RANK TWO WEAK FANO BUNDLES ON DEL PEZZO THREECFOLDS OF PICARD RANK 1

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Abstract. We classify rank two vector bundles on a del Pezzo threefold $X$ of Picard rank one whose projectivizations are weak Fano. We also investigate the moduli spaces of such vector bundles when $X$ is of degree five, especially whether it is smooth, irreducible, or fine.

1. Intro

1.1. Background. To deepen the classification theory of Fano manifolds, Szurek and Wisniewski \cite{29} introduced the notion of a Fano bundle, which is defined to be a vector bundle $\mathcal{E}$ on a smooth projective variety $X$ such that $\mathbb{P}_X(\mathcal{E})$ is Fano. They showed that the base space $X$ of a Fano bundle $\mathcal{E}$ must be a Fano manifold [THEOREM, ibid]. Thus, in papers including \cite{29, 21, 22, 23}, the classification problem of Fano bundles have been treated on a fixed base Fano manifold. Especially, rank 2 Fano bundles on a Fano 3-fold of Picard rank 1 were classified by \cite{22}. On the other hand, Langer introduced the notion of weak Fano bundles as a natural generalization of that of Fano bundles \cite{21}. Namely, a vector bundle $\mathcal{E}$ on $X$ is said to be weak Fano if $\mathbb{P}_X(\mathcal{E})$ is weak Fano, i.e., $-K_{\mathbb{P}_X(\mathcal{E})}$ is nef and big.

As opposed to Fano bundles, rank 2 weak Fano bundles on a Fano 3-fold of Picard rank 1 have not yet been classified.

Our main purpose is to classify rank 2 weak Fano bundles on a del Pezzo 3-fold $X$ of Picard rank 1. It was known that a del Pezzo 3-fold is of Picard rank 1 if and only if the degree of $X$, say $\text{deg} X$, is less than or equal to 5 \cite{15, 6}. In our previous articles \cite{14, 8}, we classify rank 2 weak Fano bundles on a del Pezzo 3-fold $X$ of $\text{deg} X \in \{3, 4\}$. In this sequel, we will classify them when the case $\text{deg} X \in \{1, 2, 5\}$.

1.2. Known Results. In our previous work \cite{8}, we treat this kind of problem when $\text{deg} X = 4$. Briefly speaking, we showed that every indecomposable weak Fano bundle is either a Fano bundle or a vector bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$ such that $\mathcal{E}(1)$ is globally generated. By the same approach as in \cite{8}, we will see that this phenomenon still holds when $\text{deg} X = 5$ (see Section 5.3).

However, when $\text{deg} X = 5$, we will encounter an instanton bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 4$ as an example of a weak Fano bundle. While instanton bundles $\mathcal{E}$ with $c_2(\mathcal{E}) \leq 3$ were studied by Sanna \cite{27, 28} in detail, those with $c_2(\mathcal{E}) = 4$ have not been studied yet. Thus, we need a different approach from our previous studies \cite{14, 8} to investigate such weak Fano bundles.

In this article, we treat weak Fano bundles in terms of resolutions consisting of natural vector bundles on a del Pezzo 3-fold of degree 5. This approach has often been taken in studies of vector bundles on projective spaces. Indeed, \cite{24, 29} and \cite{23} gave the classification of vector bundles in terms of linear resolutions of them. This form of classification is useful for studying various properties of vector bundles such as its global generation or its moduli space.
1.3. Main Results. In this article, we classify rank 2 weak Fano bundles on del Pezzo 3-folds of Picard rank 1 and give some applications of that classification. The main results of this article are broadly divided into three parts.

1.3.1. Classification on a del Pezzo threefold of degree 5. Let \( X \) be a del Pezzo 3-fold of degree 5. The first result of this article is the classification of rank 2 weak Fano bundles on \( X \).

By classical classification theory of del Pezzo 3-fold due to Fujita \cite{Fujita} and Iskovskikh \cite{Iskovskikh}, \( X \) is a codimension 3 linear section of \( \text{Gr}(2,5) \). Let \( M \) and \( X \) denote the restriction of the universal rank 2 subbundle and the universal quotient bundle of rank 3 on \( \text{Gr}(2,5) \), respectively. The main result of this article is the following explicit resolution of the rank 2 weak Fano bundles on \( X \).

**Theorem 1.1.** Let \( X \) be a del Pezzo 3-fold of degree 5. For a normalized bundle \( E \) of rank 2, \( E \) is a weak Fano bundle if and only if \( E \) satisfies one of the following.

(i) \( E \simeq O_X(1) \oplus O_X(-1) \). In this case, \( c_1(E) = 0 \) and \( c_2(E) = -5 \).

(ii) \( E \simeq O_X \oplus O_X(-1) \). In this case, \( c_1(E) = -1 \) and \( c_2(E) = 0 \).

(iii) \( E \simeq O_X^\oplus 2 \). In this case, \( c_1(E) = 0 \) and \( c_2(E) = 0 \).

(iv) \( E \simeq \mathcal{R} \). In this case, \( c_1(E) = -1 \) and \( c_2(E) = 2 \).

(v) \( E \) fits into \( 0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{R}^\oplus 2 \to E \to 0 \). In this case, \( c_1(E) = 0 \) and \( c_2(E) = 1 \).

(vi) \( E \) fits into \( 0 \to \mathcal{Q}(-1) \to \mathcal{O} \oplus \mathcal{R}^\oplus 4 \to E \to 0 \). In this case, \( c_1(E) = 0 \) and \( c_2(E) = 2 \).

(vii) \( E \) fits into \( 0 \to \mathcal{O}(-1) \oplus \mathcal{Q}(-1) \to \mathcal{R}^\oplus 5 \to \mathcal{O}^\oplus 8 \to E(1) \to 0 \). In this case, \( c_1(E) = 0 \) and \( c_2(E) = 3 \).

(viii) \( E \) fits into \( 0 \to \mathcal{O}(-1)^\oplus 2 \to \mathcal{Q}(-1)^\oplus 2 \to \mathcal{O}^\oplus 6 \to E(1) \to 0 \). In this case, \( c_1(E) = 0 \) and \( c_2(E) = 4 \).

In the above results, we regard \( H^2(X, \mathbb{Z}) \) and \( H^4(X, \mathbb{Z}) \) as \( \mathbb{Z} \) by taking the effective classes generating these cohomology groups.

Furthermore, on an arbitrary del Pezzo 3-fold of degree 5, there exist examples for each case of (i)–(viii).

**Remark 1.2.** The resolutions given in Theorem 1.1 are not unique. For example, there exist the following alternative resolutions.

(v) The exact sequence in (v) can be replaced to \( 0 \to \mathcal{R} \to \mathcal{Q}' \oplus \mathcal{O} \to E \to 0 \).

For more precise, see Remark 3.2.

(vi) The exact sequence in (vi) can be replaced to \( \mathcal{R}^\oplus 2 \to (\mathcal{Q}')^\oplus 2 \to E \to 0 \).

For more precise, see Section 4.2 and the sequence (3.6).

(vii) The exact sequence in (vii) can be replaced to \( \mathcal{O}(-1) \to \mathcal{R}^\oplus 2 \oplus \mathcal{Q}' \to \mathcal{O}^\oplus 8 \to E(1) \to 0 \).

For more precise, see Remark 3.3.

**Remark 1.3.** By our classification and Hoppe’s criterion \cite[Theorem 2.10]{Hoppe}, \( E \) is slope stable (resp. slope semi-stable, not slope semi-stable) if and only if \( E \) is of type (iv), (vi) or (viii) (resp. (iii) or (v), (i) or (ii)).

1.3.2. Moduli spaces. Our description of rank 2 weak Fano bundles on a del Pezzo 3-fold \( X \) of degree 5 (=Theorem 1.1) is useful to investigate their moduli spaces. Let \( M_{wF}^k \) over \( k \) be the coarse moduli space of rank 2 weak Fano bundles \( E \) on \( X \) with \( c_1(E) = c_1 \) and \( c_2(E) = c_2 \) (see Definition 6.1). By Theorem 1.1, the coarse moduli spaces \( M_{wF,1}^5, M_{wF,1}^6, M_{wF,1}^8, M_{wF,1}^{12} \) are isomorphic to the point, and \( M_{wF,1}^9 \) is isomorphic to \( \mathbb{P}^2 \) (see Proposition 6.3(2)). Among the above moduli spaces, only \( M_{wF,1}^{12} \) is fine as a moduli space (see Proposition 6.3(3)).

Thus we are interested in the moduli spaces \( M_{wF,n}^k \) with \( n \in \{2, 3, 4\} \). Since every rank 2 weak Fano bundle \( E \) with \( c_1(E) = 0 \) and \( c_2(E) \in \{2, 3, 4\} \) is a stable instanton
bundle (see Remark 1.3 and [8 Corollary 4.7]), \( M_{0, n}^{\text{ef}} \) is an open subscheme of the coarse moduli space \( M_{0, n}^{\text{ins}} \) of instanton bundles \( E \) on \( X \) with \( c_1(E) = 0 \) and \( c_2(E) = n \) (for the definition of \( M_{0, 2}^{\text{ef}} \) we refer to [5 (13)]. When \( n \in \{2, 3\} \), it was proved by [27, 28] that \( M_{0, n}^{\text{ins}} \) is smooth, irreducible, but not projective. When the case \( n = 2 \), it was known that \( M_{0, 2}^{\text{ef}} = M_{0, 2}^{\text{ins}} \) since every minimal instanton bundle is a weak Fano bundle (c.f. [8 Section 1.3]). Hence \( M_{0, 2}^{\text{ef}} \) is not fine as a moduli space by [28 Proposition 5.12]. On the other hand, Sanna shows that \( M_{0, 3}^{\text{ef}} \) is fine as a moduli space [28, Remark 4.11], which contains \( M_{0, 3}^{\text{ef}} \) as a proper open subvariety (see [28 Section 8] and Section 7). Hence \( M_{0, 3}^{\text{ef}} \) is a fine moduli space.

In this article, we use our result to investigate \( M_{0, 4}^{\text{ef}} \). In conclusion, we obtain the following theorem.

**Theorem 1.4.** \( M_{0, 4}^{\text{ef}} \) is an irreducible smooth quasi-projective variety of dimension 13. Moreover, \( M_{0, 4}^{\text{ef}} \) is not fine as a moduli space. In particular, the coarse moduli space \( M_{0, 4}^{\text{ef}} \) is fine if and only if \( c_2 = 3 \).

1.3.3. **Classifications on del Pezzo 3-folds of degree \( \leq 2 \).** By Theorem 1.1 and our previous studies [14, 8], we obtain the classification result on del Pezzo 3-folds \( X \) of degree \( 3 \leq \text{deg } X \leq 5 \). To complete our classification, we show there exist no non-trivial weak Fano bundles when the case \( \text{deg } X \leq 2 \), which is namely the following theorem.

**Theorem 1.5.** Let \( X \) be a del Pezzo 3-fold of degree \( d \leq 2 \). Then every rank 2 weak Fano bundle on \( X \) is the direct sum of some line bundles.

1.4. **Plan of Proof.** In our proof of Theorem 1.5 we employ almost the same strategies that were taken in [13, 8]. In contrast, we need some new ideas for proving Theorems 1.4 and 1.5. In this section, we explain an abstract of our proof of these two theorems in this article.

1.4.1. **Plan of Proof of Theorem 1.1.** Let \( X, R, \) and \( Q \) be as in Section 1.3 Let \( E \) be a weak Fano bundle on \( X \). When \( c_1(E) = -1 \), we have \( E_1 \cong O_{P_1}(-1) \oplus O_{P_1} \), for every line \( l \) on \( X \). Hence we can characterize \( E \) by using the Hilbert scheme of lines on \( X \) and its description which is mainly done by Furushima-Nakayama [9] (see Section 1). When \( c_1(E) = 0 \), we observe \( E \) for each possible value of its 2nd Chern class \( c_2(E) \). If \( E \) is indecomposable, then \( c_2(E) \geq 1 \) (see Proposition 5.1). When \( c_2(E) = 1 \), then we have the resolution in Theorem 1.1 (vi) by using the sequence \( 0 \to O \to E \to T \to 0 \). When \( c_2(E) = 2 \), then \( E \) is an instanton bundle by [8 Corollary 4.7]. Moreover, Faenzi [5] and Kuznetsov [19] gave a monad for each instanton bundle. Using this monad, we can obtain Theorem 1.1 (vi) for \( c_2(E) = 2 \). Thus, the non-trivial part of the proof of Theorem 1.1 is for \( c_1(E) = 0 \) and \( c_2(E) \geq 3 \).

To treat these cases, we will use the derived category of \( X \). This approach is inspired by Ohno’s work, which gave linear resolutions of nef vector bundles on the projective space \( \mathbb{P}^n \) using a full strong exceptional collection of its derived category [23]. In general, if a given smooth projective variety \( X \) has a full strong exceptional collection \( G_0, G_1, \ldots, G_n \in \text{Coh}(X) \), then any coherent sheaf \( E \) has a resolution consisting of \( G_0, \ldots, G_n \) provided with \( \text{Ext}^1(G_i, \mathcal{O}) = 0 \) [Section 2, ibid.]. Since Orlov [24] showed that \( (\mathcal{O}(-1), \mathcal{O}(-1), R, \mathcal{O}) \) is a full strong exceptional collection of the derived category of \( X \), \( E(d_{\text{min}}) \) has a resolution which consists of \( \mathcal{O}(-1), \mathcal{O}(-1), R, \mathcal{O} \), where \( d_{\text{min}} := \min \{d \mid \text{Ext}^2(T, E(d)) = 0 \} \) and \( T := \mathcal{O}(-1) \oplus \mathcal{Q}(-1) \oplus R \oplus \mathcal{O} \).

