DISCRETE PAINLEVÉ TRANSCENDENT SOLUTIONS TO THE
MULTIPlicative TYPE DISCRETE Kdv EquATIONS

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ABSTRACT. Hirota’s discrete KdV equation is an integrable partial difference equation on
Z^2, which approaches the Korteweg-de Vries (KdV) equation in a continuum limit. In this
paper, we show that its multiplicative-discrete versions have the special solutions given by
the solutions of q-Painlevé equations of types A^j (j = 3, 4, 5, 6).

1. Introduction

The Korteweg-de Vries (KdV) equation [18]:

\[ u_t + 6uu_x + u_{xxx} = 0, \]  

where \( u = u(t, x) \in \mathbb{C} \) and \( (t, x) \in \mathbb{C}^2 \), is known as a mathematical model of waves on
shallow water surfaces. The KdV equation is an important equation that has been studied
extensively in physics, engineering and mathematics, especially in the field of integrable
systems (see, i.e., [2, 4] and references therein). In 1977, Hirota found the following inte-
grable discrete version of the KdV (dKdV) equation [8]:

\[ u_{l+1,m+1} - u_{l,m} = \frac{1}{u_{l,m+1}} - \frac{1}{u_{l+1,m}}, \]  

where \( u_{l,m} \in \mathbb{C} \) and \( (l, m) \in \mathbb{Z}^2 \). Indeed, Equation (1.2) has the similar properties with the
KdV equation (e.g., soliton solution, Lax pair and so on) and approaches Equation (1.1) in
a continuum limit. Moreover, in 1991, Capel et al. found the non-autonomous generation
of the dKdV equation [3, 17, 19, 28]:

\[ u_{l+1,m+1} - u_{l,m} = q_{m+1} - p_{l+1} \frac{1}{u_{l,m+1}} - \frac{1}{u_{l+1,m}}, \]  

where \( p_l, q_m \in \mathbb{C} \) are arbitrary functions of \( l \) and \( m \), respectively.

In this paper, we focus on the two multiplicative-discrete type P\Delta Es. One is

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and the other is

\[ u_{l+1,m+1} - u_{l,m} = \frac{1}{u_{l,m+1}} - \frac{1}{u_{l+1,m}}, \]  

where

\[ A_j = \frac{(1 - \alpha_j)(\gamma - \alpha_j)}{\alpha_j}, \quad B_m = \frac{(1 - \beta_m)(1 - \gamma \beta_m)}{\beta_m}. \]  

Here,

\[ \alpha_j = \epsilon^j \alpha_0, \quad \beta_m = \epsilon^m \beta_0, \]  

and \( \alpha_0, \beta_0, \gamma \in \mathbb{C} \) and \( \epsilon \in \mathbb{C}^* \) are parameters. Each of equations (1.4) and (1.5) is a
special case of Equation (1.3). Therefore, these are identified as multiplicative type dKdV
equations.

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Gambier classified all the ordinary differential equations, especially where $F$ approaches 0. The Painlevé equations are numbered beginning with one: $P_1$ equations are referred to as the Painlevé transcendents. (See Theorems [12, 15].) The motivation for the discovery of such solutions is as follows:

**Remark 1.1.** As multiplicative-discrete type difference equations are also called $q$-difference equations, it is common to use the parameter $q$ for their shift parameters. However, in this paper, the parameter $q$ is used to denote the shift parameters of $q$-Painlevé equations, and we also consider the correspondence between the shift parameters of the $q$-Painlevé equations and those of the multiplicative $dKdV$ equations. Therefore, we use the parameter $e$ instead of the parameter $q$ for the multiplicative $dKdV$ equations [14] and [15] to avoid confusion.

In this paper, we study the special solutions of Equations (1.4) and (1.5). The distinctive feature of the special solutions given in this paper is that a long each of the directions confusion.

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**1.1. $q$-Painlevé equations.** In this subsection, we briefly explain the discrete Painlevé equations, especially $q$-Painlevé equations.

In the early 20th-century, in order to find new transcendental functions, Painlevé and Gambier classified all the ordinary differential equations of the type

$$u_{xx} = F(u, u_x; x),$$

(1.9)

where $F$ is a function meromorphic in $x$ and rational in $u$ and $u_x$, with the Painlevé property (the locations of possible branch points and essential singularities of the solution do not depend on the initial data) [6, 23]. As a result, they obtained six new equations. The resulting equations are now collectively referred to as the Painlevé equations, and the solutions of the Painlevé equations are referred to as the Painlevé transcendents. The Painlevé equations are numbered beginning with one: $P_1, \ldots, P_6$. Starting from $P_6$, we can through appropriate limiting processes obtain $P_J$ ($J = I, \ldots, V$). Note that $P_6$ was found by Fuchs before Painlevé et al.

Discrete Painlevé equations are nonlinear ordinary difference equations of second order, which include discrete analogues of the Painlevé equations. There are only six Painlevé equations, but there are an infinite number of discrete Painlevé equations. Moreover, there are three discrete types: elliptic-, multiplicative- and additive- types. Discrete Painlevé equations of the multiplicative-type are especially referred to as $q$-Painlevé equations. In a similar fashion as the Painlevé equations, we also refer to the solutions of discrete Painlevé equations as the discrete Painlevé transcendent.

In 2001, Sakai gave the geometric description of discrete Painlevé equations, based on types of space of initial values [16, 26]. The spaces of initial values are constructed by the blow up of $\mathbb{P}^2$ at nine base points (i.e. points where the system is ill defined because it approaches 0/0). $q$-Painlevé equations are classified into 9 types: $A_0^{(1)}, A_1^{(1)}, \ldots, A_6^{(1)}$, $A_7^{(1)}$ according to the configuration of the base points, and they relate to the (extended) affine Weyl group of the types $E_8^{(1)}, E_6^{(1)}, E_7^{(1)}, D_5^{(1)}, A_4^{(1)}$, $(A_2 + A_1)^{(1)}, (A_1 + A_1)^{(1)}, A_3^{(1)}$, $A_1^{(1)}$, respectively. These responses for $q$-Painlevé equations of types $A_J^{(1)}$ ($J = 3, 4, 5, 6$) are presented specifically in [3] and [4]. Moreover, some typical examples of $q$-Painlevé equations of types $A_J^{(1)}$ ($J = 3, 4, 5, 6$) are displayed in Appendix [3].

Together with the Painlevé equations, the discrete Painlevé equations are now regarded as one of the most important classes of equations in the theory of integrable systems (see, e.g., [7, 16]). From the point of view of special function, using the Painlevé/discrete
Painlevé transcendent for solving other differential/difference equations is as important as investigating their properties.

1.2. Main results. In this subsection, we show the main results of this paper. Theorems 1.2, 1.4 give special solutions of Equation (1.4), while Theorem 1.5 gives a special solution of Equation (1.3).

**Theorem 1.2** ($A_6^{(1)}$-type). The multiplicative dKdV equation (1.4) has the special solution

\[ u_{l,m} = \frac{e^{3/2}h_{l,m}^{1/2}(\alpha_l - \beta_m)}{\alpha_l^{1/2}(h_{l,m} + \beta_m)g_{l,m}} \]  

(1.10)

Here, the functions $g_{l,m}$ and $h_{l,m}$ satisfy the system of ordinary difference equations for the l-direction:

\[ h_{l+1,m} = \frac{\beta_m(1 + g_{l,m})}{g_{l,m}^2}, \quad g_{l+1,m} = \frac{\beta_m(h_{l+1,m} + \alpha_{l+1})}{\alpha_{l+1}(h_{l+1,m} + \beta_m + 1)}, \]  

(1.11)

and that for the m-direction:

\[ f_{l,m+1} = \frac{\alpha_l(1 + g_{l,m+1})}{\beta_l h_{l,m+1} + \alpha_{l+1}}, \quad h_{l,m+1} = \frac{\alpha_{l+1}(1 + f_{l,m+1})}{f_{l,m+1}^2}, \]  

(1.12)

where

\[ f_{l,m} = \frac{1}{g_{l,m} h_{l,m}}. \]  

(1.13)

Note that each of equations (1.11) and (1.12) is equivalent to the q-$P_{III}^{f_{1,4}}$ (A.3), which is a q-Painlevé equation of $A_6^{(1)}$-type. (See Appendix A for details.)

The proof of Theorem 1.2 is given in §2.

