How do people navigate the exploration-exploitation (EE) trade-off when making repeated choices with unknown rewards? We study this question through the lens of multi-armed bandit problems and introduce a novel behavioral model, Quantal Choice with Adaptive Reduction of Exploration (QCARE). It generalizes Thompson Sampling, allowing for a principled way to quantify the EE trade-off and reflect human decision-making patterns. The model adaptively reduces exploration as information accumulates, with the reduction rate serving as a parameter to quantify the EE trade-off dynamics. We theoretically analyze how varying reduction rates influence decision quality, shedding light on the effects of “over-exploration” and “under-exploration.” Empirically, we validate QCARE through experiments collecting behavioral data from human participants. QCARE not only captures critical behavioral patterns in the EE trade-off but also outperforms alternative models in predictive power. Our analysis reveals a behavioral tendency toward over-exploration.

Key words: multi-armed bandit, Thompson sampling, quantal choice models, experiments, exploration-exploitation trade-off

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1. Introduction

In many business contexts, decision-makers often face the challenge of making repeated choice decisions, where the rewards of actions are not known beforehand and need to be learned from past experiences. This pervasive scenario ranges from consumers selecting among unfamiliar brands of a certain product to managers selecting suppliers without precise knowledge of reliability. The decision-making process can be very intricate. In particular, it necessitates a delicate balance between exploration – trying different options to learn about potential rewards – and exploitation
expanding the best-known options to collect rewards. The exploration-exploitation (EE) trade-off is usually modeled by the multi-armed bandit problem (MAB). In this problem, the decision maker faces several slot machines (or “arms”), each with an unknown reward probability distribution. The decision maker’s goal is to maximize their reward over time by choosing which arms to pull. This problem has drawn attention from many communities such as economics, computer science, and management science. Significant advancements in algorithmic solutions have been made, such as the Gittens Index policy, Thompson Sampling, and Upper Confidence Bound methods. (See more details in the literature review.)

However, the algorithms designed to maximize rewards for MAB problems may behave very differently from human behavior when they face similar problems. For example, Gans et al. (2007) demonstrated that Gittins index-based policies may not describe human behavioral data as well as seemingly naive models, such as hot-hand and exponential smoothing policies. Given that many repeated decisions in business practices are not dictated by algorithms but rather managed by human beings, our paper is driven by the following questions:

• How do human decision-makers balance the trade-off between exploration and exploitation?
• How does varying the balance between exploration and exploitation impact decision quality?

1.1. Summary of Results and Contributions

Modeling contributions. In this paper, we introduce a novel model that bridges algorithmic and behavioral modeling to shed light on the dynamic interplay between exploration and exploitation. Our model leverages concepts from the quantal choice framework, where the decision-maker relies on a (randomized) score system and chooses the arm with the highest attraction score. On a high level, the score is arm- and history-dependent and takes the following form:

\[ \text{Score} = \text{Historical Performance} + \text{Weight} \times \text{Random Shock}. \]

We relay to (1) for a precise formula. This score system explicitly encodes the trade-off between exploration and exploitation:

• The first is purely determined by the historical performance of the arm. It signifies exploitation as it leads the decision maker to choose the arm with the best historical performance.
• The second component consists of random shocks. It enables exploration by trying arms that have not performed well historically. The weight term thus controls the balance between exploration and exploitation.

An important feature of our model is that the weight will shrink with the number of times the arm has been pulled. In other words, exploitation will be emphasized over exploration over time as the decision maker’s experience accumulates. This feature is inspired by qualitative evidence
from experiments (see Section 3.2). It is also why we term of our model as Quantal Choice with Adaptive Reduction of Exploration (QCARE).

We parameterize the decision maker’s EE trade-off dynamics into a structural parameter. We refer to it as the reduction rate of exploration and denote it as $\alpha$. The larger the value of $\alpha$, the faster the reduction rate of exploration, and therefore exploitation dominates exploration more quickly. The specific form of the weight draws inspiration from the online learning literature: QCARE can be viewed as a generalization of the well-celebrated Thompson Sampling (TS) method. When $\alpha = 0.5$, our model reduces to Gaussian TS; see Proposition 1. When $\alpha < 0.5$ (resp. $\alpha > 0.5$), our model captures a policy that explores more (resp. less) aggressively than TS. Empirically, the value of $\alpha$ can then be estimated from behavioral data.

The marriage between online learning and behavioral modeling offers our model a few advantages, which we summarize below.

- First, it captures the learning effect for dynamic choices in a parsimonious way. When it comes to dynamic choice models, the traditional approach determines choice probabilities in a “subgame-perfect” manner (e.g., Gittins 1979, Rust 1987, Erdem and Keane 1996). However, the computation of value functions suffers from the curse of dimensionality. In comparison, the scores in our model admit myopic forms and thus allow simplicity for both theoretical analysis and empirical estimation.

- Second, it provides an interpretable yet principled quantification of the EE trade-off. We theoretically characterize how EE trade-off affects decision quality, thus justifying key concepts such as over- and under-exploration. (More details are provided in our summary of technical contributions below.)

- Third, the QCARE policy family includes not only (asymptotically) optimal policies such as Thompson Sampling, but also suboptimal ones due to over- and under-exploration. This allows QCARE to capture choice behavior with potentially bounded rationality. The benefit of this flexibility is reflected by how it displays better empirical performance on behavioral data, as well as unlocking novel behavioral patterns. (More details are provided in our summary of experimental and empirical contributions below.)

Technical contributions. We study both non-asymptotic and asymptotic properties of QCARE. In Proposition 2 we show that QCARE displays comparative statistics properties that are qualitatively consistent with our lab evidence. In Theorem 1 we show that all $\alpha > 0$ enables the decision maker to converge to the optimal arm in the long run. That makes QCARE distinctively different from many common behavioral policies, such as $\varepsilon$-greedy. Furthermore, it suggests that every $\alpha > 0$ is “plausible” – at least when $T$ is large – since they all correspond to long-run-average optimal policies.
Besides intuitive qualitative properties, QCARE offers a principled way to quantify the EE trade-off using $\alpha$. We develop an asymptotic theory regarding how different alpha values lead to different decision qualities, measured by regret.

- When $\alpha = 0.5$, QCARE achieves the optimal regret order of $O(\sqrt{T})$. In other words, in the asymptotic regime where $T = \infty$ represents the “optimal” balance of exploration vs. exploitation.
- When $\alpha < 0.5$, the regret order gradually deteriorates to $\Omega(T^{1-\alpha})$; see Theorem 2.
- When $\alpha > 0.5$, the regret order worsens drastically to $\Omega(T^{1-\varepsilon})$ for every $\varepsilon > 0$; see Theorem 3.

The analysis above characterizes the effects of “over” and “under” exploration asymptotically. We extend our analysis in two dimensions. First, to deepen the theoretical understanding of the asymptotic results above, we study a general family of Markovian MAB policies and identify conditions under which the same asymptotic pattern of over- and under- exploration emerge; see Section 5.3 (Theorems 4 through 6 and Proposition 3). Second, we conduct extensive numerical studies and find that the insight from asymptotic analysis generalizes to the non-asymptotic setting where $T$ is fixed to be a moderate value; see Section 5.4.

**Experimental and Empirical contributions.** We conduct experiments to collect behavioral data regarding how real human beings make decisions in the MAB problem. We use the data to empirically validate our model. Particularly, we compare QCARE with other models drawn from different streams of literature. QCARE displays strong capabilities of capturing human behavior, especially in terms of out-of-sample prediction power. We also illustrate how the QCARE-generated reward distributions align well with the real data after controlling for $\alpha$.

Through our analysis, we discover an interesting behavioral pattern: people tend to “over-explore” in the sense that they settle at leading arm more slowly than the expected-reward-maximizing rate. While such bias could be partly rationalized by the risk aversion of the participants, their behavior of not choosing the leading arm appears to reflect a mix of random behavioral errors (in the same spirit of Su 2008) layered on top of the “intrinsic” exploration required for dynamic learning.

2. Literature Review

Despite the salient role of the EE trade-off in decision quality, investigation into decision makers’ behavior under the MAB framework is limited. The underlying difficulty is the dual challenge of (i) the intricate EE trade-off in the MAB problem itself (for the decision maker) plus (ii) the need for machinery to “learn the learning” (for the observer). With this in mind, our work lies at the intersection between MAB policy design and analysis, as well as behavioral science and operations management.
MAB policy design and analysis. The MAB problem is a classical model that inherently captures the exploration-exploitation trade-off. Various algorithms for solving stochastic MAB problems have been proposed, including the Gittins Index (Gittins 1979, Gittins et al. 2011), Upper Confidence Bound (UCB) (Lai and Robbins 1985, Auer et al. 2002), Thompson Sampling (Thompson 1933), among others. The main goal of those algorithms is to maximize the expected reward (or minimize the regret) in different senses. A particularly relevant algorithm to our paper is Thompson Sampling. It is asymptotically order optimal in terms of the instance-independent regret (Agrawal and Goyal 2013, 2017) as $T$ grows, as well as a few other regret metrics such as Bayesian regret (Russo and Van Roy 2014) and instance-specific regret (Kaufmann et al. 2012). There is also an emerging literature on a deeper understanding of the system dynamics and reward distribution of those policies, such as diffusion-scaling behavior (Kalvit and Zeevi 2021, Fan and Glynn 2024a, Xu and Wager 2024) and tail risk (Simchi-Levi et al. 2023, Fan and Glynn 2024b), to name a few.

However, (asymptotically) optimal policies may not represent people’s behavior well. Motivated by the need to capture people’s highly heterogeneous behavior, we develop a class of MAB policies (which may or may not be asymptotically optimal) that span the wide EE trade-off spectrum, and analyze their system dynamics and decision qualities. As such, we believe our analysis makes a theoretical contribution to the analysis of MAB policies in its own right.

(Dynamic) quantal choice modeling and behavioral science. As mentioned previously, this paper involves developing MAB policy families to understand human behavior in the decision-making process. That essentially boils down to developing a dynamic discrete choice model with the learning effect. A natural choice model framework is the quantal choice theory. While the literature on static quantal choice model is very rich (e.g., McFadden et al. 1976 for models with full rationality, and McKelvey and Palfrey 1995, Su 2008, Chen et al. 2012, Li et al. 2020 for models with bounded rationality), there is much less literature for dynamic models (e.g., Rust 1987 for a represented rational model and McKelvey and Palfrey 1998 for a behavioral one).

Dynamic choice models that capture learning effects are even more limited. Among those that do, one approach is to incorporate learning into Rust-type models, i.e., the agent is modeled as a forward-looking rational decision maker who solves a Bayesian dynamic program (Erdem and Keane 1996, Ching et al. 2013). This approach has good theoretical foundations, but it has a few major limitations that motivate us to consider other directions. First, it suffers from the curse of dimensionality and can be computationally prohibitive.\(^1\) Hence, not surprisingly, the actual implementation needs various numerical heuristics such as simulation and value function interpolation.

\(^1\) To put things into perspective, solving the Bayes optimal policy for the MAB problem is challenging itself (see Min et al. 2019). Empirical estimation from the data would be even harder. (Here an analogy is the structural estimation problem in Rust 1987 is much more complicated than the bus engine replacement problem itself.)
(Erdem and Keane 1996). Plus, those models assume that the observer/researcher only observes the decision maker’s actions. In our experimental environment, the realized arm rewards are also observed, which is valuable information that should be taken advantage of. Perhaps a more fundamental limitation of this approach is that the decision maker is assumed to be rational (utility maximizing). It is well documented in the literature that incorporating broader behavioral patterns can better capture real data (e.g., Banks et al. 1997, Gans et al. 2007). Many researchers have also used experiments to test the qualitative implications of Bayes optimal policies (e.g., Gittens index policy) and found various ways of deviations (Horowitz 1975, Meyer and Shi 1995, Anderson 2001).

That naturally leads us to consider the behavioral approach. There are well-celebrated behavioral learning models in the literature, such as Experience-Weighted Attraction (EWA) (Camerer and Ho 1999). Unfortunately, EWA is not informationally compatible with MAB: using online-learning terminologies, EWA is based on full-information feedback (i.e., the decision maker observes the realized rewards of all arms) while MAB requires bandit feedback (i.e., the decision maker only observes the realized rewards of the pulled arm). Therefore, behavioral models that build on EWA do not fully satisfy MAB’s information structure for the decision maker (Ferecatu and De Bruyn 2022). Stylized behavioral policies have been considered in various streams of literature, such as behavioral operations management (Gans et al. 2007), cognitive psychology (Bouneffouf et al. 2017), and computer science (Zhang and Yu 2013). Usually those policies are heuristic and lack formal theoretical understandings. Within this stream, our main innovation regards applying the toolbox and concepts in online learning to behavioral modeling. To our knowledge, this direction is quite new. The only exception is the work of Mauersberger (2022), who use Thompson Sampling as a behavioral model and observe promising empirical evidence.

3. The Multi-Armed Bandit Problem and Lab Experiments

Formulation. We first describe the standard definition of the multi-armed bandit problem. In this problem, the decision-maker (DM) faces \( N \) choices, which are referred to as arms. Each arm \( i \) corresponds to an i.i.d. sequence of stochastic rewards, denoted by \( \{r_i(t)\} \), which is generated from a fixed reward distribution. We refer to the ground truth reward distributions of the arms as an instance \( \Theta \) of the MAB problem, which is unknown to the DM. We assume that the rewards are supported on \([0,1]\). We use \( \mu_i \) to denote the expected reward of arm \( i \), and let \( i^* := \arg \max_i \mu_i \) be the optimal arm with the highest mean. For technical simplicity, we focus on cases where the optimal arm is unique. We use \( \mu^* := \max_i \mu_i \) to denote the expected reward of the optimal arm, and let \( \Delta_i := \mu^* - \mu_i \) represent the reward gap between arm \( i \) and the optimal arm.

At each time \( t \in [T] = \{1, \ldots, T\} \), the DM chooses one arm to pull, which we denote by his/her action \( a(t) \). After that, the DM receives the reward of the pulled arm, \( r_{a(t)}(t) \). As such, an admissible
policy of the DM is governed by a mapping from a historical path \( h_{t-1} := (a(1), r_{a(1)}(1), \ldots, a(t-1), r_{a(t-1)}(t-1)) \) to a (possibly randomized) action \( a(t) \). The DM’s goal is to select the arms to maximize the total rewards over the time horizon \( T \). We follow the standard convention in the online learning literature and adopt regret to measure decision qualities in terms of collecting rewards. One could understand this concept as a way to properly normalize the expected rewards of a policy so that it is easier to measure its performance across instances. The regret of a given policy is defined as the difference in the total expected rewards between always pulling the optimal arm \( i^* \) and the policy. Specifically, for every MAB instance \( \Theta \), the (instance-specific) regret \( R^\Theta(T) \) is defined by

\[
R^\Theta(T) = \mu^* T - \mathbb{E} \left[ \sum_{t=1}^{T} r_{a(t)} \right].
\]

We also define the problem-independent (worst-case) regret as \( R(T) := \max_\Theta R^\Theta(T) \). An effective policy that achieves higher expected reward corresponds to a smaller regret.

3.1. The MAB Experiments

To capture how people actually balance the EE trade-off in the MAB problem, we collected real human behavioral data based on a collection of experiments. Let us start by explaining the details of our experimental design. Later, we will present descriptive findings from the experimental data that will further motivate our quantitative behavioral model.

**Setup.** The baseline experiments involve physical experiments conducted within a controlled lab environment. Following common practice in both behavioral economics and behavioral operations literature \cite{Bolton2012}, we recruit students from a behavioral economics lab of a university as proxies for human behavior. We investigate two-armed MAB problems with Bernoulli rewards and explored two MAB instances, namely \((\mu_1, \mu_2) = (0.4, 0.5)\) and \((\mu_1, \mu_2) = (0.6, 0.4)\), respectively. We also consider two different time horizons, i.e., \( T = 100 \) and \( T = 200 \). As such, our baseline experiments include a total of four distinct configurations.

We also conduct multiple rounds of auxiliary experiments to test the robustness of the baseline experiments.

1. First, the baseline experiment is based on just two arms. Therefore, we also conducted experiments on three arms.
2. Second, the baseline experiment is just based on university students in one university. To account for potential limitations in the subject sampling, we conducted an online experiment with a more diverse participant population, including nationality, gender, and occupation, among others. The other configurations were largely the same except for a slightly different
point-to-cash conversion rate. The online experiment was facilitated by the Prolific platform, which is recognized for acquiring high-quality data in online human-subject research (Douglas et al. 2023, Albert and Smilek 2023).

3. Since the previous results were surprisingly consistent, we moved on to an extra round of replication, where we replicated the online experiments at a larger scale. In this round, we preregistered the experiments so as to credibly confirm the robustness of our findings.

To summarize, we implement 14 experiment configurations on dimensions such as reward distribution, time horizon, number of arms, subject base, and stake for incentives. We recruit a total of 610 subjects, adding up to roughly 90,000 observations of actions. The sample size aligns with (if not exceeds) the tradition; see the representative examples in the Management Science replication project (Davis et al. 2023). Since our findings for all those experiments are nearly identical, we will focus on reporting the results of our baseline experiments in the main paper. The data analysis procedure (i.e., the summary statistics, empirical validation, and behavioral pattern verification) will be repeated verbatim for all auxiliary experiments, and we relay the details in Appendix C.

