Mean curvature flow of symmetric double graphs only develops singularities on the hyperplane of symmetry

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By a symmetric double graph we mean a hypersurface which is mirror-symmetric and the two symmetric parts are graphs over the hyperplane of symmetry. We prove that there is a weak solution of mean curvature flow that preserves these properties and singularities only occur on the hyperplane of symmetry. The result can be used to construct smooth solutions to the free Neumann boundary problem on a supporting hyperplane with singular boundary.

For the construction we introduce and investigate a notion named “vanity” and which is similar to convexity. Moreover, we rely on Sáez’ and Schnürer’s “mean curvature flow without singularities” to approximate weak solutions with smooth graphical solutions in one dimension higher.

1 Introduction

Symmetry is one of the most important concepts in mathematics. As such it appears often and is worth studying. Our result demonstrates that under certain circumstances one can delimit the region where the occurrence of singularities in mean curvature flow is possible in a way naturally related to an existing symmetry. It is well-known that singularities occur in the mean curvature flow of closed hypersurfaces in Euclidean space. This can be easily seen by placing the closed hypersurface inside a sphere, which shrinks to a point under the mean curvature flow, and using the fact that by the comparison principle the evolving hypersurface must stay inside the sphere. For

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graphical hypersurfaces, however, the situation is different and no singularities occur (cf. [2]). In this note, we investigate a situation where these two regimes get in touch. Consider a mirror-symmetric closed hypersurface, i.e., a hypersurface that is symmetric with respect to some hyperplane. Furthermore, assume that the two symmetric parts are graphical over that hyperplane. In the title we called such a surface a symmetric double graph. Since the hypersurface is closed, singularities inevitably arise for the mean curvature flow. But due to the graphical properties, these occur only on the hyperplane of symmetry. Or at least, we show that there exists a weak solution, the singularity resolving solution of [6], with this behavior.

One motivation for this problem arose from the need to approximate a flow of non-compact graphs by flows of closed hypersurfaces. In this context symmetric double graphs appear as a reasonable means of approximation. In these situations it is highly desirable to exclude singularities on the graphical parts, that is to say that singularities only develop on the hyperplane of symmetry. Usually, this is done by a convexity assumption because convex hypersurfaces don’t develop singularities until they vanish in a point. For the mean curvature flow in particular, this is a well-known result of [3]. Another interesting aspect is that the graphical parts may be viewed as solutions of the free Neumann boundary value problem to mean curvature flow where the hypersurfaces meet a supporting hyperplane perpendicularly. Each symmetric part is a graphical solution to this problem that is smooth up to the (free) boundary which may be singular.

To deal with the symmetric double graphs we introduce a notion which is similar to convexity and which we call vanity. In a convex set any two points can see each other; in a vain set each point can see its mirror-image, hence the name. We will give a few results on vain sets and functions which are reminiscent of corresponding results for convex sets and functions.

We also heavily rely on the results of [6]. In that paper Sáez and Schnürer established the existence for mean curvature flow of complete graphs and interpreted the projections, the domains of definition of the functions representing the graphs, as a weak solution for mean curvature flow. They dubbed it a singularity resolving solution. The complete graphs do not show any singularities, but the singularity resolving solution can. So one has a weak solution at hand which flows through singularities and which is backed by a smooth solution in one dimension higher. We will exploit this and work with the smooth graphical solutions to investigate the weak solution that appears in the projections.

This note is organized as follows. In the next section we will introduce the notion of vanity and prove a few related results. Next, in Section 3 we investigate the graphical mean curvature flow for vain functions. The results lay the foundation for Section 4 where we show that for a singularity resolving solution of symmetric double graphs, the two graphical parts stay smooth. In particular, singularities only appear on the hyperplane of symmetry.
2 Vanity

Definition 1. For \((x^1, \ldots, x^n) \in \mathbb{R}^n\) we denote by \(\overline{x}\) the reflection of \(x\) in the first direction. So \(\overline{x}\) is given by

\[
\overline{x} := (-x^1, x^2, \ldots, x^n).
\]

Moreover, we will write

\[
x_\lambda := (\lambda x^1, x^2, \ldots, x^n) = \frac{x + \overline{x}}{2} + \lambda \frac{x - \overline{x}}{2} \quad \text{for } \lambda \in [-1, 1].
\]

Remark 2. We have \(x_1 = x\) and \(x_{-1} = \overline{x}\).

Definition 3.

