Weyl formula for the eigenvalues of the dissipative acoustic operator

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Abstract
We study the wave equation in the exterior of a bounded domain $K$ with dissipative boundary condition $\partial_xu - \gamma(x)\partial_tu = 0$ on the boundary $\Gamma$ and $\gamma(x) > 0$. The solutions are described by a contraction semi-group $V(t) = e^{tG}$, $t \geq 0$. The eigenvalues $\lambda_k$ of $G$ with $\text{Re} \lambda_k < 0$ yield asymptotically disappearing solutions $u(t, x) = e^{\lambda_k t}f(x)$ having exponentially decreasing global energy. We establish a Weyl formula for these eigenvalues in the case $\min_{x \in \Gamma} \gamma(x) > 1$. For strictly convex obstacles $K$, this formula concerns all eigenvalues of $G$.

Keywords: Dissipative boundary conditions, Eigenvalues asymptotics

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1 Introduction
Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a bounded non-empty domain. Let $\Omega = \mathbb{R}^d \setminus \tilde{K}$ be connected and $K \subset \{x \in \mathbb{R}^d : |x| \leq \rho_0\}$. We suppose that the boundary $\Gamma$ of $K$ is $C^\infty$. Consider the boundary problem

$$
\begin{aligned}
&u_{tt} - \Delta_x u = 0 \text{ in } \mathbb{R}_+^d \times \Omega, \\
&\partial_x u - \gamma(x)\partial_t u = 0 \text{ on } \mathbb{R}_+^d \times \Gamma, \\
&u(0, x) = f_1, \quad u_t(0, x) = f_2
\end{aligned}
$$

(1.1)

with initial data $(f_1, f_2) \in H^1(\Omega) \times L^2(\Omega) = \mathcal{H}$. Here, $\nu(x)$ is the unit outward normal to $\Gamma$ pointing into $\Omega$ and $\gamma(x) \geq 0$ is a $C^\infty$ function on $\Gamma$. The solution of the problem (1.1) is given by $V(t)f = e^{tG}f$, $t \geq 0$, where $V(t)$ is a contraction semi-group in $\mathcal{H}$ whose generator

$$
G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}
$$

has a domain $D(G)$ which is the closure in the graph norm

$$
\|f\| = (\|f\|^2_{\mathcal{H}} + \|Gf\|^2_{\mathcal{H}})^{1/2}
$$

of functions $f = (f_1, f_2) \in C^\infty_0(\mathbb{R}^d) \times C^\infty_0(\mathbb{R}^d)$ satisfying the boundary condition $\partial_x f_1 - \gamma f_2 = 0$ on $\Gamma$. It is well known that the spectrum of $G$ in $\text{Re} \ z < 0$ is formed by isolated eigenvalues with finite multiplicity (see [7] for $d$ odd and [12] for all $d \geq 2$). Moreover,
For dissipative boundary problems the relation (1.3) in general is not true and
and energy. Such solutions are called asymptotically disappearing. On the other hand, the
solutions \( u(t, x) = V(t)f = e^{tf}(x) \) is a solution of (1.1) with exponentially decreasing global
energy is conserved in time and the unperturbed and perturbed problems are associated
disappearing (see [8]). For \( t_0 > 0 \), the closed linear space

\[
H(t_0) = \{ g \in H : V(t)g = 0 \text{ for } t \geq t_0 \}
\]

is invariant under the action of \( V(t) \) and if \( H(t_0) \neq \{0\} \), then \( H(t_0) \) has infinite dimension.
If \( H(t_0) \) is not trivial, the scattering system is non-controllable (see section 4 in [8] for the
definition and details). Majda proved in [8] that for obstacles with analytic boundary \( \Gamma \)
and analytic \( \gamma(x) \) the condition \( \gamma(x) \neq 1, \forall x \in \Gamma \), implies that there are no disappearing
solutions.

In this paper, in the case \( \min_{x \in \Gamma} \gamma(x) > 1 \), we show that there exists a subspace \( \mathcal{H}_{sp} \subseteq \mathcal{H} \)
with infinite dimension generated by eigenfunctions of \( G \) such that \( V(t)g \in \mathcal{H}_{sp} \) is
asymptotically disappearing. The eigenvalues \( \lambda_k \) sufficiently close to \( \mathbb{R}^- \) with \( \text{Re} \lambda_k \to -\infty \)
present a particular interest for applications since they correspond to solutions decreasing
sufficiently fast as \( t \to +\infty \). It is important to know that such eigenvalues exist and to have
their asymptotic. It was proved in [2] that if we have at least one eigenvalue \( \lambda \) of \( G \) with
\( \text{Re} \lambda < 0 \), then the wave operators \( \mathcal{W}_{\pm} \) are not complete, that is \( \text{Ran} \mathcal{W}_- \neq \text{Ran} \mathcal{W}_+ \). Hence,
we cannot define the scattering operator \( S \) related to the Cauchy problem for the free wave
equations and the boundary problem (1.1) by the product \( \mathcal{W}_+^{-1} \circ \mathcal{W}_- \). When the global
energy is conserved in time and the unperturbed and perturbed problems are associated
to unitary groups, the corresponding scattering operator \( S(z) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1}) \)
satisfies the identity

\[
S^{-1}(z) = S^*(\hat{z}), \quad z \in \mathbb{C}, \tag{1.3}
\]

providing \( S(z) \) invertible at \( z \). Since \( S(z) \) and \( S^*(z) \) are analytic in the “physical” half plane
\( \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) (see [6]) the above relation implies that \( S(z) \) is invertible for \( \text{Im} z > 0 \).
For dissipative boundary problems the relation (1.3) in general is not true and \( S(z_0) \) may
have a non-trivial kernel for some \( z_0, \text{Im} z_0 > 0 \). For odd dimensions \( d \) Lax and Phillips [7]
proved that this implies that \( z_0 \) is an eigenvalue of \( G \). Thus, the analysis of the eigenvalues
of \( G \) is important for the location and the existence of points, where the kernel of \( S(z) \)
is not trivial. A similar connection occurs in the analysis of the interior transmission eigenvalues (see [1] for the definition and more references). More precisely, consider the
far-field operator

\[
(F(k)f)(\theta) = \int_{\mathbb{S}^{d-1}} a(k, \theta, \omega) f(\omega) d\omega, \quad (\theta, \omega) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}.
\]

Here, \( a(k, \theta, \omega) \) is the scattering amplitude for the Helmholtz equation \( \Delta + k^2 n(x) u = 0, \quad x \in K \) with contrast function \( n(x) > 0 \) and for \( d \) odd the scattering operator has the representation

\[
S(k) = \text{Id} + \left( \frac{ik}{2\pi} \right)^{(d-1)/2} F(k), \quad k \in \mathbb{R}.
\]
Therefore, if the kernel of $F(k)$ is non-trivial, $k$ is an interior transmission eigenvalue [1].

The location in $\mathbb{C}$ of the eigenvalues of $G$ has been studied in [12] improving previous results of Majda [9]. It was proved in [12] that for the case when $K$ is the unit ball $B_3 = \{ x \in \mathbb{R}^3 : |x| \leq 1 \}$ and $\gamma \equiv 1$, the operator $G$ has no eigenvalues. For this reason, we study the cases

$$(A) : \max_{x \in \Gamma} \gamma(x) < 1, \quad (B) : \min_{x \in \Gamma} \gamma(x) > 1.$$ 

The results in [12] say that in the case (B) for every $0 < \epsilon \ll 1$ and every $M \in \mathbb{N}$, $M \geq 1$ the eigenvalues lie in $\Lambda_{\epsilon} \cup \mathcal{R}_M$, where

$$\Lambda_{\epsilon} = \{ z \in \mathbb{C} : |\text{Re} \ z| \leq C_\epsilon (1 + |\text{Im} \ z|^{1/2+\epsilon}), \ \text{Re} \ z < 0 \},$$

$$\mathcal{R}_M = \{ z \in \mathbb{C} : |\text{Im} \ z| \leq A_M (1 + |\text{Re} \ z|)^{-M}, \ \text{Re} \ z < 0 \}.$$ 

Moreover, for strictly convex obstacles $K$ there exists $R_0 > 0$ such that the eigenvalues lie in $\mathcal{R}_M \cup \{|z| \leq R_0 \}$. In the case (A), the eigenvalues lie in $\Lambda_{\epsilon}$. By using the results in [18], it is possible to improve the eigenvalue free regions replacing $\Lambda_{\epsilon}$ by $\{ z \in \mathbb{C} : -A_0 \leq \text{Re} \ z < 0 \}$ with sufficiently large $A_0 > 0$.

The existence of eigenvalues has been proved (see Appendix in [12]) only for the ball $B_3$ and $\gamma \equiv \text{const} > 1$ and in this particular case we have

$$\sigma_p(G) \subset \left( -\infty, -\frac{1}{\gamma - 1} \right].$$

Moreover, we have infinite number of real eigenvalues and as $\gamma \searrow 1$ one gets a large strip $\{ z \in \mathbb{C} : -\frac{1}{\gamma} < \text{Re} \ z < 0 \}$ without eigenvalues.

