\textbf{\lambda\text{-ANALOGUES OF \textit{r}\text{-STIRLING NUMBERS OF THE FIRST KIND}}}

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\textsc{Abstract.} In this paper, we study \lambda\text{-analogues of the \textit{r}\text{-Stirling numbers of the first kind which have close connections with the \textit{r}\text{-Stirling numbers of the first kind and \lambda\text{-Stirling numbers of the first kind. Specifically, we give the recurrence relations for these numbers and show their connections with the \lambda\text{-Stirling numbers of the first kind and higher-order Dahee polynomials.}}}

1. Introduction

It is known that the Stirling numbers of the first kind are defined as

\[(x)_n = \sum_{l=0}^{\text{n}} S_1(n, l)x^l, \quad \text{(see [1, 2, 6 \text{--} 9, 14])}, \quad (1.1)\]

where \((x)_0 = 1, (x)_n = x(x - 1) \cdots (x - n + 1), \quad (n \geq 1).\)

For \(\lambda \in \mathbb{R},\) the \lambda\text{-analogue of falling factorial sequence is defined by

\[(x)_0,\lambda = 1, (x)_n,\lambda = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda), \quad (n \geq 1), \quad \text{(see [2, 10, 14, 15, 17])}, \quad (1.2)\]

In view of (1.1), we define \lambda\text{-analogues of the Stirling numbers of the first kind as

\[(x)_n,\lambda = \sum_{k=0}^{n} S_1,\lambda(n, k)x^k, \quad \text{(see [2, 11 \text{--} 13, 16, 17])}. \quad (1.3)\]

It is not difficult to show that

\[(1 + \lambda t)^x = \sum_{l=0}^{\infty} \left(\begin{array}{c} x \\ l \end{array}\right)_{\lambda} \frac{x^l}{l!}t^l, \quad \text{(see [4, 7 \text{--} 17])}, \quad (1.4)\]

where \left(\begin{array}{c} x \\ l \end{array}\right)_{\lambda}, \text{are the \lambda\text{-analogues of binomial coefficients \left(\begin{array}{c} x \\ l \end{array}\right)} given by \left(\begin{array}{c} x \\ l \end{array}\right)_{\lambda} = \frac{(x)_l,\lambda}{l!}.\n
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The \( r \)-Stirling numbers of the first kind are defined by the generating function
\[
\frac{1}{k!} \left( \log(1 + t) \right)^k (1 + t)^r = \sum_{n=k}^{\infty} S_1^{(r)}(n, k) \frac{t^n}{n!}, \quad \text{(see [3, 20 – 23]).} \tag{1.5}
\]
where \( k \in \mathbb{N} \cup \{0\} \) and \( r \in \mathbb{R} \).

The unsigned \( r \)-Stirling numbers of the first kind are defined as
\[
(x + r)(x + r + 1) \cdots (x + r + n - 1) = \sum_{k=0}^{n} \left[ \begin{array}{c} n + r \\ k + r \end{array} \right] x^k, \quad \text{(see [1, 17, 22]).} \tag{1.6}
\]
Thus, by (1.5), we get
\[
(x + r)_n = (x + r)(x + r - 1) \cdots (x + r - n + 1) = \sum_{k=0}^{n} S_1^{(r)}(n, k) x^k, \quad \text{(see [1]).} \tag{1.7}
\]
From (1.6) and (1.7), we note that
\[
S_1^{(r)}(n, k) = (-1)^{n-k} \left[ \begin{array}{c} n + r \\ k + r \end{array} \right] . \tag{1.8}
\]

The higher-order Daehee polynomials are defined by
\[
\left( \frac{\log(1 + t)}{t} \right)^k (1 + t)^x = \sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{t^n}{n!}, \quad \text{(see [5, 18, 19, 24]).} \tag{1.9}
\]
When \( x = 0 \), \( D_n^{(k)} = D_n^{(k)}(0) \) are called the higher-order Daehee numbers. In particular, for \( k = 1 \), \( D_n(x) = D_n^{(1)}(x) \), \( (n \geq 0) \), are called the ordinary Daehee polynomials.

