VIRTUAL ALGEBRAIC FIBRATIONS OF SURFACE-BY-SURFACE GROUPS AND ORBITS OF THE MAPPING CLASS GROUP

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Abstract. We show that a conjecture of Putman–Wieland, which posits the nonexistence of finite orbits for higher Prym representations of the mapping class group, is equivalent to the existence of surface-by-surface and surface-by-free groups which do not virtually algebraically fiber. While the question about the existence of such groups remains open, we will show that there exist free-by-free and free-by-surface groups which do not algebraically fiber (hence fail to be virtually RFRS), and study some properties of the normal subgroups of those groups, as well as those of potential counterexamples to the Putman–Wieland conjecture.

1. Introduction

In this paper we will be concerned with the study of homological and group-theoretical properties of group extensions of the form

\[(1) \quad 1 \longrightarrow K \longrightarrow G \xrightarrow{f} \Gamma \longrightarrow 1,\]

where \(K\) and \(\Gamma\), the fiber and base groups, are fundamental groups of closed, compact, orientable surfaces of genus at least 2, or finitely generated nonabelian free groups. The study of this type of extensions has quite a long story, originating in the realm of free-by-free groups and, in the broader context that includes surface groups, in several papers of F.E.A. Johnson, see e.g. [Jo99] and references therein.

The original motivation of our interest stems from a topological framework. Let \(X\) be a surface bundle with fiber \(F\) over a surface \(B\), both having genus at least 2. Then its fundamental group \(G = \pi_1(X)\) is an extension as in Equation (1) with \(K := \pi_1(F)\), \(\Gamma := \pi_1(B)\) and where \(f: G \to \Gamma\) is the map induced on the fundamental groups (after picking a basepoint) by the fibration \(X \to B\). This sequence is the nontrivial part of the long exact sequence of the homotopy groups for the fibration. In this case, we will refer to \(G\) as a surface-by-surface group. In the other cases, while the extensions are not the fundamental group of a closed aspherical 4-manifold, they share similarities with surface-by-surface groups. Note that when the base \(\Gamma\) is the free group \(F_n\), \(G\) can always be thought of as a semidirect product \(K \rtimes F_n\), while this may fail in general.

We will focus on those extensions with the property that the induced map in homology with rational coefficients \(f: H_1(G; \mathbb{Q}) \to H_1(\Gamma; \mathbb{Q})\) is an isomorphism. (This map is always surjective.) This has been referred to by saying that the sequence of Equation (1) has no excessive homology [KW19]. This is a property that depends on \(G\) alone, and not on the choice of the extension.

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Given any group $G$ as in Equation (1), we can ask whether it admits a finite-index subgroup with excessive homology. By the work of [FV19, KW19] this is equivalent to the property that $G$ virtually algebraically fibers, namely a finite index subgroup of $G$ admits an epimorphism to $\mathbb{Z}$ with finitely generated kernel. Our main aim is to show that in the case where $K = \pi_1(F)$ is a surface group there is a natural relation between this question and the so-called Putman–Wieland conjecture [PW13, Conjecture 1.2]. We refer the reader to Section 2 and the original source for more detail, but the case labeled as NFO($g$, 0, 0) of that conjecture posits that the action of the mapping class group $\text{Mod}_{1g}$ of the surface $F$ (where $g$ is the genus of $F$) on the rational homology of finite-index characteristic covers of $F$ (sometimes referred to as higher Prym representation) has no finite orbits. We will show the following.

**Theorem 3.3.** For every $g \geq 2$ the Putman–Wieland conjecture NFO($g$, 0, 0) holds if and only if there exists a surface-by-surface or a surface-by-free group $G$ with fiber genus $g$ and no virtual excessive homology or, equivalently, that is not virtually algebraically fibered.

Note that the surface-by-surface and surface-by-free group mentioned in the statement, if they exist, will fail to be virtually RFRS in light of [K20].

Assuming the Putman-Wieland conjecture, this theorem would differentiate the behavior of 3- and 4-dimensional surface bundles. Marković [Mark22] has shown that the conjecture fails when the genus of the surface is 2. Thus by Theorem 3.3 all genus 2 surface-by-surface and genus 2 surface-by-free groups are virtually algebraically fibered.

The proof of Theorem 3.3 shows that the base of $G$ can be assumed to be the fundamental group of a surface of genus 2, or the free group $F_2$. The fact that these bases are optimal is not quite obvious, see Proposition 3.4. Additionally, the corresponding surface bundle over a surface $X$ can be assumed to have signature zero.

The “only if” part of Theorem 3.3 uses an epimorphism from the base of the extension to the mapping class group $\text{Mod}_{1g}$, so that we can make the construction of the groups deciding the conjecture $NF(g, 0, 0)$ very explicit, especially in the case where $g \geq 3$. We illustrate this in the surface-by-free case: Let $F$ be a surface of genus $g \geq 3$; this admits a cyclic automorphism of order $4g + 2$, and denote by $\Pi$ the fundamental group of the mapping torus of this automorphism. (This is a Seifert–fibered manifold, finitely covered by a product.) Any two such automorphisms are conjugate in $\text{Mod}_{1g}$, so that $\Pi$ is uniquely determined as a group. By [Kor05] $\text{Mod}_{1g}$ can be generated by two such automorphisms, related by conjugation by the automorphism $\delta: K \rightarrow K$ induced by a Dehn twist along a nonseparating curve. (See Section 3 for references and more details).

**Proposition 3.5.** Let $v: K \rightarrow K$ be the generator of a cyclic subgroup of order $4g + 2$ of $\text{Mod}_{1g}$ for $g \geq 3$, and denote by $\Pi$ the corresponding mapping torus. Let $\Pi *_{\delta} \Pi$ be the amalgamated free product determined by the automorphism $\delta: K \rightarrow K$. Then the Putman–Wieland conjecture NFO($g$, 0, 0) holds if and only if the surface-by-F2 group $\Pi *_{\delta} \Pi$ fails to virtually algebraically fiber.

The group $\Pi *_{\delta} \Pi$ surjects onto $\text{Mod}_{1g}$. In a sense, the latter is “one Dehn twist away” from being the product $K \times \text{Mod}_{1g}$.

In analogy with Theorem 3.3 one may ask about the existence of free-by-free or free-by-surface groups with no virtual excessive homology hence which do not virtually algebraically fiber.
fiber. In this realm we can reach in most cases an affirmative answer, which naturally extends also to the case where the fiber group is free abelian.

**Theorem 4.1.** For each \( n \geq 2 \), and each \( m \geq 4 \) (respectively \( m \geq 2 \)) there exist groups of the form \( F_m \rtimes \Gamma \) (respectively \( \mathbb{Z}^m \rtimes \Gamma \)), where \( \Gamma \) is a copy of \( F_n \) or a the fundamental group of a surface of genus \( n \), with no virtual excessive homology or, equivalently, that are not virtually algebraically fibered. These groups are not virtually RFRS.

Note, in contrast, that these groups are virtually RFS (residually finite solvable), see Lemma 5.3. We point out, however, that Theorem 4.1 has no implication on the suitable analog of the Putman-Wieland conjecture in the context of automorphisms of free groups, which is known to be true in light of [FH17].

In section 5 we state and prove some results, on the nature of the finitely generated normal subgroups and the virtual homological torsion for the class of non-virtually fibered extensions determined (in part conditionally) in Theorem 3.3 and Theorem 4.1. Section 6 proves some results on coherence and incoherence and speculates on the geometry of subgroups.

This paper is organized as follows: In Section 2 we give background on the homology of the extensions we are considering, and of their finite index subgroups, and state the Putman-Wieland conjecture. Theorem 3.3 and Proposition 3.5 are proven in Section 3 while Theorem 4.1 is proven in Section 4. The results on subgroups and virtual homological torsion of non-virtually fibered extensions are presented in Section 5. Incoherence and coherence is discussed in Section 6.