(i) A subset \(\Omega \subset \mathbb{R}^n\) is called vain if all of its points can see their mirror image, i.e.,

\[
\forall x \in \Omega \forall \lambda \in [-1, 1]: x_\lambda \in \Omega \quad \text{holds.}
\]

(ii) If \(\Omega \subset \mathbb{R}^n\) is vain, then a function \(u: \Omega \to \mathbb{R} = [-\infty, +\infty]\) is called vain if

\[
\forall x \in \Omega \forall \lambda \in [-1, 1]: u(x_\lambda) \leq u(x) \quad \text{holds.}
\]

Remark 4.

(i) Every vain set is mirror-symmetric.

(ii) Every vain function \(u: \Omega \to \mathbb{R}\) is mirror-symmetric, i.e., \(u(x) = u(\overline{x})\) holds for all \(x \in \Omega\).

(iii) If \(\Omega\) is vain, then the fibers of the projection \(\Omega \ni x \mapsto (x^2, \ldots, x^n)\) are convex, i.e., they are lines.

(iv) A function \(u: \Omega \to \mathbb{R}\) is vain if and only if the set

\[
\{(x^1, \ldots, x^{n+1}) \in \Omega \times \mathbb{R}: u(x^1, \ldots, x^n) \leq x^{n+1}\}
\]

is vain.

Proposition 5. If \(u: \Omega \to \mathbb{R}\) is vain, then the sets \(u^{-1}([-\infty, a])\) and \(u^{-1}([-\infty, a])\) are vain for any \(a \in \mathbb{R}\).

Proof. Obvious from the definitions.

Proposition 6. If \(u\) is a vain function and \(m\) is a monotonically increasing function, then \(m \circ u\) is vain.

Furthermore, if \(m: \mathbb{R}^k \to \mathbb{R}\) is a monotonically increasing function with respect to the partial ordering given by \((z_1, \ldots, z_k) \leq (z'_1, \ldots, z'_k) \iff \forall i \in \{1, \ldots, k\}: z_i \leq z'_i\) and if \(u_1, \ldots, u_k\) are vain functions on a vain set \(\Omega\), then \(\Omega \ni x \mapsto m(u_1(x), \ldots, u_k(x))\) is vain, too.
Proof. The inequalities \( u_i(x_λ) \leq u_i(x) \) immediately imply \( m(u_1(x_λ), \ldots, u_k(x_λ)) \leq m(u_1(x), \ldots, u_k(x)) \).

Corollary 7. For two vain functions \( u, v : \Omega \to \mathbb{R} \), their sum \( u + v \) is vain again.

Proposition 8. Let \( \Omega \) be a vain set. A function \( u : \Omega \to \mathbb{R} \) is vain if and only if \( u \) is mirror-symmetric and for any \( (x^2, \ldots, x^n) \in \mathbb{R}^{n-1} \), \( u(., x^2, \ldots, x^n) \) is monotonically increasing on \( \{ x^1 : (x^1, x^2, \ldots, x^{n-1}) \in \Omega, x^1 \geq 0 \} \).

Proof. Firstly, let \( u \) be vain. Then \( u \) is mirror-symmetrical (Remark 3). Let \( (x^2, \ldots, x^n) \in \mathbb{R}^{n-1} \). Let \( 0 \leq a \leq b \) be such that \( x^{(b)} := (b, x^2, \ldots, x^n) \in \Omega \), and consequently \( x^{(a)} := (a, x^2, \ldots, x^n) \in \Omega \). With \( \lambda := \frac{a}{b} \in [0, 1] \) we can write \( x^{(a)} = x^{(b)} \).

Hence, by the vanity of \( u \), \( u(x^{(a)}) \leq u(x^{(b)}) \) holds, which proves the monotonicity of \( u(., x^2, \ldots, x^n) \) on the set \( \{ x^1 : (x^1, x^2, \ldots, x^n) \in \Omega, x^1 \geq 0 \} \).

Now we assume the symmetry and the monotonicity property and prove that \( u \) is vain. Let \( x \in \Omega \) and \( \lambda \in [-1, 1] \). We shall prove \( u(x_λ) \leq u(x) \). Symmetry is the reason why we only need to consider \( x^1 \geq 0 \) and \( \lambda \geq 0 \). Then \( u(x_λ) \leq u(x) \) follows from the monotonicity property.  

