Random Phase Approximation without Bogoliubov Quasi-particles

L. Y. Jia\footnote{Electronic address: jialiyuan84@gmail.com} \textsuperscript{1, 2,∗}

\textsuperscript{1}Department of Physics, University of Shanghai for Science and Technology, Shanghai 200093, P. R. China
\textsuperscript{2}Department of Physics, Hebei Normal University, Shijiazhuang, Hebei 050024, P. R. China

(Dated: May 7, 2014)

A new version of random phase approximation is proposed for low-energy harmonic vibrations in nuclei. The theory is not based on the quasi-particle vacuum of the BCS/HFB ground state, but on the pair condensate determined in Ref. \textsuperscript{4}. The current treatment conserves the exact particle number all the time. As a first test the theory is considered in two special cases: the degenerate model (large pairing limit) and the vanished-pairing limit.

PACS numbers: 21.60.Ev, 21.10.Re,

Keywords:

I. INTRODUCTION

The random phase approximation (RPA) is first proposed in plasma physics to describe plasma oscillations \textsuperscript{1}. In low-energy nuclear structure usually the extended version, quasi-particle random phase approximation (QRPA), is used owing to the existence of pairing correlations \textsuperscript{2, 3}. In the mean-field order, we introduce Bogoliubov quasi-particles for pairing and solve the Hartree-Fock-Bogoliubov (HFB) equation (advanced version of BCS), and the unperturbed ground state is written as the quasi-particle vacuum. On top of that, small-amplitude vibration is described by QRPA in terms of linear combinations of two quasi-particle excitations.

The QRPA has problems inherent from its starting point – BCS mean field. The wavefunction does not have definite particle number, which is conserved only on average. Also, for nuclei near the critical pairing strength of BCS, the mean-field description fails itself and certainly QRPA is inapplicable. Complicated projection techniques to good particle number are needed to restore the broken symmetry.

Recently we proposed a number-conserving theory for nuclear pairing \textsuperscript{4}. Without introducing quasi-particles, the unperturbed ground state is written as a \( N \)-pair condensate for a system with \( 2N \) particles. Within a similar computing time to that of BCS, we solve the pair structure, occupation numbers, and pair-transition amplitudes. The theory is shown to be valid at arbitrary pairing strength including those below the critical point of BCS.

In this work we try to develop a new version of number-conserving random phase approximation (N-RPA) based on the above number-conserving ground state. The derivation of N-RPA is quite straightforward following that of conventional (Q)RPA. In the end a linear homogeneous equation is resulted, and the frequency \( \omega \) is determined by requiring the determinant to be zero. The computing time cost is similar to that of conventional (Q)RPA. As a first test, the theory is considered in two special cases: the degenerate model (large pairing case) and the vanished-pairing case. In the latter the theory goes over to the conventional RPA. Further tests and applications are needed in the future.

II. FORMALISM

First we briefly repeat the essential result from Ref. \textsuperscript{4} as the starting point of this work. The antisymmetrized fermionic Hamiltonian is

\begin{equation}
H = \sum_{12} \epsilon_{12} a_{1}^{\dagger} a_{2} + \frac{1}{4} \sum_{1234} V_{1234} a_{12}^{\dagger} a_{23} a_{34} .
\end{equation}

The unperturbed ground state of the \( 2N \)-particle system is assumed to be a \( N \)-pair condensate,

\begin{equation}
|\phi_{N} \rangle = \frac{1}{\sqrt{\chi_{N}}} (P^{\dagger})^{N} |0\rangle ,
\end{equation}

where \( \chi_{N} \) is the normalization factor, and \( P^{\dagger} \) is the pair creation operator

\begin{equation}
P^{\dagger} = \frac{1}{2} \sum_{i} v_{i} a_{i}^{\dagger} a_{i}^{\dagger} .
\end{equation}

By the main equation (19) in Ref. \textsuperscript{4}, we determine the pair structure \( v_{1} \), the density matrices

\begin{equation}
\rho_{12} \equiv \langle \phi_{N} | a_{2}^{\dagger} a_{1} | \phi_{N} \rangle = \delta_{12} n_{1} ,
\end{equation}

\begin{equation}
\kappa_{12} \equiv \langle \phi_{N-1} | a_{2} a_{1} | \phi_{N} \rangle = \delta_{12} s_{1} ,
\end{equation}

and the mean fields

\begin{equation}
f_{12} \equiv \epsilon_{12} + w_{12} \equiv \epsilon_{12} + \sum_{34} V_{1324} \rho_{34} = \delta_{12} \epsilon_{1} ,
\end{equation}

\begin{equation}
\delta_{12} \equiv \frac{1}{2} \sum_{34} V_{1324} \kappa_{43} = \delta_{12} g_{1} .
\end{equation}

