Block-diagonal reduction of matrices over commutative rings I.  
(Decomposition of modules vs decomposition of their support)

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**Abstract.** Take a rectangular matrix over a commutative ring \( A \in \text{Mat}_{m \times n}(R) \). Assume the ideal of maximal minors factorizes, \( I_m(A) = J_1 \cdot J_2 \subset R \). When is \( A \) left-right equivalent to a block-diagonal matrix? (When does the module/sheaf \( \text{Coker}(A) \) decompose as \( \text{Coker}(A)|_{V(J_1)} \oplus \text{Coker}(A)|_{V(J_2)} \)?) If \( R \) is not an elementary divisor ring (i.e., a close relative of a principal ideal ring) one needs additional assumptions on \( A \). No necessary and sufficient criterion for such block-diagonal reduction is known.

In this paper we establish the following results.

- The persistence of (in)decomposability under the change of rings. For example:
  - the passage to Noetherian/local/Henselian/complete rings;
  - the decomposability of the module \( \text{Coker}(A) \) over a graded ring \( R \) vs the decomposability of the sheaf \( \text{Coker}(A) \) locally at the points of \( \text{Proj}(R) \), (this is the matrix version of Noether’s \( AF + BG \));
  - the restriction to a subscheme in \( \text{Spec}(R) \).
- The necessary and sufficient condition for decomposability of square matrices in the case: \( \det(A) = f_1 \cdot f_2 \) is not a zero divisor and \( f_1, f_2 \) are co-prime in \( R \).

As an immediate application we give new criteria of simultaneous (block-)diagonal reduction for tuples of matrices over a field, i.e., linear determinantal representations.

1. Introduction

1.1. Let \( R \) be a commutative unital ring. Consider the matrices, \( A \in \text{Mat}_{m \times n}(R), 2 \leq m \leq n \), up to the left-right equivalence, \( A \sim U \cdot A \cdot V^{-1} \), here \( U \in \text{GL}(m, R), V \in \text{GL}(n, R) \). If \( R \) is a principal ideal ring (PIR) then any matrix is equivalent to a diagonal one. This is the well known Smith normal form. More generally, this holds for the elementary divisor rings. These rings were studied in numerous works, they are all close to being of Krull dimension one. (See, e.g., [Brandal] and [Karr-Wiegand].) The problem over non-commutative rings was addressed, e.g., in [Ar.Go.O’M.Pa].

See also [H.K.K.W., Wiegand, Vamos-Wiegand, Zabavsky].

For rings of larger Krull dimension, e.g., \( k[[x, y]] \), many matrices are not equivalent to block-diagonal. For example, in the square case, \( A \in \text{Mat}_{n \times n}(R) \), the determinant \( \det(A) \in R \) can be irreducible, already this obstructs such a block-diagonal reduction (see example 1.6).

Our paper grew from the question:

\[
1. \quad \text{Suppose the ideal of maximal minors factorizes, } I_m(A) = J_1 \cdots J_r, \text{ for some ideals } J_i \subset R.
\]

How to ensure the reduction \( A \sim \oplus A_i \), with \( A_i \in \text{Mat}_{m_i \times n_i}(R) \), \( I_{m_i}(A_i) = J_i \), \( \sum m_i = m? \)

One can ask this question also for the congruence, \( A \congr U A U^t \), and for the conjugation, \( A \sim U A U^{-1} \), here \( U \in \text{GL}(n, R) \). More generally, one needs decomposability criteria of quiver representations over commutative rings. (There is a considerable body of results about containment of quiver representations, see, e.g., [Smalø] and the references therein.)

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Matrices over rings appear frequently in pure and applied mathematics, as linear matrix families (or tuples of matrices), as matrices of differentiable/analytic functions/power series, as the presentations of modules/sheaves/vector bundles and their homomorphisms, as integer matrices (or matrices over rings of integers), and so on. The determinantal ideals and determinantal representations have been intensively studied through decades, see, e.g., \[ \text{[K.C.V]}, \text{[Brunner-Vetter]}, \text{[K.M.R]}, \text{[Miro-Roig]} \]. The question of (block-)diagonalization is among the most basic, natural, and important. Surprisingly it has been untouched for rings of Krull dimension bigger than one, with the only exception \[ \text{[Laksov]} \].

1.2. This paper is the first in our study of block-diagonal reduction/block-diagonalization of matrices over commutative unital rings. Here we present the results in two directions:

- \text{(Obstruction to decomposability and change of base ring)} Take a matrix \( A \in \text{Mat}_{m \times n}(R) \) with \( I_m(A) = J_1 : J_2 \), or the module \( \text{Coker}(A) \in \text{mod}(R) \) supported on \( V(J_1) \cup V(J_2) \subset \text{Spec}(R) \). We identify the obstruction to the decomposition \( \text{Coker}(A) \cong \text{Coker}(A)|_{V(J_1)} \oplus \text{Coker}(A)|_{V(J_2)} \). This obstruction (an \( R \)-module) is functorial under the change of base ring.

It is often useful to change the ring. For example, one wants to pass to a Noetherian subring \( S \subset R \) that contains the entries of \( A \), or to localize (to check the block-diagonalization on stalks), or to pass to Henselization/completion (i.e., to work with matrices of power series), or to take a quotient (i.e., to restrict \( A \) to a subscheme in \( \text{Spec}(R) \)). Under weak (and natural) assumptions we ensure: \( A \) is decomposable iff its image (over \( S \)) is decomposable.

- \text{(Decomposability in the square case, \( A \in \text{Mat}_{n \times n}(R) \))} Assuming \( \det(A) = f_1 \cdots f_r \in R \), an obvious necessary condition to ensure the decomposability as in \( \{I_j \} \) is the following bound on the ideal of \( (n - 1) \times (n - 1) \) minors: \( I_{n-1}(A) \subseteq \sum_{j=1}^r \prod_{i \neq j} f_i \). We prove that this condition is also \textit{sufficient} under rather weak assumptions: \( \det(A) \in R \) is a regular element and the elements \( \{f_i\} \) are pairwise co-prime. Geometrically (in the local regular case) the condition is: the hypersurface germs \( V(f_i) \subset \text{Spec}(R) \) have no common components and intersect (pairwise) properly.

This solves the decomposition problem for Cohen-Macaulay modules supported on such tuples of hypersurfaces. In the linear algebra language, we extend the primary decomposition for matrices over a field to matrices over a ring.

Our criteria imply: the decomposability of matrices is controlled by their determinantal ideals \( \{I_j(A)\}_{j} \). Recall that \( \{I_j(A)\} \) are very naive/rough invariants of \( A \). It was a surprise (for us) that the decomposability question can be treated in quite general case via these ideals. We remark that controlling the ideals \( \{I_j(A)\} \) is much simpler than controlling the module \( \text{Coker}(A) \).

Due to the space limitations we give only the first immediate applications to the old problem of simultaneous diagonalization of tuples of matrices/linear determinantal representations, decomposition of sheaves on reducible curves.

In the subsequent paper, \[ \text{[Kerner.II]} \], we use these decomposability criteria to obtain the criteria for decomposability by conjugation (decomposability of representations of groups/algebras), by congruence (decomposability of quadratic/skew-symmetric forms), and more generally decomposability of quiver representations (over fields and rings). The decomposability criterion for rectangular matrices, also in \[ \text{[Kerner.II]} \], is more delicate. It involves controlled cohomology of a determinantal complex of the morphism \( A \otimes R/I_m(A) \).

Our interest in block diagonal reduction of matrices over commutative rings of higher Krull dimension originated in our study of linear determinantal representations of hypersurfaces \[ \text{[Kerner-Vinnikov.2012]} \]

\[
f(x_0, x_1, \ldots, x_n) = \det(x_0 A_0 + x_1 A_1 + \cdots + x_n A_n), \quad \text{where} \ A_0, \ldots, A_n \in \text{Mat}_{n \times n}(k) \quad \text{with} \ k \ \text{a field}.
\]

This is an old topic in algebraic geometry, see, e.g., \[ \text{[Dolgachev], [Beauville]} \], which is closely related to matrix factorizations \[ \text{[Eisenbud.1980]}, \text{[B.H.S.]} \text{[Backelin-Herzog.1989]} \]; we refer to \[ \text{[Kerner-Vinnikov.2012]} \] for detailed references and to \[ \text{[Vinnikov.2012]}, \text{[C.S.T.]} \] for some recent developments and relations. In that setting, a corollary of our results is that for \( f = f_1 f_2 \) with \( f_1, f_2 \) relatively prime, a determinantal representation of \( f \) is globally equivalent to the direct sum of determinantal representations.
of \( f_1 \) and of \( f_2 \) iff it decomposes locally at every point of intersection of the corresponding hypersurfaces. In the case of curves \((n = 2)\) and assuming that the two curves \( f_1 = 0 \) and \( f_2 = 0 \) have no common tangents at every point of intersection, a determinantal representation decomposes iff it is maximally generated (the dimension of its kernel equals the multiplicity of the point on the curve \( f = 0 \), also called Ulrich maximal [Ulrich]) at every point of intersection, see Corollary 4.7 and §4.5.

We mention in this context also the recent results [Klep-Volčič, H.K.V.] on equivalence and decomposition of matrices of linear forms, when these forms are viewed as polynomials in free noncommuting variables and are evaluated on square matrices of all sizes over the ground field.

1.3. The structure/contents of the paper.

§2 contains the relevant background. In §2.1 we set the notations and define the (stable) decomposability of matrices.

Then we recall the (trivial) decomposability of modules with non-connected support and the openness of decomposability locus.

§2.3 gives some homological versions of “co-regularity” of ideals \( J_1 \cap J_2 = J_1 \cdot J_2, \) e.g., via \( \text{Tor}_1(R/J_1,R/J_2) \) and \( H^{0}_{J_1+J_2}(M) \).

§3 treats the obstruction to decomposability and the reduction to “convenient” rings.

§3.1. We prove: \( A \) is decomposable over \( R \) iff \( A \) is decomposable over certain Noetherian subring \( S \subset R \) that contains the entries of \( A \). Moreover, (in the square case) for \( A \in \text{Mat}_{n\times n}(R) \) the assumptions “\( \det(A) = \prod f_i \) with \( \{f_i\} \) co-prime, and \( I_{n-1}(A) \subseteq \sum_{j=1}^{n} (\prod_{i\neq j} f_i) \)” hold over \( R \) iff they hold over \( S \), assuming a particular chain stabilization condition, much weaker than Noetherianity.

§3.2. When the support of a module is reducible, \( \text{Supp}(\text{Coker}(A)) = V(J_1) \cup V(J_2) \subset \text{Spec}(R) \), it is natural to separate the components, \( V(J_1) \bigcup V(J_2) \cong V(J_1) \cup V(J_2) \). Then one compares \( \text{Coker}(A) \) to (roughly) \( \pi_\ast \pi^\ast \text{Coker}(A) \). The module \( \pi_\ast \pi^\ast \text{Coker}(A) \) is obviously decomposable. Comparing the two modules one gets the obstruction to the decomposability,

\[
0 \to H^{0}_{J_1+J_2}(\text{Coker}(A)) \to \text{Coker}(A) \to \pi_\ast \pi^\ast \text{Coker}(A) \to Q \to 0.
\]

§3.3. We prove: this obstruction \( Q \) is functorial under base changes that are compatible with certain local cohomology objects.

§3.4. As an example we get: \( A \) is decomposable over \( R \) iff \( A \) is decomposable over all the localizations, \( R_m \). For a local ring, \( (R, m) \), \( A \) is decomposable over \( R \) iff it is decomposable over the Henselization, \( \hat{R}^h \). Moreover, one can pass to the completion, \( \hat{R}^h(m) \), assuming the completion functor \( \otimes \hat{R}^h(m) \) is exact.

§3.5. For the rings of Analysis, e.g., \( C^\infty(U) \) or \( C^\infty(\mathbb{R}^n, o) \), the functor \( \otimes \hat{R}^h(m) \) is not exact. Thus we give a separate argument for the reduction to complete rings.

§3.6. In many cases one needs a non-flat base change \( R \to S \), e.g., for \( S = R/d \). In this case we give simple conditions to ensure: \( A \) is decomposable over \( R \) iff \( A \otimes S \) is decomposable.

§3.7. Suppose the ring is graded, \( R = \oplus_{d \in \mathbb{R}} R_d \), and the matrix is graded, see §2.1.v. Denote by \( m \) the maximal homogeneous ideal in \( R \). We prove: \( A \) is \( GL(m, R) \times GL(n, R) \)-decomposable iff its \( m \)-localization is \( GL(m,R_m) \times GL(n, R_n) \)-decomposable.

§3.8. Suppose \( R, A \) are graded. Using the standard correspondence of graded \( R \)-modules to coherent sheaves, \( \text{mod}_{gr}-R \cong \text{Coh}(\text{Proj}(R)) \), we can consider \( \text{Coker}(A) \) as the sheaf of modules on the projective scheme \( \text{Proj}(R) \). The decomposition of \( A \) over \( R \) is reduced to the local decompositions of the stalks of the sheaf \( \text{Coker}(A) \) at the points of the subscheme \( PV(J_1) \cap PV(J_2) \subset \text{Proj}(R) \). (This is the matrix version of the fundamental AF + BG-theorem by M. Noether.) Such a reduction is perhaps unexpected, as the scheme \( PV(J_1) \cap PV(J_2) \) can be non-connected, e.g., a finite set of closed points. (Alternatively, one could expect monodromy-type effects.)

This Noether-type result gives the traditional reduction in dimension: “A question over a graded ring \( R \) is reduced to many questions over local rings of dimension \( \text{dim}(R) - 1 \).”

The results of §3 are used both in §4 and in [Kerner.II].
§4 gives the decomposition criterion for square matrices, theorem 4.1. This is a complete solution of the decomposition problem for matrices with det(A) = \( \prod f_i \), here \( \{ f_i \} \) are regular and pairwise co-prime.

§4.1. (a preparation) We establish the block-diagonalization (by conjugation) of projectors and "almost projectors" over local/Henselian rings. (By §3.3 we can always assume \( R \) is local/Henselian, though not necessarily Noetherian.)

§4.2. Theorem 4.1 is proved by creating such almost projectors from \( A \) and its adjugate \( \text{Adj}(A) \). The proof is completely down-to-earth, using only the \( R \)-linear algebra.

In Kerner,II we give a shorter proof of theorem 4.1 via certain determinantal complex. But the current down-to-earth proof has its advantages. It is easily adapted to decomposability by the congruence, \( A \to UAV' \), and is useful when extending theorem 4.1 to the case of non-commutative rings.

§4.3. The first examples. As a trivial application of theorem 4.1 we derive the "first half" of the Smith normal form over PID's. Then we give decomposability criteria for determinantal representations of maximal corank (Ulrich-maximal modules).

§4.4. We discuss the geometry of the assumptions in theorem 4.1.

§4.5. Decomposability criteria for (non-linear) determinantal representations and for torsion-free sheaves on reducible plane curves.

§4.6. We apply the criterion to the simultaneous diagonal reduction of tuples of matrices over a field, i.e., linear determinantal representations. First we reduce the question to the diagonal reduction of triples of matrices, i.e., determinantal representations of plane curves. In this later case the only obstruction to this diagonal reduction is "the total defect" of kernel-dimensions, properly counted. The triple of matrices admits the diagonal reduction iff this total defect has the "expected value".

No criteria of such type could be imagined with the previous (classical) methods.

