ACCURATE NUMERICAL POTENTIAL AND FIELD IN RAZOR-THIN, AXISYMMETRIC DISKS

JEAN-MARC HURÉ
LUTh/Observatoire de Paris–Meudon (UMR 8102 CNRS), Place Jules Janssen, 92195 Meudon Cedex; and Université Paris 7 Denis Diderot, 2 Place Jussieu, 75251 Paris Cedex 05, France; jean-marc.huro@obspm.fr

AND

ARNAUD PIERENS
LUTh/Observatoire de Paris–Meudon (UMR 8102 CNRS), Place Jules Janssen, 92195 Meudon Cedex, France; arnaud.pierens@obspm.fr

Received 2004 November 5; accepted 2005 January 13

1. INTRODUCTION AND MOTIVATION

Disks are ubiquitous objects in the universe and span different velocity and length scales: galactic (stellar) disks, active galactic nuclei disks, circumstellar disks, binary and circumbinary disks, and subnebulae. For many of them, self-gravity plays a role in their structure and dynamics, and so the gravitational potential and associated accelerations are required at a certain level of disk modeling. Solving the Poisson equation in extended, continuous media like gas disks is, however, not trivial practically, and it has occupied astrophysicists for many decades. For time-dependent simulations, fast but low-accuracy algorithms are generally preferred. For steady state analysis, accuracy is more critical than computing time, as it allows, for instance, to characterize the precise connection between various branches of solutions (e.g., Hachisu 1986; Ansorg et al. 2003).

Many numerical methods have been proposed, but very few use the integral formalism. In a previous paper (Pierens & Huré 2004, hereafter Paper I), we have outlined a method to avoid the singularity in the Poisson kernel so that the field in the plane of razor-thin, axisymmetric disks can easily be accurately computed from elliptic integrals by a single radial quadrature. The motivation of the present paper is twofold. First, Paper I just touches the problem at the theoretical level without giving numerical examples and discussing the implementation and possible performances of the method. Also, the potential was not considered. This is done here. Second, we have recently realized that the classical fast Fourier transform (FFT) method (e.g., Binney & Tremaine 1987), which is among the most widespread methods, has a much lower precision and a lower order of convergence in comparison. We show here that the density splitting method can exhibit a high convergence order, depending on the quadrature rule, and the precision can easily reach the machine precision using a few tens of source points, at least for smooth surface density profiles.

We outline the splitting method for the gravitational potential and radial field in § 2. We then stress the nonderivability of Poisson kernels in § 3 and propose a space mapping. In § 4 we illustrate the possible performances of the method on a test case (namely, a finite-size disk with exponentially decreasing surface density profile) with three different quadrature rules. We compare the accuracy of the method with the classical FFT method in § 5. A few concluding remarks are found in § 6.

2. OUTLINE OF THE DENSITY SPLITTING METHOD

The gravitational potential \( \Psi \) and radial component of the field \( g_R = -\partial \Psi / \partial r \) in the plane of a razor-thin axisymmetric disk (see Fig. 1) is\(^1\) (e.g., Durand 1964)

\[
\left( \begin{array}{c} g_R \\ \Psi \end{array} \right) = \int_{a_{in}}^{a_{out}} \Sigma \left( \begin{array}{c} \kappa \phi \\ \kappa \rho \end{array} \right) da,
\]

where \( a_{in} \) is the inner edge, \( a_{out} \) is the outer edge, \( \Sigma(a) \) is the surface density,

\[
\left( \begin{array}{c} \kappa \phi \\ \kappa \rho \end{array} \right) = -2G \sqrt{\frac{a}{R}} \left[ \frac{1}{2R} \left( \frac{K(k)}{E(k)} - \frac{E(k)}{E(k)} \right) \right]
\]

are the Poisson kernels, \( K \) and \( E \) denote the complete elliptic integral of the first and second kinds, respectively, \( k = 2(aR)^{1/2}/(a + R) \leq 1 \) is their modulus, and \( \varpi = (a - R)/(a + R) \). As is well known, these kernels diverge when \( a \rightarrow R \), with the result that, in practice, neither \( \Psi \) nor \( g_R \) can properly be determined by direct integration for \( R \in [a_{in}, a_{out}] \). As outlined\(^2\) in Paper I, such a singular behavior can be avoided if the surface density profile is split into two components according to

\[
\Sigma(a) = \Sigma_0 + \delta \Sigma(a, R),
\]

\(^1\) Matrix notation is employed only for compactness.

