THETA LIFTINGS ON WEAK MAASS FORMS

YOUNGJU CHOIE AND SUBONG LIM

Abstract. We construct theta liftings from half-integral weight weak Maass forms to even integral weight weak Maass forms by using regularized theta integral. Moreover it gives an extension of Niwa’s theta liftings on harmonic weak Maass forms. And we obtain the similar results to those by Niwa.

1. Introduction

Weak Maass forms are Maass forms which allow exponential growth at cusps. We denote the space of weak Maass forms of weight $k$ and eigenvalue $s$ for a congruence subgroup $\Gamma$ and a character $\chi$ as $WMF_{k,s}(\Gamma,\chi)$. And harmonic weak Maass forms are weak Maass forms whose eigenvalue is zero. We denote the space of harmonic weak Maass forms of weight $k$ for a congruence subgroup $\Gamma$ and a character $\chi$ as $H_k(\Gamma,\chi)$. Harmonic weak Maass forms are related to Ramanujan’s mock theta functions. A mock theta function was introduced by Ramanujan in his last letter to Hardy. He gave some examples and definition. In 2001, Zwegers discovered that mock theta functions are holomorphic parts of nonholomorphic modular forms of weight $1/2$. More generally, Zagier defined mock modular forms. And it turns out that mock modular forms are holomorphic parts of harmonic weak Maass forms of any weight.

In this paper, we construct even-integral weight weak Maass forms from half-integral weight weak Maass forms. To construct maps, we shall use the indefinite theta series. This theta series was used by Shintani, Niwa and Cipra. Using this theta series, Shintani[9] constructed an inverse map of Shimura correspondence and Niwa[7] gave another method to construct even-integral weight cusp forms from half-integral weight cusp forms and proved the conjecture about the level in the Shimura correspondence. And Cipra[6] extended Niwa’s lifting to all holomorphic modular forms of all positive, half-integral weight. And we will regularize theta integral to make theta liftings well-defined in the case of weak Maass forms. Borcherds[1] used a regularized theta integral and constructed theta liftings on weakly holomorphic modular forms. And Bruinier[3] defined theta lifting on harmonic weak Maass forms of weight $1/2$.

Let $\mathbb{H}$ be the upper half-plane and let $z = u + iv, w = \xi + i\eta \in \mathbb{H}$. Following Cipra’s method in [6] for given $N \in \mathbb{N}$ and a character $\chi$ for $\Gamma_0(4N)$ we obtain theta functions $\theta(z, w; f_{k,m})$ where $k, m \in \mathbb{Z}$. This theta function has an important property: $\theta(z, w; f_{k,m})$ is a nonholomorphic modular

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form of weight $k/2$ for $\Gamma_0(4N)$ and $\chi$ as a function of $z$ and $\overline{\theta(z, w; f_{k,m})}$ is a nonholomorphic modular form of weight $2m$ for $\Gamma_0(2N)$ and $\chi^2$ as a function of $w$. For a given weak Maass form $g$ of weight $k/2$ for $\Gamma_0(4N)$ and $\chi$ we define a function on $\mathbb{H}$

$$\Phi(g)(w) = \int_{\text{reg}} \sum_{\alpha \in \Gamma_0(4N) \setminus \mathbb{H}} g_\alpha(z) \theta_\alpha(z, w; f_{k,m}) \frac{dudv}{v^2}$$

and

$$\Phi_D(g)(w) = \Phi(g_D)(\frac{w}{D})$$

where $(g_\alpha(z)) = (cz + d)^{-k/2} g\left(\frac{az + b}{cz + d}\right)$ for $\alpha = (a b \ \frac{c}{d} \ \frac{e}{f})$ and $g_D(z) = g(Dz)$. Here $\int_{\text{reg}} \sum_{\Gamma_0(2N) \setminus \mathbb{H}}$ is a regularized integral which was used by Borcherds [1]. Then this gives a lifting from weak Maass forms of weight $k/2$ to weak Maass forms of weight $2m$. The following is the precise statement:

**Theorem 1.1.** Let $k$ and $m$ be any integers and let $g(z) \in \text{WMF}_k^{2,2}(\Gamma_0(4N), \chi)$. Assume that $\chi(0) = 0$. Then $\Phi_D(g)(w)$ is a weak Maass form of weight $2m$ and eigenvalue $4m(m - 1) - 3 - k(k - 4) + 4s$ for $\Gamma_0(2N, D)$ and $\chi^2$. But it may have singularities at Heegner points of the form $w = \frac{b \sqrt{-4Nc} - ac}{2Nd} \in \mathbb{H}$ for $a, b, c \in \mathbb{Z}$, while $\Phi_D(g)$ does not have any singularities in the case when $g$ is a usual Maass form and $k \neq 1$.

**Remark 1.2.** Shimura correspondence is the map from cusp forms of weight $k/2$ to cusp forms of weight $2\lambda = k - 1$. Unlike the Shimura correspondence, we can choose $k$ and $m$ independently. And $k$ and $m$ have any sign.

Here $\text{WMF}_k^{2,2}(\Gamma_0(4N), \chi)$ is a subspace of $\text{WMF}_k^{2,2}(\Gamma_0(4N), \chi)$, the space of weak Maass forms, whose precise definition will be given in Section 2. And $\Gamma_0(2N, D) = \{(a b \ \frac{c}{d} \ \frac{e}{f}) \in \Gamma_0(2N) \mid b \equiv 0(D)\}$. In particular, Harmonic weak Maass forms are weak Maass forms whose eigenvalues are zero. The space of harmonic weak Maass forms are denoted by $H_{k/2}(\Gamma_0(4N), \chi)$, that is, $H_{k/2}(\Gamma_0(4N), \chi) = \text{WMF}_k^{2,2}(\Gamma_0(4N), \chi)$. And we have subspaces $H_{k/2}^+(\Gamma_0(4N), \chi)$ and $H_{k/2}^{+\ast}(\Gamma_0(4N), \chi)$ of $H_{k/2}(\Gamma_0(4N), \chi)$, which will be defined in Section 2. Note that if $g(z) \in H_{k/2}^+(\Gamma_0(4N), \chi)$ then $g$ has a Fourier expansion of the form

$$g(z) = \sum_{n \in \mathbb{Q}} a^+(n)e^{-\frac{\lambda}{4}n^2} + \sum_{n \in \mathbb{Q}} a^-(n)W_{k/2}(2\pi n\lambda)e^{-\frac{\lambda}{4}n^2},$$

where $W_{k}(x) := \int_{-\infty}^{\infty} e^{-t^2} e^{t^k} dt = \Gamma(1 - k, 2|x|)$ for $x < 0$ and $e(z) = e^{2\pi iz}$.

**Theorem 1.3.** Let $k \geq 1$ and $\lambda = \frac{k-1}{2}$. Assume $\chi(0) = 0$ and that $a_+^+(0) = 0$ if $k = 1$ and $\chi$ is a principal character. And let $g \in H_{k/2}^+(\Gamma_0(4N), \chi)$. If $g(z)$ has a Fourier expansion as above then $\Phi_D(g)(w) \in H_{k/2}^{+\ast}(\Gamma_0(2N, D), \chi^2)$ with the same possible singularities at Heegner points as in Theorem [7]. Moreover if $g \in H_{k/2}^+(\Gamma_0(4N), \chi)$ then we have the following results:

1. If $k \geq 3$ then we have

$$\Phi_D(g)(i\infty) = \lim_{\eta \to \infty} (\Phi_D(g)(i\eta)) = C_D(\lambda) \frac{a_+^+(0)}{2} L(1 - \lambda, \chi_D)$$

where $C_D(\lambda)$ is a constant depending only on $D$.
and if \( k = 1 \) then we have
\[
\Phi(g)(i\infty) = 4N^{1/4}a^+(0) \sum_{m=1}^{\infty} \frac{\chi(m)}{m}.
\]

(2) If \( k \geq 3 \) then we have
\[
\int_{0}^{\infty} \eta^{s-1}(\Phi_{D}(g)(i\eta) - \Phi_{D}(g)(i\infty))d\eta = C_{D}(\lambda)(2\pi)^{-3}G(s)L(s+1,\chi_{D}) \sum_{n=1}^{\infty} \frac{a^+(Dn^2)}{n^s}
\]
and if \( k = 1 \) then we have
\[
\int_{0}^{\infty} \eta^{s-1}(\Phi(g)(i\eta) - \Phi(g)(i\infty))d\eta = C_{1}(\lambda)(2\pi)^{-3}G(s)L(s+1,\chi) \sum_{n=1}^{\infty} \frac{a^+(n^2)}{n^s}
\]
where \( C_{D}(\lambda) = (-1)^{1/2}2^{-3/4}(4N)^{1/2} \) and \( \chi_{D} = \chi(\frac{1}{\lambda})^{1/2} \).

