THE NONLINEAR STEEPEST DESCENT METHOD FOR Riemann-Hilbert Problems of Low Regularity

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Abstract. We prove a nonlinear steepest descent theorem for Riemann-Hilbert problems with Carleson jump contours and jump matrices of low regularity and slow decay. We illustrate the theorem by deriving the long-time asymptotics for the mKdV equation in the similarity sector for initial data with limited decay and regularity. Our approach is slightly different from the original approach of Deift and Zhou: By isolating the dominant contributions of the critical points directly in an appropriately rescaled Riemann-Hilbert problem, we find the asymptotics using Cauchy’s formula. This hopefully leads to a more transparent presentation.

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1. Introduction

The method of nonlinear steepest descent was introduced in the early 1990’s in a seminal paper by Deift and Zhou [4], building on earlier work of Manakov [13] and Its [9]. Whereas the classical steepest descent method yields asymptotic expansions as $t \to \infty$ of scalar integrals of the form

$$I(t) = \int_C f(z) e^{t\Phi(z)} dz$$ (1.1)

where $C$ is a contour in the complex $z$-plane, the nonlinear steepest descent method yields expansions of solutions of matrix Riemann-Hilbert (RH) problems. In the same way that the solutions of a large class of problems involving linear differential equations can be represented by scalar integrals of the form (1.1), the solutions of many nonlinear problems can be represented in terms of solutions of matrix RH problems.

The nonlinear steepest descent method relies on the same idea as its classical analog. The jump matrix of the RH problem contains basic exponentials of the form $e^{\pm t\Phi(z)}$. By deforming the contour so that the jump involves only $e^{t\Phi(z)}$ when $z$ lies in the domain $\{\text{Re } \Phi(z) < 0\}$ whereas it involves only $e^{-t\Phi(z)}$ when $z$ lies in the domain $\{\text{Re } \Phi(z) > 0\}$, it is ensured that the jump is small for large $t$ except near a small number of ‘critical points’ at which $\text{Re } \Phi(z) = 0$. Near each critical point the RH problem converges as $t \to \infty$ to a RH problem which can be explicitly solved. Hence, as in the classical method, an asymptotic expansion of the solution can be obtained by summing up the contributions from the individual critical points.

The purpose of this paper is to implement the nonlinear steepest descent method for RH problems of low regularity. More precisely, we prove a nonlinear steepest descent
theorem applicable to Riemann-Hilbert problems with Carleson jump contours and jump matrices of low regularity and slow decay (both as \( t \to \infty \) and as \( z \to \infty \)). We illustrate the theorem by deriving the long-time asymptotics for the mKdV equation

\[
  u_t - 6u^2 u_x + u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t \geq 0,
\]

in the similarity sector for initial data with limited decay and regularity. In [4], formulas were established for the asymptotics of the solution of (1.2) under the assumption that the initial data \( u_0(x) = u(x,0) \) lie in the Schwartz class \( S(\mathbb{R}) \) of rapidly decreasing functions. Even though the main idea of our approach is the same as in [4], the proof proceeds along somewhat different lines: By isolating the dominant contributions of the critical points directly in an appropriately rescaled RH problem, we can easily find the asymptotics using Cauchy’s formula. In this way, we avoid having to establish a number of operator identities related to the restriction and decoupling of various parts of the jump contour. Our goal is to provide a rigorous and yet accessible treatment; precise and uniform error estimates are presented throughout the paper.

The class of Carleson contours is very large; for example, it includes contours with cusps and nontransversal intersections. This means that our approach can be used to rigorously analyze asymptotics of a large class RH problems. We mention in this regard that RH problems with contours involving nontransversal intersections arise in the study of initial-boundary problems for integrable evolution equations, see e.g. [11]. Here, for pedagogical reasons, we only consider equation (1.2) on the line, but more complicated examples for initial and initial-boundary value problems will be analyzed elsewhere.

In Section 2, we prove a nonlinear steepest descent theorem for RH problems with Carleson jump contours. In Section 3, we recall how the solution of the mKdV equation can be expressed in terms of the solution of a RH problem. In Section 4, we derive the long time behavior of (1.2) in the similarity sector for a large class of initial data. Some results on \( L^2 \)-RH problems are collected in Appendix A. Appendix B contains a derivation of the exact solution of the model RH problem which is relevant near the critical points.

2. A NONLINEAR STEEPEST DESCENT THEOREM

Our first theorem provides an implementation of the nonlinear steepest descent method for RH problems with Carleson jump contours and jump matrices of low regularity and slow decay. Although the theorem is formulated, for definiteness, under the assumption that there are two critical points related by reflection in the imaginary axis (this is the situation relevant for the similarity sector of the mKdV equation), it can be readily generalized to scenarios with multiple critical points and different symmetries.

Let \( X \) denote the cross \( X = X_1 \cup \cdots \cup X_4 \subset \mathbb{C} \), where the rays

\[
  X_1 = \{ se^{\pi i s} \mid 0 \leq s < \infty \}, \quad X_2 = \{ se^{\frac{3\pi i}{4} s} \mid 0 \leq s < \infty \},
\]

\[
  X_3 = \{ se^{-\frac{3\pi i}{4} s} \mid 0 \leq s < \infty \}, \quad X_4 = \{ se^{-\pi i s} \mid 0 \leq s < \infty \}
\]

are oriented as in Figure 1. For \( r > 0 \), we let \( X^r = X_1^r \cup \cdots \cup X_4^r \) denote the restriction of \( X \) to the disk of radius \( r \) centered at the origin, i.e. \( X^r = X \cap \{ \left| z \right| < r \} \). Given a Carleson jump contour \( \Gamma \) and \( a, b \in \mathbb{R} \) with \( a < b \), we call \( W_{a,b} = \{ a \leq \arg k \leq b \} \) a nontangential sector at \( \infty \) if there exists a \( \delta > 0 \) such that \( W_{a-\delta,b+\delta} \) does not intersect \( \Gamma \cap \{ \left| z \right| > R \} \) whenever \( R > 0 \) is large enough. If \( f(k) \) is a function of \( k \in \mathbb{C} \setminus \Gamma \), we say that \( f \) has nontangential limit \( L \) at \( \infty \), written

\[
  \lim_{k \to \infty} f(k) = L,
\]

\footnote{We refer to Appendix A for precise definitions of Carleson jump contours and \( L^2 \)-RH problems.}
if \( \lim_{k \to \infty} f(k) = L \) for every nontangential sector \( W_{a,b} \) at \( \infty \). Throughout the paper, complex powers and logarithms are defined using the principal branch: If \( z, a \in \mathbb{C} \) and \( z \neq 0 \), then
\[
\ln z := \ln |z| + i \text{Arg } z \quad \text{and} \quad z^a := e^{a \ln z},
\]
where \( \text{Arg } z \in (-\pi, \pi] \) denotes the principal value of \( \text{arg } z \). We use \( C \) to denote a generic constant that can change within a computation. The Riemann sphere is denoted by \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \).

**Theorem 2.1** (Nonlinear steepest descent). Let \( \mathcal{I} \subset \mathbb{R} \) be a (possibly infinite) interval. Let \( \rho, \epsilon, k_0 : \mathcal{I} \to (0, \infty) \) be bounded strictly positive functions such that \( \epsilon(\zeta) < k_0(\zeta) \) for \( \zeta \in \mathcal{I} \). We henceforth drop the \( \zeta \) dependence of these functions and write simply \( \rho, \epsilon, k_0 \) for \( \rho(\zeta), \epsilon(\zeta), k_0(\zeta) \), respectively.

Let \( \Gamma = \Gamma(\zeta) \) be a family of oriented contours parametrized by \( \zeta \in \mathcal{I} \) and let \( \hat{\Gamma} = \Gamma \cup \{ |k \pm k_0| = \epsilon \} \) denote the union of \( \Gamma \) with the circles of radius \( \epsilon \) centered at \( \pm k_0 \) oriented counterclockwise. Assume that, for each \( \zeta \in \mathcal{I} \):

1. \( \Gamma \) and \( \hat{\Gamma} \) are Carleson jump contours up to reorientation of a subcontour.
2. \( \Gamma \) contains the two crosses \( \pm k_0 + X^\epsilon \) as oriented subcontours.
3. \( \Gamma \) is invariant as a set under the map \( k \mapsto -\bar{k} \). Moreover, the orientation of \( \Gamma \) is such that if \( k \) traverses \( \Gamma \) in the positive direction, then \( -\bar{k} \) traverses \( \Gamma \) in the negative direction.
4. The point \( \infty \in \hat{\mathbb{C}} \) can be approached nontangentially, i.e., there exists a sector \( W_{a,b} \) which is a nontangential sector at \( \infty \) for \( \Gamma \).

Moreover, assume that the Cauchy singular operator \( S_{\hat{\Gamma}} \) defined by
\[
(S_{\hat{\Gamma}} h)(z) = \lim_{r \to 0} \frac{1}{\pi i} \frac{1}{\Gamma \setminus \{ |z' - z| < r \}} \frac{h(z')}{z' - z} dz',
\]
is uniformly\(^2\) bounded on \( L^2(\hat{\Gamma}) \), i.e.,
\[
\sup_{\zeta \in \mathcal{I}} \| S_{\hat{\Gamma}} \|_{B(L^2(\hat{\Gamma}))} < \infty. \tag{2.2}
\]

Consider the following family of \( L^2 \)-RH problems parametrized by the two parameters \( \zeta \in \mathcal{I} \) and \( t > 0 \):

\[
\begin{align*}
&\begin{cases}
m(\zeta, t, \cdot) \in I + \dot{\mathcal{E}}^2(\hat{\mathbb{C}} \setminus \Gamma), \\
m_+(\zeta, t, k) = m_-(\zeta, t, k) v(\zeta, t, k) \quad \text{for a.e. } k \in \Gamma,
\end{cases}
\end{align*}
\tag{2.3}
\]

\(^2\)For any fixed \( \zeta \in \mathcal{I} \), \( S_{\hat{\Gamma}} \) is bounded on \( L^2(\hat{\Gamma}) \) as a consequence of (\( \Gamma 1 \)), see e.g. \cite{12}.
where the jump matrix \( v(\zeta, t, k) \) satisfies
\[
\begin{align*}
    w(\zeta, t, \cdot) := v(\zeta, t, \cdot) - I & \in L^2(\Gamma) \cap L^2(\Gamma) \cap L^\infty(\Gamma), & \zeta \in \mathcal{I}, & t > 0, \\
    \det v(\zeta, t, \cdot) = 1 & \text{ a.e. on } \Gamma, & \zeta \in \mathcal{I}, & t > 0,
\end{align*}
\]
and
\[
\begin{align*}
    v(\zeta, t, k) &= v(\zeta, t, -k), & \zeta \in \mathcal{I}, & t > 0, & k \in \Gamma.
\end{align*}
\]

Let \( \tau := t\rho^2 \). Let \( \Gamma_X \) be the union of the two crosses \( \pm k_0 + \mathcal{X} \) and let \( \Gamma' = \Gamma \setminus \Gamma_X \).
Suppose
\[
\begin{align*}
    \|w(\zeta, t, \cdot)\|_{L^p(\Gamma')} &= O(\epsilon^p \tau^{-1}), & \tau \to \infty, & \zeta \in \mathcal{I}, & p = 1, 2, \\
    \|w(\zeta, t, \cdot)\|_{L^\infty(\Gamma')} &= O(\tau^{-1}), & \tau \to \infty, & \zeta \in \mathcal{I},
\end{align*}
\]
uniformly with respect to \( \zeta \in \mathcal{I} \). Moreover, suppose that near \( k_0 \) the normalized jump matrix
\[
v_0(\zeta, t, z) = v(\zeta, t, k_0 + \frac{e^z}{\rho}), & z \in \mathcal{X}^\rho,
\]
has the form
\[
\begin{align*}
    v_0(\zeta, t, z) = \\
    \begin{cases}
    1 & z \in X_1^\rho, \\
    R_1(\zeta, t, z)z^{-2i\nu(\zeta)}e^{i\phi(\zeta, z)} & z \in X_2^\rho, \\
    1 - R_2(\zeta, t, z)z^{2i\nu(\zeta)}e^{-i\phi(\zeta, z)} & z \in X_3^\rho, \\
    - R_3(\zeta, t, z)z^{-2i\nu(\zeta)}e^{i\phi(\zeta, z)} & z \in X_4^\rho,
    \end{cases}
\end{align*}
\]
where:

- The phase \( \phi(\zeta, z) \) is a smooth function of \( (\zeta, z) \in \mathcal{I} \times \mathbb{C} \) such that
\[
\begin{align*}
    \phi(\zeta, 0) & \in i\mathbb{R}, & \frac{\partial \phi}{\partial z}(\zeta, 0) & = 0, & \frac{\partial^2 \phi}{\partial z^2}(\zeta, 0) & = i, & \zeta \in \mathcal{I}, \\
    \text{and} & & & & \text{and}
\end{align*}
\]
\[
\begin{align*}
    \text{Re} \phi(\zeta, z) & \leq -\frac{|z|^2}{4}, & z \in X_1^\rho \cup X_3^\rho, & \zeta \in \mathcal{I}, \\
    \text{Re} \phi(\zeta, z) & \geq \frac{|z|^2}{4}, & z \in X_2^\rho \cup X_4^\rho, & \zeta \in \mathcal{I}, \\
    \left| \phi(\zeta, z) - \phi(\zeta, 0) - \frac{iz^2}{2} \right| & \leq C \frac{|z|^3}{\rho}, & z \in \mathcal{X}^\rho, & \zeta \in \mathcal{I},
\end{align*}
\]
where \( C > 0 \) is a constant.
Remark 2.2. The conclusion of Theorem 2.1 can be stated more explicitly as follows:

\[ T > \text{depend discontinuously on } \zeta, t, k \]

Proof of Theorem 2.1. Let \( \rho, \epsilon, k_0 \) be the solution of Theorem B.1 and let \( m(X) \) be the solution of (2.3) such that the functions \( m(X) \) of (2.3) exists, and the inequality

\[ \lim_{k \to \infty} (km(\zeta, t, k))_{12} + \frac{2i\epsilon \text{Re } \beta(\zeta, t)}{\sqrt{\tau}} \leq K \epsilon \tau^{-\frac{1+\alpha}{2}} \]

holds for all \( (\zeta, t) \in \mathcal{I} \times [0, \infty) \) such that \( \tau = t \tau^2 > T \).

Remark 2.3. We emphasize that Theorem 2.1 allows for jump matrices \( v(\zeta, t, k) \) that depend discontinuously on \( (\zeta, t) \). It also allows for contours \( \Gamma \) and functions \( \rho, \epsilon, k_0 \) that depend discontinuously on \( \zeta \).

