Sigma-model approaches to exact solutions in higher-dimensional gravity and supergravity

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Abstract

Classical gravitating field theories reduced to three dimensions admit manifest gauge invariances and hidden symmetries, which together make up the invariance group $G$ of the theory. If this group is large enough, the target space is a symmetric space $G/H$. New solutions may be generated by the action of invariance transformations on a seed solution. Another application is the construction of multicenter solutions from null geodesics of the target space. After a general introduction on this sigma-model approach, I will discuss the case of five-dimensional gravity, with invariance group $SL(3,R)$, and minimal five-dimensional supergravity, with invariance group $G_{2(2)}$. I will also describe recent attempts at the generation of new charged rotating black rings.

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1 Introduction

An important field in classical general relativity in four or higher dimensions is the search for exact solutions describing black objects [1] (black holes, black rings, etc.). While various techniques to derive particular solutions are available, it may be very difficult to find the general solution (of a given type), i.e. with the maximum number of independent physical parameters.

For example, the most general black ring [2] solution to minimal five-dimensional supergravity (five-dimensional Einstein-Maxwell theory with a Chern-Simons term, hereafter abbreviated as EM5) should have five independent parameters: a mass $M$, two angular momenta $J_\psi$ and $J_\phi$, an electric charge $Q$, and a magnetic dipole charge $D$. At present, only black ring solutions with three independent parameters are known:

- The non-supersymmetric black rings constructed in [3] (Sect. 4). These apparently have five parameters, but two constraints to ensure the absence of Dirac-Misner strings and of conical defects leave only three independent parameters.

- The black rings with two angular momenta constructed in [4] (three parameters $M$, $J_\psi$ and $J_\phi$).

- The charged black rings with two angular momenta constructed in [5] apparently have four independent parameters $M$, $J_\psi$, $J_\phi$, and $Q$, but are singular due to the presence of Dirac-Misner strings.

A solution to the problem of generating exact solutions is provided by the sigma-model approach. Usually the solutions of interest are stationary (Killing vector $\partial_t$) and rotationally symmetric (Killing vector $\partial_\phi$), allowing e.g. the dimensional reduction of a five-dimensional gravitating field theory to a three-dimensional gravitating sigma model. If one is lucky enough, the target space $T$ of this sigma model is a symmetric space, or coset $G/H$, where $G$ is the group of isometries of $T$ ($G_{2(2)}$ for EM5), and $H \subset G$ the local isotropy subgroup ($SL(2,R) \times SL(2,R)$ for EM5). It is then possible to generate new solutions by applying group transformations to the coset representative of a seed solution. Another application is the construction of multicenter solutions as totally geodesic submanifolds of the target space.

In the next section I will give a brief overview of the sigma-model approach. This will be followed by the application to five-dimensional general relativity (E5) in section 3, and to minimal five-dimensional supergravity (EM5), including recent progress in the generation of charged rotating black rings, in section 4.
2 Pedestrian overview of the sigma-model approach

2.1 The SL(2,R)/SO(2) sigma model

There are many good reviews on the subject of gravitating sigma models \[6, 7\]. Let me introduce the basic ideas on the well-known example of four-dimensional vacuum gravity \(\text{E}4\) with one Killing vector \(\partial_t\) [8]. The standard Kaluza-Klein (KK) reduction from four to three dimensions leads to the line element

\[
ds^2 = -f(dt + a_i dx^i)^2 + f^{-1}h_{ij}dx^i dx^j ,
\]

where the gravitational potential \(f\) and the three-dimensional reduced metric \(h_{ij}\) \((i = 1, 2, 3)\) depend only on the \(x^i\). The reduced vacuum Einstein equations \(R_{\mu \nu} = 0\) split in three components:

- a vector component \(R_{0i} = 0\), written symbolically as
  \[
  \nabla \wedge (f^2 \nabla \wedge \vec{a}) = 0 ;
  \]
  this is solved by the duality equation
  \[
  \nabla \wedge \vec{a} = f^{-2} \nabla \omega ,
  \]
  which enables us to trade the Kaluza-Klein vector potential \(\vec{a}\) for a scalar potential \(\omega\), solving identically the field equation
  \[
  \nabla (f^{-2} \nabla \omega) = 0 ;
  \]
- a scalar component \(R_{00} = 0\), leading to the field equation
  \[
  f \nabla^2 f = (\nabla f)^2 - (\nabla \omega)^2 ;
  \]
- and a tensor component \(R^{ij} = 0\), leading to the field equation
  \[
  R_{(3)ij}(h) = \frac{1}{2f^2} \left( \partial_i f \partial_j f + \partial_i \omega \partial_j \omega \right) .
  \]

The equations \(2.2\), \(2.5\) and \(2.6\) derive from the reduced action

\[
S_{(3)} = \int d^3x \sqrt{|h|} \left[ -R_{(3)}(h) + G_{AB}(X) \partial_i X^A \partial_j X^B h^{ij} \right] ,
\]

3
which defines the three-dimensional (coordinates $x^i$, metric $h_{ij}$) gravitating sigma model for the two-dimensional target space (also called potential space \cite{6}) $T$ (coordinates $X^A$, metric $G_{AB}$) with the line element

$$dS^2 = G_{AB}dX^AdX^B = \frac{1}{2f^2}(df^2 + d\omega^2).$$

(2.8)

Two obvious isometries of the target space line element (2.8) follow from the definitions of the potentials $f$ and $\omega$. The line element (2.1) is form-invariant under time rescalings (a relic of the original spacetime invariance under diffeomorphisms, or gauge invariance) $t \rightarrow \alpha^{-1}t$, provided the potentials scale as $f \rightarrow \alpha^2f$, $\omega \rightarrow \alpha^2\omega$. These rescalings are generated by the Killing vector of (2.8)

$$M = 2(f\partial_f + \omega\partial_\omega).$$

(2.9)

Also, the duality equation (2.3) defines the potential $\omega$ only up to translations, generated by the Killing vector

$$N = \partial_\omega.$$ 

(2.10)

Besides these manifest symmetries, the line element (2.8) also admits the hidden symmetry under infinitesimal transformations

$$L = (\omega^2 + f^2)\partial_\omega + 2\omega f\partial_f.$$ 

(2.11)

These three Killing vectors generate the Lie algebra

$$[M,N] = -2N,$$

$$[M,L] = 2L,$$

$$[N,L] = M,$$

(2.12)

which we recognize as Lie[$SL(2,R)$].

