A UNIFORM DISCRETE INF-SUP INEQUALITY FOR FINITE ELEMENT HYDRO-ELASTIC MODELS

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Abstract. A seminal result concerning finite element (FEM) approximations of the Stokes equation was the discrete inf-sup inequality that is uniform with respect to the mesh size parameter. This inequality leads to optimal error estimates for the FEM scheme. The original version pertains to the Stokes system with non-slip boundary condition on the entire boundary. On the other hand, in fluid-structure interaction problems, the interface dynamics between the fluid and the solid satisfies velocity and stress matching constraints. As a result, the pressure variable is no longer determined up to a constant and becomes subject to non-homogeneous Dirichlet conditions on the common interface. In this context, we establish a uniform discrete inf-sup estimate for a fluid-structure FEM implementation based on Taylor-Hood elements, and use this inequality to verify some stability and error estimates of this numerical scheme. An added benefit of this framework is that it does not require the Poisson-equation approach to solve for the pressure variable.

1. Introduction. The finite element method (FEM), formulated in the mid-1950's (though with many ideas dating back to early 1940's), was originally introduced for structural analysis problems. During the active development of this technique in the subsequent years, its success with approximations of potential flows (e.g., [54, 27]) rendered the FEM as a prospective valuable tool for computational fluid dynamics. Rather than attempting yet another historical overview, for a list of seminal contributions and further developments, we refer the reader to the introduction and the references in [35] and the book [34]. It goes without saying that the subject has vastly expanded in the course of the following decades.

In step with the applications of the FEM to fluid flows, interest in hydro-elastic models was also gaining momentum. From the PDE systems proposed in [23, pp. 328–329] and [44, p. 121], originally motivated by biological applications, fluid-elasticity models emerged into a research field of their own; see, for instance, [46, 28] and the many references therein. Besides numerous papers accounting for the nonlinear aspects of the dynamics (to name a few: [36, 51, 24, 16, 9, 42, 13, 40, 38]; for plates and shells in fluid flows see, e.g., [21, 47, 20, 48] and the citations therein),
research efforts continue to pursue linearized systems as well; among the relatively more recent contributions are [1, 3, 4, 19, 14, 2, 12].

This work investigates the convergence properties of FEM approximations to fluid-elasticity models. In this context, let us mention a few, albeit by no means exhausting the background literature, foundational articles. A widely popular mixed $P_2/P_1$ FEM scheme for the Stokes flow was among the pioneering contributions to the subject developed by Hood and Taylor in [52] (other element types have also been considered starting with the contributions by Fortin [31] and by Crouzeix & Raviart [25]; for subsequent developments and a comprehensive discussion see the tractate Boffi, Brezzi & Fortin [15]). An interesting variation—fundamentally different in several aspects—of the Taylor-Hood approach and the corresponding error estimates were developed by Glowinski and Pironneau in [35], which in part served as a basis for rigorous error analysis of the original Taylor-Hood scheme: see Bercovier and Pironneau [10] and the subsequent simplification by Verfürth [53] (also Fortin [32]).

The quantification of the convergence rates of the FEM for the Stokes system relies on the uniform, with respect to the mesh parameter, discrete “inf-sup” inequality for the associated bilinear form. The inf-sup framework itself could be chronologically attributed to Banach in the form of the Closed Range Theorem, Nečas [49, Thm. 3.1], Ladyzhenskaya in the context of the Stokes problem (though the inequality is not explicitly stated, see for instance the essential technical result [41, Sec 2.1, p. 24]), Babuška [6] and Brezzi [17] with applications to finite element problems.

The same inf-sup estimate is needed in the more involved case of fluid-elasticity interactions. However, this time the functional framework is different. The non-slip condition cannot be enforced on the interface between the fluid and the solid; this feature enlarges the functional fluid-pressure framework from $H^1_0(\Omega_f) \times (L^2(\Omega_f)/\mathbb{R})$ to $H^1_{\Gamma_f \subseteq \partial \Omega_f}(\Omega_f) \times L^2(\Omega_f)$. The pressure variable is no longer determined up to a constant and becomes subject to non-homogeneous Dirichlet conditions on the common interface as dictated by the matching of the stresses. The discrete inf-sup condition for the corresponding finite element scheme was explored by Du et al. [29, Thm. 3.4, p. 12] based on piecewise-linear approximations of the velocity variable. Another numerical algorithm was later developed by Liu [45, Sec. 4, p. 7256] using the associated Poisson equation to determine the pressure. In this article, we establish a discrete inf-sup inequality for a fluid-structure model based on $P_2/P_1$ Taylor-Hood elements and without employing the pressure-Poisson-equation approach.

1.1. Goals and outline. The purpose of this note is to establish the uniform inf-sup inequality as presented below in Theorem 3.1 and apply it to error estimates for the so called “semi-discrete” FEM fluid-structure model—Corollary 1. The argument borrows its strategy from the classical results of [10] and [53] and some of the requisite technical lemmas can actually be reused, such as the “weak” Brezzi inequality in Section 4. However, several adjustments are necessary to complete the proof in the new functional setting as shown in Section 5.

In addition, a fully discrete implementation must also approximate the elliptic bilinear form itself in the variational formulation. This complication allows for stability estimates, however, does not yield convergence in the original norms—see
Section 3.2. Interestingly, the few particular numerical examples considered in the existing studies [3, 11] do reproduce the rates of Corollary 1.

1.2. Notation. Vector-value functions with components in a generic Banach space $X$ will be denoted using bold-face symbols, such as $L^2(X) = [L^2(X)]^d$, for $d = 2, 3$, etc. Given the fluid domain $\Omega_f$, the form $(\cdot, \cdot)$ will stand for the standard inner product on $L^2(\Omega_f)$ or $L^2(\Omega_f)$. In turn, $(\cdot, \cdot)_s$ will indicate the same for the solid domain $\Omega_s$. The space $H^1_f(\Omega_f)$ will denote the subspace of the $W^{1,2}(\Omega_f)$ whose functions have zero trace on $\Gamma_f \subset \partial\Omega_f$; analogously for vector-valued functions $H^1_f(\Omega_f)$.

