ON MAXIMALLY OSCILLATING PERFECT SPLINES
AND SOME OF THEIR EXTREMAL PROPERTIES

OLEG KOVALENKO

Abstract. In this paper we study analogues of the perfect splines
for weighted Sobolev classes of functions defined on the half-line.
Maximally oscillating splines play important role in the solution of
some extremal problems. In particular, using these splines, we
characterize the modulus of continuity of the differential operator.

1. Introduction

Let $I$ be a finite interval or the positive half-line $\mathbb{R}_+$. Let $X$ be a
normed space of real-valued functions defined on $I$ and $f_{\pm} \in C(I)$ be
positive functions. For $x \in X$ set

$$
\|x\|_{X, f_{\pm}}, f_{\pm} := \max \{x(\cdot), 0\}_{f_{\pm}} + \max \{-x(\cdot), 0\}_{f_{\pm}}.
$$

For positive functions $f_{\pm}, g \in C(I)$ and natural $r$ set

$$
W_r^{f_{\pm}, f_{\pm}, g}(I) := \{x \in C(I) : x^{(r-1)} \in AC_{\text{loc}}, \|x\|_{C(I), f_{\pm}, f_{\pm}} < \infty,
\|x^{(r)}\|_{L_{\infty}(I), g} \leq 1\}.
$$

If $f_{-} = f_{+}$, then we write $W_r^{f, g}(I)$ instead of $W_r^{f_{\pm}, f_{\pm}, g}(I)$. In the case,
when $f \equiv 1$ we write $W_r^{\infty, g}(I)$ instead of $W_r^{f_{\pm}, g}(I)$.

For $k = 1, \ldots, r-1$, we call the function

$$
\omega(W_{f_{\pm}, f_{\pm}, g}(I), D^k, \delta) := \sup_{x \in W_{f_{\pm}, f_{\pm}, g}(I), \|x\|_{C(I), f_{\pm}, f_{\pm}} \leq \delta} \|x^{(k)}\|_{C(I)}, \delta \geq 0
$$

a modulus of continuity of $k$-th order differentiation operator on the
class $W_{f_{\pm}, f_{\pm}, g}(I)$.

The study of the function $\omega$ is closely related to the sharp Landau–
Kolmogorov type inequalities. The first results in this topic were ob-
tained in the 1910s by Landau [6], Hadamard [3], Hardy and Little-
wood [11]. Since then the topic was intensively studied. For a more
detailed overview of the history of the question see [11] and references
therein.

Let $I$ denote a segment or a half-line. Suppose that two functions
$\Phi, \varphi \in C(I)$ such that $\Phi(t) > \varphi(t)$ for all $t \in I$ are given.
Definition 1. We say that a function $x \in C(I)$ has $n \in \mathbb{N}$ points of oscillation between the functions $\phi$ and $\Phi$, if $\phi(t) \leq x(t) \leq \Phi(t)$ for all $t \in I$ and there exist points $s_k \in I$, $k = 1, \ldots, n$, $s_1 < s_2 < \ldots < s_n$, such that for $k = 1, \ldots, n$

\begin{equation}
 x(s_k) = \begin{cases} 
 \Phi(s_k), & k \text{ is odd,} \\
 \phi(s_k), & k \text{ is even.}
 \end{cases}
\end{equation}

For the case, when $I = [0, \infty)$ and the weights $f_\pm$ and $g$ are constants, the information about the functional $\omega$ follows from the results of Schoenberg and Cavaretta [8]. We sketch their approach to build the extremal for (1) functions.

One can prove, that for each $A > 0$ there exists $\delta = \delta(A, r, n) > 0$ and a perfect spline $G = G_{r,n,A}$ of order $r \in \mathbb{N}$ with $n$ knots, defined on the segment $[0, A]$ that has $n + r + 1$ points of oscillation between the constant functions $-\delta$ and $\delta$. For fixed $r$ and $n$, $\delta$ is a continuous increasing function of $A$, $\delta \to 0$ as $A \to +0$ and

\begin{equation}
 \delta \to \infty, \text{ as } A \to \infty.
\end{equation}

Moreover, for fixed $r$ and $A$, $\delta$ is a decreasing sequence (numbered by the parameter $n$) and $\delta \to 0$ as $n \to \infty$.

For each $\delta > 0$, these properties allow to define a sequence of numbers $A_n$ such that corresponding perfect spline $G_n = G_{r,n,A_n}$ has $n$ knots and oscillates between $-\delta$ and $\delta$ for all $n \in \mathbb{N}$. Then $A_n \to \infty$ as $n \to \infty$ and the sequence $G_n$ generates a limiting perfect spline $G_\delta$ defined on the whole half-line with infinite number of knots and oscillation points between $-\delta$ and $\delta$. The spline $G_\delta$ is extremal in problem (1).

In the case, when $g$ is non-constant, the natural substitution for the polynomial splines are so-called $g$-splines.

Let $I = (a, b)$, where $a \in \mathbb{R}$ and $b$ denotes either a real number or the positive infinity. Let a positive function $g \in C(I)$ be given.

Definition 2. The function $G \in C^{r-1}(I)$ is called a perfect $g$-spline of the order $r$ with $n \in \mathbb{N}$ knots $a < t_1 < \ldots < t_n < b$, if on each of the intervals $(t_i, t_{i+1}), i = 0, 1, \ldots, n$, $t_0 := a, t_{n+1} := b$, there exists derivative $G^{(r)}$ and $\frac{G^{(r)}(t)}{g(t)} \equiv \epsilon \cdot (-1)^i$ on the intervals $(t_i, t_{i+1}), i = 0, 1, \ldots, n$, where $\epsilon \in \{1, -1\}$.

One can repeat the steps described above in the general case of rather arbitrary (we will specify the restrictions for them below) functions $g$ and $f_\pm$, substituting the polynomial perfect splines that oscillate between constant functions, by perfect $g$-splines that oscillate between the functions $-\delta_A \cdot f_- $ and $\delta_A \cdot f_+$, see [1], where the symmetric case, when $f_- = f_+$, was considered.

An important difference between the cases of constant and non-constant functions $f_\pm$ and $g$ is in the fact, that (3) may not hold in the
latter case. More precisely, in the case \( f_- = f_+ \), the following result was proved in \([1]\).

Set \( g_0 := g \) and \( g_k(t) := \int_0^t g_{k-1}(s) \, ds, \ t \geq 0, \ k = 1, 2, \ldots, r. \)

**Theorem 1.** Let \( r \in \mathbb{N} \) and \( f, g \in C[0, \infty) \) be non-increasing positive functions. Relation (3) does not hold if and only if

\[
A_0 := \int_0^\infty g(t) \, dt < \infty,
\]

for \( k = 1, \ldots, r - 1 \)

\[
A_k := \int_0^\infty \left[ \sum_{s=0}^{k-1} \frac{(-1)^{k-s-1} A_s t^{k-s-1}}{(k-s-1)!} + (-1)^k g_k(t) \right] \, dt < \infty,
\]

and

\[
\sup_{t \in [0, \infty)} \frac{|P_r(t)|}{f(t)} < \infty,
\]

where

\[
P_r(t) := (-1)^r \sum_{s=0}^{r-1} \frac{(-1)^{r-s-1} A_s t^{r-s-1}}{(r-s-1)!} + g_r(t).
\]

In \([1]\), the case \( f_- = f_+ \) was considered, and the following results were obtained.

In the case, when (3) holds, using Schoenberg and Cavaretta’s approach, the values \( \omega(W_{f,g}^r[0, \infty), D^k, \delta) \) were characterized in terms of the limit \( g \)-splines with infinite number of knots and points of oscillation for all \( \delta > 0 \).

In the case, when (3) does not hold, the values of the functional \( \omega(W_{f,g}^r[0, \infty), D^k, \delta) \) were characterized only for a non-increasing sequence of numbers \( \{\delta_{r,n}\}_{n=0}^\infty \). Moreover, this sequence may not tend to 0 as \( n \to \infty \).

In this article we characterize the values of \( \omega(W_{f_-,f_+,g}^r[0, \infty), D^k, \delta) \) for all \( \delta > 0 \) in the case when (3) does not hold. However, we impose stricter restrictions on the functions \( f_\pm \) and \( g \), then those described in Theorem \([1]\). Namely, we characterize \( \omega(W_{f_-,f_+,g}^r[0, \infty), D^k, \delta) \) under the following three assumptions.

**Assumption 1.** The function \( g \in C[0, \infty) \) is positive non-increasing and such that conditions (4) and (5) hold.

**Assumption 2.** The functions \( f_\pm \in C[0, \infty) \) are non-increasing positive and such that

\[
f_\pm(\infty) > 0
\]
Assumption 3.

\[
\lim_{{t \to \infty}} \frac{f_\pm(t) - f_\pm(\infty)}{|P_r(t)|} = 0,
\]

where the function \(P_r\) is defined in (7).

Note, that Assumption 1 is the same as in Theorem 1, but the assumption on the functions \(f_\pm\) are stricter than the ones in Theorem 1; in particular Assumptions 1 and 2 imply (6) in the case \(f_\pm = f_\pm\). However, the important case, when \(f_\pm\) are constant, is included.

