Spontaneous Interlayer Exciton Coherence in Quantum Hall Bilayers at $\nu = 1$ and $\nu = 2$: A Tutorial

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Abstract

This tutorial paper reviews some of the physics of quantum Hall bilayers with a focus on the case where there is low or zero tunnelling between the two layers. We describe the interlayer coherent states at filling factors $\nu = 1$ and $\nu = 2$ as exciton condensates and discuss some of the theory associated with these states.

Key words:

PACS:

Over the last decade, a number of extremely important advances — both experimental[1,2,3,4,5] and theoretical[1,6,7,8,9,10,11,12,13,14,15,16,17,18] — have occurred in the field of quantum Hall bilayer physics. In this paper, I hope to convey a small subset of the interesting and important theoretical ideas of this field, deferring many other fascinating aspects, as well as any detailed discussion the experiment, to other reviews. The main focus of the paper will be on the case where the tunnelling between the two layers is vanishingly small and “coherence” between the two layers occurs due to interactions alone. We will also em-
phasize how this physics can be related to the concept of an exciton condensate.

1. Physics at $\nu = 1$

1.1. The Double Well: Let us start by imagining a single spinless electron in a double well. We consider two basis states, $|L\rangle = c^\dagger_L |0\rangle$ and $|R\rangle = c^\dagger_R |0\rangle$ corresponding to the electron in the left and right wells respectively. The Hilbert space for a single electron is just the space of superpositions

$$u |L\rangle + v |R\rangle = \left( uc^\dagger_L + vc^\dagger_R \right) |0\rangle$$

with the normalization $|u|^2 + |v|^2 = 1$. We can also think of the two-vector $(u, v)$ as an “isospin” spinor with the mapping that the $|L\rangle$ state corresponds to isospin-up and the $|R\rangle$ state corresponds to isospin-down

$$\begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} V_\Delta & -t \\ -t & -V_\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix} \tag{1}$$

where we will assume $t \geq 0$ and real. For “balanced” layers ($V_\Delta = 0$), the eigenstates are the symmetric state $u = v$ with energy $E_- = -t$ and the antisymmetric state $u = -v$ with energy $E_+ = t$. More generally, with $V_\Delta \neq 0$ we will have

$$E_\pm = \pm \sqrt{V_\Delta^2 + t^2} \tag{2}$$

with eigenstates such that $|u| \neq |v|$. We will abuse nomenclature and always refer to the the lower energy state, which has $v/u$ negative and real, as being the “symmetric” state and the higher energy state, which has $v/u$ negative and real, as being the “antisymmetric” state. In the absence of tunnelling, the energy of any state is always independent of the complex phase $\phi$ of $v/u$. However, when tunnelling is added, the energy is lowest when $v/u$ is positive real, and any attempt to change the complex phase of $v/u$ will incur an energy cost.

1.2. Introduction to $\nu = 1$ Bilayers: We now turn our attention to quantum Hall bilayers. The double well we were considering above now becomes extended into two parallel 2-dimensional electron sheets, as shown in Fig. 2. When a magnetic field $B$ is applied perpendicular to the layers, the non-interacting electron states become quantized into Landau levels with energies $(n + \frac{1}{2}) e\omega_c$, where $\omega_c = eB/mc$ with each Landau level having $B/\phi_0$ states per unit area (here $m$ is the electron mass, $c$ the speed of light, $e$ the electron charge, and $\phi_0 = 2\pi\hbar/e$ the flux quantum with $\hbar$ Planck’s constant). At high enough magnetic field, low energy states are restricted to the Lowest Landau level ($n = 0$), and we specify the 2D positional degree of freedom within the lowest Landau level (LLL) by a single variable $X$. At each possible value of $X$ there are 2 states available for the electron, one in each of the wells, with tunnelling between the wells analogous to that discussed above. (For now we neglect the actual spin of the electron, assuming that the spin is fully polarized, which is a good assumption in high enough magnetic field). In addition to the simple physics of an electron in a double well at each position $X$, now there will also be Coulomb interactions between electrons at...
different positions $X$. At filling $^2 \nu = 1$ there is precisely one electron in the system for each possible value of $X$. If the tunnelling between the layers is sufficiently strong, so that the energy of the symmetric state is much lower than that of the antisymmetric state, then at each position $X$ we should fill only the symmetric state and we should leave the antisymmetric state empty. The multiparticle wavefunction can then be written as