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2. Preparations

2.1. Notations and conventions. Unless stated otherwise, \( R \) is a commutative unital ring, not necessarily Noetherian, while \( k \) denotes a field, of any characteristic. When writing "\((R,m)\) is a local ring", we do not assume \( R \) is Noetherian. The ideals are not assumed finitely generated.

i. Denote the (square) unit matrix by \( \mathbb{I} \), the zero matrix (possibly non-square) by \( 0 \). Let \( A \in \text{Mat}_{m \times n}(R) \), we always assume \( 2 \leq m \leq n \).

ii. Sometimes we change the ring, \( R \to S \). Then we take the images, \( R \ni f \to \bar{f} \in S \) and \( \text{Mat}_{m \times n}(R) \ni A \to \bar{A} \in \text{Mat}_{m \times n}(S) \). For an ideal \( I \subset R \) denote by \( \bar{I} \subset S \) the ideal generated by the image of \( I \). (No confusion with the integral closure, as we do not use it in this paper.)

For \( M = \text{Coker}(A) \) we denote \( \bar{M} = \text{Coker}(\bar{A}) \).

Applying \( S \otimes - \) to a submodule \( M \subset N \) we denote by \( S \cdot M \) the image of \( S \otimes M \to S \otimes N \).

iii. The determinantal ideal of a matrix, \( I_r(A) \), is generated by all the \( r \times r \) minors of \( A \). Usually we denote \( M := \text{Coker}(A) \). Then the determinantal ideals coincide with the Fitting ideals: \( I_j(A) = \text{Fitt}_{m-j}(M) \). In particular, as \( \text{Fitt}_0(M) \cdot M = 0 \), the image of \( A \) satisfies: \( \text{Im}(A) \subseteq I_{m+1}(A) \). This chain is invariant under the \( GL(m,R) \times GL(n,R) \) action, \( I_j(A) = I_j(UAV^{-1}) \).

Consider \( A \) as a presentation matrix of its cokernel, \( R^n \xrightarrow{A} R^m \to \text{Coker}(A) \to 0 \). Usually we denote \( M := \text{Coker}(A) \). Then the determinantal ideals coincide with the Fitting ideals: \( I_j(A) = \text{Fitt}_{m-j}(M) \). In particular, as \( \text{Fitt}_0(M) \cdot M = 0 \), the image of \( A \) satisfies: \( \text{Im}(A) \subseteq I_{m+1}(A) \).

In fact a stronger property holds: \( I_{m-1}(A) \cdot \text{Im}(A) \supseteq I_m(A) \cdot R^m \).

The determinantal ideals are functorial, for \( R \to S \) one has \( I_j(\bar{A}) = I_j(A) \cdot S \).


iv. The adjugate of a square matrix \( A \in \text{Mat}_{n \times n}(R) \) is the matrix of cofactors, \( \text{Adj}(A) \in \text{Mat}_{n \times n}(R) \). Usually in the paper \( \det(A) \in R \) is not a zero divisor, then \( \text{Adj}(A) \) is determined by the condition \( \text{Adj}(A) \cdot A = \det(A) \cdot I = A \cdot \text{Adj}(A) \). The entries of \( \text{Adj}(A) \) generate the ideal \( I_{n-1}(A) \) and one has \( \det(\text{Adj}(A)) = \det(A)^{n-1} \).

v. We use two kinds of matrix equivalence:

- the left-right equivalence: \( A \sim B \) if \( A = UBV^{-1} \) for some \( U \in GL(m, R), V \in GL(n, R) \);
- the stable left-right equivalence: \( A \overset{\text{st}}{\sim} B \) if \( U(A \oplus \mathbb{I}_{r \times r})V^{-1} = B \oplus \mathbb{I}_{r \times r} \) for some \( r, \bar{r} \) and \( U \in GL(m + r, R), V \in GL(n + r, R) \). (Here \( A, B \) can be of different sizes.)

The stable equivalence often implies the ordinary equivalence, e.g., this happens when
- \( R \) is a local ring;
- \( R \) is graded, \( R = \oplus_{d \in \mathbb{N}} R_d \), with \( R_0 \) local, Noetherian, and \( A, B \) are graded, i.e., their entries are homogeneous and the degrees satisfy \( \deg(a_{ij}) + \deg(a_{il}) = \deg(a_{ij}^l) + \deg(a_{kj}) \);
- \( R \) is a principal ideal ring.

This follows by the uniqueness of projective resolution of \( R \)-module \( \text{Coker}(A) \). [Eisenbud, §20].

Over a local ring \((R, m)\) one can pass to the minimal resolution, therefore \( A \sim I \oplus A \), with \( A \in \text{Mat}_{n \times n}(R) \).

vi. (The decomposability)

**Definition 2.1.** Let \( A \in \text{Mat}_{m \times n}(R) \) with the factorized ideal of maximal minors \( I_m(A) = J_1 \cdots J_r \subset R, 2 \leq m \leq n \).

1. \( A \) is called \((J_1, \ldots, J_r)\)-decomposable if \( A \sim \oplus_{i=1}^r A_i \), with \( I_m(A_i) = J_i \subset R \).
2. \( A \) is called stably-\((J_1, \ldots, J_r)\)-decomposable if \( A \overset{\text{st}}{\sim} (\oplus_{i=1}^r A_i) \), with \( I_m(A_i) = J_i \subset R \).
3. A module \( M \) with the factorized zeroth Fitting ideal \( \text{Fitt}_0(M) = J_1 \cdots J_r \subset R \) is called \((J_1, \ldots, J_r)\)-decomposable if \( M = \oplus M_i \), with \( \text{Fitt}_0(M_i) = J_i \).

We use the standard fact:

(3) \( A \) is stably-\((J_1, \ldots, J_r)\)-decomposable iff \( \text{Coker}(A) \) is \((J_1, \ldots, J_r)\)-decomposable.

This follows directly by Fitting’s lemma for the resolutions of the modules \( \text{Coker}(A) \), \( \oplus \text{Coker}(A_i) \), see [Eisenbud, §A.3].

vii. (Restriction onto the support) Take a local ring \((R, m)\), two matrices \( A, B \in \text{Mat}_{m \times n}(R) \) with \( 2 \leq m \leq n \), and the quotient homomorphism \( R \rightarrow R/I_m(A) \). Then \( A \sim B \) iff \( \bar{A} \sim \bar{B} \in \text{Mat}_{m \times n}(R/I_m(A)) \).

**Proof.** The part \( \Rightarrow \) is trivial, we prove the part \( \Leftarrow \).

Suppose \( \bar{A} \sim \bar{B} \) then \( A = UBV^{-1} + Q \), for some \( U \in GL(m, R), V \in GL(n, R) \) and \( Q \in \text{Mat}_{m \times n}(I_m(A)) \). By \[2.1\] we get: \( Q = A - Q' \), for a matrix \( Q' \in \text{Mat}_{m \times n}(m) \). Therefore \( \bar{A} = UBV^{-1} + (I - Q')^{-1} \).

The locality is needed e.g., because of the trivial example: \( R = \mathbb{K}[x], A = x \mathbb{I}, B = x(x-1) \mathbb{I} \).

The assumption \( 2 \leq m \) is needed because of the trivial example \( B = \mathbb{O} \in \text{Mat}_{1 \times n}(R), A \neq \mathbb{O} \).

Therefore the decomposability of \( A \) is testable by restricting to \( V(I_m(A)) \subset \text{Spec}(R) \):

(4) \( A \overset{\text{st}}{\sim} A_1 \oplus A_2 \quad \text{iff} \quad \bar{A} \overset{\text{st}}{\sim} \bar{A}_1 \oplus \bar{A}_2 \in \text{Mat}_{m \times n}(R/I_m(A)) \).

2.2. Decomposability of modules with non-connected support.

**Lemma 2.2.** Let \( A \in \text{Mat}_{m \times n}(R) \) and assume \( I_m(A) = J_1 \cdot J_2 \), for some ideals \( J_1, J_2 \subset R \) satisfying \( J_1 + J_2 = R \). Then \( A \) is stably \((J_1, J_2)\)-decomposable.

Geometrically, if \( \text{Supp}(M) = V(J_1) \cap V(J_2) \subset \text{Spec}(R) \) then \( M \) decomposes, \( M \cong M|_{V(J_1) \cap V(J_2)} \).

**Proof.** We have: \( \text{Coker}(A) = J_1 \cdot \text{Coker}(A) + J_2 \cdot \text{Coker}(A) \). This sum is direct:

(5) \( J_1 \cdot \text{Coker}(A) \cap J_2 \cdot \text{Coker}(A) = (J_1 + J_2) \cdot [J_1 \cdot \text{Coker}(A) \cap J_2 \cdot \text{Coker}(A)] = 0. \)

(Note that \( J_1 J_2 \cdot \text{Coker}(A) = 0 \), by [2.3] iii.) Now invoke (3).

This statement is for stable decomposability, the ordinary one does not hold, see remark [3.20].
2.3. Openness of the decomposability locus.

Lemma 2.3. Let $I_m(A) = J_1 \cdot J_2$ and suppose the localization $A_m \in \text{Mat}_{m \times n}(R_m)$ at a maximal ideal $m \supseteq J_1 + J_2$ is $(J_1)_m, (J_2)_m$-decomposable. Then there exists an open neighborhood $V(m) \in \mathcal{U} \subseteq \text{Spec}(R)$ such that $A_m$ is decomposable.

Namely, there exists a (non-nilpotent) element $g \in R \setminus m$ such that for the base change $R \to R[\frac{1}{g}]$ the matrix $A \in \text{Mat}_{m \times n}(R[\frac{1}{g}])$ is $(J_1, J_2)$-decomposable.

Proof. Suppose $U_m \cdot A_m \cdot V_m = A_{1,m} \oplus A_{2,m}$. Present the entries of $U_m, V_m$ as fractions. Let $g \in R \setminus m$ be the product of all the denominators of $U_m, V_m$. Take $\mathcal{U} := \text{Spec}(R) \setminus \{g \in \text{Spec}(R[\frac{1}{g}])\}$. Then $\mathcal{U} \cdot A \cdot V = A_1 \oplus A_2 \in \text{Mat}_{m \times n}(R[\frac{1}{g}])$.

2.4. Co-prime elements and co-regular ideals. Two regular elements $f_1, f_2 \in R$ (i.e., neither invertible, nor zero divisors) are called co-prime if $(f_1) \cap (f_2) = (f_1 \cdot f_2) \subset R$. This can be stated also as: both $f_1, f_2$ and $f_2, f_1$ are regular sequences in $R$. This is equivalent to: for every presentation $\{f_i = g_i \cdot h_i\}$ the element $h \in R$ is invertible.

Lemma 2.4. 1. The regular elements $\{f_i\} \subset R$ are pairwise co-prime iff the pairs $\{f_i, \prod_{j \neq i} f_j\}$ are co-prime for all $i$.

2. Assume $f_1, \ldots, f_r \in R$ are regular and pairwise co-prime. Then $\sum_{j=1}^r (\prod_{i \neq j} f_i) = \cap_{j=1}^r (f_j)$.

Proof.

1. The part $\Rightarrow$. If $a f_i \in (\prod_{j \neq i} f_j)$ then $a = a_i \cdot f_i$, for some $k \neq i$. Therefore $a_i f_i \in (\prod_{j \neq i, k} f_j)$. And so on.

The part $\Leftarrow$. If $a f_i \in (f_k)$ then $(\prod_{j \neq i, k} f_j) a f_i \in (\prod_{j \neq i} f_j)$. Therefore $(\prod_{j \neq i, k} f_j) a f_i \in (f_f)$, and hence $a \in (f_f)$.

2. The inclusion $\subseteq$ is obvious. For the inclusion $\supseteq$ take an element $a_k f_k + b_k \frac{\text{det}(A)}{f_k} \in \cap_{j=1}^r (f_j)$. It is enough to prove: $a_k f_k \in \sum_{j=1}^r (\frac{\text{det}(A)}{f_j})$. By the assumption $a_k f_k = a_i f_i + b_i \frac{\text{det}(A)}{f_i}$, thus (as $f_k, f_i$ are co-prime) $a_k \in a_i f_i + b_i \frac{\text{det}(A)}{f_i}$. Thus it is enough to prove: $a_i f_i f_k \in \sum_{j=1}^r (\frac{\text{det}(A)}{f_j})$. And so on.

We often impose this “co-regularity” condition on general ideals: $J_1 \cap J_2 = J_1 \cdot J_2 \subset R$. Geometrically, for $(R, m) \text{ local} \text{ Noetherian}$, this implies: the subschemes $V(J_1), V(J_2) \subset \text{Spec}(R)$ intersect properly and contain no embedded components supported on $V(J_1 + J_2)$.

Lemma 2.5. 1. $J_1 \cap J_2 = J_1 \cdot J_2$ iff $\text{Tor}_1^R(R/J_1, R/J_2) = 0$.

2. Let $A \in \text{Mat}_{n \times n}(R)$. Suppose $I_m(A) = J_1 \cdot J_2 = J_1 \cap J_2 \subset R$ and take the $R$-module $M := \text{Coker}(A)$.

i. $H^0_{J_1, J_2}(M) \supseteq J_1 M \cap J_2 M$.

ii. If $J_1 M \cap J_2 M = 0$ then $(J_i M) \cdot_M J_j = 0 \cdot_M J_j$ for $i \neq j$.

iii. (Square case) Suppose $\text{det}(A) = f_1 \cdot f_2$, regular and co-prime. Then $H^0_{J_1, J_2}(M) = 0$.

Proof. (These facts are standard, we recall the proof.)

1. Apply $\otimes R/J_2$ to the exact sequence $0 \to J_1 \to R \to R/J_1 \to 0$ to get:

$$0 = \text{Tor}_1^R(R/J_1, R/J_2) \to \text{Tor}_1^R(R/J_1, R/J_2) \to J_1 \otimes R/J_2 \xrightarrow{\phi} R/J_2 \to R/J_1 + J_2 \to 0.$$  

Therefore $\text{Tor}_1^R(R/J_1, R/J_2) = 0$ iff the map $\phi$ is injective iff $J_1 \cap J_2 = J_1 \cdot J_2$.

2. i. We have $(J_1 + J_2) \cdot (J_1 M \cap J_2 M) = J_1 \cdot J_2 \cdot M = 0$.

ii. The inclusion $\supseteq$ is obvious. For the part $\subseteq$ we note: $J_1 \cdot (J_1 M :_M J_2) \subseteq J_1 M \cap J_2 M = 0$.

iii. It is enough to prove: $0 :_M (f_1, f_2) = 0$. Take an element of $0 :_M (f_1, f_2)$ and its representative $\xi \in R^n$, see [241]. Then $(f_1, f_2) \cdot \xi \in \text{Im}(A)$. Therefore $(f_1, f_2) \cdot \text{Adj}(A) \cdot \xi \subset f_1 f_2 \cdot R^n$. As each of $f_i$ is regular, we get: $\text{Adj}(A) \cdot \xi \in (f_1 \cdot R^n) \cap (f_2 \cdot R^n) = f_1 f_2 \cdot R^n$. Therefore $\xi \in \text{Im}(A)$, i.e., $\xi$ represents $0 \in M$. 



3. THE OBSTRUCTION TO DECOMPOSABILITY AND REDUCTION TO “CONVENIENT” RINGS

While establishing the decomposability conditions it is often useful to change the base ring, e.g., to pass to a subring $S \subset R$ (that contains the entries of $A$), or to extend/to take the quotient, $R \rightarrow S$. To trace the behaviour of $A$ and $M := \text{Coker}(A)$ under this change we establish results of two types:

i. $A$ is decomposable over $R$ iff $\tilde{A}$ is decomposable over $S$.

ii. Consider the following conditions (they are needed for the decomposability, see §1):

\[(7) \quad I_m(A) = J_1 \cdot J_2 = J_1 \cap J_2 \subset R \text{ each contains a regular element}, \quad I_{m-1}(A) \subseteq J_1 + J_2.\]

We prove: if these conditions hold for $A \in \text{Mat}_{m \times n}(R)$ then they hold for $\tilde{A} \in \text{Mat}_{m \times n}(S)$.