Procedure, information, and incentives. We conduct our experiments using the o-Tree system (Chen et al. 2016). Once recruited, every subject is first presented with an overview of the game. In this overview, the subject will learn basic information about the game, such as the rule, the number of arms, the fact that the reward follows a fixed binary distribution, etc. However, no other information about the reward (e.g., mean reward for either arm) is provided. In this way, we respect the information structure required by a standard MAB problem. The subject is also informed of the goal of the MAB game (i.e. to accumulate as many points as possible), as well as how much money they can make for every additional point. Every subject is only routed to one of the four configurations and plays the MAB at most once to avoid spillover/cross learning issues. We also employ attention tests to ensure the quality of the responses.

Once the MAB game starts, the participant is presented with a complete history of actions and realized rewards of each period, so that the participant could – at least in principle – implement any admissible MAB policy. Summary statistics information of the history is also provided separately, including (i) the action and realized reward of the previous period, (ii) the cumulative reward, and (iii) the number of plays and average points of each arm. Finally, the subject is reminded of the period he/she is in and how many periods in total. Based on that information, the participant is asked to choose an arm, and then the MAB game moves on to the next period. Once the MAB game finishes, the subjects’ payments are settled. They earn one point each time the selected arm

Pre-registration link: [https://osf.io/ukew8/?view_only=37cca7563fa44da7bbaac7124459684](https://osf.io/ukew8/?view_only=37cca7563fa44da7bbaac7124459684) The pre-registration platform was based on Open Science Framework (OSF). We chose it because it is reportedly more robust to activities such as “preregistration hacking” compared to other platforms (Haroz 2022).
results in a realized reward of one. The total earnings consisted of two components: a show-up fee and a bonus proportional to the points earned. We refer the reader to Appendix C.1 for more details about the procedure, such as the interface, and payment calculations, among others.

3.2. Lab Evidence of the EE Trade-off

To motivate our model, we start with some high-level evidence regarding whether and how real people navigate the EE trade-off. We illustrate that by subjects’ tendency to choose the leading arm, i.e., the arm with the higher empirical mean. (Note that the leading arm does not have to be the optimal arm, but just the one with better performance at the time of choosing.) Naturally, one would expect that if decisions are purely driven by exploitation, the actions should be purely “greedy,” i.e., the leading arm is always chosen. Conversely, if subjects are solely driven by exploration, the leading arm will be as likely as, if not less likely than, the other arms. Next, we describe what we observe from the experiments.

Overall frequency of choosing the leading arm. Our experiment data reveals that only 2 of 123 subjects chose the leading arm throughout the entire time horizon and 100 out of 123 subjects chose the leading arm with at least 50% of the rounds. On average, subjects chose the leading arm 64.70% of the time.\(^3\) That ratio is strictly less than one, yet significantly larger than 0.5 (Wilcoxon signed-rank test right tail, \(p\)-value=8.6836e-15). These findings indicate that, on the aggregate level, the participants are choosing the arms in a way that is neither purely exploration nor exploitation, as expected.

What influences choosing the leading arm? The choice probability of the leading arm may not be a constant. Intuitively, the EE trade-off would move towards exploitation (i.e., exploration is reduced) if the subject is faced with stronger evidence that one arm is the superior one, and vice versa. To this end, we also provide some qualitative evidence regarding factors that impact the probability of choosing the leading arm.

We summarize our findings in Figure 1. For every fixed subject, we partition the time horizon into two groups: the first group contains periods corresponding to the highest 50% of empirical mean gaps at that time, and the second group corresponds to the lowest 50%. We then calculate the fractions of choosing the leading arm for both groups; see Appendix B for a concrete example of how we calculate the fraction. The distributions of both fractions across subjects are reported in Panel (a). We found a higher average fraction in the first group (\(p\)-value being 3.514e-07 using a two-sided Wilcoxon signed-rank test). In Panel (b), we go through a similar process but divide the

\(^3\) This number is calculated by first recording each individual’s fraction of rounds of choosing the leading arm and then taking an average across individuals. One would get a similar figure if taking the average at the individual-period level, which is 64.17%.
time horizon into the first 50% and last 50% of periods. We observed a higher average fraction in the second group (p-value being 3.618e-05 using a two-sided Wilcoxon signed-rank test). Finally, in Panel (c), we partition the time horizon by both empirical mean gap and time horizon and report the average fraction in each quadrant. Quite intuitively, we found that the leading arm is more likely to be chosen amid a wider gap in the empirical mean reward, or more observations that the empirical mean gap is based on. That is consistent with our expectation that the subjects are balancing the EE trade-off throughout the process.

(a) By empirical mean difference  (b) By period  (c) By both

Figure 1  Fraction of periods choosing the leading arm

Remark 1. Although the data reflect a fairly intuitive trade-off between exploration and exploitation, we observed an interesting behavioral anomaly. In the last period, only 69.92% of subjects chose the leading arm. While this is higher than the overall average, it clearly deviates from any Bayesian optimal policy, which would imply that 100% of subjects should choose the leading arm in the final period, as it is dominantly optimal to do so regardless of the prior beliefs or historical observations. This suggests that, at least in the last period, subjects exhibit a tendency to “over-explore” compared to the optimal policy. We will revisit and further elaborate on this point later in the paper.

4. QCARE: A Quantitative Behavioral Model
We now proceed to explain our quantitative behavioral model to help provide deeper knowledge about people’s EE trade-offs beyond descriptive statistics.

4.1. Model Definition
When faced with the task of choosing among multiple options, most popular policies (regardless of algorithmic or behavioral) are based on scoring systems. That is, a policy assigns scores to each choice and selects the one with the highest score. In the context of the MAB problem, there are many ways such scores can be generated. For instance, the Gittins index is a scoring system that corresponds to a Bayes optimal policy in the infinite-horizon discounted setting (given that it can be computed). The Thompson Sampling policy assigns a score to each arm by randomly drawing
from the posterior distribution of its reward parameters. Similarly, as suggested by Gans et al. (2007), different mechanisms assign the score for each arm by humans.

In this paper, we follow the convention of basing the choices on scoring rules. At any period $t$ and given any arm $i$, we consider two quantities below.

1. We use $k_i(t) := \sum_{\ell=1}^{t-1} I\{a(\ell) = i\}$ to denote the number of times this arm has been pulled. It quantifies the length of experiences the learner has with the arm.
2. We represent this arm’s historical reward by $\hat{\mu}_i(t) := \frac{\sum_{\ell=1}^{t-1} r_i(\ell)}{k_i(t)+1}$, where $r_i(\ell)$ is this arm’s (observed) reward at period $\ell$. This term quantifies arm $i$’s historical performance.

The notation for both terms is consistent with the earlier convention (e.g., Agrawal and Goyal 2017). The Quantal Choice with Adaptive Reduction of Exploration (QCARE) model we propose depicts a scoring system defined below:

$$\theta_i(t) = \hat{\mu}_i(t) + \frac{1}{(k_i(t)+1)^\alpha} \varepsilon_{it},$$  

(1)

where $\varepsilon_{it}$ represents independent random shocks following the standard normal distribution. The subject thus chooses the arm with the highest score value at every period, i.e., $\arg\max_i \{\theta_i(t)\}$.

Let us walk through every component of the score $\theta_i(t)$. The first term, $\hat{\mu}_i(t)$, gives higher scores to the arms with better historical performances. In this way, it signifies the exploitation aspect of behavior. The second term corresponds to a random shock that enables exploration behavior. The random shock shrinks with $k_i(t)$, i.e., the decision maker’s number of interactions in this particular arm. When the experience is limited, pulling this arm can still be encouraged – even if the historical performance is suboptimal – as reflected in the large random shock term. When there has been sufficient interactions, i.e., when $k_i(t)$ is large, the attractiveness of this arm is almost purely determined by its historical performance.

4.2. Model Interpretation and Discussion

Under our model, the balance between exploration and exploitation is captured through the relative magnitude between the exploitation and exploration terms, or more specifically, how fast the random shock shrinks. If the random shock shrinks too fast, the decision maker might be misled by the historical performances of an arm based on too few interactions. On the contrary, if the random shock shrinks too slowly (or even does not shrink at all), the decision maker may end up spending too much time on suboptimal arms despite overwhelming evidence that a certain arm is the best.

The reduction rate of exploration. We use a parametric form to present how the random shock shrinks over time, namely, $(k_i(t) + 1)^{-\alpha}$. This form is parameterized by a single parameter, $\alpha$, which we call the reduction rate of exploration. As its name suggests, a larger $\alpha$ corresponds to
a faster shrinkage of the random shock term, and hence a higher weight on exploitation given any system state. In the extreme case, if $\alpha$ approaches infinity, the exploration term becomes negligible and hence QCARE becomes purely *greedy*. That is, one simply chooses the leading arm exclusively. On the contrary, if $\alpha \leq 0$, the random shocks remain significant regardless of the learner’s historical experience. In this case, QCARE never settles on any arm, thus representing “endless” exploration.

The introduction of $\alpha$ allows us to quantify the EE trade-off using a single easy-to-interpret parameter. In the later sections, we will formally discuss how different values of $\alpha$ create different system dynamics and decision qualities. In addition, we will calculate the implied-$\alpha$ values from the behavioral data collected from the experiments. That helps shed light on the behavioral patterns when *human beings* manage the balance between exploration and exploitation.

**Relating to Thompson Sampling.** QCARE has an intimate connection to Thompson Sampling, a well-celebrated policy from the online learning literature. The basic idea of TS is to maintain the posterior distributions of the arm reward parameters and randomly pull an arm according to its posterior probability of being optimal; Appendix A for a more detailed algorithmic description. TS is asymptotically optimal in terms of the instance-independent regret \cite{Agrawal2013, Agrawal2017} as $T$ grows, as well as a few other regret metrics such as Bayesian regret \cite{Russo2014} and instance-specific regret \cite{Kaufmann2012}. QCARE generalizes TS; see the following result for a formal statement.

**Proposition 1.** When $\alpha = 0.5$, QCARE is equivalent to Gaussian Thompson Sampling.

This observation has two implications. First, it enables us to view QCARE as a parameterized generalization of (Gaussian) Thompson Sampling: when $\alpha < 0.5$ (resp. $\alpha > 0.5$), our model captures a policy that explores more (resp. less) aggressively than TS. In this way, we establish a novel connection between Thompson Sampling and quantal response models, thus bridging the seemingly distinct online learning and behavioral literature. Second, Gaussian Thompson Sampling is known to achieve the optimal regret order of $O(\sqrt{T})$, i.e., it is order optimal when $T$ diverges to infinity. As a result, we may interpret $\alpha = 0.5$ as the optimal benchmark for the EE trade-off when $T$ is infinite. We will show later how other values of $\alpha$ lead to distinctively different system dynamics and reward performances.

In passing, it is worth mentioning that a few other generalizations of TS have been considered in the online learning/psychology literature. For example, \cite{Chapelle2011} reshaped the posterior of TS and showed that shrinking the posterior to reduce exploration can benefit the non-asymptotic performance of TS. \cite{Bouneffouf2017} extended TS to a parametric version that allows to incorporation of biases associated with mental disorders. Although in principle, they could also be used to parameterize the EE trade-off and fit into behavioral data, it is important
to note that those methods are numerical and heuristic in nature. In particular, none of them lack the rigorous treatment of quantifying over/under exploration and how that affects the decision qualities.

**Relating to quantal choice models.** Quantal choice theories are driven by the query to describe a human’s quantal response facing discrete choice sets. A representative example is the classic attraction model (Luce 1959). Many variations and extensions have been proposed afterward, and a popular framework to unify them is called the random utility model (RUM); see also McFadden et al. (1976). Under this model, an individual assigns a random score $\theta_i$ to every alternative $i$ in the form of $\theta_i = u_i + \epsilon_i$, where $u_i$ and $\epsilon_i$ represent the expected utility and random utility shocks for alternative $i$, respectively. The individual then chooses the alternative with the highest realized score. Different distributions of the random shock term correspond to different models. For example, the multinomial logit (MNL), probit, and exponential choice models (Alptekinouglu and Semple 2016) correspond to independent Gumbel, Gaussian, and exponentially-distributed noises, respectively.

Quantal choice models often display favorable explanation and prediction powers on real behavioral data, partly because they allow structural errors brought by random shocks. There are usually two channels for the random shocks. One possibility, represented by RUMs, is to assume that the agents are fully rational (utility maximizing), and random shocks are merely a component of the realized utility unobserved by the researcher. This is further developed in the dynamic setting. For example, under Rust (1987)’s dynamic choice model framework, the individual solves a dynamic program while optimally accounting for future utility realizations. Rust (1987) also discussed how to structurally estimate the individual’s preference parameters from observed data.

Another possibility emphasizes that individuals may not always select the optimal choice due to bounded rationality. Therefore, the random shocks are attributed to behavioral errors. For example, Su (2008) studied decision makers’ actions for a newsvendor problem. They employed a scoring system in the form of $\theta_i = u_i + \epsilon_i/\beta$, where the noise term $\epsilon_i$ is scaled by the parameter $\beta$ that represents the extent of cognitive limitation of the decision-maker. This model successfully explains the systematic deviation from the optimal newsvendor order quantity, a pattern robustly identified in many laboratory experiments (Schweitzer and Cachon 2000, thereafter). The behavioral quantal model can also be extended to multi-agent settings. For example, McKelvey and Palfrey (1995) observed frequent errors in subjects’ choices of Nash equilibrium actions during experiments and proposed the quantal response equilibrium (QRE) framework. Under QRE, each player randomizes according to a quantal choice model rather than choosing the utility-maximizing action. QRE can be further extended to dynamic games under the framework of agent quantal response equilibrium (AQRE); see McKelvey and Palfrey (1998).
Unfortunately, both channels miss an important feature when learning comes into play: the decision makers cannot know the utilities of their choices in priori. Therefore, it is not appropriate to think about the random shocks as unobserved utilities as in Rust (1987). Similarly, one cannot purely think about the random shocks as behavioral errors as in Su (2008) since when the rewards for each arm are unknown. (To put things into perspective, the counterpart of Su (2008)’s rational benchmark in the dynamic learning setting would be always choosing the best arm according to the DM’s best estimate. But that becomes a greedy policy and highly suboptimal in the MAB problem.) Instead, an appropriate amount of exploration is intrinsically needed for dynamic learning. Therefore, the random shocks inevitably need to include this new channel, which is the “intrins-ic” exploration. The existence of such a trade-off between exploration and exploitation makes the MAB setup distinctively different from traditional quantal choice model settings and is precisely what we wish to capture in this paper.

5. Theoretical Characterizations of QCARE
5.1. Qualitative Properties
In this subsection, we derive some basic properties of QCARE to better understand it intuitively. Recall from (1) that at every time period $t$, the distribution of the arm-pulling scores $\{\theta_i(t)\}$ are determined by their empirical means $\{\hat{\mu}_i(t)\}$ and the number of pulls $\{k_i(t)\}$. In other words, which arm is pulled only depends on the state $S(t) = (k_1(t), ..., k_N(t), \hat{\mu}_1(t), ..., \hat{\mu}_N(t))$ as a sufficient statistic. To this end, for every given arm $i \in [N]$ and scalar vectors $\kappa := (\kappa_1, ..., \kappa_N) \in \mathbb{Z}_+^N$ and $u := (u_1, ..., u_N) \in [0, 1]^N$, let us introduce

$$Q_i(\kappa, u) = \Pr(\theta_i(t) \geq \theta_j(t), \forall j | S(t) = (\kappa, u))$$

to be the probability of pulling arm $i$ as a function of the state $S = (\kappa, u)$. In particular, let $\hat{i}^* := \arg \max_i \{u_i\}$ be the arm with the highest empirical mean, i.e., the leading arm, and $\hat{Q}(\kappa, u) = Q_{\hat{i}^*}(\kappa, u)$ be the probability of pulling the leading arm. The following result summarizes some comparative statistical properties of QCARE regarding $\hat{Q}$.

**Proposition 2.** (Comparative statistics) Pick any $\alpha > 0$, $N > 1$, and state $S = (\kappa_1, ..., \kappa_N, u_1, ..., u_N)$. The following facts about QCARE hold.

1. For every $\delta > 0$, we have $\hat{Q}(S) \leq \hat{Q}(\kappa_1, ..., \kappa_N, u_1, ..., u_{\hat{i}^*} + \delta, ..., u_N)$.
2. For every $\delta > 0$ and $j \neq \hat{i}^*$, we have $\hat{Q}(S) \leq \hat{Q}(\kappa_1, ..., \kappa_j + \delta, ..., \kappa_N, u_1, ..., u_N)$.

Proposition 2 reveals some structural properties regarding how the QCARE balances exploration with exploitation. The emphasis on exploitation (i.e., probability of choosing the leading arm) becomes more salient when the leading arm has established a larger empirical mean gap compared to the other arms, or when the gap is based on more observations of the non-leading arms. These
properties are qualitatively consistent with the descriptive findings from the experimental evidence summarized in Figure 1.