2.1. Proof of Theorem 2.1. Since \( \text{det } v = 1 \), uniqueness of \( m \) follows from Lemma A.3.

Let \( m^X \) be the solution of Theorem B.1 and let

\[ D(\zeta, t) = \begin{pmatrix} e^{-\frac{\tau(\zeta, 0)}{2} - \frac{\epsilon \nu(\zeta)}{2}} & 0 \\ 0 & e^{\frac{\tau(\zeta, 0)}{2} - \frac{\epsilon \nu(\zeta)}{2}} \end{pmatrix}. \]

Define \( m_0(\zeta, t, k) \) near \( k = k_0 \) by

\[ m_0(\zeta, t, k) = D(\zeta, t)m^X \left( \frac{q(\zeta)}{\epsilon}(k - k_0) \right)D(\zeta, t)^{-1}, \quad |k - k_0| \leq \epsilon, \]

and extend it to a neighborhood of \( k = -k_0 \) by symmetry:

\[ m_0(\zeta, t, k) = \overline{m_0(\zeta, t, -k)}, \quad |k + k_0| \leq \epsilon. \quad (2.15) \]

Lemma A.5 implies that \( m \) satisfies the \( L^2 \)-RH problem (2.3) if and only if the function \( \hat{m}(\zeta, t, k) \) defined by

\[ \hat{m}(\zeta, t, k) = \begin{cases} m(\zeta, t, k)m_0(\zeta, t, k)^{-1}, & |k \pm k_0| < \epsilon, \\
м(\zeta, t, k), & \text{otherwise,} \end{cases} \]
satisfies the $L^2$-RH problem

$$
\begin{cases}
\hat{m}(\zeta,t,\cdot) \in I + \hat{H}^2(\hat{C} \setminus \hat{\Gamma}), \\
\hat{m}_+(\zeta,t,k) = \hat{m}_-(\zeta,t,k)\hat{v}(x,t,k) \text{ for a.e. } k \in \hat{\Gamma},
\end{cases}
$$

where the jump matrix $\hat{v}$ is given by

$$
\hat{v}(\zeta,t,k) = \begin{cases}
m_0^-(\zeta,t,k)v(\zeta,t,k)m_0^+(\zeta,t,k)^{-1}, & |k| < \epsilon, \\
m_0^+(\zeta,t,k)^{-1}, & |k| = \epsilon, \\
v(\zeta,t,k), & \text{otherwise}.
\end{cases}
$$

It follows from (2.6) and (2.15) that $\hat{w} = \hat{v} - I$ obeys the symmetry

$$
\hat{w}(\zeta,t,k) = \hat{w}(\zeta,t,-k), \quad k \in \hat{\Gamma}.
$$

**Claim 1.** The function $\hat{w} = \hat{v} - I$ satisfies

$$
\hat{w}(\zeta,t,k) = O(\tau^{-\frac{\rho}{2}}e^{-\frac{\tau}{24\epsilon^2}[|k|+k_0]^2}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad k \in \pm k_0 + X^\epsilon,
$$

where the error term is uniform with respect to $(\zeta,k)$ in the given ranges.

**Proof of Claim 1.** We assume $k \in k_0 + X^\epsilon$; the case of $k \in -k_0 + X^\epsilon$ follows by symmetry. Then

$$
\hat{w}(\zeta,t,k) = m_0^-(\zeta,t,k)v(\zeta,t,k)m_0^+(\zeta,t,k)^{-1} - I = m_0^-(\zeta,t,k)u(\zeta,t,k)m_0^+(\zeta,t,k)^{-1},
$$

where

$$
u(\zeta,t,k) := v(\zeta,t,k) - D(\zeta,t)v^X \left( q(\zeta), \frac{\sqrt{\tau}}{\epsilon}(k-k_0) \right) D(\zeta,t)^{-1}.
$$

The functions $m_0^+(\zeta,t,k)$ and $m_0^-(\zeta,t,k)$ are uniformly bounded for $t > 0$, $\zeta \in \mathcal{I}$, and $k \in k_0 + X^\epsilon$. Therefore, it is enough to prove that

$$
u(\zeta,t,k) = O(\tau^{-\frac{\rho}{2}}e^{-\frac{\tau}{24\epsilon^2}[|k|+k_0]^2}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad k \in k_0 + X^\epsilon,
$$

uniformly with respect to $(\zeta,k)$. Introducing the function $u_0$ by

$$
u_0(\zeta,t,z) = u(\zeta,t,k_0 + \frac{\epsilon z}{\rho}) = v_0(\zeta,t,z) - D(\zeta,t)v^X (q(\zeta), \sqrt{\tau}z) D(\zeta,t)^{-1},
$$

we can rewrite the condition (2.20) as follows:

$$
u_0(\zeta,t,z) = O(\tau^{-\frac{\rho}{2}}e^{-\frac{t|q(\zeta)|^2}{24}}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in X^\rho,
$$

uniformly with respect to $(\zeta,z)$ in the given ranges. Using that

$$
D(\zeta,t)v^X (q(\zeta), \sqrt{\tau}z) D(\zeta,t)^{-1} = \begin{cases}
\begin{pmatrix}
1 & 0 \\
q(\zeta)z^{-2iv(\zeta)}e^{\frac{i\alpha}{2} e^{t\phi(\zeta,0)}} & 1
\end{pmatrix}, & z \in X_1, \\
\begin{pmatrix}
1 & 0 \\
\frac{q(\zeta)}{|1-q(\zeta)|^2}z^{-2iv(\zeta)}e^{-\frac{i\alpha}{2} e^{-t\phi(\zeta,0)}} & 1
\end{pmatrix}, & z \in X_2, \\
\begin{pmatrix}
1 & 0 \\
\frac{q(\zeta)}{|1-q(\zeta)|^2}z^{-2iv(\zeta)}e^{\frac{i\alpha}{2} e^{t\phi(\zeta,0)}} & 1
\end{pmatrix}, & z \in X_3, \\
\begin{pmatrix}
1 & 0 \\
\frac{q(\zeta)}{|1-q(\zeta)|^2}z^{2iv(\zeta)}e^{-\frac{i\alpha}{2} e^{-t\phi(\zeta,0)}} & 1
\end{pmatrix}, & z \in X_4,
\end{cases}
$$

equation (2.21) follows from the assumptions (2.9)-(2.12). Indeed, we will give the details of the proof of (2.21) in the case of $z \in X^\rho_1$; the other cases are similar.
Let \( z \in X^p_1 \). In this case only the (21) entry of \( u_0(\zeta, t, z) \) is nonzero and using that \( \arg z = \frac{\pi}{2} \) and \( \sup_{\zeta \in \mathbb{I}} |q(\zeta)| < 1 \), we find

\[
|\langle u_0(\zeta, t, z) \rangle_{21}| = |R_1(\zeta, t, z)z^{-2i\nu(\zeta)}e^{t\phi(\zeta,z)} - q(\zeta)z^{-2i\nu(\zeta)}e^{\frac{4t}{z^2}}e^{t\phi(\zeta,0)}| \\
= |z^{-2i\nu(\zeta)}| |R_1(\zeta, t, z)e^{t\phi(\zeta,z)} - q(\zeta)||e^{t\phi(\zeta,0)}e^{\frac{-4t}{z^2}}| \\
\leq e^{-\frac{\nu(\zeta)}{2}} \left( |R_1(\zeta, t, z) - q(\zeta)|e^{t\phi(\zeta,z)} + |q(\zeta)||e^{t\phi(\zeta,z)} - 1| \right) e^{-\frac{4t}{z^2}}, \quad \zeta \in \mathbb{I}, \quad t > 0, \quad z \in X^p_1, \tag{2.22}
\]

where \( \hat{\phi}(\zeta, z) = \phi(\zeta, z) - \phi(\zeta, 0) - \frac{i\zeta^2}{2} \). The simple estimate

\[
|e^w - 1| = \left| \int_0^1 w e^{sw} ds \right| \leq |w| \max_{s \in [0,1]} e^{sRe w}, \quad w \in \mathbb{C},
\]

yields the inequality

\[
|e^w - 1| \leq |w| \max(1, e^{Re w}), \quad w \in \mathbb{C}. \tag{2.23}
\]

On the other hand, by (2.10) and (2.11a),

\[
\text{Re} \hat{\phi}(\zeta, z) = \text{Re} \phi(\zeta, z) + \frac{|z|^2}{2} \leq \frac{|z|^2}{4}, \quad \zeta \in \mathbb{I}, \quad z \in X^p_1. \tag{2.24}
\]

Using (2.23), (2.24), and the fact that \( \sup_{\zeta \in \mathbb{I}} |q(\zeta)| < 1 \) in (2.22), we find

\[
|\langle u_0(\zeta, t, z) \rangle_{21}| \leq e^{-\frac{\nu(\zeta)}{2}} \left( |R_1(\zeta, t, z) - q(\zeta)| + t|\hat{\phi}(\zeta, z)| \right) e^{-\frac{4t}{z^2}}, \quad \zeta \in \mathbb{I}, \quad t > 0, \quad z \in X^p_1.
\]

By (2.11c), (2.12), and the fact that \( \sup_{\zeta \in \mathbb{I}} |\nu(\zeta)| < \infty \), the right-hand side is of order

\[
O \left( \left( \frac{L|z|^3 e^{-\frac{t|z|^2}{2}}}{\rho^p} + tC|z|^3 \right) e^{-\frac{t|z|^2}{4}} \right) = O \left( \left( \frac{(t|z|^2)^{\alpha/2}}{\tau^{\alpha/2}} + \frac{(t|z|^2)^{3/2}}{\tau^{1/2}} \right) e^{-\frac{t|z|^2}{24}} \right), \quad \tau \to \infty, \quad \zeta \in \mathbb{I}, \quad z \in X^p_1, \tag{2.25}
\]

uniformly with respect to \((\zeta, z)\) in the given ranges. This proves (2.21) in the case of \( z \in X^p_1 \).

\[ \nabla \]

Claim 2. We have

\[
\| \hat{w}(\zeta, t, \cdot) \|_{L^2(\mathbb{I})} = O(e^{\frac{1}{2} \tau^{-\frac{\alpha}{2}}}), \quad \tau \to \infty, \quad \zeta \in \mathbb{I}, \tag{2.26a}
\]

\[
\| \hat{w}(\zeta, t, \cdot) \|_{L^\infty(\mathbb{I})} = O(\tau^{-\frac{\alpha}{2}}), \quad \tau \to \infty, \quad \zeta \in \mathbb{I}, \tag{2.26b}
\]

and, for any \( p \in [1, \infty) \),

\[
\| \hat{w}(\zeta, t, \cdot) \|_{L^p(\pm k_0 + X^\nu)} = O(e^{\frac{1}{2} \tau^{-\frac{1}{2p} - \frac{\alpha}{2}}}), \quad \tau \to \infty, \quad \zeta \in \mathbb{I}. \tag{2.27}
\]

where all error terms are uniform with respect to \( \zeta \).

\[ \text{Proof of Claim 2. We have} \]

\[
\| \hat{w}(\zeta, t, \cdot) \|_{L^2(\mathbb{I})} \leq \| \hat{w}(\zeta, t, \cdot) \|_{L^2(\mathbb{I})} + \| m_0(\zeta, t, \cdot)^{-1} - I \|_{L^2(\pm k_0 = \epsilon)} \\
+ \| m_0(\zeta, t, \cdot)^{-1} - I \|_{L^2(\pm k_0 = \epsilon)} + \| \hat{w}(\zeta, t, \cdot) \|_{L^2(k_0 + X^\nu)} \\
+ \| \hat{w}(\zeta, t, \cdot) \|_{L^2(-k_0 + X^\nu)}. \tag{2.28}
\]
On \( \Gamma' \), the matrix \( \hat{w} \) is given by either \( v - I \) or \( m_0(v - I) m_0^{-1} \). Hence \( \| \hat{w}(\zeta, t, \cdot) \|_{L^2(\Gamma')} = O(\epsilon^2 \tau^{-1}) \) by the assumption (2.7). Moreover, by (B.2), \( m^X(q, z) = I + O\left(\frac{1}{\epsilon}\right) \) as \( z \to \infty \) uniformly with respect to the argument of \( z \) and \( q \) in compact subsets of \( \mathbb{D} \). Hence, as the entries of \( D(\zeta, t) \) have unit modulus,

\[
\| m_0(\zeta, t, k)^{-1} - I \|_{L^p([k-k_0]=\varepsilon)} \leq \left\| D(\zeta, t) \left[ m^X \left( \frac{\sqrt{\tau}}{\epsilon} (k-k_0) \right) - I \right] D(\zeta, t)^{-1} \right\|_{L^p([k-k_0]=\varepsilon)} = \begin{cases} O(\epsilon^{1/p} \tau^{1/2}), & p \in [1, \infty), \\ O(\tau^{-1/2}), & p = \infty, \end{cases}
\]

(2.29)

uniformly with respect to \( \zeta \in \mathcal{I} \); the third term on the right-hand side of (2.28) satisfies a similar estimate. The last two terms in (2.28) can be estimated using (2.19). This yields (2.26a). The proof of (2.26b) is similar. In order to prove (2.27), we note that (2.19) implies

\[
\| \hat{w}(\zeta, t, \cdot) \|_{L^p(k_0+X')} = O\left(\tau^{-\frac{q}{2}} \left( \int_{k_0+X'} e^{-\frac{p\tau}{24\pi^2} |k-k_0|^2 |dk|} \right)^{\frac{1}{p}} \right) = O\left(\tau^{-\frac{q}{2}} \left( \int_{0}^{\epsilon} e^{-\frac{p\tau}{24\pi^2} u^2} du \right)^{\frac{1}{p}} \right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}.
\]

(2.30)

Letting \( v = \frac{p\tau}{24\pi^2} u^2 \) we find

\[
\int_{0}^{\epsilon} e^{-\frac{p\tau}{24\pi^2} u^2} du \leq \int_{0}^{\infty} e^{-\frac{p\tau}{24\pi^2} u^2} du = \frac{\sqrt{6}}{\sqrt{p\tau}} \int_{0}^{\infty} e^{-v} dv = \frac{6\pi}{\sqrt{p\tau}}.
\]

(2.31)

Equations (2.30) and (2.31) yield (2.27). □

Let \( \hat{C} \) denote the Cauchy operator associated with \( \hat{\Gamma} \):

\[
(\hat{C} f)(z) = \frac{1}{2\pi i} \int_{\hat{\Gamma}} \frac{f(s)}{s - z} ds, \quad z \in \mathbb{C} \setminus \hat{\Gamma}.
\]

We define \( \hat{C}_{\hat{w}} : L^2(\hat{\Gamma}) + L^\infty(\hat{\Gamma}) \to L^2(\hat{\Gamma}) \) by \( \hat{C}_{\hat{w}} f = \hat{C}_- (f \hat{w}) \), i.e. \( \hat{C}_{\hat{w}} \) is defined by equation (A.4) where we have chosen, for simplicity, \( \hat{w}^+ = \hat{w} \) and \( \hat{w}^- = 0 \).

**Claim 3.** There exists a \( T > 0 \) such that \( I - \hat{C}_{\hat{w}(\zeta, t, \cdot)} \in B(L^2(\hat{\Gamma})) \) is invertible for all \( (\zeta, t) \in \mathcal{I} \times (0, \infty) \) with \( \tau > T \).