Prefix $D$ will stand for the differential, so $Du \ldots Dv$, for example, is the Frobenius product of the Jacobi matrices $D_u$ and $D_v$ of vector-valued functions $u, v$. The product $(Du, Dv)$ will correspond to $\int_{\Omega_f} Du \ldots Dv d\Omega_f$. Similarly, $(D\phi, D\psi)_s := \int_{\Omega_s} D\phi \ldots D\psi d\Omega_s$.

The norm $\| \cdot \|$ will be the one in $L^2(\Omega_f)$ or $L^2(\Omega_f)$; otherwise, an appropriate subscript will be used to indicate which space the norm pertains to. We will also invoke the semi-norm for the space $H^1(\Omega_f)$ given by

$$|u|^2_{1, \Omega_f} := (Du, Du) = \sum_{i=1}^d \|\nabla u_i\|^2.$$

Symbol $S_X$ will denote the unit sphere in the indicated Banach space $X$.

2. The model. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be the bounded open control volume encasing the fluid with an elastic body floating therein. By $\Omega_f \subset \Omega$ we denote the open portion occupied by the fluid and $\Omega_s$ will stand for the subset of $\Omega$ occupied by the elastic solid:

$$\Omega = \Omega_f \cup \Omega_s.$$

We consider the model linearized around rest, whence the domains $\Omega_s$ and $\Omega_f$ are time-independent (for a linearization about a solution to a free moving boundary problem see [14]). Denote their boundaries by

$$\Gamma_s := \partial\Omega_s \quad \text{and} \quad \Gamma_f := \partial\Omega_f \setminus \Gamma_s.$$

The vector field $n_f$ will denote the outward unit normal field to the fluid domain on $\partial\Omega_f$, whereas $n_s$ will denote the outward unit normal to the solid domain on $\Gamma_s$ as depicted in Figure 1.
The system of interest is the coupling of the Stokes and the linearized isotropic elasticity equations ([44, p. 121], [28], [1]):

\[
\begin{cases}
    \mathbf{u}_t - \Delta \mathbf{u} + \nabla p = 0 & \text{in } (0, T) \times \Omega_f \\
    \text{div}(\mathbf{u}) = 0 & \text{in } (0, T) \times \Omega_f \\
    \mathbf{w}_{tt} - \sigma(\mathbf{w}) = 0 & \text{in } (0, T) \times \Omega_s,
\end{cases}
\]

(1)

The stress operator \(\sigma\) is either vectorial Laplacian \(\Delta\) or the isotropic elastic stress tensor

\[
\text{div}(\varepsilon(\mathbf{w})), \quad \varepsilon(\mathbf{w}) = \frac{1}{2}[Dw + (Dw)^*], \quad \mathcal{C}(\mathbf{M}) = \lambda \text{Tr}(\mathbf{M}) + 2\mu \mathbf{M}
\]

with the Lamé parameters \(\lambda, \mu\). The difference between the stress operators ultimately amounts to just using a suitable equivalent inner product on \(H^1(\Omega_s)\) and the associated normal strain in the boundary conditions, but the analysis remains conceptually identical. Henceforth, for ease of presentation we will focus on the “diagonal” version:

\[
\sigma(\mathbf{w}) = \Delta \mathbf{w}.
\]

The dynamics equations are accompanied by the non-slip conditions on the outer boundary \(\Gamma_f\) and the appropriate stress and velocity matching conditions on the interface \(\Gamma_s\):

\[
\begin{cases}
    \mathbf{u} = \mathbf{0} & \text{on } (0, T) \times \Gamma_f \\
    \mathbf{u} = \mathbf{w}_t & \text{on } (0, T) \times \Gamma_s \\
    \frac{\partial \mathbf{u}}{\partial \mathbf{n}_f} - \frac{\partial \mathbf{w}}{\partial \mathbf{n}_f} = \mathbf{p} \mathbf{n}_f & \text{on } (0, T) \times \Gamma_s.
\end{cases}
\]

(2)

The initial data comes from the state space

\[
[w(0), \mathbf{w}_t(0), \mathbf{u}(0)] \in \mathcal{H} := H^1(\Omega_s) \times L^2(\Omega_s) \times H_{fl}
\]

(3)

with

\[
H_{fl} = \{ \mathbf{z} \in L^2(\Omega_f) : \text{div} \mathbf{z} = 0, \quad \mathbf{z} \cdot \mathbf{n}_f \big|_{\Gamma_f} = 0 \}.
\]

We stress again that in the definition of \(H_{fl}\) the zero normal component condition is enforced only on the exterior non-slip boundary \(\Gamma_f\), but not on the entire boundary of the fluid domain \(\partial \Omega_f\).

The semigroup well-posedness of (1)–(3) was discussed at length in [1, 3, 4] (see also [12] for the semigroup result on a related, but more complex model linearized around a steady regime).

2.1. Geometry of the domain. For the well-posedness theory it is sufficient to assume that \(\Omega_f\) and \(\Omega_s\) are of class \(C^2\), but polygonal boundaries should also be considered when dealing with the FEM framework.

The variational formulation of the problem discussed here only requires the Stokes component of the solution to be compatible with the velocity \(\mathbf{u}\) in \(H^1(\Omega_f)\) and the pressure \(p\) in \(L^2(\Omega_f)\). For the Stokes operator with non-slip conditions a polyhedral domain is sufficient [26, p. 76]. Note, however, that by the nature of the problem, as shown in Figure 1, it is no longer expected that the fluid domain \(\Omega_f\) is convex, thus, for instance, the results [26, p. 76] do not guarantee that under a smooth forcing one can expect the fluid velocity with regularity \(\mathbf{u} \in H^3(\Omega_f)\) and the pressure \(p \in H^2(\Omega_f)\). This detail bears possible limitations on the applicability of the convergence results of Corollary 1.
Remark 1 (Elliptic regularity in strong formulation). The definition of the semigroup generator for the full hydro-elastic system involves “pressure-determining” operators [3, p. 264] given by harmonic extension maps \( H^{-1/2}(\Gamma_s) \rightarrow L^2(\Omega_f) \) and \( H^{-3/2}(\Gamma_f) \rightarrow L^2(\Omega_f) \), for the Dirichlet and Neumann boundary data respectively. By duality, following the transposition argument of Lions and Magenes [43, Ch 2, Sec. 6], or the weak formulation of Babuška et al. [7, 8], these operators exist if the Poisson problems with homogeneous Neumann or Dirichlet conditions define topological isomorphisms of \( L^2(\Omega_f) \) onto the corresponding subspace of \( H^2(\Omega_f) \). However, even in 2D polygonal setting, such elliptic smoothing in general holds only for convex domains [37, Coro. 4.4.3.8 and Lem. 4.4.35, pp. 232-33]. So the strong semigroup interpretation of the problem holds on any domain which admits such harmonic extension maps, but in general it may, possibly, be lacking regularity in polyhedral approximations of domains like the one depicted in Figure 1. In particular, any polygonal approximation of the boundary interface \( \Gamma_s \) will have re-entrant corners with respect to the fluid domain \( \Omega_f \).