Corollary 9. Let \( u \in C^1(\Omega) \) be a mirror-symmetrical function on a vain set \( \Omega \). Then \( u \) is vain if and only if \( \partial_1 u(x) \geq 0 \) for \( x^1 > 0 \) holds.

The following explains the relation of vanity to symmetric double graphs.

Corollary 10. Let \( \Omega \) be an open, vain set. Then it is of the form

\[
\Omega = \{(x^1, \hat{x}) \in \mathbb{R}^n : \hat{x} \in U, |x^1| < h(\hat{x})\}
\]  

(1)

for a function \( h \) defined on an open set \( U \subset \mathbb{R}^{n-1} \).

Proof. Let \( U \) be the projection of \( \Omega \) to \( \mathbb{R}^{n-1} \), where we use the projection \( p \) that drops the first coordinate. Because \( \Omega \) is open and vain, the fibers of that projection \( p \) are lines of the form

\[
p^{-1}(\hat{x}) = \{\hat{x}\} \times (-h(\hat{x}), h(\hat{x})).
\]

Defining \( h \) on \( U \) in this way shows that \( \Omega \) has the asserted form. 

Remark 11. For any continuous function \( h \) on an open subset \( U \subset \mathbb{R}^{n-1} \), we can define an open, vain set via \( \Omega \).

Lemma 12. For a vain set \( \Omega \subset \mathbb{R}^n \) the negative distance function \( -d := -d_{\partial\Omega} \), defined on \( \Omega \), is vain.

Proof. Let \( x \in \Omega \) and \( \lambda \in [-1, 1] \) be given. Because \( \Omega \) is mirror-symmetrical, we have \( d(x) = d(\pi) \). For \( v \in \mathbb{R}^n \) with \( |v| < d(x) = d(\pi) \), there hold \( x + v \in \Omega \) and \( \pi + v \in \Omega \). Noting Remark 3, we deduce \( x_λ + v \in \Omega \). Because \( v \) was an arbitrary vector subject to the condition \( |v| < d(x) \), it follows \( d(x_λ) \geq d(x) \).
Mollification of Vain Functions. We check whether the standard mollification by convolution with a mollifying kernel preserves vanity. For special kernels we can give an affirmative answer. However, this is not expected for general kernels. For instance, convolution with a mollifying kernel preserves vanity. For special kernels we can give

\[ M(\eta) \text{ defined by } y = x/\varepsilon \text{ and has the same properties as } \eta \text{ does.} \]

Let \( u: \mathbb{R}^n \to \mathbb{R} \) be a vain function. The mollification of \( u \) is defined by \( u_\varepsilon(x_0) := \int u(x) \eta_\varepsilon(x_0 - x) \, dx \). We check \( u_{\varepsilon}(x_0) = u_{\varepsilon}(0) \) and \( \partial_1 u_{\varepsilon}(x_0) \geq 0 \) for \( x_0^1 > 0 \): (We use transformation variables \( y = \eta \) and \( z = \eta + 2 \eta_1 e_1 \).)

\[
\begin{align*}
\partial_1 u_{\varepsilon}(x_0) &= \int u(x) \partial_1 \eta_\varepsilon(x_0 - x) \, dx \\
&= \int_{\{x^1 < x_0^1\}} u(x) \partial_1 \eta_\varepsilon(x_0 - x) \, dx + \int_{\{x^1 > x_0^1\}} u(x) \partial_1 \eta_\varepsilon(x_0 - x) \, dx \\
&= \int_{\{x^1 > x_0^1\}} u(\eta + 2 \eta_1 e_1) \partial_1 \eta_\varepsilon(x_0 - \eta - 2 \eta_1 e_1) \, dz \\
&\quad + \int_{\{x^1 > x_0^1\}} u(x) \partial_1 \eta_\varepsilon(x_0 - x) \, dx \\
&= \int_{\{x^1 > x_0^1\}} (u(x) - u(\eta + 2 \eta_1 e_1)) \partial_1 \eta_\varepsilon(x_0 - x) \, dx.
\end{align*}
\]

We have used \( \partial_1 \eta_\varepsilon(x_0 - z) = -\partial_1 \eta_\varepsilon(x_0 - z) \) and have renamed the integration variable from \( z \) to \( x \) in the last step. For \( 0 < x_0^1 < x^1 \) we have \( -x^1 < \eta^1 + 2 \eta_1 - \eta^1 + 2 \eta_1 = x^1 \) and \( \partial_1 \eta_\varepsilon(x_0 - x) \geq 0 \). We deduce from the vanity of \( u \) that \( u(x) \geq u(\eta + 2 \eta_1 e_1) \) and therefore obtain from \( x_0^1 > 0 \).

