On Learning to Rank Long Sequences with Contextual Bandits

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Abstract

Motivated by problems of learning to rank long item sequences, we introduce a variant of the cascading bandit model that considers flexible length sequences with varying rewards and losses. We formulate two generative models for this problem within the generalized linear setting, and design and analyze upper confidence algorithms for it. Our analysis delivers tight regret bounds which, when specialized to vanilla cascading bandits, results in sharper guarantees than previously available in the literature. We evaluate our algorithms on a number of real-world datasets, and show significantly improved empirical performance as compared to known cascading bandit baselines.

1 Introduction

A well-known problem in content recommendation is the generation of slates of items whereby, given a set of available items and a limited number of available slots, the goal of the system is to come up with an ordered sequence of items to be arranged in the slots so as to best fulfill some goal, like improving the experience of the user at hand. Applications are ubiquitous, from web search to news recommendation, from computational advertising to web page content optimization. These are among the most prominent motivating applications behind the more abstract problem called learning to rank.

The cascade model (e.g., Chuklin et al. [2015]) for learning-to-rank has emerged as a simple and effective way to model user behavior in a number of applications. In this model, the user scans the slate sequentially from top to bottom and clicks on the first item they find attractive, disregarding all subsequent items in the slate. The length of the slate may vary widely across applications, ranging from a few items in computational advertising to dozens in news recommendation to hundreds in web search. In these and many other dynamic domains, one has to deal with a near continuous stream of new items to be recommended, along with new users to be served. Out of the collected user feedback, and in the face of a constantly evolving content universe and set of targeted users, the learning system is expected to maintain over time a good mapping between user/item features and item rankings.

In order to encompass a variety of learning-to-rank applications for dynamic environments, we introduce a generalized version of the well-known cascading bandit model of Kveton et al. [2015a]. Our model considers flexible sequence length with varying rewards and losses. The problem is broadly described by position-dependent rewards $r_j$ and losses $\ell_j$. These parameters measure how well the ranking system is doing depending on the position $j$ of the first positive signal (e.g., the first click) as well as the potential loss associated with a sequence of $j$ negative signals. Since rewards are positive and losses are negative, and the two sequences are decreasing with $j$ (in particular, $\ell_j$ becomes more and more negative as $j$ increases), this model is intended to capture a natural trade-off in decision making.

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If we commit to a long sequence, we may increase our chance of success (positive reward), but also expose ourselves to the risk of a very negative loss if all signals on that sequence turn out to be negative.

This trade-off is typical in scenarios where each negative signal in the sequence is indeed a cost for the system. As a relevant example, suppose we want to deploy our ranking algorithm within a payment system (e.g., Stripe) where, at each round we process one transaction, and the goal is to find routes to fulfill the transaction. Here, each payment attempt comes with a cost for the system, the positive signal on a route corresponds to payment fulfillment through that route, while the negative signal corresponds to a payment failure. Every unsuccessful attempt reduces the net reward gathered by a subsequent success, and may translate into bigger losses if in the end the payment is not fulfilled. This provides a classic use case of cascade models since we have to predict a ranked sequence of routes for the payment to be fulfilled with as few retries as possible. Note that the length of the ranked sequence can be large and flexible which further aligns this application to our setting.

Our contribution. In this paper, we describe two contextual upper confidence bandit algorithms for this problem, specifically focusing on the case of long ranked sequences. We analyze the two algorithms both theoretically and experimentally. Our theoretical analysis delivers tighter regret guarantees than previous investigations. In particular, we obtain a regret bound of the form $\sqrt{T}$, where $T$ is the time horizon and $b$ is the length of the ranked sequences, as opposed to $b\sqrt{T}$ achieved by prior work in cascading bandits. We then validate our algorithms experimentally on well-known benchmark datasets, and show significantly improved performance as compared to the state-of-the-art algorithms.

Related work. The study of cascading bandit models for ranking problems has been initiated by Kveton et al. [2015a]. The authors study the problem of learning to rank items on a fixed number of slots under the so-called cascade click model of user behavior. Li et al. [2016], Zong et al. [2016], Li and Zhang [2018] investigate large-scale variants where the reward of an item follows (generalized) linear structure. Cheung et al. [2019] gives an analysis for Thompson sampling. Cascading bandits have also been studied under more general click models, which can recover the standard cascade click model as well as other classical click models in the literature of online learning to rank (e.g., Zoghi et al. [2017], Lattimore et al. [2018], Li et al. [2019]), Li and De Rijke [2019] considers cascading bandits in non-stationary environments, and Hiranandani et al. [2020] studies more comprehensive cascading models of user behavior that account for both position bias and diversity of recommendations. All these works consider the case of sequences with fixed length and, when specialized to the original cascading bandit model of Kveton et al. [2015a] or generalized linear variants thereof, their analysis delivers a suboptimal dependence on the length of the sequence, which is a main theoretical concern in this paper. An in-context regret bound comparison to many of these works is carried out in Section 3. Further related work is discussed in Appendix C.

2 Setting and Main Notation

We formalize our problem of contextual bandits with long sequences as follows. Learning proceeds in a discrete sequence of time steps (or rounds or trials). At each time $t$, the learner processes a transaction having at its disposal a (finite) set of actions (or items) $A_t = \{x_{1,t}, x_{2,t}, \ldots, x_{k,t}\} \subseteq A = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$, each action being described by a $d$-dimensional feature vector of (Euclidean) norm at most one. Set $A_t$ is our context information at time $t$, while set $A$ is the universe of all possible actions. Collectively, $A_t$ may include information about the specific context in which learning is applied. In a payment scenario, this will typically include the transaction amount, the buyer and seller identities (or features), the credit card company identity (or features), etc. In a news recommendation problem this may include user features, news-of-the-day topic features, and so on. Each action corresponds to an item available at time $t$.

