Cosmological Evolution in $f(R, T)$ theory with Collisional Matter

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We study the evolution of the cosmological parameters, namely, the deceleration parameter $q(z)$ and the parameter of effective equation of state in a universe contains, besides the ordinary matter and dark energy, a self-interacting (collisional matter), in the generalized $f(R, T)$ theory of gravity, where $R$ and $T$ are the curvature scalar and the trace of the energy-momentum tensor, respectively. We use the generalized FRW equations the equation of continuity and obtain a differential equation of second order in $H(z)$, and solve it numerically for studying the evolution of the cosmological parameters. Two $f(R, T)$ models are considered and the results with collisional matter are compared with the ones of the $Λ$CDM model, and also with the model where there exists only non-collisional matter. The curves show that the models are acceptable because the values found for $w_{eff}$ are consistent with the observational data.

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I. INTRODUCTION

It is well known nowadays that our current universe is experiencing an accelerated expansion [1]-[7]. There are two general ways to explain this accelerated expansion of the universe [8]-[12]. The first ways is considering that the universe is essentially filled by an exotic fluid with negative pressure, responsible of it acceleration, called the dark energy. The second way is modifying the gravitational action from the General Relativity without the need of dark energy. The second way is modifying the gravitational action from the General Relativity without the need of dark energy. The second way is modifying the gravitational action from the General Relativity without the need of dark energy. The second way is modifying the gravitational action from the General Relativity without the need of dark energy. The second way is modifying the gravitational action from the General Relativity without the need of dark energy.

Another theories, view as alternative to the GR, have also been undertaken, still in the way to better explain the obscure content of the universe, as, $f(R, T)$[14]-[15], $f(R, G)$[16]-[17], and $f(R, T, R_{\mu\nu}T^{\mu\nu})$[18]-[19], where $G$, $T$, $R_{\mu\nu}$ and $T^{\mu\nu}$ are the invariant of Gauss-Bonnet, the trace of energy-momentum tensor, the Ricci’s tensor and the energy-momentum tensor corresponding to the ordinary content of the universe. In this paper, we focus our attention on $f(R, T)$ theory of gravity. This theory has been considered first by Harko and collaborators [20]. Another authors also have considered this present theory and important results have been found [21]-[29].

However, any one of the works developed in these papers does not solve the coincidence problem, that is, how does the universe transits from the decelerated phase to the current accelerated phase? [30]. The $f(R, T)$ theory can successfully predict the transition from the matter dominated phase to the current accelerated one and several works have described this transition [31]. The reconstruction of $f(R, T)$ models describing the matter dominated and accelerated phases had been performed in [30]. Various works also have been performed, still in the optic to better explore this transition with interesting results( see [34], [35] and [40]).

In this paper, we focus our attention on the epoch after the recombination where, beside the well known ordinary matter (the dust), there is the self-interacting matter, called collisional matter. The model of collisional matter has been studied in some works within others theories of gravity, leading to interesting results [38], [39], [41], [42]. This approach of considering new form of matter, instead of the cold dark matter can edify us on the choice of the models of modified gravity. Oikonomou and collaborators [43] have studied the time evolution of the cosmological parameters.
during the late time acceleration of the universe with the presence of the collisional matter in the framework of modified $f(R)$ gravity. In this paper, we have extended the same idea to the $f(R, T)$. Some $f(R, T)$ models have been considered and the behaviours of the cosmological parameters have been performed and compared with the ΛCDM model. We see within many results that depending on the input parameters according the model under consideration, the inclusion of collisional matter may lead to a better explanation of the phase transition, comparatively to the model where just the usual ordinary matter is considered.

The paper is organized as follows: In section 2 we describe the general formalism of $f(R, T)$ theory of gravity. The collisional latter that self-interacts is presented in the section 3. The section 4 is devoted to the study of time evolution of the cosmological parameters where the universe is considered to be filled by the usual ordinary matter and the collisional one. Here, for the study of these cosmological parameters, we focus our attention on the transition from the decelerated phase to the accelerated one. In the section 5 we examine the evolution of the equation of state of the dark energy where the matter content is assumed as a fluid is composed by the collisional matter and the radiation. The conclusion and perspectives are presented in the section 6.

II. GENERAL FORMALISM IN $f(R, T)$ GRAVITY

In this section we present the generality of $f(R, T)$ theory by substituting the curvature scalar $R$ of the GR by a function of $R$ and the trace $T$, and writing the action as

$$S = \int \sqrt{-g} dx^4 \left[ \frac{1}{2\kappa^2} f(R, T) + \mathcal{L}_m \right],$$

where $R, T$ denote the curvature scalar and the trace of the energy-momentum tensor, respectively, and $\kappa^2 = 8\pi G$, $G$ being the gravitation constant.

