A pseudospectral study of a potentially singular solution of the three-dimensional axisymmetric incompressible Euler equation:
ygers and thermalization

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Abstract

Luo and Hou [1] have investigated a potentially singular solution of the radially bounded, 3D, axisymmetric Euler equations via a hybrid, 6th-order Galerkin and 6th-order finite-difference method on an adaptive (moving) mesh. Given the importance of this problem, we have developed a pseudospectral, Fourier-Chebyshev scheme to study this solution by an independent numerical technique, which leads to new insights into this (potentially) singular solution: (a) the time of singularity is preceded, in any, practical, spectrally truncated direct numerical simulation (DNS), by the formation of structures called tygers, first investigated in the one-dimensional (1D) Burgers and two-dimensional (2D) Euler equations; (b) these tygers herald the thermalization of our spectrally truncated system of equations. We demonstrate how we can analyse this (potential) singularity by using singularity-detection criteria based on (a) the Beale-Kato-Majda theorem [2, 3] and (b) a generalization of the analyticity-strip method [4]. We find that the width of the analyticity strip, in the axial direction, decays faster than an exponential and, in the radial direction, it decays linearly; this linear decay can be extrapolated to obtain an estimate for the time at which the (potential) singularity occurs; our estimate is consistent with that of Ref. [1]. We examine the DNS-resolution dependence of (a) the tyger-formation time, (b) thermalization, (c) energy and helicity conservation, and (d) our estimate for the singularity time.

Two hundred and sixty five years ago, Euler introduced the equations for an inviscid, incompressible, three-dimensional (3D) fluid in Principes généraux du mouvement des fluides [5–7]. The incompressible Euler partial differential equation (PDE) and its viscous descendant, namely, the incompressible Navier-Stokes PDE [8, 9] govern, respectively, ideal and viscous fluid flows at low Mach numbers. They are, therefore, among the most prominent equations in theoretical physics; and their solutions are of importance in a variety of engineering, fluid-dynamical, astrophysical, geophysical, and laboratory settings. Furthermore, these equations pose challenges for mathematicians: In particular, we do not know with certainty if any solutions of these 3D equations develop a singularity in a finite time, if we start with analytic initial data (for non-analytic initial data, see Ref. [10]). The answer to this grand-challenge problem has important implications for turbulence in fluids, even if we use the 3D Euler PDE, as conjectured by Onsager [11] (see, e.g., Ref. [12, 13]); for the Navier-Stokes case, this is one of the Clay mathematics problems [14]. Explorations of this problem (for the Euler case see, e.g., Refs. [7, 15]) often use direct numerical simulations (DNSs), which have not, so far, yielded unambiguous results for or against a finite-time singularity in the 3D Euler PDE with any analytic initial velocity field.

A study by Luo and Hou [1] has explored a potentially singular solution of the radially bounded, 3D, axisymmetric Euler equations via a hybrid, 6th-order Galerkin and 6th-order finite-difference method on an adaptive (moving) mesh. Given the importance of this problem, it behooves us to study this potentially singular solution by a completely different numerical scheme. Therefore, we have developed a pseudospectral, Fourier-
Chebyshev scheme to study this problem; in any numerical implementation of such a scheme, we can only use a finite number of Fourier-Chebyshev modes, i.e., we have a spectrally truncated system. Our method leads to new insights that include the formation of structures called tygers, first introduced in the context of the one-dimensional (1D) Burgers and two-dimensional (2D) Euler equations, en route to thermalization, which has been studied in spectrally truncated direct numerical simulations (DNSs) of a variety of hydrodynamical equations including the 1D Burgers \cite{16–19}, 3D Euler \cite{20}, and 3D and 2D Gross-Pitaevskii \cite{21, 22} equations.

To analyse this (potential) singularity we use singularity-detection criteria based on (a) the Beale-Kato-Majda \cite{2, 3} (BKM) theorem and (b) the analyticity-strip method \cite{3, 4, 23–25}. The latter method is well suited to our pseudospectral study. Before the appearance of tygers, our method yields spectral convergence to the 3D Euler PDE we consider; soon after the birth of tygers, our spectrally truncated system moves towards thermalization and it does not provide a good representation of this PDE; nevertheless, we show below that a careful application of the analyticity-strip method helps us to uncover signatures of this potential singularity. Our results are, indeed, consistent with a finite-time singularity for this initial condition, which was first proposed in Ref. \cite{1}.

The remainder of the paper is organised as follows: In Section I we define the models that we use. In Section II we describe our numerical schemes, and singularity-detection methods. Section III is devoted to a description of our results. We discuss the significance of our results in Section IV.

\section{Models}

The 3D Euler PDEs, for an incompressible, inviscid fluid can be written in the following vorticity-stream-function form:

\[
\begin{align*}
\omega_t + u \cdot \nabla \omega &= \omega \cdot \nabla u; \\
\omega &= \nabla \times u; \\
u &= \nabla \times \psi; \\
\end{align*}
\]

(1)

here, \(\omega\) is the vorticity and \(\psi\) the vector-valued stream function.

