An analogue of a theorem of Kurzweil

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Abstract
A theorem of Kurzweil (’55) on inhomogeneous Diophantine approximation states that if $\theta$ is an irrational number, then the following are equivalent: (A) for every decreasing positive function $\psi$ such that $\sum_{q=1}^{\infty} \psi(q) = \infty$, and for almost every $s \in \mathbb{R}$, there exist infinitely many $q \in \mathbb{N}$ such that $\|q\theta - s\| < \psi(q)$, and (B) $\theta$ is badly approximable. This theorem is not true if one adds to condition (A) the hypothesis that the function $q \mapsto q\psi(q)$ is decreasing. In this paper we find a condition on the continued fraction expansion of $\theta$ which is equivalent to the modified version of condition (A). This expands on a recent paper of Kim (2014 Nonlinearity 27 1985–97).

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An irrational number $\theta$ is said to be badly approximable (or of bounded type) if there exists $\epsilon > 0$ such that for every rational $p/q \in \mathbb{Q}$,

$$\left| \theta - \frac{p}{q} \right| \geq \frac{\epsilon}{q^2}.$$ 

It is well known that an irrational number $\theta$ is badly approximable if and only if the partial quotients of $\theta$ form a bounded sequence. Another equivalent condition was given by Kurzweil [6]. To state it, let us define the set

$$W(\theta, \psi) = \{s \in \mathbb{R} : \exists q \in \mathbb{N} \|q\theta - s\| < \psi(q)\},$$

where $\| \cdot \|$ denotes distance to the nearest integer. Then Kurzweil’s result may be stated as follows: $\theta$ is badly approximable if and only if for every decreasing function $\psi : \mathbb{N} \to (0, \infty)$ such that $\sum_{q=1}^{\infty} \psi(q) = \infty$, the set $W(\theta, \psi)$ has full measure. (Note that if $\sum_{q=1}^{\infty} \psi(q) < \infty$, then the set $W(\theta, \psi)$ has measure zero by the Borel–Cantelli lemma.)
Rather than considering all decreasing functions $\psi$, one may consider the smaller class of **Khinchin sequences**: a function $\psi : \mathbb{N} \to (0, \infty)$ is called a Khinchin sequence if, in addition to the divergence condition $\sum_{q=1}^{\infty} \psi(q) = \infty$, the function $q \mapsto q \psi(q)$ is non-increasing. Although less natural than the condition that $\psi$ is decreasing, the hypothesis that a sequence is a Khinchin sequence is significant both for historical reasons (Khinchin first proved his eponymous theorem [3] in the setting of Khinchin sequences, although his theorem was later generalized) and because such sequences are often easier to work with.

Let $\theta$ be an irrational number and let $\psi$ be a Khinchin sequence. A recent paper of Kim [5] gives a criterion, based on the continued fraction expansion of $\theta$, for the set $W(\theta, \psi)$ to have full measure\(^1\). However, his paper leaves open the question of finding an analogue of Kurzweil’s theorem in the setting of Khinchin sequences, although he proves several results in that direction [5, section 3]. The aim of this paper is to complete the work of Kim by proving such an analogue.

1. Statement of results

We first recall the main theorem of [5], rephrased slightly\(^2\).

**Theorem 1.1 ( [5, Theorem 2.1]).** Fix $\theta \in \mathbb{R}/\mathbb{Q}$ and let $(q_k)_{k=0}^{\infty}$ be the sequence of the denominators of the convergents of $\theta$. Let $\psi : \mathbb{N} \to (0, \infty)$ be a Khinchin sequence, and let $\phi(q) = 1/(q \psi(q))$. Then the following are equivalent:

(A) $W(\theta, \psi)$ has full measure.

(B) The series
\[
\sum_{k=0}^{\infty} \log(\phi(q_k)) \wedge \log(q_{k+1}/q_k) / \phi(q_k) = \infty
\]

1.1 diverges. (In this paper, $\wedge$ and $\vee$ denote minimum and maximum, respectively.)

To state our main theorem, we use the notation
\[
\Sigma((a_i)_{i=1}^{n} : m)
\]
to denote the sum of the $m$ largest elements of the sequence $(a_i)_{i=1}^{n}$, with $\Sigma((a_i)_{i=1}^{n} : m) = \sum_{i=1}^{n} a_i$ if $m \geq n$. For $a \geq 0$, we let $\Sigma((a_i)_{i=1}^{n} : a) = \Sigma((a_i)_{i=1}^{\lfloor a \rfloor} : a)$.

