On soliton solutions of multi-component semi-discrete short pulse equation

H Wajahat A Riaz and Mahmood ul Hassan
Department of Physics, University of Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan
E-mail: ahmed.phyy@gmail.com and mhassan@physics.pu.edu.pk
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Abstract
The short pulse (SP) equation is an integrable equation. Multi-component generalizations of the SP equation are important for describing the polarization or anisotropic effects in optical fibers. An integrable semi-discretization of multi-component SP equation via Lax pair and Darboux transformation (DT) has been presented. We derive a Lax pair representation for the multi-component semi-discrete short pulse (sdSP) equation in the form of a block matrices by generalizing the $2 \times 2$ Lax pair matrices to the case of $2^N \times 2^N$. A DT is studied for the multi-component sdSP equation and is used to compute soliton solutions of the system. Further, by expanding quasideterminants, we compute cuspon-soliton, smooth-soliton and loop-soliton solutions of the complex sdSP equation.

1. Introduction
Integrable discretization of nonlinear ordinary and partial differential equations and their multi-component as well as non-abelian matrix generalizations have attracted a great deal of interest in nonlinear dynamics. Such discrete or lattice systems appear in quantum mechanics, quantum field theory, statistical mechanics, mathematical biology, economics etc. They also provide numerical schemes for the differential equations. Different systematic methods have been used to obtain an integrable discretization for a given integrable system. One approach to get a discrete analog of a given integrable system is the Ablowitz–Ladik method. In this method, a proper discretization is performed on the linear system of equations (also known as Lax pair) associated with a given integrable system so that its integrability structure is not spoiled. Various examples such as Korteweg–de Vries (KdV), modified Korteweg–de Vries (mKdV), nonlinear Scrodinger equation and Toda lattice etc. have been discretized by Ablowitz–Ladik method. All these discrete equations preserve their integrability properties such as existence of zero-curvature representation, soliton solutions and the existance of infinite sequence of conserved quantities etc [1–4]. On the other hand, in Hirota method an integrable discretization of a given integrable system can be obtained via bilinearization approach [5, 6].

In the past years, short-pulse (SP) equation and its generalizations have received great attention by the researchers because of their nice integrability structure. The SP equation exhibits many integrability properties such as existence of an infinite sequence of conserved quantities, Lax pair representations, bi-Hamiltonian structure, soliton solutions and so on [7–17].

The SP equation for a scalar function $u = u(x, t)$ is given by

$$\partial_x \partial_t u = u + \frac{1}{6} \partial_x(u^3).$$

Equation (1.1) was initially studied by Shafer–Wayne for the propagation of ultra-short waves in nonlinear optics [7, 8]. A hodograph transformation from independent variables $(x, t)$ to new variables $(X, T)$ given by

$$dX = \omega dx + \frac{1}{2} u^2 dt, \quad dT = dt.$$
with \( \omega^2 = 1 + (\partial_x u)^2 \) transforms (1.1) in the following form
\[
\partial_x \partial_T x = -\frac{1}{2} \partial_x (u^2),
\]
(1.2)
\[
\partial_x \partial_T u = u \partial_x u.
\]
(1.3)
The SP equation is linked with the mKdV equation, coupled dispersionless integrable system and sine–Gordon equation by a suitable transformation [11–16]. The complex generalization of the equations (1.2), (1.3) is given by
\[
\partial_x \partial_T x = -\frac{1}{2} \partial_x (|u|^2),
\]
(1.4)
\[
\partial_x \partial_T u = u \partial_x u.
\]
(1.5)
The set of equations (1.4), (1.5) are known as complex SP equations and can be reduced to complex generalization of equation (1.1) by hodograph transformation [15, 16].

Along with continuous model, fully discrete and semi-discrete (or lattice) SP equation and its generalizations have been studied in the articles such as [18–21]. The integrability properties of such equations are exhibited by the existence of Lax pair representations, Bäcklund transformation, Hirota bilinear method and soliton solutions [18–21]. In the present work, we define a Lax pair for the complex semi-discrete short-pulse (sdSP) equations (1.4), (1.5) which is a set of differential-difference equations where the space coordinate is taken as one-dimensional lattice and time is taken as continuous. The Lax pair representation of complex sdSP equation is given by
\[
\Omega_{j+1} = A_j \Omega_j,
\]
(1.6)
\[
\frac{d}{dT} \Omega_j = B_j \Omega_j,
\]
(1.7)
with the matrices \( A_j \) and \( B_j \) given by
\[
A_j = I + \lambda (Q_{j+1} - Q_j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} (x_{j+1} - x_j) & (u_{j+1} - u_j) \\ (\bar{u}_{j+1} - \bar{u}_j) & -(x_{j+1} - x_j) \end{pmatrix},
\]
(1.8)
\[
B_j = S_j + \lambda^{-1} S_0 = \begin{pmatrix} 0 & -\frac{1}{2} u_j \\ \frac{1}{2} \bar{u}_j & 0 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} \frac{i}{4} & 0 \\ 0 & -\frac{i}{4} \end{pmatrix},
\]
(1.9)
where \( x_j \) is a real and \( u_j \) is a complex function. The compatibility condition i.e., \( \frac{d}{dT} A_j + A_j B_j - B_{j+1} A_j = 0 \) gives the sdSP equations
\[
\frac{d}{dT} (x_{j+1} - x_j) + \frac{|u_{j+1}^2 - |u_j|^2|}{2} = 0,
\]
(1.10)
\[
\frac{d}{dT} (u_{j+1} - u_j) - (x_{j+1} - x_j) \left( \frac{u_{j+1} + u_j}{2} \right) = 0.
\]
(1.11)

Many integrable equations are extended to their multi-component or vector generalizations. Such multi-component systems include nonlinear Schrödinger (NLS) equation, KdV equation, mKdV equation, coupled integrable dispersionless system and the SP equation [1, 22–24]. From the physical context, multi-component or vector generalizations of soliton equations play an important role for describing the polarization or anisotropic effects in waveguides and optical fibers. For example, the two-component NLS equation is a physical model that describes nonlinear phenomena, where the field has two components transverse to the direction of propagation. Therefore, the multi-component soliton equations have received great interest in many areas such as plasma physics, fluid dynamics, nonlinear optics etc [1, 22–27].

In the present work, we study multi-component generalizations of sdSP equation by extending the \( 2 \times 2 \) Lax matrices to the case of \( 2^N \times 2^N \) Lax matrices. Dorboux transformation (DT) is used to compute soliton solutions to the multi-component generalization of sdSP equation. Furthermore, multisoliton solutions are expressed in terms of quasideterminants, as well as ratios of simple determinants.

