Annihilator large-superfluous submodules

M. S. ABBAS
AL-Zahraa University for women, College of Education, Karbala, Republic of Iraq.

Email: mhdsabass@gmail.com

ABSTRACT: In this paper, we investigate certain class of submodules which contains that of superfluous submodules. A submodule W of an R-module M is annihilator large-superfluous, if $\ell_S(V) \neq 0$ implies that $W + V \neq M$ where V is a large in M and $S = \text{End}_R(M)$. Several properties and characterizations of such submodules are consider. For $\alpha \in S$, we study under what conditions the image of $\alpha$, $\text{Im}(\alpha)$ being annihilator large-superfluous submodule in M. We show that $W_S(M) = \{ \alpha \in S \mid \text{Im}(\alpha) \text{ is annihilator large-superfluous in } M \}$ equal to $\{ \alpha \in S \mid \ell_S(\alpha) \text{ is large-superfluous } \}$ under certain class of projectivity. The sum $E_R(M)$ of all such submodules of M contains $J_e(M)$ and $Z_s(M)$. If M is cyclic, then $E_R(M)$ is the unique largest annihilator large-superfluous in M. MSC (2010): Primary :16010; Secondary :16080.

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1. Introduction

Throughout, all rings consider are associative with identity, and modules are unitary right modules. The ring of endomorphism $\text{End}(M)$ is denoted by S,. We abbreviate the Jacobson radical $J(M)$ of M. For simplicity, $N \leq_e M$ ( resp. $N \leq_{\max} M$ ) mean N is a large (resp. maximal) in M. $\ell_S(X)$, (resp. $\gamma_M(\alpha)$) where X is a submodule of M (resp. $\alpha \in S$) represents the left (resp. right) annihilator of X (resp. $\alpha$) in S (resp. in M).

In the field of modules and rings, the concept of superfluity is very important. A submodule W of M is superfluous ($N \leq_s M$ for short) if $W + A$ is proper in M for all proper submodules A in M. A lot of author’s generalized this concept in the literature. In [1] introduced the notion of superfluity with respect to large submodules. A submodule N of a module M is called large-superfluous ($1 - \text{superfluous}$) in M (denoted by $N \leq_{ls} M$) if for each proper large submodule L of M, $N + L$ is proper in M. In [1], they write $N \ll_e M$. Also, they studied the large Jacobson radical (denoted by $J_e(M)$) as the intersection of all large and
maximal submodules and showed that \( J_e(M) \) is equal to the sum of all large – superfluous submodules in \( M \).

T. Amouzegor-kalati and D. Keskin- Tutuncu [2] consider other type of generalizations. Let \( M \) be a module with \( N \leq M \). \( N \) is called annihilator – superfluous, if \( \ell_S(X) \neq 0 \), then \( N + X \) is proper in \( M \), for all \( X \leq M \). In this case we write \( N \leq_{als} M \). They consider the sum \( A_R(M) \) of all such submodules which includes \( J(M) \) and the left singular submodule \( Z_S(M) \).

This work organized by the above two generalizations of superfluous submodules. We investigate the following. Let \( M \) be an \( R \)-module and \( K \leq M \). We say that \( K \) is annihilator large – superfluous in \( M \), if for each large submodule \( X \) in \( M \) with \( \ell_S(X) \neq 0 \), we have \( K + X \) is proper in \( M \). Obviously, the class of such submodules contains both that of annihilator (resp. large) – superfluous submodules. We prove that under coretractibility or quasi – duality of the modules, annihilator large– superfluous concides with large – superfluous submodules.

Let \( M \) be a semi – projective \( R \)-module and \( \alpha \in S \). Then we prove the following. The submodule \( \text{Im}(\alpha) \) is annihilator large - superfluous in \( M \) if and only if \( \beta\alpha(M) \leq \beta(M) \) for all nonzero \( \beta \in S \) whose kernel is large in \( M \) if and only if \( \ell_S(1 - \alpha s) \cap L_S(M) = 0 \) for all \( s \in S \) if and only if \( \ell_S(1 - s\alpha) \cap L_S(M) = 0 \) for all \( s \in S \) where \( L_S(M) = \{ f \in S | \ker(f) \) is large in \( M \} \). Also we prove that if \( M \) is epiprojective \( R \)-module. Then \( \{ f \in S | \text{Im}(f) \) is annihilator large - superfluous in \( M \} \). Finally we study \( E_R(M) \) which is the sum of all annihilator large – superfluous submodules of an \( R \)-module \( M \).

