PHASE TRANSITIONS IN INDUCED QCD

A.A. Migdal

Laboratoire de Physique Théorique, de L’Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris CEDEX 05, France

Abstract

The variety of the phase transitions in Induced QCD are studied. Depending upon the parameters in the scalar field potential, there could be infinite number of fixed points, with different critical behavior. The integral equation for the density of the eigenvalues of the scalar field are generalized to the weak coupling phases, with the gap at the origin. We find a wide class of the massive solutions of these integral equations in the strong coupling phases, and derive an explicit eigenvalue equation for the scalar branch of the mass spectrum.
1 Introduction

Induced QCD was suggested as a possible model of hadrons two months ago [1]. This is the lattice model of the scalar field $\Phi$ in adjoint representation of the $SU_N$ gauge group, interacting with the usual lattice gauge field $U_l$, defined at links $l$. The unusual feature, which allows one to solve the model exactly in the large $N$ limit [2, 3] for arbitrary dimension $D$ of the lattice, is the absence of the bare self-interaction for the gauge field.

The idea is, that this self-interaction would be induced at larger distances, where the scalar field decouples, being heavy. Such induction often takes place in two-dimensional models of gauge and gravitational fields, such as CP$_N$ models, or the Liouville theory.

The parameters of the scalar potential $U(\Phi)$ should be carefully adjusted for this miracle to occur in more than two dimensions. The bare mass $m_0^2 = U''(0)$ should be chosen so, that the effective scalar mass $m_{\text{eff}}^2$ becomes much less than the lattice cutoff. This effective mass $m_{\text{eff}}$ would serve as the ultraviolet cutoff for perturbative QCD.

At the scales less than this mass, the scalar field could be integrated out, which yields the Yang-Mills term $N\beta \text{tr} F_{\mu,\nu}^2$ in effective Action, along with a variety of unwanted higher order terms. The bare quartic coupling should tend to a hypothetical critical point, to provide a large positive coefficient $\beta$, which could serve as the bare coupling of perturbative QCD. Then the physical mass scale $m_{\text{phys}}$ would automatically come out $m_{\text{phys}} \sim m_{\text{eff}} \exp(-A\beta) \beta^B$, with coefficients $A, B$ known from perturbative QCD.

The computation of induced gauge coupling $\beta$ as a function of the bare scalar mass and the bare quartic coupling $\lambda_0 = U'''(0)$ is not an easy task. Apparently, the quartic coupling should tend to some ultraviolet fixed point of the renormalization group, corresponding to both gauge and scalar charges. The recent computer simulations [4] for the $SU_2$ model showed the line of the first order phase transitions in the $m_0^2, \lambda_0$ plane, with the endpoint, which might correspond to the second order phase transition.

The general equations of [3] were studied in great detail for for special cases $D = 1$ and quadratic potential $U(\phi)$ [5]. In agreement with computer simulations and theoretical expectations, there is no critical point of the QCD type for the quadratic potential. At $D = 1$ in continuum limit the familiar fermionic solution was reproduced. The more general solution, for the finite lattice spacing at $D = 1$ was found in [6], again in complete agreement with previous work on matrix models.

There are some lattice artifacts in this model, which it is not yet clear how to remove. Namely, there is an extra local $Z_N$ symmetry, which must break to allow quarks to move inside hadrons. The general mechanisms of this symmetry breaking were found long ago by Khokhlaev and Makeenko [7], and later confirmed by numerical experiments and mean field analysis [8, 11, 9, 10].

It was recently conjectured in [12], that the same mechanisms spontaneously break $Z_N$ symmetry in this model at the critical point, which corresponds to the continuum limit of the lattice theory. In a subsequent paper [13] it was argued on the basis of the mean field analysis, that the $Z_N$ transition must take place before the critical point, i.e. still in the strong coupling phase of our model, if it really induces QCD.

Various properties of the generalized Wilson loops in this model were studied in [14].
Some interesting mathematical structures were found, which could be used in gauge and string models regardless conjectured induction of QCD.

There is the general heuristic argument [2], that the nontrivial fixed point of this model could be nothing but QCD. The argument is based on the common belief, that there is only one nontrivial theory in four dimensions: the asymptotically free, quark confining QCD. Even if this argument fails, the confining solution of Induced QCD would be an exciting alternative to the usual QCD. This would be the first solvable model of the QFT in four dimensions.

In the second paper [2] we came very close to this goal, by reducing the solution of the model to the following nonlinear integral equation for the vacuum density \( \rho(\lambda) \) of the scalar field eigenvalues

\[
\varphi \int_{-\infty}^{+\infty} d\lambda' \left( \frac{\pi \rho(\lambda')}{\lambda' - \lambda} + \arctan \frac{\pi \rho(\lambda')}{\lambda - R(\lambda')} \right) = 0,
\]  

where

\[
R(\lambda) = \frac{1}{2D} U''(\lambda) + \frac{D - 1}{D} \varphi \int_{-\infty}^{\infty} d\lambda' \rho(\lambda') \frac{\lambda' - \lambda}{\lambda' - \lambda'},
\]

and \( U(\phi) \) is the scalar field potential. We call this equation the master field equation, or MFE, because this density could be regarded as the long-sought master field of QCD. We found exact powerlike solution at \( \lambda \to 0 \)

\[
\rho(\lambda) \propto |\lambda|^\alpha; \cos \pi \alpha = -\frac{D}{3D - 2}; \alpha > 1,
\]

which was quite encouraging, since the scaling index \( \alpha \) showed no pathologies, like those of the string models.