Then this is the analogous result with Niwa\(^7\) and Cipra\(^6\). So this is the extension of Niwa’s theta lifting to harmonic weak Maass forms.

**Remark 1.4.** If \( g \) is a usual Maass form and \( k \neq 1 \), then we do not need to regularize integral because \( \theta(z, w; f_{k,\lambda}) \) is rapidly decreasing at all cusps (see Theorem 2.6 in \(^6\)).

**Example 1.5.** We consider a holomorphic Eisenstein series (see Proposition 2.10 in \(^6\)): Let \( k \geq 3 \). Define
\[
E_{k/2}(z, s) = E_{k/2}(z, s, 4N, \chi) = \sum_{\gamma \in \Gamma_{0}(4N)} \bar{\chi}(\gamma) \frac{\text{Im}(yz)^s}{j(\gamma, z)^k}.
\]
Then it is a Maass form of weight \( k/2 \) with eigenvalue \(-s(s-1)-ks/2\) for \( \Gamma_{0}(4N) \) and \( \chi \). Assume that \( \chi_1 = \chi(\frac{1}{\lambda})^4 \) is primitive mod \( 4N \). Recall that \( \lambda = (k-1)/2 \). Then
\[
\Phi_1(E_{k/2}(z, s))(w) = C(s)E_{2\lambda}(w, 2s)
\]
where
\[
E_{2\lambda}(w, 2s) = E_{2\lambda}(w, 2s, 2N, \chi^{2}) = \sum_{\gamma \in \Gamma_{0}(2N)} \bar{\chi}^{2}(\gamma) \frac{\text{Im}(yz)^{2s}}{(cw + d)^{2k}}.
\]
Then it is a Maass form of weight \( 2\lambda \) with eigenvalue \(-2s(2s-1)-4\lambda s\) for \( \Gamma_{0}(2N) \) and \( \chi^2 \).

**Theorem 1.6.** Let \( g \) be the same as that in Theorem \(^7,3\). If \( \Phi_{D}(g)(w) \in H_{2\lambda}^{+}(\Gamma_{0}(2N, D), \chi^{2}) \) and we write its Fourier expansion as follows
\[
\Phi_{D}(g)(w) = \sum_{n \geq 0} A_{D}^{+}(n)e(nw) + \sum_{n < 0} A_{D}^{-}(n)W(2\pi\eta)e(nw)
\]
then we have
\[
\sum_{n=1}^{\infty} \frac{A_+^+(n)}{n^s} + \frac{1}{\Gamma(s)} \left( \int_0^{\infty} \int_{2y}^{\infty} e^{y-x} x^{-k/2} y^{s-1} dx dy \right) \sum_{n=1}^{\infty} \frac{A_-^-(n)}{n^s} = C_D(\lambda)L(s - \lambda + 1, \chi) \sum_{n=1}^{\infty} \frac{a^+(Dn^2)}{n^s}.
\]

So we have the similar result with Shimura’s correspondence in [8] but we have extra terms.

2. Weak Maass forms

In this section, we review weak Maass forms and harmonic weak Maass forms, which were introduced in [4] and [5]. For \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) and \( z \in \mathbb{H} \), let
\[
J(\gamma, z) = cz + d, \quad j(\gamma, z) = c^{-1}J(\gamma, z)'/2 \quad \text{and} \quad \gamma z = \frac{az + b}{cz + d},
\]
where \( \epsilon_i = 1 \) or \( i \) as \( d \equiv 1 \) or \( 3 \), and \( (\frac{c}{d}) \) is the quadratic residue symbol as defined in [8].

If \( k \in \mathbb{Z}, \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \), and \( f \) is a function on \( \mathbb{H} \), define
\[
(f|k\gamma)(z) = (cz + d)^{-k}f(\gamma z).
\]
When \( k \in \mathbb{Z} \) is odd and \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(4) \), define
\[
(f|k/2\gamma)(z) = j(\gamma, z)^{-k}f(\gamma z).
\]

Let \( f \) be a smooth function on \( \mathbb{H} \) and let \( k \) be an integer or half-integer. And let the congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) and let \( \chi \) be a character for \( \Gamma \). And let
\[
G_k(\Gamma, \chi) = \{ f \in C^\infty(\mathbb{H}) | f|k\gamma = \chi(d)f \quad \text{for all} \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \}.
\]

If \( f \in G_k(\Gamma, \chi) \) then we say that \( f \) is a nonholomorphic modular form of weight \( k \) for \( \Gamma \) and \( \chi \).

Define \( \Delta_k = -\partial^2(\frac{\partial}{\partial u^2} + \frac{\partial}{\partial v^2}) + ik\partial(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}) \). This is the usual hyperbolic Laplace operator in weight \( k \).

**Definition 2.1.** A twice continuously differentiable function on \( \mathbb{H} \) is called a Maass form of weight \( k \) and eigenvalue \( s \) for \( \Gamma \) and \( \chi \) if
\[
\begin{align*}
(1) \quad & (f|k\gamma) = \chi(\gamma)f \quad \text{for all} \quad \gamma \in \Gamma, \\
(2) \quad & \Delta_k f = sf \quad \text{for} \quad s \in \mathbb{C}, \\
(3) \quad & \text{For all} \quad \gamma \in \text{SL}_2(\mathbb{Z}), \quad (f|k\gamma)(z) = O(v^\delta) \quad \text{as} \quad v \to \infty \quad \text{for some} \quad \delta > 0.
\end{align*}
\]
We denote the vector space of those Maass forms by \( \text{Maass}_{k,s}(\Gamma, \chi) \). If we change the third condition, which is a growth condition, into
\[
(3') \quad \text{For all} \quad \gamma \in \text{SL}_2(\mathbb{Z}), \quad (f|k\gamma)(z) = O(e^{\delta v}) \quad \text{as} \quad v \to \infty \quad \text{for some} \quad \delta > 0.
\]
then \( f \) is called a weak Maass form. Especially, if \( s = 0 \), then it is called a harmonic weak Maass form.
Let $\text{WMF}_{k,s}(\Gamma, \chi), H_k(\Gamma, \chi)$ denote respectively the spaces of weak Maass forms and harmonic weak Maass forms.

If $f$ is a weak Maass form for $\Gamma$, then it satisfies $f(z + l) = f(z)$ since $(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}) \in \Gamma$ for some positive integer $l$. Hence there is a Fourier expansion
\begin{equation}
(2.1) \quad f(z) = \sum_{n \in \mathbb{Q}} a(n, v)e(nu).
\end{equation}
Then we define
\begin{equation}
\text{WMF}_{k,s}^*(\Gamma, \chi) = \{ f \in \text{WMF}_{k,s}(\Gamma, \chi) | a(n, v)e^{2\pi nu} = O(v^\delta) \text{ for some } \delta > 0 \forall n \}.
\end{equation}
This is the subspace of $\text{WMF}_{k,s}(\Gamma, \chi)$.

