Moduli of bounded holomorphic functions in the ball

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Abstract

We prove that there is a continuous non-negative function $g$ on the unit sphere in $\mathbb{C}^d$, $d \geq 2$, whose logarithm is integrable with respect to Lebesgue measure, and which vanishes at only one point, but such that no non-zero bounded analytic function $m$ in the unit ball, with boundary values $m^*$, has $|m^*| \leq g$ almost everywhere. The proof analyzes the common range of co-analytic Toeplitz operators in the Hardy space of the ball.

0 Introduction

Let $B_d$ be the unit ball in $\mathbb{C}^d$, $S_d$ be the boundary of $B_d$, and $\sigma_d$ be normalized Lebesgue measure on $S_d$. Every function $m$ in $H^\infty(B_d)$, the space of bounded analytic functions in $B_d$, has radial limits $\sigma_d$-almost everywhere on $S_d$, defining a function $m^*$ on the sphere; conversely, $m$ can be recovered from $m^*$ by integrating against the Szegö kernel. The problem which this paper addresses is when a given non-negative bounded function $g$ on $S_d$ is the modulus $|m^*|$ of some function $m$ in $H^\infty(B_d)$.

When $d = 1$, the problem was completely solved by Szegö [10]: a necessary and sufficient condition that $g$ be the modulus of a non-zero function in $H^\infty(B_1)$ is

$$\int_{S_1} \log(g) d\sigma_1 > -\infty$$  \hspace{1cm} (0.1)

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For $d > 1$, condition (0.1) is necessary and sufficient for $g$ to be the modulus of a function in the larger Nevanlinna class $N(B_d)$, consisting of those holomorphic functions $f$ on the ball for which

$$T(f, 1) := \sup_{0 < r < 1} \int_{S_d} \log^+ |f(r\zeta)| d\sigma_d(\zeta) < \infty$$

[8, Theorem 10.11]. It is no longer sufficient, however, for $g$ to be the modulus of a bounded analytic function, because the function

$$\zeta \mapsto \operatorname{ess} \sup_{-\pi \leq \theta \leq \pi} |m^*(e^{i\theta} \zeta)|$$

must be lower semi-continuous on $S_d$ if $m$ is in $H^\infty(B_d)$ [8]. In [8, Theorem 12.5], Rudin proves that if $g$ is log-integrable, and there exists some non-zero $f$ in $H^\infty(B_d)$ with $g \geq |f^*|$ a.e. and $g/|f^*|$ lower semi-continuous, then there does exist $m$ in $H^\infty(B_d)$ with $g = |m^*|$ a.e.

The main result of this paper is the following:

**Theorem 3.4** Let $d \geq 2$. There is a non-negative continuous function $g$ on $S_d$, with $\int_{S_d} \log(g) d\sigma_d > -\infty$, and which vanishes at only one point, but such that for no non-zero function $m$ in $H^\infty(B_d)$ is $|m^*| \leq g$ almost everywhere $[\sigma_d]$.

The proof involves the analysis of co-analytic Toeplitz operators. If $\mu$ is a compactly supported measure on $\mathbb{C}^d$, let $P^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$, and let $P$ denote the orthogonal projection from $L^2(\mu)$ onto $P^2(\mu)$. If $m$ is a bounded analytic function on the support of $\mu$, the co-analytic Toeplitz operator $T_{\bar{m}}^{P^2(\mu)}$ is defined by

$$T_{\bar{m}}^{P^2(\mu)} f = P\bar{m}f.$$

When $\mu$ is $\sigma_d$, the space $P^2(\mu)$ is called the Hardy space $H^2(B_d)$.

The idea of the proof is as follows. A function $f$ is in the range of the co-analytic Toeplitz operator $T_{\bar{m}}^{H^2(B_d)}$ if and only if the linear map

$$\Gamma : p \mapsto \langle p, f \rangle_{H^2(B_d)}$$

is bounded on $P^2(|m^*|^2 \sigma)$. So $f$ is in the range of all co-analytic Toeplitz operators if and only if $\Gamma$ is a continuous linear functional on

