Bayesian Analysis of Sparse Multivariate Matched Proportions

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Abstract

Multivariate matched proportions (MMP) data appears in a variety of contexts—including post-market surveillance of adverse events in pharmaceuticals, disease classification, and agreement between care providers—and consists of multiple sets of paired binary measurements taken on the same subject. While recent work proposes non-Bayesian methods to address the complexities of MMP data, the issue of sparsity, where no or very few responses are recorded for one or more sets, is undressed. However, the presence of sparse sets results in underestimates of variance, loss of coverage, and bias in existing methods. Additionally, Bayesian methods have not previously been considered for MMP data. In this work, we propose a Bayesian marginal probability model for MMP data with robust t-priors that adjusts for sparsity using targeted informative priors on the variance components of sparse sets and half-Cauchy priors on non-sparse sets. In simulation, we demonstrate our method’s ability to handle sparsity and show that it outperforms existing methods in terms of coverage and bias, while maintaining comparable power. Finally, we analyze data from a study of care coordination within a System of Care framework and provide additional insights that the original univariate analysis missed.
1 Introduction

Multivariate matched proportions (MMP) data arises when multiple sets of paired binary measurements are taken on the same subject. An example of such data comes from a study of care coordination within a System of Care (SOC) framework by Knutson et al. (2018). An SOC informs the treatment of children with severe mental and behavioral disorders by coordinating care between six different components: mental health, primary care, education, child welfare, juvenile justice, and developmental disabilities. However, primary care frequently provides the sole treatment for children with mental health disorders making it unclear if those children are receiving the potential benefits of the SOC. In particular, primary care may not be initiating contact with SOC components at the same rate as psychiatric care. To begin to investigate primary care’s tendency to initiate coordination within an SOC, Knutson et al. (2018) present the results of a retrospective chart study of youth referred by pediatricians to a child psychiatrist at an urban community health center. The authors assess coordination by examining the documented contacts in the medical record between primary care and each of the remaining components of the SOC. The study then compares those contacts to the documented contacts between the psychiatrist, or specialty care, and the SOC components made after the initial psychiatric evaluation. We summarize the contact data for all of the SOC components in Table 1 where the column headers denote the assessor, primary care or specialty care, and the column sub-headers denote the components. The original analysis of this data did not account for the multivariate nature of the data nor did it account for sparsity in contacts made with two components. Sparsity is most noticeable for contact made with developmental disabilities by specialty care where no contacts were made (see Table 1). The sparsity of juvenile justice is also noticeable when looking at primary care contacts with this component as only one was made. Alternatively, we can represent the data in a series of population averaged tables, as in Table 2 and diagnose sparsity by noting the zero cell counts in both developmental disabilities and juvenile justice tables. Since each component has a corresponding set of paired binary measurements, two of the five sets in this data are sparse.

Several authors consider non-Bayesian approaches for analyzing MMP data. Klingenberg and Agresti (2006) use marginal probability models to test for simultaneous marginal heterogeneity. They employ generalized estimating equations (GEE) along with robust standard errors to estimate effects and construct a multivariate version of McNemar’s test (McNemar, 1947). Others propose multiple testing approaches to account for correlation between sets of matched proportions (Westfall et al., 2010; Xu and Yu, 2013). For example, Westfall et al. (2010) obtain loose control of the family wise error rate using Bonferroni-Holm on bootstrapped p-values. Lui and Chang (2013) show that mixed-effects exponential risk models can be used to estimate the risk ratios for each set of matched proportions based on consistent estimators of the risk. In more recent work by the same authors, Lui and Chang (2016) consider a mixed-effects logistic regression model with separate random effects of pair and set. The authors show the estimated effect of treatment for each outcome using this approach reduces to the log of the well-known univariate Cochran-Mantel-Haenszel (CMH) estimator for stratified $2 \times 2$ tables (Cochran, 1950; Mantel and Haenszel, 1959), after integrating out the random effects. Jiang and Xu (2017) explore power and sample
Table 1: Multivariate matched proportions data from the SOC study for care-specific contact with developmental disabilities (DD), mental health (MH), juvenile justice (JJ), child welfare (CW), and education (ED). Table values are 1 if contact was made with the agency in the column sub-header by the assessor in the column header, and 0 if no contact was made. The Pattern Count column denotes the number of times the pattern appears in the data. This table excludes patterns that do not appear in the data.

