Dephasing of entangled qubits

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March 22, 2022

Abstract

We investigate the influence of nearby two-level systems on the dynamics of a qubit. The intrinsic decoherence is given by a coupling of both the qubit and the two-level systems to a heat bath. Assuming weak interactions between the qubit and the two-level systems we show that the dephasing of the qubit follows a multi-exponential decay. For a flat distribution of energy splittings of the two-level systems the multi-exponential behavior simplifies to a good approximation to a single exponential decay. The rate is given by the sum of the dephasing rate of the isolated qubit plus all relaxation rates for the two-level systems out of their present state. This latter contribution sums up to a rate approximately given by the mean coupling between the qubit and the two-level systems times the number of thermally active neighbors.

In recent years, extensive studies have been done on quantum computing. One of the major problems to overcome in that field is the decoherence of the qubit dynamics. Besides the dephasing due to experimental imperfections the interactions with the environment yields an intrinsic source of decoherence [1, 2, 3, 4]. The so-called quality factor of the qubit, the number of quantum operations performed during the qubit coherence time, should be at least $10^4$ to allow for quantum error correction [5]. We will not discuss the various physical sources of decoherence. Instead we investigate a model where the qubit is weakly coupled to defects described as two-level systems (TLSs). Since the qubits are also described as TLSs, we do formally not distinguish between qubits and defects. All TLSs/qubits are coupled to a heat bath represented by a set of harmonic oscillators providing the basic source of relaxation and decoherence. Both result from the energy exchange with the heat bath like in the spin-boson model [6] which is different from the model proposed by Unruh [7]. Our investigation focuses on the question how neighboring TLSs influence the dephasing behavior of a given qubit. In the following we use qubit to address the TLS of interest and TLS for all the systems which form a background for the qubit. Qubits which are not used for an actual calculation contributes in that picture as a background.

We restrict our investigation to longitudinally coupled TLSs leading to the Hamiltonian

$$H_S = -\sum_i \frac{\Delta_i}{2} \sigma_{z}^{(i)} - \sum_{i \neq j} J_{ij} \sigma_{z}^{(i)} \sigma_{z}^{(j)}$$

(1)

with the Pauli matrices $\sigma_i$, the energy splittings $\Delta_i$ and the interaction constants $J_{ij}$. We allow for distributed splittings and couplings. The many-body eigenstates are given by the product states of the isolated TLSs. Nevertheless, the longitudinal coupling changes the energies of the excitations, thus leading to entanglement between the quantum states of the noninteracting qubits.

The bosons of the heat bath, $H_B = \sum_k \omega_k b_k^\dagger b_k$, couples with their strain field linear to the dipole operator $\sigma_{z}^{(i)}$ of the two-level systems

$$H_{SB} = \sum_i \sigma_{z}^{(i)} \cdot \sum_k \lambda_k^{(i)} (b_k + b_{-k}^\dagger).$$

(2)

This coupling reflects the natural picture that the two-level system can change its state via an absorption or an emission of a boson.
For a single qubit this is the widely investigated spin-boson model\cite{6, 7}. The case of a transversely coupled pair is investigated by Terzidis et al. \cite{8} and Dubé et al. \cite{9}. It is well known that a coupling to bosons leads to an additional transversal coupling between the two-level systems of the form \( \tilde{J}_{ij}\sigma_z^{(i)}\sigma_z^{(j)} \) through virtual phonon exchange \cite{12, 13}. We assume in the following that we can neglect that additional coupling. We focus on the dissipative dynamics generated by the coupling to the environment and thereby, on the dephasing times. The renormalization of the excitation energies are beyond the scope of the present paper. We treat the coupling within a resumed perturbative approach \cite{10, 11} as introduced by Würger which is a weak coupling approach.