This reduces the initial decomposability question (over $R$) to that over $S$. The study goes via the “obstruction to decomposability”, as explained in §1.3.

3.1. Reduction to Noetherian rings.

**Proposition 3.1.** Let $A \in \text{Mat}_{m \times n}(R)$. Then $I_m(A) = J_1 \cdots J_r$ and $A$ is $(J_1, \ldots, J_r)$-decomposable

iff there exists a Noetherian subring $S \subset R$ that contains the entries of $A$, with $I_m(A) = \prod J_i \subset S$, such that $A$ is $(\bar{J}_1, \ldots, \bar{J}_s)$-decomposable over $S$. \hspace{1cm} (The ideals $J_i \subset S$ are defined in the proof.)

Similar statement holds for the stable-decomposability.

**Proof.** The part $\Rightarrow$ is trivial. We prove the part $\Leftarrow$.

Assume $\bar{U} \cdot A \cdot V^{-1} = \oplus A_i$ for some $U \in \text{GL}(m, R)$, $V \in \text{GL}(n, R)$ and $A_i \in \text{Mat}_{m \times n_i}(R)$. Take the $\mathbb{Z}$-subalgebra $S \subset R$ generated by $1 \in R$ and the entries of $A$, $\{A_i\}$, $U$, $U^{-1}$, $V$, $V^{-1}$. We keep all the polynomial relations (with coefficients in $S$) among these elements that hold in $S$. Then $S$ is finitely generated, in particular Noetherian.

The ideal $J_i = I_m(A_i) \subset R$ is (finitely) generated by the maximal minors of $A_i$. Define $\bar{J}_i \subset S$ by these maximal minors. Then $U \cdot A \cdot V^{-1} = \oplus A_i$, for $U \in \text{GL}(m, S)$, $V \in \text{GL}(n, S)$, and $I_m(A) = \bar{J}_i$. \hspace{1cm} $\blacksquare$

For square matrices, $A \in \text{Mat}_{n \times n}(R)$, the conditions (7) read:

\[(8) \quad \det(A) = f_1 \cdot f_2 \text{ is regular in } R, \quad (f_1) \cdot (f_2) = (f_1) \cap (f_2) \subset R, \quad I_{n-1}(A) \subseteq (f_1, f_2).\]

We can often pass to a Noetherian subring $S \subset R$ while preserving these conditions:

**Proposition 3.2.** Assume the conditions (8) hold for $A \in \text{Mat}_{n \times n}(R)$. Assume for any finitely generated subring $S \subset R$, such that $A \in \text{Mat}_{n \times n}(S)$, the chain of ring extensions

$S \subseteq S : R (f_1f_2) \subseteq S : R (f_1f_2)^2 \subseteq \cdots = \lim \rightarrow S : R (f_1f_2)^d.$

stabilizes. Then there exists a Noetherian subring $S \subset R$ such that $A \in \text{Mat}_{n \times n}(S)$, and the conditions (8) hold over $S$.

**Proof.**

**Step 1.** Take the entries $\{a_{ij}\}$ of $A$. For each $(n-1)$-block expand $\det(A_{\square}) = d_{\square}^{(1)} \cdot f_1 + d_{\square}^{(2)} \cdot f_2$.

Take the subring $S \subset R$ generated by the (finite) collection $\{a_{ij}\}$, $f_1$, $f_2$, $\{d_{\square}^{(1)}\}$, $\{d_{\square}^{(2)}\}$.

We have:

\[(9) \quad A \in \text{Mat}_{n \times n}(S), \quad \det(A) = f_1 \cdot f_2, \quad I_{n-1}(A) \subseteq S \cdot f_1 + S \cdot f_2 \subset S.\]

The ring $S$ is Noetherian, being a finitely-generated $\mathbb{Z}$-algebra.

But the condition $(S \cdot f_1) \cdot (S \cdot f_2) = (S \cdot f_1) \cap (S \cdot f_2)$ does not necessarily hold, as $S$ is not necessarily a UFD, and we do not have the submodule/ideal contraction property, and cannot use $S \cdot f_i = (Rf_i) \cap S$.

**Step 2.** Note that conditions (8) hold also over any further extensions of $S$, i.e., if $S \subset \tilde{S}$ then $\det(\tilde{S} \otimes A) = (f_1f_2) \subset S$, $I_{n-1}(\tilde{S} \otimes A) \subseteq \tilde{S}f_1 + \tilde{S}f_2$.

We claim: $Sf_1 \cap Sf_2 = (f_1f_2) \cdot (S : R (f_1, f_2))$. Indeed, any element of $Sf_1 \cap Sf_2$ is presentable as $f_1f_2c$, for some $c \in R$ satisfying $f_1c, f_2c \in S$. And vice-versa.
Part 3.
If Part 2. $\rightarrow$
Equation (12) gives the morphism of Part 1.
$R \pi e R$ 2.

Moreover, by our assumption the sum $\sum_{d} S : R (f_1 \cdot f_2)^d$ stabilizes. Therefore, to show that $S'$ is finitely generated (hence Noetherian), it is enough to verify: the subring $S : R (f_1 \cdot f_2) \subset R$ is finitely generated. Indeed, the ideal $(f_1 f_2) : (S : R (f_1 f_2)) \subset S$ is finitely generated, as $S$ is Noetherian. And $f_1 f_2 \in R$ is a non-zero divisor. Thus $S : R (f_1 f_2) \subset R$ is finitely generated.

\textbf{Remark 3.3.}  
\textbf{i.} The assumption “the chain $S : S : R (f_1 f_2) \subset S : R (f_1 f_2)^2 \subset \cdots$ stabilizes” is satisfied: if the Artin-Rees condition (for any regular element $f \in R$) holds: $S \cap (f^N R) \subset f^{N-d} S$ for $N \gg 1$. And this holds for numerous non-Noetherian rings.

\textbf{ii.} This assumption is non-empty. For example, let $R = \mathbb{k}[t, \frac{1}{t}, \frac{1}{t^2}, \ldots]$ and $S = \mathbb{k}[t]$. Then $\sum S : R t^d = R$, non-Noetherian, and the chain $\{S : R t^d\}$ does not stabilize.

Another example is $R = \mathbb{k}[x, y]$, $S = \mathbb{k}[xy]$. Here $\sum S : R y^d = R$, but the chain $\{S : R y^d\}$ does not stabilize.

3.2. The obstruction to decomposability. Let $A \in \text{Mat}_{m \times n}(R)$ with $I_m(A) = J_1 \cdot J_2 = J_1 \cap J_2$. We construct the obstruction to decomposability, as is explained in §1.3. Let $M := \text{Coker}(A)$.

\textbf{Proposition 3.4.} The exact sequence of $R$-modules (constructed in the proof)
\begin{equation}
0 \rightarrow H^0_{J_1+J_2}(M) \rightarrow M \rightarrow M_1 \oplus M_2 \rightarrow Q \rightarrow 0
\end{equation}
satisfies:

1. The modules $M_1, M_2, Q$ are finitely generated and satisfy: $J_1 \cdot M_i = 0$, $(J_1 + J_2) \cdot Q = 0$.
2. If $M$ is $(J_1, J_2)$-decomposable then $Q = 0$.
3. If $Q = 0$ and the submodule $H^0_{J_1+J_2}(M) \subset M$ splits off as a direct summand, then $M$ is $(J_1, J_2)$-decomposable.

\textbf{Proof.} Separate the components of the support: $\prod \text{Spec}(R/J_i) \xrightarrow{\pi} \text{Spec}(R/\prod J_i)$. Here $\pi$ is defined via the idempotents, $e_i \in R/J_i$:
\begin{equation}
R/J_m(A) \xrightarrow{\pi^*} \prod R/J_i, \quad x \rightarrow (e_1 \cdot x, e_2 \cdot x).
\end{equation}
Note that $\pi^*$ is injective. Indeed, $\text{Ker}(\pi^*) = \cap J_i = \prod J_i = 0 \subset R/\prod J_i$.

Restrict the module $M$ to $V(J_i)$, i.e., define $M'_i := \text{Coker}(A \otimes R/J_i)$. Kill the unwanted torsion,
\begin{equation}
0 \rightarrow H^0_{J_1+J_2}(M'_i) = H^0_{J_i}(M'_i) \rightarrow M'_i \rightarrow M_i \rightarrow 0, \quad \text{for } i \neq j.
\end{equation}

Now push the $R/J_i$-modules $M_i$ back to $\text{Spec}(R)$, i.e., consider these as $R$-modules.

\textbf{Part 1.} Equation $\mathbf{(12)}$ gives the morphism of $R$-modules $M \xrightarrow{\phi_j} \oplus M_i$. Here $\text{ker}(\phi_i) = H^0_{J_i}(M)$ for $j \neq i$. Therefore $\text{ker}(\phi_1 \oplus \phi_2) = H^0_{J_1+J_2}(M)$. Hence the sequence $\mathbf{(11)}$ is exact at $M$. By construction we have $J_i \cdot M_i = 0$.

The module $Q$ is defined as the quotient by $\mathbf{(11)}$. Thus $M_1, M_2, Q$ are finitely generated. We verify: $(J_1 + J_2) \cdot Q = 0$. Fix an element of $Q$ and take its representative $z_1 \oplus z_2 \in M_1 \oplus M_2$. As $\phi_1, \phi_2$ are surjective, we have: $z_i = \phi_i(z_i)$ for some $z_i, z_2 \in M_i$.

\begin{equation}
M_1 \oplus M_2 \supset (J_1 + J_2)(z_1 + z_2) = J_2 z_1 + J_1 z_2 = J_2 \cdot \phi_1(z_1) + J_1 \cdot \phi_2(z_2) = J_2(\phi_1(z_1) + \phi_2(z_2)) + J_1(\phi_1(z_1) + \phi_2(z_2)) \in (J_1 + J_2) \cdot \text{Im}(\phi_1 \oplus \phi_2).
\end{equation}

Therefore $(J_1 + J_2)(z_1 + z_2)$ goes to $0 \in Q$, by the exactness of $\mathbf{(11)}$.

\textbf{Part 2.} Suppose $M = N_1 \oplus N_2$ with $\text{Fitt}_0(N_i) = J_i$. Equation $\mathbf{(13)}$ reads:
\begin{equation}
0 \rightarrow (R/J_i \otimes N_j) \xrightarrow{H^0_{J_i}(R/J_i \otimes N_i)} R/J_i \otimes (N_1 \oplus N_2) \rightarrow M_i \rightarrow 0 \quad \text{for } i \neq j.
\end{equation}

This gives: $M_i = R/J_i \otimes N_i/H^0_{J_i}(R/J_i \otimes N_i)$. In particular, $M \rightarrow M_1 \oplus M_2$, hence $Q = 0$.

\textbf{Part 3.} If $Q = 0$ (and the submodule $H^0_{J_1+J_2}(M) \subset M$ splits off) then $M = H^0_{J_1+J_2}(M) \oplus M_1 \oplus M_2$. ■
3.3. Functoriality of the obstruction $Q$. Given a morphism of rings $R \to S$, take the images $A \in \text{Mat}_{m \times n}(S)$, $I_m(A) = J_1 \cdot J_2 \subset S$, $M := \text{Coker}(A)$. Restrict $M$ to $V(J_i) \subset \text{Spec}(R)$ and take the torsion on $V(J_1 + J_2)$, i.e. $H^0_{J_1+J_2}(R/I_i \otimes M) \subset R/I_i \otimes M$. We can pass to $V(J_i) \subset \text{Spec}(S)$ in two ways: either by $S \otimes H^0_{J_1+J_2}(R/I_i \otimes M) \to S \cdot H^0_{J_1+J_2}(R/I_i \otimes M) \subseteq S/J_i \otimes M$ or as $H^0_{J_1+J_2}(M \otimes S/J_i) \subseteq S/J_i \otimes M$.

**Proposition 3.5.** Suppose $\tilde{J}_1 \cdot \tilde{J}_2 = \tilde{J}_1 \cap \tilde{J}_2 \subset S$ and let $i \in \{1,2\}$. Then:

1. The sequence \((11)\) induces the diagram:

$$
\begin{align*}
S \otimes M & \to \oplus(S \otimes M_i) \to S \otimes Q \to 0 \\
\downarrow \delta_0 & \downarrow \phi \downarrow & 0
\end{align*}
$$

(16)

2. The (natural) morphism $S \cdot H^0_{J_1+J_2}(R/I_i \otimes M) \to H^0_{J_1+J_2}(S/I_i \otimes M)$ is an embedding. It is an isomorphism iff $\phi: S \otimes Q \to Q$ is an isomorphism.

**Proof.**

1. One can either apply $S \otimes$ to the sequence \((13)\) or write the corresponding sequence for the module $\bar{M}$. This gives the two rows of the diagram:

$$
\begin{align*}
S \cdot H^0_{J_1+J_2}(M'_i) & \xrightarrow{\xi} S \otimes M'_i \to S \otimes M_i \to 0 \\
0 & \to H^0_{J_1+J_2}(M'_i) \to M'_i \to M_i \to 0
\end{align*}
$$

(17)

Here the isomorphism $\delta_1$ is the natural composition: $S \otimes M'_i = S \otimes \text{Coker}(A \otimes R/I_i) \cong \text{Coker}(A \otimes R/I_i \otimes S) = \text{Coker}(\hat{A} \otimes S/J_i) = M'_i$.

To define $\delta_2$ we observe that the morphisms $\delta_1, \epsilon$ in \((17)\) are $S$-linear. Therefore for any $\xi \in S \cdot H^0_{J_1+J_2}(M'_i)$ and a corresponding $d \gg 1$ one has: $(\tilde{J}_1 + \tilde{J}_2)^d \cdot \delta_1(\epsilon(\xi)) = 0$. Thus $(\delta_1 \circ \epsilon)(S \cdot H^0_{J_1+J_2}(M'_i)) \subseteq H^0_{J_1+J_2}(M'_i)$. Hence the morphism $\delta_1 \circ \epsilon$ factorizes through $H^0_{J_1+J_2}(M'_i)$, thus defining $\delta_2$. Moreover, $\delta_2$ is injective, as $\delta_1, \epsilon$ are injective.

Finally, $\delta_0$ is defined (and is surjective) by the standard diagram chasing.

Now the surjectivity of $\delta_0$ gives the claimed diagram \((16)\). The existence/surjectivity of $\phi$ follows by the standard diagram chasing.

2. By the diagram chasing in \((17)\) we get: $\delta_2$ is an isomorphism iff $\delta_0$ is an isomorphism iff $\phi$ is an isomorphism. ⊢

3.4. Reduction of decomposability to local/Henselian/complete rings.

**Corollary 3.6.** Let $A \in \text{Mat}_{m \times n}(R)$ with $I_m(A) = J_1 \cdot J_2 = J_1 \cap J_2$. Assume the submodule $H^0_{J_1+J_2}(M) \subset M$ splits off as a direct summand.

1. $A$ is stably-$(J_1, J_2)$-decomposable iff for every localization $R \to R_m$ at a maximal ideal $J_1 + J_2 \subseteq m \subset R$ the matrix $\hat{A} \in \text{Mat}_{m \times n}(R_m)$ is $(\hat{J}_1, \hat{J}_2)$-decomposable.