$$\Psi = \prod_X \left( uc^\dagger_{LX} + vc^\dagger_{RX} \right) |0\rangle \quad (3)$$

where $(u, v)$ are the coefficients of the symmetric eigenstate of the double well which must have $v/u$ real and positive. (As discussed above, if there is no bias between the two wells, we would have $u = v$ in the symmetric state). The resulting state (Eq. 3) is simply a filled Landau level of electrons in the symmetric superposition. This wavefunction is certainly the exact ground state in the limit where $t$ is large such that the symmetric state is much lower energy than the antisymmetric state. Interestingly, as we will discuss further below, it is thought that this state is also exact in certain circumstances for small, or even vanishing $t$.

1.3. The $\nu = 1$ Quantum Hall Ferromagnet: It is extremely interesting to examine the limit where the tunnelling $t$ between the layers becomes small. As discussed above for the case of the double quantum well, in this limit the energy of any superposition state is independent of the complex phase $\phi$ of $v/u$. Similarly, when $t \to 0$, the energy of Eq. 3 becomes independent of this phase. However, it should not be immediately obvious that Eq. 3 is the ground state in this limit. If we recall our argument that we should fill the symmetric state, but leave the antisymmetric state empty, we realize that this is no longer a valid argument in the limit of $t \to 0$ and $V_\Delta \to 0$ where the symmetric and antisymmetric states both have the same energy. Despite this degeneracy the electron-electron interaction can strongly favor all of the electrons being in the same superposition, in which case Eq. 3 remains a valid wavefunction, and indeed can be essentially exact in the limit of small spacing between the two layers $^3$.

We now think more generally in the case where the interlayer bias $V_\Delta$ is not zero. For a single electron in a double quantum well with $t \to 0$ and nonzero $V_\Delta$, the ground state will consist of the electron completely in the well of lower energy. (What we call the “symmetric” state — the eigenstate with lower energy — in the limit of $t = 0$ is just an electron in the lower well). However, when one considers a system of many electrons, due to electron-electron interaction, the self-consistent ground state will have some density of electrons in each of the wells. Indeed, the ground state will again be of the form of Eq. 3 where the relative magnitude of $|u|^2$ and $|v|^2$ indicates the relative densities in the left and right wells respectively. Furthermore, in the limit of $t = 0$ the energy of Eq. 3 must remain independent $^4$ of the phase $\phi$ of $v/u$ so long as this phase is chosen the same at all points $X$ in space.

As mentioned above, the two vector $(u, v)$ with $|u|^2 + |v|^2 = 1$ can be thought of a spin-$\frac{1}{2}$ isospinor. In the language of spin $^5$, the $z$-component $\langle S_z \rangle \propto |u|^2 - |v|^2$ represents the density imbalance between the layers. This quantity is fixed by the interlayer bias $V_\Delta$ and the interaction between electrons. Variation of $\langle S_z \rangle$ from its preferred value will cost “capacitive” energy. On the other hand, the azimuthal angle of the spin is given by the phase $\phi$ of $v/u$. As discussed above, when $t = 0$, no particular value of this phase is preferred, so long as the phase (or equivalently the iso-spin direction) is chosen the same at all points in space. This alignment of the iso-spin vector at all points in space gives the wavefunction Eq. 3 the name “quantum Hall ferromagnet.”

$^3$ Changing the layer spacing, of course, changes the interaction between the layers, and hence can change the ground state

$^4$ When $t = 0$, the interacting Hamiltonian has a precise gauge symmetry which can be expressed as $c^\dagger_{RX} \to e^{i\phi} c^\dagger_{RX}$ with the same $\phi$ at all points in space. This guarantees the independence of the energy of the wavefunction Eq. 3 as the phase $\phi$ is varied changed.

