A Non-Comprehensive Survey Of Integration Algorithms In Discrete Geometry

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Abstract

The paper suggests is a suggestion for a theoretical basis to integration algorithms which evolved since 1982, hence a short survey to these algorithms is depicted in the previous work part.
Part I
Prologue

“Beginnings always form and ending to a passing period, that will remain with you. Another sunrise, another sunset, another pain that forever will remain with you.”
– Tzuf Philosof.

1 Introduction

Ever since 1982, the Discrete Geometry community has issued integration algorithms that form discrete versions to the integration theorems of the advanced Calculus (the discrete Green’s theorem and the discrete Stokes’ theorem). The discrete versions of these theorems were required to enable computational efficiency: for example, the discrete Green’s theorem enables to calculate the double integral of a function in a discrete domain (a domain whose boundary is parallel to the axes, as is usually the case in Discrete Geometry) in a more efficient manner with respect to the regular Green’s theorem, due to the fact that the calculation is taken place based on the corners of the discrete region, and there is no need to pass through the entire boundary - as opposed to the regular Green’s theorem, that points out the connection between the double integral and the line integral.

In this paper we will suggest a theory - in \( \mathbb{R} \) and in \( \mathbb{R}^2 \) - to the origin of these discrete theorems. The paper is divided as follows. In part 1, depicted is a short survey of early work in Discrete Geometry. In part 2 the basic operators are defined. The operators are shown to be simple tools for the analysis of monotonic regions of any function. In part 3 a mathematical discussion is held, where: the connection between the properties of these operators to those of the familiar derivative is surveyed; analogous versions to some of the most fundamental theorems of Calculus are depicted; the geometric interpretation of one of the operators is drawn; a discussion is held regarding the cases where it is impossible to apply one or more of the operators to functions; a general function that demonstrates the operators is shown; and a structure to the Calculus is suggested. In part 4, an engineering-oriented discussion is held regarding the computational cost of the usage of the suggested operators. Amongst others, a proof is suggested to the computational preferability of the usage in these operators, when compared to the derivative. Afterwards, terminology and theorems are suggested, whose aim is to generalize the discrete Green’s theorem to any type of domain. In part 5 the paper is sealed; the appendix suggests some other results in Calculus, such as another definition to the limit process, and another type of continuousness.
2 Previous Work

In this section the author chose to depict basic concepts from discrete geometry.

2.1 Integral Image

Followed is an introduction to probably one of the most stunning breakthroughs in the field of computational-gain driven integral calculus over discrete domains, which was first introduced (to the author’s knowledge) by Tang in 1982. The idea of “summed area tables”, was later introduced by Lance Williams and Franklin Crow in ([6]). Yet, the most influential paper in this area is Viola and Jones’ “Integral Image” concept ([1]), which applies summed are tables to fast calculations of sums of squares in an image.

The idea is as follows. Given a function $i$ over a discrete domain $\prod_{j=1}^{2} [m_j, M_j] \in \mathbb{Z}^2$, define a new function ($sat$ stands for summed area table, and $i$ stands for image):

$$sat(x, y) = \sum_{x' \leq x \land y' \leq y} i(x', y'),$$

and now the sum of all the values that the function $i$ accepts on the grid $[a, b] \times [c, d]$, where $m_1 \leq a, b \leq M_1$ and $m_2 \leq c, d \leq M_2$, equals:

$$\sum_{x' \leq a} \sum_{y' \leq b} i(x', y') = sat(b, d) + sat(a, c) − sat(a, d) − sat(b, c).$$

2.2 Integral Video

The idea of Integral Image was extended by Yan Ke, Rahul Sukthankar and Martial Hebert in [3]. This concept was named “Integral Video”, for it aims to calculate the sum of volumetric features defined over a video sequence. It generalizes the Integral Image concept in the sense, that the cumulative function is generalized to deal with three dimensions. Namely, given a function $i$ over a discrete domain $\prod_{i=1}^{3} [m_i, M_i] \in \mathbb{Z}^3$, define a new function:

$$sat(x, y, z) = \sum_{x' \leq x \land y' \leq y \land z' \leq z} i(x', y', z'),$$

and now the sum of all the values that the function $i$ accepts on the grid $[a, b] \times [c, d] \times [e, f]$, where $m_1 \leq a, b \leq M_1$, $m_2 \leq c, d \leq M_2$, and $m_3 \leq e, f \leq M_3$, equals:

$$\sum_{x' \leq a} \sum_{y' \leq b} \sum_{z' \leq c} i(x', y', z') = sat(b, d, f) − sat(b, d, e) − sat(b, c, f) + sat(b, c, e) + sat(a, d, f) − sat(a, d, e) − sat(a, c, f) − sat(a, c, e).$$
2.3 Rotated Integral Image

Lienhart and Maydt presented a rotated version of the Integral Image concept, in [8].

2.4 Wang et al.’s Formula

Wang et al. ([7]) suggested to further generalize the Integral Image concept, in 2007. They issued the following argument, without a proof:

“Given a function $f(x): \mathbb{R}^k \rightarrow \mathbb{R}^m$, and a rectangular domain $D = [u_1, v_1] \times \ldots \times [u_k, v_k] \subset \mathbb{R}^k$. If there exists an antiderivative $F(x): \mathbb{R}^k \rightarrow \mathbb{R}^m$, of $f(x)$, then:

$$\int_D f(x) \, dx = \sum_{\nu \in B^k} (-1)^{\nu^T_1} F(\nu_1 u_1 + \nu_2 u_2, \ldots, \nu_k u_k + \nu_k u_k),$$

where $\nu = (\nu_1, \ldots, \nu_k)^T$, $\nu^T_1 = \nu_1 + \ldots + \nu_k$, $\nu^T_k = 1 - \nu_k$, and $B = \{0, 1\}$. If $k = 1$, then $\int_D f(x) \, dx = F(v_1) - F(u_1)$, which is the Fundamental Theorem of Calculus. If $k = 2$, then

$$\int_D f(x) \, dx = F(v_1, v_2) - F(v_1, u_2) - F(u_1, v_2) + F(u_1, u_2),$$

and so on.”

This formula suggests a tremendous computational power in many applications, such as in the probability and the computer vision field, as was shown to hold in [7].

2.5 Discrete Version of Stokes’ Theorem

Back in 1982, Tang ([10]) suggested a discrete version of Green’s theorem. In their paper from 2007, Wang et al.’s ([7]) suggested to further generalize Tang’s theorem to any finite dimension. This is in fact the first time, known to the author, that a discrete version to Stokes’ theorem was published. In their paper from 2009, Labelle and Lacasse ([9]) suggested a different proof to the same theorem.
Part II
Basic Terminology in Calculus

“To the Master’s honor all must turn, each in its track, without a sound, forever tracing Newton’s ground.”
– Albert Einstein.

3 Pseudo continuousness

**Definition.** Pseudo continuousness. Given a function \( f \colon \mathbb{R} \to \mathbb{R} \), we will say that it is weak pseudo-continuous from right or left in a point \( x \in \mathbb{R} \), if the following right or left limits exist accordingly:

\[
\exists \lim_{h \to 0^s} f(x + h), \ s \in \{\pm 1\}.
\]

**Examples.**

1. The function:

\[
f(x) = \begin{cases} 
|\sin\left(\frac{1}{x}\right)|, & x \neq 0 \\
0, & x = 0
\end{cases}
\]

is not pseudo continuous in \( x = 0 \). Neither is Dirichlet’s function.

2. From the definition of the continuousness, every continuous function is pseudo continuous.

4 Definition of Operators

**Definition.** The sign operator. Given a constant \( r \in \mathbb{R} \), we will define \( sgn(r) \) as follows:

\[
sgn(r) \equiv \begin{cases} 
1, & r > 0 \\
-1, & r < 0 \\
0, & r = 0
\end{cases}
\]

**Definition.** Detachable function in a point. Given a function \( f : \mathbb{R} \to \mathbb{R} \), we will say that \( f \) is tendable in a point \( x \in \mathbb{R} \) if the following limit exists:

\[
\exists \lim_{h \to 0} \{sgn[f(x + h) - f(x)]\}.
\]

**Definition.** Right-detachable function in a point. Given a function \( f : \)
R \rightarrow \mathbb{R}$, we will say that it is right-detachable in a point $x \in \mathbb{R}$, if the following limit exists:

$$\exists \lim_{h \to 0^+} \{ \text{sgn} [f(x + h) - f(x)] \}.$$ 

**Definition.** Left-detachable function in a point. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will say that it is left-detachable in a point $x \in \mathbb{R}$, if the following limit exists:

$$\exists \lim_{h \to 0^-} \{ \text{sgn} [f(x + h) - f(x)] \}.$$ 

**Claim.** A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is detachable in a point $x_0 \in \mathbb{R}$ iff it is both left and right detachable in $x_0$, and the limits are equal. 

**Proof.** Immediate from the fact that a limit of a function in a point exists iff both directions limits exist. 

**Definition.** Detachable function in an interval. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will say that $f$ is detachable in an interval $I$ if one of the following holds:

1. $I = (a, b)$, and $f$ is detachable in each $x \in (a, b)$.
2. $I = [a, b)$, and $f$ is detachable in each $x \in (a, b)$ and right detachable in $a$.
3. $I = (a, b]$, and $f$ is detachable in each $x \in (a, b)$ and left detachable in $b$.
4. $I = [a, b]$, and $f$ is detachable in each $x \in (a, b)$, left detachable in $b$ and right detachable in $a$.

**Definition.** Detachment in a point. Given a detachable function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will define the detachment operator applied for $f$ as:

$$f : \mathbb{R} \rightarrow \{ +1, -1, 0 \}$$

$$f^i(x) \equiv \lim_{h \to 0} \{ \text{sgn} [f(x + h) - f(x)] \}.$$ 

Applying the detachment operator to a function will be named: “detachment of the function”.

**Definition.** Left or right detachment in a point. Given a left or right detachable function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will define the left or right detachment operators applied for $f$ as:

$$f^s : \mathbb{R} \rightarrow \{ +1, -1, 0 \}$$

$$f^s_s(x) \equiv \lim_{h \to 0} \{ \text{sgn} [f(x + h) - f(x)] \}, \ s \in \{ \pm 1 \}.$$ 

Applying the detachment operator to a function will be named: “left or right detachment of the function”.

**Definition.** Detachment in an open interval point. Given function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will say that it is detachable in an open interval $I$ if one of the following holds:

...
\( R \), if it is detachable in an open interval \((a, b)\), we will define its detachment in each point of the interval as the detachment in a point, that was defined earlier.

**Definition.** Detachment in semi-closed or closed interval. Given function \( f : R \to R \), if it is detachable in a semi-closed interval or a closed interval, \((a, b]\) or \([a, b]\), then we will define its detachment in each point of the interval \((a, b)\) as the detachment in a point, that was defined earlier. The detachment in the end points \(a\) or \(b\) will be taken as the right or left detachment respectively.

**Remark.** The detachment of the function hides the information regarding the rate of change of the function. It is a trade-off between efficiency and information level, as will be discussed later on.

**Definition.** Signposted detachable function in a point. Given a function \( f : R \to R \), we will say that \( f \) is signposted detachable in a point \( x \in R \) if the following limit exists:

\[
\exists \lim_{h \to 0} \{ \operatorname{sgn} [h \cdot (f(x + h) - f(x))] \}.
\]

**Definition.** Right signposted detachable function in a point. Given a function \( f : R \to R \), we will say that it is right signposted detachable in a point \( x \in R \), if the following limit exists:

\[
\exists \lim_{h \to 0^+} \{ \operatorname{sgn} [h \cdot (f(x + h) - f(x))] \},
\]
i.e., if it is right detachable in the point.

**Definition.** Left signposted detachable function in a point. Given a function \( f : R \to R \), we will say that it is left signposted detachable in a point \( x \in R \), if the following limit exists:

\[
\exists \lim_{h \to 0^-} \{ \operatorname{sgn} [h \cdot (f(x + h) - f(x))] \},
\]
i.e., if it is right detachable in the point.

**Claim.** A function \( f : R \to R \) is signposted detachable in a point \( x_0 \in R \) iff it is both left and right signposted detachable in \( x_0 \), and the limits are equal.

**Proof.** Immediate from the fact that a limit of a function in a point exists iff both directions limits exist. \( \Box \)

**Definition.** Signposted detachable function in an interval. Given a function \( f : R \to R \), we will say that \( f \) is signposted detachable in an interval \( I \) if one of the following holds:

1. \( I = (a, b) \), and \( f \) is signposted detachable in each \( x \in (a, b) \).
2. $I = [a, b]$, and $f$ is signposted detachable in each $x \in (a, b)$ and right (signposted) detachable in $a$.

3. $I = (a, b]$, and $f$ is signposted detachable in each $x \in (a, b)$ and left (signposted) detachable in $b$.

4. $I = [a, b]$, and $f$ is signposted detachable in each $x \in (a, b)$, left (signposted) detachable in $b$ and right (signposted) detachable in $a$.

**Definition.** Signposted detachment in a point. Given a signposted detachable function $f : \mathbb{R} \to \mathbb{R}$, we will define the signposted detachment operator applied for $f$ as:

$$f^\uparrow : \mathbb{R} \to \{+1, -1, 0\}$$

$$f^\uparrow (x) \equiv \lim_{h \to 0} \{ \text{sgn} [h \cdot (f (x + h) - f (x))] \} .$$

Applying the signposted detachment operator to a function will be named: “signposted detachment of the function”.

**Definition.** Left or right signposted detachment in a point. Given a left or right signposted detachable function $f : \mathbb{R} \to \mathbb{R}$, we will define the left or right signposted detachment operators applied for $f$ as:

$$f^s : \mathbb{R} \to \{+1, -1, 0\}$$

$$f^s (x) \equiv \lim_{h \to 0} \{ \text{sgn} [h \cdot (f (x + h) - f (x))] \} , \ s \in \{+1, -1\} .$$

Applying the signposted detachment operator to a function will be named: “left or right signposted detachment of the function”.

**Definition.** Signposted detachment in an open interval point. Given function $f : \mathbb{R} \to \mathbb{R}$, if it is signposted detachable in an open interval $(a, b)$, we will define its signposted detachment in each point of the interval as the signposted detachment in a point, that was defined earlier.

**Definition.** Signposted detachment in semi-closed or closed interval. Given function $f : \mathbb{R} \to \mathbb{R}$, if it is signposted detachable in a semi-closed interval or a closed interval, $(a, b], [a, b]$ or $[a, b]$, then we will define its signposted detachment in each point of the interval $(a, b)$ as the signposted detachment in a point, that was defined earlier. The signposted detachment in the end point $a$ or $b$ will be taken as the right or left signposted detachment respectively.

**Definition.** Null disdetachment. Given a function $f : \mathbb{R} \to \mathbb{R}$, let $x \in \mathbb{R}$ be a point there. We will say that $f$ is null disdetachable there if it is detachable from both sides, but not detachable nor signposted detachable there.

**Definition.** Upper detachable function. Given a function $f : \mathbb{R} \to \mathbb{R}$,
we will say that it is upper detachable in a point \( x \in \mathbb{R} \), if the following partial limit exist:
\[
\exists \limsup_{h \to 0} \{ \text{sgn}\left[(f(x) + h) - f(x)\right]\}.
\]

**Definition.** The upper detachment operator. Given a function \( f : \mathbb{R} \to \mathbb{R} \) (not necessarily upper detachable), we will define the upper detachment operators applied for \( f \) as:
\[
\sup f : \mathbb{R} \to \{+1, -1, 0\}
\]
\[
\sup f^s(x) \equiv \limsup_{h \to 0^+} \{ \text{sgn}\left[(f(x) + h) - f(x)\right]\}, \ s \in \{\pm 1\}.
\]

Applying the upper detachment operator to a function will be named: “upper detachment of the function”.

**Definition.** Lower detachable function. Given a function \( f : \mathbb{R} \to \mathbb{R} \), we will say that it is lower detachable in a point \( x \in \mathbb{R} \), if the following partial limit exist:
\[
\exists \liminf_{h \to 0} \{ \text{sgn}\left[(f(x) + h) - f(x)\right]\}.
\]

**Definition.** The lower detachment operator. Given a function \( f : \mathbb{R} \to \mathbb{R} \) (not necessarily lower detachable), we will define the lower detachment operators applied for \( f \) as:
\[
in f : \mathbb{R} \to \{+1, -1, 0\}
\]
\[
in f^s_s(x) \equiv \liminf_{h \to 0^-} \{ \text{sgn}\left[(f(x) + h) - f(x)\right]\}, \ s \in \{\pm 1\}.
\]

Applying the lower detachment operator to a function will be named: “lower detachment of the function”.

**Remark.** The matching definitions for upper and lower signposted detachment are skipped.

**Examples.**

1. Let us consider the following function:
\[
f : \mathbb{R} \to \mathbb{R}
\]
\[
f(x) = \begin{cases} 
x, & x \in \mathbb{Q} 
x^2, & x \notin \mathbb{Q}.
\end{cases}
\]

Then \( f \) is not continuous in \( x = 0 \), however it is right detachable and right signposted detachable there, and \( \dot{f}_+^0(0) = +1 \). However, it is not left detachable there.
2. Let us consider the following function:

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = \begin{cases} 
  x \sin \left( \frac{1}{x} \right), & x \neq 0 \\
  0, & x = 0.
\end{cases} \]

Then \( f \) is not right nor left detachable nor signposted detachable in \( x = 0 \) since the limits \( \lim_{h \to 0^\pm} \text{sgn} [f(h) - f(0)] \) do not exist, although it is continuous in \( x = 0 \).