For sufficiently dilute two-level systems, the time evolution of the boson operators is not affected by the two-level systems and the effect of the heat bath is entirely characterized by the coupled density of states

\[
J_i(\omega) = \frac{\pi}{2} \sum_k |\lambda_k^{(i)}|^2 \delta(\omega - \omega_k) 
\]

\[
\simeq \pi \alpha^{(i)} \omega_c^{2-s} \omega^s \Theta(\omega_C - \omega).
\]

with the cut-off frequency \( \omega_c \) of the bath and a dimensionless coupling constant \( \alpha^{(i)} \). The right hand side of eq. (3) assumes a linear dispersion of the bosons with a sharp upper cut-off. An exponent \( s = 1 \) describes systems exhibiting ohmic dissipation \cite{1} and accordingly the bath is called ohmic. An exponent \( s > 1 \) is usually called a super-ohmic bath where especially \( s = 3 \) is relevant for a defect in a solid interacting with acoustic phonons.

Let’s assume that TLS 1 is our qubit and we prepare it in a superposition between up and down. Accordingly TLS 2 is a defect. For an isolated qubit the decay of coherence is given by the probability of a boson absorption or emission. At temperature below the energy splitting \( \Delta_1 \) of the qubit, phonons with energy \( \Delta_1 \) are rare and the main effect is given by spontaneous emission. However, for a pair of coupled TLSs it is necessary to have information on the state of the second TLS. If it is in its down-state, boson absorption of the second qubit would also lead to decoherence for the first qubit since the original (coupled) state is altered. If the second qubit is in its up-state, boson emission of the second qubit contributes to the decoherence of the first one. This reflects the entanglement between the two TLSs. For two entangled qubits performing a quantum calculation, it is obvious that the calculation is lost if one of the TLSs dephases. Accordingly the dephasing rate of the pair is the sum of the rates for the two isolated qubits. (This argument holds only true as long as the interaction does not change the quantum mechanical evolution of the qubits meaning \( J \ll \Delta_i \).) Since the above argument was independent of the strength of the interaction between the TLSs, the question arises, how we can avoid entanglement of a qubit with surrounding TLSs which do not take part in the actual computation.

To answer this question we derive the memory kernel, resulting from the coupling to the heat bath, for the dynamics of the dipole operator \( \sigma_z^{(i)} \) of the first qubit. Following the approach in ref. \cite{10, 11} we calculate the memory kernel within the lowest contributing order of the coupling between system and bath and apply a Markov approximation. Neglecting the real part of the memory kernel, which represents the renormalization of the energy splittings, we end up with the rates for the various dynamic processes. We restrict ourselves to the subspace of the operators, \(|\uparrow\downarrow\rangle\langle\downarrow\downarrow|\) and \(|\uparrow\uparrow\rangle\langle\downarrow\uparrow|\), which correspond to a transition from down to up for the first qubit or, in other words, to a preparation in a superposition between up- and down-state. The dynamics of this subspace is independent from the rest space and the time evolution of the subspace of the complex conjugate operators is identical.

The dynamics of the operators \(|\uparrow\downarrow\rangle\langle\downarrow\downarrow|\) and \(|\uparrow\uparrow\rangle\langle\downarrow\uparrow|\) are not independent from each other since the off-diagonal components of the memory kernel are non zero.

The time-evolution operator in Laplace space for this subspace (in the basis of the above given operators) is
given by
\[ U(z) = \left( z - \omega_+ + i(\Gamma_1 + 2\Gamma_{2\uparrow}) \right)^{-1} \]

with the excitation energies \( \omega_{\mp} = \Delta_{1\mp} \mp 2J \approx \Delta_1 \), the usual one-phonon rate of the first qubit \( \Gamma_1 = J_1(\Delta_1) \coth(\beta\Delta_1/2) \), the decay rate of the second qubit due to phonon emission \( \Gamma_{2\downarrow} = J_2(\Delta_2)(1+n(\Delta_2)) \) with the Bose factor \( n(\Delta_2) \) and the decay rate of the second qubit due to phonon absorption \( \Gamma_{2\uparrow} = J_2(\Delta_2)n(\Delta_2) \). Thereby \( J_{1/2}(\Delta_{1/2}) \) are the coupled density of states for the two TLSs (compare eq. (3)). The phase coherence rates are given by the imaginary parts of the two poles of the time evolution operator.

