A New Approach in Quantum Gravity 
and its Cosmological Implications

Simone Mercuri

Giovanni Montani

ICRA—International Center for Relativistic Astrophysics
Dipartimento di Fisica (G9),
Università di Roma, “La Sapienza”,
Piazzale Aldo Moro 5, 00185 Rome, Italy.
e-mail: mercuri@icra.it, montani@icra.it

PACS: 04.60.Ds – 98.80.Qc

Abstract

This work concerns a new reformulation of quantum geometrodynamics, which allows to overcome a fundamental ambiguity contained in the canonical approach to quantum gravity: the possibility of performing a (3+1)-slicing of space-time, when the metric tensor is in a quantum regime. Our formulation provides also a procedure to solve the problems connected to the so-called frozen formalism. In particular we fix the reference frame (i.e. the lapse function and the shift vector) by introducing the so-called kinematical action; as a consequence, the new Hamiltonian constraints become parabolic, so arriving to evolutive (Schrödinger-like) equations for the quantum dynamics. The kinematical action can be interpreted as the action of a pressure less, but, in general, non geodesic perfect fluid, so in the semi classical limit our theory leads to the dynamics of the gravitational field coupled to a dust which represents the material reference frame we have introduced fixing the slicing. We also investigate the cosmological implications of the presence of the dust, which, in the WKB limit of a cosmological problem, makes account for a dark matter component and could play, at present time, a dynamical role.
1 Introduction

Time has a special role in all the classical\textsuperscript{1} theories of physics. Newton’s time is an external parameter, respect to which we describe the dynamics of the system. In non relativistic quantum mechanics time is not a physical observable in the usual sense, i.e. it does not exist an operator associated to the time variable, but it is an external parameter as well as in classical mechanics. The construction of the theory is deeply influenced by the concept of an external time, for example, to construct the Hilbert space of quantum states we have to choose a complete set of observables, which commute at equal instants of time. It follows that the dynamical equation for the non relativistic quantum mechanics has an explicit time dependence, which reflects on the evolutive character of the quantum states represented by wave functionals. Moreover the special role of time is also the reason why the time-energy indetermination,

\[ \Delta t \Delta E \geq \hbar, \]

has a different meaning respect to the one usually associated to position and momentum.

All these simple ideas can be generalized to those systems compatible with the special theory of relativity. In this case Newton’s time is replaced by the time measured in a set of relativistic inertial frames, but the space-time is an external non dynamical structure yet, profoundly different with respect to the dynamical one of general relativity.

To construct a consistent quantum field theory (in the canonical approach) we need the hamiltonian function, which is the conjugate momentum to the relativistic temporal coordinate. So, in order to apply the canonical quantization procedure to the gravitational field, we must, first of all, extract a possible time parameter from the classical theory; but this is not a simple task for a diffeomorphisms invariant theory like general relativity. Moreover the canonical quantization procedure leads, as well known, to the so called Wheeler-DeWitt approach \cite{7, 8}, which does not contain any evolution with respect to the time parameter, reflecting the invariance under diffeomorphisms of the classical theory, i.e. the lack of any external temporal parameter in general relativity.

In this work we introduce the canonical quantization program, dedicating a wide review either to the Arnowitt-Deser-Misner formalism (ADM) \cite{1, 2, 3, 4}, either to the Wheeler-DeWitt equation (WDE).

ADM formalism is the way to extract a “time” dependence from the gravitational theory. It is based on the (3+1)-\textit{slicing} of the space-time, where “time” plays the role of parameter for the foliation, singling out the different elements of a particular family of hypersurfaces, which fill the whole space-time. It is worth noting that, in the classical theory, the slicing procedure is well defined and gives a time dependence to the events by spacial frames represented by the hypersurfaces, but, in the quantum formulation of the theory it generates

\textsuperscript{1}With the adjective “classical” we intend “not general relativistic”.
ambiguities; in fact, when the metric tensor is in a quantum regime, defining the space or time like character of a vector field (necessary to develop the slicing procedure) becomes an impossible task: it seems possible to distinguish between space or time like vector field only in average (expectation values) sense.

It is just in these ambiguities that our criticism to the canonical quantization of gravity arises. We claim that to give sense to the slicing procedure also in a quantum regime it is necessary to fix a reference frame with respect to which to operate the slicing [26, 23].

Another important feature of general relativity for this discussion is the so called \textit{relationalism}. We know that the diffeomorphisms invariance of the classical theory requires the absence of any non dynamical object in the theory, in particular, in general relativity, the space-time itself is a dynamical field. So differently from the Newtonian mechanics we have not a fixed background on which we can localize the physical events; but the localization is fully relational, in other words a dynamical object can be localized only with respect to another one. These considerations lead to think that a reference frame in the gravitational theory has to be a dynamical physical entity coupled to the gravitational field.

There exist two different way to introduce a reference system in a classical or quantum gravity theory, the first one consists in adding to the gravitational field a dynamical fluid or fields [30, 31] and [33], the other one in fixing the frame in geometrical way [22, 5] (see also [19, 20, 21]), i.e. fixing the splitting. We stress that in a recent paper is shown that these two approaches lead to an equivalent evolutive quantum dynamics, in other words there exists a dualism between introducing a physical frame and breaking down time diffeomorphisms invariance [24]. It is worth noting that all these approaches lead to a Schrödinger-Einstein quantum dynamics, i.e. the introduction of a material reference frame in the classical dynamics leads to an evolutive quantum equation for the dynamics of the coupled system.

Our point of view is the geometrical one: we fix the reference frame choosing a particular family of hypersurfaces, assigning particular values to the lapse function and to the shift vector [26, 23]. It is clear that in this way we loose the hamiltonian constraints, and so the possibility to canonically quantize the system. But using the so called \textit{kinematical action}, already introduced in the quantum field theory on curved background [18] to reparametrize the gravitational action, we obtain new hamiltonian constraints; so the introduction of the kinematical action is the price we have to pay to perform the canonical quantization after having fixed the slicing.

The kinematical action is a geometrical object, which links the choice of the lapse function and the shift vector to a particular family of hypersurfaces and to their normal vector field, but it has also a clear physical interpretation. In fact, in section 3.2 it is possible to show that the kinematical term is, the action of a non relativistic dust, which couples with a gravito-electromagnetic-like field. An important outcome of our theory is the appearance in the new hamiltonian constraints of a linear term, strictly connected to the kinematical action, which gives an evolutive character to the quantum equation, i.e. the equations become
parabolic. This feature is not so unexpected, because, fixing the lapse function and the shift vector, we are breaking the diffeomorphisms invariance of the theory, so arriving to evolutive equations along the fixed slicing.

The physical interpretation of the kinematical action allows us to recognize in the temporal parameter the conjugate variable to the energy density of the dust. The evolutive theory, moreover, overcomes the well known shortcomings of the WDE approach as shown in this paper as well as in. Moreover to study the phenomenology connected with the appearance of this additional energy term we apply our theory to a cosmological model. In particular we make some estimations to understand if this new energy term has something to do with the observed dark matter of the Universe.

Section 2 is completely dedicated to a review of the canonical quantization procedure, in particular in the first paragraph, we explain how to slice the spacetime to arrive to the Arnowitt-Deser-Misner (ADM) form for the gravitational action, the second paragraph concerns the Wheeler-De Witt equation (WDE), with a brief list of critics moved to this approach to quantum gravity, which, though consistent, is of course ambiguous.

The main part of our work is, of course, contained in section 3, where we give a detailed explanation of our theory. In paragraph 3.1 it is treated the very simple case of the quantization of a non relativistic particle, which introduce to the concept of reparametrization of the action as a way to extract the right hamiltonian constraint to perform the quantization. In paragraph 3.2, instead, we introduce the kinematical action, giving its physical interpretation. The aim of paragraph 3.3 is to give more insight into the reparametrization of a classical action in view of the canonical quantization. In fact, when the scalar field is coupled to the gravitational one, the dynamics of the background provides automatically the hamiltonian constraints for the system, but when the background is fixed, in order to obtain the the right constraints, it is necessary to reparametrize the action of the scalar field. The reparametrization is performed by the kinematical action, in the way which has inspired our reformulation of quantum geometrodynamics described in paragraph 3.4; postulating the presence of the kinematical term also in the action we start from in order to quantize the gravitational field. We show that the hamiltonian operator is an hermitian one and so the bracket of the quantum states of the system provides a conserved density of probability during the evolution. The eigenvalues problem and the smiclassical limit of the theory are faced too.

In section 4 we apply our theory to a cosmological Friedmann-Robertson-Walker model (FRW) obtaining the quantum equation containing also the term due to the density of energy of the dust. After having found a general wave functional for this model, which overcome the not physical initial singularity, replacing it with a more physical peaked density of probability, we give some phenomenological calculations to explore the possibility that the dust be a component of the observed dark matter.

Finally, in the appendix, there is a brief explanation of the so called “multi time” approach, which represent another interesting way to arrive to a Schrödinger-like quantum dynamics, but profoundly different from the one presented.
2 Canonical Quantization

The implementation of the canonical quantization formalism to the gravitational field, leads to the so-called Wheeler-De Witt equation (WDE) [7,25], consisting of a functional approach in which the states of the theory are represented by wave functionals taken on the 3-geometries and, in view of the requirement of general covariance, they do not possess any real time dependence.

Due to its hyperbolic nature, the WDE is characterized by a large number of unsatisfactory features [8], which strongly support the idea that is impossible any straightforward extension to the gravitational phenomena of procedures well-tested only in limited ranges of energies; however in some contexts, like the very early cosmology [12,16] (where a suitable internal time variable is provided by the volume of the Universe) the WDE is not a dummy theory and give interesting information about the origin of our classical universe, see [15], which may be expected to remain qualitatively valid even for the outcomes of a more consistent approach. In the following two paragraphs we give a brief review of the canonical method of quantization for the gravitational field.

2.1 (3+1)-Slicing Procedure

To obtain the hamiltonian constraints, which are the starting point for the canonical quantization of gravity, we have to write the Einstein-Hilbert action into a (3 + 1) formulation. To this aim, we have to perform a slicing of the 4-dimensional space-time, on which a metric tensor $g_{\mu \nu}$ is defined.

We consider a space-like hypersurface having a parametric equation $y^\rho = y^\rho (x^i)$ (Greek labels run from 0 to 3, while Latin ones run from 1 to 3) and in each point we define a 4-dimensional vector base composed by its tangent vectors $e^\mu_i = \partial_i y^\mu$ and by the normal unit vector $n^\mu$; as just defined, these vectors base satisfy, by construction, the following relations

$$g_{\mu \nu} e^\mu_i n^\nu = 0, \quad g_{\mu \nu} n^\mu n^\nu = -1.$$  \hspace{1cm} (2)

Now if we deform this hypersurface through the whole space-time, via the parametric equation $y^\rho = y^\rho (t, x^i)$, we construct a one-parameter family of space-like hypersurfaces slicing the 4-dimensional manifold; thus each component of the adapted base, acquiring a dependence on the time-like parameter $t$, becomes a vector field on the space-time.

Let us introduce the deformation vector $N^\mu = \partial_t y^\mu (t, x^i)$, which connects two points with the same spatial coordinates on neighboring hypersurfaces (i.e. corresponding to values of the parameter $t$ and $t + dt$).

This vector field can be decomposed with respect to the base $(n^\mu, e^\mu_i)$, obtaining the following representation:

$$N^\mu = \partial_t y^\rho = N n^\mu + N^i e^\mu_i,$$  \hspace{1cm} (3)

where $N$ and $N^i$ are, respectively, the lapse function and the shift vector, so this expression is known as lapse-shift decomposition of the deformation vector.
It is easy to realize how the space-like hypersurfaces are characterized by the following 3-dimensional metric tensor $h_{ij} = g_{\mu\nu}e^i_\mu e^j_\nu$. Since the hypersurface is deformed through space-time, it changes with a rate, which taken with respect to the label time $t$, can be decomposed into its normal and tangential contributions

$$\partial_t h_{ij} = -2Nk_{ij} + 2\nabla_{(j}N_{i)}$$

(4)

where $N_i = h_{ij}N^j$, the covariant derivative is constructed with the 3-dimensional metric and $k_{ij} = -\nabla_i N_j$ denotes the extrinsic curvature.

Now we define the co-base vectors $(n_\mu, e^i_\mu)$, as follows

$$n_\mu = g_{\mu\nu}n^\nu, \quad e^i_\mu = h_{ij}g_{\mu\nu}e^\nu_j,$$

(5)

where $h_{ij}$ is the inverse 3-metric: $h_{ij}h^{jk} = \delta^i_j$.