**Proof of Claim 3.** By (A.5) and (2.26b),

\[
\| \hat{C}_{\hat{w}} \|_{B(L^2(\hat{\Gamma}))} \leq C \| \hat{w} \|_{L^\infty(\hat{\Gamma})} = O(\tau^{-\frac{2}{2}}), \quad \tau \to \infty.
\]

(2.32)

This proves the claim. □

In view of Claim 3, we may define the \( 2 \times 2 \)-matrix valued function \( \hat{\mu}(\zeta, t, z) \) whenever \( \tau > T \) by

\[
\hat{\mu} = I + (I - \hat{C}_{\hat{w}})^{-1} \hat{C}_{\hat{w}} I \in I + L^2(\hat{\Gamma}).
\]

(2.33)

**Claim 4.** The function \( \hat{\mu}(\zeta, t, k) \) satisfies

\[
\| \hat{\mu}(\zeta, t, \cdot) - I \|_{L^2(\hat{\Gamma})} = O\left(\epsilon^2 \tau^{-\frac{2}{2}} \right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]

(2.34)

where the error term is uniform with respect to \( \zeta \in \mathcal{I} \).
Proof of Claim 4. Utilizing the Neumann series representation
\[
(I - \hat{\mathcal{C}}_w)^{-1} = \sum_{j=0}^{\infty} \hat{\mathcal{C}}_w^j
\]
we obtain
\[
\|(I - \hat{\mathcal{C}}_w)^{-1}\|_{B(L^2(\hat{\Gamma}))} \leq \sum_{j=0}^{\infty} \|\hat{\mathcal{C}}_w\|_{B(L^2(\hat{\Gamma}))}^j = \frac{1}{1 - \|\hat{\mathcal{C}}_w\|_{B(L^2(\hat{\Gamma}))}}.
\]
Assumption (2.2) together with the Sokhotski-Plemelj formula \(\hat{\mathcal{C}}_- = \frac{1}{2}(-I + S_{\hat{\Gamma}})\) show that \(\sup_{\zeta \in I} \|\mathcal{C}_-\|_{B(L^2(\hat{\Gamma}))} < \infty\). Thus,
\[
\|\hat{\mu} - I\|_{L^2(\hat{\Gamma})} = \|(I - \hat{\mathcal{C}}_w)^{-1}\hat{\mathcal{C}}_w I\|_{L^2(\hat{\Gamma})} \leq \|(I - \hat{\mathcal{C}}_w)^{-1}\|_{B(L^2(\hat{\Gamma}))}\|\hat{\mathcal{C}}_-(\hat{\nu})\|_{L^2(\hat{\Gamma})} \leq \frac{C\|\hat{\nu}\|_{L^2(\hat{\Gamma})}}{1 - \|\hat{\mathcal{C}}_w\|_{B(L^2(\hat{\Gamma}))}}.
\]
In view of (2.26a) and (2.32), this gives (2.34).
\[
\nabla
\]
By (13) and (2.18), if \( f \) obeys the symmetry \( f(k) = \overline{f(-k)} \), then so does \( \hat{C}_\omega(f) \). In view of (2.35), this implies that the operator \( (I - \hat{C}_\rho)^{-1} \) also preserves this symmetry. Thus \( \hat{\mu}(\zeta, t, k) = \hat{\mu}(\zeta, t, -k) \). Together with (2.18) this yields
\[
\left| \int_{|k-k_0| = \epsilon} \hat{\mu}(\zeta, t, k) \hat{w}(\zeta, t, k) dk \right| = \left| \int_{|k-k_0| = \epsilon} \hat{\mu}(\zeta, t, k) \hat{w}(\zeta, t, k) dk \right|.
\]
Hence, recalling that \( \hat{w} = m_0^{-1} - I \) on the circle \( |k-k_0| = \epsilon \),
\[
\lim_{k \to \infty} k(m(\zeta, t, k) - I) = -\frac{1}{2\pi i} \text{Re} \left( \int_{|k-k_0| = \epsilon} \hat{\mu}(\zeta, t, k)(m_0(\zeta, t, k)^{-1} - I) dk \right),
\]
and
\[
\lim_{|k-k_0| \to \infty} m_0(\zeta, t, k)^{-1} = \text{const.}
\]
for \( \tau \to \infty \).

By (B.2),
\[
m_0(\zeta, t, k)^{-1} = D(\zeta, t)m^X \left( q(\zeta), \frac{\sqrt{\tau}}{\epsilon}(k-k_0) \right)^{-1} D(\zeta, t)^{-1}
= I + \frac{B(\zeta, t)}{\sqrt{\tau}(k-k_0)} + O(\tau^{-1}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad |k-k_0| = \epsilon,
\]
where \( B(\zeta, t) \) is defined by
\[
B(\zeta, t) = -ie \left( 0 \quad -\beta^X(q(\zeta))e^{-t\phi(\zeta, 0)t^{-i\nu(\zeta)}} \right).
\]
Using (2.34) and (2.39) we find
\[
\left| \int_{|k-k_0| = \epsilon} \hat{\mu}(\zeta, t, k)(m_0(\zeta, t, k)^{-1} - I) dk \right| = \int_{|k-k_0| = \epsilon} (m_0(\zeta, t, k)^{-1} - I) dk
+ \int_{|k-k_0| = \epsilon} (\hat{\mu}(\zeta, t, k) - I)(m_0(\zeta, t, k)^{-1} - I) dk
= \frac{B(\zeta, t)}{\sqrt{\tau}} \int_{|k-k_0| = \epsilon} \frac{dk}{k-k_0} + O(\epsilon \tau^{-1}) + O \left( \|\hat{\mu}(\zeta, t, \cdot) - I\|_{L^2(\Gamma)} e^{\frac{3}{4} \tau^{-\frac{1}{2}}} \right)
= \frac{2\pi i B(\zeta, t)}{\sqrt{\tau}} + O(\epsilon \tau^{-\frac{1+\alpha}{2}}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]
uniformly with respect to \( \zeta \in \mathcal{I} \).

On the other hand,
\[
\left| \int_\Gamma \hat{\mu}(\zeta, t, k) \hat{w}(\zeta, t, k) dk \right| = \left| \int_\Gamma (\hat{\mu}(\zeta, t, k) - I) \hat{w}(\zeta, t, k) dk + \int_\Gamma \hat{w}(\zeta, t, k) dk \right|
\leq \|\hat{\mu} - I\|_{L^2(\Gamma)} \|\hat{w}\|_{L^2(\Gamma)} + \|\hat{w}\|_{L^1(\Gamma)}.
\]
The \( L^1 \)-norm of \( \hat{w} \) is \( O(\epsilon \tau^{-1}) \) on \( \Gamma' \) by (2.7) and is \( O(\epsilon \tau^{-\frac{1+\alpha}{2}}) \) on \( \{k_0 + x'\} \) by (2.27). Hence \( \|\hat{w}\|_{L^1(\Gamma)} = O(\epsilon \tau^{-\frac{1+\alpha}{2}}) \). Similarly, \( \|\hat{w}\|_{L^2(\Gamma)} = O(\epsilon^{1/2} \tau^{-1} + \epsilon^{\frac{3}{2}} \tau^{-\frac{1}{2} - \frac{\alpha}{2}}) \) by (2.7) and (2.27). Since \( \|\hat{\mu} - I\|_{L^2(\Gamma)} = O(\epsilon^{1/2} \tau^{-\frac{3}{2}}) \) by (2.34) and \( 1/2 \leq \alpha < 1 \), we infer that
\[
\left| \int_\Gamma \hat{\mu}(\zeta, t, k) \hat{w}(\zeta, t, k) dk \right| = O(\epsilon \tau^{-\frac{1+\alpha}{2}}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]
uniformly with respect to \( \zeta \in \mathcal{I} \). Equations (2.38), (2.40), and (2.41) imply (2.13). \( \Box \)
Remark 2.4. Certain steps in the above proof were inspired by the nice presentation of [8] (see also [10]). In particular, the definition (2.17) of \( \hat{v} \) is similar to equation (5.13) of [8]. However, whereas [8] utilizes a solution \( M_j \) of a RH problem with jump obtained by restricting \( v \) to a small cross in an \( \epsilon \) neighborhood of the critical point, we instead compare the solution \( m \) directly to \( m_0 \). This leads to a more straightforward presentation and circumvents some implicit difficulties in [8] related to the fact that \( M_j \) in general has singularities at the endpoints of the small cross.

3. INVERSE SCATTERING FOR THE mKDV EQUATION

Before we apply Theorem 2.1 to derive asymptotic formulas for the mKdV equation (1.2), we need to review how the solution of (1.2) with initial data \( u_0(x) \) can be expressed in terms of the solution of a Riemann-Hilbert problem.

Let

\[
U_0(x) = \begin{pmatrix} 0 & u_0(x) \\ u_0(x) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let \( X^+(x, k) \) and \( X^-(x, k) \) be the \( 2 \times 2 \) matrix valued solutions of the linear Volterra integral equations

\[
X^\pm(x, k) = I + \int_{\pm\infty}^x e^{ik(x'-x)\hat{\sigma}_3}(U_0X^\pm)(x, k)dx,
\]

where \( \hat{\sigma}_3 \) acts on a \( 2 \times 2 \) matrix \( A \) by \( \hat{\sigma}_3 A = [\sigma_3, A] \), i.e. \( e^{i\hat{\sigma}_3} A = e^{\sigma_3} A e^{-\sigma_3} \). Define the spectral function \( r(k) \) by

\[
r(k) = \frac{-b(k)}{a(k)}, \quad k \in \mathbb{R},
\]

where \( a(k) \) and \( b(k) \) are determined by the relation

\[
X^+(x, k) = X^-(x, k)e^{-ikx\hat{\sigma}_3}\begin{pmatrix} a(k) & b(k) \\ b(k) & a(k) \end{pmatrix}, \quad k \in \mathbb{R}.
\]

The inverse scattering transform formalism [1] [2] [6] implies that the solution \( u(x, t) \) of (1.2) with initial data \( u(x, 0) = u_0(x) \) is given by

\[
u(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{12},
\]

where \( M(x, t, k) \) is the unique solution of the \( L^2 \)-RH problem

\[
\begin{cases}
M(x, t, \cdot) \in I + \hat{E}^2(\mathbb{C} \setminus \mathbb{R}), \\
M_+(x, t, k) = M_-(x, t, k)J(x, t, k) \quad \text{for a.e. } k \in \mathbb{R},
\end{cases}
\]

with

\[
J(x, t, k) = \begin{pmatrix} 1 - |r(k)|^2 & -r(k)e^{-2ikx-8ik^3t} \\ r(k)e^{2ikx+8ik^3t} & 1 \end{pmatrix}, \quad k \in \mathbb{R}.
\]

The function \( r(k) \) satisfies

\[
r(k) = r(-k), \quad k \in \mathbb{R},
\]

and

\[
\sup_{k \in \mathbb{R}} |r(k)| < 1.
\]
An elaborate analysis of (3.1) shows that if
\[
\begin{align*}
    u_0 & \in C^{m+1}(\mathbb{R}), \\
    (1 + x)^n u_0^{(j)}(x) & \in L^1(\mathbb{R}), \quad i = 0, 1, \ldots, m + 1, \tag{3.8}
\end{align*}
\]
for some integers \( n, m \geq 1 \), then \( r \in C^m(\mathbb{R}) \) and
\[
    r^{(j)}(k) = O(k^{-m-j}), \quad |k| \to \infty, \quad k \in \mathbb{R}, \quad j = 0, 1, \ldots, n. \tag{3.9}
\]
If the initial data \( u_0(x) \) satisfy (3.8) for \( n = 1 \) and \( m = 4 \), then the limit in (3.4) exists for each \((x, t) \in \mathbb{R} \times [0, \infty)\) and defines a classical solution \( u(x, t) \) of (1.2) with initial data \( u(x, 0) = u_0(x) \), but lower regularity requirements are also possible.

Since the matrix \( \text{Re} J = \frac{1}{2}(J + J^T) \) is positive definite for \( k \in \mathbb{R} \), the homogeneous RH problem determined by \( (\mathbb{R}, J) \) has only the zero solution (see Theorem 9.3 of [15]). Hence Lemma A.2 implies that the \( L^2 \)-RH problem (3.5) has a unique solution whenever \( r \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \).

4. Asymptotics in the similarity sector

We use Theorem 2.1 to derive the asymptotics of the solution of the mKdV equation in the similarity sector. Let \( \zeta = x/t \) and define the variables \( k_0 = k_0(\zeta) \) and \( \tau = \tau(x, t) \) by
\[
    k_0 = \sqrt{\frac{-\zeta}{12}}, \quad \tau = tk_0^3. \tag{4.1}
\]

**Theorem 4.1.** Suppose \( r(k) \) is a function in \( C^{11}(\mathbb{R}) \) which satisfies (3.7) and
\[
    r^{(j)}(k) = O(k^{-4+2j}), \quad |k| \to \infty, \quad k \in \mathbb{R}, \quad j = 0, 1, 2.
\]
Then, for any \( \alpha \in [\frac{1}{2}, 1) \) and \( N > 0 \), the function \( u(x, t) \) defined by (3.4) satisfies
\[
    u(x, t) = \frac{1}{\sqrt{tk_0}} \left[ u_a(x, t) + O(\tau^{-\frac{\alpha}{2}}) \right], \quad \tau \to \infty, \quad -Nt < x < 0, \tag{4.2}
\]
where the error term is uniform with respect to \( x \) in the given range and the function \( u_a \) is defined by
\[
    u_a(x, t) = \sqrt{\frac{\nu(\zeta)}{3}} \cos \left( 16tk_0^3 - \nu(\zeta) \ln(192tk_0^3 + \phi(\zeta)) \right) \tag{4.3}
\]
with
\[
    \phi(\zeta) = \arg \Gamma(i\nu(\zeta)) + \frac{\pi}{4} - \arg r(k_0) - \frac{1}{\pi} \int_{-k_0}^{k_0} \ln \left( \frac{1 - |r(s)|^2}{1 - |r(k_0)|^2} \right) \frac{ds}{s - k_0}, \tag{4.4}
\]
\[
    \nu(\zeta) = -\frac{1}{2\pi} \ln(1 - |r(k_0)|^2). \tag{4.5}
\]

**Remark 4.2.** The conclusion of Theorem 4.1 can be stated more explicitly as follows: Given any \( \alpha \in [\frac{1}{2}, 1) \) and \( N > 0 \), there exist constants \( T > 0 \) and \( K > 0 \) such that the limit in (3.4) exists and the function \( u(x, t) \) defined by (3.4) satisfies
\[
    \left| u(x, t) - \frac{u_a(x, t)}{\sqrt{tk_0}} \right| \leq \frac{K}{\tau^{\frac{\alpha}{2}}}, \tag{4.6}
\]
whenever \((x, t) \in \mathbb{R} \times [0, \infty)\) satisfy \(-Nt < x < 0\) and \( \tau > T \).

