On a classical solution to the Abelian Higgs model

N. Mohammedi *

Institut Denis Poisson (CNRS - UMR 7013),
Université de Tours,
Faculté des Sciences et Techniques,
Parc de Grandmont, F-37200 Tours, France.

Abstract

A particular solution to the equations of motion of the Abelian Higgs model is given. The solution involves the Jacobi elliptic functions as well as the Heun functions.

*e-mail: noureddine.mohammedi@univ-tours.fr
1 Introduction

The search for analytical solutions to the classical equations of motion of field theories which are of relevance to certain domains of physics is of great interest. One of these field theories is the Abelian Higgs model whose importance embraces particle physics, condensed matter physics and cosmology [1, 2, 3, 4, 5]. The full fledged Euler-Lagrange equations of motion of this model have so far resisted attempts to solve them. This is because these are highly non-linear coupled second order partial differential equations. In the literature, the classical studies of the Abelian Higgs model are mostly dedicated to its topological properties. In this context, vortices have been identified and a great deal has been learned through numerical analyses (see [6] for a sample). An account of the problem of constructing stationary topological and non-topological classical solutions in various field theories can be found in [7, 8, 9, 10].

In order to reduce the degree of complexity of the equations of motion of the Abelian Higgs model some simplifying strategies have to be adopted. In this area, the authors of ref.[11] explicitly constructed periodic sphaleron solutions (minima and saddle points of the energy functional) of the (1 + 1)-dimensional Abelian Higgs model on a circle. They found some analytical solutions (for some special values of the Higgs mass) for the the small perturbations (normal modes) around the sphaleron solutions. Their work is directly inspired by an earlier investigation by Manton and Samols [12] who found sphaleron solutions in the scalar theory $\phi^4$ defined on a circle.

In this note we have identified another situation where the equations of motion of the four-dimensional Abelian Higgs model become relatively simple. If the complex scalar field is parametrised as $\phi = \rho e^{i\theta}$ and we define the gauge invariant quantity $\tilde{A}_\mu = A_\mu + \frac{1}{c} \partial_\mu \theta$, with $A_\mu$ being the gauge field, then imposing the constraint $\tilde{A}_\mu \tilde{A}^\mu = 0$ renders the equations of motion tractable\footnote{Upon publication of the present work, I became aware that the authors of refs.[13, 14, 15, 16, 17, 18, 19, 20] have used, among others, the condition $A_\mu A^\mu = 0$, to build analytic solutions in numerous field theories. I am greatful to Fabrizio Canfora for pointing out this to me.}. As a matter of fact, the equation of motion of the complex scalar field decouples and reduces to the usual equation of motion of a $\phi^4$ scalar field theory which is known to possess kink solutions. The problem is then brought to solving the gauge field equation in the presence of a kink background.

We have first solved the gauge field equation of motion when the scalar field $\rho$ takes the usual kink profile $\rho = \pm v \tanh (p_\mu x^\mu + w_0)$, where $p_\mu$ is a space-like four vector. The gauge field $A_\mu$ is, up to a gauge function, expressed in terms of associated Legendre functions and has two independent polarisations. However, the equation of motion of a $\phi^4$ scalar field theory is also solved by the twelve Jacobi elliptic functions (the kink solution is a very special case of this). Next, we solved the gauge field equation of motion when the scalar field $\rho$ is represented by one of the Jacobi elliptic functions. Here we found that the gauge field is determined by means of Heun functions.

In passing, we should mention that some analytical solutions involving gauge fields have been explicitly constructed in different contexts. Brihaye [21], in the case of the $SU(2)$ Yang-Mills theory coupled to a triplet Higgs field, has found (for particular values of the ratio of the two coupling constants) solutions involving the Jacobi elliptic functions. The authors of ref.[22], studying a (1 + 1)-dimensional Abelian Higgs model, have obtained exact solutions
(subject to the approximation that the two wells of the Higgs potential are deep). These were consequently used to build approximate analytical solution, in the form of oscillons and oscillating kinks, for the dynamics of both the gauge and Higgs fields. Similar studies can also be found in [23, 24]. Along these lines, one finds in [25] a numerical solutions corresponding to a kink (domain wall) in a theory consisting of two interacting complex scalars coupled to two independent gauge fields. The stability of this solution was later analysed in [26].

The partial relevance of all of these solutions to the present work is worth investigating. Finally, various analytical solutions involving a generalised Maxwell-Higgs models (theories with non-standard kinetic terms) have been obtained in [27, 28]. Nevertheless, we should insist on the fact that none of the above mentioned solutions in [11, 22, 23, 24, 25, 26] is exact. The closest study to our analyses is ref.[11] (in that it involves the Jacobi elliptic functions). Their sphaleron solution (and the perturbation about it) to not fit in the caterory $\tilde{A}_\mu \tilde{A}^\mu = 0$ considered here.

The paper is organised as follows: In the next section we briefly describe the Abelian Higgs model and lay out its simple classical solutions. In section three, we solve the equation of motion of the gauge field in the background of a kink scalar field. We then review, in section four, the classical solution to the equation of motion of a $\phi^4$ scalar field theory in terms of the Jacobi elliptic functions (solution involving the Weierstrass elliptic function are presented in an appendix). These are used in section five to build the solution to the gauge field equation of motion. Our main results are summarised in the last section.

2 The Abelian Higgs model

The Abelian Higgs model is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_{\mu} \phi^* - ieA_{\mu} \phi^*) (\partial^\mu \phi + ieA^\mu \phi) - \frac{\lambda}{2} (\varphi^* \varphi - v^2)^2. \quad (2.1)$$

Here $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the field strength of the gauge field $A_{\mu}$ and $\phi$ is the complex scalar field. It is understood that the two parameters $v^2$ and $\lambda$ are both positive.

It is convenient for our purpose to parametrise the complex scalar field $\phi$ as

$$\phi = \rho e^{i\theta}. \quad (2.2)$$

The Lagrangian (2.1) becomes then

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^2 \rho^2 \left( A_{\mu} + \frac{1}{e} \partial_{\mu} \theta \right) \left( A^\mu + \frac{1}{e} \partial^\mu \theta \right)$$

$$+ \partial_{\mu} \rho \partial^\mu \rho - \frac{\lambda}{2} (\rho^2 - v^2)^2 \quad (2.3)$$

---

2I am greatful to an anonymous referee for bringing some of these works to my attention.