| Primary Care | Specialty Care | Pattern Count |
|-------------|----------------|---------------|
| DD | MH | JJ | CW | ED | DD | MH | JJ | CW | ED | Count |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 17 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 5 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 5 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 3 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
Table 2: Population averaged tables for all SOC components. ‘Yes’ denotes contact was made with the component, ‘No’ indicates contact was not made. The diagonal counts within each table represent the number of concordant pairs. The off-diagonal counts within each table represent the number of discordant pairs.

| Development Disabilities |       |        | Mental Health |       |        |
|--------------------------|-------|--------|---------------|-------|--------|
|                         | Specialty | Primary Care | Specialty | Primary Care | Specialty | Primary Care |
|                         | Care | Yes | No | Care | Yes | No | Care | Yes | No |
| No | 211  | 3 | 71 | 212  | 2 | 50 |
| Yes | 114  | 3 | 7 | 115  | 11 | 31 |
| No | 214  | 4 | 60 | 215  | 2 | 30 |

| Child Welfare |       |        | Education |       |        |
|---------------|-------|--------|-----------|-------|--------|
|               | Specialty | Primary Care | Specialty | Primary Care | Specialty | Primary Care |
|               | Care | Yes | No | Care | Yes | No | Care | Yes | No |
| No | 211  | 3 | 71 | 212  | 2 | 50 |
| Yes | 114  | 3 | 7 | 115  | 11 | 31 |
| No | 214  | 4 | 60 | 215  | 2 | 30 |

| Juvenile Justice |       |        |
|------------------|-------|--------|
| Specialty | Primary Care | Specialty | Primary Care |
| Care | Yes | No | Care | Yes | No |
| No | 211  | 3 | 71 | 212  | 2 | 50 |
| Yes | 114  | 3 | 7 | 115  | 11 | 31 |
| No | 214  | 4 | 60 | 215  | 2 | 30 |
size calculations for this and the GEE-based marginal probability model. As is common in
the analysis of matched proportions, each of these methods exclude the concordant pairs,
only relying on information from the discordant pairs (Agresti, 2013).

The literature on Bayesian methods for matched proportions is limited to the univariate
case. For example, Altham (1971) examines closed form results for a multinomial model
with Dirichlet priors while Broemeling and Gregurich (1996) discuss the Gibbs sampling
approach to this model. Ghosh et al. (2000) present an item response model with logit,
probit, and complementary log-log links. This formulation allows for a hierarchical Bayesian
modeling of the data. The authors show that, for such models, the concordant pairs also
contribute to estimation and inference. While these approaches are useful for univariate
matched proportions, to our knowledge, no previous work considers Bayesian methods for
the analysis of MMP data.

More importantly, two of the existing methods for MMP data cannot directly handle
sparse data, as we will now illustrate. The GEE-based approach of Klingenberg and Agresti
(2006) estimates marginal probabilities under the assumption of working independence. Us-
ing a model with one parameter for each marginal probability, the authors show that the
estimates equal the sample probabilities of the columns of Table 1. But when a column
is sparse, as is the case for the DD component of the SOC data, the estimated probabil-
ity is zero. The estimated variance will also be zero, as will the associated covariances.
The variance of the difference in marginal probabilities between care-type depends on both
this variance and the corresponding covariance of the sparse column, resulting in under-
estimation of the variance. Westfall et al. (2010) base their bootstrap method off of the
differences between the paired outcomes, i.e. the difference between paired columns of Ta-
ble 1, and use this to estimate differences in the marginal probabilities. In general, these
differences can take on the values −1, 0, and 1. But when the data is sparse, as in the
DD component, the differences can only be 0 or 1 meaning the lower bound of a boot-
strapped interval is bounded below by 0. Similarly, for the JJ component, only −1 and
0 are possible realizations resulting in a bootstrapped interval that is bound above by 0.
Once again, the uncertainty of the difference in marginal probabilities for sparse outcomes
is underestimated.

Lui and Chang (2013) and Lui and Chang (2016) do both provide adjustments for
sparse data in the exponential risk model (ERM) and mixed effects regression approaches,
respectively. The ERM approach, for example, bases its estimation off of the K-specific
tables in Table 2. Specifically, the estimate is the ratio of the counts of discordant
pairs in each table: \( n_{21k}/n_{12k} \). When one of these elements is zero, the authors suggest
adding 1/2 to the numerator and the denominator making their sparsity adjusted estimator
\( \left( n_{21k} + \frac{1}{2} \right) / \left( n_{12k} + \frac{1}{2} \right) \). Neither Lui and Chang (2013) nor Lui and Chang (2016) evaluate
the performance of their estimators in the presence of sparsity and instead evaluate
the performance of their testing procedure in non-sparse simulated data settings. Such a
correction is inherently Bayesian in thinking as it recognizes that the true population pa-
rameter is not zero (or infinite) and adjusts the estimate accordingly. In fact, this estimator
is effectively an empirical Bayes estimate of the marginal risk under Jeffreys’ prior when
using the Dirichlet model proposed by Altham (1971).