The explicit expression for the 4-metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ assume, in the system $(t, x^i)$, respectively, the form:

$$g_{\mu\nu} = \left( N_iN^i - N^2 \frac{N_i}{h_{ij}} \right), \quad g^{\mu\nu} = \left( \begin{array}{cc} \frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N_i}{N^2} & -\frac{N_iN^i}{N^2} \end{array} \right).$$

(6)

Moreover, the normal unit vector $n^\mu$ has the following components $(\frac{1}{N}, -\frac{N^i}{N})$, and this implies that the covariant normal vector be $n_\mu = (-N, 0)$; below we will use to indicate the components of the vectors in the system $(t, x^i)$, with Greek barred labels as: ¯\mu, ¯\nu, ¯\rho.... We also note that in this system of coordinates, the square root of the determinant of the metric tensor assumes the form $\sqrt{-g} = N\sqrt{h}$.

It is possible to show that the Einstein-Hilbert action can be rewritten as follows [4, 35, 34]:

$$S = \int_{\Sigma^3 \times \mathbb{R}} dt d^3x N\sqrt{h} \left[ (3) R + k_{ij}k^{ij} - k^2 \right],$$

(7)

which is the most appropriate to construct the “ADM action” for the gravitational field.

Now, defining the conjugate momenta to the dynamical variables, which are the component of the 3-metric tensor, we can rewrite the gravitational action in its hamiltonian form. The gravitational Lagrangian $L^g$ does not contain the time derivative of the lapse function $N$ and of the shift vector $N^i$, so their conjugate momenta are identically zero and the Lagrangian is said singular. Summarizing, we have for the conjugate momenta:

$$p^{ij}(t, x^i) = \frac{\partial L^g}{\partial (\partial_t h_{ij})} = \sqrt{h} \left( k^{ij} - kh^{ij} \right),$$

(8)

$$\pi(t, x^i) = \frac{\partial L^g}{\partial (\partial_t N^i)} = 0, \quad \pi_i(t, x^i) = \frac{\partial L^g}{\partial (\partial_t N^i)} = 0.$$

(9)
By the above definition, we can perform the Legendre dual transformation and, with few algebra, then obtaining the below final form for the gravitational action

\[
S^g \left( h_{ij}, p^{kl}, N, N^a, \pi, \pi_b \right) = \int_{\Sigma \times \mathbb{R}} dt d^3 x \left\{ p^{ij} \partial_t h_{ij} + \pi \partial_t N + \pi_k \partial_t N^k - \left( \lambda \pi + \lambda^j \pi_j + NH^g + N^i H^g_i \right) \right\},
\]

where the so-called super-hamiltonian \( H^g \) and super-momentum \( H^g_i \), read respectively as

\[
H^g = G_{ijkl} p^{ij} p^{kl} - \sqrt{h} R, \quad H^g_i = -2 \nabla_j p^i, \tag{11}
\]

where (using geometrical units) \( G_{ijkl} = \frac{1}{2\sqrt{h}} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}) \) is the so-called super-metric (Wheeler 1968).

Now, before calculating the other dynamical equations, we want to add to this picture, also a matter field, which, for simplicity, is represented by a self-interacting scalar field \( \phi \). This lead us to the following expression for the action of the gravitational and matter field:

\[
S^g\phi = \int_{\Sigma \times \mathbb{R}} dt d^3 x \left\{ p^{ij} \partial_t h_{ij} + \pi \partial_t N + \pi_k \partial_t N^k + p_\phi \partial_t \phi \right. \\
- \left( \lambda \pi + \lambda^j \pi_j + N \left( H^g + H^\phi \right) + N^i \left( H^g_i + H^\phi_i \right) \right) \right\}, \tag{12}
\]

where the hamiltonian terms \( H^\phi \) and \( H^\phi_i \) read explicitly as:

\[
H^\phi = \frac{1}{2\sqrt{h}} p_\phi^2 + \frac{\sqrt{h}}{2} h^{ij} \partial_i \phi \partial_j \phi + \sqrt{h} V(\phi) \quad H^\phi_i = p_\phi \partial_i \phi \tag{13}
\]

where \( h = \det \{ h_{ij} \} \) and \( V(\phi) \) denotes a self-interaction potential energy.

Varying the action \( S^g\phi \) with respect to the Lagrange multipliers \( \lambda \) and \( \lambda_i \), we obtain the first class constraints:

\[
\pi = 0, \quad \pi_k = 0; \tag{14}
\]

to assure that the dynamics be consistent, the Poisson parentheses, between the constraints and the hamiltonian, have to be zero, so we must require that the second class constraints

\[
H^g + H^\phi = 0, \quad H^g_i + H^\phi_i = 0, \tag{15}
\]

be satisfied.

Moreover varying the action with respect the two conjugate momenta \( \pi \) and \( \pi_i \), we obtain the two equations:

\[
\partial_t N = \lambda, \quad \partial_t N^i = \lambda^i, \tag{16}
\]
which assure that the trajectories of the lapse function and of the shift vector in the phase space are completely arbitrarily.

The action (12) has to be varied with respect to all the dynamical variables and this gives us the Hamiltonian equations for the scalar and gravitational field, which take the following form:

\[
\frac{d}{dt} h_{ab} = 2NG_{abkl}p^{kl} + 2\nabla \frac{\nabla}{(\nabla N_b)} ,
\]

\[
\frac{d}{dt} p_{ab} = \frac{1}{2} N \frac{h^{ab}}{\sqrt{h}} \left( p^{ij} p_{ij} - \frac{1}{2} p^2 \right) - \frac{2N}{\sqrt{h}} \left( p_{ai} p^b_i - \frac{1}{2} pp_{ab} \right) +
\]

\[
- N \sqrt{h} \left( \frac{3}{2} R_{ab} - \frac{1}{2} Rh_{ab} \right) +
\]

\[
+ \sqrt{h} \left( \nabla a \nabla b N - h_{ab} \nabla i \nabla \nabla N \right) +
\]

\[
- 2\nabla i \left( p_{(a} N^{b)} \right) + \nabla i \left( N^i p_{ab} \right) +
\]

\[
+ \frac{N}{4\sqrt{h}} h^{ab} p^a_\phi - \frac{N}{2} \sqrt{h} h^{ab} \left( \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi + V(\phi) \right) ,
\]

\[
\frac{d}{dt} \phi = \frac{N}{\sqrt{h}} p_\phi + N^i \partial_i \phi ,
\]

\[
\frac{d}{dt} p_\phi = N \sqrt{h} h^{ij} \partial_i \phi \partial_j \phi + \partial_j \left( N \sqrt{h} h^{ij} \right) \partial_i \phi +
\]

\[
- N \sqrt{h} \frac{\partial V(\phi)}{\partial \phi} + \partial_i \left( N^i p_\phi \right) .
\]

The complete dynamics of the coupled gravito-scalar system is represented by the above dynamical equations together with equation (16) and the first and second class constraints (14) and (15), which tell us we can not choose the fields and their conjugate momenta arbitrarily.

### 2.2 ADM Action and Wheeler-De Witt Equation

Now we briefly recall how the Wheeler-De Witt approach [7, 18] faces the problem of quantizing a coupled system consisting of gravity and a real scalar field, which implies also the metric field now be a dynamical variable. The action describing this coupled system reads

\[
S^{\phi \phi} = \int \Sigma d^3 x \left\{ p^{ij} \partial_i h_{ij} + \pi \partial_i N + \pi_k \partial_i N^k + p_\phi \partial_i \phi +
\right.
\]

\[
- \left( \lambda \pi + \lambda^i \pi_j + N \left( H^g + H^\phi \right) + N^i \left( H^g_i + H^\phi_i \right) \right) \right\} ,
\]
where \( p^{ij} \) denotes the conjugate momenta to the 3-dimensional metric tensor \( h_{ij} \) and the super-hamiltonian and super-momentum terms take the form contained in equations (11) and (13).

Since now \( N \) and \( N^i \) are, in principle, dynamical variables, they have to be varied, so leading to the constraints \( H^g + H^\phi = 0 \) and \( H^g_i + H^{\phi} = 0 \) which are equivalent to the \( \mu - 0 \)-components of the Einstein equations and therefore play the role of constraints for the Cauchy data. It is just this restriction on the initial values problem, a peculiar difference between the previous case, at fixed background, and the present one: in fact, now, on the regular hypersurface \( t = t_0 \), the initial conditions \( \{ \phi_0(x^i), p_0(x^i), h_{ij}(x^i), p^{kl}(x^i) \} \) can not be assigned freely, but they must verify on \( \Sigma_{t_0} \), the four relations \( (H^g + H^\phi) \mid_{t_0} = 0 \) and \( (H^g_i + H^{\phi}) \mid_{t_0} = 0 \).

Indeed behaving like Lagrange multipliers, the lapse function and the shift vector have not a real dynamics and their specification corresponds to assign a particular slicing of \( M^4 \), i.e. a system of reference.

In order to quantize this system we assume that its states be represented by a wave functional \( \Psi(N, N^k, h_{ij}, \phi) \) and implement the canonical variables to operators acting on this wave functional (in particular we set \( h_{ij} \to \hat{h}_{ij}, p^{ij} \to \hat{p}^{ij} \equiv -i\hbar \delta( ) / \delta h_{ij} \)).

The quantum dynamics of the system is then induced by imposing the translation of the classical constraints, which leads to the following quantum equations:

\[
\hat{\pi} \Psi = 0, \quad \hat{\pi}_k \Psi = 0, \quad (\hat{H}_i^g + \hat{H}_i^\phi) \Psi = 0, \quad (\hat{H}^g + \hat{H}^\phi) \Psi = 0, \quad (22)
\]

which to be solved it would require a specific choice for the normal ordering of the operators. The first seven quantum equations can be simply solved: they restrict the dependence of the wave functional only on a class of 3-geometries, which we indicate with \( \{ h_{ij} \} \). The last one is the Wheeler-De Witt equation, which, in view of what just said, we rewrite as \( (\hat{H}^g + \hat{H}^\phi) \Psi(\phi, \{ h_{ij} \}) = 0 \).

Due to its hyperbolic nature this formulation of the quantum dynamics has some limiting feature, which we summarize by the following three points:

i) It does not exist any general procedure allowing to turn the space of the solutions into an Hilbert one and so any appropriate general notion of functional probability distribution is prevented.

ii) The WDE does not contain any dependence on the variable \( t \) or on the function \( y^\mu \), so loosing its evolutive character along the slicing \( \Sigma_{t_0}^3 \). Moreover individualizing an internal variable which can play the role of “time” is an ambiguous procedure which does not lead to a general prescription.

iii) At last we stress what is to be regarded as an intrinsic inconsistency of the approach above presented: the WDE is based on the primitive notion of space-like hypersurfaces, i.e. of a time-like normal field, which is in clear contradiction with the random behavior of a quantum metric field; indeed the space or time character of a vector becomes a precise notions only in the limit of a
perturbative quantum gravity theory. This remarkable ambiguity leads us to infer that there is inconsistency between the requirement of a wave equation (i.e. a wave functional) invariant, like the WDE one, under space diffeomorphisms and time displacements on one hand, and, on the other one, the (3 + 1)-slicing representation of the global manifold.

The existence of these shortcomings in the WDE approach, induces us to search for a better reformulation of the quantization procedure which addresses the solution of the above indicated three points as prescriptions to write down new dynamical quantum constraints.

3 Reformulation of Quantum Dynamics

Our reformulation of the canonical quantum gravity is based on a fundamental criticism about the possibility to speak of a unit time-like normal field and of space-like hypersurfaces, which are at the ground of the ADM formalism, when referring to a quantum space-time; in fact, in this case, either the time-like nature of a vector field, either the space-like nature of the hypersurfaces can be recognized at most in average sense, i.e. with respect to expectation values. This consideration makes extremely ambiguous to apply the (3+1)-splitting on a quantum level and leads us to claim that the canonical quantization of gravity has sense only when referred to a fixed slicing, or in other words, when referred to a fixed reference frame, i.e. only after the notion of space and time are physically distinguishable. To fix the slicing we have to choose a particular family of hypersurfaces and this means we have to fix the lapse function $N$ and the shift vector $N^i$. However, so doing, we loose the hamiltonian constraints (14), (15) and, with them, the standard procedure to quantize the dynamics of the system; as a solution to this problem, we propose to reparametrize the gravitational action using the so called Kinematical Action, obtaining new hamiltonian constraints and going on toward the canonical quantization of the system.

