{Z}_n} Z[i, f(i), \text{lex}(\sigma, \gamma)] - \prod_{j \in \mathbb{Z}_n} Y[j, f(j), \text{lex}(f)] \right\},
\]
\[
(\sigma, \gamma) \in \text{GL}_{\text{ign}}(\mathbb{F}_2) \times \text{GL}_{\text{ign}}(\mathbb{F}_2)
\]
\[
f \in \text{End}_{\text{ign}}(\mathbb{F}_2)
\]
yields a system of at most \((1 + \log n)\) equations in \(\rho \cdot d \cdot (1 + n^2)\) unknown entries for \(H \in \mathbb{C}^{\rho \times d \times n^2}\) respectively of the form

\[
\begin{align*}
1 = \prod_{f \in \text{End}_{\log n} (\mathbb{F}_2)} K_f(H) : t \in S & \quad \cup \quad 0 = \prod_{f \in \text{End}_{\log n} (\mathbb{F}_2)} K_f(H) : t \notin S
\end{align*}
\]

We know from the method of elimination via resultants that the corresponding system necessarily admits a solution whenever the number unknowns \(\rho \cdot d \cdot (1 + n^2)\) matches or exceeds the number of algebraically independent constraints \(\leq 1 + \log n\). Hence, when

\[
\rho = \left\lceil \frac{1 + \log n}{(1 + n^2) d} \right\rceil = 1,
\]

the number of variables matches or exceeds the number of algebraically independent constraints. It follows from the degree lower bound \(d \geq n\) that the desired claim holds.

In particular if we take \(S = \{\log n\}\) then the corresponding Boolean function is

\[
F_{\text{GL}_{\log n}(\mathbb{F}_2)}(f) = \begin{cases} 1 & \text{if } f \in \text{GL}_{\log n}(\mathbb{F}_2) \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } f \in \text{End}_{\log n}(\mathbb{F}_2).
\]

By Thrm. (10) \(F_{\text{GL}_{\log n}(\mathbb{F}_2)}\) admits PDPs which can be specified via a polynomial \(Q_{\text{GL}_{\log n}(\mathbb{F}_2)}(A)\) whose Chow-rank is at most \((1 + \log n)\). If the orbital argument used in the proof of Thrm. (20) is carried out for a Boolean function \(F_T\) defined in Eq. (5.1) via the \(n \times n\) orbital matrix

\[
O_{Z}[i,j] = \prod_{(\sigma, \gamma) \in S_n \times S_n} Z^{\sigma(i), \gamma(j), \text{lex}_{S_n \times S_n}(\sigma, \gamma)}, \quad \text{for all } 0 \leq i, j < n,
\]

then it means that \(F_T\) admits a PDP specified via a polynomial \(Q_T(A)\) (up to binary algebraic relations) of Chow-rank

\[
\rho \leq \left\lceil \frac{\text{Pa}(n)}{(1 + n^2)n} \right\rceil
\]

where \(\text{Pa}(n)\) denotes the integer partition function.

6. Cauchy relations and PDP relaxations.

We describe two optimal PDPs differing in their exponent parameter for the Boolean function \(F_{S_n}\) defined such that

\[
F_{S_n}(f) = \begin{cases} 1 & \text{if } f \in S_n \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } f \in \mathbb{Z}_n^2.
\]

Our proposed PDPs will have exponent parameter 2 and 1 respectively and will be specified to modulo a new set of polynomial size algebraic relations presented in their expanded form.

**Theorem 21.** There exist optimal PDPs for \(F_{S_n}\) with exponent parameters 2 specified via \(\sum \prod \sum\) depth-3 arithmetic formulas whose underlying hypermatrices are of size \(1 \times (n+1) \times n^2\).
Proof. The proof follows from the observation that PDEs of $F_{S_n}$ where the exponent parameter is $m = 2$, include expressions of the form

$$F_{S_n}(f) = \left( \frac{\partial^n \det(D_0AD_1)}{\prod_{i \in \mathbb{Z}_n} \partial a_i, f(i)} \right)^2,$$

where $D_0$ and $D_1$ denote arbitrary diagonal matrices whose diagonal entries are either 1 or $-1$. By reducing modulo Cauchy’s algebraic relations

$$\begin{align*}
  \{ a_{i,0} (a_{i,1})^j \equiv a_{i,j} \\
  \text{ s.t. } & \\
  i \in \mathbb{Z}_n \\
  0 < j < n
\},
\end{align*}$$

we optimally express $\det(A)$ via the Vandermonde determinant identity

$$\begin{align*}
(6.1) \quad 
\det(A) \equiv \left( \prod_{k \in \mathbb{Z}_n} a_{k,0} \right) \left( \prod_{0 < i < j < n} (a_{j,1} - a_{i,1}) \right) \mod \left\{ \begin{array}{l}
  a_{i,0} (a_{i,1})^j - a_{i,j} \\
  \text{ s.t. } & \\
  i \in \mathbb{Z}_n \\
  0 < j < n
\end{array} \right\}. 
\end{align*}$$

Crucially, reductions modulo Cauchy’s algebraic relations must be performed in decreasing order of degrees of variables $\{a_{i1} : i \in \mathbb{Z}_n\}$. Thus completing the proof. \qed

**Theorem 22.** There is an optimal PDPs for $F_{S_n}$ with exponent parameters 1 specified via $\sum \prod \sum$ depth-3 arithmetic formulas whose underlying hypermatrices are of size $1 \times n \times n^2$.