**Theorem 1.3** ($A_5^{(1)}$-type). The multiplicative dKdV equation (1.4) has the special solution

\[ u_{l,m} = \frac{d(\alpha_l - \beta_m)}{(\alpha_l^{1/2} + \beta_m^{1/2})g_{l,m} h_{l,m}}, \]  

(1.14)

where $d \in \mathbb{C}^*$ is an arbitrary parameter. Here, the functions $g_{l,m}$ and $h_{l,m}$ satisfy the system of ordinary difference equations for the l-direction:

\[ g_{l+1,m} = \frac{d^2(\alpha_l^{1/2} + f_{l,m})}{f_{l,m}(1 + \alpha_l^{1/2}f_{l,m})}, \quad f_{l+1,m} = \frac{d^2(\alpha_{l+1}^{1/2} + \beta_m^{1/2}g_{l+1,m})}{g_{l+1,m}(\beta_m^{1/2} + \alpha_{l+1}^{1/2}g_{l+1,m})}, \]  

(1.15)

where

\[ f_{l,m} = \frac{d^2}{g_{l,m} h_{l,m}}. \]  

(1.16)

and that for the m-direction:

\[ h_{l+1,m} = \frac{d^2(\beta_m^{1/2} + \alpha_l^{1/2}g_{l,m})}{g_{l,m}(\alpha_l^{1/2} + \beta_m^{1/2}g_{l,m})}, \quad g_{l+1,m} = \frac{d^2(\beta_m^{1/2} + h_{l+1,m})}{h_{l+1,m}(1 + \beta_m^{1/2}h_{l+1,m})}, \]  

(1.17)

Note that each of equations (1.15) and (1.17) is equivalent to the q-$P_{III}^{f_{1,4}}$ (A.3), which is a q-Painlevé equation of $A_5^{(1)}$-type. (See Appendix A for details.)

The proof of Theorem 1.3 is given in §3.1.

**Theorem 1.4** ($A_4^{(1)}$-type). The multiplicative dKdV equation (1.4) has the special solution

\[ u_{l,m} = \frac{\beta_m^{1/2}(\alpha_l - \beta_m)h_{l,m}}{e^{1/4}(\alpha_l^{1/2}d_1 d_2 + 1 + \beta_md_{l,m})}, \]  

(1.18)
where \(d_1, d_2 \in \mathbb{C}^*\) are arbitrary parameters. Here, the functions \(g_{l,m}\) and \(h_{l,m}\) satisfy the system of ordinary difference equations for the l-direction:

\[
\begin{align*}
g_{l+1,m}g_{l,m} &= \frac{(\beta_{l-1}d_2^2 + \alpha_l f_{l,m})(\beta_l + \alpha_{l+1}f_{l,m})}{\alpha_l^2 \beta_l^2 d_2^2 (d_1^2 \beta_l + f_{l,m})}, \\
f_{l+1,m}f_{l,m} &= \frac{\beta_{l-1}^2 d_2^2 (1 + \alpha_{l+1}g_{l+1,m})(1 + \alpha_{l+1} \beta_{l+1} g_{l+1,m})}{\alpha_{l+1}^2 (1 + \beta_{l+1} g_{l+1,m})},
\end{align*}
\] (1.19)

where

\[
f_{l,m} = \frac{d_1^2 d_2^2 (1 + \alpha_l \beta_m g_{l,m})}{h_{l,m}},
\] (1.20)

and that for the m-direction:

\[
\begin{align*}
h_{l,m+1}h_{l,m} &= \frac{\alpha_{l+1}^2 d_2^2 d_2^2 (1 + \beta_{l+1} g_{l,m})(1 + \alpha_l \beta_m g_{l,m})}{\beta_{l+1}^2 (1 + \alpha_l \beta_m g_{l,m})}, \\
g_{l,m+1}g_{l,m} &= \frac{(\alpha_{l+1} + \beta_{l+1} h_{l,m+1})(\alpha_{l+1} d_2^2 d_2^2 + \beta_{l+1} h_{l,m+1})}{\alpha_{l+1}^2 \beta_{l+1}^2 d_2^2 (d_1^2 d_1^2 + h_{l,m+1})}.
\end{align*}
\] (1.21)

Note that each of equations (1.19) and (1.21) is equivalent to the q-Painlevé \(A_3^{(1)}\)-type. (See Appendix A for details.)

The proof of Theorem 1.4 is given in §3.2.

**Theorem 1.5** \((A_3^{(1)}\text{-type})\). The multiplicative dKdV equation (1.5) has the special solution

\[
u_{l,m} = \frac{e^{1/4}(\beta_m \gamma - \alpha_l)}{\alpha_l^2 \beta_m^2} \left( f_{l,m}(d_1 + x_{l,m}) + d_2^{1/2}(1 + d_1 d_2^{1/2} \beta_m x_{l,m}) \right),
\] (1.22)

where \(d_1, d_2 \in \mathbb{C}^*\) are arbitrary parameters. Here, the functions \(f_{l,m}\) and \(x_{l,m}\) satisfy the system of ordinary difference equations for the l-direction:

\[
\begin{align*}
g_{l+1,m}g_{l,m} &= \frac{(f_{l,m} + d_1^{-1} d_2^{1/2} \alpha_l^{-1} \gamma)(f_{l,m} + d_2^{-1/2} \beta_m x_{l,m})}{(f_{l,m} + d_1^{-1} d_2^{1/2})(f_{l,m} + d_1^{-1} d_2^{-1/2})}, \\
f_{l+1,m}f_{l,m} &= \frac{(g_{l+1,m} + d_2^{-1/2} \beta_m^{-1} \alpha_l^{-1} \gamma)(g_{l+1,m} + d_2^{1/2} \alpha_l \beta_m^{-1} \gamma^{-1})}{(g_{l+1,m} + d_2^{-1/2} \beta_m^{-1})(g_{l+1,m} + d_2^{1/2} \beta_m \gamma^{-1})},
\end{align*}
\] (1.23)

where

\[
g_{l,m} = \frac{d_2^{1/2} \gamma + d_1 \alpha_l f_{l,m}}{\alpha_l \beta_m^{-1/2} \gamma^{1/2}(d_1 + d_2^{1/2} f_{l,m}) x_{l,m}},
\] (1.24)

and that for the m-direction:

\[
\begin{align*}
y_{l,m+1}y_{l,m} &= \frac{(x_{l,m} + d_1^{-1} d_2 \beta_m^{-1})(x_{l,m} + d_1 d_2^{-1} \beta_m^{-1} \gamma^{-1})}{(x_{l,m} + d_1^{-1})(x_{l,m} + d_1^{-1})}, \\
x_{l,m+1}x_{l,m} &= \frac{(y_{l,m+1} + \alpha_l \beta_m^{-1})(y_{l,m+1} + \alpha_l \beta_m \gamma^{-1})}{(y_{l,m+1} + \alpha_l)(y_{l,m+1} + \alpha_l)},
\end{align*}
\] (1.25)

where

\[
y_{l,m} = \frac{\alpha_l^{1/2} f_{l,m}(d_1 + d_2 \beta_m \gamma x_{l,m})}{d_2^{1/2} \beta_m^{-1} \gamma (1 + d_1 x_{l,m})},
\] (1.26)

Note that each of equations (1.23) and (1.25) is equivalent to the q-Painlevé \(A_3^{(1)}\)-type. (See Appendix A for details.)

The proof of Theorem 1.5 is given in §3.3.
1.3. Plan of the paper. This paper is organized as follows. In §2 using the birational representation of the extended affine Weyl group of type $(A_1 + A_1)^{(1)}$, we give a proof of Theorem 1.2. In §3 using exactly the same process in §2 we prove Theorems 1.3–1.5. Some concluding remarks are given in §4. In Appendix A we list some typical examples of $q$-Painlevé equations of types $A_J^{(1)}$ ($J = 3, 4, 5, 6$) and give the correspondences between those $q$-Painlevé equations and the $q$-Painlevé equations in Theorems 1.2, 1.3, 1.4, 1.5.

2. Proof of Theorem 1.2.

In this section, using the birational representation of the extended affine Weyl group of type $(A_1 + A_1)^{(1)}$, which gives rise to the $A_0^{(1)}$-type $q$-Painlevé equations (A.2) and (A.3), we give a proof of Theorem 1.2.

2.1. Birational action of $\tilde{W}(A_1 + A_1^{(1)})$. In this subsection, we show the birational actions of transformation group $\tilde{W}(A_1 + A_1^{(1)})$.

