Shear Viscosity of hadronic matter at finite temperature and magnetic field

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We calculate the transport coefficient of hadronic matter in the presence of temperature and magnetic field using the linear sigma model. In the relaxation time approximation, we estimate the shear viscosity over entropy density $\eta/s$. The point-like interaction rates of hadrons are evaluated through the $S$-matrix approach in the presence of a magnetic field to obtain the temperature and magnetic field-dependent relaxation time. We observe that the transport coefficients are anisotropic in the presence of the magnetic field. We calculate the temperature and magnetic field-dependent anisotropic shear viscosity coefficients by incorporating the estimated relaxation time. The value of viscosity over entropy density is lower in the presence of a magnetic field than the value of it in a thermal medium. The behavior of the perpendicular components of the shear-viscous coefficient is also discussed. We consider the temperature-dependent hadron masses from mean-field effects in this work.

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I. INTRODUCTION

In relativistic heavy-ion collision experiments at the Large Hadron Collider (LHC) and Relativistic Heavy Ion Collider (RHIC), a novel state of quarks and gluons, i.e., quark-gluon plasma (QGP) \[1\] is produced as a near-perfect fluid \[2–4\]. The elliptic flow \[5, 6\] data indicates the smallest viscosity to entropy density ratio (\(\eta/s\)) of the QGP medium. The produced QGP medium shows the collective behavior, and it undergoes space-time evolution and finally emanates to the hadronic phase. The transverse momentum spectra and the collectivity of the produced particles can be studied from the hydrodynamical modeling \[6\]. The transport coefficients are used as the input parameters for the hydrodynamic simulations.

More research interests have grown in the non-central heavy-ion collisions through the last decade. Several studies \[7, 8\] suggest that a strong magnetic field is produced in non-central heavy-ion collisions in the perpendicular direction to the reaction plane. Initially, at the time of the collisions, the magnitude of the produced magnetic field can be of the order \(10^{18}\) G at RHIC and \(10^{19}\) G at LHC \[9, 10\]. The magnitude of the produced magnetic field depends on several parameters such as impact parameter, the conductivity of the medium, collision energy, etc. The strong field created in the heavy-ion collisions (HIC) decreases sharply with time \[11\]. However, some studies \[12–14\] have proclaimed that the presence of finite electric conductivity of the medium can extend the lifetime of the magnetic field. In recent times, the various properties of hot and dense matter have been investigated in the presence of a finite magnetic field. Different novel phenomena like chiral magnetic effect (CME) \[15\], magnetic catalysis \[16\], inverse magnetic catalysis at finite temperature, thermodynamic properties \[17, 18\], properties of quarkonia \[19, 20\], dilepton production \[12, 21–23\], chiral susceptibility \[24\], photon damping rate \[25\] and so on have been studied over the last few years. In the presence of a magnetic field, magnetohydrodynamics (MHD) simulations \[26–28\] have been developed to describe the fluid dynamics of strongly interacting matter. In this context, the transport coefficients are relevant quantities to study, namely, shear viscosity \[29–32\], bulk viscosity \[33–35\] and electrical conductivity \[29, 36–38\] in hadronic and quark matter in presence of the constant magnetic field. All the transport coefficients become anisotropic in the magnetic field, and one gets five coefficients of shear viscosity, three coefficients for conductivity, and two bulk viscosity coefficients \[38\].

This article evaluates the shear viscosity coefficients of hadronic matter in a strong magnetic field using the linear sigma model (LSM). The LSM is one of the simple models to study the hadronic system and was first introduced by Gell-Mann and Lévy \[39\]. Several works have been
done considering this as a low-energy effective model during the last few years as it mimics the low-energy QCD region. Recent attempts have extended the LSM by including quarks [40, 41] and vector mesons in this model [42]. Chiral phase transition [43], pion condensate [44] and neutral pion mass [41] in the presence of the external magnetic field and so on have been studied using the LSM. In Refs. [45, 46] the authors have calculated the transport coefficients of hadronic matter at finite temperature using the LSM. The results show that the shear viscosity to entropy density ratio ($\eta/s$) has a minimum at the crossover temperature. In contrast, the bulk viscosity to entropy density ratio ($\xi/s$) has a maximum at the crossover temperature. As a first attempt, we study the viscous shear coefficient of hadronic matter in a magnetic field for zero chemical potential in this present work. There are five shear-viscous coefficients in a nonzero magnetic field, and we have studied all the coefficients. In the presence of a magnetic field, the relaxation times are estimated through the S-matrix approach. In these calculations, we would get the expressions of the matrix elements in terms of the Landau level summation. Our study considers only the lowest Landau level (LLL) contribution.

The paper is arranged as follows: In Sec. II we review the formalism for the estimation of the shear viscosity coefficients in the presence of the external magnetic field within relaxation time approximation. In Sec. III we discuss the basics of the linear sigma model (LSM) and the thermodynamic quantities that are used in the calculations. We have calculated the scattering amplitude and interaction rate in the presence of the magnetic field In subsection III B. Expressions for the interaction rate in the pure thermal medium are also discussed here. Incorporating the interaction rate, we finally obtain the anisotropic shear viscous coefficients, and presented is the result Sec. IV. Finally, we summarize in section V.

II. ANISOTROPIC VISCOSITY COEFFICIENTS IN NON ZERO MAGNETIC FIELD

We will study the transport properties of a hadronic medium in the presence of a magnetic field. Transport coefficients can be calculated using two popular approaches: kinetic theory [47] and Kubo framework [48–50]. Here we follow the former approach and briefly discuss this formalism in the relaxation approximation (RTA) technique [30, 51]. In a magnetic field, the transport coefficients for the charged particles become anisotropic, whereas the neutral particles contribute to the isotropic coefficients only.

In a magnetic field, the Boltzmann equation for single hadron species is written as

$$p^\mu \partial_\mu f_a + q E^{\mu\nu} p_\nu \frac{\partial f_a}{\partial p^\mu} = C[f_a],$$

(1)
where $F^{\mu\nu}$ is the electromagnetic field tensor. In absence of electric field, $F^{\mu\nu} = -B b^{\mu\nu}$ where $b^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} b_{\alpha} u_{\beta}$ with fluid four velocity $u^\mu$. The unit vector $b^\mu$ is defined as $b^\mu = \frac{B^\mu}{B}$. $C[f_a]$ is known as the collision integral, $p^\mu$ is the four momenta of the particle and $q$ is the electric charge of the particle.