The target space $T$ is the symmetric space $SL(2,R)/SO(2)$. A familiar representation of this coset is in terms of the complex Ernst potential

$$E = f + i\omega,$$

(2.13)

leading to the well-known Ernst equations \cite{8}. Less well known, but better suited for generalization to other sigma models, is the representation in terms of a symmetrical $2 \times 2$ matrix

$$\mathcal{M} = \begin{pmatrix} f + f^{-1}\omega^2 & -f^{-1}\omega \\ -f^{-1}\omega & f^{-1} \end{pmatrix},$$

(2.14)

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which allows to write the target space metric (2.8) as
\[ dS^2 = \frac{1}{4} \text{Tr}(\mathcal{M}^{-1} d\mathcal{M} \mathcal{M}^{-1} d\mathcal{M}), \]  
and the vacuum Einstein equations (2.4), (2.5) and (2.6) as
\[ \nabla (\mathcal{M}^{-1} \nabla \mathcal{M}) = 0, \quad \]  
\[ R_{(3)ij}(h) = \frac{1}{4} \text{Tr}(\mathcal{M}^{-1} \partial_i \mathcal{M}^{-1} \partial_j \mathcal{M}). \]  

Other examples of gravitating sigma models obtained by dimensional reduction to three dimensions include:

- \( D = 4 \) Einstein-Maxwell theory (EM4) with one Killing vector, leading to the coset \( T = SU(2, 1)/SU(2) \times U(1) \).

- \( D = p + 3 \) vacuum Einstein gravity (ED) with \( p \) commuting Killing vectors, leading to the coset \( T = SL(p + 1, R)/SO(2, p - 1) \).

- \( D = 4 \) Einstein-Maxwell-dilaton-axion theory (EMDA) with one Killing vector, for which the coset is \( T = Sp(4, R)/U(2) \).

Note that this example is intimately related to the preceding, as EMDA has been shown to be a sector of \( D = 6 \) vacuum Einstein gravity \( T = E_8(8)/SO(16) \) (see also [16]).

- \( D = 11 \) supergravity with 8 commuting Killing vectors, leading to the coset \( T = E_8(8)/SO(16) \).

2.2 Applications of gravitating sigma models

2.2.1 Generation of new solutions.

It is clear that a global group transformation
\[ \mathcal{M}(x) \to \mathcal{M}'(x) = P^T \mathcal{M}(x) P \quad (P \in G) \]  
(2.18)
leaves invariant the field equations (2.16) and the right-hand side of (2.17), and so preserves the reduced three-dimensional metric $h_{ij}$. After reconstructing from the matrix $\mathcal{M}'(x)$ the local fields of the full (non-reduced) theory, this transformation thus leads to a new solution of this theory. Asymptotic flatness will be preserved iff $P \in H_\infty$ (the isotropy subgroup at infinity).

Consider again the example of E4 with the Minkowskian signature $-++$. For an asymptotically flat solution, $f(\infty) = 1$, and $\omega = 0$, leading to

$$\mathcal{M}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  (2.19)

In this case $H_\infty$ is the rotation group $SO(2)$, with only one element which may be parametrized as

$$P = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$  (2.20)

Applying the corresponding transformation (2.18) to the Schwarzschild solution $f = 1 - 2m/r$, $\omega = 0$, we find that asymptotically $\omega' \sim 2m \sin 2\alpha \cdot r^{-1}$, which dualizes to $a'_\phi \sim -2m \sin 2\alpha \cos \theta$, corresponding to the Schwarzschild-NUT solution.

In the case of Einstein-Maxwell theory (EM4), similar three-parameter group transformations applied to the Kerr solution reproduce the well-known dyonic Kerr-Newman NUT solutions. More interestingly, we shall see in the next section that, in the case of E5, the same approach was used by Rasheed [18] to generate previously unknown rotating black string solutions.

### 2.2.2 Multicenter solutions.

In the case of precise equality between mass $m$ and electric charge $q$ (in gravitational units), Newtonian attraction and Coulombian repulsion exactly compensate so that stationary multiparticle systems are possible. This non-relativistic argument carries over to relativistic Einstein-Maxwell theory, in the framework of which static multicenter solutions were first constructed by Papapetrou and Majumdar [19]. These were later generalized to stationary multicenter solutions by Neugebauer, Perjes, and Israel and Wilson [20, 21]. I first showed in 1986 how to construct such multicenter solutions from null geodesics of the target space for five-dimensional gravity (E5) [22]. This method, which was later generalized to the case of other gravitating sigma models [23], shall be described in section 3.
2.2.3 Geroch group.

Often the number of commuting Killing vectors of the spacetime is larger than \( D - 3 \), i.e. not all the Killing vectors are used for reduction to three dimensions. For instance, a particularly interesting subclass of stationary solutions of four-dimensional Einstein-Maxwell theory is that of stationary axisymmetric solutions, with two Killing vectors \( \partial_t \) and \( \partial_\varphi \). In that case, one can combine transformations of the global symmetry group \( G \) with transformations in the hyperplane of Killing vectors, leading to the so-called infinite-dimensional “Geroch group” \[24\]. These transformations allow in principle the generation of all solutions of the stationary axisymmetric Einstein–Maxwell problem, which is thus completely integrable. Practically, this generation of stationary axisymmetric solutions is achieved via inverse–scattering transform methods \[25\]. It is only recently that finite Geroch transformations allowing the direct generation of rotating solutions from static solutions were found \[26, 27\]. I shall describe these approaches in subsections 3.6 and 13.

3 The case of five-dimensional general relativity (E5)

3.1 The Maison approach

The reduction of five-dimensional vacuum gravity to three dimensions and the determination of the isometries of the resulting target space were first achieved by Neugebauer \[20\], who found an eight-parameter symmetry group. Neugebauer’s approach was subsequently applied by Matos \[28\] to solution generation. However it is less transparent than Maison’s approach \[13\], which has the advantage of being manifestly covariant under \( GL(2, R) \) transformations in the plane of the two Killing vectors.

