Symmetries of almost complex structures and pseudoholomorphic foliations

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Abstract
Contrary to complex structures, a generic almost complex structure $J$ has no local symmetries. We give a criterion for finite-dimensionality of the pseudogroup $G$ of local symmetries for a given almost complex structure $J$. It will be indicated that a large symmetry pseudogroup (infinite-dimensional) is a signature of some integrable structure, like a pseudoholomorphic foliation. We classify the sub-maximal (from the viewpoint of the size of $G$) symmetric structures $J$. Almost complex structures in dimensions 4 and 6 are discussed in greater details in this paper.

1 Introduction and Results

Let $(M^{2n}, J)$ be a connected almost complex manifold, $J^2 = -1$. In this paper we focus mainly on local geometry and assume $n = \frac{1}{2} \dim M > 1$. In this case the Nijenhuis tensor $N_J(\xi, \eta) = [J\xi, J\eta] - J[J\xi, \eta] - J[J\eta, \xi] - [\xi, \eta]$ (which is a skew-symmetric $(2,1)$-tensor) is generically non-zero.

It is known that all differential invariants can be expressed via the jets of the Nijenhuis tensor $[K_1]$. In this paper we will be concerned with vector distributions canonically associated with $N_J$. We will see that their behavior is related to the amount of local symmetries of the almost complex structure.

Of course, the most symmetric case corresponds to $N_J = 0$, and this vanishing is equivalent to local integrability of $J$ $[NN]$. On the other pole we know that generically $J$ has no local symmetries at all $[K_2]$. What happens in between?

Globally, if the manifold $M$ is closed, the symmetry group $\text{Aut}(M, J)$ is a finite-dimensional Lie group $[BKW]$. For the pseudogroup of local symmetries $G = \text{Sym}_{\text{loc}}(M, J)$ we will prove the following criterion of finite-dimensionality.

**Theorem A.** Suppose that the almost complex structure $J$ is non-degenerate in the sense that the Nijenhuis tensor as the map $N_J : \Lambda^2 TM \to TM$ is epimorphic for $n > 2$ and that its image $\Pi = \text{Im}(N_J) \subset TM$ is a non-integrable vector distribution for $n = 2$. Then $\dim G < \infty$.

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In general the pseudogroup of local symmetries $G = \text{Sym}_{\text{loc}}(M, J)$ can be infinite-dimensional even in the almost complex case. By the Cartan test we can calculate the functional dimension and rank of $G$, i.e. determine that its elements formally depend on $\sigma$ functions of $p$ arguments (and some number of functions with fewer arguments). For instance, complex local biholomorphic maps depend on $n$ holomorphic functions of $n$ arguments.

In the other (non-integrable) cases we will show that the number of arguments $p \leq n-1$, and we will study the structures that maximize $\sigma$. We shall prove (this shall be understood either in analytic category or formally):

**Theorem B.** If $N_J \neq 0$, then the local transformations from the pseudogroup $G$ depend on at most $\sigma$ functions of $(n-1)$ arguments, where $\sigma = n-1$ for $n = 2, 3$ and $\sigma = n - 2$ for $n > 3$.

For the sub-maximal symmetric structures, which realize the above dimensional characteristics, some vector distributions, associated to $J$, will be shown integrable. Thus $M$ comes naturally equipped with pseudoholomorphic foliations. This feature is lacking generally, namely $J$-holomorphic foliations of complex dimension $> 1$ are non-existent for generic $J$; they do exist if the fibers are holomorphic curves, but generically $J$ is not locally projectible along the fibers, which is though the case with our canonical foliations.

In the case $n = 2$ the sub-maximal symmetric structure is locally unique, and is the only nonintegrable $J$ that admits an infinite-dimensional transitive symmetry pseudogroup $G$. It corresponds to almost holomorphic vector bundle over a holomorphic curve. We also investigate the geometry of the canonical rank 2 Nijenhuis tensor characteristic distribution $\Pi^2 = \text{Im}(N_J)$. It is shown that in general this rank 2 vector distribution can be arbitrary, but it has to be integrable if $\dim G = \infty$. In dimension 4 the existence of an almost complex structure $J$ with $N_J \neq 0$ (so that the field of 2-planes $\Pi^2$ is non-singular) implies global topological restrictions on $M$.

In the case $n = 3$ we describe the sub-maximal symmetric structures, which are again locally unique. We discuss the almost complex structures $J$ with non-degenerate Nijenhuis tensors $N_J$ (then $J$ is called non-degenerate too). We will prove that in this case $\dim G$ is at most 14 and the maximal symmetric non-degenerate almost complex structure is locally isomorphic to the $G_2$-invariant structure $J$ on $S^6$ (Calabi almost complex structure). This uniqueness is not a corollary of any general theorem and is a new result (some technical details of the proof are delegated to appendices). We will also indicate that existence of non-degenerate structures is a global topological restriction on $M$.

The analysis for $n > 3$ is harder due to rapidly increasing number of cases, but we describe the sub-maximal symmetric structures that are no longer unique. Namely the two sub-maximal structures $J$ in dimensions 4 and 6, naturally extended to dimension $2n$, give rise to two different sub-maximal almost complex structures (with the same bulk of symmetries depending upon $(n-2)$ complex functions of 2 variables). We also discuss at the end of the paper the topological obstructions for construction of non-degenerate almost complex structures $J$. 

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The zoo of all almost complex structures with infinite pseudogroup \( G \) is too vast to be handled in general, but our results show that these cases contain some integrability features, like existence of pseudoholomorphic foliations. Thus, similar to non-degenerate situation, we also observe topological obstructions for existence of highly symmetric almost complex structures.

Summarizing, this paper makes the first steps in understanding the pseudogroup \( G \) of local symmetries (automorphisms) of almost complex structures \( J \) that lie between the two poles – integrable and generic.

1 Background: Formal integrability

Let us summarize here the basics from the geometric theory of differential equations, see [Sp, KL2] for details. We consider a vector bundle \( \pi : E \to M \) with fiber \( F \) and the space \( J^k\pi \) of \( k \)-jets of its local sections. Notice that \( J^0\pi = E \) and there are natural projections \( \pi_{k,k-1} : J^k\pi \to J^{k-1}\pi \).

A system of differential equations of order \( r \) is a subbundle \( E \subset J^r\pi \). In this paper the order of \( E \) will always be \( r = 1 \). The symbol of this equation is the family of subspaces \( g_1 \subset T^*M \otimes F \).

Regularity of \( E \) means that \( E_k \) are vector subbundles of \( J^k\pi \) with respect to the projections \( \pi_{k,k-1} \) for all \( k \). \( E \) is formally integrable (all compatibility conditions hold) if and only if \( \pi_{k,k-1} : E_k \to E_{k-1} \) are submersions.

The symbols of \( k \)-th order \( g_k = \text{Ker}(d\pi_{k,k-1} : T^*E_k \to T^*E_{k-1}) \subset S^k T^*M \otimes F \) are subspaces of the Spencer-Sternberg prolongations \( g_k \), and we have equality \( g_k = g_k \) iff \( E \) is formally integrable.

Thus purely algebraic considerations allow to bound the Hilbert polynomial

\[
P_E(k) = \sum_{i \leq k} \dim g_i = \sigma k^p + \ldots\]

The degree \( p \) is called the functional dimension and the leading coefficient \( \sigma \) is called the functional rank of the system \( E \) [KL1]. According to Cartan test [BCG3] these quantities determine the amount of initial data needed to specify formal solutions of \( E \). Namely the solutions depend on \( \sigma \) functions of \( p \) arguments (and possibly some functions with fewer arguments).

These numbers can be calculated from the complexification \( g_1^\mathbb{C} \) of the symbol as follows (we restrict to the order \( r = 1 \), see [BCG3, KL1] for the general case). Consider rank one elements \( \vartheta \otimes v \in g_1^\mathbb{C} \setminus 0 \). The set of all such \( \vartheta \) form the complex affine characteristic variety \( \text{Char}_{\mathbb{C}}(E) \subset T^*M \), and the set of all \( v \) form the

\footnote{We can change this to 'local' in the analytic and sometimes also in the smooth category.}
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kernel bundle $K$ over it. We can also calculate these through prolongations (higher symbols) as follows: $g^k \otimes v \in \mathfrak{g}_k^E \setminus 0$.

Now $p = \dim \mathbb{C} \text{Char}_{\text{aff}}^C(\mathcal{E})$ and $\sigma = \text{rank}(K) \cdot \deg \text{Char}_{\text{aff}}^C(\mathcal{E})$ provided the characteristic variety is irreducible. If the characteristic variety is reducible, then $p$ is the maximal dimension among the irreducible components, and $\sigma$ is the sum of the corresponding quantities for irreducible pieces of dimension $p$.

Example. Consider the Cauchy-Riemann equation for the holomorphic map $w : \mathbb{C}^n \to \mathbb{C}^m$, $z \mapsto w(z)$. It is given as the system $\partial z_k w^j = 0$, where $2 \partial z_k = \partial z_k + i \partial y_k$. The covector $\xi = (\xi_1^i \cdot + i \xi_1^i \cdot, \ldots, \xi_n^i \cdot + i \xi_n^i \cdot)$ is characteristic if the $2 \times 2n$ real matrix with blocks

\[
\begin{pmatrix}
\xi_k^i & \xi_k^i \\
-\xi_k^i & \xi_k^i
\end{pmatrix}
\]

has rank 1 (the actual matrix of the Cauchy-Riemann operator has size $2n \times 2n$, but its rows have repetitions). Complexification of these equations writes as $(\xi_1'' \cdot, \ldots, \xi_n'' \cdot) = \pm i(\xi_1' \cdot, \ldots, \xi_n' \cdot)$. Thus $\text{Char}_{\text{aff}}^C(\mathcal{E})$ is reducible and consists of 2 planes of dimensions $p = n$, and $K$ is a bundle of rank $n$ over each of them. Thus $\sigma = 2n$.

We conclude that the solution of the Cauchy-Riemann equation depends on $2n$ (real-valued) functions of $n$ arguments. In fact, it depends on $n$ holomorphic functions of $n$ complex arguments.

In formal calculus of dimensions developed below we can interpret parameterization of the solution space via solutions of a (formally integrable) non-linear Cauchy-Riemann equation $\partial(f) = \Phi(z, f)$ (instead of holomorphic functions); the space of such $f$ is isomorphic to the space of complex-analytic functions via the initial value problem. But for simplicity we will just mention that the solutions depend on $\frac{1}{2} \sigma$ complex functions of $p$ (complex) arguments.

2 Finiteness for almost complex automorphisms

Contrary to the global situation (when closedness, Kobayashi hyperbolicity or other conditions guarantee the group of symmetries to be small), locally $(M, J)$ can have an infinite-dimensional Lie algebra sheaf of symmetries. We begin with an effective criterion to check finite-dimensionality.

Let us endow the tangent space $T = T_x M$ with the structure of module over $\mathbb{C}$ by the rule $(\alpha + i \beta) \cdot v = \alpha \cdot v + \beta \cdot Jv$ for $\alpha, \beta \in \mathbb{R}, v \in T$. The Nijenhuis tensor is then an element of $\Lambda^2 T^* \otimes \mathbb{C} T$ (the wedge product is over $\mathbb{C}$). The image of the Nijenhuis tensor is the $\mathbb{C}$-linear subspace $\text{Im}(N_J) = N_J(T \wedge T) \subset T$.

We call the Nijenhuis tensor non-degenerate if its rank as a map $N_J : \Lambda^2 \mathbb{C} T \to T$ is maximal. This means $N_J \neq 0$ for $n = 2$ and $\text{Im}(N_J) = T$ for $n > 2$.

Theorem 1. (i) If $n > 2$ and $N_J$ is non-degenerate, then the the pseudogroup $G = \text{Sym}_{\text{loc}}(M, J)$ is finite-dimensional (and so is a Lie group).

(ii) If $N_J \neq 0$, then the pseudogroup $G$ depends on at most $n - 1$ complex functions in $n - 1$ variables.

