On Stable $\mathcal{H}^\infty$ Controllers for Time-Delay Systems

Suat Gümişsoy$^*$ and Hitay Özbay$^{†\ddagger}$

Abstract

In this paper, we study the stability of suboptimal $\mathcal{H}^\infty$ controllers for time-delay systems. The optimal $\mathcal{H}^\infty$ controller may have finitely or infinitely many unstable poles. A stable suboptimal $\mathcal{H}^\infty$ controller design procedure is given for each of these cases. The design methods are illustrated with examples.

1 Introduction

A strongly stabilizing controller is a stable controller in a stable feedback, [1]. In many practical applications, strongly stabilizing controllers are desired, see e.g. [2, 3, 4, 5, 6, 7, 8, 9] and their references. In these papers, direct design methods are given for $\mathcal{H}^\infty$ strong stabilization for finite dimensional plant case. The necessary and sufficient condition for strong stabilization, parity interlacing property, is shown in [10] for single input single output delay systems. A design method to find strongly stabilizing controller for single input single output systems with time delays is given [11] in which the stable controller is constructed by using the unit satisfying some interpolation conditions.

An indirect approach to design stable controller achieving a desired $\mathcal{H}^\infty$ performance level for time delay systems is given in [12]. This approach is based on stabilization of $\mathcal{H}^\infty$ controller by another $\mathcal{H}^\infty$ controller in the feedback loop. In [12], stabilization is achieved and the sensitivity deviation is minimized. There are two main drawbacks of this method. First, the solution of sensitivity deviation brings conservatism because of finite dimensional approximation of the infinite dimensional weight. Second, the stability of overall sensitivity function is not guaranteed. Also, overall system does not achieve the exact performance level, since the optimal $\mathcal{H}^\infty$ controller is perturbed by deviation.

$^*$This work was supported in part by the National Science Foundation under grant ANI-0073725

$^†$Collaborative Center of Control Science, Department of Electrical Engineering, The Ohio State University, 2015 Neil Avenue, Columbus, OH 43210

$^\ddagger$Collaborative Center of Control Science, Department of Electrical Engineering, The Ohio State University, 2015 Neil Avenue, Columbus, OH 43210

$^\ddagger$Part of this work was done at Bilkent University, Department of Electrical and Electronics Engineering, Bilkent, Ankara 06800, Turkey
Our paper focuses on strong stabilization problem for infinite dimensional plants such that the stable controller achieves the pre-specified suboptimal $\mathcal{H}^\infty$ performance level. When the optimal controller is unstable (with infinitely or finitely many unstable poles), two methods are given based on a search algorithm to find a stable suboptimal controller. However, both methods are conservative. In other words, there may be a stable suboptimal controller achieving a smaller performance level, but the designed controller satisfies the desired overall $\mathcal{H}^\infty$ norm. The stability of optimal and suboptimal controller is discussed and necessity conditions are given.

It is known that a $\mathcal{H}^\infty$ controller for time-delay systems with finitely many unstable poles can be designed by the methods in [13, 14, 15, 16]. In general, weighted sensitivity problem results in an optimal $\mathcal{H}^\infty$ controller with infinitely unstable modes, [17, 18].

We assume that the plant is single input single output (SISO) and admits the representation as in [16],

$$P(s) = \frac{m_n(s)N_o(s)}{m_d(s)}$$

where $m_n(s) = e^{-hs}M(s)$, $h > 0$, and $M(s)$, $m_d(s)$ are finite dimensional, inner, and $N_o(s)$ is outer, possibly infinite dimensional. The optimal $\mathcal{H}^\infty$ controller, $C_{opt}$, stabilizes the feedback system and achieves the minimum $\mathcal{H}^\infty$ cost, $\gamma_{opt}$:

$$\gamma_{opt} = \left\| \begin{bmatrix} W_1(1 + PC_{opt})^{-1} \\ W_2PC_{opt}(1 + PC_{opt})^{-1} \end{bmatrix} \right\|_\infty = \inf_{C \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1 + PC)^{-1} \\ W_2PC(1 + PC)^{-1} \end{bmatrix} \right\|_\infty$$

(1.2)

where $W_1$ and $W_2$ are finite dimensional weights for the mixed sensitivity minimization problem.

In the next section, the structure of optimal and suboptimal $\mathcal{H}^\infty$ controllers will be summarized. The optimal controller with infinitely many unstable poles case is considered in Section 3. The conditions and a design method for stable suboptimal $\mathcal{H}^\infty$ controller is given in the same section. Similar work is done in Section 4 for the optimal controller with finitely many unstable poles. Examples related for these design methods are presented in Section 5, and concluding remarks can be found in Section 6.

### 2 Structure of $\mathcal{H}^\infty$ Controllers

Assume that the problem (1.2) satisfies $(W_2N_o), (W_2N_o)^{-1} \in \mathcal{H}^\infty$, then optimal $\mathcal{H}^\infty$ controller can be written as, [19],

$$C_{opt}(s) = E_{\gamma_{opt}}(s)m_d(s) \frac{N_o^{-1}(s)F_{\gamma_{opt}}(s)L(s)}{1 + m_n(s)F_{\gamma_{opt}}(s)L(s)}$$