In this paper, we consider \( \lambda \)-analogues of \( r \)-Stirling numbers of the first kind which are derived from the \( \lambda \)-analogues of the falling factorial sequence and investigate some properties for these numbers. Specifically, we give some identities and recurrence relations for the \( \lambda \)-analogues of \( r \)-Stirling numbers of the first kind and show their connections with the \( \lambda \)-Stirling numbers of the first kind and higher-order Daehee polynomials.

2. \( \lambda \)-analogues of \( r \)-Stirling numbers of the first kind

From (1.3) and (1.4), we have
\[
(1 + \lambda t)^x = \sum_{k=0}^{\infty} (x)_k,_{\lambda} \frac{t^k}{k!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} S_{1,\lambda}(k, n) x^n \right) \frac{t^k}{k!} \\
= \sum_{n=0}^{\infty} \left( n! \sum_{k=0}^{\infty} S_{1,\lambda}(k, n) \frac{t^k}{k!} \right) \frac{x^n}{n!}. \tag{2.1}
\]
On the other hand, we also have
\[
(1 + \lambda t)^{\lambda} = e^{\frac{\lambda}{\lambda} \log(1 + \lambda t)} = \sum_{n=0}^{\infty} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^n \frac{x^n}{n!},
\]
(2.2)

Therefore, by (2.1) and (2.2), we get the generating function for \( S_{1,\lambda}(n, k), (n, k \geq 0) \), which is given by
\[
\frac{1}{n!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^n = \sum_{k=n}^{\infty} S_{1,\lambda}(k, n) \frac{t^k}{k!},
\]
(2.3)

Now, we define \( \lambda \)-analogues of \( r \)-Stirling numbers of the first kind as
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{\lambda} = \sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n, k) \frac{t^n}{n!},
\]
(2.4)

where \( k \in \mathbb{N} \cup \{0\} \), and \( r \in \mathbb{R} \).

From (2.3) and (2.4), we note that \( S_{1,\lambda}^{(0)}(n, k) = S_{1,\lambda}(n, k), (n \geq k \geq 0) \). Also, it is easy to show that
\[
(1 + \lambda t)^{\lambda} (1 + \lambda t)^{\lambda} = \sum_{n=0}^{\infty} (x + r)_{n,\lambda} \frac{t^n}{n!}.
\]
(2.5)

By (2.5), we get
\[
\sum_{n=0}^{\infty} (x + r)_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{x + r}{n} \right) \frac{t^n}{n!} = (1 + \lambda t)^{\lambda} e^{\frac{\lambda}{\lambda} \log(1 + \lambda t)}
\]
\[
= \sum_{k=0}^{\infty} x^k \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{\lambda}
\]
(2.6)

Therefore, by comparing the coefficients on both sides of (2.6), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have
\[
(x + r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n, k) x^k.
\]
Now, we observe that
\[
\sum_{k=0}^{\infty} x^k \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t) \]
\[
= \left( \sum_{k=0}^{\infty} x^k \sum_{m=k}^{\infty} S_{1,\lambda}(m, k) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \frac{(r)_{l,\lambda}}{l!} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{m} S_{1,\lambda}(m, k)(r)_{n-m,\lambda} x^k \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{m} S_{1,\lambda}(m, k)(r)_{n-m,\lambda} x^k \right) \frac{t^n}{n!}.
\]

Thus, by (2.6) and (2.7), we get
\[
\sum_{k=0}^{\infty} S_{1,\lambda}^{(r)}(n, k) x^k = \sum_{k=0}^{n} \left( \sum_{m=0}^{k} \binom{n}{m} S_{1,\lambda}(m, k)(r)_{n-m,\lambda} \right) x^k.
\]

Therefore, by comparing the coefficients on both sides of (2.8), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have
\[
S_{1,\lambda}^{(r)}(n, k) = \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m, k)(r)_{n-m,\lambda}.
\]

Now, we define \( \lambda \)-analogues of the unsigned \( r \)-Stirling numbers of the first kind as follows:
\[
(x + r)(x + r + \lambda)(x + r + 2\lambda) + \cdots + (x + r + (n-1)\lambda) = \sum_{k=0}^{n} [\frac{n+r}{k+r}]_{r,\lambda} x^k. \quad (2.9)
\]

Note that \( \lim_{\lambda \to 1} [\frac{n+r}{k+r}]_{r,\lambda} = [\frac{n+r}{k+r}]_{r} \), \( (n \geq k \geq 0) \).

