Anatomy of double beta decay nuclear matrix elements

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Abstract. The necessary ingredients for a realistic evaluation of the $0\nu\beta\beta$ nuclear matrix elements are reviewed. It is argued that the short range nucleon correlations, nucleon finite size, and higher order nuclear currents need to be included in the calculation, even though a consensus on the best way to treat all of these effects has not been reached. Another positive development is the realization that the two alternative and complementary methods, the Quasiparticle Random Phase Approximation and the Nuclear Shell Model, agree on many aspects of the calculation, in particular on the competition, or cancelation, between the contribution of nuclear pairing on one hand, and the other pieces of interaction that result in admixtures of broken pairs or higher seniority states on the other hand. The relatively short range ($r \leq 2-3$ fm) of the effective $0\nu\beta\beta$ operator found in both methods is a consequence of that competition.

1. Introduction

Thanks to the success of neutrino oscillation experiments we know that neutrinos have a finite mass and that the lepton flavor is not a conserved quantity. Future oscillation studies will aim at refining our knowledge of the oscillation parameters, at determining the as yet unknown mixing angle $\theta_{13}$ and the so-called hierarchy (i.e. the sign of the mass square difference $\Delta m^2_{\text{atm}}$), and at searching for the $CP$ violation in the neutrino sector.

However, oscillation experiments will not help in fixing the absolute values of the neutrino masses, and even more importantly, will tell us nothing about the charge conjugation properties of neutrinos. Different approaches are needed for that task.

Regarding the mass scale, we can combine the oscillation results and the results of tritium beta-decay experiments to conclude that the masses of at least two (out of the total of three active) neutrinos are bracketed by $10$ meV $\leq m_\nu \leq 2 - 3$ eV. Therefore, neutrino masses are six or more orders of magnitude smaller than the masses of the other fermions and their mass pattern, i.e. the mass ratios of neutrinos, is clearly rather different than the pattern of masses of the charged fermions. This suggests that, perhaps, the origin of the neutrino mass is different from the origin (presumably coupling to the Higgs) of the masses of the other charged fermions.

The most popular explanation of the smallness of neutrino mass is the see-saw mechanism which implies, among other things, that neutrinos are Majorana particles, and consequently that the total lepton number should not be conserved. Hence tests of lepton number conservation acquire a fundamental importance. Among the possible tests of lepton number conservation the study of $0\nu\beta\beta$ decay is by far the most sensitive.
Observing the $0\nu\beta\beta$ decay would guarantee that neutrinos are massive Majorana fermions. This qualitative statement (or theorem), however, does not in general allow us to deduce the magnitude of the neutrino mass once the rate of the $0\nu\beta\beta$ decay have been determined. It is important to stress, however, that quite generally an observation of any total lepton number violating process, not only of the $0\nu\beta\beta$ decay, would necessarily imply that neutrinos are massive Majorana fermions.

In the following I will leave aside the important issue of the $0\nu\beta\beta$ mechanism. Instead, I will assume that the only way the decay can occur is through the exchange of a virtual light, but massive, Majorana neutrino (the mass eigenstate neutrinos $\nu_1$, $\nu_2$ and $\nu_3$ of the oscillation theory) between the two nucleons undergoing the transition, and that these neutrinos interact by the standard left-handed weak currents. Observation of the $0\nu\beta\beta$ decay, independently of its mechanism, would constitute a proof that the lepton number is not conserved and hence that neutrinos are massive Majorana particles.

Moreover, if we accept the above assumption about the mechanism of the decay, we can relate the $0\nu\beta\beta$-decay rate to a quantity containing information about the absolute neutrino mass. That relation is given by the well known formula

$$\frac{1}{T_{0\nu\beta\beta}^{1/2}} = G^{0\nu}(Q, Z)|M^{0\nu}|^2 \langle m_{\beta\beta} \rangle^2,$$

where $G^{0\nu}(Q, Z)$ is an easily calculable phase space factor that depends on the transition $Q$ value and on the nuclear charge $Z$. The $Z$ dependence is caused by the Coulomb interaction of the emitted electrons. $M^{0\nu}$ is the nuclear matrix element that can be evaluated in principle, although with a considerable uncertainty and is discussed in detail later, and finally the quantity $\langle m_{\beta\beta} \rangle$ which is the effective neutrino Majorana mass.