In relaxation time approximation (RTA), the Boltzmann equation 1 is given by

$$p^\mu \partial_\mu f_a + q F_{\mu\nu} p_\nu \frac{\partial f_a}{\partial p^\mu} = - \omega_a (u \cdot p) \delta f_a ,$$

where $\omega_a$ is frequency of interaction defined as the inverse of the equilibration time i.e.

$$\omega_a(E) = \tau_a^{-1}(E).$$

(3)

Assuming that the system is meagerly out of equilibrium, we can write the distribution function as,

$$f_a(x, p) = f^0_a (1 + \phi_a(x, p)) = f^0_a + \delta f_a .$$

(4)

For a small deviation from equilibrium, we can write the Boltzmann equation as

$$p^\mu \partial_\mu f^0_a = \left( - \frac{u \cdot p}{\tau_a} \right) \left( 1 - \frac{qB}{p} b^{\mu\nu} p_\nu \frac{\partial}{\partial p^\mu} \right) \delta f_a ,$$

where $f^0_a = \exp ( -u_\alpha p^\alpha / T )$ is equilibrium distribution function.

Now, in general, the energy-momentum tensor is written as

$$T^{\mu\nu} = T^\mu_0^{\nu} + \Delta T^{\mu\nu} ,$$

where $T^\mu_0^{\nu}$ represents the energy-momentum tensor in local equilibrium and $\Delta T^{\mu\nu}$ is the deviation from the equilibrium. $T^{\mu\nu}$ is given as

$$T^{\mu\nu} = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu P_a^{\nu}}{E_a} f_a + \sum_a \frac{|qB|}{2\pi} \int \frac{dp_3}{2\pi} \frac{p^\mu_3 p^{\nu}_3}{E_a} f_a ,$$

(7)

where the sum is over uncharged and charged particles in the first and second term respectively. In the second term phase factor is modified in presence of strong magnetic field.

For shear viscosity, in presence of magnetic field we can express $\delta f_a$ in terms of fourth rank projection tensors as

$$\delta f_a = \sum_{m=0}^{2} \epsilon^{\mu\nu\alpha\beta} C^{(m)}_{\mu\nu\alpha\beta} p^\mu p^\nu V^{\alpha\beta} ,$$

(8)

where $V^{\alpha\beta} = \frac{1}{2} (\partial^\alpha u^\beta + \partial^\beta u^\alpha)$. There are five complex coefficients $c_m$. 
Shear viscous tensor \( \pi^{\mu\nu} \) is given as

\[
\pi^{\mu\nu} = \eta^{\mu\nu\alpha\beta} V_{\alpha\beta},
\]

where the general form of \( \eta^{\mu\nu\alpha\beta} \) in presence of the magnetic field could be expressed with the fourth rank projection tensors as

\[
\eta^{\mu\nu\mu'\nu'} = \sum_{m=-2}^{2} c_m C^{(m)\mu\nu\mu'\nu'}. \tag{10}
\]

Here we introduce three second-rank projection tensors given as

\[
P^{(0)}_{\mu\nu} = b_\mu b_\nu, \tag{11}
\]
\[
P^{(1)}_{\mu\nu} = \frac{1}{2} (\Delta_{\mu\nu} - b_\mu b_\nu + i b_{\mu\nu}), \tag{12}
\]
\[
P^{(-1)}_{\mu\nu} = \frac{1}{2} (\Delta_{\mu\nu} - b_\mu b_\nu - i b_{\mu\nu}), \tag{13}
\]

where \( \Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \). Fourth-rank tensor defined in eq. (10) can be written in terms of two second-rank tensors as \([52, 53]\)

\[
P^{(m)}_{\mu\nu\alpha\beta} = \sum_{m_1=-1}^{1} \sum_{m_2=-1}^{1} P^{(m_1)}_{\mu\nu\alpha} P^{(m_2)}_{\mu\nu\beta} \delta(m,m_1 + m_2). \tag{14}
\]

In terms of real coefficients eq. (9) can be written as

\[
\eta_{\mu\nu\alpha\beta} = c^0 \mathcal{P}^{(0)}_{\mu\nu\alpha\beta} + \sum_{m=1}^{2} \left\{ c^{m+} \left( \mathcal{P}^{(m)}_{\mu\nu\alpha\beta} + \mathcal{P}^{(-m)}_{\mu\nu\alpha\beta} \right) + ic^{m-} \left( \mathcal{P}^{(m)}_{\mu\nu\alpha\beta} - \mathcal{P}^{(-m)}_{\mu\nu\alpha\beta} \right) \right\}, \tag{15}
\]

where \( \mathcal{P}^{(m)}_{\mu\nu\alpha\beta} = \mathcal{P}^{(m)}_{\mu\nu\beta\alpha} + \mathcal{P}^{(m)}_{\mu\nu\alpha\beta} \). The coefficients \( c^{m+} \) and \( c^{m-} \) are real and imaginary parts of coefficients \( c^m \). Three coefficients \( c^0, c^{1+} \) and \( c^{2+} \) are even functions of the magnetic field whereas other two coefficients \( c^{1-} \) and \( c^{2-} \) are odd functions of magnetic field. The shear viscosity obeys the condition \( \eta^{\mu\nu\mu'\nu'}(B^\alpha) = \eta^{\mu\nu\mu'\nu'}(-B^\alpha) \), the symmetry principle for transport coefficients.