\(^2\) Only the field was considered in Paper I.
where $\Sigma_0 \equiv \Sigma(R)$ is the local value, and $\delta \Sigma$ is the remainder (a function that depends on $a$ and $R$). Consequently, the potential and the radial field in the disk are given by

$$
(\Psi) = \left(\Psi_{\text{homo}}^{\text{homo}}\right) + \left(\Psi_{\text{res}}^{\text{res}}\right),
$$

(4)

where $\Psi_{\text{homo}}^{\text{homo}}$ and $g_{\text{homo}}^{\text{homo}}$ are analytical functions (proportional to $\Sigma_0$; see Appendix A in Paper I for the field and Appendix A in this paper for the potential). Terms $\Psi_{\text{res}}^{\text{res}}$ and $g_{\text{res}}^{\text{res}}$ correspond to the departure from the homogeneous disk. These are simply given by

$$
(\Psi_{\text{res}}^{\text{res}}) = \int_{a_\text{in}}^{a} \delta \Sigma (\kappa_{\Psi}^{\text{res}} \kappa_{\Psi}^{\text{res}}) \, da.
$$

(5)

The point is that both $\delta \Sigma \kappa_{\Psi}$ and $\delta \Sigma \kappa_{\Psi}$ are finite when $a = R$, although $\kappa_{\Psi} \to \infty$ and $\kappa_{\Psi} \to \infty$ (see Appendix B in Paper I for a proof), with

$$
|\delta \Sigma \kappa_{\Psi}|_{a=R} = 0,
$$

$$
|\delta \Sigma \kappa_{\Psi}^{\text{res}}|_{a=R} = 2G \left(\frac{d \Sigma}{da}\right)_{a=R}.
$$

(6)

Note that, in this procedure, the radial derivative of the surface density $d \Sigma / da$ is needed at each field point when computing $g_{\text{res}}^{\text{res}}$ from equation (1). This is quite uncomfortable if the surface density is not defined analytically but on a grid as in many simulations.

3. RESIDUAL INTEGRANDS ARE NOT

$C^\infty$-CLASS FUNCTIONS

Accuracy of residual terms is set by the scheme performing the numerical quadrature, provided the integrand is a well-behaved function. In the present problem a difficulty arises, because the residual integrands $\delta \Sigma \kappa^{\text{res}}$ and $\delta \Sigma \kappa_{\Psi}^{\text{res}}$ are continuous, but not $C^\infty$-class functions (i.e., differentiable for all degrees of differentiation). Even these are nonderivable (i.e., not $C^0$). This is easily understood from the first derivative

$$
\frac{d}{da} \delta \Sigma \kappa = \frac{d \Sigma}{da} \kappa + \delta \Sigma \frac{d \kappa}{da},
$$

(7)

where $\kappa$ denotes either $\kappa_{\Psi}$ or $\kappa_{\Psi}$,

$$
\frac{d \kappa}{da} = \frac{d \kappa}{dk} \frac{dk}{da},
$$

(8)

and

$$
\frac{dk}{da} = \frac{\sqrt{R (R - a)}}{a (a + R)^2} = \frac{\infty}{2a}.
$$

(9)

