Isotomic and Isogonal Conjugates Tangent Lines of Lines at Vertices of Triangle

Min Young Seo¹ and Young Joon Ahn²

Abstract

In this paper we consider the two tangent lines of isogonal and isotomic conjugates of the line at both vertices of a given triangle. We find the necessary and sufficient condition for the two tangent lines of isogonal or isotomic conjugates of the line at both vertices and the median line to be concurrent. We also prove that every line whose isotomic conjugate tangent lines at both vertices are concurrent with the median line intersects at a unique point. Moreover, we show that the three intersection points correspond to the vertices of triangle are collinear.

Keywords: Barycentric Coordinates, Isotomic Conjugate, Isogonal Conjugate, Tangent Line, Ceva’s Theorem, Concurrency of Three Lines

1. Introduction

The barycentric coordinates with respect to a triangle are widely used in CAGD (Computer Aided Geometric Design) as well as in Euclidean Geometry. In particular, the use of barycentric coordinates has played an important rule in a lot of methods for the conic representation and conic approximation¹⁻⁶. In Euclidean Plane Geometry, so many works have been done based on the use of them⁷⁻¹¹.

Recently, Akopyan¹² presented the properties of the tangency of isotomically and isogonally conjugate lines of some special lines with respect to a triangle. Yoon and Ahn¹³ showed that the isogonal and isotomic conjugates of conic tangent to two side lines at vertices are again conic tangent to the two side lines at vertices and classified them into ellipses, parabolas and hyperbolas using the barycentric coordinates.

In this paper we study the two tangent lines of isotomically and isotomic conjugates of the line L at both vertices B, C of the reference triangle ∆ABC. We find the necessary and sufficient condition for the two tangent lines of isogonal or isotomic conjugates of the line at both vertices and a median line AM₁ to be concurrent, where M₁ is the midpoint of side line BC. We also prove that all lines whose isotomic conjugate tangent lines at both vertices are concurrent with the median line AM₁ intersect at a unique point X₁. Moreover we show that the three intersection points X₁, X₂, X₃ are collinear. All of our results are based on the barycentric coordinates.

I suggest that The contents of our paper are organized as follows. In Section 2, the basic facts in elementary plane geometry are provided, and in Section 3, our main results are presented. The results in the first half of Section 3 improves of the MS Thesis of the first author of this paper¹⁴.

2. Preliminaries for Elementary Plane Geometry

In this section we remind the definitions of barycentric coordinates, homogeneous barycentric coordinates, isotomic conjugate, and isogonal conjugate⁴⁻¹⁰,¹³,¹⁵,¹⁶.

Every point P in a reference triangle ∆ABC satisfies

\[ \overrightarrow{OP} = \frac{1}{\triangle ABC} (\triangle BCP \cdot \overrightarrow{OA} + \triangle CAP \cdot \overrightarrow{OB} + \triangle ABP \cdot \overrightarrow{OC}) \]

(2.1)
where $O$ is the origin on the plane containing the triangle and $\Delta XYZ$ is the area of triangle $\Delta XYZ^{10,13}$ In Eq. (2.1), the ordered triple

$$
\left( \frac{\Delta BCP}{\Delta CAP}, \frac{\Delta CAP}{\Delta ABC}, \frac{\Delta ABC}{\Delta ABP} \right)
$$

is called by the barycentric coordinates of $P$ with respect to $\Delta ABC$. The definition of barycentric coordinates can be extended to all points on the plane from $\Delta ABC$ as follows$^{14}$. For every point $P$ on the plane, $OP$ is uniquely expressed by

$$
OP = \tau_0 OA + \tau_1 OB + \tau_2 OC
$$

$$
\tau_0 + \tau_1 + \tau_2 = 1.
$$

At this time, $(\tau_0, \tau_1, \tau_2)$ is called by the barycentric coordinates of $P$. The relationship between the signatures of barycentric coordinates and the position of $P$ outside of $\Delta ABC$ is well-known$^{13,17}$.

