Current commutator anomalies in finite-element quantum electrodynamics

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Abstract

Four-dimensional quantum electrodynamics has been formulated on a hyper-cubic Minkowski finite-element lattice. The equations of motion have been derived so as to preserve lattice gauge invariance and have been shown to be unitary. In addition, species doubling is avoided due to the nonlocality of the interactions. The model is used to investigate the lattice current algebra. Regularization of the current is shown to arise in a natural and nonarbitrary way. The commutators of the lattice current are calculated and shown to have the expected qualitative behavior. These lattice results are compared to various continuum calculations.

11.15.Ha, 11.15.Tk, 12.20.Ds, 11.40-q
I. INTRODUCTION

The linear finite-element approach has previously been successfully applied to quantum electrodynamics [1,2]. The equations of motion comprise a self-consistent lattice gauge theory, that is, they are gauge covariant to all orders of the lattice spacing. Also, the lattice Dirac equation has been shown to be unitary, i.e. the fermion canonical anticommutation relations are preserved in time. This is true even in the presence of background electromagnetic fields. The lattice Maxwell’s equations are shown to preserve unitarity in the absence of interactions. The interacting case is less clear due to anomalous behavior which this article will begin to address. In addition, it has been shown that this model avoids the common lattice problem of fermion species doubling [3]. The no-go theorem [4] is avoided by the nonlocality of the interaction term in the Dirac equation.

The finite-element approach has been successfully applied to massless two-dimensional quantum electrodynamics (the Schwinger model) [3,5]. The axial-vector divergence anomaly was calculated to be

$$\langle \partial_\mu j_5^\mu \rangle = -\frac{e^2}{M \sin(\pi/M)} E. \quad (1.1)$$

Here, $j_5^\mu$ is the axial-vector current, $e$ is the electric charge, $M$ is the number of spatial lattice sites, and $E$ is the electric field strength. The derivative is taken according to the finite-element prescription: a forward difference in the direction of the derivative, and a forward average in the other directions. The relative error is of order $M^{-2}$ as is expected from a linear finite-element approach. The anomaly in this model has also recently been evaluated by a lattice loop calculation of the vacuum polarization [5]

In this paper, we calculate the commutators of the vector current in four dimensions. These are shown to be roughly consistent with results obtained in the continuum. In addition, the regularization of the fermion bilinear $\bar{\psi} \gamma \psi$ is shown to be a direct result of requiring gauge invariance and unitarity.

In Sec. II we briefly review several continuum calculations of these commutators. The ambiguity inherent in these results is illustrated. In Sec. III our model is introduced in the
form of the four-dimensional lattice equations of motion. Several important features of these are presented. In Sec. IV we present results for vector-vector commutators, and in Sec. V we present our conclusions.

II. CURRENT COMMUTATOR ANOMALIES IN THE CONTINUUM

It is well known that there are anomalies in the commutators of the electromagnetic currents. Schwinger first calculated a non-zero value for the commutator of the charge density and vector current using a point-splitting regularization of the current [7],

\[ j_\mu(x) \equiv \lim_{\epsilon \to 0} e \psi^\dagger \left( x - \frac{1}{2} \epsilon \right) \gamma^0 \gamma_\mu \psi \left( x + \frac{1}{2} \epsilon \right), \]

(2.1)

where \( \epsilon \) is space-like and the limit is taken symmetrically in space. Using this current, he arrives at a value for the commutator of

\[ i\langle [J^0(0,x), J^\mu(0)] \rangle = -S \nabla \delta(x), \]

(2.2)

where \( S \) is the divergent limit of \((2e^2/3\pi^2)(\epsilon^2)^{-1}\).

Later calculations [8–11] revealed an additional finite contribution to this commutator. Its value depends on the type of regularization chosen to regulate the bilinear current. Hence, Eq. (2.2) is modified to read

\[ i\langle [J^0(0,x), J^\mu(0)] \rangle = -S \nabla \delta(x) - \frac{d}{\pi^2} \nabla^2 \delta(x), \]

(2.3)

where \( S \) is the divergent Schwinger term and \( d \) depends on the method of calculation.