Proof. The proof follows from the observation the PDE of $F_{S_n}$ where the exponent parameter is $m = 1$, is given by

$$F_{S_n}(f) = \frac{\partial^n \text{Per}(A)}{\prod_{i \in \mathbb{Z}_n} \partial a_i, f(i)}.$$

By reducing modulo algebraic relations

$$\begin{align*}
\{ a_{i,k} a_{j,k} \equiv 0 \\
\text{ s.t. } & \\
0 \leq i < j < n \\
0 \leq k < n \\
\text{ and } & \\
\{ a_{i,0} (a_{i,1})^j \equiv a_{i,j} \\
\text{ s.t. } & \\
 i \in \mathbb{Z}_n \\
 0 < j < n
\},
\end{align*}$$
we optimally express $\text{Per}(A)$ as follows

$$
(6.2) \quad \text{Per}(A) \equiv \left( \prod_{k \in \mathbb{Z}_n} a_{k,0} \right) \prod_{i \in \mathbb{Z}_n} \left( a_{i,1} \exp\left\{ \frac{2\pi j \sqrt{-1}}{n} \right\} \right) \mod \begin{cases}
(a_{u,1}a_{v,1})^w \\
s.t. \\
0 \leq u < v < n \\
0 < w < n \\
\text{and} \\
a_{i,0}(a_{i,1})^j - a_{i,j} \\
s.t. \\
i \in \mathbb{Z}_n \\
0 < j < n
\end{cases}.
$$

Thus completing the proof. \hfill \Box

The argument used to prove Thrm. (21) and Thrm. (22) captures an important difference separating the permanent from the determinant. Namely, the determinant is obtained by reducing a Chow-rank one polynomial modulo $2^{n\choose 2}$ polynomial size algebraic relations presented in their expanded form. Whereas the permanent is obtained by reducing a Chow-rank one polynomial of total degree $\frac{n(n+1)}{2}$ in $n$ variables taken from $\{a_{i,j} : 0 \leq i, j < n\}$ modulo $(n+1) n\choose 2$ algebraic relations presented in their expanded form.

**Conjecture 23.** There exists no Chow–rank one polynomial $Q_{S_n}(A)$ of total degree $O\left(n^2\right)$ depending asymptotically only on $O(n)$ variables taken from $\{a_{i,j} : 0 \leq i, j < n\}$ which reduces to $\text{Per}(A)$ modulo $O\left(n^2\right)$ polynomial size algebraic relations presented in their expanded form.

By analogy to the determinant case, natural candidates for refuting Conj. (23) are polynomial constructions devised from the permanent of $(n-1) \times (n-1)$ Vandermonde matrix given explicit as

$$
\sum_{\sigma \in S_{n-1} \subset (\mathbb{Z}_n \setminus \{0\})^{2n} \setminus \{0\}} \prod_{i \in \mathbb{Z}_n \setminus \{0\}} (a_{\sigma(i),1})^{i-1}.
$$

Unfortunately when $n > 3$, in contrast to the determinant setting, the permanent of $(n-1) \times (n-1)$ Vandermonde matrix has a non trivial Galois group over the field of fractions $Q\left(a_{1,1}, \cdots , a_{(k-1),1}, a_{(k+1),1}, \cdots , a_{(n-1),1}\right)$ when viewed as a univariate polynomial in the variable $a_{k,1}$ for all $0 < k < n$. Consequently it does not split into linear factors over $Q\left(a_{1,1}, \cdots , a_{(k-1),1}, a_{(k+1),1}, \cdots , a_{(n-1),1}\right)$. Alternatively, we may consider the Chow-rank one polynomial construction

$$
\text{Per}(A) \equiv \left( \prod_{k \in \mathbb{Z}_n} a_{k,0} \right) \left( \prod_{0 < i < j < n} (a_{j,1} + a_{i,1}) \right) \mod \begin{cases}
(a_{u,1}a_{v,1})^w \\
s.t. \\
0 \leq u < v < n \\
0 < w < n \\
\text{and} \\
a_{i,0}(a_{i,1})^j - a_{i,j} \\
s.t. \\
i \in \mathbb{Z}_n \\
0 < j < n
\end{cases}.
$$

Unfortunately we see that the construction above requires the same number of algebraic relations.

Having obtained an optimal implicit description of the determinant polynomial, we used it to devise other efficient PDPs.
Theorem 24. Let \( T \subset \mathbb{Z}_n^m \) be defined such that
\[
T = \left\{ f \in \mathbb{Z}_n^m : \left| f^{(n-1)}(\mathbb{Z}_n) \right| = 1 \right\},
\]
then there exists a PDP with exponent parameter \( m = 1 \), for the Boolean function
\[
F_T(f) = \begin{cases} 1 & \text{if } \left| f^{(n-1)}(\mathbb{Z}_n) \right| = 1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{for all } f \in \mathbb{Z}_n^m,
\]
specified via a polynomial which admits a Chow decomposition of rank at most \( n \).

Proof. The proof of the claim follows from Tutte’s Directed Matrix Theorem \[\text{Zei85, Tut48}\] which asserts that
\[
\begin{aligned}
P_T(A) &= \left( \sum_{f \in T} \prod_{i \in \mathbb{Z}_n} a_{i, f(i)} \right) \\
&= \sum_{i \in \mathbb{Z}_n} A[i, i] \det \left( \text{diag} \left( A1_{1 \times 1} \right) - A \right) \left[ \begin{array}{cccc}
0 & \cdots & i-1 & i+1 \\
0 & \cdots & i-1 & i+1 \\
& & \ddots & \cdots \\
& & & \ddots
\end{array} \right]
\end{aligned}
\]
using the optimal implicit description for the determinant in Eq. (6.1), the desired PDP stems from the identity
\[
F_T(f) = \left( \frac{\partial^n}{\prod_{i \in \mathbb{Z}_n} \partial a_{i, f(i)}} \right) \sum_{i \in \mathbb{Z}_n} A[i, i] \det \left( \text{diag} \left( A1_{1 \times 1} \right) - A \right) \left[ \begin{array}{cccc}
0 & \cdots & i-1 & i+1 \\
0 & \cdots & i-1 & i+1 \\
& & \ddots & \cdots \\
& & & \ddots
\end{array} \right].
\]

\(\square\)

Theorem 25. Let \( T \subset \mathbb{Z}_n^{m-1} \) defined such that
\[
T = \left\{ f \in \mathbb{Z}_n^{m-1} : f^{(n-2)}(\mathbb{Z}_n) = \left\{ i : i \in \mathbb{Z}_n^{-1} \right\} \right\},
\]
then there exists a PDP with exponent parameter \( 1 \) for the Boolean function
\[
F_T(f) = \begin{cases} 1 & \text{if } G_f \text{ contains no cycle of length } > 1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{for all } f \in \mathbb{Z}_n^{m-1},
\]
specified via a polynomial which admits a Chow decomposition of rank at most \( n \).

