Constructing a Nested Chain in James Abacus Diagram

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Abstract. James abacus diagram with $e$ column and $r$ rows is a combinatorial model used to represent any partition of a positive integer. In this paper, nested chain in James diagram is constructed which depended on $e$ and $r$. Three forms of nested chain are found which are $e < r$, $e > r$ and $e = r$. Then, characteristics for each form are provided. Furthermore, position numbers in each chain are then determined. In addition, a new function is established in order to determine the number of chains. Also sequences of the number of chains in nested chain are finally obtained.

Keywords: Partition, James abacus diagram, beta number, Sequence.

1. Introduction

James diagram is a graphical representation of special type of non-increasing sequence $(\mu_1, \mu_2, ..., \mu_b)$ called partition of $t$. This sequence can as well be associated with an abacus diagram. For any number of column, $e$ also known as runner, James abacus diagram is seen as an important component in the modern algebra which plays a key role in Iwahori–Hecke and $q$-Schur algebras as mentioned in in 1, 2, 3, 7, 11. The James diagram configuration for $\{\beta_1, \beta_2, ..., \beta_b\}$ can be created by rearranging the bead positions on the runners, where $\beta_i = \mu_i + b - i$, $1 \leq i \leq b$, $b$ is the number of partition parts. There are infinite numbers of James diagram since one or more zero(s) will be added to the partition. The diagram $\beta_j$ is obtained after adding $s$-1 zeros to $\mu$ where $\beta_j = b + s - 1$ 13 Several movement of beta number have been constructed by previous researchers such as by moving all beads as high as possible on runner 3, addition of empty runner 8, reflecting removing runner 2, addition of full runner 1, scan movement 15, justified positions movement 16. Further study also defined special type of James diagram called main diagrams and proved the existence of common bead positions among main diagram 13.

2. Preliminaries

This section will briefly discuss some basic definitions and theorems that will be used in the next section. Each partition of James abacus diagram can be connected with $e$ column which labeled from left to right, 0 to $e-1$. Bead positions are located across theses runner from top to bottom, 0, 1, 2, ... as shown in Figure 1. Let $(\mu_1, \mu_2, ..., \mu_j, ..., \mu_b)$ be a partition of integer number $t$ such that
\[ |\mu| = \sum_{j=1}^{b} \mu_j = t, \quad \mu_{b+1} = 0. \] In order to construct \( \beta \)-number, \( b \) greater than or equal to the number of partition parts is chosen and defined by \( \beta(\mu, b) = (\mu_1 + b - 1, \mu_2 + b - 2, ..., \mu_b + b - b) \), the bead positions \( ne, ne+1, \ldots, ne+e-1 \) location on row \( n \) of the diagram. The abacus configuration for \( \mu \) is called James diagram. For example \( \mu = (6, 4, 3, 3, 3, 2) \) is a partition of 21, the set of \( \beta \)-numbers is \{11, 8, 6, 5, 4, 2\}. Every \( \beta \)-number will be represented by a bead (o) position which takes its location in the diagram. The label diagram and final diagram if \( e = 3 \) are shown in Figure 1,

|   |   | - | - | 0 |
|---|---|---|---|---|
|   | - | o | o |   |
| o | - | o |   |   |
| - | - | o | o |   |

**Figure 1.** James abacus diagram where \( \mu = (6, 4, 3, 3, 3, 2) \) and \( e = 3 \).

The number of rows in the James abacus diagram can be obtained as given in the following theorem.

**Theorem 1.** (Mohommed, 2016) Let \( \{\beta_1, \beta_2, ..., \beta_b\} \) is a set of beta numbers of a partition \( \mu \) and \( r \) is the number of the rows in James abacus diagram then

\[ r = \left\lceil \frac{\beta_1}{e} \right\rceil + 1. \]

**Lemma 1.** (Mohommed, 2016) Suppose that \( \{\beta_1, \beta_2, ..., \beta_b\} \) is a set of beta numbers of a partition \( \mu \) of a positive integer \( t \) where \( b \) is a partition parts such that \( \beta_i = me + n \), then every James abacus diagram position can be converted to a matrix \( A_{r \times e} \) by

\[ me + n \rightarrow a_{mn} \]

where \( 0 \leq m \leq r \) and \( 0 \leq n \leq e \).

3. **Nested Chain**

A nested chain inscribes in a James abacus diagram which consists of outer chain and inner chains. These chains are numbered from 1 to \( i \) where \( i \) is a positive integer and chain 1 is the outer chain. There is no intersection between any two of the chains. Figure 2 illustrates a nested chain for \( e = 6, \quad \mu = (2^4, 16^2, 15, 10, 5^5, 2^3, 1) \) and \( \beta \)-number sequences is \{36,35,34,33,28,27,25,19,13,12,11,10,9,5,4,3,1\}

|   |   | - | - | 0 | 0 | 0 |
|---|---|---|---|---|---|---|
|   | - | - | o | o | 0 |   |
| o | o | - | - | - | - |   |
| o | - | o | - | - | - |   |
| o | - | o | - | o | - |   |
| o | - | - | - | 0 | 0 | 0 |

**Figure 2** Nested chain James abacus diagram.
There are three forms of inner chains such as complete chain, path chain and singleton chain. For outer chain, it will always be a complete chain. The structure of James abacus diagram change according to whether \( e < r \), \( e > r \) or \( e = r \) where \( e \) is the number of columns, \( e \geq 2 \) and \( r \) is the number of the rows will be effect on the number of the chains and the forms of nested chain.