In particular, $H_k(\Gamma, \chi)$ can be characterized by the differential operator $\xi_k$, which is studied by Bruinier and Funke\cite{BF}: We define a differential operator $\xi_k = 2iu^k \frac{\partial}{\partial u}$. This gives a map
$$
\xi_k : H_k(\Gamma, \chi) \rightarrow M_{2-k}(\Gamma, \bar{\chi}),
$$
where $M_{2-k}(\Gamma, \chi)$ is the space of weakly holomorphic modular forms of weight $k$ for $\Gamma$ and $\chi$. It is easy to see that $H_k^*(\Gamma, \chi) = \xi_k^{-1}(M_{2-k}(\Gamma, \bar{\chi}))$. Here $M_k(\Gamma, \chi)$ is the space of holomorphic modular forms of weight $k$ for $\Gamma$ and $\chi$.

The Fourier expansion of any $f \in H_k^*(\Gamma, \chi)$ gives a unique decomposition $f = f^+ + f^−$, where
\begin{equation}
(2.2) \quad f^+(z) = \sum_{n \in \mathbb{Q}^+, n \neq 0} a^+(n)e(nz),
\end{equation}
\begin{equation}
(2.3) \quad f^−(z) = \sum_{n \in \mathbb{Q}^−} a^−(n)W_k(2\pi nu)e(nz) + a^−(0)u^{1-k/2},
\end{equation}
and $W_k(x) := \int_{x-2}^{\infty} e^{-t}t^{-k}dt = \Gamma(1-k, 2|x|)$ for $x < 0$ and $e(z) = e^{2\pi iz}$. Here $f^+$ and $f^−$ are called the holomorphic part and nonholomorphic part of $f$, respectively. Note that $f$ has a Fourier expansion of the form (2.2) at all cusps. And the Fourier polynomial
$$
P(f) = \sum_{n \geq 0} a^+(n)q^n \text{ with } q = e^{2\pi iz}
$$
is called the principal part of $f$.

We define another subspace $H_k^*(\Gamma, \chi)$ of $H_k^*(\Gamma, \chi)$ as the inverse image of $S_{2-k}(\Gamma, \bar{\chi})$ under the map $\xi_k$. So it has a Fourier expansion
\begin{equation}
(2.3) \quad f(z) = \sum_{n \in \mathbb{Z}^+, n \neq 0} a^+(n)e(nz) + \sum_{n \in \mathbb{Z}^−} a^−(n)W_k(2\pi nu)e(nz).
\end{equation}

In summary we have introduced the following notations:
$$
\text{WMF}_{k,s}^*(\Gamma, \chi) \subset \text{WMF}_{k,s}(\Gamma, \chi) \subset \text{Maass}_{k,s}(\Gamma, \chi) \subset G_k(\Gamma, \chi)
$$
$$
H_k^*(\Gamma, \chi) \subset H_k^*(\Gamma, \chi) \subset H_k(\Gamma, \chi) = \text{WMF}_{k,0}(\Gamma, \chi)
$$
3. Construction of Theta lifting

3.1. Indefinite theta functions. In this section we define theta functions by using Cipra’s method in [6]. The most of results in this section are in [6]. We begin by defining the Weil representations on $\text{SL}_2(\mathbb{R})$ on the space of Schwartz functions $S(\mathbb{R}^n)$. Let $Q$ be a rational symmetric matrix of signature $(p, q), p + q = n$. For $x, y \in \mathbb{R}^n$, define the inner product

$$<x, y> = iTxy.$$

For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ and a Schwartz function $f \in S(\mathbb{R}^n)$, define the Weil representation

$$(r(\gamma, Q)f)(x) = \begin{cases} |d|^{n/2}e^{\frac{b}{c} \langle x, x \rangle}f(ax) & \text{if } c = 0, \\ |\det Q|^{-1/2}e^{-\frac{1}{2c} \langle x, y \rangle}f(y)dy & \text{if } c \neq 0. \end{cases}$$

Proposition 3.1. Let $\gamma \in \text{SL}_2(\mathbb{R})$ and $\sigma_z = \begin{pmatrix} 1/2 & iw^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix}$ for $z = u + iv \in \mathbb{H}$. Define $\phi(\text{mod } 2\pi)$ by $e^{-i\phi} = J(\gamma, z)/|J(\gamma, z)|$, and let $\kappa(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$.

Then

(1) $\gamma \sigma_z = \sigma_{\gamma z} \kappa(\phi),$ (2) $r(\gamma, Q)r(\sigma_{z}, Q) = r(\sigma_{\gamma z}, Q)r(\kappa(\phi), Q).$

Proof This is Proposition 1.3 of [6].

Corollary 3.2. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, and $t \in \mathbb{R}$, let $\gamma_t = \begin{pmatrix} a & bt^2 \\ c & dt^2 \end{pmatrix} = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}\gamma\begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}$. Let $\kappa(\phi)$ be as before. Then

$$\gamma_t \sigma_{z} = \sigma_{\gamma_t z} \kappa(\phi)$$

and

$$r(\gamma, Q)r(\sigma_{z}, Q) = r(\sigma_{\gamma z}, Q)r(\kappa(\phi), Q).$$

Proof This follows since $J(\gamma_t, z)/|J(\gamma_t, z)| = J(\gamma, z)/|J(\gamma, z)|$. □

Let $L$ be an even lattice in $\mathbb{R}^n$ and let $L^*$ be the dual lattice. Denote by $v(L)$ the volume of a fundamental parallelepiped of $L$ in $\mathbb{R}^n$:

$$v(L) = \int_{\mathbb{R}^n/L} dx.$$

Let $\{\lambda_1, \cdots, \lambda_n\}$ be a $\mathbb{Z}$-basis for $L$, and define $B = \det(<\lambda_i, \lambda_j>)$.

Definition 3.3. (1) We say that a function $\omega : L^*/L \rightarrow \mathbb{C}$ has the first permutation property for $\Gamma_0(4N)$ with a character $\chi$ if it satisfies

(a) $\omega(\kappa) = 0$ if $\kappa, \kappa \notin \mathbb{Z}$,
(b) $\omega(d\kappa) = \chi(d)\omega(\kappa)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$,

where $\chi$ is a character mod $4N$. 
(2) We say that a function $f \in S(\mathbb{R}^n)$ has the first spherical property for weight $k/2$ if it satisfies
\[ r(\kappa(\phi), Q)f = e(\kappa(\phi))^{d-q} \sqrt{e^{i\phi^{-k}}} f, \]
for all $\kappa(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$ where $k \in \mathbb{Z}$ and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
\[ \epsilon(\gamma) = \begin{cases} \sqrt{i} & c > 0, \\ i^{(1-\text{sgn}d)/2} & c = 0, \\ \sqrt{-1} & c < 0. \end{cases} \]

Let $f \in S(\mathbb{R}^n)$ and define, for $h \in L^*/L$
\[ \theta(f, h) := \sum_{x \in L} f(x + h). \]
Then, by Shintani\cite{9}:

**Proposition 3.4.** Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. Then
\[ \theta(r(\gamma, Q)f, h) = \sum_{j \in L^*/L} c(h, j)_\gamma \theta(f, j) \]
where
\[ c(h, j)_\gamma = \begin{cases} \delta_{h,j} e(i_{\mathbb{Z}} ^{\gamma} < h, j >) & \text{if } c = 0, \\ | \det Q |^{1/2} v(L^{-1})^{-1/2} \sum_{r \in L \cap cL} e\left( \frac{1}{2c} (a < h + r, h + r > -2 < j, h + r > + d < j, j >) \right) & \text{if } c \neq 0. \end{cases} \]

Take $f$ having the first spherical property for weight $k/2$, and let $\omega$ have the first permutation property for $\Gamma_0(4N)$ with character $\chi$. Define
\[ \theta(z, f, h) := v^{-k/4} \theta(r(\sigma_z, Q)f, h), \quad h \in L^*/L \]
and
\[ \theta(z, f; \omega) := \sum_{h \in L^*/L} \omega(h) \theta(z, f, h). \]

**Theorem 3.5** (Shintani). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. Then
\[ (cz + d)^{-k/2} \theta(\gamma z, f, h) = \sqrt{i^{-(p-q)\text{sgn} c}} \sum_{j \in L^*/L} c(h, j)_\gamma \theta(z, f, j) \]
where $c(h, j)_\gamma$ as in Proposition 3.4.