**Remark 4.3.** The asymptotic regime considered in Theorem 4.1 is denoted by II in [4]. Our expression for the leading asymptotics (4.3) coincides with that given in [4], except that the expression for \( \phi(\zeta) \) in [4] contains \(-\pi/4\) instead of \(+\pi/4\). The discrepancy arises because our spectral function \( r(k) \) is related to the spectral function \( r_{DZ}(k) \) of [4] by \( r(k) = ir_{DZ}(k) \).
4.1. **Proof of Theorem 4.1** Let $N > 0$ be given and let $\mathcal{I}$ denote the interval $\mathcal{I} = [-N, 0)$. Let $M(x,t,\cdot) \in \mathcal{I} + \tilde{E^2}(\mathbb{C} \setminus \mathbb{R})$ denote the unique solution of the RH problem (3.5). The jump matrix $J$ defined in (3.6) involves the exponentials $e^{\pm \Phi(\zeta,k)}$ where

$$\Phi(\zeta,k) = 2ik\zeta + 8ik^3.$$ 

It follows that there are two stationary points located at the points where $\frac{d\Phi}{dk} = 0$, i.e. at $k = \pm k_0$. The real part of $\Phi$ is shown in Figure 2. In order to apply the steepest descent result of Theorem 2.1 we need to transform the RH problem in such a way that the jump matrix has decay everywhere as $t \to \infty$ except near the two stationary points. This can be achieved by performing an appropriate triangular factorization of the jump matrix followed by a contour deformation. For $|k| > k_0$, it is easy to achieve an appropriate factorization. By conjugating the RH problem (3.5), we can achieve an appropriate factorization also for $|k| < k_0$.

**Step 1: Conjugate.** Let

$$\Delta(\zeta,k) = \begin{pmatrix} \delta(\zeta,k)^{-1} & 0 \\ 0 & \delta(\zeta,k) \end{pmatrix},$$

where

$$\delta(\zeta,k) = e^{\frac{1}{2\pi} \int_{k_0}^{k_0} \ln(1-|r(s)|^2) \frac{ds}{s-k}} , \quad k \in \mathbb{C} \setminus \mathbb{R}. \quad (4.6)$$

The function $\delta$ satisfies the following jump condition across the real axis:

$$\delta_+(\zeta,k) = \begin{cases} \delta_-(\zeta,k), & \text{if } |k| > k_0, \\ \delta_-(\zeta,k)(1-|r(k)|^2), & \text{if } |k| < k_0, \end{cases} \quad k \in \mathbb{R}. \quad (4.7)$$
Moreover, the symmetry (3.7a) implies that
\[ \delta(\zeta, k) = \delta(\zeta, -k) = \delta(\zeta, -k)^{-1}. \] (4.8)

It follows that \( \Delta \) obeys the symmetries
\[ \Delta(\zeta, k) = \Delta(\zeta, -k) = \Delta(\zeta, -k)^{-1}. \] (4.9)

**Lemma 4.4.** The \( 2 \times 2 \)-matrix valued function \( \Delta(\zeta, k) \) satisfies
\[ \Delta(\zeta, \cdot), \Delta(\zeta, \cdot)^{-1} \in I + \hat{E}^2(\mathbb{C} \setminus \mathbb{R}) \cap E^\infty(\mathbb{C} \setminus \mathbb{R}), \]
for each \( \zeta \in \mathcal{I} \).

**Proof.** First note that
\[ \delta(\zeta, k) = \left( \frac{k - k_0}{k + k_0} \right)^{i\nu} e^{\chi(\zeta, k)}, \] (4.10)
where \( \nu(\zeta) \) is given by (4.5),
\[ \chi(\zeta, k) = \frac{1}{2\pi i} \int_{\mathbb{R}} \psi(\zeta, s) \frac{ds}{s - k}, \] (4.11)
and the function \( \psi(\zeta, s) \) is defined by
\[ \psi(\zeta, s) = \begin{cases} \ln \left( \frac{1 - |r(s)|^2}{1 - |r(k_0)|^2} \right), & -k_0 < s < k_0, \\ 0, & \text{otherwise}. \end{cases} \]

Since \( \psi(\zeta, \cdot) \in H^1(\mathbb{R}) \), we have \( \chi(\zeta, \cdot) \in E^\infty(\mathbb{C} \setminus \mathbb{R}) \) for each \( \zeta \in \mathcal{I} \); in fact, see Lemma 23.3 in [2],
\[ \sup_{\zeta \in \mathcal{I}} \sup_{k \in \mathbb{C} \setminus \mathbb{R}} |\chi(\zeta, \cdot)| < \sup_{\zeta \in \mathcal{I}} \|\psi(\zeta, \cdot)\|_{H^1(\mathbb{R})} < \infty. \] (4.12)

Hence \( \delta(\zeta, \cdot), \delta(\zeta, \cdot)^{-1} \in E^\infty(\mathbb{C} \setminus \mathbb{R}) \) for each \( \zeta \in \mathcal{I} \).

Since \( \chi(\zeta, k) = O(k^{-1}) \) uniformly as \( k \to \infty \), Lemma A.4 shows that \( \delta(\zeta, \cdot), \delta(\zeta, \cdot)^{-1} \in 1 + \hat{E}^2(\mathbb{C} \setminus \mathbb{R}). \)

By Lemma 4.4 and Lemma A.5, the function \( \bar{M} \) defined by
\[ \bar{M}(x, t, k) = M(x, t, k)\Delta(\zeta, k) \]
satisfies the \( L^2 \)-RH problem
\[ \begin{cases} \bar{M}(x, t, \cdot) \in I + \hat{E}^2(\mathbb{C} \setminus \mathbb{R}), \\ \bar{M}(x, t, k) = \bar{M}(x, t, k)\bar{J}(x, t, k) \end{cases} \]
for a.e. \( k \in \mathbb{R}, \)
where
\[ \bar{J}(x, t, k) = \Delta^{-1}(\zeta, k)\bar{J}(x, t, k)\Delta(\zeta, k) \]
\[ = \begin{pmatrix} \frac{\delta_-(\zeta, k)(1 - |r(k)|^2)}{\delta_+(\zeta, k)} & \frac{-\delta_-(\zeta, k)\delta_+(\zeta, k)r(k)e^{-i\Phi(\zeta, k)}}{\delta_+(\zeta, k)} \\ \frac{-\delta_+(\zeta, k)r(k)e^{i\Phi(\zeta, k)}}{\delta_+(\zeta, k)} & \frac{\delta_+(\zeta, k)}{\delta_-(\zeta, k)} \end{pmatrix}, \quad k \in \mathbb{R}. \] (4.13)

In view of the jump (4.7) of \( \delta(\zeta, k) \), this gives, for \( k \in \mathbb{R}, \)
\[ \bar{J}(x, t, k) = \begin{cases} \begin{pmatrix} 1 - |r(k)|^2 & -\delta(\zeta, k)^2r(k)e^{-i\Phi(\zeta, k)} \\ \delta(\zeta, k)^2r(k)e^{i\Phi(\zeta, k)} & 1 \end{pmatrix}, & |k| > k_0, \\ \begin{pmatrix} \frac{1}{1 - |r(k)|^2} & \delta_+(\zeta, k)^2r(k)e^{-i\Phi(\zeta, k)} \\ \delta_-(\zeta, k)^2r(k)e^{i\Phi(\zeta, k)} & 1 - |r(k)|^2 \end{pmatrix}, & |k| < k_0. \end{cases} \]
The upshot of the above conjugation is that we can now factorize the jump matrix as follows:

\[
\tilde{J} = \begin{cases} 
B_u^{-1}B_l, & |k| > k_0, \quad k \in \mathbb{R}, \\
B_l^{-1}b_u, & |k| < k_0, \quad k \in \mathbb{R}, 
\end{cases} 
\] (4.14)

where

\[
B_l = \begin{pmatrix} 1 & 0 \\
\delta_-(\zeta, k)^{-2}r_1(k)e^{\Phi(\zeta, k)} & 1 \end{pmatrix}, \quad b_u = \begin{pmatrix} 1 & -\delta_+(\zeta, k)^2r_2(k)e^{-\Phi(\zeta, k)} \\
0 & 1 \end{pmatrix}, 
\]

\[
b_l = \begin{pmatrix} 1 & 0 \\
\delta_-(\zeta, k)^{-2}r_3(k)e^{\Phi(\zeta, k)} & 1 \end{pmatrix}, \quad B_u = \begin{pmatrix} 1 & \delta_+(\zeta, k)^2r_4(k)e^{-\Phi(\zeta, k)} \\
0 & 1 \end{pmatrix}, 
\] (4.15)

and the functions \( \{ r_j(k) \}_{j=1}^4 \) are defined by

\[
\begin{align*}
  r_1(k) &= r(k), & r_2(k) &= \frac{r(k)}{1 - r(k)r(k)}, \\
  r_3(k) &= \frac{r(k)}{1 - r(k)r(k)}, & r_4(k) &= \overline{r(k)}. 
\end{align*}
\]

Our next goal is to deform the contour. However, we first need to introduce analytic approximations of \( \{ r_j(k) \}_{j=1}^4 \).

**Step 2: Introduce analytic approximations.** The following lemma describes how to decompose \( r_j, j = 1, \ldots, 4 \), into an analytic part \( r_{j,a} \) and a small remainder \( r_{j,r} \). We introduce open domains \( U_j = U_j(\zeta), j = 1, \ldots, 4 \), as in Figure 3 so that

\[
\{ k \mid \text{Re} \Phi(\zeta, k) < 0 \} = U_1 \cup U_3, \quad \{ k \mid \text{Re} \Phi(\zeta, k) > 0 \} = U_2 \cup U_4. 
\] (4.16)

**Lemma 4.5.** There exist decompositions

\[
r_j(k) = \begin{cases} 
  r_{j,a}(x, t, k) + r_{j,r}(x, t, k), & j = 1, 4, \quad |k| > k_0, \quad k \in \mathbb{R}, \\
  r_{j,a}(x, t, k) + r_{j,r}(x, t, k), & j = 2, 3, \quad |k| < k_0, \quad k \in \mathbb{R}, 
\end{cases}
\]

where the functions \( \{ r_{j,a}, r_{j,r} \}_{j=1}^4 \) have the following properties:

(a) For each \( \zeta \in \mathcal{I} \) and each \( t > 0 \), \( r_{j,a}(x, t, k) \) is defined and continuous for \( k \in \bar{U}_j \) and analytic for \( k \in U_j, j = 1, \ldots, 4 \).
(b) The functions \( \{r_{j,a}\}^4_1 \) satisfy
\[
|r_{j,a}(x, t, k) - r_j(k_0)| \leq C|k - k_0| \epsilon^{\frac{4}{3}}|\text{Re} \Phi(\zeta, k)|,
\]
where the constant \( C \) is independent of \( \zeta, t, k \).

(c) The functions \( r_{1,a} \) and \( r_{4,a} \) satisfy
\[
|r_{j,a}(x, t, k)| \leq \frac{C}{1 + |k|} \epsilon^{\frac{2}{3}}|\text{Re} \Phi(\zeta, k)|, \quad k \in \bar{U}_j, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad j = 1, 4,
\]
where the constant \( C \) is independent of \( \zeta, t, k \).

(d) The \( L^1, L^2, \) and \( L^\infty \) norms on \((-\infty, -k_0) \cup (k_0, \infty)\) of the functions \( r_{1,r}(x, t, \cdot) \) and \( r_{4,r}(x, t, \cdot) \) are \( O(t^{-3/2}) \) as \( t \to \infty \) uniformly with respect to \( \zeta \in \mathcal{I} \).

(e) The \( L^1, L^2, \) and \( L^\infty \) norms on \((-k_0, k_0)\) of the functions \( r_{2,r}(x, t, \cdot) \) and \( r_{3,r}(x, t, \cdot) \) are \( O(t^{-3/2}) \) as \( t \to \infty \) uniformly with respect to \( \zeta \in \mathcal{I} \).

(f) The following symmetries are valid:
\[
r_{j,a}(\zeta, t, k) = r_{j,a}(\zeta, t, -k), \quad r_{j,r}(\zeta, t, k) = r_{j,r}(\zeta, t, -k), \quad j = 1, \ldots, 4.
\]

Proof. We will derive decompositions of \( r_1(k) \) and \( r_3(k) \); the decompositions of \( r_2(k) \) and \( r_4(k) \) can be obtained from these by Schwartz conjugation.

**Decomposition of** \( r_1(k) \). Let \( U_1 = U_1^+ \cup U_1^- \), where \( U_1^+ \) and \( U_1^- \) denote the parts of \( U_1 \) in the right and left half-planes respectively. We will derive a decomposition of \( r_1 \) in \( U_1^+ \) and then extend it to \( U_1^- \) by symmetry.

Since \( r \in C^{1,1}(\mathbb{R}) \), there exist functions \( \{p_j(\zeta)\}^6_0 \) such that
\[
r_1^{(n)}(k) = \frac{dn}{dk^n} \left( \sum_{j=0}^6 p_j(\zeta)(k - k_0)^j \right) + O((k - k_0)^{7-n}), \quad k \to k_0, \quad k \in \mathbb{R}, \quad n = 0, 1, 2.
\]

Let
\[
f_0(\zeta, k) = \frac{1}{10} \sum_{j=1}^{10} \frac{a_j(\zeta)}{(k - i)^j},
\]
where \( \{a_j(\zeta)\}^{10}_4 \) are such that
\[
f_0(\zeta, k) = \sum_{j=0}^6 p_j(\zeta)(k - k_0)^j + O((k - k_0)^7), \quad k \to k_0,
\]
for each \( \zeta \in \mathcal{I} \). It is easy to verify that \( f_0(\zeta, k) \) imposes seven linear conditions on the \( a_j(\zeta) \)'s that are linearly independent for each \( \zeta \in \mathcal{I} \); hence the coefficients \( a_j(\zeta) \) exist and are unique. The \( a_j(\zeta) \)'s are polynomials in \( \{p_j(\zeta)\}^6_0 \) with coefficients that are polynomials in \( k_0 \). Thus, since the interval \( \mathcal{I} \) is bounded, we have \( \sup_{\zeta \in \mathcal{I}} |a_j(\zeta)| < \infty \) for each \( j \).

Let \( f(\zeta, k) = r_1(k) - f_0(\zeta, k) \). The following properties hold:

\( (i) \) For each \( \zeta \in \mathcal{I} \), \( f_0(\zeta, k) \) is a rational function of \( k \in \mathbb{C} \) with no poles in \( U_1^+ \).

\( (ii) \) \( f_0(\zeta, k) \) coincides with \( r_1(k) \) to order six at \( k_0 \) and to order three at \( \infty \); more precisely
\[
\frac{\partial^n f}{\partial k^n}(\zeta, k) = \begin{cases} O((k - k_0)^{7-n}), & k \to k_0, \\ O(k^{-4+2n}), & k \to \infty, \end{cases} \quad k \in \mathbb{R}, \quad \zeta \in \mathcal{I}, \quad n = 0, 1, 2,
\]
where the error terms are uniform with respect to \( \zeta \in \mathcal{I} \).

