CONDITIONAL EXPECTATION ON NON-COMMUTATIVE
$H^{(r,s)}_p(A; \ell_\infty)$ AND $H_p(A; \ell_1)$ SPACES : SEMIFINITE CASE

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Abstract. In this paper we investigate the conditional expectation on the
non-commutative $H^{(r,s)}_p(A; \ell_\infty)$ and $H_p(A; \ell_1)$ spaces associated with semifi-
nite subdiagonal algebra, and prove the contractibility of the underlying con-
ditional expectation on these spaces.

1. Introduction

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal
semifinite trace $\tau$ and let $\mathcal{D}$ be a subalgebra of the $\mathcal{M}$. Let $L_p(\mathcal{M})$ and $L_p(\mathcal{D})$ be the
corresponding non-commutative $L_p$-spaces, respectively. A conditional expectation
$E : \mathcal{M} \to \mathcal{D}$ is a unique normal faithful map such that $\tau \circ E = \tau$. It is well-
known that the conditional expectation $E$ extends to a contractive projection from
$L_p(\mathcal{M})$ onto $L_p(\mathcal{D})$ for every $1 \leq p \leq \infty$. In general, $E$ cannot be continuously
extended to $L_p(\mathcal{M})$ for $p < 1$. In [1], Arveson introduced the notion of finite, maximal, subdiagonal algebras $\mathcal{A}$ of $\mathcal{M}$, as non-commutative analogues of weak*-Dirichlet algebras, for von Neumann algebra $\mathcal{M}$ with a faithful normal finite trace. Subsequently several authors studied the (non-commutative) $H_p$-spaces associated with such algebras. For references see [1, 2, 3, 4, 5, 8, 12, 13, 15, 16, 17, 19, 20] (see also [5, 6, 21, 22, 23] for recent studies) and many other sources, whereas more references on previous works can be found in the survey paper [19]. It was proved in [2] that $E$ is a contractive projection from $H_p(\mathcal{A})$ onto $L_p(\mathcal{D})$ for $p < 1$. Moreover, in [3] it was studied non-commutative $H_p$-spaces associated with semifinite subdiagonal algebra. The main objective of such paper is to obtain a generalization from above cited papers for the semi-finite case.

In [6], authors obtained that there is a contractive projection from $H^{(r,s)}_p(A; \ell_\infty)$ onto $L^{(r,s)}_p(D; \ell_\infty)$ for $0 < p, r, s \leq \infty$ (respectively, from $H_p(A; \ell_1)$ onto $L_p(D; \ell_1)$ for $0 < p \leq \infty$) and when subdiagonal algebra is finite. The main goal of this paper is to extend those results to the semifinite case. Addressing precisely this framework, our main results, Theorems [6] and [7] provide a complete description of these results, thereby complementing [6, Theorem 3 and Theorem 4]. The main technical tool of the paper, which enables us to obtain such results, is provided by Proposition 3.1 in [3].

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2. Preliminaries

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a separable Hilbert space $H$ equipped with a faithful normal semifinite trace $\tau$. A closed and densely defined operator $x$ affiliated with $\mathcal{M}$ is called $\tau$-measurable if $\tau(E_{|x|^2}(s, \infty)) < \infty$ for sufficiently large $s$. We denote the set of all $\tau$-measurable operators by $S(\mathcal{M}, \tau)$. Let $\mathcal{D}$ be a von Neumann subalgebra of $\mathcal{M}$, and let $\mathcal{E} : \mathcal{M} \to \mathcal{D}$ be the unique normal faithful conditional expectation such that $\tau \circ \mathcal{E} = \tau$.

**Definition 1.** A semifinite subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$ is a $w^*$-closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfying the following conditions

(i) $\mathcal{A} + J(\mathcal{A})$ is $w^*$-dense in $\mathcal{M}$;
(ii) $\mathcal{E}$ is multiplicative on $\mathcal{A}$, i.e., $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$ for all $x, y \in \mathcal{A}$;
(iii) $\mathcal{A} \cap J(\mathcal{A}) = \mathcal{D}$, where $J(\mathcal{A})$ is the family of all adjoint elements of the element of $\mathcal{A}$, i.e., $J(\mathcal{A}) = \{x^* : x \in \mathcal{A}\}$.