This adjustment is not generalizable either: it is not applicable to the GEE or Bootstrap
approaches as it adjusts the counts from Table 2 whereas both the GEE and Bootstrap rely on the individual-level data as in Table 1. Other common adjustments like data augmentation, e.g. adding one success and one failure in the case of binary data, do not easily generalize to the multivariate case. Augmenting sparse cells in Table 2 separately would ignore the multivariate nature of the data, breaking potential underlying correlations. Adding covariate patterns to Table 1 must take into account the impact on the multivariate structure but can also lead to severe sample size inflation. At the extreme, augmentation would involve adding one of every possible type of covariate pattern to Table 1. For even a small number of sets, such an addition would greatly inflate the sample size far beyond what augmentation strategies aim for, potentially biasing the results. Thus, there is a need for an MMP method that can accommodate sparse data more adequately in addition to the need for a Bayesian approach.

In this work, we propose a Bayesian sparsity adjusted marginal, or BSpAM, model for MMP data. This model penalizes sparseness in two ways: first, we place penalty priors to shrink the estimated probabilities toward a prior mean. For brevity, we present results from only the $t$-prior with 5 df, but we also consider two normal priors and a Laplace prior in simulation and application. Second, we place informative priors on the variances of sparse components. This prior specification is necessary to prevent underestimation of the uncertainty regarding the estimated sparse probabilities and a distinct advantage of the Bayesian approach. Next, we compare the BSpAM model to the GEE, ERM, and Bootstrap in simulation and demonstrate that, regardless of prior choice, the BSpAM outperforms the existing MMP methods on sparse components while having similar, and often better, properties for non-sparse components. Finally, we demonstrate the BSpAM in application using the SOC data as a case study. Our analysis using BSpAM provides additional insights on the nature of the System of Care that the original analysis missed.

2 Methods

At the pair-level, let $x_{ijk}$ denote the response for pair $i$, at observation $j$, for the $k$th set where $i = 1, \ldots, n$, $j = 1, 2$, and $k = 1, \ldots, K$. In context of the SOC data, $i$ denotes the subjects and $j$ denotes care-type, primary care or specialty care. The index $k$ denotes the SOC component with which contact is made, $x_{ijk} = 1$, or not made, $x_{ijk} = 0$. Combining across $k$, we let $x_i = [x_{ij1} \cdots x_{ijK}]'$ be the vector of responses at observation $j$ for pair $i$. Subject $i$’s stacked vector of responses is then $x_i = [x_{i1} \ x_{i2}]'$. Realizations of this vector make up the rows of Table 1.

Our primary parameters of interest are the $k$-specific differences in the marginal probability of contact, $P(x_{ijk} = 1)$, which both Klingenberg and Agresti (2006) and Westfall et al. (2010) estimate. First let $\theta = [\theta_{11} \cdots \theta_{1K} \ \theta_{21} \cdots \theta_{2K}]$ be the vector of marginal probabilities. Then let $\delta_k$ be the $k$th difference in the marginal probabilities, thus $\delta_k = \theta_{1k} - \theta_{2k}$. The vector of the $\delta_k$’s is $\delta$, $\delta = [\delta_1 \cdots \delta_K]$. Alternatively, define $L$ to be a block matrix of the form $L = (I_{K \times K} | -I_{K \times K})$ where $I_{K \times K}$ is a $K \times K$ identity matrix. The vector representing the differences in marginal probabilities is then $\delta = L \theta$.

Similar to Klingenberg and Agresti (2006), we consider a marginal model of the form
To the marginal probabilities, we assign a $t_5$ prior on $\theta_{jk}$:

$$\theta_{jk} \sim t_5(\mu_0, \tau).$$

We check the sensitivity of the model to this choice by also considering a normal prior, a parameter expanded normal prior, and a Laplace prior—all of which we describe in the Supplementary Material. Regardless of the choice, the prior on $\theta_{jk}$ gives some level of shrinkage toward $\mu_0$. For components with limited information, this can mean moving the estimate of $\theta_{jk}$ away from zero and toward a null value, e.g. 0.5.

