On singularities of stationary isometric deformations

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Abstract

Unstretchable thin elastic plates such as paper can be modelled as intrinsically flat \( W^{2,2} \) isometric immersions from a domain in \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \). In previous work it has been shown that if such an isometric immersion minimizes the elastic energy, then it is smooth away from a singular set consisting of three different subsets. In the present paper, we show that each of these singular subsets can indeed occur and that regularity may indeed fail there.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In this article we are address the regularity properties of stationary points of the Kirchhoff plate (or “elastic bending energy”) functional from nonlinear elasticity. This is the functional

\[
\mathcal{W}(u; S) = \int_S |A|^2
\]

(1)
on the class of isometric deformations \( u : S \to \mathbb{R}^3 \). Here \( S \subset \mathbb{R}^2 \) is a bounded domain and \( A \) is the second fundamental form of the immersion \( u \), and by an isometric deformation we mean a \( W^{2,2} \) isometric immersion, i.e. a map \( u \) in the Sobolev space \( W^{2,2}(S, \mathbb{R}^3) \) which satisfies the constraint

\[
(\nabla u)^\top \nabla u = I \text{ almost everywhere on } S.
\]

(2)
Here $I \in \mathbb{R}^{2 \times 2}$ is the unit matrix. Such maps model elastic deformations with finite bending energy and with zero membrane energy, i.e. which neither stretch nor shear the material. For such $u$ the second fundamental form is the matrix field $A : S \to \mathbb{R}^{2 \times 2}$ with entries

$$A_{ij} = (\partial_i u \times \partial_j u) \cdot \partial_i \partial_j u$$

for $i, j = 1, 2$,

where the dot denotes the Euclidean scalar product and the cross the cross product in $\mathbb{R}^3$.

Kirchhoff’s plate functional models the behaviour of inextensible thin elastic plates such as a sheet of paper. It was derived rigorously from nonlinear three dimensional elasticity in [7]. More precisely, it was shown in [7] that the functionals which associate with a deformation $z : S \times (0, h) \to \mathbb{R}^3$ of a thin film with thickness $h$ the scaled elastic energy

$$\frac{1}{h^3} \int_{S \times (0, h)} W(\nabla z)$$

converge, in the sense of $\Gamma$ convergence, to Kirchhoff’s plate functional.

Kirchhoff’s plate functional agrees (up to a constant prefactor) with the Willmore functional from differential geometry [25, 26] restricted to the class of isometric immersions. This is because the Gauss equation asserts that $\det A = 0$ for isometric immersions, so $|A|^2$ is proportional to the squared mean curvature.

A nontrivial variational problem for (1) is obtained by prescribing boundary conditions. However, due to the isometry constraint the problem is quite rigid. Therefore, boundary conditions are typically prescribed only on a (strict) closed subset $\partial_c S$ of $\partial S$ with nonzero length. In order to prescribe values which are compatible with the isometry constraint, one fixes some isometric immersion $u_0 \in W^{2,2}(S; \mathbb{R}^3)$ and seeks minimizers of (1) within the class of $W^{2,2}$ isometric immersions $u$ satisfying $u = u_0$ on $\partial_c S$.

Existence of minimizers in this setting is quite direct, see e.g. [12, 14]. The Euler–Lagrange equations satisfied by minimizers are derived in [12] and, in the same paper, a regularity result is established that can be summarized as follows: outside a closed singular set the minimizer in question is $C^3$ and outside a larger singular set it is $C^\infty$.

The singular set arising in [12] consists of line segments. This is due to the fact that $W^{2,2}$ isometric immersions are developable. This means that through any point $x \in S$ around which $u$ is not planar, there exists a unique line segment with endpoints on $\partial S$, such that $u$ is affine along this segment.

More precisely, the singular set consists of essentially three families of such segments. Each family is defined in a geometric way. And each of them arises in [12] for apparently technical reasons. So it is a priori not clear if they occur at all and if regularity actually fails there, i.e. if the regularity result from [12] is sharp.

In the present work we essentially confirm that these results are sharp. We provide explicit examples of smooth and simply connected domains $S$ and boundary data $\partial_c S$ and $u_0$, such that the corresponding minimizing isometric immersions $u$ display the singular set in question and fail to be more regular than asserted in [12]. Observe that the results in [11, 13], in contrast, show that a ‘generic’ $W^{2,2}$ isometric immersion (under similar boundary conditions) is in $C^\infty$.

Both in the mathematical and in the applied literature there has been considerable interest in isometric deformations with singularities. For instance, conical singularities were studied in [1, 2, 19, 27]. Another instance is folded or crumpled paper. This topic was addressed, e.g. in [4, 16, 18, 23, 24, 27].

One main question in those situations is about the scaling of the three dimensional elastic energy (3) of a thin sheet with respect to its thickness $h$. The singularities of the asymptotic deformations are reflected by the fact that the energy of the actual deformations $z$ will be much greater asymptotically than that of deformations with finite bending energy.
A somewhat related question is about the rigidity of isometric immersions. It is possible to construct very flexible isometric deformations with continuous deformation gradient [17, 20] and even with Hölder continuous deformation gradient [3, 6]. Such constructions must create singularities on the level of the second fundamental form. The singularities addressed here are different from those just discussed in two main respects. First, the admissible deformations in the present paper always have finite bending energy. This is not the case for cones, where the bending energy concentrates in a point, nor for folds, where the bending energy concentrates along folds, and even less so for convex integration solutions, which differ dramatically from deformations with finite bending energy. In particular, they fail to be developable and there is no meaningful way of defining their second fundamental form. In contrast, the admissible deformations considered here are developable. In [23] the authors study the extent to which developability leads to a lower bound on the extrinsic diameter of the deformed configuration. Such a bound shows that they are fundamentally different from the flexible isometric immersions in [17, 20].

Our work is related to [7], in that the setting is precisely the asymptotic zero thickness theory—Kirchhoff’s plate theory—derived in that article. The second difference is that the deformations constructed here, in addition to having finite bending energy, are minimizers of Kirchhoff’s plate functional (1).

Finally, notice that our results depend entirely on the developability of the admissible deformations and therefore differ from situations in which the prescribed metric is not intrinsically flat. However, somewhat related observations to ours are made in [8, 9] for the case of hyperbolic metrics. There it is shown that nonsmooth isometric deformations with finite bending energy may have lower energy than smooth ones.

This article is organised as follows. In the remainder of this Introduction we provide some more details about the regularity result in [12]. In section 2 we present some further facts about isometric deformations, including a result about failure of regularity. Finally, in section 3 we construct examples of stationary points which display the singular sets encountered in [12] and which fail to be regular. The appendix contains some more technical proofs.

1.1. Developability of isometric immersions and partial regularity

We begin by discussing some basic properties of the admissible deformations that are required to formulate the main (positive) regularity result from [12].

Throughout this article $S \subset \mathbb{R}^2$ models the reference configuration of an (infinitely thin) inextensible elastic sheet. We take $S$ to be a bounded, simply connected domain with boundary of class $C^\infty$. The admissible deformations are those with finite Kirchhoff bending energy, i.e. they belong to the set

$$W^{2,2}_{iso}(S; \mathbb{R}^3) = \{ u \in W^{2,2}(S; \mathbb{R}^3) : \partial_i u \cdot \partial_j u = \delta_{ij} \text{ a.e. in } S, \ i, j \in \{1, 2\} \},$$

of $W^{2,2}$-isometric immersions from $S$ to $\mathbb{R}^3$. Here $\delta_{ij}$ is the Kronecker symbol. As shown in [15], every $u \in W^{2,2}_{iso}(S; \mathbb{R}^3)$ belongs to $C^1(S; \mathbb{R}^3)$.

Let $u$ be a $W^{2,2}$-isometric immersion. In [15, 21, 22] it is shown that then $u$ is ‘developable’ away from the open set

$$C_{\nabla u} = \{ x \in S : \nabla u \text{ is constant in a neighbourhood of } x \}.$$

This means that through every point $x \in S \setminus C_{\nabla u}$ there exists a unique line segment $[x] \subset S$ with both endpoints on the boundary $\partial S$ of $S$ such that the deformation gradient $\nabla u$ is constant on
As a result the image of \([x]\) under the deformation \(u\) will still be a straight segment. We refer to [10] for a classical version of this developability result. Developability will be essential throughout this article.

We will next state the regularity result proven in [12]. For a given \(u_0 \in W^{2,2}_{iso}(S; \mathbb{R}^3)\) and a closed subset \(\partial_c S \subset \partial S\) with positive length, we define the class

\[
A_{u_0}(S, \partial_c S) = \tilde{u} \in W^{2,2}_{iso}(S; \mathbb{R}^3) : (\tilde{u}, \nabla \tilde{u}) = (u_0, \nabla u_0) \text{ on } \partial_c S.
\]

The equality \(\nabla u = \nabla u_0\) is understood in the trace sense. These are the deformations satisfying on \(\partial_c S\) the clamped boundary conditions determined by \(u_0\). We are interested in the minimizers of \((1)\) within the class \(A_{u_0}(S, \partial_c S)\). The existence of such minimizers is easy to show, cf [12] or, for a more general context, [14]. From now on the word ‘minimizer’ will always refer to a minimizer of this kind.

The singular set encountered in [12] consists of the following subsets

\[
\Sigma_\tau = \{ x \in S \setminus C_{\nu u} : [x] \text{ intersects } \partial S \text{ tangentially at one or both end-points} \},
\]

\[
\Sigma_c = \text{closure of } \{ x \in S \setminus C_{\nu u} : [x] \text{ intersects } \partial_c S \}
\]

as well as the relative boundary \(S \cap \partial \hat{C}_{\nu u}\) of the set

\[
\hat{C}_{\nu u} = \text{union of all connected components } U \text{ of } C_{\nu u} \text{ whose relative boundary } S \cap \partial U \text{ consists of at least three connected components.}
\]

Later we will see that the connected components of \(S \cap \partial U\) are segments of the form \([x]\). By ([13], proposition 14) the set \(\Sigma_\tau\) is relatively closed in \(S\).

