ON DIAGONALIZABLE OPERATORS IN MINKOWSKI SPACES WITH THE LIPSCHITZ PROPERTY

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Abstract. A real semi-inner-product space is a real vector space \( M \) equipped with a function \([.,.]: M \times M \rightarrow \mathbb{R}\) which is linear in its first variable, strictly positive and satisfies the Schwartz inequality. It is well-known that the function \(|x| = \sqrt{x, x}\) defines a norm on \( M \), and vice versa, for every norm on \( X \) there is a semi-inner-product satisfying this equality. A linear operator \( A \) on \( M \) is called adjoint abelian with respect to \([.,.]\), if it satisfies \([Ax, y] = [x, Ay]\) for every \( x, y \in M \). The aim of this paper is to characterize the diagonalizable adjoint abelian operators in finite dimensional real semi-inner-product spaces satisfying a certain smoothness condition.

1. Introduction and preliminaries

A real semi-inner-product space is a real linear space \( M \) equipped with a function \([.,.]: M \times M \rightarrow \mathbb{R}\), called a semi-inner-product, such that

1. \([.,.]\) is linear in the first variable,
2. \([x, x] \geq 0\) for every \( x \in M \), and \([x, x] = 0\) yields that \( x = 0\),
3. \([x, y]^2 \leq [x, x] \cdot [y, y]\) for every \( x, y \in M \).

These spaces were introduced in 1961 by Lumer \[8\], and have been extensively studied since then (cf., for example \[1\]). It was remarked in \[8\] that in a real semi-inner-product space \( M \), the function \(|x| = \sqrt{x, x}\) defines a norm. The converse also holds, i.e. if \( M \) is a real linear space, then for every real norm \(||.||: M \rightarrow \mathbb{R}\), there is a semi-inner-product \([.,.]: M \times M \rightarrow \mathbb{R}\) satisfying \(|x| = \sqrt{x, x}\). Furthermore, the semi-inner-product determined by a norm is unique if, and only if, its unit ball is smooth; that is, if the unit sphere has a unique supporting hyperplane at its every point. By \[4\], in this case the semi-inner-product is homogeneous in the second variable; i.e., \([x, \lambda y] = \lambda [x, y]\) for any \( x, y \in M \) and \( \lambda \in \mathbb{R}\).

We say that a real semi-inner-product is continuous, if for every \( x, y, z \in M \) with \([x, x] = [y, y] = [z, z] = 1\), \( \lambda \rightarrow 0 \) yields that \([x, y + \lambda z] \rightarrow [x, y]\) (cf. \[4\] or \[5\]). It is well-known that the semi-inner-product determined by a smooth norm is continuous; it follows, for example, from \( E^* \) on page 118 of \[11\] and Theorem 3 of \[4\].

A linear operator \( A \) is called adjoint abelian with respect to a semi-inner-product \([.,.]\), if it satisfies \([Ax, y] = [x, Ay]\) for every \( x, y \in M \) (cf., for instance \[2\] and \[3\]).

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In the following, \( M \) denotes a smooth Minkowski space; that is, a real finite dimensional smooth normed space, and \( ||\cdot|| \) and \([\cdot, \cdot]\) denote the norm and the induced semi-inner-product of \( M \), respectively. We denote by \( S \) the unit sphere with respect to the norm, i.e., we set \( S = \{ x \in M : ||x|| = 1 \} \). We say that the semi-inner-product \([\cdot, \cdot]\) has the Lipschitz property, if for every \( x \in S \), there is a real number \( \kappa \) such that for every \( y, z \in S \), we have \( ||[x, y] - [x, z]|| \leq \kappa||y - z|| \). We note that in a similar way, a differentiability property of semi-inner-products was defined in [3], and that any semi-inner-product satisfying that differentiability property satisfies also the Lipschitz property.

The aim of this paper is to characterize the diagonalizable adjoint abelian operators in finite dimensional spaces with a semi-inner-product that satisfies the Lipschitz property.

To formulate our main result, we need the following notions and notations. An isometry of \( M \) is an operator \( A : M \to M \) satisfying \( ||Ax|| = ||x|| \) for every \( x \in M \), or, equivalently, \([Ax, Ay] = [x, y] \) for every \( x, y \in M \) (cf. [9]). For the properties of isometries in Minkowski spaces, the interested reader is referred to [9].

For the following definition, see also [4].