For axisymmetric flows, we use

\[
u(r, z) = u^r(r, z) \hat{e}_r + u^\theta(r, z) \hat{e}_\theta + u^z(r, z) \hat{e}_z,
\]

(2)

where \(\hat{e}_r, \hat{e}_\theta,\) and \(\hat{e}_z\) are unit vectors in the cylindrical coordinate system. The axisymmetric Euler PDEs can be reduced to a system of equations, for the angular components \(u^\theta, \omega^\theta,\) and \(\psi^\theta.\) These equations have a formal singularity at \(r = 0,\) which can be removed by introducing the following \cite{26}:

\[
\begin{align*}
u^1 &= u^\theta / r, \\
\omega^1 &= \omega^\theta / r, \\
\psi^1 &= \psi^\theta / r,
\end{align*}
\]

(3)
in terms of which the 3D Euler PDEs are

\begin{align}
    u_1^t + u^r u_1^r + u^z u_1^z &= 2u_1^1 \psi_1^z, \\
    \omega_1^t + u^r \omega_1^r + u^z \omega_1^z &= (u_1^1)^2_z, \\
    -\left(\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2\right) \psi_1 = \omega_1,
\end{align}

with, respectively, the following radial and axial velocity components:

\begin{align}
    u^r &= -r \psi_z^1, \\
    u^z &= 2 \psi_1^1 + r \psi_r^1.
\end{align}

The variables $u^1, \omega^1$ and $\psi^1$ are well defined, so long as the solutions to (4) are smooth; $u^\theta, \omega^\theta$, and $\psi^\theta$ must all vanish at $r = 0$ for these solutions to remain smooth [27].

We solve (4) in the domain

\[ D(1, L) = \{(r, z) : 0 \leq r \leq 1, 0 \leq z \leq L\}; \]

we use periodic boundary conditions in $z$:

\begin{align}
    u^1(r, 0, t) &= u^1(r, L, t); \\
    \omega^1(r, 0, t) &= \omega^1(r, L, t); \\
    \psi^1(r, 0, t) &= \psi^1(r, L, t);
\end{align}

we use the no-flow boundary condition at $r = 1$:

\[ \psi^1(1, z, t) = 0; \]

and we enforce the pole condition [1] at the symmetry axis $r = 0$:

\[ \partial_r u^1(0, z, t) = \partial_r \omega^1(0, z, t) = \partial_r \psi^1(0, z, t) = 0. \]

We employ the initial data of Ref. [1]:

\begin{align}
    u^1(r, z) &= 100 e^{-30(1-r^2)^4} \sin\left(\frac{2\pi z}{L}\right); \\
    \omega^1(r, z) &= \psi^1(r, z) = 0.
\end{align}

Reference [1] utilizes the symmetry properties of this initial condition to study the Euler PDEs in the domain $D(1, L/4)$; in our Fourier-Chebyshev method we use the full length $L$ of the domain.

We have also studied the following one-dimensional (1D) model, which has been introduced in Ref. [1] to model the potential singularity in a solution of (4). This 1D model can be obtained if we (a) restrict the 3D axisymmetric Euler equations (4) to the boundary $r = 1$ and (b) then make the identifications $u(z) \rightarrow (u^1)^2(1, z), \omega(z) \rightarrow \omega^1(1, z)$ and $v(z) \rightarrow \partial_r \psi^1(1, z)$. 

FIG. 1. A section of our cylindrical simulation domain with the heat-map of $\omega^1$ at a representative time $t = 0.003094$ ($N_r = 512, N_z = 1024$). Chebyshev collocation points are shown schematically in the $r-z$ plane for a constant value of $\theta$; these are spaced more closely near $r = 0$ and $r = 1$ than in the middle of the domain.

With these restrictions, the flow field is negative for $z > 0$ and positive for $z < 0$; this creates a compression flow at $z = 0$. Eventually, there is a finite-time singularity in this 1D model [28], which we analyse below.

$$
\partial_t u + v \partial_z u = 0; \quad (7a)
$$
$$
\partial_t \omega + v \partial_z \omega = \partial_z u; \quad (7b)
$$

here $\partial_z v = \mathcal{H}(\omega)$, with $\mathcal{H}(.)$ the Hilbert transform; we use periodic boundary conditions [28, 29] and the initial data

$$
u_0(z) = a \sin^2(\mu z); \quad a > 0 \quad (8a)
$$
$$\omega_0(z) = 0. \quad (8b)$$

We use a Fourier pseudospectral DNS to study this 1D model, with $N = 1024$ collocation points along the $z$ axis; from this DNS we obtain the spatiotemporal evolution of $u$ and $\omega$ for $a = 1$ and $\mu = 2\pi/L$ where $L = 1/6$.

The analyticity-strip method is well suited for pseudospectral DNSs of hydrodynamical PDEs like the 3D Euler or 1D Burgers equations. For a DNS in a domain with periodic boundary conditions in all spatial directions, this method [3, 4, 23–25, 30, 31] proposes that the solution of the PDE can be continued analytically to complex space variables $z = x + iy$, inside the analyticity strip $|y| < \delta(t)$, where the time $t$ is real and $\delta(t)$, the width of this strip, follows from the spatial Fourier transform of the solution, which decays, at large wave numbers $k$, as $\exp(-k\delta(t))$ (this has an algebraic prefactor). For any initial condition, we can, therefore, obtain $\delta(t)$ and estimate if this width goes to zero at a finite time $t_*$; at this time the solution shows a finite-time singularity because singularities, in the complex plane for $t < t_*$, hit the real axis. In any DNS, the resolution is finite, so the
physical-space grid has a non-zero mesh width; therefore, our determination of the width of the analyticity strip is accurate up until times at which $\delta(t)$ remains larger than a few mesh widths. For such times, we have spectral convergence of the Fourier expansion, a condition that is necessary to resolve, precisely, the smallest scales of the exact solution.