To prevent the variance from being underestimated, we impose group-specific priors on the variance components, grouped by sparse and non-sparse sets. Assume the $e_{ijk}$ are normal, $e_{ijk} \sim N(0, \lambda_k)$. For sets that are sparse, we implement an informative inverse-gamma prior, $\lambda_k \sim IG(2.8, 0.45)$ where $k^*$ indexes the sparse sets. The parameters of the prior are chosen to make the prior mean on the variance 0.25 (a corresponding standard deviation of 0.5), the maximum of the Bernoulli variance. In simulation, we consider different choices for the parameters of this prior. For non-sparse sets, we place a half-Cauchy prior with scale equal to 0.5 on $\lambda_k$, $\sqrt{\lambda_k} \sim HC(0.5)$ where $k^*$ indexes the non-sparse sets. Setting the scale to 0.5 weakly constrains $\lambda_k$ by the largest standard deviation allowed for a Bernoulli variance (Gelman, 2006).

Noting two mixture distributions results regarding the $t$ and half-Cauchy distributions, the full conditionals under are identifiable. The mixture-normal representation of the $t_5$ prior is $\theta_{jk} \sim N[\mu_0, \sigma_{jk}^2]$ and $\sigma_{jk}^2 \sim IG(\frac{5}{2}, \frac{5}{2}\tau)$ where $IG$ denotes the inverse-gamma distribution (Gelman et al., 2013). We use Jeffreys’ prior for the hyper-prior on $\tau$, $\pi(\tau) \propto \tau^{-1}$. The half-Cauchy can be represented as a mixture of two inverse gamma distributions (Wand et al., 2011). If $\lambda_k \sim IG(1/2, 1/\zeta_k)$ and $\zeta_k \sim IG(1/2, 1/A^2)$, then $\sqrt{\lambda_k} \sim HC(A)$ where we set $A = 0.5$.

For each subject, the BSpAM model is $x_i = \theta + e_i$. Let $x$ be the vector of all $x_i$s combined into one and define $D$ to be the matrix of $n \times K \times K$ identity matrices stacked on top of each other. This vector and matrix have the forms

$$x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
d\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}
I_{K \times K} \\
I_{K \times K} \\
\vdots \\
I_{K \times K}
\end{bmatrix}.$$
The full conditionals for our primary BSpAM model are then

\[ \theta | \text{rest} \sim N \left( \Sigma_\theta \left[ D'\Lambda^{-1} Y + \text{diag} \{ \sigma^2_{jk} \}^{-1} \mu_0 \right], \Sigma_\theta \right), \]

for \( k^\circ = 1, \ldots, K^\circ \)

\[ \lambda_{k^\circ} | \text{rest} \sim IG \left( n + 2.8, 0.45 + \frac{1}{2}[X_{(k^\circ)} - D_{(k^\circ)}'\theta_{(k^\circ)}][X_{(k^\circ)} - D_{(k^\circ)}'\theta_{(k^\circ)}]' \right), \]

for \( k^\circ = 1, \ldots, K^\circ \)

\[ \zeta_{k^*} | \text{rest} \sim IG \left( 1, \frac{1}{0.5^2} + \frac{1}{\lambda_{k^*}} \right) \]

\[ \lambda_k | \text{rest} \sim IG \left( n + \frac{1}{2}, \frac{1}{\zeta_{k^*}} + \frac{1}{2}[X_{(k^*)} - D_{(k^*)}'\theta_{(k^*)}][X_{(k^*)} - D_{(k^*)}'\theta_{(k^*)}]' \right), \]

for \( k = 1, \ldots, K \) and \( j = 1, 2 \)

\[ \sigma^2_{jk} | \text{rest} \sim IG \left( \frac{5 + 1}{2}, \frac{5\tau}{2} + \frac{1}{2}[\theta_{jk} - \mu_0]^2 \right), \quad \text{and} \]

\[ \tau | \text{rest} \sim \text{gamma} \left( 5K, \frac{5}{2} \sum_{k=1}^{K} \sum_{j=1}^{2} \frac{1}{\sigma^2_{jk}} \right) \]

where \( \Sigma_\theta = \left[ D'\Lambda^{-1} D + \text{diag} \{ \sigma^2_{jk} \}^{-1} \right]^{-1} \), \( K^\circ \) is the total number of non-sparse sets, and \( K^\circ \) is the total number of sparse sets. The notation \((k^\circ)\) and \((k^\circ)\) denotes the corresponding matrix or vector subset to elements that correspond to the respective non-sparse or sparse sets.

