Parameter estimators of sparse random intersection graphs with thinned communities

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Abstract. This paper studies a statistical network model generated by a large number of randomly sized overlapping communities, where any pair of nodes sharing a community is linked with probability $q$ via the community. In the special case with $q = 1$ the model reduces to a random intersection graph which is known to generate high levels of transitivity also in the sparse context. The parameter $q$ adds a degree of freedom and leads to a parsimonious and analytically tractable network model with tunable density, transitivity, and degree fluctuations. We prove that the parameters of this model can be consistently estimated in the large and sparse limiting regime using moment estimators based on partially observed densities of links, 2-stars, and triangles.

1 Introduction

Networks often display transitivity or clustering, the tendency for nodes to be connected if they share a mutual neighbor. Random graphs can statistically model networks with clustering after adding a community structure of small relatively dense subgraphs. Triangles, or other short cycles, then occur predominantly within and not between the communities, and clustering becomes tunable through adapting the community structure.

There are various ways to install community structure, for instance by locally adding small dense graphs [1,2,3,4]. This creates nonoverlapping communities. Another way is to introduce overlapping communities through a random intersection graph (RIG) which can be defined as the 2-section of a random inhomogeneous hypergraph where hyperedges correspond to overlapping communities [5]. RIGs have attractive analytical features, for example admitting tunable transitivity (clustering coefficient) and power-law degree distributions [6,7,8]. However, by construction the RIG community structure is rigid, in the sense that every community corresponds to a clique. In this paper we relax this property and consider an extension of the RIG, a thinned RIG where nodes within the same community are linked with some probability $q \in [0,1]$ via that community, independently across all node pairs.

The RIG and thinned RIG are known to generate high levels of transitivity, even in sparse regimes where nodes have finite mean degrees in the large-network limit [6,9]. In [9] it is shown that the community density $q$ can be exploited to
tune both triangle and 4-cycle densities. In this paper we also exploit the additional freedom offered by $q$, but for controlling the density of 2-stars instead of 4-cycles. We derive scaling relations between the model parameters to create large, sparse, clustered networks, in which the number of links grows linearly in the number of nodes $n$, and the numbers of 2-stars and triangles grow quadratically in $n$. We investigate a special instance of the sparse model parameterized by a triplet $(\lambda, \mu, q)$ where $\lambda$ corresponds to the mean degree and $\mu$ to the mean number of community memberships of a node. By analyzing limiting expressions for the link, 2-star and triangle densities, we derive moment estimators for $\lambda$, $\mu$, and $q$ based on observed frequencies of 2-stars and triangles. Taken together, the densities of links, 2-stars and triangles prove sufficient to produce tunable sparsity (mean degree), degree fluctuations and transitivity.

This work is part of an emerging area in network science that connects high-order local network structure such as subgraphs with statistical estimation procedures. The triangle is the most studied subgraph, because it not only describes transitivity, but also signals hierarchy and community structure [10]. Other subgraphs, however, such as 2-stars, bifans, cycles, and cliques are also relevant for understanding network organization [11,12]. In this paper we exploit a direct connection between the model parameters and the frequencies of links, 2-stars and triangles. A key technical challenge is to characterize the mean and variance of the subgraph frequencies, where the latter requires frequencies of all subgraphs that can be constructed by merging two copies of the subgraph at hand [13,14,15,16]. A byproduct of our analysis yields a rigorous proof of the graph-ergodic theorem (analogous to [17, Theorem 3.2]) stating that the observed transitivity (a large graph average) of a large graph sample is with high probability close to the model transitivity (a probabilistic average).

Notation. For a probability distribution $\pi$ on the nonnegative integers, we denote the moments by $\pi_r = \sum x^r \pi(x)$ and the factorial moments by $(\pi)_r = \sum x(x-1)\cdots(x-r+1)\pi(x)$, where $(x)_r = x(x-1)\cdots(x-r+1)$. For sequences $a_n$ and $b_n$, we denote $a \lesssim b$ when $a_n \leq cb_n$ for some $c > 0$ and all $n$. $a \asymp b$ means $"a \lesssim b$ and $b \lesssim a"$. For $a_n = (1 + o(1))b_n$ we use the shorthand notation $a \sim b$, and for $a_n/b_n \to 0$ we use $a \ll b$.

2 Model description

We will study a statistical network model with $n$ nodes (individuals, users, vertices) and $m$ overlapping communities (attributes, blocks, groups, layers). The model is parameterized by $(n, m, \pi, q)$, where $\pi$ is a probability distribution on $\{0, \ldots, n\}$ such that $\pi(x)$ corresponds to the proportion of communities of size $x$, and $q \in [0, 1]$ is the probability that two nodes are linked via a particular community.