3. Let us consider the following function:

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = \begin{cases} 
  x^2 \sin \left( \frac{1}{x} \right), & x \neq 0 \\
  0, & x = 0.
\end{cases} \]

Then \( f \) is not right nor left detachable nor signposted detachable in \( x = 0 \) since the limits \( \lim_{h \to 0^\pm} \text{sgn} [f(h) - f(0)] \) do not exist, although it is differentiable in \( x = 0 \).

4. The function \( f : \mathbb{R} \to \mathbb{R}, \ f(x) = |x| \) is detachable in \( x = 0 \), and signposted detachable in \( \mathbb{R} \setminus \{0\} \).

5. It is not true that if \( f \) is detachable in a point \( x \) then there exists a neighborhood \( I_\varepsilon(x) \) such that \( f \) is signposted detachable in \( I_\varepsilon \). Let \( \{q_1, q_2, \ldots\} \) be an ordering of the rational numbers amongst \( \mathbb{R}^+ \cup \{0\} \). Let us consider the following function:

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = \begin{cases} 
  7, & x \in \mathbb{R} \setminus \mathbb{Q} \\
  |x|(-1)^n, & |x| = q_n \in \mathbb{Q}.
\end{cases} \]

Then \( f(0) = -1 \), however it is easy to see that for any other point \( x \in \mathbb{R} \), there are infinitely many points in any punctured neighborhood where \( f \) receives higher values, and infinitely many points in any punctured neighborhood where \( f \) receives lower values, than in the point \( x \). Hence, the only point where \( f \) is detachable is \( x = 0 \), and it is not detachable nor signposted detachable anywhere else.

6. Let us consider the function:

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ f(x) = \begin{cases} 
  0, & x \in \mathbb{R} \setminus \mathbb{Z} \\
  1, & x \in \mathbb{Z}.
\end{cases} \]
Then the detachment of $f$ is:

$$f^+: \mathbb{R} \to \mathbb{R}$$

$$f^+ (x) = \begin{cases} 
0, & x \in \mathbb{R} \setminus \mathbb{Z} \\
-1, & x \in \mathbb{Z},
\end{cases}$$

hence it exists in any point, although $f$ is not continuous in infinitely many points. The signposted detachment of $f$ is:

$$f^\dagger: \mathbb{R} \setminus \mathbb{Z} \to \{0\}$$

$$f^\dagger (x) = 0,$$

since for all the points $x \in \mathbb{Z}$ it holds that $f^+_\uparrow (x) \neq -f^\downarrow (x)$.

7. Let us consider the function: $f : \mathbb{R} \to \mathbb{R}$, $f (x) = x^2 + x$. Then:

$$f^+_\uparrow (x) = \lim_{h \to 0^+} \text{sgn} \left[ (x + h)^2 + x + h - x^2 - x \right]$$

$$= \lim_{h \to 0^+} \text{sgn} \left[ x^2 + 2xh + h^2 + x + h - x^2 - x \right]$$

$$= \lim_{h \to 0^+} \text{sgn} \left[ 2hx + h^2 + h \right]$$

$$= \lim_{h \to 0^+} \text{sgn} \left[ 2hx + h \right]$$

$$= \text{sgn} \left[ 2x + 1 \right]$$

$$= \begin{cases} 
-1, & x < -\frac{1}{2} \\
+1, & x \geq -\frac{1}{2}.
\end{cases}$$

8. The function:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f (x) = \begin{cases} 
\tan (x), & x \neq \frac{\pi}{2} + \pi k, \\
0, & x = \frac{\pi}{2} + \pi k \quad k \in \mathbb{Z}
\end{cases}$$

is everywhere signposted detachable and nowhere detachable.

9. Riemann’s function:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f (x) = \begin{cases} 
\frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \\
0, & x \in \mathbb{R} \setminus \mathbb{Q}
\end{cases}$$

is nowhere signposted detachable. It is detachable on the rationals, since for any point $x \in \mathbb{Q}$ it holds that $f^+ (x) = -1$, and it is not detachable on the irrationals, for any point $x \notin \mathbb{Q}$, the terms $f^\uparrow (x)$ do not exist. Further, Riemann’s function is upper and lower detachable in any point, since for the irrationals it holds that $\sup f^+ (x) = +1$ and $\inf f^+ (x) = 0$. It is an easy exercise to show that it is nowhere upper nor lower signposted detachable.
10. Dirichlet’s function:

\[ f : \mathbb{R} \to \{0, 1\} \]

\[ f(x) = \begin{cases} 
1, & x \in \mathbb{Q} \\
0, & x \notin \mathbb{Q}.
\end{cases} \]

is not detachable at any point. However, it is upper and lower detachable in any point, and:

\[ \sup f^i(x) = \begin{cases} 
0, & x \in \mathbb{Q} \\
+1, & x \notin \mathbb{Q},
\end{cases} \]

\[ \inf f^i(x) = \begin{cases} 
-1, & x \in \mathbb{Q} \\
0, & x \notin \mathbb{Q}.
\end{cases} \]

11. The function:

\[ f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\} \]

\[ f(x) = \begin{cases} 
\frac{1}{x}, & x \in \mathbb{Q} \\
-\frac{1}{x}, & x \in \mathbb{R} \setminus \mathbb{Q}
\end{cases} \]

is detachable from the right for any point \( x > 0 \), and detachable from the left for any point \( x < 0 \). However, this function is discontinuous anywhere.

12. Both the directions of the argument:

\[ f^\pm_i(x_0) = k \iff \lim_{x \to x_0} f^i(x) = k \]

are incorrect. For example, the function:

\[ f : [0, 1] \to \mathbb{R} \]

\[ f(x) = \begin{cases} 
1, & x = 0 \\
0, & 0 < x \leq 1
\end{cases} \]

satisfies \( \lim_{x \to 0^+} f^i(x) = 0 \) although \( f^+_{i}(0) = -1 \). Further, the function (due to Fridyandy):

\[ f : (-1, 1) \to \mathbb{R} \]

\[ f(x) = \begin{cases} 
\sin \left( \frac{1}{x} \right), & x \neq 0 \\
17, & x = 0.
\end{cases} \]

satisfies that \( f^+_{i}(0) = -1 \) although \( \lim_{x \to 0^+} f^i(x) \) does not exist.

**Definition.** Indicator function of a function with respect to domains. Given a function \( f : \mathbb{R} \to \mathbb{R} \) and a set of disjoint domains \( A = \{A_n\}_{1 \leq n \leq N} \subseteq \mathbb{R} \) such
that $\bigcup A_n = \mathbb{R}$, and a set of scalars $\{r_n\}_{1 \leq n \leq N}$, we will define the indicator function of $f$ with respect to $A$ in the following manner:

$$\chi_A f : \mathbb{R} \rightarrow \bigcup_n \{r_n\}$$

$$\chi_A f (x) = r_n \iff f (x) \in A_n.$$ 

**Definition.** Generalized detachable function. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will say that it is generalized-detachable in a point $x \in \mathbb{R}$ with respect to the set of domains $A = \{A_n\}_{1 \leq n \leq N} \subseteq \mathbb{R}$ and a set of scalars $\{r_n\}_{1 \leq n \leq N}$, if the following limit exists:

$$\exists \lim_{h \rightarrow 0} \chi_A [f (x + h) - f (x)].$$ 

**Example.** Any detachable function is generalized-detachable with respect to $A = \{0\} \cup (-\infty, 0) \cup (0, \infty)$.

**Definition.** The generalized detachment operator. Given a left or right generalized detachable function $f : \mathbb{R} \rightarrow \mathbb{R}$ with respect to a set of domains $A = \{A_n\}_{1 \leq n \leq N} \subseteq \mathbb{R}$ and a set of scalars $\{r_n\}_{1 \leq n \leq N}$, we will define the left or right generalized detachment operators applied for $f$ with respect to $A$ as:

$$f_s^{(A)} : \mathbb{R} \rightarrow \bigcup_n \{r_n\}$$

$$f_s^{(A)} (x) \equiv \lim_{h \rightarrow 0^s} \chi_A [f (x + h) - f (x)], \quad s \in \{\pm 1\}.$$ 

Applying the detachment operator to a function will be named: “generalized detachment of the function”.
Part III
Mathematical Discussion

“Do not know what I may appear to the world, but to myself I seem to have been
only like a boy playing on the sea-shore, and diverting myself in now and then
finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of
truth lay all undiscovered before me.”
– Sir Isaac Newton.

5 Classification of dissdetachment points

**Remark.** In order for a function \( f : \mathbb{R} \to \mathbb{R} \) to be detachable in a point \( x \in \mathbb{R} \), it should satisfy the equalities: \( \sup f^+_+(x) = \inf f^-_+(x) = \sup f^-_-(x) = \inf f^+_-(x) \). In order for it to be signposted detachable in the point, it should satisfy that: \( \sup f^+_+(x) = \inf f^-_+(x) \), \( \sup f^-_-(x) = \inf f^+_-(x) \) and \( \sup f^-_+(x) = -\sup f^+_-(x) \) and \( \inf f^-_+(x) = -\inf f^+_-(x) \). In other words, there are 6 causes for disdetachment in a point. Hence the following definition of classification of disdetachment points.

**Definition.** Classification of disdetachment points. Given a function \( f : \mathbb{R} \to \mathbb{R} \), we will classify its disdetachment points as follows:

1. First type (upper signposted) disdetachment in a point \( x \in \mathbb{R} \), if:
   \[
   \sup f^+_+(x) \neq -\sup f^-_-(x).
   \]

2. Second type (lower signposted) disdetachment in a point \( x \in \mathbb{R} \), if:
   \[
   \inf f^+_+(x) \neq -\inf f^-_-(x).
   \]

3. Third type (upper) disdetachment in a point \( x \in \mathbb{R} \), if:
   \[
   \sup f^+_+(x) \neq \sup f^-_-(x).
   \]

4. Fourth type (lower) disdetachment in a point \( x \in \mathbb{R} \), if:
   \[
   \inf f^+_+(x) \neq \inf f^-_-(x).
   \]

5. Fifth type (right) disdetachment in a point \( x \in \mathbb{R} \), if:
   \[
   \sup f^+_+(x) \neq \inf f^+_-(x).
   \]

6. Sixth type (left) disdetachment in a point \( x \in \mathbb{R} \), if:
   \[
   \sup f^-_+(x) \neq \inf f^-_-(x).
   \]
**Corollary.** A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is detachable in a point \( x \) iff it is only a first and second disdetachment point, and is signposted detachable there iff \( x \) is a third and fourth disdetachment point.

**Examples.**

1. Let us consider the following function:

   \[
   f : [0, 2] \rightarrow \mathbb{R}
   \]

   \[
   f(x) = \begin{cases} 
   1, & 0 \leq x < 1 \\
   2, & 1 \leq x \leq 2.
   \end{cases}
   \]

   Then \( x = 1 \) is a first, second, third and fourth type disdetachment point of \( f \). The function is also null disdetachable there.

2. Let us consider the following function:

   \[
   f : \mathbb{R} \rightarrow \mathbb{R}
   \]

   \[
   f(x) = \begin{cases} 
   x \sin \left( \frac{1}{x} \right), & x \neq 0 \\
   0, & x = 0.
   \end{cases}
   \]

   Then \( x = 0 \) is a first, second, fifth and sixth type disdetachment point of \( f \).

**Definition.** Tendency indicator vector of a function. Let \( f : X \rightarrow Y \) be a function, and let \( x \in \text{int}(X) \) be a point in the interior of \( X \). Denote by \( \omega^{\pm} \) the partial limits of the term \( \text{sgn} \left[ f(x + h) - f(x) \right] \), where \( h \to 0^{\pm} \) respectively. Then the tendency indicator vector of \( f \) in the point \( x \) is defined as following manner:

\[
\vec{s}(f, x) : \{1, \ldots, 6\} \rightarrow \{+1, -1, 0\}^6
\]

\[
s_i \equiv \chi_{s(i)} \left( \omega^{r(i)} \right),
\]

where

\[
\chi_{x}(X) = \begin{cases} 
1, & x \in X \\
0, & x \notin X
\end{cases},
\quad r(i) = \begin{cases} 
-1, & 1 \leq i \leq 3 \\
+1, & 4 \leq i \leq 6
\end{cases},
\quad s(i) = \begin{cases} 
+1, & i = 1, 4 \\
0, & i = 2, 5 \\
-1, & i = 3, 6.
\end{cases}
\]

The aim of the algorithm found in 1 is to determine the type of the disdetachment of a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) in a point \( x \). A Matlab code of the algorithm is available in the appendix. Note that the checks of the cases in the algorithm are arranged according to the multiplicity of the case, which is viable in the attached table. This is done for efficiency, in case the program breaks somehow.
Algorithm 1 Classification of disdetachment points

Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, a point $x$ and the tendency indicator vector of $f$ in the point $x$, $\vec{s} = \vec{s}(f, x)$, do:

1. Extract $\text{sup}_{f^+}(x), \text{inf}_{f^+}(x), \text{sup}_{f^-}(x), \text{inf}_{f^-}(x)$ via the entries of the vector $\vec{s}$, in the following manner (sup fits min and inf fits max, due to the nature of the definition of the vector $\vec{s}$):

   \[
   \begin{align*}
   \text{sup}_{f^+}(x) &= \varphi\left(\text{argmin}_i \{s_i^+ : s_i^+ = 1\}\right) \\
   \text{inf}_{f^+}(x) &= \varphi\left(\text{argmax}_i \{s_i^+ : s_i^+ = 1\}\right) \\
   \text{sup}_{f^-}(x) &= \varphi\left(\text{argmin}_i \{s_i^- : s_i^- = 1\}\right) \\
   \text{inf}_{f^-}(x) &= \varphi\left(\text{argmax}_i \{s_i^- : s_i^- = 1\}\right),
   \end{align*}
   \]

   where $\varphi$ is a function defined as follows:

   \[
   \varphi : \{1, \ldots, 6\} \rightarrow \{+1, -1, 0\} \\
   \varphi(n) = \begin{cases} +1, & n \in \{1, 4\} \\
   -1, & n \in \{3, 6\} \\
   0, & n \in \{2, 5\}. \end{cases}
   \]

2. If $\text{sup}_{f^+}(x) \neq -\text{sup}_{f^-}(x)$ classify $f$ as having a first type disdetachment in $x$.

3. If $\text{inf}_{f^+}(x) \neq -\text{inf}_{f^-}(x)$ classify $f$ as having a second type disdetachment in $x$.

4. If $\text{sup}_{f^+}(x) \neq \text{sup}_{f^-}(x)$ classify $f$ as having a third type disdetachment in $x$.

5. If $\text{inf}_{f^+}(x) \neq \text{inf}_{f^-}(x)$ classify $f$ as having a fourth type disdetachment in $x$.

6. If $\text{sup}_{f^+}(x) \neq \text{inf}_{f^+}(x)$ classify $f$ as having a fifth type disdetachment in $x$.

7. If $\text{sup}_{f^-}(x) \neq \text{inf}_{f^-}(x)$ classify $f$ as having a sixth type disdetachment in $x$. 
6 Analysis of the Weather Vane Function

Definition. Elaborated function. Let $D$ be a domain and let $\{D_n\}_{n=1}^N$ be pairwise disjoint sub-domains of $D$ such that $D = \bigcup_{1 \leq n \leq N} D_n$. Let $\{f_n : D_n \rightarrow \mathbb{R}\}_{n=1}^N$ be a set of functions. Then we shall define the elaborated function of them by:

$$\bigcup_n f_n : D \rightarrow \mathbb{R}$$

$$\bigcup_n f_n (x) \equiv f_n (x), \quad x \in D_n.$$  

Definition. Weather vane function. The following function’s aim is to illustrate the relationships between the detachment operators. Let us consider six functions, $\{f^{(i)} : \mathbb{R} \rightarrow \mathbb{R}\}_{i=1}^6$ in the following manner:

$$f^{(1)} (x) = f^{(6)} (x) = -x$$

$$f^{(3)} (x) = f^{(4)} (x) = +x$$

$$f^{(2)} (x) = f^{(5)} (x) = 0.$$  

Let $\vec{v} = (v_1, \ldots, v_6) \in \{0,1\}^6$ be a vector whose at least one of the first three elements and at least one of last three elements is 1. Thus, there are $2^6 - 1 - 2 \cdot (2^3 - 1) = 49$ options to select $\vec{v}$. Let us define:

$$D_1^+ = \mathbb{R}^+, \quad D_2^+ = \sqrt{2} \mathbb{Q}^+, \quad D_3^+ = \sqrt{3} \mathbb{Q}^+,$$

$$D_4^+ = \mathbb{R}^+ \setminus \sqrt{2} \mathbb{Q}, \quad D_5^+ = \mathbb{R}^+ \setminus (\sqrt{2} \mathbb{Q} \cup \sqrt{3} \mathbb{Q}), \quad D_6^+ = \emptyset.$$  

It is easy to see that there exists a unique transformation $k : \{1, \ldots, 6\} \rightarrow \{1, \ldots, 6\}$ such that:

$$\mathbb{R}^+ = \bigcup_{1 \leq i \leq 3} D_{k(i)}^+,$$

$$\mathbb{R}^- = \bigcup_{4 \leq i \leq 6} D_{k(i)}^-,$$

where $D_{k(i)}^\pm$ are pairwise disjoint. Let us define a vector of domains, $\vec{D} (\vec{v})$, by:

$$\vec{D} (v_i) \equiv D_{r(i)}^{(i)},$$

where $r(i) = \begin{cases} -1, & 1 \leq i \leq 3 \\ +1, & 4 \leq i \leq 6 \end{cases}$. Then the weather vane function is defined thus:

$$\ast (x, \vec{v}) \equiv \bigcup_i v_i f^{(i)} |_{\vec{D}(v_i)}.$$  

Example. We shall now analyse the weather vane function, denoted by $\ast (x, \vec{s})$, in the point $x = 0$. We will examine some of the 49 cases possible for $\ast$. 
• If \(s_2 = s_5 = 1\) and all the other \(s_i\)'s equal zero, then \(\ast\) is the zero function, hence both detachable and signposted detachable. (1 case).