3.1. Nested Chain in Case of \( e < r \)

For the case of \( e < r \) we have two structures of nested chain depending on whether \( e \) is even or odd.

When \( e \) is even, the nested chain consists of complete chains while if \( e \) is odd the nested chain consists of complete chains and a path chain.

Example 1 and example 2 provide the illustration of complete and path of outer or inner nested chain. Then we will formulate a general structure of nested chain for the case of \( e < r \).

**Example 1.** Let \( \mu = (20^4, 16^2, 15, 10, 5^3, 2^3, 1) \) be a partition of 169 and \( \beta_1 = 36 \) as in Figure 2. Then the nested chain of James abacus diagram has three complete chains.

From Figure 3, we observe that if \( e = 6 \), and

\[
r = \left\lfloor \frac{\beta_1}{e} \right\rfloor + 1 = \left\lfloor \frac{36}{6} \right\rfloor + 1 = 7
\]

the nested chain has three chains

Chain 1 = \( \{ a_{m1}, a_{m6}, a_{1n}, a_{6n} : 1 \leq m \leq 6, 1 < n < 6 \} \).

Chain 2 = \( \{ a_{m2}, a_{m5}, a_{2n}, a_{5n} : 2 \leq m \leq 5, 2 < n < 5 \} \).

Chain 3 = \( \{ a_{m3}, a_{m4}, 3 \leq m \leq 4 \} \).

**Example 2.** Let \( \mu = (10^7, 9, 5, 4^2, 3, 2^2, 1) \) be a partition of 120 and \( \beta_1 = 29 \). Then the nested chain of James abacus diagram has two complete chains and a path chain, as shown in Figure 4.
From Figure 4, we observe that if $e = 5$ and

\[ r = \left\lfloor \frac{\beta_1}{e} \right\rfloor + 1 = \left\lfloor \frac{29}{5} \right\rfloor + 1 = 6 \]

the nested chain has three chains

Chain 1 = \{a_{m1}, a_{m5}, a_{1n}, a_{6n} : 1 \leq m \leq 6, 1 < n < 5\}.

Chain 2 = \{a_{m2}, a_{m4}, a_{2n}, a_{5n} : 2 \leq m \leq 5, 2 < n < 4\}.

Chain 3 = \{a_{m3} : 3 \leq m \leq 4\}.

Next, the generalization for $e < r$ is given as follows.

**Definition 1.** Let $a_{mn}$ is a position whether is a beta number or empty beta number position in James abacus diagram formatted by $e$ column numbered from 1 to $e$ and $r$ rows numbered from 1 to $r$ such that $e < r$. A complete chain is a sequence of positions in the James abacus diagram of the form

\[ \{a_{mi}, a_{m(e-i+1)}, a_{in}, a_{(r-i)n} : i \leq m \leq r - i + 1, i < n < e - i + 1\} \].

Such that $i$ is the number of chains in the diagram and

\[ r = \left\lfloor \frac{\beta_1}{e} \right\rfloor + 1 \]
Let $i$ is the number of chains in James abacus diagram then $i < e-i+1$, see Table 1.

**Table 1. James abacus diagram with $e$ columns and $e$ rows**

| $e$ | 0 | 1 | $...$ | $i$ | $e-i+1$ | .. | $e-2$ | $e-1$ |
|-----|---|---|-----|-----|-------|----|-----|------|
| $e$ | $e+1$ | $...$ | $e+i$ | $...$ | $2e-i+1$ | .. | $2e-2$ | $2e-1$ |
| $2e$ | $2e+1$ | $...$ | $2e+i$ | $...$ | $3e-i+1$ | .. | $3e-2$ | $3e-1$ |
| $3e$ | $3e+1$ | $...$ | $3e+i$ | $...$ | $4e-i+1$ | .. | $4e-2$ | $4e-1$ |
| $4e$ | $4e+1$ | $...$ | $4e+i$ | $...$ | $5e-i+1$ | .. | $5e-2$ | $5e-1$ |
| $5e$ | $5e+1$ | $...$ | $5e+i$ | $...$ | $6e-i+1$ | .. | $6e-2$ | $6e-1$ |

**Remark 1.** Let $i$ is the number of chain in James abacus diagram with $e$ columns, $r$ rows and $e < r$ then

1. $i$ is a positive integers.
2. Every complete chain derived by two columns $i$ and $e-i+1$.