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## 2. Preliminaries

It will be useful in what follows to tie the sequence in Equation (1) with the monodromy representation determined by the extension (1). Namely, we have

\[
\begin{array}{ccc}
K & \xrightarrow{\zeta} & \text{Aut}(K) \\
\cong & & p \\
\downarrow & & \downarrow \\
G & \xrightarrow{f} & \Gamma \\
\eta & & \\
\end{array}
\]

Here \( \eta \) characterizes the action of \( \Gamma \) on \( K \), well-defined up to conjugation. The map \( K \to \text{Aut}(K) \) is given by the conjugation action, and it is injective as \( K \) has trivial center. This allows us to identify \( G \) as the pullback

\[
G \cong \{(\psi, \gamma) \in \text{Aut}(K) \times \Gamma \mid p(\psi) = \eta(\gamma)\},
\]

with the group structure obtained by restriction of that on \( \text{Aut}(K) \times \Gamma \). The fibration map \( f \) is induced by projection onto the second factor, and the fiber subgroup is given by the
normal subgroup

\[ K \times \{1\} \leq G \leq \text{Aut}(K) \times \Gamma. \]

With this identification, the conjugation action of \( G \) on its normal subgroup \( K \) can be written in terms of the conjugation action of \( \text{Aut}(K) \times \Gamma \) on \( K \times \{1\} \). Note that when \( K \) is a surface group of genus \( g > 1 \), we will be interested in the case where the monodromy representation \( \eta: \Gamma \rightarrow \text{Out}(K) \) has values in \( \text{Mod}_g \leq \text{Out}(K) \), and \( \zeta: G \rightarrow \text{Aut}(K) \) has values in \( \text{Mod}_1 \leq \text{Aut}(K) \) (with both modular groups subgroups of index 2 determined by orientation-preserving homeomorphisms of a surface). This condition, which is equivalent to the fact that the corresponding surface bundle is oriented, is not too restrictive and can be achieved by passing to an index-two subgroup of the base of the extension. In such case, the bottom row of Equation (2) can be interpreted as Birman’s short exact sequence.

2.1. Coinvariants and homology. We wish to understand the homology of \( G \) in terms of that of \( K \) and \( \Gamma \). If \( G \) is a group and \( M \) is a \( G \)-module, the co-invariants of \( M \), \( M^G \), is the quotient of \( M \) obtained by taking the quotient generated by elements of the form \( gm - m \), for all \( g \in G \), \( m \in M \). The invariants of \( M \), \( M^G \), is the largest submodule of \( M \) on which \( G \) acts trivially.

**Definition.** Let \( 1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1 \) be an extension. The excessive homology of this extension is the kernel of the map \( H_1(G; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q}) \).

In the cases that we will be interested in, the excessive homology can be conveniently expressed in terms of (co)invariant homology. Specifically, we have the following two Lemmata. The first applies, in generality, for any semidirect product of finitely presented groups.

**Lemma 2.1.** Let \( G = K \rtimes \Gamma \). Then the excessive homology of \( G \) is \( H_1(K; \mathbb{Q})^G \).

**Proof.** Choose a presentation for \( K \) and \( \Gamma \) of the form \( K = \langle Y | S \rangle \) and \( \Gamma = \langle X | R \rangle \); then a semidirect product of \( K \) and \( \Gamma \) admits a presentation

\[ \langle X, Y | R, S, x^{-1}yx = \phi(x)(y), x \in X, y \in Y \rangle \]

where the monodromy map \( \phi: \Gamma \rightarrow \text{Aut}(K) \) encodes the structure of semidirect product (see e.g. [JoD97 Section 10.3(S)]). The abelianization of this presentation yields

\[ H_1(G; \mathbb{Z}) = H_1(\Gamma; \mathbb{Z}) \oplus H_1(K; \mathbb{Z})/\langle (\Phi - I)(H_1(K; \mathbb{Z})) \rangle, x \in X = H_1(\Gamma; \mathbb{Z}) \oplus H_1(K; \mathbb{Z})^\Gamma \]

where \( \Phi \) denotes the homological monodromy map \( \Phi: \Gamma \rightarrow \text{Aut}(K) \rightarrow GL(n, \mathbb{Z}) \) where \( n = \text{rk}(K) \). The group \( K \) acts trivially on its homology, so that the excessive homology is \( H_1(K; \mathbb{Q})^G \).

The second yields a more refined result for any extension as in Equation (1), as long as the fiber is a surface group.

**Lemma 2.2.** Let \( G \) be an extension as in Equation (1) and assume that \( K \) is a surface group. Then the excessive homology of \( G \) is \( H_1(K; \mathbb{Q})^G \); furthermore it is isomorphic to \( H_1(K; \mathbb{Q})^G \).
Proof. The excessive homology of surface-by-free groups is given above in Lemma 2.1. Surface-by-surface groups do not always split. However, the same result holds in general. One way to do this is to compute the homology of the 4-manifold $E$, which is the associated surface bundle over a surface. By Johnson [JoF79], see also [Hi02, Theorem 5.2], every extension of a surface group by a surface group is realized by a surface bundle over a surface.

A surface bundle over a surface can be decomposed into a surface bundle over a surface with a puncture (with a surface-by-free fundamental group) and a surface bundle over a disk. We denote the surface bundle over the punctured surface by $A$ and the surface bundle over the disk by $B$. Using that $E = A \cup B$ and Mayer-Vietoris, it is proven in [Campa16, Proposition 3] that $H_1(E; \mathbb{Z}) = H_1(A; \mathbb{Z})$. In particular, the homology is completely determined by the map $\Gamma \to \text{Mod}_g \to \text{Sp}(2g, \mathbb{Z})$.

As $K$ is a surface group, $G$ acts on $H_1(K; \mathbb{Q})$ preserving its intersection form. The symplectic structure on $H_1(K; \mathbb{Q})$ induces therefore an isomorphism of $G$–spaces between $H_1(K; \mathbb{Q})$ and its dual $H_1(K; \mathbb{Q})^G$. But $H^1(K; \mathbb{Q})^G$ is dual to $H_1(K; \mathbb{Q})_G$ (see e.g. [PW13, Lemma 2.1]) so the last part of the statement follows. □

It is not too hard to verify that the existence of excessive homology is preserved by passing to finite index subgroups:

Lemma 2.3. Let $1 \to K \to G \to \Gamma \to 1$ be an extension, and let $\widetilde{G} \triangleleft G$ be a finite index subgroup. Then the excessive homology of the induced extension on $\widetilde{G}$ surjects onto that of $G$.

Proof. There exists a commutative diagram

$$
\begin{array}{cccccc}
\widetilde{K} & \xrightarrow{} & \widetilde{G} & \xrightarrow{} & \tilde{\Gamma} \\
K & \xleftarrow{} & G & \xrightarrow{} & \Gamma \\
\end{array}
$$

with self-explaining notation. This induces a commutative diagram for the homology with rational coefficients

$$
\begin{array}{cccc}
H_1(\widetilde{K}; \mathbb{Q}) & \xrightarrow{} & H_1(\widetilde{G}; \mathbb{Q}) & \xrightarrow{} & H_1(\widetilde{\Gamma}; \mathbb{Q}) \\
H_1(K; \mathbb{Q}) & \xrightarrow{} & H_1(G; \mathbb{Q}) & \xrightarrow{} & H_1(\Gamma; \mathbb{Q}) \\
\end{array}
$$

All the vertical maps in the diagram are corestriction maps on the homology with rational coefficients of a finite-index subgroup (see e.g. [Br94, Chapter III]), and they are epimorphisms because of the existence of a right–inverse, a suitable multiple of the restriction map. (This necessitates the use of rational coefficients.) Therefore there exists a surjection

$$
\text{Ker}(H_1(\tilde{G}; \mathbb{Q}) \to H_1(\tilde{\Gamma}; \mathbb{Q})) \to \text{Ker}(H_1(G; \mathbb{Q}) \to H_1(\Gamma; \mathbb{Q}))
$$

which means that the excessive homology of the sequence $1 \to K \to G \to \Gamma \to 1$ cannot decrease passing to finite index subgroups of $G$. □
Lemma 2.3 implies that we can reduce the question of the existence of virtual excessive homology to finite regular covers, so we focus on these covers, which are determined by an epimorphism $\alpha : G \to S$ onto a finite group $S$, completing the sequence in Equation (1) to the following commutative diagram of short exact sequences:

\[
\begin{array}{ccccccc}
\tilde{K} & \to & \tilde{G} & \to & \tilde{\Gamma} \\
\downarrow & & \downarrow & & \downarrow \\
K & \to & G & \to & \Gamma \\
\downarrow & & \downarrow & & \downarrow \\
\alpha(K) & \to & S & \to & S/\alpha(K)
\end{array}
\]

In general, the action of $\tilde{G}$ on $H_1(\tilde{K}; \mathbb{Q})$ does not extend to $G$, but it will do so when $\tilde{K} \triangleleft K$ is characteristic: such action is induced by restriction of the $G$–action on $K$, which preserves $\tilde{K}$ as well as $[\tilde{K}, \tilde{K}]$ whenever $\tilde{K}$ is characteristic. Conversely, given any finite index normal subgroup $\tilde{K} \triangleleft K$, it is known from (Mo87, Lemma 4.1) that there exists a finite index normal subgroup $\tilde{G} \triangleleft G$ whose intersection with $K$ is the subgroup $\tilde{K} \triangleleft K$. (Mo87, Lemma 4.1) is stated only for surface groups, but the proof applies to free group as well.) Again, the action of $\tilde{G}$ on $H_1(\tilde{K}; \mathbb{Q})$ may fail to extend to $G$, but will do so whenever $\tilde{K}$ is characteristic. As any finite index normal subgroup $\tilde{K} \triangleleft K$ contains a finite index subgroup which is characteristic in $K$, we can further reduce the study of excessive homology to covers of $X$ which induce fiberwise characteristic subgroups of $K$.

