Birth-Death Processes in Interactive Random Environments

GUODONG PANG, ANDREY SARANTSEV, AND YURI SUHOV

Abstract. We study birth-death processes in interactive random environments, where not only the birth and death rates depend on the state of the environment, but also the evolution dynamics of the environment depends on the state of the birth-death process. Two types of random environments are considered, either pure jump (finite or infinite countable) Markov process or a reflected (jump) diffusion process. The joint Markov process is constructed in such a way that the interactive impact is explicitly characterized, and the joint invariant measure takes an explicit form which may be regarded as a type of product form. We provide a few examples in queueing (multi-server and infinite-server models) and population growth models, to illustrate the construction of the joint Markov process and the conditions under which the explicit product-form invariant measure can be derived.

We then study the rate of convergence to stationarity of these models. We consider two settings which lead to either an exponential rate or a polynomial rate. In both settings, we assume that the underlying environmental Markov process has an exponential rate of convergence, however, the convergence rates of the joint Markov process depend on the certain conditions on the birth and death rates. We use a coupling approach to prove these results on the rates of convergence.

1. Introduction

Birth-death processes are fundamental stochastic models in applied probability and queueing. Birth-death processes in random environments have been extensively studied [34, 35, 4, 5] and used in various applications (see, e.g., queueing [14, 15, 10], inventory [23], population and biology [1] and epidemiology [25]). Most of the studies have been on models where the transitions of birth and death processes are affected by the environment (see, e.g., [34, 35, 4]). However, in applications, the influence of the birth-death processes and the environment can be in both directions. For example, service systems can be often modeled as multi-server queues, where customer arrivals may depend on performance ratings whose dynamics may be modeled as a Markov process, while ratings may depend on the service quality indicated by congestion (queueing state). For another example, population growth can be often modeled as a birth-death process in a random environment while the environmental changes, modeled as a Markov process, can be also influenced by the population size (say, in the case of overpopulation). In the modeling of epidemiology, the infection rate may depend on a switching environment, while transitions of the environmental states may depend on the number of infections because of different invention policies at different levels of infections.

In this paper, we study a family of birth-death processes in an interactive random environment which are constructed in such a way that the joint Markov process has a type
of product-form invariant measure. Consider a birth-death process \( N = \{N(t), t \geq 0\} \) on \( \mathbb{N} = \{0, 1, 2, \ldots\} \) with birth rates \( \lambda_n \) and death rates \( \mu_n \) depending on an environment variable \( z \) taking values in an environment space \( \mathcal{Z} \). This variable, in turn, evolves as a continuous-time Markov process \( Z = \{Z(t), t \geq 0\} \), with transition functions depending on the current state \( N(t) \) of this birth-death process. Then a combined stochastic process \( (N, Z) = \{(N(t), Z(t)), t \geq 0\} \) is called a birth-death process in an interactive random environment.

The generator of the combined process \( (N, Z) \) is constructed such that

\[
\mathcal{L}f(n,z) = \mathcal{M}_z f(n,z) + r_n(z)^{-1} \mathcal{A}_n f(n,\cdot) .
\] (1.1)

Here \( \mathcal{M}_z \) is the generator of the birth-death process with fixed environment variable \( z \) and \( \mathcal{A}_n \) is the generator of the environment process with fixed birth-death value \( n \). The parameter \( r_n(z) \) is the birth-death rate ratio parameter given in (2.2).

For the environment process, we consider two cases:

(i) the environment space \( \mathcal{Z} \) is also countable, and the environment process is a continuous-time Markov chain with generating matrix depending on the state of \( N(t) \); we call it jump environment;

(ii) the environment space \( \mathcal{Z} \) is a domain in \( \mathbb{R}^d \), and the environment process is a reflected (jump) diffusion in this domain, with the drift vector and diffusion matrix dependent on the state of \( N(t) \); we call it diffusive environment.

In this article, we study the long-time behavior of these models: (a) existence and uniqueness of a stationary distribution, that is, a probability distribution \( \pi \) on the product space \( X = \mathbb{N} \times \mathcal{Z} \) such that if \( (N(0), Z(0)) \sim \pi \), then \( (N(t), Z(t)) \sim \pi \) for all \( t \geq 0 \), and an explicit form of this stationary distribution; and (b) the fact of convergence \( (N(t), Z(t)) \rightarrow \pi \) as \( t \rightarrow \infty \) in the total variation distance, and the rate of this convergence (exponential or polynomial). We adapt and generalize the methods of our previous article [24] devoted to \( M/M/1 \) queues in an interactive random environment. We identify conditions on the birth and death rates and the underlying Markov process under which the rate of convergence can be either exponential or polynomial.

We discuss a few examples that are of interest on their own. For example, we have studied infinite-server queues with the arrival and/or service rates being an RBM or reflected Ornstein-Uhlenbeck diffusion (Examples 3.1–3.6). We have also discussed the finite-server queues (infinite-waiting space, blocking/loss model or with abandonment) where the arrival, service and/or abandonment rates are an RBM or reflected diffusion in Examples 3.8 and 3.9. Another example is the population growth model in biology with the growth and death rates dependent on the environment (see Example 2.6 in a jump environment and Example 3.10 with the rates being a three-dimensional RBM in an orthant). We have also briefly discussed how the population growth model can be extended to study growth stocks in finance in Examples 2.7 and 3.10. In all these models, we discuss how the conditions for the existence of stationary distributions are verified and provide the explicit expressions for the invariant measures.

1.1. Literature review. Birth-death processes in random environments have been widely studied, see, e.g., [34, 35, 4]. In these models, the birth and death rates are affected by the environments. Economou [8] studied continuous-time Markov chains (CTMC) in random environments where not only the the transitions rates of the CTMC depend on the environment, but also a change in the environment can trigger an immediate transition of
the CTMC. He identified conditions under which a (generalized) product form stationary distribution may exist. In [3], more general Markov chains in random environments are studied, where the transition probabilities of the Markov chains are affected by the environments. Bacaër and Ed-Darraz [1] studied a linear birth-death process in a finite-state random environment with biology applications, and derived the probability of extinction. In all these studies, the interaction with the environment is one-sided, that is, the dynamics of the environment is not influenced by the state of the Markov chains.

Cornez [5] first studied birth-death processes in random environment with feedback (that is, feedback to the environment process from the state of the birth-death process) and provided sufficient conditions under which the birth-death process component goes extinct or not. In that model, the environment process takes values in a general measurable space, and no explicit stationary distribution is derived. In [14], loss queues with interactive (Markov jump) random environments are considered, and a product-form steady state distribution of the joint queueing-environment process is derived which results in a strong insensitivity property. In [15], the authors consider Jackson networks with interactive (Markov jump) random environments, where customers departing from the network may enforce the environment to jump immediately. In [23], single server queues with state dependent arrival and service rates which are also interactively affected by a Markov jump environment are studied, and both cases of an explicit product-form (separable) steady state distribution and of a non-separable steady-state distribution are considered. In [6], another construction is provided for Markov processes in interactive random environments (pure Markov jump process) that allows simultaneous transitions for the Markov chain and environment states, for which a product form invariant measure is derived and applications to queueing and neural avalanches are discussed. We note the main differences in the construction of the joint Markov process in [14, 15, 23] from our paper: they allow simultaneous changes in the queueing and environment states, while our construction does not. Moreover, the environments in those papers are only a Markov jump process.

We also refer to [10], where a random walk interacting with a random environment of a Jackson/Gordon-Newell network is considered, and an explicit stationary distribution of a product-form type is derived. In [25], an epidemic SIS model in an interactive switching environment is studied, where the infection and recovery rates depend on a finite-state Markov jump process whose transitions also depend on the number of infectives. Large population scaling limits and the associated long-time behaviors are studied. Our work generalizes the previous work in [24], where an $M/M/1$ queue in an interactive random environment is studied, with both jump and diffusive environments. The models considered in this paper are more general, and a few new stochastic models are introduced as discussed above. In addition to exponential rate of convergence, we also establish polynomial rate of convergence to stationarity.

This paper also contributes to the understanding of rate of convergence of birth-death processes. Lindvall [17] developed the coupling approach to estimate the exponential rate of convergence for birth-death processes, which we follow and further develop for our model. Van Doorn [36] identified conditions on the birth and death rates under which the the chain is exponentially ergodic by investigating the spectral representation of the transition probabilities, and bounds on the decay parameter were also established. Ziefman [42] used the methods of theory of differential equations to derive explicit estimates for the rate of convergence for birth-death processes, which was used for some queueing examples including
the \(M/M/N\) and \(M/M/N/0\) models. That method was extended to nonhomogeneous birth-death processes for which upper and lower bounds on the rate of convergence were derived in [43], and queueing examples including \(M_t/M_t/N\) and \(M_t/M_t/N/0\) were studied. Van Doorn and Zeifman [38] study the rate of convergence of the Erlang loss system. We also refer to [39, 37, 44] for further studies on the related topics and queueing models.

1.2. Organization of the article. In Section 2, we state the model and results for the jump environment. In Section 3, we do the same for the diffusive environment. A few examples are provided in both sections. The proofs for both sections are given in Section 5. In Section 4, we state and prove the results on the exponential and polynomial rates of convergence to stationarity.

1.3. Notation. We let \(N = \{0, 1, 2, \ldots \}\) and \(\mathbb{R}_+ := [0, \infty)\) be the sets of all nonnegative integer and real numbers, respectively. The total variation distance between two probability measures \(P\) and \(Q\) on the same space \(E\) is defined as

\[ \|P - Q\|_{TV} = \sup_{A \subseteq E} |P(A) - Q(A)|. \]

The space of bounded twice continuously differentiable functions with bounded first and second derivatives on the space \(E\) is denoted by \(C^2_b(E)\).

2. Jump Environment

2.1. Model construction. Consider a birth-death process in an interactive jump environment described as follows. Let \(Z\) be a finite or countable state space for the environment. We define a two-component Markov process \((N, Z)\) taking values in the countable state space \(N \times Z\), by the following generator matrix \(R = (R[(n, z), (n', z')]):\)

\[
\begin{align*}
R[(n, z), (n + 1, z)] &= \lambda_n(z), & R[(n, z), (n - 1, z)] &= \mu_n(z), \\
R[(n, z), (n, z')] &= r_n(z)^{-1}r_n(z, z') \\
R[(n, z), (n', z')] &= 0, & n \neq n', & z \neq z'.
\end{align*}
\]  

(2.1)

where for each \(z \in Z\), we define:

\[
r_n(z) = \prod_{k=1}^{n} \frac{\lambda_{k-1}(z)}{\mu_k(z)} \quad \text{for} \quad n \geq 1, \quad \text{and} \quad r_0(z) \equiv 1,
\]  

(2.2)

and \(T_n = (\tau_n(z, z'))_{z, z' \in Z}\) is a well-defined generator for an irreducible continuous-time Markov chain on \(Z\) (see the definition of irreducibility in e.g., [21]). Here \(N = \{N(t) : t \geq 0\}\) represents the state of the birth-death process, taking values in \(\mathbb{N}\), and \(Z = \{Z(t) : t \geq 0\}\) represents a jump process (the environment) taking values in \(Z\). It can be easily checked that for each \(z \in Z\), we have the local balance equations for the birth-death process \(N(t)\):

\[
\begin{align*}
\kappa_n(z)(\lambda_n(z) + \mu_n(z)) &= \kappa_{n-1}(z)\lambda_{n-1}(z) + \kappa_{n+1}\mu_{n+1}(z), \quad n \geq 1, \\
\kappa_0(z)\lambda_0(z) &= \kappa_1(z)\mu_1(z),
\end{align*}
\]  

(2.3)

(2.4)
Birth-Death Processes in Interactive Random Environments

where
\[ \kappa_n(z) = \kappa_0(z) \prod_{k=1}^{n} \frac{\lambda_{k-1}(z)}{\mu_k(z)}, \quad n \geq 1, \]
\[ \kappa_0(z) = \left[ 1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{\lambda_{k-1}(z)}{\mu_k(z)} \right]^{-1}. \tag{2.5} \]

For the quantity \( \kappa_0(z) \) to be well defined, we make the following assumption.

