Spiral Delone sets in relative metric

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Abstract

A general Archimedean spiral lattice is a Delone set in the relative distance if and only if its rotation angle is badly approximable.

Keywords: Delone sets, relative metric, irrational rotation, badly approximable number, continued fraction, spiral, phyllotaxis.

1 Introduction

Phyllotaxis is a subject in biology which is related to crystallography and mathematics. It studies the arrangements of leaves, seeds, and other organs of plants. In the 19th century, botanical spiral patterns were considered as living crystals by A. Bravais and L. Bravais, who assumed that their rotation angles are irrational numbers.

The arrangement of seeds in the sunflower head is modeled by the Bernoulli spiral lattice \( \{ r^n e^{2\pi n \tau \sqrt{-1}} | n \in \mathbb{Z} \} \), \( \tau = (1 + \sqrt{5})/2 \), \( 0 < r < 1 \), and the general Archimedean spiral lattice \( \{ n^\alpha e^{2\pi n \theta \sqrt{-1}} | n \in \mathbb{Z}_{\geq 0} \} \), \( \alpha > 0 \) (the case \( \alpha = 1/2 \) is called Fermat, \( \alpha = 1 \) Archimedean).

Let

\[
\Gamma_{\alpha,\theta} = \{ n^\alpha e^{2\pi n \theta \sqrt{-1}} | n \in \mathbb{Z}_{\geq 0} \}, \quad \alpha > 0, \ \theta \in \mathbb{R}.
\]

Akiyama \( \square \) showed that (i) if \( \alpha > \frac{1}{2} \) then \( \Gamma_{\alpha,\theta} \) is not relatively dense, (ii) if \( \alpha < \frac{1}{2} \) then \( \Gamma_{\alpha,\theta} \) is not uniformly discrete, (iii) if \( \alpha = \frac{1}{2} \) and \( \theta \) is irrational, then \( \Gamma_{\frac{1}{2},\theta} \) is relatively dense \( \Leftrightarrow \) \( \Gamma_{\frac{1}{2},\theta} \) is uniformly discrete \( \Leftrightarrow \theta \) is badly approximable. His proof is based on the Three Gap Theorem. A Delone set is a set which is both relatively

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dense and uniformly discrete. So a Fermat spiral lattice is a Delone set if and only if \( \theta \) is badly approximable.

This paper addresses the relative denseness and uniform discreteness of \( \Gamma_{\alpha, \theta} \) with respect to the relative distance

\[
d_\beta(z, w) := \frac{|z - w|}{|z|^\beta + |w|^\beta}, \quad -\infty < \beta < 1.
\]

It is known (\cite{2}) that if \( \frac{1}{2} \leq \beta \leq 1 \), the function \( d_\beta \) satisfies the triangle inequality, so that it is a metric on \( \mathbb{C} \). In this paper we define the notions of asymptotic \( \beta \)-relative denseness and asymptotic \( \beta \)-uniform discreteness with respect to the relative distance \( d_\beta \). We show that

**Lemma 1.** If \( \Gamma_{\alpha, \theta} \) is asymptotically \( \beta \)-relatively dense, then \( 1 + 2\alpha \beta \geq 2\alpha \).

**Lemma 2.** If \( \Gamma_{\alpha, \theta} \) is asymptotically \( \beta \)-uniformly discrete, then \( 1 + 2\alpha \beta \leq 2\alpha \).

**Lemma 3.** Suppose that \( 1 + 2\alpha \beta = 2\alpha \). If \( \Gamma_{\alpha, \theta} \) is asymptotically \( \beta \)-relatively dense or asymptotically \( \beta \)-uniformly discrete, then \( \theta \not\in \mathbb{Q} \).

**Theorem 4.** Suppose that \( 1 + 2\alpha \beta = 2\alpha \). Then the following statements are mutually equivalent.

(i) \( \Gamma_{\alpha, \theta} \) is asymptotically \( \beta \)-relatively dense.

(ii) \( \Gamma_{\alpha, \theta} \) is asymptotically \( \beta \)-uniformly discrete.

(iii) \( \theta \) is badly approximable.

A main idea of the proof of Theorem 4 is the approximation by (linear) lattices. Denote by \( -\frac{1}{2} \leq \langle x \rangle < \frac{1}{2} \) be the fractional part of \( x \in \mathbb{R} \), where \( x - \langle x \rangle \in \mathbb{Z} \). By Taylor’s theorem, there exists a constant \( C > 0 \) depending only on \( \alpha \), such that

\[
(\nu + k)^\alpha e^{2\pi k\theta \sqrt{-1}} = \nu^\alpha \left( 1 + \frac{k}{\nu} \right)^\alpha e^{2\pi k\theta \sqrt{-1}}
\]

\[
= \nu^\alpha \left( 1 + \frac{\alpha k}{\nu} + 2\pi \langle \theta k \rangle \sqrt{-1} \right) + \epsilon
\]

\[
= \nu^\alpha + \nu^{\alpha - \frac{1}{2}} \left( \frac{\alpha k}{\sqrt{\nu}} + 2\pi \langle \theta k \rangle \sqrt{\nu \sqrt{-1}} \right) + \epsilon,
\]

where

\[
|\epsilon| \leq C \nu^\alpha \left( \frac{\alpha k}{\nu}^2 + |2\pi \langle \theta k \rangle|^2 \right).
\]
Before proving Theorem 4, we consider the family of linear lattices
\[
\Lambda(t) = \left\{ \left( (m\theta - n)\sqrt{t}, \frac{m}{\sqrt{t}} \right) \mid m, n \in \mathbb{Z} \right\}, \quad t \geq 1.
\]
(2)

It can be regarded as a scenery flow (3) of the lattice \( \Lambda = \{(m\theta - n, m) \mid m, n \in \mathbb{Z} \} \).

We define the notions of relative denseness and uniform discreteness of the family \( \{\Lambda(t)\}_{t \geq 1} \), and show the following Proposition.

**Proposition 5.** Let \( \theta \in \mathbb{R} \setminus \mathbb{Q} \). Let \( \{\Lambda(t)\}_{t \geq 1} \) be a family of lattices defined in (2). Then the following conditions are mutually equivalent.

(i) \( \{\Lambda(t)\}_{t \geq 1} \) is relatively dense.

(ii) \( \{\Lambda(t)\}_{t \geq 1} \) is uniformly discrete.

(iii) \( \theta \) is badly approximable.

Proposition 5 works as a linear prototype of Theorem 4. Note that a key tool in the proof of Proposition 5 is Richards’ formula, which is known in the phyllotaxis theory [4].

Section 2 defines the notions of asymptotically \( \beta \)-relative denseness and asymptotically \( \beta \)-uniform discreteness, and show Lemmas 1-3. Section 3 prepares notations in the continued fraction expansions and rational approximations. Section 4 proves Proposition 5 by using Richards’ formula. Section 5 proves Theorem 4.

See [4, 5] for the history of the study of phyllotaxis, and [6, 7] for recent surveys. In [8], it was shown that the area of Voronoi cells for a general Archimedean spiral lattice has a convergence under some scale normalization. In [9], the combinatorial structures of the grains and grain boundaries of the Voronoi tessellations for Archimedean spiral lattices were described. In [10], it was shown that, in the family of Bernoulli spiral lattices, the bifurcation diagram of Voronoi tessellations is a dual graph of the bifurcation diagram of circle packings, by using the relative metric \( d(z, w) = |z - w|/(|z| + |w|) \). Marklof [11] showed that the point set \( \{\sqrt{n}e^{2n\theta}, \sqrt{n}e^{-1} \mid n \in \mathbb{Z}_\geq 0\} \) is a Delone set for any \( \theta > 0 \).