3.1 Non Relativistic Particle

As an helpful example for the analysis below developed, we review the case of the one-dimensional non-relativistic (parametrized) particle, whose action reads

$$S = \int \{ p\dot{q} - h(p,q) \} dt ,$$

where $t$ denotes the Newton time and $h$ the hamiltonian function. In order to quantize this system, we parameterize the Newton time as $t = t(\tau)$, so getting the new action as

$$S = \int \{ p\frac{dq}{d\tau} - h(p,q) \frac{dt}{d\tau} \} d\tau .$$
Now we set \( p_0 \equiv -h \) and add this relation to the above action by a Lagrangian multiplier \( \lambda \), i.e.

\[
S = \int \left\{ \frac{p}{\tau} d\tau + p_0 \frac{dt}{\tau} - h(p, q, p_0, \lambda) \right\} d\tau \quad h \equiv \lambda(h + p_0).
\]

By varying this action with respect to \( p \) and \( q \), we get the Hamilton equations

\[
\frac{dq}{d\tau} = \lambda \frac{\partial h}{\partial p} \quad \text{and} \quad \frac{dp}{d\tau} = -\lambda \frac{\partial h}{\partial q},
\]

while the variations of \( p_0 \) and \( t \) yield

\[
\frac{dt}{\tau} = \lambda \quad \text{and} \quad \frac{dp_0}{d\tau} = 0;
\]

all together, these equations describe the same Newton dynamics, having the energy as constant of the motion. But now, by varying \( \lambda \), we get the (desired) constraint \( h + p_0 = 0 \), which, in terms of the operators \( \hat{p}_0 = -i\hbar \partial_t \) and \( \hat{h} \), provides the Schrödinger equation \( i\hbar \partial_\tau \psi = \hat{h}\psi \), as taken for the system state function \( \psi(t, q) \). Finally we remark that, when retaining the relation \( dt/d\tau = \lambda \), we are able to write the wave equation in the parametric time as

\[
i\hbar \partial_\tau \psi(\tau, q) = \lambda(\tau) \hat{h}\psi(\tau, q),
\]

where \( \lambda(\tau) \) is to be assigned.

In spite of its simplicity, this example is a naive, but very good prototype of our approach to the canonical quantum gravity.

### 3.2 Kinematical Action and its Physical Interpretation

We have introduced in the previous section the lapse-shift decomposition of the deformation vector \( \xi \). It is worth noting that we can obtain such equation varying an action built to this aim. It is the so-called kinematical action and takes the following form:

\[
S = \int_{\Sigma \times \mathbb{R}} dt d^3x \left( p_\mu \partial_\mu y^\rho - N p_\mu n^\rho - N^i p_\mu e_i^\rho \right).
\]

If we now vary the action \((27)\) with respect to the dynamical variables \( p_\mu \) and \( y^\rho \), and we put these two variations equal to zero, we obtain respectively:

\[
\partial_\mu y^\rho = N n^\rho + N^i \partial_i y^\rho, \quad \partial_\mu p_\rho = -N p_\rho \partial_\mu n^\rho + \partial_i \left( N^i p_\mu \right).
\]

The first one of such equations is the lapse-shift decomposition of the deformation vector, while the second one provides the dynamical evolution for \( p_\mu \), which is the conjugate momenta to the vector \( y^\rho \).

The kinematical action is used in quantum field theory on curved spacetime, in order to reparameterize the field action \([9, 18]\), but it will be clear in the next section how, in our approach, it plays an important role also in the reformulation of the canonical quantum gravity.

In this section we want to investigate the physical meaning of the “kinematical term”, which will outline either the main aspects of our reformulation of the canonical quantum gravity, either the meaning of the reparameterization in quantum field on curved space.
To get the searched physical insight, let us rewrite the equations \(28\) in a covariant form. To this aim we recall to denote the coordinates \((t,x^i)\) by barred Greek labels: \(\bar{\mu}, \bar{\nu}, \bar{\rho}, \ldots\) and we also remark that the following relations take place:

\[
\partial_t = \partial_t y^\mu \partial_\mu, \quad \partial_i = \partial_i y^\mu \partial_\mu, \quad n^\bar{\mu} \partial_\bar{\mu} = n^\mu \partial_\mu.
\]

Now remembering that the normal vector \(n^\mu\) has components \(n^\mu \equiv (N, -N^i)\) in the system \((t,x^i)\), it is possible to rewrite the first one of equations \(28\) in the following form:

\[
n^\rho \partial_\rho y^\mu = n^\rho \partial_\rho y^\mu; \quad \text{this equation ensures that, after the variation } n^\mu \text{ is a real unit time-like vector, i.e.}
\]

\[
g_{\mu\nu} n^\mu n^\nu = g_{\rho\sigma} n^\rho n^\sigma = -1,
\]

the last equality being true by construction of \(g_{\mu\nu}\) and \(n^\mu\). Moreover, since \(n^\mu\) is in any system of coordinates normal to the hypersurfaces \(\Sigma^3\), then we see how the use of the kinematical action allows to overcome the ambiguity in the existence of a real time-like normal vector field, we have spoken about in the introduction of this paper.

Now using the relations \(\partial_t = \partial_t y^\mu \partial_\mu, \partial_i = \partial_i y^\mu \partial_\mu, n^\bar{\mu} \partial_\bar{\mu} = n^\mu \partial_\mu\) and the first one of equations \(28\), we may rewrite the second kinematical equation, concerning the momentum dynamics as follows:

\[
n^\rho [\partial_\rho (N p_\mu) - \partial_\mu (N p_\rho)] = -\partial_\mu (N p_\rho n^\nu + p_\mu (n^\nu \partial_\rho N + \partial_i N^i)); \quad (30)
\]

we note that \(p_\mu\) is not a vector, but it is a vector density of weight 1/2; thus we can rewrite it as \(p_\mu = -\sqrt{-\varepsilon} \pi_\mu\), where \(\varepsilon\) is a real 3-scalar and \(\pi_\mu\) is a vector, such that it satisfies the relation \(n^\mu \pi_\mu = -1\). Using this new expression for \(p_\mu\), equation \((30)\) rewrites:

\[
\varepsilon n^\rho (\partial_\rho \pi_\mu - \partial_\mu \pi_\rho) = -\pi_\mu \sqrt{-g} \partial_\rho (\sqrt{-g} \varepsilon n^\rho); \quad (31)
\]

which covariantly reads

\[
\varepsilon n^\rho (\nabla_\rho \pi_\mu - \nabla_\mu \pi_\rho) + \pi_\mu \nabla_\rho (\varepsilon n^\rho) = 0. \quad (32)
\]

Then, multiplying equation \((32)\) for \(n^\nu\), we get

\[
\nabla_\rho (\varepsilon n^\rho) = 0. \quad (33)
\]

A perfect fluid, with entropy density \(\sigma\) and velocity \(u_\mu\), satisfies the equation \(\nabla_\mu (\sigma u^\mu) = 0\), but for a dust case the density of entropy is proportional to the density of energy \((\sigma \propto \varepsilon)\), so that equation \((33)\) is the one for a dust fluid of density of energy \(\varepsilon\) and 4-velocity \(n_\mu\).

Now, using equation \((33)\), we can rewrite the relation \((32)\) as

\[
n^\rho (\nabla_\rho \pi_\mu - \nabla_\mu \pi_\rho) = 0. \quad (34)
\]

Setting now \(\pi_\mu = n_\mu + s_\mu\), with \(n^\mu s_\mu = 0\), from above, we arrive to

\[
n^\rho \nabla_\rho n_\mu = n^\rho (\nabla_\rho s_\mu - \nabla_\mu s_\rho) = \gamma n^\rho F_{\mu\rho}, \quad (35)
\]
with \( s_\rho = \gamma A_\rho \), where \( \gamma \) is a constant and \( F_{\mu\rho} = \nabla_\mu A_\rho - \nabla_\rho A_\mu \) (obviously \( n^\rho A_\rho = 0 \)).

Thus equation (35), together with (33) are the field equations of a dust fluid with density of energy \( \varepsilon \), whose 4-velocity \( n^\mu \) is tangent to a space-time curve associated to the presence of an “electromagnetic-like” field (say a gravito-electromagnetic field). So, on a classical level, the kinematical action is equivalent to the action of such a dust fluid and, in this sense, it is upgraded from its geometrical nature to a physical one.

The condition \( n^\rho A_\rho = 0 \) can be written in the system \((t, x^i)\) as \( n^\rho A_\rho = 0 \), from which it follows \( \nabla^\tau A^\tau = 0 \) and this means that in the fluid reference we have to do with a gauge condition such that \( A^\mu \equiv (0, A) \), i.e. with a simple 3-vector potential for the gravito-electromagnetic field.

Now let us come back to the kinematical action (27): the corresponding super-hamiltonian and super-momentum of the kinematical term are:

\[
H^k = p_\mu n^\mu, \quad H^k_i = p_\mu e^\mu_i, \quad (36)
\]

Using the definitions above introduced for \( p_\mu \) and \( s_\mu \) we have:

\[
H^k = \sqrt{h} \varepsilon, \quad H^k_i = -\sqrt{h} \varepsilon \gamma A^\mu e^\mu_i. \quad (37)
\]

It is clear that \( A^\mu e^\mu_i = A^_i \frac{\partial y^\mu}{\partial x^i} \) is a transformation of coordinates from the generic system \( y^\mu \) to the system of the hypersurface, that is the one which we have before indicated with barred labels. So we write \( A^\mu e^\mu_i = A_i \), that is we introduce the projection of the field \( A_\mu \) on the spatial hypersurfaces. So equations (37) rewrites as:

\[
H^k = \sqrt{h} \varepsilon, \quad H^k_i = -\sqrt{h} \varepsilon \gamma A_i. \quad (38)
\]

In [26] is shown that the energy-momentum tensor of the dust is orthogonal to the hypersurfaces \( \Sigma^\Delta \); this is the reason why such tensor contributes only to the super-hamiltonian constraint, by its energy density term. Moreover, it is possible to show, via a simple model, why the presence of the field \( A_\mu \) has, instead, effects only on the super-momentum. To this end, let us consider an interaction between a current \( j^\mu \) and a field \( B_\mu \); then the hamiltonian of interaction will be:

\[
H_{int} = \int d^4 x \sqrt{-g} j^\mu B_\mu. \quad (39)
\]

Since \( H_{int} \) is obviously a scalar, we can rewrite it in the system of coordinates with barred labels, as follows

\[
H_{int} = \int d^4 \bar{x} \sqrt{\bar{h}} j^\tau \bar{B}_\tau, \quad (40)
\]

taking now \( j^\tau = \varepsilon n^\tau \) (current of matter) and \( B_\tau = \gamma A_\tau \), we have, remembering also that \( n^\tau = \left( \frac{1}{N}, \frac{-N^i}{N} \right) \),

\[
H_{int} = \int d^4 \bar{\tau} \sqrt{\bar{h}} \varepsilon (A_\tau - \frac{N^i}{N} A^i). \quad (41)
\]
This expression no more depends on the lapse function, so that it does not contribute to the super-hamiltonian, while the contribution to the super-momentum is just the one in equation

Above we have introduced the projection of the field $A_\mu$ on the spatial hypersurfaces, i.e. $A_i = A_\mu e_i^\mu$; this is of course a simple transformation of coordinates, but it does not assure $A_i$ is a 3-vector. To show this, we define $A^i = A^\mu e_\mu^i$, it is worth noting that it is not a transformation of coordinates, but this choice on how to project the contravariant 4-vector $A^\mu$, is sufficient to show that $A_i = h_{ik}A^k$, which ensures $A_i$ is a 3-vector on the hypersurfaces, which lowers and raises its index by the induced 3-metric.

In fact starting from the expression of $A_i$ and recalling that $e_\mu^i = h^{ik}g_{\mu\nu}e_k^\nu$, we can write:

$$A_i = A_\mu e_\mu^i = A_\mu h_{ik}g^{\mu\nu}e_\nu^k = h_{ik}A^\nu e_\nu^i = h_{ik}A^k,$$

where, in the last equality, we have used the definition of $A^k$.

To conclude this section, we want to study the behaviors of $\varepsilon$ and $A_i$; to this end we start from equations (28), multiplying the second one by $n^\mu$, and remembering that $n^\mu \partial_\mu = n^\mu \partial_\mu$, we arrive to

$$\partial_t (\sqrt{h}\varepsilon) - \partial_i (\sqrt{h}\varepsilon N^i) = 0;$$

Moreover, by multiplying the second one with $e_i^\mu$ and considering also the first kinematical equation, we get an expression of the form:

$$\partial_t (\sqrt{h}\varepsilon \gamma A_i) - \partial_k (\sqrt{h}\varepsilon \gamma N^k A_i) = \sqrt{h}\varepsilon \gamma A_k \partial_i N^k - \sqrt{h}\varepsilon \partial_i N.$$

To treat these two equations in a general reference frame, it is a very difficult task, but it becomes very simple in a synchronous reference, where $N = 1$ and $N^i = 0$; in this particular case we have:

$$\partial_t (\sqrt{h}\varepsilon) = 0, \quad \partial_t (\sqrt{h}\varepsilon \gamma A_i) = 0.$$

The first one of the above equations means that $\sqrt{h}\varepsilon = -\omega (x^i)$ where $\omega$ is a scalar density of weight $1/2$, which depends only on $x^i$; we note that $\varepsilon = -\frac{\omega (x^i)}{\sqrt{h}}$, this means $\varepsilon$ is the density of energy of a non relativistic dust. While from the second one we obtain $\gamma A_i \omega (x^i) = -k_i (x^k)$, which is a 3-vector density of weight $1/2$ and depends only on $x^i$ (we have to do with a simple magnetic term). It is clear that we can now write the super-hamiltonian and super-momentum of the kinematical term as follows

$$H^k = -\omega (x^i), \quad H_i^k = -k_i (x^i).$$

We will return on the above expression in the next section, when treating the eigenvalues problem and the classical limit of the quantized theory; indeed we will find a connection between the density of energy of the dust and the eigenvalue of the super-hamiltonian operator as well as between the eigenvalues of the super-momentum operator and the presence of the field $A_i$.  

