INTERSECTIONS OF SCHUBERT VARIETIES AND OTHER PERMUTATION ARRAY SCHEMES

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Abstract. Using a blend of combinatorics and geometry, we give an algorithm for algebraically finding all flags in any zero-dimensional intersection of Schubert varieties with respect to three transverse flags, and more generally, any number of flags. In particular, the number of flags in a triple intersection is also a structure constant for the cohomology ring of the flag manifold. Our algorithm is based on solving a limited number of determinantal equations for each intersection (far fewer than the naive approach). These equations may be used to compute Galois and monodromy groups of intersections of Schubert varieties. We are able to limit the number of equations by using the permutation arrays of Eriksson and Linusson, and their permutation array varieties, introduced as generalizations of Schubert varieties. We show that there exists a unique permutation array corresponding to each realizable Schubert problem and give a simple recurrence to compute the corresponding rank table, giving in particular a simple criterion for a Littlewood-Richardson coefficient to be 0. We describe pathologies of Eriksson and Linusson’s permutation array varieties (failure of existence, irreducibility, equidimensionality, and reducedness of equations), and define the more natural permutation array schemes. In particular, we give several counterexamples to the Realizability Conjecture based on classical projective geometry. Finally, we give examples where Galois/monodromy groups experimentally appear to be smaller than expected.

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1. Introduction

A typical Schubert problem asks how many lines in three-space meet four generally chosen lines. The answer, two, may be obtained by computation in the cohomology ring of the Grassmannian variety of two-dimensional planes in four-space. Such questions were considered by H. Schubert in the nineteenth century. During the past century, the study of the Grassmannian has been generalized to the flag manifold where one can ask analogous questions.

The flag manifold $F_l_n(K)$ parameterizes the complete flags

$$F_\bullet = \{ \{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = K^n \}$$

where $F_i$ is a vector space of dimension $i$. (Unless otherwise noted, we will work over an arbitrary base field $K$. The reader, and Schubert, is welcome to assume $K = \mathbb{C}$ throughout. For a general field, we should use the Chow ring rather than the cohomology ring, but they agree for $K = \mathbb{C}$. For this reason, and in order not to frighten the reader, we will use the term “cohomology” throughout.)

A modern Schubert problem asks how many flags have relative position $u, v, w$ with respect to three generally chosen fixed flags $X_\bullet, Y_\bullet$ and $Z_\bullet$. The solution to this problem, due to Lascoux and Schützenberger [Lascoux and Schützenberger, 1982], is to compute a product of Schubert polynomials and expand in the Schubert polynomial basis. The coefficient indexed by $u, v, w$ is the solution. This corresponds to a computation in the cohomology (or Chow) ring of the flag variety. (Caution: this solution is known to work only in characteristic 0, due to the failure of the Kleiman-Bertini theorem, cf. [Vakil, 2003b, Sect. 2].) The quest for a combinatorial rule for expanding these products is a long-standing open problem.

The main goal of this paper is to describe a method for directly identifying all flags in $X_u(F_\bullet) \cap X_v(G_\bullet) \cap X_w(H_\bullet)$ when $\ell(u) + \ell(v) + \ell(w) = (\frac{n}{2})$, thereby computing $c_{u,v,w}$ explicitly. This method extends to Schubert problems with more than three flags. To do this, we use the permutation arrays defined by Eriksson and Linusson to obtain a table of intersection dimensions. These permutation arrays are closely related to the checker boards used in [Vakil, 2003a, Vakil, 2003b]. Eriksson and Linusson introduced permutation array varieties as natural generalizations of Schubert varieties to an arbitrary number of flags. We show that they may be badly behaved. For example, their equations are not reduced, so we argue that the “correct” generalization of Schubert varieties are permutation array schemes. We describe pathologies of these varieties/schemes, and show that they are not irreducible nor even equidimensional in general, making a generalization of the Bruhat order problematic. We also give counterexamples to the Realizability Conjecture [4.1]. Returning to the task at hand, we use the data from the permutation array to identify and solve a collection of determinantal equations for the permutation array schemes, allowing us to solve Schubert problems explicitly and effectively, for example allowing us to compute Galois/monodromy groups.

The outline of the paper is as follows. In Section 2 we review Schubert varieties and the flag manifold. In Section 3 we review the construction of permutation arrays and the Eriksson-Linusson algorithm for generating all such arrays. In Section 4 we describe permutation varieties and their pathologies, and explain why their correct definition is as schemes. In Section 5 we describe how to use permutation arrays to solve Schubert problems and give equations for certain intersections of
Schubert varieties. In Section 4, we give two examples of an algorithm for computing triple intersections of Schubert varieties and thereby computing the cup product in the cohomology ring of the flag manifold. The equations we give also allow us to compute Galois and monodromy groups for intersections of Schubert varieties; we describe this application in Section 7. Our computations lead to examples where the Galois/monodromy group is smaller than expected.

2. The Flag Manifold and Schubert varieties

In this section we briefly review the notation and basic concepts for flag manifolds and Schubert varieties. We refer the reader to one of the following books for further background information: [Fulton, 1997, Macdonald, 1991, Manivel, 1998, Gonciulea and V. Lakshmibai, 2001, Kumar, 2002].

As described earlier, the flag manifold $F(l_n) = F(l_n)(K)$ parametrizes the complete flags $F\bullet = \{\{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = K^n\}$ where $F_i$ is a vector space of dimension $i$ over the field $K$. $F(l_n)$ is a smooth projective variety of dimension $\binom{n}{2}$. A complete flag is determined by an ordered basis $(f_1, \ldots, f_n)$ for $K^n$ by taking $F_i = \text{span}(f_1, \ldots, f_i)$.

Two flags $[F\bullet], [G\bullet] \in F(l_n)$ are in relative position $w \in S_n$ when $\dim(F_i \cap G_j) = \text{rank } w[i,j]$ for all $1 \leq i, j \leq n$ where $w[i,j]$ is the principal submatrix of the permutation matrix for $w$ with lower right hand corner in position $(i,j)$. We use the notation $\text{pos}(F\bullet, G\bullet) = w$.

Warning: in order to use the typical meaning for a principal submatrix we are using a nonstandard labeling of a permutation matrix. The permutation matrix we associate to $w$ has a 1 in the $w(i)$th row of column $n - i + 1$ for $1 \leq i \leq n$. For example, the matrix associated to $w = (5, 3, 1, 2, 4)$ is

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

If $\text{pos}(F\bullet, G\bullet) = (5, 3, 1, 2, 4)$ then $\dim(F_2 \cap G_3) = 2$ and $\dim(F_3 \cap G_2) = 1$.

Define a Schubert cell with respect to a fixed flag $F\bullet$ in $F(l_n)$ to be

$$X_w(F\bullet) = \{G\bullet \mid F\bullet \text{ and } G\bullet \text{ have relative position } w\} = \{G\bullet \mid \dim(F_i \cap G_j) = \text{rk } w[i,j]\}.$$

Using our labeling of a permutation matrix, the codimension of $X_w$ is equal to the length of $w$ (the number of inversions in $w$), denoted $\ell(w)$. In fact, $X_w$ is isomorphic to the affine space $K^{\binom{n}{2}} - \ell(w)$. We say the flags $F\bullet$ and $G\bullet$ are in transverse position if $G\bullet \in X_{id}(F\bullet)$. A randomly chosen flag will be transverse to any fixed flag $F\bullet$ with probability 1 (using any reasonable measure).

The Schubert variety $X_w(F\bullet)$ is the closure of $X_w(F\bullet)$ in $F(l_n)$. Schubert varieties may also be written in terms of rank conditions:

$$X_w(F\bullet) = \{G\bullet \mid \dim(F_i \cap G_j) \geq \text{rk } w[i,j]\}.$$
If the flags $F_\bullet$ and $G_\bullet$ are determined by ordered bases for $K^n$ then these inequalities can be rephrased as determinantal equations on the coefficients in the bases [Fulton, 1997] 10.5, Ex. 10, 11. Of course this allows one in theory to solve all Schubert problems, but the number and complexity of the equations grows quickly to make this prohibitive in any reasonable case.