However, our method to find resolutions in Theorem 1.1 is slightly different from Ohno’s approach [23]: we use mutations over these exceptional objects and...
compute them by using Serre functors of specific admissible subcategories. More precisely, we consider the object $R\mathcal{O}_{(−1)}L\mathcal{O}(E(1))$, where $L\mathcal{O}$ is the left mutation functor over $\mathcal{O}$ and $R\mathcal{O}_{(−1)}$ is the right mutation functor over $\mathcal{O}(−1)$. Since $E(1)$ is globally generated by $\mathbb{S}$ Theorem 1.7, we can show that there is a vector bundle $V = R\mathcal{O}_{(−1)}L\mathcal{O}(E(1))[−1]$ fitting into the following exact sequence

$$0 \to \mathcal{O}(−1)\oplus c_{2}(E)^{−2} \to V \to \mathcal{O}(14−2c_{2}(E)) \to E(1) \to 0.$$ 

Therefore, our problems are reduced to compute $V$. Since $V$ is an object in the subcategory $\langle Q(1), \mathcal{R} \rangle$, we have a distinguished triangle

$$R\text{Hom}(\mathcal{R}, V) \otimes \mathcal{R} \to V \to R\text{Hom}(V, Q(−1))^{∨} \otimes Q(−1) \to 1.$$ 

From computations using Serre functors of subcategories of $D^{b}(X)$, we will have isomorphisms $R\text{Hom}(\mathcal{R}, V) \simeq R\text{Hom}(Q(−1), E)$ and $R\text{Hom}(V, Q(−1))^{∨} \simeq R\text{Hom}(\mathcal{R}, E)[1]$ (see Lemma 3.9). Finally, using Sanna’s results and some geometric observations, we compute the dimensions of $\text{Ext}^{i}(Q(−1), E)$ and $\text{Ext}^{i}(\mathcal{R}, E)$, and describe $V$ explicitly in terms of the vector bundles $\mathcal{R}$ and $Q(−1)$.

1.4.2. Plan of Proof of Theorem 1.4. To show Theorem 1.4, we will use the resolution as in Theorem 1.1 (viii). Using this resolution, we will construct an open embedding $M_{0,4}^{\text{wF}} \to M_{(2,2)}^{\text{stab}}(Q)$, where $\Theta$ is a certain stability and $M_{(2,2)}^{\text{stab}}(Q)$ is the moduli space of $\Theta$-stable representations of the $5$-Kronecker quiver $Q$ with dimension vector $(2, 2)$. Thus the irreducibility of $M_{0,4}^{\text{wF}}$ follows from that of $M_{(2,2)}^{\text{stab}}(Q)$. Moreover, Chung-Moon $\mathbb{S}$ showed that the moduli space of the semi-stable representation $M_{(2,2)}^{\text{stab}}(Q)$ is isomorphic to $T_{4}$ that was introduced by Hosono-Takagi $\mathbb{S}$.

This space $T_{4}$ is given as the Stein factorization of $U_{4} \to S_{4}$, where $S_{4} \subset |O_{P^{4}}(2)|$ is the closed subscheme that parametrizes the singular hyperquadrics in $P^{4}$, and $U_{4} := \{(\Pi, Q) \in \text{Gr}(2, 5) \times S_{4} | \Pi \subset Q\}$ is the incidence variety of $2$-planes and $S_{4}$. Using this fibration structure $U_{4} \to T_{4}$, we will conclude that there is no universal bundle on $M_{0,4}^{\text{wF}} \times X$.

1.5. Organization of this article. In Section 2, we prepare our notation for materials about bounded derived categories of coherent sheaves. In Section 3 (resp. 4), we study weak Fano bundles $E$ on a del Pezzo $3$-fold of degree $5$ with $c_{1}(E) = 0$ (resp. $−1$). In Section 5, we prove Theorems 1.1 and 1.5 and obtain our classification result for rank $2$ weak Fano bundles on del Pezzo $3$-folds of Picard rank $1$. In Section 6, we apply our result to observe moduli spaces of rank $2$ weak Fano bundles on a del Pezzo $3$-fold of degree $5$ and prove Theorem 1.4. To conclude this article, we will obtain a cohomological characterization of weak Fano bundles in instanton bundles on a del Pezzo $3$-fold of degree $5$ in Section 7.

Notation and Convention. Throughout this article, we will work over an algebraically closed field $k$ of characteristic zero. We also adopt the following convention.

- We regard vector bundles as locally free sheaves. For a vector bundle $E$ on a smooth projective variety $X$, we define $\mathbb{P}_{X}(E) := \text{Proj Sym}E$.
- In this article, a del Pezzo $3$-fold is a smooth Fano $3$-fold $X$ whose Fano index $r(X) = \min\{r | −K_{X} \sim rH \text{ for some } H \in \text{Pic}(X)\}$ is $2$. For a del Pezzo $3$-fold $X$, $\text{O}_{X}(1) \in \text{Pic}(X)$ denotes the polarization such that $\text{O}_{X}(2) \simeq \text{O}_{X}(−K_{X})$.
- When $X$ is a smooth Fano $3$-fold of Picard rank $1$, we often identify $N^{i}(X)_{Z} \simeq \mathbb{Z}$ by taking the effective generator class for each $i \in \{0, 1, 2, 3\}$, where $N^{i}(X)_{Z}$ is the numerical class group of the codimension $i$ cycles with
Z-coefficients. In this article, the i-th Chern class \( c_i(\mathcal{E}) \) is often regarded as a numerical class.

- We also say that a rank 2 vector bundle \( \mathcal{E} \) on a Fano 3-fold \( X \) of Picard rank 1 is **normalized** when \( c_1(\mathcal{E}) \in \{0, -1\} \).
- On a del Pezzo 3-fold \( X \) of Picard rank 1, an **instanton bundle** \( \mathcal{E} \) is defined to be a rank 2 stable vector bundle such that \( c_1(\mathcal{E}) = 0 \) and \( h^1(\mathcal{E}(-1)) = 0 \) \footnote{For details, see \cite{11}}.
- For the Grassmannian varieties, we employ the classical notation as follows; for a \( n \)-dimensional vector space \( V = k^n \) and a positive integer \( m < n \), we define the Grassmannian variety \( \text{Gr}(m, V) = \text{Gr}(m, n) \) as the parameter space of \( m \)-dimensional linear subspaces of \( V = k^n \). In particular, if we regard \( V \) as a locally free sheaf on \( \text{Spec} k \), then \( \mathbb{P}(V) \) is canonically isomorphic to \( \text{Gr}(n - 1, V) \) in our notation.

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2. Preliminaries for derived category

In this section, we set up our notation for materials for triangulated categories. The triangulated categories we are mainly interested in are the derived categories of coherent sheaves on smooth projective varieties \( X \), denoted by \( D^b(X) \). We adopt the terminology of \cite{12}. In this article, we define mutations of objects over exceptional objects as follows.

**Definition 2.1.** Let \( \mathcal{D} \) be an Ext-finite \( k \)-linear triangulated category. For objects \( A, B \in \mathcal{D} \), we denote \( \bigoplus \text{Hom}_{\mathcal{D}}(A, B[i])[-i] \) by \( \text{Ext}^i(A, B) \). Let \( E \) be an exceptional object of \( \mathcal{D} \) and \( F \) an object of \( \mathcal{D} \).

1. The **left mutation** \( \text{L}_E(F) \) of \( F \) over \( E \) is defined to be the cone of the natural evaluation morphism \( \text{ev} : \text{Ext}^i(E, F) \otimes E \to F \):

\[
\text{Ext}^i(E, F) \otimes E \xrightarrow{\text{ev}} F \to \text{L}_E(F)^{\perp, \perp}.
\]

2. The **right mutation** \( \text{R}_E(F) \) of \( F \) over \( E \) is defined to be \( \text{Cone}(\text{coev} : F \to \text{Ext}^i(F, E)^\vee \otimes E)[-1] \), where \( \text{coev} \) is the natural co-evaluation morphism:

\[
\text{R}_E(F) \to F \xrightarrow{\text{coev}} \text{Ext}^i(F, E)^\vee \otimes E^{\perp, \perp}.
\]

For a full subcategory \( \mathcal{A} \subset \mathcal{D} \), we define

\[
\langle A \rangle^\perp = \{ F \in \mathcal{D} \mid \text{Ext}^i(A, F) = 0 \text{ for all object } A \in \mathcal{A} \},
\]

\[
\langle A \rangle^\perp = \{ F \in \mathcal{D} \mid \text{Ext}^i(\mathcal{F}, A) = 0 \text{ for all object } A \in \mathcal{A} \}.
\]

**Remark 2.2.** If \( \mathcal{D} = \langle E_1, \ldots, E_r \rangle \) is a full exceptional collection, then for any \( 1 \leq j \leq k \leq r \), we have

\[
\langle E_1, \ldots, E_{j-1} \rangle \cap \langle E_{k+1}, \ldots, E_r \rangle^\perp = \langle E_j, \ldots, E_k \rangle.
\]

Furthermore, we collect the following well-known facts for later convenience.

1. Let \( E \in \mathcal{D} \) be an exceptional object. Then the mutations over \( E \) define functors \( \text{L}_E : \mathcal{D} \to \langle E \rangle^\perp \) and \( \text{R}_E : \mathcal{D} \to \langle E \rangle^\perp \). It is easy to see that the left mutation \( \text{L}_E \) (resp. the right mutation \( \text{R}_E \)) over \( E \) is the left (resp. right) adjoint functor of the inclusion \( \langle E \rangle^\perp \hookrightarrow \mathcal{D} \) (resp. \( \langle E \rangle^\perp \hookrightarrow \mathcal{D} \)).
(2) Mutations act on exceptional collections as follows. For a given exceptional collection \((E_1, \ldots, E_i, E_{i+1}, \ldots, E_r)\), the sequences \((E_1, \ldots, E_{i-1}, L_E, (E_{i+1}), E_i, E_{i+2}, \ldots, E_r)\) and \((E_1, \ldots, E_{i-1}, E_{i+1}, R_E, (E_{i+2}), E_i, \ldots, E_r)\) are also exceptional collections. These three collections generate the same category.

(3) Related to (2), we observe that Serre functors also act on exceptional collections. If \(D = \langle E_1, E_2, \ldots, E_r \rangle\) is a full exceptional collection and there is a Serre functor \(S_D\) of \(D\), then \(\langle S_D(E_1), E_1, \ldots, E_r-1 \rangle\) and \(\langle E_2, \ldots, E_r, S_D^{-1}(E_1) \rangle\) are also full exceptional collections of \(D\).

We also prepare the following easy lemma.

**Lemma 2.3.** Let \((E_1, E_2)\) be an exceptional collection and \(F \in \langle E_1, E_2 \rangle\) be an object. Then \(R_{E_1}(F) \simeq \text{Ext}^\bullet(E_2, F) \otimes E_2\). In particular, there is a distinguished triangle

\[
\text{Ext}^\bullet(E_2, F) \otimes E_2 \to F \to \text{Ext}^\bullet(F, E_1)^\vee \otimes E_1 \to 1.
\]

**Proof.** By Remark 2.2 (1), we have \(R_{E_1}(F) \in \langle E_1 \rangle = \langle E_2 \rangle\). Since \(E_2\) is an exceptional object, there exist some finite dimensional vector spaces \(V_i\) such that \(R_{E_1}(F) \simeq E_2 \oplus \bigoplus_i V_i[i]\). It follows from the definition of the mutation that \(\text{Ext}^\bullet(E_2, R_{E_1}(F)) \simeq \text{Ext}^\bullet(E_2, F)\). Since the left hand side of the above isomorphism is \(\bigoplus_i V_i[i]\), we have \(R_{E_1}(F) \simeq \text{Ext}^\bullet(E_2, F) \otimes E_2\). \(\Box\)

### 3. Rank 2 weak Fano bundles with \(c_1 = 0\)

Throughout Sections 3 and 4, we let \(X\) denote a del Pezzo 3-fold of degree 5. We regard \(X\) as a codimension 3 linear section of \(\text{Gr}(2, 5)\) under the Plücker embedding. Let \(R\) and \(Q\) be the restriction of the universal rank 2 subbundle and rank 3 quotient bundle respectively.

The main purpose of this section is to show the following theorem, which is an essential part of Theorem 1.1.

**Theorem 3.1.** Let \(E\) be an indecomposable rank 2 weak Fano bundle on \(X\) with \(c_1(E) = 0\). If \(c_2(E) = 1, 2, 3, 4\), then \(E\) satisfies (vi), (vi), (vii), (viii) in Theorem 1.1, respectively.

As the first preliminary, we recall the following exact sequence, which is given by [24] or [19] Equalities (9) and (10):

\[
0 \to Q(-1)^{\text{ev}} \to \text{Hom}(Q(-1), R)^\vee \otimes R \simeq R^\oplus 3 \simeq \text{Hom}(R, Q^\vee) \otimes R \simeq Q^\vee \to 0.
\]

We quickly review the above exact sequence. First, we note that \(\text{hom}(R, Q^\vee) = 3\) by [19] Lemma 4.1. Moreover, if we set \(A := \text{Hom}(R, Q^\vee)\), then the Hilbert scheme of lines on \(X\) is isomorphic to \(\mathbb{P}(A^\vee)\). As explained in [19] Section 4.1, there is a non-degenerate symmetric form on \(A\) and hence an isomorphism \(A^\vee \simeq A\). Under the identification \(\text{Hom}(Q(-1), R) \simeq \text{Hom}(R^\vee, Q^\vee(1)) \simeq \text{Hom}(R, Q^\vee) = A\), the isomorphism \(A^\vee \simeq A\) induces an isomorphism \(\text{Hom}(Q(-1), R)^\vee \simeq \text{Hom}(R, Q^\vee)\) in \([5.1]\).

