q-Supercongruences on Triple and Quadruple Sums

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Abstract. Inspired by the recent work of El Bachraoui, we present some new $q$-supercongruences on triple and quadruple sums of basic hypergeometric series. In particular, we give a $q$-supercongruence modulo the fifth power of a cyclotomic polynomial, which is a $q$-analogue of the quadruple sum of Van Hamme’s supercongruence (G.2).

Mathematics Subject Classification. Primary 33D15; Secondary 11A07, 11B65.

Keywords. Basic hypergeometric series, $q$-supercongruences, creative microscoping, Chinese remainder theorem.

1. Introduction

In 1913, Ramanujan (see [3]) proposed some well-known hypergeometric series without any proof, including

$$
\sum_{k=0}^{\infty} (8k + 1) \frac{\left(\frac{3}{4}\right)^k}{k!^4} = \frac{2\sqrt{2}}{\sqrt{\pi} \Gamma\left(\frac{3}{4}\right)^2}.
$$

Here $(a)_n = a(a + 1) \cdots (a + n - 1)$ is the Pochhammer symbol and $\Gamma(x)$ is the gamma function. In 1997, Van Hamme [15] studied partial sums of

This work is supported by Natural Science Foundation of Shanghai (Grant No. 22ZR1424100).

\(\text{Birkhäuser} \)
Ramanujan’s summation formular for $1/\pi$ and conjectured 13 Ramanujan-type supercongruences, involving
\[(G.2) \sum_{k=0}^{p-1} (8k + 1) \left(\frac{3}{4}\right)_k^4 \equiv \frac{p\Gamma_p \left(\frac{1}{4}\right) \Gamma_p \left(\frac{1}{2}\right)}{\Gamma_p \left(\frac{3}{4}\right)} \pmod{p^3}, \quad \text{if} \quad p \equiv 1 \pmod{4}.
\]
Here $p$ is an odd prime and the $p$-adic gamma function is defined as
\[\Gamma_p(x) = \lim_{m \to x} (-1)^m \prod_{0 < k < m} k, \]
where the limit is for $m$ tending to $x$ $p$-adically in $\mathbb{Z}_{\geq 0}$.

The Ramanujan-type supercongruences and their $q$-analogues have attracted a number of experts. The reader interested in $q$-analogues of supercongruences is referred to [4, 12, 16, 18, 19]. Especially, a $q$-analogue of Van Hamme’s (G.2) was proposed by Liu and Wang [10] as follows: for integers $n \equiv 1 \pmod{4}$,
\[\sum_{k=0}^{n-1} [8k + 1] (q; q^4)_k^4 q^{2k} \equiv \frac{(q^2; q^4)^{n-1}}{(q^4; q^4)^{n-1}} [n]_q q^{\frac{1}{4} - \frac{n}{4}} \pmod{[n] \Phi_n(q^2)}. \quad (1.1)
\]
We now need to introduce some necessary definitions and concepts. Let $q$ be an indeterminate and $n$ a positive integer. The $q$-integer is defined as
\[[n]_q = 1 + q + \cdots + q^{n-1}. \]
And the $q$-shifted factorial is denoted by
\[(a; q)_0 = 1 \quad \text{and} \quad (a; q)_n = (1 - a)(1 - aq)\cdots(1 - aq^{n-1}). \]
Usually, the multiple $q$-shifted factorial is directly written as
\[(a_1, a_2, \ldots, a_r; q)_m = (a_1; q)_m(a_2; q)_m\cdots(a_r; q)_m, \]
where $r \in \mathbb{Z}^+$ and $m$ is a nonnegative integer. Moreover, the $n$-th cyclotomic polynomial in $q$ is represented by $\Phi_n(q)$:
\[\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(n, k) = 1} (q - \zeta^k), \]
where $\zeta$ is an $n$-th primitive root of unity. And there is a fact that
\[[n] = \prod_{s|n, s > 1} \Phi_s(q). \quad (1.2)
\]
By transforming $q$-supercongruences on double sums into the ones from squares of truncated basic hypergeometric series, El Bachraoui [1] obtained the following result: for any positive odd integer $n$ with $\gcd(n, 6) = 1$,
\[\sum_{k=0}^{n-1} \sum_{j=0}^{k} c_q(j)c_q(k - j) \equiv q^{1-n}[n]^2 \pmod{[n] \Phi_n(q^2)}, \]
where \( c_q(k) = [8k + 1] \left( q; q^2 \right)_k^2 \left( q; q^2 \right)_{2k} q^{2k^2} \left( (q^6; q^6)^2 \left( q^2; q^2 \right)_{2k} \right) \). Recently, many experts have taken attention to El Bachraoui’s work and obtained some new results (see, for example, \([5,9,13]\)).