The purpose of this paper is to establish a Weyl formula for the eigenvalues in $\mathcal{R}_M \cap \{ z \in \mathbb{C} : \text{Re} \ z < -C_0 \leq -1 \}$ in the case (B). Introduce the set

$$\Lambda = \{ \lambda \in \mathbb{C} : |\text{Im} \ \lambda| \leq C_1 (1 + |\text{Re} \ \lambda|)^{-2}, \ \text{Re} \ \lambda \leq -C_0 \leq -1 \}$$

containing $\mathcal{R}_M, \forall M \geq 2$, modulo a compact set and denote by $\sigma_p(G)$ the point spectrum of $G$. Increasing the constant $C_0 > 0$ in the definition of $\Lambda$, we subtract a compact set and this is not important for the asymptotic (1.5) below. In the following, we assume that $C_0 \geq 2C_1$. Given $\lambda \in \sigma_p(G)$, we define the algebraic multiplicity of $\lambda$ by

$$\text{mult} (\lambda) = \frac{1}{2\pi i} \int_{|z-\lambda|=\epsilon} (z-G)^{-1} \, dz$$

with $0 < \epsilon \ll 1$ sufficiently small. Our main result is the following

**Theorem 1** Assume $\gamma(x) > 1$ for all $x \in \Gamma$. Then, the counting function of the eigenvalues in $\Lambda$ taken with their multiplicities has the asymptotic

$$\sharp \{ \lambda_j \in \sigma_p(G) \cap \Lambda : |\lambda_j| \leq r, \ r \geq C_\gamma \} = \frac{o_{d-1}}{(2\pi)^{d-1}} \left( \int_{\Gamma} (\gamma^2(x) - 1)^{d-1/2} \, ds_x \right) r^{d-1} + O(r^{d-2}), \ r \to \infty,$$

(1.5)

$o_{d-1}$ being the volume of the unit ball $\{ x \in \mathbb{R}^{d-1} : |x| \leq 1 \}$.

The example concerning the ball $B_3$ and (1.4) show that the condition $r \geq C_\gamma$ is natural since the coefficient before $r^{d-1}$ in (1.5) goes to $0$ as $\max_{x \in \Gamma} \gamma(x) \searrow 1$. Notice that for strictly convex obstacles $K$ in the case (B) we obtain a Weyl formula for all eigenvalues of $G$. For Maxwell's equations with dissipative boundary conditions in the particular case $K = B_3, \gamma \equiv \text{const} \neq 1$, the formula (1.5) has been obtained in [4]. Weyl formula for
the transmission eigenvalues has been obtained by several authors. We refer to \[11,13\] for more references. It is important to note that in \[13\] the Weyl formula is established with remainder which depends on the eigenvalue free region. In \[11\] the relation with the eigenvalues free regions is not exploited and the argument is based on a Tauberian theorem which yields a weak remainder. In the present paper, we apply the eigenvalue free results in \[12\] and the remainder in \(1.5\) is optimal.

To prove Theorem 1, we apply the approach of \[15\] and the construction of a semi-classical parametrix \(T(h, z), 0 < h \leq h_0, z = -\frac{1}{(1+ih)^2}, |\eta| \leq h^2\) for the semi-classical exterior Dirichlet-to-Neumann map \(N(h, z)\) given in \[12,17\]. For \(z = -1\) the operator \(P(h) := T(h, -1) - \gamma(x)\) is self-adjoint and we denote by \(\mu_1(h) \leq \mu_2(h) \leq \cdots\) its eigenvalues counted with their multiplicities. The points \(0 < h_k \leq h_0\) for which \(\mu_k(h_k) = 0\) correspond to points \(h\) for which \(P(h)\) is not invertible. For large fixed \(k_0\), depending on \(h_0\), the eigenvalues \(\mu_k(h_0)\) are positive, whenever \(k > k_0\). Thus, if \(\mu_k(r^{-1}) < 0, k > k_0\), we have \(\mu_k(h_k) = 0\) for some \(r^{-1} < h_k < h_0\) and by a more fine analysis we prove that such a \(h_k\) is unique. The operator \(P(h)\) can be extended as holomorphic one for complex \(\tilde{h} = h(1 + i\eta) \in L\) with \(|\eta| \leq h^2\) and \(L\) defined in \(2.12\). For the resolvent \((\lambda - G)^{-1}\) a trace formula has been established in \[12\] (see Proposition 3). Similarly, a trace formula involving \(P^{-1}(\tilde{h})\) and the derivative \(\tilde{P}(\tilde{h})\) can be proved. These two trace formulas differs by negligible terms and this leads to a map between the points \(h_k \in L\), where \(P(h_k)\) is not invertible and the eigenvalues of \(G\). To obtain \(1.5\), one counts the number of the negative eigenvalues of \(P(r^{-1}), r \geq C_y\) which is given by well-known formula.

The analysis of the counting function of the eigenvalues of \(G\) lying in a strip \(z \in \mathbb{C} : -A_0 \leq \text{Re} \ z \leq 0, A_0 > 0\), as well as the study of the case \((A)\) are open problems. There is a conjecture that there exists a sequence of eigenvalues \(\lambda_k, |\text{Im} \ \lambda_k| \to \infty\). For the investigation of these problems it seems convenient to use the semi-classical parametrix \(T(h, z)\) for the exterior Dirichlet-to-Neumann problem constructed in \[16\] for strictly convex obstacles in the hyperbolic region \(z \in \mathbb{C} : z = 1 + ihw, |w| \leq B_0\).

The paper is organised as follows. In Sect. 2 we collect some facts concerning the operator \(C(\lambda) = \widetilde{N}(\lambda) - \lambda \gamma\) for \(\text{Re} \ \lambda < 0\), where \(\widetilde{N}(\lambda)\) is exterior Dirichlet-to-Neumann map defined in the beginning of Sect. 2. We recall the trace formula involving the resolvent \((G - \lambda)^{-1}\) established in \[12\]. In Sect. 3, one presents some information for the semi-classical parametrix for \(N(h, z)\) and \(z \in \mathbb{C} : z = -\frac{1}{(1 + ih)^2}, |\eta| \leq h^2\) based on the construction in \[17,19\]. The properties of the operator \(P(h)\) for \(h\) real are treated in Sect. 4. In Sect. 5, we compare the trace formulas for \(C(\lambda)\) and for \(P(h)\) and we prove Theorem 1. Finally, in Sect. 6 we discuss some generalisations and a dissipative boundary problem for Maxwell’s equations.

## 2 Preliminaries

We start with some facts which are necessary for our exposition (see \[12\]). For \(\text{Re} \ \lambda < 0\) introduce the exterior Dirichlet-to-Neumann map

\[
\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \longrightarrow \partial_n u|_\Gamma \in H^{s+1}(\Gamma),
\]

where \(u\) is the solution of the problem

\[
\begin{align*}
(-\Delta + \lambda^2)u &= 0 \quad \text{in } \Omega, \quad u \in H^2(\Omega), \\
u &= f \quad \text{on } \Gamma, \\
u : (i\lambda) - \text{outgoing}.
\end{align*}
\]

\[
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\]
A function \( u(x) \) is \((i\lambda)\)-outgoing if there exists \( R > \rho_0 \) and \( g \in L^2_{\text{comp}}(\mathbb{R}^d) \) such that
\[
u(x) = (-\Delta_0 + \lambda^2)^{-1}g, \quad |x| \geq R,
\]
where \( R_0(\lambda) = (-\Delta_0 + \lambda^2)^{-1} \) is the outgoing resolvent of the free Laplacian \(-\Delta_0\) in \( \mathbb{R}^d \) which is analytic in \( \mathbb{C} \) for \( d \) odd and on the logarithmic covering of \( \mathbb{C} \) for \( d \) even. The resolvent \( R_0(\lambda) \) has kernel
\[
R_0(\lambda, x - y) = -\frac{i}{4} \left( \frac{-i\lambda}{2\pi |x - y|} \right)^{(n-2)/2} \left( H^{(1)}_{n/2}(u) \right) \bigg|_{u = -i\lambda |x - y|}, \tag{(2.2)}
\]
where \( H^{(1)}_{n/2}(z) \) being the Hankel function of first kind and we have the asymptotic
\[
H^{(1)}_{n/2}(z) = \left( \frac{2}{\pi r} \right)^{1/2} e^{i(z - \frac{\pi}{4} - \frac{n}{2} i)} + O(r^{-3/2}), \quad -\pi < \arg z < 2\pi, |z| = r \to +\infty. \tag{(2.3)}
\]

The solution of the problem (2.1) with \( f \in H^{3/2}(\Gamma) \) has the representation
\[
u = e(f) + (-\Delta_D + \lambda^2)^{-1}((-\Delta - \lambda^2)(e(f))),
\]
where \( e(f) : H^{3/2}(\Gamma) \ni f \to e(f) \in H_{\text{comp}}^2(\Omega) \) is an extension operator and \( R_D(\lambda) = (-\Delta_D + \lambda^2)^{-1} \) is the outgoing resolvent of the Dirichlet Laplacian \( \Delta_D \) in \( \Omega \). The cut-off resolvent \( R_\chi(\lambda) = \chi(x) R_D(\lambda) \chi(x) \) with \( \chi(x) \in C_0^\infty(\mathbb{R}^d) \) equal to 1 in a neighbourhood of \( K \cup \supp e(f) \) is analytic for \( \Re \lambda < 0 \) and meromorphic in \( \mathbb{C} \) for \( d \) odd and on the logarithmic covering of \( \mathbb{C} \) for \( d \) even. Consequently, \( \mathcal{N}(\lambda) : H^{3/2}(\Gamma) \to H^{1/2}(\Gamma) \) is a meromorphic operator-valued function with the same poles as \( R_\chi(\lambda) \). The same result holds for the action of \( \mathcal{N}(\lambda) \) on other Sobolev spaces. Consider the set \( \Lambda \subset \{ z \in \mathbb{C} : \Re z < -C_0 \leq -1 \} \) introduced in Sect. 1. By using the estimates for \( R_\chi(\lambda) \) for \( \Re \lambda < -C_0 \), we obtain
\[
\|\mathcal{N}(\lambda)\|_{H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)} \leq A_0|\lambda|^2, \quad \lambda \in \Lambda. \tag{(2.4)}
\]

