Existence of invariant measures for the stochastic damped KdV equation

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Abstract

We address the long time behavior of solutions of the stochastic Korteweg-de Vries equation
du + (\partial_x^3 u + u \partial_x u + \lambda u)dt = fdt + \Phi dW_t \quad \text{on} \quad \mathbb{R}
where \( f \) is a deterministic force. We prove that the Feller property holds and establish the existence of an invariant measure. The tightness is established with the help of the asymptotic compactness, which is carried out using the Aldous criterion.

Mathematics Subject Classification:

Keywords: Invariant measures, Stochastic KdV equation, White noise, Long time behavior, Asymptotic compactness, tightness, Feller property, Aldous criterion.

1 Introduction

In this paper, we investigate the long time behavior of solutions of the stochastic damped KdV equation
\[
du + (\partial_x^3 u + u \partial_x u + \lambda u)dt = fdt + \Phi dW_t,
\]
with a nonzero deterministic force, by establishing the existence of an invariant measure.

Invariant measures play a crucial role in understanding the long time dynamics of solutions of stochastic partial differential equations \cite{[6, 13, 20, 17, 18]}. In particular, they were constructed for the stochastic Navier-Stokes system \cite{17}, the stochastic conservation laws \cite{15}, the stochastic primitive equations \cite{20}, and for many other equations and systems in mathematical physics. However, as far as we know, the existence of an invariant measure for the stochastic damped KdV equation is open, the difficulties being the non-compactness of the domain and the asymptotic compactness of the semigroup.

The existence and uniqueness of solutions for the stochastic KdV equation has been established by de Bouard and Debussche in \cite{[2]} (cf. also \cite{[1], [11], [12], [14], [31]}). However, the existence of invariant measure, which by the Krylov-Bogoliubov procedure requires the Feller property and the tightness property
of the time averages, has not been established except when including additional dissipative terms and in bounded domains \cite{27,28}.

There are two main difficulties in carrying out the Krylov-Bogoliubov procedure for of the stochastic damped KdV equation. The first difficulty is related to establishing the Feller property, whose proof usually follows from a priori estimates on the solutions and the dominated convergence theorem. However, in the case of the KdV equation, there are no a priori estimates up to deterministic times. In order to circumvent this difficulty, we use the results in \cite{9} and establish a priori estimates up to some stopping times, which are then used to show the Feller property of the transition semigroup.

The second and the main difficulty in establishing the existence of an invariant measure for (1.1) is the tightness of the time averages. In fact all known approaches fail due to the lack of compactness and dissipation. Thus, in order to obtain the tightness of averages, we are led to give a unconventional proof. Indeed, we study the equation on the whole domain which is unbounded with non-compact Sobolev embeddings. To show the tightness, we first use the existence results in \cite{9} in order to establish uniform estimates on the solutions of the equation. These bounds give us tightness of measures on the space $L^2_{\text{loc}}(\mathbb{R})$ of locally square integrable functions. To pass from tightness in $L^2_{\text{loc}}(\mathbb{R})$ to tightness in $L^2(\mathbb{R})$, one intuitively needs to show that there is no mass escaping to infinity, which in this stochastic framework means the convergence of the expectation of the square of the $L^2(\mathbb{R})$ norm to the expectation of the square of the $L^2(\mathbb{R})$ norm of the limiting measure. We then use a result in \cite{33} on the convergence in measure in Hilbert spaces, to obtain the tightness in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$.

In the deterministic case, there is a vast literature on the well-posedness of solutions of the KdV equation. Starting with the seminal of Temam \cite{36}, who established the global existence of weak solutions in $H^1$, then the existence and uniqueness in Sobolev spaces was established by Kato (cf. \cite{24}). The well-posedness theory was further studied by Bona-Smith \cite{2,3}, Saut-Temam \cite{35}, Bourgain \cite{4}, Kenig et al \cite{20,24}, Colliander et al \cite{5}, and by many other authors. The long time behavior of the KdV equations was initially studied by Ghidaglia in \cite{19}, who also established the existence and $H^2$ regularity of global attractors thus showing compactness at the infinite time. Further works by Moise, Rosa, Goubet, and Laurencot lowered the regularity of the force and showing infinite time compactness in periodic setting as well \cite{21,22,23,29,30}.

The existence and uniqueness of strong solutions of the stochastic KdV equation on the domain $\mathbb{R}$ with additive noise is established in \cite{9}. The authors provide estimates on the solutions of the linear KdV equation and use these estimates to show local in time existence of solutions for the nonlinear equation. Then using the estimates in $H^1(\mathbb{R})$, the authors show global existence of strong solutions. We also mention that the problem was also studied in \cite{32} on weighted Sobolev spaces.

The paper is organized as follows. In Section 2 we introduce the main notation, while Section 4 contains the main results. Section 5 contains the main steps of the proof, including the Feller property and the asymptotic compactness. The Appendix contains the proof of the convergence of norms, which is the crucial step in showing compactness in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$. 

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2 The notation and the main result

2.1 The stochastic Korteweg-de Vries equation

Fix a stochastic basis \((\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P})\). With \((e_i)_{i \in \mathbb{N}}\), an orthonormal basis of \(L^2(\mathbb{R})\), consisting of smooth compactly supported elements and \((\beta_i)_{i \in \mathbb{N}}\), a sequence of mutually independent one dimensional Brownian motions, denote by

\[ W(t) = \sum_{i \in \mathbb{N}} \beta_i(t)e_i \quad (2.1) \]

a cylindrical Wiener process on \(L^2(\mathbb{R})\). Consider the stochastic weakly damped Korteweg-de Vries equation

\[ du + (\partial_x^3 u + u \partial_x u + \lambda u) \, dt = f \, dt + \Phi \, dW(t), \quad (2.2) \]

where \(\lambda > 0\), with the initial condition

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \quad (2.3) \]

2.2 Notation

For functions \(u, v \in L^2(\mathbb{R})\), denote by \(\|u\|_{L^2}\) the \(L^2(\mathbb{R})\) norm of \(u\) and by \((u, v)\), the \(L^2\)-inner product of \(u\) and \(v\). For a Banach space \(B\) and with \(T > 0\) and \(p > 0\), denote by \(L^p([0, T]; B)\) the space of functions from \([0, T]\) into \(B\) with integrable \(p\)-th power over \([0, T]\) and by \(C([0, T]; B)\) the set of continuous functions from \([0, T]\) into \(B\).

The Fourier transform (resp. the inverse Fourier transform) of \(u\) is denoted by \(\mathcal{F}(u)\) (resp. \(\mathcal{F}^{-1}(u)\)). The Sobolev space \(H^\sigma(\mathbb{R})\) is the set of real functions \(u\) verifying

\[ \|u\|_{H^\sigma}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^{\sigma/2} |\mathcal{F}(u)(\xi)|^2 \, d\xi < \infty \quad (2.4) \]

and \(B(H^1(\mathbb{R}))\) is the set of Borel measurable subsets of \(H^1(\mathbb{R})\).

Similarly to functional spaces, for \(p > 0\) we denote by \(L^p(\Omega, B)\) the space of random variables with values in \(B\) and finite \(p\)-th moment.

2.3 Well-posedness of the equation

The equation \((2.2)\) was studied in \([9]\) in the case \(\lambda = 0\). The arguments carry over to the case \(\lambda > 0\) with only slight modifications.

For all \(\lambda \in \mathbb{R}\), we denote by \(\{U_\lambda(t)\}_{t \geq 0}\) the solution operator of the partial differential equation

\[ du(t) + (\partial_x^3 u + \lambda u) \, dt = 0. \quad (2.6) \]
Note that \( U_\lambda(t) = e^{-\lambda t}U_0(t) \). We then write the equation (2.2) in the mild form

\[
u(t) = U_\lambda(t)u_0 - \int_0^t U_\lambda(t-s)u(s)\partial_x u(s)\,ds + \int_0^t U_\lambda(t-s)f\,ds + \int_0^t U_\lambda(t-s)\Phi\,dW(s).
\]

(2.7)

Throughout the paper we assume that \( f \in H^3(\mathbb{R}) \) (2.8) and \( v \to (v,f) \) is continuous in \( L^2_{\text{loc}}(\mathbb{R}) \). (2.9)

We also require

\[
\Phi \in \text{HS}(L^2(\mathbb{R}); H^{3+}(\mathbb{R})).
\]

(2.10)

By \( \text{HS}(L^2(\mathbb{R}); H^{3+}(\mathbb{R})) \) we mean \( \text{HS}(L^2(\mathbb{R}); H^\sigma(\mathbb{R})) \) for some \( \sigma > 3 \). Recall that \( u \) is a mild solution of (2.2) if \( u \) verifies (2.7) for all \( t \geq 0, \mathbb{P} \text{-a.s.} \)

The following statement addresses existence and uniqueness of solutions.

**Theorem 2.1.** Assume that \( u_0 \in L^2(\Omega; H^1(\mathbb{R})) \cap L^4(\Omega; L^2(\mathbb{R})) \) is \( \mathcal{G}_0 \)-measurable. Then there exists a unique mild solution of (2.2) with paths almost surely in \( C([0, \infty); H^1(\mathbb{R})) \) and with \( u \in L^2(\Omega; L^\infty(0,T; H^1(\mathbb{R}))) \) for all \( T > 0 \). Additionally, if \( u_0 \in L^2(\Omega; H^3(\mathbb{R})) \) then \( u \in L^2(\Omega; L^\infty(0,T; H^3(\mathbb{R}))) \) for all \( T > 0 \).

The theorem follows as in [9] (Theorem 3.1 and Lemma 3.2) which treats the case \( \lambda = 0 \) and it is thus omitted. The inclusion in \( C([0, \infty); H^1(\mathbb{R})) \) is not explicitly mentioned in [9]. However with the assumption (2.10), we can use Theorem 3.2 and Proposition 3.5 of [9] to obtain this inclusion.

2.4 The Semigroup

Let \( u_0 \in H^1(\mathbb{R}) \) be a deterministic initial condition, and let \( u \) be the corresponding solution of (2.2). For all \( B \in \mathcal{B}(H^1(\mathbb{R})) \) we define the transition probabilities of the equation by

\[
P_t(u_0, B) = \mathbb{P}(u_t \in B).
\]

(2.11)

For any function \( \xi \in C_b(H^1; \mathbb{R}) \) and for \( t \geq 0 \) we denote

\[
P_t \xi(u_0) = \mathbb{E}[\xi(u_t)] = \int_{H^1} \xi(u)P_t(u_0, du).
\]

(2.12)

3 The main results

We shall rely on the Krylov-Bogoliubov procedure (cf. [8 Corollary 3.1.2]) to show the existence of invariant measures for the semigroup. The following statement is our main result.

**Theorem 3.1.** Suppose \( \lambda > 0 \), and assume that \( f \) and \( \Phi \) verify (2.8)–(2.10). Then there exists an invariant measure of the equation (2.2).