The cohomology (or Chow) ring of $\mathcal{F}l_n$ is isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]/(e_1, e_2, \ldots, e_n)$ where $e_i$ is the $i$th elementary symmetric function on $x_1, \ldots, x_n$ [Fulton, 1997] 10.2, B.3. The cycles $[X_u]$ corresponding to Schubert varieties form a $\mathbb{Z}$-basis for the ring. The product is defined by

$$[X_u] \cdot [X_v] = [X_u(F_\bullet) \cap X_v(G_\bullet)]$$

where $F_\bullet$ and $G_\bullet$ are in transverse position. Speaking somewhat informally, $X_u(F_\bullet) \cap X_v(G_\bullet)$ can be decomposed into irreducible components which are again translates of Schubert varieties. Therefore the expansion

$$[X_u] \cdot [X_v] = \sum_{\ell(u)=\ell(u)+\ell(v)} c_{u,v}^w [X_w]$$

automatically has nonnegative integer coefficients.

A simpler geometric interpretation of the coefficients $c_{u,v}^w$ may be given in terms of triple intersections [Fulton, 1997] 10.2. There exists a perfect pairing on $H^*(\mathcal{F}l_n)$ such that

$$[X_u] \cdot [X_v] = \begin{cases} [X_{w_0}] & y = w_0 w \\ 0 & y \neq w_0 w, \ell(y) = (\frac{n}{2}) - \ell(w). \end{cases}$$

Here $w_0 = (n, n-1, \ldots, 1)$ is the longest permutation in $S_n$, of length $(\frac{n}{2}) = \dim(\mathcal{F}l_n)$, and $[X_{w_0}]$ is the class of a point. Combining equations (1) and (2) we have

$$[X_u] \cdot [X_v] \cdot [X_{w_0}] = c_{u,v}^w [X_{w_0}].$$

In characteristic 0, $c_{u,v}^w$ counts the number of points $[E_\bullet] \in \mathcal{F}l_n$ in the variety

$$X_u(F_\bullet) \cap X_v(G_\bullet) \cap X_{w_0,w}(H_\bullet)$$

when $\ell(u) + \ell(v) + \ell(w_0 w) = (\frac{n}{2})$ and $F_\bullet, G_\bullet, H_\bullet$ are three generally chosen flags. Note, it is not sufficient to assume the three flags are pairwise transverse in order to get the expected number of points in the intersection. There can be additional dependencies among the subspaces of the form $F_i \cap G_j \cap H_k$.

The main goal of this article is to describe a method to find all flags in a general $d$-fold intersection of Schubert varieties when the intersection is zero-dimensional. Enumerating the flags found explicitly in a triple intersection would give the numbers $c_{u,v}^w$. We will use the permutation arrays defined in the next section to identify a different set of equations defining the intersections of Schubert varieties which are easier to solve.

### 3. Permutation arrays

In [Eriksson and Linusson, 2000a] [Eriksson and Linusson, 2000b], Eriksson and Linusson develop a $d$-dimensional analog of a permutation matrix. One way to generalize permutation matrices is to consider all $d$-dimensional arrays of 0’s and 1’s with a single 1 in each hyperplane. They claim that a better way is to consider a permutation matrix to be a two-dimensional array of 0’s and 1’s such that the rank of any principal minor is equal to the number of occupied rows in that submatrix.
The following are equivalent:  

1. $P$ is totally rankable.  
2. Every two dimensional projection of every principal subarray is totally rankable.
3. Every redundant position is covered by dots in $P$.

4. If there exist dots in $P$ in positions $y$ and $z$ and integers $i, j$ such that $y_i < z_i$ and $y_j = z_j$, then there exists a dot in some position $x \preceq (y \lor z)$ such that $x_i = z_i$ and $x_j < z_j$.

Define a permutation array in $[n]^d$ to be a totally rankable dot array of rank $n$ with no redundant dots (or equivalently, no covered dots). The permutation arrays are the unique representatives of each rank equivalence class of totally rankable dot arrays with no redundant dots. These arrays are Eriksson and Linusson's analogs of permutation matrices.

The definition of permutation arrays was motivated because they include the possible relative configurations of flags:

Theorem 3.2. [Eriksson and Linusson, 2000b, Thm. 3.1] Given flags $E_1, E_2, \ldots, E_d$, there exists an $[n]^d$-permutation array $P$ describing the table of all intersection dimensions as follows. For each $x \in [n]^d$,
\begin{equation}
\text{rk}(P[x]) = \dim \left( E_{x_1} \cap E_{x_2} \cap \cdots \cap E_{x_d} \right).
\end{equation}

A special case is the permutation array corresponding to $n$ generally chosen flags, which we denote the transverse permutation array
$$T_{n,d} = \left\{ (x_1, \ldots, x_d) \in [n]^d \mid \sum x_i = (d-1)n + 1 \right\},$$
which corresponds to
\begin{equation}
\text{rk}(T_{n,d}[x]) = \max \left( 0, n - \sum_{i=1}^{d} (n - x_i) \right).
\end{equation}

Eriksson and Linusson give an algorithm for producing all permutation arrays in $[n]^d$ recursively from the permutation arrays in $[n]^{d-1}$. We review their algorithm, as this is key to our algorithm for intersecting Schubert varieties.

Let $A$ be any antichain of dots in $P$ under the dominance order. Let $C(A)$ be the set of positions covered by dots in $A$. Define the downsizing operator $D(A, P)$ with respect to $A$ on $P$ to be the result of the following process.
1. Set $Q_1 = P \setminus A$.
2. Set $Q_2 = Q_1 \cup C(A)$.
3. Set $D(A, P) = Q_2 \setminus R(Q_2)$ where $R(Q)$ is the set of redundant positions of $Q$.

The downsizing is successful if the resulting array is totally rankable of rank $\text{rk}(P) - 1$.

Theorem 3.3 (The EL-Algorithm, [Eriksson and Linusson, 2000b, Sect. 2.3]). Every permutation array in $[n]^d$ can be obtained uniquely in the following way.
1. Choose a permutation array $P_n$ in $[n]^{d-1}$.
2. For each $n \geq i > 1$, choose an antichain $A_i$ of dots in $P_i$ such that the downsizing $D(A_i, P_i)$ is successful. Set $P_{i-1} = D(A_i, P_i)$.
3. Set $A_1 = P_1$.
4. Set $P = \{(x_1, \ldots, x_{d-1}, i) \mid (x_1, \ldots, x_{d-1}) \in A_i\}$.

For example, starting with the 2-dimensional array $\{(1,4), (2,3), (3,1), (4,2)\}$ corresponding to the permutation $w = (1,2,4,3)$, we run through the algorithm as
follows. (In the figure, dots correspond to elements in $P$ and circled dots correspond to elements in $A$.)

\begin{align*}
P_1 &= \{(1, 4), (2, 3), (3, 1), (4, 2)\} & A_4 &= \{(1, 4), (2, 3)\} \\
P_3 &= \{(2, 4), (3, 1), (4, 2)\} & A_3 &= \{(3, 1)\} \\
P_2 &= \{(2, 4), (4, 2)\} & A_2 &= \{(2, 4), (4, 2)\} \\
P_1 &= \{(4, 4)\} & A_1 &= \{(4, 4)\}
\end{align*}

This produces the 3-dimensional array

\[ P = \{(4, 4, 1), (2, 4, 2), (4, 2, 2), (3, 1, 3), (1, 4, 4), (2, 3, 4)\}. \]

We prefer to display 3-dimensional dot-arrays as 2-dimensional number-arrays as in [Eriksson and Linusson, 2000b; Vakil, 2003a] where a square $(i, j)$ contains the number $k$ if $(i, j, k) \in P$. Note that there is at most one number in any square if the number-array represents a permutation array. The previous example is represented by

\[
\begin{array}{ccc}
4 & 4 & 2 \\
3 & & \\
2 & & 1
\end{array}
\]

**Corollary 3.4.** In Theorem 3.3, each $P_i$ is an $[n]^{d-1}$-permutation array of rank $i$. Furthermore, if $P$ determines the rank table for flags $E_{i+1}^1, \ldots, E_{i+1}^d$, then $P_i$ determines the rank table for $E_{i+1}^1, \ldots, E_{i+1}^{d-1}$ intersecting the vector space $E_{i+1}^d$, i.e.

\[ \text{rk} (P_i|x) = \dim \left( \cap_{x_1, x_2, \ldots, x_{d-1}, x_d} E_{i+1}^1 \cap E_{x_2}^2 \cap \cdots \cap E_{x_{d-1}}^{d-1} \cap E_{x_d}^d \right). \]

**Proof.** $P_i$ is the permutation array obtained from the projection

\[ \{(x_1, \ldots, x_d) \mid (x_1, \ldots, x_d, x_{d+1}) \in P \text{ and } x_{d+1} \leq i\} \]

by removing all repeated or covered elements. \hfill \Box

We finish this section with a substantial improvement on the speed to the Eriksson-Linusson algorithm. In Step 2 of Theorem 3.3 one must find all positions covered by a subset of points in the antichain $A_i$. This appears to require on the order of $2^{|A_i|}$ computations. However, here we show that subsets of size at most $d$ are sufficient.