$^5$ The expectation of the components of the spin are given by $\langle S_i \rangle = (u^*, v^*) \sigma_i (u, v)$ with $\sigma_i$ the Pauli spin matrices and $i = x, y, z$. 

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$^2$ The filling fraction $\nu$ is defined here to be the total number of electrons divided by the number of states per layer, which is given by $\nu = n0e/B$ with $n$ the density

$^3$ Changing the layer spacing, of course, changes the interaction between the layers, and hence can change the ground state

$^4$ When $t = 0$, the interacting Hamiltonian has a precise gauge symmetry which can be expressed as $c^\dagger_{RX} \to e^{i\phi} c^\dagger_{RX}$ with the same $\phi$ at all points in space. This guarantees the independence of the energy of the wavefunction Eq. 3 as the phase $\phi$ is varied changed.

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romagnet.” When the system chooses a particular value of \( \phi \) among all of the physically equivalent possibilities we say there has been a spontaneous breaking of symmetry.

In this limit of no tunnelling between the two layers, there is a thus a one parameter family of physically equivalent ground states parameterized by the phase \( \phi \). This phase is a quantum mechanical variable that is conjugate to the difference in the number of particles between the two layers. In Eq. 3 the phase \( \phi \) of \( u/v \) is well defined, but the number of particles in each layer is maximally uncertain\(^6\). Conversely, by making the phase completely uncertain, we can construct a wavefunction that precisely specifies the number of particles in each layer. To do so, we integrate over the phase freedom

\[
\Psi_{NR} = \int \frac{d\phi}{2\pi} e^{-iNR\phi} \prod_X (uc^\dagger_{lx} + ve^{i\phi}c^\dagger_{rx}) |0\rangle (4)
\]

and obtain a wavefunction with precisely \( NR \) electrons on the right side and \( NL = N - NR \) on the left with \( N \) the total number of electrons, which at \( \nu = 1 \) is equal to the number of spatial orbitals.

As in all cases, when a spontaneously broken symmetry (here \( \phi \)) can vary as a function of position, there must be a Goldstone mode corresponding to long wavelength, low energy variations of the broken variable — analogous to the spin wave of a ferromagnet. We can study this possibility by proposing a wavefunction that allows the phase to vary locally

\[
\Psi(t) = \prod_X (uc^\dagger_{lx} + ve^{i\phi(x,t)}c^\dagger_{rx}) |0\rangle (5)
\]

and without loss of generality, we can now take \( u \) and \( v \) both real and positive. As we will argue below, \( \phi \) is a superfluid phase for this system, where the superfluid mode corresponds to equal and opposite currents propagating in the two opposite layers.

The local order parameter for this symmetry broken state is

\[
\psi_X = \langle c^\dagger_{lx} c_{rx} \rangle = uv e^{i\phi(X)} (6)
\]

which is the expectation of an operator that takes a particle out of the right layer and puts it in the left layer. It is quite interesting that the expectation of this operator can be large (order unity) even for infinitesimally small tunnelling between the two layers so long as the ground state is still described by Eq. 3.

Further, we can write an expression for the tunnelling current operator between the two layers as a function of position

\[
jl_{lx} = -it(c^\dagger_{rx}c_{lx} - c^\dagger_{lx}c_{rx}) (7)
\]

whose expectation is given by

\[
\langle jl_{lx} \rangle = -it(\psi^*_x - \psi_x) = 2tuv \sin[\phi(X)] (8)
\]

which appears to be similar to a Josephson current — depending on the superfluid phase, and not in any explicit way on the potential bias between the two layers. Indeed, in experiments a greatly enhanced tunnelling current is observed at very low bias\([2]\). However, this enhanced current also has some notable differences with classic Josephson tunneling\([7]\) and there appears to be no tunnelling current at precisely zero bias.

We note that when the number of particles in each layer is strictly conserved (when \( t \) is zero, and the system is isolated from the leads), as in Eq. 4, the expectation in Eq. 6 is clearly zero. Nonetheless, essentially the same broken symmetry still exists, which can be seen by examining a slightly more complicated quantity such as

\[
G_{x,x'} = \langle c^\dagger_{lx} c_{rx} c^\dagger_{rx} c_{lx} \rangle (9)
\]

which conserves the particle number in each layer, is equal to \( \psi^*_x \psi^*_{x'} \) for the wavefunction Eq. 5, and is nonzero (order unity) even for the wavefunction Eq. 4.