By Theorem 2.1 and (2.7), we get
\[
(x - r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(-r)}(n, k) x^k, \quad (2.10)
\]
and
\[
(x - r)_{n,\lambda} = \sum_{k=0}^{n} (-1)^{n-k} [\frac{n+r}{k+r}]_{r,\lambda} x^k. \quad (2.11)
\]
From (2.10) and (2.11), we can easily derive the following equation (2.12).

\[ S_{1,\lambda}^{(-r)}(n, k) = (-1)^{n-k} \binom{n+r}{k+r}, \quad (n \geq k \geq 0). \]  

(2.12)

For \( n \geq 1 \), by Theorem 2.1, we get

\[ (x + r)_{n+1,\lambda} = \sum_{k=0}^{n+1} S_{1,\lambda}^{(r)}(n + 1, k)x^k = \sum_{k=1}^{n+1} S_{1,\lambda}^{(r)}(n + 1, k)x^k + (r)_{n+1,\lambda}. \]  

(2.13)

On the other hand, by (1.2), we get

\[ (x + r)_{n+1,\lambda} = (x + r)_{n,\lambda}(x + r - n\lambda) \]

\[ = x \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n, k)x^k - (n\lambda - r) \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n, k)x^k \]

\[ = \sum_{k=1}^{n} S_{1,\lambda}^{(r)}(n, k-1)x^k - \sum_{k=1}^{n} (n\lambda - r)S_{1,\lambda}^{(r)}(n, k)x^k + (r - n\lambda)(r)_{n,\lambda} + x^{n+1} \]

\[ = \sum_{k=1}^{n} \left\{ S_{1,\lambda}^{(r)}(n, k-1) - (n\lambda - r)S_{1,\lambda}^{(r)}(n, k) \right\} x^k + (r)_{n+1,\lambda} + x^{n+1}. \]

(2.14)

Therefore, by Theorem 2.1 and (2.14), we obtain the following theorem.

**Theorem 2.3.** For \( 1 \leq k \leq n \), we have

\[ S_{1,\lambda}^{(r)}(n + 1, k) = S_{1,\lambda}^{(r)}(n, k-1) - (n\lambda - r)S_{1,\lambda}^{(r)}(n, k). \]

From (2.4), we note that

\[ \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^z = \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k \sum_{l=0}^{\infty} \frac{r_l}{l!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^l \]

\[ = \sum_{l=0}^{\infty} \left( \frac{k + l}{l} \right)^r \frac{1}{(k+l)!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^{k+l} \]

\[ = \sum_{l=0}^{\infty} \left( \frac{k + l}{l} \right)^r \sum_{n=k+l}^{\infty} S_{1,\lambda}(n, k+l) \frac{t^n}{n!} \]

\[ = \sum_{l=0}^{\infty} r_l \left( \frac{k + l}{l} \right)^r \frac{n!}{(n+k)!} \sum_{n=k+l}^{\infty} S_{1,\lambda}(n, k+l) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{n!t^k}{(n+k)!} \right) \sum_{l=0}^{n} r_l \left( \frac{k + l}{l} \right) S_{1,\lambda}(n + k, k + l) \frac{t^n}{n!} \]

