We consider Bogoliubov de Gennes equation on metric graphs. The vertex boundary conditions providing self-adjoint realization of the Bogoliubov de Gennes operator on a metric star graph are derived. Secular equation providing quantization of the energy and the vertex transmission matrix are also obtained. Application of the model for Majorana wire networks is discussed.

I. INTRODUCTION

Quantum graphs, which are the one- or quasi-one-dimensional branched quantum wires, have attracted much attention in different contexts of contemporary physics (see, e.g., Refs. [1]-[6]). Particle dynamics in such systems are described in terms of quantum mechanical wave equations on metric graphs by imposing the boundary conditions at the branching points (vertices) and bond ends. The metric graphs are the set of bonds which are assigned length and connected to each other at the vertices. The connection rule is called topology of a graph and given in terms of the adjacency matrix [1, 4]. Quantum graphs were first introduced by Exner and Seba to describe free quantum motion on branched structures [5]. Later Kotyrykin and Schrader derived the general boundary conditions providing self-adjointness of the Schrödinger operator on graphs [6]. Bolte and Harrison extended such boundary conditions for the Dirac operator on metric graphs [7]. Hul et al considered experimental realization of quantum graphs in optical microwave networks [2]. An important topic related to quantum graphs was studied in the context of quantum chaos theory and spectral statistics [1, 4, 9–11]. Spectral properties and band structure of periodic quantum graphs also attracted much interest [12, 13]. Different aspects of the Schrödinger operators on graphs have been studied in the Refs. [2, 14, 15, 17, 18].

Despite the growing interest to quantum graphs which are described by linear wave equation on metric graphs, within such approach one is restricted by modeling linear wave dynamics only. For modeling of nonlinear waves and soliton dynamics in branched structures one should consider nonlinear wave equations on metric graphs. During the past decade the studies of particle and wave dynamics in branched structures have been extended to nonlinear evolution equations by considering nonlinear Schrödinger and sine-Gordon equations on metric graphs [19–28]. For such equations, one should derive the vertex boundary conditions from fundamental conservation laws such as energy, momentum, charge and mass conservation [19–28]. Due to the numerous applications of metric graphs based approach for wave dynamics in branched systems and networks, one can expect further extension of the studies to the case of other evolution equations.

In this paper we consider quantum graphs described by Bogoliubov de Gennes (BdG) equation. The latter can be used for modeling of quasiparticle dynamics in superconductors [29] and Majorana fermions in superconducting quantum wires [30]. Here we derive the vertex boundary conditions which keep the BdG operator on metric graphs as self-adjoint. Explicit solutions for such boundary conditions are obtained. Also, we derive vertex transmission matrix describing of waves through the graph branching point.

The paper is organized as follows. In the next section we give formulation of the problem and derive vertex boundary conditions providing self-adjointness of the problem. Section II presents derivation of the transmission matrix describing wave transmission through the graph branching point. In section IV we discuss zero-mode solutions of BdG equation on metric graph and possible application to Majorana wire networks. Finally, section V presents some concluding remarks.

II. VERTEX BOUNDARY CONDITIONS AND EXPLICIT SOLUTIONS

First order BdG equation which is often used in condensed matter physics can be written as

$$H_{\text{BdG}} \Psi = E \Psi,$$

(1)

where $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^T$ and

$$H_{\text{BdG}} = \begin{pmatrix}
0 & -i \frac{\partial}{\partial x} & \Delta_0 & 0 \\
-i \frac{\partial}{\partial x} & 0 & 0 & \Delta_0 \\
\Delta_0 & 0 & 0 & i \frac{\partial}{\partial x} \\
0 & \Delta_0 & i \frac{\partial}{\partial x} & 0
\end{pmatrix}$$