• If \(s_2 + s_5 = 1 \pmod{2}\) then \(\ast\) can be upper or lower detachable. For example if \(\vec{s} = (1, 0, 0, 0, 1, 1)\) then \(\ast\) is not lower nor upper detachable. However, in the case where \(\vec{s} = (1, 0, 0, 1, 1, 0)\), the function is upper detachable, but not detachable from any other kind. (\(2 \cdot 2^4 = 32\) cases).

• If \(s_i = 1\) for all \(i\), then \(\ast\) is not detachable nor signposted detachable, however it is both upper and lower detachable. (1 case).

• If either \(s_1 = s_4 = 1\) or \(s_3 = s_6 = 1\) (and all of the other entries are null), then \(\ast\) is detachable and not signposted detachable. (2 cases).

• If either \(s_1 = s_6 = 1\) or \(s_3 = s_4 = 1\) (and all of the other entries are null), then \(\ast\) is signposted detachable and not even upper or lower detachable. These are in fact the only cases where the function is signposted detachable and not detachable. (2 cases).

• If \(s_1 = s_3 = s_4 = s_6 = 1\) (and \(s_2 = s_5 = 0\)), then \(\ast\) is not detachable nor signposted detachable. However, it is both upper and lower detachable. (1 case).

Notice the similarity between this definition of the tendency indicator vector of a function and the weather vane function. It is easy to verify that for any \(\vec{v}\) amongst the 49 possible cases:

\[
\vec{s}(\ast (x, \vec{v}), 0) = \vec{v}.
\]

7 TENDENCY AND EXTREMUM INDICATOR

7.1 Definition of operators

**Definition.** Tendable function in a point. Given a function \(f : \mathbb{R} \to \mathbb{R}\), we will say that \(f\) is tendable in a point \(x \in \mathbb{R}\) if it is detachable from right and left there.

**Example.** Let us consider the following function:

\[
f : \mathbb{R} \to \{0, 1, 2\}
\]

\[
f(x) = \begin{cases} 
2, & x \in \mathbb{Z} \\
1, & x \in \mathbb{Q}\setminus\mathbb{Z} \\
0, & x \in \mathbb{R}\setminus\mathbb{Q}.
\end{cases}
\]

Then \(f\) is discontinuous anywhere, however it is detachable (and especially tendable) in infinitely many points - the integers.
Remark. Due to the definition of the tendency, a function which is detachable from both sides will be abbreviated “tendable”.

Definition. Tendable function in an interval. Given a function $f : \mathbb{R} \to \mathbb{R}$, we will say that $f$ is tendable in an interval $I$ if one of the following holds:

1. $I = (a, b)$, and $f$ is tendable in each $x \in (a, b)$.
2. $I = [a, b)$, and $f$ is tendable in each $x \in (a, b)$ and right detachable in $a$.
3. $I = (a, b]$, and $f$ is tendable in each $x \in (a, b)$ and left detachable in $b$.
4. $I = [a, b]$, and $f$ is tendable in each $x \in (a, b)$, left detachable in $b$ and right detachable in $a$.

Definition. Tendency in a point. Given function $f : \mathbb{R} \to \mathbb{R}$, if it is tendable in a point $x \in \mathbb{R}$, we will define its tendency in the following manner:

\[ \tau_f : \mathbb{R} \to \{+1, -1, 0\} \]

\[ \tau_f (x) \equiv \begin{cases} 0, & f_+^+ (x) = f^- (x) \\ f_+^+ (x), & f_+^+ (x) \neq f^- (x) \end{cases} \]

Definition. Tendency in an open interval point. Given function $f : \mathbb{R} \to \mathbb{R}$, if it is tendable in an open interval $(a, b)$, we will define its tendency in each point of the interval as the tendency in a point, that was defined earlier.

Definition. Tendency in semi-closed or closed interval from right. Given function $f : \mathbb{R} \to \mathbb{R}$, if it is tendable in a semi-closed interval or a closed interval, $(a, b]$ or $[a, b]$ respectively, then we will define its tendency in each point of the interval which is not $b$ as the tendency in a point, that was defined earlier. The tendency in $b$ will be defined as follows:

\[ \tau_f (b) = f^- (b). \]

Remark. The rationalization behind these definition is that the tendency is supposed to imitate the operation of the derivative for tendable function, that is, to predict the monotonous behavior of the function. Thus, for example in a closed interval from right, when considering the right end of the interval, the tendency predicts that the function is to behave in the right side of this end as it did in the left neighborhood of the end point - hence the signposted detachment in the definition.

Definition. Uniformly tended function. Given a tendable function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$ is an open domain, we will say that it is uniformly tended in an closed interval $I \subseteq X$ if there exists a constant $\beta$ such that:

\[ \tau_f (x) = \beta \]
for every point $x$ in the interval perhaps, maybe, its endpoints.

**Examples.**

1. Every strictly monotone function is uniformly tended in its definition domain.

2. Let us consider the function:

$$f : \mathbb{R} \to \mathbb{R},$$

$$f(x) = x^2.$$  

then the tendency of $f$ is:

$$\tau_f : \mathbb{R} \to \mathbb{R}$$

$$\tau_f (x) = \begin{cases} 
-1, & x < 0 \\
0, & x = 0 \\
+1, & x > 0 
\end{cases}$$

hence, $f$ is uniformly tended in $(-\infty, 0]$ and in $[0, \infty).$

**Definition.** Monotony indicator. Given an function $f : \mathbb{R} \to \mathbb{R}$, we will define its monotony indicator in the following manner:

$$\wedge_f : \mathbb{R} \to \{\{0\}, \{+1\}, \{-1\}, \emptyset\}$$

$$\wedge_f (x) \equiv \begin{cases} 
\{0\}, & s(f, x) \in \{(1, 0, 0, 0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0)\} \\
\{+1\}, & s(f, x) = (0, 0, 1, 1, 0, 0) \\
\{-1\}, & s(f, x) = (1, 0, 0, 0, 0, 1) \\
\emptyset, & otherwise 
\end{cases}$$

where $s(f, x)$ is the tendency indicator vector of $f$ in the point $x$ defined earlier. It is easy to see that the monotony indicator is defined for any point of any function.

**Claim.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then:

1. $f$ has an extremum in a point $x_0$ iff $\wedge_f (x_0) = \{0\}$.

2. $f$ is strictly increasing in a point $x_0$ iff $\wedge_f (x_0) = \{+1\}$.

3. $f$ is strictly decreasing in a point $x_0$ iff $\wedge_f (x_0) = \{-1\}$.

**Proof.** Clear from the definition. $\square$

**Algorithm.** Finding extremum in general functions. Given a function $f : \mathbb{R} \to \mathbb{R}$, to find its extremum points, one need calculate $\wedge_f$ for any point in $X$. All the extremum points are received where $\wedge_f (x) = \{0\}$.  

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Remark. These algorithms are a simple version of finding extremums of differentiable functions, where the condition for “suspicious” points is \( f'(x) = 0 \). It is clear that the monotony indicator operator allows to analyze the monotonous regions of functions which are not necessarily differentiable. Further, such functions do not have a tangent in any point, hence the analysis of the domains where they are convex is not well defined; hence, for such functions, the monotony indicator operator will do.

7.2 Geometric interpretation of the tendency

Definition. An interval. Given two points \( x_1 \neq x_2 \in \mathbb{R} \), the set:

\[ \{ x \in \mathbb{R} : \min \{ x_1, x_2 \} \leq x \leq \max \{ x_1, x_2 \} \} \]

will be denoted by \([x_1, x_2] \).

Definition. Intervals of a tendable function \( f : \mathbb{R} \rightarrow \mathbb{R} \) in a point \( x \in \mathbb{R} \). Given a right and left detachable function \( f : \mathbb{R} \rightarrow \mathbb{R} \), we shall define its intervals in a point \( x \) by:

\[
I^+_f(x) = [x, x - \delta^+ f(x) \cdot h]
\]
\[
I^-_f(x) = [x, x - \delta^- f(x) \cdot h],
\]

where \( h > 0 \) is an arbitrary constant.

Definition. Signs of vertices in an interval. Given an interval \([x_1, x_2] \), we will define the vertices’ ((\(x_1, x_2\))) signs thus:

\[
\text{sgn}_{[x_1, x_2]}(x_i) = \begin{cases} 
1, & x_i > x_{2-i} \\
-1, & x_i < x_{2-i}
\end{cases}
\]

Theorem. Given a function \( f : X \rightarrow \mathbb{R} \) (where \( X \subseteq \mathbb{R} \)), which is tendable in a point \( x \in \mathbb{R} \), it holds that:

\[
\tau_f(x) = \sum_{s \in \{\pm\} \text{ and } f_s(x) = f_s^+(x)} \text{sgn}_{I_s^f(x)}(x).
\]

Proof. Let us observe the possible values for \( f_+^+(x), f_-^- (x) \). If \( f_+^+(x) = f_-^- (x) \) then \( \tau_f(x) = 0 \), in which case:

\[
\sum_{s \in \{\pm\} \text{ and } f_s(x) = f_s^+(x)} \text{sgn}_{I_s^f(x)}(x) = \text{sgn}_{I_+^f(x)}(x) + \text{sgn}_{I_-^f(x)}(x) = 0 = \tau_f(x).
\]
If on the other hand \( f^+ (x) \neq f^- (x) \) then according to the definition, \( \tau_f (x) = f^+_x(x) \). Hence:

\[
\sum_{s \in \{ \pm 1 \} \text{ and } f_s (x) = f^+_x(x)} \text{sgn} t_f (x) (x) = \text{sgn} t_f^+ (x) (x) = \begin{cases} +1, & f \text{ is increasing in } x \\ -1, & f \text{ is decreasing in } x \end{cases}
\]

\[= f^+_x(x) = \tau_f (x) \ \square.\]

**Remark.** Clearly the above theorem states the geometric connection between the tendency of a function and its intervals in a point, in a similar manner that the derivative is the slope of the tangent to a function in a point.

### 8 Continuity and Detachment

**Note.** If a function \( f : (a, b) \to \mathbb{R} \) is right-continuous everywhere in \( (a, b) \), then it is continuous there in infinitely many points.

**Note.** The above statement does not hold for detachment for right. Consider the following function:

\[
f : (0, 1) \to \mathbb{R}
\]

\[
f(x) = \begin{cases} \frac{1}{x}, & x \in \mathbb{Q} \\ -\frac{1}{x}, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}
\]

Then \( f \) is detachable from right everywhere in \( (0, 1) \), however it is also discontinuous and disdetachable everywhere there. It is easy to see that any function that is detachable from right and not detachable from left consists of two “pieces” (in this example, the pieces are \( \frac{1}{x} \) and \( -\frac{1}{x} \)). The following note refers to a simple generalization of the detachment.

**Note.** Let us consider the following generalization of the detachment, where:

\[
A_1 = (-\infty, -\epsilon), \quad r_1 = -1
\]

\[
A_2 = (-\epsilon, +\epsilon), \quad r_2 = 0
\]

\[
A_3 = (+\epsilon, \infty), \quad r_3 = +1,
\]

where \( \epsilon \) is as small as we desire. Then one can think of a function which is discontinuous in any point, and yet detachable in that sense in any point, for example:

\[
f : (0, 1) \to (-1, 1)
\]

\[
f(x) = \begin{cases} -\frac{1}{x}, & x \in \mathbb{Z} \\ 0, & x \in \mathbb{Q} \setminus \mathbb{Z} \\ +\frac{1}{x}, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}
\]

It is easy to see that one can build functions with as many “pieces” as desired.
9 MONOTONY AND DETACHMENT

9.1 Tendency

**Note.** The following claim is incorrect: “If a function is tendable in a neighborhood of the point, then there exists left and right neighborhoods of that point where the function is monotoneous”. Followed is a counter example, due to Friedlyand:

\[ f : (−1, 1) \to \mathbb{R} \]

\[ f(x) = \begin{cases} 
\sin \left( \frac{1}{x} \right), & x \neq 0 \\
17, & x = 0.
\end{cases} \]

Then \( f \) is tendable in \((-1, 1)\), however due to its discontinuity in \( x = 0 \), the existence of an interval where \( f \) is monotoneous is not guaranteed, and indeed there does not exist such in that example.

**Note.** The following claim is also incorrect: “If a function is tendable and continuous in a neighborhood of the point, then there exists left and right neighborhoods of that point where the function is monotoneous”. Followed is a counter example, due to Kaplan:

\[ f : (−1, 1) \to \mathbb{R} \]

\[ f(x) = \begin{cases} 
x \sin \left( \frac{1}{x} \right) - x, & x \neq 0 \\
0, & x = 0.
\end{cases} \]

Then \( f \) is tendable in \((-1, 1)\), further it is continuous there, however there does not exist any - left nor right - neighborhood of \( x = 0 \) where \( f \) is continuous.

**Claim.** If a function \( f : \mathbb{R} \to \mathbb{R} \) is tendable in a point \( x_0 \in \mathbb{R} \), then \( f \) is pseudo continuous there.

**Proof.** Since \( f \) is tendable, the function is monotoneous from both sides, hence as shown in calculus, the right and left limits of \( f \) exist, hence \( f \) is pseudo continuous according to the definition. □

**Conjecture.** If \( f : \mathbb{R} \to \mathbb{R} \) is continuous in \([a, b]\) and tendable in \((a, b)\), then it there exists an interval where \( f \) is monotoneous.

9.2 Detachment

**Definition.** Step function. We will say that a function \( f : \mathbb{R} \to \mathbb{R} \) is a step function if there exist a sequence of disjoint intervals \( \{A_k\} \) with \( \bigcup_k A_k = \mathbb{R} \), and a sequence of scalars \( \{r_k\} \) such that:

\[ f(x) = \sum_k r_k \cdot \chi_{A_k}(x), \]
where $\chi_{A_k}$ is the indicator function of $A_k$.

**Theorem.** If a function $f$ is detachable in the interval $[a, b]$ then it is a step function there.

**Proof.** First direction. Given that $f$ is detachable, we show that it is a step function. A similar proof to a slightly different claim was given by Behrends and Geschket in [11]. The fact that $f$ is detachable implies, according to the definition of the detachment, that any point in $[a, b]$ is a local extremum. Given $n > 0$, let us denote by $M_n$ the set of all points $x \in [a, b]$ for which $f$ receives a local maximum, and similarly denote $m_n$ the set of all points in the interval where $f$ receives a local minimum. Clearly $M_n \cap m_n$ is not necessarily empty, in case $f$ is constant in a sub-interval of $[a, b]$. Now, since each $x \in [a, b]$ is a local extremum of $f$, we obtain:

$$[a, b] = \bigcup_{n \in \mathbb{N}} \left[ m_n \cup M_n \right],$$

hence

$$f([a, b]) = \bigcup_{n \in \mathbb{N}} \left[ f(m_n) \cup f(M_n) \right].$$

To prove the argument we need to show that for each $n \in \mathbb{N}$, the set $f(m_n) \cup f(M_n)$ is countable. Without loss of generality, let us show that $f(m_n)$ is countable. Let $D_y$ be a $\frac{1}{2n}$-neighborhood of $f^{-1}(y)$. Let $z \in f(m_n)$ with $z \neq y$, and let $x \in D_y \cap D_z$. Then there exist $x_y, x_z \in m_n$ such that $f(x_y) = y$, $f(x_z) = z$ and:

$$|x_y - x| < \frac{1}{2n}, \quad |x_z - x| < \frac{1}{2n}.$$

Hence, $|x_y - x_z| < \frac{1}{n}$. Since in both the $n$-neighborhoods of $x_y, x_z$, $f$ receives its largest value in $x_y$ and $x_z$, it must hold that $f(x_y) = f(x_z)$, contradicting the choice of $y \neq z$. Hence, $D_y \cap D_z = \emptyset$. Now, let us observe the set $C = [a, b] \cap \mathbb{Q}$. Any set $D_y$, for $y \in f(m_n)$, contains an element of $C$. Since $D_y, D_z$ are disjoint for any $y \neq z$ and since $C$ is not countable, then $f(m_n)$ is also countable. Hence, $f$ is a step function. \qed

**Remarks.**

1. The second direction is not true: a step function may not be detachable in the entire interval, for example:

$$f : [0, 2] \to \mathbb{R}$$

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

which is not detachable in $x = 1$. 24
2. It is not true that if a function \( f : \mathbb{R} \to \mathbb{R} \) is detachable in an interval \([a, b]\) then it is constant there except, maybe, in a countable set of points. For example, consider the function:

\[
f : [0, 2] \to \mathbb{R} \\
f (x) = \begin{cases} 
0, & 0 < x < 1 \\
2, & x = 1 \\
1, & 1 < x < 2.
\end{cases}
\]

Then \( f \) is detachable in \([0, 2]\) although it is not constant there a.e.w.