(2.15)
On the other hand, we have
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{r} = \frac{t^k}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda t} \right)^k (1 + \lambda t)^\frac{k}{r} \\
= \left( \sum_{l=0}^{\infty} \binom{k}{l} \frac{\lambda^l t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \binom{r}{m, \lambda} \frac{t^m}{m!} \right) \frac{t^k}{k!} \quad (2.16)
\]
\[
= \left( \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \binom{k}{l} \frac{\lambda^l t^l}{l!} \binom{r}{n-m, \lambda} \frac{t^{n-m}}{(n-m)!} \right) \frac{t^k}{k!}.
\]
Thus, by (2.15) and (2.16), we get
\[
\sum_{l=0}^{n} \binom{n}{l} \frac{\lambda^l t^l}{l!} S_{1, \lambda}(n+k, k+l) = \sum_{l=0}^{n} \binom{n}{l} D_{l}^{(k)} \lambda^l (r)_{n-l, \lambda} t^n (2.17)
\]
Therefore, by (2.17), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have
\[
\sum_{l=0}^{n} \binom{n}{l} D_{l}^{(k)} \lambda^l (r)_{n-l, \lambda} t^n = \sum_{l=0}^{(k+l)} \binom{k+l}{n} (n+k) S_{1, \lambda}(n+k, k+l).
\]

Now, we observe that
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{r} = \left( \sum_{l=0}^{\infty} \binom{r}{l, \lambda} \frac{t^l}{l!} \right) \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k \\
= \sum_{n=k}^{\infty} \left( \sum_{m=k}^{n} \binom{n}{m} S_{1, \lambda}(m, k)(r)_{n-m, \lambda} \right) \frac{t^n}{n!}.
\]
Therefore, by (2.14) and (2.18), we obtain the following theorem.

**Theorem 2.5.** For \( n, k \geq 0 \), with \( n \geq k \), we have
\[
S_{1, \lambda}^{(r)}(n, k) = \sum_{m=k}^{n} \binom{n}{m} (r)_{n-m, \lambda} S_{1, \lambda}(m, k).
\]
From (2.4), we note that
\[
\frac{1}{m!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^m \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{r} \\
= \binom{m+k}{m} \frac{1}{m!} \binom{r}{m, \lambda} \frac{t^m}{m!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^{m+k} (1 + \lambda t)^\frac{k}{r} \quad (2.19)
\]
\[
= \binom{m+k}{m} \sum_{n=m+k}^{\infty} S_{1, \lambda}^{(r)}(n, m+k) \frac{t^n}{n!}.
\]
On the other hand,
\[
\frac{1}{m!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^m \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{\frac{r}{\lambda}}
\]
\[
= \left( \sum_{l=m}^{\infty} S_{1,\lambda}(l, m) \frac{t^l}{l!} \right) \left( \sum_{j=k}^{\infty} S_{1,\lambda}(j, k) \frac{t^j}{j!} \right)
\]
\[
= \sum_{n=m+k}^{\infty} \left( \sum_{l=k}^{n-m} \binom{n}{l} S_{1,\lambda}(l, k) S_{1,\lambda}(n - l, m) \right) \frac{t^n}{n!}.
\]

(2.20)

Therefore, by (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.6.** For \(m, n, k \geq 0\) with \(n \geq m + k\), we have
\[
\binom{m+k}{m} S_{1,\lambda}^{(r)}(n, m+k) = \sum_{l=k}^{n-m} \binom{n}{l} S_{1,\lambda}(l, k) S_{1,\lambda}(n - l, m).
\]

By (2.21), we get
\[
\sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{\frac{r}{\lambda}} (1 + \lambda t)^{-\frac{r}{\lambda}}
\]
\[
= \left( \sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l, k) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \binom{-\frac{r}{\lambda}}{m} \lambda^m t^m \right)
\]
\[
= \left( \sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l, k) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (-1)^m (r + (m - 1)\lambda)_{m,\lambda} \frac{t^m}{m!} \right)
\]
\[
= \sum_{n=k}^{\infty} \left( \sum_{l=k}^{n} \binom{n}{l} S_{1,\lambda}^{(r)}(l, k) (-1)^n (r + (n - l - 1)\lambda)_{n-l,\lambda} \right) \frac{t^n}{n!}.
\]

(2.21)

Comparing the coefficients on both sides of (2.21), we have the following theorem.