2. Let $(R, m)$ be a local ring and take the Henselization $R \to \hat{R}$. Then $A$ is $(J_1, J_2)$-decomposable iff $\hat{A} \in \text{Mat}_{m \times n}(\hat{R})$ is $(\hat{J}_1, \hat{J}_2)$-decomposable.

3. Let $(R, m)$ be a local ring and assume that the completion functor $\otimes \hat{R}(m)$ is exact on finitely generated $R$-modules. Then $A$ is $(J_1, J_2)$-decomposable iff $\hat{A} \in \text{Mat}_{m \times n}(\hat{R}(m))$ is $(\hat{J}_1, \hat{J}_2)$-decomposable.

**Proof.** It is enough to observe that the functors of localization, $\otimes R_m$, Henselization, $\otimes \hat{R}(h)$, and (by our assumption) completion, $\otimes \hat{R}(m)$, are exact. Then part 2 of proposition 3.5 gives: $S \otimes Q \cong \hat{Q}$. Finally:

1. The module $Q$ vanishes iff all its maximal localizations vanish. And it is enough to localize at the ideals satisfying $J_1 + J_2 \subseteq m$, as $Q_m = 0$ for $m \not\supseteq J_1 + J_2$.

2. The module $Q$ over a local ring vanishes iff its $m$-Henselization vanishes.
3. The (finitely generated) module $Q$ over a local ring vanishes iff $\hat{R}^m \otimes Q = 0$. ■

**Remark 3.7.** The functor $\otimes \hat{R}^m$ is exact on finitely generated modules over local Noetherian rings. But for local non-Noetherian rings $\otimes \hat{R}^m$ is not always exact, see example 2.8.7 in [Schenzel-Simon].

We restate the conclusion for modules.

**Corollary 3.8.** Take a finitely presented $R$-module $M$ with factorized Fitting ideal $\text{Fitt}_0(M) = J_1 \cdot J_2 = J_1 \cap J_2$. Suppose the submodule $\hat{H}^0_{J_1+J_2}(M) \subset M$ splits as a direct summand. TFAE:

i. $M$ is $(J_1, J_2)$-decomposable;

ii. the localization $M_m$ is $((J_1)_m, (J_2)_m)$-decomposable for any maximal ideal $m \supseteq J_1 + J_2$;

iii. (for $(R, m)$, assuming exactness of $- \otimes \hat{R}^m$) the completion $M \otimes \hat{R}^m$ is decomposable.

**Example 3.9.** (Semi-local rings/matrice over multi-germs of spaces) Let $R = R_1 \times \cdots \times R_k$, where $\{(R_i, m_i)\}$ are local rings. Fix the corresponding idempotents,

$$1 = \sum_{l=1}^k e_l, \quad e_l \cdot e_i = \delta_{l,i} \cdot e_l.$$  \hfill (18)

Let $A \in \text{Mat}_{m \times n}(R)$ with $I_m(A) = \prod J_i = \cap J_i$. Corollary 3.6 gives: $A$ is stably $(J_1, \ldots, J_r)$-decomposable iff $e_l A \in \text{Mat}_{m \times n}(R_l)$ is $(e_l J_1, \ldots, e_l J_r)$-decomposable, $e_l A \sim \oplus A^{(l)}_i$, for each $l = 1 \ldots k$. We do not assume that all $\{A^{(l)}_i\}$ (for a fixed $i$) are of the same size. In fact, by 3.2.iv. (and also directly via the idempotents) one gets: $A \sim \oplus_{l=1}^k e_l A$. Assuming the decomposability for each $l$, i.e., $e_l A \sim \oplus_{i=1}^r A^{(l)}_i$, we get: $A \sim \oplus_{i=1}^r \oplus_{l=1}^k A^{(l)}_i$.

Geometrically, for $\text{Spec}(R) = \bigsqcup \text{Spec}(R_i)$ (the finite union of germs of spaces), $A$ is decomposable iff each restriction $\{A|_{\text{Spec}(R_i)}\}$ is decomposable.

Finally we address the behaviour of the conditions (7) under base change.

**Lemma 3.10.** Assume the conditions (7) hold for $A \in \text{Mat}_{m \times n}(R)$. Then they hold for localizations, $\tilde{A} \in \text{Mat}_{m \times n}(R_m)$, Henselizations, $\tilde{A} \in \text{Mat}_{m \times n}(R h)$, and (when the functor $\otimes \hat{R}^m$ is exact) completions, $\hat{A} \in \text{Mat}_{m \times n}(\hat{R}^m)$.

**Proof.** The persistence of factorization, $I_m(A) = J_1 \cdot J_2$, and inclusion, $I_{m-1}(A) \subseteq J_1 + J_2$, is obvious. If $x \in J_i$ is regular then $\bar{x} \in \bar{J}_i$ is regular. (Apply the needed flat base change to the exact sequence $0 \to R \xrightarrow{\times x} R \to R/(x) \to 0$.) For the condition $J_1 \cap J_2 = J_1 \cdot J_2$ use its equivalent form, lemma 2.25, and observe that $\text{Tor}_1$ is functorial under flat base-change. ■

### 3.5. Reduction to completion for $C^\infty$-rings

Let $R$ be one of the standard (non-Noetherian) rings of Analysis:

- the smooth functions on an open set, $C^\infty(U)$, for $U \subseteq \mathbb{R}^p$;
- the germs of smooth functions along a closed subset, $C^\infty(U, Z)$;
- $C^\infty(U)/I$, $C^\infty(U, Z)/I$, for an ideal $I$, i.e., the rings of functions on the “subscheme” $V(I) \subseteq U$.

In this case the completion functor $\otimes \hat{R}$ is far from being exact/faithful. For example, the sequence $(0 \to (f) \to R) \otimes \hat{R}$ is not exact for any $0 \neq f \in m^\infty$, and $m^\infty \otimes \hat{R} = 0$. While our main decomposability criterion (theorem 1.1) is applicable to such rings (see 3.1.iv.), the “reduction-to-completion” is still important. Indeed, smooth functions are often studied via their Taylor expansions. Below we give an independent reduction-to-completion criterion.

Take a closed subset $Z \subseteq U$, and the corresponding completion $\hat{R}^Z := \varprojlim R/(I(Z)^{(j)})$. Here the differential power of ideal $I(Z)^{(j)}$ consists of functions vanishing at all the points of $Z$ to the order at least $j$:

$$I(Z)^{(j)} := \cap x \in Z m_x^j \subset R. \hfill (19)$$

The kernel of the map $R \to \hat{R}^Z$ is the ideal $I(Z)^{(\infty)}$ of the functions flat on $Z$.
The completion at $Z$ erases the complement of the formal neighbourhood of $Z$, therefore we always assume $V(I_m(A)) \subseteq Z \subseteq U$. Just a slight strengthening of this condition binds the decomposability over $R$ to that over $\hat{R}(Z)$. Let $R$ be $C^\infty(U)/I$ or $C^\infty(U, Z)/I$. Take the completion $R \to \hat{R}(Z)$ and accordingly $\text{Mat}_{m \times n}(R) \to \hat{\text{Mat}}_{m \times n}(\hat{R}(Z))$.

**Lemma 3.11.** Assume $I_m(A) \supseteq I(Z)^{\infty}$. Then $A \sim_{X} A_i$ iff $\phi(A) \sim_{X} \hat{A}_i$. (And then $\hat{A}_i \sim \phi(A_i)$.)

**Proof.** (the direction $\subseteq$) Whitney’s extension theorem ensures the surjectivity of completion, $R \to \hat{R}(Z)$, see e.g., §1.5 of Narasimhan and Bel.Ker., §2]. Therefore we choose $R$-representatives $\{A_i\}$ of $\{\hat{A}_i\}$ and have: $\phi(A) = \hat{U}_m \cdot (\oplus \phi(A_i)) \cdot \hat{U}_n^{-1}$, for some $\hat{U}_m \in \text{GL}(m, \hat{R}(Z))$, $\hat{U}_n \in \text{GL}(n, \hat{R}(Z))$.

We claim the surjectivity, $\text{GL}(n, R) \to \text{GL}(n, \hat{R}(Z))$. Indeed, take some Whitney representative $(\hat{U}_n \in \text{Mat}_{n \times n}(R)$ of $(\hat{U}_n \in \text{GL}(n, \hat{R}(Z))$. Then $V(\det(U_n)) \cap Z = \emptyset$. Thus $U_n$ is invertible in a small neighbourhood of $Z$. Modify $U_n$ outside of this neighbourhood (e.g. by cutoff functions) to achieve the invertibility on the whole $U$.

Finally, take some representatives $U_m \in \text{GL}(m, R)$, $U_n \in \text{GL}(n, R)$ of $\hat{U}_m, \hat{U}_n$. We get:

$$A = U_m \cdot (\oplus A_i) \cdot U_n^{-1} + A^\infty, \quad \text{where } A^\infty \in \text{Mat}_{m \times n}(I(Z)^{\infty}).$$

As $I_m(A) \supseteq I(Z)^{\infty}$, we can present $A^\infty = A \cdot C$, for some $C \in \text{Mat}_{m \times n}(R)$. Moreover, we can assume $C \in \text{Mat}_{m \times n}(I(Z))$. Indeed, take the $l_2$-matrix norm, $\|A^\infty\| \in \mathbb{R}$, then $A^\infty = A \cdot C$. Hence $A^\infty = A \cdot C \cdot \sqrt{\|A^\infty\|}$.

Therefore the matrix $\mathbb{I} - C$ is invertible in a small neighbourhood of $Z$. Using cutoff functions (as before), we can assume $\mathbb{I} - C \in \text{GL}(n, R)$. Therefore we get: $A = U_m \cdot (\oplus A_i) \cdot U_n^{-1} \cdot (\mathbb{I} - C)^{-1}$.

**Example 3.12.** Let $R = \mathbb{C}^{\infty}(\mathbb{P}^p)$ and assume $I_m(A) \supseteq (\mathbb{x})^{\infty}$, the ideal of functions flat at $a \in \mathbb{P}^p$. Then $A$ is decomposable iff its Taylor expansion at $a$ is decomposable.

We recall the standard way to ensure the assumption $I_m(A) \supseteq I(Z)^{\infty}$ of the last lemma.

**Lemma 3.13.** Let $R = \mathbb{C}^{\infty}(U)$, $A \in \text{Mat}_{m \times n}(R)$. Assume the function $\det(AA^T) : U \to \mathbb{R}$ satisfies the Lojasiewicz-type inequality: $|\det(AA^T)(x)| \geq C \cdot \text{dist}(x, Z)^{\delta}$, for some constants $C, \delta > 0$ and any $x \in U$. Then $I_m(A) \supseteq I(Z)^{\infty}$.

**Proof.** Recall that $I_m(A) \supseteq \det(AA^T)$. Therefore for any $h \in I(Z)^{\infty}$ it is enough to verify: $\frac{h}{\det(AA^T)} \in R$. By our assumptions, all the derivatives of $\frac{h}{\det(AA^T)} \in R$ tend to $0$ on $Z$. Therefore this ratio extends to a flat $\mathbb{C}^{\infty}$ function on $Z$. \hfill \blacksquare

### 3.6. Passage to the quotient ring, the case of non-flat base change.

In many cases the (in)decomposability is preserved under the base-change even when the morphism $R \to S$ is non-flat.

**Proposition 3.14.** Suppose the functor $S \otimes$ is faithful on finitely generated $R$-modules. Suppose $I_m(A) = J_1 \cdot J_2 = J_1 \cap J_2 \subseteq R$ and $J_1 \cdot J_2 = J_1 \cap J_2 \subseteq S$, see ii. Suppose the submodule $H^0_{j_1+j_2}(M) \subset M$ splits off, and moreover $J_1 M \cap J_2 M = 0$ and $J_1 M \cap J_2 M = 0$, and moreover $S \cdot H^0_{j_2}(M) = H^0_{j_2}(M)$ for $j = 1, 2$. Then $M$ is $(J_1, J_2)$-decomposable iff $\hat{M}$ is $(\hat{J}_1, \hat{J}_2)$-decomposable.

**Proof.** (of the part $\subseteq$) The (natural) morphism $R_{J_1} \otimes H^0_{j_1}(M) \to R_{J_1} \cdot H^0_{j_1}(M) \subseteq R_{J_1} \otimes M$ defines the embedding $R_{J_1} \cdot H^0_{j_1}(M) \to H^0_{j_1}(R_{J_1} \otimes M)$. We claim: this is an isomorphism. Indeed, take an element of $H^0_{j_1}(R_{J_1} \otimes M)$ and let $\xi \in M$ be its representative. Then $J^d_1 \cdot \xi = J \cdot J_1 \cap J_2 M$ for some $d > 1$. By our assumption we get: $J^d_1 \cdot \xi = 0$. Thus $\xi \in H^0_{j_1}(M)$, proving the claim.

In the same way one gets: $S/J_1 \cdot H^0_{j_1}(\hat{M}) \cong H^0_{j_1}(S/J_1 \otimes \hat{M})$.

Therefore we have:

$$S \cdot H^0_{j_1+j_2}(R_{J_1} \otimes M) = S \cdot H^0_{j_1}(R_{J_1} \otimes M) \cong (S \otimes H^0_{j_1})(M) = S/J_1 \cdot H^0_{j_1}(M) \cong H^0_{j_1}(S/J_1 \otimes \hat{M}) = H^0_{j_1+j_2}(S/J_1 \otimes \hat{M}).$$
Applying part 2 of proposition 3.5 we get: the morphism \( \phi : S \otimes Q \to \bar{Q} \) is an isomorphism. By our assumption \( \bar{Q} = 0 \), thus \( S \otimes Q = 0 \). By our assumption the functor \( S \otimes \) is faithful, therefore \( Q = 0 \). Finally, by proposition 3.4 we get: \( M \) is \((J_1, J_2)\)-decomposable.

\textbf{Corollary 3.15.} \textit{(Square case)} Let \( S = R/y \) and \( A \in \text{Mat}_{n \times n}(R) \) with \( \det(A) = f_1 \cdot f_2 \). Suppose \( f_1, f_2 \) are coprime, and \( f_1, f_2 \) are coprime. Suppose \( S \cdot (\text{Im}(A) : f_j) = \text{Im}(\bar{A}) : \bar{f}_j \) for \( j = 1, 2 \). Then \( A \) is \( f_1, f_2 \)-decomposable if \( \bar{A} \) is \( f_1, f_2 \)-decomposable.

\textbf{Proof.} By lemma 2.5 \( f_1 \cdot M \cap f_2 \cdot M = 0 \) and \( \bar{f}_1 \cdot M \cap \bar{f}_2 \cdot M = 0 \). The assumption \( S \cdot (\text{Im}(A) : f_j) = \text{Im}(\bar{A}) : \bar{f}_j \) implies \( S \cdot H^0_{(f_j)}(M) = H^0_{(f_j)}(\bar{M}) \). Now apply proposition 3.4 \( \blacksquare \)

\textbf{Remark 3.16.} The assumption \( S \cdot (\text{Im}(A) : f_j) = \text{Im}(\bar{A}) : \bar{f}_j \) is needed here. For example, let \( A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \), and \( S = R/(y) \). Then \( \bar{A} \) is \( \bar{x}, \bar{z} \)-decomposable, though \( A \) is indecomposable.

\textbf{3.7. Graded-to-local reduction.} Take an \( N \)-graded ring, \( R = \oplus_{d \in \mathbb{N}} R_d \), with \( R_0 \) local, Noetherian. Denote by \( m \subset R \) the ideal generated by \( R_{>0} \) and by the maximal ideal of \( R_0 \). Thus \( m \) is a homogeneous maximal ideal. It is also the largest among the homogeneous ideals. The localization of \( R \to R_m \) is injective.