(2.3)

where $E_{\gamma} = \left( \frac{W_i(-s)W_i(s)}{\gamma^2} - 1 \right)$, and for the definition of the other terms, let the right half plane zeros of $E_{\gamma}(s)$ be $\beta_i$, $i = 1, \ldots, n_1$, the right half plane poles of $P(s)$ be $\alpha_i$, $i = 1, \ldots, l$.
and that of $W_1(-s)$ be $\eta_i \ i = 1, \ldots, n_1$. Then, $F_\gamma(s) = G_\gamma(s) \prod_{i=1}^{n_1} \frac{s-\eta_i}{s+\eta_i}$ where
\[
G_\gamma(s)G_{\gamma}(-s) = \left(1 - \frac{W_2(-s)W_2(s)}{\gamma^2} - 1\right)E_\gamma^{-1}
\] (2.4)
and $G_\gamma, G_{\gamma}^{-1} \in \mathcal{H}_\infty$, and $L(s) = \frac{L_2(s)}{L_1(s)}$ , $L_1(s)$ and $L_2(s)$ are polynomials with degrees less than or equal to $(n_1+\ell-1)$ and they are determined by the following interpolation conditions,
\[
0 = L_1(\beta_i) + m_n(\beta_i)F_{\gamma}(\beta_i)L_2(\beta_i) \quad i = 1, \ldots, n_1
\] (2.5)
\[
0 = L_1(\alpha_i) + m_n(\alpha_i)F_{\gamma}(\alpha_i)L_2(\alpha_i) \quad i = 1, \ldots, l
\]
\[
0 = L_2(-\beta_i) + m_n(\beta_i)F_{\gamma}(\beta_i)L_1(-\beta_i) \quad i = 1, \ldots, n_1
\]
\[
0 = L_2(-\alpha_i) + m_n(\alpha_i)F_{\gamma}(\alpha_i)L_1(-\alpha_i) \quad i = 1, \ldots, l.
\]
The optimal performance level, $\gamma_{\text{opt}}$, is the largest $\gamma$ value such that spectral factorization (2.4) exists and interpolation conditions (2.5) are satisfied.

Similarly, the suboptimal controller achieving the performance level, $\rho$, can be defined as,
\[
C_{\text{subopt}}(s) = E_\rho(s)m_d(s)\frac{N^{-1}_\rho(s)F_\rho(s)L_U(s)}{1 + m_n(s)F_\rho(s)L_U(s)}
\] (2.6)
where $\rho > \gamma_{\text{opt}}$ and $L_U(s) = \frac{L_U(s)}{L_U(s)} = \frac{L_2(s)+L_1(s)L_U(s)}{L_1(s)+L_2(s)L_U(s)}$ with $U \in \mathcal{H}_\infty, \|U\|_{\infty} \leq 1$. The polynomials, $L_1(s)$ and $L_2(s)$, have degrees less than or equal to $n_1 + l$. Same interpolation conditions are valid with $\rho$ instead of $\gamma$. Moreover, there are two additional interpolation conditions for $L_1(s)$ and $L_2(s)$:
\[
0 = L_2(-a) + (E_\rho(a) + 1)F_\rho(a)m_n(a)L_1(-a)
\] (2.7)
\[
0 \neq L_1(-a)
\] (2.8)
where $a \in \mathbb{R}^+$ is arbitrary. The above terms and notations are the same as in [19].

Note that the unstable zeros of $E_{\gamma_{\text{opt}}}$ and $m_d$ are always cancelled by the denominator in (2.3). Therefore, $C_{\text{opt}}$ is stable if and only if the denominator in (2.3) has no unstable zeros except the unstable zeros of $E_{\gamma_{\text{opt}}}$ and $m_d$ (multiplicities considered). Same conclusions are valid for the suboptimal case, $C_{\text{subopt}}$ is stable provided that the denominator in (2.6) has unstable zeros only at the unstable zeros of $E_\rho$ and $m_d$ (again, multiplicities considered).

It is clear that the optimal, respectively suboptimal, controllers have infinitely many unstable poles if and only if there exists $\sigma_o > 0$ such that the following inequality holds
\[
\lim_{\omega \to \infty} |F_{\gamma_{\text{opt}}}(\sigma_o + j\omega)L_{\text{opt}}(\sigma_o + j\omega)| > 1, \quad (2.9)
\]
respectively,
\[
\lim_{\omega \to \infty} |F_{\rho}(\sigma_o + j\omega)L_U(\sigma_o + j\omega)| > 1.
\] (2.10)
The controller may have infinitely many poles because of the delay term in the denominator. All the other terms are finite dimensional.
Even when the optimal controller has infinitely many unstable poles, a stable suboptimal controller may be found by proper selection of the free parameter $U(s)$. In Section 3 this case is discussed.

Note that the previous case covers one and two block cases (i.e., $W_2 = 0$ and $W_2 \neq 0$ respectively). When $F_{\gamma_{opt}}$ is strictly proper, then the optimal and suboptimal controllers may have only finitely many unstable poles. Existence of stable suboptimal $H^\infty$ controllers and their design will be discussed in Section 4 for this case.

3 Stable suboptimal $H^\infty$ controllers, when the optimal controller has infinitely many unstable poles

The following lemma gives the necessary condition for a suboptimal controller to have finitely many unstable poles.

Lemma 3.1. Assume that the optimal controller has infinitely many unstable poles and $U(s)$ is finite dimensional, the suboptimal controller has finitely many unstable poles if and only if

$$\lim_{\omega \to \infty} |F_\rho(j\omega)L_U(j\omega)| \leq 1$$ (3.11)

Proof Assume that the suboptimal controller has infinitely many unstable poles, then the equation

$$1 + e^{-h(\sigma + j\omega)}M(\sigma + j\omega)F_\rho(\sigma + j\omega)L_U(\sigma + j\omega) = 0$$

has infinitely many zeros in the right half plane, i.e., there exists $\sigma = \sigma_o > 0$ and for sufficiently large $\omega$,

$$1 + e^{-h(\sigma_o + j\omega)} \lim_{\omega \to \infty} (F_\rho(\sigma_o + j\omega)L_U(\sigma_o + j\omega)) = 0$$ (3.12)

will have infinitely many zeros. Since $F_\rho$ and $L_U$ are finite dimensional,

$$\lim_{\omega \to \infty} F_\rho(j\omega) = \lim_{\omega \to \infty} F_\rho(\sigma + j\omega)$$

$$\lim_{\omega \to \infty} L_U(j\omega) = \lim_{\omega \to \infty} L_U(\sigma + j\omega) \ \forall \sigma > 0.$$ 

By using this fact, we can rewrite (3.12) as,

$$1 + e^{-h(\sigma_o + j\omega)} \lim_{\omega \to \infty} (F_\rho(j\omega)L_U(j\omega)) = 0$$ (3.13)

which implies that in order to have infinitely many zeros, the condition in lemma should satisfied. Conversely, a similar idea can be used to show that (3.11) implies finitely many unstable poles. \[\square\]
Note that this lemma is valid not only for only finite dimensional \( U(s) \) term, but also for any \( U \in \mathcal{H}^\infty, \|U\|_\infty \leq 1 \) provided that
\[
\lim_{\omega \to \infty} U(j\omega) = \lim_{\omega \to \infty} U(\sigma + j\omega) = u_\infty, \quad \forall \sigma > 0.
\] (3.14)
is satisfied where \( u_\infty \in \mathbb{R} \). Also, we can find conditions on \( U \) which guarantees finitely many unstable poles by using the lemma.