**Theorem 2.7.** For \(n, k \geq 0\), with \(n \geq k\), we have
\[
S_{1,\lambda}(n, k) = \sum_{l=k}^{n} \binom{n}{l} S_{1,\lambda}^{(r)}(l, k) (-1)^{n-l} (r + \lambda(n - l - 1))_{n-l,\lambda}.
\]
From (1.9), we have
\[
\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t) \tilde{\tau} = \frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k (1+\lambda t) \tilde{\tau}
\]
\[
= \frac{t^k}{k!} \left(\sum_{m=0}^{\infty} D_m^{(k)} \lambda^m \frac{t^m}{m!}\right) \left(\sum_{l=0}^{\infty} (r)_{l,\lambda} \frac{t^l}{l!}\right)
\]
\[
= \frac{t^k}{k!} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} D_m^{(k)} \lambda^m (r)_{n-m,\lambda}\right) \frac{t^n}{n!}.
\]
On the other hand, by (2.4), we get
\[
\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t) \tilde{\tau} = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!}
\]
\[
= \frac{t^k}{k!} \sum_{n=0}^{\infty} S_{1,\lambda}(n + k, k) \frac{n!}{(n+k)!} \frac{t^n}{n!}.
\]
Thus, by comparing the coefficients on both sides of (2.22) and (2.23), we get
\[
\sum_{m=0}^{\infty} \binom{n}{m} D_m^{(k)} \lambda^m (r)_{n-m,\lambda} = \frac{1}{(n+k)!} S_{1,\lambda}(n, k). \quad (2.24)
\]
Therefore, by (2.24), we obtain the following theorem.

**Theorem 2.8.** For \(n, k \geq 0\), we have
\[
S_{1,\lambda}(n + k, k) = \binom{n + k}{n} \sum_{m=0}^{n} \binom{n}{m} D_m^{(k)} \lambda^m (r)_{n-m,\lambda}.
\]
From (1.9), we note that
\[
\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t) \tilde{\tau} = \frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k (1+\lambda t) \tilde{\tau}
\]
\[
= \frac{t^k}{k!} \sum_{n=0}^{\infty} \lambda^n D_n^{(k)} \frac{t^n}{n!}.
\]
By (2.23) and (2.25), we get
\[
S_{1,\lambda}(n + k, k) = \lambda^n \frac{(n+k)!}{n!k!} D_n^{(k)} \frac{t^n}{n!} = \lambda^n \binom{n+k}{n} D_n^{(k)} \frac{t^n}{n!}, \quad (n \geq 0). \quad (2.26)
\]
In particular, for \(r = 0\), from (2.21) and (2.26) we have
\[ \lambda^n \binom{n+k}{k} D_n^{(k)} = S_{1,\lambda}(n+k, k) \]

\[ = \sum_{l=k}^{n+k} \binom{n+k}{l} S_{1,\lambda}^{(r)}(l, k)(-1)^{n+k-l}(r + (n + k - l - 1)\lambda)_{n+k-l,\lambda}, \tag{2.27} \]

where \( n, k \geq 0 \).

Therefore, by (2.27), we obtain the following theorem.

**Theorem 2.9.** For \( n, k \geq 0 \), we have

\[ \lambda^n \binom{n+k}{k} D_n^{(k)} = \sum_{l=k}^{n+k} \binom{n+k}{l} S_{1,\lambda}^{(r)}(l, k)(-1)^{n+k-l}(r + (n + k - l - 1)\lambda)_{n+k-l,\lambda}. \]

In addition,

\[ D_n^{(k)} = \frac{1}{(n+k)} \sum_{l=k}^{n+k} S_{1,\lambda}^{(r)}(l, k)(-1)^{n+k-l}(r + (n + k - l - 1)\lambda)_{n+k-l,\lambda}. \]

Now, we observe that

\[ \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{\frac{t}{\lambda}} e^{-\frac{t}{\lambda} \log(1 + \lambda t)} \]

\[ = \left( \sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l, k) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (-1)^m r^m \frac{1}{m!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^m \right) \]

\[ = \left( \sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l, k) \frac{t^l}{l!} \right) \left( \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} S_{1,\lambda}(j, m) \frac{t^j}{j!} \right) \]

\[ = \sum_{n=k}^{\infty} \sum_{j=0}^{n-k} \binom{n}{j} (-1)^{n-k-j} S_{1,\lambda}(j, m) S_{1,\lambda}(n-j, k) \frac{t^n}{n!}. \tag{2.28} \]

Therefore, by comparing the coefficients on both sides of (2.28), we obtain the following theorem.