Next, we explore another important property of QCARE. To put it briefly, QCARE can “learn” the optimal arm and converge to it, at least in the long run. We formally state this property in the result below.

**Theorem 1.** (Complete learning) Pick any $\alpha > 0$, $N > 1$, and the bandit problem instance $\Theta$. The following facts about QCARE hold.

- The arm-pulling process converges to the optimal arm. That is, $\Pr(a(t) = i^*) \to 1$ as $t \to \infty$;
- The regret satisfies $R_\Theta(T) = o(T)$. As a result, $\alpha$ represents a long-run-average optimal policy, i.e., one that maximizes $\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} r_{a(t)} \right]$.

The result above suggests that all values of $\alpha > 0$ correspond to “plausible” behavioral MAB policies as they all achieve complete learning and are long-run-average optimal. This sense of optimality is also referred to as consistency by Lai and Robbins (1985). While not a too stringent property, one should not take it for granted. For example, the $\varepsilon$-greedy policy is a very popular benchmark behavioral policy and one very similar to QCARE at first sight. But it does not satisfy consistency because it never concentrates on any arm. In contrast, QCARE possesses a minimally reasonable level of exploration to gradually learn the optimal arm. The fact that people do tend to learn over time is not only reflected in the aforementioned descriptive evidence but also in the fact that $\varepsilon$-greedy underperforms QCARE in terms of explanation and prediction power of behavioral data, which we will explain more later.

While QCARE has demonstrated some intuitive qualities, we believe one of its distinctive advantages is that it offers a principled (and also parsimonious) way to quantify the EE trade-off by measuring how one keeps exploring in the face of accumulating evidence. We will elaborate on the theoretical side of this next while leaving the formal empirical analysis to Section 6.

### 5.2. Quantifying the EE Trade-off: An Asymptotic Theory

In this subsection, we provide a formal treatment of how $\alpha$ relates to the decision qualities, especially the effects of “over” and “under” exploration.

**Illustrative plots.** To build intuition, we present some numerical evidence regarding how $\alpha$ influences the reward distribution under QCARE. Figure 2 depicts the histograms of the average rewards per period in a two-armed bandit problem with different values of $\alpha$ and $T$ based on a simulation study of 10,000 sample paths. The arms’ reward distributions are fixed to be Bernoulli with mean $(0.6, 0.4)$. In the figure, Panels (a) and (c) correspond to $\alpha = 0.2$, representing high exploration. We can observe that the realized reward is distributed around a mean value less than 0.6, and the
distribution tightens as $T$ becomes large. That reflects how heavy exploration, although helping to learn the optimal arm eventually, may spend too much time on the inferior arm. Conversely, Panel (b) and (d) correspond to $\alpha = 2$, representing low exploration. Here we observe a different distinct binary-type pattern in the reward distribution: around 15% of the sample paths display an average reward of around 0.4, the expected reward of the inferior arm. This indicates that while most samples quickly concentrate around the optimal value of 0.6, a significant portion of the samples are stuck in the inferior arm due to insufficient exploration - an “incomplete learning” phenomenon that is related to earlier literature; see den Boer and Zwart (2014), Keskin and Zeevi (2018), Li et al. (2024) for a few examples in various contexts.

Formalizing the intuition. Now, we present our main theoretical results characterization of the QCARE’s performance in the asymptotic setting, i.e., when $T$ approaches infinity. To this end,
we find it helpful to formally define the standard Big O notation. For every pair of functions 
\( f(\cdot), g(\cdot) : \mathbb{Z}_+ \to \mathbb{R} \), we say that:

\[
f(T) = \begin{cases} 
O(g(T)), & \text{if } \limsup_{T \to \infty} f(T)/g(T) < \infty; \\
o(g(T)), & \text{if } \limsup_{T \to \infty} f(T)/g(T) = 0; \\
\Omega(g(T)), & \text{if } \liminf_{T \to \infty} f(T)/g(T) > 0; \\
\omega(g(T)), & \text{if } \liminf_{T \to \infty} f(T)/g(T) = \infty.
\end{cases}
\]

Recall that when \( \alpha = 0.5 \), QCARE reduces to Thompson Sampling and hence achieves the optimal
regret order of \( O(\sqrt{T}) \). When \( \alpha \) deviates from the optimal benchmark of 0.5, distinctively different
patterns emerge. We will start from the case where \( \alpha < 0.5 \). The proofs of all the following theorems
can be found in Appendix D.

**Theorem 2.** Suppose \( \alpha \in (0, 0.5) \). For every \( N > 1 \), the regret of QCARE satisfies
\[
\mathcal{R}(T) = \Omega(T^{1-\alpha}).
\]

The result above indicates that every \( \alpha < 0.5 \) leads to a suboptimal regret order, and it gradually
deteriorates further as \( \alpha \) becomes smaller. Intuitively, that is because the arm-pulling probabilities
concentrate on the leading arm too slowly. In fact, QCARE can refuse to settle on any arm even
when all arms have been pulled \( \Omega(T) \) times and their empirical reward differences on the order of
\( \Omega(T^{-\alpha}) \); see Section 5.3 for more technical details. That makes the policy to spend too much time
on suboptimal arms and make it suffer from over-exploration.

Next, we turn to the case where \( \alpha > 0.5 \).

**Theorem 3.** Suppose \( \alpha > 0.5 \). Even when \( N = 2 \), there exists an instance \( \Theta \) under which an
\( \Omega(T) \) regret is incurred with probability \( \Omega(T^{-\alpha(1)}) \). Therefore, the regret of QCARE satisfies
\[
\mathcal{R}(T) = \Omega(T^{1-\varepsilon}), \quad \text{for every } \varepsilon > 0.
\]

The result above indicates that every \( \alpha > 0.5 \) leads to a suboptimal regret order, too, albeit in
a different pattern from the \( \alpha < 0.5 \) case. Here, the regret suddenly deteriorates from \( O(\sqrt{T}) \) to
nearly \( \Omega(T) \). Intuitively, that is because the arm-pulling probabilities concentrate on the leading
arm too fast. It is possible that even both arms are only pulled \( (\log T)^{1/\alpha} \) times but the arm pulling probability of the non-leading arm already diminishes with an order of \( O\left(\frac{(\log T)^{1/\alpha}}{T}\right) \); see
Section 5.3 for more technical details. As a result, there is a non-negligible event – happening with
probability on the order of \( \Omega(T^{-\alpha(1)}) \) – that QCARE is stuck in the suboptimal arm, in which
case suffering from \( \Omega(T) \) regret. Combining Theorems 2 and 3, our asymptotic analysis covers the
whole spectrum of \( \alpha > 0 \), and we illustrate the relationship between \( \alpha \) and asymptotic regret in
Figure 3.
Insights from the asymptotic analysis. In summary, our analysis quantifies how decision quality is related to the EE-trade-off. An important insight is that there is a sweet spot of the EE trade-off, supporting the concepts of ‘over-’ and ‘under-’ exploration. It is also interesting to note that over- and under-exploration lead to different decision dynamics and harm the decision quality in different ways. The former results in insufficient focus on the leading arm, even with overwhelming supporting evidence, while the latter causes a nonnegligible probability that the decision maker is trapped in the inferior arm.

5.3. Key Steps and Generalizations
To analyze the decision dynamics and regret performances of QCARE, we take the following approach. First, we consider a much more general space of Markovian MAB policies. Then we derive general conditions for respective system dynamics patterns and regret performances. Finally, we verify the conditions QCARE satisfies with different \( \alpha \) values. In this way, we distill the theoretical insights regarding the EE trade-off. Note that we differ from the traditional approach of focusing on order-optimal policies, and cover a wide range of behavioral policies that are potentially not order-optimal. As such, we believe our analysis makes a theoretical contribution to the analysis of MAB policies in its own right.

Setup and basic properties. Let us first describe the family of MAB policies our analysis extends to. Recall that \( k_i(t) \) and \( \hat{\mu}_i(t) \) represent the pull count and empirical reward of arm \( i \) up to period \( t - 1 \). We consider MAB policies that only depend on \( S(t) := (k_1(t), \ldots, k_N(t), \hat{\mu}_1(t), \ldots, \hat{\mu}_N(t)) \) as a sufficient statistic, which takes values in \( S := \mathbb{Z}_+^N \times [0, 1]^N \). We also index a typical state as \( S = (\kappa, u) \), where \( \kappa := (\kappa_1, \ldots, \kappa_N) \in \mathbb{Z}_+^N \) and \( u := (u_1, \ldots, u_N) \in [0, 1]^N \) are scalar vectors. In this way, an admissible policy can be represented by a probability function \( Q = (Q_1, \ldots, Q_N) : S \to \Delta_N \), where

\[
Q_i(S) = \Pr(a(t) = i \mid S(t) = S)
\]
is the probability to pull arm $i$ given state $S$. With a slight abuse of notation, we also find it convenient to write $Q_i(S)$ as $Q_i(S) = Q_i(S^i; S^{-i})$, where $S^i = (\kappa_i, u_i)$ and $S^{-i} = (\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+1}, \kappa_N, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$ are the state values for arm $i$ and the rest of the arms, respectively. The family of MAB policies we consider is equivalent to the Sequentially Randomized Markov Experiments studied by Xu and Wager (2024). It is easy to verify that QCARE belongs to this family, and we refer the reader to Xu and Wager (2024) for many other members of this family.

We will now proceed to define the relevant properties of the probability function.

**Definition 1.** The function $Q$ is anonymous if it is invariant to relabeling. That is, for every permutation $\sigma : [N] \rightarrow [N]$, $i \in [N]$, and state $S = (\kappa_1, \ldots, \kappa_N, u_1, \ldots, u_N) \in S$, $Q_i(\kappa_{\sigma(1)}, \ldots, \kappa_{\sigma(N)}, u_{\sigma(1)}, \ldots, u_{\sigma(N)}) = Q_{\sigma(i)}(\kappa_1, \ldots, \kappa_N, u_1, \ldots, u_N)$.

This property means that the probability function needs to be symmetric to the relabeling of the arm. In other words, it cannot “cheat” by depending on the arm identity information. This is a minimal requirement for $Q$, and we will assume it throughout the section.

**Definition 2.** The function $Q$ is recurrent if for every arm 1's state $S^1 = (\kappa_1, u_1) \in \mathbb{Z}_+ \times [0, 1]$, there exists $\varepsilon > 0$ (possibly dependent on $S^1$) so that $Q_1(S^1; S^{-1}) \geq \varepsilon$ for all $S^{-1}$.

This property means that after an arm is pulled, there is always a non-diminishing probability that it is re-pulled (regardless of the history of other arms). The term “recurrent” is formally justified by the result below.

**Lemma 1.** If the probability function $Q$ is recurrent, then with probability one, every arm is pulled infinitely many times, i.e., $k_i(t) \rightarrow \infty$ as $t \rightarrow \infty$ for every $i \in [N]$.

**Remark 2.** Note that Definition 2 only nominally describes a condition for $Q_1$ (i.e., arm 1). Since the probability function $Q$ is anonymous, similar conditions apply to other arms, too. To be more precise, for every arm $i$'s state $S^i = (\kappa_i, u_i) \in \mathbb{Z}_+ \times [0, 1]$, there exists $\varepsilon > 0$ (possibly dependent on $S^i$) so that $Q_i(S^i; S^{-i}) \geq \varepsilon$ for all $S^{-i}$. We will keep the anonymity of $Q$ in mind when defining the conditions below.

**Definition 3.** The function $Q$ is leading-arm convergent if for every sequence $(\kappa, u)$ satisfying $\kappa_i = \omega(1)$ for every $i \in [N]$ and $u_1 - u_i = \Omega(1)$ for every arm $i \geq 2$, it holds that $1 - Q_1((\kappa, u)) = o(1)$. 
Roughly speaking, a policy with a leading-arm convergent probability function $Q$ will concentrate the pulling probability towards the leading arm in light of sufficiently strong evidence that it is the optimal arm, i.e., every arm has been pulled infinitely many times and the advantage of the leading arm is still non-diminishing. The two properties in Definitions 2 and 3 are fairly mild and intuitive. That said, they are sufficient to guarantee a reasonable performance of the policy, which is stated in the result below.

**Theorem 4.** If the probability function $Q$ is recurrent and leading-arm convergent, then for every problem instance $\Theta$, $\Pr(a(t) = i^*) \to 1$ as $t \to \infty$. As a result, the regret satisfies $R^\Theta(T) = o(T)$ and the policy is long-run-average optimal.

**Conditions for “over” and “under” exploration.** Let us now proceed to two key novel measures we use to quantify “over” and ”under” exploration. Let us start with the one corresponding to over-exploration.

**Definition 4.** Let $\alpha \in (0, 0.5)$ be fixed. The probability function $Q$ is $\alpha$-exploratory if there exists constants $\varepsilon > 0$, $\delta \in (0, 1)$ and a sequence of numbers $\{\Delta_1, \Delta_2, \ldots\}$, where $\Delta_T = \Omega(T^{-\alpha})$, such that for all sufficiently large $T$, $u_1 \leq \Delta_T$, and $\kappa$ such that $\sum_{i>1} \kappa_i < \delta T$, it holds that

$$1 - Q_1(\kappa_1, \kappa_2, \ldots, \kappa_N, u_1, 0, 0, \ldots, 0) \geq \varepsilon.$$

Roughly speaking, a policy with an $\alpha$-exploratory function $Q$ allows situations where even all arms have been pulled $\Omega(T)$ times and their empirical reward differences on the order of $\Omega(T^{-\alpha})$, the arm-pulling probabilities do not concentrate on the leading arm. In other words, an $\alpha$-exploratory function $Q$ with a small $\alpha$ is a sign of heavy exploration. Intuitively, if the exploration is too heavy, then the policy may fail to concentrate on any particular arm despite overwhelming evidence that a certain arm the the best. In the result below, we formally characterize the consequence of such “over-exploration.”

**Theorem 5.** Let $\alpha \in (0, 0.5)$ and $N > 1$ be fixed. If the probability function $Q$ is $\alpha$-exploratory, then the regret satisfies $R(T) = \Omega(T^{1-\alpha})$.

To elaborate on the consequence of under-exploration, we need a technical condition below to rule out some extreme cases.

**Definition 5.** The function $Q$ is unradical if for for every $\kappa \in \mathbb{Z}_+^N$ and $u \in [0, 1]^N$ such that $u_1 \leq \min\{u_2, \ldots, u_N\}$, $Q_1(\kappa, u) \leq \frac{1}{2}$. 
The property means that the weakest-performing arm (i.e., the arm with the lowest empirical mean) is pulled with a probability no larger than $1/2$. This is fairly intuitive since one would expect that the better performing arms would be pulled with higher probabilities. So we call this property “unradical.” The next property provides a quantitative measure of how a policy could potentially “under explore.”

**Definition 6.** Let $\alpha \in (0.5, +\infty)$ and $N > 1$ be fixed. The function $Q$ is $\alpha$-irreversible if for any following list of quantities: (i) arbitrarily small constant $\delta > 0$, (ii) arm $j > 1$, (iii) $u \in [0, 1]^N$ satisfying $u_j - u_1 = \Omega(1)$, and (iv) $\kappa \in \mathbb{Z}_+^N$ satisfying $\kappa_1, \kappa_j \geq (\log T)^{\frac{1}{2\alpha}}$ for sufficiently large $T$, it holds that

$$Q_1(\kappa, u) \leq \frac{(\log T)^{\frac{1}{2\alpha}}}{T}.$$ 

Roughly speaking, a policy with an $\alpha$-irreversible function $Q$ allows situations where there are two arms where both are pulled only $(\log T)^{\frac{1}{2\alpha}}$ times, while the worse-performing arm’s pulling probability already diminishes on the order of $O \left( \frac{(\log T)^{\frac{1}{2\alpha}}}{T} \right)$. After a preliminary calculation, this loosely translates to the probability of pulling the non-leading arms decaying super-exponentially fast with the pull count. Hence that corresponds to a sign of aggressive concentration (or lack of exploration). Intuitively, if the concentration is too aggressive, then the policy may be mistakenly stuck in the inferior arm in a “bad” event where it performs better empirically during the initial periods. This intuition is also where the name “irreversibility” comes from. In the result below, we formally characterize the performance consequence for “under-exploration.”

**Theorem 6.** Even when $N = 2$, for every policy with an unradical and $\alpha$-irreversible function $Q$, there exists an instance under which an $\Omega(T)$ regret is incurred with probability $\Omega(T^{-\alpha(1)})$. Therefore, the regret satisfies $R(T) = \Omega(T^{1-\alpha(1)})$.

The theorem reveals that without sufficient exploration, the aforementioned “bad” event happens with probability $\Omega(T^{-\alpha(1)})$ and the regret conditional that bad event is on the order of $\Omega(T)$. That makes the overall regret deteriorate almost linear in $T$. Finally, we reveal the connection between the QCARE policy family and our general framework by showing that QCARE satisfies all properties defined above.