The midpoints of the side lines $BC$, $CA$, $AB$ are denoted by $M_A$, $M_B$, $M_C$, respectively. For the point $P$ inside the triangle $\Delta ABC$, let the points $P_A$, $P_B$, $P_C$ be the intersection points of the lines $AP$, $BP$, $CP$ and side lines $BC$, $CA$, $AB$, respectively, and let $P'_A$, $P'_B$, $P'_C$ be the symmetric point of $P_A$, $P_B$, $P_C$ with respect to $M_A$, $M_B$, $M_C$, respectively. Then the three lines $AP'_A$, $BP'_B$, $CP'_C$ are concurrent at a point, which is called by the isotomic conjugate of $P$ and denoted by $P'$. It is also well-known$^{10,15,16}$ that

$$
\Delta BCP : \Delta CAP : \Delta ABP = \frac{1}{\Delta BCP} : \frac{1}{\Delta CAP} : \frac{1}{\Delta ABP}
$$

and its homogeneous barycentric coordinates is

$$
\left( \frac{1}{\tau_0}; \frac{1}{\tau_1}; \frac{1}{\tau_2} \right).
$$

The angle bisectors at the vertices $A$, $B$, $C$ of $\Delta ABC$ are denoted by $L_A$, $L_B$, $L_C$, respectively, which are concurrent at the incenter of $\Delta ABC$. For the point $P$ inside the triangle $\Delta ABC$, let the lines $L'_A$, $L'_B$, $L'_C$ be the symmetric lines of the lines $AP$, $BP$, $CP$ with respect to $L_A$, $L_B$, $L_C$, respectively. The three lines $L'_A$, $L'_B$, $L'_C$ are concurrent at a point, which is called by the isogonal conjugate of $P$ and denoted by $P''$. It is also well-known$^{10,15,16}$ that

$$
\Delta BCP : \Delta CAP : \Delta ABP = \frac{\alpha^2}{\Delta BCP} : \frac{\beta^2}{\Delta CAP} : \frac{\gamma^2}{\Delta ABP}
$$

and $P''$ has the homogeneous barycentric coordinates

$$
\left( \frac{{\alpha}}{\tau_0}; \frac{\beta}{\tau_1}; \frac{\gamma}{\tau_2} \right).
$$

Ceva’s theorem$^{14}$ will be used to prove our main theorems.

**Theorem 2.1 (Ceva’s Theorem)**

Let the points $X, Y, Z$ be on the side lines $BC, CA, AB$ of a triangle $\Delta ABC$, respectively. The lines $AX$, $BY$, $CZ$ are concurrent if and only if

$$
AZ : BX : CY = 1.
$$

**3. Tangent Lines of Isotomic and Isogonal Conjugates of Line at Vertices of Triangle**

In this section, we consider a line $L$ which intersects the side lines $AB$ and $AC$ of a reference triangle $\Delta ABC$ at two points $D, E$, respectively.

**Theorem 3.1**

The two tangent lines of the isotomic conjugate curve of $L$ at the vertices $B, C$ and the median line $AM_A$ are concurrent if and only if the line $L$ is parallel to the side line $BC$.

**Proof.**

Let $p_0 = \overline{OA}$, $p_1 = \overline{OB}$, $p_2 = \overline{OC}$. There are real numbers $\delta_1, \delta_2 \in (0, 1)$ such that

$$
\overline{OD} = (1 - \delta_1) \overline{OA} + \delta_1 \overline{OB}
$$

$$
\overline{OE} = (1 - \delta_2) \overline{OA} + \delta_2 \overline{OC}.
$$

The line $L$ has the parametric equation

$$
\mathbf{r}(t) = (1 - \delta_1 + t(\delta_1 - \delta_2)) \mathbf{p}_0 + (1 - t) \delta_1 \mathbf{p}_1 + \delta_2 \mathbf{p}_2
$$

(3.1)
and the homogeneous barycentric coordinates

\[
(1 - \delta_1 + t(\delta_1 - \delta_2)) : (1 - t)\delta_1 : t\delta_2
\]

(3.2)

So, the isotomic conjugate curve \( r^*(t) \) has the homogeneous barycentric coordinates

\[
\left( \frac{1}{1 - \delta_1 + t(\delta_1 - \delta_2)} : \frac{1}{1 - t}\delta_1 : \frac{1}{t}\delta_2 \right)
\]

and the parametric equation of \( r^*(t) \) is

\[
r^*(t) = \left[ (1 - t)\delta_1 t\delta_2 \mathbf{p}_0 + (1 - \delta_1 + t(\delta_1 - \delta_2))t\delta_2 \mathbf{p}_1 \\
+(1 - \delta_1 + t(\delta_1 - \delta_2))(1 - t)\delta_1 \mathbf{p}_2 \right] / w(t)
\]

where

\[
w(t) = (1 - t)\delta_1 \delta_2 + (1 - \delta_1 + t(\delta_1 - \delta_2))\delta_2 \\
+(1 - \delta_1 + t(\delta_1 - \delta_2))(1 - t)\delta_1.
\]