3The space-time coordinates are $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$ and the indices are raised and lowered with the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. A four-vector is $V^\mu = (V^0, V^1, V^2, V^3) = (V^0, \vec{V})$.

4One must be aware that $\phi = \rho e^{i\theta} = \rho e^{i(\theta + 2\pi N)}$ with $N \in \mathbb{Z}$. In this note we assume that $\partial_{\mu} \partial_{\nu} \theta - \partial_{\nu} \partial_{\mu} \theta = 0$. 

---

3
and the local gauge symmetry is
\[ A_\mu \longrightarrow A_\mu + \frac{1}{e} \partial_\mu \Lambda , \quad \theta \longrightarrow \theta - \Lambda , \quad \rho \longrightarrow \rho . \] (2.4)

One could use this gauge freedom to set the field \( \theta \) to zero. Nevertheless, we will keep \( \theta \) throughout.

The equations of motion for the Abelian Higgs model are
\[ \partial_\mu \tilde{F}^{\mu \nu} + 2e^2 \rho^2 \tilde{A}^\nu = 0 , \] (2.5)
\[ \partial_\mu \partial_\mu \rho + \lambda \rho (\rho^2 - v^2) - e^2 \rho \tilde{A}_\mu \tilde{A}^\mu = 0 , \] (2.6)
\[ \partial_\mu \left( \rho^2 \tilde{A}^\mu \right) = 0 . \] (2.7)

We have defined the gauge invariant variable
\[ \tilde{A}_\mu = A_\mu + \frac{1}{e} \partial_\mu \theta . \] (2.8)

and \( \tilde{F}_{\mu \nu} = F_{\mu \nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \). The last equation (2.7) corresponds to the field \( \theta \) and is also a consequence of (2.5). Notice that we have expressed the equations of motion in terms of the two gauge invariant variables \( \tilde{A}_\mu \) and \( \rho \).

The simplest known solution to the equations of motion is when the scalar field \( \rho \) is frozen at one of the two minima of the potential energy. That is, \( \rho^2 = v^2 \). In this case the equations of motion become
\[ \partial_\mu \tilde{F}^{\mu \nu} + 2e^2 v^2 \tilde{A}^\nu = 0 , \] (2.9)
\[ \tilde{A}_\mu \tilde{A}^\mu = 0 , \] (2.10)
\[ \partial_\mu \left( \rho^2 \tilde{A}^\mu \right) = 0 . \] (2.11)

These equations have, for instance, the plane wave solution
\[ \tilde{A}_\mu = \varepsilon_\mu \cos (p_\mu x^\mu + w_0) , \quad \varepsilon_\mu \varepsilon^\mu = \varepsilon_\mu p^\mu = 0 , \quad p_\mu p^\mu = 2e^2 v^2 . \] (2.12)

The polarisation vector \( \varepsilon_\mu \) has two independent components for a given wave vector \( p_\mu \). The mass-shell relation \( p_\mu p^\mu = 2e^2 v^2 \) is that of a massive particle with a positive mass squared.

The other known solution is the "kink" solution for which \( \tilde{A}_\mu = 0 \). The equations of motion come then to the single equation
\[ \partial_\mu \partial_\mu \rho + \lambda \rho (\rho^2 - v^2) = 0 . \] (2.13)

This is solved by
\[ \rho = \pm v \tanh (p_\mu x^\mu + w_0) , \quad p_\mu p^\mu = -\frac{\lambda v^2}{2} . \] (2.14)

Here we have a mass-shell condition of a relativistic particle of negative mass squared.
In this note, we will look for a solution which satisfies the gauge invariant condition
\[ \tilde{A}_\mu \tilde{A}^\mu = 0 \quad . \quad (2.15) \]

The equations of motion reduce then to
\[
\begin{align*}
\partial_\mu \tilde{F}^{\mu \nu} + 2e^2 \rho^2 \tilde{A}^\nu &= 0 \quad , \\
\partial_\mu \rho + \lambda \rho (\rho^2 - v^2) &= 0 \quad , \\
\partial_\mu (\rho^2 \tilde{A}^\mu) &= 0 \quad . \quad (2.16)
\end{align*}
\]

We notice that the equation of motion for the scalar field \( \rho \) decouples from the rest. We will report here on a non-trivial solution obeying the condition (2.15).

3 A gauge field corresponding to the “kink” scalar field

The second equation (2.17) admits the "kink" solution as written in (2.14). We would like here to find the gauge field corresponding to it. We assume the following form for the gauge field \( \tilde{A}_\mu \):
\[ \tilde{A}_\mu (w) = \epsilon_\mu h (w) \quad . \quad (3.1) \]

Here (and in the rest of the paper) we will use the notation
\[ w = p_\mu x^\mu + w_0 \quad , \quad p^2 = p_\mu p^\mu \quad (3.2) \]

with \( w_0 \) a constant. The four-vector \( p_\mu \) obeys the mass-shell relation \( p^2 = -\frac{\lambda v^2}{2} \). The polarisation vector \( \epsilon_\mu \) is required to satisfy
\[ \epsilon_\mu \epsilon^\mu = p_\mu \epsilon^\mu = 0 \quad . \quad (3.3) \]

Equations (2.15) and (2.18) are then automatically obeyed.

Substituting (3.1) into (2.16) results in the differential equation
\[ \frac{d^2 h}{dw^2} (w) - \frac{4e^2}{\lambda} \tanh^2 (w) h (w) = 0 \quad . \quad (3.4) \]

By the change of variables
\[ z = \tanh (w) \quad (3.5) \]

one transforms the differential equation (3.4) into
\[ \left( 1 - z^2 \right) \frac{d^2 h}{dz^2} (z) - 2z \frac{dh}{dz} (z) + \left[ l (l + 1) - \frac{m^2}{(1 - z^2)} \right] h (z) = 0 \quad . \quad (3.6) \]

This differential equation is known as associated Legendre equations [29, 30, 31]. It is, in general, singular at the points \( z = \pm 1, \pm \infty \). In the case at hand, the variable \( z \) is real and lies in the domain \([-1, +1]\) and the real constants \( l \) and \( m \) are defined as\(^5\)
\[ l = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{4e^2}{\lambda}} \quad , \quad m^2 = \frac{4e^2}{\lambda} \quad . \quad (3.7) \]

\(^{5}\)Greek indices \( \nu \) and \( \mu \) are usually used instead of \( l \) and \( m \).
The general solution to (3.6) is
\[ h(z) = a P^m (z) + b Q^m_l (z) \quad , \] (3.8)
where \( P^m (z) \), \( Q^m_l (z) \) are associated Legendre functions \([29, 30, 31]\) of the first and second kind, respectively. The two constants \( a \) and \( b \) are arbitrary.