\textbf{Lemma 3.17.} Assume \( A \in \text{Mat}_{m \times n}(R) \) is graded \((2.1)\). Then \( A \sim \oplus A_i \) by \( GL(m, R) \times GL(n, R) \) iff \( A \sim \oplus A_i \) by \( GL(m, R_m) \times GL(n, R_m) \).

\textbf{Proof.} (of the part \( \Leftarrow \)) One has \( \text{Coker}(A) \otimes R_m \cong \text{Coker}(\oplus A_i) \otimes R_m \). Recall: two graded modules are isomorphic iff their localizations at the maximal among the homogeneous ideals are isomorphic. Thus \( \text{Coker}(A) \cong \text{Coker}(\oplus A_i) \). Now invoke \( 2.7) \).

\textbf{Remark 3.18.} \textit{i.} Assume \( A \in \text{Mat}_{m \times n}(R) \) is homogeneous, i.e., all its entries are homogeneous, and of the same degree. Then we get a stronger statement: if \( A \otimes R_m \sim \oplus A_i \) (for some \( \{A_i\} \) over \( R_m \)) then \( A \sim \oplus A_i \) by \( GL(m, R_0) \times GL(n, R_0) \).

\textit{ii.} The locality of \( R_0 \) is important and the lemma does not hold if we localize just at the ideal \( R_{>0} \).

For example, suppose \( R_0 \) is a domain and there exists \( A \in \text{Mat}_{m \times n}(R_0) \), with \( \det(A) \neq 0 \), that is indecomposable. Then \( (R_0)_{R_{>0}} \) is a field (in \( R_{R_{>0}} \)) and \( A \otimes R_{R_{>0}} \sim I \otimes R_{R_{>0}} \).

As another example, suppose \( R_0 \) is a domain, and \( 0 \neq a_0 \in R_0 \), \( 0 \neq a_1 \in R_{>0} \) satisfy: \( a_0 \cdot a_1 = 0 \). The \( R_{>0} \)-localization sends \( a_1 \) to 0 in \( R_{R_{>0}} \). Then, for any matrix \( A \in \text{Mat}_{m \times n}(R) \), the \( R_{>0} \)-localization of \( a_1 \cdot A \) is (trivially) diagonalizable.

\textbf{3.8. Decomposability over Spec\( (R) \) vs that over Proj\( (R) \) for graded rings.} In this subsection \( R = \oplus_{d \in \mathbb{N}} R_d \), with \( R_0 \) a field. Assume the ideal \( R_{>0} \) is generated (not necessarily finitely) by \( R_1 \). Take the projective scheme \( \text{Proj}(R) \), with the structure sheaf \( \mathcal{O}_{\text{Proj}(R)} \), see pg. 76-77, 116-123 of [Stacks], or \( \S 27.8-27.12 \) of [Stacks].

Recall some details. The points of \( \text{Proj}(R) \) correspond to non-maximal homogeneous primes, \( p \subset R_{>0} \). The stalks of the structure sheaf at such points can be presented explicitly, \( \mathcal{O}_{\text{Proj}(R)}(p) = R(p) \subset R_p \), where \( R(p) \) consists of the (equivalence classes of) fractions \( a/b \), here \( a \in R, b \in R \setminus p \) are homogeneous and satisfy \( \deg(a) = \deg(b) \). The basic affine charts are \( U_g := \text{Proj}(R) \setminus V(g) \), for non-nilpotent elements \( g \in R_1 \). Their structure rings are \( \mathcal{O}_{\text{Proj}(R)}(U_g) = R(g) \subset R[1/g] \), where \( R(g) := \{ \sum a_g g^d | \deg(a_g) = \deg(g^d) \} \).

The correspondence \( \text{mod}_{A, p} R \to \text{Coh}(\text{Proj}(R)) \) associates to the graded module \( M := \text{Coker}(A) \) the coherent sheaf \( \bar{M} := \text{Coker}(A) \in \text{Coh}(\text{Proj}(R)) \). For \( A = \sum A_d \in \text{Mat}_{m \times n}(\oplus R_d) \) and an affine chart \( U_g \subset \text{Proj}(R) \) we get the presentation
\[
(22) \quad \mathcal{O}_{\text{Proj}(R)}(U_g)^n \xrightarrow{A(g)} \mathcal{O}_{\text{Proj}(R)}(U_g)^m \to \bar{M}(U_g) \to 0,
\]
where \( A(g) = \sum A_d g^d \).

Localizing this sequence at a point \( p \in U_g \) we get the presentation of the stalk of the sheaf:
\[
(23) \quad \mathcal{O}_{\text{Proj}(R), p}^n \xrightarrow{A(p)} \mathcal{O}_{\text{Proj}(R), p}^m \to \bar{M}_p \to 0.
\]
Here $A^{(p)}$ was obtained through the choice of the embedding $p \in \mathcal{U}_g \subset \text{Proj}(R)$, but this presentation is well defined up to isomorphism. If $I_m(A) = J_1 \cdot J_2$ then $I_m(A^{(g)}) = J_1^{(g)} \cdot J_2^{(g)} \subset R^{(g)}$ and $I_m(A^{(p)}) = J_1^{(p)} \cdot J_2^{(p)} \subset \mathcal{O}_{(\text{Proj}(R),p)} = R^{(p)}$.

**Theorem 3.19.** Suppose $A \in \text{Mat}_{m \times n}(R)$ is graded $(\mathcal{A},v)$ and $I_m(A) = J_1 \cdot J_2 = J_1 \cap J_2$ for some homogeneous ideals.

1. If the conditions (7) hold for $A \in \text{Mat}_{m \times n}(R)$ then they hold for all the stalks $A^{(p)} \in \text{Mat}_{m \times n}(\mathcal{O}_{(\text{Proj}(R),p)})$.

2. Suppose $H^0_{J_1+J_2}(M) = 0$. Assume there exists $x \in R_{>0}$ that is $M$-regular and the base change $R \to R^{(x)}$ satisfies: $I_m(\tilde{A}) = \tilde{J}_1 \cdot \tilde{J}_2 = \tilde{J}_1 \cap \tilde{J}_2$ and $H^0_{\tilde{J}_1+\tilde{J}_2}(\tilde{M}) = 0$. Then $A$ is $(J_1,J_2)$-decomposable iff for each point $p \in \mathbb{P}(V(J_1) \cap \mathbb{P}(V(J_2)) \subset \text{Proj}(R)$ the stalk $A^{(p)}$ is $(J_1^{(p)},J_2^{(p)})$-decomposable.

Thus the graded decomposability problem in dimension $\dim(R)$ is reduced to (many) local decomposability problems in dimension $\dim(R) - 1$.

One can restate the conclusion for modules: “A graded $R$-module $M$ is decomposable iff all the stalks of the corresponding sheaf $\tilde{M}$ are decomposable”.

**Proof.**

1. The Fitting ideal sheaves of $\tilde{M}$ are the sheaves associated to the Fitting ideals of $M$. Indeed by (22) we identify the ideals on the basic affine opens:

$$Fitt_*(\tilde{M}(\mathcal{U}_g)) = Fitt_*(M^{(g)}) = Fitt_*(M^{(g)})(\mathcal{U}_g) \subset \mathcal{O}_{\text{Proj}(R)}(\mathcal{U}_g) = R^{(g)}.$$  

This identification of ideals is compatible with all the restrictions onto open subsets of $\mathcal{U}_g$. Hence the identification of the sheaves of ideals, $Fitt_*(\tilde{M}) = Fitt_*(M) \subset \mathcal{O}_{\text{Proj}(R)}$. In particular we get: $Fitt_0(\tilde{M}) = \tilde{J}_1 \cdot \tilde{J}_2$.

Let $f_i \in J_i$ be regular homogeneous elements. We prove: their images in the stalks $J_i^{(p)} \subset R^{(p)}$ are regular. It is enough to verify the regularity on charts, as the localization $R^{(g)} \to R^{(p)}$ is exact. Take a chart, $p \in \mathcal{U}_g \subset \text{Proj}(R)$, for some $g \in R_1$. Suppose \( \frac{f_i}{g^{m}} : \sum l \frac{g}{g} = 0 \in R^{(g)} = \mathcal{O}_{\text{Proj}(R)}(\mathcal{U}_g) \). Then $f_i : \sum l N g^{N-l} = 0 \in R[\frac{1}{g}]$. But $f_i \in R$ is regular in $R[\frac{1}{g}]$, therefore $\sum N l g^{N-l} = 0 \subset R[\frac{1}{g}]$. And thus $\sum N l g^{N-l} = 0 \subset R^{(g)}$.

Finally we claim: $\tilde{J}_1 \cap \tilde{J}_2 = \tilde{J}_1 \cdot \tilde{J}_2 \subset \mathcal{O}_{\text{Proj}(R)}$, i.e., these sheaves are co-regular. Namely, for each stalk: $J_i^{(p)} \cap J_j^{(p)} = J_i^{(p)} \cdot J_j^{(p)} \subset R^{(p)}$. By lemma (23) it is enough to verify (locally): $\text{Tor}_1^{R^{(p)}}(R^{(p)}/J_i^{(p)},R^{(p)}/J_j^{(p)}) = 0$ for all the points $p \in \text{Proj}(R)$. As the localization is exact, it is enough to verify this vanishing on the basic opens: $\text{Tor}_1^{R^{(g)}}(R^{(g)}/J_i^{(g)},R^{(g)}/J_j^{(g)}) = 0$. But this is the degree=0 part of $\text{Tor}_1^{R^{(g)}}(R[\frac{1}{g}]/J_i[\frac{1}{g}],R[\frac{1}{g}]/J_j[\frac{1}{g}])$. And the morphism $R \to R[\frac{1}{g}]$ is flat, therefore:

$$\text{Tor}_1^{R[\frac{1}{g}]}(R[\frac{1}{g}]/J_i[\frac{1}{g}],R[\frac{1}{g}]/J_j[\frac{1}{g}]) = R[\frac{1}{g}] \otimes_R \text{Tor}_1^{R}(R,J_i,J_j) = 0.$$  

2. The part $\Rightarrow$ is trivial. We prove the part $\Leftarrow$. First we prove that the obstruction module is supported on $V(R_{>0}) \subset \text{Spec}(R)$ only. Then we prove: $\text{depth}(Q) > 0$. Together this implies: $Q = 0$.

**Step 1.** For each point $p \in \text{Proj}(R)$ take the stalk $\tilde{M}_p = M^{(p)}$. By Step 1: $\text{Fitt}_0(\tilde{M}_p) = \tilde{J}_1^{(p)} \cap \tilde{J}_2^{(p)} = \tilde{J}_1^{(p)} \cdot \tilde{J}_2^{(p)}$. The functor $\text{mod}_g R \to \text{Coh}(\text{Proj}(R))$ is exact, §27.8-27.12 of [Stacks]. Therefore equation (11) gives the exact sequence:

$$H^0_{J_1+J_2}(M^{(p)}) \to \tilde{M}_p = M^{(p)} \to M_{1,(p)} \oplus M_{2,(p)} = (\tilde{M}_1)_p \oplus (\tilde{M}_2)_p \to \tilde{Q}_p = Q^{(p)} \to 0.$$  

Here $M_{i,(p)} = (\mathcal{M}^{i}/H_{\mathcal{M}}^{0}(\mathcal{M})^{(p)}) = M^{i}/h^{0}_{\mathcal{M}}(M^{i,p}) = (M^{(p)})_{i}$. Therefore, as $M^{(p)}$ is decomposable, proposition (3.3) gives: $\tilde{Q}_p = 0$.  

Thus the sheaf $\hat{Q} \in \text{Coh(Proj}(R))$ vanishes. Therefore $\text{Supp}(Q) \subseteq V(R_{>0}) \subset \text{Spec}(R)$. In particular, either $\text{depth}(Q) = 0$ or $Q = 0$.

**Step 2.** Take $x \in R$ as in the assumptions. As in the proof of proposition 3.19, we can either restrict equation (11) to the hypersurface $V(x) \subset \text{Spec}(R)$ (by applying $R/(x) \otimes$) or write (11) directly for $\bar{M}$. Then the diagram (16) becomes:

\[
\begin{array}{c}
\oplus \text{Tor}_{1}(R/(x), M_{i}) \rightarrow \text{Tor}_{1}(R/(x), Q) \rightarrow R/(x) \otimes M \xrightarrow{\delta} \oplus (R/(x) \otimes M_{i}) \rightarrow R/(x) \otimes Q \rightarrow 0 \\
0 \rightarrow \bar{M} \xrightarrow{\bar{\pi}} \oplus \bar{M}_{i} \rightarrow \bar{Q} \rightarrow 0
\end{array}
\]

(27)

Then $\pi$ is injective, as $\bar{\pi}$ is injective.

We claim: $x$ is $M_{i}$-regular, i.e., $\text{Tor}_{1}(R/(x), M_{i}) = 0$. Indeed, start from the presentation (13). Suppose $x \cdot [\xi + J_{i}M] = 0 \in M_{i}$ for some $\xi \in M$. Then $x \cdot \xi \in H^{0}_{J_{i}}(M'_{i})$. Therefore $J'_{i} \cdot x \cdot \xi = 0 \in M'_{i}$ for some $d \gg 1$. And then $J'_{i+1} \cdot x \cdot \xi = 0 \in M$. But $x$ is $M$-regular, thus $J'_{i+1} \cdot \xi = 0 \in M$. Hence $\xi \in H^{0}_{J_{i}}(M'_{i})$, i.e., $[\xi + J_{i}M] = 0 \in M_{i}$.

Combining the vanishing $\text{Tor}_{1}(R/(x), M_{i}) = 0$ and the injectivity of $\pi$ the last diagram gives: $\text{Tor}_{1}(R/(x), Q) = 0$. Thus if $Q \neq 0$ then $x$ is $Q$-regular, in contradiction with $\text{depth}(Q) = 0$.

Altogether $Q = 0$. By proposition 3.5, $\text{Coker}(A) = \oplus M_{i}$. Thus (2.2.1v) $A$ is decomposable over $R$.

**Remark 3.20.** The assumption $J_{1} \cdot J_{2} = J_{1} \cap J_{2}$ implies $\emptyset \neq \mathbb{P}V(J_{1}) \cap \mathbb{P}V(J_{2}) \subset \text{Proj}(R)$. The proposition does not hold when $\emptyset = \mathbb{P}V(J_{1}) \cap \mathbb{P}V(J_{2}) \subset \text{Proj}(R)$. For example, let $R = k[x, y]$, for a field $k = k$, and take $A \in \text{Mat}_{2 \times 2}(R_{j})$, whose elements are generic homogeneous polynomials of degree $d$. Then $\det(A)$ splits into co-prime linear factors. Then the stalks $\{A^{p}\}$ are trivially decomposable. But for $d \geq 3$ the matrix is indecomposable. (Note that $I_{1}(A)$ is minimally generated by 4 elements.) Compare this to the decomposition of modules with non-connected support, 2.2.

The technical assumptions of part 2 of theorem 3.19 become simple in the case of square matrices.

**Corollary 3.21.** Let $\det(A) = f_{1} \cdot f_{2}$, regular and coprime. Suppose $\text{depth}R/(f_{1}, f_{2}) > 0$. Then $M$ is $f_{1}, f_{2}$-decomposable iff $\bar{M}_{p}$ is $(f_{1})^{p}(f_{2})^{p}$-decomposable at each point $p \in \mathbb{P}V(f_{1}) \cap \mathbb{P}V(f_{2}) \subset \text{Proj}(R)$.