2. Annihilator large – superfluous submodules.

**Definition (2.1).** A submodule \( W \) of an \( R \) – module \( M \) is said to be annihilator large - superfluous( for short al- superfluous), if given large submodule \( X \) of \( M \) with \( \ell_S(X) \neq 0 \), then \( W + X \) is proper in \( M \). For this case we write \( W \leq_{als} M \).

The module \( M \) is never al – superfluous in \( M \). In semisimple modules every submodule is al - superfluous, but \( 0 \) is the only superfluous submodule. In particular, in \( Z_6 \) as \( Z \)-module, every submodule is al - superfluous, but \( 2Z_6 \) is not a-superfluous. It is clear the following implications hold.

\[
\begin{align*}
N \leq M & \quad \text{implies that} \quad \ell_S(X + Y) = 0 \quad \text{for every complement } Y \text{ of } X \text{ in } M.
\end{align*}
\]

In the \( Z \)-module \( Z_{24} \), \( 4Z_{24} \) ( \( 8Z_{24} \) and \( 12Z_{24} \) ) is large - superfluous in \( Z_{24} \), but not a-superfluous in \( Z_{24} \).

We give a characterization of al – superfluous submodules in the following

**Proposition (2.2).** For a submodule \( W \) of a module \( M \), the following are equivalent

1. \( N \leq_{als} M \)

2. \( W + X = M \), implies that \( \ell_S(X + Y) = 0 \) for every complement \( Y \) of \( X \) in \( M \).
Proof. (1) → (2) Let Y be a complement of X in M. Then X + Y is large in M. Since W + X + Y = M, then by (1), we have \( \ell_S(X + Y) = 0 \). (2) → (1). Let X be a large submodule in M with W + X = M. Then \( \ell_S(X + Y) = 0 \). But X is large, then Y = 0 and hence \( \ell_S(X) = 0 \).

Directly from the definition, we have

Lemma (2.3). If \( K \leq N \) are submodules of M with \( N \leq_{als} M \), then \( K \leq_{als} M \).

An R-module M is coretractable, if for every proper submodule \( H \) of M satisfies \( \alpha(H) = 0 \) for some \( \alpha ( \neq 0) \in S \). equivalently, for every proper large submodule \( H \) of M, there exists a nonzero \( \alpha \) in S with \( \alpha(H) = 0 \) [3]. An R-module M is called quasi-dual, if for every large submodule \( N \) of M we have \( N = \gamma_M(\ell_S(N)) \) [4]

Proposition (2.4). Let M be R-module which satisfies any one of the following

1. M is coretractable
2. M is quasi-dual.

If \( N \) is a submodule of M, then \( N \leq_{ls} M \) if and only if \( N \leq_{als} M \).

Proof. The "only if" part is clear from the definition.

1. Let X be a large submodule of M with \( N + X = M \). Then \( \ell_S(X) = 0 \). But M is coretractable, then X is not proper and hence \( X = M \).
2. Assume \( N + X = M \) where X is large in M. Then \( \ell_S(X) = 0 \). Since M is quasi-dual, then \( X = \gamma_M(\ell_S(X)) \). Thus \( X = M \).

Corollary (2.5). Let N be a submodule of quasi-dual module M. Then \( N \leq_{als} M \) if and only if \( \gamma_M(\ell_S(N)) \leq_{als} M \).

Proof. The "if" part follows from lemma (2.3). Let X be a large submodule of M with \( \gamma_M(\ell_S(N)) + X = M \). Then \( 0 = \ell_S(N) \cap \ell_S(X) = \ell_S(N + X) \) and so \( M = N + X \). But \( N \leq_{als} M \), then \( \ell_S(X) = 0 \).