**Theorem 1.2.** Fix $\theta \in \mathbb{R}/\mathbb{Q}$ and let $(q_k)_{k=0}^{\infty}$ be the sequence of the denominators of the convergents of $\theta$. Then the following are equivalent:

(A) For every Khinchin sequence $\psi : \mathbb{N} \to (0, \infty)$, the set $W(\theta, \psi)$ has full measure.

(B) For some $\epsilon > 0$,
\[
\limsup_{k \to \infty} \frac{1}{\log(q_k)} \Sigma \left( \left( \log \frac{q_{i+1}}{q_i} \right)^{k-1} \right) \left( \log(q_k) / \log \log(q_k) \right) < 1.
\]

**Remark.** Since condition (B) of theorem 1.2 is not equivalent to the condition that the sequence $(q_k)_{k=0}^{\infty}$, it follows from Kurzweil’s theorem that condition (A) is not equivalent to the condition that for every decreasing positive function $\psi : \mathbb{N} \to (0, \infty)$ such that $\sum_{q=1}^{\infty} \psi(q) = \infty$.

\(^1\) After this paper was written, Kim extended his result to all positive decreasing sequences in a joint paper with Fuchs [2].

\(^2\) Technically, the result of [5] applies to the sets $\bigcap_{\epsilon > 0} W(\theta, \epsilon \psi)$ and not directly to the sets $W(\theta, \psi)$. But since the convergence or divergence of the series (1.1) is invariant under a slight perturbation of $\psi$, [5, theorem 2.1] and theorem 1.1 are equivalent.
the set $W(\theta, \psi)$ has full measure. In particular, there exists a decreasing positive function $\psi : \mathbb{N} \to (0, \infty)$ such that $\sum_{q=1}^{\infty} \psi(q) = \infty$ and such that there is no Khinchin sequence $\psi' : \mathbb{N} \to (0, \infty)$ with $\psi'(q) \leq \psi(q)$ for all $q$. An example of such a sequence is given by the formula

$$\psi(q) = 2^{-n_k} \quad (2^{n_{k-1}} \leq q < 2^{n_k})$$

where $(n_k)_{k=1}^{\infty}$ is any sequence of integers such that $n_k - n_{k-1} \geq k$ for all $k$.

2. Proof of theorem 1.2

**Convention.** The symbol $\approx$ will denote a coarse multiplicative asymptotic, i.e. $A_n \approx B_n$ means that there exists a constant $C > 0$ (the implied constant) such that $C^{-1}B_n \leq A_n \leq CB_n$.

**Proof of (A) \implies (B).** By contradiction, suppose that (B) is false. Then for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$1 \log(q_{kn})/\Sigma_1^{k_n-1} (\log(q_{i+1}) - \log(q_i)) \geq 1 - \frac{1}{2^n}.$$ 

Without loss of generality, suppose that the sequence $(k_n)_{n=1}^{\infty}$ is increasing, and let $k_0 = 0$. For each $n \geq 1$, let $S_n'$ be a subset of $\{0, \ldots, k_n - 1\}$ of cardinality at most $2^n \log(q_{kn})$ such that

$$\sum_{k \in S_n'} \log \left(\frac{q_{k+1}}{q_k} \right) \geq (1 - 2^{-n}) \log(q_{kn}).$$

Then let $S_n = S_n'/\{0, \ldots, k_{n-1} - 1\}$ and $T_n = \{k_{n-1}, \ldots, k_n - 1\}/S_n$. Then

$$\#(S_n) \leq \frac{1}{2^n \log(q_{kn})}$$

and

$$\sum_{k \in T_n} \log \left(\frac{q_{k+1}}{q_k} \right) \leq \log(q_{kn}) - \sum_{k \in S_n} \log \left(\frac{q_{k+1}}{q_k} \right) \leq 2^{-n} \log(q_{kn}). \quad (2.2)$$

Now define the function $\phi : \mathbb{N} \to (0, \infty)$ by the formula

$$\phi(q) = \log(q_{kn}) \forall q_{kn-1} \leq q < q_{kn}.$$ 

Then $\phi$ is non-decreasing, and

$$\sum_{q=1}^{\infty} \frac{1}{\phi(q)} = \sum_{n=1}^{\infty} \frac{1}{\log(q_{kn})} \sum_{q=q_{kn-1}}^{q_{kn}-1} \frac{1}{q} \approx \sum_{n=1}^{\infty} \frac{\log(q_{kn}/q_{kn-1})}{\log(q_{kn})} = \sum_{n=1}^{\infty} \left[1 - \frac{\log(q_{kn-1})}{\log(q_{kn})}\right] \approx \sum_{n=1}^{\infty} 1 \log \left(\frac{\log(q_{kn})}{\log(q_{kn-1})}\right) = \infty.$$ 