**Proof.** Take $x \in R$ that is $R/(f_{1}, f_{2})$-regular. We can assume that $x$ is homogeneous. Then $f_{1}, f_{2}, x$ is a regular sequence. As $R$ is graded, and $f_{1}, f_{2}$ are homogeneous, the sequence $x, f_{1}, f_{2}$ is regular as well. Therefore $\bar{f}_{1}, \bar{f}_{2} \in R/(x)$ are coprime. Therefore $H^{0}_{(f_{1}, f_{2})}(M) = 0$ and $H^{0}_{(f_{1}, f_{2})}(\bar{M}) = 0$, by lemma 2.5.

Moreover, $x$ is $M$-regular. Indeed, if $x \xi \in \text{Im}(A)$ then $x \cdot \text{Adj}(A)\xi \in (f_{1}f_{2}) \cdot R^{n}$, thus $\xi \in \text{Im}(A)$.

Now apply theorem 3.19.

**Remark 3.22.** Recall Max Noether’s fundamental theorem. Given homogeneous polynomials $f, g, h \in \mathbb{k}[x, y, x_{2}], \mathbb{k} = \mathbb{k}$, with $f, g$ coprime. Then $h \in (f, g) \subset \mathbb{k}[x, y, x_{2}]$ iff this holds for the localizations $h_{p} \in (f_{p}, g_{p})$ at all the points of the intersection $V(f) \cap V(g) \subset \mathbb{P}^{2}$. Theorem 3.19 is the natural analogue of this theorem for the decomposability question of matrices over graded rings.

**4. Decomposability criteria for square matrices**

**Theorem 4.1.** Let $R$ be a commutative, unital ring and let $A \in \text{Mat}_{\alpha \times \alpha}(R)$. Suppose $\det(A) = f_{1} \cdots f_{r} \in R$, where $\{f_{j}\}$ are regular and pairwise co-prime, see 2.4.  

1. $A$ is stably-($f_{1}, \ldots, f_{r}$)-decomposable iff $I_{n-1}(A) \subseteq \sum_{j=1}^{r} (\prod_{i \neq j} f_{i})$.

2. Assume $R$ is a local ring, or $R$ is graded and $A$ is graded. Then $A$ is ($f_{1}, \ldots, f_{r}$)-decomposable iff $I_{n-1}(A) \subseteq \sum_{j=1}^{r} (\prod_{i \neq j} f_{i})$. 


For another presentation of the condition \( I_{n-1}(A) \subseteq \sum_{i=1}^{r}(\prod_{i \neq j} f_i) \) see lemma 24.

First we prove this theorem, then give examples and discuss the (necessity of the) assumptions. Then come the first applications to the diagonal reduction of linear determinantal representations/tuples of matrices.

### 4.1. Block-diagonalization of (“almost”-)projectors over a ring.

In this subsection we impose the condition on \( R \):

\[(28) \quad \text{any finitely-generated projective module over } R \text{ is free.}\]

It holds, e.g., for local rings and for \( S[x] \), with \( S \) local Noetherian, \( \dim(S) \leq 2 \), see [Eisenbud, §A.3.2).

**Lemma 4.2.** Suppose \( R \) satisfies (28), while \( \{P_i \in \text{Mat}_{n \times n}(R)\} \) satisfy: \( \sum P_i = 1 I \) \( n \times n \) and \( P_i P_j = 0 \) for \( i < j \). Then \( \{P_i\}_i \) are simultaneously diagonalizable. Namely, there exists \( U \in \text{GL}(n, R) \) satisfying: \( \sum x_i U P_i U^{-1} = \oplus x_i U P_i U^{-1} = \oplus x_i 1 I_n \). (Here \( \{x_i\} \) are indeterminates.)

**Proof.**

First we establish the case \( r = 2 \).

**Step 1.** We verify the standard projector properties: \( P_1^2 = P_1 \) and \( P_1 P_2 = 0 \) for \( i \neq j \). Indeed:

\[(29) \quad P_1^2 = P_1 (P_1 + P_2) = P_1 \cdot 1 I = P_1, \quad P_2^2 = (P_1 + P_2) P_2 = 1 I \cdot P_2 = P_2.
\]

Then from here one gets: \( P_1 = (P_1 + P_2) \cdot P_1 = P_1^2 + P_2 P_1 = P_1 + P_2 P_1 \), hence \( P_2 P_1 = 0 \).

**Step 2.** Consider \( P_1, P_2 \) as endomorphisms of the free module \( F = R^n \). Define \( F_1 = F_1(F) \). We claim: \( F_1 \oplus F_2 = F \). Indeed, by the definition: \( P_i(F_1) = P_i(F) = 0 \) for \( i \neq j \). Thus: \( P_i(F_1 \cap F_2) = 0 \) for any \( i \). But then: \( 1 I(F_1 \cap F_2) = (P_1 + P_2)(F_1 \cap F_2) = 0 \).

Besides: \( F_1 \oplus F_2 = P_1(F) \oplus P_2(F) = 1 I(F) = F \). Therefore \( F = F_1 \oplus F_2 \), hence \( F_1, F_2 \) are projective submodules of \( F \). But then, by the initial assumption on \( R \), they are free.

Finally, take some bases \( \{v^{(j)}_i\} \), of \( F_j \), for \( j = 1, 2 \). The change from the standard basis of \( F \) to the basis \( \{v^{(1)}_i\}, \{v^{(2)}_i\} \) gives the needed diagonalizing transformation \( P_i \to U P_i U^{-1} \).

For the case \( r > 2 \) we apply the \( (r = 2) \)-argument to \( P_1, \sum_{i=2}^{r} P_i \). Then restrict to the submodule \( \sum_{i=2}^{r} P_i(F) \) and iterate. ■

**Remark 4.3.** The assumption (28) is necessary. Suppose there exists a (f.g.) projective but non-free module \( F_1 \) \( \bmod R \). Take its complementary, \( F_1 \oplus F_2 = R^n \). Define the projection homomorphisms \( P_i : R^n \to F_i \subset R^n \) by \( R^n \ni s \mapsto s_i + s_2 \mapsto s_i \in F_i \). (Here the decomposition \( s = s_1 + s_2 \) is unique.) Then \( P_1 P_2 = 0 \) and \( P_1 + P_2 = 1 I \). But \( \{P_i\}_i \) cannot be brought to the prescribed form. (This would imply the freeness of \( F_1 \).)

As an explicit example take \( R = \mathbb{K}[x]/(x^2 - 1) \), \( \text{char}([\mathbb{K}]) \neq 2 \), and \( P_2 := (\frac{1+2}{2}) + (\frac{1+2}{2}) \in \text{Mat}_{2 \times 2}(R) \).

In lemma 3.2 we treated projectors. Now we treat the “almost”-projectors. Let \( a \subseteq R \) be an ideal in a local ring, and suppose \( (R, a) \) is a Henselian pair, see [Stacks §15.11].

**Lemma 4.4.** Suppose \( \{P_i \in \text{Mat}_{n \times n}(R)\} \) satisfy: \( \sum P_i = 1 I \) \( n \times n \) and \( P_i P_j \in \text{Mat}_{n \times n}(a) \) for \( i < j \). Then \( \{P_i\}_i \) are simultaneously block-diagonalizable. Namely there exists \( U \in \text{GL}(n, R) \) satisfying

\[(30) \quad \sum x_i U P_i U^{-1} = \oplus (x_i 1 I_n + \sum j x_j A_{ij}).
\]

Here \( \{x_i\} \) are indeterminates, \( A_{ij} \in \text{Mat}_{n \times n}(a) \) and \( \sum j A_{ij} = 0 \) for each \( i \).

**Proof.**

First we establish the case \( r = 2 \). Take the quotient \( R \twoheadrightarrow R/a \). We get the projectors:

\( \phi(P_1) + \phi(P_2) = 1 I, \phi(P_1) \cdot \phi(P_2) = 0 \). We can assume \( \phi(P_1) = 1 I 0 0 \) and \( \phi(P_2) = 0 0 1 I \), (by lemma 3.2) (Note that the ring \( R/a \) is local, thus the condition (28) holds.)

Now we remove the off-diagonal blocks of \( P_i \). For this it is enough to resolve for \( (P_1) \) the condition

\[(31) \quad (1 I + \tilde{U}) P_1 (1 I + \tilde{U})^{-1} = 1 I 0 0 \quad \text{for} \quad \tilde{U} \in \text{Mat}_{n \times n}(a).
\]
Present $P_1$ in the block-form, then we are to resolve the equation

$$
\begin{bmatrix}
\mathbb{I} & \tilde{U}_{12} \\
\tilde{U}_{21} & \mathbb{I}
\end{bmatrix}
\begin{bmatrix}
\mathbb{I} + P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
= 
\begin{bmatrix}
D_1 & \mathbb{O} \\
\mathbb{O} & A_{12}
\end{bmatrix}
\begin{bmatrix}
\mathbb{I} & \tilde{U}_{12} \\
\tilde{U}_{21} & \mathbb{I}
\end{bmatrix}.
$$

Here all the entries of $\{P_{ij}\}$, $\tilde{U}_{12}, \tilde{U}_{21}$ belong to $a$. We get $D_1 = \mathbb{I} + P_{11} + \tilde{U}_{12}P_{21}$, $A_{12} = \tilde{U}_{21}P_{12} + P_{22}$.

Substitute these $D_1, A_{12}$ into the off-diagonal equations, then we have to resolve:

$$
P_{12} + \tilde{U}_{12}P_{22} = (\mathbb{I} + P_{11} + \tilde{U}_{12}P_{21})\tilde{U}_{12}, \quad \tilde{U}_{21}(\mathbb{I} + P_{11}) + P_{21} = (\tilde{U}_{21}P_{12} + P_{22})\tilde{U}_{21}.
$$

Both equations are resolvable by (the implicit function theorem) as $(R, a)$ is a henselian pair. Indeed, e.g., for the map $F(X) = P_{12} + XP_{22} - (\mathbb{I} + P_{11} + XP_{21})X$ the derivative $\frac{dF}{dX}|_{X = \mathbb{I}}$ is invertible, being of the form $\mathbb{I} + (a)$.

Thus $P_1$ is brought by conjugation to the form $(\mathbb{I}_{n_1 + 1} + A_{11}) \oplus A_{12}$, where $A_{11} \in \text{Mat}_{n_1 \times n_1}(a)$, $A_{12} \in \text{Mat}_{n_2 \times n_2}(a)$. The prescribed block-diagonal form of $P_2$ follows, as the conjugation preserves the conditions $P_1 + P_2 = \mathbb{I}$, $P_1P_2 \in \text{Mat}_{n \times n}(a)$. Thus we have reached the form of (33).

For the case $r > 2$ apply the $(r = 2)$ argument to $P_1, \sum_{i=2}^r P_i$, and iterate. 

**Remark 4.5.** The condition “the pair $(R, a)$ is Henselian” cannot be weakened to “$R$ is a local ring”. For example, consider the $2 \times 2$ case:

$$
P_1 = \begin{bmatrix} 1 - f & g \\ g & f \end{bmatrix}, \quad P_2 = \begin{bmatrix} f & -g \\ -g & 1 - f \end{bmatrix}, \quad f, g \in (\mathfrak{a}) \subset \mathbb{K}[\mathfrak{a}]/(\mathfrak{a}), \quad \text{char}(\mathbb{K}) \neq 2.
$$

Then block-diagonalization means:

$$
P_1 \sim \begin{bmatrix} 1 + \sqrt{1+4(g^2 + f^2 - f)} & 0 \\ 0 & 1 - \sqrt{1+4(g^2 + f^2 - f)} \end{bmatrix}.
$$

The resulting matrix is over $\mathbb{K}[x]$ but not over $\mathbb{K}[\mathfrak{a}]/(\mathfrak{a})$, regardless of how large are the orders of $f, g$.

4.2. **The proof of theorem 4.1.** The direction $\Rightarrow$ is trivial, we prove the direction $\Leftarrow$. We prove part 1, then part 2 follows by §2.1v.

By corollary 3.0 and lemma 2.3 we can assume $(R, \mathfrak{m})$ is local and Henselian. Then, by §2.1v we can assume $A \in \text{Mat}_{n \times n}(\mathfrak{m})$.

**Step 1.** By the assumption, $\text{Adj}(A) = \sum_j(\prod_{i \neq j} f_i)B_i$, for some matrices $B_i \in \text{Mat}_{n \times n}(R)$. This decomposition is not unique due to the freedom

$$
B_i \rightarrow B_i + f_iZ_i \quad \text{for} \quad \sum Z_i = \mathbb{I}.
$$

We will use this freedom later.

From this presentation of $\text{Adj}(A)$ we get: $\det(A)\cdot \mathbb{I} = A\cdot \text{Adj}(A) = \sum_j(\prod_{i \neq j} f_i)AB_i$. As $\{f_i\}$ are regular and co-prime we have $(\prod_{i \neq j} f_i) \cap (f_j) = (\prod_{i=1}^r f_i)$, lemma 2.3. Therefore we get $(\prod_{i \neq j} f_i)AB_i \in (\det(A))\cdot \text{Mat}_{n \times n}(R)$, and thus $AB_i \subset (f_j)\cdot \text{Mat}_{m \times n}(R)$. Therefore we define the matrices $\{P_i\}, \{Q_i\}$ by $f_iP_i := AB_i$ and $f_iQ_i := B_iA$. By their definition: $\sum P_i = \mathbb{I}$ and $\sum Q_i = \mathbb{I}$.

We want to achieve: $\oplus P_i = \mathbb{I}$ and $\oplus Q_i = \mathbb{I}$. The key ingredient is the identity:

$$
f_iB_jP_i = B_jAB_i = f_jQ_jB_i, \quad \text{for} \quad i \neq j.
$$

As $f_i, f_j$ are regular and co-prime, we get that $B_jP_i$ is divisible by $f_j$, i.e., $B_jP_i = 0$ for any $i \neq j$. Thus $\{P_i\}$ and $\{Q_i\}$ are almost projectors in the sense of lemma 4.4 for $a = \mathfrak{m}$.

**Step 2.** Assume $r = 2$. The transformation $A \rightarrow UAV$ results in $B_i \rightarrow Adj(V)\cdot B_i \cdot Adj(U)$, which implies:

$$
P_i \rightarrow \det(UV) \cdot UP_iU^{-1} \quad \text{and} \quad Q_j \rightarrow \det(UV) \cdot V^{-1}Q_jV.
$$
The conditions $P_1P_2 = AZ'$ and $Q_1Q_2 = Z'A$ transform into:

$$P_1P_2 = \frac{1}{\det(UV)}UAZ'U^{-1}, \quad Q_1Q_2 = \frac{1}{\det(UV)}VZ'AV^{-1}.$$  

Thus, using lemma 4.4, we can assume:

$$P_1 = \begin{bmatrix} \mathbb{I} - Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} Z_1 & 0 \\ 0 & \mathbb{I} - Z_2 \end{bmatrix},$$

and similarly (independently) for $Q_1, Q_2$. Here $Z_i \in \text{Mat}_{n_i \times n_i}(m)$.