When \( z \) is real and belonging to the interval \([-1, +1]\), the associated Legendre function \( P^m_l (z) \) is real and is expressed as \([29, 30, 31]\)
\[ P^m_l (z) = \frac{1}{\Gamma (1 - m)} \left( \frac{1 + z}{1 - z} \right)^{\frac{m}{2}} F \left( -l, l + 1; 1 - m; \frac{1 - z}{2} \right) \quad , \] (3.9)
where \( F (a, b; c; x) \) is the hypergeometric function and \( \Gamma (\nu) \) is the gamma function. Similarly, the associated Legendre function \( Q^m_l (z) \) is given by
\[ Q^m_l (z) = \frac{\pi}{2 \sin (m \pi)} \left[ \cos (m \pi) P^m_l (z) - \frac{\Gamma (l + m + 1)}{\Gamma (l - m + 1)} P^{-m}_l (z) \right] \quad . \] (3.10)

To summarise, a particular solution to the equations of motion of the Abelian Higgs model (2.1) is given by
\[ \phi = \pm v \tanh (w) e^{i \theta} \quad , \]
\[ A_\mu = - \frac{1}{e} \partial_\mu \theta + \varepsilon_\mu \left[ a P^m_l (\tanh (w)) + b Q^m_l (\tanh (w)) \right] \quad , \]
\[ p^2 = - \frac{\lambda v^2}{2} \quad , \quad \varepsilon_\mu \varepsilon^\mu = p_\mu \varepsilon^\mu = 0 \quad . \] (3.11)

The parameters \( l \) and \( m \) are given (3.7). The field \( \theta (x) \) is any arbitrary smooth function (obeying \( \partial_\mu \partial_\nu \theta - \partial_\nu \partial_\mu \theta = 0 \)). The gauge field \( A_\mu \) has two independent polarisations for a given vector \( p_\mu \). Furthermore, the solution carries both electric and magnetic fields.

4 Solutions to \( \phi^4 \) in terms of the Jacobi elliptic functions

The equation of motion for a phi-to-the-four scalar field theory is given by
\[ \partial_\mu \partial^\mu \rho + \lambda \rho (\rho^2 - v^2) = 0 \quad . \] (4.1)

We will look for solution which depend on the single variable \( w = p_\mu x^\mu + w_0 \). That is, \( \rho = \rho (w) \). The equation of motion becomes then
\[ \frac{d^2 \rho}{dw^2} + \frac{\lambda}{p^2} \rho (\rho^2 - v^2) = 0 \quad . \] (4.2)

Notice that any solution to this equation is obviously subject to the following remark:
\[ \text{if } \rho (w, p^2) \text{ is a solution then } \rho \left( s w, \frac{p^2}{s^2} \right) \text{ is also a solution } \quad , \] (4.3)
where $s$ is a constant. Therefore, the mass-shell relation, $p^2$, will be determined up to a constant.

It is well-known that equation (4.2) is solved by the twelve Jacobi elliptic functions [29, 30, 31]. Indeed, The scalar field $\rho(w)$ satisfies the first order differential equation

$$\left( \frac{d\rho}{dw} \right)^2 = -\frac{\lambda}{p^2} \rho^2 (\rho^2 - 2v^2) + av^2$$

(4.4)

with $av^2$ a constant of integration. By writing

$$\rho(w) = v \varepsilon y(w)$$

(4.5)

where $\varepsilon$ is a constant, one obtains the first order differential equation

$$\left( \frac{dy}{dw} \right)^2 = -\frac{\lambda v^2 \varepsilon^2}{2p^2} y^4 + \frac{\lambda v^2}{p^2} y^2 + \frac{a}{\varepsilon} .$$

(4.6)

The twelve Jacobi elliptic functions are obtained as solutions to this last equation [29, 30, 31] for different choices of the three constants $p^2$, $\varepsilon^2$ and $a$ (and boundary conditions). The table in Appendix B gather these values. For instance, the solution given in terms of the “sine” Jacobi elliptic function $\text{sn}(x, m)$ corresponds to the choice

$$a = \varepsilon^2 = \frac{2m^2}{1 + m^2} ,$$
$$p^2 = -\frac{\lambda v^2}{1 + m^2}$$

(4.7)

and the differential equation (4.6) takes the form

$$\left( \frac{dy}{dw} \right)^2 = (1 - y^2) (1 - m^2 y^2) .$$

(4.8)

The general solution [29, 30, 31] to this last equation is $y(w) = \text{sn}(w + d, m)$, where $d$ is an arbitrary constant. Hence our scalar field $\rho$ is given by

$$\rho(w) = \pm v \sqrt{\frac{2m^2}{1 + m^2}} \text{sn}(w + d, m) ,$$
$$p^2 = -\frac{\lambda v^2}{1 + m^2} .$$

(4.9)

The parameter $m$ must be different from zero here. The mass-shell relation $p^2 = -\frac{\lambda v^2}{1 + m^2}$ is that of a particle with negative mass squared. It is worth mentioning that the “kink” solution (2.14) corresponds to $m = 1$ as

$$\text{sn}(w + d, 1) = \tanh (w + d) .$$

(4.10)

5 The general solution to the gauge field equation

The gauge field, $A_\mu = \varepsilon_\mu h(w)$ with $\varepsilon_\mu \varepsilon^\mu = p_\mu \varepsilon^\mu = 0$, is determined by the differential equation

$$\frac{d^2 h}{dw^2} (w) + \frac{2e^2}{p^2} \rho^2 (w) h(w) = 0 .$$

(5.1)
For the moment, the expression of $\rho^2$ is not fixed. This allows one to treat all possible mass-shell conditions at the same time.