The forbidden interval here is \( \frac{1}{2} < D < 1 \), and the only rational values are \( \alpha = n + \frac{1}{2} \) at \( D = 0 \), \( \alpha = 2n \) at \( D = \frac{1}{2} \), \( \alpha = 2n + 1 \) at \( D = 1 \) and \( \alpha = 2n + 1 \pm \frac{1}{3} \) at \( D = 2 \). The solutions at \( D = 0, \frac{1}{2} \) are unphysical, as the assumption of vanishing density at the origin is never satisfied in the corresponding matrix models. \( D = 0 \) corresponds to the one matrix model, and \( D = \frac{1}{2} \) correspond to the two matrix model (the number \( 2D \) of links meeting in each cite equals 1 here, as there is only one link, connecting two cites). \( D = 1 \) solution is already physical, as we discuss in more detail below. The \( |\lambda|^{2n+1} \) singularity comes about as the singularity of the tip of the upside-down even potential in the usual solution of the \( D = 1 \) matrix models. As for the first nontrivial case \( D = 2 \), unfortunately, the adjoint scalar field model cannot be solved by conventional methods even at \( D = 2 \), so there is nothing to compare with this solution.

The \( \frac{1}{N} \) expansion was considered in the third paper [3], where we found the integral equation for the propagator of the effective field theory with \( \rho(\lambda, x) \) as dynamical field. This linear equation involves the vacuum density in its kernel, and for the powerlike density, the powerlike solutions for the corresponding wave functions in \( \lambda \) space were found.

\[^3\text{The eigenvalues } \lambda \text{ have dimension } m^{\frac{4D-1}{2}} \text{ so that at } D > 2 \text{ the physical region is } \lambda \to 0 \text{ in the lattice units we are using.}\]
Still, the solution is incomplete, as there is no mass scale. This is the solution exactly at the critical point, where there are scaling laws in the $\lambda$ space. The physical solution of Induced QCD must involve the mass scale, which requires the more general solution of the MFE.

In this paper we find an infinite family of such solutions, which turns out to be a particular superposition of the previous powerlike terms. The implications of this simple observation are very interesting. Now, there is a calculable mass spectrum with nontrivial scaling indices in arbitrary dimension $D > 1$.

We generalize the Riemann-Hilbert problem for the weak coupling phase, where there is a gap at the origin in the density of eigenvalues. We present the new derivation, which, as we hope, is easier to comprehend, than that of [3].

2 Massive Solution of the Riemann-Hilbert Problem

The classical equation (1.1) in the local limit, when $r(\lambda) \equiv R(\lambda) - \lambda \sim \rho(\lambda) \ll \lambda$ was reduced in the previous paper [4] to the following nonlinear boundary problem.

Let us introduce two functions

$$P(z) = \frac{U'(z) - 2Dz}{2(1 - D)} + \int_{-\infty}^{\infty} d\mu \frac{\rho(\mu)}{\mu - z}, \quad (2.1)$$

$$Q(z) = \text{polynomial} + \pi \int_{-\infty}^{\infty} d\mu \frac{\rho^2(\mu)}{\mu - z}. \quad (2.2)$$

At $z \to \lambda + i0$, we have

$$P(z) \to \frac{D}{1 - D} r(\lambda) + i\pi \rho(\lambda). \quad (2.3)$$

Here $U(\phi)$ is the bare potential, but the contributions from the large eigenvalues (of the order of the lattice cutoff) in the integral effectively renormalize this potential. In the local limit, at $\lambda \ll 1$ in lattice units, the density $\rho \sim |\lambda|^\alpha$, so that formally the integral diverges if we substitute the local density, i.e., it is dominated by the lattice scales $\lambda \sim 1$, where the solution is not universal. The corresponding number of subtractions should be made, as usual in dispersion relations, or, which is more convenient, one could make the analytic continuation of these integrals in the scaling dimension $\alpha$ from the convergence region $\alpha < \frac{1}{2}$. The subtraction polynomial renormalize the bare potential $U$.

Both functions are analytic in the upper half plane and have the symmetry property

$$P(-\bar{z}) = -\bar{P}(z); \quad Q(-\bar{z}) = -\bar{Q}(z). \quad (2.4)$$

In other words, real(imaginary) parts are odd(even) with respect to the real part of $z$.

