A SINGULARLY PERTURBED AGE STRUCTURED SIRS MODEL WITH FAST RECOVERY

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ABSTRACT. Age structure of a population often plays a significant role in the spreading of a disease among its members. For instance, childhood diseases mostly affect the juvenile part of the population, while sexually transmitted diseases spread mostly among the adults. Thus, it is important to build epidemiological models which incorporate the demography of the affected populations. Doing this we must be careful as many diseases act on a time scale different from that of the vital processes. For many diseases, e.g. measles, influenza, the typical time unit is one day or one week, whereas the proper time unit for the vital processes is the average lifespan in the population; that is, 10-100 years. In such a case, the epidemiological model with vital dynamics becomes a multiple time scale model and thus it often can be significantly simplified by various asymptotic methods. The presented paper is concerned with an SIRS type disease spreading in a population with a continuous age structure modelled by the McKendrick-von Foerster equation. We consider a disease with a quick recovery rate in a large population. Though it is not too surprising that in such a model the introduced disease quickly vanishes, the result is mathematically interesting as the error estimates are uniform on the whole infinite time interval, in contrast to the typical results based on the Tikhonov theorem and classical asymptotic expansions.

1. Introduction. In many diseases the transmission rates vary significantly with age. In fact, for the exanthematic diseases (measles, scarlet fever) the transmission mainly involves early ages, while for sexually transmitted diseases the principal mechanism of infection only involves mature individuals. Thus, it can be expected that the interplay of the demographic processes with the infection mechanism will produce a nontrivial dynamics.

In this paper we consider the SIRS model introduced in [9]. In this model, the total population can be modelled by the linear McKendrick-von Foerster model
describing the evolution in time of the density of the population with respect to age
$a \in [0, \omega], \omega < \infty$, denoted by $n(a, t)$. The demography of the population is driven
by death and birth processes with vital rates $\mu(a)$ and $\beta(a)$, respectively. Due to
the epidemics, we split the population into susceptibles, infectives and recovered

$$n(a, t) = s(a, t) + i(a, t) + r(a, t),$$

so that the scalar McKendrick equation for $n$ splits into the system

\[
\begin{align*}
\partial_t s(a, t) + \partial_a s(a, t) + \mu(a)s(a, t) &= -\Lambda(a, i(\cdot, t)) s(a, t) + \delta(a)i(a, t), \\
\partial_t i(a, t) + \partial_a i(a, t) + \mu(a)i(a, t) &= \Lambda(a, i(\cdot, t)) s(a, t) - (\delta(a) + \gamma(a))i(a, t), \\
\partial_t r(a, t) + \partial_a r(a, t) + \mu(a)r(a, t) &= \gamma(a)i(a, t),
\end{align*}
\]

(1)

where $\gamma(a)$ and $\delta(a)$ are age specific removal and recovery rates, respectively (here
the removal refers to the recovery with permanent immunity). The function $\Lambda$ is
the infection rate (or the force of infection). In this paper we consider the so-called
intercohort model

$$\Lambda(a, i(\cdot, t)) = \int_0^\omega K(a, a')i(a', t) da',$$  \hspace{0.5cm} (2)

where $K$ is a nonnegative bounded function which accounts for the age dependence
of the infections. For instance, for typical childhood diseases $K$ should be large for
small $a$ and $a'$ and close to zero for large $a$ or $a'$.

System (1) is supplemented by the boundary conditions

\[
\begin{align*}
s(0, t) &= \int_0^\omega \beta(a)s(a, t) + (1 - p)i(a, t) + (1 - q)r(a, t) da, \\
i(0, t) &= p \int_0^\omega \beta(a)i(a, t) da, \\
r(0, t) &= q \int_0^\omega \beta(a)r(a, t) da,
\end{align*}
\]

(3)

where $p, q \in [0, 1]$ are the vertical transmission parameters of infectiveness and
immunity, respectively. Finally, we prescribe the initial conditions

$$s(a, 0) = \overset{\circ}{s}(a), \quad i(a, 0) = \overset{\circ}{i}(a), \quad r(a, 0) = \overset{\circ}{r}(a).$$  \hspace{0.5cm} (4)

However, in building (1) some care should be taken not to mix different time scales.
In fact, if we consider a human population, then the death and birth rates are
measured in units 1/year or even the time unit often is taken to be the average life-
span in the population, \cite{14}. Then, in a human population with average life-span
of, say, 70 years, averaged $\mu$ would be 1 and averaged $\beta$ also would be 1 (2 children
per couple in the lifetime).

On the other hand, if we model the disease such as flu or measles, then the
recovery rate is measured in units 1/day (average duration of the disease, which is
the inverse of the recovery rate, is 2-7 days). Thus, if we are to use 70 years as the
unit of time in (1), then the numerical values of $\delta$ and $\gamma$, used in the literature for
the SIRS compartmental model, should be multiplied by $70 \times 365$. Certainly, for
other diseases, such as HIV/AIDS, the duration of the disease is at the same time
scale as the vital dynamics and the above argument cannot be applied.

As far as the scaling of the infection force $\Lambda$ is concerned, the situation can
vary. For a simple compartmental SIR model $\Lambda = \lambda i$, where $\lambda$ is constant. Then,
\cite{8}, in the population with vital dynamics (that is, with the constant size of the
(population) we have $\lambda = \gamma R_0/N$, where $R_0$ is the so-called basic reproduction rate and $N$ is the size of the population. The constant $R_0$ for, say, measles, is 18, so the relative magnitude of $\lambda$ depends on the size of the population: if the population is large, then this term can be included in the ‘small’ terms, while if it is small, it is a 'large' term. In this paper we shall be concerned with the former case and, for the sake of (mathematical) completeness, we allow the population to vary without affecting the value of $\lambda$. Thus we consider

$$
\partial_t u_\epsilon = \mathcal{S}u_\epsilon + Mu_\epsilon + \mathcal{F}(u_\epsilon) + \frac{1}{\epsilon}Cu_\epsilon,
$$

$$
u_\epsilon(0,t) = \mathcal{B}[u_\epsilon(\cdot,t)],
$$

$$
u_\epsilon(a,0) = \tilde{u}_\epsilon,
$$

where $\nu_\epsilon = (s_\epsilon, i_\epsilon, r_\epsilon), \mathcal{S} = \text{diag}\{-\partial_a, -\partial_a, -\partial_a\}$ on $D(\mathcal{S}) = W^1_1([0,\omega], \mathbb{R}^3), \mathcal{M}(a) = -\text{diag}\{\mu(a), \mu(a), \mu(a)\}$ on $D(\mathcal{M}) = \{u \in L_1([0,\omega], \mathbb{R}^3); \mu u \in L_1([0,\omega], \mathbb{R}^3)\}$ and $\epsilon$ is a small parameter reflecting the ratio of the typical time scales of the vital and epidemiological processes. Further, the bounded operator $\mathcal{B} : L_1([0,\omega], \mathbb{R}^3) \to \mathbb{R}^3$ is defined by

$$
\mathcal{B}u = \int_0^\omega B(a)u(a)da,
$$

with

$$
B(a) = \begin{pmatrix}
\beta(a) & (1-p)\beta(a) & (1-q)\beta(a) \\
0 & p\beta(a) & 0 \\
0 & 0 & q\beta(a)
\end{pmatrix},
$$

$$
[\mathcal{F}(u)](a) = \begin{pmatrix}
-s(a)\int_0^\omega K(a,a')i(a')da' \\
\int_0^\omega K(a,a')i(a')da' & s(a)\int_0^\omega K(a,a')i(a')da' \\
0 & 0
\end{pmatrix},
$$

and

$$
[\mathcal{C}u](a) = \begin{pmatrix}
0 & \delta(a) & 0 \\
0 & -(\delta(a) + \gamma(a)) & 0 \\
0 & \gamma(a) & 0
\end{pmatrix} u(a).
$$

The main aim is to analyse the behaviour of the solution $(s_\epsilon, i_\epsilon, r_\epsilon)$ as $\epsilon \to 0$. The well-posedness of this problem is discussed in Section 2.

