CERTAIN NON-HOMOGENEOUS MATRICIAL DOMAINS AND PICK–NEVANLINNA INTERPOLATION PROBLEM

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Abstract. In this article, we consider certain matricial domains that are naturally associated to a given domain of the complex plane. A particular example of such domains is the spectral unit ball. We present several results for these matricial domains. Our first result shows—generalizing a result of Ransford-White for the spectral unit ball—that the holomorphic automorphism group of these matricial domains does not act transitively. We also consider 2-point and 3-point Pick–Nevanlinna interpolation problem from the unit disc to these matricial domains. We present results providing necessary conditions for the existence of a holomorphic interpolant for these problems. In particular, we shall observe that these results are generalizations of the results provided by Bharali and Chandel related to these problems.

1. Introduction and statement of results

Let \( \mathbb{D} \) denote the open unit disc in the complex plane \( \mathbb{C} \) centered at 0. Given a domain \( \Omega \subseteq \mathbb{C} \) and \( n \in \mathbb{N}, n \geq 2 \), we define:

\[
S_n(\Omega) := \{ A \in M_n(\mathbb{C}) : \sigma(A) \subset \Omega \}.
\]

Here, \( M_n(\mathbb{C}) \) denotes the set of all \( n \times n \) complex matrices, and \( \sigma(A) \) of a matrix \( A \in M_n(\mathbb{C}) \) denotes the set of eigenvalues of \( A \). It is not difficult to check that \( S_n(\Omega) \) is an open, connected subset of \( M_n(\mathbb{C}) \). Hence, by definition, considered as a subset of \( \mathbb{C}^{n^2} \), it is a domain. In the case \( \Omega = \mathbb{D} \), the set \( S_n(\Omega) \) is known in the literature as the spectral unit ball and is denoted by \( \Omega_n \).

In the first part of this article, we shall establish that the holomorphic automorphism group of \( S_n(\Omega) \), denoted by \( \text{Aut}(S_n(\Omega)) \), does not act transitively on \( S_n(\Omega) \) for certain planar domains \( \Omega \) including all bounded domains. In particular, this result generalizes an analogous result due to Ransford-White (see [16]) about the spectral unit ball \( \Omega_n, n \geq 2 \). To state this result precisely, we need to introduce a few more objects associated naturally to \( S_n(\Omega) \).

First, we consider the symmetrization map \( \pi_n : \mathbb{C}^n \rightarrow \mathbb{C}^n \) defined by:

\[
\pi_n(z) := (\pi_{n,1}(z), \ldots, \pi_{n,j}(z), \ldots, \pi_{n,n}(z)),
\]

where \( \pi_{n,j}(z) \) is the \( j \)-th elementary symmetric polynomial. In other words, if we write \( \pi_{n,j}(z_1, \ldots, z_n) = J_j(z_1, \ldots, z_n) \), then \( J_j \) satisfy:

\[
\prod_{j=1}^{n}(t - z_j) = t^n + \sum_{j=1}^{n}(-1)^j J_j(z_1, \ldots, z_n) t^{n-j}, \quad t \in \mathbb{C}.
\]

The symmetrization map \( \pi_n \) is a proper holomorphic map from \( \mathbb{C}^n \) to \( \mathbb{C}^n \). Consider the \( n \)-th symmetrized product of \( \Omega \), defined by:

\[
\Sigma^n(\Omega) := \pi_n(\Omega^n).
\]
It is easy to see that $\Sigma^n(\Omega)$ is a domain in $\mathbb{C}^n$.

We need to introduce a few more objects before we state our first result. Given a holomorphic self-map $\phi$ of $\Omega$, there is a holomorphic self-map that $\phi$ induces on $\Sigma^n(\Omega)$; namely:

$$\Sigma^n \phi : \Sigma^n(\Omega) \rightarrow \Sigma^n(\Omega)$$

defined by

$$\Sigma^n \phi (\pi_n(z_1, \ldots, z_n)) := \pi_n(\phi(z_1), \ldots, \phi(z_n)) \quad \forall (z_1, \ldots, z_n) \in \Omega^n,$$

where $\pi_n$ is the symmetrization map as before. Given $n \in \mathbb{N}$, $n \geq 2$ and a family $\mathcal{A}_n \subset O(\Sigma(\Omega), \Sigma^n(\Omega))$ of holomorphic self-maps of $\Sigma^n(\Omega)$, consider the following:

(P) for every $\Phi \in \mathcal{A}_n$, there exists a $\phi \in O(\Omega, \Omega)$ such that $\Phi \equiv \Sigma^n \phi$.

Observe that if $\mathcal{B}_n \subset \mathcal{A}_n \subset O(\Sigma^n(\Omega), \Sigma^n(\Omega))$ then it is obvious that if $\mathcal{A}_n$ satisfies the property (P) above then so does $\mathcal{B}_n$. We are now in a position to state our first main theorem as alluded to in the second paragraph above:

**Theorem 1.1.** Let $\Omega \subset \mathbb{C}$ be a domain and let $n \in \mathbb{N}$, $n \geq 2$, be such that we have $\#(\mathbb{C} \setminus \Omega) \geq 2n$. Suppose the holomorphic automorphism group of the $n$-th symmetrized product of $\Omega$, $Aut(\Sigma^n(\Omega))$, has the property (P) above. Then for every $\Psi \in Aut(S_n(\Omega))$ there exists a holomorphic automorphism $\psi$ of $\Omega$ such that

$$\sigma(\Psi(A)) = \psi(\sigma(A)) \quad \forall A \in S_n(\Omega).$$

In particular, $Aut(S_n(\Omega))$ does not act transitively on $S_n(\Omega)$.

We shall present a proof of the above theorem in Section 2. We first record the following important remark regarding Theorem 1.1.

**Remark 1.2.** Let $\Omega \subset \mathbb{C}$ be a bounded domain, then the cardinality condition in the above theorem is trivially satisfied. It is a nontrivial result of Chakrabarty–Gorai [4 Corollary 1.3] — who generalized the analogous result of Edigarian–Zwonek [3] Theorem 1 for $\Sigma^n(\mathbb{D})$ — that the family of proper holomorphic self-maps of $\Sigma^n(\Omega)$ satisfies the property (P) above. Since the later family contains $Aut(\Sigma^n(\Omega))$, we see that for every bounded domain $\Omega$, the conclusion of the above theorem holds true.

The essence of the cardinality condition that features in the statement of the above theorem is that it is the condition that guarantees the Kobayashi hyperbolicity of the domains $\Sigma^n(\Omega)$ (see Section 2 for the definition of Kobayashi hyperbolicity). In fact, as it turns out not only the cardinality condition implies the Kobayashi hyperbolicity of these domains but they also become Kobayashi complete; see [19 Theorem 16]. At this point, we do not know if the cardinality condition itself implies that $Aut(\Sigma^n(\Omega))$ satisfies the condition (P) above.

One could also ask what happens if the map $\Psi$ in the statement of Theorem 1.1 is a proper holomorphic map instead of a holomorphic automorphism? We address this question too here, and we show that under certain restrictions on the domain $\Omega$; namely: $\Omega$ is a hyperconvex domain for which the family of proper holomorphic self-maps of $\Sigma^n(\Omega)$ satisfies the condition (P) above then an analogous result similar to the conclusion of Theorem 1.1 holds true.

Before we state this result, we recall that a domain $D \subset \mathbb{C}^n$ is called hyperconvex if there exists a negative plurisubharmonic exhaustion function on $D$. We shall see in Section 2 that if a domain $\Omega \subset \mathbb{C}$ is hyperconvex then the cardinality of $\mathbb{C} \setminus \Omega$ cannot be finite. In particular, hyperconvex domains satisfy the condition on the cardinality of $\mathbb{C} \setminus \Omega$ as in the statement of Theorem 1.1.

Now we state our second result related to the proper holomorphic self-maps of $S_n(\Omega)$.

**Theorem 1.3.** Let $\Omega \subset \mathbb{C}$ be a hyperconvex domain and let $n \in \mathbb{N}$, $n \geq 2$, be given. Suppose the family of proper holomorphic self-maps of $n$-th symmetrized product of $\Omega$, $\Sigma^n(\Omega)$, satisfies
the condition (P) above. Then for every proper holomorphic map \( \Psi : S_n(\Omega) \to S_n(\Omega) \), there exists a proper holomorphic self-map \( \psi \) of \( \Omega \) such that
\[
\sigma(\Psi(A)) = \psi(\sigma(A)) \quad \forall A \in S_n(\Omega).
\] (1.1)

We shall present a proof of Theorem 1.3 in Section 2. Our proof is motivated from the proof of Theorem 17 in [8]. We state here some important observations relevant to Theorem 1.3.

**Remark 1.4.** Observe due to the result by Chakrabarty–Gorai (see Remark 1.2), the above theorem applies to any bounded hyperconvex domain. The punctured disc \( \mathbb{D}^* := \{ \zeta \in \mathbb{D} : \zeta \neq 0 \} \) is an example of a bounded domain that is not hyperconvex. So, although the family of proper holomorphic self-maps of \( \Sigma^n(\mathbb{D}^*) \) satisfies the property (P), we cannot infer — appealing to the above theorem — that every proper holomorphic self-map \( \Psi \) of \( S_n(\mathbb{D}^*) \) satisfies (1.1) for some proper holomorphic self-map \( \psi \) of \( \mathbb{D}^* \). Indeed, it would be interesting to find a counterexample in this case. We also do not know whether \( \Omega \) being hyperconvex itself implies that the property (P) is satisfied for the family of proper holomorphic self-maps of \( \Sigma^n(\Omega) \).

It is also interesting at this point to recall an Alexander-type of result due to Zwonek for the spectral unit ball \( \Omega_n \). Alexander in [1] proved that all proper holomorphic self-maps of Euclidean unit ball \( \mathbb{B}^n, n \geq 2 \), are biholomorphisms. Zwonek in [19] proved that all proper holomorphic self-maps of \( \Omega_n, n \geq 2 \) are biholomorphisms. The proof as given by Zwonek of this result crucially uses the relation (1.1) and the properties of proper holomorphic maps between analytic varieties. It will be of interest to find out if an Alexander-type result holds true for \( S_n(\Omega) \) where \( \Omega \) is a bounded hyperconvex domain.