The proof is based on the following two lemmas. The first lemma states that the Feller property holds for the stochastic damped KdV equation.
Lemma 3.2. (Feller Property) Under the assumptions of Theorem 3.1, the semigroup $P_t$ is Feller on $H^1(\mathbb{R})$. Namely, for $\xi \in C_b(H^1, \mathbb{R})$ and with $u_0, u^2, \ldots \in H^1(\mathbb{R})$ satisfying $\|u_n^0 - u_0\|_{H^1} \to 0$ as $n \to \infty$, where $u_0 \in H^1(\mathbb{R})$, the convergence

$$P_t \xi(u^0_n) \to P_t \xi(u^0), \quad n \to \infty$$

holds for all $t \geq 0$.

The second lemma asserts tightness of averages originating from the initial datum $u_0 = 0$.

Lemma 3.3. (Tightness) Under the assumptions of Theorem 3.1, the family of measures on $H^1(\mathbb{R})$

$$\mu_n(\cdot) = \frac{1}{n} \int_0^n P_t(0, \cdot) dt, \quad n = 1, 2, \ldots$$

is tight.

The proof of Lemma 3.2 is provided in Section 4.3, while the proof of tightness is given in Section 4.4.

4 Proofs

4.1 Uniform bounds in $L^2(\mathbb{R})$

The next statement establishes bounds on $E[\|u(t)\|_{L^2}^2]$ which are uniform in $t$.

Lemma 4.1. Under the assumptions of Theorem 3.1, there exists a sequence $\{C_k\}_{k \geq 1}$ depending on $f$, $\Phi$, and $\lambda$ such that

$$\sup_{t \geq 0} E[\|u(t)\|_{L^2}^{2k}] \leq C_k(\lambda, \Phi)(E[\|u_0\|_{L^2}^{2k}] + 1)$$

holds for all $k \in \mathbb{N}$ for which $E[\|u_0\|_{L^2}^{2k}] < \infty$.

From here on, we shall consider $\lambda$, $f$, and $\Phi$ fixed and thus omit indicating the dependence of the constants on these two quantities.

Proof of Lemma 4.1. We define

$$\sigma_n = \left\{ t \geq 0 : \int_0^t \|u_s\|_{L^2}^2 ds \geq n \right\}$$

and apply Ito’s lemma to $\|u(t)\|_{L^2}^2$ in order to obtain

$$\|u_t\|_{L^2}^2 + 2\lambda \int_0^t \|u_s\|_{L^2}^2 ds = \|u_0\|_{L^2}^2 + t\|\Phi\|_{HS(L^2, L^2)}^2 + \int_0^t 2(u_s, f) ds + 2M_t$$

for $0 \leq t \leq \sigma_n$, where

$$M_t = \int_0^t \sum_i \int_{\mathbb{R}} u_x \Phi e_i dx d\beta_i^x.$$
We compute
\[ \mathbb{E}[M_t^{l_n \wedge n}] = \mathbb{E} \left[ \int_0^{t \wedge \sigma_n} \sum_i (u_s, \Phi e_i)^2 ds \right] \leq \|\Phi\|_{HS(L^2, L^2)}^2 \mathbb{E} \left[ \int_0^{t \wedge \sigma_n} \|u_s\|^2 ds \right] \leq n \|\Phi\|_{HS(L^2, L^2)}^2 t < \infty, \] (4.5)
and thus \( M_t^{l_n \wedge n} \) is a martingale. Taking the expectation of both sides of the equation at \( t \wedge \sigma_n \), we get
\[ \mathbb{E}[u(t \wedge \sigma_n)]^2 + 2\lambda \mathbb{E} \left[ \int_0^{t \wedge \sigma_n} \|u(s)\|^2 ds \right] = \mathbb{E}[u_0]^2 + \mathbb{E}[t \wedge \sigma_n]\|\Phi\|_{HS(L^2, L^2)}^2 + 2\mathbb{E} \left[ \int_0^{t \wedge \sigma_n} (u_s, f) ds \right]. \] (4.6)
Therefore, we obtain an upper bound
\[ \mathbb{E}[u(t \wedge \sigma_n)]^2 + \lambda \mathbb{E} \left[ \int_0^{t \wedge \sigma_n} \|u(s)\|^2 ds \right] \leq \mathbb{E}[u_0]^2 + t\|\Phi\|_{HS(L^2, L^2)}^2 + \frac{t\|f\|_{L^2}^2}{\lambda}, \] (4.7)
which is uniform in \( n \). Note that we have
\[ \int_0^{t \wedge \sigma_n} \|u(s)\|^2 ds = \int_0^t \|u(s)\|^2 ds 1_{\{\sigma_n > t\}} + n 1_{\{\sigma_n \leq t\}} \] (4.8)
from where
\[ \lambda n P(\sigma_n \leq t) \leq \lambda \mathbb{E} \left[ \int_0^{t \wedge \sigma_n} \|u(s)\|^2 ds \right] \leq C(t). \] (4.9)
Taking the limit as \( n \) goes to infinity, we see that the stopping time \( \sigma^* = \lim_{n \to \infty} \sigma_n \) verifies \( P(\sigma^* = \infty) = 1 \). We now return to the equality (4.4). Using the dominated convergence theorem and the fact that \( \|u(s)\|_{L^2} \) is continuous, we obtain
\[ \mathbb{E}[u(t)]^2 + 2\lambda \mathbb{E} \left[ \int_0^t \|u(s)\|^2 ds \right] = \mathbb{E}[u_0]^2 + t\|\Phi\|_{HS(L^2, L^2)}^2 + 2\mathbb{E} \left[ \int_0^t (u_s, f) ds \right]. \] (4.10)
We differentiate the previous equality and use the \( \epsilon \)-Young’s inequality to control \( \mathbb{E}(u_s, f) \) and obtain
\[ \frac{d}{dt} \mathbb{E}[u(t)]^2 + \lambda \mathbb{E}[u(t)]^2 \leq \|\Phi\|_{HS(L^2, L^2)}^2 + \frac{\|f\|_{L^2}^2}{\lambda}. \] (4.11)
Solving the resulting equation, we get
\[ \mathbb{E}[\|u(t)\|_{L^2}^2] \leq e^{-\lambda t} \mathbb{E}[\|u_0\|_{L^2}^2] + \left( \|\Phi\|_{HS(L^2, L^2)}^2 + \frac{\|f\|_{L^2}^2}{\lambda} \right) \int_0^t e^{-\lambda(t-s)} ds \leq C(\mathbb{E}[\|u_0\|_{L^2}^2] + 1), \] (4.12)
where the constant \( C \) depends on \( \Phi \) only through \( \|\Phi\|_{HS(L^2, L^2)}^2 \).
In order to use the induction for \( k \geq 1 \), we need to control \( \mathbb{E}[\sup_{0 \leq s \leq t} \|u(s)\|_{L^2}^2] \). To achieve this, we return to (4.3) and obtain
\[ \sup_{s \in [0,t]} \|u(s)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + C(t) + C \sup_{s \in [0,t]} |M_s|. \] (4.13)
Then, by the Burkholder-Davis-Gundy inequality
\[ \mathbb{E}[\sup_{s \in [0,t]} |M_s|] \leq C \mathbb{E} \left[ \left( \int_0^t \sum_i (u(s), \Phi_{e_i})^2 ds \right)^{1/2} \right] \]
\[ \leq C \left( \mathbb{E} \left[ \int_0^t \|u(s)\|_{L^2}^2 ds \right] \right)^{1/2} \leq C(t), \tag{4.14} \]
which then gives \( \mathbb{E}[\sup_{s \in [0,t]} \|u(s)\|_{L^2}^2] \leq C(t) \).

In order to use induction, suppose that for some \( k \in \mathbb{N} \)
\[ \sup_{t \geq 0} \mathbb{E} \left[ \|u(t)\|_{L^2}^{2k} \right] \leq C(\mathbb{E} \left[ \|u_0\|_{L^2}^{2k} \right] + 1) < \infty \tag{4.15} \]
and
\[ \mathbb{E}[\sup_{s \in [0,t]} \|u(s)\|_{L^2}^{2k}] \leq C_k(t). \tag{4.16} \]
Recall that, but the assumptions of the theorem,
\[ \mathbb{E} \left[ \|u_0\|_{L^2}^{2(k+1)} \right] < \infty. \tag{4.17} \]
Denote \( X_t = \|u(t)\|_{L^2}^2 \) and, similarly to the previous case, let
\[ \sigma_n = \inf \left\{ t \geq 0 : \int_0^t \|X_s^{k+1}\|_{L^2} ds \geq n \right\}. \tag{4.18} \]
Applying Ito’s lemma leads to
\[ dX_t^{k+1} = (k + 1)X_t^k \left(-2\lambda X_t dt + \|\Phi\|_{H^1(L^2)}^2 dt + 2(u(t), f) dt + 2dM_t \right) + 2(k + 1)kX_t^{k-1}\|\Phi u(t)\|_{L^2}^2 dt \]
\[ \leq -\lambda (k + 1)X_t^{k+1} dt + C(X_t^k + 1) dt + 2(k + 1)X_t^{k} dM_t. \tag{4.19} \]
The quadratic variation of the stochastic integral is proportional to
\[ \int_0^{T \wedge \sigma_n} X_s^{2k} d(M)_t \leq C \int_0^{T \wedge \sigma_n} X_s^{2k} \sum_i (u(t), \Phi_{e_i})^2 dt \leq C \int_0^{T \wedge \sigma_n} X_t^{2k+1} dt \leq C n \sup_{t \in T \wedge \sigma_n} \|u(t)\|_{L^2}^{2k} dt \tag{4.20} \]
where the brackets denote the quadratic variation and the last term is integrable due to (4.19). Thus \( \int_0^{T \wedge \sigma_n} X_s^{k} dM_s \) is a square integrable martingale. We take the expectation of (4.19) and obtain an upper bound
\[ \mathbb{E}[X_t^{k+1}] + 2(k + 1)\lambda \mathbb{E} \left[ \int_0^{T \wedge \sigma_n} X_s^{k+1} ds \right] \leq C(t) \tag{4.21} \]
which is uniform in \( n \). Similarly to (4.19), we have
\[ \frac{1}{\sqrt{n}} \left( \int_0^{T \wedge \sigma_n} X_s^{2k+1} ds \right)^{1/2} \geq 1_{\{\sigma_n \leq t\}}. \tag{4.22} \]
Thus
\[
\mathbb{P}(\sigma_n \leq t) \leq \frac{1}{\sqrt{n}} \mathbb{E} \left[ \left( \int_0^{t \land \sigma_n} X^k_{s} ds \right)^{1/2} \right] \\
\leq \frac{1}{\sqrt{n}} \mathbb{E} \left[ \left( \sup_{s \in [0,t]} X^k_{s} \right)^{1/2} \left( \int_0^{t \land \sigma_n} X^k_{s} ds \right)^{1/2} \right] \\
\leq \frac{C}{\sqrt{n}} \left( \mathbb{E} \sup_{s \in [0,t]} X^k_{s} \right) + \mathbb{E} \left[ \int_0^{t \land \sigma_n} X^k_{s} ds \right] \leq \frac{C(t)}{\sqrt{n}} \quad (4.23)
\]

which converges to 0 as \( n \to \infty \) due to \( 4.21 \) and the inductive assumption. Thus the stochastic integral in \( 4.19 \) is a martingale on \([0, \infty)\). Taking the expectation of \( 4.19 \), we get
\[
d\mathbb{E}[X^{k+1}_t] + \lambda (k+1) \mathbb{E}[X^{k+1}_t] dt \leq C dt \quad (4.24)
\]

which implies
\[
\mathbb{E}[X^{k+1}_t] \leq e^{-\lambda(k+1)t} \mathbb{E}[X^{k+1}_0] + C \int_0^t e^{-\lambda(k+1)(t-s)} ds \\
\leq C \left( \mathbb{E} \left[ \|u_0\|_{L^2}^{2(k+1)} \right] + 1 \right). \quad (4.25)
\]