According to Definition 1, the points of oscillation are 'positively orientated', that is \(x(s_1) = \Phi(s_1)\). We can define 'negatively orientated' points of oscillation, substituting condition (2) by

\[
x(s_k) = \begin{cases} 
\Phi(s_k), & k \text{ is even,} \\
\varphi(s_k), & k \text{ is odd.}
\end{cases}
\]

Remark 1. Let a function \(x_+ = x_+(\varphi, \Phi)\) have \(n\) positively orientated points of oscillation between the functions \(\varphi\) and \(\Phi\). Then the function \(x_- = -x_+(-\Phi, -\varphi)\) has \(n\) negatively orientated points of oscillation between the functions \(\varphi\) and \(\Phi\).

In particular, if \(\Phi\) is positive and \(\varphi \equiv -\Phi\), then \(x_- = -x_+\). However, there is no such relation in general situation.

Definition 3. We call a function \(x \in C([0, \infty)) n\)-piecewise monotone, \(n \in \mathbb{N}\), if there exists \(\epsilon \in \{1, -1\}\) and \(0 = t_\epsilon < t_\epsilon < \ldots < t_n = \infty\), so that \(\epsilon \cdot (-1)^k x\) is increasing on the interval \((t_{k-1}, t_k)\), \(k = 1, \ldots, n\).

For a locally absolutely continuous function \(x\) defined on an interval, the notations \(\text{sgn} \ x'(t) = 1 \ (\text{sgn} \ x'(t) = -1)\) will mean that the function \(x\) is increasing (decreasing) in some neighborhood of the point \(t\).

Definition 4. A primitive \(G\) of the order \(r\) of the function \(g\) or \(-g\) on \(I\) will be called a perfect \(g\)-spline of the order \(r\) with \(0\) knots.

Definition 5. Denote by \(\Gamma^r_{n,g}(I)\) the set of all perfect \(g\)-splines \(G\) defined on \(I\) of the order \(r\) with not more than \(n \in \mathbb{Z}_+\) knots.

To obtain our main result, we study some properties of the maximally oscillating perfect \(g\)-splines. We prove the following result, which also has an independent interest. In particular, statement (c) of the theorem states, that maximally oscillating splines are least deviating from zero in non-symmetric weighted norm among the \(g\)-splines of the class \(\Gamma^r_{n,g}(I)\). Such property for polynomial splines is well known.

Theorem 2. Let Assumptions 1 2 and 3 hold.

(a) There exist two non-increasing sequences \(\{a_\pm n\}_{n=1}^\infty\) of positive numbers such that \(\lim_{{n \to \infty}} a_\pm n = 0\), and for each \(\tau \in [a_\pm n, a_\pm n], n \in \mathbb{N}\).
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\[ Z_+, a_0^\pm := \infty, \] there exists a perfect \( g \)-spline \( G_\tau^\pm \in \Gamma_{r,n,g}[0, \infty) \) with exactly \( n \) knots and \( n + 1 \) (positively oriented for \( G_\tau^+ \) and negatively oriented for \( G_\tau^- \)) points of oscillation between \(-\tau f_-\) and \( \tau f_+\).

(b) For all \( \tau \in [a_{n+1}^+, a_n^-) \) and \( s = 0, 1, \ldots, r - 1 \), \( (G_\tau^\pm)^{(s)} \) is \((n + 1)\)-piecewise monotone. Moreover, for \( s = 0, \ldots, r \),

\[ \text{sgn} (G_\tau^\pm)^{(s)}(0) = \pm (-1)^s. \]

(c) For each \( n \in \mathbb{Z}_+ \)

\[ \inf_{G \in \Gamma_{r,n,g}(0, \infty)} \|G\|_{C[0, \infty), f_- f_+} = \min \{a_{n+1}^+, a_{n+1}^-\}. \]

If \( a_{n+1}^+ < a_{n+1}^- \), then the infimum is attained on \( G_{a_{n+1}^+}^+ \), otherwise it is attained on \( G_{a_{n+1}^-}^- \).

Remark 2. It will follow from the proof, that the \( g \)-splines \( G_{a_{n+1}^+}^\pm \) satisfy

\[ G_{a_{n+1}^+}^+(\infty) = \begin{cases} -a_{n+1}^+ f_-(\infty), & \text{if } n \text{ is even,} \\ a_{n+1}^+ f_+(\infty), & \text{if } n \text{ is odd,} \end{cases} \]

and

\[ G_{a_{n+1}^-}^- (\infty) = \begin{cases} -a_{n+1}^- f_-(\infty), & \text{if } n \text{ is odd,} \\ a_{n+1}^- f_+(\infty), & \text{if } n \text{ is even.} \end{cases} \]

These \( g \)-splines can be viewed as the analogues of the perfect Chebyshev splines, having an 'oscillation point at \( \infty \}'. Here and below \( f(\infty) \) denotes \( \lim_{t \to +\infty} f(t) \).

In the theorem above, the perfect \( g \)-splines are indexed by their non-symmetric weighted norms. The same splines admit indexation based on their values at infinity. We formulate the result for the splines with positively oriented oscillation points; a similar results holds in the case of negatively oriented oscillation points.

Theorem 3. Let Assumptions 1 and 2 hold and \( n \in \mathbb{Z}_+ \) be given. For all

\[ \alpha \in \begin{cases} [0, \infty), & \text{if } n \text{ is odd,} \\ (0, \infty], & \text{if } n \text{ is even}, \end{cases} \]

there exists \( \tau = \tau_n(\alpha) > 0 \) and a perfect \( g \)-spline \( G = G(n, \alpha) \in \Gamma_{r,n,g}[0, \infty) \) with exactly \( n \) knots that has \( n + 1 \) positively oriented points of oscillation between \(-\tau f_-\) and \( \tau f_+\) and such that

\[ \frac{\tau f_+(\infty) - G(\infty)}{\tau f_-(\infty) + G(\infty)} = \alpha, \]

where \( [13] \) is understood as \( G(\infty) = -\tau f_-(\infty) \) if \( \alpha = \infty \). For \( n \in \mathbb{Z}_+ \), the function \( \tau_n \) is continuous, non-decreasing in the case of odd \( n \), and
is non-increasing in the case of even $n$. For all $n \in \mathbb{N}$ and $\alpha, \beta > 0$ one has $\tau_n(\alpha) \leq \tau_{n-1}(\beta)$.

Remark 3. We do not prove the uniqueness of the maximally oscillating perfect $g$-splines in Theorem 2 and 3. However, the correctness of the definition of the functions $\tau_n$ in Theorem 3 will be proved below.

Remark 4. Note, that Assumption 3 is needed in Theorem 2, but not needed in Theorem 3. It guarantees that there is no ‘gap’ between the possible values of the non-symmetric weighted norms of the perfect $g$-splines with $n − 1$ and $n$ knots, $n \in \mathbb{N}$. 

The main result of this article is the following theorem.

Theorem 4. Let $r \in \mathbb{N}$, $r \geq 2$ and Assumptions 1, 2, and 3 hold. Then for all $k = 1, \ldots, r − 1$ and $\delta > 0,$

$$\omega(W^r_{f−,f+,[0,\infty)}, D^k, \delta) = \max\{\|(G^+)(k)(0)\|, \|(G^-)(k)(0)\|\},$$

where $G^\pm_\delta$ are perfect $g$-splines from Theorem 2. If

$$\|(G^+)(k)(0)\| > \|(G^-)(k)(0)\|,$$

then the supremum in the definition of $\omega(W^r_{f−,f+,[0,\infty)}, D^k, \delta)$ is attained on $G^+_\delta$, otherwise it is attained on $G^-_\delta$.

Remark 5. Note, that if $\delta \geq \min\{a_1^+, a_1^−\}$, then $G^\pm_\delta$ have 0 knots, and hence $G^\pm_\delta = \pm\|P_r\| + C^\pm_\delta \in \mathbb{R}$. This implies that the functional $\omega(W^r_{f−,f+,[0,\infty)}, D^k, \cdot)$ becomes constant for all big enough values of $\delta$.

The article is organized as follows. In Section 2 we prove the existence of maximally oscillating $g$-splines on the finite segments and the half-line. In Section 3 we study the non-symmetrical weighted norms of the maximally oscillating $g$-splines. Section 4 is devoted to the proofs of the main results.

2. Perfect g-splines

2.1. Auxiliary definitions. In this article, we will often count or estimate the number of zeros and sign changes of the functions. Below we adduce necessary definitions.

Let $x$ be a continuous on an interval function and $t_1 < \ldots < t_n$ be its zeros, $n \geq 2$. The points $t_1, \ldots, t_n$ are called separated zeros, if the function $x$ is not identical zero on each of the intervals $(t_i, t_{i+1})$, $i = 1, \ldots, n − 1$.

We say that a function $x$ has essential sign change on the interval $I$, if both sets \{t \in I: x(t) > 0\} and \{t \in I: x(t) < 0\} have positive measures.