The paper is organized as follows. In section 2, we present Lax pair representations of the multi-component sdSP equation. From the zero-curvature condition, we obtain multi-component generalization of sdSP equation. In section 3, a DT is defined on the solutions to the Lax pair and on the solutions to multi-component sdSP equation. Further, the solutions are expressed in terms of quasideterminants. In section 4, one- and two-cuspon-soliton, smooth-soliton and loop-soliton solutions are obtained for the complex sdSP equation by expanding the quasideterminants. In the last section, we make concluding remarks.
2. Lax pair representation

We start with the Lax pair representation of the multi-component sdSP equation. The Lax pair is given by

\[ \begin{align*}
\Omega_{j+1} &= A_j \Omega_j = (\mathcal{I} + \lambda (Q_{j+1} - Q_j)) \Omega_j, \\
\frac{d}{dT} \Omega_j &= B_j \Omega_j = (S_j + \lambda^{-1} S_0) \Omega_j,
\end{align*} \]

(2.1)

(2.2)

where \( j \) in the subscript is the discrete index and \( \mathcal{I}, \ Q_j, \ S_j \) and \( S_0 \) are the \( 2^N \times 2^N \) block matrices given by

\[ \begin{align*}
\mathcal{I} &= \begin{pmatrix} I & 0 \\
0 & I \end{pmatrix}, \quad Q_j = \begin{pmatrix} X_j & U_j \\
V_j & -X_j \end{pmatrix}, \quad S_j = \begin{pmatrix} O & -\frac{1}{2} U_j \\
\frac{1}{2} V_j & O \end{pmatrix}, \quad S_0 = \begin{pmatrix} \frac{1}{2} I & O \\
O & -\frac{1}{2} I \end{pmatrix},
\end{align*} \]

(2.3)

where \( X_j, \ U_j \) and \( V_j = \rho U_j^\dagger \) (here \( \dagger \) in the superscript denotes Hermitian conjugation) are the \( 2^{N-1} \times 2^{N-1} \) block matrices and \( I, \ O \) are the \( 2^{N-1} \times 2^{N-1} \) identity and null matrices respectively. From the compatibility condition of the Lax pair (2.1), (2.2) i.e., \( \frac{d}{dT} A_j + A_j B_j - B_j^T A_j = O \), we obtain the matrix sdSP equation as

\[ \frac{d}{dT} (Q_{j+1} - Q_j) + (Q_{j+1} - Q_j) S_j - S_{j+1} (Q_{j+1} - Q_j) = O. \]

(2.4)

By substituting the expression of \( Q_j \) and \( S_j \) from (2.3) into (2.4), we obtain

\[ \begin{align*}
\frac{d}{dT} (X_{j+1} - X_j) + \frac{1}{2} (U_j V_{j+1} - U_j V_j) = O, \\
\frac{d}{dT} (U_{j+1} - U_j) - \frac{1}{2} [(X_{j+1} - X_j) U_j + U_{j+1} (X_{j+1} - X_j)] = O.
\end{align*} \]

(2.5)

(2.6)

For a particular case, when \( X_j \equiv x_j, \ U_j \equiv u_j \), the matrix sdSP equations (2.5), (2.6) reduce to the sdSP equations (1.10), (1.11). For higher generalizations, we take

\[ \begin{align*}
X_j^{(2)} &= x_j I_{2 \times 2}, \quad U_j^{(2)} = \begin{pmatrix} u_j^{(1)} & u_j^{(2)} \\
-\bar{u}^{(1)}_j & \bar{u}^{(2)}_j \end{pmatrix}_{2 \times 2}, \\
X_j^{(3)} &= x_j I_{4 \times 4}, \quad U_j^{(3)} = \begin{pmatrix} u_j^{(1)} & u_j^{(2)} & 0 & 0 \\
-\bar{u}^{(1)}_j & \bar{u}^{(2)}_j & 0 & 0 \\
0 & 0 & u_j^{(3)} & u_j^{(4)} \\
0 & 0 & -\bar{u}^{(3)}_j & -\bar{u}^{(4)}_j \end{pmatrix}_{4 \times 4}, \\
X_j^{(4)} &= x_j I_{8 \times 8}, \quad U_j^{(4)} = \begin{pmatrix} u_j^{(1)} & u_j^{(2)} & 0 & 0 & 0 & 0 & u_j^{(3)} & u_j^{(4)} \\
-\bar{u}^{(1)}_j & \bar{u}^{(2)}_j & 0 & 0 & 0 & 0 & -\bar{u}^{(3)}_j & -\bar{u}^{(4)}_j \\
0 & 0 & u_j^{(3)} & u_j^{(4)} & 0 & 0 & \bar{u}^{(3)}_j & \bar{u}^{(4)}_j \\
0 & 0 & -\bar{u}^{(3)}_j & -\bar{u}^{(4)}_j & 0 & 0 & u_j^{(1)} & u_j^{(2)} \end{pmatrix}_{8 \times 8}.
\end{align*} \]

(2.7)

(2.8)

(2.9)

By substituting the expressions of (2.7)–(2.9) into the set of equations (2.5), (2.6), one can obtain respectively, the 2-component, 3-component and 4-component sdSP equations. The 2-component complex sdSP equation is a set

\[ \begin{align*}
\frac{d}{dT} (x_{j+1} - x_j) + \frac{1}{2} \sum_{n=1}^{N} (\bar{u}_j^{(n)} \theta_n u_{j+1}^{(n)} - \bar{u}_j^{(n)} \theta_n u_j^{(n)}) &= 0, \\
\frac{d}{dT} (u_{j+1}^{(n)} - u_j^{(n)}) - \frac{1}{2} (u_{j+1}^{(n)} + u_j^{(n)}) (x_{j+1} - x_j) &= 0, \quad n = 1, 2.
\end{align*} \]

(2.10)

(2.11)
By substituting the expressions (2.14), (2.15), where

\[ \vartheta_n = 1, \]

Similarly, \( N \)-component complex sdSP equations is given by the following set of equations

\[ \frac{d}{dT}(u_{j+1}^{(n)} - u_j^{(n)}) - \frac{1}{2}(u_{j+1}^{(n)} + u_j^{(n)})(x_{j+1} - x_j) = 0, \quad n = 1, 2, 3. \]

In general, the \( 2^{N-1} \times 2^{N-1} \) matrices \( \mathcal{X}_j^{(N)} \) and \( \mathcal{U}_j^{(N)} \) can be written in the following form

\[ \mathcal{X}_j^{(N)} = x_j I_{2^{N-1} \times 2^{N-1}}, \quad \mathcal{U}_j^{(N)} = \begin{pmatrix} U_j^{(1)} & U_j^{(2)} \\ U_j^{(3)} & U_j^{(4)} \end{pmatrix}, \]

where \( U_j^{(i)} \) (\( i = 1, 2, 3, 4 \)) are all \( 2^{N-2} \times 2^{N-2} \) square block matrices, and \( U_j^{(i)} = (U_j^{(i)})^\dagger, \quad U_j^{(3)} = -(U_j^{(2)})^\dagger \).