**Assumption 2.1.** For each \( z \in \mathcal{Z} \), \( \lambda_n(z) \) and \( \mu_n(z) \) are positive such that
\[ \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{\lambda_{k-1}(z)}{\mu_k(z)} < \infty. \]

Observe that the local balance equations in (2.3) and (2.4) also hold by replacing \( \kappa_n(z) \) with \( r_n(z) \). This is not surprising, since \( r_n \) is proportional to \( \kappa_n \). Indeed, \( \kappa_n \) are normalized \( r_n \) so that their sum is equal to 1. We refer to this matrix \( T_n \) as the **nominal jump intensity matrix** for the birth-death process in state \( n \). When the environment is in state \( z \), the birth rate of \( N(t) \) in the state \( n \) is \( \lambda_n(z) \) while the death rate of \( N(t) \) in the state \( n \) is \( \mu_n(z) \). When the birth-death process is in state \( n \), the transition of the environment from state \( z \) to state \( z' \) occurs at the rate \( \tau_n(z,z')/r_n(z) \). Note that the last equation in (2.1) forbids simultaneous jumps for the subprocesses \( N \) and \( Z \). It is easy to verify that the pair \( (N,Z) \) is a well-defined Markov process in \( \mathbb{N} \times \mathcal{Z} \) with the generator \( R \). Denote its transition kernel by \( P^t((n,z),\cdot) \).

### 2.2. Main results on the stationary distribution.

We make the following assumption on the nominal jump intensity matrix \( T_n \).

**Assumption 2.2.** There exists a deterministic function \( v : \mathcal{Z} \to \mathbb{R}_+ \) satisfying
\[ v(z) \sum_{z' \in \mathcal{Z}} \tau_n(z,z') = \sum_{z' \in \mathcal{Z}} v(z') \tau_n(z',z), \quad \forall \quad n \in \mathbb{N}, \ z \in \mathcal{Z}; \tag{2.6} \]
\[ \Xi := \prod_{(n,z)} \prod_{k=1}^{n} \frac{\lambda_{k-1}(z)}{\mu_k(z)} v(z) < \infty. \]

**Theorem 2.1.** Under Assumptions 2.1 and 2.2, the Markov process \( (N,Z) \) is irreducible, aperiodic, and positive recurrent. It has an invariant probability measure
\[ \pi(n,z) := \eta(n,z)/\Xi, \quad \forall \quad (n,z) \in \mathbb{N} \times \mathcal{Z}, \tag{2.7} \]
\[ \eta(n,z) := r_n(z) v(z), \quad \forall \quad (n,z) \in \mathbb{N} \times \mathcal{Z}. \tag{2.8} \]

The transition kernel converges to this invariant measure in the total variation norm:
\[ \|P^t((n,z),\cdot) - \pi(\cdot)\|_{TV} \to 0 \quad \text{as} \quad t \to \infty, \quad \forall \quad (n,z) \in \mathbb{N} \times \mathcal{Z}. \tag{2.9} \]

**Remark 2.1.** We observe that the function \( v \) is independent of \( n \) in the above assumption, which is crucial to the product-form invariant measure. The condition in (2.6) will hold if \( \tau_n(z,z') \) takes a multiplicative form \( \tau_n(z,z') = \beta_n \tau(z,z') \) for some constant \( \beta_n > 0 \). However, one can construct examples of \( \tau_n(z,z') \) of more complicated forms that still guarantee the existence of a function \( v \) satisfying (2.6) (see, e.g., Examples 2.1 and 2.2 in [24]). We also refer to Section 3.5 on discussions on the diffusive setting (similar constructions can be done in the discrete setting too).
2.3. Examples. We start with classic queues: \(M/M/1, M/M/\infty, M/M/K, M/M/K/0\).

**Example 2.1.** \(M/M/1\) queue was already studied in [24]: \(\lambda_n(z) = \lambda(z)\) and \(\mu_n(z) = \mu(z)\) for all \(n \geq 0\) and \(z \in \mathbb{Z}\). Hence, \(r_n(z) = (\rho(z))^n\) for \(n \geq 1\) where \(\rho(z) := \lambda(z)/\mu(z)\) is the traffic intensity satisfying \(\rho(z) \in (0, 1)\) for all \(z \in \mathbb{Z}\). Also, \(\kappa_0(z) = 1 - \rho(z)\), and

\[
\kappa_n(z) = \kappa_0(z)\rho(z)^n = (1 - \rho(z))\rho(z)^n, \quad n \geq 1, \quad z \in \mathbb{Z}.
\]

The condition \(\Xi < \infty\) means that, as in [24, Assumption 2.2],

\[
\Xi = \sum_{(n,z)} \rho(z)^n v(z) = \sum_z \frac{v(z)}{1 - \rho(z)} < \infty.
\]

The invariant measure \(\eta(n, z) = \rho(z)^n v(z)\) for all \(n \geq 0\) and \(z \in \mathbb{Z}\).

**Example 2.2.** For \(M/M/\infty\) queue, \(\lambda_n(z) = \lambda(z)\) and \(\mu_n(z) = n\mu(z)\) for all \(n\) and \(z \in \mathbb{Z}\). Hence,

\[
r_n(z) = \frac{\rho(z)^n}{n!}, \quad n \geq 1, \quad \text{where} \quad \rho(z) := \frac{\lambda(z)}{\mu(z)}
\]

is the offered load. The sum of all these \(r_n(z)\) is \(\sum_{n=0}^{\infty} r_n(z) = e^{\rho(z)}\). Thus, after normalizing \(r_n(z)\) so that their sum is equal to 1, we get:

\[
\kappa_0(z) = e^{-\rho(z)}, \quad \text{and} \quad \kappa_n(z) = e^{-\rho(z)} \frac{\rho(z)^n}{n!}, \quad n \geq 1, \quad z \in \mathbb{Z}.
\]

The condition \(\Xi < \infty\) means that

\[
\Xi = \sum_{(n,z)} \frac{\rho(z)^n}{n!} v(z) = \sum_z e^{\rho(z)} v(z) < \infty.
\]

The invariant measure \(\eta(n, z) = \frac{\rho(z)^n}{n!} v(z)\) for all \(n \geq 0\) and \(z \in \mathbb{Z}\).

**Example 2.3.** For \(M/M/K\) queue, \(\lambda_n(z) = \lambda(z)\) and \(\mu_n(z) = \mu(z)(n \wedge K)\) for all \(n \geq 0\) and \(z \in \mathbb{Z}\). Hence,

\[
r_n(z) = \begin{cases} 
\frac{\mu(z)^n}{n!} & \text{for} \quad n < K, \\
\frac{\rho(z)^n}{n!} & \text{for} \quad n \geq K,
\end{cases}
\]

(2.10)

where \(\rho(z) = \lambda(z)/\mu(z)\) is the offered load, and \(\varrho(z) = \rho(z)/K\) is the traffic intensity. Also,

\[
\kappa_0(z) = \left( \sum_{n=0}^{K-1} \frac{\rho(z)^n}{n!} + \frac{\rho(z)^K}{K!} \frac{1}{1 - \rho(z)} \right)^{-1};
\]

\[
\kappa_n(z) = \begin{cases} 
\kappa_0(z) \frac{\rho(z)^n}{n!}, & n < K; \\
\kappa_0(z) \frac{\rho(z)^n}{K! \left( K \wedge n \right)}, & n \geq K.
\end{cases}
\]

We assume that the traffic intensity \(\varrho(z) = \rho(z)/K < 1\) for all \(z\), and the condition \(\Xi < \infty\) means that

\[
\Xi = \sum_{(n,z)} r_n(z)v(z) = \sum_z \left( \sum_{n=0}^{K-1} \frac{\rho(z)^n}{n!} + \sum_{n=K}^{\infty} \frac{\rho(z)^n}{K! \left( K \wedge n \right) !} \right) v(z) < \infty.
\]

The invariant measure is given by the following formula for all \(z \in \mathbb{Z}\):

\[
\eta(n, z) = \begin{cases} 
\frac{\rho(z)^n}{n!} v(z), & \text{for} \quad n < K, \\
\frac{\rho(z)^n}{K! \left( K \wedge n \right)} v(z), & \text{for} \quad n \geq K.
\end{cases}
\]
It is clear that the \( M/M/1 \) model in [24] is a special case of this model.

**Example 2.4.** For \( M/M/K/0 \) queue, \( \lambda_n(z) = \lambda(z) \) and \( \mu_n(z) = n\mu(z) \) for all \( 0 \leq n \leq K \) and \( z \in \mathcal{Z} \). Hence,

\[
\begin{align*}
r_n(z) &= \frac{\rho(z)^n}{n!}, \quad n \leq K; \quad \rho(z) = \frac{\lambda(z)}{\mu(z)} \\
\kappa_0(z) &= \left( \sum_{n=0}^{K} \frac{\rho(z)^n}{n!} \right)^{-1} \\
\kappa_n(z) &= \kappa_0(z) \left( \frac{\rho(z)^n}{n!} \right), \quad n = 1, \ldots, K.
\end{align*}
\]

The condition \( \Xi < \infty \) means that

\[
\Xi = \sum_{(n,z)} r_n(z)v(z) = \sum_{z} \left( \sum_{n=0}^{K} \frac{\rho(z)^n}{n!} \right) v(z) < \infty.
\]

The invariant measure is given by

\[
\eta(n, z) = \frac{\rho(z)^n}{n!} v(z), \quad n = 0, \ldots, K, \quad z \in \mathcal{Z}.
\]

**Example 2.5.** For \( M/M/K + M \) queue, \( \lambda_n(z) = \lambda(z) \), \( \mu_n(z) = \mu(z)(n \wedge K) + \gamma(z)(n - K)^+ \) for each \( n \geq 0 \) and \( z \in \mathcal{Z} \). For each \( z \), the rates \( \lambda(z), \mu(z), \gamma(z) \) represent the arrival, service and abandonment rates, respectively. Hence,

\[
r_n(z) = \begin{cases} 
\frac{\rho(z)^n}{n!}, & \text{for} \ n < K, \\
\frac{\rho(z)^n}{n!} K^{n-K} \beta(z)^n, & \text{for} \ n \geq K,
\end{cases}
\]

(2.11)

where \( \rho(z) = \lambda(z)/\mu(z) \) is the offered load, \( g(z) = \rho(z)/K \) is the traffic intensity, and \( \beta(z) = \lambda(z)/\gamma(z) \). For this model, the traffic intensity \( g(z) \) is allowed to take any positive value, less than 1 (underloaded), equal to 1 (critically loaded) or larger than 1 (overloaded).

Also,

\[
\kappa_0(z) = \left( \sum_{n=0}^{K-1} \frac{\rho(z)^n}{n!} + \frac{\rho(z)^K}{K!} e^{\beta(z)} \right)^{-1};
\]

\[
\kappa_n(z) = \begin{cases} 
\kappa_0(z) \left( \frac{\rho(z)^n}{n!} K^{n-K} \beta(z)^n \right), & \text{for} \ n < K; \\
\kappa_0(z) \left( \frac{\rho(z)^K}{K!} \beta(z)^n \right), & \text{for} \ n \geq K.
\end{cases}
\]

The condition \( \Xi < \infty \) means that

\[
\Xi = \sum_{(n,z)} r_n(z)v(z) = \sum_{z} \left( \sum_{n=0}^{K-1} \frac{\rho(z)^n}{n!} + \frac{\rho(z)^K}{K!} e^{\beta(z)} \right) v(z) < \infty.
\]

The invariant measure is given by the following formula for all \( z \in \mathcal{Z} \):

\[
\eta(n, z) = \begin{cases} 
\frac{\rho(z)^n}{n!} v(z), & \text{for} \ n < K, \\
\frac{\rho(z)^K}{K!} \beta(z)^n \left( \frac{1}{(n-K)!} \right) v(z), & \text{for} \ n \geq K.
\end{cases}
\]

We next give an example in biological reproduction and population growth.
Example 2.6. Consider a linear growth model with immigration ([26, Example 6.4]): for each \( z \in \mathcal{Z} \), \( \lambda_n(z) = n\lambda(z) + \theta(z) \) for \( n \geq 0 \) and \( \mu_n(z) = n\mu(z) \) for \( n \geq 1 \). Each individual gives birth at a rate \( \lambda(z) \) in the environmental state \( z \), and in addition, there is an exponential rate of growth of the population \( \theta(z) > 0 \) due to an external source such as immigration. The death rate is given by \( n\mu(z) \). Hence,

\[
r_n(z) = \frac{\prod_{k=1}^{n}((k-1)\lambda(z) + \theta(z))}{n!\mu(z)^n} = \frac{\rho(z)^n}{n!} \prod_{k=1}^{n} (k-1 + \theta(z)/\lambda(z)), \quad \text{for} \quad n \geq 0, \ z \in \mathcal{Z},
\]

where \( \rho(z) = \lambda(z)/\mu(z) \). We also have

\[
\kappa_0(z) = \left(1 + \sum_{j=1}^{\infty} \frac{\rho(z)^j}{j!} \prod_{k=1}^{j} ((k-1) + \theta(z)/\lambda(z)) \right)^{-1},
\]

\[
\kappa_n(z) = \kappa_0(z) \frac{\rho(z)^n}{n!} \prod_{k=1}^{n} ((k-1) + \theta(z)/\lambda(z)), \quad n \geq 1.
\]