We say that a function $x$ has essential change of sign at the point $z$, if there exists $\varepsilon > 0$ such that for almost all $0 < u < \varepsilon$, $\text{sgn} x(z − u) = −\text{sgn} x(z + u) \neq 0$. 


We say that a function $x$ has exactly $n$ essential sign changes on the interval $I$, if there exist $n+1$ points $t_1 < \ldots < t_{n+1}$ from this interval and a number $\varepsilon > 0$ such that for almost all $u \in (-\varepsilon, \varepsilon)$ one has $\text{sgn} \, x(t_i + u) = -\text{sgn} \, x(t_{i+1} + u) \neq 0$, $i = 1, \ldots, n$, and there exist no system of $n+2$ points with such property.

For continuous functions we use the term 'sign change' instead of 'essential sign change'.

We will often use the following analogue of the Rolle theorem. Between two separated zeros of a locally absolutely continuous function $x$, there is an essential change of sign of the function $x'$.

2.2. **Perfect $g$-splines with given zeros.** We need the following topological result.

**Theorem 5** (Borsuk [2]). Let $S^n = \{ \xi \in \mathbb{R}^{n+1}: \|x\| = r \}$, where $\| \cdot \|$ is some norm in $\mathbb{R}^{n+1}$, and $\phi: S^n \to \mathbb{R}^n$ be a continuous odd function. Then there exists a point $\xi^* \in S^n$ such that $\phi(\xi^*) = 0$.

According to Velikin [9], the idea to use the Borsuk theorem in the proof of existence of perfect splines with given zeros, belongs to Ruban.

**Lemma 1.** Let $A > 0$, $n \in \mathbb{N}$ and $0 \leq s_1 < s_2 < \ldots < s_n < A$. There exists a perfect $g$-spline $G \in \Gamma_{n,g}[0,A]$ such that

$$G^{(k)}(A) = 0, \ k = 0,1,\ldots, r - 1.$$  \hspace{1cm} (14)

and

$$G(s_k) = 0, k = 1,\ldots,n.$$ \hspace{1cm} (15)

Moreover, conditions (14) and (15) imply that $G$ changes sign in each of the points $s_k$, $k = 1,\ldots,n$, and has exactly $n$ knots.

Consider the sphere $S^n := \left\{ (\xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1}: \sum_{k=1}^{n+1} |\xi_k| = A \right\}$. For arbitrary $\xi \in S^n$ consider the partition of the segment $[0,A]$ by the points $t_m = \sum_{k=1}^{m} |\xi_k|$, $m = 1,\ldots,n$. Let $G(\xi) \in C^{r-1}[0,A]$ be the function that satisfies boundary conditions (14) and such that $G^{(r)}(\xi,t) = \text{sgn} \, \xi_k \cdot g(t)$ on the interval $(t_{k-1},t_k)$, $k = 1,\ldots,n+1$, $t_0 := 0$, $t_{n+1} = A$; such function is uniquely determined by the imposed conditions. Moreover, $G(\xi) = -G(-\xi)$ for arbitrary $\xi \in S^n$ and $G(\xi)$ uniformly converges to $G(\xi_0)$ provided by $\xi \to \xi_0$.

Consider a map $\phi: S^n \to \mathbb{R}^n$, $\phi(\xi) = (G(\xi;s_1),\ldots,G(\xi;s_n))$. It is continuous and odd. By the Borsuk theorem, there exists $\xi^* \in S^n$ such that $\phi(\xi^*) = 0$. The function $G(\xi^*)$ satisfies (14) and (15). $G^{(r)}(\xi^*)$ is non-zero almost everywhere. Hence all zeros $s_1,\ldots,s_n$ of $G(\xi^*)$ are separated, and thus the function $G^{(r)}(\xi^*)$ has at least $n$ sign changes due to the Rolle theorem. From the construction it follows, that the function $G^{(r)}(\xi^*)$ can not have more than $n$ sign changes, hence $G(\xi^*) \in$
The function $G(\xi^*)$ is a desired perfect $g$-spline. The lemma is proved.

**Lemma 2.** Let $A > 0$, $n \in \mathbb{N}$ and $0 < s_1 < s_2 < \ldots < s_n < A$. If two perfect $g$-splines $G_1, G_2 \in \Gamma_{n,g}[0,A]$ satisfy conditions (14) and (15), then either $G_1 \equiv G_2$, or $G_1 \equiv -G_2$.

We can assume that both $G_1$ and $G_2$ are positive on $[0,s_1)$. Hence $G_1$ and $G_2$ have the same signs on each of the intervals $(s_k, s_{k+1})$, $k = 0,\ldots,n$, $s_0 := 0$, $s_{n+1} := A$.

Let $k \in \{0,\ldots,n\}$ and $s \in (s_k, s_{k+1})$ be fixed. We prove that

$$G_1(s) = G_2(s).$$

Assume the contrary, without loss of generality we may assume that $|G_1(s)| < |G_2(s)|$. There exists $\varepsilon \in (0,1)$ such that $G_1(s) = (1 - \varepsilon) \cdot G_2(s)$. The difference $G := G_1 - (1 - \varepsilon) \cdot G_2$ satisfies conditions (14) and (15); moreover, it has an additional zero at the point $s$. From the definition of the function $G$ it follows that the function $G^{(r)}$ is non-zero almost everywhere and changes its sign only at the knots of the perfect $g$-spline $G_1$. Hence all zeros $s_1,\ldots,s_n$ and $A$ of the function $G$ are separated and thus the Rolle theorem implies that the function $G^{(r)}$ has at least $n + 1$ sign changes. This contradiction proves (16). Due to arbitrariness of the point $s$, this implies that $G_1 \equiv G_2$. The lemma is proved.

**Lemma 3.** Let $A > 0$ and $n \in \mathbb{N}$ be given. Suppose that for each $m = 0,1,\ldots$ a point $s^{(m)} = (s_{1,m},\ldots,s_{n,m}) \in \mathbb{R}^n$, $0 \leq s_{1,m} < \ldots < s_{n,m} < A$ is given. Let $G_m \in \Gamma_{n,g}[0,A]$ be a perfect $g$-spline that satisfies (14) and vanishes at the points $s^{(m)}$, $m = 0,1,\ldots$. Denote by $t^{(m)} = (t_{1,m},\ldots,t_{n,m})$, $0 < t_{1,m} < \ldots < t_{n,m} < A$ the knots of $G_m$. Then $s^{(m)} \to s^{(0)}$ as $m \to \infty$ implies $t^{(m)} \to t^{(0)}$ as $m \to \infty$.

Assume, that the sequence $\{t^{(m)}\}_{m=1}^{\infty}$ has two different limit points $u = (u_1,\ldots,u_n)$ and $v = (v_1,\ldots,v_n)$. Consider the perfect $g$ splines $G_u$ and $G_v$ with knots at the point $u$ and $v$ that satisfy boundary conditions (14); each of the splines is determined up to the sign. Moreover, since the perfect spline continuously (in the sense of uniform convergence) depends on its knots, both $G_u$ and $G_v$ vanish at the points $s^{(0)}$. However, this contradicts to Lemma 2. Hence the sequence $\{t^{(m)}\}_{m=1}^{\infty}$ has a limit. It can’t be different from $t^{(0)}$ due to Lemma 2. The lemma is proved.

2.3. Maximally oscillating perfect $g$-splines on a segment.

**Remark 6.** Everywhere below for brevity we write ’a function has $n$ points of oscillation’ instead of ’a function has $n$ positively orientated points of oscillation’.
The technics that involve Brouwer fixed-point theorem in the proof of existence of oscillating functions, can be found for example in \[5, \S 10, \text{Chapter 2}\].

**Lemma 4.** Let \(A > 0\) and two functions \(f_+, f_- \in C[0, A]\) be given. Assume \(f_+(t) > 0\) for all \(t \in [0, A]\). Then for each \(n \in \mathbb{Z}_+\) there exists \(\tau > 0\) and a perfect \(g\)-spline \(G \in \Gamma_{n,g}[0, A]\) that satisfies (14), has exactly \(n\) knots, and has \(n + 1\) points of oscillation between the functions \(-\tau \cdot f_-\) and \(\tau \cdot f_+\).

For all \(\varepsilon \in (0, \frac{A}{n+1})\) consider the \(n\)-dimensional simplex

\[
\Xi^n_\varepsilon := \left\{ (\xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1} : \xi_1, \ldots, \xi_{n+1} \geq \varepsilon, \sum_{k=1}^{n+1} \xi_k = A \right\}.
\]

Set

\[
\Xi^n_0 := \left\{ (\xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1} : \xi_1, \ldots, \xi_{n+1} > 0, \sum_{k=1}^{n+1} \xi_k = A \right\}.
\]

For each point \(\xi \in \Xi^n_\varepsilon\) consider the partition of the segment \([0, A]\) by the points \(s_m = \sum_{k=1}^{m} |\xi_k|, m = 1, \ldots, n\). Due to the definition of the simplex, \(0 < s_1 < \ldots < s_n < A\). Hence, in virtue of Lemma 1 there exists a perfect \(g\)-spline \(G(\xi)\) with exactly \(n\) knots that satisfies (14) and (15). The sign of \(G(\xi)\) is chosen in such a way, that \(G(\xi)\) is positive on \((0, s_1)\). For each \(k = 1, \ldots, n + 1\), set

\[
\delta_k(\xi) := \min \left\{ \delta \geq 0 : |G(\xi; t)| \leq \delta \cdot f_k(t), t \in (s_{k-1}, s_k) \right\},
\]

where \(s_0 := 0, s_{n+1} := A, f_k = f_+\) for odd \(k\), and \(f_k = f_-\) for even \(k\).