(2)
Using the notations for $\Omega$ we have

\[
\begin{align*}
\psi_1(x, E) &= C_{11} e^{i \kappa x} + C_{12} e^{-i \kappa x}, \\
\psi_2(x, E) &= C_{21} e^{i \kappa x} + C_{22} e^{-i \kappa x}, \\
\psi_3(x, E) &= \frac{E}{\Delta_0} C_{11} e^{i \kappa x} + \frac{E}{\Delta_0} C_{12} e^{-i \kappa x} \\
&\quad - \frac{\kappa}{\Delta_0} C_{21} e^{i \kappa x} + \frac{\kappa}{\Delta_0} C_{22} e^{-i \kappa x}, \\
\psi_4(x, E) &= - \frac{\kappa}{\Delta_0} C_{11} e^{i \kappa x} + \frac{\kappa}{\Delta_0} C_{12} e^{-i \kappa x} \\
&\quad + \frac{E}{\Delta_0} C_{21} e^{i \kappa x} + \frac{E}{\Delta_0} C_{22} e^{-i \kappa x},
\end{align*}
\]

where $\kappa = \sqrt{E^2 - \Delta_0^2}$ and $C_{11}, C_{12}, C_{21}, C_{22}$ are constants which can be found, e.g. from the normalization conditions which can be imposed, e.g. from the normalization conditions which can be found, e.g. from the normalization condition

\[
\psi, \phi \Omega(x) = \psi, \phi
\]

Current for such system is determined as

\[
J = \psi^* \left( \begin{array}{cc} \sigma_x & 0 \\ 0 & \sigma_x \end{array} \right) \psi.
\]

\[
\Omega(\psi, \phi) = \langle D \psi, \phi \rangle - \langle \psi, D \phi \rangle
\]

\[
\begin{align*}
&= i \sum_{j=1}^{N} \left( -\psi_2^{(j)}(L_j) \phi_1^{(j)}(L_j) - \psi_1^{(j)}(L_j) \phi_2^{(j)}(L_j) + \psi_4^{(j)}(L_j) \phi_3^{(j)}(L_j) + \psi_3^{(j)}(L_j) \phi_4^{(j)}(L_j) \\
&\quad + \bar{\phi}_2^{(j)}(0) \phi_1^{(j)}(0) + \bar{\phi}_1^{(j)}(0) \phi_2^{(j)}(0) - \bar{\psi}_4^{(j)}(0) \phi_3^{(j)}(0) - \bar{\psi}_3^{(j)}(0) \phi_4^{(j)}(0) \right)
\end{align*}
\]

Using the notations

\[
\psi_1 = \left( \psi_1^{(1)}(0); \ldots; \psi_1^{(N)}(0); \psi_3^{(1)}(0); \ldots; \psi_3^{(N)}(0); \psi_1^{(N)}(1); \ldots; \psi_1^{(N)}(L_N); \psi_3^{(1)}(1); \ldots; \psi_3^{(N)}(L_N) \right)^T,
\]

\[
\psi_2 = \left( \psi_2^{(1)}(0); \ldots; \psi_2^{(N)}(0); -\psi_4^{(1)}(0); \ldots; -\psi_4^{(N)}(0); -\psi_2^{(1)}(1); \ldots; -\psi_2^{(N)}(L_N); \psi_4^{(1)}(1); \ldots; \psi_4^{(N)}(L_N) \right)^T,
\]

for $\Omega$ we have

\[
\Omega(\psi, \phi) = i \left( \begin{array}{cc} \phi_1^\dagger & \phi_2^\dagger \end{array} \right) \left( \begin{array}{cc} 0 & I_{4N} \\ I_{4N} & 0 \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right).
\]