## 2 Delone sets in relative metric

In this section we define the notions of asymptotically \( \beta \)-relative denseness and asymptotically \( \beta \)-uniform discreteness, and show Lemmas 1-3.
Lemma 6. Let $r', r > 0$, $0 < \beta < 1$, $z, \zeta \in \mathbb{C}$. Suppose that $r < r' < 2r$. If $|z - \zeta| < r|z|^\beta$ and $|z| \geq M := \left(\frac{rr'}{2(r-r')}\right)^{1/(1-\beta)}$, then we have $|z - \zeta| < \frac{r'}{2}|z|^\beta + \frac{r}{2}|\zeta|^\beta$.

Proof. If $|\zeta| \geq |z|$, we have $|z - \zeta| < r|z|^\beta < r'|z|^\beta \leq \frac{r'}{2}|z|^\beta + \frac{r}{2}|\zeta|^\beta$. So we assume that $|z| > |\zeta|$. Since $|z| \geq M = \frac{rr'}{2(r-r')})^{1/(1-\beta)}$, we have $|z| \geq \frac{rr'}{2(r-r')}|z|^\beta$, and $r|z|^\beta \leq \frac{2(r-r')}{r'}|z|$. Hence $|z - \zeta| \leq |z - \zeta| < r|z|^\beta \leq \frac{2(r-r')}{r'}|z|$. This implies that $0 < \frac{2r-r'}{r'} < \frac{|\zeta|}{|z|} < 1$, and $\frac{2r-r'}{r'} < \frac{|\zeta|}{|z|} < \frac{|\zeta|^\beta}{|z|^\beta}$ since $0 < \beta < 1$. So we obtain $(2r - r')|z|^\beta < r'|\zeta|^\beta$, and $|z - \zeta| < r|z|^\beta < \frac{r'}{2}(|z|^\beta + |\zeta|^\beta)$.

Lemma 7. Let $r' > r > 0$, $0 < \beta < 1$, $z, \zeta \in \mathbb{C}$. Suppose that $|z| \geq M := \left(\frac{rr'}{r-r'}\right)^{1/(1-\beta)}$. If either

\begin{equation}
|z - \zeta| < \frac{r}{2}|z|^\beta + \frac{r}{2}|\zeta|^\beta
\end{equation}

or

\begin{equation}
|z - \zeta| < r|z|^\beta,
\end{equation}

then we have $|z - \zeta| < r'|\zeta|^\beta$.

Proof. If $|\zeta| \geq |z|$, we have $r|z|^\beta \leq \frac{r}{2}|z|^\beta + \frac{r}{2}|\zeta|^\beta \leq r|\zeta|^\beta < r'|\zeta|^\beta$. So we assume that $|z| > |\zeta|$. Then we have $\frac{r}{2}|z|^\beta + \frac{r}{2}|\zeta|^\beta < r|z|^\beta$, so the assumption (1) implies (2). Since $|z| \geq M = \frac{rr'}{r-r'})^{1/(1-\beta)}$, we have $|z| \geq \frac{rr'}{r-r'}|z|^\beta$, and $r|z|^\beta < \frac{r'}{r-r'}|z|$. Hence $|z - \zeta| \leq |z - \zeta| < r|z|^\beta \leq \frac{r}{r-r'}|z|$. This implies that $\frac{r}{r-r'} < \frac{|\zeta|}{|z|} < 1$, and $\frac{r}{r-r'} < \frac{|\zeta|}{|z|} < \frac{|\zeta|^\beta}{|z|^\beta}$ since $0 < \beta < 1$. So we obtain $|z - \zeta| < r|z|^\beta < r'|\zeta|^\beta$.

Lemma 8. If $\alpha > 0$ and $0 < x \leq \frac{1}{\alpha^2}$, then $(1 + x)^{\alpha} < 1 + 2\alpha x$.

Proof. If $0 < \alpha \leq 1$, then $(1 + x)^{\alpha} \leq 1 + \alpha x < 1 + 2\alpha x$. So suppose that $\alpha > 1$. Since the function $t \mapsto e^t$ is convex, we have $\frac{e^{1/\alpha} - 1}{1/\alpha} \leq 1 + \alpha x < 2\alpha$. Since the function $x \mapsto (1+x)^{\alpha}$ is convex, we have $\frac{(1+x)^{\alpha} - 1}{x} \leq \frac{e^{1/\alpha} - 1}{1/\alpha} \leq 2\alpha$, which completes the proof.

Lemma 9. If $\delta > 0$ and $0 < h < \frac{1}{2}$, then $(1 - h)^\delta > 1 - 2\delta h$.

Proof. If $\delta \geq 1$, then $(1 - h)^\delta \geq 1 - \delta h \geq 1 - 2\delta h$. Suppose that $0 < \delta < 1$. By the Mean Value Theorem, we have $(1 - (1 - h)^\delta)/h = \delta(1 - eh)^{\delta-1}$ for some $0 < \epsilon < 1$, and $(1 - eh)^{\delta-1} < (1 - h)^{-1} \leq 2$ for $0 < h < \frac{1}{2}$.

Lemma 10. Let $r' > r > 0$, $-\infty < \beta < 0$, $\delta = -\beta > 0$, $z, \zeta \in \mathbb{C}$. Suppose that $|z| \geq M := \max\{1, r\delta^2, \frac{2r'\delta}{r-r'}, 2r\}$ and $|\zeta| \geq 1$. Then the following statements hold.

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(1) If $|z - \zeta| < r|z|^\beta$, then we have $|z - \zeta| < r'|\zeta|^\beta$ and $|z - \zeta| < \frac{r'}{2}(|z|^\beta + |\zeta|^\beta)$.

(2) If $|z - \zeta| < \frac{r}{2}(|z|^\beta + |\zeta|^\beta)$, then $|z - \zeta| < r'|\zeta|^\beta$.

(3) If $|z - \zeta| < r|\zeta|^\beta$, then $|z - \zeta| < r'|\zeta|^\beta$.

Proof. (1). If $|\zeta| \leq |z|$, we have $|z - \zeta| < r|z|^{-\delta} < \frac{r'}{2}(|z|^{-\delta} + |\zeta|^{-\delta}) < r'|\zeta|^{-\delta}$. So assume that $|\zeta| > |z|$. Since $|\zeta| - |z| \leq |z| - \zeta < r|z|^{-\delta} \leq r$, we have $|\zeta| = 1 + \frac{r}{M} < 1 + r/M$. Since $\frac{r}{M} \leq \frac{1}{\delta}$, Lemma 5 applies to see that

$$\frac{|\zeta|^\delta}{|z|^\delta} < \left(1 + \frac{r}{M}\right)^\delta \leq 1 + \frac{2r\delta}{M} \leq \frac{r'}{r},$$

so we obtain $|z - \zeta| < r|z|^{-\delta} < r'|\zeta|^{-\delta} < \frac{r'}{2}(|z|^{-\delta} + |\zeta|^{-\delta})$.

(2). If $|\zeta| \leq |z|$, we have $|z - \zeta| < \frac{r}{2}(|z|^{-\delta} + |\zeta|^{-\delta}) < r'|\zeta|^{-\delta}$. If $|\zeta| > |z|$, we have $|z - \zeta| < \frac{r}{2}(|z|^{-\delta} + |\zeta|^{-\delta}) < r|z|^{-\delta}$, so (1) applies to see that $|z - \zeta| < r'|\zeta|^{-\delta}$.