To formulate the main result, we have to introduce some notation. Denote $X = L_1([0,\omega], \mathbb{R}^3)$ with $X_+ = L_1([0,\omega], \mathbb{R}^3_+)$ being the positive cone of $X$. The norm $\|\cdot\|$ will refer to the relevant norm in $L_1$. If necessary, we indicate the norm in a specific space $X$ by writing $\|\cdot\|_X$. For any measurable function $\phi$ on $[0,\omega]$ we introduce

$$
\bar{\phi} = \text{ess sup}_{a \in [0,\omega]} \phi(a), \quad \underline{\phi} = \text{ess inf}_{a \in [0,\omega]} \phi(a).
$$

Then we assume

**A1:** $\mu \geq 0$ is a measurable function on $[0,\omega]$ and $\mu > 0$;

**A2:** $0 \leq \beta \in L_\infty([0,\omega])$;

**A3:** $\beta, \gamma \in W^1_\infty([0,\omega])$ with $\bar{\delta} > 0, \bar{\gamma} > 0$.

We note that for the model to be realistic, the death rate $\mu$ must satisfy additional condition (15). Since this condition does not have any bearing on the formulation of main results, it will be discussed later. Next, we note that if $(s_\epsilon, i_\epsilon, r_\epsilon)$ satisfies
there are constants $n$ population (5) then, by adding the equations and the side conditions, we find that the total population $n(t, a) = s_r(t, a) + i_r(t, a) + r_r(t, a)$ satisfies
\begin{align*}
\partial_t n(a, t) &= -\partial_a n(a, t) - \mu(a)n(a, t), \\
n(0, t) &= \int_0^\omega \beta(a)n(a, t)da, \\
n(a, 0) &=: \overset{\circ}{n}(a) = \overset{\circ}{s}(a) + \overset{\circ}{i}(a) + \overset{\circ}{r}(a),
\end{align*}
and is independent of $\epsilon$. It is known, \cite{9}, that there is a dominant eigenvalue $\lambda_\mu \leq \overline{\beta} - \mu$, defined as the unique real solution to
\begin{align}
\int_0^\omega e^{-\lambda a} \beta(a)\Pi_\mu(a)da = 1,
\end{align}
where, see \cite{1, 9, 14},
\begin{align}
\Pi_\mu(a) := e^{-\int_0^a \mu(s)ds}
\end{align}
is the probability of survival of an individual till age $a$, and a constant $m$ such that
\begin{align}
\|n(t)\| \leq me^{\lambda_\mu t} \|\overset{\circ}{n}\|,
\end{align}
where $\lambda_\mu$ is negative, zero or positive if and only if the net reproduction rate $R_\mu = \int_0^\omega \beta(a)\Pi_\mu(a)da$ is, respectively, smaller, equal or greater than 1. As we mentioned earlier, the considered scaling is realistic for models in which there are no significant changes of the total population. Hence, throughout the paper we will assume
\begin{align}
R_\mu \leq 1,
\end{align}
though some results will be formulated for a general $R_\mu$. Further, let
\begin{align}
\gamma^\#(a) = \frac{\gamma(a)}{\gamma(a) + \delta(a)}, \quad \delta^\#(a) = \frac{\delta(a)}{\gamma(a) + \delta(a)}
\end{align}
and let $\overset{\circ}{w}$ be the solution to the McKendrick problem
\begin{align*}
\partial_t \overset{\circ}{w}(a, t) &= -\partial_a \overset{\circ}{w}(a, t) - \mu(a)\overset{\circ}{w}(a, t), \\
\overset{\circ}{w}(0, t) &= q \int_0^\omega \beta(a)n(a, t)da, \\
\overset{\circ}{w}(a, 0) &=: \overset{\circ}{w}(a) = \gamma^\#(a) \overset{\circ}{i}(a) + \overset{\circ}{r}(a).
\end{align*}
Then we have
\begin{thm}
Let the coefficients of the problem (5) satisfy $A1$–$A3$ together with (11), $(\overset{\circ}{s}, \overset{\circ}{i}, \overset{\circ}{r})$ be such that $(s_r, i_r, r_r)$ is a classical solution to (5). Then, for any $\delta < \overset{\circ}{\beta} + \gamma$ there are constants $C_1, C_2, C_3$, depending only on the coefficients of the problem and the $D(S) \cap D(M)$ norm of the initial conditions, such that for all sufficiently small $\epsilon > 0$
\begin{align}
\left\|s_r(t) - \left(n(t) - \overset{\circ}{w}(t) - \delta^\# \overset{\circ}{i} e^{-\frac{(\delta + \gamma) t}{\epsilon}}\right)\right\| &\leq C_1 \epsilon, \\
\left\|i_r(t) - \overset{\circ}{i} e^{-\frac{(\delta + \gamma) t}{\epsilon}}\right\| &\leq C_2 \epsilon e^{-\frac{\delta t}{\epsilon}}, \\
\left\|r_r(t) - \overset{\circ}{r} e^{-\frac{(\delta + \gamma) t}{\epsilon}}\right\| &\leq C_3 \epsilon,
\end{align}

\end{thm}
uniformly for \( t \in \mathbb{R}_+ \). Furthermore, if \( R_\mu < 1 \), then \( C_i \) can be replaced by \( C_i e^{\lambda_i t} \), \( i = 1, 3 \), so that the error decays to zero as \( t \to \infty \).

We emphasize that this result shows that the solution of the nonlinear problem (5) can be approximated by the solution of two scalar linear problems and explicitly given initial layer corrector, uniformly for any time.

2. Notation, assumptions and well-posedness results. Problems like (1)–(4) have been relatively well-researched, even in a much more complex setting involving nonlinear dependence of \( \mu \) and \( \beta \) on the total population, see [15, Section 5.3] or [7, 13] and references therein. Since, however, the results are scattered and refer to two distinct cases (with \( \omega < +\infty \) and \( \omega = \infty \)), we summarize basic facts in the form relevant to the problem at hand, see [11]. Though the model most resembles that discussed in [15], the main difference is that in op.cit. the maximum age is infinite but the death rate is bounded, which simplifies some arguments. The boundedness of \( \omega \) affects the analysis of the linear part of (1)–(4). We note that the biologically realistic assumption is that \( \omega < \infty \); that is, that no individual can live indefinitely. This requires the survival probability, given by (9), to satisfy \( \Pi_\mu(\omega) = 0 \) which, in turn, yields

\[
\int_0^\omega \mu(s)ds = \infty.
\]  

(15)

Hence, \( \mu \) cannot be bounded as \( a \to \omega^- \). This is in contrast with the case \( \omega = \infty \), where commonly it is assumed that \( \mu \) is a bounded function on \( \mathbb{R}_+ \) (see however [12]) and introduces another unbounded operator, the multiplication by \( \mu \), in (1). In some papers, e.g. [10], this was circumvented by assuming that there is a maximum reproductive age \( a_r < \omega \), so that the birth rate satisfies \( \beta(a) = 0 \) for \( a > a_r \), and hence ignoring the post-reproductive population by performing the analysis for \( a \in [0, a_r] \). Note that in this way we lose conservativeness of the model. The analysis of the model without any simplifying assumption in the scalar linear case was done in [9] by reducing it to an integral equation along characteristics. It is known that the solutions obtained in this way generate a strongly continuous semigroup on \( L_1([0, \omega]) \), see [2, 6, 15] (though in [15] it is assumed that \( \omega = \infty \)).

The technical details needed to handle the case with unbounded \( \mu \) on \([0, \omega]\) with \( \omega < \infty \) can be found in [11]. In particular, it follows that the realization of the operator \( \mathcal{A} := \mathcal{S} + \mathcal{M} \) on the domain

\[
D(\mathcal{A}) = \{ u \in D(\mathcal{S}) \cap D(\mathcal{M}); \ u(0) = Bu \}
\]  

(16)
generates a positive \( C_0 \)-semigroup, denoted by \( (e^{t \mathcal{A}})_{t \geq 0} \). Since, for a fixed \( \epsilon \), (5) is a quadratic perturbation of the linear system, a standard argument, see [5, 11], shows that if \( \tilde{\mathbf{u}} = (s, \vec{i}, \vec{r}) \in \mathbf{X}_+ \), then there exists a unique global positive mild solution \( t \to \mathbf{u}_\epsilon(t) = (s_\epsilon(t), i_\epsilon(t), r_\epsilon(t)) \in C([0, \infty), \mathbf{X}) \) to (5). This solution becomes a classical solution if \( \tilde{\mathbf{u}} \in D(\mathcal{A}) \). In such a case we obtain, in particular, that \( \mathbf{u}_\epsilon \) is continuous on \([0, \omega] \times [0, T]\) for any \( 0 \leq T < \infty \), \( \mathbf{u}_\epsilon \in D(\mathcal{S}) \cap D(\mathcal{M}) \) and (5) is satisfied termwise almost everywhere on \([0, \omega] \times [0, T]\). Standard calculations, see e.g. [10], show that \( (e^{t \mathcal{A}})_{t \geq 0} \) satisfies the estimate

\[
\|e^{t \mathcal{A}}\| \leq e^{(\beta - \omega)t}.
\]  