Proof. The proof of the claim follows from Tutte’s Directed Matrix Theorem \[\text{Zei85, Tut48}\] from which we get
\[
P_T(A) = \left( \sum_{f \in T} \prod_{i \in \mathbb{Z}_n} a_{i, f(i)} \right) = \\
\det \left( \text{diag} \left( A1_{1 \times 1} \right) - A \right) \prod_{j \in \mathbb{Z}_n} \left[ A[j, n-1] - A[j, j] \right] \mod \left[ A[j, n-1] - A[j, j] \right]_{j \in \mathbb{Z}_n^{-1}}
\]
using the optimal implicit description for the determinant in Eq. (6.1), the desired PDP stem from the identity
\[
F_T(f) = \left( \frac{\partial^n P_T(A)}{\prod_{i \in \mathbb{Z}_n} \partial a_{i, f(i)}} \right).
\]

\(\square\)
Note that the sets $S_n$,
$$\left\{ f \in \mathbb{Z}_n^Z : |f^{(n-1)}(Z_n)| = 1 \right\},$$
as well as
$$\left\{ f \in \mathbb{Z}_n^Z : f^{(n-1)}(Z_n) = \left\{ i : i \in \mathbb{Z}_n \quad f(i) = i \right\} \right\},$$
are normal subsets $\mathbb{Z}_n^Z$ of size $n^{n-1}$ and $(n+1)^{n-1}$ respectively.

### 6.1. PDP relaxations

Recall that if $T$ denotes some arbitrary subset of the transformation monoid $\mathbb{Z}_n^Z$, then the corresponding relaxation $F_T : \mathbb{Z}_n^Z \rightarrow \mathbb{N}$ is such that
$$F_T(f) = \begin{cases} \neq 0 & \text{if } f \in T \\ 0 & \text{otherwise} \end{cases}, \quad \forall f \in \mathbb{Z}_n^Z.$$

We see that the non-vanishing support of $F_T$ implicitly tests for membership of $f$ into $T$. Therefore relaxed PDPs of $F_T$ are of the form
$$F_T(f) = \left( \frac{\partial^n}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \left( Q_T(A) \mod \left\{ h_u(A) : 0 \leq u < k \right\} \right) \right)^m, \quad \forall f \in \mathbb{Z}_n^Z,$$
where $\left\{ h_u(A) : 0 \leq u < k \right\}$ denotes polynomial size set of algebraic relations presented in their expanded form. By construction it must be the case that
$$Q_T(A) \mod \left\{ h_u(A) : 0 \leq u < k \right\} \equiv P_T(A) \in \left\{ \sum_{g \in T} \omega_g \prod_{i \in \mathbb{Z}_n} a_{i,g(i)} : c_g \in \mathbb{N}, \quad (\omega)^m = 1 \right\}.$$

**Theorem 26.** Let $T$ denotes the largest subset of permutations in $S_n \subset \mathbb{Z}_n^Z$ whose graphs are a Spanning Union of Directed Even Cycles (SUDEC for short) then
$$F_T : \mathbb{Z}_n^Z \rightarrow \{0,1\}$$
defined such that
$$F_T(f) = \begin{cases} 1 & \text{if } G_f \text{ is a SUDEC} \\ 0 & \text{otherwise} \end{cases}, \quad \forall f \in \mathbb{Z}_n^Z$$
admits an efficient PDP relaxation with exponent parameter 2.

**Proof.** The proof follows from Tutte’s skew symmetric matrix construction \[^4\]. Using the optimal implicit description for the determinant in Eq. \[(6.1)\], the desired relaxed PDP is
$$\left( \left( \frac{\partial^n}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \left( \det (A - A^\top) \mod \left\{ a_{k,i}a_{k,j} : 0 \leq i, j, k < n \right\} \right) \right)^2 \right)^2.$$

Examples discussed thus far above were either PDPs of PDP relaxations. We describe here an optimal PDE relaxation which tests for a fixed $g \in \mathbb{Z}_n^Z$, wether or not the input function $f$ lies in the left $g$–coset of $\mathbb{Z}_n^Z$. 

\[\square\]
Theorem 27. For an arbitrary $g \in \mathbb{Z}_n^n$ let

$$T_g = \{ f \in \mathbb{Z}_n^n : \exists h \in \mathbb{Z}_n^n, \ s.t. \ f = gh \}$$

and then the Boolean function

$$F_{T_g}(f) = \begin{cases} 1 & \text{if } f \in R_g \\ 0 & \text{otherwise} \end{cases} \text{ for all } f \in \mathbb{Z}_n^n$$

admits an optimal PDE relaxation with exponent parameter 1.

Proof. The proof of the claim follows from the identity

$$\prod_{i \in \mathbb{Z}_n} \left( \sum_{j \in \mathbb{Z}_n} a_{i,j(i)} \right) = \sum_{f \in T_g} \left| \left\{ h \in \mathbb{Z}_n^n : f = gh \right\} \right| \prod_{i \in \mathbb{Z}_n} A[i, f(i)]$$

The desired PDE relaxation is thus given by

$$F_{T_g}(f) = \left( \frac{\partial^n}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \right) \prod_{u \in \mathbb{Z}_n} \sum_{v \in \mathbb{Z}_n} a_{u,g(v)}$$

□

Theorem 28. Let $T$ denotes the largest subset of $\mathbb{Z}_n^n$ whose graphs are connected (i.e. unicyclic)

$$F_T : \mathbb{Z}_n^n \rightarrow \{0, 1\}$$

defined such that

$$F_T(f) = \begin{cases} 1 & \text{if } G_f \text{ is unicyclic} \\ 0 & \text{otherwise} \end{cases} \text{ for all } f \in \mathbb{Z}_n^n$$

admits an efficient relaxed PDP with exponent parameter 1.