The Bjorken-Johnson-Low method defines the equal-time commutator in terms of the high-energy limit of the time-ordered product

\[ \lim_{p_0 \to \infty} -ip_0 \int d^4x e^{ipx} \langle TA(x) B(0) \rangle \equiv \int d^3x e^{-ip \cdot x} \langle [A(0,x), B(0)] \rangle. \]

(2.4)

This is shown [9,11] to be equivalent to the following position-space prescription:
\[ \langle [A \ (0, \mathbf{x}), B \ (0)] \rangle \]
\[ = \lim_{\eta \to 0^+} [\langle A (\eta, \mathbf{x}) \ B \ (0) \rangle - \langle B \ (0) \ A \ (-\eta, \mathbf{x}) \rangle] . \] (2.5)

Using this prescription, the value of \( d \) in Eq. (2.3) is calculated to be \( d = 1/12 \) \[10,11\].

The value of the finite piece of Eq. (2.3) can also be calculated using a point-split current such as defined in Eq. (2.1). Boulware and Jackiw use this to derive a value of \( d = 1/60 \) \[10\]. These authors also use a generalized point-split current, allowing \( \epsilon \) to be a general four-vector, to derive a value of \( d = 1/96 \). To further illustrate the ambiguity in this calculation, we give the results of Chanowitz \[11\]. He shows that is one uses an unsymmetrical definition of the current

\[ J^\mu (x) = \lim_{\epsilon \to 0} \psi^\dagger (x + \epsilon) \gamma^0 \gamma^\mu \psi (x) , \] (2.6)

with \( \epsilon \) spacelike, then a value of \( d = 1/15 \) is derived.

Evidently, the results for Eq. (2.3) depend on the definition one chooses for the regulation of the current. Of course, this is to be expected, since the quantity is divergent. Presumably, the exact value of the coefficients of the Schwinger terms will have no physical consequence. However, their very existence proves to be problematic in establishing the formal properties of the theory.

### III. THE LATTICE EQUATIONS OF MOTION AND REGULARIZATION OF THE ELECTROMAGNETIC CURRENT

#### A. Lattice Dirac equation

The finite-element formulation of the gauge-invariant Dirac equation was carried out in Refs. \[12\]. We will merely quote the results here, expressed in terms of the spatially averaged fermionic field:

\[ \frac{i \gamma^0}{\Delta} (\psi_{\mathbf{m}, n+1} - \psi_{\mathbf{m}, n}) + \frac{2i \gamma^j}{\Delta} \left[ \sum_{m_j = 1}^{M-1} - \sum_{m_{j^{'}} = m_j + 1}^{M} \right] (-1)^{m_j + m_{j^{'}}} \psi_{\mathbf{m}_j, \mathbf{m}_j^{'}, \mathbf{m}_{j^{'}}}, \]
where a sum over the repeated index $j$ is understood, and the overbar represents a forward average over that coordinate:

$$x_m \equiv \frac{1}{2} (x_{m+1} + x_m).$$  \hspace{1cm} (3.2)

(Recall [1] that with $M$ odd, $\psi$ is periodic on the spatial lattice.) We have chosen a hypercubic lattice with lattice spacing $\Delta$.

Eq. (3.1) was explicitly constructed to be invariant under the local gauge transformation

$$\psi_{m,n} \to e^{ieA_{m,n}} \psi_{m,n},$$

$$A^0_{m,n} \to A^0_{m,n} + \frac{1}{\Delta} (\Lambda_{m,n+1} - \Lambda_{m,n}),$$

$$A^j_{m,n} \to A^j_{m,n} + \frac{1}{\Delta} (\Lambda_{m,j+1,n} - \Lambda_{m,j,n}).$$ \hspace{1cm} (3.3)

This particular choice of transformation is made so that the mass term in the Dirac equation (3.1) is automatically covariant.