Let us assume that

$$h(w) = h(Z), \quad Z = \frac{1}{\tau^2} \rho^2 (w),$$  \hspace{1cm} (5.2)$$

where $\tau$ is a constant to be properly chosen later. Then by using the fact that the field $\rho(w)$ satisfies the two equations

$$\frac{d^2 \rho}{dw^2} = -\frac{\lambda}{\rho^2} \rho (\rho^2 - v^2),$$

$$\left(\frac{d\rho}{dw}\right)^2 = -\frac{\lambda}{2\rho^2} \rho^2 (\rho^2 - 2v^2) + av^2$$  \hspace{1cm} (5.3)$$

we arrive at the differential equation

$$\frac{d^2 h}{dZ^2} + \left(\gamma - \frac{\delta}{1-Z} - \frac{e\kappa^2}{1-k^2 Z}\right) \frac{dh}{dZ} + \frac{(s + \alpha\beta k^2 Z)}{Z(1-Z)(1-k^2 Z)} h = 0.$$  \hspace{1cm} (5.4)$$

The different constants are given by

$$s = 0,$$

$$\delta = \gamma = \epsilon = \frac{1}{2},$$

$$\alpha = \alpha_{\pm} = \frac{1}{4} \left(1 \pm \sqrt{1 + 16\frac{e^2}{\lambda}}\right),$$

$$\beta = \beta_{\mp} = \frac{1}{4} \left(1 \mp \sqrt{1 + 16\frac{e^2}{\lambda}}\right),$$

$$\tau^2 = \frac{2k^2}{(1 + k^2)} v^2.$$  \hspace{1cm} (5.5)$$

They satisfy the relation

$$\gamma + \delta + \epsilon = \alpha_{\pm} + \beta_{\mp} + 1.$$  \hspace{1cm} (5.6)$$

We also have the mass-shell relation

$$p^2 = -\frac{2\lambda v^2 k^2}{a(1 + k^2)^2}.$$  \hspace{1cm} (5.7)$$

The constant $a$ will be fixed later.

The equation (5.4) is a Fuchsian differential equation and its general solution is given by

$$h(Z) = C_1 \operatorname{Hn} \left(k^2, 0; \alpha_{\pm}, \beta_{-}, \frac{1}{2}, \frac{1}{2}; Z\right) + C_2 \operatorname{Hn} \left(k^2, 0; \alpha_{-}, \beta_{+}, \frac{1}{2}, \frac{1}{2}; Z\right),$$

$$Z = \frac{1}{\tau^2} \rho^2 (w) = \frac{(1 + k^2)}{2k^2} \frac{1}{v^2} \rho^2 (w).$$  \hspace{1cm} (5.8)$$
where $H_n(k^2, s; \alpha, \beta, \gamma, \delta; z)$ is the Heun function\(^6\) and $C1$ and $C2$ are two arbitrary constants. It is important to emphasise at this point that the scalar field $\rho(w)$ is any solution to the equation of motion (4.2), that is, any of the twelve Jacobi elliptic functions.

When, for instance, the scalar field $\rho(w)$ is expressed in terms of the Jacobi elliptic function $\text{sn}(w + d, m)$ in (4.9) then the expression of $p^2$ there has to match that written in (5.7). This leads to the identifications

$$a = \frac{2k^2 (1 + m^2)}{(1 + k^2)^2} = \frac{2m^2}{(1 + m^2)} \implies k^2 = m^2 \quad \text{or} \quad k^2 = \frac{1}{m^2} \ . \quad (5.9)$$

If we choose $k = m$, for example, then, using (4.9) and (5.8), the two fields of the Abelian Higgs model (2.1) are found to be given by

$$\phi = \pm v \sqrt{\frac{2m^2}{1 + m^2}} \text{sn}(w + d, m) e^{i\theta} ,$$

$$A_\mu = -\frac{1}{e} \partial_\mu \theta + \varepsilon_\mu \left[ C_1 H_n \left( m^2, 0; \alpha_+, \beta_-, \frac{1}{2}, \frac{1}{2} ; \text{sn}^2(w + d, m) \right) + C_2 H_n \left( m^2, 0; \alpha_-, \beta_+, \frac{1}{2}, \frac{1}{2} ; \text{sn}^2(w + d, m) \right) \right] ,$$

$$p^2 = -\frac{\lambda v^2}{1 + m^2} , \quad \varepsilon_\mu \varepsilon^\mu = p_\mu \varepsilon^\mu = 0 \ . \quad (5.10)$$

The two constants $\alpha_\pm$ and $\beta_\pm$ depend on the Abelian Higgs parameters and are listed in (5.5). The field $\theta(x)$ is arbitrary.

Finally, when $m = 1$ one has $\text{sn}(w + d, 1) = \tanh(w + d)$, and the particular solution given in (3.11) is recovered by converting the Heun functions into hypergeometric functions with the help of the relation [32]

$$\begin{cases} H_n(1, s; \alpha, \beta, \gamma, \delta; z) = (1 - z)^r F(r + \alpha, r + \beta; \gamma; z) , \\ r = \xi - \sqrt{\xi^2 - \alpha \beta - s} \ , \quad \xi = \frac{1}{2} (\gamma - \alpha - \beta) \ . \end{cases} \quad (5.11)$$

We have represented in Figure 1 the Jacobi elliptic function $\text{sn}(x, 2)$ and in Figure 2 the Heun function $H_n \left( 4, 0; \frac{3}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{2} ; \text{sn}^2(x, 2) \right)$.