(3). If $|\zeta| \geq |z|$, we have $|z - \zeta| < r|\zeta|^{-\delta} < r'|z|^{-\delta}$. Next assume that $1 \leq |\zeta| < |z|$. Since $|z| - |\zeta| \leq |z - \zeta| < r|\zeta|^{-\delta} \leq r$, we have $|\zeta| \geq 1 - \frac{r}{|z|} \geq 1 - \frac{r}{M}$. So

$$\frac{|\zeta|^\delta}{|z|^\delta} \geq \left(1 - \frac{r}{M}\right)^\delta \geq 1 - \frac{2r\delta}{M} \geq \frac{r}{r'},$$

and we obtain $|z - \zeta| < r|\zeta|^{-\delta} \leq r'|z|^{-\delta}$.

Lemma 11. Let $-\infty < \beta < 1$, $r > 0$. Let $\Gamma \subset \mathbb{C}$. Let $\Gamma' = \{\gamma \in \Gamma \mid |\gamma| \geq 1\}$. The following conditions are mutually equivalent.

(i) For any $r_1 > r$ there exists $M_1 > 0$ such that for any $z \in \mathbb{C}$ with $|z| \geq M_1$, there exists $\zeta \in \Gamma'$ such that $|z - \zeta| < r_1|z|^\beta$.

(ii) For any $r_2 > r$ there exists $M_2 > 0$ such that for any $z \in \mathbb{C}$ with $|z| \geq M_2$, there exists $\zeta \in \Gamma'$ such that $|z - \zeta| < r_2|\zeta|^\beta$.

(iii) For any $r_3 > r$ there exists $M_3 > 0$ such that for any $z \in \mathbb{C}$ with $|z| \geq M_3$, there exists $\zeta \in \Gamma'$ such that $|z - \zeta| < \frac{r_3}{2}(|z|^\beta + |\zeta|^\beta)$.

Proof. (i)$\Rightarrow$(ii). The case $\beta = 0$ is trivial, so suppose that $-\infty < \beta < 0$ or $0 < \beta < 1$. Let $r_2 > r$. Take $r_1$ such that $r < r_1 < r_2$. There exists $M_1 > 0$ such that for each $z \in \mathbb{C}$ with $|z| \geq M_1$, there exists $\zeta \in \Gamma$ such that $|z - \zeta| < r_1|z|^\beta$. Let $M_2 = \max \left\{ M_1, \left(\frac{r_1}{r_2-r_1}\right)^{1/(1-\beta)} \right\}$ if $0 < \beta < 1$, or let $M_2 = \max \left\{ M_1, r_1\delta, \frac{2r_1r_2\delta}{r_2-r_1} \right\}$ if $-\infty < \beta < 0$, where $\delta = -\beta > 0$. If $|z| \geq M_2$, then we have $|z - \zeta| < r_2|\zeta|^\beta$ by Lemmas 7-10.

(ii)$\Leftrightarrow$(iii)$\Rightarrow$(i). All the other arguments are given in a similar way.
A point set $\Gamma \subset \mathbb{C}$ is called locally finite if $\#\{\zeta \in \Gamma \mid |\zeta| < r \} < \infty$ for any $r > 0$.

**Lemma 12.** Let $-\infty < \beta < 1$, $s > 0$. Suppose that a point set $\Gamma \subset \mathbb{C}$ is locally finite. Let $\Gamma' = \{\gamma \in \Gamma \mid |\gamma| \geq 1\}$. The following conditions are mutually equivalent.

(i) For any $0 < s_1 < s$, there exists $M_1 > 0$ such that for any $z \in \mathbb{C}$ with $|z| \geq M_1$, we have $\#\{\zeta \in \Gamma' \mid |z - \zeta| < s_1|z|^\beta\} \leq 1$.

(ii) For any $0 < s_2 < s$, there exists $M_2 > 0$ such that for any $z \in \mathbb{C}$ with $|z| \geq M_2$, we have $\#\{\zeta \in \Gamma' \mid |z - \zeta| < s_2|\zeta|^\beta\} \leq 1$. 

(iii) For any $0 < s_3 < s$, there exists $M_3 > 0$ such that for any $z \in \mathbb{C}$ with $|z| \geq M_3$, we have $\#\{\zeta \in \Gamma' \mid |z - \zeta| < s_3(z^\beta + |\zeta|^\beta)\} \leq 1$.

**Proof.** (i)$\Rightarrow$(ii). The case $\beta = 0$ is trivial, so assume that $-\infty < \beta < 0$ or $0 < \beta < 1$. Suppose that (ii) does not hold. There exists $s_2 < s$ and a sequence $z_i, i \in \mathbb{Z}_{>0}$, such that $\lim_{i \to \infty} |z_i| = +\infty$ and $\#\{\zeta \in \Gamma' \mid |z_i - \zeta| < s_2|\zeta|^\beta\} \geq 2$ for each $i \in \mathbb{Z}_{>0}$. Take $s_1$ such that $s_2 < s_1 < s$. Let $M = \left(\frac{s_1s_2}{s_1-s_2}\right)^{1/(1-\beta)}$ if $0 < \beta < 1$, or let $M = \max\left\{s_2\delta^2, \frac{2s_1s_2}{s_1-s_2}, 2s_2\right\}$ if $-\infty < \beta < 0$, where $\delta = -\beta > 0$. If $|z_i| \geq M$, then we obtain $\#\{\zeta \in \Gamma' \mid |z_i - \zeta| < s_1|z|^\beta\} \geq 2$ by Lemmas [7] [10]. So (i) does not hold.

(ii)$\Leftrightarrow$(iii)$\Rightarrow$(i). All the other arguments are given in a similar way. \hfill $\square$

**Definition 13.** We say that a point set $\Gamma \subset \mathbb{C}$ is asymptotically $\beta$-relatively dense if there exists $r > 0$ that satisfies one (and hence all) of the conditions in Lemma 12.

**Definition 14.** We say that $\Gamma \subset \mathbb{C}$ is asymptotically $\beta$-uniformly discrete if there exists $r > 0$ that satisfies one (and hence all) of the conditions in Lemma 12. $\Gamma$ is called an asymptotically $\beta$-Delone set if it is both asymptotically $\beta$-relatively dense and asymptotically $\beta$-uniformly discrete.

Now we prove Lemmas [13]. Let $B(z, r) = \{\zeta \in \mathbb{C} \mid |\zeta - z| < r\}$, $z \in \mathbb{C}$, $r > 0$, be an open disk.

**Proof of Lemma [1].** If $\Gamma_{\alpha, \theta}$ is asymptotically $\beta$-relatively dense, then there exist $r > 0$ and $m_0 \in \mathbb{Z}_{>0}$ such that the region $\{\zeta \in \mathbb{C} \mid \zeta \geq m_0 + rm_0^{\alpha\beta}\}$ is covered by the family of disks $\{B(n\alpha e^{2\pi n\theta}, rn^{\alpha\beta}) \mid n \in \mathbb{Z}, n \geq m_0\}$.

First suppose that $0 \leq \beta < 1$. Then we have

$$\sum_{m_0 < n \leq m} \pi(rn^{\alpha\beta})^2 \geq \pi(m^{\alpha} - rm^{\alpha\beta})^2 - \pi(m_0^{\alpha} + rm_0^{\alpha\beta})^2$$
for any $m > m_0$, so
\[(m - m_0)\pi(r m^{\alpha \beta})^2 \geq \pi(m^\alpha - r m^{\alpha \beta})^2 - \pi(m_0^\alpha + r m_0^{\alpha \beta})^2\]
By taking $m \to \infty$, we obtain $1 + 2\alpha \beta \geq 2\alpha$.