Inspired by the above work, in this paper, we shall investigate several \( q \)-supercongruences on triple and quadruple sums.

**Theorem 1.1.** Let \( d \) and \( n \) be positive integers with \( d \geq 3 \) and \( n \equiv 1 \pmod{d} \). For integers \( k \geq 0 \), let

\[
q \equiv \frac{(q^d)^4}{(q^d)^4_k} (q^{(d-2)k})_k. \tag{1.1}
\]

Then, we have

\[
\sum_{i+j+k \leq n-1} c_q(i)c_q(j)c_q(k) \equiv [n]^3 q^{\frac{3(1-n)}{d}} \frac{(q^2; q^d)^3_{d-1}}{(q^d; q^d)^3_{d-1}} \pmod{[n] \Phi_n(q)^3}. \tag{1.3}
\]

Letting \( n = p \) be a prime, \( d = 3 \) and \( q \rightarrow 1 \) in Theorem 1.1, and using the result [11, Equation (4.4)]:

\[
\frac{\left( \frac{2}{3} \right)^{p-1}}{(1-p)^{p-1}} = - \Gamma_p \left( \frac{1}{3} \right)^3 \left\{ 1 + \frac{p^2}{9} \left( G_2 \left( \frac{1}{3} \right) - G_1 \left( \frac{1}{3} \right)^2 \right) \right\} \pmod{p^3},
\]

where \( G_k(a) := \Gamma_p^{(k)}(a)/\Gamma_p(a), \Gamma_p^{(k)}(a) \) is the \( k \)-th derivative of \( \Gamma_p(a) \), we get the following result.

**Corollary 1.2.** Let \( p \) be a prime with \( p \equiv 1 \pmod{3} \). For nonnegative integers \( i, j, k \geq 0 \), we have,

\[
\sum_{i+j+k \leq p-1} (6i + 1)(6j + 1)(6k + 1) \left( \frac{1}{3} \right)_i^4 \left( \frac{1}{3} \right)_j^4 \left( \frac{1}{3} \right)_k^4 \equiv -p^3 \Gamma_p \left( \frac{1}{3} \right)^3 \pmod{p^4}. \tag{1.4}
\]

Note that He [8] obtained the following Ramanujan-type supercongruence: for primes \( p \geq 5 \),

\[
\sum_{k=0}^{p-1} (6k + 1) \left( \frac{1}{3} \right)_k^4 \equiv -p \Gamma_p \left( \frac{1}{3} \right)^3 \pmod{p^4}, \quad \text{if } p \equiv 1 \pmod{3},
\]

and Liu and Wang [11] presented several different \( q \)-analogues of He’s supercongruence.

Similarly, setting \( n = p \) be a prime, \( d = 4 \) and \( q \rightarrow 1 \) in Theorem 1.1, we find a triple sum related to Van Hamme’s (G.2) supercongruence.

**Corollary 1.3.** Let \( p \) be a prime with \( p \equiv 1 \pmod{4} \). For nonnegative integers \( i, j, k \geq 0 \), we have,

\[
\sum_{i+j+k \leq p-1} (8i + 1)(8j + 1)(8k + 1) \left( \frac{1}{4} \right)_i^4 \left( \frac{1}{4} \right)_j^4 \left( \frac{1}{4} \right)_k^4 \equiv p^3 \left( \frac{1}{2} \right)^3 \frac{p^2}{(1)^3} \pmod{p^4}. \tag{1.4}
\]
From [14], we know that, for primes\( p \),
\[
\left( -\frac{1}{2} \right) \equiv \frac{\Gamma_p \left( \frac{1}{4} \right)^2}{\Gamma_p \left( \frac{1}{2} \right)} \pmod{p^2}.
\]
Since \( \Gamma_p(x) \) has the property:
\[
\Gamma_p \left( x \right) \Gamma_p \left( 1-x \right) = (-1)^{a_0(x)}
\]
where \( a_0(x) \in \{1, 2, \cdots, p\} \) satisfies \( x - a_0(x) \equiv 0 \pmod{p} \), we can see that (1.4) is equivalent to the following congruence:
\[
\sum_{i+j+k \leq p-1} (8i + 1)(8j + 1)(8k + 1) \left( \frac{1}{4} \right)_i^4 \left( \frac{1}{4} \right)_j^4 \left( \frac{1}{4} \right)_k^4 \equiv -p^3 \frac{\Gamma_p \left( \frac{3}{4} \right)^3 \Gamma_p \left( \frac{1}{2} \right)}{\Gamma_p \left( \frac{3}{4} \right)^3} \pmod{p^4}.
\]
Moreover, we shall give a \( q \)-analogue of the quadruple sum of (G.2) as follows.