(17)
This estimate can be improved. Indeed, as noted earlier, if \((s_\epsilon(t), i_\epsilon(t), r_\epsilon(t))\) is a classical solution to (5), then \(n(t,a) = s_\epsilon(t,a) + i_\epsilon(t,a) + r_\epsilon(t,a)\) is a classical solution to (7). We denote by \((A,D(A))\) the generator of the solution semigroup \((e^{tA})_{t \geq 0}\) for (7), with the domain defined analogously to (16). Hence, since \(\dot{u}_\epsilon \geq 0\) yields \(u_\epsilon(t) = (s_\epsilon(t), i_\epsilon(t), r_\epsilon(t)) \geq 0\), we see that each component of \(u_\epsilon\) is controlled by the \(\epsilon\)-independent solution \(n\) of (7):

\[
0 \leq s_\epsilon(a,t) \leq n(a,t), \quad 0 \leq i_\epsilon(a,t) \leq n(a,t) \quad \text{and} \quad 0 \leq r_\epsilon(a,t) \leq n(a,t)
\]

for each \(t \geq 0\) and almost every \(a \in [0,\omega]\) and thus satisfy inequality (10).

3. **Formal asymptotic expansion.** Here we justify the terms of the approximation which appear in Theorem 1.1. Following the general approach of the asymptotic analysis, see e.g. [4], we are looking for the so-called hydrodynamic space \(V\) of the singularly perturbed equation (5) which, in this case is given by the null-space of \(C\).

Since here \(a\) is a parameter, we can carry out the calculations for a fixed \(a\). Then it is easy to see that \(V\) is a two dimensional subspace

\[
V = \{ u \in \mathbb{R}^3; \ u = (u_1, u_2), u_1, u_2 \in \mathbb{R} \} = \text{Span}\{e_1, e_2\},
\]

where \(e_1, e_2\) are an appropriate basis for \(V\). The complementary spectral space \(W\), called the kinetic space, corresponding to the eigenvalue \(\lambda(a) = -(\delta(a) + \gamma(a))\), is spanned by \(e_3(a) = (-\delta(a), 1, -\gamma(a))\), see (12). To find the decomposition \(u = c_1e_1 + c_2e_2 + c_3e_3\) into the hydrodynamic and kinetic space, we have to find the coefficients \(c_i, i = 1, 2, 3\) which will give us the aggregated variables. Using standard results from linear algebra, see e.g. [2], \(c_i = f_i \cdot u,f_i \cdot e_i\), where \(f_i\) are left eigenvectors of \(C\) corresponding to the same eigenvalue as \(e_i, i = 1, 2, 3\). Here, the situation is slightly more complicated as \(V\) is two dimensional and we have to select its basis in a convenient way. Since \(f_1 = (1, 1, 1)\) is a left eigenvector of \(C\) corresponding to the zero eigenvalue and \(f_1 \cdot u = u_1 + u_2 + u_3\), it should play an important role in the analysis due to (7). Then, for convenience of calculations, we take \(e_2 = (-1, 0, 1) \in V\), which is orthogonal to \(f_1\). We have some freedom in selecting \(f_2\): taking \(f_2 = (0, \gamma(a), 1)\) yields \(e_1 = (1, 0, 0)\). Finally \(f_3 = (0, 1, 0)\) so that

\[
u = (u_1 + u_2 + u_3)e_1 + (\gamma u_2 + u_3) e_2 + u_2 e_3.\]

Next, we use this decomposition to change variables in (5). Accordingly, we define \(n = s_\epsilon + i_\epsilon + r_\epsilon, w_\epsilon = \gamma i_\epsilon + r_\epsilon\) and leave \(i_\epsilon\) unchanged. This yields the system

\[
\begin{align*}
\partial_t n &= -\partial_n n - \mu n, \\
\partial_t i_\epsilon &= -\partial_n i_\epsilon - \mu i_\epsilon + \Lambda(i_\epsilon)(n - w_\epsilon - \delta i_\epsilon) - \frac{\delta + \gamma}{\epsilon} i_\epsilon, \\
\partial_t w_\epsilon &= -\partial_n w_\epsilon - \mu w_\epsilon + \gamma \Lambda(i_\epsilon)(n - w_\epsilon - \delta i_\epsilon) + \partial_n \gamma i_\epsilon,
\end{align*}
\]

\[
\begin{align*}
n(0,t) &= \int_0^\omega (a)n(a,t)da, \\
i_\epsilon(0,t) &= p \int_0^\omega (a)i_\epsilon(a,t)da, \\
w_\epsilon(0,t) &= q \int_0^\omega (a)w_\epsilon(a,t)da + \int_0^\omega (a)\gamma i_\epsilon(a,t)da.
\end{align*}
\]
where $\gamma^*(a) = (p\gamma^#(0) - q\gamma^#(a))$, and

\[
\begin{align*}
n(a,0) &= \overset{\circ}{n}(a) := \overset{\circ}{s}(a) + \overset{\circ}{r}(a), \quad i_e(a,0) = \overset{\circ}{i}(a), \\
w_e(a,0) &= \overset{\circ}{w}(a) := \gamma^#(a) \overset{\circ}{i}(a) + \overset{\circ}{r}(a).
\end{align*}
\] (21)

Thus, the pair $(n, w_e)$ belongs to the hydrodynamic space and $i_e$ to the kinetic space. Since the total population $n$ decouples from the system, there is no need to approximate it and, in the last two equations, it can be treated as a known function. Thus we shall focus on the pair $(w_e, i_e)$ and, following the Chapman-Enskog procedure, we only expand the kinetic part:

\[i_e = \bar{\tau}_0 + \epsilon \bar{\tau}_1 + \cdots.\]

Inserting it into the second equation of (20) and denoting the bulk approximation of $w_e$ by $\bar{w}$, we have

\[
\begin{align*}
\partial_i \bar{\tau}_0 + \epsilon \partial_i \bar{\tau}_1 &= -\partial_i \bar{\tau}_0 - \epsilon \partial_i \bar{\tau}_1 - \mu \bar{\tau}_0 - \epsilon \mu \bar{\tau}_1 + (\Lambda(\bar{\tau}_0) + \epsilon \Lambda(\bar{\tau}_1))(n - \bar{w} - \delta \bar{\tau}_0 - \epsilon \delta \bar{\tau}_1) \\
&= -\frac{\delta + \gamma}{\epsilon} \bar{\tau}_0 - (\delta + \gamma) \bar{\tau}_1 + O(\epsilon^2).
\end{align*}
\]

Comparing coefficients at like powers of $\epsilon$ and using $\Lambda(0) = 0$ we find $\bar{\tau}_k = 0$ for all $k \geq 0$. A similar behaviour of the bulk expansion recently has been observed in an alignment model, see [3].

