Möbius transform, moment-angle complexes and Halperin-Carlsson conjecture

— A joint work with Xiangyu Cao

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December 12, 2009
§1 Background—A triangle

Combinatorics
—Abstract simplicial complexes

Algebra
—Stanley-Reisner face rings

Topology
—Moment-angle complexes

Combinatorics
—Abstract simplicial complexes

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References

For the edge $a$, see
[1] Stanley, Richard P, *Combinatorics and commutative algebra. Second edition*, Progress in Mathematics, **41**, Birkhäuser Boston, Inc., Boston, MA, 1996.
[2] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Math. **227**, Springer, 2005.

For other two edges $b, c$, see
[3] V. M. Buchstaber and T. E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, Vol. **24**, Amer. Math. Soc., Providence, RI, 2002.
References

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  [3] V. M. Buchstaber and T. E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, Vol. 24, Amer. Math. Soc., Providence, RI, 2002.
Let \([m] = \{1, \ldots, m\}\).

**Abstract simplicial complexes on \([m]\)**

- **An abstract simplicial complex** \(K\) **on** \([m]\) **is a collection of some subsets in** \([m]\) **such that for each** \(a \in K\), **any subset (including** \(\emptyset\) **) of** \(a\) **belongs to** \(K\).

- Each \(a\) **in** \(K\) **is called a simplex of** \(\dim = |a| - 1\), **and** \(\dim K = \max_{a \in K} \{\dim a\}\).
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Möbius transform, moment-angle complexes and Halperin-Carlsson conjecture
Notion—Stanley-Reisner face ring

$K$: an abstract simplicial complex on $[m]$  
$k$: a field.

**Stanley-Reisner face ring**

$$k(K) = k[v_1, \ldots, v_m]/I_K$$

is called the **Stanley-Reisner face ring** of $K$, and $I_K$ is the ideal generated by all square-free monomials $v_{i_1} \cdots v_{i_s}$ with $\sigma = \{i_1, \ldots, i_s\} \notin K$.

RK: write $k[v] = k[v_1, \ldots, v_m]$. 

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§1 Background—A triangle  
§2 Notions and known results  
§3 Further development  
§4 Application to Halperin-Carlsson conjecture  

Notion—Abstract simplicial complex  
§2.1 Notion—Abstract simplicial complex  
§2.1 Notion—Stanley-Reisner face ring and Tor-algebra  
§2.1 Notion—Moment-angle complex  
§2.2 Known result on edge a—Hochster Theorem  
§2.2 Known result on edge c—Buchstaber-Panov Theorem

Notion—Stanley-Reisner face ring: $K$: an abstract simplicial complex on $[m]$, $k$: a field.

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Notion—Betti numbers of Stanley-Reisner face ring $k(K)$

It is well-known that $k(K)$ is a finitely generated $\mathbb{N}^m$-graded $k[v]$-module and it has an minimal free resolution

$$0 \leftarrow k(K) \leftarrow F_0 \leftarrow^\phi_1 F_1 \leftarrow \cdots \leftarrow F_{h-1} \leftarrow^\phi_h F_h \leftarrow 0 \quad (1)$$

Write $F_i = \bigoplus_{a \in \mathbb{N}^m} \left( k[v](-a) \oplus \cdots \oplus k[v](-a) \right)$ where $k[v](-a)$ is the ideal $\langle v^a \rangle$, and $v^a = v_1^{a_1} \cdots v_m^{a_m}$ for $a = (a_1, \ldots, a_m) \in \mathbb{N}^m$.

**Betti number**

$\beta_{i,a}^{k(K)} \in \mathbb{N}$ is called the $(i, a)$-th Betti number of $k(K)$. 
Notion—Tor-algebra of Stanley-Reisner face ring $k(K)$

Applying the functor $\otimes_{k[v]} k$ to the sequence (1) above, one may obtain the following chain complex of $\mathbb{N}^m$-graded $k[v]$-modules:

$$
0 \leftarrow F_0 \otimes_{k[v]} k \leftarrow F_1 \otimes_{k[v]} k \leftarrow \cdots \leftarrow F_h \otimes_{k[v]} k \leftarrow 0.
$$