**Proposition 3.** For every $\alpha > 0$, QCARE with parameter $\alpha$ is anonymous, recurrent, leading-arm convergent, and unradical. When $\alpha \in (0, 0.5)$, it is $\alpha$-exploratory. When $\alpha \in (0.5, +\infty)$, it is $\alpha$-irreversible.

As a result, Theorems 1, 2 and 3 follow from combining Proposition 3 with Theorems 4, 5 and 6, respectively.
5.4. From Asymptotics ($T = \infty$) to Finite-Horizon ($T < \infty$)

So far, our asymptotic analysis of QCARE explains how $\alpha$ affects decision qualities when $T = \infty$. In this section, we test whether our insights from the asymptotic theory can carry through to the finite horizon setting, i.e., $T < \infty$. To do so, we simulate the reward distributions of QCARE with a range of reward parameters ($\mu \in \{(0.4, 0.5), (0.6, 0.4)\}$), $\alpha$ values ($\alpha \in \{0.01, 0.02, ..., 1.99, 2\}$), and time horizons ($T \in \{100, 200, 500, 2500\}$). Each simulation is based on 100,000 sample paths. We present our result in Figure 4. In this figure, the purple curve represents the average rewards generated by QCARE with the corresponding $\alpha$ values. The blue dashed curves depict the 95%, 97.5%, 99% percentiles of the reward distribution, and the orange dashed curves represent the 1%, 2.5%, 5% percentiles. The vertical line represents the “optimal” $\alpha$ for every given configuration, which we obtained from grid search.

Let us summarize our observations as we move from the asymptotic setting ($T = \infty$) to finite horizons ($T < \infty$). First, the mean reward is unimodular: it first increases and then decreases in $\alpha$. As a result, there is still an “optimal” level of $\alpha$ to maximize the expected reward. Once again, it represents the sweet spot for the EE trade-off and justifies the notion of over- and under-exploration measured by $\alpha$. The main difference here is that the optimal $\alpha$ is generally larger than 0.5. That is intuitive since as the time horizon gets shorter, there is less time to explore and the decision maker should exploit more aggressively. This observation is consistent with the literature, which finds that although TS (i.e., $\alpha = 0.5$) is an order optimal policy, it tends to over-explore in the finite horizon; see Min et al. (2019), Jin et al. (2023) for related works. In general, while there is a simple answer regarding how much to explore in the infinite horizon, the optimal $\alpha$ in the finite time horizon can...

Figure 4 Reward distribution of QCARE
be configuration-specific. That is quite expected since while the infinite-horizon setting tends to focus on the overall landscape of the EE trade-off and makes many factors negligible, those factors become material in the finite-horizon setting.

Another interesting finding regards how the reward distribution (e.g., its tail) is influenced by $\alpha$. When $\alpha$ is larger than the optimal level (and hence in the under-exploration region), both the dispersion and tail risk of the reward are significantly increased. That is because of the nonnegligible probability of the extremely detrimental event where the decision maker is stuck in the inferior arm. In comparison, when $\alpha$ is in the over-exploration region, there can be a noticeable drop in the mean reward for every fixed $T$, especially when $\alpha$ and $T$ are small. However, both the dispersion and tail of the reward distributions are milder, and the drop in the mean reward can be quickly mitigated as $T$ grows. Our observations are consistent with our insights from the previous analysis, and we will revisit them when we investigate the behavioral patterns from the experimental data.

6. Empirical Validation and Implications

Empirically, the value of $\alpha$ is a behavioral parameter that can be estimated from data. In this section, we validate the QCARE model from various perspectives. First, we use synthetic data to demonstrate its ability to recover the ground truth parameter values. Then, we use real data to highlight QCARE’s favorable capabilities of capturing human behavior, especially in terms of out-of-sample prediction power. We also illustrate how the QCARE-generated reward distributions align well with the behavioral data after controlling for $\alpha$. Finally, we discuss a behavioral pattern we discover: people tend to “over-explore” compared to the optimal level that maximizes the expected reward.

6.1. Empirical Analysis

Estimation. Let us first explain how we can estimate the $\alpha$ values, through maximum likelihood estimation (MLE), from the behavioral data collected in the aforementioned experiments. Since we collected human behavior data based on the two-armed MAB problem setting, we will also explain our MLE assuming that $N = 2$. The $N > 2$ can be generalized accordingly. To fully capture the heterogeneity among the subjects, we estimated the model parameter $\alpha$ by individual.

Now, suppose we have gathered data for a specific subject denoted as $j$. It can be expressed in the form of $\{\hat{\mu}_j^i(t), k_j^i(t), y_{jt}\}_{ijt}$, which we also refer to as a sample path. Here, $\hat{\mu}_j^i(t)$ and $k_j^i(t)$ are the state variables that subject $j$ faces for arm $i$ at period $t$, and $y_{jt}^j$ is an indicator variable representing whether arm $i$ is pulled by subject $j$ at period $t$. Recall that $\epsilon_{jt}^j$ are i.i.d, draws from the standard
normal distribution. We denote the cumulative distribution function of a standard normal random variable as $\Phi(\cdot)$. Thus, the log-likelihood function for the parameter $\alpha$ under QCARE is given by

$$LL^j(\alpha) = \sum_t \sum_{i \in \{1, 2\}} y_{it} \log \left( \Pr \left( \hat{\mu}^j(t) + \frac{\epsilon^j_{it}}{(k^j(t) + 1)^\alpha} \geq \hat{\mu}^j_{3-i}(t) + \frac{\epsilon^j_{(3-i)t}}{(k^j_{3-i}(t) + 1)^\alpha} \right) \right)$$

$$= \sum_t \sum_{i \in \{1, 2\}} y_{it} \log \left( \Phi \left( \frac{\hat{\mu}^j(t) - \hat{\mu}^j_{3-i}(t)}{\sqrt{\frac{1}{(k^j(t)+1)^\sigma} + \frac{1}{(k^j_{3-i}(t)+1)^\sigma}}} \right) \right).$$

The estimated value of $\alpha$ is thus calculated by solving $\max_\alpha LL^j(\alpha)$.

To check whether the estimated $\alpha$ recovers the ground true value when the subject indeed adheres to the QCARE model, we simulate the behavioral data according to QCARE for various $(T, \alpha)$ combinations, each in 10,000 sample paths. We estimate a separate $\hat{\alpha}$ for every single sample path. In Table 1 we report the distributions of $\hat{\alpha}$ for every combination of $(T, \alpha)$ by showing their average values, 5th and 95th percentiles. As this table demonstrates, the ground truth $\alpha$ is always within the 5th and 95th percentiles of $\hat{\alpha}$, and the average $\hat{\alpha}$ is very close to the ground truth $\alpha$, especially when the ground truth $\alpha$ becomes larger.

| $T$ | $\alpha$ | $\mu = (0.4, 0.5)$ | $\mu = (0.6, 0.4)$ |
|-----|--------|-----------------|-----------------|
|     | average $\hat{\alpha}$ | 5th percentile of $\hat{\alpha}$ | 95th percentile of $\hat{\alpha}$ | average $\hat{\alpha}$ | 5th percentile of $\hat{\alpha}$ | 95th percentile of $\hat{\alpha}$ |
| 100 | 0.2   | 0.2015         | 0.0139         | 0.4881         | 0.1951         | 0.0139         | 0.4181         |
| 200 | 0.2   | 0.1937         | 0.0139         | 0.3948         | 0.1924         | 0.0139         | 0.3171         |
| 1000 | 0.2   | 0.1967         | 0.1227         | 0.2549         | 0.1999         | 0.1694         | 0.2315         |
| 10000 | 0.2   | 0.2004         | 0.1927         | 0.2082         | 0.2005         | 0.1927         | 0.2082         |
| 100 | 0.5   | 0.4827         | 0.2082         | 0.6591         | 0.4996         | 0.3637         | 0.6280         |
| 200 | 0.5   | 0.4949         | 0.3948         | 0.5813         | 0.5014         | 0.4414         | 0.5658         |
| 1000 | 0.5   | 0.5006         | 0.4803         | 0.5192         | 0.5011         | 0.4803         | 0.5192         |
| 10000 | 0.5   | 0.5008         | 0.4958         | 0.5036         | 0.5005         | 0.4958         | 0.5036         |
| 100 | 1     | 1.0388         | 0.8923         | 1.2421         | 1.0531         | 0.8845         | 1.3276         |
| 200 | 1     | 1.0258         | 0.9234         | 1.1644         | 1.0371         | 0.9078         | 1.2265         |
| 1000 | 1     | 1.0115         | 0.9467         | 1.0866         | 1.0216         | 0.9312         | 1.1371         |
| 10000 | 1     | 1.0067         | 0.9622         | 1.0555         | 1.0138         | 0.9545         | 1.0983         |

**Validation on real behavioral data.** Apart from simulations, we conduct a comprehensive study on how well QCARE performs in terms of explanation/prediction power in behavioral data from the experiments.

We draw from different streams of literature and consider seven MAB policies as benchmarks. In the behavioral domain, a represented work is that by [Gans et al. (2007)](Gans et al. (2007)), who also investigated...
human behavior in a two-armed MAB problem in laboratory experiments. To this end, we also included the behavioral models in their study: Last-\(n\), hot hand, and exponential smoothing. Under those models, the probability of selecting an arm for subject \(j\) is determined through a soft-max formula:

\[
P_j\{\text{choose arm } i \text{ at period } t\} = \frac{e^{\beta_j I_{jt}^i}}{\sum_{l \in N} e^{\beta_j I_{jt}^l}},
\]

where \(\beta_j\) is a parameter that needs to be estimated for subject \(j\), and \(I_{jt}^i\) is an index for subject \(j\) and arm \(i\) at time \(t\). Each model corresponds to a different formula for \(I_{jt}^i\), which we explain below.

- **Last-\(n\) model (L-\(n_j\))**: This model operates on the assumption that the subject \(j\) possesses limited memory of their past trials. The index \(I_{jt}^i\) is determined by averaging the rewards obtained from the previous \(n_j\) pulls of arm \(i\). Here \(n_j\) is a behavioral parameter to be estimated for every subject \(j\).

- **Hot hand model (HH-\(n_j\))**: This model operates on the “hot hand” concept, that is, once an arm has won recently, it is more likely to win again. Here the index \(I_{jt}^i\) in the following way. For arm \(i\) sampled at period \(t-1\), \(I_{jt}^i\) is set to be 0 if subject \(j\) encounters consecutive losses on arm \(i\) within the last \(n_j\) pulls, and 1 otherwise. For other arms, the index equals \(1 - I_{jt}^i\). Here \(n_j\) is a parameter to be estimated for every subject \(j\).

- **Exponential smoothing model (ES-\(\gamma_j\))**: In this model, the index of arm \(i\) is based on a time-discounted average of historical rewards. Specifically, let \(t_i^-\) denote the last time arm \(i\) was pulled before period \(t\). Then we set \(I_{jt}^i = \gamma_j r_i(t_i^-) + (1 - \gamma_j) I_i(t_i^-)\). Here the parameter \(\gamma_j\) takes values in \((0, 1)\) and is to be estimated for every subject \(j\).

To investigate whether we can achieve the same descriptive/predictive power by a “simpler” model than QCARE, we consider a variation of QCARE.

- **Q CARE-0**: We adopt QCARE but restricting \(\alpha = 0\). In this way, the exploration term does not shrink over time.

Finally, we include the following classical bandit algorithms from the online learning literature.

- **\(\varepsilon_j\)-greedy**: In this model, at time period \(t\), subject \(j\) chooses the leading arm with probability \(1 - \varepsilon_j\) and other arms with probability \(\frac{\varepsilon_j}{(N-1)}\). The value of \(\varepsilon_j\) is to be estimated for each subject.

- **Thompson sampling (TS-B, TS-G)**: We consider two types of Thompson Sampling algorithms: one with Beta prior (TS-B) and the other with Gaussian prior (TS-G). In these models, the subjects follow the Thompson Sampling approach with respective prior distributions; see Appendix \[A\] for a more detailed algorithmic description.
Whenever applicable, the model parameters are estimated through MLE.

We compare both the in-sample and out-of-sample performances of all models. To accommodate the heterogeneity among individuals, we split the data into training and testing sets along the temporal dimension. That is, we use the first three quarters of periods to train/calibrate models for each subject, and we use the last quarter of periods for out-of-sample evaluation. We use two evaluation metrics: (i) the average log-likelihood values and (ii) head-to-head comparisons in terms of log-likelihood between QCARE and benchmark models, where we report the fraction of subjects that QCARE outperforms the benchmark. We also investigate whether the performance differences are statistically meaningful. To summarize, we have considered multiple factors such as various MAB policies, the performance metric, and the data to evaluate, and included $8 \times 2 \times 2 = 32$ combinations of configurations in the comparison.

The results are detailed in Table 2. In short, we find that QCARE meaningfully dominates most of the models (e.g., QCARE-0, HH-$n_j$, $\epsilon_j$-greedy, TS-B, and TS-G) both in terms of in-sample explanation power and out-of-sample prediction power. When it comes to comparisons with L-$n_j$ and ES-$\gamma_j$, the latter models display better in-sample performance, which is not surprising since those models involve two parameters to estimate, while QCARE only has one parameter. However, better fitting power does not translate to strong out-of-sample generation power due to potential over-fitting issues. To this end, we find it important to note that QCARE meaningfully dominates all benchmark models in terms of out-of-sample performances. The results presented in Table 2 offer compelling evidence that QCARE not only offers strong interpretability but also captures people’s real behavior well.

Table 2 Comparison Between QCARE and Other Models

| Data | Metric | QCARE | L-$n_j$ | HH-$n_j$ | ES-$\gamma_j$ | QCARE-0 | $\epsilon_j$-greedy | TS-B | TS-G |
|------|--------|-------|--------|---------|--------------|--------|---------------------|------|------|
| IS   | AvgLL  | -64.6 | -65.0  | -70.5** | -62.5        | -71.8** | -64.9               | -93.8** | -71.2** |
|      | Pct    | –     | 43.1%  | 56.9%   | 32.5%        | 100.0%**| 63.4%**             | 98.4%**| 98.4%** |
| OoS  | AvgLL  | -20.5 | -40.7**| -27.9** | -25.2**      | -23.7** | -20.5               | -28.4**| -22.5* |
|      | Pct    | –     | 66.7%**| 71.5%** | 64.2%**      | 84.6%** | 63.4%**             | 75.6%**| 60.2%* |

Note: “IS” means in-sample (i.e., evaluated on the training data), and “OoS” means out-of-sample (i.e., evaluated on the test data). “AvgLL” refers to the average log-likelihood at the subject level by each approach. “Pct” indicates the corresponding percentage of subjects that QCARE outperforms the benchmark. The remarks “**” and “*” indicate $p$-values less than 0.01 and 0.05, respectively.

Implied alpha and reward distributions. Finally, we use our QCARE model to address our original question, which is how human beings actually balance the EE trade-off. Figure 5 summarizes our results. In this figure, each green dot represents an individual subject and contains

4 To do so, we employ one-tailed pairwise $t$-test for the average log-likelihood comparison ($H_0$: the average log-likelihood of QCARE equals that of the benchmark; $H_1$: the average log-likelihood of QCARE is higher than that of the benchmark) and proportional test for the head-to-head comparison ($H_0$: the proportion of subjects that QCARE outperforms equals 50%; $H_1$: the proportion that QCARE outperforms the benchmark exceeds 50%).
information about the subject’s implied $\hat{\alpha}$ value and realized reward. The green dots are then organized by experiment configurations (i.e., mean reward vector $\mu$ and time horizon $T$). For each configuration, we also follow the same method in Figure 4 to display the reward distribution generated by all other $\alpha$ values. That is, we report the mean rewards, the 95%, 97.5%, 99%, 1%, 2.5%, and 5% percentiles of the reward distribution based on 10,000 simulated paths. Finally, the instance-specific optimal $\alpha$ value for each configuration is represented by the blue vertical line. Comparing the realized and simulated data, we find that the majority of the green points fall within the confidence bands under QCARE, further validating it as a reasonable model to capture human behavior in the MAB problem.

A perhaps even more interesting (and surprising) finding from Figure 5 is that a large majority of the green points lie on the left vertical line that represents the expected-reward-maximizing $\alpha$. In fact, a significant portion of subjects have implied $\alpha$ values less than 0.5, the optimal level in the asymptotic regime when $T = \infty$. It suggests that most subjects tend to exhibit an over-exploration behavior. We find this pattern surprising since if every subject has his/her own idiosyncratic behavioral preferences/biases, one would expect a distribution of $\alpha$ to be more balanced regarding the optimal level.

The behavioral pattern towards over-exploration in Figure 5 is robust across the configurations, such as time horizon and reward distributions. The results suggest that while those factors could potentially influence the EE trade-off, which is reflected in the different values of the optimal $\alpha$, the fact that people tend to explore more aggressively than the reward-maximizing level seems consistent. The patterns are also robust based on other auxiliary experiments we run; see Appendix C for more details.

