Abstract. We leverage an algorithm of Deming \cite{Deming} to decompose a matchable graph into subgraphs with a precise structure: they are either spanning even subdivisions of blossom pairs, spanning even subdivisions of the complete graph $K_4$, or a König-Egerváry graph. In each case, the subgraphs have perfect matchings; in the first two cases, their independence numbers are one less than their matching numbers, while the independence number of the KE subgraph equals its matching number. This decomposition refines previous results about the independence structure of an arbitrary graph and leads to new results about $\alpha$-critical graphs.

Keywords: matching, independence, König-Egerváry, Egerváry, Birkhoff-von Neumann

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1. Introduction

König-Egerváry graphs generalize bipartite graphs: they are defined by the condition that their independence number $\alpha$ and their matching number $\nu$ sum to their order $n$. These graphs have the property that they can be identified efficiently and their independence numbers can be calculated efficiently. This investigation is motivated by an attempt to generalize König-Egerváry graphs to a larger class of graphs with similar attractive properties.

In this paper, we leverage an algorithm of Deming \cite{Deming} to produce a useful decomposition of a matchable graph into subgraphs with a precise structure: they are either spanning even subdivisions of blossom pairs, spanning even subdivisions of the complete graph $K_4$, or a König-Egerváry graph. In all cases, the subgraphs have perfect matchings; in the first two cases, their independence numbers are one less than their matching numbers, while the independence number of the König-Egerváry subgraph equals its matching number. This decomposition refines previous results about the independence
structure of an arbitrary graph and leads to new results about \(\alpha\)-critical graphs.

1.1. **Matching Structure, König-Egerváry & Egerváry Graphs.** An independent set is a set of vertices that is pairwise nonadjacent; the independence number \(\alpha\) of a graph is the cardinality of a largest independent set. A matching is a set of pairwise non-incident (or independent) edges; the matching number \(\nu\) is the cardinality of a largest matching. The order \(n\) of a graph is the number of its vertices. A König-Egerváry graph (henceforth, KE graph) is defined by the condition that \(\alpha + \nu = n\). While the definition of an Egerváry graph and our initial theorems apply to graphs with perfect matchings, we will ultimately show that many of our results extend to arbitrary graphs.

Egerváry graphs generalize KE graphs. A graph is Egerváry if it has a perfect matching but it doesn’t contain a spanning subgraph whose components are independent edges together with at least one odd cycle (parity considerations show that “at least one” really means “a positive even number of”). These graphs are also known as ‘non-Edmonds’ graphs (e.g., in [11]) and ‘Birkhoff-von Neumann’ (BvN) graphs (in [4]). The definition above is the most convenient one for our purposes, but it’s worth noting that these graphs also admit a polyhedral description depending on their perfect matching polytope. See [11] for details. That Egerváry graphs indeed generalize KE graphs will become apparent with the statement of Corollary 1.4.

One should observe the similarity of our definition of (non-)Egerváry graphs with the notion of a basic perfect 2-matching of a graph \(G\), namely a spanning subgraph whose components are independent edges and odd cycles; see, e.g., [14]. For matchable graphs, being Egerváry is almost the negation of admitting a basic perfect 2-matching, the difference stemming from the stipulation ‘at least one’ in the first definition. We don’t make much use of this connection, but it does appear in the proof of Lemma 6.19 below.

Egerváry graphs are also related to tangled graphs, i.e., those graphs containing no two vertex-disjoint odd cycles; see, e.g., [17] or [10] and the references cited there. The definitions show that matchable tangled graphs are Egerváry, but the converse is not true. In fact, tangled graphs admit a characterization in terms of ‘super-Egerváry’ graphs, whose definition is beyond our scope. But we note that the omitted definition generalizes the polyhedral description of Egerváry graphs to which we alluded above. We hope to return to this thread in a follow-up article.

The class of Egerváry graphs includes, for instance, the complete graph \(K_4\) but not the graph \(T\) consisting of two 3-cycles and a single edge incident to one vertex of each (see Figure 1).

Deming [5] and Sterboul [18] (independently) gave the first characterizations of KE graphs. In particular, they showed that this property is in
co-NP. There are now many characterizations of KE graphs; the following one is particularly useful for the current investigation. Following [14], we call a subgraph $H$ of a matchable graph $G$ *nice* if $G - V(H)$ has a perfect matching.

**Theorem 1.1** (Lovász & Plummer, [14]). *A graph with a perfect matching is KE if and only if it does not contain an even subdivision of either $K_4$ or $T$ as a nice subgraph.*

Even subdivisions of $T$ are sometimes called *blossom pairs*. It’s easy to check that no blossom pair is KE (and that neither is any even $K_4$-subdivision). Theorem 1.1 shows that for matchable graphs, these are the only obstructions.

The proof of the next theorem imitates the proof of Theorem 1.1 since it restricts the KE characterization to fewer forbidden subgraphs, it generalizes the characterized class of graphs.

**Theorem 1.2.** *A graph with a perfect matching can be covered by a collection of independent edges and a positive number of odd cycles if and only if it contains an even subdivision of $T$ as a nice subgraph.*

**Proof.** Let $G$ be a graph with a perfect matching $M_1$. If $G$ contains an even subdivision of $T$ as a nice subgraph, then $G$ can certainly be covered by a collection of independent edges and a positive number of odd cycles.

Now suppose that $G$ can be covered by a collection $M_0$ of independent edges and a positive number of odd cycles $C_1, \ldots, C_k$. Then $M_0 \cup M_1$ consists of the edges in $M_0 \cap M_1$, some alternating (even) cycles, and some paths connecting vertices in the odd cycles $C_1, \ldots, C_k$. These paths start and end with edges of $M_1$. At least one such path must connect vertices in different cycles since they cannot pair off the vertices within a single odd cycle.

So let $P$ be a path-component of $M_0 \cup M_1$ connecting two odd cycles $C_i$ and $C_j$. The subgraph $P \cup C_i \cup C_j$ is an even subdivision of $T$, and it is also a nice subgraph of $G$. \hfill \Box

**Corollary 1.3.** *A graph with a perfect matching is Egerváry if and only if it does not contain an even subdivision of $T$ as a nice subgraph.*
Figure 2. This matchable graph contains a spanning even subdivision of $K_4$ (so is not KE by Theorem 1.1) and a nice even subdivision of $T$ (so is not Egerváry by Corollary 1.3).

Figure 2 illustrates Theorem 1.1 and Corollary 1.3. These results also give a new proof of the following theorem, originally proved by the first author of the present work.

**Corollary 1.4 ([11]).** If a graph with a perfect matching is KE, then it is Egerváry.

2. **Deming Subgraphs**

We shall show that every graph with a perfect matching admits a decomposition into subgraphs with attractive properties. Given such a graph $G$, Deming’s Algorithm [5] either produces a maximum independent set $I$ that certifies $G$ being KE or produces an even subdivision of $K_4$ or $T$ as a nice subgraph. We can push this algorithm further so that, when $G$ is not KE, we get either a ‘Deming-$K_4$ subgraph’—with a spanning even $K_4$-subdivision—or a ‘Deming-BP subgraph’ with a spanning even $T$-subdivision, both of which are, in a precise sense, almost-KE. We describe these first and then discuss our extension of Deming’s Algorithm.

A graph $K$ is a **Deming-$K_4$ graph** if it contains a spanning even subdivision of $K_4$ and $K - \{x, y\}$ is KE for every edge $xy$ in some perfect matching of $K$. Analogously, a graph $B$ is a **Deming-BP graph** (for ‘Deming-Blossom-Pair’) if it contains a spanning even subdivision of the graph $T$ (in Figure 1) and $B - \{x, y\}$ is KE for every edge $xy$ in some perfect matching of $B$. Finally, a graph $D$ is a **Deming graph** if it is either a Deming-$K_4$ graph or a Deming-BP graph.

Figure 3 shows an example of a Deming-BP graph. Notice that for the center edge 6–7, the subgraph $B - \{6, 7\}$ is not KE, but 6–7 is not in a perfect matching of $B$. It is also easy to check that the Petersen graph $P$ is a Deming-BP graph: it has a pair of 5-cycles connected by a single edge
as a spanning subgraph, every edge $xy$ is in some perfect matching, and $P - \{x, y\}$ is KE.

Note that whether a graph $D$ with a perfect matching $M$ is Deming can be efficiently determined: run Deming’s Algorithm on $D$ (with respect to $M$). Deming’s Algorithm will produce either a maximum independent set when $D$ is KE, or a nice even subdivision of either $K_4$ or $T$ when it is not. If $D$ is not KE and the produced even subdivision spans $D$, then the graph may be Deming. It now remains to identify which edges $xy$ are in some perfect matching (by checking, for instance if $D - \{x, y\}$ has a perfect matching), and then, for each such edge $xy$, checking if $D - \{x, y\}$ is KE. Of course, this algorithm leverages the fact that it is efficient to find a maximum matching in a graph [6].

One output of Deming’s Algorithm is subgraphs which have a spanning even subdivision of $K_4$. We dig deeper into the subgraphs produced by Deming’s Algorithm. Even subdivisions of $K_4$ have the property that every edge is in a perfect matching and that the removal of any edge yields a KE graph (and they are ‘α-critical’). The subgraphs produced by Deming’s Algorithm may have ‘extra’ edges. The definition here is designed to efficiently reduce a given output of Deming’s Algorithm into a similar substructure with even nicer properties.

Deming-$K_4$ graphs have even subdivisions of $K_4$; as these are central in this paper, we record some relevant properties.

**Proposition 2.1.** An even subdivision of $K_4$ has a perfect matching.

**Proof.** An even subdivision $K$ of $K_4$ consists of four ‘corner’ vertices and six odd paths between them. Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and let the edges of $K_4$ be $e_{i,j}$, one for each pair of distinct vertices $v_i, v_j$ in $K_4$. Call the corresponding corner vertices in $K$: $v_1, v_2, v_3, v_4$. And for each pair $v_i, v_j$ of distinct corner vertices, let $P_{i,j}$ be the odd path from $v_i$ to $v_j$ formed by subdividing $e_{i,j}$ in $K_4$ an even number of times. The odd-length paths $P_{1,2}$ and $P_{3,4}$ can be matched as can the odd-length paths strictly between vertices $v_1$ and $v_3$ on $P_{1,3}$, strictly between $v_2$ and $v_4$ on $P_{2,4}$, strictly between $v_1$ and $v_4$ on $P_{1,4}$, and (finally) strictly between $v_2$ and $v_3$ on $P_{2,3}$. \[\square\]
Corollary 2.2. Every edge of an even subdivision of $K_4$ is in some perfect matching.

Corollary 2.3. A Deming-$K_4$ graph has a perfect matching.

Corollary 2.4. A Deming graph has a perfect matching.

Proof. Deming-$K_4$ graphs have a perfect matchings by the preceding corollary. A Deming-BP graphs has a perfect matching as it contains a spanning even $T$ subdivision: the odd path connecting the blossom tips has a perfect matching, and the remaining non-blossom-tip vertices in both blossoms also form odd paths. □

The following related results are needed for the discussion of $\alpha$-critical graphs in Section 6.

Proposition 2.5. If $K$ is an even subdivision of $K_4$ and $e$ is an edge of $K$, then $K - e$ is $KE$.

Proof. We again view $K$ as having corner vertices $v_1, v_2, v_3, v_4$ with odd-length paths $\{P_{i,j}\}$ between them. Assume that $e = xy$ is on $P_{1,2}$ with $x$ closer to $v_1$ than is $y$. We shall first show that $K$ contains a perfect matching not containing $e$.

With $P_{1,2}$ being of odd length, the parities of the distances from $x$ to $v_1$ and $y$ to $v_2$ are necessarily the same.

If the parities are both odd, then taking the remaining edge of $P_{1,2}$ incident to vertex $x$ and every alternate edge from $x$ to $v_1$ will include an edge that saturates $v_1$. Include these edges in a matching $M$. Similarly, taking the remaining edge of $P_{1,2}$ incident to $y$ and every other edge from $y$ to $v_2$ will include an edge that saturates $v_2$. Add the edge incident to $v_3$ to $M$ along with every other edge from $v_3$ to $v_4$; this will include an edge that covers $v_3$. Now all corner vertices are covered by $M$ as well as every vertex on $P_{1,2}$ and $P_{3,4}$. It remains to add edges that saturate the interior vertices of the other four paths $P_{1,3}, P_{2,4}, P_{1,4},$ and $P_{2,3}$. In each case, the remaining paths between (and not including) the corresponding corner vertices have odd length, and these vertices can all be covered by matching edges added to $M$.

