Coulomb gauge gluon propagator and the Gribov formula

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**Abstract**

We analyze the lattice SU(2) Yang-Mills theory in Coulomb gauge. We show that the static gluon propagator is multiplicative renormalizable and takes the simple form $D(|\vec{p}|)^{-1} = \sqrt{|\vec{p}|^2 + M^4/|\vec{p}|^2}$, proposed by Gribov through heuristic arguments many years ago. We find $M = 0.88(1)$GeV $\simeq 2\sqrt{\sigma}$.

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I. INTRODUCTION

Non-Abelian gauge theories in Coulomb gauge have recently been the subject of extensive research. In particular, estimates for the ground state wave function have been derived through variational methods \[1, 2\]. A key ingredient in such procedure is the static transverse gluon propagator $D(|\vec{p}|)$, which turns out to be compatible with the Gribov-Zwanziger scenario \[3, 4\]. Unfortunately up to now \textit{ab initio} lattice calculations have failed to deliver a conclusive comparable result. A first study carried out for SU(2) at fixed $\beta = 2.2$ \[5, 6\] indicated compatibility with Gribov’s formula \[3\] in the IR but was inconclusive in the UV. Later studies in SU(2) and SU(3) showed strong scaling violations and contradictory IR results for $D(|\vec{p}|)$ \[7, 8, 9\]. All featured a UV behaviour at odds with simple dimensional arguments. Except for \[5, 6\], where the pole structure of the full propagator was analyzed, all these studies \[7, 8, 9\] have simply calculated the static propagator relying on the fact that once Coulomb gauge is fixed $D(|\vec{p}|) = \sum_{p_0} D(|\vec{p}|, p_0)$ does not depend on the residual gauge freedom of temporal fields. One takes, however, \textit{de facto} for granted that multiplicative renormalizability holds for the full propagator, although perturbative results seem to point at a more complex picture \[10\], and that cutoff effects do not influence the result. To verify these assumptions we will fix analytically the residual temporal gauge and study the renormalization of the full propagator $D(|\vec{p}|, p_0)$. We will show that: i) the latter is not multiplicatively renormalizable; ii) the static propagator $D(|\vec{p}|)$ is renormalizable only in the limit of continuous time, i.e. in the lattice Hamiltonian formulation. The resulting $D(|\vec{p}|) = \int_{-\infty}^{+\infty} dp_0 D(|\vec{p}|, p_0)$ satisfies the Gribov formula for all momenta \[3\].

II. SETUP

We discretize the SU(2) Yang-Mills theory through the Wilson plaquette action $S = \beta \sum_{x,\mu\nu}(1 - \frac{1}{2} \text{Tr} U_{\mu\nu}(x))$, employing the standard Creutz algorithm (heat-bath plus over-relaxation). We take $2.15 \leq \beta \leq 2.6$ and lattices $L=12, 16, 20$ and 24, with 200 to 1000 measurements, depending on $L$. The lattice gluon propagator $D^{ab}_{ij}(p)$ is
the Fourier transform of the gluon two-point function:

\[ D_{ij}^{ab}(p) = \left\langle \tilde{A}_i^a(k)\tilde{A}_j^b(-k) \right\rangle_U = \delta^{ab}\delta_{ij}D(p) , \quad p_\mu = p(k_\mu) = \frac{2}{a}\sin\left(\frac{\pi k_\mu}{L}\right) . \]  

Here \( \tilde{A}_i^a(k) \) is the Fourier transform of the lattice gauge potential \( A_i^a(x + \hat{\mu}/2) \) (see Eq. (3)), \( a \) is the lattice spacing, \( p = (\vec{p}, p_0) \) denotes the four-momentum and \( k_\mu \in (-L/2, +L/2] \) are the integer-valued lattice momenta. Results for the temporal component \( D_{00}^{ab}(p) \) will be presented in a forthcoming paper \[11\]. The Coulomb gauge is fixed for each configuration maximizing for every time slice

\[ F_g(t) = \frac{1}{6L^3} \sum_{\vec{x},i} \text{Tr}\ U_i^g(\vec{x}, t) , \quad U_i^g(\vec{x}, t) = g(\vec{x})\ U_i(\vec{x}, t)\ g^\dagger(\vec{x} + \hat{i}) \]  

with respect to local gauge transformations \( g(\vec{x}) \in \text{SU}(2) \). The local maxima of \( F_g(t) \) satisfy for each \( t \) the differential lattice Coulomb gauge transversality condition for the gauge potentials:

\[ \partial_i A_i^\mu(\vec{x}, t) = A_i^\mu(\vec{x} + \hat{i}/2, t) - A_i^\mu(\vec{x} - \hat{i}/2, t) = 0 , \quad A_\mu(x + \hat{\mu}/2) = \frac{1}{2i}(U_\mu(x) - U_\mu(x)^\dagger) . \]  