Hence, denoting by $\bar{\tau}$ the bulk approximation of $i_e$, we arrive at the (formal) bulk approximation $(n, \bar{w}, \bar{\tau}) = (n, \bar{w}, 0)$, where $n$ and $\bar{w}$ satisfy (7) and (13), respectively. Note that (13) is obtained by substituting the approximation $i_e \approx \bar{\tau} = 0$ in the equations for $w_e$ in (20). To check how good the obtained approximation is, we have to find the error of the approximation. Denoting the error

\[E = (e_w, e_i) = (w_e - \bar{w}, i_e - \bar{\tau}) = (w_e - \bar{w}, i_e)\] (22)

and inserting it to the equations, we find

\[
\begin{align*}
\partial_t e_w &= -\partial_a e_w - \mu e_w - \gamma^# e_i + \gamma^# \Lambda(e_i)(e_w + \delta^# e_i) + \partial_a \gamma^# e_i + \gamma^# \Lambda(e_i)n - \bar{w}, \\
\partial_t e_i &= -\partial_a e_i - \mu e_i - \Lambda(e_i)(e_w + \delta^# e_i) - \frac{\gamma + \delta}{\epsilon} e_i + \Lambda(e_i)n - \bar{w},
\end{align*}
\]

\[e_w(0,t) = q \int_0^\omega \beta(a)e_w(a,t)da + \int_0^\omega \beta(a)\gamma^*(a)e_i(a,t)da,
\]

\[e_i(0,t) = p \int_0^\omega \beta(a)e_i(a,t)da,
\]

\[e_w(a,0) = 0, \quad e_i(a,0) = \overset{\circ}{i}(a).\] (23)

We see that there is at least one reason why the error cannot be of order $\epsilon$ – the initial condition is of order 1. Thus we have to introduce the initial layer correction by blowing up time according to $\tau = t/\epsilon$ and looking for the approximation

\[w_e(t) \approx \bar{w}(t), \quad i_e(t) = \overset{\circ}{i}(\tau) = \bar{\tau}_0(\tau) + O(\epsilon),\] (24)

were we anticipate that there is no need to introduce the initial layer for $w_e$ as $\bar{w}$ satisfies the exact initial condition. Rescaling time in (20) we see that due to $\partial_t = \epsilon^{-1} \partial_\tau$, the only equation for the terms at the $\epsilon^{-1}$ level is

\[\partial_\tau \bar{\tau}_0 = - (\delta + \gamma) \bar{\tau}_0\]
which, subject to the initial condition \( \tilde{\phi}_0(0) = \tilde{i}_0 \), gives the initial layer in the form

\[
\tilde{\phi}_0(a, \tau) = \tilde{i}(a)e^{-\frac{\tau(t_0(1-a) + i(a))}{\epsilon}}.
\]

The new error is given by

\[
\tilde{\mathbf{E}} = (\tilde{e}_w, \tilde{e}_i) = (w_\epsilon - \overline{w}, i_\epsilon - \tilde{\phi}_0) = (\tau_w, \tau_i - \tilde{\tau}_0).
\]

The error equations for \( \tilde{\mathbf{E}} \) can be obtained from (23) by expressing \( \tilde{w}_w \) and \( \tilde{e}_i \) in terms of \( \tilde{e}_w \) and \( \tilde{e}_i \), according to (26). Since \( \tilde{\tau} \in W_{1}^{1}([0, \omega]) \) and \( \tilde{\mu} \tilde{\tau} \in L_1([0, \omega]) \), we obtain

\[
\frac{\partial_t \tilde{e}_w}{\partial e_w} = -\partial_a \tilde{e}_w - \mu \tilde{e}_w - \gamma \Lambda(\tilde{e}_i)(\tilde{e}_w + \delta \tilde{e}_i) + \partial_a \gamma \tilde{e}_i - \gamma \Lambda(\tilde{e}_i)(n - \tilde{w}) - \gamma \delta \Lambda(\tilde{\tau}_0) + \partial_a \gamma \tilde{e}_i + \gamma \Lambda(\tilde{\tau}_0)(n - \tilde{w}),
\]

\[
\frac{\partial_t \tilde{e}_i}{\partial e_i} = -\partial_a \tilde{e}_i - \mu \tilde{e}_i - \Lambda(\tilde{e}_i)(\tilde{e}_w + \delta \tilde{e}_i) - \gamma \Lambda(\tilde{e}_i)(n - \tilde{w}) - \Lambda(\tilde{\tau}_0) \tilde{e}_i - \delta \Lambda(\tilde{\tau}_0) \tilde{e}_i + \Lambda(\tilde{\tau}_0)(n - \tilde{w}) - \partial_a \tilde{\tau}_0 - \mu \tilde{\tau}_0,
\]

\[
\tilde{e}_w(0, t) = q \int_{0}^{\infty} \beta(a) \tilde{e}_w(a, t)da + q \int_{0}^{\infty} \beta(a) \gamma(a) \tilde{e}_i(a, t)da + \int_{0}^{\infty} \beta(a) \gamma(a) \tilde{\tau}_0 \left(a, \frac{t}{\epsilon}\right) da \quad \text{and} \quad \tilde{e}_i(0, t) = p \int_{0}^{\infty} \beta(a) \tilde{e}_i(a, t)da + p \int_{0}^{\infty} \beta(a) \gamma(a) \tilde{\tau}_0 \left(a, \frac{t}{\epsilon}\right) da - \tilde{\tau}_0 \left(0, \frac{t}{\epsilon}\right),
\]

\[
\tau_w(a, 0) = 0, \quad \tau_i(a, 0) = 0.
\]

We see that on the boundary we still have terms which are of an order lower than \( \epsilon \). Fortunately, both inhomogeneities on the boundary have an exponential decay in \( t \to \infty \) and \( \epsilon \to 0 \). Thus, we anticipate that we do not need the boundary layer, compare [6], but only the corner layer which is obtained by simultaneous rescaling of time and age according to \( \tau = t/\epsilon, \alpha = a/\epsilon \). Unfortunately, the standard approach to the corner layer, [6], will not suffice here as the corner layer equation will not incorporate the unbounded operator \( M \). Thus the classical corner layer will not belong to \( D(M) \) and we would not be able to substitute the error terms into the equation, as in (27). To alleviate the problem, we define the corner corrector to be the solution to

\[
\frac{\partial_t \tilde{w}}{\partial w} = -\partial_a \tilde{w} - \mu \tilde{w},
\]

\[
\frac{\partial_i \tilde{\tau}}{\partial i} = -\partial_a \tilde{\tau} - \mu \tilde{\tau} - \frac{\gamma + \delta}{\epsilon} \tilde{\tau},
\]

\[
\tilde{w}(0, t) = q \int_{0}^{\infty} \beta(a) \tilde{w}(a, t)da + q \int_{0}^{\infty} \beta(a) \gamma(a) \tilde{i}(a, t)da + \int_{0}^{\infty} \beta(a) \gamma(a) \tilde{\tau}_0 \left(a, \frac{t}{\epsilon}\right) da \quad \text{and} \quad \tilde{i}(0, t) = p \int_{0}^{\infty} \beta(a) \tilde{i}(a, t)da + p \int_{0}^{\infty} \beta(a) \gamma(a) \tilde{\tau}_0 \left(a, \frac{t}{\epsilon}\right) da - \tilde{\tau}_0 \left(0, \frac{t}{\epsilon}\right),
\]

\[
\tilde{w}(a, 0) = \phi \left(\frac{a}{\epsilon}\right), \quad \tilde{i}(a, 0) = 0,
\]

where \( \phi \) is a function to be determined. The reason for introducing this function is that for the classical solvability of the problem with inhomogeneous boundary
conditions one needs the equality of the boundary condition and the initial condition at \((a, t) = (0, 0)\). This is easy to see here, first for \(i\), as the equation for \(i\) is decoupled and can be solved separately. Then, solving it by reducing it to a Volterra equation, as in [9], shows that if the inhomogeneity is zero at \(t = 0\), then the total birth rate is 0 at \(t = 0\) which, together with \(i(0, a) = 0\), ensures the continuity of the solution across the diagonal \(t = a\), see [11]. In our case we have

\[
p \int_0^\omega \beta(a) \tilde{\gamma}^*(a, 0) da - \tilde{\gamma}_0 (0, 0) = p \int_0^\omega \beta(a) \tilde{\gamma}^*(a) da - \tilde{\gamma}^*(0) = 0
\]
as we assumed that the initial condition for the original problem satisfies \((\check{s}, \check{i}, \check{r}) \in D(A)\). On the other hand, typically

\[
\Theta := \int_0^\omega \beta(a) \gamma^*(a) \tilde{\gamma}_0 (a, 0) da = \int_0^\omega \beta(a) \gamma^*(a) \tilde{\gamma}^*(a) da \neq 0
\]

(it will be zero for any \(\tilde{\gamma}\), if \(p = q\) and \(\gamma^\#\) is a constant function). Thus, we consider

\[
\phi(a) = c_\epsilon e^{-a/\epsilon},
\]

where \(c_\epsilon\) is given by

\[
c_\epsilon = \frac{\Theta}{1 - q \int_0^\omega \beta(a) e^{-\frac{a}{\epsilon}} da}.
\]

Since \(\int_0^\omega \beta(a) e^{-\frac{a}{\epsilon}} da \to 0\) as \(\epsilon \to 0\), \(c_\epsilon\) is bounded as \(\epsilon \to 0\).

