The right acute angles problem?

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Abstract

The Danzer–Grünbaum acute angles problem asks for the largest size of a set of points in \( \mathbb{R}^d \) that determines only acute angles. Recently, the problem was essentially solved thanks to the results of the second author and of Gerencsér and Harangi: now, the lower and the upper bounds are \( 2^{d-1} + 1 \) and \( 2^d - 1 \), respectively. The lower-bound construction is surprisingly simple.

In this note, we suggest the following variant of the problem, which is one way to “save” the problem. Put \( F(\alpha) = \lim_{d \to \infty} f(d, \alpha)^{1/d} \), where \( f(d, \alpha) \) is the largest set of points in \( \mathbb{R}^d \) with no angle greater than \( \alpha \). Then the question is to find \( c := \lim_{\alpha \to \pi/2} F(\alpha) \). Although one may expect that \( c = 2 \) in view of the result of Gerencsér and Harangi, the best lower bound we could get is \( c \geq \sqrt{2} \).

We also solve a related problem of Erdős and Füredi on the “stability” of the acute angles problem and refute another conjecture stated in the same paper.

1 Introduction

A set of points \( X \subset \mathbb{R}^d \) is called acute (non-obtuse) if any three points from \( X \) form an acute (acute or right, respectively) triangle. In 1962, Danzer and Grünbaum [DG] confirmed a conjecture of Erdős from 1957 that any non-obtuse set of points in \( \mathbb{R}^d \) has cardinality at most \( 2^d \), moreover, the only examples of non-obtuse sets of cardinality \( 2^d \) are the hypercube and some of its affine images. They then modified the question and asked to determine the maximum size \( f(d) \) of an acute set in \( \mathbb{R}^d \) for any \( d \geq 2 \). Danzer and Grünbaum obtained the first bounds on \( f(d) \):

\[
2d - 1 \leq f(d) \leq 2^d - 1,
\]

where the upper bound immediately follows from the aforementioned result on non-obtuse sets. They conjectured that the lower bound is tight.

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As it turned out recently, the value of \( f(d) \) is actually very close to the upper bound in (1). While the only improvement upon the upper bound in (1) made so far is the inequality \( f(3) \leq 5 \) proved in [C], there were surprisingly many improvements for the lower bound. From (1) we get \( f(3) = 5 \) and this is the only non-trivial exact value of \( f(d) \) known so far (along with the trivial equality \( f(2) = 3 \)).

In 1983, Erdős and Füredi [EF] provided a probabilistic construction of an acute set on \( \left[ \frac{1}{2} \left( \frac{2}{\sqrt{3}} \right)^d \right] \) points, thus disproving the conjecture of Danzer and Grünbaum. The underlying idea was to consider a random subset of the vertices of the hypercube \( \{0, 1\}^d \) (see the next section for details). During 1983-2009, the improvements of the lower bound were very moderate: the constant \( \frac{1}{2} \) in front of the exponent \( \left( \frac{2}{\sqrt{3}} \right)^d \) was improved in several steps, resulting in the inequality \( f(d) \gtrsim 0.770 \cdot \left( \frac{2}{\sqrt{3}} \right)^d \) [B]. In 2009, Ackerman and Ben-Zwi [AB] improved the Erdős–Füredi bound by a factor of \( c \sqrt{d} \) using a certain general result concerning the independence numbers of sparse hypergraphs. In 2001, Harangi [H] made the first exponential improvement: the constant \( \frac{2}{\sqrt{3}} \approx 1.155 \) was replaced by \( \left( \frac{144}{23} \right)^{0.1} \approx 1.201 \). Harangi’s idea was to consider random subsets of the set of the form \( X_0^n \subset \mathbb{R}^{d_0^n} \), rather than \( \{0, 1\}^d \), as it was done in the proof by Erdős and Füredi. Here, \( X_0 \subset \mathbb{R}^{d_0} \) is a low-dimensional acute set, which is typically constructed by hand or with the help of computer. For example, if one takes \( X_0 \) to be an acute triangle on the plane then one gets the bound \( f(d) \gtrsim 1.158^d \), which is slightly better than the Erdős-Füredi bound. Harangi used a 12-point acute subset of \( \mathbb{R}^5 \) in his proof.

The next round of development was triggered in the spring of 2017, when the first explicit exponential acute sets were constructed by the second author [Z]. The obtained bound on \( f(d) \) was also much better than the previously known ones: \( f(d) \geq F_{d+1} \approx 1.618^d \), where \( F_d \) is the \( d \)-th Fibonacci number. The proof used induction and certain slight perturbations of the point set to make the right angles in the arising product-type constructions acute. In the fall of 2017 Gerencsér and Harangi [GH] proved that

\[
 f(d) \geq 2^{d-1} + 1. \tag{2}
\]

The proof was inspired by constructions of 9-point and 17-point acute sets in \( \mathbb{R}^4 \) and \( \mathbb{R}^5 \), respectively, made by an Ukranian mathematics enthusiast. The idea of Gerencsér and Harangi’s bound is to carefully perturb the vertices of the hypercube \( \{0, 1\}^{d-1} \) using one extra dimension to get rid of all right angles. One extra point can then be added to the construction.