**Corollary.** If a function \( f : \mathbb{R} \to \mathbb{R} \) is detachable in an interval \((a, b)\) and its detachment is constant there, \( f^* \equiv 0 \), then \( f \) is constant there.

**Proof.** Immediate from the fact that \( f \) is a step function, because had there been any jumps in the values of \( f \), then \( f^* (x) \neq 0 \) in any such jump point \( x \).

**9.3 Signposted Detachment**

**Definition.** Slide function. We will say that a function \( f : \mathbb{R} \to \mathbb{R} \) is a slide function if there exist a sequence of disjoint (perhaps withered) intervals \( \{A_k\} \) with \( \bigcup_k A_k = \mathbb{R} \), and a sequence functions \( \{f_k\} \) such that:

\[
f (x) = \sum_k f_k (x) \cdot \chi_{A_k} (x),
\]

where \( \chi_{A_k} \) is the indicator function of \( A_k \), and for all \( n \) such that \( A_n \) is an interval, \( f_n |_{A_n} \) is strictly monotonous.

**Examples.**

1. Every step function is a slide function, hence every function which is detachable in any point is a slide function.

2. The following function is a slide function:

\[
f : [0, 2] \to \mathbb{R} \\
f (x) = \begin{cases} 
-x, & 0 \leq x < 1 \\
-x + 1, & 1 \leq x \leq 2.
\end{cases}
\]

It is detachable, and not signposted detachable, in \( x = 1 \), and signposted detachable and not detachable elsewhere.
3. The following function is a slide function:

\[ f : [0, 2] \to \mathbb{R} \]

\[
f(x) = \begin{cases} 
-x, & 0 \leq x < 1 \\
-x + 1, & 1 < x \leq 2 \\
\frac{1}{2}, & x = 1.
\end{cases}
\]

It is signposted detachable everywhere.

4. The following function is a slide function:

\[ f : [0, 2] \to \mathbb{R} \]

\[
f(x) = \begin{cases} 
x, & 0 \leq x < 1 \\
-x + 4, & 1 < x \leq 2 \\
2, & x = 1.
\end{cases}
\]

It is signposted detachable everywhere. It is an example to the fact that
the monotony of the intervals in the slide function is not necessarily
the same type for all the intervals.

**Conjecture.** Any function which is signposted detachable in an interval
\((a, b)\) is a slide function there.

**Lemma.** (Kaplan). If a function \(f : \mathbb{R} \to \mathbb{R}\) is signposted detachable in an
interval \((a, b)\) and its signposted detachment is constant there then
\(f\) is stricly monotenous there.

**Proof.** If \(f^\prime \equiv 0\) in the interval then so is \(f\); and we’ve shown that in this
case, \(f\) is constant in the interval. Without loss of generality, let us assume
that \(f^\prime \equiv +1\) in the interval. Let \(x_1, x_2 \in (a, b)\) such that \(x_1 < x_2\). We
would like to show that \(f(x_1) < f(x_2)\). From the definition of the signposted
detachment, there exists a left neighborhood of \(x_2\) such that \(f(x) < f(x_2)\) for
each \(x\) in that neighborhood. Let \(t \neq x_2\) be an element of that neighborhood.
Let \(s = \sup \{x | x_1 \leq x < t, \ f(x) \geq f(x_2)\}\). On the contrary, let us assume
that \(f(x_1) \geq f(x_2)\). Then \(s \geq x_1\). If \(f(s) \geq f(x_2)\) (that is, the
supremum is accepted in the defined set), then since for any \(x > s\) it holds that
\(f(x) < f(x_2) \leq f(s)\), then \(f^\prime_+(s) = -1\), contrading \(f^\prime_+ \equiv 1\) in \((a, b)\). Hence
the maximum is not accepted. Especially it implies that \(s \neq x_1\). Therefore
according to the definition of the supremum, there exists a sequence \(x_n \to s\)
with \(\{x_n\} \subset (x_1, s)\) such that:

\[ f(x_n) \geq f(x_2) > f(s), \]

that is, \(f(x_n) > f(s)\), contradicting our assumption that \(f^\prime(s) = +1\). Hence
\(f(x_1) < f(x_2)\). □

**Theorem.** If a function \(f : \mathbb{R} \to \mathbb{R}\) is signposted detachable in an interval
\((a, b)\) and is continuous there then \(f\) is stricly monotenous there.
**Proof.** According to Kaplan’s lemma, it is enough to show that $\mathbf{f}$ is constant in $(a, b)$. Let $x, y \in (a, b)$. Assume $x < y$. On the contrary, suppose that $f^\uparrow(x) \neq f^\uparrow(y)$. Let us distinguish two main cases, where the rest of the cases are handled similarly:

1. $f^\uparrow(x) = +1, f^\uparrow(y) = -1$. That is, $f^- (y) = f^+ (x) = +1$, hence $\argmax_{t \in [x,y]} f(t) \notin \{x, y\}$. $f$ is continuous in $[x, y]$, hence there exists $t \in (x, y)$ where $f$ receives its maximum, hence $f$ is detachable, and not signposted detachable in $t$, a contradiction.

2. $f^\uparrow(x) = +1, f^\uparrow(y) = 0$. Let us denote:

$$s = \sup \{t | x < t < y, f(t) \neq f(y)\}.$$  

If $s = -\infty$ then $f$ is constant in $[x, y]$, hence $f^+ (x) = 0$, hence $f$ is either not signposted detachable in $x$ or $f^\uparrow (x) = 0$, a contradiction. That is, $x < s < y$. Hence there exists a left neighborhood of $s$ where $f(t) \neq f(y)$ for each $t$ in that neighborhood, and a right neighborhood of $s$ where $f(t) = f(y)$ for each $t$ in that neighborhood. Hence $f^- (t) \neq 0, f^+ (t) = 0$, which implies that $f$ is null-disdetachable, and especially not signposted detachable, in $x = t$, a contradiction. □

**9.4 General notes**

**Note.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f$ is constant in an open interval $(a, b)$ iff it is both detachable and signposted detachable there.

**Examples.**

1. The function:

$$f : (0, 2) \to \mathbb{R}$$

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \leq 2 \end{cases}$$

satisfies that $f^- (x) = -f^+ (x)$ for each $x \in (0, 2) \setminus \{1\}$, however: $f^+ (1) = 0, f^- (1) = -1$ and is indeed not strictly monotonous there.

2. The function:

$$f : [-1, 1] \to \mathbb{R}$$

$$f(x) = \begin{cases} 1, & |x| > 0 \\ 0, & x = 0 \end{cases}$$

is almost everywhere constant, hence detachable.
3. Weirstrass’ function is both non-detachable almost everywhere (due to uncountably many non-extremum points where its detachment is not defined) and detachable almost everywhere (due to uncountably many extremum points).

**Claim.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a pseudo continuous, not continuous and detachable from left or right in a point \( x_0 \in \mathbb{R} \). Then:

\[
\lim_{h \to 0^\pm} \text{sgn} \left[ f(x + h) - f(x) \right] = \text{sgn} \lim_{h \to 0^\pm} \left[ f(x + h) - f(x) \right].
\]

**Proof.** Since \( f \) is pseudo continuous and not continuous, then \( \lim_{h \to 0^\pm} \left[ f(x + h) - f(x) \right] \) exists and does not equal 0. Further, since \( f \) is detachable from left and right in \( x_0 \), there is a neighborhood of \( x_0 \) where \( f \) is monotonic. □

**Remark.** A function may be detachable even if it is not pseudo continuous. For example:

\[
f : \mathbb{R} \to \mathbb{R} \\
f(x) = \begin{cases} 
\text{sgn } \left( \frac{1}{x} \right), & x \neq 0 \\
0, & x = 0,
\end{cases}
\]
in \( x = 0 \).

**10 Differentiability and Detachment**

**Claim.** If a function \( f : \mathbb{R} \to \mathbb{R} \) is tendable in a point \( x_0 \in \mathbb{R} \) then:

\[
\lim_{h \to 0^\pm} \left\{ \text{sgn} \left( \frac{f(x + h) - f(x)}{h} \right) \right\} = \pm f_\pm'(x_0).
\]

**Proof.**

\[
\lim_{h \to 0^\pm} \left\{ \text{sgn} \left( \frac{f(x + h) - f(x)}{h} \right) \right\} = \pm \lim_{h \to 0^\pm} \left\{ \text{sgn} \left[ f(x + h) - f(x) \right] \right\} \\
= \pm f_\pm'(x_0). \quad \Box
\]

**Claim.** If a function \( f : \mathbb{R} \to \mathbb{R} \) is detachable and differentiable in a point \( x_0 \in \mathbb{R} \) and \( f'(x_0) = 0 \), then:

\[
f_+'(x_0) \cdot f_-'(x_0) \neq -1.
\]

**Proof.** \( f \) is detachable in \( x_0 \), hence \( x_0 \) is an extremum, hence \( f'(x_0) = 0 \). The only cases where \( f_+'(x_0) \cdot f_-'(x_0) = -1 \) are those where \( f \) is strictly increasing or decreasing, which is not the case. □
**Claim.** If a function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable in \( x_0 \in \mathbb{R} \) and satisfies that \( f'(x_0) \neq 0 \) for \( x_0 \in \mathbb{R} \), then \( f \) is signposted detachable in \( x_0 \).

**Proof.** Since \( f'(x_0) \neq 0 \), then \( f \) is strictly monotonous there, hence according to the definition of the left and right detachments, \( f_+ (x_0) = -f_- (x_0) \neq 0 \), thus \( f \) is signposted detachable in \( x_0 \). \( \square \)

**Definition.** Joint point. Given a function \( f : \mathbb{R} \to \mathbb{R} \), we will say that \( x_0 \in X \) is a joint point of \( f \) if \( f \) is continuous, tendable, and not differentiable in \( x_0 \).

**Definition.** First type joint point. Given a function \( f : \mathbb{R} \to \mathbb{R} \), we will say that \( x_0 \in X \) is a first type joint point of \( f \) if \( x_0 \) is a joint point of \( f \), and \( f_+ (x_0) = f_- (x_0) \).

**Definition.** Second type joint point. Given a function \( f : \mathbb{R} \to \mathbb{R} \), we will say that \( x_0 \in X \) is a second type joint point of \( f \) if \( x_0 \) is a joint point, \( f_+ (x_0) \neq f_- (x_0) \) and \( f_+ (x_0) \cdot f_- (x_0) \neq 0 \).

**Definition.** Third type joint point. Given a function \( f : \mathbb{R} \to \mathbb{R} \), we will say that \( x_0 \in X \) is a third type joint point of \( f \) if \( x_0 \) is a joint point, \( f_+ (x_0) \neq f_- (x_0) \) and \( f_+ (x_0) \cdot f_- (x_0) = 0 \).

**Examples.**

1. Consider the function:

   \[
   f : \mathbb{R} \to \mathbb{R} \\
   f(x) = |x|
   \]

   then \( x = 0 \) is a first type joint point of \( f \).

2. Consider the function:

   \[
   f : [0,2] \to \mathbb{R} \\
   f(x) = \begin{cases} 
   x, & 0 \leq x < 1 \\
   2x - 1, & 1 \leq x < 2.
   \end{cases}
   \]

   then \( x = 1 \) is a second type joint point of \( f \).

3. Consider the function:

   \[
   f : [0,2] \to \mathbb{R} \\
   f(x) = \begin{cases} 
   1, & 0 \leq x < 1 \\
   x, & 1 \leq x < 2.
   \end{cases}
   \]

   then \( x = 1 \) is a third type joint point of \( f \).
**Note.** If a function is brokenly both continuous and signposted detachable, then it is differentiable almost everywhere.

**Proof.** According to a previous lemma, this condition assures that the function is brokenly monotoneous, which in turn insures differentiability almost everywhere by Lebesgue’s theorem. □

**Corollary.** If a function is tendable everywhere and both its right and left detachments are locally constant, then it is differentiable almost everywhere.

**Proof.** Immediate from a previous claim that assures that in these conditions the function is monotoneous in the neighborhood of each point, and Lebesgue’s theorem. □

**Note.** A function is differentiable everywhere ⇒ it is tendable everywhere. For example consider the following functions:

\[
  f : [-1, 1] \to \mathbb{R} \\
  f(x) = \begin{cases} 
  \sin \left( \frac{1}{x} \right), & x \neq 0 \\
  17, & \text{otherwise}
  \end{cases}
\]

and:

\[
  g : [-1, 1] \to \mathbb{R} \\
  g(x) = \begin{cases} 
  x^2 \sin \left( \frac{1}{x} \right), & x \neq 0 \\
  0, & x = 0
  \end{cases}
\]

then \( f \) is tendable everywhere and not differentiable in \( x = 0 \), and \( g \) is differentiable everywhere and not tendable in \( x = 0 \).

### 11 FUNDAMENTAL THEOREMS FOR THE DETACHMENT

#### 11.1 Closure

**Note.** The tendable, detachable and signposted detachable functions are closed under multiplication by a scalar.

**Note.** The tendable functions are not closed under addition. For example, consider the following functions:

\[
  f : [-1, 1] \to \mathbb{R} \\
  f(x) = \begin{cases} 
  -1, & x = 0 \\
  n, & x = \frac{1}{n}, \ n \in \mathbb{Z} \\
  0, & \text{otherwise},
  \end{cases}
\]
and:

\[ g : [-1, 1] \to \mathbb{R} \]
\[ g(x) = \begin{cases} +1, & x = 0 \\ 0, & \text{otherwise.} \end{cases} \]

Then \( f, g \) are tendable (and even detachable) in \([-1, 1]\), however \( f + g \) is not tendable in \( x = 0 \).

In a similar manner, the detachable functions are not closed under addition neither, and so do the signposted detachable functions.

### 11.2 Arithmetic rules

**Claim.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a tendable function. Let \( c \in \mathbb{R} \) be a constant. Then:

\[ [cf] \pm = sgn(c) f. \]

**Proof.**

\[
 [cf] \pm (x) = \lim_{h \to \pm 0} sgn \{ [cf] (x + h) − [cf] (x) \} = \lim_{h \to \pm 0} sgn \{ c[f(x + h) − f(x)] \} = sgn(c) f. \quad \Box
\]

**Definition.** Pointwise-incremented function. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. We will define its \( a \)-incremented function, where \( a \in \mathbb{R} \), in the following manner:

\[ f^{(a)} : \mathbb{R} \to \mathbb{R} \]
\[ f^{(a)}(x) = f(x) + f(a). \]

**Claim.** Let \( f, g : \mathbb{R} \to \mathbb{R} \) be functions. Set \( x \in \mathbb{R} \). If \( f^{(x)}, g^{(x)} \) are tendable then so is \( (fg)^{(x)} \), and:

\[ [(fg)^{(x)}] \pm = \left( f^{(x)} \right) \pm \cdot \left( g^{(x)} \right) \pm. \]

**Proof.** Since \( f^{(x)} \) and \( g^{(x)} \) are tendable, then:

\[
 \left\{ \left( f^{(x)} \right) \pm \cdot \left( g^{(x)} \right) \pm \right\}(x_0) = \lim_{h \to \pm 0} sgn [f(x_0 + h)] \cdot sgn[g(x_0 + h)]
\]
\[ = \lim_{h \to \pm 0} sgn [(f \cdot g)(x_0 + h)] = \left[ (fg)^{(x)} \right] \pm (x_0). \quad \Box
\]

**Corollary.** Let \( f : \mathbb{R} \to \mathbb{R} \) be tendable function and let \( n \in \mathbb{N}, x \in \mathbb{R} \). Then:

\[ [(f^n)^{(x)}] \pm = \left[ \left( f^{(x)} \right)^n \right] \pm. \]
11.3 Analogous versions to the even\odd theorems (of the derivative)

**Lemma.** Let $f : \mathbb{R} \to \mathbb{R}$ be a tendable function. If $f$ is even then:

$$f_+^\prime (-x) = f_-^\prime (x).$$

**Proof.**

$$f_+^\prime (-x) = \lim_{h \to 0^+} sgn [f (-x + h) - f (-x)] = \lim_{h \to 0^+} sgn [f (x - h) - f (x)]$$

$$= \lim_{h \to 0^-} sgn [f (x + h) - f (x)] = f_-^\prime (x). \square$$

**Lemma.** Let $f : \mathbb{R} \to \mathbb{R}$ be a tendable function. If $f$ is odd then:

$$f_+^\prime (x) = -f_-^\prime (-x).$$

**Proof.**

$$-f_-^\prime (-x) = -\lim_{h \to 0^-} sgn [f (-x + h) - f (-x)] = -\lim_{h \to 0^-} sgn [-f (x - h) + f (x)]$$

$$= \lim_{h \to 0^-} sgn [f (x - h) - f (x)] = \lim_{h \to 0^-} sgn [f (x + h) - f (x)]$$

$$= f_+^\prime (x). \square$$

**Theorem.** If a detachable function $f$ is even, so is its detachment. If $f$ is odd, then $f^\prime$ is odd (hence, $f$ is constant).