Another set arising naturally in [12] is

\[
\Sigma_0 = \{ x \in S \setminus (\overline{C_{\nu u}} \cup \Sigma_\tau \cup \Sigma_c) : \lim_{r \to 0} r^{-4} \int_{B_r(x)} |H|^2 = 0 \}.
\]

The regularity result ([12], theorem 1.3) is the following:

**Theorem 1.1.** Let \(S \subset \mathbb{R}^2\) be a bounded \(C^\infty\)-domain, let \(\partial_c S \subset \partial S\) be closed and let \(u\) be a minimizer of \((1)\) within \(A_{u_0}(S, \partial_c S)\). Then

\[
u u \in C^3(S \setminus (\Sigma_\tau \cup \Sigma_c \cup \partial \hat{C}_{\nu u}); \mathbb{R}^3),
\]

and

\[
u u \in C^\infty(S \setminus (\Sigma_0 \cup \Sigma_\tau \cup \Sigma_c \cup \partial \hat{C}_{\nu u}); \mathbb{R}^3).
\]

The singular sets in \((6)\) can be visualised as follows: the set denoted by \(\Sigma_c\) consists of those segments on which \(u\) is determined (due to it being affine along those segments) by the boundary conditions prescribed on \(\partial_c S\). The set \(\Sigma_\tau\) consists of those segments which intersect the boundary \(\partial S\) tangentially. And the set \(S \cap \partial \hat{C}_{\nu u}\) consists of those segments which constitute the boundary (in \(S\)) of certain regions on which \(u\) is planar. In section 2.3 we recall the relevant
Nonlinearity 33 (2020) 4900
P Eberhard and P Hornung

Figure 1. The reference configuration $S$ can be divided into sets on which the deformation gradient $\nabla u$ is constant: straight segments and regions such as the green triangle $U$, which is a connected component of $C_{\nabla u}$. The deformation $u$ is affine on $U$ and along each of the segments. The brown segments are generic ones, while the blue segments constitute the singular set. The red part of the boundary is $\partial S$.

facts from [5, 13] about the shape of these planar regions. Essentially, they are polygons with vertices on the boundary of $S$. Figure 1 depicts this description.

1.2. Failure of regularity

As mentioned earlier, in section 3 we will provide examples showing that theorem 1.1 is essentially optimal, in the sense that one can find appropriate data $S, \partial S$ and $u_0$ such that minimizers of (1) within the corresponding class $A_{u_0}(S, \partial S)$ of admissible deformations actually have nonempty sets $\Sigma_c$, $\Sigma_r$ or $\tilde{C}_{\nabla u}$ and indeed fail to be $C^\infty$ across these sets.

It is clear that the set $\Sigma_c$ in general will be an obstruction to regularity since the boundary datum $u_0$ is not assumed to be regular. In section 3.2 we spell out a straightforward example regarding this simple singular set $\Sigma_c$.

Regarding the set $\tilde{C}_{\nabla u}$, notice that it may well be empty in general. In fact, in [5] it was shown that it will be empty unless $\partial S$ consists of at least three subintervals of the boundary.

In section 3.3 we provide an explicit and intuitive example which shows that, under suitable (and realistic) boundary conditions, minimizers must have a planar region $U$ bounded by at least three straight line segments (therefore $U \subset \tilde{C}_{\nabla u}$), and that $u$ fails to be smooth at each of these segments.

If the domain $S$ is convex, the set $\Sigma_r$ is clearly empty. In section 3.4 we give an example showing that, if the reference domain $S$ is nonconvex, then minimizers can have a nonempty set $\Sigma_r$ and that they fail to be smooth.

2. Preliminaries

In this section we provide some technical results that will be essential in the construction of the examples mentioned earlier. In section 2.1 we prove some simple but useful properties related to the metric properties of isometric immersions. In section 2.3 we recall some facts about the set $C_{\nabla u}$ and its connected components. The main non-regularity result is stated in section 2.4.

2.1. Metric lemmas

By definition we have $u = u_0$ and $\nabla u = \nabla u_0$ on $\partial S$ for all $u \in A_{u_0}(S, \partial S)$. Due to the isometry constraint, these boundary conditions may determine $u$ in the interior of $S$ as well.
Lemma 2.1. Let $l \subset \mathbb{R}^2$ be a line segment with endpoints $a$ and $b$. Let $u : l \to \mathbb{R}^3$ be 1-Lipschitz on $l$, i.e.

$|u(z) - u(y)| \leq |z - y|$ for all $z, y \in l$,

and assume that

$|u(b) - u(a)| = |b - a|.$

Then $u$ is an affine isometry on $l$, i.e.

$u((1-t)a + tb) = (1-t)u(a) + tu(b)$ for all $t \in [0, 1]$.

Proof. Let $x \in l$ be a point on the line $(ab)$. Then, since $a$, $x$ and $b$ are parallel, $|b - x| + |x - a| = |b - a|$. The triangle inequality, combined with the assumptions of the lemma, give

$|b - a| = |u(b) - u(a)| \leq |u(b) - u(x)| + |u(x) - u(a)|$

$\leq |b - x| + |x - a| = |b - a|.$

Since the leftmost and rightmost terms agree, both inequalities must in fact be equalities. Together with the Lipschitz condition, the second (in)equality implies that $|u(b) - u(x)| = |b - x|$ and $|u(x) - u(a)| = |x - a|$. Then by the triangle inequality, we get for any $y, z \in l$:

$|u(z) - u(y)| = |u(z) - u(a) + u(a) - u(y)| \geq ||z - a| - |y - a|| = |z - y|.$

Together with the Lipschitz condition, the assertion follows. 

This sort of metric results is relevant in the context of the present paper because clearly every $u \in \mathcal{W}^{1,\infty}_{0,0}(S)$ belongs to $C^0(\overline{S}, \mathbb{R}^3)$ and is 1-Lipschitz on $\overline{S}$. An application of the previous lemma therefore yields the following:

Remark 2.2. Let $u_0 \in \mathcal{W}^{2,\infty}_{0,0}(S)$ and let $x_1, x_2 \in \overline{S}$ be such that $|u_0(x_1) - u_0(x_2)| = |x_1 - x_2|$. (This is the case if $x_1$ and $x_2$ belong to a segment of constancy of $u_0$, cf below.) If $u \in \mathcal{W}^{2,\infty}_{0,0}(S)$ agrees with $u_0$ in $x_1$ and $x_2$, then in fact $u = u_0$ on the whole segment with endpoints $x_1$ and $x_2$.

The next lemma is essentially a two-dimensional version of lemma 2.1. In what follows, $B_r(x)$ denotes the open disk of radius $r$ centred at $x$.

Lemma 2.3. Let $u : B_r(x) \to \mathbb{R}^3$ be 1-Lipschitz and let $a_1, b_1, a_2, b_2 \in \partial B_r(x)$ such that the segments $(a_1 b_1)$ and $(a_2 b_2)$ have exactly the point $x$ in common. In addition, let $u$ be an affine isometry on $(a_1 b_1)$ and on $(a_2 b_2)$. Then $u$ is an affine isometry on the whole convex hull $\text{conv}\{a_1, a_2, b_1, b_2\}$.

Proof. Without loss of generality we assume $x = 0$ and $u(0) = 0$ Then the isometry condition states that $|u(a_i)| = |a_i|$ and $|u(b_i)| = |b_i|$ for $i = 1, 2$.

We will now show that

$u(b_1) \cdot u(b_2) = b_1 \cdot b_2,$

that is, $u$ preserves the angle between $(a_1 b_1)$ and $(a_2 b_2)$, or $(0 b_1)$ and $(0 b_2)$. Indeed, suppose that equation is wrong, say,

$u(b_1) \cdot u(b_2) > b_1 \cdot b_2.$  

(8)
Then we can calculate
\[
|u(b_2) - u(a_1)|^2 = |u(b_2)|^2 + |u(a_1)|^2 - 2u(b_2) \cdot u(a_1)
\[
= |u(b_2)|^2 + |u(a_1)|^2 + 2u(b_2) \cdot u(b_1) \frac{|u(a_1)|}{|u(b_1)|}
\[
> |b_2|^2 + |a_1|^2 + 2b_2 \cdot b_1 \frac{|a_1|}{|b_1|}
\[
= |b_2 - a_1|^2,
\]
but this contradicts the Lipschitz-condition.

Now let \(y_1 \in (0b_1)\) and \(y_2 \in (0b_2)\). The previous lemma shows that \(|u(y_i)| = |y_i|\). With a similar calculation to the above, we can show that \(u\) is affine on \((y_1, y_2)\):
\[
|u(y_2) - u(y_1)|^2 = |u(y_2)|^2 + |u(y_1)|^2 - 2u(y_2) \cdot u(y_1)
\[
= |u(y_2)|^2 + |u(y_1)|^2 - 2u(b_2) \cdot u(b_1) \frac{|u(y_2)|}{|u(b_2)|} \frac{|u(y_1)|}{|u(b_1)|}
\[
= |y_2|^2 + |y_1|^2 - 2b_2 \cdot b_1 \frac{|y_2|}{|b_2|} \frac{|y_1|}{|b_1|}
\[
= |y_2 - y_1|^2.
\]
With analogous inferences for \(y_1 \in (0a_1)\) and so on, we finally conclude that \(u\) is affine on \(\text{conv}\{a_1, a_2, b_1, b_2\}\).

**Lemma 2.4.** Let \(u \in W^{2,2}_{\text{loc}}(S)\) and \(l = (zy) \subset S\) with \(|u(z) - u(y)| = |z - y|\). Then either \(l \subset C_v u\) or \(l \subset S \setminus C_v u\). In the latter case, \(l \subset [x]\) for any \(x \in l\).