**Lemma 3.5.** A position $x \in [n]^d$ is covered (or equivalently, redundant) in a permutation array $P$ if and only if there exists a subset $S$ with $|S| \leq d$ which cover $x$.

**Proof.** Assume $x$ is covered by a set $Y = \{y_1, y_2, \ldots, y^k\}$ for $k > d$. That is,

- For each position $1 \leq j \leq d$, there exists a $y^i$ such that $y^i_j < x_j$ and there exists a $y^j$ such that $y^j_j = x_j$. 

For each \( y^i \in Y \), there exists a \( j \) such that \( y^i_j < x_j \) and there exists an \( l \) such that \( y^i_l = x_l \).

Consider a complete bipartite graph with left vertices labeled by \( Y \) and right vertices labeled by \( \{x_1, \ldots, x_d\} \). Color the edge from \( y^i \) to \( x \) red if \( y^i_j = x_j \), and blue if \( y^i_j < x_j \). Since \( x = \bigvee Y \), \( y^i_j > x_j \) is not possible. This is a complete bipartite graph such that each vertex meets at least one red and one blue edge, and conversely any such complete bipartite graph with left vertices chosen from \( P \) and right vertices \( \{x_1, \ldots, x_d\} \) corresponds to a covering of \( x \).

We can easily bound the minimum size of a covering set for \( x \) to be at most \( d + 1 \) as follows. Choose one red and one blue edge adjacent to \( x_1 \). Let \( S \) be the left end-points of these two edges. Vertex \( x_2 \) is connected to both elements of \( S \) in the complete bipartite graph. If the edges connecting \( x_2 \) to \( S \) are different colors, proceed to \( x_3 \). If the edges agree in color, choose one additional edge of a different color adjacent to \( x_2 \). Add its left endpoint to \( S \). Continuing in this way for \( x_3, \ldots, x_d \), we have \( |S| \leq d + 1 \) and that \( x \) is covered by \( S \).

Given a covering set \( S \) of size \( d + 1 \), we now find a subset of size \( d \) which covers \( x \). Say \( x_1, x_2, \ldots, x_{d+1} \) are all the right vertices which are adjacent to a unique edge of either color. Let \( T \) be the left endpoints of all of these edges; these are necessary in any covering subset. Choose one vertex in \( Y \setminus T \), say \( y \). Each remaining \( x_j \) has at least two edges of each color, so we can choose one of each color which is not adjacent to \( y \). The induced subgraph on \( (S \setminus \{y\}, \{x_1, \ldots, x_d\}) \) is again a complete bipartite graph where every vertex is adjacent to at least one red and one blue edge, hence \( S \setminus \{y\} \) covers \( x \). \( \square \)

4. Permutation array varieties (or schemes) and their pathologies

In analogy with Schubert cells, for any \([n]^d\)-permutation array \( P \), Eriksson and Linusson define the permutation array variety \( X^o_P \) to be the subset of \( \mathcal{F}^d_n = \{(E_1^*, \ldots, E_d^*)\} \) in “relative position \( P \)” [Eriksson and Linusson, 2000b §1.2.2]. We will soon see why \( X^o_P \) is a locally closed subvariety of \( \mathcal{F}^d_n \); this will reinforce the idea that the correct notion is of a permutation array scheme. These varieties/schemes will give a convenient way to manage the equations of intersections of Schubert varieties.

Based on many examples, Eriksson and Linusson conjectured the following.

**Realizability Conjecture 4.1** [Eriksson and Linusson, 2000b Conj. 3.2]. Every permutation array can be realized by flags. Equivalently, every \( X^o_P \) is nonempty.

This question is motivated by more than curiosity. A fundamental question is: what are the possible relative configurations of \( d \) flags? In other words: what intersection dimension tables are possible? For \( d = 2 \), the answer leads to the theory of Schubert varieties. By Theorem 3.2, each achievable intersection dimension table yields a permutation array, and the permutation arrays may be enumerated by Theorem 3.3. The Realizability Conjecture then says that we have fully answered this fundamental question. Failure of realizability would imply that we still have a poor understanding of how flags can meet.

The Realizability Conjecture is true for \( d = 1, 2, 3 \). For \( d = 1 \), the only permutation array variety is the flag variety. For \( d = 2 \), the permutation array varieties are the "generalized" Schubert cells (where the reference flag may vary). The case \( d = 3 \) follows from Shapiro et al., 1997 (as described in Eriksson and Linusson, 2000b).
The case $n \leq 2$ is fairly clear, involving only one-dimensional subspaces of a two-dimensional vector space (or projectively, points on $P^1$), cf.

[Eriksson and Linusson, 2000b, Lemma 4.3]. Nonetheless, the conjecture is false, and we give examples below which show the bounds $d \leq 3$ and $n \leq 2$ are maximal for such a realizability statement. We found it interesting that the combinatorics of permutation arrays prevent some naive attempts at counterexamples from working; somehow, permutation arrays see some subtle linear algebraic information, but not all.

**Fiber permutation array varieties.** If $P$ is an $[n]^{d+1}$ permutation array, then there is a natural morphism $X^P \to Fl^n_d$ corresponding to “forgetting the last flag”. We call the fiber over a point $(E^1_1, \ldots, E^d_1)$ a fiber permutation array variety, and denote it $X^P(E^1_1, \ldots, E^d_1)$. If the flags $E^1_1, \ldots, E^d_1$ are chosen generally, we call the fiber permutation array variety a generic fiber permutation array variety. Note that a generic fiber permutation array variety is empty unless the projection of the permutation array to the “bottom hyperplane of $P$” is the transverse permutation array $T_{n,d}$, as this projection describes the relative positions of the first $d$ flags.

The Schubert cells $X^w_{\bullet}(E^1_1)$ are fiber permutation array varieties, with $d = 2$. Also, any intersection of Schubert cells $X^w_{\bullet}(E^1_1) \cap X^w_{\bullet}(E^2_2) \cap \cdots \cap X^w_{\bullet}(E^d_d)$ is a disjoint union of fiber permutation array varieties, and if the $E^i_\bullet$ are generally chosen, the intersection is a disjoint union of generic fiber permutation array varieties.

Permutation array varieties were introduced partially for this reason, to study intersections of Schubert varieties, and indeed that is the point of this paper. It was hoped that they would in general be tractable and well-behaved (cf.

the Realizability Conjecture 4.1), but sadly this is not the case. The remainder of this section is devoted to their pathologies, and is independent of the rest of the paper.

**Permutation array schemes.** We first observe that the more natural algebro-geometric definition is of permutation array schemes: the set of $d$-tuples of flags in configuration $P$ comes with a natural scheme structure, and it would be naive to expect that the resulting schemes are reduced. In other words, the “correct” definition of $X^P_{\bullet}$ will contain infinitesimal information not present in the varieties. More precisely, the $X^P_{\bullet}$ defined above may be defined scheme-theoretically by the equations (4), and these equations will not in general be all the equations cutting out the set $X^P_{\bullet}$ (see the “Further Pathologies” discussion below). Those readers preferring to avoid the notion of schemes may ignore this definition; other readers should re-define $X^P_{\bullet}$ to be the scheme cut out by equations (4), which is a locally closed subscheme of $Fl^n_d$.

We now give a series of counterexamples to the Realizability Conjecture 4.1.

**Counterexample 1.** Eriksson and Linusson defined their permutation array varieties over $C$, so we begin with a counterexample to realizability over $K = C$, and it may be read simply as an admonition to always consider a more general base field (or indeed to work over the integers). The Fano plane is the projective plane over the field $F_2$, consisting of 7 lines $\ell_1, \ldots, \ell_7$ and 7 points $p_1, \ldots, p_7$. We may name them so that $p_i$ lies on $\ell_i$, as in Figure 1. Thus we have a configuration of 7 flags over $F_2$. (This is a projective picture, so this configuration is in affine dimension $n = 3$, and the points $p_i$ should be interpreted as one-dimensional subspaces, and the lines $\ell_j$ as two-dimensional subspaces, of $K^3$.) The proof of Theorem 3.2 is
independent of the base field, so the table of intersection dimensions of the flags yields a permutation array. However, a classical and straightforward argument in projective coordinates shows that the configuration of Figure 1 may not be achieved over the complex numbers (or indeed over any field of characteristic not 2). In particular, this permutation array variety is not realizable over $\mathbb{C}$. In order to patch this counterexample, one might now restate the Realizability Conjecture 4.1 by saying that there always exists a field such that $X_o P$ is nonempty. However, the problems have only just begun.