#### 3.1. Proof of Theorem 3.1 with \(c_2 = 1\)

Suppose \(c_2(E) = 1\). Then there exists a line \(l\) on \(X\) such that \(E\) fits in the exact sequence \(0 \to O \to E \to I_l \to 0\) by Proposition 5.1 (2), which we will independently prove in Section 5 without using any results from other sections. Moreover, by [19] Lemma 4.2, there exists the following exact sequence for each line \(l\):

\[
0 \to R \oplus Q^\vee \to I_l \to 0.
\]
Note that \( a_t : \mathcal{R} \to \mathcal{Q}^\vee \) factors as follows: \( \mathcal{R} \xrightarrow{id \otimes a_t} \mathcal{R} \otimes \text{Hom}(\mathcal{R}, \mathcal{Q}^\vee) \cong \mathcal{Q}^\vee \). Using \([3,1]\), we obtain another exact sequence
\[
(3.3) \quad 0 \to \mathcal{Q}(-1) \to \mathcal{R}^{\oplus 2} \to \mathcal{I}_t \to 0
\]
Pulling back the exact sequence \( (3.3) \) by the surjection \( \mathcal{E}' \to \mathcal{I}_t \), we have an extension
\[
0 \to \mathcal{Q}(-1) \to \mathcal{E}' \to \mathcal{E} \to 0,
\]
where \( \mathcal{E}' \) is the extension of \( \mathcal{R}^{\oplus 2} \) by \( \mathcal{O} \). Since \( \text{Ext}^1(\mathcal{R}, \mathcal{O}) = 0 \), the bundle \( \mathcal{E}' \) is isomorphic to \( \mathcal{O} \oplus \mathcal{R}^{\oplus 2} \) and we have the desired resolution
\[
(3.4) \quad 0 \to \mathcal{Q}(-1) \to \mathcal{O} \oplus \mathcal{R}^{\oplus 2} \to \mathcal{E} \to 0.
\]

**Remark 3.2.** Pulling back the exact sequence \([3,2]\) by the surjection \( \mathcal{E} \to \mathcal{I}_t \), we have an extension
\[
0 \to \mathcal{R} \to \mathcal{E}'' \to \mathcal{E} \to 0, \quad \text{where } \mathcal{E}'' \text{ is the extension of } \mathcal{Q}^\vee \text{ by } \mathcal{O}.
\]
Thus \( \mathcal{E}'' \cong \mathcal{Q}^\vee \oplus \mathcal{O} \) and we obtain another exact sequence
\[
(3.5) \quad 0 \to \mathcal{R} \to \mathcal{Q}^\vee \oplus \mathcal{O} \to \mathcal{E} \to 0.
\]

### 3.2. Proof of Theorem 3.1 with \( c_2 = 2 \)
Next, let us assume that \( c_2(\mathcal{E}) = 2 \). In this case, \( \mathcal{E} \) is a minimal instanton bundle by \([3, \text{ Corollary 4.7}]\). Then by \([19, \text{ Theorem 4.7}]\), we have an exact sequence
\[
(3.6) \quad 0 \to \mathcal{R}^{\oplus 2} \xrightarrow{\gamma'} (\mathcal{Q}^\vee)^{\oplus 2} \to \mathcal{E} \to 0.
\]

On the other hand, there is an exact sequence
\[
0 \to \mathcal{Q}(-1)^{\oplus 2} \to \mathcal{R}^{\oplus 6} \to (\mathcal{Q}^\vee)^{\oplus 2} \to 0.
\]
Since \( \text{Ext}^1(\mathcal{R}, \mathcal{Q}(-1)) = 0 \), the injection \( \gamma' \) uniquely lifts to an injection \( \tilde{\gamma} : \mathcal{R}^{\oplus 6} \hookrightarrow \mathcal{R}^{\oplus 6} \) along this surjection \( \mathcal{R}^{\oplus 6} \to (\mathcal{Q}^\vee)^{\oplus 2} \):
\[
0 \to \mathcal{R}^{\oplus 6} \xrightarrow{\tilde{\gamma}} (\mathcal{Q}^\vee)^{\oplus 2} \to 0.
\]

Thus we obtain an exact sequence
\[
(3.7) \quad 0 \to \mathcal{Q}(-1)^{\oplus 2} \to \mathcal{R}^{\oplus 4} \to \mathcal{E} \to 0.
\]

### 3.3. Preliminaries for \( c_2 \geq 3 \)
When \( c_2(\mathcal{E}) \geq 3 \), we need another technique for finding our resolutions. The key ingredient is the following Orlov’s theorem.

**Theorem 3.3.** The derived category \( D^b(X) \) of a del Pezzo 3-fold of degree 5 admits a full strong exceptional collection
\[
(3.8) \quad D^b(X) = \langle \mathcal{O}(-1), \mathcal{Q}(-1), \mathcal{R}, \mathcal{O} \rangle.
\]

**Remark 3.4.** This full strong exceptional collection can be seen as the exceptional collection \( c_2 \) in \([24, \text{ Theorem}]\) tensored by \( \mathcal{O}(-1) \).

Using Theorem 3.3, we prepare the following lemma for instanton bundles as a tool for finding the desired resolutions.

**Lemma 3.5.** For an instanton bundle \( \mathcal{E} \) on \( X \), there is a distinguished triangle
\[
(3.9) \quad R\Gamma(\mathcal{E}(1)) \otimes \mathcal{O} \to \mathcal{E}(1) \oplus H^1(\mathcal{E}) \otimes \mathcal{O}(-1)[1] \to R\mathcal{O}_\mathcal{O}(-1)L\mathcal{O}(\mathcal{E}(1)) \cong 1.
\]
Moreover, if \( \mathcal{E} \) is weak Fano, then \( R\mathcal{O}_\mathcal{O}(-1)L\mathcal{O}(\mathcal{E}(1)) \cong \mathcal{V}[1] \) for some vector bundle \( \mathcal{V} \) and we have an exact sequence
\[
(3.10) \quad 0 \to H^1(\mathcal{E}) \otimes \mathcal{O}(-1) \to \mathcal{V} \to H^0(\mathcal{E}(1)) \otimes \mathcal{O} \cong \mathcal{E}(1) \to 0.
\]
Proof. First, we consider the left mutation
\[(3.11) \quad R\Gamma(E(1)) \otimes O \to E(1) \to L_O(E(1))^\perp.\]
Taking \(R\text{Hom}(O,-(1))\) to the triangle (3.11), we have \(R\text{Hom}(L_O(E(1)), O(-(1)) \simeq R\text{Hom}(E(1), O(-(1)) \simeq R\text{Hom}(E, \omega_X)^\vee \simeq R\Gamma(E)^\vee[-3] \simeq H^1(E)[-2]\), where the final isomorphism follows from [19] Lemma 3.1. Then the right mutation \(R_O(-1)L_O(E(1))\)
fits into the following distinguished triangle:
\[(3.12) \quad R_{O(-1)}L_O(E(1)) \to L_O(E(1)) \to H^1(E) \otimes O(-1)[2]^\perp.\]
From the triangles (3.11) and (3.12), we obtain the following distinguished triangles from the octahedral axiom:
\[(3.13) \quad R_{O(-1)}L_O(E(1)) \to R\Gamma(E(1)) \otimes O[1] \to G^\perp \text{ and} \]
\[H^1(E) \otimes O(-1)[2] \to G \to E[1]^\perp,\]
where \(G \in D^b(X)\). Since the Serre duality gives \(\text{Hom}_{D^b(X)}(E(1)[1], H^1(E) \otimes O(-1)[2]) = \text{Ext}^1(E(1), O(-(1)) \otimes H^1(E) \simeq H^2(E)^\vee \otimes H^1(E) = 0\), we have \(G = H^1(E) \otimes O(-1)[2] \oplus E[1]\). Thus we obtain the distinguished triangle (3.9).

If we additionally assume that \(E\) is weak Fano, then by [8] Theorem 1.7, \(E(1)\) is globally generated. Thus \(L_O(E(1)) \simeq \text{Ker}(H^0(E(1)) \otimes O_X \to E(1))[1]\). Then by [8,12], we have \(R_{O(-1)}L_O(E(1)) \simeq H^{-1}(R_{O(-1)}L_O(E(1)))[1]\). Therefore, putting \(V := H^{-1}(R_{O(-1)}L_O(E(1)))\), we have the exact sequence (3.11) as the long exact sequence derived from (3.9).

Therefore, if we can compute \(V = H^{-1}(R_{O(-1)}L_O(E(1)))\) in the exact sequence (3.10), then we have a resolution of \(E(1)\). For computing \(V\), we need the following lemma.

**Lemma 3.6.** Let \(E \in D^b(X)\) be an object. Then we have the following isomorphisms.

1. \(\text{RHom}(R_{O(-1)}L_O(E(1)), Q(-1)) \simeq \text{RHom}(R, E)^\vee[-2]\).
2. \(\text{RHom}(R, R_{O(-1)}L_O(E(1))) \simeq \text{RHom}(Q(-1), E)[1]\).

In particular, applying Lemma 3.6 to the object \(R_{O(-1)}L_O(E(1)) \in \langle O(-1) \rangle \cap \langle O \rangle^\perp = (Q(-1), R)\), we have the following distinguished triangle
\[(3.14) \quad \text{RHom}(Q(-1), E)[1] \otimes R \to R_{O(-1)}L_O(E(1)) \to \text{RHom}(R, E)[2] \otimes Q(-1)^{\perp}.\]

**Proof.** First, we show the following claim.

**Claim 3.7.** Set \(A = (Q(-1), R)\) and \(B := (O(-1), Q(-1), R)\). Note that \(A \subset B\) and \(B \subset D^b(X)\) are admissible subcategories. Let \(S_A\) and \(S_B\) be the Serre functors of \(A\) and \(B\) induced by the Serre functor of \(D^b(X)\) respectively.

Then we have the following assertions.

1. \(S_A^{-1}(Q(-1)) \simeq Q^\vee[-1]\).
2. \(S_B(Q^\vee) \simeq R(-1)[2]\).
3. \(S_B(R) \simeq Q(-2)[2]\).

**Proof of Claim.** Since there is a semi-orthogonal decomposition \(D^b(X) = \langle A, O, O(1) \rangle\), the inverse Serre functor \(S_A^{-1}\) can be computed as \(S_A^{-1} \simeq L_OO(1)\). Thus \(S_A^{-1}(Q(-1)) \simeq L_OO(1)Q(1)[-3]\), and exact sequences \(0 \to R(1) \to O(1)[3] \to Q(1) \to 0\) and \(0 \to Q^\vee \to O^\vee \to R(1) \to 0\) give computations \(L_OO(1)Q(1)[-3] \simeq L_O(R)(1)[3] \simeq Q^\vee[-1]\). This shows (1).

Similarly, to prove (2) and (3), let us consider a semi-orthogonal decomposition \(D^b(X) = \langle O(-2), B \rangle\), which gives the functor isomorphism \(S_B \simeq R_{O(-2)}(- \otimes\)
Then the exact sequences $0 \to \mathcal{O}(2)[3] \to \mathcal{O}(2)[5] \to \mathcal{R}(-1) \to 0$ and $0 \to \mathcal{R}(2) \to \mathcal{O}(2)[5] \to \mathcal{Q}(2) \to 0$ give computations $\mathcal{S}_{\mathcal{R}}(\mathcal{Q}^{\vee}) \simeq \mathcal{R}_{\mathcal{O}(2)}(\mathcal{Q}^{\vee}(-2))[3] \simeq \mathcal{R}_{\mathcal{R}}(-2)$ and $\mathcal{S}_{\mathcal{R}}(\mathcal{R}) \simeq \mathcal{R}_{\mathcal{O}(2)}(-2) \simeq \mathcal{Q}(2)$ respectively. We complete the proof of Claim 3.7. □

Using Claim 3.7, we prove Lemma 3.8.

(1) Since $\mathcal{R}_{\mathcal{O}(1)}\mathcal{L}_{\mathcal{O}}(\mathcal{E}(1)), \mathcal{Q}(-1) \in \mathcal{A}$, we have

\[
\text{RHom}_{\mathcal{D}(\mathcal{X})}(\mathcal{R}_{\mathcal{O}(1)}\mathcal{L}_{\mathcal{O}}(\mathcal{E}(1)), \mathcal{Q}(-1)) \\
= \text{RHom}_{\mathcal{A}}(\mathcal{R}_{\mathcal{O}(1)}\mathcal{L}_{\mathcal{O}}(\mathcal{E}(1)), \mathcal{Q}(-1)) \\
= \text{RHom}_{\mathcal{A}}(\mathcal{S}_{\mathcal{A}}^{-1}(\mathcal{Q}(-1)), \mathcal{R}_{\mathcal{O}(1)}\mathcal{L}_{\mathcal{O}}(\mathcal{E}(1))) (\text{Serre duality in } \mathcal{A}) \\
= \text{RHom}_{\mathcal{A}}(\mathcal{Q}(-1), \mathcal{L}_{\mathcal{O}}(\mathcal{E}(1))) (\text{Remark 2.2 (1)}) \\
= \text{RHom}_{\mathcal{B}}(\mathcal{E}(1), \mathcal{Q}(-1)) (\text{Serre duality in } \mathcal{B}) \\
= \text{RHom}_{\mathcal{D}(\mathcal{X})}(\mathcal{E}(1), \mathcal{Q}(-1)) (\text{Claim 3.7}) \\
= \text{RHom}_{\mathcal{D}(\mathcal{X})}(\mathcal{E}(1), \mathcal{Q}(-1)[1]) (\text{Serre duality on } \mathcal{X}).
\]

We complete the proof of Lemma 3.6. □

In particular, the computation of $\mathcal{V}$ can be reduced to computing the cohomologies of $\text{RHom}(\mathcal{Q}(-1), \mathcal{E})$ and $\text{RHom}(\mathcal{R}, \mathcal{E})$. In order to compute them, we make sure that the following vanishing holds.