Now we can represent eq. (8) with real coefficients i.e.

\[
\delta f_a = \left[ c^0 \mathcal{P}^{(0)}_{\mu\nu\alpha\beta} + \sum_{m=1}^{2} \left\{ c^{m+} \left( \mathcal{P}^{(m)}_{\mu\nu\alpha\beta} + \mathcal{P}^{(-m)}_{\mu\nu\alpha\beta} \right) + ic^{m-} \left( \mathcal{P}^{(m)}_{\mu\nu\alpha\beta} - \mathcal{P}^{(-m)}_{\mu\nu\alpha\beta} \right) \right\} \right] p^\mu p'^\nu V_{\alpha\beta}. \tag{16}
\]

In integral form we can write the shear viscous tensor as

\[
\pi^{\mu\nu} = \frac{1}{15} \sum_a \sum_{m=-2}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{(\vec{p})^4}{E_a} c_m C^{(m)\mu\nu\alpha\beta} V_{\alpha\beta}. \tag{17}
\]
The left-hand side of the eq. (5) has been written in terms of the projection operators as

\[- \frac{f^0}{2T} \langle \mu \nu \rangle_{\alpha \beta} \left[ \mathcal{D}_{\langle \mu \nu \rangle_{\alpha \beta}}^{(0)} + \mathcal{D}_{\langle \mu \nu \rangle_{\alpha \beta}}^{(1)} + \mathcal{D}_{\langle \mu \nu \rangle_{\alpha \beta}}^{(-1)} + \mathcal{D}_{\langle \mu \nu \rangle_{\alpha \beta}}^{(2)} + \mathcal{D}_{\langle \mu \nu \rangle_{\alpha \beta}}^{(-2)} \right]. \tag{18}\]

Substituting \(\delta f_a\) on the right hand side of eq (5), we get

\[- \frac{u \cdot p}{\tau_a} \left( 1 - \frac{qB\tau_a}{u \cdot p} \frac{\partial}{\partial p^\alpha} \right) \sum_{m=-2}^2 c^\alpha \mathcal{C}^{(m)}_{\mu \nu \alpha \beta} p^\mu p^\nu \mathcal{D}_{\langle \mu \nu \rangle_{\alpha \beta}}^{(m)}. \tag{19}\]

Equating eq. (18) and eq. (19) after writing the fourth rank tensors in terms of second tensors, we get (see Refs. [29, 30, 52])

\[c^0 = \frac{1}{2T (u \cdot p)} f^0 \tau_a, \]
\[c^{1+} = \frac{1}{2T (u \cdot p)^2 + (qB\tau_a)^2} f^0 \tau_a, \]
\[c^{2+} = \frac{1}{2T (u \cdot p)^2 + (2qB\tau_a)^2} f^0 \tau_a, \]
\[c^{1-} = \frac{1}{2T (u \cdot p)^2 + (qB\tau_a)^2} f^0 \tau_a, \]
\[c^{2-} = \frac{1}{2T (u \cdot p)^2 + (2qB\tau_a)^2} f^0 \tau_a. \tag{20}\]

Employing eq. (17) we can find the shear viscosity coefficients as

\[\eta_{\parallel} = \frac{2}{15} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a} c^0 = \frac{1}{15T} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a} f^0 \tau_a, \tag{21}\]
\[\eta_{\perp} = \frac{2}{15} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a} c^{1+} = \frac{1}{15T} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{(u \cdot p)^2 + (qB\tau_a)^2} f^0 \tau_a, \tag{22}\]
\[\eta'_{\perp} = \frac{2}{15} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a} c^{2+} = \frac{1}{15T} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{(u \cdot p)^2 + (2qB\tau_a)^2} f^0 \tau_a, \tag{23}\]
\[\eta_{\times} = \frac{2}{15} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a} c^{1-} = \frac{1}{15T} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a (u \cdot p)^2 + (qB\tau_a)^2} f^0 \tau_a, \tag{24}\]
\[\eta'_{\times} = \frac{2}{15} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a} c^{2-} = \frac{2}{15T} \sum_a \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^4}{E_a (u \cdot p)^2 + (2qB\tau_a)^2} f^0 \tau_a. \tag{25}\]

In the presence of a nonzero magnetic field, the shear stress tensor is written using the available basis (as discussed in this section), having a component parallel to the magnetic field. The subscript \(\parallel\) denotes this parallel component, the subscripts \(\perp\) and \(\times\) are the perpendicular and Hall components. In the absence of a magnetic field, the Hall component is zero, whereas the perpendicular component becomes the same as the parallel component.

From eq. A14, the solution of eq. A1 for charged particles looks like \(e^{-iP \cdot X} \exp[-iB/2(x - P/\hbar)] H_\nu(x - P/\hbar). \) To find out the density of state we consider box of length \([L_1, L_2, L_3]\) and infinite
The particle is localized in \( x \sim p_y/eB \). So the number of states in transverse area is \( \frac{L_1 L_2 eB}{2\pi} \). Now the number of states per unit volume in \( \Delta p_z \) interval becomes \( \sim \frac{L_1 L_2 eB}{(2\pi)^2} \). Using this argument, for the case of the charged particles we use the phase space [54] as

\[
\int \frac{d^3p}{(2\pi)^3} \rightarrow \frac{|eB|}{(2\pi)^2} \int dp_z. \quad (26)
\]

For the neutral particle, the Hall components of the viscosity coefficients are zero, and the perpendicular components are equal to the parallel parts.

In the presence of a finite magnetic field, the momentum of the charged particles becomes anisotropic. The energy dispersion relation of the charged scalar particle of charge \( \sigma \) gets modified as

\[
E_n = \sqrt{p_z^2 + (2n + 1)|qB| + m^2}, \quad (27)
\]

where \( p_z \) is the momentum of the particle parallel to the direction of the magnetic field, and \( m \) is the mass of that particle. Here \( n = 0, 1, 2, 3... \) denotes the Landau levels. In our case, we consider the magnetic field to be in the z-direction. We also work in a strong magnetic field limit, i.e., the magnetic field is much greater than the temperature square scale. In this approximation, we can safely consider the confinement of charged particles in the lowest Landau level (LLL), and the energy dispersion in LLL is given as

\[
E = \sqrt{p_3^2 + m^2 + |qB|}. \quad (28)
\]

Here, we define the notation \( \tilde{p}^\mu = (E, p^3) \). For the charged pions equilibrium distribution takes the form as \( f_a^0 = \exp \left(-u_\alpha \tilde{p}^\alpha/T\right) \). Now, the uncharged particles are not affected by the magnetic field, therefore the distribution is given by \( f_a^0 = \exp \left(-u_\alpha p^\alpha/T\right) \), where \( p^\mu = (E, \tilde{p}) \) with \( \omega = \sqrt{\tilde{p}^2 + m^2} \).