A realization of the model corresponds to a collection of random subsets $V_k$ of $\{1, \ldots, n\}$ indexed by $k = 1, \ldots, m$ representing the communities, and a collection of $\{0,1\}$-valued random variables $C_{ij,k}$ indexed by unordered pairs of
integers $i, j = 1, \ldots, n$, and integers $k = 1, \ldots, m$. These objects are used to define an undirected random graph $G$ on node set $\{1, \ldots, n\}$ with adjacency matrix

$$G_{ij} = \max_{k=1,\ldots,m} \{B_{i,k}B_{j,k}C_{ij,k}\}, \quad i \neq j,$$

(1)

where $B_{i,k} = 1_{V_k(i)}$ indicates whether node $i$ belongs to community $k$, and $C_{ij,k} = 1$ means that $i$ and $j$ are linked via community $k$, given that both $i$ and $j$ are members of community $k$. We assume that $V_1, \ldots, V_m$ are independent random sets with a common probability density $\pi(|A|)(\frac{n}{|A|})^{-1}$, and that $C_{ij,k}$ are independent $\{0, 1\}$-valued random integers with mean $q$. Moreover, the arrays $(V_k)$ and $(C_{ij,k})$ are assumed independent.

The special case where $q = 1$ corresponds to the so-called passive random intersection graph model [18,7]. The special case where $\pi$ is a Dirac measure has been recently studied in [9]. The binomial community size distribution $\pi(x) = \binom{n}{x}(1-p)^{n-x}p^x$ gives another important special case of the model (referred to as Bernoulli model), which allows to smoothly interpolate between a standard Erdős–Rényi random graph (setting $p = 1$) and a binomial random intersection graphs [19] (with $q = 1$).

3 Analysis of local model characteristics

3.1 Sparse parameter regime

In this section we analyze how the model behaves when the number of nodes $n$ is large. We view a large network as a sequence of models with parameter quadruples $(n, m, \pi, q) = (n_\nu, m_\nu, \pi_\nu, q_\nu)$ indexed by a scale parameter $\nu = 1, 2, \ldots$ such that $n_\nu \to \infty$ as $\nu \to \infty$. For simplicity we omit the scale parameter from the notation.

Let $p_r = (\pi)_r/(n)_r$ denote the probability that a particular community contains a given set of $r$ nodes. Then $mp_r$ equals the mean number of communities common to a particular set of $r$ nodes, and $(\binom{n}{r})p_r = (\pi)_r/r!$ equals the expected number of $r$-sets of nodes contained in a single community. Because $mp_2q$ equals the number of communities through which a given node pair is linked, it is natural to assume that $mp_2q \ll 1$ when modeling a large and sparse network. The following result confirms this.

**Proposition 1.** The probability that any particular pair of distinct nodes is linked equals $\mathbb{P}(\text{link}) = 1 - (1 - qp_2)^n$. Furthermore, $\mathbb{P}(\text{link}) \ll 1$ if and only if $mp_2q \ll 1$, in which case

$$\mathbb{P}(\text{link}) = (1 + O(mp_2q)) mp_2q.$$  

(2)

3.2 Subgraph densities

For an arbitrary graph $R$, the $R$-covering density of the model is defined as the expected proportion of (induced or noninduced) subgraphs of $G$ among all
$R$-isomorphic subgraphs of the complete graph on $\{1, \ldots, n\}$. By symmetry, this quantity equals the probability that $G$ contains $R$ as a subgraph, when we assume that $V(R) \subset \{1, \ldots, n\}$. Note that the $K_2$-covering density of the model is just the link density analyzed in Proposition 1. The following result describes the covering densities of connected three-node graphs.

**Proposition 2.** The probabilities that the model in the sparse regime $mp_2q \ll 1$ contains as subgraph the 2-star and triangle are approximately

$$
P(\text{2-star}) = (1 + O(mp_2q)) q^2 \left( mp_3 + (m)_2 p_2^2 \right),$$

$$
P(\text{triangle}) = (1 + O(mpq_2)) q^3 \left( mp_3 + 3(m)_2 p_2 p_3 + (m)_3 p_3^2 \right).$$

### 3.3 Model transitivity

The transitivity (or global clustering coefficient) of a graph usually refers to the proportion of triangles among unordered node triplets which induce a connected graph. The *model transitivity* of a random graph is usually defined by replacing the numerator and the denominator in the latter expression by their expected values. In our case, by symmetry, the model transitivity equals $\tau = P(\text{triangle})/P(\text{2-star})$, and is characterized by the following result in the sparse parameter regime.

**Proposition 3.** The model transitivity in the sparse regime $mp_2q \ll 1$ satisfies

$$
\tau = \frac{p_3 q}{p_3 + (m-1)p_2^2} + o(1).
$$

**Remark 1.** In the special case with $q = 1$ the above result coincides with [20, Corollary 1] and [7, Theorem 3.2].

### 3.4 Degree mean and variance

**Proposition 4.** The degree $D$ of any particular node of the model in the sparse regime $mp_2q \ll 1$ satisfies

$$
E(D) \sim mnp_2q, \quad \text{Var}(D) \sim mnp_2q \left( 1 + nq \left( \frac{p_3}{p_2} - p_2 \right) \right).
$$

### 4 Parameter estimation of sparse models

Our goal is to fit the model parameters to a sparse and large graph sample of known size $n$ in a consistent way. For this we impose assumptions on the parameter sequence $(n_\nu, m_\nu, \pi_\nu, q_\nu)$, called the balanced sparse regime.