$$\bigcup_{m \in H^\infty(B_d)} P^2(|m^*|^2 \sigma_d). \quad (0.2)$$

The Smirnov class $N^+(B_d)$ consists of those functions $f$ in $N(B_d)$ for which $\{\log^+ |f(r \cdot)| : 0 < r < 1\}$ is a uniformly integrable family on $S_d$. Equipped with the metric $\rho(f, g) =$
\[
\int_{S_d} \log(1 + |f - g|) d\sigma_d, \text{ it becomes a topological vector space that is not locally convex. When } d = 1, \text{ (1.2) coincides (as a set) with the Smirnov class. In [2] it was shown that the locally convex inductive limit topology on (1.2) was the Mackey topology of } (N^+(B_1), \rho), \text{ and so a function } f \text{ is in the common range of all co-analytic Toeplitz operators in } H^2(B_1) \text{ if and only if it is in the dual of } (N^+(B_1), \rho); \text{ the dual has been characterized by Yanagihara as those functions for which } \hat{f}(n) = O(e^{-c\sqrt{n}}) \text{ for some } c > 0 \text{ [11] (see [3] for a simpler proof).}
\]

In [4], Nawrocki characterized the dual of \( N^+(B_d) \) as those functions \( f \) whose Taylor coefficients at zero satisfy
\[
\hat{f}(\alpha) = O(e^{-c|\alpha|^{d/(d+1)}}).
\]

We prove that (1.3) does not characterize the common range of co-analytic Toeplitz operators in \( H^2(B_d), d \geq 2; \)

**Theorem 2.1** Let \( f(z_1, \ldots, z_d) = f_1(z_1) = \sum_{n=0}^\infty a_n z_1^n, \) let \( \varepsilon > 0, \) and suppose that \( a_n = O(e^{-cn^{d+i}}) \) for some \( c > 0. \) Then \( f \) is in the range of the Toeplitz operator \( T_m H^2(B_d) \) for every non-zero \( m \) in \( H^\infty(B_d). \)

It follows, therefore, that for \( d \geq 2, \) the functional induced by \( f(z_1, \ldots, z_d) = \sum_{k=1}^\infty e^{-k^4} z_1^k \) is not bounded on some \( P^2(w \sigma_d). \) In the proof of Theorem 3.4 we construct such a \( w \) that is continuous and vanishes at only one point.

# 1 Preliminary Lemmata

We need to know explicitly the projection from \( L^2(B_d) \) onto \( H^2(B_d). \) Let \( \alpha = (\alpha_1, \ldots, \alpha_d) \) be a multi-index and \( \zeta = (z_1, \ldots, z_d) \) a point in \( \mathbb{C}^d. \) The function \( \zeta^\alpha \) then maps \( \zeta \) to \( z_1^{\alpha_1} \ldots z_d^{\alpha_d}. \) The notation \( |\alpha| \) stands for \( \alpha_1 + \ldots + \alpha_d, \) and \( \alpha! = \alpha_1! \ldots \alpha_d!. \)

**Lemma 1.1**
\[
\int_{S_d} \zeta^\alpha \overline{\zeta}^\beta d\sigma_d = \delta_{\alpha,\beta} (d-1)! \alpha! \frac{(d-1)! |\alpha|!}{(d-1 + |\alpha|)!}.
\]

Moreover, if \( P_{H^2(B_d)} \) denotes the projection from \( L^2(\sigma_d) \) onto \( H^2(B_d), \) then
\[
P_{H^2(B_d)}|z_2^{\alpha_2}|^2 \ldots |z_d^{\alpha_d}|^2 z_1^i \overline{z_1}^j = \begin{cases} 0 & \text{if } i < j \\ \frac{(d-1+i-j)! \alpha_1! \ldots \alpha_d!}{(i-j)! (d-1+i+\alpha_1+\ldots+\alpha_d)!} z_1^{i-j} & \text{if } i \geq j. \end{cases}
\]

**Proof:** Formula (1.2) is proved in [7]. The expression on the left-hand side of (1.3) is orthogonal to every monomial except \( z_1^{i-j}; \) taking inner products gives the constant. \( \square \)
We wish to be able to transfer information about co-analytic Toeplitz operators with the same symbol on different spaces. To do this we use the following lemma, whose proof is immediate:

**Lemma 1.4** Let \( \mathcal{H} \) be a Hilbert space of holomorphic functions on \( B_d \) in which the monomials are mutually orthogonal. Let \( m(z_1, \ldots, z_d) = \sum_{\beta \in \mathbb{N}^d} b_\beta \zeta^\beta \). Then
\[
\frac{T_{\bar{m}} \zeta^\alpha}{\| \zeta^\alpha \|_\mathcal{H}^2} = \sum_{\beta \leq \alpha} \bar{b}_{\alpha - \beta} \frac{\zeta^\beta}{\| \zeta^\beta \|_\mathcal{H}^2} \quad (1.5)
\]
This lemma also allows us to define Toeplitz operators with an unbounded conjugate analytic symbol. The formal definition (1.5) defines an upper triangular operator, with respect to the orthonormal basis of normalized monomials. It therefore has a domain which contains all the polynomials; we extend its domain to include all functions on which \( T_{\bar{m}} \), thought of as a formal operator on the power series, converges in each entry to the Taylor coefficients of some function in \( \mathcal{H} \).

Let \( A^{-n} \) consist of all holomorphic functions \( m \) in the unit disk, satisfying \( |m(z)| = O((1 - |z|)^{-n}) \). The space \( A^0 \) is \( H^\infty(B_1) \). For \( n = 0 \), the following result is proved in [1].

**Lemma 1.6** Let \( f \) be in \( A^{-n} \) for some \( n \), and \( 0 < \alpha < 2 \). Then
\[
\int_{B_1} (\log - |f|)^\alpha dA < \infty.
\]
**Proof:** We can assume that \( f \) has no zeroes in \( \{ z : |z| < \frac{1}{2} \} \). It follows from [1] and standard Nevanlinna theory that for any function \( g \) in the Nevanlinna class \( N(B_1) \), with \( g(0) \neq 0 \), and any \( a < 2 \),
\[
\int_{B_1} (\log - |g|)^a dA \leq K(T(g, 1), |g(0)|, a).
\]
Fix \( p \) strictly between 1 and \( \frac{2}{\alpha} \), let \( a = \alpha p < 2 \), let \( q = \frac{p}{p-1} \) and let \( N > q \). Let \( D_1 \) be a smoothly bounded convex domain inside the disk, containing \( \{ z : |z| < \frac{1}{2} \} \), whose closure touches the unit circle only at 1, and which has a high degree of tangency at 1: let the boundary of \( D_1 \) be \( \{ \rho(\theta) e^{i\theta} : -\pi \leq \theta \leq \pi \} \), and assume \( 1 - \rho(\theta) \sim |\theta|^N \). For any other point \( \zeta = e^{i\theta_0} \) on the boundary of the unit disk, let \( D_\zeta = e^{i\theta_0} D_1 \).

Let \( \psi_\zeta \) be the Riemann map of \( D_\zeta \) onto \( B_1 \) that takes 0 to 0 and \( \zeta \) to \( \zeta \). As the boundary of \( D_\zeta \) is smooth, it follows from the Kellog-Warschawski theorem (see e.g. [3]) that \( \psi_\zeta \) and its derivatives extend continuously to the closure of \( D_\zeta \), so distances before and after the conformal mapping are comparable.
If \( r < \frac{1}{N} \), then \( f \) is in \( H^r(D_\zeta) \), and \( \sup_{\zeta \in S_1} \| f \circ \psi_\zeta^{-1} \|_{H^r} < \infty \). Thus
\[
\int_{D_{e^{i\theta}}} (\log^{-|f|})^a dA \leq C, \quad \text{for all } e^{i\theta}.
\]
Integrating with respect to \( \theta \) and changing the order of integration yields
\[
\int_{B_1} (\log^{-|f|(re^{i\phi})})^a (1-r)^\frac{s}{n} rdrd\phi < \infty.
\]
Now
\[
\int_{B_1} (\log^{-|f|})^a dA \leq \left[ \int_{B_1} (\log^{-|f|})^a (1-r)^\frac{s}{n} dA \right]^{\frac{1}{p}} \left[ \int_{B_1} (1-r)^{-\frac{s}{n}} dA \right]^{\frac{1}{q}} < \infty.
\]
Let \( \mu_n \) be the measure on the unit disk given by \( d\mu_n(z) = \frac{1}{\pi} (1-|z|^2)^n Area(z) \), and let \( H_n \) be \( P^2(\mu_n) \). It is routine to verify that in \( H_n \) the monomials are mutually orthogonal, and
\[
\|z^k\|_{H_n}^2 = \frac{1}{(k+1) \ldots (k+n+1)}.
\]
The space \( H_0 \) is the usual Bergman space for the disk. The following lemma is proved in \([1]\) for \( n = 0 \); for \( n > 0 \), basically the same proof works (given Lemma \([1,6]\), though some care must be taken as \( T_\bar{m} \) is no longer bounded. We include a proof for completeness.