Let M be a right R-module. We set \( Z_S(M) = \{ m \in M \mid \ell_S(m) = \ell_S(mR) \text{ is large - superfluous left ideal in } S \} \)

Proposition (2.6). Let M be an R-module and N a submodule of M. If \( N \leq_{als} M \) and \( m \in Z_S(M) \), then \( N + mR \leq_{als} M \).

Proof. Assume that \( N + mR + X = M \) for some \( X \leq M \). Then \( 0 = \ell_S(mR + X) = \ell_S(mR) \cap \ell_S(X) \). Since \( \ell_S(mR) \leq_{ls} S \), then \( \ell_S(X) = 0 \).

Corollary (2.7). Let N be a submodule of an R-module M. If \( N \leq_{ae} M \) and \( Z_S(M) \) is countably generated, then \( N + Z_S(M) \leq_{als} M \).
**Proposition (2.8).** Let M be a finitely generated $R$–module with submodules $K$ and $N$. If $N \leq_{als} M$ and $K \leq_{ls} M$, then $N + K + Z_S(M) \leq_{als} M$.

**Proof.** Assume that $N + K + Z_S(M) + X = M$ for some large submodule $X$ in $M$. Then $N + Z_S(M) + X = M$, since $K \leq_{ls} M$. Let $\{m_1, m_2, \ldots, m_n\}$ be a generating set of $M$. Then $m_i = n_i + z_i + x_i$ for some $n_i \in N$, $z_i \in Z_S(M)$, $x_i \in X$ and $i = 1, 2, \ldots, n$. Thus $M = N + \sum_{i=1}^{n} z_i R + X$. Since $N \leq_{als} M$, then Proposition (2.6), $N + \sum_{i=1}^{n} z_i R \leq_{als} M$ and hence $\ell_S(X) = 0$.

$\square$

**Corollary (2.9).** Let $M$ be a finitely generated $R$-module. If $N \leq_{als} M$, then so is $N + J(M) + Z_S(M)$.

**Corollary (2.10).** The sum of al–superfluous submodule and l–superfluous submodule is al–superfluous.

Recall that an $R$-module $M$ is semi-projective if for any given $N \leq M$ and $R$-epimorphism $f : M \to N$, for any $R$-homomorphism $g : M \to N$, there exists $h \in S$ such that $fh = g$.

We define $L_S(M)$ be the set of all elements of $S$ whose kernel large in $M$.

**Lemma (2.11).** Consider the following assertion for an $R$ – module $M$ and $\alpha \in S$

1. $\text{Im}(\alpha) \leq_{als} M$
2. $\beta \alpha(M) \trianglelefteq \beta(M)$ for all nonzero $\beta \in S$ whose kernel is large in $M$,
3. $\ell_S(1 - \alpha s) :\ell_S(M) = 0$ for all $s \in S$
4. $\ell_S(1 - \alpha s) :\ell_S(M) = 0$ for all $s \in S$
5. $\ell_S(\alpha - \alpha s) :\ell_S(M) \subseteq \ell_S(\alpha)$ for all $s \in S$.

Then $(1) \to (2) \to (3) \to (4) \to (5).$ And $(5) \to (1)$ if $M$ is semi-projective.