We see that the first term in the right-hand side of equation (7) brings a diverging contribution for $a = R$, since $d \Sigma / da$ cannot be zero on the whole integration range. We thus have $d / da) \delta \Sigma \kappa \to \infty$ at the field point. This point is illustrated in Figure 2, which displays the function $\delta \Sigma \kappa$ for both the potential and field as well as their first derivatives in a disk with exponentially decreasing surface density profile. Although not visible at the scale of the graphs, there is a small “knee” just at the field point where the first derivatives are infinite. In addition, because successive derivatives of elliptic integrals inevitably produce the $K$-function, which is logarithmically diverging as $a \to R$, we conclude that

$$
\left|\frac{d^n}{da^n} \delta \Sigma \kappa\right|_{a=R} \to \infty \quad \text{for any } n \geq 1.
$$

Values of the integrands $\delta \Sigma \kappa$ and their $a$-derivatives at $a = R$ are summarized in Table 1. It means that, if residual terms are

| Order of Derivative | $\delta \Sigma \kappa$ at $a = R$ | $\delta \Sigma \kappa_{\Psi}$ at $a = R$ |
|---------------------|---------------------------------|---------------------------------|
| 0th (function)      | 0                               | $2G (\frac{d \Sigma}{da})_{a=R}$ |
| 1st                 | $\infty$                        | $\infty$                        |
| $\geq 2$nd          | $\infty$                        | $\infty$                        |
numerically determined following equation (1) (i.e., by integration in the a-space), then most classical quadrature rules that are based on a certain fitting of the integrand by polynomials cannot behave in an optimal manner. This problem especially concerns high-order quadrature rules. We have no idea how to totally remove such a difficulty. It is possible, however, to reduce it partially by resampling the integrand in an appropriate way. For instance, if we consider the function \( w(a) \) defined by

\[
\psi^{\text{res}}_R = 2a_{\text{out}} \int_{a_{\text{in}}}^{a_{\text{out}}} \delta \Sigma (\kappa_y \kappa_g) \frac{\partial}{\partial a} w(a) da,
\]

where \( \psi^{\text{res}}_R \) is the reference value obtained by computing \( \psi \) at \( a_{\text{out}} \) for a certain number of source points. The “new integrands” \( \delta \Sigma \kappa_y w(a) \) and \( \delta \Sigma \kappa_g w(a) \) are obviously regarded as a function of the new variable \( w \). The advantage of this space mapping is two-fold. First, it makes the first derivative finite; second, third derivatives in the same conditions as for Figure 2. Second, \( \delta \Sigma \kappa_g w(a) \) vanishes at the field point (since \( w = 0 \)). Hence, \( da/\partial a \) is no longer needed.

4. EXAMPLE OF PERFORMANCE

We briefly show the possible performances of the method through a typical example, namely a disk with \( (a_{\text{in}}, a_{\text{out}}) = (0, 1) \) and an exponentially decreasing surface density profile, as already considered above. Tests have concerned a large number of disk models (various surface density profiles, various axis ratios \( a_{\text{out}}/a_{\text{in}} \)). For simplicity, we discuss only the potential; results for the field are similar. We consider three different schemes to determine \( \psi^{\text{res}}_R \) at various radii \( R \) inside the disk from equation (12), namely:

1. The composite trapezoidal rule (hereafter the CT rule) with \( N \) source points. This is a second-order–accurate scheme (e.g., Press et al. 1992).
2. A sixth-order, regular-spacing, quadrature rule from Gill & Miller (1972) with \( N \) source points (hereafter the GM rule).
3. The Gauss-Chebycheff-Lobatto approximation (hereafter the GCL rule) with \( N \) Chebyshev polynomials. This collocation method would be the most efficient technique (with a spectral convergence) in the presence of a \( C^\infty \)-class integrand (e.g., Boyd 2001).