Let $a_0, a_1, b, q$ be complex parameters and $f_0, f_1, f_2$ be complex variables satisfying

$$a_0a_1 = q, \quad f_0f_1f_2 = 1.$$  \hfill (2.1)

We define the transformation group $\tilde{W}(A_1 + A_1^{(1)}) = \langle s_0, s_1, w_0, w_1, \pi \rangle$ as follows: each element of $\tilde{W}(A_1 + A_1^{(1)})$ is an isomorphism from the field of rational functions $K(f_0, f_1)$, where $K = C(a_0, a_1, b)$, to itself. Note that for each element $w \in \tilde{W}(A_1 + A_1^{(1)})$ and function $F = F(a_1, b, f_j)$, we use the notation $wF$ to mean $wF = F(aw, wb, wbf_j)$, that is, $w$ acts on the arguments from the left. The actions of $\tilde{W}(A_1 + A_1^{(1)})$ on the parameters are given by

$$s_0 : (a_0, a_1, b) \mapsto (1 - a_0^2 a_1, \frac{b}{a_0}), \quad s_1 : (a_0, a_1, b) \mapsto (a_0a_1^2, \frac{1}{a_1}, a_1b),$$

$$w_0 : (a_0, a_1, b, q) \mapsto (1 - a_1, \frac{1}{a_0}, \frac{b}{a_0}, \frac{1}{q}), \quad w_1 : (a_0, a_1, b, q) \mapsto (1 - a_1, \frac{1}{a_0}, a_0a_1^2, \frac{1}{q}),$$

$$\pi : (a_0, a_1, b, q) \mapsto (1 - a_1, \frac{1}{a_0}, a_0a_1^2, \frac{1}{q}),$$

while those on the variables are given by

$$s_0 : (f_0, f_1, f_2) \mapsto \left(\frac{f_0(a_0f_0 + a_0 + f_1)}{f_0 + f_1 + 1}, \frac{f_1(a_0f_0 + f_1 + 1)}{a_0f_0 + a_0 + f_1}, \frac{a_0f_2(a_0f_0 + f_1 + 1)}{(a_0f_0 + a_0 + f_1)(a_0f_0 + f_1 + 1)}\right),$$

$$s_1 : (f_0, f_1) \mapsto \left(\frac{f_0(a_0f_0 + a_0 + f_1)}{a_0f_0 + a_0 + f_1}, \frac{f_1(a_0f_0 + f_1 + 1)}{a_0f_0 + a_0 + f_1}, \frac{a_0f_2(a_0f_0 + f_1 + 1)}{(a_0f_0 + a_0 + f_1)(a_0f_0 + f_1 + 1)}\right),$$

$$w_0 : (f_0, f_1, f_2) \mapsto \left(\frac{a_0f_0 + f_1}{f_0 + f_1}, \frac{a_0f_0 + a_0 + f_0f_1}{a_0a_1f_0f_2}, \frac{b_2f_0}{b_0f_0f_2} \right),$$

$$w_1 : (f_0, f_1) \mapsto \left(\frac{f_1}{f_0}, \frac{f_1}{f_0} \right),$$

$$\pi : (f_1, f_2) \mapsto \left(\frac{a_0f_0 + f_1}{a_0f_0 + f_1}, \frac{b_2f_0}{a_0f_0 + f_1} \right).$$

Remark 2.1. We follow the convention that the variables and parameters not explicitly including in the actions listed in the equations above are the ones that remain unchanged under the action of the corresponding transformation. That is, the transformation acts as an identity on those variables and parameters.

The transformation group $\tilde{W}(A_1 + A_1^{(1)})$ forms the extended affine Weyl group of type $(A_1 + A_1)^{(1)}$ [11,26]. Namely, the transformations satisfy the following fundamental relations:

$$s_0^2 = s_1^2 = (s_0s_1)^{\infty} = 1, \quad w_0^2 = w_1^2 = (w_0w_1)^{\infty} = 1, \quad (2.2a)$$

$$\pi^2 = 1, \quad \pi s_0 = s_1 \pi, \quad \pi w_0 = w_1 \pi. \quad (2.2b)$$
and the action of $W(A^{(1)}_1) = \langle s_0, s_1 \rangle$ and that of $W(A^{(1)}_2) = \langle w_0, w_1 \rangle$ commute. We note that the relation $(vw)^N = 1$ for transformations $w$ and $w'$ means that there is no positive integer $N$ such that $(ww')^N = 1$.

We define the translations in $\tilde{W}(A_1 + A^{(1)}_1)$ by

\[ T_1 = w_0 w_1, \quad T_2 = \pi s_1 w_1, \quad T_3 = \pi s_0 w_0, \]

whose actions on the parameters are given by the translational motions:

\[ T_1 : (a_0, a_1, b) \mapsto (a_0, a_1, qb), \]
\[ T_2 : (a_0, a_1, b) \mapsto (qa_0, q^{-1} a_1, b), \]
\[ T_3 : (a_0, a_1, b) \mapsto (q^{-1} a_0, qa_1, q^{-1} b). \]

Note that the following hold:

\[ T_i T_j T_i = 1, \quad T_i T_j = T_j T_i \quad (i, j = 1, 2, 3), \]

and the parameter $q$ is invariant under the action of each translation.

### 2.2. Proof of Theorem 1.2

In this subsection, using the birational action of $\tilde{W}(A_1 + A^{(1)}_1)$ we give a proof of Theorem 1.2.

Let us firstly derive the $A^{(1)}_k$-type $q$-Painlevé equations (A.2) and (A.3) from the birational action of $\tilde{W}(A_1 + A^{(1)}_1)$. Define the $f$-functions by

\[ f_0^{l_1,l_2,l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3} (f_0), \quad f_1^{l_1,l_2,l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3} (f_1), \quad f_2^{l_1,l_2,l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3} (f_2), \]

where $l_1, l_2, l_3 \in \mathbb{Z}$. From the condition in (2.4) and relations (2.5), we have

\[ f_0^{l_1,l_2,l_3} f_1^{l_1,l_2,l_3} f_2^{l_1,l_2,l_3} = 1, \quad f_i^{l_1,l_2,l_3} = f_i^{l_1,l_2,l_3} (i = 1, 2, 3). \]

The $f$-functions satisfy the second order ordinary difference equation for $l_1$-direction:

\[ \left( f_0^{l_1+1,l_2,l_3} - \frac{1}{q^{l_1-b} b} \right) \left( f_0^{l_1-1,l_2,l_3} - \frac{1}{q^{l_1+b} b} \right) = \frac{q^{-l_1-2h} a_1 f_0^{l_1,l_2,l_3}}{b (1 + f_0^{l_1,l_2,l_3})}, \]

the system of first order ordinary difference equations for $l_2$-direction:

\[
\begin{cases}
    f_2^{l_1+1,l_2,l_3} = \frac{q^{l_2-1} b}{f_0^{l_1+1,l_2,l_3} (1 + f_0^{l_1,l_2,l_3})}, \\
    f_1^{l_1+1,l_2,l_3} = \frac{q^{l_2-1} a_0 (q f_0^{l_1,l_2+1,l_3} + q^{l_1-b} b)}{f_2^{l_1,l_2+1,l_3} (q f_0^{l_1,l_2+1,l_3} + q^{l_1-b} b)},
\end{cases}
\]

and that for $l_3$-direction:

\[
\begin{cases}
    f_0^{l_1,l_2+1,l_3} = \frac{q^{l_3-1} b}{f_1^{l_1,l_2+1,l_3} (1 + f_1^{l_1,l_2+1,l_3})}, \\
    f_2^{l_1,l_2+1,l_3} = \frac{q^{l_3-1} a_1 b}{f_0^{l_1,l_2+1,l_3} (q f_0^{l_1,l_2+1,l_3} + q^{l_1-b} b)}.
\end{cases}
\]

Note that Equations (2.8), (2.9) and (2.10) respectively follow from

\[ T_1 (f_0) f_0 = \frac{1}{b} + \frac{f_0 f_1}{a_0 (f_0 + 1)}, \quad T_1^{-1} (f_0) f_0 = \frac{1}{q f_1} + \frac{1}{q^{-1} b f_1}, \]
\[ T_2 (f_2) f_2 = \frac{q f_2}{f_1 (1 + f_1)}, \quad T_2 (f_1) f_1 = \frac{a_0 (b + q T_2 (f_2))}{T_2 (f_1) (q a_0 T_2 (f_2) + b)}, \]
\[ T_3 (f_0) f_0 = \frac{a_1 b + q f_2}{f_2 (b + q f_2)}, \quad T_3 (f_2) f_2 = \frac{q^{-1} a_1 b}{T_3 (f_0) (T_3 (f_0) + 1)}. \]
Remark 2.2. Equation (2.22) is equivalent to the $q$-P$_{III}$ (A.2), and each of equations (2.29) and (2.10) is equivalent to the $q$-P$_{III}$ (A.3). Indeed, the correspondence between Equations (2.8) and (A.3) is given by

$$F = f_{0}^{1, t, s}, \quad F = f_{0}^{1, t, s}, \quad F = f_{0}^{1, t, s}, \quad t = q^{h^{-1}b}, \quad c_{1} = q^{l_{2}^{-1}a_{1}},$$