Assume that \( U(s) \) is finite dimensional and bi-proper, and define
\[
f_\infty = \lim_{\omega \to \infty} |F_\rho(j\omega)| > 1
\]
\[
u_\infty = \lim_{\omega \to \infty} U(j\omega)
\]
\[
k = \lim_{\omega \to \infty} \frac{L_2(j\omega)}{L_1(j\omega)}
\]

**Lemma 3.2.** The suboptimal controller has finitely many unstable poles if and only if the following inequalities hold:
\[
|k| \leq \frac{1}{f_\infty}, \quad |u_\infty| \leq \frac{1 - f_\infty|k|}{f_\infty - |k|}
\] (3.15)
when \((n_1 + l)\) is odd (even) and \( ku_\infty < 0, (ku_\infty > 0)\), and
\[
|k| < 1, \quad \frac{f_\infty|k| - 1}{f_\infty - |k|} < |u_\infty| < \frac{f_\infty|k| + 1}{f_\infty + |k|}
\] (3.16)
when \((n_1 + l)\) is odd (even) and \( ku_\infty > 0, (ku_\infty < 0)\).

**Proof** By using Lemma 3.1 when \((n_1 + l)\) is odd (even) and \( ku_\infty < 0, (ku_\infty > 0)\), we can re-write (3.11) as
\[
f_\infty \frac{|k| + |u_\infty|}{1 + |k||u_\infty|} \leq 1.
\]
After algebraic manipulations and using \( f_\infty > 1 \), we can show that (3.15) satisfies this condition. Similarly, when \((n_1 + l)\) is odd (even) and \( ku_\infty > 0, (ku_\infty < 0)\), (3.11) is equivalent to
\[
f_\infty \frac{|k| - |u_\infty|}{1 - |k||u_\infty|} \leq 1.
\]
, and (3.10) satisfies this condition.

Note that \( u_\infty \) is a design parameter and the range can be determined, by given \( f_\infty \) and \( k \).

**Theorem 3.1.** Assume that the optimal and central suboptimal controller (when \( U = 0 \)) has infinitely many unstable poles, if there exists \( U \in \mathcal{H}^\infty, \|U\|_\infty < 1 \) such that \( L_U \) has no \( \mathbb{C}_+ \) zeros and \( |L_U(j\omega)F_\rho(j\omega)| \leq 1, \forall \omega \in [0, \infty) \), then the suboptimal controller is stable.
Proof Assume that there exists $U$ satisfying the conditions of the theorem. By maximum modulus theorem,

$$|1 + e^{-h_0}M(s_o)F_\rho(s_o)L_U(s_o)| > 1 - e^{-h_0}|F_\rho(j\omega)L_U(j\omega)| > 0,$$

therefore, there is no unstable zero, $s_o = \sigma + j\omega$ with $\sigma > 0$. Since, all imaginary axis zeros are cancelled by $E_\rho$, the suboptimal controller has no unstable poles.

The theorem has two disadvantages. First, there is no information for calculation of an appropriate parameter, $U$. Second, the inequality brings conservatism and there may exist stable suboptimal controllers even when the condition is violated. It is difficult to reveal the first problem, therefore it is better to use first order bi-proper function for $U$. For the second problem, define $\omega_{\text{max}}$ and $\eta_{\text{max}}$ as,

$$\omega_{\text{max}} = \max_{|L_U(j\omega)F_\rho(j\omega)| = 1} \omega,$$

$$\eta_{\text{max}} = \max_{\omega \in [0, \infty)} |L_U(j\omega)F_\rho(j\omega)|.$$

It is important to design $\omega_{\text{max}}$ and $\eta_{\text{max}}$ as small as possible by the choice of $U$. Otherwise, at high frequencies the delay term will generate unstable zeros when $\omega_{\text{max}}$ is large. Similarly, when $\eta_{\text{max}}$ is large, although $\omega_{\text{max}}$ is small, it may cause unstable zeros. The design method given below searches for a first order $U$, and it is based on the above ideas. An example will be given in Section 5.

Algorithm

Define $U(s) = u_\infty \left(\frac{u_p + s}{u_p + s}\right)$ such that $u_\infty, u_p, u_z \in \mathbb{R}$, $|u_\infty| < 1$, $u_p > 0$ and $u_p \geq u_\infty|u_z|$,.

1) Fix $\rho > \gamma_{\text{opt}}$,

2) Obtain $f_\infty$ and $k$ from the central suboptimal controller,

3) Calculate admissible values of $u_\infty$ by using Lemma (3.2),

4) Search admissible values for $(u_\infty, u_p, u_z)$ such that $L_1U(s)$ is stable,

5) Find the minimum $\omega_{\text{max}}$ and $\eta_{\text{max}}$ for all admissible $(u_\infty, u_p, u_z)$.

6) Check in the region $D = \{s = \sigma + j\omega, \sigma \geq 0 : |e^{-h_0}M(s)F_\rho(s)L_U(s)| > 1\}$ whether $1 + e^{-h_0}M(s)F_\rho(s)L_U(s)$ has no $C_+$ zeros except unstable zeros of $E_\rho$ and $m_d$.