13
3.3 Quantum Fields on Curved Background

Now, within the ADM formalism, we analyze the quantization of a self-interacting scalar field $\phi(t, x^i)$ described by a potential term $V(\phi)$ on a fixed gravitational background; its dynamics is summarized by the action

$$
S^\phi(\pi, \phi) = \int_{M^4} \left\{ p_\phi \partial_t \phi - N H^\phi - N^i H_i^\phi \right\} d^3x dt, \quad (47)
$$

where $p_\phi$ denotes the conjugate field to the scalar one and the hamiltonian terms $H^\phi$ and $H_i^\phi$ are those contained in equations (13).

This action should be varied with respect to $p_\phi$ and $\phi$, but not $N$, $N^i$ and $h_{ij}$ since the metric background, in this case, is assigned; but if we want to apply to this system the canonical quantization formalism we have to extract the hamiltonian constraints by a reparametrization of the action for the scalar field. This aim is reached by adding to $S^\phi$ the kinematical action (27), moreover, this additional term has a geometrical as well as a physical interpretation as seen above. The total action is

$$
S^{\phi_k} \equiv S^\phi + S^k = \int_{M^4} \left\{ p_\phi \partial_t \phi + p_\mu \partial_t y^\mu - N(H^\phi + H^k) - N^i(H_i^\phi + H_i^k) \right\} d^3x dt, \quad (48)
$$

In the above action $n^\mu$ and $h_{ij}$ are to be regarded as assigned functionals of $y^\mu(t, x^i)$; background is now fixed by the hypersurfaces $y^\mu$ and their normal vector $n^\mu$, so we can consider $N$ and $N^i$ as generic Lagrange multipliers, the addition of the kinematical action does not affect the field equation for the scalar field, while the variations with respect to $p_\mu$ and $y^\mu$ provide the equation (28) and the evolution of the kinematical momentum.

Finally, by varying, now even, with respect to $N$ and $N^i$ we get the constraints

$$
H^\phi = -p_\mu n^\mu, \quad H_i^\phi = -p_\mu e_i^\mu, \quad (49)
$$

Clearly is to be assigned the following Cauchy problem assigned on a regular initial hypersurface $\Sigma_{t_0}^3$, i.e. $y^\mu(t_0, x^i) = y_0^\mu(x^i)$

$$
\phi(t_0, x^i) = \phi_0(x^i), \quad \pi_\phi(t_0, x^i) = \pi_0(x^i), \quad (50)
$$

$y^\mu(t_0, x^i) = y_0^\mu(x^i), \quad p_\mu(t_0, x^i) = p_\mu 0(x^i)$, $p_\mu(t_0, x^i)$

At last, to complete the scheme of the field equations, we have also to specify the lapse function and the shift vector by the first of equations (28), but also the metric tensor $h_{ij}$ by the relation $h_{ij} = g_{\mu\nu} \partial_i y^\mu \partial_j y^\nu$.

This system can be easily quantized in the canonical formalism by assuming the states of the system be represented by a wave functional $\Psi(y^\mu(x^i), \phi(x^i))$ and implementing the canonical variables $\{y^\mu, p_\mu, \phi, p_\phi\}$ to operators $\{\hat{y}^\mu, \hat{p}_\mu\}$, but also the field operators $\hat{\phi}, \hat{p}_\phi = -i\hbar \delta(\ )/\delta \phi$, $\hat{\pi}_\phi = -i\hbar \delta(\ )/\delta \phi$. Then the quantum dynamics is described.
by the equations
\[ i\hbar n_\mu \frac{\delta \Psi}{\delta y_\mu} = \hat{H}_\phi \Psi = \left[ -\frac{\hbar^2}{2\sqrt{h}} \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi} + \frac{1}{2} \sqrt{\hbar} h^{ij} \partial_i \phi \partial_j \phi + \sqrt{\hbar} V(\phi) \right] \Psi, \quad (51) \]
\[ i\hbar e_\mu \frac{\delta \Psi}{\delta y_\mu} = \hat{H}_i \phi \Psi = -i\hbar \partial_i \phi \frac{\delta \Psi}{\delta \phi}. \quad (52) \]

These equations have $5 \times \infty^3$ degrees of freedom, corresponding to the values taken by the four components of $y^\mu$ and the scalar field $\phi$ in each point of a spatial hypersurface. In (51) and (52) $y^\mu$ plays the role of “time variable”, since it specifies the choice of a particular hypersurface $y^\mu = y^\mu(x^i)$.

In view of their parabolic nature, equations (51) and (52) have a space of solutions that, by an heuristic procedure, can be turned into an Hilbert space, the inner product of which reads
\[ \langle \Psi_1 | \Psi_2 \rangle \equiv \int_{y^\mu = y^\mu(x^i)} \Psi_1^* \Psi_2 D\phi \delta y_\mu = 0, \quad (53) \]
where $\Psi_1$ and $\Psi_2$ denote two generic solutions and $D\phi$ the Lebesgue measure defined on the $\phi$-function space. The above inner product induces the conserved functional probability distribution $\rho \equiv \langle \Psi | \Psi \rangle$.

The semiclassical limit of this equations (51) and (52) is obtained when taking $\hbar \to 0$ and, by setting the wave functional as
\[ \Psi = \exp \left\{ \frac{1}{\hbar} \Sigma(y^\mu, \phi) \right\} \quad (54) \]
and then expanding $\Sigma$ in powers of $\hbar/i$, i.e.
\[ \Sigma = \Sigma_0 + \frac{\hbar}{i} \Sigma_1 + \left( \frac{\hbar}{i} \right)^2 \Sigma_2 + ... \quad (55) \]
By substituting (54) and (55) in equations (51) and (52), up to the zero-order approximation, we find the Hamilton-Jacobi equations
\[ -n^\mu \frac{\delta \Sigma_0}{\delta y_\mu} = \frac{1}{2\sqrt{\hbar}} \left( \frac{\delta \Sigma_0}{\delta \phi} \right)^2 + \sqrt{\hbar} \left( \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi + V(\phi) \right), \quad e_\mu \frac{\delta \Sigma_0}{\delta y_\mu} = -\partial_i \phi \frac{\delta \Sigma_0}{\delta \phi}, \quad (56) \]
which lead to the identification $\Sigma_0 \equiv S^{0k}$.

### 3.4 Reformulation of Quantum Geometrodynamics

We start by observing that, within the framework of a functional approach, a covariant quantization of the 4-metric field is equivalent to take the wave amplitude $\Psi = \Psi(g_{\mu\nu}(x^\rho))$; in the WDE approach, by adopting the ADM slicing of the space-time, the problem is restated in terms of the following replacement
\[ \Psi(g_{\mu\nu}(x^\rho)) \to \Psi(N(t, x^i), N^1(t, x^i), h_{ij}(t, x^i)). \quad (57) \]
Then, since the lapse function $N$ and the shift vector $N^i$ are cyclic variables, i.e. their conjugate momenta $p_N$ and $p_{N^i}$ vanish identically, we get, on a quantum level, the following restrictions:

$$\pi = 0, \quad \pi_k = 0 \quad \Rightarrow \quad \frac{\delta \Psi}{\delta N} = 0, \quad \frac{\delta \Psi}{\delta N^i} = 0;$$

(58)

by other words, the wave functional $\Psi$ should be independent of $N$ and $N^i$. Finally, the super-momentum constraint leads to the dependence of $\Psi$ on the 3-geometries $\{h_{ij}\}$ (instead on a single 3-metric tensor $h_{ij}$).

The criticism to the WDE approach, developed at the point iii) of section 2.2 concerns with the ill-defined nature of the replacement (57). The content of this section is entirely devoted to reformulate the quantum geometrodynamics, by preserving the (3+1)-representation of the space-time, but avoiding the ambiguity above outlined in the WDE approach.

As outlined in the introduction to this section, we claim that the canonical quantization of gravity has sense only when referred to a fixed slicing, in which the notion of space or time like character of a vector field be physically distinguishable.

To this aim we fix the lapse function and the shift vector (now the slicing is fixed) and then we reparametrize the gravitational action using the kinematical term (as in the assigned background field theory), obtaining the total action:

$$S_{\phi k} = \int_{\Sigma \times \mathbb{R}} dt dx \left\{ p^{ij} \partial_t h_{ij} + \pi \partial_t N + \pi_k \partial_t N^k + p_\phi \partial_t \phi + p_{\mu} \partial_t y^\mu + \right.$$  

$$- \left( \lambda \pi + \lambda^i \pi_i + N \left( H^g + H^\phi + H^k \right) + N^i \left( H_i^g + H_i^\phi + H_i^k \right) \right) \right\}.$$  

(59)

Now the lapse function $N$ and the shift vector $N^i$ are to be again regarded as dynamical variables (the slicing remain fixed by the hypersurfaces parametric equations $y^\mu = y^\mu (t, x^i)$ and by the vector $n^\mu$); the new hamiltonian constraints are

$$\pi = 0, \quad \pi_k = 0,$$

(60)

$$H^g + H^\phi + H^k = 0, \quad H_i^g + H_i^\phi + H_i^k = 0.$$  

(61)

We note that the variation with respect to the dynamical field $y^\mu = y^\mu (t, x^i)$ and its conjugate momentum $p_\mu = p_\mu (t, x^i)$ leads to the kinematical equation (28).

Though from a mathematical point of view, to fix the reference frame is, in view of the reparametrization which restores the canonical constraints, a well defined procedure, it requires a physical interpretation; indeed the open question is: which are the physical consequences of fixing the slicing?

The complete answer to this question will be clear at the end of this section, but now we can say that fixing the reference frame we modify the physical system: the dynamical equations and the constraints, describe no more the dynamics of
the initial system formed by gravitational and scalar field, but the addition of the kinematical term introduces a new physical field, which, as shown in section 3.2, can be interpreted as a dust interacting with a gravito-electromagnetic-like field. We remark that in a purely classical system it is not necessary to introduce this additional term to the gravity-matter action and therefore we expect that the dust has effects on the dynamics of those systems which evolve from a quantum state.

Now to quantize the new constraints (60), (61) we use the canonical procedure, by implementing the canonical variables to quantum operators. We assume that the state of the gravitational and matter system be described by a wave functional \( \Psi = \Psi (y^\mu, \phi, h_{ij}, N, N^i) \). Then the new quantum dynamics of the whole system is now described by the functional differential system:

\[
\frac{\delta \Psi}{\delta N} = 0, \quad \frac{\delta \Psi}{\delta N^i} = 0,
\]

\[
i h n^\mu \frac{\delta \Psi}{\delta y^\mu} = \left( \hat{H}^g + \hat{H}^\phi \right) \Psi, \quad i h \partial_I y^\mu \frac{\delta \Psi}{\delta y^\mu} = \left( \hat{H}^g_i + \hat{H}^\phi_i \right) \Psi,
\]

being \( \hat{H}^g + \hat{H}^\phi \) and \( \hat{H}^g_i + \hat{H}^\phi_i \) the Hamiltonian operators after the quantum implementation of the canonical variables. By the first line equations, the wave functional does not depend on the lapse function \( N \) and the shift vector \( N^i \), so, since now, we limit our attention on the other two equations, considering that the wave functional \( \Psi \) depends only on the 3-metric \( h_{ij} (x^k) \), the scalar field \( \phi (x^k) \) and the new field \( y^\mu (x^k) \), which plays the role of a time variable, by specifying the hypersurface on which the wave functional is taken (we stress how its spatial gradients behaves like potential terms).

Moreover, the second of equation (63) still assures the invariance of the wave functional under the spatial diffeomorphism. Then, denoting by \( \{ h_{ij} \} \) a whole class of 3-geometries (i.e. connected via 3-coordinates reparameterization), the wave functional should yet be taken on such more appropriate variable instead of a special realization of the 3-metric. In the first of equations (63) the vector field \( n^\mu (y^\rho) \) is an arbitrary one, without any peculiar geometrical meaning; but when taking into account the first of kinematical equation (28), \( n^\mu \) becomes a real unit normal vector field, since, once fixed \( N \) and \( N^i \), \( y^\mu (t, x^i) \) pays the price for its geometrical interpretation. These considerations lead us to claim that the first of equation (28) should be included in the dynamics even on the quantum level. The physical justification for this statement relies on the fact that no information about the dynamic of the kinematical dust comes from such an equation has discussed in the previous section; in fact there we have shown how the whole dynamics of the dust be entirely contained in the momentum equation. In agreement to what we said in the introduction to this work, the surviving of this classical equation on a quantum level, reflects the classical nature of the “device” operating the \( (3 + 1) \)-splitting.