Set \(\Delta_k(\xi) := \delta_k(\xi) - \min_{j=1,\ldots,n+1} \delta_j(\xi), k = 1, \ldots, n + 1\), and \(\Delta(\xi) := \sum_{k=1}^{n+1} \Delta_k(\xi)\).

If for some \(\varepsilon > 0\) there exists \(\xi \in \Xi^n_\varepsilon\) such that \(\Delta(\xi) = 0\), then corresponding perfect \(g\)-spline \(G(\xi)\) is a desired one.

Assume that for all \(\varepsilon > 0\) and \(\xi \in \Xi^n_0\)

\[
(17) \quad \Delta(\xi) \neq 0.
\]

In virtue of Lemma 3 \(\delta_k\) and \(\Delta_k\) are continuous functions of \(\xi\), \(k = 1, \ldots, n + 1\), hence \(\Delta\) is also a continuous function of \(\xi\).

Note, that since the functions \(f_+\) and \(f_-\) are separated from zero,

\[
\sup_G \|G''\|_{C[0, A]} < \infty,
\]

where the supremum is taken over all splines \(G \in \Gamma_{n,g}[0, A]\) that satisfy condition (14). From this fact it follows, that there exists \(\alpha > 0\) that does not depend on \(\xi\) and \(\varepsilon\) such that

\[
(18) \quad \delta_k(\xi) \leq \alpha \cdot \xi_k, k = 1, \ldots, n + 1.
\]
From $\inf_{G \in \Gamma_{\alpha,g}[0,A]} \|G\|_{C[0,A]} > 0$ it follows that

\begin{equation}
\inf_{\varepsilon > 0} \min_{\xi \in \Xi_{\varepsilon}^n} \max_{k=1,\ldots,n+1} \delta_k'(\xi) =: \beta > 0.
\end{equation}

We claim that

\begin{equation}
\inf_{\varepsilon > 0} \min_{\xi \in \Xi_{\varepsilon}^n} \Delta(\xi) =: \gamma > 0.
\end{equation}

Indeed, if (20) does not hold, then, due to (17), for arbitrary $\varepsilon > 0$ there exists $\xi_{\varepsilon} \in \Xi_{\varepsilon}^n \setminus \Xi_{\varepsilon}^n$ such that $\Delta(\xi) < \frac{\beta}{2}$. Hence

\[ \frac{\beta}{2} > \Delta(\xi) \geq \max_{k=1,\ldots,n+1} \Delta_k(\xi) = \max_{k=1,\ldots,n+1} \delta_k(\xi) - \min_{k=1,\ldots,n+1} \delta_k(\xi), \]

and due to (19) this implies that $\min_{k=1,\ldots,n+1} \delta_k(\xi) > \frac{\beta}{2}$. However, this contradicts to (18).

Define the map $\phi: \Xi_{\varepsilon}^n \to \Xi_{\varepsilon}^n$,

\[ \phi(\xi) = \varepsilon + \frac{A - (n+1)\varepsilon}{\Delta(\xi)} (\Delta_2(\xi), \Delta_3(\xi), \ldots, \Delta_{n+2}(\xi)), \]

where $\Delta_{n+2}(\xi) := \Delta_1(\xi)$. By the construction, $\phi$ is continuous, hence by the Brouwer fixed point theorem there exists $\xi^* \in \Xi_{\varepsilon}^n$ such that

\begin{equation}
\phi(\xi^*) = \xi^* = (\xi_1^*, \ldots, \xi_{n+1}^*).
\end{equation}

Hence for $k = 1, \ldots, n+1$, due to (18) and (20), one has

\begin{equation}
\xi_k^* = \varepsilon + \frac{A - (n+1)\varepsilon}{\Delta(\xi^*)} \Delta_{k+1}(\xi^*) < \varepsilon + \frac{A}{\gamma} \delta_{k+1}(\xi^*) < \varepsilon + \frac{\alpha \cdot A}{\gamma} \varepsilon.
\end{equation}

From the construction of the functions $\Delta_k$ it follows that there exists $k \in \{1, \ldots, n+1\}$ such that $\Delta_k(\xi^*) = 0$. Let for definiteness $\Delta_1(\xi^*) = 0$. Then, due to (21), $\xi_{n+1}^* = \varepsilon$. Consecutive application of estimate (22) implies that there exist positive numbers $\eta_1, \ldots, \eta_n$ that are independent of $\varepsilon$ and $\xi^*$ and such that $\xi^*_k < \eta_k \varepsilon$, $\xi^*_{n-1} < \eta_{n-1} \varepsilon$ and so on, $\xi^*_{n} < \eta_{n} \varepsilon$. However, if $\varepsilon$ is chosen to be enough small, this contradicts to the fact that $\sum_{k=1}^{n+1} \xi_k^* = A$. The lemma is proved.

2.4. **Auxiliary results.** We write $W^r_{\infty,\infty}[0,\infty)$ instead of $W^r_{\infty,\infty}[0,\infty)$, if $g \equiv 1$. The following result is essentially due to Landau [7]. We will give its proof for completeness.

**Lemma 5.** Let $r \in \mathbb{N}$, $r \geq 2$ and $x \in W^r_{\infty,\infty}[0,\infty)$ be given. If $\lim_{t \to \infty} x(t)$ exists, then $\lim_{t \to \infty} x^{(k)}(t) = 0$, $k = 1, \ldots, r - 1$.

The function $x^{(k)}$ is bounded, since $x$ and $x^{(r)}$ are bounded, $k = 1, \ldots, r - 1$. The lemma can be proved by induction on $k$. 
Let \( M > 0, t > M \) and \( \varepsilon > 0 \). Then \( x(t + \varepsilon) - x(t) = \varepsilon x'(t) + \frac{\varepsilon^2}{2} x''(\xi), \)
\( \xi \in (t, t + \varepsilon) \), hence
\[
|x'(t)| \leq \frac{|x(t + \varepsilon) - x(t)|}{\varepsilon} + \frac{\varepsilon}{2} |x''(\xi)|. \tag{23}
\]
Since \( \lim x(t) \) exists, there exists \( M = M(\varepsilon) > 0 \) such that \( |x(s) - x(t)| < \varepsilon^2 \) for all \( t, s > M \). Hence, due to the fact, that \( x'' \) is bounded, the right-hand side of (23) can be made arbitrarily small. This implies the statement of the lemma in the case \( k = 1 \).

Induction step can be done using the same arguments. The lemma is proved.

**Lemma 6.** Let \( r \in \mathbb{N}, r \geq 2 \) and \( x \in W^{r}_{0,\infty}[0, \infty) \) be given. If \( x^{(r)} \) has a finite number of essential sign changes, then the limit \( \lim_{t \to \infty} x(t) \) exists and \( \lim_{t \to \infty} x^{(k)}(t) = 0, k = 1, \ldots, r - 1 \).

Due to conditions of the lemma, \( x' \) has a finite number of sign changes. Hence \( x \) is monotone in some neighborhood of infinity. Since it is bounded, the limit \( x(\infty) \) exists. To finish the proof, it is enough to apply Lemma 5. The lemma is proved.

**Lemma 7.** Let \( x \in W^{r}_{0,\infty}[0, \infty) \) be such, that \( x^{(r)} \) is non-zero almost everywhere and has at most \( n \in \mathbb{Z}_+ \) essential sign changes. Assume \( x' \) has at least \( n \) sign changes. Then each of the functions \( x^{(k)}, k = 0, \ldots, r - 1 \), is \((n + 1)\)-piecewise monotone. Moreover, if \( \epsilon \in \{1, -1\} \) and \( \epsilon \cdot x \) is increasing on the first monotonicity interval, then
\[
\text{sgn } x^{(k)}(0) = \epsilon(-1)^{k+1}, k = 1, \ldots, r. \tag{24}
\]

From the conditions of the lemma it follows, that the function \( x \) and all its derivatives are non-zero almost everywhere. By Lemma 6 \( x' (\infty) = 0 \). Hence \( x'' \) has at least \( n \) sign changes and, by Lemma 6 \( x''(\infty) = 0 \). Continuing the same way, we obtain that \( x^{(r)} \) has at least \( n \) essential sign changes. Since \( x^{(r)} \) has at most essential \( n \) sign changes, each of the derivatives \( x^{(k)} \) has exactly \( n \) sign changes, \( k = 1, \ldots, r \).

Then each of the functions \( x^{(k)}, k = 0, \ldots, r - 1 \), is \((n + 1)\)-piecewise monotone. Moreover, the function \( x^{(k)}, k = 1, \ldots, r - 1 \), changes sign on each of its monotonicity interval, except the last one (where it tends to 0). This implies (24). The lemma is proved.

