Real time heat sources identification by a branch eigenmodes reduced model

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Abstract. This study presents a method to solve a 3D inverse heat conduction problem in real time. The aim is to identify sequentially the time varying strength of two independent heat sources from non intrusive noisy temperatures measured by an infrared camera. Due to the geometry and the boundary conditions of the experimental setup, the modelling of the system leads to a high dimensional nonlinear model. The size of this detailed model forbids to solve the inverse problem. A reduced model is then built using the branch eigenmodes technique. This study shows that the identification of the heat sources strengths is possible using a very low order reduced model. Moreover, the computation times are very small, allowing the sequential identification in real time.

1. Introduction

Among the several kinds of Inverse Heat Conduction Problems (IHCP) [1-4], this paper deals with the identification of two time varying heat source strengths in a sequential manner from the knowledge of non intrusive temperature measurements. To be used efficiently in industrial context, the IHCP methods have to take into account complex and multi-dimensional geometries, to consider nonlinear governing equations and to give a real time estimation of the unknowns. For multi-dimensional heat conduction problems, a detailed description of the system by classical modelling (finite elements, …) leads to a model of large dimension with a proportional requirement of computation time and memory. In order to lighten this drawback, the boundary element method or the model reduction are very efficient [5-9]. In nonlinear cases, the inverse algorithms are iterative and require repeated computations of governing equations before getting a solution. Hence, it is necessary to adopt efficient techniques to overcome the large amount of computation time and memory required to solve IHCP [10-12]. In the real time computation field of research, the aim is to propose numerical schemes which decrease significantly the computation time necessary for the solution of the optimization problem [13,14]. This paper is structured as follows. First, the 3D experimental device and its modelling are described. Then, the principles of the branch eigenmodes reduction method are briefly presented. The last part focuses on the inverse algorithm and comparisons between the IHCP solutions are made.

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2. Experimental device and modelling

2.1. Description of the 3D system

The studied system is a parallelepiped block (0.164 m × 0.098 m × 0.098 m) composed of steel. The block is drilled in its length by two circular ducts (0.016 m in diameter) as shown in figure 1. The external surfaces of the block are insulated with ceramic sheets (0.024 m thick), except for the two largest vertical faces \( \Gamma_1 \) and \( \Gamma_2 \), painted in black in order to increase the radiation heat transfer. Two cylindrical heat sources (20 mm in diameter and 60 mm long) are placed in the block, as shown in figure 1.

![Figure 1. The 3D diffusive system.](image1)

The temperature measurements are given by an infrared camera positioned 1 m away from the surface \( \Gamma_1 \). A general view of the apparatus is shown in figure 2. The heat sources are driven by a power modulator. Each channel of this power modulator can be adjusted from 1% to 100% of the maximum power of the heat source. This power modulator is controlled from a PC, as well as the power modulation timing. The source strength is then computed from the voltage and the resistance of the heat source. The heat extraction is realized with a high-rate oil flow in a closed circuit with a heat exchanger in such a way that the oil input temperature in the block is constant. The input and output oil temperatures are given by thermocouples connected to the data acquisition system.

![Figure 2. Schematic view of the experiment.](image2)

2.2. Modelling of the system

The transient Detailed Model (DM) of the system is obtained solving the following set of equations:

\[ \forall M \in \Omega_i, \quad i \in \{1,2,3\} \quad \frac{\partial T}{\partial t} = \nabla \cdot \left( k_i \nabla T \right) + Q_i(t) \quad (1) \]

\[ \forall M \in \Gamma_i, \quad i \in \{2\} \quad \hat{n} \cdot \left( k_3 \nabla T \right) = h_{ext} \left( T_{ext} - T \right) + \varepsilon \sigma \left( T_{ext}^4 - T^4 \right) \quad (2) \]

\[ \forall M \in \Gamma_i, \quad i \in \{3,4,5,6\} \quad \hat{n} \cdot \left( k_3 \nabla T \right) = h_{eq} \left( T_{ext} - T \right) \quad (3) \]

\[ \forall M \in \Gamma_7 \quad \hat{n} \cdot \left( k_3 \nabla T \right) = h_{oil} \left( T_{oil}(x) - T \right) \quad (4) \]

where \( T(x,y,z,t) \) is the temperature and \( Q_i(t) \) represents the time dependent strength of the heat source in the domain \( \Omega \). Table 1 summarizes the thermophysical properties of the heterogeneous system. \( T_{ext} \) is the ambient temperature, \( h_{ext} = 8 \text{ W.m}^{-2}\text{K}^{-1} \) and \( h_{eq} = 2.5 \text{ W.m}^{-2}\text{K}^{-1} \) are convective exchange coefficients, \( \varepsilon = 0.95 \) is the emissivity of the black painted surfaces \( \Gamma_1 \) and \( \Gamma_2 \), and \( \sigma \) is the Stefan-
Boltzmann constant, \( \hat{n} \) being the normal unit vector. The oil temperature \( T_{oil}(x,t) \) is supposed to be linearly dependent of \( x \) and the convective exchange coefficient for the established laminar oil flow in the two ducts is equal to \( h_{oil} = 140 \text{ W.m}^{-2}\text{.K}^{-1} \). The nonlinear behaviour of the system is due to the radiative exchange boundary condition. The initial condition is given by \( T(t=0) = 295.7 \text{ K} \).