When prior information is available for each component, \( \mu_0 \) can be replaced by \( \mu_{jk,0} \) and \( \mu_0 \) is a vector containing the prior means. Our motivating dataset is, to our knowledge, the first of its kind and therefore does not have informative prior information to draw upon. We consider two choices for the prior mean in the absence of prior data. First, we select \( \mu_0 = 0.5 \) to represent a “null” prior for modeling the probabilities \( \theta_{jk} \). Second, we consider \( \mu_0 = 1/(n+2) \) to represent a weakly informative prior where the values of \( \theta_{jk} \) are expected to be close, but not equal, to zero. This second approach incorporates the concept of adding one success and one failure to each column in Table II into a prior. To facilitate implementation of the BSpAM model, we make available user-friendly \texttt{R} code available at \url{https://github.com/markjmeyer/bspam}.

### 3 Simulation Study

We base our simulation study, in part, on the SOC data. For each dataset, we generate data for \( n = 75 \) subjects where \( K = 2 \) sets of binary measurements are taken on each subject. We aim to induce both sparsity and within observation, or care-type, correlation. To accomplish this, we first generate probabilities for \( \theta \) from a multivariate normal with one of three different correlation structures. In the event a draw from this distribution is negative, it is set to zero. Values of \( x_i \) are then drawn from a Bernoulli distribution. The
“true” value of $\boldsymbol{\theta}$ is the vector $\boldsymbol{\theta} = \begin{bmatrix} 0.05 & \theta_{12} & \theta_{21} & 0.001 \end{bmatrix}$. To evaluate sparsity as the degree of difference between observations changes, we vary $\theta_{12}, \theta_{21} = \{0.03, 0.05, 0.07, 0.09\}$. And to evaluate the models on a non-sparse set, we consider values of $\theta_{21}$ ranging from 0.05 to 0.25, $\theta_{21} = \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4\}$. These values for $\boldsymbol{\theta}$ encompass the range of values we see when estimating $\delta$ for the SOC data. Finally, the underlying correlation structure has the form

$$0.01^2 \times \begin{bmatrix} 1 & r & 0 & 0 \\ r & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & r & 1 \end{bmatrix},$$

where $r = 0, 1/3,$ or $2/3$. We select this structure to represent possible within care-type correlation while assuming that between care-type correlation is zero, since they are different evaluators.

We base our evaluation on the bias and coverage for both $\delta_1 = \theta_{11} - \theta_{21}$ and $\delta_2 = \theta_{21} - \theta_{22}$, additionally considering power and false discovery rate for $\delta_1$. We compare our BSpAM model, under the four different prior specifications for $\theta_{jk}$, to the GEE-based approach of Klingenberg and Agresti (2006), the Bootstrap-based method of Westfall et al. (2014), and a modified ERM approach under the framework of Lui and Chang (2013). The ERM as described in their paper estimates risk ratios, however, for comparison with the GEE, Bootstrap, and BSpAM we estimate the risk difference under their model. All BSpAM estimates are based on 20,000 total samples, discarding the first 10,000 while all Bootstrap estimates are based on 10,000 resamples. Additional details on the GEE, ERM, and Bootstrap approaches are in the Supplementary Material.

We vary $\theta_{12}$ and $\theta_{21}$, considering each in combination with differing choices of $\mu_0$, $r$, and the inverse-gamma parameters for the variance of the sparse components. When varying $\theta_{12}, \theta_{21}$ is fixed at $\theta_{21} = 0.25$. When varying $\theta_{21}$, we fix $\theta_{12}$ at 0.05. For each, we generate 1000 simulated datasets, retaining only those datasets that exhibit sparsity for the second outcome—as determined by the sparsity exhibited in either Table 1 or 2. Because the value of $\theta_{22} = 0.001$, the $k = 2$ set in the simulation is the sparse set. The parameters of the simulation result in approximately 700 datasets exhibiting sparsity on each run. We split the evaluation up by $\delta_1$ and $\delta_2$, i.e. non-sparse and sparse. Tables 3 and 4 contain bias and coverage results for $\delta_2$, by value of $\theta_{12}$, while Table 5 and Figure 1 contain bias and coverage for $\delta_1$, respectively. Figure 2 contains results of the power and false discovery rate analysis for $\delta_1$. All results are based on the BSpAM model specified in Section 2 where a $t_5$ prior is placed on $\theta_{jk}$. Other prior combinations produce similar results, thus we focus our investigation on this model. Further, all BSpAM results we present here have $\mu_0 = 0.5$ as changing this quantity produces similar bias, coverage, and power for all $\theta_{jk}$ priors. Additional results not presented here, including varying sparse variance priors, $\theta_{jk}$ priors, and the $\mu_0 = 1/(n+2)$ case, are available in the Supplementary Material.