Thus $\psi(q) = 1/\phi(q)$ is a Khinchin sequence. So by (A) together with theorem 1.1, the series (1.1) diverges. On the contrary, we show that (1.1) converges:

$$\sum_{k=0}^{\infty} \log \left(\frac{\phi(q_k)}{\phi(q_{k+1})} / \phi(q_k) \right) \leq \sum_{n=1}^{\infty} \left[\sum_{k \in S_n} \log \phi(q_k) / \phi(q_k) + \sum_{k \in T_n} \log(q_{k+1}/q_k) / \phi(q_k)\right]$$

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\[
\begin{align*}
\sum_{n=1}^{\infty} \left[ \frac{\log(q_n)}{\log(q_{n+1})} \cdot (S_n) + \frac{1}{\log(q_{n+1})} \sum_{k \in T_n} \log(q_{k+1}/q_k) \right] \\
\leq \sum_{n=1}^{\infty} \left[ \frac{1}{2^n} + \frac{1}{2^n} \right] \\
\leq 2 < \infty.
\end{align*}
\]

This contradiction completes the proof. \(\square\)

**Proof of (B) \(\Rightarrow\) (A).** Let \(\psi : \mathbb{N} \to \infty\) be a Khinchin sequence, and by contradiction suppose that \(W(\theta, \psi)\) does not have full measure. Then by theorem 1.1, the series (1.1) converges, where \(\phi(q) = 1/(q\psi(q))\) is non-decreasing. Let
\[
S = \{ k : \phi(q_k) \leq q_{k+1}/q_k \}, \quad T = \mathbb{N}/S,
\]
so that
\[
\sum_{k=0}^{\infty} \frac{\log \phi(q_k)}{\phi(q_k)} = \sum_{k \in S} \frac{\log \phi(q_k)}{\phi(q_k)} + \sum_{k \in T} \frac{\log(q_{k+1}/q_k)}{\phi(q_k)}.
\]
For each \(m \in \mathbb{N}\), let \(Q_m\) be the largest integer such that \(\phi(Q_m) \leq 2^m\). Then
\[
\frac{1}{\phi(q)} \leq \sum_{m \in \mathbb{N} : q \geq 2^m} \frac{1}{2^m} = \sum_{m \in \mathbb{N} : q \leq Q_m} \frac{1}{2^m},
\]
and thus
\[
\sum_{m=0}^{\infty} \frac{\log \phi(q_k)}{\phi(q_k)} \leq \sum_{m=0}^{\infty} \frac{\log(q_{k+1}/q_k)}{\phi(q_k)}
\]
\[
\times \sum_{m=0}^{\infty} \left[ \frac{m}{2^m} \cdot \# \{ k \in S : q_k \leq Q_m \} + \frac{1}{2^m} \sum_{k \in T : q_k \leq Q_m} \log \left( \frac{q_{k+1}}{q_k} \right) \right].
\]
It follows that if
\[
\lambda_m = \frac{m \# \{ k \in S : q_k \leq Q_m \} + \frac{1}{2^m} \sum_{k \in T : q_k \leq Q_m} \log \left( \frac{q_{k+1}}{q_k} \right)}{\log(Q_m) \cdot 2^m}
\]
then
\[
\liminf_{m \to \infty} \lambda_m = 0.
\]
On the other hand, if
\[ \kappa_m = \frac{m}{2^m} \# \{ k \in S : q_k \leq Q_m \}, \]
then
\[ \lim_{m \to \infty} \kappa_m = 0. \]

Fix \( \epsilon > 0 \), and choose \( m \geq 2 \) such that \( \lambda_m, \kappa_m \leq \epsilon \). Then
\[ \frac{m}{2^m} \vee \frac{m}{\log(Q_m)} \leq \# \{ k \in S : q_k \leq Q_m \}. \]

Consider the function
\[ f(x) = \frac{x}{2^x} \vee \frac{x}{\log(Q_m)} \quad \text{(} x \geq 2 \text{)}. \]