Applying Green’s representation for the solution \( u(y) \) of (2.1) and taking the limit \( \Omega \ni y_n \to x \in \Gamma \), we have
\[
(C_{00}(\lambda)f)(x) - (C_{01}(\lambda)\mathcal{N}(\lambda)f)(x) = \frac{f(x)}{2}, \quad x \in \Gamma,
\]
where
\[
(C_{00}(\lambda)f)(x) = \int_{\Gamma} f(y) \frac{\partial}{\partial v(y)} R_0(\lambda, x - y) dS_y,
\]
\[
(C_{01}(\lambda)g)(x) = \int_{\Gamma} g(y) R_0(\lambda, x - y) dS_y
\]
are the Calderón operators or double and single layer potentials which have the same analytic properties as \( R_0(\lambda, x - y) \). Melrose showed ([10], Section 3) that there exists an entire family \( P_D(\lambda) \) of compact pseudo-differential operators of order \(-1\) on \( \Gamma \) such that
\[
-2(-\Delta_\Gamma + 1)^{1/2} C_{01}(\lambda) = Id + P_D(\lambda),
\]
\( \Delta_\Gamma \) being the Laplace–Beltrami operator on \( \Gamma \) equipped with the Riemannian metric induced by the Euclidean one in \( \mathbb{R}^d \). In fact, \(-C_{01}(\lambda)\) is a pseudo-differential operator of order \(-1\) with principal symbol \( \frac{1}{2}(-\Delta_\Gamma)^{-1/2} \) (see [10]) and one takes the composition of the operators \( \sqrt{-\Delta_\Gamma + 1} \) and \( (-\Delta_\Gamma)^{-1/2} \). Consequently, \((Id + P_D(\lambda))^{-1}\) is a meromorphic operator-valued function and for \( \Re \lambda < 0 \) one deduces
\[
\mathcal{N}(\lambda) = (Id + P_D(\lambda))^{-1}(-\Delta_\Gamma + 1)^{1/2}(Id - 2C_{00}(\lambda)). \tag{(2.5)}
\]
Since $\mathcal{N}(\lambda)$ is analytic for $\Re \lambda < 0$, 1 is not an eigenvalue of $P_D(\lambda)$ for $\Re \lambda < 0$. On the other hand, $C_{00}(\lambda)$ is a pseudo-differential operator of order $-1$, hence it is compact one.

The Neumann problem

\[
\begin{cases}
(-\Delta + \lambda^2)u = 0 \quad \text{in } \Omega, \\ u = (\Omega) - \text{outgoing}.
\end{cases}
\]

has a non-trivial solution if the operator $2C_{00}(\lambda)$ has eigenvalue 1 and this occurs only if $\lambda$ coincides with a resonance $\nu_j$, of the Neumann problem (see [6]). By Fredholm theorem, one deduces that

\[\mathcal{N}(\lambda)^{-1} = (Id - 2C_{00}(\lambda))^{-1}(-\Delta_{\Gamma} + 1)^{-1/2}(Id + P_D(\lambda)) : H^1(\Gamma) \to H^{1/2}(\Gamma)\]

is meromorphic with poles $\nu_j$.

Going back to the problem (1.2), for $\Re \lambda < 0$ we write the boundary condition as follows:

\[C(\lambda)\nu := (\mathcal{N}(\lambda) - \lambda \nu)\nu = \mathcal{N}(\lambda)(Id - \lambda \mathcal{N}(\lambda)^{-1}\nu)\nu = 0, 
\nu = f_{\Gamma} \in H^{1/2}(\Gamma).\]

Clearly, for $\Re \lambda < 0$ the operator $C(\lambda)$ has the same singularities as $\mathcal{N}(\lambda)$, hence $C(\lambda) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is analytic and satisfies the estimate (2.4) with another constant $A_0$.

The operator $\mathcal{N}(\lambda)^{-1}$ is compact and by the results in [12] there are points $\lambda_0$, $\Re \lambda_0 < 0$, for which $Id - \lambda_0 \mathcal{N}(\lambda_0)^{-1}\gamma$ is invertible. Applying the analytic Fredholm theorem for the operator $\left(Id - \lambda \mathcal{N}(\lambda)^{-1}\gamma\right)$ in the half plane $\Re \lambda < 0$, one concludes that

\[C(\lambda)^{-1} = \left(Id - \lambda \mathcal{N}(\lambda)^{-1}\gamma\right)^{-1} \mathcal{N}(\lambda)^{-1} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\]

(2.7) is a meromorphic operator-valued function. Notice that for $\lambda \in \mathbb{R}^-$ the operators $\mathcal{N}(\lambda), C(\lambda)$ are self-adjoint. This follows from the Green formula for $(-\Delta + \lambda^2)$.

**Remark 2** It is important to note that the analyticity of the resolvent $(-\Delta_D + \lambda^2)^{-1}$ for $\Re \lambda < 0$ and the absence of resonances of the Neumann problem in the half plane $\{z \in \mathbb{C} : \Re z < 0\}$ imply that $C(\lambda)^{-1}$ is meromorphic for $\Re \lambda < 0$ and (2.5) is not necessary for the proof of this statement.

For the resolvent $(\lambda - G)^{-1}$ in [12] the following trace formula has been proved.

**Proposition 3** Let $\delta \subset \{\lambda \in \mathbb{C} : \Re \lambda < 0\}$ be a closed positively oriented curve without self-intersections. Assume that $C(\lambda)^{-1}$ has no poles on $\delta$. Then,

\[
\begin{align*}
\text{tr}_{\mathcal{H}} &\frac{1}{2\pi i} \int_{\delta} (\lambda - G)^{-1} d\lambda = \text{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi i} \int_{\delta} C(\lambda)^{-1} \frac{\partial C}{\partial \lambda}(\lambda) d\lambda. \\
\end{align*}
\]

(2.8)

Since $G$ has only point spectrum in $\Re \lambda < 0$, the left hand term in (2.8) is equal to the number of the eigenvalues of $G$ in the domain $\omega$ bounded by $\delta$ counted with their algebraic multiplicities. Setting $\tilde{C}(\lambda) = \frac{\mathcal{N}(\lambda)}{\lambda} - \gamma$, we write the right hand side of (2.8) as

\[
\begin{align*}
\text{tr}_{\mathcal{H}} &\frac{1}{2\pi i} \int_{\delta} \tilde{C}(\lambda)^{-1} \frac{\partial \tilde{C}}{\partial \lambda}(\lambda) d\lambda. \\
\end{align*}
\]

(2.9)

Set $\lambda = \frac{-1}{\bar{h}}, 0 < \Re \bar{h} \ll 1$ and consider the problem

\[
\begin{cases}
(-\bar{h}^2\Delta + 1)u = 0 \quad \text{in } \Omega, \\
-\bar{h}\partial_{\nu}u - \gamma u = 0 \quad \text{on } \Gamma, \\
u = \text{outgoing}.
\end{cases}
\]

(2.10)
We introduce the operator \( C(\hat{h}) := -\hat{h}N(-\hat{h}^{-1}) - \gamma \) and using (2.9), the trace formula (2.8) becomes
\[
\text{tr}_N \frac{1}{2\pi i} \int_\delta (\lambda - G)^{-1} d\lambda = \text{tr}_{h^{1/2}(\Gamma)} \frac{1}{2\pi i} \int_\delta C(\hat{h})^{-1} \hat{C}(\hat{h}) d\hat{h},
\]
(2.11)
where \( C \) denote the derivative with respect to \( \hat{h} \) and \( \hat{\delta} \) is the curve \( \hat{\delta} = \{ z \in \mathbb{C} : z = -\frac{1}{m}, w \in \delta \} \).

Obviously, for \( \lambda \in \Lambda \) one has \( |\text{Im} \lambda| \leq 1 \) and this implies \( \hat{h} \in L \), where
\[
L := \{ \hat{h} \in C : |\text{Im} \hat{h}| \leq C_1|\hat{h}|^4, |\hat{h}| \leq C_0^{-1}, \text{Re} \hat{h} > 0 \}.
\]
(2.12)
We write the points in \( L \) as \( \hat{h} = h(1 + i\eta) \) with \( 0 < h \leq h_0 \leq C_0^{-1}, \eta \in \mathbb{R} \). Recall that \( \frac{2C_1}{C_0} \leq 1 \). Then, \( \frac{C_1}{C_0} \leq 1/3 \) and for \( \hat{h} \in L \) we get
\[
|\eta| \leq \frac{1}{2} \sqrt{1 + \eta^2},
\]
hence \( \eta^2 \leq 1/3 \). This implies
\[
|\eta| \leq C_1 h(1 + \eta^2)^{3/2} \leq h^2, h(1 + i\eta) \in L,
\]
since \( \frac{16C_1}{9} \leq 1 \). Therefore, the problem (2.10) becomes
\[
\begin{cases}
(-h^2 \Delta - z) u = 0 \text{ in } \Omega, \\
-(1 + i\eta)h \partial_\nu u - \gamma u = 0 \text{ on } \Gamma, \\
u \text{ - outgoing}
\end{cases}
\]
with \( z = -\frac{1}{(1+i\eta)^2} = -1 + s(\eta), |s(\eta)| \leq (2 + h^2)h^2 \leq 3h^2 \). On the other hand,
\[
C(\hat{h}) = -(1 + i\eta)hN(-h^{-1}) - \gamma(x).
\]

3 Parametrix for \( N(h, z) \) in the elliptic region

In our exposition we will use \( h \)-pseudo-differential operators and we refer to [5] for more details. Let \( X \) be a \( C^\infty \) smooth compact manifold without boundary with dimension \( d - 1 \geq 1 \). Let \( (x, \xi) \) be the coordinates in \( T^*(X) \) and let \( a(x, \xi, h) \in C^\infty(T^*(X) \times (0, h_0]) \).

