Structure-preserving Galerkin approximation for a class of nonlinear port-Hamiltonian partial differential equations on networks

Björn Liljegren-Sailer1,* and Nicole Marheineke1

1 Universität Trier, FB IV - Mathematik, Lehrstuhl Modellierung und Numerik, D-54286 Trier, Germany

The development of structure-preserving approximation methods, which regard fundamental underlying physical principles, is an active field of research. Especially when the application of model reduction is desirable, systematic and rather generic approaches are of great interest. In this contribution we discuss a structure-preserving Galerkin approach for a prototypical class of nonlinear partial differential equations on networks. Its derivation is guided by port-Hamiltonian-type modeling and appropriate variational principles. Also complexity-reduction schemes can be integrated in a structure-preserving way, which becomes crucial in the context of model reduction for nonlinear systems.

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1 Model equations

Let \( p = (v, \tilde{w}) \subset \mathbb{R} \) be an interval and \( T > 0 \). We consider a class of prototypical hyperbolic partial differential equations with port-Hamiltonian structure: The state \( z : [0, T] \times p \to \mathbb{R}^2 \) is governed by

\[
\partial_t z(t, x) = \begin{bmatrix} -\partial_x & -\partial_z \end{bmatrix} \nabla_z h(z(t, x)), \quad x \in p, \quad t \in [0, T].
\] (1a)

To close the system, data \( z_0 : p \to \mathbb{R}^2 \) and \( u^w : [0, T] \to \mathbb{R} \) for \( \tilde{w} \in \{v, \tilde{w}\} \) is assumed to be given to prescribe initial conditions \( z(0, x) = z_0(x) \) for \( x \in p \), and one boundary condition per boundary, each of one of the following types,

\[
\text{Type 1: } \nabla_{z_1} h(z(t, \tilde{w})) = u^w, \quad \text{Type 2: } \nabla_{z_2} h(z(t, \tilde{w})) = u^w \quad t \in [0, T],
\] (1b)

using the notation \( z = [z_1; z_2] \) and \( \nabla_z h(z) = [\nabla_{z_1} h(z); \nabla_{z_2} h(z)] \) for the respective column vectors. Throughout the paper we assume \( h \) to be continuously differentiable, convex, \( h(z) > 0 \), and \( r(z) \geq 0 \) for all considered \( z \). In the port-Hamiltonian context, cf. [7, 8] and references therein, the state \( z \) is referred to as energy variables and \( h \) as Hamiltonian density. The latter induces the Hamiltonian \( \mathcal{H}(z) = \int_p h(z) dx \), which can be interpreted as energy functional. The term \( r \) models energy dissipation. We furthermore require the initial- and boundary-data to be such that (1) is well-posed and omits a smooth solution.

**Example 1.1** All these assumptions posed on (1) are, for example, fulfilled in the modeling of gas network systems with one of the following set of equations [1, 2, 4–6].

- **Isentropic Euler equations**: The equations for density \( \rho \) and velocity \( v \) for \( x \in p, t \geq 0 \) read

\[
\partial_t \rho + \partial_x (\rho v) = 0, \quad \partial_t (\rho v) + \partial_x (\rho v^2 + \tilde{p}(\rho)) = -c_f \frac{\rho v}{\rho} \rho v
\]

with pressure \( \tilde{p}(\rho) = c_p \rho^\gamma \) for constants \( \gamma > 1, c_f, c_p > 0 \). To rewrite the system as in (1a), we introduce

\[
z = [\rho; v], \quad h(z) = \frac{\rho v^2}{2} + \frac{1}{\gamma-1}\tilde{p}(\rho), \quad r(z) = c_f \frac{\rho |v|}{\rho}.
\]

- **Simplified isentropic Euler equations**: These equations result from the isentropic Euler equations by neglecting the term \( \partial_x (\rho v^2) \). Reformulation (1a) is obtained for \( m = \rho v \) and

\[
z = [\rho; m], \quad h(z) = \frac{\rho m^2}{2} + \frac{\rho}{\gamma-1}\tilde{p}(\rho), \quad r(z) = c_f \frac{m |v|}{\rho}.
\]

- **Damped wave equation**: A damped wave equation is obtained for

\[
h(z) = 1/2(z_1^2 + z_2^2), \quad r(z) > 0 \quad \text{for all } z.
\]

The system is linear, when \( r \) is chosen to be constant.