In order to complete the proof, we also need to control \( \sup_{t \in [0,T]} X^{k+1}_t \). Note first that \( X_t \geq 0 \). From \( 4.19 \), we get
\[
\sup_{t \in [0,T]} X^{k+1}_t \leq C(T) \left( X^{k+1}_0 + \sup_{t \in [0,T]} X^k_t \right) + \sup_{t \in [0,T]} \left| \int_0^t 2(k+1)X^k_s dM_s \right|. \quad (4.26)
\]

We use the BDG inequality to bound the expectation of the last term with the expectation of the square root of the quadratic variation of the stochastic integral computed in \( 4.20 \) to obtain
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} X^{k+1}_t \right] \leq C(T) \quad (4.27)
\]

and the proof is complete. \( \square \)

### 4.2 Uniform bounds in \( H^1(\mathbb{R}) \)

We now present certain uniform bounds on the \( L^2 \) norm of \( \partial_x u \). Denote by
\[
I(v) = \int_{\mathbb{R}} \left( \partial_x v(x)^2 - \frac{v(x)^3}{3} \right) dx, \quad v \in H^1(\mathbb{R}) \quad (4.28)
\]

the second invariant of the deterministic KdV equation. For ease of notation we define
\[
\alpha(t) = \frac{\lambda}{3} \int u(t, x)^3 dx + \|\partial_x \Phi\|_{\text{HS}(L^2, L^2)}^2 - \sum_i \int u(t, x)(\Phi e_i)(x)^2 dx + 2(\partial_x u(t), \partial_x f) - (u^2(t), f).
\]

2
Lemma 4.2. The evolution of $I(u(t))$ is given by

$$dI(u(t)) + 2\lambda I(u(t))dt = \alpha(t)dt + 2(\partial_x u(t), \partial_x \Phi dW(t)) - (u^2(t), \Phi dW_t)$$

(4.29)

Moreover,

$$\sup_{t \geq 0} \mathbb{E} \left[ \|\partial_x u(t)\|^2_{L^2} \right] \leq C \left( \mathbb{E} \left[ \|u_0\|^2_{H^1} + \|u_0\|^4_{L^2} \right] + 1 \right)$$

(4.30)

for $k = 1$ or $2$, where $C$ is a constant.

Proof of Lemma 4.2. The identity (4.29) follows by Ito’s formula as in Lemma 3.3 of [9]. The quadratic variations of the stochastic integrals in (4.29) equal

$$\langle (\partial_x u(t), \partial_x \Phi dW(t)) \rangle = \sum_i (\partial_x u(t), \partial_x \Phi e_i)^2 dt \leq \|\partial_x u(t)\|^2_{L^2} \|\Phi\|^2_{HS(L^2, H^1)} dt$$

(4.31)

(the brackets denoting the quadratic variation) and

$$\langle (u^2(t), \Phi dW_t) \rangle = \sum_i (u^2(t), \Phi e_i)^2 dt \leq \|u(t)\|^4_{L^2} \sum_i \|\Phi e_i\|^2_{\infty} dt$$

$$\leq \|u(t)\|^4_{L^2} \sum_i \|\Phi e_i\|_{L^2} \|\partial_x \Phi e_i\|_{L^2} dt \leq \|u(t)\|^4_{L^2} \|\Phi\|^2_{HS(L^2, H^1)} dt.$$  (4.32)

Theorem 2.1 combined with the previous inequalities, shows that the stochastic integrals define martingales.

Now, we estimate $\alpha$. Using Agmon’s inequality

$$\|u\|_{L^\infty} \leq \|u\|_{L^2}^{1/2} \|\partial_x u\|_{L^2}^{1/2}.$$  (4.33)

we obtain

$$|\alpha(t)| \leq C \left( \|u\|_{L^2}^{5/2} \|\partial_x u\|_{L^2} \right)$$  (4.34)

$$\|u\|_{L^2} + \|\partial_x u\|_{L^2}.$$  (4.35)

Then we use the $\epsilon$-Young inequality and obtain

$$|\alpha(t)| \leq \lambda \|\partial_x u\|_{L^2}^2 + C(\|u\|_{L^2}^{10/3} + 1) \leq \lambda \|\partial_x u\|_{L^2}^2 + C(\|u\|_{L^2}^4 + 1).$$  (4.36)

Inserting these inequalities into (4.29) and taking expectation, we obtain

$$d\mathbb{E}[I(u(t))] + 2\lambda \mathbb{E}[I(u(t))] dt \leq \lambda \mathbb{E}[\|\partial_x u(t)\|^2_{L^2}] dt + C \left( 1 + \mathbb{E} \left[ \|u_0\|^4_{L^2} \right] \right) dt.$$  (4.37)

For all $v \in H^1(\mathbb{R})$ the inequalities in [33, Section 5] give

$$\frac{2}{3} \|\partial_x v\|^2_{L^2} - C \|v\|_{L^2}^{10/3} \leq I(v) \leq \frac{4}{3} \|\partial_x v\|^2_{L^2} + C \|v\|_{L^2}^{10/3}.$$  (4.38)

Using the left inequality, we get

$$\mathbb{E}[I(u(t))] \leq e^{-\lambda t/2} \mathbb{E}[I(u_0)] + C \int_0^t e^{-\lambda (t-s)/2} \left( 1 + \mathbb{E}[\|u_0\|^4_{L^2}] \right) ds$$

$$\leq C(\mathbb{E}[I(u_0)] + \mathbb{E}[\|u_0\|^4_{L^2}] + 1).$$  (4.39)

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By the second part of (4.37) we obtain
\[
\sup_{t \geq 0} \mathbb{E}[\|\partial_x u(t)\|_{L^2}^2] \leq C(\mathbb{E} [I(u_0) + \|u_0\|_{L^2}^4] + 1). \tag{4.39}
\]
Finally, combining all the inequalities above we conclude
\[
\sup_{t \geq 0} \mathbb{E}[\|u(t)\|_{H^1}^2] \leq C(\mathbb{E} [\|u_0\|_{H^1}^2 + \|u_0\|_{L^2}^4] + 1) \tag{4.40}
\]
which gives (4.30).

In order to obtain (4.30) for \( k = 2 \), we apply Itô’s Lemma to \( I^2(u(t)) \) and get
\[
dI^2(u(t)) + 4\lambda I^2(u(t))dt = 2I(u(t))\alpha(t)dt + d\widetilde{M}_t + \sum_i \left( 2(\partial_x u(t), \partial_x \Phi e_i) - (u^2(t), \Phi e_i) \right)^2 dt, \tag{4.41}
\]
where
\[
d\widetilde{M}_t = 2I(t) \sum_i \left\{ 2(\partial_x u(t), \partial_x \Phi e_i) - (u^2(t), \Phi e_i) \right\} dB^i_t.
\]
Similarly to the previous case, we have
\[
2I(u(t))\alpha(t) \leq 2\lambda I^2(u(t)) + C(1 + \|\partial_x u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^{20/3}).
\]
We also estimate the quadratic variation term as
\[
\sum_i \left( 2(\partial_x u(t), \partial_x \Phi e_i) - (u^2(t), \Phi e_i) \right)^2
\leq 8\|\partial_x u(t)\|_{L^2}^2 \|\partial_x \Phi\|_{HS(L^2,L^2)}^2 + 2\|u\|_{L^4}^4 \|\Phi\|_{HS(L^2,L^2)}^2
\leq C(\|\partial_x u(t)\|_{L^2}^2 + \|u\|_{L^2}^4), \tag{4.42}
\]
where the right hand side is bounded in expectation.

Finally, we compute the quadratic variation of \( \widetilde{M} \),
\[
d(\widetilde{M})_t = 4I^2(u(t)) \sum_i \left\{ 2(\partial_x u(t), \partial_x \Phi e_i) - (u^2(t), \Phi e_i) \right\}^2 dt
\leq CI^2(u(t))(\|\partial_x u(t)\|_{L^2}^2 + \|u(t)\|_{L^4}^4)dt
\leq CI^2(u(t))(\|\partial_x u(t)\|_{L^2}^2 + \|u\|_{L^2}^6).
\]
where we used (4.38) and the \( \epsilon \)-Young inequality. For all \( n \in \mathbb{N} \), we define the stopping time
\[
\tau_n = \inf \left\{ t \geq 0 : \int_0^t I^2(u(s))ds \geq n \right\} \tag{4.43}
\]
and
\[
\tau^* = \lim_n \tau_n. \tag{4.44}
\]
Integrating the evolution of $I^2(u(t))$ we obtain for all $T > 0$

$$\sup_{t \in [0,T]} I^2(u(t)) - I^2(u(0)) + 4\lambda \int_0^T I^2(u(s))ds$$

$$\leq \sup_{t \in [0,T]} \widetilde{M}_t + 2\lambda \int_0^T I^2(u(s))ds + C \int_0^T \left(\|\partial_x u(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^{20/3}\right)ds$$

We now take the expectation and use the Burkholder-Davis-Gundy inequality to obtain

$$\mathbb{E}\left[\sup_{t \in [0,T]} I^2(u(t)) - I^2(u(0)) + 4\lambda \int_0^T I^2(u(s))ds\right]$$

$$\leq 2\lambda \mathbb{E}\left[\int_0^T I^2(u(s))ds\right] + C\mathbb{E}\left[\left(\int_0^T I^2(u(t))(\|\partial_x u(t)\|_{L^2}^2 + 1)dt\right)^{1/2}\right]$$

$$+ C\mathbb{E}\left[\int_0^T \|\partial_x u(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^{20/3}ds\right],$$

We can use the $\epsilon$-Young inequality to have

$$\mathbb{E}\left[\left(\int_0^T I^2(u(t))(\|\partial_x u(t)\|_{L^2}^2 + 1)dt\right)^{1/2}\right] \leq \lambda \mathbb{E}\left[\int_0^T I^2(u(s))ds\right] + C\mathbb{E}\left[\sup_{t \in [0,T]} \|\partial_x u(t)\|_{L^2}^2\right].$$

Using this inequality in the previous estimate, we obtain that for all $T > 0$ we have

$$\mathbb{E}\left[\int_0^T I^2(u(s))ds\right] < C \tag{4.45}$$

for a constant independent of $n$ which implies that $\widetilde{M}$ is a martingale. Therefore,

$$\mathbb{E}\left[I^2(u(t))\right] + 2\lambda \int_0^t \mathbb{E}\left[I^2(u(s))\right]ds \leq \mathbb{E}\left[I^2(u(t))\right] + C \int_0^t \mathbb{E}\left[\|\partial_x u(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^{20/3}\right]ds,$$

where the function $s \mapsto \mathbb{E}\left[\|\partial_x u(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^{20/3}\right]$ is bounded. Proceeding as above, we obtain

$$\sup_{t \geq 0} \mathbb{E}\left[I^2(u(t))\right] < C\mathbb{E}\left[\|\partial_x u(0)\|_4^4 + \|u(0)\|_{L^2}^{20/3} + 1\right]$$

and the lemma is established. \hfill $\square$

In order to prove the Feller property, we need the following improvement of our estimates.