The necessary estimates of the corner layer solution will be provided later. Here, assuming that it is sufficiently regular, we consider the new approximation \(w_\epsilon \approx \bar{w} + \check{w}\) and \(i_\epsilon \approx \tilde{i}_0 + \check{i}\). The corresponding error,

\[
\check{E} = (\check{e}_w, \check{e}_i) = (w_\epsilon - \bar{w}, i_\epsilon - \tilde{i}_0 - \check{i}) = (\check{e}_w - \check{w}, \check{e}_i - \check{i}),
\]

satisfies the system

\[
\partial_t \check{e}_w = -\partial_a \check{e}_w - \mu \check{e}_w - \gamma^\# \Lambda(\check{e}_w) \check{e}_w + \delta^\# \check{e}_i + \partial_a \gamma^\# \check{e}_i
\]

\[
-\gamma^\# \Lambda(\check{\theta}_0) \check{e}_w + \delta^\# \check{e}_i - \gamma^\# \Lambda(\tilde{i}_0) \check{e}_w + \delta^\# \check{e}_i - \gamma^\# \delta^\# \Lambda(\tilde{i}_0) \tilde{i}_0
\]

\[
+ \gamma^\# \Lambda(\check{\theta}_0)(n - \bar{w}) - \gamma^\# \Lambda(\check{e}_i)(\check{w} + \delta^\# \check{i}) - \gamma^\# \delta^\# \Lambda(\tilde{i}_0) \tilde{i}_0 + \partial_a \gamma^\# \check{i}
\]

\[
+ \gamma^\# \Lambda(\check{\theta}_0)(n - \bar{w}) + \partial_a \gamma^\# \check{i} - \gamma^\# \Lambda(\tilde{i}_0)(\check{w} + \delta^\# \check{i}) - \gamma^\# \delta^\# \Lambda(\tilde{i}_0) \tilde{i}_0
\]

\[
+ \gamma^\# \Lambda(\tilde{i}_0)(n - \bar{w}) - \gamma^\# \Lambda(\tilde{i})(\check{w} + \delta^\# \check{i}),
\]

\[
\partial_t \check{e}_i = -\partial_a \check{e}_i - \mu \check{e}_i - \gamma^\# \Lambda(\check{e}_w) \check{e}_w + \delta^\# \check{e}_i
\]

\[
- \Lambda(\check{\theta}_0)(\check{e}_w + \delta^\# \check{i}) - \Lambda(\tilde{i}_0)(\check{e}_w + \delta^\# \check{i}) - \delta^\# \Lambda(\tilde{i}_0) \tilde{i}_0 + \Lambda(\tilde{i}_0)(n - \bar{w})
\]

\[
- \Lambda(\check{\theta}_0)(\check{w} + \delta^\# \check{i}) - \delta^\# \Lambda(\tilde{i}_0) \tilde{i}_0 + \Lambda(\tilde{i}_0)(n - \bar{w})
\]

\[
- \Lambda(\check{\theta}_0)(\check{w} + \delta^\# \check{i}) - \delta^\# \Lambda(\tilde{i}_0) \tilde{i}_0 + \Lambda(\tilde{i}_0)(n - \bar{w}) - \Lambda(\tilde{i})(\check{w} + \delta^\# \check{i})
\]

\[
- \partial_a \tilde{\gamma}_0 - \mu \tilde{\gamma}_0,
\]

with

\[
\check{e}_w(0, t) = q \int_0^\omega \beta(a) \check{e}_w(a, 0) da + \int_0^\omega \beta(a) \gamma^*(a) \check{e}_i(a, t) da
\]
\[ \dot{e}_i(0, t) = p \int_0^\infty \beta(a) \dot{e}_i(a, t) da, \]
\[ \dot{e}_w(a, 0) = -\phi(a), \quad \dot{e}_i(a, 0) = 0. \]

4. Error estimates. We split this section into the corner corrector estimates and the main error estimates.

4.1. Corner corrector estimates. Though the following estimates can be done in a general case, we shall take advantage of the fact that we can obtain the solution by solving two scalar McKendrick problems. We begin with a general observation. Let \( 0 \leq M \in L_{\text{loc, } \infty}([0, \omega]) \) be such that
\[ \Pi_M(\omega) = e^{-\int_0^\omega M(s) ds} = 0, \]
and \( \lambda_M \) be the solution of the equation (8) with \( \mu \) replaced by \( M \). Further, let \( \psi \) be a continuous function satisfying \( \psi(t) = O(e^{-ct}) \) for some constant \( c \) such that
\[ -c < \lambda_M. \] (33)

Lemma 4.1. If \( \phi \) is the solution of
\[ \partial_t \phi = -\partial_a \phi - M \phi, \quad \phi(0, t) = \int_0^\infty B(a) \phi(a, t) da + \psi(t), \quad \phi(a, 0) = \phi(a), \] (34)
where
\[ \phi(0) = \int_0^\infty B(a) \phi(a) da + \psi(0), \]
then there are constants \( K \) and \( B \) such that
\[ \|\phi(t)\| \leq K \left( e^{\lambda_M t} \|\phi\| + B \right) + \int_0^{\min\{t, \omega\}} |\psi(t-s)\Pi_M(s) ds \). \] (35)

Proof. The proof is very similar to that in [9] for proving the AEG property of the McKendrick problem (with \( \psi = 0 \)), hence we only provide a brief sketch of it.

Using the linearity, we can write the total birth rate \( B(t) = \phi(0, t) \) as \( B(t) = B_\phi(t) + B_\psi(t) \) and focus on the case \( \phi = 0 \). Thus
\[ B_\psi(t) = \int_0^{\min\{t, \omega\}} K_M(t-a) B_\phi(a) da + \psi(t), \] (36)
where \( K_M(a) = \beta(a) \Pi_M(a) \). To use the uniform notation for the Laplace transform, all functions defined on \([0, \omega]\) are considered to be extended by 0 outside this interval. Taking the Laplace transform of (36), we obtain
\[ \hat{B}_\psi(\lambda) = \frac{\hat{\psi}(\lambda)}{1 - K_M(\lambda)} = \hat{\psi}(\lambda) + \frac{\hat{\psi}(\lambda)K_M(\lambda)}{1 - K_M(\lambda)}. \] (37)
The difference with the calculations of [9], which will become relevant later, is that \( \hat{\psi} \) is not an entire function. It is, however, analytic in \( \Re \lambda > -c \), hence the only singularities of \( \hat{B}_\psi \) are due to the zeroes of \( 1 - K_M \) or are contained in the half-plane.
\{\Re \lambda \leq -c\}. Since \(\hat{K}_M\) is an entire function, its zeroes are isolated of finite order (thus giving rise to poles of \(B_\psi\) which do not have finite accumulation point). As mentioned earlier, \(\lambda_M\) is the only real root of multiplicity 1 of (8) and in each strip \(a < \Re \lambda < b\), there is at most a finite number of other roots. By (33), \(\lambda_M\) is in the domain of analyticity of \(\hat{\psi}\).

Let us consider the second term in the last formula of (37), \(H := \hat{\psi}\hat{K}_M/(1-\hat{K}_M)\). As in [9], on any line \(\{\sigma + iy; \ y \in \mathbb{R}\}\) which does not meet any root of (8), we have

\[
\inf_{y \in \mathbb{R}} |1 - \hat{K}_M(\sigma + iy)| = m_\sigma > 0, \quad \int_{-\infty}^{\infty} \left| \frac{\hat{\psi}(\sigma + iy)\hat{K}_M(\sigma + iy)}{1 - \hat{K}_M(\sigma + iy)} \right| dy < \infty. \quad (38)
\]