**Proof.** If $f$ is even, then according to the first lemma above it holds that $f^\prime (x) = f^\prime (-x)$, hence the detachment is even. If $f$ is odd, then according to the second lemma, for each $x$ it holds that: $f_+^\prime (x) = -f_-^\prime (x)$. Hence $\partial f (x) \equiv 0$, and $f$ is constant, and especially odd. $\square$

**Theorem.** If a signposted detachable function $f$ is even, then its signposted detachment is odd. If $f$ is odd, then its signposted detachment is even.

**Proof.** By considering the cases, via the previous lemmas. $\square$

11.4 Analogous versions to Weirstrass’ theorem

**Claim.** Let $f$ be a detachable function in a finite closed interval $[a, b]$. Then for any $\epsilon > 0$, $f$ is bounded in $[a + \epsilon, b - \epsilon]$.

**Proof.** Immediate from the fact that $f$ is constant almost everywhere there. $\square$

**Claim.** Let $f$ be a signposted detachable function in a finite closed interval $[a, b]$. Then for any $\epsilon > 0$, $f$ is bounded in $[a + \epsilon, b - \epsilon]$.

**Proof.** Immediate from the fact that $f$ is strictly monotone there. $\square$
11.5 An analogous version to Fermat’s theorem

**Lemma.** Let \( f : (a, b) \rightarrow \mathbb{R} \). Then it is detachable in a point \( x_0 \in (a, b) \) iff \( x \) is a local extremum.

**Proof.** Immediate from the definition of the detachment. □

**Theorem.** Let \( f : (a, b) \rightarrow \mathbb{R} \) and let \( x_0 \in (a, b) \) be an extremum of \( f \). Then the function is detachable in \( x_0 \), and:

\[
\tau_f (x_0) = 0.
\]

**Proof.** \( x_0 \) is an extremum, hence according to the lemma, \( f \) is detachable there. Further, according to the definition of the detachment, \( f_+ (x_0) = f_- (x_0) \), hence according to the definition of the dendency, \( \tau_f (x_0) = 0 \). □

11.6 An analogous version to Rolle’s theorem

**Theorem.** Let \( f \) be a function defined on a closed interval \([a, b] \subseteq \mathbb{R}\). Suppose that \( f \) satisfies the following:

1. \( f \) is continuous in \([a, b]\).
2. \( f (a) = f (b) \).

Then, there exists a point \( c \in (a, b) \) where \( f \) is detachable, and especially, \( \tau_f (c) = 0 \).

**Proof.** \( f \) is continuous in a closed interval, hence according to Weistrass’ theorem, it receives there a maximum \( M \) and a minimum \( m \). In case \( m < M \), then since it is given that \( f (a) = f (b) \), then one of the values \( m \) or \( M \) must be an image of one of the points in the open interval \((a, b)\). Denote (one of) this point(s) by \( c \). Hence \( f \) receives an extremum in \( c \), which implies that \( f \) is detachable there, and especially, \( \tau_f (c) = 0 \). In case \( m = M \), then \( f \) is constant and the claim holds trivially. □

11.7 Analogous versions to Lagrange’s theorem

**Note.** Let \( f \) be tendable in \((a, b)\) and suppose that \( f (a) \neq f (b) \). Then it is not always true that there exists a point \( c \in [a, b] \) such that:

\[
\tau_f (c) = \text{sgn} [f (b) - f (a)], \quad \forall x \in I.
\]

For example, consider the function:

\[
f : [0, 2] \rightarrow \mathbb{R}
\]

\[
f (x) = \begin{cases} 
0, & 0 \leq x < 1 \\
1, & 1 \leq x \leq 2
\end{cases}
\]
Then $f(2) > f(0)$, further $f$ is tendable in $[0,2]$ however $\tau_f \equiv 0$ there.

**Theorem.** Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and tendable in $(a, b)$. Assume $f(a) \neq f(b)$. Then for each $v \in (f(a), f(b))$ there exists a point $c_v \in f^{-1}(v)$ such that:

$$
\tau_f(c_v) = \text{sgn} \left[ f(b) - f(a) \right].
$$

**Proof.** Without loss of generality, let us assume that $f(a) < f(b)$. Let $v \in (f(a), f(b))$. According to Cauchy's intermediate theorem, $f^{-1}(v) \neq \emptyset$. On the contrary, let us assume that $f^+(x) = -1$ for each $x \in f^{-1}(v)$. Let $x_{\text{max}} = \sup f^{-1}(v)$. The maximum is accepted since $f$ is continuous, hence $f(x_{\text{max}}) = v$. Then according to our assumption $f^+(x_{\text{max}}) = -1$, and especially there exists a point $t > x_{\text{max}}$ such that $f(t) < f(x_{\text{max}}) = v$. But $f$ is continuous in $[t, b]$, thus according to Cauchy's intermediate theorem, there exists a point $s \in [t, b]$ for which $f(s) = v$, which contradicts the choice of $x_{\text{max}}$. In the same manner it is impossible that $f^-(x) = 0$ for each point $x \in f^{-1}(v)$, because then the same contradiction will rise from $f^-(x_{\text{max}}) = +1$. Hence, $S = f^{-1}(v) \cap \{ x | f^+(x) = +1 \} \neq \emptyset$. Now we will show that $S$ must contain a point $x$ for which $f^-(x) \neq +1$. Let us observe $x_{\text{min}} = \inf S$. We will now show that $f^+(x_{\text{min}}) = +1$. On the contrary, if $f^+(x_{\text{min}}) = 0$, then $f$ is constant in a right neighborhood of $x_{\text{min}}$, which contradicts the definition of $x_{\text{min}} = \inf S$. Further, if we assume that $f^+(x_{\text{min}}) = -1$, then from the definition of $x_{\text{min}}$, it follows that

$$
\lim_{x \to x_{\text{min}}^-} f(x) < f(x_{\text{min}}),
$$

with contradiction to the function’s continuousness. Hence $f^+(x_{\text{min}}) = +1$. On the contrary, suppose that $f^-(x_{\text{min}}) = +1$. Then especially there exists $t < x_{\text{min}}$ with $f(x_{\text{min}}) < f(t)$. But $f$ is continuous in $[a, t]$, hence according to Cauchy’s intermediate theorem, $f^{-1}(v) \cap (a, t) \neq \emptyset$. Let us observe $s = \max [f^{-1}(v) \cap (a, t)]$. Then it can be shown in a similar manner that $f^-(s) = +1$, hence $s \in S$, which forms a contradiction since $s < x_{\text{min}}$. Further $f(x_{\text{min}}) = v$ since $f$ is continuous, and from the definition of $x_{\text{min}}$. Thus $c_v = x_{\text{min}}$ satisfies that $f(c_v) = v$, $f^+(c_v) = +1$, and $f^-(c_v) \neq +1$. Thus, $\tau_f(c_v) = +1$. \(\Box\)

**Note.** The above theorem is no longer true if we demand that for any value $v$ there exists $c_v$ where:

$$
\tau_f(c_v) = \text{sgn} \left[ f(b) - f(a) \right].
$$

Consider the function:

$$
f : [0, 2] \to \mathbb{R}
$$

$$
f(x) = \begin{cases} 
x, & 0 \leq x \leq 1 \\
2x - 1, & 1 \leq x \leq 2.
\end{cases}
$$

Then for $v = 1$, $f$ is not even differentiable in $f^{-1}(v) = \{1\}$. 

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**Conjecture.** Let $f$ be continuous in $[a, b]$ and tendable in $(a, b)$. Then there exists a closed (perhaps withered) interval, $I \subseteq (a, b)$ such that:

$$\tau_f(x) = \text{sgn}[f(b) - f(a)], \quad \forall x \in I.$$ 

**Proof Suggestion.** If $f(a) = f(b)$, then the analogous version of Rolle’s theorem proves the claim, since the point $c$ in that theorem forms the closed interval $I = \{c\}$. Otherwise, suppose without loss of generality that $f(a) < f(b)$. Then, we ought to point out an interval $I \subseteq (a, b)$ for which:

$$\tau_f(x) = +1, \quad \forall x \in I.$$ 

Now, since $f$ is continuous in $[a, b]$ and tendable in $(a, b)$, then it there exists an interval where $f$ is increasing (according to a previous conjecture), hence according to the definition of the tendency, $\tau_f(x) = +1$ for each $x$ in that interval. $\square$

11.8 **An analogous version to Darboux’s theorem**

**Theorem.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous in a neighborhood of the point $x_0$. If $x_0$ is a local maximum or a local minimum then there exists a neighborhood $I(x_0)$ such that $\text{Im}(\tau_f|_{I(x_0)}) = \{0, \pm 1\}$.

**Proof.** Since $x_0$ is an extremum, then $\tau_f(x_0) = 0$. Assume without loss of generality that $x_0$ is a local maximum, then from the function’s continuousness there exists a left neighborhood of $x_0$ where $f$ increases, thus $\tau_f(x) = +1$ for each $x$ in that neighborhood, and there exists a right neighborhood of $x_0$ where $f$ decreases, thus $\tau_f(x) = -1$ for each $x$ in that neighborhood. The union of these neighborhoods forms the desired neighborhood $I$. $\square$

**Note.** If $f$ is not continuous then the above theorem does not hold. Consider the following function:

$$f : [0, 2] \to \mathbb{R}$$

$$f(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 2. \end{cases}$$

Then $x = 2$ is a local maximum, however $\tau_f \in \{0, -1\}$.

11.9 **An analogous version to the Fundamental Theorem of Calculus**

**Definition.** Antiderivative. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Then its antiderivative is defined as follows:

$$F : \mathbb{R} \to \mathbb{R}$$

$$F(x) = \int_{-\infty}^{x} f(t) \, dt.$$ 

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**Definition.** Local antiderivative. Let \( f : \mathbb{R} \to \mathbb{R} \) be an integrable function, and let \( p \in \mathbb{R} \). Then its local antiderivative is defined as follows:

\[
F_p : [p, \infty) \to \mathbb{R} \\
F_p(x) = \int_p^x f(t) \, dt.
\]

**Definition.** Extended local antiderivative. Let \( f : \mathbb{R} \to \mathbb{R} \) be an integrable function, and let \( p \in \mathbb{R} \cup \{-\infty\} \). Then its local antiderivative is defined as follows:

\[
F_p : \mathbb{R} \to \mathbb{R} \\
F_p(x) = \int_p^x f(t) \, dt.
\]

**Theorem.** Let \( f : \mathbb{R} \to \mathbb{R} \) be an integrable function, and let \( F_p \) be its extended local antiderivative, where \( p \in \mathbb{R} \cup \{-\infty\} \). Let \( x_0 \in \mathbb{R} \). Suppose that the pointwise incremented function \( f(x_0) \) is tendable in \( x_0 \). Then \( F_p \) is tendable in \( x_0 \), and:

\[
(F_p)^\pm (x_0) = \left(f(x_0)^\pm \right)(x_0),
\]

where \( f(x_0) \) is the pointwise incremented function of \( f \) defined earlier.

**Proof.** Without loss of generality, let us prove the theorem for a right-neighborhood of \( x_0 \). If \( f(x_0) = 0 \), then \( f(x_0) = f \), and the proof of the claim is trivial: Since \( f \) is tendable in \( x_0 \), there exists a right neighborhood of \( x_0 \) where \( f \) is monotoneous. Since \( f(x_0) = 0 \), then \( f \) is entirely positive or entirely negative in that neighborhood, hence \( F_p \) is also monotoneous in this neighborhood, and it shares the same type of monotony with \( f \).

Let us assume now that \( f(x_0) \neq 0 \). As done in the proof of the Fundamental theorem of Calculus, while assuming \( h > 0 \) and \( \epsilon > 0 \), it holds that:

\[
h \cdot f(x_0) - \epsilon \leq \int_{x_0}^{x_0+h} f(t) \, dt \leq h \cdot f(x_0) + \epsilon.
\]

By applying the \( \text{sgn} (\cdot) \) operator on both handsides and recalling that \( h > 0 \), we get:

\[
\text{sgn}[f(x_0) - \epsilon] \leq \text{sgn} \int_{x_0}^{x_0+h} f(t) \, dt \leq \text{sgn}[f(x_0) + \epsilon].
\]

Hence:

\[
\text{sgn}[f(x_0) - \epsilon] \leq \text{sgn}[F_p(x_0 + h) - F_p(x_0)] \leq \text{sgn}[f(x_0) + \epsilon].
\]

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From our assumption that $f(x_0) \neq 0$ and from the fact that $f^{(x_0)}$ is tendable in $x_0$, it is derived that there exist left and right neighborhoods of $x_0$ where $\text{sgn} [f(x)] \in \{\pm 1\}$ is constant for each $x$ in that neighborhood. Hence, for small enough $\epsilon$’s it holds that $\text{sgn} [f (x_0) + \epsilon] < \text{sgn} [f (x_0)] + \epsilon$ and $\text{sgn} [f (x_0)] - \epsilon < \text{sgn} [f (x_0) - \epsilon]$, thus:

$$\text{sgn} [f (x_0)] - \epsilon \leq \text{sgn} [F_p (x_0 + h) - F_p (x_0)] \leq \text{sgn} [f (x_0)] + \epsilon.$$  

According to the definitions and the Sandwich theorem, we get $(F_p)_+ (x_0) = \lim_{x \to x_0} \text{sgn} [f (x)] = (f^{(x_0)})_+ (x_0).$  $\Box$
Part IV
Engineering-Oriented Discussion

Nature does nothing in vain, and more is in vain when less will serve; for nature is pleased with simplicity, and affects not the pomp of superfluous causes.”
— Sir Isaac Newton.

12 APPROXIMATION OF PARTIAL LIMITS

**Definition.** Approximation of a partial limit of a sequence. Given a sequence \( \{a_n\}_{n \in \mathbb{N}} \), we will say that it has an approximated partial limit \( \tilde{P} \), if there exists a randomly chosen sub-sequence of \( \{a_n\}_{n \in \mathbb{N}} \), namely \( \{a_{n_k}\}_{k \in \mathbb{N}} \), for which there exist two numbers, \( 0 < M_{\text{min}} < M_{\text{max}} \), such that for any \( M_{\text{min}} < m < M_{\text{max}} \) there exist two numbers \( K_{\text{min}}, K_{\text{max}} \) with \( 0 < K_{\text{min}} < K_{\text{max}} \) such that for all \( K_{\text{min}} < k < K_{\text{max}} \) it holds that:

\[
|a_{n_k} - P| < \frac{1}{m}.
\]

We will denote the set of approximated partial limits by \( \tilde{\text{plima}}_n \). Hence in the discussed case:

\( P \in \tilde{\text{plima}}_n. \)

**Examples.**

1. Let \( a_n = (-1)^n, \ n \in \mathbb{N} \). Then \( \tilde{\text{plima}}_n = \{\pm 1\} \), while the set of partial limits of \( a_n \) is \( \{\pm 1\} \).

2. Let:

\[
a_n = \begin{cases} 
17, & n < 10^{100} \\
(-1)^n, & n \geq 10^{100}.
\end{cases}
\]

Then \( \tilde{\text{plima}}_n = \{17, \pm 1\} \), although the set of partial limits of \( a_n \) is \( \{\pm 1\} \).

3. Let \( a_n = \frac{1}{n} \). Then \( \tilde{\text{plima}}_n = \bigcup_{M_{\text{max}} \gg 1} \left\{ \frac{1}{M_{\text{max}}} \right\} \), although the limit of \( a_n \) is 0.

**Conjecture.** The set of all sequences \( \left\{ \{a_n(\omega)\}_{n \in \mathbb{N}} \right\}_\omega \) for which the set \( \tilde{\text{plima}}_{n(\omega)} \) does not intersect the set of partial limits of \( \{a_n(\omega)\}_{n \in \mathbb{N}} \) is a negligible with respect to the set of all possible sequences.

**Corollary.** If the above conjecture is shown to hold, then almost always partial limits can be found by the limit approximation process, and it will do
for any engineering requirement.

**Definition.** Approximation of a partial limit of a function. Given a function \( f : \mathbb{R} \rightarrow \mathbb{R} \), we will say that it has an approximated partial limit \( P \) in a point \( x_0 \in \mathbb{R} \), if there exists a random sequence, \( \{x_n\}_{n \in \mathbb{N}} \) that satisfies \( x_n^{(k)} \rightarrow x_0 \), such that:

\[
P \in \text{plim} f (x_n).
\]

We will then denote:

\[
P \in \text{plim} \ f (x).
\]

**Remark.** In order to approximate the limit of a sequence (rather than just its partial limit), or the limit of a function, one may sample \( S \gg 1 \) sub-sequences and approximate the partial limits for each of them.

