DEGENERATE PARAMETRIC AMPLIFICATION OF SQUEEZED PHOTONS: EXPLICIT SOLUTIONS, STATISTICS, MEANS AND VARIANCES

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Abstract. In the Schrödinger picture, we find explicit solutions for two models of degenerate parametric oscillators in the case of multi-parameter squeezed input photons. The corresponding photon statistics and Wigner's function are also derived in coordinate representation. Their time evolution is investigated in detail. The unitary transformation and an extension of the squeeze operator are briefly discussed.

1. Introduction

A model of the non-degenerate parametric amplifier was introduced and studied in detail in classical papers [73], [50], [94], [95] (see also reviews [55] and [119] for a historical perspective and textbooks/reviews [10], [62], [63], [64], [66], [79], [107] and [116] for a standard paradigm in quantum optics). In a nonlinear dielectric medium, one adds to the linear susceptibility tensor the second and third terms in the power expansion of the polarization in the electric field. This nonlinear polarization couples back to the electric field and a subsequent field quantization results in interaction Hamiltonians that are cubic in the field. Among typical nonlinear effects of this kind are the optical parametric amplification, the second-harmonic generation (both third-order), and the degenerate four-wave mixing (fourth-order) (see [94], [10], [51], [55] and the references therein for more details). Nonlinear media are essential for the generation of squeezed and entangled states of light [5], [16], [17], [48], [76], [78], [91]. The squeezed states and the quantum non-demolition experiment are expected to be utilized in the detection of gravitational waves [18], [11], [30] and also to enhance the performance of optical communication systems [10], [51].

The degenerate parametric amplifier was investigated in [113], [92], [99], [60], [98], [100], [110], [120], [101], [57], [118], [9], [8] in the so-called parametric approximation, when the pump mode is treated classically. Quantum description beyond this approximation can be found in [99], [58], [23], [55] (see also the references therein). Various aspects of the corresponding photon statistics and photon-counting were studied in [49], [108], [109], [93], [18], [19], [24], [114], [6], [25], [115], [7], [41], [42], [61], [85], [86], [87], [88], [38] in detail. Connections with the experimentally observed dynamical Casimir effect [80], [34], [31], [117], [70] are discussed in [33], [31], [59], [46], [96]. All these results are also of interest to the theory of quantum noise and measurement (see, for example, [22], [20], [21] and the references therein). Nowadays, advanced experimental techniques allow one to measure photon correlation functions of input microwave signals [13], to do quantum tomography on itinerant microwave photons [39], and to study squeezing of microwave fields [40], [83] (see also [15], [47] for experimental study of a single-mode thermal field using a microwave parametric amplifier).

1991 Mathematics Subject Classification. Primary 81Q05, 35Q05; Secondary 42A38.

Key words and phrases. Time-dependent Schrödinger equation, degenerate parametric amplifier, Ermakov-type system, Wigner function, Ince equation, differential Galois theory, Kovacic algorithm, interaction picture.
In spite of the considerable literature on the degenerate parametric amplifiers, the general case of multi-parameter squeezed input photons (corresponding, say, to a cascade of nonlinear crystals), to the best of our knowledge, has never been discussed. Traditionally, the interaction picture is commonly used [92], [107], [116] even though the statistics is postulated in the Schrödinger picture [63], [65]. In this article, we study the statistical properties of output squeezed quanta in terms of explicit solutions of certain Ermakov-type system introduced in [71]. In particular, explicit formulas for the mean number and variance of generated photons are found together with the corresponding time-dependent photon statistics. We elaborate on the dynamical aspects of this problem related to evolution of the corresponding photon states in Fock’s space. In order to achieve this goal, we utilize a unified approach to generalized harmonic oscillators discussed in detail in several recent publications [26], [89], [28], [29], [71], [102], [74], [68] (see also [82], [35], [37], [81] for the classical accounts). A similar treatment may be useful for the Josephson metamaterial dynamical Casimir effect [59], [96], [117], [70], [46] (see also [32], [33], [72], [14]), for experimental recognition of squeezed microwave photons [40], [83], and for study of a single-mode thermal microwave field [15], [47].

The paper is organized as follows. In sections 2 and 3, we describe two exactly solvable models of optical degenerate parametric amplifiers. The generalized Fock states are constructed in section 4. The mean and variance of the number operator are evaluated in Schrödinger’s picture in section 5. The eigenfunction expansions of the generalized harmonic states of light in terms of the standard Fock ones are derived in section 6 in coordinate representation. In section 7, the Wigner and Moyal functions of the multi-parameter squeezed states are evaluated directly from the corresponding wavefunctions and their time evolution is verified with the help of a computer algebra system. A brief summary is provided in the end. Our explicit solutions of the corresponding Ermakov-type systems are given in Appendix A together with the means and variances of position and momentum operators. A convenient expansion for the single photon mode Hamiltonian is presented in Appendix B, solutions in interaction picture are briefly discussed in Appendix C, and a canonical transformation of creation and annihilation operators is derived in Appendix D. An attempt to collect relevant references is made.

2. Degenerate Parametric Amplifiers

In this article, we consider the quantization of radiation field in a variable dielectric medium in the Schrödinger picture, as outlined in [54], [12] (in vacuum), by using the method of dynamical invariants originally developed in [35], [37], [82], [81] and recently revisited in [27], [111], [102]. To that end, we follow the mathematical technique of the field quantization for a variable quadratic system in an abstract (Fock-)Hilbert space discussed in [68] and concentrate on a single mode of the radiation field. In this picture, the time evolution of degenerate parametric amplifier is governed by the time-dependent Schrödinger equation for the state vector $| \psi (t) \rangle$:

$$i \frac{d}{dt} | \psi (t) \rangle = \hat{H} (t) | \psi (t) \rangle \quad (2.1)$$

with a certain variable quadratic Hamiltonian considered in original publications [92], [99], [110]. In a more general setting, the degenerate parametric amplification with time-dependent amplitude and phase was discussed by Raiford [100]. The corresponding Hamiltonian, without damping and neglecting high-frequency terms, has the form

$$\hat{H} (t) = \frac{\omega}{2} (\hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a}) - \frac{\lambda (t)}{2} \left( e^{i (2 \omega t + \phi (t))} \hat{a}^2 + e^{-i (2 \omega t + \phi (t))} (\hat{a}^\dagger)^2 \right). \quad (2.2)$$
In this model, the phenomenological coupling parameter $\lambda(t)$, which describes the strength of the interaction between the quantized signal of frequency $\omega$ and the classical pump of frequency $2\omega$, and the pump phase $\phi(t)$ are in general functions of time. (It includes the special case of the pump and signal being off-resonance by a given amount $\epsilon$, i.e., the pump frequency being $2\omega + \epsilon$, by letting $\phi(t) = \epsilon t$ and $\lambda(t) = \lambda$, a constant [100].) One can use the standard annihilation and creation operators for a given mode $\omega$, 

$$\hat{a} = \frac{1}{\sqrt{2\omega}} (\omega \hat{q} + i \hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}} (\omega \hat{q} - i \hat{p}), \quad [\hat{a}, \hat{a}^\dagger] = 1,$$

where $\hat{q}$ and $\hat{p}$ are time-independent operators in an abstract Hilbert space with the canonical commutation relation $[\hat{q}, \hat{p}] = i$ (in the units of $\hbar$). Then

$$\hat{H}(t) = \frac{1}{2} \left( 1 + \frac{\lambda(t)}{\omega} \cos (2\omega t + \phi(t)) \right) \hat{p}^2 + \omega^2 \left( 1 - \frac{\lambda(t)}{\omega} \cos (2\omega t + \phi(t)) \right) \hat{q}^2 + \frac{\lambda(t)}{2} \sin (2\omega t + \phi(t)) (\hat{p} \hat{q} + \hat{q} \hat{p})$$

and the corresponding characteristic equation (classical equation of motion [27], [71]) takes the form [29]:

$$\mu'' + \frac{\lambda \sin (2\omega t + \phi) (2\omega + \phi') - \lambda' \cos (2\omega t + \phi)}{\omega + \lambda \cos (2\omega t + \phi)} \mu' + \frac{\omega (\omega^2 - 3\lambda^2) - \lambda \phi' - \lambda (\omega^2 + \lambda^2 + \omega \phi') \cos (2\omega t + \phi) - \lambda' \omega \sin (2\omega t + \phi)}{\omega + \lambda \cos (2\omega t + \phi)} \mu = 0,$$

which can be thought of as an extension of Ince’s equation [77], [90]. Here, we present two explicit solutions of this model when $\lambda' = 0$ and $\phi = 0, \pi/2$ as usually accepted in the literature. A general case can be discussed in a similar fashion.