Then the vertex boundary conditions can be written in the following form:

\[
H_{BdG} \psi^{(j)}(E) = E \psi^{(j)}(E),
\]

where $j = 1, 2, \ldots, N$ is the bond number, $\psi^{(j)}$ and $H_{BdG}$ are the wave function and BdG operator (respectively), which are given by

\[
\psi^{(j)} = \left( \psi_1^{(j)}, \psi_2^{(j)}, \psi_3^{(j)}, \psi_4^{(j)} \right)^T
\]

and

\[
H_{BdG} = \left( \begin{array}{cccc} 0 & -i \frac{\partial}{\partial x} & \Delta_0 & 0 \\ -i \frac{\partial}{\partial x} & 0 & 0 & \Delta_0 \\ \Delta_0 & 0 & 0 & \frac{i}{2} \frac{\partial}{\partial x} \\ \frac{i}{2} \frac{\partial}{\partial x} & 0 & \Delta_0 & 0 \end{array} \right)
\]

Eq. (3) describes quasiparticle dynamics in branched superconductors [31, 32] and (for $E = 0$) Majorana wire networks [33, 36, 38, 40]. The latter has attracted much attention in the context of topological quantum computation recently [39, 42]. For simplicity, in the following we assume that $\Delta_0 = \text{const}$. To solve Eq. (3) on metric graph, we need to impose the boundary conditions at the graph vertex. Such boundary conditions should not break self-adjointness of the BdG operator. Here we use the prescription proposed first in [8] for the Schrodinger equation on metric graph and developed later for Dirac equation in [3]. We construct the scalar product of two functions on the graph, $\phi$ and $\psi$ which given by

\[
\langle \phi, \psi \rangle = \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^{L_j} \phi_k^{(j)}(x) \bar{\phi}_k^{(j)}(x) dx.
\]

Also, for a given differential operator, $D$ on a graph we define so-called skew-Hermitian form which is given as $[8, 9]$.
a linear subspace of $\mathbb{C}^{8N}$ as

$$\mathbf{A}\psi_1 + \mathbf{B}\psi_2 = 0,$$  

(11)

with complex $4N \times 4N$ matrices $\mathbf{A}$ and $\mathbf{B}$. Using the relations

$$\psi_1 = -\mathbf{A}^{-1}\mathbf{B}\psi_2, \quad \phi_1^\dagger = -\phi_2^\dagger\mathbf{B}^\dagger(\mathbf{A}^{-1})^\dagger$$

we get

$$\Omega(\psi, \phi) = iu \begin{pmatrix} \phi_1^\dagger & \phi_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{B}^\dagger(\mathbf{A}^{-1})^\dagger \\ -\mathbf{A}^{-1}\mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$  

(12)

Then we have from $\Omega(\phi, \psi) = 0$

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 4N,$$

(13)

$$\mathbf{A}\mathbf{B}^\dagger = -\mathbf{B}\mathbf{A}^\dagger.$$  

(14)

Solution of Eq. (5) for the positive energy ($H_{\text{BdG}}\Psi^{(j)} = E\Psi^{(j)}$) can be written as

$$\Psi^{(j)}(x) = \mu^{(j)}_\alpha \begin{pmatrix} 1 \\ 0 \\ \frac{e^{i\kappa L}}{\Delta_0} \\ \frac{\kappa}{\Delta_0} \end{pmatrix} e^{ixx} + \mu^{(j)}_\beta \begin{pmatrix} 0 \\ 1 \\ -\frac{e^{i\kappa L}}{\Delta_0} \\ \frac{-\kappa}{\Delta_0} \end{pmatrix} e^{ixx}$$

$$+ \mu_\alpha^{(j)} \begin{pmatrix} 1 \\ 0 \\ \frac{e^{-i\kappa L}}{\Delta_0} \\ \frac{-\kappa}{\Delta_0} \end{pmatrix} e^{-ixx} + \mu_\beta^{(j)} \begin{pmatrix} 0 \\ 1 \\ \frac{-e^{-i\kappa L}}{\Delta_0} \\ \frac{\kappa}{\Delta_0} \end{pmatrix} e^{-ixx}.$$  

(15)