2.2. Variational formulation. System (1)-(3) can be recast [3] as a saddle-point problem, where in process one must invoke a different from classical pressure-elimination procedure since the Leray projector is not compatible with the space \( H\text{fl} \), due to the lack of the boundary conditions on \( \Gamma_s \).

First, express (1)-(3) as an evolution system

\[
\begin{align*}
\text{Y}_t &= \triangle \text{Y} \\
\end{align*}
\]

on state space \( \mathcal{H} \), as in (3), for a dissipative semigroup generator \( \triangle \), with appropriate domain \( \mathcal{D}(\triangle) \) [1, 3, 4]. The resolvent problem of finding \( \text{Y} \in \mathcal{D}(\triangle) \) such that

\[
(\triangle - \lambda) \text{Y} = \text{Y}^*,
\]

for \( \lambda > 0 \) and given \( \text{Y}^* = (w_1^*, w_2^*, u^*) \in \mathcal{H} \), reduces to finding the velocity \( u \in H^1(\Omega_f) \) and pressure \( p \in L^2(\Omega_f) \) functions that satisfy

\[
\begin{align*}
\begin{cases}
  a_\lambda(u, v) + b(v, p) &= F(v) &\text{for all } v \in H^1(\Omega_f) \\
  b(u, q) &= 0 &\text{for all } q \in L^2(\Omega_f).
\end{cases}
\end{align*}
\]

The symmetric bilinear form \( a_\lambda \) is given by [3, (4.24), p. 272]:

\[
a_\lambda(u, v) := \lambda(u, v) + (Du \cdot Dv) + \frac{1}{\lambda} \left( D\mathcal{D}_\lambda(u|_{\Gamma_s}) \cdot D\mathcal{D}_\lambda(v|_{\Gamma_s}) \right)_s
\]

on \( H^1(\Omega_f) \times H^1(\Omega_f) \), where the restriction to \( \Gamma_s \) indicates the trace map, \( \mathcal{D}_\lambda \), for \( \lambda > 0 \), is the solution map to the Dirichlet harmonic extension problem

\[
\mathcal{D}_\lambda : L^2(\Gamma_s) \rightarrow L^2(\Omega_s),
\]

for which \( g = \mathcal{D}_\lambda h \) if and only if

\[
\begin{align*}
\begin{cases}
  (\lambda^2 - \Delta)g &= 0 &\text{in } \Omega_s \\
  g &= h &\text{on } \Gamma_s.
\end{cases}
\end{align*}
\]

The second bilinear form is

\[
b(u, q) := -\langle q, \text{div} u \rangle &\text{ on } H^1(\Omega_f) \times L^2(\Omega_f).
\]

And the functional \( F \) is defined by [3, (4.25), p. 273]:

\[
F(v) := (w_1^*, v) + \left( w_2^* + \lambda w_1^*, \mathcal{D}_\lambda(v|_{\Gamma_s}) \right)_s - (\lambda^2 + 1) \left( W^*, \mathcal{D}_\lambda(v|_{\Gamma_s}) \right)_s
\]

for which

\[
F(v) = -\left( Dw^*, D\mathcal{D}_\lambda(v|_{\Gamma_s}) \right)_s
\]
wherein
\[ W^* := \frac{1}{\lambda} D_\lambda(w_1^*|_{\Gamma_s}) + \Delta_\lambda^{-1}(w_2^* + \lambda w_1^*) \]
using
\[ \Delta_\lambda := (\lambda^2 + I - \Delta) \text{ on domain } H^2(\Omega_s) \cap H^1_0(\Omega_s). \]
Thus \( \Delta_\lambda^{-1} \) is a solution operator for a Poisson-type problem.

An important distinction from the non-slip Stokes flow model is that \( \lambda \) is uniquely determined, whereas the Stokes model determines the pressure only up to a constant.

If this form vanishes for all \( v \in H^1_f(\Omega_f) \), then \( q \) must not merely be constant, but, in fact, zero in \( \Omega_f \). Hence solutions to the hydro-elastic system (5) are determined uniquely, whereas the Stokes model determines the pressure only up to a constant.

System (5) has a solution if (e.g., see [39, Thm. 3.1.5, p. 116]) \( \alpha_\lambda \) is elliptic (in fact, the ellipticity is only needed on an appropriate subspace with respect to \( b \)) and for some \( c > 0 \)
\[ \sup_{0 \neq u \in H^1_f(\Omega_f)} \frac{b(u, q)}{\|u\|_{H^1(\Gamma_f)}} \geq c\|q\| \quad \text{for all } q \in L^2(\Omega_f). \]
The latter can be equivalently restated as
\[ \inf_{q} \sup_{u} b(u, q) \geq c > 0 \quad \text{on } \mathcal{H}^1(\Omega_f) \times \mathcal{L}^2(\Omega_f). \]

The ellipticity of \( \alpha_\lambda \) is apparent from the definition (6) for \( \lambda > 0 \), and the verification of the continuous inf-sup inequality for (5) can be found in [3].

In the FEM version of (5), one has to find a single \( c \) satisfying (10) for every mesh in the appropriate family on domain \( \Omega_f \).

3. **Main results.** Assume that \( \Omega_f \) is a polyhedron. By the structure of the domain, that implies that \( \Omega_s \) is a polyhedron as well. For a shorthand, denote
\[ \mathcal{X} := H^1_f(\Omega_f), \quad \mathcal{Y} := H^1(\Omega_s) \quad \text{and} \quad \mathcal{M} := L^2(\Omega_f). \]
Let \( \{T_h\}_{h>0} \) denote a quasi-uniform ([30, Def. 1.140, p. 76]) family of affine meshes on \( \Omega = \Omega_f \cup \Omega_s \) such that every element \( K \in T_h \) resides either in \( \Omega_f \) or in \( \Omega_s \).