The condition \( \Xi < \infty \) means that

\[
\Xi = \sum_{(n,z)} r_n(z)v(z) = \sum_{(n,z)} \frac{\rho(z)^n}{n!} \prod_{k=1}^{n} (k-1 + \theta(z)/\lambda(z))v(z) < \infty.
\]

The invariant measure is given by

\[
\eta(n, z) = r_n(z)v(z) = \frac{\rho(z)^n}{n!} \prod_{k=1}^{n} ((k-1) + \theta(z)/\lambda(z))v(z), \quad \text{for} \quad n \geq 0, \ z \in \mathcal{Z}.
\]

Example 2.7. The previous model can be also used to model growth stocks such as Internet or biotech as proposed in [13]. In that setting, the parameters \( \lambda(z) \) and \( \mu(z) \) represent the instantaneous appreciation and depreciation of the stock price due to market fluctuations, and the parameter \( \theta(z) > 0 \) represents the rate of increase in the stock price due to non-market factors such as the effect of additional shares via public offering. One can also allow the death rate to have an external effect parameter, that is, \( \mu_n(z) = n\mu(z) + \vartheta(z) \), where \( \vartheta(z) \) captures the rate of decrease in the stock price due to nonmarket factors such as dividend payments (for most growth stocks, dividends are zero). In this case, we have

\[
r_n(z) = \prod_{k=1}^{n} \frac{(k-1)\lambda(z) + \theta(z)}{k\mu(z) + \vartheta(z)}.
\]

The steady-state distribution can be used to study the size distribution of growth stocks. Our model captures the seasonal or any environmental/external effects that may impact the growth of stock values.

3. Diffusive environment

In this section, we consider birth-death processes with diffusive rates, where the environment process can be a general reflected jump diffusion process. We include the case of oblique reflection, and consider piecewise smooth domains. There is a well-developed theory of these reflected diffusion processes, see [32, 33, 28, 29] and references therein.
3.1. **Reflected diffusion process.** We consider a jump diffusion process $\tilde{Z}$ taking values in a piecewise smooth domain $Z \subset \mathbb{R}^d$ with drift function $b(z)$ and variance function $\sigma(z)$ and with reflections described below.

A **domain** in $\mathbb{R}^d$ is the closure of an open connected subset. A domain is called smooth if its boundary is a $(d-1)$-dimensional $C^2$ manifold. Consider an intersection of $m$ smooth domains $D_1, \ldots, D_m$: $Z = \cap_{i=1}^m D_i$, and assume it has a boundary $\partial D$ with $m$ ($d-1$)-dimensional faces: $F_i := \partial Z \cap \partial D_i$. Then $Z$ is called a piecewise smooth domain in $\mathbb{R}^d$.

Define by $n_i(z)$ the inward unit normal vector to $\partial D_i$ at $z \in F_i$. An example is a convex polyhedron with $D_i$ being half-spaces.

Define a continuous function $\gamma_i : F_i \to \mathbb{R}^d$ satisfying $\gamma_i(z) \cdot n_i(z) > 0$. Let $\ell_i = \{\ell_i(t) : t \geq 0\}$ be continuous nondecreasing processes such that $\ell_i$ can only grow on $F_i$, for $i = 1, \ldots, m$. For each $z \in Z$, let $\varpi(z, \cdot)$ be a finite measure on $Z$ such that $\varpi(z, \cdot) \Rightarrow \varpi(z^0, \cdot)$ as $z \to z^0$ in $Z$. Let $J(t)$ be a process that is right continuous piecewise constant, with jump measure $\varpi(\tilde{Z}(t) -, \cdot)$. We can write

$$d\tilde{Z}(t) = b(\tilde{Z}(t))dt + \sigma(Z(t))dW(t) + dJ(t) + \sum_{i=1}^m \gamma_i(\tilde{Z}(t))d\ell_i(t),$$

(3.1)

where $W(t)$ is a standard Wiener process adapted to the natural filtration. The generator $\mathcal{A}$ of the process $\tilde{Z}$ takes the form

$$\mathcal{A}g(z) = b(z) \cdot \nabla g(z) + \frac{1}{2} \text{tr}(\sigma(\sigma)'\sigma(z)\nabla^2 g(z)) + \int_Z (g(z') - g(z))\varpi(z, dz'),$$

(3.2)

for any function $g$ in the domain of $\mathcal{A}$:

$$\mathcal{D}_z = \{g \in C^2_b(Z) | \gamma_i(\tilde{z}) \cdot \nabla g(\tilde{z}) = 0, \tilde{z} \in F_i, i = 1, \ldots, m\}.$$

3.2. **Joint generator.** Consider a birth-death process in a diffusive environment described as above. We construct the joint Markov process $X = (N, Z)$ on $\mathbb{N} \times Z$ via the following generator:

$$\mathcal{L}f(n, z) = \mathcal{M}_zf(n, z) + \beta_n r_n(z)^{-1} \mathcal{A}f(n, z),$$

(3.3)

for any function $f$ in the domain of $\mathcal{L}$:

$$\mathcal{D} = \{f : \mathbb{N} \times Z \to \mathbb{R} | f(n, \cdot) \in \mathcal{D}_z \forall n \in \mathbb{N}\}.$$

Here, $\beta_n$ is the variability coefficient for the diffusive environment depending on the state $n$, while $r_n(z)$ is the impact factor from the birth-death process as defined in (2.2). Also,

$$\mathcal{M}_z g(n) = \lambda_n(z)(g(n+1) - g(n)) + 1_{n\neq 0} \mu_n(z)(g(n-1) - g(n)),$$

(3.4)

for any function $g$ in the domain of $\mathcal{M}_z$ for each given $z \in Z$. It can be regarded as the generator of a birth-death process $\tilde{N}$ with birth rate $\lambda_n(z)$ and death rate $\mu_n(z)$ for each $z \in Z$. Denote by $P^t((n, z), \cdot)$ the transition kernel of $(\tilde{N}, Z)$ for $(n, z) \in \mathbb{N} \times Z$. The joint Markov process $(N, Z)$ evolves as follows:

- If $N(t) = n \in \mathbb{N}$, then $Z$ evolves as a reflected jump diffusion in $Z$ with generator $\beta_n(r_n(z))^{-1} \mathcal{A}$ and the reflection vector fields $\gamma_1, \ldots, \gamma_m$ on faces $F_1, \ldots, F_m$ in $\mathbb{R}^d$.
- If $Z(t) = z \in Z$, then $N(t)$ jumps from $n$ to $n+1$ with rate $\lambda_n(z)$ and to $n-1$ with rate $\mu_n(z)$ (if $n \neq 0$), that is, evolving with the generator $\mathcal{M}_z$ in (3.4).
3.3. Main results on the stationary distribution. We make the following assumptions on the invariant measures of the reflected jump diffusion process.

**Assumption 3.1.** The reflected jump diffusion with generator $A$ is positive recurrent, and has a unique invariant measure $\nu$, together with boundary measures $\nu_F$, $i = 1, \ldots, m$. That is, there exists a stationary version of the process $\tilde{Z}^* = \{\tilde{Z}^*(t) : t \geq 0\}$ such that $\tilde{Z}^*(t) \sim \nu$ for all $t \geq 0$ and for each $i = 1, \ldots, m$ and bounded function $f : F_i \rightarrow \mathbb{R}$,

$$\mathbb{E} \int_0^t f(\tilde{Z}^*(s))d\ell_i(s) = t \int_{F_i} f(z)\nu_F(dz).$$

Moreover, the measure $\nu$ satisfies

$$\Xi := \sum_{n=0}^{\infty} \int_\mathcal{Z} r_n(z)v(dz) < \infty. \quad (3.5)$$

**Theorem 3.1.** Under Assumptions 2.1 and 3.1, there is a unique invariant measure for $(N, Z)$:

$$\eta(\{n\}, dz) = r_n(z)\nu(dz), \quad (3.6)$$

and the corresponding probability measure is

$$\pi(\{n\}, dz) = \Xi^{-1}r_n(z)\nu(dz). \quad (3.7)$$

The boundary measures $\pi_i$ for $F_i$ are given by

$$\pi_i(\{n\}, dz) = \Xi^{-1}r_n(z)\nu_F(dz). \quad (3.8)$$

The joint Markov process $(N, Z)$ is ergodic: for each $(n, z) \in \mathbb{N} \times \mathcal{Z}$,

$$\|P^t((n, z), \cdot) - \pi(\cdot)\|_{TV} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \quad (3.9)$$

**Remark 3.1.** We remark that the coefficient $\beta_n$ does not appear in the invariant measure $\eta(\{n\}, dz)$ in (3.6). In the jump environment, in the joint generator (2.1), we had $R[(n, z), (n, z')] = r_n(z)^{-1}r_n(z, z')$ where $r_n(z, z')$ depends on $n$. In that setting, the measure $\nu$ in Assumption 2.2 is also independent on $n$. In the construction of the joint generator $\mathcal{L}$ in (3.3), the second component $\beta_n r_n(z)^{-1}A_f(n, z)$ is purposely made in the multiplicative form $\beta_n A_f(n, z)$ such that the dependence on $n$ is through the constant $\beta_n$. However, this multiplicative construction does not entail any effect of $\beta_n$ upon the invariant measure. In Section 3.5, we discuss a more general construction where the generator $A_n$ of the reflected (jump) diffusion process depends on the state of the birth-death process through the reflection domains. It would be interesting to study more general constructions of $A_n$ with dependence on $n$.

3.4. Examples. The $M/M/1$ queue with diffusive rates was already considered in [24, Section 3] and can be regarded a special case of the $M/M/K$ queue below, so it is omitted for brevity. We first start with some cases of $M/M/\infty$ queues with diffusive rates in the following Examples 3.1–3.6.

**Example 3.1.** Let $\lambda(\cdot)$ be an RBM with a negative drift $-c$ and diffusion $\sigma^2$ in $\mathbb{R}_+$: $\lambda(\cdot) = \tilde{Z}$ where $\tilde{Z}(t) = -ct + \sigma W(t) + \ell(t)$. Next, let $\mu(t) \equiv \mu$ be a constant. The process $\tilde{Z}$ has the exponential stationary distribution

$$\nu(dz) = \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2}z\right)dz. \quad (3.10)$$
Thus \( r_n(z) = (z/\mu)^n/n! \). The condition \( \Xi < \infty \) means that
\[
\Xi = \sum_{n=0}^{\infty} \int_{0}^{\infty} r_n(z) \nu(dz) = \int_{0}^{\infty} \left( \sum_{n=0}^{\infty} \frac{(z/\mu)^n}{n!} \right) \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2} z\right) dz \\
= \int_{0}^{\infty} e^{z/\mu} \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2} z\right) dz = \frac{2c}{\sigma^2} \int_{0}^{\infty} e^{-\left(\frac{2c}{\sigma^2} -1/\mu\right) z} dz \\
= \frac{2c}{\sigma^2} \left( \frac{2c}{\sigma^2} - 1/\mu \right)^{-1} < \infty,
\]
provided that
\[
\frac{2c}{\sigma^2} - \frac{1}{\mu} > 0. \tag{3.11}
\]
Under condition (3.11), the invariant measure of the joint process \((N, Z)\) is given by:
\[
\eta(\{n\}, dz) = r_n(z) \nu(dz) = \frac{(z/\mu)^n}{n!} \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2} z\right) dz.
\]

**Example 3.2.** Let \( \lambda(\cdot) \equiv \lambda \) be a constant and \( \mu(t) \) be an RBM with a negative drift in \([\mu_0, \infty)\) for \( \mu_0 > 0 \), similarly to \( \lambda \) in the previous example. We have the same invariant measure \( \nu(dz) \) from (3.10), shifted by \( \mu_0 \); it has Lebesgue density
\[
\nu(dz) = \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2}(z-\mu_0)\right) dz. \tag{3.12}
\]
Next, \( r_n(z) = (\lambda/z)^n/n! \). The condition \( \Xi < \infty \) means that
\[
\Xi = \sum_{n=0}^{\infty} \int_{\mu_0}^{\infty} r_n(z) \nu(dz) = \int_{\mu_0}^{\infty} \left( \sum_{n=0}^{\infty} \frac{(\lambda/z)^n}{n!} \right) \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2}(z-\mu_0)\right) dz \\
= \int_{\mu_0}^{\infty} e^{\lambda/z} \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2}(z-\mu_0)\right) dz < \infty.
\]
(If we took \( \mu_0 = 0 \), then this integral would be infinite.) The invariant measure of the joint process \((N, Z)\) is
\[
\eta(\{n\}, dz) = r_n(z) \nu(dz) = \frac{(\lambda/z)^n}{n!} \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2}(z-\mu_0)\right) dz.
\]