One feature of all known explicit exponential-sized constructions is that the largest angle among their points is just barely smaller than \( \frac{\pi}{2} \), and the constructions break completely if we require the largest angle to be, say \( \frac{\pi}{2} - 0.001 \). On the other hand, as we shall see below, random constructions can be usually modified so that the largest angle would be separated from \( \frac{\pi}{2} \). This suggests a certain interesting direction for research, but let us first introduce a couple of definitions.

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1Here \( F_0 = F_1 = 1 \).
Definition 1. Denote by \( f(d, \alpha) \) the size of the largest set of points in \( \mathbb{R}^d \) with no three points forming an angle at least \( \alpha \). Put

\[
F(\alpha) := \limsup_{d \to \infty} f(d, \alpha)^{1/d}
\]

Thus, for instance, \( f(d) = f(d, \frac{\pi}{2}) \), and the result of Gerencsér-Harangi now implies that \( F(\frac{\pi}{2}) = 2 \). In [Kup], the first author have shown that \( \lim_{\alpha \to \pi/2^+} f(d, \alpha) = 2^d \).

Note that \( f(d, \alpha) \) is consistent only for \( \alpha \in [\frac{\pi}{3}, \pi] \), and that obviously \( f(d, \frac{\pi}{3}) = d + 1 \). Some further results about \( f(d, \alpha) \) for \( \alpha \) close to \( \frac{\pi}{3} \) or to \( \pi \) can be found in [EF].

Results of Erdős-Füredi ([EF], Theorem 3.6) translate to the following.

\[
F(\frac{\pi}{3} + \delta) \in [1 + \delta^2, 1 + 4\delta]
\]

In the range \( \alpha > \frac{\pi}{2} \) it turns out that \( f(d, \alpha) \) grows surprisingly fast. The following result is essentially due to Erdős-Füredi (Theorem 4.3 [EF]) but their formulation is inaccurate (their bounds are valid only for \( \alpha \) close to \( \pi \)).

Proposition 1. For any \( \alpha \in (\frac{\pi}{2}, \pi) \) there are constants \( C, c > 1 \) such that for all sufficiently large \( d \)

\[
2^{cd-1} < f(d, \alpha) < 2^{C^{d-1}}.
\]

Sketch of the proof. To prove the lower bound, we construct a set \( \{v_1, \ldots, v_m\} \) of \( m \geq c^d \) unit vectors in \( \mathbb{R}^d \) such that the angle between any two of them lies in \( (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon) \), where \( 2\varepsilon = \alpha - \frac{\pi}{2} \). This can be done by taking a random subset on the unit sphere and applying a concentration inequality. Now take a sufficiently large number \( \lambda \) and consider the set \( X = \{v_I = \sum_{i \in I} \lambda^i v_i \mid I \subset [m]\} \). Note that \( |X| = 2^{cd} \). For any two points \( v_I, v_J \in X \) we have \( v_I - v_J \approx \pm \lambda^t v_t \), where \( t \) is the largest element of \( I \Delta J \). So the angle between \( v_I - v_J \) and \( v_I - v_K \) is approximately equal to the angle between some vectors \( \pm v_i \) and \( \pm v_j \), and therefore, it is at most \( \alpha \).

To prove the upper bound, we construct a set \( \{v_1, \ldots, v_m\} \) of \( m \leq C^d \) vectors such that any vector determines an angle less than \( \frac{\pi - \alpha}{2} \) with one of them. This can be done by a greedy algorithm or deduced from known results for the sphere packing problem. Take a set \( X \) of more than \( 2^m \) points. For \( x, y \in X \), color a pair \( (x, y), x \neq y \), in color \( i \) if the angle between \( v_i \) and \( x - y \) is at most \( \frac{\pi - \alpha}{2} \). In what follows, we show that, since \( |X| > 2^m \), there exists a triple \( x, y, z \) such that \( (x, y) \) and \( (y, z) \) received the same color (i.e., there is a monochromatic oriented 2-path). But then the angle between \( y - x \) and \( y - z \) is at least \( \alpha \).