Proof. The proof of the claim follows from Tutte's Directed Matrix Theorem [Zei85, Tut48], from which we get

$$Q_T(A) = \left( \sum_{0 \leq i, j < n} a_{i,j} \right) \det \left\{ \text{diag} \left( (A + A^T) 1_{n \times 1} \right) - (A + A^T) \right\} [ : n-1, : n-1]$$

$$= \left( \sum_{f \in T} \left| f^{(n-1)}(\mathbb{Z}_n) \right| \prod_{i \in \mathbb{Z}_n} a_{i,f(i)} \right) \mod \left\{ \begin{array}{c} a_{k,i} a_{k,j} \\ 0 \leq i, j, k < n \end{array} \right\}.$$

Using the optimal implicit description for the determinant in Eq. (6.1), the desired relaxed PDP is

$$\left( \frac{\partial^n}{\prod_{i \in \mathbb{Z}_n} \partial a_{i,f(i)}} \right) \left( Q_T(A) \mod \left\{ \begin{array}{c} a_{k,i} a_{k,j} \\ 0 \leq i, j, k < n \end{array} \right\} \right)$$

□

There are of course natural examples of Boolean functions over $\mathbb{Z}_n^n$ which expectedly admit no efficient PDPs relaxation.
Conjecture 29. The Boolean function

\[ F_{\text{comp}} : \mathbb{Z}_n^2 \rightarrow \mathbb{N} \]

defined such that

\[
F_{\text{comp}}(f) = \begin{cases} 
1 & \text{if there exist } g \in \mathbb{Z}_n^2 \text{ such that } f = g \circ g \\
0 & \text{otherwise}
\end{cases}
\]

admits no efficient PDP relaxations.

We may broaden slightly the computational model to include arithmetic circuits whose gates are restricted to operations

\[ \{ \text{add(\textbullet,\textbullet), mul(\textbullet,\textbullet), exp(\textbullet,\textbullet), } \frac{\partial}{\partial \textbullet}, \log(\textbullet), \text{mod(\textbullet,\textbullet)} \} . \]

The gates above respectively correspond to addition, multiplication, exponentiation, partial differentiation, logarithm and modular gates. For simplicity each gate has fan-in equal to two. A partial differentiation gate outputs the partial derivative of its left input with respect to its right input. Whereas the output of addition, multiplication gates are single-valued, the output of other gates may be multivalued. For instance, a logarithm gate outputs the multivalued logarithm of its right input taken with respect to the logarithmic basis specified by its left input. Modular gates output the remainder the Euclidean division. We conclude this section by describing small circuits in this broader computational model for construction akin to PDPs expressing matrix inversion.

Theorem 30. In the proposed model of computation there are constructions akin to PDPs for expressing matrix inversion modulo Cauchy’s algebraic relations and specified via a Chow–rank 1 polynomial

Proof. The proof follows from the optimal expression of the determinant described in Eq (6.1) and the well known identity

\[ A^{-1} = \nabla_{A^\top} \ln(\det A). \]

where

\[ (\nabla_{A^\top} F(A))[i,j] = \frac{\partial F(A)}{\partial a_{j,i}}, \forall 0 \leq i, j < n. \]

It follows that the desired PDP is given by

\[
A^{-1} = \nabla_{A^\top} \ln \left\{ \left( \prod_{0 \leq i < n} a_{i,0} \right) \left( \prod_{0 \leq i < j < n} (a_{j,1} - a_{i,1}) \right) \right\} \mod \left\{ \begin{array}{c}
a_{i,0} (a_{i,1})^j - a_{i,j} \\
\text{s.t.} \\
i \in \mathbb{Z}_n \\
0 < j < n
\end{array} \right. \}
\]

Note that optimal PDP like constructions for inverting matrices yield asymptotically optimal PDP like construction for multiplying matrices via the well known reduction identity

\[
\begin{pmatrix}
I_n & X & 0_{n \times n} \\
0_{n \times n} & I_n & Y \\
0_{n \times n} & 0_{n \times n} & I_n
\end{pmatrix}^{-1} = \begin{pmatrix}
I_n & -X & XY \\
0_{n \times n} & I_n & -Y \\
0_{n \times n} & 0_{n \times n} & I_n
\end{pmatrix}.
\]
7. Orbital Bound for Graph Isomorphism and Sub-Isomorphism Instances via Group Actions.

We introduce conjugacy class variants of Boolean functions $F_{\leq S}$, $F_{\geq S}$ and $F_S$ as Boolean functions defined with respect to some given graph $G$ such that

$$F_{\leq G} (A_H) = \begin{cases} 1 & \text{if } H \leq G, \\ 0 & \text{otherwise} \end{cases}, \quad F_{\geq G} (A_H) = \begin{cases} 1 & \text{if } H \geq G, \\ 0 & \text{otherwise} \end{cases},$$

and

$$F_S (A_H) = \begin{cases} 1 & \text{if } H \sim G, \\ 0 & \text{otherwise} \end{cases},$$

where $A_H \in \{0, 1\}^{n \times n}$ denotes the adjacency matrix of the $n$-vertex graph $H$. Let $OZ$ denote the $n \times n$ orbital matrix whose entries (are monomials in entries of a symbolic $n \times n \times (n!)$ hypermatrix $Z$) depict edge orbits induced by the action of the symmetric group on the vertex set

$$OZ [i, j] = \prod_{\sigma \in S_n} Z[\sigma (i), \sigma (j), \text{lex}_{S_n} (\sigma)], \quad \forall (i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

where

$$\text{lex}_{S_n} (\sigma) = \sum_{k \in \mathbb{Z}_n} (n - 1 - k)! [\{\sigma (i) > \sigma (k) : 0 \leq i < k < n\}].$$

A lower bounds on the number of terms per factor in an optimal PDE/PDP follows from the prime factorization of the number of non-vanishing terms occurring in the expanded form of multilinear polynomials used to specify a PDE. For instance, consider the Boolean functions $F_{\sim G}$ where $G$ is an arbitrary rigid $n$-vertex graph. Then PDPs for $F_{\sim G}$ are of the form

$$F_{\sim G} (A_H) = \left( \prod_{(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n} \left( \frac{\partial}{\partial a_{i, j}} \right)^{A_H [i, j]} P_{\sim G} (A) \right)^m_{A=0_{n \times n}},$$

where

$$P_{\sim G} (A) \in \left\{ \sum_{\sigma \in S_n / \text{Aut}(G)} \omega_{G \sigma^{-1}} \prod_{(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n} a_{i, j}^{A_G [\sigma (i), \sigma (j)]} : (\omega_{G \sigma^{-1}})^m = 1 \right\},$$