If $-\infty < \beta < 0$, we have
\[\sum_{m < n \leq m_0} \pi(r n^{\alpha \beta})^2 \geq \pi((2m)^\alpha - r(2m)^{\alpha \beta})^2 - \pi(m_0^\alpha + r m_0^{\alpha \beta})^2\]
for any $m \geq m_0$. We may assume $m$ is so large that $(2m)^\alpha - r(2m)^{\alpha \beta} \geq (\frac{3}{2})^\alpha(m_0^\alpha + r m_0^{\alpha \beta})$. Then
\[m\pi(r m^{\alpha \beta})^2 \geq \pi\left(\left(\frac{3}{2}\right)^\alpha - 1\right)(m_0^\alpha + r m_0^{\alpha \beta})^2\]
By taking $m \to \infty$, we obtain $1 + 2\alpha \beta \geq 2\alpha$.

Proof of Lemma 2. If $\Gamma_{\alpha, \theta}$ is asymptotically $\beta$-uniformly discrete, then there exist $r > 0$ and $m_0 \in \mathbb{Z}_{>0}$ such that the family of disks $\{B(n^\alpha e^{2\pi n \theta \sqrt{-1}}, r n^{\alpha \beta}) \mid n \in \mathbb{Z}, n > m_0\}$, is distinct. We assume that $m_0$ is sufficiently large and $m_0^\alpha - r m_0^{\alpha \beta} > 0$.

If $0 \leq \beta < 1$, we have
\[\sum_{m < n \leq m_0} \pi(r n^{\alpha \beta})^2 \leq \pi((2m)^\alpha + r(2m)^{\alpha \beta})^2 - \pi(m_0^\alpha + r m_0^{\alpha \beta})^2\]
for any $m \geq m_0$. So $m\pi(r m^{\alpha \beta})^2 \leq \pi((2m)^\alpha + r(2m)^{\alpha \beta})^2$. By taking $m \to \infty$, we obtain $1 + 2\alpha \beta \leq 2\alpha$.

Next suppose that $-\infty < \beta \leq 0$. For any $m > m_0$, we have
\[\sum_{m_0 < n \leq m} \pi(r n^{\alpha \beta})^2 \leq \pi(m_0^\alpha + r m_0^{\alpha \beta})^2 - \pi(m_0^\alpha - r m_0^{\alpha \beta})^2\]
so
\[(m - m_0)\pi(r m^{\alpha \beta})^2 \leq \pi(m_0^\alpha + r m_0^{\alpha \beta})^2\]
By taking $m \to \infty$, we have $1 + 2\alpha \beta \leq 2\alpha$.

Proof of Lemma 3. Suppose that $\theta = p/q$ is an irreducible fraction. Then for any $r > 0$ and any $t > t_0 := r^{1/(1-\beta)}$, we have $rt^\beta < t$ and
\[B\left(te^{\pi \sqrt{-1}/q}, rt^\beta \sin \frac{\pi}{q}\right) \cap \Gamma_{\alpha, \theta} \subset B\left(te^{\pi \sqrt{-1}/q}, t \sin \frac{\pi}{q}\right) \cap \{se^{2\pi k \sqrt{-1}/q} \mid s \geq 0, k \in \mathbb{Z}\} = \emptyset.\]
This implies that $\Gamma_{\alpha,\theta}$ is not asymptotically $\beta$-relatively dense. Let $z_n = n^{\alpha}e^{2\pi i \theta \sqrt{-1}}$. We have
\[
\frac{|z_{(j+1)q} - z_{jq}|}{|z_{jq}|^{\beta}} = \frac{(j + 1)^{\alpha}q^{\alpha} - (jq)^{\alpha}}{(jq)^{\alpha - \frac{1}{2}}} = \sqrt{jq} \left( \left(1 + \frac{1}{j}\right)^{\alpha} - 1 \right) \to 0
\]
as $j \to \infty$. This implies that $\Gamma_{\alpha,\theta}$ is not asymptotically $\beta$-uniformly discrete. 

3 Continued fractions and rational approximations

This section prepares some properties of continued fractions and rational approximations. A fraction $\frac{a}{q} \in \mathbb{Q}$ always assumes that $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$. A pair of fractions $\frac{a}{m}, \frac{b}{n}$ is called a Farey pair if $mb - na = 1$. An open interval $(\frac{a}{m}, \frac{b}{n})$ is called a Farey interval of if its endpoints $\frac{a}{m}, \frac{b}{n}$ are a Farey pair.

Let
\[
x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = [a_0, a_1, a_2, \ldots], \quad a_0 \in \mathbb{Z}, \quad a_i \in \mathbb{Z}_{>0}, \quad i \in \mathbb{Z}_{>0}
\]
be a continued fraction expansion of $x \in \mathbb{R}$. An irrational $x$ is called badly approximable if the set of the partial quotients $\{a_i \mid i \in \mathbb{Z}_{>0}\}$ is bounded. Define the sequences $\{p_i\}_{i \geq -1}$ and $\{q_i\}_{i \geq -1}$ by $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0a_1 + 1$, $q_1 = a_1$, and $p_{i+1} = a_{i+1}p_i + p_{i-1}$, $q_{i+1} = a_{i+1}q_i + q_{i-1}$, $i \geq 1$. Let $p_{i,k} = kp_i + p_{i-1}$, $q_{i,k} = kq_i + q_{i-1}$ for $i \geq 0$, $0 \leq k \leq a_{i+1}$. Note that $p_{i,0} = p_{i-1}$, $q_{i,0} = q_{i-1}$, $p_{i,a_{i+1}} = p_{i+1}$, $q_{i,a_{i+1}} = q_{i+1}$. The fraction $p_{i}/q_{i} = [a_0, a_1, \ldots, a_i]$, $i \geq 0$, is called a (principal) convergent of $x$, and $p_{i,k}/q_{i,k} = [a_0, a_1, \ldots, a_i, k]$, $i \geq 0$, $0 < k < a_{i+1}$, is called an intermediate convergent of $x$. An induction shows that if $i$ is odd and $0 \leq k \leq a_{i+1}$, then $(\frac{p_{i,k}}{q_{i,k}}, \frac{p_{i}}{q_{i}})$ is a Farey interval and $\frac{p_{i,k}}{q_{i,k}} < x < \frac{p_{i}}{q_{i}}$. If $i$ is even, $0 \leq k \leq a_{i+1}$ and $(i, k) \neq (0, 0)$, then $(\frac{p_{i,k}}{q_{i,k}}, \frac{p_{i}}{q_{i}})$ is a Farey interval and $\frac{p_{i}}{q_{i}} < x < \frac{p_{i,k}}{q_{i,k}}$.

Lemma 15. Let $(\frac{a}{m}, \frac{b}{n})$ be a Farey interval containing $x$. Then we have either $(\frac{a}{m}, \frac{b}{n}) = (\frac{p_{i,k}}{q_{i,k}}, \frac{p_{i}}{q_{i}})$ for some $i$ odd and $0 \leq k < a_{i+1}$, or $(\frac{a}{m}, \frac{b}{n}) = (\frac{p_{i,k}}{q_{i,k}}, \frac{p_{i}}{q_{i}})$ for some $i$ even and $0 \leq k < a_{i+1}$.

Proof. See [3] Lemma 4.

Lemma 16. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. In the continued fraction expansion of $\theta$, the following inequalities hold.

1. $q_{i,k}|q_{i,k}\theta - p_{i,k}| > \frac{1}{2\pi q_{i,k}}$ for $i \geq 0$, $0 \leq k < a_{i+1}$.
2. \( q_{i,k} |q_{i,k} \theta - p_{i,k}| < k + 1 \) for \( i \geq 0, 0 \leq k \leq a_{i+1} \).

3. \( q_i |q_i \theta - p_i| < \frac{1}{a_{i+1}} \) for \( i \geq 0 \).

4. If \( i \geq 1 \) and \( k = \lfloor \frac{1}{2} a_{i+1} \rfloor \), then \( q_{i,k} |q_{i,k} \theta - p_{i,k}| > \frac{a_{i+1} - 1}{4} \).