Define $\text{Tor}^i_{k[v]}(k(K), k) := \frac{\ker \phi_i}{\text{im} \phi_{i+1}} = F_i \otimes_{k[v]} k$ so

$$
\dim_k \text{Tor}^i_{k[v]}(k(K), k) = \text{rank} F_i = \sum_{a \in \mathbb{N}^m} \beta^i_{k(K)}.
$$

Tor-algebra

$$
\text{Tor}^i_{k[v]}(k(K), k) = \bigoplus_{i=0}^h \text{Tor}^i_{k[v]}(k(K), k) = \bigoplus_{i \in [0, h] \cap \mathbb{N}} \bigoplus_{a \in \mathbb{N}^m} \text{Tor}^i_{k[v]}(k(K), k)_a
$$
It is well-known that if $\mathbf{a} \in \mathbb{N}^m$ is not a vector in $\{0,1\}^m$, then $\text{Tor}_{i}^{k[v]}(k(K), k)_{\mathbf{a}} = 0$, so $\beta_{i,\mathbf{a}}^{k(K)} = 0$.

$$\{0,1\}^m \leftrightarrow 2^m$$

write

$$\beta_{i,\mathbf{a}}^{k(K)} := \beta_{i,\mathbf{a}}^{k(K)}$$

where $2^m \ni \mathbf{a} \leftrightarrow \mathbf{a} \in \{0,1\}^m$. 

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Möbius transform, moment-angle complexes and Halperin-Carlsson conjecture — A joint work with Xiangyu Cao
A general construction

$K$: a simplicial complex on vertex set $[m] = \{1, \ldots, m\}$

$(X, W)$: a pair of top. spaces with $W \subset X$.

$$K(X, W) := \bigcup_{\sigma \in K} \left( \prod_{i \in \sigma} X \times \prod_{i \notin \sigma} W \right) \subseteq X^m.$$ 

- $\mathcal{Z}_K := K(D^2, S^1) \subset (D^2)^m$ is called the \textit{moment-angle complex} on $K$.

- $\mathbb{R}\mathcal{Z}_K := K(D^1, S^0) \subset (D^1)^m$ is called the \textit{real moment-angle complex} on $K$. 
A general construction

\[ K : \text{a simplicial complex on vertex set } [m] = \{1, \ldots, m\} \]
\[(X, W) : \text{a pair of top. spaces with } W \subset X. \]

\[ K(X, W) := \bigcup_{\sigma \in K} \left( \prod_{i \in \sigma} X \times \prod_{i \notin \sigma} W \right) \subseteq X^m. \]

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- \( \mathbb{R}\mathcal{Z}_K := K(D^1, S^0) \subset (D^1)^m \) is called the \textit{real moment-angle complex} on \( K \).
A canonical action on $\mathbb{Z}_K$

$D^2 = \{ z \in \mathbb{C} | |z| \leq 1 \}$ and $S^1 = \partial D^2$.
Since $(D^2)^m \subset \mathbb{C}^m$ is invariant under the standard action of $T^m$ on $\mathbb{C}^m$ given by

$((g_1, \ldots, g_m), (z_1, \ldots, z_m)) \mapsto (g_1z_1, \ldots, g_mz_m),$

$(D^2)^m$ admits a natural $T^m$-action whose orbit space is the unit cube $I^m \subset \mathbb{R}^m_{\geq 0}$. The action $T^m \curvearrowright (D^2)^m$ then induces a canonical $T^m$-action $\Phi$ on $\mathbb{Z}_K$.

Similarly

A canonical action on $\mathbb{R}\mathbb{Z}_K$

$\mathbb{R}\mathbb{Z}_K$ admits a canonical $(\mathbb{Z}_2)^m$-action $\Phi_\mathbb{R}$ on $\mathbb{R}\mathbb{Z}_K$
Hochster Theorem

On the edge $a$ of the triangle, there is the following essential result:

\[ \tilde{H}^{|a|-i-1}(K_a; k) \cong \text{Tor}_i^{k[v]}(k(K), k)_a \]

where $K_a = \{ \sigma \in K \mid \sigma \subseteq a \}$. 
On the edge $c$ of the triangle, there is the following essential result:

**Buchstaber-Panov Theorem**

As $k$-algebras,

$$H^*(Z_K; k) \cong \text{Tor}^k[v](k(K), k)$$

where $k(K) = k[v]/I_K = k[v_1, \ldots, v_m]/I_K$ with $\deg v_i = 2$, and $k$ is a field.
Further development—A viewpoint of analysis

- Let $2^m \ast = \{ f \mid f : 2^m \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \}$. $2^m \ast$ forms an algebra over $\mathbb{Z}/2\mathbb{Z}$ in the usual way, and it has a natural basis $\{ \delta_a \mid a \in 2^m \}$ where $\delta_a$ is defined as follows:
  \[ \delta_a(b) = 1 \iff b = a. \]
- Given a $f \in 2^m \ast$, set $\text{supp}(f) := f^{-1}(1)$.
- $f$ is said to be nice if $\text{supp}(f)$ is an abstract simplicial complex.

A one-one correspondence

$\{\text{all nice functions in } 2^m \ast\} \longleftrightarrow \{ \text{all abst. sim. subcpxes in } 2^m \}$. 
Further development—An algebra-combinatorics formula

Möbius transform

On $2^{[m]}^*$, define a $\mathbb{Z}/2\mathbb{Z}$-valued Möbius transform

$$\mathcal{M} : 2^{[m]}^* \rightarrow 2^{[m]}^*$$

by the following way: for any $f \in 2^{[m]}^*$ and $a \in 2^{[m]}$,

$$\mathcal{M}(f)(a) = \sum_{b \subseteq a} f(b)$$
Further development—An algebra-combinatorics formula

The following result indicates an essential relationship between $\mathcal{M}(f)$ and the Betti numbers of $k(K_f)$.

**Algebra–combinatorics formula (Cao-Lü)**

Suppose that $f \in 2^{[m]}^*$ is nice such that $K_f = \text{supp}(f)$ is an abstract simplicial complex on $[m]$. Then

$$\mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{k(K_f)} \delta_a$$

where $h$ denotes the length of the minimal free resolution of $k(K_f)$, and $\beta_{i,a}^{k(K_f)}$'s denote the Betti numbers of $k(K_f)$. 
An algebra-combinatorics formula

Corollary

\[ |\text{supp}(\mathcal{M}(f))| \leq \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{k(K_f)}. \]

Proof.

\[ \mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{k(K_f)} \delta_a = \sum_{a \in 2^{[m]}} \left( \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \right) \delta_a \]

\[ \implies \text{for any } a \in \text{supp}(\mathcal{M}(f)), \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \text{ must be odd so} \]

\[ \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \geq 1. \]

Therefore

\[ \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{k(K_f)} \geq \sum_{a \in \text{supp}(\mathcal{M}(f))} \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \geq \sum_{a \in \text{supp}(\mathcal{M}(f))} 1 = |\text{supp}(\mathcal{M}(f))|. \]
Generalized moment-angle complex

Given an abstract simplicial complex $K$ on $[m]$, let $(X, W) = \{(X_i, W_i)\}_{i=1}^m$ be $m$ pairs of CW-complexes with $W_i \subset X_i$. Then the generalized moment-angle complex is defined as follows:

$$K(X, W) = \bigcup_{\sigma \in K} B_\sigma(X, W) \subset \prod_{i=1}^m X_i$$

where $B_\sigma(X, W) = \prod_{i=1}^m H_i$ and $H_i = \begin{cases} X_i & \text{if } i \in \sigma \\ W_i & \text{if } i \in [m] \setminus \sigma. \end{cases}$
A class of generalized moment-angle complexes

Take \((X, W) = (\mathbb{D}, S) = \{(\mathbb{D}_i, S_i)\}_{i=1}^m\) with each CW-complex pair \((\mathbb{D}_i, S_i)\) subject to the following conditions:

1. \(\mathbb{D}_i\) is acyclic, that is, \(\tilde{H}_j(\mathbb{D}_i) = 0\) for any \(j\).
2. There exists a unique \(\kappa_i\) such that \(\tilde{H}_{\kappa_i}(S_i) = \mathbb{Z}\) and \(\tilde{H}_j(S_i) = 0\) for any \(j \neq \kappa_i\).