**Proof.** $(1) \to (2).$ Let $\beta \in \ell_S(M)$ with $\beta \alpha(M) = \beta(M).$ Then for each $m \in M$, there exists $m' \in M$ such that $\beta(m) = \beta \alpha(m').$ This implies that $m - \alpha(m') \notin \gamma_M(\beta)$ and hence $M = \alpha(M) + \gamma_M(\beta)$. Since $\text{Im}(\alpha) \leq_{als} M$, then $\ell_S(\gamma_M(\beta)) = 0$ and hence $\beta = 0$. $(2) \to (3)$. Let $s \in S$ and $\beta \in \ell_S(1 - \alpha s) :\ell_S(M)$. Then $\beta = \beta \alpha s$ and this implies that $M = \beta \alpha s(M) \subseteq \beta \alpha(M).$ Thus $\beta = 0$, by(2) and hence $\ell_S(1 - \alpha s) :\ell_S(M) = 0$. $(3) \to (4)$. Let $\beta \in S$ and $\beta \in \ell_S(1 - \alpha s) :\ell_S(M)$. Then $(1 - \alpha s) = 0$ implies that $\beta s(1 - \alpha s) = \beta s - \alpha s = \beta(1 - \alpha s)s = 0$. Thus $\beta S = 0$, by(3) and so $\beta = \beta \alpha s = 0$. $(4) \to (5)$. Let $s \in S$ and $\beta \in \ell_S(1 - \alpha s) :\ell_S(M)$. Since $\beta \alpha \in L_S(M)$. By $(4)$ $\beta \alpha = 0$ and hence $\beta \in \ell_S(\alpha)$. $(5) \to (1)$. Assume that $M$ is semi-projective and $X$ a large submodule of $M$ with $\text{Im}(\alpha) + X = M$. Let $\beta \in \ell_S(X)$ and $m \in M$. Then there exists $m' \in M$ and $x \in X$ such that $m = \alpha(m') + x$. Now $m = \beta \alpha(m')$ and so $\beta(M) = \beta \alpha(M)$. By semi-projectivity of $M$, there exists $s \in S$ such that $\beta \alpha s = \beta$ and hence $\beta \alpha s \alpha = \beta \alpha$, so $\beta \in \ell_S(\alpha - \alpha s \alpha) :\ell_S(M)$. By $(5)$ $\beta \alpha = 0$ and hence $\beta = 0$.

**Corollary (2.12).** The following statements are equivalent for an element $k$ in $R$

1. $k R \leq_{als} R$
2. $b k R \leq_{als} b K$ for all nonzero $b \in Z_k(R)$
3. For all \( r \in \mathbb{R} \), \( \ell_R(1 - kr) \cap Z_r(R) = 0 \)
4. For all \( r \in \mathbb{R} \), \( \ell_R(1 - rk) \cap Z_r(R) = 0 \)
5. For all \( r \in \mathbb{R}, \ell_R(k - krk) \cap Z_r(R) \subseteq \ell_R(k) \)

**Corollary (2.13).** The following are equivalent of a right singular ring \( R \) and element \( k \in R \).

1. \( kR \leq_{als} R \)
2. For all \( b (\neq 0) \in R \), \( bkR \supseteq bR \)
3. For all \( r \in \mathbb{R} \), \( \ell_R(1 - kr) = 0 \)
4. For all \( r \in \mathbb{R} \), \( \ell_R(1 - rk) = 0 \)
5. For all \( r \in \mathbb{R}, \ell_R(k - krk) = \ell_R(k) \)

The following Corollary follows (2.13) and Corollary (2.8) in [2]

**Corollary (2.14).** Let \( R \) be a right singular ring and \( k \in R \). Then \( kR \leq_{als} R \) if and only if \( kR \leq_{als} R \).

**Corollary (2.15).** Let \( M \) be an \( M \) - injective \( R \)-module and \( \alpha \in S \). Consider the following condition

1. \( \text{Im}(\alpha) \subseteq_{als} M \)
2. \( \beta(M) \supseteq \beta(M) \) for all \( \beta (\neq 0) \in J(S) \)
3. \( \ell_S(1 - \alpha s) \cap J(S) = 0 \) for all \( s \in S \)
4. \( \ell_S(1 - s\alpha) \cap J(S) = 0 \) for all \( s \in S \)
5. \( \ell_S(\alpha - \alpha s) \cap J(S) \leq \ell_S(\alpha) \) for all \( s \in S \).

Then \( 1 \) \(\rightarrow\) \( 2 \) \(\rightarrow\) \( 3 \) \(\rightarrow\) \( 4 \) \(\rightarrow\) \( 5 \) if \( M \) is semi-projective.

In [2], the authors consider for any module \( M \), the set \( K_S(M) = \{ \alpha \in S \mid \text{Im}(\alpha) \subseteq_{als} M \} \).