The learning problem is parameterized by a decreasing (or non-increasing) sequence of rewards $r_{1,t}, r_{2,t}, \ldots$ and a decreasing (or non-increasing) sequence of losses $\ell_{0,t}, \ell_{1,t}, \ell_{2,t}, \ldots$, where

$$1 \geq r_{1,t} \geq r_{2,t} \geq \ldots > 0 \quad \text{and} \quad 0 > \ell_{0,t} \geq \ell_{1,t} \geq \ell_{2,t} \geq \ldots > -1 .$$

The rewards are positive, while the losses are negative. The dependence on $t$ of these quantities emphasizes the potential dependence of these values on the current context. For instance, in the payment scenario, $r_{1,t}$ is often proportional to the amount of the current transaction. Moreover, to set the scale of these parameters, we shall assume throughout that $r_{1,t} \in [0, 1] \text{ and } \ell_{1,t} \in [-1, 0]$

1This normalization is done for notational convenience only; any bounded action space would work here.
After playing \( \ell \) we "give up" and incur loss. We simply suffer loss (or negative reward) for all \( i \) and \( t \). Finally, each transaction may be accompanied by a budget value \( b_t \) that bounds from above the number of allowed retries, as defined next.

In round \( t \), the algorithm is compelled to play an ordered sequence of actions \( J_t = \langle x_{j_1,t}, x_{j_2,t}, \ldots, x_{j_{s_t},t} \rangle \), where each component vector \( x_{j,t} \) is taken from \( A_t \). We call \( J_t \) a retry sequence or simply a sequence. The set of all such sequences \( J_t \) corresponds to the action space available to the learner at time \( t \). Notice that the length \( s_t \) of \( J_t \) is part of the action selected by the learner (that is, the algorithm has to decide the length of the sequence as well). This length \( s_t \) determines the number of retries on the transaction at time \( t \). \( J_t \) can also be empty; in such a case we have \( s_t = 0 \) and write \( J_t = \langle \rangle \). The budget constraint \( b_t \) requires \( s_t \) to satisfy \( s_t \leq b_t \). In general, \( b_t \) may depend on time, and there are practical scenarios where this is indeed advisable, e.g., a payment system where the number of attempts depends on the transaction amount.

Sequence \( J_t \) has associated rewards and losses as detailed next. Upon committing to \( J_t \), if \( J_t = \langle \rangle \) we simply suffer loss (or negative reward) \( \ell_{0,t} \) and go to the next round. Otherwise, the first item \( x_{j_{1,t}} \) is attempted. If \( x_{j_{1,t}} \) is successful we gather reward \( r_{1,t} \) and stop, going to the next round. If \( x_{j_{1,t}} \) is unsuccessful, \( x_{j_{2,t}} \) is attempted. If \( x_{j_{2,t}} \) is successful we gather reward \( r_{2,t} \) and again stop. In this way, finally, \( x_{j_{s_t,t}} \) is attempted. If \( x_{j_{s_t,t}} \) is successful we gather reward \( r_{s_t,t} \) and stop. Otherwise, we "give up" and incur loss \( \ell_{s_t,t} \). A pictorial illustration is given in Figure 1.

The effort behind this parametrization for rewards and losses is to capture the tension between a potentially small reward of a successful late retry and a potentially small loss incurred by an early give up. On one hand, the earlier is the success in a sequence \( J_t \) the higher the reward is likely to be. On the other, the later we give up (after many unsuccessful attempts) the higher is the loss we incur.

For simplicity, in this model rewards and losses incurred at time \( t \) only depend on the position of the items in sequence \( J_t \), rather than the actually played item in that position. Also, upon processing the transaction at time \( t \), the algorithm has to commit to the entire sequence \( J_t \), that is, this sequence cannot be changed on the fly based on partial observations we are gathering on that sequence. So, this is indeed a (parametric) cascading bandit model.

After playing \( J_t \) at time \( t \), the algorithm observes the reward associated with \( J_t \), which is generated as follows. Let the outcome vector \( Y_t \) be a Boolean vector \( Y_t = \langle y_{1,t}, \ldots, y_{|A_t|,t} \rangle \in \{0,1\}^{|A_t|} \). Then we can define the reward \( R_t(J_t, Y_t) \) of sequence \( J_t \) at time \( t \) (i.e., on the transaction occurring at time \( t \)) w.r.t. outcome \( Y_t \) as follows (for ease of notation, we drop subscript \( t \) and leave the

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A sequence might have repeated actions, but for simplicity we assume here each component of \( J_t \) is distinct.

This is typically the case when the system is serving ranked content to (human) users.
We will soon give (2) a parametric form. For the moment, observe that, based on the above generative model, we can define the expected reward \( R_t(J_t, Y_t) \) of \( J_t \) on \( A_t \) w.r.t. the random draw of \( Y_t \). Specifically, if we take an expectation of (1) we obtain (we again drop subscript \( t \) for readability):

\[
E_Y[R(J, Y)] = r_1 p(x_{j_1}) + \ldots + r_s p(x_{j_s}, x_{j_{s+1}}, \ldots, x_{j_{s-1}}) \prod_{i=1}^{s-1} \left( 1 - p(x_{j_i} \mid x_{j_1}, \ldots, x_{j_{i-1}}) \right) + \ell_s \prod_{i=1}^{s} \left( 1 - p(x_{j_i} \mid x_{j_1}, \ldots, x_{j_{i-1}}) \right),
\]

and \( E_Y[R(J, Y)] = \ell_0 \) if \( J = \emptyset \). The involved conditional probabilities (2) are the only ones that matter in computing the expected reward \( E_Y[R(J, Y)] \). Moreover, the expected reward can be either positive or negative, due to the fact that the last term is negative.