**Proof.**

1. \( |\theta - \frac{p_{i,k}}{q_{i,k}}| > |\frac{p_{i+1}}{q_{i+1}} - \frac{p_{i,k}}{q_{i,k}}| = \frac{a_{i+1} - k}{q_{i+1}q_{i,k}} \). So \( q_{i,k} |q_{i,k} \theta - p_{i,k}| > \frac{q_{i,k}(a_{i+1} - k)}{q_{i+1}} = \frac{(kq_i + q_{i+1})(a_{i+1} - k)}{a_{i+1}q_i + q_{i+1}} \geq \frac{1}{q_{i+1}} \) (\( a_{i+1} \)).

2. \( |\theta - \frac{p_{i,k}}{q_{i,k}}| < |\frac{p_i}{q_i} - \frac{p_{i,k}}{q_{i,k}}| = \frac{1}{q_iq_{i,k}} \). So \( q_i |q_i \theta - p_i| < \frac{q_i}{q_{i+1}} < \frac{1}{a_{i+1}} \).

3. \( \frac{kq_i + q_{i+1}}{a_{i+1}q_i + q_{i+1}} > \frac{a_{i+1} + 1}{q_{i+1}q_{i,k}} \). So \( q_{i,k} |q_{i,k} \theta - p_{i,k}| > \frac{q_{i,k}(a_{i+1} - k)}{q_{i+1}} > \frac{k(a_{i+1} - k)}{a_{i+1}} \).

4. If \( a_{i+1} \) is even, we have \( k = \frac{a_{i+1}}{2} \), and \( \frac{k(a_{i+1} - k)}{a_{i+1}} = \frac{a_{i+1} - 1}{4} \). If \( a_{i+1} \) is odd, we have \( k = \frac{a_{i+1} - 1}{2} \), and \( \frac{k(a_{i+1} - k)}{a_{i+1}} > \frac{a_{i+1} - 1}{4} \). \( \square \)

4 Delone families of linear lattices

This section considers relative denseness and uniform discreteness of a family of (linear) lattices, and proves Proposition 5.

Let \( T \subset \mathbb{R} \) be a parameter set. A family of point sets \( \{ \Lambda_t \subset \mathbb{R}^2 \mid t \in T \} \) is called relatively dense if there exists \( r > 0 \) such that for any \( t \in T \) and any \( \zeta \in \mathbb{R}^2 \), we have \( B(\zeta, r) \cap \Lambda_t \neq \emptyset \). The family \( \{ \Lambda_t \}_{t \in T} \) is called uniformly discrete if there exists \( s > 0 \) such that for any \( t \in T \) and any \( \zeta \in \mathbb{R}^2 \) we have \( \#(B(\zeta, s) \cap \Lambda_t) \leq 1 \). The family \( \{ \Lambda_t \} \) is called a Delone family if it is relatively dense and uniformly discrete.

Let \( \theta \in \mathbb{R} \), \( z = (\theta, 1) \in \mathbb{R}^2 \). Consider the continued fraction expansion of \( \theta \), as in the previous section. Let \( z_i = (q_i, \theta - p_i, q_i) \) for \( i \geq -1 \), and \( z_{i,k} = (q_{i,k} \theta - p_{i,k}, q_{i,k}) \) for \( i \geq 0, 0 \leq k \leq a_{i+1} \). We have \( z_{-1} = z_0 = (-1, 0), z_0 = (\theta - a_0, 1), z_{0,1} = (\theta - a_0 - 1, 1), z_{0,a_1} = z_1 = (q_1 \theta - p_1, q_1) \). Let \( \Lambda = z_1 \mathbb{Z} + z_0 \mathbb{Z} = \{(m \theta - n, m) \mid m, n \in \mathbb{Z} \} \) a lattice. Let \( T = \{ t \in \mathbb{R} \mid t \geq 1 \} \) from now on. Let \( A(t) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix} \) be a \( 2 \times 2 \) matrix, and

\[
\Lambda(t) := A(\sqrt{t}) \Lambda = \left\{ \left( (m \theta - n) \sqrt{t}, \frac{m}{\sqrt{t}} \right) \mid m, n \in \mathbb{Z} \right\}.
\]
Let
\[ t_i = \frac{q_i}{|q_i \theta - p_i|}, \quad i \geq -1, \]
\[ t_{i,k} = \frac{q_{i,k}}{|q_{i,k} \theta - p_{i,k}|}, \quad i \geq 0, \quad 0 \leq k \leq a_{i+1}. \]

We have \( t_{-1} = t_{0,0} = 0, \) \( t_0 = \frac{1}{a_{-1}}, \) \( t_{0,1} = \frac{1}{a_{-1}|a_0|}, \) \( t_{0,a_1} = 1 = \frac{1}{|a_0/a_1|}, \) and
\[ t_{i-1} < t_i, \]
\[ t_{i,0} = t_{i-1} < t_{i,1} < \ldots < t_{i,a_{i+1}} = t_{i+1}, \]
\[ \sqrt{t_{i-1}t_i} = \sqrt{t_{i,0}t_{i,1}} < \ldots < \sqrt{t_{i,a_{i+1}}t_{i+1}} = \sqrt{t_{i+1}t_{i+2}} \]
for \( i \geq 0. \)

Denote by
\[ z_i(t) := A(t)z_i = ((q_i \theta - p_i)\sqrt{t}, q_i/\sqrt{t}), \]
\[ z_{i,k}(t) := A(t)z_{i,k} = ((q_{i,k} \theta - p_{i,k})\sqrt{t}, q_{i,k}/\sqrt{t}). \]

We have
\[ z_i(t_i) = ((-1)^i \sqrt{q_i|q_i \theta - p_i|}, \sqrt{q_i|q_i \theta - p_i|}), \]
\[ z_{i,k}(t_{i,k}) = ((-1)^{i+1} \sqrt{q_{i,k}|q_{i,k} \theta - p_{i,k}|}, \sqrt{q_{i,k}|q_{i,k} \theta - p_{i,k}|}). \]

**Lemma 17** (Richards’ formula). Let \( i \geq 0, \) \( 0 \leq k \leq a_{i+1}. \) Let
\[ t := \sqrt{t_{i,k}} = \sqrt{\frac{q_{i,k}}{|q_{i,k} \theta - p_{i,k}|}}. \]
Then the parallelogram \( Q = \square(0, z_i(t), z_i(t) + z_{i,k}(t), z_{i,k}(t)) \) is a rectangle.

**Proof.** \( z_i(t) = ((q_i \theta - p_i)\sqrt{t}, q_i/\sqrt{t}), \) \( z_{i,k}(t) = ((q_{i,k} \theta - p_{i,k})\sqrt{t}, q_{i,k}/\sqrt{t}), \) and we have
\[ (q_i \theta - p_i)\sqrt{t} \cdot (q_{i,k} \theta - p_{i,k})\sqrt{t} + q_i \cdot \frac{q_{i,k}}{\sqrt{t}} = 0 \]
when \( t = \sqrt{t_{i,k}}. \)

**Lemma 18.** If \( \{A(t)\}_{t \geq 1} \) is relatively dense, then \( \theta \) is badly approximable.
Proof. Suppose that $\sup_i a_i = \infty$. Let $i \geq 1$, $k = \lceil \frac{a_i+1}{2} \rceil$, $r_i := \sqrt{q_{i,k}q_{i,k}\theta - p_{i,k}}$. If $i$ is odd, let $Q(i) = [0, r_i] \times [0, r_i]$ be a square. If $i$ is even, let $Q(i) = [-r_i, 0] \times [0, r_i]$. In either case, we have $\Lambda(t_{i,k}) \cap Q(i) = \{0, z_{i,k}(t_{i,k})\}$. If $i$ is odd, we have $Q(i) \supset B(r_i(1 + \sqrt{-1})/2, r_i/2)$ and $B(r_i(1 - \sqrt{-1})/2, r_i/2) \cap \Lambda(t_{i,k}) = \emptyset$. If $i$ is even, we have $Q(i) \supset B(r_i(-1 + \sqrt{-1})/2, r_i/2)$ and $B(r_i(-1 - \sqrt{-1})/2, r_i/2) \cap \Lambda(t_{i,k}) = \emptyset$. By Lemma 16, we have $\sup_i r_i \geq \sup_i \sqrt{a_{i+1} - 1}/2 = \infty$, so $\{\Lambda(t)\}_{t \geq 1}$ is not relatively dense. \qed

Lemma 19. If $\{\Lambda(t)\}_{t \geq 1}$ is uniformly discrete, then $\theta$ is badly approximable.