**Proof.** Assume that \(l\) is not contained in \(C_v u\) and let \(x \in l \setminus C_v u\). Then \([x]\) is well-defined and is contained in \(S \setminus C_v u\). If \([x]\) were not parallel to \(l\), then lemma 2.3 would imply that \(x \in C_v u\), a contradiction. Hence \(l\) is parallel to \([x]\) and therefore \(l \subset [x]\) by maximality of \([x]\). □

Recall that \(W^{2,2}_{\text{loc}}(S)\) embeds into \(C^4(S)\), so pointwise values of \(\nabla u\) are well-defined.

**Definition 2.5.** A segment \(l \subset S\) is said to be a *segment of constancy* of \(\nabla u\) if both endpoints of \(l\) lie on \(\partial S\) and \(\nabla u\) is constant on \(l\).

**Remark 2.6.** Let \(u \in W^{2,2}_{\text{loc}}(S)\). Then the following are true:
(a) If \(x \in S \setminus C_v u\), then \([x]\) is the unique segment of constancy containing \(x\).
(b) If there is no segment of constancy containing \(x\), then \(x \in \hat{C}_v u\).

**Proof.** The first assertion is in fact part of our definition of developability. The second one follows from ([13], proposition 1) and ([13], proposition 9). □

**Remark 2.7.** Let \(u \in W^{2,2}_{\text{loc}}(S)\) and let \(l \subset S\) be a nondegenerate segment with endpoints on \(\partial S\). Then the following are equivalent:
(a) The segment \(l\) is a segment of constancy of \(\nabla u\).
(b) Denoting by \(x_1, x_2\) the endpoints of \(l\), we have \(|u(x_1) - u(x_2)| = |x_1 - x_2|\).
**Proof.** If $l$ is a segment of constancy, then clearly $u$ is an affine isometry on $l$. Conversely, if $|u(x_1) - u(x_2)| = |x_1 - x_2|$ then lemma 2.4 implies that $l$ is a segment of constancy. □

**Lemma 2.8.** Let $u_0 \in W^{2,2}_{iso}(S; \mathbb{R}^3)$, let $\partial S$ be a closed subset of $\partial S$ and let $u \in A_{u_0}(S; \partial_c S)$. Let $x \in S \setminus C_{u_0}$ be such that both endpoints of $[x]_{a_0}$ lie in $\partial_c S$. Then $[x]_{a_0}$ is a line of constancy of $\nabla u$ and $u = u_0$ on $[x]_{a_0}$. If, moreover, $x \in S \setminus C_{u_0}$ then $[x]_{a_0} = [x]_{b_0}$.

**Proof.** By hypothesis $|u_0(x_2) - u_0(x_1)| = |x_2 - x_1|$ and $u(x_i) = u_0(x_i)$ for $i = 1, 2$. Hence $u = u_0$ on $[x]_{a_0}$ by remark 2.2 and $[x]_{a_0}$ is a segment of constancy of $\nabla u$ by remark 2.7, which must agree with $[x]_{b_0}$ if $x \in S \setminus C_{u_0}$. □

### 2.2. Developability

In the language of [11, 13], the developability of maps $u \in W^{2,2}_{iso}(S; \mathbb{R}^3)$ can be stated as follows: there is a ruling for $\nabla u$, i.e. a map $q : S \setminus C_{u_0} \to S^1$ such that $\nabla u$ is constant on $[x]_{a_0}$ for all $x \in S \setminus C_{u_0}$. Here $[x]_l$, is the maximal line segment contained in $S$, with $x \in [x]_l$, and whose direction is given by $e \in \mathbb{R}^2$. The ruling $q$ is unique (after identifying antipodal points). Moreover, for all $x, y \in S \setminus C_{u_0}$ either $[x]_{a_0} \cap [y]_{a_0} = \emptyset$ or $[x]_{a_0} = [y]_{a_0}$. This condition implies that $q$ is Lipschitz on compact subsets of $S \setminus C_{u_0}$ if regarded as a mapping into the projective space $\mathbb{P}^1$, with Lipschitz constant near $x$ dominated by $(\text{dist}(x, \partial S))^{-1}$ [see ([13], remark 1 p 959) and ([15], proposition 2.30(i))].

In what follows we will write $[x]$ instead of $[x]_{a_0}$ unless there is a danger of confusion. In cases where it is not clear which immersion $u \in W^{2,2}_{iso}(S; \mathbb{R}^3)$ the ruling refers to, we will write $[x]_u$.

#### 2.3. The set $C_{u_0}$

We recall here some results from [5, 13] that will be essential in later sections.

**Lemma 2.9.** The set $C_{u_0}$ consists of countably many connected components. Every connected component $U$ of $C_{u_0}$ satisfies

(a) $S \cap U \subset S \setminus C_{u_0}$, and if $x \in S \cap \partial U$, then $[x]_l \subset S \cap \partial U$.

(b) for every $x \in S \cap \partial U$ there exists $r > 0$ such that one component of $B_r(x) \setminus [x]$ is contained in $U$ and the other one is contained in $S \setminus U$.

**Proof.** This is proven in ([13], lemma 5); see also ([5], proposition 2.3). □

In what follows, if $U$ is a connected component of $C_{u_0}$, then for each $x \in S \cap \partial U$ the segment $[x]$ is called an edge of $U$.

**Lemma 2.10.** Let $l_1, l_2 \subset S$ be segments of constancy of $\nabla u$ with $l_1 \neq l_2$. Then the following are true:

(a) if $l_1 \cap l_2 \neq \emptyset$, then there exists a connected component $U$ of $C_{u_0}$ such that $l_1 \cup l_2 \subset U$.

(b) if $l_1$ and $l_2$ disjoint but have a common endpoint, then there exists a connected component $U$ of $C_{u_0}$ such that $l_1 \cup l_2 \subset U$.

In particular, if $U$ is a connected component of $C_{u_0}$ and $x \in S \setminus C_{u_0}$ is such that $[x]$ intersects $\overline{U}$, then in fact $[x] \subset \partial U$.

**Proof.** Part (a) is ([5], lemma 2.6), but it also follows directly from lemma 2.4 above.
Part (b) is contained in ([5], proposition 2.8).
To prove the final assertion, notice that if \([x]\) is not contained in \(\partial U\), then it must be disjoint from all edges of \(U\). Since, however, \([x]\) intersects \(\overline{U}\), the segment \([x]\) must have precisely one endpoint in common with an edge \(l\) of \(U\). Hence, by part (b), \([x]\) and \(l\) belong to the boundary of a connected component \(V\) of \(\nabla u\). Since \(V\) and \(U\) have zero distance from each other (both contain \(l\) in their boundary), the continuity of \(\nabla u\) and the maximality of \(U\) imply that \(V \subset U\). Hence \([x]\) \(\subset \partial V \subset \partial U\), a contradiction. 

It is useful to notice that if \(u\) is a minimizer in \(A_u(S, \partial S)\), then it is also a minimizer under its own boundary conditions on suitable subsets of \(S\).

**Proposition 2.11.** Let \(S\) be a bounded simply connected domain, let \(l_1, \ldots, l_k \subset S\) be pairwise disjoint line segments such that there is a connected component \(S_0\) of \(S \setminus \bigcup_{i=1}^{k} l_i\) with

\[
S \cap \partial S_0 = \bigcup_{i=1}^{k} l_i.
\]

Then for every minimizer \(u\) within \(A_u(S, \partial S)\) such that each \(l_i\) (with \(i = 1, \ldots, k\)) is a segment of constancy of \(\nabla u\), the restriction \(u|_{S_0}\) is a minimizer within

\[
A_u \left( S_0, (\partial S \cap \partial S_0) \cup \bigcup_{i=1}^{k} T_i \right).
\]

**Proof.** The result follows from the observation that if \(\tilde{u} \in A_u(S_0, (\partial S \cap \partial S_0) \cup \bigcup_{i=1}^{k} T_i)\) then

\[
\tilde{u}(x) = \begin{cases} 
\tilde{u}(x) & \text{if } x \in S_0 \\
u(x) & \text{otherwise}
\end{cases}
\]

belongs to the set \(A_u(S, \partial S)\). Details can be found in ([5], proposition 2.11). 

**Remark.** Notice that

\[
A_u \left( S_0, (\partial S \cap \partial S_0) \cup \bigcup_{i=1}^{k} T_i \right) = A_u \left( S_0, \bigcup_{i=1}^{k} T_i \right)
\]

if \(S_0 \cap \partial S \subset \bigcup_{i=1}^{k} T_i\).

**Proposition 2.12.** Let \(u \in W^{2,2}_{iso}(S; \mathbb{R}^3)\), let \(l_1, \ldots, l_k \subset S\) be pairwise disjoint segments of constancy of \(\nabla u\), and let \(S_0\) be a connected component of \(S \setminus \bigcup_{i=1}^{k} l_i\). Assume that \(u\) is a minimizer within

\[
A_u \left( S_0, \bigcup_{i=1}^{k} T_i \right).
\]

Then the following are true:

(a) \(\hat{C}_{\nabla u}\) has at most \(k - 2\) connected components contained in \(S_0\).

(b) Each connected component \(U\) of \(\hat{C}_{\nabla u}\) contained in \(S_0\) has at most \(k\) edges.

(c) If \(U \subset S_0\) is a connected component of \(\hat{C}_{\nabla u}\) with fewer than three edges, then each edge of \(U\) intersects \(\partial S\) tangentially at one endpoint, or it coincides with some \(l_i\).

**Proof.** A proof can be found in ([5], theorem 4.3) and ([5], proposition 4.9).
2.4. Negative regularity result

The Euler–Lagrange equations derived in [12] lead to the (positive) regularity result theorem 1.1. However, they can also be used to obtain a negative regularity result which will be essential in the examples below. It is based on the following lemma.