If the distance parities from $x$ to $v_1$ and $y$ to $v_2$ in $P_{1,2}$ are both even, then taking the remaining edge of $P_{1,2}$ incident to $x$ and every other edge from $x$ to $v_1$ will not include an edge that saturates $v_1$. Include these edges in $M$. Similarly, taking the remaining edge of $P_{1,2}$ incident to $y$ and every other edge from $y$ to $v_2$ will not include an edge that saturates $v_2$. Also include in $M$ a matching that covers the odd-length paths $P_{1,4}$ and $P_{2,3}$. These matching edges will include all corner vertices. It remains to add edges that saturate the interior vertices of the other three paths $P_{1,3}, P_{2,4},$ and $P_{3,4}$. In each case, the remaining paths between (and not including) the corresponding corner vertices have odd lengths, and these vertices can all be covered by matching edges added to $M$. 6
So in both the odd- and even-parity cases, $K - e$ has a perfect matching. Using $M$ to denote that matching in either case, we shall construct an independent set $I$ with $|I| = |M|$. As this condition implies that $K - e$ is KE, arranging for it will suffice to complete the proof.

In the first case, $x$ is an odd distance to $v_1$. Along $P_{1,2}$, build $I$ by including $x$, including every alternate vertex along the $(x, v_1)$-segment of $P_{1,2}$, and including $y$ and every alternate vertex along the $(y, v_2)$-segment of $P_{1,2}$. Continuing around the cycle $P_{1,2}$ followed by $P_{2,3}$, $P_{3,4}$, and $P_{4,1}$, extend $I$ by including $v_2$'s neighbor along $P_{2,3}$ and then every alternate vertex (moving away from $v_2$) up to but not including $v_1$. If that cycle is $C$, then around $C$, we so far have selected one vertex for $I$ from each $M$-edge of $C$—including $v_3$ but not $v_1$, $v_2$, or $v_4$. It remains to make analogous choices for vertices of $I$ along $P_{1,3}$ and $P_{2,4}$. Because $v_3 \in I$, we extend $I$ on $P_{1,3}$ by choosing all vertices at even distance from $v_3$ along $P_{1,3}$; notice that this puts $v_1$'s neighbor on $P_{1,3}$ into $I$—and not $v_1$. This is important because $v_1$'s neighbor on $P_{1,2}$ is already in $I$, and we're building an independent set. Because $v_2, v_4 \notin I$, we have some choice for extending $I$ on $P_{2,4}$. For definiteness, we select for $I$ all vertices at odd distance from $v_2$ along $P_{2,4}$, except for $v_4$ (whose neighbors along both $P_{4,1}$ and $P_{3,4}$ are already in $I$).

One can check that the described set $I$ is independent and contains exactly one vertex of each $M$-edge when $M$ is the perfect matching defined for this case. Thus, $|I| = |M|$ as desired.

In the second case, $x$ is an even distance to $v_1$ and $y$ is an even distance to $v_2$. Similarly to the odd-parity case above, we may again construct an independent set $I$—necessarily containing $x$ and $y$—with $|I| = |M|$, for $M$ now the perfect matching defined for the even case. Here we omit a detailed description. □

One motivation for our definition of Deming-$K_4$ and Deming-BP graphs is that an arbitrary graph can be efficiently decomposed into subgraphs of three types with a precise independence and matching structure. In particular, Deming graphs satisfy $\alpha = \nu - 1$.

**Theorem 2.6.** If $D$ is a Deming graph, then $\alpha(D) = \nu(D) - 1$.

**Proof.** Corollary 2.3 shows that $D$ has a perfect matching, and thus $\nu(D) = n(D)/2$. Deming graphs are not KE, and thus $\alpha(D) < n(D)/2 = \nu(D)$; and so $\alpha(D) \leq \nu(D) - 1$. Let $xy$ be an edge in a perfect matching of $D$. Consider the graph $D' = D - \{x, y\}$ (formed by removing vertices $x$ and $y$ and all incident edges). Since $xy$ is in a perfect matching of $D$, the subgraph $D'$ also has a perfect matching, and $\nu(D') = \nu(D) - 1$. By definition, $D'$ is KE; this and the fact that $D'$ is matchable give $\alpha(D') = \nu(D')$. Then $\alpha(D) \geq \alpha(D') = \nu(D') = \nu(D) - 1$. So $\alpha(D) = \nu(D) - 1$. □

By definition, a Deming-$K_4$ graph $K$ may contain edges not participating in an even $K_4$-subdivision spanning it. For example, in Figure 1 the edge 5–6 lies outside the $K_4$-subdivision with corner vertices $\{0, 1, 2, 3\}$. So deleting
Figure 4. A Deming-$K_4$ graph $K$ that is not an even subdivision of $K_4$. Notice that neither $K - \{5-6\}$ nor $K - \{0-6\}$ is KE because both contain spanning even $K_4$-subdivisions. Theorem 2.7 characterizes when all such edge-deleted subgraphs are KE.

proof. Denote a perfect matching in $K$ by $M$. First suppose that $K$ is an even subdivision of $K_4$, with corner vertices $v_1, v_2, v_3, v_4$ and paths $\{P_{i,j}\}$ defined as in the proof of Proposition 2.1. Let $e$ be an edge of $K$. We showed in Proposition 2.5 that $K - e$ is KE.

Next suppose that $K - e$ is KE for every edge $e$ of $K$. By the definition of a Deming-$K_4$ graph, $K$ has a spanning even $K_4$-subdivision. We'll show that the subdivision edges are in fact the only edges of $K$. Again, let the corner vertices of this subdivision be $\{v_1, v_2, v_3, v_4\}$ and the six odd-length paths between them be $\{P_{i,j}\}$. Suppose that $xy$ is an edge not in any of the paths $P_{i,j}$. We may assume that $x \in P_{1,2}$.

As a first case, suppose that $x = v_1$ and $y \in P_{1,2}$. Either the distance from $x$ to $y$ on $P_{1,2}$ is odd or it is even. If this distance is even, then there is a nice even subdivision of $T$ in $K$. There are odd cycles: (1) the path from $x$ to $y$ in $P_{1,2}$ and back to $x$; and (2) the path $P_{2,3}$, followed by $P_{3,4}$ and then $P_{4,2}$, with blossom tips $x = v_1$ and $v_3$ joined by the odd path $P_{1,3}$. When the distance from $x$ to $y$ on $P_{1,2}$ is odd, at least one of the edges $e$ on the path from $x$ to $y$ in $P_{1,2}$ is not in $M$. Now $K - e$ still contains a nice even subdivision of $K_4$ and thus cannot be KE.
Next we assume that while both \(x\) and \(y\) are in \(P_{1,2}\), neither of them is \(v_1\) or \(v_2\); assume that \(x\) is closer to \(v_1\) than \(y\). Again, either the distance from \(x\) to \(y\) on \(P_{1,2}\) is odd or it is even. If this distance is even, then there is a nice even subdivision of \(T\) in \(K\). Assume the distance from \(x\) to \(v_1\) along \(P_{1,2}\) is even (either it is or the distance from \(y\) to \(v_2\) must be). There are odd cycles: (1) the path from \(x\) to \(y\) in \(P_{1,2}\) and back to \(x\); and (2) the path \(P_{2,3}\), followed by \(P_{3,4}\) and then \(P_{4,2}\), with blossom tips \(x\) and \(v_3\) joined by the odd path from \(x\) to \(v_1\), finally followed by \(P_{1,3}\). If the distance from \(x\) to \(y\) on \(P_{1,2}\) is odd, then at least one of the edges \(e\) on the path from \(x\) to \(y\) in \(P_{1,2}\) is not in \(M\). Now \(K - e\) still contains a nice even subdivision of \(K_4\) and thus cannot be KE.

The next case we consider is when \(x\) and \(y\) are on adjacent subdivision paths, say \(P_{1,2}\) and \(P_{2,3}\). Either the distance from \(x\) to \(v_1\) in \(P_{1,2}\) and from \(y\) to \(v_3\) in \(P_{2,3}\) have the same parity, or they do not. If the parities are the same, then there is a nice even subdivision of \(T\) in \(K\). There are odd cycles: (1) the path from \(x\) to \(v_2\) in \(P_{1,2}\), from \(v_2\) to \(y\) in \(P_{2,3}\), and back to \(x\); and (2) the path \(P_{1,3}\), followed by \(P_{3,4}\), and then \(P_{4,1}\), with blossom tips \(v_2\) and \(v_4\) joined by the odd path \(P_{2,4}\). In the special case where \(x = v_1\) and \(y = v_3\) (so the distances are 0 and the parities are even), at least one of the edges \(e\) on the path from \(v_1\) to \(v_3\) in \(P_{1,3}\) is not in \(M\). Now \(K - e\) still contains a nice even subdivision of \(K_4\) and thus cannot be KE. If the parities from \(x\) to \(v_1\) in \(P_{1,2}\) and from \(y\) to \(v_3\) in \(P_{2,3}\) are different (assume the distance from \(y\) to \(v_3\) in \(P_{2,3}\) is odd), at least one of the edges \(e\) on the path from \(x\) to \(v_2\) in \(P_{1,2}\) and from \(v_2\) to \(y\) in \(P_{2,3}\) is not in \(M\). Now \(K - e\) still contains a nice even subdivision of \(K_4\) and thus cannot be KE.

The last case we consider is when \(x\) and \(y\) are on opposite subdivision paths, say \(P_{1,2}\) and \(P_{3,4}\). Let \(e\) be an edge in \(P_{2,3}\). By assumption, \(K - e\) is KE. We will show that in fact (and contrary to our assumption), \(K\) contains a nice even subdivision of \(K_4\). We may assume that neither \(x\) nor \(y\) is a corner vertex—otherwise these fall into cases that we handled previously. The distances from \(x\) to the corner points of \(P_{1,2}\) necessarily have different parities. Similarly, the distances from \(y\) to the corner points of \(P_{3,4}\) have different parities. Assume that the distances from \(x\) to \(v_1\) and from \(y\) to \(v_4\) are both odd. We now construct an even subdivision of \(K_4\) with corner points \(v_1, x, y,\) and \(v_4\). There is an odd path from \(x\) to \(v_1\) consisting of the even path from \(x\) to \(v_2\) in \(P_{1,2}\) followed by \(P_{2,4}\). And there is also an odd path from \(y\) to \(v_3\) consisting of the even path from \(y\) to \(v_3\) in \(P_{3,4}\) followed by \(P_{3,1}\).

A graph \(G\) is \(\alpha\)-critical if \(\alpha(G - xy) = \alpha(G) + 1\) for each edge \(xy\) in \(G\). (We include \(K_1\) as (vacuously) \(\alpha\)-critical; it is not only true but also makes the statements of some of our results cleaner.) The study of \(\alpha\)-critical graphs has a long history [8, 2, 13, 16]. We record one fact here and return to this topic in Section 6.
Theorem 2.6 implies both that \( \alpha \) has a spanning even so from Theorem 2.7 that \( K \) only if \( K \).

**Proof.** Suppose that \( K \) is an even subdivision of \( K_4 \). It immediately follows from Theorem 2.7 that \( K \) is \( \alpha \)-critical.

Suppose then that \( K \) is a Deming-\( K_4 \) graph and is \( \alpha \)-critical. Then \( K \) has a spanning even \( K_4 \)-subdivision \( F \). Note that \( F \) is also a Deming-\( K_4 \) graph and that \( K \) and \( F \) must both have the same matching number. Theorem 2.6 implies both that \( \alpha(K) = \nu(K) - 1 \) and that \( \alpha(F) = \nu(F) - 1 \). So \( \alpha(K) = \alpha(F) \). Now suppose that \( xy \) is a non-\( F \) edge of \( K \). Clearly \( \alpha(K) \leq \alpha(K - xy) \leq \alpha(F - xy) = \alpha(F) = \alpha(K) \). So we have both that \( \alpha(K) = \alpha(K - xy) \) and that \( \alpha(K - xy) = \alpha(K) + 1 \). Since this is impossible, \( K \) cannot contain a non-\( F \) edge. \( \square \)

**Proposition 2.9.** If \( K \) is an even \( T \)-subdivision, then it has a unique perfect matching \( M \), and for every edge \( e \) not in \( M \), the graph \( K - e \) is KE and has a unique perfect matching.

