On Computable Geometric Expressions in Quantum Theory

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Abstract. Geometric Algebra and Calculus are mathematical languages encoding fundamental geometric relations that theories of physics seem to respect. We propose criteria given which statistics of expressions in geometric algebra are computable in quantum theory, in such a way that preserves their algebraic properties. They are that one must be able to arbitrarily transform the basis of the Clifford algebra, via multiplication by elements of the algebra that act trivially on the state space; all such elements must be neighbored by operators corresponding to factors in the original expression and not the state vectors. We explore the consequences of these criteria for a physics of dynamical multivector fields.

Keywords. Geometric algebra, Quantum theory, Clifford bundle, Electroweak.

1. Introduction

Multivectors are elements of a Clifford algebra atop $\mathbb{R}^n$ that may represent formal sums of oriented geometric extents in a generalized tangent space (see Appendix A for a brief overview, and e.g. [1, 2, 3] for in-depth coverage). In [4, 5, 6], a distinction is made between multivectors whose components transform under rotations via one- versus two-sided Clifford multiplication:

$$\psi \rightarrow S\psi \quad \text{vs.} \quad \mathcal{F} \rightarrow S\mathcal{F}\mathcal{S}^\dagger$$

where $S \in \{ \exp(\frac{1}{4}\epsilon_{ijkl}\theta_i e^j e^k) \mid \theta_i \in \mathbb{R} \}$ is a rotor and $\dagger$ denotes a reverse involution. Though the two-sided transformation of the latter is standard in geometric algebra, it is suggested in [4] that the former are the more fundamental objects because the latter can be constructed from the former

$$\psi\psi^\dagger \rightarrow S(\psi\psi^\dagger)S^\dagger$$

but not vice versa.

In order for expectation values and higher moments of random variable expressions in geometric algebra to be “computable” in terms of state vectors
in the probabilistic framework of quantum theory, one must remain free to distort the basis of the Clifford algebra by applying linear operators—the rotors introduced above. Rotors can only act on elements of the geometric algebra, which must first be appended to the state vector by the action of a field operator (or a non-field "catalog" operator). Since rotors cannot be allowed to act directly on state vectors living in a separate Hilbert space, it would seem that multivectors transforming by one-sided multiplication (like \( \psi \) above) must occupy the outer positions\(^1\) adjacent to the state vectors in any expression that is computable by the above criterion.

We present a scheme in which dynamical multivector fields whose components for all particular intents transform by left-multiplication alone may be constructed from those transforming natively with the standard two-sided rule. These fields are coupled in one of two configurations to a gauge-like connection that transforms non-covariantly under basis rotations applied to fibers of the Clifford bundle. We argue that expressions of multivector-valued fields and their derivatives must conform to these configurations to be computable in the linear associative framework of quantum theory. We draw a comparison between this scheme and the fermion-gauge field couplings prescribed by the Weinberg-Salam electroweak model.

2. Mixing Geometric Algebra and State Vectors

Assembling a generic multivector \( \mathcal{F} \) in the geometric algebra of three spatial dimensions, \( \mathcal{C}(\mathbb{R}) \), we have

\[
\mathcal{F} = b_0 \mathbf{1} + b_i \mathbf{e}^i + b_{jk} \mathbf{e}^j \mathbf{e}^k + b_{123} \mathbf{e}^1 \mathbf{e}^2 \mathbf{e}^3
\]

(3)

\[
\rightarrow b_0 \varsigma^0 + b^i \varsigma^i + i \epsilon_{ijk} b_{jk} \varsigma^i + i b_{123} \varsigma^0
\]

(4)

Here we take \( \mathbf{e}^i \) to be orthonormal vector basis elements of that algebra, with matrix representations \( \varsigma^i \) satisfying \( \varsigma^1 \varsigma^2 = i \varsigma^3 \) and \( -i \varsigma^1 \varsigma^2 \varsigma^3 = \mathbb{I} \); all coefficients are real-valued. Consider expressions taking the form

\[
\langle \mathcal{F}^\dagger \mathcal{O} \mathcal{F} \rangle_0 = \langle \mathcal{F}^\dagger \mathcal{O} \mathcal{F} \rangle_0 = \langle \mathcal{O} \mathcal{F} \mathcal{F} \rangle_0
\]

(5)

for some multivector element \( \mathcal{O} \), where \( \langle \cdot \rangle_0 \) denotes the grade-0 projection. Adapting a matrix representation, the grade-0 projection is obtained up to a constant factor as the trace of the Hermitian part. Using \( S^\dagger = S^{-1} \) and the cyclic property of \( \langle \cdot \rangle_0 \), the outer transformation operators acting on \( \mathcal{F} \) cancel out, leaving the one-sided effective transformation \( \mathcal{F} \rightarrow S \mathcal{F} \).

\[
\langle \mathcal{F}^\dagger \mathcal{O} \mathcal{F} \rangle_0 \rightarrow \langle (S S^\dagger)^\dagger \mathcal{O} (S S^\dagger)^\dagger \rangle_0 = \langle \mathcal{F}^\dagger S^\dagger \mathcal{O} S \mathcal{F} \rangle_0
\]

(6)

Now suppose one exports information about the configuration of \( \mathcal{F} \) to a state vector \( |\Psi\rangle \). In place of \( \mathcal{F} \) one substitutes a linear operator \( \hat{\mathcal{F}} \) mapping

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\(^1\)If the expression in geometric algebra is \( ABC \), and its expectation value given \( |\Psi\rangle \) is computed as \( \text{E}[ABC] = \langle \Psi | \hat{A} \hat{B} \hat{C} |\Psi\rangle \), then \( \hat{A} \) and \( \hat{C} \) occupy “outer positions.”
between the Hilbert space of states $\mathcal{H}$ and a composite space $(\mathcal{C}\ell_3(\mathbb{R}), \mathcal{H})$ containing the geometric algebra, e.g.