**Lemma 2.13.** Let \( \partial_t S, u_0 \) and \( u \) satisfy the hypotheses of theorem 1.1. Let \( x_0 \in S \setminus C_{\Sigma_0} \) and \( R > 0 \) be such that one connected component of \( B_R(x_0) \setminus \{x_0\} \), denoted by \( B^+_R \), is contained in \( S \setminus (\Sigma_e \cup \Sigma_t \cup \partial C_{\Sigma_0}) \). Denote by \( \zeta \) the inner unit normal to \( B^+_R \) in \( x_0 \) and denote by \( \partial_t \) the directional derivative in the direction \( \zeta \). Then \( u \in C^3(B^+_R) \) and one of the following mutually exclusive assertions is true:

(a) \( \partial_t \partial_t u(x_0 + t\zeta) \) does not converge to zero as \( t \downarrow 0 \).

(b) \( \partial_t \partial_t u(x_0 + t\zeta) \rightarrow 0 \) as \( t \downarrow 0 \) and there is \( r \in (0, R) \) and \( c > 0 \) such that

\[
|\partial_t \partial_t u(x_0 + t\zeta)| \geq ct \quad \text{for all } t \in (0, r)
\]

(c) there is \( r \in (0, R) \) and \( C, c > 0 \) such that

\[
c + \frac{t}{\log t} \leq |\partial_t \partial_t u(x_0 + t\zeta)| \leq C + \frac{t}{\log t} \quad \text{for all } t \in (0, r)
\]

(d) \( u \) is affine on \( B^+_R \).

The proof of lemma 2.13 depends heavily upon [12, 13] and is therefore postponed to the appendix.

In what follows, for a given \( \alpha \in (0, 1) \) and an open set \( V \subset \mathbb{R}^2 \) we denote by \( C^{3,\alpha}(V) \) the space of all \( u \in C^3(V) \) which satisfy the Hölder condition

\[
\sup \left\{ \frac{|\nabla^3 u(x) - \nabla^3 u(y)|}{|x - y|^\alpha} : x, y \in V \text{ and } x \neq y \right\} < \infty.
\]

Let us note some consequences of lemma 2.13.

**Lemma 2.14.** Let \( \partial_t S, u_0 \) and \( u \) satisfy the hypotheses of theorem 1.1. Let \( \alpha \in (0, 1) \), let \( r > 0 \) and let \( x_0 \in S \) be such that \( B_r(x_0) \subset S \) and assume that \( u \in C^{3,\alpha}(B_r(x_0)) \). Assume, moreover, that \( u \) is affine on one half-ball (of \( B_r(x_0) \)) and that the other half-ball does not intersect \( \Sigma_e \cup \Sigma_t \cup \partial C_{\Sigma_0} \). Then \( u \) is affine on all of \( B_r(x_0) \).

**Proof.** We assume without loss of generality that \( x_0 = 0 \) and that \( u \) is affine on \( B^+_r = \{ x \in B_r(0) : x \cdot e_1 < 0 \} \). Since \( u \in C^3(B_r(0)) \), it is affine along \( B_r(0) \cap \{0\} \times \mathbb{R} \). Hence the hypotheses of lemma 2.13 are satisfied.

Notice that all derivatives of \( u \) of order at least two vanish on \( B^-_r \). Since \( u \in C^2(B_r(0)) \), we therefore have \( |\partial_t \partial_t u(x)| \rightarrow 0 \) as \( x \rightarrow 0 \). Hence (a) from lemma 2.13 does not hold.

Moreover, since \( u \in C^{3,\alpha}(B_r(0)) \), for all \( t \in (0, r) \) we have

\[
|\partial_t \partial_t u(t, 0)| \leq \int_0^t |\partial_t^2 u(s, 0)|, \quad ds = \int_0^t |\partial_t^2 u(s, 0) - \partial_t^2 u(0, 0)|, \quad ds \leq Ct^{1+\alpha}.
\]

Hence neither (b) nor (c) of lemma 2.13 are satisfied. Hence statement (d) from that lemma must be true.

**Proposition 2.15.** Let \( \partial_t S, u_0 \) and \( u \) satisfy the hypotheses of theorem 1.1 and let \( \alpha \in (0, 1) \). Let \( U \) be a connected component of \( C_{\Sigma_0} \), let \( x_0 \in S \cap \partial U \) and denote the outer unit
normal to $U$ in $x_0$ by $\zeta$. Assume that there is an $R > 0$ such that
\[
\{ x \in B_R(x_0) : (x - x_0) \cdot \zeta > 0 \} \cap (\Sigma_r \cup \Sigma_c \cup \partial \hat{C}_{\nu^u}) = \emptyset.
\]
Then $u$ fails to be $C^{3,\alpha}$ in any neighbourhood of $x_0$.

**Proof.** Due to lemma 2.9 there is $r \in (0, R)$ such that one connected component of $B_r(x_0) \setminus \{x_0\}$ is contained in $U$, while the other one, namely the half-disk
\[
B^+_r = \{ x \in B_r(x_0) : (x - x_0) \cdot \zeta > 0 \},
\]
does not intersect $U$.

So if we had $u \in C^{3,\alpha}(B_r(x_0))$ for some $\alpha \in (0, 1)$, then $B_r(x_0)$ would satisfy the hypotheses of lemma 2.14. Hence $u$ would have to be affine on all of $B_r(x_0)$, so by maximality of $U$ the whole disk $B_r(x_0)$ would have to be contained in $U$, contradicting the properties of $B^+_r$.

**Corollary 2.16.** Let $\partial \Sigma, \nu_0$ and $u$ satisfy the hypotheses of theorem 1.1 and let $\alpha \in (0, 1)$. Let $U$ be a connected component of $\hat{C}_{\nu^u}$ and let $x_0 \in (S \cap \partial U) \setminus (\Sigma_r \cup \Sigma_c)$. Assume, moreover, that $x_0$ has a positive distance from $\hat{C}_{\nu^u} \setminus U$ (this is satisfied, e.g. if $\hat{C}_{\nu^u}$ has only finitely many connected components). Then $u$ fails to be $C^{3,\alpha}$ near $x_0$.

**Proof.** We use the notation from the proof of proposition 2.15. Since $x_0$ has a positive distance from $\hat{C}_{\nu^u} \setminus U$, after possibly shrinking $r$ we may assume that $B^+_r$ does not intersect $\hat{C}_{\nu^u}$.

Since $S \setminus (\Sigma_r \cup \Sigma_c)$ is open and contains $x_0$, we may further assume that $B^+_r$ is contained in this set as well. Hence the claim follows from proposition 2.15.

3. Counterexamples to regularity

We are now ready to construct the examples announced in section 1.2.

3.1. Cylindrical building block

For the examples given in the next sections, it is useful to have the following building block at hand: with any arclength parametrised curve $\Xi$ from some interval $I$ into $\mathbb{R}^2$ and with any $\xi \in \mathbb{S}^1$ we associate the corresponding cylindrical deformation
\[
Z_{\Xi, \xi}(x) = \Xi_1(x \cdot \xi) \xi + (x \cdot \xi^\perp) \xi^\perp + \Xi_2(x \cdot \xi) e_3,
\]
defined on $J_{\Xi, \xi} = \{ x \in \mathbb{R}^2 : x \cdot \xi \in I \}$, with values in $\mathbb{R}^3$ and using the standard identification $\mathbb{R}^2 \equiv \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. So $Z_{\Xi, \xi}(x)$ is a local parametrisation of the cylinder generated by the regular curve $\Xi$. For the particular case $\Xi(t) = (\sin t, 1 - \cos t)$, we will briefly write $Z_\xi$ instead of $Z_{\Xi, \xi}$.

Observe that $Z_{\Xi, \xi}$ is an isometric immersion. Indeed, one computes
\[
\nabla Z_{\Xi, \xi}(x) = \Xi_1'(x \cdot \xi) \xi \otimes \xi + \xi^\perp \otimes \xi^\perp + \Xi_2'(x \cdot \xi) e_3 \otimes \xi
\]
from which directly follows that $Z_{\Xi, \xi}$ is an isometric immersion on its domain; its regularity is clearly determined by that of the curve $\Xi$. 4910
3.2. Example concerning $\Sigma_c$

Here we give a simple example of a smooth convex domain $S$ and boundary data $u_0 \in W^{2,2}_\text{iso}(S; \mathbb{R}^3)$ and $\partial_c S$ such that the minimizer is not $C^2$ and for which we can fully characterise $\Sigma_c$.

Let $S = B_1(0) \subset \mathbb{R}^2$ and $B_1^+(0) = B_1(0) \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$.

Using the cylindrical deformation from section 3.1 with the vector $\xi = e_1$, we define

$$u_0(x) = \begin{cases} Z_\xi(x) & x \in B_1^+(0), \\
 x & x \in B_1(0) \setminus B_1^+(0), \end{cases}$$

where we identify $\mathbb{R}^2$ with the set $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. We can write this explicitly as

$$u_0(x) = \begin{cases} (\sin(x_1), x_2, 1 - \cos(x_1))^T & \text{for } x_1 > 0, \\
 (x_1, x_2, 0)^T & \text{for } x_1 \leq 0. \end{cases}$$

Clearly $u_0 \in C^1(\overline{S}, \mathbb{R}^3)$. Notice that

$$C_{\nabla u_0} = B_1(0) \setminus B_1^+(0) \quad \text{and} \quad q_{u_0}(x) = e_2 \text{ for all } x \in B_1^+(0).$$

Set $R = [-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R}$. We set $\partial_c S = \partial S \cap R$. The situation is depicted in figure 2.

**Proposition 3.1.** Let $u \in \mathcal{A}_{u_0}(S, \partial_c S)$. Then the following are true:

(a) $u = u_0$ on $S \cap R$.
(b) $u \notin C^2(S)$.
(c) If, moreover, $u$ is a minimizer within $\mathcal{A}_{u_0}(S, \partial_c S)$, then $\Sigma_c = B_1^+(0) \cap R$.