The energy-momentum tensor associated to the matter is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}.$$

Let us assume that the matter Lagrangian density $\mathcal{L}_m$ only depends on the components of the metric tensor, but not on its derivatives. Thereby, one gets

$$T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_m - \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}}.$$

Within the metric formalism, varying the action (1) with respect to the metric, one obtains the following field equations

$$f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R, T) + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R = \kappa^2 T_{\mu\nu} - f_T (T_{\mu\nu} + \Theta_{\mu\nu}),$$

where $f_R, f_T$ are the partial derivatives of $f(R, T)$ with respect to the $R, T$ respectively. The tensor $\Theta_{\mu\nu}$ is determined by

$$\Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = -2 T_{\mu\nu} + g_{\mu\nu} \mathcal{L}_m - 2 g^{\alpha\beta} \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}.$$

As mentionned in our introduction, we assume that the whole content of the universe is a perfect fluid, and then, the energy-momentum tensor takes the following expression

$$T_{\mu\nu} = (\rho_m + p_m) u_\mu u_\nu - p_m g_{\mu\nu}.$$

In this way, the simple setting of the matter Lagrangian is taking $\mathcal{L}_m = -p_m$. The four-velocity $u_\mu u^\mu = 1$ and $u^\mu \nabla_\nu u_\mu = 0$. Hence, Eq (5) takes a new form as

$$\Theta_{\mu\nu} = -2 T_{\mu\nu} - p_m g_{\mu\nu}.$$

Making use of Eq (7), the field equation (4) can be reformulated as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2_{eff} T_{\mu\nu}^{eff},$$

where $\kappa^2_{eff}$ is the effective gravitational constant. 

As for the field equations, they include in addition the standard field equations for the universe filled with a perfect fluid and a collisional matter, which are presented below.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2_{eff} (\rho_m + p_m) g_{\mu\nu} - \kappa^2_{eff} p_m g_{\mu\nu} - \kappa^2_{eff} p_m g_{\mu\nu},$$

where $\kappa^2_{eff}$ is the effective gravitational constant.
where $\kappa_{\text{eff}}^2 = \frac{\kappa^2 f_T}{f_R}$ is the effective gravitational constant and

$$T_{\mu\nu}^{\text{eff}} = \left[ T_{\mu\nu} + \frac{1}{\kappa^2 + f_T} \left( \frac{1}{2} g_{\mu\nu} (f - R f_R) + f_T p_m g_{\mu\nu} - (g_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}) f_R \right) \right],$$

(9)

representing the effective energy-momentum tensor of the matter.

Here we are interested to the spatially flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a(t)^2 [dx^2 + dy^2 + dz^2].$$

(10)

Thus, the Ricci scalar in this background is given by

$$R = -6(2H^2 + \dot{H}).$$

(11)

### III. COLLISIONAL MATTER MODEL WITHIN $f(R,T)$ THEORY

This kind of matter has been introduced firstly in [38, 39] in the framework of GR and $f(R)$. The evolution of the collisional matter depends on the source that drives the whole content (matter and energy) of the universe.

The study of the model of collisional matter with perfect fluid has been performed widely in [40]. In this work, we recall that our basic assumption is that the matter has a total mass-energy density, denoted by $\varepsilon_m$, which is assumed to be depending on two contributions as

$$\varepsilon_m = \rho_m + \rho_m \Pi.$$  

(12)

Here, $\rho_m$ refers to the part that does not change due to the fact of characterising the usual matter content [38, 39]. Concerning the term $\rho_m \Pi$, it expresses the energy density part of the energy momentum tensor associated with thermodynamical content of the collisional matter. This fluid could not be dust, but possesses a positive pressure and satisfying the following equation of state

$$P_m = w \rho_m,$$

(13)

where $w$ denotes the parameter of equation of state of the collisional matter whose values is such that $0 < w < 1$. The potential energy density is assumed to have the form [38, 39],

$$\Pi = \Pi_0 + w \ln \left( \frac{\rho_m}{\rho_{m0}} \right),$$

(14)

where the constants $\rho_{m0}$ and $\Pi_0$ are their current values.

Accordingly, after elementary transformation the total-energy density of the universe can be written as

$$\varepsilon_m = \rho_m \left( 1 + \Pi_0 + w \ln \left( \frac{\rho_m}{\rho_{m0}} \right) \right).$$

(15)

Through the continuity equation, the motions of the volume elements in the interior of a continuous medium can be traduced

$$\nabla_{\nu} T_{\mu\nu} = 0,$$

(16)

and the energy-momentum tensor takes the following form

$$T_{\mu\nu} = (\varepsilon_m + p_m) u_{\mu} u_{\nu} - p_m g_{\mu\nu},$$

(17)

where $u_{\mu} = dx_{\mu}/ds$ is the four velocity, satisfying the relation $u_{\mu} u_{\nu} = 1$. We note here that $p_m = P_m$ because the pressure of ordinary matter is negligible. Making use of the FRW line element [10], the conservation law of the Equation (16)

$$\dot{\varepsilon}_m + 3 \frac{\dot{a}}{a} (\varepsilon_m + p_m) = 0,$$

(18)
which, upon consideration of the equations (13) and (15), leads to

\[ \rho_m = \rho_{m0} \left( \frac{a_0}{a} \right)^3, \]  

(19)

where \( a_0 \) is the current scale factor.

