Fisher information and spin squeezing in the Lipkin-Meshkov-Glick Model

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Fisher information, lies at the heart of parameter estimation theory, was recently found to have a close relation with multipartite entanglement (Pezzé and Smerzi, Phys. Rev. Lett. 102, 100401). We use Fisher information to distinguish and characterize behaviors of ground state of the Lipkin-Meshkov-Glick model, which displays a second-order quantum phase transition between the broken and symmetric phases. We find that the parameter sensitivity of the system attains the Heisenberg limit in the broken phase, while it is just around the shot-noise limit in the symmetric phase. Based on parameter estimation, Fisher information provides us an approach to the quantum phase transition.

I. INTRODUCTION

Parameter estimation of probability distributions is one of the most basic tasks in information theory, and has been generalized to quantum regime [1, 2] since the description of quantum mechanics is essentially probabilistic. How to improve the precision of parameter estimation has been focused for many years, and is of important applications in quantum technology like quantum frequency standards [3, 4], measurement of gravity accelerations [5], and clock synchronization [6] etc.

Consider a quantum state \( \rho = U_\theta \rho_{in} U_\theta^\dagger \), where \( U_\theta = \exp \left( i \theta \hat{K} \right) \) and \( \hat{K} \) is a generator. We estimate parameter \( \theta \) through proper measurements, however, the precision of our estimation is limited by the quantum Cramer-Rao (QCR) bound [1, 2],

\[
\Delta \hat{\theta} \geq (\Delta \theta)_{QCR} = \frac{1}{\sqrt{\nu F(\rho_{in}, \hat{K})}},
\]

where \( \nu \) is the number of trails, \( \hat{\theta} \) is the so-called unbiased estimator, and \( F(\rho_{in}, \hat{K}) \) is the quantum Fisher information (QFI) [1, 2, 7, 8]. In a sense, parameter estimation is equivalent to distinguishing neighboring states along the path in parameter space. We know QFI has close relation with Bures distance [9], the most studied distance in quantum-state space, and Bures distance is directly related to the Uhlmann fidelity [10]. The QFI is proportional to the Bures distance [11, 12]. For pure states, the QFI, as well as the Bures distance \( ds_B^2 \), is just proportional to the variance of \( \hat{K} \) [8], that is \( F(\rho_{in}, \hat{K}) = 4ds_B^2 = 4(\Delta \hat{K})^2 \). Therefore, besides increasing experimental times \( \nu \), we can improve the estimation precision \( \Delta \hat{\theta} \) by choosing proper states \( \rho_{in} \) for a given \( \hat{K} \). In general, entangled states are more sensitive than separable states, i.e., the variance of \( \hat{K} \) is large.

In the past, many works have been devoted to improvement of parameter sensitivity by using entangled states [13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24].

Quite recently, Pezzé and Smerzi [25] found an interesting application of QFI in multipartite entanglement and the sub-shot-noise phase sensitivity in the estimation of a collective rotation angle. Consider an ensemble of spin-half particles in the state \( \rho_{in} \), they introduced a quantity

\[
\chi^2 = \frac{N}{F(\rho_{in}, S_{\vec{n}})},
\]

and prove that \( \chi^2 < 1 \) implies multipartite entanglement. Here, the generator of \( \theta \) is \( S_{\vec{n}} = \vec{S} \cdot \vec{n} \) that denotes the collective spin operator along direction \( \vec{n} \). Namely, a sufficient condition is given for quantum entanglement. We may define a mean Fisher information as \( F_m = F(\rho_{in}, S_{\vec{n}})/N \). Then, \( \chi^2 \) and \( F_m \) are reciprocal to each other. The relation between \( \chi \) and QCR bound is

\[
\Delta \hat{\theta} \geq \frac{1}{\sqrt{F(\rho_{in}, S_{\vec{n}})}} = \frac{\chi}{\sqrt{N}} = \chi (\Delta \theta)_{SN},
\]

where \( (\Delta \theta)_{SN} = 1/\sqrt{N} \) is the shot-noise limit and we set \( \nu = 1 \). Thus, it is evident that \( \chi^2 < 1 \) becomes a necessary and sufficient condition for sub-shot-noise phase estimation.