**Example 3.3.** Let \((\lambda, \mu)\) form a two-dimensional RBM in the shifted positive quadrant \( \mathbb{R}_+ \times [\mu_0, \infty) \) (the condition on \( \mu_0 \) to be specified below). Specifically, \((\lambda(\cdot), \mu(\cdot)) = (Z_1, Z_2)\), where \( Z = (Z_1, Z_2) \) is a 2-d RBM in \( \mathbb{R}_+ \times [\mu_0, \infty) \), given by \( Z(t) = ct + \sigma W(t) + RY(t) \), with \( c = (c_1, c_2)' \) being the drift, \( \sigma \) being the covariance coefficient, \( R \) being the reflection matrix, and \( Y(t) \) being the regulating process (continuous and nondecreasing, \( Y(0) = 0 \) and \( Y_i \) increases at times for which \( Z_i = 0 \) for \( i = 1, 2 \)). It is shown in [11] that if \( Z \) is positive recurrent and the skew-symmetry condition is satisfied: \( 2\Sigma = RD + DR' \) with \( \Sigma = \sigma \Sigma' \) and \( D = \text{diag}\{\Sigma_{ii}\} \), then the invariant measure has an explicit product form, that is, \( \nu(dz_1, dz_2) = \alpha_1 e^{-\alpha_1 z_1 - \alpha_2 (z_2 - \mu_0)} dz_1 dz_2 \), where \( \alpha_i = 2c_i \xi_i/\Sigma_{ii} \) and \( \xi = R^{-1}c \). See also the survey [41]. Assume that \( \mu_0 > 1/\alpha_1 \).

We have \( r_n(z) = \rho(z)^n/n! = (z_1/z_2)^n/n! \). We also obtain
\[
\Xi = \sum_{n=0}^{\infty} \int_{0}^{\infty} \int_{\mu_0}^{\infty} \frac{1}{n!} \left( \frac{z_1}{z_2} \right)^n \alpha_1 \alpha_2 e^{-\alpha_1 z_1 - \alpha_2 (z_2 - \mu_0)} dz_1 dz_2
\]
\[
\begin{align*}
&= \int_0^\infty \int_0^\infty e^{z_1/z_2} \omega_1 \omega_2 e^{-\omega_1 z_1 - \omega_2 (z_2 - \mu_0)} dz_1 dz_2 \\
&= \omega_1 \omega_2 \int_0^\infty \left( \int_0^\infty e^{-(\omega_1 - 1/z_2) z_1} dz_1 \right) e^{-\omega_2 (z_2 - \mu_0)} dz_2 \\
&= \omega_1 \omega_2 \int_0^\infty \frac{1}{\omega_1 - 1/z_2} e^{-\omega_2 (z_2 - \mu_0)} dz_2 \\
&= \frac{\omega_1 \omega_2}{\omega_1 - 1/\mu_0} \times \frac{1}{\omega_2} = \frac{\omega_1}{\omega_1 - 1/\mu_0} < \infty.
\end{align*}
\]

Here in the fourth and fifth lines we have used the condition \( z_2 > \mu_0 > 1/\omega_1 \). Thus, the invariant measure of the joint process \((N, Z)\) is given by

\[
\eta(\{n\}, dz_1, dz_2) = r_n(z) \nu(dz) = \frac{(z_1/z_2)^n}{n!} \omega_1 \omega_2 e^{-\omega_1 z_1 - \omega_2 (z_2 - \mu_0)} dz_1 dz_2.
\]

**Example 3.4.** Let us modify Example 3.1 to make the arrival rate a reflected Ornstein-Uhlenbeck process: \( \lambda(\cdot) \equiv Z \), with \( Z \) satisfying the stochastic differential equation (SDE) with reflection on \( \mathbb{R}_+ \):

\[
dZ(t) = -cZ(t) \, dt + \sigma dW(t) + d\ell(t). \tag{3.13}
\]

By [40], its stationary distribution is one-sided Gaussian: Its density is proportional to

\[
v(dz) = \exp \left( -\frac{2c}{\sigma^2} z^2 \right) dz. \tag{3.14}
\]

Similarly to Example 3.1, \( r_n(z) = (z/\mu)^n/n! \). The condition \( \Xi < \infty \) is satisfied since

\[
\begin{align*}
\int_0^\infty \sum_{n=0}^\infty r_n(z) v(z) \, dz &= \int_0^\infty \exp \left( \frac{z}{\mu} - \frac{2c}{\sigma^2} z^2 \right) \, dz \\
&= \exp \left( \frac{\sigma^2}{2cm^2} \right) \int_0^\infty \exp \left( -\frac{2c}{\sigma^2} \left( z - \frac{\sigma^2}{4cm} \right)^2 \right) \, dz \\
&= \sigma \sqrt{\frac{\pi}{2c}} \Phi \left( -\frac{\sigma^2}{4cm} \right) < \infty,
\end{align*}
\]

where \( \Phi(\cdot) \) is the c.d.f. of the standard normal distribution. Thus, the invariant measure of the joint process \((N, Z)\) is given by

\[
\eta(\{n\}, dz) = r_n(z) \nu(dz) = \frac{(z/\mu)^n}{n!} \exp \left( -\frac{2c}{\sigma^2} z^2 \right) \, dz.
\]

**Example 3.5.** Next, let us modify Example 3.2 to make the service rate a reflected Ornstein-Uhlenbeck process (3.13), but on \([\mu_0, \infty)\) for some \( \mu_0 > 0 \). Its stationary distribution has density proportional to \( v \) from (3.14). We have \( r_n(z) = (\lambda/z)^n/n! \). So the condition \( \Xi < \infty \) is satisfied by verifying

\[
\begin{align*}
\int_{\mu_0}^\infty \sum_{n=0}^\infty r_n(z) v(z) \, dz &= \int_{\mu_0}^\infty \exp \left[ \frac{\lambda}{z} - \frac{2c}{\sigma^2} (z - \mu_0)^2 \right] \, dz \\
&= < \infty.
\end{align*}
\]

(The integral is finite since \( z > \mu_0 > 0 \).) Then the invariant measure of the joint process \((N, Z)\) is given similarly.

Of course one could also consider the model of the arrival and service rates being a two-dimensional reflected Ornstein-Uhlenbeck process in the positive orthant. However, the explicit expression for its invariant measure is unknown.
This example and the previous one can be extended to one-dimensional reflected diffusions with piecewise linear drift, which will have truncated Gaussian invariant measures, see [2].

**Example 3.6.** In the previous examples of $M/M/\infty$ queues with diffusive rates, we have considered the cases of the parameters being reflected diffusions with drifts. One can consider the parameters being diffusions constrained in compact sets. For example, we can take a constant arrival rate and a service rate being a RBM without drift on $[\mu_0, \mu_1] \subset (0, \infty)$. Alternatively, we can take a constant service rate and arrival rate a RBM without drift on $[\lambda_0, \lambda_1] \subset (0, \infty)$. Yet another alternative: we can take the arrival and service rates as a two-dimensional RBM on a piecewise smooth region which is a compact subset of $\mathbb{R}_+^2$. In all these cases, the stationary distribution is uniform in one or two dimensions, the ratio of the arrival and service rates $\lambda(z)/\mu(z)$ is bounded, and thus the condition $\Xi < \infty$ is satisfied, and the invariant measures are then explicitly given accordingly.

**Example 3.7.** Let us modify Example 3.1 to include Poisson jumps. Consider an RBM on the half-line with a negative drift $-c$ and diffusion $\sigma^2$, and with i.i.d. jumps with intensity $\lambda$ and distribution of jump size $\Lambda$ (so that $\lambda \Lambda(\cdot)$ is the spectral measure), supported on $\mathbb{R}_+$. Then the generator of the process is given by

$$\mathcal{L} f(x) = -cf'(x) + \frac{\sigma^2}{2} f''(x) + \lambda \int_0^\infty [f(x+y) - f(x)] \Lambda(dy),$$

(3.15)

for $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ in $C^2$ with condition $f'(0) = 0$. The implied drift is the sum of continuous drift $-c$ and the average shift of the process due to jumps, which, in turn, is equal to the intensity of jumps times average size $\overline{\Lambda}$ of each jump: $M := -c + \lambda \overline{\Lambda}$. If this drift is negative: $M < 0$, then the process is ergodic, see [27, Section 6]. Its stationary distribution in general does not have an explicit density (like an exponential). But we can find its moment generating function (MGF) $M_\nu$, rewriting (3.15) in terms of Laplace image. Denote by $M_\Lambda$ the MGF of the jump measure (=normalized spectral measure). After adjusting [19, Example 4.3] for notation, the MGF of this stationary distribution is

$$M_\nu(u) := \frac{M u}{F(u)}, \quad F(u) := cu - \frac{1}{2} \sigma^2 u^2 - \lambda M_\Lambda(u) + \lambda.$$

(3.16)

Note that $F$ is a concave function, and it has two zeros: $u = 0$ and $u = u_0 > 0$. The latter is true since $F'(0) = c - \lambda M'\Lambda(0) = c - \lambda \overline{\Lambda} = -M > 0$. Similarly to Example 3.1, the quantity $\Xi$ is equal to the value of the MGF of this stationary distribution: $\Xi = M_\nu(u_0)$. Thus it is finite if $\mu^{-1} < u_0$. This condition is an analogue of (3.11).

Similar extensions of continuous diffusive cases to jump-diffusion cases could be done for the previous examples.

**Example 3.8.** We consider $M/M/K$ queues with diffusive arrival and/or service rates:

(a) Let $\lambda$ be an RBM taking values in $[0, \mu K]$ and $\mu =$ const.

(b) Let $\lambda =$ const and $\mu$ be an RBM with a negative drift on $[\mu_0, \infty)$ for $\mu_0 > \lambda/K$.

(c) Let $(\lambda, \mu)$ be a two-dimensional RBM in a wedge $\{z = (z_1, z_2) \in \mathbb{R}_+^2 : z_1 \leq z_2 K\}$.

In all these cases, we have $r_n(z)$ given in (2.10) in Example 2.3. Case (a) is similar to those discussed in the previous example. Case (b), we have the invariant measure $\nu(dz)$ given in (3.12), so the condition $\Xi < \infty$ requires that

$$\Xi = \sum_n \int_{\mu_0}^\infty r_n(z) \nu(dz).$$
= \int_{\mu_0}^{\infty} \left( \sum_{n=0}^{K-1} \frac{(\lambda/z)^n}{n!} + \sum_{n=K}^{\infty} \frac{(\lambda/z)^n}{K!K^{n-K}} \right) 2c \sigma^2 \exp \left( - \frac{2c}{\sigma^2}(z - \mu_0) \right) dz < \infty.

Case (c) seems much more challenging, which require a good understanding of the invariant measure of two-dimensional RBMs in a wedge (see, e.g., [7]).

For $M/M/K/0$ queue, since there is no stability condition and the queueing state is finite, the underlying diffusive environment for either arrival or service rates or both can be any reflected diffusion as discussed above as long as the invariant measure $\nu$ exists. Then the invariant measure for the joint process $(N, Z)$ is $\eta(n, dz) = r_n(z) \nu(dz)$ for the $r_n(z)$ given in Example 2.4.

Example 3.9. For $M/M/K+M$ queue, as discussed in Example 2.5, the arrival, service and abandonment rates can all depend on the environment. One interesting case is that the rates $(\lambda(\cdot), \mu(\cdot), \gamma(\cdot)) = (Z_1, Z_2, Z_3)$ where $Z = (Z_1, Z_2, Z_3)$ is an RBM in $\mathbb{R}_+ \times [\mu_0, \infty) \times [\gamma_0, \infty) \subset \mathbb{R}_+^3$, taking the form $Z(t) = ct + \alpha W(t) + \beta Y(t)$ as in Example 3.3 but in three dimensions. The positive constants $\mu_0 > 0$ and $\gamma_0 > 0$ satisfy $\gamma_0 > 1/\alpha_1$. Under the positive recurrence and skew-symmetry conditions as shown in [11], the process $Z = (Z_1, Z_2, Z_3)$ has a product-form invariant measure $\nu(dz_1, dz_2, dz_3) = \alpha_1 \alpha_2 \alpha_3 e^{-\alpha_1 Z_1 - \alpha_2 (z_2 - \mu_0) - \alpha_3 (z_3 - \gamma_0)} dz_1 dz_2 dz_3$.