To take into account this equation is equivalent to reduce \( y^\mu \) to a simple \( \infty \)-dimensional parameter for the system dynamics.
In agreement with this point of view, we can smear the quantum dynamics on a whole 1-parameter family of spatial hypersurfaces $\Sigma^3_t$ filling the space-time; as soon as we introduce the notation
\[
\partial_t = \int_{\Sigma^3_t} \partial_t \delta y^\mu \frac{\delta}{\delta y^\mu},
\]
then equations (63) acquire the Schrödinger form
\[
i\hbar \partial_t \Psi = \hat{H} \Psi,
\]
where
\[
\hat{H} = \int_{\Sigma^3_t} d^3x \left[ N \left( \hat{H}^g + \hat{H}^\phi \right) + N^i \left( \hat{H}^g_i + \hat{H}^\phi_i \right) \right].
\]

In this new framework the wave functional can be taken directly on the label time (i.e. $\Psi = \Psi (t, \phi, h_{ij})$) (where we have removed the curl bracket from $h_{ij}$ because, now, the wave functional is no longer invariant under 3-diffeomorphism), since the latter becomes a physical clock via the correspondence, we show below, between the eigenvalue problem of the equation (65) and the energy-momentum of the dust discussed in the previous section.

In order to construct the Hilbert space associated to the Schrödinger-like equation we must prove the hermitianity of the hamiltonian operator; since the hermitian character of the $\phi$ term was proved in [18], as well as of the operator $\hat{H}^g$ in [26] under the following choice for the normal ordering
\[
G_{ijkl} \rightarrow -\hbar^2 \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta}{\delta h_{kl}} \right),
\]
then it remains to be shown the hermitian character of the operator $\hat{h} = \int_{\Sigma^3_t} d^3x N^i \hat{H}^g_i$. In Dirac notation we have to show that:
\[
\langle \Psi_1 | \hat{h} | \Psi_2 \rangle = \langle \Psi_2 | \hat{h} | \Psi_1 \rangle^*.
\]
To this aim we write down the explicit expression of the above bracket:
\[
\langle \Psi_1 | \hat{h} | \Psi_2 \rangle = 2i\hbar \int_{\mathcal{F}_1} Dh \int_{\Sigma^3_t} d^3x \Psi_1^* N^i h_{ik} \nabla_j \frac{\delta}{\delta h_{kj}} \Psi_2,
\]
where $Dh$ is the Lebesgue measure in the 3-geometries functional space.Now integrating by parts, considering that the hypersurfaces $\Sigma^3_t$ are compact and using, in view of the functional Gauss theorem, the following relation:
\[
\int_{\mathcal{F}_1} Dh \int_{\Sigma^3_t} d^3x \frac{\delta}{\delta h_{kj}} (....) = 0,
\]
we can rewrite the expression (69) in the following form:

\[
\langle \Psi_1 | \hat{h} | \Psi_2 \rangle = 2i\hbar \int_{\mathcal{F}_t} \frac{\delta}{\delta h_{kj}} \left( \nabla_j (N^i) h_{ik} \right) \Psi_2.
\] 

(71)

It is possible to show that two of the terms, which come from the right side of (71) when the functional derivative operates on the quantities in the parenthesis, are zero. In fact, acting with the functional derivative on the 3-metric, we obtain:

\[
2i\hbar \int_{\mathcal{F}_t} \frac{\delta}{\delta h_{kj}} \left( \nabla_j (N^i) \right) \Psi_2 = -2i\hbar \int_{\mathcal{F}_t} \frac{\delta}{\delta h_{kj}} \left( \nabla_j (\Psi_1 \Psi_2) N^j \right),
\]

(72)

where we have integrated by parts and used the compactness of the hypersurfaces \(\Sigma^3_3\). But the right hand side of (72) is zero, because \(\Psi\) is a functional, so it does not depend on \(x\).

When the functional derivative in expression (71) acts on the covariant derivative of the shift vector, we obtain:

\[
2i\hbar \int_{\mathcal{F}_t} \frac{\delta}{\delta h_{kj}} \left( \nabla_j (\Psi_1 \Psi_2) N^i \right) = -2i\hbar \int_{\mathcal{F}_t} \frac{\delta}{\delta h_{kj}} \left( \Gamma^i_{jm} N^m \right),
\]

(73)

since in the right side term, the derivative operator is applied to a function of \(x\) and not to a functional, thus, like in the case of the variation with respect a dynamical variable, the ordinary derivative operator and the functional one commute, so it is simple to show that \(\frac{\delta}{\delta h_{kj}} \left( \Gamma^i_{jm} N^m \right) = 0\), thus the term (73) is identically zero.

Finally the expression (71) can be rewrite:

\[
\langle \Psi_1 | \hat{h} | \Psi_2 \rangle = 2i\hbar \int_{\mathcal{F}_t} \frac{\delta}{\delta h_{kj}} \left( \nabla_j (N^i) \right) h_{ik} \Psi_2 = -2i\hbar \int_{\mathcal{F}_t} \frac{\delta}{\delta h_{kj}} \left( \nabla_j (\Psi_2 N^i h_{ik} \nabla_j \nabla^i) \right) = \langle \Psi_2 | \hat{h} | \Psi_1 \rangle^*.
\]

(74)

The above equality assures \(\hat{H}\) is an Hermitian operator. Defining the following inner product:

\[
\langle \Psi_1 | \Psi_2 \rangle = \int_{y_t} Dh D\phi \Psi^*_1 \Psi_2,
\]

(75)

where \(Dh D\phi\) is the Lebesgue measure for the functional space of all the dynamical variables and \(y_t\) is the corresponding functional domain, we can turn
the space of solutions of the Schrödinger-like equation into an Hilbert space. We interpret the above bracket as the probability that a state $|\Psi_1\rangle$ falls into another state $|\Psi_2\rangle$ and, defining the density of probability $\rho = \Psi^*\Psi$, we can also construct the amplitude for the system lying in a field configuration. By the hermitian character of the operator $\hat{H}$, it is possible to show that the probability is constant in time, in fact:

$$\partial_t \langle \Psi_1 | \Psi_2 \rangle = \int d^3x \partial_t y^\mu \frac{\delta}{\delta y^\mu} \langle \Psi_1 | \Psi_2 \rangle = \frac{i}{\hbar} \left( \langle \hat{H}\Psi_1 | \Psi_2 \rangle - \langle \Psi_1 | \hat{H}\Psi_2 \rangle \right) = 0,$$

(76)

the general character of the deformation vector allows us to write the fundamental conservation law

$$\frac{\delta \langle \Psi_1 | \Psi_2 \rangle}{\delta y^\mu} = 0,$$

(77)

which assures the probability does not depend on the choice of the hypersurface.

The density of probability $\rho$ satisfies a continuity equation, which can be obtained multiplying the Schrödinger-like equation times the complex conjugate wave function $\Psi^*$ and the complex conjugate equation times the wave function $\Psi$, i.e.

$$i\hbar \Psi^* \partial_t \Psi = \Psi^* \hat{H}\Psi, \quad -i\hbar \partial_t \Psi^* = \Psi \hat{H}^* \Psi^*, \quad (78)$$

subtracting the second of equation (78) from the first one, we obtain:

$$i\hbar \partial_t (\Psi \Psi^*) = \int d^3x \left\{ -\hbar^2 \left( \Psi^* \frac{\delta}{\delta h_{ij}} G_{ijkl} \frac{\delta}{\delta h_{kl}} \Psi - \Psi \frac{\delta}{\delta h_{ij}} G_{ijkl} \frac{\delta}{\delta h_{kl}} \Psi^* \right) + \right.$$

$$- \hbar^2 \left( \Psi^* \frac{2\sqrt{h}}{\delta \phi} \frac{\delta}{\delta \phi} \Psi - \Psi \frac{2\sqrt{h}}{\delta \phi} \frac{\delta}{\delta \phi} \Psi^* \right) +$$

$$+ 2i\hbar \left( \Psi^* N^i h_{ik} \nabla_j \frac{\delta}{\delta h_{kj}} \Psi + \Psi N^i h_{ik} \nabla_j \frac{\delta}{\delta h_{kj}} \Psi^* \right) +$$

$$- i\hbar \left( \Psi^* N^i \partial_i \phi \frac{\delta}{\delta \phi} \Psi + \Psi N^i \partial_i \phi \frac{\delta}{\delta \phi} \Psi^* \right) \right\}, \quad (79)$$

defining now the tensor probability current $A_{ij}$, which is connected with the 3-metric tensor field, in the following way:

$$A_{ij} = -i\hbar \left( \Psi^* N G_{ijkl} \frac{\delta}{\delta h_{kl}} \Psi - \Psi N G_{ijkl} \frac{\delta}{\delta h_{kl}} \Psi^* \right) + 2h_{ki} (\nabla_j N^j) \Psi^* \Psi, \quad (80)$$

and the scalar probability current $A$, connected, instead, to the presence of the scalar field $\phi$, as:

$$A = -i\hbar \frac{N}{2\sqrt{h}} \left( \Psi^* \frac{\delta}{\delta \phi} \Psi - \Psi \frac{\delta}{\delta \phi} \Psi^* \right) - i\hbar (\phi \partial_i N^i \Psi^* \Psi), \quad (81)$$

the equation (78) takes the following form:

$$\partial_t \rho + \int d^3x \left( \frac{\delta A_{ij}}{\delta h_{ij}} + \frac{\delta A}{\delta \phi} \right) = 0,$$

(82)

20
integrating on the functional space $y_t$, using the generalized Gauss theorem (70), the continuity equation assures that the probability is constant in time as above.

Let us now reconsider the Schrödinger dynamics in terms of a time independent eigenvalues problem. To this end we expand the wave functional as follows:

$$\Psi (t, \phi, h_{ij}) = \int_{y_t^*} D\Omega D K \Theta (\Omega, K_i) \chi_{\Omega, K_i} (\phi, h_{ij}) \cdot$$

$$\cdot \exp \left\{ -\frac{i}{\hbar} \int_{t_0}^t dt' \int \frac{d^3 x}{\Sigma_3} (N\Omega + N^i K_i) \right\}, \quad (83)$$

being $t_0$ an assigned initial “instant”. Where $D\Omega D K$ denotes the Lebesgue measure in the functional space $y_t^*$ of the conjugate function $\Omega (x^i)$ and $K_i (x^i)$, $\Theta = \Theta (\Omega, K_i)$ a functional valued in this domain, whose form is determined by the initial conditions $\Psi_0 = \Psi (t_0, \phi, h_{ij})$. When we substitute the expansion $(83)$ of the wave functional into $(65)$, such equation is satisfied only if the following $\infty^3-$dimensional eigenvalues problem takes place:

$$\left( \hat{H}^9 + \hat{H}^\phi \right) \chi_{\Omega, K_i} = \Omega (x^j) \chi_{\Omega, K_i}, \quad \left( \hat{H}^9_i + \hat{H}^\phi_i \right) \chi_{\Omega, K_i} = K_i (x^j) \chi_{\Omega, K_i}. \quad (84)$$

Now to characterize the physical meaning of the above eigenvalues, we construct the semi-classical limit of the Schrödinger-like equation, by splitting the wave functional into its modulus and phase, as follows:

$$\Psi = \sqrt{\rho e^{i \sigma / \hbar}}. \quad (85)$$

Then in the limit $\hbar \rightarrow 0$ we obtain for $\sigma$ an Hamilton-Jacobi equation of the form:

$$-\partial_t \sigma = \int d^3 x \frac{\delta N}{\Sigma_3} \left( G_{ijkl} \frac{\delta^2 \sigma}{\delta h_{ij} \delta h_{kl}} - \sqrt{\hbar} (3) R + \right.$$

$$+ \frac{1}{2 \sqrt{\hbar}} \frac{\delta \sigma}{\delta \phi} \delta \sigma + \frac{\sqrt{\hbar}}{2} h^{ij} \partial_i \phi \delta \sigma + \sqrt{\hbar} V (\phi) \right) +$$

$$\left. - \int d^3 x \left( 2 h_{ik} \nabla_j \delta \sigma \delta h_{kj} - \partial_i \phi \delta \sigma \delta \phi \right) \right), \quad (86)$$

The non vanishing of the $\sigma$ time derivative reflects the evolutive character appearing in the constructed theory and makes account for the presence, on the classical limit, of the dust matter discussed in the previous section. To clarify this feature, we set

$$\sigma (t, \phi, h_{ij}) = \tau (\phi, h_{ij}) + \int_{t_0}^t dt' \int \frac{d^3 x}{\Sigma_3} (N\Omega + N^i K_i). \quad (87)$$
When we substitute this expression in the Hamilton-Jacobi equation, and identify \( p_{ij} = \frac{\delta \tau}{\delta h_{ij}} \), \( p_{\phi} = \frac{\delta \tau}{\delta \phi} \), then the equation (86) becomes equivalent to the \( \infty \)-dimensional ones:

\[
\begin{align*}
(H^g + H^\phi) &= \Omega (x^j) \\
(H^g_i + H^\phi_i) &= K_i (x^j)
\end{align*}
\] (88)

We stress how these equations coincides with those ones obtainable by the eigenvalues problem, as soon as we choose the classical limit of \( \chi \sim e^{\frac{i}{\hbar} \tau} \) thus, at the end of this analysis, recalling expressions (16) and (41), we can identify the super-hamiltonian eigenvalue \( \Omega \) with \( \omega \) and the super-momentum eigenvalues \( K_i \) with \( k_i \). On the other hand by equations (16) and (38), the above identification implies that: \( \Omega = -\sqrt{\hbar \varepsilon} \) and \( K_i = -\gamma A_i \omega \).