Lemma 3.8. It holds that $\text{Ext}^{\geq 2}(\mathcal{Q}(-1), \mathcal{E}) = \text{Ext}^{\geq 2}(\mathcal{R}, \mathcal{E}) = 0$ for an instanton bundle $\mathcal{E}$.

Proof. First of all, we know $H^{\geq 2}(\mathcal{E}(1)) = 0$ by [19] Lemma 3.1. Therefore, we have $H^{\geq 1}(\mathcal{R}_{\mathcal{O}(1)}\mathcal{L}_{\mathcal{O}}(\mathcal{E}(1))) = 0$ by (3.7). Hence by (3.11), it suffices to show $\text{Ext}^{i}(\mathcal{R}, \mathcal{E}) \simeq H^{i}(\mathcal{R}^{\vee} \otimes \mathcal{E}) = 0$ for $i \geq 2$.

Let $s: \mathcal{O} \to \mathcal{R}^{\vee}$ be a general section and consider the following exact sequence:

\[
0 \to \mathcal{E} \xrightarrow{s} \mathcal{R}^{\vee} \otimes \mathcal{E} \to \mathcal{I}_{\mathcal{C}}(1) \otimes \mathcal{E} \to 0,
\]

where $\mathcal{C}$ is the conic corresponding to $s$. It is well-known that every conic $\mathcal{C}$ on $\mathcal{X}$ can be realized as the zero scheme of a global section $\mathcal{R}^{\vee}$ (see [4] Theorem 5.6 and its reference). Since $H^{\geq 2}(\mathcal{E}) = 0$ by [19] Lemma 3.1, it is enough to show that $H^{\geq 2}(\mathcal{I}_{\mathcal{C}}(1) \otimes \mathcal{E}) = 0$ for a general conic $\mathcal{C}$. This vanishing holds since it was shown by [27] Theorem 4.24 that $\mathcal{E}_{|\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}^{\oplus 2}$ for a general conic $\mathcal{C}$. We complete the proof. □
3.4. **Proof of Theorem 3.1 with $c_2 = 3$**. Let $\mathcal{E}$ be a rank 2 weak Fano bundle with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 3$. We will see in Proposition 5.1 that $h^0(\mathcal{E}) = 0$ in this case, and hence $\mathcal{E}$ is an instanton bundle as discussed in Remark 1.3. Moreover, since $\mathcal{E}(1)$ is nef, it holds that $\mathcal{E}(1) \simeq \mathcal{O}^2$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ for every line $l$ on $X$.

Then it follows from [28, Definition 8.2 and Proposition 8.7] that $\text{RHom}(\mathcal{R}, \mathcal{E}) \simeq k$.

Taking the cohomologies of the distinguished triangle (3.14), we have an exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{R}^{\oplus 5} \to \mathcal{K} \to 0$$

where $\mathcal{V} := H^{-1}(\mathcal{R}(\mathcal{E}))$ and $a = \text{hom}(\mathcal{Q}(1), \mathcal{E})$. Note that $R\Gamma(\mathcal{E}) = k$ by the Hirzebruch-Riemann-Roch theorem and the Kawamata-Viehweg vanishing theorem, and $R\Gamma(\mathcal{E}) = k[-1]$ by [19, Lemma 3.1]. Then Lemma 3.5 gives $\text{rk} \mathcal{V} = 7$, which implies $a = 5$. Set $K := \text{Ker}(\text{ev} : H^0(\mathcal{E})) \to \mathcal{E}(1))$ and $K' := \text{Ker}(\mathcal{R} \to \mathcal{V} \to K)$. Since $h^1(\mathcal{E}) = 1$, we have the following diagram by (3.10):

![Diagram](image.png)

Since $\text{Ext}^1(\mathcal{O}(1), \mathcal{Q}(1)) = 0$, we have $K' = \mathcal{O}(1) \oplus \mathcal{Q}(1)$. Regarding this exact sequence $0 \to \mathcal{O}(1) \oplus \mathcal{Q}(1) \to \mathcal{R}^5 \to \mathcal{K} \to 0$ as a resolution of $K$, we have the desired resolution

$$0 \to \mathcal{O}(1) \oplus \mathcal{Q}(1) \to \mathcal{R}^5 \to \mathcal{O}^8 \to \mathcal{E}(1) \to 0. \quad \Box$$

**Remark 3.9.** We can give another resolution by showing $\mathcal{V} \simeq \mathcal{R}^{\oplus 2} \oplus \mathcal{Q}'$. By the universality of the co-evaluation map (see (3.11)), the map $\alpha : \mathcal{Q}(1) \to \mathcal{R}^{\oplus 5}$ factors as follows:

![Diagram](image.png)

If $\alpha$ is not injective, then $\text{Im}(\alpha) \simeq \mathcal{R}^{\oplus 2} \subset \mathcal{R}^{\oplus 5}$. Then it follows from (3.3) that $\mathcal{V} \simeq \mathcal{I}_l \oplus \mathcal{R}^{\oplus 8}$ for some line $l$, which contradicts that $\mathcal{V}$ is locally free. Hence $\alpha$ is injective. Then we have $0 \to \mathcal{Q}' \to \mathcal{V} \to \mathcal{R}^{\oplus 2} \to 0$. Since $\text{Ext}^1(\mathcal{R}, \mathcal{Q}') = 0$ (c.f. [24, 19, Lemma 4.1]), we have $\mathcal{V} = \mathcal{R}^{\oplus 2} \oplus \mathcal{Q}'$. By Lemma 3.5, we obtain another resolution

$$0 \to \mathcal{O}(1) \to \mathcal{R}^{\oplus 2} \oplus \mathcal{Q}' \to \mathcal{O}^8 \to \mathcal{E}(1) \to 0.$$

3.5. **Proof of Theorem 3.1 with $c_2 = 4$**. Let $\mathcal{E}$ be a rank 2 weak Fano bundle with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 4$. In this case, we need the following inequality, which will be used to indicate that there is no morphism between certain vector bundles.

**Lemma 3.10.** Let $X$ be a smooth projective 3-fold. Let $\mathcal{F}$ be a nef locally free sheaf and $\mathcal{G} \subset \mathcal{F}$ a torsion free subsheaf of rank 2. Let $\pi : \mathbb{P}_X(\mathcal{F}) \to X$ be the projectivization of $\mathcal{F}$ and $\xi$ a tautological divisor.
Suppose that $\text{Hom}(G(B), F) = 0$ for every non-zero effective divisor $B > 0$ on $X$. Then we have

\begin{equation}
0 \leq (c_1(F))^3 - 2c_1(F)c_2(F) + c_3(F)) - (c_1(F)^2 - c_2(F))c_1(G)
+ \frac{1}{2}c_1(F)(ch(G)^2 - 2ch_2(G))
\end{equation}

\textbf{Proof.} Let $\pi^*G \to O_{F}(\xi)$ be the morphism corresponding to $G \to F$. Tensoring by $O_{D}(\xi)$, we obtain a map $\pi^*G(\xi) \to O_{D}(\xi)$. Let $I$ be its image, which is an ideal sheaf. Then there is an effective divisor $D$ and a closed subscheme $Z$ with $\text{codim}_X Z \geq 2$ such that $I = I_Z/\pi(\xi)(-D)$. Since $G$ is a rank 2 subsheaf of $F$, $D$ vanishes over the generic point of $X$. Hence $D = \pi^*B$ for some effective Cartier divisor $B$ on $X$. Now we obtain a surjection $\pi^*(G(B))(\xi) \to I_Z$. Tensoring by $O_{D}(\xi)$, we have a surjection $\alpha := \pi^*(G(B)) \to I_Z(\xi)$, which gives a non-zero morphism $G(B) \to F$. Then $B = 0$ follows from our assumption.

Hence we obtain a surjection $\alpha := \pi^*G \to I_Z(\xi)$. We also note that $\pi^*G$ is torsion free and so is $\text{Ker} \alpha$. Since we suppose that $\text{rk} G = 2$, there is an exact sequence

\[ 0 \to \pi^* \text{det} G \otimes O(-\xi) \otimes W \to \pi^*G \to I_Z(\xi) \to 0, \]

where $W \subset X$ is a closed subscheme of $\text{codim}_X W \geq 2$ and $\text{det} G$ is the determinant invertible sheaf of the torsion free sheaf $G$ defined as in [10]. Hence we obtain that $\text{ch}(\pi^*G) = \text{ch}(\pi^* \text{det} G \otimes O(-\xi) \otimes W) + \text{ch}(I_Z(\xi))$. Note that $\text{ch}(I_Z) = 1 - \text{ch}(O_Z)$, $\text{ch}(I_W) = 1 - \text{ch}(O_W)$, and $\text{ch}_k(O_Z) = ch_k(O_W) = 0$ for $k \leq 1$ since $\text{codim}_X Z \geq 2$ and $\text{codim}_X W \geq 2$. Thus we obtain the following equality by direct computation:

\begin{equation}
\text{ch}_2(O_Z) + \text{ch}_2(O_W) = \xi^2 - \xi \cdot \text{ch}_1(G) + \frac{1}{2} (\text{ch}_1(G)^2 - 2 \text{ch}_2(G)).
\end{equation}

Hence $\xi^2 - \xi \cdot \text{ch}_1(G) + \frac{1}{2} (\text{ch}_1(G)^2 - 2 \text{ch}_2(G))$ is an effective codimension 2 cycle. Since $\pi$ is nef, letting $n := \text{dim} \mathbb{P}X(F) = \text{dim} X + \text{rk} F - 1$, we have $0 \leq \xi^n - \xi^{n-1} \cdot \text{ch}_1(G)(B)) + \xi^{n-2} \cdot 1/2 (\text{ch}_1(G)(B)) - 2 \text{ch}_2(G(B)))$ by [3.10]. Since $\pi \cdot G^n = c_1(F)^3 - 2c_1(F)c_2(F) + c_3(F), \pi \cdot \xi^{n-2} = c_1(F)^2 - c_2(F),$ and $\pi \cdot \xi^{n-2} = c_1(F),$ we obtain the inequality [3.15].

Now Lemma [3.10] enables us to compute the following derived Hom spaces.

\textbf{Lemma 3.11.} $\text{RHom}(\mathcal{R}, E) \simeq \mathcal{O}[-1]$ and $\text{RHom}(Q(-1), E) \simeq 0$.

\textbf{Proof of Lemma 3.11.} The Hirzebruch-Riemann-Roch theorem gives $\chi(Q(-1), E) = \chi(Q^\vee(-1) \otimes E) = \int_X \text{ch}(Q^\vee) \text{ch}(O(1)) \text{ch}(E) \text{td}(X) = 0$ and $\chi(R, E) = \chi(R^\vee \otimes E) = \int_X \text{ch}(R^\vee) \text{ch}(E) \text{td}(X) = -2$. Since $\text{Ext}^{\leq 2}(Q(-1), E) = \text{Ext}^{\leq 2}(R, E) = 0$ by Lemma [3.8], it is enough to show $\text{Hom}(R, E) = \text{Hom}(Q(-1), E) = 0$.

Now we set $R := E(1)$. Then the above equality is equivalent to $\text{Hom}(R^\vee, F) = \text{Hom}(Q, F) = 0$.

\textbf{Step 1.} In this step, we show $\text{Hom}(R^\vee, F) = 0$.

Assume there is a non-zero morphism $s : R^\vee \to F$. If $\text{Im} s$ is of rank 1, then $(\text{Im} s)^{\vee} \simeq \mathcal{O}(a)$ for some integer $a$. Then we have $\text{Hom}(R^\vee, \mathcal{O}(a)) \neq 0$, which implies $a > 0$, and $\text{Hom}(\mathcal{O}(a), F) \neq 0$, which implies $a \leq 0$. This is a contradiction and hence we have $\text{rk} \text{Im} s = 2$, which implies $s$ is injective. Since $X$ is of Picard rank 1 and $c_1(F) - c_1(R^\vee) = H$, there is no non-zero effective divisor $B$ such that $\text{Hom}(R^\vee(B), F) \neq 0$. Then Lemma [3.10] implies $0 \leq c_1(F)^3 - 2c_1(F)c_2(F) - c_1(R^\vee)c_1(F)^2 - c_2(F)c_1(F)^2 + c_2(R^\vee)c_1(F)^2$. Since $c_1(F)^2 = 2, c_2(F) = 9, c_1(R^\vee) = 1, c_2(R^\vee) = 2$, the right hand side is equal to $-3$. This is a contradiction.

\textbf{Step 2.} In this step, we show $\text{Hom}(Q, F) = 0$.

Assume the contrary and let $s : Q \to F$ be a non-zero morphism. We set $G := \text{Im}(s), K := \text{Ker}(s),$ and $T := F/G$. Then $K$ is reflexive and $G$ is torsion free. Let
Moreover, if \( \eta \) denotes a tautological divisor of \( \mathbb{P}_2(\mathcal{G}) \) and \( L \) the pull-back of a line on \( \mathbb{P}^2 \) under \( \pi : \mathbb{P}_2(\mathcal{G}) \to \mathbb{P}^2 \), then

1. \( e^*H \sim \eta + L \) for a hyperplane section \( H \) on \( X \), and
2. \( e^*L \sim \eta.L - L^2 \) for a line \( l \) on \( X \).

We refer to [9] for the proof of Theorem 4.1 and more details. Using Theorem 4.1 we show the following proposition.