Proof. (i) Let us consider the Lie equation on the 1-jets of infinitesimal symmetries $X \in \mathfrak{X}_M$ (space of vector fields on $M$) at various points $x \in M$.
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preserving the structure \( J \):

\[ \mathfrak{Lie}(J) = \{ [X^1]_J : L_X(J)_x = 0 \} \subset J^1(TM). \]

Its symbol is \( T^* \otimes_{\mathbb{C}} T \) and this equation is formally integrable iff \( J \) is integrable. So we consider the first prolongation-projection of this equation \( E = \pi_1(\mathfrak{Lie}(J)_1) \), which is the Lie equation for the pair \((J, N_J)\) consisting of 1-jets of vector fields preserving both tensors. The symbol of the equation \( E \) is

\[ \mathfrak{g}_1 = \{ f \in T^* \otimes_{\mathbb{C}} T : N_J(f \xi, \eta) + N_J(\xi, f \eta) = f N_J(\xi, \eta) \forall \xi, \eta \in T \}. \]

The Spencer-Sternberg prolongation \( \mathfrak{g}_1^{(1)} \) of this space equals (the symmetric tensor product \( S^k \) everywhere below is over \( \mathbb{C} \)):

\[ \mathfrak{g}_2 = \{ h \in S^2T^* \otimes_{\mathbb{C}} T : N_J(h(\xi, \eta), \zeta) + N_J(\eta, h(\xi, \zeta)) = h(\zeta, N_J(\eta, \zeta)) \}. \]

We can substitute \( J \xi \) instead of \( \xi \) in the defining relation for \( h \). Then antilinearity of \( N_J \) implies \( h(W, \cdot) = 0 \) for \( W = \text{Im}(N_J) \). If \( N_J \) is non-degenerate, this implies that \( \mathfrak{g}_2 = 0 \).

The same holds in a more general case, when \( W^\perp = \{ v \in T : N_J(v, W) = 0 \} \) is zero. Indeed, the defining relation yields

\[ N_J(h(\xi, \eta), \zeta) = N_J(h(\zeta, \eta), \xi), \quad (1) \]

so taking \( \zeta \in W \) we get \( h(\xi, \eta) \in W^\perp \).

When \( \mathfrak{g}_2 = 0 \), the system \( \mathfrak{Lie}(J) \) has finite type \[ \text{Sp} \; \text{ALV} \; \text{KL}_2 \]. Consequently \( G \) is a Lie group \[ \text{KO} \], and \( \dim G = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1 < 2(n^2 + n) \).

(ii) In the general case let us observe that the first part of the proof implies that the higher prolongations satisfy \( (k > 1) \)

\[ \mathfrak{g}_k \subset S^k \text{Ann}(W) \otimes_{\mathbb{C}} W^\perp, \]

where \( \text{Ann}(W) = \{ \varrho \in T^* : \varrho(W) = 0 \} \) is the annihilator. Indeed, for \( k = 2 \) this follows from the analysis of \( \mathfrak{g}_2 \), and the claim for \( k > 2 \) follows by prolongation.

Thus the characteristic variety of the Lie equation \( E = \mathfrak{Lie}(J, N_J) \) (and so also of \( \mathfrak{Lie}(J) \)) satisfies

\[ \text{Char}^C_{\text{aff}}(E) \subset (\text{Ann } W)_C^{1,0} \cup (\text{Ann } W)_C^{0,1} \subset C^* T^* \]

and so the maximal value of \( p \) is \( n - 1 \).

To calculate the kernel space \( K_\varrho \) over \( \varrho \in \text{Char}^C_{\text{aff}}(E) \) consider \( h = \varrho^2 \otimes v \in \mathfrak{g}_2^C, v \in K_\varrho \). Setting \( \eta \in \text{Ann}(\varrho) = \{ u \in T : \varrho(u) = 0 \} \), \( \zeta = \zeta \notin \text{Ann}(\varrho) \) into \( \Box \) we obtain: \( h(\xi, \eta) = 0 \Rightarrow N_J(v, \eta) = 0 \). Therefore

\[ K_\varrho = \{ v \in T : N_J(v, \text{Ann}(\varrho)) = 0 \} \subset W^\perp. \]

This space is \( J \)-invariant and is strictly smaller than \( T \). Indeed since \( \text{Ann}(\varrho) \) is a hypersurface, equality \( \text{Ann}(\varrho)^{\perp} = T \) would imply \( N_J = 0 \). Therefore \( \frac{1}{2} \sigma \leq (n - 1) \).

In \[ \text{KO} \] it is explained how to check that \( G \) is not only finite-dimensional, but actually finite. However the criterion is more complicated.

For \( n = 2 \) the image of the Nijenhuis tensor never spans \( T \), and a finer criterion is provided in the next section.
3 Almost complex structures in dimension 4

In dimension $2n = 4$ the induced $GL(2, \mathbb{C})$ action on the space $\Lambda^2 T^* \otimes \mathbb{C} T$ of Nijenhuis tensors has exactly two orbits: zero and its compliment. For non-zero tensor $N_J$ its image as a map $\Lambda^2 T \to T$ is a rank 2 distribution in $M^4$ ([Ko]).

**Definition 1.** $\Pi^2 = \text{Im}(N_J) \subset T$ is called the Nijenhuis tensor characteristic distribution.

Provided $J$ is generic, $\Pi^2$ is generic as well (see [3]). In particular, there is the derived rank 3 distribution $\Pi^3 = \partial \Pi^2$ formed by the brackets of sections of $\Pi$.

This leads to the invariant e-structure $\{\xi_i\}_{i=1}^4, \xi_i \in \mathcal{D}_M$, as follows:

$$\xi_1 \in C^\infty(\Pi^2), \quad \xi_3 \in C^\infty(\Pi^3), \quad N_J(\xi_1, \xi_3) = \xi_1, \quad \xi_2 = J\xi_1, \quad [\xi_1, \xi_2] = \xi_3, \quad \xi_4 = J\xi_3.$$  

These formulae define the pair $(\xi_1, \xi_2)$ up to multiplication by $\pm 1$ and the pair $(\xi_3, \xi_4)$ absolutely canonically ([Ko]).

**Theorem 2.** If $\partial \Pi^2$ is a rank 3 distribution, then the group $G$ of local symmetries is at most 4-dimensional. Moreover $G$ is finite-dimensional unless $\Pi$ is integrable and $J$ is projectible along the fibers of the corresponding foliation.

If $G$ is infinite-dimensional, then its orbits contain the leaves of $\Pi$, and $G$ is formally parametrized by 1 complex function of 1 argument.

**Proof.** It is known ([Ko]) that the automorphism group of an e-structure on a manifold $M$ is a Lie group of dimension at most $\dim M$, which is 4 in our case. The additional $\pm$ can only change the amount of connected components. Thus in the case of non-integrable $\Pi$ we get $\dim G \leq 4$.

For general distribution $\Pi$ from the proof of Theorem 1 we have

$$\mathfrak{g}_2 = \{h \in S^2 T^* \otimes \mathbb{C} T : N_J(h(\xi, \eta), \zeta) = N_J(h(\xi, \zeta), \eta), \quad h(\xi, N_J(\eta, \zeta)) = 0\}.$$  

These relations imply $h(\Pi, \cdot) = 0$ and $h(\xi, \eta) \in \Pi$. Thus $\mathfrak{g}_2 = S^2 \text{Ann}(\Pi) \otimes \mathbb{C} \Pi$ and more generally $\mathfrak{g}_k = S^k \text{Ann}(\Pi) \otimes \mathbb{C} \Pi$, $k > 1$. The characteristic variety equals

$$\text{Char}_{\text{aff}}(\mathcal{E}) = (\text{Ann} \Pi)^{1,0}_C \cup (\text{Ann} \Pi)^{0,1}_C \subset C T^*.$$

The kernel bundle over $\text{Char}_{\text{aff}}(\mathcal{E})$ is the 1-dimensional complex line $\Pi$, and this (together with the fact that the symmetries are real) implies the result on functional dimension.

Now let us note that $\nu = T/\Pi$ is a Riemannian bundle over $M$. The orthogonality is given by $J$ and the unit circle $S^1 \subset \nu_x$ ($x \in M$ is an arbitrary point) consists of vectors $v \in \nu_x$ satisfying the condition $N_J(v, \cdot) \in \text{SL}(\Pi_x)$.

Let $\mathcal{O}$ be the orbit of the pseudogroup $G$ action through $x$ and $V = T_x \mathcal{O}$ the tangent space, which is obtained by evaluation of all infinitesimal symmetry

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3 Denoting the (local) projection along fibers by $\pi$, we get that $J$ is projectible if $\pi_* J = J_0 \pi_*$ for some almost complex structure $J_0$ on the quotient. Similarly, $N_J$ is projectible if $\pi_* N_J = N_0 \pi_*$ for some $(2,1)$-tensor $N_0$ on the quotient.
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fields at \( x \). If \( V \cap \Pi = 0 \), then the pseudogroup is at most 3-dimensional as it preserves the Riemannian metric on \( O \).

Consider the case \( \dim V \cap \Pi \neq 0 \), and suppose \( G \) is infinite-dimensional. Then (as the transversal symmetries are at most 3-dimensional) we can take a vector field \( \xi \) tangent to \( V \cap \Pi \) which coincides with a symmetry at \( x \) up to the second order of smallness. Then for a vector field \( \eta \) transversal to \( \Pi \) we have at \( x \):

\[
[x, J\eta] = J[x, \eta] \mod \Pi, \quad [Jx, J\eta] = J[Jx, \eta] \mod \Pi \text{ at } x. \]

Thus \( L_X(J)(\eta) = 0 \mod \Pi \forall X \in \Pi, \eta \in \nu \), so that \( J \) is projectible along \( \Pi \).

□

**Remark 1.** If \( J \) is projectible along the leaves of \( \Pi \), then \( N_J \) is also projectible. The inverse is not true, see an example in [K3].

The theorem holds under the regularity assumptions (in a neighborhood of a regular point), when all the involved bundles have constant ranks. It does not obviously extend to singular points, because there are examples of linear equations with sheaf of solutions being finite-dimensional at every regular point, yet infinite-dimensional at some singular points [K6]. The claim is however true globally in analytic category.

### 4 Sub-maximal symmetric structures for \( n = 2 \)

Now we describe when the Lie equation \( E = \mathfrak{Lie}(J, N_J) \) from [2] is compatible (formally integrable) in the case \( M \) is 4-dimensional.

In [LS] almost holomorphic vector bundles \( \pi : (E, J) \rightarrow (\Sigma, J_0) \) were defined as such pseudoholomorphic bundles that the fiber-wise addition \( E \times \Sigma E \rightarrow E \) is a pseudoholomorphic map. In [K4] we gave a constructive version and related it to the theory of normal pseudoholomorphic bundles.

**Theorem 3.** Suppose \( J \) is non-integrable everywhere, i.e. \( N_J \mid x \neq 0 \forall x \in M \). Then \( G = \text{Sym}_\text{loc}(M, J) \) is infinite-dimensional iff \( J \) is (locally) isomorphic to an almost holomorphic vector bundle structure \( (M^4, J) \rightarrow (\Sigma^2, J_0) \) over a Riemannian surface.

If \( G \), in addition, acts transitively, then in local coordinates we have the normal form

\[
J\partial_z = i \partial_z + w \partial_w, \quad J\partial_w = i \partial_w. \]

**Proof.** Let \( \dim G = \infty \). From the necessary conditions of Theorem [2] we know that projection along the leaves of the integrable distribution \( \Pi \) gives us locally the pseudoholomorphic map \( \pi : (M, J) \rightarrow (\Sigma, J_0) \). Choosing a complex coordinate \( z \) for the complex structure \( J_0 \) on the holomorphic curve \( \Sigma \) and a complex coordinate \( w \) along the fibers of \( \pi \) we get the local description of \( J \)

\[
J\partial_z = i \partial_z + a \partial_w + b \partial_\bar{w}, \quad J\partial_w = i \partial_w, \tag{2}
\]

(action on \( \partial_\bar{w}, \partial_w \) is obtained by conjugation due to reality of \( J \); cf. the real version of the above description in [K3]). The condition \( J^2 \partial_z = -\partial_z \) yields...
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\( a = 0 \). Then we calculate \( N_J(\partial_z, \partial_w) = -2i b_w \partial_{\bar{w}} \), whence \( \Pi = \text{Re}(C \cdot \partial_w) \) according to our setup and \( b_w \neq 0 \) by the assumption.