---

4As was discussed already in the first paper, the renormalization group analysis tells us, that the logarithmic laws of the asymptotic freedom translate into the power laws in the induced QCD models. The critical indices depend upon the effective quartic scalar interaction, which is not calculable by perturbative methods.
At $\Im z \to +0$ by construction
\[ \Im Q = (\Im P)^2. \] (2.5)

On the other hand, as shown in [2], in virtue of the classical equation (1.1), up to $O(P^3)$ terms,
\[ \Re Q = \frac{1 - D}{D} \Im (P^2). \] (2.6)

The last three equations represent the nonlinear Riemann-Hilbert problem.

In the previous papers only the massless solutions were found
\[ P = iA(-iz)^\alpha \cos \frac{\pi\alpha}{2}, \] (2.7)
\[ Q = iA^2(-iz)^{2\alpha} \cos \frac{\pi\alpha}{2}. \] (2.8)

Various values of $\alpha = 2n + 1 \pm \frac{1}{\pi} \arccos \frac{D}{3D - 2}$ correspond to various fixed points of the model.

We could not find any general theory of the nonlinear Riemann-Hilbert problem, but in this particular one we found the class of exact solutions, which are built from the above power terms,
\[ P = iA(-iz)^\alpha \cos \frac{\pi\alpha}{2} + iB(-iz)^\beta \cos \frac{\pi\beta}{2}; \quad \alpha + \beta = 2k, \] (2.9)
\[ Q = iA^2(-iz)^{2\alpha} + B^2(-iz)^{2\beta} \cos \frac{\pi\alpha}{2} + 2iABz^{\alpha+\beta}. \] (2.10)

It is not difficult to check this solution. The symmetry property is manifest, and so is the first equation. As for the second equation, the key point is that at $z = \lambda + i0$
\[ \Im P = A|\lambda|^\alpha + B|\lambda|^\beta; \quad \Re P = \frac{\lambda}{|\lambda|} \tan \frac{\pi\alpha}{2} \left( A|\lambda|^\alpha - B|\lambda|^\beta \right), \] (2.11)
so that the cross terms in $\Im (P^2) = 2\Im P \Re P$ are absent.

In general, these $A$ and $B$ could be arbitrary polynomials of $z^2$. This ambiguity reflects the ambiguity in the choice of the initial scalar potential $U(\phi)$. However, the critical phenomena, which we are interested in, are universal, as they take place in the infinitesimal vicinity of the origin, in the lattice units we are using.

In this limit, only the two leading terms in $A, B$ can be left. With proper redefinition of $\alpha, \beta$ this corresponds to constant $A, B$. We assume, that $\alpha > \beta$, then the critical region corresponds to
\[ z \sim z_0 \equiv \left( \frac{B}{A} \right)^\omega; \quad \omega = \frac{1}{\alpha - \beta}, \] (2.12)
so, that $B = 0$ at the critical point. The leading index $\alpha$ must be greater than 1, as it follows from original derivation. As for the subleading index $\beta$, it should be greater than $-1$ for the density to be integrable at the origin.

5These terms are down by a power of the ultraviolet cutoff in the local limit.
Using the language of renormalization group, the $\beta$-term represents the perturbation of the UV-stable fixed point $\alpha$ by the relevant operator of the lower scaling dimension. In general, the coefficient $B$ linearly vanishes at the critical point. At $B < 0$ the solution becomes unstable, as the density changes sign near the origin. In this case, there would be the phase transition to the weak coupling phase. The local limit of the MFE is different in this phase, as we shall see in the next Section.

The lowest solution for $\alpha, \beta$ in the strong coupling phase would be $1 < \alpha < 1.5$, and $\beta = 2 - \alpha$. In four dimensions

$$\{\alpha, \beta\} = \{1.36901, 0.63099\}. \quad (2.13)$$

The next one is

$$\{\alpha, \beta\} = \{2.63099, -0.63099\}. \quad (2.14)$$

Note that the singularity in $\rho$ is still integrable for this solution. These are the only physical solutions for $k = 1$. At the next level, $k = 2$, there are three solutions

$$\{\alpha, \beta\} = \{\{2.63099, 1.36901\}, \{3.36901, 0.63099\}, \{4.63099, -0.63099\}\}. \quad (2.15)$$

We discuss the choice of the solution later, when we study the wave equation. As we shall see, the first solution at $k = 1$ have tachyons.

It is worth mentioning, that at $D = 1$ the equations degenerate. The real part of $Q$ vanishes, so that the most general solution with proper symmetry would be $iW(z^2)$, where $W$ is some polynomial with real coefficients. Then, from the first equation we find

$$\pi \rho(\lambda) = \sqrt{W(\lambda^2)}. \quad (2.16)$$

This is in complete agreement with the solution of the lattice MFE, found recently by D.Gross [5]. In this case $W(z^2) = 2\pi^2(E - U(z))$ where $U(z)$ is initial potential, and the chemical potential $E$ is to be determined from the normalization of density. This solution also agrees with conventional solution of the $D = 1$ model in terms of effective fermi gas. Note, that the generic singularity is $|\lambda|^{2n+1}$, for $W(\lambda) \propto \lambda^{4n+2}$. This agrees with above powerlike solution.