**Theorem 1.4.** Let \( n \) be a positive integer with \( n \equiv 1 \pmod{4} \). For integers \( k \geq 0 \), let
\[
c_q(k) = [8k + 1] \left( \frac{q^4}{(q^4; q^4)_k^4} q^{2k} \right).
\]
Modulo \([n] \Phi_n(q)^4\), we have
\[
\sum_{j_1+j_2+j_3+j_4 \leq n-1} c_q(j_1)c_q(j_2)c_q(j_3)c_q(j_4) \equiv [n]^8 q^{4(1-n)} \sum_{k=0}^{n-1} \frac{(q^4, q^4, q^5; q^4)_k^4}{(q^4, q^4, q^5; q^4)_k^4} q^{4k}.
\]
(1.5)

The rest of the paper is organized as follows. In the next section, we shall prove Theorem 1.1 by some essential tools including the ‘creative microscoping’ method introduced by Guo and Zudilin [7] and the Chinese remainder theorem for coprime polynomials. Finally, in Sect. 3, we complete the proof of Theorem 1.4 through establishing its parametric generalization.

### 2. Proof of Theorem 1.1

We first give a result due to El Bachraoui [2], which plays an important role in our whole proof.

**Lemma 2.1.** Let \( d, n, t \) be positive integers with \( d \geq t \geq 2 \) and \( n \equiv 1 \pmod{d} \). Let \( \{c(k)\}_{k=0}^{n-1} \) be a sequence of complex numbers. If \( c(k) = 0 \) for \( (n-1)/d < k < n \), then
\[
\sum_{k_1+\cdots+k_t \leq n-1} c(k_1) \cdots c(k_t) = \left( \sum_{k=0}^{n-1} c(k) \right)^t.
\]
(2.1)
Furthermore, if \( c(ln+k)/c(ln) = c(k) \) for all nonnegative integers \( k \) and \( l \) such that \( 0 \leq k \leq n - 1 \), then,
\[
\sum_{k_1 + \cdots + k_t = l+n} c(k_1)c(k_2) \cdots c(k_t)
= \sum_{k_1 + \cdots + k_t = k} c(k_1)c(k_2) \cdots c(k_t)
\times \sum_{s_1=0}^{l-s_1} c(s_1n) \sum_{s_2=0}^{l-s_1-s_1} c(s_2n) \cdots \sum_{s_{t-1}=0}^{l-s_1-\cdots-s_{t-2}} c(s_{t-1}n)c((l-s_1-\cdots-s_{t-1})n).
\]
(2.2)

Next, with the help of the result presented in [6], we can prove the following two auxiliary lemmas.

Lemma 2.2. Let \( n, d \) be positive integers with \( d \geq 3 \) and \( n \equiv 1 \) (mod \( d \)), \( a \) and \( b \) indeterminates. For integers \( k \geq 0 \), let
\[
z_q(k) = [2dk + 1] \frac{(q, aq, q/a, q/b; q^d)_k}{(q^d, aq^d, q^d/a, bq^d; q^d)_k} b^k q^{d-2k}.
\]
Then, we have, modulo \([n](1 - aq^n)(a - q^n)\),
\[
\sum_{i+j+k \leq n-1} z_q(i)z_q(j)z_q(k) \equiv [n]^3 \frac{(b/q)^{\frac{3(n-1)}{d}} (q^2/b; q^d)^{\frac{n-1}{d}}}{(bq^d; q^d)^{\frac{n-1}{d}}}.
\]
(2.3)