3. Virtual excessive homology and the Putman–Wieland conjecture

3.1. The Putman-Wieland Conjecture. In [PW13], the authors connect the study of the orbits of the mapping class group acting on the first homology of a surface and its finite covers to the classical conjecture that mapping class groups do not virtually surject $\mathbb{Z}$. Denote by $\text{Mod}_{g,n}^p$ the mapping class group of a surface $\Sigma_{g,n}^p$ with genus $g$, $p$ punctures and $n$ boundary components. (We use $\text{Mod}_1^1$ for the case $p = 1, n = 0$ and $\text{Mod}_g$ for the closed case, which is the relevant case here.) When $\tilde{K} \triangleleft K = \pi_1(\Sigma_{g,n}^p)$ is a characteristic subgroup, the group $\text{Mod}_{g,n}^{p+1}$ acts on $V_{\tilde{K}} := H_1(\tilde{K}; \mathbb{Q})$. They posit the following:

Conjecture 3.1. [PW13, Conjecture 1.2] Fix $g \geq 2$ and $n, p \geq 0$. Let $\tilde{K} \triangleleft K = \pi_1(\Sigma_{g,n}^p)$ be a finite-index characteristic subgroup. Then for all nonzero vectors $v \in V_{\tilde{K}}$, the $\text{Mod}_{g,n}^{p+1}$-orbit of $v$ is infinite.

For fixed $g \geq 2$ and $n, p \geq 0$, the conjecture above will be referred to as NFO$(g, n, p)$.

3.2. Connections with virtually excessive homology. Our first result connects virtual excessive homology of a surface-by-surface or surface-by-free group $G$ with the the orbits on the homology of characteristic subgroups of the fiber group $K$.

Lemma 3.2. Let $G$ be a surface-by-surface or a surface-by-free group. Then the following two properties are equivalent:

(1) $G$ has no virtual excessive homology;
(2) for any finite index characteristic subgroup $\tilde{K} \leq K$, and any nonzero $v \in H_1(\tilde{K}; \mathbb{Q})$, the orbit $G \cdot v \subset H_1(\tilde{K}; \mathbb{Q})$ is infinite.

Proof. $(1) \Rightarrow (2)$: Assume by contradiction that there exists a finite index characteristic subgroup $\tilde{K} \leq K$, and a nonzero $v \in H_1(\tilde{K}; \mathbb{Q})$ with finite orbit $G \cdot v$. As discussed before, $G$ admits a normal finite index subgroup $\tilde{G} \leq G$ whose intersection with $K$ is the subgroup $\tilde{K} \leq K$. As $G \cdot v$ is finite, there is a finite index subgroup $H \leq G$ such that the orbit $H \cdot v = \{v\}$. As $\tilde{K}$ acts trivially on its own homology, $\tilde{K} \leq H$. We now replace $\tilde{G}$ with its finite index subgroup $H \cap \tilde{G}$; namely, we take a finite index subgroup of $\tilde{G}$ defined by the pull–back of a suitable finite index subgroup of $\tilde{K}$. Going to the normal core of $H \cap \tilde{G}$ in $G$ if needed, we can assume $H \cap \tilde{G} \leq G$ is normal; this normal core will still contain $\tilde{K}$ by the assumption that the latter is characteristic in $K$.

Hoping that no risk of confusion arises, we maintain the notation $1 \rightarrow \tilde{K} \rightarrow \tilde{G} \rightarrow \tilde{\Gamma} \rightarrow 1$ for the ensuing normal subgroup of $G$: we stress that $\tilde{K}$ has not changed in the process. With this notation in place, we have that $\tilde{G} \cdot v = \{v\}$, whence the space of invariants $H_1(\tilde{K}; \mathbb{Q})^\tilde{G}$ is nontrivial. As $\tilde{G}$ acts on $H_1(\tilde{K}; \mathbb{Q})$ preserving its intersection form, the symplectic structure on $H_1(\tilde{K}; \mathbb{Q})$ induces an isomorphism of $G$–spaces between $H_1(\tilde{K}; \mathbb{Q})$ and its dual $H^1(\tilde{K}; \mathbb{Q})$; consequently, $H^1(\tilde{K}; \mathbb{Q})^\tilde{G} \cong H_1(\tilde{K}; \mathbb{Q})^\tilde{G}$. The latter vector space is nontrivial, as it contains the span of $v$. But $H^1(\tilde{K}; \mathbb{Q})^\tilde{G}$ is dual to $H_1(\tilde{K}; \mathbb{Q})_G^\ast$ (see e.g. [PW13 Lemma 2.1]), hence $\dim H_1(\tilde{K}; \mathbb{Q})_G > 0$. It follows that $\tilde{G}$ has excessive homology.

$(2) \Rightarrow (1)$: Let $\tilde{G}$ be any finite index normal subgroup of $G$. As any surface group $\tilde{K}$ contains a finite index subgroup that is characteristic in $K$, we can assume without loss of generality that $\tilde{K}$ is characteristic. By assumption, for any nonzero $v \in H_1(\tilde{K}; \mathbb{Q})$ the orbit $G \cdot v$ is infinite. As $\tilde{G} \leq G$, so must be the orbit $\tilde{G} \cdot v$. The space of invariants $H_1(\tilde{K}; \mathbb{Q})^\tilde{G}$ is trivial, and proceeding as above so is the space of coinvariants $H_1(\tilde{K}; \mathbb{Q})_G$, hence $\tilde{G}$ has no excessive homology.

Next, we show the equivalence of the existence of a surface-by-surface or surface-by-free group with no virtual excessive homology and the case NFO($g$, 0, 0) (Conjecture 1.2 of [PW13] for $\Sigma^0_{g,0}$) of the Putman–Wieland Conjecture.

**Theorem 3.3.** For every $g \geq 2$ the Putman–Wieland conjecture NFO($g$, 0, 0) holds if and only if there exists a surface-by-surface or a surface-by-free group $G$ with fiber genus $g$ and no virtual excessive homology or, equivalently, that is not virtually algebraically fibered.

Proof. Let $G$ be an extension as in the statement. We claim that for any finite index characteristic subgroup $\tilde{K} \leq K$, and any nonzero $v \in H_1(\tilde{K}; \mathbb{Q})$ the orbit $\text{Mod}^1_g \cdot v \subset H_1(\tilde{K}; \mathbb{Q})$ is infinite. Indeed, by Lemma 3.2 we know that for any finite index characteristic subgroup $\tilde{K} \leq K$ and any nonzero $v \in H_1(\tilde{K}; \mathbb{Q})$ the orbit $G \cdot v \subset H_1(\tilde{K}; \mathbb{Q})$ is infinite, which shows that the orbit $\text{Mod}^1_g \cdot v \supset G \cdot v$ is infinite as well.