1.4. Exciton Language: The state Eq. 3 can equally well be thought of as a condensate of excitons\([6]\). To see this, we imagine an effective vacuum state given by one of the quantum wells being completely empty and the other being a completely full LLL. For example, if we write

\[
|\text{Filled-Left}\rangle = \prod_X c^\dagger_{lx} |0\rangle (10)
\]
we then have the wavefunction of Eq. 3 being given by
\[ \Psi = \prod_X \left( u + v c_{RX}^{\dagger} c_{LX} \right) |\text{Filled-Left}\rangle \quad (11) \]
This then appears essentially identical to an exciton condensate in the standard BCS form where the excitons \( c_{RX}^{\dagger} c_{LX} \) are made by taking an electron out the filled left band and placing it in the previously empty right band. The excitons should then be thought of as an electron in the right layer bound to a hole in the left layer. Superfluid motion of these bound objects now clearly corresponds to counter-propagating charge currents as claimed above. Indeed, experiments have observed vanishing resistance for counter-propagating currents.

It is interesting to note that, as usual with wavefunctions of BCS form, one can rewrite Eq. 11 in a form reminiscent of a coherent state of bosons\(^7\)
\[ \Psi = \prod_X \left( e^{(v/u) v X} \right) |\text{Filled-Left}\rangle \quad (12) \]
where \( b_X^{\dagger} = c_{RX}^{\dagger} c_{LX} \) is the exciton-boson. Thus, we speak of this wavefunction as being a “coherent” state. The term “spontaneous coherence” is often used for the case of zero tunnelling to mean that the global formation of this coherent state is purely due to electron-electron interaction. This is in contrast with the case of finite tunnelling where the symmetric superposition (and hence wavefunctions of the form of Eq. 3) is energetically favored even in the absence of interaction.

We note that, similarly to the above manipulations, one could have started with a filled right LLL band and created excitons of the form \( c_{LX}^{\dagger} c_{RX} \) to obtain the very same state as follows
\[ \Psi = \prod_X \left( u c_{LX}^{\dagger} c_{RX} + v \right) |\text{Filled-Right}\rangle \quad (13) \]
In this language, the exciton-boson is instead written as \( b_X^{\dagger} = c_{LX}^{\dagger} c_{RX} \) which is just \( b_X \) in the above definition we used in describing Eq. 11. Thus, creating an exciton in the language of Eq. 13 is the same as annihilating an exciton in the language of Eq. 11. This duality should not surprise us given the particle-hole symmetry of a half-filled Lowest Landau level. Here, we have precisely half of the available states filled and it is equally valid to think about these states as being half-full of electrons or half-full of holes.

1.5. First Quantized Language: Bilayer wavefunctions were first proposed in the language of first quantized trial wavefunctions. The wavefunction of Eq. 3 is described as a filled Landau level of electrons in a particular superposition described by the spinor \((u, v)\). We can write this wavefunction in first quantized form as
\[ \Psi = \prod_i \left( u |L_i\rangle + v |R_i\rangle \right) \otimes \Psi_{\text{FilledLLL}}(r_1, \ldots, r_N) \quad (14) \]
where the second term is the wavefunction for the positional degree of freedom and the first term is the wavefunction of the iso-spin or layer degree of freedom that puts each electron in the \((u, v)\) superposition.

Of course, when the tunnelling between layers is strictly zero, as discussed above, the number of electrons in each layer should be a conserved quantity. As discussed above in Eq. 4 this restriction can be enforced properly by integrating over the the phase degeneracy \( \phi \) of \( v/u \) to obtain
\[ \Psi_{NR} = \hat{P}_{NR} \Psi_{\text{FilledLLL}}(r_1, \ldots, r_N) \quad (15) \]
where \( \hat{P}_{NR} \) is a projection operator that enforces that precisely \( N_R \) electrons should be on the right side.

