The Variational Field Equations of Cosmological Topologically Massive Supergravity

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We discuss the formulation of cosmological topologically massive (simple) supergravity theory in three-dimensional Riemann-Cartan space-times. We use the language of exterior differential forms and the properties of Majorana spinors on 3-dimensional space-times to explicitly demonstrate the local supersymmetry of the action density involved. Exact coupled field equations that are complete in both of their bosonic and fermionic sectors are derived by a first order variational principle subject to a torsion-constraint imposed by the method of Lagrange multipliers. Cotton and Cottino 2-forms that are complete to all orders in the gravitino field are derived and their properties such as trace are investigated.

1. Introduction

It is well-known that Einstein’s theory of general relativity has no propagating gravitational degree of freedom in (1 + 2)-dimensions.[11] In a similar way its simple (N = 1) locally supersymmetric generalization has no propagating degree of freedom either. It may be possible to introduce a cosmological constant Λ and a mass constant m for the gravitino as well and it is still possible to maintain local supersymmetry provided Λ = −m².[14]

Then for Λ ≠ −m², this would mean that there is one massive gravitino degree of freedom with a non-dynamical background metric and an algebraic torsion that is quadratic in the gravitino field.[15]

A propagating graviton degree of freedom in (1 + 2)-dimensions can be induced topologically by introducing a gravitational Chern-Simons density 3-form in the action of Einstein’s general relativity theory directly,[2,3] or with the cosmological constant present.[13] Then the Einstein field equations obtained from the infinitesimal variations of the TMG action with respect to the components of the metric tensor include the Cotton tensor with components that involve third derivatives of the metric components. These higher derivatives are the reason why the theory is now dynamical. Despite having third order field equations in metric components, the theory turns out to be ghost-free and implies causal propagation and has been extensively studied in the literature. It has a viable quantum description for the wrong sign of the Einstein-Hilbert term.[5] It admits the celebrated BTZ black-hole solution[6,7] that is asymptotically AdS₃. A formulation using a first order constrained variational formalism in Riemann-Cartan spaces in the language of exterior differential forms has been described by us before.[8–10]

An obvious question could be raised at this point: Is there a simple (N = 1) cosmological topologically massive supergravity theory? The answer is yes. This theory is discovered many years ago, without the cosmological constant in [16] and with the cosmological constant in [17]. Both in [16] and [17], a locally supersymmetric action density 3-form with both of its bosonic and fermionic sectors is written down and treated using second order variational formalism. That is, the connection coefficients were assumed to be fixed in terms of the dreibein and gravitino fields and not treated as independent variables. Some families of its supersymmetric solutions and several generalizations of it were studied since then [18, 20–23]. Yet the complete, explicit expressions for the variational field equations of the theory are still missing to the best of our knowledge.

Our aim in this paper is a pedagogical one. Firstly, we explicitly demonstrate the local supersymmetry of cosmological topologically massive supergravity action. Next we derive the consistent set of its variational field equations. All the details of calculation will be shown. Even though we are not going to present striking new results here, the fact that long time has passed since the theory has been introduced, and yet there is an the apparent lack of explicit derivation of the complete set of variational field equations may be considered sufficient justification for this review.

In what follows, we are going to use the coordinate-free and local frame independent language of exterior differential forms over a 3-dimensional space-time manifold. Neither a coordinate chart (xµ) nor a local Lorentz frame (Xα) need to be introduced in this formalism, unless one would like to consider a specific family of solutions and/or conservation relations. In Section 2 we give the basic definitions of a metric-affine space-time geometry with torsion as well as the definition of (real) Majorana 2-spinors and several spinorial identities involving them which we are going to use later on. The simple Einsteinian supergravity is introduced and discussed in Section 3.1. The infinitesimal variations are written down and local supersymmetry of the action is proven. The space-time torsion is found to be given by an algebraic relation in terms of a quadratic expression in the gravitino field. The same techniques above are used subject to a torsion constraint by the method of Lagrange multipliers on the

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DOI: 10.1002/prop.202100009
action for the cosmological topologically massive supergravity in Section 3.2. We explicitly prove the local supersymmetry of the action. We also derived and simplified the variational field equations that follow from this action. We reserve Section 3.3 to a discussion of the significance of the Lagrangian constraint that we impose on the space-time torsion. We point out that without such a constraint term, the Cotton 2-forms and their super-partner Cottino 2-form in the field equations which are responsible for the propagating degrees of freedom wouldn’t arise. Some identities respected by the Cotton 2-forms and the Cottino 2-form are discussed in Section 3.4. A brief Section 4 is devoted to concluding remarks.

2. Mathematical Preliminaries

2.1. Riemann-Cartan Space-times

A 3-dimensional Riemann-Cartan space-time is determined by the triplet \( \{ M, g, V \} \) where \( M \) is a smooth 3-manifold, modeled on a Lorentzian vector space \((\mathbb{R}^{1,2}, \eta)\), equipped with a non-degenerate Lorentzian metric \( g \) and a linear connection \( V \) that is compatible with the metric, \( \nabla g = 0 \). The metric on \( M \) is related to the Lorentzian metric of the model space as

\[
g(X_a, X_b) = \eta_{ab} = \text{diag}(−, +, +). \tag{2.1}\]

Here \( \{ X_a \} \) denotes a set of \( g \)-orthonormal frames on \( M \) which are dual to the co-frame 1-forms \( \{ \epsilon^a \} \) by the canonical pairing \( \delta^a(X_b) = t_a^b \). \( \imath \) denotes the interior product operation with respect to a frame vector \( X_a \). We will use a shorthand \( t_a^b = t_b^a \) that is totally skew-symmetric in its indices. The orientation of \( M \) is fixed by the volume 3-form \( * = \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \) where \( * : \Lambda^3(M) \to \Lambda^1(M) \) is the Hodge duality operator. The linear connection \( V \) on \( M \) takes values in the Lorentz algebra and may be expressed in terms of a set of totally anti-symmetric connection 1-forms \( \{ \omega^{a}_{bc} \} \) such that

\[
V_{X_a} X_b = \omega^{a}_{bc}(X_a) X_b. \tag{2.2}\]

The torsion 2-forms and the curvature 2-forms of the connection are defined through the first and second Cartan structure equations given by

\[
d \epsilon^a + \omega^{a}_{bc} \wedge \epsilon^b = T^a, \tag{2.3}\]

and

\[
d \omega^{a}_{bc} + \omega^{a}_{de} \wedge \omega^{e}_{bc} = R^a_{b}, \tag{2.4}\]

respectively. We note that the connection 1-forms may be decomposed uniquely into two parts according to

\[
\omega^{a}_{b} = \tilde{\omega}^{a}_{b} + K^{a}_{b}, \tag{2.5}\]

where \( \{ \tilde{\omega}^{a}_{b} \} \) denotes the set of torsion-free Levi-Civita connection 1-forms that satisfy

\[
d \epsilon^a + \tilde{\omega}^{a}_{bc} \wedge \epsilon^b = 0, \tag{2.6}\]

and \( \{ K^{a}_{b} \} \) is the set of contortion 1-forms such that

\[
K^{a}_{b} \wedge \epsilon^b = T^a. \tag{2.7}\]

The Bianchi identities

\[
DT^a = R^a_{b} \wedge \epsilon^b, \quad DR^a_{b} = 0. \tag{2.8}\]

are derived as integrability conditions from the Cartan structure equations. In the formulas above, \( d : \Lambda^q(M) \to \Lambda^{q+1}(M) \) and \( D : \Lambda^q(M) \to \Lambda^{q+1}(M) \) denote the exterior derivative and covariant exterior derivative respectively with respect to connection 1-forms \( \{ \omega^{a}_{bc} \} \), respectively. We also present the contracted version of the first Bianchi identity (2.8) because it will be relevant for our calculations later:

\[
2(t_{abc} R_{abc} - t_{abd} R_{bad}) = t_{abc}(DT_a) + t_{abd}(DT_b) + t_{abc}(DT_c) + t_{abc}(DT_d). \tag{2.9}\]

The contracted Bianchi identity (2.9) shows us that in the presence of torsion, the two-two symmetry of the components of the Riemann curvature tensor fails in general.