Proof. For \( n = 1 \), the result is clearly true. We now assume that \( n \) is an integer with \( n > 1 \). Recall that Guo and Schlosser [6, Equation (4.3)] gave the following result: modulo \([n](1 - aq^n)(a - q^n)(b - q^n)\),
\[
\sum_{k=0}^{M} z_q(k) \equiv [n] \frac{(b/q)^{\frac{n-1}{d}} (q^2/b; q^d)^{\frac{n-1}{d}} (b - q^n)(ab - 1 - a^2 + aq^n)}{(bq^d; q^d)^{\frac{n-1}{d}}} (a-b)(1-ab)
+ [n] \frac{(q, q^{d-1}; q^d)^{\frac{n-1}{d}} (1 - aq^n)(a - q^n)}{(aq^d, q^d/a; q^d)^{\frac{n-1}{d}}} (a-b)(1-ab),
\]
(2.4)
where \( M = (n-1)/d \) or \( n-1 \). It is natural to see that \( (q; q^d)_k \equiv 0 \) (mod \( \Phi_n(q) \)) for \((n-1)/d < k < n\), which means that \( z_q(k) \equiv 0 \) (mod \( \Phi_n(q) \)) for \((n-1)/d < k < n\). Therefore, employing Lemma 2.1 with \( t = 3 \) produces
\[
\sum_{i+j+k \leq n-1} z_q(i)z_q(j)z_q(k) \equiv 0 \quad (\text{mod} \ \Phi_n(q)).
\]
(2.5)
Let \( \zeta \neq 1 \) be an \( n \)-th root of unity, not necessarily primitive. Namely, \( \zeta \) is a primitive \( m \)-th root of unity with \( m \mid n \) and \( m > 1 \). From (2.5), we get
\[
\sum_{i+j+k \leq m-1} z_{\zeta}(i)z_{\zeta}(j)z_{\zeta}(k) = 0.
\]
(2.6)
It is not difficult to check that $\zeta(lm+s)/\zeta(lm) = \zeta(s)$ for nonnegative integers $l$ and $s$ with $0 \leq s \leq m - 1$. Applying Lemma 2.1 with $t = 3$ again, we have

$$\sum_{i+j+k \leq n-1} z_\zeta(i)z_\zeta(j)z_\zeta(k)$$

$$= \sum_{r=0}^{n-1} \sum_{i+j+k=r} z_\zeta(i)z_\zeta(j)z_\zeta(k)$$

$$= \sum_{l=0}^{m-1} \sum_{s=0}^{m-1} \sum_{i+j+k=lm+s} z_\zeta(i)z_\zeta(j)z_\zeta(k)$$

$$= \sum_{l=0}^{m-1} \sum_{a=0}^{l-1} z_\zeta(\zeta^l) \sum_{t=0}^{l-a} z_\zeta((l-a-t)m) \sum_{s=0}^{m-1} \sum_{i+j+k=s} z_\zeta(i)z_\zeta(j)z_\zeta(k)$$

$$= 0.$$

Since the above equality is true for any $n$-th root of unity $\zeta \neq 1$, we conclude that

$$\sum_{i+j+k \leq n-1} z_\zeta(i)z_\zeta(j)z_\zeta(k) \equiv 0 \pmod{[n]}, \quad (2.7)$$

where we have used (1.2).

On the other hand, letting $a = q^n$ or $q^{-n}$ in (2.4), we obtain

$$\sum_{k=0}^{M} \tilde{z}_\zeta(k) = [n]_{b/q} \frac{(q^2/b; q^d)^{n-1}}{(bq^d; q^d)^{n-1}}$$

where

$$\tilde{z}_\zeta(k) = [2dk + 1] \frac{(q, q^{1+n}, q^{1-n}, q/b; q^d)_k}{(q^d, q^{d+n}, q^{d-n}, bq^d; q^d)_k}.$$ 

Note the fact that $\tilde{z}_\zeta(k) = 0$ for $(n-1)/d < k < n$. Using Lemma 2.1 with $t = 3$ again, we arrive at

$$\sum_{i+j+k \leq n-1} \tilde{z}_\zeta(i)\tilde{z}_\zeta(j)\tilde{z}_\zeta(k) = [n]^3 \frac{(b/q)^{3(n-1)}}{(bq^d; q^d)^{3(n-1)/d}}.$$

Then, we have the following $q$-congruence involving a variable $a$: modulo $(1 - aq^n) (a - q^n)$,

$$\sum_{i+j+k \leq n-1} z_\zeta(i)z_\zeta(j)z_\zeta(k) \equiv [n]^3 \frac{(b/q)^{3(n-1)}}{(bq^d; q^d)^{3(n-1)/d}}. \quad (2.8)$$

From (2.7) and (2.8), we conclude that (2.8) is true modulo $[n] (1 - aq^n) (a - q^n)$, as desired. $\blacksquare$
Lemma 2.3. Let \( n, d \) be positive integers with \( d \geq 3 \) and \( n \equiv 1 \pmod{d} \), \( a \) and \( b \) indeterminates. For integers \( k \geq 0 \), let

\[
z_q(k) = [2dk + 1] \frac{(q, aq, q/a, q/b; q^d)_k}{(q^d, aq^d, q^d/a, bq^d; q^d)_k} b^k q^{(d-2)k}.
\]