3. Two Integrable Cases

For the Hamiltonian (2.4) with $\lambda =$constant and $\phi = 0$ :

$$\hat{H}(t) = \frac{1}{2} \left( 1 + \frac{\lambda}{\omega} \cos (2\omega t) \right) \hat{p}^2 + \omega^2 \left( 1 - \frac{\lambda}{\omega} \cos (2\omega t) \right) \hat{q}^2 + \frac{\lambda}{2} \sin 2\omega t (\hat{p} \hat{q} + \hat{q} \hat{p}),$$

the corresponding Ince’s equation [29]

$$(\omega + \lambda \cos (2\omega t)) \mu'' + 2\lambda \omega \sin (2\omega t) \mu' + (\omega (\omega^2 - 3\lambda^2) - \lambda (\omega^2 + \lambda^2) \cos (2\omega t)) \mu = 0$$

has the following standard solutions:

$$\mu_0(t) = (\sinh \lambda t \cos \omega t + \cosh \lambda t \sin \omega t) / \omega, \quad \mu_1(t) = \cosh \lambda t \cos \omega t + \sinh \lambda t \sin \omega t,$$
which have been recently found with the aid of differential Galois theory \cite{4}, in particular, by using techniques for solving the one-dimensional stationary Schrödinger equation such as algebrization procedure and Kovacic algorithm \cite{2}, \cite{3}. The Wronskian is given by

$$W(\mu_0, \mu_1) = -\frac{1}{\cos 2\omega t}.$$  

Traditionally, Ince’s equation was studied for the sake of periodic solutions, which do not exist for the degenerate parametric oscillators under consideration \cite{29}, \cite{77}.

In the second case, when $\phi = \pi/2$ and

$$\hat{H}(t) = \frac{1}{2} \left( 1 - \frac{\lambda}{\omega} \sin 2\omega t \right) \hat{p}^2 + \frac{\omega^2}{2} \left( 1 + \frac{\lambda}{\omega} \sin 2\omega t \right) \hat{q}^2 + \frac{\lambda}{2} \cos 2\omega t \left( \hat{p} \hat{q} + \hat{q} \hat{p} \right),$$  

we find standard solutions of the corresponding characteristic equation:

$$(\omega - \lambda \sin 2\omega t) \mu'' + 2\lambda \omega \cos 2\omega t \mu'$$

$$+ \left( \omega^2 - 3\lambda^2 \right) + \lambda \left( \omega^2 + \lambda^2 \right) \sin 2\omega t \mu = 0$$

in a similar fashion:

$$\mu_0(t) = \frac{1}{\omega} e^{-\lambda t} \sin \omega t,$$

$$\mu_1(t) = e^{\lambda t} \cos \omega t - \frac{\lambda}{\omega} e^{-\lambda t} \sin \omega t$$

with the Wronskian $W(\mu_0, \mu_1) = 1 - (\lambda/\omega) \sin 2\omega t$.

4. Generalized Fock States for Multi-Parameter Squeezed Photons

The linear dynamical invariants have the form \cite{102} (see also Theorem 1 of Ref. \cite{68} for an abstract setting, which is adapted here): \cite{4}

$$\hat{b}(t) = e^{-2i\gamma(t)} \sqrt{2} \left( \beta(t) \hat{q} + \varepsilon(t) \right) + i \frac{\hat{p} - 2\alpha(t) \hat{q} - \delta(t)}{\beta(t)},$$

$$\hat{b}^\dagger(t) = e^{2i\gamma(t)} \sqrt{2} \left( \beta(t) \hat{q} + \varepsilon(t) \right) - i \frac{\hat{p} - 2\alpha(t) \hat{q} - \delta(t)}{\beta(t)},$$

The real-valued solution of the corresponding Ermakov-type system \cite{71}, \cite{74} (subject to arbitrary real-valued initial data $\alpha(0), \beta(0) \neq 0, \gamma(0), \delta(0), \varepsilon(0), \kappa(0)$), which provides a natural multi-parameter description of the squeezing at $t = 0$ \cite{69}) is given by

$$\alpha(t) = \alpha_0(t) - \beta_0^2(t) \frac{\alpha(0) + \gamma_0(t)}{\beta^4(0) + 4(\alpha(0) + \gamma_0(t))^2},$$

$$\beta(t) = -\frac{\beta(0) \beta_0(t)}{\sqrt{\beta^4(0) + 4(\alpha(0) + \gamma_0(t))^2}},$$

$$\gamma(t) = \gamma(0) - \frac{1}{2} \arctan \frac{\beta^2(0)}{2(\alpha(0) + \gamma_0(t))},$$

\footnote{It represents a general time-dependent Bogoliubov’s transformation of the creation and annihilation operators.}
and

\[ \delta(t) = -\beta_0(t) \frac{\varepsilon(0) \beta^3(0) + 2(\alpha(0) + \gamma_0(t)) \delta(0)}{\beta^4(0) + 4(\alpha(0) + \gamma_0(t))^2}, \tag{4.5} \]

\[ \varepsilon(t) = \frac{2\varepsilon(0)(\alpha(0) + \gamma_0(t)) - \beta(0) \delta(0)}{\sqrt{\beta^4(0) + 4(\alpha(0) + \gamma_0(t))^2}}, \tag{4.6} \]

\[ \kappa(t) = \kappa(0) - \varepsilon(0) \beta^3(0) \frac{\delta(0)}{\beta^4(0) + 4(\alpha(0) + \gamma_0(t))^2} \]

\[ + (\alpha(0) + \gamma_0(t)) \frac{\varepsilon^2(0) \beta^2(0) - \delta^2(0)}{\beta^4(0) + 4(\alpha(0) + \gamma_0(t))^2}, \tag{4.7} \]

in terms of the fundamental solutions:

\[ \alpha_0(t) = \frac{1}{4a(t)} \mu_0'(t) - \frac{d(t)}{2a(t)}, \quad \beta_0(t) = -\frac{1}{\mu_0(t)}, \tag{4.8} \]

\[ \gamma_0(t) = \frac{\mu_1(t)}{2\mu_0(t)} + \frac{d(0)}{2a(0)}. \]