Now we begin the derivation of N-RPA. Here we do not include angular momentum because in some cases
(for example the deformed QRPA) rotational symmetry is not conserved. We assume that the correlated ground state $|0_N\rangle$ and the excited state $|1_N\rangle$ are related by

$$\langle 1_N | A^\dagger | 0_N \rangle = 1 \equiv \text{Tr}[c R] = \sum_{12} c_{12} a_{12}^\dagger a_{21},$$  

(8)

where $R_{12} \equiv a_{12}^\dagger a_{21}$ are density matrix operators. $c_{12}$ in the phonon creation operator $A^\dagger$ are parameters to be determined later by the main equation (13). The correlated ground state $|0_N\rangle$ is not the unperturbed pair condensate $|\phi_N\rangle$ (2), but has "2-particle-2-hole components" in it.

Let us calculate the following quantity

$$\langle 0_N | R_{12} | 1_N \rangle = \sum_{3} c_{13} \langle 0_N | a_{12}^\dagger a_{34} | 0_N \rangle - \sum_{34} c_{43} \langle 0_N | a_{a4}^\dagger a_{13} | 0_N \rangle,$$  

(9)

where we have used Eq. (8). Here we make the main approximation of the theory, the so-called "linearization". We assume that

$$\langle 0_N | a_{12}^\dagger a_{21} | 0_N \rangle \approx \langle \phi_N | a_{12}^\dagger a_{21} | \phi_N \rangle = \rho_{12},$$  

(10)

and

$$\langle 0_N | a_{14}^\dagger a_{42}^\dagger a_{21} | 0_N \rangle \approx \langle \phi_N | a_{14}^\dagger a_{42}^\dagger a_{21} | \phi_N \rangle = \langle \phi_N | a_{14}^\dagger a_{21}^\dagger a_{42} | \phi_N \rangle = \langle a_{14}^\dagger a_{21} | \langle a_{14}^\dagger a_{21} | \phi_N \rangle = \rho_{14} \rho_{23} - \rho_{24} \rho_{13} + (\epsilon^1 s_{34}) \kappa_{12},$$  

(11)

where $\langle a_{14}^\dagger a_{21} | \phi_N \rangle$ means $\langle a_{14}^\dagger a_{21} | | \phi_N \rangle$, $\langle a_{21} a_{14} | | \phi_N \rangle$, etc. In Eqs. (10) and (11) we made the usual approximation (see Ref. [3]) that the correlated ground state $|0_N\rangle$ does not differ much from the unperturbed ground state $|\phi_N\rangle$ if the vibrational amplitudes are small. Under the approximations (10) and (11), Eq. (9) becomes

$$\langle 0_N | R_{12} | 1_N \rangle = c_{12} n_2 (1 - n_1) + c_{31} s_1 s_2.$$  

(12)

The exact Heisenberg equation of motion for the density matrix operators $R_{12} = a_{12}^\dagger a_{21}$ is calculated as

$$-\frac{1}{2} \sum_{34} V_{5432} a_{12}^\dagger a_{34} a_{34} a_{12} + \frac{1}{2} \sum_{34} V_{1345} a_{12}^\dagger a_{34} a_{45}.$$  

(13)

We bra-ket Eq. (13) with "|0_N\rangle" and "|1_N\rangle". On the left-hand side we have $\langle 0_N | R_{12}, H | 1_N \rangle = (E_K - E_K' \langle 0_N | R_{12} | 1_N \rangle$, where $E_K - E_K' \equiv \omega$ is the N-RPA frequency, and $\langle 0_N | R_{12} | 1_N \rangle$ is already given in Eq. (12). The right-hand side is handled in a way similar to that of Eqs. (9)-(12); after substituting Eq. (8), we again use Eqs. (10), (11), and their extension to three-body density matrix operators $\langle 0_N | a_{14}^\dagger a_{21}^\dagger a_{42} | 0_N \rangle$, which is approximated by all possible "fully contracted terms" (there are 15 terms in Wick's theorem (although normal ordering of operators may be hard to define). In the end we get

$$\omega [c_{12} n_2 (1 - n_1) + c_{31} s_1 s_2] = \sum_3 \epsilon_3 \sum_3 c_{32} n_2 (1 - n_3) + c_{23} s_2 s_3 - \sum_3 [c_{13} n_3 (1 - n_1) + c_{31} s_1 s_3] \epsilon_{32}$$

$$+ \sum_3 w_3 [c_{32} n_2 (1 - n_3) - c_{23} s_2 s_3] - \sum_3 [c_{13} n_3 (1 - n_1) - c_{31} s_1 s_3] w_{32}$$

$$- (n_1 - n_2) \sum_{34} V_{1432} c_{34} n_4 (1 - n_3) + c_{33} s_3 s_4 + c_{12} g_2 s_2 (1 - n_1) + g_1 s_1 n_2 + c_{31} [g_2 s_1 (1 - n_2) + g_1 s_2 n_1]$$

$$+ \sum_{34} V_{1345} c_{34} s_4 s_4 (1 - n_3)$$  

(14)

$$\approx c_{31} s_1 (1 - n_2) - c_{12} (1 - n_1) s_2.$$  

In the above two equations approximations (10) and (11) are used.