The equation for pseudoholomorphic curves in \( M \) transversal to \( \Pi \) writes

\[
\bar{w}z = \psi, \quad \text{where} \quad \psi = \frac{1}{2} i \bar{b},
\]

which we couple with the conjugate into \( \mathcal{E} = \{ \bar{w}z = \psi, \bar{w}z = \bar{\psi} \} \subset J^1(\Sigma, M)_\mathbb{C} \) (the latter is the complexified space of jets of \( \pi \)). The almost complex structure \( J \) on \( M \) reads off (up to \( \pm \)) from the collection of all these \( J \)-holomorphic curves, so the symmetries of \( J \) coincide with the (real) symmetries of the equation \( \mathcal{E} \).

Passing to vector fields, the Lie algebra sheaf of \( G \) corresponds to the space of infinitesimal point symmetries of \( \mathcal{E} \), which we search by the method of S.Lie, see [KL2] and references therein. Since we know that most of the symmetries (\( \infty \) with the possible exception of 3) are tangent to \( \Pi \) (fibers of \( \pi \)), we write the point symmetry as

\[
\xi = f \partial_w + \bar{f} \partial_{\bar{w}} + (\ldots) \partial_{w_z} + (\ldots) \partial_{\bar{w}_z} + (f \bar{w} + w f_\bar{w} + \bar{w} z f_w) \partial_{w_z} + (\ldots) \partial_{\bar{w}_z}
\]

acts as follows (the second equation of \( \mathcal{E} \) gives the conjugate relation)

\[
\hat{\xi}(\bar{w}z - \psi)|_{\mathcal{E}} = f \bar{w} + \psi f_w + \bar{w} z f_w - f \bar{w} - \bar{f} \psi_{\bar{w}} = 0.
\]

Since \( \bar{w}z \) is arbitrary, this obviously decouples into

\[
f_{\bar{w}} = 0 \quad \text{and} \quad f \bar{w} + \psi f_w = f \bar{w} + \bar{f} \psi_{\bar{w}}.
\]

We will investigate this overdetermined system \( \mathcal{S}_J \) by uncovering the compatibility conditions. Differentiating (5) by \( \bar{w} \) yields

\[
f_w - f = (\log |\psi_{\bar{w}}|)w f + (\log |\psi_{\bar{w}}|)_{\bar{w}} \bar{f}.
\]

Taking the sum of (9) and its conjugate results in

\[
(\log |\psi_{\bar{w}}|^2)_w f + (\log |\psi_{\bar{w}}|^2)_{\bar{w}} \bar{f} = 0.
\]

If this relation is nontrivial, then \( f = h \cdot f_0 \), where \( f_0 \) is a fixed complex function and \( h \) a real indeterminate. Substitution of this into \( \mathcal{S}_J \) gives a finite type system, so the symmetry algebra is finite-dimensional.

Thus we must assume that both coefficients of (7) vanish, so that \( |\psi_{\bar{w}}| = k = \text{const by} \ w \). We write \( \psi_{\bar{w}} = k e^{\theta} \), where \( \theta \) is a real variable (mod \( 2\pi \)). Differentiation of (4) by \( w \) and by \( \bar{z} \) gives:

\[
f_{ww} = (\log \psi_{\bar{w}})_w f_w + (\log \psi_{\bar{w}})_{ww} f + (\log \psi_{\bar{w}})_{w\bar{w}} \bar{f}.
\]

\[
(\log \psi_w)_{w\bar{w}} f_{\bar{z}} = \Psi(f, \bar{f}, f_w, \bar{f}_w),
\]

where
where $\Psi$ is some linear function in the indicated arguments with coefficients depending differentially on $\psi$. Differential corollaries \[8\] and \[9\] together with the original equations \[1\] - \[5\] form a finite type system unless

\[(\log \psi_w)_{w\bar{w}} = 0 \iff (\log \psi_{\bar{w}})_{w\bar{w}} = 0. \tag{10}\]

So we obtain another restriction on $\psi$: $\theta$ is harmonic.

Now the choice of complex coordinate $w$ in the leaves of $\Pi$ was arbitrary, and we wish to normalize it. We can suppose that the infinite pseudogroup $G$ contains a fiber-wise symmetry with no stationary points. Then we can fix it to be translation in each leaf, i.e. the generating vector field is $q\partial_w + q\bar{\partial}_{\bar{w}}$, where $q = q(z, \bar{z})$. In other words we can assume (locally) that $S_f$ has a nonzero solution $f$ with $f_w = 0$. Then derivatives of \[8\] imply that the system of vectors

\[(\theta_w, \theta_{\bar{w}}), \quad (\theta_{w\bar{w}}, 0) \quad (0, \theta_{\bar{w}w})\]

has rank $\leq 1$. Together with \[10\] this implies $\theta = \theta_0 + \alpha w + \bar{\alpha} \bar{w}$ with real $\theta_0$ and complex $\alpha$ functions of $z, \bar{z}$. Moreover \[8\] yields $\psi = \varphi_0(z, \bar{z}) + \varphi_1(z, \bar{z})w + \Phi_{-1}$, where $\Phi_{-1} = \varphi_{-1}(z, \bar{z})\bar{w}$ for $\alpha = 0$ and $\Phi_{-1} = \frac{1}{\theta} e^{i\theta}$ else.

Let us consider the case $\alpha = 0$ first. Substitution of the expression for $\psi$ into \[8\] and \[9\] implies that $f_w$ is real and constant by $w$, i.e. we have: $f = f_0(z, \bar{z}) + f_1(z, \bar{z})w$ ($f_1 \in \mathbb{R}$). This together with $w$-derivative of \[10\] give us $f_1\bar{w} = 0$, i.e. $f_1 = c$ is a real constant.

Since the first term in $\psi = \varphi_0(z, \bar{z}) + \varphi_1(z, \bar{z})w + \varphi_{-1}(z, \bar{z})\bar{w}$ can be eliminated by a shift in $w$, $J$ is an almost pseudoholomorphic vector bundle structure. The general symmetry $f = f_0(z, \bar{z}) + cw$ is a combination of the scaling symmetry $Re(w\partial_w)$ and the shift by solution $Re(f_0(z, \bar{z})\partial_w)$, where $f_0$ satisfies the linear (nonhomogeneous) Cauchy-Riemann equation

\[w_z = -c \varphi_0(z, \bar{z}) + \varphi_1(z, \bar{z})w + \varphi_{-1}(z, \bar{z})\bar{w}.\]

The solutions of the latter are parametrized by 1 function in 1 argument.

Via shift by a solution and variation of constants the linear Cauchy-Riemann equation takes the normal form

\[w_z = \lambda(z, \bar{z})\bar{w}, \tag{11}\]

where $\lambda \neq 0$ corresponds to $N_f \neq 0$. The gauge changes preserving this type of equation are $z \mapsto Z(z)$, $w \mapsto U(z)w$ (functions $Z, U$ are holomorphic), and so $\Lambda = (\log \lambda)z d\bar{w} + d\bar{z} zd\bar{w}$ is an invariant quadratic form.

Consider now the case $\alpha \neq 0$. Here substitution of $\psi = \varphi_0(z, \bar{z}) + \varphi_1(z, \bar{z})w + \frac{k}{\theta} e^{i\theta}$ in \[9\] gives $f_w = i\alpha f + r$, where $r = r(z, \bar{z})$ is a real function. This implies $f = \frac{w}{\alpha} + ce^{i\omega w}$, where $c = c(z, \bar{z})$ is a complex function. Substituting both this and the expression for $\psi$ into \[10\] yields

\[(\frac{\bar{w}}{\alpha})_z + (c\bar{z} + ic\alpha\bar{z}w) e^{i\omega w} + i\alpha(\varphi_0 + \varphi_1 w) c e^{i\omega w} = \frac{w}{\alpha} \varphi_1 + c\varphi_1 e^{i\omega w} + k e^{i\theta_0 + i\omega w}.\]

---

\[\text{A general fiber-wise diffeomorphism preserves the symmetry algebra but changes the coordinate form of the Cauchy-Riemann equation} \[8\]: \emph{linear to nonlinear etc. An alternative way to the normalization is to find all restrictions on $\psi$ and then to check linearization of the Cauchy-Riemann equation.}
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Splitting this equation according to $w$-parts we get $\varphi_1 = -\frac{\alpha}{r}, \ r_z = 0$ (so that $r$ is a real constant) and a Cauchy-Riemann type equation on $c$.

Let us change the variable $w \mapsto \sigma e^{-i\alpha w}$, where $\sigma$ satisfies $(\log \sigma)\bar{z} = i\alpha \varphi_0$. Then direct calculation shows that the equation $w\bar{z} = \psi$ is transformed into the equation of type (11), so we can proceed as in the case $\alpha = 0$.

Notice that in addition to $\Lambda$ there is another canonical quadric (Riemannian metric) $Q = |\lambda|^4 dz d\bar{z}$ on $\Sigma = M/\Pi \simeq \mathbb{C}(z)$, discussed in the proof of Theorem 2 ($T\Sigma = \nu$). The condition that the symmetry group $G$ acts transitively implies that the space of (local) Killing fields for this Riemannian metric is 3-dimensional, and hence it is of constant curvature. This implies that $|\lambda/\lambda_0|^2 = (1 + \epsilon z\bar{z})^{-1}$ for real constants $\epsilon, \lambda_0$, whence $\Lambda = \frac{\epsilon^2}{2|\lambda_0|^4} Q$.

However for $\epsilon \neq 0$ the motion group of $Q$ does not act by symmetries on $J$ (direct calculation shows that 2 out of 3 Killing fields fail to do so), and so to achieve transitivity of $G$-action we must assume $\epsilon = 0$.

In this case $\Lambda = 0$, i.e. we can get $\lambda = \text{const}$ by a gauge transformation in (11). Thus we obtain the biggest sub-maximal symmetric model, i.e. the model with maximal symmetry group among all non-integrable structures $J$: the symmetry group $G$ is a semi-direct product of the 3-dimensional group of motions $U(1) \ltimes \mathbb{C}$ of the plane $\mathbb{C}(z)$, the scaling of the $w$-variable and the infinite-dimensional group of shifts by solutions $w \mapsto w + f_0$ discussed above. □

5 Nijenhuis tensor characteristic distribution

It was shown in [K3] that rank 2 distributions $\Pi^2$ in $\mathbb{R}^4$ of fixed type (Engel, quasi-contact, Frobenius) are realizable as the Nijenhuis tensor characteristic distribution. Moreover if $\Pi^2$ is analytic, it is realizable as well: $\Pi = \text{Im}(N_J)$.

In this section we remove the assumption of analyticity.

Lemma 1. An almost complex structure in $M^4$ has the following normal form in local coordinates $(z, w)$ with some smooth complex-valued functions $\alpha, \beta$.

$$J\partial_z = i\sqrt{1 + |\alpha|^2} \partial_z + \alpha \partial_{\bar{z}} + \frac{i\alpha \beta}{1 + \sqrt{1 + |\alpha|^2}} \partial_w + \beta \partial_{\bar{w}}, \quad J\partial_w = i \partial_{\bar{w}}.$$

Proof. We can foliate a neighborhood $U \subset (M^4, J)$ by $J$-holomorphic discs $U = \cup_r B_r$ (see [NW, M, AL]) and choose a transversal pseudoholomorphic disk $D \subset U$. Let’s introduce a complex $J$-holomorphic coordinate $z$ on $D$ and pull-back it to $U$ using the projection along the foliation $B_r$. On every leaf $B_r$ we introduce a complex $J$-holomorphic coordinate $w$. This local coordinate system $(z, w)$ on $U \simeq \mathbb{C}^2$ is not $J$-holomorphic; the almost complex structure $J$ writes in it:

$$J\partial_z = a \partial_z + \alpha \partial_{\bar{z}} + b \partial_w + \beta \partial_{\bar{w}}, \quad J\partial_w = i \partial_{\bar{w}}.$$  

The maximal symmetric model is obviously $\mathbb{C}^n$ in any dimension $n$. 

