Representations of quantum groups arising from the Stokes phenomenon and applications

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Abstract

In this paper, we prove that the Stokes matrices, of certain "universal" meromorphic linear system of ordinary differential equations of Poincaré rank one, give rise to a family of representations of quantum group \( U_q(\mathfrak{gl}_n) \). We then apply the representation theory of quantum groups to get algebraic understanding of some analysis problems. In particular, motivated by the theory of canonical basis and representations at roots of unity, we give conjectural characterizations of the WKB approximation and soliton solutions of the meromorphic differential equations respectively. Along the way, we point out that the asymptotic expansion of Stokes matrices obtained in our previous work gives rise to an explicit Drinfeld isomorphism.

1 Introduction and main results

The Stokes phenomenon states that a solution of a meromorphic linear ordinary differential equation may have different asymptotic expressions as \( z \) approaches to an irregular singularity \( z_0 \) from different sectorial regions around \( z_0 \). The discontinuous jump of the asymptotics around \( z_0 \) can be measured by the Stokes matrices.

In the past decades, the Stokes phenomenon/matrices of meromorphic linear systems of differential equation with Poincaré rank one play an important role in many subjects of mathematics and physics. The aim of our project, started in this paper and \[43\], is to first develop algebraic understanding of various aspects of the Stokes phenomenon, and then apply the algebraic methods/ideas to solve difficult analysis problems in the study of Stokes phenomenon, the isomonodromy deformation equations and related fields.

Let us take the Lie algebra \( \mathfrak{gl}_n \) over the field of complex numbers, and its universal enveloping algebra \( U(\mathfrak{gl}_n) \) generated by \( \{e_{ij}\}_{1 \leq i,j \leq n} \) subject to the relation \([e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}\). Let us take the \( n \times n \) matrix \( T = (T_{ij}) \) with entries valued in \( U(\mathfrak{gl}_n) \)

\[ T_{ij} = e_{ij}, \quad \text{for } 1 \leq i,j \leq n. \]

Let \( h_{\text{reg}} \) denote the set of \( n \times n \) diagonal matrices with distinct eigenvalues. Given any finite-dimensional irreducible representation \( L(\lambda) \) of \( \mathfrak{gl}_n \) with a highest weight \( \lambda \), let us consider the linear system of differential equation

\[ \frac{dF}{dz} = h\left(u + \frac{T}{z}\right) \cdot F, \]  

for \( F(z) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \) an \( n \times n \) matrix function with entries in \( \text{End}(L(\lambda)) \). Here \( h \) is a complex parameter, \( u \in h_{\text{reg}} \) is seen as a \( n \times n \) matrix with scalar entries in \( U(\mathfrak{gl}_n) \), and the action of the coefficient matrix on \( F(z) \) is given by matrix multiplication and the representation of \( \mathfrak{gl}_n \).

1.1 Stokes matrices and representation of quantum groups

1.1.1 Nonresonant case \( h \notin \mathbb{Q} \)

Let us first assume \( h \notin \mathbb{Q} \). The equation (1) is then nonresonant and thus has a unique formal solution \( \hat{F} \) around \( z = \infty \). See Proposition [2.1] for a proof. The standard theory of resummation states that there exist certain sectorial regions around \( z = \infty \), such that on each of these sectors there is a unique (therefore canonical) holomorphic solution with the prescribed asymptotics \( \hat{F} \). These solutions are in general different (that reflects the Stokes phenomenon), and the transition between them can be measured by a pair of Stokes matrices \( S_{\pm}(u; h) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \). See Section [2] for more details.
Theorem 1.1. For any fixed $h \notin \mathbb{Q}$ and $u \in \mathfrak{h}_{\text{reg}}$, the map (with $q = e^{\pi i h}$)

$$S_q(u) : U_q(\mathfrak{g}l_n) \to \text{End}(L(\lambda)); e_i \mapsto \frac{S_+(u)_{i,i}^{-1} \cdot S_+(u)_{i,i+1}}{q - q^{-1}}, f_i \mapsto \frac{S_-(u)_{i+1,i} \cdot S_-(u)_{i,i}^{-1}}{q - q^{-1}}, q^h_i \mapsto S_+(u)_{i,i}$$ (2)

defines a representation of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g}l_n)$ on the vector space $L(\lambda)$. Here recall that $U_q(\mathfrak{g}l_n)$ is a unital associative algebra with generators $q^{\pm h_i}, e_j, f_j, 1 \leq j \leq n - 1, 1 \leq i \leq n$ and relations:

- for each $1 \leq i \leq n, 1 \leq j \leq n - 1$,
  $$q^{h_i}q^{-h_i} = q^{-h_i}q^{h_i} = 1, \quad q^{h_i}e_jq^{-h_i} = q^{h_i}q^{-h_i}q^{h_i}j\epsilon_j, \quad q^{-h_i}f_jq^{h_i} = q^{-h_i}q^{h_i}j\epsilon_jf_j;$$

- for each $1 \leq i, j \leq n - 1$,
  $$[e_i, f_j] = \delta_{ij}\frac{q^{h_i-h_i+1} - q^{-h_i+h_i+1}}{q - q^{-1}};$$

- for $|i - j| = 1$,
  $$e_i^2e_j - (q + q^{-1})e_i\epsilon_je_i + e_j\epsilon_i^2 = 0, \quad f_i^2f_j - (q + q^{-1})f_i\epsilon_if_i + f_j\epsilon_jf_i = 0,$$
  and for $|i - j| \neq 1, [e_i, e_j] = 0 = [f_i, f_j].$

The theorem associates to any representation $L(\lambda)$ of $U(\mathfrak{g}l_n)$ a representation $S_q(u)$ of $U_q(\mathfrak{g}l_n)$ on the same vector space $L(\lambda)$. And the construction of $S_q(u)$ is universal and actually gives a family of Drinfeld isomorphism parameterized by $u$. See Section 6.4 for more details.

1.1.2 Resonant case $h \in \mathbb{Q}$

Now let us assume $h = h_0 \in \mathbb{Q}$. Then the corresponding differential equation (1) becomes resonant. In this case, the uniqueness of the formal fundamental solution $\hat{F}(z)$ is not valid. Instead, the equation (1) has a family of formal solution around $z = \infty$ depending on a finite set $c$ of complex parameters. Accordingly, there are a family of Stokes matrices $S_{\pm}(u; c)$ depending on the same set of parameters. However, there is a pair of distinguished Stokes matrices, denoted by $S^\ast_{\pm}(u)$, among the family $S_{\pm}(u; c)$. They are actually the continuous extension of $S_{\pm}(u)$ from $h \in \mathbb{C} \setminus \mathbb{Q}$ to $h_0 \in \mathbb{Q}$. See Section 5 for more details.

Theorem 1.2. For any fixed $h_0 \in \mathbb{Q}$ and $u \in \mathfrak{h}_{\text{reg}}$, the map $S_{q_0}(u)$ given in (2) defines a representation of $U_{q_0}(\mathfrak{g}l_n)$ at $q_0 = e^{\pi i h_0}$ a root of unity.

1.2 Two applications in the study of Stokes phenomenon

The aim of our project was to develop algebraic understanding of various aspects of the Stokes phenomenon. By pursuing the heuristics of Theorem 1.1 and Theorem 1.2, we are led to rather interesting conjectures. In the following, let us show that how Theorem 1.1 and Theorem 1.2 indicate respectively a characterization of the Stokes phenomenon in the WKB approximation via the theory of canonical basis introduced by Lusztig and Kashiwara, and an algebraic characterization of the soliton/polynomial solutions of confluent hypergeometric type equations.

1.2.1 Nonresonant case: canonical basis and WKB analysis

In the discussion below, let us assume $h \in i\mathbb{R}$ the set of purely imaginary numbers, and $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$ the set of $n \times n$ diagonal matrices with distinct real eigenvalues.

Let $U^+$ be the subalgebra of $U_q(\mathfrak{g}l_n)$ generated by the elements $\{e_i\}$ and let $B$ be the canonical basis in $U^+$. We refer the reader to [30] for the construction of $B$. Upon acting on the lowest weight vector, the image of the canonical basis $B$, under the map $S_q(u)$ given in Theorem 1.1 defines a set $B_q(u; \lambda)$ of vectors in $L(\lambda)$. 

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Conjecture 1.3. The set $B_q(u; \lambda)$ is a basis of $L(\lambda)$ for all $q \in (0, \infty)$, whose leading asymptotics as $q \to 0$ correspond to an eigenbasis $E(u; \lambda)$ of the action of the shift of argument subalgebra $\mathcal{A}(u) \subset U(\mathfrak{gl}_n)$ on $L(\lambda)$. 

Here the shift of argument subalgebra $\mathcal{A}(u)$ of $U(\mathfrak{gl}_n)$ is a maximal commutative subalgebras parameterized by $u$, and its action on $L(\lambda)$ has simple spectral for $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$. See e.g., [15, 18] for more details. In the case $n = 2$, the conjecture can be verified directly using the closed formula of the Stokes matrices. In the case $n = 3$ the canonical basis is given explicitly in [30, Example 3.4]. Then using the method of isomonodromy deformation stated in Section 6, Conjecture 1.3 for $\mathfrak{gl}_n$ can be verified using the properties of the Painlevé VI function. However, other than the conjecture, in this paper we are more interested in the indication of the algebraic structures of canonical bases in the study of the Stokes phenomenon itself.

On the one hand, as $q = 1$, $U_q(\mathfrak{gl}_n)$ becomes the undeformed $U(\mathfrak{gl}_n)$, and the representation in (2) coincides with the given representation of $U(\mathfrak{gl}_n)$ on $L(\lambda)$. On the other hand, by specializing $q = 1$ and acting on $L(\lambda)$, the canonical basis $\mathcal{B}$ of $U^+$ recovers the canonical basis $B(\lambda)$ in $L(\lambda)$, i.e., $B_q(u; \lambda)$ coincides with $B(\lambda)$. Thus, the conjecture enables us to get a one parameter family of basis $B_q(u; \lambda)$ connecting the canonical basis $B(\lambda)$ and eigenbasis $E(u; \lambda)$ in $L(\lambda)$, by varying $q = e^{nh}$ from 1 to 0 (i.e., varying $h$ from 0 to $+i\infty$ along the imaginary axis). In this way, the eigenbasis $E(u; \lambda)$ will inherit the combinatorial structure of canonical basis, which plays a role in the study of WKB analysis as follows.

The WKB analysis, named after Wentzel, Kramers, and Brillouin, is for approximating solutions of a differential equation whose highest derivative is multiplied by a small parameter (other names, including Liouville, Green, and Jeffreys are sometimes attached to this method). In a first approximation, Conjecture 1.3 amounts to the characterization of the WKB approximation as $h \to +i\infty$ of the Stokes matrices of the equation (1). Let us give a brief discussion here.

First, the action of $S_+(u; h)_{k,k+1}$ on the eigenbasis vectors $\{v_i(u)\}_{i \in I}$ of $E(u; \lambda)$ should have the asymptotic behaviour

$$S_+(u; h)_{k,k+1} \cdot v_i(u) = \sum_{j \in I} e^{h\phi^{(k)}_{ij}(u)+i\theta^{(k)}_{ij}(u,h)}(v_j(u) + O(h^{-1})), \quad \text{as } h \to +i\infty,$$

(3)

where $\phi^{(k)}_{ij}(u)$ are real valued functions independent of $h$, and $\theta^{(k)}_{ij}(u,h)$ are real valued functions for all $1 \leq i, j \leq k \leq n - 1$. An element $v_i(u)$ of $E(u; \lambda)$ is called generic if there exists only one index $j \in I$ such that $\phi^{(k)}_{ij}(u)$ is the biggest in the collection $\{\phi^{(k)}_{ij}(u)\}_{i \in I}$ of real numbers.

Thus, the WKB approximation of $s_{k+1,k}(u; h)$ naturally defines an operator $\tilde{\epsilon}_k$ on generic elements of $B(\lambda, u)$ by picking the unique leading term in (3), i.e.,

$$\tilde{\epsilon}_k(v_i(u)) := v_j(u), \quad \text{if } \phi^{(k)}_{ij}(u) = \max\{\phi^{(k)}_{il}(u) | l \in I\}.$$

(4)

Similarly, by considering the WKB approximation of $S_-(u; h)_{k+1,k}$, one defines an operator $\tilde{f}_k$ on (some other set of) generic elements of $E(u; \lambda)$. In a universal sense, the operators $\{\tilde{\epsilon}_k, \tilde{f}_k\}_{k=1,\ldots,n-1}$ uniquely extend to the whole set $B(\lambda, u)$. See [45] for more details on a particular $u$.

In this heuristic spirit, Conjecture 1.3 predicts that $\{\tilde{\epsilon}_k, \tilde{f}_k\}_{k=1,\ldots,n-1}$ are crystal operators on the finite set $E(u; \lambda)$, i.e., the WKB approximation of the Stokes matrices is characterized by a crystal structure. To be more precise,

Conjecture 1.4. [45] For any $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$, $h \in i\mathbb{R}$ and each $k = 1, ..., n - 1$, there exists canonical operators $\tilde{\epsilon}_k(u)$ and $\tilde{f}_k(u)$ acting on the finite set $E(u; \lambda)$ such that for any generic element $\xi \in E(u; \lambda)$, there exist real valued functions $c_{kj}(\xi)$ and $h_{kj}(h, u, \xi)$ with $j = 1, 2$ such that

$$\lim_{h \to +i\infty} \left( S_+(u; h)_{k,k+1} \cdot e^{c_{k1}(\xi)h+i\theta_{k1}(h,u,\xi)}\xi \right) = \tilde{\epsilon}_k(\xi),$$

(5)

$$\lim_{h \to +i\infty} \left( S_-(u; h)_{k+1,k} \cdot e^{c_{k2}(\xi)h+i\theta_{k2}(h,u,\xi)}\xi \right) = \tilde{f}_k(\xi).$$

(6)

Furthermore, the WKB datum $(E(u; \lambda), \tilde{\epsilon}_k(u), \tilde{f}_k(u))$ is a $\mathfrak{gl}_n$-crystal, and coincides with the crystal structure on the eigenbasis $E(u; \lambda)$ found by Halacheva-Kamnitzer-Rybnikov-Weeke [18]. In the meanwhile, the discrete choices in the definition of $S_\pm(u)$ correspond to the cactus group action on the $\mathfrak{gl}_n$-crystals.
Remark 1.5. The functions $c_{k1}(\xi)$ and $c_{k2}(\xi)$ are also determined by the representation theoretic data in the $gl_n$-crystal. But we do not need it in this paper.

We refer the reader to [18] [43] for the definition of a $gl_n$-crystal. Here we just remark that a $gl_n$-crystal is a finite set along with some operators called crystal operators satisfying certain conditions, where the finite set models a weight basis for a representation of $gl_n$, and crystal operators indicate the leading order behaviour of the simple root vectors on the basis under the crystal limit $q \to 0$ in quantum group $U_q(gl_n)$. And the cactus group action on the $gl_n$-crystal was introduced by Berenstein and Kirillov [6]. See e.g., [18] for more discussions.

In Section 6 an isomonodromy deformation approach to this conjecture is given. It decomposes the proof of the conjecture into a problem of quantitative analysis and a problem of qualitative analysis. The quantitative analysis problem was solved via the asymptotic analysis of the isomonodromy equation, as well as the explicit determination of the conjecture into a problem of quantitative analysis and a problem of qualitative analysis. The quantitative analysis part models a weight basis for a representation of $gl_n$, and has been understood in the context of physics by [16] (where the Stokes graphs are known as spectral networks). The above conjecture and its proof for the special case, see Theorem 6.9, help shed lights on the discussion.

1.2.2 Resonant case: soliton/polynomial solutions of confluent hypergeometric type equations

In the end, we propose a characterization of differential equations which allow soliton solutions, the Conjecture 1.4 which seems to us a striking example of the indication of the algebraic understanding of the Stokes matrices.

Let us consider the case $q_0 = e^{\pi i h_0}$ a primitive $p$-th root of unity where $p$ is odd. The representation theory of the quantum group $U_{q_0}(gl_n)$ is much richer than generic case. We refer the reader to the book [25] for an introduction to the representation theory in the roots of unity case. In particular, any irreducible representation of $U_{q_0}(gl_n)$ has dimension less than or equal to $\frac{n(n-1)}{2}$, and an irreducible representation can be neither a highest nor a lowest representation. Now let us focus on the representation $S_{q_0}(u)$ of $U_{q_0}(gl_n)$ on the vector space $L(\lambda)$ obtained in Theorem 1.2.

On the one hand, following [18] (or from the viewpoint of the isomonodromy deformation in the WKB approximation, see Section 6), there exists a parametrization of the eigenbasis $E(u; \lambda)$ of $L(\lambda)$ by the Gelfand-Tsetlin patterns $\Lambda$ (up to a cactus group action). Denote by the $n$-tuples of numbers $\lambda = (\lambda_1^{(n)}, ..., \lambda_n^{(n)})$ parameterizing the highest weight $\lambda$. Then such a pattern $\Lambda$ is a collection of numbers $\{\lambda_j^{(i)}(\Lambda)\}_{1 \leq j \leq i \leq n}$ with the fixed $\lambda_k^{(n)}(\Lambda) = \lambda_k^{(n)}$ satisfying the interfacing conditions

$$\lambda_j^{(i)}(\Lambda) - \lambda_j^{(i-1)}(\Lambda) \in \mathbb{Z}_{\geq 0}, \quad \lambda_j^{(i-1)}(\Lambda) - \lambda_{j+1}^{(i)}(\Lambda) \in \mathbb{Z}_{\geq 0}. \tag{7}$$

Let us denote by $\xi_\Lambda(u) \in E(u; \lambda)$ the basis vector labelled by the Gelfand-Tsetlin pattern $\Lambda$.

Let us assume the highest weight $\lambda$ is such that $\lambda_j^{(n)} - \lambda_j^{(n)}$ is bigger than $p$ for $i = 2, ..., n$. On the other hand, since the representation $S_{q_0}(u)$ of $U_{q_0}(gl_n)$ is the continuous extension of the irreducible representation $S_q(u)$, one can argue that the actions of $S_{q_0}^\infty(u; h_0)_{k,k+1}$ for all $k$ on the lowest vector of $L(\lambda)$ generate an invariant subspace $L^\omega(\lambda)$ of dimension $p^{n(n-1)/2}$. The invariant subspace $L^\omega(\lambda)$ inherits an irreducible representation with highest weight and lowest weight. Furthermore, the invariant subspace is expected to be spanned by all basis vectors $\xi_\Lambda(u)$ satisfying $\lambda_j^{(i)}(\Lambda) - \lambda_j^{(i+1)}(\Lambda) \leq 0$ for $1 \leq j \leq i \leq n - 1$. Here it is a useful idea to compare the asymptotic expansion of Stokes matrices given in Theorem 6.7 and the representation of quantum groups at roots of unity in the Gelfand-Tsetlin basis [25] Section 7.5.

The above discussion inspires

Conjecture 1.6. Let $\Lambda^\omega$ be the pattern determined by

$$\lambda_j^{(i)}(\Lambda^\omega) - \lambda_j^{(i+1)}(\Lambda^\omega) = p - 1, \quad \text{for } 1 \leq j \leq i \leq n - 1. \tag{8}$$

then the actions of $S_{q_0}^\infty(u; h_0)_{k,k+1}$ on $\xi(\Lambda^\omega)$ (the highest vector of $L(\lambda)$) are zero for all $k = 1, ..., n - 1$. 

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Remark 1.7. In order to prove this conjecture, in [39] we prove that the Stokes matrices of (1) can be obtained from an infinite products of representations of Yangians. Then the idea is to prove Conjecture 1.6 via the representation of Yangians, which we leave it to a next paper.

Since $S^0_+(u; h_0)$ is the evaluation of $S_+(u; h)$ at a rational $h = h_0$, expanding $h$ around 0, the conjecture predicts that

$$S_+(u; h)_{k,k+1} \cdot \xi_A(u) = \sum_{\xi_{A'} \in E(u, \lambda)} f_k(A', u, h)\xi_{A'}(u)$$  \hspace{1cm} (9)

where for fixed $u$ and $h$, $f_k(A', u, h)$ is a special function of the variables $\{\lambda^{(i)}_j(A')\}$ of the Gelfand-Tsetlin pattern $A'$, vanish whenever $hA^{(i)}_j(A') - hA^{(j+i+1)}_j(A') \in \mathbb{Z}_{>0}$. The expression of $f_k(A', u, h)$ is given at a special $u$ (called a caterpillar point) and for $h \in i\mathbb{R}$, see Theorem 6.7 and 6.8 which can be seen as a perfect model.

Semiclassical limit in the setting of quantum algebras. The semiclassical limit of the equation (1) as $h \to 0$ is understood as the equation for a function $f(z) \in GL(n, \mathbb{C})$

$$\frac{df}{dz} = \left( u + \frac{A}{z} \right) \cdot f.$$  \hspace{1cm} (10)

where $u \in h\text{reg}$, and $A = (a_{ij}) \in gl_n$. Roughly speaking, it follows from the identity $[he_{ij}, he_{kl}] = h(\delta_{jk}he_{il} - \delta_{il}he_{kj})$, thus as $h \to 0$ any power series in $he_{ij}$ corresponds to a power series of same form but in the commutative variables $a_{ij}$. Here recall that $\{e_{ij}\}_{1 \leq i, j \leq n}$ are the generators of $U(g_{n})$. See Remark 5.3 or [41] for more details.

As far as the quantum–classical correspondence is concerned, the Conjecture 1.6 and prediction (9) yield the following conjecture. Let us denote by $S_\pm(u, A)$ the Stokes matrices of (10), and let $\Gamma(u, A) \subset T^*\mathbb{P}^1$ denote the associated spectral curve defined by the characteristic polynomial

$$\det \left[ \omega \cdot \text{Id} - \left( u + \frac{A}{z} \right) dz \right] = 0.$$  \hspace{1cm} (11)

Here $\Gamma(u, A)$ is a punctured curve living in $T^*\mathbb{P}^1$ and gives a $n - 1 - 1$ cover of $\mathbb{P}^1 \setminus \{0, \infty\}$. For generic $u$ and $A$, it will be smooth and has genus $\frac{(n-1)(n-2)}{2}$. The pull-back of the canonical tautological 1-form on $T^*\mathbb{P}^1$ to $\Gamma(u, A)$ gives the canonical 1-form $\omega$ on $\Gamma(u, A)$. Then

**Conjecture 1.8.** There exists a set of cycles $\{L^{(k)}_i\}_{1 \leq i \leq k \leq n}$ on the spectral curve $\Gamma(u, A)$ such that the Stokes matrices $S_\pm(u, A)$ are the identity matrix if and only if the set of periods $\{\int_{L^{(k)}_i} \omega\}_{1 \leq k \leq n}$ constitutes a Gelfand-Tsetlin pattern, i.e., satisfies the interlacing conditions

$$\int_{L^{(k)}_i} \omega - \int_{L^{(k+1)}_i} \omega \in \mathbb{Z}_+, \quad \int_{L^{(k+1)}_i} \omega - \int_{L^{(k)}_i} \omega \in \mathbb{Z}_+. \hspace{1cm} (12)$$

**Remark 1.9.** The periods are studied in [1], as the WKB approximation of the Stokes matrices of the equation (10). In particular, a concrete choice of the cycles for some $u$ are proposed in [1], with a relation to the cluster algebra structure on the space of Stokes matrices. We remark that there are different choices of the set of cycles, which correspond to the cactus group action on the Gelfand-Tsetlin pattern.

