Schrödinger Representation of $CP(N)$ Model for Large $N$

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Abstract

We examine the 1+1 dimensional $CP(N)$ model in the large $N$ limit by using the Schrödinger representation. Starting from the Hamiltonian analysis of the model, we present the variational gap equation resulting from the Gaussian trial wave functional. The renormalization of the theory is performed with insertion of mass and energy counter-terms, and the dynamical generation of mass and the energy eigenvalue are derived.

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I. INTRODUCTION

The Schrödinger representation [1,2] has been proven to be a powerful method for probing the non-perturbative aspects of quantum field theory [3–6]. It provides a natural way of extracting physical quantities as the eigenvalues of the equivalent operators acting on wave functionals, which describe the theory. An effective way of computing the Hamiltonian eigenvalue is to use the variational method [7–9]. Especially, the Gaussian trial wave functional has been extensively used in a variety of models, and it reproduces the one-loop results [10].

In this paper, we reexamine the 1+1 dimensional $CP(N)$ model [11] for large $N$ [12,13]. The purpose is to test the Gaussian approximation in a geometric theory with non-polynomial interactions, which is known to be exactly soluble in the large $N$ limit. The study also gives some insight into gauge theories where similar features such as dynamical mass generation and asymptotic freedom arise. We approximate the vacuum wave functional with a Gaussian and calculate the energy eigenvalue by minimizing the expectation value of the Hamiltonian. The ensuing gap equation can be solved with a mass counter-term, and the solution exhibits the asymptotic freedom as expected. The renormalization of the Hamiltonian is performed with the insertion of an extra energy counter-term and the result is in agreement with the path integral treatment [13].

II. LARGE $N CP(N)$ MODEL

We first use the coadjoint orbit method to formulate the $CP(N)$ model [14] in the large $N$ limit. Let us introduce

$$Q = gKg^{-1},$$

where $g \in G = SU(N + 1)$ and $K = \frac{i}{N} \text{diag}(N, -1, -1, \ldots, -1)$. The action of the $CP(N)$ model is given by

$$S = -\frac{1}{2\lambda} \int d^2x \text{Tr}(\nabla_\mu Q \nabla^\mu Q),$$

while $\lambda$ is coupling constant, and $\nabla_\mu = \partial/\partial x^\mu$. Parameterizing $g$ by $(N + 1)$ vectors as $g = (\vec{Z}_1, \vec{Z}_2, \ldots, \vec{Z}_{N+1})$ with $\vec{Z}_p \in \mathbb{C}^{N+1}$ ($p = 1, \ldots, N + 1$), such that $\vec{Z}_p^\ast \cdot \vec{Z}_q = \delta_{pq}$, and $\det(\vec{Z}_1, \vec{Z}_2, \ldots, \vec{Z}_{N+1}) = 1$, we obtain

$$S = \frac{1}{\lambda} \int d^2x (\nabla_\mu \vec{Z}_1^\ast \cdot \nabla^\mu \vec{Z}_1 + (\vec{Z}_1^\ast \cdot \nabla_\mu \vec{Z}_1)^2),$$

with the constraint $\vec{Z}_1^\ast \cdot \vec{Z}_1 = 1$, for $\vec{Z}_1 = (z_1, \cdots, z_{N+1})$. This constraint can be solved by assuming $z_{N+1} = z_N^\ast$. Introducing the Fubini-Study coordinate:

$$\psi^a = \sqrt{\frac{N}{z_{N+1}}} \frac{z_a}{z_N},$$

we obtain

$$1$$
\[ z_a = \frac{\psi^a}{\sqrt{N + |\psi|^2}}, \quad z_{N+1} = \frac{\sqrt{N}}{\sqrt{N + |\psi|^2}}. \] (5)

Substitution into (3) gives
\[ S = \frac{1}{\lambda} \int d^2x g_{ab} \partial_\mu \psi^a \partial^\mu \bar{\psi}^b, \] (6)

where \( g_{ab} \) is the standard Fubini-Study metric on \( CP(N) \):
\[ g_{ab} = \frac{(N + |\psi|^2)\delta_{ab} - \bar{\psi}^a \psi^b}{(N + |\psi|^2)^2}, \quad |\psi|^2 = |\psi_1|^2 + \cdots + |\psi_N|^2, \] (7)

where \( \bar{\psi}^a \) is an unconstrained \( N \)-component field. Note that \( g_{ab} \) can be obtained from the Kähler potential \( K(\psi, \bar{\psi}) = \ln(N + |\psi|^2) \) by
\[ g_{ab} = \frac{\partial^2 K(\psi, \bar{\psi})}{\partial \psi^a \partial \bar{\psi}^b}. \] (8)