The magnitude of $\langle m_{\beta\beta} \rangle$, in turn, is constrained by the known parameters governing the neutrino oscillation phenomena (mixing angles and $\Delta m^2$) for the two possible mass ordering (hierarchies) and by the absolute mass scale. Depending on that mass scale, one can divide the range of the quantity $\langle m_{\beta\beta} \rangle$ into three regions:

- The degenerate mass region where all $m_i \gg \sqrt{\Delta m^2_{\text{atm}}}$. In that region $\langle m_{\beta\beta} \rangle \geq 0.1$ eV, corresponding crudely to $0\nu$ half-lives of $10^{26-27}$ years. To explore it (in a realistic time frame), $\sim$ 100 kg of the decaying nucleus is needed. Several experiments aiming at such sensitivity are being built and should run very soon and give results within the next $\sim$ 3 years. Moreover, this mass region (or a substantial part of it) will be explored, in a similar time frame, by the study of ordinary $\beta$ decay and by the observational cosmology. These latter techniques are independent of the Majorana nature of neutrinos.

- The so-called inverted hierarchy mass region where $20 < \langle m_{\beta\beta} \rangle < 100$ meV and the $0\nu\beta\beta$ half-lives are about $10^{27-28}$ years. (The name is to some extent a misnomer. In that interval one could encounter not only the inverted hierarchy but also a quasi-degenerate but normal neutrino mass ordering. Successful observation of the $0\nu\beta\beta$ decay will not be able to distinguish these possibilities.) To explore this mass region, $\sim$ ton size sources would be required. Proposals for the corresponding experiments exist, but none has been funded as yet, and presumably the real work will begin depending on the experience with the various $\sim$ 100 kg size sources. Timeline for exploring this mass region is $\sim$ 10 years.

- Normal mass hierarchy region where $\langle m_{\beta\beta} \rangle \leq 10-20$ meV. To explore this mass region, $\sim$ 100 ton sources would be required. There are no realistic proposals for experiments of this size at present.

Obviously, relating the measured decay half-life to $\langle m_{\beta\beta} \rangle$ requires a knowledge of the nuclear matrix elements $M^{0\nu}$. Any error or uncertainty in their value will be reflected in the
corresponding error and uncertainty in \( \langle m_{\beta\beta} \rangle \). Moreover, planning a \( 0\nu\beta\beta \) decay experiment, and comparing the results of experiments with different nuclei also relies on the knowledge of the nuclear matrix elements \( M^{0\nu} \).

2. What is needed in order to evaluate \( M^{0\nu} \) nuclear matrix elements

The matrix elements \( M^{0\nu} \) must be evaluated theoretically; they cannot be extracted from any other nuclear experiments. Since all \( 0\nu\beta\beta \) decay candidate nuclei are relatively heavy, the corresponding many body problem cannot be solved without approximations. The fundamental issue is, in that case, how to estimate the corresponding uncertainty. But even aside from this problem, in order to perform the calculation, and do a good job, one needs to use a proper effective transition operator. In the past, various authors used different formulations, and consequently got different results. Only recently has a consensus emerged in the community of practitioners, and most recent evaluations use the same basic ingredients.

Ordinary beta decay, and also the \( 2\nu\beta\beta \) decay, are low momentum transfer processes. The long wavelength approximation is valid in that case and the relevant operators are just the Fermi \( (\tau^\pm) \) and Gamow-Teller \( (\bar{\sigma}\tau^\pm) \) operators. In analogy, the early evaluations of the \( 0\nu\beta\beta \) decay matrix elements used transition operators based on the two-nucleon generalization of these basic operators. Since the \( 0\nu\beta\beta \) decay involves the exchange of a virtual Majorana neutrino, its propagator should be also included. In the momentum representation the transition operator is then

\[
\Omega = \tau^+ \tau^+ \quad \frac{-g_\nu^2 + g_A^2 \sigma_{12}}{q(q + E_m - (M_i + M_f)/2)} , \quad \sigma_{12} = \bar{\sigma}_1 \cdot \bar{\sigma}_2 . \tag{2}
\]