We can describe the collisional matter by equations (15) and (19) and use them in the rest of the manuscript in the next section. The value of \( \Pi_0 \) is equal to

\[ \Pi_0 = \left( \frac{1}{12M} - 1 \right). \]  

(20)

We will use the same value \( \Pi_0 = 2.58423 \) for making the numerical study.

IV. LATE TIME COSMOLOGICAL EVOLUTION IN F(R,T) THEORY

A. Deceleration Parameter

In this subsection we examine the deceleration parameter \( q(z) \) in \( f(R,T) \) considering that the universe, beside the ordinary matter and dark energy, also is filled with collisional matter at late time cosmological evolution. In this optic, we rewrite the field equation (4) in the following form

\[ 3H^2 = \frac{1 + f_T}{f_R} \varepsilon_m + \frac{1}{f_R} \left[ \frac{1}{2} (f - Rf_R) - 3H f_{RR} + p_m f_T \right], \]  

(21)

\[ - 2\dot{H} - 3H^2 = \frac{1 + f_T}{f_R} p_m + \frac{1}{f_R} \left[ 2H \dot{R} f_{RR} + \dot{R} f_{RR} + \dot{R}^2 f_{RRR} - \frac{1}{2} (f - Rf_R) - p_m f_T \right]. \]  

(22)

where we have assumed \( \kappa^2 = 1 \). The parameter \( H = \frac{\dot{a}}{a} \) denotes the Hubble parameter, the “dot”, the time derivative of \( \partial/\partial t \). We propose to reformulate the right side of (21) and (22) respectively in term of effective energy density \( \rho_{eff} \) and pressure \( p_{eff} \), and we write (21) and (22) as

\[ 3H^2 = \kappa_{eff}^2 \rho_{eff}, \]  

(23)

\[ - 2\dot{H} - 3H^2 = \kappa_{eff}^2 p_{eff}. \]  

(24)

Here, the effective energy density \( \rho_{eff} \) and effective pressure \( p_{eff} \) are respectively defined as

\[ \rho_{eff} = \varepsilon_m + \frac{1}{1 + f_T} \left[ \frac{1}{2} (f - Rf_R) - 3H f_{RR} + p_m f_T \right], \]  

(25)

\[ p_{eff} = p_m + \frac{1}{1 + f_T} \left[ 2H \dot{R} f_{RR} + \dot{R} f_{RR} + \dot{R}^2 f_{RRR} - \frac{1}{2} (f - Rf_R) - p_m f_T \right]. \]  

(26)

In this way, we have a matter fluid representation of the so-called geometrical dark energy in \( f(R,T) \) gravity with the energy density \( \rho_{DE} = \rho_{eff} - \varepsilon_m \) and pressure \( p_{DE} = p_{eff} - p_m \).

From the conservation law, the effective energy density evolves as

\[ \frac{d(\kappa_{eff}^2 \rho_{eff})}{dt} + 3H \kappa_{eff}^2 (\rho_{eff} + p_{eff}) = 0 \]  

(27)

By making use of the equations (21) and (22), (27) can be rewritten as

\[ 18 \frac{f_{RR}}{f_R} H (\dot{H} + 4H \dot{H}) + 3(\dot{H} + H^2) + \frac{1 + f_T}{f_R} \varepsilon_m + p_m \frac{f_T}{f_R} + \frac{f}{2f_R} = 0. \]  

(28)
One can rewrite the above equation using the redshift \( z = \frac{1}{\alpha} - 1 \) as follows

\[
\frac{d^2 H}{dz^2} = \frac{3}{1 + z} \frac{dH}{dz} - \frac{1}{H} \left( \frac{dH}{dz} \right)^2 - \frac{3f_R \left( H^2 - (1 + z)H \frac{dH}{dz} \right) + \frac{f_R}{2} + \rho_m f_T + (1 + f_T) \varepsilon_m}{18H^3 f_{RR}(1 + z)^2}.
\]  

(29)

As convention in this paper, we set to unity the current value of the scale factor. We also assume the algebraic function as a sum of two independent functions, such that, \( f(R, T) = f(R) + f(T) \). The \( f(T) \) model is the one obtained by imposing the conservation of the energy-momentum tensor. The dynamics and stability of this model are studied in [37] and interesting results have been found. This model reads

\[
f(T) \propto T^\alpha,
\]

(30)

with \( \alpha = \frac{1 + 3w}{2(1 + w)} \). Here the trace of the energy-momentum tensor [17] reads \( T = \varepsilon_m - 3p_m \).