Conjecture 1.8 characterizes those $u$ and $A$ such that the corresponding equation (10) has a soliton solution taking the form

$$f(z) = (1 + h_{1}z^{-1} + \cdots + h_{m}z^{-m}) \cdot e^{uz}z^{\delta A},$$

where $h_{1}$ is an $n \times n$ matrix, $m$ is a positive integer and $\delta A$ is the diagonal part of $A$. It is because the existence of soliton solutions amounts to determine when the Stokes matrices $S_\pm(u, A)$ are identity. Here the name soliton solution is after [22, Section 6].

It is interesting to notice that the existence of soliton solutions is related to the Bohr-Sommerfeld quantization condition of the integrable systems on the moduli space of meromorphic differential equations. See e.g., [17] for the example of Gelfand-Tsetlin system.
In the end, we mention that the situation of resonant is perpendicular to the WKB approximation in Section 1.2.1 for \( h \to \infty \) along the real axis, the WKB approximation of Stokes matrices in general do not exist, as can be seen from the explicit formula in the \( n = 2 \) case. It corresponds to the degeneration of spectral networks or Stokes graphs, studied by Gaiotto-Moore-Neitzke [16], of the hypergeometric type equation. Based on the above discussion, see also [1], we expect that the WKB approximation of (the Stokes matrices of) the confluent hypergeometric type equation (10) produces an example of (a variation of) the BPS structures introduced by Bridgeland [10], with a close relation to the theory of Gelfand-Tsetlin and canonical basis. We leave a detailed study of the BPS structure and the associated Riemann-Hilbert problem [10] to a future work.

2 Stokes matrices of the equation (1)

In Section 2.1-2.3 we introduce the unique formal solution, the canonical solutions and Stokes matrices of the equation (1) respectively.

2.1 The unique formal fundamental solution of equation (1) in nonresonant case

**Proposition 2.1.** For any \( h \not\in \mathbb{Q} \) and \( u \in \mathfrak{h}_{\text{reg}} \), the ordinary differential equation

\[
\frac{dF}{dz} = h\left(u + \frac{T}{z}\right) \cdot F
\]

has a unique formal fundamental solution taking the form

\[
\hat{F}(z) = \hat{H}(z)e^{\frac{hu}{2}z^2}z^{\delta T}, \quad \text{for } \hat{H} = 1 + H_1 z^{-1} + H_2 z^{-2} + \cdots,
\]

where each coefficient \( H_m \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \), and \( \delta T \) denotes the diagonal part of \( T \), i.e., \( \delta T = \sum_k e_{kk} \otimes E_{kk} \).

**Proof.** Plugging \( \hat{F} \) into the equation (13) gives rise to the equation for \( \hat{H} \),

\[
\frac{1}{h} \frac{d\hat{H}}{dz} + \hat{H} \cdot \left(u + \frac{\delta T}{z}\right) = \left(u + \frac{T}{z}\right) \cdot \hat{H}.
\]

Comparing the coefficients of \( z^{m-1} \), we see that \( H_m \) satisfies

\[
[H_{m+1}, u] = \left(\frac{m}{h} + T\right) \cdot H_m - H_m \cdot \delta T.
\]

Set \( \{E_{kl}\}_{1 \leq k,l \leq n} \) the standard basis of \( \text{End}(\mathbb{C}) \). Then

\[
T = \sum_{k,l} e_{kl} \otimes E_{kl}, \quad \text{and } u = \sum_i 1 \otimes u_i E_{ii}.
\]

Plugging \( H_m = \sum_{k,l} H_{m,kl} \otimes E_{kl} \), with \( H_{m,kl} \in \text{End}(L(\lambda)) \), into the equation (16) gives rise to

\[
\sum_{k,l} (u_l - u_k)H_{m+1,kl} \otimes E_{kl} = \sum_{k,l} \frac{m}{h} H_{m,kl} \otimes E_{kl} + \sum_{k,l,l'} e_{kl'}H_{m,l'l} \otimes E_{kl} - \sum_{k,l} H_{m,kl} e_{kl} \otimes E_{kl}.
\]

Here \( e_{kl} \) is understood as elements in \( \text{End}(L(\lambda)) \) via the given representation. That is for \( k \neq l \)

\[
(u_l - u_k)H_{m+1,kl} = \frac{m}{h} H_{m,kl} + \sum_{l'} e_{kl'} H_{m,l'l} - H_{m,kl} e_{ll} \in \text{End}(L(\lambda)),
\]

and for \( k = l \),

\[
0 = \sum_{l' \neq k} e_{kl'} H_{m,l'k} + \frac{m}{h} H_{m,kk} + [e_{kk}, H_{m,kl}] \in \text{End}(L(\lambda)).
\]

To see the above recursive relation have a unique solution, first note that, since \( u_k \neq u_l \) for \( k \neq l \), the identity (17) uniquely defines the "off-diagonal" part \( H_{m+1,kl} (k \neq l) \) of \( H_{m+1} \) from \( H_m \). Furthermore, since \( h \not\in \mathbb{Q} \), we have \( m/h \mathbf{1} + \text{ad}_{e_{kk}} \) is invertible on \( \text{End}(L(\lambda)) \) for any integer \( m \). Thus, the condition (18) (replacing \( m \) by \( m + 1 \)) uniquely defines the "diagonal" part \( H_{m+1,kk} \) of \( H_{m+1} \) from the off diagonal part. 

**Remark 2.2.** The recursive relation (16) can be solved explicitly. See Proposition 5.1.
2.2 The canonical solutions with a prescribed asymptotics

The radius of convergence of the formal power series $\hat{H}(z)$ is in general zero. However, it follows from the general principal of differential equations with irregular singularities that (see e.g., [4, 29, 32] or the proof of Proposition 2.13) the Borel resummation (Borel-Laplace transform) of $\hat{H}$ gives a holomorphic function in each Stokes supersector around $z = \infty$. In this way, one gets actual solutions of (13) on each Stokes supersector. These sectors are determined by the irregular term $hu$ of the differential equation as follows.

For any two real numbers $a, b$, an open sector and a closed sector with opening angle $b - a > 0$ are respectively denoted by

$$S(a, b) := \{ z \in \mathbb{C} \mid a < \arg(z) < b \}, \quad \overline{S(a, b)} := \{ z \in \mathbb{C} \mid a \leq \arg(z) \leq b \}.$$

**Definition 2.3.** The anti-Stokes rays of the equation (13) are the directions along which $e^{h(u_i - u_j)z}$ decays most rapidly as $z \to \infty$ for some $u_i \neq u_j$. The Stokes supersectors are the open regions of $\mathbb{C}$

$$\text{Sect}_i := S\left(d_i - \frac{\pi}{2}, d_{i+1} + \frac{\pi}{2}\right),$$

bounded by two adjacent anti-Stokes rays $d_i$ and $d_{i+1}$. Here we label the anti-Stokes rays $d_1, d_2, \ldots, d_{2l}$ (modulo $2l$, i.e., $d_0 = d_{2l}$) going in a positive sense (the indices are taken modulo $2l$), and use the same letter $d_i$ to denote the argument of the ray/direction $d_i$.

In this paper we denote a ray by its argument, thus the anti-Stokes rays of (13) are $-\arg(hu_i - hu_j)$ for all the possible $i \neq j$. Let us make a choice of branch of $\log(z)$ on Sect0, and by convention the branch of $\log(z)$ Sect0 is extended to the other sectors in a negative sense. Then the following theorem follows from the general principal of differential equations with irregular singularities that (see e.g., [4, 29, 32] or the proof of Proposition 2.13).

**Theorem 2.4.** For fixed $h \not\in \mathbb{Q}$ and $u \in h_{\text{reg}}$, there is a unique (therefore canonical) holomorphic function $H_i : \text{Sect}_i \to \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n)$ such that the function

$$F_i(z) := H_i(z) \cdot e^{hu_i z} \cdot e^{\theta z}$$

satisfies the equation (1), and at the same time

$$H_i(z) \sim \hat{H}(z), \quad \text{as } z \to \infty \text{ within } \text{Sect}_i.$$

We stress that, the actual solution of the differential equation, with the prescribed asymptotic expansion as $z \to \infty$ in a sector whose opening is larger than $\pi$, is unique and this fact is important in what follows later.

2.3 Stokes matrices

**Definition 2.5.** For any $h \not\in \mathbb{Q}$ and $u \in h_{\text{reg}}$, the Stokes matrices $S_{\pm}(u) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n)$ of the equation (1) (with respect to the chosen sector Sect0) are the unique matrices such that:

- If $F_0$ is continued in a positive sense to Secti then $F_i(z) = F_0(z) \cdot S_+(u)e^{\pi i \theta},$ and
- If $F_i$ is continued in a positive sense to Sect0 then $F_0(ze^{-2\pi i}) = F_i(z) \cdot e^{-\pi i \theta} S_-(u)^{-1}.$

**Remark 2.6.** The definition of the Stokes matrices involves the auxiliary choices of initial sector Sect0 and a choice of branch of $\log(z)$ on Sect0, and the formal monodromy $e^{2\pi i \theta}$ usually is separated from the Stokes matrices (thus the latter have the identities along the diagonal). Here we take a different convention that the definition of Stokes matrices includes the data of formal monodromy. The advantage is that in our convention the Stokes matrices themselves satisfy the RLL relations. See Section 3.

It is convenient to think of $S_{\pm}(u)$ as $n \times n$ matrices with entries in $\text{End}(L(\lambda))$. Let us introduce the permutation matrix $1 \otimes J \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n)$ associated to the choice of Sect0 defined by $J_{ij} = \delta_{\sigma(i)j}$, where $\sigma$ is the permutation of $\{1, \ldots, n\}$ corresponding to the dominance ordering of $\{e^{hu_1 z}, \ldots, e^{hu_n z}\}$ along the direction $\theta$ bisecting the sector Sect($d_1, d_l$), that is $\sigma(i) < \sigma(j)$ if and only if $e^{h(u_i - u_j)z} \to 0$ as $z \to \infty$ along $\theta$. Then the prescribed asymptotics of $F_0(z)$ and $F_i(z)$ at $z = \infty$ ensures that...
Lemma 2.7. The matrices $S_{\pm}(u)$ are triangular, up to the permutation matrix $1 \otimes J$. Furthermore, their diagonal elements are

$$S_{+}(u)^{-1}_{kk} = S_{-}(u)_{kk} = e^{\pi i \epsilon_{kk}} \in \text{End}(L(\lambda)), \quad \text{for } k = 1, \ldots, n.$$  

Proof. By the asymptotics (19) of $H_{\pm}(z)$, the identity (106) leads to

$$e^{huz}z^{-h\delta T} \cdot \left(e^{-\pi i h\delta T}S_{+}(u)^{-1}\right) \cdot e^{-huz}z^{-h\delta T} = H_{i}(z)^{-1}H_{0}(z) \to 1, \quad \text{as } z \to \infty \text{ within } \text{Sect}_{0} \cap \text{Sect}_{t}.$$  

Let us write $S_{\pm}(u)^{-1} = \sum_{ij}(S_{\pm}^{-1})_{ij} \otimes E_{ij} \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^{n})$. Since the exponentials $e^{huz}$ dominate, we must have for any $i \neq j$, $e^{h(u_{i}-u_{j})z}(S_{\pm}^{-1})_{ij} \otimes E_{ij} \to 0$. It implies that $(e^{-\pi i h\delta T}S_{+})_{ij} = \delta_{ij}$ unless $e^{zh(u_{i}-u_{j})} \to 0$ as $z \to \infty$ within $\text{Sect}_{0} \cap \text{Sect}_{t}$. Thus $S_{+}(u)^{-1}$ is triangular, up to the permutation matrix $1 \otimes J$. The argument for $S_{-}$ is the same once the change of the branches of $\log(z)$ is accounted for. \(\blacksquare\)

As a matter of convenience, in the rest of the paper, we will assume that $u \in \mathfrak{h}_{\text{reg}}$ and the sector $\text{Sect}_{0}$ are chosen such that the corresponding permutation matrix is the identity. Under this assumption, the Stokes matrices $S_{+}(u)$ and $S_{-}(u)$ are upper and lower triangular respectively. That is, $S_{+}(u)_{ij} = 0 \in \text{End}(L(\lambda))$ if $i > j$.

Remark 2.8. There is a factorization of the Stokes matrices $S_{\pm}(u)$ as an ordered product of the Stokes factors $S_{i}(u)$ for $i = 0, 1, \ldots, l - 1$ (up to a multiplication by $e^{\pi i h\delta T}$). Here each $S_{i}(u) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^{n})$ is determined by $F_{i+1}(z) = F_{i}(z) \cdot S_{i}(u)$ within $\text{Sect}_{t} \cap \text{Sect}_{t+1}$. Furthermore, there is an action of the braid group $B_{n}$ on the Stokes matrices $S_{\pm}(u)$ simply arising from the discrete choices of initial Stokes sector $\text{Sect}_{0}$. See e.g., [9]. Based on Theorem 1, the factorization of the Stokes matrices are analogous to the factorization of the universal R–matrices of Drinfeld-Jimbo quantum groups [26, 28], while the braid group action on the Stokes matrices $S_{\pm}(u)$ are analogous to the quantum Weyl group action on the quantum groups [26, 31, 28].

3 Stokes matrices satisfy the RLL relation

Let us take the standard R-matrix $R \in \text{End}(\mathbb{C}^{n}) \otimes \text{End}(\mathbb{C}^{n})$, see e.g., [20, 36].

$$R = \sum_{i \neq j, i, j = 1}^{n} E_{ii} \otimes E_{jj} + e^{\pi i h} \sum_{i = 1}^{n} E_{ii} \otimes E_{ii} + (e^{\pi i h} - e^{-\pi i h}) \sum_{1 \leq j < i \leq n} E_{ij} \otimes E_{ji}. \quad (20)$$  

The following theorem is the main result of this section.

Theorem 3.1. For any $h \notin \mathbb{Q}$ and $u \in \mathfrak{h}_{\text{reg}}$, the Stokes matrices $S_{\pm}(u)$ of (11) satisfy

$$R^{12}S_{\pm}^{(1)}(u)S_{\pm}^{(2)}(u) = S_{\pm}^{(2)}(u)S_{\pm}^{(1)}(u)R^{12} \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^{n}) \otimes \text{End}(\mathbb{C}^{n}), \quad (21)$$  

$$R^{12}S_{\pm}^{(1)}(u)S_{\pm}^{(2)}(u) = S_{\pm}^{(2)}(u)S_{\pm}^{(1)}(u)R^{12} \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^{n}) \otimes \text{End}(\mathbb{C}^{n}). \quad (22)$$  

Note that the Stokes matrices $S_{\pm}(u)$ are $n \times n$ matrices with entries $S_{\pm}(u)_{ij}$ in $\text{End}(L(\lambda))$, i.e., $S_{\pm}(u) = \sum_{i,j}S_{\pm}(u)_{ij} \otimes E_{ij} \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^{n})$. Here $S_{\pm}^{(1)}(u) := \sum_{i,j}S_{\pm}(u)_{ij} \otimes E_{ij} \otimes 1$, $S_{\pm}^{(2)}(u) := \sum_{i,j}S_{\pm}(u)_{ij} \otimes 1 \otimes E_{ij}$, and $R^{12} := 1 \otimes R \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^{n}) \otimes \text{End}(\mathbb{C}^{n})$.

This theorem is closely related to the result in [44]. To prove it, we introduce a partial differential equation for a function $Y(z, t)$ with two complex variables. We then introduce six holomorphic solutions $Y_{\pm k}(z, t)$ of the equation in six different connected regions $D_{\pm k}$ for $k = 1, 2, 3$, specified by a common prescribed asymptotics. In the end, we prove that after the analytic continuation of $Y_{+3}$ from the domain $D_{+3}$ to the domain $D_{-3}$, along two homotopic paths $D_{+3} \rightarrow D_{+2} \rightarrow D_{+1} \rightarrow D_{-3}$ and $D_{+3} \rightarrow D_{-1} \rightarrow D_{-2} \rightarrow D_{-3}$, we have respectively

$$Y_{-1}(z, t) = Y_{+3}(z, t) \cdot S_{+}^{(2)}(u)S_{+}^{(1)}(u)R^{(12)}, \quad \text{for } (z, t) \in D_{-3}.$$  

and

$$Y_{-1}(z, t) = Y_{+3}(z, t) \cdot R^{(12)}S_{+}^{(2)}(u)S_{+}^{(1)}(u), \quad \text{for } (z, t) \in D_{-3}.$$  

It implies $S_{+}^{(2)}S_{+}^{(1)}R^{(12)} = R^{(12)}S_{+}^{(2)}S_{+}^{(1)}$. Similarly, we obtain the other identities in Theorem 3.1.
3.1 Solutions of the Knizhnik–Zamolodchikov equations with irregular singularities

Again, let \( L(\lambda) \) be a finite dimensional highest weight \( U(\mathfrak{gl}_n) \)-module. Let us consider a system of equations, for a \( \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \) valued function \( Y(z_1, z_2) \) of two complex variables,

\[
\frac{1}{h} \frac{\partial Y}{\partial z} = \left( u^{(1)} + tu^{(2)} + \frac{T^{(1)} + T^{(2)} + P}{z} \right) \cdot Y, \tag{23}
\]

\[
\frac{1}{h} \frac{\partial Y}{\partial t} = \left( zu^{(2)} + \frac{T^{(2)}}{t} + \frac{P}{t-1} \right) \cdot Y. \tag{24}
\]

Here \( h \) is a complex parameter, \( u = \text{diag}(u_1, \ldots, u_n) \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), and

\[
u^{(1)} = \sum_{i} 1 \otimes u_i E_{ii} \otimes 1, \quad u^{(2)} = \sum_{i} 1 \otimes 1 \otimes u_i E_{ii},
\]

\[
T^{(1)} = \sum_{k,l} e_{kl} \otimes E_{kl} \otimes 1, \quad T^{(2)} = \sum_{k,l} e_{kl} \otimes 1 \otimes E_{kl}, \quad P = - \sum_{k,l} 1 \otimes E_{kl} \otimes E_{lk}, \tag{25}
\]

are elements in \( U(\mathfrak{gl}_n) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \). The action of the coefficient matrix of the system on \( Y(z_1, z_2) \) is given by the matrix multiplication and the representation of \( \mathfrak{gl}_n \).

**Proposition 3.2.** The system of equations (23) and (24) is compatible.

**Proof.** It follows from a direct computation and the identities \( [T^{(1)} + T^{(2)}, P] = 0, [T^{(1)}, T^{(2)} + P] = 0 \). Here we remark that there is a minus sign in the expression (25) of \( P \). ■

**Remark 3.3.** The system is equivalent to the two variables Knizhnik–Zamolodchikov (KZ) equations with irregular singularities [35, 14]. The canonical solutions of the KZ equation with prescribed asymptotics was first studied in a somewhat different setting by Toledano Laredo [40]. The proof of this section is mainly motivated by [40].

In the rest of this section, we fix a \( u = \text{diag}(u_1, \ldots, u_n) \in \mathfrak{h}_{\text{reg}} \). For any fixed \( t \), the first equation (23) becomes a meromorphic ordinary differential equation with an irregular singularity at \( z = \infty \). In the following, we first follow the standard way to produce solutions of equation (23) with a prescribed asymptotics at \( z = \infty \), and then apply the resulting solutions to solve the system of equations (23) and (24).

3.1.1 The unique formal solution of equation (23) in the nonresonant case

For any \( u \in \mathfrak{h}_{\text{reg}} \) let us introduce the \( u \) dependent region in \( \mathbb{C} \),

\[
D_u := \left\{ t \in \mathbb{C} \mid t \neq \frac{u_i - u_j}{u_k - u_l}, \text{ for any } i, j, k, l \right\}.
\]

In the following, we will first consider the case \( t \in D_u \), and then treat the leftover case by means of analytic continuation.