From Table 5 the BSpAM, GEE, and Bootstrap have similar bias for all values of $\theta_{12}$—the GEE and Bootstrap actually have the same bias. The ERM, on the other hand, has bias that is typically the largest and even an order of magnitude larger under several values of $\theta_{12}$ and $r$. As the underlying correlation increases, the bias tends to increase for
Table 3: Bias of $\delta_2$ by value of $\theta_{12}$ (see column sub-headers) and $r$ value, averaged over all simulated datasets.

| $r$ | Model | $\theta_{12}$ | 0.03   | 0.05   | 0.07 | 0.09 |
|-----|-------|---------------|--------|--------|------|------|
| 0   | BSpAM | 0.0022        | 0.0005 | 0.0003 | 0.0002 |
|     | ERM   | 0.0039        | 0.0060 | 0.0069 | 0.0064 |
|     | GEE   | 0.0022        | 0.0005 | 0.0002 | 0.0003 |
|     | Boot. | 0.0022        | 0.0005 | 0.0002 | 0.0003 |
| 1/3 | BSpAM | 0.0034        | 0.0019 | 0.0007 | 0.0006 |
|     | ERM   | 0.0026        | 0.0058 | 0.0045 | 0.0070 |
|     | GEE   | 0.0035        | 0.0019 | 0.0008 | 0.0004 |
|     | Boot. | 0.0035        | 0.0019 | 0.0008 | 0.0004 |
| 2/3 | BSpAM | 0.0020        | 0.0015 | 0.0018 | 0.0023 |
|     | ERM   | 0.0038        | 0.0050 | 0.0047 | 0.0042 |
|     | GEE   | 0.0021        | 0.0015 | 0.0019 | 0.0024 |
|     | Boot. | 0.0021        | 0.0015 | 0.0019 | 0.0024 |

Table 4: Coverage of $\delta_2$ by value of $\theta_{12}$ (see column sub-headers) and $r$ value, averaged over all simulated datasets.

| $r$ | Model | $\theta_{12}$ | 0.03   | 0.05   | 0.07 | 0.09 |
|-----|-------|---------------|--------|--------|------|------|
| 0   | BSpAM | 96.02%        | 95.01% | 96.84% | 96.58% |
|     | ERM   | 96.17%        | 89.47% | 89.85% | 90.14% |
|     | GEE   | 96.02%        | 89.34% | 89.44% | 90.14% |
|     | Boot. | 95.43%        | 87.81% | 89.44% | 95.62% |
| 1/3 | BSpAM | 95.98%        | 95.45% | 96.21% | 94.97% |
|     | ERM   | 96.28%        | 90.90% | 90.04% | 88.81% |
|     | GEE   | 95.98%        | 90.90% | 89.62% | 88.81% |
|     | Boot. | 95.68%        | 89.76% | 89.62% | 93.99% |
| 2/3 | BSpAM | 94.33%        | 94.76% | 97.60% | 95.47% |
|     | ERM   | 94.48%        | 89.66% | 90.95% | 90.23% |
|     | GEE   | 94.33%        | 89.38% | 90.52% | 90.23% |
|     | Boot. | 94.04%        | 88.10% | 90.38% | 94.48% |
Table 5: Bias of $\delta_1$ by value of $\theta_{21}$ (see column sub-headers) and $r$ value, averaged over all simulated datasets.

| $r$ | Model  | 0.05  | 0.10  | 0.15  | 0.20  | 0.25  | 0.30  | 0.35  | 0.40  |
|-----|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0   | BSpAM  | 0.0005| 0.0015| 0.0010| 0.0000| 0.0010| 0.0008| 0.0001| 0.0002|
|     | ERM    | 0.0005| 0.0014| 0.0011| 0.0002| 0.0015| 0.0001| 0.0009| 0.0014|
|     | GEE    | 0.0005| 0.0016| 0.0014| 0.0005| 0.0018| 0.0002| 0.0013| 0.0019|
|     | Boot.  | 0.0005| 0.0016| 0.0014| 0.0005| 0.0018| 0.0002| 0.0013| 0.0019|
| 1/3 | BSpAM  | 0.0001| 0.0001| 0.0001| 0.0009| 0.0021| 0.0033| 0.0029| 0.0041|
|     | ERM    | 0.0001| 0.0002| 0.0001| 0.0007| 0.0018| 0.0027| 0.0022| 0.0031|
|     | GEE    | 0.0001| 0.0001| 0.0003| 0.0003| 0.0013| 0.0021| 0.0014| 0.0023|
|     | Boot.  | 0.0001| 0.0001| 0.0003| 0.0003| 0.0013| 0.0021| 0.0014| 0.0023|
| 2/3 | BSpAM  | 0.0012| 0.0013| 0.0009| 0.0012| 0.0002| 0.0006| 0.0002| 0.0018|
|     | ERM    | 0.0012| 0.0014| 0.0009| 0.0010| 0.0002| 0.0000| 0.0005| 0.0009|
|     | GEE    | 0.0012| 0.0011| 0.0006| 0.0007| 0.0006| 0.0005| 0.0012| 0.0001|
|     | Boot.  | 0.0012| 0.0011| 0.0006| 0.0007| 0.0006| 0.0005| 0.0012| 0.0001|