**III. LINEAR SIGMA MODEL**

The LSM model is a simplistic effective model of pions. Here we use it to calculate transport coefficients. In general, the LSM Lagrangian consists of \( N \) bosonic fields. For \( N = 4 \), it represents the theory of three pions (\( \pi_i \)) and one sigma (\( \sigma \)) fields. The LSM Lagrangian density [45, 55] for \( N = 4 \) is

\[
\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 - V(\sigma, \pi), \quad (29)
\]
where the potential term

\[ V(\sigma, \pi) = \frac{\lambda}{4}(\sigma^2 + \pi^2 - f^2)^2 - H\sigma. \]  

(30)

Here \( H\sigma \) is the explicit chiral symmetry breaking term that gives the pion mass. The scalar field \( \sigma \) takes the vacuum expectation value \( v \) as \( \sigma = v + \Delta \), where \( \Delta \) is the fluctuation and \( v \) is determined by the symmetry breaking term as

\[ \lambda v(v^2 - f^2) = H. \]  

(31)

Other parameters \( \lambda, H \) and \( f \) are expressed in terms of pion decay constant \( f_\pi \), pion masses \( (m_\pi) \) and sigma masses \( (m_\sigma) \) i.e.

\[ \lambda = \frac{m_\sigma^2 - m_\pi^2}{2f_\pi^2}, \]
\[ H = f_\pi m_\sigma^2, \]
\[ f^2 = f_\pi^2 \frac{m_\sigma^2 - 3m_\pi^2}{m_\sigma^2 - m_\pi^2}. \]  

(32)

For calculation we are taking vacuum pion mass \( m_\pi = 140 \text{ MeV} \), vacuum \( \sigma \) masses \( m_\sigma = \{500, 700\} \) MeV and decay constant \( f_\pi = 93 \text{ MeV} \).

We would continue our calculations in the isospin pion basis representing the physical pions. The physical pions can be expressed in terms of Cartesian pion fields as,

\[ \pi^0 = \pi_3, \]  
\[ \pi^\pm = \frac{1}{\sqrt{2}}(\pi_1 \pm i\pi_2). \]  

(33)

(34)

In physical pion basis interaction Lagrangian can be written as

\[ \mathcal{L}_{int} = \frac{\lambda}{4}\left(\sigma^4 + (\pi^0)^4 + (\pi^+)^4 + (\pi^-)^4 + 2(\pi^0)^2(\pi^+)^2 + 2(\pi^0)^2(\pi^-)^2 + 2(\pi^0)^2\sigma^2 + 2(\pi^+)^2(\pi^-)^2 + 2(\pi^+)^2\sigma^2 + 2(\pi^-)^2\sigma^2 + 4\nu\sigma(\pi^0)^2 + 4\nu\sigma(\pi^+)^2 + 4\nu\sigma(\pi^-)^2 + 4\nu^2\sigma^2\right). \]  

(35)

From the above interaction Lagrangian, one can find the probable interactions between the mesons.

As we are considering a magnetic field in \( z \)-direction, the magnetic field \( \vec{B} = B\hat{z} \). The covariant four-derivative \( D_\mu = \partial_\mu + QA_\mu \) replaces the four-derivative \( \partial_\mu \) in the kinetic terms of the Lagrangian for the charged pions. Here, \( Q = e \) for \( \pi^\pm \) and \( A^\mu = \{0, 0, XB, 0\} \).
A. Thermodynamics

The temperature dependence of the effective masses of pions and the condensate \( \nu \) is rigorously discussed in Refs. [56–58]. There is a significant difference between meson masses at low temperature, i.e., chiral symmetry is broken, and symmetry is restored at around 245 MeV. We are also considering only temperature dependence on effective masses in our case.

\[
\epsilon_B = \sum_{\alpha=\sigma,\pi^0} \int \frac{d^3p}{(2\pi)^3} \frac{|p|^2}{E_a} f^0(E_a/T) + \sum_{\alpha=\pi^\pm} \frac{|eB|}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{p_z^2}{E_a} f^0(E_a/T),
\]

where \( E_\sigma = \sqrt{p^2 + \bar{m}_\sigma^2}, \quad E_{\pi^0} = \sqrt{p^2 + \bar{m}_{\pi^0}^2} \) and \( E_{\pi^\pm} = \sqrt{p_z^2 + \bar{m}_{\pi^\pm}^2 + |eB|} \). Here \( \bar{m}(T) \) is considered as temperature dependent effective mass which comes from the mean field. Similarly, the pressure \( P \) and the entropy density \( s \) can be written as

\[
P_B = \sum_{\alpha=\sigma,\pi^0} \int \frac{d^3p}{(2\pi)^3} \frac{|p|^2}{E_a} f^0(E_a/T) + \sum_{\alpha=\pi^\pm} \frac{|eB|}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{p_z^2}{E_a} f^0(E_a/T),
\]

and

\[
s_B = \frac{1}{3T^2} \sum_{\alpha=\sigma,\pi^0} \int \frac{d^3p}{(2\pi)^3} |p|^2 f^0(E_a/T) + \frac{1}{3T^2} \sum_{\alpha=\pi^\pm} \frac{|eB|}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{p_z^2}{E_a} f^0(E_a/T).
\]
In fig. 1 energy density $\epsilon$ scaled with $T^4$ and entropy density $s$ scaled with $T^3$ are plotted for $m_\sigma = 500$ MeV and 700 MeV respectively. The dashed lines represent the values of the corresponding thermodynamic quantities in the presence of the magnetic field. In the presence of a magnetic field, both the energy and entropy density decrease.

B. Scattering amplitudes and interaction frequency

1. Thermal case

Here we present the interaction frequency for pure thermal medium. For the pure thermal medium, the matrix elements are \[ M_{fi}(\sigma \sigma|\pi \pi) = -6\lambda, \]
\[ M_{fi}(\pi^a \pi^b|\pi^a \pi^b) = -6\lambda, \quad \{a = 0, +, -\} \]
\[ M_{fi}(\pi^+ \pi^-|\pi^+ \pi^-) = -2\lambda, \]
\[ M_{fi}(\pi^0 \pi^0|\sigma \sigma) = -2\lambda, \]
\[ M_{fi}(\pi^b \pi^0|\pi^b \pi^0) = -2\lambda, \quad \{b = +, -\}. \]

The poles in the $s$ and $u$ channels cause issues in the scattering amplitudes, resulting in divergent integrals. The divergence can be cured by introducing the thermal width of mesons violating the crossing symmetries. These terms come from the three-point vertices, and we have excluded them in the equation of state. So we are avoiding those terms taking the infinity limits of $s$, $t$, and $u$.