**Proof.** Let \( K \) be formed from two odd cycles \( C_1, C_2 \) with respective blossom tips \( v, w \) joined by an odd-length path \( P \). Let \( V(C_1) = \{v = v_1, \ldots, v_{2k+1}\} \) and \( V(C_2) = \{w = w_1, \ldots, w_{2l+1}\} \). Since every perfect matching \( M \) of \( K \) must include the edges of \( P \) that cover \( v \) and \( w \) (and every alternate edge between them), the remaining uncovered vertices of \( C_1 \) form the odd path from \( v_2 \) to \( v_{2k+1} \). As there is a unique matching \( M_1 \) covering these vertices, \( M \) must include the edges of \( M_1 \). Similarly, the remaining uncovered vertices in \( C_2 \) form the odd path from \( w_2 \) to \( w_{2l+1} \), and there is a unique matching covering these vertices. Thus, \( M \) must include these edges as well. So \( G \) has a unique perfect matching.

Now let \( e = xy \) be a non-\( M \) edge. There are two cases: either \( e \) is on the path \( P \) or \( e \) belongs to one of the cycles (which we may assume to be \( C_1 \)). Suppose first that \( e \) belongs to \( P \). Assume that \( x \) is the end of \( e \) closer to \( v \) (so \( y \) is the end closer to \( w \)). So there is an odd-length path from \( v \) to \( x \) and one from \( y \) to \( w \). The graph \( K - e \) then comprises two components, each consisting of an odd cycle together with an odd-length path adjoined to the blossom tip. Since both of these components are KE, so too is \( K - e \).

Now suppose that \( e \) is a non-\( M \) edge in \( C_1 \). Assume that \( x \) precedes \( y \) on the path from \( v_2 \) to \( v_{2k+1} \). Then \( x \) must equal \( v_i \) for some odd index \( i \) (and thus \( y = v_{i+1} \) with \( i + 1 \) even). Let \( I \) be the independent set containing the vertices \( w_2, w_4, \ldots, w_{2l} \), the vertices at odd distance from \( w \) on \( P \) (including \( v \)), the vertices at even distance from \( v \) on the path \( v = v_1, v_2, \ldots, x \) (including \( x \)), and finally the vertices at even distance from \( y \) along the path \( y = v_{i+1}, \ldots, v_{2k} \) (including \( y \)). The set \( I \) thus includes exactly one end of each \( M \)-edge; whence \( |I| = |M| \) and \( G - e \) is KE.

Of course, \( K - e \) has a unique perfect matching because \( K \) does. \( \square \)

**Corollary 2.10.** If \( D \) is an \( \alpha \)-critical Deming graph, then \( D \) is an even subdivision of \( K_4 \).
Proof. First note that the graph $T$ is not $\alpha$-critical; neither is any blossom pair $D$: if $x$ and $y$ are blossom tips and $xx'$ is the edge incident to $x$ on the path from $x$ to $y$, then $\alpha(D) = \alpha(D - xx')$. Extending this observation, we can show that no Deming-BP graph $D$ is $\alpha$-critical. For such a $D$ is either a blossom pair (in which case we’re done) or contains an edge $uv$ that is not an edge in $D$’s spanning blossom pair. Let $D' = D - uv$. It is easy to check that $D'$ is also a Deming-BP graph. Theorem 2.6 then implies that $\alpha(D) = \nu(D) - 1$ and $\alpha(D') = \nu(D') - 1$. Since $D'$ has a perfect matching, whence $\nu(D) = \nu(D')$, it follows that $\alpha(D) = \alpha(D') = \alpha(D - uv)$, and thus indeed $D$ is not $\alpha$-critical.

So if $D$ is an $\alpha$-critical Deming graph, it must be a Deming-$K_4$ graph, and now Corollary 2.8 shows that $D$ is an even subdivision of $K_4$. □

3. Extending Deming’s Algorithm and Deming Decompositions

Given a graph $G$ with a perfect matching, Deming’s original algorithm [5] either produces a maximum independent set $I$ certifying that $G$ is KE or produces an even subdivision of either $K_4$ or $T$ as a nice subgraph. In the later cases, we test if the produced obstructions are Deming graphs and, if not, re-apply Deming’s Algorithm to certain subgraphs.

Suppose that Deming’s Algorithm is applied to a graph $G$, and the output is an obstruction $H$, i.e., an even subdivision of $T$ or $K_4$. In either case, we find the edges $xy$ which are in some perfect matching of $H$ and, for each of these, test whether $H - \{x, y\}$ is KE by re-applying Deming’s Algorithm. If there is an edge $xy$ such that $H - \{x, y\}$ is not KE, then this graph must itself contain an obstruction $H'$: a nice even subdivision of $T$ or $K_4$. We then iterate on $H - \{x, y\}$. Since we remove two vertices on each iteration, we must terminate (in no more than $n(H)/2$ steps). If $H$ is not KE, then this final output obstruction $D$ must be a Deming graph.

Now the vertices of $D$ can be removed from $H$. The resulting graph $H - D = G[V(H) \setminus V(D)]$ must itself have a perfect matching (as the obstruction $D$ contains a nice even subdivision of either $T$ or $K_4$), and this extended Deming Algorithm can be applied to the subgraph $H - D$. Since we remove at least four vertices on each iteration, we must terminate (in no more than $n(H)/4$ steps). The final output graph, possibly null, is necessarily KE. So this process results in a finite sequence of Deming graphs together with a single KE graph whose vertices partition the vertices of the parent graph $G$ (allowing some parts to be empty).

It is easy either to keep track, or efficient to test directly, whether the output Deming graphs are Deming-$K_4$ subgraphs or Deming-BP subgraphs, all of which defines a decomposition of $G$ into subgraphs with well-defined properties. We call this a ‘Deming decomposition’; to formalize, a Deming decomposition of a matchable graph $G$ is:

1. a collection of Deming-BP subgraphs $\{B_i\}_{i=1}^r$ of $G$ ($r \geq 0$);
2. a collection of Deming-$K_4$ subgraphs $\{K_i\}_{i=1}^\ell$ of $G$ ($\ell \geq 0$);
Figure 5. A Deming decomposition of $G$ in (A) is \{\{B\}, \{K\}, R\}, where $B$ is a Deming-BP subgraph, $K$ is a Deming-$K_4$ subgraph, and $R$ is KE.

(3) a KE graph $R$ (possibly null); and such that
(4) the vertices of these subgraphs partition $V(G)$.

Figure 5 depicts a Deming decomposition in which each 'part' has a single constituent.

It is worth emphasizing that any graph with a perfect matching has a Deming decomposition $\{\{B_i\}_{i=1}^r, \{K_j\}_{j=1}^\ell, R\}$, but a Deming decomposition is not necessarily unique. The input graph $G$ is KE if and only if $r = \ell = 0$. If $G$ is Egerváry, then it has no Deming-BP subgraphs and $r = 0$. A Deming decomposition is then a tool for investigating whether a graph is Egerváry. We shall see that it yields necessary conditions for whether a graph is Egerváry. We shall also see that this decomposition can be extended to general graphs.

In computing a Deming decomposition, start by computing an initial perfect matching $M$ (which won’t necessarily induce a perfect matching in each of the Deming subgraphs). Consider, for instance, the graph in Figure 6 and the perfect matching $\{1-5, 3-4, 0-2\}$. A Deming decomposition consists of the Deming-$K_4$ graph induced on the vertices \{0, 1, 2, 3\} and the KE graph induced on the vertices \{4, 5\}. So $M$ doesn’t induce a perfect matching on either subgraph. Of course, since each subgraph in a Deming decomposition has a perfect matching, these define a perfect matching $M'$ of the parent
Figure 6. An initial matching \{1–5, 3–4, 0–2\} may differ from a Deming decomposition induced perfect matching \{0–2, 1–3, 4–5\}. We call such an \(M'\) a *Deming decomposition induced perfect matching*. (See the second paragraph following the statement of Corollary 6.8 for related remarks.)

For future reference, we collect several of our results on Deming decompositions into a single theorem.

**Theorem 3.1.** If \(G\) is a matchable graph with a Deming decomposition \(\{\{B_i\}_{i=1}^r, \{K_j\}_{j=1}^\ell, R\}\), then:

1. \(\alpha(B_i) = \nu(B_i) - 1\), for \(i \in [r]\);
2. \(\alpha(K_j) = \nu(K_j) - 1\), for \(j \in [\ell]\);
3. \(\alpha(R) = \nu(R)\);
4. \(\nu(G) = \sum_{i=1}^r \nu(B_i) + \sum_{j=1}^\ell \nu(K_j) + \nu(R)\); and
5. \(\alpha(G) \leq \nu(G) - (r + \ell)\).

Before moving to the next section, where we employ our basic results in further understanding Egerváry graphs, let us consider another concrete example: the so-called ‘Buckminster Fullerene’; see Figure 7. While Theorem 3.1 (part (4)) shows that the matching number \(\nu\) is additive across the parts of a Deming decomposition, part (5) guarantees only that the independence number \(\alpha\) is subadditive. Nevertheless, at least for the graph \(C_{60}\), the invariant \(\alpha\) does behave additively for the decomposition mentioned in Figure 7’s caption.

**4. Characterizing & Recognizing Egerváry Graphs**

Deming’s Algorithm can be used to identify whether an input graph \(G\) with a perfect matching \(M\) is KE. If it is, the algorithm also produces a maximum independent set \(I\), which yields a certificate (in this case \(|M| + |I|\)
Figure 7. The Buckminster Fullerene $C_{60}$ has 60 vertices. Of its 32 faces, 12 are pentagons which are pairwise vertex disjoint. One Deming decomposition consists of 6 pairs of pentagons, each pair joined by a single edge. Interestingly, $\alpha(C_{60}) = 24$, and the independence number of each of these 6 Deming subgraphs is 4.

must equal the order). In the case where $G$ is not KE, the algorithm either produces a nice even $T$-subdivision, and this provides a certificate that $G$ is not Egerváý. Deming’s Algorithm may also terminate with the production of a nice even $K_4$-subdivision $F$. While the production of $F$ provides a certificate that $G$ is not KE, it may or may not be the case that $G$ is Egerváý. It remains an open question to find a certificate that a graph $G$ is Egerváý. We record some conditions that may be relevant.

Theorem 4.1. A graph $G$ with a perfect matching is Egerváý if and only if $G$ is not an even $T$-subdivision and, for every edge $e$ not in all perfect matchings of $G$, the graph $G - e$ is Egerváý.

Proof. Suppose first that $G$ is Egerváý. So $G$ does not have a nice even $T$-subdivision. In particular, $G$ is not itself a $T$-subdivision. If every edge is in every perfect matching of $G$, then $G$ must consist of isolated edges and the condition that—for every edge $e$ not in all perfect matchings of $G$, the graph $G - e$ is Egerváý—is vacuously satisfied. Assume then that there is an edge $e$ that is not in every perfect matching of $G$. Note that $G - e$ must then have a perfect matching and that $G - e$ is Egerváý: if $G - e$ had a nice even $T$-subdivision, then $G$ would as well.

Conversely, suppose that $G$ is not an even $T$-subdivision, and, for every edge $e$ not in all perfect matchings of $G$, the graph $G - e$ is Egerváý. If $G$ has a nice even $T$-subdivision $H$, then, since $G$ does not equal $H$, the parent $G$ must contain an edge $e$ incident to some vertex of $H$. Since $H$ is nice, it has a perfect matching $M$ that can be extended to all of $G$. Since $e \notin M$,
the edge $e$ is not in every perfect matching. But then $G - e$ contains the nice even $T$-subdivision $H$, contradicting the fact that $G - e$ is Egerváry. Therefore, $G$ contains no such $H$ and by Corollary 1.3 is Egerváry. 

Given a matchable graph $G$ with a Deming decomposition $\{\{B_i\}_{i=1}^r, \{K_j\}_{j=1}^\ell, R\}$, one obvious necessary condition for $G$ to be Egerváry is that there cannot be any Deming-BP subgraphs: if $G$ has a Deming-BP subgraph, then it has a nice even $T$-subdivision and is not Egerváry (by Corollary 1.3). We will establish other necessary conditions.