Proof. Suppose that $\sup_i a_i = \infty$. We have $\text{dist}(z_i(t_i), 0) = |z_i(t_i)| = \sqrt{2q_i |q_i \theta - p_i|} < \sqrt{2/a_{i+1}}$, so $\inf_i \text{dist}(z_i(t_i), 0) = 0$, and the family $\{\Lambda(t)\}_{t \geq 1}$ is not uniformly discrete. \qed

Lemma 20. If $\theta$ is badly approximable, then $\{\Lambda(t)\}_{t \geq 1}$ is uniformly discrete.

Proof. Suppose that $\sup_i a_i = M < \infty$. We have

$$|z_{i,k}(t)|^2 = t(q_{i,k}\theta - p_{i,k})^2 + q_{i,k}^2/t$$

$$\geq 2q_{i,k}|q_{i,k}\theta - p_{i,k}| \geq \frac{2}{2 + a_i} \geq \frac{2}{2 + M}.$$ 

This implies that $\text{dist}(\lambda, 0) = |\lambda| \geq \sqrt{2/2+M}$ for any $t \geq 1$, $\lambda \in \Lambda(t)$ \setminus \{0\}, and that $\text{dist}(\lambda, \lambda') = |\lambda - \lambda'| \geq \sqrt{2/2+M}$ for any distinct $\lambda, \lambda' \in \Lambda(t)$. So $\{\Lambda(t)\}_{t \geq 1}$ is uniformly discrete. \qed

Lemma 21. If $\theta$ is badly approximable, then $\{\Lambda(t)\}_{t \geq 1}$ is relatively dense.

Proof. Let $M = \sup_i a_i < \infty$. Fix $t \geq 1$. There exist $i \geq -1$ such that $t_i \leq t \leq t_{i+1}$. Let 

$$Q = \Box(0, z_i(t), z_i(t) + z_{i+1}(t), z_{i+1}(t))$$

be a closed parallelogram. The family of translated parallelograms $\lambda + Q$, $\lambda \in \Lambda(t)$, covers the plane, $\bigcup_{\lambda \in \Lambda(t)} (\lambda + Q) = \mathbb{C}$. For any $\zeta \in \mathbb{R}^2$, there exists $\lambda \in \Lambda(t)$ such that $\zeta \in \lambda + Q$. Let 

$$Q' = \left[-|\langle q_i, \theta \rangle|\sqrt{t}, |\langle q_i, \theta \rangle|\sqrt{t}\right] \times \left[0, \frac{q_i + q_{i+1}}{\sqrt{t}}\right] \supset Q$$

be a rectangle. Since $\zeta \in \lambda + Q'$, we have 

$$|\zeta - \lambda| \leq |\langle q_i, \theta \rangle|\sqrt{t} + \frac{q_i + q_{i+1}}{\sqrt{t}},$$
where
\[ |q_i \theta - p_i| \sqrt{t} \leq |q_i \theta - p_i| \sqrt{t_{i+1}} = \frac{(q_{i+1}(q_i \theta - p_i))^2}{|q_{i+1} \theta - p_{i+1}|} \leq a_{i+2} + 1 \leq \sqrt{M+1}, \]
\[ \frac{q_i + q_{i+1}}{\sqrt{t}} \leq \frac{q_i + q_{i+1}}{\sqrt{t_i}} = \frac{(q_i + q_{i+1})^2|q_i \theta - p_i|}{q_i} \leq \frac{(M+2)^2}{M+1} \leq 2\sqrt{M+1}. \]

Thus, for any \( \zeta \in \mathbb{R}^2 \), there exist \( \lambda \in \Lambda(t) \) such that \( |\zeta - \lambda| \leq 3\sqrt{M+1} \). So \( \{\Lambda(t)\}_{t \geq 1} \) is relatively dense. \( \square \)

**Proof of Proposition 5.** The proof is given by Lemmas 18-21.

5 Spiral Delone set

This section proves Theorem 4. Let \( \alpha > 0, -\infty < \beta = 1 - \frac{1}{2\alpha} < 1 \). Let \( \Gamma_{\alpha, \theta} = \{F(n) \mid n \in \mathbb{Z}_{>0}\} \), where \( F(n) := n^{\alpha}e^{2\pi n \theta \sqrt{-1}} \).

**Lemma 22.** Let \( c > 0, 0 < \alpha < 1 \). If \( 0 < x \leq 1 \), then we have
\[ (1 + cx)^\alpha - 1 \geq x((1 + c)^\alpha - 1). \]

**Proof.** Let \( f(x) = (1 + cx)^\alpha \). Since the function \(-f(x)\) is convex, we have
\[ \frac{f(x) - f(0)}{x - 0} = \frac{(1 + cx)^\alpha - 1}{x} \geq \frac{f(1) - f(0)}{1 - 0} = (1 + c)^\alpha - 1 \]
for \( 0 < x \leq 1 \). \( \square \)

**Lemma 23.** If \( 0 < \alpha < 1 \) and \( 0 < x \leq \frac{1}{1 - \alpha} \), then \( (1 + x)^\alpha \geq 1 + \frac{\alpha x}{2} \).

**Proof.** By Taylor’s theorem, there exists \( 0 < \xi < x \) such that \( (1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2}(1 + \xi)^{\alpha - 2}x^2 \). So we have
\[ (1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2}(1 + \xi)^{\alpha - 2}x^2 \]
\[ > 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2}x^2 \]
\[ \geq 1 + \frac{\alpha x}{2} \]
for \( x \leq 1/(1 - \alpha) \). \( \square \)
Lemma 24. Suppose that \( \theta \) is badly approximable. Then \( \Gamma_{\alpha, \theta} \) is asymptotically \( \beta \)-uniformly discrete.

**Proof.** Suppose that \( \sup_i a_i = M < \infty \). By Lemma 16, we have \( \frac{q_{i,k} \theta - p_{i,k}}{\nu} > \frac{1}{2 + a_i} \geq \frac{1}{2 + M} \) for \( i \geq 0, 0 \leq k < a_{i+1} \). This implies that \( \frac{q}{\nu} \cdot |q \theta - p| \sqrt{\nu} > \frac{1}{2 + M} \) for any \( \nu, q \in \mathbb{Z}_{>0} \) and \( p \in \mathbb{Z} \).