**Proposition 4.2.** Let \( \mathcal{E} \) be a rank 2 bundle on \( X \) with \( c_1(\mathcal{E}) = -1 \). Assume that \( \mathcal{E}|_l \simeq \mathcal{O} \oplus \mathcal{O}(-1) \) for every line \( l \) on \( X \). Then \( \mathcal{E} \) is \( \mathcal{O} \oplus \mathcal{O}(-H_X) \) or \( \mathcal{R} \).
Proof. Set $\mathcal{E}_U := e^*\mathcal{E}$. Then $\mathcal{E}_U|_{\mathbb{P}^2} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ for every $p \in \mathbb{P}^2$. Hence there is an integer $a \in \mathbb{Z}$ such that $\mathcal{O}(-a) \simeq \pi_\mathcal{E}_U$. Note that the cokernel of the natural injection $\pi_\mathcal{O}_\mathcal{P}(-a) \to \mathcal{E}_U$ is invertible. Since $\det \mathcal{E}_U \simeq e^*\mathcal{O}_X(-1) \simeq \mathcal{O}(-n + L)$, we obtain the following exact sequence:
\[
0 \to \mathcal{O}_U(-(a) \to \mathcal{E}_U \simeq e^*\mathcal{E} \to \mathcal{O}_U(-\eta + (a - 1)L) \to 0.
\]
Thus $c_2(\mathcal{E}_U) = -(a)L(\eta + (a - 1)L) = a\eta L - a(a - 1)L^2$. On the other hand, by Theorem 4.1 (2), $c_2(\mathcal{E}_U) = e^*c_2(\mathcal{E})$ must be divisible by $\eta L - L^2$ in $H^2(U, \mathbb{Z})$. Hence $a \in \{0, 2\}$ and $c_2(\mathcal{E}) = a$.

When $a = c_2(\mathcal{E}) = 0$, we have $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ by Proposition 5.1.

When $a = c_2(\mathcal{E}) = 2$, we have $\text{R}^1(\mathcal{U}, e^*\mathcal{E}) = 0$ from (4.1) with $a = 2$. Since $e^*\mathcal{E}$ has $\mathcal{E}$ as a direct summand, we have $\text{R}^1(X, \mathcal{E}) = 0$. By the Serre duality and the natural isomorphism $\mathcal{E} \simeq \mathcal{E}^*(-1)$, we obtain $\text{R}^1(\mathcal{E}, \mathcal{O}(-1)) \simeq \text{R}^1(\mathcal{E}^*(-1)) \simeq \text{R}^1(\mathcal{E}) \simeq 0$. Hence $\mathcal{E} \in \langle \mathcal{O} \rangle \cap \langle \mathcal{O}(-1) \rangle$ in $\text{D}^b(X)$, which means that $\mathcal{E} \in \langle \mathcal{Q}(-1), \mathcal{R} \rangle$. By Lemma 2, we obtain the following distinguished triangle:
\[
\text{RHom(}\mathcal{R}, \mathcal{E}) \otimes \mathcal{R} \to \mathcal{E} \to \text{RHom}(\mathcal{E}, \mathcal{Q}(-1)^\vee) \otimes \mathcal{Q}(-1)^{\otimes 3}.
\]
We note that $\mathcal{E}$ is slope stable since $h^0(\mathcal{E}) = 0$. Thus we have $\text{Hom}(\mathcal{E}, \mathcal{Q}(-1)) = 0$.

Taking the long exact sequence from the above distinguished triangle, we have
\[
0 \to \mathcal{Q}(-1)^{\otimes a} \to \mathcal{R}^{\otimes b} \to \mathcal{E} \to 0,
\]
where $a = \text{ext}^1(\mathcal{E}, \mathcal{Q}(-1))$ and $b = \text{hom}(\mathcal{R}, \mathcal{E})$. Computing the rank and the 1st Chern class, we have equalities $3a + 2 = 2b$ and $-2a - 1 = -b$, which implies $a = 0$ and $b = 1$. Thus $\mathcal{E} \simeq \mathcal{R}$. We complete the proof. □

Corollary 4.3. Let $X$ be a del Pezzo 3-fold of degree 5. Then every rank 2 weak Fano bundle $\mathcal{E}$ on $X$ with $c_1(\mathcal{E}) = -1$ is isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ or $\mathcal{R}$.

Proof. Since $\mathcal{E}$ is weak Fano, $\mathcal{E}(2)$ is ample. Hence $\mathcal{E}(2)|_l \simeq \mathcal{O}_l(1) \oplus \mathcal{O}_l(2)$ for every line $l \subset X$. Then Proposition 4.2 implies the result. □

5. Classifications of rank 2 weak Fano bundles of del Pezzo threefolds of Picard rank 1

In this section, we complete our classification of weak Fano bundles on a del Pezzo 3-fold of Picard rank 1. To deduce the conclusion, we prepare the following proposition, which is a detailed version of [13, Lemma 3.2].

Proposition 5.1. Let $X$ be a del Pezzo 3-fold of degree $d \leq 5$. Let $\mathcal{E}$ be a normalized rank 2 weak Fano bundle on $X$.

(1) $c_2(\mathcal{E}) \leq 0$ if and only if $\mathcal{E}$ is the direct sum of line bundles.

(2) The following are equivalent.

(a) $c_2(\mathcal{E}) = 1$.

(b) $c_2(\mathcal{E}) > 0$ and $h^0(\mathcal{E}) \neq 0$.

(c) $d = \text{deg} X \geq 3$ and there is an exact sequence $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_l X \to 0$, where $l$ is a line on $X$.

In particular, if $\text{deg} X \leq 2$, then $c_2(\mathcal{E}) \neq 1$.

Proof. Let $c_1 \in \mathbb{Z}$ be the integer such that $\Delta = \mathcal{O}_X(-c_1)$. Then $c_2(\mathcal{E}) = c_2(\mathcal{E}) = c_2 \cdot l \in \text{H}^2(X, \mathbb{Z})$. Before starting the proof, we note that $h^0(\mathcal{E}(2)) = h^0(\mathcal{E}(2)) = h^0(\mathcal{E}(2)) = 0$ by the Serre duality and the Le-Potier vanishing for the ample bundle $\mathcal{E}(2)$.

(1) It suffices to show that if $c_2 \leq 0$, then $\mathcal{E}$ is decomposed into line bundles. First, we treat the case when $h^0(\mathcal{E}(1)) \neq 0$. Since $h^0(\mathcal{E}(1)) = 0$ and $h^0(\mathcal{E}(1)) \neq 0$, we have $0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{I}_{Z, -1} \to 0$, where $Z$ is a closed subscheme with $\text{codim} Z \geq 2$. If $c_1 = -1$, then $\mathcal{E}(2)$ is ample and there is a surjection $\mathcal{E}(2) \to \mathcal{I}_Z$,
which is a contradiction. Hence $c_1 = 0$. Then $E(1)$ is nef and there is a surjection $E(1) \to I_Z$, which implies $Z = \emptyset$. Thus we have $E \cong O(1) \oplus O(-1)$.

Next, we treat the case $h^0(E(-1)) = 0$. By the Le- Potier vanishing and the Hirzebruch-Riemann-Roch theorem, we have

$$h^0(E) \geq \chi(E) = \begin{cases} 
1 - \frac{1}{2}c_2 & \text{if } c_1 = -1 \\
2 - c_2 & \text{if } c_1 = 0 
\end{cases}.$$

Since we assume $c_2 \leq 0$ and $h^0(E(-1)) = 0$, we have $h^0(E) > 0$ and hence an exact sequence $0 \to O \to E \to I_Z(c_1) \to 0$, where $Z$ is a closed subscheme with $\text{codim}_X Z \geq 2$. Then $Z \equiv c_2(E)$ implies that $Z$ is empty. Hence $E \cong O \oplus O(c_1)$.

(2) The implication (a) $\Rightarrow$ (b) follows from (5.1). We show (b) $\Rightarrow$ (c). If $c_2(E) > 0$ and $h^0(E) \neq 0$, then by (1) and its proof, $E$ is indecomposable and $h^0(E(-1)) = 0$.

As discussed in (1), we obtain an exact sequence $0 \to O \to E \to I_Z(c_1) \to 0$ with $\text{codim}_X Z \geq 2$. Let $\pi_E: P(E) \to X$ be the projectivization of $E$ and $\xi_E$ be a tautological divisor. Since $h^0(E) > 0$, $\xi_E$ is effective, which implies $0 \leq -K_{P(E)} = -4c_1 - 24c_2 + c_1^2 + 6c_1^2 + 12c_1 + 8d$. Since $c_2 > 0$ and $c_1 \in \{0, 1\}$, we have $c_1 = 0$, $c_2 = 1$, and $d \geq 3$. Then $Z$ is of degree $c_1 = 1$, which implies $Z$ is a line. Hence (b) $\Rightarrow$ (c) holds. The implication (c) $\Rightarrow$ (a) is obvious. We complete the proof.

5.1. Proof of Theorem 1.1 Let $X$ be a del Pezzo 3-fold of degree 5 and $E$ a normalized rank 2 vector bundle. It is easy to see that $E$ is a weak Fano bundle if $E$ is one of (i) – (viii) in Theorem 1.1. Hence, it suffices to show that every normalized rank 2 weak Fano bundle $E$ satisfies one of (i) – (viii).

If $E$ is decomposable, then it is easy to see $E$ satisfies one of (i), (ii), or (iii). Assume $E$ is decomposable. If $c_1(E) = -1$, then Corollary 1.3 gives that $E \cong R$, i.e., $E$ is of type (iv). Hence we may assume that $c_1(E) = 0$. Then by Proposition 5.1, we have $c_2(E) \geq 1$. On the other hand, since $-K_{P(E)}^4 = 16(20 - 4c_2(E)) > 0$, we have $c_2(E) \leq 4$. Thus $E$ is of type (v), (vi), (vii), or (viii) if $c_2(E) = 1, 2, 3, 4$ respectively.

Finally, we show the existence of an example for each of (i) – (viii). This statement is trivial for the cases (i) – (iv). Hence it suffices to show that, for a given $c \in \{1, 2, 3, 4\}$, there exists weak Fano bundle $E$ such that $c_1(E) = 0$ and $c_2(E) = c$.

When $c = 1$, we have such an $E$ as a unique extension of $I_l/X$ by $O_X$, where $l$ is a line on $X$ (c.f. Proposition 5.1). When $c = 2$, we have such an $E$ as $F(-1)$, where $F$ is a special Ulrich bundle on $X$, which was constructed by 2 Proposition 6.1. Hence we may assume that $c \in \{3, 4\}$. Note that all del Pezzo 3-folds of degree 5 are isomorphic [15]. Hence in this case, we can apply 11 Theorem 5.8 and obtain an elliptic curve $C$ of degree $c + 5$ such that $-K_{BIC X}$ is nef and big, where $BIC X$ is the blowing-up of $X$ along $C$. Let $F$ be a unique non-trivial extension of $I_C(-K_X)$ by $O_X$. Then $BIC X$ is a member of $|O_{P(F)}(1)|$ and $-K_{BIC X} \cong O_{P(F)}(1)|_{BIC X}$. Thus we can conclude that $O_{P(F)}(1)$ is nef and big and so is $-K_{P(F)}$. Letting $E := F(-1)$, we obtain an example of a rank 2 weak Fano bundle $E$ with $c_1(E) = 0$ and $c_2(E) = c$. We complete the proof of Theorem 1.1. 

5.2. Proof of Theorem 1.5 Let $X$ be a del Pezzo 3-fold of degree less than 3. By Proposition 5.1, it suffices to show $c_2(E) < 2$. If $c_1(E) = 0$, then we have $0 < (-K_{P(E)})^4 = 4(d - c_2(E))$, which implies $2 \geq d > c_2(E)$. If $c_1(E) = -1$, then we have $c_2(E) \leq d$ by the inequality $(-K_{P(E)})^4 = 80d - 64c_2(E) > 0$. Hence we have $d = 2$ and $c_2(E) = 2$. Then the Le-Potier vanishing and the Hirzebruch-Riemann-Roch theorem implies $h^0(E(1)) \geq \chi(E(1)) = 2$. Hence, letting $\pi_E: P(E) \to X$ be the projectivization, we conclude that the divisor $\xi_E + \pi_E^*XH_X$ is linearly equivalent to an effective divisor. Since $-K_{P(E)} \sim 2\xi_E + \pi_E^*(3H_X)$ is nef and big, we obtain...
0 \leq (-K_{\mathcal{F}})^{\beta}(\xi_{\mathcal{E}} + \pi^{*}H_{X}) = -2, \text{ which is a contradiction. We complete the proof of Theorem 1.3.} \quad \square

5.3. Summary. By Theorems 1.4 and 1.6, we classified rank 2 weak Fano bundles on a del Pezzo 3-fold of degree 1, 2, 5. Therefore, we complete our classification of weak Fano bundles on a del Pezzo 3-fold of Picard rank 1 by [14], [8], and this article. As a corollary of these classification results, every rank 2 weak Fano bundle \( \mathcal{F} \) on a del Pezzo 3-fold of Picard rank 1 is a globally generated bundle \( \mathcal{F} \) with \( c_{1}(\mathcal{F}) \in \{1, 2\} \) up to twist. Moreover, if we additionally assume that \( \mathcal{F} \) is not a Fano bundle, then \( c_{1}(\mathcal{F}) = 2 \).

6. MODULI SPACES OF RANK 2 WEAK FANO BUNDLES ON A DEL PEZZO 3-FOLD OF DEGREE 5

Let \( X \) be a del Pezzo 3-fold of degree 5. As an application of Theorem 1.1, we study moduli spaces of rank 2 weak Fano bundles on \( X \). First of all, we define the moduli functors that we want to handle by following [13].