Finally, we are left with the constant scattering amplitudes, and those are given as,

\[ M_{fi}(\sigma \sigma|\sigma \sigma) = -6\lambda, \]
\[ M_{fi}(\pi^a \pi^b|\pi^a \pi^b) = -6\lambda, \quad \{a = 0, +, -\} \]
\[ M_{fi}(\pi^+ \pi^-|\pi^+ \pi^-) = -2\lambda, \]
\[ M_{fi}(\pi^0 \pi^0|\sigma \sigma) = -2\lambda, \]
\[ M_{fi}(\pi^b \pi^0|\pi^b \pi^0) = -2\lambda, \quad \{b = +, -\}. \]

For $a + b \to c + d$ type interaction, interaction frequency $\omega_a = 1/\tau_a$ is written as \[ \omega_{th}(E_a) = E_a^{-1} - \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int \frac{d^3p_a d^3p_b d^3p_c d^3p_d |M(ab \to cd)|^2}{(2\pi)^8} \delta^4(p_a + p_b - p_c - p_d) f_b^{eq}. \]
In the centre of mass frame, the interaction frequency can be written in simplified form from eq. (46) as

\[
\omega_{\text{cm}}^2 = \frac{1}{256\pi^3 E_a} \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int_{m_n}^{\infty} dE_b \sqrt{E_b^2 - m_b^2} \int_{-1}^{1} \frac{dx}{p_{ab} \sqrt{s}} (t_{\text{max}} - t_{\text{min}}) |\mathcal{M}|^2 f_{\text{eq}}(E_b),
\]

where

\[
p_{ab}(s) = \frac{1}{2\sqrt{s}} \sqrt{\lambda(s, m_a^2, m_b^2)},
\]

with the kinematic function \(\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)\). Other quantities are defined as

\[
s = 2E_a E_b \left(1 + \frac{m_a^2 + m_b^2}{2E_a E_b} - \frac{p_a p_b}{E_a E_b} x \right),
\]

\[
t_{\text{max, min}} = m_a^2 + m_c^2 - \frac{1}{2s} (s + m_a^2 - m_b^2)(s + m_c^2 - m_d^2) \pm \frac{1}{2s} \sqrt{\lambda(s, m_a^2, m_b^2)\lambda(s, m_c^2, m_d^2)}. \tag{50}
\]

2. In the presence of magnetic field

A finite magnetic field would affect the scattering amplitudes containing charged pions as the charged pions interact with the magnetic field. In this section, we calculate the magnetic field-affect interaction rates. Here we also consider the four-point interactions. Starting from the S-matrix elements, we end up with the interaction rates of corresponding processes. The calculations of the matrix elements involve the Klein-Gordon solutions of the charged scalar particles in the presence of the magnetic field. The solutions to the Klein-Gordon equation are discussed in Appendix A. As we are confining ourselves in the strong magnetic field case, we have obtained the interaction rates only for the lowest Landau level.

The S-matrix element for \(\pi^b(\vec{p}_\pi, m) + \pi^b(\vec{k}_\pi, n) \rightarrow \pi^b(\vec{p}_\pi', m') + \pi^b(\vec{k}_\pi', n')\) scattering in presence of magnetic field is written as

\[
S_{fi} = 4! \frac{\lambda}{4} \int d^4 X (\pi^b(n', \vec{k}_\pi') \pi^b(m', \vec{p}_\pi')) (\pi^b(n, \vec{k}_\pi) \pi^b(m, \vec{p}_\pi)), \quad \{b = +, -\}
\]

\[
= 4! \frac{\lambda}{4} \int d^4 X \frac{e^{-i(P+K-P'-K') \cdot X}}{\sqrt{16E_n E_m E_n' E_m'(L_x L_z)^4}} f_n(x, \vec{k}_\pi)f_m(x, \vec{p}_\pi)f_{n'}(x, \vec{K}_\pi')f_{m'}(x, \vec{p}_\pi')
\]

\[
= (2\pi)^3 \delta^{(3)}(p + k' - k) \frac{1}{\sqrt{16E_n E_m E_n' E_m'(L_x L_z)^2}} \mathcal{M}_{fi}, \tag{51}
\]

where \(\delta^{(3)}\) implies the \(\delta\)-function for all the space-time coordinates except \(x\). In this case four-momentum conservation is not appearing through the delta function as the \(x\)-component of the momentum is not a good quantum number. The matrix element \(\mathcal{M}_{fi}\) from eq. (51) can be read as

\[
\mathcal{M}_{fi}(\pi^b n | \pi^b m) = 4! \frac{\lambda}{4} \int dx f_n(x, \vec{k}_\pi)f_m(x, \vec{p}_\pi)f_{n'}(x, \vec{K}_\pi')f_{m'}(x, \vec{p}_\pi'), \quad \{b = +, -\}. \tag{52}
\]
and similarly we can write the scattering amplitudes for $\pi^+(k') + \pi^-(p') \rightarrow \pi^+(k) + \pi^-(p)$ as

$$\mathcal{M}_{f_1}(\pi^+\pi^- | \pi^+\pi^-) = 2\lambda \int dx f_n(x, \vec{k}_x) f_m(x, \vec{p}_x) f_n^*(x, \vec{k}_x') f_m^*(x, \vec{p}_x'). \quad (53)$$

Other scattering amplitudes affected by magnetic fields are

$$\mathcal{M}_{f_1}(\pi^b\sigma | \pi^b\sigma) = 2\lambda \int dx e^{i(k_z-k'_z)x} f_n(x, \vec{k}_x) f_m^*(x, \vec{p}_x), \quad \{b = +, -\} \quad (54)$$

$$\mathcal{M}_{f_1}(\pi^b\pi^0 | \pi^b\pi^0) = 2\lambda \int dx e^{i(k_z-k'_z)x} f_n(x, \vec{k}_x) f_m^*(x, \vec{p}_x). \quad \{b = +, -\} \quad (55)$$