Specifically we consider approximations of \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{M} \) based on triangular or tetrahedral Taylor-Hood elements \( \mathbb{P}_2/\mathbb{P}_1 \). For \( K \in T_h \) let \( P^m(K) \) denote the set of polynomials of degree \( \leq m \) on \( K \). Then define the approximating spaces by
\[ \mathcal{X}_h := \{ v_h \in C^0(\Omega_f) : v_h|_{\Gamma_f} = 0 \quad \text{and for all } K \in T_h \cap \Omega_f, v_h|_K \in P^2(K) \}, \]
\[ \mathcal{Y}_h := \{ w_h \in C^0(\Omega_s) : \text{ for all } K \in T_h \cap \Omega_s, w_h|_K \in P^2(K) \}, \]
\[ \mathcal{M}_h := \{ q_h \in C^0(\Omega_f) : \text{ for all } K \in T_h \cap \Omega_f, q_h|_K \in P^1(K) \}. \]
(The slight abuse of notation \( K \in T_h \cap \Omega_f \) or \( s \) means \( K \cap \Omega_f \) or \( s \) has positive measure.) Note that the space \( \mathcal{Y}_h \) does not explicitly enter the variational formulation (5), though it will come up within the “fully discrete” model discussed below in Section 3.2.
3.1. “Semi-discrete” scheme. To start with, assume that the bilinear form $a_{\lambda}$ (6) and the functional $F$ (8) can be evaluated exactly. The discretization, thus, shows up only in the function spaces. Accordingly, consider the following approximation of system (5): find $u_h \in X_h$ and $p_h \in M_h$ such that

$$
\begin{align*}
\begin{cases}
a_{\lambda}(u_h, v_h) + b(v_h, p_h) &= F(v_h) & \text{for all } v_h \in X_h \quad (11) \\
b(u_h, q_h) &= 0 & \text{for all } q_h \in M_h.
\end{cases}
\end{align*}
$$

For exposition purposes we present the result in the two-dimensional formulation, but there are no technical obstacles to its extensions to three dimensions. Thus, henceforth, $\Omega \subset \mathbb{R}^2$ and each $K \in T_h$ is a triangle.

**Theorem 3.1** (Discrete uniform inf-sup inequality). Assume that for each $h > 0$, every element in $T_h$ that is supported in $\Omega_f$ has at least one vertex not in $\Gamma_f$. Then there is a constant $C^* > 0$ independent of $h \in (0, h_0)$, some $h_0 > 0$, such that

$$
S_{q_h} := \sup_{0 \neq v_h \in X_h} \frac{b(v_h, q_h)}{|v_h|_{1, \Omega_f}} \geq C^*
$$

for every $q_h \in M_h$, with $\|q_h\| = 1$. Equivalently, via the Poincaré inequality, there is $C^*_p > 0$, such that for all $h \in (0, h_0)$,

$$
\inf_{0 \neq q_h \in M_h} \sup_{0 \neq v_h \in X_h} \frac{b(v_h, q_h)}{\|v_h\|_{H^1(\Omega_f)} \|q_h\|} \geq C^*_p.
$$

Before addressing the proof we can present some convergence estimates for the hydro-elastic model:

**Corollary 1.** If functions $u \in X$ and $p \in M$ solve (5) and $u_h \in X_h$, $p_h \in M_h$ solve (11), then under the assumptions of Theorem 3.1, there exists $c > 0$ independent of $h > 0$ small, such that

$$
\|u - u_h\|_{H^1(\Omega_f)} + \|p - p_h\|_{L^2(\Omega_f)} \leq c \left( \inf_{v_h \in X_h} \|u - v_h\|_{H^1(\Omega_f)} + \inf_{q_h \in M_h} \|p - q_h\| \right).
$$

(12)

Thus, if a priori $u \in H^3(\Omega_f)$ and $p \in H^2(\Omega_f)$, we recover the classical error estimates, as in [10]:

$$
\|u - u_h\|_{H^1(\Omega_f)} + \|p - p_h\|_{L^2(\Omega_f)} \leq C h^2 \left( \|u\|_{H^3(\Omega_f)} + \|p\|_{H^2(\Omega_f)} \right),
$$

(13)

and

$$
\|\nabla (p - p_h)\| \leq C h \left( \|u\|_{H^3(\Omega_f)} + \|p\|_{H^2(\Omega_f)} \right).
$$

(14)

**Proof.** The first inequality follows directly from [18, Thm. 2.1, p. 60], noting that in this case the pressure is in fact determined in the space $L^2(\Omega_f)$ rather than $L^2(\Omega_f)/\mathbb{R}$. The ensuing convergence estimate (13) is the direct consequence of [30, Prop. 1.134, p. 73].

The inequality (14) can be obtained with just a bit of extra work. Start with,

$$
\|\nabla (p - p_h)\| \leq \|\nabla (p - q_h)\| + \|\nabla (q_h - p_h)\| \quad \text{for all } q_h \in M_h.
$$

The bound on $\|\nabla (p - q_h)\|$ for the piecewise-linear interpolant $q_h \in M_h$ of $p \in H^2(\Omega_f)$ is of the order $o(h)$. Hence it suffices to establish a suitable bound on $\|\nabla (q_h - p_h)\|$. The difference of the equations (11) and (5) with the test function $v_h \in X_h$ gives

$$
a_{\lambda}(u_h - u, v_h) = b(v_h, p - p_h)
$$

or, after adding $b(v_h, q_h)$ on each side,

$$
b(v_h, q_h - p_h) = a_{\lambda}(u_h - u, v_h) - b(v_h, p - q_h).
$$

From the inequality (e.g., [30, Rem. 1.143, p. 77])
\[ |v_h|_{1,\Omega} \leq ch^{-1} \|v_h\| \quad \text{for all } v_h \in X_h, \]
in combination with the Poincaré inequality, it leads to
\[ \frac{|b(v_h, q_h - p_h)|}{\|v_h\|} \leq \frac{c}{\|v_h\|_{H^1(\Omega)}} \big(o_{\lambda}(u_h - u, v_h) + |b(v_h, p - q_h)|\big). \]
By the “weak” Brezzi inequality of Lemma 4.2 take the supremum over \( v_h \in X_h \) to conclude
\[ \|\nabla (q_h - p_h)\| \leq \frac{c}{h} \left( \|u_h - u\|_{H^1(\Omega)} + \|p - q_h\| \right). \]
By the convergence result (13) and according to the interpolation estimate for the interpolant, this is well-posed [30, Pro. 3.26, p. 125], but the convergence estimate (15) is of the order \( h^{-1} o(h^2) = o(h) \), which completes the proof.