To prove the reverse implication, we recall that, given any finite presentation $\Theta$ of $\text{Out}(K)$, there exists a surface bundle $X$ of fiber $F$ over a surface $B$ (whose genus equals the rank $r$ of the presentation) induced by an epimorphism $\eta: \Gamma \rightarrow \text{Mod}_g$. This construction is
due to Kotschick in [Ko98, Proposition 4]; the map $\eta$ is defined by sending the first $r$ generators of $\Gamma = \pi_1(B) = \langle \alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r | \prod_{i=1}^{r} [\alpha_i, \beta_i] \rangle$ to the set of generators of $\text{Out}(K)$, while the remaining $r$ generators are sent to the trivial element. We denote this surface bundle by $X_\Theta$ so that $G = \pi_1(X_\Theta)$ will be the desired surface-by-surface group. Similarly, we can consider the presentation epimorphism $\eta: \Gamma = F_r \to \text{Mod}_{g}$, with $G$ being the induced surface-by-free group. By construction, these extensions are of type I in Johnson’s trichotomy (namely, the monodromy homomorphisms $\eta: \Gamma \to \text{Mod}_{g}$ have infinite kernel and image, see [JoF93]) and, in the surface-by-surface case $X_\Theta$ has signature zero. The virtual excessive homology is determined by the behavior of the orbits of $G$ on the homology of the characteristic subgroups of $K$. As $\eta: \Gamma \to \text{Mod}_{g}$ is surjective, so is $\zeta: G \to \text{Mod}_{1g}$, so the $G$–orbits coincide with the orbits of $\text{Mod}_{1g}$. It follows that $\text{NFO}(g,0,0)$ is true if and only if $G$ has no virtual excessive homology. Note that as there exists presentations of $\text{Mod}_{g}$ with two generators ([Wa96, Kor05]), we can assume that $r = 2$, in particular the base $B$ of the surface bundle over a surface can be chosen to have genus 2.

It is quite straightforward to see that the result above is optimal as far as the base genus is concerned. In fact we have the following:

**Proposition 3.4.** Let $F \hookrightarrow X \xrightarrow{f} T^2$ be a surface bundle over a torus with fiber genus greater or equal than 2. Then $G$ has nonzero virtual excessive homology.

**Proof.** In [FV13] it is proven that the fundamental group of a surface bundle of the type described is large. In particular, this implies that $vb_1(X) = \infty$. Then let $\tilde{X} \to X$ be a finite regular cover, as described in the diagram of Equation (5), with $b_1(\tilde{X}) > 2$. As $\tilde{B}$ is also a torus, the fibration $\tilde{F} \to \tilde{X} \to \tilde{B}$ has excessive homology. □

It is quite interesting, at this point, to ask whether there exist classes of surface bundles for which there is always virtual excessive homology, and hence virtual algebraic fibrations. For instance, this is the case when the fibration is a holomorphic bundle, see e.g. [BHPV04]. In particular, it would be interesting decide this case for the class of Kodaira fibrations, or of surface bundles of type III in Johnson’s trichotomy (injective monodromy). (The surface bundles discussed in Theorem 3.3 cannot be Kodaira fibrations, as these have strictly positive signature, see e.g. [BHPV04].)

In the case where $g \geq 3$, we can use a result of Korkmaz to give a quite explicit description of the type of surface bundles that are involved in the statement of Theorem 3.3. For sake of concreteness, we limit ourselves to the surface-by-free case, that is somewhat more striking. In [Kor05, Section 5] Korkmaz shows that the mapping class group $\text{Mod}_{g}$ can be assumed to be generated by two elements, generators of two cyclic subgroups of order $4g+2$ of $\text{Mod}_{g}$. These generators are conjugated in $\text{Mod}_{g}$ by a Dehn twist along a nonseparating curve. We’ll denote by $\delta: K \to K$ the corresponding automorphism. We have the following:

**Proposition 3.5.** Let $\nu: K \to K$ be the generator of a cyclic subgroup of order $4g+2$ of $\text{Mod}_{g}$ for $g \geq 3$, and denote by $\Pi$ the corresponding mapping torus. Let $\Pi *_{\delta} \Pi$ be the amalgamated free product determined by the automorphism $\delta: K \to K$. Then the Putman–Wielandt conjecture $\text{NFO}(g,0,0)$ holds if and only if the surface-by-$F_2$ group $\Pi *_{\delta} \Pi$ fails to virtually algebraically fiber.
Proof. The proof of this Proposition is a specialization of an argument used in the proof of Theorem 3.3. By [Kor05] there exists a presentation of \( \text{Mod}_g \) of rank 2 in which the two generators \( x, y \) each generate a cyclic subgroup of order \( 4g + 2 \). All these generators are conjugate in \( \text{Mod}_g \) (see e.g. [FM12, Section 7.2.4]), so that in particular the mapping tori of the induced automorphisms of \( K \) are isomorphic. In the case at hand we can assume that the conjugating element is induced by a Dehn twist along a nonseparating curve (see [Kor05, Section 5]). It follows that there exists a commutative diagram of the form

\[
\begin{array}{ccc}
K & \xrightarrow{f} & F(x, y) \\
\downarrow \cong & & \downarrow \eta \\
K & \xrightarrow{\text{Mod}_g^1} & \text{Mod}_g
\end{array}
\]

where \( \eta: F(x, y) \to \text{Mod}_g \) is the presentation quotient. By fiat, \( G \) is the free product of the mapping tori of two automorphisms of \( K \), amalgamated along \( K \). These mapping tori arise as pull-back of the monodromies determined by the two generators \( x \) and \( y \) of \( F(x, y) \), namely they are the unique (up to conjugation) cyclic monodromies of order \( 4g + 2 \) on \( K \). Denoting by \( \Pi \) the resulting mapping torus, well-defined up to isomorphism, the group \( G \) is isomorphic to the free amalgamated product \( \Pi * \delta \Pi \), where the amalgamation is determined by the automorphism \( \delta: K \to K \). The rest of the proof follows exactly as the proof of Theorem 3.3. \( \square \)

Remark. The five term sequence of the Lyndon–Hochschild–Serre spectral sequence tells us that \( H_1(\Pi) = \mathbb{Z} \oplus H_1(K)_{\mathbb{Z}_{4g+2}} \) where we make explicit that the periodic monodromy factorizes through the quotient map \( \mathbb{Z} \to \mathbb{Z}_{4g+2} \). The action of \( \mathbb{Z}_{4g+2} \) on \( K \) determines an orbisurface cover (or, if preferred, a branched cover) whose quotient is an orbisphere with 3 orbifold points of order \( 2, 2g + 1, 4g + 2 \) whose orbifold fundamental group we denote \( \Delta \). We have a short exact sequence

\[
1 \longrightarrow K \longrightarrow \Delta \longrightarrow \mathbb{Z}_{4g+2} \longrightarrow 1.
\]

As \( \mathbb{Z}_{4g+2} \) is torsion, the coinvariant homology \( H_1(K)_{\mathbb{Z}_{4g+2}} \) has the same rank as \( H_1(\Delta) \), namely it is torsion. It follows that \( b_1(\Pi) = 1 \) and \( b_1(\Pi * \delta \Pi) = 2 \).

4. Extensions with no virtual excessive homology

In this section we will show that there are extension of free (and free abelian) groups that have no virtual excessive homology. As we already observed, this does not imply an analog of the NFO conjecture in the realm of free groups. However, it is interesting that the proof hinges on Property (T) for suitable automorphism groups, a theme related with the circle of ideas at the origin of [PW13]. Regarding Property (T), we recall the definitions here; for full details see [BdlHV08].

Definition. [BdlHV08 Def 1.1.3, 1.4.3] A group \( G \) has Property (T), if every unitary representation of \( G \) with almost invariant vectors has a non-trivial invariant vector. A pair of discrete groups \( (G, H) \) with \( H \leq G \) has Relative Property (T), if every unitary representation of \( G \) with almost invariant vectors has a non-trivial \( H \) invariant vector.
(For the precise definition of almost invariant vectors see Definition 1.1 in [BdlHV08].) If $G$ has Property (T), then any quotient of $G$ also has Property (T). Therefore, any amenable discrete quotient of $G$ must be finite. Likewise, if $(G, H)$ has Relative Property (T), then in any amenable discrete quotient of $G$ the image of $H$ must be finite. This follows since an amenable group $K$ has almost invariant vectors in the left-regular representation on $L^2(K)$ (Reitner’s condition). In particular, this holds for abelian quotients, since abelian groups are amenable.

**Theorem 4.1.** For each $n \geq 2$, and each $m \geq 4$ (respectively $m \geq 2$) there exist groups of the form $F_m \rtimes \Gamma$ (respectively $\mathbb{Z}^m \rtimes \Gamma$), where $\Gamma$ is a copy of $F_n$ or a the fundamental group of a surface of genus $n$, with no virtual excessive homology or, equivalently, that are not virtually algebraically fibered. These groups are not virtually RFRS.