The algebra $\mathcal{D}$ is called the diagonal of $\mathcal{A}$. It’s proved by Ji [13] (see also [12] for the finite case) that a semifinite subdiagonal algebra $\mathcal{A}$ is automatically maximal in the sense that if $\mathcal{B}$ is another subdiagonal algebra with respect to $\mathcal{E}$ containing $\mathcal{A}$, then $\mathcal{B} = \mathcal{A}$. This maximality yields the following useful characterization of $\mathcal{A}$:

$$A = \{x \in \mathcal{M} : \tau(xy) = 0, \forall y \in \mathcal{A}_0\}$$

where $\mathcal{A}_0 = \mathcal{A} \cap \ker \mathcal{E}$ (see [1]).

Let $0 < p < \infty$. Then we define the noncommutative $L_p$-spaces associated with $(\mathcal{M}, \tau)$ as follows

$$L_p(\mathcal{M}) = \{x \in S(\mathcal{M}, \tau) : \tau(|x|^p) < \infty\}$$

with the quasi-norm

$$\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}.$$

Recall that $L_\infty(\mathcal{M}) = \mathcal{M}$, equipped with the operator norm. For more details on these spaces we refer the reader to [19]. Now we will define noncommutative $H_p$-space (see [3]).

**Definition 2.** For $0 < p \leq \infty$ we define the noncommutative $H_p$-space:

$$H_p(\mathcal{A}) = \text{closure of } \mathcal{A} \cap L_p(\mathcal{M}) \text{ in } L_p(\mathcal{M}),$$

$$H_p^0(\mathcal{A}) = \text{closure of } \mathcal{A}_0 \cap L_p(\mathcal{M}) \text{ in } L_p(\mathcal{M}).$$

These are so called Hardy spaces associated with semifinite subdiagonal algebra $\mathcal{A}$. They are noncommutative extensions of the classical Hardy space on the torus $\mathbb{T}$. These noncommutative Hardy spaces have received a lot of attention since Arveson’s pioneer work. For references see [1] [3] [4] [5] [6] [7] [8] [17] [18] [19] [20] [21] [22] [23], whereas more references on previous works can be found in the survey paper [19].

The theory of vector-valued noncommutative $L_p$-spaces are introduced by Pisier in [18] for the case $\mathcal{M}$ is hyperfinite. Junge introduced these spaces for general setting in [9] (see also [10] [11] [13]). Let $0 < p, r, s \leq \infty$ such that $1/p = 1/r + 1/s$. Define the space

$$L_p^{(r,s)}(\mathcal{M}, \ell_\infty)$$

of all sequences $g = \{g_n\}_{n \geq 1}$ of operators in $L_p(\mathcal{M})$ for which there is a bounded sequence $x = \{x_n\}_{n \geq 1}$ in $\mathcal{M}$ and operators $y \in L_r(\mathcal{M})$ and $z \in L_s(\mathcal{M})$ such that

$$g_n = yx_nz, \quad \forall \ n \geq 1.$$
Definition 3. Let $0 < p, r, s \leq \infty$ such that $1/p = 1/r + 1/s$.

(i) We define $H^{(r,s)}_p(A; \ell_\infty)$ as the space of all sequences $g = \{g_n\}_{n \geq 1}$ in $H_p(A)$ which admit a factorization of the following form:

there are $y \in H_r(A), z \in H_s(A)$ and a bounded sequence $x = \{x_n\}_{n \geq 1} \subset A$ such that

$$g_n = yx_nz, \quad \forall n \geq 1.$$ 

Put

$$\|g\|_{H^{(r,s)}_p(A; \ell_\infty)} = \inf \{\|y\|_{H_r(A)} \sup_n \|x_n\|_{H_s(A)} \|z\|_{H_s(A)}\},$$

where the infimum runs over all factorizations of $\{g_n\}_{n \geq 1}$ as above. The spaces

$H^{right}_p(A; \ell_\infty) = H^{(\infty,p)}_p(A; \ell_\infty)$

and

$H^{left}_p(A; \ell_\infty) = H^{(p,\infty)}_p(A; \ell_\infty)$

for all $n$

$$\sum_{k,n=1}^\infty u_{kn}u_{kn}^* \in L_p(M) \quad \sum_{n,k=1}^\infty v_{nk}v_{nk}^* \in L_p(M).$$