**Assumption 1 (Balanced sparse regime).** The ratio $m/n$, the factorial moments $(\pi)_1, (\pi)_2, (\pi)_3$, and the parameter $q$ converge to nonzero finite constants as the scale parameter tends to infinity.
Propositions 3 and 4 imply that in the balanced sparse regime, the mean degree $\lambda$, the degree variance $\sigma^2$, and the model transitivity $\tau$ converge to nonzero finite constants which are related to the model characteristics via the formulas

$$
\lambda \sim (m/n)(\pi)_2 q, \quad \sigma^2 \sim \lambda \left(1 + q \frac{(\pi)_3}{(\pi)_2}\right), \quad \tau \sim \frac{(\pi)_3 q}{(\pi)_3 + (m/n)(\pi)_2^2}.
$$

These are the three model characteristics we wish to fit to real data. Single-parameter distributions $\pi$ are of special interest, as the parameter then determines both $(\pi)_2$ and $(\pi)_3$, reducing the number of unknowns by one.

### 4.1 Empirical subgraph counts

Consider the model $G = (n, m, q, \pi)$ and assume that we have observed a subgraph $G^{(n_0)}$ induced by $n_0$ nodes. We wish to estimate one or more model parameters using the empirical subgraph counts in $G^{(n_0)}$ and the asymptotic relations developed in Section 3.

Denote by $N_{K_2}(G^{(n_0)})$ the number of links, by $N_{S_2}(G^{(n_0)})$ the number of (induced or noninduced) subgraphs which are isomorphic to the 2-star, and by $N_{K_3}(G^{(n_0)})$ the number of triangles in the observed graph $G^{(n_0)}$. These are asymptotically close to the expected subgraph counts by the following theorem.

**Theorem 2.** Consider the model in the balanced sparse regime (Assumption 1). If $(\pi)_4 \lesssim 1$ and $n_0 \gg n^{1/2}$, then number of links in the observed graph $G^{(n_0)}$ satisfies

$$
N_{K_2}(G^{(n_0)}) = (1 + o_P(1)) E N_{K_2}(G^{(n_0)}).
$$

(5)

If also $(\pi)_6 \lesssim 1$, and $n_0 \gg n^{2/3}$, then

$$
N_{S_2}(G^{(n_0)}) = (1 + o_P(1)) E N_{S_2}(G^{(n_0)}),
$$

(6)

$$
N_{K_3}(G^{(n_0)}) = (1 + o_P(1)) E N_{K_3}(G^{(n_0)}).
$$

(7)

### 4.2 Parameter estimation in the Bernoulli model

The binomial community size distribution $\pi(x) = \binom{n}{x}(1-p)^{n-x}p^x$ with $p \in (0, 1)$ gives $(\pi)_r = p^r$ for all integers $r \geq 1$. We parameterize the model with three positive constants $(\lambda, \mu, q)$ (with $q$ not depending on scale) and choose

$$
m = \left\lfloor \frac{\mu^2 q}{\lambda} n \right\rfloor, \quad p = \frac{\lambda}{\mu q} n^{-1},
$$

(8)

where $\mu$ can be interpreted as the mean number of communities of a node. The following (asymptotic) relations follow from the results in Section 3:

$$
\lambda = nqmp^2, \quad \sigma^2 = nqmp^2 (1 + nqp), \quad \tau = \frac{q}{1 + mp}.
$$
from which one may solve

$$\mu = \frac{\lambda^2}{\sigma^2 - \lambda} \quad \text{and} \quad q = \tau(1 + \frac{\lambda^2}{\sigma^2 - \lambda}).$$

After substituting the asymptotic densities from Section 3 and estimating them using empirical counts we obtain (after some algebra) the estimators

$$\hat{\lambda} = (n-1)\binom{n_0}{2}^{-1} N_{K_2}(G^{(n_0)}),$$

$$\hat{\mu} = \frac{2N_{K_2}(G^{(n_0)})^2}{n_0N_{S_2}(G^{(n_0)}) - 2N_{K_2}(G^{(n_0)})^2}, \quad \hat{q} = \frac{3n_0N_{K_3}(G^{(n_0)})}{n_0N_{S_2}(G^{(n_0)}) - 2N_{K_2}(G^{(n_0)})^2}.$$  

To summarize, we estimate the parameters $\mu$ and $q$ by counting the numbers of links, 2-stars, and triangles from an induced subgraph of $n_0$ nodes. Alternatively, this can be seen as a way of fitting the transitivity and the mean and variance of the degrees. The theoretical justification is given by the following theorem.

**Theorem 3.** $\hat{\lambda}$, $\hat{\mu}$, and $\hat{q}$ converge in probability to the true values $\lambda$, $\mu$, and $q$, under the Bernoulli model defined by (8) given $n_0 \gg n^{2/3}$.  

**Proof.** The assumptions of Theorem 2 and Propositions 1 and 4 are satisfied by (8), which establishes the claim for $\hat{\lambda}$. Dividing and multiplying both $\hat{\mu}$ and $\hat{q}$ by $n_0^3$ yields rational expressions where the numerators and denominators converge in probability to nonzero constants by Theorem 2 and Propositions 1 and 2. The claim now follows from the continuous mapping theorem.