In this work, we study the Fisher information of the ground state of the Lipkin-Meshkov-Glick (LMG) model [26], which has a second-order quantum phase transition (QPT) [27], between a symmetric (polarized, \( h \geq 1 \)) phase and a broken (collective, \( h < 1 \)) phase. Some works have been devoted to study the LMG model with concurrence [28] and entropy [29, 30]. In our work we find that, besides indicating the critical point and entanglement, \( \chi^2 \) reflects the performances of ground states of these two phases in the sense of parameter sensitivity. In the symmetric phase, \( \chi^2 \) approaches to 1 with the increasing of \( h \), and is independent of \( N \), that means \( (\Delta \theta)_{QCR} \sim (\Delta \theta)_{SN} \). In the broken phase, we find \( \chi^2 \simeq 1/N \), thus \( (\Delta \theta)_{QCR} \simeq 1/N \) attaining the Heisenberg limit.
This paper is organized as follows. In Sec. II, we give brief discussions about the relations between spin squeezing and $\chi^2$. Then in Sec III, we study $\chi^2$ and spin squeezing for the ground state of the LMG model in both isotropic ($\gamma = 1$) and anisotropic ($\gamma \neq 1$) cases. In isotropic case, the LMG model is diagonal in Dicke states. For Dicke states, we find that, $\chi^2$ and the spin squeezing parameter by Kitagawa and Ueda \cite{31}, are reciprocal to each other. In anisotropic case, we use Holstein-Primakoff transformation and derive $\chi^2$ in the thermodynamic limit. The finite size behaviors of $\chi^2$ and the spin squeezing parameter in the critical point are also obtained. The numerical results coincide well with the analytical ones.

II. FISHER INFORMATION AND SPIN SQUEEZING PARAMETERS

Fisher information is related to spin squeezing, and there are two spin squeezing parameters respectively given by Kitagawa and Ueda \cite{31}, and Wineland \cite{32},

$$\xi^2 = \frac{4 (\Delta S_{\hat{n}_\perp})^2}{N}, \quad \eta^2 = \frac{N (\Delta S_{\hat{n}_\perp})^2}{|\langle S_{\hat{n}} \rangle|^2},$$

where subscript $\hat{n}_\perp$ refers to an arbitrary axis perpendicular to the mean spin $\langle \hat{S} \rangle$, where the minimum value of $(\Delta S)^2$ is obtained. The inequality $\xi^2 < 1$ ($i = 1, 2$) indicates that the state is spin squeezed. Spin squeezed states can be used to reduce the measurement uncertainty \cite{31, 32}, and improve the measurement precision of the atomic clock transition \cite{34, 35}. The spin squeezing inequality is a criteria for multipartite entanglement \cite{36, 37}. For an arbitrary multiqubit separable states, it was found that $\xi^2 \geq 1$, and thus $\xi^2 < 1$ implies quantum entanglement.

As proved in \cite{25},

$$F (\rho, S_{\hat{n}_\perp}) (\Delta S_{\hat{n}_\perp})^2 \geq |\langle S_{\hat{n}} \rangle|^2,$$

where directions $\hat{n}_\perp, \hat{n}_\parallel, \hat{n}$ are orthogonal to each other. $F (\rho, S_{\hat{n}_\perp}) = 4 (\Delta \hat{R})^2$, where $\hat{R}$ is determined by $\hat{R} \rho + \rho \hat{R} = i (\rho S_{\hat{n}_\parallel} - S_{\hat{n}_\parallel} \rho)$. In general, $(\Delta \hat{R})^2 \leq (\Delta S_{\hat{n}_\perp})^2$, and the equality is obtained only for pure states. Then Eq. (5) reduces to the usual uncertainty relation,

$$(\Delta S_{\hat{n}_\perp})^2 (\Delta S_{\hat{n}_\parallel})^2 \geq \frac{|\langle S_{\hat{n}} \rangle|^2}{4},$$

for pure states. The above inequality can be written in terms of the inverse of the mean QFI and the squeezing parameter $\xi^2$ as

$$\xi^2 = \frac{N (\Delta S_{\hat{n}_\perp})^2}{|\langle S_{\hat{n}} \rangle|^2} \geq \frac{N}{4(\Delta S_{\hat{n}_\perp})^2} = \frac{1}{F_m} = \chi^2.$$