Here all series are required to be convergent in $L_p(M)$ (relative to the w*-topology in the case of $p = \infty$). $L_p(M; \ell_1)$ is a quasi-Banach space (Banach space whenever $p \geq 1$) when equipped with the norm

$$\|x\|_{L_p(M; \ell_1)} = \inf \{\| \sum_{k,n=1}^\infty u_{kn}u_{kn}^* \|_p^{1/2}, \| \sum_{n,k=1}^\infty v_{nk}v_{nk}^* \|_p^{1/2}\},$$

where the infimum runs over all decompositions of $(x_n)$ as above.

We now define the spaces $H^{(r,s)}_p^('A; \ell_\infty)$ and $H_p(A; \ell_1)$ (see [5] for the finite case) by a similar way.
of all sequences \( \{g_n\}_{n \geq 1} \) which allow uniform factorizations \( g_n = x_n z \) and \( g_n = y z_n \) with \( y, z \in H_p(A) \) and a bounded sequence \( \{x_n\}_{n \geq 1} \subset A \), respectively. Moreover, in the multiplicity of \( x \),

\[
H_p(A; \ell_\infty) = H_p^{(2p, 2p)}(A; \ell_\infty).
\]

(ii) Let \( 0 < p \leq \infty \). We define \( H_p(A; \ell_1) \) as the space of all sequences \( x = \{x_n\}_{n \geq 1} \) in \( H_p(A) \) which can be decomposed as

\[
x_n = \sum_{k=1}^{\infty} u_{kn} v_{nk}, \forall n \geq 1
\]

for two families \( \{u_{kn}\}_{k,n \geq 1} \) and \( \{v_{nk}\}_{n,k \geq 1} \) in \( H_{2p}(A) \) such that

\[
\sum_{k,n=1}^{\infty} u_{kn}^* u_{kn} \in L_p(M) \quad \text{and} \quad \sum_{n,k=1}^{\infty} v_{nk}^* v_{nk} \in L_p(M).
\]

Here all series are required to be convergent in \( L_p(M) \) (relative to the \( w^\ast \)-topology in the case of \( p = \infty \)). It is equipped with the norm

\[
\|x\|_{H_p(A; \ell_1)} = \inf \left\{ \| \sum_{k,n=1}^{\infty} u_{kn}^* u_{kn} \|_{L_p(M)}^{1/2} \| \sum_{n,k=1}^{\infty} v_{nk}^* v_{nk} \|_{L_p(M)}^{1/2} \right\},
\]

where the infimum runs over all decompositions of \( x \) as above.

3. Contractibility of \( \overline{E} \) on \( H_p^{(r,s)}(A; \ell_\infty) \) and \( H_p(A; \ell_1) \) spaces

In this section, we define a conditional expectation on \( H_p^{(r,s)}(A; \ell_\infty) \) and \( H_p(A; \ell_1) \) spaces defined in the Definition \( \ref{defn:spaces} \) and prove that it is a contractive projection on such spaces. First, we need the following result.

**Lemma 4.** \( \mathbb{E} \) is multiplicative on Hardy spaces. More precisely, \( \mathbb{E}(xy) = \mathbb{E}(x)\mathbb{E}(y) \) for all \( x \in H_p(A) \) and \( y \in H_q(A) \) with \( 0 < p, q \leq \infty \).

**Proof.** Let \( x \in H_p(A) \) and \( y \in H_q(A) \). Then, by Definition \( \ref{defn:Hardy} \) there exist sequences \( x_n \in L_p(M) \cap A \) and \( y_n \in L_q(M) \cap A \) such that \( \lim_{n \to \infty} \|x_n - x\|_{L_p(M)} = 0 \) and \( \lim_{n \to \infty} \|y_n - y\|_{L_q(M)} = 0 \). Hence, it follows from Hölder inequality (see \( \ref{holder} \)) that \( x_n y_n \in L_r(M) \cap A \), where \( r \) is determined by \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Letting \( n \to \infty \), we obtain \( xy \in H_r(A) \). Thus, \( \mathbb{E}(xy) \) is well defined. Then, the result follows immediately from the multiplicativity of \( \mathbb{E} \) on \( A \) and \( \ref{defn:Hardy} \) Proposition 3.1].