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The condition $J^2 = -1$ is equivalent to

$$(a + \bar{a})\alpha = 0, \ a^2 + |\alpha|^2 = -1, \ (a + i)b + \alpha\bar{\beta} = 0, \ \bar{\alpha}b + (\bar{a} + i)\bar{\beta} = 0.$$ 

If $\alpha = 0$ then $a = \pm i$, so we can take the general solution $a = ik, \ k \in \mathbb{R}$, $k^2 = |\alpha|^2 + 1$. Without loss of generality we can assume $k \geq 1$ (because $a \approx i$ near $D$), then the last equation can be omitted and we compute $b$. 

**Remark 2.** The choice $\alpha = 0$ corresponds to the normal form (3) of $J$ with integrable Nijenhuis tensor characteristic distribution. Another normal form in complex dimension $n = 2$ is considered in [10].

**Lemma 2.** The Nijenhuis tensor characteristic distribution is $\mathbb{C}$-generated by the vector $v = \Xi_\perp \cdot \xi$, where

$$\xi = \partial_z - i \frac{1 + \sqrt{1 + |\alpha|^2}}{\bar{\alpha}} \partial_z + \frac{\bar{\beta}}{\alpha} \partial_w + \left( \frac{i\beta}{1 + \sqrt{1 + |\alpha|^2}} \frac{\Xi_-}{\Xi_+} - 2i\beta \frac{\partial_w}{\Xi_+} \right) \partial_w$$

and

$$\Xi_\pm = \frac{\alpha w \bar{\alpha} + \alpha \bar{\alpha} w}{2\sqrt{1 + |\alpha|^2}} \pm \frac{\alpha w \bar{\alpha} - \alpha \bar{\alpha} w}{2}.$$ 

At the points where $\alpha = 0$ we have

$$v = -i\alpha_w \partial_z + \bar{\beta} \alpha_w \partial_w - 2i\beta_w \partial_{\bar{w}}.$$ 

**Proof.** This is a straightforward calculation. Notice that $\Xi_+ \neq 0, \ \alpha \neq 0$ is a sufficient condition for $\Pi = \text{Im}(N_J)$ to be a non-singular distribution. \hfill $\square$

Notice that the coordinate system $(z, w)$ from Lemma 1 is very special, for instance the absolute value of the coefficient of $\partial_z$ in $\xi$ is bigger than that of $\partial_z$.

**Theorem 4.** Locally any smooth rank 2 distribution $\Pi$ is the Nijenhuis tensor characteristic distribution: $\Pi = \text{Im}(N_J)$. 

**Proof.** A rank 2 distribution can be written in the coordinate system $(z, w)$ as

$$\Pi^C = (\partial_z - A\partial_w - B\partial_{\bar{w}}, \partial_z - \bar{B}\partial_w - \bar{A}\partial_{\bar{w}})$$

(to obtain real $\Pi$ one has to take the real and imaginary parts of the first vector).

In the notations of Lemma 2 we have:

$$\xi \mod \Pi = \left( A - i \frac{1 + \sqrt{1 + |\alpha|^2}}{\bar{\alpha}} B + \frac{\bar{\beta}}{\alpha} \right) \partial_w$$

$$+ \left( B - i \frac{1 + \sqrt{1 + |\alpha|^2}}{\bar{\alpha}} A + \frac{i\beta}{1 + \sqrt{1 + |\alpha|^2}} \frac{\Xi_-}{\Xi_+} - 2i\beta \frac{\partial_w}{\Xi_+} \right) \partial_{\bar{w}}$$

Vanishing of the $\partial_w$-coefficient yeilds

$$\beta = -\alpha A - i(1 + \sqrt{1 + |\alpha|^2}) B$$ \hspace{1cm} (12)
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(this is also obtainable from the condition of invariance of $\Pi$ with respect to the almost complex structure $J$ from Lemma 1).

Substituting this into the $\partial \bar{w}$-coefficient of $\xi \mod \Pi$ we observe that all derivatives of $\alpha$ cancel and the resulting relation is

$$i\alpha \bar{A}_w - (1 + \sqrt{1 + |\alpha|^2})B_w = 0.$$  

This is easily solvable provided $|A_w| > |B_w|:

$$\alpha = \frac{-2iA_wB_w}{|A_w|^2 - |B_w|^2}. \quad (13)$$

Now we can construct $J$. The coordinate system $(z, w)$ is at our hands. We can choose it so that the above inequality is satisfied (even integrable $\Pi$ can be written with non-constant $A, B$). We calculate using (13)

$$\Xi_+ = \frac{4A_w\bar{B}_w(\bar{A}_wB_{ww} - \bar{A}_{ww}B_w)}{|A_w|^2 - |B_w|^2}.$$  

Thus the condition $\Xi_+ \neq 0$ translates into the inequalities $|A_w| > |B_w| > 0$ and $\bar{A}_wB_{ww} \neq \bar{A}_{ww}B_w$. We adapt these into the choice of $(z, w)$.

Then we define the complex-valued functions $\alpha, \beta$ by (13)-(12) and the almost complex structure $J$ from Lemma 1 realizes $\Pi$ as the Nijenhuis tensor characteristic distribution. \hfill $\Box$

Remark 3. It is surprising that the construction is purely algebraic. This is due to the choice of special coordinates. In [K3] the problem was solved via a PDE system in Cauchy-Kovalevskaya form, that is why analyticity was assumed.

6 Global obstructions for non-degeneracy

Non-degeneracy of the Nijenhuis tensor is a generic condition locally, but globally on a closed 4-manifold it is a topological constraint.

Definition 2. An almost complex structure $J$ on a 4-dimensional manifold $M$ is called non-degenerate if $N_J \neq 0$ everywhere.

There are obstructions to existence of non-degenerate $N_J$, some depending on the homotopy class of the almost complex structure $J$ and some purely topological. Indeed, the Nijenhuis tensor determines an anti-linear isomorphism of the complex vector bundles

$$N_J : \Lambda^2 T \to \Pi \subset T.$$  

This implies $c_1(\Pi) = -c_1(\Lambda^2 T) = -c_1(M)$. Denoting by $\Pi^\perp$ the normal vector bundle equipped with the natural complex structure we have: $c_1(M) = c_1(\Pi) + c_1(\Pi^\perp)$, $c_2(M) = c_1(\Pi)c_1(\Pi^\perp)$, whence

$$c_2(M) + 2c_1(M)^2 = 0.$$
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Since the Euler characteristic is $\chi = c_2(M) [M]$, the signature is $\tau = \frac{1}{2} p_1(M) [M]$, with $p_1(M) = c_1^2(M) - 2c_2(M)$ (implying the classical Wu’s criterion for almost complex structures in dimension four: $2\chi + 3\tau = c_1^2$; see [De]), and $c_1(M)^2 = KK$ is the self-intersection of the canonical class $K$ of $M$, we get the main topological restrictions for existence of non-degenerate almost complex structure on $M$:

$$5\chi + 6\tau = 0, \quad \chi = -2 KK.$$

**Remark 4.** Reference [HI] relates the existence of an almost complex structure on $M^4$ to the existence of a rank 2 distribution. We see that in the case these two are compatible (through $N_J$) we get stronger restrictions on $M$.

In addition there are divisibility constraints [H]. Indeed by Max Noether theorem $\chi + \tau \equiv 0 \mod 4$ (equivalently $c_1^2 + c_2 \equiv 0 \mod 12$), so we must have

$$\chi \equiv 0 \mod 24.$$

These restrictions applied to a simply connected closed 4-manifold yield $b_- = 10 + 11b_+$ and $b_+ \equiv 1 \mod 2$, where $b_+$ and $b_-$ are the dimensions of the maximal positive and negative definite subspaces of $H^2(M)$. For instance (the same for any type I manifold), if

$$\#_r \mathbb{C}P^2 \#_s \overline{\mathbb{C}P^2}$$

possesses a non-degenerate almost complex structure, then $r = 2k + 1$, $s = 22k + 21$, $k \in \mathbb{Z}$ (notice that this manifold possesses an almost complex structure iff $r$ is odd). In particular, the projective plane blown-up at $\leq 20$ points does not possess a non-degenerate almost complex structure.

For type II manifolds with the intersection form $m \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + n E_8$, the above relations imply $n = \frac{4}{3}(m + 1) > 0$, $m \equiv 3 \mod 4$.

In particular, there exists no non-degenerate almost complex structures on K3 surfaces (then $5\chi + 6\tau = 24$), on Enriques surfaces ($\chi = 12$ not divisible by 24) and on complex surfaces of Kodaira dimension 2 (surfaces of general type, then both $\chi$ and $K\cdot K > 0$).

**Remark 5.** In other words, for any almost complex structure $J$ on any of these manifolds there will exist points $x$ with $N_J|_x = 0$.

There are however closed 4-manifolds with non-degenerate almost complex structures, for instance Abelian surfaces. Indeed on the torus $T^4 = \mathbb{C}^2(z, w)/\mathbb{Z}^4$ with almost complex structure given by

$$J\partial_z = i \partial_z + e^{2\pi i \text{Re}(w)}\partial_w, \quad J\partial_w = i \partial_w$$

the Nijenhuis tensor nowhere vanishes.

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7 Almost complex structures in dimension 6

For $n = 3$ non-degeneracy of the Nijenhuis tensor means $\mathbb{C}$-antilinear isomorphism
\[ N_J : \Lambda^2T \to T. \] (14)
Nijenhuis tensors were classified in [K3]. There are 4 non-degenerate types:

- **NDG(1)** $N(X_1, X_2) = X_2, N(X_1, X_3) = \lambda X_3, N(X_2, X_3) = e^{i\varphi} X_1,$
- **NDG(2)** $N(X_1, X_2) = X_2, N(X_1, X_3) = X_3 + X_2, N(X_2, X_3) = e^{i\varphi} X_1,$
- **NDG(3)** $N(X_1, X_2) = e^{-i\varphi} X_3, N(X_1, X_3) = -e^{i\varphi} X_2, N(X_2, X_3) = e^{i\varphi} X_1,$
- **NDG(4)** $N(X_1, X_2) = X_1, N(X_1, X_3) = X_2, N(X_2, X_3) = X_2 + X_3.$

Parameters $\lambda, \varphi, \psi$ are real. For non-exceptional values of the parameters ($\lambda \neq \pm 1, -\lambda \neq e^{\pm 2i\varphi}, e^{\pm 2i\varphi} \neq \pm 1, e^{\pm 2i\varphi} \neq \pm 1$ and $\psi \pm 2\varphi \neq \pi Z$) there is precisely one fixed point $\Pi$ of the composition of maps $\Phi_2 : \text{Gr}_2^C(3) \to \mathbb{C}P^2$, $\Pi \mapsto N_J(\Lambda^2\Pi)$ and $\Phi_1 : \mathbb{C}P^2 \to \text{Gr}_2^C(3)$, $L \mapsto \text{Im}(N_J(L, :))$ that satisfies $N_J(\Lambda^2\Pi) \cap \Pi = 0$ in cases NDG(1-2), three such $\Pi$ in case NDG(3) and exactly one $\Pi$ with $N_J(\Lambda^2\Pi) \subset \Pi$ in case NDG(4).

Isomorphism (14) imposes global obstructions for existence of a non-degenerate almost complex structure $J$. Indeed, the total Chern class $c(T)$ is related to $c(\Lambda^2T) = 1 + (2c_1(T)) + (c_2(T) + c_1(T)^2) + (-c_3(T) + c_1(T)c_2(T))$ through (14), so (these formulae were also obtained in [B])
\[ 3c_1(J) = 0, c_1(J)^2 = 0, c_1(J)c_2(J) = 0. \] (15)

We write $c_i(J)$ instead of $c_i(T) = c_i(M)$ to stress that, contrary to 4-dimensional situation, the Chern numbers, like $c_1c_2$, obtained through these characteristic classes depend on the homotopy class of $J$ (and not only on the diffeomorphism type of $M$). The obstructions for existence of almost complex structure in dimension 6 [H, Ge] are $c_1c_2 \equiv 0 \mod 24$, $c_1^2 \equiv c_3 \equiv 0 \mod 2$, and only the last requirement $\chi(M) \in 2\mathbb{Z}$ does not follow from (15).