The Ricci 1-forms \( \{ R_a \} \) and the curvature scalar \( R \) are obtained by the following contractions of the curvature 2-forms:

\[
R_a = \omega^{a}_{bc} R^{bc}, \quad R = \epsilon^a R_a. \tag{2.10}\]

Finally the Einstein 2-forms \( \{ G_{ab} \} \) are defined in terms of the Ricci 1-forms and the curvature scalar as

\[
G_{ab} = G_{ab} * \epsilon^c = - \left( R_a - \frac{1}{2} R_{a} \right) = - \frac{1}{2} \epsilon_{abc} R^{bc}. \tag{2.11}\]

The last equality above shows that the Einstein 2-forms are directly proportional to the curvature 2-forms and vice versa. This property holds true only in 3-dimensions.

Before we move on to describe the spinors that we will work with, we are going to give some identities regarding the exterior algebra that will be helpful in the following sections:

\[
t_a \xi = (-1)^a (e_a \wedge \xi), \quad \xi \in \Lambda^q(M) \tag{2.12}\]

\[
\epsilon^{abc} \epsilon_{klm} = - \eta_{a}^{k} (\eta_{l}^{b} \eta_{m}^{c} - \eta_{l}^{c} \eta_{m}^{b}) + \eta_{b}^{k} (\eta_{l}^{a} \eta_{m}^{c} - \eta_{l}^{c} \eta_{m}^{a}) - \eta_{c}^{k} (\eta_{l}^{a} \eta_{m}^{b} - \eta_{l}^{b} \eta_{m}^{a}). \tag{2.13}\]

\[
\epsilon^a \wedge e_{ij} = - \eta_{a}^{i} e_{j} + \eta_{a}^{j} e_{i}, \tag{2.14}\]

\[
\epsilon^{abc} e_{ij} = - 2 \eta_{a}^{[b} \eta_{ij}^{c]} + 2 \eta_{a}^{[c} \eta_{ij}^{b]} - 2 \eta_{a}^{[b} \eta_{ij}^{c]}, \tag{2.15}\]

where \( e_{ij} = * e_{ijkl} \) denotes the totally anti-symmetric Levi-Civita symbol with the choice \( e_{012} = 1 \) and square bracket around some indices means the normalized total anti-symmetrization of those indices.

\[1\] \( \Lambda(M) = \bigoplus_{q=0}^{3} \Lambda^q(M) \) stands for the exterior algebra over \( M \).
2.2. Majorana Spinors

The spinor fields on $M$ are related to the spinors on the model space $(\mathbb{R}^{1,2}, \eta)$ whose Clifford algebra is denoted by $\text{Cl}(1, 2)$. We are going to use a real realization generated by the Pauli matrices given by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.16)$$

These generators satisfy the Clifford product rule

$$\gamma_a \gamma_b = \eta_{ab} I + \epsilon_{abc} \gamma^c,$$ \quad (2.17)

so that

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} I, \quad [\gamma_a, \gamma_b] = 2\epsilon_{abc} \gamma^c.$$ \quad (2.18)

The Clifford algebra $\text{Cl}(1, 2)$ can be spanned by a basis $\{I, \gamma_0, 2\sigma_{ab}, \gamma_5\}$ where $I$ is the $2 \times 2$ identity operator, $\{\sigma_{ab}\}$ and $\gamma_5$ are the Lorentz generators and the volume element, given explicitly by

$$\sigma_{ab} = \frac{1}{4} \epsilon_{abc} \gamma^c, \quad \gamma_5 = \gamma_0 \gamma_1 \gamma_2 = I,$$ \quad (2.19)

respectively. The following identities are satisfied by the generators of the Clifford algebra:

$$2\gamma_a \sigma_{bc} = \eta_{ab} \gamma_c - \eta_{ac} \gamma_b + \epsilon_{abc},$$ \quad (2.20)

$$2\sigma_{ab} \gamma_c = -\eta_{ac} \gamma_b + \eta_{bc} \gamma_a + \epsilon_{abc} I,$$ \quad (2.21)

$$[\sigma_{ab}, \sigma_{cd}] = -\eta_{ad} \sigma_{bc} + \eta_{bd} \sigma_{ac} + \eta_{cd} \sigma_{ab} - \eta_{ab} \sigma_{cd}.$$ \quad (2.22)

Furthermore the following summation identities are also satisfied:

$$\gamma^a \gamma_a = 3, \quad \gamma^a \gamma_0 \gamma_a = -\gamma_0, \quad \gamma^a \gamma_0 \gamma_1 \gamma_a = 3\gamma_0 - 2\sigma_{bc},$$

$$\gamma^a \sigma_{bc} \gamma_a = -\sigma_{bc}, \quad \gamma^a \sigma_{ab} \gamma_a = \gamma_0, \quad 2\sigma_{ab} \sigma_{cd} = -3, \quad 2\sigma^a \gamma_1 \sigma_{ab} = \gamma_c.$$ \quad (2.23)

Since the rank-2 and rank-3 elements of the Clifford basis are linearly dependent on the rank-0 and rank-1 elements, we use the basis $\{\gamma_a\} = \{I, \gamma_0\}$.

The spin group of our model space, $\text{Spin}(1, 2) \cong \text{SL}(2, \mathbb{R})$ is the double cover of the local Lorentz group $\text{SO}(1, 2)$. $\text{Spin}(1, 2)$ is generated by the elements $\{I, 2\sigma_{ab}\}$ of the even Clifford subalgebra $\text{Cl}_0(1, 2)$ and the spinors carry its irreducible representations. In our case the representation space will be $\mathbb{R}^2$, however, components of the spinors should be odd-Grassmann valued. That is, given $\psi = (\psi_1, \psi_2)^T \in \mathbb{R}^2$, both components are nilpotent and they anti-commute:

$$\psi_i \psi_j = 0 = \psi_j \psi_i, \quad \psi_i \psi_j = -\psi_j \psi_i.$$ \quad (2.24)

An adjoint spinor is defined as an element of the dual space of the spinor space. The map between the representation space $\mathbb{R}^2$ and its dual space $(\mathbb{R}^2)^*$ is given by an anti-symmetric operator $C: \mathbb{R}^2 \rightarrow (\mathbb{R}^2)^*$

$$\psi \mapsto \bar{\psi} = \psi^T C \quad (2.25)$$
called the charge conjugation operator. In the Majorana realization that we use,

$$C = \gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ \quad (2.26)

which satisfies $C^{-1} = C^T = -C$ and $C^2 = -I$. The inverse map acts on the conjugate spinors from the left and defines charge conjugated spinors

$$C^{-1}: (\mathbb{R}^2)^* \rightarrow \mathbb{R}^2$$ \quad (2.27)

$$\psi \mapsto \psi_C = C(\bar{\psi})^T.$$ \quad (2.28)

Note that $C^{-1} C = \text{id}$ as expected. Because we are using real Clifford generators, our spinors are self-charge conjugate, $\psi_C = \psi$. That is to say, all our spinors will be odd-Grassmann valued Majorana (real) spinors.