Together with \( (2) \), the symmetry of \( u_\varepsilon \), the vanity of the mollification \( u_\varepsilon \) now follows from Corollary \( (9) \).

3 Graphical mean curvature flow

For a function \( (x,t) \mapsto u(x,t) \) of space and time, its graphs \((u(\cdot,t))_t \) flow by mean curvature if and only if \( u \) solves the graphical mean curvature flow

\[
\partial_t u = \left( \delta^{ij} - \frac{u^i u^j}{1 + |Du|^2} \right) u_{ij}.
\]
In this section we examine whether vanity is preserved under the graphical mean curvature flow. Proposition 13 proves that for a bounded situation with constant boundary values. In Proposition 14, we demonstrate that for the graphical mean curvature flow of complete graphs, there exists a solution which preserves vanity.

**Proposition 13.** Let \( \Omega \subset \mathbb{R}^n \) be a vain, open, bounded, and smooth set. Let \( u \in C^{2,1}(\Omega \times [0, T)) \) be a solution of graphical mean curvature flow such that \( u \leq a \) (\( a \in \mathbb{R} \)) and \( u(x, t) = a \) for \( x \in \partial \Omega, \ t \in [0, T) \) hold.

If \( u(\cdot, 0) \) is vain, then \( u(\cdot, t) \) is vain for all \( t \in [0, T) \).

**Proof.** Because of the uniqueness of the solution, \( u(\cdot, t) \) is mirror-symmetrical for all \( t \in [0, T) \).

Let \( \nu(\cdot, t) \) be the downwards pointing normal to \( M_t := \text{graph}\, u(\cdot, t) \subset \mathbb{R}^{n+1} \). If we track points on \( M_t \) in time along the normal direction, then on \( M_t \) holds

\[
\partial_t \nu - \Delta \nu = |A|^2 \nu ,
\]

where \(|A|^2\) is the squared norm of the second fundamental form and \( \Delta \) denotes the Laplace-Beltrami operator of \( M_t \). Accordingly, for \( w := \nu_1 = \langle \nu, e_1 \rangle \equiv \frac{\partial_u u}{\sqrt{1 + |Du|^2}} \) holds

\[
\partial_t w - \Delta w = |A|^2 w . \tag{4}
\]

Let \( M_t^+ := \{ X \in M_t : X^1 > 0 \} \). By the mirror-symmetry of \( u(\cdot, t) \), we have \( w(\cdot, t) = 0 \) on the part of \( \partial M_t^+ \) which lies on \( \{ X^1 = 0 \} \). The vanity of \( \Omega \) and \( u(\cdot, t) \leq a \) as well as \( u(x, t) = a \) for \( x \in \partial \Omega \) imply that \( w(\cdot, t) \geq 0 \) holds on the remaining part of \( \partial M_t^+ \) (cf. Proposition 10). Furthermore, the vanity of \( u(\cdot, 0) \) implies that \( w(\cdot, 0) \geq 0 \) on \( M_0^+ \) (cf. Corollary 9).

The parabolic maximum principle yields \( w(p, t) \geq 0 \) for all \( p \in M_t^+ \) and for all \( t \in [0, T) \). This in turn implies the vanity of \( u(\cdot, t) \) for all \( t \in [0, T) \) (Corollary 9).

**Proposition 14.** Let \( \Omega_0 \subset \mathbb{R}^n \) be an open, vain set and let \( u_0 : \Omega_0 \rightarrow \mathbb{R} \) be a vain, locally Lipschitz function such that for any \( a \in \mathbb{R} \) the set \( \{ x : u_0(x) \leq a \} \) is compact.

Then there exists a relatively open set \( \Omega = \bigcup_{t \geq 0} \Omega_t \times \{ t \} \subset \mathbb{R}^n \times [0, \infty) \) compatible with \( \Omega_0 \) from above and such that \( \Omega_t \) is vain for every \( t \geq 0 \). And there exists a continuous function \( u : \Omega \rightarrow \mathbb{R} \) which is smooth on \( \bigcup_{t \geq 0} \Omega_t \times \{ t \} \) and which is a solution of the graphical mean curvature flow starting from the initial condition \( u_0 \) such that \( u(\cdot, t) : \Omega_t \rightarrow \mathbb{R} \) is vain for every \( t \geq 0 \). Moreover, for any \( a \in \mathbb{R} \) the set \( \{ (x, t) : u(x, t) \leq a \} \) is compact.