From Equations (15) and (19), the model \( f(T) \) and its first derivative are

\[
f_T = \alpha(1 + z)^3(\alpha - 1) \rho_m(1 + z) \left( 1 + \Pi_0 - 3w(1 - \ln(1 + z)) \right)^{(\alpha - 1)},
\]

(31)

By using (31) and (32), one rewrite (29) as

\[
\frac{d^2 H}{dz^2} = \frac{3}{1 + z} \frac{dH}{dz} - \frac{1}{H} \left( \frac{dH}{dz} \right)^2 - J_1(z) - J_2(z) - J_3(z),
\]

(33)

where

\[
J_1(z) = \frac{3f_R \left( H^2 - (1 + z)H \frac{dH}{dz} \right) + \frac{f_R}{2}}{18H^3 f_{RR}(1 + z)^2},
\]

(34)

\[
J_2(z) = \frac{\rho_m(1 + z)(1 + \Pi_0 + 3w \ln(1 + z))}{18H^3 f_{RR}},
\]

(35)

\[
J_3(z) = \left( \alpha(1 + w) + \frac{1}{2}(1 - 3w) \right) \rho_m(1 + z)^{\alpha - 1} \left( 1 + \Pi_0 - 3w(1 - \ln(1 + z)) \right)^{(\alpha - 1)}J_2(z)
\]

(36)

The equation (33) is the main tool to be used for analysing the cosmological evolution of the considered \( f(R, T) \) models. The numerical analysis also will be performed in order to find the Hubble parameter. In a first step, by using the expression

\[
q(z) = \frac{(1 + z) \frac{dH}{dz} - 1}{H(z) \frac{dH}{dz}},
\]

(37)

we can perform the study of the deceleration parameter with the account of the collisional matter, this latter assumed as a function of the redshift \( z \). In this rubric we will focus our attention on the transition from the decelerated phase to the accelerated one. Therefore, we will compare the results with the case where the non-collisional matter (pressure-less) is considered, and also compare with the \( \Lambda CDM \) case.

We recall that in [41], the corresponding deceleration parameter of the expansion in the flat friedmann cosmology for the \( \Lambda CDM \) model is given by

\[
q(z) = \left[ \frac{\Omega_m(1 + z)^3 - L_0}{2} \right] \left[ \Omega_m(1 + z)^3 + L_0 \right]^{-1},
\]

(38)
where \(\Omega_{m0} = 0.279\) and \(L_0 = 0.721\)\(^{41}\).

Moreover, the effective equation of state parameter \(w_{eff}\) is defined as,

\[
w_{eff} = \frac{p_{eff}}{\rho_{eff}}  \tag{39}
\]

\[
w_{eff} = -1 + \frac{2(1 + z)}{3H(z)} \frac{dH}{dz} \tag{40}
\]

and we present its evolution versus the redshift. A comparison with the results to be found will be done with the results coming from pure \(f(R,T)\) models with non-collisional matter.

\[B.\] Study of particular cases \(f(R)\) models in modified \(f(R,T)\) gravity universe with collisional matter

In this subsection we discuss two \(f(R)\) models, performed in \(^{40, 42}\) for describing the late time cosmological evolution of our universe under the consideration that it is filled with collisional (self-interacting) matter. Thereby, we solve numerically the second order differential equation about \(H(z)\).

The task here is to find the models that should be in agreement with the observational data, namely the deceleration parameter \(q(z)\) and the parameter of effective equation of state \(w_{eff}\).

The first model considered is a modified power-law model of the curvature scalar

\[
f(R) = \lambda_0(\lambda + R)^n;  \tag{41}
\]

where \(\lambda_0, \lambda\) and \(n\) are positive constants.

The second, is an exponential model

\[
f(R) = R_0e^{\beta R}, \tag{42}
\]

where \(R_0\) and \(\beta\) are constant parameters.

Numerical plots of the deceleration parameter \(q(z)\) and the effective equation of state \(w_{eff}\) are presented from \(^{88}\) with collisional matter \((w = 0.6)\) and non-collisional matter \((w = 0)\) for each model.

\[\text{Figure 1: The graphs show the evolution of } q(z) \text{ and } w_{eff} \text{ versus } z \text{ for the model } f(R) = \lambda_0(\lambda + R)^n. \text{ The graphs are plotted for } \lambda_0 = 1, \lambda = 13.5 \text{ and } n = 0.5. \text{ The red, blue and the magenta refer to non-collisional, collisional matter and } \Lambda CDM \text{ model, respectively.}\]

For Fig. 1, related to the first \(f(R,T)\) model, we see that, at the left hand side, traducing the evolution of the parameter of deceleration the curve representative of the collisional matter is nearby that of \(\Lambda CDM\) than the curve representative of the non-collisional matter. We also see that, in the order, collisional matter, \(\Lambda CDM\) and non-collisional matter, the transition from the decelerated phase to the accelerated phase is realized from high to the low
The graph shows the evolution of $q(z)$ and $w_{\text{eff}}$ versus $z$ for the model $f(R) = R_0 e^{\beta R}$. The graph are plotted for $R_0 = 1$ and $\beta = 1.5$. The red, blue and the magenta refer to non-collisional matter, collisional matter and $\Lambda\text{CDM}$ model, respectively.

value of the redshift. Concerning the curve representative of $w_{\text{eff}}$ the same behaviours of the phase transition order is confirmed.