The condition $P_1P_2 = AZ'$ gives:

$$\left(\begin{bmatrix} \mathbb{I} - Z_1 & 0 \\ 0 & Z_2(\mathbb{I} - Z_2) \end{bmatrix}\right) = AZ', \quad \text{and thus} \quad \begin{bmatrix} Z_1 & 0 \\ 0 & -Z_2 \end{bmatrix} = AZ'\left(\begin{bmatrix} \mathbb{I} - Z_1 & 0 \\ 0 & -(\mathbb{I} - Z_2) \end{bmatrix}^{-1} \right).$$

Now apply the freedom of equation 366, $B_1 \to B_1 + f_1Z, B_2 \to B_2 - f_2Z$, it amounts to: $P_1 \to P_1 + AZ$ and $P_2 \to P_2 - AZ$. Thus we choose

$$Z = Z'\left(\begin{bmatrix} \mathbb{I} - Z_1 & 0 \\ 0 & -(\mathbb{I} - Z_2) \end{bmatrix}^{-1} \right) \text{ to get: } P_1 \to \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \quad P_2 \to \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{bmatrix}.$$

Take $P_1, P_2$ in this form, thus $\mathbb{O} = P_1P_2 = \begin{bmatrix} AB_1 & AB_2 \\ f_1 & f_2 \end{bmatrix}$. Therefore $B_1AB_2 = \mathbb{O}$ and $Q_1Q_2 = \mathbb{O}$.

By lemma 4.2 we can assume $Q_1 = \mathbb{I} \oplus \mathbb{O}$ and $Q_2 = \mathbb{O} \oplus \mathbb{I}$.

Finally, use the original definition of $P_i$ and $Q_i$, to write: $\frac{1}{f} \text{Adj}(A)P_i = B_i = \frac{1}{f}Q_i\text{Adj}(A)$.

This gives $\text{Adj}(A) = f_2B_1 \oplus f_1B_2$. Therefore $A = A_1 \oplus A_2$, with $\text{Adj}(A_i) = B_i$.

For $r > 2$, apply the previous argument to $P_i, \sum_{i=1}^r P_i$. Decompose $A = A_1 \oplus A_2$. Then restrict to the subspace corresponding to $A_2$ and iterate.

### 4.3. The first examples.

**Example 4.6.**

i. Suppose $R$ is a principal ideal domain, e.g., $R = \mathbb{k}[x]$, for a field $\mathbb{k}$. Let $A \in \text{Mat}_{n \times n}(R), \det(A) = f_1f_2$, with $f_1, f_2$ regular and co-prime. In this case $(f_1) + (f_2) = R$, as this ideal must be principal and cannot be proper (otherwise $f_1, f_2$ are not co-prime). Thus the condition $I_{n-1}(A) \subseteq (f_1) + (f_2)$ is empty, hence $A \not\simeq A_1 \oplus A_2$ with $\det(A_i) = f_i$.

By iterating this procedure we get: if $\det(A) = \prod_i f_i^{p_i}$ is the decomposition into irreducible (non-invertible) pairwise co-prime elements then $A \not\simeq A_1 \oplus A_2$, with $\det(A_i) = f_i^{p_i}$. This establishes “a half” of the classical Smith normal form of matrices over PID’s.

ii. Let $A = \begin{bmatrix} y & x^k \\ x^l & y \end{bmatrix}$ with $R = \mathbb{S}[x, y]$, $S$ being any (commutative, unital) ring. Then $\det(A) = y^2 - x^{k+l}$ is reducible iff $k + l \in 2\mathbb{Z}$. Suppose $k + l \in 2\mathbb{Z}$, and $2 \in R$ is invertible. Then $A$ is decomposable iff $k = l$. (This goes by verifying the condition $I_1(A) \subseteq (y - x^{k/2}) + (y + x^{k/2})$.)

iii. A bit more generally, let $R = \mathbb{S}[x], \mathbb{S}$ with $x$ a multi-variable. Take a matrix of monomials, $A = \{a_{ij}\} = \{x^{d_{ij}}\} \in \text{Mat}_{2 \times 2}(R)$. Here $\{d_{ij}\}$ are 2-vectors of natural numbers. Assume all the coordinates of the vectors $d_{11} + d_{22}, d_{12} + d_{21}$ are even numbers, then

$$\det(A) = \left(\frac{1}{2}d_{11}^2 + d_{12}d_{21} - \frac{1}{2}d_{22}^2 + d_{11}d_{22}\right)\left(\frac{1}{2}d_{11}^2 + d_{12}d_{21} + \frac{1}{2}d_{22}^2 + d_{11}d_{22}\right) =: f_- \cdot f_+ \in R.$$

Assume $f_-, f_+$ are co-prime. We claim: $A$ is $(f_-, f_+)$-decomposable iff $(d_{11} = d_{22} \text{ and } d_{12} = d_{21})$. Indeed, $(f_-) + (f_+) = (\frac{1}{2}d_{11}^2 + d_{12}d_{21}) + (\frac{1}{2}d_{22}^2 + d_{11}d_{22})$. Then the necessary condition for the decomposability, $I_1(A) \subseteq (f_-) + (f_+)$, reads:

$$(44) \quad \forall(i, j) : \text{ either } d_{ij} \geq \frac{d_{11} + d_{22}}{2} \text{ or } d_{ij} \geq \frac{d_{12} + d_{21}}{2}.$$

(Here $\geq$ means inequality in each coordinate of these vectors.) Observe:

- if $d_{11}, d_{22} \geq \frac{d_{11} + d_{22}}{2}$ then $f_-, f_+$ are not coprime;
- if $d_{11} \geq \frac{d_{11} + d_{22}}{2}$ and $d_{22} \geq \frac{d_{12} + d_{21}}{2}$ then $d_{11} + d_{22} \geq d_{12} + d_{21}, \text{ thus } f_-, f_+ \text{ are not coprime};$
- the only remaining case is: $d_{11}, d_{22} \geq \frac{d_{11} + d_{22}}{2}$, implying $d_{11} = d_{22};$
- and similarly for $d_{12}, d_{21}$.  

Altogether we get: $d_{11} = d_{21}$ and $d_{12} = d_{22}$.

Vice versa, if $d_{11} = d_{22}$ and $d_{12} = d_{21}$ then $I_1(A) = (f_-) + (f_+)$. And thus $A$ is $(f_-, f_+)$-decomposable, by theorem 4.3.

iv. Yet more generally, suppose $A \in \text{Mat}_{2 \times 2}(R)$ satisfies: $\det(A) = f_1f_2$, with $f_1, f_2$ regular and co-prime. Then $A$ is (stably-$(f_1, f_2)$-decomposable iff the entries $a_{11}, a_{12}, a_{21}, a_{22}$ all belong to the ideal $(f_1) + (f_2) \subset R$.

An immediate consequence of theorem 4.3 is:

**Corollary 4.7.** Let $(R, \mathfrak{m})$ be a local ring and $A \in \text{Mat}_{n \times n}(\mathfrak{m})$. Suppose $\det(A) = f_1f_2$ with $f_1, f_2$ regular and co-prime, and $(f_1) + (f_2) \supseteq \mathfrak{m}^{n-1}$. Then $A$ is $(f_1, f_2)$-decomposable.

We recall some cases when the condition $(f_1) + (f_2) \supseteq \mathfrak{m}^{n-1}$ holds. As $A \in \text{Mat}_{n \times n}(\mathfrak{m})$ the $\mathfrak{m}$-order of $\det(A)$ is at least $n$, assume $\text{ord}_R(\det(A)) = n$. (This corresponds to the maximal determinantal representations, or a representation of maximal corank, see Kerner-Vinnikov, 2012. In this case the module $\text{Coker}(A)$ is Ulrich-maximal, being minimally generated by $n$ elements, see Ulrich.)

Assume $\text{ord}_\mathfrak{m} f_1 = n_1$, with $n = n_1 + n_2$. Take the images $\text{jet}_{n_1}(f_1), \text{jet}_{n_2}(f_2) \in \text{gr}_R(R)$ in the associated graded ring. By Nakayama it is enough to verify:

$$\frac{\mathfrak{m}^{n-1}}{\mathfrak{m}^n} \subseteq (\text{jet}_{n_1}(f_1)) + (\text{jet}_{n_2}(f_2)) \subseteq \text{gr}_R(R).$$

Thus we study the $R/\mathfrak{m}$-vector subspace

$$\text{jet}_{n_1}(f_1) \cdot \frac{\mathfrak{m}^{n-1-n_1}}{\mathfrak{m}^{n-n_1}} + \text{jet}_{n_2}(f_2) \cdot \frac{\mathfrak{m}^{n-1-n_2}}{\mathfrak{m}^{n-n_2}} \subseteq \frac{\mathfrak{m}^{n-1}}{\mathfrak{m}^n}.$$

Assume $\text{jet}_{n_1}(f_1), \text{jet}_{n_2}(f_2) \in \text{gr}_R(R)$ is a regular sequence, then this vector subspace is the direct sum, of dimension $\dim \frac{\mathfrak{m}^{n-1-n_1}}{\mathfrak{m}^{n-n_1}} + \dim \frac{\mathfrak{m}^{n-1-n_2}}{\mathfrak{m}^{n-n_2}}$.

**Example 4.8.** i. Let $(R, \mathfrak{m})$ be a regular local Noetherian ring of Krull dimension 2. Then the Hilbert function $p_R(j) := \dim \frac{\mathfrak{m}^{n-1-j}}{\mathfrak{m}^n}$ satisfies $p_R(n_1) + p_R(n_2) = p_R(n_1 + n_2)$. For $\det(A) = f_1f_2$ we get two curve-germs, $V(f_1), V(f_2) \subset \text{Spec}(R)$. The condition “$\text{jet}_{n_1}(f_1), \text{jet}_{n_2}(f_2) \in \text{gr}_R(R)$ are co-prime” means: the tangent cones of $V(f_1), V(f_2)$ have no common line. Suppose the $\mathfrak{m}$-order of $\det(A)$ equals $n = n_1 + n_2$, then the condition (45) holds. Then $A \sim A_1 \oplus A_2$, with $\det(A_1) = f_1$.

ii. More generally, suppose for $(R, \mathfrak{m})$ the Hilbert function $p_R(j)$ satisfies $p_R(n_1) + p_R(n_2) = p_R(n_1 + n_2)$ for $n_1, n_2 \gg 1$. (A typical example is a two-dimensional local ring whose integral closure is regular.) Then the condition (45) holds for $n_1, n_2 \gg 1$, when $\text{jet}_{n_1}(f_1), \text{jet}_{n_2}(f_2)$ are co-prime.

**Remark 4.9.** i. The assumption of maximal corank, $\text{ord}_R(\det(A)) = n$, is vital. For example, let $R = \mathbb{k}[x]$ and $A \in \text{Mat}_{2 \times 2}(\mathbb{m}^N)$ be a matrix of homogeneous forms of degree $N$ in two variables and $\mathbb{k} = \mathbb{k}$. Then $\det(A)$ necessarily splits. But if $N \geq 3$ then usually $I_1(A)$ cannot be generated by fewer than 4 elements. Hence $A$ is indecomposable, not even equivalent to an upper-block-triangular form.

ii. The regularity of $R$ is important. If $\dim_k(\mathfrak{m}/\mathfrak{m}^2) > 2$ then usually $(f_1) + (f_2) \not\supseteq \mathfrak{m}^{n-1}$, even when the ideal $(f_1) + (f_2)$ is $\mathfrak{m}$-primary.

iii. The condition $(f_1) + (f_2) \supseteq \mathfrak{m}^{n-1}$ does not hold when $\dim(R) > 2$, even for $n \gg 1$.

4.4. **Remarks on the assumptions of theorem 4.3.**

i. The condition “$f_1, f_2$ are co-prime in $R$” is essential, it is the analogue of the condition on distinct eigenvalues when diagonalizing a matrix over a field. The theorem does not hold if $f_1, f_2$ are not co-prime. As an example, take some matrix factorization $A_1A_2 = f I$, for a non-zero divisor $f \in R$, such that $\det(A_i) = f^{p_i}$. Then $A := \begin{bmatrix} A_1 & B \\ \emptyset & A_2 \end{bmatrix}$ satisfies: $\det(A) = f^{p_1+p_2}$ and

$$\text{Adj}(A) = \begin{bmatrix} \text{Adj}(A_1) & -\text{Adj}(A_1) \cdot B \cdot \text{ Adj}(A_2) \\ \emptyset & \text{Adj}(A_2) \end{bmatrix} \begin{bmatrix} f^{p_1+p_2-1}A_2 & -f^{p_1+p_2-2}A_2B A_1 \\ f^{p_1+p_2-1}A_1 & 0 \end{bmatrix}.$$  

Thus, for $p_1, p_2 \geq 2$, we have the inclusion $I_{n-1}(A) \subset (f^{p_1+p_2-2}) \subset (f^{p_1}, f^{p_2})$. But $A$ is not equivalent to a block-diagonal matrix, as we make no assumptions on $B$. 


ii. The condition \((f_1) \cap (f_2) = (f_1f_2)\) implies: the hypersurfaces \(V(f_1), V(f_2) \subseteq \text{Spec}(R)\) have no common component. Here \(V(f_i)\) are taken as subschemes, not just the zero sets. And the condition is on all the components, including the embedded components.

For example, for \(R = \mathbb{k}[x, y, z]/(z^2, x - y)\) take \(f_1 = x, f_2 = y\). The intersection of the sets \(V(x) \cap V(y) \subseteq \mathbb{k}^3\) is proper, but \((x) \cap (y) \neq (x - y)\).

Another example (showing that all the closed points of a scheme should be checked) is \(R = \mathbb{R}[x, y]/(x, y)\) with \(f_1 = x(x^2 + y^2), f_2 = y(x^2 + y^2)\). Here \((f_1) \cap (f_2) \supseteq (f_1f_2)\).

iii. The condition \(I_{n-1}(A) \subseteq (f_1) + (f_2)\) is necessary for decomposability, by the direct check of \(I_{n-1}(A_1 \oplus A_2)\).

The geometric meaning of this condition, when the subscheme \(V(f_1, f_2) \subseteq \text{Spec}(R)\) is reduced, is: \(A\) is of \(\text{corank} \geq 2\) on the locus \(V(f_1) \cap V(f_2) \subseteq \text{Spec}(R)\).

iv. We do not assume that \(R\) is Noetherian thus \(\text{theorem } 4.1\) works e.g., for the rings of continuously-differentiable functions, \(C^r(U),\) for \(1 \leq r \leq \infty,\) and \(U \subset \mathbb{R}^p,\) or their germs along closed subsets, \(C^r(U, \mathcal{Z}),\) resp. their quotients by ideals, \(C^r(U)/I, C^r(U, \mathcal{Z})/I\). Here the condition “\(f\) is regular” is easy to verify. Recall that \(f \in C^r(U)\) is a non-zero divisor iff its zero locus, \(V(f) \subset U,\) has empty interior. But the co-primeness of \(f_1, f_2\) is a restrictive condition. For example, let \(R = C^\infty(\mathbb{R}^p, 0)\) and \(f_1, f_2 \in \mathfrak{m}^\infty,\) then \(\frac{f_1}{|x|^2} \in C^\infty(\mathbb{R}^p, 0).\) And thus \(\frac{f_1}{|x|^2}, \frac{f_2}{|x|^2} \in (f_1) \cap (f_2)\) but \(\frac{f_1}{|x|^2}, \frac{f_2}{|x|^2} \not\in (f_1f_2)\). Thus any two elements of \(\mathfrak{m}^\infty\) are not co-prime.

For the ring of continuous functions, \(R = C^0(U)\), the theorem is useless, as any elements \(f_1, f_2 \in R\) with \(V(f_1) \cap V(f_2) \neq \emptyset\) are not co-prime. Indeed, both \(f_1\) and \(f_2\) are divisible by \(\sqrt{|f_1| + |f_2|}\), thus \(\frac{f_1}{|f_1|^2}, \frac{f_2}{|f_2|^2} \in (f_1) \cap (f_2) \subset R,\) thus \((f_1) \cap (f_2) \supseteq (f_1 \cdot f_2)\). See [Grove-Pedersen] for diagonalization criteria in this case.