For the negative energy ($H_{\text{BdG}}\Psi^{(j)} = -E\Psi^{(j)}$) we have

$$\Psi^{(j)}(x) = \mu^{(j)}_\alpha \begin{pmatrix} \frac{-e^{i\kappa L}}{\Delta_0} \\ \frac{1}{\Delta_0} \\ 0 \\ 1 \end{pmatrix} e^{ixx} + \mu^{(j)}_\beta \begin{pmatrix} \frac{1}{\Delta_0} \\ \frac{-e^{i\kappa L}}{\Delta_0} \\ 0 \\ 1 \end{pmatrix} e^{ixx}$$

$$+ \mu_\alpha^{(j)} \begin{pmatrix} \frac{-e^{-i\kappa L}}{\Delta_0} \\ \frac{1}{\Delta_0} \\ 0 \\ 1 \end{pmatrix} e^{-ixx} + \mu_\beta^{(j)} \begin{pmatrix} \frac{1}{\Delta_0} \\ \frac{-e^{-i\kappa L}}{\Delta_0} \\ 0 \\ 1 \end{pmatrix} e^{-ixx}.$$  

(16)

For these solutions the vertex boundary conditions given by Eq. (11) can be written as (for $E > 0$)

$$\mathbf{A}\Theta_1 + \mathbf{B}\Theta_2 = 0$$  

(17)

where

$$\Theta_1 = \begin{pmatrix} I_N & e^{-i\kappa L}I_N \\ \frac{e^{i\kappa L}}{\Delta_0}I_N & 0 \\ \frac{\kappa}{\Delta_0}I_N & 0 \\ \frac{\kappa}{\Delta_0}I_N & 0 \end{pmatrix}$$

and (for $E < 0$)

$$(\mathbf{A}\Theta_3 + \mathbf{B}\Theta_4) \begin{pmatrix} \mu_\alpha \\ \mu_\beta \\ \hat{\mu}_\alpha \\ \hat{\mu}_\beta \end{pmatrix} = 0$$  

(18)

where

$$\Theta_3 = \begin{pmatrix} \frac{e^{i\kappa L}}{\Delta_0}I_N & \frac{\kappa}{\Delta_0}I_N & 0 \\ \frac{e^{i\kappa L}}{\Delta_0}I_N & 0 & \frac{e^{-i\kappa L}}{\Delta_0}I_N \\ \frac{\kappa}{\Delta_0}I_N & 0 & \frac{e^{-i\kappa L}}{\Delta_0}I_N \\ 0 & \frac{\kappa}{\Delta_0}I_N & 0 \end{pmatrix}$$

$$\Theta_4 = \begin{pmatrix} \frac{e^{i\kappa L}}{\Delta_0}I_N & \frac{\kappa}{\Delta_0}I_N & \frac{e^{-i\kappa L}}{\Delta_0}I_N \\ \frac{\kappa}{\Delta_0}I_N & \frac{\kappa}{\Delta_0}I_N & \frac{e^{-i\kappa L}}{\Delta_0}I_N \\ \frac{e^{i\kappa L}}{\Delta_0}I_N & 0 & \frac{e^{-i\kappa L}}{\Delta_0}I_N \\ 0 & \frac{\kappa}{\Delta_0}I_N & 0 \end{pmatrix}.$$  

Eqs. (17) and (18) leads to quantization conditions for finding the eigenvalues from the following secular equations:

$$\text{det} (\mathbf{A}\Theta_1 + \mathbf{B}\Theta_2) = 0$$  

for the positive energy and

$$\text{det} (\mathbf{A}\Theta_3 + \mathbf{B}\Theta_4) = 0$$  

for the negative energy. Here $e^{i\kappa L} = \text{diag}\{e^{i\kappa L_1}, \ldots, e^{i\kappa L_N}\}$, $\mu_\alpha = \{\mu^{(1)}_\alpha, \ldots, \mu^{(N)}_\alpha\}$, $\mu_\beta = \{\mu^{(1)}_\beta, \ldots, \mu^{(N)}_\beta\}$, $\hat{\mu}_\alpha = \{\hat{\mu}^{(1)}_\alpha, \ldots, \hat{\mu}^{(N)}_\alpha\}$, $\hat{\mu}_\beta = \{\hat{\mu}^{(1)}_\beta, \ldots, \hat{\mu}^{(N)}_\beta\}$, $I_N$ is the identity matrix with the Nth order.