Note that in contrast to the isentropic Euler equations, the simplified version inherits a Hamiltonian density, which is separable into \( h(z) = h_1(z_1) + h_2(z_2) \) with quadratic \( h_2 \). For the damped wave equation, the Hamiltonian density is a fully quadratic function and \( z = \nabla_z h(z) \).

* Corresponding author: e-mail bjoern.sailer@uni-trier.de, phone +49 651 201 3468
2 A variational principle

The aim of this section is to introduce the variational principle our approximation is based on. The function spaces $L^2(p)$ and $H^1(p)$ are defined as the Sobolev space of square integrable functions on $p$ and the Sobolev space with additionally square integrable weak derivatives, respectively. The $L^2$-scalar product on $p$ is written as $(\cdot, \cdot)$, and the boundary terms for $b \in H^1(p)$ are denoted as $b[w]$ and $b[\dot{w}]$. Accordingly, we define the boundary operator $T : H^1(p) \to \mathbb{R}^2$ as $Tb = [-b[\dot{w}]; b[w]]$, which is well-defined by the trace theorem.

For a strong solution $z = [z_1; z_2]$ of (1), the following variational principle holds for $t > 0$

$$(\partial_t z_1, b_1) = - (\partial_z \nabla_{z_2} h(z), b_1) \quad \text{for } b_1 \in L^2(p)$$

$$(\partial_t z_2, b_2) = (\nabla_{z_1} h(z), \partial_z b_2) + e_B \cdot T b_2 - (r(z) \nabla_{z_2} h(z), b_2) \quad \text{for } b_2 \in H^1(p)$$

with $e_B = [\nabla_{z_1} h(z)[\dot{w}]; \nabla_{z_1} h(z)[w]]$. The boundary term in the second equation originates from partial integration. To characterize the exchange with the boundary, we additionally define $f_B = [f_B[\dot{w}]; f_B[w]]$ by $f_B = T \nabla_{z_1} h(z)$.

**Theorem 2.1** Let $z = [z_1; z_2]$ be a strong solution of (1). Then the following energy dissipation and global mass conservation hold

$$\frac{d}{dt} \int_p z_1 dx = f_B[\dot{w}] + f_B[w].$$

**Proof.** Both relations follow by testing the variational principle with appropriate $b = [b_1; b_2]$. To derive the energy dissipation, we use the chain rule and apply the variational principle with $b = \nabla_z h(z)$ to get

$$\frac{d}{dt} \int_p H(z) = (\partial_t z_1, \nabla_z h(z)) = e_B \cdot T \nabla_{z_2} h(z) - (r(z) \nabla_{z_2} h(z), \nabla_{z_2} h(z)).$$

By assumption, the damping term involving $r(z)$ is bounded by zero from above, which shows the first relation.

Global mass conservation is derived as follows: First the variational principle with $b = [1; 0]$, being the constant one function, is used and then partial integration and $T \nabla_{z_1} h(z) = [f_B[\dot{w}]; f_B[w]]$ is employed, which together gives

$$\frac{d}{dt} \int_p z_1 dx = (\partial_t z_1, [1; 0]) = - (\partial_z \nabla_{z_2} h(z), 1) = f_B[\dot{w}] + f_B[w].$$

\[\square\]

Note that local mass conservation can be shown accordingly by testing with constant functions with local support.

**Remark 2.2** For the isentropic Euler equations, our variational principle reads for $m = \rho v$ and $t > 0$

$$(\partial_t \rho, b_1) = - (\partial_z m, b_1)$$

$$= - \frac{m}{\rho^2} \partial_t \rho + \frac{1}{\rho} \partial_t m, b_2) = \left( \frac{m^2}{2\rho^2} + \frac{\gamma}{\gamma - 1} \frac{\dot{p}(\rho)}{\rho} \partial_z b_2 \right) + e_B \cdot T b_2 = \left( \frac{c_f |m|}{\rho^2} m, b_2 \right)$$

for $b_1 \in L^2(p)$

$$(\partial_z V_2) = V_1$$

for $b \in L^2(p)$ : It exists $b_2 \in V_2$, such that $b = \partial_z b_2$.

$$(\partial_t V_2) = \{ b \in L^2(p) : \text{It exists } b_2 \in V_2, \text{ such that } b = \partial_z b_2 \}. \quad (2)$$

Besides the already discussed variational principle, the essential is the parametrization in which the solution is discretized in.

