MAKING RESEARCH ON SYMMETRIC FUNCTIONS WITH MUPAD-COMBINAT

FRANCOIS DESCOUENS

Abstract. We report on the 2005 AIM workshop “Generalized Kostka Polynomials”, which gathered 20 researchers in the active area of $q,t$-analogues of symmetric functions. Our goal is to present a typical use-case of the open source package MuPAD-Combinat in a research environment.

1. Introduction

The project MuPAD-Combinat (see http://mupad-combinat.sf.net/ and [4]), born in spring 2001 under the leadership of F. Hivert and N. Thiéry, is an open source package for making algebraic combinatorics using the computer algebra system MuPAD (see www.mupad.de and [3] for more details). The main goal of this package is to bring an open source flexible and easily extensible toolbox for checking conjectures in a short programming time. This package contains a large collection of tools as implementations of classical combinatorial objects (partitions, Young tableaux, trees, . . .), computations in combinatorial Hopf algebras (in particular symmetric functions), manipulations of graphs, automata, . . .

The study of symmetric functions is a major historical field in algebraic combinatorics ([8]) and some people are mainly interested in generalizations of Kostka polynomials. In section 2, we give briefly mathematical definitions about different objects we work on. We illustrate how a computer algebra system which contains classical combinatorial objects, evolutive implementation of symmetric functions and easy way to incorporate program written in C++ can be used for our research.

In section 3, we explain the design of our implementation of symmetric functions and some of our technical choices. We also give some examples of computations and show some advanced functionalities as adding new bases on the fly (using different characterizations) or implementing new operators on symmetric functions. Finally, in section 4 we describe how we incorporate programs into a coherent design using MAPITL library. The technical concepts used in section 3 and 4 are essentially classical, but we want to stress on their integration into the package in order to have a powerful tool for making efficient. The difficulty is to find the appropriate combination that yields an intuitive yet flexible and powerful research tool. We show that our choices are promising in a real situation of collaboration at a workshop on generalized Kostka polynomials in Palo Alto, California organized by the American Institute of Mathematics in July 2005 (see http://www.aimath.org/WWN/kostka/ for more informations on this event).

2. Symmetric functions and Kostka polynomials

2.0.1. Basic definitions. A symmetric polynomial in variables $X = \{x_1, \ldots, x_n\}$ is a polynomial in $X$ invariant under permutations of variables. When is infinite, we call symmetric function such a polynomial. The set of symmetric functions with coefficients in $\mathbb{C}(t)$, denoted $\Lambda_t$, is a graded algebra with respect to the degree of polynomials, i.e

$$\Lambda_t = \bigoplus_{n \geq 0} \Lambda_t^n.$$

For all $n \geq 0$, the dimension of $\Lambda_t^n$ is the number of partitions of $n$ (a partition of a positive integer $n$, written $\lambda \vdash n$, is a decreasing sequence of positive integers with sum $n$). One main basis of this algebra is constituted with monomial functions defined for all partitions by $\lambda = (\lambda_1, \ldots, \lambda_n)$, by

$$m_{\lambda} = \sum_{v \in O(\lambda)} x_1^{v_1} \cdots x_n^{v_n},$$

where $O(\lambda)$ is the set of ordered partitions of $\lambda$. The technical concepts used in section 3 and 4 are essentially classical, but we want to stress on their integration into the package in order to have a powerful tool for making efficient. The difficulty is to find the appropriate combination that yields an intuitive yet flexible and powerful research tool. We show that our choices are promising in a real situation of collaboration at a workshop on generalized Kostka polynomials in Palo Alto, California organized by the American Institute of Mathematics in July 2005 (see http://www.aimath.org/WWN/kostka/ for more informations on this event).
where \( O(\lambda) \) represents the set of all the permutations of \( \lambda \). Another interesting basis consists of the symmetric powersums, defined for all partitions \( \lambda \) by
\[
p_\lambda(X) = p_{\lambda_1} \cdots p_{\lambda_n} \quad \text{where} \quad p_{\lambda_i} = \sum_{j=1}^{n} x_j^{\lambda_i}.
\]
A scalar product on \( \Lambda_t \) can be uniquely defined by
\[
(p_\lambda, p_\mu) = \delta_{\lambda, \mu} \prod_{i \geq 1} (m_i)! \ i^{m_i(\lambda)},
\]
where \( m_i(\lambda) \) represents the multiplicity of part \( i \) in partition \( \lambda \). Applying Gram-Schmidt process of orthonormalization on the monomial basis with respect to this scalar product yields us the basis of Schur functions \((s_\lambda)_\lambda\). These functions are intensively studied (from an algebraic and combinatorial point of view) and one of the beautiful results is the Littlewood-Richardson rule, a combinatorial interpretation of the product of two Schur functions.