4. EXAMPLE OF PERFORMANCE

We measure the relative precision \( \Delta \psi/\psi \) on potential values \( \psi \) with the \( \epsilon \)-parameter defined as

\[
\epsilon \equiv \log \left( \frac{|\psi - \psi^{\text{ref}}|}{\psi^{\text{ref}}} \right),
\]

where \( \psi^{\text{ref}} \) is the reference value. Since the exact potential is not known for the case considered, reference values \( \psi^{\text{ref}} \) are obtained by considering a larger number of source points, as commonly done (e.g., Cohl & Tohline 1999). We use double precision (DP) computer calculus so that the precision is limited to \( \sim 2 \times 10^{-16} \) (that is, \( \epsilon \geq 10^{-16} \)).

We measure the relative precision \( \Delta \psi/\psi \) on potential values \( \psi \) with the \( \epsilon \)-parameter defined as

\[
\epsilon \equiv \log \left( \frac{|\psi - \psi^{\text{ref}}|}{\psi^{\text{ref}}} \right),
\]

where \( \psi^{\text{ref}} \) is the reference value. Since the exact potential is not known for the case considered, reference values \( \psi^{\text{ref}} \) are obtained by considering a larger number of source points, as commonly done (e.g., Cohl & Tohline 1999). We use double precision (DP) computer calculus so that the precision is limited to \( \sim 2 \times 10^{-16} \) (that is, \( \epsilon \geq 10^{-16} \)).

5. COMPARISON WITH THE CLASSICAL FFT METHODS

It is interesting to compare the accuracy of the density splitting method with that of the classical FFT method (e.g.,
Binney & Tremaine (1987) based on $N \times N$ polar cells. Results obtained on the same test model are reported in Figure 4 as open and filled circles (see below). We see that the precision is rather poor even for large $N$ and is definitively lower than that of the splitting method with the CT rule.

Let us remind that the FFT method is probably the most widely method used in simulations of self-gravitating disks to compute the gravitational potential (Binney & Tremaine 1987). The only advantage of the FFT method seems to be its great rapidity, $N \log(N)$ order in time. Low computing time is a fundamental requirement if the Poisson equation is to be coupled with other equations, as is generally the case.

However, one must realize that the FFT method has a few major drawbacks. First, it is first-order accurate, as can be seen in Figure 4, that is, one order less than the splitting method with the CT rule. This means that a huge amount of source points is required before reaching great accuracy. If we extrapolate data shown in the plot (and ignoring the loss of significance and round-off errors that would impose a saturation of $\epsilon$ well above $\epsilon_{CP}$), we find that the FFT method would need $N \approx 10^{14}$ to reach the computer precision (that is, $10^{28}$ polar cells). Second, it is a particle-type method (e.g., Hockney & Eastwood 1988): each cell is made homogeneous and converted into a particle with arbitrary assignment of both location and mass density. The precision of the FFT method is apparent and fortunate (assignment errors almost cancel or compensate). Third, the use of the FFT method in cylindrical coordinates requires a logarithmic spacing of grid points. This means that (1) the origin cannot be included, (2) the outer disk has always had a much lower resolution than the inner disk (by a factor $N$), and (3) many interpolations are necessary when other equations are solved on a different grid (one has to pass from one grid to the other). It is true that matter in astrophysical disks is generally concentrated at the inner edge, but disk self-gravity concerns regions located near the outer edge. Figure 5 shows $\epsilon$ versus $R$ for $N = 128$ in the example considered before. We see that the accuracy is better in the inner disk than in the outer disk with the FFT method, whereas it is uniform with the density splitting method (whatever the quadrature rule). Fourth, the local contribution to the gravitational potential (which is the contribution of the cell containing the field point $R$) is treated in an artificial way. A smoothing length $\lambda$ is often introduced, and its value is chosen arbitrarily. Results obtained for $\lambda = 0.01$ are shown in Figure 4 (open circles). Hence, potential values are dependent upon smoothing length. A slightly better accuracy is obtained with the prescription by Binney & Tremaine (1987) (Fig. 4, filled circles), for which the local contribution is estimated analytically, without any $\lambda$.