**Proof** This is Theorem 1.5 in [6]. \qed

**Corollary 3.6** (Shintani). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$. Then
\[ j(\gamma, z)^{-k} \theta(\gamma z, f; \omega) = \chi'(d) \theta(z, f; \omega) \]
where $\chi'(d) = \frac{-1}{d} \binom{k-2}{d}(-1)^{\nu(d)}B,d)_{\infty} \chi(d)$ with the Hilbert symbol

$$(x,y)_{\infty} = \begin{cases} -1 & \text{if } x,y < 0, \\ 1 & \text{otherwise}. \end{cases}$$

Let $O(Q)$ be the orthogonal group of $Q$: $O(Q) = \{ g \mid gQg = Q \}$. Let $SO(Q)$ denote the connected component of the identity in $O(Q)$, consisting of those matrices $g$ with $\det g = 1$. We define a unitary representation of $SO(Q)$ on $L^2(\mathbb{R}^n)$ by letting $(p(g)f)(x) = f(g^{-1}x)$. By definition of $SO(Q)$, $p(g)$ commutes with the Weil representation (See [6] page 64):

$$(3.2) \quad p(g)(r(\gamma, Q)f) = r(\gamma, Q)(p(g)f).$$

Now we introduce theta kernel as in [6]. Take the following special $Q$: let $YOUNGJU CHOE AND SUBONG LIM$

$$Q = \frac{2}{N} \begin{pmatrix} 2 & \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & cd & 2b \end{pmatrix} \\ c^2 & d \end{pmatrix},$$

be a matrix with signature $(2, 1)$. Let $L = 4NZ \oplus NZ \oplus NZ/4$. Then $\nu(L) = N^3$. Also, $L^\ast = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/16$ and $B = -32N^3$.

As a quadratic form, $Q$ is given by the determinant of a matrix, for $x = (x_1, x_2, x_3)$

$$Q(x) = xQx = \frac{2}{N} (x_2^2 - 4x_1x_3) = -\frac{8}{N} \left| \begin{array}{cc} x_1 & \frac{x_2}{2} \\ \frac{x_2}{2} & x_3 \end{array} \right|. $$

And there is a map from $SL_2(\mathbb{R})$ to $SO(Q)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & cd & 2b \end{pmatrix}.$$ 

This map gives an isomorphism of $SO(Q)$ with $SL_2(\mathbb{R})/\pm I$.

**Definition 3.7.**

1. Let $\Gamma_Q$ be a discrete subgroup of $SO(Q)$ which leaves $L$ invariant. Let $\Gamma_Q^\ast$ be the (normal) subgroup of $\Gamma_Q$ which fixes $L^\ast/L$. Let $\chi$ be a character of $\Gamma_Q$ which is trivial on $\Gamma_Q^\ast$. We say that $\omega : L^\ast/L \to \mathbb{C}$ has the second permutation property for $\Gamma_Q$ with character $\chi$ if it satisfies

$$\omega(\gamma \kappa) = \chi(\gamma)\omega(\kappa), \quad \gamma \in \Gamma_Q, \kappa \in L^\ast.$$ 

2. Let $m \in \mathbb{Z}$. We say that a function $f \in S(\mathbb{R}^3)$ has the second spherical property for weight $2m$ if it satisfies, by identifying $\kappa(\phi)$ as an element of $SO(Q)$

$$f(\kappa(\phi)^{-1}x) = e^{-2im\phi}f(x) \text{ for all } \phi \in \mathbb{R} \text{ and } x \in \mathbb{R}^3.$$ 

Let us define Hermite polynomials: for $0 \leq \nu \in \mathbb{Z}$, define

$$H_\nu(x) = (-1)^\nu \exp(x^2/2) \frac{d^\nu}{dx^\nu} \exp(-x^2/2).$$

**Theorem 3.8.** Let $m$ and $\lambda$ be integers. Then for every positive integer $\mu$ such that $|m| \leq \lambda + \mu$, there is a function $L_{m,\lambda,\mu}$ such that

$$(3.4) \quad f_{m,\lambda,\mu}(x) = L_{m,\lambda,\mu}(x)H_\mu\left(\frac{\sqrt{8\pi}}{N}(x_1 + x_3)\right) \exp\left(-\frac{2\pi}{N}(2x_1^2 + x_2^2 + 2x_3^2)\right)$$
has the first spherical property for weight \( k/2 = \lambda + 1/2 \), and the second spherical property for weight \( 2m \). The function \( L_{m,\lambda,\mu} \) is defined by
\[
L_{m,\lambda,\mu}(x) = \frac{1}{2\pi} \int_{\phi} e^{2m\phi} L_{\lambda,\mu}(\kappa(\phi))^{-1} x d\phi
\]
where \( L_{\lambda,\mu}(x) = H_{\nu_1}(\sqrt{8\pi/\lambda}(x_1 - x_3)) H_{\nu_2}(\sqrt{8\pi/\mu}x_2) \) for any choice of \( \nu_1 \) and \( \nu_2 \) such that \( \nu_1 + \nu_2 - \mu = \lambda \). In particular, we may take
\[
L_{1,1,0}(x) = (x_1 - ix_2 - x_3)^4.
\]

**Proof**
This is Theorem 2.1 in [6].

Let \( f_{k,m} = f_{m,\lambda,\mu} \) for some fixed \( \mu \) as in Theorem 3.8 where \( \frac{k}{2} = \lambda + \frac{1}{2} \). Consider theta function associated with \( f_{k,m} \): for \( z = u + iw, w = \xi + i\eta \in \mathbb{H} \) and given character \( \chi \) for \( \Gamma_0(4N) \) we define a theta function
\[
\theta(z, w; f_{k,m}) := (32N^3)^{-1/4} i^{k/4}(4\eta)^{-m} \sum_{x \in L_N^*} \tilde{\chi}_1(4x_1)(r(\sigma_{4Nz}, Q)p(\sigma_{2Nw})f_{k,m})(x),
\]
where \( \chi_1 = \chi(z^{-1}), \tilde{\chi}_1(l) = \sum_{h=1}^{4N} \tilde{\chi}_1(h) e^{2\pi h/4N} \), and \( L_N^* = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \), the dual lattice to \( L_N = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

**Theorem 3.9.** The above theta function is a two-variable nonholomorphic modular form:

1. \( \theta(z, w; f_{k,m}) \in G_{k/2}(\Gamma_0(4N), \chi) \) as a function of \( z \),
2. \( \theta(z, w; f_{k,m}) \in G_{2m}(\Gamma_0(2N), \chi^2) \) as a function of \( w \).