For example, there is no non-degenerate $J$ in the homotopy class of the standard complex structure of
\[ \mathbb{C}P^3 \# \mathbb{C}P^3 \# \ldots \# \mathbb{C}P^3. \]

The canonical $G_2$-invariant almost complex structure $J$ on $S^6$ is non-degenerate and it corresponds to NDG(3) $\varphi = \psi = 0$. But if we blow up $S^6$, then the resulting $S^6 \# \mathbb{C}P^3 \simeq \mathbb{C}P^3$ (orientation forgetting diffeomorphism) has no non-degenerate $J$ in the respective homotopy class.

More generally, according to [Th] the almost complex structures on $\mathbb{C}P^3$ are bijective with the total Chern classes
\[ 1 + 2rx + 2(r^2 - 1)x^2 + 4x^3, \]
\(^6\)The third type is obtained from the equation in [K2] by changing the basis $(X_2, X_3) \mapsto (X_2 \pm JX_3)$ and complex scaling.

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where $x$ is the standard generator of $H^2(\mathbb{C}P^3)$ and $r \in \mathbb{Z}$ ($r = 2$ corresponds to the standard complex structure and $r = -1$ corresponds to the blown-up $S^6$). There are no non-degenerate $J$ unless $r = 0$ (where we don’t know).

If we impose a stronger requirement of $N_J$ being non-exceptional of type NDG(1-2), then $N_J : \Lambda^2\Pi \simeq L$, $T = L \oplus \Pi$ imply $c_1(J) = 0$, i.e. the canonical class vanishes $K = 0$. Moreover, $c_3(J) = \lambda(c_2(J) + \lambda^2)$ for an integer cohomology class $\lambda \in H^2(M)$ ($\lambda = -c_1(\Pi)$).

For non-exceptional NDG(3) we also have $K = 0$ and in addition $-c_2(J) = \lambda^2 + \lambda \mu + \mu^2$, $-c_3(J) = \lambda \mu(\lambda + \mu)$ for some integer classes $\lambda, \mu \in H^2(M)$ (these conditions imply the above condition for the type NDG(1-2), but not otherwise).

If $N_J$ is of type NDG(4), then $c_2(J) = \lambda^2 + c_1(J)\lambda$, $c_3(J) = -2\lambda^3$ for some integer cohomology class $\lambda \in H^2(M)$ ($\lambda = -c_1(\Pi)$).

For instance $\mathbb{C}P^3$ contains neither non-exceptional type NDG(3) nor non-exceptional type NDG(4) structures; the sphere $S^6$ has no non-exceptional almost complex structure (of any type).

### 8 Nondegenerate symmetric structures for $n = 3$

By Theorem 1 an almost complex structure on a 6-dimensional manifold has a finite-dimensional symmetry algebra provided $N_J$ is non-degenerate.

An alternative approach to prove this is via the Tanaka theory. Namely, the complexified tangent bundle $\mathcal{C}T$ has the canonical subbundle $V = T^{1,0}$ with $\bar{V} = T^{0,1}$. Non-degeneracy of $N_J$ is equivalent to step-2 bracket-generating property: $[V, V] = \mathcal{C}T$. Thus the growth of the distribution is $(3, 6)$ and consequently the symmetry group is finite-dimensional by [14, K].

The Tanaka prolongation of the corresponding graded nilpotent Lie algebra $m = g_{-2} \oplus g_{-1}$ (with $g_{-1} = V$ and $g_{-2} = \mathcal{C}T/V$) is equal to

$$g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 = \Lambda^2V \oplus V \oplus gl(V) \oplus V^* \oplus \Lambda^2V^*.$$

This is the 21-dimensional Lie algebra $so(3, 4)$, but it is not the most symmetric case for non-degenerate almost complex structures. Indeed, we have not used the natural isomorphism $N_J : \Lambda^2V \simeq V$, which reduces the algebra of derivations of $m$ to at most 14 dimensions:

**Theorem 5.** For non-degenerate $J$ the symmetry group satisfies $\dim G \leq 14$. The equality is attained only for the exceptional Lie group $G_2$.

Note that here we treat only the germ of the structure $J$, so the conclusion concerns the local group (equivalently, its Lie algebra).

**Proof.** $g_2 = 0$ for a non-degenerate $J$ and so $\dim G \leq \dim \mathcal{O} + \dim g_1$, where $\mathcal{O}$ is the orbit of $G$ of dimension $\leq \dim M = 6$. We claim that $\dim g_1 \leq 8$.

---

7 The standard theory concerns real distributions in the tangent bundle but can be generalized to real distributions in the complexified tangent bundle. This is however a formal calculus. For real symmetries the Lie algebra of the compact $G_2$ (that act on $S^6$) lies in $so(7)$.
This is based on the case by case analysis of the normal forms from \[\text{(7)}\]
Consider for instance NDG(1). For non-exceptional parameters there exists a unique \(\mathbb{C}\)-line \(L = \langle X_1, JX_1 \rangle \subset T\) that is invariant with respect to \(\Phi_2 \Phi_1\) and satisfies \(T = L \oplus \Pi, \Pi = \Phi_1(L)\). An endomorphism \(f \in \mathfrak{g}_1\) preserves this splitting and is an arbitrary complex map on \(L\), while it is given by at most 4 real parameters on \(\Pi\). Thus in this case \(\dim \mathfrak{g}_1 \leq 6\).

In some exceptional cases the splitting is also invariant with \(L = \langle X_1 \rangle\) distinguished by the condition that \(\Phi_1(L) = \langle X_2, X_3 \rangle\) is generated by fixed points \(\tilde{L}\) of \(\Phi_2 \Phi_1\) satisfying \(\Phi_1(\tilde{L}) \supset \tilde{L}\). Then the conclusion is the same. But for \(e^{2i\varphi} = -\lambda = \pm 1\) all complex lines \(L \in \mathbb{C}P^2\) are fixed points of \(\Phi_2 \Phi_1\) (those satisfying the condition \(\Phi_1(L) \supset L\) form a nondegenerate quadratic cone).

Here the group \(\mathfrak{g}_1\) acts transitively on \(T\) by complex transformations preserving
\[
N(X_1, X_2) = X_2, \quad N(X_1, X_3) = -X_3, \quad N(X_2, X_3) = X_1, \quad \text{(16)}
\]
This tensor is isomorphic to \[\text{(17)}\] and so will be discussed below.

Similarly, in the cases NDG(2), NDG(4) we get \(\dim \mathfrak{g}_1 \leq 2\), because \(\langle X_1, X_2 \rangle\) is \(G\)-invariant etc. For NDG(3) we have \(\dim \mathfrak{g}_1 \leq 8\), and the equality is strict except for two cases:
\[
N(X_1, X_2) = X_3, \quad N(X_1, X_3) = X_2, \quad N(X_2, X_3) = X_1 \quad \text{(17)}
\]
and
\[
N(X_1, X_2) = X_3, \quad N(X_3, X_1) = X_2, \quad N(X_2, X_3) = X_1, \quad \text{(18)}
\]
This last case is the (anti-)complexification of the usual vector product in \(\mathbb{R}^3\), and it is realized by the \(G_2\)-invariant almost complex structure \(J\) on \(S^6\).

Case (17), equivalent to (16), is another candidate for \(\dim G = 14\), but it is not realizable. The proof uses completely different technique (representation theory) and so is delegated to the Appendices.

It follows that an almost complex structure \(J\) with the transitive automorphism group, \(\dim \mathfrak{g}_0 = 6\), and having the stabilizer of \(\dim \mathfrak{g}_1 = 8\) must be locally isomorphic to the Calabi structure on \(S^6\) (coming from the octonionic vector product on \(\mathbb{R}^7\)) corresponding to (18), see the Appendices for more details. □

\textbf{Corollary 1.} \textit{Calabi octonionic structure} \(J\) on \(S^6\) \textit{is the most symmetric non-degenerate almost complex structure in dimension 6.}

\section{Sub-maximal symmetric structures for \(n = 3\)}

Now consider degenerate tensors \(N_f\). By Theorem 1 in the case \(\dim_{\mathbb{C}} \text{Im}(N_f) = 2\) (denoted \(\text{DG}_1\) in [K2]) the symmetry group \(G\) depends on \(\leq 2\) complex functions of 1 (complex) argument.

---

8The symmetry group is the real compact form of the exceptional Lie group \(G_2\). The split real form of \(G_2\) is the largest symmetry group of nondegenerate rank 2 distributions in \(\mathbb{R}^5\), with the corresponding maximal model being the Hilbert-Cartan equation (E.Cartan, 1910). Informally, the ”mirror” of the Calabi structure \(J\) in 6D is the Cartan 2-distribution in 5D.
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More symmetries (complex functions of 2 argument) can occur in the case \( \dim \mathfrak{g}_2 \mathfrak{g} \mathfrak{t} \mathfrak{m} = 1 \), which has two normal forms:

\[
\begin{align*}
\text{DG}_2(1) & \quad N(X_1, X_2) = X_1, N(X_1, X_3) = 0, N(X_2, X_3) = 0, \\
\text{DG}_2(2) & \quad N(X_1, X_2) = X_3, N(X_1, X_3) = 0, N(X_2, X_3) = 0.
\end{align*}
\]

In the first case \( \text{Char}^C_{\mathfrak{g}_2 \mathfrak{a} \mathfrak{f} \mathfrak{f}} = (\text{Ann}(X_1))^{1,0} \cup (\text{Ann}(X_1))^{0,1} \). Indeed for any \( h \in \mathfrak{g}_2 \) we have \( h(X_1, \cdot) = 0 \). Moreover the condition \( N_J(h(\xi, \eta), \theta) = N_J(h(\xi, \theta), \eta) \) gives \( h(\cdot, \cdot) \in \langle X_1, X_3 \rangle^C \) (via substitution \( \eta = X_1 \)) and \( h(X_1, \cdot) \in \langle X_3 \rangle^C \) (via substitution \( \eta = X_3 \)). There are no more relations, and we get \( \mathfrak{g}_2 = S^2 \text{Ann}(\langle X_1 \rangle^C) \otimes \langle X_3 \rangle^C + S^2 \text{Ann}(\langle X_1, X_3 \rangle^C) \otimes \langle X_1 \rangle^C \).

Now we can calculate the characteristic variety through \( \mathfrak{g}_2 \) as discussed in \([4]\). The kernel bundle \( K \) over \( \text{Char}^C_{\mathfrak{g}_2 \mathfrak{a} \mathfrak{f} \mathfrak{f}} \) has complex dimension 1 on a Zariski open set \( \text{Ann}(\langle X_1 \rangle^C) \setminus \text{Ann}(\langle X_1, X_3 \rangle^C) \), and so formally the general symmetry depends on at most 1 complex function of 2 arguments.

In order for this to be realized the two \( G \)-invariant distributions \( \text{Im}(N_J) = \langle X_1 \rangle^C \) and \( \text{Ker}(N_J) = \langle X_3 \rangle^C \) have to be integrable (otherwise their curvature tensor is another defining relation in \( G \), see \([4]\), and also their sum \( \langle X_1, X_3 \rangle^C \) shall be a holomorphic foliation of \( \dim \mathfrak{g}_2 = 2 \). Furthermore the biggest (in the sense of dimension theory) symmetry group \( G \) corresponds to pseudoholomorphic projection along the foliation \( \langle X_3 \rangle^C \) onto a 4-manifold, which has to be the sub-maximal symmetric model \( M^4 \) for \( n = 2 \).