Both the inequalities, $\xi^2 < 1$ and mean QFI $F_m > 1$ ($\chi^2 < 1$), indicate the presence of entanglement.

Furthermore, $\xi^2 \leq \eta^2$, and there is no similar relation between $\chi^2$ and $\xi^2$ like Eq. (7). However, we find that

$$\xi^2 \chi^2 = \frac{(\Delta S_{\hat{n}_\perp})^2}{(\Delta S_{\hat{n}_\parallel})^2} \leq 1,$$

since $(\Delta S_{\hat{n}_\perp})^2 (\Delta S_{\hat{n}_\parallel})^2$ is the maximum (minimum) variance. As proved in \cite{37}, if the pure state is of exchange symmetry, $\xi^2 < 1$ implies entanglement. Then, from the above equation $\chi^2 > 1$ implies entanglement. We know that $\chi^2 < 1$ indicates entanglement too. Therefore, a pure symmetric state is entangled iff $\chi^2 \neq 1$.

When the mean spin direction is along z direction, the squeezing parameter $\xi_1^2$ and $\chi^2$ becomes

$$\xi_1^2 = \frac{4 \min \langle S_{\hat{z}} \rangle}{N}, \quad \chi^2 = \frac{N}{4 \max \langle S_{\hat{z}} \rangle^2},$$

where

$$S_{\hat{z}} = \cos \theta S_x + \sin \theta S_y.$$  

Furthermore, if $\langle S_x, S_y \rangle = 0$, for instance, in the LMG model \cite{28}, we have

$$\xi_1^2 = \frac{4 \min \langle S_{\hat{z}}^2 \rangle}{N}, \quad \chi^2 = \frac{N}{4 \max \langle S_{\hat{z}}^2 \rangle^2},$$

thus we only need to compute $\langle S_{\hat{z}}^2 \rangle$ and $\langle S_{\hat{z}}^2 \rangle$ to determined the squeezing parameter and quantity $\chi^2$ in the following discussions of QPTs in LMG model.

III. FISHER INFORMATION AND SQUEEZING IN THE LMG MODEL

The LMG model, originally introduced in nuclear physics and has found applications in a broad range of other topics: statistical mechanics of quantum spin system \cite{38}, Bose-Einstein condensates \cite{39}, or magnetic molecules such as Mn$_{12}$ acetate \cite{40}. Recently, some quantum-information concepts, such as quantum entanglement \cite{29, 30} and quantum fidelity \cite{41, 42}, have been studied in this model, aim at characterizing its QPT. It is an exactly solvable \cite{43, 44} many-body interacting quantum system as well as one of the simplest to show a quantum transition in the regime of strong coupling. The quantum phase transition of this model is also clear and interest: the ground state becomes degenerate and a macroscopic change in the ground state energy takes place.
A. LMG Hamiltonian

The Hamiltonian of the LMG model reads

\[ H = -\frac{1}{N} (S_x^2 + \gamma S_y^2) - hS_z, \]  

(12)

where \( S_\alpha = \sum_{i=1}^N \sigma_\alpha^i/2 \) are the total spin operators in the direction \( \alpha = x, y, z; \sigma_\alpha^i \) are the Pauli matrices; \( N \) is the total spin number, \( \gamma \) is the anisotropic parameter and \( h \) is the effective strength of the external field. Without loss of generality, we assume \( 0 \leq \gamma \leq 1 \) and \( h \geq 0 \).