\(^6\)Sometimes the notation $a = 1/k^2$ and $q = -s/k^2$ is used. A useful note on Heun’s functions and further references can be found in [32].
6 Conclusion

We have presented classical solutions to the equations of motion of the Abelian Higgs model (2.1). The complex scalar field $\phi$ and the gauge field $A_\mu$ are given by

$$
\phi = \pm v \varepsilon_{pq}(w + d, m) e^{i\theta},
$$

$$
A_\mu = -\frac{1}{e} \partial_\mu \theta + \varepsilon_\mu \left[ C_1 \text{Hn} \left( k^2, 0; \alpha_+, \beta_-, \frac{1}{2}, \frac{1}{2}; Z \right) + C_2 \text{Hn} \left( k^2, 0; \alpha_-, \beta_+, \frac{1}{2}, \frac{1}{2}; Z \right) \right],
$$

$$
Z = \varepsilon^2 \left( 1 + k^2 \right) \frac{p^2 (w + d, m)}{2k^2}, \quad \varepsilon_\mu \varepsilon^\mu = p_\mu \varepsilon^\mu = 0.
$$

Here $pq(w + d, m)$, with $pq$ any pair of the letters (c,d,n,s), is one of the twelve Jacobi elliptic functions and $\text{Hn} (k^2, s; \alpha, \beta, \gamma, \delta; z)$ is the Heun function. We have listed in table 1 the values $p^2$ (the mass-shell relation) and $\varepsilon$ for each function $pq(w + d, m)$. Table 1 gives also the expression of $k^2$ in terms of the parameter $m$ entering the Jacobi elliptic function $pq(w + d, m)$. Finally, $\alpha_\pm = \frac{1}{4} \left( 1 \pm \sqrt{1 + 16 \varepsilon^2 \frac{e^2}{X}} \right)$ and $\beta_\mp = \frac{1}{4} \left( 1 \mp \sqrt{1 + 16 \varepsilon^2 \frac{e^2}{X}} \right)$.

Notice that the differential equation (5.1) is linear in $h(w)$. Therefore, the gauge field $A_\mu$ is in fact a linear combination of all the polarisation vectors $\varepsilon_\mu$ satisfying $\varepsilon_\mu \varepsilon^\mu = p_\mu \varepsilon^\mu = 0$.
Finally, the solution (6.1) could have been expressed in terms of the Weierstrass elliptic function as shown in Appendix A.

There are two immediate questions that one might ask regarding the solutions found in this article. The first regards their physical relevance. Unfortunately, our solution cannot find applications in condensed matter physics. This is because the 'trick' used to decouple the complex scalar field equation of motion does not apply in the non-relativistic version of the Abelian Higgs model (Ginzburg-Landau theory, gauged non-linear Schrödinger, London theory, ...). The equivalent of $\tilde{A}_\mu \tilde{A}^\mu = 0$ would be $A_0 = c A^2$, $c$ being a constant. However, this is not compatible with Maxwell’s equations. It is though plausible that our solution might be of use in a theory of gravity coupled to the Abelian Higgs model. This is currently under investigation.

The second (hard) question concerns the stability of the solution presented here. Our solution is not protected by any topological argument (unlike the vortex solution) and therefore is expected to be unstable. The only way to set one’s mind at rest is by carrying a perturbation expansion around our exact solutions. This is done through the substitution\(^7\) (in the spirit of refs.[11, 26])

\[
\begin{align*}
\rho & \rightarrow \rho (w) + \kappa (w) \ e^{i\omega_1 t} \\
\tilde{A}_\mu & \rightarrow \tilde{A}_\mu (w) + a_\mu (w) \ e^{i\omega_2 t}.
\end{align*}
\]

Here $\rho (w)$ and $\tilde{A}_\mu (w)$ are solutions to the equations of motion of the Abelian Higgs model. The perturbations $\kappa (w)$ and $a_\mu (w)$ are accompanied by their normal modes represented by the two constants $\omega_1$ and $\omega_2$, respectively. If $\omega_1$ and $\omega_2$ are real then we have an oscillatory regime and the solution is stable. The next step is to plug the above substitution into the full equations of motion (2.5), (2.6) and (2.7). To first order in the perturbation, one obtains then the differential equations which, in principle, allow one to determine the normal modes. The differential equations obtained here involve the Jacobi elliptic functions and Heun functions and are, at the moment, difficult to analyse.

It might be easier, though, to choose a perturbation which respects the constraint $\tilde{A}_\mu \tilde{A}^\mu = 0$. This can be achieved for instance by writing $a_\mu (w) = v_\mu g (w)$ and demanding that $v_\mu v^\mu = \varepsilon_\mu v^\mu = 0$, where $\varepsilon_\mu$ is the polarisation vector of $\tilde{A}_\mu (w)$. The substitution (6.2) is then carried out in the simpler equations of motion (2.16), (2.17) and (2.18). We hope to report on the progress in this case in the near future.

As a last remark, we mention that our solution is easily adapted to the case of a $(1 + 1)$-dimensional Abelian Higgs model and the complex scalar field $\phi$ could be compactified on a circle of length $L$. This would make close contact with the works in [11, 12, 41] and might be of help in the determination of the normal modes.

\(^7\)Our solution can be make static by setting $p_0 = 0$ in the variable $w = p_\mu x^\mu + w_0$.  

11
APPENDICES

A Solutions to $\phi^4$ in terms of the Weierstrass elliptic function

The Weierstrass elliptic function is a general solution to the first order differential equation [29, 30, 31] (see also [33] for some lectures on the subject)

$$\left( \frac{d\wp}{dz} \right)^2 = 4 \wp^3(z) - g_2 \wp(z) - g_3$$
$$\quad = 4 (\wp - e_1)(\wp - e_2)(\wp - e_3) . \quad (A.1)$$

The constants $g_2$ and $g_3$ are known as the lattice invariants and $e_i, i = 1, 2, 3$ are the roots of the Weierstrass normal cubic equation $4z^3 - g_2z - g_3 = 0$.