**Proof**
Basically we follow the proof of Theorem 2.3 of [6]. Let \( z = u + iw, w = \xi + i\eta \) for \( u, v, \xi, \eta \in \mathbb{R} \). We define
\[
\Theta(z, w; f_{k,m}) = (4\eta)^{-m} v^{-k/4} \sum_{x \in L'} \tilde{\chi}_1(x_1)(r(\sigma_z, Q)p(\sigma_{4w})f_{k,m})(x),
\]
where \( L' = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \). With the notation of Corollary 3.6
\[
\Theta(z, w; f_{k,m}) = (4\eta)^{-m} \theta(z, p(\sigma_{4w})f_{k,m}; \omega)
\]
with \( \omega : L^*/L \to \mathbb{C} \) defined by

1. \( \omega(\kappa) = 0 \) if \( \kappa \notin L' \),
2. \( \omega(\kappa) = \tilde{\chi}_1(\kappa) \) if \( \kappa \in (\kappa_1, \kappa_2, \kappa_3) \in L' \),

where \( L = 4\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \) and its dual is \( L^* = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/16 \). Note that \( \omega \) has the first permutation property for \( \Gamma_0(4N) \) with character \( \tilde{\chi}_1 \), and the second permutation property for \( \Gamma_Q = \left( \begin{array}{c} 2 \\ 1/2 \end{array} \right) \Gamma_0(2N) \left( \begin{array}{c} 1/2 \\ 2 \end{array} \right) \) with character \( \chi^2 \) (See Proposition 2.2 in [6]). Note that \( p(\sigma_{4w})f_{k,m} \) has the first spherical property of weight \( k/2 \) since \( p \) commutes with the Weil representation (See 3.2). Then by the Corollary 3.6 \( \Theta(z, w; f_{k,m}) \in G_{k/2}(\Gamma_0(4N), \tilde{\chi}(N)) \) as a function of \( z \).
If we use equation (3.1) then we see that for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N) \)
\[
p(\sigma_{4w})(x) = p(\gamma_2 \sigma_{4w} \gamma^{-1})(x)
\]
where \( \gamma_2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \) and \( e^{-i\theta} = J(\gamma, w)/|J(\gamma, w)| \). Note that \( \gamma_2 \in \Gamma_0 \). So if we use the second spherical property of \( f_{k,m} \) and second permutation property of \( \omega \) then we get the desired transformation property for the second variable \( w \).

We define the Fricke involution \( W(N) \):
\[
(f|kW(N))(z) = \begin{cases} N^{-k/2}(-iz)^{-k}f(-1/Nz) & k = \text{half-integer}, \\
N^{-k/2}z^{-k}f(-1/Nz) & k = \text{even-integer}.
\end{cases}
\]
Let \( |k/2 W(4N) \) act on the variable \( z \) and \( |2m W(2N) \) act on \( w \) (See [6]). Then
\[
\theta(z, w; f_{k,m}) = (\Theta|k/2 W(4N)|2m W(2N))(z, w; f_{k,m})
\]
and hence transformation formulas of \( \theta(z, w; f_{k,m}) \) follow. \( \square \)

This theta function also has a good property about differential operators.

**Proposition 3.10.** The theta function \( \theta(z, w; f_{k,m}) \), defined in (3.3), satisfies the PDE
\[
4[v^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) - i \frac{k}{2} v(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v}) + \frac{k}{4}(k - 1)]\theta(z, w; f_{k,m}) = [\eta^2(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}) + 2m\eta(\frac{\partial}{\partial \xi} - i \frac{\partial}{\partial \eta}) + m(m - 1) - \frac{3}{4}]\theta(z, w; f_{k,m}).
\]

**Proof** This is Proposition 2.13 of [6]. \( \square \)

3.2. **Regularized theta lifting.** In this section we explain how to regularize the integral and we define theta lifting using that regularization. We will use Borcherds’ **regularized integral**. Throughout we use the setup of [11]. If we use the weak Maass form, the integral in the theta lifting may be divergent. So it has to be regularized as follows: We integrate over the region \( \tilde{\Delta}_t \), where \( \tilde{\Delta}_\infty = \{ z \in \mathbb{H} | |z| \geq 1, |\text{Re}(z)| \leq 1/2 \} \) is the usual fundamental domain of \( SL_2(\mathbb{Z}) \) and \( \tilde{\Delta}_t \) is the subset of \( \tilde{\Delta}_\infty \) of points \( z \) with \( \text{Im}(z) \leq t \). Suppose that
\[
(3.6) \lim_{t \to \infty} \int_{\tilde{\Delta}_t} F(z)w^{-s}\frac{dudv}{v^2}
\]
exists for \( \text{Re}(s) \gg 0 \) and can be continued to a meromorphic function defined for all complex \( s \). Then we define
\[
\int_{SL_2(\mathbb{Z})\mathbb{H}}^{\text{reg}} F(z)\frac{dudv}{v^2}
\]
to be the constant term of the Laurent expansion of the function (3.6) at \( s = 0 \). If we use this regularized integral, we can define the theta lifting even though we use weak Maass forms.
This is a regularized integral for $SL_2(\mathbb{Z})$ but we need a regularized integral for $\Gamma_0(4N)$. Note that for a modular form $g \in S_{k/2}(\Gamma_0(4N), \chi)$
\[
\int_{\Gamma_0(4N) \setminus \mathbb{H}} v^{k/2} g(z) \overline{\theta(z, w; f_{k,m})} \frac{du dv}{v^2} = \int_{SL_2(\mathbb{Z}) \setminus \mathbb{H}} \sum_{\alpha \in \Gamma_0(4N) \setminus SL_2(\mathbb{Z})} g_{\alpha}(z) \overline{\theta_{\alpha}(z, w; f_{k,m})} \frac{du dv}{v^2}
\]
where $g_{\alpha}(z) = (cz + d)^{-k/2} g(\alpha z)$ for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $g \in WMF^*_{k/2,\alpha}(\Gamma_0(4N), \chi)$ define a function on $\mathbb{H}$
\[
\Phi(g)(w) = \int_{SL_2(\mathbb{Z}) \setminus \mathbb{H}} v^{k/2} \sum_{\alpha \in \Gamma_0(4N) \setminus SL_2(\mathbb{Z})} g_{\alpha}(z) \overline{\theta_{\alpha}(z, w; f_{k,m})} \frac{du dv}{v^2}.
\]

And for square-free, positive integer $D$ and $g \in WMF^*_{k/2,\alpha}(\Gamma_0(4N), \chi)$ let $g_D(x) = g(Dx)$. Then $g_D \in WMF^*(\Gamma_0(4ND), \chi_D)$. Define
\[
\Phi_D(g)(w) = \Phi(g_D)(\frac{w}{D}).
\]
But in this case, we use $\theta(z, w; f_{k,m})$ such that its level is $4ND$ and its character is $\chi_D = \chi(\frac{-1}{D})$ and use a regularized integral for $\Gamma_0(4ND)$.

4. Proof of Theorem 1.1

First we prove the convergence of $\Phi_D(g)$ for $g \in WMF^*_{k/2,\alpha}(\Gamma_0(4N), \chi)$. Since $\Phi_D(g)$ is defined by using $\Phi(g_D)$, it is enough to show the convergence of $\Phi(g)$. Note that since $\sum_{\alpha \in \Gamma_0(4N) \setminus SL_2(\mathbb{Z})}$ is a finite sum, we can exchange sum and integration. And since $\int_{\mathbb{H}} = \int_{\mathbb{H}_1} + \int_{\mathbb{H}_2} \int_{\mathbb{H}_3}$ we only need to check
\[
(4.1) \quad \lim_{t \to \infty} \int_{t}^{\infty} \int_{u=1/2}^{u+1/2} v^{k/2} g_{\alpha}(z) \overline{\theta_{\alpha}(z, w; f_{k,m})} \frac{du dv}{v^2}
\]
for each $\alpha \in \Gamma_0(4N) \setminus SL_2(\mathbb{Z})$. And if $g_{\alpha}(z) \overline{\theta_{\alpha}(z, w; f_{k,m})}$ has a Fourier expansion as $\sum_{n \in \mathbb{Z}} a(n, v, w)e^{2\pi inu}$ then
\[
\int_{1}^{\infty} \int_{u=1/2}^{u+1/2} v^{k/2} g_{\alpha}(z) \overline{\theta_{\alpha}(z, w; f_{k,m})} \frac{du dv}{v^2} = \int_{1}^{\infty} v^{k/2-2} a(0, v, w) dv.
\]
So we need to check the Fourier coefficients of the constant term of $g_{\alpha}(z) \overline{\theta_{\alpha}(z, w; f_{k,m})}$. For this we need Fourier expansions of $g_{\alpha}(z)$ and $\theta_{\alpha}(z, w; f_{k,m})$ with respect to $z$.