**Figure 1.** The Fano plane, and a bijection of points and lines (indicated by arrows from points to the corresponding line).

**Counterexample 2.** We next sketch an elementary counterexample for $n = 3$ and $d = 9$, over an arbitrary field, with the disadvantage that it requires a computer check. Recall Pappus’ Theorem in classical geometry: if $A, B, and C$ are collinear, and $D, E, and F$ are collinear, and $X = AE \cap BD$, $Y = AF \cap CD$, and $Z = BF \cap CE$, then $X, Y,$ and $Z$ are collinear [Coxeter and Greitzer, 1967, §3.5]. The result holds over any field. A picture is shown in Figure 2 (Ignore the dashed arc and the stars for now.)

**Figure 2.** Pappus’ Theorem, and a counterexample to Realizability in dimension $d = 3$ with $n = 9
We construct an unrealizable permutation array as follows. We imagine that line $YZ$ does not meet $X$. (In the figure, the starred line $YZ$ “hops over” the point marked $X$.) We construct a counterexample with nine flags by letting the flags correspond to the nine lines of our “deformed Pappus configuration”, choosing points on the lines arbitrarily. We then construct the rank table of this configuration, and verify that this corresponds to a valid permutation array. (This last step was done by computer.) This permutation array is not realizable, by Pappus’ theorem.

**Counterexample 3.** Our next example shows that realizability already fails for $n = 4$, $d = 4$. The projective intuition is as follows. Suppose $\ell_1$, $\ell_2$, $\ell_3$, $\ell_4$ are four lines in projective space, no three meeting in a point, such that we require $\ell_i$ and $\ell_j$ to meet, except (possibly) $\ell_3$ and $\ell_4$. This forces all 4 lines to be coplanar, so $\ell_3$ and $\ell_4$ must meet. Hence we construct an unrealizable configuration as follows: we “imagine” (as in Figure 3) that $\ell_3$ and $\ell_4$ don’t meet. Again, we must turn the projective picture in $\mathbb{P}^3$ into linear algebra in 4-space, so the projective points in the figure correspond to one-dimensional subspaces, the projective lines in the figure correspond to two-dimensional subspaces of their respective flags, etc. Again, the tail of each arrow corresponds with the point which lies on the line the arrow follows. We construct the corresponding dot array:

Here the rows represent the flag $F_1^*$, columns represent the flag $F_2^*$, numbers represent the flag $F_3^*$, and the boards represent the flag $F_4^*$. This is readily checked to be a permutation array. The easiest way is to compare it to the dot array for the “legitimate” configuration, where $F_2^*$ and $F_3^*$ do meet, and using the fact that this second array is a permutation array by Theorem 3.2. The only difference between the permutation array above and the “legitimate” one is that the circled 3 should be a 2.

**Figure 3.** A counterexample to realizability with $n = d = 4$

*Remark.* Eriksson and Linusson have verified the Realizability Conjecture 4.1 for $n = 3$ and $d = 4$ [Eriksson and Linusson, 2000b] [§3.1]. Hence the only four
open cases left are \( n = 3 \) and \( 5 \leq d \leq 8 \). These cases seem simple, as they involve (projectively) between 5 and 8 lines in the plane. Can these remaining cases be settled?

**Further pathologies from Mnëv’s universality theorem: failure of irreducibility and equidimensionality.** Mnëv’s universality theorem shows that permutation array schemes will be “arbitrarily” badly behaved in general, even for \( n = 3 \). Informally, Mnëv’s theorem states that given any singularity type of finite type over the integers there is a configuration of projective lines in the plane such that the corresponding permutation array scheme has that singularity type. By a singularity type of finite type over the integers, we mean up to smooth parameters, any singularity cut out by polynomials with integer co-efficients in a finite number of variables. See \cite{Mnëv1985, Mnëv1988, Vakil2004} for the original sources, and \cite{Vakil2004, §3} for a precise statement and for an exposition of the version we need. (Mnëv’s theorem is usually stated in a different language of course.)

In particular, (i) permutation array schemes need not be irreducible, answering a question raised in \cite{ErikssonLinusson2000, §1.2.3}. They can have arbitrarily many components, indeed of arbitrarily many different dimensions. (ii) Permutation array schemes need not be reduced, i.e. they have genuine scheme-theoretic (or infinitesimal) structure not present in the variety. In other words, the definition of permutation array schemes is indeed different from that of permutation array varieties, and the equations \( \mathfrak{A} \) do not cut out the permutation array varieties scheme-theoretically. (iii) Permutation array schemes need not be equidimensional. Hence the hope that permutation array varieties/schemes might be well-behaved is misplaced. In particular, the notion of Bruhat order is problematic. We suspect, for example, that there exist two permutation array schemes \( X \) and \( Y \) such that \( Y \) is reducible, and some but not components of \( Y \) lie in the closure of \( X \).

Although Mnëv’s theorem is constructive, we have not attempted to explicitly produce a reducible or non-reduced permutation array scheme.

5. **Intersecting Schubert varieties**

In this section, we consider a Schubert problem in \( Fl_n \) of the form

\[
X = X_{w_1}(E^1_1) \cap X_{w_2}(E^2_2) \cap \cdots \cap X_{w_d}(E^d_d)
\]

with \( E^1_1, \ldots, E^d_d \) chosen generally and \( \sum_i \ell(w_i) = \binom{n}{2} \). We show there is a unique permutation array \( P \) for this problem if \( X \) is nonempty, and we identify it. In Theorem \ref{thm:main} we show how to use \( P \) to write down equations for \( X \). These equations can also be used to determine if \( E^1_1, \ldots, E^d_d \) are sufficiently general for computing intersection numbers. The number of solutions will always be either infinite or no greater than the expected number. The expected number is achieved on a dense open subset of \( Fl^d_n \). It may be useful for the reader to refer to the examples in Section \ref{sec:examples} while reading this section.

**Theorem 5.1.** If \( X \) is 0-dimensional and nonempty, there exists a unique permutation array \( P \subset [n]^{d+1} \) such that

\[
\dim \left( E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}} \right) = \rk P[x]
\]

for all \( F_x \in X \) and all \( x \in [n]^{d+1} \). Hence, \( X \) is equal to the fiber permutation array variety \( X_P(E^1_1, \ldots, E^d_d) \).
As remarked in Section 5.2, the projection of $P$ onto the first $d$ coordinates must be the transverse permutation array $T_{n,d}$. We will prove the theorem by explicitly constructing $P$. As an immediate consequence, as the permutation array corresponding to $d$ generally chosen flags $E^1, \ldots, E^d$ is given by $T_{n,d}$, we have the following.

**Corollary 5.2.** If $P_n \neq T_{n,d}$, then $X$ is the empty set.

When $d = 4$, this corollary can often be used to detect when the coefficients $c_{u,v}^w$ are zero in Equation [4]. This criterion catches 7 of the 8 zero coefficients in 3 dimensions, 373 of the 425 in 4 dimensions, and 28920 of the 33265 in dimension 5. The dimension 3 case missed by this criterion is presumably typical of what the criterion fails to see: there are no 2-planes in 3-space containing three general 1-dimensional subspaces. However, given a 2-plane $V$, three general flags with 1-subspaces contained in $V$ are indeed transverse.

The corollary is efficient to apply. For example, consider the following three anagrams of the name “Richard P. Stanley”:

- $u = \text{A Children’s Party}$
- $v = \text{Hip Trendy Rascal}$
- $w = \text{Raid Ranch Let Spy}$

Using a computer, we can easily compute $P_n$ corresponding to $X = X_u \cap X_v \cap X_w$:

![Permutation Table](image)

Clearly, $P_{15} \neq T_{15,3}$ so $X$ is empty and $c_{u,v}^w = 0$.

**Remark.** The array $T_{n,d}$ is an antichain under the dominance order on $[n]^d$ so each element corresponds to a 1-dimensional vector space. These lines will provide a “skeleton” for the given Schubert problem.

The proof of Theorem 5.1 follows directly from the next lemma.