We have the same formula of $r_n(z)$ in (2.11) with $\rho(z) = z_1/z_2$ and $\beta(z) = z_1/z_3$. Thus, the condition $\Xi < \infty$ requires that

$$\Xi = \sum_{n \in \mathbb{R}_+ \times [\mu_0, \infty) \times [\gamma_0, \infty)} r_n(z) \nu(dz)$$

$$= \int_{\mathbb{R}_+ \times [\mu_0, \infty) \times [\gamma_0, \infty)} \left( \sum_{n=0}^{K-1} \frac{(z_1/z_2)^n}{n!} + \frac{(z_1/z_2)^K}{K! e^{z_1/z_3}} \right)$$

$$\times \alpha_1 \alpha_2 \alpha_3 e^{-\alpha_1 z_1 - \alpha_2 (z_2 - \mu_0) - \alpha_3 (z_3 - \gamma_0)} dz_1 dz_2 dz_3.$$

It is then easy to check similarly to Example 2.5 that the first component of the integral is finite since $z_2 > \mu_0 > 0$ and the second component of the integral is finite under the conditions $z_3 > \gamma_0 > 1/\alpha_1$ and $z_2 > \mu_0 > 0$. Thus, the invariant measure of $(N, Z)$ is

$$\eta(n, dz_1, dz_2, dz_3) = \left( \sum_{n=0}^{K-1} \frac{(z_1/z_2)^n}{n!} + \frac{(z_1/z_2)^K}{K! e^{z_1/z_3}} \right)$$

$$\times \alpha_1 \alpha_2 \alpha_3 e^{-\alpha_1 z_1 - \alpha_2 (z_2 - \mu_0) - \alpha_3 (z_3 - \gamma_0)} dz_1 dz_2 dz_3.$$

The cases where one or two of the three parameters are diffusive can be similarly considered.

Example 3.10. For the linear population growth model with immigration in Example 2.6, the parameters $\lambda, \mu, \theta$ can be all diffusive, as a three-dimensional RBM in $\mathbb{R}_+ \times [\mu_0, \infty) \times \mathbb{R}_+$ discussed in the above example. The constant $\mu_0 > 0$ satisfies $\mu_0 > 1/\alpha_1$. The process $Z = (Z_1, Z_2, Z_3)$ has a product-form invariant measure $\nu(dz_1, dz_2, dz_3) = \alpha_1 \alpha_2 \alpha_3 e^{-\alpha_1 Z_1 - \alpha_2 (z_2 - \mu_0) - \alpha_3 z_1} dz_1 dz_2 dz_3$. We have the formula of $r_n(z)$ in (2.12) with $\rho(z) = z_1/z_2$. The condition $\Xi < \infty$ requires that

$$\Xi = \int_{\mathbb{R}_+ \times [\mu_0, \infty) \times \mathbb{R}_+} \sum_{n} \frac{(z_1/z_2)^n}{n!} \prod_{k=1}^{n} \left( k - 1 + \frac{z_3}{z_1} \right) \alpha_1 \alpha_2 \alpha_3 e^{-\alpha_1 z_1 - \alpha_2 (z_2 - \mu_0) - \alpha_3 z_3} dz_1 dz_2 dz_3.$$
which can be verified to be finite under the condition $z_2 > \mu_0 > 1/\alpha_1$, similarly as the example above. Thus the invariant measure of $(N, Z)$ is

$$\eta\{n\}, dz_1, dz_2, dz_3 = \frac{(z_1/z_2)^n}{n!} \prod_{k=1}^{n} (k - 1 + \frac{z_3}{z_1}) \alpha_1 \alpha_2 \alpha_3 e^{-\alpha_1 z_1 - \alpha_2 (z_2 - \mu_0) - \alpha_3 z_3} dz_1 dz_2 dz_3.$$

The growth stock model in Example 2.7 with four parameters can be also considered analogously with the parameters being a four-dimensional RBM in a subset of $\mathbb{R}^4_+$.

3.5. A more general reflected (jump) diffusion environment. As mentioned in Remark 3.1, in this section we consider a more general setup where the reflection domains of the (jump) diffusion depend on the state $n$ of the birth-death process. Such a setup was considered in Section 3.4 of [24], so we present the idea on the construction and some examples.

For each $n \in \mathbb{N}$, let $D_n \subset \mathcal{Z}$ be a piecewise smooth domain with $m_n$ faces $F_1(n), \ldots, F_{m_n}(n)$ of the boundary $\partial D_n$ and reflection vector fields $f_i(n) : F_i(n) \to \mathbb{R}^d$. Assume that $\mathcal{Z} = \bigcup_{n \in \mathbb{N}} D_n$.

We construct the joint Markov process $X = (N, Z)$ on $\mathbb{N} \times \mathcal{Z}$ via the following generator and its domain:

$$\mathcal{L} f(n, z) = \mathcal{M}_z f(n, z) + \beta_n r_n(z)^{-1} A_n f(n, z),$$

$$D = \{ f : \mathbb{N} \times \mathcal{Z} \to \mathbb{R} | f(n, \cdot) \in \mathcal{D}_n^0 \ \forall n \in \mathbb{N} \}, \quad (3.17)$$

And we construct the auxiliary generator $A_n$ and its domain as follows:

$$A_n g(z) = b(z) \cdot \nabla g(z) + \frac{1}{2} \text{tr}(\sigma(z) \cdot \sigma(z) \nabla^2 g(z)) + \int_{D_n} (g(z') - g(z)) \omega(z, dz'),$$

$$\mathcal{D}_n^0 = \{ g \in C_b^0(D_n) | \gamma_i(\bar{z}) \cdot \nabla g(\bar{z}) = 0, \ \bar{z} \in F_i(n), \ i = 1, \ldots, m \}. \quad (3.18)$$

We modify the conditions in Assumption 3.1 as follows. We assume that the reflected jump diffusion with generator $A_n$ is positive recurrent, and has a unique invariant measure $\nu_{D_n}^{(n)}$, with boundary measures $\nu_{F_i(n)}^{(n)}$, $i = 1, \ldots, m_n$. Moreover, the measure $\nu_{D_n}^{(n)}$ satisfies

$$\Xi := \sum_{n=0}^{\infty} \int_{D_n} r_n(z) \nu_{D_n}^{(n)}(dz) < \infty.$$ 

Then by slightly modifying the proof of Theorem 3.1, we can show that under the above conditions, the joint Markov process $(N, Z)$ has a unique invariant measure

$$\pi\{n\}, dz = \Xi^{-1} r_n(z) \nu_{D_n}^{(n)}(dz),$$

and the corresponding boundary measures $\nu_{F_i(n)}^{(n)}$ for $F_i(n)$ (if there is reflection) are given by

$$\pi_{F_i(n)}\{n\}, dz = \Xi^{-1} r_n(z) \nu_{F_i(n)}^{(n)}(dz), \quad i = 1, \ldots, m_n.$$

We next discuss some examples.

**Example 3.11.** Recall that in Examples 3.1–3.6, we have discussed the $M/M/\infty$ queues with arrival and/or service rates being a reflected diffusion (Brownian motion or Ornstein-Uhlenbeck) on a proper domain. However, one may allow the domains to depend on the
state of the queue. In Example 3.2, one can consider the service rate being an RBM with a negative drift in \([\mu_n, \infty)\) for each \(n \in \mathbb{N}\) with \(\mu_n > 0\). We then have the invariant measure
\[
\nu^{(n)}(dz) = \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2}(z - \mu_n)\right) dz.
\]
The condition \(\Xi < \infty\) means that
\[
\Xi = \sum_{n=0}^{\infty} \int_{\mu_n}^{\infty} r_n(z) \nu^{(n)}(dz) = \sum_{n=0}^{\infty} \int_{\mu_n}^{\infty} \left(\frac{\lambda}{z}\right)^n \frac{2c}{\sigma^2} \exp\left(-\frac{2c}{\sigma^2}(z - \mu_n)\right) dz < \infty.
\]
The invariant measure of the joint process \((N, Z)\) is
\[
\eta(\{n\}, dz) = r_n(z) \nu^{(n)}(dz) = \left(\frac{\lambda}{z}\right)^n \frac{2c}{n!} \exp\left(-\frac{2c}{\sigma^2}(z - \mu_n)\right) dz.
\]
Similarly extensions can be done in the other examples in the previous subsection.

4. Rate of Convergence to Stationarity

In this section we study the rate of convergence to stationarity of the joint Markov processes \((N, Z)\) constructed in the previous two sections. We will focus on the diffusive random environment since the jump environment can be studied similarly. We consider two scenarios where the convergence rate is either exponential or polynomial. In both scenarios, we make an assumption on the rate of convergence for the generator associated with the random environment, which is exponential. We impose two sets of conditions on the birth and death rates which result in either an exponential rate or a polynomial rate. We use the coupling approach to establish the rates of convergence by adapting the approach by Lindvall [17] for birth-death processes and also by generalizing the coupling approach in [24] for \(M/M/1\) queues in random environment.

4.1. Exponential convergence rate. We impose the following condition on the birth and death rates. Define \(q_i(z) := \lambda_i(z) + \mu_i(z)\) and \(p_i(z) = \lambda_i(z)/q_i(z)\) for each \(i \in \mathbb{N}\) and \(z \in \mathbb{Z}\).

**Assumption 4.1.** Assume that
\[
\overline{\lambda} := \inf\inf_{z \in \mathbb{Z}} q_i(z) > 0, \quad \overline{\lambda} := \sup\sup_{z \in \mathbb{Z}} p_i(z) < 1/2.
\]  
(4.1)

In addition,
\[
\sup_{z \in \mathbb{Z}} \sup_{i} \frac{\lambda_{i-1}(z)}{\mu_i(z)} \leq \frac{\overline{\lambda}}{1 - \overline{\lambda}}.
\]  
(4.2)

**Remark 4.1.** The conditions in (4.1) in Assumption 4.1 was imposed for the study of exponential rate of convergence of birth-death processes (without environment states \(z\)) in [17, Proposition 2]. For the queuing examples with a finite number of servers, the second condition \(\overline{\lambda} < 1/2\) in (4.1) implies that the traffic intensity is less than one. On the other hand, for infinite-server queues and finite-server models with abandonment, this condition is not particularly used to guarantee stability. However, this condition is used in the proof using the coupling approach where the dominating embedded Markov chain with \(\overline{\lambda}\) as a parameter is used. This is critical in Step 4 of the proof of Theorem 4.1 below. The condition on the ratio of birth and death rates in (4.2) is required in Step 1 of that proof.
We then make the following assumption on the exponential rate of convergence associated with the generator \( A \) (which is the same as [24, Assumption 4.1]). It imposes a condition on the coupling time with an exponential tail for the environment process. Examples of the environment processes satisfying this assumption are also given in [24, Section 4.3.2]. For instance, an RBM on \([0,a]\) in [12, Chapter 2, Problem 8.2].