**Lemma 1.7** Let \( n \geq 0 \), and \( m \) be a function in \( A^{-n} \), not identically zero. Suppose \( f(z) = \sum_{k=0}^\infty a_k z^k \) where \( a_k = O(e^{-ck^{\frac{1}{2}} + \varepsilon}) \) for some \( \varepsilon \) and \( c \) greater than 0. Then for any \( s \geq 2n \) there exists \( g \) in \( H_s \) such that \( T_{\bar{m}} g = f \).

**Proof:** First, observe that \( f = T_{\bar{m}} g \) for some \( g \) if and only if there is a constant \( C \) such that for all polynomials \( p \)
\[
|\langle p, f \rangle_{H_s}| \leq C \sqrt{\int |p|^2 |m|^2 d\mu_s}.
\]
So it is sufficient to prove that
\[
|\sum_{k=0}^\infty \bar{a}_k \hat{p}(k) \frac{1}{(k+1) \ldots (k+s+1)}| \leq C \sqrt{\int |p|^2 |m|^2 d\mu_s}.
\]
This in turn will follow from the Banach-Steinhaus theorem if we can show that for any function \( h \) in \( P^2(|m|^2 \mu_s) \),
\[
\hat{h}(k) = O(e^{ck^{\frac{1}{2}} + \varepsilon}). \quad (1.8)
\]
Now Stoll showed in [9] that if $h$ satisfies
\[ \int_{B_1} (\log^+ |h|)^\alpha d\text{Area} < \infty \]
for some $\alpha > 0$ then $\hat{h}(k) = O(e^{\alpha^2/k})$. We can assume $\varepsilon$ is small, and take $\alpha = \frac{2 - 4\varepsilon}{1 + 2\varepsilon}$. As $h$ is in $P^2(|m|^2\mu_s)$, $h(z)m(z)(1 - |z|^2)^{s/2}k(z)$ is in $L^2(d\text{Area})$, and
\[ \log^+ |h| \leq \log^+ |k| + \log^- |(1 - |z|^2)^{s/2}| + \log^- |m|. \]
The first two terms on the right are clearly integrable to the $\alpha^{th}$ power, and so is the third by Lemma [1.6]; therefore $h$ satisfies (1.8) as desired. \[ \square \]

We want to be able to restrict functions in the ball to planes and factor out zeros without losing control of the size of the function; the next lemma allows us to do this.

**Lemma 1.9** Let $m$ be holomorphic on $B_d$ and satisfy
\[ |m(z_1, \ldots, z_d)| \leq C(1 - \sqrt{|z_1|^2 + \ldots |z_d|^2})^{-s}. \]
Suppose also that
\[ m(z_1, \ldots, z_d) = z_d^t m_2(z_1, \ldots, z_d) + z_d^{t+1} m_3(z_1, \ldots, z_d), \]
where $m_2$ and $m_3$ are analytic. Let $m_1(z_1, \ldots, z_{d-1}) = m_2(z_1, \ldots, z_{d-1}, 0)$. Then
\[ |m_1(z_1, \ldots, z_{d-1})| \leq (3d)^{s+t} C(1 - \sqrt{|z_1|^2 + \ldots |z_{d-1}|^2})^{-(s+t)}. \]

**Proof:** Let $(z_1, \ldots, z_{d-1})$ be in $B_{d-1}$, and let $\varepsilon = \frac{1}{3d}(1 - \sqrt{|z_1|^2 + \ldots + |z_{d-1}|^2})$. Then the polydisk centered at $(z_1, \ldots, z_{d-1}, 0)$ with multi-radius $(\varepsilon, \ldots, \varepsilon)$ is contained in $(1 - \varepsilon)B_d$. Integrating on the distinguished boundary of the polydisk we get
\[ \left| m_1(z_1, \ldots, z_{d-1}) \right| = \left| m_2(z_1, \ldots, z_{d-1}, 0) \right| \]
\[ = \left| \int_{(z_1, \ldots, z_{d-1}, 0) + \varepsilon T_d} \frac{m(\zeta_1, \ldots, \zeta_d)}{\zeta_3} d\zeta_3 \right| \]
\[ \leq \frac{C}{\varepsilon^{s+t}}. \]
\[ \square \]
2 Common Range of $T_m$

We can now prove that a function that depends on only one variable is in the range of every $T_{H^2(B_d)}$ if its Taylor coefficients decay like $e^{-ck^{1+\varepsilon}}$.