$$\hat{F} = b_0 \xi^0 \otimes \hat{a}_0(b_0) + b_i \xi^i \otimes \hat{a}_i(b_i) + b_{jk} \xi^j \xi^k \otimes \hat{a}_{jk}(b_{jk}) + \cdots$$  \hspace{1cm} (7)

or $F : (\mathcal{C}\ell_3(\mathbb{R}), \mathcal{H}) \rightarrow (\mathcal{C}\ell_3(\mathbb{R}), \mathcal{H})$ mapping among elements of that composite space

$$\hat{F} = b_0 \xi^0 \otimes \hat{a}_0(b_0) + b_i \xi^i \otimes \hat{a}_i(b_i) + b_{jk} \xi^j \xi^k \otimes \hat{a}_{jk}(b_{jk}) + \cdots$$  \hspace{1cm} (8)

Here $\hat{a}_{\{\}} : \mathcal{H} \rightarrow \mathcal{H}$ transform vectors in the Hilbert space, and $\otimes$ denotes that $\hat{F}$ maps between $\mathcal{H}$ on the right and a composition $(\mathcal{C}\ell_3(\mathbb{R}), \mathcal{H})$ on the left. We refer to $\hat{F}$ as a "catalog" operator, by analogy to a store catalog that maps codes in an inventory record to representations of concrete objects; elements of $(\mathcal{C}\ell_3(\mathbb{R}), \mathcal{H})$ are multivectors appended to state vectors. (From this point forward, a hat (\hat{\quad}) denotes a rectangular operator that maps between the Hilbert space and a larger composite space; a caron (\ˇ{\quad}) denotes a square operator that acts non-trivially on $\mathcal{H}$, but maps between identical spaces; and operators like $S$ that act trivially on $\mathcal{H}$ have no ornament.)

One can transform each domain of the catalog operator separately

$$\hat{F} \rightarrow \begin{cases} b_0 S \xi^0 S^\dagger \otimes \hat{a}_0(b_0) + b_i S \xi^i S^\dagger \otimes \hat{a}_i(b_i) + \cdots & \text{(Clifford basis)} \\ b_0 \xi^0 \otimes U \hat{a}_0(b_0) U^\dagger + b_i \xi^i \otimes U \hat{a}_i(b_i) U^\dagger + \cdots & \text{(Hilbert space map)} \end{cases}$$

with the state vector transforming under a unitary representation of $SO(3)$ as $|\Psi\rangle \rightarrow U |\Psi\rangle$. This work relies on an insistence that one be able to compute expectation values of transformed expressions after a change of the Clifford basis in terms of the original catalog operators, in a manner consistent with the linear formulation of quantum theory. We denote the transformation acting on the Clifford basis as simply

$$\hat{F} \rightarrow \hat{S} \hat{F} \hat{S}^\dagger$$  \hspace{1cm} (9)

Note that the rectangular hatted form $\hat{F} : \mathcal{H} \rightarrow (\mathcal{C}\ell_3(\mathbb{R}), \mathcal{H})$ cannot be acted upon from the right by any operator not merely mapping $\mathcal{H} \rightarrow \mathcal{H}$, without changing the input space of the resulting operator.

Computing the expectation value of the quantity $[5]$ for a given state $|\Psi\rangle$ after a transformation of the Clifford basis (not of $\mathcal{H}$), we are free to omit transformation operators that cancel out in $[6]$

$$\langle \Psi | \text{Tr}_c \left[ \hat{F}^\dagger \hat{O} \hat{F} |\Psi\rangle \right] + \text{h.c.} \rightarrow \langle \Psi | \text{Tr}_c \left[ \hat{F}^\dagger S^\dagger \hat{O}' S \hat{F} |\Psi\rangle \right] + \text{h.c.}$$  \hspace{1cm} (10)

Here $\text{Tr}_c : (\mathcal{C}\ell_3(\mathbb{R}), \mathcal{H}) \rightarrow \mathcal{H}$ denotes a partial trace over the matrix representation of the geometric algebra, and $\hat{O} \rightarrow \hat{O}'$ under basis rotation. We henceforth suppress addition of the Hermitian conjugate when computing grade-0 projections in the matrix representation.

From a geometric standpoint, nothing changes if one cycles the order of multiplication as in $[5]$ (while accordingly reassigning the ladder operators...
\( \tilde{a}_\{\cdot\} : \mathcal{H} \to \mathcal{H} \), after which the transformation becomes

\[
\langle \Psi | \text{Tr}_\varsigma \left[ \tilde{O} \tilde{F} \tilde{F}^\dagger | \Psi \rangle \right] = \langle \Psi | \text{Tr}_\varsigma \left[ \tilde{O}' (S \tilde{F} S^\dagger) (S \tilde{F}^\dagger S) | \Psi \rangle \right]
\]  

(11)

If one takes associativity of the operations in brackets literally, one must allow for the interpretation of the transformation operator \( S \) as acting directly on the state vector to the right. But \( | \Psi \rangle \) does not contain an element of the geometric algebra in the representation on which \( S \) is to act. Nor does \( S \) constitute a map between the Hilbert space and a larger composite space containing the geometric algebra in the representation \( \varsigma \), as does \( \tilde{F} \).