6.2. Implications and Further Discussion

Prescriptive implications. The QCARE model and the behavioral pattern have a few prescriptive implications. First, the pattern of over-exploration suggests that managers' instincts about the exploration-exploitation trade-off in MAB environments may be inaccurate. Instead, it seems that they could benefit from leaning more aggressively toward exploitation, especially when faced with limited time constraints.

The development of QCARE, or dynamic choice models with learning effects in general, offers opportunities for the platforms and companies, too. More specifically, platforms and companies could utilize the dynamic choice model for consumer/user behavior to improve assortment and recommendation strategies. This is in the same spirit as how static choice models (e.g., MNL) provide guidance on assortment optimization. Of course, the consumer/user choice behavior (e.g.,
the mental optimization problem among the finite set of alternatives) would be much simpler under the traditional static setting.

Further discussion. From the behavioral side, it would also be interesting to investigate the underlying mechanisms behind such behavioral patterns. Here let us discuss a few plausible explanations. First, the pattern could be partially rationalized by the subject’s own goals and preferences, such as risk aversion. In fact, there is an emerging literature discussing risk-return trade-off in MAB policies, and how more aggressive exploration could reduce the risk; see Keller et al. (2019), Zhu and Tan (2020), Simchi-Levi et al. (2023), Fan and Glynn (2024b) for related works. Our model displays the same pattern: with a slightly smaller $\alpha$ than the optimal level, the risk can be reduced (based on various measures of risk, such as the dispersion of the reward distribution, the 1%/2.5%/5% tail risks, etc.) at an expense of lower expected reward; see Figure 5. In this sense, at least some subjects’ behavior could be explained by their using over-exploration to trade expected rewards for less risk.

However, risk aversion can only explain part of the observations. When exploration is too heavy (i.e., when $\alpha$ is too small), both the expected reward and the risk level deteriorate. In the real data,
a noticeable portion of subjects could have simultaneously reduced their risk and increased their expected reward by using a larger $\alpha$. In light of our model-free observation in Remark 1, it seems that the over-exploration pattern should also be attributed to the subjects’ bounded rationality and cognitive limitations.

In other words, subjects’ behavior of not choosing the leading arm appears to reflect a mix of random behavioral errors (in the same spirit of Su (2008)) layered on top of the “intrinsic” exploration required for dynamic learning.

We believe our finding opens a door to future research opportunities. For example, one could extend the QCARE model by exploring alternative parametrizations of exploration to handle different decision-making environments, such as those with higher stakes or more complex reward structures. Furthermore, applying QCARE to decision-making tasks in real-world scenarios with richer operational contexts could help validate the findings from the lab environments and deepen the insight. Finally, it would be interesting to see if one can use “nudging” to mitigate the over-exploration.

7. Conclusion

This paper introduces a dynamic quantal choice model, QCARE, that parametrically extends Thompson Sampling to represent people’s decision behavior in multi-armed bandit settings. The model quantifies the exploration-exploitation trade-off by incorporating a reduction rate of exploration (i.e., $\alpha$) as more information becomes available. The relationship between different values of $\alpha$ and the decision qualities is mathematically established through an asymptotic theory, which could further extend to a more general space of Markovian MAB policies. The analysis thus helps us better understand the effects of “over” and “under” exploration. Based on behavioral data we collected from experiments, QCARE demonstrates a compelling ability to capture people’s decision-making process: it replicates key qualitative patterns observed in subjects’ behavior and also displays favorable explanation/predictive power compared to other models. Finally, empirical analysis through QCARE offers a novel perspective on how over-exploration tends to be a behavioral pattern.

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This is reasonable given the complexity of MAB problems. Even with advanced computational power, finding the Bayes optimal solution to the MAB problem can be challenging as $T$ increases, due to the curse of dimensionality.
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Appendix A  Algorithmic Descriptions of QCARE and TS

We provide an algorithmic description of QCARE below.

**Quantal Choice with Adaptive Reduction of Exploration (QCARE)**

**Input:** $\alpha > 0$.

**Procedure:** Initialize with $k_i(1) = 0, \hat{\mu}_i(1) = 0$ for each arm $i \in [N]$.

At each time $t \in [T]$:
1. For each arm $i \in [N]$, sample $\epsilon_{it}$ independently. Set $\theta_i(t) := \hat{\mu}_i(t) + \epsilon_{it}/(k_i(t) + 1)^\alpha$.
2. Play arm $a(t) := \arg \max \theta_i(t)$ and observe reward $r_{a(t)}(t)$.
3. Update states:
   - For arm $i = a(t)$, set $k_i(t+1) = k_i(t) + 1$ and $\hat{\mu}_i(t+1) = \frac{\hat{\mu}_i(t)(k_i(t)+1)+r_{i(t)}}{k_i(t+1)+1}$;
   - For other arms $i \neq a(t)$, set $k_i(t+1) = k_i(t)$ and $\hat{\mu}_i(t+1) = \hat{\mu}_i(t)$.

We also describe TS with Gaussian priors and TS with Beta priors, following the version in Agrawal and Goyal (2013).

**Thompson Sampling using Gaussian Priors**

For each arm $i \in [N]$ set $k_i = 0, \hat{\mu}_i = 0$.

At each time $t \in [T]$:
1. For each arm $i \in [N]$, sample $\theta_i(t)$ independently from the $\mathcal{N}(\hat{\mu}_i, \frac{1}{k_i+1})$ distribution.
2. Play arm $a(t) := \arg \max \theta_i(t)$ and observe reward $r_t$.
3. Set $\hat{\mu}_{a(t)} := \frac{\hat{\mu}_{a(t)} k_{a(t)} + r_t}{k_{a(t)} + 1}$, $k_{a(t)} := k_{a(t)} + 1$.

**Thompson Sampling using Beta priors**

For each arm $i \in [N]$ set $S_i = 0, F_i = 0$.

At each time $t \in [T]$:
1. For each arm $i \in [N]$, sample $\theta_i(t)$ independently from the $\text{Beta}(S_i+1, F_i+1)$ distribution.
2. Play arm $a(t) := \arg \max_i \theta_i(t)$ and observe reward $r_t$.
3. If $r_t = 1$, then $S_{a(t)} := S_{a(t)} + 1$, else $F_{a(t)} := F_{a(t)} + 1$.

Appendix B  An Example for Aggregating the Choice Frequencies of the Leading Arm

In the descriptive analysis of our experiment data, we consider a collection of metrics to evaluate subjects’ tendency to choose the leading arm. Here we present a concrete example of how to construct the metrics. Suppose one subject plays a 10-round two-armed bandit game and the historical record is summarized in Table 3 below. Let us walk through the calculations and values of our metrics.

1. In periods 4, 7, 8, and 10, the subject chose the leading arm. The choice fraction of the leading arm is 40%.
2. In periods 1, 2, 8, 9, 10, the empirical mean gaps are in the low group, while in periods 3-7, gaps are in the high group. The choice fraction of the leading arm is 40% in the low group and also 40% in the high group.
3. In first half periods 1-5, the choice fraction of the leading arm is 20%. In the second half, the fraction is 60%.
Table 3  Example: historical table of one subject

| t  | action | reward | $\hat{\mu}_1(t)$ | $\hat{\mu}_2(t)$ | Whether choose leading arm |
|----|--------|--------|------------------|------------------|--------------------------|
| 1  | 2      | 0      | 0.000            | 0.000            | N                        |
| 2  | 2      | 1      | 0.000            | 0.000            | N                        |
| 3  | 1      | 1      | 0.000            | 0.333            | N                        |
| 4  | 1      | 1      | 0.500            | 0.333            | Y                        |
| 5  | 2      | 1      | 0.667            | 0.333            | N                        |
| 6  | 2      | 0      | 0.667            | 0.500            | N                        |
| 7  | 1      | 0      | 0.667            | 0.400            | Y                        |
| 8  | 1      | 0      | 0.500            | 0.400            | Y                        |
| 9  | 1      | 1      | 0.400            | 0.400            | N                        |
| 10 | 1      | 1      | 0.500            | 0.400            | Y                        |

In this way, for other subjects, we can calculate these metrics and then calculate the average among subjects.

Appendix C  Further Details about the Experiments

C.1  The Baseline Experiments

We present in Table 4 a detailed breakdown by configuration and in Figures 6 the experiment interfaces.

Table 4  Number of Subjects by Configuration (Baseline Experiments)

| $\mu = (0.4, 0.5)$ | $\mu = (0.6, 0.4)$ |
|-------------------|-------------------|
| $T=100$           | 35                | 29                |
| $T=200$           | 33                | 26                |

(Total observations: 18,200)

Figure 6  Interface of the baseline experiment
Let us also explain the payoff details. The show-up fee is 5 Chinese Yuan (CNY). The point-to-reward conversion rule is normalized to 0.5 CNY per point when $\mu_1 = 0.5$ and 0.4 CNY per point when $\mu_2 = 0.6$. In this way, the earnings of playing the clairvoyant policy (i.e., always selecting the arm with a larger mean) are roughly the same across parameter configurations. Based on those settings, the average total earning per subject was approximately 5.23 US Dollars (USD). On average, subjects took 13 minutes to complete the experiment, with a maximum duration of 24 minutes.

C.2 The Three-armed Bandit Setting

We extend the setting to three arms and present our experiment and analysis details. We conduct lab experiments involving three-armed MAB problems. The rewards were set to follow a Bernoulli distribution with mean values of 0.6, 0.4, and 0.2, respectively. In total, we recruited 61 subjects; see Table 5 for a breakdown by configuration. The show-up fee is 5 CNY and each point is worth 0.4 CNY. In the following, we replicate our empirical analysis for the baseline experiments to show that the findings under the two-armed MAB problem remain robust in the three-arm bandit setting.

| $\mu = (0.6, 0.4, 0.2)$ | T=100 | 30 |
|--------------------------|-------|----|
| T=200                    | 31    |
| (Total observations: 9,200) |

Descriptive analysis. Our experiment data reveals that only 2 of 61 subjects chose the leading arm throughout the entire time horizon. On average, subjects chose the leading arm 62.52% of the time. That ratio is strictly less than one, yet significantly larger than 0.5 (Wilcoxon signed-rank test right tail, p-value =4.9013e-6). Hence, on the aggregate level, the participants are choosing the arms in a way that is neither purely exploration nor exploitation, as expected. We also calculate the percentage of rounds in which each individual subject selected the second good performing arm and the third good performing arm. The result shows that on average, subjects chose the second-performing (resp. third-performing) arm in 22.02% (resp. 14.71%) rounds. This result indicates that subjects chose the second performing arm more often than the third performing arm. This difference is found to be statistically significant by a two-sided Wilcoxon signed-rank test with a $p$-value of 1.1683e-6. These findings suggest that human decision-making in this context does not

\[6\text{If taking the average at the individual-period level, the figure would be 63.77%}.)\]
follow the $\varepsilon$-greedy model, which predicts that these empirically suboptimal arms have an equal probability of being selected.

Next, we use Figure 7 based on the three-armed experimental data as a replication for Figure 1 in the main body. In Panel (a), we calculate the difference between the leading arm and the second best performing arm. Overall, all the plots in Figure 7 indicate the findings in Section 3.2 are robust.

![Figure 7 Percentage of rounds choosing the leading arm](image)

(a) By empirical mean difference  (b) By period  (c) By both

Validation using simulated data. We verify whether the estimated $\alpha$ can recover the true $\alpha$ value when subject $j$ indeed adheres to the QCARE model. Specifically, we simulate subject $j$’s behavior based on the QCARE model for a given $T$ and $\alpha$ in 10,000 samples. The mean vector of three arms is $(0.2, 0.4, 0.6)$, which is aligned with that used in the lab experiment. Table 6 presents the summary of the estimated values of $\hat{\alpha}$. Again, the ground truth $\alpha$ is always within the 5th and 95th percentiles of $\hat{\alpha}$, and the average $\hat{\alpha}$ is very close to the ground truth $\alpha$, especially when the ground truth $\alpha$ becomes larger.

Empirical validation of real behavioral data. Similar to Section 5, we compare the QCARE model with the same benchmark models in the main body, but under the three-arm setting. The detailed comparison results are presented in Table 7. Similar to the two-armed setting, QCARE continues to display favorable performance, especially in terms of out-of-sample prediction power.

Realized rewards and behavioral pattern. We replicate the analysis in Section 6 and examine the relationship between the realized reward and the reduction rate of exploration. We present a comprehensive illustration of reward distribution with different $\alpha$ values, as well as the subjects’ behavior in Figure 8. Similar to the two-armed setting, the green points fall within the confidence bands generated by the QCARE model, further validating the ability of the QCARE model to capture the actual human behavior. In addition, the majority of the green points are positioned to the left of the blue vertical line, suggesting a tendency towards over-exploration.
Table 6  Comparison of the ground truth ($\alpha$) and estimated values ($\hat{\alpha}$)

| $T$  | $\alpha$ | average $\hat{\alpha}$ | 5th percentile of $\hat{\alpha}$ | 95th percentile of $\hat{\alpha}$ |
|------|----------|-------------------------|----------------------------------|----------------------------------|
| 100  | 0.2      | 0.1978                  | 0.0139                           | 0.3637                           |
| 200  | 0.2      | 0.1978                  | 0.0916                           | 0.2860                           |
| 1000 | 0.2      | 0.2003                  | 0.1771                           | 0.2238                           |
| 10000| 0.2      | 0.2005                  | 0.2004                           | 0.2004                           |
| 100  | 0.5      | 0.5041                  | 0.4026                           | 0.6047                           |
| 200  | 0.5      | 0.5020                  | 0.4570                           | 0.5503                           |
| 1000 | 0.5      | 0.5009                  | 0.4881                           | 0.5114                           |
| 10000| 0.5      | 0.5007                  | 0.4958                           | 0.5036                           |
| 100  | 1        | 1.0193                  | 0.9001                           | 1.1721                           |
| 200  | 1        | 1.0119                  | 0.9234                           | 1.1177                           |
| 1000 | 1        | 1.0051                  | 0.9467                           | 1.0633                           |
| 10000| 1        | 1.0019                  | 0.9622                           | 1.0400                           |

Table 7  Comparison Between QCARE and Other Models

| Data | Metric | QCARE | L-$n_j$ | HH-$n_j$ | ES-$\gamma_j$ | QCARE-0 | $\varepsilon_j$-greedy | TS-B | TS-G |
|------|--------|-------|---------|----------|---------------|---------|------------------------|------|------|
| IS   | AvgLL  | -91.7 | -91.8   | -95.0    | -85.9         | -111.1**| -92.8                  | -163.2**| 104.4**|
|      | Pct    | 50.8% | 52.5%   | 18.0%    | 100.0%**      | 55.7%   | 95.1%**                | 98.4%**|      |
| OoS  | AvgLL  | -27.0 | -61.6** | -29.1*   | -37.0**       | -35.4** | -27.4                  | -44.7**| -32.1**|
|      | Pct    | 78.7%**| 62.3%* | 75.4%**  | 82.0%**       | 63.9%*  | 86.9%**                | 67.2%**|      |

Note: “IS” means in-sample (i.e., evaluated on the training data), and “OoS” means out-of-sample (i.e., evaluated on the test data). “AvgLL” refers to the average log-likelihood at the subject level by each approach. “Pct” indicates the corresponding percentage of subjects that QCARE outperforms the benchmark. The remarks “**” and “*” indicate $p$-values less than 0.01 and 0.05, respectively.

C.3  The Online Experiments

We conducted an online experiment facilitated by the Prolific platform to replicate our empirical analysis of the baseline experiments. In total, we recruited 123 subjects for four scenarios; see Table 8 for a breakdown by configuration.
Table 8  Number of Subjects by Configuration (Online Experiments)

\[
\begin{array}{ccc}
\mu = (0.4, 0.5) & \mu = (0.6, 0.4) \\
T=100 & 32 & 33 \\
T=200 & 30 & 28 \\
\end{array}
\]

(Total observations: 18,100)

(a) Introduction Page (\(T = 200\))

(b) A Typical Round Page

Figure 9  Interface of the online experiment

The experimental procedure followed the same methodology as described in Section 3.1. We show the interface of the introduction page and the round page during the experiment in Figure 9. At the beginning of the experiment, the subjects were told the number of rounds and the basic information about the reward mechanism. Each participant’s total earnings comprised two components: a show-up fee of 1.5 pounds and a bonus based on realized points. These points were subsequently converted into cash at the end of the session, with a conversion rate of 1 point equal to 0.03 British pounds. Based on this setting, the final average earnings per subject were 3.68 pounds (approximately 3.94 USD). On average, subjects took 14.85 minutes to complete the experiment, with a maximum duration of 36 minutes.

Descriptive analysis.

Similar to our baseline experiments, the results from our online study demonstrated that only 1 of 123 subjects chose the leading arm throughout the entire time horizon, and 87 out of 123 subjects chose the leading arm with at least 50% of the rounds. On average, subjects chose the leading arm in 59.76% of the time.\(^7\) That ratio is strictly less than one, yet significantly larger than 0.5 (Wilcoxon signed-rank test right tail, \(p\)-value=8.4440e-10). These findings suggest that human

\(^7\) If taking the average at the individual-period level, the figure is 59.10%.
decision-making involves a balance between exploration and exploitation, rather than favoring either exploration or exploitation exclusively.

