Holographic Mapping of Many-Body Localized System by Spectrum Bifurcation Renormalization Group

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Entanglement 15
MBL 15
Introduction

• When we talk about quantum many-body physics, we usually think of ground states.

• Magnets, superconductors, topological insulators …
• Quantum phase transitions between ground states
• Highly-excited states (finite energy density $E/V$) are typically thermalized, described by statistical mechanics.
Introduction

- Eigenstate Thermalization Hypothesis (ETH) Deutsch 91, Srednicki 94
  - System serves as its own heat bath
  - Density matrix of a subsystem
    \[ \rho_A = \text{Tr}_\overline{A} \langle \Psi | \Psi \rangle \sim e^{-\beta H_A} \]
  - Volume-law entanglement entropy
    \[ S_A = -\text{Tr}_A \rho_A \ln \rho_A \sim s |A| \]
    In contrast to ground states (area-law)

- Are highly-excited states always thermalized? - No.
- Localization in disordered system violates ETH
  - Lack of energy diffusion \( \rightarrow \) fail to thermalize
**Introduction**

- **Single-particle: Anderson localization**
  
  \[ H = \sum_i -t(c_i^\dagger c_{i+1} + h.c.) - \epsilon_i n_i \quad \text{random} \, \epsilon_i \in [-W, W] \]

- **Fock-space: Many-Body Localization (MBL)**
  
  \[ H = \sum_i -t(c_i^\dagger c_{i+1} + h.c.) - \epsilon_i n_i - V n_i n_{i+1} \]

  Localization can survive interaction. Both fermion & spin systems.

- **Experimental Realizations**

- **Figure 1:** Coherence and mobility. (a) Measured width of \[ \Delta (E_R) \] and \[ \phi \] (units of m/s) vs. \[ \Delta_c \]. (b) Momentum acquired after an applied impulse for varying interaction strength and three disorder 

- **Graphs:**
  - DeMarco group 1305.6072
  - Inguscio/Modugno group 1405.1210
  - Bloch group 1501.05661

**References:**
- Anderson 1958
- Basko, Aleiner, Altshuler 06
- Gorny, Mirlin, Polyakov 05
- Znidaric, Prosen, Prelovsek 08
- Imbrie 14
**Introduction**

- **Full MBL**: all energy eigenstates are localized
  - Extensive number of LIOMs $\hat{n}_a$
  - Effective Hamiltonian in terms of LIOMs
    - Fermionic systems
      $$H_{\text{eff}} = \sum_a \epsilon_a \hat{n}_a + \sum_{a,b} \epsilon_{ab} \hat{n}_a \hat{n}_b + \sum_{a,b,c} \epsilon_{abc} \hat{n}_a \hat{n}_b \hat{n}_c + \ldots$$
      $$[H_{\text{eff}}, \hat{n}_a] = 0$$
    - Bosonic/Spin systems:
      $$H_{\text{eff}} = \sum_a \epsilon_a \tau^z_a + \sum_{a,b} \epsilon_{ab} \tau^z_a \tau^z_b + \sum_{a,b,c} \epsilon_{abc} \tau^z_a \tau^z_b \tau^z_c + \ldots$$
      ($\tau^z_a = \pm 1$) stabilizer like Landau Fermi liquid as RG fixed point
  - Area-law entanglement entropy (like ground states)
  - Quantum many-body physics in highly-excited states

Bauer, Nayak 13; Huse, Nandkishore, Oganesyan, Pal, Sondhi 13; Bahri, Vosk, Altman, Vishwanath 13; Chandran, Khemani, Laumann, Sondhi 14; Potter, Vishwanath 15; Slagle, Bi, You, Xu 15
Introduction

- **Marginal MBL**: quantum phase transition at finite $T$
- **Thermalization** transition: emergence of statistical mechanics
- **Thermalization of marginal MBL system** (e.g. thermalization of MBL-SPT boundary)