For all the two-dimensional fields in our study, we use the Fourier-Chebyshev representation, in which a function $f(r, z)$ is approximated by the truncated series

$$f_{N_r,N_z}(r, z) = \sum_{k=0}^{N_z} \sum_{m=0}^{N_r} \hat{f}_{k,m} e^{ikz} T_m(2r - 1),$$

where $T_m$ is the Chebyshev polynomial (of the first kind) of order $m$. Given this Fourier-Chebyshev expansion, we must extend the analyticity-strip method: (a) We first work with a fixed value of $r$; we evaluate the Fourier transform of the components of the velocity in the $z$ direction, for which we use periodic boundary conditions; the wave-number dependence of this transform yields the width of this analyticity strip, as we have discussed above. (b) Next, we work with a fixed value of $z$; we evaluate the Chebyshev transform of the components of the velocity in the $r$ direction; we then examine the dependence of the Chebyshev-expansion coefficients [32–36] on the order $m$; if these coefficients decrease as $\exp(-m\alpha)$, for large $m$, then the velocity field is analytic in the Bernstein ellipse $E_{\rho_*} = \{ z \in \mathbb{C} \mid z = (\rho_* e^{i\theta} - \rho_*^{-1} e^{-i\theta})/2, 0 \leq \theta \leq 2\pi \}$, with $\rho_* = \exp(\alpha)$; the width of this analyticity strip is $\delta_r = (\rho_* - \rho_*^{-1})/2$. Therefore, we can obtain $\delta_r$ from the Chebyshev-transformed velocity spectrum for different values of $z$ (see subsection III E).

II. NUMERICAL METHODS

In the schematic diagram in Fig. 1, we display the collocation points in our Fourier-Chebyshev DNS; these points are distributed uniformly in the periodic (axial) direction $z$, with Fourier modes labelled by the wave number $k$; in the radial direction $r$, these points coincide with the roots of the highest-order Chebyshev polynomial in our basis. We use a finer resolution in the $z$ direction than in the radial one, because, for a given number of collocation points, the Chebyshev nodes are spaced more closely near the boundary at $r = 1$ than the Fourier wave numbers $k$. This prevents excessive elongation of the cells in our simulation grid, in physical space near this boundary. If these cells are very elongated and narrow in the radial direction, it becomes difficult to satisfy the Courant-Friedrichs-Lewy (CFL) condition at every time-integration step. We use a CFL number $C = 0.2$ and adjust the time step $dt$, to ensure that the CFL condition is satisfied. For the temporal evolution of (4a) and (4b), we use the explicit fourth-order Runge-Kutta scheme in physical space; we evaluate the derivatives in Fourier-Chebyshev space and, subsequently, compute the nonlinear terms in physical space. We solve the Poisson equation (4c) in domain (5a) with the boundary conditions (5). We have checked that, for a simple smooth function, the relative error of the solution obtained using our Poisson solver is (a) $\approx 10^{-10}$ and (b) is only weakly dependent on the spatial resolution of our DNS (see the Tau solver in the Supplementary Information). We use the 2/3 truncation method.
for dealiasing both Fourier and Chebyshev modes. We have checked the robustness of our results, with the Tau Poisson solver, by comparing them with those from a scheme that employs a Galerkin Poisson solver [37, 38], adapted to our boundary conditions. We have also validated our code by comparing its results with those for stationary solutions, which can be obtained analytically [39](see the Supplementary Information).

The two invariants of the 3D Euler equation, namely, the total energy and helicity are, respectively,

\[
E = \frac{1}{2} \int_0^1 \int_0^L \left( |u_r|^2 + |u_z|^2 + |u^\theta|^2 \right) r dr dz; \quad (10a)
\]

\[
H = \int_0^1 \int_0^L \mathbf{u} \cdot \mathbf{\omega} r dr dz. \quad (10b)
\]

We calculate these by using the Fourier-Chebyshev coefficients of \( \mathbf{u} \) and \( \mathbf{\omega} \). Fourier and Chebyshev transforms, over \( z \) and \( r \), respectively, yield the fixed-\( r \) and fixed-\( z \) spectra

\[
E(r, k) = \frac{g(k)}{2 N_z} \left( |\hat{u}^\theta|^2 + |\hat{u}^r|^2 + |\hat{u}^z|^2 \right)_{r, k}, \quad (11a)
\]

\[
E(m, z) = \frac{N_r}{2 g(m)} \left( |\hat{u}^\theta|^2 + |\hat{u}^r|^2 + |\hat{u}^z|^2 \right)_{z, m}, \quad (11b)
\]

where \( g(i = 0) = 1 \) and \( g(i > 0) = 2 \) (\( i \) is \( k \) or \( m \)); similarly, a simultaneous Fourier-Chebyshev transform gives us the spectrum \( E(m, k) \). All these spectra depend on the time \( t \); we suppress the argument \( t \) to simplify the notation.