Furthermore we note that under the action of the charge conjugation operator, the Clifford basis elements $\{\gamma_a\}$ are transposed:

$$\gamma_a \mapsto \gamma_a^T.$$ \quad (2.29)

Another useful way to think about the charge conjugation matrix is to consider it as a metric on space of the spinors. Using this property, we may pair spinors to obtain objects which have tensorial behavior under local Lorentz transformations. In fact, in 3-dimensions there are only two spinor bi-linears that one may write:

$$\psi \phi, \quad \psi \gamma_a \phi.$$ \quad (2.30)

The first bi-linear is a pseudoscalar and the second one is a Lorentz vector. The other bi-linears $\psi \sigma_{ab} \phi$ and $\psi \gamma_a \phi$ may be expressed in terms of the above ones. Any two arbitrary spinors $\psi$ and $\phi$ satisfy the Majorana flip identities, given by

$$\bar{\psi} \phi = \bar{\phi} \psi, \quad \bar{\psi} \gamma_a \phi = -\bar{\phi} \gamma_a \psi.$$ \quad (2.31)

The complex conjugation operation is an anti-linear anti-involution acting on the Clifford algebra. Consequently, the spinor bi-linears in (2.30) are pure imaginary. In order to obtain real quantities instead, we have to introduce a factor of complex unit $i$ into these expressions. Therefore

$$i(\bar{\psi} \phi) \in \mathbb{R}, \quad i(\bar{\psi} \gamma_a \phi) \in \mathbb{R}^{1,2}.$$ \quad (2.32)

Considering the product of three or more spinors, the order of the products may be arranged according to the Fierz rearrangement formula. Suppose $U$ and $V$ are real valued $2 \times 2$ matrices and $\alpha, \beta, \phi, \psi$ are arbitrary Majorana 2-spinors. Then

$$\langle \bar{a} U \beta \rangle \langle \bar{\phi} V \psi \rangle = -\frac{1}{2} \sum_{A=1}^5 \langle \bar{a} U \gamma_A V \psi \rangle \langle \bar{\phi} \gamma_A \beta \rangle.$$ \quad (2.33)
When considering spinor fields on $M$, we consider sections of the spin bundle on $M$ which takes values in our spinor space and is acted upon by the spin group $SL(2, \mathbb{R})$ fiberwise. We define the spin covariant exterior derivative operation that acts on a Majorana spinor valued $p$-form section, for instance the gravitino 1-form over the Riemann-Cartan space-time as:

$$D\chi = d\chi + \frac{1}{2} \omega^{ab} \sigma_{ab} \wedge \chi$$

(2.33)

where $\{\omega^{ab}\}$ is a set of metric compatible connection 1-forms on $M$ and $\chi$ is a section of Majorana spinor valued 1-forms. The Ricci’s identity takes the following form on the spinor valued differential form fields:

$$D^2 \chi = \frac{1}{2} R^{ab} \sigma_{ab} \wedge \chi$$

(2.34)

where $\{R^{ab}\}$ are the curvature 2-forms on $M$.

### 3. Supergravity Theories in 3-Dimensions

#### 3.1. Cosmological Supergravity

The action for cosmological supergravity theory[14,15]

$$S[e^a, \omega^{ab}, \chi] = \int_M \mathcal{L}_{CSG}$$

(3.1)

is going to be varied in a first order variational formulation with respect to the co-frames $\{e^a\}$, connection 1-forms $\{\omega^{ab}\}$ and the Majorana spinor valued 1-form gravitino field $\chi$ taken as independent field variables. The Lagrangian density 3-form $\mathcal{L}_{CSG} = \mathcal{L}_{SG} + \mathcal{L}_C$ can be decomposed in terms of the action densities for the simple supergravity and the cosmological sectors given by

$$\mathcal{L}_{SG} = -\frac{1}{2} R^{ab} \wedge * \epsilon_{ab} - \frac{i}{2} \tilde{\chi} \wedge D\chi,$$

(3.2)

and

$$\mathcal{L}_C = \Lambda * -\frac{im}{4} \tilde{\chi} \wedge \gamma^a \wedge \chi,$$

(3.3)

respectively. Here we set the gravitational constant $\kappa = 1$, $\Lambda$ is a cosmological constant and $m$ is a mass parameter. The fermionic part of the supergravity and cosmological sectors are the kinetic and non-topological mass terms for the Rarita-Schwinger (gravitino) field. The gravitino field $\chi$ and its field strength $D\chi$ are Majorana spinor valued 1- and 2-form fields, respectively:

$$\chi = (t_a \chi) e^a = \chi_a e^a, \quad D\chi = \frac{1}{2} (t_{b a} D\chi) e^{ab} = (D\chi)_{a b c d} e^{a b},$$

(3.4)

Furthermore we introduced a gamma matrix valued 1-form $\gamma = \gamma_a e^a$ to write down the mass term for the gravitino field. The cosmological constant and the ”mass” of the gravitino field shall be related below via $\Lambda = -m^2$ for local supersymmetry.

Then the total variation of the action reads (upto a closed form)

$$\delta \mathcal{L} = \delta \hat{e}^a \wedge \left\{ -\frac{1}{2} \epsilon_{abc} \tilde{R}^{ac} + \frac{m}{4} \tilde{\chi} \wedge \gamma^a \gamma^b \chi + \Lambda * \epsilon_a \right\} + \omega^{ab} \wedge \left\{ -\frac{1}{2} \epsilon_{abc} \left( T^c - \frac{i}{4} \tilde{\chi} \wedge \gamma^c \chi \right) \right\} + \tilde{\chi} \wedge \left\{ -iD\chi - \frac{im}{2} \gamma^a \chi \right\}.$$  

(3.5)

We determine from this expression the coupled field equations of the cosmological supergravity theory:

$$G_a + \Lambda * \epsilon_a + \frac{im}{4} \tilde{\chi} \wedge \gamma_a \chi = 0,$$

(3.6)

$$D\chi + \frac{m}{2} \gamma^a \wedge \chi = 0,$$

(3.7)

$$T^a = \frac{i}{4} \tilde{\chi} \wedge \gamma^a \chi.$$

(3.8)

The (infinitesimal) local supersymmetry transformations of the supergravity multiplet are given as usual by:

$$\delta e^a = i\tilde{\alpha} \gamma^a \chi, \quad \delta \tilde{\chi} = 2D\alpha + m\gamma a$$

(3.9)

where the local supersymmetry parameter $\alpha = \alpha(x)$ is an arbitrary odd-Grassmann valued Majorana spinor. In order to determine the supersymmetry transformation law for the connection field, we look at the variation of the first Cartan structure equation (2.3) which yields

$$\delta \omega^{ab} \wedge \hat{e}^a = -i\tilde{\alpha} \gamma^a (D\chi) + T^a = \frac{im}{2} \tilde{\alpha} \gamma_a \wedge \chi - i\tilde{\alpha} D\chi =: Z_a.$$  

(3.10)

The final simplification follows from the field equations (3.8).