With our ”S-matrix” approach we immediately find the solution, but cannot relate the parameters to those of original lattice theory. Fortunately, this is never needed. What is really needed, is to check internal consistency of the solution, such as absence of ghosts and tachyons, which is not a priori guaranteed in the ”S-matrix” approach.

Another comment. At any $D$ there always exists a trivial solution, without critical behavior

$$\mathcal{P} = za(z^2) + i\sqrt{b(z^2)}; \quad Q = ib^2(z^2) + 2\frac{1 - D}{D} za(z^2)\sqrt{b(z^2)}, \quad (2.17)$$

with the support of eigenvalues at $b(\lambda^2) > 0$. In the simplest case of constant $a$ and linear $b$ this is the semicircle solution. As was recently shown in [5], for the case of quadratic potential $U$ this is the only solution in the strong coupling phase.

Unfortunately, the simplest nontrivial scaling solution would take at least two adjustable parameters, so the higher order terms in a potential are required. In this case, as we suspect, the explicit solution of the lattice model is unavailable, and one either has to rely upon the above Riemann-Hilbert approach, or use the numerical methods to solve the lattice MFE.
3 Phase Transition

The Riemann-Hilbert problem was derived under assumption, that there was infinite support of the eigenvalues, without any gap at the origin. As was mentioned in the first paper [1], we expect this model to undergo the phase transition from above strong coupling phase to the weak coupling phase, with the gap in the support of the eigenvalues. In that paper we could not find nontrivial spectrum, because the kinetic term in the effective Lagrangean for the density fluctuations vanished at \( N = \infty \).

The more recent solution [3] produces such term regardless the phase of the model. The term proves to be positive definite, which means that this solution is different from the first one. Perhaps, there was something wrong with the assumptions of the orthogonal polynomial method in this case.[4]

Anyway, let us assume, that there is a gap from \(-a\) to \(a\) in the vacuum density \( \rho(\lambda) \). The basic equation (1.1) remains the same, but the dispersion relation between real and imaginary parts of all analytic functions modifies. The simplest way to account for these changes is to note that conformal transformation

\[
\zeta(z) = \sqrt{z^2 - a^2}
\]

maps the upper half of \( z \) plane onto the upper half plane of \( \zeta \), eliminating the gap from \(-a\) to \(a\).

We could use old dispersion relations for the even functions of \( z \) with \( \zeta \) instead of \( z \). For the odd functions there would be fictitious singularity at \( z^2 = 0 \). In particular, the odd function \( \mathcal{P}(z) \) was reconstructed in [2] from the real part. The derivative of this relation reads (so far, at \( a = 0 \)),

\[
\mathcal{P}'(z) = \frac{U''(z) - 2D}{2(1-D)} + i \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \left( \frac{1}{z - C(\nu)} - \frac{1}{z - R(\nu)} \right),
\]

where

\[
C(\nu) \equiv R(\nu) - i\pi \rho(\nu).
\]

This is the difference of two Cauchy integrals, the first one going over the complex curve \( z = C(\nu) \) in the lower half plane, and the second one going backwards over the real axis \( z = R(\nu) \). One could write this as a single Cauchy integral over the complex contour \( \mathcal{C} = \{C, R\} \),

\[
\mathcal{P}'(z) = \frac{U''(z) - 2D}{2(1-D)} + i \oint_{\mathcal{C}} \frac{dW_{\mathcal{C}}(y)}{\pi} \frac{1}{z - y},
\]

where the density \( W_{\mathcal{C}} \) is given by the parametric equation

\[
W_{\mathcal{C}}(C(\nu)) = \frac{1}{C'(\nu)}
\]

\[\text{6} \quad \text{In some cases, the even and odd coefficients of expansion in orthogonal polynomials tend to different analytic functions of } \frac{N}{N}. \text{ Certainly, this issue must be further analyzed.}\]
Transforming this relation from $z$ to $\zeta$, we find

$$
\mathcal{P}'(z) = \frac{U''(z) - 2D}{2(1-D)} + i \int_C \frac{dy}{\pi} \frac{W_C(y)}{\zeta(z) - y},
$$

$$
W_C(\zeta(C(\nu))) = \frac{1}{C'(\nu)}.
$$

In terms of original variables

$$
\mathcal{P}'(z) = \frac{U''(z) - 2D}{2(1-D)} + i \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \left( \frac{\zeta'(C(\nu))}{\zeta(z) - \zeta(C(\nu))} - \frac{\zeta'(R(\nu))}{\zeta(z) - \zeta(R(\nu))} \right),
$$

Note, that the integrand vanishes inside the gap, as $C(\nu) = R(\nu)$ there. Also, note, that in virtue of the symmetry $R(-\nu) = -R(\nu), \rho(-\nu) = \rho(\nu)$ the real (imaginary) part of this function is odd(even) function of $\Re z$, as it should be. One can readily check that this function is analytic in the upper half plane, that its imaginary part at the real axis vanishes at $z^2 < a^2$, and that its real part at $z^2 > a^2$ agrees with derivative of MFE.