To simplify things, we choose the temporal gauge ($A^0 = 0$). In this gauge, $\mathcal{T}^0$ vanishes identically [1], and Eq. (3.1) contains only fields at time $n$. The explicit form of the spatial parts of the interaction is

$$\mathcal{T}^j_{m_1,m_j,m_j',n} = \epsilon_{m_j,m_j'} (-1)^{m_j+m_j'} \left[ -1 + \cos \left( \sum_{m_j''=1}^M \text{sgn} (m_j'' - m_j) \text{sgn} (m_j'' - m_j') \zeta^j_{m_1,m_j'',n} \right) \sec \zeta^j_{m_1,n} \right]$$

$$+ i (-1)^{m_j+m_j'} \sin \left( \sum_{m_j''=1}^M \text{sgn} (m_j'' - m_j) \text{sgn} (m_j'' - m_j') \zeta^j_{m_1,m_j'',n} \right) \sec \zeta^j_{m_1,n}. \hspace{1cm} (3.4)$$

We have used the abbreviations

$$\zeta^j_{m_1,m_j,n} = \frac{e\Delta}{2} A^j_{m_1,m_j-1,n}, \quad \zeta^j_{m_1,n} = \sum_{m_j=1}^M \zeta^j_{m_1,m_j,n},$$ \hspace{1cm} (3.5)

and
\[ \text{sgn}(x) = \begin{cases} +1, & x > 0 \\ -1, & x \leq 0 \end{cases} \]  \hspace{1cm} (3.6)

\[ \epsilon_{m_j,m'_j} = \begin{cases} +1, & m_j > m'_j \\ 0, & m_j = m'_j \\ -1, & m_j < m'_j \end{cases} \]  \hspace{1cm} (3.7)

Eq. (3.1) can be written in a more compact matrix notation:

\[ \frac{i}{\Delta} (\phi_{n+1} - \phi_n) + \frac{2i\gamma_0}{\Delta} \cdot (Q - I_n) \phi_n + \mu \phi_n = 0. \] \hspace{1cm} (3.8)

Here, \( Q \) and \( I \) are matrices in the spatial indices

\[ (Q^j)_{m,m'} = \delta_{m,m'} (-1)^{m_j + m'_j} \epsilon_{m_j,m'_j}, \] \hspace{1cm} (3.9)

and \( \phi_n \) is a vector of the spatially-averaged fields

\[ (\phi_n)_m = \psi_{m,n}. \] \hspace{1cm} (3.10)

Notice that \( Q \) is related to the lattice derivative,

\[ \frac{2}{\Delta} Q \leftrightarrow \nabla, \] \hspace{1cm} (3.11)

and the lattice version of the covariant derivative is

\[ \frac{2}{\Delta} (Q - I_n) \leftrightarrow \mathcal{D}. \] \hspace{1cm} (3.12)

Eq. (3.8) can be solved for the transfer matrix defined by

\[ \phi_{n+1} \equiv T_n \phi_n, \] \hspace{1cm} (3.13)

giving
\[ T_n = \frac{1 - \gamma^0 (Q - I_n) + i\nu \gamma^0}{1 + \gamma^0 (Q - I_n) - i\nu \gamma^0}, \]  
(3.14)

with \( \nu = \frac{\mu \Delta}{2} \). From Eqs. (3.4) and (3.9), it is clear that both \( Q \) and \( I_n \) are anti-hermitian. Therefore, the transfer matrix defined in Eq. (3.14) is unitary. Hence, the spatially averaged fields \( \phi_n \) can be taken to be the canonical fermionic fields.

**B. Lattice Maxwell equations**

Using the notation of Eq. (3.9), the lattice Maxwell’s equations are written as

\[
\begin{align*}
E_{\pi} &= \frac{1}{\Delta} (A_{n+1} - A_n), \\
2 \Delta Q \cdot E_{\pi} &= J_n^0, \\
E_{n+1} - E_n &= -J_n - 2 \Delta Q \cdot F, \\
F_{ij} &= -2 \Delta (Q_i A_j^n - Q_j A_i^n),
\end{align*}
\]

(3.15)

where

\[
E_n \equiv E_{m,n}, \\
A_n \equiv A_{m,n},
\]

(3.16)

Gauge invariance of Eqs. (3.15) under (3.3) is assured as long as the current \( J_{m,n}^\mu \) is constructed to be gauge invariant.