**Lemma 4.3.** For all $R_0, T > 0$ there exists a constant $C(R_0, T)$ such that we have

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|u(t)\|_{H^1}^2\right] \leq C(R_0, T) \tag{4.46}$$

for all initial conditions $u_0 \in H^1(\mathbb{R})$ verifying $\|u_0\|_{H^1} \leq R_0$. 

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Proof. We write (4.29) in the form

\[ I(u(t)) = I(u_0) + M_t - 2\lambda \int_0^t I(u(s)) + ||\partial_x \Phi||^2_{HS(L^2, L^2)} ds \]
\[ + \int_0^t \left( \frac{\lambda}{3} \int u^3(s, x) dx - \sum \int_{\mathbb{R}} u(s, x)(\Phi e_i)(x)^2 dx \right. \]
\[ \left. + 2(\partial_x u(s), \partial_x f) - (u^2(s), f) \right) ds \] (4.47)

where \( M_t \) is the martingale term. Then

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |I(u(t))| \right] \leq \mathbb{E} |I(u_0)| + C \int_0^T \mathbb{E} ||I(u(s))|| ds \]
\[ + C \int_0^T \mathbb{E} \left[ |u^3(s, x)| dx + ||\partial_x \Phi||^2_{HS(L^2, L^2)} + \sum \int_{\mathbb{R}} |u(s, x)|(\Phi e_i)(x)^2 dx \right] ds \]
\[ + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t| \right] \leq \mathbb{E} |I(u_0)| + CT(\|u_0\|^2_{H^1} + \|u_0\|^4_{L^2} + 1) + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t| \right]. \] (4.48)

The Burkholder-Davis-Gundy inequality, together with (4.31) and (4.32), gives the required bound for \( \mathbb{E} [\sup_{0 \leq t \leq T} |M_t|] \).

\[ \square \]

4.3 Proof of the Feller property

For all \( T \geq 0 \) we, as in [9], define

\[ X_1(T) = \left\{ u \in C(0, T; H^1(\mathbb{R})) \cap L^2(\mathbb{R}; L^\infty([0, T])); \right. \]
\[ \nabla \partial_x u \in L^\infty(\mathbb{R}, L^2([0, T])), \partial_x u \in L^4([0, T]; L^\infty(\mathbb{R})) \left. \right\} \] (4.49)

and denote

\[ \pi_T = \int_0^T U_\lambda(T - s) \Phi dW_s. \] (4.50)

For a stopping time \( \tau \) we define a shifted process by

\[ \pi_T^\tau = 0 \quad \text{for } T \leq \tau \] (4.51)

and

\[ \pi_T^\tau = \int_0^T U_\lambda(T - s) \Phi dW_s, \quad T \geq \tau. \] (4.52)

We start with the following auxiliary result.

Lemma 4.4 ([9]). Under the assumption (2.10), \( ||\pi||_{X_1(s)} \) is a continuous process.
Proof. Since $\pi$ is a solution of a linear equation, it is sufficient to prove that
\[
\|\pi\|_{X_1(s)} \to 0, \quad \text{P-a.s.} \tag{4.53}
\]
as $s \to 0$. Now, $X_1(s)$ is defined in (4.49) as the intersection of four spaces and thus we need to show convergence to $0$ for all four norms.

Note that $\|\pi(s)\|_{H^1}$, for $s \in [0, T]$ is a continuous uniformly integrable semi-martingale and thus
\[
E \left[ \sup_{0 \leq r \leq s} \|\pi(r)\|_{H^1} \right] \to 0 \quad \text{as} \quad s \to 0. \tag{4.54}
\]
For the norms associated to $L^2(\mathbb{R}; L^\infty([0, T]))$ and $L^4([0, T]; L^\infty(\mathbb{R}))$, the convergence follows by the monotone convergence theorem. The only issue is for the $L^\infty(\mathbb{R}, L^2([0, T]))$ norm. In order to show convergence, we modify the proof of [9, Proposition 3.3] to obtain
\[
E \left[ \sup_{x \in \mathbb{R}} \int_0^s |D\partial_x \pi(r)|^2 \, dr \right] \leq C(\lambda, s)\|\Phi\|_{HS(L^2, H^1)}^2 \tag{4.55}
\]
where $C(\lambda, s) \to 0$ as $s \to 0$, which completes the proof. □

We now return to the proof of the Feller property. Fix $u_0 \in H^1(\mathbb{R})$ and $t > 0$. Also, let $\xi \in C_b(H^1, \mathbb{R})$ and $\epsilon > 0$. Denote $R_0 = \|u_0\|_{H^1} + 1$ and
\[
M = \sup_{v \in H^1(\mathbb{R})} |\xi(v)|. \tag{4.56}
\]

Step 1: For all $v_0$ such that $\|v_0 - u_0\|_{H^1} \leq 1$ with the associated solution $v$ of (2.2), Lemma 4.3 gives
\[
P \left( \max \left\{ \sup_{s \in [0, t]} \|u(s)\|_{H^1}, \sup_{s \in [0, t]} \|v(s)\|_{H^1} \right\} \geq R \right)
\leq \frac{1}{R} E \left[ \sup_{s \in [0, t]} \|u(s)\|_{H^1}^2 + \sup_{s \in [0, t]} \|v(s)\|_{H^1}^2 \right]
\leq \frac{C(R_0)}{R}. \tag{4.57}
\]
(Note that $t > 0$ is fixed and thus the dependence of all constants on $t$ is not indicated.) Fix $R > 0$ so that the last term verifies
\[
\frac{C(R_0)}{R} \leq \frac{\epsilon}{6M} \tag{4.58}
\]

Step 2: Using results in [9], we obtain a non-decreasing function $\tilde{C}$ such that we have the inequalities
\[
E\|\pi\|_{X_1(T)} + \left\| \int_0^T U_\lambda(\cdot - r) f \, dr \right\|_{X_1(T)} \leq \tilde{C}(T) \tag{4.59}
\]
with
\[
\left\| \int_0^T U_\lambda(\cdot - s) h(s) \partial_x g(s) \, ds \right\|_{X_1(T)} \leq \tilde{C}(T) T^{1/2}\|h\|_{X_1(T)}\|g\|_{X_1(T)} \tag{4.60}
\]
for all \( h, g \in X_1(T) \) and
\[
\| U_\lambda(\cdot)v_0 \|_{X_1(T)} \leq \tilde{C}(T)\|v_0\|_{H^1}
\] (4.61)
for \( v_0 \in H^1(\mathbb{R}) \). Since \( \tilde{C} \) is non-decreasing, we may increase it so that it is also continuous. For all \( s \leq t \) the solutions \( u \) and \( v \) verify
\[
u(s) = T_{u_0}(u)(s)
\] (4.62)
and
\[
v(s) = T_{v_0}(v)(s)
\] (4.63)
where
\[
T_h(g)(s) = U_\lambda(s)h - \int_0^s U_\lambda(s - r)g(r)\partial_2 g(r) \, dr + \int_0^s U_\lambda(s - r)f \, dr + \Pi(s)
\] (4.64)
for \( h \in H^1(\mathbb{R}) \) and \( g \in X_1(t) \). Define
\[
\tau = \inf \left\{ s \geq 0 : 8\tilde{C}(t)s^{1/2} \left( \tilde{C}(t)R + \|\Pi\|_{X_1(s)} + \left\| \int_0^s U_\lambda(\cdot - r)f \, dr \right\|_{X_1(s)} \right) > 1 \right\}.
\] (4.65)
By Lemma 4.4 \( \tau \) is a stopping time. Note that
\[
P(\tau < s) \leq P \left( 8\tilde{C}(t)s^{1/2} \left( \tilde{C}(t)R + \|\Pi\|_{X_1(s)} + \left\| \int_0^s U_\lambda(\cdot - r)f \, dr \right\|_{X_1(s)} \right) > 1 \right) \leq E \left[ 8\tilde{C}(t)s^{1/2} \left( \tilde{C}(t)R + \|\Pi\|_{X_1(s)} + \left\| \int_0^s U_\lambda(\cdot - r)f \, dr \right\|_{X_1(s)} \right] \leq \left( 8\tilde{C}^2(t)s^{1/2}(R + 1) \right) \leq C(R)s^{1/2}
\] (4.66)
and thus
\[
E[\tau] = \int_0^\infty P(s \leq \tau) \, ds = \int_0^\infty (1 - P(s > \tau)) \, ds 
\geq \int_0^{1/C(R)^2} \left( 1 - 1 \land (C(R)s^{1/2}) \right) \, ds \geq \frac{1}{3C(R)^2}
\] (4.67)
where the dependence on \( t \) is understood.

**Step 3:** We now inductively define a sequence of stopping times. We start with
\[
\tau_0 = \tau
\] (4.68)
and then for \( k = 0, 1, \ldots \) let
\[
\tau_{k+1} = \inf \left\{ s \geq \tau_k : 8\tilde{C}(t)(s - \tau_k)^{1/2} \left( \tilde{C}(t)R + \|\Pi\|_{X_1(\tau_k,s)} + \left\| \int_{\tau_k}^s U_\lambda(\cdot - r)f \, dr \right\|_{X_1(\tau_k,s)} \right) > 1 \right\}
\] (4.69)
where
\[ \mathbb{P}^s = \int_{r_k}^s U_x(s-r) \Phi dW_r \] (4.70)
and \( X_1(\tau_k, s) \) is defined similarly to \( X_1(T) \) for shifted process (defined on \([\tau_k, s])\). For simplicity of notation, set \( \tau_{-1} = 0 \)

Note that \( \mathbb{P}^s = 0 \) for \( s \leq \tau_k \). For \( s \geq \tau_k \), we have that \( \mathbb{P}^s \) is \( \sigma(W_r - W_{r_k}, r \in [\tau_k, s]) \)-measurable. Therefore, \( \tau_{k+1} - \tau_k \) is independent from \( G_{\tau_k} \) and has the same distribution as \( \tau \).