Then, we have

\[
\sum_{i+j+k \leq n-1} z_q(i)z_q(j)z_q(k) \equiv [n]^3 \frac{(q, q^{d-1}; q^d)^{n-1}}{(aq^d, q^d/a; q^d)^{n-1}} (\mod b - q^n).
\]

Proof. Letting \( b = q^n \) in (2.4), we get

\[
\sum_{k=0}^M \hat{z}_q(k) = [n] \frac{(q, q^{d-1}; q^d)^{n-1}}{(aq^d, q^d/a; q^d)^{n-1}},
\]

where

\[
\hat{z}_q(k) = [2dk + 1] \frac{(q, aq, q/a, q^{1-n}; q^d)_k}{(q^d, aq^d, q^d/a, q^{d+n}; q^d)_k} q^{(n+d-2)k}.
\]

Note that \( \hat{z}_q(k) = 0 \) for \((n - 1)/d < k < n\). From Lemma 2.1 with \( t = 3 \), we obtain

\[
\sum_{i+j+k \leq n-1} \hat{z}_q(i)\hat{z}_q(j)\hat{z}_q(k) = [n]^3 \frac{(q, q^{d-1}; q^d)^{n-1}}{(aq^d, q^d/a; q^d)^{n-1}},
\]

which means that

\[
\sum_{i+j+k \leq n-1} z_q(i)z_q(j)z_q(k) \equiv [n]^3 \frac{(q, q^{d-1}; q^d)^{n-1}}{(aq^d, q^d/a; q^d)^{n-1}} (\mod b - q^n)
\]

is true. \( \square \)

Proof of Theorem 1.1. It is easy to see that \([n] (1 - aq^n) (a - q^n)\) and \( b - q^n \) are relatively prime polynomials. Noting the relations

\[
\frac{(b - q^n) (ab - 1 - a^2 + aq^n)}{(a - b)(1 - ab)} \equiv 1 (\mod (1 - aq^n) (a - q^n)),
\]

\[
\frac{(1 - aq^n)(a - q^n)}{(a - b)(1 - ab)} \equiv 1 (\mod b - q^n),
\]

and employing the Chinese remainder theorem for coprime polynomials, we derive the following result from (2.3) and (2.9): modulo \([n] (1 - aq^n) (a - q^n) (b - q^n),\)
\[
\sum_{i+j+k \leq n-1} z_q(i)z_q(j)z_q(k) \\
\equiv [n]^3 \left\{ \frac{(b - q^a)(ab - 1 - a^2 + aq^n)}{(a-b)(1-ab)} \frac{(b/q)^{\frac{3(n-1)}{d}} (q^2/b; q^d)^{\frac{3-n}{d}}}{(bq^d; q^d)^{\frac{3-n}{d}}} \\
+ \frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)} \frac{(q, q^{d-1}; q^d)^{\frac{3-n}{d}}}{(aq^d/q^d/a; q^d)^{\frac{3-n}{d}}} \right\}.
\] (2.10)

Taking \(b \to 1\) in (2.10), we deduce that, modulo \(\Phi_n(q)^2(1-aq^n)(a-q^n)\),
\[
\sum_{i+j+k \leq n-1} z_q(i)z_q(j)z_q(k) \equiv [n]^3 q^{\frac{3(1-n)}{d}} \frac{(q^2; q^d)^{\frac{3-n}{d}}}{(q^d; q^d)^{\frac{3-n}{d}}}. \tag{2.11}
\]

Here we have employed the relation:
\[(1-q^n)(1+a^2-a-aq^n) = (1-a)^2 + (1-aq^n)(a-q^n).
\]

Finally, letting \(a \to 1\) in (2.11) and noting \(1-q^n\) has the factor \(\Phi_n(q)\), we see that the congruence (1.3) is true modulo \(\Phi_n(q)^4\). On the other hand, our proof of (2.7) is still valid for \(a = b = 1\). Namely, (1.3) is also correct modulo \([n]\). Since the least common multiple of \(\Phi_n(q)^4\) and \([n]\) is \([n]\Phi_n(q)^3\), we finish the proof of Theorem 1.1. \(\square\)

3. Proof of Theorem 1.4

We first establish a parametric extension of Theorem 1.4 as follows.