the BSpAM, GEE, and Bootstrap while staying relatively static for ERM. Notably, as $\theta_{12}$ gets larger and, therefore, $\delta_2$ increases, the bias tends to decrease for the BSpAM, GEE, and Bootstrap but it tends to increase for the ERM.

Table 4 contains coverage results for intervals constructed at the 95% level and shows that, regardless of the value of $\theta_{12}$ or $r$, the BSpAM either exceeds nominal coverage or is quite close to it. This is not the case for the remaining methods. While the smallest value of $\theta_{12}$ results in nominal to near-nominal coverage for the GEE, ERM, and Bootstrap, coverage is often below 90% for values of $\theta_{12} \geq 0.05$. The Bootstrap is the exception for the largest value of $\theta_{12}$ but also exhibits the lowest coverage value in the Table, 87.81%. Unlike their bias results, the coverage of $\delta_2$ for the GEE and Bootstrap differs with the GEE-based approach performing slightly better. Changing the prior on either $\theta_{jk}$ or $\lambda_k$ (or both) produces similar coverage values for the BSpAM, see the Supplementary Material.

The BSpAM performs similarly to the other methods for the non-sparse outcome $\delta_1$ in terms of bias, see Table 5. When $r = 0$, the BSpAM tends to have the smallest bias. The GEE and Bootstrap again have identical bias under the non-sparse setting. The effect of $r$ varies more in the non-sparse setting as some methods have lower bias for higher values of $\theta_{21}$ at the largest value of $r$ than at the lowest value of $r$. As it did for the sparse outcome, the coverage for the BSpAM exceeds or is close to the nominal level of 95%, regardless of $r$, see Figure 1. In comparison to the other methods, BSpAM has the highest coverage for all but one value of $\delta_1$. Varying the prior on $\theta_{jk}$ does not impact the coverage, see the Supplementary Material. The ERM and Bootstrap can, in particular, have coverage over a full percentage point below nominal. Moreover, for the smallest value of $\theta_{21}$, coverage is below 90% for the Bootstrap and can be as low as 85%, depending on the value of $r$. The GEE typically has the second closest-to-nominal coverage after the BSpAM, and
Figure 1: Coverage curves for $\delta_1$, varying by $r$. The upper left panel displays $r = 0$ setting, the upper right panel has $r = 1/3$, and the lower panel shows $r = 2/3$. 
Figure 2: Power curves for \( \delta_1 \), varying by \( r \). The upper left panel displays \( r = 0 \) setting, the upper right panel has \( r = 1/3 \), and the lower panel shows \( r = 2/3 \). At \( \delta_1 = 0 \), the graphs give the false discovery rate.

All methods have coverage nearing, or reaching, nominal as \( \delta_1 \) increases. Power is nearly identical for all models as seen in Figure 2. The Bootstrap tends to have the smallest power while ERM usually has the largest, with the BSpAM and GEE falling in-between. All methods have false discovery rates at or below 5%, see Figure 2 when \( \delta_1 = 0 \). The value of \( r \) does not impact power in general and the power of the BSpAM for \( \delta_1 \) is not affected by varying the prior type on \( \theta_{jk} \), see the Supplementary Material.

4 System of Care Data

The goal of the SOC data is to determine if primary care physicians are initiating contact with the same agencies within the SOC framework, and with similar frequency, to psychiatric, or specialty, care. Knutson et al. (2018) argues that it is important to identify services for improved coordination to help the overburdened mental health system and their original findings suggest there is room for improvement among primary care givers. Knutson et al. (2018) separately analyzed only the non-sparse sets using standard univariate methods including logistic regression. Because of the sparsity in the juvenile justice (JJ) and developmental disabilities (DD) components, the authors did not separately examine these outcomes. Of the three outcomes they analyzed, only contact with the education
Table 6: Analysis of SOC Data.