For a given pair \((A_t, b_t)\), a natural benchmark to compare to is the Bayes optimal sequence \( J_t^* = (x_{j_1}^{b_t}, x_{j_2}^{b_t}, \ldots, x_{j_{s-1}}^{b_t}) \), that is, the sequence \( J_t \) that maximizes \( E_{Y_t}[R_t(J_t, Y_t)] \) over all possible sequences built on \( A_t \), of length at most \( b_t \). Recall that \( J_t^* \) is computed by knowing beforehand all probabilities (2) for all candidate sequences \( J_t \). Consequently, we define the
time-$t$ (pseudo) regret of an algorithm that commits to $J_t$ on $A_t$ as $E_{Y_t}[R_t(J_t^*, Y_t) − E_{Y_t}[R_t(J_t, Y_t)]]$, and its cumulative regret over $T$ rounds on the sequence of pairs $(A_1, b_1), (A_2, b_2), \ldots, (A_T, b_T)$ as
\[
T \sum_{t=1}^{T} E_{Y_t}[R_t(J_t^*, Y_t) − E_{Y_t}[R_t(J_t, Y_t)]].
\]
Our goal is to make the above quantity as small as possible (with high probability). Next, we formulate a parametric model for the conditional probabilities \footnote{4Such coverage vectors can be obtained based on domain knowledge. E.g., they may be obtained as a latent probability distribution after training a Gaussian Mixture Model where the $d'$ Gaussian centroids represent the latent topics, and $c_i(x)$ is the probability that $x$ belongs to topic $i$ according to the mixture model. This is essentially what we do in our experiments in Section \footnote{5As the reader can easily see, the content of this paper can be seamlessly extended to more general link functions (see, e.g., the treatment in \cite{gentile2012less}) but, for simplicity of presentation, we restrict to the sigmoidal link.}} $p$, and show: (i) how to compute $J_t^*$, and (ii) how to define the contextual bandit algorithms that determines $J_t$ so as to make the cumulative regret small.

2.2 Parametric model

Given our universe of actions $A = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$, we associate each item $x$ with a so-called coverage vector $c(x) = (c_1(x), \ldots, c_d'(x)) \in [0, 1]^{d'}$, where $d'$ is the dimensionality of a latent space of topics.\footnote{4} The coverage $c_i(A')$ of a (finite) set $A' \subseteq A$ of items on topic $i$ is a monotone and sub-modular function on sets, e.g., $c_i(A') = 1 − \prod_{x \in A'} (1 − c_i(x))$, with $c_i(\emptyset) = 0$. Here we slightly abuse the notation and set $c_i(x) = c_i(\{x\})$. Following, e.g., \cite{yue2011contextual}, \cite{hiranandani2020contextual}, we then define the $d'$-dimensional vector $c'(x_j | x_{i_1}, \ldots, x_{i_k})$ of coverage differences, whose $i$-th component is
\[
c_i(\{x_{i_1}, \ldots, x_{i_k}, x_j\}) − c_i(\{x_{i_1}, \ldots, x_{i_k}\}) \in [0, 1].
\]
Since such vectors have only positive components, we shift them to their center so as both positive and negative components exist, and then divide by a constant that makes their norm at most 1. For instance, we may set $\bar{c}_i(x_j | x_{i_1}, \ldots, x_{i_k}) = \frac{1}{d'} (2c_i'(x_j | x_{i_1}, \ldots, x_{i_k}) − 1)$ to be the $i$-th component of the transformed vector $\bar{c}(\{x_{i_1}, \ldots, x_{i_k}\})$ of coverage differences.

Our parametric model is represented by a $d'$-dimensional vector $u \in \mathbb{R}^d$ with the link function $\sigma : \mathbb{R} \to [0, 1], \sigma(z) = \frac{\exp(z)}{1 + \exp(z)}$. Specifically we set the conditional probability as
\[
p(x_j | x_{i_1}, \ldots, x_{i_k}) = \sigma(\bar{c}(x_j | x_{i_1}, \ldots, x_{i_k})^\top u). \tag{5}
\]
Hence the marginal probabilities $p(x)$ and conditional probabilities $p(x_j | x_{i_1}, \ldots, x_{i_k})$ are encoded as generalized linear functions with unknown parameter vector $u$. The idea behind this model is that if the additional topic-wise diversity brought up by $x_j$ as compared to the already selected $x_{i_1}, \ldots, x_{i_k}$ is relevant w.r.t. the weight vector $u$, then the probability that $x_j$ is successful given that $x_{i_1}, \ldots, x_{i_k}$ has failed should be large. The opposite happens if the additional diversity contributed by $x_j$ is indifferent w.r.t. $u$.

We now separate two cases: (i) the independent outcome case, where only marginal probabilities $p(x)$ are needed, and (ii) the more general dependent outcome case, where also the conditional probabilities $p(x_j | x_{i_1}, \ldots, x_{i_k})$ have to be considered. As we will see in the sequel, (ii) reduces to (i), up to the computation of $J_t^*$. For the independent case we can simply set $\bar{c}(x | x_{i_1}, \ldots, x_{i_k}) = x$, for all $x, x_{i_1}, \ldots, x_{i_k}$, and $d' = d$ to save notations, which makes $p(x) = \sigma(u^\top x)$.

3 Independent Outcomes

This is the simplest possible setting where the Boolean vector $Y_t$ has independent components. In this case, in \footnote{4} we have $p(x_j | x_{i_1}, \ldots, x_{i_k}) = p(x_j)$ for all $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$, and $x_j$. Hence there is no reason to model conditional probabilities, and we restrict to modeling $p(x) = \sigma(u^\top x)$. Moreover, in this case, Bayes is formulated only by means of marginal probabilities $p(x_i)$, and reduce to sorting items in $A_t$ in decreasing order of $p(x_j)$ and stopping when a suitable condition is met. We now claim that, in this specific case, the Bayes optimal sequence $J_t^*$ can be computed fairly easily. Due to space limitations, all proofs are postponed to the appendix.
Algorithm 1: The contextual bandit algorithm in the independent case. Here the link function
\[ \sigma(\cdot) = \frac{\exp(\cdot)}{1 + \exp(\cdot)} \]

**Lemma 1.** Let \( p_\cdot(A) = \prod_{j=1}^{|A|} p(x_j) \), and \( b \) be the budget length. Then \( J^* \) can be computed as follows. Set \( s^* = \arg \max_{s=0,1,\ldots,b} \mathbb{E}_Y[R(J^*_s, Y)] \), where \( J^*_s = \langle x^*_1, x^*_2, \ldots, x^*_s \rangle \), \( x^*_1, x^*_2, \ldots, x^*_s \) the items associated with the \( s \) largest marginal probabilities \( p(x_j), \ x_j \in A \), sorted in non-increasing order. Then \( J^* = J^*_s^* \), with \( J^*_s^* = () \) if \( s^* = 0 \).