**Proof.** In [KNO19, KKN21, Nit20], it is shown that $\text{Aut}(F_m)$ has property (T) for $m \geq 4$. Thus by [KNO19 Prop 10] we see that $F_m \rtimes \text{Aut}(F_m)$ has property (T). Thus if $Q$ is an abelian quotient of a finite index subgroup $H$ of $G = F_m \rtimes \text{Aut}(F_m)$, then $Q$ is finite. In particular, any finite index subgroup of $G$ has finite abelianization. Similarly, the pair $(\mathbb{Z}^m \rtimes SL_m(\mathbb{Z}), \mathbb{Z}^m)$ has Relative Property (T) when $m \geq 2$ [dlHV89 Ex. 1.7.4, 4.2.2], thus given any abelian quotient $Q$ of a finite index subgroup $H$ of $\mathbb{Z}^m \rtimes SL_m(\mathbb{Z})$, then $\mathbb{Z}^m \cap H$ has finite image in $Q$. These are the key properties that make our proof work.

We start with the case where the fiber $K$ is the free group $F_m$ for $m \geq 4$. Let $\eta : \Gamma \to \text{Aut}(F_m)$ be a surjection and build the associated extension $G = F_m \rtimes \Gamma$. Note we can take $\Gamma = F_n, \pi_1(S_g)$ with $g, n = 2$ since $\text{Aut}(F_m)$ is generated by 2 elements [N33]. Thus we have a commutative diagram

\[
\begin{array}{cccccccc}
F_m & \xrightarrow{\zeta} & F_m \rtimes \text{Aut}(F_m) & \xrightarrow{p} & \text{Aut}(F_m) \\
\sim & \Uparrow & \sim & \downarrow \eta & \sim & \downarrow \eta \\
G & \xrightarrow{f} & \Gamma & & & \end{array}
\]

The top row of this diagram is split so we have a splitting $s : \Gamma \to G$.

The proof proceeds with a variation on the proof of Theorem 3.3 as we need to control the virtual coinvariant homology of $G$ without resorting to the invariant homology.

Let $\tilde{G}$ be an arbitrary finite index subgroup of $G$. Then we have the short exact sequence

\[
1 \to F_k \to \tilde{G} \to \tilde{\Gamma} \to 1
\]

where $\tilde{\Gamma}$ is the image of $\tilde{G}$ under $f : G \to \Gamma$ and $F_k = \tilde{G} \cap F_m$. We note that $\tilde{\Gamma}$ may not stabilize $F_k$ under the action given by $\eta(\tilde{\Gamma})$. If this is the case, then $s(\tilde{\Gamma})$ is not contained in $\tilde{G}$. However, $s(\tilde{\Gamma}) \cap \tilde{G}$ has finite index in $s(\tilde{\Gamma})$. Therefore we can consider the subgroup $\hat{G}$ of $G$ generated by $F_k$ and $s(\tilde{\Gamma}) \cap \tilde{G}$ this has finite index in $\tilde{G}$, hence in $G$ as well. Thus we obtain a commutative diagram as follows:

\[
\begin{array}{cccccccc}
F_k & \xrightarrow{s} & G & \xrightarrow{f} & \tilde{\Gamma} \\
\sim & \Uparrow & \sim & \downarrow \eta & \sim & \downarrow \eta \\
H & \xrightarrow{p} & T & & & \end{array}
\]
where \( H = \zeta(\tilde{G}) \) is finite index in \( F_m \rtimes \text{Aut}(F_m) \); furthermore \( \tilde{\Gamma} = f(\tilde{G}) = f(s(\Gamma) \cap \tilde{G}) \) is finite index in \( \Gamma \) and \( T \), the image of \( \tilde{\Gamma} \) under \( \eta \), is finite index in \( \text{Aut}(F_m) \).

By construction, the top horizontal sequence in Eq (5) splits: any element \( \gamma \in f(s(\Gamma) \cap \tilde{G}) \) is mapped to \( s(\gamma) \in \tilde{G} \), which sits by construction in \( \tilde{G} \). The image \( H \) is also a semidirect product; it is the image of \( F_k \times \tilde{\Gamma} \) in \( F_m \rtimes \text{Aut}(F_m) \) where the action of \( \zeta(s(\tilde{\Gamma})) \) on \( F_m \) stabilizes \( F_k \leq F_m \).

So, as excessive homology is non-decreasing over finite index subgroups by Lemma 2.3 we can restrict ourselves to the case where the finite index subgroup of \( G \) is a semidirect product of the form \( \tilde{G} = F_k \rtimes \tilde{\Gamma} \) for some finite index subgroup \( \tilde{\Gamma} \leq \Gamma \), and its image \( H \leq F_m \rtimes \text{Aut}(F_m) \) is an extension of \( F_k \) by a finite index subgroup \( T \leq \text{Aut}(F_m) \). By Lemma 2.1 the excessive homology of \( \tilde{G} \) is given by the coinvariant homology group \( H_1(F_k; \mathbb{Q})_{\tilde{\Gamma}} \), and as the action of \( \tilde{\Gamma} \) on \( F_k \) factors through \( T \) by construction, this coinvariant homology group coincides with \( H_1(F_k; \mathbb{Q})_T \). By Lemma 2.1 again, the excessive homology of \( H \) is given by \( H_1(F_k; \mathbb{Q})_T \) as well.

At this point we can invoke the fact that abelian quotients of any finite index subgroups of \( F_m \rtimes \text{Aut}(F_m) \), in particular \( H \), are finite. This implies that \( H_1(H; \mathbb{Z}) \) is torsion, hence \( H_1(F_k; \mathbb{Q})_T \) is trivial. This entails that \( b_1(\tilde{G}) = b_1(\tilde{\Gamma}) \).

The proof for \( K = \mathbb{Z}^m \) is slightly different from the above (although we could have done the proof above using relative property \( T \), see Corollary 4.2). The pair \((\mathbb{Z}^m \rtimes \text{SL}_m(\mathbb{Z}), \mathbb{Z}^m)\) has Relative Property \( (T) \) when \( m \geq 2 \) [HV89, Ex. 1.7.4, 4.2.2], thus given any abelian quotient \( Q \) of a finite index subgroup \( H \) of \( \mathbb{Z}^m \rtimes \text{SL}_m(\mathbb{Z}) \), then \( \mathbb{Z}^m \cap H \) has finite image in \( Q \).

It is classically known that there exist presentations of \( \text{SL}_m(\mathbb{Z}) \) with two generators (see [N33]); we can therefore again choose an epimorphism \( \eta : \Gamma \to \text{SL}_m(\mathbb{Z}) \) where \( \Gamma \) is either a free or a surface group of rank or genus at least 2 and consider the extension \( G \) the pull-back group of the semidirect product \( \mathbb{Z}^m \rtimes \text{SL}_m(\mathbb{Z}) \) under the projection onto the base. The groups in question fit in the commutative diagram:

\[
\begin{array}{ccc}
\mathbb{Z}^m \subset & \overset{f}{\longrightarrow} & G \\
\mathbb{Z}^m \overset{\zeta}{\cong} & & \overset{\gamma}{\longrightarrow} \\
& \overset{\eta}{\longrightarrow} & \Gamma \\
\end{array}
\]

Given a finite index subgroup \( \tilde{G} \) of \( G \), we can assume as above that \( \tilde{G} \) has the form \( \tilde{K} \times \tilde{\Gamma} \) for some finite index subgroup \( \tilde{\Gamma} \leq \Gamma \) and finite index \( \tilde{K} \leq \mathbb{Z}^m \) (where of course \( \tilde{K} \) itself is abstractly isomorphic to \( \mathbb{Z}^n \)) and, much as above, we get the commutative diagram:

\[
\begin{array}{ccc}
\tilde{K} \subset & \overset{f}{\longrightarrow} & \tilde{G} \\
\tilde{K} \overset{\zeta}{\cong} & & \overset{\gamma}{\longrightarrow} \\
& \overset{\eta}{\longrightarrow} & \tilde{\Gamma} \\
\end{array}
\]
where $H = \zeta(\hat{G}) \leq \mathbb{Z}^m \rtimes SL_m(\mathbb{Z})$. The horizontal sequence is a split extensions, so by Lemma 2.1 its excessive homology is given by the coinvariant homology of the fiber. We can now proceed as in the previous proof, using Relative Property (T) for the bottom row.