2.0.2. Kostka polynomials. One can introduce a \( t \)-deformation of the previous scalar product by
\[
(p_\lambda, p_\mu)_t = \delta_{\lambda, \mu} \prod_{i \geq 1} (m_i)! \ i^{m_i(\lambda)} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1-t^i}.
\]
The orthogonalization of the monomial basis with respect to this scalar product defines the Hall-Littlewood functions \( P_\lambda(X; t) \). There exist two other families of Hall-Littlewood functions: on the one hand, the \( Q_\lambda(X; t) \) which are the dual elements of \( P_\lambda(X; t) \) with respect to the scalar product \(( , )_t\) and on the other hand \( Q'_\lambda(X; t) \) which are the dual elements of \( P_\lambda(X; t) \) with respect to \(( , )_t\). The expansion of the \( Q'_\lambda \) on Schur functions is an algebraic way to define the Kostka polynomials \( K_{\lambda, \mu}(t) \)
\[
Q'_\lambda(X; t) = \sum_{\mu \vdash |\lambda|} K_{\lambda, \mu}(t) s_{\mu}.
\]
These polynomials have also a combinatorial interpretation, using the charge on semi-standard Young tableaux in [7], or the rigged configurations introduced by Kerov, Kirillov and Reshetikhin in [5].

2.0.3. Generalizations of Kostka polynomials. Using \( k \)-ribbon tableaux introduced by Lascoux, Leclerc and Thibon in [6], we define for each positive integer \( k \) and each partition a particular symmetric functions \( H_\lambda^{(k)}(X; t) \). Their expansion on Schur functions gives us a way to define an increasing filtration of Kostka polynomials \( K_{\lambda, \mu}^{(k)}(t) \)
\[
H_\lambda^{(k)}(X; t) = \sum_{\mu \vdash |\lambda|} K_{\lambda, \mu}^{(k)}(t) s_{\mu}.
\]
The expansion of \( K_{\lambda, \mu}^{(k)}(t) \) on the monomial basis is a way to define unrestricted generalized Kostka polynomials. In order to generalize Kostka polynomials, there exist other combinatorial ways (with unrestricted rigged configurations recently introduced by L. Deka and A. Schilling in [11] and algebraic ways (using the theory of crystal bases for quantum groups of type \( A_n \)) to generalize Kostka polynomials. Other generalizations are given by M. Zabrocki using creation operators in [16] and by L. Lapointe and J. Morse introducing a \( t \)-deformation of \( k \)-Schur functions in [9]. An interesting problem, explained in [10], is to show that these generalizations coincide in some particular cases.

3. Implementation of symmetric functions

3.1. Design goals. Symmetric functions can be represented in many different ways, and in particular in different basis (powersum, elementary, Schur, monomials, . . .). As usual in computer science, it is essential at each point to use the appropriate representation, both for efficiency and interpretation of the results. What make the situation specific is the number of those representations. In particular, it is neither practical nor sometimes possible to implement explicitly all conversions. Instead we want to be able to only implement
a few and deduce the others by compositions or linear transformations such as inversion or transposition. Furthermore to that end, the user and also programmers should not need to know which conversion are implemented because this information is too volatile. For example, the addition of symmetric function not given in the same basis is possible

```plaintext
>> S::s([2,1]) + S::QP([2,1]) + S::p([2,1]);
(t + 4) m[1, 1, 1] + (t + 1) m[3] + (t + 3) m[2, 1]
```

The monomial basis has been chosen by the system because it minimizes the number conversions (the cost of the conversion is not taken into account). The same process occurs for making conversion between two bases as the expansion of the Hall-Littlewood function $Q_{311}^\prime$ on the monomial basis

```plaintext
>> S::m(S::QP([2,1]));
(t + 2) m[1, 1, 1] + (t + 1) m[2, 1] + t m[3]
```

We also consider operators on symmetric functions. In most cases, they are easier to define (and consequently to implement) on one of the bases but not on all of them. Consequently, the system uses implicit conversions between bases in order to apply operators on any given basis. The main technologies used are linear algebra and overloading mechanisms which are not essentially new, but a great effort is made in order to make manipulation intuitive.