The relation we obtained show how super-hamiltonian and super-momentum eigenvalues are directly connected with the dust fields introduced in section 3.2. Even starting from a quantum point of view we recognize the existence of a dust fluid playing the role of a physical clock for the gravity-matter dynamics.

4 A Simple Cosmological Model

If the theory here proposed is a predictive one, we should expect to observe the trace of this reference fluid energy density from all those systems which underwent a classical limit; such a situation is surely true for our actual Universe and, indeed, we really observe (in the synchronous reference of our galaxy) an unidentified dust energy, the so-called dark matter; in the next two sub-sections, we will try to understand if it can exist a correlation between our dust fluid and the observed “matter component” of the Universe.

4.1 3-Diffeomorphisms Invariant Theory

Before to discuss the application of our theory to a FRW Universe, we want to rewrite the above reformulation of quantum geometrodynamics preserving the 3-diffeomorphisms invariance. This means that the quantum equation take the following form:

\[
i\hbar \frac{\delta \Psi}{\delta y^\mu} = (\hat{H}^g + \hat{H}^\phi) \Psi, \quad (\hat{H}^g_i + \hat{H}^\phi_i) \Psi = 0, \quad \Psi = \Psi(\{h_{ij}\}, y^\mu),
\] (89)

where now the wave functional is taken again on the 3-geometries (\( \{h_{ij}\} \)) related by the 3-diffeomorphisms.

These \( (4 \times \infty^3) \) equations, which correspond to a natural extension of the Wheeler-De Witt approach, have the fundamental feature that again the first of them is parabolic and it is just this property which still allows to overcome the limits of the WDE above discussed. Though this set of equations provides a
satisfactory description of the 3-geometries quantum dynamics, nevertheless it turns out convenient and physically meaningful to take, by (28), the wave functional evolution along a one-parameter family of spatial hypersurfaces, filling the Universe.

By the first of equations (28), the above (89) can be rewritten as follows:

\[ i\hbar \frac{\delta \Psi}{\delta y_\mu} \partial_t y_\mu = N(\hat{H}^g + \hat{H}^\phi) \Psi. \]  
(90)

Now this set of equations can be (heuristically) rewritten as a single one by integrating over the hypersurfaces \( \Sigma_1^t \), i.e.

\[ i\hbar \partial_t \Psi = i\hbar \int_{\Sigma_1^t} \left\{ \frac{\delta \Psi}{\delta y_\mu} \partial_t y_\mu \right\} d^3x = \hat{H} \Psi \equiv \int_{\Sigma_1^t} N(\hat{H}^g + \hat{H}^\phi) d^3x \]  
(91)

The above equations (91) and (89) show how in the present approach the wave functional is still no longer invariant under infinitesimal displacements of the time variable.

It is possible to show that, like above, the operator \( \hat{H} \) is an hermitian one, so we still have the fundamental conservation law

\[ \frac{\delta \langle \Psi_1 | \Psi_2 \rangle}{\delta y_\mu} = 0. \]  
(92)

Substituting the usual expansion

\[ \Psi(y_\mu, \{h_{ij}\}, \phi) = \int_{\mathcal{Y}_t} D\omega \Theta(\omega) \chi_\omega(\{h_{ij}\}, \phi) \exp \left\{ \frac{i}{\hbar} \int_{\Sigma_1^t} d^3x \int dt' \partial_{t'} y_\mu(\omega n_\mu) \right\} \]  
(93)

into equations (89) we get the eigenvalues problems

\[ (\hat{H}^g + \hat{H}^\phi) \chi_\omega = \omega \chi_\omega \quad (\hat{H}_i^g + \hat{H}_i^\phi) \chi_\omega = 0 \]  
(94)

Here \( \omega(x^i) \) is not a 3-scalar, but it transforms, under 3-diffeomorphisms, like \( \hat{H}^g \) or \( \hat{H}^\phi \), so ensuring that \( \omega d^3x \), as it should, be an invariant quantity.

Now we observe that, by (28), equation (89) rewrites

\[ \Psi(y_\mu, \{h_{ij}\}, \phi) = \int_{\mathcal{Y}_t} D\omega \Theta(\omega) \chi_\omega(\{h_{ij}\}, \phi) \exp \left\{ \frac{i}{\hbar} \int_{\Sigma_1^t} d^3x \int_{t_0}^t dt' \partial_{t'} y_\mu(\omega n_\mu) \right\} = \right. \]

\[ \quad = \left. \int_{\mathcal{Y}_t} D\omega \Theta(\omega) \chi_\omega(\{h_{ij}\}, \phi) \exp \left\{ \frac{i}{\hbar} \int_{t_0}^t dt' \int_{\Sigma_1^t} d^3x (N\omega) \right\} \right], \]  
(95)

being \( t_0 \) an assigned initial “instant.”

To the same result we could arrive by choosing, without any loss of generality, the coordinates system \((t, x^i)\), i.e. \( y^0 \equiv t, y^i \equiv x^i \); indeed, for this system,
the spatial hypersurfaces have equation $t = \text{const}$, i.e. $dy^\mu \to (dt, 0, 0, 0)$ and we have $n_0 = N$. By other words the wave functional is to be interpreted directly in terms of the time variable $t$, i.e. $\Psi(\{h_{ij}\}, \phi, t)$ and, in fact, it turns out solution of the wave equation

$$i\hbar \partial_t \Psi(\{h_{ij}\}, \phi, t) = \hat{H} \Psi(\{h_{ij}\}, \phi, t)$$

(96)

The expansion of the wave functional and the eigenvalues problems completely describe the quantum dynamics of the 3-geometries.

In this 3-diffeomorphisms invariant approach it is very simple to show that the fluid of reference reduces to a real dust with the energy momentum tensor

$$T^{\mu\nu} = \varepsilon n^\mu n^\nu.$$  

(97)

To conclude, it is worth remarking how, the main difference between our approach and others interesting ones, that lead to the same formal issue (see the discussion in the appendix about the comparison with the so-called “multi-time approach” as well as the formulations presented in [30, 31] and [33, 32]), consists of, in the latter, the super-hamiltonian is preliminary reduced to a linear form, and, overall, of setting ad hoc fields which play the role of time (for instance in [33, 32] is postulated, in the theory, the presence of a real mass-less scalar field), in the former, in stead, we simply extend to the 3-metric dynamics the kinematical (embedding-like) action to provide physical meaning in the splitting procedure, and then interpret it as a dust fluid (with the role of time). In this scheme the 3-metric is related to the space-time one by the dynamical field $y^\mu$, so, heuristically, we can say to bypass the theory background independence.

### 4.2 FRW Quantum Universe

Since the clock by which we are measuring the age of the Universe is (essentially) a synchronous one, and we expect the cosmological dynamics became a classical one, then the contribution of the “dust fluid” energy density must appear in the galaxies recession. Below we will face the questions about the modifications introduced, by our approach, in the quantum evolution of the Universe, and about the actual value of the dust energy density.

We investigate the quantum dynamics predicted, in a synchronous reference, by equation (22) for the closed Friedmann-Robertson-Walker model, whose line element reads (below we adopt the standard notations for the fundamental constants)

$$ds^2 = -c^2 dt^2 + R_c^2(t)[d\xi^2 + \sin^2 \xi(d\eta^2 + \sin^2 \eta d\phi^2)],$$

(98)

where $0 \leq \xi < \pi$, $0 \leq \eta < \pi$, $0 \leq \phi < 2\pi$. Here $R_c$ denotes the radius of curvature of the Universe, measurable, in principle, via the relation $R_c = c/(H\sqrt{\Omega - 1})$ (being $H$ the Hubble function, $\Omega$ the critical parameter and $R_c(\text{today}) \sim O(10^{28}\text{cm})$).
In the very early phases of the Universe evolution, it is expected a space filled by a thermal bath, involving all the fundamental particles; since, at very high temperatures, all the massive particles are ultra relativistic ones, then the most appropriate phenomenological representation of the matter-radiation thermal bath, is provided by an energy density of the form \( \mu^2/R^4 \).

Furthermore, the idea that the Universe underwent an inflationary scenario, leads us to include \textit{ab initio} in the dynamics a real self-interacting scalar field \( \phi \), described by a “finite-temperature” potential \( V_T(\phi) \) (here \( T \) denotes the thermal bath temperature), which we may take, for instance, in the Coleman-Weinberg form

\[
V_T(\phi) = \frac{B\sigma^4}{2\hbar^3c^3} + B\phi^4 \left[ \ln \left( \frac{l_{Pl}^2\phi^2}{\sigma^2} \right) - \frac{1}{2} \right] + \frac{1}{2}m_T^2\phi^2 \quad m_T = \sqrt{\lambda T^2 - m^2},
\]

with \((m, \lambda) = \text{const.}, B\) is a parameter related to the fundamental constraints of the theory (estimated \( O(10^{-3}) \)), \( \sigma \) corresponds to the energy scale associated with the symmetry breaking process (i.e. \( \sigma \sim O(10^{15})\text{GeV} \)), while \( m \) and \( l_{Pl} \) denote, respectively, the inverse of a characteristic length and \( l_{Pl} \) the Planck length \( l_{Pl} \equiv \sqrt{G\hbar/c^3}; \) the temperature dependence of the potential term can be also regarded as a time evolution of the model.

The dynamics of such a cosmological model is summarized, as shown when developing the Einstein-Hilbert action under the present symmetries, by the hamiltonian function

\[
\frac{\mathcal{H}}{\hbar} = -\frac{l_{Pl}^2}{3\pi\hbar} \frac{p_R^2}{R_c} + \frac{c}{4\pi^2} \frac{p_{\phi}^2}{R_c^3} + \frac{\mu^2}{4l_{Pl}^2} R_c + 2\pi^2 R_c^3 V_T(\phi),
\]

with \( p_R \) and \( p_{\phi} \) being the conjugate momenta to \( R_c \) and \( \phi \).

Thus, the Schrödinger equation reads, once turned the above hamiltonian into an operator (which possesses the right normal ordering), as follows

\[
\frac{i\hbar}{\hbar} \partial_t \Psi(t, R_c, \phi) = \left\{ \frac{l_{Pl}^2}{3\pi} \frac{1}{R_c} \partial_{R_c} - \frac{\hbar^2 c}{4\pi^2} \frac{1}{R_c} \partial_{\phi}^2 + \frac{\mu^2}{4l_{Pl}^2} R_c + 2\pi^2 R_c^3 V_T(\phi) \right\} \Psi(t, R_c, \phi),
\]

Before going on with the analysis of this equation, we need to precise some aspects concerning the potential term relevance during the Universe evolution.

It is well-known that the classical scalar field dynamics is governed by the following equation

\[
\ddot{\phi} + 3H \dot{\phi} + c^2 \hbar^2 \frac{dV_T}{d\phi} = 0.
\]

The presence of the potential term is surely crucial to generate the inflationary scenario, but, sufficiently close to the initial “Big-Bang”, its dynamical role is expected to be very limited; in fact, if we neglect the potential term in (102), then, remembering that for early times \( R_c \sim \sqrt{t} \rightarrow H \sim 1/2t \), we get the free
field solution $\phi \propto \ln t$. Now the terms we retained to solve equation (102) are potentially of the order $O(1/t^2)$; in the limit toward the “Big-Bang” ($t \to 0$), the potential term (99) (we recall that $T \propto 1/R_c \propto 1/\sqrt{t}$) can be clearly negligible, i.e. $t^2 V_T(\phi(t)) \to 0$. Apart from very peculiar stiff cases, all the inflationary potentials result to be negligible at very high temperatures.