This system undergoes a second-order QPT at \( h = 1 \), between a symmetric \((h \geq 1)\) phase and a broken \((h < 1)\) phase, which are associated with single-particle and collective behaviors, respectively. These two phases are well described by a mean-field approach [28]. The classical state is fully polarized in the field direction \( \langle \sigma_z^i \rangle = 1 \) for \( h \geq 1 \), and is twofold degenerate with \( \langle \sigma_z^i \rangle = h \) for \( h < 1 \). However, since the Hamiltonian is of spin-flip symmetry, i.e., \( \left[ H, \prod_{i=1}^N \sigma_z^i \right] = 0 \), we have

\[ \langle S_x \rangle = \langle S_y \rangle = 0, \quad \langle S_x S_z \rangle = \langle S_y S_z \rangle = 0. \]  

(13)

Thus the mean spin direction is along the \( z \)-axis for finite size case. In addition, \( \left[ H, S^2 \right] = 0 \), and the ground state lies in the \( S = N/2 \) symmetric section.

B. Isotropic case and Dicke state

We begin with the simple isotropic case, \( \gamma = 1 \). The Hamiltonian reduces to

\[ H = -\frac{1}{N} (S_x^2 - S_z^2) - hS_z, \]

(14)

which is diagonal in the standard eigenbasis \( \{ |S,M\rangle \} \) of \( S^2 \) and \( S_z \). For \( S = N/2 \) the energy eigenvalue is

\[ E(M,h) = \frac{2}{N} \left( - \frac{hN}{2} \right)^2 - \frac{N}{2} \left( 1 + h^2 \right), \]  

(15)

and the ground state \( |S,M_0\rangle \) is readily obtained when

\[ M_0 = \begin{cases} N/2 & \text{for} \quad h \geq 1, \\ N/2 - R[N(1-h)/2] & \text{for} \quad 0 \leq h < 1, \end{cases} \]  

(16)

where \( R(x) \equiv \text{round}(x) \) gives the nearest integer of \( x \). Then one can see level crossings exist at \( h = h_j \), where \( h_j = 1 - (2j + 1)/N \), between the two states \( |S,S - j\rangle \) and \( |S,S - j - 1\rangle \).

As the ground state is actually a Dicke state \( |S,M\rangle \), \( \langle S_x^2 \rangle = \langle S_y^2 \rangle = (S^2 + S - M^2)/2 \), then

\[ \chi^2 = \frac{N}{2} \left( S^2 + S - M^2 \right) \]

\[ = \frac{N/2 + 1 - M^2/N}{1}, \]

(17)

the equality is obtained for \( M = \pm S \). Immediately, we have

\[ \xi_1^2 = 1/\chi^2 \geq 1. \]

As we know that, when \( M \neq \pm S \), the Dicke states are entangled but not spin squeezed, since \( \xi_2^2 > \xi_1^2 > 1 \). Numerical results of \( \xi_1^2 \) and \( \chi^2 \) for the isotropic LMG model are shown in Fig. 1(d). We can see that, in the broken phase, \( M < S, \xi_1^2 = 1/\chi^2 \), while in the symmetric phase, the ground state is \( |S,S\rangle \), thus \( \chi^2 = \xi_1^2 = 1 \).

By considering \( \chi^2 \) in Eq. (17), when \( M \) is close to \( \pm S \), \( \chi^2 \) is just a bit lower than \( 1 \), thus \( \Delta \theta \) is not improved much than \( \langle \Delta \theta \rangle_{\text{SN}} \). When \( |M| \leq \sqrt{S} \), we have

\[ 2/N \leq \chi^2 \leq 2/(N + 1), \]

(18)

and thus

\[ \langle \Delta \theta \rangle_{\text{QCR}} = \chi/\sqrt{N} \sim 1/N, \]