If we are to be free in principle to evaluate (5) with any ordering, then a basis transformation cannot require multiplication by \( S \) from the right on the right-most catalog operator \( \hat{F}^\dagger \) in (11), between the latter and the state \( | \Psi \rangle \). Only if \( \tilde{O} \) simultaneously transforms as \( \tilde{O} \to \tilde{O}' = S \tilde{O} S^\dagger \), or commutes with all \( S \), is the transformed expression computable.

\[
\langle \Psi | \text{Tr}_\varsigma \left[ S \tilde{O} S^\dagger \tilde{F} \tilde{F}^\dagger S^\dagger | \Psi \rangle \right] = \langle \Psi | \text{Tr}_\varsigma \left[ \tilde{O} \tilde{F} \tilde{F}^\dagger | \Psi \rangle \right]
\]  

(12)

This amounts to a constraint on the geometric quantities about which information can be retrieved from state vectors. By this reasoning, for example, the quantity \( \langle F^\dagger e^i F \rangle_0 \) —where \( e^i \) is a constant reference vector that is left invariant under basis rotations—would not be admissible. (For multivector fields, we will find that including a right-acting derivative operator in the mix imposes an order of multiplication that cannot be permuted without changing the content of the expression, allowing constant reference elements \( e^k \) to appear alongside \( \partial_k \).

Having verified that the transformed expression in terms of geometric algebra is free of outer basis transformation operators, we are free to discard the curious notation adopted above. We can substitute a vector-only notation—replacing \( \tilde{F} \) with a set of catalog operators mapping \( | \Psi \rangle \) to a tensor product space containing columns of the matrix representation—with no loss of fidelity.

\[
E \left[ \langle F^\dagger O F \rangle_0 \right] \sim \sum_A \langle \Psi | \hat{\psi}^\dagger A \tilde{O} \hat{\psi} A | \Psi \rangle, \quad A \in \{1, \ldots, \dim \varsigma^0\}
\]  

(13)

If the \( 2 \times 2 \) complex representation is adopted, then the \( \hat{\psi} A \) for \( A \in \{1, 2\} \) are the usual 2-component Weyl spinors, effectively transforming as \( \hat{\psi} A \to S \hat{\psi} A \).

### 3. Dynamical Multivector Fields

Let us attempt to construct a physics of elementary multivector fields, transforming with the standard two-sided Clifford multiplication rule. A candidate free Hamiltonian for such a theory is given by

\[
dH = \langle d^3x \mathcal{H}(x) \rangle_0 = \langle d^3x F(x)^\dagger e^k \partial_k F(x) \rangle_0
\]

(14)

where \( \langle \cdot \rangle_0 \) again denotes the grade-0 projection, and the 3-volume element \( d^3x \equiv d^3x(x) \) is taken to be a homogeneous multivector field of grade 3. It
represents the oriented volume of an infinitesimal voxel at \( x \), against which the orientation of the multivector integrand is evaluated.

\[
\mathbf{d}^3 \mathbf{x} \equiv \bigwedge_{k=1}^{3} \mathbf{d}x^k(x) \mathbf{e}_k = S \mathbf{d}^3 \mathbf{x} S^\dagger \tag{15}
\]

The \( e^k \) in (14) are constant reference vector elements of the geometric algebra; they do not transform with the basis of the local Clifford algebra as does the field. The Hamiltonian is the integral of (14) over all of space; for now we only consider configurations for which the integral is positive.

3.1. Rotations

For multivector fields, two transformations are involved in a change of reference frames: one of the coordinates on the manifold and another of the basis of multivectors spanning the Clifford algebra at each event (the union of such spaces forms the Clifford bundle). For brevity, we will often refer to the latter as simply basis transformations.

It is convenient to adopt a holonomic basis for the Clifford bundle, in which the vector basis elements of the algebra at each point in the manifold are taken to align with the directions along which the spatial coordinates increase. However, non-holonomic bases are admissible and should describe the same physics when reflected in the state vector. The Hamiltonian must be invariant under complementary coordinate and basis transformations, but not under transformations between holonomic and non-holonomic bases. (The latter change the geometric interpretation of the information contained in the state vector (e.g. spin orientation relative to field gradient), and so must correspondingly give a different time evolution for that new physical system.)

Let new coordinates \( \bar{x} \) relate to the old as \( \bar{x} = \Lambda x \), and let the multivector \( S = \exp(\frac{1}{4} \epsilon_{ijk} \theta^i e^j e^k) \) operate on elements of the geometric algebra as \( S(\cdot)S^\dagger \) to enact a proper basis rotation. Under the combined rotation

\[
\mathbf{d}H(x) \to \langle (S \mathbf{d}^3 \mathbf{x} S^\dagger)(S \mathcal{F}(\Lambda^{-1}x)^\dagger S^\dagger) e^k \partial_k (S \mathcal{F}(\Lambda^{-1}x)S^\dagger) \rangle_0 \tag{16}
\]

where we assume \( \mathbf{d}^3 \mathbf{x}(\Lambda^{-1}x) \equiv \mathbf{d}^3 \mathbf{x}(x) \). When the rotation is homogeneous \((S(x) = S)\), with the cyclic property of \( \langle \cdot \rangle_0 \) this becomes

\[
\mathbf{d}H(x) \to \mathbf{d}H_{\text{hom}}(x) = \langle \mathbf{d}^3 \mathcal{F}(\Lambda^{-1}x)^\dagger S^\dagger e^k S \partial_k \mathcal{F}(\Lambda^{-1}x) \rangle_0 \tag{17}
\]

If the basis transformation coded by \( S \) is not homogeneous (e.g. transforming between holonomic bases corresponding to Cartesian and curvilinear coordinates), then we get additional terms from the transformation

\[
\mathbf{d}H(x) \to \mathbf{d}H_{\text{hom}}(x) + \langle \mathbf{d}^3 \mathcal{F}(\Lambda^{-1}x)^\dagger S^\dagger e^k (\partial_k S) \mathcal{F}(\Lambda^{-1}x) \rangle_0 +
\langle \mathbf{d}^3 \mathcal{F}(\Lambda^{-1}x)^\dagger S^\dagger e^k S \partial_k \mathcal{F}(\Lambda^{-1}x)(\partial_k S^\dagger) \rangle_0 \tag{18}
\]

This transformed Hamiltonian is of course still well defined as a sequence of operations involving multivectors followed by a grade projection.