In order to obtain a Weierstrass differential equation in the case of $\phi^4$ theory, we start from equation (4.6)

$$\left( \frac{dy}{dw} \right)^2 = -\frac{\lambda v^2 \varepsilon^2}{2p^2} y^4 + \frac{\lambda v^2}{p^2} y^2 + \frac{a}{\varepsilon^2} , \quad (A.2)$$

and multiply both sides with $4y^2$ to get

$$\left( \frac{df}{dw} \right)^2 = -2\lambda v^2 \varepsilon^2 f^3 + \frac{4\lambda v^2}{p^2} f^2 + \frac{4a}{\varepsilon^2} f ,$$
$$\quad f(w) \equiv y^2(w) . \quad (A.3)$$

The next step is to get rid of the term proportional to $f^2$. This is achieved by the change of functions

$$f(w) = g(w) + \frac{2}{3} \frac{1}{\varepsilon^2} . \quad (A.4)$$

The resulting differential equation is given by

$$\left( \frac{dg}{dw} \right)^2 = \frac{2\lambda v^2 \varepsilon^2}{p^2} g^3 + \frac{4}{\varepsilon^2} \left( \frac{2}{3} \frac{\lambda v^2}{p^2} + a \right) g + \frac{8}{3} \frac{1}{\varepsilon^4} \left( \frac{4\lambda v^2}{9p^2} + a \right) . \quad (A.5)$$

In order to obtain a Weierstrass differential equation, we choose the constant $\varepsilon$ such that

$$p^2 = -\varepsilon^2 \frac{\lambda v^2}{2} . \quad (A.6)$$

This leads to

$$\left( \frac{dg}{dw} \right)^2 = 4g^3 + \frac{4}{\varepsilon^4} \left( \varepsilon^2 a - \frac{4}{3} \right) g + \frac{8}{3} \frac{1}{\varepsilon^6} \left( \varepsilon^2 a - \frac{8}{9} \right) ,$$
$$\quad = 4 \left[ g - \frac{1}{\varepsilon^2} \left( \frac{1}{3} + \sqrt{1 - \varepsilon^2 a} \right) \right] \left[ g - \frac{1}{\varepsilon^2} \left( \frac{1}{3} - \sqrt{1 - \varepsilon^2 a} \right) \right] \left[ g + \frac{2}{3} \frac{1}{\varepsilon^2} \right] . \quad (A.7)$$
The solution to this differential equation is given by the Weierstrass elliptic function

$$g(w) = \wp \left( w + d; -\frac{4}{\varepsilon^4} \left( \varepsilon^2 a - \frac{4}{3} \right), -\frac{8}{3} \frac{1}{\varepsilon^6} \left( \varepsilon^2 a - \frac{8}{9} \right) \right),$$  \hspace{1cm} (A.8)

where \(d\) is a constant. Recalling that \(\rho(w) = v\varepsilon y(w)\), the square of our scalar field \(\rho(w)\) is finally given by

$$\rho^2(w) = v^2 \left[ \frac{2}{3} + \varepsilon^2 \wp \left( w + d; -\frac{4}{\varepsilon^4} \left( \varepsilon^2 a - \frac{4}{3} \right), -\frac{8}{3} \frac{1}{\varepsilon^6} \left( \varepsilon^2 a - \frac{8}{9} \right) \right) \right]$$

$$= v^2 \left[ \frac{2}{3} + \wp \left( \frac{1}{\varepsilon} (w + d); -4 \left( \varepsilon^2 a - \frac{4}{3} \right), -\frac{8}{3} \left( \varepsilon^2 a - \frac{8}{9} \right) \right) \right],$$

$$p^2 = -\varepsilon^2 \frac{\lambda v^2}{2}. \hspace{1cm} (A.9)$$

In the last equality we have used the homogeneity relation [29, 30, 31, 33]

$$\wp(z; g_2, g_3) = \mu^{-2} \wp \left( \mu^{-1} z; \mu^4 g_2, \mu^6 g_3 \right). \hspace{1cm} (A.10)$$

Therefore we could have used the Weierstrass elliptic function (instead of the Jacobi elliptic functions) to express the solution to the Abelian Higgs model given in (6.1). In this case, the identification of the two expressions of \(p^2\) in (A.9) and (5.7) leads to

$$\varepsilon^2 a = \frac{4k^2}{(1 + k^2)^2}. \hspace{1cm} (A.11)$$

Finally, we should mention that the Weierstrass elliptic function could be converted into Jacobi elliptic functions [29, 30, 31, 33].

### B The twelve Jacobi elliptic functions Solutions to \(\phi^4\)

The table below summarises the solutions to the equation

$$\left( \frac{dy}{dw} \right)^2 = -\frac{\lambda v^2 \varepsilon^2}{2p^2} y^4 + \frac{\lambda v^2}{p^2} y^2 + \frac{a}{\varepsilon^2} \hspace{1cm} (B.1)$$

and gives the corresponding values of the parameters \(p^2\) (the mass-shell relation), \(\varepsilon^2\) and \(a\) (see also [34, 35] and [36] for some particular cases).

It is important to notice that \(\varepsilon^2\) has to be strictly positive (for \(\rho(w)\) to be a real field and different from zero). This, consequently, puts restrictions on the allowed values of the parameter \(m\) for some of the solutions.
The general solution to this equation is expressed in terms of the twelve Jacobi elliptic functions solutions to (B.1) and (4.2) and their corresponding parameters $p^2$ (the mass-shell relation), $\varepsilon^2$ and $a^2$. The table gives also the relation between $k^2$ and $m^2$ appearing in (6.1).

### Table 1: The twelve Jacobi elliptic functions solutions to (B.1) and (4.2) and their corresponding parameters $p^2$ (the mass-shell relation), $\varepsilon^2$ and $a^2$. The table gives also the relation between $k^2$ and $m^2$ appearing in (6.1).