When the central suboptimal controller has infinitely many unstable poles, it is not possible to obtain a stable suboptimal controller by a choice of $U$ as strictly proper or inner function. Once we find a $U$ from the above algorithm, the resulting stable suboptimal $\mathcal{H}^\infty$ controller can be represented as cascade and feedback connections of finite dimensional terms and a finite impulse response filter that does not have unstable pole-zero cancellations in the controller, as explained in [20].
4 Stable suboptimal $\mathcal{H}^\infty$ controllers, when the optimal controller has finitely many unstable poles

In this section, we will derive the conditions for the $\mathcal{H}^\infty$ controllers to have finitely many unstable poles. A sufficient condition for the existence of stable suboptimal $\mathcal{H}^\infty$ controllers is given, and a design method will be derived.

The optimal and suboptimal controllers have infinitely many unstable poles, when $F_{\gamma_{\text{opt}}}L_{\text{opt}}$ and $F_{\rho}L_U$ has magnitude greater than one as $\omega \to \infty$. It is not difficult to see that controllers will have finitely many unstable poles if $F_{\gamma_{\text{opt}}}$ and $F_{\rho}$ are strictly proper. Since, these terms decrease as $\omega \to \infty$ and delay term decays as $\sigma$ increases, only finitely many unstable poles may appear. Clearly, there may be $\mathcal{H}^\infty$ controllers (depending on parameter values) with finitely many poles while $F_{\gamma_{\text{opt}}}$ and $F_{\rho}$ are bi-proper. However, it is important to find the sufficient conditions when they are strictly proper, which results in controllers with finitely many unstable poles regardless of parameters.

**Lemma 4.1.** The $\mathcal{H}^\infty$ controller has finitely many unstable poles if the plant is strictly proper and $W_1$ is proper (in the sensitivity minimization problem) and, $W_1$ is proper and $W_2$ is improper (in the mixed sensitivity minimization problem).

**Proof** Transfer function $F(s)$ can be written as ratio of two polynomials, $N_F$ and $D_F$, with degrees $m$ and $n$ respectively. We can define relative degree function, $\phi$, as

$$\phi(F(s)) = \phi \left( \frac{N_F(s)}{D_F(s)} \right) = n - m.$$  

Note that $\phi(F_1(s)F_2(s)) = \phi(F_1(s)) + \phi(F_2(s))$ and $\phi(F(s)F(-s)) = 2\phi(F(s))$.

The optimal controller has finitely many unstable poles if $F_{\gamma_{\text{opt}}}$ is strictly proper, i.e. $\phi(F_{\gamma_{\text{opt}}}(s)) > 0$. To show this, we can write by using definition of $F_{\gamma_{\text{opt}}}$ and (2.4),

$$\phi(F_{\gamma_{\text{opt}}}(s)) = \phi(G_{\gamma_{\text{opt}}}(s)),$$

$$= \frac{1}{2} \phi((W_1(s)W_1(-s) + W_2(s)W_2(-s) - \gamma_{\text{opt}}^{-2}W_1(s)W_1(-s)W_2(s)W_2(-s))^{-1}),$$

$$= -\frac{1}{2} \phi((W_1(s)W_1(-s) + W_2(s)W_2(-s) - \gamma_{\text{opt}}^{-2}W_1(s)W_1(-s)W_2(s)W_2(-s))),$$

$$= -\frac{1}{2} \min \{\phi(W_1(s)W_1(-s)), \phi(W_2(s)W_2(-s)), \phi(W_1(s)W_1(-s)W_2(s)W_2(-s))\},$$

$$= -\min \{\phi(W_1(s)), \phi(W_2(s)), \phi(W_1(s)) + \phi(W_2(s))\}.$$  

Strictly properness of $F_{\gamma_{\text{opt}}}$ implies,

$$\min \{\phi(W_1(s)), \phi(W_2(s)), \phi(W_1(s)) + \phi(W_2(s))\} < 0.$$  

(4.17)

We know that $\phi(W_1(s)) \geq 0$ and $\phi(W_2(s)) \leq 0$. Therefore, the inequality (4.17) is satisfied if and only if $\phi(W_1(s)) \geq 0$ and $\phi(W_2(s)) < 0$ are valid which means that $W_1(s)$ is
proper and $W_2(s)$ is improper. Since we have $(W_2N_o)^{-1} \in \mathcal{RH}^\infty$ [19], we can conclude that the plant is strictly proper. Same proof is valid for the suboptimal case.

We know that the suboptimal controllers are written as [26],

$$C_{subopt}(s) = E_\rho(s)m_d(s)\frac{N_o^{-1}(s)F_\rho(s)L_U(s)}{1 + m_n(s)F_\rho(s)L_U(s)}$$

we can rewrite the suboptimal controllers as,

$$C_{subopt}(s) = \left( \frac{N_o^{-1}(s)F_\rho(s)}{dE_\rho(s)dm_d(s)} \right) \left( \frac{L_2(s) + L_1(-s)m_n(s)F_\rho(s)}{P_1(s) + P_2(s)U(s)} \right)$$

where

$$P_1(s) = \frac{L_1(s) + L_2(s)m_n(s)F_\rho(s)}{dE_\rho(s)dm_d(s)}$$

$$P_2(s) = \frac{L_2(-s) + L_1(-s)m_n(s)F_\rho(s)}{nE_\rho(s)nm_d(s)}$$

and $nE_\rho, dE_\rho$ and $nm_d, dm_d$ are numerator and denominator of $E_\rho$ and $m_d$ respectively. Denominators of $P_1$ and $P_2$ are cancelled by numerators.

Note that unstable poles of $C_{subopt}$ are the zeros of $P_1 + P_2U$. If there exists a $U \in \mathcal{RH}^\infty$ with $\|U\|_\infty < 1$, such that $P_1 + P_2U$ has no unstable zeros, then the corresponding suboptimal controller is stable.