But what happens when we try to compute statistics of this Hamiltonian in terms of data externalized to a state vector, while taking seriously the requirement that the probability theory be linear and associative? We promote \( \mathcal{F}(x) \) to a field operator \( \hat{\mathcal{F}}(x) \), which acts on the state vector \( |\Psi\rangle \).
serving as a dictionary between the Hilbert space and sections of the Clifford bundle. We can cast the derivatives as right-acting linear operators, requiring only that $\mathcal{F}$ appears alone to the right of the derivative operator in the untransformed expression (and so fixing the order of multiplication).

When the transformation of the field values is homogeneous, we can use the cyclic property of the grade-0 projection in (14), to cancel the left-most $S$ and the right-most $S^\dagger$ in the basis transformed version (16). We are free to omit those operators that do not survive in the transformed expression

$$\langle \Psi | H | \Psi \rangle \rightarrow \cdots + \langle \Psi | Tr_{\xi} \int d^3x \hat{\mathcal{F}}^\dagger S^\dagger \zeta^k \mathbf{\hat{\tau}}_k \hat{\mathcal{F}} | \Psi \rangle$$

leaving a field operator at the right-most position mapping the state to an element of a composite space $p C_3 p R q H q$ that includes a copy of the geometric algebra in the representation specified by the reference vector $\zeta^k$. If the basis transformation is not uniform, then we get a term

$$\langle \Psi | \hat{H} | \Psi \rangle \rightarrow \cdots + \langle \Psi | Tr_{\xi} \int S d^3x \hat{\mathcal{F}}^\dagger S^\dagger \zeta^k \mathbf{\hat{\tau}}_k \hat{\mathcal{F}} | \Psi \rangle$$

in which we would have to allow for the interpretation of $S^\dagger$ as acting on $| \Psi \rangle$. But as before, the basis transformation operator $S^\dagger$ cannot be allowed to act directly on the bare quantum state (or on $\hat{\mathcal{F}}$ from the right), since an element of the Clifford algebra in the representation consistent with $\zeta^k$ has not yet been appended to the state vector. The Hamiltonian must be constructed such that all outer basis transformation operators (OBTOs) can be omitted without consequence.

In order to accommodate a kinetic term resembling (14) in a computable quantum field theory, one would have to introduce an auxiliary grade-2 field with a coordinate index, whose components transforms non-covariantly under basis rotations:

$$\mathcal{W}_k(x) \rightarrow \begin{cases} S(x) \mathcal{W}_k(x) S(x)^\dagger - S(x) \partial_k S(x)^\dagger & \text{(Clifford basis)} \\ \Lambda^k_{\mu} \mathcal{W}_k(\Lambda^{-1}x) & \text{(coordinates)} \end{cases}$$

Using (21) there are two available modifications to (14) that resolve the issue:

$$\mathcal{H}_+ (x) = \mathcal{F}^\dagger e^k \partial_k \mathcal{F} + \mathcal{F}^\dagger e^k [\mathcal{F}, \mathcal{W}_k] \pm \mathcal{F}^\dagger e^k \mathcal{W}_k \mathcal{F}$$

Using $S \partial_\mu S^\dagger = -S^\dagger \partial_\mu S$, the second term first removes all dependence on $\partial_k S$, and then the third term restores an inner dependence up to a sign:

$$\mathcal{H}_- (x) = \mathcal{F}^\dagger e^{k'} S^\dagger \partial_k (S \mathcal{F}) + \mathcal{F}^\dagger e^{k'} \mathcal{F} \mathcal{W}_k$$

Here $e^{k'} = S^\dagger e^k S$, and we suppress a left-most $S$ and right-most $S^\dagger$, since $\mathcal{H}(x)$ always appears (with a factor $d^3x$) under the grade projection in (14).

When computing expectation values of these Hamiltonians in terms of state vectors, the surviving basis transformation operators all then properly
act on \((\mathcal{L}_3(\mathbb{R}), \mathcal{H})\) of which \(\hat{\mathcal{F}} |\Psi\rangle, \hat{\mathcal{W}}_k \hat{\mathcal{F}} |\Psi\rangle,\) and \(\hat{\mathcal{F}} \hat{\mathcal{W}}_k |\Psi\rangle\) are elements:

\[
\hat{\mathcal{F}} S^\dagger \xi^k \partial_k S \hat{\mathcal{F}} |\Psi\rangle \quad (\mathcal{H}_+)
\]

\[
\hat{\mathcal{F}} S^\dagger \xi^k S^2 \partial_k S S^\dagger \hat{\mathcal{F}} |\Psi\rangle \quad (\mathcal{H}_-)
\]

\[
\hat{\mathcal{F}} S^\dagger \xi^k S \hat{\mathcal{F}} \hat{\mathcal{W}}_k |\Psi\rangle
\]

\[
\hat{\mathcal{F}} S^\dagger \xi^k S \hat{\mathcal{W}}_k \hat{\mathcal{F}} |\Psi\rangle
\]

While \(\mathcal{H}_+\) appears to be the minimal satisfactory correction to the problem of OBTOs—conforming more or less to the form of a covariant derivative—one cannot accept \(\mathcal{H}_+\) while rejecting \(\mathcal{H}_-\) based only on arguments presented thus far. Both are well defined and consistent with our criteria for computable geometric expressions. If (22) are to define physical field theories and our computability criteria carry weight, then all observables in the corresponding theories must be computable—not just the Hamiltonian. These include first and foremost the linear and angular momenta.