**Remark 15.** The assumption that \( \{ x : u_0(x) \leq a \} \) be compact implies the completeness of the graph of \( u_0 \) as well as a boundedness of \( u_0 \) from below. The compactness of \( \{ (x, t) : u(x, t) \leq a \} \) combines the completeness of the graphical hypersurfaces with a maximality condition: One cannot stop the solution at some arbitrary time without destroying that property.

**Proof.** One uses the construction for the mean curvature flow without singularities from [6] and checks whether the vanity is preserved in the steps taken there.
Firstly, we extend \( u_0 \) to a function \( \overline{u}_0 : \mathbb{R}^n \to \mathbb{R} \) by setting \( \overline{u}_0(x) = +\infty \) for \( x \notin \Omega_0 \). Then \( \overline{u}_0 \) is vain. Next, we cut off at some height \( a \in \mathbb{R} \) by concatenation with the monotonically increasing function \( \min\{\cdot, a\} \). By Proposition 6 \( \min\{\overline{u}_0, a\} \) is a vain function. Because \( \{x : u_0(x) \leq a\} \) is compact, the resulting function \( \min\{\overline{u}_0, a\} \) is constantly equal to \( a \) outside a ball. A subsequent mollification as described in the last section does not destroy the vanity nor this constancy property.

Having prepared the initial data in this way we can consider with these the Dirichlet problem on a large ball for the graphical mean curvature flow. This gives us an approximating solution. Proposition 13 ensures that the approximating solution is still vain.

Of course we let \( a \to \infty \), let the mollification parameter tend to zero, and take larger and larger balls. The resulting sequence of approximating solutions converges pointwise to a spacetime function \( \overline{u} : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) as asserted except for the vanity assertions (cf. [6]).

\[ \Omega = \overline{u}^{-1}(\langle -\infty, \infty \rangle) = \overline{u}^{-1}(\langle -\infty, \infty \rangle). \]

Therefore, the vanity of \( \Omega \) and thus the vanity of the time slices \( \Omega_t \) follows from Proposition 5. Because the property of being vain is preserved under pointwise limits, \( \overline{u}(\cdot, t) \) is vain for any \( t \geq 0 \). Hence, the same holds for \( u(\cdot, t) \).

### 4 Singularity resolving solutions

**Corollary 16.** Let \( \Omega_0 \subset \mathbb{R}^n \) be open and vain. Then there is a weak solution \((\Omega_t)_{t \in [0, \infty)}\) of the mean curvature flow in the sense of a singularity resolving solution that starts with \( \Omega_0 \) and such that \( \Omega_t \) is vain for all \( t \geq 0 \).

**Remark 17.** Singularity resolving solutions are introduced in [6]. A singularity resolving solution is the projection, or likewise the domains of definition, of a mean curvature flow of complete graphical hypersurfaces. It has been proven in [6] that in the case of non-fattening of the level-set flow starting from the boundary \( \partial \Omega_0 \), \( \Omega_t \) coincides \( H^n \)-almost everywhere with the corresponding set of the level-set flow. In [5], the author has shown that \((\Omega_t)_{t \geq 0}\) is always a weak solution in the sense that any smooth mean curvature flow starting inside of \( \Omega_0 \) will stay inside of \( \Omega_t \) for \( t \geq t_0 \) and analogous for “outside”.

**Proof.** Without loss of generality we assume \( \Omega_0 \neq \emptyset \). Let \( d := \text{dist}_{\partial \Omega_0} \) be the positive distance function to the boundary on \( \Omega_0 \). We set \( u_0(x) = \frac{1}{d(x)} + |x|^2 \) for \( x \in \Omega_0 \). By Lemma 12 and Propositions 6 and 7 \( u_0 \) is a vain function and \( u_0 \) satisfies the hypothesis of Proposition 13. Let \( u : \Omega \to \mathbb{R} \) be a solution from this Proposition (uniqueness of the solution is not proven). Then the \( \Omega_t \) are vain and they form a singularity resolving solution by definition.

