RELATION BETWEEN THE DIMENSIONS OF THE
RING GENERATED BY A VECTOR BUNDLE OF
DEGREE ZERO ON AN ELLIPTIC CURVE AND A
TORSOR TRIVIALIZING THIS BUNDLE

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1. Introduction and Notations

Let $X$ be a complete, connected, reduced scheme over a perfect field $k$. We define $\text{Vect}(X)$ to be the set of isomorphism classes $[V]$ of vector bundles $V$ on $X$. We can define an addition and a multiplication on $\text{Vect}(X)$:

$$[V] + [V'] = [V \oplus V']$$
$$[V] \cdot [V'] = [V \otimes V'].$$

The (naive) Grothendieck ring $K(X)$ (see [1]) is the ring associated to the additive monoid $\text{Vect}(X)$, that means

$$K(X) = \frac{\mathbb{Z}[\text{Vect}(X)]}{H},$$

where $H$ is the subgroup of $\mathbb{Z}[\text{Vect}(X)]$ generated by all elements of the form $[V \oplus V'] - [V] - [V']$.

The indecomposable vector bundles on $X$ form a free basis of $K(X)$. Since $H^0(X, \text{End}(V))$ is finite dimensional, the Krull-Schmidt theorem ([3]) holds on $X$. This means that a decomposition of a vector bundle in indecomposable components exists and is unique up to isomorphism.

We want to generalize a theorem of M. Nori on finite vector bundles. A vector bundle $V$ on $X$ is called finite, if the collection $S(V)$ of all indecomposable components of $V^\otimes n$ for all integers $n \in \mathbb{Z}$ is finite. In the following, we denote by $R(V)$ the $\mathbb{Q}$-subalgebra of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the set $S(V)$. Thus a vector bundle $V$ is finite if and only if the ring $R(V)$ is of Krull dimension zero.

In [1], Nori proves the following theorem:

For every finite vector bundle $V$ on $X$ there exists a finite group scheme $G$ and a principal $G$-bundle $\pi : P \to X$, such that $\pi^*V$ is trivial on $P$. In particular, the equality

$$\dim R(V) = \dim G (= 0)$$
holds.
As every vector bundle $V$ on $X$ of rank $r$ trivializes on its associated principal $\text{GL}(r)$-bundle, we can look for a group scheme $G$ of smallest dimension and a principal $G$-bundle on which the pullback of the vector bundle $V$ is trivial. We might also compare the dimension of the group scheme to $\dim R(V)$.
In this article we consider the family of vector bundles of degree zero on an elliptic curve. We will prove in propositions 2 and 3 that they trivialize on a principal $G$-bundle with $G$ a group scheme of smallest dimension one.
As in the situation of Nori’s theorem, this dimension turns out to be equal to the dimension of the ring $R(V)$.
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2. Dimension relation for vector bundles of degree zero on an elliptic curve

Let $X$ be an elliptic curve over an algebraically closed field $k$ of characteristic zero. We consider vector bundles of degree zero on $X$ which can be classified according to Atiyah (see [2]). By $E(r, 0)$ we denote the set of indecomposable vector bundles of rank $r$ and degree zero.

**Theorem 1.** (Atiyah [2])

1. There exists a vector bundle $F_r \in E(r, 0)$, unique up to isomorphism, with $\Gamma(X, F_r) \neq 0$.
   Moreover we have an exact sequence
   $\begin{align*}
   0 & \rightarrow \mathcal{O}_X \rightarrow F_r \rightarrow F_{r-1} \rightarrow 0.
   \end{align*}$

2. Let $E \in E(r, 0)$, then $E \cong L \otimes F_r$ where $L$ is a line bundle of degree zero, unique up to isomorphism (and such that $L' \cong \det E$.)

**Proposition 2.**

i) The $\mathbb{Q}$-subalgebra $R(F_r)$ of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $S(F_r)$ is $\mathbb{Q}[x]$, where $x = [F_2]$, if $r$ is even, and $x = [F_3]$, if $r$ is odd. In particular, $R(F_r)$ is of Krull dimension zero.

ii) There exists a principal $\mathbb{G}_a$-bundle $\pi : P \rightarrow X$ such that $\pi^*(F_r)$ is trivial for all $r \geq 2$.