**Remark 1.5.** Let \( \Omega \subset \mathbb{C} \) be a domain such that \( \mathbb{C} \setminus \Omega \) has finitely many connected components none of which is a single point. It follows that for a fixed point \( p \in \Omega \), the function \( -g_\Omega(\cdot, p) \), where \( g_\Omega(\cdot, \cdot) \) denotes the Green’s function for \( \Omega \), is a negative subharmonic exhaustion function on \( \Omega \) whence \( \Omega \) is hyperconvex (see [17] Chapter 4). Futhermore, by Koebe’s uniformization theorem for finitely connected domains, these domains are biholomorphic to a domain all of whose boundary components are circle. Hence, by [2, Corollary 1.6], the proper holomorphic self-maps of \( \Sigma^n(\Omega) \) satisfies the property (P) above.

**Remark 1.6.** Let \( \Omega \) be a domain as in Remark 1.5 and let \( p \in \mathbb{N} \) denote the number of connected components of \( \mathbb{C} \setminus \Omega \). Mueller-Rudin showed in [14] that when \( p \geq 3 \) then the set of all proper holomorphic self-maps of \( \Omega \) coincides with the set of all holomorphic automorphism of \( \Omega \), and that the later set is a finite set. In the case when \( p = 2 \), it is a fact that \( \Omega \) is biholomorphic to an annulus \( A_r := \{ z \in \mathbb{C} : r < |z| < 1 \} \) for some \( r, 0 < r < 1 \). In the case of annulus it is also known (see [14] for a reference) that every proper holomorphic self-map is a holomorphic automorphism. In [14], the authors also constructed interesting domains of infinite connectivity that possess non-trivial proper holomorphic self-mappings. The problem about the form of proper holomorphic self-mappings of their symmetrized product is interesting in its own.

The second part of this article is devoted to the following Pick-Nevanlinna interpolation type problem.

- Given \( \{(\zeta_j, W_j) \in \mathbb{D} \times S_n(\Omega) : 1 \leq j \leq N \}, N \geq 2 \) and \( \zeta_j \)'s being distinct, find necessary and sufficient conditions for the existence of a holomorphic map \( F \in \mathcal{O}(\mathbb{D}, S_n(\Omega)) \) such that \( F(\zeta_j) = W_j \) for all \( j = 1, \ldots, N \).

In the case when such a function \( F \) exists, we shall say that \( F \) is an interpolant of the data \( \{(\zeta_j, W_j) \in \mathbb{D} \times S_n(\Omega) : 1 \leq j \leq N \} \).

In this article, we shall only consider the above problem when \( N = 2 \) or \( N = 3 \). Starting with \( N = 2 \), we shall provide a necessary condition for the existence of a holomorphic
interpolant. This necessary condition will remind the reader of the classical Schwarz lemma in one complex variables. But before we state this result, we need to introduce the following pseudo-distance.

The Carathéodory pseudo-distance, denoted by \( C_\Omega \), on a domain \( \Omega \) in \( \mathbb{C} \) is defined by:

\[
C_\Omega(p, q) := \sup \{ \mathcal{M}_\Omega(f(p), f(q)) : f \in \mathcal{O}(\Omega, \mathbb{D}) \}.
\]

(1.2)

Here and elsewhere in this article \( \mathcal{M}_\Omega(z_1, z_2) \) is the Möbius distance between \( z_1 \) and \( z_2 \), defined as:

\[
\mathcal{M}_\Omega(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|} \quad \forall z_1, z_2 \in \mathbb{D}.
\]

The reader will notice that we have defined \( C_\Omega \) in terms of the Möbius distance rather than the hyperbolic distance on \( \mathbb{D} \). This is done purposely because most conclusions in metric geometry that rely on \( C_\Omega \) are essentially unchanged if \( \mathcal{M}_\Omega \) is replaced by the hyperbolic distance on \( \mathbb{D} \) in (1.2), and because the Möbius distance arises naturally in the proofs of our theorems.

A domain \( \Omega \subset \mathbb{C} \) will be called Carathéodory hyperbolic if \( C_\Omega \) is a distance in the sense of metric spaces. It is easy to see that \( \Omega \) is Carathéodory hyperbolic if and only if \( H^\infty(\Omega) \), the set of all bounded holomorphic functions in \( \Omega \), separates points in \( \Omega \); e.g. every bounded domain is Carathéodory hyperbolic. In what follows, we shall always consider domains \( \Omega \) that are Carathéodory hyperbolic.

We now present our first result concerning the interpolation problem above when \( N = 2 \).

**Theorem 1.7.** Let \( F \in \mathcal{O}(\mathbb{D}, S_n(\Omega)) \), \( n \geq 2 \), and let \( \zeta_1, \zeta_2 \in \mathbb{D} \). Write \( W_j = F(\zeta_j) \), and if \( \lambda \in \sigma(W_j) \), then let \( m(\lambda) \) denote the multiplicity of \( \lambda \) as a zero of the minimal polynomial of \( W_j \). Then:

\[
\max \left\{ \max_{\mu \in \sigma(W_1)} \prod_{\lambda \in \sigma(W_1)} C_\Omega(\mu, \lambda)^{m(\lambda)}, \max_{\mu \in \sigma(W_2)} \prod_{\lambda \in \sigma(W_2)} C_\Omega(\lambda, \mu)^{m(\mu)} \right\} \leq \frac{|\zeta_1 - \zeta_2|}{1 - \overline{\zeta_2}\zeta_1}.
\]

(1.3)

We shall present our proof of Theorem 1.7 in Section 4. When \( \Omega = \mathbb{D} \), we know that \( C_\Omega(z_1, z_2) = \mathcal{M}_\Omega(z_1, z_2) \). Substituting this into the inequality (1.3) establishes Theorem 1.5 in [3] which is an important result related to the 2-point interpolation problem from \( \mathbb{D} \) to \( \Omega_n \). Hence Theorem 1.7 gives a generalization of Theorem 1.5 in [3] for all matricial domains \( S_n(\Omega) \).

It is shown in [3] that when \( \Omega = \mathbb{D} \) the above result gives a necessary condition for the existence of an interpolant for the 2-point interpolation problem that is inequivalent to the necessary condition that are known in the literature [3] [15]. Moreover, in this case \( (\Omega = \mathbb{D}) \) when \( n \geq 3 \), there exists a 2-point data set for which (1.3) implies that the data cannot admit an interpolant whereas the condition in [3] [15] are inconclusive.

At this point we wish to discuss an important tool that plays a crucial role in establishing the inequality (1.3) and is at the heart of our next theorem related to the 3-point interpolation problem. We begin with the extremal problem associated to the Carathéodory pseudo-distance \( C_\Omega \) on a domain \( \Omega \) in \( \mathbb{C} \). Recall:

\[
C_\Omega(p, q) := \sup \{ \mathcal{M}_\Omega(f(p), f(q)) : f \in \mathcal{O}(\Omega, \mathbb{D}) \}
\]

(1.4)

\[
= \sup \{ |f(p)| : f \in \mathcal{O}(\Omega, \mathbb{D}) : f(p) = 0 \}.
\]

The equality in (1.4) is due to the fact that the automorphism group of \( \mathbb{D} \) acts transitively on \( \mathbb{D} \) and the Möbius distance is invariant under its action. Applying Montel’s Theorem, it is easy to see that there exists a function \( g \in \mathcal{O}(\Omega, \mathbb{D}) \) such that \( g(p) = 0 \) and \( g(q) = C_\Omega(p, q) \). Such a function is called an extremal solution for the extremal problem determined by (1.4),
It is a fact that for Carathéodory hyperbolic domains there is a unique extremal solution (see the last two paragraphs in [10]). Let us denote by $G_{\Omega}(p, q; \cdot)$ the unique extremal solution determined by the extremal problem (1.4).

**Definition 1.8.** Given $A \in S_n(\Omega)$ and $z \in \Omega \setminus \sigma(A)$, consider the function:

$$B(A, z; \cdot) := \prod_{\lambda \in \sigma(A)} G_{\Omega}(\lambda, z; \cdot)^{m(\lambda)}$$

(1.5)

where $G_{\Omega}(\lambda, z; \cdot)$ is the unique extremal solution corresponding to the pair $(\lambda, z)$ as discussed above and $m(\lambda)$ is the multiplicity of $\lambda \in \sigma(A)$ as a zero of the minimal polynomial of $A$.

Observe that for every $z \in \Omega \setminus \sigma(A)$, $B(A, z; \cdot) \in \mathcal{O}(\Omega, \mathbb{D})$ and has a zero at each $\lambda \in \sigma(A)$ of multiplicity at least $m(\lambda)$. For each $z \in \Omega \setminus \sigma(A)$, $B(A, z; \cdot)$ induces, via the holomorphic functional calculus (which we will discuss in Section 3), a holomorphic map from $S_n(\Omega)$ to $\Omega_n$ that maps $A$ to $0 \in M_n(\mathbb{C})$. This simple trick turns out to be quite important in addressing the 3-point interpolation problem.

Now we are ready to present the result of this article related to the 3-point interpolation problem from $\mathbb{D}$ to $S_n(\Omega)$. In what follows, $B_{j,z}$ will denote the function as defined in (1.5) — as well as its extension to $S_n(\Omega)$ — associated to the matrix $W_j$, $j = 1, 2, 3$.