Since \( f \) is the maximum of an increasing function and a decreasing function, \( f \) has a unique minimum, which occurs when the two inputs to the maximum agree, namely at \( x = \log_2 \log(Q_m) \). Thus
\[ \frac{\epsilon}{\# \{ k \in S : q_k \leq Q_m \}} \geq f(m) \geq \min(f) = \frac{\log_2 \log(Q_m)}{\log(Q_m)} \geq \frac{\log \log(Q_m)}{\log(Q_m)} \]
i.e.
\[ \# \{ k \in S : q_k \leq Q_m \} \leq \frac{\epsilon \log(Q_m)}{\log \log(Q_m)}. \]

On the other hand, since \( \lambda_m \leq \epsilon \),
\[ \sum_{k \in T \atop q_k \leq Q_m} \log \left( \frac{q_{i+1}}{q_i} \right) \leq \epsilon \log(Q_m). \]

Let \( k_m \) be the smallest integer such that \( Q_m < q_{k_m} \). Then
\[ \# \{ k \in S : k < k_m \} \leq \frac{\epsilon \log(q_{k_m})}{\log \log(q_{k_m})} \]
\[ \sum_{k \in T \atop k < k_m} \log \left( \frac{q_{i+1}}{q_i} \right) \leq \epsilon \log(q_{k_m}) \]
and thus
\[ \sum \left( \log \left( \frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k_m-1} \frac{\epsilon \log(q_{k_m})}{\log \log(q_{k_m})} \geq (1 - \epsilon) \log(q_{k_m}). \]

Since \( \epsilon \) was arbitrary and \( k_m \to \infty \), for all \( \epsilon > 0 \) we have
\[ \limsup_{k \to \infty} \frac{1}{\log(q_k)} \sum \left( \log \left( \frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k-1} \frac{\epsilon \log(q_i)}{\log \log(q_i)} = 1, \]
contradicting (B). \( \square \)
3. Consequences of theorem 1.2

In this section we use theorem 1.2 to prove some necessary and sufficient conditions on $\theta$ for $W(\theta, \psi)$ to be full measure for every Khinchin sequence $\psi$, including reproving some results from [5, section 3]. For convenience let

$\Omega = \{ \theta \in \mathbb{R} : \text{for every Khinchin sequence } \psi, \text{the set } W(\theta, \psi) \text{ has full measure} \}$.

In other words, $\Omega$ is the set of all $\theta$ such that the equivalent conditions of theorem 1.2 hold.

**Theorem 3.1.** Fix $\theta \in \mathbb{R}/\mathbb{Q}$ and let $(q_k)_{k=0}^{\infty}$ be the sequence of the denominators of the convergents of $\theta$.

(i) If

$$\limsup_{k \to \infty} \frac{\log(q_k)}{k} < \infty,$$

then $\theta \in \Omega$.

(ii) If

$$\limsup_{k \to \infty} \frac{\log(q_k)}{k \log(k)} = \infty,$$

then $\theta \notin \Omega$.

(iii) If

$$\sum_{k=2}^{\infty} \frac{1}{\log(q_k)} < \infty,$$

then $\theta \notin \Omega$.

(iv) If

$$\limsup_{k \to \infty} \frac{q_{k+1}/q_k}{\log(q_k)} < \infty,$$

then $\theta \in \Omega$.

(v) If

$$\limsup_{k \to \infty} \frac{\log(q_{k+1}/q_k)}{\log(q_k)} = \infty,$$

then $\theta \notin \Omega$.

**Remark.** Parts (i), (iii) and (iv) correspond to [5, theorem 3.1 and proposition 3.2]. Although in some cases the new proofs are not shorter than the old proofs, having two proofs may bring further insight.

**Remark.** By well-known facts about continued fractions (e.g. [4, theorems 9 and 13]), the conditions (3.4) and (3.5) have interpretations in terms of Diophantine approximation:

- $\theta$ satisfies (3.4) if and only if for some $\epsilon > 0$, $\theta$ is not $\psi$-approximable, where

$$\psi(q) = \frac{\epsilon}{q^2 \log(q)}.$$

We recall that a number $\theta$ is called $\psi$-approximable if there exist infinitely many rationals $p/q \in \mathbb{Q}$ such that

$$\left| \theta - \frac{p}{q} \right| < \psi(q).$$
• $\theta$ satisfies (3.5) if and only if $\theta$ is a Liouville number. We recall that a number $\theta$ is called Liouville if for all $n \in \mathbb{N}$, $\theta$ is $\psi_n$-approximable, where $\psi_n(q) = q^{-n}$.