**Remark:** As in Nori’s case we have a correspondence of dimensions

$$\dim R(F_r) = \dim \mathbb{G}_a = 1.$$
Proof:
As proved by Atiyah in [2], the vector bundles \( F_r \) are self-dual and fulfill the formula
\[
F_r \otimes F_s = F_{r-s+1} \oplus F_{r-s+3} \oplus \cdots \oplus F_{(r-s)+(2s-1)} \quad \text{for } s \leq r.
\]
For even \( r \), it follows by induction that there exist integers \( a_i(n) \) such that
\[
F_r^\otimes_n = a_2(n)F_2 \oplus a_4(n)F_4 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}
\]
for odd \( n \geq 3 \), and
\[
F_r^\otimes_n = a_1(n)O_X \oplus a_3(n)F_3 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}
\]
for even \( n \geq 2 \).
Therefore we obtain
\[
S(F_r) = \{ F_i \mid i = 1, 2, 3, \ldots \}, \text{ if } r \text{ even,}
\]
and \( S(F_r) \) generates the subring \( \mathbb{Q}[F_2] \) of \( K(X) \otimes \mathbb{Q} \), because inductively we can write every vector bundle \( F_i \) as \( p(F_2) \) for some polynomial \( p \in \mathbb{Z}[x] \).
For odd \( r \), Atiyah’s multiplication formula gives
\[
F_r^\otimes_n = a_1(n)O_X \oplus a_3(n)F_3 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}
\]
for all \( n \geq 2 \). It follows that
\[
S(F_r) = \{ F_i \mid i \text{ odd} \}, \text{ if } r \text{ odd.}
\]
For odd \( r \), the set \( S(F_r) \) generates the ring \( R(F_r) = \mathbb{Q}[F_3] \), as for odd \( i \) each \( F_i \) is \( p(F_3) \) for a polynomial \( p \in \mathbb{Z}[x] \).

The vector bundle \( F_2 \) is an element of \( H^1(X, GL(2, \mathcal{O})) \). Because of the exact sequence
\[
0 \rightarrow \mathcal{O}_X \rightarrow F_2 \rightarrow \mathcal{O}_X \rightarrow 0,
\]
\( F_2 \) is even an element of \( H^1(X, \mathbb{G}_a) \). Here we embed \( \mathbb{G}_a \) into \( GL(2, \mathcal{O}) \) via \( u \rightarrow \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \). Hence \( F_2 \) trivializes on a principal \( \mathbb{G}_a \)-bundle. As \( F_r = S^{r-1}F_2, r \geq 3 \), each \( F_r \) trivializes on the same principal \( \mathbb{G}_a \)-bundle as \( F_2 \).

As the classes \([ F_r ]\) are not torsion elements in \( H^1(X, GL(2, \mathcal{O})) \), none of the bundles \( F_r \) can trivialize on a principal \( G \)-bundle with \( G \) a finite group scheme. \( \square \)

Remark: In the given examples of vector bundles \( E \) there was so far not only a correspondence of the dimensions of the group scheme
and the ring $R(E)$. The algebra $R(E)$ was also the Hopf algebra corresponding to the group scheme. The following proposition shows that this is not true in general.

**Proposition 3.** Let $E \cong L \otimes F_r \in \mathcal{E}(r,0)$ (see theorem 1).

1. If $L$ is not torsion, the ring $R(E)$ is isomorphic to $\mathbb{Q}[x, x^{-1}] \otimes \mathbb{Q}[y]$ and $E$ trivializes on a principal $\mathbb{G}_m \times \mathbb{G}_a$-bundle.

2. If $L$ is torsion, let $n \in \mathbb{N}, n \geq 1$, be the minimal number such that $L^\otimes n \cong \mathcal{O}_X$. If $n$ and $r$ are both even, the ring $R(E)$ is isomorphic to

$$\mathbb{Q}[x]/ < x^{n/2} - 1 > \otimes \mathbb{Q}[y]$$

and $E$ trivializes on a principal $\mu_n \times \mathbb{G}_a$-bundle. There is no principal $\mu_{n/2} \times \mathbb{G}_a$-bundle where $E$ is trivial.

If $n$ and $r$ are not both even, the ring $R(E)$ is isomorphic to

$$\mathbb{Q}[x]/ < x^n - 1 > \otimes \mathbb{Q}[y]$$

and $E$ trivializes on a principal $\mu_n \times \mathbb{G}_a$-bundle.

Proof: Let $E \in \mathcal{E}(r,0)$ with $\Gamma(X, E) = 0$. (If $\Gamma(X, E) \neq 0$, then $E \cong F_r$. This case was already dealt with in proposition 2.)