**Theorem 1.9.** Let $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{D}$ be distinct points and let $W_1, W_2, W_3 \in S_n(\Omega)$, $n \geq 2$. Let $m(j, \lambda)$ denote the multiplicity of $\lambda$ as a zero of the minimal polynomial of $W_j$, $j \in \{1, 2, 3\}$. Given $j, k \in \{1, 2, 3\}$ such that $j \neq k$, $z \in \Omega \setminus \sigma(W_k)$ and $\nu \in \mathbb{D}$, we write:

$$q_z(\nu, j, k) := \max \left\{ \frac{m(j, \lambda) - 1}{\text{ord}_\lambda B_{k,z}'(\nu)} + 1 : \lambda \in \sigma(W_j) \cap B_{k,z}^{-1}(\nu) \right\}.$$

Finally, for each $k \in \{1, 2, 3\}$ let

$$G(k) := \max\{\{1, 2, 3\} \setminus \{k\}\}, \quad L(k) := \min\{\{1, 2, 3\} \setminus \{k\}\}.$$

If there exists a map $F \in \mathcal{O}(\mathbb{D}, S_n(\Omega))$ such that $F(\zeta_j) = W_j$, $j \in \{1, 2, 3\}$, then for each $k \in \{1, 2, 3\}$, and $z \in \Omega \setminus \sigma(W_k)$ we have:

- either $\sigma(B_{k,z}(W_{G(k)})) \subset D(0, |\psi_k(\zeta_{G(k)})|)$, $\sigma(B_{k,z}(W_{L(k)})) \subset D(0, |\psi_k(\zeta_{L(k)})|)$ and

$$\max \left\{ \max_{\nu \in \sigma(B_{k,z}(W_{G(k)}))} \prod_{\mu \in \sigma(B_{k,z}(W_{G(k)}))} \mathcal{M}_{D} \left( \frac{\mu}{\psi_k(\zeta_{G(k)})}, \frac{\nu}{\psi_k(\zeta_{G(k)})} \right) q_z(\nu, G(k), k) \right\} \leq \mathcal{M}_{D} \left( \zeta_{G(k)}, \zeta_{G(k)} \right)$$

$$\max \left\{ \max_{\nu \in \sigma(B_{k,z}(W_{L(k)}))} \prod_{\mu \in \sigma(B_{k,z}(W_{L(k)}))} \mathcal{M}_{D} \left( \frac{\mu}{\psi_k(\zeta_{G(k)})}, \frac{\nu}{\psi_k(\zeta_{G(k)})} \right) q_z(\nu, L(k), k) \right\} \leq \mathcal{M}_{D} \left( \zeta_{L(k)}, \zeta_{G(k)} \right)$$

- or there exists a $\theta_z \in \mathbb{R}$ such that

$$B_{k,z}^{-1}\{e^{i\theta_z} \psi_k(\zeta_{G(k)})\} \subseteq \sigma(W_{G(k)}) \quad \text{and} \quad B_{k,z}^{-1}\{e^{i\theta_z} \psi_k(\zeta_{L(k)})\} \subseteq \sigma(W_{L(k)}).$$

Here, $\psi_j$ denotes the automorphism $\psi_j(\zeta) := (\zeta - \zeta_j)/(1 - \overline{\zeta_j}\zeta)^{-1}$, $\zeta \in \mathbb{D}$, of $\mathbb{D}$ and $[\cdot]$ denotes the greatest-integer function. Given $a \in \mathbb{C}$ and a function $g$ that is holomorphic in a neighbourhood of $a$, $\text{ord}_a g$ will denote the order of vanishing of $g$ at $a$ (with the understanding that $\text{ord}_a g = 0$ if $g$ does not vanish at $a$).

**Remark 1.10.** We shall present our proof of Theorem 1.9 in Section 6. The proof is strongly motivated from the proof of Theorem 1.4 in [5]. In fact, Theorem 1.9 above is a generalization of Theorem 1.4 in [5]. This is shown in Observation 6.1 by explicitly computing the function $B_{k,z}(\cdot)$. It turns out that when $\Omega = \mathbb{D}$ the statement of Theorem 1.9 coincides with the
statement of Theorem 1.4 in [5]. We also refer the reader to Remark 1.5 in [5] for the discussion of how Theorem 1.4 in [5] is different from the other results that are present in the literature related to the 3-point interpolation problem.

2. PRELIMINARIES AND PROOFS OF THEOREM 1.1 AND THEOREM 1.3

In this section, we shall present our proofs of Theorem 1.1 and Theorem 1.3. At the heart of our proofs is a result that itself is quite interesting. It is Lemma 2.2 below. But before we prove this lemma, we shall take a digression and recall the definition of Kobayashi pseudo-distance and Kobayashi hyperbolicity.

2.1. The Kobayashi pseudo-distance. Let $D \subset \mathbb{C}^n$ be a domain and let $h$ denote the hyperbolic distance on $\mathbb{D}$ induced by the Poincaré metric on $\mathbb{D}$. The Kobayashi pseudo-distance $K_D : D \times D \rightarrow [0, \infty)$ is defined by: given two points $p, q \in D$,

$$K_D(p, q) := \inf \left\{ \sum_{i=1}^{n} h(\zeta_{i-1}, \zeta_i) : (\phi_1, \ldots, \phi_n; \zeta_0, \ldots, \zeta_n) \in \mathfrak{A}(p, q) \right\}$$

where $\mathfrak{A}(p, q)$ is the set of all analytic chains in $D$ joining $p$ to $q$. Here, $(\phi_1, \ldots, \phi_n; \zeta_0, \ldots, \zeta_n)$ is an analytic chain in $D$ joining $p$ to $q$ if $\phi_i \in \mathcal{O}(\mathbb{D}, D)$ for each $i$ such that

$$p = \phi_1(\zeta_0), \; \phi_n(\zeta_n) = q \; \text{and} \; \phi_i(\zeta_i) = \phi_{i+1}(\zeta_i)$$

for $i = 1, \ldots, n - 1$.

It follows from the definition that $K_D$ is a pseudo-distance. Using the Schwarz lemma in one complex variable one could see that $K_D \equiv h$. One of the most important properties that the Kobayashi pseudo-distance enjoys is the following: if $F : D_1 \rightarrow D_2$ is a holomorphic map, then $K_{D_2}(F(p), F(q)) \leq K_{D_1}(p, q)$ for all $p, q \in D_1$.

A domain $D \subset \mathbb{C}^n$ is called Kobayashi hyperbolic if the pseudo-distance $K_D$ is a true distance, i.e., $K_D(p, q) = 0$ if and only if $p = q$. The collection of all bounded domains is an example of Kobayashi hyperbolic domains. We refer the interested reader to [12] Chapter 3 for a comprehensive account on Kobayashi pseudo-distance. It is a fact that $K_D \equiv 0$ for all $d \geq 1$. This is not difficult to prove but we skip the proof of this fact here (see [12] Chapter 3). The following is a generalization of Liouville’s Theorem and is obvious:

**Result 2.1.** Let $D \subset \mathbb{C}^n$ be a Kobayashi hyperbolic domain. Let $F : \mathbb{C}^d \rightarrow D$ be a holomorphic map then $F$ is a constant function.

We need one more tool to state the lemma alluded to at the beginning of this section. Given a matrix $A \in M_n(\mathbb{C})$, we write its characteristic polynomial as $\chi(A)(t) = t^n + \sum_{k=1}^{n} (-1)^k \chi_k(A) t^{n-k}$, where $\chi_k(A)$ are polynomials in the entries of $A$. It is obvious that this naturally leads to a map $\chi : S_n(\Omega) \rightarrow \Sigma^n(\Omega)$ defined by:

$$\chi(A) = (\chi_1(A), \ldots, \chi_n(A)),$$

where $\chi_k(A)$’s are as above. Notice that $\chi$ is a holomorphic map from $S_n(\Omega)$ to $\Sigma^n(\Omega)$.

Now we can state the lemma which implies that a holomorphic self-map of $S_n(\Omega)$ preserves the spectra of matrices. More precisely:

**Lemma 2.2.** Let $\Omega \subset \mathbb{C}$ be a domain and let $n \in \mathbb{N}$, $n \geq 2$, be such that $\#(\mathbb{C} \setminus \Omega) \geq 2n$. Let $F : S_n(\Omega) \rightarrow S_n(\Omega)$ be a holomorphic self-map. Then for every $A, B \in S_n(\Omega)$ such that $\chi(A) = \chi(B)$, we have $\chi(F(A)) = \chi(F(B))$.

**Proof.** Let $D$ be a diagonal matrix such that $\chi(D) = \chi(A)$. We know that there exists $C \in M_n(\mathbb{C})$ and an strictly upper triangular matrix $U$ such that

$$A = \exp(-C) (D + U) \exp(C)$$
Now consider the map $f : \mathbb{C} \to M_n(\mathbb{C})$ defined by

$$f(\zeta) := \exp(-C \zeta) (D + \zeta U) \exp(C \zeta) \quad \forall \zeta \in \mathbb{C}. $$

Notice that $\chi(f(\zeta)) = \chi(D + \zeta U) = \chi(D)$, hence $f(\mathbb{C}) \subset S_n(\Omega)$. This lets us define the map $\Psi(\zeta) := \chi \circ F \circ f(\zeta)$ for all $\zeta \in \mathbb{C}$. It is obvious that $\Psi$ is a holomorphic map from $\mathbb{C}$ to $\Sigma^n(\Omega)$.

Under the condition on $\Omega$ as in the statement of the lemma, a result of Zwonek [19] Theorem 16 implies that $\Sigma^n(\Omega)$ is Kobayashi hyperbolic. Then Result 2.1 implies that $\Psi$ is a constant function. Hence, $\Psi(0) = \chi(F(D)) = \Psi(1) = \chi(F(A))$. Proceeding similarly we get $\chi(F(D)) = \chi(F(B))$. This establishes the lemma. \hfill \Box

The proof of the above lemma is motivated from that of Theorem 1 in [16] by Ransford–White. But it is a far reaching generalization of Theorem 1 in [16]; e.g., when $\Omega$ is any Carathéodory hyperbolic domain, then $\#(\mathbb{C} \setminus \Omega)$ cannot be finite. In particular, Carathéodory hyperbolic domains satisfy the condition as in the statement of Lemma 2.2.

We shall now present our proof of Theorem 1.1.