Finally the most symmetric model of type \( \text{DG}_2(1) \) is \( M^6 = M^4(z, w) \times \mathbb{C}(\zeta) \). Provided \( G \) acts transitively, the almost complex structure \( J \) in local coordinates \( z, w, \zeta \) (corresponding to the numeration \( X_2, X_1, X_3 \) above) is given by

\[
J \partial_z = i \partial_z + w \partial_{\bar{w}}, \quad J \partial_w = i \partial_w, \quad J \partial_{\zeta} = i \partial_{\zeta}.
\] (19)

Notice that there is a 4-dimensional \( J \)-holomorphic foliation \( \mathbb{C}^2(z, w) \) with non-integrable restriction of \( J \), and also lots of transversal 4-dimensional \( J \)-holomorphic foliations with integrable \( J \) of the type \( \Sigma^2 \times \mathbb{C}(\zeta) \), where \( \Sigma \subset \mathbb{C}^2(z, w) \) is a pseudoholomorphic curve.

The bulk of symmetries form the holomorphic transformations \( \zeta \mapsto \Xi(z, \zeta) \) (but there are some other symmetries depending on a function of 1 argument).

The other case \( \text{DG}_2(2) \) is more symmetric:

**Theorem 6.** Suppose \( J \) is non-integrable everywhere, i.e. \( N_J \vert_x \neq 0 \) \( \forall x \in M \). Then \( G \) has the biggest functional dimension iff the almost complex structure is given by the following normal form:

\[
J \partial_z = i \partial_z + \zeta \partial_{\bar{w}}, \quad J \partial_{\zeta} = i \partial_{\zeta}, \quad J \partial_w = i \partial_w.
\] (20)

We shall see that the symmetries depend on 2 complex functions of 2 arguments, but there are also functions with fewer arguments arising via Cartan’s test. The structure of the theorem maximizes the amount of this initial value data (otherwise, as in Theorem \([3]\) there are symmetric models with slightly smaller pseudogroup \( G \); they correspond to change \( J \partial_z = i \partial_z + \lambda \partial_{\bar{w}} \) in \((20)\)).
**Proof.** The characteristic variety of $DG_2(2)$ is

$$\text{Char}^C_{\text{aff}} = (\text{Ann}(X_3)_C)^{1,0} \cup (\text{Ann}(X_3)_C)^{0,1},$$

since $h(X_3, \cdot) = 0$ for any $h \in g_2$. Moreover $\text{rank}(K) = 2$.

To see this we can use the defining relation for $g_1$ from [2]. If $f = \rho \otimes v \in g_1^C$ with $\rho \in \text{Char}^C_{\text{aff}}, v \in K$, then $\rho(g(\xi)v(\eta) + g(\eta)N_J(\xi, v)) = v \cdot g(N_J(\xi, \eta))$. Substituting $\eta = X_3$ we get $v \in \langle X_3 \rangle_C$ or $g(X_3) = 0$, but the first possibility implies the second. Thus, denoting by $\theta_i$ the coframe dual to the frame $X_i$, we get $\rho = \alpha_1 \theta_1 + \alpha_2 \theta_2 \neq 0 \Rightarrow K_\rho = \langle \alpha_2 X_1 - \alpha_1 X_2, X_3 \rangle_C$.

We can also see this by calculating $g_2$ via (1) or as the prolongation of $g_1$. Denoting $\Pi = \langle X_1, X_2 \rangle_C$, $\Pi^* = \langle \theta_1, \theta_2 \rangle_C = \text{Ann}(X_3)_C$ and $sl_2(\mathbb{C}) \subset \Pi^* \otimes \Pi$, we get $g_1 = sl_2(\mathbb{C}) \otimes \Pi^* \otimes \langle X_3 \rangle_C$ (tensor products over $\mathbb{C}$).

Thus we obtain the biggest possible functional dimension of $G$ for non-integrable $J$, so if it is realized the case will be sub-maximal.

Compatibility conditions include integrability of the invariant distribution $\text{Im}(N_J) = \ker(N_J) = \langle X_3 \rangle_C$. This yields a $J$-holomorphic projection of $(M^6, J)$ onto $\mathbb{C}^3$ with (integrable) complex structure and it also implies constancy of the Nijenhuis tensor (otherwise the pseudogroup $G$ does not act transitively). The arguments similar to those used in Theorem [3] show that in order to keep the size of $G$ the structure $J$ must come from an almost holomorphic vector bundle, and then we justify the normal form of the theorem.

To calculate the symmetries of the model let us introduce complex coordinates $(z, \zeta, w)$ according to the order $(X_1, X_2, X_3)$ in which $J$ takes the form \cite{20}. From the above description it is clear that $J$-holomorphic change of coordinates has the form $z \mapsto Z(z, \zeta), \zeta \mapsto \Xi(z, \zeta), w \mapsto W(w, \bar{w}, z, \bar{z}, \zeta, \bar{\zeta})$.

The condition that $J$ has the same normal form in the new coordinate system writes as the system of PDEs

$$W_w = 0, \quad \bar{W}_z = -\frac{i}{2}Z_{\zeta} \Xi, \quad \bar{W}_\zeta = -\frac{i}{2}(Z_{\Xi} \Xi - \zeta \bar{W}_w).$$

This system is not yet formally integrable, and we need to add the compatibility conditions. The resulting involutive system is ($c$ is a constant):

$$Z_{\zeta} \Xi - Z_{\Xi} \zeta = c, \quad W_w = \bar{c}, \quad W_\zeta = 0, \quad W_\zeta = \frac{i}{2}(Z_{\Xi} \Xi - \bar{c} \bar{\zeta}), \quad W_\zeta = \frac{i}{2}Z_{\Xi} \Xi.$$

Thus the form $\Omega = (Z_\Xi \Xi - c \zeta) dz + Z_{\Xi} \Xi d\zeta$ is closed and we can integrate it $\Omega = dW_\theta$, where $W_\theta = W_\theta(z, \zeta)$. This implies $W = \bar{c} w + \frac{i}{2} W_\theta + \Psi$, where $\Psi = \Psi(z, \zeta)$ is an arbitrary holomorphic function.

Finally, the change in $(z, \zeta)$ is a biholomorphism with constant holomorphic Jacobian, i.e. it involves 1 complex function of 2 arguments; the other such function $\Psi$ comes from the change in $w$ (also a function with 1 argument and a constant are involved into the general symmetry). \hfill \Box

### 10 Dimensions $2n = 8$ and higher

In the case $\dim C M = n > 3$ the orbit space of $GL(n, \mathbb{C})$-action on $\Lambda^2 T^* \otimes \mathbb{C} T$ is quite complicated (as well as its real analog - normal forms of $(2,1)$-tensors).
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However we can give a sharper estimate for the dimensional bound of $G$.

**Theorem 7.** For $n \geq 4$ in the case $N_J \neq 0$, the pseudogroup $G$ depends on at most $(n-2)$ complex functions of $(n-1)$ arguments.

**Proof.** By Theorem 1 $p \leq \dim_C \text{Ann}(W)$ for $W = \text{Im}(N_J)$ and so the functional dimension of $G$ is $(n-1)$ if $\dim_C W = 1$. In this case $N_J$ can be identified with a $C$-valued 2-form $\omega$. Let $Z \subset T$ be the kernel of $\omega$. Denote by $2m = \text{rank}_C(\omega)$ (rank is even by Cartan’s lemma). Then $\dim_C Z = n-2m$.

We have $\text{Ann}(\varrho) \supset W$ for a characteristic covector $\varrho \in \text{Char}^C_{\text{aff}}$ and the restriction $\omega|_{\text{Ann}(\varrho)}$ can have $C$-rank $2m$ or $2(m-1)$ depending on whether $\text{Ann}(\varrho)$ is transversal to $Z$ or not. The first situation is generic with respect to the choice of $\varrho$ unless $n = 2m$ or $W = Z$.

Therefore the kernel space $K_\varrho = \text{Ann}(\varrho)^\perp$ (skew-orthogonality with respect to $\omega$) is $Z$ or $Z \oplus L^1$ respective the value of the rank $2m$ or $2(m-1)$ (in the latter case $L^1 \in \text{Ker}(\omega|_{\text{Ann}(\varrho)})$ is $C$-generated by one vector not belonging to $Z$).

In other words, the functional $C$-rank $\frac{1}{2}\sigma$ is equal to $n-2m \leq n-2$ (achieved for $m = 1$) or $n - 2m + 1$ (achieved either for $n = 2m$, when $\frac{1}{2}\sigma = 1$, or for $W = Z$ being 1-dimensional, when $\frac{1}{2}\sigma = 2$; in any case for $n \geq 4$ this branch gives the value $< n - 2$).

We see from the proof that the symmetry pseudogroup $G$ is largest in the case $N_J \neq 0$, $n \geq 4$ only if $m = \text{rank}_C N_J = 1$, i.e. the only non-trivial relation for the Nijenhuis tensor can be either $N_J(X_1, X_2) = X_1$ ($W \cap Z = 0$) or $N_J(X_1, X_2) = X_3$ ($W \subset Z$). As before we must impose integrability assumptions on the involved distributions in order to realize this $G$.

Thus for $n \geq 4$ there are 2 different cases with the largest symmetry: the 6-dimensional manifolds described by (19) or (20), multiplied by $C^{n-3}$. In other words, the the sub-maximal symmetric models are either $M^{2n}_1 = M^4 \times C^{n-2}$, where $(M^4, J)$ is the submaximal model from Theorem 3 or $M^{2n}_2 = M^6 \times C^{n-3}$, where $(M^6, J)$ is the submaximal model from Theorem 4.

Notice that both cases are described by almost holomorphic vector bundle $\pi_1: M^{2n} \to C^{n-1}$ (with $J$ being minimal in the sense of [GS]), and that both models contain a canonical pseudoholomorphic foliation by the kernel of the Nijenhuis tensor $\pi_2: M^{2n} \to C^2$.

The other structures $J$ with large symmetry pseudogroup $G$ must also have the involved invariant distributions integrable, so that these structures give rise to canonical pseudoholomorphic foliations. The hierarchy of intermediate size groups $G$ is immense, but also the totality of almost complex structures $J$ with non-degenerate $N_J$ is vast. To study the latter the following idea is useful.

Consider the $C$-antilinear map $N_J: \Lambda^2 T \to T, T \simeq C^n$. For non-degenerate $N_J$ the pre-image $N^{-1}_J(0)$ has complex dimension $d = \frac{1}{2}n^2 - \frac{3}{2}n$. The Plücker embedding $\rho: \text{Gr}^C(2, n) \hookrightarrow P^C \Lambda^2 T$ has image of codimension $d + 3 - n$. Therefore generically $\Sigma = \varpi(N^{-1}_J(0)) \cap \text{Im} \rho$ has dimension $n - 4$, where $\varpi: \Lambda^2 \to P^C \Lambda^2$ is the projectivization map. The degree of subvariety $\Sigma$ is the same as the degree of a Plücker embedding of the Grassmannian $\text{Gr}^C(2, n)$, namely the Catalan number $\frac{1}{n-1} \binom{2n-4}{n-2}$. 

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Let us consider in more details the case \( n = 4 \), when \( \Sigma \) is zero-dimensional of degree 2. Thus \( \Sigma \) consists of two 4-dimensional \( J \)-invariant subspaces of \( T \), which we assume transversal \( T = \Pi_1 \oplus \Pi_2 \) (\( \Pi_i \) are characterized uniquely by the condition \( N_J|_{\Pi_i} = 0 \)). We will call such almost complex structure \( J \) on 8-dimensional manifold \( M \) transversally non-degenerate.

This assumption imposes topological restrictions on \( M \) and the homotopy type of \( J \). Indeed, \( N_J : \Pi_1 \otimes \Pi_2 \to T \) is an anti-isomorphism and therefore the total Chern class of the tangent bundle is equal to

\[
c(T, J) = \frac{c(\Pi_1 \oplus \Pi_2)}{c(\Pi_1 \oplus \Pi_2)}
\]

Denoting \( c' = c_k(\Pi_1), c''_k = c_k(\Pi_2) \) the Chern classes of the complex rank 2 bundles \( \Pi_i \) we can use the formulae \( c(\Pi_1 \oplus \Pi_2) = 1 + 2(c'_1 + c''_1) + (c'_1 + c''_1)^2 + c'_1 c''_1 + 4(c'_1 + c''_1) + ((c'_1 + c''_1)^2 + (c'_1 - c''_1)^2 + (c'_1 + c''_1)(c'_1 c''_1 + c''_1 c'_1)) \) and \( c(T, J) = (1 + c'_1 + c''_1)(1 + c'_1 + c''_1) \). They imply the following restrictions for the Chern classes \( c_i = c_i(T, J) \) of transversally non-degenerate \( J \):

\[
3c_1 = 0, \quad 3c_2 = -3q^2, \quad 3c_3 = 0, \quad 15c_4 = 0
\]

for some \( q \in H^4(M) \) (here \( q = c_2(\Pi_1) \)). In particular, \( \chi(M) = 0 \).