Here,

\[ a(t) = \begin{cases} 
(1 + (\lambda/\omega) \cos 2\omega t) / 2, \\
(1 - (\lambda/\omega) \sin 2\omega t) / 2 
\end{cases} \quad d(t) = \begin{cases} 
(\lambda \sin 2\omega t) / 2, \\
(\lambda \cos 2\omega t) / 2 
\end{cases} \tag{4.9} \]

and

\[ \alpha_0(t) = \begin{cases} 
\frac{\omega \cosh \lambda t \cos \omega t - \sinh \lambda t \sin \omega t}{2 \cosh \lambda t \sin \omega t + \sinh \lambda t \cos \omega t}, \\
\frac{\omega}{2} \cot \omega t 
\end{cases} \tag{4.10} \]

\[ \beta_0(t) = \begin{cases} 
-\frac{\cosh \lambda t \sin \omega t + \sinh \lambda t \cos \omega t}{\omega} \\
-e^{\lambda t} \frac{\omega}{\sin \omega t} 
\end{cases} \tag{4.11} \]

\[ \gamma_0(t) = \begin{cases} 
\frac{\omega \cosh \lambda t \cos \omega t + \sinh \lambda t \sin \omega t}{2 \cosh \lambda t \sin \omega t + \sinh \lambda t \cos \omega t}, \\
\frac{\omega}{2} e^{2\lambda t} \cot \omega t 
\end{cases} \tag{4.12} \]

for the Hamiltonians (3.1) and (3.4), respectively. Equations (4.10)–(4.12) define the corresponding Green’s functions (see [4], [26], [29] and [112] for more details; in the limit \( \omega \to 0 \) for the second Hamiltonian, we obtain an interesting model of a strong coupling; the propagator in the interaction picture is traditionally used in the physical literature [92], [107], [116]). Explicit forms of solutions (4.2)–(4.7) for both Hamiltonians are presented in Appendix A; see (A.1)–(A.8) and (A.9)–(A.14), respectively.

The corresponding dynamical Fock states \( |\psi_n(t)\rangle \), where the phase \( \kappa(t) \) finally shows up, can be obtained from now on in a standard fashion [44], [45], [12] with the aid of our variable creation and annihilation operators [41] (see also [67], in particular, dialogues 8 and 9 and section 3.4, and (D.11)). Under a certain condition, they do satisfy the time-dependent Schrödinger equation

\(^2\text{See also Appendix C for a brief summary.}\)
(2.1) (see [43], Lemma 2 of Ref. [68] and (D.11) for more details). Moreover, the wave functions of degenerate parametric oscillators in coordinate representation are given by equation (18) of [71] in terms of our explicit solutions of the Ermakov-type system; see also (6.1) below.

5. Mean Photon Number, Variances and Identities

The time-dependent variances [68]:
\[
\sigma_p = \langle (\Delta \hat{p})^2 \rangle = \left( n + \frac{1}{2} \right) \frac{4\alpha^2 + \beta^4}{\beta^2}, \quad \sigma_q = \langle (\Delta \hat{q})^2 \rangle = \left( n + \frac{1}{2} \right) \frac{1}{\beta^2},
\]
(5.1)
\[
\sigma_{pq} = \frac{1}{2} \langle \Delta \hat{p} \Delta \hat{q} + \Delta \hat{q} \Delta \hat{p} \rangle = \left( n + \frac{1}{2} \right) \frac{2\alpha}{\beta^2}
\]
with an invariant [38]:
\[
\left| \begin{array}{cc}
\sigma_p & \sigma_{pq} \\
\sigma_{pq} & \sigma_q
\end{array} \right| = \sigma_p \sigma_q - \sigma_{pq}^2 = \left( n + \frac{1}{2} \right)^2
\]
(5.2)
can be evaluated in terms of solutions of the Ermakov-type system for the generalized Fock states (described in general by Lemma 2 of [68]; see also (D.11)). Expressions (4.9)–(4.12) of [68] provide a convenient generic form for all quadratic operators under consideration.

As a result, directly in the Schrödinger picture, the average number of photons for these states \(|\psi_n(t)\rangle\):
\[
\langle \hat{N} \rangle(t) = \langle \hat{a}^\dagger \hat{a} \rangle = \langle \psi_n(t) \left| \frac{1}{2\omega} \left( \hat{p}^2 + \omega^2 \hat{q}^2 \right) - \frac{1}{2} \right| \psi_n(t) \rangle
\]
(5.3)
is given by
\[
\langle \hat{N} \rangle = \left( n + \frac{1}{2} \right) \frac{4\alpha^2 + \beta^4 + \omega^2}{2\omega\beta^2} - \frac{1}{2}
\]
\[
+ \frac{1}{2\omega} \left[ \left( \delta - \frac{2\alpha \varepsilon}{\beta^2} \right)^2 + \frac{\omega^2 \varepsilon^2}{\beta^2} \right].
\]
(5.4)

For the reader’s convenience, a useful expansion of the single photon mode Hamiltonian, \(\hat{H} = (\hat{p}^2 + \omega^2 \hat{q}^2)/2\), in terms of our variable creation and annihilation operators and solutions of the corresponding Ermakov-type system is given in Appendix B.

The general expression for the mean photon number can be significantly simplified for the Hamiltonians (3.1) and (3.4) of the degenerate parametric oscillators under consideration. Indeed, one can get in terms of “slow” variables only:
\[
A(t) = \frac{4\alpha^2 + \beta^4 + \omega^2}{\beta^2}
\]
(5.5)
\[
= \left\{ \begin{array}{l}
\left( 2\alpha(0) + \omega \right)^2 + \beta^4(0) e^{2\lambda t} \\
2\beta^2(0) \\
+ \left( 2\alpha(0) - \omega \right)^2 + \beta^4(0) e^{-2\lambda t} \\
2\beta^2(0) \\
(4\alpha^2(0) + \beta^4(0)) e^{-2\lambda t} + \omega^2 e^{2\lambda t} \\
\beta^2(0)
\end{array} \right.
\]
and
\[ B(t) = \left( \delta - \frac{2\alpha \varepsilon}{\beta} \right)^2 + \frac{\omega^2 \varepsilon^2}{\beta^2} \]
(5.6)

\[ = \begin{cases} 
\frac{1}{2} \left( \delta(0) - \frac{2\alpha(0) \varepsilon(0)}{\beta(0)} - \frac{\omega \varepsilon(0)}{\beta(0)} \right)^2 e^{2\lambda t} \\
+ \frac{1}{2} \left( \delta(0) - \frac{2\alpha(0) \varepsilon(0)}{\beta(0)} + \frac{\omega \varepsilon(0)}{\beta(0)} \right)^2 e^{-2\lambda t}, \\
\left( \delta(0) - \frac{2\alpha(0) \varepsilon(0)}{\beta(0)} \right)^2 e^{-2\lambda t} + \frac{\varepsilon^2(0) \omega^2}{\beta^2(0)} e^{2\lambda t}
\end{cases} 
\]
for (3.1) and (3.4), respectively, with the common invariants
\[ \varepsilon^2 + \frac{\delta^2}{\beta^2} = \varepsilon^2(0) + \frac{\delta^2(0)}{\beta^2(0)} = C, \]
(5.7)
\[ \kappa - \frac{\delta \varepsilon}{2\beta} = \kappa(0) - \frac{\delta(0) \varepsilon(0)}{2\beta(0)} = D. \]
(5.8)

In compact form,
\[ \langle \hat{N} \rangle = \left( n + \frac{1}{2} \right) \frac{A(t)}{2\omega} + \frac{B(t)}{2\omega} - \frac{1}{2} \]
(5.9)

(see also [17], [38], [69] and the references therein).