Assume that $F_\rho$ is strictly proper which implies $P_1$ and $P_2$ has finitely many unstable zeros. The suboptimal controller is stable if and only if $S_U := (1 + \tilde{P}U)^{-1}$ is stable where $\tilde{P} = \frac{P_2}{P_1}$. Note that since $P_1$ and $P_2$ has finitely many unstable zeros, we can write $\tilde{P}$ as,

$$\tilde{P} = \frac{\tilde{M}}{\tilde{M}_d}N_o$$

where $\tilde{M}$ and $\tilde{M}_d$ are inner, finite dimensional and $N_o$ is outer and infinite dimensional. Finding stable $S_U$ with $U \in \mathcal{H}^\infty$ is a sensitivity minimization problem with stable controller which is considered in [9]. However, in our case, $U$ has a norm restriction as $\|U\|_\infty \leq 1$ and we can write $U$ as,

$$U(s) = \left( \frac{1 - S_U(s)}{S_U(s)} \right) \left( \frac{P_1(s)}{P_2(s)} \right).$$

Define $\mu_{opt}$ as,

$$\mu_{opt} = \inf_{U \in \mathcal{H}^\infty} \|S_U\|_\infty = \inf_{U \in \mathcal{H}^\infty} \|(1 + \tilde{P}U)^{-1}\|_\infty.$$
Lemma 4.2. Assume that $W_1$ and $W_2$ are proper and improper respectively. If there exists $\mu_o > \mu_{opt}$ and $Q_o$ with $Q_o \in \mathcal{H}_\infty$ and $\|Q_o\|_\infty \leq 1$ satisfying

$$\left| \left(\frac{1 - S_{U,\mu_o}(Q_o(j\omega))}{S_{U,\mu_o}(Q_o(j\omega))}\right) \left(\frac{P_1(j\omega)}{P_2(j\omega)}\right) \right| \leq 1,$$

(4.18)

then the suboptimal controller, $C_{subopt}$, achieves the performance level $\rho$ by selecting the parameter $U$ as,

$$U(s) = \left(\frac{1 - S_{U,\mu_o}(Q_o(s))(s)}{S_{U,\mu_o}(Q_o(s))}\right) \left(\frac{P_1(s)}{P_2(s)}\right).$$

(4.19)

Proof The result of theorem is immediate. Since $Q_o$ satisfies the norm condition of $U$ and makes $S_{U,\mu}(Q_o)$ stable, the suboptimal controller has no right half plane poles by selection of $U$ as shown in theorem.

A stable suboptimal controller can be designed by finding $Q_o$ for $\mu_o$. By using a search algorithm, we can find $Q_o$ satisfying the norm condition for $U$. Instead of finding $U$ resulting stable suboptimal controller, the problem is converted finding $Q_o$ satisfying the norm condition. First problem needs to check whether a quasi-polynomial has unstable zeros. However, by using the theorem, this problem reduced into searching stable function with infinity norm less than one and satisfying norm condition for $U$. Conservatively, the search algorithm for $Q_o$ can be done for first order bi-proper functions such that $Q_o(s) = u(\alpha + \beta s)/(\alpha + \gamma s + \delta)$ where $\alpha > 0$, $\beta, \gamma, \delta \in \mathbb{R}$, and $|u(\alpha)| \leq \max \{1, |\beta/\gamma|\}$. The algorithm for this approach is explained below.

Algorithm

Assume that the optimal and central suboptimal controllers have finitely many unstable poles. We can design a stable suboptimal $\mathcal{H}_\infty$ controller by using the following algorithm:

1) Fix $\rho > \gamma_{opt}$,

2) Obtain $P_1$ and $P_2$. If $P_1$ has no unstable zeros, then the suboptimal controller is stable for $U = 0$. If not, go to step 3.

3) Define the right half plane zeros of $P_1$ and $P_2$ as $p_i$ and $s_i$ respectively. Note that these are right half plane zeros of $\tilde{M}_d(s)$ and $\tilde{M}(s)$ respectively. Calculate $w_i = \frac{\alpha - \beta}{\alpha + \gamma}$ and $z_i = \frac{\alpha - \beta}{\alpha + \gamma}$ where $\alpha > 0$.

4) Search for minimum $\mu$ which makes the Pick matrix positive semi-definite,

$$QP\{\mu\}_{(i,k)} = \left(\frac{-\ln \frac{\alpha}{\mu} - \ln \frac{\beta}{\mu} + j2\pi(n_k - n_i)}{1 - z_i \bar{z}_k}\right)$$

(4.20)

where $n_{[\cdot]}$ is integer. Note that most of the integers will not result in positive semi-definite Pick matrix. Therefore, for each integer set, we can find the smallest $\mu$ and $\mu_{opt}$ will be the minimum of these values. For details, see [6].
5) After the integer set and $\mu_{opt}$ is found, the function $g(z) \in \mathcal{H}^\infty$ can be obtained satisfying interpolation conditions,

$$g(z_i) = -\ln \frac{w_i}{\mu_{opt}} - j2\pi n_i$$

(4.21)

by Nevanlinna-Pick interpolation approach [19],[21]. Then, we can write $S_U(s) = \mu_{opt} \tilde{M}_d(s)e^{-G(s)}$ where $G(s) = g(\frac{\omega}{s+a})$ and obtain $U(s)$. Check the norm condition $\|U\|_\infty \leq 1$. If it is satisfied, then, $U(s)$ results in stable suboptimal controller achieving performance level $\rho$. If not, go to next step.

6) Increase $\mu$ such that $\mu > \mu_{opt}$. For all possible integer set, obtain $g(z) \in \mathcal{H}^\infty$ with interpolation conditions,

$$g(z_i) = -\ln \frac{w_i}{\mu} - j2\pi n_i.$$ 

(4.22)

Note that since $g(z)$ has a free parameter $q(z)$ ($q \in \mathcal{H}^\infty$ and $\|q\|_\infty \leq 1$), we can write the function as $g(z, q)$. Then, search for parameters $(u_\infty, z_u, p_u)$ satisfying

$$\left| \left( \frac{1 - \mu \tilde{M}_d(j\omega)e^{-G(j\omega, Q)}}{\mu \tilde{M}_d(j\omega)e^{-G(j\omega, Q)}} \right) \left( \frac{P_1(j\omega)}{P_2(j\omega)} \right) \right| \leq 1, \quad \forall \omega \in [0, \infty)$$

(4.23)

where $G(s, Q(s)) = g(\frac{s+a}{s+p_u}, q(\frac{s+a}{s+p_u}))$ and $Q(s) = u_\infty \left( \frac{s+z_u}{s+p_u} \right)$ as defined before. If one of the parameter set satisfies the inequality, then $Q_o = u_{\infty, o} \left( \frac{s+z_o}{s+p_{u, o}} \right)$ and corresponding $U$ results in a stable suboptimal $\mathcal{H}^\infty$ controller. If no parameter set satisfies the inequality, go to step 6, and repeat the procedure for sufficiently high $\mu$, until a pre-specified maximum is reached, in which case go next step.