**Lemma 5.3.** Let $E^1, \ldots, E^d$ be generally chosen flags. Let $w^1, \ldots, w^d$ be permutations in $S_n$ such that $\sum \ell(w^i) = \dim (F_{l_n})$. Let $F_i$ be a flag such that $\pos(E^i, F_i) = w^i$ for each $1 \leq i \leq d$. Then the rank table of intersection dimensions among the components of the $d + 1$ flags is determined by the recurrence

$$
\dim \left( E^{s_1}_{x_{s_1}} \cap E^{s_2}_{x_{s_2}} \cap \cdots \cap E^{s_k}_{x_{s_k}} \cap F_j \right)
= \max \left\{ \dim \left( E^{s_1}_{x_{s_1}} \cap F_j \right) + \dim \left( E^{s_2}_{x_{s_2}} \cap \cdots \cap E^{s_k}_{x_{s_k}} \cap F_j \right) - j \right\}
$$

where $1 \leq s_1 < s_2 < \cdots < s_k \leq d$, $k \geq 2$, $1 \leq x_{s_i} \leq n - 1$ for each $1 \leq i \leq k$, $\dim (F_0) = 0$, and $\dim (E^i \cap F_j)$ is determined by the rank table corresponding to the permutation $w^i$.

\[1\] Anagrams of the name “Richard P. Stanley”.
Note that this recurrence determines the full intersection table since
\[
\dim \left( E_{x_1}^1 \cap E_{x_2}^2 \cap \cdots \cap E_{x_d}^d \cap F_j \right) = \dim \left( E_{x_1}^{s_1} \cap E_{x_2}^{s_2} \cap \cdots \cap E_{x_k}^{s_k} \cap F_j \right)
\]
if \( x_i = n \) for each \( i \in [d] \setminus \{s_1, \ldots, s_k\} \).

**Proof.** Set \( U = E_{x_1}^{s_1} \cap F_j \) and \( V = E_{x_2}^{s_2} \cap \cdots \cap E_{x_k}^{s_k} \cap F_j \). Since \( F_{j-1} \subset F_j \) and \( \dim(F_j) = 1 + \dim(F_{j-1}) \), we have
\[
\dim(U \cap V \cap F_{j-1}) \leq \dim(U \cap V) \leq 1 + \dim(U \cap V \cap F_{j-1}).
\]
We also know that \( \dim(U \cap V) \geq \dim(U) + \dim(V) - j \) since \( U, V \subset F_j \) and \( \dim(F_j) = j \). We need to show \( \dim(U \cap V \cap F_{j-1}) < \dim(U \cap V) \) if and only if \( \dim(U \cap V) = \dim(U) + \dim(V) - j \).

Let \( W = U \cap V \cap F_{j-1} \) and choose \( U' \) and \( V' \) so that \( U = W \oplus U' \) and \( V = W \oplus V' \). By the assumption that \( E_1^1, \ldots, E_4^d \) are general we have
\[
\dim(U' \cap V') = \max \left\{ \dim(U') + \dim(V') - (j - \dim(W)) \right\}.
\]
Therefore, \( \dim(U \cap V) > \dim(W) \) if and only if \( \dim(U' \cap V') > 0 \) if and only if \( (\dim(U') + \dim(V') - (j - \dim(W))) > 0 \) if and only if
\[
\dim(U \cap V) = \dim(W) + \dim(U' \cap V')
= \dim(W) + \dim(U') + \dim(V') - (j - \dim(W))
= \dim(U) + \dim(V) - j.
\]
\[\square\]

**Theorem 5.4.** Let \( X = X_{w_1}(E^1_1) \cap X_{w_2}(E^2_2) \cap \cdots \cap X_{w_d}(E^d_d) \) be a \( 0 \)-dimensional intersection, with \( E^1_1, \ldots, E^d_d \) general. Let \( P \subset [n]^{d+1} \) be the unique permutation array associated to this intersection by the recurrence in Lemma 5.3. Let \( V(E^1_1, \ldots, E^d_d) = \{v_x \mid x \in T_{n,d}\} \) be a collection of vectors chosen such that \( v_x \in E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d} \). Then polynomial equations defining \( X \) can be determined simply by knowing \( P \) and \( V(E^1_1, \ldots, E^d_d) \).

**Proof.** Given \( P \in [n]^{d+1} \), let \( P_1, \ldots, P_n \) be the sequence of permutation arrays in \([n]^d\) defined by the EL-algorithm in Theorem 3.6. If \( F_x \in X \), then by Corollary 3.7 \( P_i \) is the unique permutation array encoding \( \dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d} \cap F_i) \). Furthermore, for each \( x \in P_i, 1 \leq i \leq n \), we could choose a representative vector in the corresponding intersection, say \( v_x^i \in E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d} \cap F_i \). Define
\[
V_i = \{v_x^i \mid y \in P_i\}
\]
\[
V_i[x] = \{v_y^i \mid y \in P_i[x]\}.
\]
In fact, we can choose the vectors \( v_x^i \) so that \( v_x^i \notin \operatorname{Span}(V_i \setminus \{v_x^i\}) \) since the rank function must increase at position \( x \). Therefore, we would have
\[
\text{v.rank}(V_i[x]) = \text{rk}(P_i[x])
\]
for all \( x \in [n]^d \) and all \( 1 \leq i \leq n \) where \( \text{v.rank}(S) \) is the dimension of the vector space spanned by the vectors in \( S \). These rank equations define \( X \).

Let \( V_n = V(E^1_1, \ldots, E^d_d) \) be the finite collection of vectors in the case \( i = n \). Given \( V_{i+1}, F_{i+1} \) and \( P_i \), we compute
\[
V_i = \{v_x^i \mid x \in P_i\}
If ever a solution implies c depends only on one indeterminate. Now the same rank equations as in (5) must hold. In fact, it is sufficient in a 0-dimensional variety X to require only
\[ v \cdot \{ v \in \mathbb{R}^d \mid y \in P_i[x] \} \leq \text{rk}(P_i[x]) \]
for all \( x \in [n]^d \) and all \( 1 \leq i \leq n \). Let \( \text{minors}_k(M) \) be the set of all \( k \times k \) determinantal minors of a matrix M. Let \( M(V_i[x]) \) be the matrix whose rows are given by the vectors in \( V_i[x] \).

Then, the equations (6) can be rephrased as
\[ \text{minors}_{k(P_i[x])+1}(M(V_i[x])) = 0 \]
for all \( 1 \leq i < n \) and \( x \in [n]^d \) such that \( \sum x_i > (d-1)n \).

For each set of solutions \( S \) to the equations in (5), we obtain a collection of vector sets by substituting solutions for the indeterminates in the formulas for the vectors. We can further eliminate variables whenever a vector depends only on one variable \( c_y^i \) by setting it equal to any nonzero value which does not force another \( c_z^i = 0 \). If ever a solution implies \( c_z^i = 0 \), then the choice of \( E_1^\bullet, \ldots, E_d^\bullet \) was not general. Let \( V_1^S, \ldots, V_n^S \) be the final collection of vector sets depending on the solutions \( S \). Since \( X \) is 0-dimensional, if \( V_1^S, \ldots, V_n^S \) depends on any indeterminate then \( E_1^\bullet, \ldots, E_d^\bullet \) was not general. Let \( F_i^S \) be the span of the vectors in \( V_i^S \). Then the flag \( F_1^S = (F_1^S \cup \ldots \cup F_n^S) \) satisfies all the rank conditions defining \( X = X_p(E_1^\bullet, \ldots, E_d^\bullet) \). Hence, \( F_i^S \in X \).

**Remark.** There are too many equations and indeterminates involved in the equations (5) to solve this system simultaneously in practice. First, it is useful to solve all equations pertaining to \( V_{i+1} \) before computing the initial form of the vectors in \( V_i \). Second, we have found that proceeding through all \( x \in [n]^d \) such that \( \sum x_i > (d-1)n \) in lexicographic order works well, with the additional caveat that if \( P_i[x] = \{x\} \) then the rank of the matrix \( M \) with rows determined by \( \{x\} \cup (P_i \cap P_{i+1}) \) must have rank at most \( i \). Solve all of the determinantal equations implying the rank condition \( v \cdot \text{rk}(V_i[x]) = \text{rk}(P_i[x]) \) simultaneously and substitute each solution back into the collection of vectors before considering the next rank condition.

**Corollary 5.5.** The equations appearing in (5) provide a test for determining if \( E_1^\bullet, \ldots, E_d^\bullet \) is sufficiently general for the given Schubert problem. Namely, the number of flags satisfying the equations (5) is the generic intersection number if each indeterminate \( c_z^i \neq 0 \) and the solution space determined by the equations is 0-dimensional.