The bandit algorithm corresponding to (or mimicking) the Bayes computation in Lemma 1 is described in Algorithm 1. In this pseudo-code and elsewhere, we use the notation \( Y_t \downarrow J_t \), henceforth called outcome projected onto the retry sequence, to denote the binary string of the form \( \langle \rangle \) which encodes the components of output vector \( \hat{Y}_t \) that are revealed by playing sequence \( J_t \). Recall Figure 1 for an
example: If \( Y_t = (0, 0, 1, 0, 1, 0, 0, 1, 0, 0) \) and \( J_t = \langle x_1, x_2, x_7, x_{10} \rangle \) we have \( Y_t \downarrow J_t = \langle 0, 0, 1 \rangle \), that is, playing \( J_t \) when the outcome is \( Y_t \) reveals the components of \( Y_t \) in the order determined by \( J_t \) up to the first 1 in \( Y_t \). In this example, we observe the 1st, the 2nd, and the 7th component of \( Y_t \).

Notice, in particular, that we do not observe \( Y_t \)'s 10th component.

Algorithm [1] replaces the true marginal probabilities \( p(x_j) = \sigma(u^T x_j) \) with upper confidence estimations \( \hat{p}_{j,t} = \sigma(\Delta_{j,t} + \epsilon_t) \), and then mimics the Bayes optimal computation to determine \( J_t \).

The update rule is a second-order descent method on an appropriate loss function (logistic, in this case) associated with the link function \( \sigma \). Notice that the items \( x_j \) which do not occur in \( Y_t \downarrow J_t \) have \( s_{j,t} = 0 \), hence they do not contribute to the update of \( M_t \) or \( w_t \). Yet, it is important to emphasize that \( s_{i,t} \) can be zero (that is, the corresponding component \( y_{t,i} \) is not observed) also due to the fact that an earlier item than \( x_i \) in \( J_t \) has been successful. The update of vector \( w_{c_i,t-1} \rightarrow w_{c_i,t} \) is done by first projecting \( w_{c_i,t-1} \) onto the set \( \{ w \in \mathbb{R}^d : w^T x_j \leq D \} \) to obtain \( w'_{c_i,t-1} \) and then computing a standard Newton step. The projection can be efficiently calculated in closed form (see Appendix A).

A convenient way of viewing the way the algorithm works is as follows. The time horizon is split into rounds \( t = 1, 2, \ldots, T \), each round containing multiple update steps. At the beginning of round \( t \), the algorithm commits to a sequence \( J_t \) of length \( s_t \) using the weight vector \( w'_t \), available at the beginning of that round. Then feedback sequence \( Y_t \downarrow J_t \) of length \( s'_t \leq s_t \) is observed and a sequence \( j = 1, \ldots, s'_t \) of updates are executed within round \( t \). The remaining \( s_t - s'_t \) are those corresponding to \( s_{j,t} = 0 \). Notice that, unlike the cascading contextual bandit algorithms available in the literature (e.g., Zong et al. [2016], Li et al. [2016], Li and Zhang [2018], Liu et al. [2018a], Li [2019], Li et al. [2019], Hiranandani et al. [2020]), our Algorithm [1] clearly tells apart through the update rule the actions in the sequence \( J_t \) that have been observed to be failures \((s_{j,t} = -1)\) and those that have not been observed at all \((s_{j,t} = 0)\). As shown in the appendix, this rich update rule helps us prove a sharper regret guarantee than those available in the literature. The next is the main result of this section.

**Theorem 1.** Assume there exists \( D > 0 \) such that \( u^T x_j \in [-D, D] \) for all \( x_j \in \mathcal{A} \). Let \( c_\sigma \) and \( c_{\sigma'} \) be two positive constants such that, for all \( \Delta \in [-D, D] \), we have \( 0 < 1 - c_\sigma \leq \sigma(\Delta) \leq c_\sigma < 1 \) and \( \sigma'(\Delta) \geq c_{\sigma'} \). Then with probability at least \( 1 - \delta \), with \( \delta < 1/e \), the cumulative regret of Algorithm [1] run with a link function \( \sigma : \mathbb{R} \rightarrow [0, 1] \) such that \( \sigma'(\Delta) \leq z \) for all \( \Delta \in \mathbb{R} \) satisfies

\[
\sum_{t=1}^{T} \mathbb{E}_{Y_t}[R(J^*_t, Y_t)] - \mathbb{E}_{Y_t}[R(J_t, Y_t)] \leq 4z \sqrt{\frac{e c_\sigma}{1 - c_\sigma}} T \alpha(b, d, T, \delta) \log(1 + T),
\]

where \( \alpha(b, d, T, \delta) \) is the log factor

\[
O \left[ bD^2 + \left( \frac{c_\sigma}{c_{\sigma'}} \right)^2 \left( 1 + \frac{1}{b} \right) \left( T c_{\sigma'} + \log \frac{T}{\delta} \right) \right] + \left( \frac{c_\sigma}{c_{\sigma'}} \right)^2 \left( 1 + \frac{1 + D}{c_{\sigma'}} \right) \log \frac{bT}{\delta} + D^2 \log \frac{bdT}{\delta},
\]

the big-oh hiding additive and multiplicative constants independent of \( T, b, D, \delta, c_\sigma, \sigma_\sigma, \) and \( c_{\sigma'} \).