4.5. Decomposition of determinantal representations/sheaves on plane curves. Let \(R = \mathbb{k}[x_0, \ldots, x_l],\) with \(l \geq 2\) and the algebraically closed field, \(\mathbb{k} = \mathbb{k}\.\) Thus \(\text{Proj}(R) = \mathbb{P}^l_k\).

i. Suppose the matrix is homogeneous, \(A \in \text{Mat}_{n \times n}(R_l),\) thus \(A\) is a (non-linear) determinantal representation of the hypersurface \(V(\det(A)) \subseteq \mathbb{P}^l\). Assume \(\det(A) = f_1 \cdot f_2,\) co-prime elements. \(\text{Corollary } 3.2\) gives: \(A\) is decomposable (as an \(R\)-matrix) iff for each point \(V(\mathfrak{m}) \in \mathbb{P}V(f_1) \cap \mathbb{P}V(f_2) \subseteq \mathbb{P}^l\) the localized version \(A(\mathfrak{m})\) is decomposable. For \(d = 1, \text{char}(\mathbb{k}) = 0,\) this recovers [Kerner-Vinnikov 2012, Theorem 3.1].

ii. For \(l = 2\) we get a reducible determinantal curve \(\mathbb{P}V(\det(A)) = C_1 \cup C_2 \subset \mathbb{P}^2\). The curves \(C_1, C_2\) intersect at a finite (non-zero) number of points. Assume all these points are ordinary multiple points of \(C_1 \cup C_2,\) and \(A(\mathfrak{m})\) is a locally maximal determinantal representation at each such point, see corollary 4.7 and example 1.8. Then \(A(\mathfrak{m})\) decomposes locally at each such point, therefore \(A\) is globally \((f_1, f_2)\)-decomposable.

iii. (Torsion-free sheaves on reducible plane curves) Take a plane curve \(C = \cup_{j} C_j \subset \mathbb{P}^2,\) here each \(C_j\) can be further reducible, but is reduced, and the intersections \(\{C_j \cap C_i\}_{i \neq j}\) are finite. Take its determinantal representation, i.e., a graded matrix (see 2.1) \(A \in \text{Mat}_{n \times n}(\mathbb{k}[x_0, x_1, x_2])\) with \(\det(A) = \prod f_j.\) Then \(\text{Coker}(A)\) is a coherent sheaf on the curve \(C.\) It is torsion free and of rank one on each \(C_j\).

Suppose all the intersection points of \(\{C_i \cap C_j\}\) are nodes. Then the stalk of \(\mathcal{F}\) at each such point is either a free module or decomposes. Therefore we get:

\[
\mathcal{F} = \oplus \mathcal{F}_j \text{ with } \text{Supp}(\mathcal{F}_j) = C_j \text{ if } \mathcal{F} \text{ is not locally free at each such node.}
\]

More generally, suppose for each point \(p \in C_i \cap C_j\) the germs have no common tangents, \(T_{(C_i, p)} \cap T_{(C_j, p)} = (0),\) and \(\text{ord}(C, p) = \text{ker}(A|_p).\) Then \(\mathcal{F}\) decomposes locally at each point of \(\{C_i \cap C_j\}\), and thus decomposes globally.

4.6. Simultaneous diagonal reduction of tuples of matrices/linear determinantal representations. Take an algebraically closed field, \(\mathbb{k} = \mathbb{k}\), and a (possibly infinite) collection of matrices \(\{A^{(a)} \in \text{Mat}_{m \times n}(\mathbb{k})\}_a\). The classical question is to ensure the simultaneous diagonalization of this tuple.

We can assume \(\cap \ker(A^{(a)}) = 0 \subset \mathbb{k}^n,\) otherwise one restricts to the complementary subspace of \(\cap \ker(A^{(a)})\) in \(\mathbb{k}^n.\)
Split the sizes, \( m = \sum n_i, n = \sum n_i \). We are looking for \( U \in GL(m, k), V \in GL(n, k) \) to ensure:

\[
U \cdot A^{(\alpha)} \cdot V^{-1} = \oplus_i A_i^{(\alpha)}, \quad \{A_i^{(\alpha)} \in \text{Mat}_{m_i \times n_i}(k)\}_\alpha, \quad \forall \alpha.
\]

First we formulate this as the decomposition problem of one matrix, over a ring. Take the vector space \( \prod \mathbb{k}[x_{\alpha}] \) (we allow infinite linear combinations) and the graded ring \( R = \mathbb{k}[\prod \mathbb{k}[x_{\alpha}]] \). In the finite case this is \( \mathbb{k}[x] \), otherwise \( R \) is non-Noetherian. Take the matrix \( A := \sum x_{\alpha} \cdot A^{(\alpha)} \in \text{Mat}_{m \times n}(R) \). The collection \( \{A^{(\alpha)}\} \) is simultaneously decomposable iff \( A \) is decomposable by \( GL(m, R) \times GL(n, R) \), see lemma 3.17. (We can even pass to the localization, \( R_{R_{>0}} \).

In the same way one transforms the simultaneous block-diagonalization (of tuples of matrices) by congruence/conjugation into that for \( A \in \text{Mat}_{n \times n}(R) \). More generally, the decomposability questions of quiver representations are reduced to those of matrices over a ring, see [Kerner II].

Below we work with square matrices.

### 4.6.1 Reduction to pairs/triples of matrices.

Consider linear forms, \( l(A) = \sum_{i=1}^r c_i A_i \), with \( r > 3 \).

**Corollary 4.10.** Suppose the restriction to generic linear forms \( l_1, l_2, l_3 \) satisfies: \( \det(\sum_{j=1}^3 y_j l_j(A)) = f_1(y) \cdot f_2(y) \), coprime polynomials, and the matrix \( \sum_{i=1}^3 y_i l_i(A) \) is \( (f_1, f_2) \)-decomposable. Then the tuple \( \{A_1, \ldots, A_r\} \) is simultaneously decomposable.

By iterating this statement one gets the criterion for the full diagonal reduction.

**Proof.** The passage \( \text{Mat}_{n \times n}(k[x]) \ni \sum x_i A_i \mapsto \sum_{j=1}^r y_j l_j(A) = \sum_{i=1}^r A_i l_i(y) \) is done by the quotient map \( k[x] \to k[x](l_1(x), \ldots, l_r(x)) \). Here \( l_1, \ldots, l_r \) are linear forms, and the plane \( V(l_1, \ldots, l_r) \subset k^r \) is generic for the given hypersurface \( V(\det(\sum x_i A_i)) \subset k^r \).

This generic plane section is reducible, \( \det(\sum x_i A_i) \big|_{V(l_1, \ldots, l_r)} = f_1(y) \cdot f_2(y) \). Therefore by Bertini theorem (in any characteristic) [H], pg.179], we get: \( \det(\sum x_i A_i) = \tilde{f}_1(x) \cdot \tilde{f}_2(x) \). Moreover, \( \tilde{f}_1(x), \tilde{f}_2(x) \) are coprime, as \( f_1(y), f_2(y) \) are.

Finally, as \( \sum_{j=1}^r y_j l_j(A) = f_1(y) \cdot f_2(y) \)-decomposable, one has: \( I_{n-1}(\sum_{j=1}^r y_j l_j(A)) \subseteq (f_1(y), f_2(y)) \).

Therefore \( I_{n-1}(\sum x_i A_i) \subseteq (\tilde{f}_1(x), \tilde{f}_2(x)) = (l_1(x), \ldots, l_r(x)) \). Here the forms \( l_1, \ldots, l_r \) are generic, and the sequence \( \{l_1, \ldots, l_r, f_1, f_2\} \) is regular, and \( \dim_k k[x](l_1, \ldots, l_r, f_1, f_2) > 0 \). Therefore \( I_{n-1}(\sum x_i A_i) \subseteq (\tilde{f}_1(x), \tilde{f}_2(x)) \). Therefore \( \sum_{i=1}^r x_i A_i \) is \( \tilde{f}_1(x), \tilde{f}_2(x) \)-decomposable. \( \blacksquare \)

### 4.6.2 The case of a pair of matrices.

**Corollary 4.11.** Let \( A_1, A_2 \in \text{Mat}_{n \times n}(k) \) with eigenvalues \( \{\lambda_1^{(1)}\} \) and \( \{\lambda_2^{(2)}\} \). Take the corresponding generalized eigenspaces and \( \{V^{(1)}_{\lambda^{(1)}}\}, \{V^{(2)}_{\lambda^{(2)}}\} \). If \( \dim(V^{(1)}_{\lambda^{(1)}} \cap V^{(2)}_{\lambda^{(2)}}) \leq 1 \) for any \( i, j \) then the pair \( \{A_1, A_2\} \) admits simultaneous diagonal reduction.

**Proof.** The homogeneous polynomial splits, \( \det(x_1 A_1 + x_2 A_2) = \prod_{j=1}^r l_j(x_1, x_2) \). The assumption \( \dim(V^{(1)}_{\lambda^{(1)}} \cap V^{(2)}_{\lambda^{(2)}}) \leq 1 \) implies: the linear forms \( \{l_i\} \) are pairwise-coprime. Therefore the homogeneous forms \( \left\{ \frac{\det(x_1 A_1 + x_2 A_2)}{l_j(x_1, x_2)} \right\} \) are \( k \)-linearly independent. Hence \( I_{n-1}(x_1 A_1 + x_2 A_2) \subseteq (x_1, x_2)^n-1 = \sum_j \left\{ \frac{\det(x_1 A_1 + x_2 A_2)}{l_j(x_1, x_2)} \right\} \). Thus \( x_1 A_1 + x_2 A_2 \) is \( \{l_i\} \)-decomposable. \( \blacksquare \)

### 4.6.3 The case of a triple.

For each point \( (x_0 : x_1 : x_2) \in \mathbb{P}_k^2 \) take the kernel \( \text{Ker}(\sum x_i A_i) \subset k^n \). This space is non-trivial iff \( (x_0 : x_1 : x_2) \in C := V(\det \sum x_i A_i) \subset \mathbb{P}^2 \), the associated determinantal curve. Recall that \( \dim_k(\text{Ker}(\sum x_i A_i)) = 1 \) at the smooth points of \( C \). Call \( (x_0, x_1, x_2) \in \mathbb{P}_k^2 \) a singular point for the triple \( \{A_0, A_1, A_2\} \) if \( \dim(\text{Ker}(\sum x_i A_i)) \geq 2 \). A singular point for the triple is necessarily a singular point for \( C \). But not every singular points of \( C \) is a singular point of the triple.

A triple admits the diagonal reduction exactly when the amount of such large kernels is maximal possible.

**Proposition 4.12.** Assume the polynomial \( \det(\sum x_i A_i) \in \mathbb{k}[x_0, x_1, x_2] \) is square-free.
1. \[ \sum \binom{\dim_k(\ker \sum x_i A_i)}{2} \leq \binom{n}{2}. \]

2. The triple \{A_0, A_1, A_2\} admits the diagonal reduction if \( \sum \binom{\dim_k(\ker \sum x_i A_i)}{2} = \binom{n}{2}. \)

Here the sum (over all the points of \( \mathbb{P}^2 \)) is finite, as only the singular points of \( C \) can contribute.

Proof. (A preparation) We recall the delta invariant of curve singularities, see e.g., [G.L.S., pg. 206]. Take the normalization of a reduced (possibly reducible) curve germ, \( \prod_{i=1}^{r}(\tilde{C}_i, x_i) \to (C, x) \).

This defines the embedding of the local rings, \( \mathcal{O}_{C,x} \to \prod_i \mathcal{O}_{\tilde{C}_i,x_i}. \) The basic invariant of a singular point is the vector space dimension of the quotient, \( \delta(C, x) := \dim_k \prod_i \mathcal{O}_{\tilde{C}_i,x_i}/\mathcal{O}_{C,x}. \) If the multiplicity of \((C, x)\) is \( p \) then \( \delta(C, x) \geq \binom{p}{2} \).

For a projective (not necessarily planar) curve one has the global normalization, \( \tilde{C} = \bigcup_{j=1}^r \tilde{C}_j \to C = \bigcup_{j=1}^r C_j. \) Take the exact sequence \( 0 \to \mathcal{O}_C \to \mathcal{O}_{\tilde{C}} \to \mathcal{O}_{\tilde{C}}/\mathcal{O}_C \cong \oplus_{x \in \text{Sing}(C)} (\mathcal{O}_{\tilde{C}}/\mathcal{O}_C)_x \to 0. \) Then the total delta invariant can be computed as:

\[
\delta(C) := \sum_{x \in \text{Sing}(C)} \delta(C, x) = \dim_k \mathcal{O}_{\tilde{C}}/\mathcal{O}_C = r - 1 + h^1(\mathcal{O}_C) - h^1(\mathcal{O}_{\tilde{C}}).
\]

Here \( h^1(\mathcal{O}_C) = \sum g(C_j) \). For a plane curve, \( C \subset \mathbb{P}^2, \) of degree \( d \), we compute \( h^1(\mathcal{O}_C): \)

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0, \quad \text{hence} \quad h^1(\mathcal{O}_C) = h^2(\mathcal{O}_{\mathbb{P}^2}(-d)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d - 3)) = \binom{d-1}{2}.
\]

Altogether, the total delta is: \( \delta(C) = \binom{d-1}{2} - 1 - \sum_{j=1}^r (g(\tilde{C}_j) - 1). \) In particular, \( \delta(C) \leq \binom{d-1}{2} - 1 + d = \binom{d}{2}. \) The equality here holds iff the curve is a line arrangement.

Part 1. For each point \( p_a = (x_0 : x_1 : x_2) \in \mathbb{P}^2 \) the multiplicity of the determinantal curve \( V(\det(\sum x_i A_i)) \) is at least \( \dim_k(\ker \sum x_i A_i) \). Therefore the local delta-invariant of the reduced plane curve singularity is \( \delta(C, p_a) \geq \binom{\dim(\ker \sum x_i A_i)}{2}. \)

Finally, as \( C \) is a reduced plane curve, the total \( \delta \)-invariant is bounded by that of the line arrangement, i.e., \( \delta(C) = \sum \delta(C, p_a) \leq \binom{n}{2}. \)

Part 2. The part \( \Leftarrow. \) The condition \( \sum \binom{\dim_k(\ker \sum x_i A_i)}{2} = \binom{n}{2} \) implies \( \delta(C) = \sum \delta(C, p_a) = \binom{n}{2}. \)

Thus the curve \( V(\det(\sum x_i A_i)) \subset \mathbb{P}^2 \) is a (reduced) line arrangement, i.e., \( \det(\sum x_i A_i) \) splits into pairwise independent linear forms. Therefore all the singular points of \( C \) are ordinary multiple points, and for each point \( \ker(\sum x_i A_i) \) is of maximal possible dimension, i.e., \( \sum x_i A_i \) is of maximal corank. Then we get the local decomposability, by example [L.LS].

Finally, as \( \sum x_i A_i \) decomposes locally at all the intersection points, it decomposes globally, by corollary 3.21.

The part \( \Rightarrow. \) If the triple \( \{A_0, A_1, A_2\} \) admits the diagonal reduction then the (square-free) polynomial \( \det(\sum x_i A_i) \) splits into pairwise independent linear factors. Thus the curve is a (reduced) line arrangement.

The multiplicity of this curve at the point \( (x_0, x_1, x_2) \) is exactly \( p_j := \dim_k(\ker \sum x_i A_i), \) and the local delta-invariant equals \( \binom{p_j}{2}. \) Finally, as the curve is a line arrangement, the total delta invariant equals \( \sum \binom{p_j}{2} = \binom{n}{2}. \)

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