As we are considering strong magnetic field, we restrict ourselves to the lowest Landau levels. For $\pi^+(k_a) + \pi^+(k_b) \rightarrow \pi^+(k_c) + \pi^+(k_d)$ and $\pi^-(k_a) + \pi^-(k_b) \rightarrow \pi^-(k_c) + \pi^-(k_d)$ scatterings, the interaction frequencies of "$\pi^b(b = \pm)$" for these processes are given by

$$\omega^b_i = \frac{1}{2} \int \frac{dk_y^b}{(2\pi)^2} \frac{dk_z^b}{(2\pi)^2} \frac{dk_z^c}{(2\pi)^2} \frac{dk_y^d}{(2\pi)^2} (2\pi)^3 \delta^{(3)}(k_a + k_b - k_c - k_d) \times \frac{1}{16E_aE_bE_cE_d} |\mathcal{M}_{f_1}(\pi^+\pi^+ | \pi^+\pi^+)|^2 f_b^{eq}, \quad (56)$$

with

$$|\mathcal{M}_{f_1}|^2 = (6\lambda)^2 N_0^8 \frac{\pi}{2|\epsilon B|} \exp \left\{ -\frac{(k_y^b + k_y^c + k_y^d)^2}{4|\epsilon B|} - 4(k_y^a)^2 - 4(k_y^a)^2 - 4(k_y^b)^2 - 4(k_y^b)^2 \right\} \quad (57)$$

After integration over $k_y^b, k_y^c, k_y^d$ we get,

$$\omega^b_i = \frac{1}{2} 6^2 \lambda^2 N_0^8 \frac{\pi^2}{2} \frac{(2\pi)^3}{16(2\pi)^6} \int \frac{dk_z^b}{(2\pi)^2} \frac{dk_z^c}{(2\pi)^2} \frac{dk_z^d}{(2\pi)^2} \delta(E_a + E_b - E_c - E_d) \delta(k_z^a + k_z^b - k_z^c - k_z^d) \frac{1}{E_aE_bE_cE_d} f_b^{eq}$$

$$= \frac{1}{2} 6^2 \lambda^2 \frac{|\epsilon B|^2}{32} \frac{\pi^2}{(2\pi)^3} \int \frac{dk_z^b}{(2\pi)^2} \frac{dk_z^c}{(2\pi)^2} \frac{dk_z^d}{(2\pi)^2} \delta(E_a + E_b - E_c - E_d) \frac{1}{E_aE_bE_cE_d^\epsilon} f_b^{eq}$$

$$= \frac{1}{2} 6^2 \lambda^2 \frac{|\epsilon B|^2}{32} \frac{\pi^2}{(2\pi)^3} \int \frac{dk_z^b}{(2\pi)^2} \frac{dk_z^c}{(2\pi)^2} \frac{dk_z^d}{(2\pi)^2} \frac{1}{E_aE_bE_cE_d^\epsilon} f_b^{eq} \quad (58)$$

Here we have used the identity

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \quad (59)$$

where $x_i$ are the roots of $g(x)$.

Similarly, for $\pi^+(k_a) + \pi^-(k_b) \rightarrow \pi^+(k_c) + \pi^- (k_d)$ type scattering the interaction frequencies of "$\pi^b(b = \pm)$" are given by,

$$\omega^b = 2^2 \lambda^2 |\epsilon B|^2 \frac{32}{3} \int_{-\infty}^{\infty} dk_z^b \frac{2}{E_a^2 E_b^2} \left( \frac{k_z^a}{E_a} - \frac{k_z^b}{E_b} \right)^{-1} f_b^{eq}. \quad (60)$$
Now we are considering the other scatterings $\pi^b(p) + \sigma(k) \rightarrow \pi^b(p') + \sigma(k')$ and $\pi^b(p) + \pi^0(k) \rightarrow \pi^b(p') + \pi^0(k')$ with $\{b = +, -\}$. In these cases interaction frequency of $\pi^b$ particle is

$$\omega_{3}^b(p) = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{d^2p}{2\pi} \frac{d^2p'}{2\pi} \frac{(2\pi)^3\delta^3(p + k - p' - k')}{16E_kE_{p'}E_{k'}} |M_{fi}|^2 f^{eq}(E_k),$$  \hspace{1cm} (61)

with

$$|M_{fi}|^2 = (2\lambda)^2 N_0^4 \frac{\pi}{|eB|} \exp \left\{ - \frac{(p_y - p_y')^2 + (k_x - k_x')^2}{2|eB|} \right\}.$$  \hspace{1cm} (62)

After integration over $p_y'$ and $k_x$ we get,

$$\omega_{3}^b = (2\lambda)^2 N_0^4 \frac{\pi}{16|eB|} \int \frac{d^3k'}{(2\pi)^3} \frac{d^2p}{2\pi} \frac{d^2p'}{2\pi} \frac{(2\pi)^3\delta^3(p + k - p' - k')}{16E_kE_{p'}E_{k'}} \left[ E_{p'} + \sqrt{k_{\perp}^2 + (p_{\perp}' + k_{\perp}' - p_{\perp})^2} \right] f^{eq} \left( \sqrt{k_{\perp}^2 + (p_{\perp}' + k_{\perp}' - p_{\perp})^2} \right)$$  \hspace{1cm} (63)

We perform the integration numerically after completing the $k_\perp$ integration using the delta function.