### 3.2. Lagrangians and Linear Momenta

In order for the Hamiltonian to remain free of OBTOs after an inhomogeneous basis rotation and subsequent boost (and for the Lagrangian density to be Lorentz invariant), \(\mathcal{W}_k\) must be the spatial components of a four-vector \(\mathcal{W}_\mu\).

\[
\mathcal{W}_\mu(x) \rightarrow \begin{cases} 
S(t,x)\mathcal{W}_\mu(x)S(t,x)^{-1} - S(t,x)\partial_\mu S(t,x)^{-1} & \text{(Clifford basis)} \\
\Lambda_\mu^\nu \mathcal{W}_\mu(\Lambda^{-1}x) & \text{(coordinates)}
\end{cases}
\]

(29)

Furthermore, in order that (21) remains consistent after an inhomogeneous translation, \(\mathcal{W}_\mu\) cannot transform as an ordinary function of the coordinates. Under a translation \(x^\mu \rightarrow x^\mu + \epsilon^\mu(t,x)\) with \(\epsilon^\mu\) small, where \(S \rightarrow S - \epsilon^\nu \partial_\nu S + \cdots\), we must have

\[
\mathcal{W}_\mu \rightarrow \mathcal{W}_\mu - \epsilon^\nu \partial_\nu \mathcal{W}_\mu - (\partial_\mu \epsilon^\nu)\mathcal{W}_\nu + \cdots \quad \text{(infinitesimal translation)}
\]

(30)

Both (29) and (30) follow from the standard form of the coordinate one-form transformation: \(\mathcal{W}_\mu \rightarrow (\partial x^\nu / \partial x^\mu)\mathcal{W}_\nu\).

We construct the Lagrangian circularly as

\[
dL(t,x) = \langle d^3x \mathcal{L}(t,x) \rangle_0, \quad \mathcal{L}(t,x) = \tilde{\mathcal{P}}_0 - \mathcal{H}
\]

(31)

where \(\tilde{\mathcal{P}}_0\) is the fictitious temporal component of the momentum four-vector density obtained by substituting \(k \rightarrow 0\) in the expression for \(\mathcal{P}_k\), to be determined. Corresponding to the Hamiltonian densities in (22), we guess

\[
\mathcal{L}_\pm(t,x) = \mathcal{F}^\dagger (\hat{\mathcal{F}} + [\mathcal{F}, \mathcal{W}_0] \pm \mathcal{W}_0 \mathcal{F}) - \mathcal{H}_\pm
\]

(32)

Under an infinitesimal translation \(x^k \rightarrow x^k + \epsilon^k\), \(\mathcal{L}_+\) transforms as

\[
\mathcal{L}_+ \rightarrow \mathcal{L}_+ - \epsilon^k (\mathcal{F}^\dagger \partial_k \mathcal{F} + \mathcal{F}^\dagger \mathcal{F} \partial_k \mathcal{W}_k) + \cdots
\]

(33)

(34)
verifying the spatial components of the momentum $P_k \equiv \partial L/\partial \dot{x}^k$

$$dP_k \equiv \langle d^3x \mathcal{P}_k(x) \rangle_0, \quad \mathcal{P}_{k,\pm}(x) = -\mathcal{F}^\dagger \partial_\pm \mathcal{F} - \mathcal{F}^\dagger \mathcal{F}\mathcal{W}_k \mp \mathcal{F}^\dagger \mathcal{W}_k \mathcal{F}$$ (35)

and hence $\mathcal{P}_{0,\pm}(x) = \mathcal{F}^\dagger (\dot{\mathcal{F}} + [\mathcal{F}, \mathcal{W}_0] \pm \mathcal{W}_0 \mathcal{F})$. The linear momentum $P_k$ is protected from OBTOs, as it must be by the same arguments applying to $H$.

$$\mathcal{P}_{k,+}(x) \rightarrow -\mathcal{F}^\dagger S^\dagger \partial_+ (S \mathcal{F}) - \mathcal{F}^\dagger S^\dagger S \mathcal{F} \mathcal{W}_k$$ (36)

$$\mathcal{P}_{k,-}(x) \rightarrow -\mathcal{F}^\dagger S^\dagger S^2 \partial_-(S^\dagger \mathcal{F}) - \mathcal{F}^\dagger S^\dagger S \mathcal{F} \mathcal{W}_k + 2 \mathcal{F}^\dagger S^\dagger S \mathcal{W}_k \mathcal{F}$$ (37)

Since $\mathcal{W}_0$ serves no purpose in the Hamiltonian ($\mathcal{H}_\pm$ contains no explicit time derivatives), we tentatively set $\mathcal{W}_0 = 0$, excluding it from $\mathcal{H}_\pm$ and the roster of dynamical degrees of freedom with representation in the state space. Then (31) is equivalent to the usual Legendre transform with $\mathcal{H}$ defined as in (22), and $e^1 e^2 e^3 \mathcal{F}^\dagger$ the conjugate momentum to $\mathcal{F}$.