For the initial coherent state, when \( n = \alpha(0) = 0 \) and \( \beta^2(0) = \omega \), one gets
\[ \langle \hat{N} \rangle = \sinh^2 \lambda t \]
\[ + \begin{cases} 
\frac{1}{2} \left( \varepsilon^2(0) + \frac{\delta^2(0)}{\omega} \right) \cosh 2\lambda t - \frac{\delta(0) \varepsilon(0)}{\sqrt{\omega}} \sinh 2\lambda t & \text{for (3.1)}, \\
\frac{1}{2} \left( \varepsilon^2(0) e^{2\lambda t} + \frac{\delta^2(0)}{\omega} e^{-2\lambda t} \right) & \text{for (3.4)}.
\end{cases} \]
(5.10)

For the vacuum state, when \( \delta(0) = \varepsilon(0) = 0 \), we obtain a familiar expression from the theory of dynamical Casimir effect and spontaneous parametric fluorescence:
\[ \langle \hat{N} \rangle = \sinh^2 \lambda t \]
(5.11)

(see, for example, [32], [33], [34], [62], [119] and the references therein).

The corresponding time-dependent variance:
\[ \text{Var} ~ \hat{N} = \left( \left( \hat{N} - \langle \hat{N} \rangle \right)^2 \right) = \left( \langle \hat{H}/\omega \rangle^2 \right) - \left( \langle \hat{H}/\omega \rangle \right)^2 \]
(5.12)
can also be evaluated in terms of “slow” variables only:
\[ \text{Var} ~ \hat{N} = \frac{A^2(t) - 4\omega^2}{8\omega^2} \left[ \left( n + \frac{1}{2} \right)^2 + \frac{3}{4} \right] + \frac{A(t) B(t) - C}{\omega^2} \left( n + \frac{1}{2} \right) \]
(5.13)
with the help of expansion (B.1). For the initial vacuum state, in particular,

$$\text{Var} \hat{N} = \frac{1}{2} \sinh^2 2\lambda t. \quad (5.14)$$

Moreover, the second-order intensity correlation function [49],

$$g^{(2)} = \frac{\langle (\hat{a}^\dagger)^2 \hat{a}^2 \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = 1 + \frac{\text{Var} \hat{N} - \langle \hat{N} \rangle}{\langle \hat{N} \rangle}, \quad (5.15)$$

is explicitly given in terms of (5.9) and (5.13) for the multi-parameter squeezed number states $$|\psi_n(t)\rangle$$ (see also [6], [115], [61] and [85] for special cases). Explicit expressions for averages of operators $$\hat{q}$$ and $$\hat{p}$$ and for their time-dependent variances are given in Appendix A.

For a complete quantum mechanical description of the nonclassical states of light generated in the process of degenerate parametric amplification, the corresponding photon statistics are required. In vacuum, an explicit connection between the multi-parameter squeezed states and photon distributions is found in our recent publication [69] (in coordinate representation, when $$\hat{q} = x$$ and $$\hat{p} = -i\partial/\partial x$$). It can be readily extended to the models of optical degenerate parametric oscillators under consideration. One should replace $$\alpha \to \alpha/\omega$$, $$\beta \to \beta/\sqrt{\omega}$$, $$\delta \to \delta/\sqrt{\omega}$$ in (6.9)–(6.11) of [69] and a required modification of identities (6.11)–(6.20) is as follows. A joint complex identity,

$$\frac{\delta}{\beta} + i\varepsilon = \left( \frac{\delta(0)}{\beta(0)} + i\varepsilon(0) \right) e^{2i\gamma}, \quad (5.16)$$

holds for both Hamiltonians (3.1) and (3.4), which implies (5.7). Moreover, by separating the “fast”, $$\omega$$, and “slow”, $$\lambda$$, variables,

$$\delta - \frac{2\alpha \varepsilon}{\beta} + i\frac{\omega \varepsilon}{\beta} = e^{-i\omega t} \xi(t), \quad (5.17)$$

where

$$\xi(t) = \left\{ \begin{array}{l}
\left( \delta(0) - \frac{2\alpha(0) \varepsilon(0)}{\beta(0)} \right) \cosh \lambda t - \frac{\omega \varepsilon(0)}{\beta(0)} \sinh \lambda t \\
+ i \left( \frac{\omega \varepsilon(0)}{\beta(0)} \cosh \lambda t - \left( \delta(0) - \frac{2\alpha(0) \varepsilon(0)}{\beta(0)} \right) \sinh \lambda t \right),
\end{array} \right. \quad (5.18)$$

for the Hamiltonian (3.1) and (3.4), respectively. As a by-product, we derive identities (5.6).

In a similar fashion,

$$\frac{\omega + \beta^2}{2} - i\alpha = \frac{1}{2} e^{i\omega t} \frac{\eta(t)}{z(t)}, \quad \omega - \beta^2 + i\alpha = \frac{1}{2} e^{-i\omega t} \frac{\xi(t)}{z(t)}, \quad (5.19)$$

where

$$z(t) = \left\{ \begin{array}{l}
\frac{\omega \cos \omega t + (2\alpha(0) + i\beta^2(0)) \sin \omega t}{\omega} \cosh 2\lambda t \\
+ \frac{(2\alpha(0) + i\beta^2(0)) \cos \omega t + \omega \sin \omega t}{\omega} \sinh 2\lambda t, \\
e^{2\lambda t} \omega \cos \omega t + 2\alpha(0) \sin \omega t + i\beta^2(0) \sin \omega t
\end{array} \right. \quad (5.20)$$
and

\[ \eta(t) = \begin{cases} (\omega + \beta^2(0)) \cosh 2\lambda t + 2\alpha(0) \sinh \lambda t \\ -i(2\alpha(0) \cosh 2\lambda t + (\omega - \beta^2(0)) \sinh \lambda t), \\ -2i\alpha(0) + \beta^2(0) + \omega e^{2\lambda t} \end{cases} \]

\[ \zeta(t) = \begin{cases} (\omega - \beta^2(0)) \cosh 2\lambda t + 2\alpha(0) \sinh \lambda t \\ +i(2\alpha(0) \cosh 2\lambda t + (\omega + \beta^2(0)) \sinh \lambda t), \\ 2i\alpha(0) - \beta^2(0) + \omega e^{2\lambda t} \end{cases} \]

for the Hamiltonians (3.1) and (3.4), respectively. (Computational details are left to the reader; we use Mathematica and Maple to verify our calculations.) Vector \( \mathbf{z}(t) \) is related to the complex parametrization found in [68], [69] (see also [36] and [52]). Having these modifications in mind, in the next section, we will be able to derive variable probability amplitudes and photon statistics in terms of hypergeometric series which are similar to those in [69] (see also [113], [114], [85], [86]).

6. Time-Dependent Probability Amplitudes and Photon Statistics

In coordinate representation, when \( \hat{q} = x \) and \( \hat{p} = -i\partial/\partial x \), the wave functions of the optical degenerate parametric oscillators under consideration take the form:

\[ \psi_n(x,t) = e^{i(\alpha x^2 + \delta x + \kappa)} e^{i(2n+1)\gamma} \sqrt{\frac{\beta}{2^n n! \sqrt{\pi}}} e^{-(\beta x + \varepsilon)^2/2} H_n(\beta x + \varepsilon), \]

where \( H_n(x) \) are the Hermite polynomials [97] and explicit solutions of the corresponding Ermakov-type system are given by (A.1)–(A.6) and (A.9)–(A.14) for the Hamiltonians (3.1) and (3.4), respectively. (An important special case \( \lambda = 0 \) was originally investigated in [84]; see also [36], [75], [74], [69] and the references therein.) In terms of the stationary harmonic oscillator wavefunctions,

\[ \Psi_n(x) = \left( \frac{\omega}{\pi} \right)^{1/4} \frac{e^{-\omega x^2/2}}{\sqrt{2^n n!}} H_n(x\sqrt{\omega}), \]

the eigenfunction expansion has the form [69]:

\[ \psi_n(x,t) = e^{i(2n+1)\gamma} \sqrt{\frac{\beta}{\omega^{1/2}}} \sum_{m=0}^{\infty} C_{mn}(t) \Psi_m(x). \]