7) Increase $\rho$, go to step 2, if a maximum pre-specified $\rho$ is reached, stop. This method fails to provide a stable $\mathcal{H}^\infty$ controller.

An illustrative example is presented in Section 5.

5 Examples

Two examples will be given in this section. In the first example, the optimal and central suboptimal controllers have infinitely many unstable poles; by using the design method, we show that there exists a stable suboptimal controller even the magnitude condition $\|L_U(j\omega)F_\rho(j\omega)\|_\infty \leq 1$ is violated for low frequencies. In other words, the example illustrates that the conditions in (3.1) are sufficient.

The second example explains the design method for stable suboptimal $\mathcal{H}^\infty$ controller whose central controller is unstable with finitely many unstable poles and implements the algorithm step by step as mentioned in section 4.
5.1 Example

Let \( P(s) = e^{-0.1s} \left( \frac{s-1}{s+1} \right) \) and choose \( W_1(s) = \frac{1+0.6s}{s+1} \) and \( W_2 = 0 \) (one-block problem). Using Skew-Toeplitz approach in [19], the minimum \( H^\infty \) value, \( \gamma_{opt} \), is 0.8108. The optimal controller has infinitely many unstable poles converging to \( s = 3.0109 \pm j\frac{(2k+1)\pi}{h} \) as \( k \to \infty \). If central suboptimal controller \((U = 0)\) is calculated for \( \rho = 0.814 \), it has infinitely many unstable poles converging to \( s = 2.445 \pm j\frac{(2k+1)\pi}{h} \) as \( k \to \infty \). The suboptimal controllers can be represented as,

\[
C_{subopt}(s) = \frac{E_{\rho}(s)L_U(s)}{1 + m_n(s)F_{\rho}(s)L_U(s)}
\]

(5.24)

where

\[
m_n(s) = e^{-0.1s} \left( \frac{s-1}{s+1} \right),
\]
\[
E_{\rho}(s) = \frac{0.3374 + 0.3026s^2}{0.6626(1-s^2)},
\]
\[
F_{\rho}(s) = 0.814 \left( \frac{1-s}{1+0.6s} \right),
\]
\[
L_U(s) = \frac{L_{2U}(s)}{L_{1U}(s)} = \frac{L_2(s) + L_1(-s)U(s)}{L_1(s) + L_2(-s)U(s)},
\]
\[
L_2(s) = -(0.9413s + 1.8716),
\]
\[
L_1(s) = (s + 1.8373).
\]

We will use the design method of the Section 3 to find a stable suboptimal controller by search for \( U \). The central suboptimal controller \((U = 0)\) has infinitely many unstable poles as mentioned before. The algorithm is tried for \( u_z = u_p = 0 \) case, i.e., \( U(s) = u_\infty \).

1) Fix \( \rho = 0.814 > \gamma_{opt} = 0.8108 \),

2) \( k = -0.9413 \) and \( f_\infty = 1.3567 \) are calculated.

3) \( n_1 = 1, l = 0, n_1 + l \) is odd and \( |k| > \frac{1}{f_\infty} \). By using Lemma (3.2), the admissible values for \( u_\infty \) are \( -0.9909 < u_\infty < -0.6668 \).

4) \( L_{1U}(s) \) is stable for \( u_\infty \in [-1, 0.98] \).

5) Overall admissible values for \( U \) are \( u_\infty \in [-0.9909, -0.6668] \). The values of \( \omega_{max} \) and \( \eta_{max} \) for all admissible \( u_\infty \) range can be seen in Figure [1]. Since \( \eta_{max} \) values do not vary much, the minimum value of \( \omega_{max} \) determines the optimal \( u_\infty \) value as \( \omega_{max} = 19.458 \) at \( u_\infty = -0.813 \).

6) Figure [2] shows the plot of \( Z(s) = |1 + e^{-hs}M(s)F_{\rho}(s)L_U(s)| \) in the right half plane. The function has only right half plane zero at \( s = \pm 1.056j \), which is right half zeros of \( E_{\rho}(s) \). Note that, only one part of right plane is graphed since the other half is same.
Therefore, we can conclude that suboptimal controller is stable for \( U(s) = -0.813 \) and achieves the \( \mathcal{H}^\infty \) norm \( \rho = 0.814 \).

### 5.2 Example

For given plant \( P(s) = e^{-3s} \) and weight functions \( W_1(s) = \frac{2.24+s}{1+s} \) and \( W_2(s) = 0.5(2.24+s) \), we can find the optimal performance level as \( \gamma_{opt} = 1.9452 \). The corresponding optimal \( \mathcal{H}^\infty \) controller can be written as,

\[
C_{opt}(s) = E_{\gamma_{opt}}(s) \frac{F_{\gamma_{opt}}(s)L_{opt}(s)}{1 + m_n(s)F_{\gamma_{opt}}(s)L_{opt}(s)}
\]

where

\[
m_n(s) = e^{-3s},
\]

\[
E_{\gamma_{opt}}(s) = \frac{1.2162 + 2.7838s^2}{3.7838(1 - s^2)},
\]

\[
F_{\rho}(s) = \frac{5.5119}{(2.24 + s)^2},
\]

\[
L_{opt}(s) = 1.
\]

The optimal controller has unstable poles at \( s = 0.0292 \pm 2.2354j \). Note that since \( W_1 \) and \( W_2 \) are proper and improper respectively, all \( \mathcal{H}^\infty \) controllers will have finitely many unstable poles by Theorem 4.2. Therefore we can apply the algorithm in section 4.