Let the prime factorization of the number of non-vanishing terms in the expanded form of $P_{\sim G}$ be given by

$$|S_n / \text{Aut}(G)| = n! = \prod_{p \in \mathbb{P}} p^\alpha_p,$$

where $\mathbb{P} \subset \mathbb{N}$ denotes the set of all primes. Given that $G$ is rigid we know that

$$\alpha_p = \sum_{j \geq 1} \left\lfloor \frac{n}{p^j} \right\rfloor, \quad \forall p \in \mathbb{P}.$$

The smallest depth–3 $\sum \prod \sum$ formula expressing a multilinear polynomial whose expanded form has $\prod_p p^\alpha_p$ non vanishing terms is of size

$$1 \times \left( \sum_{p \in \mathbb{P}} \alpha_p \right) \times (1 + n^2).$$
This lower-bounds is seldom achievable, as seen from the fact that \( P_{\sim G} \) typically has Chow–Rank \( > 1 \). Using the orbital argument we derive upper bound on the Chow–Rank of polynomial used to specify PDPs of \( F_{\sim G} \), \( F_{\sim G} \) and \( F_{\sim G} \) prescribed modulo binary algebraic relations.

**Theorem 31.** Let \( G \) be a given graph on \( n \) vertices. Let PDPs for Boolean functions \( F_{\sim G} \) and \( F_{\sim G} \) be given by

\[
F_{\sim G} (A_H) = \left( \frac{\partial^{\#(E(H))} \left( Q_{\sim G} (A) \mod \{ (a_{i,j})^2 - a_{i,j} \} \right)} {\prod_{(i,j) \in E(H)} \partial a_{i,j}} \right)^m_{A=0_{n \times n}}
\]

and

\[
F_{\sim G} (A_H) = \left( \frac{\partial^{\#(E(H))} \left( Q_{\sim G} (A) \mod \{ (a_{i,j})^2 - a_{i,j} \} \right)} {\prod_{(i,j) \in E(H)} \partial a_{i,j}} \right)^m_{A=0_{n \times n}}
\]

Let optimal Chow–decompositions over \( C \) of \( Q_{\sim G} \) as well as \( Q_{\sim G} \) be given by

\[
Q_{\sim G} (A) = \sum_{0 \leq u < \rho \ 0 \leq v < d} \prod_{0 \leq i,j < n} \left( B [u, v, 0] + \sum_{0 \leq i,j < n} B [u, v, 1 + n i + j] a_{i,j} \right),
\]

\[
Q_{\sim G} (A) = \sum_{0 \leq u < \rho' \ 0 \leq v < d'} \prod_{0 \leq i,j < n} \left( B' [u, v, 0] + \sum_{0 \leq i,j < n} B' [u, v, 1 + n i + j] a_{i,j} \right).
\]

Then bounds on the sizes of hypermatrices \( B \in \mathbb{C}^{\rho \times d \times (1+n^2)} \) and \( B' \in \mathbb{C}^{\rho' \times d' \times (1+n^2)} \) which underly depth–3 arithmetic formulas used to express \( Q_{\sim S} \) and \( Q_{\sim S} \) are such that

\[
\rho \leq \left[ \{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \subseteq G \} \right] + \left[ \{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \subseteq G \} \right] \left( 1 + n^2 \right) d
\]

and

\[
\rho' \leq \left[ \{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \supseteq G \} \right] + \left[ \{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \supseteq G \} \right] \left( 1 + n^2 \right) d'.
\]

**Proof.** It suffices to work out the upper bound for the size of \( B \in \mathbb{C}^{\rho \times d \times (1+n^2)} \), for the argument is identical for \( B' \in \mathbb{C}^{\rho' \times d' \times (1+n^2)} \). By definition, PDPs with exponent parameter \( m = 1 \) prescribed modulo Boolean relations are such that

\[
\sum_{H \subseteq G (i,j) \in E(H)} \prod_{0 \leq u < \rho \ 0 \leq v < d} \left( B [u, v, 0] + \sum_{0 \leq i,j < n} B [u, v, 1 + n i + j] a_{i,j} \right) \mod \{ (a_{i,j})^2 - a_{i,j} \}.
\]
By expanding the expression on the right-hand side and reducing it modulo prescribed relations we get the equality
\[
\sum_{H \in G} \prod_{(i,j) \in E(H)} a_{i,j} \equiv \sum_{H \subseteq \mathbb{K}_n} K_H(B) \prod_{(i,j) \in E(H)} a_{i,j},
\]
the multivariate polynomial \( K_H(B) \) is given by
\[
K_H(B) = \left( \sum_{\{d_{ij} \geq 1: (i,j) \in E(H)\}} \prod_{(i,j) \in E(H)} \left( \frac{\partial}{\partial a_{i,j}} \right)^{d_{ij}} \prod_{0 \leq u < \rho \leq u < d} \left( B[u, v, 0] + \sum_{0 \leq i, j < n} B[u, v, 1 + n i + j] a_{i,j} \right) \right)_{A=0_{n \times n}}
\] (7.1),
substituting entries of \( A \) with the corresponding entries of the orbital matrix \( O_Z \) yields
\[
\sum_{H \subseteq G} \prod_{(i,j) \in E(H)} O_Z[i, j] = \sum_{H \subseteq \mathbb{K}_n} K_H(B) \prod_{(i,j) \in E(H)} O_Z[i, j].
\]
Now we do modulo operations on both sides of this equation. Equating corresponding coefficients on both sides of the equal sign, which are coefficients in respective canonical representative congruence classes
\[
\sum_{H \subseteq G} \prod_{(i,j) \in E(H)} O_Z[i, j] \mod \left\{ \prod_{(i,j) \in E(H)} Z[i, j, \text{lex}_S(\sigma)] - \prod_{(i,j) \in E(H)} Y[i, j, \text{lex}(H)] \right\} \equiv \sum_{H \subseteq G} K_H(B) K_{\sigma \in S_n} H \subseteq G, H \subseteq \mathbb{K}_n \}
\]
where
\[
\text{lex}(H) = \sum_{(i,j) \in E(H)} 2^{n-i+j}, \text{ for all } H \subseteq \mathbb{K}_n
\]
and the corresponding coefficients in the canonical representative of the congruence class
\[
\sum_{H \subseteq \mathbb{K}_n} K_H(B) \prod_{(i,j) \in E(H)} O_Z[i, j] \mod \left\{ \prod_{(i,j) \in E(H)} Z[i, j, \text{lex}_S(\sigma)] - \prod_{(i,j) \in E(H)} Y[i, j, \text{lex}(H)] \right\}_{\sigma \in S_n, H \subseteq \mathbb{K}_n} \equiv \sum_{H \subseteq G} K_H(B) K_{\sigma \in S_n} H \subseteq G, H \subseteq \mathbb{K}_n \}
\] yields a system of
\[
\left\{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \subseteq G \right\} + \left\{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \not\subseteq G \right\}
\]
equations in the \( \rho \cdot d \cdot (1 + n^2) \) unknown entries for \( B \in \mathbb{C}^{\rho \times d \times (1 + n^2)} \) after merging the same terms on both sides. \( \left\{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \subseteq G \right\} \) stands for the number of graphs \( H \) that are subgraph-isomorphic to \( G \) and the number of terms with a non-zero coefficient in the canonical representative of the congruence class, while 
\( \left\{ A_H \in \{0,1\}^{n \times n} / \text{Iso} : H \not\subseteq G \right\} \) stands for the number of graphs that are not subgraph-isomorphic to \( G \), which is also the number of terms with 0 coefficients in the canonical representative of the congruence class. Clearly \( d \geq |E(G)| \) and we know that by eliminating variables via the method of resultants, the latter system
of equations necessarily admits a solution whenever the number unknowns \( \rho \cdot d \cdot (1 + n^2) \) matches or exceeds the number of algebraically independent constraints. We see that setting