**Proposition 3.4.** Let us assume that \( h \notin \mathbb{Q} \). Then for any fixed \( u \in \mathfrak{h}_{\text{reg}} \) and \( t \in D_u \), the ordinary differential equation (23) has a unique formal fundamental solution taking the form

\[
\hat{Y}(z, t) = \hat{Q}(z, t)e^{hz(u^{(1)} + tu^{(2)})}z^{h(T^{(1)} + T^{(2)} + P)}, \tag{26}
\]

where \( \hat{Q} = 1 + Q_1 z^{-1} + Q_2 z^{-2} + \cdots \), and each coefficient \( Q_m(t) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \). Here \( \delta \) takes the diagonal part, i.e.,

\[
\delta T^{(1)} = \sum_k e_{kk} \otimes E_{kk} \otimes 1, \quad \delta T^{(2)} = \sum_k e_{kk} \otimes 1 \otimes E_{kk}, \quad \delta P = - \sum_k 1 \otimes E_{kk} \otimes E_{kk}.
\]

**Proof.** Plugging \( \hat{Y} \) in (26) into the equation (23) gives rise to the equation for the formal power series \( \hat{Q} \),

\[
\frac{1}{h} \frac{d \hat{Q}}{dz} + \hat{Q} \cdot \left( u^{(1)} + tu^{(2)} + \frac{\delta T^{(1)} + \delta T^{(2)} + \delta P}{z} \right) = \left( u^{(1)} + tu^{(2)} + \frac{T^{(1)} + T^{(2)} + P}{z} \right) \cdot \hat{Q}. \tag{27}
\]
Comparing the coefficients of \(z^{-m-1}\) in the two sides of (27), we get that the recursion relation

\[
[Q_{m+1}(t), u^{(1)} + tu^{(2)}] = \left(m/h + T^{(1)} + T^{(2)} + P\right) \cdot Q_m(t) - Q_m(t) \cdot \left(\delta T^{(1)} + \delta T^{(2)} + \delta P\right). \tag{28}
\]

Let us write \(Q_m = \sum_{i,j,k,l} Q_{i,j,k,l} \otimes E_{i,j} \otimes E_{k,l}\) in terms of a basis \(\{E_{i,j} \otimes E_{k,l}\}_{1 \leq i,j,k,l \leq n}\) of \(\End(\mathbb{C}^{\otimes 2})\), where each \(Q_{i,j,k,l} \in \End(\mathcal{L}(\lambda))\). Then using the expressions in (25), a direct computation shows that the relation (28) gives rise to

(1) For \(i \neq j\) or \(k \neq l\):

\[
(u_j - u_i + tu_l - tu_k)Q_{m+1,jkl} = \sum_{j'} e_{ij'} Q_{m,j'jkl} + \sum_{k'} e_{kk'} Q_{m,ijkl} + \delta_{ij} Q_{m,kjl} - Q_{m,kjl} e_{jj} - Q_{m,ijkl} e_{ll} + \frac{m}{h} Q_{m,ijkl}. \tag{29}
\]

(2) For \(i = j\) and \(k = l\):

\[
0 = \sum_{i' \neq i} e_{i'i} Q_{m,i'ikk} + \sum_{k' \neq k} e_{kk'} Q_{m,ikkk} + \delta_{ik} Q_{m,ikki} - Q_{m,ikki} + [e_{ii} + e_{kk}, Q_{m,ikki}] + \frac{m}{h} Q_{m,ikki}. \tag{30}
\]

Let us check that the above recursive relation have a unique solution. First, by the assumption \(t \in D_u\), i.e., \(t \neq \frac{u_i - u_j}{u_k - u_l}\) for all \(i, j, k, l\), the identity (29) uniquely defines the "off-diagonal" part \(Q_{m+1,i,jkl}\) (\(i \neq j\) or \(k \neq l\)) of \(Q_m\). Second, replacing \(m\) by \(m+1\) in the identity (30) (the solvability condition of the recursion relation), we get a linear system

\[
[Q_{m+1,ikki}, e_{ii} + e_{kk}] - (m+1)/h \cdot Q_{m+1,ikki} = \sum_{i' \neq i} e_{i'i} Q_{m+1,i'ikki} + \sum_{k' \neq k} e_{kk'} Q_{m+1,ikki} + \delta_{ik} Q_{m+1,ikki} - Q_{m+1,ikki} \tag{31}
\]

for the unknown \(Q_{m+1,ikki} \in \End(\mathcal{L}(\lambda))\). Note that the off-diagonal part of \(Q_{m+1}\) on the right hand side of (31) has been determined by the identity (29). Since \(h \notin \mathbb{Q}\), we have \((m+1)/h\Id + \ad_{e_{ii}+e_{kk}}\) is invertible on \(\End(\mathcal{L}(\lambda))\). That is, the linear system for \(Q_{m+1,ikki}\) is nondegenerate and has a unique solution. It finishes the proof. ■

3.1.2 Canonical solutions of the equation (23)

In the rest of this section, let us fix \(h \notin \mathbb{Q}\) and \(u \in \mathfrak{h}_{\text{reg}}\).

Similar to Section 2.2, the following results follow from the general principal of the Borel resummation of the formal solutions of the meromorphic differential equations. First, the anti-Stokes rays of the equation (23) are the directions along which \(e^{h(u_i - u_j + tu_k - tu_l)}z\) decays most rapidly as \(z \to \infty\) for some \(u_i \neq u_j\) or \(u_k \neq u_l\). Let \(d(t)\) and \(d'(t)\) be two adjacent anti-Stokes rays (going in a positive sense). Then the corresponding Stokes supersectors are the open regions of the complex plane

\[
\text{Sect}_{d,d'}(t) := S\left(d(t) - \frac{\pi}{2}, d'(t) + \frac{\pi}{2}\right).
\]

Here we stress the dependence of the anti-Stokes rays and sectors on \(t\). The Borel resummation of \(\hat{Q}(z)\) defines holomorphic functions in each such sector, which in turn determines the holomorphic solutions of (23) with the prescribed asymptotics. In summary, we have

**Theorem 3.5.** For fixed \(t \in D_u\), there is a unique (therefore canonical) holomorphic function \(Q_{d,d'}(z;t) : \text{Sect}_{d,d'}(t) \to \End(\mathcal{L}(\lambda)) \otimes \End(\mathbb{C}^n)\) such that the function

\[
\hat{Y}_{d,d'}(z;t) := Q_{d,d'}(z;t) \cdot e^{hz(u^{(1)} + tu^{(2)})} \cdot e^{h(\delta T^{(1)} + \delta T^{(2)} + \delta P)}
\]

satisfies the equation (23), and at the same time

\[
Q_{d,d'}(z;t) \sim \hat{Q}(z;t), \quad \text{as } z \to \infty \text{ within } \text{Sect}_{d,d'}(t). \tag{32}
\]
3.1.3 Solutions of the system of equations (23) and (24)

Let \( \mathcal{R} \in D_u \) be any region of the \( t \) plane such that there exist two adjacent anti-Stokes rays \( d(t) \) and \( d'(t) \) that continuously dependent on \( t \in D_u \) and never collide. For any \( t \in \mathcal{R} \), let \( \tilde{Y}_{d,d'} \) be the function given in Theorem 3.5 defined on the corresponding \( t \)-dependent sector \( \text{Sect}_{d,d'}(t) \). Then we have

**Theorem 3.6.** The function

\[
Y_{d,d'}(z,t) = \tilde{Y}_{d,d'}(z,t) \cdot t^{h \delta T^{(2)}(t-1)^{h \delta P}},
\]

defined for \( t \in \mathcal{R} \) and \( z \in \text{Sect}_{d,d'}(t) \), satisfies the system of equations (23) and (24).

**Proof.** By the compatibility of the equations (23) and (24), we have that the functions

\[
U_{d,d'}(z,t) := \frac{dY_{d,d'}(z,t)}{dt} - h\left(zu(2)^2 + \frac{T^{(2)}}{t} + \frac{P}{t-1}\right) \cdot Y_{d,d'},
\]

satisfy the equation (23) in the corresponding domains. It implies that the ratio

\[
C := Y_{d,d'}^{-1} \cdot U_{d,d'} = Y_{d,d'}^{-1} \frac{dY_{d,d'}}{dt} - Y_{d,d'}^{-1} \cdot h\left(zu(2)^2 + \frac{T^{(2)}}{t} + \frac{P}{t-1}\right) \cdot Y_{d,d'}
\]

is independent of \( z \). To show that \( C \) is zero, let us rewrite

\[
Y_{d,d'} = \tilde{Y}_{d,d'}(z,t) \cdot t^{h \delta T^{(2)}(t-1)^{h \delta P}} = Q_{d,d'}(z,t)e^{hzu(1)+tu(2)}z^{h \delta T^{(1)}(t)}z^{h \delta T^{(2)}(t)}(t\cdot z)^{h \delta P}.
\]

Differentiating the above identity with respect to \( t \) gives

\[
\frac{dY_{d,d'}}{dt} \cdot Y_{d,d'}^{-1} = \frac{dQ_{d,d'}}{dt}Q_{d,d'}^{-1} + Q_{d,d'}^{-1} \cdot h\left(zu(2)^2 + \frac{T^{(2)}}{t} + \frac{\delta P}{t-1}\right),
\]

Following from (32), we have

\[
Q_{d,d'}(z,t) \sim \tilde{Q}(z,t) = 1 + Q_1(t)z^{-1} + Q_2(t)z^{-2} + \cdots, \text{ as } z \to \infty \text{ within } \text{Sect}_{d,d'}(t).
\]

By (28), the first term \( Q_1(t) \) should satisfy

\[
[Q_1(t),tu(2)] = T^{(1)} + T^{(2)} + P - \delta T^{(1)} - T^{(2)} - \delta P.
\]

Solving (38) gives the off diagonal part of \( Q_1(t) = \sum_{i,j,k,l} H_{1,ijkl} \otimes E_{ij} \otimes E_{kl} \),

\[
Q_1(t) - \sum_{i,k} Q_{1,ikk} \otimes E_{ii} \otimes E_{kk} = \sum_{1 \leq i,j \leq n} \left( c_{ij} \otimes E_{ij} \otimes E_{ij} \otimes E_{ij} + e_{ij} \otimes E_{ij} \otimes E_{ij} + \frac{1}{t-1} \otimes E_{ij} \otimes E_{ij}\right).
\]

Just as in the proof of Proposition 3.4, the assumption \( t \in D_u \) makes the right hand side of (39) well-defined. Since the diagonal part \( \sum_{i,k} Q_{1,ikk} \otimes E_{ii} \otimes E_{kk} \) of \( Q_1 \) commutes with \( u(2) \), the identity (39) implies that

\[
[Q_1(t),u(2)] = T^{(2)} - \frac{T^{(2)}}{t} + \frac{P - \delta P}{t-1}.
\]

Therefore, the above identity and the asymptotics (32) of \( Q_1 \) lead to

\[
Q_{d,d'} \cdot h\left(zu(2)^2 + \frac{T^{(2)}}{t} + \frac{\delta P}{t-1}\right) \cdot Q_{d,d'}^{-1} = h\left(zu(2)^2 + \frac{T^{(2)}}{t} + \frac{P}{t-1}\right) + O(z^{-1}), \text{ as } z \to \infty \text{ within } \text{Sect}_{d,d'}(t).
\]

That is,

\[
Y_{d,d'} \cdot C \cdot Y_{d,d'}^{-1} = \frac{dY_{d,d'}}{dt} \cdot Y_{d,d'}^{-1} - h\left(zu(2)^2 + \frac{T^{(2)}}{t} + \frac{P}{t-1}\right) = O(z^{-1}) \text{ as } z \to \infty \text{ within } \text{Sect}_{d,d'}(t).
\]
The above identity can be rewritten as

\[ e^{hz(u^{(1)}+tu^{(2)})}z^{-h\delta T^{(1)}}(t)z^{-h\delta T^{(2)}}(z)Q_{d,d'}(z,t)^{-1} \cdot O(z^{-1}) \cdot Q_{d,d'}(z,t) \]  

Since \( Q_{d,d'}(z,t) \sim 1 \), we get the left hand side of the above identities is \( O(z^{-1}) \) as \( z \to \infty \) within \( \text{Sect}_{d,d'}(t) \).

Let us write \( C = Y_{d,d'}^{-1} \cdot U_{d,d'} = \sum_{a,b,c,d} E_{ab} \otimes E_{cd}. \) Since the exponentials term \( e^{hz(u^{(1)}+tu^{(2)})} \) dominate, the asymptotics \( (40) \) forces that for any \( a, b, c, d, \)

\[ e^{hz(u_a-u_b+u_c-u_d)}C_{i,abcd} \otimes E_{ij} \otimes E_{kl} \to 0, \quad z \to \infty \quad \text{within} \quad \text{Sect}_{d,d'}(t). \]

Since the sector \( \text{Sect}_i \) has an opening angle bigger than \( \pi \), it implies that the all the elements \( C_{i,abcd} \) must vanish. In the meanwhile, for the diagonal elements, we have

\[ \left(z^{h(e_{aa}+e_{cc})} \cdot C_{i,aace} \cdot z^{-h(e_{aa}+e_{cc})}\right) \otimes E_{aa} \otimes E_{cc} = O(z^{-1}), \quad z \to \infty \quad \text{within} \quad \text{Sect}_{d,d'}(t). \]  

(41)

If \( h \) is a purely imaginary number, the formula \( (41) \) implies \( C_{i,iikk} = 0 \). For a generic \( h \in \mathbb{C} \setminus \mathbb{Q} \), \( C_{i,iikk} = 0 \) by means of analytic continuation. Note that \( Y_{d,d'} \) is fundamental solutions, it implies that \( U_{d,d'}(z,t) = 0 \), and thus finishes the proof. \( \blacksquare \)

3.1.4 Stokes factors

For fixed \( u \) and \( h \), let us denote by \( d(t), d'(t), d''(t) \) three adjacent anti-Stokes rays of the equation \( (23) \) in a positive sense, at a point \( t \in D_u \). Following \( (33) \), denote by \( Y_{d,d'} \) and \( Y_{d',d''} \) the solutions on the sectors \( \text{Sect}_{d,d'}(t) \) and \( \text{Sect}_{d',d''}(t) \) respectively.

**Definition 3.7.** The Stokes factor \( S_{d'} \) associated to the ray \( d'(t) \) is defined by

\[ Y_{d',d''}(z,t) = Y_{d,d'}(z,t) \cdot S_{d'} \quad \text{in} \quad \text{Sect}_{d,d'}(t) \cap \text{Sect}_{d',d''}(t). \]  

(42)

**Lemma 3.8.** If we write \( S_{d'} = \sum_{1 \leq i,j,k,l \leq n} c_{ijkl} \otimes E_{ij} \otimes E_{kl} \), then the element \( c_{ijkl} \in \text{End}(L(\lambda)) \) must be zero unless \( d'(t) \) coincides with the ray \(-\arg(h(u_i - u_j + t(u_k - u_l)))\) with the same index \( i, j, k, l \).

**Proof.** The proof of Lemma \( 2.7 \) can be applied. \( \blacksquare \)

Note that as \( t \) varies, the configuration of the anti-Stokes rays varies accordingly. In particular, as \( t \) crosses from one side of some point to the other, some rays may collide and some new rays may emerge. In the following, we will determine all the Stokes factors of \( (23) \) for \( t \in \mathbb{R} \), and thus solve the connection problem between the solutions defined on different regions given in Theorem \( 3.6 \).

3.2 Solutions of the system of equations \( (23) \) and \( (24) \) for \( t \in \mathbb{R} \setminus \{0, 1\} \)

In this subsection and Section \( 3.3 \) we fix two (separate) adjacent anti-Stokes rays \( d_i \) and \( d_{i+1} \) of \( (1) \).

Recall that all the anti-Stokes rays of \( (23) \) at \( t \) have the arguments \(-\arg(u_i - u_j + t(u_k - u_l))\) for some index \( i, j, k, l \). Any of them will concentrate to one of the anti-Stokes rays \(-\arg(h(u_i - u_j))\) of \( (1) \) as \( t \to 0 \) or \( t \to \infty \) (without causing ambiguity we also denote a ray by its argument). Associated to the two fixed rays \( d_i \) and \( d_{i+1} \) of \( (1) \), there exists a unique pair of (separate) adjacent anti-Stokes rays \( d(t) \) and \( d'(t) \) of \( (23) \) defined for all \( t \in (-\delta, 0) \cup (0, \delta) \cup (1/\delta, +\infty) \) (where \( \delta \) is a small enough real number), and such that \( d(t) \) and \( d'(t) \) approach to the rays \( d_i \) and \( d_{i+1} \) of \( (1) \) as \( t \to 0-, t \to 0+ \) and \( t \to +\infty \) respectively. We remark that the rays \( d(t), d'(t) \) depend continuously on \( t \in (-\delta, 0) \cup (0, \delta) \), but may have a discontinuous jump as \( t \) passes through \( 0 \).

Then we can take a real number \( \theta > 0 \) such that the sector

\[ \text{Sect}_\theta := (d_i + \theta - \pi/2, d_{i+1} - \theta + \pi/2) \]

has opening angle bigger than \( \pi \), and is contained in \( \text{Sect}_{d,d'}(t) \) for all \( t \in (-\delta, 0) \cup (0, \delta) \cup (1/\delta, +\infty) \). By Theorem \( 3.3 \), we get functions \( Q_{d,d'}(z,t) \) defined on the three regions \( \text{Sect}_\theta \times (-\delta, 0), \text{Sect}_\theta \times (0, \delta) \) and \( \text{Sect}_\theta \times (1/\delta, +\infty) \). According to Theorem \( 3.6 \) we then get the corresponding solutions \( Y_{d,d'}(z,t) \) of the system of equations \( (23) \) and \( (24) \) on the three different regions.
Definition 3.9. The analytic continuation of the solutions

\[ Y_{d,d'}(z,t) = Q_{d,d'}(z,t) \cdot e^{hz(u^{1(1)} + tu^{1(2)})} z^{h(\delta T(1) z t \delta T(2)) (zt - z)^{h \delta P}}, \]

from Sect_{\theta} \times (-\delta, 0), Sect_{\theta} \times (0, \delta) and Sect_{\theta} \times (1/\delta, +\infty) to Sect_{\theta} \times (-\infty, 0), Sect_{\theta} \times (0, 1) and Sect_{\theta} \times (1, +\infty), with respect to the t variable, will be denoted by

\[ Y_{i1}(z,t), \ Y_{i2}(z,t) \text{ and } Y_{i3}(z,t) \]

respectively.

We emphasize that for \(1 \leq k \neq l \leq 3\) the solution \( Y_{ik}(z,t) \), continued to the domain of \( Y_{il}(z,t) \) along a path on the complex t plane, in general does not equal to \( Y_{i \bar{l}}(z,t) \). See Lemma 3.28 for the connection formula between these solutions.

3.3 Factorization properties of the solutions \( Y_{ik} \) for \( k = 1, 2, 3 \) at \( t = 0, 1, \infty \)

Under the coordinates change \( z_1 = z \) and \( z_2 = zt \), the system of equations (23) and (24) becomes

\[
\frac{1}{h} \frac{\partial Y}{\partial z_1} = \left( u^{1(1)} + \frac{T^{(1)}}{z_1} + \frac{P}{z_1 - z_2} \right) \cdot Y,
\]

\[
\frac{1}{h} \frac{\partial Y}{\partial z_2} = \left( u^{2(1)} + \frac{T^{(2)}}{z_2} + \frac{P}{z_2 - z_1} \right) \cdot Y.
\]

And under the coordinate change \( \omega_1 = z \) and \( \omega_2 = zt - z \), the system becomes

\[
\frac{1}{h} \frac{\partial Y}{\partial \omega_1} = \left( u^{1(1)} + u^{2(1)} + \frac{T^{(1)}}{\omega_1} + \frac{T^{(2)}}{\omega_1 + \omega_2} \right) \cdot Y,
\]

\[
\frac{1}{h} \frac{\partial Y}{\partial \omega_2} = \left( u^{2(2)} + \frac{T^{(2)}}{\omega_1 + \omega_2} + \frac{P}{\omega_2} \right) \cdot Y.
\]

Using the same method as in Section 3.1, we can find canonical solutions of the system with prescribed asymptotics (in different asymptotic regions, see Theorem 3.10) in terms of the new coordinates \((z_1, z_2)\) and \((\omega_1, \omega_2)\) respectively.

In this subsection, we compute the connection formula between these solutions. In particular, we prove

Theorem 3.10. Let \( Y_{ik}(z,t) \) for \( k = 1, 2, 3 \) be the solutions given in Definition 3.19

(a) If \( t \in (-\infty, 0) \), then (in their common domain of definition)

\[
\bar{W}_i(z,zt) F_{i+1}^{(1)}(z) = Y_{i1}(z,t) = W_i(z,zt) F_{i+1}^{(2)}(zt) e^{\pi i h \delta P}.
\]

(b) If \( t \in (0, 1) \), then

\[
W_i(z,zt) F_{i}^{(2)}(zt) e^{\pi i h \delta P} = Y_{i2}(z,t) = \bar{W}_i(z,zt - z) X_{i+1}^{(2)}(zt - z).
\]

(c) If \( t \in (1, \infty) \), then

\[
\bar{W}_i(z,zt - z) X_i^{(2)}(zt - z) = Y_{i3}(z,t) = \bar{W}_i(z,zt) F_{i}^{(1)}(z).
\]

Here

- \( F_i^{(1)}(z) \) and \( F_i^{(2)}(z) \) are the unique solutions, defined on \( \text{Sect}_i = S \left( d_i - \frac{\pi}{2}, d_{i+1} + \frac{\pi}{2} \right) \), of the equations for \( \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \) valued functions

\[
\frac{1}{h} \frac{dF}{dz} = \left( u^{1(1)} + \frac{T^{(1)}}{z} \right) \cdot F,
\]

and

\[
\frac{1}{h} \frac{dF}{dz} = \left( u^{2(2)} + \frac{T^{(2)}}{z} \right) \cdot F.
\]
respectively, with the prescribed asymptotics
\[ F_i^{(1)}(z) \cdot e^{-hu_1(z)}z^{-\delta T(1)} \sim 1 \text{ and } F_i^{(2)}(z) \cdot e^{-hu_2(z)}z^{-\delta T(2)} \sim 1, \quad \text{as } z \to \infty \text{ within } \text{Sect}_i. \]

Actually, \( F_i^{(1)} \) and \( F_i^{(2)} \) are the obvious extensions of the function \( F_i(z) \in \text{End}(L(\lambda)) \otimes \text{End}(C^n) \) given in Theorem [2.4] respectively. Recall from Definition [2.3] that there are \( 2l \) anti-Stokes rays in total, and \( \text{Sect}_i \) and \( \text{Sect}_{i+1} \) are two opposite Stokes supersectors of \( (\Pi) \).

- \( X_i^{(2)}(z) = K_i^{(2)}(z)e^{hu_2(z)z^{-\delta P}} \) is the unique solution, defined on \( \text{Sect}_i \), of the equation for a \( \text{End}(L(\lambda)) \otimes \text{End}(C^n) \otimes \text{End}(C^n) \) valued function
\[
\frac{1}{h} \frac{dX}{dz} = \left( u^{(2)} + \frac{P}{z} \right) \cdot X,
\]
with the holomorphic part \( K_i^{(2)}(z) \) tending to 1 as \( z \to \infty \) within \( \text{Sect}_i \).