We now turn to writing the wavefunction out explicitly. In radial gauge, the single electron spatial eigenstates of the lowest Landau level are given by \( \phi_m(\mathbf{r}) = z^{m-1} e^{-|\mathbf{r}|^2/(4\ell^2)} \) where \( \ell \) is the magnetic length, \( m \) ranges from 1 to the number of orbitals and \( z = x + iy \) is the position \( \mathbf{r} = (x, y) \) written as a complex number. The wavefunction of a filled Lowest Landau level is then given by the Slater, or Vandermonde, determinant
\[ \Psi_{\text{FilledLLL}} = \det[\phi_m(\mathbf{r}_n)] = \det[z^{m-1}] \quad (16) \]
where we have dropped the Gaussian factors for simplicity of notation.

In the limit where there is strictly no tunnelling between the layers, we need only project this wavefunction such that there are precisely $N_R$ particles on the right. In this limit of no tunnelling, the particles on the right and the left become distinguishable particles, in which case it is conventional to write the coordinates of the $N_R$ particle on the right as $z_i$ and the coordinates of the $N_L = N - N_R$ particles on the left as $w_i$. Performing the projection, we then obtain an explicit expression for wavefunction Eq. 4

$$
\Psi_{NR} = \prod_{i<j\leq NR} (z_i - z_j) \prod_{i<j \leq NL} (w_i - w_j) \prod_{i \leq NL; j \leq NL} (z_i - w_j) \tag{18}
$$

which is the form first written down over twenty years ago[8]. Interesting generalizations of this wavefunction that may occur at other filling fractions fields include the possibility of raising the first two factors to some odd power $n$ and the last factor to some power $m$ to generate the so-called $mmn$-state. For this reason, the wavefunction Eq. 18, and sometimes the wavefunctions of the form Eq. 3, are often called the 111-state.

Naturally, as required by Fermi statistics, due to the two factors on the first line of Eq. 18, this wavefunction vanishes when any two particles in the same layer come together (any two $z$’s come to the same position or any two $w$’s come to the same position). The interesting physics of this wavefunction is from the factor on the second line — the term that makes the wavefunction also vanish when any $z$ comes to the same location as any $w$. This means that wherever there is a particle in one layer, the space directly opposite it in the other layer cannot have an electron, or equivalently must have a hole. This binding of electron in one layer to a hole in the other is, of course, the same exciton that was discussed above in Section 1.4 in second quantized language.

1.6. Quantum Disordering the 111 State, In Brief: Although the above described 111 state (Eq. 3 or Eq. 18) is clearly the exact ground state at high enough tunnelling strength, at small tunnelling strength it is not so clear that this should be exact. As mentioned above, it turns out that for Coulomb interactions, even at $t = 0$ the 111 state is indeed an exact ground state when the distance between the two layers goes to zero. In the opposite limit, however, where the layers are very very far apart, the interaction between the layers drops to zero and we should clearly have two separate and uncorrelated layers. In the “balanced” case, where the density in the two layers is the same, we expect that we should end up with two independent layers (not interacting with each other) each with half the density ($\nu = 1/2$). Fortunately, the ground state of such a $\nu = 1/2$ single layer system is well understood to be described as a composite fermion fermi liquid[8][20] — a state with very strong electron-electron correlations within the single layer. Indeed, the strong electron-electron correlation within the layer can be thought of as binding of electron to a correlation hole within the same layer[20,21]. Thus, the crossover between the 111 state at small layer spacing and the composite fermion liquid at large layer spacing is one where the inter-layer excitons are replaced by intra-layer electron-hole binding. Simultaneously, we expect that, as the layer spacing is increased, the coherence order parameter $\psi$ (Eq. 6) must fall from its finite value in the 111 phase to a zero value in the composite fermion liquid phase. Exact diagonalization studies[11] indeed show the continuous reduction of the order parameter as the layer spacing is increased. This reduction of the order in the ground state as the parameter of layer spacing is changed, is sometimes known as “quantum disorder.”