2.2. The proof of Theorem 1.1

Proof. Consider a relation $G$ from $\Sigma^n(\Omega)$ into $\Sigma^n(\Omega)$ defined by:

$$G(X) := \chi \circ \Psi \circ \chi^{-1}(\{X\}) \quad \forall X \in \Sigma^n(\Omega). \quad (2.1)$$

Here $\Psi$ is as in the statement of Theorem 1.1. From Lemma 2.2 it follows that for each $X \in \Sigma^n(\Omega)$, $G(X)$ is a singleton set. Hence $G : \Sigma^n(\Omega) \to \Sigma^n(\Omega)$ is a well defined map.

Claim. $G$ is holomorphic.

To see this, choose an arbitrary $X \in \Sigma^n(\Omega)$ and fix it. Now consider the polynomial $P_X[ t ] := t^n + \sum_{j=1}^{n} (-1)^j X_j t^{n-j}$. We define a map $\tau : \Sigma^n(\Omega) \to M_n(\mathbb{C})$ by setting:

$$\tau(X) := C(P_X) \quad (2.2)$$

where $C(P_X)$ denotes the companion matrix of the polynomial $P_X$. Recall: given a monic polynomial of degree $k$ of the form $p[ t ] = t^k + \sum_{j=1}^{k} a_j t^{k-j}$, where $a_j \in \mathbb{C}$, the companion matrix of $p$ is the matrix $C(p) \in M_k(\mathbb{C})$ given by

$$C(p) := \begin{bmatrix}
0 & a_k \\
1 & 0 & -a_k \\
& \ddots & \ddots \\
& & 1 & -a_{k-1} \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & 1 & -a_1 \\
& & & & & & \ddots & \ddots \\
& & & & & & & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots \\
& & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & & & & & \ddots
\end{bmatrix}_{k \times k}
$$

It is a fact that $\chi(C(p))(t) = p(t)$. From this, it follows that $\tau$ is holomorphic and $\chi \circ \tau = \mathbb{I}$ on $\Sigma^n(\Omega)$. This, in particular, implies that $\tau(X) \in \chi^{-1}\{X\}$. Applying Lemma 2.2 again, we see that $\chi \circ \Psi \circ \chi^{-1}(\{X\}) = \chi \circ \Psi \circ \tau(X)$, i.e., $G(X) = \chi \circ \Psi \circ \tau(X)$. Since each of the maps $\chi, \Psi, \tau$ are holomorphic, the claim follows.

Claim. $G \in Aut(\Sigma^n(\Omega))$.

To see this, consider $H : \Sigma^n(\Omega) \to \Sigma^n(\Omega)$ defined by $H(X) := \chi \circ \Psi^{-1} \circ \chi^{-1}(\{X\})$ for all $X \in \Sigma^n(\Omega)$. Exactly the same argument as above shows that $H(X) = \chi \circ \Psi^{-1} \circ \tau(X)$. From this and that $\chi \circ \tau = \mathbb{I}$ on $\Sigma^n(\Omega)$, we get that $G \circ H = \mathbb{I}, \ H \circ G = \mathbb{I}$. This establishes that $G$ as defined in (2.1) is a holomorphic automorphism of $\Sigma^n(\Omega)$.

Since $Aut(\Sigma^n(\Omega))$ satisfies the property (P), as in the statement of Theorem 1.1 it is not difficult to see that there exists $\psi$, a holomorphic automorphism of $\Omega$, such that:

$$G \circ \pi_n(z_1, \ldots, z_n) := \pi_n(\psi(z_1), \ldots, \psi(z_n))$$

for all $(z_1, \ldots, z_n) \in \Omega^n$. 


Now let $A \in S_n(\Omega)$ be given and suppose $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$ is a list of eigenvalues of $A$ and $\Psi(A)$ respectively, repeated according to their multiplicity as zeros of characteristic polynomial. Then from the definition of $G$ and $\chi$, we have $G \circ \pi_n(\lambda_1, \ldots, \lambda_n) = \pi_n(\mu_1, \ldots, \mu_n)$. On the other hand from the equation above we get $G \circ \pi_n(\lambda_1, \ldots, \lambda_n) = \pi_n(\psi(\lambda_1), \ldots, \psi(\lambda_n))$. This implies $\pi_n(\psi(\lambda_1), \ldots, \psi(\lambda_n)) = \pi_n(\mu_1, \ldots, \mu_n)$. This in particular implies that $\sigma(\Psi(A)) = \psi(\sigma(A))$. Since $A$ is arbitrary, this establishes the conclusion of our theorem.

2.3. A few more Preliminaries. In this subsection, we shall gather a few more tools that are crucial to our proof of Theorem 1.3. We shall present our proof of Theorem 1.3 in the next subsection.

Recall the construction of the function $f$ in the proof of Lemma 2.2. In particular, it shows that for any $A \in S_n(\Omega)$ there exists an entire function $f$ into $S_n(\Omega)$ such that $f(1) = A$ and $f(0) = D$, where $D$ is a diagonal matrix such that $\chi(D) = \chi(A)$. Using this property, we shall prove a proposition regarding the pluri-complex Green function for the domain $S_n(\Omega)$. Before we do this, let us first recall the pluri-complex Green function for a domain in $\mathbb{C}^n$.

Let $D \subset \mathbb{C}^n$ be a domain and let $P := \{(p_j, m_j) \in D \times \mathbb{R}_+ : 1 \leq j \leq N\}$ be a set of poles with $p_j \neq p_k$ when $j \neq k$. Following Lelong [13], we define the pluri-complex Green function with poles in $P$ by:

$$g_D(P; w) := \sup \{ \nu(w) : \nu \in PSH(D, [-\infty, 0]) \text{ and such that } \nu(z) - m_j \log||z - p_j|| \text{ is bounded from above in a neighborhood of } p_j, j = 1, \ldots, N\}$$

Here, $PSH(D, [-\infty, 0])$ denotes the set of all negative pluri-subharmonic functions on $D$. In case $m_j = 1$ for all $j$, we shall write $g_D(p_1, \ldots, p_N; \cdot)$ in place of $g_D(P; \cdot)$. Now we can state and prove the proposition alluded to in the previous paragraph regarding the pluri-complex Green function for $S_n(\Omega)$.

**Proposition 2.3.** Let $A, B \in S_n(\Omega)$. Then for any $D$, a diagonal matrix with $\chi(D) = \chi(B)$ we have:

$$g_{S_n(\Omega)}(A; B) = g_{S_n(\Omega)}(A; D).$$

**Proof.** As discussed above, let $f : \mathbb{C} \rightarrow S_n(\Omega)$ be a holomorphic map such that $f(0) = D$ and $f(1) = B$. Consider now $u : \mathbb{C} \rightarrow [-\infty, 0]$ defined by $u(\zeta) := g_{S_n(\Omega)}(A; f(\zeta))$. Then $u$ is a bounded subharmonic function defined on $\mathbb{C}$. Hence $u$ has to be constant. In particular, $g_{S_n(\Omega)}(A; B) = g_{S_n(\Omega)}(A; f(1)) = g_{S_n(\Omega)}(A; f(0)) = g_{S_n(\Omega)}(A; D)$. \hfill \(\Box\)

Let $\Omega \subset \mathbb{C}$ be a hyperconvex domain. By definition, there is a negative subharmonic exhaustion function $u$ on $\Omega$. In particular, $u$ is a non-constant, bounded above, subharmonic function on $\Omega$. This implies if we let $E = \mathbb{C} \setminus \Omega$ then $E \neq \emptyset$. Moreover, $E$ cannot be a polar set and hence has to be uncountable (see, for instance, [17] Chapter 3).

From the above discussion we see that given a hyperconvex domain $\Omega \subset \mathbb{C}$, and a positive integer $n$, $n \geq 2$, $\#(\mathbb{C} \setminus \Omega) \geq 2n$. Hence, $\Sigma^n(\Omega)$ is Kobayashi hyperbolic for every hyperconvex domain $\Omega$. We also wish to state that Zwonek in the article [19] Proposition 11] proved that $\Sigma^n(\Omega)$ is hyperconvex if and only if $\Omega$ is hyperconvex. Putting all this together what we have is that if $\Omega$ is hyperconvex then for each $n \in \mathbb{N}, n \geq 2$, the domains $\Sigma^n(\Omega)$ are Kobayashi hyperbolic and hyperconvex.

Now we are in a position to present our proof of Theorem 1.3.
2.4. The proof of Theorem 1.3

Proof. We consider the function $G : \Sigma^n(\Omega) \rightarrow \Sigma^n(\Omega)$ defined by $G(X) = \chi \circ \Psi \circ \chi^{-1}\{X\}$. From the discussion above, we know that $\Omega$ satisfies the hypothesis as in Lemma 2.2. Hence by Lemma 2.2 $G$ is well defined. Exactly proceeding as in the proof of Theorem 1.1 we also see that $G$ is holomorphic.

Claim. $G$ is a proper holomorphic self-map of $\Sigma^n(\Omega)$.

To establish this it will be sufficient to prove that if $\{X_\nu\} \subset \Sigma^n(\Omega)$ is a sequence, having no limit points in $\Sigma^n(\Omega)$, then $\{G(X_\nu)\}$ has no limit points in $\Sigma^n(\Omega)$. Assume, on the contrary, a sequence $\{X_\nu\} \subset \Sigma^n(\Omega)$, having no limit points in $\Sigma^n(\Omega)$, such that $\{G(X_\nu)\}$ has a limit point in $\Sigma^n(\Omega)$. This implies that there is a subsequence of $\{G(X_\nu)\}$, that we continue to denote with $\{G(X_\nu)\}$, such that $\{G(X_\nu)\}$ converges to a point $X_0 \in \Sigma^n(\Omega)$.