**Theorem 3.1.** Let \(d\) and \(n\) be positive integers with \(d \geq 4\) and \(n \equiv 1 \pmod{d}\), \(c\) an indeterminate. For integers \(k \geq 0\), let
\[
c_q(k) = [2dk + 1] \frac{(q; q^d)_k^5 (cq; q^d)_k}{(q^d; q^d)_k^5 (q^d/c; q^d)_k} \left( \frac{q^{2d-3}}{c} \right)^k.
\]

Then, we have, modulo \([n]\Phi_n(q)^4\),
\[
\sum_{j_1+j_2+j_3+j_4 \leq n-1} c_q(j_1)c_q(j_2)c_q(j_3)c_q(j_4) \\
\equiv [n]^4 (cq)^{\frac{4(1-n)}{d}} \frac{(cq^2; q^d)^{\frac{4}{d-1}}}{(q^d/c; q^d)^{\frac{4}{d-1}}} \left( \sum_{k=0}^{\frac{n-1}{d}} \frac{(q, q, cq, q^{d-1}; q^d)_k}{(q^d, q^d, cq^2; q^d)_k} \right)^4. \tag{3.1}
\]

Clearly, Theorem 1.4 can be gotten by setting \(c = q^{d-1}\) and \(d = 4\) in Theorem 3.1.

In order to confirm Theorem 3.1, we need to prove the following parametric result.
Lemma 3.2. Let \( d \) and \( n \) be positive integers with \( d \geq 4 \) and \( n \equiv 1 \pmod{d} \), \( a, b \) and \( c \) indeterminates. For integers \( k \geq 0 \), let

\[
\nu_{q}(k,a,b) = \left[ 2dk + 1 \right] \frac{(aq,q/a,bq,q/b,cq,q;q^{d})_{k}}{(q^{d}/a,aq^{d},q^{d}/b,bq^{d},c^{d}/c,q^{d};q^{d})_{k}} \left( \frac{q^{2d-3}}{c} \right)^{k}.
\]

Then, modulo \([n]\)(1 - \( aq^{n} \))(1 - \( q^{n} \))(1 - \( bq^{n} \))(b - \( q^{n} \)),

\[
\sum_{j_{1}+j_{2}+j_{3}+j_{4} \leq n-1} z_{q}(j_{1},a,b)z_{q}(j_{2},a,b)z_{q}(j_{3},a,b)z_{q}(j_{4},a,b)
\equiv [n]^{4} \left( c \right) \frac{4(1-n)}{d} \left( c^{2};q^{d} \right)^{\frac{n-1}{d}} \left( q^{d}/c; q^{d} \right)^{\frac{n-1}{d}} \times \left( \begin{array}{c}
(1 - bq^{n})(b - q^{n})(-1 - a^{2} + aq^{n})
\end{array} \right)
\frac{}{(a-b)(1-ab)} \left( \begin{array}{c}
\left( \begin{array}{c}
\sum_{k=0}^{n-1} (aq,q/a,cq,q^{d-1};q^{d})_{k} q^{dk}
\end{array} \right)^{4}
\end{array} \right)
\times \left( \begin{array}{c}
(1 - aq^{n})(a - q^{n})(-1 - b^{2} + bq^{n})
\end{array} \right)
\frac{}{(b-a)(1-ab)} \left( \begin{array}{c}
\left( \begin{array}{c}
\sum_{k=0}^{n-1} (bq,q/b,cq,q^{d-1};q^{d})_{k} q^{dk}
\end{array} \right)^{4}
\end{array} \right)
\right) \quad (3.2)
\]

Proof. For \( n = 1 \), the result is obviously true. We now assume that \( n \) is an integer with \( n > 1 \). Setting \( r = 1 \) in [17, Theorem 5.7], we have, modulo \([n]\)(1 - \( aq^{n} \))(1 - \( q^{n} \))(1 - \( bq^{n} \))(b - \( q^{n} \)),

\[
\sum_{k=0}^{M} z_{q}(k,a,b)
\equiv [n] \left( c \right) \frac{1-n}{d} \left( c^{2};q^{d} \right)^{\frac{n-1}{d}} \left( q^{d}/c; q^{d} \right)^{\frac{n-1}{d}} \times \left( \begin{array}{c}
(1 - bq^{n})(b - q^{n})(-1 - a^{2} + aq^{n})
\end{array} \right)
\frac{}{(a-b)(1-ab)} \left( \begin{array}{c}
\left( \begin{array}{c}
\sum_{k=0}^{n-1} (aq,q/a,cq,q^{d-1};q^{d})_{k} q^{dk}
\end{array} \right)^{4}
\end{array} \right)
\times \left( \begin{array}{c}
(1 - aq^{n})(a - q^{n})(-1 - b^{2} + bq^{n})
\end{array} \right)
\frac{}{(b-a)(1-ab)} \left( \begin{array}{c}
\left( \begin{array}{c}
\sum_{k=0}^{n-1} (bq,q/b,cq,q^{d-1};q^{d})_{k} q^{dk}
\end{array} \right)^{4}
\end{array} \right)
\right) \quad (3.3)
\]
where \( M = (n-1)/d \) or \( n-1 \). Then, from (3.3), we can deduce that

\[
\sum_{k=0}^{M} z_{q}(k,a,b) \equiv 0 \pmod{[n]}.
\quad (3.4)
\]