- For any fixed \( z_2, W_i(z_1, z_2) \) is the unique solution of the equation for an \( \text{End}(L(\lambda)) \otimes \text{End}(C^n) \otimes \text{End}(C^n) \) valued function
\[
\frac{1}{h} \frac{dW}{dz_1} = \left( u^{(1)} + \frac{T^{(1)}}{z_1} + \frac{P}{z_1 - z_2} \right) \cdot W_i,
\]
defined for \( z_1 \in \text{Sect}_i \) (the sector \( S(d_i - \pi/2, d_{i+1} + \pi/2) \) of the \( z_1 \) plane) and \( |z_1| > |z_2| \), and at the same time
\[
W_i(z_1, z_2) \cdot e^{-hu_1(z_1)z_1^{-\delta T}}(z_1 - z_2)^{-\delta P} \sim 1, \quad \text{as } z_1 \to \infty \text{ within } \text{Sect}_i.
\]

See Lemma [3.11] for the existence of such a \( W_i \). Similarly, for any fixed \( z_1, \tilde{W}_i(z_1, z_2) \) is the unique solution of
\[
\frac{1}{h} \frac{dW}{dz_2} = \left( u^{(2)} + \frac{T^{(2)}}{z_2} + \frac{P}{z_2 - z_1} \right) \cdot W_i,
\]
defined for \( z_2 \in \text{Sect}_i \) (the sector \( S(d_i - \pi/2, d_{i+1} + \pi/2) \) of the \( z_2 \) plane) and \( |z_2| > |z_1| \), and at the same time
\[
\tilde{W}_i(z_1, z_2) \cdot e^{-hu_2(z_2)z_2^{-\delta T}}(z_2 - z_1)^{-\delta P} \sim 1, \quad \text{as } z_2 \to \infty \text{ within } \text{Sect}_i.
\]

- For any fixed \( \omega_2, \tilde{W}_i(\omega_1, \omega_2) \) is the unique solution of the equation
\[
\frac{1}{h} \frac{dW}{d\omega_1} = \left( u^{(1)} + u^{(2)} + \frac{T^{(1)}}{\omega_1} + \frac{T^{(2)}}{\omega_1 + \omega_2} \right) \cdot W
\]
defined on \( \text{Sect}_i \) (the sector \( S(d_i - \pi/2, d_{i+1} + \pi/2) \) of the \( \omega_1 \) plane), with the prescribed asymptotics
\[
\tilde{W}_i(\omega_1, \omega_2) \cdot e^{-h(u_1^{(1)} + u_2^{(2)})\omega_1 - \delta T(1)\omega_1} - \delta T(2) \omega_2 \sim 1, \quad \text{as } \omega_1 \to \infty \text{ within } \text{Sect}_i.
\]

In the following we only give a proof of (b) in the proposition. The part (a) and (c) follow from a similar treatment. The proof is decomposed to five steps.

### 3.3.1 Step 1: Existence of \( W_i(z_1, z_2) \)

**Lemma 3.11.** There exists a unique function \( W_i(z_1, z_2) \) satisfying (49) and (50).

**Proof.** Let us follow the standard procedure: first to show that for any fixed \( z_2 \in C \setminus \{0\} \), the equation (49) has a unique formal fundamental solution taking the form
\[
\tilde{W}(z_1; z_2) = \left(1 + \sum_{m \geq 1} K_m(z_2)z_1^{-m}\right) \cdot e^{hu_1(z_1)z_1^{-\delta T(1)}}(z_1 - z_2)^{-\delta P},
\]
where each coefficient $K_m(z_2) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$.

Expanding $\frac{P}{z_1 - z_2} = \frac{P}{z_1}(1 + \sum_{k \geq 1} \left(\frac{z_2}{z_1}\right)^k)$ and comparing the coefficients of $z_1^{-m-1}$ in the two sides of (49), we get the recursion relation

$$[K_{m+1}(z_2), u^{(1)}] = \left(m/h + T^{(1)} + P\right) \cdot K_m(z_2) - K_m(z_2) \cdot \left(\delta T^{(1)} + \delta P\right) + \sum_{r=1}^{m} (P - \delta P) K_{m-r}(z_2) z_2^r.$$  (55)

Let us write $K_m = \sum_{i,j,k,l} K_{m,ijkl} \otimes E_{ij} \otimes E_{kl}$, where each $K_{m,ijkl} \in \text{End}(L(\lambda))$. Then

1. For $i \neq j$ or $k \neq l$: 

$$(u_j - u_i) K_{m+1,i,jkl} = \sum_{j'} e_{ij'} K_{m',j'jkl} + \delta_{ij} K_{m,jijkl} - K_{m,kjil} - K_{m,ijkl} e_{jj'} + \frac{m}{h} K_{m,ijkl} + \sum_{r=1}^{m} K_{m-r,kjil} z_2^r.$$  

2. For $i = j$ and $k = l$:

$$0 = \sum_{i' \neq i} e_{ii'} K_{m,i'ikk} + \delta_{ik} K_{m,iiii} - K_{m,iiik} + [e_{ii}, K_{m,iiik}] + \frac{m}{h} K_{m,iiik} + \sum_{r=1}^{m} (K_{m-r,iiik} - \delta_{ik} K_{m,iiii}) z_2^r.$$  

Since $h \notin \mathbb{Q}$, a same argument as in the proof of Proposition 2.1 shows the existence and uniqueness of $K_m$ for all $m \geq 1$.

After knowing the formal solution, we note that the anti-Stokes rays of the equation (49) are $-\arg(h u_i - h u_j)$ for all the possible $i \neq j$, which coincide with the rays of the equation (1). Again, we label the corresponding Stokes supersectors $\text{Sect}_0, \ldots, \text{Sect}_{m-1}$ in the same way. Then, for any fixed $z_2$, the Borel resummation of $K = 1 + \sum_{m \geq 1} K_m z_1^{-m}$ defines holomorphic functions $K_i$ in each domain $\text{Sect}_i \cap \{z_1 \in \mathbb{C} \mid |z_1| > |z_2|\}$, which in turn determines the solutions of (49) with the prescribed asymptotics (50).  

Lemma 3.12. The product $W_i(z_1, z_2) F_i^{(2)}(z_2)$ satisfies the compatible system of equations

$$\frac{1}{h} \partial W_i(z_1, z_2) F_i^{(2)}(z_2) = \left( u^{(1)} + \frac{T^{(1)}}{z_1} \right) - \frac{P}{z_1 - z_2} \cdot W_i(z_1, z_2) F_i^{(2)}(z_2),$$  (56)

$$\frac{1}{h} \partial W_i(z_1, z_2) F_i^{(2)}(z_2) = \left( u^{(2)} + \frac{T^{(2)}}{z_2} \right) - \frac{P}{z_2 - z_1} \cdot W_i(z_1, z_2) F_i^{(2)}(z_2).$$  (57)

Proof. Since $F_i^{(2)}(z_2)$ is independent of $z_1$, the equation (56) follows from the equation (49) satisfied by $W_i$.

By the compatibility of the equations (56) and (57), we have that the function

$$N_{dd'}(z_1, z_2) := \frac{\partial (W_i F_i^{(2)})}{\partial z_2} - h \left( u^{(2)} + \frac{T^{(2)}}{z_2} + \frac{P}{z_2 - z_1} \right) \cdot W_i F_i^{(2)}$$

satisfies the equation (56). It implies that the ratio $C := (W_i F_i^{(2)}(z_2))^{-1} \cdot N_{dd'}$ is independent of $z_1$. To show that $C$ is zero, let us rewrite

$$W_i(z_1, z_2) F_i^{(2)}(z_2) = K_i(z_1, z_2) \cdot e^{hu^{(1)} z_1 h \delta T^{(1)}} (z_1 - z_2) h \delta P F_i^{(2)}(z_2) = K_i \cdot (1 - z_2 / z_1)^{h \delta P} F_i^{(2)}(z_2).$$

Here we use the fact that $e^{hu^{(1)} z_1 h \delta T^{(1)} + h \delta P}$ commutes with the coefficient matrix of the equation (47) and the initial value as $z_2 \to \infty$, therefore commutes with the solution $F_i^{(2)}(z_2)$. Set $L_i = K_i \cdot (1 - z_2 / z_1)^{h \delta P}$. Differentiating the above identity with respect to $z_2$ gives

$$\frac{\partial W_i F_i^{(2)}}{\partial z_2} \cdot (W_i F_i^{(2)})^{-1} = \frac{\partial L_i}{\partial z_2} L_i^{-1} + L_i \cdot \left( u^{(2)} + \frac{T^{(2)}}{z_2} \right) \cdot L_i^{-1}.$$

It leads to

$$\frac{1}{h} W_i F_i^{(2)} \cdot C \cdot (W_i F_i^{(2)})^{-1} = \frac{\partial L_i}{\partial z_2} L_i^{-1} + L_i \cdot \left( u^{(2)} + \frac{T^{(2)}}{z_2} \right) \cdot L_i^{-1} - \left( u^{(2)} + \frac{T^{(2)}}{z_2} + \frac{P}{z_2 - z_1} \right).$$  (58)

Recall that $K_i(z_1, z_2) \sim 1$ as $z_1 \to \infty$ within $\text{Sect}_i$. Then the right hand side of (58) is $O(z_1^{-1})$ as $z_1 \to \infty$ within $\text{Sect}_i$. Similar to the argument in Theorem 3.6 one shows that $C = 0$. It finishes the proof.  

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3.3.2 Step 2: the $t \to 0$ asymptotic property of the function $Q_{d,d'}$ given in Theorem 3.5

Following Section 3.2 let us take the functions $Q_{d,d'}(z,t)$ defined on the regions $\text{Sect}_1^0 \times (-\delta, 0)$ and $\text{Sect}_1^0 \times (0, \delta)$. Let us study the $t \to 0$ asymptotics of $Q_{d,d'}(z,t)$. First by the expression (29), in the case of $i = j$ and $k \neq l$, the coefficient before $Q_{m+1,ijkl}$ becomes $t(u_l - u_k)$, which will blow up as $t \to 0$. Thus, for fixed $z$ the function $Q_{d,d'}(z,t)$, obtained by the Borel-Laplace transform of $\hat{Q}$, may not have limit as $t \to 0$. However, in the following, let us show that $Q_{d,d'}(z,t)$ has limit as $t \to 0$ along the path $zt = z_2$ a constant.

**Proposition 3.13.** (a) Let $z_2 \in \text{Sect}_1^0$ be any none zero constant. Then the function $Q_{d,d'}$ satisfies

$$Q_{d,d'}(z,t) \sim f_{i+}(z_2), \quad \text{as } t \to 0 + \text{ along the path } zt = z_2 \text{ living in } \text{Sect}_1^0 \times (0, \delta),$$

where $f_{i+}(z_2)$ is a continuous function, and is asymptotic to 1 as $z_2 \to \infty$.

(b) Let $z_2 \in \overline{\text{Sect}_{i+1}}$ be any none zero constant, where $\overline{\text{Sect}_i}$ is the opposite sector of $\text{Sect}_i$. Then the function $Q_{d,d'}$ satisfies

$$Q_{d,d'}(z,t) \sim f_{i-}(z_2), \quad \text{as } t \to 0 - \text{ along the path } zt = z_2 \text{ living in } \text{Sect}_i^0 \times (-\delta, 0),$$

where $f_{i-}(z_2)$ is a continuous function, and is asymptotic to 1 as $z_2 \to \infty$.

**Proof.** Since the proofs of (a) and (b) are the same, here we only prove (a). First note that the path $zt = z_2$ for $t \in (0, \delta)$ is in the domain $\text{Sect}_1^0 \times (0, \delta)$.

Let us introduce $Q_m' := t^m Q_m$. Then multiplying by $t^m$ on the two sides of the identity (29) leads to

$$\frac{u_j - u_i + tu_k - tu_j}{t} Q_{m+1,ijkl}' = \sum_{j'} e_{ij} Q_{m,j'kl}' + \sum_{j'} e_{kl} Q_{m,ijkl}' - Q_{m,ijkl}'(\delta_{ij} + e_{jj}) - Q_{m,ijkl}'e_{il} + \frac{m}{h} Q_{m,ijkl}'. \quad (61)$$

Note that we can take a real number $r > 0$ such that

$$r \leq |(u_j - u_i + tu_k - tu_j)/t|, \quad \text{for all } i, j, k, l \text{ and } 0 < t < \delta. \quad (62)$$

Then multiplying by $t^m$ on the two sides of the identity (29), and using the above inequality, leads to

$$|Q_{m+1,ijkl}'| \leq \frac{1}{r} \sum_{j'} e_{ij} Q_{m,j'kl}' + \sum_{j'} e_{kl} Q_{m,ijkl}' + |Q_{m,ijkl}'(\delta_{ij} + e_{jj}) - Q_{m,ijkl}'e_{il} + \frac{m}{h} Q_{m,ijkl}'|.$$  

Again, from the inequality it is a standard fact that there exists constants $c' > 0$ and $K > 0$ independent of $t$ such that

$$|Q_m'(t)| \leq c' K^m m!, \quad \text{for all } t \in (0, \delta). \quad (63)$$

Let us think of $\hat{Q} = 1 + \sum_{m \geq 1} Q_m'(t) \cdot (zt)^{-m}$ as a power series in the new variable $z_2 = zt$. In the following, let us first recall the Borel resummation $Q_{d,d'}(z_2; t)$ of $\hat{Q}$ as a holomorphic function of $z_2 = zt$ on each sector $\text{Sect}_i$, and then prove that $Q_{d,d'}(z_2; t) \sim 1$, as $z_2 \to \infty$ within $\text{Sect}_1^0$ uniformly with respect to $t \in (0, \delta)$.

**Borel transform.** Denote by $B_{z_2}(\hat{Q})$ the formal Borel transform of the power series $\hat{Q} - 1$ in the variable $z_2 = zt$ (in the Borel plane with complex variable $\xi$), i.e.,

$$B_{z_2}(\hat{Q})(\xi; t) := \sum_{m \geq 1} \frac{Q_m'(t)}{\Gamma(m)} \xi^{m-1}.$$  

**Remark 3.14.** Note that $\hat{Q}$ is a formal power series in the variable $z_2^{-1}$ (negative power), while its Borel transform is a power series in $\xi$ (positive power). Here, in order to be consistent with the convention in (29), we implicitly make a coordinate change switching the zero and infinite point. Accordingly, the change of the argument should be accounted for. For example, it leads to the minus sign of the arguments $d_i$ and $d_{i+1}$ in terms of the variable $\xi$ in Lemma 3.15.
Thus by (63), the Borel transform is convergent in a small neighborhood of $\xi = 0$. Recall that $\theta > 0$ is a chosen real number such that the sector $\text{Sect}_\theta^d = S(d_i + \theta - \pi/2, d_i - \theta + \pi/2)$ has opening angle bigger than $\pi$, and is contained in $\text{Sect}_{d,e}(t)$ for all $t \in (0, \delta)$. Then

**Lemma 3.15.** For any $t \in (0, \delta)$, the Borel transform $B_{z_2}(\hat{Q})(\xi; t)$ can be analytically continued to the sector $S(-d_{i+1} + \theta, -d_i - \theta)$ of the $\xi$ plane, and there exist constants $\alpha, \beta > 0$ (independent of $t$) such that

$$|B_{z_2}(\hat{Q})(\xi; t)| \leq \alpha e^{\beta |\xi|}, \text{ for all } \xi \in S(-d_{i+1} + \theta, -d_i - \theta) \text{ and } t \in (0, \delta). \quad (64)$$

Furthermore, the limit $B_{z_2}(\hat{Q})(\xi; 0)$ of $B_{z_2}(\hat{Q})(\xi; t)$ as $t \to 0+$ exists and is a holomorphic function on $S(-d_{i+1} + \theta, -d_i - \theta)$ satisfying the same inequality (64).

**Proof.** Set $L = hT^{(1)} + hT^{(2)} + hP$, then the identity (27) is formally equivalent to the integral equation

$$B_{z_2}(\hat{Q})(\xi; t) = B_{z_2}(\hat{Q})(\xi; 0) + \int_0^\xi \left( L \cdot B_{z_2}(\hat{Q})(\xi; t) - B_{z_2}(\hat{Q})(\xi; t) \cdot \delta L \right) dx. \quad (65)$$

To study the integral equation, we employ an iteration, by beginning $B_{z_2}^{(0)}(\hat{Q})(\xi; t) \equiv 0$, and plugging $B_{z_2}^{(m)}(\hat{Q})(\xi; t)$ into the right hand side of (65) and determining $B_{z_2}^{(m+1)}(\hat{Q})(\xi; t)$ from the left hand side. The sequence so obtained is holomorphic in $G$, the largest star-shaped region that does not contain any point $\xi$ for which $hu_t(1)/t + hu_2(2) + \xi$ have an eigenvalue in common. In particular, the set $G$ contains the sector $S(-d'(t), -d(t))$ in the $\xi$ plane, and therefore contains the sector $S(-d_{i+1} + \theta, -d_i - \theta)$ for the sufficiently small $\theta > 0$.

Now let $\rho$ be any ray in the sector $S(-d_{i+1} + \theta, -d_i - \theta)$ of $\xi$ plane, and let $\varepsilon > 0$ be small enough such that $S(\rho - \varepsilon, \rho + \varepsilon)$ is inside $S(-d_{i+1} + \theta, -d_i - \theta)$. Let us show that each $B_{z_2}^{(m)}(\hat{Q})(\xi; t)$ has exponential growth of order $1$ at $\infty$. For this purpose, let us set $a = |A|$. By comparing the coefficient of $\xi^{k-1}$ of the iterated formula of $B_{z_2}^{(m)}(\hat{Q})(\xi; t)$ for all $m$, inductively we get the estimates of the form

$$|B_{z_2}^{(m)}(\hat{Q})(\xi; t)| \leq \sum_{k \geq 1} b_k^{(m)} |\xi|^{k-1}/\Gamma(k), \quad (66)$$

where $b_k^{(m)}$ are determined by the recursive relation

$$b_k^{(m+1)} = c \cdot (a + 2ab_k^{(m)}), \quad (67)$$

where $c$ is a constant which arises when solving the left hand side of (65). Let $(1)$ be any ray in the sector $S(\rho - \varepsilon, \rho + \varepsilon)$ for some constant $\beta > 0$.

Now for every $k$, the numbers $b_k^{(m+1)}$ monotonically increase with respect to $m$ and become constant when $m \geq k$ (due to the fact that the initials $b_k^{(0)} = 0$ for all $k$). The limiting values $b_k$ satisfy the same recursion equation as (67), which implies that $b_k$ cannot grow faster than $\beta^k$ for some constant $\beta > 0$. Therefore, each $B_{z_2}^{(m)}(\hat{Q})(\xi; t)$ is estimated by $\alpha e^{\beta |\xi|}$ with suitably $\alpha, \beta$ independent of $t \in (0, \delta)$.

Next let us show the convergence of $B_{z_2}^{(m)}(\hat{Q})(\xi; t)$ as $m \to \infty$. It can be done by deriving estimates for the differences $B_{z_2}^{(m)}(\hat{Q})(\xi; t) - B_{z_2}^{(m-1)}(\hat{Q})(\xi; t)$ and turning the sequence into a telescoping sum: similar to the estimate (66), inductively we get, from the iterated formula (65) for $B_{z_2}^{(m)}(\hat{Q})(\xi; t) - B_{z_2}^{(m-1)}(\hat{Q})(\xi; t)$ and $B_{z_2}^{(m-1)}(\hat{Q})(\xi; t) - B_{z_2}^{(m-2)}(\hat{Q})(\xi; t)$, that

$$|B_{z_2}^{(m)}(\hat{Q})(\xi; t) - B_{z_2}^{(m-1)}(\hat{Q})(\xi; t)| \leq \sum_{k \geq m} w^{k-1} |\xi|^{k-1}/\Gamma(k), \quad (68)$$

where $w > 0$ is a sufficient large constant that can be chosen independent of $t$.

The estimate (68) shows that $B_{z_2}^{(m)}(\hat{Q})(\xi; t)$ converges uniformly on every compact subset of the region $S(\rho - \varepsilon, \rho + \varepsilon) \subset S(-d_{i+1} + \theta, -d_i - \theta)$, and uniformly with respect to $t \in (0, \delta)$. Since the direction $\rho$ is arbitrary, it proves the first part.
For the second part, note that the limit $B^{(m)}_{-2}(\varphi)(\xi; 0)$ of each iterated term $B^{(m)}_{-2}(\varphi)(\xi; t)$ as $t \to 0^+$ exists and is a holomorphic function on $S(-d_{i+1} + \theta, -d_i - \theta)$. Furthermore, $B^{(m)}_{-2}(\varphi)(\xi; t)$ converges as $t \to 0^+$ uniformly on every compact subset of the region $S(\rho - \varepsilon, \rho + \varepsilon)$. In the end, since the constants $b^{(m)}_k$ in (66) and the constant $w$ in (68) are independent of $t \in (0, \delta)$, we get the convergence of $B^{(m)}_{-2}(\varphi)(\xi; 0)$ as $m \to \infty$, as well as the estimate of the resulting function $B^{(m)}_{-2}(\varphi)(\xi; 0)$ as $m \to \infty$. 

**Laplace transform.** The Laplace transform of the function $B_{-2}(\varphi)(\xi; t)$ in the direction $\rho$ is a function (in the Laplace plane of the initial variable $z_2$) defined by

$$Q_\rho(z_2; t) = 1 + \int_{\xi=0}^{+\infty} e^{-z_2 t} B_{-2}(\varphi)(\xi; t)d\xi.$$ 

Here the line of integration is the line with argument $\rho$. The inequality (64) ensures that for any $-d_{i+1} + \theta \leq \rho \leq -d_i - \theta$, the integrand is indeed defined on the integral path, and that for any fixed $\beta' > \beta$ the integral exists for all $z_2 \in R(\rho, \beta')$ and $t \in (0, \delta)$, where the domain

$$R(\rho, \beta') := \{ z_2 \in \mathbb{C} \mid \text{Re}(z_2 e^{\rho}) > \beta', |\rho + \arg(z_2)| < \pi/2 \}.$$ 

By Lemma 3.15, there exists a continuous function $f_{\rho+}(z_2)$ such that $Q_\rho(z_2, t) \sim f_{\rho+}(z_2)$ as $t \to 0^+$. Next, let us show that $f_{\rho+}(z_2)$ approaches 1 as $z_2 \to \infty$. To see this, we only need to show that

**Lemma 3.16.** There exist constants $C, D > 0$ (independent of $t$) such that

$$|Q_\rho(z_2; t) - \sum_{m=0}^{N-1} Q'_m(t) z_2^{-m}| \leq C N e^{-N|z_2|} D^N, \quad \text{for all } z_2 \in R(\rho, \beta'), \, t \in (0, \delta), \quad N \in \mathbb{N}_+.$$ 

**Proof.** For a fixed $t$, the proof of the inequality (69) is standard, e.g., [29, Theorem 5.3.9]. In the following, we will go through the proof given in [29], and show that the involved constants $D$ and $L$ can be chosen independent of $t \in (0, \delta)$.