Recall the map $\tau : \Sigma^n(\Omega) \rightarrow S_n(\Omega)$ as in the proof of Theorem 1.1. Let $\{\lambda_{1,\nu}, \ldots, \lambda_{n,\nu}\}$ be a list of eigenvalues of $\Psi(\tau(X_\nu))$ repeated according to their multiplicity as a zero of the characteristic polynomial of $\Psi(\tau(X_\nu))$. Then, using the property that $\chi \circ \tau \equiv \mathbb{I}$, we have:

$$G(X_\nu) = \chi(\Psi(\tau(X_\nu))) = \chi(\text{diag}[\lambda_{1,\nu}, \ldots, \lambda_{n,\nu}]) = \pi_n(\lambda_{1,\nu}, \ldots, \lambda_{n,\nu})$$

(2.3)

where $\text{diag}[\lambda_{1,\nu}, \ldots, \lambda_{n,\nu}]$ denotes the diagonal matrix with entries $\lambda_{j,\nu}$. Now using the properness of $\pi_n|_{\Omega^n} : \Omega^n \rightarrow \Sigma^n(\Omega)$ and that $\{G(X_\nu)\}$ converges to $X_0 \in \Sigma^n(\Omega)$—there exists a subsequence of $\{\lambda_\nu \in \Omega^n : \lambda_\nu = (\lambda_{1,\nu}, \ldots, \lambda_{n,\nu})\}$, which we continue to denote by $\{\lambda_\nu\}$, such that $\{\lambda_\nu\}$ converges to $\Lambda_0 = (\lambda_{1,0}, \ldots, \lambda_{n,0}) \in \Omega^n$. This, owing to equation (2.3) above, implies:

$$\pi_n(\lambda_{1,0}, \ldots, \lambda_{n,0}) = \chi(\text{diag}[\lambda_{1,0}, \ldots, \lambda_{n,0}]) = X_0.$$  

Let $A \in S_n(\Omega)$ be such that $A$ is not a critical value of $\Psi$. Let $N$ be the multiplicity of $\Psi$ and suppose $\{B_1, \ldots, B_N\} = \Psi^{-1}\{A\}$. The upper semicontinuity of $g_{S_n(\Omega)}$ implies

$$g_{S_n(\Omega)}(A; \text{diag}[\lambda_{1,0}, \ldots, \lambda_{n,0}]) \geq \lim_{\nu \to \infty} g_{S_n(\Omega)}(A; \text{diag}[\lambda_{1,\nu}, \ldots, \lambda_{n,\nu}]).$$

(2.4)

Now by Proposition 2.3 and the behaviour of Green function under proper holomorphic mappings (see [9, Theorem 1.2]) we get:

$$g_{S_n(\Omega)}(A; \text{diag}[\lambda_{1,\nu}, \ldots, \lambda_{n,\nu}]) = g_{S_n(\Omega)}(A; \Psi(\tau(X_\nu)))$$

$$= g_{S_n(\Omega)}(B_1, \ldots, B_N; \tau(X_\nu))$$

$$\geq \sum_{j=1}^{N} g_{S_n(\Omega)}(B_j; \tau(X_\nu)) \geq \sum_{j=1}^{N} g_{\Sigma^n(\Omega)}(\chi(B_j); X_\nu).$$

(2.5)

Since $\Sigma^n(\Omega)$ is hyperconvex and $\{X_\nu\}$ is a sequence that does not have a limit point in $\Sigma^n(\Omega)$, we have $\lim_{\nu \to \infty} g_{\Sigma^n(\Omega)}(\chi(B_j); X_\nu) \rightarrow 0$ for every $j, 1 \leq j \leq N$; see e.g. [13]. From this, inequality (2.5) and (2.4) it follows that $g_{S_n(\Omega)}(A; \text{diag}[\lambda_{1,0}, \ldots, \lambda_{n,0}]) = 0$, which is a contradiction. Hence our assumption that $\{G(X_\nu)\}$ has a subsequence that converges in $\Sigma^n(\Omega)$ is false. This establishes that $G$ is a proper holomorphic self-map of $\Sigma^n(\Omega)$.

Now since the proper holomorphic maps on $\Sigma^n(\Omega)$ satisfy the property (P), there exists $\psi$, a proper holomorphic self-map of $\Omega$, such that $G = \Sigma^n\psi$. Using this and proceeding similarly as in the last paragraph in the proof of Theorem 1.1 we get the desired result. \hfill \Box

3. A brief survey of holomorphic functional calculus

A very essential part of our proofs of Theorem 1.7 and Theorem 1.9 below is the ability, given a domain $\Omega \subset \mathbb{C}$ and a matrix $A \in S_n(\Omega)$, to define $f(A)$ in a meaningful way for each $f \in \mathcal{O}(\Omega)$. Most readers will be aware that this is what is known as the holomorphic
functional calculus. We briefly recapitulate the holomorphic functional calculus and its basic properties in a setting which will be relevant to our proofs in the coming sections.

Throughout this section, \( \mathcal{X} \) will denote a finite dimensional complex Banach space and \( T \) a linear operator in \( \mathcal{B}(\mathcal{X}) := \) the set of all bounded linear operators on \( \mathcal{X} \). The symbol \( \mathbb{I} \) will denote the identity operator and we will interpret \( T^0 = \mathbb{I} \). In what follows, we shall denote by \( \mathbb{C}[t] \) the set of all polynomials with complex coefficients in the indeterminate \( t \). Given a polynomial \( P \in \mathbb{C}[t], \) if we write \( P(t) := \sum_{i=0}^{n} \alpha_i t^i \) with \( \alpha_i \in \mathbb{C}, \) then by \( P(T) \) we will mean the sum \( \sum_{i=0}^{n} \alpha_i T^i \).

For a fix \( T \in \mathcal{B}(\mathcal{X}) \) and \( \lambda \in \mathbb{C}, \) we consider the set \{ \( (\lambda \mathbb{I} - T)^j : j \in \mathbb{N} \). \} If \( \lambda \notin \sigma(T) \), we notice that \( \text{Ker}((\lambda \mathbb{I} - T)^j) = \{0\} \) for each \( j \in \mathbb{N}. \) On the other hand if \( \lambda \in \sigma(T) \) and if we define \( V_\lambda^j := \text{Ker}((\lambda \mathbb{I} - T)^j) \) then we have \( \{0\} \subseteq V_\lambda^1 \subseteq V_\lambda^2 \subseteq \cdots \subseteq V_\lambda^j \subseteq \cdots \). Since \( \mathcal{X} \) is finite dimensional, there is a \( k, k \leq \dim(\mathcal{X}) \), such that \( V_\lambda^k = V_\lambda^{k+1} = V_\lambda^j \) for all \( j \geq k + 1 \).

**Definition 3.1.** Let \( T \in \mathcal{B}(\mathcal{X}) \) and let \( \lambda \in \mathbb{C}. \) Then the index of \( \lambda \), \( m(\lambda) \), is defined by:

\[
m(\lambda) := \min \{ j \in \mathbb{N} : \text{Ker}((\lambda \mathbb{I} - T)^j) = \text{Ker}((\lambda \mathbb{I} - T)^{j+1}) \}.
\]

Notice that \( m(\lambda) = 0 \) if and only if \( \lambda \notin \sigma(T) \). A question arises at this point: given polynomials \( P, Q \in \mathbb{C}[t], \) and \( T \in \mathcal{B}(\mathcal{X}), \) when \( P(T) = Q(T) \)? We state a very important result that among other things primarily provides answer to this question:

**Result 3.2.** [7, Chapter 7, Section 1] Let \( P, Q \in \mathbb{C}[t], \) and let \( T \in \mathcal{B}(\mathcal{X}). \) Then we have \( P(T) = Q(T) \) if and only if each \( \lambda \in \sigma(T) \) is a zero of \( P - Q \) of order \( m(\lambda) \).

We refer the reader to [7, Chapter 7, Section 1] for a proof of this result. The proof as given in [7] also shows that \( m(\lambda) \) is the multiplicity of \( \lambda \) as a zero of the minimal polynomial of \( T \).

Given \( T \in \mathcal{B}(\mathcal{X}), \) let \( \mathcal{F}(T) \) be the set of all holomorphic functions in a neighbourhood of \( \sigma(T). \) Given \( f \in \mathcal{F}(T), \) we define:

\[
f(T) := P(T), \text{ where } P \in \mathbb{C}[t] \text{ with } f^{(j)}(\lambda) = P^{(j)}(\lambda), 0 \leq j \leq m(\lambda) - 1, \forall \lambda \in \sigma(T).
\]

Here, \( f^{(j)} \), \( P^{(j)} \) denotes the \( j \)-th derivative of \( f \) and \( P \) respectively. It follows from Result 3.2 that the definition above is unambiguous and the function \( \Theta_T : \mathcal{F}(T) \to \mathcal{B}(\mathcal{X}) \) defined by \( \Theta_T(f) := f(T) \) has following properties: if \( f, g \in \mathcal{F}(T) \) and \( \alpha, \beta \in \mathbb{C} \) then:

- \( \alpha f + \beta g \in \mathcal{F}(T), \) and \( \Theta_T(\alpha f + \beta g) = \alpha \Theta_T(f) + \beta \Theta_T(g), \)
- \( fg \in \mathcal{F}(T) \) and \( \Theta_T(fg) = \Theta_T(f) \Theta_T(g), \)
- \( \sigma(\Theta_T(f)) = \sigma(f(T)) = f(\sigma(T)). \)

The last property above is called the Spectral Mapping Property in literature. We also note that from the second property above, it follows that \( f(T)g(T) = g(T)f(T) \) for all \( f, g \in \mathcal{F}(T). \)

Given \( p \in \mathbb{C}, \) let \( e_p(\cdot) \) be a function that is identically equal to 1 in a neighbourhood of \( p, \) and identically equal to 0 in a neighbourhood of each point of \( \sigma(T) \cap (\mathbb{C} \setminus \{p\}) \). Set \( E(p) = e_p(T). \) Then we have following:

**Result 3.3.** [7, Chapter 7, Section 1] Given \( T \in \mathcal{B}(\mathcal{X}) \) and \( p \in \mathbb{C}, \) let \( E(p) \in \mathcal{B}(\mathcal{X}) \) be as defined above then we have:

- \( E(p) \neq 0 \) if and only if \( p \in \sigma(T). \)
- \( E(p)^2 = E(p) \) and \( E(p)E(q) = 0 \) for \( p \neq q. \)
- \( \mathbb{I} = \sum_{p \in \sigma(T)} E(p). \)
Remark 3.4. If \( \{\lambda_1, \ldots, \lambda_k\} \) be an enumeration of \( \sigma(T) \), and let \( X_i = E(\lambda_i)X \). Result 3.3 implies that

\[
X = X_1 \oplus \cdots \oplus X_k.
\]

Moreover, since \( TE(\lambda_i) = E(\lambda_i)T \), it follows that \( TX_i \subseteq X_i, i = 1, \ldots, k \). Thus, to the decomposition of the spectrum \( \sigma(T) \) into \( k \) points there corresponds a direct sum decomposition of \( X \) into \( k \) invariant subspaces of \( T \). Thus the study of the action of \( T \) on \( X \) may be reduced to the study of the action of \( T \) on each of the subspaces \( X_i \).