**Definition 6.1.** Let \((\text{Sch}/k)\) be the category of schemes of finite type over the base field \( k \). Let \((\text{Sets})\) be the category of sets. For a given \((c_{1}, c_{2}) \in \{(0, -1)\} \times \mathbb{Z}\), we consider the following functor \( \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}} : (\text{Sch}/k)^{\text{op}} \to (\text{Sets}) \) defined by

\[
\mathcal{M}_{c_{1}, c_{2}}^{\text{wF}}(S) = \left\{ \begin{array}{l}
\mathcal{E}_{S} : \text{a coherent sheaf on } X \times S, \\
\forall s \in S, \mathcal{E}_{s} := \mathcal{E}_{X_{s} \otimes k(s)} \text{ is a rank } 2 \text{ weak Fano bundle on } X, \\
\mathcal{E}_{s}\text{ is flat over } S, \\
c_{1}(\mathcal{E}_{s}) = c_{1} \text{ and } c_{2}(\mathcal{E}_{s}) = c_{2}, \\
\text{where } \mathcal{E}_{S} \sim \mathcal{E}_{S}' \iff \exists \mathcal{E}_{S} \in \text{Pic}(S) \text{ such that } \mathcal{E}_{S}' \cong \mathcal{E}_{S} \otimes \mathcal{O}_{S}. 
\end{array} \right\}/\sim,
\]

where \( \mathcal{E}_{S} \sim \mathcal{E}_{S}' \iff \exists \mathcal{E}_{S} \in \text{Pic}(S) \) such that \( \mathcal{E}_{S}' \cong \mathcal{E}_{S} \otimes \mathcal{O}_{S}. \) In the above definition, we note that being weak Fano is preserved under base change for a smooth projective variety over a field (c.f. [17] Lemma 2.18).

As usual, we say that a scheme \( \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}} \) over \( k \) is the coarse (resp. fine) moduli space of \( \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}} \) if there is a natural transformation \( \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}} \to \text{Hom}(\mathcal{E}, \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}}) \) which corepresents (resp. represents) the functor \( \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}} \) and the map \( \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}}(\text{Spec } k) \to \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}}(\text{Spec } k) \) is isomorphic.

**Remark 6.2.** Let \( \mathcal{E} \) be a rank 2 weak Fano bundle on \( X \). By Theorem 1.4.

When the case \((c_{1}, c_{2}) \neq (0, 4)\), each moduli space \( \mathcal{M}_{c_{1}, c_{2}}^{\text{wF}} \) was essentially observed by previous studies and our discussion so far. We summarize results in the following proposition.

**Proposition 6.3.** (1) The coarse moduli spaces \( \mathcal{M}_{0, -5}^{\text{wF}}, \mathcal{M}_{-1, 0}^{\text{wF}}, \mathcal{M}_{0, 0}^{\text{wF}}, \text{ and } \mathcal{M}_{-1, 2}^{\text{wF}} \) are isomorphic to the point \( \text{Spec } k \).

(2) The coarse moduli space \( \mathcal{M}_{1, 1}^{\text{wF}} \) is isomorphic to \( \mathbb{P}^{2} \).

(3) Among the moduli spaces in (1) and (2), only \( \mathcal{M}_{1, 1}^{\text{wF}} \) is fine as a moduli space.

(4) The coarse moduli \( \mathcal{M}_{0, 2}^{\text{wF}} \) of \( \mathcal{M}_{0, 2}^{\text{wF}} \) is a smooth irreducible variety. Moreover, it is not fine (resp. fine) as a moduli space.

**Proof.** (1) This proposition follows from Theorem 1.4.

(2) First, we construct a morphism \( \tau : \mathcal{M}_{0, 1}^{\text{wF}} \to \text{Hom}(-, \mathbb{P}^{2}) \). Let \( S \) be a scheme of finite type over \( k \) and \( \mathcal{E}_{S} \in \mathcal{M}_{0, 1}^{\text{wF}}(S) \) an object. Then \( \varepsilon : \text{pr}_{2}^{*}\mathcal{E}_{S} \to \mathcal{E}_{S} \) is injective and \( \text{Cok } \varepsilon \) is flat over \( S \). Moreover, \( L_{1} := \text{pr}_{2}^{*}\mathcal{E}_{S} \) and \( L_{2} := \text{pr}_{2}^{*}\text{Hom}(\text{Cok } \varepsilon, \mathcal{O}_{X \times S}) \) are invertible by Cohomology and Base Change. Under the isomorphisms \( H^{0}(\mathcal{E}_{S}) \cong H^{0}(L_{2}) \cong \text{pr}_{2}^{*}\text{Hom}(\text{Cok } \varepsilon, \mathcal{O}_{X \times S}) \cong \mathcal{O}_{X \times S} \).
\[ \text{Hom}(pr_2^* \mathcal{L}^{-1}_T \otimes \text{Hom}(\text{Cok} \varepsilon, \mathcal{O}_{X \times S})) = H^0(\text{Cok} \varepsilon \otimes pr_2^* \mathcal{L}_2, \mathcal{O}_{X \times S}) \],

there is a morphism \( t \): \( \text{Cok} \varepsilon \otimes pr_2^* \mathcal{L}_2 \to \mathcal{O} \) corresponding to the unit of \( H^0(S, \mathcal{O}_S) \). Since \( \text{Cok} \varepsilon \) is flat over \( S \), \( t \) is injective and its cokernel, say \( \mathcal{O}_Z \) where \( Z \) denotes a closed subscheme of \( X \times S \), is flat over \( S \). Hence there is an exact sequence \( 0 \to \mathcal{L}_S \to \mathcal{E}_S \to \mathcal{L}_2 \to 0 \). Replacing \( \mathcal{E}_S \) to \( \mathcal{E}_S \otimes pr_2^* \mathcal{L}_2 \) and putting \( \mathcal{L}_S := \mathcal{L}_1 \otimes \mathcal{L}_2 \), we obtain the following exact sequence

\[ (6.1) \]

\[ 0 \to \mathcal{L}_S \to \mathcal{E}_S \to \mathcal{I}_{Z/X} \to 0. \]

It is easy to see that \( Z_s \) is a line on \( X \) for every \( s \in S \). Now let us consider the following spectral sequence:

\[ H^j(S, \mathcal{E}xt^j_{pr_2}(\mathcal{I}_Z, \mathcal{L}_S)) \Rightarrow \text{Ext}^{i+j}_{X \times S}(\mathcal{I}_Z, \mathcal{L}_S), \]

where \( \mathcal{E}xt^j_{pr_2}(\mathcal{F}, -) \) denotes the \( j \)-th cohomology of the left exact functor \( \text{pr}_2^* \text{Hom}(\mathcal{F}, -) : \text{Coh}(X \times S) \to \text{Coh}(S) \). Since \( \mathcal{E}xt^0_{pr_2}(\mathcal{I}_Z, \mathcal{L}_S) = \mathcal{L}_S \), we have an exact sequence

\[ 0 \to H^1(\mathcal{L}_S) \to \text{Ext}^1(\mathcal{I}_Z, \mathcal{L}_S) \to H^0(\mathcal{E}xt^1_{pr_2}(\mathcal{I}_Z, \mathcal{L}_S)) \to H^2(\mathcal{L}_S). \]

Note that \( \mathcal{E}xt^1_{pr_2}(\mathcal{I}_Z, \mathcal{L}_S) \) is an invertible sheaf (c.f. [21, Theorem 1.4]). Moreover, the global section \( \varepsilon := \delta([\mathcal{E}_S]) \) is a nowhere vanishing section. Hence \( \mathcal{E}xt^1_{pr_2}(\mathcal{I}_Z, \mathcal{L}_S) \cong \mathcal{O}_S \), which implies \( \mathcal{L}_S \cong \mathcal{E}xt^1_{pr_2}(\mathcal{I}_Z, \mathcal{O}_S)^\vee \). By this argument, we have a natural identification

\[ \mathcal{M}_{0,1}^{ref}(S) = \left\{ (f : S \to \text{Hilb}_{t+1}(X), [\mathcal{E}_S]) \mid [\mathcal{E}_S] \in \text{Ext}^1(\mathcal{I}_{Z/X \times S}, \mathcal{L}_S), \right. \]

\[ Z := S \times \text{Hilb}_{t+1}(X), \mathcal{L}_S := \mathcal{E}xt^1_{pr_2}(\mathcal{I}_Z, \mathcal{O}_X), \mathcal{E}_S \text{ fits into the exact sequence} \]

\[ \left. \beta : \text{Ext}^1(\mathcal{I}_{Z/X \times S}, \mathcal{L}_S) \to H^0(S, \mathcal{O}_S). \right\} \]

where \( (f, [\mathcal{E}_S]) \sim (f', [\mathcal{E}_S']) : \iff f' = f \) and \( \mathcal{E}_S \) is isomorphic to \( \mathcal{E}_S' \) as an \( \mathcal{O}_{X \times S} \)-module. In particular, there is a functor \( \tau : \mathcal{M}_{0,1}^{ref} \to \text{Hom}(\mathcal{L}_S, \text{Hilb}_{t+1}(X)) \cong \text{Hom}(\mathcal{L}_S, \mathbb{P}^2) \) (c.f. Theorem [11]).

Next we construct a functor \( \sigma : \text{Hom}(\mathcal{L}_S, \mathbb{P}^2) \to \mathcal{M}_{0,1}^{ref} \) such that \( \tau \circ \sigma = \text{id} \) and \( \sigma(T) = \tau(T)^{-1} \) for every affine scheme \( T \). To construct \( \sigma \), we consider the case when \( S = \text{Hilb}_{t+1}(X) \cong \mathbb{P}^2 \) and \( Z \subset \mathbb{P}^2 \times X \) is the universal family \( \text{Hilb}_{t+1}(X) \cong \mathbb{P}^2(\mathcal{G}) \), where \( \mathcal{G} \) is defined in Theorem [11].

To construct a locally free sheaf \( \mathcal{E}_p \) on \( X \times \mathbb{P}^2 \) which fits into the exact sequence (6.1) using the Hartshorne-Serre correspondence, it suffices to check that \( H^2(\mathcal{L}_p^{-1}) = 0 \). Recall that \( \mathcal{L}_p = \mathcal{E}xt^1_{pr_2}(\mathcal{I}_Z, \mathcal{O}_{X \times \mathbb{P}^2}) \). Since \( \mathcal{E}xt^1_{pr_2}(\mathcal{O}_{X \times S}, \mathcal{O}_{X \times S}) = 0 \) for each \( i \geq 0 \), we have \( \mathcal{E}xt^1_{pr_2}(\mathcal{I}_Z, \mathcal{O}_{X \times S}) = \mathcal{E}xt^1_{pr_2}(\mathcal{O}_Z, \mathcal{O}_{X \times S}). \) Since \( R\text{Hom}(\mathcal{O}_Z, \mathcal{O}_{X \times S}) \cong \omega_Z \otimes \omega_{X \times S}|_{\mathbb{P}^2}^{-1} \), we have \( \mathcal{E}xt^1_{pr_2}(\mathcal{O}_Z, \mathcal{O}_X) = (pr_2)|_{\mathbb{P}^2} (\omega_Z \otimes \omega_{X \times \mathbb{P}^2}|_{\mathbb{P}^2}) \). Since \( \omega_{X \times \mathbb{P}^2}|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-2\mathcal{G} - 5H_{\mathbb{P}^2}) \) and \( \omega_Z = \mathcal{O}_Z(-2\mathcal{G}) \), we have \( \omega_Z \otimes \omega_{X \times \mathbb{P}^2}|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(5H_{\mathbb{P}^2}) \), which implies that \( \mathcal{L}_p = \mathcal{O}_{\mathbb{P}^2}(-5) \). Hence we have \( H^2(\mathcal{L}_p^{-1}) = 0 \) and a locally free sheaf \( \mathcal{E}_p \) on \( X \times \mathbb{P}^2 \) fitting into (6.1). Then we obtain a functor \( \sigma : \text{Hom}(\mathcal{L}_p, \mathbb{P}^2) \to \mathcal{M}_{0,1}^{ref} \) such that \( \sigma(S) \) sends \( f : S \to \mathbb{P}^2 \) to \( [f, (\text{id}_X \times f)^* \mathcal{E}_p] \in \mathcal{M}_{0,1}^{ref}(S) \). Note that \( \sigma = \text{id}_{\text{Hom}(\mathcal{L}_p, \mathbb{P}^2)} \) and \( \tau(S) = \sigma(S)^{-1} \) for every affine scheme \( S \).

Let us show that \( \tau \) is the corepresentation of the functor \( \mathcal{M}_{0,1}^{ref} \). Let \( B \) be an arbitrary \( k \)-scheme and \( \alpha : \mathcal{M}_{0,1}^{ref} \to \text{Hom}(\mathcal{L}_p, \mathbb{P}^2) \) a functor. It suffices to show that \( \alpha \circ \sigma \circ \tau = \alpha \). Let \( g : \mathbb{P}^2 \to B \) be the image of \([\mathcal{E}_p]\) under
\( \alpha(\mathbb{P}^2): \mathcal{M}_{0,1}^{\text{reg}}(\mathbb{P}^2) \to \text{Hom}(\mathbb{P}^2, B) \). Then the functor \( \alpha \circ \sigma \) is induced from \( g \).

Let \( S \) be a scheme of finite type over \( k \) and \( S = \bigcup_i S_i \) be an affine open covering. Then, for any element \([f, e] \in \mathcal{M}_{0,1}^{\text{reg}}(S)\), we have \( \tau(S)([f, e]) = f: S \to \mathbb{P}^2 \) and \( \alpha(S)(f) = g \circ f: S \to B \). We set \( h := \alpha(S)([f, e]): S \to B \). Since \( \sigma(S_i)^{-1} = \tau(S_i) \), we have \( h|_{S_i} = (g \circ f)|_{S_i}: S_i \to B \). Thus \( h = g \circ f \) holds. We complete the proof.