**Theorem 2.1** Let $f(z_1, \ldots, z_d) = f_1(z_1) = \sum_{n=0}^{\infty} a_n z_1^n$, and suppose that $a_n = O(e^{-cn^{1+\varepsilon}})$ for some $\varepsilon, c > 0$. Then $f$ is in the range of the Toeplitz operator $T_{m}^{H^2(B_d)}$ for every non-zero $m$ in $H^\infty(B_d)$.

**Proof:** For $d = 1$, this is proved (without the $\varepsilon$) in \cite{2}, so assume $d \geq 2$. Fix $m$ in $H^\infty(B_d)$;

$$m(z_1, \ldots, z_d) = \sum_{i_1, \ldots, i_d=0}^{\infty} b_{i_1, \ldots, i_d} z_1^{i_1} \ldots z_d^{i_d}.$$

Let

$$S = \{(i_2, \ldots, i_d) : \text{for some } i_1, b_{i_1, \ldots, i_d} \neq 0\}.$$

Define

$$t_d = \inf\{i_d : \text{for some } i_2, \ldots, i_{d-1}, (i_2, \ldots, i_{d-1}, i_d) \in S\},$$

and define $t_k$ inductively by

$$t_k = \inf\{i_k : \text{for some } i_2, \ldots, i_{k-1}, (i_2, \ldots, i_{k-1}, i_k, t_{k+1}, \ldots, t_d) \in S\}.$$

Let $n = t_2 + \ldots + t_d$.

Case (a): $n = 0$.

Then the function

$$m_1(z_1) = m(z_1, 0, \ldots, 0)$$

is not identically zero, and is in $H^\infty(B_1)$. By Lemma \[.1\],

$$T_{m_1}^{H^2(B_d)} z_1^i = \sum_{j} b_{j,0,\ldots,0} \frac{(i - j + 1) \ldots (i - j + d - 1)}{(i + 1) \ldots (i + d - 1)} z_1^{i-j}.$$

So by Lemma \[.4\] if one can solve the equation

$$T_{m_1}^{H_{d-2}} g_1 = f_1$$

for some $g_1$ in $\mathcal{H}_{d-2}$, then $g(z_1, \ldots, z_d) = g_1(z_1)$ solves

$$T_{m}^{H^2(B_d)} g = f_1.$$
and by Equation (1.2) \( \|g\|_{H^2(B_d)} = \sqrt{(d-1)!\|g_1\|_{H^{d-1}} < \infty} \). By Lemma 1.7, equation (2.2) has a solution.

Case (b): \( n > 0 \).

One can decompose \( m \) as

\[
m(z_1, \ldots, z_d) = z_2^{t_2} \cdots z_d^{t_d} m_2(z_1, \ldots, z_d) + m_3(z_1, \ldots, z_d),
\]

where each term in the expansion of \( m_3 \) is divisible by some \( z_k^{t_k+1} \). Applying Lemma 1.9 inductively, \( m_1(z) = m_2(z, 0, \ldots, 0) \) is in \( A^{-n} \), and by the choice of \( t_2, \ldots, t_d \), it is not identically zero. Consider the function

\[
f_2(z) = \sum_{k=0}^{\infty} a_k (k + d)(k + d + 1) \cdots (k + dn + 1) z^k.
\]

As \( d \geq 2 \), we can apply Lemma 1.7 with \( s = dn \), so there is

\[
g_2(z) = \sum_{k=0}^{\infty} \gamma_k (k+1)(k+2) \cdots (k+dn+1) z^k
\]
in \( H_{dn} \) with

\[
T_{m_1}^{H_{dn}} g_2 = f_2.
\]

Define \( g \) by

\[
g(z_1, \ldots, z_d) = \frac{1}{t_2! \cdots t_d!} z_2^{t_2} \cdots z_d^{t_d} \sum_{k=0}^{\infty} \gamma_k (k+1)(k+2) \cdots (k+n+d-1) z_1^k.
\]