Remark 3.5. If we restrict \( T \) on \( X_i \) and write \( T = \lambda_i I + (T - \lambda_i I) \) on \( X_i \). Then since the holomorphic function \( (t - \lambda_i)^m(\lambda_i) e_{\lambda_i}(t) \) has a zero of order \( m(\lambda_i) \) at each point of \( \sigma(T) \), we have \( (T - \lambda_iI)^m(\lambda_i) E(\lambda_i) = 0 \). Clearly \( (T - \lambda_iI)^{m(\lambda_i)-1} E(\lambda_i) \neq 0 \). This shows that restricted to \( X_i \), the operator \( T - \lambda_i I \) is a nilpotent operator of order \( m(\lambda_i) \). Thus, in each space \( X_i \), the operator \( T \) is the sum of a scalar multiple \( \lambda_i I \) of the identity and a nilpotent operator \( T - \lambda_i I \) of order \( m(\lambda_i) \).

We now state a result that gives an explicit formula for computing \( f(T) \) in terms of the projections \( E(\lambda), \lambda \in \sigma(T) \).

**Result 3.6.** Let \( T \in B(X) \) and let \( f \in \mathcal{F}(T) \) be given. Then

\[
f(T) = \sum_{\lambda \in \sigma(T)} \sum_{j=0}^{m(\lambda) - 1} \frac{(T - \lambda I)^j}{j!} f^{(j)}(\lambda) E(\lambda). \tag{3.1}
\]

The formula (3.1) follows immediately from the properties of the function \( \Theta_T \) described above; once we observe the following: given \( f \in \mathcal{F}(T) \), consider the function \( g \in \mathcal{F}(T) \) defined by:

\[
g(t) := \sum_{\lambda \in \sigma(T)} \sum_{j=0}^{m(\lambda) - 1} \frac{(t - \lambda)^j}{j!} f^{(j)}(\lambda) e_{\lambda}(t),
\]

then we have \( f^{(i)}(\lambda) = g^{(i)}(\lambda), i \leq m(\lambda) - 1 \), for \( \lambda \in \sigma(T) \).

4. **The proof of Theorem 1.7**

We shall present our proof of Theorem 1.7 in this section. Before we present our proof, we shall need certain complex analytic tools related to the spectral unit ball. We first discuss them in the next subsection.

4.1. Complex analytic properties of the spectral unit ball. For \( n \in \mathbb{Z}_+ \), recall the spectral unit ball, \( \Omega_n \subset \mathbb{C}^n \), is the collection of all matrices \( A \in M_n(\mathbb{C}) \) whose spectrum \( \sigma(A) \) is contained in \( \mathbb{D} \). We also recall the definition of spectral radius \( \rho \) of a matrix \( A \) defined by \( \rho(A) := \max \{|\lambda| : \lambda \in \sigma(A)\} \). We first state the result:

**Result 4.1** (Janardhanan, [11]). The spectral unit ball, \( \Omega_n \), is an unbounded, balanced, pseudo-convex domain with Minkowski function given by the spectral radius \( \rho \).

It is a fact that the Minkowski function of a balanced pseudo-convex domain is pluri-subharmonic (see [12], Appendix B.7.6). Hence, it follows from this result that \( \rho|_{\Omega_n} \) is pluri-subharmonic. The pluri-subharmonicity of spectral radius function also follows from another important result due to Vesentini [18]. This latter result regarding the spectral radius function is for a general Banach algebra.

We now state an important lemma for holomorphic functions in \( \mathcal{O}(\mathbb{D}, \Omega_n) \). This lemma could be considered as a generalization of the Schwarz lemma for holomorphic functions in \( \mathcal{O}(\mathbb{D}, \mathbb{D}) \).
**Lemma 4.2.** Let $F \in \mathcal{O}(\mathbb{D}, \Omega_n)$ be such that $F(0) = 0$. Then there exists $G \in \mathcal{O}(\mathbb{D}, \Omega_n)$ such that $F(\zeta) = \zeta G(\zeta)$ for all $\zeta \in \mathbb{D}$. In particular, we have $\rho(F(\zeta)) \leq |\zeta|$ for all $\zeta \in \mathbb{D}$.

The lemma is a consequence of the fact that $\rho|_{\Omega_n}$ is pluri-subharmonic. We do not wish to present a proof of the above lemma here; we refer the interested reader to [5] Lemma 4.3 for a proof. We also wish to state another another lemma which is a consequence of the pluri-subharmonicity of the spectral radius function.

**Lemma 4.3.** Let $\Phi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ be such that there exists a $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \mathbb{D}$ satisfying $e^{i\theta_0} \in \sigma(\Phi(\zeta_0))$. Then $e^{i\theta_0} \in \sigma(\Phi(\zeta))$ for all $\zeta \in \mathbb{D}$.

Lemma 4.3 is not needed in the proof of Theorem 1.7 although it will be an important tool in the proof of Theorem 4.2. We stated it here since it follows from the pluri-subharmonicity of spectral radius function; see e.g. [5] Section 4. We are now ready to present our proof of Theorem 1.7.

### 4.2. The proof of Theorem 1.7

**Proof.** Let $F \in \mathcal{O}(\mathbb{D}, S_n(\Omega))$ be such that $F(\zeta_j) = W_j$, $j = 1, 2$. For each $k \in \{1, 2\}$, consider $\Phi_k \in \mathcal{O}(\mathbb{D}, S_n(\Omega))$ defined by:

$$\Phi_k(\zeta) = F \circ \psi_k^{-1}(\zeta) \quad \forall \zeta \in \mathbb{D}. \quad (4.1)$$

Here, $\psi_k$ is the automorphism $\psi_k(\zeta) := (\zeta - \zeta_k)(1 - \zeta_k \zeta)^{-1}$, $\zeta \in \mathbb{D}$, of $\mathbb{D}$. Then $\Phi_k(0) = W_k$ and $\Phi_k(\psi_k(\zeta_j)) = W_j$, $j \neq k$. Now for an arbitrary but fixed $z \in \Omega \setminus \sigma(W_k)$, consider $B_{k,z} \in \mathcal{O}(\Omega, \mathbb{D})$ defined by:

$$B_{k,z}(\cdot) := \prod_{\lambda \in \sigma(W_k)} G_{\Omega}(\lambda, z; \cdot)^{m(\lambda)}. \quad (4.2)$$

Observe that $B_{k,z}(\cdot) = B(W_k, z; \cdot)$ is as in Definition 1.8.

As $B_{k,z} \in \mathcal{O}(\Omega)$, it induces — via the holomorphic functional calculus — a map (which we continue to denote by $B_{k,z}$) from $S_n(\Omega)$ to $M_n(\mathbb{C})$. The Spectral Mapping Theorem tells us that $\sigma(B_{k,z}(X)) = B_{k,z}(\sigma(X)) \subset \mathbb{D}$ for every $X \in S_n(\Omega)$. Hence $B_{k,z}(X) \subset \Omega_n$ for every $X \in S_n(\Omega)$.

**Claim.** $B_{k,z}(\Phi_k(0)) = 0$.

To see this we write:

$$B_{k,z}(\zeta) = \left( \prod_{\lambda \in \sigma(W_k)} (\zeta - \lambda)^{m(\lambda)} \right) g_z(\zeta)$$

for some $g_z \in \mathcal{O}(\Omega)$ and for every $\zeta \in \Omega$. Hence, since — by the holomorphic functional calculus — the assignment $f \mapsto f(\Phi_k(0))$, $f \in \mathcal{O}(\Omega)$, is multiplicative, as discussed in Section 3 we get

$$B_{k,z}(\Phi_k(0)) = \left( \prod_{\lambda \in \sigma(W_k)} (\Phi_k(0) - \lambda)^{m(\lambda)} \right) g_z(\Phi_k(0)).$$

Now since the minimal polynomial for $W_k = \Phi_k(0)$ is given by $\prod_{\lambda \in \sigma(W_k)} (t - \lambda)^{m(\lambda)}$, we see that the product term in the right hand side of the above equation is zero, whence the claim.

Consider the map $\Psi_{k,z}$ defined by:

$$\Psi_{k,z}(\zeta) := B_{k,z} \circ \Phi_k(\zeta), \quad \zeta \in \mathbb{D}.$$

It is a fact that $\Psi_{k,z} \in \mathcal{O}(\mathbb{D})$. Moreover, from the above claim and the discussion just before it, we have $\Psi_{k,z} \in \mathcal{O}(\mathbb{D}, \Omega_n)$ with $\Psi_{k,z}(0) = 0$. By Lemma 4.2 we get that $\rho(\Psi_{k,z}(\zeta)) \leq |\zeta|$ for all $\zeta \in \mathbb{D}$. 