To complete the proof of the statement, observe that the vanishing of virtual excessive homology for $G$ entails that it does not virtually algebraically fiber, as the BNS invariant of all its covers coincide with that of of a free group or a surface, hence it is empty. But this implies by [K20, Theorem 5.3] that $G$ is not virtually RFRS as for such groups the only obstruction to virtual algebraic fibration is the vanishing of the first $\ell^2$–Betti number, which we have since there is an infinite index normal finitely generated non-trivial subgroup in all of our cases of $G$. □

The core of the above proof shows the following corollary.

**Corollary 4.2.** Let $N$ be a finitely generated group such that $\text{Aut}(N)$ is finitely generated. Suppose $(N \rtimes \text{Aut}(N), N)$ has relative property (T). Then there exists a group of the form $N \rtimes F$ which does not algebraically fiber, where $F$ can be non-abelian free group or a surface group of sufficient genus.

**Remark.** With the result [KW19, Theorem 6.1] that $F_2 \rtimes F_n$ groups virtually algebraically fiber, we are left with the case of whether or not all groups of the form $F_3 \rtimes F_n$ virtually algebraically fiber. It is known that $\text{Aut}(F_3)$ is large and hence does not have property (T). However it may be the case that $(F_3 \rtimes \text{Aut}(F_3), F_3)$ has relative property (T) and then Corollary 4.2 could be applied.

5. **Subgroups and virtual homological torsion**

Nonabelian surface groups and free groups satisfy one important property, namely all their finitely generated normal subgroups are finite index. This is false for surface-by-surface and free-by-free groups, as witnessed by the normal subgroup $K \lhd G$, but that property has some vestigial effects on $G$. We want to analyze which properties surface-by-surface and free-by-free groups satisfy, in terms of their finitely generated subgroups.

We review first what is known about finitely generated infinite index normal subgroups of surface-by-surface groups. To the best of our understanding, three types of such groups occur, at least for some choices of $G$.

1. **Surfaces:** Any normal $FP_3$ subgroup $C \leq G$ (in particular, any surface subgroup) is a (perhaps virtual) fiber group. Indeed, we have a short exact sequence

   $$1 \rightarrow C \rightarrow G \rightarrow D \rightarrow 1.$$

   We can then apply [Hi02, Theorem 3.10] to deduce that $C$ is a surface group and $D$ is a virtual surface group, so there exists an epimorphism $D \rightarrow S$ to a finite group whose kernel $\hat{D} \leq D$ is a surface group. This epimorphism induces an epimorphism $G \rightarrow S$ whose kernel $\hat{G} \leq G$ can be written as

   $$1 \rightarrow C \rightarrow \hat{G} \rightarrow \hat{D} \rightarrow 1.$$

   It follows that $C$ is the fundamental group of a virtual fiber. There are examples where $X$ admits multiple fibrations as a surface-by-surface group: this is the case
for the Atiyah–Kodaira fibrations, or more recently for Salters’s construction of manifolds with arbitrarily many fibrations, see [Sa15a]. By [Sa15b] multiple fibrations can arise only if the sequence in Equation (1) has excessive homology.

Virtual algebraic fibers:

If the sequence of Equation (1) has excessive homology, there exist a cone of primitive elements in $\text{Hom}(G, \mathbb{Z}) \setminus f^* \text{Hom}(\Gamma, \mathbb{Z})$ with the property that for any $\phi: G \to \mathbb{Z}$ in that cone, $\Lambda = \ker \phi \leq G$ is finitely generated (but not finitely presented), hence there are infinitely many such groups.

In this case, we say that $G$ algebraically fibers, and $\Lambda$ is the algebraic fiber. The existence of such subgroups was independently proven in [FV19, KW19]. We can observe the following. As $f(\Lambda) \leq \Gamma$ is a normal finitely generated subgroup, it is either trivial or finite index. It cannot be trivial (lest $\Lambda \leq K$ which would entail that $\mathbb{Z}$ surject on $\Gamma$), hence $f(\Lambda)$ is finite index. It follows that $\Lambda$ surjects to a surface group (with kernel $F_\infty$) hence has a rich collection of finite quotients. In particular we can construct finite index characteristic subgroups $\tilde{\Lambda} \leq_f \Lambda$, which are therefore finitely generated normal subgroups of $G$ of infinite index. The quotients $G/\tilde{\Lambda}$ are commensurable with $\mathbb{Z}$, hence virtually $\mathbb{Z}$, so much as for the case of surface fibers, we can find a finite index subgroup of $G$ where $\tilde{\Lambda}$ is an algebraic fiber.

Pencil kernels:

In the case where $X$ is a Kodaira fibration with excessive homology $b_1(G) - b_1(\Gamma)$ equal to 2 or 4, Bregman proved in [Br21] the existence of an irregular pencil (in the sense of algebraic geometry) $g: X \to \Sigma$ where $\Sigma$ is a surface of genus respectively 1 or 2. (For higher excessive homology the situation is unknown.) In those cases, $G$ is an extension

$$1 \to C \to G \xrightarrow{f^*} \pi_1^{orb}(\Sigma) \to 1$$

where $\pi_1^{orb}(\Sigma)$ is the orbifold fundamental group associated to the pencil (a virtual surface group) and $C$ is finitely generated, see e.g. [Ca03].

There are known cases (that arise when the Kodaira fibration is a surface bundle of type $I$ in Johnson’s trichotomy) where this pencil does not correspond, even going to a cover, to a second structure of Kodaira fibration (in which case $C$ will fail to be finitely presented). We are not aware of similar phenomena in the general (non holomorphic) setting.

In the case of free-by-free groups, we have some similar results: the classification of $FP_2$ normal subgroups of infinite index is dealt by [Br81, Corollary 8.6], which results only in free groups with properties similar to Case (1) above. Regarding Case (2), [FV19, KW19] show finitely generated examples whenever $G$ has virtual excessive homology. We are not aware of examples as the ones in Case (3), but they would require excessive homology (see the proof of Proposition 5.2 below).

We will focus then on the case where $G$ has no virtual excessive homology. By definition, this happens if and only if for any $\tilde{G} \leq_f G$ the top sequence in Equation (1) has no excessive homology, namely $b_1(\tilde{G}) = b_1(\tilde{\Gamma})$.

Parsing through the description of the known types of finitely generated normal subgroups of a surface-by-surface and free-by-free groups, we can deduce that if $G$ has no virtual
excessive homology, then only virtual fiber subgroups \( C \trianglelefteq K \), normal in \( G \), are known to occur. It is natural to ask if they are the only examples.

**Question 5.1.** Let \( G \) be a surface-by-surface or a free-by-free group with nonabelian base and fiber group. Assume that \( G \) has no virtual excessive homology. Let \( \Lambda \trianglelefteq G \) be a finitely generated normal subgroup; is \( \Lambda \trianglelefteq K \), where \( K \) is the fiber group?

The next proposition gives some support to an affirmative answer to Question 5.1. The strategy of its proof draws from work of Johnson \[JoF93\] (see also \[Ca03\] \[Sa15b\]) and is based on the aforementioned fact that finitely generated normal subgroups of nonabelian surface groups or free groups are either trivial or finite index.

**Proposition 5.2.** Let \( G \) be a surface-by-surface, surface-by-free, or a free-by-free group with nonabelian base and fiber group. Assume that \( G \) has no virtual excessive homology or, equivalently, for any epimorphism \( \alpha : G \to S \) as in Equation (5), \( b_1(\tilde{G}) = b_1(\tilde{\Gamma}) \). Let \( \Lambda \trianglelefteq G \) be a finitely generated normal subgroup; then either \( \Lambda \trianglelefteq K \) and \( \Lambda \) is a virtual fiber group, or \( Q := G/\Lambda \) has \( \text{vb}_1(Q) = 0 \).