Now without lose of generality, let us assume that $\rho = 0$. By the inequality (64), there exist a constant $\varepsilon$ such that $B_{-2}(\varphi)(\xi; t)$ is holomorphic in the union of $\{ \xi : |\xi| \leq 1/K \}$ and the sector $\{ -\varepsilon < \arg(\xi) < \varepsilon \}$. Just as in [29, Theorem 5.3.9], let us take a point $b$ with argument $\pi/4$ and small enough norm $|b| < 1/K$ such that the path, following a straight line from 0 to $b$ and continues along a horizontal line from $b$ to $+\infty$, lies in the domain $\{ \xi : |\xi| \leq 1/K \} \cup \{ -\varepsilon < \arg(\xi) < \varepsilon \}$ of the $\xi$ plane. Since the path is homotopy to $[0, +\infty)$, by the Cauchy’s theorem, the Laplace integral $Q_{\rho=0}(z_2; t)$ decomposes to $Q_{\rho=0}(z_2; t) = 1 + Q^b(z_2; t) + Y^b(z_2; u)$ along the path, where

$$Q^b(z_2; t) = \int_{\xi=0}^{b} e^{-z_2 \xi} B_{-2}(\varphi)(\xi; t)d\xi, \quad Y^b(z_2; t) = \int_{\xi=b}^{+\infty} e^{-z_2 \xi} B_{-2}(\varphi)(\xi; t)d\xi.$$ 

On the one hand, given $0 < \gamma < \pi/2$, following [29, Lemma 1.3.2], we have

$$|Q^b(z_2; t) - \sum_{m=0}^{N-1} Q'_m(t) z_2^{-m}| \leq C' N e^{-N|z_2|} D^N, \quad \text{for } -\pi/4 + \gamma < \arg(z_2) < 3\pi/4 - \gamma, \quad N \in \mathbb{N}_+,$n

where for any given $t \in (0, \delta)$ the constants $C', D'$ are

$$C' := \sum_{m \geq 1} \frac{|Q'_m(t)|}{\Gamma(m)} b^{-m}, \quad D' := b \cdot \sin(\gamma).$$ 

On the other hand, let us take the constant $\beta' > \beta$, then following the proof of [29, Theorem 5.3.9], we have

$$|Y^b(z_2; t)| \leq pe^{-cz_2}, \quad \text{for } z \in \{ -\pi/4 + \gamma < \arg(z_2) < 3\pi/4 - \gamma \} \cap R(\rho = 0, l),$$

with the constants given by

$$p = \frac{\alpha e^{\beta|b|}}{\beta' - \beta}, \quad c = |b| \cos(\pi/2 - \gamma).$$ 

(72)
The estimation (71) further implies, see e.g., [29 Proposition 1.2.17], for \( z_2 \in \{ -\pi/4 + \gamma < \arg(z_2) < 3\pi/4 - \gamma \} \cap R(\rho = 0, \beta') \)

\[
|Y^b(z_2; t)| \leq C'' N^N e^{-N|z_2|^N/D''}, \quad N \in \mathbb{N},
\]

with the constants \( C'' \) and \( D'' \) determined by \( p \) and \( c \). In conclusion, if we take \( C = \max(C', C'') \) and \( D = \max(D', D'') \), then \( Q_{\rho=0}(z_2; t) \) satisfies the inequality (69) on the domain

\[
\{ -\pi/4 + \gamma < \arg(z_2) < 3\pi/4 - \gamma \} \cap R(\theta = 0, \beta').
\]

According to [29 Theorem 5.3.9], an argument using the symmetry with respect to the real axis, i.e., by choosing \( b \) instead of \( b \) and the corresponding path, shows that \( Q_{\rho=0}(z_2; t) \) satisfies the inequality (69) on the symmetric domain

\[
\{ -3\pi/4 + \gamma < \arg(z_2) < \pi/4 - \gamma \} \cap R(\theta = 0, 0).
\]

In the end, let us check the independence of constants \( C = \max(C', C'') \) and \( D = \max(D', D'') \) on \( t \in (0, \delta) \). First, following the inequality (63), we have \( C' \leq \frac{Kb}{1-Kb} \). Thus we can set \( C' = \frac{Kb}{1-Kb} \), and then the constants \( C, D \) are determined by \( \varepsilon, K, \alpha, \beta, b \) and \( \delta \). By (63) and (64), as a sufficiently small \( \varepsilon \) fixed, those constants can be chosen independent of \( t \in (0, \delta) \). It therefore verifies (69).

Now the Borel-Laplace transform \( Q_{\rho} \) and \( Q_{\rho'} \) of \( \hat{Q}(z_2; t) \), with respect to two directions satisfying \(-d_{1+1} + \theta \leq \rho, \rho' \leq -d_i - \theta \), coincide in the overlapping of their defining domains. See e.g., [29 Proposition 5.3.7] or [4 Section 6.2]. (While if \( \rho \) and \( \rho' \) is not in a same sector bounded by two adjacent Stokes rays, \( Q_{\rho} \) is in general not equal to \( Q_{\rho'} \) at the points where both functions are defined.) Thus the functions \( Q_{\rho} \) for all \(-d_{1+1} + \theta \leq \rho \leq -d_i - \theta \) in the \( \xi \) plane glue together into a holomorphic function \( Q_{d,d'}(z_2; t) \) defined on the domain \( \text{Sect}_i^\theta \times (0, \delta) \). By (63) and (64), there exists a continuous function \( f_{i+}(z_2) \) (the gluing of \( f_{p+}(z_2) \) for all \( \rho \) with \(-d_{1+1} + \theta \leq \rho \leq -d_i - \theta \)) such that \( Q_{d,d'}(z_2; t) \sim f_{i+}(z_2) \) as \( t \rightarrow 0^+ \). Furthermore, by (69), we have

\[
Q_{d,d'}(z_2; t) \sim 1, \quad \text{as} \quad z_2 \rightarrow \infty \quad \text{within} \quad \text{Sect}_i^\theta
\]

uniformly for \( t \in (0, \delta) \). It therefore proves the proposition.

### 3.3.3 Step 3: Proof of the first identity in the part (b) of Theorem 3.10

Let \( Y_{d,d'} \) be the function as in Section 3.2 and \( Y_{d,d'} \) the corresponding solution given by (56).

#### Proposition 3.17. The solution satisfy

\[
Y_{d,d'}(z, t) = W_i(z, zt)F_i^{(2)}(z) e^{\pi i h\delta P}, \quad \text{on the domain} \quad z \in \text{Sect}_i^\theta \text{and} \quad t \in (0, \delta), \tag{73}
\]

\[
Y_{d,d'}(z, t) = W_i(z, zt)F_i^{(2)}(z) e^{\pi i h\delta P}, \quad \text{on the domain} \quad z \in \text{Sect}_i^\theta \text{and} \quad t \in (\delta, 0). \tag{74}
\]

**Proof.** Let us only prove (73). Note that the system of equations (23) and (24) becomes the system of (56) and (57) in terms of the coordinates transform \( z_1 = z \) and \( z_2 = zt \). Following Lemma 3.12, \( Y_{d,d'}(z, t) \) and \( W_i(z, zt)F_i^{(2)}(z) e^{\pi i h\delta P} \) satisfy the same system. Thus, we have \( Y_{d,d'} = W_i F_i^{(2)} e^{\pi i h\delta P} \) for a constant \( C \) (independent of \( z_1 \) and \( z_2 \)).

Recall that \( F_i^{(2)}(z_2) = H_i^{(2)}(z_2) \cdot e^{hu^{(2)}z_2} z_2^{\theta T^{(2)}}, \) and \( W_i = K_i \cdot e^{h z_1 u^{(1)}(z_1 h\delta T^{(1)}(z_1 - z_2))}. \) To proceed, let us write

\[
Y_{d,d'}(z_1, z_2) = e^{-\pi i h\delta P} F_i^{(2)}(z_2) W_i(z_1, z_2)^{-1}
\]

\[
Q_{d,d'}(z_1, z_2) \cdot (z_2/z_1 - 1)^{h\delta P} e^{-\pi i h\delta P} e^{hu^{(1)}(z_1) z_1^{\theta T^{(1)}(z_1 - z_2))} + h\delta P \cdot H_i^{(2)}(z_2)^{-1} e^{-hu^{(1)}(z_1) z_1^{-\theta T^{(1)}(z_1 - z_2))} - h\delta P \cdot K_i^{-1}.
\]

Here in the last identity we use the fact that \( e^{hu^{(1)}(z_1) z_1^{\theta T^{(1)}(z_1 - z_2))} \) commutes with the coefficient matrix of the equation (47) and the initial value as \( z_2 \rightarrow \infty \), therefore commutes with the function \( H_i^{(2)}(z_2) \).
Now following the prescribed asymptotics in the definition of canonical solutions, for sufficiently large \( z_2 \) the function \( H^{(2)}_i(z_2) \) is sufficiently close to 1. Furthermore, for the sufficiently large but fixed \( z_2 \), both \( Q_{d,d'} \) (following from Proposition \( 3.13 \)), \( K_1(z_1,z_2) \) (following from Lemma \( 3.11 \)) and \( (1 - z_2/z_1)^{h\delta P} \) are asymptotic to 1 as \( z_1 \to \infty \) within \( \text{Sect}_t^\theta \subset \text{Sect}_t \). It thus gives \( W_i F_i^{(2)} e^{\pi i h\delta P} C e^{-\pi i h\delta P} (W_i F_i^{(2)})^{-1} \to 1 \) for \( z_2 \to \infty \) and \( z_2/z_1 = t \to 0 \) in proper domains. Since the exponential terms dominate and the opening angle of \( \text{Sect}_t^\theta \) is bigger than \( \pi \), the constant \( C \) must be 1.

Note that the solution \( W_i(z,t) F_i^{(2)}(z,t) e^{\pi i h\delta P} \) is associated to the two adjacent rays \( d_i \) and \( d_{i+1} \) of \( \text{Sect}_t^\theta \), while the solution \( Y_{d,d'}(z,t) \) is associated to the two rays \( d(t) \) and \( d'(t) \) of \( (23) \). However, there are many other adjacent pair of rays of \( (23) \) that are bounded by \( d_i \) and \( d_{i+1} \), \( d(t) \) and \( d'(t) \) is only one of them. Actually, we can generalize Proposition \( 3.17 \)

**Proposition 3.18.** Assume that \( l(t), l'(t) \) are any pair of adjacent rays of \( (23) \) that are bounded by \( d_i \) and \( d_{i+1} \) for all \( t \in (-\delta,0) \cup (0,\delta) \), and \( Y_{l,l'}(z,t) \) the corresponding solutions on the domain \( t \in (-\delta,0), z \in \text{Sect}_{l,l'}(t) \) and \( t \in (0,\delta), z \in \text{Sect}_{l,l'}(t) \) as in Theorem \( 3.6 \). Then there exists a sufficiently small positive real number \( 1/\beta' > 0 \) such that

\[
Y_{l,l'}(z,t) = W_i(z,t) F_i^{(2)}(z,t) e^{\pi i h\delta P} \quad \text{for } z \in \text{Sect}_{l,l'}(t) \text{ and } t \in (0,\delta) \cap (0,1/\beta'),
\]

\[
Y_{l,l'}(z,t) = W_i(z,t) F_i^{(2)}(z,t) e^{\pi i h\delta P} \quad \text{for } z \in \text{Sect}_{l,l'}(t) \text{ and } t \in (-\delta,0) \cap (-1/\beta',0).
\]

**Proof.** As a consequence of Proposition \( 3.17 \) (mainly based on Proposition \( 3.13 \)), the function \( W_i(z_1,z_2) \) given in Lemma \( 3.11 \) has the following asymptotics

\[
W_i(z_1,z_2) \cdot e^{-h z_1 u^{(1)}(1)/z_1^3 h T(1)} (z_1 - z_2)^{-h\delta P} \sim 1
\]

as \( z_1 \to \infty \) along the path \( z_2/z_1 \) being a constant inside \( (-\delta,0) \cup (0,\delta) \). Actually we can prove the asymptotics \( (77) \) holds in a larger region. To see this, let us check the Borel-Laplace transform of \( \hat{W} \) in details. following Lemma \( 3.11 \) the unique formal fundamental solution of the equation \( (19) \) takes the form

\[
\hat{W}(z_1,z_2) = (1 + \sum_{m \geq 1} K_m(z_2) z_1^{-m}) \cdot e^{h z_1 u^{(1)}(1)/z_1^3 h T(1)} (z_1 - z_2)^{h\delta P},
\]

where \( K_m \) is recursively determined by \( (55) \). Let us introduce \( K'_m := z_2^{-m} K_m \). Then multiplying by \( z_2^{-m} \) on the two sides of the identity \( (55) \) leads to

\[
z_2 \cdot [K'_{m+1}(z_2), u^{(1)}] = \left( m/h + T(1) + P \right) K'_m(z_2) - K'_m(z_2) \cdot \left( \delta T(1) + \delta P \right) + \sum_{r=1}^m (P - \delta P) K'_m(z_2).
\]

Let us think of \( \hat{K} = 1 + \sum_{m \geq 1} K'_m(z_2) \cdot (z_2/z_1)^m \) as a power series in the new variable \( t = z_2/z_1 \). From the above recursive relation, we see that the norm of its coefficient \( K'_{m+1} \) is proportional to \( 1/|z_2| \).

Denote by \( B_t(\hat{K}) \) the formal Borel transform of the power series \( \hat{K} - 1 \) in the variable \( t = z_2/z_1 \) (in the Borel plane with complex variable \( \xi \), i.e.,

\[
B_t(\hat{K})(\xi; z_2) := \sum_{m \geq 1} K'_m(z_2) \Gamma(m) \xi^m - 1.
\]

Set \( V = h T(1) + h P \), then the identity \( (55) \) is formally equivalent to the integral equation

\[
B_t(\hat{K})(\xi; z_2) \cdot h z_2 u^{(1)}(1) - (h z_2 u^{(1)}(1) + \xi) \cdot B_t(\hat{K})(\xi; z_2) = V - \delta V + \int_{x=0}^\xi \left( V \cdot B_t(\hat{K})(\xi; z_2) - B_t(\hat{K})(\xi; z_2) \cdot \delta V + B_t(\hat{K})(\xi; z_2) \cdot \sum_{k \geq 1} \frac{(\xi - x)^k}{k!} \right) dx.
\]

Here we remark that \( \hat{K} \) is a formal power series in the variable \( z_1^{-1} \) (negative power), while its Borel transform is a power series in \( t \) (positive power). It explains the minus sign of the arguments \( d_i \) and \( d_{i+1} \) in the following lemma.
Lemma 3.19. For any \( z_2 = |z_2|e^{i\delta} \) and any small positive real number \( \varepsilon \), the Borel transform \( B_t(\hat{K})(\xi; z_2) \) can be analytically continued to the sector \( S(-d_{i+1} + \delta + \varepsilon, -d_i + \delta - \varepsilon) \) of the \( \xi \) plane, and if \( z_2 \) keeps a fixed positive distance from 0 then there exist constants \( \alpha, \beta > 0 \) (independent of \( |z_2| \)) such that

\[
|B_t(\hat{K})(\xi; z_2)| \leq \frac{1}{|z_2|} e^{\beta|\xi|}, \quad \text{for all } \xi \in S(-d_{i+1} + \delta + \varepsilon, -d_i + \delta - \varepsilon). \tag{78}
\]

It follows from the similar argument as in Lemma 3.15, one applies the same estimate of the integral equation for the product function \( z_2 \cdot B_t(\hat{K})(\xi; z_2) \) to find the constants \( \alpha \) and \( \beta \). Here we just stress that the constant \( \alpha \) in (78) relies on the distance from the numbers \( hu_i z_2 - hu_j z_2 \) and the sector \( S(-d_{i+1} + \delta + \varepsilon, -d_i + \delta - \varepsilon) \), which is independent of the norm \( |z_2| \) (as long as \( |z_2| \) is bigger than a fixed positive real number as in our assumption) but depends on the number \( \varepsilon \) and the argument \( \delta \) of \( z_2 \).

Now the Laplace transform of the function \( B_t(\hat{K})(\xi; z_2) \) along the direction \( \rho \), with \(-d_{i+1} + \delta + \varepsilon < \rho < -d_i + \delta - \varepsilon\), is a function (in the Laplace plane of the initial variable \( t = z_2/z_1 \)) defined by

\[
K_\rho(t; z_2) = 1 + \int_{\xi=-\infty}^{+\infty} e^{-\xi t} B_t(\hat{K})(\xi; z_2) d\xi.
\]

The inequality (78) ensures that for any fixed real number \( \beta’ > \beta > 0 \), the integral exists for all \( t \in R(\rho, \beta’) \), where the domain

\[
R(\rho, \beta’) := \{ t \in \mathbb{C} | \Re(te^{i\rho}) > \beta’, |\rho + \arg(t)| < \pi/2 \}.
\]

Similar to the standard estimate (69), we can use the estimate (78) to determine the asymptotics of \( K(t; z_2) \) as \( t \in R(\rho, \beta’) \) and the constants involved. In particular,

Lemma 3.20. There exist constants \( C, D > 0 \) (independent of \( |z_2| \)) such that

\[
|K_\rho(t; z_2) - \sum_{m=0}^{N-1} K_m’(z_2) t^{-m}| \leq \frac{1}{|z_2|} C N^N e^{-N|t|} D^N, \quad \text{for all } t \in R(\rho, \beta’), \quad N \in \mathbb{N}_+ . \tag{79}
\]

All the functions \( K_\rho(t; z_2) \) coincide with each other in their common domain as \(-d_{i+1} + \delta + \varepsilon < \rho < -d_i + \delta - \varepsilon\), and thus glue to a function \( K_\rho(t; z_2) \) which satisfies the inequality (79) as

\[
t \in S(-d_{i+1} + \delta + \varepsilon - \pi/2, -d_i + \delta - \varepsilon + \pi/2) \quad \text{and } |t| < \varepsilon_1, \quad \text{for a fixed sufficiently small number } \varepsilon_1.
\]

Therefore, when the argument \( \delta \) of \( z_2 \) satisfies

\[
d_i + \varepsilon - \pi/2 < \delta < d_{i+1} - \varepsilon + \pi/2,
\]

the above sector in the \( t \) plane includes the positive real axis.

Since the function \( W_i(z_1, z_2) \) given in Lemma 3.11 is obtained by the Borel-Laplace transform, thus by definition

\[
W_i(z_1, z_2) = K_i(z_2/z_1; z_2) e^{h z_1 u^{(1)}_1(z_2 - z_1)^{h \delta P}}.
\]

By (79) and (80), for any fixed \( t = z_2/z_1 \in (0, 1/\beta’) \) the function

\[
K(z_2/z_1; z_2) \to 1, \quad \text{as } z_2 \to \infty \text{ within } S(d_i + \varepsilon - \pi/2, d_{i+1} - \varepsilon + \pi/2).
\]

Note that \( \varepsilon \) can be any small positive real number. In particular, for any fixed \( t \in (0, \delta) \cap (0, 1/\beta’) \) the number \( \varepsilon \) can be small enough such that \( S(d_i + \varepsilon - \pi/2, d_{i+1} - \varepsilon + \pi/2) \) contains the sector \( \text{Sect}_{l,P}(t) \). Using the same argument as in Proposition 3.17, it implies that for any fixed \( t \in (0, \delta) \cap (0, 1/\beta’) \), the two solutions \( Y_i(t, z) \) and \( W_i(z, zt) F_i^{(2)}(zt) \cdot e^{v_1 h \delta P} \) have the same prescribed asymptotics as \( z \to \infty \) within \( \text{Sect}_{l,P}(t) \subset \text{Sect}_i \), Then by the uniqueness statement of the solution in sectorial region \( \text{Sect}_{l,P}(t) \) with opening bigger than \( \pi \), the identity (75) holds.
3.3.4 Step 4: Triviality of the Stokes factors and extension with respective to the \( t \) variable

Let us denote by \( d(t), d(t'), d''(t) \) three separate adjacent anti-Stokes rays of the equation (23) (in a positive sense) for \( t \in (0, \delta) \) that are bounded between \( d_i \) and \( d_{i+1} \). Then Proposition 3.18 implies that for any \( t \in (0, \delta) \)

\[
Y_{d,d'} = Y_{d',d''} \text{ on } \text{Sect}_{d,d'} \cap \text{Sect}_{d',d''}.
\]  

(81)

Recall that the Stokes factor \( S_{d'} \) of (23) is defined by \( Y_{d',d''} = Y_{d,d'} \cdot S_{d'} \). As a consequence, we get

**Corollary 3.21.** If \( t \in (0, \delta) \), then \( S_{d'} \) must be \( \text{Id} \) unless the anti-Stokes ray \( d'(t) \) has the argument \(-\arg(h(u_i - u_j))\) for some \( i \neq j \).

The proof of Proposition 3.17 implies the triviality of some Stokes factors of the equation (23). That is

**Proposition 3.22.** If \( t \in (0, 1) \), then \( S_{d'} \) must be \( \text{Id} \) unless the anti-Stokes ray \( d'(t) \) equals to \(-\arg(h(u_i - u_j))\) for some \( i \neq j \).

**Proof.** Let us move \( t \) from the neighbourhood of 0 to 1. Let us assume that as \( t \) crosses from one side of a point \( t_0 \in (0, 1) \) to the other, only two rays \( d_-(t), d'_-(t) \) collide and two new rays \( d_+(t), d'_+(t) \) emerge. That is if we denote by \( d_1(t), d_-(t), d'_+(t), d_+(t) \) four separate adjacent anti-Stokes rays of (23) in a positive sense for \( t \in (t_-, t_0) \), and \( d_1(t), d_+(t), d'_+(t), d_-(t) \) four separate adjacent rays for \( t \in (t_0, t_+) \), then \( d_1(t_0) \neq d_{t_0}(t_0) = d'_+(t_0) \neq d_-(t_0) \) at \( t = t_0 \). We can think of \( d_-(t) \) becomes \( d'_+(t) \) after \( t \) passes through \( t = t_0 \). In the following, we show that if the Stokes factor \( S_{d'} \) associated to \( d_-(t) \) for \( t_+ < t < t_0 \) is identity, then so is the Stokes factor \( S_{d'} \) associated to \( d'_+(t) \) for \( t_0 < t < t_+ \).