By the law of large numbers, a.s.
\[ \frac{\tau_n}{n} = \frac{1}{n} \sum_{i=0}^{n} (\tau_i - \tau_{i-1}) \rightarrow \mathbb{E}[\tau] \geq \frac{1}{3C^2(R)} \] (4.71)
as \( n \rightarrow \infty \), where the constant is the same as in \( 1.67 \). Thus the sequence of random variables
\[ 1\{\tau_n \leq t\} = 1\{\tau_n/n - \mathbb{E}[\tau] \leq t/n - \mathbb{E}[\tau]\} \] (4.72)
converges \( \mathbb{P} \)-a.s. to 0 as \( n \rightarrow \infty \). By the dominated convergence theorem \( \mathbb{P}(\tau_n \leq t) \rightarrow 0 \) as \( n \rightarrow \infty \).

Hence, there exists \( n > 0 \) depending only on \((R, \epsilon, M)\) such that
\[ \mathbb{P}(\tau_n \leq t) \leq \frac{\epsilon}{6M}. \] (4.73)
Therefore, for all \( v_0 \) satisfying \( \|u_0 - v_0\|_{H^1} \leq 1 \), we have
\[
\mathbb{E}[|\xi(u(t)) - \xi(v(t))|] \\
\leq \mathbb{E}\left[|\xi(u_t) - \xi(v_t)|1_{\max\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|v(s)\|_{H^1}^2 \geq R}\right] \\
+ \mathbb{E}\left[|\xi(u_t) - \xi(v_t)|1_{\max\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|v(s)\|_{H^1}^2 \leq R}\right] 1_{\{\tau_n \leq t\}} \\
+ \mathbb{E}\left[|\xi(u_t) - \xi(v_t)|1_{\max\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|v(s)\|_{H^1}^2 \leq R\}} \leq R\right] 1_{\{\tau_n \geq t\}} \\
= T_1 + T_2 + \mathbb{E}\left[|\xi(u_t) - \xi(v_t)|1_{\max\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|v(s)\|_{H^1}^2 \leq R\}} \right] 1_{\{\tau_n \geq t\}} \right). \] (4.74)

Note that by the choice of \( R \), we have \( T_1 \leq \epsilon/3 \). Similarly, by \( 4.73 \), \( T_2 \leq \epsilon/3 \). Thus for all \( v_0 \) such that \( \|u_0 - v_0\|_{H^1} \leq 1 \), we have
\[
\mathbb{E}[|\xi(u(t)) - \xi(v(t))|] \\
\leq \frac{2\epsilon}{3} + \mathbb{E}\left[|\xi(u_t) - \xi(v_t)|1_{\max\{\sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|v(s)\|_{H^1}^2 \leq R\}} \right] 1_{\{\tau_n \geq t\}} \right). \] (4.75)

In order to continue our analysis, we need the following lemma.

**Lemma 4.5.** Denote the event
\[ A = \left\{ \max \left\{ \sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|v(s)\|_{H^1}^2 \right\} \leq R \right\}. \] (4.76)
Then for all \( k \in \mathbb{N}_0 \) we have the inequality
\[
\sup_{s \in [0,t]} \|u(t) - v(t)\|_{H^1} \leq (2C(t))^{k+1} \|u_0 - v_0\|_{H^1} \] (4.77)
on the event \( A \cap \{\tau_k \geq t\} \).
**Proof of Lemma 4.5.** We start the induction with $k = 0$. Fix $s \in [0,t]$. Then on the set $\{\tau_0 > s\}$, we have

$$\sup_{r \in [0,s]} \|u(r) - v(r)\|_{H^1} \leq 2\tilde{C}(t)\|u_0 - v_0\|_{H^1}. \quad (4.78)$$

Indeed, let

$$L_0 = \tilde{C}(t)R + \|\Xi\|_{X_1(s)} + \left\| \int_0^s U_{\lambda}(\cdot - r) f \, dr \right\|_{X_1(s)} \quad (4.79)$$

and let $g \in X_1(s)$ be such that $\|g\|_{X_1(s)} \leq 2L_0$. Then on the set $\{\tau_0 > s\}$

$$\|\mathcal{T}_{u_0}(g)\|_{X_1(s)} \leq \tilde{C}(s)\|u_0\|_{H^1} + \left\| \int_0^s U_{\lambda}(\cdot - r)g(r)\partial_x g(r) \, dr \right\|_{X_1(s)} + \left\| \tilde{L}_1^s \right\|_{X_1(s)} + \left\| \Xi \right\|_{X_1(s)}$$

$$\leq \tilde{C}(t)R + \|\Xi\|_{X_1(s)} + \left\| \tilde{L}_1^s \right\|_{X_1(s)} + \left\| \tilde{L}_1^s \right\|_{X_1(s)} + \tilde{C}(t)s^{1/2}\|g\|_{X_1(s)}^2 \leq L_0 + 4L_0^2\tilde{C}(t)s^{1/2} \leq 2L_0 \quad (4.80)$$

where the last inequality holds due to inclusion $\{\tau_0 > s\} \subset \{L_0 \leq 1/8\tilde{C}(t)s^{1/2}\}$.

Note that $u$ is a fixed point of $\mathcal{T}_{u_0}$ and that $\mathcal{T}_{u_0}$ maps the ball of radius $2L_0$ of $X_1(s)$ into itself. Then $\|u\|_{X_1(s)} \leq 2L_0$. Similarly, $\|v\|_{X_1(s)} \leq 2L_0$. Now, observe that $u$ (resp. $v$) is a fixed point of $\mathcal{T}_{u_0}$ (resp. $\mathcal{T}_{v_0}$). Therefore, on the set $A \cap \{\tau_0 > s\}$,

$$\|u - v\|_{X_1(s)} = \|\mathcal{T}_{u_0}(u) - \mathcal{T}_{v_0}(v)\|_{X_1(s)}$$

$$= \|U_{\lambda}(\cdot)(u_0 - v_0)\|_{X_1(s)} + \left\| \int_0^s U_{\lambda}(\cdot - r)((u(r) - v(r))\partial_x u(r) + v(r)\partial_x(u(r) - v(r))) \, dr \right\|_{X_1(s)}$$

$$\leq \tilde{C}(t)\|u_0 - v_0\|_{H^1} + \tilde{C}(t)s^{1/2}\|u - v\|_{X_1(s)}(\|u\|_{X_1(s)} + \|v\|_{X_1(s)})$$

$$\leq \tilde{C}(t)\|u_0 - v_0\|_{H^1} + 4\tilde{C}(t)s^{1/2}L_0\|u - v\|_{X_1(s)}$$

$$\leq \tilde{C}(t)\|u_0 - v_0\|_{H^1} + \frac{1}{2}\|u - v\|_{X_1(s)} \quad (4.81)$$

which implies that on the set $\{\tau_0 > s\}$,

$$\sup_{r \in [0,s]} \|u(r) - v(r)\|_{H^1} \leq \|u - v\|_{X_1(s)} \leq 2\tilde{C}(t)\|u_0 - v_0\|_{H^1}. \quad (4.82)$$

By the continuity in time of the processes, we have

$$\sup_{r \in [0,\tau_0]} \|u(r) - v(r)\|_{H^1} \leq 2\tilde{C}(t)\|u_0 - v_0\|_{H^1}. \quad (4.83)$$

We finish the proof by induction. Assume that for some $k \leq n - 1$, we have on the set $A \cap \{\tau_k \geq t\}$

$$\sup_{r \in [0,\tau_k]} \|u(r) - v(r)\|_{H^1} \leq (2\tilde{C}(t))^{k+1}\|u_0 - v_0\|_{H^1}. \quad (4.84)$$

In order to obtain the heredity, we need to give the upper bound on $A \cap \{\tau_{k+1} \geq t > \tau_k\}$.

Note that by the strong Markov property for all $s \geq \tau_k$

$$u_s = U_{\lambda}(s - \tau_k)u_{\tau_k} - \int_{\tau_k}^s U_{\lambda}(s - r)u_r\partial_x u_r \, dr + \int_{\tau_k}^s U_{\lambda}(s - r)f \, dr + \Xi_s^{\tau_k} \quad (4.85)$$
Using the dominated convergence theorem and the previous lemma, we have

$$v_s = U_\lambda(s - \tau_k)v_{\tau_k} - \int_{\tau_k}^s U_\lambda(s - r)v_r \partial_x v_r \, dr + \int_{\tau_k}^s U_\lambda(s - r)f \, dr + \pi_k^x.$$ (4.86)

On the set \( A \cap \{ \tau_{k+1} \geq t > \tau_k \} \), we have \( \| u_{\tau_k} \|_{H^1} \leq R \) and \( \| v_{\tau_k} \|_{H^1} \leq R \). Hence, we may define \( L_{k+1} = \tilde{C}(t)R + \| \pi_k^x \|_{X_1(\tau_{k+1}, s)} + \left\| \int_{\tau_k}^s U_\lambda(s - r)f \, dr \right\|_{X_1(\tau_{k+1}, s)} \) for \( s \in [\tau_k, t] \).

Similarly, we can prove that on the set \( A \cap \{ s < \tau_k \} \) we have \( \| u \|_{X_1(\tau_k, s)} \leq 2L_{k+1} \) and \( \| u \|_{X_1(\tau_k, s)} \leq 2L_{k+1} \) and proceed as in the proof for \( k = 0 \) that

$$\sup_{r \in [\tau_k, \tau_{k+1}]} \| u(r) - v(r) \|_{H^1} \leq 2\tilde{C}(t)\| u(\tau_k) - v(\tau_k) \|_{H^1} \leq (2\tilde{C}(t))^{k+2} \| u_0 - v_0 \|_{L^2}$$ (4.87)

and the lemma is established.

We now continue the proof of the Feller property starting from the inequality \( (4.78) \). Using the previous lemma we have

$$\| u(t) - v(t) \|_{H^1} \leq (2\tilde{C}(t))^{k+1} \| u_0 - v_0 \|_{H^1} \quad \text{on } A \cap \{ \tau_n \geq t \}.$$ (4.88)

Therefore, as \( v_0 \to u_0 \) we have

$$\| \xi(u) - \xi(v) \|_{\mathbb{H}^1} \leq \left\| \left\{ \sup_{s \in [0, t]} \| u(s) \|_{H^1}^2, \sup_{s \in [0, t]} \| v(s) \|_{H^1}^2 \right\} \leq R \right\} \mathbb{1}_{\{ \tau_n \geq t \}} \to 0 \text{ a.s.}$$ (4.89)

Using the dominated convergence theorem

$$\mathbb{E} \left[ \| \xi(u) - \xi(v) \|_{\mathbb{H}^1} \leq \left\{ \sup_{s \in [0, t]} \| u(s) \|_{H^1}^2, \sup_{s \in [0, t]} \| v(s) \|_{H^1}^2 \right\} \leq R \right] \mathbb{1}_{\{ \tau_n \geq t \}} \to 0$$ (4.90)

as \( v_0 \to u_0 \) in \( H^1 \). Note that the choice of \( R \) and \( n \) does not depend on \( v_0 \).

### 4.4 Asymptotic compactness of the semi-group

We use the distributional convergence over various Sobolev spaces. In order to fix the ideas, we first recall the definition.