To perform Hamiltonian analysis, we introduce the conjugate momenta, \( \pi_a \) and \( \bar{\pi}_a \) of \( \psi^a \) and \( \bar{\psi}^a \) respectively by
\[ \pi_a = \frac{1}{\lambda} g_{ab} \dot{\psi}^b, \quad \bar{\pi}_a = \frac{1}{\lambda} g_{ba} \dot{\bar{\psi}}^b. \] (9)

Then the Hamiltonian density \( \mathcal{H} \) is given by
\[ \mathcal{H} = \pi_a \dot{\psi}^a + \bar{\pi}_a \dot{\bar{\psi}}^a - \mathcal{L} = \lambda g^{ab}\pi_a \pi_b + \frac{1}{\lambda} g_{ab} \nabla \psi^a \nabla \bar{\psi}^b, \] (10)

where \( g^{ab} \) is the inverse of \( g_{ab} \):
\[ g^{ab} = \frac{1}{N(N + |\psi|^2)}(N\delta_{ab} + \bar{\psi}^a \psi^b). \] (11)

The canonical quantization gives the following equal time commutators:
\[ [\psi^a(x), \pi_b(y)] = i\hbar \delta^a_b \delta(x - y), \]
\[ [\bar{\psi}^a(x), \bar{\pi}_b(y)] = i\hbar \delta^a_b \delta(x - y), \]
\[ [\psi^a(x), \bar{\pi}_b(y)] = [\bar{\psi}^a(x), \pi_b(y)] = 0, \] (12)

where \( x \) denotes only a space variable, and time is fixed at a common value. Using the differential representation of the momentum implied by the Schrödinger representation
\[ \pi_a(x) = -i\hbar \frac{\delta}{\delta \psi^a(x)}, \quad \bar{\pi}_a(x) = -i\hbar \frac{\delta}{\delta \bar{\psi}^a(x)}, \] (13)

the quantum Hamiltonian is given by \( (g = N\lambda) \)
\[ H = \frac{g}{N^2} \int dx \left[ -\hbar^2 (N + |\psi|^2) (N \delta_{ab} + \bar{\psi}^a \psi^b) \frac{\delta^2}{\delta \psi_a(x) \delta \bar{\psi}^b(x)} + \frac{N^2 (N + |\psi|^2) \delta_{ab} - \bar{\psi}_a \psi_b \nabla \psi_a \nabla \bar{\psi}_b}{(N + |\psi|^2)^2} \right]. \] (14)

As the \( CP(N) \) model generates a divergent mass in the quantum level, (see equation (19) and (20)), it is necessary to regularize by introducing a mass counter-term. In addition, the eigenvalue of the Hamiltonian turns out to diverge. Hence, we add the following part to the Hamiltonian

\[ H_{c.t.} = \frac{1}{g} \int dx \bar{\mu}_0^2 (N + |\psi|^2) + \mathcal{E}_{c.t.}, \] (15)

where \( \bar{\mu}_0^2 \) and \( \mathcal{E}_{c.t.} \) will be determined appropriately in the following in order to subtract the divergences of the Hamiltonian. Note that the mass counter-term of (15) breaks the original \( SU(N+1) \) invariance of \( CP(N) \) model, (1), to its linearly realized subgroup \( SU(N) \times U(1) \).

III. VARIATIONAL METHOD

The Schrödinger equation will be studied within the variational approach \( [7,8] \). Let us take a Gaussian trial wave functional

\[ \Psi[\psi, \bar{\psi}] = \exp \left\{ -\int dx dy (\bar{\psi}_a(x) - \bar{\psi}_a(x)) \frac{G^{-1}_{ab}(x,y)}{2\hbar} (\psi_b(y) - \psi_b(y)) \right\}, \] (16)

for some specific configuration \( \bar{\psi}, \psi \), and some propagator, \( G^{-1}_{ab}(x,y) \), to be determined. We suppressed the normalization constant. Then, the following expectation values result

\[ \langle \psi_a(x) \rangle = \bar{\psi}_a(x), \quad \langle \bar{\psi}_a(x) \rangle = \bar{\psi}_a(x), \quad \langle \bar{\psi}_a(x) \psi_b(y) \rangle = \bar{\psi}_a(x) \bar{\psi}_b(y) + \hbar G^{-1}_{ba}(y,x), \]

\[ \langle \psi_a(x) \psi_b(y) \rangle = \bar{\psi}_a(x) \bar{\psi}_b(y), \quad \langle \bar{\psi}_a(x) \bar{\psi}_b(y) \rangle = \bar{\psi}_a(x) \bar{\psi}_b(y), \quad \langle \pi_a(x) \pi_b(y) \rangle = \frac{\hbar}{4} G^{-1}_{ba}(y,x). \] (17)