### 5 Numerical experiments

#### 5.1 Attainable regions in the Bernoulli model

The relations $\sigma^2 \geq \lambda$, $\tau \in (0,1)$ and $\tau \leq (1 + \lambda^2/(\sigma^2 - \lambda))^{-1}$ restrict the attainable combinations $(\lambda, \tau, \sigma^2)$; see Figure 1. To obtain a model with a large asymptotic transitivity coefficient, one may choose a low mean degree and a large degree variance. The flexibility gained by allowing $q \leq 1$ is also illustrated in Figure 2. The discreteness of the attainable points $(P(\text{link}), P(\text{triangle}))$ is obvious with $q = 1$, whereas the points with $q \leq 1$ fill a large part above the curve $P(\text{triangle}) = P(\text{link})^3$. 
Fig. 1: Attainable combinations of \((\tau, \sigma)\) for a range of values of \(\lambda\). Combinations with \(q = 1\) lie on the curves. The points under the curves are obtained by setting \(q \leq 1\).

Fig. 2: Attainable combinations of link and triangle probabilities in Bernoulli models with different values of \((\lambda, \tau)\) and (a) \(q = 1\) (exact probabilities) and (b) \(q \leq 1\) (averages of 1000 Monte Carlo samples). The solid curves represent theoretical bounds, and the thick black curve the Erdős–Rényi graph.

5.2 Real data

Ten data sets of different sizes were analyzed using the Bernoulli model. The whole data sets were used for estimation, i.e., \(n_0 = n\). The obtained estimates are listed in Table 1. Because we essentially fit \(\tau\) and \(\lambda\), these values are listed in Table 1 only for illustration purposes. In the largest data sets the estimates
of $q$ are very small, which might suggest that the structure of the model is not strongly supported by the data. For one of the data sets, Dolphin, the estimate of $q$ is outside the allowed range $(0,1)$. This may be related to the denseness of the network. On the other hand, simulation results in [17] suggest that the size $n = 62$ may not be sufficient for estimators based on asymptotic moment equations.

| Data set          | $n$     | $\hat{\lambda}$ | $\hat{\tau}$ | $\hat{q}$ | $\hat{m}$ | $\hat{\sigma}$ | $\hat{m}_{q=1}$ | $\hat{\sigma}_{q=1}$ |
|-------------------|---------|------------------|---------------|-----------|-----------|-----------------|----------------|-------------------|
| ca-AstroPh$^1$    | 18772   | 21.1             | 0.32          | 0.47      | 100       | 30.6           | 4092           | 15.1              |
| ca-HepPh$^1$      | 12008   | 19.7             | 0.66          | 0.78      | 15        | 46.6           | 162            | 28.2              |
| Dolphin$^2$       | 62      | 5.1              | 0.31          | - (2.36)  | 1255      | 3.0            | 61             | 4.1               |
| email-Eu-core$^1$ | 1005    | 32.0             | 0.27          | 0.47      | 8         | 37.0           | 236            | 20.1              |
| Facebook$^2$      | 63731   | 25.6             | 0.15          | 0.21      | 90        | 40.0           | 82756          | 11.8              |
| Flickr$^1$        | 105938  | 43.7             | 0.40          | 0.46      | 23        | 115.6          | 5377           | 36.4              |
| Flixster$^2$      | 2523386 | 6.3              | 0.01          | 0.014     | 4         | 36.6           | 2.1*10$^9$     | 2.6               |
| Twitter           | 2919613 | 8.8              | 0.006         | <0.001    | 77        | 20.9           | 9.3*10$^9$     | 3.0               |
| USAir97$^3$       | 332     | 12.8             | 0.40          | 0.56      | 2         | 20.1           | 60.1           | 11.0              |
| wiki-talk$^1$     | 2394385 | 3.9              | 0.002         | 0.002     | <1        | 102.5          | 1.27*10$^{11}$ | 2.0               |

Table 1: Parameter estimates of the Bernoulli model for collaboration networks in astrophysics and high energy physics, a social network of bottleneck dolphins, an e-mail network from a research institution, a geographically local Facebook network, a Flickr image network, a social network of Flixster users, a Twitter network of users who mention each other in their tweets, a US airport network, and a Wikipedia communications network. Data sets from $^1$[21], $^2$[22], and $^3$[23].

The rightmost two columns in Table 1 display reference values of $m$ and $\sigma$ estimated for the RIG model ($q = 1$) using the estimators introduced in [17]. These estimators give very large values for $m$ and grossly underestimate $\sigma$ in the largest data sets. These observations speak for the significantly improved model fit when using the thinned RIG model instead of the classical RIG model.