Define a conditional expectation on \( H_p^{(r,s)}(A; \ell_\infty) \) and \( H_p(A; \ell_1) \) spaces as follows

\[
\mathbb{E} : h = \{h_n\}_{n \geq 1} \to \left\{ \mathbb{E}(h_n) \right\}_{n \geq 1},
\]

where \( \mathbb{E} \) is the conditional expectation on \( H_p(A) \) such that \( \mathbb{E} : H_p(A) \to L_p(D), 0 < p \leq \infty \).

**Theorem 5.** Let \( 0 < p, r, s \leq \infty \). Then

\[
\|\mathbb{E}(h)\|_{H_p^{(r,s)}(D; \ell_\infty)} \leq \|h\|_{H_p^{(r,s)}(A; \ell_\infty)}, \quad \forall h = \{h_n\}_{n \geq 1} \in H_p^{(r,s)}(A; \ell_\infty).
\]
Proof. Let \( h = \{h_n\}_{n \geq 1} \in H_p^{(r,s)}(A; \ell_\infty) \). Then for \( \varepsilon > 0 \) there exist \( x \in H_r(A) \), \( y \in H_s(A) \) and a bounded sequence \( \{z_n\}_{n \geq 1} \subset A \) such that for all \( n \), \( h_n = y x_n z \), and

\[
\|\{h_n\}_{n \geq 1}\|_{H_p^{(r,s)}(A; \ell_\infty)} + \varepsilon \geq \|y\|_{H_r(A)} \sup_n \|x_n\|_{H_s(A)} \|z\|_{H_s(A)}.
\]

Hence, by Lemma 4 and [3, Proposition 3.1], we have

\[
\mathbb{E}(h_n) = \mathbb{E}(y x_n z) = \mathbb{E}(y) \mathbb{E}(x_n) \mathbb{E}(z),
\]

where \( \mathbb{E}(y) \in L_r(D) \), \( \mathbb{E}(x_n) \in D \), \( \mathbb{E}(z) \in L_s(D) \) and

\[
\|\mathbb{E}(y)\|_{L_r(D)} \leq \|y\|_{H_r(A)}, \quad \|\mathbb{E}(x_n)\|_{L_s(D)} \leq \|x_n\|_{H_s(A)}, \quad \|\mathbb{E}(z)\|_{L_s(D)} \leq \|z\|_{H_s(A)}.
\]

Therefore,

\[
\|\mathbb{E}(h)\|_{L_p^{(r,s)}(D; \ell_\infty)} = \|\{\mathbb{E}(h_n)\}_{n \geq 1}\|_{L_p^{(r,s)}(D; \ell_\infty)} \\
\leq \|\mathbb{E}(y)\|_{L_r(D)} \sup_n \|\mathbb{E}(x_n)\|_{L_s(D)} \|\mathbb{E}(z)\|_{L_s(D)} \\
\leq \|y\|_{H_r(A)} \sup_n \|x_n\|_{H_s(A)} \|z\|_{H_s(A)} \\
\|\{h_n\}_{n \geq 1}\|_{H_p^{(r,s)}(A; \ell_\infty)} + \varepsilon.
\]

Letting \( \varepsilon \to 0 \) we obtain the desired inequality. \( \square \)

Let \( x = \{x_n\}_{n \geq 0} \) be a finite sequence in \( L_p(M) \), define

\[
\|x\|_{L_p(M; \ell_\infty^2)} := \left\| \left( \sum_{n=0}^{\infty} |x_n|^2 \right)^{1/2} \right\|_{L_p(M)}.
\]

**Lemma 6.** Let \( 0 < p \leq \infty \). Then

\[
\left\| \left( \sum_{n=0}^{\infty} |x_n|^2 \right)^{1/2} \right\|_{L_p(D)} \leq \left\| \left( \sum_{n=0}^{\infty} |x_n|^2 \right)^{1/2} \right\|_{H_p(A)} , \quad \forall \{x_n\}_{n \geq 1} \subset H_p(A).
\]