3.2. Fully discrete scheme. Another crucial difference from the Stokes problem is that the bilinear form \( a_\lambda \) in (6) and functional \( F \) in (8) incorporate the Dirichlet harmonic extension operator \( D \) into the solid domain \( \Omega_s \), as well as the solution to the Poisson-type problem for \( \Delta \lambda \). In practice these cannot be exactly evaluated. Hence the fully discrete problem, rather than having the form (11), actually reads
\[ \begin{cases} a_\lambda^h(u_h, v_h) + b(v_h, p_h) = F^h(v_h) & \text{for all } v_h \in X_h, \\ b(u_h, q_h) = 0 & \text{for all } q_h \in M_h, \end{cases} \]
with \( a_\lambda^h \) and \( F^h \) depend on \( h \). Then instead of (12) we, following [18, Prop. 2.16, p. 67], acquire
\[ \begin{align*}
&\inf_{v_h \in X_h} \|u - v_h\|_{H^1(\Omega)} + \inf_{q_h \in M_h} \|p - q_h\| \\
&\leq \sup_{v_h \in X_h} \frac{|a_\lambda^h(u, v_h) + b(v_h, p) - F(v_h)|}{\|v_h\|_{H^1(\Omega)}} + \sup_{0 \neq v_h \in X_h} \frac{|F(v_h) - F^h(v_h)|}{\|v_h\|_{H^1(\Omega)}}. 
\end{align*} \]
The critical part in estimating the suprema on the right of (17) boils down to considering the difference
\[ ||D D^h \lambda(v_h|_{\Gamma_s}) - D D^h \lambda(v_h|_{\Gamma_s})||_{L^2(\Omega_s)} \]
where \( D \) is the harmonic extension (7), and \( D^h \) is its finite element approximation. Specifically, \( g_h := D^h(z_h) \) for \( z_h \in X_h \) is the solution in the space \( Y_h \) (the finite element subspace of \( H^1(\Omega_s) \)) to the problem
\[ \lambda^2 (g_h, \psi_h)_s + (D g_h, D \psi_h)_s = 0 \quad \text{for all } \psi_h \in Y_h, \]
\[ g_h|_{\Omega_s} = z_h|_{\Omega_s}. \]
The problem \( D^h \lambda(v_h|_{\Gamma_s}) \) is well-posed [30, Pro. 3.26, p. 125], but the convergence as \( h \to 0 \) requires a more careful consideration.
If we define \( \gamma^h_s(z) \) to be the trace approximation on \( \Gamma_s \) obtained from a suitable interpolant, e.g., [50], of the function \( z \in H^1(\Omega_f) \), then we have the classical convergence estimate, for instance [30, Coro. 3.29, p. 126],
\[ ||D D^h \lambda^h_s(z) - D D^h \lambda(z|_{\Gamma_s})||_{L^2(\Omega_s)} \leq Ch ||D^h \lambda(z|_{\Gamma_s})||_{H^2(\Omega_s)}. \]
However, with respect to the $H^1(\Omega_f)$ norm only, as the normalization on the right-hand side of (17) necessitates, we merely have an upper bound:

$$\|D\mathcal{D}_h^b(v_h|\Gamma_s) - D\mathcal{D}_{\lambda}(v_h|\Gamma_s)\|_{L^2(\Omega_s)} \leq C_1\|\mathcal{D}_h(v_h|\Gamma_s)\|_{H^1(\Omega_s)} \leq C_2\|v_h\|_{H^1(\Gamma_s)} \leq C_3\|v_h\|_{H^1(\Omega_f)}.$$  

Thus we can infer that the supremum terms on the right hand side of (17) are bounded uniformly in $h$, thus verifying a stability estimate for the numerical scheme. But unless one considers convergence in lower-order norms, there is no expectation of decay of these bounds as $h \searrow 0$ in general.

Despite this discrepancy in the convergence analysis between the semi-discrete model (5) and the fully discrete model (11), the few particular examples tested in [3] and [11] do exhibit the decay rates of the semi-discrete scenario, as furnished by Corollary 1.

The remainder of the paper is devoted to the proof of Theorem 3.1.

4. Revisiting the “weak” Brezzi inequality. We will henceforth only deal with the elements in the fluid subdomain $\Omega_f$. For such an element $K \in \mathcal{T}_h$, denote by $p_1, p_2, p_3$ its locally numbered vertices. We assume the numbering goes counterclockwise around the element. For the edge from $p_i$ to $p_{(i \mod 3)+1}$, define $m_i$ to be its midpoint, and let $\tau_i$ be a CCW-oriented unit tangent vector along the edge:

$$\tau_i := \frac{p_{(i \mod 3)+1} - p_i}{|p_{(i \mod 3)+1} - p_i|}.$$  

See a sketch of a sample element in Figure 2.

![Figure 2](image-url)

**Figure 2.** Element $K \in \mathcal{T}_h$. The numbers in the parentheses denote the local node numbers.

We start with an elementary result:

**Lemma 4.1.** There exists $\kappa > 0$ independent of $h > 0$ such that for each $q_h \in \mathcal{M}_h$ and each element $K \in \mathcal{T}_h$, the function $\nabla q_h$ (which is, in fact, constant on $K$) satisfies

$$|\nabla q_h|_K^2 \leq \kappa \sum_{\text{any two } \tau_i} (\nabla q_h \cdot \tau_i(m_i))^2.$$  

In particular, we can choose $\kappa = 2/(\sin \theta_{\text{min}})^2$ where $\theta_{\text{min}}$ is a lower bound on the angles between any adjacent element edges for the elements of the mesh family $\{\mathcal{T}_h\}_{h>0}$. 