We define the set \( W_S(M) = \{ \alpha \in S \mid \text{Im}(\alpha) \subseteq_{als} M \} \). Trivially \( K_S(M) \subseteq W_S(M) \).

**Corollary (2.16).** Let \( \alpha \in W_S(M) \). Then \( \alpha S \subseteq W_S(M) \). And \( S\alpha \subseteq W_S(M) \) if \( M \) is semi-projective.

**Proof.** Lemma (2.3) implies that \( \alpha S \subseteq W_S(M) \). Assume that \( M \) is semi-projective and \( s \in S \).

We prove that \( s\alpha \in W_S(M) \). Let \( \beta \in S \). Then by lemma (2.12), \( \ell_S(1 - gs\alpha) \cap L_S(M) = 0 \), since \( \alpha \in W_S(M) \). Again by lemma (2.12), \( s\alpha(M) \leq_{als} M \) and hence \( S\alpha \subseteq W_S(M) \).
Proposition (2.17). \( W_S(M) \subseteq \gamma_S(\text{soc}(S) \cap L_S(M)) \) for any \( R \)-module \( M \), where \( \text{soc}(S) \) is the sum of all simple right ideals of \( S \).

Proof. Let \( s \in W_S(M) \) and \( t \) a nonzero element in \( \text{soc}(S) \cap L_S(M) \). Then \( t \in (S_i \cap L_S(M) \) \( \bigoplus (S_j \cap L_S(M) \) \( \bigoplus \ldots \bigoplus (S_n \cap L_S(M)) \) where \( S_i \) is a simple right ideal of \( S \) (\( i = 1, 2, \ldots, n \)). We prove that \( ts = 0 \), if not \( t = t_L + t_2 + \ldots + t_n \) where \( t_i \in S_i \cap L_S(M) \) for \( i = 1, 2, \ldots, n \). There exists \( j \in \{ 1, 2, \ldots, n \} \) such that \( t_j s \neq 0 \). Since \( S_j \) is simple, then \( t_j S_j = S_j \cap L_S(M) \) and there is \( \alpha \in S \) such that \( t_j s = t_j \) and hence \( t_j = \ell_S(1 - \alpha) \cap L_S(M) \). Since \( s(M) \subseteq \text{als} \) \( M \), then by Lemma (2.12), \( \ell_S(1 - \alpha) \cap L_S(M) = 0 \) and hence \( t_j = 0 \), a contradiction. Thus \( ts = 0 \) and so \( W_S(M) \subseteq \gamma_S(\text{soc}(S) \cap L_S(M)) \).

Recall that an \( R \)-module \( M \) is epi–projective, if any \( R \)-epimorphisms \( q, f : M \rightarrow A \) where \( A \) is any module then \( q \circ f' = f \) for some \( f' \in S \) \([5]\).

It is well-known that if \( M \) is epi–projective, then \( J(S) = \nabla(M) = \{ \varphi \in S \mid \text{Im}(\varphi) \leq_{I_k} M \} \) for any \( R \)-module \( M \). It is clear that \( \nabla(M) \subseteq \nabla_I(M) \).

Proposition (2.18). Let \( M \) be epi–projective \( R \)-module. Then \( W_S(M) = \nabla_I(M) \).

Proof. Let \( f \in W_S(M) \) and \( f(M) + N = M \) for some large submodule \( N \) of \( M \). Let \( \pi : M \rightarrow M/N \) be the natural epimorphism. Then \( \pi f(M) = f(M) + N = M \) and hence \( \pi f = \text{epimorphism} \). So there exists \( \beta \in S \) such that \( \pi = \pi f \beta \), and hence \( \pi = \ell_S(1 - f \beta) \cap L_S(M) \). Since \( f(M) \leq_{\text{als}} M \), then by Lemma (2.12), \( \ell_S(1 - f \beta) \cap L_S(M) = 0 \). Thus \( \pi = 0 \) and hence \( N = M \). Therefore \( W_S(M) \subseteq \nabla_I(M) \) and the other inclusion is clear.