Assuming the existence of two Killing vectors \( \partial_4 \) (timelike) and \( \partial_5 \) (spacelike), the \( GL(2, R) \)-covariant five-to-three reduction is achieved by

\[
\begin{aligned}
ds^2 = \lambda_{ab}(dx^a + a_i^a dx^i)(dx^b + a_j^b dx^j) + \tau^{-1}h_{ij}dx^idx^j, \\
\end{aligned}
\]

where \( a, b = 4, 5 \), \( i, j = 1, 2, 3 \), and \( \tau \equiv -\det \lambda \). The reduction of the five-dimensional Einstein equations follows the same path as in the case of E4, the duality equation \(2.3\) generalizing to

\[
\tau \lambda_{ab} \nabla \wedge \bar{a}^b = \nabla \omega_a.
\]

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The result is a gravitating sigma model for a five-dimensional target space (three “gravielectric” potentials $\lambda_{ab}$ and two “gravimagnetic” potentials $\omega_a$). Maison determined the eight Killing vectors of this target space, and identified the symmetry group as $SL(3, R)$. He then showed that this group acts bilinearly on the symmetric, unimodular matrix potential

$$\chi = \begin{pmatrix} \lambda - \tau^{-1} \omega \omega^T & \tau^{-1} \omega \\ \tau^{-1} \omega^T & -\tau^{-1} \end{pmatrix},$$

(3.3)

where $\lambda$ is a $2 \times 2$ block, and $\omega$ a 2-component column matrix.

In terms of $\chi$, the reduced field equations

$$\nabla(\chi^{-1} \nabla \chi) = 0,$$

(3.4)

$$R_{ij}(h) = \frac{1}{4} \text{Tr}(\chi^{-1} \partial_i \chi \chi^{-1} \partial_j \chi),$$

(3.5)

are manifestly invariant under $G = SL(3, R)$. Assuming the five-dimensional metric (3.1) to be asymptotically Minkowskian with signature $-++++$, the Maison matrix (3.3) goes asymptotically to the constant matrix

$$\chi_\infty = \eta_{BS} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

(3.6)

At this point we should stress that our seemingly natural assumption of an asymptotically Minkowskian five-metric implies $g_{55}(\infty) = \text{constant}$. This is indeed the case for five-dimensional black strings (BS), but not for five-dimensional black holes (BH), in which case the fifth coordinate is an angle on the three-sphere; we shall return to this question in subsection 3.6. The asymptotic behavior (3.6) is preserved by the isotropy subgroup at infinity $H = SO(2, 1)$. Thus, the five-dimensional target space for E5 is $SL(3, R)/SO(2, 1)$.

### 3.2 First applications

The first application of this formalism was given in 1982 by Dobiasch and Maison [29], who obtained all the 2-stationary spherically symmetric solutions of E5 by direct solution of the matrix differential equations (3.4) and (3.5), generalizing previous static spherically symmetric solutions given by Leutwyler [30] and Chodos and Detweiler [31].

In 1986 I proposed a scheme [32] to relate stationary solutions of five-dimensional general relativity (invariance group $SL(3, R)$) and stationary solutions of four-dimensional Einstein-Maxwell theory (invariance group $SU(2, 1)$).
Stationary solutions of EM4 depending on at most two real potentials belong to sectors invariant under subgroups of $SU(2,1)$. These are locally isomorphic to subgroups of $SL(3,R)$:

\[
\begin{array}{ccc}
SU(2,1) & \approx & SL(3,R) \\
SU(1,1) & \approx & SL(2,R) \\
SO(2,1) & = & SO(2,1) \\
SU(2) & \approx & SO(3) \\
U(1) & \approx & O(2),
\end{array}
\]

implying a correspondence between stationary solutions of the two theories. In the axisymmetric case, this correspondence could be made precise, leading to theorems relating exact solutions of EM4 and E5. Several of these relations were already known, however the investigation of the case $SU(2) \approx SO(3)$ led to a new class of 2-stationary five-dimensional space-times generated from the class $E = +1$ of solutions of the Einstein-Maxwell equations. This construction was applied to generate from the massless Kerr-Newman solution a three-parameter family of geodesically complete, asymptotically flat solutions of E5, rotating generalizations of static axisymmetric Lorentzian wormhole solutions constructed from the Chodos-Detweiler wormhole [31] in [33].

### 3.3 Multicenter solutions

Let me first introduce the basic procedure [22, 23] for a generic gravitating sigma model, before describing briefly its application to E5. Consider the case of solutions depending on a single real potential, $M = M[\sigma(x)]$. As shown in [10], this potential can be chosen to be harmonic,

\[
\nabla^2 \sigma = 0.
\]

Then the equations (2.16) and (2.17) reduce to

\[
\frac{d}{d\sigma} \left( M^{-1} \frac{dM}{d\sigma} \right) = 0,
\]

\[
R_{ij} = \frac{1}{4} \text{Tr} \left( M^{-1} \frac{dM}{d\sigma} \right)^2 \partial_i \sigma \partial_j \sigma.
\]

The first of these equations is the geodesic equation for the target space metric (2.15) with $\sigma$ the affine parameter. It is solved by

\[
M = \eta e^{\lambda \sigma},
\]
where $\eta \in G/H$ and $A \in \text{Lie}(G)$ are constant matrices. If $\sigma(\infty) = 0$, then $\eta = M_{\infty}$. In terms of the solution (3.10), the target space metric (2.15) and the three-dimensional Einstein equations (2.17) become

$$dl^2 = \frac{1}{4} \text{Tr}(A^2) d\sigma^2, \quad R_{ij} = \frac{1}{4} \text{Tr}(A^2) \partial_i \sigma \partial_j \sigma,$$

showing that the sign of the spatial curvature, hence the nature of the three-geometry, depends on the sign of the constant $\text{Tr}(A^2)$. Black holes necessarily belong to the class of spacelike target space geodesics ($\text{Tr}(A^2) > 0$), while Lorentzian wormholes [34] necessarily belong to the class of timelike target space geodesics ($\text{Tr}(A^2) < 0$). Null target space geodesics,

$$\text{Tr}(A^2) = 0,$$

lead to solutions with a flat reduced three-space. The harmonic condition (3.7) reduces in this case to the linear Laplace equation, the solutions of which can be linearly superposed, yielding multicenter solutions

$$\sigma(\vec{x}) = \sum_{\alpha} \frac{c_{\alpha}}{|\vec{x} - a_{\alpha}|}.$$  

(3.13)

In the case of E5, the symmetry and unimodularity of the Maison matrix imply the conditions

$$A^T = \eta A \eta, \quad \text{Tr} A = 0.$$  

(3.14)

With $\eta$ given by (3.6), the generic matrix $A$ may be parametrized as

$$A = 2 \begin{pmatrix} -M - \Sigma/\sqrt{3} & -Q & N \\ Q & 2\Sigma/\sqrt{3} & P \\ N & -P & M - \Sigma/\sqrt{3} \end{pmatrix},$$  

(3.15)

where the parameters $M, N, \Sigma, Q$ and $P$ are proportional to the mass, NUT charge, scalar charge, electric charge, and magnetic charge. The constraint (3.12) translates into

$$M^2 + N^2 + \Sigma^2 - Q^2 - P^2 = 0,$$  

(3.16)

which generalizes the antigravity condition of Sherk [35, 36] expressing the balance between “scalar” (attractive) forces and “vector” (repulsive) forces.