| $p^2$          | $\varepsilon^2$ | $a^2$ | $y(w)$                  | $\rho(w)$                              | $k^2$ or $\frac{1}{k^2}$ |
|----------------|-----------------|-------|-------------------------|----------------------------------------|----------------------------|
| $-\lambda v^2$ | $\frac{2m^2}{1+m^2}$ | 1     | $\text{sn}(w,m)$        | $\pm v \varepsilon \text{sn}(w+d,m)$ | $m^2$                      |
| $-\lambda v^2$ | $-\frac{2m^2}{1-2m^2}$ | $1-m^2$ | $\text{cn}(w+d,m)$      | $\pm v \varepsilon \text{cn}(w+d,m)$ | $\frac{m^2-1}{m^2}$       |
| $-\lambda v^2$ | $\frac{2}{2-m^2}$ | $m^2-1$ | $\text{dn}(w+d,m)$      | $\pm v \varepsilon \text{dn}(w+d,m)$ | $1-m^2$                    |
| $\lambda v^2$  | $\frac{2m^2-1}{1+m^2}$ | 1     | $\text{cd}(w,m)$        | $\pm v \varepsilon \text{cd}(w+d,m)$ | $m^2$                      |
| $\lambda v^2$  | $\frac{2m^2(1-m^2)}{2m^2-1}$ | $1$   | $\text{sd}(w,m)$        | $\pm v \varepsilon \text{sd}(w+d,m)$ | $\frac{m^2-1}{m^2}$       |
| $\lambda v^2$  | $\frac{2(1-m^2)}{2-m^2}$ | $-1$  | $\text{nd}(w,m)$        | $\pm v \varepsilon \text{nd}(w+d,m)$ | $1-m^2$                    |
| $-\lambda v^2$ | $\frac{2}{1+m^2}$ | $m^2$  | $\text{dc}(w,m)$        | $\pm v \varepsilon \text{dc}(w+d,m)$ | $m^2$                      |
| $\lambda v^2$  | $\frac{2(1-m^2)}{2m^2-1}$ | $-m^2$ | $\text{nc}(w,m)$        | $\pm v \varepsilon \text{nc}(w+d,m)$ | $\frac{m^2-1}{m^2}$       |
| $\lambda v^2$  | $\frac{2(1-m^2)}{2-2m^2}$ | $1$    | $\text{sc}(w,m)$        | $\pm v \varepsilon \text{sc}(w+d,m)$ | $1-m^2$                    |
| $-\lambda v^2$ | $\frac{2}{1-2m^2}$ | $m^2$  | $\text{ns}(w,m)$        | $\pm v \varepsilon \text{ns}(w+d,m)$ | $m^2$                      |
| $\lambda v^2$  | $\frac{-2}{2m^2-1}$ | $-m^2(1-m^2)$ | $\text{ds}(w,m)$        | $\pm v \varepsilon \text{ds}(w+d,m)$ | $\frac{m^2-1}{m^2}$       |
| $\lambda v^2$  | $\frac{-2}{2-m^2}$ | $1-m^2$ | $\text{cs}(w,m)$        | $\pm v \varepsilon \text{cs}(w+d,m)$ | $1-m^2$                    |

### Solutions to $\phi^4$ in terms of trigonometric and hyperbolic functions

We have seen that the equation of motion for the field $\rho(w)$ is given by the differential equation

$$\frac{d^2\rho}{dw^2} + \frac{\lambda}{p^2} \rho (\rho^2 - v^2) = 0 \quad . \quad (C.1)$$

The general solution to this equation is expressed in terms of the twelve Jacobi elliptic functions depending on a parameter $m$. On the other hand, the Jacobi elliptic functions reduce to ordinary trigonometric or hyperbolic functions for the two special values $m = 0$ and $m = 1$ [29, 30, 31].

There are twelve different trigonometric and hyperbolic functions corresponding to the values $m = 0$ and $m = 1$. However, only five of them are solutions the above differential

---

8The other seven functions, for $m = 0$ and $m = 1$, lead either to $\rho(w) = 0$ or to a complex $\rho(w)$. For example, $\text{ds}(w+d,1) = \frac{1}{\sinh(w+d)}$ but $\varepsilon^2 = -2$ for $m = 1$, as can be seen from table 1. This leads to a complex scalar field $\rho(w)$.

---
The solutions depend on a parameter $\tau$ and a constant $\tau\alpha + \beta$. However, sometimes these solutions are found under different writings [37, 38, 39, 40]. For instance, the first solution (C.2) can be expressed as

$$\rho (w) = \pm v \tanh [\tau (w + \alpha) + \beta] = \pm v \tanh [\frac{\mu + \tanh [\tau (w + \alpha)]}{1 + \mu \tanh [\tau (w + \alpha)]}], \mu = \tanh (\beta). \quad (C.7)$$

Other expressions can be reached by simply expanding the arguments of the trigonometric and hyperbolic functions.

References

[1] H. B. Nielsen and P. Olesen, Vortex-line models for dual strings, Nucl. Phys. B 61 (1973) 45-61.

[2] Edward Witten, Superconducting strings, Nucl. Phys. B 249, Issue 4, (1985) 557-592.

[3] E. B. Bogomolny, The stability of classical solutions, Soviet Journal of Nuclear Physics, vol. 24, (1976) pp. 449–454.

[4] A. Vilenkin and E. P. S. Shellard, Cosmic strings and other topological defects, in Cambridge Monographs in Mathematical Physics, Cambridge University Press (2000).

[5] N. Manton and P. Sutcliffe, Topological solitons, Cambridge University Press (2004).

[6] J. Klačka, Metod Saniga and Ján Rybák, Numerical analysis of a static cylindrically symmetric Abelian Higgs sunspot, Contributions of the Astronomical Observatory Skalnate Pleso, vol. 22, p. 107-115 (1992).

[7] T. D. Lee and Y. Pang, Nontopological solitons, Phys. Rept. 221, (1992) 251-350.

[8] Ya. M. Shnir, Topological and non-topological solitons in scalar field theories, Cambridge University Press (2018).
[9] E. Ya. Nugaev and A. V. Shkerin, Review of nontopological solitons in theories with $U(1)$ symmetry, JETP 130, (2020) 301-320, arXiv:1905.05146 [hep-th].

[10] Eugen Radu and Mikhail S. Volkov, Stationary ring solitons in field theory-knots and vortons, Phys. Rept. 468 (2008) 101–151, arXiv:0804.1357 [hep-th].

[11] Yves Brihaye, Stefan Giller, Piotr Kosinski and Jutta Kunz, Sphalerons and normal modes in the (1+1)-dimensional Abelian Higgs model on the circle, Phys. Lett. B 293 (1992) 383-388.

[12] N. S. Manton and T. M. Samols, Sphalerons on a circle, Phys. Lett. B 207 (1988) 179.

[13] F. Canfora, A. Cisterna, D. Hidalgo and J. Oliva, Exact pp-waves, (A)dS waves and Kundt spaces in the Abelian-Higgs model, Phys. Rev. D 103 (2021) 8, 085007, arXiv:2102.05481 [hep-th].