**Proof.** Notice that

$$|u_0(x_1, x_2) - u_0(x_1, -x_2)| = |(x_1, x_2) - (x_1, -x_2)|$$
whenever \((x_1, x_2) \in \partial_s S\). Hence (a) follows from remark 2.2.

To prove (b), notice that \(u_0\) is not \(C^2\) at the origin. In fact, \(\nabla^2 u_0 \equiv 0\) on \(S \setminus B_1^+(0)\) whereas

\[
\partial_1 \partial_2 u_0(x) = (-\sin(x_1), 0, -\cos(x_1))^T,
\]

so \(|\partial_1 \partial_2 u_0(x)| = 1\) in the interior of \(B_1^+(0)\). From this follows that \(u \notin C^2(S)\) since the origin is a point in the interior of \(S \cap R\) and \(u = u_0\) on \(S \cap R\).

To prove (c), notice that by (a) we know that \(u\) is affine on \((S \cap R) \setminus B_1^+(0)\). Since \((\overline{S} \setminus R) \cap \partial_3 S = \emptyset\) and \(u\) is a minimizer, it follows that \(u\) is also affine on \(S \setminus R\). Hence

\[
C_{uv} = (S \setminus B_1^+(0)) \cup (B_1^+(0) \setminus R).
\]

Now \(S \setminus C_{uv} = B_1^+(0) \cap R\) and in this region \(u = u_0\) and \(\overline{x}|_0 \cap \partial_3 S \neq \emptyset\). Necessarily \(\Sigma_e = B_1^+(0) \cap R\). \(\square\)

3.3. Example concerning \(\partial C_{uv}\)

In this section we give an example of a domain and of boundary data which show that the set \(C_{uv}\) can really be nonempty for a minimizer (and indeed for any admissible immersion \(u \in A_{u_0}(S, \partial_3 S)\)). We then show that regularity may indeed fail at the singular set \(S \cap \partial C_{uv}\).

3.3.1. Existence of \(C_{uv}\). We give an example of a smooth convex domain \(S\) and boundary data \(u_0 \in W^{2,2}_{\text{loc}}(S; \mathbb{R}^3)\) and \(\partial_3 S\) such that \(C_{uv} \neq \emptyset\) for all \(u \in A_{u_0}(S, \partial_3 S)\).

Let \(S = B_1(0) \subset \mathbb{R}^2\) and let \(\Delta\) denote the interior of the equilateral triangle with baricenter at the origin and vertices on \(\partial S\) given by \(e^{-\pi}, e^{\frac{\pi}{2}}\) and \(e^{\frac{\pi}{3}}\); see figure 3. Here and in what follows we use the standard identification of \(\mathbb{R}^2\) with \(\mathbb{C}\). Setting \(\varphi_k = \frac{\pi}{3} + \frac{2\pi}{3}k\) and \(v_k = e^{i\varphi_k}\), we have

\[
\Delta = \bigcap_{k=0}^{2} \left\{ x \in \mathbb{R}^2 : x \cdot v_k < \frac{1}{2} \right\}.
\]

The idea is to choose \(u_0 \in W^{2,2}_{\text{loc}}(S; \mathbb{R}^3)\) such that \(u_0|_{\Delta}\) is the identity, while on the three connected components of \(S \setminus \Delta\) the deformation \(u_0\) consists of pieces of cylinders that are \(C^1\)-attached to \(\Delta\). By suitably choosing \(\partial_3 S \subset \partial S\), any element \(u\) in \(A_{u_0}(S, \partial_3 S)\) will necessarily satisfy \(C_{uv} \neq \emptyset\).

We define \(u_0 : S \to \mathbb{R}^3\) explicitly by setting (we use the canonical injection of \(\mathbb{R}^2\) into \(\mathbb{R}^3\))

\[
u_0(x) = \begin{cases} 
Z_{\alpha} \left( \frac{x - \frac{v_k}{2}}{2} \right) + \frac{v_k}{2} & \text{if } x \cdot v_k \geq \frac{1}{2} \\
x & \text{if } x \in \Delta.
\end{cases}
\]

Here \(Z_{\alpha}\) the cylindrical deformation defined in section 3.1. The map \(u_0\) is perhaps the simplest isometric deformation displaying the set \(C_{uv}\); as such it resembles, e.g. a figure in [23], where the set \(C_{uv}\) is discussed in the context of rigidity. Notice that one could easily modify \(u_0\) to make the transition from the affine triangle to each cylindrical part \((C^\infty\)-smooth. For minimizers, however, we will see that this transition is not smooth.
Figure 3. The disk is $S$. The yellow triangle is $\Delta$. The dashed triangle is $\Delta'$. The three red arcs constitute $\partial S$. The blue segments belong to $\Sigma_c$; here we have $u = u_0$.

Our first main result in this section is the following one:

**Proposition 3.2.** Let $v_k$ and $u_0$ as defined above, let $\alpha \in (\frac{\pi}{3}, \frac{2\pi}{3})$ and set $\rho = 2 \cos \left(\frac{\alpha}{2}\right)$.

Define

$$\partial_S = \bigcup_{k=0}^{2} \left\{ x \in S^1 : x \cdot v_k \geq \frac{\rho}{2} \right\}. \quad (13)$$

Then for every $u \in A_{\partial u}(S, \partial S)$ the set $\hat{C}_u$ is nonempty and contains the origin.

The upper bound on $\alpha$ ensures that $\rho > 1$ while the lower bound is needed in the proof of lemma 3.3 below.

We split the proof of proposition 3.2 into two lemmas. Set

$$\Delta' = \bigcup_{k=0}^{2} \left\{ x \in \mathbb{R}^2 : x \cdot v_k < \frac{\rho}{2} \right\}$$

and define $S' = S \cap \Delta'$. For $k = 0, 1, 2$ set $A_k = \{x \in S : x \cdot v_k = \rho/2\}$. We first prove that any segment with both endpoints on $\partial S$ and containing the origin has to intersect $S \setminus S'$.

**Lemma 3.3.** Let $l \subset S$ be a segment containing the origin and intersecting $\partial S$ at both endpoints. Then there is $k \in \{0, 1, 2\}$ and there are $y^{(1)}, y^{(2)} \in l$ such that

$$y^{(1)} \cdot v_k, \quad y^{(2)} \cdot v_k \in (\rho, 1) \text{ yet } y^{(1)} \cdot v_k \neq y^{(2)} \cdot v_k.$$

**Proof.** First we notice that the claim will follow directly once we prove that $l$ intersects $S \setminus S'$, because $S \setminus S'$ is open.

We parameterised $l$ by the map $(-1, 1) \ni t \mapsto t e^{i\psi}$ for some $\psi \in \mathbb{R}$. By the choice of $\rho$ we see that the ray $\mathbb{R} e^{i\psi}$ intersects $S \setminus S'$ if and only if

$$\psi \in \left(\phi_k - \frac{\alpha}{2}, \phi_k + \frac{\alpha}{2}\right).$$
for some $k$. So if $l \subset S$ is an arbitrary segment through the origin with end-points $e^{i\varphi}, e^{i(\varphi + \pi)}$ on $\partial S$, then $l$ intersects $S \setminus \overline{S}$ if and only if $\psi$ is contained in
\[
\bigcup_{k=0}^{2} \left( \frac{\alpha}{2} \cdot \varphi_k + \frac{\alpha}{2} \right) + \pi \mathbb{Z} = \left( \frac{\pi}{6} - \frac{\alpha}{2} \right) \mathbb{Z}.
\]
The equality arises from the definition of the $\varphi_k$. But the right-hand side is the whole real line because $\alpha > \frac{\pi}{2}$. Hence the condition on $\psi$ is trivially satisfied. \hfill \Box

**Lemma 3.4.** Let $S'$ be defined as in (11). Then, $u = u_0$ on $S \setminus S'$ for any $u \in \mathcal{A}_{\alpha}(S, \partial S)$.

**Proof.** Clearly, $C_{\nabla u_0} = \Delta$ and the ruling $q_{u_0}$ of $\nabla u_0$ is such that $q_{u_0}(x) = v_k^\perp$ whenever $x \cdot v_k \geq 1/2$. The claim follows from lemma 2.8. \hfill \Box

**Proof of proposition 3.2.** Let $l$ be a segment in $S$ with both endpoints on $\partial S$ and containing the origin. Let $y^{(1)}, y^{(2)} \in l$ as in the conclusion of lemma 3.3. From the definition of $u_0$ and from (9) it is clear that $\nabla u_0(y^{(1)}) \neq \nabla u_0(y^{(2)})$ since $y^{(1)}, v_k \neq y^{(2)}, v_k$. On the other hand, by lemma 3.4 we also have $u = u_0$ on the open set $S \setminus \overline{S}$, to which $y^{(1)}$ and $y^{(2)}$ belong. So $\nabla u(y^{(1)}) \neq \nabla u(y^{(2)})$. We conclude that $\nabla u$ is not constant on $l$.

Hence there exists no segment of constancy of $\nabla u$ that contains the origin. So by remark 2.6 the origin is contained in $\hat{C}_{u}$. \hfill \Box

3.3.2. **Failure of regularity for minimizers.** We will now see that minimizers may indeed fail to be $C^\infty$ at $\partial \hat{C}_{u}$. With notation as above, we have the following result.

We begin with a general lemma.

**Lemma 3.5.** Let $S$ be convex, $\partial S \subset \partial S$ closed and let $u \in W^{2,2}(S)$. Then
\[
\Sigma_c = \{ x \in S \setminus C_{\nabla u} : [x] \cap \partial S \neq \emptyset \}.
\] (14)

**Remark.** For nonconvex $S$ the assertion is false in general.

**Proof.** We must show that the set on the right-hand side of (14) is closed in $S$. Let $x_n \in S \setminus C_{\nabla u}$ be such that $[x_n] \cap \partial S \neq \emptyset$ for all $n$ and assume that the $x_n$ converge to some $x \in S$. Then $x \notin C_{\nabla u}$ because $C_{\nabla u}$ is open. And by ([13], lemma 2(iii)) the segments $[x_n]$ converge to $[x]$ in the Hausdorff sense. Hence indeed $[x] \cap \partial S \neq \emptyset$ as well. \hfill \Box

**Proposition 3.6.** Let $S = B_1(0)$, define $\partial_S$ as in (13) and $u_0$ as in (12), and let $u$ be a minimizer within $\mathcal{A}_{\alpha}(S, \partial S)$.