We emphasize that, unlike the conventional RPA or QRPA, in the current theory $A |0_N\rangle$ $\neq 0$. Considering the special case of infinitesimal residual interaction (but pairing interaction has normal strength), $|0_N\rangle$ would be almost $|\phi_N\rangle$, and $A |\phi_N\rangle$ does not vanish for any one-body phonon operator $A$ (8).
III. DEGENERATE MODEL

We consider the degenerate model (degeneracy $2\Omega$) with “quadrupole-plus-pairing” force:

$$\epsilon_{12} = \delta_{12}\epsilon, \quad V_{1234} = -G\delta_{12}\delta_{34} - \kappa q_{14}q_{23} + \kappa q_{13}q_{24}$$

in the Hamiltonian \[1\]. The occupation number \( n_1 = N/\Omega \), pair-transition amplitude \( s_1 = \sqrt{n(1-n)} \), mean fields \( \epsilon_1 = \epsilon - nG \) and \( \eta_1 = G\Omega s_1 \) are already calculated in Ref. \[4\]. Consequently Eq. (14) becomes

$$(G\Omega - \omega)(c_{12} + c_{21}) = \kappa(qc)_{12} + \kappa(qc)_{21},$$

where we have used that operator \( q \) is time-even \( (q_{12} = q_{21}) \). At infinitesimal \( \kappa \), the N-RPA frequency \( \omega \approx G\Omega \), which is the correct pairing gap.

IV. ZERO-PAIRING LIMIT

In this section we show that the N-RPA equation (14) goes over to the usual RPA equation in the case of vanished pairing interaction. In this case, \( |\phi_N\rangle \) is a Slater determinant with a sharp Fermi surface (F). The single-particle levels below F are fully occupied and are empty above F. The pair-emission amplitudes \( s_1 \) for \( 1 \neq F \) vanishes, and \( s_F \approx \sqrt{n_F(1-n_F)} = 0 \). Introducing \( r_{12} \equiv \langle 0_N|R_{12}|1_N\rangle = c_{12}n_2(1-n_1) \), Eq. (14) becomes

$$\omega r_{12} = \epsilon_{12}r_{12} - n_1\sum_{34}V_{1432}r_{34}.$$  \hspace{1cm}  (15)

This is the usual RPA equation. \( r_{12} \) is related to the conventional \( X, Y \) amplitudes (see Ref. \[3\]) by \( r_{mi} = X_{mi}, \quad r_{im} = Y_{mi} \), where \( i < F \) is a hole level and \( m > F \) is a particle level.

In summary, a new version of number-conserving random phase approximation is proposed following the procedure of deriving the conventional (quasi-particle) random-phase approximation. The theory is considered in two limits: the degenerate model (large pairing case), and the vanished-pairing case. More applications/tests are needed in the future.

In the next step anharmonicities could be included in the following way. Using the “factorizations” \[10\], \[11\] and their extensions, we could calculate the Hamiltonian matrix in the collective subspace \( \langle m|H|n\rangle \equiv \langle 0_N|A^mH(A^\dagger)^n|0_N\rangle \). Then we diagonalize the matrix \( \langle m|H|n\rangle \). Odd-mass nuclei could be treated similarly by calculating the matrix \( \langle mj|H|nj\rangle = \langle 0_N|A^ma_jH(a_j^\dagger)^n|0_N\rangle \) using the “factorization”.

It may also be possible to generalize the conventional particle-particle RPA (see Ref. \[3\]) by treating the Heisenberg equation of motion \([K_{12}, H]\) of the density matrix operator \( K_{12} \equiv a_2a_1 \) in a way similar to that of Eq. (13).

[1] David Bohm, and David Pines, Phys. Rev. 92, 609 (1953).
[2] A. Bohr and B. Mottelson, Nuclear Structure (Benjamin, New York, 1975), Vol. 2.
[3] P. Ring and P. Schuck, The Nuclear Many-Body Problem (Springer-Verlag, New York, 1980).
[4] L. Y. Jia, arXiv:1305.2697 [nucl-th].