\[
\rho = \frac{\left[ \left\{ A_H \in \{0,1\}^{n \times n}/\text{iso} : H \subset G \right\} + \left\{ A_H \in \{0,1\}^{n \times n}/\text{iso} : H \not\sim G \quad \left| E(H) \right| \leq |E(G)| \right\} \right]}{(1 + n^2) d},
\]

the number of variables matches or exceeds the number of algebraically independent constraints. It follows from the degree lower bound \( d \geq |E(G)| \) that we can take

\[
\rho = \frac{\left[ \left\{ A_H \in \{0,1\}^{n \times n}/\text{iso} : H \subset G \right\} + \left\{ A_H \in \{0,1\}^{n \times n}/\text{iso} : H \not\sim G \quad \left| E(H) \right| \leq |E(G)| \right\} \right]}{(1 + n^2) |E(G)|}.
\]

\[ \square \]

If we restrict the discussion to PDEs which test whether or not the graph of the input function is isomorphism to the graph of the given function \( g \in \mathbb{Z}_n^n \), then the corresponding Boolean function is of the form

\[
F_{\geq G_g}(f) = \begin{cases} 
1 & \text{if there exist } \sigma \in S_n \text{ such that } \sigma f \sigma^{-1} = g \\
0 & \text{otherwise}
\end{cases} \quad \text{for all } f \in \mathbb{Z}_n^n.
\]

PDEs of \( F_{\geq G_g} \) are of the form

\[
F_{\geq G_g}(f) = \left( \frac{\partial^n P_{\geq G_g}(A)}{\prod a_{i,f(i)}} \right)^m, \quad \text{for all } f \in \mathbb{Z}_n^n,
\]

where

\[
P_{\geq G_g}(A) = \left\{ \sum_{\sigma \in S_n/\text{Aut} G_g} \omega_\sigma \prod_{i \in \mathbb{Z}_n} a_{i,\sigma g \sigma^{-1}(i)} : (\omega_\sigma)^m = 1 \right\}.
\]

The orbital argument yields PDP specified in term of a polynomial \( Q_{\geq G_g}(A) \) subject to

\[
P_{\geq G_g}(A) \equiv \left( Q_{\geq G_g}(A) \mod \left\{ \left( a_{i,j} \right)^2 - a_{i,j} \quad 0 \leq i, j < n \right\} \right),
\]

\( Q_{\geq G_g}(A) \) is of Chow-rank

\[
\rho \leq \left[ \frac{\kappa^n}{(n + n^3) \sqrt{n}} \right],
\]

for some real number \( \kappa > 1 \).
8. Orbital hypergraph isomorphism and sub-isomorphism PDPs.

We describe hyperedges of $k$–uniform $n$-vertex hypergraph $H$ as a fixed subset subset of $E(H) \subseteq \mathbb{Z}_n^k$. The monomial hyperedge list description of $H$ is

$$\prod_{f \in E(H)} A \left[ f (0), \cdots, f (k-1) \right],$$

where $A$ denotes a symbolic side length $n$ hypermatrix of order $m$ such that

$$A [i_0, \cdots, i_{k-1}] = a_{i_0 \cdots i_{k-1}}, \quad 0 \leq i_0, \cdots, i_{k-1} < n.$$ 

At the limit where $k \to n$, an arbitrary hypergraph $H$ is specified by providing a fixed subset subset of $E(H) \subseteq \mathbb{Z}_n^k$. Their orbit list generating polynomial yields an optimal PDP for the Boolean function

$$F_{\leq H} (A_{H'}) = \begin{cases} 1 & \text{if } H' \simeq H \\ 0 & \text{otherwise} \end{cases},$$

where $A_H \in \{0,1\}^{n \times \cdots \times n}$ denote the adjacency hypermatrix of $H$. Let $\mathcal{O}_Z$ denote the orbital hypermatrix whose order is $n + 1$ and side length is equal to $n$. Entries of $\mathcal{O}_Z$ depicts hyperedge orbits induced by the action of the symmetric group on the vertex set

$$\mathcal{O}_Z [i_0, \cdots, i_{n-1}] = \prod_{\sigma \in S_n} Z [\sigma (i_0), \cdots, \sigma (i_{n-1}), \text{lex}_S (\sigma)].$$