The decomposition of \( r_1(k) \) can now be derived as follows. For each \( \zeta \in \mathcal{I} \), the map \( k \mapsto \phi = \phi(\zeta, k) \) where
\[
\phi = -i\Phi(\zeta, k) = -24k_0^2k + 8k^3.
\]
is a bijection \([k_0, \infty) \to [-16k_0^3, \infty)\), so we may define a function \(F(\zeta, \phi)\) by
\[
F(\zeta, \phi) = \begin{cases} \frac{(k-i)^3}{k-k_0} f(\zeta, k), & \phi \geq -16k_0^3, \\ 0, & \phi < -16k_0^3, \end{cases} \quad \zeta \in \mathcal{I}, \quad \phi \in \mathbb{R}.
\] (4.23)

For each \(\zeta \in \mathcal{I}\), the function \(F(\zeta, \cdot)\) is smooth for \(\phi \neq -16k_0^3\) and
\[
\frac{\partial^n F}{\partial \phi^n}(\zeta, \phi) = \left( \frac{1}{24(k^2-k_0^2)} \frac{\partial}{\partial k} \right)^n \left[ \frac{(k-i)^3}{k-k_0} f(\zeta, k) \right], \quad \phi \geq -16k_0^3.
\] (4.24)

Equations (4.21) and (4.24) together with the trivial inequalities
\[
\frac{k}{k+k_0} \leq 1 \quad \text{and} \quad \frac{k-k_0}{k+k_0} \leq 1 \quad \text{for} \quad k \geq k_0,
\] (4.25)
show that \(F(\zeta, \cdot) \in C^1(\mathbb{R})\) for each \(\zeta\) and that
\[
\left| \frac{\partial^n F}{\partial \phi^n}(\zeta, \phi) \right| \leq \frac{C}{(1+|\phi|)^\frac{3}{2}}, \quad \phi \in (-16k_0^3, \infty), \quad \zeta \in \mathcal{I}, \quad n = 0, 1, 2.
\] Hence
\[
\sup_{\zeta \in \mathcal{I}} \left\| \frac{\partial^n F}{\partial \phi^n}(\zeta, \cdot) \right\|_{L^2(\mathbb{R})} < \infty, \quad n = 0, 1, 2.
\] (4.26)

In particular, \(F(\zeta, \cdot)\) belongs to the Sobolev space \(H^2(\mathbb{R})\) for each \(\zeta \in \mathcal{I}\). We conclude that the Fourier transform \(\hat{F}(\zeta, s)\) defined by
\[
\hat{F}(\zeta, s) = \frac{1}{2\pi} \int_\mathbb{R} F(\zeta, \phi) e^{-i\phi s} d\phi,
\] (4.27)
satisfies
\[
F(\zeta, \phi) = \int_\mathbb{R} \hat{F}(\zeta, s) e^{i\phi s} ds
\] (4.28)
and
\[
\sup_{\zeta \in \mathcal{I}} \|s^2 \hat{F}(\zeta, s)\|_{L^2(\mathbb{R})} < \infty.
\] (4.29)

Equations (4.23) and (4.28) imply
\[
k-k_0 (k-i)^3 \int_\mathbb{R} \hat{F}(\zeta, s) e^{s\Phi(\zeta,k)} ds = \begin{cases} f(\zeta, k), & k \geq k_0, \\ 0, & k < k_0, \end{cases} \quad \zeta \in \mathcal{I}.
\]
Writing
\[
f(\zeta, k) = f_a(x, t, k) + f_r(x, t, k), \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \geq k_0,
\]
where the functions \(f_a\) and \(f_r\) are defined by
\[
f_a(x, t, k) = \frac{k-k_0}{(k-i)^3} \int_{-\frac{t}{4}}^{\frac{t}{4}} \hat{F}(\zeta, s) e^{s\Phi(\zeta,k)} ds, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \in \bar{U}_1^+,
\]
\[
f_r(x, t, k) = \frac{k-k_0}{(k-i)^3} \int_{-\frac{t}{4}}^{-\frac{t}{4}} \hat{F}(\zeta, s) e^{s\Phi(\zeta,k)} ds, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \geq k_0,
\]
we infer that \(f_a(x, t, \cdot)\) is continuous in \(\bar{U}_1^+\) and analytic in \(U_1^+\). Furthermore,
\[
|f_a(x, t, k)| \leq \frac{|k-k_0|}{|k-i|^3} \|\hat{F}(\zeta, \cdot)\|_{L^1(\mathbb{R})} \sup_{s \geq -\frac{t}{4}} e^{s\text{Re} \Phi(\zeta,k)} \leq \frac{C|k-k_0|}{|k-i|^3} \frac{e^{\frac{t}{4}|\text{Re} \Phi(\zeta,k)|}}{t}, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \in \bar{U}_1^+,
\] (4.30)
and

\[ |f_r(x, t, k)| \leq \left| \frac{k - k_0}{k - k_0^2} \right| \int_{-\infty}^{-\frac{t}{2}} s^2 |\hat{F}(\zeta, s)| s^{-2} ds \leq \frac{C}{1 + |k|^2} \|s^2 \hat{F}(\zeta, s)\|_{L^2(\mathbb{R})} \int_{-\infty}^{-\frac{t}{2}} s^{-4} ds \]

\[ \leq \frac{C}{1 + |k|^2} t^{-3/2}, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \geq k_0. \] (4.31)

Hence the \( L^1, \) \( L^2, \) and \( L^\infty \) norms of \( f_r \) on \( (k_0, \infty) \) are \( O(t^{-3/2}) \) uniformly with respect to \( \zeta \in \mathcal{I}. \) Letting

\[ r_{1,a}(x, t, k) = f_0(\zeta, k) + f_a(x, t, k), \quad k \in \bar{U}_1^+, \]
\[ r_{1,\tau}(x, t, k) = f_\tau(x, t, k), \quad k \geq k_0. \]

we find a decomposition of \( r_1 \) for \( k > k_0 \) with the properties listed in the statement of the lemma. We use the symmetry (4.19) to extend this decomposition to \( k < -k_0. \)

**Decomposition of \( r_3.** Following [4], we split \( r_3(k) \) into even and odd parts as follows:

\[ r_3(k) = r_+(k^2) + kr_-(k^2), \quad k \in \mathbb{R}, \]

where \( r_\pm : [0, \infty) \to \mathbb{C} \) are defined by

\[ r_+(s) = \frac{r_3(\sqrt{s}) + r_3(-\sqrt{s})}{2}, \quad r_-(s) = \frac{r_3(\sqrt{s}) - r_3(-\sqrt{s})}{2\sqrt{s}}, \quad s \geq 0. \]

The symmetry \( r_3(k) = \overline{r_3(-k)} \) implies

\[ r_+(s) = \text{Re} r_3(\sqrt{s}), \quad r_-(s) = \frac{i \text{Im} r_3(\sqrt{s})}{\sqrt{s}}, \quad s \geq 0. \]

If \( \{q_j\}_{0}^{10} \) denote the coefficients in the Taylor series expansion

\[ r_3(k) = \sum_{j=0}^{10} q_j k^j + \frac{1}{10!} \int_{0}^{k} r_3^{(11)}(t) (k - t)^{10} dt, \]

then

\[ r_+(s) = \sum_{i=0}^{5} q_{2i}s^i + \frac{1}{2 \cdot 10!} \int_{0}^{\sqrt{s}} (r_3^{(11)}(t) - r_3^{(11)}(-t))(\sqrt{s} - t)^{10} dt, \] (4.32a)
\[ r_-(s) = \sum_{i=0}^{4} q_{2i+1}s^i + \frac{1}{2 \cdot 10! \sqrt{s}} \int_{0}^{\sqrt{s}} (r_3^{(11)}(t) + r_3^{(11)}(-t))(\sqrt{s} - t)^{10} dt. \] (4.32b)

Since \( r_3 \in C^{11}(\mathbb{R}), \) the equations (4.32) show that the derivatives \( r_\pm^{(j)}(s) \) are bounded on \( [0, \infty) \) for \( 0 \leq j \leq 5. \) The Taylor representations

\[ r_\pm(k^2) = \sum_{j=0}^{4} p_j^\pm(\zeta)(k^2 - k_0^2)^j + \frac{1}{4!} \int_{k_0^2}^{k^2} r_\pm^{(5)}(t)(k^2 - t)^{4} dt, \]

then show that the function \( f_0(\zeta, k) \) defined by

\[ f_0(\zeta, k) = \sum_{j=0}^{4} p_j^+(\zeta)(k^2 - k_0^2)^j + k \sum_{j=0}^{4} p_j^-\zeta)(k^2 - k_0^2)^j \]

has the following properties:

(i) \( f_0(\zeta, k) \) is a polynomial in \( k \in \mathbb{C} \) whose coefficients are bounded functions of \( \zeta \in \mathcal{I}. \)
(ii) The difference \( f(\zeta, k) = r_3(k) - f_0(\zeta, k) \) satisfies
\[
\frac{\partial^n f}{\partial k^n}(\zeta, k) \leq C|k^2 - k_0^2|^{5-n}, \quad \zeta \in \mathcal{I}, \quad -k_0 \leq k \leq k_0, \quad n = 0, 1, 2,
\]
where \( C \) is independent of \( \zeta \) and \( k \).

(iii) \( f_0(\zeta, k) = f_0(\zeta, -k) \) for \( k \in \mathbb{C} \).

The decomposition of \( r_3(k) \) can now be derived as follows. The function \( k \mapsto \phi \) defined in (4.22) is a bijection \([-k_0, k_0] \rightarrow [-16k_0^3, 16k_0^3] \), so we may define a function \( F(\zeta, \phi) \) by
\[
F(\zeta, \phi) = \begin{cases} 
\frac{1}{k^2 - k_0^2} f(\zeta, k), & |\phi| \leq 16k_0^3, \\
0, & |\phi| > 16k_0^3,
\end{cases} \quad \zeta \in \mathcal{I}, \quad \phi \in \mathbb{R}.
\]

For each \( \zeta \in \mathcal{I} \), the function \( F(\zeta, \phi) \) is smooth for \( \phi \neq \pm 16k_0^3 \) and
\[
\frac{\partial^n F}{\partial \phi^n}(\zeta, \phi) = \left( \frac{1}{24(k^2 - k_0^2)} \frac{\partial}{\partial k} \right)^n f(\zeta, k), \quad |\phi| \leq 16k_0^3.
\]
Equations (4.33) and (4.35) show that \( F(\zeta, \cdot) \in C^1(\mathbb{R}) \) for each \( \zeta \) and that
\[
\frac{|\partial^n F|}{|\partial \phi^n|}(\zeta, \phi) \leq C, \quad |\phi| \leq 16k_0^3, \quad \phi \in \mathbb{R}, \quad n = 0, 1, 2.
\]

Hence \( F \) satisfies (4.26) and the Fourier transform \( \hat{F}(\zeta, s) \) defined by (4.27) satisfies (4.28) and (4.29). Equations (4.28) and (4.34) imply
\[
(k^2 - k_0^2) \int_{\mathbb{R}} \hat{F}(\zeta, s)e^{s\Phi(\zeta,k)}ds = \begin{cases} 
f(\zeta, k), & |k| \leq k_0, \\
0, & |k| > k_0,
\end{cases} \quad \zeta \in \mathcal{I}, \quad k \in \mathbb{R}.
\]

Writing
\[
f(\zeta, k) = f_a(x, t, k) + f_r(x, t, k), \quad \zeta \in \mathcal{I}, \quad t > 0, \quad -k_0 \leq k \leq k_0,
\]
where the functions \( f_a \) and \( f_r \) are defined by
\[
f_a(x, t, k) = (k^2 - k_0^2) \int_{-\frac{1}{s}}^{\frac{1}{s}} \hat{F}(\zeta, s)e^{s\Phi(\zeta,k)}ds, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \in \bar{U}_3,
\]
\[
f_r(x, t, k) = (k^2 - k_0^2) \int_{-\infty}^{\frac{1}{s}} \hat{F}(\zeta, s)e^{s\Phi(\zeta,k)}ds, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad -k_0 \leq k \leq k_0,
\]
we infer that \( f_r(x, t, \cdot) \) is continuous in \( \bar{U}_3 \) and analytic in \( U_3 \). Estimating \( f_a \) and \( f_r \) as in (4.30) and (4.31), we find
\[
|f_a(\zeta, t, k)| \leq C|k^2 - k_0^2|e^{\frac{1}{4}|\text{Re}\Phi(\zeta,k)|}, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \in \bar{U}_3,
\]
\[
|f_r(\zeta, t, k)| \leq Ct^{-3/2}, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad -k_0 \leq k \leq k_0.
\]

It follows that
\[
r_{3,a}(x, t, k) = f_0(\zeta, k) + f_a(x, t, k), \quad r_{3,r}(x, t, k) = f_r(x, t, k),
\]
provides a decomposition of \( r_3 \) with the properties listed in the statement of the lemma. The symmetries (4.19) are satisfied since \( F(\zeta, \phi) = \overline{F(\zeta, -\phi)} \) for \( \phi \in \mathbb{R} \).

\[\nabla\]

**Step 3: Deform.** Let \( \Gamma \) be the contour consisting of \( \mathbb{R} \) together with the four half-lines
\[
\{k_0 + u e^{i\phi_0} | -\sqrt{2}k_0 < u < \infty\}, \quad \{-k_0 + u e^{i\phi_0} | -\sqrt{2}k_0 < u < \infty\},
\]
oriented as in Figure 4. Let \( \{V_j\}_0 \) be the open sets shown in Figure 4. Write
\[
B_t = B_{t,a}B_{t,a}, \quad b_u = b_{u,r}b_{u,a}, \quad b_l = b_{l,r}b_{l,a}, \quad B_u = B_{u,r}B_{u,a},
\]
where \( \{B_{l,a}, b_{u,a}, b_{l,a}, B_{u,a}\} \) and \( \{B_{l,r}, b_{u,r}, b_{l,r}, B_{u,r}\} \) denote the matrices \( \{B_{l}, b_{u}, b_{l}, B_{u}\} \) with \( \{r_j(k)\}_4^1 \) replaced with \( \{r_{ja}(k)\}_4^1 \) and \( \{r_{jr}(k)\}_4^1 \), respectively. The estimates (4.17) and (4.18) together with Lemma A.4 imply that

\[
\begin{align*}
B_{l,a}(x,t,\cdot), B_{l,a}^{-1}(x,t,\cdot) &\in I + \dot{\mathcal{E}}^2(V) \cap E^\infty(V), \\
b_{l,a}(x,t,\cdot), b_{u,a}^{-1}(x,t,\cdot) &\in I + \dot{\mathcal{E}}^2(V) \cap E^\infty(V), \\
b_{l,a}(x,t,\cdot), b_{u,a}^{-1}(x,t,\cdot) &\in I + \dot{\mathcal{E}}^2(V) \cap E^\infty(V), \\
B_{u,a}(x,t,\cdot), B_{u,a}^{-1}(x,t,\cdot) &\in I + \dot{\mathcal{E}}^2(V) \cap E^\infty(V),
\end{align*}
\]

for each \( \zeta \in \mathcal{I} \) and each \( t > 0 \). Hence, we may apply Lemma A.5 to deduce that the function \( m(x,t,k) \) defined by

\[
m(x,t,k) = \begin{cases} 
\tilde{M}(x,t,k)B_{l,a}(x,t,k)^{-1}, & k \in V_1, \\
\tilde{M}(x,t,k)b_{u,a}(x,t,k)^{-1}, & k \in V_3, \\
\tilde{M}(x,t,k)b_{l,a}(x,t,k)^{-1}, & k \in V_4, \\
\tilde{M}(x,t,k)B_{u,a}(x,t,k)^{-1}, & k \in V_6, \\
\tilde{M}(x,t,k), & \text{elsewhere,}
\end{cases}
\]

satisfies the \( L^2 \)-RH problem

\[
\begin{align*}
m(x,t,\cdot) &\in I + \dot{\mathcal{E}}^2(\mathcal{C} \setminus \Gamma), \\
m_+(x,t,k) = m_-(x,t,k)v(x,t,k) &\text{ for a.e. } k \in \Gamma,
\end{align*}
\]