Our parametrization relies on the mapping $a \mapsto z_a(a) = [z_{a_1}(a); z_{a_2}(a)]$ for $a = [a_1; a_2] \in V_1 \times V_2$ to be continuously differentiable and have the inverse $z \mapsto [z_1; \nabla_{z_2} h(z)]$, i.e., $z_{a_1}(a) = a_1$ and $\nabla_{z_2} h(z_{a}(a)) = a_2$. The proposed Galerkin approximation then reads:

\[\text{www.gamm-proceedings.com}\]
Find \( \mathbf{a} = [a_1; a_2] \in C^1([0,T], V_1 \times V_2) \) and \( \mathbf{e}_B, \mathbf{f}_B \in C^0([0,T], \mathbb{R}^2) \) such that for \( t \in [0,T] \)

\[
(\partial_t a_1, b_1) = -(\partial_x a_2, b_1) \quad \text{for } b_1 \in V_1 \tag{3a}
\]

\[
(\partial_t z_{a2}(a), b_2) = (\nabla_z h(z_a(a)), \partial_x b_2) + \mathbf{e}_B \cdot \mathbf{T} b_2 - (r(z_a(a)) a_2, b_2) \quad \text{for } b_2 \in V_2 \tag{3b}
\]

\[
\mathbf{f}_B = \mathbf{T} \partial_t a_2. \tag{3c}
\]

To close the system, initial conditions and boundary conditions have to be chosen. In conformity with the port-Hamiltonian framework [7,8], we added the boundary effort \( \mathbf{e}_B = [\mathbf{e}_B[\bar{w}]; \mathbf{e}_B[w]] \) and boundary flow \( \mathbf{f}_B = [\mathbf{f}_B[\bar{w}]; \mathbf{f}_B[w]] \) to the dependent variables to describe interaction at boundaries. The two types of boundary conditions in (1b) relate to

\[
\text{Type 1: } \mathbf{e}_B[\bar{w}] = u^\bar{w}, \quad \text{Type 2: } \mathbf{f}_B[w] = u^w, \quad t \in [0,T] \tag{3d}
\]

for given data \( u^\bar{w} : [0,T] \to \mathbb{R} \). As mentioned, we assume one boundary condition to be chosen for \( \bar{w} \in \{w, \bar{w}\} \) each. Type 1 resembles weakly imposed boundary conditions and Type 2 essential boundary conditions. The approximation inherits energy dissipation and mass conservation, as the following counterpart to Theorem 2.1 shows:

**Theorem 3.1** For a solution \( \mathbf{a} = [a_1; a_2] \) of (3) it holds for \( t \in [0,T] \)

\[
\frac{d}{dt} H(z_a(a)) = -(r(z_a(a)) a_2, a_2) + \mathbf{e}_B \cdot \mathbf{f}_B \leq \mathbf{e}_B \cdot \mathbf{f}_B
\]

\[
\frac{d}{dt} \int_p a_1 dx = f_B[\bar{w}] + f_B[w].
\]

**Proof.** As a preliminary step, we show the existence of \( \xi(a) \in V_1 \) such that

\[
(\nabla_z h(z_a(a)), b_1) = \langle \xi(a), b_1 \rangle \quad \text{for all } b_1 \in V_1, \quad \tag{4}
\]

for solutions \( \mathbf{a} \) of (3). Considering (3b), it follows that \( (\nabla_z h(z_a(a)), \partial_x b_2) \) is well-defined for all \( b_2 \in V_2 \) and \( t \in [0,T] \). The compatibility \( V_1 \subset \partial_x V_2 \) then implies that \( b_1 \mapsto (\nabla_z h(z_a(a)), b_1) \) is well-defined as an element of the dual space of \( V_1 \). The existence of \( \xi(a) \) fulfilling (4) can now be deduced from the Riesz representation theorem, as \( V_1 \) with \( \langle -,- \rangle \) is a Hilbert space. Moreover, the chain rule gives

\[
\frac{d}{dt} H(z_a(a)) = (\partial_t z_{a1}(a), \nabla_x h(z_a(a))) = (\partial_t a_1, \nabla_z h(z_a(a))) + (\partial_t z_{a2}(a), a_2).
\]