**Remark.** Any badly approximable number $\theta$ satisfies both (3.1) and (3.4), so $\mathbb{B} A \subset \Omega$. This can also be seen from Kurzweil’s theorem.

**Remark.** The continued fraction expansion of $e$ (see e.g. [1]) satisfies (3.4), so $e \in \Omega$.

**Proof of (i).** Choose $M < \infty$ so that for all $k$, $\log(q_k) \leq Mk$. Let $\epsilon > 0$ be arbitrary (e.g. $\epsilon = 1$). Then for sufficiently large $k$,

$$\frac{\epsilon \log(q_k)}{\log(q_k)} \leq \frac{\epsilon M}{\log(Mk)} \leq \frac{k}{8}.$$ 

Let $S \subset \{0, \ldots, k - 1\}$ be a subset of cardinality at most $k/8$, and let $T = \{0, \ldots, k - 1\}/S$. A counting argument shows that

$$\sum_{i \in T} \log(q_{i+1}/q_i) \geq \sum_{i \text{ even}} \log(q_{i+2}/q_i) \geq (k/4) \log(2) \geq \frac{\log(2)}{4M} \log(q_k).$$

It follows that

$$\frac{1}{\log(q_k)} \sum_{i=0}^{k-1} \log \left( \left( \frac{q_{i+1}}{q_i} \right)^{k-1} : \epsilon \log(q_k) \right) \leq 1 - \frac{\log(2)}{4M}.$$ 

To complete the proof, we take the limsup as $k \to \infty$ and then apply theorem 1.2.

**Proof of (ii).** Fix $\epsilon > 0$. By assumption, there exist infinitely many $k$ satisfying

$$\log(q_k) \geq \frac{2}{\epsilon} k \log(k).$$

For such $k$,

$$\frac{\epsilon \log(q_k)}{\log(q_k)} \geq \frac{\epsilon (2/\epsilon) k \log(k)}{\log((2/\epsilon) k \log(k))} \geq \frac{2k \log(k)}{\log(k^2)} = k,$$

where the middle inequality holds for all $k$ sufficiently large. But then

$$\sum_{i=0}^{k-1} \log \left( \left( \frac{q_{i+1}}{q_i} \right)^{k-1} : \epsilon \log(q_k) \right) = \sum_{i=0}^{k-1} \log \left( \frac{q_{i+1}}{q_i} \right) = \log(q_k).$$

To complete the proof, we divide by $\log(q_k)$, take the limsup as $k \to \infty$, and apply theorem 1.2.

**Proof of (iv).** Choose $M < \infty$ such that for all $k$, $q_{k+1}/q_k \leq M \log(q_k)$. Then for all $\epsilon > 0$ and $k \in \mathbb{N}$,

$$\sum_{i=0}^{k-1} \log \left( \left( \frac{q_{i+1}}{q_i} \right)^{k-1} : \epsilon \log(q_k) \right) \leq \frac{\epsilon \log(q_k)}{\log(q_k)} \max \left\{ \log \left( \frac{q_{i+1}}{q_i} \right) : i = 0, \ldots, k - 1 \right\} \leq \frac{\epsilon \log(q_k)}{\log(q_k)} \log(M \log(q_k)) \leq 2\epsilon \log(q_k).$$

Since (3.3) implies (3.2), (iii) does not require a separate proof.
where the last inequality holds for all $k$ large enough such that $q_k \geq e^M$. To complete the proof, we let $\epsilon = 1/4$, divide by $\log(q_k)$, take the limsup as $k \to \infty$, and apply theorem 1.2. □

**Proof of (v).** The assumption (3.5) implies that

$$\limsup_{k \to \infty} \frac{\log(q_k/q_{k-1})}{\log(q_k)} = 1.$$ 

Fix $\epsilon > 0$. By assumption, there exist infinitely many $k$ such that

$$\log(q_k/q_{k-1}) \geq (1 - \epsilon) \log(q_k).$$

For such $k$, if we assume that $k$ is chosen large enough so that $\frac{\epsilon \log(q_k)}{\log(q_k)} \geq 1$, then

$$\sum \left( \log \left( \frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k-1} : \frac{\epsilon \log(q_k)}{\log(q_k)} \geq \log \left( \frac{q_k}{q_{k-1}} \right) \geq (1 - \epsilon) \log(q_k).$$

To complete the proof, we divide by $\log(q_k)$, take the limsup as $k \to \infty$, use the fact that $\epsilon$ was arbitrary, and apply theorem 1.2. □

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