**Definition 1.** If \( x, y \in M \) and \([x, y] = 0 \), we say that \( x \) is transversal to \( y \), or \( y \) is normal to \( x \). If \( X, Y \subset M \) such that \([x, y] = 0 \) for every \( x \in X \) and \( y \in Y \), we say that \( X \) is transversal to \( Y \) or \( Y \) is normal to \( X \).

**Definition 2.** Let \( U \) and \( V \) be linear subspaces of \( M \) such that \( M = U \oplus V \). If for every \( x_u, y_u \in U \) and \( x_v, y_v \in V \), we have

\[
[x_u + x_v, y_u + y_v] = [x_u, y_u] + [x_v, y_v],
\]

then we say that the semi-inner-product \([\cdot, \cdot]\) is the direct sum of \([\cdot, \cdot]_U \) and \([\cdot, \cdot]_V \), and denote it by \([\cdot, \cdot] = [\cdot, \cdot]_U + [\cdot, \cdot]_V \). If there are no such linear subspaces of \( M \), we say that \([\cdot, \cdot]\) is non-decomposable.

We remark that if \([\cdot, \cdot] = [\cdot, \cdot]_U + [\cdot, \cdot]_V \) for some linear subspaces \( U \) and \( V \), then \( U \) and \( V \) are both transversal and normal, and that the converse does not hold. We note also that any two semi-inner-product spaces can be added in this way (cf. [5]). Definition 2 can be formulated for finitely many subspaces as well in the natural way. For the simplicity of notation, we mean that every semi-inner-product space is the direct sum of itself.

Let \( A : M \to M \) be a linear operator, and let \( \lambda_1 > \lambda_2 > \ldots > \lambda_k \geq 0 \) be the absolute values of the eigenvalues of \( A \). If \( \lambda_i \) is an eigenvalue of \( A \), then \( E_i \) denotes the eigenspace of \( A \) belonging to \( \lambda_i \), and if \( \lambda_i \) is not an eigenvalue, we set \( E_i = \{0\} \). We define \( E_{-i} \) similarly with \(-\lambda_i \) in place of \( \lambda_i \), and set \( \bar{E}_i = \text{span}(E_i \cup E_{-i}) \). Our main theorem is the following.

**Theorem 1.** Let \( M \) be a smooth Minkowski space such that the induced semi-inner-product \([\cdot, \cdot] : M \times M \to \mathbb{R} \) satisfies the Lipschitz condition, and let \( A : M \to M \) be a diagonalizable linear operator. Then \( A \) is adjoint abelian with respect to \([\cdot, \cdot]\) if, and only if, the following hold.

1. \([\cdot, \cdot]\) is the direct sum of its restrictions to the subspaces \( \bar{E}_i \), \( i = 1, 2, \ldots, k \);
2. for every value of \( i \), the subspaces \( E_i \) and \( E_{-i} \) are both transversal and normal;
for every value of $i$, the restriction of $A$ to $\bar{E}_i$ is the product of $\lambda_i$ and an isometry of $\bar{E}_i$.

From Theorem 1 we readily obtain the following corollary.

Corollary 1. Let $\mathbb{M}$ be a smooth Minkowski space such that the induced semi-inner-product $[.,.]$ satisfies the Lipschitz condition. Then the following are equivalent.

1. $[.,.]$ is non-decomposable;
2. every diagonalizable adjoint abelian linear operator of $\mathbb{M}$ is a scalar multiple of an isometry of $\mathbb{M}$.

Note that if $A$ is not diagonal, then we may apply Theorem 1 for the span of the eigenspaces of $A$.

Corollary 2. Let $\mathbb{M}$ be a smooth Minkowski space such that the induced semi-inner-product $[.,.] : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ satisfies the Lipschitz condition, and let $A : \mathbb{M} \to \mathbb{M}$ be an adjoint abelian linear operator with respect to $[.,.]$. Then (1), (2) and (3) in Theorem 1 hold for $A$.

If $[.,.] = [.,.]|_U + [.,.]|_V$ for some subspaces $U$ and $V$, and $u \in U$ and $v \in V$, then, by Theorem 1, $S \cap \text{span}\{u, v\}$ is an ellipse. This observation is proved, for example, in Statement 1 of [5]. Thus, we have the following.

Corollary 3. Let $\mathbb{M}$ be a smooth Minkowski space such that the induced semi-inner-product satisfies the Lipschitz condition. If no section of the unit sphere $S$ with a plane is an ellipse with the origin as its centre, then every diagonalizable adjoint abelian operator of $\mathbb{M}$ is a scalar multiple of an isometry of $\mathbb{M}$.