6. **The key example: Triple intersections**

We now implement the algorithm of the previous section in an important special case. Our goal is to describe a method for directly identifying all flags in \( X = X_u(E_1^4) \cap X_v(E_2^4) \cap X_w(E_3^4) \) when \( \ell(u) + \ell(v) + \ell(w) = \left( \frac{d}{2} \right) \) and \( E_1^*, E_2^*, \) and \( E_3^* \)
are in general position. This gives a method for computing the structure constants in the cohomology ring of the flag variety from equations (11) and (13).

There are two parts to this algorithm. First, we use the recurrence of Lemma 5.3 to find the unique permutation array \( P \subset [n]^4 \) with position vector \((u, v, w)\) such that \( P_n = T_{n,3} \). Second, given \( P \) we use the equations in (8) to find all flags in \( X \).

As a demonstration, we explicitly compute the flags in \( X \) in two cases. For convenience, we work over \( \mathbb{C} \), but of course the algorithm is independent of the field. In the first there is just one solution which is relatively easy to see “by eye”. In the second case, there are two solutions, and the equations are more complicated. The algorithm has been implemented in Maple and works well on examples where \( n \leq 8 \).

**Example 6.1.**

Let \( u = (1, 3, 2, 4), v = (3, 2, 1, 4), w = (1, 3, 4, 2) \). The sum of their lengths is \( 1 + 3 + 2 = 6 = (\frac{n}{2}) \). The unique permutation array \( P \in [n]^4 \) determined by the recurrence in Lemma 5.3 consists of the following dots:

\[
(4421) \quad (4142) \quad (2442) \quad (4233) \quad (3243) \\
(3433) \quad (4414) \quad (4324) \quad (3424) \quad (3334) \\
(2434) \quad (2344) \quad (1444)
\]

The EL-algorithm produces the following list of permutation arrays \( P_1, P_2, P_3, P_4 \) in \([n]^3\) corresponding to \( P \):

\[
\begin{array}{cccc}
\begin{array}{cccc}
& & & \\
& 1 & 2 & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
& \begin{array}{cccc}
& & & \\
& 3 & 4 & \\
& & & \\
& & & \\
& & & \\
\end{array}
& \begin{array}{cccc}
& & & \\
& 4 & 1 & \\
& & & \\
& & & \\
& & & \\
\end{array}
& \begin{array}{cccc}
& & & \\
& 4 & 1 & \\
& & & \\
& & & \\
& & & \\
\end{array}
\end{array}
\]

Notice that \( P_4 \) is the transverse permutation array \( T_{n,3} \). Notice also how to read \( u, v, \) and \( w \) from \( P_1, \ldots, P_4 \): \( P_i \) has one less row than \( P_{i+1} \); listing these excised rows from right to left yields \( u \). Similarly, listing the excised columns from right to left yields \( v \), and listing the excised numbers from right to left yields \( w \) (see the example immediately above).

We want to specify three transverse fixed flags \( E_1^*, E_2^*, E_3^* \). It will be notationally convenient to represent a vector \( v = (v_1, \ldots, v_n) \) by the polynomial \( v_1 + v_2 x + \cdots + v_n x^{n-1} \). We choose three flags, or equivalently three “transverse” ordered bases, as follows:

\[
E_1^* = (1, x, x^2, x^3) \\
E_2^* = (x^3, x^2, x, 1) \\
E_3^* = ((x+1)^3, (x+1)^2, (x+1), 1)
\]

We will show that the only flag in \( X_u(E_1^*) \cap X_v(E_2^*) \cap X_w(E_3^*) \) is

\[
F^* = (2 + 3x - x^3, x^3, x^2, 1).
\]

For each element \((i, j, k)\) in \( P_4 \), we choose a vector in the corresponding 1-dimensional intersection \( E_1^* \cap E_2^* \cap E_3^* \cap F^* \) and put it in position \((i, j)\) in the matrix below:

\[
V(E_1^*, E_2^*, E_3^*) = V_4 = \\
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & x & x+1 \\
0 & x^2 & x(x+1) & (x+1)^2 \\
x^3 & x^2(x+1) & x(x+1)^2 & (x+1)^3
\end{bmatrix}
\]
In $P_3$, every element in the 4th column is covered by a subset in the antichain removed from $P_4$. This column adds only one degree of freedom so we establish $V_3$ by adding only one variable in position (2, 4) and solving all other rank two equations in terms of this one:

$$V_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1 + x) + cx \\ 0 & x^2 & 0 & 1 + x + (1 + c)x(1 + x) \\ x^3 & x^2(x + 1) & 0 & (x + 1)^2 + cx(x + 1)^2 \end{bmatrix}.$$  

According to equation (6) the entry in position (4, 2) can have two indeterminates: $b(1 + x) + cx$, where $b, c \neq 0$. As any two linearly dependent ordered pairs $(b, c)$ yield the same configuration of subspaces, we may normalize $b$ to 1.

Once $V_3$ is determined, we find the vectors in $V_2$. In $P_2$, every element is contained in $P_3$, so $V_2$ is a subset of $V_3$:

$$V_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + (2 + c)x \\ 0 & 0 & 0 & 0 \\ x^3 & 0 & 0 & (x + 1)^2 + cx(x + 1)^2 \end{bmatrix}. $$

The rank of $P_2$ is 2, so all $3 \times 3$ minors of the following matrix must be zero:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 + c & 0 & 0 \\ 1 & 2 + c & 1 + 2c & c \end{pmatrix}. $$

In particular, $1 + 2c = 0$, so the only solution is $c = -\frac{1}{2}$. Substituting for $c$, we have

$$V_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + \frac{3}{2}x \\ 0 & 0 & 0 & 0 \\ x^3 & 0 & 0 & (x + 1)^2 - \frac{1}{2}x(x + 1)^2 \end{bmatrix}. $$

Finally $P_1$ is contained in $P_2$, so $V_1^S$ contains just the vector

$$v_{(4,4,2)}^1 = v_{(4,4,2)}^2 = (x + 1)^2 - \frac{1}{2}x(x + 1)^2 = \frac{1}{2}(2 + 3x - x^3). $$

Therefore, there is just one solution, namely the flag spanned by the collections of vectors $V_1^S, V_2^S, V_3^S, V_4^S$ which is equivalent to the flag in (6).

If we choose an arbitrary general collection of three flags, we can always change bases so that we have the following situation:

$$E_1^1 = (1, x, x^2, x^3) $$
$$E_2^2 = (x^3, x^2, x, 1) $$
$$E_3^3 = (a_1 + a_2x + a_3x^2 + x^3, b_1 + b_2x + x^2, c_1 + x, 1). $$

Using these coordinates, the same procedure as above will produce the unique solution

$$F_* = \langle (a_1 - a_3b_1) + (a_2 - b_2a_3)x - x^3, x^3, x^2, 1 \rangle. $$

Example 6.2.

This example is of a Schubert problem with multiple solutions. Let $u = (1, 3, 2, 5, 4, 6)$, $v = (3, 5, 1, 2, 4, 6)$, $w = (3, 1, 6, 5, 4, 2)$. If $P$ is the unique permutation array in [6] determined by the recurrence in Lemma 6.2 for $u, v, w$ then the EL-algorithm
produces the following list of permutation arrays $P_1, \ldots, P_6$ in $[6]^3$ corresponding to $P$:

\[
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
& & \\
\hline
& & \\
\hline
& & \\
\hline
2 & 4 & 2 \\
\hline
6 & 6 & 5 \\
\hline
6 & 5 & 4 \\
\hline
6 & 5 & 4 \\
\hline
6 & 5 & 4 \\
\hline
6 & 5 & 4 \\
\hline
\end{array}
\]

We take the following triple of fixed flags:

\[
E^1_2 = \langle 1, x, \ldots, x^5 \rangle \\
E^2_3 = \langle x^3, \ldots, x, 1 \rangle \\
E^3_4 = \langle (1 + x)^6, (1 + x)^4, \ldots, 1 \rangle
\]

The third flag is clearly not chosen generally but leads to two solutions to this Schubert problem which is the generic number of solutions. We prefer to work with explicit but simple numbers here to demonstrate the computation without making the formulas too complicated.