(2.14)

that between Equations (2.9) and (A.3) is given by

$$F = \frac{1}{f_{0}^{1, t, s}}, \quad G = \frac{1}{f_{0}^{1, t, s}}, \quad \overline{F} = \frac{1}{f_{0}^{1, t, s}}, \quad \overline{G} = \frac{1}{f_{0}^{1, t, s}},$$

(2.15)

and that between Equations (2.10) and (A.3) is given by

$$F = \frac{1}{f_{0}^{1, t, s}}, \quad G = \frac{1}{f_{0}^{1, t, s}}, \quad \overline{F} = \frac{1}{f_{0}^{1, t, s}}, \quad \overline{G} = \frac{1}{f_{0}^{1, t, s}},$$

(2.16)

We are now in a position to prove Theorem 1.2. Letting

$$u = \frac{1 - a_{1}}{q^{l_{2}^{-1}a_{1}^{-1}f_{0}(b + qf_{2})}},$$

(2.17)

we can verify that the following relation holds:

$$T_{2}T_{3}(u) - u = \frac{b^{-1}(1 - q^{-1}a_{1}^{-1}) - b^{-1}(q^{-1} - a_{1}^{-1})}{T_{3}(u)},$$

(2.18)

Applying $T_{2}^{l_{2}}T_{3}^{m}$ to the equation above and setting

$$U_{l_{1}, m} = T_{2}^{l_{2}}T_{3}^{m}(u) = \frac{1 - q^{-l_{2}^{-1}a_{1}^{-1}f_{0}(q^{-m}b + qf_{2}^{0, l_{2}^{1}, m})}{q^{l_{2}^{-1}a_{1}^{-1}f_{0}(q^{-m}b + qf_{2}^{0, l_{2}^{1}, m})}},$$

(2.19)

we obtain

$$U_{l_{1}+1, m+1} - U_{l_{1}, m} = \frac{q^{l_{2}^{1}m}b^{-1} - q^{-l_{2}^{-1}a_{1}^{-1}f_{0}(q^{-m}b + qf_{2}^{0, l_{2}^{1}, m})}{U_{l_{1}+1, m}} = \frac{q^{l_{2}^{1}m}b^{-1} - q^{-l_{2}^{-1}a_{1}^{-1}f_{0}(q^{-m}b + qf_{2}^{0, l_{2}^{1}, m})}{U_{l_{1}, m}}.$$  

(2.20)

Equation (2.20) is equivalent to the multiplicative dKdV equation (1.4) and the correspondence is given by the following:

$$a_{l_{1}} = U_{l_{1}, m}, \quad a_{l_{1}} = q^{l_{2}^{-1}a_{1}^{-1}f_{0}(q^{-m}b + qf_{2}^{0, l_{2}^{1}, m}), \quad \beta_{m} = q^{l_{2}^{1}m}b^{-1}, \quad \epsilon = q.$$  

(2.21)

From the way $f_{0}^{0, l_{1}, m}$ and $f_{2}^{0, l_{2}, m}$ are constructed, we know that they are rational functions over $K = \mathbb{C}(a_{0}, a_{1}, b)$ of the two unknown variables $f_{0}$ and $f_{1}$, that is,

$$f_{0}^{0, l_{1}, m}, f_{2}^{0, l_{2}, m} \in K(f_{0}, f_{1}).$$  

(2.22)

Since the functions $f_{0}^{0, l_{1}, m}$ and $f_{2}^{0, l_{2}, m}$ collectively satisfy Equations (2.9) and (2.10) and the number of initial values of a second-order ordinary difference equations is 2, the functions $f_{0}^{0, l_{1}, m}$ and $f_{2}^{0, l_{2}, m}$ give the general solution to each of equations (2.9) and (2.10) with $l = l_{2}^{-1}l_{1}$ and $m = l_{1} - l_{1}$. Moreover, from the relation (2.19), we find that the functions $f_{0}^{0, l_{1}, m}$ and $f_{2}^{0, l_{2}, m}$ also give the special solutions to Equation (2.20). Therefore, setting

$$f_{l_{1}, m} = \frac{1}{f_{0}^{0, l_{1}, m}}, \quad g_{l_{1}, m} = \frac{1}{f_{1}^{1, l_{1}, m}}, \quad h_{l_{1}, m} = \frac{1}{f_{2}^{0, l_{2}, m}},$$

(2.23)

and using Equations (2.7), (2.9), (2.10) and (2.19) with the correspondence (2.21), we have completed the proof of Theorem 1.2.
3. Proofs of Theorems 1.3–1.5

In this section, using the transformation groups $\tilde{W}((A_2 + A_1)^{(1)})$, $\tilde{W}(A_1^{(1)})$ and $\tilde{W}(D_4^{(1)})$, which respectively relate to $A_2^{(1)}$, $A_1^{(1)}$ and $A_3^{(1)}$-type $q$-Painlevé equations, we give proofs of Theorems 1.3–1.5. Since each process for demonstrating the results is exactly the same as the process in §2 we omit a detailed discussion for brevity.

3.1. Proof of Theorem 1.3

In this subsection using the transformation group $\tilde{W}((A_2 + A_1)^{(1)})$ we give a proof of Theorem 1.3.

Let $a_0$, $a_1$, $a_2$, $c$, $q$ be complex parameters and $f_0$, $f_1$, $f_2$ be complex variables satisfying

$$a_0a_1a_2 = q, \quad f_0f_1f_2 = qe^2. \quad (3.1)$$

The transformation group $\tilde{W}((A_2 + A_1)^{(1)}) = \langle s_0, s_1, s_2, \pi, w_0, w_1, r \rangle$ is defined by the actions on the parameters:

$$s_i(a_j) = a_ja_0^{-c_{ij}}, \quad \pi(a_i) = a_i + 1, \quad w_0(c) = c^{-1}, \quad w_1(c) = q^{-2}c^{-1}, \quad r(c) = q^{-1}c^{-1},$$

where $i, j \in \mathbb{Z}/3\mathbb{Z}$ and $C_{ij}^2_{i,j=0}$ is the Cartan matrix of type $A_1^{(1)}$:

$$(C_{ij}^2)_{i,j=0} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

and those on the variables:

$$s_i(f_{i-1}) = f_{i-1} \frac{1 + a_if_i}{a_i + f_i}, \quad s_i(f_i) = f_i, \quad s_i(f_{i+1}) = f_{i+1} \frac{a_i + f_i}{1 + a_if_i}, \quad \pi(f_i) = f_{i+1},$$

$$w_0(f_i) = a_ai_1(a_i-1a_i + a_{i-1}f_i + f_{i-1}f_i),$$

$$w_1(f_i) = \frac{1 + a_if_i + a_{i+1}f_i + f_if_{i+1}}{a_{i+1}f_{i+1}(1 + a_{i+1}f_{i+1} + a_{i-1}f_{i-1}f_{i-1})}, \quad r(f_i) = f_{i-1}^{-1},$$

where $i \in \mathbb{Z}/3\mathbb{Z}$. See Remark 2.1 for the convention on how to write these actions. The transformation group $\tilde{W}((A_2 + A_1)^{(1)})$ satisfies the fundamental relations of the extended affine Weyl group of type $(A_2 + A_1)^{(1)}$ [14]:

$$s_i^2 = (s_is_{i+1})^3 = 1, \quad \pi^3 = 1, \quad \pi s_i = s_{i+1} \pi, \quad (3.2a)$$

$$w_0^2 = w_1^2 = (w_0w_1)^{\infty} = 1, \quad r^2 = 1, \quad rw_0 = w_1r, \quad (3.2b)$$

where $i \in \mathbb{Z}/3\mathbb{Z}$, and the action of $\tilde{W}(A_1^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$ and that of $\tilde{W}(A_1^{(1)}) = \langle w_0, w_1, r \rangle$ commute.

Define the translations $T_1$ and $T_2$ by

$$T_1 = \pi s_2 s_1, \quad T_2 = \pi s_0 s_2, \quad (3.3)$$

whose actions on the parameters are given by

$$T_1 : (a_0, a_1, a_2, c) \mapsto (qa_0, q^{-1}a_1, a_2, c), \quad (3.4a)$$

$$T_2 : (a_0, a_1, a_2, c) \mapsto (a_0, qa_1, q^{-1}a_2, c). \quad (3.4b)$$

Note that the translations $T_1$ and $T_2$ commute with each other and the parameter $q$ is invariant under the action of each translation.