Eqs. (15), (16), (19) and (20) present complete set of the eigenfunctions and eigenvalues of Eq. (4) for the vertex boundary conditions (17) and (18). It is clear that these solutions provide current conservation in the form of Kirchhoff rules at the vertex. This can be directly checked from the definition of the current given by Eq. (4).

III. VERTEX TRANSMISSION

Here we treat the problem of wave transmission through the graph branching point. Defining the vectors for outgoing and incoming waves at the vertex

$$\mathbf{\hat{\mu}} = \begin{pmatrix} \hat{\mu}_\alpha \\ \hat{\mu}_\beta \end{pmatrix}, \quad \mathbf{\hat{\mu}} = \begin{pmatrix} \hat{\mu}_\alpha \\ \hat{\mu}_\beta \end{pmatrix},$$

where

$$\mathbf{\hat{\mu}} = \begin{pmatrix} \mu_\alpha \\ \mu_\beta \end{pmatrix},$$

$$\mathbf{\hat{\mu}} = \begin{pmatrix} \hat{\mu}_\alpha \\ \hat{\mu}_\beta \end{pmatrix}.$$
from the vertex boundary conditions we have

\[ \overline{\mu} = - (AA' + BB')^{(-1)} (AA'' + BB'') \overline{\mu}, \]

where

\[ A' = \begin{pmatrix} I_N & 0 & 0 & 0 \\ \frac{E}{\Delta_0} I_N & - \frac{\kappa}{\Delta_0} I_N & 0 & 0 \\ 0 & 0 & I_N & 0 \\ 0 & 0 & \frac{\kappa}{\Delta_0} I_N & - \frac{E}{\Delta_0} I_N \end{pmatrix}, \]

\[ A'' = \begin{pmatrix} I_N & 0 & 0 & 0 \\ \frac{E}{\Delta_0} I_N & - \frac{\kappa}{\Delta_0} I_N & 0 & 0 \\ 0 & 0 & I_N & 0 \\ 0 & 0 & - \frac{\kappa}{\Delta_0} I_N & \frac{E}{\Delta_0} I_N \end{pmatrix}, \]

\[ B' = \begin{pmatrix} 0 & I_N & 0 & 0 \\ \frac{\kappa}{\Delta_0} I_N & - \frac{E}{\Delta_0} I_N & 0 & 0 \\ 0 & 0 & I_N & 0 \\ 0 & 0 & \frac{\kappa}{\Delta_0} I_N & - \frac{E}{\Delta_0} I_N \end{pmatrix}, \]

\[ B'' = \begin{pmatrix} 0 & I_N & 0 & 0 \\ - \frac{\kappa}{\Delta_0} I_N & \frac{E}{\Delta_0} I_N & 0 & 0 \\ 0 & 0 & I_N & 0 \\ 0 & 0 & - \frac{\kappa}{\Delta_0} I_N & \frac{E}{\Delta_0} I_N \end{pmatrix}, \]

and \( \det(AA' + BB') \neq 0 \). Then the vertex transition matrix can be written as

\[ T = -(AA' + BB')^{(-1)} (AA'' + BB''). \]

Following the Refs. [1, 8], one can construct the bond scattering matrix in terms of the transmission matrix as

\[ S_{(ij)(kl)} = \delta_{ik} T_{(ij)(kl)} e^{i\kappa L_{(kl)}}, \]

where \((mn)\) is the bond connected vertices \(m\) and \(n\).