III. RESULTS

We first present our DNS results for the time evolution of \( E \) and \( H \) and singularity detection that uses the BKM theorem. Then we describe (a) the temporal development of spectra, (b) the birth of tygers and the subsequent thermalization of our spectrally truncated system of equations, and (c) singularity detection by the analyticity-strip method. We also present similar results for the 1D system (7).

A. Energy and Helicity

In Fig. 2 we show plots of \( E \) and \( H \) versus \( t \) for the initial condition (6b), on the domain \( D(1, \frac{1}{3}) \); we follow the solution for as long as the relative error in energy is lesser than \( 10^{-5} \). The higher the resolution of our DNS, the better the energy conservation (Fig. 1 of the Supplementary Information). The expression for \( H \) requires \( \mathbf{\omega} \), so it is not as well conserved as is \( E \) in a DNS; this is apparent in Fig. 2 (\( H(t = 0) = 0 \)). Movie S6 in the Supplementary Material shows the evolution of the spectra \( \ln(H(k, m)) \) with time.
FIG. 2. Plots versus time $t$ of the energy $E$ (in red) and the helicity $H$ (in blue), from our DNS with $N_z = 4096$ and $N_r = 512$. As $t$ increases and approaches the time of the (potential) singularity (at $t \approx 0.0035056$), the conservation of $E$ and $H$ deteriorate. For $t \lesssim 0.0033095$, the relative error in energy is lesser than $10^{-5}$. We show in Fig.1 in the Supplementary Information that, as we increase $N_z$ and $N_r$, energy and helicity conservation improve.

FIG. 3. Singularity-detection method based on the BKM theorem [2, 3]: Plot versus time $t$ of $\log(\log(|\omega|_{\infty}))$ (base 10) (blue full line) for $N_z = 4096$ and $N_r = 512$. The red (blue) dashed line indicates the time of the birth of a tyger (see text) in $u^\theta(\theta)$; the black dashed line denotes the estimate for the time of the (potential) singularity, from Ref. [1]. In Fig. 1 of the Supplementary Information, we give similar plots for other values of $N_z$ and $N_r$; the higher the values of $N_z$ and $N_r$ (especially $N_z$), the better our scheme captures the rapid growth of $\log(\log(|\omega|_{\infty}))$.

B. Singularity Detection: BKM Method

The detection of a (potential) singularity based on the BKM theorem [2, 3] uses a plot of $\log(\log(|\omega|_{\infty}))$ versus time $t$. We show such a plot (blue full line) in Fig. 3, from our DNS with $N_z = 4096$ and $N_r = 512$; the red (blue) dashed line indicates the time of the birth of a tyger (see below) in $u^\theta(\theta)$; the black dashed line denotes the estimate for the time of the (potential) singularity, from Ref. [1]. [In Fig.1 of the Supplementary Information, we give similar plots for other values of $N_z$ and $N_r$; the higher the values of $N_z$ and $N_r$ (especially $N_z$), the better our scheme captures the rapid growth of $\log(\log(|\omega|_{\infty}))$.] As we show below, in our pseudospectral DNS, it is better to use the analyticity-strip
method, to uncover the (potential) singularity than the BKM criterion.

C. Spectra

Given our initial condition, the spectrum $E(m, k)$, at $t = 0$, has significant weight at low values of $m$ and $k$; with the passage of time, energy cascades to large values of $m$ and $k$ (see the movie S7 in the Supplementary Information). For intermediate times, we have spectral convergence of our Fourier-Chebyshev expansions, and $E(m, k)$ decays at large values of $m$ and $k$. This allows us to employ the analyticity-strip method for the detection of a (potential) finite-time singularity. Henceforth, we concentrate on the spectra $E(r, k)$ and $E(m, z)$ (see (11b)).

In Fig. 4 we plot $\ln(E(k, r = 1))$ versus $k$, at different times $t$ (full temporal evolution in the movie S1 in the Supplementary Information), with $N_r = 512$, $N_z = 1024$, and the dealiasing cutoff at $k = 341$.

Our initial condition has the following symmetries: (a) $u^1$ is even at $z = \frac{1}{4}L, \frac{3}{4}L$ and odd at $z = 0, \frac{1}{2}L$; and (b) both $\omega^1$ and $\psi^1$ are odd at $z = 0, \frac{1}{4}L, \frac{1}{2}L, \frac{3}{4}L$. These symmetries are preserved by the evolution equations (4), so, for all $t$, the Fourier representations (for fixed $r$) of $u^r$ and $u^\theta$ ( $u^\theta$ ) contain only modes with even (odd) values of $k$. This leads to even-odd $k$ oscillations in, e.g., $E(k, r = 1)$ (black, blue, and red curves in Fig. 4). At small and intermediate values of $t$, these oscillations have exponentially decaying envelopes at large $k$. The envelope for odd $k$ lies above its even-$k$ counterpart and the separation between these envelopes increases with $t$. The natural logarithmic decrements of these envelopes, $\delta_{odd}(t)$ and $\delta_{even}(t)$, respectively, decrease as $t$ increases; and they play a crucial role in the analyticity-strip method (see below).