Let us now go to the local limit in above complex contour integral for $\mathcal{P}'(z)$, using the same expansion, as in ref. [3].

$$
R(\nu) = \nu + r(\nu); \ C(\nu) = \nu + c(\nu); \ c(y) = r(y) - i\pi\rho(y) \ll \nu.
$$

Iterating implicit equation for $W_C$, we find

$$
W_C(\zeta(z)) = 1 - c'(z) + \frac{1}{2} (c^2(z))'' + O(c^3)
$$

We are interested in $\pi\rho'(\lambda) = \Im \mathcal{P}'(\lambda + i0)$. One contribution to this imaginary part comes from $\Im W(\zeta(\lambda))$ times the $\delta$-function term at $y = \zeta(\lambda)$. Subtracting the same terms with $\rho = 0$, to account for the integral with $C(\nu)$ replaced by $R(\nu)$, we find

$$
\Im W_C(\zeta(\lambda)) - \Im W_R(\zeta(\lambda)) = \pi\rho'(\lambda) - \pi (r(\lambda)\rho(\lambda))''
$$

We see, that the left side exactly cancels with the first term, so that we have to keep the $O(c^3)$ terms.

There is the second contribution to $\Im \mathcal{P}'(\lambda + i0)$, coming from the principal value of the integral of $\Re W_C$. As before, subtracting the integrals with $C$ and $R$, we find

$$
-\frac{\pi}{2} \varphi \int_{-\infty}^{\infty} dy \frac{1}{(\sqrt{\lambda^2 - a^2 - y})} \left( \rho^2(\nu) \right)''_{\nu=\sqrt{a^2+y^2}},
$$

or, symmetrizing in $y \rightarrow -y$ and transforming $ydy = \nu d\nu$,

$$
-\pi \sqrt{\lambda^2 - a^2} \varphi \int_{a}^{\infty} \frac{d\nu}{\sqrt{\nu^2 - a^2}} \left( \frac{\nu}{\lambda^2 - \nu^2} \right) \left( \rho^2(\nu) \right)'' =
$$

$$
-\frac{\pi}{2} \sqrt{1 - \frac{a^2}{\lambda^2}} \varphi \int_1^{\frac{a}{\lambda}} \frac{d\nu}{\sqrt{1 - \frac{a^2}{\nu^2}}} \left( \frac{1}{\lambda - \nu} \right) \left( \rho^2(\nu) \right)''
$$

The sceptical reader is invited to check this formula by means of usual dispersion relations in the $z^2$ plane, with extra factors $\sqrt{a^2 - z^2}$ to convert real part to imaginary. In differentiating the MFE, one should take into account the $\delta$-function terms, coming from the discontinuity of the actangent at $\pm \infty$. 

7
where $S = \{(-\infty, -a), (a, \infty)\}$ is the support of eigenvalues.

Collecting the terms, we arrive at the following equation

$$2 (r(\lambda) \rho(\lambda))'' = -\sqrt{1 - \frac{a^2}{\lambda^2}} \varphi \int_S \frac{d\nu}{\sqrt{1 - \frac{a^2}{\nu^2}}} \frac{\left(\rho^2(\nu)\right)''}{\lambda - \nu}$$  (3.14)

At $a = 0$ this equation reduces to the old one, after two integrations by parts. We drop the divergent polynomial terms, keeping in mind the corresponding number of subtractions in dispersion relations.

The dispersion relation for $r(\lambda)$ reads, as before

$$r(\lambda) = \frac{U'(\lambda) - 2D\lambda}{2D} + \frac{D - 1}{D} \varphi \int_S \frac{d\nu}{\lambda - \nu} \rho(\nu).$$  (3.15)

Let us now reduce these equations to the Riemann-Hilbert problem. The first function, $\mathcal{P}(z)$ is the same as before. The second function, $\mathcal{Q}(z)$ is introduced as follows

$$\mathcal{Q}''(z) = \pi \int_S \frac{d\nu}{\sqrt{1 - \frac{a^2}{\nu^2}}} \frac{\left(\rho^2(\nu)\right)''}{\nu - z}$$  (3.16)

This function has the same symmetry and analyticity properties, as the first one, including the gap in imaginary part at $z = \lambda + i0$

$$\sqrt{1 - \frac{a^2}{\lambda^2}} \Im \mathcal{Q}'' = \theta \left(\lambda^2 - a^2\right) [(\Im \mathcal{P})^2]''$$  (3.17)

As for the real parts, they are related at $\lambda^2 > a^2$ as follows

$$\sqrt{1 - \frac{a^2}{\lambda^2}} \Re \mathcal{Q}'' = \frac{1}{D} \left[(\Re \mathcal{P})^2\right]'$$  (3.18)

At finite gap, we cannot eliminate the derivatives, because we cannot include the factor $\sqrt{1 - \frac{a^2}{z^2}}$ in $\mathcal{Q}''(z)$ without introducing the singularity at $z = 0$. Still, the equations are so simple and universal, that one may hope to find the analytic solution, like the one we found in the strong coupling phase.