It is very clear experimentally, in both tunnelling and drag experiments, that some sort of crossover or phase transition between two very different lim-
iting states is occurring\cite{1,2,3,4}. However, the nature of this crossover is still a topic of theoretical discussion\cite{1,10,11,12,13}. One proposal (made by the current author and collaborators\cite{12,13}) is to describe the transition by a set of trial wavefunctions that allow a variable number of inter- versus intra-layer excitons. The crossover is then easily described by varying the amount of inter- versus intra layer binding. Overlaps of these trial states with exact ground states is found to be quite good for small systems\cite{12}.

2. Some Physics at $\nu = 2$

2.1. The Double Well With Spin: We now consider the more rich\cite{14,15,16,17,18} physics of bilayers at $\nu = 2$. Here, we will consider both the layer degree of freedom (which we have called the isospin) as well as the actual spin degree of freedom of the electron. Resorting to our simple model of a single electron problem as above in section 1.1, we now study a single electron with spin in a double well. As discussed above, the spatial wavefunction has two eigenstates: the lower energy symmetric $|S\rangle$ state and the higher energy antisymmetric $|A\rangle$ state with energies $\pm \sqrt{t^2 + V_\Delta^2}$. When the electron has spin, the spin can be in a spin-up eigenstate, which we write as $|\uparrow\rangle$ or a spin-down state, which we write as $|\downarrow\rangle$. In the presence of a magnetic field, the spin couples to the magnetic field and there will be a Zeeman energy splitting between the up and down states which is usually written as $2V_z = g\mu_B B$. Thus, the spinful electron in the double well has a total of four possible eigenstates (See Fig. 3) given by $|S\uparrow\rangle$, $|S\downarrow\rangle$, $|A\uparrow\rangle$ and $|A\downarrow\rangle$ with energies

$$E = \pm \sqrt{t^2 + V_\Delta^2} \pm V_z \quad (19)$$

where the first $\pm$ is for the symmetric state and $+$ for the antisymmetric state, and the second $\pm$ is for the spin-down state and $+$ for the spin-up state. Thus, the symmetric, spin-down state $|S\downarrow\rangle$ is always the lowest energy state, and is always the ground state of the single spinful electron in the double quantum well.

![Energy Diagram](image)

We now turn our attention to the more interesting case where there are two spinful electrons in the double quantum well, and for simplicity, let us turn off the interaction between the two electrons. Now we should fill the two lowest of the four single electron states in the well. Again, the symmetric spin-down state $|S\downarrow\rangle$ is always the lowest energy state, and should always be filled. If $\sqrt{t^2 + V_\Delta^2} > V_z$, the second lowest state is the symmetric, spin-up state. Filling the lowest two states in this case (spin-up and spin-down symmetric states) we obtain the so-called the $S$ state, where $S$ stands for "Symmetric" or "Singlet". On the other hand, if $\sqrt{t^2 + V_\Delta^2} < V_z$, the second lowest state is the antisymmetric, spin-down state. Filling the lowest two states in this case (symmetric and antisymmetric spin-down) we obtain the $F$ or ferromagnetic state. Thus, as the Zeeman energy is varied compared to the symmetric-antisymmetric splitting there is a transition from a singlet to a ferromagnetic state, as shown in Figure 3.

2.2. $\nu = 2$ states Again we now imagine extending our double well with two electrons to a two dimensional bilayer. Including spin, at each position $X$ in space, there are now four possible state
that we can fill, and if we fix our attention to the filling fraction \( \nu = 2 \) we now have precisely two electrons per position \( X \). In analogy with the case of two electrons in a double well (See Fig 3) we might imagine having only two phases, the ferromagnetic phase at high ratio of Zeeman energy to symmetric-antisymmetric splitting and the singlet phase at low ratio of Zeeman energy to symmetric-antisymmetric splitting. The wavefunction of the ferromagnetic phase naturally can be written as

\[
|F\rangle = \prod_X c_{L,LX}^\dagger c_{R,LX}^\dagger |0\rangle
\]

(20)

whereas the wavefunction in the singlet phase is written as

\[
|S\rangle = \prod_X (uc_{L,LX}^\dagger + vc_{R,LX}^\dagger)(uc_{L,LX}^\dagger + vc_{R,LX}^\dagger)|0\rangle
\]

(21)

where \((u,v)\) is the iso-spinor representing the symmetric superposition.