**Definition 6.** Let Assumption 4 hold. Set \( P_0 := g \) and for \( k = 1, \ldots, r \) and \( t \in [0, \infty) \) set
\[
P_k(t) := (-1)^k \sum_{s=0}^{k-1} \frac{(-1)^{k-s-1} A_s}{(k-s-1)!} t^{k-s-1} + g_k(t).
\]
Note, that \( P_k \) is the \( k \)-th order primitive of the function \( g \) such that \( P_k(\infty) = 0, k = 1, \ldots, r \). Moreover, \( P_k \) is monotone and does not change sign on \([0, \infty)\).
Lemma 8. Let Assumption \([\text{II}\]) hold, \(r \in \mathbb{N}\), \(x \in W_{\infty, g}^r[0, \infty)\) and \(\lim_{t \to \infty} x(t) = A \in \mathbb{R}\). Then for all \(t \in \mathbb{R}\), \(|x(t) - A| \leq |P_r(t)|\).

It is enough to prove the lemma in the case when \(A = 0\). This, in turn, can be done by induction on \(r\). Really, using induction hypothesis for \(r > 1\) (which can be applied, due to Lemma \([\text{III}\]) or the definition of the class \(W_{\infty, g}^r[0, \infty)\) for \(r = 1\), we have

\[
|x(t)| = |x(t) - x(\infty)| = \left| \int_t^\infty x'(s) ds \right| \leq \int_t^\infty |x'(s)| ds \leq \int_t^\infty |P_{r-1}(s)| ds
\]

\[
= \left| \int_t^\infty P_{r-1}(s) ds \right| = |P_r(t) - P_r(\infty)| = |P_r(t)|.
\]

The lemma is proved.

2.5. Maximally oscillating perfect \(g\)-splines on a half-line. In what follows we assume that Assumptions \([\text{II}\] and \([\text{II}\]) hold.

Lemma 9. Let \(n \in \mathbb{Z}_+\) be given. For all \(\alpha \in (0, \infty)\) there exists \(\tau > 0\) and a perfect \(g\)-spline \(G = G(n, \alpha) \in \Gamma_{n, g}^r[0, \infty)\) with exactly \(n\) knots that has \(n+1\) points of oscillation between \(-\tau \cdot f_-\) and \(\tau \cdot f_+\) and such that

\[
\frac{\tau f_+(\infty) - G(\infty)}{\tau f_-(\infty) + G(\infty)} = \alpha.
\]

For all \(0 < \alpha_1 < \alpha_2 < \infty\) there exists a number \(C = C(\alpha_1, \alpha_2) > 0\) such that \(\tau \leq C\), provided by \(\alpha \in [\alpha_1, \alpha_2]\).

Set \(a := \frac{\tau}{1+\tau} f_+(\infty) - \frac{-\alpha}{1+\alpha} f_-(\infty)\). Then \(a \in (-f_-(\infty), f_+(\infty))\) and hence both \(f_+ - a\) and \(f_- + a\) are positive continuous non-increasing on \([0, \infty)\) functions.

For arbitrary \(A > 0\) consider a perfect \(g\)-spline \(G_A \in \Gamma_{n, g}^r[0, A]\) that oscillates maximally between the functions \(-\tau_A(f_+ + a)\) and \(\tau_A(f_+ - a)\), where \(\tau_A > 0\) and the existence of such spline is guaranteed by Lemma \([\text{II}\]). Let \(0 \leq s_1^A < \ldots < s_{n+1}^A \leq A\) be the oscillation points of \(G_A\) and \(0 < t_k^A < \ldots < t_n^A < A\) be its knots.

Letting \(A \to \infty\) and switching to a subsequence, if needed, we may assume that each of the sequences \(s_k^A, k = 1, \ldots, n+1\), \(t_k^A, k = 1, \ldots, n\) and \(\tau_A\), has a finite or infinite limit \(s_k, t_k\) and \(\tau\) respectively.

Since \(G_A\) satisfies \([\text{II}\] it can be continued to a function from the class \(W_{\infty, g}^r[0, \infty)\) by setting it equal to 0 on \([A, \infty)\). Then, due to Lemma \([\text{IV}\]) for all \(A > 0\) and \(t \in [0, A]\)

\[
|G_A(t)| \leq |P_r(t)|.
\]

The latter inequality implies that for all \(A > 0\) one has \(\tau_A \leq \|P_r\|_{C[0, \infty), f_- + a, f_+ - a}\). Hence \(\tau \leq \|P_r\|_{C[0, \infty), f_- + a, f_+ - a}\). If \(0 < \alpha_1 <
\( \alpha_2 < \infty \) are fixed, then both functions \( f_- + a \) and \( f_+ - a \) are separated from zero by a constant independent of \( \alpha \in [\alpha_1, \alpha_2] \) due to (8). Hence \( \|P_r\|_{C(0, \infty), f_- + a, f_+ - a} \) is uniformly bounded from above for such values of \( \alpha \).

Note, that \( \tau > 0 \), since otherwise for each \( \varepsilon > 0 \) we could find \( A = A(\varepsilon) > 1 \), such that \( \|G_A\|_{C([0, 1])} < \varepsilon \), which is impossible, since the restriction of the \( g \)-spline \( G_A \) to \([0, 1]\) belongs to \( \Gamma_{n,g}^{-}([0, 1]) \) and
\[
\inf_{G \in \Gamma_{n,g}^{-}([0, 1])} \|G\|_{C([0, 1])} > 0.
\]

Since (25) holds for \( t = s_{n+1}^A \), from (8) and the fact that \( P_r(\infty) = 0 \), it follows that
\[ s_{n+1} < \infty. \]

From (8) and the fact that the derivative \( G'_A \) cannot become arbitrarily large, it follows that \( s_k \neq s_j, k \neq j \).

Taking into account (26) and using standard compactness arguments, we can extract a limit \( g \)-spline \( G = \lim_{A \to \infty} G_A \), where the convergence of the function and all derivatives of order \( \leq r - 1 \) is uniform on any bounded interval.

Then \( G \) has \( n + 1 \) points of oscillation between \( -\tau \cdot (f_- + a) \) and \( \tau \cdot (f_+ - a) \). Therefore \( G \) has \( n \) knots, i.e. \( t_k \neq t_j \), \( k \neq j \). Really, \( G \) has \( n \) zeros \( z_1 < \ldots < z_n \) on \([0, s_{n+1}]\) and \( G(\infty) = 0 \). Hence \( G' \) has \( n - 1 \) zeros on \([0, z_n]\) and one on \((z_n, \infty)\). Continuing the same way, we obtain that \( G^{(r)} \) has \( n \) essential sign changes, hence \( G \) has \( n \) knots.

This means that \( G \in \Gamma_{n,g}^{-}([0, \infty)) \).

The function \( G + \tau a \) is a desired one. Really, it is a perfect \( g \)-spline with exactly \( n \) knots that has \( n + 1 \) points of oscillation between \( -\tau \cdot f_- \) and \( \tau \cdot f_+ \). Moreover, condition (25) holds, due to the choice of \( a \). The lemma is proved.

**Lemma 10.** Let \( n \in \mathbb{Z}_+ \), \( G \) be a perfect \( g \)-spline from Lemma 8 with some \( \alpha > 0 \) and \( x \in W_{r,g}^{-}([0, \infty)) \) be such that
\[ \|x\|_{C([0, \infty)), f_- + a} \leq \|G\|_{C([0, \infty)), f_- + f_+}. \]

If \( \varepsilon \in (-1, 1) \) and \( \Delta = G - \varepsilon \cdot x \), then \( \Delta' \) has at least \( n \) sign changes.

Let \( s_1 < \ldots < s_{n+1} \) be the oscillation points of \( G \).

We prove the partial case, when \( x \equiv 0 \), first. Since \( \text{sgn} G(s_k) = (-1)^{k+1}, k = 1, \ldots, n + 1 \), there exist intervals \( I_k \subset (s_k, s_{k+1}) \), such that \( \text{sgn} G'(t) = (-1)^k \) for \( t \in I_k \), \( k = 1, \ldots, n \). Since \( f_{\pm} \) are non-increasing functions, there exists interval \( I_{n+1} \subset (s_{n+1}, \infty) \) such that \( G'(t) > 0 \) if \( n \) is odd or \( G'(t) < 0 \) if \( n \) is even, i.e. \( \text{sgn} G'(t) = (-1)^{n+1}, t \in I_{n+1} \). This implies that \( G' \) has at least \( n \) sign changes.

Moreover, by Lemma 4, \( G' \) has exactly \( n \) sign changes and thus
\[
\text{sgn} G'(t) = (-1)^{n+1}, t > \inf I_{n+1}.
\]

Now we return to the proof of the lemma in the general case.
From the conditions of the lemma it follows that $\Delta^{(r)}$ is non-zero almost everywhere and changes sign only in the knots of $G$, hence has exactly $n$ essential sign changes. Then $\Delta'(\infty) = 0$ due to Lemma 6. Thus $x'(\infty) = 0$, and hence for all $t > 0$

\begin{equation}
|x'(t)| \leq |P_{r-1}(t)|, \tag{28}
\end{equation}

due to Lemma 8.