**Definition 4.1.** Let \( \Gamma \) be a topological vector space, and let \( \{ X_n \}_{n \geq 0} \) and \( X_\infty \) be random variables taking values in \( \Gamma \), possibly defined in different probability spaces. We say that \( X_n \) converges to \( X_\infty \) in distribution in \( \Gamma \) if

$$\mathbb{E}[F(X_n)] \to \mathbb{E}[F(X_\infty)] \text{ as } n \to \infty$$ (4.91)

for all continuous bounded functions \( F: \Gamma \to \mathbb{R} \).

We shall exploit the classical results on the asymptotic compactness of the solution operator of the KdV equation in order to prove the following lemma.

**Lemma 4.6.** For any sequence of deterministic initial condition \( u_0^n \) satisfying

$$R := \sup_n \left\{ \| u_0^n \|_{H^1}^2 \right\} < \infty$$

and a sequence of nonnegative numbers \( t_1, t_2, \ldots \) such that \( \lim_{n \to \infty} t_n = \infty \), the set of probabilities \( \{ P_n(u_0^n, \cdot) : n \in \mathbb{N} \} \) is tight in \( H^1 \).
Proof of Lemma 4.6. Without loss of generality, we may assume that \( t_1, t_2, \ldots \) is increasing. Let \( \{u^n_0\}_{n=1}^\infty \) be a sequence of initial conditions as above. We denote by \( \{u^n(t)\}_{n=1}^\infty \) the respective solutions of \( \text{\ref{eq:2.2}} \).

We intend to show that there is a subsequence of \( \{u^n_0\} \) that converges in distribution in \( H^1 \).

By Lemma 4.2 we have the bound
\[
\sup_n \mathbb{E}[\|u^n(t_n)\|^2_{H^1}] \leq C(R). \tag{4.92}
\]

**Step 1: convergence in distribution in \( L^2_{\text{loc}}(\mathbb{R}) \)**

Bounded sets of \( H^1(\mathbb{R}) \) are relatively compact in \( L^2_{\text{loc}}(\mathbb{R}) \). Thus the inequality \( \text{\ref{eq:4.92}} \) and Prokhorov’s theorem in \( L^2_{\text{loc}}(\mathbb{R}) \) allow us to conclude that there exists an \( L^2_{\text{loc}} \) valued random variable \( \xi \) (possibly defined on another probability space) and a subsequence of \( \{u^n_0\} \) such that
\[
u^n_0 \to \xi \text{ in distribution in } L^2_{\text{loc}} \text{ as } n \to \infty. \tag{4.93}
\]

Let \( \{f_i\}_{i=1}^\infty \) be an orthonormal basis of \( H^1(\mathbb{R}) \) with \( f_i \) smooth and compactly supported. For all \( i \in \mathbb{N} \) and \( M > 0 \) we define the mapping
\[
v \to \psi_{i,M}(v) := |(v, f_i)_{H^1}| \wedge M = |(v, (1 - \partial_x^2)f_i)| \wedge M \]

which is continuous in \( L^2_{\text{loc}}(\mathbb{R}) \) and bounded. Therefore the distributional \( L^2_{\text{loc}}(\mathbb{R}) \) convergence imply
\[
\mathbb{E} \left[ \sum_{i=1}^N \left( (u^n_{t_n}, f_i)_\mathcal{H}^1 \wedge M^2 \right) \right] \to \mathbb{E} \left[ \sum_{i=1}^N \left( (\xi, f_i)_\mathcal{H}^1 \wedge M^2 \right) \right].
\]

Therefore, for all \( N \in \mathbb{N} \) and \( M > 0 \)
\[
\mathbb{E} \left[ \sum_{i=1}^N \left( (\xi, f_i)_\mathcal{H}^1 \wedge M^2 \right) \right] \leq C(R). \tag{4.94}
\]

Sending \( M \) to infinity, by Fatou’s Lemma we obtain that
\[
\mathbb{E} \left[ \sum_{i=1}^N \left( (\xi, f_i)_\mathcal{H}^1 \wedge M^2 \right) \right] \leq C(R) \tag{4.95}
\]

and similarly sending \( N \) to infinity, we get
\[
\mathbb{E}[\|\xi\|_{\mathcal{H}^1}^2] \leq \liminf_N \mathbb{E} \left[ \sum_{i=1}^N (\xi, f_i)_\mathcal{H}^1 \right] \leq C(R). \tag{4.96}
\]

This shows that \( \xi \) is \( H^1(\mathbb{R}) \)-valued.

**Step 2: convergence in distribution in \( L^2 \).** In order to prove this, we use
\[
\lim_n \mathbb{E}[\|u^n_{t_n}\|^2_{\mathcal{H}^1}] = \mathbb{E}[\|\xi\|_{\mathcal{H}^1}^2]. \tag{4.97}
\]

The proof of this fact is given in the appendix.

Recall from Section 2 that \( \{e_i\} \) denotes an orthonormal basis of \( L^2(\mathbb{R}) \) consisting of smooth compactly supported functions. For all \( N \in \mathbb{N} \) and \( M > 0 \), the convergence in distribution in \( L^2_{\text{loc}}(\mathbb{R}) \) of \( u^n_{t_n} \) implies that
\[
\mathbb{E} \left[ \sum_{i=1}^N (u^n_{t_n}, e_i)^2 \wedge M^2 \right] \to \mathbb{E} \left[ \sum_{i=1}^N (\xi, e_i)^2 \wedge M^2 \right]. \tag{4.98}
\]
Using the inequality (4.1), we obtain that the family \( \|u^n_t\|_{L^2}^2 \) is uniformly integrable. Thus we can take \( M \) to infinity and obtain
\[
E \left[ \sum_{i=1}^{N} (u^n_{t_n}, e_i)^2 \right] \to E \left[ \sum_{i=1}^{N} (\xi, e_i)^2 \right].
\]
(4.99)

Therefore, combined with (4.97), we get
\[
E \left[ \sum_{i=N+1}^{\infty} (u^n_{t_n}, e_i)^2 \right] \to E \left[ \sum_{i=N+1}^{\infty} (\xi, e_i)^2 \right].
\]
(4.100)

Now, fix \( \epsilon > 0 \). There exists \( N_0 \in \mathbb{N} \) such that
\[
E \left[ \sum_{i=N_0+1}^{\infty} (\xi, e_i)^2 \right] \leq \epsilon / 2.
\]
(4.101)

Then, using (4.100), there exists \( n_\epsilon \in \mathbb{N} \) such that
\[
\sup_{n \geq n_\epsilon} E \left[ \sum_{i=N_0+1}^{\infty} (u^n_{t_n}, e_i)^2 \right] \leq \epsilon.
\]
(4.102)

By the uniform second moment bounds (4.92),
\[
\lim_{N \to \infty} E \left[ \sum_{i=N}^{\infty} (u^n_{t_n}, e_i)^2 \right] = 0, \quad n \leq n_\epsilon - 1.
\]
(4.103)

By (4.102) and (4.103), there exists \( N_1 \geq N_0 \) such that
\[
\sup_{n \in \mathbb{N}} E \left[ \sum_{i=N_1+1}^{\infty} (u^n_{t_n}, e_i)^2 \right] \leq \epsilon.
\]
(4.104)

Thus we have proven that
\[
\lim_{N \to \infty} \sup_{n} E \left[ \sum_{i=N}^{\infty} (u^n_{t_n}, e_i)^2 \right] = 0.
\]
(4.105)

By [32, Theorem 1.13] this implies tightness in distribution in \( L^2 \) of measures of \( \{u^n_{t_n}\} \). Note that any limiting measure can only be the measure of \( \xi \). Thus
\[
u^n_{t_n} \to \xi
\]
(4.106)
in distribution in \( L^2 \).

We emphasize that we have not taken any further subsequence to pass from (4.93) to (4.106). We have proven that any limit in distribution in \( L^2_{loc}(\mathbb{R}) \) of \( \{u^n_{t_n}\} \) is also its limit in distribution in \( L^2(\mathbb{R}) \).  

**Step 3: Convergence in distribution in \( H^1 \)** 

The fundamental tool is the fact
\[
E[I(\xi)] = \lim_n E[I(u^n_{t_n})]
\]
(4.107)
the proof of which is given in the appendix. Note that we have the uniform bounds (4.92) and convergence in distribution in \( L^2 \). Using Agmon’s inequality, we obtain that the mapping \( v \to \int v^3(x) dx \) is continuous.
Given the choice of \( R > 0 \) (independent of \( n \)), we define the events

\[
A^n = \left\{ \max_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|u^n(s)\|_{H^1}^2 \leq R \right\}.
\]

in distribution in \( H^1 \). Thus, using again the uniform integrability of the families \( \|\partial_x u\|_{L^2}^2 \) and \( \|u\|_{L^2}^2 \),

\[
\mathbb{E} \left[ \int (u^n_k(x))^3 \, dx \right] \to \mathbb{E} \left[ \int (\xi(x))^3 \, dx \right].
\]  

Combined with (4.107) this implies

\[
\lim_{n} \mathbb{E}[\|\partial_x u^n_k\|_{L^2}^2] = \mathbb{E}[\|\partial_\xi\|_{L^2}^2].
\]  

Note that the inequality (4.30), for \( k = 2 \) gives the uniform integrability of \( \|u^n_k\|_{H^1}^2 \). Repeating for the space \( H^1 \) the same ideas that allowed us to obtain the convergence in distribution in \( L^2 \) we obtain

\[
u^n_k \to \xi
\]

in distribution in \( H^1 \). \( \Box \)

Before proving the lemma 5.3 we prove the following lemma.

**Lemma 4.7.** Let \( K \) be a compact subset of \( H^1(\mathbb{R}) \). Then the set of measures on \( H^1(\mathbb{R}) \)

\[
\{ P_s(v, \cdot) : s \in [0,1], v \in K \}
\]

is tight.

**Proof.** We will take a countable subset \( \{ P_s(v, \cdot) : s \in [0,1], v \in K \} \) and show that it has a convergent subsequence. Let \( (s^n, v^n) \in [0,1] \times K \). By compactness of the sets there exists a subsequence of \( (s^n, v^n) \) (still denoted \( (s^n, v^n) \)) the converges to \( (s, v) \in [0,1] \times K \). Denote by \( u^n \) the solution of (1.1) with initial data \( v^n \) and by \( u \) the solution of (1.1) with initial data \( v \).