Proof. For $q \in \mathcal{M}_h$ and $K \in \mathcal{T}_h$ the gradient $\nabla q_h$ is a constant vector field on $K$ so denote it by $\nabla q_h|_K = (c_i)$. Pick unit tangential vectors on any two boundaries of the element. For a more compact notation label them as $\tau$ and $\tilde{\tau}$. Of course, $\nabla q_h$ can be reconstructed in terms of these two linearly independent vectors. We only need to remark why the relationship is uniform in $\nabla$ of the element. For a more compact notation label them as $\tau$ and $\tilde{\tau}$ so denote it by $\eta$.

\begin{align*}
\alpha(c_1 \tau_1 + c_2 \tau_2) + \beta(c_1 \tilde{\tau}_1 + c_2 \tilde{\tau}_2) &= c_1 ,
\end{align*}

whence

\begin{align*}
\begin{bmatrix} \tau_1 & \tilde{\tau}_1 \\ \tau_2 & \tilde{\tau}_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\tau_1 \tilde{\tau}_2 - \tau_2 \tilde{\tau}_1} \begin{bmatrix} \tilde{\tau}_2 \\ -\tau_2 \end{bmatrix} .
\end{align*}

Thus, from (19) we infer

\begin{align*}
c_1 = \frac{\tilde{\tau}_2}{\tau_1 \tilde{\tau}_2 - \tau_2 \tilde{\tau}_1} (\nabla q_h \cdot \tau) - \frac{\tau_2}{\tau_1 \tilde{\tau}_2 - \tau_2 \tilde{\tau}_1} (\nabla q_h \cdot \tilde{\tau}) .
\end{align*}

The determinant $\tau_1 \tilde{\tau}_2 - \tau_2 \tilde{\tau}_1$ is the area of the parallelogram spanned by the unit vectors $\tau_1, \tilde{\tau}_2$, so it is the sine of the acute angle between them. So if $\theta_0$ is the minimal angle in $K$ then we have

$$
\max\{|\alpha|, |\beta|\} \leq |\sin \theta_0|^{-1} .
$$

Similar analysis holds for the second component $c_2$ of $\nabla q_h$. Thus, when the mesh family is shape-regular with $\theta_{\text{min}}$ being a lower bound on the angle, the conclusion of the Lemma follows with $\kappa = 2(\sin \theta_{\text{min}})^{-2}$, independently of $h$.

**Remark 2.** Because the finite element pressure subspace is that of piecewise linear functions, and since in our case the supremum is taken over a larger space $X_h$ (now allowing for functions that are not necessarily zero on the boundary portion $\Gamma_s$), this Lemma actually follows from [10, Prop 1., p. 214]. We present a slightly modified proof for completeness and also to point out that the conditions on the mesh can be relaxed a little admitting elements with two edges on $\Gamma_s$, albeit then at the obvious expense of convexity of $\Omega$, since with triangular elements it would acquire a reentrant corner.

**Proof. Step 1. Test function $v_h$.** We employ the fluid test functions, with a slight modification, that were used in the proof of [30, Lemma 4.23, p. 193]. Fix any $q_h \in \mathcal{M}_h$. Define $v_h \in X_h$ on each element $K \in \mathcal{T}_h$ via

\begin{align*}
\begin{cases}
v_h = 0 & \text{on the vertices of } K \\
v_h(m_i) = 0 & \text{if } m_i \in K \cap \Gamma_f \\
v_h(m_i) = (\nabla q_h \cdot \tau_i) \tau_i & \text{if } m_i \in K \setminus \Gamma_f .
\end{cases}
\end{align*}

Piecewise quadratic $v_h$ is uniquely determined by its values at the vertices and the edge midpoints, hence the definition is consistent. It is helpful to note the following
property of $v_h$. Let $\{\phi_r^{(K)}\}$ denote the standard local quadratic finite element basis [5, p. 177] on $K$. Then each $v_h = [v_1, v_2]^*$ admits the expansion

$$v_1|_K = \sum_{r=1}^{6} \alpha_r^{(K)} \phi_r^{(K)} \quad \text{and} \quad v_2|_K = \sum_{r=1}^{6} \beta_r^{(K)} \phi_r^{(K)}$$

Note that all the coefficients $\alpha_r^{(K)}$ and $\beta_r^{(K)}$ are either 0 or bounded by $|\nabla q_h \cdot \tau_i|$ according to the definition of $v_h$. From here,

$$\int_{\Omega_f} (v_1)^2 \, d\Omega_f = \sum_{K \in T_h} \int_{K} \left| \sum_{r=1}^{6} \alpha_r^{(K)} \phi_r^{(K)} \right|^2 \, d\Omega_f$$

Since the considered mesh family is, in particular, shape-regular, then the isomorphisms between the elements and the reference element are uniformly (operator-norm) bounded in either direction independently of $h$ and $K$. Hence the Lagrange local basis functions are pointwise bounded independently of $h$ or $K$. Therefore, we have

$$\int_{\Omega_f} (v_1)^2 \, d\Omega_f \leq C_1 \sum_{K \in T_h} \int_{K} \left| \sum_{r=1}^{6} \alpha_r^{(K)} \phi_r^{(K)} \right|^2 \, d\Omega_f \leq C_2 \sum_{K \in T_h} |K| \sum_{m \in K \setminus \Gamma_f} |\nabla q_h \cdot \tau_i|^2,$$

and likewise for $v_2$. Because $|K| |\nabla q_h|^2 = \int_K |\nabla q_h|^2 \, d\Omega_f$, we conclude

$$\|v_h\|^2 \leq C_2 \sum_{K \in T_h} |K| \sum_{m \in K \setminus \Gamma_f} |\nabla q_h|^2 \leq C_3 \int_{\Omega_f} |\nabla q_h|^2$$

for $C_3 = 3 \cdot C_2$ independent of $h$. We restate this as

$$\frac{1}{\sqrt{C_3}} \|v_h\| \leq \|\nabla q_h\| . \quad (21)$$

**Step 2. The estimate.** Integrate by parts:

$$b(v_h, q_h) := -\int_{\Omega_f} q_h \text{div}(v_h) \, d\Omega_f = \int_{\Omega_f} \nabla q_h \cdot v_h \, d\Omega_f - \int_{\Gamma_s} q_h n_f \cdot v_h \, d\Gamma_s \quad (22)$$

where recall $n_f$ is the exterior normal field to the fluid domain. Label by $E_j$ the edges $\{E_j = [d_j, f_j]\}$ that comprise $\Gamma_s$. Then the boundary integral in $(22)$ gives:

$$\int_{\Gamma_s} q_h n_f \cdot v_h \, d\Gamma_s = \sum_{E_j \in \Gamma_s} \int_{d_j}^{f_j} q_h n_f \cdot v_h \, d\Gamma_s .$$