**Remark 2.** Let $i$ is the number of chain in James abacus diagram with $e$ even columns, $r$ rows and $e < r$ then the last chain will derived by two consecutive columns.

**Definition 2.** Let $a_{mn}$ is a position whether beta number position or empty beta number position in James abacus diagram formatted by $e$ column, $r$ rows where $e$ is odd number and $e < r$. Path chain is a sequence of the positions in the James abacus diagram of the form

$$\{a_{mn}: n = \frac{e+1}{2}, \frac{e+1}{2} \leq m \leq \frac{2r-e+1}{2}\}$$

such that $r = \left\lfloor \frac{\beta_1}{e} \right\rfloor + 1$.

**Remark 3.**

1. If $e < r$ and $e = 2x$ (even number) then the nested chain consists of complete chains.
2. If $e < r$ and $e = 2x + 1$ (odd number) then the nested chain consists of complete chains and a path chain.

Where $x$ is a positive integer.

**Theorem 2.** Let $e$ be the column number in the James abacus diagram and $e < r$. Then, the number of chains($i$) in the diagram for any partition $\mu$ is
Proof
1. By Remark 1 and 2, every chain is derived from two columns, which are \( i \) and \( e - i + 1 \). Since the last chain is derived from two consecutive columns, the difference between these two column numbers is
\[
(e - i + 1) - i = 1.
\]
Thus,
\[
i = \frac{e}{2}
\]
2. Since every complete chain derived from two columns and the number of columns equal to \( 2x + 1 \) then there are
\[
x = \frac{e - 1}{2}
\]
complete chain and one path chain. Hence
\[
x + 1 = \frac{e - 1}{2} + 1 = \frac{e + 1}{2}.
\]
In the next definition, a new function maps from the number of columns \( e \) to the number of the chains in James abacus diagram.

**Definition 3.** Let \( F_e \) is a function, \( F_e: \mathbb{N} \rightarrow \mathbb{N} \) such that
\[
F_e = \begin{cases} 
\frac{e}{2} & \text{if } e \text{ is even number} \\
\frac{e + 1}{2} & \text{if } e \text{ is odd number}
\end{cases}
\]
see Table.2.

| \( r \) | \( F_2 \) | \( F_3 \) | \( F_4 \) | \( F_5 \) | \( F_6 \) | \( F_7 \) | \( F_8 \) | \( F_9 \) | \( F_{10} \) | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ... |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | ... |
| 4 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | ... |
| 5 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | ... |
| 6 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | ... |

A new sequence of chain numbers called (C.N.s.) in James abacus diagram is obtain if \( e \geq r \)
\[
\text{C.N.s.} = \{1, 2, 2, 3, 4, 4, 5, 5, 6, 6, 7, 7, ... \}
\]

**Definition 7.** Let \( e > r \) and \( F_j \) is the number of the chains in James abacus diagram with \( e \) column where and \( e \geq 2 \). The C.N.s. of James abacus diagram is the series of the chain numbers in the James abacus diagram with two initial \( F_1 = 1 \) and \( F_2 = 1 \),
\[
F_n + F_{n+1} = F_{2n+1} = F_{2n+2}.
\]
3.2. Nested chain in case of $e = r = 2x + 1$

In this case $e$ and $r$ are odd the nested chain consist of complete chains and a singleton chain.

**Example 4.** Let $\mu = (2^2, 1^4)$ be a partition of 8 and $\beta$ - number = $\{7, 6, 3, 4, 2, 1\}$. Then the nested chain of James abacus diagram consists of one complete chains and a singleton chain, as shown in Figure 7.

From Figure 7, we observe that If $e = 3$ and $\beta_1 = 7$ the nested chain has two chains

Chain 1 = $\{a_{m1}, a_{m3}, a_{1n}, a_{3n}: 1 \leq m \leq 3, 1 < n < 3\}$.

Chain 3 = $\{a_{33}\}$.

Where $r = 3$.

The generalization for $e = r = 2x+1$ is given as follows.

**Definition 8.** Let $a_{mn}$ is a position whether beta number position or empty beta number position in the James abacus diagram formatted by $e$ column where $e$ is odd number numbered from 1 to $e$ and $r$ rows numbered by from 1 to $r$ such that $r = e$. Singleton chain is a one position in the James abacus diagram located in column $\frac{r+1}{2}$ and row $\frac{r+1}{2}$. See Figure 7.

4. Conclusions

In this paper, James abacus diagram with $e$ columns as a nested chain of beta numbers will be redefined. The characteristics of the nested chain change according to either $e < r$, $e > r$, $e = r$. We obtain three forms of nested chain for James abacus diagram where each form has its own characteristics Furthermore, the establishment sequences among the nested chains of the diagram position is considered. At this stage, two questions can be asked:

• Could we use James abacus diagram to classify classes for polyominoes?
• Could we use the design structure of James abacus diagram to construct the generating function for polyominoes?

5. References

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