Were there not electron-electron interactions, the \( S \) and \( F \) phases would be the entire story. However, in the presence of interactions, near the transition point between these two phases, it turns out that other phases exist[14,15,16,17,18], including so-called “canted” phases, where the spins in the two layers are partially aligned — i.e., the spins are neither fully aligned as in the \( F \) phase, nor fully antialigned as in the \( S \) phase. Spectroscopic experiments have indeed seen very suggestive indications of phase transitions in \( \nu = 2 \) bilayers[5].

2.3. The Interlayer Coherent I-Phase: Once again, we will focus our attention on the limit of vanishing tunnelling between the layers. Reiterating, if \( V_z \) is large enough, then both spins will be in the spin-down state, and we will have the ferromagnetic phase (Eq. 20). On the other hand, if energy difference between the two wells \( V_{\Delta} \) is large enough, then both of the electrons will go into a single well and we will have a trivial singlet phase with wavefunction

\[
|S\rangle = \prod_X c_{R,LX}^\dagger c_{R,RX}^\dagger |0\rangle
\]

(22)

where we have assumed here that the right well \((R)\) has the lower of the two potentials. Note, Eq. 22 is just the \( t \to 0 \) limit of the \( S \) phase in Eq. 21 since what we would call the “symmetric” state (the lower of the two single particle eigenstates) is just the electron residing in the lower of the two wells in this limit.

As suggested above, the transition between the \( F \) and \( S \) phase is not direct. An interesting phase also occurs[15] for intermediate values of \( V_z / V_{\Delta} \) which we call the I-phase which stands for “Interlayer Coherent” phase.\(^9\) The wavefunction of the I phase is given by

\[
|I\rangle = \prod_X (\hat{u}c_{L,LX}^\dagger + \hat{v}c_{R,RX}^\dagger)c_{R,LX}^\dagger |0\rangle
\]

(23)

where the spinor \((\hat{u}, \hat{v})\) here allows continuous interpolation between the \( F \) phase \((\hat{u} = 1, \hat{v} = 0)\) and the \( S \) phase \((\hat{u} = 0, \hat{v} = 1)\).

This wavefunction is clearly quite analogous to the \( \nu = 1 \) wavefunction Eq. 3, and much of the physics is also the same. As in that case, here there is a (Goldstone) phase freedom (the phase of \( \hat{v} / \hat{u} \)), and an order parameter

\[
\psi_X = \langle c_{L,LX}^\dagger c_{R,RX}^\dagger \rangle
\]

(24)

As in the \( \nu = 1 \) case, this order parameter indicates interlayer coherence that is “spontaneous” in the sense that it would not occur without electron-electron interactions.

It is worth noting, however, that the coherent interlayer tunnelling current here,

\[
j \sim -i(c_{L,LX}^\dagger c_{R,RX}^\dagger - c_{R,LX}^\dagger c_{R,RX}^\dagger)
\]

(25)

analogous to Eq. 7 involves a spin-flip, and therefore should be extremely suppressed in experiments. To understand this suppression we need only realize that, in the absence of spin-orbit terms, which are usually weak, the total spin angular momentum of the system is precisely conserved which completely forbids spin-flip tunnelling current. When one adds back in the weak spin-orbit terms, the spin angular momentum is “almost” conserved meaning that spin-flips can only occur very occasionally — which strongly limits the interlayer spin-flip current even if there is such a low-bias Josephson-like contribution.

\(^9\) The \( I \) phase is the \( t \to 0 \) limit of a canted phase[15,16].
2.4. Exciton Language} Analogous to the \( \nu = 1 \) wavefunction Eq. 3 the I-state can be written in the language of excitons. We can write the wavefunction for the I-phase in either of two forms. The first form is built on top of the ferromagnetic phase Eq. 20

\[
|I\rangle = \prod_X (\hat{u}c_{R,X}d_{L,X})|F\rangle
\]

and the exciton \( \hat{b}_X = c_{R,X}d_{L,X} \) now carries spin. The other equivalent possibility is to build the state on top of the singlet phase (Eq. 22)