Note that \( r^*(t) \) is passing through the points \( C, B \) when \( t = 0, 1 \), respectively. For \( i = 0, 1 \), let \( T_i \) be the tangent line of \( r^*(t) \) at \( t = i \), and let \( F, G \) be the intersection points of \( T_0 \) and \( AB, T_1 \) and \( AC \), respectively. Since

\[
r^*(0) = \frac{\delta_2}{(1 - \delta_1)\delta_2}((\delta_1 \mathbf{p}_0 + (1 - \delta_1) \mathbf{p}_1 - \mathbf{p}_2))
\]

\[
r^*(1) = -\frac{\delta_1}{(1 - \delta_1)\delta_2}((\delta_1 \mathbf{p}_0 - \mathbf{p}_1 + (1 - \delta_1) \mathbf{p}_2)),
\]

we have

\[
AF : FB = 1 - \delta_1 : \delta_1 \\
AG : GC = 1 - \delta_2 : \delta_2.
\]

By Ceva’s Theorem, the three lines, \( AM_A, T_0, \) and \( T_1 \) are concurrent if and only if

\[
AF \cdot \frac{BM_A}{M_A C} \cdot \frac{CG}{GA} = 1.
\]

Since

\[
AF \cdot \frac{BM_A}{M_A C} \cdot \frac{CG}{GA} = \frac{1 - \delta_1}{\delta_1} \cdot \frac{\delta_2}{(1 - \delta_2) = 1}
\]

is equivalent to \( \delta_1 = \delta_2 \), the two tangent lines \( T_0, T_1 \), and

the median line \( AM_A \) are concurrent if and only if the line \( L \) is parallel to the side line \( BC \).

Figs. 1-2 illustrate Theorem 3.1. In Figs. 1-5, \( a = 10, b = \sqrt{50} \) and \( c = \sqrt{150} \). In the case that the line \( L \) (orange color) and the side line \( BC \) are parallel, the two tangent lines \( T_0, T_1 \) (blue line) of the isotomic conjugate curve \( r^*(t) \) (magenta color) at vertices \( C, B \), and the median line \( AM_A \) (green color) are concurrent, as shown in Fig. 1. If the line \( L \) and the side line \( BC \) are not parallel, then \( T_0, T_1, \) and \( AM_A \) are not concurrent, as shown in Fig. 2.

Theorem 3.2

The two tangent lines of the isogonal conjugate curve of the line \( DE \) at the vertices \( B, C \) of triangle \( \triangle ABC \), and the median line \( AM_A \) are concurrent if and only if the line \( DE \) satisfies

\[
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\]
\[ \frac{b^2DB}{AD} = \frac{c^2EC}{AE} \]  

(3.3)

**Proof.**

By Eqs. (3.1)-(3.2), the isogonal conjugate curve \( r^*(t) \) of the line \( L \) has the homogeneous barycentric coordinates

\[ \left( \frac{a^2}{1 - \delta_1 + t(\delta_1 - \delta_2)}, \frac{b^2}{1 - t \delta_1}, \frac{c^2}{t \delta_2} \right) \]

and the parametric equation of \( r^*(t) \) is

\[ r^*(t) = [a^2(1-t)\delta_1, b^2(1-t)\delta_1 + b^2(1-t)\delta_1(t(\delta_1 - \delta_2))\delta_2 + c^2(1-t)\delta_1(t(\delta_1 - \delta_2))\delta_2]/w^*(t) \]

where

\[ w^*(t) = a^2(1-t)\delta_1 + b^2(1-t)\delta_1(t(\delta_1 - \delta_2))\delta_2 + c^2(1-t)\delta_1(t(\delta_1 - \delta_2))\]

For \( i = 0,1 \), let \( T_i^* \) be the tangent line of \( r^*(t) \) at \( t = 0,1 \), and let \( F,G \) be the intersection points of \( T_i^* \) and \( AB, T_i^* \) and \( AC \), respectively. Since

\[ r^*(0) = \frac{-\delta_1}{a^2(1-\delta_1)\delta_1} \]
\[ (a^2 \delta_p_0 + b^2(1-\delta_1)\delta_1) \]
\[ r^*(1) = \frac{-\delta_1}{a^2(1-\delta_1)\delta_1} \]
\[ (a^2 \delta_p_0 - (a^2 \delta_2 + c^2(1-\delta_2))\delta_1) + c^2(1-\delta_2)\delta_2 \]

we have

\[ AF : FB = b^2(1-\delta_1) : a^2 \delta_1 \]
\[ AG : GC = c^2(1-\delta_2) : a^2 \delta_2 \]