As shown in the preliminary step, \( \nabla_z h(z_a(a)) \) can be replaced by \( \xi(a) \) here. Applying the variational principle then with \( b = [\xi(a); a_2] \) shows the energy dissipation with a similar calculation as in the proof of Theorem 2.1.

For the proof of mass conservation, we define \( 1 \) as constant one function, and as possibly \( 1 \not\in V_1 \), also its orthogonal projection onto \( V_1 \) as \( P_1 \). Using that \( \partial_t a_1, \partial_t a_2 \in V_1 \), and the variational principle with \( b = [P_1; 0] \), we get

\[
\frac{d}{dt} \int_p a_1 dx = (\partial_t a_1, P_1) = (\partial_t a_1, P_1) = -(\partial_t a_2, P_1) = -d(\partial_t a_2, 1) = f_B[\bar{w}] + f_B[w].
\]

In the last step, partial integration has been applied. \( \square \)

**Remark 3.2** For the linear case, similar schemes have been developed in [2,6]. In contrast to the linear case, the parametrization of the solution plays a crucial role in the general nonlinear case. Another Galerkin scheme as the one discussed in this paper, is obtained, when our variational principle is combined with the parametrization of the solution in the effort variable \( e = \nabla_z h(z) \). For the resulting scheme, energy dissipation can be derived similarly as in Theorem 3.1. The proof for mass conservation from Theorem 3.1, however, does not hold any more. Moreover, that scheme is not applicable for the isentropic Euler equations, which is our motivating example. The reason is that the mapping \( z \mapsto \nabla_z h(z) \) has no explicit description and the efforts are not the variables of interest. Our parametrization, however, is particularly well-suited for that example, as \( a = [z_1; \nabla_z h(z)] = [\rho; \rho v] \) and \( z = [\rho; v] \), cf. Remark 2.2.

### 4 Network case

For the generalization to a network, we use a similar notation as in [4,6]. We assume the network to be described by a directed connected graph \((V, E)\) of a set of nodes \( V \) and edges \( E \), which could, e.g., model pipes in a gas network. With slight abuse of notation, we identify edges \( p = (w, \bar{w}) \in E \) for \( w, \bar{w} \in V \) with one-dimensional spatial domains as before. Nodes are...
distinguished into inner nodes $V_I$ and boundary nodes $V_B = V\setminus V_I$. The set of edges incident to a node $w$ is denoted as $\mathcal{E}(w)$. Given that the dynamics on every edge $p \in \mathcal{E}$ are governed by equations (1a), we define edge-wise quantities $a^p$, $V^p_1$, $V^p_2$ with compatibility $\partial_z V^p_1 = V^p_2$ as in Section 3. The Galerkin approximation of the network then reads:

Find $a^p = [a^p_1; a^p_2] \in C^1([0, T], V^p_1 \times V^p_2)$, and $e^p, f^p \in C^0([0, T], \mathbb{R}^2)$, and $\lambda^w \in C^0([0, T], \mathbb{R})$ for all $p \in \mathcal{E}$, $w \in V_0$ such that for $t \in [0, T]$ there hold the edge-wise equations

\[
\begin{align*}
(\partial_t a^p_1, b_1) &= - (\partial_z a^p_2, b_1) & \text{for } b_1 \in V^p_1 \\
(\partial_t z_{a^p}(a^p), b_2) &= (\nabla_z h(z_a(a^p)), \partial_z b_2) + e^p \cdot f^p b_2 - (r(z_a(a^p)) a^p_2, b_2) & \text{for } b_2 \in V^p_2
\end{align*}
\]

and the coupling conditions

\[
\sum_{p \in \mathcal{E}(w)} f^p_B[w] = 0 \quad \text{for } w \in V_0, \quad e^p_B[w] = \lambda^w \quad \text{for } p \in \mathcal{E}(w), \ w \in V_0.
\]