**Proof.** Let \( \left\| \sum_{n=0}^{\infty} |x_n|^2 \right\|_{H_p(A)} < \infty \) and let \( \mathcal{N} = M_n(D) \) be the algebra of \( n \times n \) matrices with entries from \( D \) and \( B = M_n(A) \) be the algebra of \( n \times n \) matrices with entries from \( A \). For \( x \in \mathcal{N} \) with entries \( x_{i,j} \), define \( \Phi(x) \) to be the matrix with entries \( \mathbb{E}(x_{i,j}) \) and \( \nu(x) = \sum_{i=1}^{n} (x_{i,i}) \). Then \( (\mathcal{N}, \nu) \) is a semifinite von Neumann algebra and \( B \) is a semifinite subdiagonal algebra of \( (\mathcal{N}, \nu) \). Let \( x' = \left\{ (x_1, x_2, \ldots, x_n, 0, \ldots) \right\} \). Then by [3, Proposition 3.1], we have

\[
\left\| \left( \sum_{k=0}^{n} |x_k|^2 \right)^{1/2} \right\|_{L_p(D)} = \|\Phi(T(x'))\|_{L_p(\mathcal{N})} \leq \|T(x')\|_{H_p(B)}
\]

(3.1)

\[
= \left\| \left( \sum_{k=0}^{n} |x_k|^2 \right)^{1/2} \right\|_{H_p(A)} , \quad \forall \{x_n\}_{n \geq 1} \subset H_p(A)
\]
where $T$ is the map (see also [3]):

$$T : \{x_n\}_{n \geq 1} \mapsto \begin{pmatrix} x_1 & 0 & \cdots \\ x_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$ 

Letting $n \to \infty$ in (3.1), we obtain desired inequality. \qed

**Theorem 7.** Let $0 < p \leq \infty$. Then

$$\|E(h)\|_{L_p(D; \ell_1)} \leq \|h\|_{H_p(A; \ell_1)}, \quad \forall h = \{h_n\}_{n \geq 1} \in H_p(A; \ell_1).$$

**Proof.** Let $\{h_n\}_{n \geq 1} \in H_p(A; \ell_1)$, then for $\varepsilon > 0$, there are $\{u_{kn}\}_{k,n \geq 1}$ and $\{v_{nk}\}_{n,k \geq 1}$ in $H_{2p}(A)$ such that

$$h_n = \sum_{k=1}^{\infty} u_{kn}v_{nk}, \quad \forall n \geq 1,$$

and $(\sum_{k,n=1}^{\infty} u_{kn}u_{kn}^*)^{1/2}, (\sum_{n,k=1}^{\infty} v_{nk}v_{nk}^*)^{1/2} \in L_{2p}(\mathcal{M})$, and

$$\|\{h_n\}_{n \geq 1}\|_{H_p(A; \ell_1)} + \varepsilon \geq \|\sum_{k,n=1}^{\infty} u_{kn}u_{kn}^*\|_{H_p(A)}^{1/2} \cdot \|\sum_{k,n=1}^{\infty} v_{nk}v_{nk}^*\|_{H_p(A)}^{1/2}.$$ 

Hence, by Lemma [4]

$$E(h_n) = E(\sum_{k=1}^{\infty} u_{kn}v_{nk}) = \sum_{k=1}^{\infty} E(u_{kn})E(v_{nk}) = \sum_{k=1}^{\infty} E(u_{kn})E(v_{nk}), \quad \forall n.$$ 

Since $J(A)$ is a subdiagonal algebra of $A$, using Lemma [6] we obtain

$$\|E(h)\|_{L_p(D; \ell_1)} = \left\|E(\{h_n\}_{n \geq 1})\right\|_{L_p(D; \ell_1)} = \left\|\sum_{k=1}^{\infty} E(u_{kn})E(v_{nk})\right\|_{L_p(D; \ell_1)} \leq \left\|\sum_{k,n=1}^{\infty} |u_{kn}|^2\right\|_{L_p(D)}^{1/2} \cdot \left\|\sum_{k,n=1}^{\infty} |v_{nk}|^2\right\|_{L_p(D)}^{1/2} \leq \|\{h_n\}_{n \geq 1}\|_{H_p(A; \ell_1)} + \varepsilon = \|h\|_{H_p(A; \ell_1)} + \varepsilon.$$