The solution to the system of equations (3.10) is obtained algebraically as

$$2\delta \omega^{ab} = \iota_a Z_b - \iota_b Z_a - \delta \iota_a Z_b.$$  

(3.11)

that explicitly yields

$$\delta \omega^{ab} = \frac{i}{4} \left( \tilde{\alpha} \gamma_a t_b (D\chi) - \tilde{\alpha} \gamma_b t_a (D\chi) + \tilde{\alpha} \gamma_{a b c} (D\chi) \right) - \frac{im}{2} \left( \epsilon_{abc} (\tilde{\alpha} \gamma^c \chi) + \epsilon_a (\alpha \gamma_b \chi) - \epsilon_b (\alpha \gamma_a \chi) \right).$$  

(3.12)

Although this does not contribute to the transformation of the action density on-shell, we give the transformation law (3.12) for the connection 1-forms for completeness. This result will be relevant when we discuss cosmological topologically massive supergravity theory in what follows.

We now prove the local supersymmetry of cosmological supergravity theory. Let us first consider the contributions that are independent of $m$ in the variations of the action density under our local supersymmetry transformations:

$$\delta \mathcal{L}_{SG}(m = 0) = -\frac{i}{2} \epsilon_{abc} (\tilde{\alpha} \gamma^a \chi) \wedge \tilde{R}^{bc} - 2iD\tilde{\alpha} \wedge D\chi.$$  

(3.13)
This particular combination may be shown to add up to a closed form by noting that:

\[-\frac{i}{2} \epsilon_{abc} (\bar{\alpha} g^a \chi) \wedge R^c = -i \bar{a} R^{ab} \omega^b \wedge \chi. \quad (3.14)\]

and

\[-2i \bar{D}a \wedge D\chi = i \bar{a} R^{ab} \sigma^b \wedge \chi + d(-2i \bar{a} D\chi). \quad (3.15)\]

The rest of the contributions \((m \neq 0)\) are given by

\[i \lambda \bar{a} \wedge \gamma \wedge \chi + \frac{im^2}{2} \bar{\alpha} \gamma \wedge \gamma \wedge \chi - \frac{m}{4} (\bar{\alpha} g^a) \wedge (\bar{\chi} \wedge \gamma_a \chi) \]

\[-i \bar{m} \bar{D}a \wedge \gamma \wedge \chi + i \bar{m} \gamma \wedge \gamma \wedge D\chi. \quad (3.16)\]

The first two terms cancel each other out when we set \(\Lambda = -m^2\) and use the identity \(\gamma \wedge \gamma = 2 \wedge \gamma\). The last two terms on the other hand can be combined to give

\[-i \bar{m} \bar{D}a \wedge \gamma \wedge \chi + i \bar{m} \gamma \wedge \gamma \wedge D\chi = -d(i \bar{m} \bar{a} \wedge \chi) - \frac{m}{4} (\bar{\alpha} g^a) \wedge (\bar{\chi} \wedge \gamma_a \chi). \quad (3.17)\]

When all the above contributions are put together, we are left with a closed form plus a non-linear spinorial expression

\[-\frac{m}{2} (\bar{\alpha} g^a) \wedge (\bar{\chi} \wedge \gamma_a \chi). \quad (3.18)\]

It is not difficult to verify that (3.18) vanishes identically by performing a Fierz rearrangement. However, some care is needed for signs during the Fierz rearrangements because we are dealing with spinor valued differential forms. One must first open up an expression in the co-frame basis, apply the Fierz rearrangement formula to the components and then bring back in the basis forms. The final outcome reads

\[-\frac{m}{2} (\bar{\alpha} g^a) \wedge (\bar{\chi} \wedge \gamma_a \chi) = -\frac{m}{4} (\bar{\alpha} \chi) \wedge (\bar{\chi} \wedge \chi) = 0. \quad (3.19)\]

With this result, the local supersymmetry of the cosmological supergravity action (3.1) is proven.

### 3.2. Cosmological Topologically Massive Supergravity

The action functional of this theory is given by

\[S[\epsilon^a, \omega^{ab}, \chi, \lambda_a] = \int_M L_{\text{Total}} \quad (3.20)\]

that will be varied independently with respect to the co-frames \(\{\epsilon^a\}\), connection 1-forms \(\{\omega^{ab}\}\) and the gravitino 1-form \(\chi\) as before. We further introduce below Lagrange multipliers 1-forms \(\{\lambda_a\}\) that are also varied as independent variables. Now our Lagrangian density 3-form decomposes according to,

\[L_{\text{Total}} = L_{\text{CS}} + L_{\text{SC}} + L_C + L_{\text{Constraint}} \quad (3.21)\]

where we added on to the cosmological supergravity action density (3.2) of the previous section, the topological Chern-Simons density 3-form

\[L_{\text{CS}} = \frac{1}{\mu} \left( \omega^a \wedge D\omega^b + \frac{2}{3} \omega^a \wedge \omega^b \wedge \omega^c \right) - \frac{i}{\mu} (D\bar{\chi} \wedge \gamma_a \wedge D\chi). \quad (3.22)\]

Here \(\mu\) is a new coupling constant. It should be noted that the fermionic part of the Chern-Simons density (3.22) contains derivatives of order 2 of the gravitino field. This is consistent with the fact that third order derivatives of metric components appear in the usual bosonic part of Chern-Simons density. Alternatively, the fermionic part of the Chern-Simons density 3-form could have been expressed as

\[-\frac{i}{\mu} (2 \bar{D}\bar{\chi} \wedge \gamma_a \wedge (\gamma \wedge D\chi)) \]

which seems more suitable for a Hamiltonian description. However, as far as the variations of the action are concerned this form is considerably harder to work with and we prefer to use our form of the topological action density. We furthermore introduced a set of Lagrange multiplier 1-forms \(\{\lambda_a\}\) that appear linearly in the constraint Lagrangian density

\[L_{\text{Constraint}} = \left( T^a - \frac{i}{4} \bar{\chi} \wedge \gamma^a \chi \right) \wedge \lambda_a. \quad (3.23)\]

Then independent variations of the action relative to the Lagrange multipliers impose the constraint that the space-time torsion 2-forms are given algebraically by (3.8) as in the previous section. The remaining variational field equations are to be solved subject to this Lagrangian constraint. In Riemannian space-times, in a similar way, one may introduce a constraint term of the form \(T^a \wedge \lambda_a\) in the action whose variations with respect to the multipliers set the space-time torsion to zero in a first order constrained variational formulation of gravitational theories. However, since we are working with a supergravity theory we don’t want the torsion to vanish, but be equal to the quadratic expression given by (3.8). As we are going to show later on, the origin of the Cotton tensor which involves third derivatives of the metric components in the Einstein field equations is due to this constraint term. The torsion constraint furthermore ensures that the supersymmetry transformation law (3.12) of the connection 1-forms remains as it is in the previous section.