**Theorem 2.10.** For \( n, k \geq 0 \), with \( n \geq k \), we have

\[ S_{1,\lambda}(n, k) = \sum_{j=0}^{n-k} \sum_{m=0}^{j} \binom{n}{j} (-1)^{m} r^m S_{1,\lambda}(j, m) S_{1,\lambda}(n-j, k). \]
For $m, n \geq 0$, we define $\lambda$-analogues of the Whitney’s type $r$-Stirling numbers of the first kind as
\[
(mx + r)_{n, \lambda} = (mx + r)(mx + r - \lambda)(mx + r - 2\lambda) \cdots (mx + r - (n - 1)\lambda)
\]
\[
= \sum_{k=0}^{n} T_{1, \lambda}^{(r)}(n, k|m)x^k.
\] (2.29)

By (2.29), we get
\[
\sum_{n=0}^{\infty} (mx + r)_{n, \lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} T_{1, \lambda}^{(r)}(n, k|m)x^k \right) \frac{t^n}{n!}
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} T_{1, \lambda}^{(r)}(n, k|m) \frac{t^n}{n!} \right) x^k.
\] (2.30)

On the other hand, by binomial expansion, we get
\[
\sum_{n=0}^{\infty} (mx + r)_{n, \lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{mx + r}{n} \frac{t^n}{n!}
\]
\[
= (1 + \lambda t)^{mx + r} = (1 + \lambda t)^{\frac{mx + r}{\lambda}} e^{mx \frac{\log(1 + \lambda t)}{\lambda}}
\]
\[
= \sum_{k=0}^{\infty} \frac{m^k}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{\frac{k}{\lambda}} x^k.
\] (2.31)

Comparing the coefficients on both sides of (2.30) and (2.31), the generating function for $T_{1, \lambda}^{(r)}(n, k|m)$, $(n, k \geq 0)$, is given by
\[
\frac{m^k}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{\frac{k}{\lambda}} = \sum_{n=k}^{\infty} T_{1, \lambda}^{(r)}(n, k|m) \frac{t^n}{n!}.
\] (2.32)

From (2.4) and (2.32), we note that
\[
S_{1, \lambda}^{(r)}(n, k) = \frac{1}{m^k} T_{1, \lambda}^{(r)}(n, k|m), \quad (n \geq k \geq 0).
\] (2.33)

It is known that the $r$-Whitney numbers are defined as
\[
(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n, k)(x)_k, \quad \text{(see [3]).}
\] (2.34)

By (1.3), we get
\((mx + r)_{n, \lambda} = \sum_{l=0}^{n} S_{1, \lambda}(n, l)(mx + r)^l\)

\[
= \sum_{l=0}^{n} S_{1, \lambda}(n, l) \sum_{j=0}^{l} m^j W_{m,r}(l, j)(x)_j
\]

\[
= \sum_{j=0}^{n} \sum_{l=0}^{n} S_{1, \lambda}(n, l)m^j W_{m,r}(l, j)(x)_j
\]

\[
= \sum_{j=0}^{n} \sum_{l=0}^{n} S_{1, \lambda}(n, l)m^j W_{m,r}(l, j)\sum_{k=0}^{j} S_{1}(j, k)x^k
\]

\[
= \sum_{k=0}^{n} \left( \sum_{j=k}^{n} \sum_{l=j}^{n} S_{1, \lambda}(n, l)S_{1}(j, k)m^j W_{m,r}(l, j) \right)x^k. \tag{2.35}
\]

Therefore, by (2.29) and (2.35), we obtain the following theorem.

**Theorem 2.11.** For \(n, k \geq 0\), with \(n \geq k\), we have

\[
T_{1, \lambda}^{(r)}(n, k|m) = \sum_{j=k}^{n} \sum_{l=j}^{n} S_{1, \lambda}(n, l)S_{1}(j, k)m^j W_{m,r}(l, j).
\]

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