In the proof of Theorem 1 we need the following lemma.

Lemma 1. Let $\mathbb{M}$ be a smooth Minkowski space. Let $||.||$, $[.,.]$ and $S$ denote the norm, the associated semi-inner-product and the unit sphere of $\mathbb{M}$. Then the following are equivalent.

1. $[.,.]$ satisfies the Lipschitz condition;
2. for every $x \in \mathbb{M}$, the function $f_x : \mathbb{M} \to \mathbb{R}$, $f_x(y) = [x,y]$ is uniformly continuous on $\mathbb{M}$; that is, for every $x \in \mathbb{M}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $y, z \in \mathbb{M}$ and $||y - z|| < \delta$ imply $||x, y|| - ||x, z|| < \varepsilon$;
3. for every $x \in \mathbb{M}$ and any sequences $\{y_n\}, \{z_n\}$ in $\mathbb{M}$, if $||y_n - z_n|| \to 0$, then $||x, y_n|| - ||x, z_n|| \to 0$.

Proof. Note that (2) and (3) are equivalent. We prove that (1) and (3) are equivalent.

First we show that (1) yields (3). Observe that since $[x,y]$ is homogeneous in $x$, it suffices to prove (3) for $x \in S$. Let $x \in S$, and assume that there is a number $\kappa \in \mathbb{R}$ such that for every $y, z \in S$, we have $||x, y|| - ||x, z|| < \kappa||y - z||$. Consider the sequences $\{y_n\}$ and $\{z_n\}$ in $\mathbb{M}$, and assume that $||y_n - z_n|| \to 0$. Since a continuous function is uniformly continuous on any compact set and since the unit ball of $\mathbb{M}$ is compact, we may assume that $||y_n|| \geq 1$ and that $||z_n|| \geq 1$ for every $n$. Let $w_n = \frac{||z_n||}{||y_n||}y_n$. Observe that, by the definition of semi-inner-product, $|[u,v]| \leq 1$
for any $u, v \in S$. Then, from $\left| x, \frac{w_n}{\|y_n\|} \right| \leq 1$ and from the triangle inequality, we obtain that
\[
\left| \left| x, y_n \right| - \left| x, z_n \right| \right| \leq \left| \left| x, y_n \right| - \left| x, w_n \right| \right| + \left| \left| x, w_n \right| - \left| x, z_n \right| \right| = \left| \|y_n\| - \|w_n\| \right| - \left| \|y_n\| - \|z_n\| \right|.
\]
\[
\left| \left| x, \frac{y_n}{\|y_n\|} \right| - \left| x, \frac{w_n}{\|w_n\|} \right| \right| + \left| \frac{z_n}{\|z_n\|} \right| \left| \left| \frac{w_n}{\|w_n\|} - \left| \frac{z_n}{\|z_n\|} \right| \right| \leq \left| \|y_n\| - \|w_n\| \right| + \|\|y_n\| - \|z_n\|\| \to 0.
\]
Note that $\|w_n - y_n\| = \||y_n\| - \|z_n\|\| \leq \|y_n - z_n\| \to 0$ and that $\|w_n - z_n\| \leq \|w_n - y_n\| + \|y_n - z_n\| \to 0$, from which it follows that $\left| x, y_n \right| - \left| x, z_n \right| \to 0$.

Assume that (1) does not hold. Then there is a point $x \in S$ and sequences $y_n, z_n \in S$ such that $\left| \left| x, y_n \right| - \left| x, z_n \right| \right| = \kappa_n \|y_n - z_n\|$ where $\kappa_n \to \infty$. We may assume that $\kappa_n > 0$ for every $n$, and since $S$ is compact, also that $y_n \to y$ and $z_n \to z$ for some $y, z \in S$. Note that $\kappa_n \to \infty$ implies $y = z$. Let $\delta_n = \|y_n - z_n\|$, and assume that $\delta_n > 0$ for every $n$. Observe that as $y_n$ and $z_n$ converge to the same point, we have $\delta_n \to 0$, and, as $[x, y]$ is continuous in $y \in S$ for every $x \in S$, we have also that $\kappa_n \|y_n - z_n\| = \kappa_n \delta_n \to 0$. Let $u_n = \frac{u_n}{\kappa_n \delta_n}$ and $v_n = \frac{z_n}{\kappa_n \delta_n}$. Then $\left| u_n - v_n \right| = \frac{1}{\kappa_n} \to 0$, and $\left| \left| x, u_n \right| - \left| x, v_n \right| \right| = 1$, and hence, (3) does not hold. \hfill \Box