1) Fix \( \rho = 1.9454 > \gamma_{opt} = 1.9452 \),

2) The suboptimal controllers can be written as,

\[
C_{subopt}(s) = E_{\rho}(s) \frac{F_{\rho}(s)L_U(s)}{1 + m_n(s)F_{\rho}(s)L_U(s)}
\]

12
where

\[
\begin{align*}
m_n(s) &= e^{-3s}, \\
E_p(s) &= \frac{1.2154 + 2.7846s^2}{3.7846(1 - s^2)}, \\
F_p(s) &= 5.5115 \frac{(1 - s)}{(2.24 + s)^2}, \\
L_U(s) &= \frac{L_{2U}(s)}{L_{1U}(s)} = \frac{L_2(s) + L_1(-s)U(s)}{L_1(s) + L_2(-s)U(s)}, \\
L_2(s) &= (2.9837 + 0.9946s), \\
L_1(s) &= (2.9829 + s),
\end{align*}
\]

and \( U \) is free parameter such that \( U \in \mathcal{H}^\infty, \|U\|_\infty \leq 1. \) We can write \( P_1 \) and \( P_2 \) as,

\[
\begin{align*}
P_1(s) &= \frac{L_1(s) + m_n(s)F_p(s)L_2(s)}{nE_p(s)}, \\
&= \frac{(2.9829 + s)(2.24 + s)^2 + 5.5115(1 - s)(2.9837 + 0.9946s)e^{-3s}}{(1.2154 + 2.7846s^2)(2.24 + s)^2}, \\
P_2(s) &= \frac{L_2(-s) + m_n(s)F_p(s)L_1(-s)}{nE_p(s)}, \\
&= \frac{(2.9837 - 0.9946s)(2.24 + s)^2 + 5.5115(1 - s)(2.9829 - s)e^{-3s}}{(1.2154 + 2.7846s^2)(2.24 + s)^2}.
\end{align*}
\]

Note that \( P_1 \) and \( P_2 \) has unstable zeros at \( 0.0287 \pm 2.2346j \) and \( 0.0297 \pm 2.2346j \) respectively. Therefore, the central controller \( (U = 0) \) for the chosen performance level, \( \rho = 1.9458, \) is unstable.

3) Define the following variables and functions as,

\[
\begin{align*}
p_i &= 0.0287 \pm 2.2346j, \quad i = 1, 2, \\
s_i &= 0.0297 \pm 2.2346j, \quad i = 1, 2, \\
\tilde{M}_d(s) &= \frac{(s - p_1)(s - p_2)}{(s + p_1)(s + p_2)} = \frac{s^2 - 0.0574s + 4.9943}{s^2 + 0.0574s + 4.9943}, \\
w_i &= \frac{1}{\tilde{M}_d(s_i)} = 58.4002 \mp 0.7501j, \quad i = 1, 2, \\
z_i &= \frac{s_i - 1}{s_i + 1} = 0.6598 \pm 0.7383i
\end{align*}
\]

where conformal mapping parameter, \( a, \) is chosen as \( a = 1. \)

4) In order to find the minimum \( \mu \) resulting in positive semi-definite Pick matrix,

\[
QP(\mu) = \begin{pmatrix}
-8.1348 + 2 \ln \mu & 0.0196 \\
0.0196 & -8.1348 + 0.0257j + 2 \ln \mu + j2\pi(n_2 - n_1) \\
(-8.1348 - 0.0257j + 2 \ln \mu + j2\pi(n_1 - n_2)) & 1.0907 - 0.9742j \\
1.0907 - 0.9742j & -8.1348 + 2 \ln \mu
\end{pmatrix}, (5.27)
\]

13
we will find the minimum \( \mu \) for all possible integer pairs \((n_1, n_2)\). It is not difficult to do this search since many integer pairs do not result in positive semi-definite Pick matrix. For each integer pair, we can find the minimum \( \mu, \mu_{\text{min}} \), and then \( \mu_{\text{opt}} \) will be smallest of all \( \mu_{\text{min}} \). Note that since Pick matrix depends on difference of integers, we can normalize the search by taking \( n_1 = 0 \). In Figure 3 we can see the minimum \( \mu \) values for integers, \( n_2 \). The minimum of all \( \mu_{\text{min}} \) values is \( \mu_{\text{opt}} = 58.4167 \).

5) The calculation of \( U(s) \) for \( \mu_{\text{opt}} \) is omitted. It does not satisfy the norm condition \( \|U\|_{\infty} \leq 1 \).

6) Fix \( \mu = 64 \) and \( n_1 = n_2 = 0 \). The interpolation conditions for \( g(z) \) can be written as,

\[
g(z_i) = 0.0915 \pm 0.0128 j, \quad i = 1, 2. \tag{5.28}
\]

By Nevanlinna-Pick approach, (see e.g. [19]),

\[
g(z, q) = \frac{(1.0878z^2 - 1.3782z + 0.9804)q(z) + (0.0724z - 0.1054)}{(0.9804z^2 - 1.3782z + 1.0878) + (0.0724z - 0.1054z^2)q(z)} \tag{5.29}
\]

where \( q(z) \) is a parameterization term such that \( q \in \mathcal{H}^\infty \) and \( \|q\|_{\infty} \leq 1 \). The search algorithm tries to find \( q_o \) satisfying the norm condition

\[
\left| \left( \begin{array}{c} 1 - \mu \tilde{M}_d(j\omega)e^{-G(j\omega, q)} \\ \mu \tilde{M}_d(j\omega)e^{-G(j\omega, q)} \end{array} \right) \left( \begin{array}{c} P_1(j\omega) \\ P_2(j\omega) \end{array} \right) \right| \leq 1, \quad \forall \omega \in [0, \infty) \tag{5.30}
\]