However, as some authors have noticed (e.g., Caunt & Tagger 2001), the FFT method can trigger numerical instabilities in hydrodynamic codes, probably because the order of this method is lower (and errors larger) than the order (generally two) of difference schemes used in other fluid equations.

Fig. 4.—Averaged $\epsilon$-parameter vs. $N$ for the splitting methods and for the FFT method with smoothing length (open circles; $\lambda = 0.01$) and without smoothing length (filled circles). For the FFT method, the disk inner edge is set to $10^{-2}$. See Fig. 5 for the $\epsilon(R)$ and $N = 128$.

Fig. 5.—The $\epsilon$-parameter vs. the radius $R$ with the splitting methods (lines) and with the FFT method (filled circles) for $N = 128$. For the FFT method, the disk inner edge is set to $10^{-2}$.

6. CONCLUDING REMARKS

In this paper, we have discussed the possible implementation and performances of the splitting method for razor-thin, axisymmetric disks. In particular, we have emphasized the role of a space mapping in order to increase the accuracy (or effective order) of advanced quadrature rules. We have noticed the important weaknesses of the classical FFT methods, especially in terms of accuracy, by direct comparisons. Obviously, our method is characterized by much longer computing times. Note, however,
that a few grid points are necessary to reach a high precision. Besides, the method is well suited to parallel computing.

Several extensions and improvements to the present method could be brought. For instance, other space mapping than that proposed can probably work well. However, it is hard to find a sampling function that makes the integrands and derivatives finite (and possibly zero) at the field point without increasing its wings, which is numerically uncomfortable (for instance, this can easily be shown with a $w$-function of the form $|a - R|^{1/n}$ for large $n$). Besides, depending on the global surface density profile in the disk, it can be more appropriate to replace the homogeneous contribution (see §2) by another analytical contribution (for instance, $1/a$ instead of a constant as done here, provided kernels can be integrated). Finally, the method can easily be extended to treat nonaxisymmetric systems (A. Pierens & J.-M. Hure´ 2005, in preparation), and even tridimensional mass distributions. We plan to open soon to the scientific community an Internet site devoted to this question.

We thank the referee for valuable comments.

APPENDIX A

SPLITTING METHOD FOR THE POTENTIAL

According to our notation, the general expression for the potential in the equatorial plane of a disk with inner edge $a_{in}$ and $a_{out}$ is (e.g., Durand 1964)

$$\Psi = -2G \int_{a_{in}}^{a_{out}} \sqrt{\frac{a}{R}} k \Sigma(a) K(k) \, da. \quad (A1)$$

For a constant surface density $\Sigma_0$, equation (A1) reads

$$\Psi_{\text{homo}}(R) = -2G \Sigma_0 \int_{a_{in}}^{a_{out}} \sqrt{\frac{a}{R}} k K(k) \, da. \quad (A2)$$

For $a_{in} \leq R \leq a_{out}$, we separate this integral into two integrals, leftward and rightward to the field point where $k = 1$, namely

$$\Psi_{\text{homo}} = -2G \Sigma_0 \left[ \int_{a_{in}}^{R} \sqrt{\frac{a}{R}} k K(k) \, da + \int_{R}^{a_{out}} \sqrt{\frac{a}{R}} k K(k) \, da \right]. \quad (A3)$$

We now set $u = a/R \leq 1$ in the first integral and $v = R/a \leq 1$ in the second one. With the Gauss transformation (Gradshteyn & Ryzhik 1980)

$$K\left(\frac{2\sqrt{u}}{1+u}\right) = (1 + u) K(u), \quad u < 1,$$

equation (A2) becomes

$$\Psi_{\text{homo}}(R) = -4G \Sigma_0 R \left[ \int_{a_{in}/R}^{1} u K(u) \, du - \int_{1}^{R/a_{out}} \frac{K(v)}{v^2} \, dv \right], \quad (A5)$$

and so

$$\Psi_{\text{homo}}(R) = -4G \Sigma_0 R \left[ \frac{a_{out}}{R} E\left(\frac{R}{a_{out}}\right) - E\left(\frac{a_{in}}{R}\right) + \left(1 - \frac{a_{in}^2}{R^2}\right) K\left(\frac{a_{in}}{R}\right) \right]. \quad (A6)$$