The matrices \( U_j^{(1)} \) and \( U_j^{(2)} \) are given by

\[ U_j^{(1)} = \begin{pmatrix} L_j^{(1)} & \cdots & 0 \\ & \ddots & \vdots \\ 0 & \cdots & L_j^{(1)} \end{pmatrix}, \quad U_j^{(2)} = \begin{pmatrix} M_j^{(1)} & M_j^{(2)} \\ M_j^{(3)} & M_j^{(4)} \end{pmatrix}, \]

where \( M_j^{(i)} = (M_j^{(i)})^\dagger, \quad M_j^{(3)} = -(M_j^{(2)})^\dagger \) are square block matrices given by

\[ M_j^{(i)} = \begin{pmatrix} L_j^{(2)} & \cdots & 0 \\ & \ddots & \vdots \\ 0 & \cdots & L_j^{(2)} \end{pmatrix}, \quad M_j^{(3)} = \begin{pmatrix} \mathbb{O} & L_j^{(3)} & \cdots & L_j^{(N)} \\ L_j^{(3)} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ L_j^{(N)} & \cdots & L_j^{(3)} & \mathbb{O} \end{pmatrix}, \]

\[ \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad L_j^{(1)} = \begin{pmatrix} u_j^{(1)} & u_j^{(2)} \\ -u_j^{(2)} & u_j^{(1)} \end{pmatrix}, \quad L_j^{(3)} = \begin{pmatrix} u_j^{(4)} & 0 \\ 0 & u_j^{(4)} \end{pmatrix}, \]

\[ n = 2, 3, \ldots, N - 1. \]

By substituting the expressions (2.16)–(2.18) into the set of equations (2.5), (2.6), one can obtain the equations (2.14), (2.15) by a straightforward computation. The set of equations (2.14), (2.15) represents an integrable semi-discrete analog of the multi-component SP equation.

Now we write the equations (2.14), (2.15) in a more general form. For this, let us write the equations (2.14), (2.15) in vector notation as

\[ \frac{d}{dT}(x_{j+1} - x_j) + \frac{1}{2}(u_{j+1}^\dagger \Upsilon u_{j+1} - u_j^\dagger \Upsilon u_j) = 0, \]

\[ \frac{d}{dT}(u_{j+1} - u_j) - \frac{1}{2}(u_{j+1} + u_j)(x_{j+1} - x_j) = 0, \]

where \( u_j = (u_j^{(1)}, u_j^{(2)}, \ldots, u_j^{(N)})^\dagger, \quad \Upsilon = \text{diag}(\vartheta_1, \vartheta_2, \ldots, \vartheta_N). \) Introduce \( u_j = P r_j \), where \( P \) is a real orthogonal matrix which satisfies \( P^\dagger P = PP^\dagger = I \), so that the equations (2.19), (2.20) can be expressed as

\[ \frac{d}{dT}(r_{j+1} - r_j) + \frac{1}{2}(r_{j+1}^\dagger G r_{j+1} - r_j^\dagger G r_j) = 0, \]

\[ \frac{d}{dT}(r_{j+1} + r_j)(x_{j+1} - x_j) = 0, \]

where \( G = P^\dagger \Upsilon P \) is a real constant \( N \times N \) matrix. We introduce \( G = \epsilon_{\text{min}} \), so that the equations (2.21), (2.22) in a more explicit form can be expressed as

\[ \frac{d}{dT}(r_{j+1} - r_j) + \frac{1}{2}(r_{j+1}^\dagger G r_{j+1} - r_j^\dagger G r_j) = 0, \]

\[ \frac{d}{dT}(r_{j+1} + r_j)(x_{j+1} - x_j) = 0, \]
\[ \frac{d}{dT}(x_{j+1} - x_j) + \frac{1}{2} \sum_{m,n=1}^{N} \epsilon_{mn}(r_{j+1}^{(m)} - r_j^{(m)}) = 0, \quad (2.23) \]
\[ \frac{d}{dT}(r_{j+1}^{(n)} - r_j^{(n)}) - \frac{1}{2}(r_{j+1}^{(n)} + r_j^{(n)})(x_{j+1} - x_j) = 0, \quad n = 1, ..., N, \quad (2.24) \]

where \( \epsilon_{mm} = \epsilon_{mm} = 0. \) The set of equations (2.23), (2.24) have also been obtained in [20]. In order to recover the equations (1.10), (1.11) from (2.23), (2.24), let us take \( N = 2, \) so that the equations (2.23), (2.24) reads
\[ \frac{d}{dT}(x_{j+1} - x_j) + \frac{1}{2}(r_{j+1} + r_j)(x_{j+1} - x_j) = 0, \quad (2.25) \]
\[ \frac{d}{dT}(r_{j+1} - r_j) - \frac{1}{2}(r_{j+1} + r_j)(x_{j+1} - x_j) = 0, \quad (2.26) \]
\[ \frac{d}{dT}(s_{j+1} - s_j) - \frac{1}{2}(s_{j+1} + s_j)(x_{j+1} - x_j) = 0, \quad (2.27) \]

where \( r_j^{(1)} = r_j, \ r_j^{(2)} = s_j, \ \epsilon_{12} = 1. \) The system (2.25)–(2.27) is the semi-discrete two-component SP equation obtained in [20]. Equations (1.10), (1.11) can be recovered from (2.25)–(2.27), when \( s_j = r_j. \)

The continuous multi-component SP equation can be recovered from the set of equations (2.14), (2.15) by applying continuum limit on the lattice parameter. For this, let us define \( \lim_{\delta \to 0} h_{i+\delta} - h_i = h_{X}\delta, \) where \( \delta \) is the lattice parameter in space-direction. Applying this to the set of equations (2.14), (2.15), we obtain
\[ \partial_x \partial_t x + \frac{1}{2} \sum_{n=1}^{N} \partial_x (|u^{(n)}|^2) = 0, \quad (2.28) \]
\[ \partial_x \partial_t u^{(n)} - u^{(n)} \partial_x x = 0, \quad n = 1, ..., N. \quad (2.29) \]

This is the set of multi-component SP equations in the continuum limit and for \( N = 1, \) it is the usual complex SP equations (1.4), (1.5). Further, with the help of hodograph transformation, the set of equations (2.28), (2.29) can be converted to the multi-component complex SP equation as that obtained in [15].