where

\[
G(s, Q(s)) = g \left( \frac{s - 1}{s + 1}, q \left( \frac{s - 1}{s + 1} \right) \right),
\]

and \( Q \in \mathcal{H}^\infty, \|Q\|_{\infty} \leq 1 \). We will search for \( Q \) satisfying the norm condition (5.30) in the form of \( Q(s) = u_{\infty} \) with \( |u_{\infty}| \leq 1 \). Note that we choose \( z_u = p_u = 0 \) and all functions in norm condition, \( P_1, P_2, \tilde{M}_d \), are defined before. After search is done, the condition (5.30) is satisfied for \( u_{\infty} = 0.323 \). The magnitude of \( U(j\omega) \) is smaller than one for all frequency values as seen in Figure. (i.e., \( \|U\|_{\infty} = 0.9924 \)). As a result, the suboptimal \( \mathcal{H}^\infty \) controller achieving the performance level, \( \rho = 1.9454 \), is stable with selection of parameter \( U \) as,

\[
U(s) = \left( \frac{s^2 + 0.0574s + 4.9943}{s^2 - 0.0574s + 4.9943} \right) e^{\left( \frac{0.1889s^2 - 0.4250s + 1.0802}{0.6794s^2 + 0.3297s + 3.4357} \right) - 1} \left( \frac{P_1(s)}{P_2(s)} \right). \tag{5.31}
\]
By the search algorithm, we can find many $u_\infty$ values for different $\mu$ resulting in stable $\mathcal{H}_\infty$ controller at $\rho = 1.94584$ provided that $U$ satisfies the norm condition for chosen $Q = u_\infty$. The various $u_\infty$ values resulting stable $\mathcal{H}_\infty$ controller can be seen in Figure 5. We can observe that as $\mu$ is increased, the range of $u_\infty$ stabilizing the controller decreases and the minimum value of $\|U\|_\infty$ in the $u_\infty$ range becomes smaller.

6 Conclusions

In this paper, for delay systems, we investigated stability of the $\mathcal{H}_\infty$ controllers whose structure is given in [16], [19]. We considered the controllers in two subsections according to their number of poles (finite, infinite). For each case, necessary conditions and design methods based on simple sufficient condition are given to find stable suboptimal $\mathcal{H}_\infty$ controllers.
References

[1] M. Vidyasagar, Control System Synthesis: A Factorization Approach, MIT Press, 1985.

[2] A. Sideris and M. G. Safonov, “Infinity-norm optimization with a stable controller,” Proc. American Control Conference, 1985, 804-805.

[3] A. E. Barabanov, “Design of $H^\infty$ optimal stable controller,” Proc. Conference on Decision and Control, 1996, 734-738.

[4] M. Jacobus, M. Jamshidi, C. Abdullah, P. Dorato and D. Bernstein, “Suboptimal strong stabilization using fixed-order dynamic compensation,” Proc. American Control Conference, 1990, 2659-2660.

[5] H. Ito, H. Ohmori and A. Sano, “Design of stable controllers attaining low $H^\infty$ weighted sensitivity,” IEEE Transactions on Automatic Control 38, 1993, 485-488.

[6] C. Ganesh and J. B. Pearson, “Design of optimal control systems with stable feedback,” Proc. American Control Conference, 1986, 1969-1973.

[7] M. Zeren and H. Özbay, “On the synthesis of stable $H^\infty$ controllers,” IEEE Transactions on Automatic Control 44, 1999, 431-435.

[8] M. Zeren and H. Özbay, “On the strong stabilization and stable $H^\infty$-controller design problems for MIMO systems,” Automatica 36, 2000, 1675–1684.

[9] D. U. Campos-Delgado and K. Zhou, “$H^\infty$ Strong stabilization,” IEEE Transactions on Automatic Control 46, 2001, 1968-1972

[10] J.L.Abedor and K.Poolla, “On the strong stabilization of delay systems,” Proc. Decision and Control Conference, 1989, 2317-2318.

[11] K.Suyama, “Strong stabilization of systems with time-delays,” Proc. IEEE Industrial Electronics Society Conference, 1991, 1758-1763.

[12] S.Gümüşsoy and H.Özbay, “Control of Systems with Infinitely Many Unstable Modes and Strongly Stabilizing Controllers Achieving a Desired Sensitivity,” Proc. Mathematical Theory of Networks and Systems, 2002.

[13] C. Foias, A. Tannenbaum and G. Zames, “Weighted Sensitivity Minimization for delay systems,” IEEE Transactions on Automatic Control, 31, 1986, 763–766.

[14] K. Zhou and P.P. Khargonekar, “On the weighted sensitivity minimization problem for delay systems,” Systems & Control Letters, 8, 1987, 307–312.
[15] H. Özbay, M.C. Smith and A. Tannenbaum, “Mixed-sensitivity optimization for a class of unstable infinite-dimensional systems,” *Linear Algebra Applications*, 178, 1993, 43–83.

[16] O. Toker and H. Özbay, “$H^\infty$ Optimal and suboptimal controllers for infinite dimensional SISO plants,” *IEEE Transactions on Automatic Control* 40, 1995, 751–755.

[17] D. S. Flamm and S. K. Mitter, “$H^\infty$ sensitivity minimization for delay systems,” *Systems & Control Letters* 9, 1987, 17-24.

[18] K. E. Lenz, “Properties of optimal weighted sensitivity designs,” *IEEE Transactions on Automatic Control* 40, 1995, 298-301.

[19] C. Foias, H. Özbay, and A. Tannenbaum, *Robust Control of infinite Dimensional Systems: Frequency Domain Methods*, No.209 in LNCIS, Springer-Verlag, 1996.

[20] G. Meinsma and H. Zwart, “On $H^\infty$ control for dead-time systems,” *IEEE Transactions on Automatic Control*, 45, 2000, 272–285.

[21] M. Zeren and H. Özbay, “Comments ‘Solutions to the combined sensitivity and complementary sensitivity problem in control systems’,” *IEEE Transactions on Automatic Control*, 43, 1998, 724.
$w_{\text{max}}$ versus $u_\infty$