Since \((q; q^{d})_{k} \equiv 0 \pmod{\Phi_{n}(q)}\) for \((n-1)/d < k < n\), we deduce that
\( z_{q}(k,a,b) \equiv 0 \pmod{\Phi_{n}(q)}\) for \((n-1)/d < k < n\). Then, applying Lemma 2.1
with \( t = 4 \), we arrive at
\[
\sum_{j_1 + j_2 + j_3 + j_4 \leq n - 1} z_q(j_1, a, b)z_q(j_2, a, b)z_q(j_3, a, b)z_q(j_4, a, b) \equiv 0 \pmod{\Phi_n(q)}.
\]

Letting \( \zeta \neq 1 \) be an \( n \)th root of unity, not necessarily primitive. Namely, \( \zeta \) is a primitive \( m \)th root of unity with \( m \mid n \) and \( m > 1 \). From (3.5), we have
\[
\sum_{j_1 + j_2 + j_3 + j_4 \leq m - 1} z_\zeta(j_1, a, b)z_\zeta(j_2, a, b)z_\zeta(j_3, a, b)z_\zeta(j_4, a, b) = 0. \tag{3.6}
\]

Observe that \( z_\zeta(lm + k)/z_\zeta(lm) = z_\zeta(k) \) for nonnegative integers \( l \) and \( k \) with \( 0 \leq k \leq m - 1 \). Using Lemma 2.1 with \( t = 4 \) again, we obtain
\[
\sum_{j_1 + j_2 + j_3 + j_4 \leq n - 1} z_\zeta(j_1, a, b)z_\zeta(j_2, a, b)z_\zeta(j_3, a, b)z_\zeta(j_4, a, b)
\]
\[
= \sum_{r=0}^{n-1} \sum_{j_1 + j_2 + j_3 + j_4 = r} z_\zeta(j_1, a, b)z_\zeta(j_2, a, b)z_\zeta(j_3, a, b)z_\zeta(j_4, a, b)
\]
\[
= \sum_{l=0}^{m-1} \sum_{k_0=0}^{l-1} \sum_{j_1 + j_2 + j_3 + j_4 = lm + k} z_\zeta(j_1, a, b)z_\zeta(j_2, a, b)z_\zeta(j_3, a, b)z_\zeta(j_4, a, b)
\]
\[
= \sum_{l=0}^{m-1} \sum_{k_1=0}^{l-k_1} \sum_{k_2=0}^{l-k_1-k_2} \sum_{k_3=0}^{l-k_1-k_2-k_3} z_\zeta(k_1m, a, b)z_\zeta(k_2m, a, b)
\]
\[
\times \sum_{k_1=0}^{m-1} \sum_{j_1 + j_2 + j_3 + j_4 = km} z_\zeta(j_1, a, b)z_\zeta(j_2, a, b)z_\zeta(j_3, a, b)z_\zeta(j_4, a, b)
\]
\[
= 0,
\]
which indicates that
\[
\sum_{j_1 + j_2 + j_3 + j_4 \leq n - 1} z_q(j_1, a, b)z_q(j_2, a, b)z_q(j_3, a, b)z_q(j_4, a, b) \equiv 0 \pmod{|n|}
\]

is true.

Moreover, letting \( a = q^n \) or \( a = q^{-n} \) in (3.3) gives
\[
\sum_{k=0}^{M} z_q(k, q^n, b) = \sum_{k=0}^{M} z_q(k, q^{-n}, b) = [n](cq)^{\frac{1-n}{a}} \frac{(cq^2; q^d)^{n-1}}{(q^d/c; q^d)^{n-1}} \sum_{k=0}^{n-1} (q^{1-n}, q^{1+n}, cq, q^{d-1}; q^d)_k q^{dk}. \tag{3.8}
\]