Now from the definition of $\Psi_{k,z}$ and from the Spectral Mapping Theorem we get $\sigma(\Psi_{k,z}(\zeta)) = \sigma(B_{k,z}(\Phi_k(\zeta)))$. This together with the above equation gives us:

$$|B_{k,z}(\mu)| \leq |\zeta| \quad \forall \zeta \in \mathbb{D} \text{ and } \mu \in \sigma(\Phi_k(\zeta)).$$

We put $\zeta = \psi_k(\zeta_j)$ in the above equation. Then, since $\Phi_k(\psi_k(\zeta_j)) = W_j$, by the above equation and (4.2), we get:

$$|B_{k,z}(\mu)| = \prod_{\lambda \in \sigma(W_k)} |G_{\Omega}(\lambda, z; \mu)|^{m(\lambda)} \leq |\psi_k(\zeta_j)| \quad \forall \mu \in \sigma(W_j). \quad (4.3)$$

Since $z$ is arbitrary, for an arbitrary but fixed $\mu \in \sigma(W_j)$, we can take $z = \mu$ in the above equation. This with the observation that $G_{\Omega}(\lambda, \mu; \mu) = C_{\Omega}(\lambda, \mu)$ gives us that:

$$\prod_{\lambda \in \sigma(W_k)} C_{\Omega}(\lambda, \mu)^{m(\lambda)} \leq \left| \frac{\zeta_1 - \zeta_2}{1 - \zeta_2\zeta_1} \right| \quad \forall \mu \in \sigma(W_j). \quad (4.4)$$

Interchanging the roles of $j$ and $k$, in the above discussion will give an inequality of the form (4.3) with $j$ and $k$ interchanged. The inequality (1.3) will follow from these two inequalities. \hfill \Box

5. Minimal polynomials under holomorphic functional calculus

In this section, we develop the key matricial tool needed in establishing Theorem 1.9 which is the computation of the minimal polynomial for $f(A)$, given $f \in \mathcal{O}(\Omega)$ and $A \in S_n(\Omega)$, $n \geq 2$. This is the content of Theorem 5.2 below. In what follows, given integers $p < q$, $[p..q]$ will denote the set of integers $\{p, p+1, \ldots, q\}$. Given $A \in M_n(\mathbb{C})$, we will denote its minimal polynomial by $M_A$. We begin with a lemma:

**Lemma 5.1** (Chandel, [5]). Let $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{C}^n$, $n \geq 2$. Let $A = \sum_{j=0}^{n-1} \alpha_j N^j$, where $N$ is the nilpotent operator of degree $n$. Then the minimal polynomial for $A$ is given by:

$$M_A(t) = (t - \alpha_0)^{[n-1]/l(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})] + 1}. \quad (5.1)$$

Here, $[\cdot]$ denotes the greatest integer function and $l(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ is defined by:

$$l(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) := \begin{cases} n, & \text{if } \alpha_j = 0 \forall j \in [1..n-1], \\ \min\{j \in [1..n-1] : \alpha_j \neq 0\}, & \text{otherwise}. \end{cases}$$

The reader is referred to [5, Lemma 3.1] for a proof of this lemma.

Given $a \in \mathbb{C}$ and $g$ a holomorphic function in a neighbourhood of $a$, $\text{ord}_a g$ will denote the order of vanishing of $g$ at $a$, as defined after the statement of Theorem 1.9. We now present the main result of this section.

**Theorem 5.2.** Let $A \in S_n(\Omega)$, $n \geq 2$, and let $f \in \mathcal{O}(\Omega)$ be a non-constant function. Suppose that the minimal polynomial for $A$ is given by

$$M_A(t) = \prod_{\lambda \in \sigma(A)} (t - \lambda)^{m(\lambda)}. \quad \text{Then the minimal polynomial for } f(A) \text{ is given by}$$

$$M_{f(A)}(t) = \prod_{\nu \in f(\sigma(A))} (t - \nu)^{k(\nu)},$$

where, $k(\nu) = \max \left\{ \left\lfloor \frac{m(\lambda) - 1}{\text{ord}_\lambda f^{-1}} + 1 : \lambda \in \sigma(A) \cap f^{-1}\{\nu\} \right\rfloor \right\}$.  

Proof. Let $\lambda \in \sigma(A)$ and let $E(\lambda)$ be the projection operator as defined in Section 3 with $T = A$. Then by Remark 3.3 we know that:

$$X = \bigoplus_{\lambda \in \sigma(A)} X_\lambda,$$

where $X_\lambda = E(\lambda)X$, $\lambda \in \sigma(A)$. \hfill (5.2)

Note that $X = \mathbb{C}^n$, but it really does not matter here. Let $x \in X$ and write $x = \bigoplus_{\lambda \in \sigma(A)} x_\lambda$, where $x_\lambda = E(\lambda)x$. Then by Result 3.6 we have:

$$f(A)x = \sum_{\lambda \in \sigma(A)} \left( \sum_{j=0}^{m(\lambda)-1} \frac{(A - \lambda I)^j}{j!} f^{(j)}(\lambda) E(\lambda) \right) x.$$

Now consider the operators $A_\lambda : X_\lambda \longrightarrow X$ defined by:

$$A_\lambda := \sum_{j=0}^{m(\lambda)-1} \frac{(A - \lambda I)^j}{j!} f^{(j)}(\lambda).$$

Since $AX \subseteq X_\lambda$ for all $\lambda \in \sigma(A)$, $A_\lambda$ leaves $X_\lambda$ invariant; i.e. $A_\lambda(X_\lambda) \subseteq X_\lambda$. Hence

$$f(A)x = \bigoplus_{\lambda \in \sigma(A)} A_\lambda x_\lambda,$$

where $x = \bigoplus_{\lambda \in \sigma(A)} x_\lambda$ is as in (5.2), \hfill (5.3)

This establishes that $f(A) = \bigoplus_{\lambda \in \sigma(A)} A_\lambda$.

Focussing our attention to $A_\lambda$, we first observe that – by Remark 3.5 – the operator $A - \lambda I$, $\lambda \in \sigma(A)$, restricted to $X_\lambda$ is a nilpotent operator of degree $m(\lambda) - 1$. Hence by Lemma 5.1 we have $M_{A_\lambda}(t) := (t - f(\lambda))^\nu$, where

$$\nu = \left[ \frac{m(\lambda) - 1}{l(f'(\lambda), f''(\lambda), \ldots, f^{m(\lambda)-1}(\lambda))} \right] + 1.$$

If $\text{ord}_A f' \leq m(\lambda) - 2$, then $l(f'(\lambda), f''(\lambda), \ldots, f^{m(\lambda)-1}(\lambda)) = \text{ord}_A f' + 1$, else $\text{ord}_A f' + 1 > (m(\lambda) - 1)$ and $l(f'(\lambda), f''(\lambda), \ldots, f^{m(\lambda)-1}(\lambda)) > (m(\lambda) - 1)$. In both cases we have:

$$\left[ \frac{m(\lambda) - 1}{l(f'(\lambda), f''(\lambda), \ldots, f^{m(\lambda)-1}(\lambda))} \right] = \left[ \frac{m(\lambda) - 1}{\text{ord}_A f' + 1} \right].$$

From the last two expressions $M_{A_\lambda}(t) = (t - f(\lambda))^\nu$, where

Let us rewrite (5.3) in another way:

$$f(A) = \bigoplus_{\nu \in f(\sigma(A))} B_\nu,$$

where $B_\nu = \bigoplus_{f(\lambda) = \nu} A_\lambda$

The minimal polynomial for $f(A)$, $M_{f(A)}$, is the least common multiple of $M_{B_\nu}$, $\nu \in f(\sigma(A))$. Notice that the polynomials $M_{B_\nu}$, $\nu \in f(\sigma(A))$ are relatively prime to each other. Hence

$$M_{f(A)}(t) = \prod_{\nu \in f(\sigma(A))} M_{B_\nu}(t).$$

Now the minimal polynomial for $B_\nu$ is the least common multiple of minimal polynomials of $A_\lambda$ such that $f(\lambda) = \nu$, $\lambda \in \sigma(A)$. It is easy to see (using the expression for $M_{A_\lambda}$) that

$$M_{B_\nu}(t) = (t - \nu)^{k(\nu)},$$

where $k(\nu) = \max \left\{ \left[ \frac{m(\lambda) - 1}{\text{ord}_A f' + 1} \right] : \lambda \in f^{-1}\{\nu\} \cap \sigma(A) \right\}$

From the last two expressions, we get the desired result. \hfill $\Box$
Remark 5.3. Theorem \[5.2\] is a generalization of Theorem 3.4 in \[5\] which dealt with the case \(\Omega = \mathbb{D}\). The proof of Theorem 3.4 as presented in \[5\] is very specific to the unit disc. This is because the proof of Lemma 3.2—which is a crucial tool in establishing Theorem 3.4 in \[5\]—exploits a property of a holomorphic function in the unit disc; namely: every holomorphic function in \(\mathbb{D}\) has a power series representation on \(\mathbb{D}\). This latter property is very specific to the unit disc.

6. The proof of Theorem 1.9

In this section, we shall present our proof of Theorem 1.9. After the proof, we shall also present an observation that, among other things, shows that when \(\Omega = \mathbb{D}\), Theorem 1.4 in \[5\] is the same as our theorem, in particular, \[5, \text{Theorem 1.4}\] is a special case of Theorem 1.9.

Before we present our proof, we wish to restate the inequality (1.3) when \(\Omega = \mathbb{D}\). In view of (6.2), we have

\[
\max \left\{ \max_{\mu \in \sigma(W_2)} |b_1(\mu)|, \max_{\lambda \in \sigma(W_1)} |b_2(\lambda)| \right\} \leq \left| \frac{\zeta_1 - \zeta_2}{1 - \zeta_1 \zeta_2} \right|.
\]

(6.1)

It is in this form that we shall use the inequality (1.3) in our proof when \(\Omega = \mathbb{D}\). We now present our proof of Theorem 1.9.