**C. Current regularization**

The simplest choice of gauge-invariant current is

\[
J_{m,n}^\mu \equiv e \bar{\psi}_{m,n} \gamma^\mu \psi_{m,n}
\]

(3.17)

\[
= e \bar{\phi}_{m,n} \gamma^\mu \phi_{m,n},
\]

(3.18)

which is invariant under the transformation (3.3). Notice that the covariant field \( \psi_{m,n} \) that is involved in the definition of \( J_{m,n}^\mu \) is not the same as the canonical field \( \phi_{m,n} \) defined in Eq. (3.13)—they differ by a temporal averaging. This forces point-splitting of the current:
\[ J^\mu_{m,n} = e \bar{\phi}_{m,n} \gamma^\mu \phi_{m,n} = e \sum_{m',m''} \phi^{\dagger}_{m',n} \left( 1 + T^\dagger (A^\mu_{m'}) \right)_{m',m} \gamma^\nu \gamma^\mu \left( 1 + T (A^\mu_{m''}) \right)_{m''} \phi_{m'',n}. \] (3.19)

The important point is that this is not introduced in an arbitrary way. The regularization of the current is mandated by requiring gauge-invariance and unitarity of the equations of motion.

**IV. VECTOR-VECTOR COMMUTATORS ON THE LATTICE**

**A. Analytical results**

Using the definition of the electromagnetic current Eq. (3.18), the commutators of various components can be calculated. To lowest order in \( e \), the commutators can be calculated using the zeroth-order contribution of the transfer matrix

\[ T_n = \frac{1 - \gamma^0 (\gamma \cdot Q) + i \nu \gamma^0}{1 + \gamma^0 (\gamma \cdot Q) - i \nu \gamma^0} \] (4.1)

and a free field Fock space expansion for the Dirac fields

\[ \phi_{m,n} = \sum_{\sigma,p} \sqrt{\frac{\mu}{\omega_p}} \left[ u^{(\sigma)}_p b^{(\sigma)}_p e^{2\pi i p \cdot m/M} + v^{(\sigma)}_p d^{(\sigma)}_p e^{-2\pi i p \cdot m/M} \right], \] (4.2)

where the spinors are normalized according to

\[ \sum_{\sigma} u^{(\sigma)}_p u^{(\sigma)\dagger} \gamma^0 = \frac{\omega \gamma^0 - \frac{2}{\Delta} (\gamma \cdot t) - \mu}{2\mu} \]

\[ \sum_{\sigma} v^{(\sigma)}_p v^{(\sigma)\dagger} \gamma^0 = \frac{\omega \gamma^0 - \frac{2}{\Delta} (\gamma \cdot t) + \mu}{2\mu} \] (4.3)

with

\[ t^i_p = \tan \left( \frac{p^i \pi}{M} \right), \quad \omega_p = \sqrt{\left( \frac{2}{\Delta} \right)^2 t^2_p + \mu^2}. \] (4.4)
Then the canonical anticommutation relations for the Dirac fields

\[ \left\{ \phi_{m,n}, \phi^{\dagger m',n} \right\} = \frac{1}{\Delta^3} \delta_{m,m'} \]  \hspace{1cm} (4.5)

are satisfied if

\[ \left\{ b^\sigma_p, b^{\sigma \dagger}_p' \right\} = \frac{1}{(M\Delta)^3} \delta_{p,p'} \delta_{\sigma,\sigma'} \]  \hspace{1cm} (4.6)

and all other anticommutators of these operators vanish.

The results of the calculations are presented below:

\[ \langle [J^{0}_{m,n}, J^{0}_{m',n}] \rangle = 0 \]  \hspace{1cm} (4.7)

\[ \langle [J^{i}_{m,n}, J^{j}_{m',n}] \rangle = 0 \]  \hspace{1cm} (4.8)

\[ i \langle [J^{0}_{m,n}, J^{i}_{m',n}] \rangle = -\frac{4ie^2}{(M\Delta)^6} \sum_{p,p'} \frac{\tan \left( \frac{p'\pi}{2} \right)}{1 + (\Delta \omega_p - \omega_{p'})^2} \left( 1 + \left( \frac{\Delta \omega_{p'}}{2} \right)^2 \right) e^{2\pi ip \cdot (m - m')/M}. \]  \hspace{1cm} (4.9)