Let us recall that the Stokes factors \( S_{d_+}, S_{d'_+} \), independent of \( t \) and \( z \), are determined by

\[
Y_{d_-,d'_-} = Y_{d_1,d_-(t)} \cdot S_{d_+} \text{ in } \text{Sect}_{d_1,d_-(t)} \cap \text{Sect}_{d_-,d'_-(t)}, \quad Y_{d'_+,d_+} = Y_{d_1,d'_+(t)} \cdot S_{d'_+} \text{ in } \text{Sect}_{d_1,d'_+(t)} \cap \text{Sect}_{d'_+,d_+(t)},
\]

and

\[
Y_{d_+,d'_+} = Y_{d_1,d_+(t)} \cdot S_{d'_+} \text{ in } \text{Sect}_{d_1,d_+(t)} \cap \text{Sect}_{d_+,d'_+(t)}, \quad Y_{d'_-,d_+} = Y_{d_1,d'_-(t)} \cdot S_{d'_+} \text{ in } \text{Sect}_{d_1,d'_-(t)} \cap \text{Sect}_{d'_-,d_+(t)}.
\]

Thus we have

\[
Y_{d'_-,d_+} = Y_{d_1,d_-(t)} \cdot S_{d_+} \cdot S_{d'_+} \quad \text{and} \quad Y_{d'_+,d_+} = Y_{d_1,d_+(t)} \cdot S_{d_+} \cdot S_{d'_+}.
\]

Since \( d_1(t), d_-(t) \) (resp. \( d_1(t), d_+(t) \)) are adjacent and do not collide for \( t \in (t_-, t_0) \) (resp. \( t \in [t_0, t_+] \)), by Theorem 3.6 we must have \( Y_{d_1,d_+} = Y_{d_1,d_+} \) after the continuation of \( t \) from \((t_-, t_0)\) to \((t_0, t_+).\) Similarly, we have \( Y_{d'_-,d_+} = Y_{d'_-,d_+} \) after the continuation of \( t \). In the end, we get

\[
S_{d_+} = S_{d_+} \cdot S_{d'_+}.
\]

Using the uniqueness of the factorization of a matrix into an ordered product of two matrices satisfying the triangular property (see Lemma 3.3), we conclude that if \( S_{d_+} \) is identity matrix, so is \( S_{d'_+}. \) We remark that in the above we assume that there are only two rays \( d_-(t), d'_-(t) \) colliding at \( t_0 \). The extension to the case of collision of many anti-Stokes rays at a point \( t_0 \) is direct.

Thus, we see that as \( t \) varies from 0 to 1, the effective anti-Stokes rays \(-\arg(h(u_i - u_j))\) (those correspond to none identity Stokes matrices) are preserved. In the meanwhile, the ineffective anti-Stokes rays (whose corresponding Stokes factors are identity as \( t \) near 0) moves on the complex plane as \( t \) moves toward 1 along the real axis, and can cross each other and cross the \((t\)-independent\) effective rays as \( t \) crosses through some particular point. And after crossing any particular point, the ineffective rays are still ineffective. It therefore finishes the proof. \[\square\]

Proposition 3.18 or the proof of Proposition 3.22 leads to a direct corollary.

**Corollary 3.23.** The solution \( Y_{12}(z, t) \) of \( (23) \) and \( (24) \) defined on \( \text{Sect}_z \times (0, 1) \), as the analytic continuation of the solution \( Y_{d,d'}(z, t) \) given in Definition 3.9, has the following asymptotics for all \( t \in (0, 1) \)

\[
Y_{12}(z, t) \cdot e^{-h(z(u^{(1)}+t_u^{(2)})z^2)-(hT^{(1)}(tz)hT^{(2)}(tz)-h^2P)} \to 1, \text{ as } z \to \infty \text{ within } \text{Sect}_z.
\]
3.3.5 Step 5: Proof of the second identity in the part (b) of Theorem 3.10

Let us consider the function \( Y_i(z, t) \) on \((z, t) \in \text{Sect}_i \times (0, 1)\). The second identity in the part (b) of Theorem 3.10 is equivalent to

**Proposition 3.24.** For fixed \( zt - z \in \text{Sect}_{i+4} \), the function \( \overline{W}_i(z, zt - z) := Y_i(z, t) X_{i+4}^{(2)}(zt - z)^{-1} \) is the unique solution of \((53)\) with the prescribed asymptotics in \((54)\).

**Proof.** Let us take the coordinates transform \( \omega_1 = z \) and \( \omega_2 = zt - z \). The anti-Stokes rays of \((53)\) take the form of \(-\text{arg}(h(u_i - u_j + u_k - u_l))\). Let us take any Stokes supersector \( \text{Sect}'_i \) of \((53)\) which is contained inside \( \text{Sect}_i \) (the sector \( S(d_i - \pi/2, d_{i+1} + \pi/2) \) of the \( \omega_1 \) plane), and denote by \( \overline{W}_i(\omega_1 = z, \omega_2 = zt - z) \) the corresponding solution of \((53)\) with prescribed asymptotics \((54)\) on \( \text{Sect}'_i \). Then based on the equation satisfied by \( \overline{W}_i \) and \( X_{i+4}^{(2)} \), we check that the function

\[
\overline{W}_i(z, zt - z) X_{i+4}^{(2)}(zt - z),
\]

defined for \( z \in \text{Sect}'_i \subset \text{Sect}_i \) and \( t \in (0, 1) \), satisfies the system of equations \((23)\) and \((24)\). Similar to the proof of Proposition 3.18, a detailed analysis of the Borel-Laplace transform shows that there exists a sufficiently small positive real number \( \varepsilon > 0 \) such that

\[
\overline{W}_i(z, zt - z) e^{-hz(1) + u(2)} z^{-h\delta T(1) - h\delta T(2)} \sim 1, \quad z \to \infty \text{ within } \text{Sect}'_i \text{ for any fixed } t \in (1 - \varepsilon, 1).
\]

Then Corollary 3.23 and the same argument as in Proposition 3.17 lead to the consequence that the solutions \( Y_i(z, t) \) and \( \overline{W}_i(z, zt - z) X_{i+4}^{(2)}(zt - z) \) have the same asymptotics, and actually

\[
Y_i(z, t) = \overline{W}_i(z, zt - z) X_{i+4}^{(2)}(zt - z) \quad \text{for } (z, t) \in \text{Sect}'_i \times (1 - \varepsilon, 1).
\]

Since \( Y_i(z, t) \) is defined for \( z \in \text{Sect}_i \) and \( t \in (0, 1) \). It particular implies that for any two Stokes supersector \( \text{Sect}'_i \) and \( \text{Sect}'_j \) of \((53)\) that are contained inside the same \( \text{Sect}_i \), the corresponding solutions \( \overline{W}_i \) and \( \overline{W}_j \) of \((53)\), defined on \( \text{Sect}'_i \) and \( \text{Sect}'_j \), coincide in their common domain \( \text{Sect}'_i \cap \text{Sect}'_j \), and therefore glue together to a unique function, denoted by \( \overline{W}_i \), with the prescribed asymptotics in \((54)\). (It shows that the Stokes factors associated to the anti-Stokes rays \(-\text{arg}(h(u_i - u_j + u_k - u_l))\) of \((53)\), that do not coincide with \( d_i \) for \( i = 0, 1, ..., 2l - 1 \), are the identity.) It finishes the proof. ■

**Remark 3.25.** From the above proof, we see that there exist sufficiently small positive real numbers \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) such that

- the \( z \to \infty \) asymptotics of \( W_i(z, zt) \) as \( t \) fixed is valid for \( t \in (-\varepsilon_1, 0) \cup (0, \varepsilon_1) \), Proposition 3.18;
- the \( z \to \infty \) asymptotics of \( \overline{W}_i(z, zt - z) \) is valid for \( t \in (1 - \varepsilon_2, 1) \cup (1, 1 + \varepsilon_2) \), Proposition 3.24;
- the \( z \to \infty \) asymptotics of \( \overline{W}_i(z, zt) \) as \( t \) fixed is valid for \( t \in (-\infty, -1/\varepsilon_3) \cup (1/\varepsilon_3, +\infty) \).

Since the interval \((-\infty, -1/\varepsilon_3)\) may not overlap with \((-\varepsilon_1, 0)\), we can not directly state that the left hand side \( \overline{W}_i(z, zt) e^{\pi i hP} \) of the identity \((43)\), in part (a) of Theorem 3.10, is equal to the right hand side \( W_i(z, zt) e^{\pi i hP} \) (after the necessary analytic continuation) by comparing their asymptotics. So it is necessary to introduce the solution \( Y_{i1}(z, t) \) connecting them, similarly, the solutions \( Y_{i2}(z, t) \) and \( Y_{i3}(z, t) \).

3.4 The connection formula on the complex \( t \) plane

Section 3.3 solves the connection problem between the solutions defined in Theorem 3.6 for real \( t \in \mathbb{R} \setminus \{0, 1\} \). In this subsection, we find the connection formula between these solutions on the complex \( t \) plane. To match up with the definition of Stokes matrices, see the proofs of Lemma 3.27 to Lemma 3.29 let us introduce the solutions modified by the corresponding half formal monodromy.
Definition 3.26. For any $k = 1, 2, 3$, we introduce the solutions $Y_{\pm k}$ on the regions $D_{\pm k}$

\[
\begin{align*}
Y_{+3}(z, t) &= Y_{03}(z, t), & D_{+3} &= \{ z \in \text{Sect}_0, \ 1 < t \}, \\
Y_{+2}(z, t) &= Y_{02}(z, t) \cdot e^{-\pi i \delta P}, & D_{+2} &= \{ z \in \text{Sect}_0, 0 < t < 1 \}, \\
Y_{+1}(z, t) &= Y_{01}(z, t) \cdot e^{-\pi i \delta P + \delta T(2)}, & D_{+1} &= \{ z \in \text{Sect}_0, \ t < 0 \}, \\
Y_{-1}(z, t) &= Y_{11}(z, t) \cdot e^{-\pi i \delta T(1)}, & D_{-1} &= \{ z \in \text{Sect}_t, \ t < 0 \}, \\
Y_{-2}(z, t) &= Y_{02}(z, t) \cdot e^{-\pi i (\delta T(1) + \delta T(2))}, & D_{-2} &= \{ z \in \text{Sect}_t, 0 < t < 1 \}, \\
Y_{-3}(z, t) &= Y_{03}(z, t) \cdot e^{-\pi i (\delta P + \delta T(1) + \delta T(2))}, & D_{-3} &= \{ z \in \text{Sect}_t, \ 1 < t \}.
\end{align*}
\]

From now on, let us switch to the complex $t$ plane. The solutions $Y_{\pm k}(z, t)$ are in general different, after being analytic continued to a common domain on the complex $t$ plane. In the following, we derive the connection formula between them.

\[
\begin{array}{ccc}
  & 0 & 1 \\[0.5cm]\hline
  \nearrow & t & \searrow
\end{array}
\]

Figure 1: Transposition of $t$ near 1 such that $t$ passes above 1.

Recall that we have made the assumption that $u \in h_{\text{reg}}$ and the sector $\text{Sect}_0$ are chosen such that the Stokes matrices $S_+(u)$ and $S_-(u)$ are upper and lower triangular respectively. The extension of the following lemma to the general case is simple, provided the permutation matrix is accounted for.

Lemma 3.27. Under the assumption, we have that

\[
Y_{+3}(z, t) = Y_{+2}(z, t) \cdot R^{(12)},
\]

after the analytic continuation of $Y_{+3}(z, t)$ to the defining domain $D_{+2}$ of $Y_{+2}$ along a simple positive path in the complex $t$ plane, as shown in Figure 2. Here $R$ is the standard $R$-matrix given in (20) and recall that $R^{(12)} = 1 \otimes R \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$.

Proof. It follows from the first identity in part (c) of Theorem 3.10 that

\[
Y_{+3}(z, t) = \overline{W}_0(z, zt - z) X_0^{(2)}(zt - z),
\]

and from the second identity in part (b) of Theorem 3.10 that

\[
Y_{+2}(z, t) = \overline{W}_0(z, zt - z) X_1^{(2)}(zt - z) e^{-\pi i \delta P}.
\]

Here the corresponding index $i$ of the anti-Stokes ray $d_i$ in Theorem 3.10 is just $i = 0$. Note that $\overline{W}_0(z_1, z_2)$ does not contribute monodromy from the continuation on $z_2$ plane. Thus, after the analytic continuation of $Y_{+3}$, we have

\[
Y_{+2}(z, t)^{-1} Y_{+3}(z, t) = e^{\pi i \delta P} X_1^{(2)}(zt - z)^{-1} X_0^{(2)}(zt - z) = S_+^{(2)}(P)^{-1}.
\]

Here $S_+^{(2)}(P)$ is simply the Stokes matrix of (48) associated to $\text{Sect}_0$, i.e., the ratio of $K_0^{(2)}$ and $K_1^{(2)}$ in the intersection $\text{Sect}_0 \cap \text{Sect}_t$. The rest is to show that the inverse of the Stokes matrix $S_+(P)^{-1}$ of the equation

\[
\frac{dX}{dz} = h(u + \frac{P}{z}) X
\]

for a $\text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n)$ valued function, equals to $R$. The canonical solutions and the Stokes matrices of the equation (48) are the obvious extensions of the ones of (83) from $\text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n)$ to $\text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$.

Let us first consider the $\mathfrak{gl}_2$ example. That is to consider the equation (83) with

\[
u = \begin{pmatrix}
u_1 & 0 & 0 & 0 \\
0 & \nu_1 & 0 & 0 \\
0 & 0 & \nu_2 & 0 \\
0 & 0 & 0 & \nu_2
\end{pmatrix} \quad \text{and} \quad P = -\begin{pmatrix}1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[24\]
It reduces to the computation of the $2 \times 2$ system

$$\frac{dX}{dz} = \begin{pmatrix} hu_1 & 0 \\ 0 & hu_2 \end{pmatrix} X + \frac{1}{z} \begin{pmatrix} 0 & -h \\ -h & 0 \end{pmatrix} X. \quad (84)$$

Applying the formula of Stokes matrices of a general system of rank $2$ in [5 Proposition 8], see also [43], the Stokes matrices of $S_{+}$ are expressed by the gamma functions of eigenvalues $\pm h$ of the residue matrix. Using the Euler’s reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ (for $z \notin \mathbb{Z}$) to rewrite the gamma function by hyperbolic function, we get the Stokes matrix $S_{+}$ of $(84)$

$$S_{+}^{-1} = \begin{pmatrix} 1 & e^{\frac{h}{t}} - e^{-\frac{h}{t}} \\ 0 & 1 \end{pmatrix}.$$ 

Here our assumption ensures that $S_{+}$ is upper triangular. Thus the Stokes matrix $S_{+}(P)$ of the $4 \times 4$ system is

$$S_{+}(P)^{-1} = \begin{pmatrix} e^{\frac{h}{t}} & 0 & 0 & 0 \\ 0 & 1 & e^{\frac{h}{t}} - e^{-\frac{h}{t}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\frac{h}{t}} \end{pmatrix}.$$

Note that it doesn’t depend on $u = \text{diag}(u_1, u_2)$, and coincides with the R-matrix $R$ defined in (20) for $\mathfrak{gl}_2$.

For general $n$, by the definition of (the permutation operator) $P$, the linear system (48) can be composed to multiple $2 \times 2$ and $1 \times 1$ systems. It thus reduces to the computation of $\mathfrak{gl}_2$ case. The lemma then follows by a direct computation, as long as the assumption on the orders of $u_1, \ldots, u_n$ are accounted for. \hfill \blacksquare

Now let us take the analytic continuation of $Y_{+2}$ from $0 < t < 1$ to $t < 0$ along a simple positive path in the complex $t$ plane, as shown in Figure 2.

![Figure 2: Transposition of $t$ near 0 such that $t$ passes above 0.](image)

**Lemma 3.28.** For the connection formula from $Y_{+2}$ to $Y_{+1}$, we have that

$$Y_{+2}(z, t) = Y_{+1}(z, t) \cdot S_{+}^{(2)}(u)^{-1}.$$ \hfill (85)

**Proof.** It follows from the first identity of the part $(b)$ and the second identity of the part $(a)$ of Theorem 3.10 that

$$Y_{+2}(z, t) = W_0(z, zt) F_0^{(2)}(zt) e^{\pi i \text{d} P} \cdot e^{-h \pi i \text{d} P} = W_0(z, zt) F_0^{(2)}(zt),$$

and

$$Y_{+1}(z, t) = W_0(z, zt) F_1^{(2)}(zt) e^{\pi i \text{d} P} \cdot e^{-h \pi i (\text{d} P + \text{d} T^{(2)})} = W_0(z, zt) F_0^{(2)}(zt) e^{-h \pi i T^{(2)}}.$$ 

Note that $W_0(z_1, z_2)$ does not contribute monodromy from the continuation on $z_2$ plane. Thus, after the analytic continuation we have

$$Y_{+1}(z, t)^{-1} Y_{+2}(z, t) = e^{-h \pi i T^{(2)}} F_1^{(2)}(zt)^{-1} F_0^{(2)}(zt) = S_{+}^{(2)}(u)^{-1}.$$ 

Here $S_{+}^{(2)}(u)$ is simply the Stokes matrix of (47) associated to $\text{Sect}_0$, i.e., the ratio of $F_0^{(2)}$ and $F_1^{(2)}$ in the intersection $\text{Sect}_0 \cap \text{Sect}_t$. \hfill \blacksquare

For any $z \in \text{Sect}_+$, let us take the analytic continuation of $Y_{+3}(z, t)$ from $t < 0$ to $t > 1$ along a simple positive path in the complex $t$ plane, as shown in Figure 3.

**Lemma 3.29.** After the analytic continuation of $Y_{+1}(z, t)$, we have

$$Y_{+1}(z, t) = Y_{-3}(z, t) \cdot S_{+}^{(1)}(u)^{-1} \quad \text{in the domain } \{z \in \text{Sect}_+ \cap \text{Sect}_-, 1 < t\}. \hfill (86)$$
Figure 3: Transposition of \( t \) near \( \infty \) such that \( t \) passes below 0 and 1.

**Proof.** It follows from the first identity of the part \((a)\) and the second identity of the part \((c)\) of Theorem 3.10 that

\[
Y_{+1}(z, t) = \widetilde{W}_0(z, zt) F_t^{(1)}(z) e^{-h\pi i(\delta P + \delta T^{(2)})},
\]

and

\[
Y_{-3}(z, t) = \widetilde{W}_0(z, zt) F_t^{(1)}(z) e^{-h\pi i(\delta P + \delta T^{(2)} + \delta T^{(1)})},
\]

respectively. Note that \( W_0(z_1, z_2) \) is defined for all \( z_2 \in \mathbb{C} \) and does not contribute monodromy from the continuation on \( z_2 \) plane. Thus, after the analytic continuation of \( Y_{+1} \) we have

\[
Y_{-3}(z, t)^{-1} Y_{+1}(z, t) = e^{h\pi i(\delta P + \delta T^{(2)})} e^{i\pi T^{(1)}} F_t^{(1)}(z)^{-1} F_0^{(1)}(z) e^{-h\pi i(\delta P + \delta T^{(2)})} = S_{+}^{(1)}(u)^{-1}.
\]

Here \( S_{+}^{(1)}(u) \) is simply the Stokes matrix of \((46)\) associated to \( \text{Sect}_0 \), i.e., the (modified by \( e^{\pi i\delta T^{(1)}} \)) ratio of \( F_0^{(1)} \) and \( F_t^{(1)} \) in the intersection \( \text{Sect}_0 \cap \text{Sect}_l \). And the second identity follows from the fact that \( e^{h\pi i(\delta P + \delta T^{(2)})} \) commutes with the coefficient matrix of the equation \((46)\) and the regularized initial value at \( z = \infty \), therefore commutes with the solutions \( F_0^{(1)} \) and \( F_t^{(1)} \) as well as the Stokes matrix \( S_{+}^{(1)}(u) \).

### 3.5 The RLL = LLR relation: a proof of Theorem 3.1

As a consequence of Lemma 3.27-3.29, Proposition 3.30.

**Proposition 3.30.** After the analytic continuation of \( Y_{+3}(z, t) \) from the domain \( D_{+3} \) to \( D_{-3} \), in the manner \( D_{+3} \rightarrow D_{+2} \rightarrow D_{+1} \rightarrow D_{-3} \) specified in Lemma 3.27-3.29 we have

\[
Y_{+3}(z, t) = Y_{-3}(z, t) \cdot S_{+}^{(1)}(u)^{-1} S_{+}^{(2)}(u)^{-1} R^{(12)}, \quad \text{for} \ (z,t) \in D_{-3}.
\]

In the meanwhile, we can also take the analytic continuation of \( Y_{+3}(z, t) \) from the domain \( D_{+3} \) to \( D_{-3} \), in a manner \( D_{+3} \rightarrow D_{-1} \rightarrow D_{-2} \rightarrow D_{-3} \) along the following three paths on the \( t \) plane given in Figure 4-6 respectively

Figure 4: Transposition of \( t \) connecting \( D_{+3} \) and \( D_{-1} \).

Figure 5: Transposition of \( t \) near 0 connecting \( D_{-1} \) and \( D_{-2} \).

Then analog to Lemma 3.27-3.29 we have

**Lemma 3.31.** After the analytic continuation of \( Y_{+3}(z, t) \) along the path given in Figure 4, we have

\[
Y_{+3}(z, t) = Y_{-1}(z, t) \cdot S_{+}^{(1)}(u)^{-1} \quad \text{in the domain} \ \{ z \in \text{Sect}_0 \cap \text{Sect}_l, \ t < 0 \}.
\]
which rest of the proof is similar to Lemma 3.27-3.29. After the analytic continuation of Lemma 3.33.

Lemma 3.32. After the analytic continuation of $Y_{-1}(z, t)$ along the path given in Figure 5, we have

$$Y_{-1}(z, t) = Y_{-2}(z, t) \cdot S^{(2)}_{-1}(u)^{-1} \text{ in } D_{-2}. \quad (89)$$

Proof. It follows from the second identity of the parts (a) and (b) of Theorem 3.10 that

$$Y_{-1}(z, t) = W_l(z, zt)F^{(2)}_0(zt),$$

and

$$Y_{-2}(z, t) = W_l(z, zt)F^{(2)}_0(zt) \cdot e^{-\pi i h \delta T^{(2)}},$$

respectively. Here we remark that the corresponding index $i$ of the anti-Stokes ray $d_i$ in Theorem 3.10 is taken as $i = 0$. The rest of the proof is similar to Lemma 3.27-3.29. ■

Lemma 3.33. After the analytic continuation of $Y_{-2}(z, t)$ along the path given in Figure 6, we have

$$Y_{-2}(z, t) = Y_{-3}(z, t) \cdot R^{(12)} \text{ in } D_{-3}. \quad (90)$$

Proposition 3.34. After the analytic continuation $D_{+3} \rightarrow D_{-1} \rightarrow D_{-2} \rightarrow D_{-3}$, we have

$$Y_{+3}(z, t) = Y_{-3}(z, t) \cdot R^{(12)}S^{(2)}_{-1}(u)^{-1}S^{(1)}_{+1}(u)^{-1}, \text{ for } (z, t) \in D_{-3}. \quad (91)$$

Since the analytic continuation in the Proposition 3.30 and 3.34 are along two homotopy paths, the comparison of the identities (87) and (91) proves part of Theorem 3.1.

