Hořava-Lifshitz modifications of the Casimir effect

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Abstract

We study the modifications induced by spacetime anisotropy on the Casimir effect in the case of two parallel plates. Nonperturbative and perturbative regimes are analyzed. In the first case the Casimir force either vanishes or it reverses its direction which, in any case, makes the proposal untenable. On the other hand, the perturbative model enables us to incorporate appropriately the effects of spacetime anisotropy.

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I. INTRODUCTION

A great deal of attention has been devoted to the Hořava-Lifshitz (HL) theory since it might give rise to the existence of a renormalizable formulation of quantum gravity. The essence of the Hořava proposal consists in attributing different scaling properties to space and time coordinates: $x^i \rightarrow bx^i$, $t \rightarrow b^z t$, where $z$ is a critical exponent characterizing the ultraviolet behavior of the theory. Power counting suffices to suggest the renormalizability of four-dimensional quantum gravity for $z = 3$. One expects to recover the Lorentz symmetry at the infrared limit. Different issues related to the HL gravity, including its cosmological aspects, exact solutions, and black holes, have already been presented in the literature.

We dedicate this work to study the consequences of the spacetime anisotropy on the Casimir effect for two parallel plates. Other consequences of the HL proposal in field and brane theories have been considered in. We consider two different models. One is essentially nonperturbative and therefore closer in spirit to the original HL proposal. It is contained in Section II. The second one is tailored in such a way that the spacetime anisotropy is a correction to the standard Casimir effect. This is our Section III. Section IV contains the conclusions.

II. THE NONPERTURBATIVE MODEL

The model is described by the following action (repeated Latin indices sum from one to $d$)

$$S = \frac{1}{2} \int dt d^d x \left( \partial_0 \phi \partial_0 \phi - \ell^2 (z-1) \partial_{i_1} \partial_{i_2} \ldots \partial_{i_z} \phi \partial_{i_1} \partial_{i_2} \ldots \partial_{i_z} \phi \right).$$

Notice that the number of spatial derivatives acting on the real scalar field $\phi$ is equal to $2z$, where $z$ is the above mentioned critical exponent. As required by consistency, the constant $\ell$ has dimension of length.

As in Ref., we start by considering two large parallel plates with area $L^2$, with a separation $a$ ($a \ll L$). Let these plates be orthogonal to the $x_3$ axis. The free scalar field $\phi$, being the solution of the equation of motion

$$\left( \partial_0^2 - \ell^2 (z-1) (-1)^z \Delta^z \right) \phi = 0,$$

is localized between these plates, and verifies, by assumption, Dirichlet boundary conditions, i.e., $\phi(x_3 = 0) = \phi(x_3 = a) = 0$. The solving of this equation leads to

$$\phi = \sqrt{\frac{2}{a}} \frac{1}{2\pi} \sum_{n=0}^{\infty} \int \frac{d^d k}{2\omega^2_{k,n}} \sin \left( \frac{\pi n x_3}{a} \right) \left( a_n \left(k\right) e^{ikx} + a_n^\dagger \left(k\right) e^{-ikx} \right),$$

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where \( n \) is an integer, \( kx \equiv k_0 x_0 - k_1 x_1 - k_2 x_2 \),

\[
k_0 = \varepsilon^{-1} \omega_{k,n}^z,
\]

and

\[
\omega_{k,n} = \sqrt{k_1^2 + k_2^2 + \left(\frac{\pi n}{a}\right)^2}.
\]

The Hamiltonian operator emerging from canonical quantization reads

\[
H = \sum_{n=1}^{\infty} \int d^2k \left( a_n a_n^\dagger + \frac{1}{2} \right) \omega_{k,n}^z,
\]

where \( a_n(\vec{k}) \) and \( a_n^\dagger(\vec{k}) \) are, respectively, annihilation and creation operators obeying the commutation relation

\[
\left[ a_n(\vec{k}), a_m^\dagger(\vec{k}') \right] = 2\omega_{k,n}^z \delta_{nm} \delta^2(\vec{k} - \vec{k}').
\]