### 3. Darboux transformation

DT is one of solution generating technique in soliton theory that allow us to express the solutions for a given integrable equation in simple explicit form [28–33]. In this section, we construct the DT of the multi-component (or matrix) sDSP equation.

In what follows, we shall apply a DT to the solutions of the Lax pair and to the solutions of the multi-component sDSP equations and then express them in terms of quasideterminants. Let us define a new solution \( \Omega_j^{(1)} \) to the Lax pair (2.1), (2.2) which is related to the old solution \( \Omega \) by means of 2\( N \times 2\)\( N \) matrix \( D_j(t; \lambda) \) called the Darboux matrix. The one-fold DT on the solution to the Lax pair is given by
\[ \Omega_j^{(1)} = D_j(t; \lambda) \Omega_j = (\lambda^{-1} I - \Xi_j) \Omega_j. \quad (3.1) \]

In the present case, \( I \) is a \( 2^N \times 2^N \) unit matrix and \( \Xi_j \) is an invertible \( 2^N \times 2^N \) matrix, to be determined. The covariance of the Lax pair under the DT requires that the new solution \( \Omega_j^{(1)} \) satisfies the same Lax pair i.e.,
\[ \Omega_{j+1}^{(1)} = A_j^{(1)} \Omega_j^{(1)} = (I^{(1)} + \lambda(Q_{j+1}^{(1)} - Q_j^{(1)}))\Omega_j^{(1)}, \quad (3.2) \]
\[ \frac{d}{dT} \Omega_j^{(1)} = A_j^{(1)} \Omega_j^{(1)} = (\Sigma_j^{(1)} + \lambda^{-1} S_j^{(1)}) \Omega_j^{(1)}. \quad (3.3) \]

By substituting the expression of \( \Omega_j^{(1)} \) from (3.1) into Lax pair equations (3.2), (3.3), the coefficients of \( \lambda \) yield the DT on the matrices \( Q_j, \ S_j, \ I \) and \( S_0 \) given by
\[ Q_j^{(1)} = Q_j - \Xi_j, \quad (3.4) \]
\[ S_j^{(1)} = S_j + [S_0, \ Z_j], \quad (3.5) \]
\[ I^{(1)} = I, \quad S_0^{(1)} = S_0, \quad (3.6) \]

and the matrix \( \Xi_j \) is required to satisfy
\[ (\Xi_{j+1} - \Xi_j) \Xi_j = (Q_{j+1} - Q_j) \Xi_j - \Xi_{j+1} (Q_{j+1} - Q_j), \quad (3.7) \]
\[ \frac{d}{dt} \Xi_j = [S_0, \ Z_j] + [S_0, \ Z_j] \Xi_j. \quad (3.8) \]

Now we construct the matrix \( \Xi_j \) in terms of solutions to the Lax pair (2.1), (2.2). For this, we proceed as follows:
Define $2^N$ distinct constant parameters $\lambda_1$, $\lambda_2$, ..., $\lambda_{2^N}$ such that for each $\lambda_i$ we have a peculiar column vector solution $|h_i^{(0)}⟩ = \Omega_i(\lambda_i)|e_i⟩$ (where $|e_i⟩$ is a constant column vector) to the Lax pair (2.1), (2.2). For $\lambda = \lambda_i (i = 1, 2, ..., 2^N)$, we write

$$|h_i^{(0)}⟩_{j+1} = |h_i^{(0)}⟩_j + \lambda_i(Q_{j+1} - Q_j)|h_i^{(0)}⟩_j, \quad (3.9)$$

$$\frac{d}{dr} |h_i^{(0)}⟩_j = S_j |h_i^{(0)}⟩_j + \lambda_i^{-1}S_0 |h_i^{(0)}⟩_j. \quad (3.10)$$

Define $2^N \times 2^N$ constant eigenvalue matrix with entries $\lambda_i$, i.e., $\Lambda = \text{diag}(\lambda_1, ..., \lambda_{2^N})$, and construct an invertible $2^N \times 2^N$ matrix $H_j$ as

$$H_j = (\Omega_i(\lambda_i)|e_i⟩, ..., \Omega_i(\lambda_{2^N})|e_{2^N}⟩) = (|h_i^{(0)}⟩_j, ..., |h_{2^N}^{(0)}⟩_j), \quad (3.11)$$

so that the Lax pair (2.1), (2.2) with the particular matrix $H_j$ as solution to (2.1), (2.2), can be written as

$$\frac{d}{dr} H_j = S_j H_j + S_0 H_j \Lambda^{-1}. \quad (3.12)$$

Further we check that the choice of the matrix $\Xi_j = H_j \Lambda^{-1} H_j^{-1}$ satisfies the conditions (3.7), (3.8) as imposed by Darboux covariance. This can be checked by a direct computation as follows

$$(\Xi_{j+1} - \Xi_j) \Xi_j = H_{j+1} \Lambda^{-1} H_{j+1}^{-1} H_j \Lambda H_j^{-1} - H_j \Lambda^{-1} H_j^{-1} H_{j+1} \Lambda H_{j+1}^{-1}$$

$$+ H_{j+1} \Lambda^{-1} H_{j+1}^{-1} H_j \Lambda H_j^{-1} - H_{j+1} \Lambda^{-1} H_{j+1}^{-1} H_{j+1} \Lambda H_{j+1}^{-1},$$

$$= (H_{j+1} \Lambda^{-1} H_{j+1}^{-1} - H_j \Lambda^{-1} H_j^{-1}) H_j \Lambda H_j^{-1}$$

$$- H_{j+1} \Lambda^{-1} H_{j+1}^{-1} (H_j \Lambda H_j^{-1} - H_{j+1} \Lambda H_{j+1}^{-1}),$$