We now prove that there exists a subsequence \( (s^{n_k}, v^{n_k}) \) such that

\[
P\text{-a.s. } \lim_{k} \|u^{n_k} - u\|_{H^1} + \|u^{n_k} - u\|_{H^1} \to 0 \quad \text{as } k \to \infty.
\]  

(4.111)

The almost sure convergence \( \|u^n - u\|_{H^1} \to 0 \) is a direct consequence of \( u \in C([0,1]; H^1) \). Fix \( \epsilon > 0 \) and \( \delta > 0 \) and similarly to the proof of the Feller property denote \( R_0 = \sup_{v \in K} \|v\|_{H^1} + 1 \). We choose \( R > 0 \) (independent of \( n \)) such

\[
P\left( \max \left\{ \sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|u^n(s)\|_{H^1}^2 \right\} \geq R \right) \leq \frac{1}{R} \mathbb{E} \left[ \sup_{s \in [0,t]} \|u(s)\|_{H^1}^2 + \sup_{s \in [0,t]} \|u^n(s)\|_{H^1}^2 \right] \leq \frac{C(R_0)}{R} \leq \epsilon/2.
\]

Given the choice of \( R \) we define the hitting times \( \tau_k \) as in (4.68) (with \( t = 1 \)). We choose \( N \) such that

\[
P(\tau_N \leq 1) \leq \epsilon/2.
\]

We define the events

\[
A^n = \left\{ \max \left\{ \sup_{s \in [0,t]} \|u(s)\|_{H^1}^2, \sup_{s \in [0,t]} \|u^n(s)\|_{H^1}^2 \right\} \leq R \right\}.
\]  

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Thanks to Lemma 4.5 on the set \( A^n \cap \{ \tau_N \geq 1 \} \) one has
\[
\sup_{s \in [0,1]} \| u(t) - u^n(t) \|_{H^1} \leq (2\tilde{C}(1))^{N+1} \| v - v^n \|_{H^1}.
\]
We choose \( n \) large enough to have \( (2\tilde{C}(1))^{N+1} \| v - v^n \|_{H^1} \leq \delta \). This implies that for all \( n \) large enough
\[
\mathbb{P} \left( \sup_{s \in [0,1]} \| u(t) - u^n(t) \|_{H^1} \leq \delta \right) \geq \mathbb{P} (A^n \cap \{ \tau_N \geq 1 \}) \geq 1 - \epsilon
\]
which is exactly \( \sup_{s \in [0,1]} \| u(t) - u^n(t) \|_{H^1} \to 0 \) as \( n \to \infty \) in probability. Hence we can take a subsequence that converges almost surely. Let \( \xi \) be a real valued uniformly continuous function on \( H^1(\mathbb{R}) \). By direct estimates
\[
| P_{s, n_k} \xi (v_{n_k}) - P_s \xi (v) | \leq \mathbb{E} \left[ | \xi(u^n_{n_k}) - \xi(u_{n_k}) | + \epsilon/2 \right].
\]
The dominated convergence theorem, the convergence (4.111) and the uniform continuity of \( \xi \) imply that the right hand side goes to 0.

\[\square\]

**Proof of Lemma 4.3.** Fix \( \epsilon > 0 \). The asymptotic compactness of the equation implies that the set of probabilities on \( H^1(\mathbb{R}) \)
\[
\{ P_n(0, \cdot); n \geq 0 \}
\]
is tight. We choose a compact set \( K_\epsilon \subset H^1(\mathbb{R}) \) such that
\[
\sup_n P_n(0, K_\epsilon^c) \leq \epsilon/2.
\]
Additionally by the lemma 4.7 the set of probabilities on \( H^1(\mathbb{R}) \)
\[
\{ P_s(v, \cdot); s \in [0,1], v \in K_\epsilon \}
\]
is also tight. We pick another compact \( A_\epsilon \subset H^1(\mathbb{R}) \) such that
\[
\sup_{s \in [0,1], v \in K_\epsilon} P_s(v, A_\epsilon^c) \leq \epsilon/2.
\]
By direct computation
\[
\mu_n(A_\epsilon^c) = \frac{1}{n} \int_0^n P(u(t) \in A_\epsilon^c) \, dt
\]
\[
= \frac{1}{n} \sum_{i=0}^{n-1} \int_{i}^{i+1} \int_{H^1} P_t(0; dy) P_{t-i}(y; A_\epsilon^c) \, dt
\]
\[
= \frac{1}{n} \sum_{i=0}^{n-1} \int_{i}^{i+1} \left\{ \int_{K_\epsilon} P_t(0; dy) P_{t-i}(y; A_\epsilon^c) + \int_{K_\epsilon^c} P_t(0; dy) P_{t-i}(y; A_\epsilon^c) \right\} \, dt
\]
\[
\leq \frac{1}{n} \sum_{i=0}^{n-1} \int_{i}^{i+1} \left\{ \int_{K_\epsilon} P_t(0; dy) \epsilon/2 + P_t(0; K_\epsilon^c) \right\} \, dt \leq \epsilon.
\]
Thus \( \mu_n(A_\epsilon^c) \leq \epsilon \), and the proof is concluded. \[\square\]
5 Appendix

5.1 Proof of $\lim_n E[\|u^n_t\|^2_{L^2}] \to E[\|\xi\|^2_{L^2}]$

Note that the inequality $\liminf_n E[\|u^n_t\|^2_{L^2}] \geq E[\|\xi\|^2_{L^2}]$ can be shown easily. In order to prove the reverse inequality, assume, contrary to the assertion, that there exists $\epsilon > 0$ and a subsequence of $\{u^n_t\}$ (still indexed by $n$) such that for all $n \geq 0$

$$E[\|u^n_t\|^2_{L^2}] - E[\|\xi\|^2_{L^2}] \geq \epsilon. \quad (5.1)$$

Fix $T > 0$ such that $3C(R)e^{-2\lambda T} \leq \epsilon$ where $C(R)$ is the constant in $C([0,T];L^2_{loc}(\mathbb{R}))$. Note that the sequence $\{u^n_{t_n-T}\}$ satisfies the same assumptions as $\{u^n_t\}$ and thus there exists a further subsequence (still indexed by $n$) and $\xi_{-T}$, an $H^1$ valued random variable, such that we also have

$$u^n_{t_n-T} \to \xi_{-T} \quad (5.2)$$

in distribution in $L^2_{loc}(\mathbb{R})$.

We shall work on the space $\mathcal{Z} = C([0,T];L^2_{loc}(\mathbb{R}))$. Denote by $z$ the canonical process on this space and $D$ its right continuous filtration.

**Definition 5.1.** A measure $\nu$ on $\mathcal{Z}$ is a solution of the equation (2.2) if for all $\phi$ smooth and compactly supported functions

$$M^\phi_t = (z_t - z_0, \phi) + \int_0^t (\partial_x^2 z_s + z_x \partial_x z_s + \lambda z_s - f, \phi) ds \quad (5.3)$$

and

$$(M^\phi_t)^2 - t \sum_i (\Phi e_i, \phi)^2 \quad (5.4)$$

are $\nu$ local-martingales.

Define the sequence of measures

$$\nu^n(dz) = \int_{\Omega} \delta_{\{u^n_{t_n-T}(\omega), \omega \in [0,T]\}}(dz)\mathbb{P}(d\omega) \quad (5.5)$$

on $\mathcal{Z}$. We shall prove by the Aldous criterion ([1] Theorem 16.10) that the sequence $\{\nu^n\}_{n=1}^\infty$ is tight in distribution in $\mathcal{Z}$. The first step is the following estimate.

**Lemma 5.1.** We have $E[\|u^n_{T_n+d_n} - u^n_{T_n}\|^2_{L^2}] \to 0$ for all stopping times $T_n$ and for all $d_n \to 0$. 


Proof of Lemma 5.1. Denote $A = (1 - \partial^2_x)$ and $U^n_s = A^{-1}(u^n_s - u^n_T)$. Applying Itô's lemma, we get

\[
\|u^n_{T_n+d_n} - u^n_{T_n}\|_{L^2}^2 = \int_{T_n}^{T_n+d_n} \left( -2(\partial^3_x u^n_s + u^n_s \partial_x u^n_s + \lambda u^n_s - f, u^n_s - u^n_{T_n}) + \|\Phi\|_{H^2(L^2; L^2)}^2 \right) \, ds \\
+ 2 \int_{T_n}^{T_n+d_n} (\Phi dW_s, u^n_s - u^n_{T_n}) \\
= \int_{T_n}^{T_n+d_n} \left( -2(\partial^3_x u^n_s + u^n_s \partial_x u^n_s + \lambda u^n_s - f, u^n_s) + \|\Phi\|_{H^2(L^2; L^2)}^2 \right) \, ds \\
+ 2 \left( u^n_{T_n+d_n} - u^n_{T_n} - \int_{T_n}^{T_n+d_n} \Phi dW_s, u^n_{T_n} \right) \\
+ 2 \int_{T_n}^{T_n+d_n} (\Phi dW_s, u^n_s - u^n_{T_n}) \\
(5.6)
\]

Now, we proceed to bound the terms on the far right side of the above equality. We first use $u^n_s \in H^3(\mathbb{R})$ and $(\partial^3_x u^n_s + u^n_s \partial_x u^n_s, u^n_s) = 0$ to get

\[
|\langle \partial^3_x u^n_s + u^n_s \partial_x u^n_s + \lambda u^n_s - f, u^n_s \rangle | = |\langle \lambda u^n_s - f, u^n_s \rangle | \leq C(\|u^n_s\|_{L^2} + 1) \\
(5.7)
\]

which is bounded in expectation. The difficult term is

\[
\left| \left( u^n_{T_n+d_n} - u^n_{T_n} - \int_{T_n}^{T_n+d_n} \Phi dW_s, u^n_{T_n} \right) \right| \\
\leq \left( \|u^n_{T_n+d_n} - u^n_{T_n}\|_{H^{-1}} + \left\| \int_{T_n}^{T_n+d_n} \Phi dW_s \right\|_{H^{-1}} \right) \|u^n_{T_n}\|_{H^1} \\
(5.8)
\]

This shows that in order to obtain an estimate in $L^2$ one only needs the estimate in $H^{-1}$. Note that the bound on $\mathbb{E}[\| \int_{T_n}^{T_n+d_n} \Phi dW_s \|_{H^{-1}}]$ can be easily obtained by the Burkholder-Davis-Gundy inequality. In order to control $\|u^n_{T_n+d_n} - u^n_{T_n}\|_{H^{-1}}$, we apply the Itô’s lemma and obtain

\[
\|u^n_{T_n+d_n} - u^n_{T_n}\|_{H^{-1}}^2 = \int_{T_n}^{T_n+d_n} \left( -2(\partial^3_x u^n_s + u^n_s \partial_x u^n_s + \lambda u^n_s - f, U^n_s) + \|A^{-1/2}\Phi\|_{H^2(L^2; L^2)}^2 \right) \, ds \\
+ 2 \int_{T_n}^{T_n+d_n} (\Phi dW_s, U^n_s) \\
(5.9)
\]

First note that $\mathbb{E}[(u^n_s \partial_x u^n_s + \lambda u^n_s - f, U^n_s)]$ is bounded due to uniform $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ bounds in (4.1) and (4.30). One may also control $\mathbb{E}[\int_{T_n}^{T_n+d_n} (\Phi dW_s, U^n_s)]$ by the Burkholder-Davis-Gundy-inequality. The main term is

\[
\mathbb{E} \left[ \int_{T_n}^{T_n+d_n} (\partial^3_x u^n_s, U^n_s) \, ds \right] \\
(5.10)
\]

which we estimate as

\[
\mathbb{E} \left[ (\partial^3_x u^n_s, U^n_s) \right] = \mathbb{E} \left[ (A^{-1} \partial^3_x u^n_s, u^n_s - u^n_{T_n+d_n}) \right] \leq \mathbb{E} \left[ \|A^{-1} \partial^3_x u^n_s\|_{L^2}^2 \right] + \mathbb{E} \left[ \|u^n_s - u^n_{T_n}\|_{L^2}^2 \right] \\
\leq \mathbb{E} \left[ \|u^n_s\|_{H^1}^2 \right] + \mathbb{E} \left[ \|u^n_s - u^n_{T_n}\|_{L^2}^2 \right] \leq 4C(R). \\
(5.11)
\]

Finally, combining all the estimates we obtain that $\mathbb{E}[\|u^n_{T_n+d_n} - u^n_{T_n}\|_{L^2}^2] \leq C d_n \to 0$ as $n \to \infty$. 

\[ \square \]
Lemma 5.2. The family of measures $\nu^n$ is tight over $Z$ and any limiting measure $\nu$ of this sequence is a solution of (2.2). Additionally, the distribution of $z_0$ (resp. $z_T$) under $\nu$ is the same as the distribution of $\xi_T$ (resp. $\xi$).

