Supplementary Material:
Lift force in odd compressible fluids

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I. INCOMPRESSIBLE DRAG FORCE

In this section we explicitly compute the response matrix in two simple incompressible scenarios. In the absence of relaxation (τ−1 → 0), the incompressible response matrix is given by

\[
(M^{-1})_{ij}(\tilde{\omega}) = \frac{\delta_{ij}}{4\pi \eta_s} \int dz \frac{z J_0(z)}{z^2 - i\tilde{\omega}} = \frac{\delta_{ij}}{8\pi \eta_s} \left[ \log \left( \frac{\tilde{\omega}}{4} \right) + 2\gamma_{EM} - i\frac{\pi}{2} \right] + O(\tilde{\omega}),
\]

(S1)

where \(\gamma_{EM}\) is the Euler-Mascheroni constant. We see that this result is divergent in the steady-state limit (\(\tilde{\omega} \to 0\)), which is a signature of the Stokes paradox [1]. Note that the shell localization result given in Eq. (S1) matches the drag force that one would obtain from solving explicitly the Stokes equation with no-slip boundary conditions [2, 3].

A second scenario is the steady-steady case \(\tilde{\omega} \to 0\) with a finite relaxation rate \(\tilde{\tau}^{-1} \neq 0\). The incompressible response matrix now reads

\[
(M^{-1})_{ij}(0) = \frac{\delta_{ij}}{4\pi \eta_s} \int dz \frac{z J_0(z)}{z^2 + 1/\tilde{\tau}} = \frac{\delta_{ij}}{4\pi \eta_s} \left[ \log \left( 2\sqrt{\tilde{\tau}} \right) - \gamma_{EM} \right] + O(\tilde{\tau}^{-1}).
\]

(S2)

In this case, the response matrix is non-divergent thanks to the momentum relaxation circumventing the Stokes paradox [4–7]. The result in Eq. (S2) can be compared to the result from works of Saffman and Delbrück [8, 9], if one matches the relaxation \(\tilde{\tau}\) as [5]

\[
\tilde{\tau} = \left( \frac{\eta_s}{2a\eta_s'} \right)^2,
\]

(S3)

where \(\eta_s'\) is the shear viscosity of the surrounding bulk fluid that is tied to the substrate in Refs. [8, 9]. We thus find that in these two instances, the shell localization approach yields the same results as in previous works where the fluid velocity profile is computed over the entire two-dimensional surface [8, 9].

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1 Note in Refs. [8, 9] the shear viscosity in the substrate \(\eta_s^{(SD)}\) is three-dimensional and therefore it has different units from the \(\eta_s\) appearing in this letter. In Eq. (S3) the two viscosities are related by taking \(\eta_s^{(SD)} \to \eta_s/h\), with \(h\) being the height of the substrate.
II. ANALYTICAL COMPUTATION OF THE RESPONSE MATRIX

In this section, we show how the integrals performed throughout this Letter can be performed using the method of residues. For a compressible fluid as described in the main text, the response coefficients are obtained by performing momentum integrals that take the form

$$I[R] = \int_0^\infty dz R(z)J_0(z),$$  \hspace{1cm} (S4)

where \( R(z) = A(z)/B(z) \) is an odd function of \( z \), and where \( A \) and \( B \) are polynomials in \( z \). We call \( z_n \) the \( n \)th root of \( B(z) \), such that \( B(z_n) = 0 \). Following Ref. [10], the integral \( I[R] \) can be computed analytically in terms of the Hankel functions of the first kind \( H_\nu^{(1)} \) and the Bessel functions of the second kind \( Y_\nu \). It reads:

$$I[R] = i\pi \sum_{z_n \in \mathbb{C} \setminus \mathbb{R}} \text{Res} \left( R(z)H_0^{(1)}(z), z_n \right) - \pi \sum_{z_n \in \mathbb{R}^+} \text{Res} \left( R(z)Y_0(z), z_n \right),$$  \hspace{1cm} (S5)

where the first sum is over the roots of \( B(z) \) whose imaginary part is strictly positive, and the second one is over the positive real roots of \( B(z) \). We denote by \( \text{Res}(f(z), z_n) \) the residue of \( f \) at point \( z_n \).

As an illustration, we consider the oscillatory incompressible case in the absence of relaxation (\( \tilde{\tau}^{-1} \to 0 \)), for which one has

$$R(z) = \frac{z}{z^2 - i\omega},$$  \hspace{1cm} (S6)

and thus for which \( A(z) = z \) and \( B(z) = z^2 - i\omega \) with the roots \( z_{1,2} = \pm \sqrt{i\omega} \). In this case, only the first term in the right-hand side of Eq. (S5) contributes, and since the Hankel function \( H_0^{(1)} \) has no pole in \( z_1 = \sqrt{i\omega} \), it yields

$$I \left[ z/(z^2 - i\omega) \right] = \frac{i\pi}{2} H_0^{(1)}(\sqrt{i\omega}) = K_0(-i\sqrt{i\omega}),$$  \hspace{1cm} (S7)

where \( K_\nu(x) \) is the \( \nu \)th modified Bessel function of the second kind. An expansion of Eq. (S7) in series of \( \tilde{\omega} \) yields the result given in Eq. (S1).

The same procedure can be applied for the compressible case, and was used the main text.

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