### 3.3. Angular Momenta

If we identify rotations of the orientation angle invoked in Noether’s theorem with global, complementary transformations of the coordinates and bases of the local Clifford algebras, then for the sake of computing angular momentum, $\mathcal{L}_+$ transforms under rotations as

$$\mathcal{L}_+(x) \rightarrow \mathcal{F}^\dagger S^\dagger \partial_0 (S \mathcal{F}) + \mathcal{F}^\dagger \mathcal{F} \mathcal{W}_0 - \mathcal{H}_+[\mathcal{F}, \mathcal{W}_\mu](\Lambda^{-1}x)$$ (38)

with all fields evaluated at $\Lambda^{-1}x$. Taking $S$ to enact an infinitesimal rotation in the 1-2 plane, with the form $S = 1 + \frac{1}{2} \theta(t) e^1 e^2$ where $\theta(t) \ll 1$

$$\mathcal{L}_+ \rightarrow \mathcal{L}_+ + \dot{\theta} \frac{1}{2} \mathcal{F}^\dagger e^1 e^2 \mathcal{F} + \cdots$$ (39)

The second Lagrangian density $\mathcal{L}_-$ transforms as

$$\mathcal{L}_- \rightarrow \mathcal{F}^\dagger S \partial_0 (S^\dagger \mathcal{F}) + \mathcal{F}^\dagger \mathcal{F} \mathcal{W}_0 - 2 \mathcal{F}^\dagger \mathcal{W}_0 \mathcal{F} - \mathcal{H}_-[\mathcal{F}, \mathcal{W}_\mu](\Lambda^{-1}x)$$ (41)

$$\mathcal{L}_- \rightarrow \mathcal{F}^\dagger (1 + \frac{1}{2} \theta e^1 e^2) \partial_0 ((1 - \frac{1}{2} \theta e^1 e^2) \mathcal{F}) + \cdots$$ (42)

$$\mathcal{L}_- \rightarrow \mathcal{L}_- - \dot{\theta} \frac{1}{2} \mathcal{F}^\dagger e^1 e^2 \mathcal{F} + \cdots$$ (43)

Differentiating with respect to $\theta$ while neglecting the $\theta$ dependence owing to spatial variation of $\mathcal{F}$, we obtain the intrinsic angular momentum.

$$L^3_\pm \equiv \partial L_\pm/\partial \dot{\theta}, \quad dL^3_\pm = \pm \frac{1}{2} \langle d^3x \mathcal{F}^\dagger e^1 e^2 \mathcal{F} \rangle_0 + \cdots$$ (44)

Since $e^1 e^2$ comes from $S^\dagger \partial_0 S$, it is not a fixed reference element but rather transforms as $e^1 e^2 \rightarrow S e^1 e^2 S^\dagger$ under subsequent rotations; the spin angular momentum is computable, as is the total angular momentum by inspection.
3.4. Discussion

With the modifications in (22), spatially inhomogeneous rotations of the Clifford bundle applied to expressions for the Hamiltonian, linear momenta, and angular momenta are computable by the criteria proposed in § 2. This allows one to change between holonomic bases corresponding to Cartesian and to curvilinear coordinates on the spacelike hypersurface on which the field operators have support, by acting on state vectors from the left with a sequence of linear operators. Can one relate $\mathcal{H}_\pm$ to the Standard Model of particle physics? How might one interpret the state space?

3.4.1. Time-dependent Basis Rotations in the Hamiltonian. Above we have included the timelike component of the Clifford connection $\mathcal{W}_0$ in the Lagrangian, accounting for its non-covariant transformation under time-dependent basis rotations when calculating spin; but we have excluded it from the Hamiltonian by tentatively setting $\mathcal{W}_0 = 0$ and leaving out the non-covariant term. Including $\mathcal{W}_0$ in $\mathcal{H}_+$ either introduces OBTOs when the basis transformation varies in time

$$\mathcal{H}_+ \rightarrow \cdots + \mathcal{F}^\dagger \mathcal{F} \mathcal{W}_0 - \mathcal{F}^\dagger \mathcal{F} S \partial_0 S^\dagger$$

or requires an ad hoc rule that the non-covariant term is to be added to the Lagrangian but not to the Hamiltonian when it involves a time derivative.

One could relax the criteria sufficient for computability to require only that the Hamiltonian be free of OBTOs accounting for spatially varying basis transformations, since the quantum state (itself a function of time in the Schrödinger picture) only directly informs field configurations on a spacelike hypersurface. Applying a time-dependent basis transformation to a field operator that only has support on a spacelike hypersurface comes across as not obviously well defined. Alternatively, the OBTO in (45) can be removed by a subsequent transformation

$$\mathcal{F} \rightarrow \mathcal{F} S \quad \mathcal{W}_\mu \rightarrow S^\dagger \mathcal{W}_\mu S - S^\dagger \partial_\mu S$$

under which $\mathcal{L}_+$ is invariant.

When $\mathcal{W}_0$ and the non-covariant term from its basis transformation are included in $\mathcal{H}_-$, the OBTO cannot be so removed by a transformation that leaves the other terms invariant.

$$\mathcal{H}_- \rightarrow \cdots + \mathcal{F}^\dagger \mathcal{F} \mathcal{W}_0 - \mathcal{F}^\dagger \mathcal{F} S \partial_0 S^\dagger - 2 \mathcal{F}^\dagger \mathcal{W}_0 \mathcal{F} + 2 \mathcal{F}^\dagger S \partial_0 S^\dagger \mathcal{F}$$

Only if the terms involving $\mathcal{W}_\mu$ were to not produce any changes in occupation number—e.g. due to conservation laws—so that one could exclude such terms outright when computing $\langle \dot{\mathcal{H}}_- \rangle$, could $\mathcal{H}_-$ be de facto free of OBTOs to the same standard as accepted in earlier sections.