This dynamics has two simple extremes. When the difference of the two excitation energies \( |\omega_+ - \omega_-| = 4J \) is bigger than the off-diagonal components, we can neglect the off-diagonal entries. In this case, the dynamics of qubit 1 has two different rates which depend on the state of qubit 2. Therefore, the rates are determined by both processes: decay of the qubit 1 and decay of the qubit 2.

In the second case, \( J < \max\{\Gamma_{2\downarrow}, \Gamma_{2\uparrow}\} = \Gamma_{2\downarrow} \), we have to taken into account the off-diagonal entries. The off-diagonal elements cause that the eigenstates of the time evolution do not correspond to the eigenstates of the Hamiltonian. Instead the dynamics of the two qubits completely decouples and the coupling is negligible (for the dynamics of the qubit 1) as long as it is smaller than the linewidth of the states of the isolated qubit 2 which is given by \((\Gamma_{2\uparrow} + \Gamma_{2\downarrow})/2\).

Thus, only when the coupling between a qubit and a second TLS is smaller than the linewidth of the second TLS the dynamics of the qubit is independent from the second TLS. We should point out that the actual quantum mechanical evolution of the states are only weakly affected since \( J/\Delta \ll 1 \). But the dephasing rate might be changed considerably since we have no knowledge about \( \Gamma_1 \gg \Gamma_{2\downarrow} \).

In the case of many qubits the above conclusion remains qualitatively valid. The decoherence rate of a qubit of interest depends on the actual states of all other TLSs. If we have a distribution of interaction energies, we have to devise all TLSs into two groups depending on the condition \( J_{\alpha i} < \Gamma_{i\downarrow} \) or \( J_{\alpha i} > \Gamma_{i\downarrow} \) where \( \alpha \) is the index of the qubit of interest.

If \( J_{\alpha i} < \Gamma_{i\downarrow} \) the i-th TLS does not influence the qubit \( \alpha \). Nevertheless a flip of the qubit \( i \) will change the resonance energy of the qubit \( \alpha \). Since that change is smaller than the linewidth, we would not be able to measure it. In the case of many neighbors fulfilling that condition, the changes might add up resulting in a change in the resonance energy which is bigger than the linewidth. In an echo experiment at an ensemble of qubits, this leads to an 'effective' dephasing where the relative phase between the qubits decays. This effect is called 'spectral diffusion' \([14]\).

All neighbors in the second group with \( J_{\alpha i} > \Gamma_{i\downarrow} \) form an entangled cluster with the qubit \( \alpha \). As in the case of a pair, the decoherence rate of a superposition of eigenstates of the qubit \( \alpha \), with all the TLSs in an eigenstate, gets:
\[ \Gamma_\ell = \Gamma_\alpha + 2 \sum_{\substack{i \in \{i: J_{\alpha i} > \Gamma_{i\downarrow}\} \setminus \{\ell\} \}} \frac{\Gamma_{i\uparrow}}{\Gamma_{i\downarrow}} \text{TLS } i \text{ in the down-state} \]
\[ \frac{\Gamma_{i\uparrow}}{\Gamma_{i\downarrow}} \text{for TLS } i \text{ in the up-state} \]

Thereby, we assumed a clear separation between \( J_{\alpha i} \) and \( \Gamma_{i\downarrow} \). If both are of the same order of magnitude, the time evolution is a complicated mixture of both extremes. This assumption should at least be valid for the majority of the TLSs. If the TLSs are qubits like the qubit \( \alpha \), the decoherence rate scales with the number of qubits. This seems to be natural for a situation where \( N \) entangled qubits perform one calculation. If only one of these qubits dephases, the calculation is lost. The main point lies in the fact that they are entangled as long as the interaction between two qubits exceeds their linewidth. For example, if one wants to send quantum information from one to another quantum computer, the interaction must be bigger than the linewidth since otherwise, the quantum computers would dephase even before they are entangled (starting from an isolated initial condition). After the transmission both computers must be separated again fulfilling the above condition.