One other easy-to-see necessary condition is that each Deming subgraph itself is Egerváry. Yet another is that there can be no edge between vertices of different Deming subgraphs: these have spanning even $K_4$ subdivisions and if there is such an edge, then a cycle from each subgraph can be paired to construct a nice even $T$-subdivision.

The following theorem resembles Theorem 4.1, specified to the Deming-$K_4$ subgraph case. In fact, Deming-$K_4$ subgraphs have significant structure and this specification utilizes that structure. In particular, Deming-$K_4$ subgraphs $D$, by definition, do not have nice even $K_4$-subdivisions with fewer vertices: if one did, there would be an edge $xy$ of $D$ not spanned by that (smaller) subdivision and then, by definition $D - \{x, y\}$ would both be KE and have a nice even $K_4$-subdivision, a contradiction. In this sense, Deming-$K_4$ subgraphs are minimal non-KE graphs with nice even $K_4$-subdivisions. The proof of this next result depends on this fact essentially.

**Theorem 4.2.** A Deming-$K_4$ graph $K$ with a spanning even $K_4$-subdivision $F$ is Egerváry if and only if for every edge $e$ of $F$, either: (1) $K - e$ is KE, or (2) $K - e$ itself has a spanning even $K_4$-subdivision and is Egerváry.

**Proof.** First assume that $K$ is Egerváry. Given an edge $e$ of $F$, we'll show that either (1) or (2) holds. Suppose that $K - e$ is not KE and thus contains either a nice even $T$-subdivision or a nice even $K_4$-subdivision. First consider the case where $K - e$ contains a nice even $T$-subdivision $H$. But then $H$ is a nice even $T$-subdivision within $K$, contradicting the assumption that $K$ is Egerváry. So it must be that $K - e$ contains a nice even $K_4$-subdivision $F'$. If $F'$ does not span $K - e$, then—since $F'$ is a nice subgraph of $K - e$—the vertices of $K - e$ that are not in $F'$ are perfectly matched by some edges of $K - e$. If $xy$ is a perfect matching edge between two vertices not in $F'$, then $xy$ must be a perfect matching edge of $K$. Since $K$ is a Deming-$K_4$ subgraph, it follows that $K - \{x, y\}$ is KE. But $K - \{x, y\}$ contains a nice even $K_4$-subdivision $F'$, contradicting the fact that $K - \{x, y\}$ is KE. So it must be that $F'$ spans $K - e$. Since $K$ is Egerváry, $K - e$ must also be Egerváry, and we have that $K - e$ itself has a spanning even $K_4$-subdivision and is Egerváry.

For the converse, suppose that $K$ is not Egerváry. By Corollary 1.3, then $K$ has a nice even $T$-subdivision $H$. The subgraph $F$ must contain an edge $e$ not in $H$, and we can assume that $e$ is incident to a vertex of $H$. The
Proof. Denote a perfect matching in $K$ of perfect matchings corresponding Deming decomposition \{\{K_j\}_{j=1}^l, R\}, and $K$ and $K'$ are Deming-$K_4$ subgraphs, then no edge is incident to both a vertex in $K$ and one in $K'$.

Theorem 4.3. If $G$ is an Egerváry graph with a perfect matching and corresponding Deming decomposition \{\{K_j\}_{j=1}^l, R\}, and $K$ and $K'$ are Deming-$K_4$ subgraphs, then no edge is incident to both a vertex in $K$ and one in $K'$.

Proof. Denote a perfect matching in $G$ by $M$, which we may assume induces perfect matchings $M_1$ in $K$ and $M_2$ in $K'$. We will identify $K$ and $K'$ with their spanning $K_4$-subdivisions—the ‘extra’ edges (edges not in these spanning subgraphs) will play no role here in our considerations.

Let $v_1, v_2, v_3, v_4$ be the corner vertices of $K$ and $w_1, w_2, w_3, w_4$ be the corner vertices of $K'$. Let $P_{i,j}$ be the odd paths connecting corner vertices in $K$ and $Q_{i,j}$ be the corresponding paths in $K'$. Suppose that $x \in V(K)$ and $y \in V(K')$, with $x$ adjacent to $y$. We will show that then $G$ contains a nice even subdivision of $T$—contradicting the fact that $G$ is Egerváry.

In the first case, assume that both $x$ and $y$ are corner vertices. We may assume that $x = v_1$, $y = w_1$, that $P_{1,2}$ and $P_{3,4}$ are $M_1$-saturated paths, and that $Q_{1,2}$ and $Q_{3,4}$ are $M_2$-saturated paths. Thus, the cycle formed from $P_{2,3}$ followed by $P_{3,4}$ and then $P_{1,2}$ is a blossom with tip $v_2$. Similarly, the cycle formed from $Q_{2,3}$ followed by $Q_{3,4}$ and then $Q_{1,2}$ is a blossom with tip $w_2$. The $M$-saturated $(v_2, w_2)$-path formed from $P_{2,1}$ followed by the edge $v_1w_1$ (which is $xy$) and then $Q_{1,2}$ is an odd path between blossom tips. Thus, the two blossoms and this path form an even subdivision of $T$. Now edges in $M$ are incident to either zero or two vertices in this subgraph, and since $M$ is perfect, we have a nice subgraph.

In the second case, assume that $x$ is not a corner vertex and that $y$ is a corner vertex. We may assume that $x$ is interior to $P_{1,2}$ and that $y = w_1$; we continue to assume that $P_{1,2}$ and $P_{3,4}$ are $M_1$-saturated and that $Q_{1,2}$ and $Q_{3,4}$ are $M_2$-saturated. The key point here is that $x$ is an odd distance from exactly one of $v_1$, $v_2$. Assume that $x$ is an odd distance from $v_2$. Again the cycle formed from $P_{2,3}$ followed by $P_{3,4}$ and then $P_{4,2}$ is a blossom with tip $v_2$, and the cycle formed from $Q_{2,3}$ followed by $Q_{3,4}$ and then $Q_{4,2}$ is a blossom with tip $w_2$. The path from $v_2$ to $x$ to $w_2$ formed from the $(v_2, x)$-segment of $P_{2,1}$, followed by the edge $xy$ and then $Q_{1,2}$ is an odd $M$-saturated path joining blossom tips, and so we again have a nice even subdivision of $T$ as a subgraph of $G$.

Finally, consider the case where neither $x$ nor $y$ is a corner vertex. Here, we may assume that $x$ is interior to $P_{1,2}$ and that $y$ is interior to $Q_{1,2}$. We may further assume that $P_{1,3}$ and $P_{2,4}$ are $M_1$-saturated paths and that $Q_{1,3}$ and $Q_{2,4}$ are $M_2$-saturated paths. Similarly to the preceding case, we may
Theorem 4.4. If $K$ is a Deming-$K_4$ graph with perfect matching $M_K$, $R$ is a vertex-disjoint $KE$ graph with perfect matching $M_R$, and $G$ is an Egerváry graph formed by adding some edges between $K$ and $R$, then no vertex $v \in V(K)$ admits an $M_R$-alternating path to itself.

Remark: Technically, we should use ‘cycle’ instead of ‘path’ in the assertion because the walk under consideration would be closed. But, as seen in the proof, it would also have odd length, hence vacuously not be $M_R$-alternating. The point of the result is not to rule out such a cycle for this (trivial) parity reason, rather to rule it out for the structural reason explored in the proof.

Proof. Let $v$ be a vertex in $K$ and $F$ be $K$’s spanning even $K_4$-subdivision containing $M_K$. We need to show that there is no $M_R$-alternating path from $v$ to itself. For a contradiction, suppose that there is such a path, say $e_1, e_2, \ldots, e_r$. As $M_R$-edges lie in $R$ and $v \notin V(R)$, the edges $e_1$ and $e_r$ are not in $M_R$. So this path (really cycle) has odd length and hence defines a blossom $B_1$ in $G$. Since $M_K$ is a perfect matching of $K$, the vertex $v$ must be incident to an $M_K$-edge $vw$ on one of the paths $P_{i,j}$ defining $F$—we can assume that it’s $P_{1,2}$ and that $w$ is closer to $v_2$ than to $v_1$. There are two cases to consider. In the first case, the distance from $w$ to $v_2$ along $P_{1,2}$ is even. Here, the last edge of the $(w, v_2)$-segment of $P_{1,2}$ is necessarily a matching edge. So the cycle formed by concatenating $P_{2,3}, P_{3,4}$, and $P_{4,2}$ is a blossom $B_2$, and the blossom $B_1$, followed by the $(v, v_2)$-segment of $P_{1,2}$ (an odd path), and followed by $B_2$ is a nice even $T$-subdivision. But then $G$ is not Egerváry, which is a contradiction. In the second case, the distance from $w$ to $v_2$ along $P_{1,2}$ is odd, so the last edge on the $(w, v_2)$-segment of $P_{1,2}$ is not in $M_K$. But $v_2$ is saturated by $M_K$; so assume that the first (and hence the last) edge of the path $P_{2,3}$ is in $M_K$. Now the concatenation of $P_{3,4}, P_{4,1}$, and $P_{1,3}$ is a blossom $B_2$ with blossom tip $v_3$. So the blossom $B_1$, followed by the path from $v$ to $v_2$ along $P_{1,2}$, followed by the path $P_{2,3}$ (an odd path), and finally followed by $B_2$ is a nice even $T$-subdivision. But then again we reach the contradiction that $G$ is not Egerváry. □

Theorem 4.5. If $K$ is an Egerváry Deming-$K_4$ graph with a spanning even $K_4$-subdivision $F$ and $vw \in E(K) \setminus E(F)$, then $vw$ lies in a perfect matching of $K$. 

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Proof. Call the edges of $K$ not in $F$ extra edges. By Corollary 2.2, the conclusion holds when $K$ has 0 extra edges, for such $K$ are just even $K_4$-subdivisions. Assume, then, that for a fixed integer $k > 0$, the conclusion holds for Egerváry-Deming-$K_4$ graphs with fewer than $k$ extra edges, and suppose that $K$ has $k$ extra edges $e_1, e_2, \ldots, e_k$.

We need to show that each of these edges is in a perfect matching of $K$, and when $k > 1$, this follows immediately by induction. For if $1 \leq i \leq k$, then the subgraph $K' = K - e_i$ is Egerváry (because $K$ is Egerváry), has $F$ as a spanning even $K_4$-subdivision, and has $k - 1 > 0$ extra edges. So all these extra edges are in perfect matchings of $K'$ (hence of $K$), and since $i$ was arbitrary, the conclusion follows.

Indeed, the crux of this proof is when $k = 1$, i.e., when $e = e_1$ is the only extra edge (so $K$ consists of an even $K_4$-subdivision with one extra edge $e = vw$).

There are three main cases to consider: (1) when $e$ is incident to a pair of vertices on a single $P_{i,j}$ path of $F$ (we can assume $P_{1,2}$); (2) when $e$ is incident to vertices on two ‘incident’ paths of $F$ (we’ll assume $P_{1,2}$ and $P_{1,3}$); and (3) when $e$ is incident to vertices on ‘opposite’ paths of $F$ (we’ll assume $P_{1,2}$ and $P_{3,4}$). In each case, we’ll show either that the case is impossible—that the existence of such an edge violates the assumption that $K$ is Egerváry—or that $K$ contains a perfect matching containing $e$.

In case (1), we assume that $e$’s endpoints $v, w$ on $P_{1,2}$ have $v$ closer to $v_1$. We first argue that the number of edges from $v$ to $w$ on $P_{1,2}$ must be odd. If this number were even, we could find a nice even $T$-subdivision in $K$, contrary to $K$ being Egerváry. First note that exactly one of the subpaths from $v$ to $v_1$ or $w$ to $v_2$ of $P_{1,2}$ is even. Let us assume that the first of these (from $v$ to $v_1$) is even (the other possibility is similar). Since the number of edges from $v$ to $w$ on $P_{1,2}$ is even, the cycle formed by these edges together with the edge $vw$ is odd. Next, there is an odd path from $v$ to $v_1$ (along $P_{1,2}$) to $v_4$ (using all of $P_{1,4}$). Finally, the paths $P_{4,2}$, followed by $P_{2,3}$, and then $P_{3,4}$ form an odd cycle. We have now formed our desired even $T$-subdivision.