Regarding the second model, whose cosmological parameters are plotted at Fig. 2, we see that for the transition from the decelerated to the accelerated phases, from the high to low redshifts, the order is non-collisional matter, $\Lambda\text{CDM}$ and collisional matter. Moreover, we see that in the accelerated phase, the collisional matter is more nearby the $\Lambda\text{CDM}$ than the non-collisional one, while in the decelerated phase, both the collisional and non-collisional matters, almost follow the same trajectory for high redshifts. This aspect is very important because reveals that the collisional takes its origin from the ordinary matter. We then conclude that at early times there is no collisional matter, but just non-collisional, and as the time evolves, the collisional matter start being created from the non-collisional matter. After crossing the transition line, the collisional matter approaches more the $\Lambda\text{CDM}$ as the low redshifts are being reached.

C. $f(R,T)$ Models with Cardassian Self-interacting Matter

The cardassian model is firstly developed by [43], [44], [45] and is another approach to describe the self interacting matter. According to the observational it is well known that the universe is flat and accelerating; the cardassian model of matter is characterised by negative pressure and there is no vacuum energy whatsoever. In this subsection we perform the same study as in the previous section and use the models $f(R)$ models [41], [42]. According to the so-called Cardassian model of matter [43], the total energy density $\varepsilon_m$ of matter is defined by

$$\varepsilon_m = \rho + \rho K(\rho)$$  \hspace{1cm} (43)

where $\rho$ is related to the ordinary matter-energy density and $\rho K(\rho)$ describes the cardassian model of matter and is general, a function of the ordinary mass-energy density $\rho$.

According to the original Cardassian model, one has

$$\rho K(\rho) = B \rho^{n'}$$  \hspace{1cm} (44)

where $B$ is a real number and it is fixed $n' < \frac{2}{3}$ in order to guarantee the acceleration.

By taking into account the relation (44), we reobtain the total energy density of matter [43] as,

$$\varepsilon_m = \rho + B \rho^{n'}.$$  \hspace{1cm} (45)

Following the same idea as [40], we assume that the late time evolution is governed by the geometric dark fluid with negative pressure coming from the $f(R,T)$ models plus a gravitating fluid of positive pressure, satisfying the following
equation of state
\[ p = w_k \rho. \]  
\[ (46) \]
We mention in this model that the pressure has to be assumed as negative quantity, \( p < 0 \), because the Cardassian model of matter possesses negative pressure. In such a situation, this negative has to be assumed as the one which generates the acceleration of our universe.

Adopting the assumption that the total energy density of matter is \([45]\), the main equation \((29)\) can be used for analysing the evolution of the viable \( f(R, T) \) models considered with Cardassian self-interacting matter, yielding
\[
\frac{d^2H}{dz^2} = \frac{3}{1+z} \frac{dH}{dz} - \frac{1}{H} \left( \frac{dH}{dz} \right)^2 - J'_1(z) - J'_2(z) - J'_3(z),
\]
where
\[
J'_1(z) = \frac{3f_R \left( H^2 - (1+z)H \frac{dH}{dz} \right) + f(R)}{18H^3f_{RR}(1+z)^2},
\]
\[
J'_2(z) = \frac{\rho_m(1+z)}{18H^3f_{RR}} \left( 1 + B\rho_m^{(n'-1)}(1+z)^3(n'-1) \right),
\]
\[
J'_3(z) = \rho_m^{\alpha-1}(1+z)^{3(\alpha-1)}(\alpha + 1/2)(1 + w_k) \left( 1 + w_k + B\rho_m^{(n'-1)}(1+z)^3(n'-1) \right)^{\alpha-1} J'_2(z).
\]
By adopting the same treatment as in the previous section, we present the evolutions of the deceleration parameter \( q(z) \) and the effective equation \( w_{eff} \) of state for the late time cosmological evolution of the universe with the \( f(R, T) \) models including Cardassian Matter.

![Graph](image)

Figure 3: The graph shows the evolution of \( q(z) \) and \( w_{eff} \) versus \( z \) for the model \( f(R) = \lambda_0(\lambda + R)^n \). The graph are plotted for \( \lambda_0 = 1 \), \( \lambda = 13.5 \) and \( n = 0.5 \) The red color, blue and the magenta refer to collision-less, Cardassian and \( \Lambda CDM \) model, respectively and we use \( B = 0.2 \), \( n' = -3 \).

Here, for the first model, Fig. 3, we see that the transition from the phase of deceleration to the acceleration one is realized, from the high to low-redshifts, in the order, \( \Lambda CDM \), collisional matter and non-collisional matter. In the decelerated phase, both the collisional and non-collisional matters are confused, while, within the accelerated phase, the collisional matter detached from the non-collisional one and start approaching the \( \Lambda CDM \) model as the low-redshifts are being reached.