It is not possible to compute the expectation value of the Hamiltonian, \( \langle H \rangle \), in a closed form, because of the non-polynomial character of \( H \). Instead, we take the large \( N \) limit (with \( g \) fixed), and keep the dominant term in \( N \). In order to simplify the result even more, we take \( \bar{\psi} \) to be \( x \)-independent. Due to the \( N \)-plicity of the fields in the \( CP(N) \) model we can set the scales such that \( |\bar{\psi}|^2 \sim N \) and \( G_{aa}(x,x) \sim N \). Hence, we obtain for large \( N \)

\[ \langle H \rangle = \frac{g}{N} \int dx \left[ -\frac{\hbar^2}{4} (N + |\psi|^2) (\delta(0))^2 + \frac{\hbar^2}{4} (N + |\psi|^2) G^{-1}_{aa}(x,x) + \frac{\hbar^2}{4} G_{aa}(x,x) G^{-1}_{bb}(x,x) \right. \]

\[ - \frac{N^2 \hbar}{g^2 (N + |\psi|^2)} \nabla^2_{x} G_{aa}(x,x') \bigg|_{x'=x} + \frac{N^2 \hbar^2}{g^2 (N + |\psi|^2)^2} G_{aa}(x,x) \nabla^2_{x} G_{bb}(x,x') \bigg|_{x'=x} \]

\[ + \frac{N^2 \hbar^2}{g^2 (N + |\psi|^2)^2} \left( \nabla_{x} G_{aa}(x,x') \bigg|_{x'=x} \right)^2 + \frac{N}{g^2 \bar{\mu}_0^2 (N + |\psi|^2 + \hbar G_{aa}(x,x))] + \mathcal{E}_{c.t.}, \] (18)
where terms are kept only to order $\hbar^2$. In the following we are going to keep $\hat{\psi}$ as an arbitrary parameter, while $G^{-1}$ will be chosen such that it minimizes the expectation value of the Hamiltonian. The variation with respect to $G_{ab}(x, y)$ gives the following equation

$$
\left(1 + \frac{\hbar G_{cc}(x, x)}{N + |\hat{\psi}|^2}\right)G_{ab}^{-2}(x, y) =
$$

$$
\frac{4}{g^2 (N + |\hat{\psi}|^2)^2} \left(1 - \frac{\hbar G_{cc}(x, x)}{N + |\hat{\psi}|^2}\right) \left(- \nabla_x^2 + m^2\right) \delta(x - y) \delta_{ab},
$$

where the finite mass, $m^2$, is given by

$$
m^2 = \frac{(N + |\hat{\psi}|^2)}{N} \left(1 - \frac{\hbar G_{cc}(x, x)}{N + |\hat{\psi}|^2}\right)^{-1} \left(\frac{\hbar g^2}{4N} G_{aa}^{-1}(x, x) + \tilde{\mu}_0^2\right) \equiv \mu^2 + \mu_0^2.
$$

This equation can be solved by performing Fourier transform of the delta function, resulting in

$$
G_{ab}^{-1}(x, y) = \frac{2N}{g(N + |\hat{\psi}|^2)} \left(1 - \frac{\hbar G_{cc}(x, x)}{N + |\hat{\psi}|^2}\right) \int \frac{dp}{2\pi} \sqrt{p^2 + m^2} e^{ip(x - y)} \delta_{ab}
$$

which yields the mass gap equation:

$$
\mu^2 = \frac{g}{2} \int \frac{dp}{2\pi} \sqrt{p^2 + m^2}.
$$

Note that the dynamically generated mass $\mu^2$ has a quadratic divergence (times bare coupling) in terms of the momentum cutoff. This is consistent with the one-loop result in perturbation theory. Substitution of the variational equation (19) into (18) leads to effective Hamiltonian. We can bring it into the following expression;

$$
\langle H \rangle = N \left[\int dx \left(\frac{m^2}{g} + \frac{\hbar}{2} \int \frac{dp}{2\pi} \sqrt{p^2 + m^2} - \frac{\hbar m^2}{2} \int \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + m^2}}\right) + \mathcal{E}_{c.t.}\right]
$$

In the above, we only kept terms up to order $\hbar$ for producing the one-loop result. In this expression the mass, $m^2$, is finite. However, there are infinities connected with the ultra-violet behavior of the momentum. In the next section we are going to derive the renormalization of the physical quantities of the theory.