And it’s nice because the omitted vertices of $K$ are those strictly between $w$ and $v_2$ along $P_{1,2}$ and those strictly between $v_1$ and $v_3$ along $P_{1,3}$, all of which can be perfectly matched along these paths.

Now we have that the number of edges from $v$ to $w$ along $P_{1,2}$ is odd. Thus, the $(v, v_1)$- and $(w, v_2)$-segments of $P_{1,2}$ have the same parity. We consider the subcase when these segments are both of even length; the odd case is similar. Here, we find a perfect matching of $K$ consisting of $vw$, independent edges between $v$ and $w$ on $P_{1,2}$, independent edges between $v$ and $v_1$—including one that is necessarily incident to $v_1$—independent edges between $w$ and $v_2$—including one that is necessarily incident to $v_2$—independent edges on $P_{3,4}$ including one incident to $v_3$ and one incident to $v_4$ that saturate all vertices on this path, and lastly independent edges saturating all subdivision vertices on $P_{1,4}$, $P_{2,3}$, $P_{1,3}$, and $P_{2,4}$.
In case (2), we assume that $e$’s endpoint $v$ is on $P_{1,2}$ and $w$ is on $P_{1,3}$. We shall argue momentarily that the number $N$ of edges on the path $P$ from $v$ to $v_1$ along $P_{1,2}$ followed by $v_1$ to $w$ on $P_{1,3}$ can be assumed odd. In this case, we construct a perfect matching of $K$ starting with $vw$ together with independent edges between $v$ and $w$ on $P$. Either the number of edges from $v$ to $v_2$ on $P_{1,2}$ is even and the number of edges from $w$ to $v_3$ on $P_{1,3}$ is odd, or vice versa. First assume the former. Then it is possible to extend the matching to saturate all remaining vertices on $P_{1,2}$ including $v_2$ and every vertex of $P_{1,3}$ except $v_3$. The matching can be further extended with independent edges along $P_{3,4}$ that saturate all these vertices and finally covering the subdivision vertices of $P_{2,4}$, $P_{1,4}$, and $P_{2,3}$. This yields a perfect matching of $K$ which includes the edge $vw$. The latter subcase—when the number of edges from $v$ to $v_2$ on $P_{1,2}$ is odd and the number of edges from $w$ to $v_3$ on $P_{1,3}$ is even—can be addressed similarly.

We turn to the case when $N$ (of the preceding paragraph) is even. There are two subcases depending on whether the number $N'$ of edges from $v$ to $v_2$ along $P_{1,2}$ is odd or even (note that the number of edges from $w$ to $v_3$ along $P_{1,3}$ must have the same parity as $N'$). Consider the subcase when $N'$ is odd. Since $N$ is even, the path $P$ together with the edge $vw$ forms an odd cycle $C$ containing $v$. From there, the $(v, v_2)$-segment of $P_{1,2}$ is also odd (because $N'$ is odd), and this path attaches at $v_2$ to the odd cycle formed by the paths $P_{2,4}$, $P_{4,3}$, and $P_{3,2}$. Thus we observe an even $T$-subdivision in $K$. The remaining vertices (not belonging to $T$) are the ones strictly between $w$ and $v_3$ on $P_{1,3}$ (an even number) and the ones strictly between $v_1$ and $v_4$ on $P_{1,4}$ (also an even number). Since these can all be saturated by a matching, we see that $T$ is nice, which is impossible.

Within case (2), it remains to consider the subcase when $N'$ is even, and here, we construct another perfect matching $M$ of $K$ containing the edge $vw$. With $N$ still being even, we see that the odd cycle $C$ from above persists. Thus, we can initialize $M$ with edges including $vw$ and saturating every vertex of $C$ except $v_1$. With $N'$ being even, the $(v, v_2)$-segment of $P_{1,2}$ and the $(w, v_3)$-segment of $P_{1,3}$ are both even. Hence we can extend $M$ with independent edges along these segments so that $v_2$ and $v_3$ are both saturated, along with the internal vertices up to but not including $v$ and $w$ (as these were already covered). To reach a perfect matching, it remains to include independent edges along $P_{1,4}$ that saturate all these vertices and finally cover the subdivision vertices of $P_{2,4}$, $P_{3,4}$, and $P_{2,3}$; this is possible because these paths are all odd.

The last case (3) is in a sense the easy case because none of the subcases lead to contradictions. Here, the edge $e = vw$ is incident to vertices on ‘opposite’ paths (we will assume $P_{1,2}$ and $P_{3,4}$, with $v$ on the first and $w$ on the second). In this case, $e$ can always be included in a perfect matching of $K$. The vertex $v$ must be either at an odd distance from $v_1$ on $P_{1,2}$ and an even distance from $v_2$ or at an even distance from $v_1$ on $P_{1,2}$ and an odd
In the first subcase, the distance from \( w \) to \( v_4 \) on \( P_{3,4} \) is odd. Then the edge \( e \) is included in the even cycle consisting of the path from \( v \) to \( v_1 \) on \( P_{1,2} \), followed by \( P_{1,4} \), followed by the path from \( v_4 \) to \( w \) on \( P_{3,4} \), and concluding with the edge \( e \) (connecting \( w \) back to starting point \( v \)). The remaining vertices consist of three paths, each with an even number of vertices. The observed even cycle can be perfectly matched with a matching containing \( e \), and the three remaining paths each can be perfectly matched, all together yielding a perfect matching of \( K \) containing \( e \).

In the remaining subcase, the distance from \( w \) to \( v_4 \) on \( P_{3,4} \) is even. Here, the edge \( e \) is included in the even cycle consisting of the path from \( v \) to \( v_1 \) on \( P_{1,2} \), followed by \( P_{1,3} \), followed by the path from \( v_3 \) to \( w \) on \( P_{3,4} \), and finishing with the edge \( e \) (connecting \( w \) back to the starting point \( v \)). The remaining vertices again comprise three odd-length paths (i.e., each with an even number of vertices). Again, the described even cycle can be perfectly matched with a matching containing \( e \), and the remaining paths can each be perfectly matched, yielding a perfect matching of \( K \) containing \( e \). □

**Corollary 4.6.** If \( K \) is a Deming-\( K_4 \) graph that is Egerváry, then every edge of \( K \) is in some perfect matching of \( K \).

The following theorem simply summarizes our previous results. The proof of condition (6) therein is analogous to that of (the more difficult) Theorem 4.4. The graph \( G[V(K) \cup V(R)] \) is the subgraph of \( G \) induced on the union of the vertices of subgraphs \( K \) and \( R \), that is, the graphs \( K \) and \( R \) together with any edges between them in \( G \).

**Theorem 4.7.** Let \( G \) be a graph with a perfect matching \( M \) and a Deming decomposition \( \{\{B_i\}_{i=1}^r, \{K_j\}_{j=1}^\ell, R\} \) (with respect to \( M \)). If \( G \) is Egerváry then:

1. there are no Deming-BP subgraphs (that is, \( r = 0 \));
2. each Deming-\( K_4 \) subgraph \( K_j \) is Egerváry;
3. every edge in each Deming-\( K_4 \) subgraph \( K_j \) is in a perfect matching of \( G \);
4. the Deming-\( K_4 \) subgraphs \( K_j \) are independent (that is, there are no edges in \( G \) incident to more than one Deming-\( K_4 \) subgraph);
5. for each Deming-\( K_4 \) subgraph \( K_j \), the graph \( G[V(K_j) \cup V(R)] \) is Egerváry; and
6. for the matching \( M_R \) consisting of \( M \) restricted to \( R \), there are no \( M_R \)-alternating paths between any pair of vertices \( v, w \) from different Deming-\( K_4 \) subgraphs.

We conjecture that this necessary condition is also sufficient—that is, this is a characterization of Egerváry graphs—and also that the independence number of \( G \) is the sum of the independence numbers of these Deming decomposition subgraphs. We record these formally.

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*Distance from \( v_2 \). We assume the former and omit the latter (which can be addressed similarly).*
Conjecture 4.8. Let $G$ be a graph with a perfect matching and a Deming decomposition $\{\{B_i\}_{i=1}^r, \{K_j\}_{j=1}^\ell, R\}$. The $G$ is Egerváry if and only if:

1. there are no Deming-BP subgraphs (that is, $r = 0$);
2. each Deming-$K_4$ subgraph $K_j$ is Egerváry;
3. every edge in each Deming-$K_4$ subgraph $K_j$ is in a perfect matching of $G$;
4. the Deming-$K_4$ subgraphs $K_j$ are independent;
5. for each Deming-$K_4$ subgraph $K_j$, the graph $K_j + R$ is Egerváry; and
6. there are no $M_R$-alternating paths between any pair of vertices $v$, $w$ from different Deming-$K_4$ subgraphs.

Conjecture 4.9. If $G$ is Egerváry with Deming decomposition $\{\{K_j\}_{j=1}^\ell, R\}$, then $\alpha(G) = \sum_{j=1}^\ell \alpha(K_j) + \alpha(R)$ (and thus $\alpha(G) = \nu(G) - \ell$).

5. Constructing Egerváry Graphs

Every KE graph is Egerváry (Corollary 1.4). But there are also non-KE graphs which are Egerváry. We give examples of some constructions. These examples are interesting not only because they provide infinite families of non-KE Egerváry graphs—but also because they are not obviously derived from Deming subgraphs with the addition of extra edges.

The graph $K_4$ is the smallest instance of what we call a weak wheel, i.e., a cycle $C$ with three or more nodes, a node $w$ not in $C$, and a path (spoke) from each node of $C$ to $w$ such that these spokes are disjoint except for $w$.

Theorem 5.1. Every weak wheel with a perfect matching is Egerváry.

Proof. A weak wheel cannot have a disjoint pair of odd cycles. \qed
A graph is a *weak banana* when it consists of two nodes, v, w, together with three or more internally disjoint vw-paths. Notice that a weak banana fails to be bipartite exactly when at least two of its vw-paths are of opposite parity.

**Theorem 5.2.** Every weak banana with a perfect matching is Egerváry.

*Proof.* A weak banana cannot have a disjoint pair of odd cycles. □

A *bracelet* consists of three distinct nodes u, v, and w, together with three weak bananas, respectively joining u to v, v to w, and w to u.

**Theorem 5.3.** Every bracelet with a perfect matching is Egerváry.

*Proof.* A bracelet cannot have a disjoint pair of odd cycles. □

Figure 8 depicts one example each of a weak wheel, a weak banana, and a bracelet.

A *bipartite extension* of a graph $G$ is obtained by replacing an edge $e = vw$ of $G$ by a connected bipartite graph $H$ so that $v$ is identified with a node in one part of $H$ and $w$ with a node in the other; we use $G : H$ to denote the resulting graph. See Figure 9 for the bipartite extension $K_4 : K_{3,3}$, in which the edge 2–5 of $K_4$ has been deleted, the vertex 2 is the identification of the vertex 2 from $K_4$ and one from $K_{3,3}$, and the vertex 5 is the identification of the vertex 5 from $K_4$ and one from the opposite part of $K_{3,3}$. It’s an exercise to confirm that a bipartite extension $G : H$ of a matchable graph $G$ is itself matchable only if the bipartite graph $H$ has parts of equal size.

The following assertion is immediate.

**Theorem 5.4.** If a matchable graph $G$ contains no disjoint pair of odd cycles, then every matchable bipartite extension of $G$ is Egerváry.

*Proof.* Under the hypotheses, a bipartite extension of $G$ cannot contain a disjoint pair of odd cycles. □

Theorem 5.4 provides a way to generate several infinite families of Egerváry graphs that are not K-E: every bipartite extension of a weak wheel (or a non-bipartite weak banana, or a bracelet) that has a perfect matching is non-KE but Egerváry.
Notice that the Egervány graphs of Theorems 5.1, 5.4 all can be obtained by starting with a graph $G$ that contains no disjoint pair of (either even or odd) cycles, then replacing some edges of $G$ by paths to get some odd cycles, and finally taking some bipartite extensions.

Here is another way to produce new Egervány graphs from old.

**Theorem 5.5.** If $H$ is Egervány, $R$ is a matchable KE graph vertex-disjoint from $H$, the set $I$ is a maximum independent set of $R$, and $G$ is formed by adding any edges from $H$ to $N = N(I)$, then $G$ is Egervány.