$$= (Q_{j+1} - Q_j) \Xi_j - \Xi_{j+1}(Q_{j+1} - Q_j). \quad (3.14)$$

$$\frac{d}{dr} \Xi_j = \left(\frac{dH_j}{dr}\right) \Lambda^{-1} H_j^{-1} - H_j \Lambda^{-1} H_j^{-1} \left(\frac{dH_j}{dr}\right) H_j^{-1},$$

$$= (S_j H_j + S_0 H_j \Lambda^{-1}) \Lambda^{-1} H_j^{-1} - H_j \Lambda^{-1} H_j^{-1} (S_j H_j + S_0 H_j \Lambda^{-1}) \Lambda^{-1} H_j^{-1},$$

$$= [S_j, \Xi_j] + [S_0, \Xi_j] \Xi_j. \quad (3.15)$$

So the conditions (3.7) and (3.8) are satisfied. Hence, one can say that the transformation

$$\Omega_j^{(1)} = D_j(t; \lambda) \Omega_j = (\Lambda^{-1} I - H_j \Lambda^{-1} H_j^{-1}) \Omega_j, \quad (3.16)$$

$$Q_j^{(1)} = Q_j - H_j \Lambda^{-1} H_j^{-1}. \quad (3.17)$$

is the required DT on the solutions of (2.1), (2.2) and (2.4).

The one-fold DT $\Omega_j^{(1)}$ and $Q_j^{(1)}$ can be written in terms of quasideterminants and then by $K$-times iteration process, one can obtain the $K$-fold DT by using the properties of quasideterminants. For the Darboux matrix $D_j(t; \lambda) = \Lambda^{-1} I - H_j \Lambda^{-1} H_j^{-1}$, equations (3.16) and (3.17) can be expressed as

$$\Omega_j^{(1)} = D_j(t; \lambda) \Omega_j = \Lambda^{-1} \Omega_j - H_j \Lambda^{-1} H_j^{-1} \Omega_j,$$

$$= \Lambda^{-1} \Omega_j + \begin{vmatrix} H_j & \Omega_j \\ H_j \Lambda^{-1} & \square \end{vmatrix} = \begin{vmatrix} H_j & \Omega_j \\ H_j \Lambda^{-1} & \Lambda^{-1} \Omega_j \end{vmatrix}, \quad (3.18)$$

$$Q_j^{(1)} = Q_j - H_j \Lambda^{-1} H_j^{-1} = Q_j + \begin{vmatrix} H_j & I \\ H_j \Lambda^{-1} & \square \end{vmatrix}. \quad (3.19)$$

For $\Lambda = \Lambda_k (k = 1, 2, ..., K)$, one can generalize one-fold DT $\Omega_j^{(1)}$ to $K$-fold DT on the solution by an iteration process, so that the $K$-fold DT $\Omega_j^{(K)}$ can be expressed as

$$\Omega_j^{(K)} = \begin{vmatrix} H_{j, 1} & H_{j, 2} & \cdots & H_{j, K} & \Omega_j \\ H_{j, 1} \Lambda_k^{-1} & H_{j, 2} \Lambda_k^{-1} & \cdots & H_{j, K} \Lambda_k^{-1} & \Lambda_k^{-1} \Omega_j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{j, 1} \Lambda_k^{-K} & H_{j, 2} \Lambda_k^{-K} & \cdots & H_{j, K} \Lambda_k^{-K} & \Lambda_k^{-K} \Omega_j \end{vmatrix}. \quad (3.20)$$

1 We use the notion of quasideterminants which have various useful properties that play important roles in constructing exact solutions of integrable equations. For details (see e.g. [34–36]).
Similarly the $K$-fold DT on the matrix $Q_j$ is

$$Q_j^{[K]} = Q_j + \mathcal{F}_j^{[K]}E^{[K]},$$

where $2^N \times 2^N$ matrix $E^{[K]}$ is a quasideterminant given by

$$E^{[K]} = \begin{pmatrix} \mathcal{F}_j \mathcal{E}^{[K]} \end{pmatrix},$$

where $\mathcal{E}^{[K]}$ are $2^N K \times 2^N$ and $\mathcal{F}_j$ are the $2^N \times 2^N K, 2^N K \times 2^N$ matrices respectively, i.e.

$$\mathcal{F}_j = \begin{pmatrix} \mathcal{H}_{j,1} \mathcal{H}_{j,2} \cdots \mathcal{H}_{j,K} \\ \mathcal{H}_{j,1} \mathcal{H}_{j,2} \cdots \mathcal{H}_{j,K} \\ \vdots \\ \mathcal{H}_{j,1} \mathcal{H}_{j,2} \cdots \mathcal{H}_{j,K} \end{pmatrix}.$$

The matrix elements of the matrix $E^{[K]}$ can be computed as

$$(E^{[K]})_{pq} = \begin{pmatrix} \mathcal{F}_j \mathcal{E}^{[K]} \end{pmatrix}_{pq} = \begin{pmatrix} \mathcal{F}_j \mathcal{E}^{[K]} \end{pmatrix}_{pq}, \quad p = q,$$

$$(E^{[K]})_{pq} = \begin{pmatrix} \mathcal{F}_j \mathcal{E}^{[K]} \end{pmatrix}_{pq} = \begin{pmatrix} \mathcal{F}_j \mathcal{E}^{[K]} \end{pmatrix}_{pq}, \quad p = q,$$

where $(\mathcal{F}_j)_p$ indicates the $p$th row of $\mathcal{F}_j$ and $(\mathcal{E}^{[K]})_q$ represents $q$th column of $E^{[K]}$ respectively. From the set of equations (3.22)–(3.25), one can compute explicit expressions of DT on the scalar solutions of the multi-component sdSP equation.

4. Explicit soliton solutions

In order to generate soliton solutions, let us take matrix valued seed solution as

$$X_{j+1} - X_j = \begin{pmatrix} \mathcal{I} \mathcal{H}_j \mathcal{I} \end{pmatrix}, \quad U_j = \begin{pmatrix} \mathcal{H}_j \mathcal{I} \mathcal{H}_j \end{pmatrix},$$

where $\varepsilon$ is a non-zero real constant, so the matrix valued solution $\Omega_j$ to the Lax pair (2.1), (2.2) can be written as

$$\Omega_j = \begin{pmatrix} \mathcal{E}_j(\lambda) \mathcal{I} \mathcal{E}_j(\lambda) \mathcal{I} \end{pmatrix},$$

where $\mathcal{E}_j(\lambda) = (1 + \varepsilon \lambda)^\mathcal{I} \mathcal{E}_j(\lambda) \mathcal{I} \mathcal{E}_j(\lambda) \mathcal{I}$ and $\mathcal{E}_j(\lambda) = (1 - \varepsilon \lambda)^\mathcal{I} \mathcal{E}_j(\lambda) \mathcal{I} \mathcal{E}_j(\lambda) \mathcal{I}$.