The vector table associated to $P_6$ is easily determined by Pascal’s formula:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & [0, 1, 0, 0, 0, 0] \\
\hline
& & & [0, 1, 0, 0, 0, 0] \\
\hline
& & [0, 1, 0, 0, 0, 0] \\
\hline
& [0, 0, 0, 1, 0, 0] \\
\hline
[0, 0, 0, 0, 0, 1] \\
\hline
[0, 0, 0, 0, 1, 1] \\
\hline
[0, 0, 0, 0, 1, 1] \\
\hline
\end{array}
\]

The vector table associated to $P_5$ has one degree of freedom. The vector in position $(3, 5)$ is freely chosen to be $x + c x^2$. Then for all other points in $P_5 \setminus P_6$ we can solve a rank 2 equation which determines the corresponding vector in terms of $c$. Therefore, $V_5$ becomes:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & [0, 1, c, 0, 0, 0] \\
\hline
& & [0, 1, c, 0, 0, 0] \\
\hline
& [0, 1, c, 0, 0, 0] \\
\hline
[0, 0, 1, \frac{6}{c - 1}, 0, 0] \\
\hline
[0, 0, 1, \frac{6}{c - 1}, 0, 0] \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x^4 & \frac{1}{c - 1} & \frac{3}{c - 1} & 2 \frac{1}{c - 1} \\
\hline
x^5 & x^4 + x^5 & \frac{1}{c - 1} & \frac{3}{c - 1} & 2 \frac{1}{c - 1} \\
\hline
\end{array}
\]
Every vector in $V_4$ appears in $V_5$, but now some of them are subject to new rank conditions:

\[
\begin{bmatrix}
10(c-1) & 0 & 0 & 0 & 0 \\
3(c^2+1) & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
10(c-1) & 0 & 0 & 0 & 0 \\
3(c^2+1) & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

or equivalently if the following nontrivial minors of the matrix are zero

\[
\begin{bmatrix}
4(10c + c^2 - 20) & -8(10c + c^2 - 20) & -8(10c + c^2 - 20) & -8(c - 1)(10c + c^2 - 20) \\
3(c - 8 + 5c) & (c - 8 + 5c)(c - 4) & (c - 8 + 5c)(c - 4) & 3(c - 8 + 5c)(c - 4) \\
\end{bmatrix}
\]

All rank 3 minors will be zero if $c^2 + 10c - 20 = 0$, or $c = -5 \pm 3\sqrt{5}$. Plugging each solution for $c$ into the vectors gives the two solutions $V_4^{S_1}$ and $V_4^{S_2}$. For example, using $c = -5 + 3\sqrt{5}$ and solving a single rank 2 equation involving $d$ gives:

\[
\begin{bmatrix}
1 & 0 & 10(c-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
10(c-1) & 0 & 0 & 0 & 0 \\
3(c^2+1) & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

or equivalently if the following nontrivial minors of the matrix are zero

\[
\begin{bmatrix}
4(10c + c^2 - 20) & -8(10c + c^2 - 20) & -8(10c + c^2 - 20) & -8(c - 1)(10c + c^2 - 20) \\
3(c - 8 + 5c) & (c - 8 + 5c)(c - 4) & (c - 8 + 5c)(c - 4) & 3(c - 8 + 5c)(c - 4) \\
\end{bmatrix}
\]

The remaining vectors in $V_4^{S_1}$, $V_4^{S_2}$, $V_3^{S_1}$ will be a subset of $V_4^{S_1}$ so no further equations need to be solved, and similarly for $V_4^{S_2}$.

7. Monodromy and Galois groups

The monodromy group of a problem in enumerative geometry captures information reflecting three aspects of algebraic geometry: geometry, algebra, and arithmetic. Informally, it is the symmetry group of the set of solutions. Three more precise interpretations are given below. Historically, these groups were studied since the nineteenth century [Jordan, 1870] [Dickson et al., 1916] [Weber, 1941]; modern interest probably dates from a letter from Serre to Kleiman in the seventies (see the historical discussion in the survey article [Kleiman, 1987] p. 325]). Their modern foundations were laid by Harris [Harris, 1979]; among other things, he showed that the monodromy group of a problem is equivalent to the Galois group of the equations defining it.

These groups are difficult to compute in general, and indeed they are known for relatively few enumerative problems. In this section, we use the computation of explicit algebraic solutions to Schubert problems (along with a criterion from...
to give a method to compute many such groups explicitly (when they are “full”, or as large as possible), and to give an experimental method to compute groups in other cases.

It is most interesting to exhibit cases where the Galois/monodromy group is unexpectedly small. Indeed, Harris writes of his calculations:

the results represent an affirmation of one understanding of the geometry underlying each of these problems, in the following sense: in every case dealt with here, the actual structure on the set of solutions of the enumerative problem as determined by the Galois group of the problems, is readily described in terms of standard algebrao-geometric constructions. In particular, in every case in which current theory had failed to discern any intrinsic structure on the set of solutions — it is proved here — there is in fact none. [Harris, 1979, p. 687-8]

We exhibit an example of a Schubert problem whose Galois/monodromy group experimentally appears to be smaller than expected — it is the dihedral group $D_4 \subset S_4$. This is the first example in which current theory fails to discern intrinsic structure. Examples of “small” Galois groups were given in [Vakil, 2003b, Sect. 5]; but there an explanation had already been given by Derksen. Here, however, we have a mystery: We do not understand geometrically why the group is $D_4$. (However, see the end of this section for a conjectural answer.)

We now describe the three interpretations of the Galois/monodromy group for a Schubert problem. The definition for a general problem in enumerative geometry is the obvious generalization; see [Harris, 1979] for a precise definition, and for the equivalence of $(A)$ and $(B)$. See [Vakil, 2003b, Sect. 2.9] for more discussion.

$(A)$ Geometry. Begin with $m$ general flags; suppose there are $N$ solutions to the Schubert problem (i.e. there are $N$ flags meeting our $m$ given flags in the specified manner). Move the $m$ flags around in such a way that no two of the solutions ever come together, returning the $m$ flags to their starting positions, and follow the $N$ solutions. The $N$ solutions are returned to their initial positions as a set, but the individual $N$ solutions may be permuted. What are the possible permutations? (See the applet http://lamar.colostate.edu/~jachter/mono.html for an illustration of this concept.)

$(B)$ Algebra. The $m$ flags are parameterized by $F^m_n$. Define the “solution space” to be the subvariety of $F^m_n \times F^m_n$ mapping to $F^m_n$, corresponding to those flags satisfying the given Schubert conditions. There is one irreducible component $X$ of the solution space mapping dominantly to $F^m_n$; the morphism has generic degree $N$. The Galois/monodromy group is the Galois group of the Galois closure of the corresponding extension of function fields. The irreducibility of $X$ implies that the Galois group $G$ is a transitive subgroup of $S_N$.

$(C)$ Arithmetic. If the $m$ flags are defined over $\mathbb{Q}$, then the smallest field of definition of a solution must have Galois group that is a subgroup of the Galois/monodromy group $G$. Moreover, for a randomly chosen set of $m$ flags, the field of definition will have Galois group precisely $G$ with positive probability (depending on the particular problem). The equivalence of this version with the previous two follows from $(B)$ by the Hilbert irreducibility theorem, as $F^m_n$ is rational ([Lang, 1983, Sect. 9.2], see also Serre, 1989 Sect. 1.5] and [Cohen, 1981]). We are grateful to M. Nakamaye for discussions on this topic.
Given any enumerative problem with $N$ solutions, we see that the Galois/monodromy group is a subgroup of $S_N$; it is well-defined up to conjugacy in $S_N$. As the solution set should be expected to be as symmetric as possible, one should expect it to be as large as possible; it should be $S_N$ unless the set of solutions has some additional geometric structure.

For example, in [Harris, 1979], Harris computed several Galois/monodromy groups, and in each case they were the full symmetric group, unless there was a previously known geometric reason why the group was smaller. The incidence relations of the 27 lines on a smooth cubic surface prevent the corresponding group from being two-transitive. There exist two of the 27 lines that intersect, and there exist another two that do not. These incidence relations can be used to show that the Galois/monodromy group must be contained in the reflection group $W(E_6) \subset S_{27}$, e.g. [Manin, 1974 Sects. 25, 26] or [Hartshorne, 1977 Prob. V.4.11]; Harris shows that equality holds [Harris, 1979 III.3].

Other examples can be computed based on permutation arrays.

**Corollary 7.1.** The explicit equations defining a Schubert problem in Theorem 5.4 can be used to determine the Galois/monodromy group for the problem as well.

As a toy example, we see that the monodromy group for Example 6.2 is $S_2$, as there are two solutions to the Schubert problem, and the only transitive subgroup of $S_2$ is $S_2$ itself. Algebraically, this corresponds to the fact that the roots of the irreducible quadratic $c^2 + 10c - 20$ in example 6.2 generate a Galois extension of $\mathbb{Q}$ with Galois group $S_2$.