Next, we use Figure 10 based on the online experimental data as a replication for Figure 1 in the main body. In Panel (a), we observe a significantly higher average ratio when the difference is larger than the median ($p$-value being $6.8480\times 10^{-8}$ using a two-sided Wilcoxon signed-rank test). From Panel (b), we find a significantly higher average ratio in the second half of the time horizon ($p$-value being $5.0270\times 10^{-5}$ using a two-sided Wilcoxon signed-rank test). Panel (c) shows that the leading arm is more likely to be chosen amid a wider gap in the empirical mean reward, or more observations that the empirical mean gap is based on. Overall, all the plots in Figure 10 indicate the findings in Section 3.2 are robust.

(a) By empirical mean difference  
(b) By period  
(c) By both

Figure 10  Percentage of rounds choosing the leading arm

**Empirical validation on real behavioral data.** Similar to Section 6, we compare the QCARE model with the same benchmark models, but using the behavioral data gathered from our online experiments. The detailed comparison results are presented in Table 9, which is quite similar to the baseline experiments. QCARE continues to display favorable performance, especially in terms of out-of-sample prediction power.

| Data | Metric | QCARE | L-$n_j$ | HH-$n_j$ | ES-$\gamma_j$ | QCARE-0 | $\varepsilon_j$-greedy | TS-B | TS-G |
|------|--------|-------|--------|--------|-------------|---------|---------------------|------|------|
| IS   | AvgLL  | -68.8 | -66.3  | -70.5**| -64.4       | -73.7** | -69.6               | -116.4** | -77.9** |
| Pct  |       |       | 33.3%  | 51.2%  | 18.7%       | 100.0%**| 60.2%*              | 97.6%** | 97.6%** |
| OoS  | AvgLL  | -21.5 | -28.8**| -27.0**| -23.8**     | -24.3** | -21.9              | -30.6** | -26.6** |
| Pct  |       |       | 62.6%**| 59.3%* | 55.3%       | 85.4%** | 59.3%*             | 87.0%** | 74.0%** |

Note: “IS” means in-sample (i.e., evaluated on the training data), and “OoS” means out-of-sample (i.e., evaluated on the test data). “AvgLL” refers to the average log-likelihood at the subject level by each approach. “Pct” indicates the corresponding percentage of subjects that QCARE outperforms the benchmark. The remarks “**” and “*” indicate $p$-values less than 0.01 and 0.05, respectively.
Realized rewards and behavioral patterns. We also investigate the relationship between the realized reward and the reduction rate of exploration, effectively replicating Figure 5 using Figure 11 based on online experiment data. Same as the lab experiment setting, most of the points lie within the confidence band implied by the QCARE model. In addition, the majority of points are positioned to the left of the optimal level. This observation suggests that the tendency towards over-exploration is robust in the online experiments.

C.4 The Preregistered Experiments
We conducted a preregistered online experiment to once again replicate our empirical analysis and confirm our findings are robust. In total, we recruited 303 subjects across four configurations; see Table 10 for a detailed breakdown.

|                | \( \mu = (0.4, 0.5) \) | \( \mu = (0.6, 0.4) \) |
|----------------|-------------------------|-------------------------|
| \( T=100 \)   | 76                      | 78                      |
| \( T=200 \)   | 75                      | 74                      |

(Total observations: 45,200)
The experimental procedure followed the same methodology outlined in Section 3.1. The interface of the introduction and round pages, and the payment rules were identical to those of the online experiment.

**Descriptive analysis.** The results from our preregistered online study showed that only 4 of 303 subjects consistently chose the leading arm across all rounds, while 229 out of 303 subjects selected the leading arm in at least 50% of the rounds. On average, subjects chose the leading arm 59.95% of the time. That ratio is strictly less than one, yet significantly larger than 0.5 (Wilcoxon signed-rank test right tail, \( p \)-value = 4.1155e-27). Next, we use Figure 12 based on the pre-registered experiments as a replication for Figure 1. In Panel (a), we observe a significantly higher average ratio when the difference exceeds the median (\( p \)-value being 1.4501e-16 using a two-sided Wilcoxon signed-rank test). Panel (b) shows a significantly higher average ratio in the second half of the time horizon (\( p \)-value being 2.3613e-10 using a two-sided Wilcoxon signed-rank test). Panel (c) indicates that the leading arm is more likely to be chosen amidst wider gaps in the empirical mean reward, or more observations that the empirical mean gap is based on. Overall, the plots in Figure 12 support the robustness of the findings in Section 3.2.

![Figure 12](image_url)

(a) By empirical mean difference  
(b) By period  
(c) By both  

**Comparison of prediction power.** As in Section 6, we compare the prediction power of the QCARE model against other benchmark models, but here using the preregistered online experiment data. The detailed comparison results are provided in Table 11. Once again, the results strongly support the QCARE model as a credible behavioral model, particularly in terms of predictive power compared to the benchmarks.

**Realized rewards and behavioral patterns.** We also examine the relationship between the realized reward and the reduction rate of exploration, replicating Figure 5 using data collected from the preregistered online experiment. We report the results in Figure 13. Once again, it is

\[ 56.5461\% \quad 68.4162\% \]
\[ 48.0719\% \quad 59.1088\% \]

If taking the average at the individual-period level, the figure would be 60.33%.
Table 11  Comparison Between QCARE and Other Models

| Data | Metric | QCARE | L-\(n_j\) | HH-\(n_j\) | ES-\(\gamma_j\) | QCARE-0 | \(\varepsilon_j\)-greedy | TS-B | TS-G |
|------|--------|-------|-----------|----------|--------------|---------|----------------|------|------|
| IS   | AvgLL  | -70.2 | -67.5     | -71.8**  | -66.0        | -74.3** | -70.6         | -118.9** | -79.1** |
|      | Pct    | 26.1% | 46.5%     | 20.5%    | 100.0%**    | 62.7%** | 98.3%**       | 98.7%** |
| OoS  | AvgLL  | -23.3 | -27.6**   | -26.0**  | -24.9**      | -24.6** | -23.4         | -29.5** | -27.3** |
|      | Pct    | 58.1%** | 57.8%**   | 55.8%*  | 76.9%**      | 56.8%** | 79.5%**       | 73.6%** |

Note: “IS” means in-sample (i.e., evaluated on the training data), and “OoS” means out-of-sample (i.e., evaluated on the test data). “AvgLL” refers to the average log-likelihood at the subject level by each approach. “Pct” indicates the corresponding percentage of subjects that QCARE outperforms the benchmark. The remarks “**” and “***” indicate p-values less than 0.01 and 0.05, respectively.

Figure 13  Comparison of estimated \(\alpha\) by individual subjects

It is clear that most green points are positioned to the left of the optimal level, suggesting that, as in previous results, the majority of subjects exhibit over-exploration behavior.

Appendix D  Proofs

Proof of Proposition 1  We note that Gaussian Thompson Sampling draws samples independently from the posterior distribution \(\mathcal{N}(\hat{\mu}_i, \frac{1}{k_i+1})\). The QCARE with \(\alpha = 0.5\) samples quantities
\[ \theta_i(t) = \hat{\mu}_i(t) + \epsilon_{it}/\sqrt{k_i(t)} + 1, \] where \( \epsilon_{it} \sim N(0,1^2) \). By the property of normal distribution, these two policies are equivalent. \( \square \)

**Proof of Proposition 2** Let \( \kappa = (\kappa_1, \ldots, \kappa_N) \) and \( u = (u_1, \ldots, u_N) \) be given. Without loss, suppose \( \hat{i}^* = 1 \), i.e., \( u_1 \geq u_j \) for all \( j \neq 1 \).

(1). Recall that the arm-pulling score is such that \( \theta_i = u_i + \frac{\epsilon_i}{(\kappa_i + 1)^\alpha} \) for every \( i \in [N] \), where \( \epsilon_i \) follows the standard normal distribution independently. Note that

\[
\hat{Q}(\kappa, u) = \Pr(\theta_1 \geq \max\{\theta_2, \ldots, \theta_N\}) = \Pr_{\epsilon}
\left(u_1 + \frac{\epsilon_1}{(\kappa_1 + 1)^\alpha} \geq \max\left\{ u_2 + \frac{\epsilon_2}{(\kappa_2 + 1)^\alpha}, \ldots, u_N + \frac{\epsilon_N}{(\kappa_N + 1)^\alpha} \right\} \right)
\]

and

\[
\hat{Q}(\kappa, u_1 + \delta, u_2, \ldots, u_N) = \Pr_{\epsilon}
\left(u_1 + \frac{\epsilon_1}{(\kappa_1 + 1)^\alpha} + \delta \geq \max\left\{ u_2 + \frac{\epsilon_2}{(\kappa_2 + 1)^\alpha}, \ldots, u_N + \frac{\epsilon_N}{(\kappa_N + 1)^\alpha} \right\} \right).
\]

Since \( u_1 + \frac{\epsilon_1}{(\kappa_1 + 1)^\alpha} + \delta > u_1 + \frac{\epsilon_1}{(\kappa_1 + 1)^\alpha} \) with probability one, the proof is finished by noting the relationship between the following two events: \( E_1 \subseteq E_2 \).

(2). Note that

\[
\hat{Q}(\kappa, u) = \Pr(\theta_1 \geq \max\{\theta_2, \ldots, \theta_N\}) = \Pr_{\epsilon}
\left(\theta_1 \geq \theta_2, \ldots, \theta_1 \geq \theta_N \right)
= \int \Pr_{\epsilon}
\left(\theta_2 \leq x, \ldots, \theta_N \leq x \right) f_{\theta_1}(x) \, dx
= \int F_{\theta_2}(x) \ldots F_{\theta_N}(x) f_{\theta_1}(x) \, dx.
\]

The last equality holds as \( \theta_2, \ldots, \theta_N \) are independent from each other. Similarly, note that

\[
\hat{Q}(\kappa_1, \ldots, \kappa_j + \delta, \ldots, \kappa_N, u_1, \ldots, u_N)
= \Pr_{\epsilon}
\left(u_1 + \frac{\epsilon_1}{(\kappa_1 + 1)^\alpha} \geq \max\left\{ u_2 + \frac{\epsilon_2}{(\kappa_2 + 1)^\alpha}, \ldots, u_j + \frac{\epsilon_j}{(\kappa_j + \delta + 1)^\alpha}, \ldots, u_N + \frac{\epsilon_N}{(\kappa_N + 1)^\alpha} \right\} \right)
= \int F_{\theta_2}(x) \ldots F_{\theta_j}(x) \ldots F_{\theta_N}(x) f_{\theta_1}(x) \, dx,
\]

where \( \theta_j' = u_j + \frac{\epsilon_j}{(\kappa_j + \delta + 1)^\alpha} \). Therefore, we have

\[
\hat{Q}(\kappa_1, \ldots, \kappa_j + \delta, \ldots, \kappa_N, u_1, \ldots, u_N) - \hat{Q}(\kappa, u)
= \int F_{\theta_2}(x) \ldots F_{\theta_j'}(x) \ldots F_{\theta_N}(x) f_{\theta_1}(x) \, dx - \int F_{\theta_2}(x) \ldots F_{\theta_j}(x) \ldots F_{\theta_N}(x) f_{\theta_1}(x) \, dx
= \int F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x) F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x) (F_{\theta_j'}(x) - F_{\theta_j}(x)) f_{\theta_1}(x) \, dx. \quad (B.1)
\]
Next, we will construct two functions to bound the terms $F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x)F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x)$ and $f_{\theta_1}(x)$. We define two new functions $\chi(x)$ and $\phi(x)$ as follows:

$$\chi(x) = \begin{cases} 
F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x)F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x), & \text{if } x \leq u_j, \\
F_{\theta_2}(2u_j - x) \ldots F_{\theta_{j-1}}(2u_j - x)F_{\theta_{j+1}}(2u_j - x) \ldots F_{\theta_N}(2u_j - x), & \text{if } x > u_j,
\end{cases}$$

and

$$\phi(x) = \begin{cases} 
f_{\theta_1}(x), & \text{if } x \leq u_j, \\
f_{\theta_1}(2u_j - x), & \text{if } x > u_j.
\end{cases}$$

It is evident that both $\chi(x)$ and $\phi(x)$ are axially symmetric about line $x = u_j$. As CDFs are non-decreasing, when $x > u_j$, we have $F_{\theta_1}(2u_j - x) < F_{\theta_1}(u_j) < F_{\theta_1}(x)$ for all $i$. Similarly, as $u_i > u_j$, it holds that $f_{\theta_i}(x) \geq f_{\theta_i}(2u_j - x)$ when $x > u_j$. Then, we have $F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x)F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x) \geq \chi(x)$ and $f_{\theta_1}(x) \geq \phi(x)$. Thus, combining Equation (B.1),

$$\hat{Q}(\kappa_1, \ldots, \kappa_j + \delta, \ldots, \kappa_N, u_1, \ldots, u_N) - \hat{Q}(\kappa, u) = \int_{-\infty}^{u_j} F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x)F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x)(F_{\theta_j}(x) - F_{\theta_j}(x))f_{\theta_1}(x)dx$$

$$+ \int_{u_j}^{\infty} F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x)F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x)(F_{\theta_j}(x) - F_{\theta_j}(x))f_{\theta_1}(x)dx$$

$$\geq \int_{-\infty}^{\infty} \chi(x)(F_{\theta_j}(x) - F_{\theta_j}(x))\phi(x)dx + \int_{u_j}^{\infty} \chi(x)(F_{\theta_j}(x) - F_{\theta_j}(x))\phi(x)dx$$

$$\geq 0.$$

The first inequality holds since when $x \leq u_j$, we get $F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x)F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x) = \chi(x)$ and $f_{\theta_1}(x) = \phi(x)$ by definitions. When $x > u_j$, as $F_{\theta_j}(x) - F_{\theta_j}(x) > 0$ and facts that $F_{\theta_2}(x) \ldots F_{\theta_{j-1}}(x)F_{\theta_{j+1}}(x) \ldots F_{\theta_N}(x) \geq \chi(x)$ and $f_{\theta_1}(x) \geq \phi(x)$, the inequality holds. The last inequality holds as $\chi(x)$ and $\phi(x)$ are both axially symmetric about line $x = u_j$. We know that $\chi(x) \geq 0$, $\phi(x) > 0$. Figure 14 plots the CDFs of $\theta_j$ and $\theta_j'$, and the difference term $F_{\theta_j}(x) - F_{\theta_j}(x)$ is highlighted by the color. We can see that the term $F_{\theta_j}(x) - F_{\theta_j}(x)$ is centrally symmetric about point $(u_j, \frac{1}{2})$, and $F_{\theta_j}(x) - F_{\theta_j}(x)$ is negative when $x < u_j$ and positive when $x > u_j$. The blue part and yellow part are axially symmetric about point $(u_j, \frac{1}{2})$. Thus, we have $\hat{Q}(\kappa_1, \ldots, \kappa_j + \delta, \ldots, \kappa_N, u_1, \ldots, u_N) - \hat{Q}(\kappa, u) \geq 0$. 

**Proof of Lemma 1** Fix an arm $i \in [N]$. Let $t_0 := 0$ and for $j \in \mathbb{Z}_+$, let $t_j := \min\{t: k_i(t) \geq j\}$ be the (first) time after the $j^{th}$ pull of the arm $i$. Define the process $\{\tau_j\}_{j \geq 0}$, where $\tau_0 := 0$ and $\tau_j := t_j - t_{j-1}$ for all $t \in \mathbb{Z}_+$. This process represents the inter-pulling times between the $(j - 1)$th and $j$th pull of arm $i$. It suffices to show that $\tau_j < \infty$ almost surely for all $j \in \mathbb{Z}_+$.

Fix an arbitrary $j \in \mathbb{Z}_+$. Through a standard induction argument, assume without loss of generality that $\tau_\ell < +\infty$ almost surely for all $\ell \in \{0, \ldots, j - 1\}$. In other words, $t_{j-1} < +\infty$. We wish
to show that \( t_j < \infty \) almost surely, too. Pick arbitrary \( u \in [0, 1] \) and it suffices to show that \( t_j < \infty \) almost surely conditional on \( \{ \hat{\mu}_i(t_{j-1}) = u \} \).

Between the \((j-1)\)th and \( j \)th pull of arm \( i \), the state value for this arm is kept at \((j-1), u\). Because the \( Q \) function is recurrent, there exists \( \varepsilon > 0 \) such that

\[
\Pr \left( a(t) = i \mid S^i(t) = (j-1, u) \right) \geq \inf_{S^{-i}} Q_i((j-1, u); S^{-i}) \geq \varepsilon > 0.
\]

For all \( t \in \{0, 1, \ldots\} \), let \( E_t := \{ a(t_{j-1} + \ell) \neq i \text{ for all } \ell \leq t \} \) be the event that arm \( i \) keeps being skipped starting from period \( t_{j-1} \) to period \( t_{j-1} + t \). Therefore, \( \Pr (E_t \mid \hat{\mu}_i(t_{j-1}) = u) \leq (1 - \varepsilon)^{t+1} \).