First we consider the case that $L$ is not torsion.

We must distinguish between odd and even $r$.

For odd $r$, Atiyah's multiplication formula (see proof of proposition 4) gives the following result:

For $m \in \mathbb{N}, m \geq 2$, the tensor power $E^{\otimes m} \cong L^{\otimes m} \otimes F_r^{\otimes m}$ has the indecomposable components $L^{\otimes m} \otimes \mathcal{O}_X, L^{\otimes m} \otimes F_3, \ldots, L^{\otimes m} \otimes F_{(r-1)m+1}$, the tensor power $E^{\otimes -m} \cong L^{\otimes -m} \otimes F_r^{\otimes m}$ has the indecomposable components $L^{\otimes -m} \otimes \mathcal{O}_X, L^{\otimes -m} \otimes F_3, \ldots, L^{\otimes -m} \otimes F_{(r-1)m+1}$.

Thus we obtain

$$S(E) = \left\{ \mathcal{O}_X, L \otimes F_r, L^{-1} \otimes F_r, L^{\otimes i} \otimes F_3, L^{\otimes i} \otimes F_5, \ldots, L^{\otimes i} \otimes F_{(r-1)i+1}, i \in \mathbb{N} \right\}.$$ 

The algebra $R(E)$ which is generated by $S(E)$ is the subalgebra of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $L, L^{-1}$ and $F_3$, thus

$$R(E) = \mathbb{Q}[L, L^{-1}] \otimes_{\mathbb{Z}} \mathbb{Q}[F_3].$$

For even $r$, a similar computation gives that

$$S(E) = \left\{ \mathcal{O}_X, L \otimes F_r, L^{-1} \otimes F_r, L^{\otimes 2i} \otimes F_3, L^{\otimes 2i} \otimes F_5, \ldots, L^{\otimes 2i} \otimes F_{(r-1)2i+1}, i \in \mathbb{N}  \\
L^{\otimes (2i+1)} \otimes F_2, L^{\otimes (2i+1)} \otimes F_4, \ldots,  \\
L^{\otimes (2i+1)} \otimes F_{(r-1)(2i+1)+1}, i \in \mathbb{N} \right\}.$$
The ring $R(E)$, generated by $S(E)$, is the subring of $K(X) \otimes \mathbb{Z} \mathbb{Q}$ which is generated by the elements $L^\otimes 2$, $L^{\otimes -2}$, $L^{-1} \otimes F_2$, therefore

$$R(E) = \mathbb{Q}[L^\otimes 2, L^{\otimes -2}] \otimes \mathbb{Z} \mathbb{Q}[L^{-1} \otimes F_2].$$

If $L$ is not a torsion bundle, it is clear that $L$ trivializes on a principal $\mathbb{G}_m$-bundle $P_L$. The vector bundle $E \cong L \otimes F_2$ trivializes on the $\mathbb{G}_m \times \mathbb{G}_a$-bundle $P_L \times_X P$, where $P$ is the principal $\mathbb{G}_a$-bundle from proposition 2, where $F_2$ and hence all the $F_r$ trivialize.

Let now $L$ be torsion and $n \in \mathbb{N}$, $n \geq 2$, the minimal number with $L^{\otimes n} \cong O_X$. As the $F_r$ are selfdual and $L^{\otimes n-1} = L^{-1}$, it suffices to consider positive tensor powers.

Again we compute the tensor powers using Atiyah’s formula to find the indecomposable components.

If $r$ is even and $n$ is odd, the set $S(E)$ contains the following bundles:

$$S(E) = \{O_X, L^{\otimes i} \otimes F_j \mid i = 0, 1, \ldots, n-1, j \in \mathbb{N}\}.$$

With the help of the multiplication formula for $F_2$ it is easy to show that all elements of $S(E)$ can be generated by $L$ and $F_2$. In addition, the relation $L^{\otimes n} \cong O_X$ holds. Hence we obtain

$$R(E) = \frac{\mathbb{Q}[L]}{< L^{\otimes n-1} >} \otimes \mathbb{Z} \mathbb{Q}[F_2].$$

If $r$ is odd and $n$ is even or odd, the result is

$$S(E) = \{L^{\otimes i} \otimes F_j \mid i = 0, 1, \ldots, n-1, j \in \mathbb{N} \text{ odd}\}.$$