B. Numerical evaluation

The matrix (in \( \mathbf{m} \) and \( \mathbf{m}' \)) represented by Eq. (4.9) is shown for lattice size \( M = 9 \) and mass \( \mu = 0 \) in Fig. 1. The abscissas of the plot correspond to one-dimensional representations of the three components of \( \mathbf{m} \) and \( \mathbf{m}' \), respectively. (That is, the base-\( M \) number \( (m_1, m_2, m_3) \) is converted to a base-10 value of the abscissa.) It is apparent that the leading-order behavior of this commutator is a first derivative of the Dirac delta function, as expected.

To make this comparison more quantitative, we fit this result to the functional form expected in the continuum, Eq. (2.3). The lattice analog of the Dirac delta function is
\[
\delta (x) \approx \frac{1}{\Delta^3} \delta_{m,0} = \frac{1}{(M\Delta)^3} \sum_{p} e^{2\pi i p \cdot m/M}, \tag{4.10}
\]

so we take as a trial function

\[
i \langle [J_{m,n}^0, J_{m',n}^r] \rangle \approx -\frac{e^2}{(M\Delta)^3/\Delta^6} \sum_{p} p_j \sum_{r=0}^{R-1} a_{2r+1} P_r (\Delta) (-1)^r \left( \frac{2\pi}{M\Delta} \right)^{2r} p^{2r} e^{2\pi i p \cdot (m-m')/M}. \tag{4.11}
\]

The coefficients \(a\) do not depend on the lattice spacing \(\Delta\) (for mass \(\mu = 0\)); they presumably remain finite in the continuum limit, \(\Delta \to 0\). The continuum behavior of the terms in (4.11) is dictated by the functions \(P_r (\Delta)\). These are inserted to force each term in the series to be of the same order in \(\Delta\) as the commutator (4.9), namely \(1/\Delta^6\):

\[
P_r (\Delta) = \Delta^{2r-2}. \tag{4.12}
\]

Therefore, in the continuum limit, the first term in Eq. (4.11) will be quadratically divergent, the second will be finite, and the rest will vanish, as expected.

To perform the fit, the data is first converted from a three-dimensional to a two-dimensional representation. (This is necessary because the generation of 3-D plots is impractical beyond \(M \approx 9\) and because 3-D fits are prone to difficulties.) Specifically, a plot such as Fig. 4 is projected onto a plane orthogonal to the \(m, m'\) plane and to the diagonal of the matrix. This yields a graph such as the solid curve in Fig. 2(a). The curve is then fit to a similar projection of the trial function in Eq. (4.11). In particular, Fig. 2 shows fit vs. data for a fit of the commutator (4.9) to a linear superposition of three spectra, i.e., \(R = 3\) in (4.11).

It turns out that a fit to five spectra produces comparable results. The addition of the last two spectra neither particularly improves the fit nor radically changes the coefficients of the original three. (That is, the fit is stable.) In addition, the coefficients of all but the first two spectra are vanishingly small. In Fig. 3 we show the results for a fit to (4.11) with \(R = 3, 4, \text{ and } 5\) for various lattice sizes. These are compared to some of the results of continuum calculations. The lattice values, though of the same order as other results, are
significantly different. This is as expected, since (3.19) is yet another definition of current regularization. (Note that the result of Chanowitz [11] is within one sigma of our result. This is interesting in that both his method and ours use a temporal averaging to regularize the current.)

A note should be added to discuss the parameterization of errors in the data: Since the data (4.9) is computer-generated, any “measurement” errors should be strictly due to round-off errors. Presumably this will be roughly the same for each data point. Therefore, it is reasonable to assume that all measurement errors are equal, \( \sigma_i^2 \equiv \sigma^2 \). If we further assume normally distributed errors, then the value of \( \chi^2 \) for the fit is

\[
\chi^2 = \frac{1}{\sigma^2} \sum_{\text{data points}} (\text{data} - \text{fit})^2.
\] (4.13)

If we now require the fit to be “good”, i.e \( \chi^2 \) per degree of freedom equal to one, then we can determine a value for \( \sigma^2 \),

\[
\sigma^2 = \frac{\sum_{\text{data points}} (\text{data} - \text{fit})^2}{\text{degrees of freedom}},
\] (4.14)

where the number of degrees of freedom is the number of data points less the number of terms in the trial function. Multiplying this by the diagonal elements of the covariance matrix \( C \) (which is determined as a by-product of our minimization process), gives the variances of the coefficients \( a_r \)

\[
\sigma^2 (a^j_r) = \sigma^2 C_{jj}.
\] (4.15)

These are plotted along with the values of the coefficients in Fig. 3.