Concerning the second model, Fig. 4, the collisional and non-collisional matter are also confused for high redshifts and as the low-redshifts are being reached, the collisional matter approximate the \( \Lambda CDM \) model.
Figure 4: The graphs show the evolution of $q(z)$ and $w_{eff}$ versus $z$ for the model $f(R) = R_0 e^{\beta R}$. The graphs are plotted for $R_0 = 0.5$ and $\beta = 1.4$. The red, blue and magenta refer to collision-less matter, cardassian matter and $\Lambda CDM$ model, respectively and we use $B = 0.2$, $n' = -3$.

V. EQUATION OF STATE FOR DARK ENERGY WITH COLLISIONAL MATTER FLUID IN F(R,T) GRAVITY

The work performed in this section is about the cosmological evolution of the universe through $f(R, T)$ models in presence of matter composed by collisional matter and radiation.

We focus our attention on the exponential $f(R)$ model \[46, 47\] given by

$$f(R) = R - \beta R_S (1 - e^{-R/R_S}),$$  

(51)

where $c_1 = -\beta R_S$ and $c_2 = R_S$ constants with curvature scalar dimension. The conditions for having a viable exponential model has been studied in \[48\].

Within flat FRW metric and considering the above content of the universe, the field equations, from the general form (4), take the following form

$$3f_R H^2 = (1 + f_T) \rho_{matt} + \frac{1}{2} (f - R f_R) - 3H f_R + P_{matt} f_T,$$

(52)

$$-2f_R H = (1 + f_T)(\rho_{matt} + P_{matt}) + f_R - H f_R,$$

(53)

where $\rho_{matt}$ and $P_{matt}$ denote the energy density and pressure of all perfect fluids of generic matter, respectively.

Concerning the component to be considered here, we assume that the universe is filled with collisional matter (self-interacting matter) and the relativistic matter (radiation). This means that in our analysis, the contribution from the radiation may play an important role. Therefore, the matter energy density, $\rho_{matt}$, is given by

$$\rho_{matt} = \varepsilon_m + \rho_{r0} a^{-4},$$

(54)

where $\rho_{r0}$ is the current energy density of radiation. The pressure of all perfect fluids of matter is given by

$$P_{matt} = p_m + p_r.$$  

(55)

The first term characterises the pressure of collisional matter, given by (13), and the second, is related to the radiation. Making use of (15) and (19), one can put Eq. (54) into the following form

$$\rho_{matt} = \rho_{m0} a^{-3} \left( 1 + \frac{H_0 + 3w \ln(a)}{f_{m0}} \right) + \rho_{r0} a^{-4}.$$  