Inverting \(H(\lambda)\), we have

\[
H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(\sigma + iy)\hat{K}_M(\sigma + iy) e^{(\sigma + iy)t} dy
\]

for any \(\sigma > \lambda_M\). Hence \(B_\psi(t) = \psi(t) + H(t)\). To estimate \(H(t)\) we note that, by properties of \(H\), we can shift the line of integration to \(\{\sigma_1 + iy; \ y \in \mathbb{R}\}\), where \(\zeta < \sigma_1 < \lambda_M\), \(\zeta = \max\{\Re \lambda_M, -c\}\) and \(\lambda_1\) is the first eigenvalue such that \(\Re \lambda_1 < \lambda_M\). Then, using the Cauchy theorem, we can write \(H(t) = H_1(t) + H_2(t)\), where

\[
H_1(t) = \text{res}_{\lambda = \lambda_M} e^{\lambda t}\hat{\psi}(\lambda)\hat{K}_M(\lambda) = B_0 e^{\lambda_M t}
\]

with

\[
B_0 = \frac{\int_{0}^{\infty} e^{-\lambda_M a} \psi(a) da}{\int_{0}^{\infty} a e^{-\lambda_M a} K_M(a) da}
\]

and \(H_2\) satisfies the estimate

\[
|H_2(t)| \leq \frac{e^{\sigma_1 t}}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\hat{\psi}(\sigma_1 + iy)\hat{K}_M(\sigma_1 + iy)}{1 - \hat{K}_M(\sigma_1 + iy)} \right| dy \leq B_1 e^{\sigma_1 t}, \quad (40)
\]

where \(B_1\) satisfies \(B_1 \leq C\|k_\sigma\|_{L_2(\mathbb{R})}\|\phi_\sigma\|_{L_2(\mathbb{R})}\) for some constant \(C\). Thus, we arrived at the representation

\[
B_\psi(t) = e^{\lambda_M t} B_0 + \psi(t) + e^{\sigma_1 t} B_1,
\]

or, taking into account that \(\sigma_1 < \lambda_M\), and denoting \(B_2 = |B_0| + |B_1|\), we can write

\[
|B_\psi(t)| \leq B_2 e^{\lambda_M t} + |\psi(t)|. \quad (41)
\]

Using the standard representation of the solution in terms of \(B\) and the estimates of [9, p. 22] for \(B_\phi\), we arrive at \((35)\). \(\square\)

We want to specify \((35)\) to the context of \((28)\). First, let us denote \(\alpha(a) = \gamma(a) + \delta(a)\) and consider an arbitrary constant \(\alpha^* < \alpha\), see \((6)\). We do not assume \((11)\) here.
Proposition 1. Let \( \tilde{i} \) and \( \tilde{w} \) be the solution to (28). Then there are constants \( C_i \) and \( C_w \), depending on the coefficients of the problem and on the \( W^1_0(\mathbb{R}_+) \) norm of the initial condition \( i \), such that for any \( t \in \mathbb{R}_+ \),
\[
\| \tilde{i}(t) \| \leq C_i e^{-\frac{\omega}{\epsilon} t} \max\{e^{\lambda_M t}, 1\},
\]
\[
\| \tilde{w}(t) \| \leq C_w \epsilon \max\{e^{\lambda_M t}, 1\}.
\]

Proof. Let us begin with some general comments and notation. First, we observe that if \( M \leq M_0 \) then, since \( \lambda_M \) and \( \lambda_{M_0} \) are the unique real solutions to (8), from monotonicity it follows
\[
\lambda_{M_0} \leq \lambda_M. \tag{44}
\]
Let \( \lambda_{M_0} \) be the (dominant) eigenvalue corresponding to \( M_0 = M_0 + \theta/\epsilon \) for any function of \( a \), or a constant, \( \theta \). Note that \( \lambda_{M_0} = \lambda_M \), the dominant eigenvalue corresponding to \( M_0 = \mu \); that is, of the problem (7). Hereafter, we shall use that later notation. Then, by the definition of \( \alpha^* \), we have
\[
\lambda_{M_0} < \lambda_{M_0}^* = \lambda_M - \frac{\alpha^*}{\epsilon}. \tag{45}
\]
Let \( \lambda' \) be the solution to (8) with the largest real part \( \Re \lambda' = \sigma' \) that is smaller than \( \lambda_\mu \). Then \( \lambda' - \alpha^*/\epsilon \) is the solution to (8) specified to \( M_{\alpha^*} \); that is, of the equation
\[
\int_0^\omega e^{-\lambda a}K_{M_{\alpha^*}}(a)da = 1 \tag{46}
\]
and there are no other solutions in \( \{ \lambda; \sigma' - \alpha^*/\epsilon < \Re \lambda < \lambda_{M_{\alpha^*}} = \lambda_M - \alpha^*/\epsilon \} \).

Next, by \( \phi_\theta \) we denote the solution to (34) with \( M_\theta \). If \( \theta_2 \leq \theta_1 \), then
\[
\phi_{\theta_1}(a, t) \leq \phi_{\theta_2}(a, t), \quad \text{for a.a. } (t, a) \in \mathbb{R}_+ \times [0, \omega]. \tag{47}
\]

First we consider the equation for \( \tilde{i} \). The problem with the direct application of (35) is that the function \( \tilde{i} \) in this case satisfies
\[
|\tilde{i}(t)| = |p \int_0^\omega \beta(a) \tilde{i}_0 \left( a, \frac{t}{\epsilon} \right) da - \tilde{i}_0 \left( 0, \frac{t}{\epsilon} \right)| \leq e^{-\frac{\omega}{\epsilon}(p \beta \| \tilde{i}_0 \| \| \tilde{i} \|_{W^1_0([0, \omega])})} \leq C_1 e^{-\frac{\omega}{\epsilon} t}
\]
for some constant \( C_1 \). Thus, \( \tilde{i} \) is analytic for \( \Re \lambda > -\alpha/\epsilon \) and, by (45), in general \( \lambda_{M_{\alpha^*}} \) will not belong to the domain of analyticity of \( \tilde{i} \) if \( \lambda_\mu \leq 0 \). Thus, in this case, we have to settle on a slightly weaker estimate. Namely, by (47), we write
\[
\phi_{\alpha} \leq \phi_{\alpha^*}
\]
and, if \( \lambda_\mu < 0 \), we assume that \( \epsilon \leq \epsilon_0 \) with \( \epsilon_0 < -(\alpha - \alpha^*)/\lambda_\mu \). Then \( \lambda_{M_{\alpha^*}} \) is in the domain of analyticity of \( \tilde{i} \) and we carry out the estimates for \( \phi_{\alpha^*} \). We have
\[
\int_0^\omega e^{-\lambda_{M_{\alpha^*}} a} |\psi(a)| da \leq C_1 \int_0^\omega e^{-\left( \lambda_\mu - \frac{\omega}{\epsilon} \right) a} e^{-\frac{\omega}{\epsilon} a} da = \frac{C_1 \epsilon}{e\lambda_\mu + (\alpha - \alpha^*)}.
\]
Similarly,
\[
\int_0^\omega a e^{-\lambda_{M_{\alpha^*}} a} K_{M_{\alpha^*}}(a) da = \int_0^\omega a e^{-\lambda_\mu a} \beta(a) e^{-\int_0^a \mu(s) ds} da.
\]
Hence, we can see that in this case
\[ |B_0| \leq C\epsilon \quad (48) \]
for some constant \( C \). Next, we need to estimate \( B_1 \). To simplify calculations for \( \sigma' < 0 \), we further assume that \( \epsilon \leq \epsilon_0 \) where \( \epsilon_0 < -(\alpha_0 - \alpha)'/\sigma' \), otherwise \( \epsilon \) can be arbitrary. Then both \( \lambda M_{\alpha, \beta} \) and \( \lambda' - \alpha' / \epsilon \) belong to the domain of analyticity of \( \hat{\psi} \).

First we observe that \( \inf_{\mu \in R} |1 - \hat{K}_{M_{\alpha, \beta}}(\sigma + i\epsilon)| \geq m > 0 \) independently of \( \epsilon \), where \( \sigma \) is between \( \lambda M_{\alpha, \beta} \) and the real part of the next solution to (46). In fact, by the considerations above, we can take \( \sigma = \tilde{\sigma} - \alpha' / \epsilon \), where \( \sigma' < \tilde{\sigma} < \lambda_\mu \) is independent of \( \epsilon \). Then
\[ \hat{K}_{M_{\alpha, \beta}}(\sigma + iy) = \int_0^\omega e^{-(\tilde{\sigma} - \frac{\alpha}{\epsilon})a\beta(a)e^{-\frac{i}{\sigma} \mu(s)ds - \frac{\alpha}{\epsilon}a}} da = \hat{K}_\mu(\tilde{\sigma} + iy) \]
and the claim follows. To complete the estimate of \( B_1 \) we calculate
\[ \int_0^\omega |e^{-\frac{2\alpha}{\epsilon}a}e^{\sigma' a}K_{M_{\alpha, \beta}}(\sigma e^{-\sigma a})| da \leq C_2 \int_0^\omega e^{-\frac{\alpha - \alpha'}{\epsilon}a} da \leq \epsilon C_3, \quad (49) \]
where \( C_3 \) is bounded independently of \( \epsilon \).