### 3.2. General overview of the implementation

In our design, for each basis of symmetric functions there is a domain (in the category `Cat::GradedHopfAlgebraWithBasis`) which represents the space of symmetric functions expanded on this basis. For example, here is the example of implementation of the complete basis domain SymT::complete(R: DOM_DOMAIN)

```plaintext
  inherits SymT::common(R);
  category Cat::GradedHopfAlgebraWithBasis(R), Cat::CommutativeRing;
  info_str := "Domain for symmetric functions expanded on complete basis";

  basisName := hold(h);

  // Implementation of multiplication (complete basis is a multiplicative basis)
  mult2Basis := dom::term@revert@sort@_concat;

  // Each domain corresponding to a basis is declared
  h := SymT::complete(dom::coeffRing, Options);

  // Implementation of different conversions between bases (stored in a table)
  basisChangesBasis :=
    table(
      ...;

    // Explicit combinatorial conversions
      (dom::QP, dom::s) = (part ->(_plus(combinat::tableaux::kostkaPol(mu, part, dom::vHL)
        * dom::s(mu) $ mu in
          combinat::partitions::list(_plus(op(part))))),
    ...

      (dom::McdP, dom::m) = (part ->((dom::GramSchmidt(dom::m, _plus(op(part)),
          dom::scalartq))\[op(part)])),
    ...

    // Dual conversions and inverse conversions

    (dom::s, dom::QP) = dom::invertBasisChange(dom::QP, dom::s),
    (dom::s, dom::m) = dom::transposeBasisChange(dom::h, dom::s, dom::s, dom::m),
    )
```

Note that only some conversions are implemented in this table; the overloading mechanism is in charge of finding the shortest number of intermediate conversions needed (it doesn’t take into account the cost of each conversion). In order to use $(q,t)$-deformation of symmetric function, we can declare
3.2.1. Adding new bases on the fly. In order to add new basis on the fly, we define a generic domain.

```plaintext
>> S::examples::SymmetricFunctions(Dom::ExpressionFieldWithDegreeOneElements([t,q]),
   vHL=t, vMcd=q);
```

In order to add a new basis named \( E \) for example, we can use

```plaintext
>> B := S::newBasis(S::coeffRing, "E");
```

and we define the change of bases we want. Let suppose that the change of basis between \( B \) and monomial basis corresponds to the function \( \text{testChange} \)

```plaintext
>> S::declareBasisChangeBasis((B, dom::m, testChange);
```

and by assuming that the action on the Schur basis is given by a function \( f \), we declare

```plaintext
>> operators::overloaded::declareSignature(
   S::newOp, [S::s, DOM_INT],
   S::s::moduleMorphism(f, dom::s));
```

and you can apply this operator on any basis due to the overloading mechanism.

4. Integration of Others Programs Written in C++

In July 2005, the American Institute of Mathematics organized in Palo Alto, California, a workshop on the generalized Kostka polynomials under the leadership of A. Schilling and M. Vazirani. During problem sessions, MuPAD-Combinat was put to use for testing conjectures. As usual in algebraic combinatorics, computations required the combination of preexisting combinatorial and algebraic functionalities (as provided by MuPAD-Combinat) with new combinatorial features (namely a highly technical bijection between \( k \)-tuples of Young Tableaux and unrestricted rigged configurations). Thanks to a dynamic module we could reuse a pre-existing robust C++ implementation of this bijection written by L. Deka; this allowed us to start manipulating large examples quickly. In general, dynamic modules permit us to reuse C++ code with two goals in mind: to avoid reimplementing nontrivial and tested code, and to get quicker computations than in pure MuPAD language. In section 4.1, we describe the implementation of combinatorial objects in MuPAD, and in section 4.2 we present our technical choices for a seamless integration of a combinatorial bijection implemented in C++.