**IV. MAJORANA WIRE NETWORK**

The above treatment concerns non-zero energy \((E \neq 0)\) solutions of BdG equation on metric graph. An important case having application in topological states of condensed matter is described by zero-energy solutions of BdG. Such solutions describe bound states of the Majorana fermions on a quantum wire (so-called Majorana wires) [30, 31, 33] which are localized at the ends of the wire. Majorana particles in quantum wires are considered as fixed (immobile), i.e. they do not carry any current [30, 31]. However, one can achieve current carrying regime by constructing Y(T)-junctions of Majorana wires, or more complicated branching topologies [34, 42]. Modeling of such systems can be done in terms of BdG equation on metric graphs. Thus the problem we want to address is the BdG equation on metric star graph given by

\[ H_{\text{BdG}} \Psi^{(j)} = 0, \]

where the spinor \(\Psi^{(j)}\) has (for Majorana fermions) the Nambu structure which is given as [31, 34]

\[ \Psi^{(j)}(x) = \left( \Psi_{\uparrow}^{(j)}, \Psi_{\downarrow}^{(j)}, \Psi_{\downarrow}^{(j)*}, -\Psi_{\uparrow}^{(j)*} \right)^T. \]

Eq. (24) describes Majorana wire networks. Such networks have attracted much attention during last few years (see, e.g., Refs. [30, 31]). Typical examples of Majorana wire networks are presented in Fig.2. For T- or Y-junction of quantum wires two types of branching is possible: First type has Majorana fermion at the vertex (Fig. 2a), while for in second type vertex does not contain Majorana fermion (Fig. 2b). Using different pairing rules and disposition of Majorana fermions in quantum wire networks one can construct a network with required property. Being far from detailed treatment of Majorana wire networks in terms of BdG equation, we will focus on finding specific solutions of the BdG equation metric star graph for zero energy. Here we impose the vertex boundary conditions providing continuity and current conservation. General solution of Eq. (21) (for Nambu spinor) can be written as

\[ \Psi^{(j)}(x) = \overline{\mu}_{\alpha}^{(j)} \begin{pmatrix} q^* \\ 0 \\ 0 \\ -q \end{pmatrix} e^{-\Delta_0 x} + \mu_{\beta}^{(j)} \begin{pmatrix} q \\ q^* \\ q \\ 0 \end{pmatrix} e^{-\Delta_0 x} \]

\[ + \overline{\mu}_{\alpha}^{(j)} \begin{pmatrix} q \\ 0 \\ 0 \\ -q^* \end{pmatrix} e^{\Delta_0 x} + \mu_{\beta}^{(j)} \begin{pmatrix} 0 \\ 0 \\ q \\ q \end{pmatrix} e^{\Delta_0 x}. \]

where \(q = 1 + i\).

Furthermore, we choose the following vertex boundary

**FIG. 2**: (Color online). Sketch of a branched (Y-junction) Majorana wire. Each \(\gamma\) denotes Majorana fermion located at the end of a branch.
conditions:
\[ \Psi_1^{(1)}(0) = \Psi_2^{(2)}(0) = \Psi_3^{(3)}(0), \]  
\[ \Psi_2^{(1)}(0) + \Psi_2^{(2)}(0) + \Psi_2^{(3)}(0) = 0, \]  
\[ \Psi_3^{(1)}(0) + \Psi_3^{(2)}(0) + \Psi_3^{(3)}(0) = 0, \]  
\[ \Psi_4^{(1)}(0) = \Psi_4^{(2)}(0) = \Psi_4^{(3)}(0), \]  
\[ \Psi_4^{(j)}(L_j) = \Psi_4^{(j)}(L_j), \]  
\[ j = 1, 2, 3, \]  
which provide self-adjointness of the BdG operator on metric star graph. Eqs. (26) and (29) provide continuity of wave function, while Eqs. (27) and (28) lead to Kirchoff rule at the vertex.