Let $t_n$ be the last knot of $G$. Then $G'(t) = \pm P_{r-1}(t)$, $t > t_n$, since both of them are primitives of either $g$ or $-g$ of order $r - 1$ that vanish in infinity together with all of their derivatives. Together with (28), this implies $|x'(t)| \leq |G'(t)|$, hence

\begin{equation}
\text{sgn} \Delta'(t) = \text{sgn} G'(t) \tag{29}
\end{equation}

for $t > t_n$.

Since $\text{sgn} \Delta(s_k) = (-1)^{k+1}$, $k = 1, \ldots, n + 1$, there exist intervals $J_k \subset (s_k, s_{k+1})$, such that $\text{sgn} \Delta'(t) = (-1)^k$ for $t \in J_k$, $k = 1, \ldots, n$. Combining (27) and (29), we obtain that there exists an interval $J_{n+1} \subset (s_{n+1}, \infty)$ such that $\text{sgn} \Delta'(t) = (-1)^{n+1}$ for $t \in J_{n+1}$. Hence $\Delta'$ has at least $n$ sign changes. The lemma is proved.

Applying Lemma 10 with $x \equiv 0$ and Lemma 7 we obtain the following lemma.

**Lemma 11.** If $n \in \mathbb{Z}_+$ and $G \in \Gamma_{n,g}^{r,[0, \infty)}$ has $n$ knots and $n + 1$ points of oscillation, then $\text{sgn} G^{(r)}(0) = (-1)^r$.

### 3. Maximally Oscillating Perfect $g$-Spline with Given Norm

#### 3.1. Definition of the functions $\tau_n$.

**Lemma 12.** Let $n \in \mathbb{Z}_+$ and assume there are two perfect $g$-splines $G_i \in \Gamma_{n,g}^{r,[0, \infty)}$ that oscillate maximally between $-\tau_i \cdot f_-$ and $\tau_i \cdot f_+$, $i = 1, 2$, and

\begin{equation}
G_1(\infty) = G_2(\infty), \tag{30}
\end{equation}

then $\tau_1 = \tau_2$.

In the case, when $n = 0$, $G_1 - G_2$ is a constant function, thus due to (30), $G_1 \equiv G_2$ and the lemma is proved in this case.

Let $n > 0$. Assume the contrary, let for definiteness $\tau_1 > \tau_2$. Let $0 \leq s_1 < \ldots < s_{n+1}$ be the oscillation points of $G_1$, $0 < t_1 < \ldots < t_n$ be the knots of $G_1$ and $0 < u_1 < \ldots < u_n$ be the knots of $G_2$. Set $\delta := G_1 - G_2$. Then $\text{sgn} \delta(s_k) = \text{sgn} G_1(s_k)$, $k = 1, \ldots, n + 1$, and

\begin{equation}
\delta^{(k)}(\infty) = 0, k = 0, \ldots, r - 1. \tag{31}
\end{equation}

Hence $\delta$ has $n$ separated zeros on $(0, s_{n+1})$. Due to (31), $\delta'$ has $n$ separated zeros on $(0, \infty)$. Continuing the same way, we obtain that $\delta^{(r)}$ has $n$ essential sign changes on $(0, \infty)$. This implies, that $\delta^{(r)}$ is not
identical zero on each of the intervals \((t_k, t_{k+1})\), \(k = 0, \ldots, n\), \(t_0 := 0\), \(t_n+1 := \infty\).

Hence \(u_1 < t_1\), since otherwise, due to Lemma \(11\) \(\delta^{(r)}(t) = 0\), \(t \in (t_0, t_1)\); \(u_2 < t_2\), since otherwise \((u_1, u_2) \supset (t_1, t_2)\), and hence \(\delta^{(r)}(t) = 0\), \(t \in (t_1, t_2)\), and so on, \(u_n < t_n\). However, we obtain that \(\delta^{(r)}(t) = 0\) for \(t \in (t_n, t_{n+1})\) and hence \(\delta^{(r)}\) has at most \(n - 1\) sign changes. Contradiction. The lemma is proved.

**Lemma 13.** The conclusion of Lemma \(12\) remains true if condition \((30)\) is substituted by one of the following conditions:

(a) \[
\frac{\tau_i f_+(\infty) - G_i(\infty)}{\tau_i f_-(\infty) + G_i(\infty)} = \alpha,
\]
\(i = 1, 2, \alpha > 0\).

(b) \(n\) is odd and \(G_i(\infty) = \tau_i f_+(\infty), \ i = 1, 2\).

(c) \(n\) is even and \(G_i(\infty) = -\tau_i f_-(\infty),\ i = 1, 2\).

We prove case (a) first. Set \(a := \frac{1}{1+i\alpha} f_+(\infty) - \frac{\alpha}{i+i\alpha} f_-(\infty)\). Then for \(i = 1, 2\), \(G_i(\infty) = \tau_i \cdot a\). The perfect \(g\)-splines \(G_i - G_i(\infty)\) satisfy \((30)\) and oscillate maximally between \(-\tau_i(f_+ + a)\) and \(\tau_i(f_+ - a),\ i = 1, 2\). Lemma \(12\) now implies that \(\tau_1 = \tau_2\).

Cases (b) and (c) can be proved similarly. We prove case (b). If \(\tau_1 > \tau_2\) and \(s_1 < \ldots < s_{n+1}\) are the oscillation points of \(G_1\), then \(G_1(s_{n+1}) = -\tau_1 f_-(s_{n+1}) < G_2(s_{n+1})\) and \(G_1(\infty) > G_2(\infty)\). Hence the difference \(G_1 - G_2\) has \(n\) zeroes between the oscillation points of \(G_1\) and an additional zero on the interval \((s_{n+1}, \infty)\), together at least \(n + 1\) separated zeros. The contradiction can now be obtained using the arguments from the proof of Lemma \(12\). The lemma is proved.

**Remark 7.** Case (a) of Lemma \(13\) implies the correctness of the following definition.

**Definition 7.** For fixed \(n \in \mathbb{Z}_+\) and \(\alpha > 0\) set
\[
\tau_n(\alpha) = \tau_{r,n,g,f_+,f_-}(\alpha) := \|G_\alpha\|_{C([0,\infty), f_+, f_-)},
\]
where \(G_\alpha\) is a spline from \(\Gamma_{n,g}^r[0, \infty)\) built according to Lemma \(9\).

**3.2. Some properties of the functions \(\tau_n\).**

**Lemma 14.** For each \(n \in \mathbb{Z}_+\), \(\tau_n\) is continuous on \((0, \infty)\).

Let \(\beta > 0\) and \(\beta_n \to \beta\), \(s \to \infty\). Denote by \(G_\beta\) and \(G_n\), \(s \in \mathbb{N}\), the perfect \(g\)-splines built according to Lemma \(9\) with \(\alpha\) in the boundary condition \((25)\) substituted by \(\beta\) and \(\beta_n\) respectively.

Assume that \(\tau_n\) is not continuous at the point \(\beta\). Due to Lemma \(9\) the sequence \(\{\tau_n(\beta_n)\}_{n=1}^\infty\) is bounded, hence switching to a subsequence, if needed, we may assume that it has a limit \(\lim_{s \to \infty} \tau_n(\beta_s) = \tau \neq \tau_n(\beta)\).

Switching to a subsequence, if needed once more, we may assume that the sequences of the knots and the oscillation points of \(G_s\) have
limits as \( s \to \infty \). Moreover, analogously to the proof of Lemma 9, we can prove that all oscillation point limits are finite and different. Hence we obtain a perfect \( g \)-spline \( G \in \Gamma_{n,d}[0,\infty) \) that oscillates maximally between \( -\tau f_- \) and \( \tau f_+ \), and satisfies (25) with \( \alpha = \beta \). However, this contradicts to statement (a) of Lemma 13. The lemma is proved.

**Lemma 15.** For \( n \in \mathbb{Z}_+ \) the function \( \tau_n \) is non-decreasing in the case of odd \( n \), and is non-increasing in the case of even \( n \).

We prove the lemma for the case of odd \( n \); the case of even \( n \) can be proved using similar arguments.

For each \( \alpha > 0 \) denote by \( G_\alpha \) a perfect \( g \)-spline built according to Lemma 9. Then, due to (25),

\[
G_\alpha(\infty) = \frac{\tau_n(\alpha)}{\alpha + 1} (f_+(\infty) - \alpha \cdot f_-(\infty)).
\]

Note, that from Lemma 14 and (32) it follows, that \( G_\alpha(\infty) \) is a continuous function of \( \alpha \).

Since \( n \) is odd, \( G_\alpha \) is increasing on the interval \( (M_\alpha, \infty) \), where \( M_\alpha > 0 \) is the last point of oscillation of the \( g \)-spline \( G_\alpha \).