(56)
In the same way, Eq. (52) can be rewritten as
\[
H^2 + H^2 f_{RR} \frac{dR}{d \ln a} - \frac{1}{6}(f-R) + (1-f_R)(H \frac{dH}{d \ln a} + H^2) = \frac{1}{3} \beta \rho_{\text{matt}} + \frac{f_T}{3}(\rho_{\text{matt}} + P_{\text{matt}}),
\]
while the scalar curvature $R$ can be expressed as
\[
R = -(12H^2 + 6H \frac{dH}{d \ln a}).
\]
Through the Eq. (56), we assume that the total matter-energy density becomes
\[
\rho_{\text{matt}} = \rho_{n0}(g(a) + \chi a^{-4}).
\]
In the previous expression, $\chi$ is defined by $\chi = \rho_{r0}/\rho_{m0} \simeq 3.1 \times 10^{-4}$, $\rho_{r0}$ being the current energy density of radiation. The parameter $g(a)$, describing the nature of the collisional matter (view as perfect fluid), is equal to
\[
g(a) = a^{-3} \left(1 + \Pi_0 + 3w \ln(a) \right).
\]
Note that for the non-collisional matter (assumed as the dust), for which the parameter $w = 0$, one gets $g(a) = a^{-3}$.
In the optic to better study the cosmological evolution of the $f(R, T)$ models in the framework of flat FLRW universe, we may introduce the variable $\bar{m}^2$
\[
\bar{m}^2 = \frac{\kappa^2 \rho_{m0}}{3}.
\]
Making use of (57), the expression $\frac{1}{m^2} \frac{dR}{d \ln a}$ yields
\[
\frac{1}{\bar{m}^2} \frac{dR}{d \ln a} = \frac{1}{H^2 f_{RR}} \left[ \frac{f_T}{3 \bar{m}^2}(\rho_{\text{matt}} + P_{\text{matt}}) + \frac{1}{3 \bar{m}^2} \rho_{\text{matt}} - \frac{H^2}{\bar{m}^2} + \frac{1}{6 \bar{m}^2} (f-R) - (1-f_R) \left( \frac{H}{m^2} \frac{dH}{d \ln a} + \frac{H^2}{m^2} \right) \right].
\]
We recall in this work that the trace of the stress tensor depends on the nature of the matter content. Therefore, using the same model (30) where $T = \rho_{n0}[g(a)(1-3w)]$, and also (55) and (56), Eq. (63) becomes
\[
\frac{1}{\bar{m}^2} \frac{dR}{d \ln a} = \frac{1}{H^2 f_{RR}} \left[ 3 \bar{m}^2 \beta (1+\beta) \left( g(a)(1-3w) \right)^\beta \left( (1+w)g(a) + \frac{4}{3} \chi a^{-4} \right) - y_H + \frac{1}{6 \bar{m}^2} (f-R) - (1-f_R) \left( \frac{H}{m^2} \frac{dH}{d \ln a} + \frac{H^2}{m^2} \right) \right].
\]
where $\beta = \alpha - 1$. Eqs. (68) and (69) are reduced to a coupled set of ordinary differential
\[
\frac{dy_H}{d \ln a} = \frac{1}{3} y_R - 4y_H - \frac{4}{3} \frac{d g(a)}{d \ln a} - 4g(a),
\]
\[
\frac{dy_R}{d \ln a} = \frac{1}{f_{RR} \bar{m}^2(y_H + g(a) + \chi a^{-4})} \left[ 3 \bar{m}^2 \beta (1+\beta) \left( g(a)(1-3w) \right)^\beta \left( (1+w)g(a) + \frac{4}{3} \chi a^{-4} \right) - y_H + \frac{1}{6 \bar{m}^2} (f-R) - (1-f_R) \left( \frac{1}{2} \frac{dy_H}{d \ln a} + \frac{1}{2} \frac{dg(a)}{d \ln a} + y_H + g(a) - \chi a^{-4} \right) \right].
\]
Moreover, the curvature scalar is expressed as
\[
R = -3 \bar{m}^2 \left( 4y_H + 4g(a) + \frac{dy_H}{d \ln a} + \frac{dg(a)}{d \ln a} \right).
\]
By operating the differentiation of the relation (65) with respect to \( \ln a \), and eliminating \( \frac{d\ln a}{d\ln a} \) from this result and (66), we obtain

\[
\frac{d^2 Y_H}{d\ln a^2} + \left( 4 - \frac{1 - f_R}{6\bar{m}^2 f_{RR}(y_H + g(a) + \chi a^{-4})} \right) \frac{dy_H}{d\ln a} + \left( \frac{f_R - 2}{3\bar{m}^2 f_{RR}(y_H + g(a) + \chi a^{-4})} \right) y_H + \left[ \frac{d^2 g(a)}{d\ln a^2} + \frac{4}{d\ln a} \right] \frac{dg(a)}{d\ln a} + \frac{3\beta^2 2(1 + \beta)(g(a)(1 - 3w))}{f_{RR}\bar{m}^2(y_H + g(a) + \chi a^{-4})} \right] = 0. (68)
\]

Taking into account the relations

\[
\frac{d}{d\ln a} = -(1 + z) \frac{d}{dz}, \quad (69)
\]

\[
\frac{d^2}{d\ln a^2} = (1 + z)^2 \frac{d^2}{dz^2} + (1 + z) \frac{d}{dz}, \quad (70)
\]

we express the equation (68) in terms of the redshift, as follows

\[
\frac{d^2 Y_H}{dz^2} + J_1'' \frac{dy_H}{dz} + J_2' y_H + J_3' = 0, \quad (71)
\]

where

\[
J_1'' = \frac{1}{(1 + z)^2} \left[ -3 + \frac{f_R - 2}{6\bar{m}^2 f_{RR}(y_H + g(z) + \chi (1 + z)^4)} \right], \quad (72)
\]

\[
J_2'' = \frac{1}{(1 + z)^2} \left[ \frac{f_R}{3\bar{m}^2 f_{RR}(y_H + g(z) + \chi (1 + z)^4)} \right], \quad (73)
\]

\[
J_3'' = \frac{d^2 g(z)}{dz^2} - \frac{3}{(1 + z)^2} \frac{dg(z)}{dz} + \frac{1}{(1 + z)^2 6\bar{m}^2 f_{RR}(y_H + g(z) + \chi (1 + z)^4)} \left[ 3\beta^2 2(1 + \beta)(g(z)(1 - 3w))^{\beta} (g(z)(1 + w) + \frac{4}{3} \chi (1 + z)^4) \right] + \frac{f_R}{3\bar{m}^2} + \frac{2g(z)}{1 + z} + 2\chi (1 + z)^4 + \frac{3\beta^2 2(1 + \beta)(g(z)(1 - 3w))^{\beta + 1}}{f_{RR}}. \quad (74)
\]

Eq. (71) characterises the equation to be used for describing the cosmological evolution of the dark energy in the universe filled with collisional matter and radiation. We also solve numerically this equation for a given \( g(a) \) from (60) and we present the cosmological evolution of the dark energy scale \( y_H \equiv \frac{\rho_{DE}}{\rho_m} \) of collisional matter \((w = 0.6)\) in comparison to the non-collisional matter as functions of the redshift.
In this rubric, we plot the evolution of the parameter of equation of state for dark energy $w_{DE}(z) \equiv P_{DE}/\rho_{DE}$, given by

$$w_{DE}(z) = -1 + \frac{1}{3}(1 + z) \frac{dy_H}{dz},$$

(75)

derived by the continuity equation

$$\rho_{DE} + 3H(1 + w_{DE})\rho_{DE} = 0,$$

(76)

for collisional matter ($w = 0.6$) in comparison to the non-collisional one.