6 Technical proofs

6.1 Analysis of link density

Proof (Proof of Proposition 1). The probability of the event $\mathcal{E}_k$ that nodes 1 and 2 are linked via community $k$ can be written as

$$P(\mathcal{E}_k) = P(V_k \supset \{i,j\}, C_{12,k} = 1) = p_2 q.$$

Because the events $\mathcal{E}_1, \ldots, \mathcal{E}_m$ are independent, it follows that

$$P(\text{link}) = P\left( \bigcup_k \mathcal{E}_k \right) = 1 - \prod_k P(\mathcal{E}_k) = 1 - (1 - p_2 q)^m.$$
The inequality $1 - x \leq e^{-x}$ and the union bound $P(\cup_k E_k) \leq \sum_k P(E_k)$ imply that $1 - e^{-mp_2q} \leq P(\text{link}) \leq mp_2q$, from which we see that $P(\text{link}) \ll 1$ if and only if $mp_2q \ll 1$. The approximation formula (2) follows from the Bonferroni’s bounds

$$mp_2q - \binom{m}{2}(p_2q)^2 \leq P(\text{link}) \leq mp_2q.$$ 

### 6.2 Analysis of 2-star covering density

**Proof (Proof of Proposition 2: equation (3)).** Consider a 2-star with node set \{1, 2, 3\} and link set \{\{1, 2\}, \{1, 3\}\}. Denote by $B_{A,k} = \{V_k \supset A\}$ the event that community $k$ covers a node set $A$, and by $C_{ij,k}$ the event that $C_{ij,k} = 1$. Then $E_{ij,k} = B_{ij,k} \cap C_{ij,k}$ is the event that node pair $ij$ is linked by community $k$. Then the probability that $G$ contains the 2-star as a subgraph is given by

$$P(\text{2-star}) = P\left(\bigcup_{k \in [m]^2} F_k\right),$$

where $F_k = E_{12,k_1} \cap E_{13,k_2}$ for an ordered community pair $k = (k_1, k_2)$. Observe that $P(F_k) = q^2p_3$ for $k_1 = k_2$ and $P(F_k) = q^2p_2^2$ otherwise. Therefore,

$$P(\text{2-star}) \leq \sum_{k \in [m]^2} P(F_k) = mq^2p_3 + (m)2q^2p_2^2.$$ 

To prove the claim using Bonferroni’s bounds, it suffices to show that

$$\sum_{(k,\ell)} P(F_k, F_\ell) \ll q^2\left(mp_3 + (m)2p_2^2\right), \quad (9)$$

where the sum on the left is over all $(k, \ell)$-pairs with $k, \ell \in [m]^2$ and $k \neq \ell$.

We will now compute the sum on the left side of (9). Note that

$$P(F_k, F_\ell) = q^{\{(k_1, \ell_1)\} + \{(k_2, \ell_2)\}}P(B_{12,k_1}, B_{13,k_2}, B_{12,\ell_1}, B_{13,\ell_2}).$$

Therefore, for example, for a $(k, \ell)$-pair of the form $(k_1, k_2, \ell_1, \ell_2) = (a, a, b, c)$ with distinct $a, b, c$ we have

$$P(F_k, F_\ell) = q^3P(B_{123,a}, B_{12,b}, B_{13,c}) = q^2p_2p_3.$$ 

The table below displays the values of $P(F_k, F_\ell)$ for all combinations of $k \neq \ell$, and the cardinalities of such combinations.

| $(k_1, k_2, \ell_1, \ell_2)$ | Cardinality $\mathbb{P}(F_k, F_\ell)$ |
|-----------------------------|-------------------------------------|
| $(a, b, c, d)$              | $(m)_4$                             |
| $(a, b, a, c)$ or $(a, b, c, b)$ | $2(m)_3$                             |
| $(a, a, b, c)$ or $(a, b, c, c)$ or $(a, b, c, a)$ or $(a, b, b, c)$ | $(4m)_3$                             |
| $(a, a, b, b)$ or $(a, b, b, a)$ | $2(m)_2$                             |
| $(a, a, b, a)$ or $(a, a, b, a)$ or $(a, b, a, a)$ or $(b, a, a, a)$ | $(4m)_2$                             |
As a consequence,

$$\sum_{(k,\ell)} \mathbb{P}(F_k, F_\ell) = (m)_4q^4p_1^4 + 2(m)_3q^3p_3^3 + 4(m)_3q^3p_2p_3 + 2(m)_2q^4p_3^3 + 4(m)_2q^3p_2p_3$$

By noting that $p_3 \leq p_2$, we see that the first three terms on the right are bounded from above by $4(mp_2q)q^2(m)_2p_2^2$, and the last two terms on the right are bounded from above by $4(mp_2q)q^3mp_3$. Hence the above sum is at most $12(mp_2q)q^2(mp_3 + (m)_2p_2^2)$, claim (9) is valid, and the claim follows.