4 The Mass Spectrum in the Strong Coupling Phase

Let us now substitute the above general solution (2.9) into the wave equation, found in the previous paper [3]. In present notations, with

$$M_{eff}^2(z) \equiv \frac{DU''(z) - 2D^2 - 2D + 2}{D - 1} + \text{polynomial} = \tau_0 + \tau_1 z^2 + \ldots,$$  (4.1)
the wave equation reads (at $z = \lambda + i0$)

$$\frac{1}{2} \left( P^2 + M_{\text{eff}}^2 \right) \Im \mathcal{F} = D \Re \mathcal{P} \Im \mathcal{F} - \Re \mathcal{P} \Re \mathcal{F}' + \frac{(D - 1)^2}{D} \Im \mathcal{P} \left( \frac{\Re \mathcal{P} \Im \mathcal{F}}{\Re \mathcal{P}} \right)', \quad (4.2)$$

where $P$ is the Euclidean 4-momentum of the vacuum excitation, corresponding to plane wave fluctuations of $\rho$,

$$\delta \rho(\lambda, x) = \frac{1}{N} e^{ipx} \phi \int_{-\infty}^{\infty} d\nu \frac{\Im \mathcal{F}(\nu + i0)}{\pi^2 \rho(\nu)} \frac{1}{\nu - \lambda}. \quad (4.3)$$

The term in the ratio of real and imaginary parts of $\mathcal{P}$ yields the $\delta(\lambda)$ term which drops provided

$$\Re \mathcal{P} \Im \mathcal{F} = 0 \text{ at } z = 0. \quad (4.4)$$

This boundary condition selects the physical solutions.

In the simplest nontrivial case of the mass term plus quartic interaction $U(\lambda) = \frac{1}{2} m_0^2 + \frac{1}{4} \lambda_0 \lambda^4$ there are two terms $\tau_0, \tau_1$ present in effective mass term. The first term $\tau_0$ must vanish as $z_0^{-1}$ to be relevant in the critical region $z \sim z_0$. This yields the equation

$$\tau_0 \propto B^{\delta_0}; \quad \delta_0 = \frac{\alpha - \frac{1}{2}}{\alpha - \beta}. \quad (4.5)$$

In terms of the original parameters of the scalar potential, above relation describes the curve of the first order phase transitions in the $m_0^2, \lambda_0$ plane, ending at the critical point, where $\tau_0 = B = 0$. This is in qualitative agreement with the simulations of [4].

When the $\alpha > 3$ solution is taken, the first $\tau_1$ correction to $M_{\text{eff}}^2$ is relevant. The similar estimate yields the scaling relation

$$\tau_1 \propto B^{\delta_1}; \quad \delta_1 = \frac{\alpha - \frac{3}{2}}{\alpha - \beta}. \quad (4.6)$$

This would correspond to the tricritical point. In general, for $\alpha > 2m + 1$ the $\tau_m \phi^{2m}$ terms in $M_{\text{eff}}^2$, coming from the $\phi^{2m+2}$ terms in $U(\phi)$, become relevant,

$$\tau_m \propto B^{\delta_m}; \quad \delta_m = \frac{\alpha - 2m - \frac{1}{2}}{\alpha - \beta}. \quad (4.7)$$

Let us denote

$$\alpha = k + \mu; \quad \beta = k - \mu; \quad \mu > 0. \quad (4.8)$$

Note that the mass indexes

$$\delta_m = \frac{1}{2} + \frac{k - 2m - \frac{1}{2}}{\mu}, \quad (4.9)$$

are trivial only for the simplest fixed point, with $k = 1, m = 0$. In general, these are transcendental numbers, which agrees with the induced QCD scenario, and contradicts the Gaussian fixed point for the scalar field.
Consider the infinite sum of power terms for $\mathcal{F}$

$$
\mathcal{F} = \sum_{\epsilon} f(\epsilon) \frac{(-iz)^{\epsilon}}{\sin \frac{\pi(\epsilon - s)}{2}},
$$

(4.10)

where $s = \{0, 1\}$ is the "$\lambda$-parity"

$$
\mathcal{F}(-z) = (-1)^s \mathcal{F}(z); \quad \delta \rho(-\lambda, x) = (-1)^s \delta \rho(\lambda, x).
$$

(4.11)

Differentiating real and imaginary parts of $\mathcal{P}, \mathcal{F}$, multiplying by $\Im \mathcal{P}$ and collecting the power terms, we find the following equation

$$
\sum_{\epsilon} f(\epsilon) \lambda^\epsilon \left[ -\frac{1}{2}(P^2 + M_{eff}^2(\lambda)) \left( A\lambda^\mu + B\lambda^{-\mu} \right) \right]
= \sum_{\epsilon} \epsilon f(\epsilon) \lambda^{\epsilon+k-1} \left[ A^2 \Phi_2(\epsilon) \lambda^{2\mu} + 2AB\Phi_0(\epsilon) + B^2 \Phi_{-2}(\epsilon) \lambda^{-2\mu} \right],
$$