Moreover generically the 6-dimensional manifold \( \Lambda^6 = \{ N_J(\xi, \eta) : \xi \in \Pi_1, \eta \in \Pi_2 \} \subset T \) meets \( \Pi_i \) at two different complex lines \( L_i^1, L_i^2 \) (degree of \( \Lambda^6 \) is 2). This means that there exist two canonical decompositions into the sum of complex lines \( \Pi_i = V^{i*}_1 \oplus V^{i*}_2 \) such that \( N_J(V^{i*}_1, V^{i*}_2) = L_i^1, 1 \leq i, j, s \leq 2 \) (the line bundles are defined up to the changes \( (i, s) \mapsto (i + 1, s + 1) \) and \( j \mapsto j + 1 \) when we consider \( i, j, s \mod 2 \)).

If the subbundles \( V^{i*}_1 \subset \Pi_i \) are transversal to one of the lines \( L_i^j \) (say to \( L_i^1 \)), we call \( J \) (resp. \( N_J \)) strongly non-degenerate. It is easy to classify strongly non-degenerate Nijenhuis tensors (the moduli space has complex dimension 8). Indeed, there exists a basis \( e_i \in L_i^j \) (these 4 vectors in \( T \) are defined up to simultaneous multiplication by \( \sqrt{V} \)) such that \( N_J \) is given by 8 complex constants \( \lambda_i^{s*} \) and the relations

\[
N_J(e_i^1 + \lambda_i^{s*} e_1^1, e_i^2 + \lambda_i^{s*} e_2^2) = e_i^s, \quad e_i^1 + \lambda_i^{s*} e_i^2 \in V_i^{s*}.
\]

Existence of strongly non-degenerate structures on \( M \) implies even stronger topological obstructions on the homotopy class of \( J \). Let for simplicity the line bundles be numerated (elsewise the torsion order has to be multiplied by 4).

Then the above relations imply that the 1st Chern classes \( c_1(V^{i*}_1) = c_1(L_i^1) = \alpha \) are independent of indices and \( 3\alpha = 0 \) (the claim that \( c_1(V^{i*}_1) = c_1(L_i^1) \) depend only on the index \( i \) follows from \( \Lambda^2 \Pi_i = L_i^1 \otimes L_i^2 = V_i^{1*} \otimes V_i^{2*} \); the rest follows from the defining equation for \( N_J \)).

Consequently, we have: \( c_1 = 4\alpha = \alpha, c_2 = 6\alpha^2 = 0, c_3 = 4\alpha^3 = c_3^1, c_4 = \alpha^4 = c_4^1 \) (if \( H^*(M) \) has no torsion, then all Chern numbers vanish).

According to [MG] the existence of an almost complex structure in dimension 8 is equivalent to such relations:

\[
-4c_2^4 + 4c_2^3 c_1 + c_1 c_3 + 3c_2^2 c_1 - c_4 \equiv 0 \mod 720,
2c_4^4 + c_1^2 c_2 - c_1 c_3^2 - c_2^4 \equiv 0 \mod 12, \quad c_1 c_3^2 - 2c_2^4 \equiv 0 \mod 4.
\]
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For strongly non-degenerate $J$ these relations specify to the only constraint $c_4 \equiv 0 \mod 720$. It would be interesting to understand the restrictions on Chern classes if $N_J$ is non-degenerate in the weak sense, $\text{Im}(N_J) = T$.

A Associated Hermitian metrics in dimension 6

In [B] an invariant Hermitian metric was associated to an almost complex structure in dimension 6. This construction depends pointwise on $J, N_J$, and can be written following [K5] so (the trace is over $\mathbb{R}$):

$$h(\xi, \eta) = \text{Tr}[N_J(\xi, N_J(\eta, \cdot)) + N_J(\eta, N_J(\xi, \cdot))].$$

Since it satisfies the property $h(J\xi, J\eta) = h(\xi, \eta)$, it has (complex) type $(1,1)$.

The hermitian metric determines the smooth volume form $\Omega = \Omega_h$ and there is also the canonical holomorphic $(3,0)$-form $\sigma$ related to $\Omega$

$$\Omega = \frac{1}{3} \omega^3 = \frac{i}{4} \sigma \wedge \bar{\sigma}$$

(the form $\sigma$ is normalization and alternation of the following $\mathbb{C}$-valued 3-tensor: $\varsigma(X, Y, Z) = h(N_J(X, Y), Z) - i h(N_J(X, Y), JZ)$).

For the cases of our current interests the metric on $m = T_x M$ in complex coordinates (naturally related to the basis in which the normal form is given) is the following. For (18) it equals (after normalization of constants)

$$h = \sum_{i=1}^3 |dz_i|^2 = \sum_{i=1}^3 dz_i \cdot d\bar{z}_i, \quad (21)$$

and for (17) it equals

$$h = |dz_1|^2 + |dz_2|^2 - |dz_3|^2 = dz_1 \cdot d\bar{z}_1 + dz_2 \cdot d\bar{z}_2 - dz_3 \cdot d\bar{z}_3. \quad (22)$$

The symplectic form on $m$ in both cases is

$$\omega = \frac{i}{2} \sum_{i=1}^3 dz_i \wedge d\bar{z}_i = \sum_{i=1}^3 dx_i \wedge dy_i,$$

so the complex structure $J = J_1 + J_2 + J_3$ for (18) and it equals $J = J_1 + J_2 - J_3$ for (17), where $J_k = i \partial_{z_k} \otimes dz_k - i \partial_{\bar{z}_k} \otimes d\bar{z}_k = \partial_{y_k} \otimes dx_k - \partial_{x_k} \otimes dy_k$ is the complex structure on the $k$-th summand in $\mathbb{C}^3$. 

\textsuperscript{9}In the most symmetric non-degenerate case the 3-form $d\omega$ is the restriction of a generic 3-form in $\mathbb{R}^7$ with stabilizer $G_2$.

\textsuperscript{10}We have changed the last summand $\mathbb{C}$ in $m = \mathbb{C}^3$ to $\bar{\mathbb{C}}$ for (17) to keep the two cases parallel from the viewpoint of $\omega$ and $\sigma$.  

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The canonical holomorphic and the smooth volume forms in both cases are
\[ \sigma = d^3z = dz_1 \wedge dz_2 \wedge dz_3, \quad \Omega = \frac{i}{4} d^3z \wedge d^3\bar{z}. \]

Now it’s easy to calculate the stabilizer \( \mathfrak{h} = \mathfrak{g}_1 \), which is the automorphism of the triple \( (\mathfrak{h}, J, \sigma) \) on \( \mathfrak{m} \), in both cases. It is important that this triple has the same automorphism group as the pair \( (J, N_J) \) [13] [K3].

We have \( \mathfrak{h} = \mathfrak{su}(3) \) for [18] and \( \mathfrak{h} = \mathfrak{su}(2, 1) \) for [17].

**B  Lie algebras in dimension 14**

Here we consider the stabilizer \( \mathfrak{h} = \mathfrak{g}_1 \) of dimension 8 from the previous section naturally acting on \( \mathfrak{m} = \mathfrak{g}_0 = T_x \) of dimension 6. The symmetry algebra has maximal dimension 14 if we can construct Lie algebra \( \mathfrak{g} \) allowing the following exact 3-sequence
\[ 0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{m} \to 0. \]

We will show that such \( \mathfrak{g} \) can be constructed for [18], but not for [17].

Since \( \mathfrak{h} \) is simple, we have direct decomposition of \( \mathfrak{h} \)-representations \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \), and \( \mathfrak{m} = \mathbb{C}^3 \) as \( \mathfrak{h} \)-module. Thus to restore the Lie algebra \( \mathfrak{g} \) we need two components of the \( \mathfrak{h} \)-morphisms \( \Lambda^2 \mathfrak{m} \to \mathfrak{g} \). The \( \mathfrak{m} \)-part is given uniquely (see below) and the only part to be determined is the \( \mathfrak{h} \)-component of \( [\mathfrak{m}, \mathfrak{m}] \).

As \( \mathfrak{h} \)-representation \( \Lambda^2 \mathfrak{m} \) is reducible, and it decomposes into three submodules of dimensions 6, 1, 8, which are better described in the complexification:
\[ \Lambda^2 \mathfrak{m} = \Lambda^{2,0}(\mathfrak{m}) \oplus \Lambda^{1,1}(\mathfrak{m}) \oplus \Lambda^{0,2}(\mathfrak{m}). \]

The piece \( \Lambda^{2,0} \oplus \Lambda^{0,2} \) is the complexification of a 6 dimensional summand, and \( \Lambda^{1,1} = \mathbb{C} \oplus \Lambda^{1,1}_0 \) corresponds to 1 + 8 dimensional piece. Here we identify via the Hermitian metric \( \Lambda^{1,1} = \text{End}(\mathfrak{m})_\mathbb{C} \) and the endomorphisms \( \mathfrak{h} \)-equivariantly split into the scalar and traceless parts (now the trace is over \( \mathbb{C} \)).

Thus the \( \mathfrak{m} \)-part of the above \( \mathfrak{h} \)-morphism \( \Lambda^2 \mathfrak{m} \to \mathfrak{m} \) is given by the projection
\[ \Lambda^2 \mathfrak{m} \supset \Lambda^{2,0}(\mathfrak{m}) \oplus \Lambda^{0,2}(\mathfrak{m}) \simeq \Lambda^{0,1}(\mathfrak{m}) \oplus \Lambda^{1,0}(\mathfrak{m}) = [\mathfrak{m}, \mathfrak{m}] \]
(we must use the conjugation to get \( \mathbb{C} \)-antilinearity, which yields the required tensor \( N_J \neq 0 \)).

The \( \mathfrak{h} \)-part \( \Lambda^2 \mathfrak{m} \to \mathfrak{h} \) is given by the complexification as the projection
\[ \Lambda^2 \mathfrak{h} \supset \Lambda^{1,1} \to \Lambda^{1,1}_0 \simeq \mathfrak{h} \mathbb{C}, \]
where the identification \( \Lambda^{1,1} = \mathfrak{m}_{0,1} \circ \mathfrak{m}_{1,0} \simeq \text{End}(\mathfrak{m})_\mathbb{C} \subset \mathfrak{m}_{1,0} \circ \mathfrak{m}_{1,0} + \mathfrak{m}_{0,1} \circ \mathfrak{m}_{0,1} \) is given by the Hermitian metric, \( \mathfrak{m}_{1,0}^* = \mathfrak{m}_{0,1} = \mathfrak{m}_{1,0} \).

Let us now specify to the case \( \mathfrak{h} = \mathfrak{su}(3) \) corresponding to [18]. The Hermitian form is given by [21] and the complex structure is \( J = J_1 + J_2 + J_3 \).

The map \( \Lambda^{1,1} \to \mathfrak{su}(3) \otimes \mathbb{C} \) is given by
\[ \xi \wedge \eta \mapsto -3\xi \otimes h(\eta, \cdot) + 3\eta \otimes h(\xi, \cdot) + h(\xi, J\eta)J. \]
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Thus the $h$-component of the bracket on $m$ is

$$[\partial_{z_k}, \bar{\partial}_{\bar{z}_k}] = i(3J_k - J), \quad [\partial_{z_k}, \bar{\partial}_{\bar{z}_\alpha}] = -3\partial_{z_{\alpha}} \otimes d\bar{z}_k + 3\bar{\partial}_{\bar{z}_k} \otimes d\bar{z}_\alpha$$

for $\alpha \neq \beta$.