### 6.3 Analysis of triangle covering density

**Proof (Proof sketch of Proposition 2: equation (4)).** Consider a triangle with node set $\{1, 2, 3\}$. Denote by $\mathcal{E}_{e,k} = \{V_k \supset e, C_{e,k} = 1\}$ the event that node pair $e$ is linked via community $k$. Then $\mathbb{P}(\text{triangle}) = \mathbb{P}(\cup_{k \in [m]} \mathcal{F}_k)$, where $\mathcal{F}_k = \mathcal{E}_{12,k} \cap \mathcal{E}_{13,k} \cap \mathcal{E}_{23,k}$ is the event that the node pairs of the triangle are linked via communities of the triplet $k = (k_1, k_2, k_3)$. Because

$$\mathbb{P}(\mathcal{F}_k) = q^3\mathbb{P}(V_{k_1} \supset 12, V_{k_2} \supset 13, V_{k_3} \supset 23) = \begin{cases} 
q^3p_3, & |\{k_1, k_2, k_3\}| = 1, \\
q^3p_2p_3, & |\{k_1, k_2, k_3\}| = 2, \\
q^3p_2^3, & |\{k_1, k_2, k_3\}| = 3,
\end{cases}$$

the union bound implies that

$$\mathbb{P}(\text{triangle}) \leq \sum_k \mathbb{P}(\mathcal{F}_k) \leq q^3\left(mp_3 + 3(m)_2p_2p_3 + (m)_3p_2^3\right).$$

By similar techniques as in the proof of (3), one can show that

$$\sum_{(k,\ell): k \neq \ell} \mathbb{P}(F_k, F_\ell) \ll (mp_2) \sum_k \mathbb{P}(F_k) \ll \sum_k \mathbb{P}(F_k),$$

and the claim follows by Bonferroni’s bounds. (The details of the lengthy computations are omitted.)

### 6.4 Analysis of model transitivity

**Proof (Proof of Proposition 3).** By applying Propositions 2 we find that

$$\tau = (1 + o(1))q\frac{mp_3 + 3(m)_2p_2p_3 + (m)_3p_2^3}{mp_3 + (m)_2p_2^2} = (1 + o(1))q\left(\frac{mp_3}{mp_3 + (m)_2p_2^2} + R\right),$$

where

$$R = \frac{3(m)_2p_2p_3 + (m)_3p_2^3}{mp_3 + (m)_2p_2^2} \leq mp_2\frac{3mp_3 + (m)_2p_2^2}{mp_3 + (m)_2p_2^2} \leq 3mp_2.$$  

The assumption $mp_2 \ll 1$ now implies that $qR = o(1)$. Hence we conclude

$$\tau = (1 + o(1))\left(\frac{qmp_3}{mp_3 + (m)_2p_2^2} + o(1)\right) = \frac{qmp_3}{p_3 + (m - 1)p_2^2} + o(1).$$
6.5 Analysis of degree moments

Proof (Proof of Proposition 4). By expressing the degree of node $i$ using the adjacency matrix as $D = \sum_{j \neq i} G_{i,j}$ and taking expectations, we find that

\[
E(D) = (n-1)\mathbb{P}(\text{link}),
\]
\[
E(D^2) = (n-1)\mathbb{P}(\text{link}) + (n-1)(n-2)\mathbb{P}(\text{2-star}).
\]

By Propositions 1 and 2 we find that

\[
\mathbb{P}(\text{link}) = (1 + O(mp_2q))mp_2q,
\]
\[
\mathbb{P}(\text{2-star}) - \mathbb{P}(\text{link})^2 = (1 + O(mp_2q))q^2(mp_3 + (m)p_2^2 - m^2p_2^3).
\]

Hence $E(D) \sim mn p_2 q$, and by the formula $\text{Var}(D) = E(D^2) - (E(D))^2$,

\[
\text{Var}(D) = (1 + O(n^{-1})\left(n\mathbb{P}(\text{link}) + n^2\left(\mathbb{P}(\text{2-star}) - \mathbb{P}(\text{link})^2\right)\right))
\]
\[
= (1 + O(n^{-1})(1 + O(mp_2q))\left(mnp_2 + mn^2q^2(p_3 - p_2^2)\right)).
\]

6.6 Analysis of observed link density

Proof (Proof of Theorem 2: equation (5)). Let us denote by $\hat{N} = N_{K_2}(G(n_0))$ the number of links in the observed graph $G(n_0)$. The assumptions $(\pi)_2 \gtrsim 1$ and $(\pi)_4 \lesssim 1$ imply that $p_2 \asymp n^{-2}$ and $p_r \lesssim n^{-r}$ for $r = 3, 4$. Because $m \asymp n$, and $q \gtrsim 1$, with the help of Proposition 1, we see that

\[
\mathbb{P}(\text{link}) = (1 + o(1))mp_2 q \asymp n^{-1},
\]
and

\[
E\hat{N} = \binom{n_0}{2}\mathbb{P}(\text{link}) \asymp n_0^2 n^{-1} \gg 1.
\]

Denote by $\mathbb{P}(\text{link}^2)$ the probability that $G$ contains any particular pair of disjoint node pairs. Note that

\[
\text{Var}(\hat{N}) = \sum_e \sum_{e'} \mathbb{P}(e \in E(G(n_0)), e' \in E(G(n_0))) - \binom{n_0}{2}^2 \mathbb{P}(\text{link})^2
\]
\[
= \binom{n_0}{2}\mathbb{P}(\text{link}) + \binom{n_0}{2}\mathbb{P}(\text{2-star}) + \binom{n_0}{2}\left(\binom{n_0}{2} - \binom{n_0}{2}\right)\mathbb{P}(\text{link})^2 - \binom{n_0}{2}^2 \mathbb{P}(\text{link})^2
\]
\[
\leq n_0^2 \mathbb{P}(\text{link}) + n_0^3 \mathbb{P}(\text{2-star}) + \frac{n_0}{2} \left(\mathbb{P}(\text{link})^2 - \mathbb{P}(\text{link}^2)\right).
\]