\[
|I\rangle = \prod_X (\hat{u}c_{L,X}d_{R,X} + \hat{v})|S\rangle
\]

where the exciton-boson is then \( \hat{b}_X = c_{L,X}d_{R,X} = b_X \) which again is just the conjugate of the above exciton definition. These two possibilities are, of course, analogous to the two ways of writing the \( \nu = 1 \) state (Eq. 11 and 13). In either language, the excitons form a coherent state, and the superfluid mode of these particles (the Goldstone mode of the phase degree of freedom of \( \hat{v}/\hat{u} \)) involves counterpropagating currents of opposite spins in the opposite layers. As of yet, there have been no experiments to try to observe such a superfluid current.

2.5. First Quantized Language} Finally, it is worth pointing out that the I-phase has a very simple first quantized form analogous to Eq. 18 above. We define \( z_{ij} \) with \( i = 1, \ldots, N \) to be the positions of the spin-down electrons in the right layer, \( z_{\gamma i} \) with \( i = 1, \ldots, N \) to be the positions of the spin-up electrons in the right layer, and \( w_{\gamma i} \) with \( i = 1, \ldots, N_R \) to be the positions of the spin-down electrons in the left layer. Here again \( N = N_R + N_L \) is the total number of positional states in the lowest Landau level. Reasoning analogous to that of section 1.5 above now yields the wavefunction

\[
\Psi_{N_R} = \prod_{i<j \leq N} (z_{ij} - z_{\gamma i}) \prod_{i<j \leq N_R} (z_{\gamma i} - z_{\gamma j}) \prod_{i<j \leq N_L} (w_{\gamma i} - w_{\gamma j}) \prod_{i \leq N_R, j \leq N_L} (z_{\gamma i} - w_{\gamma j})
\]

Here, the first term (the first line) gives a filled Landau level of spin down electrons in the right layer, and the terms on the second and third line are analogous to the pieces of the wavefunction of Eq. 18. As required by Fermi statistics, the wavefunction vanishes whenever two electrons with the same spin state come to the same position in the same layer (if any two \( z_{\gamma i} \)’s come to the same position or if any two \( z_{\gamma i} \)’s come to the same position, or if any two \( w_{\gamma i} \)’s come to the same position). Again, the interesting piece of this wavefunction is the final term which forces the wavefunction to vanish if an up-spin electron in one layer \( z_{\gamma} \) comes to the same positions as a down-spin electron in the opposite layer \( w_{\gamma} \) — which is just the up-spin electron binding to the down-spin hole in the opposite layer.

3. The Tip of The Iceberg

The physics discussed above in this paper just scratches the surface of the interesting physics that can occur in bilayers — the possible phase space of states that one can explore in this system is quite large. In the current paper we have looked at \( \nu = 1 \) and \( \nu = 2 \) and considered mainly the limit of \( t \to 0 \). We have considered (albeit briefly) variation of several parameters including the interlayer bias \( V_z \), the Zeeman energy \( V_x \), and the distance between the two layers. More generally, one could imagine exploring a much bigger phase space where we also imagine varying the filling fraction \( \nu \), the tunnelling \( t \), the Zeeman energy \( V_x \), the interlayer bias \( V_{\Delta} \), the temperature, the disorder, the in-plane magnetic field as well as a number of other less obvious experimental parameters (For example, we could tweak the effective interaction by changing the quantum well width, or we could change the spin-orbit coupling by using holes rather than electrons). Some pieces of this larger phase space have been studied already[1-18], but others still are waiting for more complete analysis.

\[\text{10 Once the tunnelling is nonzero, magnetic field directed in the plane of the sample can greatly change the nature of ground state, since electrons accumulate Aharanov-Bohm phases when they tunnel between the layers.}\]
It is worth emphasizing, however, that the interest in bilayers does not rest solely on the fact that there are so many parameters to play with. Much more important is the fact that the physics of these bilayer systems is new and exciting — blending the effects of quantum mechanics, statistical physics, and many body interactions in unique and novel ways. Over the last decade, experiment and theory have had a rare and delightful synergy in pushing knowledge forward. It seems quite clear that this will remain a productive and active field for years to come.

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