APPENDIX B

SUCCESSIVE DERIVATIVES

From equation (11) we have

$$\frac{da}{dw} = 2\sqrt{a_{out}|R - a|} = \begin{cases} 
-2w a_{out} > 0 & \text{if } R > a, \\
0 & \text{if } R = a, \\
2w a_{out} > 0 & \text{if } R < a.
\end{cases} \quad (B1)$$
Letting $\kappa$ be either $\kappa_q$ or $\kappa_g$, we have
\[
\frac{d}{dw} \delta \Sigma |w| = |w| \frac{d}{dw} \delta \Sigma \kappa \pm \delta \Sigma \kappa
\]
\[
= |w| \frac{d}{da} \delta \Sigma \kappa \frac{da}{dw} \pm \delta \Sigma \kappa
\]
\[
= 2w^2 a_{out} \frac{d}{da} \delta \Sigma \kappa \pm \delta \Sigma \kappa
\] (B2)

for the first derivative,
\[
\frac{d^2}{dw^2} \delta \Sigma |w| = \frac{d}{dw} \left( 2w^2 a_{out} \frac{d}{da} \delta \Sigma \kappa \pm \delta \Sigma \kappa \right)
\]
\[
= 4wa_{out} \frac{d}{da} \delta \Sigma \kappa + 4|w|^3 a_{out}^2 \frac{d^2}{da^2} \delta \Sigma \kappa \pm 2|w|a_{out} \frac{d}{da} \delta \Sigma \kappa
\]
\[
= 2(2w \pm |w|)a_{out} \frac{d}{da} \delta \Sigma \kappa + 4|w|^3 a_{out}^2 \frac{d^2}{da^2} \delta \Sigma \kappa
\] (B3)

for the second derivative, and
\[
\frac{d^3}{dw^3} \delta \Sigma |w| = \frac{d}{dw} \left[ 2(2w \pm |w|)a_{out} \frac{d}{da} \delta \Sigma \kappa + 4|w|^3 a_{out}^2 \frac{d^2}{da^2} \delta \Sigma \kappa \right]
\]
\[
= 6a_{out} \frac{d}{da} \delta \Sigma \kappa + 8wa_{out}^2 (|w| \pm 2w) \frac{d^2}{da^2} \delta \Sigma \kappa
\]
\[
+ 8w^2 a_{out}^3 \frac{d^3}{da^3} \delta \Sigma \kappa
\] (B4)

for the third derivative.

REFERENCES

Ansorg, M., Kleinwächter, A., & Meinel, R. 2003, MNRAS, 339, 515
Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton Univ. Press), 747
Boyd, J. P. 2001, Chebyshev and Fourier Spectral Methods (2nd ed.; New York: Dover)
Caunt, S. E., & Tagger, M. 2001, A&A, 367, 1095
Cohl, H. S., & Tohline, J. E. 1999, ApJ, 527, 86
Durand, E. 1964, in Electrostatique I: Les Distributions (Paris: Masson Ed.), chap. 6
Gill, P. E., & Miller, G. F. 1972, Comput. J., 15, 80
Gradshteyn, I. S., & Ryzhik, I. M. 1980, Table of Integrals, Series, and Products (New York: Academic Press)
Hachisu, I. 1986, ApJS, 62, 461
Hockney, R. W., & Eastwood, J. W. 1988, Computer Simulation Using Particles (Bristol: Hilger)
Nelson, A. F., & Benz, W. 2003, ApJ, 589, 578
Pierens, A., & Huré, J.-M. 2004, ApJ, 605, 179
Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, in Numerical Recipes in FORTRAN: The Art of Scientific Computing (2nd ed.; Cambridge: Cambridge Univ. Press)