4.1. Implementation of combinatorial objects in MuPAD-Combinat. We implement each combinatorial class as a domain in the MuPAD category \( \text{Cat::CombinatorialClassWith2DBoxedRepresentation} \). This category provides, among other things, a pretty printing method using ASCII characters. Let us illustrate this design in the case of skew riggings implemented in the domain \( \text{combinat::skewRiggings} \).

```plaintext
domain combinat::skewRiggings
   inherits Dom::BaseDomain;
   category Cat::CombinatorialClassWith2DBoxedRepresentation;
   axiom As::canonicalRep;
   info_str := "Combinatorial class for rigged skew partitions";
end_domain:
```

We implement the constructor \( \text{combinat::skewRiggings::new} \) which builds an object from the list of its operands:
The first element of the internal representation is the type

```plaintext
>> a[0]:

combinat::skewRiggings
```

Ribbon rigged configurations are particular sequences of skew riggings, implemented as plain lists of typed MuPAD objects. We implement them in the following domain

```plaintext
domain combinat::riggedConfigurations::RcRibbonsTableaux
inherits Dom::BaseDomain;
category Cat::CombinatorialClassWith2DBoxedRepresentation,
// Elements of this domain are represented using a plain MuPAD data structure
Cat::FacadeDomain(DOM_LIST);

info_str := "Combinatorial class for rigged configurations";
```

Here is an example of the bijection applied on the set of all 3-ribbon tableaux of shape (432) and evaluation (111)

```plaintext
>> rc := map(combinat::ribbonsTableaux::list([4,3,2],[1,1,1],3),
        combinat::riggedConfigurations::RcRibbonsTableaux::fromRibbonTableau);
```

L. Deka and A. Schilling introduced in [1] a new kind of rigged configurations, namely the unrestricted ones. They were implemented as an independent C++ program (file: FromOneCrystalPath.cc, headers: FromOneCrystalPath.h). On each rigged configurations (ribbons one and unrestricted one) we can compute a statistic. The interesting question is to find, in a special case, a bijection which preserves the statistic between these two kinds of rigged configurations. In order to manipulate these two objects in a single program, we decided in collaboration with A. Schilling and L. Deka, to also integrate this C++ program into MuPAD-Combinat. In order to make the integration of this version of rigged configurations in a transparent way for the user, we kept the same design we used for ribbon rigged configurations.
4.2. Implementation of combinatorial objects in a dynamic module using the MAPITL library.

We describe now the integration of the previous C++ program using MAPITL library (MuPAD Application Programming Interface Template Library (included in MuPAD-Combinat)). This library provides:

- Wrappers to use MuPAD lists as standard containers
- Easy C++ ←→ MuPAD conversions with one single overloaded template for each direction:
  - C++ object → MuPAD object: \texttt{CtoM(c)}
  - MuPAD object → C++ object: \texttt{MtoC(c)}

This includes conversions to/from containers and recursive containers.

The C++ program computes the rigged configurations corresponding to a list of Young tableaux also called path which is denoted by the class \texttt{path}. We want to call from MuPAD \texttt{void path::build_rigged_for_path} using the procedure \texttt{combinat::RiggedConfigurations::newRiggedConfigurations}.

The building of the interaction is divided into three steps which are realized in the file \texttt{RiggedConfigurationsPaths.mcc}:

- convert a list of Young Tableaux in MuPAD to a C++ object of the class \texttt{path}
- call the function \texttt{void path::build_rigged_for_path}
- convert the result into a MuPAD object of type \texttt{DOM\_LIST}

Original files are \texttt{FromOneCrystalPath.cc} containing declarations of different classes. We now explain some parts of the file \texttt{RiggedConfigurationsPaths.mcc} which is entirely given in appendix. The beginning of the file is the inclusion of the header file of the independent program and MAPITL.

4.2.1. Conversion MuPAD to C++.

\texttt{MFUNC( newRiggedConfigurations, MCnop )}

\{
\texttt{MFargsCheck(3);}
\texttt{MFargsCheck(1, DOM\_INT);}  
\texttt{MFargsCheck(2, DOM\_INT);}  
\texttt{MFargsCheck(3, DOM\_LIST);}  
\texttt{n = MtoC<int>(MFarg(1));}
\texttt{int path_len = MtoC<int>(MFarg(2));}
\texttt{Cell input(MFarg(3));}
\texttt{...}
\texttt{Cell *toto = input.toArray<Cell>();}
\texttt{...}
\}