**Proof.** We proceed by contradiction. If $G$ is not Egervány, then $G$ contains a nice, even T-subdivision $S$. As $H$ is Egervány, it doesn’t contain $S$, and likewise the KE graph $R$ doesn’t contain $S$. So $V(S)$ meets both $V(H)$ and $V(R)$; let us call edges of $S$ with one end in $V(H)$ and one end in $V(R)$ crossing edges.

Because $H$ is Egervány (hence matchable), $R$ is matchable, and $S$ is nice, $G$ contains a perfect matching $M$ with $M_S = M \cap E(S)$ a perfect matching of $S$ and $M_R = M \cap E(R)$ a perfect matching of $R$. Notice that every perfect matching of $G$ necessarily matches $I$ into $N$ because $I$ is independent and $G$ contains no edges from $I$ to $H$. In particular, $M$ matches $I$ into $N$. Also, $M$ matches $N$ into $I$ because $M_R$ is a perfect matching of $R$.

Now consider a crossing edge $xy$ of $S$ with $x \in V(R)$ and $y \in V(H)$. Then $x \in N$, and since $M$ matches $N$ into $I$, the edge $xy$ is not in $M$. We distinguish two cases. 

Case (i): $xy$ is in a cycle $C$ of $S$.

As $M_S$ is a perfect matching of $S$ and $xy \notin M$, the next edge $xz$ of $C$, starting at $x$ and moving away from $y$, is in $M$, and, knowing $M$ matches $N$ into $I$, we see that $z \in I$. Thus, the next vertex $w$ of $C$ after $z$ has to be in $N$ and so cannot be $y$ (which is in $V(H)$, not $N$). Continuing around $C$ from $w \in N$, we can argue similarly as we did from $x \in N$ to see that the vertices of $C$ alternate between $N$ and $I$. But $C$ is an odd cycle, so when we finally return to $y$, say, along an edge $uy$, the vertex $u$ must be in $I$, and this forces $y$ to be in $N$. This is a contradiction because $y \in V(H)$ and $V(H) \cap N = \emptyset$.

Case (ii): $xy$ belongs to the path $P$ joining the two cycles of $S$.

As in Case (i), the next vertex $z$ of $P$ moving away from $y$ must lie in $I$ and the edge $xz$ must be in $M$. Continuing along $P$ from $z \in I$, the edges must alternate out and in of $M$ because $M_S$ is a perfect matching of $S$. And because $M$ perfectly matches $I$ with $N$, the vertices $x, z, \ldots$ of $P$ must lie alternately in $N$ and $I$. The structure of the perfect matching $M_S$ within $S$ puts the blossom tip $t$ reached along $P$ in this direction incident with an $M$-edge of $P$. Thus, the $N-I$ alternation implies that $t$ belongs to $I$. Being a tip, $t$ lies in one odd cycle $C'$ of $S$. With $t \in I$, we see that one neighbor $v$ of $t$ on $C'$ must belong to $N$. Now again, with $N$ matched to $I$ under $M$, the next vertex adjacent to $v$ on $C'$ must lie in $I$. As in Case (i), the vertices of $C'$ must alternate between $I$ and $N$ as we traverse $C'$ starting a sequence
$t \in I, v \in N$... Thus, with $C'$ being odd, the other neighbor of $t$ on $C'$ lies in $I$ together with $t$, which contradicts the independence of $I$. □

6. Independence Structure

Some structure of maximum independent sets was given in [12]: here we show that every graph can be decomposed into a KE graph and a 2-bicritical graph with certain attractive properties. A Deming decomposition of the (unique) 2-bicritical subgraph extends this earlier decomposition. It is worth noting that Pulleyblank [15] proved that almost all graphs are 2-bicritical.

Following a characterization in [15], we call a graph $G$ 2-bicritical if and only if $|N(I)| > |I|$ for every nonempty independent set $I$ of $G$. An independent set $I$ is critical if every independent set $J$ satisfies the relation $|N(I)| - |I| \geq |N(J)| - |J|$. The neighbors $N(I)$ of a critical independent set $I$ can necessarily be matched into $I$ (else it can be shown that $I$ is not critical). A maximum critical independent set is a critical independent set of largest cardinality. The following theorem implies that a graph is 2-bicritical if and only if it has no nonempty critical independent set. We can use this fact to provide a certificate that a graph is 2-bicritical.

Theorem 6.1 (Larson, [12]). For every graph $G$, there is a unique set $X \subseteq V(G)$ such that:

1. $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$;
2. $G[X]$ is a Kőnig-Egerváry graph;
3. $G[X^c]$ is 2-bicritical; and
4. each maximum critical independent set $J_c$ of $G$ satisfies $X = J_c \cup N(J_c)$.

A Deming decomposition of $G[X^c]$ then extends this decomposition of $G$. Note that if $G$ has a perfect matching, it immediately follows that the vertices of every maximum critical independent set $J_c$ must be matched to $N(J_c)$ and $|J_c| = |N(J_c)|$; it also necessarily follows that $G[X^c]$ has a perfect matching.

Proposition 6.2. If $G$ has a critical independent set $I$ with $|I| = |N(I)|$, then $G$ is Egerváry if and only if $G - I - N(I)$ is Egerváry.

Proof. For the necessity, suppose that $G - I - N(I)$ is not Egerváry. By Corollary [13], then this graph has a nice even subdivision $T_0$ of $T$. Because $I$ is perfectly matched with $N(I)$, the subgraph $T_0$ is also nice in $G$, and so $G$ is likewise not Egerváry.

Conversely, suppose that $G$ is not Egerváry. For a contradiction, suppose that $G - I - N(I)$ is Egerváry. With an eye toward applying Theorem [5.5], now let $H = G - I - N(I)$ and $R = G[I \cup N(I)]$ (the induced subgraph). Then the triple $(H, R, I)$ satisfies the hypotheses of Theorem [5.5], and the present graph $G$ is formed by adding some (maybe zero) edges from $H$ to $N(I)$ and so, by the same theorem, is Egerváry. This is a contradiction.

Therefore, $G - I - N(I)$ is not Egerváry, and the proof is complete. □
The class of $\alpha$-critical graphs was mentioned earlier (see the paragraph preceding Corollary 2.8); in particular, a Deming-$K_4$ graph is $\alpha$-critical if and only if it is an even subdivision of $K_4$ (Corollary 2.8). It is not the case that $\alpha$-critical graphs admit nontrivial decompositions with independence additivity.

**Proposition 6.3.** If $G$ is a connected $\alpha$-critical graph, if $V_1, V_2$ partition $V(G)$, and if $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$, then either $V_1$ or $V_2$ is empty.

**Proof.** Suppose that both $V_1$ and $V_2$ are nonempty. Since $G$ is connected, there must be some pair of adjacent vertices $v \in V_1$ and $w \in V_2$; and since $G$ is $\alpha$-critical, it is also the case that $\alpha(G-vw) = \alpha(G)+1$. But $\alpha(G-vw) \leq \alpha(G[V_1]) + \alpha(G[V_2])$. So $\alpha(G)+1 = \alpha(G-vw) \leq \alpha(G[V_1]) + \alpha(G[V_2]) = \alpha(G)$, which is impossible. □

**Corollary 6.4.** If $G$ is a connected $\alpha$-critical graph with maximum critical independent set $I$, and $X = I \cup N(I)$, then either $X$ or $X^c$ is empty.

**Proof.** The sets $X$ and $X^c$ partition $V(G)$, and Theorem 6.1 yields $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$. □

**Corollary 6.5.** If a connected graph $G$ is $\alpha$-critical, then $G$ is either KE or 2-bicritical.

**Extension to unmatchable graphs.** We have so far focused on graphs with perfect matchings. While random graphs of even order indeed have perfect matchings (see, e.g., [3, p. 178]), it’s natural to ask what can be said about graphs of odd order or other graphs admitting no perfect matching. Do our results yield anything about the independence structure of unmatchable graphs? Indeed they do.

Deming [5] gave a useful extension of a general graph $G$ to one with a perfect matching. Let $M$ be a maximum matching in $G$, and let $S$ be the set of $M$-unsaturated vertices of $G$. For each $v \in S$, add a new vertex $v'$ adjacent to $v$, and for each edge $vw$ in $G$, add an edge $v'w$; denote the resulting graph by $G'$. (Notice that the closed neighborhoods $N[v]$ and $N[v']$ in $G'$ are the same for each $v \in S$.) Now $G'$ has a perfect matching $M'$ (consisting of $M$ together with the edges $vv'$, for $v \in S$), and any maximum independent set in $G$ is necessarily a maximum independent set in $G'$ (see the following paragraph). We call the supergraph $G'$ the Deming extension of $G$ with respect to $M$ and $M'$ its standard perfect matching.

Without specific reference to Deming extensions, two vertices $v$, $v'$ in a graph $G$ are twins if they have the same closed neighborhoods; i.e., $N_G[v] = N_G[v']$—the definition implies that $v$ and $v'$ are adjacent. Twins arise in independence theory because a twin can be removed from $G$ without changing its independence number; to wit, if $v$ and $v'$ are twins in $G$ and $\tilde{G} = G - v'$, then $\alpha(\tilde{G}) = \alpha(G)$ (for if $I$ is a maximum independent set of $G$ containing $v'$, then $v \notin I$ and $I' = I - v' + v$ has $|I'| = |I|$ and is a maximum independent set of $\tilde{G}$). Because a Deming extension $G'$ is obtained from $G$ by adding certain twins, this proves the following
**Proposition 6.6.** If $G'$ is a Deming extension of $G$ (with respect to any maximum matching $M$), then $\alpha(G') = \alpha(G)$.

Theorem 6.9 below expands on this observation. Using Proposition 6.6, it is easy to check that $G'$ is KE if and only if $G$ is KE.

So the theory developed in this paper for matchable graphs can be applied to a Deming extension $G'$: $G'$ can be decomposed into Deming subgraphs and a KE subgraph; each of these parts defines a corresponding part in $G$—that will not necessarily be a Deming subgraph or a KE subgraph of $G$, but will be a subgraph of one of these in $G'$. These subgraphs interact with their parent graphs in convenient ways as the next two results show.

**Proposition 6.7.** Each Deming subgraph $D'$ in a Deming extension $G'$ of a graph $G$ and corresponding subgraph $D$ of $D'$ in $G$ satisfy $\alpha(D') = \alpha(D)$.

**Proof.** Let $D'$ be in $G'$, and let $D = D' \cap G$ (that is, the part of $D'$ that’s in $G$ or, more formally, $D = G[V(D') \cap V(G)]$, the graph induced on the common vertices). If $D = D'$ then there is nothing to show. So assume that there is a vertex $v' \in V(D') \setminus V(D)$ and also that $v'$ is in some maximum independent set $I'$ of $D'$. The edge $vv'$ is by definition in the standard matching $M'$, and, by the design of Deming’s Algorithm, $vv'$ is an edge in $D'$. Also, $v$ cannot be in $I'$ because $v'$ is in this independent set. Now $v$ and $v'$ have the same neighbors in $G'$ and thus the same neighbors in $D'$. So $v'$ can be replaced with $v$ to create a new maximum independent set $I = I' - v' + v$. As this can be done for every such vertex, we obtain a maximum independent set in $D'$ consisting only of vertices in $D$. And we can conclude that $\alpha(D) = \alpha(D')$. \qed

An analogous argument gives a parallel result for the KE-part of $G'$.

**Corollary 6.8.** The KE subgraph $R'$ in a Deming extension $G'$ of a graph $G$ and corresponding subgraph $R$ of $R'$ in $G$ satisfy $\alpha(R') = \alpha(R)$.

Thus, the independence structure of a general graph $G$ with no perfect matching is mirrored (in the sense given) in the independence structure of the matchable supergraph $G'$.

Moving forward, the reader should keep in mind the compendium of perfect matchings introduced by now. We have the standard perfect matching of $G'$, which may or may not coincide with an ‘initial’ perfect matching used to initialize the computation of a Deming decomposition. It’s possible that either of these matchings may not induce perfect matchings for the parts of the Deming decomposition. But these parts each have perfect matchings, the union of which yields a (third) perfect matching for the parent graph. In considering Deming decompositions, it is usually most convenient to work with these last ‘Deming decomposition induced’ perfect matchings.