Proof of Lemma 5.2. The tightness follows directly from the Lemma 5.1 and the Aldous criterion, [1] Theorem 16.10.

We first show the solution property of the limiting measure. Let $\phi$ be a smooth compactly supported function, let $0 \leq s_1 \leq \cdots \leq s_k \leq s < t$, and assume that $g: \mathbb{R}^k \to \mathbb{R}$ continuous and bounded. Since $u^n$ is a solution under $\nu^n$, we have

$$
E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_t^\phi \right] = E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_s^\phi \right]
$$

(5.12)

and

$$
E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi)((M_t^\phi)^2 - t \sum_i (\Phi e_i, \phi)^2) \right] = E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi)((M_s^\phi)^2 - s \sum_i (\Phi e_i, \phi)^2) \right].
$$

(5.13)

The mappings that we are integrating are continuous in $z$ under the topology of $Z$, but they are not bounded. However, the bound (4.92) allows us to truncate them, obtain a uniform estimate on the remainder, and pass to the limit in $n$. We obtain

$$
E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_t^\phi \right] = E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_s^\phi \right]
$$

(5.14)

and

$$
E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi)((M_t^\phi)^2 - t \sum_i (\Phi e_i, \phi)^2) \right] = E^n \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi)((M_s^\phi)^2 - s \sum_i (\Phi e_i, \phi)^2) \right].
$$

(5.15)

Thus $\nu$ is a solution of (2.2). We already had the convergence in distribution in $L^2_{loc}(\mathbb{R})$. Now,

$$
u^n \rightharpoonup \xi
$$

(5.16)

and

$$
u^n_{t_n \to T} \rightharpoonup \xi_{T-T}.
$$

(5.17)

The distribution of $z_0$ (resp. $z_T$) under $\nu$ is the distribution of $\xi_{T-T}$ (resp. $\xi$), which completes the proof of Lemma 5.2. \qed

We now finish the proof of (4.97). Under $\nu$, given the quadratic variation of $z_t - z_0 + \int_0^t (\partial_x z_0 + z_s \partial_x z_s + \lambda z_s - f) ds$, there exists a sequence of $\nu$ Brownian motions $\tilde{B}^i$ such that $z_t - z_0 + \int_0^t (\partial_x z_0 + z_s \partial_x z_s + \lambda z_s - f) ds = \sum_i \Phi e_i \tilde{B}_t^i$.

Under $\nu$, the process $z$ is $H^1(\mathbb{R})$-valued. Let $z^n_0$ be a sequence in $H^3(\mathbb{R})$ converging to $z_0$ in $H^1(\mathbb{R})$ and let $z^k$ be the associated solutions of (2.2) in the probability space $(Z, \mathcal{D}, \nu)$. By [9] Lemma 3.2, $z^k_s \in H^3(\mathbb{R})$ for all $s \in [0, T]$. Applying Ito’s Lemma to $z^k$, we get that the difference

$$
\|z^k_t\|_{L^2}^2 - \|z^k_0\|_{L^2}^2 + 2\lambda \int_0^t \|z^k_s\|_{L^2}^2 ds - \int_0^t (z^k_s, f) ds - \|\Phi\|_{H^3}^2 t
$$

(5.18)
defines a martingale. Taking the expectation under \( \nu \) we find that

\[
E'[\|z_k^T\|_{L^2}^2] - e^{-2\lambda T}E'[\|z_0^b\|_{L^2}^2] = \int_0^T e^{-2\lambda (T-s)} \left( E'[\langle z_k^b, f \rangle] + \|\Phi\|_{HS}^2 \right) ds.
\]  

(5.19)

By the Feller property and the convergence of \( z_k^b \), taking the limit as \( k \) goes to \( \infty \), we obtain that

\[
E'[\|z_T\|_{L^2}^2] - e^{-2\lambda T}E'[\|z_0\|_{L^2}^2] = \int_0^T e^{-2\lambda (T-s)} \left( E'[\langle z_s, f \rangle] + \|\Phi\|_{HS}^2 \right) ds.
\]  

(5.20)

By an assumption the mapping \( v \to (v, f) \) is continuous in \( L^2_{loc}(\mathbb{R}) \). Thus

\[
E'[\|\xi\|_{L^2}^2] - e^{-2\lambda T}E'[\|\xi - x\|_{L^2}^2] = E'[\|z_T\|_{L^2}^2] - e^{-2\lambda T}E'[\|z_0\|_{L^2}^2] = \int_0^T e^{-2\lambda (T-s)} \left( E'[\langle u_{t_n}^n, T+s, f \rangle] + \|\Phi\|_{HS}^2 \right) ds
\]

\[
= \lim_{n} \int_0^T e^{-2\lambda (T-s)} \left( E'[\langle u_{t_n}^n, T+s, f \rangle] + \|\Phi\|_{HS}^2 \right) ds
\]

\[
= \lim_{n} E[\|u_{t_n}^n\|_{L^2}^2] - e^{-2\lambda T}E'[\|u_{t_n}^n\|_{L^2}^2].
\]  

(5.21)

Note that \( e^{-2\lambda T}E'[\|\xi - x\|_{L^2}^2] + e^{-2\lambda T}E'[\|u_{t_n}^n\|_{L^2}^2] \leq 2e^{-2\lambda T}C(R) \). Thus using the previous inequality and the choice of \( T \)

\[
2\epsilon/3 \geq 2C(R)e^{-2\lambda T} \geq \lim_{n} \sup E[\|u_{t_n}^n\|_{L^2}^2] - \lim_{n} E[\|\xi\|_{L^2}^2] \geq \epsilon
\]  

(5.22)

which is a contradiction, thus concluding the proof of \( (1.17) \).

5.2 Proof of \( E[I(\xi)] = \lim_n E[I(u_{t_n}^n)] \)

Recall that

\[
u^n \to \nu
\]  

(5.23)
in distribution in \( L^2 \). We assume again that the convergence \( E[I(\xi)] = \lim_n E[I(u_{t_n}^n)] \) does not hold. As in the previous section, there exists a further subsequence (denoted similarly) and \( \epsilon > 0 \) such that

\[
|E[I(\xi)] - E[I(u_{t_n}^n)]| \geq \epsilon.
\]  

(5.24)

Given the uniform estimates, one can prove that there exists a constant dependent on \( R \) such that

\[
\sup |E[I(u_t)]| + |E[u_{t_n}^n|_{L^2}^2] \leq C. \]

We fix \( T \) such that \( 3Ce^{-2\sqrt{T}} \leq \epsilon \).

Using the same notation and results in the first part of the appendix, we have that \( \nu^n \to \nu \) in distribution in \( Z \).

Thus for all \( s \in [0, T] \), we have \( u_{t_n, T+s}^n \to z_s \) in distribution in \( L^2_{loc}(\mathbb{R}) \). Note that to pass from convergence \( (1.93) \) to the convergence \( (1.106) \) we haven’t needed to pass to a subsequence. Thus one may show similarly that for all \( s \in [0, T] \) the convergence of \( u_{t_n, T+s}^n \) to \( z_s \) is in fact in distribution in \( L^2(\mathbb{R}) \). Additionally, the mapping \( v \to (\int v^3(x)dx, (\partial_x v, \partial_x f) - (v^2, f), \int v(x) \sum_i |\Phi e_i|^2 dx) \) is continuous in \( L^2(\mathbb{R}) \) on bounded sets of \( H^1(\mathbb{R}) \).
Therefore, as $n$ goes to infinity,
\[
\int_0^T e^{-2\lambda(T-s)} \left( \mathbb{E} \left[ \frac{\lambda}{3} \int \mathbb{E}^n(t_n - T + s, x)^3 dx + 2(\partial_x u^n(t_n - T + s, x), \partial_x f) - ((u^n(t_n - T + s))^2, f) \right] + \|\partial_x \Phi\|_{L^2, L^2}^2 - \sum_i \int \mathbb{E}^n(t_n - T + s, x) |\Phi e_i|^2 dx \right) ds
\]
converges to
\[
\int_0^T e^{-2\lambda(T-s)} \left( \mathbb{E}^\nu \left[ \frac{\lambda}{3} \int z(s, x)^3 dx + 2(\partial_x z(s), \partial_x f) - (z^2(s), f) \right] + \|\partial_x \Phi\|_{L^2, L^2}^2 - \sum_i \int \mathbb{E}^\nu[z(s, x)] |\Phi e_i|^2 dx \right) ds.
\]

We have that $\nu$ is a solution of (2.2) and similarly to the previous section we can approximate the process $\{z_s\}$ by $\{z^k_s\}$, appeal to the Feller property in $H^1(\mathbb{R})$, and show that the equality
\[
\mathbb{E}^\nu[I(z_T)] - e^{-2\lambda T} \mathbb{E}^\nu[I(z_0)] = \int_0^T e^{-2\lambda(T-s)} \left( \mathbb{E}^\nu \left[ \frac{\lambda}{3} \int z(s, x)^3 dx + 2(\partial_x z(s), \partial_x f) - (z^2(s), f) \right] + \|\partial_x \Phi\|_{L^2, L^2}^2 - \sum_i \int \mathbb{E}^\nu[z(s, x)] |\Phi e_i|^2 dx \right) ds
\]
holds. This finally gives
\[
\mathbb{E}[I(\xi)] - e^{-2\lambda T} \mathbb{E}^\nu[I(\xi - T)] = \mathbb{E}^\nu[I(z_T)] - e^{-2\lambda T} \mathbb{E}^\nu[I(z_0)] = \lim_n \mathbb{E}[I(u^n_{t_n})] - e^{-2\lambda T} \mathbb{E}^\nu[I(u^n_{t_n-T})].
\]

We finish similarly to the previous section. Namely,
\[
\epsilon \leq \liminf_n \left| \mathbb{E}[I(u^n_{t_n})] - \mathbb{E}[I(\xi)] \right| \leq e^{-2\lambda T} \liminf_n \left| \mathbb{E}[I(u^n_{t_n-T})] - \mathbb{E}[I(\xi - T)] \right|
\leq e^{-2\lambda T} 2C \leq 2\epsilon/3
\]
which gives the contradiction.

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