We show that such a triple exists by induction on \( m \). The statement is clear for \( m = 1 \) and \( |X| = 3 \). Next, for \( m \)-colorings, take any color, say, red, and consider all edges of this color. If there is no red oriented 2-path, then each vertex either has only incoming or only outgoing red edges, and so red edges span a bipartite graph. (The vertices with degree 0 we are free to assign to any of the two parts.) Take the bigger part of this bipartite graph. It has size at least \( \lceil (2^m + 1)/2 \rceil = 2^{m-1} + 1 \) and is colored in \( m - 1 \) color. Thus, it contains a monochromatic 2-path. \( \square \)
Now we can formulate our main question.

**Question 1.** Is it true that
\[
\lim_{\alpha \to \pi/2} F(\alpha) = 2? \tag{6}
\]
Equivalently, is it true that for any \( \varepsilon > 0 \) there is \( \delta > 0 \) so that for any large \( d \) there is a set \( X \subset \mathbb{R}^d \) of cardinality at least \( (2 - \varepsilon)^d \) and such that any three points from \( X \) determine an angle less than \( \frac{\pi}{2} - \delta \)?

Although the problem is very close to the acute angles problem, the current methods that use explicit constructions fail completely, and the gap between the bounds is still exponential. We can only prove the following theorem.

**Theorem 1.** We have
\[
\lim_{\alpha \to \pi/2} F(\alpha) \geq \sqrt{2}. \tag{7}
\]
That is for every \( \varepsilon > 0 \) and any sufficiently large \( d \) there is a set \( X \subset \mathbb{R}^d \) of cardinality at least \( (\sqrt{2} - \varepsilon)^d \) determining only angles less that \( \frac{\pi}{2} - \delta \) for some \( \delta \).

Our proof is a combination of the method of Erdős-Füredi with the recent construction of acute sets by Gerencsér-Harangi.

The second result gives a non-trivial upper bound on \( F(\alpha) \) for any \( \alpha < \pi/2 \).

**Theorem 2.** For \( \alpha > 0 \) small enough we have \( F(\pi/2 - \alpha) \leq 2 - \alpha^2 \).

Theorem 2 confirms a conjecture of Erdős-Füredi (Conjecture 3.5, [EF]). The proof is a modification of the proof of the inequality \( f(d) \leq 2d \) due to Danzer and Grünbaum.

## 2 The proofs

**Proof of Theorem 1.** Fix an arbitrary \( \varepsilon > 0 \). Take a sufficiently large \( d_0 \) and an acute set \( X_0 \subset \mathbb{R}^{d_0} \) of size \( 2^{d_0-1}+1 \) (which exists by [2]). Let \( R > 0 \) be the diameter of \( X_0 \) and denote by \( s \) the smallest scalar product \( \langle x - y, x - z \rangle \) over all triples \( x, y, z \in X_0 \) such that \( x \neq y, z \). By the definition of an acute set, we have \( s > 0 \).

W.l.o.g., assume that \( d_0 \) divides \( d \). Let \( m = 2^{1-\varepsilon nd_0} \) where \( n = d/d_0 \). Choose \( 2m \) uniformly random points \( p_1, \ldots, p_{2m} \in X_0^n \subset \mathbb{R}^{d_0n} \), and denote \( p_i = (p_{i1}, \ldots, p_{in}) \). Let us estimate the expectation of the number of triples \( (i, j, k) \) such that \( \langle p_i - p_j, p_i - p_k \rangle \leq \frac{s}{2} n s \).

If for some \( i, j, k \) we have \( \langle p_i - p_j, p_i - p_k \rangle \leq \frac{s}{2} n s \) then there are at least \( (1-\frac{s}{2})n \) coordinates \( t \in \{1, \ldots, n\} \) for which \( p_{it} = p_{jt} \) or \( p_{it} = p_{kt} \). The probability of the latter event is at most
\[
\left( \frac{n}{2^{d_0n}} \right) \left( \frac{2}{|X_0|} \right)^{(1-\frac{s}{2})n} \leq 2^{n-(1-\frac{s}{2}(d_0-2)n}. \quad \text{So, the expectation of the number of such triples is at most}
\]
\[
m^3 2^{n-(1-\frac{s}{2}(d_0-2)n} \leq m 2^{(1-\varepsilon)nd_0} 2^{-n(1-\frac{s}{2})d_0+3n} \ll m. \tag{8}
\]
Thus, there are points $p_1, \ldots, p_{2m}$ with at most $m$ "bad" triples. Remove one point from each of these triples and obtain a set $X \subset X^n_0 \subset \mathbb{R}^{nd_0}$ of cardinality at least $m = \sqrt{2}^{(1-\varepsilon)nd_0}$ such that for any two points $x, y \in X$ we have $|x - y|^2 \leq R^2n$ and for any three points $x, y, z \in X$ we have $\langle x - y, x - z \rangle > \frac{-\varepsilon}{10}ns$. This means that the angle $\alpha$ between vectors $x - y, x - z$ satisfies $\cos \alpha \geq \frac{-\varepsilon}{10}s/R^2$ and thus depends on $\varepsilon$ only. \qed

In the proof of Theorem 2, we shall need the following lemma.