Proposition 6.6 and the results following it show that Deming extensions $G'$—which are motivated by matching considerations—preserve simple independence properties of a given graph $G$. The next result implies that the
connection runs still deeper, to \( \alpha \)-criticality. Thus, we believe that Deming decompositions of \( \alpha \)-critical graphs will be applicable to further investigation of this important graph class.

The result bears a resemblance to the ‘Replication Lemma’ of Lovász and Plummer [14, Lemma 12.2.2], which substitutes the property of being a perfect graph for being \( \alpha \)-critical. (Of course, graphs can have either of these properties without the other; consider a 4-cycle [not \( \alpha \)-critical] and a 5-cycle [not perfect]. So the theorems are not related.) This result follows from a theorem of Wessel [19], but we give an independent proof.

**Theorem 6.9.** If \( v \) and \( v' \) are twin vertices in a graph \( G \), and \( \tilde{G} = G - v' \), then \( \tilde{G} \) is \( \alpha \)-critical if and only if \( G \) is \( \alpha \)-critical.

**Proof.** We use a couple of times the well-known fact that if a graph \( G \) is \( \alpha \)-critical and \( xy \in E(H) \), then there is a maximum independent set of \( H \) containing \( x \) and every maximum independent set of \( H - xy \) contains both \( x \) and \( y \) (II.3 Problem 8.12).

Suppose first that \( \tilde{G} \) is \( \alpha \)-critical. Let \( xy \) be an edge in \( G \). If \( xy \) is an edge in \( \tilde{G} \), then \( \alpha(\tilde{G} - xy) = \alpha(\tilde{G}) + 1 = \alpha(G) + 1 \). It remains to show that \( \alpha(G - xy) = \alpha(\tilde{G} - xy) \). Since \( \tilde{G} - xy \) is an induced subgraph of \( G - xy \), we have \( \alpha(\tilde{G} - xy) \leq \alpha(G - xy) \). Now let \( I \) be a maximum independent set in \( G - xy \). If \( v' \) is not in \( I \), then \( I \) is an independent set in \( \tilde{G} - xy \) and \( \alpha(G - xy) = \alpha(\tilde{G} - xy) \). If \( I \) does contain \( v' \), then \( I' = I - v' + v \) must also be an independent set in \( G - xy \) with cardinality \( |I| \), and \( \alpha(G - xy) = |I'| \leq \alpha(\tilde{G} - xy) \).

If \( xy \) is not an edge in \( \tilde{G} \), then either \( x \) or \( y \) must be \( v' \). Suppose that \( x = v' \). Then \( y \) is either \( v \) or a common neighbor of \( v \) and \( v' \). Suppose that \( y \) is \( \alpha \)-critical, there is a maximum independent set \( I' \) of \( \tilde{G} \) containing \( v \). Then \( I = I' + v \) is an independent set in \( G - v'v = G - xy \). So \( \alpha(G) + 1 = \alpha(\tilde{G}) + 1 = |I'| + 1 = |I| \leq \alpha(G - xy) \). Suppose then that \( y \) is a common neighbor of \( v \) and \( v' \). Then \( vy \) is an edge in \( \tilde{G} \), and since \( \tilde{G} \) is \( \alpha \)-critical, we have \( \alpha(G - vy) = \alpha(\tilde{G}) + 1 = \alpha(G) + 1 \). So it’s enough to show that

\[
\alpha(G - v'y) = \alpha(\tilde{G} - vy).
\]

Let \( I \) be a maximum independent set in \( G - v'y \). If \( I \) does not contain \( v' \), then \( I \) is independent in \( \tilde{G} \) and hence independent in \( \tilde{G} - vy \). So \( \alpha(G - v'y) = |I| \leq \alpha(\tilde{G} - vy) \). If \( I \) does contain \( v' \), it cannot contain any neighbor of \( v' \) except \( y \); then \( I' = I - v' + v \) must be an independent set in \( \tilde{G} - vy \). Thus \( \alpha(G - v'y) = |I| = |I'| \leq \alpha(\tilde{G} - vy) \). We’ve shown that in either case—i.e. \( I \) does not, or does, contain \( v' \)—the relation

\[
\alpha(G - v'y) \leq \alpha(\tilde{G} - vy)
\]

holds. To establish the reverse inequality, consider a maximum independent set \( I \) in \( \tilde{G} - vy \). As \( \tilde{G} \) is \( \alpha \)-critical, the set \( I \) contains both of \( v, y \). Thus,
I' = I - v + v' must be independent in G - v'y. So now we have \( \alpha(G - v'y) \geq |I'| = |I| = \alpha(G - vy) \), and combining this with (6.2) gives (6.1).

Conversely, suppose that G is \( \alpha \)-critical. Let \( xy \) be an edge in \( \tilde{G} \). Since \( \tilde{G} \) is a subgraph of G, the edge \( xy \) lies in \( G \) and \( \alpha(G - xy) = \alpha(G) + 1 \). As \( \alpha(G) = \alpha(\tilde{G}) \), it’s enough to show that \( \alpha(G - xy) = \alpha(\tilde{G} - xy) \). But \( \tilde{G} - xy \) is an induced subgraph of \( G - xy \), whence \( \alpha(\tilde{G} - xy) \leq \alpha(G - xy) \). Now let \( I \) be a maximum independent set in \( G - xy \), so that \( |I| = \alpha(G - xy) \). If \( I \) does not contain \( v' \), then \( I \) is an independent set in \( \tilde{G} - xy \) and \( \alpha(\tilde{G} - xy) \geq |I| = \alpha(G - xy) \). If \( I \) does contain \( v' \), then \( I' = I - v' + v \) must also be an independent set in \( \tilde{G} - xy \) with cardinality \( |I| \), and \( \alpha(G - xy) \geq |I'| = \alpha(G - xy) \).

It is a curiosity that a large number of \( \alpha \)-critical graphs appearing in the literature have twin vertices; these can be reduced by Theorem 6.9 to smaller twin-free \( \alpha \)-critical graphs. For instance, all complete graphs are in this sense ‘twin-equivalent’. \( K_1 \) is \( \alpha \)-critical, and since \( K_2 \) is the extension of \( K_1 \) by a twin, it is \( \alpha \)-critical, and so on: since \( K_{n-1} \) is \( \alpha \)-critical and \( K_n \) is a twin-extension of \( K_{n-1} \), it too is \( \alpha \)-critical.

**Corollary 6.10.** If \( G' \) is a Deming extension of a graph \( G \), then \( G' \) is \( \alpha \)-critical if and only if \( G \) is \( \alpha \)-critical.

**Proof.** A Deming extension can be viewed as the successive addition of a twin vertex \( v' \) for each vertex \( v \) that is left unsaturated by a maximum matching.

**Gallai class number.** The Gallai class number \( \delta = n - 2\alpha \)—a measure of complexity of critical graphs—has been central in \( \alpha \)-critical graph investigations; see, e.g., [14, Chapter 12]. Chvátal conjectured, and Sewell and Trotter proved [16], that every \( \alpha \)-critical graph with \( \delta \geq 2 \) contains an even subdivision of \( K_4 \). Matchable graphs \( G \) satisfy \( n = 2\nu \), so we have \( \delta = 2\nu - 2\alpha \). Since \( \nu \geq \alpha \) for such graphs, we have \( \delta \geq 0 \) in this case, and \( \delta \) is necessarily even. Also, \( \delta = 0 \) if and only if \( G \) is KE, and \( \delta = 2 \) if and only if \( \alpha = \nu = 1 \).

**Proposition 6.11.** If \( G \) is a connected \( \alpha \)-critical KE graph with a perfect matching \( M \), then \( G \) is \( K_2 \).

**Proof.** The hypotheses give \( \alpha = \nu \). For a contradiction, suppose that \( \nu > 1 \), so that \( |M| \geq 2 \) and, since \( G \) is connected, it contains a non-\( M \) edge \( e \). Now criticality gives \( \alpha(G - e) = \alpha(G) + 1 = \nu(G) + 1 = \nu(G - e) + 1 \), which is impossible. Thus \( \nu = 1 \), and since \( G \) has a perfect matching, \( G = K_2 \).

**Corollary 6.12.** If \( G \) is a connected \( \alpha \)-critical KE graph, then \( G \) is \( K_1 \) or \( K_2 \).

**Proof.** Let \( G \) be a nontrivial connected \( \alpha \)-critical KE graph. If \( G \) is matchable, then Proposition 6.11 shows that \( G = K_2 \), so we may assume that \( G \)}
contains no perfect matching. We shall see that this leads to a contradiction. If we form the Deming extension $G'$ of $G$, then Corollary 6.10 shows that $G'$ is $\alpha$-critical and Proposition 6.6 that $\alpha(G') = \alpha(G)$. By the definition of a Deming extension, in forming $G'$, we add $(n(G) - 2\nu(G))$ new vertices to $G$; so $n(G') = n(G) + (n(G) - 2\nu(G))$ and $\nu(G') = \nu(G) + (n(G) - 2\nu(G))$. With $G$ being KE, we have $\alpha(G) + \nu(G) = n(G)$, and it follows that $\alpha(G') + \nu(G') = n(G')$, so that $G'$ is also KE. But then Proposition 6.11 implies that $G'$ is $K_2$, which contradicts the fact that $G$ is nontrivial. □

**Corollary 6.13.** If $G$ is a connected $\alpha$-critical graph, then $G$ is either $K_1$, $K_2$, or is 2-bicritical.

**Proof.** Corollary 6.5 shows that either $G$ is KE or $G$ is 2-bicritical. Now Corollary 6.12 shows that if $G$ is KE, then it is $K_1$ or $K_2$. □

We now obtain new proofs of three basic facts about $\alpha$-critical graphs. For the first, cf. [13, Problem 8.21].

**Corollary 6.14.** If $I$ is an independent set in an $\alpha$-critical graph $G$ with no isolated vertices, then $|N(I)| \geq |I|$.

**Proof.** It’s enough to establish the conclusion for a component $H$ of $G$ and an independent set $I_H = I \cap V(H)$ of $H$. We may assume that $I_H$ is nonempty. As $H$ is also $\alpha$-critical, the hypotheses and Corollaries 6.5 6.12 together imply that $H$ is either $K_2$ or 2-bicritical. If $H$ is $K_2$, then $I_H$ consists of a single vertex which has a neighbor. If $H$ is 2-bicritical, then $|N(I_H)| > |I_H|$ is part of the defining condition for such graphs. □

For the second, cf. [14, Corollary 12.1.11].

**Corollary 6.15.** Every $\alpha$-critical graph $G$ without isolated vertices contains a perfect 2-matching.

**Proof.** As in the proof of Corollary 6.14 each component of $G$ is either $K_2$ or 2-bicritical. Both cases admit perfect 2-matchings, all of which can be assembled into a perfect 2-matching of $G$ ([14 Corollary 6.2.2] addresses the 2-bicritical case). □

For the third, cf. [14, Corollary 12.1.12].

**Corollary 6.16.** If $G$ is $\alpha$-critical and contains no isolated vertices, then $\alpha(G) \leq n/2$.

**Proof.** Since $\alpha$ is additive across $G$’s components, it again suffices to obtain the bound for each component $H$. If $H = K_2$, then this is clear, and if $H$ is 2-bicritical, then the bound follows from the defining condition ($|N_H(I)| > |I|$) applied to a maximum independent set $I$ of $H$. □

The following result is indicated in [14, p. 453] and obtained using a result of Hajnal [9] linking the Gallai class number $\delta$ to maximum degree. We include it because it also follows from the theory developed here.

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Proposition 6.17. If a graph $G$ is connected, $\alpha$-critical, and $\delta(G) = 0$, then $G = K_2$.

Proof. As $0 = \delta(G) = n - 2\alpha$ and $\alpha \geq 1$, we see that $G$ has even order $n \geq 2$. As $G$ is connected and $\alpha$-critical, Corollary 6.5 implies that $G$ is either KE or 2-bicritical. But $G$ cannot be 2-bicritical: if $I$ is a maximum independent set (with $\alpha = |I|$), it is impossible for both $n = 2\alpha$ and $|N(I)| > |I|$. Then Corollary 6.12 implies that $G$ is $K_1$ or $K_2$; and since $n \geq 2$, we have $G = K_2$. □

The following theorem about graphs with $\alpha = \nu - 1$ implies a characterization of those connected $\alpha$-critical graphs with Gallai class number 2 (Theorem 6.20 below).