**Remark 1.** Here and throughout, since \( \sigma'(x) = \frac{\exp(x)}{1 + \exp(x)} \), we have \( c_{\sigma'} = \frac{D}{1 + e^{-D}} \) (so that \( c_{\sigma'} = e^D \)), \( c_{\sigma'} = e^{-D}/(1 + e^{-D})^2 \geq e^{-D}/4 \), and \( z = 1 \). The dependence on \( e^D \) is common to all logistic bandit bounds [7] and is due to the nonlinear shape of \( \sigma(\cdot) \) (see, e.g., Filippi et al. [2010], Gentile and Orabona [2012], Zhang et al. [2016], Li et al. [2017], Faury et al. [2020]), where it takes the form of an upper bound on \( 1/\sigma(\cdot) \)). Also notice that \( D \) is meant to be a constant here. As for the dependence on the sequence length \( b \), our bound has the form \( \tilde{O}(\sqrt{bt}) \). Yet, we would like to emphasize that if we are willing to pay an extra additive term of the form \( e^b \) in the regret guarantee, there is a simple way to obtain a bound of the form \( \tilde{O}(e^b + \sqrt{T \log b}) \) through a more careful tuning of \( b \) in Algorithm [1].

Specifically, following Li [2019], we can set

\[
b = \arg \min_{b \geq \max_t, b_t} \left( bD^2 + \left( \frac{c_\sigma}{c_{\sigma'}} \right)^2 \left( 1 + \frac{1}{b} \right) \left( T e^D + \log \frac{T}{\delta} \right) \right)
\]

to achieve the claimed guarantee.

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*Notice that since we have assumed \( ||x_j||_2 \leq 1 \) for all vectors \( x_j \), we also have \( ||u||_2 \leq D \).*

*This actually applies only to the so-called frequentist regret bounds, which are the ones considered here. Switching to a Bayesian regret guarantee allows one to give bounds which, under some conditions, are independent of \( D \); see Zong et al. [2019]. Staying within the realm of frequentist guarantees, it might be possible to improve Theorem 1 by following the more refined self-concordant analysis contained in Faury et al. [2020]. This analysis allows one to move the multiplicative dependence on \( e^D \) from \( \sqrt{T} \) to a lower order term in \( T \).*
**Regret bound comparison.** Many papers have tackled the problem of cascading bandits with contextual information, some of them adopting a linear model assumption (e.g., Zong et al. [2016], Li et al. [2016, 2019], Hiranandani et al. [2020]), others a generalized linear model assumption (e.g., Li and Zhang [2018], Liu et al. [2018a, b], Li [2019]). Most of these papers have been chiefly motivated by learning-to-rank tasks applied to recommendation problems. Our usage of cascading bandits may be motivated by widely different application domains, where the sequence $J_t$ can potentially be far longer than the ranked list of items typically served to the user of an online content provider. So, we are interested in both the dependence on the time horizon $T$ and the maximal length $b$ of the form $\sqrt{bT}$ improves on past results in contextual cascading bandits, where the dependence on $b$ is either of the form $b \sqrt{T}$ (Zong et al. [2016], Li et al. [2019, 2019], Hiranandani et al. [2020]) or of the form $b \sqrt{T}$ (Li et al. [2018a]) or of the form $p^* + b \sqrt{T}$ (Li and Zhang [2018, 2019]) or even of the form $\frac{1}{p^*} \sqrt{bT}$ (Li et al. [2016]), where $p^*$ is the smallest probability of any sequence of length $b$, which can easily be exponentially small in $b$, even in the case of independent outcomes considered here.

### 4 Dependent Outcomes

Starting from the parametric model of Section 2.3, we can write the conditional probabilities as

$$p(x_{j_{k+1}} | x_{j_1}, \ldots, x_{j_k}) = \sigma(\Delta_{j_1, \ldots, j_{k+1}}, u),$$

where $\Delta_{j_1} = c(x_{j_1})^T u$ and $\Delta_{j_1, \ldots, j_{k+1}} = c(x_{j_{k+1}} | x_{j_1}, \ldots, x_{j_k})^T u$, for all $k \geq 1$. With this notation, the expected regret (4) can be written as

$$\mathbb{E}_Y[R(J, Y)] = \begin{cases} E(\Delta_{j_1}, \ldots, \Delta_{j_1, \ldots, j_s}) & \text{if } J \neq \langle \rangle \\ \ell_0 & \text{otherwise} \end{cases},$$

(6)

where $E(\cdot, \ldots, \cdot)$ is defined in (9) (see Appendix A) with $p(\cdot)$ therein replaced by $\sigma(\cdot)$.

The algorithm operating with the above generative model is an adaptation of the one we presented for the independent case. The main difference here is that we use conditional probabilities computed from coverage difference vectors. Notice that calculating $J^*$ may be computationally intractable. Yet, having at our disposal an oracle that maximizes (6) over $J$, we could clearly carry out a formal regret analysis similar to the one in Theorem 1. As in Hiranandani et al. [2020], we resort to a greedy algorithm to reduce the computational complexity. Specifically, we give an order over all candidate items based on their coverage difference vectors $c(\cdot | x_{j_1}, \ldots, x_{j_{k-1}})$ w.r.t. the already listed items. Then the empirical mean and upper confidence levels are computed based on these difference vectors, while the length of the sequence is chosen based on a search over all possible length values with the computed upper confidence levels.

Below we describe a simple greedy algorithm (henceforth called GREEDY) operating on true probabilities $p(x_{j_{k+1}} | x_{j_1}, \ldots, x_{j_k})$, and give the pseudocode of its bandit counterpart in Appendix B. This bandit GREEDY will be tested in our experimental comparison in Section 5.