Given \( \ell, m \in \mathbb{R} \), one denotes by \( S^{\ell,m} \) the set of symbols so that
\[
|\partial_\alpha^\xi \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha, \beta} h^{-\ell}(1 + |\xi|)^{m-|\beta|}, \forall \alpha, \forall \beta, \quad (x, \xi) \in T^*(X).
\]

If \( \ell = 0 \), we denote \( S^{\ell,m} \) by \( S^m \). The \( h \)-pseudo-differential operator with symbol \( a(x, \xi, h) \) is defined by
\[
(Op_h a)(f)(x) := (2\pi h)^{-d+1} \int_{T^*X} e^{i(x-y, \xi)}/h a(x, \xi, h)f(y)dy d\xi.
\]

We define the space of symbols \( S^{\ell,m}_{cd} \) which have an asymptotic expansion
\[
a(x, \eta, h) \sim \sum_{j=0}^{\infty} h^{\ell-j} a_j(x, \eta), \quad a_j \in S^{m-j}
\]
and the corresponding classical pseudo-differential operator is given by
\[
(Op a)(f)(x) := (2\pi)^{-d+1} \int_{T^*X} e^{i(x-y, \eta)} a(x, \eta, h)f(y)dy d\eta.
\]

It is clear that by a change of variable \( \xi = h\eta \) we may write a \( h \)-pseudo-differential operator as a classical one with parameter \( h \). We will use this fact in Sect. 4. The operators with
symbols in $S_{\ell,m}^c, S_{\ell,m}^e$ are denoted by $L_{\ell,m}^c, L_{\ell,m}^e$, respectively. The wave front $WF(A) \subset T^*\Gamma$ of an operator $A \in L_{\ell,m}^c$ is defined as in [15], where $T^*\Gamma$ is the compactification of $T^*\Gamma$.

We will recall some results for the **exterior** semi-classical Dirichlet-to-Neumann map (see [12,16,17]). Consider the operator

$$P(h, z)u = (-h^2 \Delta - z)u, \quad z = -1 + s(\eta).$$

In local normal geodesic coordinates $(y_1, y')$, $y_1 = \text{dist}(y, \Gamma)$ in a neighbourhood $U$ of $x_0 \in \Gamma$ the operator $P$ has the form (see [14])

$$P(h, z) = h^2 D_{y_1}^2 + r(y, hD_y) + h^2 q(x)D_{y_1} - z, \quad D_j = -i\partial_{y_j}$$

with $r(y, \eta') = \langle R(y)\eta', \eta' \rangle, \quad q(y) \in C^\infty$. Here

$$R(y) = \left\{ \sum_{k=1}^d \frac{\partial y_m}{\partial x_k} \frac{\partial y_j}{\partial x_k} \right\}_{m,j=2}^d = \left\{ \left( \frac{\partial y_m}{\partial x_k} \frac{\partial y_j}{\partial x_k} \right)_{m,j=2}^d \right\}$$

is a symmetric $((d-1) \times (d-1))$ matrix and $r(0, y', \eta') = r_0(y', \eta')$, where $r_0(y', \eta')$ is the principal symbol of the Laplace–Beltrami operator $-\Delta_\Gamma$ on $\Gamma$ equipped with the Riemannian metric induced by the Euclidean one in $\mathbb{R}^d$. For $z = -1 + s(\eta)$ introduce $\rho(y', \eta', z) = \sqrt{z - r_0(y', \eta')} \in C^\infty(T^*\Gamma)$ as the root of the equation

$$\rho^2 + r_0(y', \eta') - z = 0$$

with $\text{Im} \rho(y', \eta', z) > 0$. We have $\rho \in S^1$ and

$$\sqrt{-1 + s(\eta)} - r_0 = i\sqrt{1 + r_0} - \frac{s(\eta)}{\sqrt{1 - s(\eta)) + r_0}}$$

which implies $\rho - i\sqrt{1 + r_0} \in S^{-1}$.

Let $u$ be the solution of the Dirichlet problem

$$\begin{cases}
(h^2 \Delta - z)u = 0 \text{ in } \Omega, \\
u = f \text{ on } \Gamma, \\
u - \text{outgoing}.
\end{cases} \quad (3.1)$$

Consider the semi-classical Sobolev spaces $H^k_h(\Gamma)$ with norm $\|(1 - h^2 \Delta)^{\nu/2}u\|_{L^2(\Gamma)}$ and introduce the exterior semi-classical Dirichlet-to-Neumann map

$$N(h, z) : H^k_h(\Gamma) \ni f \longrightarrow -h\partial_\nu u|_{\Gamma} \in H^{k-1}_h(\Gamma).$$

Vodev [17] established for bounded domains $K \subset \mathbb{R}^d, \quad d \geq 2$, with $C^\infty$ boundary and solutions $u$ of the Helmholtz equation $(-h^2 \Delta - z)u = 0, \quad x \in K$, an approximation of the interior Dirichlet-to-Neumann map. With some modifications his results can be applied for the exterior Dirichlet-to-Neumann map $N(h, z)$ (see [12]). We need some information for the parametrix build in [17,19] in the elliptic region $Z_e := \{z \in \mathbb{C} : \text{Im} z = -1 + s(\eta))$.

For the reader convenience, we recall some points of the construction in [17,19] for $z \in Z_e$. Let $\psi \in C^\infty_0(U_0), \quad \psi = 1$ in a neighbourhood $U_0$ of $x_0 \in \Gamma$. Denote the local normal geodesic coordinates by $(x_1, x')$ and the dual variables by $(\xi_1, \xi')$. We search a parametrix $u_{\psi}$ of the problem (3.1) with boundary data $\psi f$ in the form

$$u_{\psi}(x) = (2\pi h)^{-d+1} \int \int e^{i\xi_1^2/\delta} e^{i\xi_1^2/\delta} \phi(\frac{x_1}{\delta}) a(x, \xi', h, z)f(y') d\xi' dy'.$$
Here, $0 < \delta \ll 1$ and $\phi(t) \in C_0^\infty(\mathbb{R})$ is equal to 1 for $|t| \leq 1$ and to 0 for $|t| \geq 2$. We write
\[
R(x) = \sum_{k=0}^{N-1} x_k^R R_k(x') + x_1^N R_N(x), \quad q(x) = \sum_{k=0}^{N-1} x_1 q_k(x') + x_1^N Q_N(x).
\]

For $\phi$, the eikonal equation modulo $x_1^N$ becomes $(\partial_{x_i} \phi)^2 + \langle R(x) \partial_x \phi, \partial_x \phi \rangle - z = x_1^N \Phi_N$ and one obtains a smooth solution having the form
\[
\phi = \sum_{k=0}^{N} x_1^k \phi_k(x', \xi', z), \quad \phi_0 = -\langle x', \xi' \rangle, \quad \partial_{x_i} \phi \big|_{x_1=0} = \phi_1 = \rho.
\]
The functions $\phi_k$ satisfy for $0 \leq K \leq N - 2$ the equalities
\[
\sum_{k+j=K} (k+1)(j+1)\phi_{k+1} \phi_{j+1} + \sum_{k+j+l=K} \langle R \nabla x' \phi_k, \nabla x' \phi_j \rangle = 0.
\] (3.2)

Clearly, we can determine $\phi_{K+1}$ from the above equality since $\rho \neq 0$. For $z = -1$ we have $\rho = i\sqrt{1 - r_0^2}$ and by recurrence one deduces $\phi_k = i\tilde{\phi}_k$ with real-valued function $\tilde{\phi}_k$. Thus, for $z = -1$ we have $\phi = -\langle x', \xi' \rangle + i\tilde{\phi}$ with real-valued function $\tilde{\phi}$. The amplitude of the parametrix has the form
\[
a = \sum_{j=0}^{N-1} (h^j a_j(x, \xi', z), a_0 \big|_{x_1=0} = \psi, a_j \big|_{x_1=0} = 0, \ j \geq 1
\]
with $a_j = \sum_{k=0}^{N} x_1^k a_{k,j}(x', \xi', z), \ a_{0,0} = \psi, \ a_{0,j} = 0, \ j \geq 1$. The functions $a_j$ satisfy the transport equations
\[
2i \frac{\partial \phi}{\partial x_1} a_j + 2i \langle R(x) \nabla x' \phi_j, \nabla x' a_j \rangle + i(\Delta \phi) a_j + \Delta a_{j-1}
\]
\[
= x_1^N A_N^{(j)}, \quad 0 \leq j \leq N - 1, \ a_{-1} = 0.
\]

We write (see Section 3 in [19])
\[
\Delta \phi = \sum_{k=0}^{N-1} x_1^k \phi_k^\Delta + x_1^N E_N(x), \quad \Delta a_{j-1} = \sum_{k=0}^{N-1} x_1^k a_{k,j-1}^\Delta + x_1^N E_N^{(j-1)}(x)
\]
with
\[
\phi_k^\Delta = (k+1)(k+2)\phi_{k+2} + \sum_{\ell+v=k} \left( \langle R \nabla x' \phi_k, \nabla x' \phi_v \rangle + q_{\ell}(v+1)\phi_{v+1} \right),
\]
\[
a_{k,j-1}^\Delta = (k+1)(k+2)a_{k+2,j-1} + \sum_{\ell+v=k} \left( \langle R \nabla x' \phi_{k,j-1}, \nabla x' a_{\ell,j-1} \rangle + q_{\ell}(v+1)a_{\ell+1,j-1} \right).
\]

This leads to the equality (see (3.18) in [19])
\[
2i \sum_{k_1+k_2=k} (k_1+1)(k_2+1)\phi_{k_1+1} a_{k_2+1,j} + 2i \sum_{k_1+k_2+k_3=k} \langle R_{k_1} \nabla x' \phi_{k_2}, \nabla x' a_{k_3,j} \rangle
\]
\[
+ \sum_{k_1+k_2=k} i\phi_{k_1}^\Delta a_{k_2,j} = -a_{k,j-1}^\Delta \quad \text{for} \ 0 \leq k \leq N - 1, \ 0 \leq j \leq N - 1. \ (3.3)
\]

We can determine $a_{k,j}$ by recurrence from the above equality so that $a_{0,0} = \psi, \ a_{0,j} = 0, \ j \geq 1, \ a_{k,-1} = 0, \ k \geq 0$. Next introduce the operator
\[
T_\psi(h, z)f = -h \left. \frac{\partial \phi}{\partial x_1} \right|_{x_1=0} = Op_k(\tau_\psi)f
\]
with 
\[ r_\psi = -i\hbar \psi - \sum_{j=0}^{N-1} h_j^{j+1} a_{1j}, \quad a_{1j} \in S^{-j}. \]