The time-dependent coefficients can be found in terms of our solutions of the Ermakov-type system as follows

\[ C_{mn}(t) = \sum_{k=0}^{\infty} M_{mk}(\alpha, \beta) T_{kn}(\varepsilon, \frac{\delta}{\beta}, \kappa), \]

\[ = \sum_{k=0}^{\infty} T_{mk} \left( \frac{\omega^{1/2} \varepsilon}{\beta}, \omega^{-1/2} \left( \frac{\delta - 2\alpha \varepsilon}{\beta} \right), \kappa - \frac{\alpha \varepsilon^2}{\beta^2} \right) M_{kn}(\alpha, \beta). \]

\(^3\)A direct Mathematica verification of these solutions is given by Christoph Koutschan.
Here,

\[ T_{mn}(A, B, \Gamma) = i^{m-n}e^{i(\Gamma-AB/2)}\frac{e^{-\nu/2}}{\sqrt{m!n!}}\left(\frac{IA+B}{\sqrt{2}}\right)^m\left(\frac{IA-B}{\sqrt{2}}\right)^n \]

\times \ \binom{2F_0}{-n, -m; -\frac{1}{\nu}}, \quad \nu = (A^2 + B^2)/2 \tag{6.5} \]

and

\[ M_{mn}(\alpha, \beta) = i^n\frac{\sqrt{2m+n}\omega}{m!n!\pi} \Gamma\left(\frac{m+n+1}{2}\right) \]

\times \ \frac{\left(\frac{\omega-\beta^2}{2} + i\alpha\right)^{m/2}}{\left(\frac{\omega-\beta^2}{2} - i\alpha\right)^{n/2}}\]

\times \ \binom{2F_1}{-m, -n; \frac{1}{2}\left(1 - m - n\right); \frac{1}{2}\left(1 \pm \frac{2i\beta\sqrt{\omega}}{4\alpha^2 + (\beta^2 - \omega)^2}\right)} \tag{6.6} \]

(We use the standard definition of generalized hypergeometric series and an integral evaluated by Bailey [11]; see [69] for more details.)

These general expressions can be significantly simplified, once again, with the help of identities found in the previous section. By separating the “fast” and “slow” variables, one gets

\[ T_{mn}(\varepsilon, \delta, \kappa) = e^{2i(m-n)\gamma} S_{mn}, \quad S_{mn} = T_{mn}(\varepsilon, \delta, \kappa)\bigg|_{t=0}, \tag{6.7} \]

\[ T_{mn}\left(\omega^{1/2}\varepsilon, \omega^{-1/2}\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right), \kappa - \frac{\alpha\varepsilon^2}{\beta^2}\right) = e^{i\omega(m-n)t} R_{mn}(t) \tag{6.8} \]

and

\[ M_{mn}(\alpha, \beta) = e^{-i\omega(m+1/2)t}e^{-i(2n+1)\gamma}\sqrt{\frac{\beta(0)}{\beta}} N_{mn}(t). \tag{6.9} \]

Here, by definition

\[ R_{mn}(t) = \frac{i^{m+n}}{\sqrt{m!n!(2\omega)^{m+n}}} \xi^m(\xi^*)^n \]

\times \ \binom{2F_0}{-n, -m; -\frac{2\omega}{B(t)}} \tag{6.10} \]

and

\[ N_{mn}(t) = i^n\frac{\sqrt{2m+n+1}\omega}{m!n!\pi} \Gamma\left(\frac{m+n+1}{2}\right) \]

\times \ \binom{2F_1}{-m, -n; \frac{1}{2}\left(1 - m - n\right); \frac{1}{2}\left(1 + 2i\sqrt{\frac{\omega}{A(t) - 2\omega}}\right)} \tag{6.11} \]
The asterisk denotes complex conjugation. (Quadratic transformation (6.8) of [69] can be used in order to complete our evaluation; see also [113], [114], [85], [86], [87] and Appendix D. The special case $\lambda = 0$, $\omega = 1$ corrects a typo in [69].)

As a result, we finally obtain

$$\psi_n(x,t) = \sqrt{\beta(0)\omega} \sum_{m=0}^{\infty} c_{mn}(t) e^{-i\omega(m+1/2)t} \Psi_m(x),$$

(6.12)

where the time-dependent probability amplitudes are given by

$$c_{mn}(t) = \sum_{k=0}^{\infty} N_{mk}(t) S_{kn} = \sum_{k=0}^{\infty} R_{mk}(t) N_{kn}(t)$$

(6.13)

in terms of our “slow” variables $\xi, \eta,$ and $\zeta$ (“adiabatic invariants”) only for all real-valued initial data/constants of motion (of the corresponding Ermakov-type system). Thus, the total probability amplitudes are associated with the product of two infinite matrices related to the Poisson and Pascal distributions; see [69] for more details. The quantities $|c_{mn}(t)|^2$ explicitly determine the corresponding variable photon statistics. In mathematical terms, our expansion (6.12) gives a mapping between two complete sets of vectors in $L^2(\mathbb{R})$; namely, the transition matrix (6.13) between the generalized harmonic, or “squeezed” states, $\{\psi_n(x,t)\}_{n=0}^{\infty}$, and the standard Fock ones, $\{e^{-i\omega(m+1/2)t} \Psi_m(x)\}_{n=0}^{\infty}$, in coordinate representation; the latter are usually being recorded by a photodetector in clever quantum nonlinear optics experiments (see, for example, [63], [64], [17], [76], [107], [65], [10], [66], [53] and the references therein).

7. The Wigner and Moyal Functions

In coordinate representation, Wigner’s functions can be derived by following our analysis in the simplest case, when $\lambda = 0$ [69] (see also [38], [56], [103] and [104] for a general approach). For the multi-parameter “dynamical vacuum state”, when $n = 0$, for example, the final result is given by

$$W(x,p,t) = \frac{1}{\pi\sqrt{\omega}} \exp(-Q(U,V)),$$

(7.1)

where

$$Q(U,V) = \frac{\beta^2(0)[\beta(0)U + \omega\xi(0)]^2 + [2\alpha(0)U - \omega(V - \delta(0))]^2}{\beta^2(0)\omega^2}$$

(7.2)

in the rotating $X = \omega x \cos \omega t - p \sin \omega t$, $P = \omega x \sin \omega t + p \sin \omega t$ and “squeezing” coordinates

$$U = \frac{[(X - P) e^{\lambda t} + (X + P) e^{-\lambda t}]}{2},$$

(7.3)

$$V = \frac{[(P - X) e^{\lambda t} + (P + X) e^{-\lambda t}]}{2},$$

(7.4)

in the quantum phase space for the Hamiltonians (3.1) and (3.4), respectively. Our result is consistent with [92], [6], [11], [38] and [110] in the case of the initial vacuum state for the second Hamiltonian. A similar consideration is valid for Moyal’s functions [69], [103]. An example of generation of the squeezed vacuum state is presented in Figure 1.
Figure 1. Phase space animation showing rotation and squeezing of contours $Q(U,V) = 2$ for the first Hamiltonian in (7.2)–(7.4) with $\omega = 1$ and $\lambda = .25$ for the initial vacuum state. The minimum-uncertainty squeezed states occur at $t_{\text{min}} = 0, \pi/4, 3\pi/4, \text{etc.}$ (The color version of this figure is available only in the electronic edition.)