**Assumption 4.2.** There exist constants \( \alpha > 1 \) and \( \gamma > 0 \) such that for all \( z_1, z_2 \in \mathbb{Z} \) we can couple two processes \( Z_1, Z_2 \) with generator \( A \), starting from \( Z_1(0) = z_1 \) and \( Z_2(0) = z_2 \), in time \( \tau_{z_1,z_2} := \inf \{ t \geq 0 \mid Z_1(t) = Z_2(t) \} \), with

\[
P(\tau_{z_1,z_2} \geq t) \leq \alpha e^{-\gamma t}.
\]

Next, we define the following auxiliary function:

\[
\theta(\alpha, \beta, \gamma, a) := \frac{\beta}{\beta - a} - \frac{a \gamma}{(\beta - a)(\beta + \gamma - a)} \alpha^{-(\beta - a)/\gamma},
\]

for any \( \alpha > 1 \), \( \beta, \gamma > 0 \) and \( a \geq 0 \). This quantity appears to be an upper bound for the MGF of the minimum of an exponential random variable and an independent variable with exponential tail (see Lemma 6.1 in the Appendix and [24, Lemma 6.1]). Also, define

\[
g(s) := \frac{1 - \sqrt{1 - bs^2}}{2ps}, \quad \text{with} \quad b := 4\overline{p}(1 - \overline{p}),
\]

\[
G(u) := g\left(\frac{\overline{q}}{\overline{q} - u}\right).
\]

**Remark 4.2.** From [9, Section 14.5], take a discrete-time random walk \( S = (S_n)_{n \geq 0} \) on \( \mathbb{Z} \) with steps \(+1\) and \(-1\) with probabilities \( \overline{p} \) and \( 1 - \overline{p} \). By Assumption 4.1, the overall direction of this random walk is down since \( \overline{p} < 1/2 \): \( \mathbb{E}[S_n - S_0] < 0 \). Starting from \( S_0 = 1 \), the hitting time \( \tau := \min \{ n \geq 0 \mid S_n = 0 \} \) is a.s. finite, and it has generating function:

\[
\mathbb{E}[s^\tau] = g(s).
\]

**Remark 4.3.** Note that \( b < 1 \) since \( \overline{p} < 1/2 \) under Assumption 4.1. The function \( g(s) \) is defined for \( s \in (0, b^{-0.5}] \). Since \( \tau \geq 1 \), this function is increasing. This maximal value is achieved at \( s = b^{-0.5} \), and is equal to

\[
g(b^{-0.5}) = \frac{1}{2pb^{-0.5}} = \left[\frac{1 - \overline{p}}{\overline{p}}\right]^{1/2}.
\]

Thus the function \( G \) is defined for \( u \in [0, u_\ast] \) with \( u_\ast < \overline{q} \) solving the equation \( \overline{q}/(\overline{q} - u_\ast) = b^{-0.5} \). This solution exists, and is unique and nonnegative.

Let us compute

\[
G(u_\ast) = g(b^{-0.5}) = (2\overline{p}b^{-0.5})^{-1} = \frac{\sqrt{\overline{p}(1 - \overline{p})}}{\overline{q}},
\]

\[
\theta(\alpha, \overline{q}, \gamma, u_\ast) = \frac{\overline{q}}{\overline{q} - u_\ast} - \frac{u_\ast \gamma}{(\overline{q} - u_\ast)(\overline{q} - u_\ast + \gamma)} \alpha^{-(\overline{q} - u_\ast)/\gamma}
\]

\[
= \frac{1}{\sqrt{4\overline{p}(1 - \overline{p})}} \left(1 - \frac{\sqrt{4\overline{p}(1 - \overline{p})} + 1}{\gamma - 1\sqrt{4\overline{p}(1 - \overline{p})}} \alpha^{-\gamma - 1\sqrt{4\overline{p}(1 - \overline{p})}}\right).
\]

Here is the main result on exponential convergence, extending [24, Theorem 4.1].
Theorem 4.1. Suppose that the stationarity condition in Assumption 3.1 for the generator $A$ holds. Under the additional Assumptions 4.1 and 4.2, we get the following convergence result. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $z \in \mathcal{Z}$,

$$\|P^t((n, z), \cdot) - \pi(\cdot)\|_{TV} \leq C(1 + G(u)^n)e^{-\kappa t},$$

(4.7)

with $\pi(\cdot)$ given in (3.7) and $z = (1 - \varepsilon)u$, where $\varepsilon \in (0, 1)$, and $u \in (0, u^*)$ satisfying

$$G(u)\theta(\alpha, q, \gamma, u) < \left(1 - \alpha^{-\gamma/\gamma} \frac{\gamma}{q + \gamma}\right)^{-\varepsilon/(1-\varepsilon)},$$

(4.8)

Remark 4.4. The equation (4.8) holds for some $u$ in the right neighborhood of zero. Indeed, its left-hand side is continuous in $u$, and its right-hand side is independent of $u$. Now, the left-hand side value at $u = 0$ is equal to 1 (since $G(1) = g(0) = 1$, and $\theta(\alpha, q, \gamma, 0) = 1$). The right-hand side is greater than 1 (since $-\varepsilon/(1-\varepsilon) < 0$).

Proof. We modify the proof of [24, Theorem 4.1] and further develop the coupling approach for the joint Markov process. The general idea of the coupling approach is to take two copies $(N_1, Z_1)$ and $(N_2, Z_2)$ of the joint process $(N, Z)$ starting from the corresponding initial states $(n_1, z_1)$ and $(n_2, z_2)$, respectively, and couple them (on the same probability space) such that the coupling time $\tau = \inf\{t \geq 0 : N_1(t) = N_2(t), Z_1(t) = Z_2(t)\}$ satisfies $\mathbb{E}[e^{\kappa \tau}] < \infty$ for some constant $\kappa > 0$, and then the Lindvall inequality can be applied to obtain the upper bound for $\|P^t((n_1, z_1), \cdot) - P^t((n_2, z_2), \cdot)\|_{TV} \leq \mathbb{E}[e^{\kappa \tau}]e^{-\kappa t}$. To construct the coupling, we need to first wait until the birth-death processes are coupled at zero and then wait either the environment processes are coupled or the birth-death processes have a birth in which case the coupling process restarts. The main task is then how to estimate $\mathbb{E}[e^{\kappa \tau}]$ for such a coupling time $\tau$ and identify the exponent $\kappa$. The proof goes in the following seven steps, the same as in [24, Theorem 4.1], while some steps require major changes as highlighted below.

Step 1. Observe that to prove (4.7) it suffices to show that

$$\|P^t((n_1, z_1), \cdot) - P^t((n_2, z_2), \cdot)\|_{TV} \leq (G(u)^{n_1} + G(u)^{n_2})e^{-\kappa t}.$$ 

This is because we can obtain (4.7) from the above by rewriting it using the definition of total variation and integrating with respect to $(n_2, z_2) \sim \pi$, and we have the integrability

$$\sum_{n=0}^{\infty} \int_{\mathcal{Z}} G(u)^n r_n(z) \nu(dz) < \infty,$$

(4.9)

under Assumption 4.1. This integrability (4.9) is proved as follows. There exists a constant $c \in (0, 1)$ such that for all $z \in \mathcal{Z}$ and $u \in (0, u^*)$, we get:

$$\lim_{n \to \infty} \frac{G(u)^n \cdot r_n(z)}{G(u)^{n-1} \cdot r_{n-1}(z)} \leq c.$$

(4.10)

Combining (4.10) with the observation that the term corresponding to $n = 0$ is $G(u)^0 r_0(z) = 1$, we get: the series inside the integral in (4.9) is estimated from above by $1 + c + c^2 + \ldots = 1/(1 - c)$. Integrating this with respect to the probability measure $\nu$, we complete the proof of (4.9). Now, let us show (4.10). We can replace $G(u) = g(s)$ for $s = q/(q - u)$. Using (2.2), we can replace

$$\frac{r_n(z)}{r_{n-1}(z)} = \frac{\lambda_n(z)}{\mu_n(z)},$$

(4.11)
Combining (4.11) with (4.2) and (4.6) in Remark 4.3, we get that the left-hand side of (4.10) is bounded by $c = \left[ \frac{\tau}{1 - \tau} \right]^{3/2}$.

**Step 2.** By Lindvall’s inequality [17], it then requires to prove that for the coupling time $\tau$ defined above, $E[e^{\lambda \tau}] < \infty$, so that

$$
\|P^t((n_1, z_1), \cdot) - P^t((n_2, z_2), \cdot)\|_{TV} \leq 2P(\tau > t) \leq 2E[e^{\lambda \tau}]e^{-\lambda t}. \quad (4.12)
$$

**Step 3.** The coupling time $\tau$ can be written as follows:

$$
\tau = \sum_{j=0}^{J-1}(\tau_j + \eta_j) + \zeta_J \quad (4.13)
$$

where the initial random times are defined as

$$
\tau_0 := \inf\{t \geq 0 \mid N_1(t) = N_2(t) = 0\},
$$

$$
\eta_0 \sim \text{Exp}(\lambda), \quad \bar{\lambda} = \sup \sup \lambda_i(z),
$$

$$
\zeta_0 := \inf\{t \geq 0 \mid Z_1(\tau_0 + \cdot) = Z_2(\tau_0 + \cdot)\},
$$

and then the random times are defined iteratively as follows: for $j \geq 1$,

$$
\tau_j := \inf\{t \geq 0 \mid N_1(t + \tau_{j-1} + \eta_{j-1}) = N_2(t + \tau_{j-1} + \eta_{j-1}) = 0\}, \quad \eta_j \sim \text{Exp}(\bar{\lambda}),
$$

$$
\zeta_j := \inf\{t \geq 0 \mid Z_1(\tau_0 + \tau_{j-1} + \cdot) = Z_2(\tau_0 + \tau_{j-1} + \cdot)\},
$$

and the stopping time $J$ is defined by

$$
J = \min\{j \geq 0 : \zeta_j < \eta_j\}. \quad (4.14)
$$

Note that $\zeta_k$ and $\eta_k$ are independent.

Observe that if $\zeta_0 < \eta_0$, then the time $S_0 = \tau_0 + \zeta_0$ is the coupling time for $(N_1, Z_1)$ and $(N_2, Z_2)$, and otherwise, if $\zeta_0 > \eta_0$ and if $\zeta_0 < \eta_1$, then the $S_1 = \tau_0 + \eta_0 + \tau_1 + \zeta_1$ is the coupling time for $(N_1, Z_1)$ and $(N_2, Z_2)$. This procedure continues until the coupling time

$$
S_J = \sum_{j=0}^{J}(\tau_j + \eta_j \wedge \zeta_j), \quad (4.15)
$$

which is the equal to the expression of $\tau$ above in (4.13).

**Step 4.** We show that

$$
E[e^{u\tau_k}] \leq G(u)^{n_1 V n_2}, \quad k = 0, 1, \ldots \quad (4.16)
$$

Consider the case of $\tau_0$, which is the coupling time if there were no random environment. Take the embedded discrete-time Markov chain (birth-death process) with transition probability from $i$ to $i+1$ given by $p_i(z)$, and the transition probability from $i$ to $i-1$ given by $1 - p_i(z)$. This process, in turn, is dominated by the discrete-time birth-death process $N = (N_n)_{n \geq 0}$, with transition probabilities $\overline{p}$ and $1 - \overline{p}$, up and down. Let $V := \min\{n \geq 0 \mid N_n = 0\}$. Then we have stochastic domination:

$$
\tau_0 \leq U_1 + \ldots + U_V, \quad U_i \sim \text{Exp}(\overline{\tau}). \quad (4.17)
$$

Indeed, waiting times for jumps have intensity greater than or equal to $\overline{\tau}$. We have the moment generating function (MGF): $E[e^{uU_{1}}] = \overline{\tau}/(\overline{\tau} - u)$. Also,

$$
E\left[e^{u(U_1 + \ldots + U_V)}\right] = \frac{E\left[E[e^{uU_1}]^V\right]}{E\left[E\left[e^{uU_1}\right]^V\right]} = \frac{\left(\frac{\overline{\tau}}{\overline{\tau} - u}\right)^V}{\frac{\overline{\tau}}{\overline{\tau} - u}}.
$$
Combining this with (4.5), we get the estimate (4.16) for \( \tau_0 \). Similarly, for \( \tau_i \), \( i = 1, 2, \ldots \) we apply (4.16) with \( n_1 \lor n_2 \) replaced by 1.

**Step 5.** Use the same argument as in Step 5 of the proof of [24, Theorem 4.1]. Apply Assumption 4.2 and Lemma 6.1. We can show that the random variable \( J \) is stochastically bounded by a geometric random variable \( \tilde{J} \) with parameter

\[
\vartheta = \frac{\gamma}{\lambda + \gamma} \alpha^{-\lambda/\gamma},
\]

which has a generation function

\[
E[s^{\tilde{J}}] = \frac{\vartheta s}{1 - (1 - \vartheta)s}, \quad s \in [0, 1/(1 - \vartheta)).
\]

**Step 6.** Under Assumption 4.2, by Lemma 6.1, we obtain

\[
E\left[e^{u(\tau_k \wedge \eta_k)}\right] \leq \theta(\alpha, q, \gamma, u), \quad k = 0, 1, \ldots
\]

**Step 7.** Finally we derive the bound for the MGF of the coupling time \( \tau = S_{\tilde{J}} \). We have from Steps 4 and 6 that

\[
E\left[e^{u(\tau_k + \zeta_k \wedge \eta_k)}\right] \leq G(u)^{n_1 \lor n_2} \theta(\alpha, q, \gamma, u) =: \psi(u).
\]

By applying the optimal stopping theorem to the martingale

\[
M_{\ell} = \exp\left(u \sum_{k=0}^{\ell} (\tau_k + \zeta_k \wedge \eta_k) - \ell \ln \psi(u)\right)
\]

with \( \tilde{J} \) being the stopping time, we obtain that

\[
E[M_{\tilde{J}}] = E[M_0] = E[e^{u(\tau_0 + \zeta_0 \wedge \eta_0)}] \leq G(u)\theta(\alpha, q, \gamma, u).
\]

Now, we have \( E[M_{\tilde{J}}] = E[\exp \left(u(S_{\tilde{J}} - J \ln \psi(u))\right)] \). Using Hölder’s inequality, we get

\[
E[\exp \left((1 - \epsilon)u S_{\tilde{J}}\right)] \leq (E[M_{\tilde{J}}])^{1-\epsilon} \cdot E[\psi(u)^{(1-\epsilon)J/\epsilon}].
\]

Using the result in Step 5, we obtain the desired bound in (4.7) with the condition (4.8). □

### 4.2. Polynomial rate of convergence

In this section, instead of the conditions on the birth and death rates in Assumption 4.1, we make the following assumption on the generator \( \mathcal{M}_z \) such that the joint Markov process \((N, Z)\) has a polynomial rate of convergence, while the generator of environment \( A \) satisfies Assumption 4.2 with an exponential rate of convergence.