Figure 4. The jump contour \( \Gamma \) and the open sets \( \{V_j\}_1^6 \).
where, in view of (4.13) and (4.14), the jump matrix $v$ is given by

$$
v = \begin{cases}
B_{t,a} &= \begin{pmatrix} 1 & \delta(\zeta,k) - 2r_{1,a}(x,t,k)e^{\Phi(\zeta,k)} & 0 \\
0 & 1 & 1 \end{pmatrix}, & k \in \mathcal{V}_1 \cap \mathcal{V}_2, \\
b_{u,a} &= \begin{pmatrix} 1 & -\delta(\zeta,k)^2 r_{2,a}(x,t,k)e^{-\Phi(\zeta,k)} \\
0 & 1 \end{pmatrix}, & k \in \mathcal{V}_2 \cap \mathcal{V}_3, \\
b_{l,a} &= \begin{pmatrix} 1 & -\delta(\zeta,k)^2 r_{3,a}(x,t,k)e^{\Phi(\zeta,k)} \\
0 & 1 \end{pmatrix}, & k \in \mathcal{V}_4 \cap \mathcal{V}_5, \\
B_{u,a} &= \begin{pmatrix} 1 & \delta(\zeta,k) - 2r_{4,a}(x,t,k)e^{-\Phi(\zeta,k)} \\
0 & 1 \end{pmatrix}, & k \in \mathcal{V}_5 \cap \mathcal{V}_6, \\
B^{-1}_{u,a}B_{l,r}, & k \in \mathcal{V}_1 \cap \mathcal{V}_6, \\
b^{-1}_{l,a}b_{u,r}, & k \in \mathcal{V}_3 \cap \mathcal{V}_4.
\end{cases}
$$

From the symmetries (4.8) and (4.19), we infer that $v$ satisfies

$$v(x,t,k) = v(x,t,-k), \quad k \in \Gamma.$$  \hspace{1cm} (4.39)

**Step 4: Apply Theorem 2.1.** We verify that Theorem 2.1 can be applied to the interval $I = [-N,0]$, the contour $\Gamma$, and the jump matrix $v$ with

$$\epsilon = \frac{k_0}{2}, \quad \rho = \epsilon \sqrt{-i \frac{\partial^2 \Phi}{\partial k^2}(\zeta,k_0)} = \epsilon \sqrt{48k_0}, \quad \tau = t \rho^2 = 12k_0^3 t,$$

$$q(\zeta) = e^{-2\chi(\zeta,k_0)}r(k_0)e^{2\nu(\zeta)\ln(2\sqrt{3k_0}^3/2)},$$

$$\phi(\zeta,z) = \Phi(\zeta,k_0 + \frac{\epsilon}{\rho}z) = -16ik_0^3 + iz^2 + \frac{iz^3}{12\rho}.$$

The contours $\Gamma$ and $\Gamma'$ are shown in Figure 3. The conditions (Γ1)-(Γ4) are clearly satisfied. Since the contour $k_0^{-1}\Gamma$ is independent of $\zeta$, a scaling argument shows that $\|S_{\Gamma}\|_{BL^2(\Gamma)}$ is independent of $\zeta$. In particular, $S_{\Gamma}$ is uniformly bounded on $L^2(\Gamma)$. Equation (2.4) follows from (4.38) and the estimates in Lemma 4.5. Clearly $\det v = 1$. The symmetry condition (2.6) follows from (4.39).

We next verify (2.7). Let $w = v - I$ and let $\Gamma'$ denote the contour obtained from $\Gamma$ by removing the crosses $\pm k_0 + X'$. By parts (d) and (e) of Lemma 4.5 the $L^1$, $L^2$, and $L^\infty$ norms of $w$ are $O(t^{-\frac{3}{4}})$ on $\Re$ uniformly with respect to $\zeta \in I$. Let $\gamma$ denote the part of $\Gamma'$ that belongs to the line $k_0 + \Re e^{\frac{i\pi}{4}}$, i.e.

$$\gamma = \left\{ k_0 + u e^{\frac{i\pi}{4}} \mid u \in \left( -\sqrt{2}k_0, -\frac{k_0}{2} \right) \cup \left[ \frac{k_0}{2}, \infty \right) \right\}.$$ 

Let $k = k_0 + u e^{\frac{i\pi}{4}}$. Then

$$\Re \Phi(\zeta,k) = -4u^2(6k_0 + \sqrt{2}u) < -16u^2k_0 \quad \text{for} \quad -\sqrt{2}k_0 < u < \infty.$$ 

(4.40)

Together with (4.18) this yields

$$|r_{1,a}(x,t,k)e^{\Phi(\zeta,k)}| \leq Ce^{-\frac{3}{4}|\Re \Phi(\zeta,k)|} \leq Ce^{-12u^2k_0} \leq \frac{Ctu^2k_0e^{-12tu^2k_0}}{tu^2k_0} \leq \frac{C}{tu^2k_0} \leq \frac{Ck_0^2}{u^2\tau}, \quad \frac{k_0}{2} < u < \infty.$$
Similarly, by (4.17),
\[ |r_{3,a}(x, t, k)e^{i\Phi(\zeta, k)}| \leq C e^{\frac{-u}{\tau}}|\text{Re} \Phi(\zeta, k)| \leq \frac{C k_0^2}{u^2}, -\sqrt{2}k_0 < u < -\frac{k_0}{2}. \]

Hence
\[ \|w\|_{L^1(\gamma)} \leq \frac{C k_0^2}{\tau} \left( \int_{-\sqrt{2}k_0}^{-\frac{k_0}{2}} + \int_{\frac{k_0}{2}}^\infty \right) u^{-2} du = O(k_0 \tau^{-1}), \]
\[ \|w\|_{L^2(\gamma)} \leq \frac{C k_0^2}{\tau} \sqrt{\left( \int_{-\sqrt{2}k_0}^{-\frac{k_0}{2}} + \int_{\frac{k_0}{2}}^\infty \right) u^{-4} du} = O(k_0^{\frac{3}{2}} \tau^{-1}), \]
\[ \|w\|_{L^\infty(\gamma)} = O(\tau^{-1}). \]

This shows that the estimates in (2.7) hold also on $\gamma$. Since similar arguments apply to the remaining parts of $\Gamma'$, this verifies (2.7).

Equation (4.38) implies that (2.8) and (2.9) are satisfied with
\[
\begin{align*}
R_1(\zeta, t, z) &= \delta(\zeta, k)^{-2}r_{1,a}(x, t, k) z^{2i\nu(\zeta)}, \\
R_2(\zeta, t, z) &= \delta(\zeta, k) r_{2,a}(x, t, k) z^{-2i\nu(\zeta)}, \\
R_3(\zeta, t, z) &= \delta(\zeta, k)^{-2}r_{3,a}(x, t, k) z^{2i\nu(\zeta)}, \\
R_4(\zeta, t, z) &= \delta(\zeta, k)^{-2}r_{4,a}(x, t, k) z^{-2i\nu(\zeta)},
\end{align*}
\]

where $k$ and $z$ are related by $k = k_0 + \frac{z}{\tau}$.

It is clear that $\phi$ satisfies (2.10) and (2.11c). The following estimate proves (2.11b):
\[ \text{Re} \phi(\zeta, z) = \frac{|z|^2}{2} \left( 1 \pm \frac{|z|}{6\sqrt{2}\rho} \right) \geq \frac{|z|^2}{4}, \quad z \in X_2^\rho \cup X_4^\rho, \quad \zeta \in \mathbb{I}, \]
where the plus and minus signs are valid for $z \in X_2^\rho$ and $z \in X_4^\rho$ respectively. The proof of (2.11a) is similar. Since $|q(\zeta)| = |r(k_0)|$, we have $\sup_{\zeta \in \mathbb{I}} |q(\zeta)| < 1$ and
\[ \nu(\zeta) = -\frac{1}{2\pi} \ln(1 - |r(k_0)|^2) = -\frac{1}{2\pi} \ln(1 - |q(\zeta)|^2). \]

Finally, we show that given any $\alpha \in [1/2, 1)$, there exists an $L > 0$ such that the inequalities (2.12) hold. Let $k = k_0 + \frac{z}{\rho}$. Using the expression (4.10) for $\delta(\zeta, k)$, we may write
\[ R_1(\zeta, t, z) = e^{-2\chi(\zeta, k)r_{1,a}(x, t, k)} e^{2i\nu(\zeta) \ln((k+k_0)\sqrt{\text{Im}k_0})}, \quad z \in \mathbb{X}_1. \]
Thus,
\[ R_1(\zeta, t, 0) = e^{-2\chi(\zeta, k_0)r_{1,a}(x, t, k_0)} e^{2i\nu(\zeta) \ln(2\sqrt{\text{Im}k_0}^3/2)}. \]
Now \( r_{1,a}(x, t, k_0) = r(k_0) \) by \((4.17)\). Hence \( R_1(\zeta, t, 0) = q(\zeta) \). Similarly, we find
\[
R_2(\zeta, t, 0) = \frac{q(\zeta)}{1 - |q(\zeta)|^2}, \quad R_3(\zeta, t, 0) = \frac{\overline{q(\zeta)}}{1 - |q(\zeta)|^2}, \quad R_4(\zeta, t, 0) = \overline{q(\zeta)}.
\]

We establish the estimate \((2.12)\) in the case of \( z \in X_1^\rho \); the other cases are similar. Note that \( z \in X_1^\rho \) corresponds to \( k \in k_0 + X_1^* \).

We have, for \( z \in X_1^\rho \),
\[
|R_1(\zeta, t, z) - q(\zeta)| \leq e^{-2\chi(\zeta, k)} - e^{-2\chi(\zeta, k_0)} \left| r_{1,a}(x, t, k)e^{2i\nu(\zeta)\ln((k+k_0)\sqrt{48k_0})} \right| \\
+ \left| e^{-2\chi(\zeta, k)} \right| |r_{1,a}(x, t, k) - r(k_0)| \left| e^{2i\nu(\zeta)\ln((k+k_0)\sqrt{48k_0})} \right| \\
+ \left| e^{-2\chi(\zeta, k)}r(k_0) \right| \left| e^{2i\nu(\zeta)\ln((k+k_0)\sqrt{48k_0})} - e^{2i\nu(\zeta)\ln(2\sqrt{48k_0}^3/2)} \right|.
\]

The functions \( e^{2i\nu(\zeta)\ln((k+k_0)\sqrt{48k_0})} \) and \( e^{-2\chi(\zeta, k)} \) are uniformly bounded with respect to \( k \in k_0 + X_1^* \) and \( \zeta \in \mathcal{I} \). Moreover, employing the estimate
\[
|\Re \phi(\zeta, ve^{\frac{it}{\rho}})| = \left| -\frac{v^2}{2} \left( 1 + \frac{v}{6\sqrt{2}\rho} \right) \right| \leq \frac{2v^2}{3}, \quad -\rho \leq v \leq \rho,
\]
we see that equations \((4.17)\) and \((4.18)\) yield
\[
|r_{1,a}(x, t, k) - r(k_0)| \leq C|k - k_0|e^{\frac{2}{\rho}|\Re \Phi(\zeta, k)|} = C\frac{|z|}{\rho}e^{\frac{4}{\rho} |\Re \phi(\zeta, z)|}
\]
\[
\leq C\frac{|z|}{\rho}e^{\frac{4}{\rho} |\Re \phi(\zeta, z)|}, \quad z \in X_1^\rho,
\]
and
\[
|r_{1,a}(x, t, k)| \leq \frac{C}{1 + |k|}e^{\frac{4}{\rho} |\Re \Phi(\zeta, k)|} \leq Ce^{\frac{4}{\rho} |\Re \phi(\zeta, z)|} \leq Ce^{\frac{4}{\rho} |\Re \phi(\zeta, z)|}, \quad z \in X_1^\rho,
\]
respectively. Thus,
\[
|R_1(\zeta, t, z) - q(\zeta)| \leq Ce^{-\frac{|z|^2}{6}} \left| e^{-2\chi(\zeta, k)} - e^{-2\chi(\zeta, k_0)} \right| + C\frac{|z|}{\rho}e^{\frac{4}{\rho} |\Re \phi(\zeta, z)|}
\]
\[
\quad + C|1 - e^{-2i\nu(\zeta)\ln(\frac{k+k_0}{2k_0})}|, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad z \in X_1^\rho.
\]

The estimate in \((2.12)\) for \( z \in X_1^\rho \) now follows from the following lemma.

**Lemma 4.6.** The following inequalities are valid for all \( \zeta \in \mathcal{I} \) and all \( k \in k_0 + X_1^* \):
\[
\left| e^{-2\chi(\zeta, k)} - e^{-2\chi(\zeta, k_0)} \right| \leq C|k - k_0|(1 + |\ln |k - k_0||), \quad (4.42)
\]
\[
1 - e^{-2i\nu(\zeta)\ln(\frac{k+k_0}{2k_0})} \leq Ck_0^{-1}|k - k_0|, \quad (4.43)
\]
where the constant \( C \) is independent of \( \zeta \) and \( k \).

**Proof.** We first prove that
\[
|\chi(\zeta, k) - \chi(\zeta, k_0)| \leq C|k - k_0|(1 + |\ln |k - k_0||), \quad \zeta \in \mathcal{I}, \quad k \in k_0 + X_1^*.
\]

Integration by parts in the definition \((4.11)\) of \( \chi \) gives
\[
\chi(\zeta, k) = -\frac{1}{2\pi i} \int_{-k_0}^{k_0} \ln(s - k)d\ln(1 - |r(s)|^2).
\]

Hence
\[
|\chi(\zeta, k) - \chi(\zeta, k_0)| = \left| \frac{1}{2\pi} \int_{-k_0}^{k_0} \ln\left( \frac{s - k}{s - k_0} \right)d\ln(1 - |r(s)|^2) \right|.
\]
This proves (4.42).

The change of variables \( v = |k - k_0|/u \) yields

\[
\int_0^{2k_0} \ln \left(1 + \frac{k - k_0}{u}\right) \, du = |k - k_0| \int_{|k - k_0|/2k_0}^{\infty} \ln(1 + ve^{i\pi}) \, \frac{dv}{v^2}.
\]

Since

\[
|\ln(1 + ve^{i\pi})| \leq C \begin{cases} \frac{v}{2}, & 0 \leq v \leq 2, \\ v, & 2 \leq v < \infty, \end{cases}
\]

we conclude that

\[
\int_0^{2k_0} \ln \left(1 + \frac{k - k_0}{u}\right) \, du \leq C|k - k_0| \left( \int_{|k - k_0|/2k_0}^2 \frac{dv}{v} + \int_2^{\infty} \frac{\ln v \, dv}{v^2} \right)
\]

\[
\leq C|k - k_0|(\ln|k - k_0| + C), \quad k \in k_0 + X_1^e.
\]

This proves (4.44).