The $m$-component of the bracket on $m$ is given by the relations

$$[\partial_{z_1}, \partial_{z_2}] = 2\bar{\partial}_{z_3}, \quad [\partial_{z_2}, \partial_{z_3}] = 2\bar{\partial}_{z_1}, \quad [\partial_{z_3}, \partial_{z_1}] = 2\bar{\partial}_{z_2}$$

(we use a normalization constant) and their conjugations.

Notice that this is the complexification of the real Lie algebra, i.e. we can write real commutator relations in the real basis of $M$:

$$[\partial_{z_k}, \partial_{z_l}]_h = [\partial_{y_k}, \partial_{y_l}]_h = -3\partial_{z_k} \otimes d\bar{z}_l + 3\partial_{z_l} \otimes d\bar{z}_k \ (\text{+ conjugated})$$

$$[\partial_{z_k}, \partial_{y_l}]_h = 3i\partial_{z_k} \otimes d\bar{z}_l + 3i\partial_{z_l} \otimes d\bar{z}_k \ (\text{+ conjugated}), \quad [\partial_{y_k}, \partial_{y_l}]_h = 2(3J_k - J)$$

(we indicate only the $h$ part of the bracket, the $m$ part is obtained similarly).

Let us check the Jacobi identity (we can use complex basis for this). If all 3 vectors have (1,0)-type, the only non-trivial relation is

$$[\partial_{z_1}, [\partial_{z_2}, \partial_{z_3}]] + [\partial_{z_2}, [\partial_{z_3}, \partial_{z_1}]] + [\partial_{z_3}, [\partial_{z_1}, \partial_{z_2}]] = 0,$$

which is equivalent to the trace-zero condition $\sum_k [\partial_{z_k}, \bar{\partial}_{\bar{z}_k}] = 0$.

If we use two (1,0)-types and one (0,1)-type, then there are 3 similar relations with 3 different indices, like

$$[\bar{\partial}_{z_1}, [\partial_{z_2}, \partial_{z_3}]] + [\bar{\partial}_{z_2}, [\partial_{z_3}, \partial_{z_1}]] + [\bar{\partial}_{z_3}, [\partial_{z_1}, \partial_{z_2}]] = 0 + 0 + 0 = 0$$

and 6 similar relations with 2 indices equal and 1 different, like

$$[\bar{\partial}_{z_1}, [\partial_{z_1}, \partial_{z_3}]] + [\partial_{z_1}, [\partial_{z_3}, \bar{\partial}_{z_1}]] + [\partial_{z_3}, [\bar{\partial}_{z_1}, \partial_{z_1}]] = 0.$$

This latter re-writes as

$$2[\bar{\partial}_{z_1}, \bar{\partial}_{z_3}] - (-3\partial_{z_2} \otimes d\bar{z}_1 + 3\bar{\partial}_{\bar{z}_1} \otimes d\bar{z}_2)(\bar{\partial}_{z_3}) + i(2J_1 - J_2 - J_3)\partial_{z_2}$$

$$= -4\partial_{z_2} + 3\partial_{z_2} + \partial_{z_2} = 0.$$  

The other relations are conjugated to these, and so $g$ has the unique Lie algebra structure corresponding to the compact form of the exceptional Lie algebra $G_2$.

**Remark 6.** With $[m, m] = 0$ we realize $g$ as a 14-dimensional Lie algebra for both choices of $h$, but in this case $N_1 = 0$ and the corresponding homogeneous space $M^6$ is flat (so that the actual symmetry is the maximal infinite group).

Now let us turn to $h = \text{su}(2,1)$. The Hermitian form is given by

$$\xi \wedge \eta \mapsto -k\xi \otimes h(\eta, \cdot) + k\eta \otimes h(\xi, \cdot) + k \cdot h(\xi, \eta)J,$$

and the complex structure is $J = J_1 + J_2 - J_3$.

The map $A^{1,1} \to \text{su}(2,1) \otimes \mathbb{C}$ is given by the following $h$-morphism (the normalization constant $k$, which can be also zero, is to be specified below)

$$\xi \wedge \eta \mapsto -k\xi \otimes h(\eta, \cdot) + k\eta \otimes h(\xi, \cdot) + k \cdot h(\xi, \eta)J.$$

\[\text{Notice that it prohibits the possibility that the } h\text{-part of } [m, m] \text{ vanishes.}\]
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With this we have the following brackets:

\[ [\partial_{z_1}, \bar{\partial}_{z_1}] = i k (J_3 - J_2), \quad [\partial_{z_2}, \bar{\partial}_{z_2}] = i k (J_3 - J_1), \quad [\partial_{z_3}, \bar{\partial}_{z_3}] = i k (J_1 + J_2 - 2J_3), \]

\[ [\partial_{z_1}, \bar{\partial}_{z_2}] = -k \partial_{z_1} \otimes dz_2 + k \bar{\partial}_{z_2} \otimes d\bar{z}_1, \quad [\partial_{z_1}, \bar{\partial}_{z_3}] = k \partial_{z_1} \otimes dz_3 + k \bar{\partial}_{z_3} \otimes d\bar{z}_1, \]

\[ [\partial_{z_2}, \bar{\partial}_{z_1}] = -k \partial_{z_2} \otimes dz_1 + k \bar{\partial}_{z_1} \otimes d\bar{z}_2, \quad [\partial_{z_2}, \bar{\partial}_{z_3}] = k \partial_{z_2} \otimes dz_3 + k \bar{\partial}_{z_3} \otimes d\bar{z}_2, \]

\[ [\partial_{z_3}, \bar{\partial}_{z_1}] = -k \partial_{z_3} \otimes dz_1 - k \bar{\partial}_{z_1} \otimes d\bar{z}_3, \quad [\partial_{z_3}, \bar{\partial}_{z_2}] = -k \partial_{z_3} \otimes dz_2 - k \bar{\partial}_{z_2} \otimes d\bar{z}_3. \]

The Jacobi relations with 3 different indices are un-changed. The constant \( k \) is specified by one of the last group of 6 relations:

\[ [\bar{\partial}_{z_1}, [\partial_{z_1}, \bar{\partial}_{z_2}]] + [\partial_{z_1}, [\bar{\partial}_{z_2}, \bar{\partial}_{z_1}]] + [\bar{\partial}_{z_2}, [\partial_{z_1}, \bar{\partial}_{z_1}]] = 0, \]

which gives \(-4k \partial_{z_2} + k \partial_{z_2} = 0\), i.e. \( k = 2 \). But the next Jacobi relation

\[ [\bar{\partial}_{z_1}, [\partial_{z_1}, \partial_{z_2}]] + [\partial_{z_1}, [\bar{\partial}_{z_2}, \bar{\partial}_{z_1}]] + [\bar{\partial}_{z_2}, [\partial_{z_1}, \partial_{z_1}]] = 0 \]

is already a contradiction: \((-4 + k - k) \partial_{z_2} = 0\). Thus we have proven

**Proposition 8.** No Lie algebra of the type \( \mathfrak{g} = \mathfrak{su}(2, 1) \ltimes \mathbb{R}^6 \) (as \( \mathfrak{h}\)-representation) can be a symmetry of an almost complex structure \( J \) on a 6-dimensional manifold with non-degenerate Nijenhuis tensor \( N_J \).

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**References**

[ALV] D. Alekseevskij, V. Lychagin, A. Vinogradov, *Basic ideas and concepts of differential geometry*, Encyclopedia of math. sciences 28, Geometry 1, Springer-Verlag (1991).

[AL] M. Audin, J. Lafontaine (eds.), *Holomorphic curves in symplectic geometry*, Birkhäuser Verlag, Progr. in Math. 117 (1994).

[B] R. Bryant, *On the geometry of almost complex 6-manifolds*, Asian J. Math. 10 (2006), no. 3, 561–605.

[BCG] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, MSRI Publications 18, Springer-Verlag (1991).

[BKW] W. M. Boothby, S. Kobayashi, H. C. Wang, ”A note on mappings and automorphisms of almost complex manifolds”, Ann. Math. 77, 329–334 (1963).

[De] A. Dessai, *Some remarks on almost and stable almost complex manifolds*, Math. Nachr. 192, 159172 (1998).

[GS] H. Gaussier, A. Sukhov, *On the geometry of model almost complex manifolds with boundary*, Math. Z. 254, no. 3, 567589 (2006).

[Ge] H. Geiges, *Chern numbers of almost complex manifolds*, Proc. Amer. Math. Soc. 129, no. 12, 37493752 (2001).

[Gr] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math., 82, 307–347 (1985).
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[H] F. Hirzebruch, Komplexe Mannigfaltigkeiten, Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York, 119136 (1960)

[HH] F. Hirzebruch, H. Hopf, Felder von Flachenelementen in 4-dimensionalen Mannigfaltigkeiten, Math. Ann. 136, 156–172 (1958).

[Ko] S. Kobayashi, Transformation groups in Differential geometry, Springer-Verlag (1972).

[K1] B. Kruglikov, Nijenhuis tensors and obstructions for pseudoholomorphic mapping constructions, Math. Notes, 63, no. 4 (1998), 541–561.

[K2] B. Kruglikov, Non-existence of higher-dimensional pseudoholomorphic submanifolds, Manuscripta Mathematica, 111 (2003), 51–69.

[K3] B. Kruglikov, Characteristic distributions on 4-dimensional almost complex manifolds, in: Geometry and Topology of Caustics - Caustics '02, Banach Center Publications, 62 (2004), 173–182.

[K4] B. Kruglikov, Tangent and normal bundles in almost complex geometry, Diff.Geom.Appl., 25, no.4 (2007).

[K5] B. Kruglikov, Invariants and submanifolds in almost complex geometry, Differential Geometry and its Applications, Proc. Conf. in Honour of Leonhard Euler [Olomouc, August 2007], World Scientific, 305–316 (2008).

[K6] B. Kruglikov, Finite-dimensionality in Tanaka theory, Annales de l’Institut Henri Poincaré (C) Non Linear Analysis 28, issue 1, 75-90 (2011).

[KL1] B. S. Kruglikov, V. V. Lychagin, Dimension of the solutions space of PDEs, ArXive e-print: math.DG/0610789; In: Global Integrability of Field Theories, Proc. of GIFT-2006, Ed. J. Calmet, W. Seiler, R. Tucker (2006), 5–25.

[KL2] B. Kruglikov, V. Lychagin, Geometry of Differential equations, Handbook of Global Analysis, Ed. D. Krupka, D. Saunders, Elsevier, 725-772 (2008).

[Ku] A. Kumpera, Invariants differentiels d’un pseudogroupe de Lie. I, J. Differential Geometry 10 (1975), no. 2, 289–345; II, ibid. 10 (1975), no. 3, 347–416.

[LS] L. Lempert, R. Szőke, The tangent bundle of an almost complex manifold, Canad. Math. Bull. 44, no. 1 (2001), 70–79.

[M] D. McDuff, The local behavior of holomorphic curves in almost complex 4-manifolds, Journ. Diff. Geom., 34, 143-164 (1991).

[MG] S. Müller, H. Geiges, Almost complex structures on 8-manifolds, Enseign. Math. (2) 46, no. 1-2, 95-107 (2000).

[NN] A. Newlander, L. Nirenberg, Complex analytic coordinates in almost-complex manifolds, Ann. Math., 65, ser. 2, issue 3, 391–404 (1957).

[NW] A. Nijenhuis, W. Woolf, Some integration problems in almost-complex and complex manifolds, Ann. Math. 77, 424–489 (1963).

[Se] W. Seiler, Involution. The formal theory of differential equations and its applications in computer algebra, Algorithms and Computation in Mathematics 24, Springer-Verlag, Berlin (2010).

[Sp] D. C. Spencer, Overdetermined systems of linear partial differential equations, Bull. Amer. Math. Soc., 75 (1969), 179–239.

[St] S. Sternberg, Lectures on Differential geometry, Prentice-Hall, New Jersey (1964).

[Ta] N. Tanaka, On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. 10, no.1, 1–82 (1970).

[Th] E. Thomas, Complex structures on real vector bundles, Amer. J. Math. 89, 887–908 (1967).

[To] J. Tonejc, Normal forms for almost complex structures, Internat. J. Math. 19, no. 3, 303–321 (2008).

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