Let

$$P_{\leq H} (A) = \prod_{f \in E(H)} a_{f (0), \cdots, f (k-1)},$$

then the desired PDP is given by

$$F_{\leq H} (A_{H'}) = \prod_{f \in E(H')} \frac{\partial |E(H')|}{\partial a_{f (0), \cdots, f (n-1)}} \left( P_{\leq H} (\mathcal{O}_Z) \mod \left\{ \prod_{g \in E(H')} Z [\sigma g (0), \cdots, \sigma g (n-1), \text{lex}_S (\sigma)] \right\} \right),$$

or alternatively

$$F_{\leq H} (A_G) = \left( \frac{\partial |\mathcal{O}_Z| |E(G)|}{\partial \omega_H} \prod_{g \in E(\sigma H), \sigma \in S_n/\text{Aut}(H)} y_g (0), \cdots, y_g (k-1), \text{lex}_E (\sigma H) \right)^m,$$

where

$$\text{lex} (E (R)) = \sum_{f \in E (R)} 2^\text{lex}_S (f).$$

The first construction is a valid PDP since we know by Stirling approximation that

$$n! \sim \left( \frac{n}{e} \right)^n \sqrt{2\pi n}.$$
is polynomial in the parameter $|\mathbb{Z}_n^2|$. The latter construction describes an optimal PDE. Unfortunately adapting the constructions above to sub-isomorphism instances does not result in PDP for the set of algebraic relations needed is no longer polynomial in the parameter $|\mathbb{Z}_n^2|$. Fortunately PDPs inspire another approach to articulating the subtle gap in complexity separating isomorphism instances from their sub-isomorphism counterparts. Typically one considers specific isomorphism or sub-isomorphism instances specified with two input hypergraphs. In such a setting one seeks to determine whether or not the specific instance is a YES instance or a NO instance. This restricted setting is very different from the PDP constructions that we have described thus far. In PDP construction that we have described we sought to construct a Boolean functions which test for isomorphism or sub-isomorphism of a given graph to any other graph. We see that determining whether or not the specific instance is a YES instance or a NO instance is an easier task.

**Theorem 32.** Given $m$-uniform hypergraphs $H$ and $G$, the corresponding isomorphism instance is a YES instance if and only if

$$0 = \text{Discriminant}_Z \left( x^2 - p_1 x + \frac{p_1^2 - p_2}{2} \right) = (2p_2 - p_1^2),$$

where

$$\prod_{f \in E(H)} \mathcal{O}_Z [f(0), \cdots, f(n-1)] + \prod_{g \in E(G)} \mathcal{O}_Z [g(0), \cdots, g(n-1)]$$

mod

$$\prod_{h \in E(R)} \mathbb{Z} [h(0), \cdots, h(n-1), \text{lex}_{S_n} (\sigma)] - \prod_{h \in E(H)} \mathbb{Y} [h(0), \cdots, h(n-1), \text{lex}(R)]^k$$

$$R \in \bigcup_{\sigma \in S_n/\text{Aut}(H)} \sigma H \bigcup \bigcup_{\sigma \in S_n/\text{Aut}(G)} \sigma G$$

$$\equiv \prod_{K \simeq H} \prod_{f \in E(K)} \mathbb{Y} [f(0), \cdots, f(n-1), \text{lex}(K)]^k + \prod_{K \simeq G} \prod_{f \in E(K)} \mathbb{Y} [f(0), \cdots, f(n-1), \text{lex}(K)]^k = p_k$$

**Proof.** The key idea here is that if $H \simeq G$ then both $p_1$ and $p_2$ are monomials and $p_2 = \frac{(p_1)^2}{2}$, while if $H$ is not isomorphic to $G$ then $p_1, p_2$ each have two terms. For a specific isomorphism instance the claim immediately follows from the observation that the canonical representative of

$$p_k \equiv \prod_{f \in E(H)} \mathcal{O}_Z [f(0), \cdots, f(n-1)] + \prod_{f \in E(G)} \mathcal{O}_Z [f(0), \cdots, f(n-1)]$$

mod

$$\prod_{f \in E(R)} \mathbb{Z} [f(0), \cdots, f(n-1), \text{lex}_{S_n} (\sigma)] - \prod_{h \in E(H)} \mathbb{Y} [f(0), \cdots, f(n-1), \text{lex}(R)]^k$$

$$R \in \bigcup_{\sigma \in S_n/\text{Aut}(H)} \sigma H \bigcup \bigcup_{\sigma \in S_n/\text{Aut}(G)} \sigma G$$

equals

$$p_k = 2 \prod_{\sigma \in S_n/\text{Aut}(H)} \left( \prod_{f \in E(\sigma H)} \mathbb{Y} [f(0), \cdots, f(n-1), \text{lex}(\sigma H)] \right)^k$$
for a YES instance and equals
\[ \prod_{\sigma \in S_n/\text{Aut}(H)} \left( \prod_{f \in E(\sigma H)} Y[f(0), \ldots, f(n-1), \text{lex}(\sigma H)] \right)^k + \prod_{\gamma \in \text{Generator of Aut}(G_f)} \left( \prod_{g \in E(\sigma G)} Y[g(0), \ldots, g(n-1), \text{lex}(\sigma G)] \right)^k \]
for a NO instance. The discriminant equation therefore follows from Newton–Girard formulas.

Note that the Chow–rank of the polynomial construction is at most 2 both before and after performing the reduction modulo prescribed algebraic relations. Also note that the number of variables appearing in the PDP can be reduced by considering an orbital matrix whose entries, instead, depict the action of cosets of some canonically chosen set of generators for the automorphism groups of hypergraphs \( H \) and \( G \) respectively. This is best illustrated with isomorphism instances defined over functional directed graphs. Let \( f, g \in Z_n^Z \) and consider two distinct orbital matrices

\[ O_Z[i, j] = \prod_{\sigma \in S_n/\text{Aut}(G_f)} Z[\sigma \gamma(i), \sigma \gamma(j), \text{lex}_{S_n}(\sigma \gamma)]. \]

\[ O'_Z[i, j] = \prod_{\sigma \in S_n/\text{Aut}(G_g)} Z[\sigma \gamma(i), \sigma \gamma(j), \text{lex}_{S_n}(\sigma \gamma)]. \]