**Theorem 18.** Let \( \Omega_0 \subset \mathbb{R}^n \) be an open and vain set. Suppose \( \Omega_0 \) is bounded in the \( x^1 \)-direction, \( N_0 := \partial \Omega_0 \cap \{x^1 > 0\} \) is of class \( C^2 \), and suppose that the curvature of
$N_0$ is bounded on sets $\{x : x^1 > \varepsilon\}$ and that $\nu_1 \geq c > 0$ is positively bounded from below on these sets. (The bounds may depend on $\varepsilon$, $\nu$ is the normal to $N_0$.)

Then, for the weak solution $(\Omega_t)_{t \in [0, \infty)}$ from Corollary 16 the $N_t := \partial \Omega_t \cap \{x : x^1 > 0\}$ are smooth submanifolds for $t > 0$ and they solve the mean curvature flow in the classical sense.

Proof. Let $u : \Omega \to \mathbb{R}$ be the mean curvature flow without singularities from the proof of Corollary 16. Let $M_t := \text{graph} u(\cdot, t)$. We will prove uniform estimates for $M_t$ in the region $X^1 > \varepsilon$. From these we infer that $(M_t \cap \{x : x^1 > 0\}) \to_{j \to \infty}^{\text{loc}} N_t \times \mathbb{R}$ converges locally smoothly with locally uniform estimates. In particular, $N_t$ is smooth.

For the estimates one would like to use the cut-off function $(X^1 - \varepsilon) +$. However, this function doesn’t cut off compact subsets from the mean curvature flow $M_t$. For this reason, one considers cut-off functions similar to those in [2] and whose supports are given by shrinking balls. One chooses larger and larger balls such that in the limit the half-space $X^1 > \varepsilon$ is obtained. More precisely, we do the following construction. For $R_0 > 0$ and $\varepsilon > 0$, we set $X_0 := (R_0 + \varepsilon, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. We define the corresponding cut-off function $\varphi : \mathbb{R}^{n+1} \times [0, \infty) \to \mathbb{R}$ by

$$\varphi(X, t) := \frac{1}{2R_0} (R_0^2 - 2nt - |X - X_0|^2)_+ .$$

The support of $\varphi$ is given by a shrinking ball around $X_0$ of initial radius $R_0$. If $(X, t)$ is fixed and $R_0 \to \infty$, in the limit one obtains the cut-off function

$$\lim_{R_0 \to \infty} \varphi(X, t) = \lim_{R_0 \to \infty} \frac{1}{2R_0} \left( -2nt - \sum_{\alpha \geq 2} (X^\alpha)^2 + R_0^2 - (X^1 - (R_0 + \varepsilon))^2 \right)_+$$

$$= \lim_{R_0 \to \infty} \left( \frac{1}{2R_0} (R_0 - (X^1 - (R_0 + \varepsilon))) (R_0 + (X^1 - (R_0 + \varepsilon))) \right)_+$$

$$= \lim_{R_0 \to \infty} \left( \frac{1}{2R_0} (2R_0 - X^1 + \varepsilon) (X^1 - \varepsilon) \right)_+$$

$$= (X^1 - \varepsilon)_+ .$$

Considering the operator $\frac{d}{dt} - \Delta$ on the surface $M_t$ and where $\varphi > 0$, the function $\varphi$ satisfies, with a local parametrization $p \mapsto X(p, t)$ of $M_t$ such that $\partial_t X$ points in normal direction,

$$\left( \frac{d}{dt} - \Delta \right) \varphi(X(p, t), t) = \frac{1}{2R_0} \left( -2n - \left( \frac{d}{dt} - \Delta \right) |X(p, t)|^2 \right)$$

$$= \frac{1}{2R_0} \left( -2n - 2 \left( \left( \frac{d}{dt} - \Delta \right) X, X \right) + 2 |\nabla X|^2 \right)$$

$$= \frac{1}{2R_0} (-2n + 2n) = 0 .$$
Here we have used \( \left( \frac{d}{dt} - \Delta \right) X = 0 \) and \( |\nabla X|^2 = g^{ij} \delta_{\alpha\beta} \nabla_i X^\alpha \nabla_j X^\beta = g^{ij} g_{ij} = n \).

In a region where \( \varphi > \delta t \) holds (\( \delta > 0 \)), we have for \( \psi := \varphi - \delta t \)
\[
\left( \frac{d}{dt} - \Delta \right) \log \psi = \left( \frac{d}{dt} - \Delta \right) \psi + \left| \nabla \psi \right|^2 = -\frac{\delta}{\psi} + \left| \nabla \psi \right|^2.
\]

(6)

At first, we estimate \( w = \nu_1 \), where \( \nu \) is the downwards pointing normal to \( M_t \). By the vanity of \( u \), we have \( w \geq 0 \) in the region \( X^1 > 0 \). The strong maximum principle implies that \( w > 0 \) in the region \( X^1 > 0 \) (note (4)).