Then $\hat{C}_{u}$ consists of precisely one connected component $U$. The component $U$ contains the origin and it has precisely three edges. At least two of these edges do not belong to $\Sigma_c$ and near any point in $S \cap \partial U \setminus \Sigma_c$ (i.e. on any edge that does not belong to $\Sigma_c$) the map $u$ fails to be $C^\infty\alpha$ for any $\alpha > 0$.

**Proof.** As $u \equiv u_0$ on $S \setminus S'$, by lemma 3.4 it is clear that $C_{\nabla u} \subset S'$, because $\nabla u_0$ is not constant on any open subset of $S \setminus S'$. By proposition 3.2 we know that there exists a connected component $U$ of $C_{\nabla u}$ containing the origin, and that in fact $U$ belongs to $\hat{C}_{u}$.

Clearly $S'$ is a connected component of $S \setminus \bigcup_{k=0}^{2} l_k$ and $S \cap \partial S' = \bigcup_{k=0}^{2} l_k$. By continuity $\nabla u$ is constant on each $l_k$ and agrees with $\nabla u_0$ on $l_k$, because both agree on $S \setminus S'$. Notice that $l_k = [x_k]$ for suitable $x_k \in S \setminus C_{\nabla u}$, because clearly $l_k$ is not contained in $C_{\nabla u}$ (since $C_{\nabla u} \subset S'$), and $\nabla u$ is constant on $l_k$. In particular, the segments $l_k$ are disjoint segments of constancy of $\nabla u$. 
Proposition 2.11 implies that \(u|_{\gamma'}\) is a minimizer within
\[
\mathcal{A}_{\gamma}(S', \bigcup_{k=0}^{2} T_k).
\] (15)

Since \(C_{\gamma_u} \subset S'_\gamma\) in particular \(U \subset S'_\gamma\), Proposition 2.12 implies that \(U\) is the only connected component of \(\tilde{C}_{\gamma_u}\) and that it has at most (hence precisely, due to the definition of \(\tilde{C}_{\gamma_u}\)) three edges.

Lemma 2.10(b) shows that for \(k = 0, 1, 2\) we have
\[
\mathcal{U} \cap T_k \neq \emptyset \implies l_k \subset \partial U.
\] (16)

Since \(\nabla u\) is constant on \(U\) and since \(\nabla u = \nabla u_0\) attains a different constant value on each of the \(l_k\), we deduce from (16) that there is at most one \(k\) such that \(U\) intersects \(T_k\) (and then in fact \(l_k \subset \partial U\)). Hence lemma 3.5 implies that the other two edges of \(U\) (they constitute the set \(S \cap \partial U \backslash \bigcup_{k=0}^{2} l_k\)) do not intersect \(\Sigma\).

Finally, let \(x_0 \in S \cap \partial U \cap \Sigma\). Since \(S\) is convex we have \(\Sigma_{\tau} = \emptyset\). Since moreover \(\tilde{C}_{\gamma_u} = U\), we conclude that \(x_0\) satisfies the hypotheses of corollary 2.16, so the claim follows.

3.4. Example concerning \(\Sigma_{\tau}\)

Define \(f: \mathbb{R} \to \mathbb{R}\) by
\[
f(x) = \frac{1}{2}(1 - x^2)(7x^2 + 1)
\]
and define
\[
S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-1.1, 1.1), x_2 \in (f(1.1), f(x_1))\}.
\] (17)

Set
\[
\partial^- S = \{x \in \partial S : x_2 < 0\}
\]
and
\[
\partial^+ S = \{x \in \partial S : x_1 < 0 \text{ and } x_2 > \frac{7}{8}\}
\]
and define
\[
\partial \gamma S = \partial^- S \cup \partial^+ S.
\] (18)

Lemma 3.7. There exists an isometric immersion \(u_0 \in W^{2,2}_{\text{iso}}(S)\) that satisfies the boundary conditions
\[
u_0(x) = \begin{cases} (x_1, x_2, 0) & \text{if } x \in \partial^- S \\ (x_1, \frac{1}{2} - x_2, \frac{1}{\pi}) & \text{if } x \in \partial^+ S. \end{cases}
\] (19)

Intuitively, any \(u \in \mathcal{A}_{u_0}(S, \partial \gamma S)\) therefore remains clamped to the plane \(\{x_3 = 0\}\) on the lower part of \(S\), while the left ‘hill’ of \(S\) is bent backwards by an angle \(\pi\).

Proof. The deformation \(u_0\) will be cylindrical. The underlying two-dimensional curve is \(\beta: \mathbb{R} \to \mathbb{R}^2\) given by
\[
\beta(\tau) = \begin{cases} \tau + \frac{\pi}{2} - i \text{ if } \tau < -\frac{\pi}{2} \\ e^{i\tau} \text{ if } \tau \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \frac{\pi}{2} - \tau + i \text{ if } \tau > \frac{\pi}{2}. \end{cases}
\]
Figure 4. The curve $\Xi$. The red region corresponds to the parameter domain $t < 0$ and the blue line to the parameter domain $t > \frac{1}{2}$. In between $\Xi$ parametrises a half-circle of radius $\frac{1}{2\pi}$ transversed in the counterclockwise direction.

Notice that $\beta \in W^{2,\infty}(\mathbb{R})$ and that it is parametrised by arclength. We define $\Xi : \mathbb{R} \to \mathbb{R}^2$ by

$$\Xi(t) = \frac{i}{2\pi} + \frac{1}{2\pi} \beta \left( 2\pi t - \frac{\pi}{2} \right).$$

This curve is depicted in figure 4. Finally, we set $u_0 = Z_{\Xi,c_2}$; the right-hand side is the cylindrical building block from section 3.1. It is easy to verify that $u_0$ has the desired properties.

As was the case in our previous example, also here we could easily modify $u_0$ to become a $C^\infty$ isometric immersion satisfying (19). This merely requires replacing the curve $\beta$ by a smooth curve with similar properties, e.g. by mollifying and reparametrising.

We will see that a typical $u \in A_{u_0}(S, \partial_c S)$ must exhibit the singular set $\Sigma_T$. Before stating this result, we collect some simple facts about $f$.

**Lemma 3.8.** The function $f$ defined above has the following properties:

- $f$ is symmetric with respect to the $x_2$-axis.
- $f$ has a local minimum in 0, and $f(0) = \frac{1}{2}$.
- $f$ has points of inflection at $x_1 = \pm a$ with $a = \frac{1}{\sqrt{7}}$, and $f(\pm a) = \frac{6}{7}$.
- $f$ is strictly concave on the intervals $(-1, -a)$ and $(a, 1)$ and strictly convex on $(-a, a)$.

Moreover, we have

$$f(-a) < f(0) < f(a),$$

and

$$f'(a) < \frac{f(a)}{1-a}.$$  

**Proof.** Most statements are readily verified. Regarding inequality (20), it follows from the fact that $\frac{2\sqrt{7}}{7} < \frac{1}{2}$. Finally, inequality (21) follows from

$$f'(-a) = \left. [-2x(7x^2 - 3)]_{x=-a} = 2a(7a^2 - 3) = -4a = -\frac{4\sqrt{7}}{7}\right.$$

4916
and

\[- \frac{f(a)}{1 - a} = - \frac{6}{7(1 - 1/\sqrt{7})} = - \frac{6\sqrt{7}(\sqrt{7} + 1)}{7(\sqrt{7} + 1)(\sqrt{7} - 1)} = - \frac{7 + \sqrt{7}}{7}.\]

since $3\sqrt{7} > 7$. \hfill \Box

**Proposition 3.9.** Let $S$ be as in (17), let $\partial_x S$ as in (18), let $u_0$ satisfy the conclusion of lemma 3.7 and assume that $u$ is a minimizer within $A_{u_0}(S, \partial_x S)$.

Then there exists an $x_0 \in S \setminus C_{u_0}$ such that $[x_0]$ intersects $\partial S$ tangentially. Moreover, $u$ fails to be $C^1,\alpha$ near $x_0$, for any $\alpha \in (0, 1)$.

**Proof.** Since $\partial_x S$ consists of only two disjoint components $\partial_x^1 S$ and $\partial_x^2 S$, an application of ([5], proposition 4.9) shows that $\hat{C}_{\partial_x S}$ is empty. Hence by ([13], proposition 1) and ([13], proposition 9), there exists a ruling for $u$ on $S \setminus \hat{C}_{\partial_x S}$, which in this case is all of $S$. We fix one of these (not uniquely determined) rulings on $S$ and call it $q$. As usual, we write $[x] = [x]_{q(x)}$.

Let $a = \frac{1}{\sqrt{7}}$ and define the interval

$Z = \{ z \in (0, f(a)) : \left[ (z', a) \right] \cap \text{graph } f|_{[1-a,a]} = \emptyset \text{ for all } z' \in (0, z) \}.$

Set $z_0 = \sup Z.$ Observe that $Z$ is nonempty and $z_0 > 0$ because $q(a, z) \to \pm e_1$ as $z \downarrow 0$. We claim that $[z_0, 0)$ intersects $\text{graph } f|_{[1-a,a]}$ tangentially, see figure 5.