In order to obtain the coordinate representation one makes the Fourier transform and arrives at the “neutrino potential”

\[
H(r, E_m) = \frac{R}{2\pi^2 g_A^2} \int \frac{d^3q}{\omega} \frac{1}{\omega + A_m} e^{i\vec{q}\cdot\vec{r}} = \frac{2R}{\pi^2 g_A^2} \int_0^\infty dq q \sin(qr) \left( \omega + A_m \right) , \tag{3}
\]

where the nuclear radius \( R = 1.2A^{1/3} \) fm has been added as an auxiliary factor so that \( H \) becomes dimensionless. \( \omega \) is just the magnitude of \( q \) and \( A_m \) is the excitation energy of the virtual intermediate state with respect to \( (M_i + M_f)/2 \).

With the operator (2) one can proceed in two ways. Either the dependence on the energy of the intermediate state \( E_m \) is taken into account explicitly (this is often done when the Quasiparticle Random Phase Approximation (QRPA) is used), or one can use the closure approximation, neglecting the dependence on \( E_m \) and thus avoiding the necessity to treat the virtual intermediate states (this is usually done in the Nuclear Shell Model (NSM)). Since the typical values of \( q \equiv \omega \) are quite high, as I will show next, the closure approximation is quite adequate and the corresponding error is small (few % at most).

What are, then, the characteristic values of the neutrino momentum \( q \)? Clearly, since the virtual neutrino connects two nucleons in the nucleus, its momentum is constrained by the uncertainty relation \( q \geq 1/R \). Already this argument shows that the long wavelength approximation is not valid. Very recently the calculation within QRPA [1] and NSM [2] showed that nuclear structure effects cause cancellation between the contribution of lower values of \( q \) or equivalently higher values of \( r \), the distance between the neutrons that undergo the transition. In Figure 1 the dependence of \( M^{0\nu} \) on \( r \) is shown. Note that only \( r \leq 2-3 \) fm contribute, i.e. distances considerably smaller than the nuclear radii \( R \). The lower panel, which will be explained below, shows the cancellation between two opposing tendencies that is almost perfect for \( r \geq 2-3 \) fm. A very similar result have been found in NSM [2].

The relatively short range of the effective operator (or equivalently relatively high \( q \) values) has important consequences. The following effects that are usually not relevant in applications
of nuclear structure theory dealing with low-lying nuclear states and transitions between them need to be considered when the matrix elements $M^{0\nu}$ are evaluated:

- Higher order nuclear currents, in particular induced pseudoscalar and weak magnetism. These currents are proportional to $q/m_\pi$ and $q/M_p$ and thus suppressed in ordinary $\beta$ decay, but need to be included when $0\nu\beta\beta$-decay is considered.

- Nucleon-nucleon short range repulsion that causes correlations between nucleons which are not included when the nuclear wave functions are constructed from products of the single-nucleon mean field wave functions.

- Nucleon composite nature or finite size. At high momentum transfer weak transitions (both the charge changing or charge conserving) that leave a nucleon intact are suppressed. This is a consequence of the finite size of the nucleon. That effect is usually taken into account by introducing nucleon form factors with the characteristic cut-off $\Lambda_{cut} \sim 1$ GeV.

In the past various authors neglected some or all of these effects. That is one of the reasons (albeit not the main one) for the variation between the published values of matrix elements $M^{0\nu}$ stressed forcefully by Bahcall and collaborators [3] and interpreted as uncertainty. Clearly, the above effects need to be included, and the proper discussion must concentrate on the most rigorous way of including them, not on the need to consider them or not.

The main issue, naturally, is the treatment of nuclear structure; in other words, how the wave functions of the ground states of the initial and final nuclei are evaluated. To understand the difficulties, let us recall that the nuclear matrix elements for both modes of the $\beta\beta$ decay (the $2\nu$
as well as the $0\nu$) depend sensitively on the competition of two opposing tendencies related to the two important components of the effective nucleon-nucleon interaction. The pairing interaction is responsible for the extra binding of even-even nuclei and thus also responsible for the existence of $\beta\beta$ decay itself. If pairing was the only relevant component of the nuclear interaction, all like nucleons would reside in Cooper-like pairs with $0^+$. The matrix elements $M^{0\nu}$ would be large and positive in our phase convention (the $J = 0$ part in Figs. 1 and 2).