The matrix $A$ is a solution of its characteristic equation which, owing to the constraints (3.12) and (3.14), reduces to

$$A^3 = \det A.$$  

(3.17)

As discussed in [22], this leads to three classes of null target space geodesics according to the rank $r(A)$ of $A$:
1. $r(A) = 3 \ (A^3 = 1)$. This class contains, among others, regular static dyon solutions.

2. $r(A) = 2 \ (A^3 = 0, \ A^2 \neq 0)$. This class contains static dyons with equal electric and magnetic charges and vanishing scalar charge sitting in a four-dimensional geometry which is (multiple) extreme Reissner-Nordström.

3. $r(A) = 1 \ (A^2 = 0)$. This class contains systems of electric or magnetic monopoles.

The construction (3.10) may be generalized to the case of several harmonic functions [23]. In the case of E5 [22], multicenter solutions depending on two real harmonic potentials $\sigma$ and $\phi$ are totally geodesic surfaces of the target space

$$\chi = r e^{A \sigma} e^{A^2 \phi},$$

(3.18)

where the matrix $A$ is of rank 2 (class [2]). Appropriate choices of this matrix lead, for harmonic functions which are the real and imaginary part of the complex potential

$$V(\vec{x}) = \sum_{\alpha} \frac{c_{\alpha}}{|\vec{x} - \vec{a}_\alpha - i\vec{b}_\alpha|},$$

(3.19)

to solutions describing systems of rotating electric or magnetic monopoles, or of rotating dyons, generalizing the classes 3 and 2 above. Multiple cosmic string solutions of the five-dimensional Einstein-Gauss-Bonnet equations [38] were also constructed from the ansatz (3.18) in [39].

### 3.4 Rotating dyonic black strings

Stationary black strings in five dimensions (or Kaluza-Klein black holes) were investigated in 1995 by Rasheed [18]. The Maison matrix for asymptotically flat static black strings with regular horizons [40] is of the form (3.10) with $\det A = 0$. This can be obtained from the Schwarzschild Maison matrix, such that $N = \Sigma = Q = P = 0$ in (3.15) and

$$\sigma = -\frac{1}{2M} \ln \left(1 - \frac{2M}{r}\right),$$

(3.20)

by an $SO(2,1)$ transformation. Similarly, rotating dyonic black strings may be generated from the Kerr metric by the global group transform

$$\chi = P^T \chi_K P,$$

(3.21)
where $\chi_K$ is the Maison matrix for the Kerr black string

\[ ds^2_{(5)} = (dx^5)^2 + ds^2_K, \quad (3.22) \]

with $ds^2_K$ the four-dimensional Kerr metric, and $P$ an $SO(2,1)$ matrix. Rasheed actually used only the subclass of transformations $P$ constrained so that the final NUT charge $N$ vanishes, yielding a four-parameter (mass, angular momentum, and electric and magnetic charges) family of rotating black strings.

### 3.5 Relating black strings and black holes

Besides black strings, with a topologically $R \times S^2$ horizon, five-dimensional general relativity also admits black hole solutions [41], with the horizon topology $S^3$. The static spherically symmetric black hole is given by the Tangherlini solution [42],

\[ ds^2_T = -(1 - \frac{\mu}{\rho^2}) dt^2 + (1 - \frac{\mu}{\rho^2})^{-1} d\rho^2 + \rho^2 d\Omega^2_3, \quad (3.23) \]

where

\[ d\Omega^2_3 = \frac{1}{4}[(d\eta - \cos \theta d\varphi)^2 + d\theta^2 + \sin^2 \theta d\varphi^2] \quad (3.24) \]

is the three-sphere line element. Putting $\rho^2 = 4mr$, $\eta = x^5/m$, with $m^2 = \mu/8$, transforms the line element (3.23) to

\[ ds^2_T = -\frac{r-2m}{r} dt^2 + \frac{r}{m}(dx^5 - m \cos \theta d\varphi)^2 + \frac{m}{r-2m} dr^2 + r(r-2m)\left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right]. \quad (3.25) \]

As observed in [43], this has the same reduced three-dimensional metric as the four-dimensional Schwarzschild black string,

\[ ds^2_S = -\frac{r-2m}{r} dt^2 + (dx^5)^2 + \frac{r}{r-2m} dr^2 + r(r-2m)\left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \quad (3.26) \]

so the two corresponding Maison matrices must be related by an $SL(3,R)$ transformation

\[ \chi_T = P^T_{ST} \chi_S P_{ST}. \quad (3.27) \]

The transformation matrix $P_{ST}$, which was determined in [43], does not belong to the subgroup $SO(2,1)$ because the metrics (3.26) and (3.25) have different asymptotic behaviors, so that the matrix $\chi_T$ goes for $r \to \infty$ to a
constant matrix $\eta_{BH}$ (given in the next subsection) different from the matrix $\eta_{BS}$ of (3.6).

As all the black string metrics have the same asymptotic behavior as (3.26), and all the black hole metrics have the same asymptotic behavior as (3.25), one expects that the transformation (3.27) will more generally transform black strings into black holes:

$$\chi_{BH} = P_{ST}^T \chi_{BS} P_{ST}. \quad (3.28)$$

Indeed, it was found in [43] that the action of this transformation on the Kerr black string (3.22) led to the Myers-Perry black hole with opposite angular momenta $a_+ = -a_-$. It was also observed in [43] (Sect. 6) that the reduced three-dimensional metric of the generic Myers-Perry black hole with arbitrary $a_+$ and $a_-$ again coincided with that of the Kerr black string. In the special case of the Myers-Perry black hole with equal angular momenta, $a_+ = a_-$, the reduced four-dimensional metric was static (dyonic), and the reduced three-dimensional metric was the same as that of the Schwarzschild black string. Unfortunately, the full significance of this observation was missed (see next section).