Denote by $m(z)$ the unique slope such that the segment $[(a, z)]$ lies on the graph of the affine function $l_z : \mathbb{R} \to \mathbb{R}$ defined by

$$l_z(t) = z + m(z)(t - a).$$

Let us prove that

$$m(z) > - \frac{z}{1 - a} \quad (22)$$

for all $z \in (0, f(a))$. In fact, otherwise $l_z < f$ on $[a, 1]$ and $l_z(1) < 0$. The former inequality implies that the graph of $l_z$ restricted to $\{ x_1 \geq a : l_z(x_1) > 0 \}$ agrees with $\left[ (a, z) \right] \cap \{ x_1 \geq a \}$. Hence the inequality $l_z(1) < 0$ would imply that $\left[ (a, z) \right]$ intersects $\mathbb{R} \times \{ 0 \}$. This would contradict the properties of the ruling $q$.

Next we claim that

$$m(z) < \frac{z}{1 + a} \text{ for all } z \in Z. \quad (23)$$

In fact, assume that this were not satisfied for some $z \in Z$. Then $l_z(-1) < 0$. On the other hand, by the definition of $Z$ we know that $\left[ (a, z) \right]$ does not intersect the graph of $f|_{[-a,a]}$. Hence $l_z < f$ on $[-1, a]$. We can argue as before to see that this would imply that $\left[ (a, z) \right]$ intersects $\mathbb{R} \times \{ 0 \}$.

For $z \in Z$ we have $l_z(0) < f(0)$, because otherwise $l_z$ would intersect the graph of $f$ in $[0, a)$. Hence we conclude using (23) that

$$z < (a + 1)f(0) \text{ for all } z \in Z. \quad (24)$$

Since $f(a)$ exceeds the right-hand side of this inequality by (20), we conclude that indeed $z_0 < f(a)$.

We claim that $l_{z_0}$ intersects $f$ tangentially in precisely one point $a_1 \in [-a, a]$, and that in fact $a_1 \in (-a, a)$. To see this, observe that by maximality of $z_0$ and compactness of $\text{graph } [1-a,a]$, it clearly must intersect $f$ in $[-a, a]$. And then it must do so at some point $a_1 \in (-a, a)$ because
\( l_{z_0}(a) = z_0 < f(a) \), so \( l_{z_0} \) cannot intersect \( f \) in \( a \). But if it intersected \( f \) in \( -a \), then its slope \( m(z_0) \) would have to be smaller than \( f'(-a) \). Therefore using (21) and (24), we would derive

\[
m(z_0) < f'(-a) < -\frac{f(a)}{1-a} < -\frac{z_0}{1-a},
\]

contradicting (22).

Finally, let us verify that \( l_{z_0} \) must intersect \( f \) tangentially in \( a_1 \). In fact, otherwise for all \( z \) near \( z_0 \) the function \( l_z \) would intersect \( f \) on \((-a,a)\) as well, because \( a_1 \in (-a,a) \) and \( l_z \rightarrow l_{z_0} \) locally uniformly on \( \mathbb{R} \) as \( z \rightarrow z_0 \). This would contradict the choice of \( z_0 \).

By strict convexity of \( f([-a,a]) \), we see that \( l_{z_0} \) intersects \( f \) precisely once on \([-a,a] \), because it is a supporting segment.

For all \( z < z_0 \) the segment \([z,a]\) intersects \( f \) outside \([-a,a] \). Since here \( f \) is strictly concave, all these segments intersect \( \partial S \) transversally. Since, moreover, \((z_0,a) \notin \Sigma_r \) and since \( \Sigma_c \) is open, we conclude that there exists an \( r > 0 \) such that

\[
\{ x \in B_r(z_0,a) : (x - (z_0,a)) \cdot (m(z_0),-1) > 0 \} \cap (\Sigma_r \cup \Sigma_c) = \emptyset.
\]

We have seen that \( l_{z_0} \) intersects \( f \) on \([-a,a] \) in precisely one point \( a_1 \in (-a,a) \). Since \( f \) is strictly concave outside \([-a,a] \), there is exactly one \( a_2 \in (a,1) \) with \( l_{z_0}(a_2) = f(a_2) \). Clearly \([a,z_0] = \text{graph} l_{z_0}|_{(a_1,a_2)} \). The set

\[
U = \{(x_1,x_2) : x_1 \in (a_1,a_2) \text{ and } x_2 \in (l_{z_0}(x_1), f(x_1))\}
\]

is a connected component of \( S \setminus [(a,z_0)] \) that has a positive distance from \( \partial_r S \). Since \( u \) is a minimizer, it is affine on \( U \).

Therefore, in view of (25), we conclude that \( x_0 = (a,z_0) \) satisfies the hypotheses of proposition 2.15. The claim follows from that proposition. \( \square \)

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Appendix. The Euler–Lagrange equation

Following [13] we define the function \( \nu : S \times (\mathbb{R}^2 \setminus \{0\}) \to (0, \infty) \) by

\[
\nu(x, \theta) = \min\{s > 0 : x + s\theta \in \partial S\}.
\]

Following [12] we describe \( u \in W^{2,2}_{loc}(S, \mathbb{R}^3) \) locally near any point \( x \in S \setminus \overline{C_T} \) via an arclength parametrised curve \( \Gamma \in \mathcal{L}^{\infty}([-T, T]; S) \) with \( \Gamma(0) = x \) and a function \( \mu \in L^2([-T, T]) \).

More precisely, let \( q : S \setminus \overline{C_T} \to \mathbb{R}^2 \) be a ruling for \( \nabla u \) and let \( \Gamma \) be the solution of \( \Gamma' = -q^{-1}(\Gamma) \) with initial value \( \Gamma(0) = x \). Since \( q \) is locally Lipschitz, there exists \( T > 0 \) such that this solution \( \Gamma \) exists on an interval \([-T, T]\). Moreover, setting \( N(t) = (\Gamma'(t))^\perp \), we see that \( u \) is affine along

\[
[\Gamma(t)]_{q(\Gamma(t))} = \{\Gamma(t) + sN(t) : s^T(t) < s < s^T(t)\}.
\]

Here we have set, following [13],

\[
s^T(t) = \pm \nu(\Gamma(t), \pm N(t)).
\]

Hence

\[
u(\Gamma(t) + sN(t)) = u(\Gamma(t)) + s\nabla u(\Gamma(t))N(t)
\]

for all \( t \in [-T, T] \) and for all

\[
s^T(t) < s < s^T(t).
\]

Finally, there exists a function \( \mu \in L^2(-T, T) \) such that

\[
\partial_3 \partial_2 \mu(\Gamma(t) + sN(t)) = \frac{\mu(t)}{1 - s\kappa(t)} \Gamma'(t)N'(t) \partial_3 u(\Gamma(t)) \times \partial_2 u(\Gamma(t)),
\]

where \( \kappa = \Gamma'' \cdot N \) is the curvature of \( \Gamma \).

For \( \Gamma \) and \( \mu \) as above, we will write \( u = (\Gamma, \mu) \). We refer to ([11], proposition 1) and ([12], lemma 2.7) as well as ([5], section 1.1) for more details on this correspondence between \( u \) and pairs \( (\Gamma, \mu) \).

The following result is ([12], theorem 1.8), taking into account ([12], remark 1.9).

**Theorem 3.10.** Let \( \partial S \) be a closed subset of \( \partial S \) with positive length and let \( u_0 \in W^{2,2}_{loc}(S, \mathbb{R}^3) \). Let \( u \) be a minimizer in \( A_{\partial S}(S, \partial S) \) and let \( x_0 \in S \setminus (\overline{C_T} \cup \Sigma_c \cup \Sigma_c) \).

Then there exist \( T > 0 \) and an arclength parametrised curve \( \Gamma \in \mathcal{L}^{\infty}([-T, T], S) \) with \( \Gamma(0) = x_0 \) and which is transversal in \([-T, T]\), and there exists \( \mu \in L^2(-T, T) \) such that \( u = (\Gamma, \mu) \) on \([\Gamma(-T, T)]\).

Moreover, there exist solutions \( \Lambda \in W^{1,2}([-T, T]; \mathbb{R}^3) \) and \( M \in W^{1,2}([-T, T]; \mathbb{R}^3) \) of the system

\[
M' = \begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & 0 \\ -\mu & 0 & 0 \end{pmatrix} M,
\]

and

\[
\Lambda' = \begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & 0 \\ -\mu & 0 & 0 \end{pmatrix} \Lambda + \begin{pmatrix} 0 & e_3 \cdot M \\ e_3 \cdot M & 0 \end{pmatrix},
\]

where
and solutions $\Omega_2, \Omega_3 \in W^{1,1}(-T,T)$ of

$$
\Omega_2' = \mu e \cdot M + \mu^2 \left( \frac{\Gamma' \cdot \partial_1 \nu(\Gamma, N)}{1 - s_1^2} + \frac{\Gamma' \cdot \partial_1 \nu(\Gamma, -N)}{1 - s_1^2} \right),
$$

$$
\Omega_3' = -\mu e \cdot A - \mu^2 \left( \frac{\Gamma' \cdot \partial_2 \nu(\Gamma, N)}{1 - s_1^2} - \frac{\Gamma' \cdot \partial_2 \nu(\Gamma, -N)}{1 - s_1^2} \right),
$$

such that the following equations are satisfied for all $t \in [-T,T]$:

$$
e_2 \cdot A(t) = -2 \int_{\Gamma(t)} \frac{\mu(t)}{1 - s_k(t)} \, ds,
$$

$$\Omega_2(t) = -\int_{\Gamma(t)} \left( \frac{\mu(t)}{1 - s_k(t)} \right)^2 \, ds,
$$

$$\Omega_3(t) = \int_{\Gamma(t)} \left( \frac{\mu(t)}{1 - s_k(t)} \right)^2 \, s \, ds.
$$

Here $\partial_1 \nu(\Gamma, N)$ respectively $\partial_2 \nu(\Gamma, N)$ denote the gradient of $\nu$ with respect to the first respectively the second argument.