### 13 Computational Cost

**Definition.** Computational cost of singular expressions. Given a singular operator \( \bowtie \), a computer \( c \), and a number \( r \), we will define the computational cost of the expression \( \bowtie (r) \) given the computer, as the period of time required for the computer to evaluate the term \( \bowtie (r) \), assuming that the computer’s memory and computational power is wholly devoted to that mission. We will denote this cost by:

\[
\Upsilon_c (\bowtie (r)).
\]

**Definition.** Computational cost of boolean expressions. Given a boolean operator \( \bowtie \), a computer \( c \), and two numbers \( \{r_1, r_2\} \), we will define the computational cost of the expression \( r_1 \bowtie r_2 \) given the computer, as the period of time required for the computer to evaluate the expression \( r_1 \bowtie r_2 \), assuming that the computer’s memory and computational power is wholly devoted to that mission. We will denote this cost by:

\[
\Upsilon_c (r_1 \bowtie r_2).
\]

**Definition.** Computational cost of assembled expressions. Given a set of singular or boolean operators \( \{\bowtie_n\}_{1 \leq n \leq N} \), a computer \( c \), and a set of numbers \( \{r_1, \ldots, r_{n+1}\} \) we will define the computational cost of the assembled expression, \( r_1 \bowtie_1 r_2 \bowtie_3 \cdots \bowtie_n r_{n+1} \) in a recursive manner as:

\[
\Upsilon_c (r_1 \bowtie_1 r_2 \bowtie_3 \cdots \bowtie_n r_{n+1}) \equiv \Upsilon_c (r_1 \bowtie_1 r_2 \bowtie_3 \cdots \bowtie_{n-1} r_n) + \Upsilon_c (r_{n-1} \bowtie_n r_n),
\]

where \( r_{n-1}' \) is the value of the expression \( r_1 \bowtie_1 r_2 \bowtie_3 \cdots \bowtie_{n-1} r_n \).

**Remark.** The evaluation of the sign operator is a very withered case of
the evaluation of singular expressions. Although the $\text{sgn}(\cdot)$ operator can be interpreted as an assembly of logical boolean expressions, i.e (in C code):

$$\text{sgn}(r) = (r > 0)? 1 : (r < 0? -1 : 0),$$
	he computer in fact may not use this sequence of boolean expressions. The computer may only check the sign bit of the already evaluated expression $r$ (if such a bit is allocated). Especially, for any computer $c$, for the “$\div$” operator and for any numbers $r, r'$, it holds that:

$$\Upsilon_c(\text{sgn}(r)) \ll \Upsilon_c(r \div r'),$$

since the evaluation of the right-side expression operator involves bits manipulation, and requires a few cycles even in the strongest arithmetic logic unit (ALU), which are spared in the evaluation of the sign.

**Examples.**

1. Computational cost of approximating a partial limit of a sequence. Let $c$ be a computer, and let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence. Say we wish to evaluate the computational cost of the approximation of one of the partial limits of the sequence. Let us assume that we have guessed a partial limit $P$, sampled a random sequence of indexes $\{n_k\}_{k \in \mathbb{N}}$, and also guessed $M_{\min}, M_{\max}$ in the limit approximation process, along with guesses for $K_{\min}(m), K_{\max}(m)$ for each $M_{\min} < m < M_{\max}$. Hence, we should evaluate the logical expression $|a_{n_k} - P| \leq \frac{1}{m}$ for all possible values of $k, m$ in the domain. Thus, the computational cost of that process would be:

$$\Upsilon_c \Bigg( P \in \text{plim} a_n \Bigg) = \sum_{M_{\min} < m < M_{\max}} \sum_{N_{\min} < n_k < N_{\max}} \Upsilon_c \left( |a_{n_k} - P| \leq \frac{1}{m} \right)$$

and each addended can be written as:

$$\Upsilon_c(a_{n_k}) + \Upsilon_c(r_1 - P) + \Upsilon_c(|r_2|) + \Upsilon_c \left( \frac{1}{m} \right) + \Upsilon_c \left( r_3 \leq r_4 \right)$$

where $r_1 = a_{n_k}$, $r_2 = a_{n_k} - P$, $r_3 = |a_{n_k} - P|$ and $r_4 = \frac{1}{m}$.

2. Computational cost of approximating a partial limit of a function in a point. Let $c$ be a computer, $f : \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$. Say we wish to evaluate the computational cost of the approximation of a partial limit of $f$ in $x_0$. Let us assume that we have guessed the partial limit, $P$. Say we already sampled a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \to x_0$. Thus, the computational cost of that process would be

$$\Upsilon_c \left( P \in \text{plim} f(x) \right) = \Upsilon_c \left( P \in \text{plim} f(x_n) \right),$$

where the right side term is evaluated via paragraph 1.
**Theorem.** Given a non-parametric differentiable function $f : \mathbb{R} \to \mathbb{R}$ (by non-parametric the author means that the formula of $f$ is unknown, and especially the derivative cannot be calculated simply by placing $x_0$ in the formula of the derivative) and a computer $c$, the computational cost of approximating its derivative in a point $x_0 \in \mathbb{R}$ is much higher than the computational cost of approximating its extremum indicator, i.e.:

$$\Upsilon_c \left( \land f (x_0) \right) \ll \Upsilon_c \left( f'(x_0) \right).$$

**Proof.** Let us analyze the cost of the two main stages of the approximation of both the derivative and the extremum indicator:

1. The set of limits the computer needs to guess from. Let us assume that a sophisticated pre-processing algorithm managed to reduce the suspected values of the derivative (all of which one should verify in the definition of the approximation of the limit, as in the example above) to a very large, however finite set. Note that in order to approximate the extremum indicator, on the paper there seems to be more work (because there are more partial limits - namely 6 - to calculate); however, they are all calculated parallely (the computer may choose the same sequences that he used to approximate the derivative, and summarize the approximated partial limits of the upper and lower left and right detachments there). Further, the set of candidate partial limits for the extremum indicator is finite and very small: $\{0, \pm 1\}$.

2. Notice that the difference between the derivative and the detachment is the $\div$ operator vs. the $\text{sgn}$ operator. As we mentioned earlier,

$$\Upsilon_c \left( \text{sgn} \left( r \right) \right) \ll \Upsilon_c \left( r \div r' \right),$$

and this is for any $r, r'$. Hence the computational cost of any of the expressions inside the limit is cheaper for the detachment (hence for the extremum indicator) than the for the derivative.

To sum up, in both stages of the approximation of the limit there is a massive computational advantage to the extremum indicator over the derivative, which proves the claim. □

14 A NATURAL GENERALIZATION TO THE FUNDAMENTAL THEOREM OF CALCULUS

**Definition.** Antiderivative. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue-integrable function $\mathbb{R}^n$. Then its antiderivative is defined as follows:

$$F : \mathbb{R}^n \to \mathbb{R}$$

$$F(x_1, \ldots, x_n) \equiv \int_a^b f d\lambda,$$
where $B \equiv \prod_{i=1}^{n} (-\infty, x_i)$.

**Definition.** Local antiderivative. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue-integrable function $\mathbb{R}^n$. Then its local antiderivative initialized at the point $p = (p_1, \ldots, p_n)$ is defined as follows:

$$F_p : \prod_{i=1}^{n} [p_i, \infty) \to \mathbb{R}$$

$$F_p (x_1, \ldots, x_n) \equiv \int \limits_{\mu_p} fd\lambda,$$

where $B \equiv \prod_{i=1}^{n} (p_i, x_i)$.

**Lemma.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a given function. Then the following holds:

$$\sum_{u \in \{0,1\}^n} (-1)^{\sum_{i=1}^{n} u_i} f\left(\frac{a_1 + b_1}{2} + (-1)^{u_1} \frac{b_1 - a_1}{2}, \ldots, \frac{a_n + b_n}{2} + (-1)^{u_n} \frac{b_n - a_n}{2}, b_{n+1}\right)$$

$$- \sum_{v \in \{0,1\}^n} (-1)^{\sum_{i=1}^{n} v_i} f\left(\frac{a_1 + b_1}{2} + (-1)^{v_1} \frac{b_1 - a_1}{2}, \ldots, \frac{a_n + b_n}{2} + (-1)^{v_n} \frac{b_n - a_n}{2}, a_{n+1}\right)$$

$$= \sum_{t \in \{0,1\}^{n+1}} (-1)^{\sum_{i=1}^{n+1} t_i} f\left(\prod_{i=1}^{n+1} \left(\frac{a_i + b_i}{2} + (-1)^{t_i} \frac{b_i - a_i}{2}\right)\right).$$

**Proof.** To show the equality, we essentially ought to show that for any given addend in the left side, there exists an equal addend in the right side, and vice versa.

Let $t = (t_1, \ldots, t_{n+1})$ be a vector representing an addend in the right side. If $t_{n+1}$ is even, then the matching addend on the left side is given by choosing $u = (t_1, \ldots, t_n)$ in the first summation (for if $t_{n+1}$ is even then the last element in the vector is $b_{n+1}$, and $\sum_{i=1}^{n+1} t_i = \sum_{i=1}^{n} t_i (mod2)$, hence the signs coefficients are equal). Else, if $t_{n+1}$ is odd, then the matching addend on the left side is given by choosing $v = (t_1, \ldots, t_n)$, since the last element in the vector is $a_{n+1}$, and $\sum_{i=1}^{n+1} t_i \neq \sum_{i=1}^{n} t_i (mod2)$, hence the minus coefficient of the second summation adjusts the signs, and the addends are equal.

Now, let us consider an addend on the left side. In case this addend was chosen from the first summation, then the matching addend on the right side is given by choosing $t \equiv (u_1, \ldots, u_n, 0)$, in which case the last element of the right-side addend is set to $b_{n+1}$, and the sign coefficient is the same as the addend on the left side, again because $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i (mod2)$. If on the other hand, the addend was chosen from the second summation, then the matching addend on the right side is given by $t = (u_1, \ldots, u_n, 1)$, such that the last element is $a_{n+1}$, and the signs are the same due to $\sum_{i=1}^{n+1} t_i \neq \sum_{i=1}^{n} u_i (mod2)$ and the fact that the second
Lemma. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a given function which is Lipschitz continuous in a box \( B = \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n \) there. Then the following function:

\[
\begin{align*}
  f : \mathbb{R}^{n+1} &\to \mathbb{R} \\
  f(t_1, \ldots, t_n, t_{n+1}) &\equiv g(t_1, \ldots, t_n), \quad \forall t_{n+1} \in \mathbb{R}
\end{align*}
\]

is Lipschitz continuous in \( \prod_{i=1}^{n+1} [a_i, b_i] \), for every choice of \( \{a_{n+1}, b_{n+1}\} \subseteq \mathbb{R} \).

Proof. Let \( B' = \prod_{i=1}^{n+1} [a_i, b_i] \subset \mathbb{R}^{n+1} \) be a box. Let \( B = \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n \) be \( B' \)'s projection on \( \mathbb{R}^n \). Let \( a = (x_1, \ldots, x_{n+1}) \) and \( b = (y_1, \ldots, y_{n+1}) \) be two points such that \( a, b \in B' \). Then:

\[
|F(a) - F(b)| = |G(x_1, \ldots, x_n) - G(y_1, \ldots, y_n)| \leq M' \prod_{i=1}^{n+1} |x_i - y_i|
\]

\[
\leq M \prod_{i=1}^{n+1} |x_i - y_i| = M |a - b|,
\]

where \( M = \sup_B |f| \) and \( M' = \sup_{B'} |g| \), and the transitions are also due to the fact that \( F(B) \supseteq G(B') \). \( \square \)

Theorem. (In this paper, this theorem will be referred to as the “main theorem”). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a Lebesgue-integrable function \( \mathbb{R}^n \). Let us consider its local antiderivative:

\[
F : \mathbb{R}^n \to \mathbb{R}
\]

\[
F_p(x_1, \ldots, x_n) \equiv \int_B f \, d\lambda,
\]

where \( p = (p_1, \ldots, p_n) \) is a given point and \( B \equiv \prod_{i=1}^{n} (p_i, x_i) \). Then, \( F_p \) is Lipschitz continuous in any box \( B = \prod_{i=1}^{n} [a_i, b_i] \subseteq \mathbb{R}^n \) such that for each \( i \) it holds that \( a_i, b_i \geq p_i \), it is true that:

\[
\int_B f \, d\lambda = \sum_{s \in \{0,1\}^n} (-1)^{\sum_{j=1}^{n} s_j} \prod_{i=1}^{n} \left( \frac{b_i + a_i}{2} + (-1)^{s_i} \frac{b_i - a_i}{2} \right),
\]

where, in each addend, \( s = (s_1, \ldots, s_n) \).

Proof. For the simplicity of the discussion, we will show the correctness of the claim for a Riemann integrable function, and the correctness is easily
derived for Lebesgue integrable function. We show that the proposition holds by induction on $n$. For $n = 1$, the claim consolidates with the fundamental theorem of (integral) calculus, in its version quoted in the early work section. Let us suppose that the claim holds for a natural number $n$, and we will show that the claim is also true for $n + 1$. That is, we want to show that given an integrable function $f$, hence its antiderivative, $F$, exists, and is defined as follows:

$$F : \mathbb{R}^{n+1} \to \mathbb{R}$$

$$F(x_1, \ldots, x_{n+1}) \equiv \int_{-\infty}^{x_{n+1}} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f(t_1, \ldots, t_{n+1}) \, dt_1 \cdots dt_n \, dt_{n+1},$$

then, $F$ is Lipschitz continuous in any box $B = \prod_{i=1}^{n+1} [a_i, b_i] \subseteq \mathbb{R}^{n+1}$, and it holds that:

$$\int_{B} \int g \, dx = \sum_{x \in (0,1)^{n+1}} (-1)^{\sum_{i=1}^{n+1} s_i} F \left[ \prod_{i=1}^{n+1} \left( \frac{b_i + a_i}{2} + (-1)^{s_i} \frac{b_i - a_i}{2} \right) \right].$$

Let us set $t_{n+1}$ to constant, and define the following function:

$$g(t_1, \ldots, t_n) \equiv f(t_1, \ldots, t_n, t_{n+1}), \forall \{t_1, \ldots, t_n\} \subseteq \mathbb{R}^n. \quad (4)$$

Let us observe the box $B' = \prod_{i=1}^{n} [a_i, b_i] \subseteq \mathbb{R}^n$. Now $g$ is defined by a projection of an integrable function $f$ in $B$, hence $g$ is integrable in $B'$. By applying the previous lemma, it holds that $F$ is Lipschitz continuous in $B$. By applying the second induction hypothesis, the function:

$$G : \mathbb{R}^n \to \mathbb{R}$$

$$G(x_1, \ldots, x_n) \equiv \int_{-\infty}^{x_n} \int_{-\infty}^{x_{n-1}} \cdots \int_{-\infty}^{x_1} g(t_1, \ldots, t_n) \, dt_1 \cdots dt_{n-1} \, dt_n,$$

is Lipschitz continuous in $B'$, and:

$$\int_{B'} \int g \, dx = \sum_{x \in (0,1)^n} (-1)^{\sum_{i=1}^{n} s_i} G \left[ \prod_{i=1}^{n} \left( \frac{b_i + a_i}{2} + (-1)^{s_i} \frac{b_i - a_i}{2} \right) \right].$$

Let us set $(x_1, \ldots, x_n)$ to constant, and integrate equation 4 in the extent of $n + 1$ times, over the unbounded box $\prod_{i=1}^{n} (-\infty, x_i) \times [a_{n+1}, b_{n+1}]$:

$$\int_{a_{n+1} - \infty}^{b_{n+1}} \cdots \int_{a_{n+1} - \infty}^{b_{n+1}} g(t_1, \ldots, t_n) \, dt_1 \cdots dt_n \, dt_{n+1} = \int_{a_{n+1} - \infty}^{b_{n+1}} \cdots \int_{a_{n+1} - \infty}^{b_{n+1}} f(t_1, \ldots, t_n, t_{n+1}) \, dt_1 \cdots dt_n \, dt_{n+1}. $$

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By the definition of $F$ and $G$,
\[
\int_{a_{n+1}}^{b_{n+1}} G(x_1, \ldots, x_n) \, dt_{n+1} = F(x_1, \ldots, x_n, b_{n+1}) - F(x_1, \ldots, x_n, a_{n+1}). \tag{5}
\]

Simple manipulations result with:
\[
\begin{align*}
&\int_{a_{n+1}}^{b_{n+1}} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(t_1, \ldots, t_n, t_{n+1}) \, dt_1 \cdots dt_n \, dt_{n+1} \\
&= \int_{a_{n+1}}^{b_{n+1}} \left( \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} g(t_1, \ldots, t_n) \, dt_1 \cdots dt_n \right) \, dt_{n+1} \\
&= \int_{a_{n+1}}^{b_{n+1}} \left( \sum_{s \in \{0, 1\}^n} (-1)^\sum_i s_i G \left[ \prod_{i=1}^n \left( \frac{b_i + a_i}{2} + (-1)^{s_i} \frac{b_i - a_i}{2} \right) \right] \right) \, dt_{n+1} \\
&= \sum_{u \in \{0, 1\}^n} (-1)^\sum_i u_i \int_{a_{n+1}}^{b_{n+1}} G \left( \frac{a_1 + b_1}{2} + (-1)^{u_1} \frac{b_1 - a_1}{2}, \ldots, \frac{a_n + b_n}{2} + (-1)^{u_n} \frac{b_n - a_n}{2} \right) \, dt_{n+1} \\
&= \sum_{u \in \{0, 1\}^n} (-1)^\sum_i u_i F \left( \frac{a_1 + b_1}{2} + (-1)^{u_1} \frac{b_1 - a_1}{2}, \ldots, \frac{a_n + b_n}{2} + (-1)^{u_n} \frac{b_n - a_n}{2}, a_{n+1} \right) \\
&\quad - \sum_{v \in \{0, 1\}^n} (-1)^\sum_i v_i F \left( \frac{a_1 + b_1}{2} + (-1)^{v_1} \frac{b_1 - a_1}{2}, \ldots, \frac{a_n + b_n}{2} + (-1)^{v_n} \frac{b_n - a_n}{2}, a_{n+1} \right) \\
&= \sum_{t \in \{0, 1\}^{n+1}} (-1)^\sum_i t_i F \left( \prod_{i=1}^{n+1} \left( \frac{a_i + b_i}{2} + (-1)^{t_i} \frac{b_i - a_i}{2} \right) \right),
\end{align*}
\]

Where the last transitions are due to the definitions of $F, G,$ equation (5), and a previous lemma. \(\square\)

15 \textbf{SLANTED LINE INTEGRAL}

15.1 Definitions of operators

**Definition.** Induced Measure. Given a Lebesgue-integrable function $f : \mathbb{R}^n \to \mathbb{R}$, and a domain $D \subset \mathbb{R}^n$, we will define the induced measure, $m_f$, of the domain $D$ given the function $f$, in the following manner:

\[ m_f(D) \equiv \int_D f \, d\lambda. \]
It real analysis it is shown that \( m_f \) is indeed a measure.