(3) When \((c_1, c_2) \in \{(0, -5), (-1, 0), (0, 0), (0, 1)\}\), it is easy to see that every object \( E \in \mathcal{M}_{c_1, c_2}^{\text{reg}}(\text{Spec} \, k) \) is not simple. Hence \( \mathcal{M}_{c_1, c_2}^{\text{reg}} \) cannot be fine as a moduli space. On the other hand, if \((c_1, c_2) = (-1, 2)\), then every element \( E \in \mathcal{M}_{0, c}^{\text{reg}}(\text{Spec} \, k) \) is isomorphic to \( \mathcal{R} \), which is stable. Hence the point \( \text{Spec} \, k \) is a fine moduli space of \( \mathcal{M}_{0, c}^{\text{reg}} \).

(4) For \( c \in \{2, 3\} \), \( \mathcal{M}_{0, c}^{\text{reg}} \) is an open subscheme of \( \mathcal{M}_{0, c}^{\text{ins}} \). Hence the assertion follows from the fact that the moduli space of instanton bundles \( \mathcal{M}_{0, c}^{\text{ins}} \) (resp. \( \mathcal{M}_{0, 3}^{\text{ins}} \)) is a smooth irreducible variety and not fine (resp. fine) as a moduli space \([28]\). For the proof, we refer the reader to \([28] \) Proposition 5.12] and \([28] \) Remark 4.11 and Section 7.

\[ \square \]

6.1. Preliminaries and setting. Our main purpose of this section is to investigate \( \mathcal{M}_{0,4}^{\text{reg}} \) and prove Theorem 1.4. To show this theorem, we associate this moduli space with the moduli space of representations of a certain quiver. For notation about quivers and its representations, we basically follow \([18][25]\). Furthermore, we work over the following notation until the end of this section.

**Notation 6.4.**

- Let \( \mathcal{H} \) be the cokernel of the natural map \( 0 \to \mathcal{O}(-1) \to \mathcal{Q}(-1) \oplus \text{Hom}(\mathcal{O}(-1), \mathcal{Q}(-1))^\vee \simeq \mathcal{Q}(-1)^{\oplus 5} \to \mathcal{H} \to 0 \).

Note that \( \mathcal{H} = \mathcal{R}(\mathcal{Q}(-1))^{\oplus 5} \) and hence \( \langle \mathcal{O}(-1), \mathcal{Q}(-1) \rangle = \langle \mathcal{Q}(-1), \mathcal{H} \rangle \).

- Set \( T := \mathcal{Q}(-1) \oplus \mathcal{H} \). Since \( \text{Hom}(\mathcal{Q}(-1), \mathcal{H}) \simeq \mathbb{R}^{\oplus 5} \), \( \text{Hom}(\mathcal{H}, \mathcal{Q}(-1)) = 0 \), and \( \mathcal{Q}(-1) \) and \( \mathcal{H} \) are simple, \( \text{End}(T) \) is isomorphic to the path algebra \( \mathbb{R} \cdot 5 \)-Kronecker quiver \( Q \).

- Let \( v_0 \) (resp. \( v_1 \)) be the vertex of \( Q \) corresponding to \( \mathcal{Q}(-1) \) (resp. \( \mathcal{H} \)). For a representation \( M \) of \( Q \) over \( \mathbb{R} \), the dimension vector \( \text{dim}_{\mathbb{R}}(M) \) is defined to be \( (\text{dim}_{\mathbb{R}} v_0 M, \text{dim}_{\mathbb{R}} v_1 M) \).

- We define a stability function \( \Theta: \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \) as \( \Theta(a, b) := b - a \).

As a preliminary, we prepare the following lemma.

**Proposition 6.5.** Let \( \mathcal{E} \) be a rank 2 weak Fano bundle with \( c_1(\mathcal{E}) = 0 \) and \( c_2(\mathcal{E}) = 4 \). Let \( \mathcal{F} := \mathcal{E}(1) \) and \( \mathcal{K}_\mathcal{F} := \text{Ker}(H^0(\mathcal{F}) \otimes \mathcal{O} \to \mathcal{F}) \). Then the following assertions hold.

1. \( \text{Ext}^i(T, \mathcal{K}_\mathcal{F}) = 0 \) for every \( i > 0 \). Moreover, \( \text{Hom}(T, \mathcal{K}_\mathcal{F}) \) is a right \( k\mathbb{Q} \) as an \( \text{End}(T) \)-module of dimension vector \( (2, 2) \).
2. \( \text{Hom}(T, \mathcal{K}_\mathcal{F}) \) is \( \Theta \)-stable.

**Proof.** (1) It suffices to show that \( \text{Ext}^i(\mathcal{Q}(-1), \mathcal{K}_\mathcal{F}) \simeq \text{Ext}^i(\mathcal{H}, \mathcal{K}_\mathcal{F}) \simeq \begin{cases} \mathbb{R}^2 & (i = 0) \\ 0 & (i \neq 0) \end{cases} \).

This directly follows the exact sequence

\[
0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{Q}(-1)^{\oplus 2} \to \mathcal{K}_\mathcal{F} \to 0
\]
which is obtained in Theorem [11] (viii).

(2) Since \( (Q(-1), \mathcal{H}) \) is a strong exceptional collection, the functor
\[
\Phi: (Q(-1), \mathcal{H}) \ni \mathcal{F} \mapsto \text{RHom}(\mathcal{T}, \mathcal{F}) \in D^b(\text{mod-kQ})
\]
gives a category equivalence. Note that the heart \( \{ M^* \in D^b(\text{mod-kQ}) \mid H^\partial(M^*) = 0 \} \) of the standard t-structure of \( D^b(\text{mod-kQ}) \) is an abelian full subcategory. Let \( \mathcal{A} \subset (Q(-1), \mathcal{H}) \) be the corresponding abelian subcategory under \( \Phi \). Then \( \mathcal{A} = \{ F^* \in (Q(-1), \mathcal{H}) \mid \text{Ext}^\partial(\mathcal{T}, F^*) = 0 \} \). For each object \( F^* \in \mathcal{A} \), we define its dimension vector as \( \dim(F^*) := \dim(\Phi(F^*)) = (\dim\text{Hom}(Q(-1), F^*), \dim\text{Hom}(H, F^*)) \).

By (1), \( K_F \in (Q(-1), \mathcal{H}) \) is an object of \( \mathcal{A} \). Therefore, to show the \( \Theta \)-stability, it suffices to show the inequality \( \Theta(M) = \dim\text{Hom}(H, M) - \dim\text{Hom}(Q(-1), M) < 0 \) for every subobject \( 0 \neq M \subsetneq K_F \in \mathcal{A} \).

Since \( \text{RHom}(\mathcal{T}, Q(-1)) = k \) and \( \text{RHom}(\mathcal{T}, O(-1)) = k[-1] \) and \( O(-1)[1] \) are objects in \( \mathcal{A} \). Taking a shift of the exact sequence (6.2), we obtain an exact sequence
\[
(6.4) \quad 0 \rightarrow Q(-1)^{\oplus 2} \rightarrow K_F \rightarrow O(-1)[1]^{\oplus 2} \rightarrow 0
\]
in \( \mathcal{A} \). Let \( 0 \neq M \subsetneq K_F \) be a subobject in \( \mathcal{A} \) and \( N := K_F/M \in \mathcal{A} \). Since \( M, N \in \mathcal{A} \subset (Q(-1), \mathcal{H}) = (O(-1)[1], \mathcal{H}) \), there are distinguished triangles
\[
\text{RHom}(Q(-1), M) \otimes Q(-1) \rightarrow M \rightarrow \text{RHom}(M, O(-1))^\vee \otimes O(-1)^{\oplus 1} \rightarrow 0\quad \text{and}
\]
\[
\text{RHom}(Q(-1), N) \otimes Q(-1) \rightarrow N \rightarrow \text{RHom}(N, O(-1))^\vee \otimes O(-1)^{\oplus 1} \rightarrow 0
\]

Since \( \text{Ext}^\partial(0, M) = \text{Ext}^\partial(Q(-1), N) = 0 \) and \( O(-1)[i] \in \mathcal{A} \) if and only if \( i = -1 \), there are \( a, b, c, d \in \mathbb{Z}_{\geq 0} \) such that \( \text{RHom}(Q(-1), M) \simeq k^{\oplus a} \), \( \text{RHom}(M, O(-1))^\vee \simeq k^{\oplus b}[-1] \), \( \text{RHom}(Q(-1), N) \simeq k^{\oplus c} \), and \( \text{RHom}(N, O(-1))^\vee \simeq k^{\oplus d}[-1] \). Now we obtain the following diagram in \( \mathcal{A} \).

\[
\begin{array}{ccccccccc}
0 & \rightarrow & Q(-1)^{\oplus a} & \rightarrow & Q(-1)^{\oplus 2} & \rightarrow & Q(-1)^{\oplus c} & \rightarrow & 0 \\
0 & \rightarrow & M & \rightarrow & K_F & \rightarrow & N & \rightarrow & 0 \\
0 & \rightarrow & O(-1)[1]^{\oplus b} & \rightarrow & O(-1)[1]^{\oplus 2} & \rightarrow & O(-1)[1]^{\oplus d} & \rightarrow & 0 \\
& & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Thus \( c = 2-a \) and \( d = 2-b \). Since \( \dim(Q(-1)) = (1, 0) \) and \( \dim(O(-1)[1]) = (0, 1) \), we have \( \dim(M) = (a, b) \) and \( \Theta(M) = b - a \).

Let us show \( \Theta(M) < 0 \). First of all, the equalities \( \text{Hom}(O(-1)[1], K_F) = \text{Ext}^{-1}_{D^b(X)}(O(-1), K_F) = 0 \) implies that \( a \neq 0 \). Next we show \( d \neq 0 \). If \( d = 0 \), then there is a surjection \( K_F \rightarrow N \simeq Q(-1)^{\oplus c} \). However, if there is a surjection \( K_F \rightarrow Q(-1) \), then the kernel is a line bundle, which must be isomorphic to \( O \), which contradicts that \( h^1(K_F) = 0 \). Hence \( c = 0 \), which implies \( N = 0 \). This contradicts our assumption \( M \subsetneq K_F \).

Therefore, we have \( a \neq 0 \) and \( d \neq 0 \), which implies \( a \in \{ 1, 2 \} \) and \( b \in \{ 0, 1 \} \). To show \( \Theta(M) < 0 \), it suffices to show that the case \( a = b = 1 \) does not occur. If \( a = b = 1 \), then we obtain an exact sequence \( 0 \rightarrow Q(-1) \rightarrow M \rightarrow O(-1)[1] \rightarrow 0 \) in \( \mathcal{A} \).
Let $s \in \text{Ext}^1(O(-1)[1], \mathcal{Q}(-1)) \simeq \text{Hom}_{D^b(X)}(O(-1), \mathcal{Q}(-1))$ be the corresponding extension class. Then the mapping cone of $s$ is isomorphic to $M$. If $s = 0$, then $M \simeq O(-1)[1] \oplus \mathcal{Q}(-1)$, which contradicts that $\text{Hom}(O(-1)[1], \mathcal{K}_X) = 0$. Hence $s \neq 0$ and the mapping cone of $s$ is isomorphic to the cokernel of $s$ in $\text{Coh}(X)$. In particular, $M$ is a coherent sheaf, which is not locally free since $(s = 0) \neq \emptyset$. Since $c = d = 1$ also holds, it follows from the exactly same argument that $N$ is a coherent sheaf which is not locally free. This contradicts that there is an exact sequence $0 \to M \to \mathcal{K}_X \to N \to 0$ where $\mathcal{K}_X$ is locally free. We complete the proof of (2).

**6.2. Irreducibility of $M_{0,4}$.** We define the moduli functor $\mathcal{M}^\Theta_{(2,2)}(Q) : (\text{Sch}/k)^{\text{op}} \to \text{(Sets)}$ as follows:

$$\mathcal{M}^\Theta_{(2,2)}(Q)(S) := \{ \mathcal{V} : \text{a locally free sheaf on } S \}$$

where $\mathcal{V} \sim \mathcal{V}' \iff \exists \mathcal{L} \in \text{Pic}(S)$ such that $\mathcal{V}' \simeq \mathcal{V} \otimes \mathcal{L}$. For the definition of families of $k\mathcal{Q}$-modules over $S$ and its dimension vectors, we refer to [15]. In the following proposition, we see a relationship between these moduli functors $\mathcal{M}^\Theta_{0,4}$ and $\mathcal{M}^\Theta_{(2,2)}(Q)$.

**Proposition 6.6.** For $\mathcal{E} \in \mathcal{M}^\Theta_{0,4}(S)$, we set $\mathcal{F} := \mathcal{E} \otimes \text{pr}_1^* \mathcal{O}_X(1)$ and define

$$\mathcal{K}_X := \text{Ker}(\text{pr}_2^* \text{pr}_2 \mathcal{F} \to \mathcal{F}).$$

Since $\mathcal{F}$ and $\text{pr}_2 \mathcal{F}$ and locally free, so is $\mathcal{K}_X$. Note that $\mathcal{K}_X$ is also flat over $S$.

Then the following assertions hold.

1. For each $i > 0$, we have $\text{Ext}^i_{\mathcal{K}_X}(\mathcal{K}_X, \mathcal{K}_X) = 0$. Moreover, the coherent sheaves $\mathcal{V}_1 := \text{Ext}^i_{\mathcal{K}_X}(\mathcal{K}_X, \mathcal{K}_X)$ are locally free sheaves on $S$ of rank 2. In particular, $\mathcal{V}_1 \simeq \mathcal{V}_2$ is a family of $k\mathcal{Q}$-modules.