Corollary (2.19). Let \( M \) be epi–projective \( R \)-module and \( \alpha \in S \). Then \( \alpha(M) \leq_{\text{als}} M \) if and only if \( \alpha(M) \leq_{I_k} M \).

Corollary (2.20). Let \( M \) be an epi–projective \( R \)-module, then \( J(S) \leq W_S(M) \).

Proposition (2.21). Let \( M \) be an epi–projective \( R \)-module in which \( \ell_S(\alpha) \cap L_S(M) = 0 \), \( \alpha \in S \) implies \( \alpha S = S \). Then \( J(S) = W_S(M) \).

Proof. First \( J(S) \subseteq W_S(M) \) follows from Corollary (2.20). Let \( k \in W_S(M) \). Then \( k(M) \leq_{\text{als}} M \), so \( \ell_S(1 - ks) \cap L_S(M) = 0 \) for all \( s \in S \), by Lemma (2.12). Hence \( (1 - ks)S = S \) by hypothesis. Therefor \( k \in J(S) \).

An element \( m \in M \) is called al - superfluous, if \( mR \leq_{\text{als}} M \) and set \( L_R(M) = \{ m \in M \mid m \text{ is al - superfluous in } M \} \).

It is clear that \( L_R(M) \) may not be closed under addition for example in the \( Z \)-module \( Z \), \(-2, 3 \in L_Z(Z) \), but \( 1 \notin L_Z(Z) \). We have proved that the sum of al – superfluous submodule and 1 – superfluous submodule is al - superfluous, but the sum of al – superfluous submodules need not be al - superfluous, for example, consider \( 3Z + (-2)Z \) in the \( Z \)-module \( Z \).

Let \( M \) be an \( R \)-module. we define \( E_R(M) = \sum \{ K \leq_{I_k} M \mid K \leq_{\text{als}} M \} \).
clearly $L_R(M) \subseteq E_R(M)$ for any $R$-module $M$, but the equality may not be true (consider the $Z$-module $Z$).

**Proposition (2.22).** Let $M$ be an $R$-module. Then

1. $E_R(M) = \{ x_1 + x_2 + \ldots + x_n \mid x_i \in L_R(M) \text{ for each } i, n > 1 \}$

2. $E_R(M) = L_R(M)$

3. $J_e(M) \subseteq L_R(M)$ and $Z_S(M) \subseteq L_R(M)$.

**Proof.**

1. Assume that $X = \{ x_1 + x_2 + \ldots + x_n \mid x_i \in L_R(M) \text{ for each } i, n > 1 \}$. If $x \in E_R(M)$, then $x = x_1 + x_2 + \ldots + x_n$ where $x_i \leq a_l M$ for each $i$. Thus if $x = x_1 + x_2 + \ldots + x_n$ where $x_i \in X$, then $x \in x_i \leq a_l M$, by lemma (2.3). Hence $x_i \in L_R(M)$ for each $i$. Thus $E_R(M) \subseteq X$ and the inclusion $X \subseteq E_R(M)$ is clear.

2. By part (1), $L_R(M) \subseteq E_R(M)$.

3. Let $x \in J_e(M)$. Then $xR \ll a_l M$, by ([1], Theorem 2.10), and hence $xR \ll a_l M$. So $x \in L_R(M)$. Thus $J_e(M) \subseteq L_R(M)$. Let $y \in Z_S(M)$. Then $yR \ll a_l S$. By ([2], lemma 2.5) $yR \ll a_l M$ and hence $yR \ll a_l M$. So $y \in L_R(M)$. Thus $Z_S(M) \subseteq L_R(M)$.

The authors in [1] define $Soc_S(M) = \sum \{ N \leq a_l M \mid N \text{ is minimal in } M \}$ for any $R$-module $M$ and they proved that $Soc_S(M) = \bigcap \{ L \leq a_l M \mid L \leq a_l M \}$ where a submodule $N$ of an $R$-module $M$ is said to be superfluous-large in $M$ (denoted by $N \leq a_l M$ ); if $N \cap L = 0$ with $L \leq a_l M$ implies that $L = 0$. Also $Soc_S(M) \subseteq J(M) \subseteq J_e(M)$ and $Soc_S(M) \subseteq Soc(M) \subseteq J_e(M)$.