4.1. $N = 1$ case

For a complex sdSP equation, we have $X_j = x_j$ and $U_j = u_j$, the matrix $Q_j$ takes the form

$$Q_j = \begin{pmatrix} x_j & u_j \\ a_j & -x_j \end{pmatrix}$$

To generate soliton solutions we proceed as follows:
The matrix $\mathcal{H}_j$ for the complex sdSP equation has the form

$$\mathcal{H}_j = (\|H^{(1)}\|_j, \|H^{(2)}\|_j) = \begin{pmatrix} h^{(1)}_{j, 11} & h^{(1)}_{j, 12} \\ h^{(1)}_{j, 21} & h^{(1)}_{j, 22} \\ h^{(2)}_{j, 11} & h^{(2)}_{j, 12} \\ h^{(2)}_{j, 21} & h^{(2)}_{j, 22} \end{pmatrix}$$

(4.4)

so the different matrix solutions $\mathcal{H}_{j, k}$ with the eigenvalue matrices $\Lambda_k$ to the Lax pair (2.1), (2.2) is written as

$$\mathcal{H}_{j, k} = \begin{pmatrix} h^{(2k-1)}_{j, 11} & h^{(2k-1)}_{j, 12} \\ h^{(2k-1)}_{j, 21} & h^{(2k-1)}_{j, 22} \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \lambda_k - 1 & 0 \\ 0 & \lambda_k \end{pmatrix}, \quad k = 1, 2, ..., K.$$

(4.5)

And the $2 \times 2$ matrix $\Theta^{(K)}_j$ is

$$\Theta^{(K)}_j = \begin{pmatrix} \Theta^{(K)}_{j, 11} & \Theta^{(K)}_{j, 12} \\ \Theta^{(K)}_{j, 21} & \Theta^{(K)}_{j, 22} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_j & \mathcal{E}^{(K)}_j \\ \mathcal{F}_j & \mathcal{E}^{(K)}_j \end{pmatrix}.$$  

(4.6)

Here in the present case, $\mathcal{E}^{(K)}$ are $2K \times 2$ and $\mathcal{F}_j$ are the $2 \times 2K, 2K \times 2K$ matrices respectively. From the equations (3.22) with (4.6) and (4.3), the $K$-fold DT on the scalar fields $x_j$ and $u_j$ are given by

$$x^{(K)}_j = x_j + \Theta^{(K)}_{j, 11},$$

(4.7)

$$u^{(K)}_j = \Theta^{(K)}_{j, 12}.$$ 

(4.8)

For one soliton $K = 1$, we have

$$\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{H}_{j, 1} = \begin{pmatrix} h^{(1)}_{j, 11} & h^{(1)}_{j, 12} \\ h^{(1)}_{j, 21} & h^{(1)}_{j, 22} \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

(4.9)

so that the matrices $\mathcal{F}_j$ and $\tilde{\mathcal{F}}_j$

$$\mathcal{F}_j = \mathcal{H}_{j, 1} = \begin{pmatrix} h^{(1)}_{j, 11} & h^{(1)}_{j, 12} \\ h^{(1)}_{j, 21} & h^{(1)}_{j, 22} \end{pmatrix}, \quad \tilde{\mathcal{F}}_j = \mathcal{H}_{j, 1}\Lambda_1 = \begin{pmatrix} \lambda_1^{-1}h^{(1)}_{j, 11} & \lambda_2^{-1}h^{(1)}_{j, 12} \\ \lambda_1^{-1}h^{(1)}_{j, 21} & \lambda_2^{-1}h^{(1)}_{j, 22} \end{pmatrix}.$$ 

(4.10)

Therefore, the matrix element $\Theta^{(1)}_{j, 11}$ in the matrix $\Theta^{(1)}_j$ can be computed as

$$\Theta^{(1)}_{j, 11} = \begin{vmatrix} \mathcal{F}_j & \mathcal{E}^{(1)}_j \\ \tilde{\mathcal{F}}_j & \end{vmatrix} = \begin{vmatrix} h^{(1)}_{j, 11} & h^{(1)}_{j, 12} & 1 \\ h^{(1)}_{j, 21} & h^{(1)}_{j, 22} & 0 \\ \lambda_1^{-1}h^{(1)}_{j, 11} & \lambda_2^{-1}h^{(2)}_{j, 12} & \end{vmatrix} = \det \begin{vmatrix} \lambda_1^{-1}h^{(1)}_{j, 11} & \lambda_2^{-1}h^{(1)}_{j, 12} \\ h^{(1)}_{j, 21} & h^{(1)}_{j, 22} \end{vmatrix} = -\frac{\lambda_1^{-1}h^{(1)}_{j, 11}h^{(1)}_{j, 22} - \lambda_2^{-1}h^{(1)}_{j, 21}h^{(1)}_{j, 12}}{h^{(1)}_{j, 11}h^{(1)}_{j, 22} - h^{(1)}_{j, 21}h^{(1)}_{j, 12}}.$$ 

(4.11)

Let us take $h^{(2)}_{j, 22} = h^{(1)}_{j, 11}, h^{(2)}_{j, 12} = -h^{(1)}_{j, 21}$ and $\lambda_2 = -\lambda_1$, we obtain

$$\Theta^{(1)}_{j, 11} = -\frac{\lambda_1^{-1}h^{(1)}_{j, 11}^2 - \lambda_2^{-1}h^{(1)}_{j, 21}^2}{h^{(1)}_{j, 11}^2 + h^{(1)}_{j, 21}^2}.$$ 

(4.12)

Similarly

$$\Theta^{(1)}_{j, 12} = -\frac{\lambda_1^{-1} + \frac{\lambda_1^{-1}}{\lambda_2^{-1}}h^{(1)}_{j, 11}h^{(1)}_{j, 21}}{h^{(1)}_{j, 11}^2 + h^{(1)}_{j, 21}^2}.$$ 

(4.13)

From (4.2), we have the column vector solutions $\Omega_j(\lambda_k)e_k$ to the Lax pair (2.1), (2.2) as

$$\Omega_j(\lambda_k)e_k = \begin{pmatrix} (1 + \varepsilon \lambda_k)e^{\pi \varepsilon \lambda_k} \\ (1 - \varepsilon \lambda_k)e^{-\pi \varepsilon \lambda_k} \end{pmatrix}.$$ 