\texttt{MFargsCheck} is the type checking of MuPAD arguments and \texttt{MtoC} permits to convert MuPAD objects into C++ ones. After we initialize an object of class \texttt{path} with values contained in \texttt{toto}

\begin{verbatim}
path_class* input_path = new path_class(path_len)
for(long a=0; a < input.size(); a++)
    {
        ...
    }
\end{verbatim}

4.2.2. Using the original function in C++. We can call now the original function in order to compute the bijection

\begin{verbatim}
input_path->build_rigged_for_path()
\end{verbatim}

4.2.3. Conversion C++ to MuPAD. Next we create a C++ vector which contained operands of the corresponding into MuPAD object

\begin{verbatim}
for(i=0; i < n; i++)
    {
        vector<int> yyy;
        j = 0;
        while(input_path->rigged[i][0][j] != UNUSED)
            {
                yyy.push_back(input_path->rigged[i][0][j]);
                yyy.push_back(input_path->rigged[i][2][j]);
                yyy.push_back(input_path->rigged[i][1][j]);
            }
    }
\end{verbatim}
4.3. Compilation of the dynamic module. The next step is the compilation of the \texttt{mcc} file giving us the dynamic module \texttt{RiggedConfigurationsPaths.mdm}. We have to load this module from MuPAD using the function

\begin{verbatim}
combinat::RiggedConfigurationsPaths := proc()
save RiggedConfigurationsPaths;
begin
if module::which("RiggedConfigurationsPaths") = FAIL then
  userinfo(1, "Dynamic module RiggedConfigurationsPaths not available");
  RiggedConfigurationsPaths := FAIL;
else
  if traperror(module("RiggedConfigurationsPaths")) <> 0 then
    warning("Error loading the dynamic module RiggedConfigurationsPaths:");
    lasterror();
  end_if;
end_if;
RiggedConfigurationsPaths;
end_proc():
\end{verbatim}

We create the domain \texttt{combinat::riggedConfigurations::RcPathsEnergy} and procedure \texttt{combinat::riggedConfigurations::RcPathsEnergy::fromOnePath} computes the bijection by calling the function implemented in the dynamic module \texttt{combinat::RiggedConfigurations::newRiggedConfigurations}

\begin{verbatim}
>> a:=[combinat::tableaux([[3,3]]), combinat::tableaux([[2,2]]), combinat::tableaux([[1,1]])];
-- +---+---+ ++---+---+ ++---+---+ --
| | 3 | 3 | | 2 | 2 | | 1 | 1 | |
-- +---+---+ ++---+---+ ++---+---+ --

We compute the bijection
\end{verbatim}

\begin{verbatim}
>> rc := combinat::riggedConfigurations::RcPathsEnergy::fromOnePath(a);
-- +---+---+ --
| | 0 | 0 +---+---+ |
| +---+---+ , | | 0 | 0 |
| | 0 | 0 +---+---+ |
-- +---+---+ --

>> op(rc[1])[0]

combinat::skewRiggings
\end{verbatim}

The object computed using the C++ program is interfacing in a transparent way for the user who can use other MuPAD functionalities on the previous result.

4.3.1. Acknowledgment. The author wants to thank A. Schilling and L. Deka for their collaboration on rigged configurations, M. Zabroki for its code on creating operators, and F. Hivert and N. Thiery for their support in the project MuPAD-Combinat.

References

[1] L. Deka and A. Schilling, \textit{New fermionic formula for unrestricted Kostka polynomials}, Journal of Combinatorial Theory, Series A, to appear (math.CO/0509194).
[2] F. Descouens, \textit{A generating algorithm for ribbon tableaux}, Journal of Automata, Languages and Combinatorics, to appear.
[3] B. Fuchssteiner and Al., \textit{MuPAD User’s Manual - MuPAD Version 1.2.2.}, John Wiley and sons, Chichester, New York, first edition, march 1996. includes a CD for Apple Macintosh and UNIX.
[4] F. Hivert and N. Thiéry, \textit{MuPAD-Combinat, an open-source package for research in algebraic combinatorics}, Sém. Lothar. Combin. \\textbf{51} (2004), 70p. (electronic).
\url{http://mupad-combinat.sourceforge.net/}
[5] S. Kerov, A. Kirillov and N. Yu. Reshetikhin, \textit{Combinatorics, the Bethe ansatz and representations of Symmetric group}, J. Soviet Math \\textbf{41} (1988) no.2, 916-924.
[6] A. Lascoux, B. Leclerc and J.-Y. Thibon, Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties, Journal of Mathematical Physics 38 (1997), 1041-1068.