2. $\mathcal{V}_1 \oplus \mathcal{V}_2$ is a geometrically $\Theta$-stable representation of the quiver $Q$ of dimension vector $(2, 2)$.

In particular, the functor $F(S) : \mathcal{M}^\Theta_{0,4}(S) \ni \mathcal{F} \mapsto \mathcal{V}_1 \oplus \mathcal{V}_2 \in \mathcal{M}^\Theta_{(2,2)}(Q)(S)$ is well-defined.

**Proof of Proposition 6.6** First of all, since $\mathcal{K}_X$ is locally free, for every locally free sheaf $\mathcal{E}$ on $X \times S$, we have $\text{Ext}^i_{\mathcal{K}_X}(\mathcal{E}, \mathcal{K}_X) \simeq \text{Ext}^i_{\mathcal{K}_X}(\mathcal{E}, \mathcal{K}_X)$. Every locally free sheaf $\mathcal{E}_X$ on $X$ and every scheme-theoretic point $s \in S$, we have a natural map $\phi^s_i : \mathcal{E}_X \otimes \text{pr}_2^* \text{Hom}(\mathcal{E}_X, \mathcal{K}_X)|_{X_s} \to \text{Ext}^i(\mathcal{E}_X, \mathcal{K}_X)|_{X_s}$. Then the Cohomology and Base Change theorem and Lemma 6.5 (1) gives (1). (2) follows from Lemma 6.5 (2).

Now we have the following proposition, which is a half part of Theorem 1.4.

**Proposition 6.7.** The coarse moduli space $M_{0,4}^\Theta$ of $M_{0,4}^\Theta$ is a smooth variety of dimension 13.

**Proof.** Let $\mathcal{E}$ be a rank 2 weak Fano bundle with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 4$ and $\mathcal{F} := \mathcal{E}(1)$. Note that $\mathcal{F}$ (and so is $\mathcal{E}$) is stable. Moreover, being weak Fano for rank 2 stable bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$ is characterized by the global generation of $\mathcal{E}(1)$ [8, Theorem 1.7]. Thus there is a coarse moduli space $M := M_{0,4}^\Theta$ of $M_{0,4}^\Theta$ (c.f. [13, Theorem 4.3.4]).
To show the smoothness of $M$ and $\dim M = 13$, it suffices to check that $\dim \Ext^1(F, F) = 13$ and $\Ext^2(F, F) = 0$ for each $[F] \in M(\mathbb{k})$. Let $s \in H^0(F)$ be a general section. Then $C = (s = 0)$ is an elliptic curve on $X$ of degree 9. There is an exact sequence $0 \to \mathcal{O} \to F \to \mathcal{I}_C(-K_X) \to 0$ which gives

$$0 \to F^\vee \to F \otimes F^\vee \to F^\vee \otimes \mathcal{I}_C(2) \to 0.$$ 

Since $R\Gamma(X, F^\vee) = R\Gamma(X, E(-1)) = 0$, we have $\Ext^i(F, F) = H^i(F^\vee \otimes \mathcal{I}_C(2)) = H^i(F \otimes \mathcal{I}_C)$. In particular, we have $h^0(F \otimes \mathcal{I}_C) = 1$. Consider

$$0 \to \mathcal{I}_C \otimes F \to F \to F|_C \to 0.$$ 

Note that $F|_C \simeq \mathcal{N}_{\mathbb{C}/X}$. Moreover, by the classification of weak Fano 3-folds of Picard rank 2 whose crepant contraction is divisorial [16], it is known that the crepant contraction from the weak Fano 3-fold $\text{Bl}_X X$ is small. Thus $\mathcal{N}_{\mathbb{C}/X} \simeq \mathcal{N}_{\mathbb{C}/X} \otimes \mathcal{O}(-K_X)|_C$ has no trivial line bundle as its quotient. Hence the Riemann-Roch theorem gives $R\Gamma(F|_C) \simeq \mathbb{k}^{\oplus 18}$. Since $h^0(F \otimes \mathcal{I}_C) = 1$, we have $\dim \Ext^1(F, F) = h^1(F \otimes \mathcal{I}_C) = 13$ and $\dim \Ext^2(F, F) = h^2(F \otimes \mathcal{I}_C) = 0$.

On the other hand, it was known that the moduli functor $M_{\Theta}^{\Theta\text{-st}}(\mathbb{Q})$ is represented by a smooth irreducible variety $M' := M_{\Theta}^{\Theta\text{-st}}(\mathbb{Q})$ ([25, Section 3.5], [18, Proposition 5.2]). Let $f : M \to M'$ be the morphism induced by the functor in Proposition 6.6. Now it suffices to show that $f$ is an open immersion. Since $M$ and $M'$ are smooth and $\dim M = \dim M' = 13$, it is enough to show that $f(\mathbb{k}) : M(\mathbb{k}) = M(\mathbb{k}) \to M'(\mathbb{k}) = M'(\mathbb{k})$ is injective. Let $E_1, E_2$ be rank 2 weak Fano bundles with $c_1(E_1) = 0$ and $c_2(E_1) = 4$. Set $F_i := E_i(1)$ for each $i$. It is enough to show that $F_1 \simeq F_2$ if and only if $H^0(T, K_{F_i}) \simeq H^0(T, K_{F_1})$. Note that $H^0(F_i) \simeq H^0(K_{F_1})$. Thus $F_1 \simeq F_2$ if and only if $K_{F_1} \simeq K_{F_2}$. By the equivalence $\Phi$ in [63], $K_{F_1} \simeq K_{F_2}$ if and only if $H^0(T, K_{F_1}) \simeq H^0(T, K_{F_2})$ as $\text{End}(T) = kQ$-modules. Therefore, $f(\mathbb{k})$ is injective, which concludes that $f : M \to M'$ is an open immersion. Hence we conclude that $M$ is irreducible since so is $M'$. We complete the proof.

\section{6.3. Geometric aspects of $M_{\Theta}^{\Theta\text{-st}}$.} In this section, we study some geometric property of the coarse moduli space $M_{\Theta}^{\text{f, r}}$ and show that it is not fine as a moduli space.

From now on, we fix a vector space $V = \mathbb{k}^5$. Recall the following varieties for $r \in \{1, 2, 3, 4\}$, which were introduced by [11]:

$$S_r := \{Q \in |O(\mathbb{P}(V))(2)| \mid \text{rk } Q \leq r\} \subset |O(\mathbb{P}(V))(2)| = \mathbb{P}(\text{Sym}^2 V^\vee)$$
$$U_r := \{([\mathbb{P}], Q) \in \text{Gr}(2, V) \times S_r \mid \Sigma \subset Q\}.$$

In particular, $S_4$ parametrizes the singular hyperquadrics on $\mathbb{P}(V)$. In the above definition, we regard $\text{Gr}(2, V)$ as the parametrizing space of 2-planes on $\mathbb{P}(V) := \text{Proj} \text{Sym}^2 V$. We also note that $U_4 \simeq \text{Proj}(\text{Gr}(2, V)(\mathcal{E}))$, where $\mathcal{E} := \text{Ker}(\text{Sym}^2 V \otimes O \to \text{Sym}^2 \mathbb{Q})$. Let $U_4 \xrightarrow{p} T_4 \xrightarrow{q} S_4$ be the Stein factorization. It was proved by [11, Proposition 2.3] that $T_4 \to S_4$ is a double covering branched along $S_3 \subset S_4$.

By [3], it was proved that $T_4$ contains the moduli space of $M'$ as an open subscheme. More precisely, $T_4$ is the moduli space $M_{\Theta}^{\Theta\text{-st}}(\mathbb{Q})$ of the semi-stable representations of the 5-Kronecker quiver whose dimension vector is $(2, 2)$. Then by Proposition 6.7, $M$ is naturally contained in $T_4$ as an open subscheme.
Corollary 7.2. was already given by Sanna [28, Proof of Theorem 8.22].

We can give an alternative proof of the following numerical characterization, which

\[ \text{Numerical characterization.} \]

\[ \text{7.2.} \]

implies (\( -20 \)) implies the nefness of \( \mathcal{O} \). We have

\[ \text{the following cohomological characterization.} \]

\[ \text{ Especially on a del Pezzo 3-fold of degree 5, we have} \]

Note that an instanton bundle \( E \) bundles in instanton bundles. An instanton bundle

\[ \text{always satisfies} \]

\[ \text{7.1. Cohomological characterization.} \]

By our previous articles [14, 8] and this sequel article, we complete our classification of weak Fano bundles on a del Pezzo 3-fold of Picard rank 1. As a further study, we discuss how to characterize weak Fano bundles in instanton bundles. An instanton bundle \( \mathcal{E} \) always satisfies \( c_2(\mathcal{E}) \geq 2 \).

Note that an instanton bundle \( \mathcal{E} \) with \( c_2(\mathcal{E}) = 2 \) is always a weak Fano bundle since \( \mathcal{E}(1) \) is 0-regular by [19, Lemma 3.1]. If \( c_2(\mathcal{E}) > 2 \), an instanton bundle \( \mathcal{E} \) might not be a weak Fano bundle. Especially on a del Pezzo 3-fold of degree 5, we have the following cohomological characterization.

Proposition 7.1. Let \( \mathcal{E} \) be an instanton bundle on a del Pezzo 3-fold of degree 5. Then \( \mathcal{E} \) is a weak Fano bundle if and only if \( \text{Ext}^1(\mathcal{Q}(-1), \mathcal{E}) = 0 \).

Proof. First, we suppose that \( \mathcal{E} \) is a weak Fano instanton bundle. Then we have \( 2 \leq c_2(\mathcal{E}) < 4 \) by [19, Corollary 3.2] and \( c_2(\mathcal{E}) \leq 4 \) by the inequality \((-K_{\mathcal{E}})^4 = 16(20 - 4c_2(\mathcal{E})) > 0 \). If \( c_2(\mathcal{E}) = 2 \), then \( \mathcal{E} \) satisfies (v), which implies \( \text{Ext}^1(\mathcal{Q}(-1), \mathcal{E}) = 0 \). If \( c_2(\mathcal{E}) = 3 \), then we have \( \text{RHom}(\mathcal{R}, \mathcal{E}) = k \) as explained in Section 3.4 and \( H^0(\mathcal{R}_{\mathcal{E}})\mathcal{L}_\mathcal{E}(\mathcal{E}(1)) = 0 \) by Lemma 3.3. Then \( \text{Ext}^1(\mathcal{Q}(-1), \mathcal{E}) = 0 \) follows from (5.14). If \( c_2(\mathcal{E}) = 4 \), then \( \text{Ext}^1(\mathcal{Q}(-1), \mathcal{E}) = 0 \) by Lemma 5.11. Hence we have \( \text{Ext}^1(\mathcal{Q}(-1), \mathcal{E}) = 0 \) for each case.

Next, we assume that \( \text{Ext}^1(\mathcal{Q}(-1), \mathcal{E}) = 0 \). Since \( \text{Ext}^2(\mathcal{R}, \mathcal{E}) = 0 \) by Lemma 3.8 we have \( H^0(\mathcal{R}_{\mathcal{E}})\mathcal{L}_\mathcal{E}(\mathcal{E}(1)) = 0 \). Hence \( \mathcal{E}(1) \) is globally generated by \( \mathcal{E} \), which implies the nefness of \(-K_{\mathcal{E}}\). Finally, since \( 0 \leq \text{hom}(\mathcal{Q}(-1), \mathcal{E}) = \chi(\mathcal{Q}, \mathcal{E}) = 20 - 5c_2(\mathcal{E}) \) by the Hirzebruch-Riemann-Roch theorem, we have \( c_2(\mathcal{E}) \leq 4 \), which implies \((-K_{\mathcal{E}})^4 = 16(20 - 4c_2(\mathcal{E})) > 0 \). Thus \( \mathcal{E} \) is a weak Fano bundle. We complete the proof.

7.2. Numerical characterization. By this cohomological characterization, we can give an alternative proof of the following numerical characterization, which was already given by Sanna [28, Proof of Theorem 8.22].

Corollary 7.2. Let \( X \) be a del Pezzo 3-fold of degree 5 and \( \mathcal{E} \) an instanton bundle with \( c_2(\mathcal{E}) = 3 \). If \( \mathcal{E}(1) \mid \ell \) is nef for every line \( \ell \), then \( \mathcal{E} \) is a weak Fano bundle.
Proof. Sanna showed that the condition $\text{RHom}(R, E) \simeq \mathbb{k}$ is equivalent to that $E(1)|_l$ is nef for every line $l$. \cite{Sanna} Definition 8.2 and Proposition 8.7. If $\text{Ext}^1(R, E) = 0$, then $H^0(R(-1)\otimes \mathcal{O}(1)) \simeq R \otimes \text{Ext}^1(Q(-1), E)$. By \cite{Sanna}, we have an exact sequence $E(1) \to R \otimes \text{Ext}^1(Q(-1), E) \to H^1(E(1)) \otimes \mathcal{O} \to 0$. Thus $\text{Ext}^1(Q(-1), E)$ must vanish since $E$ is stable. Hence Proposition 7.1 implies $E$ is a weak Fano bundle.

$\square$

Remark 7.3. For a given instanton bundles $E$ with $c_2(E) = 3$, Sanna \cite{Sanna} calls $E$ non-special if $E(1)|_l$ is nef for every line $l$. \cite{Sanna} Definition 8.2 and Proposition 8.7.

On the other hand, instanton bundles $E$ with $c_2(E) = 4$ seem much more complicated. When the case $c_2(E) = 4$, we do not know a criterion for the nefness of $E(1)$ like Corollary 7.2. We leave this problem to future work.

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