By using the outgoing resolvent \((h^2 \Delta_D - z)^{-1}\) for the Dirichlet Laplacian in \(\Omega\), we obtain a parametrix \(u_\psi\) in \(\Omega\) and for \(z \in Z_\epsilon\) we have (see Prop. 2.2 in [12] and [17])
\[
\|N'(h, z)(\psi f) - T_\psi f\|_{L^2(\Omega)} \leq C_N h^{\gamma_d + N} \| f \|_{L^2(\Omega)}, \quad \forall N \in \mathbb{N}
\] (3.4)
with \(C_N > 0, \gamma_d > 0\) independent of \(f, h\) and \(z\) and \(\gamma_d\) independent of \(N\). Taking a partition of unity \(\sum_{j=1}^{M} \psi_j(x') \equiv 1\) on \(\Gamma\), we construct a parametrix and define the operator 
\[ T(h, z) = \sum_{j=1}^{M} T_\psi(h, z). \]
For \(z = -1\) the symbol \(\sqrt{1 + r_0 + \sum_{j=0}^{N-1} h_j^{j+1} a_{1j}}\) of \(T(h, -1)\) is real-valued and we have the estimate (3.4) with \(T_\psi(h, z)\) replaced by \(T(h, z)\). Clearly, we may extend the symbol of \(T(h, z)\) holomorphically for \(h \in L\).

4 Properties of the operator \(P(h)\)

In this section we assume that \(\gamma(x) > 1, \forall x \in \Gamma\), and we study the operator 
\[ P(h) = T(h, -1) - \gamma(x) \]
when \(h\) is real. Set
\[ \min_{x \in \Gamma} \gamma(x) = c_0 > 1, \quad \max_{x \in \Gamma} \gamma(x) = c_1 \geq c_0 \]
and choose a constant \(C = \frac{2}{c_1}\). As we mentioned in Sect. 3, we can consider the operator 
\(P(h)\) as a classical pseudo-differential operator \(Op(P)\) with parameter \(h\) with classical symbol \(P = \sqrt{1 + h^2r_0 - \gamma + hP_0(x, \hbar \xi)}\), \(P_0(x, \xi) \in S^0\). We denote by \((., .)\) the scalar product in \(L^2(\Gamma)\) and for two self-adjoint operators \(L_1, L_2\) the inequality \(L_1 \geq L_2\) means \((L_1u, u) \geq (L_2u, u), \forall u \in L^2(\Gamma)\).

Proposition 4
Let \((h\Delta) = (1 - h^2\Delta_D)^{1/2}\) and let \(\epsilon = C(c_0 - 1)^2 < 2\). Then, for \(h\) sufficiently small we have
\[ \frac{\partial P(h)}{\partial h} + CP(h)(h\Delta)^{-1/2} P(h) \geq \epsilon(1 - C_2 h)(h\Delta) \] (4.1)
with a constant \(C_2 > 0\) independent of \(h\).

Proof
The principal symbol of the operator on the left hand side in (4.1) has the form
\[
q_1 = 2h^2 r_0 (1 + h^2 r_0)^{-1/2} + C \sqrt{1 + h^2 r_0} - 2C\gamma(x) + C\gamma^2(x)(1 + h^2 r_0)^{-1/2}
\]
\[
= (2 + C - \epsilon) \sqrt{1 + h^2 r_0} - 2C\gamma(x) + (C\gamma^2(x) - 2)(1 + h^2 r_0)^{-1/2}
\]
\[
+ \epsilon \sqrt{1 + h^2 r_0}. \quad (4.2)
\]
Clearly,
\[
((2 + C - \epsilon)(hD)u, u) \geq ((2 + C - \epsilon)u, u)
\]
and
\[ C\gamma^2(x) - 2 \leq Cc_1^2 - 2 = 0. \]

Therefore,
\[
((C\gamma^2(x) - 2)(hD)^{-1/2}u, u) = ((C\gamma^2(x) - 2)((hD)^{-1/2} - 1)u, u)
\]
\[
+ ((C\gamma^2(x) - 2)u, u) \geq ((C\gamma^2(x) - 2)u, u) - hC_1 \|u\|^2, \quad 0 < h \leq h_0.
\]
Here, the operator $(C\gamma^2(x)-2)(\langle hD\rangle^{-1/2} - 1)$ has non-negative (classical) principal symbol

$$\frac{(2 - C\gamma^2(x))h^2r_0}{1 + h^2r_0 + \sqrt{1 + h^2r_0}}$$

and we applied the semi-classical sharp Gårding inequality (see for instance, [5], Theorem 7.12). Taking into account (4.2) and the inequality $C(\gamma(x)-1)^2 - \epsilon \geq C(c_0 - 1)^2 - \epsilon = 0$, one deduces

$$(\text{Op}(q_1)u, u) \geq ((C(\gamma(x)-1)^2 - \epsilon)u, u) + \epsilon(\langle hD\rangle u, u) - hC_1\|u\|^2$$

$$\geq \epsilon(\langle hD\rangle u, u) - hC_1\|u\|^2.$$ 

The full symbol of the operator on the left hand side of (4.1) has the form $q_1 + hq_0$. The term $h(\text{Op}(q_0)u, u) - hC_1\|u\|^2$ can be absorbed by $\epsilon C_2 h(\langle hD\rangle u, u)$ taking $C_2 \geq \frac{1}{\epsilon}(C_1 + \|\text{Op}(q_0)|_{L^2\to L^2}) = \frac{C_1}{\epsilon}$ and this completes the proof. \hfill \Box

Remark 5 \quad The values of $\epsilon$ depends on $(c_0 - 1)^2$ and $\epsilon \searrow 0$ as $c_0 \searrow 1$. On the other hand, we must have $(1 - C_2 h) > 0$, hence $0 < h < \frac{1}{c_0} \leq \frac{\epsilon}{\sqrt{\epsilon}} = o((c_0 - 1)^2)$. In the case when $\gamma \equiv \text{const}$ and $K$ is the ball $\{x : \|x\| \leq 1\}$ the operator $G$ has no eigenvalues if $\gamma \equiv 1$ (see [12]). Moreover, in this case for $\gamma > 1$ the eigenvalues of $G$ lie in the interval $(-\infty, -\frac{1}{\gamma - 1}]$. Thus, the choice of $h = o((c_0 - 1)^2)$ agrees with eigenvalues free region for the ball.

Next we follow the argument of Sect. 4, [15] with some modifications. Consider the semi-classical Sobolev space $H^s(\Gamma)$ with norm $\|u\|_s = \|\langle hD\rangle^s u\|_{L^2}$. The operator $P(h) : H^1 \to L^2$ has derivative $\dot{P}(h) = O(h^{-1}) : H^1 \to L^2$. Denote by

$$\mu_1(h) \leq \mu_2(h) \leq \ldots \leq \mu_k(h) \leq \ldots$$

the eigenvalues of $P(h)$ repeated with their multiplicities.

Let $h_1$ be small and let $\mu_k(h_1)$ have multiplicity $m$. For $h$ close to $h_1$ one has exactly $m$ eigenvalues and we denote by $F(h)$ the space spanned by them. We can find a small interval $(a, \beta)$ around $\mu_k(h_1)$, independent on $h$, containing the eigenvalues spanning $F(h)$. Given $h_2 > h_1$ close to $h_1$, consider a normalised eigenfunction $e(h_2)$ with eigenvalue $\mu_k(h_2)$.

Let $\pi(h) = E_{(a, \beta)}$ be the spectral projection of $P(h)$, hence $F(h) = (hD)^{L^2(\Gamma)}$. Then, $(\pi(h) - I)\pi(h) = 0$ yields $\pi(h)\pi(h)\pi(h) = 0$ and we deduce $\pi(h)|_{F(h)} = 0$. We construct a smooth extension $e(h) \in F(h)$, $h \in [h_1, h_2]$ of $e(h_2)$ with $\|e(h)\| = 1$, $\dot{e}(h) \in F(h)$. Obviously, $e(h_1)$ will be normalised eigenfunction with eigenvalue $\mu_k(h_1)$.

Considering the eigenvalues $\mu_k(h)$ of $P(h)$ in a small interval $[-\delta, \delta]$, $\delta > 0$, one gets $\|P(h)e(h)\| \leq \delta$. On the other hand,

$$h\dot{P}(h) = h^2\Delta(hD)^{-1} + hL_0 = P(h) - \langle hD\rangle^{-1} + hL_1$$

with zero order operators $L_0, L_1$ and this implies $|\langle \dot{P}(h)e(h), e(h)\rangle| \leq C_0h^{-1}$, $h \in [h_1, h_2]$. Therefore,

$$|\mu_k(h_2) - \mu_k(h_1)| = \left| \int_{h_1}^{h_2} \frac{d}{dh}(P(h)e(h), e(h))dh \right| \leq C_0 \int_{h_1}^{h_2} h^{-1}dh \leq \frac{C_0}{h_1}(h_2 - h_1).$$