The variation of the total action with respect to the independent field variables turns out to be, modulo a closed form,

\[\delta L_{\text{Total}} = \delta \epsilon^a \wedge \left\{ -\frac{1}{2} \epsilon_{abc} R^{bc} - m^2 \epsilon_a + \frac{m}{4} \bar{\chi} \wedge \gamma_a \chi + \frac{i}{\mu} \left( t_a D\bar{\chi} \wedge \gamma_a \wedge D\chi - D\bar{\chi} t_a \wedge D\chi \wedge \gamma_a \wedge D\chi \right) + 2t_a D\bar{\chi} \wedge \gamma_a \wedge (\gamma \wedge D\chi) \right\} + \omega^{ab} \wedge \left\{ -\frac{1}{2} \epsilon_{abc} T^c - \frac{i}{4} \bar{\chi} \wedge \gamma^a \chi \right\} \]

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\[-\frac{2}{\mu} R_{ab} - \frac{1}{2} (\lambda_a \wedge e_b - \lambda_b \wedge e_a) + \frac{i}{2\mu} \epsilon_{abc} (\tilde{\lambda} \wedge \gamma^c \wedge D\chi + \tilde{\chi} \wedge \gamma^c \wedge (\gamma \wedge D\chi)) \}
\]
\[+ \tilde{\lambda} \wedge \left\{ -i D\chi + \frac{i}{2} \lambda_\mu \wedge \gamma^\mu \chi \right\} \]
\[= -\frac{2}{\mu} \frac{1}{2} (D \wedge D\chi + D \wedge (\gamma \wedge D\chi)) + \frac{1}{2} \lambda_\mu \wedge \gamma^\mu \chi \] 

\[\lambda_\mu \wedge \{ T^a - \frac{i}{4} \tilde{\chi} \wedge \gamma^a \chi \}. \quad (3.24)\]

We will first demonstrate the local supersymmetry of the action (3.20) under the usual transformations of the co-frame, connection and gravitino fields given by (3.9) and (3.12). The explicit supersymmetry transformations of the Lagrange multiplier 1-forms \( \lambda^a \) are not necessary because transformation of the connection is obtained using the torsion field. Therefore the last term in (3.24) does not make any contribution to the variations on-shell.

Under local supersymmetry transformations (3.9) and (3.12), the variation of the action density decomposes as follows:

\[\mathcal{L}_{\text{Total}} = \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{C}} + \mathcal{L}_{\text{Constraint}}, \quad (3.25)\]

where the contribution \( \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{C}} \) from the cosmological supergravity sector is already shown to yield a closed form given by (3.15). We are left to deal with contributions coming from the topological sector and the constraint term. The contributions from the constraint term can be seen to produce just a closed form by a straightforward computation:

\[\mathcal{L}_{\text{Constraint}} = i (\bar{\lambda} \gamma^a \chi) \wedge D\lambda_a + i D\bar{\alpha} \wedge \alpha^a \gamma_a \chi - \frac{i}{2} \frac{1}{\mu} \bar{\gamma} \gamma^a \chi \wedge \lambda_a \]
\[-\frac{i}{2} \left( \bar{\alpha} \gamma^a \chi \right) \wedge D\lambda_a + \bar{\lambda} \wedge \gamma_a \chi \wedge \lambda_a \]
\[-\frac{i}{2} \left( \bar{\alpha} \gamma^a \chi \right) \wedge D\lambda_a + \bar{\lambda} \wedge \gamma_a \chi \wedge \lambda_a \]
\[= iD(\bar{\alpha}) \wedge \gamma_a \chi - i \bar{\alpha} \wedge \gamma_a \chi = iD(\bar{\alpha}) \wedge \gamma_a \chi. \quad (3.26)\]

We note that the result (3.26) does not depend on the explicit form of the Lagrange multiplier 1-forms. We therefore must only check those contributions coming from the topological Chern-Simons density. In order to ease the discussion, we are going to deal separately with terms obtained when \( \mu = 0 \) and rest of the terms for \( \mu \neq 0 \).

Case: \( \mu = 0 \)

The supersymmetry transformation of the Chern-Simons density gives

\[\mathcal{L}_{\text{CS}}(\mu = 0) = -\frac{i}{\mu} \left( 2\bar{\alpha} \gamma^a \chi \wedge D\lambda_a + \bar{\lambda} \wedge \gamma^a \chi \wedge D\lambda_a \right) + \frac{i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{4i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{i}{\mu} (\bar{\lambda} \wedge \gamma_a \chi \wedge D\chi - D\bar{\lambda} \wedge \gamma_a \chi + \bar{\lambda} \wedge \gamma_a \chi \wedge D\chi) \]

\[\mathcal{L}_{\text{CS}}(\mu = 0) = -\frac{i}{\mu} \left( 2\bar{\alpha} \gamma^a \chi \wedge D\lambda_a + \bar{\lambda} \wedge \gamma^a \chi \wedge D\lambda_a \right) + \frac{i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{4i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{i}{\mu} (\bar{\lambda} \wedge \gamma_a \chi \wedge D\chi - D\bar{\lambda} \wedge \gamma_a \chi + \bar{\lambda} \wedge \gamma_a \chi \wedge D\chi) \]

\[\mathcal{L}_{\text{CS}}(\mu = 0) = -\frac{i}{\mu} \left( 2\bar{\alpha} \gamma^a \chi \wedge D\lambda_a + \bar{\lambda} \wedge \gamma^a \chi \wedge D\lambda_a \right) + \frac{i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{4i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{i}{\mu} (\bar{\lambda} \wedge \gamma_a \chi \wedge D\chi - D\bar{\lambda} \wedge \gamma_a \chi + \bar{\lambda} \wedge \gamma_a \chi \wedge D\chi) \]

\[\mathcal{L}_{\text{CS}}(\mu = 0) = -\frac{i}{\mu} \left( 2\bar{\alpha} \gamma^a \chi \wedge D\lambda_a + \bar{\lambda} \wedge \gamma^a \chi \wedge D\lambda_a \right) + \frac{i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{4i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{i}{\mu} (\bar{\lambda} \wedge \gamma_a \chi \wedge D\chi - D\bar{\lambda} \wedge \gamma_a \chi + \bar{\lambda} \wedge \gamma_a \chi \wedge D\chi) \]

\[\mathcal{L}_{\text{CS}}(\mu = 0) = -\frac{i}{\mu} \left( 2\bar{\alpha} \gamma^a \chi \wedge D\lambda_a + \bar{\lambda} \wedge \gamma^a \chi \wedge D\lambda_a \right) + \frac{i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{4i}{\mu} D^2 \bar{\alpha} \wedge (\gamma \wedge D\chi) \]
\[-\frac{i}{\mu} (\bar{\lambda} \wedge \gamma_a \chi \wedge D\chi - D\bar{\lambda} \wedge \gamma_a \chi + \bar{\lambda} \wedge \gamma_a \chi \wedge D\chi) \]
of the fact that a 4-form vanishes identically. In the final equality we make use of the contracted Bianchi identity (2.9) to show $e_{ab}^i R^i_{bc} = 2 \star (i_i D^T T)$ and used the torsion expression (3.8). We note that the last term in (3.29) cancels the first term in the total variation (3.27) and we have no terms remaining proportional to curvature 2-forms.\footnote{By looking at these cancellations, we fixed the relative sign between the bosonic and fermionic terms in Chern-Simons action density (3.22).}

The remaining terms can be brought into a generic form $(\bar{a} \psi)(D \bar{f} D \chi)$ by applying the Fierz rearrangement formula (2.32). While doing such calculations, we make ample use of the coframe identities (2.13)-(2.15). Finally bringing everything back together, the relevant piece of the supersymmetry transform of the topological Chern-Simons density will be put into the following form:

$$\mathcal{L}_{CS}(m = 0) = \frac{1}{\mu} \left\{ (\bar{a} \gamma^a \chi) \wedge \left[ -\frac{1}{4} (i_i D \bar{f} \chi \wedge * D \chi) - \frac{1}{2} (D \bar{f} \gamma \wedge * D \chi) \right] - \frac{1}{4} (\bar{a} \chi \wedge \chi) \wedge (\bar{t}_i D \bar{f} \gamma \wedge * D \gamma) \right. $$

$$ \left. + (\bar{a} \gamma^a \chi) \wedge \left[ D \bar{f} \wedge (\chi \wedge D \chi) - \frac{5}{4} D \bar{f} \wedge (\gamma \wedge * D \gamma) \right] + \frac{1}{4} \gamma^a \wedge (\bar{a} \gamma \wedge (\chi \wedge D \gamma \wedge * D \gamma \wedge * D \gamma) \right) $$