Assume the contrary, let \( 0 < \gamma < \beta \) be such that \( \tau_n(\gamma) > \tau_n(\beta) \). We claim that (33)

\[
G_\beta(\infty) > G_\gamma(\infty).
\]

Indeed, otherwise we get an extra zero of \( G_\gamma - G_\beta \) on the interval \( (M_\alpha, \infty) \) and obtain a contradiction using arguments similar to the proof of Lemma 12.

For arbitrary \( 0 < \alpha < \beta \) one has \( G_\alpha(\infty) \neq G_\beta(\infty) \). Really, otherwise Lemma 12 implies \( \tau_n(\alpha) = \tau_n(\beta) \), and hence, due to (32), \( \alpha = \beta \), which is impossible.

From the continuity of \( G_\alpha(\infty) \) and (33) it now follows, that (34)

\[
G_\beta(\infty) > G_\alpha(\infty)
\]

for all \( 0 < \alpha < \beta \).

Note, that equality (32) can be rewritten as

\[
G_\alpha(\infty) = \tau_n(\alpha)f_+(\infty) - \frac{\alpha}{1 + \alpha} \tau_n(\alpha)(f_+(\infty) + f_-(\infty)).
\]

This implies that \( G_\beta(\infty) < \tau_n(\beta)f_+(\infty) \). Let \( \varepsilon > 0 \) be such, that \( G_\beta(\infty) + \varepsilon < \tau_n(\beta)f_+(\infty) \). Then \( \frac{\alpha}{1 + \alpha} \tau_n(\alpha)(f_+(\infty) + f_-(\infty)) < \varepsilon \) for small enough \( \alpha > 0 \) and hence for such \( \alpha \)

\[
\tau_n(\alpha)f_+(\infty) = G_\alpha(\infty) + \frac{\alpha}{1 + \alpha} \tau_n(\alpha)(f_+(\infty) + f_-(\infty)) < G_\alpha(\infty) + \varepsilon < G_\beta(\infty) + \varepsilon < \tau_n(\beta)f_+(\infty).
\]

Thus for all small enough \( \alpha > 0 \) one has \( \tau_n(\alpha) < \tau_n(\beta) \). However, the latter inequality implies that the difference \( G_\beta - G_\alpha \) has \( n \) separated zeros between oscillation points of \( G_\beta \) and additional zero on the
interval \((M, \infty)\) due to (31). Using arguments similar to the proof of Lemma 12, we obtain a contradiction. The lemma is proved.

**Lemma 16.** For all \(n \in \mathbb{N}\) and \(\alpha, \beta > 0\) one has \(\tau_n(\alpha) \leq \tau_{n-1}(\beta)\).

Assume the contrary, let \(\tau_n(\alpha) > \tau_{n-1}(\beta)\). Let \(G_\alpha\) and \(G_\beta\) be the maximally oscillating splines built according to Lemma 9 with \(n\) and \(n-1\) knots respectively. Set \(\Delta := G_\alpha - G_\beta\).

Let \(s_1 < \ldots < s_{n+1}\) be the oscillation points of \(G_\alpha\) and let for definiteness \(n = 2k-1\) be odd, \(k \in \mathbb{N}\). Then for each \(m = 1, \ldots, 2k-1\), there exists \(s_m^k \in (s_{m}, s_{m+1})\) such that \((-1)^m \Delta'(s_m^k) > 0\). Moreover, \(G_\alpha\) increases on \((s_{2k}, \infty)\) and \(G_\beta\) decreases on \((M, \infty)\), where \(M\) is the last oscillation point of \(G_\beta\). Hence there exists \(s_{2k}^k > s_{2k-1}^k\) such that \(\Delta'(s_{2k}^k) > 0\). Thus \(\Delta'\) has \(2k-1\) sign changes and moreover, \(\Delta'(+0) = 0\). Hence \(\Delta''\) has \(2k-1\) sign changes and moreover, \(\Delta''(+\infty) = 0\). Continuing the same way, we obtain that \(\Delta^{(r)}\) has at least \(2k-1 = n\) essential sign changes. However, \(\Delta^{(r)}\) can change sign only in the knots of \(G_\beta\), hence not more than \(n-1\) times. Contradiction. The lemma is proved.

### 3.3. Asymptotic behavior of the functions \(\tau_n\).

**Lemma 17.** Let \(n \in \mathbb{N}\), \(G \in \Gamma_{n,g}^{r}[0, \infty)\) have \(n+1\) points of oscillation between \(-f_-\) and \(f_+\). Assume \(s_1 < \ldots < s_n\) are all zeros of \(G'\) and \(t_1 < \ldots < t_n\) are its knots. Then \(t_k \geq s_k\), \(k = 1, \ldots, n\).

Assume the contrary, let \(k \in \{1, \ldots, n\}\) be such that \(t_k < s_k\). Then there exists \(\varepsilon > 0\) such that \(G\) has at most \(n-k\) knots on the interval \((s_k - \varepsilon, \infty)\). Since all zeros of \(G'\) are simple, on the interval \((s_k - \varepsilon, \infty)\) the function \(G'\) changes sign in each of the points \(s_l\), \(l = k, \ldots, n\), totally \(n-k+1\) times. Since \(G'(\infty) = 0\), \(G''\) has at least \(n-k+1\) sign changes on \((s_k - \varepsilon, \infty)\). Continuing in a similar way, we obtain that \(G^{(r)}\) has at least \(n-k+1\) sign changes on \((s_k - \varepsilon, \infty)\). However, this is impossible, since it has only at most \(n-k\) knots on this interval. The lemma is proved.

The following two lemmas describe the limits of the splines from Lemma 9 as \(\alpha \to +0\) and \(\alpha \to \infty\). For brevity we write \(\tau_n(+0)\) instead of \(\lim_{\alpha \to +0} \tau_n(\alpha)\).

The limits \(\tau_n(+0), n \in \mathbb{N}\), and \(\tau_n(\infty) := \lim_{\alpha \to \infty} \tau_n(\alpha), n \in \mathbb{Z}_+,\) exist, since \(\tau_n\) is monotone due to Lemma 15. Moreover, these limits are positive and finite, due to Lemmas 15 and 16.

**Lemma 18.** If \(n\) is odd, then there exists a perfect \(g\)-spline \(G_0 \in \Gamma_{n,g}^{r}[0, \infty)\) that has exactly \(n\) knots, \(n+1\) points of oscillation between \(-\tau_n(+0)f_-\) and \(\tau_n(+0)f_+\), and such that \(G_0(\infty) = \tau_n(+0)f_+(\infty)\).

If additionally

\[
(35) \quad \lim_{t \to \infty} \frac{f_-(t) - f_-(\infty)}{|P_r(t)|} = 0,
\]
then there exists a perfect $g$-spline $G_\infty \in \Gamma_r^n \[0, \infty\)$ with exactly $n - 1$
 knots that has $n$ points of oscillation between $-\tau_n(\infty)f_-$ and $\tau_n(\infty)f_+$, and such that

$$G_\infty(\infty) = -\tau_n(\infty) \cdot f_-(\infty).$$

For each $\alpha > 0$ denote by $G_\alpha$ a perfect $g$-spline with $n$ knots built according to Lemma 8. Let $s_1^\alpha < \ldots < s_{n+1}^\alpha$ be its oscillation points. Since $n$ is odd, $G_\alpha$ is increasing on the interval $(s_{n+1}^\alpha, \infty)$. We sketch the proof of the existence of $G_0$. Let $G_0$ be the limiting $g$-spline of appropriately chosen sequence of $G_\alpha$, $\alpha \to +0$, and let $s_{n+1} = \lim_{\alpha \to +0} s_{n+1}^\alpha$. Since

$$G_\alpha(\infty) - G_\alpha(s_{n+1}^\alpha) \geq G_\alpha(\infty) + \tau_n(\alpha) f_-(\infty),$$

the addend $\tau_n(\alpha) f_-(\infty)$ is separated from zero by a constant independent of $\alpha$, then due to Lemma 8, $s_{n+1} < \infty$. Then, repeating the arguments from Lemma 8 one can prove that $G_0$ is the desired $g$-spline.

Next we prove the existence of the $g$-spline $G_\infty$.

Switching to a subsequence, if needed, let $s_1 \leq \ldots \leq s_{n+1}$ be the (finite or infinite) limits of the sequences $s_1^\alpha, \ldots, s_{n+1}^\alpha$ as $\alpha \to \infty$. Let $G_\infty$ be a limiting $g$-spline in the sequence $G_\alpha$.

Let $z_n^\alpha$ denote the last zero of $G_\alpha'$ and $z_n = \lim_{\alpha \to \infty} z_n^\alpha$ (if needed, we switch to a subsequence once more). We prove that

$$z_n = \infty. \tag{36}$$

Assume the contrary, let $z_n < \infty$. Then $G_\infty$ increases on $(z_n, \infty)$, and hence $G_\infty(t) = G_\infty(\infty) - |P_r(t)|$ for all big enough $t$, since its restriction to the interval $(M, \infty)$ with enough big $M$ is a primitive of the order $r$ of either $g$ or $-g$.