Assuming $\mu_k(h) \in [-\delta, \delta]$, we deduce that $\mu_k(h)$ is locally Lipschitz function in $h$ and its almost defined derivative satisfies $|\frac{d}{dh}\mu_k(h)| \leq C_0h^{-1}$. 
To estimate \( h^2 \frac{\partial \mu_k(h)}{\partial h} \) from below, we exploit Proposition 4 and apply (4.1). For \( h \leq h_0 \leq \frac{1}{8C} \) and \( \mu_k(h) \in [-\delta, \delta] \) we have
\[
\frac{\partial \mu_k(h)}{\partial h} = (hP(h)e(h), e(h)) \geq \epsilon(1 - C_2h)(hD)e(h)) - C(hD)^{-1}P(h)e(h), P(h)e(h) \\
\geq \epsilon(1 - C_2h) - C \delta^2 \geq \frac{3\epsilon}{4},
\]
choosing
\[
\delta = (c_0 - 1)\sqrt{\frac{1}{4} - C_2h_0} \geq \frac{(c_0 - 1)}{2\sqrt{2}}.
\]
Consequently, for \( h \in [h_1, h_2] \) one has
\[
\frac{3\epsilon}{4} \geq \mu_k(h_2) - \mu_k(h_1) \geq \frac{3\epsilon}{4} \int_{h_1}^{h_2} h^{-1} dh \geq \frac{3\epsilon}{4h_2}(h_2 - h_1)
\]
and we obtain
\[
\frac{3\epsilon}{4} \leq h \frac{d\mu_k(h)}{dh} \leq C_0.
\]
Fixing \( h_0 > 0 \) small, we conclude that the eigenvalue \( \mu_k(h) \) increases when \( h \) increases and \( \mu_k(h) \in [-\delta, \delta] \). It is well known (see for instance, [5]) that
\[
\mu_k(h) \leq 0 \Rightarrow \kappa_0 = \frac{1}{(2\pi h_0)^d-1} \int_{p_1(x, \xi) \leq 0} dx d\xi + O(h_0^{d+2}),
\]
p_1(x, \xi) being the principal symbol of \( P(\text{Re} h) \). Then, for \( k > \kappa_0 \) we have \( \mu_k(h_0) > 0 \) and if for \( h < h_0 \) one has \( \mu_k(h) < 0 \), then there exists a point \( h < h_k < h_0 \) with the properties \( \mu_k(h_k) = 0 \). \( \mu_k(h) < 0 \) for \( 0 < h < h_k \). This implies that there exists a sequence \( h_k \geq h_k+1 \geq \cdots \) of values \( 0 < h \leq h_0 \) such that \( \mu_k(h_k) = 0 \), \( h_0 > \kappa_0 \). These values \( h_k \) are precisely those for which \( P(h) \) is not invertible. Next we choose \( p > d \) and construct the intervals \( J_{k,p} \) containing \( h_k \) with length \( |J_{k,p}| \sim \hbar^{p+1} \) and \( \mu_k(h) \geq \hbar^p \) for \( h \in (0, h_0) \setminus J_{k,p} \). As in [15], one constructs the disjoint intervals \( J_{k,p} \) and we obtain the following

**Proposition 6** (Prop. 4.1, [15]) Let \( p > d \) be fixed. The inverse operator \( P(h)^{-1} : L^2 \rightarrow L^2 \) exists and has norm \( O(h^{-p}) \) for \( h \in (0, h_0) \setminus \Omega_p \), where \( \Omega_p \) is a union of disjoint closed intervals \( J_{1,p}, J_{2,p}, \ldots \) with \( |J_{k,p}| = O(h^{p+2-d}) \) for \( h \in J_{k,p} \). Moreover, the number of such intervals that intersect \( [h/2, h] \) for \( 0 < h \leq h_0 \) is at most \( O(h^{1-p}) \).

### 5 Relations between the trace integrals for \( C(h) \) and \( P(h) \)

In this section we study the operators \( C(h) \) and \( P(h) \) for complex \( h \in L \). We use the notation \( h \) instead of \( \hbar \) used in Sects. 2 and 3. For \( z = -1 \) the operator \( T(\text{Re} h, -1) \) constructed in Sect. 3 has principal semi-classical symbol \( \sqrt{T + r_0} \), so it is elliptic. The ellipticity holds also for the operator \( T(h, z) \), \( h \in L, z = -1 + s(\eta) \), holomorphic with respect to \( h \), provided \( |\eta| \) is small enough. On the other hand, \( P(h) = (1 + i\eta)T(h, z) - \gamma(x) \) is not elliptic and for \( h \in \mathbb{R}, \eta = 0, z = -1 \) its semi-classical principal symbol vanishes on the set
\[
\Sigma = \{(x, \xi) \in T^*(\Gamma) : r_0(x, \xi) = \gamma^2 - 1\}.
\]
For the symbol \( r_0(x, \xi) \) of the Laplace–Beltrami operator on \( \Gamma \) there exists a constant \( C_3 > 0 \) such that \( r_0(x, \xi) \geq C_3\|\xi\|^2 \), \( (x, \xi) \in T^*(\Gamma) \). Choose a constant \( B_0 > 0 \) so that
\[ \sqrt{C_3}B_0 \geq 2c_1 \] and consider a symbol \( \chi(x, \xi) \in C_0^\infty(T^*(\Gamma)) \), \( 0 \leq \chi(x, \xi) \leq 2 \) such that
\[
\chi(x, \xi) = \begin{cases} 
2, & x \in \Gamma, \|\xi\| \leq B_0, \\
0, & x \in \Gamma, \|\xi\| \geq B_0 + 1.
\end{cases}
\]

Introduce the operator
\[ \tilde{M}(h) = P(\Re h) + \gamma(x)\chi(x, hD_x) = T(\Re h, -1) + \gamma(x)\chi(x, hD_x) - 1. \]

The principal symbol of \( \tilde{M}(h) \) has the form
\[ \hat{m}(x, \xi) = \sqrt{1 + r_0 + \gamma(x)\chi(x, \xi)} - 1. \]

Clearly, \( \tilde{M}(h) \) is elliptic since \( \|\xi\| \leq B_0 \) one gets \( \Re \hat{m}(x, \xi) \geq c_0 \), while for \( \|\xi\| > B_0 \) we have
\[ |\hat{m}(x, \xi)| \geq \sqrt{C_3\|\xi\|} - c_1 \geq \frac{\sqrt{C_3}}{2}\|\xi\| + \frac{\sqrt{C_3}}{2}B_0 - c_1 \geq \frac{\sqrt{C_3}}{2}\|\xi\|. \]

Consequently, \( \hat{m}(x, \xi) \in S^1_{0,0} \), the operator \( \tilde{M}(h)^{-1} : \dot{H}^s - \dot{H}^{s+1} \) is bounded by \( O_s(1) \) and \( \hat{WF}(P(\Re h) - \tilde{M}(h)) \cap \{\|\xi\| \gg B_0 + 1\} = \emptyset. \) Since \( \chi(x, \xi) \) vanishes for \( \|\xi\| \geq B_0 + 1 \), by applying Proposition A.1 in [15], we can extend holomorphically \( \chi(x, hD_x) \) to \( \eta(x, \hat{h}D_x) \) in the domain \( L \). As we mentioned in Sect. 3, the operator \( P(h) \) also has a holomorphic extension for \( \hat{h} \in L \). Thus, \( \tilde{M}(h) \) has a holomorphic extension
\[ M(h) = P(h) + \gamma(x)\eta(x, \hat{h}D_x) - 1 \]
for \( \hat{h} \in L \) and \( \hat{WF}(P(h) - M(h)) \cap \{\|\xi\| \gg B_0 + 1\} = \emptyset. \) The last relation implies \( P(h) - M(h) : O_s(1) : \dot{H}^{-s} \rightarrow \dot{H}^{s}, \forall s. \)

Now we can repeat without any change the proof of Lemma 5.1 in [15], exploiting Proposition 4. First, we obtain
\[ \|P(h)^{-1}\|_{\mathcal{L}(H^{-1/2},H^{1/2})} \leq C_{\Re h} \frac{\Re h}{|\Im h|}, \Re h > 0, \Im h \neq 0. \]

Next, one deduces the estimate
\[ \|P(h)^{-1}\|_{\mathcal{L}(H^s,H^{s+1})} \leq C_{\Re h} \frac{\Re h}{|\Im h|}, \Re h > 0, \Im h \neq 0 \]
applying (5.1) and the representation
\[ P^{-1} = M^{-1} - M^{-1}(P - M)M^{-1} + M^{-1}(P - M)P^{-1}(P - M)M^{-1}, \]
combined with the property of \( P(h) - M(h) \) mentioned above. Following [15], introduce a piecewise smooth simply positively oriented curve \( \gamma_{k,p} \) as a union of four segments: \( \Re h \in \mathcal{J}_{k,p}, \Im h = \pm(\Re h)^{\alpha+1} \) and \( \Re h \in \partial \mathcal{J}_{k,p}, |\Im h| \leq (\Re h)^{\alpha+1}, \) where \( \mathcal{J}_{k,p} \) is one of the intervals in \( \Omega_p \) defined in Proposition 6. Then, we have

**Proposition 7** (Prop. 5.2, [15]) For every \( h \in \gamma_{k,p} \) the inverse operator \( P(h)^{-1} \) exists and
\[ \|P(h)^{-1}\|_{\mathcal{L}(H^s,H^{s+1})} \leq C_{\Re h} (\Re h)^{-\alpha}, \Re h \in \gamma_{k,p}. \]

To estimate \( C(h)^{-1} \), we write
\[ C(h) = -(1 + i\eta)hN(\Re h, z) - \gamma(x) = (1 + i\eta)T(\Re h, z) - \gamma(x) + \mathcal{R}_m(\Re h, z) \]
\[ = P(h) + \mathcal{R}_m(\Re h, z), \mathcal{R}_m \gg 2p \]
with \( \mathcal{R}_m(\Re h, z) : O((\Re h)^m) : H^s \rightarrow H^{s+m-1} \). Therefore,
\[ C(h)P(h)^{-1} = Id + \mathcal{R}_m(\Re h, z)P(h)^{-1} \]
(5.3)
and Proposition 7 imply
\[ \| R_m(\text{Re } h, z) P(h)^{-1} \|_{L(H^1, H^{s+1})} \leq C_s(\text{Re } h)^{-p+m}. \]

For small \text{Re } h, the operator on the right hand side of (5.3) is invertible and
\[ C(h) P(h)^{-1} \left( \text{Id} + R_m(\text{Re } h, z) P(h)^{-1} \right)^{-1} = \text{Id}. \]