(19)

which attains the Heisenberg limit. Although \( |S,\pm S\rangle \) is not entangled, "cat state" (or GHZ state)

\[ |\psi\rangle = (|S,S\rangle + |S,-S\rangle)/\sqrt{2}, \]

is entangled and useful in phase estimation [16]. Under \( |\psi\rangle \), \( \langle S_\alpha \rangle = 0 \), for \( \alpha = x, y, z \), thus there is no spin squeezing. We find the maximum variance \( \langle \Delta S_z \rangle^2 = S^2 \), thus

\[ \chi_{|\psi\rangle}^2 = 1/N, \quad \langle \Delta \theta \rangle_{\text{QCR}} = 1/N, \]

(20)

beating the Heisenberg limit. From the above analysis we know that, for typical symmetry multipartite states, Dicke states, there are no spin squeezing, while \( \chi^2 < 1 \) indicates that they are entangled and are useful resources for phase estimation.

C. Anisotropic case

Now we consider the anisotropic case, \( 0 \leq \gamma < 1 \). The spin expectation values \( \langle S_\alpha^2 \rangle \) can not be obtained analytically. By treating the quantum effect as small fluctuations, approximate results can be obtained by using the Holstein-Primakoff (H-P) transformation [45] in the thermodynamic limit, and by using the continues unitary transformation method [46, 47, 48] for finite size case.

In the thermodynamic limit, the quantum fluctuations are small, we can use the H-P approximation. This method requires one to determine the semiclassical magnetization \( \langle \vec{S} \rangle \), which is not along \( z \)-axis in the broken phase in the thermodynamic limit. Following conventional steps, we first employ a mean field approach, define a spin coherent state

\[ |\theta, \phi\rangle = \bigotimes_{i=1}^N \left[ e^{-i\phi/2} \cos \frac{\theta}{2} |0\rangle_l + e^{i\phi/2} \sin \frac{\theta}{2} |1\rangle_l \right], \]  

(21)
under which

$$\langle \theta, \phi | \tilde{S} | \theta, \phi \rangle = \frac{N}{2} (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$  (22)

The Hamiltonian is rewritten as

$$H = -N \left[ \frac{1}{2} \sin^2 \theta \left( \cos^2 \phi + \gamma \sin^2 \phi \right) + h \cos \theta \right].$$  (23)

As max $[\cos^2 \phi + \gamma \sin^2 \phi] = \max (1, \gamma) = 1$, $(\gamma \leq 1)$, we have

$$\min H = -N \max \left[ \frac{1}{2} \sin^2 \theta + h \cos \theta \right],$$  (24)

then we conclude: (i), symmetric phase, $h \geq 1$, $\theta_0 = 0$, for all $\gamma$; (ii), broken phase, $0 \leq h < 1$, $\theta_0 = \arccos h$, $\phi = 0, \pi$, for $\gamma \neq 1$. We emphasize that, the mean spin direction is along the $z$-axis when the system is finite.

We rotate the $z$-axis to the semiclassical magnetization,

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos \theta_0 & 0 & \sin \theta_0 \\ 0 & 1 & 0 \\ -\sin \theta_0 & 0 & \cos \theta_0 \end{pmatrix} \begin{pmatrix} \tilde{S}_x \\ \tilde{S}_y \\ \tilde{S}_z \end{pmatrix}.$$  (25)

As presented in [28], $\theta_0 = 0$ for $h > 1$ so that $S = \tilde{S}$, and $\theta_0 = \arccos h$ for $h \leq 1$. The transformed Hamiltonian reads

$$\tilde{H} = -\hbar m \tilde{S}_z - \frac{1}{N} \left[ \frac{m^2 + \gamma}{2} \tilde{S}_x^2 - \frac{3m^2 + \gamma - 2}{2} \tilde{S}_z^2 \right] + \frac{h\sqrt{1-m^2}}{2} (\tilde{S}_+ + \tilde{S}_-) - \frac{m^2 - \gamma}{4N} (\tilde{S}_+^2 + \tilde{S}_-^2) - \frac{m \sqrt{1-m^2}}{2N} (\tilde{S}_+ \tilde{S}_z + \tilde{S}_- \tilde{S}_z + \tilde{S}_z \tilde{S}_+ + \tilde{S}_z \tilde{S}_-),$$  (26)

where $m = \cos \theta_0$. Then we introduce the H-P transformation

$$\tilde{S}_z = N/2 - a^1 a, \quad \tilde{S}_- = \sqrt{N} a^1 \sqrt{1 - a^1 a}/N = (\tilde{S}_+)\dagger.$$  (27)