The expression of interest:

\[ p_k \equiv \left( \prod_{i \in Z_n} O_Z[i, f(i)] + \prod_{j \in Z_n} O'_Z[j, g(j)] \right) \mod \left\{ \prod_{i \in Z_n} Z[i, h(i), \text{lex}_{S_n}(\sigma)] - \left( \prod_{j \in Z_n} Y[j, h(j), \text{lex}_{S_n}(\sigma)] \right)^k \right\}. \]

The isomorphism instance is thus a YES instance if and only if

\[ 0 = \text{Discriminant}_x \left( x^2 - p_1 x + \frac{p_1^2 - p_2}{2} \right) = (2p_2 - p_1^2). \]

The number of substitutions prescribed by the search and replacement procedure reduces in this setting to

\[ |S_n/\text{Aut}(G_f)| \cdot |\text{Generator of Aut}(G_f)| + |S_n/\text{Aut}(G_g)| \cdot |\text{Generator of Aut}(G_g)|. \]

We now contrast the analysis above to sub-isomorphism instances. In order to check sub-isomorphism, we may construct two polynomials

\[ d(x) = \prod_{K \subseteq G} (x + \prod_{f \in E(K)} Y[f(0), \ldots, f(n-1), \text{lex}(K)]) \]

\[ g(x) = \prod_{K \subseteq G \text{ or } K \not\sim H} (x + \prod_{f \in E(K)} Y[f(0), \ldots, f(n-1), \text{lex}(K)]) \]
If $H$ is sub-isomorphic to $G$, then $g(x)$ divides $(d(x))^2$. For we see that every monomial in the entries of $Y$ occurring in factors of $g(x)$ appears at most twice in $(d(x))^2$. Otherwise, some monomial in the entries of $Y$ occurring in factors of $g(x)$ never occurs in a factor of $(d(x))^2$. Meanwhile, an orbital construction yields $d(x)$ and $g(x)$ by reducing modulo relations introduced in the previous theorem. Using the fundamental theorem of symmetric polynomials we devise an explicit expression for the the expanded form of $d(x)$ and $g(x)$.

**Theorem 33.** Given $m$-uniform hypergraphs $H$ and $G$, the corresponding sub-isomorphism instance is a YES instance if and only if the polynomial

$$g(x) = \left( x^{1+2|E(G)|} + \sum_{0<k\leq 1+2|E(G)|} (-1)^k x^{1+2|E(G)|-k} \sum_{m_1+2m_2+\cdots+km_k=1+2|E(G)|} \prod_{0<i\leq 1+2|E(G)|} \frac{(-q_k)^{m_i}}{m_i! i^{m_i}} \right)$$

divides the polynomial

$$(d(x))^2 = \left( x^{2|E(G)|} + \sum_{0<k\leq 2|E(G)|} (-1)^k x^{2|E(G)|-k} \sum_{m_1+2m_2+\cdots+km_k=2|E(G)|} \prod_{0<i\leq \eta} \frac{(-p_k)^{m_i}}{m_i! i^{m_i}} \right)^2,$$

where

$$\prod_{f\in E(G)} (1 + O_Z[f(0), \cdots, f(n-1)])$$

$$\mod \left\{ \prod_{f\in E(R)} \mathbb{Z}[f(0), \cdots, f(n-1), \text{lex}_{S_n}(\sigma)] - \left( \prod_{f\in E(R)} \mathbb{Y}[f(0), \cdots, f(n-1), \text{lex}(R)] \right)^k \right\}$$

$$\equiv \sum_{K\subseteq G} \prod_{f\in E(K)} \mathbb{Y}[f(0), \cdots, f(n-1), \text{lex}(R)]^k = p_k$$

$$\left( \prod_{f\in E(H)} O_Z[f(0), \cdots, f(n-1)] + \prod_{f\in E(G)} (1 + O_Z[f(0), \cdots, f(n-1)]) \right)$$

$$\mod \left\{ \prod_{f\in E(R)} \mathbb{Z}[f(0), \cdots, f(n-1), \text{lex}_{S_n}(\sigma)] - \left( \prod_{f\in E(R)} \mathbb{Y}[f(0), \cdots, f(n-1), \text{lex}(R)] \right)^k \right\}$$

$$\equiv \left( \sum_{K\subseteq G} \prod_{f\in E(K)} \mathbb{Y}[f(0), \cdots, f(n-1), \text{lex}(R)]^k + \sum_{K\supset H} \prod_{f\in E(K)} \mathbb{Y}[f(0), \cdots, f(n-1), \text{lex}(R)]^k \right) = q_k$$

**Proof.** The claim follows from the observation that for a YES sub-isomorphism instance the monomial support of the canonical representative of the congruence class

$$\prod_{f\in E(G)} (1 + O_Z[f(0), \cdots, f(n-1)])$$
ON PARTIAL DIFFERENTIAL ENCODINGS OF BOOLEAN FUNCTIONS

\[
\begin{align*}
\mod \left\{ \prod_{f \in E(R)} Z[f(0), \cdots, f(n-1), \text{lex}_{\sigma}(\sigma)] - \prod_{f \in E(R)} Y[f(0), \cdots, f(n-1), \text{lex}(R)] \right\}, \\
R \subseteq \mathbb{Z}_n^{2^n}
\end{align*}
\]
matches the monomial support of the canonical representative of the congruence class

\[
\prod_{f \in E(H)} O_z[f(0), \cdots, f(n-1)] + \prod_{f \in E(G)} (1 + O_z[f(0), \cdots, f(n-1)])
\]

\[
\mod \left\{ \prod_{f \in E(R)} Z[f(0), \cdots, f(n-1), \text{lex}_{\sigma}(\sigma)] - \prod_{f \in E(R)} Y[f(0), \cdots, f(n-1), \text{lex}(R)] \right\}, \\
R \subseteq \mathbb{Z}_n^{2^n}
\]

Furthermore, for such an instance the two polynomial differ by exactly one of the non-vanishing integer coefficient being incremented by one. Using the Newton-Girard formulas we derive the polynomial division property. □

We see that the Chow–rank of polynomials start out having Chow–Rank at most 2 prior to the reduction and increases to at least \(2^n\) and at most \(2^n + 1\) after performing the reduction. The PDP construction therefore exhibits an unconditional exponential separation between isomorphism and sub-isomorphism instances.

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