As performed in the previous section, we are also interested to the evolution of others parameters. Theses parameters, the Hubble parameter $H(z)$, the curvature scalar $R(z)$ and the parameter $w_{eff}$ of the effective equation of state, can be described by still considering that the universe is filled by collisional matter and radiation. In the same way, we find the numerical solution for $y_H$ (71) for both collisional and non-collisional matters, and compare them. These parameters can be expressed as follows

$$H(z) = \sqrt{\bar{m}^2(y_H + g(z) + \chi(1 + z)^4)}$$

(77)

obtained from equation (61),

$$R = 3\bar{m}^2 \left( 4y_H + 4g(z) - (1 + z) \frac{dy_H}{dz} - (1 + z) \frac{dg(z)}{dz} \right)$$

(78)

obtained by combining the equations (67) and (69),

$$w_{eff} = -1 + \frac{2(1 + z) \frac{dH(z)}{dz}}{3H(z) \frac{dz}{dz}}.$$
Figure 6: Cosmological evolutions of $\omega_{DE}$ as functions of the redshift $z$ for $b = 0.19$, $Rs = 1.5$ of collisional matter for $w=0.6$ (blue) and the non-collisional matter (red).

Figure 7: Comparison of the Hubble parameter $H(z)$ over $z$ (left) and of the Ricci scalar $R(z)$ (right), for $b = 0.19$, $Rs = 1.5$. The red line corresponds to non-collisional matter while the blue corresponds to collisional matter for $w=0.6$.

For the Figs. 5, 6, 7 and 8, curves traducing the evolution of $y_H(z)$, $w_{DE}(z)$, $H(z)$, $R(z)$ and $w_{eff}(z)$, present some values strongly consistent with the cosmological observational data. These features prove that the fact of considering the coexistence of the collisional matter and radiation do not change the well know behaviour the different parameters, and confirm that such consideration may be made.
VI. CONCLUSION

In this paper we undertake the $f(R, T)$ theory of gravity. We focus our attention on the cosmological evolution of the parameter of deceleration $q(z)$ and also the parameter $w_{\text{eff}}$ of the effective equation of state. The spatial aspect in this paper is that, besides the usual ordinary and the dark energy, new matter is assumed to participate to the matter contribution of the universe. This matter is self-interacting and called collisional matter, but with positive pressure. What we now generally is that the ordinary matter interacts with the dark energy, but any of them does self interacts. From the cosmological considerations, its quite reasonable to introduce such a matter, i.e, a collisional, and search for its effects on the cosmological evolution of the universe.

To this end, we considered two $f(R, T)$ models as a sum of two gravitational models $f_1(R)$ and $f_2(T)$, respectively two functions depending the curvature scalar $R$ and the trace $T$ of the energy-momentum tensor. The general form of the $f(R, T)$ models is constrained to the conservation of the energy leading to the form \( (30) \). The task has been to choose a suitable expression for the function $f(R)$. Therefore, we considered, first, a generalized power-law in $R$, and the second expression is an exponential form in $R$. We plot the evolutions of $q(z)$ and $w_{\text{eff}}$ for both the collisional matter, non-collisional matter and the $\Lambda$CDM and compare them. Our results show that, depending on the type of the model under consideration, the curve characteristic of the collisional matter approaches the $\Lambda$CDM one that the curve that describes the evolution of the dust. In the same way, the curve representative of $w_{\text{eff}}$ shows that its values are consistent with the observational, confirm the importance of considering the collisional matter in the study of the evolution of our universe.

Moreover, we point out other type of matter, the so-called cardassian self-interacting matter, possessing a negative pressure and its corresponding parameter of equation of state has to be different from $-1$. This means that if such a matter is included in the content of the universe, there does not have vacuum energy. In this case, the same analysis have been done, as in the case of collisional matter, comparing the curves representative of this later to those of the dust and also the $\Lambda$CDM. Our results show that, for both the first and the second $f(R, T)$ models, there is quite reasonable to consider the existence of the cardassian matter since the curve characteristic of this latter approached more the $\Lambda$CDM that the dust for some values of the redshift, confirming also that this kind of matter may be considered as a component of the universe, because also in this, the parameter $w_{\text{eff}}$ takes values strongly consistent with the observational data.

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