**Definition.** Corners of a straight surface. Let \( \Pi : \pi(t) \subset \mathbb{R}^n \), \( t \in [0,1]^n \) be a straight surface, in the sense that it is perpendicular to one of the axes of \( \mathbb{R}^n \), i.e., there exists at least one dimension \( d \) which satisfies that \( \pi(t_1, \ldots, t_d, \ldots, t_n) \) is constant for any choice of \( t_d \in [0,1] \). We will say that the point \( x \in \mathbb{R}^n \) is a corner of the straight surface if there exists a vector \( s \in \{0,1\}^n \) such that \( x = \pi(s) \). We will denote the set of corners of a straight surface by:

\[
\nabla \cdot \Pi.
\]

**Definition.** Generalized rectangular domain. A generalized rectangular domain \( D \subset \mathbb{R}^n \) is a domain that satisfies:

\[
\partial D = \bigcup_{\omega \in \Omega} \Pi_\omega,
\]

where each \( \Pi_\omega \) is perpendicular to one of the axes of \( \mathbb{R}^n \). In this paper we will sometime abbreviate “generalized rectangular Domain” by “GRD”.

**Definition.** Corners of a generalized rectangular domain. Given a generalized rectangular domain \( D \), we will say that a point \( x \in \partial D = \bigcup_{\omega \in \Omega} \Pi_\omega \) is a corner of \( D \) if there exist at least two numbers, \( \{\omega_1, \omega_2\} \subseteq \Omega \) such that:

\[
x \in \nabla \cdot \Pi_{\omega_1} \cap \nabla \cdot \Pi_{\omega_2}.
\]

**Remark.** The following is a version of Wang et al.'s theorem, for Lebesgue Integral. The formulation of the original theorem is found in [7].

**Theorem. (The Discrete Green’s Theorem).** Let \( D \subset \mathbb{R}^n \) be a generalized rectangular domain, and let \( f \) be a Lebesgue-Integrable function in \( \mathbb{R}^n \). Let \( F \) be the antiderivative of \( f \), in the same terms of this paper's theorem 2. Then:

\[
\int_D f \, d\lambda = \sum_{x \in \nabla \cdot D} \alpha_D(x) F(x),
\]

where \( \alpha_D : \mathbb{R}^n \rightarrow \mathbb{Z} \), is a map that depends on \( n \). For \( n = 2 \) it is such that \( \alpha_D(x) \in \{0, \pm1, \pm2\} \), according to which of the 10 types of corners, depicted in figure 1 in Wang et al.'s paper, \( x \) belongs to.

For the simplicity of the discussion, let us observe \( \mathbb{R}^2 \) throughout the rest of this section, although natural generalizations can be also built for \( \mathbb{R}^n \). Further, we will assume that the coordinates system is known, which is the case in computers calculation. Let us also observe that the connection between eight of the corners in this theorem and the detachments of the functions that form the curve (in case these exist).

**Definition.** Tendable curve. Let \( C = \gamma(t) = (x(t), y(t)) \), \( 0 \leq t \leq 1 \) be a curve, where \( x, y : [0,1] \rightarrow \mathbb{R} \). It will be said to be tendable if the functions that
form the curve, i.e. $x$ and $y$, are both tendable for $0 \leq t \leq 1$.

**Definition.** Tendency indicator vector of a tendable curve. Let $C = \gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$ be a tendable curve, and let $z = \gamma(t_0) = (x(t_0), y(t_0)) \in C$ be a point on the curve. We will define the tendency indicator vector of the curve $C$ in the point, as:

$$
\vec{s}(C, t_0) : \{1, 2, 3, 4\} \rightarrow \{+1, -1, 0\}^4
$$

$$
\vec{s}(C, t_0) \equiv (x^+_1, x^-_1, y^+_1, y^-_1) \mid t_0.
$$

**Definition.** Tendency of a tendable curve. Let $C = \gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$ be a tendable curve. We will define the tendency of the curve $C$ in the point $z = \gamma(t_0) = (x(t_0), y(t_0)) \in C$, as:

$$
\tau_C : C \rightarrow \{+1, -1, 0\}
$$

$$
\tau_C(z) = \begin{cases} +1, & (s_1 \cdot s_4 \neq 0) \land (s_2 = s_3) \\ -1, & (s_2 \cdot s_3 \neq 0) \land (s_1 = -s_4) \\ 0, & \text{otherwise}, \end{cases}
$$

where $\vec{s}(C, t_0) \equiv (x^+_1, x^-_1, y^+_1, y^-_1) \mid t_0$ is the tendency indicator vector of the curve.

**Example.** The following values of the tendency indicator vector imply a positive tendency in the point:

$$
\tau_C = +1 \iff \vec{s} \in \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} +1 \\ 0 \\ 0 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ +1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} +1 \\ 0 \\ 0 \\ +1 \end{pmatrix} \right\},
$$

and the following values imply a negative tendency in a point:

$$
\tau_C = -1 \iff \vec{s} \in \left\{ \begin{pmatrix} 0 \\ +1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ +1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ +1 \\ 1 \end{pmatrix}, \begin{pmatrix} +1 \\ +1 \\ -1 \end{pmatrix}, \begin{pmatrix} +1 \\ +1 \\ 0 \end{pmatrix} \right\}.
$$

**Definition.** Uniformly tended curve. Given a tendable curve $C$, if it holds that the tendency of the curve is a constant $\beta$ for each point on the curve apart perhaps its two end-points, then we will say that the curve is tended uniformly, and denote: $C_{\beta} \equiv C$.

**Lemma.** Let $C = \gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$ be a given tendable curve. If the four left and right detachments, $\delta^-, \delta^+, \delta^-, \delta^+$ are constant on the curve for each $0 < t < 1$, then $C$ is totally contained in a square whose
opposite vertices are the given curve’s endpoints.

**Proof.** According to a claim regarding signposted detachable functions in \( \mathbb{R} \), both the functions \( x \) and \( y \) are monotone there, hence for each \( 0 < t < 1 \) it holds that:

\[
\begin{align*}
x(0) &< x(t) < x(1) \\
y(0) &< y(t) < y(1),
\end{align*}
\]

hence the curve’s points are fully contained in the square \([x(0), y(0)] \times [x(1), y(1)]\). \(\Box\)

**Definition.** A straight path between two points. Given two points, \( \{ x = (a_1, b_1), y = (a_2, b_2) \} \subset \mathbb{R}^2 \), we will define the following curves:

\[
\begin{align*}
\gamma_1^+ &:= \{ x(t) = ct + a_1 (1 - t) \\
y(t) &= b \\
\gamma_2^+ &:= \{ x(t) = c \\
y(t) &= dt + b (1 - t) \\
\gamma_1^- &:= \{ x(t) = a \\
y(t) &= dt + b (1 - t) \\
\gamma_2^- &:= \{ x(t) = ct + a (1 - t) \\
y(t) &= d,
\end{align*}
\]

where, in each term, it holds that \( 0 \leq t \leq 1 \). Then, we will sat that \( \gamma^+ (\{x, y\}) \equiv \gamma_1^+ \cup \gamma_2^+ \) and \( \gamma^- (\{x, y\}) \equiv \gamma_1^- \cup \gamma_2^- \) are the straight paths between the two points. We will refer to \( \gamma^+ (\{x, y\}) \), \( \gamma^- (\{x, y\}) \) as the positive and negative straight paths of \( \{x, y\} \), respectively.

**Definition.** Paths of a curve. Given a curve \( C = \gamma(t) \), Let us consider its end points, \( \{ \gamma(0), \gamma(1) \} \), and let us consider the straight paths between the points, \( \gamma^+ \) and \( \gamma^- \), as suggested in a previous definition. We will define the paths of the curve \( C \) in the following manner:

\[
C^+ \equiv \gamma^+(\{\gamma(0), \gamma(1)\}), \quad C^- \equiv \gamma^-(\{\gamma(0), \gamma(1)\}).
\]

We will refer to \( C^+, C^- \) as the curve’s positive and negative paths respectively.

**Definition.** Partial domains of a continuous uniformly tended curve. Given a continuous uniformly tended curve \( C_\beta \) whose orientation is \( s \), we will define the partial domains of \( C_\beta \), namely \( D^+(C_\beta) \) and \( D^-(C_\beta) \), as the closed domains whose boundaries satisfy:

\[
\partial D^+(C_\beta) \equiv C^*_\beta, \quad \partial D^-(C_\beta) \equiv C^-_\beta^s,
\]

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where $C_\beta, C_{\beta^-}$ are the paths of the $C_\beta$. We will refer to $D_+(C_\beta)$ as the selected domain of the continuous uniformly tended curve.

**Definition.** A square of a point on a curve. Given a curve $C = \gamma(t)$, with orientation $s$ ($s \in \{\pm 1\}$, to denote that the curve is either positively or negatively oriented), we will say that $S(C, x, s)$ is a square of the point $x = \gamma(t_0) \in C$, if $S(C, x, s)$ is a square whose edges are parallel to the axes, $x \in \nabla \cdot S(C, x, s)$ and $S(C, x, s) \cap D_-^s(x, \gamma(t)) \subseteq C$, i.e., the square $S(C, x, s)$ is fully contained in one side of the curve $C$, including the curve itself, where the side is determined according to the sign of $s$. We will define $\alpha(S(C, x, s))$ in the following manner:

$$\alpha(S(C, x, s)) \equiv \alpha_{D}(S(C, x, s))|_x.$$ 

**Remark.** It is trivial to show that $\alpha_C(x)$ is independent of the sizes of the squares in the chosen set $\{S^{(i)}(C, x, s)\}_i$, hence the tendency of the curve is well defined.

**Theorem.** Let $C$ be a tendable curve. The tendency of $C$ in a point $z \in C$ satisfies:

$$\tau_C(z) = \sum \alpha\left(S^{(i)}(C, z, s)\right),$$

where $\{S^{(i)}(C, z, s)\}_i$ is a maximal set of squares such that the measure of the set $\bigcap_i S^{(i)}(C, z, s)$ is zero. If the set $\{S^{(i)}(C, z, s)\}_i$ is empty, we will define $\sum \alpha(S^{(i)}(C, z, s)) = 0$.

**Proof.** By division to cases, in a similar manner that is done in $\mathbb{R}$. □

**Definition.** Selected partial domains of a continuous curve. Given a continuous curve $C = \bigcup_{\omega} C_{\beta}(\omega)$, where $\{C_{\beta}(\omega)\}$ is a set of pairwise disjoint continuous uniformly tended curves, we will define the selected partial domains for each $\omega$ as $D(\omega)(C_{\beta}(\omega)) \equiv D_+(C_{\beta}(\omega))$.

**Definition.** Slanted line integral of a Lebesgue-Integrable function’s antiderivative on a uniformly tended curve in $\mathbb{R}^2$. Let us consider a uniformly tended curve $C_\beta = \gamma(t)$, $0 \leq t \leq 1$, whose orientation is $s$. Let us consider a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is Lebesgue-Integrable there. Let us consider its local antiderivative, $F_p$, where $p \in \mathbb{R}^2$. Then the slanted line integral of $F_p$ on $C_\beta$ is defined as follows:

$$\int_{C_{\beta}} F_p \equiv m_f \left(D_+(C_{\beta}) - \beta F_p(\gamma(1)) + \frac{1}{2}[\beta_0 F_p(\gamma(0)) + \beta_1 F_p(\gamma(1))]\right),$$

where $\gamma^+_1$ is either $\gamma^+$ or $\gamma^-$ according to the sign of $s$, and $\beta_0, \beta_1$ are the curve’s tendencies in the points $\gamma(0)$ and $\gamma(1)$ respectively.

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15.2 Properties of the slanted line integral and an analogous theorem to Green’s theorem

**Lemma.** (Additivity). Let $C_1 = C^{(1)}_\beta, C_2 = C^{(2)}_\beta$ two uniformly tended curves, that satisfy:

$$\exists ! x \in \mathbb{R}^2 : x \in C^{(1)}_\beta \cap C^{(2)}_\beta,$$

and let us also assume that both the curves share the same orientation $s$. Let us consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is Lebesgue-Integrable there. Let us consider its local antiderivative, $F_p$, where $p \in \mathbb{R}^2$. Let us consider the curve $C_\beta \equiv C^{(1)}_\beta \cup C^{(2)}_\beta$. Let us denote the curves by $C_1 = \gamma_1, C_2 = \gamma_2$ and $C = \gamma$ accordingly. Then:

$$\int_{C_\beta} F_p = \int_{C^{(1)}_\beta} F_p + \int_{C^{(2)}_\beta} F_p.$$

**Proof.** Without loss of generality, let us assume that $C_1 \cap C_2 = \gamma_1 (1) = \gamma_2 (0)$. Let us denote the paths of the curves $C_1, C_2$ and $C$ by $\gamma_1, \gamma_2$ and $\gamma$ accordingly, where $i \in \{1, 2\}$. According to the definition of the slanted line integral, we obtain:

$$\int_{C_1} F_p = m (D^+ (C_1)) - \beta F_p (\gamma_1 (1)) + \frac{1}{2} [\beta_0 F_p (\gamma_1 (0)) + \beta_1 F_p (\gamma_1 (1))],$$

$$\int_{C_2} F_p = m (D^+ (C_2)) - \beta F_p (\gamma_2 (1)) + \frac{1}{2} [\beta_0 F_p (\gamma_2 (0)) + \beta_1 F_p (\gamma_2 (1))].$$

Now according to this paper’s main theorem, it holds that:

$$m (D) = m (D_1) + m (D_2) + \beta \left\{ [F_p (\gamma_1^s (1)) + F_p (\gamma_2^s (1))] - [F_p (\gamma_1^s (1)) + F_p (\gamma_2^s (1))] \right\}.$$

Now the desired result is derived from the above formula by applying the assumption in the beginning of the proof. □

**Definition.** Slanted line integral of a Lebesgue-Integrable function’s antiderivative on a curve in $\mathbb{R}^2$. Let us consider a curve $C = \bigcup \{C^{(w)}_\beta\}$, where each $\{C^{(w)}_\beta\}$ are pairwise disjoint uniformly tended curves. Let us consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is Lebesgue-Integrable there. Let us consider its local antiderivative, $F_p$, where $p \in \mathbb{R}^2$. Then the slanted line integral of $F_p$ on $C$ is defined as follows:

$$\int_C F_p = \int_{C^{(w)}_\beta} F_p \, d\lambda.$$
**Lemma.** Let us consider a uniformly tended curve \( C_\beta = \gamma(t) \), \( 0 \leq t \leq 1 \) whose orientation is positive. Let us consider the curve \(-C_\beta\) which consolidates with \( C_\beta \) apart from the fact that its orientation is negative. Let us consider a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) which is Lebesgue-Integrable there. Let \( F_p \) be its local antiderivative, where \( p \in \mathbb{R}^2 \). Then it holds that:

\[
\int_{C_\beta} F_p = -\int_{-C_\beta} F_p.
\]

**Proof.** Since it holds that

\[
\int_{D^+(C_\beta)} f d\lambda + \int_{D^-(C_\beta)} f d\lambda = \int_{D^+(C_\beta) \cup D^-(C_\beta)} f d\lambda,
\]

since the “main theorem” claims that:

\[
\int_{D^+(C_\beta) \cup D^-(C_\beta)} f d\lambda = \beta \left\{ F(\gamma^+_1(1)) + F(\gamma^-_1(1)) - [F(\gamma(1)) + F(\gamma(1))] \right\},
\]

then by considering all the cases of \( \beta \), rearranging the terms and applying the definition of the slanted line integral for \( C_\beta \), the corollary is trivially derived. □

**Theorem.** Let us consider a curve \( C = \gamma(t) \), \( 0 \leq t \leq 1 \) whose orientation is constant. Let us consider the curve \(-C\) which consolidates with \( C \) apart from the fact that its orientation is the opposite to the given curve’s orientation. Let us consider a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) which is Lebesgue-Integrable there. Let \( F_p \) be its local antiderivative, where \( p \in \mathbb{R}^2 \). Then it holds that:

\[
\int_{-C} F_p = -\int_{C} F_p.
\]

**Proof.** Immediate from the previous lemma by rearranging the terms. □

**Lemma.** Given a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) which is Lebesgue-Integrable there, and a continuous uniformly tended curve \( C_0 \), then the slanted line integral of its antiderivative, \( F \) on \( C_0 \) satisfies that:

\[
\int_{C_0} F = 0
\]

**Proof.** Since it is easily shown via measure theory that the measure of the set \( D^\pm(C_0) \) is zero (since the curve consolidates with its paths), and since \( \beta = 0 \), then corollary is derived from the slanted line integral’s definition. □

**Definition.** Generalized rectangular curve. We will say the \( C \subset \mathbb{R}^2 \) is a generalized rectangular curve if \( C = \bigcup_{\omega \in \Omega} L_\omega \) is a union of intervals which are parallel to the axes.
**Definition.** Corners of a generalized rectangular curve. We will say the point \( x \in C = \bigcup_{\omega \in \Omega} I_\omega \subset \mathbb{R}^2 \) is a corner of the generalized rectangular square \( C \) if there exist at least two numbers \( \{\omega_1, \omega_2\} \) such that \( x \in I_{\omega_1} \cap I_{\omega_2} \). We will denote the set of corners of a generalized rectangular curve by: \( \nabla \cdot C \).