**Proof.** We have a diagram

\[
\begin{array}{c}
\Lambda \\
\downarrow f \\
G \\
\downarrow f \\
\Gamma \\
\end{array}
\]

with the property that \( H^1(G; \mathbb{R}) = f^*H^1(\Gamma; \mathbb{R}) \). First, consider the finitely generated normal subgroup \( f(\Lambda) \trianglelefteq \Gamma \). A finitely generated normal subgroup of a nonabelian surface group or a free group is either trivial or finite index. If the former happens, we have immediately \( \Lambda \trianglelefteq K \). If the latter happens, we have a short exact sequence

\[
1 \to K \cdot \Lambda \to G \to \Gamma/f(\Lambda) \to 1,
\]

which shows that the Frobenius product \( K \cdot \Lambda \) is a finite index subgroup of \( G \). Let \( \phi \in \pi^*H^1(Q; \mathbb{R}) \subset H^1(G; \mathbb{R}) \): by definition, \( \phi \) is trivial on elements of \( \Lambda \trianglelefteq G \). By assumption, \( H^1(G; \mathbb{R}) = f^*H^1(\Gamma; \mathbb{R}) \), hence \( \phi \) is simultaneously trivial on elements of \( K \trianglelefteq G \). But then \( \phi \) is trivial on the entire finite index subgroup \( K \cdot \Lambda \trianglelefteq G \). As \( H^1(G; \mathbb{R}) \) does not contain any torsion, this entails that \( H^1(Q; \mathbb{R}) = 0 \), hence \( b_1(Q) = 0 \). Given any normal finite index subgroup \( \tilde{Q} \trianglelefteq f(\tilde{Q}) \), we take the corresponding epimorphism \( Q \to S \) and the induced epimorphism \( \alpha : G \to Q \to S \). Using the notation of Equation (5), the subgroup \( \tilde{G} \trianglelefteq f(\tilde{G}) \) sits in a diagram as the one in Equation (10) where again, by assumption, \( H^1(\tilde{G}; \mathbb{R}) = f^*H^1(\tilde{\Gamma}; \mathbb{R}) \). Applying the same argument as before we deduce as well that \( b_1(\tilde{Q}) = 0 \).

**Remark.** Note that by \[Bi81\] Corollary 8.6 (in the free-by-free case) and \[Hi02\] Theorem 3.10 (in the surface-by-surface case) we can supplement the previous proposition by observing that, unless \( \Lambda \) is a virtual fiber group, then it is not of type \( FP_2 \) (in the free-by-free case) or \( FP_3 \) (in the surface-by-surface case).

The next results focus on a problem that is in a sense dual to that discussed above, and describes some properties of the abelianization of finite index subgroups of \( G \), where
$G$ has no virtual excessive homology, as the potential examples discussed in Theorem 3.3 and Proposition 3.5 or the free-by-free extensions with no virtual excessive homology we constructed in Section 4.

We will require first the following lemma. Recall that a group $G$ is residually $p$, where $p$ is a prime number, if for any element $g \in G \setminus \{e\}$ there exist a $p$-group quotient where the image of $g$ is nontrivial.

**Lemma 5.3.** Let $G$ be a group that fits into a short exact sequence $1 \to K \to G \xrightarrow{f} \Gamma \to 1$. Suppose $K$ has trivial center and $K, \Gamma$ are virtually residually $p$, then $G$ is virtually residually $p$.

**Proof.** Without loss of generality, we can assume that $\Gamma$ is already residually $p$.

By [Lub80], we have that if $K$ is virtually residually $p$, then so is $\text{Aut}(K)$. Let $\zeta : G \to \text{Aut}(K)$ be the representation of $G$ covering the monodromy map $\eta : \Gamma \to \text{Out}(K)$ as in Equation (2). Since $K$ has trivial center, $\zeta$ is injective when restricted to $K$. Let $A \leq \text{Aut}(K)$ be a finite index subgroup of $\text{Aut}(K)$ which is residually $p$.

Let $\tilde{G} = \zeta^{-1}(A) \cap G \leq G$. There is a short exact sequence $1 \to \tilde{K} \to \tilde{G} \to \tilde{\Gamma} \to 1$, where $\tilde{K}$ is a finite index subgroup of $K$.

Let $g \in \tilde{G} \setminus \{e\}$. If $f(g)$ is non-trivial then we can find a $p$-group quotient of $\tilde{\Gamma}$ where the image of $f(g)$ is non-trivial. Thus we have a $p$-group quotient of $\tilde{G}$ where the image of $g$ is non-trivial.

If $f(g) = e$, then $g \in \tilde{K}$. In this case $\zeta(g)$ is non-trivial. Thus we can find a $p$-group quotient of $A$, hence of $\tilde{G}$, under which the image of $\zeta(g)$ is non-trivial. Thus in either case, we can find a $p$-group quotient where the image of $g$ is non-trivial and hence $\tilde{G}$ is residually $p$ and $G$ is virtually residually $p$. \hfill □

We know that nonabelian free and surface groups satisfy the assumptions for $K$ and $\Gamma$ in Lemma 5.3; we are then in position now to discuss our next result, that asserts that extensions without virtual excessive homology have nontrivial virtual homological torsion.

**Proposition 5.4.** Let $G$ be a surface-by-surface, a surface-by-free, or a free-by-free group without virtual excessive homology. Then for any $n \in \mathbb{N}$ and any prime $p$, there is a subgroup $\tilde{G} \leq G$ of finite index at least $n$ such that $H_1(\tilde{G})$ has nontrivial $p$-torsion.

**Proof.** By Lemma 5.3 the group $G$ is virtually residually $p$. Because of the form of the statement, it is not restrictive to assume that $G$ itself is residually $p$. Since $G$ is residually $p$, there exists a filtration $\{G_i \mid i \geq 0\}$ of finite index normal subgroups whose index is a power of $p$ with $\bigcap_i G_i = \{1\}$ and where the successive quotient maps $\alpha_i : G_i \to G_i/G_{i+1} = S_i$ factorize through the maximal abelian quotient:

\[
\begin{array}{c}
G_{i+1} \xrightarrow{\alpha_i} G_i \xrightarrow{\alpha_i} S_i \\
\downarrow \quad \quad \downarrow \\
H_1(G_i) \end{array}
\]

Now let $\kappa \in K$ be an element with the property that $\kappa \in G_i \setminus G_{i+1}$; denoting $K_i = K \cap G_i$ and combining the diagram in Equation (5) with the residually $p$ assumption, we get the diagram
As $b_1(G_i) = b_1(\Gamma_i)$, the image of $H_1(K_i)\Gamma_i$ in $H_1(G_i)$ is a torsion subgroup. And as $\alpha_i(\kappa) \neq 1 \in S_i$, the class $[\kappa] \in H_1(G_i)$ is nonzero, hence the torsion subgroup is nontrivial. Moreover, $\alpha(\kappa)$ is non-trivial in $S_i$ and has order $p^l$ for some $l$. We can consider the class $[\kappa]$ in $H_1(G_i)$, this has finite order and maps onto $\alpha(\kappa)$, thus we see that $[\kappa]$ has order $p^l r$ for some $r$. We conclude that $H_1(G_i)$ contains an element of order $p$.

6. Coherence and incoherence

A question related to virtual fibering is that of coherence. The witnesses to incoherence that are found in [KW19, FV19] are virtual fibers. We record here the following fact relating algebraic fibers and PD(4).

**Lemma 6.1.** If $G$ is a PD(4) group with $\chi(G) \neq 0$ which algebraically fibers, then $G$ is incoherent.

**Proof.** By assumption there exists an epimorphism to $\mathbb{Z}$ with finitely generated kernel $K$. By [HK07, Corollary 1.1], if the kernel is finitely presented, it is a PD_3 group. Hence the kernel has an Euler characteristic and by [Br94, IX, 7.3(b)], the Euler characteristic of $G$ is $\chi(K) \cdot \chi(\mathbb{Z})$ which is 0. □

A closed aspherical 4-manifold with word-hyperbolic fundamental group cannot have a normal infinite index 3-manifold subgroup by Mostow rigidity. Any normal infinitely index subgroup of a hyperbolic group is non-quasi-convex, so we may ask the following:

**Question 6.2.** Can a closed aspherical 4-manifold with word-hyperbolic fundamental group contain a non-quasiconvex 3-manifold subgroup?

This also relates to the groups studied here as a negative answer would show that surface-by-surface groups cannot be word-hyperbolic. Indeed, these contain a hyperbolic 3-manifold group which contains the fiber subgroup.

Although virtual algebraic fibers often imply incoherence, we can show several similar classes of group extensions that are coherent. Here we show that (1-ended hyperbolic)-by-abelian groups are coherent. We begin by studying the case of surface bundles. Since coherence is invariant under commensurability we can pass to a finite index subgroup and assume that the abelian quotient is a free abelian group.