Theorem 6.18. A graph $G$ with a perfect matching $M$ satisfies $\alpha(G) = \nu(G) - 1$ if and only if every Deming decomposition of $G$ consists of a KE graph $R$ and a single Deming subgraph $D$ (with perfect matching $M_D = M \cap E(D)$) such that for some edge $xy \in M_D$, the graph $G - \{x, y\}$ is KE.

Proof. Write a given Deming decomposition (with respect to $M$) as $\{\{B_i\}_{i=1}^r, \{K_j\}_{j=1}^\ell, R\}$. Theorem 3.1 shows that $\alpha(G) \leq \nu(G) - (r + \ell)$.

Suppose first that $\alpha(G) = \nu(G) - 1$. Then $r + \ell = 1$, so either $r = 1$ or $\ell = 1$, whence $G$ has exactly one Deming subgraph $D$, and it must be the case that $\alpha(D) = \nu(D) - 1$ (see Theorem 3.1 parts (1, 2, 5)). Let $I$ be a maximum independent set of $G$ (so $|I| = \nu(G) - 1$), and let $I_D = I \cap V(D)$ and $I_R = I \cap V(R)$; $I_D$ is then a maximum independent set of $D$ and $I_R$ is a maximum independent set of $R$. Note that no more than one vertex of $I_D$ is incident to any edge of $M_D$, and—because $\alpha(D) = \nu(D) - 1$—there must be one edge $xy \in M_D$ that is not incident with any vertex in $I_D$. Let $G' = G - \{x, y\}$. By construction, $I \subseteq V(G')$, and $M' = M - \{xy\}$ is a perfect matching of $G'$ with cardinality $|I|$. Thus $G'$ is KE.

Now suppose that every Deming decomposition consists of a KE graph $R$ and a single Deming subgraph $D$ (with perfect matching $M_D = M \cap E(D)$) such that for some edge $xy \in M_D$, the graph $G' = G - \{x, y\}$ is KE. Since $M$ is a perfect matching of $G$ and $xy \in M$, the graph $G'$ is also matchable and $\nu(G') = \nu(G) - 1$. Now $G'$, being KE, has $\alpha(G') = \nu(G')$, which implies that $G'$ has an independent set $I \subseteq V(G')$ of order $\nu(G')$. Thus, $I$ consists of exactly one end of each edge in $M - xy$, and therefore $I$ is also independent in $G$. So we have

$$\alpha(G) \geq |I| = \nu(G') = \nu(G) - 1.$$  \hspace{1cm} (6.3)

But $G$ is not KE because its Deming decomposition contains a non-null Deming subgraph $(D)$. This means that the chain (6.3) is sharp; i.e., $\alpha(G) = \nu(G) - 1$. □

We’re soon to recover the result of Andrásfai [11] on connected $\alpha$-critical graphs with $\delta = 2$ (to which we alluded prior to the statement of Theorem 6.18); first we need to show that such graphs are matchable.
Lemma 6.19. If a graph $G$ is connected, $\alpha$-critical, and $\delta(G) = 2$, then $G$ contains a perfect matching.

Proof. As $2 = \delta(G) = n - 2\alpha$ and $\alpha \geq 1$, we see that $G$ has even order $n \geq 4$. With $G$ being connected and $\alpha$-critical, Corollary 6.5 shows that $G$ is either KE or 2-bicritical. But it’s not the first of these, for if so, then Corollary 6.12 would give $G \in \{K_1, K_2\}$ and thus contradict $n \geq 4$. Since $G$ is 2-bicritical, it contains a spanning subgraph $H$ consisting of a matching $M$ together with an even number of odd cycles $C_1, C_2, \ldots, C_\ell$ (cf. [14, Corollary 6.2.2], where $H$ is termed a ‘basic perfect 2-matching’ of $G$). As $n \geq 4$ and $G$ is connected, $G$ must contain an edge $e \notin E(H)$. Because $G$ is $\alpha$-critical, we have

$$\alpha(G - e) = \alpha(G) + 1 = \frac{n - 2}{2} + 1 = \frac{n}{2}.$$  

Now consider an independent set $I$ in $G - e$ with $|I| = n/2$. Since $e \notin E(H)$, the subgraph $H$ spans $G - e$, and so the set $I$ is also independent in $H$. Such a set can occupy at most one end of each edge of $M$ and at most $(|C_i| - 1)/2$ vertices of each cycle $C_i$. Thus,

$$|I| \leq |M| + \sum_{i=1}^{\ell} \left|C_i\right| - 1 \geq \frac{n}{2} - \frac{\ell}{2},$$  

and since $|I| = n/2$, the bound (6.4) shows that $\ell = 0$. That is, $H$ contains no cycle components and is therefore a perfect matching of $G$. \hfill \Box

Perhaps surprisingly, the hypotheses of Lemma 6.19 guarantee not just a perfect matching in $G$ but an exact, concise description of this graph. At this point, it may be instructive to review our earlier and much simpler Proposition 2.1.

Theorem 6.20 (Andrásfai, [1]). If a graph $G$ is connected, $\alpha$-critical, and $\delta(G) = 2$, then $G$ is an even subdivision of $K_4$.

Proof. Denote by $M$ the perfect matching guaranteed by Lemma 6.19. We showed above that $\delta(G) = 2$ is equivalent to $\alpha(G) = \nu(G) - 1$. Then Theorem 6.18 implies that a Deming decomposition of $G$ consists of a KE graph $R$ and a single Deming subgraph $D$ (with perfect matching $M_D = M \cap E(D)$).

While $R$ (which is KE) may not have any vertices, $D$ must (if $V(D) = \emptyset$, then $\alpha(G) = \alpha(R) = \nu(R) = \nu(G)$, contrary to the fact that $\alpha(G) = \nu(G) - 1$). Assume then that $V(R) \neq \emptyset$. The relation $\alpha(G) = \nu(G) - 1$ implies that $\alpha(R) = \nu(R)$ and $\alpha(D) = \nu(D) - 1$. And $\nu(G) = \nu(R) + \nu(D)$ implies that $\alpha(G) = \alpha(R) + \alpha(D)$. Since $G$ is connected, there must be an edge from some vertex $x$ in $R$ to some vertex $y$ in $D$. Notice that $\alpha(G - xy) = \alpha(G)$. (Otherwise, $\alpha(G - xy) > \alpha(G) = \alpha(R) + \alpha(D)$; but then a maximum independent set in $G - xy$ would, by the Pigeonhole Principle, contain either more than $\alpha(R)$ vertices of $R$ or more than $\alpha(D)$ vertices of $D$.
and so could not be independent.) But this contradicts \( G \) being \( \alpha \)-critical. So it must be that \( R \) is a null graph and \( G = D \).

With \( D \) being an \( \alpha \)-critical Deming graph, Corollary 2.10 shows that \( D(= G) \) is an even subdivision of \( K_4 \). \( \square \)

The following theorem suggests a tantalizing connection between the structure of \( \alpha \)-critical graphs and Deming decompositions. Is it true that, for instance, any connected \( \alpha \)-critical graph with a perfect matching has an associated Deming decomposition with a Deming-\( K_4 \) graph (and hence the required even \( K_4 \)-subdivision)?

**Theorem 6.21** (Sewell and Trotter, [16]). If a graph \( G \) is connected, \( \alpha \)-critical, and \( \delta(G) \geq 2 \), then \( G \) contains an even subdivision of \( K_4 \).

What more can be said about the independence structure of the 2-bicritical subgraph of an arbitrary graph? We assume that it has a perfect matching. It is a simple consequence of this and the defining condition of 2-bicritical graphs that any independent set can be matched and thus that \( \alpha < \nu \).

We also assume that some Deming decomposition of the graph has \( k \) Deming graphs (either Deming-BP or Deming-\( K_4 \) graphs) \( D_i \). We know (from Theorem 2.6) that these Deming graphs satisfy \( \alpha(D_i) = \nu(D_i) - 1 \). An interesting feature of the following claims and related algorithms is that a perfect matching \( M \) must be computed only once—it can be reused for all further computations.

The following result generalizes Theorem 6.18—needed for our investigation of \( \alpha \)-critical graphs—to matchable graphs with \( k \) Deming subgraphs in a Deming decomposition.

**Theorem 6.22.** Let \( G \) be a graph with a perfect matching \( M \) and a Deming decomposition consisting of a KE graph \( R \) and Deming subgraphs \( \{D_i\}_{i=1}^k \) with \( k \geq 1 \). Let \( M_i \) be the edges of \( M \) in \( D_i \). Then \( \alpha(G) = \nu(G) - k \) if and only if there is an edge \( x_iy_i \in M_i \), for each \( i \in [k] \), such that \( G - \{x_1, \ldots, x_k, y_1, \ldots, y_k\} \) is KE.

**Proof.** The independence number of \( G \) is no more than the sum of the independence numbers of the KE and Deming subgraphs from a Deming decomposition. Thus,

\[
\alpha(G) \leq \alpha(R) + \sum_{i=1}^{k} \alpha(D_i) = \nu(R) + \sum_{i=1}^{k} (\nu(D_i) - 1) = \nu(G) - k,
\]

the first equality following from Theorem 2.6 and the fact that \( R \) is KE.

Suppose first that \( \alpha(G) = \nu(G) - k \), and let \( I \) be a maximum independent set of \( G \) (so \( |I| = \nu(G) - k \)). With \( I_i = I \cap V(D_i) \) and \( I_R = I \cap V(R) \), the sets \( I_i \) are maximum independent sets of their respective \( D_i \)'s, and \( I_R \) is a maximum independent set of \( R \). Note that no more than one vertex of \( I_i \) is incident to any edge in \( M_i \), and—because \( \alpha(D_i) = \nu(D_i) - 1 \)—there must be one edge \( x_iy_i \in M_i \) that is not incident with any vertex in \( I_i \).
Let $G' = G - \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. By construction, $I \subseteq V(G')$ and $M' = M - \{x_1y_1, \ldots, x_ky_k\}$ is a perfect matching of $G'$ with cardinality $|I|$. Thus $G'$ is KE.

Suppose now that there is an edge $x_iy_i \in M_i$ for each $i \in [k]$ such that $G' = G - \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ is KE. Since $G'$ has the perfect matching $M' = M - \{x_1y_1, \ldots, x_ky_k\}$ with $|M'| = |M| - k = \nu(G) - k$, it follows that $\alpha(G') = \nu(G') = \nu(G) - k$, which shows that $G'$ has an independent set $I \subseteq V(G')$ of order $\nu(G) - k$. Thus, $I$ consists of exactly one end of each edge in $M - \{x_1y_1, \ldots, x_ky_k\}$, and so $I$ is also independent in $G$. We therefore have

$$\alpha(G) \geq |I| = \nu(G) - k,$$

and combining (6.5) with (6.6) gives $\alpha(G) = \nu(G) - k$. □

It may not be the case that $\alpha(G) = \nu(G) - 1$ for a graph $G$ with a perfect matching $M$ and a Deming decomposition with a single Deming subgraph $D$. Theorem 6.18 identifies a condition that guarantees this; namely, for some edge $xy \in M_D$, the graph $G - \{x,y\}$ is KE. Failing this, $\alpha(G) = \nu(G) - k$ for some $k > 1$. Can this $k$ be efficiently identified? This (open) question provides a natural segue to our closing section.

7. (More) Open Problems

The motivating questions of this investigation were:

1. If a graph is Egerváry, is there an efficiently checkable certificate?
2. Is there an NP ∩ co-NP description (i.e., a good characterization) of graphs that have no disjoint pair of odd cycles? Or is it NP-complete to decide?
3. Can Egerváry graphs be recognized in polynomial-time?
4. Is it possible to identify a maximum independent set in an Egerváry graph in polynomial-time?

Our investigations herein raise a new question:

5. If $G$ is a graph with a perfect matching $M$ and a Deming decomposition $\{\{K_j\}_{j=1}^{\ell}, R\}$ (with respect to $M$), is it true that $\alpha(G) = \sum_{j=1}^{\ell} \alpha(K_j) + \alpha(R)$?

These questions remain open.

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