To show that the right hand side vanishes, we group the terms into three generic types which read as follows:

1. $(\bar{a} \gamma^a \chi)(D \bar{f} D \chi)$

2. $(\bar{a} \chi)(D \bar{f} \gamma D \chi)$

3. $(\bar{a} \gamma^a \chi)(D \bar{f} \gamma D \chi)$

Again these generic types do not mix with each other under Fierz rearrangements and each group of terms vanish on their own. In particular, the terms of the types (3.31) and (3.32) can be shown to vanish by expanding each term in the coframe basis and then using coframe identities, taking $* 1$ out of these expressions. However, when applying the same method for terms of the type (3.33), some simplifications occur and we obtain the following combination:

$$ \frac{1}{\mu} \left\{ (\bar{a} \gamma^a \chi) \wedge \left[ 3 \gamma_a \wedge * D \bar{f} \gamma * D \chi - \frac{3}{2} \gamma \wedge * D \bar{f} \chi \wedge \gamma_a * D \chi \right] $$

$$ - \frac{3}{2} \gamma_a \wedge * D \bar{f} \gamma \wedge * D \gamma \wedge \gamma_a * D \chi \wedge * D \chi \right) \right\}. $$

(3.34)

To show that this combination vanishes we move an interior product operation in the first three terms and use the fact that a 4-form field identically vanishes. Then the resulting combination cancels out the fourth term. Thus the local supersymmetry of the Chern-Simons density (3.27) under the local supersymmetry transformations (3.9) and (3.12) with $m = 0$ is established.

Case: $m \neq 0$

Now we move on to take care of $m \neq 0$ terms coming from topological sector. They explicitly read

$$ \left\{ \frac{im}{2} [e_{ab}(\bar{a} \gamma^a \chi) + 2 e_{ab}(\bar{a} \chi)] \wedge \left[ \frac{1}{2} R^b - \frac{1}{2} \gamma \wedge * D \chi \right] $$

$$ + \bar{a} \wedge \gamma_a \wedge (\gamma \wedge * D \chi) \right\} + \frac{2im}{\mu} D(\bar{a} \gamma) \wedge (D \chi + \gamma \wedge * D \chi) $$

$$ = \frac{m}{2\mu} [\bar{a} \gamma^a \chi] \wedge \left[ \bar{D} (\bar{a} \chi) \wedge (\gamma \wedge * D \chi) + (\bar{D} \gamma^i \bar{t}_i D \chi) \right] $$

$$ - \frac{m}{2\mu} [\bar{a} \chi \gamma_a \wedge (\gamma \wedge * D \chi)] $$

$$ - \frac{m}{2\mu} \bar{a} \gamma^a \bar{t}_i D \chi \wedge (\gamma \wedge * D \chi) + \frac{im}{\mu} e_{ab}(\bar{a} \gamma^a \chi) \wedge R^{ab} $$

$$ + \frac{2im}{\mu} D \bar{D} \wedge \gamma \wedge * D \chi + \frac{2im}{\mu} D \bar{D} \wedge \gamma \wedge * D \chi. $$

(3.35)

To show that the right hand side of the above equality adds up to a closed form, we will start by manipulating the very last term. Using $\gamma \gamma^a = e^a + e^{a b} e_{b}$, we can write

$$ \frac{2im}{\mu} D \bar{D} \wedge \gamma \wedge * D \chi $$

$$ = \frac{4im}{\mu} D \bar{D} \wedge * D \chi + \frac{2im}{\mu} (D \bar{D} \gamma \wedge * D \chi) \wedge e^{a b} \wedge e^b $$

$$ = \bar{D} \left\{ \frac{4im}{\mu} D \bar{D} \wedge * D \chi - \frac{4im}{\mu} \bar{D} \gamma \wedge \gamma \wedge * D \chi \right\} $$

$$ = \bar{D} \left\{ \frac{4im}{\mu} D \bar{D} \wedge * D \chi \right\}. $$

(3.36)

Above in the second equality we distributed a covariant derivative in the first term and made use of the identity (2.14). Then in the third equality we used the Ricci identity and (2.34) with the interior product identity (2.12). The result (3.36) shows that the last three terms in (3.35) combine to yield a closed form.
The remaining terms of (3.35) can be brought into the form \((\tilde{a} D X) (\tilde{\chi}_X)\) by Fierzing once. The resulting expression can be shown to vanish identically after taking * 1 out of each term. We have thus proven, with this final observation, the local supersymmetry of the cosmological topologically massive supergravity action (3.20).

Now we are ready to derive the complete set of field equations of cosmological topologically massive supergravity. The field equations are read off from the variations of the total action (3.24) and they consist of the Einstein field equations

\[
G_a - m^2 \ast e_a + D \lambda_a = 0
\]

(3.37)

\[
+ \frac{i}{\mu} \left( -i D \tilde{\chi} \ast D X + D \tilde{\chi} (t_a \ast D X) + D \tilde{\chi} \ast (\gamma \ast D X) \right)
\]

subject to the constraint that the space-time torsion is given by

\[
T^a = \frac{i}{4} \tilde{\chi} \ast \gamma^a \chi.
\]

(3.39)

We note that the \(D \lambda_a\) term in the Einstein field equations (3.37) is precisely the term that produces the Cotton tensor in topologically massive gravity theory. It is obtained by solving the Lagrange multiplier 1-forms \(\lambda_a\) from the connection field equations and then substituting the result back in the other field equations. Here we do the same and solve the connection variational field equations below algebraically for the Lagrange multiplier 1-forms:

\[
\lambda_a \wedge e_b - \lambda_b \wedge e_a = 2 \Sigma_{ab}
\]

(3.40)

where

\[
\Sigma_{ab} = -\frac{2}{\mu} R_{ab} + \frac{i}{2\mu} e_{ab} \tilde{\chi} \ast (D X + \gamma \ast D X).
\]

(3.41)

The result turns out to be

\[
\lambda_a = -\frac{4}{\mu} \left( Y_a + \frac{i}{4} W_a \right)
\]

(3.42)

where

\[
Y_a = Ric_a - \frac{1}{4} \nabla_a
\]

(3.43)

are the Schouten curvature 1-forms and

\[
W_a = t_a \ast (\tilde{\chi} \wedge \gamma) \ast (D X + \gamma \ast D X)
\]

(3.44)

The Schouten curvature 1-forms \(Y_a\) and their fermionic counterparts \(W_a\) are given by the expressions (3.43) and (3.44), respectively.

3.3. The Role of Lagrangian Constraints in the Action

Before we go any further, we wish to emphasize the vital importance of the Lagrangian constraint (3.23) for our first order variational formulation of the cosmological topologically massive supergravity theory.

Let us first look at the Lagrangian constraint for the formulation of topologically massive gravity theory using the first order formalism. In [10, 11], it is shown that in order to obtain the Cotton tensor in the Einstein field equations of the topologically massive gravity using first order variational techniques, it is essential to impose the zero-torsion constraint term of the form \(\lambda_a \wedge T^a\) in the action. This term constrains the torsion in the geometry to vanish identically and its contribution to the Einstein field equations yields the Cotton 2-forms. The components of the Cotton tensor involve third order derivatives of the metric components which makes the theory dynamical. Without the zero-torsion constraint, the first order variational field equations do not produce a dynamical system.