**Corollary (2.23).** Let $M$ be an $R$-module. Then $Soc_S(M)$, $Soc(M)$, and $J(M)$ is contained in $L_R(M)$.

**Proposition (2.24).**

1. Let $M$ be a coretractable or quasi-dual $R$-module. Then $J_e(M) = E_R(M) = L_R(M)$.

2. Let $M$ be a coretractable and semi-injective. Then

$$\gamma_M(Soc(\quad S)) \subseteq J_e(M) = E_R(M) = L_R(M).$$

**Proof.** By Proposition (2.4) $J_e(M) = E_R(M) = L_R(M)$. This proves (1). Now Assume that $M$ is coretractable semi-injective $R$-module. Then by ([3], Corollary 4.7) we have $J(M) = \gamma_M(Soc(\quad S))$. Then by (1) we have $\gamma_M(Soc(\quad S)) \subseteq J_e(M) = E_R(M) = L_R(M)$.

**Proposition (2.25).** Let $M$ be an $R$-module. Consider the following conditions

1. If $K, L \leq a_l M$, then $K + L \leq a_l M$.

2. $L_R(M)$ is closed under addition.

3. $E_R(M) = L_R(M)$.

4. $E_R(M) \leq a_l L_R(M)$.

5. Clearly $L_R(M) \subseteq E_R(M)$ for any $R$-module $M$, but the equality may not be true (consider the $Z$-module $Z$).

6. **Proposition (2.22).** Let $M$ be an $R$-module. Then

1. $E_R(M) = \{ x_1 + x_2 + \ldots + x_n \mid x_i \in L_R(M) \text{ for each } i, n > 1 \}$

2. $E_R(M) = L_R(M)$

3. $J_e(M) \subseteq L_R(M)$ and $Z_S(M) \subseteq L_R(M)$.

**Proof.**

1. Assume that $X = \{ x_1 + x_2 + \ldots + x_n \mid x_i \in L_R(M) \text{ for each } i, n > 1 \}$. If $x \in E_R(M)$, then $x = x_1 + x_2 + \ldots + x_n$ where $x_i \leq a_l M$ for each $i$. Thus if $x = x_1 + x_2 + \ldots + x_n$ where $x_i \in X$, then $x_i \leq a_l M$, by lemma (2.3). Hence $x_i \in L_R(M)$ for each $i$. Thus $E_R(M) \subseteq X$ and the inclusion $X \subseteq E_R(M)$ is clear.

2. By part (1), $L_R(M) \subseteq E_R(M)$.

3. Let $x \in J_e(M)$. Then $xR \ll a_l M$, by ([1], Theorem 2.10), and hence $xR \ll a_l M$. So $x \in L_R(M)$. Thus $J_e(M) \subseteq L_R(M)$. Let $y \in Z_S(M)$. Then $yR \ll a_l S$. By ([2], lemma 2.5) $yR \ll a_l M$ and hence $yR \ll a_l M$. So $y \in L_R(M)$. Thus $Z_S(M) \subseteq L_R(M)$.

The authors in [1] define $Soc_S(M) = \sum \{ N \leq a_l M \mid N \text{ is minimal in } M \}$ for any $R$-module $M$ and they proved that $Soc_S(M) = \bigcap \{ L \leq a_l M \mid L \leq a_l M \}$ where a submodule $N$ of an $R$-module $M$ is said to be superfluous-large in $M$ (denoted by $N \leq a_l M$ ); if $N \cap L = 0$ with $L \leq a_l M$ implies that $L = 0$. Also $Soc_S(M) \subseteq J(M) \subseteq J_e(M)$ and $Soc_S(M) \subseteq Soc(M) \subseteq J_e(M)$.