Using the inequality (2.23) together with (4.12) and (4.44), we estimate

\[
|e^{-2\chi(\zeta,k)} - e^{-2\chi(\zeta,k_0)}| \leq |e^{-2\chi(\zeta,k_0)}||e^{-2|\chi(\zeta,k) - \chi(\zeta,k_0)|} - 1|
\]

\[
\leq C|\chi(\zeta,k) - \chi(\zeta,k_0)|e^{2|\Re(\chi(\zeta,k) - \chi(\zeta,k_0))|}
\]

\[
\leq C|k - k_0|(1 + |\ln|k - k_0||), \quad \zeta \in I, \quad k \in k_0 + X_1^e.
\]

This proves (4.42).

By (2.23)

\[
\left|1 - e^{-2\nu(\zeta) \ln \left(\frac{k + k_0}{2k_0}\right)}\right| \leq 2\nu(\zeta) \ln \left(\frac{k + k_0}{2k_0}\right) e^{\left|\Re \left(2\nu(\zeta) \ln \left(\frac{k + k_0}{2k_0}\right)\right)\right|}
\]

\[
\leq C \ln \left(1 + \frac{k - k_0}{e^2}\right) \leq Ck_0^{1-\epsilon} |k - k_0|, \quad \zeta \in I, \quad k \in k_0 + X_1^e,
\]

which proves (4.43).

**Step 5: Find asymptotics.** Theorem 2.1 implies that the limit (3.4) defining \( u(x,t) \) exists for all sufficiently large \( \tau \). Moreover,

\[
u(\zeta) = 2i \lim_{k \to \infty} \text{Im} (kM(x,t,k))_{12} = 2i \lim_{k \to \infty} \text{Im} (km(x,t,k))_{12}.
\]

Equation (2.13) of Theorem 2.1 then yields

\[
u(\zeta) = \frac{4\epsilon \Re \beta(\zeta, t)}{\sqrt{\tau}} + O(\epsilon \tau^{-1+\alpha}) = \frac{\Re \beta(\zeta, t)}{\sqrt{\epsilon k_0}} + O(\epsilon \tau^{-1+\alpha}),
\]

where \( \beta(\zeta, t) \) is defined by (2.14). We have

\[
\Re \beta(\zeta, t) = \sqrt{\nu(\zeta)} \cos \left(\frac{\pi}{4} - \arg q(\zeta) + \arg \Gamma(i\nu(\zeta)) + 16tk_0^3 - \nu(\zeta) \ln t\right).
\]

where

\[
\arg q(\zeta) = 2\nu(\zeta) \ln(2k_0 \sqrt{4k_0}) + \arg r(k_0) + \frac{1}{\pi} \int_{-k_0}^{k_0} \ln \left(\frac{1 - |r(s)|^2}{1 - |r(k_0)|^2}\right) \frac{ds}{s - k_0}.
\]
Thus,

$$u(x, t) = \sqrt{\frac{\nu(\zeta)}{3t k_0}} \cos \left(16 t k_0^2 - \nu(\zeta) \ln(192 t k_0^2) + \phi(\zeta)\right) + O(\epsilon r^{-1/3}),$$

where $$\phi(\zeta)$$ is defined by (4.4). This proves (4.2) and completes the proof of Theorem 4.1.

**Appendix A.** \textit{L}^2\textit{-Riemann-Hilbert problems}

A theory of \textit{L}^p\textit{-RH} problems with jumps across Carleson contours was presented in [12]; here we collect a number of relevant definitions and results. In the context of smooth contours, more information on \textit{L}^p\textit{-RH} problems can be found in [3, 5, 7, 15].

Let \( \mathcal{J} \) denote the collection of all subsets \( \Gamma \) of the Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) such that \( \Gamma \) is homeomorphic to the unit circle and

$$\sup_{z \in \Gamma \cap \mathbb{C}} \sup_{r > 0} \frac{\left|\Gamma \cap D(z, r)\right|}{r} < \infty,$$

(A.1)

where \( D(z, r) \) denotes the disk of radius \( r \) centered at \( z \). Curves satisfying (A.1) are called Carleson curves. Let \( 1 \leq p < \infty \). If \( D \) is the bounded component of \( \hat{\mathbb{C}} \setminus \Gamma \) where \( \Gamma \in \mathcal{J} \) and \( \infty \notin \Gamma \), then a function \( f \) analytic in \( D \) belongs to the Smirnoff class \( \text{E}^p(D) \) if there exists a sequence of rectifiable Jordan curves \( \{C_n\}^\infty_{n=1} \) in \( D \), tending to the boundary in the sense that \( C_n \) eventually surrounds each compact subdomain of \( D \), such that

$$\sup_{n \geq 1} \left|\int_{C_n} |f(z)|^p |dz| \right| < \infty.$$

(A.2)

If \( D \) is a subset of \( \hat{\mathbb{C}} \) bounded by an arbitrary curve in \( \mathcal{J} \), \( \text{E}^p(D) \) is defined as the set of functions \( f \) analytic in \( D \) for which \( f \circ \varphi^{-1} \in \text{E}^p(\varphi(D)) \), where \( \varphi(z) = \frac{1}{z - z_0} \) and \( z_0 \) is any point in \( \hat{\mathbb{C}} \setminus D \). The subspace of \( \text{E}^p(D) \) consisting of all functions \( f \in \text{E}^p(D) \) such that \( z f(z) \in \text{E}^p(D) \) is denoted by \( \hat{E}^p(D) \). If \( D = D_1 \cup \cdots \cup D_n \) is the union of a finite number of disjoint subsets of \( \hat{\mathbb{C}} \) each of which is bounded by a curve in \( \mathcal{J} \), then \( \text{E}^p(D) \) and \( \hat{E}^p(D) \) denote the set of functions \( f \) analytic in \( D \) such that \( f|_{D_j} \in \text{E}^p(D_j) \) and \( f|_{D_j} \in \hat{E}^p(D_j) \) for each \( j \), respectively. We let \( \text{E}^\infty(D) \) denote the space of bounded analytic functions in \( D \). A **Carleson jump contour** is a connected subset \( \Gamma \) of \( \hat{\mathbb{C}} \) such that:

(a) \( \Gamma \cap \mathbb{C} \) is the union of finitely many oriented arcs each of which have at most endpoints in common.

(b) \( \hat{\mathbb{C}} \setminus \Gamma \) is the union of two disjoint open sets \( D_+ \) and \( D_- \) each of which has a finite number of simply connected components in \( \hat{\mathbb{C}} \).

(c) \( \Gamma \) is the positively oriented boundary of \( D_+ \) and the negatively oriented boundary of \( D_- \), i.e., \( \Gamma = \partial D_+ = - \partial D_- \).

(d) If \( \{D_j^+\}_{1}^{m} \) and \( \{D_j^-\}_{1}^{m} \) are the components of \( D_+ \) and \( D_- \), then \( \partial D_j^+ \in \mathcal{J} \) for \( j = 1, \ldots, n \), and \( \partial D_j^- \in \mathcal{J} \) for \( j = 1, \ldots, m \).

Let \( n \geq 1 \) be an integer and let \( \Gamma \) be a Carleson jump contour. Given an \( n \times n \)-matrix valued function \( v : \Gamma \rightarrow GL(n, \mathbb{C}) \), a solution of the \( \textit{L}^p \textit{-RH problem determined by} \ (\Gamma, v) \) is an \( n \times n \)-matrix valued function \( m \in I + \hat{E}^p(\mathbb{C} \setminus \Gamma) \) such that the nontangential boundary values \( m_\pm \) satisfy \( m_+ = m_- \) a.e. on \( \Gamma \). If \( \hat{\mathbb{C}} \) denotes the Carleson jump contour \( \Gamma \) with the orientation reversed on a subset \( \Gamma_0 \subset \Gamma \) and \( \hat{v} = v \) on \( \Gamma \setminus \Gamma_0 \) and

---

\(^3\text{A subset } \Gamma \subset \mathbb{C} \text{ is an arc if it is homeomorphic to a connected subset } I \text{ of the real line which contains at least two distinct points.}\)
\( \hat{v} = v^{-1} \) on \( \Gamma_0 \), then we say that \( m \in I + \hat{E}^p(D) \) satisfies the \( L^p \)-RH problem determined by \( (\hat{\Gamma}, \hat{v}) \) if and only if \( m \) satisfies the \( L^p \)-RH problem determined by \( (\Gamma, v) \).

We next list some facts about Smirnoff classes and \( L^2 \)-RH problems; detailed proofs of can be found in \([12]\). We assume that \( \Gamma = \partial D_+ = -\partial D_- \) is a Carleson jump contour.

A.1. Basic facts. If \( f \in \hat{E}^2(D_+) \) or \( f \in \hat{E}^2(D_-) \), then the nontangential limits of \( f(z) \) as \( z \) approaches the boundary exist a.e. on \( \Gamma \) and the boundary function belongs to \( L^2(\Gamma) \). If \( h \in L^2(\Gamma) \), then the Cauchy transform \( \mathcal{C}h \) defined by

\[
(\mathcal{C}h)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(s)}{s - z} ds, \quad z \in \mathbb{C} \setminus \Gamma,
\]

satisfies \( \mathcal{C}h \in \hat{E}^2(D_+ \cup D_-) \). We denote the nontangential boundary values of \( \mathcal{C}f \) from the left and right sides of \( \Gamma \) by \( \mathcal{C}_+ f \) and \( \mathcal{C}_- f \) respectively. Then \( \mathcal{C}_+ \) and \( \mathcal{C}_- \) are bounded operators on \( L^2(\Gamma) \) and \( \mathcal{C}_+ - \mathcal{C}_- = I \).

Given two functions \( w^\pm \in L^2(\Gamma) \cap L^\infty(\Gamma) \), we define the operator \( \mathcal{C}_w : L^2(\Gamma) + L^\infty(\Gamma) \rightarrow L^2(\Gamma) \) by

\[
\mathcal{C}_w(f) = \mathcal{C}_+(fw^-) + \mathcal{C}_-(fw^+).
\]

Then

\[
\|\mathcal{C}_w\|_{B(L^2(\Gamma))} \leq C \max \{ \|w^+\|_{L^\infty(\Gamma)}, \|w^-\|_{L^\infty(\Gamma)} \},
\]

where \( C = \max\{\|\mathcal{C}_+\|_{B(L^2(\Gamma))}, \|\mathcal{C}_-\|_{B(L^2(\Gamma))}\} < \infty \) and \( B(L^2(\Gamma)) \) denotes the Banach space of bounded linear maps \( L^2(\Gamma) \rightarrow L^2(\Gamma) \).

The next lemma shows that if \( v = (v^-)^{-1}v^+ \) and \( w^\pm = \pm v^\mp \mp I \) then the \( L^2 \)-RH problem determined by \( (\Gamma, v) \) is equivalent to the following singular integral equation for \( \mu \in I + L^2(\Gamma) \):

\[
\mu - I = \mathcal{C}_w(\mu) \quad \text{in} \quad L^2(\Gamma).
\]

Lemma A.1. Given \( v^\pm : \Gamma \rightarrow GL(n, \mathbb{C}) \), let \( v = (v^-)^{-1}v^+ \), \( w^+ = v^+ - I \), and \( w^- = \mp v^- \mp I \). Suppose \( v^\pm, (v^-)^{-1} \in I + L^2(\Gamma) \cap L^\infty(\Gamma) \). If \( m \in I + \hat{E}^2(D) \) satisfies the \( L^2 \)-RH problem determined by \( (\Gamma, v) \), then \( \mu = m_+(v^-)^{-1} = m_-(v^-)^{-1} \in I + L^2(\Gamma) \) satisfies \( [A.6] \). Conversely, if \( \mu \in I + L^2(\Gamma) \) satisfies \( [A.6] \), then \( m = I + \mathcal{C}(\mu(w^+ + w^-)) \in I + \hat{E}^2(D) \) satisfies the \( L^2 \)-RH problem determined by \( (\Gamma, v) \).

Lemma A.2. Given \( v^\pm : \Gamma \rightarrow GL(n, \mathbb{C}) \), let \( v = (v^-)^{-1}v^+ \), \( w^+ = v^+ - I \), and \( w^- = \mp v^- \mp I \). Suppose \( v^\pm, (v^-)^{-1} \in I + L^2(\Gamma) \cap L^\infty(\Gamma) \) and \( v^\pm \in C(\Gamma) \). If \( w^\pm \) are nilpotent matrices, then each of the following four statements implies the other three:

(a) The map \( I - \mathcal{C}_w : L^2(\Gamma) \rightarrow L^2(\Gamma) \) is bijective.
(b) The \( L^2 \)-RH problem determined by \( (\Gamma, v) \) has a unique solution.
(c) The homogeneous \( L^2 \)-RH problem determined by \( (\Gamma, v) \) has only the zero solution.
(d) The map \( I - \mathcal{C}_w : L^2(\Gamma) \rightarrow L^2(\Gamma) \) is injective.

Lemma A.3 (Uniqueness). Suppose \( v : \Gamma \rightarrow GL(2, \mathbb{C}) \) satisfies \( \det v = 1 \) a.e. on \( \Gamma \). If the solution of the \( L^2 \)-RH problem determined by \( (\Gamma, v) \) exists, then it is unique and has unit determinant.

Lemma A.4. Let \( D \) be a subset of \( \hat{\mathbb{C}} \) bounded by a curve \( \gamma \in \mathcal{J} \). Let \( f : D \rightarrow \mathbb{C} \) be an analytic function. Suppose there exist curves \( \{C_n\}_1^\infty \subset \mathcal{J} \) in \( D \), tending to \( \gamma \) in the sense that \( C_n \) eventually surrounds each compact subset of \( D \subset \hat{\mathbb{C}} \), such that \( \sup_{n \geq 1} \|f(z)\|_{L^2(C_n)} < \infty \). Then \( f \in \hat{E}^2(D) \).
**Lemma A.5** (Contour deformation). Let $\gamma \in \mathcal{J}$. Suppose that, reversing the orientation on a subcontour if necessary, $\hat{\Gamma} = \Gamma \cup \gamma$ is a Carleson jump contour, see Figure 6. Let $B_+$ and $B_-$ be the two components of $\hat{\mathbb{C}} \setminus \gamma$. Let $\hat{D}_{\pm}$ be the open sets such that $\hat{\mathbb{C}} \setminus \hat{\Gamma} = \hat{D}_+ \cup \hat{D}_-$ and $\partial \hat{D}_+ = -\partial \hat{D}_- = \hat{\Gamma}$. Let $\hat{D} = \hat{D}_+ \cup \hat{D}_-$. Let $\gamma_+$ and $\gamma_-$ be the parts of $\gamma$ that belong to the boundary of $\hat{D}_+ \cap B_+$ and $\hat{D}_- \cap B_+$, respectively. Suppose $v : \Gamma \to GL(n, \mathbb{C})$. Suppose $m_0 : \hat{D} \cap B_+ \to GL(n, \mathbb{C})$ satisfies

$$m_0, m_0^{-1} \in I + \dot{E}^2(\hat{D} \cap B_+) \cap E^\infty(\hat{D} \cap B_+).$$

Define $\hat{v} : \hat{\Gamma} \to GL(n, \mathbb{C})$ by

$$\hat{v} = \begin{cases} m_0 - \nu m_0^{-1} & \text{on } \Gamma \cap B_+, \\ m_0^{-1} & \text{on } \gamma_+, \\ m_0 & \text{on } \gamma_-, \\ v & \text{on } \Gamma \cap B_. \end{cases}$$

Let $m$ and $\hat{m}$ be related by

$$\hat{m} = \begin{cases} mm_0^{-1} & \text{on } \hat{D} \cap B_+, \\ m & \text{on } \hat{D} \cap B_. \end{cases}$$

Then $m(z)$ satisfies the $L^2$-RH problem determined by $(\Gamma, v)$ if and only if $\hat{m}(z)$ satisfies the $L^2$-RH problem determined by $(\hat{\Gamma}, \hat{v})$.