In detail we: i) apply for the whole lattice a flip preconditioning \[12, 13\] as in Ref. \[8\]; ii) maximize \( F_g(t) \) via several standard over-relaxation and/or simulated annealing cycles after a random gauge copy for each time separately; iii) choose at every slice the copy with the highest \( F_g(t) \); iv) perform a random gauge transformation on the whole lattice and repeat from step i). The number of total recursions varies between 10-40, depending on \( \beta \) and \( L \). The copy of the whole lattice with the best functional value \( \sum_t F_g(t) \) is chosen to select the gauge-fixed configuration. This procedure remains relatively efficient for every volume at higher \( \beta \), but in the strong coupling (low \( \beta \)) region Gribov noise is no longer under control as the lattice size increases. This prevents us from going beyond \( L = 24 \). Future studies on larger lattices should rather adapt the improved algorithm proposed in Ref. \[13\].

Once the Coulomb gauge has been fixed, only space independent gauge transformations \( g(t) \) can be performed at each time slice. In the continuum and on the lattice...
one can choose:
\[ 0 = \int d^3 x \partial_\mu A_\mu(\vec{x}, t) = \partial_0 \int d^3 x A_0(\vec{x}, t) = 0 \Leftrightarrow u(t) = \frac{1}{L^3} \sum_{\vec{x}} U_0(\vec{x}, t) \to \text{const.} \]

Define \( \hat{u}(t) = u(t)/\text{Det}[u(t)] \in \text{SU}(2) \). Periodic boundary conditions make \( \text{Tr} \prod_t \hat{u}(t) = \text{Tr} P \) invariant under \( g(t) \). Eq. (4) can thus be fixed recursively through \( g(t) \hat{u}(t) g(t+1) = \tilde{u} \equiv P^{1/L} \), up to a \( g(0) \) satisfying \( [g(0), P] = 0 \). We can choose \( g(0) = 1 \). Of course \( \tilde{u} \to 1 \) only for \( L \to \infty \). We find \( 1 - \text{Tr} \tilde{u} \simeq 10^{-3} \) already for \( L = 24 \), with leading corrections \( \propto L^{-2} \).

We always select spatial momenta indices \( \vec{k} \) satisfying a cylinder cut [9, 14] to minimize violations of rotational invariance. Time-like momenta are unconstrained. The scale is set re-expressing the lattice spacing \( a(\beta) \) in units of the string tension \( \sigma = (440\text{MeV})^2 \) [15].

III. RESULTS

We find that the lattice bare propagator \( D_\beta(|\vec{p}|, p_0) \) satisfies over the whole \( p \) range:
\[ D_\beta(|\vec{p}|, p_0) = f_\beta(|\vec{p}|) \frac{g_\beta(z)}{|\vec{p}|^2} \frac{1}{1+z^2} z = \frac{p_0}{|\vec{p}|}, \quad (5) \]
where \( |\vec{p}|^2(1 + z^2) \) is explicitly factorized since for large \( p^2 \) one expects \( D_\beta(|\vec{p}|, p_0) \simeq (|\vec{p}|^2 + p_0^2)^{-1} \) on dimensional grounds. \( f_\beta \) and \( g_\beta \) are dimensionless functions and without loss of generality we can choose \( g_\beta(0) = 1 \). From Eq. (5) it is clear that a dynamical mass generation must occur in \( f_\beta \), i.e. it must be a function of \( |\vec{p}|/M \). We can check that taking \( \sum_{p_0} D_\beta(|\vec{p}|, p_0) \) reproduces the results of Ref. [5]. The data for \( g_\beta(z) = (1+z^2)D_\beta(|\vec{p}|, p_0)/D_\beta(|\vec{p}|, 0) \) are shown in Fig. [1] for \( L = 24 \). Their leading behaviour is well described by a power law \( (1+z^2)^\alpha \). Sub-leading corrections can be parameterized equally well through different Ansätze, with all fits giving \( \chi^2/\text{d.o.f.} \) between 0.8 and 1.2. Without further theoretical input we have no reason to prefer one upon the other. For \( \beta \gtrsim 2.3 \) the functions \( g_\beta \) vary quite consistently with \( L \). In Table [1] we give our best estimates for \( \alpha(\beta) \) in the large \( L \) limit, assuming again a leading correction \( \propto L^{-2} \). Within one or two standard deviations all values of \( \alpha \) are compatible with
FIG. 1: Data for $g_\beta(z)$ vs $1 + z^2$ in log-log scale, $L = 24$. For sake of readability not all $\beta$ are shown.