To be complete, let me mention an alternate black string-black hole correspondence which was also given in [43]. This proceeds via the standard five-to-four Kaluza-Klein reduction to $\alpha^2 = 3$ Einstein-Maxwell-dilaton theory, according to the diagram

\[
\begin{array}{ccc}
\text{5D static BH} & \Downarrow & \text{4D NAF static} \\
\text{(Tangherlini)} & & \text{magnetic BH} \\
\text{(Schwarzschild')} & \Uparrow & \text{electric BH} \\
\end{array}
\]

where the downarrow (uparrow) stands for Kaluza-Klein reduction (oxidation), and the rightarrow stands for the four-dimensional electric-magnetic duality relating the non-asymptotically flat (NAF) magnetic and electric black holes. The Schwarzschild' metric is

$$ds^2 = -2\frac{r-2m}{r}(dt-\frac{1}{2}dx^5)^2+\frac{1}{2}(dx^5)^2+r-2m\left[dr^2+r(r-2m)(d\theta^2+\sin^2\theta d\varphi^2)\right].$$

This correspondence was extended in [43, 16] to relate Myers-Perry black holes with two angular momenta and Rasheed dyonic black strings with NUT charge.

\footnote{To conform with the conventions of [27], we use here angular momentum parameters related to those (primed) of [43] by $a_\pm = \mp a'_\pm$.}
3.6 Relating static and rotating solutions

Giusto and Saxena independently observed in [27] that the asymptotic Maison matrix is different for black strings and for black holes. For the T ancherlini metric (3.25), $\omega_4 = 0$, but $\omega_5 = r/m + b$ (where $b$ is some additive constant) diverges at infinity with $r^{-1} \omega_5 \to 1$. The Maison matrix $\chi_T$ goes to a finite non-diagonal limit $\eta_{BH}$ at infinity, and the constant $b$ may be chosen so that $\chi_{55}(\infty) = 0$, leading to

$$
\eta_{BH} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
$$

The two matrices $\eta_{BH}$ and $\eta_{BS}$ are of course related by the general black string/black hole transformation (3.27)

$$
\eta_{BH} = P_{ST}^T \eta_{BS} P_{ST},
$$

with the transformation matrix (for the choice of the additive constant $b$ just mentioned)

$$
P_{ST} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/\sqrt{2} & 1/\sqrt{2} \\
0 & -1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}.
$$

The isotropy group $SO(2,1)$ preserving $\eta_{BH}$ is generated by three transformations $M_\alpha$, $M_\beta$ and $M_\gamma$, different from the Rasheed transformations preserving $\eta_{BS}$. While the transformations $M_\beta$ and $M_\gamma$ are trivial (generalized gauge transformations), Giusto and Saxena showed that the transformation $M_\alpha$ could be used to generate the Myers-Perry black hole from the static T ancherlini black hole in three steps:

1. Act with $M_\alpha$ on the T ancherlini metric (3.23) (written in a form which exhibits the symmetry between the $S^3$ angles $\eta$ and $\varphi$

$$
ds_T^2 = -\left(1 - \frac{\mu}{\rho^2}\right) dt^2 + \left(1 - \frac{\mu}{\rho^2}\right)^{-1} dr^2 + \frac{\rho^2}{4} (d\theta^2 + d\eta^2 + d\varphi^2 - 2 \cos \theta d\eta d\varphi)
$$

reduced with respect to the Killing vectors $\partial_t$ and $\partial_\eta$. This leads to the Myers-Perry metric with equal angular momenta, $a_+ = a_-$. 

2. ”Flip” the angles $\eta \leftrightarrow \varphi$. This amounts to reducing the Myers-Perry metric with $a_+ = a_-$ with respect to the Killing vectors $\partial_t$ and $\partial_\varphi$ instead of $\partial_t$ and $\partial_\eta$ (a finite Geroch transformation!) or, equivalently, to replacing the Myers-Perry metric with equal angular momenta by the Myers-Perry metric with opposite angular momenta.
3. Act on this with a second transformation \( M_{\alpha'} \), leading to a generic Myers-Perry metric with \( a_+ \neq \pm a_- \).

More generally, as discussed in [27], the combined transformation \( M_{\alpha'} \ast \text{Flip} \ast M_{\alpha} \) can be used to generate a 2-rotating solution from any given seed static solution. An interesting (but perhaps technically involved) exercise would be take as a seed the static (singular) black ring of Emparan and Reall [44] with \( S^1 \times S^2 \) horizon, and generate \( \text{à la} \) Giusto-Saxena:

- the black ring rotating along \( S^1 \) [45];
- the black ring rotating along \( S^2 \) [46];
- the black ring with two angular momenta [4].

3.7 Summary

We have seen that E5 reduced to three dimensions leads to the \( SL(3, R)/SO(2, 1) \) gravitating sigma model. The two main applications of this sigma model are:

1. The generation by \( SL(3, R) \) transformations of rotating dyonic black strings from the Kerr black string (Rasheed), black holes from black strings (Clément-Leygnac), and rotating black holes from static black holes (Giusto-Saxena). So, all rotating five-dimensional black strings and black holes can be generated from the four-dimensional Schwarzschild solution.

2. The construction of rotating multicenter solutions from null totally geodesic surfaces of the target space.

4 The case of minimal five-dimensional supergravity (EM5)

4.1 The \( G_{2(2)} \)-based sigma model

The bosonic sector of five-dimensional minimal supergravity [47, 48] is defined by the Einstein-Maxwell-Chern-Simons action (see also Jutta Kunz’s talk at this Seminar)

\[
S_5 = -\frac{1}{16\pi G_5} \int d^5 x \left[ \sqrt{|g|} \left( R + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) + \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A_\lambda \right],
\]

(4.1)

where \( F = dA \), and \( \epsilon^{\mu\nu\rho\sigma\lambda} \) is the five-dimensional antisymmetric symbol. It has been known for some time [49, 50, 51] that reduction of (4.1) to three
Euclidean dimensions leads to the $G_{2(2)}/SL(2, R) \times SL(2, R)$ coset. I will here outline the five-to-three reduction, the identification of the coset and the construction of coset representatives following the approach of [52].