The rest part of Theorem 3.1 can be proved in a similar way, i.e., by the comparison of the connection formula of $Y_{\pm k}$ along various homotopy paths, which we simply skip here.

4 Representation of quantum groups arising from the Stokes matrices

4.1 The Faddeev-Reshetikhin-Takhtajan (FRT) realization of quantum groups

Let us recall the FRT realization [36] of the quantized universal enveloping algebra $U(R)$ by means of solutions of the Yang-Baxter equation. The algebra $U(R)$ is generated by elements $l_{ij}^{(+)}$, $l_{ij}^{(-)}$, $1 \leq i \leq j \leq n$: set

$$L_{\pm} = \sum_{i,j} l_{ij}^{(\pm)} \otimes E_{ij} \in U(R) \otimes \text{End}(\mathbb{C}^n),$$

with $l_{ij}^{(+)0} = l_{ij}^{(-)} = 0$ for $1 \leq j < i \leq n$, then the defining relations are given in matrix form

$$R^{12}L_{\pm}^{(1)}L_{\pm}^{(2)} = L_{\pm}^{(2)}L_{\pm}^{(1)}R^{12}, \quad (92)$$

$$R^{12}F^{+1}_{\pm}F^{-1}_{\pm} = L_{\pm}^{(2)}F^{+1}_{\pm}R^{12}, \quad (93)$$

and

$$l_{ii}^{(+)0} = l_{ii}^{(-)} = 1, \text{ for } i = 1, \ldots, n.$$

Following [36], the two different realization $U(R)$ and $U_q(gl_n)$ (given in Theorem 1.1) of quantum groups are isomorphic. In particular, following [13], there is an explicit isomorphism $I : U_q(gl_n) \cong U(R)$, under which

$$l_{ii}^{(+)} = q^{-h_i}, \quad l_{i,i+1}^{(+)} = -(q - q^{-1})q^{-h_i}e_i, \quad l_{i+1,i}^{(-)} = (q - q^{-1})f_i q^h_i. \quad (94)$$
4.2 A proof of Theorem 1.1

Theorem 3.1 states that the Stokes matrices $S_\pm(u)$ satisfy the RLL relation (92)-(93). Thus, as a consequence of the explicit isomorphism $I : U_q(\mathfrak{gl}_n) \cong U(R)$ given in (94), we get a representation of $U_q(\mathfrak{gl}_n)$ from the Stokes matrices. It gives a proof of Theorem 1.1.

5 The stokes phenomenon of the equation (1) in the resonant case $h \in \mathbb{Q}$

In the case of $h \in \mathbb{Q}$, the equation (1) becomes resonant, and uniqueness of its formal solution is not valid. Accordingly, there exists a family of formal/actual solutions, as well as Stokes matrices $S$. Theorem 3.1 states that the Stokes matrices $S$ give a description of all solutions, and gives a description of all $S_\pm(u;c)$ on a finite set $c$. This section introduces a pair of distinguished Stokes matrices $S^0_\pm(u)$ among the family, and gives a description of all $S^\pm(u;c)$ via $S^0_\pm(u)$. In the end, it shows that $S^0_\pm(u)$ gives rise to a representation of $U_q(\mathfrak{gl}_n)$ at $q = e^{h/2}$ a root of unity, i.e., gives a proof of Theorem 1.2.

5.1 A distinguished formal solution

In the resonant case, i.e., $h \in \mathbb{Q}$, the proof of Proposition 2.1 fails. Thus, to study the existence of formal solutions, one needs a better understand of the recursive relation (16). For this purpose, in [39] we solve explicitly the recursive relation (16) for all $h$. Let us define $T(\zeta) := \zeta I_T - hT$ for $\zeta$ an indeterminate. Given any elements $(a_1, a_2)$ and $(b_1, b_2)$ of $\{1, ..., n\}$, let us introduce the $2 \times 2$ quantum minor (formed from rows $i, j$ and columns $k, l$) of $T(\zeta)$.

$$\Delta^i_j(T(\zeta)) := (\delta_{ik}\zeta - he_{ik})(\delta_{jl}\zeta - he_{jl}) - (\delta_{jk}(\zeta + h) - he_{jk})(\delta_{il}(\zeta + h) - he_{il}),$$

which is a degree two polynomial in $\zeta$ with coefficients in $U(\mathfrak{gl}_n)$. Suppose $X \in U(\mathfrak{gl}_n)$ is an element commuting with $e_{ik}, e_{jl}, e_{jk}$ and $e_{il}$, then $\Delta^i_j(T(X))$ is defined without ambiguity. For example, if $\Delta^i_j(T(\zeta)) = a_2^2 + a_1\zeta + a_0$, then we have $\Delta^i_j(T(X)) = a_2X^2 + a_1X + a_0$.

**Proposition 5.1.** [39] For all $h \in \mathbb{C} \setminus \{0\}$, the ordinary differential equation (1) has a formal fundamental solution

$$\hat{F}(z) = \hat{H}(z)e^{huz}\zeta^{h\delta T},$$

with $\hat{H} = 1 + H_1z^{-1} + H_2z^{-1} + \cdots$, (95)

where every coefficient $H_{m+1} = \sum_{i,j}(H_{m+1})_{ij} \otimes E_{ij} \in \operatorname{End}(L(\lambda)) \otimes \operatorname{End}(\mathbb{C}^n)$ is recursively determined by

$$(H_{m+1})_{ik} = \begin{cases} \frac{1}{m} \frac{1}{huk - hu_i} \left( \sum_{j \neq k} \Delta^i_j(T(he_{kk} - m)) \cdot (H_m)_{jk} \right), & \text{if } i \neq k \\ -\frac{1}{m+1} \left( \sum_{j \neq k} he_{kj} \cdot (H_{m+1})_{jk} \right), & \text{if } i = k. \end{cases}$$

(96)

In particular, this proposition states that the formal solution $\hat{F}(z)$ defined in Proposition 2.1 extends to $h \in \mathbb{Q}$. To distinguish between the resonant and nonresonant cases, we denote by

$$\hat{F}^0(\zeta) = \hat{H}^0(\zeta)e^{huz}\zeta^{h\delta T}$$

the formal solution in Proposition 5.1 as $h \in \mathbb{Q}$.

**Remark 5.2.** The coefficient matrix of the recursive relation in (96) actually encodes the defining relation of $Y(\mathfrak{gl}_{n-1})$. In [39], we prove for any $u$ (not necessary regular), the recursive relation is equivalent to a difference equation

$$X(z + 1) = L(z)X(z),$$

where $L(z)$ satisfies the defining relation of Yangian $Y(\mathfrak{gl}_m)$ for an integer $m$ determined by $u$. We also generalize the result to the analogous equation (1) for classical Lie algebras and twisted Yangians. We refer the reader to [33] for a general theory on the quantum minors and Yangians.
We follow a general theory of resonant meromorphic differential equations. We fix $\hat{h}$ with the Borel resummation, and thus the semiclassical limit of the Stokes matrices of (1) becomes the Stokes matrices of (10). See [41] for more details.

5.2 The family of formal fundamental solutions

We follow a general theory of resonant meromorphic differential equations. We fix $h \in \mathbb{Q}$. For any positive integer $m$, let $I_m \subset \{1, \ldots, n\}$ denote the set of all $k$ such that $m \text{Id} - h\delta T_{kk}$ is not invertible on $\text{End}(L(\lambda))$. Then let $U_{mk,s} \in \text{End}(L(\lambda))$, for $s = 1, 2, \ldots, d_{mk}$, be a set of linearly independent elements satisfying

$$mU_{mk,s} - h[\epsilon_{kk}, U_{mk,s}] = 0. \tag{99}$$

Let $I$ be set of the positive integers $m$ such that the above set $I_m$ is not empty. Associated to a family $c = \{c_{ik,s}\}$ of complex parameters for $m \in I$, $k \in I_m$ and $s = 1, \ldots, d_{mk}$, we introduce a Laurent polynomial

$$U(z; c) = 1 + \sum_{m \in I} \sum_{k \in I_m} \sum_{s=1}^{d_{mk}} c_{mk,s} U_{mk,s} \otimes E_{kk} z^{-m} \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n). \tag{100}$$

Proposition 5.4. For $h \in \mathbb{Q}$, the ordinary differential equation (1) has a family of formal fundamental solutions taking the form

$$\hat{F}(z; c) = \hat{H}^\alpha(z) \cdot U(z; c) e^{huz} z^{h\deltaT}. \tag{101}$$

Here $\hat{H}^\alpha(z)$ is the formal series from (97). Furthermore,

$$U(z; h,c) e^{\epsilon_{kk} z^{h\deltaT}} = e^{z^{h\deltaT}} D(c) \tag{102}$$

where $D(c) = \sum_k D_{kk} \otimes E_{kk}$ is the diagonal constant given by

$$D(c) = 1 + \sum_{m \in I} \sum_{k \in I_m} \sum_{s=1}^{d_{mk}} c_{mk,s} U_{mk,s} \otimes E_{kk} \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n).$$

Proof. To verify the function in (101) is a solution, we only need to check

$$\frac{dU(z; c)}{dz} + \left[ U(z; c), \frac{h\delta T}{z} \right] = 0. \tag{103}$$

Recall that $\delta T = \sum_k \epsilon_{kk} \otimes E_{kk}$, then the above identity follows from (99). In the meanwhile, a direct computation using the identity (99) shows that (102) holds. $\blacksquare$
5.3 A family of canonical solutions with a prescribed asymptotics and Stokes matrices

As in the case of $h \notin \mathbb{Q}$, the Borel resummation of $\hat{H}^{0}(z)$ can be used to construct the unique holomorphic function $H^{0}(z)$ with the asymptotics $\hat{H}^{0}(z)$ as $z \to \infty$ within each Stokes supersector $\text{Sect}_i$.

**Theorem 5.5.** For any choice of the parameter set $c$, the function $F_{i}(z; c) = H^{0}_{i}(z)U(z; c) \cdot e^{huz \cdot z^{\delta T}}$ is the unique (therefore called canonical) holomorphic solution of (1) defined on $\text{Sect}_i$, such that $F_{i}e^{-huz \cdot z^{-\delta T}} \sim \hat{H}^{0}_{i} \cdot U(z; c)$ within $\text{Sect}_i$.

**Definition 5.6.** Given $h \in \mathbb{Q}$, $u \in h_{\text{reg}}$ and any choice of the parameter set $c$, the Stokes matrices of the equation (1) (with respect to the chosen sector $\text{Sect}_0$ and the branch of $\log(z)$) are the unique matrices such that:

- If $F_{0}$ is continued in a positive sense to $\text{Sect}_i$ then $F_{i}(z; c) = F_{0}(z; c) \cdot S_{+(u; c)}e^{\pi ih\delta T}$, and
- If $F_{1}$ is continued in a positive sense to $\text{Sect}_0$ then $F_{0}(ze^{-2\pi i}; c) = F_{1}(z; c) \cdot e^{-\pi ih\delta T}S_{-(u; c)}^{-1}$.

5.4 Comparison of Stokes matrices in the resonant and nonresonant cases

On the one hand, for $h \notin \mathbb{Q}$, $u \in h_{\text{reg}}$ and a representation $L(\lambda)$ of $\mathfrak{gl}_{n}$, there is only one pair of Stokes matrices $S_{\pm}(u)$, based on the uniqueness of the formal solution of (1). Then following Proposition 5.1, for any fixed $u$ the unique formal solution (therefore its Borel resummation and the corresponding Stokes matrices) extends from $h \notin \mathbb{Q}$ to rational numbers. For $h \in \mathbb{Q}$ the resulting Stokes matrices from the natural extension are denoted by $S^{0}_{\pm}(u)$.

On the other hand, for the same $h \in \mathbb{Q}$, $u$ and $L(\lambda)$, there is a family of Stokes matrices $S_{\pm}(u; c)$ of (1) parameterized by the set $c$, where $c$ depends on the chosen $h$ and $L(\lambda)$. If all the parameters $c_{mk,s}$ in (100) of the set $c$ are chosen to be zero, then the corresponding Stokes matrices $S_{\pm}(u; c = 0)$ coincide with the natural extension $S^{0}_{\pm}(u)$ from the resonant cases.

Following the definition of Stokes matrices and the identity

$$F_{i}(z; c) = H^{0}_{i}(z)U(z; c) \cdot e^{huz \cdot z^{\delta T}} = H^{0}_{i}(z) \cdot e^{huz \cdot z^{\delta T}}D(c)$$

on each $\text{Sect}_i$, we have

**Proposition 5.7.** For $h \in \mathbb{Q}$, $u \in h_{\text{reg}}$ and any choice of parameter set $c$, the Stokes matrices

$$S_{i}(u; c) = D(c) \cdot S^{0}_{\pm}(u) \cdot D(c)^{-1},$$

where $D(c)$ is the diagonal element given in Proposition 5.4.

Proposition 5.7 gives a complete description of the family of Stokes matrices of (1) in the case of $h$ a rational number, and reduces the family to the study of $S^{0}_{\pm}(u)$.

5.5 A proof of Theorem 1.2

Following Theorem 3.1 for all $h \notin \mathbb{Q}$, the Stokes matrices $S_{\pm}(u)$ satisfy the RLL relations. The Stokes matrices $S^{0}_{\pm}(u)$, corresponding to a rational number $h_{0} \in \mathbb{Q}$, are the continuous extension of $S_{\pm}(u)$ from $h \in \mathbb{C} \setminus \mathbb{Q}$ to $h_{0}$. Therefore, Theorem 1.2 follows from the fact that the RLL relations are preserved under the continuous extension of $h$.

It can also be seen from the explicit expression in $n = 2$ example, where clearly the case of $q$ is a root of unity also defines a representation.

6 An isomonodromy deformation approach to Conjecture 1.4 $\mathfrak{gl}_{n}$-crystals and Drinfeld isomorphisms

In this section, let us assume that $u \in h_{\text{reg}}(\mathbb{R})$ and $h \in i\mathbb{R}$. In this case, the equation (1) has only two Stokes supersectors

$$\text{Sect}_0 = \{z \in \mathbb{C} \mid -\pi < \arg(z) < \pi\}, \quad \text{Sect}_1 = \{z \in \mathbb{C} \mid -2\pi < \arg(z) < 0\}.$$  (105)
Accordingly, the Stokes matrices are the elements $S_{\pm}(u,A) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n)$ determined by

\begin{align*}
F_1(z) &= F_0(z) \cdot S_{+}(u)e^{\pi i h \delta T}, \quad \text{in } \{-\pi < \arg(z) < 0\} \quad (106) \\
F_0(ze^{-2\pi i}) &= F_1(z) \cdot e^{-\pi i h \delta T} S_{-}(u)^{-1}, \quad \text{in } \{-2\pi < \arg(z) < -\pi\} \quad (107)
\end{align*}

where the second identity is understood to hold after $F_1$ has been analytically continued anticlockwise.

In [43], the explicit expression of Stokes matrices, of meromorphic linear systems of ordinary differential equations of Ponceau rank 1, was derived via the study of the asymptotics of the associated isomonodromy deformation equation. It partially generalizes Jimbo’s connection formula [19] of Painlevé VI equation (that can be equivalent to a $3 \times 3$ JMMS system) to the case of a general $n \times n$ JMMS system, see [45] for an explanation. In the following, we will use the explicit expression of Stokes matrices to propose an isomonodromy approach to Conjecture [1,4].

### 6.1 The regularized limit of Stokes matrices

It follows, from the general theory of isomonodromic deformation of meromorphic linear system of ordinary differential equations [22], that

**Proposition 6.1.** As a function of $u \in h_{\text{reg}}(\mathbb{R})$, the entries of the Stokes matrix $S_{\pm}(u;h)$ satisfy

\begin{equation}
\frac{\partial S_{kl}}{\partial u_i} = h \sum_{j \neq i} \frac{[e_{ij}e_{ji}^*]}{u_i - u_j} \in \text{End}(L(\lambda)).
\end{equation}

Here $e_{ij}$ is seen as an element in $\text{End}(L(\lambda))$ via the given representation of $\text{gl}_n$ on $L(\lambda)$.

For each $k = 1, \ldots, n$, let us take $V_k := \sum_{1 \leq j \leq k-1} e_{kj}^* e_{jk}$. The elements $V_k$ commute with each other, and are the analog of the Jucys-Murphy elements in the group algebra of $S_n$. Let $G(u) \in \text{End}(L(\lambda))$ be the solution of the equation

\begin{equation}
\frac{\partial G}{\partial u_i} = h \sum_{j \neq i} \frac{e_{ij}^* e_{ji}}{u_i - u_j} \cdot G, \quad i = 1, \ldots, n,
\end{equation}

with the prescribed asymptotics

\begin{equation}
G(u) \cdot \prod_{k=1}^{n} u_k^{hV_k} \to 1, \quad \text{as } u_1 \ll \cdots \ll u_n.
\end{equation}

Here $u_1 \ll \cdots \ll u_n$ stands for $\frac{u_k - u_{k+1}}{u_k - u_1} \to 0$ for all $k$. The existence of $G(u)$ follows from the general theory that a formal solution of differential equations with regular singularities is in fact analytic. This equation was called the Casimir equation, see [40].

It follows from [108] and [109] that for any $k, l$ the element $G(u)^{-1} \cdot S_{\pm}(u;h)_{kl} \cdot G(u) \in \text{End}(L(\lambda))$ is a constant. Since the infinite point $u_1 \ll \cdots \ll u_n$ is a caterpillar point $u_{\text{cat}}$ on the De Cocini-Procesi space of $h_{\text{reg}}(\mathbb{R})$ (see e.g., [18]), we name the constant as the Stokes matrices at $u_{\text{cat}}$.

**Definition 6.2.** [43] The Stokes matrices $S_{\pm}(u_{\text{cat}};h) \in \text{End}(L(\lambda)) \otimes \text{End}(\mathbb{C}^n)$ at the caterpillar point $u_{\text{cat}}$ are defined by

\begin{equation}
S_{\pm}(u_{\text{cat}};h)_{kl} := G(u)^{-1} \cdot S_{\pm}(u;h)_{kl} \cdot G(u), \quad k, l = 1, \ldots, n.
\end{equation}

By the asymptotics [110], the constant $S_{\pm}(u_{\text{cat}};h)_{kl}$ can be seen as the regularized limit of $S_{\pm}(u;h)_{kl}$ as $0 < u_1 \ll \cdots \ll u_n$. The importance of $S_{\pm}(u_{\text{cat}};h)$ follows from the following fact.

**Proposition 6.3.** [43] The Stokes matrices $S_{\pm}(u_{\text{cat}};h)$ at the caterpillar point $u_{\text{cat}}$ satisfy the RLL relations [21] and [22].

The proposition follows from the fact that the gauge transformation $G(u)$ preserves the RLL relations. For example, the transformation by $G(u)$ preserves the defining relation in $U_q(\text{gl}_n)$,

\begin{equation}
S_{\pm}(u)_{k,k+1} S_{\pm}(u)_{k-1,k} - (q + q^{-1}) S_{\pm}(u)_{k,k+1} S_{\pm}(u)_{k,k+1} S_{\pm}(u)_{k-1,k} + S_{\pm}(u)_{k-1,k} S_{\pm}(u)_{k,k+1}^2 = 0.
\end{equation}

It concludes that $S_{\pm}(u_{\text{cat}};h)$ satisfy the same relation as $S_{\pm}(u;h)$. 

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Corollary 6.4. Associated to any representation $L(\lambda)$ of $U(\mathfrak{gl}_n)$, the map

$$S_q(u_{\text{cat}}) : U_q(\mathfrak{gl}_n) \to \text{End}(L(\lambda)) ; \ e_i \mapsto S_+ (u_{\text{cat}})^{\frac{1}{q} - 1} \cdot S_+ (u_{\text{cat}})_{i,i+1}^{-1}, \ f_i \mapsto S_- (u_{\text{cat}})^{\frac{1}{q} - 1} \cdot S_- (u_{\text{cat}})_{i,i}^{-1}$$

defines a representation of $U_q(\mathfrak{gl}_n)$ on $L(\lambda)$.

Remark 6.5. More generally, in [43] we introduce the regularized limit of the Stokes matrices $S_{\pm}(u)$, as some components $u_i$ of $u = \text{diag}(u_1, \ldots, u_n)$ collapse in a comparable speed. The prescription of the regularized limit is proved to be controlled by the geometry of the De Concini-Procesi space $\mathfrak{h}_{\text{reg}}(\mathbb{R})$. Here, by the construction of [11], the space $\mathfrak{h}_{\text{reg}}(\mathbb{R})$ replaces the set of diagonal hyperplanes in $\mathbb{R}^n$ by a divisor with normal crossings, and leaves $\mathfrak{h}_{\text{reg}}(\mathbb{R})$ unchanged as an open part.

6.2 Expression of Stokes matrices at caterpillar points in terms of Gelfand-Tsetlin basis

Denote by $\mathfrak{gl}_k$ the subalgebra of $\mathfrak{gl}_n$ spanned by the elements $\{e_{ij}\}_{i,j=1,\ldots,k}$, and denote by the $n$-tuples of numbers $(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ parameterizing the highest weight $\lambda$. Then the Gelfand-Tsetlin basis in $L(\lambda)$, associated to the chain of subalgebras

$$\mathfrak{gl}_1 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$$

is an orthonormal basis $E_{GZ}(\lambda) = \{\xi_\lambda(u_{\text{cat}})\}$, parameterized by the Gelfand-Tsetlin patterns $\Lambda$. Recall that such a pattern $\Lambda$ is a collection of numbers $\{\lambda_{ij}^{(k)}(\Lambda)\}_{1 \leq j \leq n}$ with the fixed $\{\lambda_{ij}^{(n)}\}$ satisfying the interlacing conditions [7]. We refer the reader to [33] for a general theory of Gelfand-Tsetlin basis.

Remark 6.6. The shift of argument subalgebras $A(u)$ of $U(\mathfrak{gl}_n)$ extend from $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$ to the de Concini-Procesi space $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$. In particular, the subalgebra $A(u)$ at $u_{\text{cat}}$ becomes the Gelfand-Tsetlin subalgebra, and the eigenbasis $E(u; \Lambda)$ at $u_{\text{cat}}$ becomes the Gelfand-Tsetlin basis $E_{GZ}(\lambda)$. This is why we denote the basis vector in $E_{GZ}(\lambda)$ corresponding to a pattern $\Lambda$ by $\xi_\Lambda(u_{\text{cat}})$.