In summary, we have investigated, as explicitly as possible in coordinate representation, all quantum statistical properties of the single mode multi-parameter squeezed photon states in two classical models of optical degenerate parametric oscillators. In principle, one can reproduce our explicit time-dependent photon statistics in a more general operator approach with the help of methods of representation theory (see, for example, [85], [87], [105], [106] and Appendix D). It will be of interest to everyone who studies quantum optics and cavity QED. From the mathematical standpoint, our results motivate further investigations of the Hamiltonian (2.2) and the corresponding generalized Ince equation (2.5). Last but not least, it is worth noting that variable linear terms in creation and annihilation operators, which correspond to a classical current [49], [51], [107], [116], can be easily incorporated into this Hamiltonian in our approach. Lossy medium models may also be taken into consideration.
Acknowledgments. We would like to thank Albert Boggess for help and encouragement. This research was partially supported by AFOSR grant FA9550-11-1-0220. The first named author was partially supported by MICIIN/FEDER MTM2009–06973, gencat 2009SGR859 and DIDI – Universidad del Norte. One of the authors (ES) was also supported by the AMS-Simons Travel Grants, with support provided by the Simons Foundation. Sergei Suslov thanks Marlan O. Scully, Wolfgang Schleich and M. Suhail Zubairy for valuable discussions during his visit to Institute for Quantum Science and Engineering, Texas A&M University. We are grateful to Victor V. Dodonov, Vladimir I. Man’ko, Geza Giedke, Boris A. Malomed, Paulina Marian, Giuseppe Ruoso, Christoph Koutschan, José M. Vega-Guzmán and Andreas Ruffing for valuable comments and important references and to Kamal Barley for graphics enhancement.

Appendix A. Explicit Solutions of Ermakov-type System, Means and Variances

The general solution of Ermakov-type system for the first Hamiltonian \((3.1)\) can be found as follows

\[
\alpha(t) = \frac{\omega}{2L(t)} \left[ (4\alpha^2(0) + \beta^4(0) - \omega^2) \sin 2\omega t \\
+ (4\alpha(0) \omega \cosh 2\lambda t + (4\alpha^2(0) \beta^4(0) + \omega^2) \sinh 2\lambda t) \cos 2\omega t \right],
\]

\[
\beta(t) = \beta(0) \omega \sqrt{\frac{2}{L(t)}},
\]

\[
\gamma(t) = -\frac{1}{2} \arctan \left( \frac{\beta^2(0) (\sinh \lambda t \cos \omega t + \cosh \lambda t \sin \omega t)}{M(t)} \right),
\]

and

\[
\delta(t) = \frac{2\omega}{L(t)} \left[ (\delta(0) \omega \cos \omega t + (2\alpha(0) \delta(0) + \beta^3(0) \varepsilon(0)) \sin \omega t) \cosh \lambda t \\
+ ((2\alpha(0) \delta(0) + \beta^3(0) \varepsilon(0)) \cos \omega t + \delta(0) \omega \sin \omega t) \sinh \lambda t \right],
\]

\[
\varepsilon(t) = \sqrt{\frac{2}{L(t)}} \left[ (\varepsilon(0) \omega \cos \omega t + (2\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) \sin \omega t) \cosh \lambda t \\
+ ((2\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) \cos \omega t + \varepsilon(0) \omega \sin \omega t) \sinh \lambda t \right],
\]

\[
\kappa(t) = \frac{1}{2L(t)} \left[ 2 \left( \beta^3(0) \delta(0) \varepsilon(0) + \alpha(0) \left( \delta^2(0) - \beta^2(0) \varepsilon^2(0) \right) \right) \cos 2\omega t \\
- \left( 2 \left( \beta^3(0) \delta(0) \varepsilon(0) + \alpha(0) \left( \delta^2(0) - \beta^2(0) \varepsilon^2(0) \right) \right) \omega \sin 2\omega t \right) \cosh 2\lambda t \\
+ \left( \left( \delta^2(0) - \beta^2(0) \varepsilon^2(0) \right) \omega \sin 2\omega t \right) \cosh 2\lambda t \\
+ 2 \left( \beta^3(0) \delta(0) \varepsilon(0) + \alpha(0) \left( \delta^2(0) - \beta^2(0) \varepsilon^2(0) \right) \right) \sin 2\omega t \sinh 2\lambda t \right],
\]

where

\[
L(t) = (4\alpha^2(0) + \beta^4(0) + \omega^2 + 4\alpha(0) \omega \sin 2\omega t) \cosh 2\lambda t \\
+ (4\alpha(0) \omega + (4\alpha^2(0) + \beta^4(0) + \omega^2) \sin 2\omega t) \sinh 2\lambda t \\
- (4\alpha^2(0) + \beta^4(0) - \omega^2) \cos 2\omega t.
\]
and

\[ M(t) = (\omega \cos \omega t + 2\alpha(0) \sin \omega t) \cosh \lambda t \]
\[ + (2\alpha(0) \cos \omega t + \omega \sin \omega t) \sinh \lambda t. \]  

For the second Hamiltonian (3.4), in a similar fashion,

\[ \alpha(t) = \frac{\alpha(0) \omega e^{2\lambda t} \cos \omega t + \sin 2\omega t \left( \beta^4(0) + 4\alpha^2(0) - \omega^2 e^{4\lambda t} \right)}{4 \left( \beta^4(0) \sin^2 \omega t + (2\alpha(0) \sin \omega t + \omega e^{2\lambda t} \cos \omega t)^2 \right)^2}, \]  

\[ \beta(t) = \frac{\omega \beta(0) e^{\lambda t}}{\sqrt{\beta^4(0) \sin^2 \omega t + (2\alpha(0) \sin \omega t + \omega e^{2\lambda t} \cos \omega t)^2}}, \]  

\[ \gamma(t) = -\frac{1}{2} \arctan \frac{\beta^2(0)}{2\alpha(0) + \omega e^{2\lambda t} \cot \omega t}, \]  

and

\[ \delta(t) = \omega e^{\lambda t} \varepsilon(0) \beta^3(0) \sin \omega t + (2\alpha(0) \cos \omega t + \omega e^{2\lambda t} \cos \omega t) \delta(0), \]
\[ e(t) = \varepsilon(0) \left( 2\alpha(0) \sin \omega t + \omega e^{2\lambda t} \cos \omega t \right) - \beta(0) \delta(0) \sin \omega t, \]
\[ \kappa(t) = \sin^2 \omega t \varepsilon(0) \beta^2(0) \left( \alpha(0) \varepsilon(0) - \beta(0) \delta(0) \right) - \alpha(0) \beta^2(0) \beta^4(0) \sin^2 \omega t + (2\alpha(0) \sin \omega t + \omega e^{2\lambda t} \cos \omega t)^2 \]
\[ + \frac{\omega e^{2\lambda t} \sin 2\omega t \left( \varepsilon^2(0) \beta^2(0) - \delta^2(0) \right)}{4 \left( \beta^4(0) \sin^2 \omega t + (2\alpha(0) \sin \omega t + \omega e^{2\lambda t} \cos \omega t)^2 \right)^2}. \]

(In both cases, we assume without loss of generality that \( \gamma(0) = \kappa(0) = 0. \))