By computing explicitly representations in the definition of $\theta(z, w; f_{k,m})$ in (3.5) it turns out that
\[
\theta(z, w; f_{k,m}) = \sum_{x \in \mathbb{Z}^3} h(x, v, w; k)e^{2\pi i \Lambda(x,w)z} e^{2\pi i (x_1^2 - x_1 x_3)}
\]
where $h$ is a polynomial of $x, v$ and $w$ and $\Lambda(x,w) = \frac{1}{4\eta}(x_1 - 4Nwx_2 + 4N^2 w^2 x_3)$. This gives a Fourier expansion with respect to $z = u + iv$. 

[Page dimensions: 612.0x792.0]
Next we will see the Fourier expansion of \( \theta_{\alpha}(z, w; f_{k,m}) \) for general \( \alpha \in \Gamma_0(4N) \setminus \text{SL}_2(\mathbb{Z}) \). Let
\[
Q_4 = \frac{1}{2} \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}.
\]
This is just the original \( Q \) in (3.3) when \( N = 4 \). Likewise let \( f_4 \) be a function \( f_{k,m} \) in (3.4) with \( N = 4 \). Then \( f_4 \) satisfies the first and second spherical properties for the weights \( k/2 \) and \( 2m \) respectively. Now let
\[
L = 4N \mathbb{Z} \oplus 2 \mathbb{Z} \oplus \mathbb{Z} \\
L' = \mathbb{Z} \oplus 2 \mathbb{Z} \oplus \mathbb{Z} \\
L^* = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/4N
\]
and define \( \omega : L^*/L \rightarrow \mathbb{C} \) by
\[
\omega(\kappa) = \begin{cases} 
0 & \kappa \notin L' \\
\tilde{\chi}(\kappa_1) & \kappa = (\kappa_1, \kappa_2, \kappa_3) \in L'.
\end{cases}
\]
Then \( \omega \) has the first permutation property for \( \Gamma_0(4N) \) with character \( \chi_1 \) and the second permutation property for \( \Gamma_Q = \left( \begin{smallmatrix} 2/1/2 \end{smallmatrix} \right) \Gamma_0(2N) \left( \begin{smallmatrix} 1/2 \end{smallmatrix} \right) \) with character \( \tilde{\chi}^2 \). Direct computation shows that, up to constant multiple,
\[
\theta(z, w; f_{k,m}) \approx \eta^{-m} \theta(z, p(\sigma_{2Nw}) f_4; \omega).
\]
By Theorem 3.5 we see that for each \( \alpha \in \Gamma_0(4N) \setminus \text{SL}_2(\mathbb{Z}) \), \( \theta_{\alpha}(z, w; f_{k,m}) \) can be written as
\[
\theta_{\alpha}(z, w; f_{k,m}) = \sum_{x \in \mathbb{Z}^3} \eta_{\alpha}(x, v, w; k) e^{\frac{-4\pi i}{N^2} |\Lambda(x, w)|^2} e^{2\pi i (x_3 - x_1 x_3)}
\]
where \( \eta_{\alpha} \) is another polynomial for \( x, v \) and \( w \).

And we know that \( g_{\alpha} \) has the Fourier expansion of the form (2.1). So the constant term of \( g_{\alpha}(z) \theta_{\alpha}(z, w; f_{k,m}) \) is a sum of terms of the form \( a(x_2^2 - x_1 x_3, v) e^{2\pi i (x_2^2 - x_1 x_3) w} h_{\alpha}(x, v, w; k) e^{4\pi i |\Lambda(x, w)|^2} \) where \( x = (x_1, x_2, x_3) \in \mathbb{Z}^3 \). By the definition of \( \text{WMF}_{\Gamma_{\chi}}(\Gamma_0(4N), \chi) \), \( a(x_2^2 - x_1 x_3, v) e^{2\pi i (x_2^2 - x_1 x_3) w} = O(v^\delta) \) for some \( \delta > 0 \). So every term is exponentially decreasing as \( v \rightarrow \infty \) except the case of \( \Lambda(x, w) = 0 \). So if \( \Lambda(x, w) \neq 0 \) for all \( x \in \mathbb{Z}^3 \) then the constant term of \( g_{\alpha}(z) \theta_{\alpha}(z, w; f_{k,m}) \) goes to zero when \( v \) goes to \( \infty \). So (4.1) converges and hence \( \Phi_{\theta}(w) \) is well defined where \( \Lambda(x, w) \neq 0 \) for all \( x \in \mathbb{Z}^3 \). And it may have singularities where \( \Lambda(x, w) = 0 \) for some \( x \in \mathbb{Z}^3 \). Since \( \chi(0) = 0 \), we don’t need to consider the case of \( x = 0 \). So the singularities may occur where \( w = \frac{b_3 \sqrt{v} - aw}{2Nc} \in \mathbb{H} \) for \( a, b, c \in \mathbb{Z} \).

And transformation properties of \( \Phi_{\theta}(w) \) come easily from the fact that \( \theta(z, w; f_{k,m}) \) is a non-holomorphic modular form of weight \( 2m \) for \( \Gamma_0(2N) \) and \( \chi^2 \) as a function of \( w \).

We have Maass differential operators on smooth functions from \( \mathbb{H} \) to \( \mathbb{C} \) (See [3], page 97):
\[
R_k = 2i \frac{\partial}{\partial \xi} + kv^{-1}, \\
L_k = 2iv^2 \frac{\partial}{\partial \bar{\xi}}.
\]
For any smooth function \( f : \mathbb{H} \to \mathbb{C} \) and \( \gamma \in \text{SL}_2(\mathbb{Z}) \) it is well known that

\[
(R_k f)|_{k+2} \gamma = R_k(f|_{k} \gamma), \quad (L_k f)|_{k-2} \gamma = L_k(f|_{k} \gamma).
\]

The operator \( \Delta_k \) can be expressed in terms of \( R_k \) and \( L_k \) by

\[
\Delta_k = L_{k+2}R_k - k = R_{k-2}L_k.
\]

We will use \( \Delta_{2m} \) as the Laplace operator with respect to \( w \) and \( \Delta_{k/2} \) as the Laplace operator with respect to \( z \). To prove that \( \Phi(g)(w) \) is an eigenfunction of \( \Delta_{2m} \) we need following lemmas, which are essentially Lemma 4.2 and Lemma 4.3 in [3].

**Lemma 4.1.** Let \( f \in G_{k/2}(\Gamma_0(4N), \chi) \) and \( g \in G_{k/2+2}(\Gamma_0(4N), \chi) \). Then

\[
\int \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)(L_{k/2+2}(g_{\alpha}))(z)u^{k/2-2}dudv - \int \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} (R_{k/2}(f_{\alpha}))(z)g_{\alpha}(z)u^{k/2}dudv = \int \left[ \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)g_{\alpha}(z)u^{k/2} \right]_{u=\bar{v}}dudv.
\]

**Proof** The assumptions imply that \( \omega = v^{k/2}(\sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)g_{\alpha}(z))d\bar{z} \) is a \( \text{SL}_2(\mathbb{Z}) \)-invariant 1-form on \( \mathbb{H} \). By Stokes’ theorem we have

\[
\int_{\partial F} v^{k/2} \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)g_{\alpha}(z)d\bar{z} = \int \left( -\frac{\partial}{\partial v} v^{k/2} \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)g_{\alpha}(z) \right) dudv + \int \left( v^{k/2-2} \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)(L_{k/2+2}(g_{\alpha}))(z) - v^{k/2} \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)(R_{k/2}(f_{\alpha}))(z)g_{\alpha}(z) \right) dudv.
\]

In the integrand over \( \partial F \) on the left hand side the contributions from \( \text{SL}_2(\mathbb{Z}) \)-equivalent boundary pieces cancel. Thus

\[
\int_{\partial F} v^{k/2} \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)g_{\alpha}(z)d\bar{z} = \int \left[ v^{k/2} \sum_{\alpha \in \Gamma_0(4N)\text{SL}_2(\mathbb{Z})} f_{\alpha}(z)g_{\alpha}(z) \right]_{u=\bar{v}}dudv.
\]
This implies the assertion.