[7] A. Lascoux and M.P. Schützenberger, Sur une conjecture de H.O. Foulkes, C.R Acad. Sci. Paris 288(1979), 95-98.

[8] I.G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., Oxford University Press, 1995.

[9] L. Lapointe and J. Morse, Schur function identities, their t-analogs, and k-Schur irreducibility, Advances in Math (2003).

[10] A. Schilling, X=M Theorem: Fermionic formulas and rigged configurations under review preprint (math.QA/0512161).

[11] A. Schilling, Crystal structure on rigged configurations IMRN, to appear (math.QA/0508107).

[12] A. Schilling, q-Supernomial coefficients: From riggings to ribbons, MathPhys Odyssey 2001, M. Kashiwara and T. Miwa (eds.), Birkhaeuser Boston, Cambridge, MA, 2002, pp. 437-454 (math.CO/0107214).

[13] J. Stembridge, Package SF, Posets, Coxeter/Weyl, http://www.math.lsa.umich.edu/~jrs/maple.html.

[14] Symmetricta , http://www.mathe2.uni-bayreuth.de/axel/symneuengl.html.

[15] S. Veigneau, ACE, an Algebraic Combinatorics Environment for the computer algebra system MAPLE: User’s Reference Manual, Version 3.0, Report 98-11, IGM. http://www.igm.univ-mlv.fr/~jrs/ACE/ACE.html

[16] M. Zabrocki, Vertex operators for standard bases of the symmetric functions, Journal of Algebraic Combinatorics, 13, No. 1 (2000), pp. 83-101.

APPENDIX: RiggedConfigurationsPaths.mcc

```cpp
#include<vector>
#include "FromOneCrystalPath.h"
#include "MAPITL.h"
#include "MAPITL_tmpl.h"
using MAPITL::MtoC;
using MAPITL::CtoM;
using MAPITL::Container;
using MAPITL::SimpleCell;
using MAPITL::Cell;
using MAPITL::ObjectInCell;
using MAPITL::ArrayInCell;

using namespace std;

MFUNC( newRiggedConfigurations, MCnop )
{
    MFargCheck(3);
    MFargCheck(1, DOM_INT);
    MFargCheck(2, DOM_INT);
    MFargCheck(3, DOM_LIST);
    n = MtoC<int>(MFarg(1));
    int path_len = MtoC<int>(MFarg(2));
    Cell input(MFarg(3));
    int i, j, k, tmp, path_index, tblu_index, col;
    
    tmp = UNUSED;
    path_index = 0;
    tblu_index = 0;
    i = 0; j = 0; k = 0; col = 0; l = 0;
    initialize_lambda();
    path_class* input_path = new path_class(path_len);
    reset_tableau();
    Cell *toto = input.toArray<Cell>();
    vector< vector<int> > res(n+1);
    vector< vector<int> > res(n+1);
    for(long a=0; a < input.size(); a++)
    {
        Cell* alpha=(toto[a]).toArray<Cell>();
        int s = (alpha[0]).size();
        tblu_class *my_tblu = new tblu_class((toto[a]).size(),s);
        my_tblu->tblu_id = tblu_index;
        tblu_index ++ 1;
        for(long d=0; dcto[a].size(); d++)
        {
            int* beta=(alpha[d]).toArray<int>();
            for(long b=0; b < s ;b++)
            {
                my_tblu->tb[d][b] = beta[b];
                my_tblu->tab_lambda[beta[b]-1] =
                my_tblu->tab_lambda[beta[b]-1] + 1;
            }
        }
    }
}
input_path->path[path_index] = my_tbleu;
reset_tableau();
path_index += 1;
}
input_path->build_rigged_for_path();
for(i=0; i<n; i++)
{
    vector<int> yyy;
    j = 0;
    while(input_path->rigged[i][0][j] != UNUSED)
    {
        yyy.push_back(input_path->rigged[i][0][j]);
        yyy.push_back(input_path->rigged[i][2][j]);
        yyy.push_back(input_path->rigged[i][1][j]);
        j++;
    }
    res[i] = yyy;
}
input_path->calculate_cocharge();
vector<int> stat;
stat.push_back(input_path->cocharge);
res[n] = stat;
MFreturn(Cell(res));
} MFEND