The function \( g \) is in \( H^2(B_d) \) because

\[
\|g\|^2_{H^2(B_d)} = \frac{(d-1)!}{t_2! \cdots t_d!} \sum_{k=0}^{\infty} |\gamma_k|^2 (k+1) \cdots (k+n+d-1) \\
\leq \frac{(d-1)!}{t_2! \cdots t_d!} \sum_{k=0}^{\infty} |\gamma_k|^2 (k+1) \cdots (k+dn+1) \\
= \frac{(d-1)!}{t_2! \cdots t_d!} \|g_2\|^2_{H^{dn}} \\
< \infty.
\]

Moreover

\[
T_{m}^{H^2(B_d)} g = T_{m}^{H^2(B_d)} z_2^{t_2} \cdots z_d^{t_d} m_1(z_1)^g
\]
is a function of \( z_1 \) only; it is, in fact, \( f \). For if \( T_{m}^{H^2(B_d)} g = \sum_{k=0}^{\infty} c_k z_1^k \), and \( m_1(z) = \sum_{k=0}^{\infty} c_k z_1^k \), then taking the inner product with \( z_1^j \) we get

\[
\frac{(d-1)!}{(j+1) \cdots (j+d-1)} c_j = \langle T_{m}^{H^2(B_d)} g, z_1^j \rangle_{H^2(B_d)} \\
= \langle g, z_2^{t_2} \cdots z_d^{t_d} m_1(z_1)^g \rangle_{H^2(B_d)} \\
= (d-1)! \sum_{k=j}^{\infty} \gamma_k \bar{c}_k - j
\]

(2.4)
Taking the inner product with $z^j$ in Equation (2.3), we get

$$\frac{1}{(j+1)\ldots(j+d-1)}a_j = \langle T_{m_1}^{H_{dn}} g_2, z^j \rangle_{H_{dn}}$$
$$= \langle g_2, m_1 z^j \rangle_{H_{dn}}$$
$$= \sum_{k=j}^{\infty} \gamma_k \tilde{c}_{k-j}$$ \tag{2.5}$$

Comparing Equations (2.4) and (2.5), we see that $T_{m_1}^{H^2(B_d)} g = f$, as desired. \hfill \square

### 3 Proof of the main theorem

Define $F_{c,w}$ by

$$F_{c,w}(z) = \exp(c \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^{d+1}})$$ \tag{3.1}

We need the following two results. The first states that $d(F_{c,w}, 0) \to 0$, and was proved by Drewnowski. A proof is given in [4, Lemma 3.2].

**Lemma 3.2 (Drewnowski)**

$$\lim_{c \to 0} \sup_{w \in B_d} \int_{S^d} \log(1 + |cF_{c,w}|) d\sigma_d = 0.$$ 

The second result, due to Nawrocki, estimates the growth of the Taylor coefficients of $F_{c,w}$. We are interested in $w = re_1 = (r, 0, \ldots, 0)$; in this case all the Taylor coefficients of $F_{c,re_1}$ are positive, and the following follows easily from the proof of [4, Lemma 3.3]:

**Lemma 3.3 (Nawrocki)** For each $c > 0$ there exists $\varepsilon > 0$ such that

$$\inf_{i \in \mathbb{N}} \sup_{0 < r < 1} \frac{(d-1+i)!}{(d-1)!i!} \hat{F}_{c,re_1}(i, 0, \ldots, 0) e^{-\varepsilon \frac{d}{d+1}} > 0.$$ 

We can now prove the main theorem.

**Theorem 3.4** Let $d \geq 2$. There is a continuous non-negative function $g$ on $S_d$, vanishing only at the point $e_1$, and satisfying $\int_{S_d} \log(g) d\sigma_d > -\infty$, with the property that the only function $m$ in $H^\infty(B_d)$ with $|m^*| \leq g$ almost everywhere $[\sigma_d]$ is the zero function.