2. Proof of Theorem 1

Assume that $A$ is adjoint abelian. Let $\mu$ and $\nu$ be two different eigenvalues of $A$ and let $x$ and $y$ be eigenvectors belonging to $\mu$ and $\nu$, respectively. Then,
\[
\mu[x, y] = [Ax, y] = [x, Ay] = \nu[x, y],
\]
which yields that $x$ is transversal to $y$. Thus, any two eigenspaces, belonging to distinct eigenvalues, are both transversal and normal, which, in particular, proves (2) (for isometries, see this observation in [7]). Recall that an Auerbach basis of a Minkowski space is a basis in which any two distinct vectors are transversal and normal to each other, and that in every norm there is an Auerbach basis. Note that the restriction of a norm to a linear subspace is also a norm, and thus, we may choose Auerbach bases in each eigenspace separately, which, by the previous observation, form an Auerbach basis in the whole space. Let $x \in \mathbb{M}$, and observe that $x$ has a unique representation of the form $x = \sum_{i=1}^k x_i$, where $x_i \in \bar{E_i}$. To prove Theorem 1 we need the following lemma.

Lemma 2. If $z \in \bar{E_i}$ for some value of $i$, then $[z, x] = [z, x_i]$.

Proof. Assume that $z \in \bar{E_i}$ for some $i \in \{1, 2, \ldots, k\}$.

Case 1, $i = 1$.

If $\lambda_1 = 0$, then $A$ is the zero operator, and the assertion immediately follows. Let us assume that $\lambda_1 > 0$. As $A$ is adjoint abelian, we have that $[A^2 z, x] = [Az, Ax] = [z, A^2 x]$. Observe that $A^2 z = \lambda_1^2 z$, and that $A^2 x = \sum_{i=1}^k \lambda_i^2 x_i$. Thus,
\[
[z, x] = \left[ z, \sum_{i=1}^k \left( \frac{\lambda_i^2}{\lambda_1^2} \right)^{\frac{n}{\lambda_1^2}} x_i \right]
\]
for every positive integer $n$. Since $[\cdot, \cdot]$ is continuous in both variables, we obtain that the limit of the right-hand side of (1) is $[z, x_1]$, and hence, $[z, x] = [z, x_1]$.\hfill \Box
other hand, Note that in this case and thus, we have for every positive integer \( n \). We prove by induction on \( n \) and its eigenvalues of \( A \). Note that \( A|_{F_i} \) is invertible, adjoint abelian, and its inverse is also adjoint abelian. Let \( B_i \) denote the inverse of \( A|_{F_i} \), and observe that the absolute values of the eigenvalues of \( B_i \) are \( \frac{1}{\lambda_j} \) and its eigenspaces are \( E_j \) and \( E_{-j} \), where \( j = 1, 2, \ldots, i \). Thus, we have \( \frac{1}{n} [z, w] = [B_i^2 z, w] = [B_i w, B_i w] = [z, B_i^2 w] \) and

\[
[z, w] = \left[ z, \sum_{j=1}^{i} \left( \frac{\lambda_j^2}{\lambda_j^2} \right)^n x_j \right]
\]

for every positive integer \( n \). By the continuity of the semi-inner-product, and since the limit of the right-hand side is \( [z, x_i] \), we have the desired equality.

Now we show that \( [z, x] = [z, x_i] \). Similarly like before, we obtain that

\[
[z, x] = \left[ z, \sum_{j=1}^{k} \left( \frac{\lambda_j^2}{\lambda_j^2} \right)^n x_j \right]
\]

for every positive integer \( n \). Observe that \( \lim_{n \to \infty} \left| \sum_{j=i+1}^{k} \left( \frac{\lambda_j^2}{\lambda_j^2} \right)^n x_j \right| = 0 \), and that, by the previous paragraph, \( [z, \sum_{j=1}^{i} \left( \frac{\lambda_j^2}{\lambda_j^2} \right)^n x_j] = [z, x_i] \) for every positive integer \( n \). Thus, by Lemma \( \text{II} \) we have that \( [z, x] = [z, x_i] \).

Case 3. \( i > 1 \) and \( \lambda_i = 0 \).