However, that is not so in reality. The nuclear interaction contains other important parts, in particular the neutron-proton force. The presence of these components of the interaction means that the ground states of the initial and final nuclei contain admixtures of pairs with $J \neq 0$. Such “broken pairs” typically give a negative contribution to $M^{0\nu}$ as also shown in Figs. 1 and 2. Moreover, the positive and negative contributions are of similar magnitude thus increasing the sensitivity of the final value of $M^{0\nu}$ to the corresponding interaction strengths. (That competition was first noted, it seems, in Ref. [4] and stressed again recently in [1] within the QRPA and in [5] within the NSM.)

![Figure 2. Contributions of different angular momenta $J$ associated with the two decaying neutrons to $M^{0\nu}$ in $^{76}$Ge, $^{100}$Mo and $^{130}$Te. The QRPA is used, with the interaction strength $g_{pp}$ adjusted so that the $2\nu\beta\beta$ lifetime is correctly reproduced](image)

In our calculations based on QRPA (and RQRPA, renormalized QRPA) calculations [1, 6, 7] we adjust the neutron-proton particle-particle interaction coupling constant, responsible for the “broken pairs” contribution, using the known $2\nu\beta\beta$-decay lifetimes or, equivalently, the corresponding $M^{2\nu}$ matrix elements. We based this procedure on the fact that the Gamow-Teller strength, the contribution of the $1^+$ virtual intermediate states that are fully responsible for the $M^{2\nu}$, is the quantity most sensitive to the corresponding parameter, usually denoted as $g_{pp}$. The nominal value, corresponding to the G-matrix based on the realistic nucleon-nucleon force, is $g_{pp} = 1$. The renormalized $g_{pp}$ has values between 0.8 and 1.2.

In the NSM there is no analog to the adjustment of the $g_{pp}$ parameter. The hamiltonian (primarily its so-called monopole part) is adjusted so that a set of nuclear spectroscopic data is optimally reproduced. There is no attempt to reproduce specifically the $2\nu\beta\beta$-decay lifetime, but the agreement with experiment is, in most cases, acceptable [2, 5, 8].

Within QRPA one can trace the main reasons for the differences between the calculated values of $M^{0\nu}$ matrix elements by different authors noted in [3] to the different ways the effective interaction was chosen, and adjusted to the chosen set of the included single particle orbits. The advantage of the above quoted procedure (choose the coupling constant $g_{pp}$ so that the known $2\nu\beta\beta$-decay lifetimes are reproduced) is that the dependence on the other chosen input parameters is greatly reduced.
3. Short range correlations

One of the items listed in the previous section is the inclusion in the calculation of the effect of the short-range nucleon-nucleon repulsion known from scattering experiments. Two nucleons strongly repel each other at distances $r \leq 0.5 - 1.0$ fm, i.e. distances very relevant to the evaluation of the $M^{0\nu}$ as shown in Fig. 1. The nuclear wave functions used in QRPA and NSM, which are products of the mean field single-nucleon wave functions, do not take into account the influence of this repulsion that is irrelevant in most standard nuclear structure theory applications. The most natural way to include this effect is to modify the radial dependence of the effective $0\nu\beta\beta$ operator so that the effect of short distances (small values of $r$) is reduced. However, no consensus on how to do this properly has been reached as yet.

![Figure 3](image.png)

**Figure 3.** The $r$ dependence of $M^{0\nu}$ in $^{76}$Ge. The four curves show the effects of different treatments of short-range correlations. The resulting $M^{0\nu}$ values are 5.32 when the effect is ignored, 5.01 when the UCOM transformation [9] is applied and 4.14 when the treatment based on the Fermi hypernetted chain [10] and 3.98 when the phenomenological Jastrow function is used. [11]. The QRPA has been used.