In an interior maximum point of \( w^{-1} \psi \), and consequently of \( -\log w + \log \psi \), there hold
\[
\frac{\nabla \psi}{w} = \frac{\nabla \psi}{w} \text{ and}
0 \leq \left( \frac{d}{dt} - \Delta \right) (-\log w + \log \psi)
= -|A|^2 - \left| \frac{\nabla w}{w} \right|^2 - \frac{\delta}{\psi} + \left| \nabla \psi \right|^2 \quad \text{(note (4) and (6))}
= -|A|^2 - \frac{\delta}{\psi} < 0.
\]

Contradiction. So there cannot be an interior maximum point, and because \( \psi \) vanishes on the lateral boundary of the region \( \{ (X, t) : \varphi(X, t) > \delta t \} \), \( w^{-1} \psi \) is bounded by the supremum of its initial values. With \( \delta \to 0 \) it follows that \( w^{-1} \varphi \) is bounded by its initial values, too. Finally, we let \( R_0 \to \infty \) and obtain the estimate
\[
w^{-1} (X^1 - \varepsilon)^+ \leq \sup_{t=0} w^{-1} (X^1 - \varepsilon)^+.
\]

(7)

For the curvature estimate we consider the test function \( f := \frac{|A|^2}{w^2 - w} \), where \( \inf w \) and the infimum is taken over the set \( \text{supp} \varphi \). Using (7) with \( \varepsilon = \frac{\varphi}{2} \), we see that \( w \geq c > 0 \).

We shall make use of (4), (6),
\[
\left( \frac{d}{dt} - \Delta \right) |A|^2 = -2 |\nabla A|^2 + 2 |A|^4,
\]
and \( |\nabla |A|^2| = |2 \langle A, \nabla A \rangle| \leq 2 |A| |\nabla A| \).

In an interior maximum point of \( f \), there holds
\[
0 \leq \left( \frac{d}{dt} - \Delta \right) \log f^2 = \left( \frac{d}{dt} - \Delta \right) \left( \log |A|^2 - 2 \log(w - w) + 2 \log \varphi \right)
= -2 \frac{\left| \nabla |A|^2 \right|^2}{|A|^2} + 2 \frac{|A|^4}{|A|^2} + \frac{2 \left| \nabla |A|^2 \right|^2}{|A|^2} - 2 \frac{\nabla w}{w - w} |A|^2 - 2 \frac{\left| \nabla w \right|^2}{w - w} + 2 \frac{\left| \nabla \varphi \right|^2}{\varphi}
\leq -\frac{1}{2} \frac{\left| \nabla |A|^2 \right|^2}{|A|^2} - 2 \frac{\nabla w}{w - w} |A|^2 = -2 \frac{\nabla w}{w - w} |A|^2 - 2 \frac{\left| \nabla w \right|^2}{w - w} + 2 \frac{\left| \nabla \varphi \right|^2}{\varphi}.
\]
For arbitrary $\delta > 0$ we deduce from the condition $\nabla \log f^2 = 0$ at the maximal point

$$
\left| \frac{\nabla |A|^2}{|A|} \right|^2 = 2 \frac{\nabla w}{w - w} - 2 \frac{\nabla \varphi}{\varphi} \leq (1 + \delta) \left( \frac{\nabla w}{w - w} \right)^2 + 4 (1 + \delta^{-1}) \left| \frac{\nabla \varphi}{\varphi} \right|^2.
$$

Therefore,

$$
0 \leq - \frac{2 w}{w - w} |A|^2 + 2 \delta \left| \frac{\nabla w}{w - w} \right|^2 + 2 (2 + \delta^{-1}) \left| \frac{\nabla \varphi}{\varphi} \right|^2.
$$

(8)

There hold

$$
|\nabla \varphi|^2 = \left( \frac{X - X_0}{R_0}, \nabla X \right)^2 \leq 1 \cdot |\nabla X|^2 = g^{ij} \delta_{\alpha \beta} \nabla_i X^\alpha \nabla_j X^\beta = g^{ij} n = n
$$

(9)

and

$$
|\nabla w|^2 = |\nabla u|^2 = g^{ij} \left( h^k_i \nabla_k X^1 \right) \left( h^l_j \nabla_l X^1 \right) \leq |A|^2 |\nabla X|^2 \leq |A|^2.
$$