**Proof:** Let $V_n = \{\zeta \in S_d : |\zeta - e_1| \geq \frac{1}{n}\}$. By Lemma 3.3, for any sequence $c_n$ tending to zero, one can choose $i_n$ and $r_n$ such that

$$\hat{F}_{c_n, r_n e_1}(i_n, 0, \ldots, 0) > \frac{n}{c_n} e^{(i_n)\frac{4}{d}} \tag{3.5}$$
(because $\frac{1}{4} < \frac{d}{d+1}$). Moreover, by passing to a subsequence, one can assume that

$$\sup_{\zeta \in V_n} c_n |F_{c_n,r_n e_1}(\zeta)| \leq \frac{1}{2^n},$$

and that

$$\int_{S_d} \log(1 + |c_n F_{c_n,r_n e_1}|) d\sigma_d \leq \frac{1}{2^n}.$$

Define $g$ by

$$g(\zeta) = \sqrt{\frac{1}{1 + \sum_{n=1}^{\infty} |c_n F_{c_n,r_n e_1}(\zeta)|^2}}.$$

It follows from (3.6) that $g$ is continuous and vanishes only at $e_1$. Moreover

$$\int_{S_d} \log g d\sigma_d = -\frac{1}{2} \int_{S_d} \log(1 + \sum_{n=1}^{\infty} |c_n F_{c_n,r_n e_1}|^2) d\sigma_d$$

$$> - \int_{S_d} \log \prod_{n=1}^{\infty} (1 + |c_n F_{c_n,r_n e_1}|^2) d\sigma_d$$

$$= -2 \sum_{n=1}^{\infty} \int_{S_d} \log(1 + |c_n F_{c_n,r_n e_1}|) d\sigma_d$$

$$\geq -2.$$

Now suppose there is a non-zero $m$ in $H^\infty(B_d)$ with $|m^*| \leq g$ a.e. Then each of the functions $c_n F_{c_n,r_n e_1}$, being analytic in the ball of radius $\frac{1}{r_n}$, is in $P^2(|m^*|^2 \sigma)$; moreover they are all of norm less than one in this space, because

$$\int_{S_d} |c_n F_{c_n,r_n e_1}|^2 |m^*|^2 d\sigma \leq \int_{S_d} |c_n F_{c_n,r_n e_1}|^2 g^2 d\sigma < 1.$$

Let

$$f(z_1, \ldots, z_d) = \sum_{k=0}^{\infty} e^{-k^4 \frac{k}{(d-1)! k!} z_1^k}.$$

By Theorem 2.1, there is a function $h$ in $H^2(B_d)$ with

$$T_{\tilde{m}}^{H^2(B_d)} h = f.$$

It follows that the linear map

$$\Gamma : p \mapsto \langle p, f \rangle_{H^2(B_d)},$$

defined a priori on the polynomials, extends by continuity to a bounded linear map on $P^2(|m^*|^2 \sigma)$, as

$$|\Gamma(p)| = |\langle p, P(\tilde{m} h) \rangle| = \int p m^* h^* d\sigma_d \leq \|h\|_{H^2(B_d)} \|p\|_{P^2(|m^*|^2 \sigma)}.$$
Moreover, each function $c_nF_{c_n,r_n e_1}$ is uniformly approximated on $S_d$ by the partial sums of its Taylor series; hence

$$\Gamma(c_nF_{c_n,r_n e_1}) = \sum_{k=0}^{\infty} c_n\hat{F}_{c_n,r_n e_1}(k)e^{-k\|e\|}. \tag{3.7}$$

But all the terms on the right-hand side of (3.7) are positive, and the $i^{th}$ term is at least $n$ by Equation (3.5). This contradicts the boundedness of $\Gamma$.

We note that the theorem is much easier to prove if $g$ is not required to be continuous. One shows, as in the one-variable case, that

$$N^+(B_d) = \bigcup P^2(w\sigma_d)$$

where $w$ ranges over all log-integrable weights. This is a strictly larger set than

$$\bigcup_{m \in H^\infty(B_d)} P^2(|m^*|^2\sigma_d)$$

because if $f$ is in some $P^2(|m^*|^2\sigma_d)$, then $fm$ is in $H^2(B_d)$, so the zero-set of $f$ is contained in an $H^2(B_d)$-zero set; but it is a result of Rudin that for any $p < 2$, there is a function in $H^p(B_d)$ whose zero set is not contained in any $H^2(B_d)$-zero set. It follows that such a function is in $P^2(w^2\sigma_d)$ for some log-integrable $w$, but not in any $P^2(|m^*|^2\sigma_d)$, so $w$ cannot dominate the modulus of any $m^*$.

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