At sufficiently large $t$, $E(k, r = 1)$ does not have exponentially decaying envelopes (e.g., the green curve in Fig. 4), because of the formation of tygers (see below), our spectrally truncated system proceeds to thermalization, and we lose spectral convergence of the Fourier expansion.

We also obtain Chebyshev spectra, at fixed values of $z$, such as the illustrative plots of $E(m, z = 0)$ in Fig. 5. At small and intermediate values of $t$ (black, blue, and red curves in Fig. 5), these spectra decay exponentially at large values of $m$, with $\delta_m(t)$ decreasing with increasing $t$. At sufficiently large $t$, $E(m, z = 0)$ does not decay at large $m$ (e.g., the green curve in Fig. 4), because of the formation of tygers (see below), thermalization, and the consequent loss of spectral convergence of the Chebyshev expansion.

D. Tygers

Given the finite resolution of any practical spectral or pseudospectral DNS, we integrate not the full hydrodynamical partial differential equation (PDE), but its Galerkin-truncated (or spectrally truncated) modification. So long as we maintain the spectral convergence mentioned above, such a DNS yields a reliable solution of the PDE. Once this convergence is lost, the truncation leads to localized, short-wavelength oscillations,
FIG. 4. Plots versus $k$ of $\ln(E(k, r = 1))$, at different times $t$ (the full temporal evolution is given in the movie S1 in the Supplementary Information); $N_r = 512$, $N_z = 1024$, and the dealiasing cutoff is at $k = 341$; the even-odd $k$ oscillations in the spectrum $E(k, r = 1)$ arise because of a symmetry in the initial condition; there is an exponentially decaying tail in this spectrum, at large $k$; the rate of this decay decreases with time (see below). At sufficiently large $t$, there is no exponential decay (e.g., for the top spectrum in green) because of the thermalization of our spectrally truncated system.

FIG. 5. Plots versus $m$ of $\ln(E(z = 0, m))$ at different times $t$ (the full temporal evolution is given in the movie S2 in the Supplementary Information); $N_r = 512$, $N_z = 1024$, and the dealiasing cutoff is at $m = 340$; there is an exponentially decaying tail in the spectrum $E(z = 0, m)$, at large $m$; the rate of this decay decreases with time (see below). At sufficiently large $t$, there is no exponential decay (e.g., for the top spectrum in green) because of the thermalization of our spectrally truncated system.

which have been christened *tygers* in Ref. [40], in the context of the 1D Burgers and 2D Euler equations.

Tygers appear when complex-space singularities come within one Galerkin wavelength $\lambda_G = 2\pi/k_G$ [24] of the real axis. As we increase the resolution of our DNS, $\lambda_G$ decreases, hence there is an increase in the time taken by the pole, nearest to the real axis, to cross
In the first three columns of Fig. 6, we present plots, versus $z$, of $u^\theta (r = 1)$ [top row] and $\omega^\theta (r = 1)$ [bottom row], for $N_r = 512$ and $N_z = 1024$ at various values of time $t$; as we go from the first to the third column, we zoom in to the region with the localized oscillatory structures called tygers [16, 40]; in the last column we plot $t_b$, the time of the birth of tygers versus $\lambda_G = 2\pi/k_G$, where $k_G$ is the dealiasing cutoff wavenumber.

Tygers appear clearly in $\omega^\theta (r = 1)$ before they become visible in $u^\theta (r = 1)$, because the vorticity is a curl of the velocity. Both tyger-birth times, for the vorticity and the velocity, precede (Fig. 3) the estimate for the singularity time given in Ref. [1]. To the best of our knowledge, the plots in Fig. 6 are the clearest examples of tygers in a 3D hydrodynamical PDE (albeit one in which we consider a solution that is axisymmetric and radially bounded).

The plots versus $z$ in Fig. 6 provide a natural motivation for studying the 1D model of (7); we return to this model in Subsection III F. We note that, as in the 1D Burgers equation [16, 40], tygers do not appear at the point where the singularity develops, as a step in $u^\theta (r = 1, z)$ at $z = 0$, but some distance away from it, where a resonant interaction occurs between the fluid particle and the truncation waves that are generated by sharp structures like the gradients in vorticity. Moreover, with the passage of time, the tygers grow, as they initiate the process of thermalization and spread through the whole domain; this is the real-space manifestation of thermalization, which we have discussed above in the context of spectra, where it leads to the equipartition of energy over all Fourier-Chebyshev modes, in the long-time limit. In the problem we study, the development of the (potential) singularity leads to numerical errors as our DNS nears the time at which it is estimated in [1] that the singularity occurs; eventually, energy and helicity conservation become poor, and this hampers us from proceeding, in our DNS, all the way to complete thermalization.
E. Singularity Detection: Analyticity-strip Method

We have discussed above how the widths of the analyticity strips follow from the natural logarithmic decrements of the spectra $E(k, r)$ and $E(z, m)$; we concentrate on $r = 1$ and $z = 0$. We discuss the details of the numerical estimations of these decrements and the reason for choosing $r = 1$ and $z = 0$ in the Supplementary Information. Here, we give the main results:

- In the top panel of Fig. 7, we plot versus time $t$, the widths $\delta_{\text{odd}}$ and $\delta_{\text{even}}$ associated with the odd- and even-$k$ envelopes, respectively, of $E(k, r = 1)$. In the bottom panel of Fig. 7, we present a log-log (base 10) plot of $\delta_{\text{even}}$ versus $t - t_*$, where $t_* = 0.0035056$ is the estimate of the time of the (potential) singularity in Ref. [1] along with the power-law fit $\delta_{\text{even}} = a|t - t_*|^b$; we find $\log a = 7.088$ and $b = 2.736$ in the gray-shaded region (the movie S3 in the Supplementary Information shows the evolution of this fit with time).