For convenience, we drop subscript $t$. On the set of available actions $A$, the algorithm builds sequence $J_s = (x_{j_1}, x_{j_2}, \ldots, x_{j_s})$ of length $s \leq b_t$ as follows. For $k = 1, \ldots, s$, append

$$x_{j_k} = \arg \max_{x \in A \setminus \{x_{j_1}, \ldots, x_{j_{k-1}}\}} p(x | x_{j_1}, \ldots, x_{j_{k-1}}),$$

(7)

to $(x_{j_1}, x_{j_2}, \ldots, x_{j_{k-1}})$.

As for the analysis, let $J^*_t$, the Bayes optimal sequence at time $t$, have length $s^*_t$. Then it can be proven that the greedy algorithm gives an approximation ratio $0 < \gamma(s^*_t) < 1$, with some mild assumptions on rewards $r_i$ and losses $\ell_i$. Such a ratio is unavoidable since the optimal offline solution (when the true probabilities are known) is computationally intractable. Like previous work on combinatorial multi-armed bandits with an approximation oracle (e.g., Hiranandani et al. [2020]), we also consider the scaled cumulative regret, where one-time regret is defined as

$$\mathbb{E}_Y[\gamma(s^*_t) R(J^*_t, Y)] - \mathbb{E}_Y[R(J_t, Y)].$$

(8)

Then by a result similar to Lemma B for the independent case, we can derive a regret bound of the form $\sqrt{\alpha(b, d^*, T, \delta) T d^* \log T}$. The detailed derivation of $\gamma(s^*_t)$ and the proof of the key lemmas are given in Appendix B.
5 Experiments

In order to demonstrate the efficacy of the proposed algorithms, we present our experimental results on ranking tasks defined on the Million Songs [Bertin-Mahieux et al. [2011]], Yelp [Yel], MovieLens-25M [Harper and Konstan [2015]], and MNIST [mnist] datasets. We compare our algorithms to exploration-exploitation baselines in the cascading bandits literature specifically to the CascadeUCB1 algorithm of [Zong et al. [2016]] (called “C-UCB1” later on), the GL-CDCM algorithm of [Liu et al. [2018b]] which relies on a generalized linear model with the original Maximum Likelihood Estimator (MLE) as in [Filippi et al. [2010]], an ϵ-greedy version of our Algorithm 1 (called “Eps” later on), and a purely random policy (called “Rand” later on).

Datasets and preprocessing. We describe the pre-processing steps used for the MovieLens-25M dataset. The Million Songs and Yelp datasets have been treated using similar steps. MovieLens-25M contains ratings of 59,047 movies by 162,541 users, and is popularly studied in the recommendation system literature. We sample 10,000 movies at random and calculate the singular value decomposition (SVD) of the corresponding 162,541 × 10,000 ratings matrix into 10 principal components. The projection matrices from the SVD are used to compute embeddings of dimension $d = 10$ for the remaining 49,047 movies for training the bandit algorithms. The embeddings are normalized to unit $L_2$-norm and the dataset is shuffled randomly. In every round of bandit learning, the algorithm is presented with a non-overlapping chunk of movies as arms ($A_t$). The chunk size is 100 (except for the last one, which is of size 47). The rate of success of an arm is decided by the mean rating received by the corresponding movie in the dataset. This mean rating is normalized by first re-centering through its mean, and then converting to a probability by passing through a sigmoidal function. As mentioned in Section 2.2, for the dependent algorithm the 49,047 SVD-projected $d$-dimensional vectors have been used to compute coverage vectors through a Gaussian Mixture Model (GMM) with $d'$ centroids. As for MNIST, this is a multi-class classification dataset. We designed 10 ranking tasks out of it, one for each of the 10 classes in the dataset. Each task has one class as the “pivot-class”.

The algorithm must rank a collection of samples to have an item of the pivot-class (if present) as high up in the list as possible. Further details on pre-processing can be found in the appendix.

Scenarios. We study two reward/loss scenarios. The first one, which we call “Vanilla”, is designed to reproduce the standard scenario studied in the traditional cascading bandit literature: $r_{j,t} = 1$, for all $t$ and $j = 1, 2, \ldots, b_t$, and $\ell_{j,t} = 0$, for all $t$ and $j = 0, 1, \ldots, b_t$. The second scenario, called “Exponential” is comprised of exponentially decaying rewards and losses, and is designed to incentivize early success:

$$r_{j,t} = \frac{1}{2^{j-t}}, \quad \text{for all } t \text{ and } j = 1, 2, \ldots, b_t, \quad \text{and} \quad \ell_{j,t} = \frac{4}{3} \times \frac{1}{2^t} - 1, \quad \text{for all } t \text{ and } j = 0, 1, \ldots, b_t.$$  

Notice that in the exponential scenario $r_{1,t} = 1$ and $\ell_{0,t} = -0.2$. The exponential scenario captures the true essence of the proposed models since it remains sensitive to early success even for larger budgets.

Tuning of Hyperparameters. We run a fine grid-search over the space of hyperparameters of each algorithm and only report the results corresponding to the combination of hyperparameters that obtains the largest final cumulative reward. We search the value of learning rate $\alpha$ in the range $1.0 - 100.0$, UCB exploration parameter $\alpha = \alpha(b, d, T, \delta)$ or $\alpha = \alpha(b, d', T, \delta)$ on a logarithmic scale between $10^{-9} - 10.0$, $\epsilon$ in $\epsilon$-greedy in the range $0.01 - 0.5$, L2 regularization weight $\lambda$ in our implementation of the GL-CDCM baseline [Liu et al. [2018b]] on a logarithmic scale between $10^{-7} - 10^1$, and the number $d'$ of latent components for the proposed dependent algorithm between 3 and 30.