6.1. The proof of Theorem 1.9 Let \(F \in \mathcal{O}(\mathbb{D}, S_n(\Omega))\) be such that \(F(\zeta_j) = W_j, j \in \{1, 2, 3\}\). Let \(k \in \{1, 2, 3\}\) and let us denote by \(B_{k, z}(\cdot)\) the function \(B(W_k, z; \cdot)\), where \(z \in \Omega \setminus \sigma(W_k)\) be any arbitrary but fixed point in \(\Omega\). Now consider the function:

\[
\tilde{F}_{k, z} := B_{k, z} \circ F \circ \psi_k^{-1}.
\]

Here \(\psi_k(\zeta) = (\zeta - \zeta_k)/(1 - \overline{\zeta_k} \zeta)\) is an automorphism of the unit disc that maps \(\zeta_k\) to 0. Then \(\tilde{F}_{k, z} \in \mathcal{O}(\mathbb{D},\Omega_n)\) such that \(\tilde{F}_{k, z}(\psi_k(\zeta)) = B_{k, z}(W_k(\zeta))\), \(\tilde{F}_{k, z}(\psi_k(\zeta(\tilde{G}(g)))) = B_{k, z}(W_G(\zeta))\) and \(\tilde{F}_{k, z}(0) = 0\). By Lemma 4.2, we get

\[
\tilde{F}_{k, z}(\zeta) = \zeta \tilde{G}_{k, z}(\zeta) \forall \zeta \in \mathbb{D}, \text{ for some } \tilde{G}_{k, z} \in \mathcal{O}(\mathbb{D}, \overline{\Omega}_n).
\]

(6.2)

Two cases arise:

Case 1. \(\tilde{G}_{k, z}(\mathbb{D}) \subset \Omega_n\).

In view of (6.2), we have

\[
\tilde{G}_{k, z}(\psi_k(\zeta L(k))) = W_L(k),k, z \text{ and } \tilde{G}_{k, z}(\psi_k(\zeta G(k))) = W_G(k),k, z
\]

(6.3)

where \(W_L(k),k, z := B_{k, z}(W_L(k))/\psi_k(\zeta L(k))\) and \(W_G(k),k, z := B_{k, z}(W_G(\zeta))/\psi_k(\zeta G(k))\). Now using the inequality (6.1), a necessary condition for (6.3) is

\[
\max \left\{ \max_{\eta \in \sigma(W_L(k),k, z)} |b_{L}(k,\eta)|, \max_{\eta \in \sigma(W_G(k),k, z)} |b_{L}(k,\eta)| \right\} \leq \mathcal{M}_D \left( \zeta G(k), \zeta L(k) \right),
\]

(6.4)
where \( b_{L(k), k, z} \) and \( b_{G(k), k, z} \) denote the minimal Blaschke product corresponding to the matrices \( W_{L(k), k, z} \), \( W_{G(k), k, z} \). Given the definitions of the latter matrices, we will need Theorem 5.2 to determine \( b_{L(k), k, z}, b_{G(k), k, z} \). By this theorem, we have

\[
b_{L(k), k, z}(t) = \prod_{\nu \in \sigma(B_{k, z}(W_{L(k)}))} \left( \frac{t - \nu/\psi_k(\zeta_{L(k)})}{1 - \nu/\psi_k(\zeta_{L(k)})} \right)^{q_\nu(L(k), k)} \quad (6.5)
\]

\[
b_{G(k), k, z}(t) = \prod_{\nu \in \sigma(B_{k, z}(W_{G(k)}))} \left( \frac{t - \nu/\psi_k(\zeta_{G(k)})}{1 - \nu/\psi_k(\zeta_{G(k)})} \right)^{q_\nu(G(k), k)} \quad (6.6)
\]

where \( q_\nu(L, k, k) \) and \( q_\nu(G, k, k) \) are as in the statement of Theorem 1.9. Now if \( \eta \in \sigma(W_{L(k), k, z}) \) or \( \eta \in \sigma(W_{G(k), k, z}) \), then \( \eta = \mu/\psi_k(\zeta_{L(k)}) \) for some \( \mu \in \sigma(B_{k, z}(W_{L(k)})) \) or \( \eta = \mu/\psi_k(\zeta_{G(k)}) \) for some \( \mu \in \sigma(B_{k, z}(W_{G(k)})) \), respectively, and conversely. This observation together with (6.6), (6.5) and (6.4) establishes the first part of our theorem.

**Case 2.** \( G_{k, z}(\mathbb{D}) \cap \partial \Omega_n \neq \emptyset \).

Let \( \zeta_0 \in \mathbb{D} \) be such that \( e^{i\theta} \in \sigma(G_{k, z}(\zeta_0)) \) for some \( \theta \in \mathbb{R} \). By Lemma 4.3, we have \( e^{i\theta} \in \sigma(G_{k, z}(\zeta)) \) for every \( \zeta \in \mathbb{D} \). By (6.2), \( e^{i\theta} \in \sigma(F_{k, z}(\zeta)) \). Let \( \Phi \equiv F \circ \psi_k^{-1} \). Then \( \Phi \in O(\mathbb{D}, S_n(\Omega)) \) and we have:

\[
e^{i\theta} \zeta \in \sigma(B_{k, z} \sigma(F(\zeta))) = B_{k, z} \{ \sigma(F(\zeta)) \} \quad \forall \zeta \in \mathbb{D},
\]

where the last equality is an application of the Spectral Mapping Theorem. For each \( \zeta \in \mathbb{D} \), let \( \omega_\zeta \in \sigma(F(\zeta)) \) be such that \( B_{k, z}(\omega_\zeta) = e^{i\theta} \). Notice if \( \zeta_1 \neq \zeta_2 \) then \( \omega_{\zeta_1} \neq \omega_{\zeta_2} \), whence \( E := \{ \omega_\zeta : \zeta \in \mathbb{D} \} \) is an uncountable set in \( \Omega \). Notice that \( \omega_\zeta \) satisfies:

\[
B_{k, z}(\omega_\zeta) = e^{i\theta} \zeta \quad \text{and} \quad (\omega_\zeta \ll \Phi(\zeta)) = 0 \quad \forall \zeta \in \mathbb{D}.
\]

As \( E \) is uncountable, it follows from the identity principle that the holomorphic map \( x \mapsto \det(x - \Phi(e^{-i\theta}B_{k, z}(x))) \) is identically 0 on \( \Omega \). As \( B_{k, z} \) maps \( \Omega \) into \( \mathbb{D} \), it follows that

\[
B_{k, z}^{-1} \{ e^{i\theta} \zeta \} \subset \sigma(F(\zeta)) = \sigma(F \circ \psi_k^{-1}(\zeta)) \quad \forall \zeta \in \mathbb{D}. \quad (6.7)
\]

Putting \( \zeta = \psi_k(\zeta_{L(k)}) \) and \( \psi_k(\zeta_{G(k)}) \) respectively in (6.7) we get

\[
B_{k, z}^{-1} \{ e^{i\theta} \psi_k(\zeta_{L(k)}) \} \subset \sigma(F(\zeta_{L(k)})) = \sigma(W_{L(k)}) \quad \text{and} \quad B_{k, z}^{-1} \{ e^{i\theta} \psi_k(\zeta_{G(k)}) \} \subset \sigma(F(\zeta_{G(k)})) = \sigma(W_{G(k)}).
\]

**Observation 6.1.** When \( \Omega = \mathbb{D} \), it is not difficult to show that

\[
G_{\mathbb{D}}(\lambda, z; \zeta) := V(\lambda, z) \frac{\zeta - \lambda}{1 - \lambda \overline{\zeta}}
\]

where \( V(\lambda, z) \in \mathbb{T} \) such that \( V(\lambda, z)((z - \lambda)/(1 - \lambda z)) = M_{\mathbb{D}}(\lambda, z) \). Hence when \( \Omega = \mathbb{D} \), and \( A \in \Omega_n \) then \( B(A, z; \cdot) \) is of the form

\[
B(A, z; \cdot) := R(B(A), z) \prod_{\lambda \in \sigma(A)} \left( \frac{\zeta - \lambda}{1 - \lambda \overline{\zeta}} \right)^{m(\lambda)}.
\]

Here, \( R(B(A), z) \) is a uni-modular constant. In particular, \( B(A, z; \cdot) \) at any point \( z \) is a scalar multiple of the minimal Blaschke product corresponding to \( A \) by a uni-modular constant depending on \( z \).

In this case, given \( W_i \in \Omega_n \), \( i \in \{1, 2, 3\} \) as in the statement of Theorem 1.9, we notice that:

\[
\text{ord}_\lambda B'_{k, z} = \text{ord}_\lambda B'_{k},
\]
where $B_k(\cdot) = B_{W_k}(\cdot)$ is the minimal Blaschke product corresponding to $W_k$. Hence the numbers $q_z(\nu, j, k)$ does not depend on $z$. Also, since $M_D(z_1, z_2) = M_D(e^{i\theta}z_1, e^{i\theta}z_2)$, we have:

$$
\max_{\mu \in \sigma(B_{k_1}(W_{L(k_1)}))} \prod_{\nu \in \sigma(B_{k_2}(W_{G(k_2)}))} M_D\left(\frac{\mu}{\psi_k(\zeta_{L(k_1)})}, \frac{\nu}{\psi_k(\zeta_{G(k_2)})}\right) q(\nu, G(k), k, z)
$$

$$
= \max_{\mu \in \sigma(B_k(W_{L(k)}))} \prod_{\nu \in \sigma(B_k(W_{G(k)}))} M_D\left(\frac{\mu}{\psi_k(\zeta_{L(k)})}, \frac{\nu}{\psi_k(\zeta_{G(k)})}\right) q(\nu, G(k), k).
$$

The above equality is also true when we replace $G(k)$ by $L(k)$ and vice-versa. It follows from this that when $\Omega = \mathbb{D}$, the statement of Theorem 1.4 in [6] reduces to that of Theorem 1.9.

**Remark 6.2.** It would be interesting to investigate for what domains $\Omega$, the statement of Theorem 1.9 does not depend on the choice of point $z \in \Omega$. As we have observed in Observation 6.1 that this is the case when $\Omega = \mathbb{D}$. This question is, of course, related to the behaviour of Carathéodory extremal $G_\Omega(\lambda, z; \cdot)$ as $z$ varies over $\Omega$ and $\lambda$ is any fixed point in $\Omega$.

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