Since

\[ \frac{AF}{FB} = \frac{BM_A}{M_A C} \]
\[ \frac{CG}{GA} = \frac{b^2(1-\delta_1)}{\delta_1} \cdot \frac{\delta_2}{c^2(1-\delta_2)} \]

the two tangent lines \( T_0^*, T_1^* \) and the median line \( AM_A \) are concurrent if and only if the line \( DE \) satisfies

\[ \frac{b^2DB}{AD} = \frac{c^2EC}{AE}. \]

Figs. 3-4 illustrate Theorem 3.2. In Fig. 3, \( \delta_i = \frac{1}{2}, \delta_i = \frac{13}{18} \) and it shows that if the line \( L \) (orange color) passes through \( D,E \) satisfying Eq. (3.3), then the two tangent lines \( T_0^*, T_1^* \) (blue line) of the isogonal conjugate curve \( r^*(t) \) (magenta color) of the line \( L \) at vertices \( B,C \), and the median line \( AM_A \) are concurrent. Fig. 4 shows that if the line \( L \) does not satisfy Eq. (3.3), then \( T_0^*, T_1^* \), and \( AM_A \) are not concurrent.

**Definition 3.3**

The tangent lines of isogonal conjugate curve of the line \( L \) at \( B,C \) are called by the isogonal conjugate tangent line of \( L \) at \( B,C \), respectively.
Theorem 3.4
All lines whose isogonal conjugate tangent lines at $B, C$ are concurrent with the median line $AM_i$ intersect at a unique point $X_A$ which is the externally dividing point of $B, C$ in the ratio of $c:a$, i.e.,

$$X_A = \left\{ \begin{array}{ll}
\frac{b^2 \cdot B - c^2 \cdot C}{b^2 - c^2} & (b \neq c) \\
\frac{b}{b^2 - c^2} & (b = c).
\end{array} \right. \quad (3.4)
$$

proof.
If $b = c$, then the assertion is clearly true. If $b \neq c$, then, by Eq. (3.4),

$$\overrightarrow{DX_A} = -\frac{DB}{AB} \overrightarrow{OA} + \left( \frac{b}{b^2 - c^2} \cdot AD \right) \overrightarrow{OB} - \frac{c}{b^2 - c^2} \overrightarrow{OC}$$

and by Eq. (3.3),

$$\overrightarrow{EX_A} = -\frac{EC}{AC} \overrightarrow{OA} + \left( \frac{b}{b^2 - c^2} \cdot OB \right) - \frac{c}{(b^2 - c^2) \cdot AC} \overrightarrow{OC}$$

Thus we have $\overrightarrow{DX_A} = \frac{\epsilon \cdot AC \cdot DA \overrightarrow{EX_A}}{\epsilon}$, and so, all lines passing through $D, E$ intersect at the point $X_A$. Since the slopes of all lines are different mutually, the intersection point $X_A$ is unique.

Similarly as $X_A$, we define the unique intersection point in Theorem 3.3, we define the point $X_B$ (or $X_C$) by the unique intersection point of all lines whose isogonal conjugate tangent lines at $C, A$ (or $A, B$) concurrent with the median line $BM_i$ (or $CM_i$). Then $X_B$ and $X_C$ are the externally dividing point of $C, A$ in the ratio of $a^2 : c^2$, and of $A, B$ in the ratio of $b^2 : a^2$, respectively.

Theorem 3.5
The three points $X_A, X_B, X_C$ are collinear.

Fig. 5. Lines satisfying Eq. (3.3).

Fig. 6. The collinearity of three points.

proof.
If $\triangle ABC$ is a isosceles triangle, then at least one of three points $X_A, X_B, X_C$ is the infinite point. Thus the three points are trivially collinear. Otherwise,

$$X_A = \frac{b^2 \cdot B - c^2 \cdot C}{b^2 - c^2}$$

$$X_B = \frac{a^2 \cdot A - c^2 \cdot C}{a^2 - c^2}$$

$$X_C = \frac{a^2 \cdot A - b^2 \cdot B}{a^2 - b^2}$$

yield that

$$\overrightarrow{X_A X_B} = \frac{a^2 \cdot (b^2 - c^2) - (a^2 - b^2) \cdot (b^2 - c^2)}{(a^2 - b^2)(a^2 - c^2)} \overrightarrow{A} - \frac{b^2}{a^2 - b^2} \overrightarrow{B} - \frac{c^2}{a^2 - c^2} \overrightarrow{C}$$

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Thus we have $X_AX_B = \frac{a^2 - b^2}{b^2 - c^2} X_BX_C$, and so, the three points $X_A, X_B, X_C$ are collinear.

Fig. 6 illustrates Theorem 3.5. It shows that the three points $X_A, X_B, X_C$ are collinear for the triangle $ABC$ with $a = 10, b = \sqrt{18}, c = \sqrt{58}$.

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