Within the picture of one qubit surrounded by TLSs, we have to specify the distributions of couplings \( P(J) \) and of the resonance energies \( Q(\Delta) \) in order to get quantitative results.
Assuming homogeneously distributed TLSs and a dipole-dipole like interaction between TLSs we obtain for the distribution of couplings

\[ P(J) = \frac{\bar{J}}{J^2} \tag{6} \]

between a minimal and a maximal value of the coupling with equal probability of both signs. \( \bar{J} \approx NJ_{\text{min}} \) normalizes the distribution to the number of TLSs \( N \). Since \( \bar{J} \approx \langle |J| \rangle = \bar{J} \ln(J_{\text{max}}/J_{\text{min}}) \), we refer to \( \bar{J} \) in the following as the mean coupling.

We want to consider a case in which all neighbors are equivalent

\[ Q(\Delta) = \delta(\Delta - \bar{\Delta}) . \tag{7} \]

Without further knowledge over the neighbors we expect them in thermal equilibrium with the environment. In that case the average decay of the qubit \( \alpha \) follows the form \( (J_i \ll \bar{\Delta}) \)

\[ \langle e^{-\Gamma t} \rangle \approx e^{-\Gamma_{\alpha} t} \cdot \left\{ p^- e^{-2\bar{J}t} + p^+ e^{-2e^{-\beta \bar{\Delta}}\bar{J}t} \right\} \tag{8} \]

with the occupation probabilities of the ground/excited state \( p^{\pm} = \exp(\pm \beta \Delta / 2) / \cosh(\beta \bar{\Delta} / 2) \).

This simple result has few remarkable features. First of all, we have no longer a simple exponential decay but a sum of two. The original decay of the qubit \( \alpha \) is still present in the prefactor. The decay of the TLSs lead to the term in the brackets. This term does not depend on the actual decay rates of the TLSs but only on their mean coupling \( \bar{J} \) to the qubit \( \alpha \) and the probabilities of these neighbors to be in their ground or excited state. If the temperature is higher than the energy splitting of the TLSs \( (k_B T \ll \bar{\Delta}) \) both rates are equal to a good approximation. In the other limit \( (k_B T \gg \bar{\Delta}) \) only one decay is relevant and its contribution from the neighboring TLSs is vanishing. Thus only thermally active TLSs are affecting the dephasing of the qubit. Actually two decays are only seen in the case where the temperature matches the energy splitting.

In the second step, you might consider neighbors with broadly distributed energy splittings. If we assume, for example, just a flat distribution \( Q(\Delta) = Q_0 \Theta(\Delta_{\text{min}} - \Delta) \Theta(\Delta - \Delta_{\text{max}}) \) between a minimal and a maximal value, we get

\[ \langle e^{-\Gamma t} \rangle \approx e^{-\Gamma_{\alpha} t} \cdot \prod_i \left\{ p_i^- e^{-2\bar{J}_i t} + p_i^+ e^{-2e^{-\beta \Delta_i} \bar{J}_i t} \right\} \tag{9} \]

where \( i \) runs over all neighbors and \( \bar{J}_i = Q_0 J_{\text{min}} \), with the normalization factor \( Q_0 \). In order to get simple analytical expressions we approximate the exponential by \( \exp(-\bar{\Delta}_i) \approx \Theta(T - \bar{\Delta}_i) \). This approximation fails for all systems with \( \Delta_i \simeq T \). If the majority of TLSs, as in the case of a flat distribution of energy splittings, does not fulfill that condition, we expect that the approximation works pretty well and we get

\[ \langle e^{-\Gamma t} \rangle \approx e^{-\Gamma_{\alpha} t - 2\bar{J}(T) t} \tag{10} \]

with \( \bar{J}(T) = Q_0 (k_B T - \Delta_{\text{min}}) NJ_{\text{min}} \). This shows an additional dephasing rate which is approximately linear in temperature as long as the temperature is between the extremal values of the distribution \( Q(\Delta) \).