In a similar way, the use of a Lagrangian constraint term in the action for the first order formulation of the cosmological topologically massive supergravity theory is vital as well.
difference here is that for the supergravity theory, torsion is constrained to the usual gravitino bilinear (3.47), so that the usual super-symmetry transformation law (3.12) for the connection field is maintained. At this point it may be said that we are using nothing but the so-called 1.5 formalism. However, the constrained variational principle that we apply here has much more general applications that couldn’t be possible with the 1.5 formalism. On the one hand, one may constrain the space-time torsion to any desired algebraic expression in terms of the field variables. See for instance[10] On the other hand, a Lagrangian constraint that is non-linear in the multipliers may also be imposed. For example, the first order variational field equations of minimal massive gravity is obtained that way.[11]

Finally we will demonstrate what would happen if the torsion constraint hasn’t been imposed by the method of Lagrange multipliers. Then by first order variations of the field variables a completely different theory both at the level of action and field equations is obtained in general. Let us consider the infinitesimal variations of the action (3.20) without the Lagrangian constraint term:

\[ R^{ab} = \Lambda e^{ab} - \frac{i}{\mu} e^{abc} \left( \iota_a D\hat{\chi} \wedge D\chi - D(\iota_a D\hat{\chi}) \right) + * D\hat{\chi} \wedge \gamma_a \wedge D\chi - 2(\iota_a D\hat{\chi}) \gamma \wedge D\chi + 2\iota_a D\hat{\chi} \wedge (\gamma \wedge D\chi) \]  

(3.48)

\[ T^a = \frac{i}{4} D\hat{\chi} \wedge \gamma_a \chi + \frac{2}{\mu} e^{abc} R_{bc} + \frac{i}{\mu} D\hat{\chi} \wedge (\gamma \wedge D\chi + * (\gamma \wedge D\chi)) \]  

(3.49)

\[ D\chi + \frac{m}{2} \gamma \wedge \chi + \frac{2}{\mu} D(\gamma \wedge D\chi + * (\gamma \wedge D\chi)) = 0. \]  

(3.50)

Here the Einstein field equations (3.48) are devoid of terms that contain the third derivatives of the metric components. Such terms would have been obtained by looking at the contributions of the Cotton 2-forms coming from the bosonic part of the topological action. Moreover the torsion field equations are a generalized version of our usual torsion expressions (3.47) with curvature contributions arising from the topological term. Thus the torsion of space-time will now be dynamical in general. Due to the nonlinearities in the torsion field equation (3.49) one may not be able to solve algebraically for the local supersymmetry transformation of the connection. Supposing that they can be solved, the local supersymmetry transformation law for the connection will include terms proportional to the Chern-Simons coupling constant \( \mu \) at different orders. It is not even apparent whether the same action without the Lagrangian constraint will be invariant or not under supersymmetry transformations because of the highly non-linear nature of the expression above for the torsion.

### 3.4. Cotton 2-Forms and Cottino 2-Form

The terms with coefficient \( 1/\mu \) that appear in the Einstein and gravitino field equations are identified as the Cotton 2-forms and their fermionic counterpart, the so-called Cottino 2-form, respectively. We wish to make a few remarks concerning these. We read off from the final version of our field equations, the Cotton 2-forms

\[ C_a = D\hat{\chi}_a + \frac{i}{4} D\hat{\omega}_a - \frac{i}{4} \left( \iota_a D\hat{\chi} \wedge D\chi - D(\iota_a D\hat{\chi}) \right) + * D\hat{\chi} \wedge \gamma_a \wedge D\chi - 2(\iota_a D\hat{\chi}) \gamma \wedge D\chi \]  

(3.51)

and the Cottino 2-form

\[ C = Y^a \wedge \gamma_a \chi + \frac{i}{4} W^a \wedge \gamma_a \chi + D \wedge D\chi + D \wedge (\gamma \wedge D\chi). \]  

(3.52)

Let us discuss the Cotton 2-forms first. The Cotton 2-forms that one obtains in the formulation of topologically massive gravity theory are given by only the first term in (3.51). The second term contains fermionic contributions coming from the Lagrange multiplier 1-forms and the third term governs the higher order contributions in the gravitino field that are due to the fermionic part of the topological action and contains second, fourth and sixth powers of the gravitino field. This may be observed by separating the connection 1-forms according to (2.5) and expanding the covariant derivatives of the gravitino field as:

\[ D\chi = \hat{D}\chi + \frac{i}{8} \left( [\hat{D}^* \chi + \hat{D}\chi] X + \hat{D} \chi X \right) \wedge \sigma_{ab} \chi \]  

(3.53)

where \( \hat{D} \) denotes the covariant exterior derivative operation with respect to Levi-Civita connection and the second term is the contribution of contortion.

An important feature of the Cotton 2-forms in the Riemannian case (that is, with no torsion present in the geometry) is that they are traceless. The trace of Cotton 2-forms can be taken by wedging them with the co-frame from the left as follows:

\[ e^a \wedge \hat{D} \hat{\chi}_a = -d(e^a \wedge \hat{\chi}_a) = -d(e^{ab} \hat{\chi}_a b) = 0. \]  

(3.54)

because the components of the Schouten 1-forms are symmetric. Again, a hat over a quantity means that it is obtained by using the Levi-Civita connection. Of course this does not hold when there is torsion present in the geometry, however, one is tempted to ask whether this property still holds for the full Cotton 2-forms given by (3.51), provided we take our background geometry to be Riemannian. Unfortunately even this is not the case. The trace of the modified Cotton 2-forms read:

\[ e^a \wedge \hat{C}_a = e^a \wedge \frac{i}{4} \left( \hat{D}\hat{\chi}_a + \iota_a \hat{D}\hat{\chi} \wedge \hat{D}\chi - \hat{D}(\iota_a \hat{D}\chi) \right) + * \hat{D}\hat{\chi} \wedge \gamma_a \wedge \hat{D}\chi - 2(\iota_a \hat{D}\chi) \gamma \wedge \hat{D}\chi + 2\iota_a \hat{D}\hat{\chi} \wedge (\gamma \wedge \hat{D}\chi) \]  

\[ = -\frac{i}{4} \left( d(e^a \wedge \hat{\omega}_a) + \hat{D}\hat{\chi} \wedge \hat{D}\chi + * \hat{D}\hat{\chi} \wedge \gamma \wedge \hat{D}\chi \right) \]  

\[ = \frac{i}{4} \left( 2d(\hat{D}\hat{\chi} \wedge \hat{\chi}) - 2 \hat{D}\hat{\chi} \wedge \gamma \wedge \hat{D}\chi \right) \]  

\[ - \hat{\chi} \wedge \hat{D}(\hat{D}\chi - \gamma \wedge \hat{D}\chi). \]  

(3.55)
There are two reasons for the modified Cotton 2-forms not to be trace-free. The first and main impediment is due to the fact that the components of the contributions \( W_a = W_a \varepsilon^a \) are not symmetric unlike the components of Schouten 1-forms. Explicitly they read:

\[
W_{a,b} = - \left[ (\hat{x}^b \gamma_a \partial_b \chi) + \frac{1}{2} e_{abc} (\hat{x}^c \partial_b \chi) \right] \gamma_b + (\hat{x}^a \gamma_b \partial_b \chi) + (\hat{x}^b \gamma_a \partial_b \chi) + (\hat{x}^c \gamma_a \partial_c \chi) \right] e_{abc} + (\hat{x}^a \gamma_b \partial_c \chi) + (\hat{x}^b \gamma_c \partial_a \chi). \tag{3.56}
\]

In general the variation of a fermionic action with respect to the co-frame field yields an asymmetric tensor. The asymmetry of (3.56) is an example to this fact. The second reason is that we included the terms coming from co-frame variations of the fermionic part of Chern-Simons 3-form. The contribution of these terms to the variation is the fermionic part of Chern-Simons 3-form itself as can be seen from the second equality in (3.55). This contribution is also asymmetric. Another important property of Cotton 2-forms is that, they are symmetric and divergence free when working in a Riemannian geometry. The modified Cotton 2-forms are neither symmetric and nor divergence-free because of the fermionic contributions. We do not write down the divergence and the anti-symmetric part of the modified Cotton 2-forms because their explicit expressions are not very instructive.