(4.14)

Now substitution of equation (4.14) into equations (4.7), (4.8) with (4.12), (4.13) yields the one-soliton solution of the complex sdSP equation as

$$x^{(1)}_j = x_j - \frac{\lambda_1^{-1}x_j^+ - \lambda_1^{-1}x_j^-}{\lambda_j^+ + \lambda_j^-},$$ 

(4.15)
where

\[ \chi_j^+ = (1 + \varepsilon \lambda_j)^4 \chi_j^+ e^{(\xi_j^+) + \lambda_j^+ \tau + 2\eta}, \quad \chi_j^- = (1 - \varepsilon \lambda_j)^4 \chi_j^- e^{-\xi_j^- - \lambda_j^- \tau - 2\eta}, \]

\[ \psi_j^+ = (1 + \varepsilon \lambda_j)^4 \psi_j^+ e^{(\psi_j^+) - \lambda_j^- \tau}. \] (4.17)

In the continuum limit, replace \( \epsilon \to \delta \epsilon \) and take \( \delta \to 0 \), applying this to equations (4.15) and (4.16), we obtain

\[ x^{(1)} = x - \frac{1}{|\Lambda_1^2|} [\lambda_{1R} \tanh(\eta_1 + 2a_1) - i \lambda_{1I}], \] (4.18)

\[ u^{(1)} = - \frac{\lambda_{1R}}{|\Lambda_1^2|} \mathrm{sech}(\eta_1 + 2a_1) e^{\imath \mu_1}, \] (4.19)

where

\[ \eta_1 = 2\lambda_{1R} x + \frac{\lambda_{1R}}{2 |\Lambda_1^2|} t, \quad \mu_1 = 2\lambda_{1I} x - \frac{\lambda_{1I}}{2 |\Lambda_1^2|} t. \] (4.20)

The one-soliton solutions (4.18), (4.19) has been depicted in figures 1–3. For two soliton, take the matrices \( \mathcal{E}^{(2)}, \mathcal{H}_{j, 1}, \mathcal{H}_{j, 2}, \Lambda_1 \) and \( \Lambda_2 \) to be

\[ \mathcal{E}^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{H}_{j, 1} = \begin{pmatrix} h_{j, 11}^{(1)} & h_{j, 12}^{(2)} \\ h_{j, 21}^{(1)} & h_{j, 22}^{(2)} \end{pmatrix}, \quad \mathcal{H}_{j, 2} = \begin{pmatrix} h_{j, 11}^{(3)} & h_{j, 12}^{(4)} \\ h_{j, 21}^{(3)} & h_{j, 22}^{(4)} \end{pmatrix}, \]

\[ \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix} \] (4.21)

so that the matrices \( \mathcal{F}_j \) and \( \tilde{\mathcal{F}}_j \) become

\[ \mathcal{F}_j = \begin{pmatrix} \mathcal{H}_{j, 1} & \mathcal{H}_{j, 2} \\ \mathcal{H}_{j, 1} \Lambda_1^{-1} & \mathcal{H}_{j, 2} \Lambda_2^{-1} \end{pmatrix}, \quad \tilde{\mathcal{F}}_j = \begin{pmatrix} \mathcal{H}_{j, 1} \Lambda_1^{-2} & \mathcal{H}_{j, 2} \Lambda_2^{-2} \end{pmatrix}. \]

The two-fold DT on the scalar fields \( x_j \) and \( u_j \) are given by

\[ x_j^{(2)} = x_j + \Theta_{j, 11}, \] (4.22)

\[ u_j^{(2)} = \Theta_{j, 12}. \] (4.23)
Now substitute equations (4.24), (4.25) into equations (4.22), (4.23) respectively, to get the two-fold DT on the fields $x_j$ and $u_j$. Further we use $h_{(2)j}^{(2)}, = (-1)^2 h_{(2j)}^{(2l+1)}$, $h_{(2j)}^{(2)l} = (-1)^{2l-1} h_{(2j)}^{(2l-1)}$ and $\lambda_{2l} = -\lambda_{2l-1}$ where $l = 1, \, 2$ in equations (4.22) and (4.23), we obtain two-soliton solutions of the complex sdSP equation. Explicit expression of two-soliton solutions by the approach of continuum limit is given by

$$\begin{align*}
x^{(2)} &= x - \frac{\psi_j}{\varphi_j}, \\
y^{(2)} &= -\frac{\phi_j}{\varphi_j},
\end{align*}$$

where the matrix elements $\Theta_{j,11}^{(2)}$, $\Theta_{j,12}^{(2)}$, can be computed as

$$\begin{align*}
\Theta_{j,11}^{(2)} &= \mathcal{F}_j \left( \mathcal{F}_j^{-1} \right) \\
\Theta_{j,12}^{(2)} &= \left| \begin{array}{cccc}
0 & h_{j,11}^{(2)} & h_{j,12}^{(2)} & h_{j,11}^{(4)} \\
h_{j,21}^{(2)} & h_{j,22}^{(2)} & h_{j,21}^{(4)} & h_{j,22}^{(4)} \\
\lambda_1^{-1} h_{j,11}^{(1)} & \lambda_2^{-1} h_{j,12}^{(1)} & \lambda_3^{-1} h_{j,11}^{(3)} & \lambda_4^{-1} h_{j,12}^{(3)} \\
\lambda_1^{-1} h_{j,21}^{(1)} & \lambda_2^{-1} h_{j,22}^{(1)} & \lambda_3^{-1} h_{j,21}^{(3)} & \lambda_4^{-1} h_{j,22}^{(3)} \\
\lambda_1^{-1} h_{j,11}^{(3)} & \lambda_2^{-1} h_{j,12}^{(3)} & \lambda_3^{-1} h_{j,11}^{(4)} & \lambda_4^{-1} h_{j,12}^{(4)} \\
\lambda_1^{-1} h_{j,21}^{(3)} & \lambda_2^{-1} h_{j,22}^{(3)} & \lambda_3^{-1} h_{j,21}^{(4)} & \lambda_4^{-1} h_{j,22}^{(4)}
\end{array} \right|.
\end{align*}$$