**Theorem 6.3.** Suppose that $G$ fits into a short exact sequence

$$1 \to \Sigma_g \to G \to T \to 1$$

where $\Sigma_g$ is the fundamental group of a closed surface and $T$ is free abelian. Then $G$ is coherent.
Proof. In the case \( g = 1 \) we are studying a polycyclic group. These are known to be coherent. Thus we will assume that \( g > 1 \).

Consider the image of \( T \) in \( \text{Out}^+(S_g) \). By Birman-Lubotzky-McCarthy \([BLM83]\) such a subgroup is reducible, namely after passing to a finite index subgroup \( \Theta(T) \) leaves a collection of simple closed curves in a surface of genus \( g \), as well as their complement, componentwise invariant. Pick a maximal such collection. On each component of the complement the monodromy subgroup is either trivial or contains a pseudo-Anosov element. Passing to a further finite index subgroup we can assume that each monodromy subgroup for each component of the complement is either trivial or is cyclic and generated by a pseudo-Anosov.

Thus, \( G \) has a finite index subgroup which splits as a graph of groups. Each vertex group fits into a short exact sequence \( S \to H \to T' \) and the edge groups \( H \) fit into a short exact sequence \( \mathbb{Z} \to H \to T' \) for some finite index \( T' < T \) and some surface group (possibly with boundary) \( S \).

Since the monodromy of each of the vertex groups is either trivial or cyclic, we can take a generating set \( \{a_1, \ldots, a_l\} \) for \( T' \) where \( a_1 \) acts trivially or by a pseudo-Anosov and \( a_i \) acts trivially for \( i > 1 \). Thus each of the vertex groups is an extension of a fibered 3-manifold group by a central subgroup and hence coherent. Moreover each of the edge groups are polycyclic and hence slender (meaning that every subgroup is finitely generated). Since \( G \) is the fundamental group of a graph of groups with coherent vertex groups and slender edge groups we can conclude that \( G \) is coherent by \([KS70]\). \( \square \)

Now we consider a more general situation, the group of automorphisms of a 1-ended hyperbolic group \( H \). Let \( Q \) be a subgroup of \( \text{Out}(H) \) and \( G \) the full pre-image in \( \text{Aut}(H) \). Since \( H \) is hyperbolic, the center is finite and characteristic and so the extension of \( H \) corresponding to \( Q \) is a finite central extension of \( G \). Thus throughout we will assume \( H \) has trivial center. So, we have the short exact sequence \( 1 \to H \to G \to Q \to 1 \). If \( H \) is incoherent, then it is clear that \( G \) is also incoherent. Thus, we will focus on the case that \( H \) is coherent.

Since \( H \) is one-ended and hyperbolic, if \( \partial H \neq S^1 \), \( H \) admits a JSJ splitting \([Bo98]\) which is determined by \( \partial H \). This splitting is an action of \( H \) on a simplicial bipartite tree where the vertex and edge groups are quasi-convex and the the edge groups are all 2-ended. The vertex groups are either two-ended, rigid, or Fuchsian. See, for example, \([KiW19]\) for elaboration. This splitting will allow us to understand the coherence of groups of automorphisms of \( H \).

**Theorem 6.4.** Let \( H \) be a one-ended non-elementary coherent hyperbolic group, and \( Q \) a subgroup of \( \text{Out}(H) \). Let \( G \) be the full pre-image of \( Q \) in \( \text{Aut}(H) \). Suppose that \( Q \) does not contain \( F_2 \), then \( G \) is coherent.

**Proof.** If \( \partial H = S^1 \), then \( H \) is virtually Fuchsian. Thus \( H \) contains a finite index characteristic group which is the fundamental group of a surface. Thus, \( G \) contains a finite index subgroup \( G' \) which is fits into a short exact sequence \( 1 \to \Sigma_g \to G' \to Q \to 1 \). By \([Mc85]\), if \( Q \) does not contain \( F_2 \), then \( Q \) is virtually abelian. Thus we can pass to a finite index subgroup and assume that \( Q \) is abelian. Thus we can now appeal to Theorem 6.3 to see that \( G' \) is coherent. Since coherence is preserved by commensurability, we see that \( G \) is coherent.
Now let $H$ be a one-ended non-elementary coherent hyperbolic group, with $\partial H \neq S^1$. Then by [Le05, Section 5] $\text{Out}(H)$ has a finite index subgroup $\text{Out}'(H)$ which fits into an exact sequence

$$1 \to \mathcal{T} \to \text{Out}'(H) \to \Pi_{v \in V_2}(\text{Mod}(H_v)) \to 1.$$  

Where $V_2$ is the set of Fuchsian vertices in the JSJ decomposition. $\text{Out}'(H)$ is the kernel of the map to the automorphism group of the rigid vertex stabilizers, which is finite. The group of partial conjugations $\mathcal{T}$ is virtually free abelian with free rank the number of edge groups with infinite center.

Let $Q'$ be the intersection of $Q$ with $\text{Out}'(H)$. Let $G'$ be the pre-image of $Q'$ in $\text{Aut}(H)$, which is finite index in $G$. We’ll show $G'$ is coherent if $Q$ does not contain $F_2$.

Now suppose that $Q'$ does not contain an $F_2$. Then by [Mc85], the projection to each $\text{Mod}(H_v)$ is virtually abelian. Hence $Q'$ is a virtually free abelian extension of a virtually free abelian group and hence virtually polycyclic. Every subgroup of $Q'$ is finitely generated ($Q'$ is slender) and $Q'$ is coherent. Now consider the splitting of $G'$ given by the fact that the automorphisms of $H$ act on the tree that is the JSJ splitting of $H$. Each vertex stabilizer $G'_v$ of this splitting fits into an exact sequence $1 \to H_v \to G'_v \to Q' \to 1$. When $H_v$ is rigid, the action of $Q'$ is trivial, so $G'_v$ splits as a product $H_v \times Q'$ and is coherent since $Q'$ is slender.

When $H_v$ is 2-ended, we can see that $G'_v$ has a finite index subgroup that is $\mathbb{Z} \times Q''$ for some finite index subgroup $Q''$ of $Q'$ and is coherent.

When $H_v$ is a virtually free Fuchsian group, up to finite index, $Q'$ preserves the finite index free group. Thus by the Tits alternative for free groups [BFH00] [BFH05] the image of $Q'$ in $\text{Out}(H_v)$ is virtually abelian. Let $P$ be the image in $\text{Out}(H_v)$, which we can assume is abelian, and $K$ be the kernel. Thus $Q'$ fits into a short exact sequence $1 \to K \to Q' \to P \to 1$. Since $Q'$ is polycyclic, $K$ is also polycyclic. We can now first extend $H_v$ by $K$ which acts trivially so we get $H_v \times K$. Then $G'_v$ fits into a short exact sequence $1 \to H_v \times K \to G'_v \to P \to 1$. Any automorphism of $H_v \times K$ preserves the product splitting. Thus, $K$ is a normal subgroup of $G'_v$ and the quotient is a group $L_v$ fitting into a short exact sequence $1 \to H_v \to L_v \to P \to 1$. By Theorem 6.3 $L_v$ is coherent as an extension of a surface group by a free-abelian group. We now have a short exact sequence $1 \to K \to G'_v \to L_v \to 1$. Since $K$ is slender and $L_v$ is coherent we can see that $G'_v$ is also coherent.

We claim that each of the edge groups $G'_e$ is slender. From above $Q'$ is polycyclic, and each group $H_e$ is virtually infinite cyclic. Thus the edge groups fit into the exact sequence $1 \to H_e \to G'_e \to Q' \to 1$ and are themselves slender. Thus we have a graph of groups with coherent vertex groups and slender edge groups so $G'$ is coherent by [KS70] and hence $G$ is coherent.

Groups with two ends are virtually cyclic and the outer automorphism group is finite. Thus the above theorem holds trivially.

In the case of infinitely many ends, the group has a splitting with finite edge groups and vertex groups which have either one or two ends. We can reduce to the case where all the groups have two ends. We end this section by conjecturing the analogue of Theorem 6.3 for free groups.
Conjecture 6.5. Let $G$ be a group fitting into a short exact sequence

$$1 \to F_n \to G \to \mathbb{Z}^n \to 1.$$ 

Then $G$ is coherent.

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