Results. Figure 2 contains an experimental comparison among all algorithms. We evaluate the algorithms in terms of their time-averaged Cumulative Reward (CR) obtained over all rounds of training by computing, for each algorithm, the fraction of reward/loss units accumulated per time step, up to time $t$, for $t = 1, \ldots, T$. If a given dataset has $T$ chunks then each algorithm is trained for exactly $T$ rounds. Figure 2 shows the variation of $\frac{\text{CR}(t)}{t}$ over rounds of training for two of the scenarios that incentivize early success. In the exponential scenario, we restrict to comparing Rand,
Figure 2: Average cumulative reward $CR(t)/t$ as a function of $t = 1, \ldots, T$ for the various algorithms on Million Songs Dataset (MSD), Yelp, MovieLens-25M, and MNIST with pivot-class 0, respectively. Vanilla scenarios are on the top row, exponential scenarios on the bottom row. $b_t = 1$ for vanilla and $b_t = 10$ for exponential. In the bottom right plot, Rand is not included for better visibility. The proposed Dependent ("Dep") algorithm performs best across the datasets, with an exception of MovieLens-25M, where the proposed Independent ("Ind") algorithm performs slightly better.

Ind, and Dep, since the other baselines are not designed to cope with it. Notice that, since the vanilla scenario does not distinguish between early and late successes in the sequence, for larger values of $b_t$ the performances of all algorithms become indistinguishable from one another. For lower values of $b_t$ in the vanilla scenario and all values of $b_t$ in the exponential scenario, achieving higher $CR$ is synonymous of early success, and we observe that the proposed dependent algorithm ("Dep") outperforms the other algorithms in these scenarios, an exception being MovieLens-25M, where the proposed Independent ("Ind") algorithm performs slightly better. GL-CDCM turns out to be a strong competitor, often at par with Ind, though it should be emphasized that the MLE estimation in GL-CDCM makes its running time far higher than that of Ind and Dep. Further experimental results are provided in the appendix, where similar trends as those reported here can be observed.

6 Conclusions

We have introduced a cascading bandit model with flexible sequences and varying rewards and losses. The model is specifically focused on learning-to-rank applications, like web search or payment systems, where the item sequence can be significantly long. We have analyzed two algorithms with improved regret guarantees, and have empirically demonstrated their competitiveness against standard baselines on a number of well-known real-world benchmark datasets.

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A Appendix

The following lemma is of preliminary importance. It delivers a monotonicity property showing that
the upper confidence scheme adopted in Algorithm 1 below is properly defined, but it also serves in
the proof of subsequent lemmas.

**Lemma 2.** For constants \( r_1 \geq r_2 \geq \ldots \geq r_s > 0, \ell_s < 0, \) and a differentiable function \( p : \mathbb{R} \to [0, 1] \)
which is monotonically increasing, the function \( E : \mathbb{R}^s \to \mathbb{R} \) defined as

\[
E(\Delta_1, \Delta_2, \ldots, \Delta_s) = r_1 p(\Delta_1) + r_2 p(\Delta_2)(1 - p(\Delta_1)) + \ldots + r_s p(\Delta_s) \prod_{i=1}^{s-1} (1 - p(\Delta_i)) + \ell_s \prod_{i=1}^s (1 - p(\Delta_i)) 
\]

enjoys the following properties:

1. \( E \) is non-decreasing in each individual variable \( \Delta_i \).
2. If, in addition, \( r_i \in [0, 1], \) for \( i = 1, \ldots, s, \ell_s \in [-1, 0], \) and \( \frac{\partial p(\Delta)}{\partial \Delta} \leq z \) for all \( \Delta \in \mathbb{R}, \) then
   \( \frac{\partial E(\Delta_1, \ldots, \Delta_s)}{\partial \Delta_i} \leq z(r_i - \ell_s) \leq 2z \) holds for all \( \Delta_1, \ldots, \Delta_s \in \mathbb{R}, \) and \( i. \)
3. Under the same assumption as in item 2 above,
   \[
   \frac{\partial E(\Delta_1, \ldots, \Delta_s)}{\partial \Delta_k} \leq 2z \prod_{j=1}^{k-1} (1 - p(\Delta_j)).
   \]

**Proof.** Define, for \( k = 1, \ldots, s, \)

\[
E_k = E_k(\Delta_k, \Delta_{k+1}, \ldots, \Delta_s)
\]

\[
= r_k p(\Delta_k) + r_{k+1} p(\Delta_{k+1})(1 - p(\Delta_k)) + \ldots + r_s p(\Delta_s) \prod_{i=k}^{s-1} (1 - p(\Delta_i)) + \ell_s \prod_{i=k}^s (1 - p(\Delta_i)),
\]

and notice that

\[
E_k \leq r_k \left( p(\Delta_k) + p(\Delta_{k+1})(1 - p(\Delta_k)) + \ldots + p(\Delta_s) \prod_{i=k}^{s-1} (1 - p(\Delta_i)) \right) + \ell_s \prod_{i=k}^s (1 - p(\Delta_i))
\]

(due to the fact that \( r_s \leq r_{s-1} \leq \ldots \leq r_{k+1} \leq r_k \))

\[
\leq r_k \left( p(\Delta_k) + p(\Delta_{k+1})(1 - p(\Delta_k)) + \ldots + p(\Delta_s) \prod_{i=k}^{s-1} (1 - p(\Delta_i)) + \prod_{i=k}^s (1 - p(\Delta_i)) \right)
\]

(since \( \ell_s \leq 0 \leq r_k \))

\[
= r_k
\]

(since the expression in braces equals 1).