**IV. RENORMALIZATION**

The renormalization of the theory requires that the infinities appearing in the Hamiltonian, are absorbed in the bare coupling, $g$, the mass, $\tilde{\mu}_0^2$ and the energy counter-term, $\mathcal{E}_{c.t.}$. Introducing a momentum cut-off, $\Lambda$, the finite mass, $m^2$, becomes

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1 Small $\psi^a, \bar{\psi}^b$ expansion of the Lagrangian given in (6) contains the derivative interaction of the form $-1/(gN)(|\psi|^2 \partial \bar{\psi}^2 + |\bar{\psi}|^2 \partial \psi^2)$. This lowest order interaction will produce quadratic divergence in the one-loop order contributing to the dynamically generated mass.
\[ m^2 = \mu_0^2 + \mu^2 = \mu_0^2 + \frac{g}{4\pi} \left( \Lambda^2 + \frac{m^2}{2} + m^2 \log \frac{2\Lambda}{m} \right) \]  

(24)

where the divergences for \( \Lambda \to \infty \), have to be absorbed in \( \mu_0^2 \) and \( g \). We have set \( \hbar = 1 \). In order to treat the infinities, we first define a finite renormalized coupling constant \( g_R \) as

\[ \frac{1}{g_R} = \frac{1}{g} + \frac{1}{4\pi} \log \frac{M}{2\Lambda}. \]  

(25)

with an arbitrary renormalization scale \( M \). We also define a renormalized mass \( \mu_R \) by

\[ \mu_R^2 = \frac{\mu_0^2 + \frac{g}{4\pi} \Lambda^2}{1 - \frac{2}{4\pi} \log \frac{2\Lambda}{M}}. \]  

(26)

In terms of (25) and (26), relation (24) becomes

\[ m^2 = \mu_R^2 + \frac{g_R m^2}{2\pi} \left( 1 + \log \frac{M^2}{m^2} \right). \]  

(27)

The definitions (25) and (26) can also be viewed as determining \( \mu_0^2 \) and \( g \) for finite values of \( \mu_R^2 \) and \( g_R \) with respect to the cut-off, \( \Lambda \). Eq. (25) shows asymptotic freedom as expected.

In terms of the renormalized coupling the Hamiltonian in (23) becomes

\[ \langle H \rangle = N \left[ \int dx \right] \left( \frac{m^2}{g_R} + \frac{m^2}{8\pi} \log \frac{m^2}{M^2} + \frac{m^2}{8\pi} \right). \]  

(28)

where we have chosen the counter-term \( \mathcal{E}_{\text{c.t.}} \) such that it cancels the appearing quadratic divergence, \( \frac{1}{16\pi} \Lambda^2 \), of the Hamiltonian. This renormalized Hamiltonian is not unique because the final expression is dependent upon the renormalization prescription in (25). If one chooses another definition by adding a constant \( C/4\pi \) term to the right-hand side of (25), the final Hamiltonian (28) changes by \( \Delta \langle H \rangle = N \int dx \left( -C m^2 / 8\pi \right) \). Note that relation (27) remains the same. The constant \( C \) can be fixed by requiring a renormalization condition. If we demand, for example,

\[ \left. \frac{d \langle H \rangle}{dm^2} \right|_{m^2=m_0^2} = 0, \]  

(29)

where \( m_0^2 \) is the solution of (25),

\[ m_0^2 = M^2 \exp \left[ -\frac{8\pi}{g_R} \right], \]  

(30)

we find \( C = 2 \), and this \( \langle H \rangle \) agrees with the one in the literature [13]. Note that \( \langle H \rangle \big|_{m^2=m_0^2} \) does not depend on the coupling constant explicitly, a phenomenon known as dimensional transmutation [13].
V. CONCLUSIONS

We have shown that the variational technique with Gaussian wave functional reproduces the known results of $CP(N)$ model in the large $N$ limit. Unlike the path integral method which uses an auxiliary field, Hamiltonian procedure necessitates two counter terms. One is a mass counter-term to treat the divergence appearing in the mass gap equation and to extract a finite mass. The other is a quadratically divergent energy counter-term. The remaining divergences in the Hamiltonian can be absorbed by renormalizing the coupling constant and mass.

The present work can be extended in a couple of ways. First, recall that nonlinear sigma model with target manifold of symmetric space can also be solved in the large $N$ limit [15]. It would be worthwhile to test the Gaussian method in this case also. Secondly, it would be interesting to perform higher-loop analysis which requires a systematic expansion of Gaussian wave functional [16].

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