**Assumption 4.3.** For every \( z \in \mathcal{Z} \), the birth-death process with generator \( \mathcal{M}_z \) is stochastically dominated by another birth-death process \( \mathcal{N} \) with the birth rates \( \overline{\lambda}_n \) and death rates \( \overline{\mu}_n \), independent of \( \mathcal{Z} \). Assume \( \text{sup}_n \left( \overline{\lambda}_n - \overline{\mu}_n \right) < -C < 0 \).

**Lemma 4.1.** Under Assumption 4.3, the process \( \mathcal{N} = (\mathcal{N}(t), t \geq 0) \) has a unique stationary distribution \( \mathcal{P} = \{\mathcal{P}_n : n \in \mathbb{N}\} \). This process is ergodic: Regardless of the initial point, this process converges to this stationary distribution in the total variation distance. This stationary distribution satisfies the property \( \sum_n n \mathcal{P}_n < \infty \). Finally, starting from \( n \), the hitting time \( \tau(n) \) of 0 for the dominating process \( \mathcal{N} \) satisfies for some constant \( C_\tau > 0 \):

\[
E[\tau(n)] =: m_\tau(n) \leq C_\tau n, \quad n = 1, 2, \ldots
\]

The proof of Lemma 4.1 is postponed to the end of the section. Now we state and prove the result on polynomial convergence.
**Theorem 4.2.** Under Assumptions 3.1, 4.2, 4.3, there exists a constant $C$ such that
\[
\|P^t((n,z),\cdot) - \pi(\cdot)\|_{TV} \leq \frac{C(1+n)}{t}, \quad t > 0, \quad (n,z) \in \mathbb{N} \times \mathbb{Z}.
\] (4.19)

**Proof.** We again modify the approach [24] with the seven steps as in the proof of the previous theorem. Instead of estimating the moment generating functions, we estimate the expectation of the coupling time directly.

**Step 1:** Assume we proved the version of (4.19) with two starting points instead of one:
\[
\|P^t((n_1,z_1),\cdot) - P^t((n_2,z_2),\cdot)\|_{TV} \leq \frac{C(n_1+n_2)}{t}.
\] (4.20)

Now we must show that the stationary distribution $\pi$ for the process $(N,Z)$ satisfies $(\pi,V) < \infty$, where $V(n,z) := n+1$. Indeed, this stationary distribution $\pi$ has marginal corresponding to $n$ which is stochastically dominated by $\pi$, the stationary distribution of the dominating process $N$ from Assumption 4.3. If you integrate $V$ with respect to this stationary distribution, the result is finite. The rest of the proof is omitted.

**Step 2:** Instead of (4.12), we use the Markov inequality:
\[
\|P^t((n_1,z_1),\cdot) - P^t((n_2,z_2),\cdot)\|_{TV} \leq 2\mathbb{P}(\tau > t) \leq \frac{2\mathbb{E}[\tau]}{t}, \quad t > 0.
\] (4.21)

**Step 3:** We obtain the same expression of the coupling time $\tau = S_\mathcal{J}$ in (4.13) and (4.15).

**Step 4:** The required property (4.18) is proved in Lemma 4.1.

**Step 5:** Recall the stopping time $\tilde{J}$ in (4.14). Note that it is a geometric random variable with parameter $\tilde{\theta} = \mathbb{P}(\zeta_k < \eta_k)$. By Lemma 6.1, we get
\[
\tilde{\theta} = \frac{\gamma}{\lambda + \gamma} e^{-\gamma/\lambda}.
\] (4.22)

Thus, the expectation of $\tilde{J}$ is given by
\[
\mathbb{E}[\tilde{J}] = (1 - \tilde{\theta})\tilde{\theta}^{-1} \leq \tilde{\theta}^{-1} = \frac{\lambda + \gamma}{\gamma} e^{-\gamma/\lambda}.
\] (4.23)

**Step 6:** Instead of estimating MGF, we get estimates of the mean:
\[
\mathbb{E}[\zeta_k] \leq \mathbb{E}[\eta_k] \leq \mathbb{A}^{-1},
\] (4.24)

Let $\xi_k = \tau_k + \zeta_k \wedge \eta_k$, $k \geq 0$, where $\tau_k$ is as defined in Step 3 in the proof of the previous theorem. By Assumption 4.3 and (4.24), we have
\[
\mathbb{E}[\xi_k] = \mathbb{E}[\tau_k + \zeta_k \wedge \eta_k] \leq m_\tau(1) + \mathbb{A}^{-1} < \infty.
\] (4.25)

for $k = 1,2,\ldots$, since after we failed to couple the processes on the first attempt and the second and so on, we start our dominating queueing process at $\mathbb{A} = 1$. But (4.25) for $k = 0$ should be with $m_\tau(n_0)$ where $\mathbb{A}(0) = n_0$. Of course, $m_\tau(n_0 \lor 1) \geq \max(m_\tau(1),m_\tau(n_0))$. Thus we can create a common bound $m_\tau(n_0 \lor 1)$.

**Step 7:** We have the following estimates of the expectation of the coupling time. Define the random walk $S_k := \xi_1 + \ldots + \xi_k$, $k \geq 0$, and let $m_\xi = \mathbb{E}[\xi_1]$. Then $M_k = S_k - m_\xi k$, $k \geq 0$, is a martingale. Applying the Optional Stopping Theorem with the stopping time $\mathcal{J}$, we get
\[
\mathbb{E}[S_\mathcal{J} - m_\xi \mathcal{J}] = \mathbb{E}[S_0].
\] (4.26)
Combining (4.22), (4.23), (4.25) and (4.26), we get
\[
E[S_J] \leq \left( m_x(n_0 \lor 1) + \overline{\lambda}^{-1} \right) \left( 1 + \frac{\overline{\lambda} + \gamma}{\gamma} \nu^{\gamma} \right).
\]
The bound on the right-hand side is independent of \(z\), but depends on \(n\) linearly, by (4.18). Applying Lindvall’s inequality from [18, Chapter 1] we complete the proof of (4.20). \(\square\)

Finally, we prove Lemma 4.1, which is used in Step 4 above.

**Proof of Lemma 4.1.** We follow the proof of [30, Theorem 1]. The generator of \(\overline{N}\) is
\[
\overline{L}f(n) = \overline{\lambda}_n(f(n + 1) - f(n)) + \overline{\nu}_n(f(n - 1) - f(n)), \quad n = 1, 2, \ldots
\]
Take the Lyapunov function \(V(n) = n + 1\). Define \(\varphi(n) := \overline{\lambda}_{n-1} - \overline{\nu}_{n-1}\) for \(n = 1, 2, \ldots\) Then \(\overline{L}\varphi(n) \leq -C\) for all \(n = 1, 2, \ldots\) and there exists a unique stationary distribution, and the process is ergodic. We can extend \(\varphi\) on positive non-integers continuously, which will enable us to apply the results of [30]. Take \(h(t)\) to be the inverse of \(H(t) = \int_0^t \varphi^{-1}(s)ds\), with \([t]\) being the integer part of \(t\), and define \(U(x) = 1\). Then the pair \((h, U)\) satisfies the following inequality, see [30, Remark 2]:
\[
h(t)U(x) \leq G(t, x), \quad t \geq 0, \quad x \geq 1,
\]
where \(G(t, x) = h(H(u) + t)\) was defined in the beginning of [30, subsection 2.2]. By [30, Lemma 4], the following process is a supermartingale: \(G(t \land \tau, V(\overline{N}(t \land \tau)))\) for \(t \geq 0\). Combining this property with the inequality, we get:
\[
E[h(t \land \tau)] \leq E[G(t \land \tau, V(\overline{N}(t \land \tau)))] \leq E[G(0, V(\overline{N}(0)))].
\]
By [30, Lemma 3], \(G(0, x) = x\) for \(x \geq 1\). Applying Fatou’s lemma to the left-hand side of (4.28), and letting \(t \to \infty\), we get:
\[
Eh(\tau) \leq \lim_{t \to \infty} E[h(t \land \tau)] \leq V(n_0) = n_0 + 1.
\]
The function \(\varphi\) is bounded. Hence \(H(t) \geq ct\) for some constant \(c\). Thus \(h(t)\) is sublinear: \(h(t) \leq t/c\). This, together with (4.29), completes the proof. \(\square\)

**Remark 4.5.** It is reasonable to conjecture that the rate of convergence of the joint Markov process will be at a slower rate out of two: the birth-death process itself and the environment process. It would be also interesting to consider more general slower-than-exponential convergence rates, such as \(t^{-\alpha}\) or \(\exp\left[-c(\ln t)^{1-\varepsilon}\right]\) for \(\alpha, \varepsilon > 0\).

5. **Proofs of Theorems 2.1 and 3.1**

**Proof of Theorem 2.1.** The irreducibility and aperiodicity properties are straightforward. For the measure \(\eta(n, z)\) in (2.8) to be finite, by Assumption 2.2,
\[
\sum_{(n, z)} \eta(n, z) = \sum_{(n, z)} r_n(z)v(z) = \Xi < \infty.
\]
To verify that \( \eta(n, z) \) in (2.8) is an invariant measure, we prove that \( \eta'\mathbf{R} = 0 \):

\[
-\eta(n, z)R[(n, z), (n, z)] = \eta(n, z)R[(n - 1, z), (n, z)] + \eta(n + 1, z)R[(n + 1, z), (n, z)] + \sum_{z' \neq z} \eta(n, z')R[(n, z'), (n, z)], \quad n = 1, 2, \ldots, \quad z \in \mathcal{Z};
\]

\( -\eta(0, z)R[(0, z), (0, z)] = \eta(1, z)R[(1, z), (0, z)] + \sum_{z' \neq z} \eta(0, z')R[(0, z'), (0, z)], \quad n = 0, \quad z \in \mathcal{Z}. \) \hspace{1cm} (5.1)

For (5.1) with \( n \geq 1 \), the left-hand side is

\[
\eta(n, z) \sum_{(n', z') \neq (n, z)} R[(n, z), (n', z')]
\]

\[
= \eta(n, z) \left( R[(n, z), (n + 1, z)] + R[(n, z), (n - 1, z)] + \sum_{z' \neq z} R[(n, z), (n, z')] \right)
\]

\[
= r_n(z)v(z) \left( \lambda_n(z) + \mu_n(z) + \sum_{z' \neq z} r_n(z) - r_n(z') \tau_n(z, z') \right)
\]

\[
= r_n(z)v(z) \left( \lambda_n(z) + \mu_n(z) \right) + v(z) \sum_{z' \neq z} \tau_n(z, z'),
\]

and the right-hand side is equal to

\[
r_{n-1}(z)v(z)\lambda_{n-1}(z) + r_{n+1}(z)v(z)\mu_{n+1}(z) + \sum_{z' \neq z} r_n(z')v(z')r_n(z')^{-1}\tau_n(z', z)
\]

\[
= v(z) \left( r_{n-1}(z)\lambda_{n-1}(z) + r_{n+1}\mu_{n+1}(z) \right) + \sum_{z' \neq z} v(z')\tau_n(z', z).
\]

We get equality thanks to the assumption in (2.6) and the local balance equation in (2.3). For (5.1) with \( n = 0 \), the left-hand side is

\[
\eta(0, z) \sum_{z' \neq z} R[(0, z), (0, z')] = r_0(z)v(z) \left( R[(0, z), (1, z)] + \sum_{z' \neq z} R[(0, z), (0, z')] \right)
\]

\[
= r_0(z)v(z) \left( \lambda_0(z) + \sum_{z' \neq z} r_0(z)^{-1} \tau_0(z, z') \right) = r_0(z)v(z)\lambda_0(z) + v(z) \sum_{z' \neq z} \tau_0(z, z'),
\]

and the right-hand side is

\[
r_1(z)v(z)\mu_1(z) + \sum_{z' \neq z} r_0(z')v(z')r_0(z')^{-1}\tau_0(z', z) = r_1(z)v(z)\mu_1(z) + \sum_{z' \neq z} v(z')\tau_0(z', z).
\]

This again leads to the equality thanks to (2.6) and (2.4) for \( n = 0 \). Thus we have shown that \( \pi(n, z) \) in (2.7) is an invariant probability measure. The positive recurrence property follows from [22, Theorem 3.5.3] (see also [31, Theorem 2.7.18]). The ergodicity property of convergence in total variation follows from [20].