1, i.e. $D_\beta(|\vec{p}|, p_0)$ might be $p_0$ independent in the thermodynamic limit. A study on larger, perhaps anisotropic lattices will be needed to clarify the matter. It is anyway clear from Eq. (5) that, as long as for our choices of $\beta$ and $L$ the functions $g_\beta(z)$ remain different, as they do, $D_\beta(|\vec{p}|, p_0)$ can not be multiplicative renormalizable. On the other hand, let us consider different lattice cut-offs for space and time $a_s/a_t =$
\[ \beta \begin{pmatrix} 2.15 & 2.2 & 2.25 & 2.3 & 2.4 & 2.5 & 2.6 \\ \alpha(\beta) & 1.01(2) & 1.02(3) & 1.02(3) & 0.99(5) & 0.96(6) & 0.8(1) & 0.6(2) \\ Z(\beta,\mu) & 0.403(4) & 0.430(5) & 0.456(6) & 0.475(7) & 0.515(10) & 0.536(12) & 0.548(13) \end{pmatrix} \]

TABLE I: \( L \to \infty \) exponents \( \alpha \) for the functions \( g_\beta(z) \); renormalization coefficients \( Z(\beta,\mu) \) as in Eq. (8).

For \( \xi > 1 \) [16]. For \( \xi \to \infty \) (lattice Hamiltonian limit) \( D_\beta(|\vec{p}|) \propto |\vec{p}|^{-1}f_\beta(|\vec{p}|) \) up to the (possibly diverging) constant \( \int_0^\infty dz g_\beta(z)/(1+z^2) \) and multiplicative renormalizability relies solely on \( f_\beta(|\vec{p}|) \). Eq. (5) also explains the scaling violations in Ref. [7, 8, 9]. Let us assume that \( f_\beta(|\vec{p}|) \) is renormalizable, define \( \hat{p} = a_s|\vec{p}| \) and for simplicity neglect sub-leading corrections in \( g_\beta(z) \). For large \( L \) we can approximate the discrete sum over \( p_0 \) by an integral and:

\[
D_\beta(|\vec{p}|) \approx \int_{-\frac{2\pi}{\xi}}^{\frac{2\pi}{\xi}} \frac{dp_0}{2\pi} D_\beta(|\vec{p}|, p_0) = f_\beta(|\vec{p}|) \int_{0}^{\frac{2\pi}{\xi}} \frac{dz}{\pi} (1 + z^2)^{\alpha - 1} = f_\beta(|\vec{p}|) I\left(\frac{2\xi}{\hat{p}, \alpha}\right);
\]

\[
I\left(\frac{2\xi}{\hat{p}, \alpha}\right) = \frac{1}{2\pi} B\left(\frac{4\xi^2}{4\xi^2 + \hat{p}^2}, \frac{1}{2}, -\alpha + \frac{1}{2}\right),
\]

where \( B(z, a, b) \) is the incomplete beta function. For \( \xi \to \infty \) \( I \) explicitly goes to a \( |\vec{p}| \) independent constant, \( B(1, 1/2, -\alpha + 1/2) \), while in a standard lattice formulation, where \( \xi \equiv 1 \), there is no way to avoid the extra \( |\vec{p}| \) dependence \( I(2/\hat{p}, \alpha) \), which will violate multiplicative renormalizability. Indeed such function times the tree level result \( |\vec{p}|^{-1} \) compares well with the behaviour observed in [7, 8, 9].

From the above discussion it is clear that the \( p_0 \) dependence is immaterial to the static propagator in the Hamiltonian limit, all the relevant information being encoded in \( f_\beta(|\vec{p}|) \). Since the limit \( \xi \to \infty \) is out of the reach of available simulations, to extract \( f_\beta \) from simulations at standard sizes and coupling (i.e. spacing) one could use only results at fixed \( p_0 \), e.g. \( f_\beta(|\vec{p}|) \equiv D_\beta(|\vec{p}|, 0) \), at the cost of losing data. Although this works, we propose here an alternative procedure which makes use of the results for all energies, increasing the statistics. We divide the \( z \) dependence out of the data using.
the fitted functional form for $g_\beta(z)$ and average the result over $p_0$:
\[
f_\beta(|\vec{p}|) := \frac{1}{L} \sum_{p_0} D_\beta(|\vec{p}|, p_0) \left[ 1 + \frac{z^2}{g_\beta(z)} \right] ; \quad D_\beta(|\vec{p}|) := \frac{f_\beta(|\vec{p}|)}{|\vec{p}|} .
\]  
(7)