3.4.2. Effective Spin is Real Spin. The components of $\mathcal{F}$ transform as Lorentz scalar, polar 3-vector, and their Hodge duals—all left invariant after rotations of $2\pi$ radians. Nonetheless, when coupled as in (22), $\mathcal{F}$ “no longer” represents an integer-spin field. Multivector fields governed by $\mathcal{H}_+$ in (22) effectively transform as Weyl spinor doublets, owing to the non-covariant
term coming from (21) cancelling the other “half” of the kinetic term’s intrinsic dependence on $d\theta/dt$ after a time-dependent rotation. Subject to $\mathcal{H}_-$, the extra coupling term containing $e^kW_k$ flips the sign of the allowed $S^1\partial_\mu S$ multiplying $F$ from the left, giving it spin opposite to fields governed by $\mathcal{H}_+$ for the same field configuration.

The single-particle states must transform with unitary representations of the Lorentz group, picking up a phase of $iL_3^\theta$ under rotation, where $L_3^\theta$ is the effective angular momentum about $e^3$. Since the linear momenta $P_{k,\pm}$ are defined in the same way when $W_\mu$ vanishes and the spin contributions (44) to $L_3^\theta$ differ by a sign, the doublets governed by $\mathcal{H}_+$ have opposite helicities. Thus the single-particle states corresponding to each must transform under different representations, $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, with opposite chiralities.

3.4.3. Electroweak Resemblance. The $SU(2)$ gauge field $W_\mu$ of the Weinberg-Salam electroweak model [7, 8] couples exclusively to left-chiral fermion fields, which are organized into doublets whose components mix under the corresponding gauge transformation. A theoretic justification for this asymmetry (and for the appearance of an $SU(2)$ gauge group in the Standard Model, more generally) has yet to be widely accepted, though a number of attempts at such have been offered (see, e.g. [4, 9, 10, 11, 12]).

The Hamiltonian $\mathcal{H}_+$ in (22) is equivalent to that of a left-chiral fermion doublet in the Standard Model under the temporal gauge $\bar{W}_0^\mu = 0$, if one substitutes $W_\mu \rightarrow i\gamma^aW_\mu^a$, adds a coupling to the $U(1)$ gauge field with the usual rationale, and adopts an anti-commuting creation/annihilation operator algebra to act on a fermionic Fock space. To recover a more familiar notation, one can replace $F$ with a sum over $\psi_A$’s transforming as $\psi_A \rightarrow S\psi_A$ representing the columns of the $2 \times 2$ complex representation, while taking $\varsigma^k \otimes W_k$ to act on the $\psi_A$’s arranged into a column doublet. The invariance of $L_+$ under (46) implies $SU(2)$ gauge invariance.

An opening for an $SU(2)$ gauge interpretation of spinorial wave equations in geometric algebra has been pointed out by others—notably [12]—but in a framework in which multivector fields transform under physical rotations by left-multiplication alone. The usual arguments about gauging a global symmetry of the Lagrangian are used to justify introducing the gauge field, which transforms under rotations only through its Lorentz coordinate index. ($\mathcal{H}_-$ is nonsensical in that framework.) Rather, we have insisted that multivector fields transform by the standard dual-sided multiplication rule, and that the Clifford connection $W_\mu$ must be introduced for the sake of computability of a first order Hamiltonian for such fields in quantum theory.

If $\mathcal{L}_+$ describes left-chiral fermions, then $\mathcal{L}_-$ describes right-chiral ones. This right-chiral Lagrangian is not invariant under (46), and does not correspond to any recognizable from the Standard Model. If $W_\mu$ is identified with the $SU(2)$ gauge field and time-dependent OBTOs could justifiably be left out of the Hamiltonian, then a necessary condition for $\mathcal{H}_-$ to describe a viable field theory is that it does not couple $F$ to massless excitations of $W_\mu$. These must be described by a gauge invariant Lagrangian, from considerations of
3.4.4. In Relation to Gauge Theory Gravity. The non-covariant term in the transformation of $W_\mu$ is added under transformations of the bases of the Clifford bundle fibers, with respect to which the components of the field values are defined. Under transformations of the coordinates (merely relabeling of events on the manifold), only the Greek index on $W_\mu$ transforms. One can imagine an object $\omega_\mu$ that behaves in the opposite way, transforming covariantly with the basis of the local Clifford algebra and non-covariantly under a coordinate transformation.

\[ \omega_\mu(x) \rightarrow \begin{cases} 
S(t, x) \omega_\mu(x) S^{-1}(t, x) & \text{(Clifford basis)} \\
\Lambda_\mu^\nu \omega_\mu(x') - S^{-1}_\Lambda(t', x') \partial_\mu S_\Lambda(t', x') & \text{(coordinates)} 
\end{cases} \] (48)

Here $S_\Lambda$ is a rotor that, if applied to an initially holonomic basis, would realign the vector basis elements of the geometric algebra with the local directions along which the coordinates increase after the coordinate transformation. If one demands that $L_\pm$ be invariant after complementary coordinate transformations and local rotations of the Clifford bundle, then one must include a coupling to $\omega_\mu$.

\[ L_\pm = F^\dagger e^\mu \partial_\mu F + F^\dagger e^\mu [F, W_\mu] \pm F^\dagger e^k \omega_\mu F \pm F^\dagger e^k \omega_\mu F \] (49)

Under just a coordinate rotation, we then have

\[ L_+ \rightarrow F^\dagger e^\mu S_\Lambda \partial_\mu (S_\Lambda^\dagger F) + F^\dagger e^\mu F W_\mu^\dagger + F^\dagger e^\mu \omega_\mu^\dagger F \] (50)

\[ L_- \rightarrow F^\dagger e^\mu S_\Lambda^\dagger \partial_\mu (S_\Lambda F) + F^\dagger e^\mu F W_\mu + 2 F^\dagger e^\mu W_\mu F - F^\dagger e^\mu \omega_\mu F \] (51)

where all fields on the RHS are evaluated at $(t', x')$. This indicates that the same apparent spin angular momentum results from a steady time-dependent rotation of the coordinates in one sense or a rotation of the Clifford basis in the opposite sense, as it must be.