Explicit solutions fulfilling these boundary conditions (for \( L_1 = L_2 = L_3 = L \)) can be written as
\[ \Psi^{(1,2)}(x) = \begin{pmatrix} q' \\ q' \\ -q \\ -q \end{pmatrix} e^{\Delta_0(L-x)} + \begin{pmatrix} -q \\ q' \\ q' \\ q' \end{pmatrix} e^{-\Delta_0(L-x)}, \]
\[ \Psi^{(3)}(x) = \begin{pmatrix} q' \\ -2q \\ -2q' \\ -q \end{pmatrix} e^{\Delta_0(L-x)} + \begin{pmatrix} -q \\ -2q' \\ -2q' \\ q' \end{pmatrix} e^{-\Delta_0(L-x)}. \]

Using the same prescription as in the previous section, one can derive vertex transmission matrix for zero-energy solution which can be written as
\[ T = -(AA' + BB')^{-1} (AA'' + BB''), \]
where
\[ A' = \begin{pmatrix} qI_N & 0 & 0 & 0 \\ 0 & qI_N & 0 & 0 \\ 0 & 0 & qI_N & 0 \\ 0 & 0 & 0 & qI_N \end{pmatrix}, \]
\[ A'' = \begin{pmatrix} qI_N & 0 & 0 & 0 \\ 0 & qI_N & 0 & 0 \\ 0 & 0 & qI_N & 0 \\ 0 & 0 & 0 & qI_N \end{pmatrix}, \]
\[ B' = \begin{pmatrix} 0 & qI_N & 0 & 0 \\ qI_N & 0 & 0 & 0 \\ 0 & 0 & -qI_N & 0 \\ 0 & 0 & -qI_N & 0 \end{pmatrix}, \]
\[ B'' = \begin{pmatrix} 0 & q^*I_N & 0 & 0 \\ q^*I_N & 0 & 0 & 0 \\ 0 & 0 & -qI_N & 0 \\ 0 & 0 & -qI_N & 0 \end{pmatrix} \]
and \( \det(\AA' + \BB') \neq 0 \). Then the vertex transition matrix can be written as
\[ T = -(AA' + BB')^{-1} (AA'' + BB''). \]

The above approach can be an effective model for branched Majoana wires with different topologies and vertex structures. Considering the appropriate vertex boundary conditions for Eq. (21), one can treat different model realizations of Majorana wire networks.

V. CONCLUSIONS

We have studied the first order Bogoliubov de Gennes equation on a metric star graph. The vertex boundary conditions providing the self-adjointness realization of the BdG operator on a metric star graph are derived for non-zero and zero energy cases. The solutions of some special types of the vertex boundary conditions providing continuity and current conservation are obtained. The secular equation for finding of eigenvalues is derived. The above results can be used for modeling of one dimensional branched superconductors and Majorana wire networks by choosing appropriate boundary conditions at the branching points.