- In the top panel of Fig. 8, we plot, versus time $t$, the width $\delta_r$, which we obtain from the natural logarithmic decrements of $E(z = 0, m)$. This is very nearly linear until just before the estimate of the time of the (potential) singularity given in Ref. [1]. From a linear fit, in the gray-shaded region in the bottom panel of Fig. 8, we find an intercept, on the horizontal axis, at $t = 0.0033058$, which is slightly less than the estimate for the time of (potential) singularity $t = 0.003505$ given in Ref. [1] (the movie S4 in the Supplementary Information shows the evolution of this fit with time).

F. The 1D Model

We have carried out similar studies for the 1D model of equations (7). The movie S5 in Supplementary Information gives the temporal evolution of the fields and the spectra in this model. We see, once again, the development of tygers, before the time at which a finite-time singularity occurs. We plot these in the Fig. 9. The last column, top row gives a plot of the analyticity-strip width $\delta(t)$ versus the time $t$. The growth of tygers in this 1D model leads to thermalization in a manner that is akin to what we have discussed above for the 3D axisymmetric and radially bounded Euler (1); this is shown clearly by the energy spectra in the last column, bottom row of Fig. 9.

IV. CONCLUSIONS

We have examined the potentially singular solution of the 3D, axisymmetric and radially bounded Euler equation, suggested in [1] by developing a pseudospectral, Fourier-Chebyshev scheme. Our method leads to new insights for it shows that, in this scheme,
FIG. 7. Top panel: plots versus time $t$ of the widths $\delta_{\text{odd}}$ and $\delta_{\text{even}}$ of the odd- and even-$k$ envelopes, respectively, of $E(k, r = 1)$. Bottom panel: log-log (base 10) plots of $\delta_{\text{even}}$ versus $t - t^*$, where $t^* = 0.0035056$ is the estimate of the time of the (potential) singularity in Ref. [1] along with the power-law fit $\delta_{\text{even}} = a|t - t^*|^b$; we find $\log a = 7.088$ and $b = 2.736$ in the grey-shaded region; here, $N_r = 512$ and $N_z = 4096$.

FIG. 8. Top panel: Plot of $\delta_r$ versus time $t$. Bottom panel: a linear fit, in the grey-shaded region in the bottom panel yields an intercept, on the horizontal axis, at $t = 0.0033058$.

the formation of tygers precedes the development of the (potential) singularity and leads eventually to the thermalization of our spectrally truncated Fourier-Chebyshev scheme. We then show how to generalise the analyticity-strip method [3, 4, 23–25] to track this (potential) singularity. Our singularity-detection method yields results that are, indeed, consistent with the finite-time proposed in Ref. [1].
FIG. 9. First three columns: Plots versus $z$ of $u$ [top row], $\omega$ [bottom row]; as we go from columns one to three, we zoom in to the region with tygers, which are localized oscillatory structures [16, 40]. The last column, top row gives a plot of the analyticity strip width $\delta(t)$ versus the time $t$. Last column, bottom row: semilog (base $e$) plots of energy spectra $E(k)$, of the even-$k$ modes in Fourier space, at different, representative times. Here, we use $N = 1024$ Fourier collocation points and the dealiasing cutoff is at $k_G = 341$.

A recent paper by D. Barkley [41] has also used a Fourier-Chebyshev method to study the Luo-Hou initial condition; this paper concentrates on the physical mechanism for the (potential) singularity and not on the issues we discuss here. Recent work by Hertel, Besse, and Frisch (to be published) has examined this (potential) singularity by combining Fourier-Chebyshev and Cauchy-Lagrange [42–46] methods. Furthermore, a new paper [47] has developed a method for purging tygers in the 1D Burgers equation. We have checked that this purging method can also be used for the tygers we have studied here as we will discuss elsewhere. We will show in future work how to generalise the methods used in Ref. [17], to study the process of thermalization in the 1D Burgers equation, to investigate thermalization in systems of equations we consider here.

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Supplementary Information for: A pseudospectral study of a potentially singular solution of the three-dimensional axisymmetric incompressible Euler equation: tygers and thermalization

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In this Supplementary Information we give details regarding (a) the conservation of energy and helicity in our direct numerical simulation (DNS), (b) the Poisson solvers that we use, (c) the validation of our code by comparing its results with those that can be obtained analytically for stationary solutions, (d) the fitting procedure that we use for the analyticity-strip method, and (e) movies from our DNSs.