We could have defined $v_h$ to be zero on all the nodes on $\Gamma_s$ which would eliminate this boundary integral. That would impose a minor restriction of each element to have one vertex outside $\Gamma_s$. Alternatively, note that the given definition works just as well. Specifically, the polynomial $q_h n_f \cdot v_h$ restricted to edge segment $(d_j, f_j)$ reduces to a cubic in one variable. For polynomial $\phi$ of degree 3, we can exactly obtain the value from the Simpson’s rule

$$\int_{d_j}^{f_j} \phi(x) \, d\Gamma_s = \frac{|f_j - d_j|}{6} [\phi(d_j) + 4\phi(m_j) + \phi(f_j)]$$

which gives

$$\int_{\Gamma_s} q_h n_f \cdot v_h \, d\Gamma_s = \frac{2}{3} \sum_j |f_j - d_j| q_h(m_j)(\nabla q_h \cdot \tau_j) \cdot n_f = 0 .$$
Thus (22) simplifies to
\[ - \int_{\Omega_f} q_h \, \text{div}(v_h) \, d\Omega_f = \int_{\Omega_f} \nabla q_h \cdot v_h \, d\Omega_f. \] (23)

For \( K \in \mathcal{T}_h \), the product \( q_h \, \text{div} v_h \big|_K \) is a quadratic polynomial. So we can use the exact quadrature
\[ \int_K \phi(x) dx = \frac{|K|}{3} \sum_{m_i \in K} \phi(m_i), \]
the sum being taken over all the edge midpoints of \( K \). Apply this identity to equation (23) and use the definition (20) of \( v_h \) at the midpoints to obtain
\[ b(v_h, q_h) = \sum_{K \in \mathcal{T}_h} \sum_{m_i \in \Gamma_f \setminus K} v_h(m_i) \cdot \nabla q_h \frac{|K|}{3} = \left( \sum_{K \in \mathcal{T}_h} \frac{|K|}{3} \sum_{m_i \in \Gamma_f \setminus K} (\nabla q_h \cdot \tau_i)^2 \right) \]

By the assumption that each \( K \) contains at least one vertex not in \( \Gamma_f \) (though it could have its vertices in \( \Gamma_f \)), we know that each sum in the preceding identity involves at least two edge midpoints on the boundary \( K \). According to Lemma 4.1 for each \( K \) the sum on the right dominates the constant field \( \nabla q_h \big|_K \). Hence
\[ b(v_h, q_h) \geq \sum_{K \in \mathcal{T}_h} \frac{|K|}{3} \left( \frac{1}{\kappa} \right) \| \nabla q_h \big|_K \|^2 = \frac{1}{3\kappa} \sum_{K \in \mathcal{T}_h} \int_K |\nabla q_h|^2 = \frac{1}{3\kappa} \| \nabla q_h \|^2. \]

Invoke (21) to arrive at
\[ b(v_h, q_h) \geq \frac{1}{3\kappa} \| \nabla q_h \| \frac{1}{\sqrt{C_3}} \| v_h \|. \]

From which the conclusion of the Lemma follows with \( C = 1/(3\kappa \sqrt{C_3}) \).

5. **Proof of Theorem 3.1.** Let \( q_h \) reside on the unit sphere in \( M_h \). From FEM inverse estimates we know (see, for instance [30, Rem. 1.143, p. 77]) that for all \( v_h \in X_h \)
\[ |v_h|_{1,\Omega_f} \leq c \frac{1}{h} \| v_h \|, \]
where \( c \) is independent of \( h > 0 \) small. So Lemma 4.2 immediately yields
\[ S_{q_h} \geq C_{\ast}^* h \| \nabla q_h \|. \] (24)

This is not quite the estimate we are after but we will take advantage of this inequality later.

Recall that \( b(v_h, q_h) = -\int_{\Omega_f} q_h \, \text{div}(v_h) \, d\Omega_f \). In order to establish the desired estimate, the goal is to find a function \( v_h \) such that \( \text{div}(v_h) \) suitably approximates \( -q_h \). First, since \( \Omega \) has Lipschitz boundary, then from [37, Lemma 1.5.1.9, p. 40] there is \( \delta > 0 \) and \( \mu \in [C^\infty(\Omega)]^n \) such that
\[ \mu \cdot n_f \geq \delta \quad \text{a.e. on } \Gamma \] (25)
Next, the second term on the right-hand side of (29) after integration by parts gives

\[
- \text{div}(v) = \left( \int_{\Gamma_s} \mu \cdot n_f \, d\Gamma_s \right) q_h \quad \text{in} \quad \Omega_f
\]
\[
v = 0 \quad \text{on} \quad \Gamma_f
\]
\[
v = - \left( \int_{\Omega_f} q_h \, d\Omega_f \right) \mu \quad \text{on} \quad \Gamma_s.
\]

Note that the compatibility condition

\[
- \int_{\Gamma_f} v \cdot n_f \, d\Gamma_s = - \int_{\Omega_f} \text{div}(v) \, d\Omega_f = \left( \int_{\Gamma_s} \mu \cdot n_f \, d\Gamma_s \right) \left( \int_{\Omega_f} q_h \, d\Omega_f \right)
\]

holds by the choice of the boundary condition on \( \Gamma_s \). By the result in [33, Sec. III.3, p. 120; also Exercise 3.4 on p. 131] the problem admits a solution \( v \in H^1(\Omega_f) \) and (from [33, (3.25), p. 132]) by our choice of the boundary condition on \( \Gamma_s \) it satisfies:

\[
|v|_{1,\Omega_f} \leq C_\mu \| q_h \|_{L^2(\Omega_f)}.
\]

Using the assumption \( \| q_h \|_{L^2(\Omega_f)} = 1 \) we can just write

\[
|v|_{1,\Omega_f} \leq C_\mu.
\]