Theorem 6.7. [43] As $u_1 << \cdots << u_n$, the leading asymptotics of the off-diagonal entries of $S_h(u)$, as elements in $\text{End}(L(\lambda))$, are given by (for $k = 1, \ldots, n - 1$)

$$S_+(u)_{k+1,k} \sim -2\pi i h^{\epsilon_{kk} - \epsilon_{k+1,k+1} - 1} \sum_{i=1}^k \prod_{l=1, l \neq i}^k \Gamma(1 + h(\zeta_{kk} - \zeta_{k+1,k+1}^{(k)} - 1)) \cdot h^{\alpha_{i}^{(k)}} f_{i}^{(k)},$$

$$S_-(u)_{k+1,k} \sim -2\pi i h^{\epsilon_{kk} - \epsilon_{k+1,k+1} - 1} \sum_{i=1}^k \prod_{l=1, l \neq i}^k \Gamma(1 - h(\zeta_{kk} - \zeta_{k+1,k+1}^{(k)} - 1)) \cdot h^{\beta_{i}^{(k)}} g_{i}^{(k)}.$$ 

Here

$$f_{i}^{(k)} = u_k^{-h \epsilon_{kk}} (u_k^{-1})^{h_{\xi_{i}^{(k)}}}, \quad g_{i}^{(k)} = u_k^{-h \epsilon_{kk} - 1} (u_k^{-1})^{h_{\xi_{i}^{(k)}}}, \quad (112)$$

and $\epsilon_{kk}$, $\zeta_{i}^{(k)}$ and $\alpha_{i}^{(k)}$ are in $\text{End}(L(\lambda))$ acting on the Gelfand-Tsetlin basis $\xi_\Lambda(u_{\text{cat}})$ of $L(\lambda)$ by

$$\epsilon_{kk} : \xi_\Lambda(u_{\text{cat}}) = \left( \sum_{i=1}^k \lambda_i^{(k)} - \sum_{i=1}^{k-1} \lambda_i^{(k-1)} \right) \xi_\Lambda(u_{\text{cat}}), \quad (113)$$

$$\zeta_{i}^{(k)} : \xi_\Lambda(u_{\text{cat}}) = (\lambda_i^{(k)} - i + 1 + (k-1)/2) \xi_\Lambda(u_{\text{cat}}), \quad (114)$$

$$\alpha_{i}^{(k)} : \xi_\Lambda(u_{\text{cat}}) = \sqrt{\frac{\prod_{l=1, l \neq i}^k (\zeta_{k}^{(k)} - \zeta_{i}^{(k)})}{\prod_{l=1}^k (\zeta_{k}^{(k)} - \zeta_{i}^{(k)} - 1)}} \cdot \xi_{\Lambda + \delta_{i}}(u_{\text{cat}}), \quad (115)$$

where the pattern $\Lambda + \delta_{i}^{(k)}$ is obtained from $\Lambda$ by replacing $\lambda_i^{(k)}$ by $\lambda_i^{(k)} + 1$. It is supposed that $\xi_\Lambda(u_{\text{cat}})$ is zero if $\Lambda$ is not a pattern. Furthermore, by Theorem [37], other entries (therefore the leading asymptotics) of Stokes matrices are given explicitly by the off-diagonal entries.
From the viewpoint of the isomonodromy deformation, $S_+(u; h)_{k,k+1} = G(u)S_+(u_{\text{cat}}; h)_{k,k+1}G(u)^{-1}$. Since $h \in \mathbb{R}$, one checks that the norm of $G(u)$ is independent of $u$ by verifying the derivative of $G(u)^\dagger G(u)$ is zero. Here $G(u)^\dagger$ is the complex conjugate transpose of $G(u)$. A manipulation of Gauss decomposition shows that the leading asymptotics \([10]\) of $G(u)$ contribute to the fast spin terms $f^{(k)}_1$ (acting on the argument of the Gelfand-Tsetlin basis) in the asymptotics of $S_+(u; h)_{k,k+1}$ given in Theorem 6.7. Therefore, taking the regularized limit of $S_+(u)$ as $u_1 \ll \cdots \ll u_n$ amounts to the average of the fast spin terms $f^{(k)}_1$ and $g^{(k)}_1$. That is

**Theorem 6.8.** [43] The off-diagonal entries of $S_{h\pm}(u_{\text{cat}})$ are given by (for $k = 1, ..., n-1$)

\[
S_+(u_{\text{cat}})_{k,k+1} = -2\pi i h \left( e^{h} \cdot c_{k+1}(\xi_{k-1}, u_{\text{cat}}; h) \right) \cdot h\alpha^{(k)}_1,
\]

\[
S_-(u_{\text{cat}})_{k,k+1} = -2\pi i h \left( e^{h} \cdot c_{k+1}(\xi_{k-1}, u_{\text{cat}}; h) \right) \cdot h\beta^{(k)}_1.
\]

### 6.3 A model of the WKB approximation of the Stokes matrices

Theorem [6.7] enables us to compute explicitly the WKB approximation of Stokes matrices in an asymptotic zone. The result leads to a realization of the gl$_n$-crystal via the Stokes phenomenon.

**Theorem 6.9.** [43] For each $k = 1, ..., n-1$, there exists canonical operators $\tilde{e}_k(u_{\text{cat}})$ and $\tilde{f}_k(u_{\text{cat}})$ acting on the finite set $E_{GZ}(\lambda)$ such that for any generic element $\xi \in E_{GZ}(\lambda)$, there exist real valued functions $c_{ki}(\xi)$ and $\theta_{ki}(h, u, \xi)$ with $i = 1, 2$ such that

\[
\lim_{h \to -\infty} \left( \lim_{u_1 \ll \cdots \ll u_n} S^{(\pm)}_{k,k+1}(u) \cdot e^{c_{k+1}(\xi)h + i\theta_{k1}(h,u,\xi)} \right) = \tilde{e}_k(\xi),
\]

\[
\lim_{h \to -\infty} \left( \lim_{u_1 \ll \cdots \ll u_n} S^{(\pm)}_{k+1,k}(u) \cdot e^{c_{k+2}(\xi)h + i\theta_{k2}(h,u,\xi)} \right) = \tilde{f}_k(\xi).
\]

Furthermore, the set $B_{GZ}(\lambda)$ equipped with the operators $\tilde{e}_k(u_{\text{cat}})$ and $\tilde{f}_k(u_{\text{cat}})$ is a gl$_n$-crystal.

Following Remark 6.6, Theorem 6.9 is essentially a special case of Conjecture 1.4 at $u = u_{\text{cat}}$. That is Theorem 6.9 can be equivalently stated as

**Theorem 6.10.** [43] For each $k = 1, ..., n-1$, we have

\[
\lim_{h \to -\infty} \left( S_+(u_{\text{cat}})_{k,k+1} \cdot e^{c_{k+1}(\xi)h + i\theta_{k1}(h,u,\xi)} \right) = \tilde{e}_k(\xi),
\]

\[
\lim_{h \to -\infty} \left( S_-(u_{\text{cat}})_{k,k+1} \cdot e^{c_{k+2}(\xi)h + i\theta_{k2}(h,u,\xi)} \right) = \tilde{f}_k(\xi).
\]

Furthermore, the set $B_{GZ}(\lambda)$ equipped with the operators $\tilde{e}_k(u_{\text{cat}})$ and $\tilde{f}_k(u_{\text{cat}})$ is a gl$_n$-crystal.

More generally, Conjecture 1.4 extends to all the $u$ in the de Concini-Procesi space $\mathfrak{h}_{\text{reg}}(\mathbb{R})$. See [43] for more details.

**Remark 6.11.** In the case $n = 3$ the canonical basis is given explicitly in [30] Example 3.4]. Using the expression in Theorem 6.7, it is direct to verify Conjecture 1.3 for gl$_3$ and $u = u_{\text{cat}}$.

### 6.4 Stokes matrices at a caterpillar point and explicit Drinfeld isomorphisms

Let us point out that in the course of finding the asymptotic expressions of Stokes matrices we actually construct an explicit Drinfeld isomorphism.

To this end, set $q = e^h$ and let us work over the ring of formal power series $\mathbb{C}[h]$. Then the (adically completed) $U_h(\mathfrak{g} \mathfrak{l}_n)$ turns out to be a formal deformation of $U(\mathfrak{g} \mathfrak{l}_n)$. Let us take the topological Hopf algebra $U(\mathfrak{g} \mathfrak{l}(n))[h]$ over $\mathbb{C}[h]$. In [12], Drinfeld pointed out that there exists an isomorphism between the $\mathbb{C}[h]$ algebras $U_q(\mathfrak{g} \mathfrak{l}(n))$ and $U(\mathfrak{g} \mathfrak{l}(n))[h]$. Such isomorphisms are not canonical, and one explicit isomorphism $\Psi_{AG}$ involving gamma function was constructed by Appel and Gautam in [2] (for $\mathfrak{g} \mathfrak{l}(n)$). Then any representation...
\( L(\lambda) \) of \( U(\mathfrak{gl}_n) \) gives rise to a representation of \( U_q(\mathfrak{gl}_n) \), by composing the map \( \Phi_{AG} \) and specializing \( q = e^h \) to a complex number. In the meanwhile, the map \( S_q(u_{cat}) \) in Corollary 6.4 defines a representation of \( U_q(\mathfrak{gl}_n) \) on the same vector space \( L(\lambda) \). By directly comparing the expression of \( \Psi_{AG} \) (see [2] Remark 2.6 (2)) with the expression of \( S_q(u_{cat}) \) given in Theorem 6.8, we get the following theorem and commutative diagram:

\[
\begin{array}{ccc}
U(\mathfrak{gl}_n)[h] & \xrightarrow{\Psi_{AG}} & \text{End}(L(\lambda)) \\
\downarrow & & \\
U_q(\mathfrak{gl}_n) & \xrightarrow{S_q(u_{cat})} & \\
\end{array}
\]

**Theorem 6.12.** Associated to any representation \( \rho : U(\mathfrak{gl}_n) \to L(\lambda) \), the representation \( S(u_{cat}) \) of \( U_q(\mathfrak{gl}_n) \) on the same vector space \( L(\lambda) \) factors through the algebraic isomorphism \( \Psi_{AG} \), i.e., \( S(u_{cat}) = \rho \circ \Psi_{AG} \) (provided specializing \( q = e^h \) to a complex number in \( \rho \circ \Psi_{AG} \)).

### 6.5 An isomonodromy deformation approach to Conjecture 1.4

Let us recall that \( \xi_\Lambda(u) \in E(u; \lambda) \subset L(\lambda) \) denotes the eigenbasis vector labelled by a Gelfand-Tsetlin pattern \( \Lambda \).

**Conjecture 6.13.** For any fixed \( u \in U_{id} \subset h_{reg}(\mathbb{R}) \),

\[
G(u; h) \cdot \xi_\Lambda(u_{cat}) \sim e^{hf(u; \Lambda)} \xi_\Lambda(u), \quad \text{as } h \to +i\infty,
\]

(116)

where \( f(u; \Lambda) \) is a real valued function.

**Remark 6.14.** Note that \( G(u; h) \) is unitary, and the basis vectors \( \xi_\Lambda(u) \) are chosen to be unit.

Conjecture 6.13 predicts that the solution \( G(u; h) \) of the isomonodromy equation as a function valued in \( \text{End}(L(\lambda)) \), in the WKB approximation as \( h \to +i\infty \), decomposes into a combination of fast/isospectral spin \( e^{hf} \) and slow/Whitham dynamics on the basis vectors \( \{\xi_\Lambda(u)\} \). Here the fast/isospectral spin \( e^{hf} \) only changes the phase of the basis vector, while the slow/Whitham dynamics transports the basis vectors \( \{\xi_\Lambda(u)\} \) along the \( u \) space.

**Proposition 6.15.** Conjecture 1.4 is true under the hypothesis of 116.

**Proof.** Since \( h \in i\mathbb{R} \) and \( f(u, \Lambda) \) is real value function, we have that the norm of \( e^{hf(u; \Lambda)} \) is 1. Then the existence of the operators \( \hat{e}_k(u) \) and \( \hat{f}_k(u) \) in Conjecture 1.4 satisfying the identities (5) and (6), follows from the identity (11), i.e., \( S_\pm(u; h) e_k = G(u; h) S_\pm(u_{cat}; h) e_k G(u; h)^{-1} \), and the asymptotics of \( G(u) \) and \( S_\pm(u_{cat}; h) \) given in (116) and Theorem 6.7 respectively.

Furthermore, it is already known that, by Theorem 6.10 the WKB approximation at \( u_{cat} \) induces a \( \mathfrak{gl}_n \)-crystal structure. Therefore, the finite set \( \{\xi_\Lambda(u)\} \) equipped with the operators \( \{\hat{e}_k(u)\}, \{\hat{f}_k(u)\} \) is also a \( \mathfrak{gl}_n \)-crystal structure, and is isomorphic to the one induced at \( u_{cat} \) as in Theorem 6.10. Thus we have the following commutative diagram, where the slow dynamics transports the isomorphic \( \mathfrak{gl}_n \)-crystals at different \( u \) reproducing the Halacheva-Kamnitzer-Rybnikov-Weekes cover 18 of \( \mathfrak{gl}_n \)-crystals over \( h_{reg}(\mathbb{R}) \).

\[
\begin{array}{ccc}
S_\pm(u; h) & \xrightarrow{\text{transformation by } G(u; h)} & S_\pm(u_{cat}; h) \\
\downarrow & & \downarrow \\
\{\xi_\Lambda(u)\}, \hat{e}_k(u), \hat{f}_k(u) & \xrightarrow{\text{slow dynamics with respect to } u} & \{\xi_\Lambda(u_{cat})\}, \hat{e}_k(u_{cat}), \hat{f}_k(u_{cat})
\end{array}
\]

\[\Box\]
**Remark 6.16.** The KZ type equations are closely related to the isomonodromic problems. The first such observation is due to Reshetikhin [35], where he showed that the KZ equation is a quantization of the Schlesinger equation. Later in [9] Boalch proved that the Casimir equations (see [40]), equivalently the dynamical equations compatible with the KZ equation with irregular singularities [14], can be viewed as a quantization of the Jimbo-Miwa-Mōri-Sato (JMMS) equation. Here the JMMS equation is the simplest generalization of the Schlesinger equation with an extra irregular singular point at infinity. Therefore, Conjecture 6.13 is seen as a quantum analog of the long standing conjectural approximation of the JMMS equation by slow modulations of an isospectral problem, see [38]. See also Krichever [27, Section 7] for a general discussion of isomonodromy deformation equations in the WKB approximation. One of our motivation in [43] and this paper is to set a rigorous formulation of the degeneration of isomonodromy to isospectral equations in the WKB approximation.

**A method of stationary phase.**

Let \( v_1(u), \ldots, v_r(u) \in E(\lambda, u) \) be the eigenbasis of the action of the shifted argument subalgebra \( \mathcal{A}(u) \) on the representation \( L(\lambda) \) of \( U(\mathfrak{gl}_r) \). Let \( \{\omega_{ik}(u)\}_{1 \leq i \leq n, 1 \leq k \leq r} \) and \( \{A_{i,kj}(u)\}_{1 \leq j, k \leq r, 1 \leq i \leq n} \) be the set of functions defined by

\[
\sum_{j \neq i} \frac{\epsilon_{ij} \epsilon_{ij}}{u_i - u_j} \cdot v_k(u) = \omega_{ik}(u)v_k(u), \quad \text{for } v_i \in B(\lambda, u),
\]

\[
\frac{\partial v_k}{\partial u_i} = \sum_{j=1}^{r} A_{i,kj}(u)v_j(u)
\]

Therefore, in terms of the (time \( u \) dependent) basis \( \{v_i(u)\}_{i=1, \ldots, r} \), the equation (109) becomes

\[
\frac{\partial \mathcal{G}}{\partial u_i} = (h\omega_i + A_i) \cdot \mathcal{G}, \quad k = 1, \ldots, n,
\]

for a function \( \mathcal{G} \in \text{End}(L(\lambda)) \). Here \( h \in \mathbb{R}, \omega_i(u_1, \ldots, u_n) \) is a diagonal real valued matrix \( \omega_i = \text{diag}(\omega_{i1}, \ldots, \omega_{ir}) \), and \( A_i \) is the \( r \times r \) matrix whose \((k, j)\) entry is \( A_{i,kj} \). In the following, we will fix \( u_1, \ldots, u_{n-1} \) and consider the ordinary differential equation with respect to the variable \( u_n \).

For that, let us fix a point \( p = \text{diag}(p_1, \ldots, p_n) \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \) and take the \( r \times r \) matrix function of \( u_n \)

\[
D(p_1, \ldots, p_{n-1}; u_n) = \exp\left(\int_{p_1}^{u_n} h\omega_1 dt, \ldots, \int_{p_n}^{u_n} h\omega_r dt\right) = \text{diag}(e^{\int_{p_1}^{u_n} h\omega_1 dt}, \ldots, e^{\int_{p_n}^{u_n} h\omega_r dt}),
\]

then \( Y(p_1, \ldots, p_{n-1}; u_n, h) := D(p_1, \ldots, p_{n-1}; u_n) \cdot \mathcal{G}(p_1, \ldots, p_{n-1}, u_n) \in \text{End}(L(\lambda)) \) satisfies the ordinary differential equation (while fixing \( u_1 = p_1, \ldots, u_{n-1} = p_{n-1} \))

\[
\frac{dY}{du_n} = DA_n D^{-1} \cdot Y.
\]

Here \( DA_n D^{-1}(u_n, h) = D(u_n, h)A_n(p_1, \ldots, p_{n-1}, u_n)D(u_n, h)^{-1} \) is a \( r \times r \) matrix function of \( u_n \) and \( h \). For simplicity, we will drop the \( p_1, \ldots, p_{n-1} \) in the expression of \( Y \), \( D \) and \( A_n \).

Expand the solution \( Y(u_n, h) \) as a convergent infinite sum of iterated integrals

\[
Y(u_n, h) = 1 + Y_1 + Y_2 + \cdots,
\]

where the \( k \)-th term is given by

\[
Y_k(u_n, h) = \int_{p_n \leq t_1 \leq \cdots \leq t_k \leq u_n} D A_n D^{-1}(t_k) \cdots D A_n D^{-1}(t_2) D A_n D^{-1}(t_1).
\]

Set \( D(t) = \text{diag}(d_1(t), \ldots, d_r(t)) \), then \( Y_k \) can be written as

\[
Y_k = \sum_{i_0, \ldots, i_k} \int_{p_n \leq t_1 \leq \cdots \leq t_k \leq u_n} A_{i_1,i_2 \ldots i_k}(t_k) \cdots A_{i_1,i_0}(t_1) e^{h(d_{i_1}(t_1) - d_{i_2}(t_2) + \cdots + d_{i_k}(t_k))},
\]

where \( A_n = (A_{ij})_{1 \leq i, j \leq r} \) and \( E_{i_0 \ldots i_k} \) is the elementary \( r \times r \) matrix. Denote by \( d_{i,j}(t) = d_i(t) - d_j(t) \) and introduce the multi-index \( I = (i_0, \ldots, i_k) \) and

\[
d_I(t_1, \ldots, t_k) = d_{i_0} i_0(t_1) + \cdots + d_{i_k} - i_k(t_k) + d_{i_k}(u_n),
\]

\[
A_I(t_1, \ldots, t_k) = E_{i_0 \ldots i_k} A_{i_k,i_{k-1} \ldots i_0}(t_k) \cdots A_{i_0}(t_1).
\]
Then multiplying $Y$ by $D^{-1}$ and plugging in these notations, we get

$$\mathcal{G}(u_n, h) = \sum_l G_l(u_n, h),$$

(122)

where each $G_l(u_n, h)$ is

$$G_l(u_n, h) = \int_{p_n \leq t_1 \leq \cdots \leq t_k \leq u_n} A_l(t_1, \ldots, t_k) e^{h d_l(t_1, \ldots, t_k)},$$

(123)

We stress that $\mathcal{G}(u_n, h)$ is a solution of (109) (fixing $u_1, \ldots, u_{n-1}$) with initial value $G(p_n, h) = 1$, while the function $G(u_n, h)$ given in Section 6.1 is a solution with the prescribed asymptotics as $u_n \to \infty$.

For a fixed finite $u_n > p_1, \ldots, p_{n-1}$, applying the general result of Simpson on the asymptotics of monodromy [37] to the equation (119) (fixing $u_1, \ldots, u_{n-1}$), one shows that the matrix function $\mathcal{G}(u_n, h)$, explained as the monodromy from $p_n$ to $u_n$, has an asymptotic expansion of the form

$$\mathcal{G}(u_n, h) \sim \sum_{i=1}^{m} e^{\lambda_i h} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} K(j) \sum_{m=0}^{\infty} C_{ijk} h^{-\frac{j}{4}} \left(\log h\right)^k,$$

(124)

as $h \to +i\infty$, where $\lambda_i$'s are real numbers. The strategy in [37] is to use the method of stationary phase to obtain asymptotic expansion for each term $G_l$ in (123), and then to add these expansions together to obtain an asymptotic expansion of $\mathcal{G}(u_n, h)$. The main problem solved using [37] is that the sum of asymptotic expansions of $G_l(u_n, h)$ converges to an asymptotic expansion for the sum $\mathcal{G}(u_n, h)$ of the functions $G_l(u_n, h)$.

In a recursive step as fixing $u_1, \ldots, u_{n-1}$, Conjecture 6.13 concerns the $h \to +i\infty$ asymptotics of the monodromy of (109) from $u_n = \infty$ to a finite $u_n = p_n$. Since $\mathcal{G}(u_n, h)$ is the monodromy from $p_n$ to $u_n$, to prove the Conjecture 6.13 in a recursive method, one needs to generalize the asymptotics (124) from a finite $u_n$ to $u_n = \infty$, and to prove that the leading constant coefficient $C_{ijk}$ is a diagonal and non-degenerate matrix $C_{i00}$. The first part should follow from an estimate of the norm of $A_n(u_n)$ as $u_n \to +\infty$, that guarantees the infinite sum (121) converges uniformly with respect to $u_n > p_n$. The second part needs a carefully study of the cycles of integration in deriving the asymptotics of (123) by the method of stationary phase. We leave a further investigation of this approach to Conjecture 6.13 (therefore Conjecture 1.4) to a future work.

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