As a consequence, \( \sum_{t=0}^{\infty} \Pr (E_t \mid \hat{\mu}_i(t_{j-1}) = u) < \infty \), and due to the Borel Contelli lemma, the events \( \{ E_t \} \) only happen finitely often. In other words, arm \( i \) eventually gets pulled after finite periods. \( \square \)

**Proof of Theorem 4** Following Lemma 1 and with a recurrent probability function \( Q \), it holds that for every \( i \in [N] \), \( k_i(t) \to \infty \) as \( t \to \infty \) almost surely. Due to the law of large numbers, \( \hat{\mu}_i(t) \to \mu_i \) for all arm \( i \). Assuming that the optimal arm is such that \( i^* = 1 \), we have \( \hat{\mu}_1(t) - \hat{\mu}_i(t) = \Omega(1) \) for every \( i \neq 1 \). Since \( Q \) is leading-arm convergent, it holds that with probability one, \( 1 - Q_1(S(t)) = o(1) \). In other words, for every \( \varepsilon > 0 \), there exists a time \( T(\omega) \) such that \( 1 - Q_1(S(t)) < \varepsilon \) for all \( t > T(\omega) \). For every \( t > 0 \), let \( E_t := \{ \omega : T(\omega) > t \} \) be the “bad” event under which we cannot guarantee \( 1 - Q_1(S(t)) < \varepsilon \).

The probability of the bad event diminishes over time. To see why, note that \( E_t \supseteq E_{t'} \) for \( t \leq t' \) and \( \cap_t E_t = \emptyset \). Therefore, \( \Pr (E_t) \downarrow 0 \) as \( t \to \infty \). With that in mind, the unconditional probability of pulling arm 1 satisfies

\[
\Pr (a(t) \neq 1) = \Pr (a(t) \neq 1 \mid E_t) \Pr (E_t) + \Pr (a(t) \neq 1 \mid E_t^c) \Pr (E_t^c) \leq \Pr (E_t) + \varepsilon \to \varepsilon
\]
as \( t \to \infty \). Since \( \varepsilon \) is arbitrary, we conclude that \( \Pr(a(t) \neq 1) = o(1) \). Therefore, the total number of pulls of suboptimal arms satisfy
\[
\sum_{i=2}^{N} \mathbb{E}[k_i(T + 1)] = \sum_{i=2}^{N} \sum_{t=1}^{T} \Pr(a(t) = i) = \sum_{t=1}^{T} \Pr(a(t) \neq 1) = o(T).
\]
Recall that \( \Delta_i = \mu^* - \mu_i \) is the expected reward gap between arm \( i \) and the optimal arm. We have \( R^\theta(T) = \mathbb{E}\left[\sum_{t=1}^{T} (\mu^* - r_{a(t)})\right] = \mathbb{E}\left[\sum_{t=1}^{T} (\mu^* - \mu_{a(t)})\right] \leq (\max_i \Delta_i) \sum_{i=2}^{N} \mathbb{E}[k_i(T + 1)] = o(T) \).

Finally, let \( \pi \) be any feasible MAB policy and we use \( \mathbb{E}_\pi \) to denote the expectation operator for the system dynamics under this policy. We have
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \left[ \sum_{t=1}^{T} r_{a(t)} \right] - \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \left[ \sum_{t=1}^{T} r_{a(t)} \right] \leq \mu^* - \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \left[ \sum_{t=1}^{T} r_{a(t)} \right] \\
\leq \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \left[ \sum_{t=1}^{T} (\mu^* - r_{a(t)}) \right] = 0.
\]
Therefore, the policy under our \( Q \) function is long-run-average optimal. \( \Box \)

**Proof of Theorem 5** Fix \( N > 1 \) and \( \alpha \in (0, 0.5) \). Suppose the \( Q \) function is \( \alpha \)-exploratory. Pick constants \( \varepsilon > 0, \delta \in (0, 1) \), and a sequence \( \{\Delta_T\} \) satisfying \( \Delta_T = \Omega(T^{-\alpha}) \) as required in Definition 4.

Therefore, for all sufficiently large \( T \geq T_0, u_1 \leq \Delta_T \), and \( \kappa \) such that \( \sum_{i>1} \kappa_i < \delta T \), it holds that
\[
1 - Q_1(\kappa_1, \kappa_2, ..., \kappa_N, u_1, 0, 0, ..., 0) \geq \varepsilon.
\]

We construct a problem instance for every \( T \geq T_0 \), which we denote by \( \Theta_T \). Under this instance, the reward distribution for each arm is deterministic. The rewards of the arms are \( \mu_1 = \min\{1, \Delta_T\} = \Omega(T^{-\alpha}) \) and \( \mu_2 = \mu_3 = \cdots = \mu_N = 0 \). Let \( \delta_2 := (1 + \delta)/2 \) so that \( \delta_2 \in (\delta, 1) \). For every \( t \geq \delta_2 T \), we introduce the event
\[
A_t := \left\{ \sum_{i>1} k_i(t) \leq \delta T \right\}.
\]
Let us discuss the consequences of this event for future reference. First, since \( \sum_{i>1} k_i(t) \leq \delta T \), it holds that \( k_1(t) = (t - 1) - \left[ \sum_{i>1} k_i(t) \right] \geq \delta_2 T - 1 - \delta T = \frac{1-\delta}{2} T - 1 > 0 \) for \( T > 2/(1-\delta) \). In other words, arm 1 has been pulled at least once before period \( t \). Because the rewards are deterministic, we must have \( \hat{\mu}_1(t) = \mu_1 \times (k_1(t)/(k_1(t) + 1) \leq \Delta_T \) and \( \hat{\mu}_2(t) = \cdots = \hat{\mu}_N(t) = 0 \). (Here we use the convention that if an arm is not pulled, the reward state is valued at zero.) Since the probability function \( Q \) is \( \alpha \)-exploratory, it holds that for all \( t \geq \delta_2 T \),
\[
\Pr(a(t) \neq 1|A_t) = 1 - Q_1(k_1(t), ..., k_N(t), \hat{\mu}_1(t), ..., \hat{\mu}_N(t)|k \in A_t) \\
= 1 - Q_1(k_1(t), ..., k_N(t), \hat{\mu}_1(t), 0, ..., 0|k \in A_t) \geq \varepsilon.
\]
Let $c_0 := 1/2$ be an absolute constant. We break our proof into two cases depending on the values of $\{\Pr(A_t)\}$.

**Case 1:** $\Pr(A_t) \geq c_0$ for all $t > \delta_2 T$. In this case,

$$R(T) \geq R^{opt}(T) = \mu_1 \sum_{t=1}^{T} \Pr(\tau(t) \neq 1) \geq \mu_1 \sum_{t=\lceil \delta_2 T \rceil + 1}^{T} \Pr(\tau(t) \neq 1 | A_t) \Pr(A_t)$$

$$= \Omega(T^{-\alpha}) \cdot (1 - \delta_2) T \cdot \varepsilon c_0 = \Omega(T^{1 - \alpha}).$$

**Case 2:** There exists $\ell \in (\delta_2 T, T]$ such that $\Pr(A_{\ell}) < c_0$. In this case, note that $\sum_{t=1}^{\ell} k_i(t)$ is non-decreasing in $t$. As a result, the event $\{A_{\ell}\}$ is non-increasing. (That is, $A_{\ell}$ is false implies $A_{\ell}$ is false for all $\ell \geq t$.) Therefore, $\Pr(A_{T+1}) \leq \Pr(A_{\ell}) < c_0$. In other words, $\Pr(A_{T+1}) \geq 1 - c_0$. Let us evaluate the cumulative regret below. Due to Markov’s inequality,

$$R(T) \geq R^{opt}(T) = \mu_1 \sum_{i=1}^{r} \varepsilon [k_i(T + 1)]$$

$$\geq \mu_1 (\delta T) \Pr(\sum_{i=1}^{r} k_i(T + 1) > \delta T)$$

$$= \mu_1 (\delta T) \Pr(A_{T+1})$$

$$\geq \mu_1 (\delta T) (1 - c_0) = \Omega(T^{-\alpha}) (\delta T) (1 - c_0) = \Omega(T^{1 - \alpha}).$$

That finishes the proof. \(\square\)

**Proof of Theorem 6** Fix $N = 2$ and $\alpha > 0.5$. We construct a problem instance $\Theta$ so that the reward of arm 1 follows the Bernoulli distribution with mean $\mu_1 = 0.9$ and the reward of arm 2 is deterministic with $\mu_2 = 0.7$. We introduce $\Delta := \mu_1 - \mu_2 = 0.2$, which is an absolute constant. Suppose the $Q$ function is $\alpha$-irreversible. Pick a sufficiently small $\delta > 0$ so that $2\alpha - 2\delta > 1$. (The specific reason for this criterion will be explained later.) Let $c := 2(\log T)^{\frac{\alpha}{\alpha - 2\delta}}$ for shorthand notation. Let $t_0 := 0$ and for $j \in \mathbb{Z}_+$, $t_j := \min\{t : k_1(t) \geq j\}$ be the (first) time after the $j$th pull of the arm 1. Define the process $\{\tau_j\}_{j \geq 0}$, where $\tau_0 := 0$ and $\tau_j := t_j - t_{j-1} - 1$ for all $t \in \mathbb{Z}_+$. This process represents the number of arm pulls between the $(j-1)$th and $j$th pull of arm 1 (excluding pulling of arm 1 itself). We also introduce two events:

$$E_1 := \{k_2(t_c) > 0.9c\} \quad \text{and} \quad E_2 := \{r_1(t_1 - 1) = \cdots = r_1(t_c - 1) = 0\}.$$ 

The first event means that by the time arm 1 is pulled $c$ times, arm 2 has been pulled at least $0.9c$ times. The second event means that the first $c$ times of pulling arm 1 all lead to zero rewards. We also consider the event $E := E_1 \cap E_2$. We break the rest of the proof into three steps.

**Step 1:** We show that $\Pr(E) = \Omega(T^{-\alpha(1)})$. 

\(\square\)
The reward of arm 1, conditional on it being pulled, follows the Bernoulli distribution with mean \( \mu_1 \), and is independent of the historical path. Therefore,
\[
\Pr(E_2) = (1 - \mu_1)^c = \exp\{2 \log(1 - \mu_1)(\log T)^{1/20}\} = \Omega(T^{-o(1)})
\]
In the derivation, the last inequality is because \( \frac{1}{20}\alpha - \alpha < 1 \) (recall how \( \delta \) was picked initially).

Next, let us evaluate \( \Pr(E_1|E_2) \). For every \( i = 1, \ldots, c \) and \( k \geq 1 \), the event \( \{a(t_{i-1}) \neq 1, a(t_{i-1} + 1) \neq 1, \ldots, \text{ and } a(t_{i-1} + k - 1) \neq 1\} \) implies that \( \{\tau_i \geq k\} \). Therefore,
\[
\Pr(\tau_i \geq k|E_2) \geq \Pr(\{a(t_{i-1}) \neq 1, a(t_{i-1} + 1) \neq 1, \ldots, \text{ and } a(t_{i-1} + k - 1) \neq 1\}|E_2)
\]
\[
= \prod_{i=0}^{k-1} \Pr(a(t_{i-1} + \ell) \neq 1|E_2 \cap \{a(t_{i-1}) \neq 1, \ldots, \text{ and } a(t_{i-1} + \ell - 1) \neq 1\})
\]
\[
\geq (1/2)^k.
\]

To see why the last inequality holds, note that for every \( \ell \in \{0, \ldots, k-1\} \), the event \( \{E_2 \cap \{a(t_{i-1}) \neq 1, \ldots, \text{ and } a(t_{i-1} + \ell - 1) \neq 1\}\} \) implies that \( \hat{\mu}_1(a(t_i + \ell)) = 0 \). That is, arm 1 has not achieved any successes up to period \( t_{i-1} + \ell - 1 \). Because the probability function \( Q \) is unradical, it holds that the probability of not pulling arm 1 is at least \( 1/2 \).

Let \( \{X_i\}_{i=1}^c \) be an i.i.d. sequence of geometrically distributed random variables, each representing the number of failures before the first success for a sequence of Bernoulli trials with a success rate of \( 1/2 \). Based on the analysis above, \( \tau_i \) stochastically dominates \( X_i \) conditional event \( E_2 \). That is, \( \Pr(\tau_i \geq k|E_2) \geq (1 - 1/2)^k = \Pr(X_i \geq k) \) for all \( k \geq 0 \). Let \( X := \sum_{j=1}^c X_i \), which follows negative binomial distribution with mean \( \frac{1 - 1/2}{1/2}c = c \) and variance \( \frac{(1 - 1/2)}{(1/2)^2}c = 2c \). Then it holds that
\[
\Pr(E_1|E_2) = \Pr(k_2(t_c) > 0.9c | E_2)
\]
\[
= \Pr\left(\sum_{i=1}^c \tau_i > 0.9c | E_2\right)
\]
\[
\geq \Pr(X > 0.9c)
\]
\[
= 1 - \Pr(X \leq 0.9c)
\]
\[
\geq 1 - \Pr(|X - c| \geq 0.1c)
\]
\[
\geq 1 - \frac{2c}{(0.1c)^2} = 1 - \frac{200}{c} = \Omega(1),
\]
where the last inequality comes from Chebyshev’s inequality. To summarize,
\[
\Pr(E) = \Pr(E_1|E_2) \cdot \Pr(E_2) \geq (1 - \mu_1)^c \left(1 - \frac{200}{c}\right) = \Omega(T^{-o(1)}).
\]

Step 2: We show that \( \mathbb{E}[k_2(T + 1)|E] = \Omega(T) \).
Let $M = 0.1c = 0.2(\log T)^{\frac{1}{2a-2\delta}}$. Due to Markov’s inequality,

$$\mathbb{E}[k_2(T + 1) | E] \geq \Pr (k_2(T + 1) > T - c - M | E) \cdot (T - c - M).$$  \hfill (B.2)

In the meanwhile, it holds that

$$\Pr (k_2(T + 1) > T - c - M | E)$$
$$= \Pr (k_1(T + 1) \leq c + M | E)$$
$$= \Pr (t_{c+M} \geq T + 1 | E)$$
$$\geq \Pr (t_{c+M} - t_c \geq T + 1 | E)$$
$$= \Pr \left( \sum_{t=1}^{M} \tau_{c+i} \geq T - M + 1 | E \right)$$
$$\geq \Pr \left( \sum_{t=1}^{M} \tau_{c+i} \geq T | E \right).$$  \hfill (B.3)

To evaluate the last term, pick an arbitrary $i \in [M]$ and $t \geq t_{c+i-1}$. The event $E \cap \{a(t_{c+i-1}) \neq 1, \ldots, a(t-1) \neq 1\}$ leads to a few facts. First, arm 1 has been pulled exactly $c + i - 1$ times, the first $c$ of which all produce zero rewards. Therefore, $k_1(t) = c + i - 1 \geq c = 2(\log T)^{\frac{1}{2a-2\delta}} > (\log T)^{\frac{1}{2a-2\delta}}$ and

$$\hat{\mu}_1(t) \leq \frac{i}{c+i} \leq \frac{M}{c+M} = \frac{0.1}{1+0.1} < 0.1.$$

Second, $\hat{\mu}_2(t) = \mu_2 = 0.7$ since arm 2’s rewards are deterministic. Therefore, $\hat{\mu}_2(t) - \hat{\mu}_1(t) > 0.5$, an absolute constant. Finally, $k_2(t) \geq k_2(t_c) > 0.9c > (\log T)^{\frac{1}{2a-2\delta}}$. Since $Q$ is $\alpha$-irreversible, for sufficiently large $T$,

$$\Pr (a(t) = 1 | E) = Q_1(k_1(t), k_2(t), \hat{\mu}_1(t), \hat{\mu}_2(t)) \leq \frac{(\log T)^{\frac{1}{2a-2\delta}}}{T} = \frac{c}{2T(\log T)^{\frac{1}{2a-2\delta}}} = g,$$

where $\delta' = \frac{1}{2a-2\delta} - \frac{1}{2a-\delta} > 0$ is a strictly positive small constant. Let $\{Y_i\}_{i=1}^{M}$ be an i.i.d. sequence of geometrically distributed random variables, each representing the number of failures before the first success for a sequence of Bernoulli trials with a success rate of $g$. Let $Y := \sum_{i=1}^{M} Y_i$, which follows negative binomial distribution with mean $\frac{1-g}{g}M < \frac{M}{g} = (0.1c)^{2T(\log T)^{\frac{1}{2a-2\delta}}} = 0.2T(\log T)^{\delta'}$ and variance $\frac{1-g^2}{g^2}M < \frac{M}{g^2} = (0.1c)^{4T^2(\log T)^{2\delta'}} = 0.4T^2(\log T)^{2\delta'}$. Following the same argument as in Step 1, $\tau_{c+i}$ stochastically dominates $Y_i$. Therefore, for sufficiently large $T$,

$$\Pr \left( \sum_{i=1}^{M} \tau_{c+i} \geq T | E \right)$$
$$\geq \Pr (Y \geq T)$$
$$= 1 - \Pr (Y < T)$$
\[ \geq 1 - \Pr \left( |Y - \mathbb{E}[Y]| > 0.2T(\log T)^{\delta'} - T \right) \]
\[ \geq 1 - \Pr \left( |Y - \mathbb{E}[Y]| > 0.1T(\log T)^{\delta'} \right) \]
\[ \geq 1 - \frac{0.4T^2(\log T)^{2\delta'}}{c} \frac{1}{0.01T^2(\log T)^{2\delta'}} \]
\[ = 1 - \frac{40}{c} = 1 - \frac{20}{(\log T)^{2\alpha}}, \quad (B.4) \]
where parts (a) holds when \(0.1(\log T)^{\delta'} > 1\). Combining \([B.2]\) and \([B.4]\), we have
\[
\mathbb{E}[k_2(T+1)|E] \geq \left( 1 - \frac{20}{(\log T)^{2\alpha}} \right) (T - c - M) = \Omega(T).
\]

Step 3: We combine everything and evaluate the regret \(R(T)\). Invoking the results from Steps 1 and 2, it holds that for every \(\varepsilon > 0\),
\[ R(T) \geq R^\Theta(T) \geq \Delta \cdot \mathbb{E}[k_2(T+1) | E] \Pr(E) = \Delta \cdot \Omega(T) \cdot \Omega(T^{-o(1)}) = \Omega(T^{1-o(1)}). \]
That finishes the proof. \(\square\)

**D.1 Proof of Proposition 3**

Let \(F_\epsilon(\cdot)\) be the cumulative distribution function for the noise term \(\epsilon\) in QCARE, which follows a standard normal distribution. We first state the following properties of \(F_\epsilon(\cdot)\) to be used in later proofs.