**Corollary (2.23).** Let $M$ be an $R$-module. Then $Soc_S(M)$, $Soc(M)$, and $J(M)$ is contained in $L_R(M)$.

**Proposition (2.24).**

1. Let $M$ be a coretractable or quasi-dual $R$-module. Then $J_e(M) = E_R(M) = L_R(M)$.

2. Let $M$ be a coretractable and semi-injective. Then

$$\gamma_M(Soc(\quad S)) \subseteq J_e(M) = E_R(M) = L_R(M).$$

**Proof.** By Proposition (2.4) $J_e(M) = E_R(M) = L_R(M)$, This proves (1). Now Assume that $M$ is coretractable semi-injective $R$-module. Then by ([3], Corollary 4.7) we have $J(M) = \gamma_M(Soc(\quad S))$. Then by (1) we have $\gamma_M(Soc(\quad S)) \subseteq J_e(M) = E_R(M) = L_R(M)$.

**Proposition (2.25).** Let $M$ be an $R$-module. Consider the following conditions

1. If $K, L \leq a_l M$, then $K + L \leq a_l M$.

2. $L_R(M)$ is closed under addition.

3. $E_R(M) = L_R(M)$.

4. $E_R(M) \leq a_l L_R(M)$. 

5. Clearly $L_R(M) \subseteq E_R(M)$ for any $R$-module $M$, but the equality may not be true (consider the $Z$-module $Z$).
Then (1) → (2) → (3) and (4) → (1). If $M$ is cyclic, then (3) → (4).

**Proof.** (1) → (2) For each $x, y \in L_R(M)$. Since $(x + y)R \subseteq R + yR$. Then by (1) $xR + yR \subseteq_{als} M$ and hence $(x + y)R \subseteq_{als} M$ by lemma (2.3). Thus $x + y \in L_R(M)$. (2) → (3). It is clear that $L_R(M) \subseteq E_R(M)$. By (2) and Proposition (2.22) we have $E_R(M) \subseteq L_R(M)$. (3) → (4). Let $M = mR$ be a cyclic $R$-module for some $m \in M$ and $X$ be a large submodule of $M$ with $E_R(M) + X = M$. By (3), $L_R(M) + X = M$. There exist $k \in L_R(M)$ and $x \in X$ such that $k + x = m$, then $kR + X = M$ and since $KR \subseteq_{als} M$, then $f_S(X) = 0$, so $E_R(M) \subseteq_{als} L_R(M)$. (4) → (1). Let $L, K \subseteq_{als} M$. Then $L, K \subseteq E_R(M)$, and hence $L + K \subseteq E_R(M)$. By (4) $E_R(M) \subseteq_{als} M$, so by lemma (2.3), $L + K \subseteq_{als} M$.

**Proposition (2.26).** Let $M = mR$ be a cyclic $R$-module and one of the condition in (2.25) holds. Then

1. $E_R(M)$ is the unique largest al – superfluous submodule of $M$
2. $E_R(M) = \cap \{ U \leq_{e} M \mid U \text{ is maximal in } M \text{ and } E_R(M) \subseteq U \}$

**Proof.** 1. is clear by (4) and the Definition of $E_R(M)$.

2. If $a \notin E_R(M)$, then by (3) $aR$ is not al - superfluous in $M$, so there is a large submodule $X$ of $M$ with $aR + X = M$ and $f_S(X) \neq 0$. Since $E_R(M) \subseteq_{als} M$, by (4), then $E_R(M) + X \neq M$, there is a maximal submodule $L$ of $M$ with $E_R(M) + X \subseteq L \subseteq M$. In fact $U$ is large in $M$, then $a \notin L$ and this proves (2)

**Proposition (2.27).** Let $M$ be a finitely generated $R$-module. If $E_R(M) \subseteq J(M) + Z_S(M)$, then the sum of any two al – superfluous submodules is al - superfluous

Proof. Let $N_1$ and $N_2$ be two al – superfluous submodule of $M$, then $N_1 + N_2 \subseteq E_R(M)$ and hence $N_1 + N_2 \subseteq J(M) + Z_S(M)$. By Corollary (2.9), $J(M) + Z_S(M) \subseteq_{als} M$ and then by lemma (2.3), $N_1 + N_2 \subseteq_{als} M$.

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