Note that $\mathbb{P}(\text{link}) \asymp n^{-1}$ and $\mathbb{P}(\text{2-star}) \lesssim n^{-2}$. Furthermore,

\[
\mathbb{P}(\text{link}^2) = \mathbb{P}(\cup_k \cup_{\ell} \{V_k \ni \{1, 2, C_{12, k} = 1, V_\ell \ni \{3, 4, C_{34, \ell} = 1\}\})
\]
\[
\leq \sum_k \sum_{\ell} \mathbb{P}(\{V_k \ni \{1, 2, C_{12, k} = 1, V_\ell \ni \{3, 4, C_{34, \ell} = 1\}\})
\]
\[
= (m)p_2^2 q^2 + mp_4 q^2 = (1 + o(1))\mathbb{P}(\text{link})^2 + O(n^{-3}),
\]
so that

\[
\text{Var}(\hat{N}) \leq n_0^2n^{-1} + n_0^3n^{-2} + n_0^4n^{-3} + o(1)(\mathbb{E}\hat{N})^2
\leq 3n_0^2n^{-1} + o(1)(\mathbb{E}\hat{N})^2 \ll (\mathbb{E}\hat{N})^2.
\]

6.7 Analysis of observed 2-star covering density

Proof (Proof sketch of Theorem 2: equation (6)). Let us denote \( \hat{N} = N_{S_2}(G^{(n_0)}) \). Note that

\[
\hat{N} = \sum_R 1_{A_R}
\]

where the sum ranges over the set of all \( S_2 \)-isomorphic subgraphs of \( K_{[n_0]} \), and \( 1_{A_R} \) is the indicator of the event \( A_R \) that \( G^{(n_0)} \) contains \( R \) as a subgraph. The assumptions \((\pi)_2 \geq 1 \) and \( (\pi)_3 \leq 1 \) imply that \( p_2 \approx n^{-2} \) and \( p_r \leq n^{-r} \) for \( r = 3, \ldots, 6 \). Because \( m \approx n \), and \( q \geq 1 \), with the help of Proposition 2, we see that

\[
\mathbb{P}(\text{2-star}) = q^2\left(mp_3 + (m)p_2^2\right) \approx n^{-2},
\]

(10)

and

\[
\mathbb{E}\hat{N} = 3\left(\frac{n_0}{3}\right)\mathbb{P}(\text{2-star}) \approx n_0^3n^{-2} \gg 1.
\]

The above relation underlines the role of assumption \( n_0 \gg n^{2/3} \). This guarantees that there are lots of (dependent) samples to sum in the observed graph.

Let us next analyze the variance of \( \hat{N} \). By applying the formula \( \text{Var}(\hat{N}) = \mathbb{E}(\hat{N}^2) - (\mathbb{E}\hat{N})^2 \) and noting that \( A_R \cap A_{R'} = A_{R \cup R'} \), we see that

\[
\text{Var}(\hat{N}) = \sum_R \sum_{R'} \mathbb{P}(A_R, A_{R'}) - \sum_R \sum_{R'} \mathbb{P}(A_R)\mathbb{P}(A_{R'}) = \sum_{i=0}^{3} M_i,
\]

(11)

where

\[
M_i = \sum_R \sum_{R': |V(R)| \leq i} \left(\mathbb{P}(A_R \cup R') - \mathbb{P}(A_R)^2\right).
\]

For \( i \geq 1 \), we approximate \( M_i \) from above by omitting the \( \mathbb{P}(A_R) \) term in (11). By generalizing the analytical technique used in [17] (details will be available in the extended version), it can be shown that for any graph \( R \) such that \( |V(R)| \leq 6 \),

\[
\mathbb{P}(A_R) \preceq n^{-\kappa(R)},
\]

(12)

where \( \kappa(R) = \min_E(||E|| - |E|) \), with the minimum taken across all partitions of \( E(R) \) into nonempty sets, where \( |E| \) is the number of parts in the partition, and we set \( ||E|| = \sum_{E \in E} |E^c| \) where \( E^c = \cup_{e \in E} e \) denotes the set of nodes covered by the node pairs of \( E \), so that for example, \( \{\{1,2\}\}^b = \{1,2\} \) and \( \{\{1,3\}, \{2,3\}\}^b = \{1,2,3\} \). Table 2 summarizes the values of \( \kappa(R) \) for the type
of graphs that can be obtained as unions of two 2-stars. By applying (12), it follows that

\[ M_1 \lesssim n_0^2 \left( P(\text{4-star}) + P(\text{4-path}) + P(\text{chair}) \right) \lesssim n_0^5 n^{-4}, \]

\[ M_2 \lesssim n_0^4 \left( P(\text{3-star}) + P(\text{3-path}) + P(\text{3-pan}) + P(\text{4-cycle}) \right) \lesssim n_0^4 n^{-3}, \]

\[ M_3 \lesssim n_0^3 \left( P(\text{2-star}) + P(\text{triangle}) \right) \lesssim n_0^3 n^{-2}. \]

Because \( n_0 \gg n^{2/3} \), it follows that \( M_i \lesssim n_0^3 n^{-2} \ll (\mathbb{E} \tilde{N})^2 \) for \( i = 1, 2, 3 \).