Next, we are calculating the interaction rate of the neutral scalar particles $\sigma$ and $\pi^0$ from $\pi^b(p) + \sigma(k) \rightarrow \pi^b(p') + \sigma(k')$ and $\pi^b(p) + \pi^0(k) \rightarrow \pi^b(p') + \pi^0(k')$ scatterings with $\{b = +, -\}$. For these types of interactions, we can write the expression of interaction frequencies of $\pi^0$ and $\sigma$ as

$$\omega_{4}^{\sigma,\pi^0}(k) = \frac{1}{L_x} \int \frac{d^3k'}{(2\pi)^3} \frac{d^2p}{2\pi} \frac{d^2p'}{2\pi} \frac{(2\pi)^3\delta^3(P + K - P' - K')}{16E_kE_{p'}E_{k'}} |M_{fi}|^2 f^{eq}(E_p),$$  \hspace{1cm} (64)

where the matrix element is same as in Eq. (57). After integration over $p_y'$ and $k_x'$ we get,

$$\omega_{4}^{\sigma,\pi^0} = (2\lambda)^2 N_0^4 \frac{\pi}{16|eB|} \int \frac{d^3k'}{(2\pi)^3} \frac{d^2p}{2\pi} \frac{d^2p'}{2\pi} \frac{(2\pi)^3\delta^3(P + K - P' - K')}{16E_kE_{p'}E_{k'}} \left[ E_{p'} + \sqrt{k_{\perp}^2 + (p_{\perp} + k_{\perp} - p_{\perp}')^2} \right] f^{eq} \left( \sqrt{k_{\perp}^2 + (p_{\perp} + k_{\perp} - p_{\perp}')^2} \right).$$  \hspace{1cm} (65)

We have performed the integration numerically after using the delta function. Here we have used $\int dp_y = |eB|L_x$.

Associated scattering processes to calculate $\pi^\pm$ relaxation time i.e. $\tau_{\pi^\pm}$ are

$$\pi^+ + \pi^0 \rightarrow \pi^+ + \pi^a \hspace{1cm} (a = +, -, 0),$$
\[ \pi^+ + \sigma \rightarrow \pi^+ + \sigma. \quad (66) \]

Total interaction frequency for \( \pi^+ \) is obtained as \( \omega_{\pi^+} = \omega_1^{\pi^+} + \omega_2^{\pi^+} + \omega_3^{\pi^+} \). The equilibration time \( \tau_{\pi^+} \) is given by \( \tau_{\pi^+} = 1/\omega_{\pi^+} \). For the only scalar interaction (for example: \( \sigma\sigma \rightarrow \sigma\sigma \)) we considered the expression of interaction rate from equation (47). In the similar fashion we can calculate the interaction frequencies for other particles. Incorporating the estimated relaxation times in eq. (22)-eq. (25), we can obtain the viscosity coefficients.

### IV. RESULTS

![Image](https://via.placeholder.com/150)

**FIG. 2:** The ratio of shear viscosity to entropy density as a function of temperature for vacuum sigma masses \( m_\sigma = 500 \text{ MeV} \) (left) and 700 MeV (right). In both the plots blue lines indicate the pure thermal case whereas the other lines represent the parallel components of shear viscosity coefficients to entropy ratio for \( 5m_\pi^2 \) (red line), \( 10m_\pi^2 \) (green line), \( 15m_\pi^2 \) (black line).

We have summarized our results for the anisotropic components of the shear viscosity coefficients in the relaxation time approximation for the nonzero magnetic field. We then briefly discussed the linear sigma model and its thermodynamics. The temperature and magnetic field-dependent nature of the thermodynamic quantities like \( s/T^3 \) and \( \epsilon/T^3 \) are plotted and discussed.

We have revisited the solution of the Klein-Gordon (KG) equation in the presence of a background magnetic field described by a particular vector potential. The quantized nature of the transverse motion of the charged particles emerges to change the particles’ energy. The solutions of the KG equation are dependent on the Landau levels. Quantizing the theory, we have calculated the matrix elements using the field operators to obtain the interaction rates. The temperature
and magnetic field-dependent interaction rates are incorporated into the thermal relaxation times. In our present study, we have done our evaluation for a strong magnetic field by considering only the lowest Landau level contributions. In fig. 2 we have compared the pure thermal \((B = 0)\) isotropic viscous coefficient with parallel component of shear viscosity of the thermo-magnetic medium for vacuum sigma masses \(m_\sigma = 500\) MeV and 700 MeV. The viscous coefficients are scaled with the entropy density. Only temperature-dependent entropy is considered for the thermal case, whereas temperature \((T)\) and magnetic field \((B)\) dependent entropy is taken for the magnetic case. The plots are shown for three magnetic field strengths, i.e., \(5m_\pi^2\) (Redline), \(10m_\pi^2\) (Green line), and \(15m_\pi^2\) (Blackline). There is a minimum at crossover temperature of 245 MeV for both thermal and magnetic cases. We can also observe that shear viscosity is reduced in the presence of the magnetic field.

![FIG. 2: Pure thermal \((B = 0)\) isotropic shear viscosity coefficient with parallel component of shear viscosity of the thermo-magnetic medium for vacuum sigma masses \(m_\sigma = 500\) MeV and 700 MeV.](image)

Now we will explore the other shear viscous coefficients. LSM has both charged and neutral hadrons. So we studied the perpendicular components for charged and neutral particles differently. As mentioned earlier, neutral particles have a single viscous coefficient, which only contributes to the isotropic shear viscosity. In fig. 3, the solid black line indicates the variation of the scaled isotropic shear viscous coefficient with temperature. The blue dotted line represents the perpendicular component of shear viscosity for the charged particles, whereas the brown dot-dashed line shows the parallel component. The dashed line (magenta) shows the isotropic contribution to the viscous coefficients coming from neutral particles is shown by the dashed line (magenta). Total

![FIG. 3: Ratio of parallel \(\eta_{||}\) and perpendicular \(\eta_{\perp}\) shear viscosity components to entropy as a function of temperature \((T)\) for vacuum sigma mass \(m_\sigma = 500\) MeV (left) and \(m_\sigma = 700\) MeV (right). Magnetic field strength is taken as \(15m_\pi^2\).](image)
parallel (solid red) and perpendicular (solid blue) shear viscous coefficients are also plotted to compare with the pure thermal case. The plots are shown for vacuum sigma mass of 500 MeV (left figure) and 700 (right figure) MeV. It is observed from the figure that the anisotropic viscous coefficients for the charged particles are quite lower than the neutral hadrons. Note that for $m_\sigma = 500$ MeV, total(parallel), total (perpendicular), and neutral hadron contributions coincide as the contribution from the charged hadrons is very small for this case. It is also noted that the Hall type shear viscosity is zero for vanishing baryon chemical potential even in a finite magnetic field.