I. ENERGY AND HELICITY CONSERVATION

We have described our Fourier-Chebyshev DNS and time-integration scheme in our paper. As this integration time approaches the estimate in Ref. [1] \( t \approx 0.0035056 \) for the (potential) singularity, the conservation of the energy \( E \) and the helicity \( H \) deteriorate; for \( t \lesssim 0.0033095 \), the energy is \( 10^{-3} \% \) less than its initial value. In Fig. 1 we show the following:

- As the spatial resolution of our DNS increases, the conservation of \( E \) and \( H \) improves.
- \( H \), whose definition requires the vorticity, is not conserved as well as \( E \).
- The higher the spatial resolution of our DNS the longer we can track the growth \( |\omega|_\infty \).

![Energy and Helicity Plots](image)

FIG. 1. Plots versus \( t \) of the Energy (top panels), Helicity (left bottom panel) and \( \log(\log(|\omega|_\infty|)) \) (right bottom panel) for different resolutions.

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II. POISSON SOLVERS FOR AXISYMMETRIC DOMAINS

To solve eq.(4c) in the main text, we use an axisymmetric Poisson solver with the appropriate boundary conditions (eqs.(5e-5f) in the main text) to be imposed on $\psi^1$:

$$-\left[ \frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right] \psi^1(r, z) = \omega^1(r, z); \quad (1a)$$

$$\psi^1(1, z) = 0; \quad \partial_r \psi^1(0, z) = 0; \quad \psi^1(r, 0, t) = \psi^1(r, L, t). \quad (1b)$$

We have used two independent Poisson solvers based on the Shen Galerkin and Tau methods, to compare their accuracies for the implementation of boundary conditions. Both methods involve the inversion of the matrix system in (1b) in spectral space. The Fourier-Chebyshev transformed system ($\partial_z^2 \rightarrow -k^2; r = (1 + x)/2; x \in [-1, 1]$) is:

$$-\left[ 4(x + 1) \partial_x^2 + 12 \frac{\partial}{\partial x} - k^2(x + 1) \right] \psi^1(x, k) = \omega^1(x, k); \quad (2a)$$

$$\psi^1(x = 1, k) = 0; \quad \partial_x \psi^1(x = -1, k) = 0. \quad (2b)$$

A. Galerkin method

This method [2, 3] involves the construction of basis functions $\phi_m(x)$, each of which satisfy the boundary conditions, and are linear combinations of Chebyshev polynomials $T_m(x) = \cos(m \cos^{-1}(x))$:

$$\phi_m(x) = T_m(x) + \frac{-4(m + 1)}{(m + 1)^2 + (m + 2)^2} T_{m+1}(x) + \frac{m^2 + (m + 1)^2}{(m + 1)^2 + (m + 2)^2} T_{m+2}(x). \quad (3a)$$

The Galerkin approximation of $\psi^1$, in terms of $\phi_m$, is

$$\psi^1(x, k) = \sum_{m=0}^{N-3} a(m, k) \phi_m(x). \quad (3b)$$

We then take the weighted inner product of (1b) with the $\phi_m$:

$$((x + 1) \partial_x^2 \psi^1, \eta \phi_m) - (2 \partial_x \psi^1, \phi_m)_{\eta} + \beta((x + 1) \psi^1, \phi_m)_{\eta} = (g, \phi_m)_{\eta}; \quad (3c)$$

where $\eta$ is the Chebyshev weight and $g = \frac{1}{4} \omega^1(x + 1)$. This matrix system can be inverted in spectral space to get $\psi^1$.

B. Tau method

In this method, the boundary conditions are explicitly enforced and the basis polynomials do not satisfy the boundary conditions inherently [4]. Here, we choose the Chebyshev polynomials as the basis;

$$\psi^1(x, k) = \sum_{m=0}^{N-1} a(m, k) T_m(x). \quad (4a)$$
The weighted inner product of the Poisson equation is:

\[-4(x + 1)\partial^2_x\psi^1, T_m)\eta - (12 \partial_x\psi^1, T_m)\eta + (k^2(x + 1)\psi^1, T_m)\eta = (\omega^1, T_m)\eta.\]  

(4b)

The last two rows of the operator matrix are replaced by the boundary conditions (5e) and (5f) in the main text. We have the following expressions for the boundary conditions:

- The no-flow boundary condition at $r = 1$:

\[M_{N_r-2,m} = \cos \left( \frac{m\lambda}{2N} \right) ; \quad m = 0, 1..N_r - 1.\]  

(4c)

- The pole condition at $r = 0$ is enforced as follows:

\[M_{N_r-1,m} = \begin{cases} 2m \sum_{n=1}^{m/2} \cos \left( \left(2n - 1\right) \cos^{-1}(-\lambda) \right) & m \text{ even;} \\ m \sum_{n=0}^{(m-1)/2} \cos \left( \left(2n\right) \cos^{-1}(-\lambda) \right) & m \text{ odd.} \end{cases}\]  

(4d)

Fig.2 compares the results that we obtain by using Galerkin and Tau solvers for the Euler equation with initial condition given by eq.(6) in the main text.