Fix \( \nu, q \in \mathbb{Z}_{>0} \). Then we have either

\[
|\langle q \theta \rangle| \sqrt{\nu} \geq \frac{1}{\sqrt{2 + M}} \tag{6}
\]

or

\[
\frac{q}{\sqrt{\nu}} \geq \frac{1}{\sqrt{2 + M}}. \tag{7}
\]

Suppose first that (6) holds. We have

\[
|F(\nu + q) - F(\nu)| \geq \nu^{\alpha} |e^{2\pi q \theta \sqrt{\nu} - 1}| \geq 2\nu^{\alpha} \sin |\langle q \theta \rangle| \pi \geq 4\nu^{\alpha} |\langle q \theta \rangle| \geq |F(\nu)|^\beta \frac{4}{\sqrt{2 + M}}.
\]

Next suppose that (7) holds. If \( \alpha \geq 1 \), we have

\[
|F(\nu + q) - F(\nu)| \geq (\nu + q)^{\alpha} - \nu^{\alpha} = |F(\nu)|^\beta \sqrt{\nu} \left( (1 + \frac{q}{\nu})^{\alpha} - 1 \right) \geq |F(\nu)|^\beta \sqrt{\nu} \left( 1 + \frac{q}{\nu} - 1 \right) = |F(\nu)|^\beta \cdot \frac{q}{\sqrt{\nu}} \geq |F(\nu)|^\beta \frac{1}{\sqrt{2 + M}}.
\]

If \( 0 < \alpha < 1 \), we have

\[
|F(\nu + q) - F(\nu)| \geq (\nu + q)^{\alpha} - \nu^{\alpha} = \nu^{\alpha} \left( (1 + \frac{q}{\nu})^{\alpha} - 1 \right) \geq \nu^{\alpha} \left( 1 + \frac{1}{\sqrt{\nu(2 + M)}} \right)^{\alpha} - 1 \geq \nu^{\alpha - \frac{1}{2}} \frac{2^{\alpha} - 1}{\sqrt{2 + M}} \geq |F(\nu)|^\beta \frac{2^{\alpha} - 1}{\sqrt{2 + M}}
\]

by Lemma 22. Thus we obtain

\[
|F(\nu + q) - F(\nu)| \geq |F(\nu)|^\beta \min\{1, 2^{\alpha} - 1\} \frac{\nu^{\alpha - \frac{1}{2}}}{\sqrt{2 + M}}.
\]

for any \( \nu, q \in \mathbb{Z}_{>0} \). This implies that \( \Gamma_{\alpha, \theta} \) is \( \beta \)-uniformly discrete. \( \Box \)
**Lemma 25.** If $\Gamma_{\alpha,\theta}$ is asymptotically $\beta$-uniformly discrete, then $\theta$ is badly approximable.

**Proof.** Let $\nu_i := \lceil t_i \rceil \in \mathbb{Z}$. We have

$$
|F(\nu_i + q_i) - F(\nu_i)| \leq |F(\nu_i + q_i) - \nu_i^\alpha e^{2\pi \theta (\nu_i + q_i) \sqrt{-1}}| + |\nu_i^\alpha e^{2\pi \theta (\nu_i + q_i) \sqrt{-1}} - F(\nu_i)|
$$

$$
\leq ((\nu_i + q_i)^\alpha - \nu_i^\alpha) + \nu_i^\alpha 2\pi |\langle q\theta \rangle|
$$

$$
= |F(\nu_i)|^\beta \left( \sqrt{\nu_i((1 + \frac{q_i}{\nu_i})^\alpha - 1)} + 2\pi |\langle q\theta \rangle| \sqrt{\nu_i} \right)
$$

where

$$
|\langle q\theta \rangle|\sqrt{\nu_i} < |\langle q\theta \rangle|\sqrt{t_i + 1} < |\langle q\theta \rangle|\sqrt{2t_i} = 2\sqrt{\nu_i |q_i \theta - p_i|} < \frac{2}{a_{i+1}}.
$$

If $i$ is large, we may assume that $\frac{q_i}{\nu_i} \leq \frac{q_i}{t_i} = |q_i \theta - p_i| \leq \frac{1}{\alpha^2}$, which implies

$$
\sqrt{\nu_i((1 + \frac{q_i}{\nu_i})^\alpha - 1)} \leq \sqrt{\nu_i(1 + 2\frac{q_i}{\nu_i} - 1)} \leq \frac{2q_i}{\sqrt{t_i}} = 2\sqrt{\nu_i |q_i \theta - p_i|} < \frac{2}{a_{i+1}}.
$$

Thus we obtain

$$
|F(\nu_i + q_i) - F(\nu_i)| \leq \frac{2 + \sqrt{2}}{a_{i+1}} |F(\nu_i)|^\beta
$$

for $i$ large. Since $\Gamma_{\alpha,\theta}$ is $\beta$-uniformly discrete, we have $\sup_i a_i < +\infty$. 

**Lemma 26.** Suppose that $\theta$ is badly approximable. Then $\Gamma_{\alpha,\theta}$ is asymptotically $\beta$-relatively dense.

**Proof.** Consider the polar coordinates of the plane, $\varphi : \mathbb{R}_{>0} \times \mathbb{R} \rightarrow \mathbb{C}$, $\varphi(s, t) = s^\alpha e^{2\pi t \sqrt{-1}}$. We have $\varphi(k, k\theta) = F(k)$ and $\varphi(s, t + k) = \varphi(s, t)$ for $k \in \mathbb{Z}$. Let $\Lambda = (1, \theta)\mathbb{Z} + (0, 1)\mathbb{Z}$ be a linear lattice. We have $\varphi(\Lambda) = \Gamma_{\alpha,\theta}$.

Suppose $i_0$ is sufficiently large that

$$
q_{i_0} \geq 2(1 + \alpha^2), \quad q_{i_0} \geq 3(2 + M). \quad (8)
$$

Let $\zeta \in \mathbb{C}$. We are going to show that if $|\zeta| \geq t_{i_0}$, then there exists $\nu \in \mathbb{Z}_{>0}$ such that

$$
|\zeta - F(\nu)| \leq |F(\nu)|^\beta (2\alpha \sqrt{2(M + 2)} + 2\pi \sqrt{M + 1}). \quad (9)
$$
If $|\zeta| \geq t_{i_0}$, there exists $i \geq i_0$ such that $t_i \leq |\zeta| < t_{i+1}$. Let $T = \square((0, 0), (q_i, (q_i\theta)), (q_i + q_{i+1}, (q_i + q_{i+1}\theta)), (q_{i+1}, (q_{i+1}\theta)))$

be a parallelogram. There exists $\nu \in \mathbb{Z}_{>0}$ such that $\zeta \in \varphi((\nu, (\nu\theta)) + T)$. We have $t_i - (q_i + q_{i+1}) \leq \nu \leq t_{i+1}$. There exist $s, t \in \mathbb{R}$ such that $\zeta = (\nu + s)^\alpha e^{2\pi((\nu\theta) + t)\sqrt{-1}}$, $0 \leq s \leq q_i + q_{i+1}$, $|t| \leq |(\nu\theta)|$. We have

$$\begin{align*}
|\zeta - F(\nu)| \\
= |(\nu + s)^\alpha e^{2\pi((\nu\theta) + t)\sqrt{-1}} - \nu^\alpha e^{2\pi\nu\theta\sqrt{-1}}| \\
\leq |(\nu + s)^\alpha e^{2\pi((\nu\theta) + t)\sqrt{-1}} - \nu^\alpha e^{2\pi\nu\theta\sqrt{-1}}| + |\nu^\alpha e^{2\pi(\nu\theta)\sqrt{-1}} - \nu^\alpha e^{2\pi\nu\theta\sqrt{-1}}| \\
\leq (\nu + s)^\alpha - \nu^\alpha + \nu^\alpha 2\pi |t| \\
= |F(\nu)|^\beta \left( \sqrt{\nu} \left( \left(1 + \frac{s}{\nu}\right)^\alpha - 1 \right) + 2\pi |t| \sqrt{\nu} \right).
\end{align*}$$

We have

$$
2\pi |t| \sqrt{\nu} \leq 2\pi |q_i\theta - p_i| \sqrt{t_{i+1}} = 2\pi \sqrt{\frac{q_{i+1}(q_i\theta - p_i)^2}{|q_{i+1}\theta - p_{i+1}|}} \\
\leq 2\pi \sqrt{a_{i+2} + 1} \\
\leq 2\pi \sqrt{M + 1}.
$$