This simplest way, used traditionally and in recent works [1, 2], includes the effect of short range correlations by multiplying the neutrino potential by the square of a Jastrow-like function first derived in [11] and in a more modern form in [10]. That phenomenological procedure reduces the magnitude of $M^{0\nu}$ by 20-25% as illustrated in Fig. 3. Recently, another procedure, based on the Unitary Correlation Operator Method (UCOM) has been proposed [9]. That procedure, still applied not fully consistently, reduces the $M^{0\nu}$ much less, only by about 5% [12]. It is prudent even though not really satisfactory, as suggested in [1], to include these two possibilities as extremes and the corresponding range as systematic error.

Another effect, also already mentioned, that needs to be taken into account is the nucleon finite size. That is included, usually again phenomenologically, by introducing the dipole form of the nucleon form factor

$$f_{V,A} = \left( \frac{1}{1 + q^2/\Lambda_{V,A}^2} \right)^2, \tag{4}$$

where the cut-off parameters $\Lambda_{V,A}$ have values (deduced in the reactions of free neutrinos with free or quasifree nucleons) $\sim 1$ GeV. This corresponds to a nucleon size of $\sim 0.5 - 1.0$ fm. Note that in our case we are dealing with bound nucleons and neutrinos far off mass shell, hence it is not obvious that the above form factors are applicable. It turns out, however, that once the short range correlations are properly included (by either of the procedures discussed above) the $M^{0\nu}$ becomes essentially independent of the adopted values provided $\Lambda_{V,A} \geq 1$ GeV is used. However, in the past various authors neglected the effect of short range correlations, and in that case a proper inclusion of nucleon form factor (or their neglect) again causes variations in the calculated $M^{0\nu}$ values.
4. Discussion
Within the QRPA (and RQRPA) the full range of the calculated nuclear matrix elements $M^{0\nu}$ is presented in Table I in Ref. [1] separated into entries with the two alternative ways of treating the short range correlations (Jastrow and UCOM). The quoted ranges reflect the differences in the $M^{0\nu}$ values evaluated in different sizes of the included single-particle basis (usually two, three or four oscillator shells), differences between QRPA and RQRPA, as well as the full range obtained with the unquenched strength of the axial current weak interaction ($g_A = 1.25$) and its quenched analog ($g_A = 1.0$). The effect of the uncertainty in the experimental $2\nu\beta\beta$ decay lifetime (even though usually small) is also included. Note that these analogs of error bars are highly correlated; if the true value for one element is, say, near the upper edge, it will be near the upper edge for the other candidate nuclei as well.

The most recent NSM results [2], obtained with Jastrow type short range correlation corrections, are noticeably lower than the QRPA values. That difference is particularly acute in the lighter nuclei $^{76}$Ge and $^{82}$Se. While the QRPA and NSM agree on many aspects of the problem, in particular on the role of the competition between “pairing” and “broken pairs” contributions and on the $r$ dependence of the matrix elements, the disagreement in the actual values remains to be explained.

When one compares the experimentally determined $2\nu$ matrix elements and the calculated $0\nu$ matrix elements one can note the fast variation in $M^{2\nu}$ when going from one nucleus to another while $M^{0\nu}$ changes only rather smoothly, in both QRPA and NSM. This is presumably related to the high momentum transfer (or short range) involved in $0\nu\beta\beta$. The smooth variation with $Z$ and $A$ of the $M^{0\nu}$ matrix elements makes the comparison of results obtained in different nuclei easier and more reliable. Altogether, we see that while a substantial progress has been achieved and calculations by different authors agree substantially better with each other now than before, we are still somewhat far from being able to evaluate the $0\nu\beta\beta$ nuclear matrix elements confidently and accurately.

Acknowledgments
It has been a great pleasure to participate in this workshop celebrating the contribution of Frank Avignone and Ettore Fiorini to neutrino physics. I look forward to many more years of collaboration and friendship with both of them. The original results reported here were obtained in collaboration with Fedor Šimkovic, Vadim Rodin, Amand Faessler and Jonathan Engel. The fruitful collaboration with them is gratefully acknowledged.

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