(4.12)

where

$$
M_{eff}^2(\lambda) = \sum_{m=0}^{[\frac{k+1}{2}]} \tau_m \lambda^{2m},
$$

(4.13)

$$
\Phi_{\pm2}(\epsilon) = \epsilon \cot \left( \frac{\pi(\epsilon - s)}{2} \right) \pm \frac{(D-1)^2}{D} \epsilon (k \pm \mu) \tan \frac{\pi \alpha}{2},
$$

$$
\Phi_0(\epsilon) = \epsilon \cot \left( \frac{\pi(\epsilon - s)}{2} \right) + \mu \frac{D^2 + 2(D-1)^2}{D} \tan \frac{\pi \alpha}{2}.
$$

Let us consider the simplest case $k = 1$, where $\delta_0 = \frac{1}{2}$, and $M_{eff}^2 = \tau_0$. In this case, it is clear from the above equation, that $f(\epsilon) = 0$ unless

$$
\epsilon = \epsilon_0 - n\mu; \quad n = 0, 1, \ldots,
$$

(4.14)

where the highest power $\epsilon_0$ is determined by the equation

$$
\Phi_2(\epsilon_0) = 0.
$$

(4.15)

This highest power term was already found in the previous paper.

We solved this equation numerically in four dimensions and we found the following values of $\epsilon_0$ for two lowest values of $\alpha^\Lambda$, in spectroscopic notations $\Lambda = (-1)^s$

$$
1.36901^+ : \epsilon_0 = \{2.08496, 4.11512, 6.13072, 8.14028, \ldots \},
$$

(4.16)

$$
1.36901^- : \epsilon_0 = \{1.05590, 3.10290, 5.12399, 7.13601, \ldots \},
$$

$$
2.63099^+ : \epsilon_0 = \{1.94572, 3.91608, 5.89759, 7.88500, \ldots \},
$$

(4.17)

$$
2.63099^- : \epsilon_0 = \{2.92897, 4.90587, 8.88011, 10.87223, \ldots \}.
$$

Let us pick up a particular $\epsilon_0$, and let us study the arising recurrent equation for the coefficients

$$
f(\epsilon) = \sum_{k=1}^{4} W_k(\epsilon) f(\epsilon + k\mu),
$$

(4.17)
where

\[ W_1(\epsilon) = -\frac{(P^2 + \tau_0)}{2A\Phi_2(\epsilon)}, \quad (4.18) \]

\[ W_2(\epsilon) = -\frac{2B\Phi_0(\epsilon)}{A\Phi_2(\epsilon)}, \]

\[ W_3(\epsilon) = -\frac{B(P^2 + \tau_0)}{2A^2\Phi_2(\epsilon)}, \]

\[ W_4(\epsilon) = -\frac{B^2\Phi_2(\epsilon)}{A^2\Phi_2(\epsilon)}. \]

We could write down the formal solution

\[ f(\epsilon_0 - n\mu) = \prod_{l=n}^{l=1} \left( \sum_{k=1}^{4} W_k(\epsilon_0 - l\mu) \exp \left( k \frac{d}{d\epsilon_0} \right) \right) f(\epsilon_0), \quad (4.19) \]

where it is implied, that \( f(\epsilon) = 0 \) at \( \epsilon > \epsilon_0 \), and the operator ordering is as indicated, i.e., from \( l = n \) to \( l = 1 \).

These coefficients should terminate at the smallest \( \epsilon > -\beta \), according to our boundary condition \((4.4)\)

\[ f(\epsilon_0 - n_0\mu) = 0; \quad n_0 = \left\lfloor \frac{\epsilon_0 + \beta}{\mu} \right\rfloor. \quad (4.20) \]

This provides us with the spectral equation, which is a polynomial in \( P^2 \). The roots \( \xi_n \) of this polynomial correspond to the particle spectrum. Restoring quantum numbers,

\[ -P^2 = \tau_0 + \sqrt{AB}\xi_n(s, \epsilon_0). \quad (4.21) \]

The corresponding roots for the lowest levels of \( \epsilon_0 \) are

\[ 1.36901^+, \epsilon_0 = 2.08496 : \xi_n = \pm\{6.94993 \pm 25.62917 i, 11.64012 \pm 7.61785 i\}, \quad (4.22) \]

\[ 1.36901^-, \epsilon_0 = 1.05590 : \xi_n = \{0, 0, 0, 0\}; \]

\[ 2.63099^+, \epsilon_0 = 1.94572 : \xi_n = 0, \]

\[ 2.63099^-, \epsilon_0 = 3.91608 : \xi_n = \{0, 0, 0\}, \]

\[ 2.63099^-, \epsilon_0 = 2.92897 : \xi_n = \pm 49.97203, \]

\[ 2.63099^-, \epsilon_0 = 4.90587 : \xi_n = \{0, 0, 0\}. \]

With large enough \( \tau_0 \) there exist tachyon-free solutions for the second fixed point, \( \alpha = 2.63099 \), but apparently, there are no solutions with infinitely rising masses, because of the sign degeneracy \( \xi \to -\xi \) in both fixed points.