### Table 1. Thermophysical properties of the domains.

| Domain | Thermal conductivity \( k \) (W.m\(^{-1}\).K\(^{-1}\)) | Volumetric heat capacity \( c \) (J.m\(^{-3}\).K\(^{-1}\)) | Heat source strength \( Q \) (W.m\(^{-3}\)) |
|--------|------------------------------------------------|-------------------------------------------------|---------------------------------|
| \( \Omega_1 \) | 10 | 3.557 \( 10^6 \) | \( Q_1 \) |
| \( \Omega_2 \) | 10 | 3.557 \( 10^6 \) | \( Q_2 \) |
| \( \Omega_3 \) | 52 | 3.532 \( 10^6 \) | 0 |

The spatial discretization of the system using the finite element method leads to a large number of mesh nodes, \( N = 23139 \), due to the 3D geometry and the cylindrical surfaces of oil ducts and heat sources. After spatial discretization, equations (1-4) can be written under matrix form:

\[
\begin{bmatrix}
C & \tilde{T}(t) = K(T)T(t) + \Phi(t) + BU(t) \\
Y(t) = S(T(t))
\end{bmatrix}
\]

where \( C \) and \( K \) (dim. \( N \times N \)) are respectively the heat capacitance and heat conductance matrices, \( \Phi \) (dim. \( N \)) the vector of thermal excitations, \( B \) (dim. \( N \times p \)) the command matrix relative to the input vector \( U(t) \). The matrix \( S \) (dim. \( q \times N \)) is the observation matrix which allows us to select a part of the whole temperature field \( T(t) \). This selection is contained in the output vector \( Y(t) \) (dim. \( q \)). In our case, \( p = 2 \), \( U(t) \) is the vector containing the heat source strengths \( Q_1 \) and \( Q_2 \).

An implicit first-order scheme with adaptative time steps is implemented and the conjugate gradient method is used to solve the high dimensional nonlinear system.

First, it is essential to verify that DM leads to responses that fit with experiment. With this aim in view, a test is realized with a set of variations for \( Q_1(t) \) and \( Q_2(t) \), as depicted in figure 7. In figure 4, the measurement of temperature evolutions of two pixels are presented (\( T_1 \) and \( T_7 \) in figure 3) and compared with DM computations. Hence, DM seems faithful to the experiment.

**Figure 3.** Position of the measurement pixels of the camera.

**Figure 4.** Comparison of camera and DM temperature evolutions for the points \( T_1 \) and \( T_7 \).
Due to the size of the matrices (dim. $N \times N$), DM is very difficult to use to solve the inverse problem. The objective of the next part is to build a Reduced Model (RM), computationally efficient.

3. Reduced model

3.1. Branch eigenmodes model reduction

The technique used in order to solve IHCP is the branch eigenmodes reduction method [15-17]. The branch eigenmodes problem applied to a thermal system is defined by the following equations:

\begin{equation}
\forall M \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \quad \vec{\nabla} \cdot \left( \vec{k} \vec{V}_i(M) \right) = \lambda_i c \vec{V}_i(M)
\end{equation}

\begin{equation}
\forall M \in \Gamma_j \quad j \in [1,2,3,4,5,6,7] \quad -k_3 \vec{\nabla} V_i(M) \cdot \vec{n}(M) = \lambda_i \zeta \vec{V}_i(M)
\end{equation}

Compared to the classical eigenmodes problem, the eigenvalue $\lambda_i$ appears in the boundary condition. Equation (7) is called the Steklov boundary condition. This condition allows to consider nonlinear problems. In the studied case, $\zeta = 180000 \text{ J.m}^{-2}.\text{K}^{-1}$. The first $N_E = 4000$ eigenmodes are computed using the Arnoldi technique [18]. The temperature field is then decomposed as follows:

\begin{equation}
T(M,t) = \sum_{i=1}^{N_E} X_i(t) V_i(M)
\end{equation}

where $N_E \leq N$ is the number of computed eigenmodes and $X_i(t)$ the state vector related to the eigenmode $V_i(M)$. The objective of RM is to compute the temperature field, with the same equation as equation (8), but with less eigenmodes, in order to decrease the computation time:

\begin{equation}
\hat{T}(M,t) = \sum_{i=1}^{n} \hat{X}_i(t) \hat{V}_i(M) \equiv \sum_{i=1}^{N_E} X_i(t) V_i(M)
\end{equation}

with $n << N_E$ and $\hat{V}_i$ being the $i$th amalgam eigenmode of the reduced model, computed with the simplified amalgam method [19]. The objective is to bring together the eigenmodes into $n$ subspaces ($n << N$), where $n$ is the order of RM. First, the $n$ major eigenmodes are chosen. Then, the distribution of the minor eigenmodes into the $n$ subspaces is performed, according to a measure of the reduction error. Finally, the $n$ amalgam eigenmodes are brought together into matrix $\hat{V}$ (dim. $N \times n$) and RM can be expressed under matrix form:

\begin{equation}
\begin{cases}
L \hat{X}(t) = M(\hat{T}) \hat{X}(t) + \hat{V}^T \Phi(t) + GU(t) \\
\hat{V}(t) = H \hat{X}(t)
\end{cases}
\end{equation}

With: $\hat{T} = \hat{V} \hat{X}$, $L = \hat{V}^T C \hat{V}$, $M(\hat{T}) = \hat{V}^T K(\hat{T}) \hat{V}$, $G = \hat{V}^T B$, $H = S \hat{V}$

Equation (10) is a system of reduced order $n$. As for DM, an implicit first-order scheme with adaptive time steps is implemented. The LDLT factorization method, well adapted to low order matrices, is used to solve the system.

3.2. RM validation

Different reduced models are tested: $n = 5, 20, 40, 60, 80$ and $100$. Note that the abbreviation RM5 refers to $n = 5$. As for DM, the same test is realized for RM with $Q_1(t)$ and $Q_2(t)$ represented in figure 7. Note that the use of RM provides a large reduction ratio, inducing an important gain in CPU time: 5 s for the whole simulation using RM5, 65 s with RM60, compared to 1120 s for DM. The
comparison of the temperature evolutions between camera and RM5 and between camera and RM60 for the points $T_1$ and $T_7$ are shown respectively in figure 5 and figure 6.

![Figure 5](image1.png) ![Figure 6](image2.png)

Figure 5. Comparison of camera and RM5 temperature evolutions for the points $T_1$ and $T_7$.

Figure 6. Comparison of camera and RM60 temperature evolutions for the points $T_1$ and $T_7$.

As expected, RM60 gives the best results, with a temperature deviation less than 2.5 K.

4. Inverse problem

4.1. Inverse algorithm

RM is now used in order to solve IHCP with a sequential procedure. Knowing the input vector $U(k)$ at the time step $k$, the vector $U(k+1)$ is identified from temperatures at the time step $(k + 1)$. Using an implicit time discretization, equation (10) leads to a relation between RM output vector and the input vector $U(k+1)$:

$$
\tilde{Y}(k + 1) = H \, \mathcal{G} \, U(k + 1) + H \, [M \, \tilde{X}(k) + L \, \Phi(k + 1)]
$$

(11)

with the following matrices: $M = [I - \Delta t \, L^{-1} \, M(T)]^{-1}$, $L = \mathcal{M} \, \Delta t \, L^{-1} \, \tilde{V}^T$, $\mathcal{G} = \mathcal{M} \, \Delta t \, L^{-1} \, G$

In order to take into account the lagging and damping effects of the diffusion process, it is necessary to obtain information using future time steps [1,4,6]. A temporary assumption is made on the additional unknowns: $U(k+1+1)$, ..., $U(k+1+nf)$, where $nf$ is the number of future time steps. In this study, a constant value of $U$ is chosen:

$$
U(k + 1 + i) = U(k + 1) = \text{constant} \quad \text{for} \quad 1 \leq i \leq nf
$$

(12)

The addition of future time steps leads to the resolution of a system of $(nf + 1) \times q$ equations where $q$ is the number of measurements. Hence, at each time step, the temperature measurements vector $\mathbf{Y}^*$ contains $(nf + 1)$ rows, as follows:
The system to solve is then overdetermined. The aim is to identify the pseudo-solution \( \hat{U}(k+1) \) of IHCP, such as \( \mathbf{Y}^* - \mathbf{Y} \cong 0 \), where \( \mathbf{Y} \) is the temperature vector computed by RM. The inversion procedure, using the least square method, leads to the sequential solution:

\[
\hat{U}(k+1) = (\mathbf{C}^T \mathbf{C})^{-1} \mathbf{C}^T (\mathbf{Y}^* - \mathbf{D})
\]

with the following matrices:

\[
\mathbf{C} = \begin{bmatrix}
    \mathbf{H} \quad \mathbf{G} \\
    \mathbf{H} [\mathbf{G} + \mathbf{M} \mathbf{G}] \\
    \vdots \\
    \mathbf{H} [\mathbf{I} + \sum_{j=1}^{nf} \mathbf{M}^j] \mathbf{G} \\
    \forall \ nf \geq 1
\end{bmatrix}
\]

\[
\mathbf{D} = \begin{bmatrix}
    \mathbf{H} \left[ \mathbf{M} \bar{X}(k) + \mathbf{L} \Phi(k+1) \right] \\
    \mathbf{H} \left[ \mathbf{M}^2 \bar{X}(k) + \mathbf{M} \mathbf{L} \Phi(k+1) + \mathbf{L} \Phi(k+2) \right] \\
    \vdots \\
    \mathbf{H} \left[ \mathbf{M}^{nf+1} \bar{X}(k) + \sum_{j=0}^{nf} \mathbf{M}^j \mathbf{L} \Phi(k+nf+1-j) \right]
\end{bmatrix}
\]

Using RM, the IHCP can be solved while it is impossible using DM due to the size of the matrices to invert.

4.2. IHCP results

4.2.1. Inversion from points \( T_1 \) and \( T_7 \). In this case, the inversion is carried out using only two points: \( T_1 \) and \( T_7 \). As these two points are located on the block surface, the thermal signal coming from the heat sources reaches the sensors with a delay. This duration can be estimated to 50 s. Hence, as the time step is equal to 10 s, the number of future temperatures used here is \( nf = 4 \). The inversion results for RM5 and RM60 are respectively presented in figure 7 and figure 8, as well as the measured strengths.
These results are satisfactory except for the identification of the peaks. It clearly appears from figure 7 and figure 8 that RM5 reduces the oscillations in the identified values of $Q_1(t)$ and $Q_2(t)$, which means that RM5 acts as a regularization procedure in IHCP.

4.2.2. Inversion from 9 points. As the camera gives the whole temperature field of the surface $\Gamma_1$ at each time step, it is easy to carry out the inversion using a larger number of temperature measurements. Hence, the 9 points represented in figure 3 are used to solve IHCP. As previously, $n_f = 4$. The inversion has been carried out using several RMs. The results obtained with RM5 and RM60 are presented respectively in figure 9 and figure 10.

The results are clearly improved, especially for RM60, which deletes oscillations. An important bias is observed for $Q_2$ from the time $t = 1350$ s. This is probably due to an imprecise knowledge of the physical parameters of the model. Table 2 summarizes the computation time per time step and the
mean quadratic criterion between the identified inputs \( \hat{U} \) and the measured ones \( U \), computed as follows:

\[
\sigma_U = \left[ \frac{1}{p \cdot (nt - nf)} \sum_{i=t}^{p(nt-nf)} (U_i - \hat{U}_i)^2 \right]^{1/2}
\]

where \( nt \) is the number of time steps and \( p \) the number of inputs. In our case, \( nt = 361 \) and \( p = 2 \).

**Table 2.** Comparison of inversion results for different RM, \( nf = 4 \).

| Reduced Model | RM5   | RM20  | RM40  | RM60  | RM80  | RM100 |
|---------------|-------|-------|-------|-------|-------|-------|
| \( \sigma_U \) (W.m\(^{-2}\)) | \(2.17 \times 10^6\) | \(2.10 \times 10^6\) | \(2.04 \times 10^6\) | \(1.99 \times 10^6\) | \(2.09 \times 10^6\) | \(2.01 \times 10^6\) |
| CPU time per step (s) | 0.013 | 0.039 | 0.086 | 0.158 | 0.243 | 0.342 |

It can be underlined that the increase of RM order does not improve significantly the inversion results, contrary to the direct problem. Moreover, due to the low dimension of RM, CPU time is very small, making real time identification possible. For example, the identification of both inputs requires only 0.013 s at each time step for RM5.

**5. Conclusion**

This study focuses on the real time identification of two thermal loads dissipated in heat sources from non intrusive temperatures measured by an infrared camera. The geometry and the boundary conditions of the system require a 3D nonlinear model of high dimension, which can not be used in an inverse procedure. A reduced model of low dimension is then built using the branch eigenmodes technique. On the one hand, even if RM order is very low (\( n = 5 \) compared to \( N = 23139 \)), the results are satisfactory. It is shown that when RM order increases, oscillations in the identified inputs can be deleted adding temperature measurement points. On the other hand, the essential feature of this work lies in the fact that the computation time is very small. This is due to the low dimension of RM state matrices, a dimension which allows to solve the IHCP in real time.

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