For eigenstates in a many-body spectrum

Nandkishore, Potter
1406.0847

You, Ludwig, Xu,
1602.06964
Finding Effective Hamiltonian

- Given a disordered many-body Hamiltonian, find
  \[ H_{\text{eff}} = \sum_a \epsilon_a \tau^z_a + \sum_{a,b} \epsilon_{ab} \tau^z_a \tau^z_b + \sum_{a,b,c} \epsilon_{abc} \tau^z_a \tau^z_b \tau^z_c + \ldots \quad (\tau^z_a = \pm 1) \]

  - Finding \( H_{\text{eff}} \sim \) diagonalization of many-body Hamiltonian

- MBL: Area-law entanglement entropy → matrix/tensor product state (MPS/TPS)
  \[ |\Psi\rangle = \sum_{\{\sigma_i\}} \Psi(\{\sigma_i\}) |\{\sigma_i\}\rangle \]
  \[ \Psi(\{\sigma_i\}) = \text{Tr} A^{\sigma_1} A^{\sigma_2} A^{\sigma_3} \ldots \]

  - Renormalization Group (RG) approach
    - Real Space RG (RSRG-X)
      - Spectrum Bifurcation RG (SBRG)
    - DMRG-X

Bauer, Nayak 1306.5753
Chandran, Carraquilla, Kim, Abanin, Vidal 1410.0687; Pekker, Clark 1410.2224; Pollmann, Khemani, Cirac, Sondhi 1506.07179
You, Qi, Xu 1508.03635
Spectrum Bifurcation RG

• Disordered Quantum Ising Model

\[ H = - \sum_i J_i \sigma_i^x \sigma_{i+1}^x + K_i \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z \quad \text{random } J_i, K_i, h_i \]

Or as interacting spinless fermions

\[ H = - \sum_i \frac{J_i}{4} (c_i^\dagger c_{i+1} + c_i c_{i+1} + h.c.) + \frac{K_i}{4} n_i n_{i+1} - \frac{h_i}{2} n_i \]

• Pick out the leading energy scale term, rotate to its diagonal basis

• Generate effective couplings within high/low-energy subspaces by 2nd order perturbation

\[ \text{Clifford} \quad \text{Schrieffer-Wolff} \]
Spectrum Bifurcation RG

- Generic Qubit Model (qubits ~ spins/fermions)

\[ H = \sum_{[\mu]} h_{[\mu]} \sigma^{[\mu]}, \quad \sigma^{[\mu]} = \sigma^{\mu_1} \otimes \sigma^{\mu_2} \otimes \sigma^{\mu_3} \ldots \quad (\mu_i = 0, 1, 2, 3) \]

- Each RG step contains two unitary transformations \( R \) and \( S \):

\[
\begin{align*}
H \xrightarrow{R} H &= H_0 + \Delta + \sum S H = H_0 + \Delta - \frac{1}{2} \sum H_0^{-1} \Sigma \\
H_0 \xrightarrow{R} H_0 &= \epsilon_a \tau^z_a \\
H_0 \Delta = \Delta H_0, \text{ in the off-diagonal block} \quad H_0 \Sigma = -\Sigma H_0, \text{ in the diagonal block}
\end{align*}
\]

- Hilbert-space-preserving (unitary) RG

\[
U = \prod_{k} R_k S_k : H \rightarrow H_{\text{eff}} = U^\dagger H U = \sum_a \epsilon_a \tau^z_a + \sum_{a,b} \epsilon_{ab} \tau^z_a \tau^z_b + \ldots
\]
Quantum Circuit and MPS

- Approx. RG transform by Clifford circuit
  \[ U = \prod_k R_k S_k \rightarrow U_{\text{Cl}} = \prod_k R_k \]

- Clifford circuit = Matrix Product Operator (MPO)

- MPS representation of MBL eigenstate

- Entanglement entropy
  \[ S_A \leq \ln D \]

- Direct-product state

- Bound dimension
  \[ D \approx 2^{\ln N} \]
Trinity of Emergent Qubits

- Emergent qubit

- LIOM

- Controls the spectrum branching

- Holographic bulk degrees of freedom

\[
H_{\text{eff}} = \sum_a \epsilon_a \tau_a^z + \ldots
\]
Holographic Mapping

- Emergent qubit
- LIOM
- Controls the spectrum branching
- Holographic bulk degrees of freedom