**Proposition 4.** (Abramowitz and Stegun 1965) The following facts about \(F_\epsilon(\cdot)\) are true.

1. \(F_\epsilon(x) \leq 1 - \frac{1}{8\sqrt{\pi}} \exp(-x^2/2)\) for any \(x \geq 0\);
2. \(F_\epsilon(x) \geq 1 - \frac{1}{8\sqrt{\pi}} \exp(-x^2/2)\) for all \(x \geq 1\);
3. \(F_\epsilon(0) = 1/2\).

In the statement above, Property (1) corresponds to an anti-concentration property while Property (2) can be interpreted as a concentration property. Property (3) is self-evident. We will show how these work in the following analysis.

Since the anonymity of QCARE is straightforward, we will next verify all other conditions of QCARE.

**Lemma 2 (QCARE is recurrent).** Let \(N > 1\) and consider the QCARE policy with \(\alpha > 0\). For every arm \(i \in [N]\) and state \(S = (\kappa,u)\), it holds that
\[ Q_i(S^i;S^{-i}) > \frac{1}{8\sqrt{\pi}} \left( \frac{1}{2} \right)^{N-1} e^{-14(\kappa_i+1)^2\alpha}. \]
Since the term on the right-hand side is independent of \(S^{-i}\), the probability function \(Q\) under QCARE is recurrent.
Proof of Lemma 2. Let an QCARE policy with coefficient $\alpha > 0$ be given. Fix a state $S = (\kappa, u)$. Let $\beta_i = 1/(\kappa_i + 1)^\alpha$ and $\theta_i = u_i + \epsilon_i \beta_i$ for every $i \in [N]$, where $\{\epsilon_i\}$ follow independent standard normal distribution. According to the definition of QCARE,

$$Q_i(S) = \Pr(\theta_i \geq \theta_j, \forall j \neq i)$$

\begin{align*}
= & \Pr(u_i + \epsilon_i \beta_i \geq u_j + \epsilon_j \beta_i, \forall j \neq i) \\
\geq & \Pr(\epsilon_i \beta_i - \epsilon_j \beta_j \geq 1, \forall j \neq i) \\
\geq & \Pr(\epsilon_i \beta_i \geq 2 \text{ and } \epsilon_j \beta_j \leq 1, \forall j \neq i) \\
\geq & \Pr(\epsilon_i \beta_i \geq 2) \left( \Pr(\epsilon_2 \beta_2 \leq 1) \right)^{N-1} \\
= & \Pr(\epsilon_i > 2(\kappa_i + 1)^\alpha) \left( \Pr(\epsilon_2 \leq (\kappa_2 + 1)^\alpha) \right)^{N-1} \\
\geq & \frac{1}{8\sqrt{\pi}} e^{-14(\kappa_i+1)^{2\alpha}} \cdot \left( \frac{1}{2} \right)^{N-1} = \frac{1}{8\sqrt{\pi}} \left( \frac{1}{2} \right)^{N-1} e^{-14(\kappa_i+1)^{2\alpha}}.
\end{align*}

In the derivations above, part (a) is because $u_\ell \in [0,1]$ for every $\ell \in [N]$. Part (b) is because $\{\epsilon_i\}$ are independent and identically distributed. Part (c) is because of two reasons. First, $\Pr(\epsilon_i > 2(\kappa_i + 1)^\alpha) > \frac{1}{8\sqrt{\pi}} e^{-14(\kappa_i+1)^{2\alpha}}$ by Proposition 4.(1). Second, $\kappa_2 + 1 > 0$, and therefore $\Pr(\epsilon_2 \leq (\kappa_2 + 1)^\alpha) > F_\epsilon(0) = 1/2$ by Proposition 4.(3).

**Lemma 3 (QCARE is leading-arm convergent).** Let $N > 1$ and consider the QCARE policy with $\alpha > 0$. For every sequence $(\kappa, u) = (\kappa(T), u(T))$ satisfying $\kappa_i = \omega(1)$ for every arm $i \in [N]$ and $u_1 - u_i = \Omega(1)$ for every arm $i \geq 2$, it holds that $1 - Q_1((\kappa, u)) = o(1)$. In other words, the function $Q$ under QCARE is leading-arm convergent.

Proof of Lemma 3. Pick an arbitrary sequence $(\kappa, u) = (\kappa(T), u(T))$ such that $\kappa_i = \omega(1)$ for every arm $i \in [N]$ and $u_1 - u_i = \Omega(1)$ for every arm $i \geq 2$. Following earlier conventions, let $\beta_i = 1/(\kappa_i + 1)^\alpha$ and $\theta_i = u_i + \epsilon_i \beta_i$ for every $i \in [N]$, where $\{\epsilon_i\}$ follow independent standard normal distribution. According to the definition of QCARE,

$$1 - Q_1(S) = 1 - \Pr(\theta_1 \geq \theta_i, \forall i > 1)$$

\begin{align*}
= & 1 - \Pr(u_1 + \epsilon_1 \beta_1 \geq u_i + \epsilon_i \beta_i, \forall i > 1) \\
= & \Pr(u_1 + \epsilon_1 \beta_1 \leq u_i + \epsilon_i \beta_i, \text{ for some } i > 1) \\
\leq & \sum_{i=2}^{N} \Pr(u_1 + \epsilon_1 \beta_1 \leq u_i + \epsilon_i \beta_i) \\
= & \sum_{i=2}^{N} \Pr(\epsilon_1 \beta_1 - \epsilon_i \beta_i \leq u_i - u_1) \\
= & \sum_{i=2}^{N} \Pr(\epsilon_1/(\kappa_1 + 1)^\alpha - \epsilon_i/(\kappa_i + 1)^\alpha \leq u_i - u_1).
\end{align*}
Note that for every $i > 1$, $\epsilon_i/\alpha - \epsilon_i/\alpha$ follows a normal distribution with zero mean and standard deviation on the order of $o(1)$. In the meanwhile, $u_1 - u_i = \Omega(1)$. Therefore, $\Pr(\epsilon_i/\alpha - \epsilon_i/\alpha < u_i - u_1) = o(1)$, which finishes the proof.

**Lemma 4 (QCARE is $\alpha$-exploratory for $\alpha < 0.5$).** Pick $N > 1$ and consider the QCARE policy with $\alpha \in (0, 0.5)$. Let $\delta := 0.1$ and $\Delta_T := T^{-\alpha}$. Then for all $T \geq 2$, $u_1 \leq \Delta_T$, and $\kappa$ such that $\sum_{i > 1} \kappa_i < \delta T$, it holds that

$$1 - Q_1(\kappa_1, \kappa_2, ..., \kappa_N, u_1, 0, 0, ..., 0) \geq \frac{1}{16\sqrt{\pi}} \exp(-3.5 \times 0.9^{2\alpha}) > 0.$$  

In other words, the probability function $Q$ under QCARE is $\alpha$-exploratory.

**Proof of Lemma 4** Pick a $T \geq 2$ so that $\delta T = 0.1T < 0.9T - 1$. Pick arbitrary $u_1, \kappa$ such that $u_1 \leq \Delta_T = T^{-\alpha}$ and $\sum_{i > 1} \kappa_i < \delta T < 0.9T - 1$. Finally, let $u_2 = \cdots = u_N = 0$. Following earlier conventions, let $\beta_i = 1/(\kappa_i + 1)^\alpha$ and $\theta_i = u_i + \epsilon_i \beta_i$ for every $i \in [N]$, where $\{\epsilon_i\}$ follow independent standard normal distribution. According to the definition of QCARE,

$$1 - Q_1(S) = 1 - \Pr(\theta_i \geq \theta_i, \forall i > 1)
= (a) 1 - \Pr(\Delta_T + \epsilon_1 \beta_1 \geq \epsilon_i \beta_i, \forall i > 1)
\geq \Pr(\Delta_T + \epsilon_1 \beta_1 \leq \epsilon_i \beta_i, \text{ for some } i > 1)
\geq (b) \Pr(\epsilon_1 \beta_1 \leq 0) \Pr(\Delta_T \leq \epsilon_i \beta_i, \text{ for some } i > 1)
\geq \Pr(\epsilon_1 \beta_1 \leq 0) \Pr(\Delta_T \leq \epsilon_2 \beta_2)
\geq \frac{1}{2} \left[ 1 - \Pr(\Delta_T (\kappa_2 + 1)^\alpha \geq \epsilon_2) \right]
\geq (c) \frac{1}{2} \cdot \frac{1}{8\sqrt{\pi}} \cdot \exp(-7\Delta_T^2 (\kappa_2 + 1)^{2\alpha}/2)
\geq (d) \frac{1}{2} \cdot \frac{1}{8\sqrt{\pi}} \cdot \exp(-7\Delta_T^2 (0.9T)^{2\alpha}/2)
= \frac{1}{16\sqrt{\pi}} \exp(-3.5 \times 0.9^{2\alpha}).$$

In the derivations above, part (a) is because $u_1 \leq \Delta_T$ and $u_2 = \cdots = u_N = 0$. Part (b) is because $\{\epsilon_i\}$ are independent and identically distributed. Part (c) is due to Proposition 4(1), which states that $F_\epsilon(x) \leq 1 - \frac{1}{8\sqrt{\pi}} \cdot e^{-7x^2/2}$ for all $x > 0$. Finally, part (d) is because $\kappa_2 \leq \sum_{i > 1} \kappa_i < \delta T < 0.9T - 1$. □

**Lemma 5 (QCARE is $\alpha$-irreversible for $\alpha > 0.5$).** Pick $N > 1$ and consider the QCARE policy with $\alpha > 0.5$. For any following list of quantities: (i) arbitrarily small constant $\delta > 0$, (ii) arm $j > 1$, (iii) $u \in [0, 1]^N$ satisfying $u_j - u_1 = \Omega(1)$, and (iv) $\kappa \in \mathbb{Z}_+^N$ satisfying $\kappa_1, \kappa_j \geq (\log T)^{1 - \alpha}$ for sufficiently large $T$, it holds that

$$Q_1(\kappa, u) \leq \frac{(\log T)^{1 - \alpha}}{T^\alpha}.$$
Therefore, the probability function $Q$ under QCARE is $\alpha$-irreversible.

**Proof of Lemma 5.** Pick arm $j > 1$, arbitrarily small $\delta > 0$, $u$ such that $u_j - u_1 \geq \zeta$ for some constant $\zeta > 0$, and $\kappa$ satisfying $\kappa_i, \kappa_j \geq (\log T)^{-2/(2\alpha-3)}$ for sufficiently large $T$. Following earlier conventions, let $\beta_i = 1/(\kappa_i + 1)^\alpha$ and $\theta_i = u_i + \epsilon_i \beta_i$ for every $i \in [N]$, where $\{\epsilon_i\}$ follow independent standard normal distribution. According to the definition of QCARE, $Q_1(S) = \Pr(\theta_1 \geq \theta_i, \forall i > 1)$

$$= \Pr(u_1 + \epsilon_1 \beta_1 \geq u_i + \epsilon_i \beta_i, \forall i > 1)$$

$$\leq \Pr(u_1 + \epsilon_1 \beta_1 \geq u_j + \epsilon_j \beta_j)$$

$$= \Pr\left(\frac{\epsilon_1}{(\kappa_1 + 1)^\alpha} - \frac{\epsilon_j}{(\kappa_j + 1)^\alpha} \geq u_j - u_1\right)$$

$$(a) \leq \Pr\left(\frac{\epsilon_1}{(\log T)^{\alpha/(2\alpha-3)}} - \frac{\epsilon_j}{(\log T)^{\alpha/(2\alpha-3)}} \geq u_j - u_1\right)$$

$$\leq \Pr\left(\frac{\epsilon_1}{(\log T)^{\alpha/(2\alpha-3)}} - \frac{\epsilon_j}{(\log T)^{\alpha/(2\alpha-3)}} \geq \zeta\right)$$

$$= \Pr\left(\epsilon_1 - \epsilon_j \geq \zeta \left(\frac{\log T}{\alpha/(2\alpha-3)}\right)\right)$$

$$(b) \leq \Pr\left(\zeta \left(\frac{\log T}{\alpha/(2\alpha-3)}\right) / \sqrt{2}\right)$$

$$(c) \leq \frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\zeta \left(\frac{\log T}{\alpha/(2\alpha-3)}\right) / \sqrt{2}\right)^2 / 2\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\zeta^2 \left(\frac{\log T}{\alpha/(2\alpha-3)}\right)^2 / 4\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{\log T}{\alpha/(2\alpha-3)}\right)^2 / 4\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{T^{-\zeta^2/(2\alpha-3)/4}}{\left(\frac{\log T}{\alpha/(2\alpha-3)}\right)^{1/2}}$$

for sufficiently large $T$. In the derivations above, part (a) is because both $\frac{\epsilon_1}{(\kappa_1 + 1)^\alpha} - \frac{\epsilon_j}{(\kappa_j + 1)^\alpha}$ and $\frac{\epsilon_1}{(\log T)^{\alpha/(2\alpha-3)}} - \frac{\epsilon_j}{(\log T)^{\alpha/(2\alpha-3)}}$ are normally distributed random variables with mean zero, but the later has a larger variance. It fact, it holds that $\kappa_i + 1 > \kappa_j \geq (\log T)^{-2/(2\alpha-3)}$ for both $i \in \{1, 2\}$. Part (b) is because $\epsilon_1 - \epsilon_2$ follows a normal distribution with standard deviation $\sqrt{2}$, and $\xi$ is introduced to represent a standard normal distribution. Part (c) follows from the standard tail bounds of the standard normal distribution. In fact, by Proposition[(2), it follows that for all $x \geq 1$,

$$\Pr(\xi \geq x) \leq \frac{e^{-x^2/2}}{x \sqrt{2\pi}} \leq \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

That finishes the proof.

**Lemma 6 (QCARE is nonradical).** Let $N > 1$ and consider the QCARE policy with $\alpha > 0$. For every $u \in [0, 1]^N$ such that $u_1 \leq \min\{u_2, \ldots, u_N\}$ and $\kappa \in \mathbb{Z}_N^N$, $Q_1(\kappa, u) \leq \frac{1}{2}$. In other words, the probability function $Q$ under QCARE is nonradical.
Proof of Lemma 6. Pick $u \in [0,1]^N$ such that $u_1 \leq \min\{u_2,\ldots,u_N\}$ and $\kappa \in \mathbb{Z}_+^N$. Following earlier conventions, let $\beta_i = 1/(\kappa_i + 1)^{\alpha}$ and $\theta_i = u_i + \epsilon_i \beta_i$ for every $i \in [N]$, where $\{\epsilon_i\}$ follow independent standard normal distribution. According to the definition of QCARE,

$$Q_1(S) = \Pr(\theta_1 \geq \theta_i, \forall i > 1) = \Pr(u_1 + \epsilon_1 \beta_1 \geq u_i + \epsilon_i \beta_i, \forall i > 1) \leq \Pr(u_1 + \epsilon_1 \beta_1 \geq u_2 + \epsilon_2 \beta_2) \leq \Pr(\epsilon_1 \beta_1 - \epsilon_2 \beta_2 \geq u_2 - u_1) \leq \Pr(\epsilon_1 \beta_1 - \epsilon_2 \beta_2 \geq 0) = \frac{1}{2},$$

where the last inequality is because $\{\epsilon_i\}$ are independent and identically distributed with standard normal distribution, combined with Proposition 4.(3). $\square$