The dephasing due to nearby local defects is given by the mean coupling to the thermal active neighbors. In fact, the above used approximation divides the TLSs sharply in thermal active and inactive ones. Thereby we loose the multi-exponential behavior of the correct solution.

We investigated the dephasing in a model of weakly coupled TLSs. The coupling was chosen only to influence the energies of the excitations and thus to ensure entanglement. It does not change the states in a sense that the eigenstates of a coupled cluster are the product states of the isolated TLSs. In order to cause dephasing we coupled all TLSs to a heat bath in a way that dephasing as well as relaxation is caused through energy exchange with the bath.

We find that the dephasing of a qubit weakly coupled to \( N \) TLSs is no longer a simple exponential decay but a complicated multi-exponential decay. We investigated two simple cases. In both, we assumed a homogeneous distribution of TLSs and a dipole type coupling between the qubit and the TLSs.

In the case of identical TLSs, we get two exponential decays instead of one. If the temperature is higher than the energy splitting of the TLSs \( (k_B T \gg \bar{\Delta}) \), both rates are equal. In the other limit \( (k_B T \ll \bar{\Delta}) \) only one
decay is relevant and its contribution from the neighboring TLSs is vanishing. Thus only thermally active TLSs are affecting the dephasing of the qubit. Actually the two decays are only seen in the case where the temperature matches the energy splitting.

On the other hand, a flat distribution of energy splittings for the TLSs leads to a multi exponential decay. Nevertheless, we end up with one major dephasing rate where all thermally active TLSs are contributing to. The dephasing through local defects is thereby given by the mean interaction with all thermally active defects. Thus, the additional dephasing is independent from the actual decay mechanisme of the TLSs. The above results are valid as long as the linewidth of the TLSs is smaller then their coupling to the qubit. This fact opens another way to avoid dephasing due to nearby TLSs by increasing the linewidth of the thermal active TLSs.

The author wants to thank H. Horner, B. Thimmel, Y. Lee and W. Harrison for many stimulating and clarifying discussion. Parts of the work were done in the Institut für Theoretische Physik at the University of Heidelberg. I want to thank the DFG which supported that part within the project HO 766/5-3 'Wechselwirkende Tunnelssysteme in Gläsern und Kristallen bei tiefen Temperaturen’. I also want to thank the Alexander-von-Humboldt foundation which supports the work done at Stanford.

References

[1] W.G. Unruh: Phys. Rev. A 51 (1995) 992
[2] A. Garg: Phys. Rev. Lett. 77 (1996) 964
[3] G. Massimo Palma, K.-A. Suominen, A. Ekert: Proc. R. Soc. Lond. A 452 (1996) 567
[4] Yu. Makhlin, G. Schöen, A. Shnirman: Rev. Mod. Phys. (2001) in press
[5] J. Preskill: Proc. R. Soc. Lond. A 454 (1998) 385
[6] A.J. Leggett, S. Chakravarty, A.T. Dorsey, M. Fisher, A. Garg, W. Zwerger: Rev. Mod. Phys. 59 (1987) 1
[7] U. Weiss: Quantum Dissipative Systems, Series in Modern Condensed Matter Physics, Vol. 2, World Scientific, Singapore (1993)
[8] O. Terzidis, A. Würger: J. Phys.: Condens. Matter 8 (1996) 7303
[9] M. Dubé, P.C.E. Stamp: Int. J. Mod. Phys. B 12 (1998) 1191;
[10] P. Nalbach: to be submitted to Phys. Rev. B
[11] A. Würger: J. Phys.: Condens. Matter 9 (1997) 5543
[12] J. Joffrin, A. Levelut: J. Phys. (Paris) 36 (1975) 811
[13] K. Kassner: Z. Phys. B 81 (1990) 245
[14] J.R. Klauder, P.W. Anderson: Phys. Rev. 125 (1962) 912
Fig. 1: The spectrum of two coupled TLSs. The states are labeled by their composition of the up and down states of the two isolated TLSs. The possible phonon transitions are shown by the arrows. The full (dashed) arrows are transitions induced by interactions with the TLS 1 (TLS 2).