The Hamiltonian can be written as

$$\tilde{H}^{(0)} = \frac{2\hbar m - 3m^2 - \gamma + 2}{2} a^1 a - \frac{m^2 - \gamma}{4} (a^{12} + a^2) + \frac{1 - m^2}{4},$$  (28)

up to the 0-th order of $N$. We neglect the terms of the 1-th order of $N$ as they are constant. Now we use the Bogoliubov transformation

$$a^\dagger = \cosh \left( \frac{\theta}{2} \right) b^\dagger + \sinh \left( \frac{\theta}{2} \right) b.$$  (29)

To diagonalize $\tilde{H}^{(0)}$, we find

$$\tanh \theta = \varepsilon = \frac{m^2 - \gamma}{2\hbar m - 3m^2 - \gamma + 2}.$$  (30)

The rotated spins are written under the H-P representation,

$$\tilde{S}_x = \sqrt{N} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{1/4} (b^\dagger + b) + O(1/N),$$

$$\tilde{S}_y = \frac{i\sqrt{N}}{2} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{1/4} (b^\dagger + b) + O(1/N),$$

$$\tilde{S}_z = \frac{N}{2} + \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 - \varepsilon^2}} \right) - \frac{1}{\sqrt{1 - \varepsilon^2}} \left[ b^\dagger b + \frac{\varepsilon}{2} (b^2 + b^2) \right].$$  (31)

For symmetric phase, $m = 1$, $S_\alpha = \tilde{S}_\alpha$, we have

$$\langle \tilde{S}_x^2 \rangle = \langle \tilde{S}_y^2 \rangle = \frac{N}{4} \sqrt{\frac{h - \gamma}{h - 1}},$$

$$\langle \tilde{S}_z^2 \rangle = \langle \tilde{S}_y^2 \rangle = \frac{N}{4} \sqrt{\frac{h - 1}{h - \gamma}},$$  (32)

while for broken phase, $m = h$, we need to rotate $\tilde{S}_x$ back to $S_x$ as

$$S_x = \tilde{S}_x \cos \theta_0 + \tilde{S}_z \sin \theta_0$$

$$= h \tilde{S}_x + \sqrt{1 - h^2} \tilde{S}_z,$$  (33)
get the spin squeezing, to 1, then \( \Delta \sim (\Delta \theta)_{SN} \). For broken phase, \( h < 1 \), we get the spin squeezing,

\[
\xi^2 = \chi^2 = \frac{1 - h^2}{1 - \gamma} < 1.
\]  

(35)

When \( h \) is far from the critical point, \( \xi^2 \) and \( \chi^2 \) approach to 1, then \( \Delta \theta \sim (\Delta \theta)_{SN} \). For broken phase, \( h < 1 \), we get the spin squeezing,

\[
\xi^2 = \chi^2 = \frac{1 - h^2}{1 - \gamma},
\]  

(36)

while

\[
\chi^2 = \frac{N}{4} (S^2) \simeq \frac{1}{(N + 2)} (1 - h^2) \simeq \frac{1}{N}.
\]  

(37)

Thus \( (\Delta \theta)_{QCR} \simeq 1/N \). When \( h \) approaches to the critical point \( h_c = 1 \), there are two limit processes in Eq.

\[
\frac{4(S^2_x)}{N^2} \bigg|_{h=1} \sim \frac{a_{xx}^{(0)}}{N^{2/3}}, \quad \frac{4(S^2_y)}{N^2} \bigg|_{h=1} \sim \frac{a_{yy}^{(0)}}{N^{4/3}},
\]  

(38)

where \( a_{xx}^{(0)} \) and \( a_{yy}^{(0)} \) are constant independent of \( N \). Now we have

\[
\xi^2 \big|_{h=1} \sim \frac{a_{xx}^{(0)}}{N^{2/3}}, \quad \chi^2 \big|_{h=1} \sim \frac{a_{yy}^{(0)}}{N^{4/3}},
\]  

(39)

then for large \( N, \xi^2 \) and \( \chi^2 \) converge to zero as \( 1/N^{2/3} \), and \( (\Delta \theta)_{QCR} \sim 1/N^{5/6} \).