On the other hand, the operator \( C(h) \) is elliptic for \(|\xi| \gg 1\) and this implies that \( C(h) : H^{1/2} \to H^{-1/2} \) is a Fredholm operator. The index of \( C(h) \) is constant for \( h \in L \) and according to the results in [12], this index is 0. Hence, the right inverse to \( C(h) \) is also a left inverse, so it is two side inverse. Thus, we obtain
\[ ||C(h)^{-1}||_{L(H^1, H^{s+1})} \leq C_s(\text{Re } h)^{-p}, \ h \in \gamma_{k,p}. \quad (5.4) \]

Moreover,
\[ C(h)^{-1} - P(h)^{-1} = P(h)^{-1} \left( (\text{Id} + R_m(\text{Re } h, z) P(h)^{-1})^{-1} - \text{Id} \right) = K(h) \quad (5.5) \]

with \( K(h) = O_s(|h|^{m-2p}) : H^s \to H^{s+m+1}, \forall s, h \in \gamma_{k,p} \). To estimate \( \dot{C}(h) - \dot{P}(h) \), notice that \( C(h) - P(h) \) is holomorphic with respect to \( h \) in \( L \) and by Cauchy formula
\[ \dot{C}(h) - \dot{P}(h) = \frac{1}{2\pi i} \int_{\gamma_{k,p}} \frac{C(\zeta) - P(\zeta)}{\zeta - h} d\zeta = \frac{1}{2\pi i} \int_{\gamma_{k,p}} \frac{R_m(\text{Re } h, z)}{\zeta - h} d\zeta = K'(h), \]

where \( \gamma_{k,p} \) is the boundary of a domain containing \( \gamma_{k,p} \) with the property dist \( (\gamma_{k,p}, \gamma_{k,p}) \geq (\text{Re } h)^p \). Thus yields
\[ K'(h) = O_s(|h|^{m-2p}) : H^s \to H^{s+m+1}, \forall s, h \in \gamma_{k,p}. \]

Concerning the operator \( P(h) \), we obtain a trace formula repeating without any change the argument in [15]. Let \( \mu_k(h_k) = 0, k \geq k_0 \). It is easy to see that \( \mu_k(h) \) has no other zeros for \( 0 < h \leq h_0 \), exploiting the fact that \( \mu_k(h) \) in increasing for \( \mu_k(h) \in [-\delta, \delta] \). One defines the multiplicity of \( h_k \) as the multiplicity of the eigenvalue \( \mu_k(h_k) \). Then, we have

**Proposition 8** (Prop. 5.3, [15]) Let \( \beta \subset L \) be a closed positively oriented \( C^1 \) curve without self-intersections which avoids the points \( h_k \) with \( \mu_k(h_k) = 0 \). Then,
\[ \text{tr} \frac{1}{2\pi i} \int_{\beta} P(h)^{-1} \dot{P}(h) dh \]

is equal to the number of \( h_k \) in the domain bounded by \( \beta \).

Now we may compare the trace formula for \( C(h) \) and \( P(h) \). First, we compare the integrals over \( \gamma_{k,p} \). We have
\[ \text{tr} \frac{1}{2\pi i} \int_{\gamma_{k,p}} C(h)^{-1} C(h) dh = \text{tr} \frac{1}{2\pi i} \int_{\gamma_{k,p}} (C(h)^{-1} - P(h)^{-1}) \dot{C}(h) dh \\
+ \text{tr} \frac{1}{2\pi i} \int_{\gamma_{k,p}} P(h)^{-1} \dot{C}(h) dh = \text{tr} \frac{1}{2\pi i} \int_{\gamma_{k,p}} P(h)^{-1} \dot{C}(h) dh + O_p((\text{Re } h)^{m-2p}). \]

Here, we have used (5.5) and the estimate
\[ \| \dot{C}(h) \|_{L(H^{1/2}, H^{-1/2})} \leq C|h|^{-2}, \ h \in L. \]

which follows from (2.4). Next the property of \( K'(h) \) yields
\[ \text{tr} \frac{1}{2\pi i} \int_{\gamma_{k,p}} P(h)^{-1} \dot{C}(h) dh = \text{tr} \frac{1}{2\pi i} \int_{\gamma_{k,p}} P(h)^{-1} \dot{P}(h) dh + O_p((\text{Re } h)^{m-2p}). \]
For small \( h \) and \( m \gg 2p \), the terms \( O_p((\text{Re } h)^{m-2p}) \) are negligible and we obtain, as in [15], a map \( \ell_p \) between the set of points \( h_k \in (0, h(p)] \) counted with their multiplicities and the eigenvalues \( \ell_p(h_k) \in \Lambda \) counted with their multiplicities. The number of points \( h_k \in I_{k,p} \) counted with their multiplicities is equal to the number of eigenvalues \( \lambda_j = \ell_p(h_k) \) of \( G \) counted with their multiplicities lying in \( \Lambda_{k,p} = \{ z \in \mathbb{C} : z = -\frac{1}{2}, \xi \in \omega_{k,p} \} \), \( \omega_{k,p} \subset L \) being the domain bounded by \( \gamma_{k,p} \). Notice that for a point \( h_k \) we could have many \( \lambda_j \in \ell_p(h_k) \subset \Lambda_{k,p} \). On the other hand, for every \( \lambda_j \in \ell_p(h_k) \) one has

\[
\left| \lambda_j + \frac{1}{h_k} \right| \leq C_p h_k^{p+2-d}.
\]

The integral over \( \beta \) in Proposition 8 can be presented as a sum of integrals over \( \gamma_{k,p} \) plus integrals over curves \( \alpha_{j,p} \) which are the boundary of domains \( \beta_{j,p} \) such that \( \beta_{j,p} \cap \Omega_p = \emptyset, \forall j \).

By Proposition 6 for \( h \in (0, h_0] \setminus \Omega_p \) the operator \( P(h) \) is invertible. Applying an argument similar to that used above, one concludes that \( P(h) \) is invertible for \( h \in \beta_{j,p} \). Consequently, there are no contributions from the integrals over \( \alpha_{j,p} \) and we must sum the contributions over the integrals over \( \gamma_{k,p} \) that is the sum of the number of the corresponding points \( h_k \).

Consider the counting function

\[ N(r) = \# \{ \lambda \in \sigma_p(G) \cap \Lambda : |\lambda| \leq r, \text{ Re } \lambda \leq -C_0, \ r > C_0 \] with \( h_0^{-1} = C_0 > 0 \) large enough. Then, for \( |\lambda| \leq r \) we must consider \( r^{-1} = |\check{h}|, \check{h} \in L \). Modulo a finite number eigenvalues (see Sect. 4 and the number \( k_0 \)), we are going to count the points \( r^{-1} < h_k \leq h_0 \) and the number of the eigenvalues \( \mu_k(h) \) for which we have \( \mu_k(h_k) = 0 \). Hence, we have \( \mu_k(r^{-1}) < 0 \), since otherwise we obtain a contradiction. The problem is reduced to find the number of the negative eigenvalues of \( P(r^{-1}) \) which is given by well-known formula

\[
\frac{r^{d-1}}{(2\pi)^{d-1}} \int_{\Omega} \alpha_{1,\xi} \leq 0 \text{ dx } d\xi + O(r^{d-2}).
\]

Clearly,

\[
\int_{\Omega} \alpha_{1,\xi} \leq 0 \text{ dx } d\xi = \int_{\eta_0(\xi,\eta) \leq \gamma^2(\xi-1)} \text{ dx } d\xi = \int_{\gamma^2(\xi-1)} \text{ dx } d\xi = \int_{\gamma^2(\xi-1)} (\gamma^2(\xi-1)-1)^{d-1}/2 \int_{\eta_0(\xi,\eta) \leq 1} \text{ dy }.
\]

For the induced Riemannian metric on \( \Gamma \), the integral over the dual variable \( \eta \) yields the volume \( \omega_{d-1} \) of the unit ball \( \{ x \in \mathbb{R}^{d-1} : |x| \leq 1 \} \) and we obtain the asymptotic (1.5). This completes the proof of Theorem 1.

6 Generalisations

We may study with some modifications a more general dissipative boundary problem

\[
\begin{align*}
\partial_t u - \Delta_x u + c(x)u_t &= 0 \quad \text{in } \Omega_t^+ \times \Omega, \\
\partial_t u - \gamma(x)\partial_t u - \sigma(x)u &= 0 \quad \text{on } \Gamma^+ \times \Gamma, \\
u(0, x) &= f_1, \quad u_t(0, x) = f_2,
\end{align*}
\]

(6.1)

where \( c(x) \geq 0, \sigma(x) \geq 0 \) are smooth functions defined respectively in \( \mathbb{R}^d \) and \( \Gamma \) and \( c(x) = 0 \) for \( |x| \geq R_0 > 0 \) (see [8]). The solution is given by a semi-group \( V(t) = e^{tf}, t \geq 0 \) with \( f = (f_1, f_2) \) in the energy space \( \mathcal{H}_E \) with norm

\[ \| f \|^2_{\mathcal{H}_E} = \int_{\Omega} (|\nabla_x f_1|^2 + |f_2|^2) \text{ dx } + \int_{\Gamma} \sigma |f_1|^2 \text{ dy } . \]
The generator of $V(t)$ has the form

$$ G = \begin{pmatrix} 0 & 1 \\ \Delta & c \end{pmatrix} $$

with a domain $\mathcal{D}(G)$ being the closure in the graph norm

$$ \|f\|_{\mathcal{E}} = (\|f\|_{\mathcal{E}}^2 + \|Gf\|_{\mathcal{E}}^2)^{1/2} $$

of functions $f = (f_1, f_2) \in C_{\infty}^0(\mathbb{R}^d) \times C_{\infty}^0(\mathbb{R}^d)$ satisfying the boundary condition $\partial_n f_1 - \gamma f_2 - \sigma f_1 = 0$ on $\Gamma$. If we have an eigenfunction $f = (f_1, f_2)$ with $Gf = \lambda f$ and $\lambda = -\frac{1}{h}$ (for simplicity, we keep the notation of Sect. 2), then $u = f_1$ is a solution of the problem

$$ \begin{align*}
(\hat{\hbar}^2 \Delta + 1 - \hat{\hbar} c)u &= 0 \text{ in } \Omega, \\
-\hat{\hbar} \partial_n u - \gamma u + \hat{\hbar} \sigma u &= 0 \text{ on } \Gamma, \\
u &= \text{outgoing}.
\end{align*} $$(6.2)