(10)

Substituting these two inequalities into (8) yields

$$
0 \leq \left( \frac{-2 w}{w - w} + \frac{2 \delta}{(w - w)^2} \right) |A|^2 + 2 (2 + \delta^{-1}) \frac{n}{\varphi^2}.
$$

(11)

With the choice $\delta = \frac{1}{2} w^2$, we obtain

$$
\left( \frac{-2 w}{w - w} + \frac{2 \delta}{(w - w)^2} \right) \leq - \frac{2 w (w - w) + w^2}{(w - w)^2} \leq - \frac{2 w^2 + w^2}{(w - w)^2} = - \frac{w^2}{(w - w)^2}.
$$

(12)

We conclude

$$
\frac{|A|^2 \varphi^2}{(w - w)^2} \leq 2 (2 + 2 w^{-2}) n w^{-2} \equiv C(w, n).
$$

(13)

So, in an interior maximum point, $f$ is bounded by a controlled constant. In particular, it follows that $|A| \varphi \leq C(w, n) \left( 1 + \sup_{t=0} (|A| \varphi) \right)$. With $R_0 \to \infty$ we obtain the estimate

$$
|A| (X^1 - \varepsilon)_+ \leq C \left( \sup_{t=0} (|A| (X^1 - \varepsilon)_+), \sup_{X^1 > \varepsilon} w^{-1}, n \right).
$$

(14)

Together with the estimate (7) on $w$, we obtain the curvature estimate

$$
\sup_{X^1 > 2 \varepsilon} |A| \leq C \left( \varepsilon, \sup_{t=0, X^1 > \varepsilon} |A|, \sup_{t=0, X^1 > 4 \varepsilon} w^{-1}, \sup X^1, n \right).
$$

(15)

The higher order estimates, i.e., estimates on $|\nabla^k A|$, are omitted because these can be obtained by following (2) from hereon.

We have proven uniform estimates for $M_t = \text{graph} u(\cdot, t)$ for all $t$ in regions $\{X : X^1 \geq \varepsilon\}$. By Proposition 10, $M_t \cap \{X^1 > 0\}$ is graphical over the hyperplane orthogonal to $e_1$. Let $h$ be the representing function. Then $h$ is bounded and $h$ is monotonically increasing in the $e_{n+1}$ direction. The inequality (10) yields a gradient estimate on $h$ where $h > \varepsilon$. The estimates on the second fundamental form (12) and on its higher derivatives provides us with estimates on the higher derivatives of $h$. We conclude that there is a smooth limit of $h(\cdot, x^{n+1})$ for $x^{n+1} \to \infty$. This limit is a graphical representation of $N_t$. Hence, $N_t$ is smooth. \qed
Figure 1: Free boundary value problem

The graphical surface moves by mean curvature flow subject to the condition of meeting the plane perpendicularly. The first picture shows the situation before, the second after the formation of a singularity. Despite the singularity at the boundary the surface stays the graph of a function which is smooth in the interior for all time.

Remark 19. As a byproduct, Theorem 18 (see also Proposition 10) provides a solution of a free boundary value problem where the boundary moves on a hyperplane and the hypersurface meets that hyperplane perpendicularly. Namely, the family \((N_t)_{t \in [0, \infty)}\) is a family of smooth graphical hypersurfaces over a hyperplane that solves the mean curvature flow \((N_t\) may be empty). The boundaries \(\partial N_t\), which reside on the hyperplane, may be singular, however. But at spacetime-points (on the hyperplane) where \(\partial \Omega_t\) is smooth \(\partial N_t\) is smooth too and by symmetry of \(\partial \Omega_t\) the normal to \(\partial \Omega_t\) lies in the hyperplane such that \(N_t = \partial \Omega_t \cap \{x : x^1 > 0\}\) meets the hyperplane perpendicularly. In this way \(N_t\) can be viewed as a smooth graphical solution to the free Neumann boundary value problem with singularities at the boundary.

As was pointed out to the author by O. Schnürer, it is not clear that \(N_t\) always meets the hyperplane perpendicularly. To explain this, one needs to think about singularities of \(\partial \Omega_t\) where it is possible to continuously extend the normal coming from one of the two sides but where the limits from the two sides disagree. In this case it may be possible that the normal for \(N_t\) is definable on the hyperplane but that it points out of the hyperplane.

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