Let us estimate \( \hat{\iota} \) from the relevant equation in (28). By (35), we have
\[ \min\{t, \omega\} \int_0^\omega |\psi(t - s)| \Pi_{M_{\alpha, \beta}}(s) ds \leq C_1 e^{-\frac{\alpha}{\epsilon}t} \left\{ \begin{array}{ll} 1 - e^{-\frac{\alpha - \alpha'}{\epsilon}t} & \text{for } t < \omega, \\ e^{\frac{\alpha}{\epsilon}t} - e^{-\frac{\alpha}{\epsilon}t} & \text{for } t \geq \omega, \end{array} \right. \]
\[ \leq C_5 e^{-\frac{\alpha}{\epsilon}t}. \quad (50) \]
Hence, taking into account that \( \hat{\iota}(0) = 0 \), (48) and (49) give
\[ \|\hat{\iota}(t)\| \leq C_4 e^{\lambda_\mu t - \frac{\alpha}{\epsilon}t} + C_5 e^{-\frac{\alpha}{\epsilon}t} \]
which yields (42).

Next we use this estimate for \( \hat{\psi} \). Here
\[ |\psi(t)| = \left| \int_0^\omega \beta(a)\gamma'(a)\hat{\iota}(a, t) da + \int_0^\omega \beta(a)\gamma'(a) \hat{i}, \left(a, \frac{t}{\epsilon}\right) \right| da \]
\[ \leq C_6 \|\hat{\iota}(t)\| + C_7 \left| \hat{i}, \left(a, \frac{t}{\epsilon}\right) \right| \leq C_8 e^{\lambda_\mu t - \frac{\alpha}{\epsilon}t} + C_9 e^{\frac{\alpha}{\epsilon}t} + C_{10} e^{-\frac{\alpha}{\epsilon}t} \]
In this case \( \lambda_M = \lambda_\mu \) and, using the above,
\[ \int_0^\infty e^{-\lambda_M a t} |\psi(a)| da \leq C_{11} \epsilon. \]
Hence, since the denominator in (39) does not depend on \( \epsilon \), we get \( B_0 \leq C_{12} \epsilon \). Similarly, \( \sigma_1 \) in the line of integration in (40) does not depend on \( \epsilon \), so that the denominator is bounded away from zero independently of \( \epsilon \) and for the numerator we have
\[ \int_0^\omega |\psi(a)K_\mu(a) e^{-2\sigma_1 a}| da \leq \epsilon C_{13}. \quad (51) \]
Hence, $|B_2| \leq C_4 \epsilon$. Therefore, by (29), (35) and (50) with $\alpha^* = 0$ and $\epsilon < \alpha^*/\lambda_\mu$ if $\lambda_\mu > 0$, we arrive at (43). \hfill $\Box$

### 4.2. Proof of Theorem 1.1.

We shall simplify the notation and subsequent calculations by introducing the rescaled errors $\tilde{e}_w = \epsilon u$ and $\tilde{e}_i = \epsilon v$; that is,

$$w_\epsilon = \tilde{w} + \tilde{w} + \epsilon u, \quad i_\epsilon = \tilde{i}_0 + \tilde{i} + \epsilon v.$$  \hspace{1cm} (52)

This converts (32) into

$$\partial_t u = -\partial_x u - \mu u - \epsilon \gamma^\# \Lambda(v)(u + \delta^\# v) + \gamma^\# F_1(i_0, \tilde{i})u + (\partial_x \gamma^\# + \gamma^\# F_2(\tilde{i}_0, \tilde{i}))v + \gamma^\# F_3(n, \tilde{w}, i_0, \tilde{w}, \tilde{i}) \Lambda(v) + \frac{1}{\epsilon} \left( \partial_x \gamma^\#(i_0 + \tilde{i}) + \gamma^\# G(n, \tilde{w}, \tilde{i}_0, \tilde{w}, \tilde{i}) \right)$$

$$\partial_t v = -\partial_x v - \mu v - \frac{\gamma + \delta}{\epsilon} v - \epsilon \gamma(v)(u + \delta^\# v) + F_1(\tilde{i}_0, \tilde{i})u + F_2(\tilde{i}_0, \tilde{i})v + F_3(n, \tilde{w}, i_0, \tilde{w}, \tilde{i}) \Lambda(v) + \frac{1}{\epsilon} G(n, \tilde{w}, \tilde{i}_0, \tilde{w}, \tilde{i}) - \partial_x \tilde{i}_0 - \mu \tilde{i}_0,$$

where

$$F_1(i_0, \tilde{i}) = -\Lambda(\tilde{i}_0) - \Lambda(\tilde{i}), \quad F_2(i_0, \tilde{i}) = -\delta^\# \Lambda(i_0) - \delta^\# \Lambda(\tilde{i}),$$

$$F_3(n, \tilde{w}, i_0, \tilde{w}, \tilde{i}) = -\delta^\# \tilde{i}_0 + (n - \tilde{w}) - (\tilde{w} + \delta^\# \tilde{i}),$$

$$G(n, \tilde{w}, i_0, \tilde{w}, \tilde{i}) = -\delta^\# \Lambda(\tilde{i}_0)\tilde{i}_0 + \Lambda(\tilde{i}_0)(n - \tilde{w}) - \Lambda(\tilde{i}_0)(\tilde{w} + \delta^\# \tilde{i}) - \delta^\# \Lambda(\tilde{i})\tilde{i}_0 + \Lambda(\tilde{i})(n - \tilde{w}) - \Lambda(\tilde{i})(\tilde{w} + \delta^\# \tilde{i}).$$  \hspace{1cm} (54)

Let us recall that we assumed (11). In this case $\lambda_\mu \leq 0$ and both $n$ and $\tilde{w}$ (since $q \in [0,1]$) satisfy (10). Then, (52), by (10) with (18), (25), (42) and (43), gives

$$\|e_v\| \leq C\|\tilde{n}\|_{\mathcal{W}_1^p(\mathbb{R}^+)} \left(1 + e^{-\frac{\tilde{w}}{\epsilon} t} + e^{-\frac{\tilde{i} + \epsilon}{\epsilon} t}\right)$$ \hspace{1cm} (55)

$$\|e_u\| \leq C\|\tilde{n}\|_{\mathcal{W}_1^p(\mathbb{R}^+)} (1 + \epsilon)$$ \hspace{1cm} (56)

for some constant $C$.

Consider now the auxiliary problem

$$\partial_t \phi = -\partial_x \phi - \mu \phi - \frac{\gamma + \delta}{\epsilon} \phi$$

$$\phi(0, t) = p \int_0^\omega \beta(a) \phi(a, t) da,$$

$$\phi(a, 0) = \phi(a).$$  \hspace{1cm} (57)

Using the fact that $(e^{tA})_{t \geq 0}$ is a positive semigroup, it is easy to see that then

$$\|\phi(t)\| \leq me^{-\frac{\tilde{w}}{\epsilon} t} \|\phi\|, \quad t \geq 0,$$  \hspace{1cm} (58)