The \( M_0 \)-term in the variance formula (11) satisfies

\[ M_0 \approx n_0^6 \left( P(\text{2-star}^2) - P(\text{2-star})^2 \right) \]

where \( P(\text{2-star}^2) \) indicates the probability that \( G \) contains a particular union of two disjoint 2-stars as a subgraph. Here we need more careful analysis because the technique used to bound \( M_i \) for \( i \geq 1 \) would only yield an upper bound for \( M_0 \) of the same order as \( (\mathbb{E} \tilde{N})^2 \). Nevertheless, a tedious but straightforward computation (details will be available in the extended version) involving all 15 partitions of the link set of a union of two disjoint 2-stars can be used to verify that

\[ P(\text{2-star}^2) \leq q^4 \left( m^2 p_3^2 + 2m^3 p_2 p_3 + m^4 p_2^2 \right) + O(n^{-5}) \]

\[ = (1 + o(1))P(\text{2-star})^2 + O(n^{-5}). \]

By comparing this with (10), we find that \( P(\text{2-star}^2) - P(\text{2-star})^2 \ll P(\text{2-star})^2 \), and

\[ M_0 \approx n_0^6 \left( P(\text{2-star}^2) - P(\text{2-star})^2 \right) \ll n_0^6 P(\text{2-star})^2 \approx (\mathbb{E} \tilde{N})^2. \]

We may now conclude that \( \text{Var}(\tilde{N}) = \sum_{i=0}^3 M_i \ll (\mathbb{E} \tilde{N})^2 \), and hence the claim follows by Chebyshev’s inequality.

| R         | |V(R)| |E(R)| |κ(R)|   |
|-----------|-----------------|-----------------|-----------------|-----------------|
| 2-star    | 3               | 2               | 2               |
| 3-cycle   | 3               | 3               | 2               |
| 3-star    | 4               | 3               | 3               |
| 3-path    | 4               | 3               | 3               |
| 3-pan     | 4               | 4               | 3               |
| 4-cycle   | 4               | 4               | 3               |
| 4-star    | 5               | 4               | 4               |
| 4-path    | 5               | 4               | 4               |
| Chair     | 5               | 4               | 4               |
| Disjoint 2-stars | 6               | 4               | 4               |

Table 2: Values of \( \kappa(R) \) (obtained using an exhaustive computer search) for graphs obtained as unions of two 2-stars.
6.8 Analysis of observed triangle density

Proof (Proof sketch of Theorem 2: equation (7)). Let us denote by $\hat{N} = N_{K_3}(G^{(n_0)})$ the number of triangles in the observed graph $G^{(n_0)}$. The assumptions $(\pi)_2 \gtrsim 1$ and $(\pi)_6 \lesssim 1$ imply that $p_2 \asymp n^{-2}$ and $p_r \lesssim n^{-r}$ for $r = 3, \ldots, 6$. Because $m \asymp n$, and $q \gtrsim 1$, with the help of Proposition 2, we see that

$$P(\text{triangle}) = (1 + o(1))mp_3q^3 \asymp n^{-2},$$

and

$$\mathbb{E}\hat{N} = \left(\frac{n_0}{3}\right)P(\text{triangle}) \asymp n_0^3n^{-2} \gg 1.$$

To show that $\hat{N}$ is with high probability close to $\mathbb{E}\hat{N}$, by Chebyshev’s inequality it suffices to verify that $\text{Var}(\hat{N}) \ll (\mathbb{E}\hat{N})^2$. By applying the formula $\text{Var}(\hat{N}) = \mathbb{E}\hat{N}^2 - (\mathbb{E}\hat{N})^2$ and noting that $A_R \cap A_{R'} = A_R \cup A_{R'}$, we see that

$$\text{Var}(\hat{N}) = \sum_R \sum_{R'} P(A_R, A_{R'}) - \sum_R \sum_{R'} P(A_R)P(A_{R'}) = \sum_{i=0}^3 M_i,$$

where

$$M_i = \sum_R \sum_{R': |V(R) \cap V(R')| = i} \left(P(A_R \cup A_{R'}) - P(A_R)^2\right).$$

In analogy with the proof of (6) one can show (details omitted) that $M_i \ll (\mathbb{E}\hat{N})^2$ for $i = 1, 2, 3$ by analyzing the subgraph containment probabilities of $G$ for unions of two triangles. Again, the $M_0$ term requires special attention. A careful analysis of the various patterns through which the communities of the model can cover the links of two disjoint triangles (details available in the extended version) shows that

$$P(\text{triangle}^2) - P(\text{triangle})^2 \ll P(\text{triangle})^2.$$

This implies $M_0 \ll (\mathbb{E}\hat{N})^2$ and allows to conclude that $\text{Var}(\hat{N}) = \sum_{i=0}^3 M_i \ll (\mathbb{E}\hat{N})^2$. Hence the claim follows by Chebyshev’s inequality.

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