From (35) it follows, that there exists arbitrarily large $t$ such that

$$\frac{f_-(t) - f_-(\infty)}{|P_r(t)|} < \frac{1}{\tau_n(\infty)}. \tag{36}$$

Hence we obtain

$$G_\infty(t) = G_\infty(\infty) - |P_r(t)| = -\tau_n(\infty) f_-(\infty) - |P_r(t)| < \tau_n(\infty) (- f_-(\infty) - f_-(t) + f_-(\infty)) = -\tau_n(\infty) f_-(t),$$

which is impossible. Thus (36) holds. Hence $s_{n+1} = \infty$.

Since by Lemma 8 for all $\alpha > 0$

$$|G_\alpha(s_{n+1}^\alpha) - G_\alpha(s_n^\alpha)| = |G_\alpha(s_{n+1}^\alpha) - G_\alpha(\infty) - (G_\alpha(s_n^\alpha) - G_\alpha(\infty))| \leq |G_\alpha(s_{n+1}^\alpha) - G_\alpha(\infty)| + |G_\alpha(s_n^\alpha) - G_\alpha(\infty)| \leq |P_r(s_{n+1}^\alpha)| + |P_r(s_n^\alpha)|,$$

then $s_n^\alpha$ can not become arbitrarily large, due to (8). Hence $s_n < \infty$.

From Lemma 17 and (36) it follows, that $G_\infty$ can not have $n$ knots. On the other hand, since $G_\infty$ has $n$ points of oscillation, it can not have less than $n - 1$ knots. Hence $G_\infty$ has exactly $n - 1$ knots.
Thus $G_\infty$ is a desired perfect $g$-spline and the lemma is proved.

**Lemma 19.** If $n$ is even, then there exists a perfect $g$-spline $G_\infty \in \Gamma_{n,g}^r[0,\infty)$ that has exactly $n$ knots, $n + 1$ points of oscillation between $-\tau_n(\infty)f_-$ and $\tau_n(\infty)f_+$, and such that $G_\infty(\infty) = -\tau_n(\infty)f_-$. If $n \geq 2$ and additionally

$$
\lim_{t \to \infty} \frac{f_+(t) - f_+(\infty)}{|P_r(t)|} = 0,
$$

then there exists a perfect $g$-spline $G_0 \in \Gamma_{n-1,g}^r[0,\infty)$ with exactly $n - 1$ knots that has $n$ points of oscillation between $-\tau_n(0)f_-$ and $\tau_n(0)f_+$, and such that

$$
G_0(\infty) = \tau_n(0) \cdot f_+(\infty).
$$

If (37) holds, then $\tau_0(0) = \infty$.

We prove the last statement. All others can be proved similar to Lemma 18.

The spline $G_\alpha$, $\alpha > 0$, from Lemma 9 with 0 knots has the form $G_\alpha(t) = |P_r(t)| + C_\alpha$, $C_\alpha \in \mathbb{R}$. If $\tau := \tau_0(0) < \infty$, then the splines $G_\alpha$ tend to $G := |P_r| + \tau f_+(\infty)$ for some sequence $\alpha \to +0$. However, condition (37) implies that for some sufficiently large $t$, $G(t) = |P_r(t)| + \tau f_+(\infty) > \tau f_+(t)$, which is impossible. The lemma is proved.

4. Proof of the main results

4.1. Proof of Theorem 2. We prove statements (a) and (b) for the $g$-splines with positively oriented points of oscillation. Due to connection between differently orientated oscillation functions given in Remark 1, the case of negatively oriented points of oscillation can be proved using the same arguments.

Let $n \in \mathbb{Z}_+$ be fixed. For $\alpha \in (0, \infty)$ denote by $G_{n,\alpha}$ a perfect $g$-spline with $n$ knots from Lemma 9. Set

$$
T_n := \left\{ \|G_{n,\alpha}\|_{C[0,\infty), f_- f_+] : \alpha \in (0, \infty) \right\}.
$$

Due to Lemma 14, $T_n$ is an interval (open, closed or semi-open). For all $t \in T_{n+1}$ and $s \in T_n$, one has $t \leq s$, due to Lemma 16. This implies that $\sup T_{n+1} \leq \inf T_n$. We claim, that $\sup T_{n+1} = \inf T_n$.

Let $n$ be even for definiteness. Then $n + 1$ is odd and we set $G_1$ to be the perfect $g$-spline $G_\infty$ from Lemma 18 with $n$ substituted by $n + 1$; by Lemma 15, $\tau_{n+1}(\infty) = \sup_{\alpha > 0} \tau_{n+1}(\alpha) = \sup T_{n+1}$.

Set $G_2$ to be the perfect $g$-spline $G_\infty$ from Lemma 19 by Lemma 15, $\tau_{n}(\infty) = \inf_{\alpha > 0} \tau_{n}(\alpha) = \inf T_n$.

Case (c) of Lemma 13 now implies that $\sup T_{n+1} = \inf T_n$.

Since

$$
\int_0^\infty G_{n,\alpha}(t)|G_{n,\alpha}'(t)| dt,
$$

...
then due to Lemma 8 applied to $G'$, all variations of the splines $G_{n,a}$ are bounded by some number independent of $n$ and $a$. Assumption 2 now implies that $\inf T_n \to 0$ as $n \to \infty$. To finish the proof of statement (a), it is sufficient to set $a_{n+1}^+ := \inf T_n$, $n \in \mathbb{Z}_+$ and note, that $\sup T_0 = \infty$, due to Lemma 19.

To prove statement (b), it is sufficient to apply Lemma 10 with $x \equiv 0$, Lemma 7 and note, the $G^+_n(0) > 0$ and it decreases on the first monotonicity interval.

To prove statement (c), we first note that equalities (11) and (12) hold due to Lemmas 18 and 19.

Assume the contrary, let $G \in \Gamma_{r,g}[0, \infty)$ be such that

$$\|G\|_{C[0,\infty), f_-f_+} < \min \|G^+_n\|_{C[0,\infty), f_-f_+}.$$ 

Denote by $G_n$ one of the $g$-splines $G^+_n$, such that

$$\text{sgn} G_n^{(r)}(0) = \text{sgn} G^{(r)}(0).$$

This can be done due to (10).

Let $s_k$, $k = 1, \ldots, n+1$, be the oscillation points of $G_n$ and $s_{n+2} = \infty$. Set $\delta := G_n - G$. Then $\text{sgn} \delta(s_k) = \text{sgn} G_n(s_k)$, $k = 1, \ldots, n+1$. Moreover, this equality also holds for $k = n+2$, due to (11) or (12). Using the arguments from the proof of Lemma 12, we obtain a contradiction.

The theorem is proved.

4.2. Proof of Theorem 3. The theorem is a combination of the Lemmas 9, 14, 15, 16, 18 and 19.

4.3. Proof of Theorem 4. Let $x \in W^r_{f_-f_+, g}[0, \infty)$, $\|x\|_{C[0,\infty), f_-f_+} \leq \delta$ and assume that

$$(38) \quad \|x^{(k)}\|_{C[0,\infty)} > \max \{ |(G^+_{\delta})^{(k)}(0)|, |(G^-_{\delta})^{(k)}(0)| \}. $$

Since $f_\delta$ and $g$ are non-increasing, for arbitrary $\alpha \geq 0$, $x(\cdot + \alpha) \in W^r_{f_-f_+, g}[0, \infty)$, and $\|x\|_{C[0,\infty), f_-f_+} \geq \|x(\cdot + \alpha)\|_{C[0,\infty), f_-f_+}$. Hence without loss of generality we can assume that $\|x^{(k)}\|_{C[0,\infty)} = \|x^{(k)}(0)\|$.

Denote by $G$ one of the functions $G^\pm_{\delta}$, chosen by the condition

$$(39) \quad \text{sgn} x^{(k)}(0) = \text{sgn} G^{(k)}(0).$$

This can be done due to (10).

Assume for definiteness, that $G = G^+_\delta$, it has $n$ knots and $n + 1$ points of oscillation $s_1 < \ldots < s_{n+1}$.

There exists $\varepsilon > 0$ such that $(1 - \varepsilon)|x^{(k)}(0)| > |G^{(k)}(0)|$. Set $\Delta = G - (1 - \varepsilon)x$.

From Lemmas 10 and 7 it follows, that the functions $\Delta^{(m)}$ and $G^{(m)}$, $m = 0, \ldots, r-1$, are $n+1$-piecewise monotone. Moreover, $\text{sgn} \Delta(s_1) = \text{sgn} G(s_1) = 1$. Applying the arguments from the proof of Lemma 10, one can prove that the functions $\Delta$ and $G$ are decreasing on their first interval of monotonicity. Hence from Lemma 7 it follows, that
\[ \text{sgn } \Delta^{(m)}(0) = \text{sgn } G^{(m)}(0) = (-1)^m, m = 0, \ldots, r - 1. \] However, these equalities with \( m = k \) contradict to (38) and (39).

The theorem is proved.

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OLEG KOVALENKO, DEPARTMENT OF MATHEMATICS AND MECHANICS, OLES HONCHAR DNIPRO NATIONAL UNIVERSITY, GAGARINA AVE., 72, DNIPRO, 49010, UKRAINE

E-mail address: olegkovalenko90@gmail.com