**Lemma 4.2.** Let \( f, g \in G_{k/2}(\Gamma_0(4N), \chi) \). Then

\[
\int_{F_1} v^{k/2} \sum_{a \in \Gamma_0(4N) \setminus \mathbb{SL}_2(\mathbb{Z})} (\Delta_{k/2}(f_a))(z)g_a(z) \frac{dudv}{v^2} - \int_{F_1} v^{k/2} \sum_{a \in \Gamma_0(4N) \setminus \mathbb{SL}_2(\mathbb{Z})} f_a(z)(\Delta_{k/2}(g_a))(z) \frac{dudv}{v^2} = \int_{-1/2}^{1/2} [v^{k/2} \sum_{a \in \Gamma_0(4N) \setminus \mathbb{SL}_2(\mathbb{Z})} f_a(z)(L_{k/2}(g_a))(z)]_{v=1} dv - \int_{-1/2}^{1/2} [v^{k/2} \sum_{a \in \Gamma_0(4N) \setminus \mathbb{SL}_2(\mathbb{Z})} (L_{k/2}(f_a))(z)g_a(z)]_{v=1} dv.
\]

**Proof** We write \( \Delta_{k/2} = R_{k/2} - L_{k/2} \) and apply Lemma 4.1 twice. □

Observe that \( \Delta_{k/2}(g_a) = sg_a \). We have

\[
\Delta_{2m}(\Phi(g))(w) = -4 \int_{\mathbb{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} v^{k/2} \sum_{a \in \Gamma_0(4N) \setminus \mathbb{SL}_2(\mathbb{Z})} g_a(z)(\frac{\partial^2}{\partial w \partial \bar{w}} + 2\min \frac{\partial}{\partial w}) \theta_a(z, w; f_{k,m}) \frac{dudv}{v^2} + 4(m(m - 1) - \frac{3}{4} - k(\frac{k}{4} - 1))\Phi(g)(w).
\]

The last equality comes from Proposition 3.10. And by Lemma 4.2

\[
\int_{\mathbb{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} v^{k/2} \sum_{a} g_a(z)(\Delta_{k/2}(\theta_a))(z, w; f_{k,m}) \frac{dudv}{v^2} = s\Phi(g)(w) - \lim_{t \to \infty} \int_{-1/2}^{1/2} v^{k/2} \sum_{a} g_a(z)(L_{k/2}(\theta_a))(z, w; f_{k,m}) du + \lim_{t \to \infty} \int_{-1/2}^{1/2} v^{k/2} \sum_{a} (L_{k/2}(g_a))(z)\theta_a(z, w; f_{k,m}) du.
\]

By the same argument in the proof of the convergence, we see that two integrals

\[
\lim_{t \to \infty} \int_{-1/2}^{1/2} v^{k/2} \sum_{a} g_a(z)(L_{k/2}(\theta_a))(z, w; f_{k,m}) du, \quad \lim_{t \to \infty} \int_{-1/2}^{1/2} v^{k/2} \sum_{a} (L_{k/2}(g_a))(z)\theta_a(z, w; f_{k,m}) du
\]

vanish.
And note that $\Phi(g_D)(w)$ transforms under the group $\Gamma_0(2ND)$. So $\Phi(g_D)(w/D)$ transforms under the group $\Gamma_0(2N,D)$.

5. Proof of Theorem 1.3

Let $\lambda = (k - 1)/2$. We consider the lifting from $H_{k/2}^+(\Gamma_0(4N),\chi)$ to $H_{2\lambda}^+(\Gamma_0(2N, D), \chi^2)$ with singularities. In this case we take $f_{k,\lambda}(x) = (x_1 - i x_2 - x_3)^i \exp(-\frac{2i}{N}(2x_1^2 + x_2^2 + 2x_3^2))$. Then $\theta(z, w; f_{k,\lambda})$ is the same theta function which was used by Niwa [7] and Cipra [6].

As a special case of Theorem 1.1 if $g \in H_{k/2}^+(\Gamma_0(4N),\chi)$ then $\Phi_D(g)(w) \in H_{2\lambda}(\Gamma_0(2N, D), \chi^2)$ with singularities. Now we will compute the image of the lifting when $k \geq 1$ by using the unfolding method. To do that we need to rewrite theta functions as follows:

**Lemma 5.1.** We have

$$\theta(z, i\eta; f_{k,\lambda}) = C \sum_{\nu=0}^{1} (\frac{1}{\nu}) (2/\pi)^{\nu} \sum_{\gamma \in \Gamma_0(4N)} \chi(d) \frac{\gamma(z)}{(\text{Im } \gamma z)^{1/2}} \sum_{m, n} \chi_1(m)m^{1-\nu}H_v(2\sqrt{2\pi \text{Im } \gamma z})e^{2\pi i \frac{m^2 \eta^2}{\text{Im } \gamma z}}$$

with $C = (-1)^{1/2} 2^{-1/4} N^{3/2-1/4}$.

**Proof** This is Theorem 2.11 of [6].

Now we will observe the behavior of $\Phi(g)(w)$ at $i\infty$. The following computations are essentially done in Theorem 2.12 in [6]. Assume that $k \geq 1$. Let

$$g(z) = \sum_{n\geq 0} a^+(n)e(nz) + \sum_{n<0} a^-(n)W_{k/2}(2\pi n) e(nz) \in H_{k/2}^+(\Gamma_0(4N),\chi).$$
Using the Lemma 5.1, we have

\[
\Phi(g)(i\eta) = \int_{\Gamma_0(4N)\backslash \mathbb{H}} \psi^{1/2} g(z) \theta(z, i\eta; f, \chi) \frac{dudv}{v^2}
\]

\[
= C \sum_{y=0}^A \left( \frac{\lambda}{y} \right) (2/\pi)^{y/2} i^{1-y} \int_0^\infty \int_0^1 \psi^{1/2} g(z) v^{y/2-1} \sum_{m,n=-\infty}^\infty \chi_1(m) \times m^{1-y} H_v(2\sqrt{2\pi v}) e^{-2\pi i v \frac{m^2}{4v}} dv dv
\]

\[
= C \sum_{y=0}^A \left( \frac{\lambda}{y} \right) (2/\pi)^{y/2} i^{1-y} \int_0^\infty v^{(y-1)/2} \sum_{n=-\infty}^\infty a^+(n^2) H_v(2\sqrt{2\pi v})
\]

\[
\times e^{-4\pi m^2 v} \sum_{m=-\infty}^\infty \chi_1(m)m^{1-y} e^{-\pi \eta^2 m^2/4v} dv/v
\]

\[
= C' \sum_{y=0}^A \left( \frac{\lambda}{y} \right) (2/\pi)^y \int_0^\infty \left[ a^+(0) H_v(0) + \sum_{n=0}^\infty a^+(n^2) H_v(n) e^{-n^2 y^2/2} \right] \times (\eta/y)^{1-y} \sum_{m=-\infty}^\infty \chi_1(m)m^{1-y} e^{-2\pi^2 m^2 (\eta/y)^2} dy/y
\]

with \( C' = 2C(8\pi)^{1/2} = (-1)^{1/2} 2^{-1/2} N^{1/2} (2\pi)^{1/2}. \) As \( \eta \to \infty, \) the only non-negligible term is that one involving \( a^+(0): \)

\[
\Phi(g)(i\infty) = a^+(0) C' \sum_{y=0}^A \left( \frac{\lambda}{y} \right) (2/\pi)^y H_v(0) \int_0^\infty y^{1-y} \sum_{m=-\infty}^\infty \chi_1(m)m^{1-y} e^{-2\pi^2 m^2 y^2} dy/y.
\]