Unfortunately, the calculations of monodromy groups for flag varieties becomes computationally infeasible as $n \to 10$ where the number of solutions becomes larger. Therefore, we have considered related problems of computing Schubert problems for the Grassmannian manifolds $G(k, n)$. Here, $G(k, n)$ is the set of $k$-dimensional planes in $\mathbb{C}^n$. Schubert varieties are defined analogously by rank conditions with respect to a fixed flag. These varieties are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$. The permutation arrays work equally well for keeping track of the rank conditions for intersecting Schubert varieties in the Grassmannian if we replace the condition that a permutation array must have rank $n$ by requiring rank $k$.

In the case of the Grassmannian, combinatorial criteria were given for the Galois/monodromy group of a Schubert problem to be $A_N$ or $S_N$ in [Vakil, 2003b]. Intersections on the Grassmannian manifold may be interpreted as a special case of intersections on the flag manifold, so our computational techniques apply. We sketch the criteria here, and refer the reader to [Vakil, 2003b] for explicit descriptions and demonstrations.

**Criterion 7.2. Schubert Induction.** Given a Schubert problem in the Grassmannian manifold, a choice of geometric degenerations yields a directed rooted tree. The edges are directed away from the root. Each vertex has out-degree between 0 and 2. The portion of the tree connected to an outward-edge of a vertex is called a branch of that vertex. Let $N$ be the number of leaves in the tree.

(i) Suppose each vertex with out-degree two satisfies either (a) there are a different number of leaves on the two branches, or (b) there is one leaf on each branch. Then the Galois/monodromy group of the Schubert problem is $A_N$ or $S_N$. 
(ii) Suppose each vertex with out-degree two has a branch with one leaf. Then the Galois/monodromy group of the Schubert problem is $S_N$.

(iii) Suppose that each vertex with out-degree two satisfies (a) or (b) above, or (c) there are $m \neq 6$ leaves on each branch, and it is known that the corresponding Galois/monodromy group is two-transitive. Then the Galois/monodromy group is $A_N$ or $S_N$.

Part (i) is [Vakil, 2003b, Thm. 5.2], (ii) follows from the proof of [Vakil, 2003b, Thm. 5.2], and (iii) is [Vakil, 2003b, Thm. 5.10]. Criterion (i) seems to apply “almost always”. Criterion (ii) applies rarely. Criterion (iii) requires additional information and is useful only in ad hoc circumstances.

The method discussed in this paper of explicitly (algebraically) solving Schubert problems gives two new means of computing Galois groups. The first, in combination with the Schubert induction rule, is a straightforward means of proving that a Galois group is the full symmetric group. The second gives strong experimental evidence (but no proof!) that a Galois group is smaller than expected.

**Criterion 7.3. Criterion for Galois/monodromy group to be full.** If $m$ flags defined over $\mathbb{Q}$ are exhibited such that the solutions are described in terms of the roots of an irreducible degree $N$ polynomial $p(x)$, and this polynomial has a discriminant that is not a square, then by the arithmetic interpretation (C) above, the Galois/monodromy group is not contained in $A_N$.

Hence in combination with the Schubert induction criterion (i), this gives a criterion for a Galois/monodromy group to be the full symmetric group $S_N$.

(In principle one could omit the Schubert induction criterion: if one could exhibit a single Schubert problem defined over $\mathbb{Q}$ whose Galois group was $S_N$, then the Galois/monodromy group would have to be $S_N$ as well. However, showing that a given degree $N$ polynomial has Galois group $S_N$ is difficult; our discriminant criterion is immediate to apply.)

The smallest Schubert problem where Criterion (i) applies but Criterion (ii) does not is the intersection of six copies of the Schubert variety indexed by the partition $(1)$ in $G(2,5)$ (and the dual problem in $G(3,5)$). Geometrically, it asks how many lines in $\mathbb{P}^4$ meet six planes. When the planes are chosen generally, there are five solutions (i.e. five lines). By satisfying the first criterion we know the Galois/monodromy group is “at least alternating” i.e. either $A_N$ or $S_N$, but we don’t know that the group is $S_N$. We randomly chose six planes defined over $\mathbb{Q}$. Maple found the five solutions, which were in terms of the solutions of the quintic $101z^5-554z^4+887z^3-536z^2+194z-32$. This quintic has non-square discriminant, so we conclude that the Galois/monodromy group is $S_5$. As other examples, the Schubert problem $(2)^2(1)^4$ in $G(2,6)$ has full Galois/monodromy group $S_6$, the Schubert problem $(2)(1)^6$ in $G(2,6)$ has full Galois/monodromy group $S_6$, and the Schubert problem $(2,2)(1)^5$ in $G(3,6)$ has full Galois/monodromy group $S_6$. We applied this to many Schubert problems and found no examples satisfying Criterion (ii) or (iii) that did not have full Galois group $S_N$.

As an example of the limits of this method, solving the Schubert problem $(1)^8$ in $G(2,6)$ is not computationally feasible (it has 14 solutions), so this is the smallest Schubert problem whose Galois/monodromy group is unknown (although Criterion (i) applies, so the group is $A_{14}$ or $S_{14}$).
Criterion 7.4. Probabilistic evidence for smaller Galois/monodromy groups.
If for a fixed Schubert problem, a large number of “random” choices of flags in $\mathbb{Q}^n$
always yield Galois groups contained in a proper subgroup $G \subset S_N$, and the group $G$
is achieved for some choice of Schubert conditions, this gives strong evidence that
the Galois/monodromy group is $G$.

This is of course not a proof — we could be very unlucky in our “random” choices
of conditions — but it leaves little doubt.

As an example, consider the Schubert problem $(2,1,1)(3,1)(2,2)\in G(4,8)$.
There are four solutions to this Schubert problem. When random (rational) choices
of the four conditions are taken, Maple always (experimentally!) yields a solution
in terms of $\sqrt{a + b}\sqrt{c}$ where $a$, $b$, and $c$ are rational. The Galois group of any
such algebraic number is contained in $D_4$: it is contained in $S_4$ as $\sqrt{a + b}\sqrt{c}$ has
at most 4 Galois conjugates, and the Galois closure may be obtained by a tower
of quadratic extensions over $\mathbb{Q}$. Thus the Galois group is a 2-subgroup of $S_4$ and
hence contained in a 2-Sylow subgroup $D_4$.

We found a specific choice of Schubert conditions for which the Galois group of
the Galois closure $K$ of $\mathbb{Q}(\sqrt{a + b}\sqrt{c})$ over $\mathbb{Q}$ was $D_4$. (The numbers $a$, $b$, and $c$
are large and hence not included here; the Galois group computation is routine.)
Thus we have rigorously shown that the Galois group is at least $D_4$, hence $D_4$ or
$S_4$. We have strong experimental evidence that the group is $D_4$.

Challenge: Prove that the Galois group of this Schubert problem is $D_4$.

We conjecture that the geometry behind this example is as follows. Given four
general conditions, the four solutions may be labeled $V_1$, $\ldots$, $V_4$ so that either
(i) $\dim(V_i \cap V_j) = 0$ if $i \equiv j \pmod{2}$ and $\dim(V_i \cap V_j) = 2$ otherwise, or (ii)
$\dim(V_i \cap V_j) = 2$ if $i \equiv j \pmod{2}$ and $\dim(V_i \cap V_j) = 0$ otherwise. If (i) or (ii)
holds then necessarily $G \neq S_4$, implying $G \cong D_4$.

This example (along with the examples of [Vakil, 2003b, Sect. 5.12]) naturally
leads to the following question. Suppose $V_1$, $\ldots$, $V_N$ are the solutions to a Schubert
problem (with generally chosen conditions). Construct a rank table

$$\left\{ \dim \left( \bigcap_{i \in I} V_i \right) \right\}_{I \subset \{1, \ldots, n\}}.$$ 

In each known example, the Galois/monodromy group is precisely the group of
permutations of $\{1, \ldots, n\}$ preserving the rank table.

Question: Is this always true?

Remark. Schubert problems for the Grassmannian varieties were among the
first examples where the Galois/monodromy groups may be smaller than expected.
The first example is due to H. Derksen; the “hidden geometry” behind the smaller
Galois group is clearer from the point of view of quiver theory. Derksen’s example,
and other infinite families of examples, are given in [Vakil, 2003b, Sect. 5.13–5.15].

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