Hence, the vacuum energy is given by

\[
E = \langle 0 | H | 0 \rangle = \varepsilon^{-1} \frac{L^2}{8\pi^2} \sum_{n=1}^{\infty} \int d^2k \omega_{k,n}^z.
\]

This last integral is divergent and its evaluation demands the introduction of a regularization procedure. In particular, we choose to replace Eq. (8) by

\[
E_{\text{Reg}} = \varepsilon^{-1} \frac{L^2}{8\pi^2} \sum_{n=1}^{\infty} \int d^2k \omega_{k,n}^z e^{-\alpha \omega_{k,n}^z},
\]

where \( \alpha \) is a semipositive real parameter to be set to zero at the end of the calculations. By performing the angular integration, one finds

\[
E_{\text{Reg}} = \frac{\varepsilon^{-1} L^2}{4\pi} \sum_{n=1}^{\infty} \int_{\frac{\pi}{a}}^{\infty} d\omega_{k,n} \omega_{k,n}^z e^{-\alpha \omega_{k,n}},
\]

so that

\[
E_{\text{Reg}} = \frac{\varepsilon^{-1} L^2}{4\pi^2} (-1)^{z+1} \left( \frac{\partial}{\partial \alpha} \right)^{z+1} \left[ \frac{a}{\alpha^2} \sum_{n=0}^{\infty} B_n \left( \frac{\pi}{\alpha} \right)^n \right] \equiv E_{\text{Reg}}^0,
\]

where the \( B_n \) are the Bernoulli numbers.

We shall next subtract from this last expression the vacuum energy in the absence of plates, namely,

\[
E_{\text{Reg}}^0 = \frac{\varepsilon^{-1} L^2}{4\pi^2} (-1)^{z+1} \left( \frac{\partial}{\partial \alpha} \right)^{z+1} \frac{a}{\alpha^z}.
\]
Thus,
\[
E_{\text{reg}} - E_{\text{reg}}^0 = \frac{\ell^z L^2}{4\pi^2} (-1)^{z+1} \left( \frac{\partial}{\partial \alpha} \right)^{z+1} \left[ \frac{a^2}{\alpha^2} \sum_{n=1}^{\infty} \frac{B_n}{n!} \left( \frac{\pi \alpha}{a} \right)^n \right].
\] (13)

However, as \(\alpha \to 0\) only the term \(n = z + 3\),
\[
\frac{\ell^z L^2}{4\pi^2} (-1)^{z+1} \frac{\pi^{z+3} B_{z+3}}{(z+3)(z+2)} \frac{1}{a^{z+2}},
\] (14)
contributes to the modified Casimir force
\[
F = -\frac{d}{d\alpha} \lim_{\alpha \to 0} \left( E_{\text{reg}} - E_{\text{reg}}^0 \right).
\] (15)

The resulting force per unit area turns out to be
\[
\frac{F_z}{L^2} = (-1)^{z-1} \frac{\ell^z \pi^{z+1} B_{z+3}}{4(z+3)} \frac{1}{a^{z+3}}.
\] (16)

For the standard case, \(z = 1\), one recovers the well-known result
\[
\frac{F_1}{L^2} = -\frac{\pi^2}{480 a^4},
\] (17)
since \(B_4 = -1/30\).

For other values of \(z\), we found instructive to compute the relative intensity of the Casimir force referred to that of the standard case, i.e.,
\[
\frac{F_z}{F_1} = (-1)^z \times 120 \frac{\pi^{z-1} B_{z+3}}{(3+z)} \left( \frac{\ell}{a} \right)^{z-1}.
\] (18)

The relevant points here are that \(F_2\) vanishes while the direction of \(F_3\) is opposite to that of \(F_1\). This indicates that the model defined in Eq. (1) is untenable.