![Comparisons of our results for the Energy E, Helicity H, percentage energy deviation and log(log(||ω∞||)) using the Tau and Galerkin Poisson solvers when implemented in the scheme. These plots are for resolution of 512 radial and 1024 axial collocation points.](image)
The relative errors of the Poisson solvers and Fourier and Chebyshev derivative sub-
routines for the function: \( f = \cos(\pi r/2)e^{\sin(z)} \) at \( N_r = 512, N_z = 4096 \) are given by

- \( \max(\frac{\partial_z f}{\partial_z f_{\text{analytical}}} - 1) = 1.6725758300641053 \times 10^{-9} \)
- \( \max(\frac{\partial_r f}{\partial_r f_{\text{analytical}}} - 1) = 8.9829913335348056 \times 10^{-6} \)
- \( \max(\nabla^2 f^{\text{analytical}} - 1) \) for Tau solver= \( 1.2604251300660698 \times 10^{-10} \)
- \( \max(\nabla^2 f^{\text{analytical}} - 1) \) for Shen Galerkin solver = \( 2.2208520394808275 \times 10^{-12} \)

III. BENCH-MARKING OF OUR 3D AXISYMMETRIC EULER CODE

To validate our code, we use the stationary analytical solution given in Ref.\[5\]. We have the following family of stationary solutions and their forms at the pole:

\[
\psi^1 = \frac{J_1(\sqrt{B^2 - \kappa^2}r)}{r} \cos(\kappa z); \quad \psi^1(r = 0) = \frac{\sqrt{B^2 - \kappa^2}}{2} \cos(\kappa z);
\]
\[
u^1 = \frac{B J_1(\sqrt{B^2 - \kappa^2}r)}{r} \cos(\kappa z); \quad \nu^1(r = 0) = \frac{B\sqrt{B^2 - \kappa^2}}{2} \cos(\kappa z); \quad (5)
\]
\[
\omega^1 = \frac{B^2 J_1(\sqrt{B^2 - \kappa^2}r)}{r} \cos(\kappa z); \quad \omega^1(r = 0) = \frac{B^2\sqrt{B^2 - \kappa^2}}{2} \cos(\kappa z).
\]

Let \( x_{\text{root}} \) be one of the roots of \( J_1 \), then \( B = \sqrt{x^2_{\text{root}} + \kappa^2} \), where \( \kappa = 0, 1, 2... \) In Fig.3, we plot versus t, the percentage deviation of the energy, from our DNS, relative to the energy of the stationary solution (5) with \( \kappa = 1 \) and \( x_{\text{root}} \) is the first root of \( J_1 \); the percentage deviation of energy is less than \( 10^{-5} \) for a DNS with resolution as low as \( (N_r, N_z) = (256, 512) \).

IV. ANALYTICITY STRIP METHOD

We extract \( \delta_{\text{even}}(r, t) \) (similarly \( \delta_{\text{odd}}(r, t) \)) by a least-squares fit using the following equation on the odd (even) envelope of \( E(k, r, t) \):

\[
\log(E(k_{\text{even}})|_{r,t}) = C - n \log(k_{\text{even}}) - 2\delta_{\text{even}}(r, t)k_{\text{even}} \quad (6)
\]

In Fig. 4, we show the time series of \( \delta_{\text{even}} \) and \( \delta_{\text{odd}} \) for different values of \( r \) as a sheet, where we see that the decay is fastest for \( r = 1 \); which we plot in Fig.7 in main text.

Similarly for \( \delta_r(z, t) \) \[6\], we carry out the least-squares fit on \( E(z, m, t) \):

\[
\log(E(m)|_{z,t}) = C - 2m\alpha. \quad (7)
\]

Here, the velocity field is analytic in the Bernstein ellipse with minor axis given by \( \rho_\ast = \exp(\alpha) \) \[7–10\]. The width of the analyticity strip for Chebyshev spectra can then be
defined as $\delta_r = (\rho_s - \rho_s^{-1})/2$. Fig.5 shows $\delta_r(z,t)$ where fastest decay is seen for $z = 0$; shown in Fig.8 (top panel). We have fit this data to a linear form given by $a|t - t_*| + b$ in Fig.8 (bottom panel) in the main text, within the gray window as explained above.

V. MOVIES

S1 Temporal evolution of $\ln(E(k,r))$.
S2 Temporal evolution of $\ln(E(m,z))$.
S3 Fitting $\ln(E(k, r = 1))$ with $t$.
S4 Fitting $\ln(E(m, z = 0))$ with $t$.
S5 Composite (in order from left to right) of temporal evolution of $\omega(z)$ and fitting of $\ln(E(k))$ and time series of $\delta$ of the 1-D model.
S6 Composite (in order from left to right) of time series of helicity, temporal evolution of helicity isosurface in real space $H(r, z)$ and spectral space $\ln(H(k, m))$, followed by the separate contributions of even and odd modes to the spectra of $\ln(H(k, m))$.
S7 Composite (in order from left to right) of time series of energy, temporal evolution of $u^1(r, z)$, $\omega^1(r, z)$ and $\log_e(E(k, m))$.

FIG. 3. Plot versus $t$ of the percentage deviation of energy for the stationary solution where $x_{root}$ is the first root of $J_1(r)$ and $\kappa = 1$.
FIG. 4. Surface plots of $\delta_z(r,t)$ for even and odd Fourier modes showing dependence on radial direction. It can be seen that, in both cases, $\delta_z$ falls fastest at the wall i.e. $r = 1$. 
FIG. 5. Surface plot of $\delta_r(z,t)$ for Chebyshev modes showing dependence on axial direction. It can be seen that $\delta_r$ falls fastest at $z = 0, L/2, L$.

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