The factor $|\vec{p}|^{-1}$ only serves to restores dimensions when comparing with the Hamiltonian result, according to Eq. (6), and is immaterial to renormalization. To minimize finite size effects we exclude spatial momenta not satisfying a cone cut \[9, 14\] for all lattices with \((a(\beta)L)^4 < (2.5\text{fm})^4\). The $D_\beta(|\vec{p}|)$ calculated form Eq. (7) are multiplicative renormalizable via $D_\mu(|\vec{p}|) = Z(\beta, \mu)D_\beta(|\vec{p}|)$, where $\mu$ defines the renormalization point. We choose it such that (see Table I):
\[
D_\mu(|\vec{p}|)|_{\mu=\infty} = Z(\beta, \mu)|_{\mu=\infty}D_\beta(|\vec{p}|)|_{|\vec{p}| \rightarrow \infty} \frac{1}{2|\vec{p}|} ,
\]  
(8)
corresponding to the naive UV continuum behaviour. Fig. 2 shows the result of our procedure. We have performed fits of $D_\mu(|\vec{p}|)$ allowing at the same time a power law in the IR, $|\vec{p}|^q/M^2 - q$, and a power law plus logarithmic corrections in the UV, $|\vec{p}|^{-r}\log|\vec{p}|^{-s}$. We get $q = 0.99(1)$, $r = 1.002(3)$, $s = 0.002(2)$ and $M = 0.88(1)$ GeV, with $\chi^2$/d.o.f. all in the range 3.2-3.3, in agreement with the UV and IR analysis in Ref. \[2, 17, 18\]. Since our data show excellent IR-UV symmetry for $p \leftrightarrow M^2/p$ (see Fig. 1), we constrain $q = r = 1$, $s = 0$ and fit the whole result through the Gribov formula
\[
D_\mu(|\vec{p}|)|_{\mu=\infty} = \frac{1}{2 \sqrt{|\vec{p}|^2 + M^4}}
\]  
(9)
We find just as good agreement ($\chi^2$/d.o.f. = 3.3) again with $M = 0.88(1)\text{GeV} \simeq 2\sqrt{\sigma}$. This compares well with $M \simeq 0.5 - 0.8$ GeV from the IR analysis in Ref. \[5, 6\], given that only $\beta = 2.2$ was used there. Apart from unavoidable residual violations of rotational invariance, the highest noise in the data occurs around the peak, $|\vec{p}| \simeq M$. Only better gauge fixing and larger volumes can help here. We have also collected data for $2.6 < \beta \leq 2.8$. They are still in perfect agreement with the above formula, however since physical volumes are very small we cannot claim to have control over systematic effects, so we omit them in the above fits. An investigation on higher volumes would be welcome.
FIG. 2: Gluon propagator. Circles are (a few) data for $|\vec{p}| \rightarrow M^2/|\vec{p}|$, the continuum line is the fit to Gribov’s formula and the dot-dashed line the result of the Hamiltonian approach of Ref. [17].

IV. CONCLUSIONS

We have shown that on the lattice the static gluon propagator in Coulomb gauge is multiplicative renormalizable only in the Hamiltonian limit. This calls for analogous
investigations in the continuum, where to our knowledge renormalizability has not been proven in Coulomb gauge. We have also given a procedure to extract the relevant information from the data at accessible lattice sizes and spacings. Summarizing this procedure: i) calculate the full bare propagator $D_\beta(|\vec{p}|,p_0)$ after fixing the residual gauge as in Eq.(4); ii) fit to the data the function $g_\beta(z)$ arising from the factorization in Eq.(5); iii) divide out $g_\beta(z)$ from $D_\beta(|\vec{p}|,p_0)$ to isolate its $p_0$ independent part $f_\beta(|\vec{p}|)$; iv) define $D_\beta(|\vec{p}|) = f_\beta(|\vec{p}|)/|\vec{p}|$. The so obtained static propagator $D_\beta(|\vec{p}|)$ is multiplicative renormalizable, UV-IR symmetric and is well described by Gribov’s formula over the whole momentum range. Its infrared and ultraviolet behaviours are in good agreement with the results obtained in the variational approach to continuum Yang-Mills theory in Coulomb gauge [2]. Both the infrared analysis of the Dyson-Schwinger equations in this approach and its numerical solution yield for the infrared exponent $q = 1$ [17, 18]. We will compare further quantities obtained in the variational approach [2] with the lattice using the procedure proposed above [11].

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