Letting $\varepsilon \to 0$ we complete the proof. \qed

**References**

[1] W.B. Arveson, Analyticity in operator algebras, *Amer. J. Math.* 89 (1967), 578-642.
[2] T.N. Bekjan and Quanhua Xu, Riesz and Szegö type factorizations for noncommutative Hardy spaces, *J. Operator Theory* 62:1 (2009), 215-231.
[3] T.N. Bekjan, Noncommutative Hardy space associated with semi-finite subdiagonal algebras, *J. Math. Anal. Appl.* 420 (2015), 1347-1369.
[4] T.N. Bekjan, Noncommutative symmetric Hardy spaces, *Integral Equations Operator Theory* 81 (2015), 191-212.

[5] T.N. Bekjan, B.K. Sageman, A property of conditional expectation, *Positivity* 22:5 (2018), 1359-1369.

[6] T.N. Bekjan, K. Tulenov, D. Dauitbek, The noncommutative $H^{(r,s)}_p(A; \ell_\infty)$ and $H_p(A; \ell_1)$ spaces, *Positivity* 19:4 (2015), 877-891.

[7] D.P. Blecher, L.E. Labuschagne, Applications of the Fuglede-Kadison determinant: Szegö’s theorem and outers for noncommutative $H^p$, *Trans. Amer. Math. Soc.* 360 (2008), 6131-6147.

[8] D.P. Blecher, L.E. Labuschagne, Characterizations of noncommutative $H^\infty$, *Integral Equations Operator Theory* 56 (2006), 301-321.

[9] M. Junge, Doob’s inequality for non-commutative martingales, *J. Reine Angew. Math.*, 549 (2002), 149-190.

[10] A. Defant and M. Junge, Maximal theorems of Menchoff-Rademacher type in noncommutative $L_q$-spaces, *J. Funct. Anal.* 206 (2004), 322-355.

[11] S. Dirksen, Noncommutative boyd interpolation theorems, *Transactions of the american mathematical society* 367: 6 (2015), 4079-4110.

[12] R. Exel, Maximal subdiagonal algebras, *Amer. J.Math.* 110 (1988), 775-782.

[13] G. Ji, Maximality of semifinite subdiagonal algebras, *J. Shaanxi Normal Univ. Nat. Sci. Ed.* 28:1 (2000), 15-17.

[14] M. Junge and Q. Xu, Noncommutative maximal ergodic theorems, *J. Amer. Math. Soc.* 20 (2007), 385-439.

[15] L.E. Labuschagne, A noncommutative Szegö theorem for subdiagonal subalgebras of von Neumann algebras, *Proc. Amer. Math. Soc.* 133 (2005), 3643-3646.

[16] M. Marsalli, G. West, Noncommutative $H^p$ spaces, *J.Operator Theory* 40 (1998), 339-355

[17] G. Pisier, Interpolation between $H^p$ spaces and non-commutative generalizations I, *Pacific J. Math.* 155 (1992), 341-368.

[18] G. Pisier, Non-commutative vector valued $L^p$-spaces and completely p-summing maps, *Astérisque* 247 (247), 1998, 5189 (1997), 667-698.

[19] G. Pisier and Q. Xu, Noncommutative $L^p$-spaces, *Handbook of the geometry of Banach spaces*, 2 (2003) 1459-1517, 2003.

[20] K.S. Saito, A note on invariant subspaces for finite maximal subdiagonal algebras, *Proc. Amer. Math. Soc.* 77 (1979), 348-352.

[21] F. Sukochev, K. Tulenov, D. Zanin, Nehari-Type Theorem for Non-commutative Hardy Spaces, *J. Geom. Anal.* 27:3 (1979), 1789-1802.

[22] K. S. Tulenov, Noncommutative vector-valued symmetric Hardy spaces, *Russian Mathematics* 59:11 (2015), 74-79.

[23] K. Tulenov, Some properties of the noncommutative $H^{(r,s)}_p(A; \ell_\infty)$ and $H_p(A; \ell_1)$ spaces, *AIP Conference Proceedings* 1676:020093 (2015).

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