Therefore, with $\hat{\hbar} = h(1 + i\eta), \eta \in \mathbb{R}, z = -\frac{1}{(1 + i\eta)^2}$ we obtain the problem

$$ \begin{align*}
(\hat{\hbar}^2 \Delta - z - \frac{\hat{\hbar}}{1 + i\eta} c)u &= 0 \text{ in } \Omega, \\
-(1 + i\eta)h \partial_n u - \gamma u + h(1 + i\eta)\sigma u &= 0 \text{ on } \Gamma, \\
u &= \text{outgoing}.
\end{align*} $$(6.3)

We need to consider the semi-classical exterior Dirichlet-to-Neumann operator $N(h, z)$ related to the operator $-\hbar^2 \Delta - z - \frac{\hbar}{1 + i\eta} c$. The construction of the semi-classical parametrix for $N(h, z)$ is the same as in [12, 17]. The term $\frac{\hbar}{1 + i\eta} c$ is lower order operator and the principal symbol of $N(h, z)$ is $\sqrt{1 + \eta^2}$. Next, we deal with the operator

$$ P(\hat{\hbar}) = (1 + i\eta)N(h, z) - \gamma - h(1 + i\eta)\sigma $$

and the self-adjoint operator $P(\hbar) = N(h, z) - \gamma - h\sigma$. Here $h(1 + i\eta)\sigma$ is a lower order operator and we may repeat the arguments of Sects. 4 and 5. Under the assumptions of Theorem 1, one obtains a Weyl formula (1.5) with the same leading term. We leave the details to the reader.

We hope that our arguments combined with the construction of a semi-classical parametrix in [20] can be applied for the analysis of the eigenvalues of Maxwell’s equations with dissipative boundary conditions

$$ \begin{align*}
\partial_t E - \text{curl} \ H &= 0, \quad \partial_t H + \text{curl} \ E = 0 \quad \text{in } \mathbb{R}^+ \times \Omega, \\
v \wedge E - \gamma(x)(v \wedge v \wedge H) &= 0 \quad \text{on } \mathbb{R}^+ \times \Gamma, \\
E(0, x) &= E_0(x), \quad H(0, x) = H_0(x),
\end{align*} $$

where $d = 3, (E_0, H_0) \in L^2(\mathbb{R}^+ \times \Omega ; \mathbb{C}^6), \gamma(x) > 0, \forall x \in \Gamma$. The solution of (6.4) is given by a contraction semi-group $V_b(t) = e^{tG_b}, t \geq 0$ (see [4] for the definition of $G_b$) and the spectrum of $G_b$ in the half plane $\{z \in \mathbb{C} : \text{Re } z < 0\}$ is formed by isolated eigenvalues with finite multiplicities [2].

We sketch briefly below the similitudes with the analysis in Sect. 2. If $(E, H) \neq 0$ is an eigenfunction of $G_b$ with eigenvalue $\lambda$, then

$$ \begin{align*}
\text{curl} \ E &= -\lambda H, \quad \text{curl} \ H = \lambda E \quad \text{in } \Omega, \\
\frac{1}{\sqrt{c}} (v \wedge v \wedge E) + v \wedge H &= 0 \quad \text{on } \Gamma, \\
(E, H) : (i\lambda) &= \text{outgoing}.
\end{align*} $$

(6.5)
Consider the problem
\[
\begin{align*}
\text{curl } E &= -\lambda H, \quad \text{curl } H = \lambda E \quad \text{in } \Omega, \\
\nu \wedge E &= f \quad \text{on } \Gamma, \\
(E, H) : (i\lambda) &= -\text{outgoing},
\end{align*}
\] (6.6)

In the space \(\mathcal{H}^1_s(\Gamma) := \{u \in H^s(\Gamma) : \langle v, u \rangle = 0\}\) introduce the operator \(\mathcal{N}_b(\lambda) : \mathcal{H}^1_s(\Gamma) \to \mathcal{H}^1_s(\Gamma)\) defined by
\[
\mathcal{N}_b(\lambda)f = \nu \wedge H|\Gamma,
\]
\((E, H)\) being the solution of the problem (6.6). The operator \(\mathcal{N}_b(\lambda)\) is the analog of the exterior Dirichlet-to-Neumann operator in Sect. 2 (see [20]) and the boundary condition on (6.5) can be written as
\[
C_b(\lambda)f = \frac{1}{\gamma(x)}(\nu \wedge f) + \mathcal{N}_b(\lambda)f = 0, \quad f = \nu \wedge E|\Gamma.
\]

The outgoing resolvent of the problem
\[
\begin{align*}
\text{curl } E &= -\lambda H + F_1, \quad \text{curl } H = \lambda E + F_2 \quad \text{in } \Omega, \\
\nu \wedge E &= 0 \quad \text{on } \Gamma, \\
(E, H) : (i\lambda) &= -\text{outgoing},
\end{align*}
\]
is analytic for Re \(\lambda < 0\) since the above problem corresponds to a self-adjoint operator. Therefore, we can prove that \(C_b(\lambda)\) is analytic for Re \(\lambda < 0\). In the same way from the fact that for Re \(\lambda < 0\), there are no non-trivial solutions of the problem
\[
\begin{align*}
\text{curl } E &= -\lambda H, \quad \text{curl } H = \lambda E \quad \text{in } \Omega, \\
\nu \wedge H &= 0 \quad \text{on } \Gamma, \\
(E, H) : (i\lambda) &= -\text{outgoing},
\end{align*}
\]
one concludes that \(\mathcal{N}_b(\lambda)^{-1}\) is analytic for Re \(\lambda < 0\). As in Sect. 2, one deduces that \(C_b(\lambda)^{-1}\) is a meromorphic operator-valued function for Re \(\lambda < 0\) (see (2.7) and Remark 2). Assuming \(\gamma(x) \neq 1, \forall x \in \Gamma\), according to the results in [3], for every \(\epsilon > 0\) and every \(M \in \mathbb{N}, M \geq 1\) the eigenvalues of \(G_b\) lie in \(\Lambda_\epsilon \cup R_M\). Next one can establish a trace formula involving \((\lambda - G_b)^{-1}\) and for the analysis of the counting function of the eigenvalue of \(G_b\) in \(\Lambda\) it is possible to apply the strategy of Sects. 4 and 5 combined with the semi-classical parametrix constructed in [20].

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References

1. Cakoni, F., Haddar, H.: Transmission Eigenvalues in Inverse Scattering Theory, Inside and Out II, MSRI Publications, p. 60 (2012)
2. Colombini, F., Petkov, V., Rauch, J.: Spectral problems for non elliptic symmetric systems with dissipative boundary conditions. J. Funct. Anal. 267, 1637–1661 (2014)
3. Colombini, F., Petkov, V., Rauch, J.: Eigenvalues of the Maxwell’s equations with dissipative boundary conditions. Asymptot. Anal. 99, 105–124 (2016)
4. Colombini, F., Petkov, V.: Weyl formula for the negative dissipative eigenvalues of Maxwell’s equations. Arch. Math. 110, 183–195 (2018)
5. Dimassi, M., Sjöstrand, J.: Spectral Asymptotics in Semi-classical Limits, London Mathematical Society, Lecture Notes Series, vol. 268, Cambridge University Press (1999)
6. Lax, P., Phillips, R.: Scattering Theory, 2nd edn. Academic Press, New York (1989)
7. Lax, P., Phillips, R.: Scattering theory for dissipative systems. J. Funct. Anal. 14, 172–235 (1973)
8. Majda, A.: Disappearing solutions for dissipative wave equation. Indiana Univ. Math. J. 24(12), 1119–1133 (1974)
9. Majda, A.: The location of the spectrum of the dissipative acoustic operator. Indiana Univ. Math. J. 25, 973–987 (1976)
10. Melrose, R.: Polynomial bound on the distribution of poles in scattering by an obstacle. Journées Equations aux Dérivées Partielles 1984, 1–8 (1984)
11. Nguyen, Hoai-Minh., Nguyen, Quoc-Hung.: The Weyl law of the transmission eigenvalues and the completeness of generalised transmission eigenfunctions. J. Funct. Anal. 281(8), 109146 (2021)
12. Petkov, V.: Location of eigenvalues for the wave equation with dissipative boundary conditions. Inverse Probl. Imaging 10, 1111–1139 (2016)
13. Petkov, V., Vodev, G.: Asymptotics of the number of the interior transmission eigenvalues. J. Spectral Theory 7, 1–31 (2017)
14. Robiano, L.: Spectral analysis of the interior transmission eigenvalue problem. Inverse Probl. 29, 104001 (2013)
15. Sjöstrand, J., Vodev, G.: Asymptotics of the number of Rayleigh resonances. Math. Ann. 309, 287–306 (1997)
16. Sjöstrand, J.: Weyl law for semi-classical resonances with random perturbed potentials. In: Mémoires de SMF, vol. 136 (2014)
17. Vodev, G.: Transmission eigenvalue-free regions. Commun. Math. Phys. 336, 1141–1166 (2015)
18. Vodev, G.: High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues. Anal. & PDF 11(1), 213–236 (2018)
19. Vodev, G.: Parabolic transmission eigenvalues in the degenerate isotropic case. Asymptot. Anal. 106, 147–168 (2018)
20. Vodev, G.: Semiclassical parametrix for the Maxwell equation and applications to the electromagnetic transmission eigenvalues. Res. Math. Sci. 8(3), 35 (2021)

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