The hypotheses of theorem 3.10 do not allow $x_0 \in \tilde{C}_{\nu} \cup \Sigma_r$. The proof of lemma 2.13, however, requires information at such points. This is why some additional work is needed.

**Proof of lemma 2.13.** The fact that $u \in C^1(B^+_R)$ follows from theorem 1.1. To simplify the notation, we assume without loss of generality that $\zeta = e_1$, $x_0 = 0$ and $R = 1$, and we write $B^+_1 = \{x \in B_1(0) : x \cdot e_1 > 0\}$.

If $B^+_1$ intersects $\tilde{C}_{\nu}$, then $B^+_1 \subset \tilde{C}_{\nu}$ because $B^+_1 \cap \partial \tilde{C}_{\nu} = \emptyset$ and $B^+_1$ is connected. Hence $u$ is affine on $B^+_1$ and we are done.

So let us assume that $B^+_1$ does not intersect $\tilde{C}_{\nu}$. Then as in section 2.2 there exists a ruling $q$ for $\nabla u$ on $B^+_1$, and $q(0) = e_2$ because $u$ is affine on $B_1 \cap \{0\} \times \mathbb{R}$. There exists $T > 0$ and a unique solution $\Gamma \in W^{2,\infty}(0,T)$ of $\Gamma' = -q^+(\Gamma)$ with initial condition $\Gamma(0) = 0$, and we have (26).

We claim that $\mu \in C^0([0,T])$ and that one of the following (mutually exclusive) assertions is true:

(a) $\mu(0) \neq 0$.
(b) $\mu(0) = 0$ and there exists $c > 0$ such that $|\mu(t)| \geq ct$ for all $t$ near 0.
(c) there exist $C, c > 0$ such that

$$c \frac{|t|}{| \log t |} \leq \left| \mu(t) \right| \leq C \frac{|t|}{| \log t |} \text{ for all } t \text{ near } 0.
$$

(d) $\mu = 0$ near 0.

To prove the claim, let $t_n \downarrow 0$ and set $\Gamma_n = \Gamma|_{[t_n,T]}$. Observe that (choosing $T$ small enough) $\Gamma([t_n,T]) \subset B^+_1$. Hence

$$\Gamma([t_n,T]) \subset S^2(\Sigma_r \cup \Sigma_r \cup \partial \tilde{C}_{\nu}) \text{ for all } n \in \mathbb{N}.$$
Hence we can apply theorem 3.10 for every $n$, so there exist $\Lambda_n \in W^{1,2}((t_n, T); \mathbb{R}^3)$ and $M_n \in W^{1,2}((t_n, T); \mathbb{R}^3)$ as well as $\Omega_{2,n}, \Omega_{3,n} \in W^{1,1}((t_n, T); \mathbb{R})$ such that (27)–(30) are satisfied (with an index $n$) on $(t_n, T)$.

Reasoning as in ([12], p. 441) one finds $\Lambda_{n+1} = \Lambda_n$ and $M_{n+1} = M_n$ and $\Omega_{2,n+1} = \Omega_{2,n}$ as well as $\Omega_{3,n+1} = \Omega_{3,n}$ on $(t_n, T)$. Hence there exist $\Lambda \in W^{1,2}_{\text{loc}}(0, T); \mathbb{R}^3)$ and $M \in W^{1,2}_{\text{loc}}(0, T); \mathbb{R}^3)$ and $\Omega_2, \Omega_3 \in W^{1,1}_{\text{loc}}(0, T)$ such that (27)–(30) are satisfied on all of $(0, T)$.

But these equations imply that then in fact $M, \Lambda \in W^{1,2}(0, T); \mathbb{R}^3)$ and $\Omega_2, \Omega_3 \in W^{1,1}(0, T)$, because $\mu \in L^2(0, T)$ and $\kappa \in L^\infty(0, T)$. In particular, $M, \Lambda$ and $\Omega_{1,2}$ are continuous on $[0, T]$.

In what follows we use results from ([12], sections 4.3 and 4.4). Although these results are stated for points inside an open interval, the proofs in fact also apply at the boundary point 0 of the interval $(0, T)$. A global assumption in ([12], section 4) is that we do not have $\mu = 0$ almost everywhere on $(0, T)$. We assume this from now on, since otherwise (d) applies so we are done.

As in [12] we now write $\Lambda_t = \epsilon_t \cdot \Lambda$ and $M_t = \epsilon_t \cdot M$.

Observe that, since we did not exclude $x_0 \in \Sigma_{\epsilon_t}$, it is not true in general that $s_{\epsilon_t}^+ \varepsilon_t$ are continuous at 0. However, ([13], proposition 14(ii)) asserts that the one-sided limits

$$\lim_{t \downarrow 0} s_{\epsilon_t}^+(t)$$

exist. We define $s_{\epsilon_t}^+(0)$ to be this limit.

Using this, by ([12], proposition 4.7) we have $\mu \in C^0([0, T])$ and

$$\kappa \in C^0 \left\{ t \in [0, T]: \Omega_2(t) \neq 0 \right\}.$$  \hspace{1cm} (36)

Let us first consider the case $\Lambda_2(0) = 0$. Then ([12], proposition 4.15) implies that $s_{\epsilon_t}^+(0)M_2(0) \neq \Lambda_1(0)$.

If $\Omega_2(0) = 0$, then ([12], proposition 4.10) shows that $|\mu| \geq c t$ in a right neighbourhood of 0. On the other hand, the $s$-integral in (31) is bounded from below by a positive constant and $\Lambda_2$ is continuous and zero in 0. Hence $\mu(0) = 0$, too. Therefore, (b) follows.

If $\Omega_2(0) \neq 0$, then (36) implies that $\kappa$ is continuous on $[0, t]$ for some $t > 0$. And ([12], proposition 4.15) shows that $s_{\epsilon_t}^-(0)\kappa(0) = 1$ or $s_{\epsilon_t}^+(0)\kappa(0) = 1$. Hence ([12], proposition 4.16) implies that (34) is satisfied in a right neighbourhood of 0 and so (c) follows.

Now consider the case $\Lambda_2(0) \neq 0$. Then ([12], remark 4.3) implies that $\Omega_2(0) \neq 0$, hence $\kappa$ is continuous down to 0 by (36), and ([12], proposition 4.15) shows that $s_{\epsilon_t}^-(0)\kappa(0) < 1$. Thus (31) implies that $\mu(0) \neq 0$, because the integral in that equation remains bounded as $t \downarrow 0$. Hence (a) follows. This concludes the proof of the claim.

It remains to restate parts (a)–(c) of the claim somewhat. After possibly shrinking $T$, these cases can be stated as follows: there is a constant $c > 0$ such that $\mu$ satisfies

$$|\mu(t)| \geq cF(t) \quad \forall t \in (0, T),$$

where either $F(t) = t$ or $F(t) = \frac{t}{1 + \log t}$ or $F \equiv 1$.

For convenience let us extend $\Gamma$ by setting $\Gamma(t) = (t, 0)$ for all $t < 0$. Then ([13], proposition 10) shows that, for $r \in (0, R)$ small enough, the map $(s, t) \mapsto \Gamma(t) + sN(t)$ is a homeomorphism from $(-2r, 2r)^2$ onto its image. Hence there exist unique (continuous) functions $\tau : [-r, r] \to (-T, T)$ and $\sigma : [-r, r] \to \mathbb{R}$ such that

$$\Gamma(\tau(x_1)) + \sigma(x_1)N(\tau(x_1)) = (x_1, 0) \quad \text{for all } x_1 \in [-r, r].$$ \hspace{1cm} (37)
There are constants $c, C > 0$ such that for all $x_1 \in [0, r]$ we have
\[
|\sigma(x_1)| \leq Cx_1^2
\quad \text{(38)}
\]
\[
cx_1 \leq \tau(x_1) \leq Cx_1
\quad \text{(39)}
\]

Of course (38) just means that $\mathbb{R} \times \{0\}$ is the tangent to $\Gamma$ in 0. We include a short proof of (38), (39) for the convenience of the reader.

Since $\Gamma'(0) = e_1$ and $\Gamma'' = \kappa N$, we have (from now on we omit the argument of $\sigma$ and $\tau$)
\[
|\Gamma'(\tau) - e_1| \leq \int_0^\tau |\kappa| \leq C\tau.
\quad \text{(40)}
\]

Hence from $\Gamma(\tau) = \int_0^\tau \Gamma'$ we deduce that
\[
|\Gamma(\tau) - (\tau, 0)| \leq C\tau^2.
\]

Projecting (37) onto $\Gamma'(\tau)$ and employing these estimates we find
\[
|x_1\Gamma'(\tau) - \tau| = |\Gamma(\tau) \cdot \Gamma'(\tau) - \tau| \leq C\tau^2.
\]

Since (40) implies $|\Gamma'(\tau) - 1| \leq C\tau$, the estimates (39) readily follow because $\tau(0) = 0$ and $\tau$ is continuous 0.

Projecting (37) onto $N(\tau)$ we find
\[
|\sigma| = |x_1N_1(\tau) - N(\tau) \cdot \Gamma'(\tau)| \leq C(t + \tau)\tau.
\]

So (38) follows from (39).

Since $\kappa$ is bounded, after possibly shrinking $r$ we deduce from (38) that $\frac{1}{1 - \sigma\kappa} \geq \frac{1}{2}$. Since by (40) we may assume that $\Gamma'(\tau) \cdot e_1 \geq \frac{1}{2}$, from (26) we find
\[
|\partial_1\partial_1u(x_1, 0)| = |\partial_1\partial_1u(\Gamma'(\tau) + \sigma N(\tau))| = (\Gamma'(\tau) \cdot e_1)^2 \left| \frac{\mu(\tau)}{1 - \sigma\kappa(\tau)} \right| \geq cF(\tau).
\]

Finally, notice that $F(\tau) \geq cF(x_1)$ for some $c > 0$ because of (39). The remaining estimate is proven similarly. \qed

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