Similarly

$$\begin{align*}
\Theta_{j,12}^{(2)} &= \left| \begin{array}{cccc}
h_{j,11}^{(2)} & h_{j,12}^{(2)} & h_{j,11}^{(4)} & h_{j,12}^{(4)} \\
h_{j,21}^{(2)} & h_{j,22}^{(2)} & h_{j,21}^{(4)} & h_{j,22}^{(4)} \\
\lambda_1^{-1} h_{j,11}^{(1)} & \lambda_2^{-1} h_{j,12}^{(1)} & \lambda_3^{-1} h_{j,11}^{(3)} & \lambda_4^{-1} h_{j,12}^{(3)} \\
\lambda_1^{-1} h_{j,21}^{(1)} & \lambda_2^{-1} h_{j,22}^{(1)} & \lambda_3^{-1} h_{j,21}^{(3)} & \lambda_4^{-1} h_{j,22}^{(3)} \\
\lambda_1^{-1} h_{j,11}^{(3)} & \lambda_2^{-1} h_{j,12}^{(3)} & \lambda_3^{-1} h_{j,11}^{(4)} & \lambda_4^{-1} h_{j,12}^{(4)} \\
\lambda_1^{-1} h_{j,21}^{(3)} & \lambda_2^{-1} h_{j,22}^{(3)} & \lambda_3^{-1} h_{j,21}^{(4)} & \lambda_4^{-1} h_{j,22}^{(4)}
\end{array} \right|.
\end{align*}$$

Figure 2. Smooth one-soliton solution of the complex sdSP equation.
where

$$\psi_l = \lambda_l \tilde{\lambda}_l^{-1} (\lambda_l^1 - \lambda_l^3) (\lambda_l^1 + \tilde{\lambda}_l^{-1}) [\lambda_l^1 e^{\alpha_l} - \tilde{\lambda}_l^{-1} e^{-\alpha_l}] [(\lambda_l^1 + \lambda_l^3) e^{\alpha_l} + (\lambda_l^1 - \lambda_l^3) e^{-\alpha_l}]$$

$$+ \lambda_l \lambda_l^3 (\lambda_l^1 + \tilde{\lambda}_l^{-1}) (\lambda_l^1 + \tilde{\lambda}_l^{-1}) [\lambda_l^1 e^{\beta_l} - \lambda_l^3 e^{-\beta_l}] [(\lambda_l^1 + \lambda_l^3) e^{\beta_l} - (\lambda_l^1 - \lambda_l^3) e^{-\beta_l}]$$

$$- \lambda_l \lambda_l^3 (\lambda_l^1 + \tilde{\lambda}_l^{-1}) (\lambda_l^1 + \tilde{\lambda}_l^{-1}) [\lambda_l^1 e^{\gamma_l} + \lambda_l^3 e^{-\gamma_l}] [(\lambda_l^1 + \lambda_l^3) e^{\gamma_l} + (\lambda_l^1 - \lambda_l^3) e^{-\gamma_l}]$$

$$\phi = (\lambda_l^1 + \tilde{\lambda}_l^{-1}) (\lambda_l^1 - \lambda_l^3) (\lambda_l^1 + \lambda_l^3) [\lambda_l^1 e^{\gamma_l} - \lambda_l^3 e^{-\gamma_l}] e^{\alpha_l + \beta_l}$$

$$+ (\lambda_l^1 + \tilde{\lambda}_l^{-1}) (\lambda_l^1 + \tilde{\lambda}_l^{-1}) [\lambda_l^1 e^{\beta_l} - \lambda_l^3 e^{-\beta_l}] e^{\alpha_l - \gamma_l}$$

$$- (\lambda_l^1 - \tilde{\lambda}_l^{-1}) (\lambda_l^1 + \lambda_l^3) (\lambda_l^1 + \lambda_l^3) [\lambda_l^1 e^{\alpha_l} + \tilde{\lambda}_l^{-1} e^{-\alpha_l}] e^{\gamma_l - \beta_l},$$

$$\varphi = \lambda_l^{-1} (\lambda_l^1 - \lambda_l^3) (\lambda_l^1 + \lambda_l^3) [e^{\gamma_l} + e^{-\gamma_l}] [e^{\alpha_l} + e^{-\alpha_l}] + \lambda_l^3 (\lambda_l^1 + \tilde{\lambda}_l^{-1}) (\lambda_l^1 - \lambda_l^3) [e^{\gamma_l} + e^{-\gamma_l}] [e^{\alpha_l} + e^{-\alpha_l}]$$

$$\times [e^{\beta_l} - e^{-\beta_l}] [e^{\gamma_l} - e^{-\gamma_l}] - \tilde{\lambda}_l^{-1} (\lambda_l^1 + \tilde{\lambda}_l^{-1}) (\lambda_l^1 - \lambda_l^3) [e^{\gamma_l} + e^{-\gamma_l}] [e^{\alpha_l} + e^{-\alpha_l}],$$

(4.28)

where $\theta_k = \lambda_k x + \frac{1}{4} t + \alpha_k$, $k = 1, 3$ and $\alpha_1 = \theta_1 + \tilde{\theta}_2$, $\beta_1 = \theta_1 - \tilde{\theta}_2$, $\gamma_3 = \theta_1 + \tilde{\theta}_2$.

The two-soliton solutions (4.26), (4.27) have been depicted in figures 4–9.

We have shown the interaction process between two-cuspon, two-soliton and two-loop solutions with their respective particular values of the parameters in figures 4–9. For example, figures 4 and 5 describes the interaction of two individual cuspons i.e., each cuspon has its own lump of energy and velocity which remains unchanged after interaction with other cuspons. And by a thrice iterated solutions, we get three-cuspon soliton solutions. The configuration of three cuspons represents elastic scattering of three cuspons with their original shapes and velocities. Similarly, by iterated quasideterminant Darboux matrix solutions to $K$-times, one can obtain multisoliton solutions of the complex sdSP equation.
5. Concluding remarks

In this paper, we have studied the multi-component or matrix generalization of the sdSP equation. We applied DT in order to calculate multi-cuspon, multi-loop and multi-soliton solutions of the system. The \( K \)-time iteration of DT has been carried out with the help of quasideterminants. The profiles of various analytic solutions have been plotted in figures. The solutions obtained in this paper have wide applications in physics and engineering because the SP equation is a physical model of describing the ultra-SPs in nonlinear media.
Moreover, semi-discrete analog of the multi-component SP equation proposed in this paper can be served as an integrable numerical scheme, i.e. self-adaptive moving mesh method used for the numerical simulation of the multi-component SP equation. Further, the kind of techniques used in this paper can also be used to calculate analytical solutions of other semi-discrete and fully discrete integrable systems.

**ORCID iDs**

H Wajahat A Riaz  @ https://orcid.org/0000-0002-9556-6968

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