There are a few features of the graphs of Fig. 3 which merit some discussion. First, there is the consistent kink in the data at lattice size \( M = 13 \). We have no explanation for this behavior except to suppose it is just a numerical quirk of our code.

Second, the scattering of the data points increases significantly with the number of fit spectra \( R \). This seems to be a problem with numerical precision in the fitting procedure—Each successive term in the trial function (L.1) is an order of magnitude or two larger than
the previous one. Thus, at higher numbers of fit spectra \((R \sim 5)\), the range in values of terms in (4.11) approaches the numerical precision of the data. For example, if we continue to \(R = 6\), our procedure breaks down—the coefficients generated by the fitting algorithm do not give a good fit to the data.

Lastly, the errors associated with the data do not seem to account for the scatter of the points (especially for \(R = 5\)). Remember, though, that the derivation used to give the errors in our data is based upon several assumptions. These are not rigorously justified. Rather these suppositions serve to compensate for our lack of knowledge of the fundamental source of errors. In order to summarize the data, incorporating the errors in a more model-independent manner, we have averaged the data points at all lattice sizes (excluding \(M = 13\)) and plotted these results alongside the other data. (Figure 3)

The data for \(R = 3\) and \(R = 4\) are in good agreement over a wide range of lattice sizes. These only disagree at low \(M\), where the model is not expected to yield good results, and at \(M \sim 31\), where the fitting procedure for \(R = 4\) begins to break down. This agreement gives support to our claim that the fit is stable with respect to the number of fit spectra \(R\). Thus, we are able to consider the best data (with \(R = 3\)) as representative of the commutator (4.9). The results in the fit of the trial function (4.11) with \(R = 3\) to the commutator (4.9) are given in Fig. 4.

V. CONCLUSIONS

Using the definition of the current (3.19), the current commutators are calculated and are seen to be anomalous. These exhibit the qualitative behavior expected from continuum calculations, having both quadratically divergent and finite contributions. The coefficients of these are of the same order of magnitude as the various continuum calculations. In fact, our results seem to be in quantitative agreement with those of Chanowitz [11] who also uses a temporal point-splitting regularization scheme.

Furthermore, the results arise from a regularization of the current that is both natural
and nonarbitrary. The requirements of gauge invariance and unitarity dictate a current which exhibits a temporal averaging of the composite fermionic fields. This averaging is distinct, yet akin, to a point-splitting regularization.

We have also begun work on vector-axial-vector commutators. These are more interesting since they can be directly related to the chiral anomaly \cite{12}

\[ A(x) = \frac{1}{4\pi^2} \epsilon^{0ijk} \{ E_i, F_{jk} \}, \]  

by the Gauss' Law constraint,

\[ \nabla \cdot E = J^0. \]  

Preliminary results are promising, but much work remains to be done.

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FIGURES

FIG. 1. Three-dimensional plot of (4.9) for $M = 9$ and $\mu = 0$.

FIG. 2. (a) Fit vs. data of two-dimensional projection of (4.9) and (4.11) with $R = 5$. ($M = 21$ and $\mu = 0$) (b) Smaller scale picture of the same. (The position variable has been normalized to range from $-1$ to $+1$.)

FIG. 3. Coefficients of a 3, 4, and 5 spectra fit for different lattice sizes. ($\mu = 0$) (a) Coefficient of first derivative of delta function. (b) Coefficient of third derivative of delta functions. The right-most data points summarize the data excluding $M = 13$. Also plotted are various continuum results for $d$ in Eq. (2.3).

FIG. 4. Summary of the results for a three spectra fit.