This is another indication of the triviality of the \( k = 1 \) fixed points (they are the only ones with rational critical index \( \delta_0 = \frac{1}{2} \) of the mass spectrum). Most likely, there is the finite number of the free scalar particles, i.e., these are just the Gaussian fixed points.

The case \( k > 1 \) is much more complicated, as there are also negative integer powers of \( \lambda \) involved, apart from powers of \( \lambda^{-\mu} \) in the expansion. We are going to study this case in the next paper.
To summarize, there are two branches of the spectrum. The odd $\lambda$-parity states represent the scalar particles, dressed by interaction with the gauge fields. The even $\lambda$-parity states represent the mixture of “glueballs” with the even number of scalar particles. The mass scale behaves as certain irrational power of initial parameters of the lattice model.

In the strong coupling phase for the two simplest fixed points with rational scaling indexes for masses, we computed the particle spectrum. One fixed point, with $\rho(0) = 0$, turned out unstable, and in the other one, with $\rho(0) = \infty$, there were stable solutions. However, the spectrum did not rise to infinity, as one would expect in QCD, which indicates triviality of these fixed points. We leave for future study the exciting numerical problem of computing masses from above equations of higher critical points in the strong coupling phase.

So far, we cannot even tell, whether the spectrum terminates, and whether there are tachyons. Perhaps, some general inequalities can be derived to answer this question. One way to guarantee the absence of tachyons in the given fixed point is is to arrive at this fixed point from the lattice MFE, with real potential, stable at infinity. This requires the serious numerical study of the MFE, and/or the simulations of the initial lattice model.

5 Acknowledgements

I would like to thank the theory groups of Ecole Normale and Jussieu in Paris for their hospitality, and David Gross, Volodja Kazakov and Ivan Kostov for interesting discussions. This work was partially supported by the National Science Foundation under contract PHYS-90-21984.

References

[1] V.A.Kazakov and A.A.Migdal, Induced QCD at large N, Paris / Princeton preprint LPTENS-92/15 / PUPT-1322 (June, 1992)
[2] A.A.Migdal, Exact solution of induced lattice gauge theory at large N, Princeton preprint PUPT-1323 (June, 1992)
[3] A.A.Migdal, 1/N expansion and particle spectrum in induced QCD, Princeton preprint PUPT-1332 (July, 1992)
[4] A.Gocksch and Y.Shen, The phase diagram of the $N = 2$ Kazakov-Migdal model, BNL preprint (July, 1992);
[5] D.Gross, Some remarks about induced QCD, Princeton preprint PUPT-1335 (August, 1992)
[6] M.Caselle, A.D.‘Adda and S.Panzeri, Exact solution of $D=1$ Kazakov-Migdal induced gauge theory, Turin preprint DFTT 38/92 (July, 1992);
[7] S.B.Khokhlachev and Yu.M.Makeenko, Phys. Lett. 101B (1981) 403; ZhETF 80 (1981) 448 (Sov. Phys. JETP 53 (1981) 228)
[8] I.G.Holliday and A.Schwimmer, Phys. Lett. 101B (1981) 327; J.Greensite and B.Lautrup, Phys. Rev. Lett 47 (1981) 9; G.Bhanot, Phys. Lett. 108B (1982) 337; M.Creutz and K.J.M.Moriarty, Nucl. Phys. B210[FS6] 50
[9] Yu.M.Makeenko and M.I.Polikarpov, Nucl. Phys. B205[FS5] (1982) 386; S.Samuel, Phys. Lett. 112B (1982) 237, 122B (1983) 287
[10] J.Greensite and B.Lautrup, *Phys. Lett.* **104B** (1981) 41; P.Cvitanović, J.Greensite and B.Lautrup, *Phys. Lett.* **105B** (1981) 197; T.-L.Chen, C.-I Tan and X.-T.Zheng, *Phys. Lett.* **109B** (1982) 383; *Phys. Rev.* **D26** (1982) 2843; M.C.Ogilvie and A.Horowitz, *Nucl.Phys.* **B215** (1983) 249

[11] I.G.Holliday and A.Schwimmer, *Phys. Lett.* **102B** (1981) 337; R.C.Brower, D.A.Kessler and H.Levine, *Nucl. Phys.* **B205[FS5]** (1982) 77; L.Caneschi, I.G.Holliday and A.Schwimmer, *Nucl. Phys.* **B200[FS4]** (1982) 409

[12] I.I.Kogan, G.W.Semenoff and N.Weiss, *Induced QCD and hidden local $Z_N$ symmetry*, UBC preprint UBCTP-92-022 (June, 1992)

[13] S.B.Khokhlachev and Yu.M.Makeenko,*The problem of large-N phase transition in Kazakov-Migdal model of induced QCD*, ITEP-YM-5-92, (July, 1992)

[14] I.I.Kogan, A.Morozov, G.W.Semenoff and N.Weiss, *Area law and continuum limit in ”induced QCD”*, UBC preprint UBCTP-92-022 (June, 1992)