Taking into account the above classical analysis, we may assume that, during the Planck epoch, when the Universe performed its quantum evolution, the potential of the scalar field plies no significant role; therefore, by choosing the following expansion for the wave function

$$\Psi(t, R_c, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\epsilon dp C(\epsilon, p) \theta(\epsilon, pR_c) \exp\left\{ i \frac{\hbar}{c}(p\phi - \epsilon t) \right\},$$

(with $C(\epsilon, p)$ denoting generic coefficients), we get, from (101), the eigenvalues problem

$$\left\{ \frac{l^2_{Pl}}{3\pi} \frac{d}{dR_c} \frac{1}{R_c} \frac{d}{dR_c} + p^2 c^2 - \frac{\mu^2}{R_c} - \frac{3\pi \hbar}{4l^2_{Pl} R_c} \right\} \theta = \epsilon \frac{c}{R_c} \theta. \quad (104)$$

with the boundary conditions $\theta(R_c = 0) = 0$ and $\theta(R_c \to \infty) = 0$.

A solution to this equation reads in the form

$$\theta \propto \sqrt{R_c} \exp\left\{ -\frac{(R_c - R_{c(0)})^2}{4\alpha^2} \right\}; \quad (105)$$

in order to be the above functional form a solution of equation (104), we have to require the relations $p = \pm \sqrt{\pi \hbar/\alpha_{Pl}}$, $\alpha = l_{Pl}/\sqrt{3\pi}$ and $\epsilon = \mp 3\pi \hbar c R_{c(0)}/2l^2_{Pl}$. Furthermore, since the ultra relativistic energy density is manifestly positive, then, from the following expression for $\mu^2$

$$\mu^2 = \frac{l^2_{Pl}}{3\pi} \frac{1}{2\alpha^2} \frac{R^2_{c(0)}}{4\alpha^4}; \quad (106)$$

we find an important restriction on the continuous eigenvalue spectrum, i.e.

$$-\sqrt{3\pi}/2M^2_{Pl}c^2 < \epsilon < \sqrt{3\pi}/2M^2_{pl}c^2,$$

(107)

being $M_{pl}$ the Planck mass, $M_{pl} = \hbar/c{l_{Pl}}$.

Thus, we get a (non-normalizable) probability amplitude, for the stationary states, of the form

$$P_{Stat} \propto \cos^2(\frac{c}{R_c} \exp\left\{ -\frac{(R_c - R_{c(0)})^2}{2\alpha^2} \right\} \cdot \phi \right) R_c.$$
stationary states for the radius of curvature; in the obtained dynamics, we see that the notion of the cosmological singularity is replaced by the more physical one of a peaked probability to find \( R_c \) near zero. The approximation of neglecting the potential term \( V_T \) can be regarded as confirmed \textit{a posteriori} by the small probability that the system penetrates regions where \( R_c \) is much greater than the Planck length and the temperature is sufficiently small to be compared with the symmetry breaking scale.

In order to construct the semiclassical limit of equation (104), we separate \( \theta \) into its modulus and phase, i.e. \( \theta = \sqrt{\alpha} \exp\{i\beta/\hbar\} \); then we get the following two, real and complex, components of equation (104)

\[
\frac{\ell_p^2}{3\pi\hbar} \frac{1}{R_c} \left( \frac{d\beta}{dR_c} \right)^2 + \frac{\mu^2 c}{4\pi^2 R_c^2} + \frac{\mu^2}{4\ell_p^2 R_c} - \frac{\hbar}{c} + \hbar^2 V_{Quantum} = 0 \quad (109)
\]

\[
\frac{1}{\sqrt{\alpha}} \frac{d}{dR_c} \left( \frac{\alpha}{R_c} \frac{d\beta}{dR_c} \right) = 0 \Rightarrow \alpha \propto R_c/(d\beta/dR_c), \quad (110)
\]

\[
V_{Quantum} \propto \frac{1}{\sqrt{\alpha}} \frac{d}{dR_c} \left( \frac{1}{R_c} \frac{d\sqrt{\alpha}}{dR_c} \right). \quad (111)
\]

In the limit \( \hbar \to 0 \), when \( V_{Quantum} \) becomes negligible, we reobtain the Hamilton Jacobi equation describing the Universe classical dynamics, but with an additional term corresponding to a non-relativistic matter contribution, which, when \( \epsilon \) is negative, acquires positive energy density; to this respect, we remark how, on the quantum level, the Universe is expected to approach the lowest, i.e. negative, energy state.

We stress how, for sufficiently large \( R_c \), if the non-relativistic term dominates (the spatial curvature being yet negligible), then we get \( d\beta/dR_c \propto \sqrt{R_c} \) and therefore \( R_c \to \infty \Rightarrow V_{Quantum} \sim 1/(R_c^3) \to 0 \); such a behavior supports the idea that, when the Universe “expands enough” (i.e. its volume fluctuating explores regions of high \( R_c \) values), it can approach a classical dynamics.

The analysis of this section answers the question about the cosmological phenomenology implied by our approach and the issue goes toward the appearance, in a synchronous reference, of a pressureless contribution to the Universe energy density. In the next section, we make some estimations in order to understand if such a new term (which is nothing more than the classical limit of the total Universe quantum energy) may have something to do with the observed dark matter component.

### 4.3 Phenomenological Considerations

Indeed, by adding a term to the gravitational action, we may expect it appears as a new kind of energy-momentum term; what makes our analysis a valuable one is in the following points:

i) The kinematical action is an embedding-like geometrical object, whose existence in quantum gravity, was postulated in [26] on the base of well-grounded statements and not invented \textit{ad hoc}. Above we have shown that it can be
interpreted, from a classical point of view, as a non-relativistic dust fluid; a non-relativistic energy density is also what appears from the quantum dynamics, when taking the classical limit.

ii) All the accepted models of cold dark matter predict the existence of a very early (decoupled) zero-pressure component, able, by this feature, to develop large-scale structures (at the present time even the heat dark matter is expected to be non-relativistic). Indeed, a non-baryonic component of this kind, is estimated (either by the supernova data, either by the cosmic microwaves background (detected) anisotropy) to be \( \sim 0.3 \) of the actual Universe critical density.

Since in equation (109) \( \beta \) plays the role of the (reduced) action function, we can write, by using Hamilton equations, the following relation

\[
\frac{d\beta}{dR_c} = p_{R_c} = -\frac{3\pi h}{2c l_P^4} \frac{dR_c}{dt}.
\]

Then, remembering that \( H = (dR_c/dt)/R_c \) and \( \Omega - 1 = \epsilon^2 / H^2 R_c^2 \), we see how equation (109) takes the simple form (with obvious notation for the different contributions) \( \sum_i X_i = 1 \), being \( X_i = \Omega_i / \Omega \) (\( i = p, \mu, dm, \text{curv} \)); thus, our dust fluid provides a component of the critical parameter \( \tilde{\Omega}_{dm} \), given by

\[
\tilde{\Omega}_{dm} = -\frac{4l_P^2 c^2 \epsilon}{3\pi h H^2 R_c^4}.
\]

Such a formula is valid in general, independently of the other kinds of matter present in the universe, and, therefore, provides a good tool to investigate the role it could play in the actual cosmology; in this respect, we stress the following three relevant points:

i) If we take for \( \epsilon \) the minimum value of the continuous spectrum obtained in the previous section, within the framework of a “pre-inflationary” scenario, i.e. \( \epsilon \sim \mathcal{O}(-M_P c^2) \), then we get

\[
\tilde{\Omega}_{dm} = \mathcal{O} \left( \frac{l_P c^2 H^{-2}}{R_c^4} \right) \sim \mathcal{O}(10^{-63}).
\]

ii) The value of \( \epsilon \), required to have \( \tilde{\Omega}_{dm} = \mathcal{O}(1) \) (so that it could make account for the real dark matter component, estimated about 0.3 of the actual critical density), corresponds to

\[
\epsilon^* \sim \mathcal{O} \left( \frac{h c R_c^3}{l_P c^2 H^{-2}} \right) \sim \mathcal{O}(10^{82} \text{GeV}) ;
\]

such a value corresponds to the present one of the total energy of the Universe, whether it admits a closed space. A crucial point is that \( \epsilon \) is a constant of the motion and therefore, since the Universe became a classical one, it was characterized by such value \( \epsilon^* \).

---

\(^2\) The same result could be directly obtained by applying the Hamilton-Jacobi method to the full action \( S = \beta(R_c) + p\phi - \epsilon t \).
iii) In order to get an inflationary scenario, able to explain the paradoxes of the Standard Cosmological Model, we need a sufficiently large “e-folding” which allows the size of an horizon, at the inflation beginning, be now of the order of the actual Hubble radius; such a value corresponds, at least, to about 60, i.e. the ratio between the scale factors, respectively, after and before the inflation, is around a factor $O(10^{26})$. This means that, if today $R_c \sim O(10^{28}\text{cm})$, then, taking into account that the redshift of the end of the inflation is about $z \sim O(10^{24})$, we see that when the de-Sitter phase started its value was $R_c \sim O(10^{-22}\text{cm})$. Thus, the total energy of the Universe, when the dynamics became dominated by the “vacuum energy” at the temperature $\sigma \sim O(10^{15}\text{GeV})$, is given by the expression

$$\epsilon_\Lambda \sim \frac{\sigma^4 R_c^3}{h^3 c^5} \sim O(10^{36}\text{GeV}) \ll \epsilon^*; \quad (116)$$

this result seems to indicate that, assuming the Universe underwent an inflationary scenario, we get the contradictory issue about the impossibility of a dominating “vacuum energy”.

5 Concluding Remarks

We have presented a reformulation of the canonical quantization of geometrodynamics with respect to a fixed reference frame; the main goal of our analysis is achieved by removing the fundamental shortcoming of the WDE stated at the point iii) in paragraph 2.2, i.e. now the quantization procedure takes place in a fixed reference frame and no ambiguity survives about the time-like character of the normal field; by other words, in this new approach it is possible to quantize the 3-geometry field on a fixed family of spatial hypersurfaces (corresponding to its evolution in the space-time), because this quantization scheme does not contradict the strong assumption of a (3+1)-slicing of the 4-dimensional manifold.

The main result obtained, including the kinematical action in the global dynamics, is the characterization of an appropriate internal physical clock. In our theory the role of clock is played by the reference fluid, comoving with the 3-hypersurfaces and its presence is necessary to distinguish between space-like and time-like geometrical objects before the canonical quantization procedure.

The fluid shows its presence through a comoving (non-positive defined) density of energy and momentum, which we have characterized either from a classical either from a quantum point of view: classically it comes from having introduced the kinematical action, but its real nature must be investigated in the classical limit of the eigenvalues equations.

It is worth noting that the considerations presented in paragraph 4.3 are against the idea that the here obtained $\bar{\Omega}_{dm}$ can make account for the dark matter, if inflation took place. The situation is different if we take the picture of the Standard Cosmological Model because, for instance, a classical estimation of the thermal bath energy at the Planck epoch is about $O((R_c/l_pl)^3 M_{Pl} c^3) \sim O(10^{112}\text{GeV})$; thus, in absence of inflation, the value of $\epsilon^*$ would have become
important only in the later stage of the Universe evolution and it could play today a relevant dynamical role.

Moreover to be applicable to a generic inhomogeneous gravitational system, the theory here presented has to be reduced, necessarily, to a formulation on a lattice; recently some interesting proposal has appeared to discretize a quantum constraint \cite{9}, \cite{10} and they are of course relevant for the discretization of the present theory. A more direct approach can be obtained applying the Regge calculus \cite{28}, \cite{29}, to the 3-geometries on the spatial hypersurfaces.

\section{Multi-Time Approach}

In this section we provide a schematic formulation of the so-called multi-time approach and of its smeared Schrödinger version, in view of a comparison with the proposal of previous section.

The multi-time formalism is based on the idea that many gravitational degrees of freedom appearing in the classical geometrodynamics have to be not quantized because are not real physical ones; indeed we have to do with $10 \times \infty^3$ variables, i.e. the values of the functions $N, N^k, h_{ij}$ in each point of the hypersurface $\Sigma^3$, but it is well-known that the gravitational field possesses only $4^3$ physical degrees of freedom in the phase space (in fact the gravitational waves have, in each point of the space, only two independent polarizations and satisfy second order equations).

The first step is therefore to extract the real canonical variables by the transformation

$$\{h_{ij}, \pi^{ij}\} \rightarrow \{\xi^\mu, \pi^\mu\} \quad \{H_r, P^r\} \quad \mu = 0, 1, 2, 3 \quad r = 1, 2, 3 \quad (117)$$

where $H_r, P^r$ are the four real degrees of freedom, while $\xi^\mu, \pi^\mu$ play the role of embedding variables.