If \( k \geq 3 \) then we invert the theta function using Poisson summation

\[
\Phi(g)(i\infty) = a^+(0) C'' \sum_{y=0}^A \left( \frac{\lambda}{y} \right) i^y H_v(0) \int_0^\infty y^{-1} \sum_{m=-\infty}^\infty \tilde{\chi}_1(m) H_{A-y}(\frac{m}{4Ny}) e^{-m^2/32N^2 y^2} dy/y
\]

with \( C'' = C'(2\pi i)^{-1} (2\pi)^{-1/2} (4N)^{-1} = i^{1/2} 2^{-5/2} N^{-1/2} \pi^{-1}. \) We can now actually sum over \( \nu: \)

\[
\sum_{y=0}^A \left( \frac{\lambda}{y} \right) i^y H_v(0) H_{A-y}(\frac{m}{4Ny}) = \left( \frac{m}{4Ny} \right)^4
\]

so

\[
\Phi(g)(i\infty) = a^+(0) C'' \int_0^\infty \sum_{m=-\infty}^\infty \tilde{\chi}_1(m)(\frac{m}{4Ny})^4 e^{-m^2/32N^2 y^2} dy/y
\]

\[
= a^+(0) C'' \int_0^\infty \sum_{m=1}^\infty \tilde{\chi}_1(m)(\frac{m}{m^2}) y^4 e^{-y^2} dy/y
\]
with \( C^* = C^*(8N)^{-1} = i^42^{-3/2}N^{(3/2)\alpha-(3/4)}\pi^{-\lambda}. \) The integral gives \( \Gamma(\lambda), \) and \( C^*\Gamma(\lambda) = C_0(\lambda). \) So

\[
\Phi(g)(i\infty) = a^+(0)C_0(\lambda) \sum_{m=1}^{\infty} \xi_1(m)m^{-\lambda}
\]

where \( C_0(\lambda) = i^42^{-3/2}N^{(3/2)\alpha-(3/4)}\pi^{-\lambda}\Gamma(\lambda) \) and \( \xi_1(m) = \sum_{h=1}^{4N} \chi_1(h)e^{\pi ihm/2N}. \) And if we use functional equations for the \( L \)-series, we get the result about the constant term.

If \( k = 1 \) then we can evaluate directly without the inversion

\[
\Phi(g)(i\infty) = a^+(0)C' \int_{0}^{\infty} y \sum_{m=0}^{\infty} \chi(m) \exp(-2\pi^2 m^2 y^2)dy/y
\]

\[
= a^+(0)C'2^{-1/2}\pi^{-1/2} \Gamma(1/2) \sum_{m=1}^{\infty} \frac{\chi(m)}{m}
\]

\[
= 4N^{1/4}a^+(0) \sum_{m=1}^{\infty} \frac{\chi(m)}{m}.
\]

From this, we see that \( \Phi(g)(w) \) is bounded at \( i\infty. \) So we see that \( \Phi(g)(w) \in H^*_2(\Gamma_0(2N), \chi^2) \) with singularities and \( P(\Phi(g)) \) is constant. We have

\[
\Phi(g)(i\eta) - \Phi(g)(i\infty) = C' \sum_{\nu=0}^{\lambda} \left( \frac{i}{\nu} \right) (2\pi)^{-\nu} \int_{0}^{\infty} \sum_{n=1}^{\infty} a^+(n^2)H_\nu(yn)e^{-\pi^2 n^2(y^2/2)}(\eta/y)^{1-\nu}
\]

\[
\times \sum_{m=0}^{\infty} \chi_1(m)m^{-\lambda-\nu} e^{-2\pi^2 m^2(\eta/y)^2}dy/y.
\]

Now

\[
\int_{0}^{\infty} \eta^{\nu-1}(\eta/y)^{1-\nu} \sum_{m=0}^{\infty} \chi_1(m)m^{-\lambda-\nu} e^{-2\pi^2 m^2(\eta/y)^2}d\eta
\]

\[
= y'(2\pi^{2})(\nu-s-1)/2 \sum_{m=1}^{\infty} \chi_1(m)m^{-(s-\lambda+1)} \int_{0}^{\infty} \mu^{(s+1-\nu)/2} e^{-\mu} d\mu/\mu
\]

\[
= y'(2\pi^{2})(\nu-s-1)/2 L(s-\lambda + 1, \chi_1) \Gamma \left( \frac{s+1-\nu}{2} \right).
\]

Thus

\[
\int_{0}^{\infty} \eta^{\nu-1}(\Phi(g)(i\eta) - \Phi(g)(i\infty))d\eta = C' L(s-\lambda + 1, \chi_1) \sum_{\nu=0}^{\lambda} \left( \frac{i}{\nu} \right) (2\pi)^{-\nu}(2\pi^{2})(\nu-s-1)/2
\]

\[
\times \Gamma \left( \frac{s+1-\nu}{2} \right) \int_{0}^{\infty} y^{\nu} \sum_{n=1}^{\infty} a^+(n^2)H_\nu(yn)e^{-\pi^2 n^2(y^2/2)}dy/y.
\]
Now

\[
\int_0^\infty y^{s-1} \sum_{n=0}^\infty \frac{a^n(n^2)}{n^s} H_\nu(y) e^{-ny^2/2} dy/y
\]

\[
= \left( \sum_{n=1}^\infty \frac{a^n(n^2)}{n^s} \right) \int_0^\infty y^{s-1} (H_\nu(y) + H_\nu(-y)) e^{-y^2/2} dy/y
\]

\[
= \begin{cases} 
0 & \text{if } \nu \text{ is odd} \\
2 \left( \sum_{n=1}^\infty \frac{a^n(n^2)}{n^s} \right) \int_0^\infty y^{s-1} (-1)\nu \left( \frac{dy}{dy} e^{-y^2/2} \right) dy & \text{if } \nu \text{ is even}
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } \nu \text{ is odd} \\
2 \left( \sum_{n=1}^\infty \frac{a^n(n^2)}{n^s} \right) (s-1) \cdots (s-\nu) \int_0^\infty y^{s-\nu-1} e^{-y^2/2} dy & \text{if } \nu \text{ is even}
\end{cases}
\]

Thus

\[
\int_0^\infty \eta^{s-1} (\Phi(g)(i\eta) - \Phi(g)(i\infty)) d\eta = \Gamma(s-\lambda + 1, \chi_1) \left( \sum_{n=1}^\infty \frac{a^n(n^2)}{n^s} \right) \sum_{\nu \text{ even}} \left( \frac{1}{\nu} \right)
\]

\[
\times \frac{\Gamma\left( \frac{s+\nu-1}{2} \right) \Gamma\left( \frac{s+\nu+1}{2} \right)}{2^{\nu+1/2} \pi^{\nu+1}} (s-1) \cdots (s-\nu).
\]

Using the identity \( \Gamma(t/2) \Gamma((t+1)/2) = 2^{1-t} \pi^{1/2} \Gamma(t) \), we get

\[
\int_0^\infty \eta^{s-1} (\Phi(g)(i\eta) - \Phi(g)(i\infty)) d\eta = \Gamma''(s-\lambda + 1, \chi_1) \left( \sum_{n=1}^\infty \frac{a^n(n^2)}{n^s} \right) \sum_{\nu \text{ even}} \left( \frac{1}{\nu} \right)
\]

\[
\times \frac{\Gamma(s-\nu)(s-1) \cdots (s-\nu)}{(2\pi)^s}
\]

with \( \Gamma'' = 2C'(2\pi)^{-1/2} = (-1)^{s-1} 2^{-s-1/2} N^{s+1/2} \). Note that \( \Gamma(s-\nu)(s-1) \cdots (s-\nu) = \Gamma(s) \) and

\[
\sum \left( \frac{1}{\nu} \right) = 2^{1-1} = 1.
\]

So

\[
\int_0^\infty \eta^{s-1} (\Phi(g)(i\eta) - \Phi(g)(i\infty)) d\eta = C_1(\lambda) (2\pi)^{-s} \Gamma(s-\lambda + 1, \chi_1) \left( \sum_{n=1}^\infty \frac{a^n(n^2)}{n^s} \right)
\]

with \( C_1(\lambda) = (-1)^{s-1} 2^{-s-1/2} N^{s+1/2} \).
THETA LIFTINGS ON WEAK MAASS FORMS

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