4.1.1 Five-to-three reduction.

The five-dimensional metric fields of (4.1) are broken down by the Kaluza-Klein ansatz (3.1) to the scalars $\lambda_{ab}$ and the Kaluza-Klein vectors $a_i$, and the five-dimensional electromagnetic potential is broken down according to

$$A_{(5)} = \sqrt{3}(\psi_a dx^a + A_i dx^i).$$

(4.2)

The space components of the Maxwell-Chern-Simons equations allow the dualization of the vector potentials $A_i$ to the magnetic scalar potential $\mu$,

$$\tau \left( \nabla \wedge \bar{A} + a^a \wedge \nabla \psi_a \right) - \epsilon^{ab} \psi_a \nabla \psi_b = \nabla \mu,$$

(4.3)

while the mixed $(ai)$ Einstein equations now lead to the duality equations for the gravimagnetic scalar potentials $\omega_a$

$$\tau \lambda_{ab} \nabla \wedge \bar{a}^b + \psi_a (3 \nabla \mu + \epsilon^{bc} \psi_b \nabla \psi_c) = \nabla \omega_a.$$

(4.4)

The remaining field equations are those of a gravitating sigma model, with the eight-dimensional target space $T$ of metric:

$$dS^2 = \frac{1}{2} \text{Tr}(\lambda^{-1}d\lambda^{-1}d\lambda) + \frac{1}{2} \tau^{-2}d\tau^2 - \tau^{-1}V^T \lambda^{-1}V + 3 \left( d\psi^T \lambda^{-1}d\psi - \tau^{-1} \eta^2 \right),$$

where

$$\eta = d\mu + \epsilon^{ab} \psi_a d\psi_b, \quad V_a = d\omega_a - \psi_a \left( 3d\mu + \epsilon^{bc} \psi_b d\psi_c \right).$$

(4.5)

(4.6)

4.1.2 Isometries of $T$.

This metric admits nine manifest Killing vectors, grouped into the $GL(2, R)$ multiplets:

- a quadruplet (generators of $gl(2, R)$ transformations in the $(x^4, x^5)$ plane)

$$M_a^b = 2 \lambda_{ac} \frac{\partial}{\partial \lambda_{cb}} + \omega_a \frac{\partial}{\partial \omega_b} + \delta_a^b \omega_c \frac{\partial}{\partial \omega_c} + \psi_a \frac{\partial}{\partial \psi_b} + \delta_a^b \mu \frac{\partial}{\partial \mu},$$

(4.7)
• a doublet and a singlet (translations of the “magnetic” coordinates)

\[ N^a = \frac{\partial}{\partial \omega^a}, \quad Q = \frac{\partial}{\partial \mu}, \] (4.8)

• and a doublet (gauge transformations of the “electric” coordinates \( \psi_a \))

\[ R^a = \frac{\partial}{\partial \psi_a} + 3\mu \frac{\partial}{\partial \omega_a} - \epsilon^{ab} \psi_b \left( \frac{\partial}{\partial \mu} + \psi_c \frac{\partial}{\partial \omega_c} \right). \] (4.9)

This is enough information to determine the full isometry group \( G \), provided one makes the two assumptions:

1. \( G \) contains the subgroup \( SL(3, R) \) (this is motivated by the fact that EM5 can be consistently truncated to E5 by taking \( A(5) = 0 \));

2. \( \text{Lie}(G) \) is minimal.

These, together with the Jacobi identities, lead to the conclusion\(^2\) that the algebra is \( g_2 \). This has 14 generators, the nine manifest Killing vectors given above, together with the five hidden Killing vectors \( L_a, P_a, \) and \( T \).

These are determined by solving the Lie brackets, up to a single integration constant, which is fixed by enforcing that \( T \) is a Killing vector of (4.5). The result is

\[
T = \left[ 2\mu \lambda_{bc} + 6\epsilon^{de} \lambda_{bd} \psi_c \psi_e \right] \frac{\partial}{\partial \lambda_{bc}} + \left[ 3\mu \omega_b + 3\tau \psi_b - \epsilon^{cd} \omega_c \psi_b \psi_d + 4\tau \lambda \psi_b \psi_c \psi_d \right] \frac{\partial}{\partial \omega_b} + \left[ \omega_b + \mu \psi_b + 2\epsilon^{cd} \lambda_{bd} \psi_c \right] \frac{\partial}{\partial \psi_b} + \left[ \mu^2 + \tau - \epsilon^{bc} \omega_b \psi_c + 2\tau \lambda \psi_b \psi_c \right] \frac{\partial}{\partial \mu}. \] (4.10)

The remaining hidden Killing vectors can be determined from \([R^a, T] = 2\epsilon^{ab} P_b\) and \([P_a, T] = 3L_a\).

Taking into account the signature of the target space metric (4.5), the isometry group is the real noncompact form \( G_{2(2)} \) of the exceptional group \( G_2 \). The root diagram of the \( g_2 \) algebra is shown in Fig. 1. The generators of the Cartan subalgebra are

\[ H_1 = (M_4^4 + M_5^5) / \sqrt{6}, \quad H_2 = (M_4^4 - M_5^5) / \sqrt{2}. \] (4.11)

The \( sl(2, R) \) subalgebra contains these together with the six outermost roots.

---

\(^2\)The question of the symmetry group for a value of the Chern-Simons coupling constant different from the supergravity value \( [54] \) in (4.1) remains open.
4.1.3 Coset representative.

The real form of $g_2$ may be represented in terms of real $7 \times 7$ matrices obtained from the $Z$ matrices of [55] by omitting $i$'s. From the structure of the $gl(2,R)$ generators, one infers that the coset matrix representative in the vacuum ($\psi = \mu = 0$) sector is a real $7 \times 7$ matrix of the block form

$$\mathcal{M}_1 = \begin{pmatrix} \chi & 0 & 0 \\ 0 & \chi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(4.12)

where $\chi$ is the Maison matrix (3.3). The coset representative for the full target space may be obtained [5, 52] by the local group transformation

$$\mathcal{M} = N^T \mathcal{M}_0 N$$

(4.13)

acting on the static ($\omega = 0$) matrix

$$\mathcal{M}_0 = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & -\tau^{-1} & 0 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & -\tau & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(4.14)
with

\[ N = e^{\psi a r a} e^{\mu a} e^{\omega a r a}, \]  