III. THE PERTURBATIVE MODEL

We now propose the action
\[
S' = \frac{1}{2} \int dt d^4x \left( \partial_0 \phi \partial_0 \phi - \partial_i \phi \partial_i \phi - \ell^{2(z-1)} \partial_1 \partial_2 \ldots \partial_z \phi \partial_1 \partial_2 \ldots \partial_z \phi \right),
\] (19)
where the spacetime anisotropy shows up as a modification of the standard free Lagrangian. The dispersion relation modifies as follows,
\[
k_0 = \omega_{k,n}^z \sqrt{1 + \left( \ell \omega_{k,n}^z \right)^{2(z-1)}},
\] (20)
where \( \omega_{k,n} \) was already defined in Eq. (5). The vacuum energy reads

\[
E = \frac{L^2}{8\pi} \sum_{n=1}^{\infty} \int_{\frac{\omega_{k,n}}{\pi}}^{\infty} d\omega_{k,n} \omega_{k,n}^2 \sqrt{1 + \left( \ell \omega_{k,n} \right)^2} \left( \ell \omega_{k,n} \right)^{2(z-1)}.
\]

(21)

By adopting the same regularization as in Section II,

\[
E_{\text{Reg}} = \frac{L^2}{8\pi} \sum_{n=1}^{\infty} \int_{\frac{\omega_{k,n}}{\pi}}^{\infty} d\omega_{k,n} \omega_{k,n}^2 \sqrt{1 + \left( \ell \omega_{k,n} \right)^2} e^{-a \omega_{k,n}} \left( \ell \omega_{k,n} \right)^{2(z-1)}
\]

(22)

we simplify the integrand by using that

\[
\sqrt{1 + \left( \ell \omega_{k,n} \right)^2} \sim 1 + \frac{1}{2} \left( \ell \omega_{k,n} \right)^2.
\]

(23)

Then, the vacuum energy turns out to be given by the sum of the usual one plus the correction

\[
\delta E_{\text{Reg}} = \frac{L^2 \ell^{2(z-1)}}{16\pi} \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial \alpha} \right)^{2z} \int_{\frac{\omega_{k,n}}{\pi}}^{\infty} d\omega_{k,n} e^{-a \omega_{k,n}} \left( \ell \omega_{k,n} \right)^{2(z-1)}.
\]

(24)

This expression can be cast

\[
\delta E_{\text{Reg}} = \frac{L^2 \ell^{2(z-1)}}{16\pi^2} \left( \frac{\partial}{\partial \alpha} \right)^{2z} \left[ \frac{a}{\alpha^2} \sum_{n=0}^{\infty} \frac{B_n}{n!} \left( \frac{\pi \alpha}{a} \right)^n \right].
\]

(25)

One is next to subtract the vacuum energy in the absence of the plates, which corresponds to the \( n = 0 \) term in the sum above. Moreover, only the term \( n = 2z + 2 \),

\[
\frac{L^2}{16\pi^2} \frac{\ell^{2(z-1)} \pi^{2z+2} B_{2z+2}}{(2z+2)(2z+1)a^{2z+1}},
\]

(26)

happens to contribute to the Casimir force \([15]\).

Thus, the total resulting force per unit area is

\[
F + \delta F = -\frac{\pi^2}{480a^4} \left[ 1 - 30 \frac{\pi^{2z-2} B_{2z+2}}{(2z+2)} \left( \frac{\ell}{a} \right)^{2(z-1)} \right].
\]

(27)

We emphasize that the correction induced by the spacetime anisotropy never vanishes. However, to make sense, the ratio \( \ell/a \) must be small enough so as to secure that the just mentioned correction remains below experimental uncertainties. To be more precise, for an experimental uncertainty of order 1\% \([13, 14]\) we estimate \(|\ell/a| \lesssim 0.17\) for \( z = 3 \) and \(|\ell/a| \lesssim 0.09\) for \( z = 2 \). In \([13, 14]\), the separation \( a \) is as small as \( 10^{-7} \)m, so in principle \( \ell \) could be at the most of order \( 10^{-8} \)m.

**IV. CONCLUSIONS**

We have shown that the Casimir force is indeed modified by the presence of spacetime anisotropy. The effect may be drastic, up to the point of changing the direction of the force, which cannot be
accommodated within current experimental observations. Alternatively, one may implement the anisotropy in such a way that the effects it produces remain acceptable from the experimental point of view.

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