The Cotton 2-form, unlike its superpartner, is not discussed abundantly in the literature. To our knowledge, the Cotton 2-form is first introduced in the references [19,20]. But only the part of the Cotton 2-form that is linear in the gravitino field is discussed. It reads in a Riemannian geometry,

\[
\hat{\mathcal{C}} = \hat{D} \ast \hat{D} \chi + \hat{D} \ast (\gamma \wedge \ast \hat{D} \chi) + \hat{\eta} \wedge \gamma_a \chi \tag{3.57}
\]

and is linear only when we are working in a background Riemannian geometry. Otherwise the full Cotton 2-form (3.52) contains terms that are to the first, third and fifth powers in the gravitino field. In a way similar to the Cotton 2-forms, some higher order contributions to the Cotton 2-form are encoded in \( W_a \varepsilon^a \). Again in the Riemannian context, the linear part (3.57) of Cotton 2-form is \( \gamma \)-traceless. This is the spinorial version of the Cotton tensor being traceless. The \( \gamma \)-trace operation is given by wedging the Cotton 2-form from the left with the \( \gamma \)-matrix valued 1-form \( \gamma = \gamma_a \varepsilon^a \):

\[
\gamma \wedge \hat{\mathcal{C}} = \gamma \wedge (\hat{D} \ast \hat{D} \chi) + \gamma \wedge (\gamma \wedge \ast \hat{D} \chi) + \gamma \wedge (\hat{\eta} \wedge \gamma_a \chi) = - \hat{D}[\gamma \wedge \sigma^{ab} \gamma_{ba} \partial_a \chi] + e_{abc} e^a \wedge \gamma^b \wedge \gamma^c \chi
\]

\[
= 2 \hat{D} \chi \wedge \gamma \wedge \chi = 0. \tag{3.58}
\]

When showing \( \gamma \)-tracelessness, in the second equality we used the fact that the connection is torsion-free together with the identity \( \hat{D} \chi \ast (\gamma \wedge \ast \hat{D} \chi) = \sigma^{ab} \gamma_{ba} \hat{D} \chi \) and the fact that Schouten tensor is symmetric. In the final equality we made use of the curvature identity (2.11) and the Ricci identity (2.34). The \( \gamma \)-tracelessness does not hold in general in the presence of torsion, but we may still evaluate the \( \gamma \)-trace of the full Cotton 2-form (3.52) when there is no torsion. The final result is

\[
\gamma \wedge \hat{\mathcal{C}} = \frac{i}{4} \gamma \wedge \hat{W}^a \wedge \gamma_a \chi
\]

\[
= \frac{i}{4} \left[ \frac{1}{2} e_{ab} (\gamma \partial^a \hat{D} \chi (\hat{D} \chi) + \eta_{a} \chi + \hat{D} \chi (\hat{D} \chi) + \eta_{a} \chi + \frac{R}{2} \gamma \wedge \chi = 0. \tag{3.59}
\]

When calculating the \( \gamma \)-trace we opened the expression in co-frame basis and Fierzed once to bring every term into the form \( D\chi (\hat{D} \chi) \). The expression (3.59) shows that, like in the case of Cotton 2-forms, the contributions due to terms \( W_a \varepsilon^a \) spoil the \( \gamma \)-tracelessness of the linear part of the Cotton 2-form (3.52).

To our knowledge, the exact expressions (3.51) and (3.52) of the Cotton 2-forms and the Cotton 2-form that are complete in all non-vanishing powers of the gravitino field have not been derived explicitly before. The definitions can be found in [19] and [20] where in both references the Cotton 2-forms are devoid of fermionic contributions altogether and the Cotton 2-form only covers contributions that are linear in the gravitino field.

4. Concluding Remarks

In the present work we formulate the cosmological topologically massive supergravity theory using a torsion-constrained first order variational formalism in the language of exterior differential forms on three dimensional Riemann-Cartan space-times. In particular, we regard the connection 1-forms as independent field variables thus treating them at the same level as local Lorentz co-frames and the gravitino field. However, the space-time torsion is constrained algebraically to its standard form by the method of Lagrange multipliers. This is an essential feature of our approach giving rise to contributions of the Lagrange multiplier fields in the final set of field equations. We first prove the invariance of the action under infinitesimal local supersymmetry transformations of the co-frame, connection and the gravitino fields. This we did in explicit detail. We also present and simplify the final set of variational field equations since the field equations in their complete form had been lacking in the previous literature.

In particular the field equations that come from the connection variations are solved algebraically for the Lagrange multiplier fields. We substitute them into the coupled Einstein and Rarita-Schwinger field equations which arise from the co-frame and gravitino field variations, respectively. We identified the Cotton 2-forms and their fermionic counterpart, the so-called, Cotton 2-form from the Einstein and gravitino field equations. We note that the variations of the complete Chern-Simons density imply some further non-linear terms besides those coming from the Lagrange multipliers.

As far as we are aware, the full set of exact field equations (3.45), (3.46) and (3.47) of the cosmological topologically massive supergravity theory as well as the exact expressions for the Cotton and Cotton 2-forms (3.51) and (3.52) has not been given explicitly before in the literature. In the references [16] and [17], the field equations of the theory are not discussed. In [19], the Einstein
Massive Gravity theory [12,13] which solves the bulk versus bound-
cate may be the supersymmetric generalisation of Minimally
theory express full set of field equations. One important can-
of differential forms is very effective method to write down the
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ilar formulations other 3D supergravity theories starting from
eto all order.
find the contributions coming to the Cotton and Cottino 2-forms
sistently from a variational principle. By doing so, we are able to
and Cottino tensor is again given by the
In [20], higher order contributions to the Einstein field equations
re second order and the exact field equations are not discussed.
The fieldequations are Taylor expanded
are not explicitly given and Cottino tensor is again given by the
all the equations are written in terms of Levi-Civita connection so
the higher order contributions in the gravitin of field are omitted.

The future directions that one may consider is to look for sim-
lar formulations other 3D supergravity theories starting from a
non-supersymmetric gravitational theory. As demonstrated in
this paper, first order variational formulation using the language
of differential forms is very effective method to write down the
theory express full set of field equations. One important can-
didate may be the supersymmetric generalisation of Minimal-
Massive Gravity theory [12,13] which solves the bulk versus bound-
ary clash problem. Another direction may be to consider the
models that generalize the CTMS model by extending the action
density [24–26] or by having extended supersymmetries [27–29] and
look for new solutions.

Acknowledgements

We dedicate this work to Stanley Deser whose insights were our inspira-
tion. We thank Özgür Sarıoğlu for useful comments and discussions.

Keywords

Massive Supergravity Theories, Riemann-Cartan Space-times

Received: January 25, 2021
Revised: March 22, 2021
Published online: May 20, 2021

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