(4.15)

where \( n^a, q, r^a \) are the matrix representatives of the manifest Killing vectors \( N^a, Q, R^a \). The resulting matrix representative has the symmetrical block structure

\[ \mathcal{M} = \begin{pmatrix} A & B & \sqrt{2}U \\ B^T & C & \sqrt{2}V \\ \sqrt{2}U^T & \sqrt{2}V^T & S \end{pmatrix}, \]  

(4.16)

where \( A \) and \( C \) are symmetrical \( 3 \times 3 \) matrices, \( B \) is a \( 3 \times 3 \) matrix, \( U \) and \( V \) are 3-component column matrices, and \( S \) a scalar. The inverse matrix is given by

\[ \mathcal{M}^{-1} = K \mathcal{M} K, \quad K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]  

(4.17)

### 4.2 Applications

The first, obvious application \cite{5} was to generate charged solutions of EM5 from seed vacuum solutions by the global group transform

\[ \mathcal{M}' = C^T \mathcal{M} C, \]  

(4.18)

with

\[ C = e^{\beta(p^4 + r^4)} \]  

(4.19)

the charge-generating transformation. This was tested on the example of a Myers-Perry vacuum seed \cite{11}, yielding the charged rotating black hole first given in \cite{57} and rederived, in a different parametrization, in \cite{58}. The action of the transformation (4.18) on the neutral black ring with two angular momenta \cite{4} led to a doubly rotating charged black ring. However the resulting five-dimensional solution \((g', A')\) was singular, due to the presence of Dirac-Misner strings. The same transformation was also applied recently to the generation of rotating black strings with Maxwell electric charge from Rasheed black strings with vanishing Kaluza-Klein electric charge \cite{59}. An alternate route to the generation of such charged black strings would be to apply the inverse black string/black hole transformation \cite{3.28}, extended to EM5 (see subsection 4.3), to charged black holes.

\footnote{Another, equivalent 7 \times 7 matrix representation of the \( G_2(2) / SL(2, R)^2 \) coset was independently given in \cite{56}.}
It has recently been pointed out that there is a shorter route to charge generation in EM5. Minimal five-dimensional supergravity reduced to four dimensions has a global $SL(2,R)$ symmetry, and a certain $O(1,1) \in SL(2,R)$ transformation does the same job as the charging transformation $C$, without all the cumbersome $G_2(2)$ machinery [60].

The potentialities of the $G_2(2)$ generating technique could be more fully exploited by extending the Giusto-Saxena approach to this case. The asymptotic limit $\eta_{BH}$ of the matrix $M$ for black holes is (4.12) with $\chi$ given by (3.29). The isotropy subgroup $SL(2,R) \times SL(2,R)$ preserving $\eta_{BH}$ [61] now contains three trivial (gauge) transformations, and the three non-trivial transformations:

\[
S = e^{\alpha(l_4 + m_5)}, \\
C = e^{\beta(p_4 + r_4)}, \\
D = e^{\gamma(p_5 - t)}.
\]

(4.20)

$S$ is the original Giusto-Saxena spin-generating transformation, $C$ is our electric charge-generating transformation (4.19), and the transformation $D$ generates a dipole charge. One could in principle combine these three transformations together with flips to generate a five-parameter black ring from the uncharged static black ring of [44]. This solution-generating technique could also be applied to the recently discovered black lens solution [62].

4.3 Generating rotating solutions via 5D Bertotti-Robinson

4.3.1 Generating rotating solutions in EM4

Let me first recall the sigma-model technique to generate rotating solutions outlined in [26]. As recalled in 2.1, four-dimensional Einstein-Maxwell theory reduced (with respect to $\partial_t$) to three dimensions leads to the $SU(2,1)/S[U(2) \times U(1)]$ sigma model. According to Geroch [24], in the stationary axisymmetric case one can in principle combine finite global $SU(2,1)$ transformations with finite linear transformations in the plane of the two Killing vectors $\partial_t$ and $\partial_\varphi$ to generate new solutions. The problem is that the only such linear transformation which does not lead to the appearance of closed timelike curves and/or conical singularities is the transformation $R(\Omega, \gamma)$ (transition to a uniformly rotating frame combined with a time dilation),

\[
d\varphi' = d\varphi - \Omega \, dt, \\
dt' = \gamma^{-1} \, dt.
\]

(4.21)
which, in the case of an asymptotically Minkowskian seed solution, drastically changes the asymptotic behavior (appearance of a “centrifugal force”). The solution to this problem, as given in [26], is to combine the transformation (4.21) with an \( SU(2,1) \) transformation \( \Pi \) also changing the asymptotic behavior.

Consider the Bertotti-Robinson solution to \( EM4 \) [63], describing the \( AdS_2 \times S^2 \) spacetime generated by a constant electric field,

\[
d\hat{s}_2 = \left(1 - \frac{x^2}{m^2}\right) dt^2 + \left(\frac{x^2}{m^2} - 1\right)^{-1} \left[ dx^2 + (x^2 - m^2) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]
\[
\hat{A} = -\frac{x}{m} dt.
\]

The reduced three-dimensional metric in (4.22) is the same as that of the Schwarzschild metric \((x = r - m)\)

\[
ds_S^2 = -\frac{x - m}{x + m} dt^2 + \frac{x + m}{x - m} \left[ dr^2 + (x^2 - m^2) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]

so the two solutions are related by a transformation \( \Pi \in SU(2,1) \), given in [26]. The global coordinate transformation \( R(\Omega, \gamma) \) acting on the Bertotti-Robinson solution now leads to a solution \((d\hat{s}_2', \hat{A}')\) (the same Bertotti-Robinson spacetime and electromagnetic field described in a different coordinate system) with the same asymptotic behavior, for instance,

\[
g'_{tt} = \gamma^2 (1 + m^2 \Omega^2 \sin^2 \theta - r^2/m^2).
\]

One can now return to the asymptotically flat world by the action on this primed Bertotti-Robinson solution of the inverse transformation \( \Pi^{-1} \), leading (for the choice\(^4 \gamma = (1 + m^2 \Omega^2)^{-1} \) to the Kerr solution:

\[
\begin{array}{c}
\begin{array}{cccc}
(\text{as. } AdS_2 \times S^2) & \text{Bertotti-Robinson} & \longrightarrow & \text{Bertotti-Robinson'} \\
(\text{as. } M_4) & \text{Schwarzschild} & \longrightarrow & \text{Kerr} \\
\end{array}
\end{array}
\]

More generally, the combined transformation \( \Sigma(\Omega, \gamma) = \Pi^{-1} \ast R(\Omega, \gamma) \ast \Pi \) acting on a static asymptotically flat solution leads to a rotating asymptotically flat solution (examples are given in [26]).