In terms of this new set of canonical variables, the gravity-“matter” action \cite{24} rewrites

$$S^{g\phi} = \int_{M^4} \left\{ \pi_\mu \partial_t \xi^\mu + P^r \partial_t H_r + \pi_\phi \partial_t \phi - N(H^g + H^\phi) - N^i(H^g_i + H^\phi_i) \right\} d^3x dt, \quad (118)$$

where $H^g = H^g(\xi^\mu, \pi^\mu, H_r, P^r)$ and $H^\phi = H^\phi(\xi^\mu, \pi^\mu, H_r, P^r)$.

Now we provide an ADM reduction of the dynamical problem by solving the hamiltonian constraint for the momenta $\pi^\mu$

$$\pi_\mu + h_\mu(\xi^\mu, H_r, P^r, \phi, \pi_\phi) = 0. \quad (119)$$

Hence the above action takes the reduced form

$$S^{g\phi} = \int_{M^4} \left\{ P^r \partial_t H_r + \pi_\phi \partial_t \phi - h_\mu(\partial_t \xi^\mu) \right\} d^3x dt. \quad (120)$$

Finally the lapse function and the shift vector are fixed by the hamiltonian equations lost with the ADM reduction, as soon as, the functions $\partial_t \xi^\mu$ are
assigned. A choice of particular relevance is to set \( \partial_t \xi^\mu = \delta^\mu_0 \) which leads to

\[
S = \int_{M^4} \left\{ \pi^r \partial_r H_r + \pi_\phi \partial_0 \phi - h_0 \right\} d^3 x dt.
\]

(121)

The canonical quantization of the model follows by replacing all the Poisson brackets with the corresponding commutators; if we assume that the states of the quantum system are represented by a wave functional \( \Psi = \Psi(\xi^\mu, H_r, \phi) \), then the evolution is described by the equations

\[
i \hbar \frac{\delta \Psi}{\delta \xi^\mu} = \hat{h}_\mu \Psi,
\]

(122)

where \( \hat{h}_\mu \) are the operator version of the classical hamiltonian densities.

In its smeared formulation the multi-time approach reduces to the following Schrödinger equation

\[
i \hbar \partial_t \Psi = \hat{\langle} \Psi \Psi = \Psi(t, H_r, \phi).
\]

(123)

Here \( \hat{\langle} \) denote the quantum correspondence to the smeared hamiltonian

\[
\langle = \int_{M^4} \{ h_\mu \partial_t \xi^\mu \} d^3 x dt.
\]

(124)

Now, observing that the first of equations can be rewritten as follows

\[
i \hbar \frac{\delta \Psi}{\delta y^\mu} = -n_\mu(\hat{H}^g + \hat{H}^\phi) \Psi,
\]

(125)

it exists a correspondence between the above multi-time approach and our proposal, viewed by identifying the formulas (59)-(121), (125)-(122) and (91)-(123).

But the following two key differences appear evident: i) the embedding variables \( y^\mu \) are added by hand, while the corresponding ones \( \xi^\mu \) come from non-physical degrees of freedom; ii) the hamiltonians \( H \) and \( \langle \) (as well as their corresponding densities) describe very different dynamical situations.

We show explicitly the parallel between these two approaches by their implementation in a minisuperspace model: a Bianchi type IX Universe containing a self-interacting scalar field. By using Misner variables \( (\alpha, \beta_+, \beta_-) \) the classical action describing this system reads:

\[
S = \int \left\{ p_\alpha \dot{\alpha} + p_{\beta_+} \dot{\beta}_+ + p_{\beta_-} \dot{\beta}_- + p_\phi \dot{\phi} - cNe^{-3\alpha} \times 
\]

\[
\times - p_\alpha^2 + p_{\beta_+}^2 + p_{\beta_-}^2 + p_\phi^2 + V(\alpha, \beta_\pm, \phi) \right\} dt, \quad c = \text{const},
\]

(126)

where \( (\ldots) \equiv \frac{d (\ldots)}{dt} \) and the precise form of the potential term \( V \) is not relevant for our discussion.
For this model, since the Hamiltonian density is independent of the spatial coordinates, then the multi-time approach and its smeared Schrödinger version overlap, the same being true in our formalism.

In the spirit of our proposal the quantum dynamic of this model is described by the equation

$$i\hbar \partial_t \Psi = c Ne^{-3\alpha} \hbar^2 \left\{ \partial_\alpha^2 - \partial_{\beta_+}^2 - \partial_{\beta_-}^2 - \partial_\phi^2 + V \right\} \Psi, \quad \Psi = \Psi(t, \alpha, \beta_\pm, \phi), \quad (127)$$

to which it should be added the restriction that the initial wave function phase $\sigma_0 = \sigma_0(\alpha, \beta_\pm, \phi)$ satisfies the Hamilton-Jacobi equation

$$\left\{ -(\partial_\alpha)^2 + (\partial_{\beta_+})^2 + (\partial_{\beta_-})^2 + (\partial_\phi)^2 \right\} \sigma_0 + V = 0. \quad (128)$$

In this scheme $N(t)$ is an arbitrary function of the label time to be specified when fixing a reference.

To set up the multi-time approach we have to preliminarily perform an ADM reduction of the dynamics (126). By solving the Hamiltonian constraint obtained varying $N$, we find the relation

$$-p_\alpha \equiv h_{ADM} = \sqrt{p_{\beta_+}^2 + p_{\beta_-}^2 + p_\phi^2 + V}. \quad (129)$$

Therefore action (126) rewrites as

$$S = \int \left\{ p_{\beta_+} \dot{\beta}_+ + p_{\beta_-} \dot{\beta}_- + p_\phi \dot{\phi} - \dot{\alpha} h_{ADM} \right\} dt. \quad (130)$$

Thus we see how $\alpha$ plays the role of an embedding variable (indeed it is related to the Universe volume), while $\beta_\pm$ are the real gravitational degrees of freedom (they describe the Universe anisotropy).

By one of the Hamiltonian equation lost in the ADM reduction (i.e. when varying $p_\alpha$ in (126)), we get

$$\dot{\alpha} = -2c Ne^{-3\alpha} p_\alpha = 2c Ne^{-3\alpha} h_{ADM}. \quad (131)$$

Hence by setting $\dot{\alpha} = 1$, we fix the lapse function as

$$N = \frac{e^{3\alpha}}{2ch_{ADM}}. \quad (132)$$

The quantum dynamics in the multi-time approach is summarized by the equation

$$i\hbar \partial_\alpha \Psi = \sqrt{-h^2(\partial_{\beta_+}^2 + \partial_{\beta_-}^2 + \partial_\phi^2) + V} \Psi, \quad \Psi = \Psi(\alpha, \beta_\pm, \phi). \quad (133)$$

We stress that in this multi-time approach the variable $\alpha$, i.e. the volume of the Universe, behaves as a “time”-coordinate and therefore the quantum dynamics cannot avoid the Universe reaches the cosmological singularity ($\alpha \to -\infty$). On the other hand, in the formalism we proposed, $\alpha$ is on the same footing of the other variables and are admissible “stationary states” for which it is distributed
in probabilistic way. This feature reflects a more general and fundamental difference existing between the two approaches: the multi-time formalism violates the geometrical nature of the gravitational field in view of real physical degrees of freedom, while the proposed quantum dynamics implements this idea only up to the lapse function and the shift vector, but preserves the geometrical origin of the 3-metric field.

References

[1] R. Arnowitt, S. Deser, C. Misner, *Dynamical structure and definition of energy in general relativity*, (1959), *Phys. Rev.* **116**, 1322.

[2] R. Arnowitt, S. Deser, C. Misner, *Canonical variables for general relativity*, (1960), *Phys. Rev.* **117**, 1595.

[3] R. Arnowitt, S. Deser, C. Misner, *Consistency of the canonical reduction of general relativity*, (1960), *J. Math. Phys.* **1**, 434.

[4] R. Arnowitt, S. Deser, C. Misner, *The dynamics of general relativity*, in L. Witten ed., ‘Gravitation: An Introduction to Current Research’, Wiley, New York, pp. 227-265, (1962).

[5] J. Bicak, K. Kuchař, *Null dust in canonical gravity*, (1995), *Phys. Rev.* **D51**, 5600.

[6] N.D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, (1982), Cambridge University Press, Cambridge.

[7] B.S. De Witt, *Quantum theory of gravity. I. The canonical theory*, (1967), *Phys. Rev.* **160**, 1113.

[8] B.S. De Witt, *The quantum and gravity: the Wheeler-De Witt equation*, Jerusalem, 22-27 June 1997, *Proc. eighth Marcell Grossmann meeting*, ed T. Piran, 6.

[9] C. DiBartolo, R. Gambini, J. Pullin *Canonical quantization of constrained theories on discrete space-time lattices*, (2002), *Class. Quant. Grav.*, textbf19, 5275, available on gr-qc/0205123.

[10] R. Gambini, J. Pullin, *Canonical quantization of general relativity in discrete space-times*, (2002), *Class. Quant. Grav.*, textbf20, 3341, available gr-qc/0206055.

[11] J.B. Hartle, in *Conceptual Problems of Quantum Gravity*, (1991), edited by A. Ashtekar and J. Stachel (Birkhauser, Boston).

[12] J.B. Hartle, in *Highlights in Gravitation and Cosmology*, (1988), eds B. Iver et al., Cambrigde Univ. Press.
[13] J.B. Hartle and S.W. Hawking, *Wave function of the Universe*, (1983), Phys. Rev. D28, 2960.

[14] C.J. Isham, *Canonical Quantum Gravity and the Problem of Time*, (1992), available gr-qc/9201011.

[15] A.A. Kirillov, G. Montani, *Origin of a classical space in quantum inhomogeneous models*, (1997), JETP Lett. 66, 7, 475;

[16] E.W. Kolb, M.S. Turner, *The Early Universe*, (1990), Adison-Wesley, Reading, 447.

[17] K. Kuchař, *Canonical quantization of gravity*, (1973), Relativity, Astrophysics and Cosmology, Reidel, Dordrecht, pp.237-288.

[18] K. Kuchař, *Canonical methods of quantisation*, (1981), C. Isham, R. Penrose, D. Sciama, eds, ‘Quantum Gravity 2: A Second Oxford Symposium’, Clarendon Press, Oxford, pp.329-374.

[19] K. Kuchař, *The problem of time in canonical quantization*, (1991), A. Ashtekar, J. Stachel, eds, ‘Conceptual Problems of Quantum Gravity’, Birkhäuser, Boston, pp.141-171.

[20] K. Kuchař, *Extrinsic curvature as a reference fluid in canonical gravity*, (1992), Phys. Rev. D45, 4443.

[21] K. Kuchař, *Time and interpretations of quantum gravity*, (1992), Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, World Scientific, Singapore.

[22] K. Kuchař, C. Torre, *Gaussian reference fluid and the interpretation of geometrodynamics*, (1991), Phys. Rev. D43, 419.

[23] S. Mercuri, G. Montani, *Revised Canonical Quantum Gravity via the Frame Fixing*, (2003), to appear on Int. Jour. Mod. Phys. D, available on gr-qc/0310077.

[24] S. Mercuri, G. Montani, *Dualism between Physical Frames and Time in Quantum Gravity*, (2003), submitted to Class. Quant. Grav., available on gr-qc/0312077.

[25] W. Misner, K. Thorne, J.A. Wheeler, *Gravitation*, (1973), Freeman, San Francisco, Ch. 43, Ch. 21.

[26] G. Montani, *Canonical quantization of gravity without “frozen formalism”*, (2002), Nucl. Phys. B634, 370, available on gr-qc/0205032.

[27] D. Oriti, *Spacetime from algebra: spin foam models for non-perturbative quantum gravity*, (2001), to appear in Rep. Prog. Phys., available gr-qc/0106091.
[28] T. Regge, *General relativity without coordinates*, (1961), Nuovo Cimento 19, 558.

[29] T. Regge, Jerusalem, 22-27 June 1997, *Discrete gravity*, Proceedings Eighth Marcell Grossmann meeting, ed T. Piran, 2.

[30] C. Rovelli, *What is observable in classical and quantum gravity?*, (1991), Class. Quant. Grav. 8, 297.

[31] C. Rovelli, *Quantum reference system*, (1991), Class. Quant. Grav. 8, 317.

[32] C. Rovelli and L. Smolin, (1993), *The physical hamiltonian in non perturbative quantum gravity*, Phys. Rev. Lett. 72, 446, available on gr-qc/9308002.

[33] L. Smolin, (1993), *Time, measurement and information loss in quantum cosmology*, available on gr-qc/9301016.

[34] T. Thiemann, *Introduction to modern canonical quantum general relativity*, (2001), available gr-qc/0110034.

[35] R.M. Wald, *General Relativity*, 1984, The university of Chicago press.

[36] J. Zinn-Justin, (1996), *Quantum Field Theory and Critical Phenomena*, third edition by Clarendon Press, Oxford, 228.