Then our objective is to calculate the cohomology of

\[
\mathcal{Z}^{(\mathbb{D}, S)}_K := K(\mathbb{D}, S) = \bigcup_{\sigma \in K} B_\sigma(\mathbb{D}, S) \subset \prod_{i=1}^m \mathbb{D}_i.
\]
Further development—Cohomology of a class of generalized moment-angle complexes

Theorem (Cao-Lü)

As graded $\mathbf{k}$-modules,

$$H^*(\mathcal{Z}_K^{(\mathbb{D},S)}; \mathbf{k}) \cong \text{Tor}^{\mathbf{k}[v]}(\mathbf{k}(K), \mathbf{k}).$$

Corollary

$$\sum_i \dim_\mathbf{k} H^i(\mathcal{Z}_K^{(\mathbb{D},S)}; \mathbf{k}) = \sum_{i=0}^{h} \sum_{a \in 2^m} \beta_{i,a}^\mathbf{k}(K).$$
Halperin-Carlsson conjecture

If a finite-dimensional paracompact Hausdorff space $X$ admits a free action of a torus $T^r$ (resp. a $p$-torus $(\mathbb{Z}_p)^r$, $p$ prime) of rank $r$, then the total dimension of its cohomology,

$$\sum_i \dim_k H^i(X; k) \geq 2^r$$

where $k$ is a field of characteristic 0 (resp. $p$).
Remark

- Historically, the above conjecture in the $p$-torus case originates from the work of P. A. Smith in 1950s.
- For the case of a $p$-torus $(\mathbb{Z}_p)^r$ freely acting on a finite CW-complex homotopic to $(S^n)^k$ suggested by P. E. Conner, the problem has made an essential progress.
- In the general case, the inequality was conjectured by S. Halperin for the torus case, and by G. Carlsson for the $p$-torus case.
- So far, the conjecture holds if $r \leq 3$ in the torus and 2-torus cases and if $r \leq 2$ in the odd $p$-torus case. Also, many authors have given contributions to the conjecture in many different aspects.
Recall that

$$\sum_i \dim_k H^i(\mathcal{Z}_{Kf}^{(D,S)}; k) = \sum_{i=0}^h \sum_{a \in 2^m} \beta^k(K_f) \geq |\text{supp}(\mathcal{M}(f))|.$$ 

We can upbuild a method of compressing $\text{supp}(f)$ to get the desired lower bound of $|\text{supp}(\mathcal{M}(f))|$.

**Theorem (Cao-Lü)**

For any nice $f \in 2^m^*$, there exists some $a \in \text{supp}(f)$ such that

$$|\text{supp}(\mathcal{M}(f))| \geq 2^m - |a|.$$
Application to free actions

**Theorem (Cao-Lü)**

Let $H$ (resp. $H_\mathbb{R}$) be a rank $r$ subtorus of $T^m$ (resp. $(\mathbb{Z}_2)^m$). If $H$ (resp. $H_\mathbb{R}$) can act freely on $\mathbb{Z}_K$ (resp. $\mathbb{R}\mathbb{Z}_K$), then

$$\sum_i \dim_k H^i(\mathbb{Z}_K; k) = \sum_i \dim_k H^i(\mathbb{R}\mathbb{Z}_K; k) \geq 2^r.$$ 

**Remark**

The action of $H$ (resp. $H_\mathbb{R}$) on $\mathbb{Z}_K$ (resp. $\mathbb{R}\mathbb{Z}_K$) is naturally regarded as the restriction of the $T^m$-action $\Phi$ to $H$ (resp. the $(\mathbb{Z}_2)^m$-action $\Phi_\mathbb{R}$ to $H_\mathbb{R}$).
Application to free actions

Corollary

The Halperin–Carlsson conjecture holds for $\mathbb{Z}_K$ (resp. $\mathbb{R}\mathbb{Z}_K$) under the restriction of the $T^m$-action $\Phi$ (resp. the $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$).

Remark

Using a different method, Yury Ustinovsky has also recently proved the Halperin’s toral rank conjecture for the moment-angle complexes with the restriction of natural tori actions, see arXiv:0909.1053.