To verify these analytical prediction, in Figs. 1 and 2, we show numerical results for \( \xi^2 \) and \( \chi^2 \) as functions of \( h \) with different \( \gamma \) for finite size system. As shown in Fig. 1, in the symmetric phase, \( \xi^2 = \chi^2 < 1 \), while in the broken phase, \( \chi^2 \) and \( \xi^2 \) behaves very differently. In the most of parameter range, \( \chi^2 \) is nearly independent of \( N \), which indicates entanglement, while for \( h \leq \sqrt{\gamma}, \xi^2 \simeq 1 \), and thus the system is not spin squeezed (Fig. 1(d)). For the isotropic case, there is no spin squeezing. In Fig. 2 we plot \( \chi^2 \) and \( \xi^2 \) for \( N = 500 \) and the thermodynamical limit, and find the numerical results coincide well with the analytical ones obtained by H-P transformation method.

As shown in Figs. 3 and 4, \( \chi^2 \) is nearly independent of larger \( N \) in the symmetric phase, and approaches to \( 1/N \) as \( h \) being away from the critical point in the broken phase. Therefore, entanglement characterized by \( \chi^2 \) is very different in the two phases, especially when we treat them as resources for quantum estimation. In the
symmetric phase, as shown in Fig. 3. \( \chi^2 \) is nearly independent of \( N \), and the parameter sensitivity is at the level of \( (\Delta \theta)_{SN} \), while in the broken phase, the ground states are more sensitive in parameter. In Fig. 4, we show numerical results for \( \chi^2 \) in the broken phase at \( h = 1/2, \gamma = 1/2 \), and we see clearly that \( \chi^2 \propto 1/N \). Therefore, the parameter estimation in the broken phase is enhanced to the Heisenberg limit.

One can use the concurrence and entropy to quantify entanglement in the LMG model and results are obtained in [28, 29, 30]. We see that, both concurrence and entropy indicate well the presence of entanglement, however, from them, we cannot tell whether the entanglement of the ground state is useful in parameter estimation. From results of \( \chi^2 \), we can see that, the entanglements in these two phases are different according to their performances in estimation. On one hand, we can use the collective behavior of the LMG model to improve the phase estimation precision, on the other hand, the differences of the parameter sensitivities can be used to characterize and distinguish the two quantum phases.

IV. CONCLUSION

We have analyzed \( \chi^2 \) and spin squeezing parameters \( \xi^2 \) in the ground state of the LMG model. For isotropic case, the Hamiltonian is diagonal in Dicke states, for which we have \( \xi^2 \geq \xi^2 = 1/\chi^2 \). For anisotropic case, our results indicate that, \( \chi^2 \) classifies states in different phases in the sense of quantum phase estimation. Hence we can use \( \chi^2 \) to distinguish and characterize the behaviors of the two phases of the LMG model. In the symmetric phase, \( \chi^2 \) is independent of \( N \) and approaches 1 with the increasing of \( h \), thus \( (\Delta \theta)_{QCR} \sim (\Delta \theta)_{SN} \), that is just a bit lower than the shot-noise limit. In the broken phase, we find \( \chi^2 \approx 1/N \) and \( (\Delta \theta)_{QCR} \approx 1/N \), which attains the Heisenberg limit.

Fisher information, being related to the Cramer-Rao inequality, is used to measure how much information that we know about some certain parameters in a probability distribution. From present results, we see that Fisher information can also characterize the QPT, by distinguishing the entangled ground states in the sense of parameter sensitivity. This approach is promising and expected to be applicable to other spin systems undergoing a QPT.

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