V. SUMMARY AND OUTLOOK

In the presence of the magnetic field, the charged particles get affected, and the system becomes anisotropic. Therefore, the transport coefficients become anisotropic. This work calculates the shear viscosity of hadronic matter in a strong magnetic field and vanishing chemical potential. We have calculated the parallel and perpendicular components of share viscosity in the relaxation time approximation. We have observed that the shear viscosity to entropy ratio for the neutral hadrons gets modified in the presence of a strong magnetic field because of their interaction with the charged particles. In addition, the shear viscosity for charged hadrons is modified in the thermomagnetic medium. We have observed that the contributions to the total shear viscosity to entropy ratio are more dominant for the charged neutral hadrons than the charged hadrons.

The present investigation is limited to the lowest Landau level (LLL) approximation, and it is valid for a very high value of the magnetic field. To study the effect of the magnetic field on the hadronic transport coefficients at a small to moderate strength of the magnetic field, one should include higher Landau levels. Such calculations are in progress and will be presented elsewhere.

VI. ACKNOWLEDGMENT

RG is supported by University Grants Commission (UGC). N.H. is supported in part by the SERB-MATRICS under Grant No. MTR/2021/000939.
Appendix A: Charged scalar field

1. wave function

We consider charged particles in a constant magnetic field. The Klein-Gordon equation becomes [60, 61]

\[
\left( \frac{i}{\partial t} - eA_0 \right)^2 \phi(x, y, z, t) = \left( (i\vec{\nabla} + e\vec{A})^2 + m^2 \right) \phi(x, y, z, t),
\]

where the wave function can be written in the following form

\[
\phi(x, y, z, t) = \phi(x, y, z) e^{-iEt}.
\]

In our case magnetic field is in z-direction i.e. \( \vec{B} = B\hat{z} \). We choose vector potential as

\[
A^\mu = (0, 0, xB, 0).
\]

Using the vector potential from Eq. (A3), Eq. (A1) becomes

\[
(E^2 - m^2) \phi(x, y, z) = -\nabla^2 + 2ieBx \frac{\partial}{\partial y} + e^2B^2x^2 \phi(x, y, z).
\]

The coordinate \( x \) appears through the derivatives, so we expect solution as

\[
\phi(x, y, z) = f(x)e^{ik_yy + ik_zz}.
\]

Putting it in above equation we get,

\[
\left( \frac{d^2}{dx^2} + 2eBxk_y - e^2B^2x^2 + \epsilon \right) f(x) = 0
\]

\[
\left[ \frac{d^2}{dx^2} - (eBx - k_y)^2 + (E^2 - k_z^2 - m^2) \right] f(x) = 0.
\]

After doing variable transformation i.e.

\[
\xi = \sqrt{|eB|} \left( x - \frac{k_y}{eB} \right),
\]

we arrive to equation

\[
\left( \frac{d^2}{d\xi^2} - \xi^2 + a \right) f(x) = 0,
\]

where \( a = \frac{E^2 - k_z^2 - m^2}{|eB|} \). The solution of above equation exists when \( a = 2\nu + 1 \) for \( \nu = 0, 1, 2, \ldots \).

Energy eigenvalues becomes,

\[
E^2 = k_z^2 + m^2 + (2\nu + 1)|eB|,
\]
and the solution for $f$ is

$$f_\nu(\xi) \equiv N_\nu e^{-\xi^2/2} H_\nu(\xi), \quad (A10)$$

where $H_\nu$ are Hermite polynomials and normalization constant is

$$N_\nu = \left( \frac{\sqrt{|eB|}}{\nu! 2^\nu \sqrt{\pi}} \right)^{1/2}. \quad (A11)$$

$f_\nu(\xi)$ satisfy the completeness relation

$$\sum_n f_n(\xi)f_n(\xi') = \delta(x - x'), \quad (A12)$$

and also

$$\int_{-\infty}^{\infty} f_\mu(x)f_\nu(x)dx = \sqrt{eB} \delta_{\mu,\nu}. \quad (A13)$$

Finally we can write

$$\phi_n(x, y, z, t) = e^{-iK \cdot \vec{x}} f_n(x, \vec{k}_\parallel), \quad (A14)$$

where $X_\parallel$ is position four vector setting $x$ component to zero. We would use $X$ to represent spacial co-ordinates.

### 2. Quantization

The scalar field operator can be written in terms of annihilation and creation operator as

$$\Phi(X) = \sum_{n=0}^{\infty} \int \frac{dk_y dk_z}{2\pi \sqrt{2E_n}} \left[ e^{-iK \cdot \vec{x}} f_n(x, \vec{k}_\parallel)a(n, \vec{k}_\parallel) + e^{iK \cdot \vec{x}} f_n^*(x, \vec{k}_\parallel)b^\dagger(n, \vec{k}_\parallel) \right]. \quad (A15)$$

The field $\Phi(X)$ and $\Pi(X) = \Phi^\dagger$ satisfy the commutation relation

$$[\Phi(X), \Pi(Y)] = \delta^{(3)}(\vec{x} - \vec{y}). \quad (A16)$$

We can obtain the commutation relation for annihilation and creation operator as

$$[a(n, p_\parallel), a^\dagger(m, p'_\parallel)] = \delta_{n,m} \delta(k_y - k'_y) \delta(k_z - k'_z), \quad (A17)$$

and similar for $b$ and $b^\dagger$. Now we define the one-particle states

$$|n, \vec{k}_\parallel\rangle = \frac{2\pi}{\sqrt{L_y L_z}} a^\dagger(n, \vec{k}_\parallel)|0\rangle. \quad (A18)$$
Here we have considered a finite box of sides \((L_x, L_y, L_z)\), which is taken to infinite volume limit at the end. The action of field operators on one-particle states reads as

\[
\Phi |\pi^-(n, \vec{k})\rangle = \frac{1}{\sqrt{2E_n L_y L_z}} e^{-iK \cdot \vec{x}} f_n(x, \vec{k}) |0\rangle,
\]

\[
\Phi^\dagger |\pi^+(n, \vec{k})\rangle = \frac{1}{\sqrt{2E_n L_y L_z}} e^{-iK \cdot \vec{x}} f_n(x, \vec{k}) |0\rangle.
\]

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