EHM: Qi 2013

Swingle 09,12; Evenbly, Vidal 11; Leigh et.al. 14; Ryu, Takayanagi et.al. 12,13,14; Lee 13, 15; Haegeman et.al. 13; Czech et. al. 15; Pastawki et.al. 15; Bao et.al. 15; Molina-Vilaplana 15 …
Holographic Mapping

- Geometric Interpretations of Entanglement Features
  - Entanglement entropy
    \[ S_A = |\gamma_A| \]  
    Ryu, Takayanagi 06
  - Correlation, Mutual Information
    \[ I_{ij} = I_0 \ e^{-d_{ij}/\xi} \]

- Full-spectrum holographic mapping for generic many-body system is challenging.

- MBL: "quasi-solvable", allows Hilbert-space-preserving RG and a controlled holographic mapping of the entire many-body Hilbert space.
Entanglement Entropy

- All states have *approximately* the same entanglement entropy, given by the **Clifford circuit**.
- Roughly: each broken Clifford gate $\rightarrow$ 1 bit entropy
- Precisely: stabilizer rank (fast algorithm)  

\[
\text{Clifford gate}
\]

\[
\text{minimal surface } \gamma_A
\]

\[
\text{A}
\]

\[
S_E \text{ [bit]}
\]

\[
\log_2 L
\]

\[
A \quad B \quad C \quad E \quad F \quad G
\]

\[
A: c' = 0.99 \ln 2
B: c' = 0.52 \ln 2
C: c' = 0.50 \ln 2
\]

\cite{Fattal et al. quant-ph/0406168}
Entanglement Entropy

\[ H = -\sum_i J_i \sigma_i^x \sigma_{i+1}^x + K_i \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z \]

- MBL (SG, PM): \( S_E \sim \text{const.} \)
- Marginal MBL: \( S_E = \frac{c'}{3} \ln L \) \( c' = c \ln 2 \) (for Ising/Majorana systems)

\( h = 0: \) two Majorana chains

\[ \chi = 0: \] two Majorana chains

\[ \chi = \ln 2: \]

\[ \chi = 0.99 \ln 2: \]

\[ \chi = 0.52 \ln 2: \]

\[ \chi = 0.50 \ln 2: \]
Local Integrals of Motion

- Holographic duality
- Bulk: Emergent qubits
- Boundary: Stabilizers

\[ \hat{\tau}_a = U_{\text{Cl}} \tau^z_a U_{\text{Cl}}^\dagger \]

\[ H_{\text{eff}} = \sum_a \epsilon_a \tau^z_a + \ldots \]
Stabilizer Locality

- Stabilizer length
- MBL phases
  \[ P(\ell) \sim e^{-\ell/\xi} \]
- Marginal MBL (Critical)
  \[ P(\ell) \sim \ell^{-\alpha} \quad (\alpha = 2) \]
- Interaction does not change \( \alpha \)

Free case:

- D.S. Fisher 95
- Pekker et al. 14
- Vasseur et al. 15
Application to Other Marginal MBL

• SBRG: good for Ising/Majorana-type models

• 1D XYZ Spin Chain  Slagle, You, Xu, 1604.04283

\[ H = - \sum_i J_{x,i} \sigma_i^x \sigma_{i+1}^x + J_{y,i} \sigma_i^y \sigma_{i+1}^y + J_{z,i} \sigma_i^z \sigma_{i+1}^z \]

\( J_{x,i}, J_{y,i}, J_{z,i} \) independently random  \( \mathbb{Z}_2 \times \mathbb{Z}_2 (D_2) \) symmetry

• Entanglement Entropy
  \[ S_E = \frac{c'}{3} \ln L \quad c' = \ln 2 \]

• Edward-Anderson Correlator
  \[ \langle \sigma_i^a \sigma_j^a \rangle^2 \sim |i - j|^{-\eta_a} \]

• Mutual Information
  \[ \mathcal{I}_{AB} \sim (x_{AB} / l)^{-k} \quad \mathcal{I}_{AB} = S_E(A) + S_E(B) - S_E(A \cup B) \]
Out-of-Time-Order Correlation