By (8), we have

$$\frac{\nu}{s} + 1 \geq \frac{t_i - (q_i + q_{i+1})}{q_i + q_{i+1}} + 1 = \frac{t_i}{q_i + q_{i+1}} \\
= \frac{q_i}{(q_i + q_{i+1})|q_i\theta - p_i|} \\
\geq \frac{q_i q_{i+1}}{q_i + q_{i+1}} \geq \frac{q_i}{2} \geq 1 + \alpha^2,
$$

so $0 < \frac{s}{\nu} \leq \frac{1}{\alpha^2}$. By Lemma 8, we have

$$
\sqrt{\nu} \left( \left(1 + \frac{s}{\nu}\right)^\alpha - 1 \right) \leq \sqrt{\nu} \left( 1 + \frac{2\alpha s}{\nu} - 1 \right) = \frac{2\alpha s}{\sqrt{\nu}}.
$$
We have
\[
\frac{\sqrt{t_i}}{s} \geq \frac{\sqrt{t_i - (q_i + q_{i+1})}}{q_i + q_{i+1}} = \frac{\sqrt{\frac{q_i q_{i+1}}{(q_i + q_{i+1})^2} - \frac{1}{q_i + q_{i+1}}}}{q_i + q_{i+1}}
\]
\[
\geq \sqrt{\frac{M + 1}{(M + 2)^2} - \frac{1}{2q_i}}
\]
\[
\geq \sqrt{\frac{M + 1}{(M + 2)^2} - \frac{1}{6(M + 2)}} = \sqrt{\frac{5M + 4}{6(M + 2)^2}}
\]
\[
= \frac{1}{\sqrt{2(M + 2)}}.
\]
Thus we obtain (9). \(\square\)

**Lemma 27.** If \(\Gamma_{\alpha,\theta}\) is asymptotically \(\beta\)-relatively dense, then \(\theta\) is badly approximable.

**Proof.** Suppose that \(\sup_i a_i = \infty\). Let \(i \geq 0, k = \lfloor \frac{a_i+1}{2} \rfloor, \nu_i := \lfloor t_{i,k} \rfloor\). Let

\[
D := \{xe^{2\pi y\sqrt{-1}} | \nu_i^\alpha \leq x \leq (\nu_i + q_i,k)\alpha, 0 \leq y - \nu_i \theta \leq q_i,k \theta - p_i,k\}
\]
if \(i\) is odd, or

\[
D := \{xe^{2\pi y\sqrt{-1}} | \nu_i^\alpha \leq x \leq (\nu_i + q_i,k)\alpha, 0 \geq y - \nu_i \theta \geq q_i,k \theta - p_i,k\}
\]
if \(i\) is even. In either case we have \(D \cap \Gamma_{\alpha,\theta} = \{F(\nu_i), F(\nu_i + q_i,k)\}\), and \(D \supset B(\xi_i, R_i)\), where

\[
\xi_i := \nu_i^\alpha + \frac{(\nu_i + q_i,k)\alpha}{2} e^{2\pi(\nu_i \theta + (q_i,k \theta - p_i,k)/2)\sqrt{-1}},
\]

\[
R_i := \min \left\{ \frac{(\nu_i + q_i,k)\alpha - \nu_i^\alpha}{2}, |\xi_i| \sin |\pi(q_i,k \theta - p_i,k)| \right\}.
\]
So we have

\[
|\xi_i - F(m)| \geq |\xi_i|^\beta r_i
\]
for any \(m \in \mathbb{Z}_{>0}\), where \(r_i := R_i|\xi_i|^{-\beta}\). If \(\Gamma_{\alpha,\theta}\) is \(\beta\)-relatively dense, then \(\sup_i r_i < +\infty\).

Since \(|\xi_i|^{1/\alpha} > \nu \geq t_{i,k}\), we have

\[
|\xi_i| \sin |\pi(q_i,k \theta - p_i,k)| \geq 2|\xi_i||q_i,k \theta - p_i,k|
\]
\[
\geq 2|\xi_i|^\beta \sqrt{t_{i,k}|q_i,k \theta - p_i,k|}
\]
\[
= 2|\xi_i|^\beta \sqrt{q_i,k|q_i,k \theta - p_i,k|}
\]
\[
\geq |\xi_i|^\beta \sqrt{a_i+1 - 1}.
\]

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We may assume that $i$ is so large that 
\[
\frac{q_{i,k}}{t_{i,k}} = |q_{i,k}\theta - p_{i,k}| \leq 3^{1/\alpha} - 1 \text{ holds, which imply that}
\]
\[
1 < \frac{|\xi_i|}{\nu^\alpha} = \frac{1}{2} \left(1 + \left(1 + \frac{q_{i,k}}{\nu}\right)^\alpha\right) \leq \frac{1}{2} \left(1 + \left(1 + \frac{q_{i,k}}{t_{i,k}}\right)^\alpha\right) \leq 2.
\]
If $\alpha \geq 1$, we have
\[
(\nu + q_{i,k})^\alpha - \nu^\alpha = \nu^\alpha((1 + \frac{q_{i,k}}{\nu})^\alpha - 1)
\geq \frac{1}{2}|\xi_i|(1 + \frac{\alpha q_{i,k}}{\nu} - 1)
> \frac{1}{2}|\xi_i|\frac{\alpha q_{i,k}}{\sqrt{\nu}} > \frac{1}{2}|\xi_i|\frac{\alpha q_{i,k}}{\sqrt{1 + \nu_{i,k}}}
\geq \frac{\alpha}{2\sqrt{2}}|\xi_i|^\beta \frac{q_{i,k}|q_{i,k}\theta - p_{i,k}|}{\sqrt{2\nu_{i,k}}}
> \frac{\alpha}{4\sqrt{2}}|\xi_i|^\beta \frac{q_{i,k}|q_{i,k}\theta - p_{i,k}|}{\sqrt{2\nu_i}}
\geq \frac{\alpha}{8\sqrt{2}}|\xi_i|^\beta \frac{\sqrt{2}}{\sqrt{a_{i+1} - 1}}.
\]
If $0 < \alpha < 1$, we further assume that \( \frac{q_{i,k}}{\nu} \leq \frac{q_{i,k}}{t_{i,k}} = |q_{i,k}\theta - p_{i,k}| \leq \frac{1}{1-\alpha} \). By Lemma 23 we have
\[
(\nu + q_{i,k})^\alpha - \nu^\alpha = \nu^\alpha((1 + \frac{q_{i,k}}{\nu})^\alpha - 1)
\geq \frac{1}{2}|\xi_i|(1 + \frac{\alpha q_{i,k}}{2\nu} - 1)
> \frac{\alpha}{4}|\xi_i|^\beta \frac{q_{i,k}}{\sqrt{\nu}} > \frac{\alpha}{4}|\xi_i|^\beta \frac{q_{i,k}}{\sqrt{2\nu_{i,k}}}
\geq \frac{\alpha}{8\sqrt{2}}|\xi_i|^\beta \frac{q_{i,k}|q_{i,k}\theta - p_{i,k}|}{\sqrt{2\nu_i}}
\geq \frac{\alpha}{16\sqrt{2}}|\xi_i|^\beta \sqrt{a_{i+1} - 1}.
\]
Thus we obtain \( r_i \geq \min\{1, \frac{\alpha}{16\sqrt{2}}\} \sqrt{a_{i+1} - 1} \). If $\Gamma_{\alpha,\theta}$ is $\beta$-relatively dense, then we have $\sup_i r_i < +\infty$ and $\sup_i a_i < +\infty$.

Proof of Theorem 4. The proof is given by Lemmas 24-27.

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