Define the $f$-functions by

$$f_0^{lm} = T_1^l T_2^m (f_0), \quad f_1^{lm} = T_1^l T_2^m (f_1), \quad f_2^{lm} = T_1^l T_2^m (f_2), \quad (3.5)$$

where $l, m \in \mathbb{Z}$, which from (3.1) satisfy

$$f_0^{lm} f_1^{lm} f_2^{lm} = qe^2. \quad (3.6)$$
The $f$-functions satisfy the following $q$-difference equation for $l$-direction:

\[ f_{l}^{i+1,m} = \frac{q c^{2}(1 + q^{l} a_{l} f_{l}^{i,m})}{f_{i}^{m}(q^{l} a_{l} + f_{l}^{i,m})}, \quad f_{l}^{i+1,m} = \frac{q c^{2}(1 + q^{-l} a_{l} f_{l}^{i+1,m})}{f_{i+1,m}(q^{-l} a_{l} a_{l} + f_{l}^{i+1,m})}, \quad (3.7) \]

and that for $m$-direction:

\[ f_{l}^{m+1} = \frac{q c^{2}(1 + q^{m-1} a_{l} f_{l}^{m})}{f_{l}^{m}(q^{m-1} a_{l} + f_{l}^{m})}, \quad f_{l}^{m+1} = \frac{q c^{2}(1 + q^{-m} a_{l} f_{l}^{m+1})}{f_{l}^{m+1}(q^{-m} a_{l} a_{l} + f_{l}^{m+1})}, \quad (3.8) \]

Note that Equations (3.7) and (3.8) follow from

\[ T_{1}(f_{l}) = \frac{q c^{2}(1 + a_{l} f_{l})}{f_{l}(a_{l} + f_{l})}, \quad T_{1}^{-1}(f_{l}) = \frac{q c^{2}(1 + q a_{l} f_{l})}{f_{l}(q a_{l} a_{l} + f_{l})}, \quad (3.9) \]

\[ T_{2}(f_{l}) = \frac{q c^{2}(1 + a_{l} f_{l})}{f_{l}(a_{l} + f_{l})}, \quad T_{2}^{-1}(f_{l}) = \frac{q c^{2}(1 + q a_{l} f_{l})}{f_{l}(q a_{l} a_{l} + f_{l})}, \quad (3.10) \]

respectively.

**Remark 3.1.** Each of equations (3.7) and (3.8) is equivalent to the $q$-PDE (A.4). The correspondence between Equations (3.7) and (A.4) is given by

\[ F = f_{l}, \quad G = f_{l}, \quad \bar{T} = f_{l}, \quad \bar{G} = f_{l}, \quad t = q a_{l}, \quad c_{1} = q c^{2}, \quad c_{2} = q^{-m} a_{l}, \quad (3.11) \]

while that between Equations (3.8) and (A.4) is given by

\[ F = f_{l}, \quad G = f_{l}, \quad \bar{T} = f_{l}, \quad \bar{G} = f_{l}, \quad t = q^{-m} a_{l}, \quad c_{1} = q c^{2}, \quad c_{2} = q^{-m} a_{l}. \quad (3.12) \]

Letting

\[ u = \frac{q^{1/2} (a_{l} - q^{-2} a_{l})}{(a_{l} + q^{-2} a_{l}) f_{l}}, \quad (3.13) \]

we can verify that the following relation holds:

\[ T_{1} T_{2}(u) - u = \frac{q^{-2} a_{l}^{2} - a_{l}^{2} - q^{-2} a_{l}^{2} - a_{l}^{2}}{T_{2}(u) \cdot T_{1}(u)} \quad (3.14) \]

Therefore, applying $T_{1} T_{2}$ to the equation above we obtain

\[ U_{i+1,m} - U_{i,m} = \frac{q^{2m-4} a_{l}^{2} - q^{-2} a_{l}^{2} - q^{2m-2} a_{l}^{2} - q^{-2} a_{l}^{2}}{U_{i+1,m} - U_{i,m}} \quad (3.15) \]

where $U_{i,m} = T_{1} T_{2}(u)$, which is equivalent to the multiplicative dKdV equation (1.4) with the following correspondence:

\[ u_{i,m} = U_{i,m}, \quad a_{l} = q^{-2} a_{l}^{2}, \quad \beta_{m} = q^{-2m} a_{l}^{2}, \quad \epsilon = q^{-2}. \quad (3.16) \]

Therefore, setting

\[ d = e^{-1/4} c, \quad f_{l,m} = f_{l}, \quad g_{l,m} = f_{l}, \quad h_{l,m} = f_{l}, \quad (3.17) \]

and using Equations (3.6), (3.7), (3.8) and (3.13) with the correspondence (3.16), we have completed the proof of Theorem 1.3.
3.2. Proof of Theorem 1.4

In this subsection using the transformation group \( \tilde{W}(A_4^{(1)}) \) we give a proof of Theorem 1.4.

Let \( a_i \) \((i = 0, \ldots, 4) \) and \( q \) be complex parameters satisfying
\[ a_0 a_1 a_2 a_3 a_4 = q, \tag{3.18} \]
and \( f_i^{(j)} \) \((i = 1, 2, j = 1, \ldots, 5) \) be complex variables satisfying
\[ f_2^{(j)} = \frac{a_2 a_4 (a_1 + a_2 f_2^{(j)})}{a_2 a_4 (a_1 + a_2 f_2^{(j)})}, \quad a_1 a_2 a_3 a_4 = q, \tag{3.19a} \]
\[ a_0 a_1^2 f_1^{(j)} = a_3 a_4 (a_1 + a_3 f_1^{(j)}), \quad a_2 a_3^2 f_1^{(j)} = a_0 a_1 (a_3 + a_0 f_1^{(j)}), \tag{3.19b} \]
where \( j \in \mathbb{Z}/5\mathbb{Z} \). Note that the number of \( f \)-variables is essentially two. Indeed, using the relations (3.19) we can express all \( f \)-variables only by \( f_1^{(1)} \) and \( f_2^{(5)} \). The transformation group \( \tilde{W}(A_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4, \sigma, \iota \rangle \) is defined by the actions on the parameters:
\[ s_i(a_j) = a_j a_i^{-1}, \quad \sigma(a_j) = a_{j+1}, \]
\[ \iota : (a_0, a_1, a_2, a_3, a_4, q) \mapsto (a_0^{-1}, a_4^{-1}, a_3^{-1}, a_2^{-1}, a_1^{-1}, q^1), \]
where \( i, j \in \mathbb{Z}/5\mathbb{Z} \) and \( (C_{ij})_{i,j=0}^{5} \) is the Cartan matrix of type \( A_4^{(1)} \):
\[ (C_{ij})_{i,j=0}^{5} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \]
and those on the variables:
\[ s_j(f_2^{(j+3)}) = f_2^{(j+3)} , \quad s_j(f_2^{(j+3)}) = f_2^{(j+3)} , \quad s_j(f_1^{(j)}) = f_1^{(j)} = \frac{a_2 a_4 (a_1 + a_2 f_2^{(j+2)})}{a_2 a_4 (a_1 + a_2 f_2^{(j+2)})}, \]
\[ s_j(f_2^{(j+2)}) = \frac{a_2 a_4 (a_2 + a_2 f_2^{(j+2)}) + g_1 (a_2 f_1^{(j+2)})}{a_2 a_4 (a_2 + a_2 f_2^{(j+2)})}, \]
\[ s_j(f_2^{(j+4)}) = \frac{a_2 a_4 (a_2 + a_3 f_2^{(j+4)})}{a_2 a_4 (a_2 + a_3 f_2^{(j+4)})} , \]
\[ s_j(f_2^{(j)}) = a_2 a_4 (a_2 + a_2 f_2^{(j+1)}) + a_2 a_4 (a_2 f_1^{(j+2)}) , \]
\[ \iota(f_1^{(j)}) = f_1^{(j+1)} , \quad \iota(f_2^{(j)}) = f_2^{(j+1)} = \frac{a_2 a_4 (a_2 + a_2 f_2^{(j+1)})}{a_2 a_4 (a_2 + a_2 f_2^{(j+1)})} , \]
where \( j \in \mathbb{Z}/5\mathbb{Z} \). See Remark 2.1 for the convention on how to write these actions. The transformation group \( \tilde{W}(A_4^{(1)}) \) satisfies the fundamental relations of the extended affine Weyl group of type \( A_4^{(1)} \) (12, 29):
\[ s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad (s_i s_j)^2 = 1, \quad j \neq i \pm 1, \tag{3.20a} \]
\[ \sigma^2 = 1, \quad \sigma s_i = s_{i+1} \sigma, \quad \iota^2 = 1, \quad \iota s_{i=0,1,2,3,1} = s_{i=0,4,3,2,1}, \tag{3.20b} \]
where \( i, j \in \mathbb{Z}/5\mathbb{Z} \).