where, as defined before Proposition 1, $\tilde{a} = \min_{a \in [0,\omega]}(\gamma(a) + \delta(a))$. Note that the assumption on $R_\mu$ is also sufficient here since $p \in [0,1]$. Using this estimate and the
Duhamel formula, we have
\[
\|v(t)\| \leq m \int_0^t e^{-\frac{\alpha}{2}(t-s)} \left\| \epsilon \Lambda(v)(u + \delta^\# v) + F_1(i_0, i)u + F_2(i_0, i)v \right\| ds + \left\| \epsilon G(n, w, i_0, w, i) + \partial_\delta i_0 + \mu i_0 \right\| ds. \tag{59}
\]
Next, by (55) and (56), we obtain
\[
\left\| \epsilon \Lambda(v)(u + \delta^\# v) \right\| \leq c_1 \|v\|. \tag{60}
\]
Further, by (55) and (42),
\[
\|F_1(i_0, i)u\| \leq c_2 \epsilon^{-1}(\|i_0\| + \|i\|) \leq c_3(e^{-\frac{\alpha}{2}t} + \epsilon^{-1} e^{-\frac{\alpha}{2}t}). \tag{61}
\]
Similarly, by (42),
\[
\|F_2(i_0, i)v\| \leq c_4(\|i_0\| + \|i\|)\|v\| \leq c_5(e^{-\frac{\alpha}{2}t} + \epsilon e^{-\frac{\alpha}{2}t})\|v\|. \tag{62}
\]
Next, by (10), (42) and (43),
\[
\|F_3(n, w, i_0, w, i)\| \leq c_6\|v\| \left( e^{-\frac{\alpha}{2}t} + 1 + \epsilon + e^{-\frac{\alpha}{2}t} \right). \tag{63}
\]
Finally,
\[
\left\| \frac{1}{\epsilon} G(n, w, i_0, w, i) \right\| \leq c_7 e^{-\frac{2\alpha}{5} t} + e^{-\frac{\alpha}{2} t} + e^{-\frac{\alpha}{2} t} (1 + e^{-\frac{\alpha}{2} t}) + e^{-\frac{2\alpha}{5} t} + e^{-\frac{\alpha}{2} t} (1 + e^{-\frac{\alpha}{2} t}) \tag{64}
\]
and
\[
\|\partial_\delta i_0 + \mu i_0\| \leq c_8 e^{-\frac{\alpha}{2} t}, \tag{65}
\]
where \(c_8\) depends, in particular, on the \(L_1\) norm of \(\partial_\delta i_0\) and \(\mu i_0\); that is, on the \(D(A)\) norm of the initial condition. Let us define \(V(t) = \| e^{\frac{\alpha}{2} t} v(t) \|\), with \(0 < \alpha' < \alpha^* < \alpha\), see the definition above (45). Then (59) can be written as
\[
V(t) \leq C \int_0^t V(s) ds + \int_0^t \Psi(s) ds, \tag{66}
\]
where \(C\) is a constant and \(0 \leq \Psi(s) \leq c_1_0 e^{-\frac{\alpha''}{2} t}(1 + e^{-\frac{\alpha''}{2} t})\) for some \(0 < \alpha'' < \alpha^* - \alpha'.\) Thus,
\[
\int_0^t \Psi(s) ds \leq c_9 \frac{1 + \epsilon}{\alpha''} (1 - e^{-\frac{\alpha''}{2} t}) \leq c_{10}(1 - e^{-\frac{\alpha''}{2} t})
\]
and the Gronwall inequality gives
\[
V(t) \leq c_{10} \left( 1 - e^{-\frac{\alpha''}{2} t} \right) + C c_{10} e^{C t} \int_0^t e^{-C s} \left( 1 - e^{-\frac{\alpha''}{2} s} \right) ds \leq c_{11} e^{C t}
\]
and hence
\[
\|v(t)\| \leq c_{12} e^{\left( C - \frac{\alpha'}{2} \right) t} \leq c_{12} e^{-\frac{\alpha'}{2} t} \tag{67}
\]
for \(\epsilon < (\alpha' - \vartheta)/C\), where \(\vartheta\) is an arbitrary positive constant smaller than \(\alpha'\) (and thus \(\vartheta\) maybe an arbitrary positive constant smaller than \(\alpha\), provided all intermediate constants have this property as well).
Let us turn our attention to the estimates of \( u \). We note that \( u \) satisfies

\[
\partial_t u = -\partial_a u - \mu u + \Theta(t),
\]

\[
u(0, t) = q \int_0^\omega \beta(a)u(a, t)da + \psi(t),
\]

\[
u(a, 0) = \frac{1}{\epsilon} \phi(a),
\]

where \( \phi \) is given by (29), \( \psi(t) = \int_0^\omega \beta(a)\gamma^r(a)v(a, t)da \) and

\[
\Theta(t) = -e\gamma^r \Lambda(v)u + \gamma^r F_1(\tilde{i}_0, \tilde{\nu})u + (\partial_a \gamma^r + \gamma^r F_2(\tilde{i}_0, \tilde{\nu}))v + \gamma^r F_3(n, \bar{\nu}, \tilde{i}_0, \bar{\nu}, \tilde{\nu})\Lambda(v)
\]

\[
+ \frac{1}{\epsilon} \left( \partial_a \gamma^r(\tilde{i}_0 + \tilde{\nu}) + \gamma^r G(n, \bar{\nu}, \tilde{i}_0, \bar{\nu}, \tilde{\nu}) \right)
\]

is a known function. Using (60)–(64) together with (67), the regularity of \( \gamma^r \) and its derivative and \( 0 < \vartheta < \alpha' < \alpha < \alpha \), we see that the function \( \Theta \) satisfies

\[
\Theta(t) \leq c_{13}e^{-\vartheta t}(1 + \epsilon^{-1}).
\]

Denote problem (68) by \( P(\Theta, \psi, -\epsilon^{-1} \phi) \). Since it is a linear problem, its solution \( u \) can be written as \( u = u_1 + u_2 \), where \( u_1 \) solves \( P(\Theta, 0, -\epsilon^{-1} \phi) \) and \( u_2 \) solves \( P(0, \psi, 0) \). Further, denote by \( (e^{tA_\epsilon})_{t \geq 0} \) the McKendrick semigroup of the problem \( P(0, 0, -\epsilon^{-1} \phi) \). Since \( 0 \leq q \leq 1 \), \( (e^{tA_\epsilon})_{t \geq 0} \) satisfies (10); that is, \( \|e^{tA_\epsilon}\| \leq m \) for \( t \geq 0 \) and hence

\[
\|u_1(t)\| \leq \epsilon^{-1}m\|\phi\| + mc_\epsilon + mc_{13}(1 + \epsilon^{-1}) \int_0^t e^{-\vartheta s}ds \leq c_{14},
\]

where \( c_{14} \) is independent of \( \epsilon \) by (30). Next, using (35) with

\[
|\psi(t)| = \left| \int_0^\omega \beta(a)\gamma^r(a)v(a, t)da \right| \leq c_{15}e^{-\vartheta t}
\]

(derived by (67)) and \( \lambda_\mu \leq 0 \), we have

\[
\|u_2(t)\| \leq c_{16} \left( 1 + \epsilon \vartheta^{-1}(e^{\frac{\min(0, \omega-1)}{\epsilon}} - e^{-\vartheta t}) \right) \leq c_{17}. \tag{69}
\]

This ends the proof of the first part of Theorem 1.1.

Next, let \( \lambda_\mu < 0 \). Then \( \|e^{tA_\epsilon}\| \leq mc_{\lambda_\mu t} \) for \( t \geq 0 \) and

\[
\|u_1(t)\| \leq mc_{\epsilon}e^{\lambda_\mu t} + mc_{13}e^{\lambda_\mu t}(1 + \epsilon^{-1}) \int_0^t e^{-\left(\lambda_\mu + \frac{2}{\epsilon}\right)s}ds
\]

\[
\leq mc_{\epsilon}e^{\lambda_\mu t} + e^{\lambda_\mu t}mc_{12}(1 + \epsilon) \frac{mc_{12}(1 + \epsilon)}{\vartheta + c_{\lambda_\mu}} \left( 1 - e^{-\left(\lambda_\mu + \frac{2}{\epsilon}\right)t} \right) \leq c_{18}e^{\lambda_\mu t} \tag{70}
\]

for \( \epsilon < -\vartheta/2\lambda_\mu \), where \( c_{18} \) is independent of \( \epsilon \).

Further, let us observe that

\[
e^{\frac{\vartheta\min(0, \omega-1)}{\epsilon}} \leq e^{-2\lambda_\mu \omega}e^{\lambda_\mu t}, \quad t \geq 0, \epsilon \leq \frac{-\vartheta}{2\lambda_\mu}.
\]
Indeed, for $0 \leq t \leq 2\omega$ the left hand side is smaller or equal than 1 and the right hand side is larger or equal than 1. Then, for $t \geq 2\omega$ we have

$$\frac{\partial}{\partial t} \frac{\omega}{t} - \frac{\vartheta}{\epsilon} - \lambda \mu \leq -\frac{\vartheta}{2\epsilon} - \lambda \mu \leq 0$$

provided $\epsilon \leq -\frac{\vartheta}{2\lambda \mu}$ and hence

$$e^{\vartheta \min(t,\omega-t)} = e^{\vartheta (\omega-t)} \leq e^{\lambda \mu t} \leq e^{-2\lambda \mu \omega} e^{\lambda \mu t}.$$ 

Thus,

$$\|u_2(t)\| \leq c_{16} \left( e^{\lambda \mu t} + e^{-\frac{\vartheta}{\epsilon} t} \int_{0}^{\min(t,\omega)} e^{\frac{\vartheta}{\epsilon} s} ds \right) \leq c_{19} e^{\lambda \mu t}. \quad (71)$$

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