The bundles $L$ and $F_3$ are in $S(E)$ and generate all elements of $S(E)$. Because of the relation $L^{\otimes n} \cong O_X$, the algebra $R(E)$ is

$$R(E) = \frac{\mathbb{Q}[L]}{< L^{\otimes n-1} >} \otimes \mathbb{Z} \mathbb{Q}[F_3].$$

If $r$ and $n$ are both even

$$S(E) = \{L^{\otimes 2i} \otimes F_{2j-1}, L^{\otimes 2i+1} \otimes F_{2j} \mid i = 0, 1, \ldots, n/2, j \in \mathbb{N}\}.$$

The algebra $R(E)$ is generated by $L^{\otimes 2}$ and $L \otimes F_2$. The generators are subject to the relation $L^{\otimes n} \cong O_X$, thus

$$R(E) = \frac{\mathbb{Q}[L^{\otimes 2}]}{< (L^{\otimes 2})^{\otimes m-1} >} \otimes \mathbb{Q}[L \otimes F_2],$$

where $m = n/2$.

Recall that $n \geq 2$ is the minimal number such that $L^{\otimes n} \cong O_X$. Thus the bundle $L$ trivializes on a $\mu_n$-bundle $P_L$ and not on a $\mu_m$-torsor for $m < n$.

The bundle $E \cong L \otimes F_r$ then trivializes on the $\mu_n \times \mathbb{G}_a$-bundle $P_L \times_X P$,
where $P$ is again the principal $\mathbb{G}_a$-bundle from proposition 2. We will now show that the bundle $E$ does not trivialize on a $\mu_{n/2} \times \mathbb{G}_a$-bundle: If $E \cong L \otimes F$, trivializes on $Q \times_X P$, where $Q$ is a $\mu_n$-torsor and $P$ a $\mathbb{G}_a$-torsor, then $\det(L \otimes F) = L$ is the identity element in the group $\text{Pic}(Q \times_X P)$. But one has $\text{Pic}(Q \times_X P) = \text{Pic}(Q)$ by homotopy invariance. Thus $L$ must trivialize on the $\mu_n$-torsor $Q$, which is impossible for $m < n$.

Remark: The correspondence between the dimension of the “minimal” group scheme and the dimension of the ring $R(E)$ also occurs in the case of vector bundles on the projective line, as one easily sees.

Let $X$ be the complex projective line $\mathbb{P}^1$ and $E := \mathcal{O}(a)$ a line bundle. If $a = 0$ we have $S(E) = \{\mathcal{O}\}$ and $R(E) = \mathbb{Q}$.

We define the group scheme $G$ to be $G = \text{Spec } \mathbb{Q}$ and the trivializing torsor is simply $\mathbb{P}^1$.

If $a \neq 0$ we can easily compute that $S(E) = \{\mathcal{O}(\lambda \cdot a) | \lambda \in \mathbb{Z}\}$ and $R(E) = \mathbb{Q}[x, x^{-1}]$. We define the group scheme to be $G = \mathbb{G}_m = \text{Spec } \mathbb{Q}[x, x^{-1}]$.

The given line bundle $E$ trivializes on a principal $\mathbb{G}_m$-bundle $P_a$, which depends on $a$.

Thus we get the correspondence of $\dim R(E)$ and $\dim G$ in the case of a line bundle on $\mathbb{P}^1$. This computation can easily be generalized to the case of vector bundles of higher rank. We illustrate this for bundles of rank two.

Let now $E$ be a vector bundle of rank 2 on $\mathbb{P}^1$, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$.

The case $(a, b) = (0, 0)$ is trivial. We can see at once that $S(E) = \{\mathcal{O}\}$ and therefore $R(E) = \mathbb{Q}$.

The vector bundle $E$ trivializes on the principal $\text{Spec } \mathbb{Q}$-bundle $\mathbb{P}^1$.

If $(a, b) \neq (0, 0)$ the computation gives that $S(\mathcal{O}(a) \oplus \mathcal{O}(b)) = S(\mathcal{O}(c))$, where $c = (a, b)$ (with $(a, 0) = a$ and $(0, b) = b$) and therefore $R(E) = \mathbb{Q}[x, x^{-1}]$. $E$ trivializes on the principal $\mathbb{G}_m$-bundle $P_c$ that belongs to $\mathcal{O}(c)$ as $\mathcal{O}(a) = \mathcal{O}(c)^\lambda$ and $\mathcal{O}(b) = \mathcal{O}(c)^\mu$ for appropriate integers $\lambda$ and $\mu$.

References

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