Define the translations \( T_1 \) and \( T_2 \) by
\[ T_1 = \sigma s_4 s_3 s_2 s_1, \quad T_2 = \sigma s_1 s_0 s_4 s_3, \tag{3.21} \]
whose actions on the parameters are given by

$$T_1 : (a_0, a_1, a_2, a_3, a_4) \mapsto (qa_0, q^{-1}a_1, a_2, a_3, a_4),$$  \hspace{1cm} (3.22a)  
$$T_2 : (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, a_1, qa_2, q^{-1}a_3, a_4).$$  \hspace{1cm} (3.22b)  

Note that the translations $T_1$ and $T_2$ commute with each other and the parameter $q$ is invariant under the action of each translation.

Define the $f$-functions by

$$f_{1m}^{(1)} = T_1^1 T_2^m (f_1^{(1)}), \quad f_{1m}^{(3)} = T_1^1 T_2^m (f_1^{(3)}), \quad f_{1m}^{(5)} = T_1^1 T_2^m (f_1^{(5)}),$$

where $l, m \in \mathbb{Z}$, from (3.19) satisfy

$$f_{1m}^{(3)} = f_{1m}^{(1)} = \frac{a_0 a_1 (q^{-m} a_3 + q^m a_3 f_{1m}^{(3)})}{q^{-m} a_2 a_3^2}.$$  \hspace{1cm} (3.24)  

The $f$-functions satisfy the $q$-difference equation for $l$-direction:

$$f_{1l+1,m}^{(3)} - f_{1l,m}^{(3)} = q^{-m} a_2 (q^{-m} a_3 + a_0 a_1 f_{1l+1,m}^{(3)})(q^{-m} a_3 + a_0 f_{1l,m}^{(3)})$$  \hspace{0.5cm} \text{and that for $m$-direction:}  

$$f_{1,m+1}^{(3)} - f_{1,m}^{(3)} = q^{-m} a_2 (q^{-m} a_3 + a_0 a_1 f_{1,m+1}^{(3)})(q^{-m} a_3 + a_0 f_{1,m}^{(3)}).$$  \hspace{1cm} (3.26)  

Note that Equations (3.25) and (3.26) follow from

$$T_1 (f_{1}^{(3)})_{1} = \frac{a_3 (a_1 + a_2 a_3 f_1^{(1)}) f_1^{(3)}}{a_0 a_1 a_2 a_3 (a_2 a_3 + a_0 f_1^{(3)})},$$
$$T_1^{-1} (f_{1}^{(1)})_{1} = \frac{a_3 (a_1 + a_2 a_3 f_1^{(1)}) f_1^{(3)}}{a_0 a_1 a_2 a_3 (a_2 a_3 + a_0 f_1^{(3)})},$$
$$T_2 (f_{1}^{(5)})_{1} = \frac{a_0 a_1 a_2 a_3 f_1^{(5)}}{a_0 a_1 a_2 a_3 f_1^{(3)}},$$
$$T_2^{-1} (f_{1}^{(3)})_{1} = \frac{a_0 a_1 a_2 a_3 f_1^{(5)}}{a_0 a_1 a_2 a_3 f_1^{(3)}},$$

respectively.

**Remark 3.2.** Each of equations (3.25) and (3.26) is equivalent to the $q$-$P_Y$ (A.3). The correspondence between Equations (3.25) and (A.3) is given by

$$F = f_{1m}^{(1)}, \quad G = f_{1m}^{(3)}, \quad \overline{F} = f_{1,m+1}^{(1)}, \quad \overline{G} = f_{1,m+1}^{(3)},$$
$$t = q^m a_1 a_2 a_3^{-1}, \quad c_1 = q^{-1} a_2 a_1, \quad c_2 = a_2 a_3^{-1}, \quad c_3 = q^{-m} a_2 a_3 a_1^{-1},$$

while that between Equations (3.26) and (A.3) is given by

$$F = f_{1m}^{(3)}, \quad G = f_{1m+1}^{(5)}, \quad \overline{F} = f_{1,m+1}^{(5)}, \quad \overline{G} = f_{1,m+1}^{(5)},$$
$$t = q^m a_1 a_2 a_3^{-1}, \quad c_1 = q^{-1} a_2 a_3, \quad c_2 = q^{-1} a_4 a_0^{-1}, \quad c_3 = q^{-m} a_2 a_3 a_0^{-1}.$$  \hspace{1cm} (3.29)  

(3.30)
Letting
\[ u = \frac{1 - \alpha \beta \gamma \delta}{\eta \zeta \eta \zeta \eta \zeta}, \] (3.31)
we can verify that the following relation holds:
\[ T_1 T_2(u) - u = \frac{\eta \zeta \eta \zeta \eta \zeta}{\eta \zeta \eta \zeta} - \frac{\eta \zeta \eta \zeta}{\eta \zeta}. \] (3.32)
Therefore, applying \( T_1 T_2 \) to the equation above we obtain
\[ U_{l+1,m+1} - U_{l,m} = \frac{\eta \zeta \eta \zeta \eta \zeta}{\eta \zeta \eta \zeta} - \frac{\eta \zeta \eta \zeta}{\eta \zeta}, \] (3.33)
where \( U_{l,m} = T_1 T_2(u) \), which is equivalent to the multiplicative dKdV equation \( (1.4) \)
with the following correspondence:
\[ u_{l,m} = U_{l,m}, \quad \alpha = \eta \zeta \eta \zeta \eta \zeta, \quad \beta = \eta \zeta \eta \zeta, \quad \epsilon = \eta \zeta. \] (3.34)
Therefore, setting\[ d_1 = \alpha \beta \gamma \delta, \quad d_2 = \alpha \beta \gamma \delta, \quad f_{l,m} = d_1 \eta \zeta (1), \quad g_{l,m} = d_1 \eta \zeta (2), \] (3.35)
and using Equations \( (3.24), (3.25), (3.26) \) and \( (3.31) \) with the correspondence \( (3.34) \), we have completed the proof of Theorem \( 1.4 \).

3.3. **Proof of Theorem 1.5**

In this subsection using the transformation group \( \hat{W}(D_5^{(1)}) \) we give a proof of Theorem 1.5.

Let \( \alpha_i (i = 0, \ldots, 5) \) and \( \beta \) be complex parameters satisfying
\[ \alpha_0 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 = \beta, \] (3.36)
and \( f_0, f_1 \) be complex variables. The transformation group \( \hat{W}(D_5^{(1)}) = (s_0, \ldots, s_5, s_1, s_2) \)
is defined by the actions on the parameters:
\[ s_i(a_i) = \alpha_i a_i^{-c_i}, \]
\[ s_1 : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta) \mapsto (\alpha_5^{-1}, \alpha_1^{-1}, \alpha_3^{-1}, \alpha_2^{-1}, \alpha_1^{-1}, \alpha_0^{-1}, \beta^1), \]
\[ s_2 : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta) \mapsto (\alpha_5^{-1}, \alpha_0^{-1}, \alpha_2^{-1}, \alpha_3^{-1}, \alpha_5^{-1}, \beta^1), \]
where \( i, j \in \mathbb{Z}/6\mathbb{Z} \) and \( (C_i)_{j=0}^5 \) is the Cartan matrix of type \( D_5^{(1)} \):
\[ (C_i)_{j=0}^5 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 2 & 2 \end{pmatrix} \]
and those on the variables:
\[ s_2(f_1) = \frac{f_1(\alpha_0 f_0 + \alpha_1 \alpha_2)}{\alpha_0 \alpha_2}, \quad s_3(f_0) = \frac{f_0(\alpha_5 \alpha_1 f_1 + \alpha_5)}{\alpha_5 \alpha_1 f_1}, \quad s_5(f_0) = \frac{f_0(\alpha_5 \alpha_1 f_1 + \alpha_5)}{\alpha_5 \alpha_1 f_1}, \]
\[ s_1(f_0) = f_1, \quad s_1(f_1) = f_0, \quad s_2(f_1) = f_1, \quad s_3(f_1) = f_0, \quad s_4(f_1) = \frac{1}{f_1}. \]
See Remark 2.11 for the convention on how to write these actions. The transformation group \( W(D_5^{(1)}) \) satisfies the fundamental relations of the extended affine Weyl group of type \( D_5^{(1)} \) [26,27,30]:

\[
s_i^2 = 1, \quad (s_i s_j)^2 = 1 \quad (\text{if } C_{ij} = 0), \quad (s_i s_j)^3 = 1 \quad (\text{if } C_{ij} = -1),
\]

\[
\sigma_1^2 = \sigma_2^2 = (\sigma_1 \sigma_2)^3 = 1, \quad \sigma_1 s_{[0,1,2,3,4,5]} = s_{[5,4,3,2,1,0]} \sigma_1,
\]

\[
\sigma_2 s_{[0,1,2,3,4,5]} = s_{[1,0,2,3,4,5]} \sigma_2.
\]

where \( i, j \in \mathbb{Z}/6\mathbb{Z} \).