\(^4\)For a generic value of \( \gamma \) one obtains the Kerr-Newman solution.
4.3.2 Generalization to EM5

Reduction of EM5 to four spacetime dimensions leads to an Einstein theory with two coupled abelian gauge fields, the reduced Maxwell field $F$ and the Kaluza-Klein field $G$, a dilaton $\phi$ and an axion $\kappa$. This theory can be consistently truncated to four-dimensional Einstein-Maxwell theory (EM4) by enforcing the constraints

$$\phi = 0, \quad \kappa = 0, \quad G = \frac{1}{\sqrt{3}} \star F. \quad (4.25)$$

After further reduction to three dimensions, one finds [61] that these constraints are preserved by eight infinitesimal transformations which generate the Lie algebra of $SU(2,1)$ (the isometry group of EM4 reduced to 3D).

Conversely, any solution of EM4 can be lifted to a solution of EM5 satisfying the constraints (4.25). Applying this lifting procedure to the four-dimensional Bertotti-Robinson solution (4.22) ($AdS_2 \times S^2$), we obtain [61] the five-dimensional Bertotti-Robinson solution ($AdS_2 \times S^3$):

$$ds_B^2 = \left(1 - \frac{x^2}{m^2}\right) dt^2 + \left(\frac{x^2}{m^2} - 1\right)^{-1} dx^2 + m^2 \left[(d\eta - \cos \theta d\varphi)^2 + d\theta^2 + \sin^2 \theta d\varphi^2\right]. \quad (4.26)$$

This has the same reduced three-dimensional metric as the Tangherlini solution (3.25) (with $r = x + m$), so there is a $G_2$ transformation $P_{TB}$ relating the two, which can be obtained as the product transformation Tangherlini $\to$ Schwarzschild black string $\to$ Bertotti-Robinson:

$$P_{TB} = P_{TS} P_{SB}. \quad (4.27)$$

The $7 \times 7$ matrix $P_{TS}$ is

$$P_{TS} = \begin{pmatrix} P_{(3)TS} & 0 & 0 \\ 0 & P_{(3)ST} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.28)$$

where $P_{(3)ST}$ is the matrix (3.31), and $P_{(3)TS} = P_{(3)ST}^{-1}$. To determine the matrix $P_{SB}$, we use the fact that the spherically symmetric Schwarzschild and Bertotti-Robinson solutions depend on the same real potential $\sigma$, and so, from subsection 3.3, can be written as

$$M_S = \eta_S e^{A_{s\sigma}}, \quad M_B = \eta_B e^{A_{b\sigma}}, \quad (4.29)$$
with
\[
\eta_B = P_{SB}^T \eta_S P_{SB}, \quad A_B = P_{BS} A_S P_{SB}.
\] (4.30)

\(A_S\) is diagonal, so \(P_{SB}\) is the matrix which diagonalizes \(A_B\) and satisfies the first equation (4.30). The resulting matrix \(P_{TB}\) is
\[
P_{TB} = \frac{1}{2}
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & \sqrt{2} \\
0 & 0 & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & -\sqrt{2} \\
0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 \\
0 & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\
\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & 0
\end{pmatrix}.
\] (4.31)

As in the case of EM4, the rotation-generating transformation is the product
\[
\Sigma(\Omega, \gamma) = P_{BT} R(\Omega, \gamma) P_{TB},
\] (4.32)

where \(R(\Omega, \gamma)\) is the coordinate transformation [4.21]. By construction, the transformation \(\Sigma(\Omega, \gamma)\) acting on an asymptotically Tangherlini solution generates an asymptotically Myers-Perry solution. We have applied this transformation to the (singular) static black ring of [44]. This complex procedure, involving repeated reductions (dualizations) and oxidations (inverse dualizations) according to the diagram
\[
\begin{array}{cccccc}
ds^2 & ds^2, \hat{A} & \rightarrow & ds^2, \hat{A}' & & ds^2, A' \\
\downarrow & \uparrow & \downarrow & \uparrow \\
\mathcal{M} & \rightarrow & \mathcal{M}' & & \mathcal{M}' & \rightarrow & \mathcal{M}' \\
\end{array}
\]
leads [61] to a complicated solution describing a rotating black ring with multipole electric and magnetic moments. A drawback of this procedure is that, as in the case of EM4, it transforms horizons into horizons, and singularities into singularities, so that our rotating black rings are singular.

### 4.4 Other recent developments

In [64], Gaiotto, Li and Padi constructed a class of multicenter solutions of EM5 as geodesics
\[
M = \eta e^{A_\sigma},
\] (4.33)
with $\sigma$ a harmonic function and

$$A^3 = 0,$$  (4.34)

implying, for $A \in g_2, \text{Tr}(A^2) = 0$. This certainly does not exhaust the subject, as EM5 can be consistently truncated to E5 which, as we have seen in subsection 3.3, admits three classes of multicenter solutions. The general analysis remains to be done.

In [65], Gal’tsov and Scherbluk discussed the hidden symmetries of five-dimensional supergravity with three Abelian gauge fields, a truncated toroidal compactification of $D = 11$ supergravity, and showed that reduction of this theory to three dimensions leads to the $SO(4,4)/SO(2,2) \times SO(2,2)$ gravitating sigma model. They constructed coset representatives as real $8 \times 8$ matrices and, as an application, derived the doubly rotating black hole solution with three independent charges.

By identification of the three vector fields, this $U(1)^3$ five-dimensional supergravity can be contracted to minimal five-dimensional supergravity which, as we have seen, admits upon reduction to three dimensions the invariance group $G_{2(2)} \subset SO(4,4)$. Contraction of the matrix representatives of the $U(1)^3$ theory thus leads to an $8 \times 8$ matrix representation [65] of the $G_{2(2)}/SL(2,R) \times SL(2,R)$ coset, alternate to the $7 \times 7$ representation given above. It is not clear which is more simple to use for solution generation.

4.5 Summary

We have seen that EM5 reduced to three dimensions leads to the $G_{2(2)}/SL(3,R) \times SL(2,R)$ gravitating sigma model. The first applications to the generation of charged solutions from neutral solutions could be generalized by extending the Giusto-Saxena approach. We have also discussed the generation of rotating solutions from static solutions via the five-dimensional Bertotti-Robinson solution.

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