Note that in this case \( i = k \) and \( \tilde{E}_k = E_k \). Let \( F = \text{span}(\bigcup_{i=1}^{k-1} \tilde{E}_i) \) and set \( x_f = \sum_{j=1}^{k-1} x_j \). By Cases 1 and 2, we have \( [x_f, x_k] = \sum_{j=1}^{k-1}[x_j, x_k] = 0 \). On the other hand,

\[
[x_k, x_f] = \left[ x_k, A^2 \sum_{j=1}^{k-1} \frac{1}{\lambda_j^2} x_j \right] = \left[ A^2 x_k, \sum_{j=1}^{k-1} \frac{1}{\lambda_j^2} x_j \right] = 0.
\]

Thus, we obtained that \( F \) and \( E_k \) are both transversal and normal. Now we let \( G = \text{span}\{x_f, x_k, z\} \).

Subcase 3.1. \( \dim G = 2 \).

Let \( e_1 = \frac{x_f}{||x_f||} \) and \( e_2 = \frac{x_k}{||x_k||} \). Since \( F \) and \( E_k \) are transversal and normal, the pair \( \{e_1, e_2\} \) is an Auerbach basis in \( G \). By Cases 1 and 2, \( [\cdot, \cdot]|_F = \sum_{j=1}^{k-1}[\cdot, \cdot]|_{\tilde{E}_j} \), which yields that \( [e_1, \alpha_1 e_1 + \alpha_2 e_2] = \alpha_1 \) for any \( \alpha_1, \alpha_2 \in \mathbb{R} \).

Now we identify \( G \) with the Euclidean plane \( \mathbb{R}^2 \) by \( \alpha_1 e_1 + \alpha_2 e_2 \mapsto (\alpha_1, \alpha_2) \); or in other words, we assume that \( e_1 \) and \( e_2 \) are the standard basis of an underlying Euclidean plane. We need to show that \( [e_2, \alpha_1 e_1 + \alpha_2 e_2] = \alpha_2 \) for any \( \alpha_1, \alpha_2 \in \mathbb{R} \), or, equivalently, that the unit circle \( S \cap G \) of the subspace \( G \) is the Euclidean unit circle.

Since \( \mathbb{M} \) is smooth, \( S \cap G \) is a convex differentiable curve. Consider the Descartes coordinate system induced by the standard basis \( e_1 \) and \( e_2 \), and note that the lines \( x = 1, x = -1, y = 1 \) and \( y = -1 \) support conv \( S \). Thus, for every value
of \( x \in (-1, 1) \), there is exactly one point of \( S \) with \( x \) as its \( x \)-coordinate and nonnegative \( y \)-coordinate. We represent the points of \( S \cap G \) with nonnegative \( y \)-coordinates as the union of the graph of a function \( x \mapsto f(x) \) with \( x \in [-1, 1] \), and (possibly) two segments on the lines with equations \( x = 1 \) and \( x = -1 \). We express the equality \([e_1, \alpha_1 e_1 + \alpha_2 e_2] = \alpha_1 \) with the function \( f \).

We may assume that \( v = \alpha_1 e_1 + \alpha_2 e_2 \in S \cap G \). Consider the case that \( v = x_0 e_1 + f(x_0) e_2 \) for some \( x_0 \in (-1, 1) \). Then we have \([e_1, v] = [e_1, x_0 e_1 + f(x_0) e_2] = x_0 \). Let \( v_p \) denote the projection of \( e_1 \) onto the line \( \{ \lambda v : \lambda \in \mathbb{R} \} \) parallel to the supporting line of \( \text{conv} \ S \) at \( v \), and let \( e_p \) denote the projection of \( v \) onto the line \( \{ \lambda e_1 : \lambda \in \mathbb{R} \} \) parallel to the supporting line of \( \text{conv} \ S \) at \( e_1 \) (cf. Figure 1). Let \( v_p = \mu v \) and observe that \( e_p = x_0 e_1 \). We note that, by the construction of the semi-inner-product described, for example in [8], we have that \([e_1, v] = \mu \) and \([v, e_1] = x_0 \). Hence the triangle with vertices \( o, e_1, v \) is similar to the triangle with vertices \( o, v_p, e_p \), with similarity ratio \( x_0 \). From this, we obtain that \( v_p = x_0 v = x_0^2 e_1 + x_0 f(x_0) e_2 \). As the line, passing through \( e_1 \) and \( v_p \), is parallel to the supporting line of \( \text{conv} \ S \) at \( v \), we have

\[
f'(x_0) = -\frac{x_0 f(x_0)}{1 - x_0^2},
\]

which is an ordinary differential equation for \( f \) with the initial condition \( f(0) = 1 \).