The means and variances are given by

\[ \langle \hat{q} \rangle = \frac{(2\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) \sin \omega t - \varepsilon(0) \omega \cos \omega t}{\beta(0) \omega} \cosh \lambda t \]  

\[ + \frac{(2\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) \cos \omega t - \varepsilon(0) \omega \sin \omega t}{\beta(0) \omega} \sinh \lambda t, \]  

\[ \langle \hat{p} \rangle = \frac{\varepsilon(0) \omega \sin \omega t - (2\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) \cos \omega t}{\beta(0) \omega} \cosh \lambda t \]  

\[ - \frac{(2\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) \sin \omega t - \varepsilon(0) \omega \cos \omega t}{\beta(0) \omega} \sinh \lambda t, \]  

\[ \sigma_q(t) = \frac{4\alpha^2(0) + \beta^4(0) + \omega^2 + 4\alpha(0) \omega \sin 2\omega t}{4\beta^2(0) \omega^2} \cosh 2\lambda t \]  

\[ + \frac{4\alpha(0) \omega + (4\alpha^2(0) + \beta^4(0) + \omega^2) \sin 2\omega t}{4\beta^2(0) \omega^2} \sinh 2\lambda t \]  

\[ - \frac{4\alpha^2(0) + \beta^4(0) - \omega^2}{4\beta^2(0) \omega^2} \cos 2\omega t, \]
\[
\sigma_p(t) = \frac{4\alpha^2(0) + \beta^4(0) + \omega^2 - 4\alpha(0)\omega \sin 2\omega t}{4\beta^2(0)} \cosh 2\lambda t
\]

\[
+ \frac{4\alpha(0)\omega - (4\alpha^2(0) + \beta^4(0) + \omega^2)\sin 2\omega t}{4\beta^2(0)} \sinh 2\lambda t
\]

\[
+ \frac{4\alpha^2(0) + \beta^4(0) - \omega^2}{4\beta^2(0)} \cos 2\omega t
\]

and

\[
\langle \hat{q} \rangle = -\frac{\varepsilon(0)}{\beta(0)} e^{\lambda t} \cos \omega t + \frac{1}{\omega} \left( \delta(0) - \frac{2\alpha(0)}{\beta(0)} \varepsilon(0) \right) e^{-\lambda t} \sin \omega t,
\]

\[
\langle \hat{p} \rangle = \left( \delta(0) - \frac{2\alpha(0)}{\beta(0)} \varepsilon(0) \right) e^{-\lambda t} \cos \omega t + \frac{\varepsilon(0)}{\beta(0)} e^{\lambda t} \sin \omega t.
\]

\[
\sigma_q(t) = \frac{(4\alpha^2(0) + \beta^4(0)) e^{-2\lambda t} + \omega^2 e^{2\lambda t}}{4\beta^2(0) \omega^2} + \frac{\alpha(0)}{\beta^2(0)} \sin 2\omega t
\]

\[
- \frac{(4\alpha^2(0) + \beta^4(0)) e^{-2\lambda t} - \omega^2 e^{2\lambda t}}{4\beta^2(0) \omega^2} \cos 2\omega t,
\]

\[
\sigma_p(t) = \frac{(4\alpha^2(0) + \beta^4(0)) e^{-2\lambda t} + \omega^2 e^{2\lambda t}}{4\beta^2(0) \omega^2} - \frac{\alpha(0)\omega}{\beta^2(0)} \sin 2\omega t
\]

\[
+ \frac{(4\alpha^2(0) + \beta^4(0)) e^{-2\lambda t} - \omega^2 e^{2\lambda t}}{4\beta^2(0) \omega^2} \cos 2\omega t
\]

for the Hamiltonians (3.1) and (3.4), respectively.

In this article, we take the initial data of the corresponding Ermakov-type systems as constants of motion (that naturally describes multi-parameter squeezing). In coordinate representation, their physical meaning is clear from the explicit wave function (6.1) when \( t = 0 \). Another useful set of integrals of motion consists of the expectation values and variances at \( t = 0 \) : \( \langle \hat{q} \rangle|_{t=0} \), \( \langle \hat{p} \rangle|_{t=0} \), \( \sigma_q(0)|_{n=0} \), \( \sigma_p(0)|_{n=0} \) and \( \sigma_{pq}(0)|_{n=0} \); the latter are related by (5.2) (see Ref. [38] for more details).

**Appendix B. Hamiltonian Expansion**

The single mode Hamiltonian, \( \hat{H} = (\hat{p}^2 + \omega^2 \hat{q}^2) / 2 \), can be rewritten as follows

\[
\hat{H} = \left( \frac{4\alpha^2 - \beta^2 + \omega^2}{4\beta^2} - i\alpha \right) \hat{a}(t)^2 + \left( \frac{4\alpha^2 - \beta^2 + \omega^2}{4\beta^2} + i\alpha \right) \hat{a}^\dagger(t)^2
\]

\[
+ \frac{4\alpha^2 + \beta^2 + \omega^2}{4\beta^2} \left[ \hat{a}(t) \hat{a}^\dagger(t) + \hat{a}^\dagger(t) \hat{a}(t) \right]
\]

\[
+ \sqrt{2} \left[ \frac{\alpha}{\beta} \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) - \frac{\varepsilon\omega^2}{2\beta^2} - \frac{i\beta}{2} \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) \right] \hat{a}(t)
\]

\[
+ \sqrt{2} \left[ \frac{\alpha}{\beta} \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) - \frac{\varepsilon\omega^2}{2\beta^2} + \frac{i\beta}{2} \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) \right] \hat{a}^\dagger(t)
\]

\[
+ \frac{1}{2} \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right)^2 + \frac{\varepsilon^2\omega^2}{2\beta^2}
\]
in terms of variable creation and annihilation operators
\( \hat{a} (t) = e^{2i\gamma t} \hat{b} (t) \) and \( \hat{a}^\dagger (t) = e^{-2i\gamma t} \hat{b}^\dagger (t) \) which is convenient for evaluation of the corresponding matrix elements in section 5. More details can be found in [68], [69], [102].

**Appendix C. Interaction Picture**

Formally, from (2.1)–(2.2) one gets,
\[
\psi = e^{-i(\hat{a}^\dagger \hat{a} + 1/2) \omega t} \chi, \quad i\chi_t = \hat{V}\chi,
\]
where
\[
\hat{V} = \frac{-\lambda(t)}{2} e^{i(\hat{a}^\dagger \hat{a})\omega t} \left( e^{i(2\omega t + \phi(t))} \hat{a}^2 + e^{-i(2\omega t + \phi(t))} (\hat{a}^\dagger)^2 \right) e^{-i(\hat{a}^\dagger \hat{a})\omega t}
\]
(C.2)
with the help of a familiar formal identity
\[
e^{i(\hat{a}^\dagger \hat{a})\omega t} \hat{a} e^{-i(\hat{a}^\dagger \hat{a})\omega t} = \hat{a} e^{-i\omega t}
\]
(C.3)
and its conjugate. As a result, in interaction picture:
\[
\hat{V} = \begin{cases}
-\lambda \left( \hat{a}^2 + (\hat{a}^\dagger)^2 \right) / 2 = \lambda \left( \hat{p}^2 - \omega^2 \hat{q}^2 \right) / (2\omega) , \\
\lambda \left( \hat{a}^2 - (\hat{a}^\dagger)^2 \right) / (2i) = \lambda (\hat{p} \hat{q} - \hat{q} \hat{p}) / 2
\end{cases}
\]
(C.4)
for the original Hamiltonians (3.1) and (3.4), respectively.

In coordinate representation, the formal substitution (C.1) takes the form
\[
\psi (x, t) = \int_{-\infty}^{\infty} G_\omega (x, y, t) \chi (y, t) \, dy
\]
(C.5)
in \( L^2 (\mathbb{R}) \), where
\[
G_\omega (x, y, t) = \frac{\omega}{2\pi i \sin \omega t} \exp \left( i\omega \frac{(x^2 + y^2) \cos \omega t - 2xy}{2 \sin \omega t} \right)
\]
(C.6)
is a familiar Green’s function for quantum harmonic oscillator.