| Component | $\delta$  | Cred. Int. 2.5% | 97.5% | $P(\delta > 0)$ | Gelman-Rubin $R$ | Upper 95% |
|-----------|-----------|-----------------|-------|----------------|-----------------|-----------|
| DD        | 0.041     | -0.019          | 0.100 | 0.911          | 1.000           | 1.001     |
| MH        | -0.108    | -0.253          | 0.038 | 0.078          | 1.001           | 1.003     |
| JJ        | -0.027    | -0.092          | 0.038 | 0.208          | 1.000           | 1.000     |
| CW        | -0.040    | -0.144          | 0.066 | 0.226          | 1.000           | 1.001     |
| ED        | -0.388    | -0.534          | -0.246| 0.000          | 1.000           | 1.000     |

The results of our analysis are in Table 6, which contains the median estimate of $\delta$, 95% credible intervals, and posterior probabilities for $\delta_k > 0$. To assess convergence, we use the Gelman-Rubin potential scale reduction factor, $\hat{R}$ (Gelman and Rubin, 1992). This value, alongside its 95% upper bound, are also in Table 6. All parameters converged on 20,000 total samples, with estimates based on the last 10,000 draws. As a sensitivity analysis, we implement both normal priors as well as the Laplace prior on the $\theta_{jk}$. Similar tables to Table 6 are in the Supplementary Material, however varying the prior does not substantially impact posterior inference.

From Table 6, we see that primary care tends to initiate contact less than specialty for four of the components: mental health (MH), juvenile justice (JJ), child welfare (CW), and the education system (ED). In the case of contacts with the education system, primary care is considerably less likely to initiate contact than specialty care. This result is highly significant with all of the posterior samples for the corresponding $\delta_k$ falling below 0—this is consistent with the findings in Knutson et al. (2018). Of the remaining two non-sparse components, the probability of contact with mental health is lower for primary care and only 7.8% percent of the posterior samples fall above zero. While Knutson et al. (2018) deemed this component non-significant, the corresponding posterior probability suggests further study is warranted as primary care may be initiating contact at a lower rate for this component, by approximately 10%. We see a higher rate of contact with the sparse outcome developmental disabilities (DD) for primary care with 91.1% of the posterior probabilities falling above zero. This result also warrants further study as primary care may be contacting that component more than necessary, about 4% more. It also suggests that primary care may not universally makes contact at lower rate and that the discrepancies may be component-specific.

5 Discussion

Methods for MMP data are limited in scope both in terms of Bayesian approaches and methods that can handle sparse outcomes. To our knowledge, no previous work on Bayesian
methods for MMP data has been done. Moreover, the existing non-Bayesian methods cannot adequately handle sparse MMP data. However, such data does exist for even relatively small value of $K$, as evidenced by the SOC data. A Bayesian approach is uniquely poised to be able to handle the issues arising because of sparsity given the flexibility inherent in the prior specification process. The BSpAM model we propose takes advantage of this flexibility for joint estimation of a model that has previously been used for MMP data, but one that now adjusts for sparsity.

In our simulation study, we demonstrate the advantage of the BSpAM over existing methods in terms of coverage for both sparse and non-sparse sets. What the coverage simulation illustrates is that the BSpAM does not underestimate the variability in the estimate of $\delta$ while the other methods, the sparsity-corrected ERM included, do underestimate it. Failing to account for this underestimation could result in misleading conclusions when sparse MMP data is analyzed. Even for non-sparse sets, the BSpAM still tends to have the closest-to-nominal coverage. Beyond coverage, the BSpAM also typically has the smallest average bias, regardless of the underlying correlation. Thus the gain in coverage does not come at loss in accuracy. While the ERM is slightly more powerful, the BSpAM maintains similar power to the GEE approach and both have higher power than the Bootstrap method.

Finally, the application of the BSpAM to our motivating SOC data reveals two additional components of interest that warrant further study. Focusing on the original results, where only contact with the education system was found to differ significantly may not be sufficient for improving the coherence of the SOC framework. By analyzing all the components simultaneously using the BSpAM, including the sparse components, we were able to show that contact with mental health (not-sparse) and developmental disabilities (sparse) may be of interest for further study. The latter component has an effect in the opposite direction of the remaining components suggesting that the differences in contact rate could be more nuanced than the case of primary care physicians simply under-contacting SOC components in general. The results from our multivariate analysis of the original SOC data can, using the BSpAM, inform the priors of follow-up studies. Inclusion of this information can occur by letting $\mu_0$ vary with $j$ and $k$, with the values based off the $\theta_{jks}$ that generate Table 6.

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