Define the translations \( T_1 \) and \( T_2 \) by

\[
T_1 = (\sigma_2 g_1 s_{[0,1,3,2,3,4]} s_{[0,1,3,2,3,4]}), \quad T_2 = (\sigma_2 g_1 s_{[1,3,2,3,4,5]} s_{[0,1,3,2,3,4]}),
\]

whose actions on the parameters are given by

\[
T_1 : (a_0, a_1, a_2, a_3, a_4, a_5) \mapsto (a_0, a_1, a_2, a_3, q^{-1} a_4, q a_5), \quad (3.39a)
\]

\[
T_2 : (a_0, a_1, a_2, a_3, a_4, a_5) \mapsto (q^{-1} a_0, q a_1, a_2, a_4, a_3, a_5). \quad (3.39b)
\]

Note that the translations \( T_1 \) and \( T_2 \) commute with each other and the parameter \( q \) is invariant under the action of each translation.

Let us introduce the additional variables \( g_0 \) and \( g_1 \) by

\[
g_0 = \frac{(a_0^3 a_1 a_2^2 + f_0) f_1}{a_0 (a_1 + a_0 a_2^2 f_0)}, \quad g_1 = \frac{a_1^2 a_3^2 a_5 + f_1}{a_1 (a_5 + a_3^3 a_4 f_1) f_0}.
\]

(3.40)

Moreover, we define

\[
f_i^{l,m} = T_1^{-1} T_2^{-m} (f_i), \quad g_i^{l,m} = T_1^{-1} T_2^{-m} (g_i), \quad (i = 0, 1)
\]

(3.41)

These functions satisfy the \( q \)-difference equation for \( l \)-direction:

\[
g_{i+1,m}^{l,m} \equiv g_{i}^{l+1,m} = \frac{a_4^3 (1 + q^{2l} a_3^2 a_5^3 f_{i}^{l,0})(1 + q^{2l} a_3^{-2} a_5^{-3} f_{i}^{l,0})}{(q^{2l} a_3 + a_3^{-1} a_5 f_{i}^{l,0})(q^{2l} a_3 + a_3^3 a_4 f_{i}^{l,0})},
\]

\[
f_{i+1,m}^{l,m} \equiv f_{i}^{l+1,m} = \frac{a_5^2 (1 + q^{2l+1} a_3 a_5 f_{i}^{l,0})(1 + q^{2l+1} a_3^{-1} a_5^{-1} a_4 f_{i}^{l,0})}{(q^{2l+1} a_3 + q^{2l+1} a_3 a_4 f_{i}^{l,0})(q^{2l+1} a_3 + q^{2l+1} a_3^3 a_4 f_{i}^{l,0})},
\]

(3.43)

and that for \( m \)-direction:

\[
g_{0,m+1}^{l,m} \equiv g_{0}^{l,m+1} = \frac{a_6^2 (1 + q^{2m} a_3 a_5 f_{0}^{l,0})(1 + q^{2m} a_3^{-1} a_5^{-1} a_4 f_{0}^{l,0})}{(q^{2m} a_3 + a_3^{-2} f_{0}^{l,0})(q^{2m} a_3 + a_3^2 a_4 f_{0}^{l,0})},
\]

\[
f_{0,m+1}^{l,m} \equiv f_{0}^{l,m+1} = \frac{a_7^2 (1 + q^{2m+1} a_3 a_5 f_{0}^{l,0})(1 + q^{2m+1} a_3^{-1} a_5^{-1} a_4 f_{0}^{l,0})}{(q^{2m+1} a_3 + q^{2m+1} a_3 a_4 f_{0}^{l,0})(q^{2m+1} a_3 + q^{2m+1} a_3^2 a_4 f_{0}^{l,0})}.
\]

(3.44)

Note that Equations (3.43) and (3.44) follow from

\[
\begin{aligned}
T_1 (g_i) g_1 &= \frac{(1 + a_2^3 a_4 a_5 f_i)(a_2^3 a_4 a_5 + f_i)}{a_2 a_5 (a_3 a_5 + a_4 f_i)(a_5 + a_3^3 a_4 f_i)}, \\
T_1^{-1} (f_i) f_1 &= \frac{a_0 (a_3 a_5 f_i)^2 + a_1 g_1 (a_0 a_3 a_5 f_i)^2 + g_1)}{a_1 (a_0 + a_1 a_3 a_5 f_i)(1 + a_3 a_5 f_i)(1 + a_3 a_5 f_i)}, \\
T_2 (g_0) g_0 &= \frac{(1 + a_0^3 a_2 a_5 f_0)(a_0 a_2 a_5 f_0)}{a_0 a_3 (a_2 a_5 + a_0 f_0)(a_3 + a_2^2 a_5 f_0)}, \\
T_2^{-1} (f_0) f_0 &= \frac{a_5 (a_2 a_5 f_0)^2 + a_5 g_0 (a_2 a_5 f_0)^2 + g_0)}{a_5 (a_2 a_5 f_0)(1 + a_2 a_5 f_0)}, \\
\end{aligned}
\]

(3.45)

(3.46)
respectively.

**Remark 3.3.** Each of equations (3.43) and (3.44) is equivalent to the q-P_v1 (A.6). The correspondence between Equations (3.43) and (A.6) is given by

\[
F = \frac{a_4}{q^{2n}a_1} f_{lm}, \quad G = \frac{a_4}{q^{2m}a_1} g_{lm}, \quad \mathcal{T} = \frac{a_4}{q^{2n+1}a_1} f_{lm+1}, \quad \mathcal{C} = \frac{a_4}{q^{2m+1}a_1} g_{lm+1},
\]

\[
t = \frac{q^\alpha a_2^{1/2}}{a_1^{1/2}}, \quad c_1 = (a_1 a_2)^2, \quad c_2 = a_2^2, \quad c_3 = q^{-2a_2^2}, \quad c_4 = q^{-2a_2^2}.
\]

while that between Equations (3.44) and (A.6) is given by

\[
F = \frac{a_0}{q^{2n-1}a_1} f_{lm}, \quad G = \frac{a_0}{q^{2m-1}a_1} g_{lm}, \quad \mathcal{T} = \frac{a_0}{q^{2n+1}a_1} f_{lm+1}, \quad \mathcal{C} = \frac{a_0}{q^{2m+1}a_1} g_{lm+1},
\]

\[
t = \frac{q^\beta a_2^{1/2}}{a_0^{1/2}}, \quad c_1 = (a_0 a_2)^2, \quad c_2 = a_2^2, \quad c_3 = q^{-2a_2^2}, \quad c_4 = q^{-2a_2^2}.
\]

Letting

\[
u = \frac{a_1 a_2^2 a_3^3 a_4^4 - 1}{a_1 a_2^2 a_3^3 a_4^4} \left(1 + \frac{a_1 a_2^2 a_3^3 a_4^4}{1 + \frac{a_1 a_2^2 a_3^3 a_4^4}{a_1 a_2^2 a_3^3 a_4^4} f_0} \right),
\]

we can verify that the following relation holds:

\[
T_1^T_2(u) - u = \frac{1 - q^\alpha a_1 a_2^2 a_3^3 a_4^4 (a_4^4 + a_3^4 - q^\alpha a_1 a_2^2 a_3^3 a_4^4)}{q^\alpha a_1 a_2^2 a_3^3 a_4^4 T_2(u)} + \frac{a_1 a_2^2 a_3^3 a_4^4}{T_1(u)}.
\]

Therefore, applying \(T_1^T_2^{-m}\) to the equation above we obtain

\[
U_{l+1, m+1} - U_{l, m} = \frac{B_{m+1} - A_{l}}{U_{l, m+1}} - \frac{B_{m} - A_{l+1}}{U_{l+1, m}},
\]

where \(U_{l, m} = T_1^T_2^{-m}(u)\) and

\[
A_{l} = q^{-2l} a_2^{2l} (1 - q^{2l} a_2^4)(a_4^4 a_3^{-4} - q^{2l} a_2^4),
\]

\[
B_{m} = q^{-2m} a_1^{-2m} a_2^{-4} a_3^{-4} (1 - q^{2m} a_1^{-2m} a_2^{-4} a_3^{-4})(1 - q^{2m} a_1^{-2m} a_2^{-4} a_3^{-4}).
\]

Equation (3.51) is equivalent to the multiplicative dKdV equation (1.5) and the correspondence is given by the following:

\[
u_{l, m} = U_{l, m}, \quad \alpha_l = q^{-2l} a_2^4, \quad \beta_m = q^{-2m} a_1^{-2m} a_2^{-4}, \quad \gamma = (a_1 a_2)^4, \quad \epsilon = q^4.
\]

Therefore, setting

\[
x_{lm} = \frac{e^{1/4} f_{lm}}{a_1^{1/2}}, \quad y_{lm} = \frac{e^{1/2} g_{lm}}{a_1^{1/2}},
\]

and using Equations (3.42), (3.43), (3.44) and (3.49) with the correspondence (3.54), we have completed the proof of Theorem 1.5.
In this paper, we have constructed the special solutions to the multiplicative dKdV equations (1.4) and (1.5). The distinctive feature of these solutions is that along each direction for \( l \in \mathbb{Z} \) and \( m \in \mathbb{Z} \) they are represented by discrete Painlevé transcedents (see Theorems 1.2-1.5). Although not explicitly mentioned, special solutions with similar features for another 2-dimensional lattice equation can be found in [20]. In [20], each of such solutions is constructed by imposing a periodic condition on a system of 2-dimensional lattice equations. It is a matter of future work to consider what kind of constraint is imposed on the special solutions obtained in this paper.