- **OTOC**
  \[ F(t) = \langle W^+(t) V^+(0) W(t) V(0) \rangle_\beta \]

- Operator growth
  \[ \langle |[W(t), V(0)]|^2 \rangle_\beta = 2 (1 - F(t)) \]

- Butterfly effect
  \[ F(t) = \langle y | x \rangle; \quad |x\rangle = W(t) V |\beta\rangle, \quad |y\rangle = V W(t) |\beta\rangle \]

- MBL and marginal MBL systems
  \[ H_{\text{eff}} = \sum_A \epsilon_A \, T_A \]
  \[ U(t) = e^{-i t H_{\text{eff}}} = \prod_A e^{-i t \epsilon_A T_A} \]
  \[ T_A = \prod_{a \in A} \tau^z_a \quad \text{commuting Pauli operators} \]

- Operator growth
  \[ W(t) = W \prod_{T_A \in \mathcal{A}_W} e^{-2 i t \epsilon_A T_A} \]

- OTOC
  \[ F(t) = W V W V \prod_{T_A \in \mathcal{A}_W \cap \mathcal{A}_V} e^{A i t \epsilon_A T_A} \]
Out-of-Time-Order Correlation

- **MBL**

  \[ \exp \text{ mean } \ln \sigma_i^x \sigma_j^y \text{ OTOC} \]

  \[ L = 256, \bar{J} = (0.125, 0.125, 1) \]

  Logarithmic light-cone \( \ln t_{sc} \sim |i - j| / \xi \)

- **Marginal MBL**

  Squared-logarithmic light-cone

  \[ \ln t_{sc} \sim |i - j|^{1/2} \]

  \[ H_{\text{eff}} = \sum_a \epsilon_a \tau_a^z + \ldots \]

  Dynamical scaling \( l \sim (-\ln \epsilon)^2 \)

  Huang, Zhang, Chen 1608.01091, Fan, Zhang, Shen, Zhai 1608.01914, Swingle, Chowdhury 1608.03280
Beyond 1D: MBL Topological Order

- Strong disorder toric code model

\[ H = \sum_v J_v A_v + \sum_p J_p B_p + \text{random } J_v, J_p \]

\[ \sum_l (h_e \sigma_l^z + h_m \sigma_l^x) \]

\[ A_v = \prod_{l \in d_v} \sigma_l^x, \quad B_p = \prod_{l \in \partial p} \sigma_l^z \]

- Long-range Mutual Information

\[ \mathcal{I}_{AB} = S_A + S_B - S_{A \cup B} \]

Jian, Kim, Qi 1508.07006

measures long-range entanglement

\[ \mathcal{I}_{AB} = \begin{cases} 
2 \text{ bit} & \text{deconfine} \\
0 \text{ bit} & \text{confining/Higgs} 
\end{cases} \]
Holographic Hamiltonian

• Geometry of the holographic bulk

\[ d_{ab} = -\xi \ln \frac{I_{ab}}{I_0} \]  mutual information

\[ I_{ab} = S_a + S_b - S_{ab} \]

• Mapping \( H \) to the bulk

\[ H_{\text{hol}} = U^\dagger_{\text{Cl}} H U_{\text{Cl}} \]

• Portion of off-diagonal terms

\[ \frac{\text{Tr}(H_{\text{hol}} - \text{diag } H_{\text{hol}})^2}{\text{Tr } H_{\text{hol}}^2} = \frac{\delta E^2}{E^2} \]

• Deep MBL: fragmented space

• Less disorder, more entangled, closer in distance.
Summary

- Spectrum Bifurcation RG
  - Numerical method to study MBL physics
  - Entanglement holographic mapping for MBL systems

- Goal: understand thermalization, the origin of Stat. Mech.
  - A random tensor network & holography based approach?

- Code available on GitHub!

- Vosk, Huse, Altman 1412.3117; Potter, Vasseur, Parameswaran 1501.03501;
- Chandran, Laumann 1501.01971;
- Chen, Yu, Cho, Clark, Fradkin 1509.03890