[1] T.Kottos and U.Smilansky, Ann.Phys., 76 274 (1999).
[2] Oleh Hul et al, Phys. Rev. E 69, 056205 (2004).
[3] P.Kuchment, Waves in Random Media, 14 S107 (2004).
[4] S.Gnutzmann and U.Smilansky, Adv.Phys. 55 527 (2006).
[5] N.Goldman and P.Gaspard, Phys. Rev. B 77, 024302 (2008).
[6] P.Exner and H.Kovarik, Quantum waveguides. (Springer, 2015).
[7] P.Exner, P.Seba, P.Stovicek, J. Phys. A: Math. Gen. 21 4009 (1988).
[8] V.Kostrykin and R.Schrader J. Phys. A: Math. Gen. 32 595 (1999)
[9] J.Bolte and J.Harrison, J. Phys. A: Math. Gen. 36 L433 (2003).
[10] S.Gnutzmann, J.P.Keating, F.Piotet, Ann.Phys., 325 2595 (2010).
[11] J.Harrison, T.Weyand, and K.Kirsten, J. Math. Phys. 57 102301 (2016).
[12] R.Band, G.Berkoaliko, Phys. Rev. Lett., 111 130404 (2013).
[13] E.Korotyaeva, N.Saburovab, J. Math.Anal.Appl., 420
576 (2014).
[14] D.Mugnolo. *Semigroup Methods for Evolution Equations on Networks.* Springer-Verlag, Berlin, (2014).
[15] G.Berkolaiko, P.Kuchment, *Introduction to Quantum Graphs, Mathematical Surveys and Monographs* AMS (2013).
[16] S.Gnutzmann, H.Schanz and U.Smilansky, Phys. Rev. Lett., 110 094101 (2013).
[17] V.Barrera-Figueroa, V.S.Rabinovich, J. Phys. A: Math. Theor. it 50 215207 (2017).
[18] J.Bolte, G.Garforth, J. Phys. A: Math. Theor. it 50 105101 (2017).
[19] Z.Sobirov, D.Matrasulov, K.Sabirov, S.Sawada, and K.Nakamura, Phys. Rev. E 81 , 066602 (2010).
[20] Z. Sobirov, D. Matrasulov, S. Sawada, and K. Nakamura, Phys.Rev.E 84 , 026609 (2011).
[21] R.Adami, C.Cacciapuoti, D.Finco, D.N., Rev.Math.Phys, 23 4 (2011).
[22] K.K.Sabirov, Z.A.Sabirov, D.Babajanov, and D.U.Matrasulov, Phys.Lett. A, 377, 860 (2013).
[23] H.Susanto, S.A.Vargils, Phys. Lett. A, 338, 239 (2005).
[24] J.-G.Caputo , D.Dutykh, Phys. Rev. E 90, 022912 (2014).
[25] H.Uecker, D.Grieser, Z.Sobirov, D.Babajanov and D.Matrasulov, Phys. Rev. E 91, 023209 (2015).
[26] D.Noja, Philos. Trans. R. Soc. A 372, 20130002 (2014).
[27] D.Noja, D.Pelinovsky, and G.Shaikhova, *Nonlinearity* 28, 2343 (2015).
[28] Z.Sobirov, D.Babajanov, D.Matrasulov, K.Nakamura, and H.Uecker, EPL 115 , 50002 (2016).
[29] P.G. de Gennes,*Superconductivity of Metals and Alloys* (New York, 1966).
[30] C.Chamon, R.Jackiw, Y.Nishida, S.-Y.Pi, and L.Santos, Phys. Rev.B. 81 224515 (2010).
[31] I.Kosztin, S.Kos, M.Stone, A.J.Leggett, Phys. Rev. B 58 9365 (1998).
[32] E.Serret, P.Butaud, B.Pannetier, EPL, 59 225 (2002).
[33] F.P.Mancini, P.Sodano, A.Trombettoni, J. Mod. Phys. B 21 1923 (2007).
[34] J.Alicea, Rep. Prog. Phys. 75 076501 (2012).
[35] M.Leijnse and K.Flensberg, Semicond. Sci. Technol. 27 124003 (2012).
[36] D.J.Clarke, J.D.Sau, and S.Tewari, Phys. Rev.B. 84 035120 (2011).
[37] J.Alicea, Y.Oreg, G.Refael, F. von Oppen and M.P.A. Fisher, Nat.Phys., 7 412 (2011).
[38] B.I.Halperin, Y.Oreg, A.Stern, G.Refael, J.Alicea, F.von Oppen, Phys. Rev.B. 85 144501 (2012).
[39] C.V.Kraus, P.Zoller, and M.A.Baranov, Phys. Rev. Lett. 111 203001 (2013).
[40] F.L.Pedrocchi and D.P.DiVincenzo, Phys. Rev. Lett. 115 120402 (2015).
[41] K.Björnson, A.M.Black-Schaffer, Phys. Rev.B. 94 100501 (2016).
[42] M.Hell, K.Flensberg, M.Leijnse, Phys. Rev.B. 96 035444 (2017).