![Figure 1. An illustration for Subcase 3.1](image)

We omit an elementary computation that shows that the solution of this differential equation is \( y = \sqrt{1 - x^2} \). Thus, we obtain that \( S \cap G \) is the Euclidean unit circle, which yields, in particular, that \([z, x f + x_k] = [z, x_k] \).

**Subcase 3.2, \( \dim G = 3 \).**

Set \( e_1 = \frac{x}{\|x\|} \) and choose an Auerbach basis \( \{e_2, e_3\} \) in \( \text{span}\{x_k, z\} \). Then the set \( \{e_1, e_2, e_3\} \) is an Auerbach basis in \( G \). Furthermore, since \( F \) and \( E_k \) are transversal and normal, \( \{e_1, v\} \) is an Auerbach basis in its span for any \( v \in \text{span}\{x_k, z\} \) with \( \|v\| = 1 \). Thus, applying the argument in Subcase 3.1 for the subspace \( \text{span}\{e_1, v\} \), we obtain that \( S \cap \text{span}\{e_1, v\} \) is the ellipse with semiaxes \( e_1 \) and \( v \). Note that this property and \( S \cap \text{span}\{e_1, e_2\} \) determines the norm.
Consider the semi-inner-product defined by \( [\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3] = \beta_1 \alpha_1 + [\beta_2 e_2 + \beta_3 e_3, \alpha_2 e_2 + \alpha_3 e_3] \). We show that \([.,.]\)' and \([.,.\]) define the same norm, which, as a smooth norm uniquely determines its semi-inner-product, yields that \([.,.] = [.,.]\).

Let \( v = \alpha_2 e_2 + \alpha_3 e_3 \) with \( ||v|| = 1 \) be arbitrary. Note that if \( ||\mu e_1 + \nu v|| = 1 \), then
\[
[\mu e_1 + \nu v, \mu e_1 + \nu v]' = \mu^2 + \nu^2 [v, v] = \mu^2 + \nu^2 = 1,
\]
which, in span \( \{e_1, v\} \), is the equation of the ellipse with semiaxes \( e_1 \) and \( v \). As the restrictions of \([.,.]\)' and \([.,.\]) to span \( \{e_2, e_3\} \) are clearly equal, we obtain that \([.,.] = [.,.]\)' , which, in particular, implies that \( [z, x_f + x_k] = [z, x_k] \). □

By Lemma 2 we have that (1) of Theorem 1 holds. Thus, it remains to show that (3) also holds. Without loss of generality, let us assume that \( k = 1 \), and that \( \lambda_1 = 1 \). Then every \( x \in \mathcal{M} \) can be decomposed as \( x = x_1 + y_1 \) with \( x_1 \in \mathcal{E}_1 \) and \( y_1 \in \mathcal{E}_{-1} \). Hence,
\[
[A(x_1 + y_1), A(x_1 + y_1)] = [x_1 + y_1, A^2(x_1 + y_1)] = [x_1 + y_1, x_1 + y_1],
\]
and thus, \( A \) is an isometry.

Finally, we show that if (1), (2) and (3) holds, then \( A \) is adjoint abelian. Let \( x = \sum_{i=1}^k x_i \) and \( y = \sum_{i=1}^k y_i \) with \( x_i, y_i \in \mathcal{E}_i \). Assume, first, that \( \lambda_k \neq 0 \), which means that \( A \) is invertible. Note that \( \mathcal{E}_i \) is an invariant subspace of \( A \) for every value of \( i \), and that \( \left( \frac{1}{\lambda_i} A \right)^{-1} = \frac{1}{\lambda_i} A = \text{id} \) on \( \mathcal{E}_i \). Hence,
\[
[Ax, y] = \sum_{i=1}^k [Ax_i, y_i] = \sum_{i=1}^k \lambda_i \left[ \frac{1}{\lambda_i} Ax_i, y_i \right] = \sum_{i=1}^k \lambda_i \left[ x_i, \left( \frac{1}{\lambda_i} A \right)^{-1} y_i \right] = \sum_{i=1}^k \lambda_i \left[ x_i, \frac{1}{\lambda_i} Ay_i \right] = \sum_{i=1}^k [x_i, Ay_i] = [x, Ay],
\]
and the assertion follows. If \( A \) is not invertible, we may apply a slightly modified argument.

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