All the \( q \)-Painlevé equations treated in this paper arise from the translations of the corresponding extended affine Weyl groups. However, \( q \)-Painlevé equations and the \( \text{lmKdV} \) equation arise not only from translations but also arise from non-translation elements [11-13]. This implies that we can also construct discrete Painlevé transcendent solutions to the multiplicative type dKdV equations using non-translations. The results in this direction will be reported in forthcoming publications.

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**Appendix A. \( q \)-Painlevé Equations of Types \( A_3^{(1)}, A_4^{(1)}, A_5^{(1)} \), and \( A_6^{(3)} \)**

We here list some typical examples of \( q \)-Painlevé equations of types \( A_J^{(1)} \) \((J = 3, 4, 5, 6)\). Note that in the following \( t \in \mathbb{C}^* \) plays the role of an independent variable, \( F(t), G(t) \in \mathbb{C} \) play the roles of dependent variables and \( c_i, q \in \mathbb{C}^* \) play the roles of parameters. Moreover, we adopt the following shorthand notations for the dependent variables:

\[
F = F(t), \quad G = G(t), \quad \overline{F} = F(qt), \quad \overline{G} = G(qt), \quad F = F(q^{-1}t). \tag{A.1}
\]

\( A_6^{(1)} \)-type

\[
q\text{-P}_\text{II} : \quad \overline{F} F - \frac{1}{t} \left( F F - \frac{q}{t} \right) = \frac{c_1 F}{t(F + 1)} \tag{A.2}
\]

\[
q\text{-P}_{\text{III}}^{(0)} : \quad \overline{G} G = \frac{1 + t^{-1} F}{F(1 + c_1^{-1} F)}, \quad \overline{F} F = \frac{c_1(1 + \overline{G})}{\overline{G}^2} \tag{A.3}
\]

\( A_5^{(1)} \)-type

\[
q\text{-P}_\text{III} : \quad \overline{G} G = \frac{c_1(1 + t F)}{F(t + F)}, \quad \overline{F} F = \frac{c_1(1 + c_2 \overline{G})}{\overline{G}(c_2 t + \overline{G})} \tag{A.4}
\]

\( A_4^{(1)} \)-type

\[
q\text{-P}_\text{V} : \quad \overline{G} G = \frac{(c_1 + t F)(c_2 + t F)}{c_3^2 (c_3 + F)}, \quad \overline{F} F = \frac{c_3^2 (c_3^{-1} + q \overline{G})(c_1 c_2 c_3^{-2} + i \overline{G})}{q t^2 (c_3^{-1} + \overline{G})} \tag{A.5}
\]

\( A_3^{(1)} \)-type

\[
q\text{-P}_{VI} : \quad \overline{G} G = \frac{(F + c_1 t^{-1})(F + c_1^{-1} t^4)}{(F + c_2)(F + c_2^{-1})}, \quad \overline{F} F = \frac{(\overline{G} + q^2 c_3 t^{-4})(\overline{G} + q^2 c_3^{-1} t^4)}{(\overline{G} + c_4)(\overline{G} + c_4^{-1})} \tag{A.6}
\]
Remark A.1. Equations (A.2)–(A.6) are known as a $q$-discrete analogue of the Painlevé II equation [24], that of the Painlevé III equation of type $D_4^{(1)}$ [25], that of the Painlevé III equation [26], that of the Painlevé V equation [27] and that of the Painlevé VI equation [28], respectively.

The following is a list of correspondences between the $q$-Painlevé equations (A.3)–(A.6) and those in Theorems 1.2–1.5.

Equations (A.3) and (1.11):

$$F = h_{l,m}, \quad G = g_{l-1,m}, \quad \overline{F} = h_{l+1,m}, \quad \overline{G} = g_{l,m}, \quad t = a_{l+2}, \quad q = \epsilon, \quad c_1 = \beta_{m+2}. \quad (A.7)$$

Equations (A.3) and (1.12):

$$F = h_{l,m}, \quad G = f_{l,m}, \quad \overline{F} = h_{l+1,m}, \quad \overline{G} = f_{l+1,m}, \quad t = \beta_{m+2}, \quad q = \epsilon, \quad c_1 = a_{l+2}. \quad (A.8)$$

Equations (A.4) and (1.15):

$$F = f_{l,m}, \quad G = g_{l,m}, \quad \overline{F} = f_{l+1,m}, \quad \overline{G} = g_{l+1,m}, \quad t = \alpha_l^{-1/2}, \quad q = \epsilon^{-1/2}, \quad c_1 = d^2, \quad c_2 = \beta_{m-1/2}. \quad (A.9)$$

Equations (A.4) and (1.17):

$$F = g_{l,m}, \quad G = h_{l,m}, \quad \overline{F} = g_{l,m+1}, \quad \overline{G} = h_{l,m+1}, \quad t = \alpha_l^{1/2} \beta_{m-1/2}, \quad q = \epsilon^{-1/2}, \quad c_1 = d^2, \quad c_2 = \alpha_l^{-1/2}. \quad (A.10)$$

Equations (A.5) and (1.19):

$$F = \frac{f_{l,m}}{d_2^2}, \quad G = \frac{d_2^2 g_{l,m}}{d_1^2}, \quad \overline{F} = \frac{f_{l+1,m}}{d_2^2}, \quad \overline{G} = \frac{d_2^2 g_{l+1,m}}{d_1^2}, \quad t = \frac{\alpha_l}{\beta_m}, \quad q = \epsilon, \quad c_1 = \frac{d_1^2}{\epsilon}, \quad c_2 = \frac{1}{d_2^2}, \quad c_3 = \frac{d_1^2 \beta_m}{d_2^2}. \quad (A.11)$$

Equations (A.5) and (1.21):

$$F = \frac{d_2^2 g_{l,m}}{d_1^2}, \quad G = \frac{h_{l,m}}{d_2^2}, \quad \overline{F} = \frac{d_2^2 g_{l,m+1}}{d_1^2}, \quad \overline{G} = \frac{h_{l,m+1}}{d_2^2}, \quad t = d_1^2 \beta_{m-1}, \quad q = \epsilon, \quad c_1 = \frac{d_2^2}{\epsilon}, \quad c_2 = \frac{d_2^2}{\alpha_l+1}, \quad c_3 = \frac{d_2^2}{d_1^2 \alpha_l}. \quad (A.12)$$

Equations (A.6) and (1.23):

$$F = f_{l,m}, \quad G = g_{l,m}, \quad \overline{F} = f_{l+1,m}, \quad \overline{G} = g_{l+1,m}, \quad t = \frac{\alpha_l^{1/4}}{\gamma^{1/8}}, \quad q = \epsilon^{1/4}, \quad c_1 = \frac{d_2^{1/2} \gamma^{1/2}}{d_1}, \quad c_2 = \frac{d_2^{1/2}}{d_1}, \quad c_3 = \frac{\beta_m^{1/2}}{d_2^{1/2}}, \quad c_4 = \frac{\epsilon^{1/2}}{d_2^{1/2} \gamma^{1/2} \epsilon^{1/8}}. \quad (A.13)$$

Equations (A.6) and (1.25):

$$F = x_{l,m}, \quad G = y_{l,m}, \quad \overline{F} = x_{l,m+1}, \quad \overline{G} = y_{l,m+1}, \quad t = \frac{\beta_m^{1/4} \gamma^{1/8}}{\epsilon^{1/8}}, \quad q = \epsilon^{1/4}, \quad c_1 = \frac{\epsilon^{1/2} d_1}{d_2 \gamma^{1/2}}, \quad c_2 = d_1, \quad c_3 = \frac{\beta_m^{1/2}}{d_2^{1/2}}, \quad c_4 = \alpha_l^{1/2}. \quad (A.14)$$
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