\textit{Proof of Theorem 3.1.} The proof follows from analogous argument as that of Theorem 3.1 in [24], so we only highlight the differences. We apply [16], and use their notation as follows:
let \(E = \mathbb{N} \times \mathcal{Z}\) and \(U = \{0, 1, \ldots, m\}\), where “0” indicates \(\mathcal{Z}\) and \(i = 1, \ldots, m\) for the faces \(F_1, \ldots, F_m\) of the boundary, and for \(n \in \mathbb{N}, z \in \mathcal{Z}\) and \(u \in U\),

\[
\mu_0(u \times \{n\} \times dz) = 1_{u=0} r_n(z) \nu(dz), \\
\mu_1(u \times \{n\} \times dz) = 1_{u \neq 0} r_n(z) \nu_i(dz), \\
\nu_0^E(\{n\} \times dz) = r_n(z) \nu(dz), \\
\nu_1^E(\{n\} \times dz) = r_n(z)(\nu_{F_1}(dz) + \cdots + \nu_{F_m}(dz)), \\
\eta_0((n, z), \{u\}) = 1_{u=0}, \\
\eta_1((n, z), \{u\}) = 1_{u \neq 0},
\]

\[
Af((n, z), u) := \beta_n^{-1} r_n(z) \mathcal{L} f(n, z), \\
Bf((n, z), u) := 1_{u \neq 0, z \in \partial D_i, i=1, \ldots, m} \gamma u(z) \cdot \nabla f(z).
\]

To check [16, Condition 1.2] on the absolutely continuous generator \(A\) and the singular generator \(B\), we can verify the conditions (i)-(v) in the same way as in the proof of [24, Theorem 3.1]. For the main condition in [16, Theorem 1.7, (1.17)], we need to show that the generators \(A\) and \(B\) satisfy

\[
\int_{E \times U} Af(x, u) \mu_0(dx \times du) + \int_{E \times U} Bf(x, u) \mu_1(dx \times du) = 0. \tag{5.2}
\]

By definition of \(A\) and \(B\), we can write the left hand side as

\[
\sum_{n=0}^{\infty} \int_{\mathcal{Z}} \beta_n^{-1} r_n(z) \mathcal{M}_z f(n, z) \nu(dz) \\
+ \sum_{n=0}^{\infty} \left( \int_{\mathcal{Z}} Af(n, z) \nu(dz) + \sum_{i=1}^{m} \int_{F_i} \gamma_i(z) \cdot \nabla f(z) \nu_{F_i}(dz) \right).
\]

The sum of the last two terms is equal to zero, because the basic adjoint relationship holds for the reflected jump diffusion process \(\tilde{Z}\) (see, e.g., [41]), that is, for \(f \in C^2_b(\mathcal{Z})\),

\[
\int_{\mathcal{Z}} Af(n, z) \nu(dz) + \sum_{i=1}^{m} \int_{F_i} \gamma_i(z) \cdot \nabla f(z) \nu_{F_i}(dz) = 0.
\]

For each \(z \in \mathcal{Z}\), the birth-death process \(N(t)\) has the stationary distribution given in (2.2) and (2.5), which satisfy

\[
r_n(z) \mathcal{M}_z f(n, \cdot) = 0, \text{ for each } z \in \mathcal{Z}, \quad n \in \mathbb{N}.
\]

Multiplying this by \(\beta_n^{-1}\) and integrating over \(z \in \mathcal{Z}\), we get:

\[
\sum_{n=0}^{\infty} \int_{\mathcal{Z}} \beta_n^{-1} r_n(z) \mathcal{M}_z f(n, z) \nu(dz) = 0.
\]

Thus we have verified that (5.2) holds. The rest of the proof follows the same argument as the proof of [24, Theorem 3.1]. \(\square\)
6. Appendix: A Technical Comparison Lemma

Lemma 6.1 (Lemma 5.1 in [24]). Fix constants $\alpha > 1$, $\beta, \gamma > 0$. Take two independent random variables $\xi \sim \text{Exp}(\beta)$ and $\eta > 0$ which satisfies $\Pr(\eta > u) \leq \alpha e^{-\gamma u}$ for $u \geq 0$. Then

$$\Pr(\eta < \xi) \geq \frac{\alpha^{-\beta/\gamma} \gamma}{\beta + \gamma}. \quad (6.1)$$

For $a \in [0, \beta + \gamma)$, the moment generating function for $\xi \wedge \eta$ satisfies

$$\mathbb{E} \left[ e^{a(\xi \wedge \eta)} \right] \leq \theta(\alpha, \beta, \gamma, a), \quad (6.2)$$

where the function $\theta$ is defined in (4.3).

Acknowledgments

G. Pang was supported in part by NSF grant DMS-2216765. A. Sarantsev thanks Department of Mathematics & Statistics at the University of Nevada, Reno for welcoming atmosphere for research. Y. Suhov thanks Department of Mathematics at Pennsylvania State university for hospitality and support.

References

[1] N. Bacaër and A. Ed-Darraz. On linear birth-and-death processes in a random environment. *Journal of Mathematical Biology*, 69(1):73–90, 2014.
[2] S. Browne and W. Whitt. Piecewise-linear diffusion processes. *Advances in Queueing: Theory, Methods, and Open Problems*, 4:463–480, 1995.
[3] R. Cogburn. Markov chains in random environments: the case of Markovian environments. *The Annals of Probability*, 8(5):908–916, 1980.
[4] R. Cogburn and W. C. Torrez. Birth and death processes with random environments in continuous time. *Journal of Applied Probability*, 18(1):19–30, 1981.
[5] R. Cornez. Birth and death processes in random environments with feedback. *Journal of Applied Probability*, 24(1):25–34, 1987.
[6] A. Das. Constructions of Markov processes in random environments which lead to a product form of the stationary measure. *Markov Processes and Related Fields*, 23(2):211–232, 2017.
[7] A. Dieker and J. Moriarty. Reflected Brownian motion in a wedge: sum-of-exponential stationary densities. *Electronic Communications in Probability*, 14:1–16, 2009.
[8] A. Economou. Generalized product-form stationary distributions for markov chains in random environments with queueing applications. *Advances in Applied Probability*, 37(1):185–211, 2005.
[9] W. Feller. *An Introduction to Probability Theory and its Applications*, volume 1. Wiley, 1950.
[10] M. Gannon, E. Pechersky, Y. Suhov, and A. Yambartsev. Random walks in a queueing network environment. *Journal of Applied Probability*, 53(2):448–462, 2016.
[11] J. M. Harrison and M. I. Reiman. On the distribution of multidimensional reflected Brownian motion. *SIAM Journal on Applied Mathematics*, 41(2):345–361, 1981.
[12] I. Karatzas and S. Shreve. *Stochastic Calculus and Brownian Motion*, 1991.
[13] S. C. Kou and S. G. Kou. Modeling growth stocks via birth-death processes. *Advances in Applied Probability*, 35(3):641–664, 2003.
[14] R. Krenzler and H. Daduna. Loss systems in a random environment: steady state analysis. *Queueing Systems*, 80(1):127–153, 2015.
[15] R. Krenzler, H. Daduna, and S. Otten. Jackson networks in nonautonomous random environments. *Advances in Applied Probability*, 48(2):315–331, 2016.
[16] T. Kurtz and R. Stockbridge. Stationary solutions and forward equations for controlled and singular martingale problems. *Electronic Journal of Probability*, 6:1–52, 2001.
[17] T. Lindvall. A note on coupling of birth and death processes. *Journal of Applied Probability*, 16(3):505–512, 1979.
[18] T. Lindvall. *Lectures on the Coupling Method*. Dover, 1992.
[19] R. R. Mazumdar and F. M. Guillemin. Forward equation for reflected diffusions with jumps. *Applied Mathematics and Optimization*, 33:81–102, 1996.

[20] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes I: Criteria for discrete-time chains. *Advances in Applied Probability*, 24(3):542–574, 1992.

[21] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes II: Continuous-time processes and sampled chains. *Advances in Applied Probability*, 25(3):487–517, 1993.

[22] J. R. Norris. *Markov Chains*. Cambridge University Press, 1997.

[23] S. Otten, R. Krenzler, H. Daduna, and K. Kruse. Queues in a random environment. *arXiv preprint arXiv:2006.15712*, 2020.

[24] G. Pang, A. Sarantsev, Y. Belopolskaya, and Y. Suhov. Stationary distributions and convergence for M/M/1 queues in interactive random environment. *Queueing Systems*, 94(3):357–392, 2020.

[25] A. Prodhomme and É. Strickler. Large population asymptotics for a multitype stochastic sis epidemic model in randomly switched environment. *arXiv preprint arXiv:2107.05333*, 2021.

[26] S. M. Ross. *Introduction to Probability Models*. Academic Press, 12th edition, 2019.

[27] A. Sarantsev. Explicit rates of exponential convergence for reflected jump-diffusions on the half-line. *ALEA Latin American Journal of Probability & Mathematical Statistics*, 13:1069–1093, 2016.

[28] A. Sarantsev. Weak convergence of obliquely reflected diffusions. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 54(3):1408–1431, 2018.

[29] A. Sarantsev. Penalty method for obliquely reflected diffusions. *Lithuanian Mathematical Journal*, 61:518–549, 2021.

[30] A. Sarantsev. Sub-exponential rate of convergence to equilibrium for processes on the half-line. *Statistics & Probability Letters*, 175:109115, 2021.

[31] I. M. Soukhov and M. Kelbert. *Probability and Statistics by Example: Markov Chains: A Primer in Random Processes and their Applications*. Cambridge University Press, 2008.

[32] D. W. Stroock and S. S. Varadhan. Diffusion processes with boundary conditions. *Communications on Pure and Applied Mathematics*, 24(2):147–225, 1971.

[33] H. Tanaka. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Mathematical Journal*, 9(1):163–177, 1979.

[34] W. C. Torrez. The birth and death chain in a random environment: instability and extinction theorems. *The Annals of Probability*, 6(6):1026–1043, 1978.

[35] W. C. Torrez. Calculating extinction probabilities for the birth and death chain in a random environment. *Journal of Applied Probability*, 16(4):709–720, 1979.

[36] E. A. Van Doorn. Conditions for exponential ergodicity and bounds for the decay parameter of a birth-death process. *Advances in Applied Probability*, 17(3):514–530, 1985.

[37] E. A. Van Doorn. Rate of convergence to stationarity of the system M/M/N/N + R. *TOP*, 19(2):336–350, 2011.

[38] E. A. Van Doorn and A. I. Zeifman. On the speed of convergence to stationarity of the Erlang loss system. *Queueing Systems*, 63(1):241–252, 2009.

[39] E. A. van Doorn, A. I. Zeifman, and T. L. Panfilova. Bounds and asymptotics for the rate of convergence of birth-death processes. *Theory of Probability & Its Applications*, 54(1):97–113, 2010.

[40] A. R. Ward and P. W. Glynn. Properties of the reflected Ornstein–Uhlenbeck process. *Queueing Systems*, 44(2):109–123, 2003.

[41] R. J. Williams. Semimartingale reflecting Brownian motions in the orthant. *IMA Volumes in Mathematics and its Applications*, 71:125–137, 1995.

[42] A. I. Zeifman. Some estimates of the rate of convergence for birth and death processes. *Journal of Applied Probability*, 28(2):268–277, 1991.

[43] A. I. Zeifman. Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes. *Stochastic Processes and their Applications*, 59(1):157–173, 1995.

[44] A. I. Zeifman and T. L. Panfilova. On convergence rate estimates for some birth and death processes. *Journal of Mathematical Sciences*, 221(4):616–623, 2017.