One boundary condition at each $\bar{w} \in V_B$ for given data $u^w : [0, T] \to \mathbb{R}$ is prescribed. Assuming there is only the edge $\bar{p}$ incident to $\bar{w}$, either of the following is chosen

Type 1: $e^p_B[\bar{w}] = u^w$, \quad Type 2: $f^p_B[\bar{w}] = u^w$.

To close the system, it has to be supplemented with initial conditions. Mass conservation and energy dissipation can be derived for the discretized network case very similarly as done for Theorem 3.1, cf. [4]. Let us mention that for the generalization of Theorem 3.1 to hold, it is essential that our coupling conditions are chosen such that they are power-neutral and mass conservative. Finding power-neutral coupling conditions may not be obvious, when neither an energy-based modeling [2, 8], nor appropriate variational principles [4] are employed. Furthermore, note that the underlying coupling structure of the discretized network is essentially the same as in the case of the linear damped wave equation [6]. There it is also commented on the differential-algebraic structure associated to the appearance of Lagrange-multipliers (here $\lambda^w$), and how they can be eliminated if desired.

## 5 Outlook and complexity-reduction

In this contribution, we discussed a systematic structure-preserving approximation framework for a class of port-Hamiltonian systems. The variational arguments used are rather generic, which makes the approach also attractive in the model reduction context. We note, however, that model reduction typically has to be complemented with so-called complexity-reduction to deal efficiently with nonlinearities. As the standard methods like DEIM would destroy the underlying Hamiltonian structure, nonlinearities in the damping term $r$ can be coped with sparse empirical quadrature with positive weights in a natural way, cf. [5]. When the system involves nonlinear Hamiltonian densities, as they occur in the simplified Euler equations, sparse approximation of the Hamiltonian density $h$ itself represents a promising approach in our eyes. The latter is basically an adaption of the complexity-reduction method discussed in [3] for port-Hamiltonian ordinary differential equations to our Galerkin approximation. We currently investigate the application of the presented framework for gas network simulations.

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**References**

[1] J. Brouwer, I. Gasser, and M. Herty. Gas pipeline models revisited: Model hierarchies, nonisothermal models, and simulations of networks. *Multiscale Modeling & Simulation*, 9(2):601–623, 2011.

[2] F. Cardoso Ribeiro, D. Maitignon, and L. Lefèvre. A structure-preserving partitioned finite element method for the 2D wave equation. *IFAC-PapersOnLine*, 5(3):119–124, 2018.

[3] S. Chaturantabut, C. Beattie, and S. Gugercin. Structure-preserving model reduction for nonlinear port-Hamiltonian systems. *SIAM J. Sci. Comput.*, 38(5):B837–B865, 2016.

[4] H. Egger. A robust conservative mixed finite element method for compressible flow on pipe networks. *SIAM J. Sci. Comput.*, 40(1):A108–A129, 2018.

[5] H. Egger, T. Kugler, and B. Liljegren-Sailer. Stability preserving approximations of a semilinear hyperbolic gas transport model. *arXiv e-prints*, 1812.03726, 2018.

[6] H. Egger, T. Kugler, B. Liljegren-Sailer, N. Marheineke, and V. Mehrmann. On structure-preserving model reduction for damped wave propagation in transport networks. *SIAM J. Sci. Comput.*, 40(1):A331–A365, 2018.

[7] O. Farle, R. Baltes, and R. Dyczij-Edlinger. Strukturierende Diskretisierung verteilt-parametrischer port-Hamiltonscher Systeme mittels Finiter Elemente. *Automatisierungstechnik*, 62(7):500–511, 2014.

[8] P. Kotyczka, B. Maschke, and L. Lefèvre. Weak form of Stokes-Dirac structures and geometric discretization of port-Hamiltonian systems. *J. Comput. Phys.*, 361:442–476, 02 2018.