Noticing that \(z_q(k, q^n, b) = z_q(k, q^{-n}, b) = 0\) for \((n-1)/d < k < n\). Then, employing Lemma 2.1 with \(t = 4\), we obtain, modulo \((1-aq^n)(a-q^n)\),
\[
\sum_{j_1+j_2+j_3+j_4 \leq n-1} z_q(j_1, a, b)z_q(j_2, a, b)z_q(j_3, a, b)z_q(j_4, a, b) 
\equiv \lfloor n \rfloor^4 (cq) \frac{4(1-n)}{a} \frac{(cq^2; q^d)^{n-1}}{(q^d/c; q^d)^{n-1}} \left( \sum_{k=0}^{n-1} (aq, q/a, cq, q^{d-1}; q^d)_k q^{dk} \right)^4.
\tag{3.9}
\]

Similarly, setting \(b = q^n\) or \(b = q^{-n}\) in (3.3) and utilizing Lemma 2.1 with \(t = 4\) again, we get, modulo \((1-bq^n)(b-q^n)\),
\[
\sum_{j_1+j_2+j_3+j_4 \leq n-1} z_q(j_1, a, b)z_q(j_2, a, b)z_q(j_3, a, b)z_q(j_4, a, b) 
\equiv \lfloor n \rfloor^4 (cq) \frac{4(1-n)}{a} \frac{(cq^2; q^d)^{n-1}}{(q^d/c; q^d)^{n-1}} \left( \sum_{k=0}^{n-1} (bq, q/b, cq, q^{d-1}; q^d)_k q^{dk} \right)^4.
\tag{3.10}
\]

Noting that the polynomials \([n], (1-aq^n)(a-q^n)\) and \((1-bq^n)(b-q^n)\) are relatively prime. Then, utilizing the Chinese remainder theorem for coprime polynomials and the relations:
\[
\frac{(1-bq^n)(b-q^n)(-1-a^2+aq^n)}{(a-b)(1-ab)} \equiv 1 \pmod{(1-aq^n)(a-q^n)}, \tag*{(mod (1-aq^n)(a-q^n))}
\]
\[
\frac{(1-aq^n)(a-q^n)(-1-b^2+bq^n)}{(b-a)(1-ab)} \equiv 1 \pmod{(1-bq^n)(b-q^n)}, \tag*{(mod (1-bq^n)(b-q^n))}
\]
we can obtain the desired result from (3.7), (3.9) and (3.10). \(\square\)

Subsequently, we start to prove Theorem 3.1.

**Proof of Theorem 3.1.** Letting \(b \to 1\) in the congruence (3.2) produces, modulo \(\Phi_n(q)^3(1-aq^n)(a-q^n)\),
\[
\sum_{j_1+j_2+j_3+j_4 \leq n-1} z_q(j_1, a, 1)z_q(j_2, a, 1)z_q(j_3, a, 1)z_q(j_4, a, 1)
\equiv [n]^4 (cq)^4 \frac{(cq^2; q^d)^4_{n+1}}{(q^d/c; q^d)^4_{n+1} \cdot 
\times \left\{ \left( \frac{(1-q^n)^2(-1-a^2+aq^n)}{(a-1)(1-a)} \sum_{k=0}^{n-1} \frac{(aq, q/a, cq, q^{d-1}; q^d)_k}{(q^d, q^d, cq^2, q^d; q^d)_k} q^{dk} \right)^4 - \left( \frac{(1-aq^n)(a-q^n)(-2+q^n)}{(a-1)(1-a)} \sum_{k=0}^{n-1} \frac{(q, q, cq, q^{d-1}; q^d)_k}{(aq^d, q^d/a, cq^2, q^d; q^d)_k} q^{dk} \right)^4 \right\}.
\]

(3.11)

Next, setting \(a \to 1\) and utilizing the L’Hospital rule in (3.11), we arrive at

\[
\sum_{j_1+j_2+j_3+j_4 \leq n-1} c_q(j_1)c_q(j_2)c_q(j_3)c_q(j_4)
\equiv [n]^4 (cq)^4 \frac{(cq^2; q^d)^4_{n+1}}{(q^d/c; q^d)^4_{n+1} \cdot 
\times \left\{ \left( \sum_{k=0}^{n-1} \frac{(q, q, cq, q^{d-1}; q^d)_k}{(aq^d, q^d/a, cq^2, q^d; q^d)_k} q^{dk} \right)^4 \right\} \mod \Phi_n(q^5).
\]

(3.12)

On the other hand, our proof of (3.7) is still valid for \(a = b = 1\), which means that (3.12) is also correct modulo \([n]\). Since the least common multiple of \(\Phi_n(q)^5\) and \([n]\) is \([n]\Phi_n(q)^4\), we arrive at Theorem 3.1. \(\square\)

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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Received: May 5, 2022.
Accepted: November 14, 2022.

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