[14] F. Canfora, Ordered arrays of Baryonic tubes in the Skyrme model in (3+1) dimensions at finite density, Eur. Phys. J. C 78 (2018) 11, 929, 1807.02090 [hep-th].

[15] L. Avilés, F. Canfora, N. Dimakis, and D. Hidalgo, Analytic topologically nontrivial solutions of the (3 + 1)-dimensional $U(1) \times U(1)$ gauged Skyrme model and extended duality, Phys. Rev. D 96 (2017) 12, 125005, 1711.07408 [hep-th].

[16] F. Canfora, M. Lagos, S. H. Oh, J. Oliva and A. Vera, Analytic (3 + 1)-dimensional gauged Skyrmions, Heun, and Whittaker-Hill equations and resurgence, Phys. Rev. D 98 (2018) 8, 085003, 1809.10386 [hep-th].

[17] F. Canfora, N. Dimakis, and A. Paliathanasis, Analytic Studies of Static and Transport Properties of (Gauged) Skyrmions, Eur. Phys. J. C 79 (2019) 2, 139, 1902.01563 [hep-th].

[18] F. Canfora, S. H. Oh and A. Vera, Analytic crystals of solitons in the four dimensional gauged non-linear sigma model, Eur. Phys. J. C 79 (2019) 6, 485, 1905.12818 [hep-th].

[19] F. Canfora, M. Lagos and A. Vera, Crystals of superconducting Baryonic tubes in the low energy limit of QCD at finite density, Eur. Phys. J. C 80 (2020) 8, 697, 2007.11543 [hep-th].

[20] F. Canfora, A. Giacomini, M. Lagos, S. H. Oh, A. Vera, Gravitating superconducting solitons in the (3 + 1)-dimensional Einstein gauged non-linear $\sigma$-model, Eur. Phys. J. C 81 (2021) 1, 55, 2001.11910 [hep-th].

[21] Y. Brihaye, Non-Abelian Plane Waves in the Higgs Model, Lett. Nuovo. Cim. 36 (1983) 275.

[22] F. K. Diakonos, G. C. Katsimiga, X. N. Maintas and C. E. Tsagkarakis, Symmetric solitonic excitations of the (1 + 1)-dimensional Abelian-Higgs classical vacuum, Phys. Rev. E 91 (2015) 2, 023202, 1404.1607 [hep-th].
[23] V. Achilleos, F. K. Diakonos, D. J. Frantzeskakis, G. C. Katsimiga, X. N. Maintas, E. Manousakis, C. E. Tsagkarakis and A. Tsapalis, Oscillons and oscillating kinks in the Abelian-Higgs model, Phys. Rev. D 88, 045015 (2013), arXiv:1306.3868 [hep-th].

[24] G. C. Katsimiga, F. K. Diakonos and X. N. Maintas, Classical dynamics of the Abelian Higgs model from the critical point and beyond, Phys. Lett. B 748 (2015) 117-124.

[25] J. S. Rozowsky, R. R. Volkas and K. C. Wali, Domain wall solutions with Abelian gauge fields, Phys. Lett. B 580 (2004) 249-256, arXiv:hep-th/0305232.

[26] Damien P. George and Raymond R. Volkas, Stability of domain walls coupled to Abelian gauge fields, Phys. Rev. D 72 (2005) 105011, arXiv:hep-ph/0508206.

[27] D. Bazeia, L. Losano, M. A. Marques and R. Menezes, Analytic vortex solutions in generalized models of the Maxwell-Higgs type, Phys. Lett. B 778 (2018) 22, arXiv:1801.01077 [hep-th].

[28] R. Casana, M. M. Ferreira Jr., E. da Hora and C. dos Santos, Analytical BPS Maxwell-Higgs vortices, Advances in High Energy Physics, vol. 2014, Article ID 210929, (2014), arXiv:1405.7920 [hep-th].

[29] Table of Integrals, Series, and Products, I. S. Gradshteyn and I. M. Ryzhik, (Alan Jeffrey, Editor), Academic Press, Fifth Edition (1994).

[30] Handbook of Mathematical Functions, Edited by M. Abramowitz and I. A. Stegun, Dover Publications, New York, Ninth Printing (1970).

[31] NIST Handbook of Mathematical Functions, Edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, Cambridge University Press, New York, First Published (2010).

[32] Galliano Valent, Heun functions versus elliptic functions, arXiv:math-ph/0512006v1, (2005).

[33] Georgios Pastras, Four Lectures on Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics, arXiv:math-ph/1706.07371v2, (2017).

[34] Khaled A. Gepreel, Explicit Jacobi elliptic exact solutions for nonlinear partial fractional differential equations, Advances in Difference Equations, Vol. 2014, 286-300, (2014).

[35] A. Daşcioğlu, S. Çulha Ünal and D. Varol Bayram, New Analytical Solutions for Space and Time Fractional Phi-4 Equation, NATURENGS, MTU Journal of Engineering and Natural Sciences 1:1 (2020) 30-46.

[36] Marco Frasca, Exact solutions of classical scalar field equations, Journal of Nonlinear Mathematical Physics, Vol. 1, No. 1 (2009) 1–7.

[37] A. Bekir, New Exact Travelling Wave Solutions for Regularized Long-wave, Phi-Four and Drinfeld-Sokolov Equations, International Journal of Nonlinear Science, 6(1), (2008) 46-52.
[38] Seyma Tuluce Demiray, Hasan Bulut, *Analytical solutions of Phi-four equation*, An International Journal of Optimization and Control: Theories & Applications, Vol.7, No.3, (2017) 275-280.

[39] A. M. Wazwaz, *A sine-cosine method for handling nonlinear wave equations*, Mathematical and Computer Modelling, 40, (2004) 499-508.

[40] Berat Karaagac, Selcuk Kutluay, Nuri Murat Yagmurlu and Alaattin Esen, *Exact solutions of nonlinear evolution equations using the extended modified Exp(-Ω(ξ)) function method*, Tbilisi Mathematical Journal 12(3), (2019) 109–119.

[41] Jiu-Qing Liang, H. J. W. Müller-Kirsten and D. H. Tchrakian, *Solitons, bounces and sphalerons on a circle*, Phys. Lett. B 282 (1992) 105-110.