**Lemma 1.** Suppose $X \subset \mathbb{R}^d$, $|X| = N \geq d + 1$ and $\text{conv}(X)$ has non-zero volume. Then for any $c \in \left[ \frac{12d \ln N}{N}, 1 \right]$ there are sets $A \subset B \subset X$ such that

1. $|B \setminus A| \geq \frac{c}{3d \log_2 N} N$.
2. $0 \neq \text{Vol}(\text{conv}(B)) \leq (1 + c)\text{Vol}(\text{conv}(A))$.

**Proof.** By Caratheodory’s theorem, every point of $\text{conv}(X)$ lies in the convex hull of some $(d + 1)$ points of $X$, so by the pigeonhole principle, there is a set $X_0 \subset X$ of size $d + 1$ and such that

$$\text{Vol}(\text{conv}(X_0)) \geq \left( \frac{N}{d + 1} \right)^{-1} \text{Vol}(\text{conv}(X)) \geq N^{-d-1} \text{Vol}(\text{conv}(X)).$$

Take any chain $X_0 \subset X_1 \subset \ldots \subset X_m = X$, such that $|X_{i+1} \setminus X_i| \in \left[ \frac{c}{3d \log_2 N}, \frac{c}{2d \log_2 N} \right]$ (it’s possible because of the restriction on $c$). We have $m \geq \frac{2d \log_2 N}{c}$, so if for any $i$ we have $\text{Vol}(\text{conv}(X_{i+1})) > (1 + c)\text{Vol}(\text{conv}(X_i))$ then

$$\text{Vol}(\text{conv}(X)) > (1 + c)^m \text{Vol}(\text{conv}(X_0)) \geq 2^{2d \log_2 N} \text{Vol}(\text{conv}(X_0)) \geq \text{Vol}(\text{conv}(X)),$$

a contradiction. \qed

**Proof of Theorem 2.** Take a set $X \subset \mathbb{R}^d$ which determines only angles at most $\frac{\pi}{2} - \alpha$ for a sufficiently small $\alpha > 0$. Put $\varepsilon = \sin \alpha$. It is easy to see that for any three different points $x, y, z \in X$

$$\langle y - x, z - x \rangle \geq 2\varepsilon \|y - x\|\|z - x\| > 2\varepsilon^2\|z - x\|^2,$$

where the last inequality follows from the fact that $\|\frac{y - x}{\|z - x\|}\| = \frac{\sin \angle zyx}{\sin \angle zyx} > \sin \angle zyx > \varepsilon$. Applying (9) for $z - x$ and $x - z$, we actually get that for any three distinct $x, y, z$ we have

$$2\varepsilon^2\|z - x\|^2 < \langle y - x, z - x \rangle < 1 - 2\varepsilon^2\|z - x\|^2.$$  \hfill (10)

Applying Lemma 1 with $c = 1$ we get sets $A \subset B$ such that $0 \neq \text{Vol}(\text{conv}B) \leq 2\text{Vol}(\text{conv}A)$ and $|B \setminus A| \geq \frac{|X|}{2^{1/2}}$. Take $\lambda = \frac{1}{2} \cdot (1 - 2\varepsilon^2)^{-1}$, from (10) we see that for any $x, y \in B \setminus A$ we have $\|(1 - \lambda)x + \text{conv}(\lambda A)\) \cap ((1 - \lambda)z + \text{conv}(\lambda A)) = \emptyset$. Indeed, for any point $y$ from the first set we have $\langle y - x, z - x \rangle < \lambda(1 - 2\varepsilon^2)\|z - x\|^2 = \frac{1}{2}\|z - x\|^2$, while for any $y'$ from the second set we have $\langle y' - x, z - x \rangle > (1 - \lambda)\|z - x\|^2 + \lambda \cdot 2\varepsilon^2\|z - x\|^2 = \frac{1}{2}\|z - x\|^2$. Moreover, $(1 - \lambda)x + \text{conv}(\lambda A) \subset \text{conv}B$ for any $x \in B$, so

$$|B \setminus A|\lambda^d \text{Vol}(\text{conv}A) \leq \text{Vol}(\text{conv}B) \leq 2\text{Vol}(\text{conv}A),$$  \hfill (11)
thus,
\[ |X| \leq 4d^2 |B \setminus A| \leq 8d^2 \lambda^d = 8d^2 2^d \left(1 - 2\varepsilon^2\right)^d \leq (2 - \alpha^2)^d, \]
provided $d$ is sufficiently large and $\alpha > 0$ is sufficiently small. (We used $\lim_{\alpha \to 0^+} \frac{\sin \alpha}{\alpha} = 1$.)

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