Now we consider a suitable approximation of \( v \) in the space \( X_h \). Using [22, Thm. 1] there exists a projection operator \( R_h : X \to X_h \) such that for some \( c > 0 \) and any \( v \in X \)

\[
|v - R_h v|_{\kappa,\Omega_f} \leq ch^{1-k} |v|_{1,\Omega_f}, \quad k = 0, 1.
\]

In particular, \( |R_h v|_{1,\Omega_f} \leq |R_h v - v|_{1,\Omega_f} + |v| \leq (c+1)|v|_{1,\Omega_f} \). So

\[
S_{q_h} \geq c_1 \frac{b(R_h v, q_h)}{|R_h v|_{1,\Omega_f}} \geq c_1 \frac{b(R_h v, q_h)}{|v|_{1,\Omega_f}} + c_1 \frac{b(q_h - v, q_h)}{|v|_{1,\Omega_f}}.
\]

Let \( \alpha := \int_{\Gamma_s} \mu \cdot n_f \, d\Gamma_s \), as the coefficient in (26). Using the property (25) of the vector field \( \mu \), the first term on the right-hand side gives

\[
c_1 \alpha \frac{b(v, q_h)}{|v|_{1,\Omega_f}} = c_1 \alpha \frac{\| q_h \|_{L^2(\Omega_f)}^2}{|v|_{1,\Omega_f}} \geq c_1 \alpha \frac{\| q_h \|_{L^2(\Omega_f)}^2}{|v|_{1,\Omega_f}}.
\]

Invoking (27), we also obtain

\[
c_1 \frac{b(v, q_h)}{|v|_{1,\Omega_f}} \geq c_1 \delta \frac{\| q_h \|_{L^2(\Omega_f)}^2}{C_\mu} = c_1 \delta \frac{\| q_h \|_{L^2(\Omega_f)}^2}{C_\mu} := C_1.
\]

Next, the second term on the right-hand side of (29) after integration by parts gives

\[
c_1 \frac{b(R_h v - v, q_h)}{|v|_{1,\Omega_f}} \geq c_1 \frac{\| \nabla q_h \|_{\Omega_f}}{|v|_{1,\Omega_f}} \| R_h v - v \|_{1,\Omega_f}^2 - c_1 \frac{\| q_h \|_{L^2(\Gamma_s)} \| R_h v - v \|_{L^2(\Gamma_s)}}{|v|_{1,\Omega_f}}.
\]
With the help of projection error estimate (28) the term $I$ in (31) can be estimated directly as:

$$I = \frac{c_1}{|v|_{1, \Omega_f}} \| \nabla q_h \| \| R_h v - v \| \leq \frac{c_1}{|v|_{1, \Omega_f}} \| \nabla q_h \| c h |v|_{1, \Omega_f} = C_2 h \| \nabla q_h \| .$$

For the term $II$ in (31), we first bound the $L^2(\Gamma_s)$-norms by $H^{1/2}(\Omega_f)$ norms and interpolate the latter between $L^2(\Omega_f)$ and $H^1(\Omega_f)$. For the $H^1(\Omega_f)$-norm of $(R_h v - v)$ we can, up to a Poincaré constant, use the seminorm $| \cdot |_{1, \Omega_f}$ since these functions vanish on $\Gamma_f$. For estimates of $q_h$ use the fact that $\| q_h \|_{L^2(\Omega_f)} = 1$. Then invoke the projection error estimate (28):

$$II = \frac{c_1}{|v|_{1, \Omega_f}} \| q_h \|_{L^2(\Gamma_s)} \| R_h v - v \|$$

$$\leq \frac{c_1}{|v|_{1, \Omega_f}} c_p \| R_h v - v \|^{1/2} |R_h v - v|_{1, \Omega_f} \times \| q_h \|^{1/2} \left( \sqrt{1 + \| \nabla q_h \|^2} \right)^{1/2}$$

$$\leq \frac{c_1}{|v|_{1, \Omega_f}} c_p \left( c^{1/2} h^{1/2} |v|_{1, \Omega_f} \right) \left( c^{1/2} |v|_{1, \Omega_f} \right) \times 1 \cdot \left( 1 + \| \nabla q_h \| \right)^{1/2}$$

$$= c_1 c_p c h^{1/2} (1 + \| \nabla q_h \|)^{1/2}$$

$$\leq \frac{\varepsilon}{2} + C_3 h^{1/2} (1 + \| \nabla q_h \|)$$

for any $\varepsilon > 0$. Thus, (31) gives us

$$b(R_h v - v, q_h) \leq C_2 h \| \nabla q_h \| + \frac{\varepsilon}{2} + C_3 \frac{h}{2 \varepsilon} (1 + \| \nabla q_h \|).$$

Now plug this estimate, along with (30), into (29):

$$S_{q_h} \geq C_1 - C_2 h \| \nabla q_h \| - \frac{\varepsilon}{2} - C_3 \frac{h}{2 \varepsilon} (1 + \| \nabla q_h \|).$$

Fix $\varepsilon < 2C_1$, let $h_0 > 0$ be small enough so that

$$C_2^* := C_1 - \frac{\varepsilon}{2} - C_3 h_0 \frac{h_0}{2 \varepsilon} > 0.$$

Further define $C_3^* := C_2 + C_3/(2 \varepsilon)$. Then for all $h \in (0, h_0)$ we have

$$S_{q_h} \geq C_2^* h \| \nabla q_h \| .$$

Next, recall from (24) that for small $h$

$$S_{q_h} \geq C_2^* h \| \nabla q_h \| .$$

Directly from the two boxed identities we get, respectively,

$$C_1^* S_{q_h} \geq C_1^* C_2^* - C_1^* C_3^* h \| \nabla q_h \| \ \text{and} \ \ C_3^* S_{q_h} \geq C_3^* C_2^* h \| \nabla q_h \|$$

Adding them together yields:

$$(C_1^* + C_3^*) S_{q_h} \geq C_1^* C_2^*$$

or

$$S_{q_h} \geq C^* \frac{C_1^* C_2^*}{(C_1^* + C_3^*)}$$

independently of $h \in (0, h_0)$ and $q_h \in M_h$, for a suitably small fixed $h_0 > 0$. This step completes the proof of Theorem 3.1.
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