**Theorem.** (The slanted line integral theorem in its \( \mathbb{R}^2 \) form). Let us consider a curve \( C \subset \mathbb{R}^2 \) whose orientation is given. Let us consider a function \( f : \mathbb{R}^2 \to \mathbb{R} \) which is Lebesgue-Integrable there. Let us consider its local antiderivative, \( F_p \), where \( p \in \mathbb{R}^2 \). Let \( \{C_{\beta}^{(\omega)}\} \) be uniformly tended, continuous, and pairwise disjoint curves such that: \( C = \bigcup_{\omega} C_{\beta}^{(\omega)} \), where each \( \omega \) is chosen from a family of real numbers, \( \Omega \). Let \( \{D^{(\omega)}\} \) be the set of chosen domains of \( C \). Let \( C_{||} \) be the union of all the straight lines which are the boundaries of \( D^{(\omega)} \), for each \( \omega \). Then the slanted line integral of \( F_p \) on \( C \) obeys the following formula:

\[
\int_{C} F_p \equiv \int_{\omega} m_f \left( D^{(\omega)} \right) d\lambda - \int_{x \in \nabla \cdot C_{||} \cap C} \beta_{C_{||}} (x) F_p (x) d\lambda + \int_{x \in \nabla \cdot C_{||} \cap C} \beta_{C_{||}} (x) F_p (x) d\lambda,
\]

where \( \beta_{C_{||}} \) is defined as follows:

\[
\beta_{C_{||}} (x) \equiv \begin{cases} \int_{\eta} \alpha_{D^{(\eta)}} (x) d\lambda, & x \in \bigcap_{\eta} \nabla \cdot D^{(\eta)} \\ \alpha_{C_{||}} (x), & x \in C \end{cases}, \quad \forall x \in \nabla \cdot C_{||}.
\]

It is trivial to show that in the terms of the above definition, \( \nabla \cdot C_{||} \subset \nabla \cdot D^{(\omega)} \cup C \), hence \( \beta_{C_{||}} \) is well defined.

**Proof.** Immediate from this paper’s “main theorem” and the definition of the slanted line integral, by rearranging the terms. \( \square \)

**Theorem.** Let \( D \subseteq \mathbb{R}^2 \) be a given domain, and let \( f \) be a Lebesgue-Integrable function in \( \mathbb{R}^2 \). Let \( F_p \) be its local antiderivative, where \( p \in \mathbb{R}^2 \). Then, in the same terms that were introduced, it holds that:

\[
\int_{D} f d\lambda = \int_{\nabla \cdot D} F_p.
\]

**Proof.** Immediate from the definition of the slanted line integral in its \( \mathbb{R}^2 \) form by rearranging the terms and applying this paper’s “slanted line integral theorem”. \( \square \)

**Notation.** The slanted line integral of a function’s local antiderivative \( F_p \),
on a curve $C$ (where $p \in \mathbb{R}^2$), calculated with a given rotation $\theta$ of the coordinates system, will be denoted by $\int_C^\theta F_p$.

**Corollary.** Let $C \subseteq \mathbb{R}^2$ be a closed and continuous curve, and let $f$ be a Lebesgue-Integrable function in $\mathbb{R}^2$. Let $F_{p_1}, F_{p_2}$ be two of its local antiderivatives, where $p \in \mathbb{R}^2$, where $F_{p_1}, F_{p_2}$ are calculated with given rotations $\theta_1, \theta_2$ of the coordinates systems. Then, in the same terms that were introduced, it holds that:

$$\int_C^\theta_1 F_{p_1} = \int_C^\theta_2 F_{p_2}.$$

**Proof.** Since $C$ is closed and continuous, it is the boundary of a domain $D$, hence both the following equalities hold:

$$\int_D f \, d\lambda = \int_C^\theta_1 F_{p_1}, \quad \int_D f \, d\lambda = \int_C^\theta_2 F_{p_2}. \quad \Box$$

**Notation.** According to the above corollary, the slanted line integral of a closed curve in $\mathbb{R}^2$ is independent of the choice of the angle of the rotation of the coordinates system or the point $p$ in the definition of the local antiderivative $F_p$. Hence, for a bounded domain $D$ we shall denote the previous quoted theorem by:

$$\oint_\partial D F = \int_D f \, d\lambda,$$

where $\oint_\partial D F \equiv \int_C^\theta F_p$ for any choice of the coordinates system's rotation $\theta$ and for any point $p \in \mathbb{R}^2$.

**Remark.** It is easy to show that Green’s theorem is bidirectional to the above theorem: the first direction is easily shown via chosing $m_f(D) = \int \int_D f \, dx \, dy$ in the definition of the induced measure and selecting $f \equiv Q'_x - P'_y$. The other direction is proved by an elaborated version of Green’s theorem that also holds for an arbitrary curve (such as fractals) in space.
Part V
Epilogue

“Whenever the road ends there begins somekind of a path, whenever the night ends -
the morning begins, when an hour has run out - another hour arrives, only in the end
of knowledge - the error begins. An ending is always a beginning of something else.
Better? Worse? I do not know which of them holds. Something else.”

– Lea Naor.

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17 APPENDIX

17.1 Series continuousness and series differentiability

**Remark.** As usual, we will say that an operator is applied on a function
$f : X \rightarrow Y$ in $X$ if it is applied to every point $x \in X$.

**Definition.** Series-continuousness. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will
say that it is series-continuous from right or left in a point $x \in \mathbb{R}$, if there exists
a number $\delta > 0$ such that for any monotoneous sequence $\{x_n^+\}_{n=1}^\infty$ or $\{x_n^-\}_{n=1}^\infty$
(with respect to the left or right side), such that $x_n \rightarrow x$, $\pm x \leq \pm x_n^\pm$ and
$0 < |x_n^\pm - x| < \delta$ for all $n$, it holds that $\sum_{n \in \mathbb{N}} |f(x_n^\pm) - f(x)|$ converges.

**Definition.** Series-differentiability. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we will
say that it is series-differentiable from right or left in a point $x \in \mathbb{R}$, if
there exists a number $\delta > 0$ such that for any monotoneous sequence $\{x_n\}_{n=1}^\infty$ or $\{x_n^+\}_{n=1}^\infty$ (with respect to the left or right side), such that $x_n \rightarrow x$, $\pm x \leq \pm x_n^\pm$ and
$0 < |x_n^\pm - x| < \delta$ for all $n$, it holds that $\sum_{n \in \mathbb{N}} \frac{|f(x_n^\pm) - f(x)|}{|x_n^\pm - x|}$ converges.

**Remark.** The above definition, although resembles the absolute continuity definition, is not a special case of it, since this definition is applied for a
single point rather than a whole interval.
EXAMPLES.

1. The function \( f : \mathbb{R} \to \mathbb{R} \), \( f(x) = x \) is not series-continuous nor series-differentiable in any point. For example, for \( x = 0 \), one can define the series \( x_n = \frac{1}{n} \), for which the series diverges.

2. The function:

\[
f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right), & x \neq 0 \\ 0, & x = 0 \end{cases}
\]

is shown to be series-continuous in \( x = 0 \) via Dirichlet’s condition.

3. The function:

\[
f(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right), & x \neq 0 \\ 0, & x = 0 \end{cases}
\]

is shown to be series-continuous and series-differentiable in \( x = 0 \) via Dirichlet’s condition.

REMARK. Not every series-continuous function \( f : \mathbb{R} \to \mathbb{R} \) is continuous. Consider for instance the function:

\[
f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right), & x \in \mathbb{Q} \\ \frac{1}{7}, & \text{o.w.} \end{cases}
\]

Then according to Dirichlet’s condition, \( f \) is series-continuous in \( x = 0 \), however it is not continuous there.

THEOREM. Every series-differentiable function \( f : \mathbb{R} \to \mathbb{R} \) is series-continuous.

PROOF. Let \( x \in \mathbb{R} \). Since \( f \) is series differentiable, then for any monotonous sequence \( \{x_n\}_{n=1}^{\infty} \) such that \( x_n \to x \) it holds that the series \( \sum_{n \in \mathbb{N}} \frac{|f(x_n) - f(x)|}{|x_n - x|} \) converges; now since there exists \( N \) such that for any \( n > N \) it holds that \( |x_n - x| < 1 \), then \( |f(x_n) - f(x)| < \frac{|f(x_n) - f(x)|}{|x_n - x|} \), hence the series \( \sum_{n \in \mathbb{N}} |f(x_n) - f(x)| \) also converges, hence \( f \) is series-continuous. \( \square \)

THEOREM. If a series-countinuous function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable, then \( f' \equiv 0 \).

PROOF. Let \( x \in \mathbb{R} \). On the contrary, suppose \( f'(x) \neq 0 \). Then by observing the sequence \( \{x_n\}_{n=1}^{\infty} \) defined by \( x_n = x + \frac{1}{n} \), it holds that \( x_n \to x \) although:

\[
\sum_{n \in \mathbb{N}} \left| f\left( x + \frac{1}{n} \right) - f(x) \right| \approx \sum_{n \in \mathbb{N}} \left| f(x) + \frac{1}{n} f'(x) - f(x) \right| = |f'(x)| \sum_{n \in \mathbb{N}} \frac{1}{n} \to \infty,
\]
where the approximation is due to Lagrange’s intermediate theorem. This contradicts the assumption that \( f \) is series-continuous. □

### 17.2 A different definition to the limit process

**Definition.** A sequence \( \{a_n\}_{n \in \mathbb{N}} \) is said to have a limit \( L \) if for any \( \epsilon > 0 \) there exists an index \( \bar{N} \) such that for any \( N_{max} > \bar{N} \) there exists \( N_{min} < N_{max} - 1 \) such that for all \( N_{min} < n < N_{max} \) it holds that:

\[
|a_n - L| < \epsilon.
\]

**Theorem.** The above definition, and Cauchy’s definition, are equivalent.

**Proof.** First direction. Suppose that Cauchy’s definition holds for a sequence. Hence, given an \( \epsilon > 0 \) there exists a number \( N \) such that for any \( n > N \) it holds that \( |a_n - L| < \epsilon \). Let us choose \( \bar{N} = N + 1 \). Then especially, given \( N_{max} > \bar{N} \), then \( N_{min} = \bar{N} \) satisfies the required condition.

Second direction. Suppose that the above definition holds for a sequence. We want to show that Cauchy’s definition also holds. Given \( \epsilon > 0 \), let us choose \( N = \bar{N} \). We would like to show now that for any \( n > N \) it holds that \( |a_n - L| < \epsilon \). Indeed, let \( n_0 > N \). Then according to the definition, if we choose \( N_{max} = n_0 + 1 \), there exists \( N_{min} < n_0 \) such that for any \( n \) that satisfies \( N_{min} < n < N_{max} \) it holds that \( |a_n - L| < \epsilon \). Especially, \( |a_{n_0} - L| < \epsilon \). □

**Remark.** Note that Cauchy’s definition for limit consists of the following argument. A sequence \( \{a_n\}_{n \in \mathbb{N}} \) is said to have a limit \( L \) if for any \( \epsilon > 0 \) (as small as we desire), there exists some \( N(\epsilon) \) such that for any \( n > N \) it holds that:

\[
|a_n - L| < \epsilon.
\]

Consider the following alternative: There exists \( \epsilon_{max} \) such that for any \( 0 < \epsilon < \epsilon_{max} \) there exists \( N(\epsilon) \) with \( |a_n - L| < \epsilon \). The author would like to point out that this alternative suggest a more rigorous terminology to the term “as small as we desire”, and also forms a computational advantage: one knows exactly what is domain from which \( \epsilon \) should be chosen. It is clear that both the definitions are equivalent, hence the proof is skipped. Following is a slightly different modification of the discussed suggestion to define the limit, where once again the proof to its equivalency to the previously known definitions is skipped.

**Definition.** A sequence \( \{a_n\}_{n \in \mathbb{N}} \) is said to have a limit \( L \) if there exists a number \( M > 0 \) such that for any \( m > M \) there exists \( \hat{N}(m) \) such that for any \( N_{max} > \hat{N} \) there exists \( N_{min} < N_{max} - 1 \) such that for all \( N_{min} < n < N_{max} \) it holds that:

\[
|a_n - L| < \frac{1}{m}.
\]
17.3 Limit of a monotonic bounded function

**Definition.** Limit of a monotonic bounded function in a point. We will say that a function \( f : \mathbb{R} \to \mathbb{R} \) satisfies that \( \lim_{x \to x_0} f(x) = f(x) \) (and say that \( f(x) \) is the monotonic limit of \( f \)), if \( f \) is monotonic and bounded in the neighborhood of \( x \), and for any monotonic sequence \( x_n \to x \) it holds that \( f(x_n) \to f(x) \).

**Remark.** Note the difference between the above definition and Heine’s definition: the term “monotonic” was added.

**Theorem.** The above definition to a limit of a monotonic bounded function in a point, and Heine’s definition, are equivalent, in the sense that if \( f \) is monotonic and bounded around \( x_0 \) then:

\[
\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x)
\]

**Proof.** The first direction is easy: if any sequence satisfies the condition, then especially any monotonic bounded function satisfies it for any monotonic sequence. Now suppose that any monotonic bounded sequence satisfies the condition. We would like to show that any sequence also satisfies it. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence, for which \( x_n \to x \). As shown in calculus, the convergence of this sequence implies that it has a monotonic sub-sequence \( \{x_{n_k}\}_{k \in \mathbb{N}} \). But this sub-sequence converges to \( x \) (otherwise, \( x_n \not\to x \)). Hence according to the assumption, \( f(x_{n_k}) \to f(x) \). But \( f \) is bounded and monotonic in a neighborhood of \( x_0 \), hence \( f(x_n) \) converges. Hence \( f(x_n) \to f(x) \) (since any converging sequence has only one partial limit). \( \square \)
17.4  Source code in matlab

Algorithm 2  Source code for determining the type of disdetachment

% DetermineTypeOfDisdetachment - given a function f and its tendency indicator vector in a point x,
% will return a vector containing the classification of the function to its detachment types, via
% the vector res, i.e: res(i) = 1 iff the function has i-th type disdetachment in x.
% Author: Amir Finkelstein, amir.f22@gmail.com
% Date: 16-February-2010

function res = DetermineTypeOfDisdetachment(v)
    res = zeros(NUM_CLASSIFICATIONS, 1);
    v_minus = v(1:3); v_plus = v(4:6);
    d_plus_sup = GetSign(min(find(v_plus)));
    d_plus_inf = GetSign(max(find(v_plus)));
    d_minus_sup = GetSign(min(find(v_minus)));
    d_minus_inf = GetSign(max(find(v_minus)));
    if d_plus_sup ~= -d_minus_sup
        res(1) = 1;
    end
    if d_plus_inf ~= -d_minus_inf
        res(2) = 1;
    end
    if d_plus_sup ~= d_minus_sup
        res(3) = 1;
    end
    if d_plus_inf ~= d_minus_inf
        res(4) = 1;
    end
    if d_plus_sup ~= d_plus_inf
        res(5) = 1;
    end
    if d_minus_sup ~= d_minus_inf
        res(6) = 1;
    end
    function phi = GetSign(index)
        switch (index)
            case {1,4}
                phi = +1;
            case {2,5}
                phi = 0;
            case {3,6}
                phi = -1;
        end
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