The bivector-valued connection $\omega_\mu$ then plays the same role as the rotation gauge field $\frac{1}{2} \Omega (e_\mu)$ in Gauge Theory Gravity [14, 15]. GTG does not separate basis and coordinate transformations, citing only a transformation law incorporating both for cases when they are holonomy-preserving:

\[ \Omega(a) \rightarrow S \Omega(a) S^\dagger - ae^\mu \partial_\mu S^\dagger - e^\mu a S \partial_\mu S^\dagger \] (basis & coordinates) (52)

Inclusion of $\omega_\mu$ in the Hamiltonian is not required on the same grounds as $W_\mu$, since it only affects inner transformation operators; an additional assumption like that of necessary general covariance must be made to justify it.

4. Concluding Remarks

We have suggested criteria for expectation values of expressions in geometric algebra (grade projections of products of multivectors) to be computable in quantum theory, considered as a generalized probability theory of linear
operators. When applied to observables in a first-order free theory of elementary multivector fields with values in \( Cl_3(\mathbb{R}) \), the Hamiltonian is found not to be computable according to these criteria unless one introduces coupling terms to a bivector-valued one-form that transforms non-covariantly under basis rotations applied to fibers of the Clifford bundle.

The result is a scheme containing left- and right-chiral spin-\( \frac{1}{2} \) fields, in which all of the geometric objects are left invariant by a global rotation of \( 2\pi \) radians, and the two chiralities couple differently to a gauge-like connection for reasons intimately tied to their geometric character. This scheme does not plausibly fill the explanatory gaps in the origins of maximal parity violation in the Weinberg-Salam electroweak model; but it offers an example of a first-principles framework in which chiral asymmetry provides as much conceptual cohesion as would the null hypothesis of exact mirror symmetry.

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Appendix A. Geometric Algebra

Consider a vector space over a field $K$ equipped with both a symmetric inner product and the exterior wedge product

$$a \cdot b = \frac{1}{2} (Q(a + b) - Q(a) - Q(b)) \in K$$

$$a \wedge b = -b \wedge a$$

for some quadratic form $Q$. Adopting the product $ab$ that satisfies

$$a \cdot b = \frac{1}{2} (ab + ba) \quad a \wedge b = \frac{1}{2} (ab - ba)$$

defines a Clifford algebra. A Clifford algebra over $\mathbb{R}^n$ with $Q$ the standard $\ell_2$ norm is a geometric algebra $C\ell_n(\mathbb{R})$, with a set of preferred orthonormal sets $\{e_i\}$ of basis generating elements spanning $\mathbb{R}^n$ that satisfy

$$e_i e_j + e_j e_i = 2\delta_{ij}$$

A generic element of $C\ell_2(\mathbb{R})$ written in terms of such a basis takes the form

$$a = a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_1 e_2$$

with real components $a_0, a_i, a_{12}$. The Pauli matrices that feature prominently in quantum mechanics provide a representation of $C\ell_3(\mathbb{R})$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

There are countless other equally valid matrix representations, e.g.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

If a multivector can be expressed as a linear combination of $(k-1)$-fold exterior products of vector basis elements, then it has a well defined grade of $k$ (0 if the multivector is just a real number) and is called a $k$-blade. In that case the grade-$k$ projection is equal to the original multivector, and all other grade projections are equal to 0. The matrix representation of the unit grade-0 element is always the identity matrix. The grade-0 projection may be computed as the trace of the matrix representation divided by its dimension.

$$\langle a \rangle_0 = \frac{1}{d} \text{Tr } A$$
Likewise, any higher grade projection can be computed in terms of the trace of a product with each of a complete basis of orthonormal elements with that grade, e.g.
\[
\langle a \rangle_1 = \sum_i e^i \langle a(e^i)\rangle_0 = \frac{1}{d} \sum_i \xi^i \text{Tr}[A(\xi^i)\dagger]
\]
where \(\dagger\) denotes the reverse involution of a multivector
\[
(ab)\dagger = b\dagger a\dagger \quad 1\dagger = 1 \quad (e^i)\dagger \equiv e^i
\]
Multivectors transform under rotations by the action of rotors
\[
S \in \{\exp(\frac{1}{4} \epsilon_{ijk} \theta_i e^j e^k) \mid \theta_i \in \mathbb{R}\}
\]
as \(F \to SF S^{-1}\) (or perhaps \(\psi \to S\psi\)). Since \(S\dagger S = SS\dagger = 1\) for rotations, the matrix representation of \(S\) is unitary, and we often write the transformation as \(F \to SF S\dagger\) in the case of rotations. If the grade projections of \(F\) are taken to be Lorentz scalar, polar 3-vector, and their Hodge duals, then a Lorentz boost is performed by substituting for the bivector \(S\) the paravector
\[
S \in \{\exp(\frac{1}{2} w_i e^k) \mid w_i \in \mathbb{R}\}
\]
This boost paravector is its own reverse: \(S = S\dagger\); its inverse is obtained by taking \(w_i \to -w_i\). The 3-volume element \(d^3x\) is the time-like component of a 4-pseudovector, transforming under boosts as
\[
d^3x \to S d^3x S\dagger = S d^3x S
\]

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