**Appendix B. Exact solution in terms of parabolic cylinder functions**

Let $X = X_1 \cup \cdots \cup X_4 \subset \mathbb{C}$ be the cross defined in (2.1) and oriented as in Figure 1. Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk and define the function $\nu : \mathbb{D} \to (0, \infty)$ by $\nu(q) = -\frac{1}{2\pi} \ln(1 - |q|^2)$. We consider the following family of $L^2$-RH problems parametrized by $q \in \mathbb{D}$:

$$\begin{array}{ll}
  \{ m^X(q, \cdot) \in I + \dot{E}^2(\mathbb{C} \setminus X), \\
  m^X_+(q, z) = m^X_-(q, z)v^X(q, z) \text{ for a.e. } z \in X, \}
\end{array}$$

Figure 6. Examples of contours $\gamma$, $\Gamma$, and $\hat{\Gamma} = \Gamma \cup \gamma$ satisfying the conditions of Lemma A.5.
where the jump matrix $v^X(q, z)$ is defined by

$$v^X(q, z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ qz^{-2i\nu(q)}e^{\frac{i\pi z^2}{2}} & 1 \end{pmatrix}, & z \in X_1, \\
\begin{pmatrix} 1 & qz^{-2i\nu(q)}e^{\frac{i\pi z^2}{2}} \\ 0 & 1 \end{pmatrix}, & z \in X_2, \\
\begin{pmatrix} 1 & 0 \\ -qz^{-2i\nu(q)}e^{\frac{i\pi z^2}{2}} & 1 \end{pmatrix}, & z \in X_3, \\
\begin{pmatrix} 1 & \frac{q}{1-|q|^2}z^{-2i\nu(q)}e^{\frac{i\pi z^2}{2}} \\ 0 & 1 \end{pmatrix}, & z \in X_4.
\end{cases}$$

The matrix $v^X$ has entries that oscillate rapidly as $z \to 0$ and $v^X$ is not continuous at $z = 0$; however $v^X(q, \cdot) - I \in L^2(X) \cap L^\infty(X)$.

The RH problem (B.1) can be solved explicitly in terms of parabolic cylinder functions [9].

\textbf{Theorem B.1.} The $L^2$-RH problem (B.1) has a unique solution $m^X(q, z)$ for each $q \in \mathbb{D}$. This solution satisfies

$$m^X(q, z) = I + \frac{i}{z} \begin{pmatrix} 0 & -\beta^X(q) \\ \beta^X(q) & 0 \end{pmatrix} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad q \in \mathbb{D}, \quad (B.2)$$

where the error term is uniform with respect to $\arg z \in [0, 2\pi]$ and $q$ in compact subsets of $\mathbb{D}$, and the function $\beta^X(q)$ is defined by

$$\beta^X(q) = \sqrt{\nu(q)}e^{\frac{i}{2}-\arg q+\arg(\nu(q))}, \quad q \in \mathbb{D}. \quad (B.3)$$

Moreover, for each compact subset $K$ of $\mathbb{D}$,

$$\sup_{q \in K} \sup_{z \in \mathbb{C}\setminus X} |m^X(q, z)| < \infty.$$

\textbf{Proof.} Since $\det v^X = 1$, uniqueness follows from Lemma A.3
Define a sectionally analytic function $\tilde{m}^X(q, z)$ by

$$\tilde{m}^X(q, z) = \begin{pmatrix} \psi_1(q, z) & (d + iz)\psi_2(q, z) \\ (d^+ + iz)\psi_1(q, z) & \beta^X(q) \psi_2(q, z) \end{pmatrix}, \quad q \in \mathbb{D}, \ z \in \mathbb{C} \setminus \mathbb{R}, \quad (B.4)$$

where $\beta^X(q)$ is given by [B.3], the functions $\psi_1$ and $\psi_2$ are defined by

$$\psi_1(q, z) = \begin{cases} e^{-\frac{3i\nu(q)}{4}}D_{iv}(q)(e^{-\frac{3i\pi}{4}}z), \quad \text{Im } z > 0, \\ e^{\frac{i\nu(q)}{4}}D_{iv}(q)(e^{\frac{i\pi}{4}}z), \quad \text{Im } z < 0, \end{cases}$$

$$\psi_2(q, z) = \begin{cases} e^{-\frac{3i\nu(q)}{4}}D_{-iv}(q)(e^{-\frac{i\pi}{4}}z), \quad \text{Im } z > 0, \\ e^{\frac{i\nu(q)}{4}}D_{-iv}(q)(e^{\frac{i\pi}{4}}z), \quad \text{Im } z < 0, \end{cases}$$

and $D_a(z)$ denotes the parabolic cylinder function. Since $D_a(z)$ is an entire function of both $a$ and $z$, $\tilde{m}^X(q, z)$ is analytic in the upper and lower halves of the complex $z$-plane with a jump across the real axis. Observe that $\tilde{m}^X(q, z)$ is regular at $q = 0$ despite the fact that $\beta^X(q)$ vanishes at $q = 0$.

For $j = 1, \ldots, 4$, we denote the open domain enclosed by $\mathbb{R}$ and $X_j$ by $\Omega_j$, see Figure 7. Let $\Omega_0 = \mathbb{C} \setminus \bigcup_{j=1}^4 \Omega_j$ and define $m^X(q, z)$ by

$$m^X(q, z) = \tilde{m}^X(q, z)D_j(q, z), \quad z \in \Omega_j, \ j = 0, \ldots, 4, \quad (B.5)$$

where

$$D_0(q, z) = \begin{pmatrix} z^{-iv(q)e^{i\frac{\pi}{4}}} & 0 \\ 0 & z^{iv(q)e^{-i\frac{\pi}{4}}} \end{pmatrix}$$

and

$$D_1(q, z) = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}D_0(q, z), \quad D_2(q, z) = \begin{pmatrix} 1 & \frac{q}{1-|q|^2} \\ 0 & 1 \end{pmatrix}D_0(q, z),$$

$$D_3(q, z) = \begin{pmatrix} \frac{1}{1-|q|^2} & 0 \\ 0 & 1 \end{pmatrix}D_0(q, z), \quad D_4(q, z) = \begin{pmatrix} 1 & -\frac{q}{1} \\ 0 & 1 \end{pmatrix}D_0(q, z).$$

**Claim 1.** The function $m^X(q, z)$ is analytic for $z \in \mathbb{C} \setminus X$ and satisfies the jump condition $m^X_+ = m^X_- v^X$ a.e. on $X$.

**Proof of Claim 1.** Clearly, the boundary values $m^X_+(q, z)$ exist for all $z \in X \setminus \{0\}$. Since $\tilde{m}^X$ is continuous across $X$, it is straightforward to verify the jump condition across $X$. For example, the jump of $m^X$ across $X_1$ is given by

$$(m^X)^{-1}m^X_+ = D_1^{-1}D_0 = \begin{pmatrix} 1 & 0 \\ qz^{-2iv(q)e^{i\frac{\pi}{4}}} & 1 \end{pmatrix}, \quad z \in X_1.$$

In order to show that $m^X$ is analytic for $z \in \mathbb{C} \setminus X$, it is enough to verify that $m^X$ does not jump across $\mathbb{R}$. We will first prove that $\tilde{m}^X$ satisfies

$$\tilde{m}^X_+(q, z) = \tilde{m}^X_-(q, z) \begin{pmatrix} 1 - |q|^2 & -q \\ q & 1 \end{pmatrix}, \quad q \in \mathbb{D}, \ z \in \mathbb{R}. \quad (B.6)$$

Since both $\tilde{m}^X_+(q, z)$ and $\tilde{m}^X_-(q, z)$ satisfy the differential equation

$$\left(\frac{d}{dz} + \frac{iz}{2}q_3\right)\tilde{m}^X(q, z) = \begin{pmatrix} 0 & \beta^X(q) \\ \beta^X(q) & 0 \end{pmatrix}\tilde{m}^X(q, z), \quad (B.7)$$
the jump matrix $\delta^X(q, z) := \tilde{m}_-^X(q, z)^{-1}\tilde{m}_+^X(q, z)$ is independent of $z$. Evaluating at $z = 0$ and using the identities

$$
\beta^X(q) = \frac{\sqrt{2\pi}e^{\frac{x\nu}{2}}e^{-\frac{x
u(q)}{2}}}{\Gamma(-i\nu(q))q}, \quad \beta^X(q)\beta^X(q) = \nu(q), \quad q \in \mathbb{D},
$$

and

$$
D_a(0) = \frac{2\sqrt{2\pi}}{\Gamma(1/2)}, \quad D_a'(0) = -\frac{2^{1+a}\sqrt{\pi}}{\Gamma(-a/2)},
$$

we find the following equation which proves (B.6):

$$
\delta^X(q, z) = \begin{pmatrix}
\frac{e^{\frac{3\nu}{4}D_4}e^{\frac{3\nu}{4}D_d}(0)}{\beta^X(q)} & e^{\frac{3\nu}{4}D_4}e^{\frac{3\nu}{4}D_d}(0) \\
\frac{e^{\frac{3\nu}{4}D_4}e^{\frac{3\nu}{4}D_d}(0)}{\beta^X(q)} & e^{\frac{3\nu}{4}D_d}(0)
\end{pmatrix}^{-1} \begin{pmatrix}
1 - |q|^2 & -q \\
q & 1
\end{pmatrix}, \quad q \in \mathbb{D}, \quad z \in \mathbb{R}.
$$

Since $z^{i\nu}$ has a branch cut along the negative real axis, the jump of $m^X$ across $\mathbb{R}$ is given by

$$(m^X_-)^{-1}m^X_+ = \begin{cases}
D_{4,1}^{-1} \begin{pmatrix}
1 - |q|^2 & -q \\
q & 1
\end{pmatrix} D_1, & z > 0, \\
D_{0,-}^{-1} \begin{pmatrix}
1 - |q|^2 & 0 \\
0 & 1/1-|q|^2
\end{pmatrix} D_{0+,} & z < 0,
\end{cases}
$$

where $D_{0,+}$ and $D_{0,-}$ denote the values of $D_0$ for $z$ just above and below the negative real axis respectively:

$$
D_{0,\pm}(q, z) = \begin{pmatrix}
e^{-i \nu(q)(\ln |z| + i \pi)}e^{\frac{i\nu^2}{2}} & 0 \\
0 & e^{i \nu(q)(\ln |z| + i \pi)}e^{-\frac{i\nu^2}{2}}
\end{pmatrix}, \quad z < 0.
$$

Simplification of (B.8) shows that $m^X$ has no jump across $\mathbb{R}$. \hfill \square

Claim 2. $m^X(q, z)$ satisfies the asymptotic formula (B.2) as $z \to \infty$.

Proof of Claim 2. Let $\delta > 0$ be an arbitrarily small positive constant. The parabolic cylinder function satisfies the asymptotic formula [14]

$$
D_a(z) = z^ae^{-\frac{z^2}{4}} \left(1 - \frac{a(a-1)}{2z^2} + O(z^{-4})\right) - \sqrt{2\pi}e^{\frac{a^2}{2}}z^{-a-1}\left(1 + \frac{(a+1)(a+2)}{2z^2} + O(z^{-4})\right)
$$

\times \begin{cases}
0, & \text{arg } z \in [-\frac{3\pi}{4} + \delta, \frac{3\pi}{4} - \delta], \\
e^{i\pi a}, & \text{arg } z \in [\frac{\pi}{4} + \delta, \frac{3\pi}{4} - \delta], \\
e^{-i\pi a}, & \text{arg } z \in [-\frac{3\pi}{4} + \delta, -\frac{\pi}{4} - \delta],
\end{cases} \quad z \to \infty, \quad a \in \mathbb{C},
where the error terms are uniform with respect to \( a \) in compact subsets and \( \arg z \) in the given ranges. It follows that
\[
\psi_{11}(q, z) = z^{i\nu} e^{-\frac{iz^2}{4}} \left( 1 - \frac{i\nu(i\nu - 1)}{2iz^2} + O(z^{-4}) \right)
\]
\[
- \sqrt{2\pi e^{\frac{iz^2}{4}}} z^{-i\nu - 1} e^{i\nu} \frac{1 + (i\nu + 1)(i\nu + 2)}{2iz^2} + O(z^{-4})
\]
\[
\times \begin{cases} 
0, & \arg z \in [\delta, \pi] \cup [-\pi + \delta, 0], \\
\frac{\pi v}{\pi} e^{\frac{\pi z}{2}}, & \arg z \in [0, \frac{\pi}{2} - \delta], \\
\frac{\pi v}{\pi} e^{-\frac{\pi z}{2}}, & \arg z \in [-\pi, -\frac{\pi}{2} - \delta],
\end{cases}
\]
and
\[
\psi_{22}(q, z) = z^{-i\nu} e^{\frac{iz^2}{4}} \left( 1 + \frac{i\nu(i\nu + 1)}{2iz^2} + O(z^{-4}) \right)
\]
\[
- \sqrt{2\pi e^{\frac{iz^2}{4}}} z^{-i\nu - 1} e^{i\nu} \frac{1 + (-i\nu + 1)(-i\nu + 2)}{2iz^2} + O(z^{-4})
\]
\[
\times \begin{cases} 
0, & \arg z \in [0, \pi - \delta] \cup [-\pi, -\delta], \\
\frac{\pi v}{\pi} e^{\frac{\pi z}{2}}, & \arg z \in [\frac{\pi}{2}, \pi + \delta], \\
\frac{\pi v}{\pi} e^{-\frac{\pi z}{2}}, & \arg z \in [-\frac{\pi}{2}, -\delta, 0],
\end{cases}
\]
uniformly with respect to \( a \) in compact subsets and \( \arg z \) in the given ranges. Using the identity
\[
\frac{d}{dz} D_a(z) = \frac{z}{2} D_a(z) - D_{a+1}(z)
\]
we find similar asymptotic formulas for the derivatives of \( \psi_{11} \) and \( \psi_{22} \). Substituting these formulas into the defining equations (B.4) and (B.5) for \( m_X \), equation (B.2) follows from a long but straightforward computation. \( \square \)

It only remains to show that \( m_X(q, \cdot) \in I + \dot{E}^2(\mathbb{C} \setminus X) \) and that \( \sup_{z \in \mathbb{C} \setminus X} |m_X(q, z)| \) is bounded uniformly with respect to \( q \) in compact subsets of \( \mathbb{D} \). This is an easy consequence of Lemma A.4, the asymptotics (B.2), and the explicit formula (B.5) for \( m_X \). \( \square \)

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