In interaction picture, solutions of the corresponding Cauchy initial value problems for wave function \( \chi (x, t) \) can be found as follows
\[
i\chi_t + \frac{\lambda}{2\omega} \left( \chi_{xx} + \omega^2 x^2 \chi \right) = 0, \quad \chi (x, t) = \int_{-\infty}^{\infty} G_\lambda (x, y, t) \psi_0 (y) \, dy,
\]
(C.7)
where
\[
G_\lambda (x, y, t) = \frac{\omega}{2\pi i \sinh \lambda t} \exp \left( i\omega \frac{(x^2 + y^2) \cosh \lambda t - 2xy}{2 \sinh \lambda t} \right),
\]
(C.8)
and
\[
\chi_t + \lambda x \chi_x + \frac{\lambda}{2} \chi = 0, \quad \chi (x, t) = e^{-\lambda t/2} \psi_0 (xe^{-\lambda t}),
\]
(C.9)
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in the first case and the second cases, respectively. This consideration gives an independent verification of our Green’s functions

\[ G(x, y, t) = \frac{e^{i(\alpha_0 x^2 + \beta_0 xy + \gamma_0 y^2)}}{\sqrt{2\pi i \mu_0}} = \begin{cases} \int_{-\infty}^{\infty} G_{\omega}(x, z, t) G_{\lambda}(z, y, t) \, dz, \\ e^{\lambda t/2} G_{\omega}(x, ye^{\lambda t}, t) \end{cases} \]  

(C.10)

for the Hamiltonians (3.1) and (3.4), respectively; cf. Eqs. (4.10)–(4.12).

APPENDIX D. Canonical Transformation

Two familiar identities are valid:

\[ e^{\xi^* \hat{a} - \xi \hat{a}^\dagger} e^{-\xi^* \hat{a} + \xi \hat{a}^\dagger} = \hat{a} + \xi, \]  

(D.1)

\[ e^{(e^{2iv} \hat{a}^2 - e^{-2iv} (\hat{a}^\dagger)^2)\tau/2} \hat{a} e^{- (e^{2iv} \hat{a}^2 - e^{-2iv} (\hat{a}^\dagger)^2)\tau/2} = (\cosh \tau) \hat{a} + e^{-2i\varphi} (\sinh \tau) \hat{a}^\dagger \]  

(D.2)

formally for the single-mode displacement \[49\] and squeeze \[108, 109\] operators, respectively (see, for example, \[105, 106\] for more details). With the aid of (C.3) and (D.1)–(D.2), one can verify that the canonical transformation (4.1), namely,

\[ \hat{b}(t) = U(t) \hat{a} U^{-1}(t) = e^{\frac{-2i\gamma}{\sqrt{2}} \left( \beta \hat{q} + \varepsilon + i (\beta - 2\alpha \hat{q} - \delta) \right)} \]  

(D.3)

holds for the unitary operator of the form:

\[ U(t) = e^{i(\hat{a}^\dagger \hat{a})\theta} e^{\left( e^{2iv} \hat{a}^2 - e^{-2iv} (\hat{a}^\dagger)^2 \right)\tau/2} e^{\xi^* \hat{a} - \xi \hat{a}^\dagger} e^{2i(\hat{a}^\dagger \hat{a})\gamma} \]  

(D.4)

provided

\[ \frac{1}{\sqrt{\omega}} \left( \beta - \frac{2i\alpha}{\beta} \right) + \frac{\sqrt{\omega}}{\beta} = 2e^{-i\theta} \cosh \tau, \]  

(D.5)

\[ \frac{1}{\sqrt{\omega}} \left( \beta - \frac{2i\alpha}{\beta} \right) - \frac{\sqrt{\omega}}{\beta} = 2e^{i(\theta - 2\varphi)} \sinh \tau, \]  

(D.6)

and

\[ \xi \sqrt{2} = \varepsilon - i \frac{\delta}{\beta} = -i \left( \frac{\delta(0)}{\beta(0)} + i \varepsilon(0) \right) e^{2iv}. \]  

(D.7)

Therefore, the time-dependent parameters of our single-mode “multi-parameter squeeze operator” \[D.4\], namely, \(\theta(t), \tau(t), \varphi(t)\) and \(\xi(t)\), are determined in terms of solutions of the corresponding Ermakov-type system as follows

\[ \tan \theta(t) = \frac{2\alpha}{\beta^2 + \omega}, \quad \tan 2\varphi(t) = \frac{4\alpha \beta^2}{4\alpha^2 + \omega^2 - \beta^2}, \]  

(D.8)

\[ 4 \left[ \cosh \tau(t) \right]^2 = \left( \frac{\beta}{\sqrt{\omega}} + \frac{\sqrt{\omega}}{\beta} \right)^2 + \frac{4\alpha^2}{\omega\beta^2}, \]  

(D.9)

\[ 4 \left[ \sinh \tau(t) \right]^2 = \left( \frac{\beta}{\sqrt{\omega}} - \frac{\sqrt{\omega}}{\beta} \right)^2 + \frac{4\alpha^2}{\omega\beta^2}. \]  

(D.10)
by (D.4) and (D.11) should be applied for our initial “multi-parameter squeezed number states” given 
[106], [108], [109] can be used, say, “stroboscopically”. In general, the variable squeeze operator in 
α
measurement similar to Refs. [114], [85], [86].

Consideration allows one to re-evaluate the photon amplitudes (6.13) in a pure algebraic fashion 
and in terms of standard Fock’s number states. In addition, one may conclude that the minimum-
uncertainty squeezed states occur when α(min) = 0 and n = 0 (see, for example, Eq. (5.5) of Ref. [68]). Indeed, only at these moments of time, by (D.8) the following identities hold, \( \theta (t_{\text{min}}) = \varphi (t_{\text{min}}) = \alpha (t_{\text{min}}) = 0 \), and a traditional definition of single-mode squeeze operator from Refs. [105], [106], [108], [109] can be used, say, “stroboscopically”. In general, the variable squeeze operator in 
(\hat{\alpha})^n |0\rangle = U (t) |n\rangle (D.11)

In this article, we use the Schrödinger picture which is more convenient in the theory of quantum 
measurement[4]. In the Heisenberg picture, one gets 
\( \hat{q} (t) = U^{-1} (t) \hat{q} U (t) = \frac{1}{\beta \sqrt{2}} (e^{2i\gamma \hat{a}} + \hat{a}^\dagger e^{-2i\gamma}) - \frac{\varepsilon}{\beta} \) (D.12)

and 
\( \hat{p} (t) = U^{-1} (t) \hat{p} U (t) = \frac{\alpha \sqrt{2}}{\beta} (e^{2i\gamma \hat{a}} + \hat{a}^\dagger e^{-2i\gamma}) \)

\[ + \frac{\beta}{i \sqrt{2}} (e^{2i\gamma \hat{a}} - \hat{a}^\dagger e^{-2i\gamma}) + \delta - \frac{2\alpha \varepsilon}{\beta} \]

with the aid of (D.3). The standard equations of motion hold,
\[
\frac{d}{dt} \hat{H} (t) = \left[ \hat{p} (t), \hat{H} (t) \right], \quad \frac{d}{dt} \hat{q} (t) = \left[ \hat{q} (t), \hat{H} (t) \right],
\]

where \( \hat{H} (t) = U^{-1} (t) \hat{H} (t) U (t) \). All information about the state of radiation field is now encoded into the operators (D.12) and (D.13) in a form of initial data/constants of motion of the corresponding Ermakov-type system. The field evolution is also completely determined in terms of explicit solutions of this system. The expectation values of operators \( \hat{q} (t) \) and \( \hat{p} (t) \) with respect to Fock’s states coincide, of course, with those found in Ref. [68] in the Schrödinger picture.

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