Then we have, for \( k \geq 2 \)

\[
E_{k-1} = \underbrace{(1 - p(\Delta_{k-1}))}_{\geq 0} E_k + r_{k-1} p(\Delta_{k-1}) \]

\[
= p(\Delta_{k-1}) \underbrace{(r_{k-1} - E_k) + E_k}_{\geq r_{k-1} - r_k \geq 0}.
\]

From (11) one can see that, viewed solely as a function of \( \Delta_{k-1}, \) the quantity \( E_{k-1} \) can be seen as a
positive constant times \( p(\Delta_{k-1}) \) (since \( r_{k-1} - E_k \geq 0 \) and \( E_k \) only depends on variables \( \Delta_k, \ldots, \Delta_s \))
plus a constant term independent of \( \Delta_{k-1} \) (again, because \( E_k \) only depends on \( \Delta_k, \ldots, \Delta_s \). We can
now proceed by backward induction on \( k = s, s-1, \ldots, 1. \) For \( k = s \) we have \( E_s = \ell_s (1 - p(\Delta_s)) \)
which is non-decreasing in \( \Delta_s \) since so is \( p(\cdot) \), and \( \ell_s < 0. \) Assuming by induction \( E_k \) is non-decreasing in
\( \Delta_k, \ldots, \Delta_s, \) we have from (11) that \( E_{k-1} \) is non-decreasing in \( \Delta_{k-1} \), thanks to
the fact that \( p(\Delta_{k-1}) \) is monotonically increasing in \( \Delta_{k-1} \), \( E_k \) only depends on \( \Delta_k, \ldots, \Delta_s \), and \( r_{k-1} - E_k \geq 0 \). Moreover, \( E_{k-1} \) is also non-decreasing in \( \Delta_k, \ldots, \Delta_s \) since, from \( \text{(10)} \), \( E_{k-1} \) is a positive constant (i.e., independent of \( \Delta_k, \ldots, \Delta_s \)) times \( E_k \) plus a constant term, again independent of \( \Delta_k, \ldots, \Delta_s \). Since by induction \( E_k \) is non-decreasing in \( \Delta_k, \ldots, \Delta_s \), so is \( E_{k-1} \).

The above holds for all \( k \), hence it holds in particular for \( k = 1 \), which concludes the proof of the first part.

As for the second part, we again proceed by backward induction on \( k = s, s - 1, \ldots, 1 \). We have
\[
\frac{\partial E_{k-1}(\Delta_j)}{\partial \Delta_j} = -\ell_j \frac{\partial p(\Delta_j)}{\partial \Delta_j} \leq z(-\ell_j) \leq z(r_i - \ell_s) \quad \text{for all } \Delta_j.
\]
Then assume by the inductive hypothesis that \( \frac{\partial E_k(\Delta_k, \ldots, \Delta_s)}{\partial \Delta_i} \leq z(r_i - \ell_s) \) for all \( \Delta_k, \ldots, \Delta_s \), and \( i = k, \ldots, s \). From \( \text{(11)} \), we can write
\[
\frac{\partial E_{k-1}(\Delta_k, \ldots, \Delta_s)}{\partial \Delta_{k-1}} = \frac{\partial p(\Delta_{k-1})}{\partial \Delta_{k-1}} (r_{k-1} - E_k) \leq z(r_{k-1} - \ell_s) \leq 2z,
\]
the first inequality deriving from \( E_k \geq \ell_s \). On the other hand, from \( \text{(10)} \) we also have, for \( i = k, \ldots, s \),
\[
\frac{\partial E_{k-1}(\Delta_i, \ldots, \Delta_s)}{\partial \Delta_i} = (1 - p(\Delta_{k-1})) \frac{\partial E_k(\Delta_k, \ldots, \Delta_s)}{\partial \Delta_i} \leq \frac{\partial E_k(\Delta_k, \ldots, \Delta_s)}{\partial \Delta_i} \leq z(r_i - \ell_s),
\]
the inequality following from the inductive hypothesis.

Again, the above holds for all \( k \), hence it holds for \( k = 1 \), which concludes the proof of the second part.

Finally, as for the third part, we first observe that, for any \( k \),
\[
\frac{\partial E(\Delta_1, \ldots, \Delta_s)}{\partial \Delta_k} = \prod_{j=1}^{k-1} (1 - p(\Delta_j)) \frac{\partial E_k(\Delta_k, \ldots, \Delta_s)}{\partial \Delta_k},
\]
and then apply the bound \( \frac{\partial E_k(\Delta_k, \ldots, \Delta_s)}{\partial \Delta_k} \leq 2z \) from \( \text{(12)} \) to obtain the claimed result. \( \square \)

**Proof of Lemma** Consider the following argument.

1. Let \( J = \langle x_{j_1}, x_{j_2}, \ldots, x_{j_k}, \ldots, x_{j_{l'}} \rangle \) be an arbitrary sequence, and let a perturbed sequence \( J' = \langle x_{j_1}, x_{j_2}, \ldots, x_{j_{k'}}, \ldots, x_{j_{l'}} \rangle \) be obtained from \( J \) just by swapping \( x_{j_k} \) with \( x_{j_{k'}} \). Moreover, suppose \( p(x_{j_k}) > p(x_{j_{k'}}) \). Then considering the difference \( E_Y[R(J', Y)] - E_Y[R(J, Y)] \) and relying on the fact that rewards \( r_j \) are non-decreasing, we want to show that \( E_Y[R(J', Y)] \geq E_Y[R(J, Y)] \). It suffices to show the claim for the case where \( x_{j_k} \) and \( x_{j_{k'}} \) are adjacent in \( J \), so that \( k' = k + 1 \).

   Let us introduce the short-hand notation \( p_i = p(x_{j_i}) \), and \( \Pi = \prod_{i=1}^{k-1} (1 - p_i) \). Our assumption then becomes \( p_{k+1} \geq p_k \). Now, since \( Y \)'s components are independent, \( E_Y[R(J', Y)] \) has the form of function \( E(\ldots, \cdot, \cdot) \) defined in Lemma \( \text{[3]} \). Then, because \( k \) and \( k + 1 \) are adjacent positions, one can easily verify that, removing common terms, the difference \( E_Y[R(J', Y)] - E_Y[R(J, Y)] \) can be written as
\[
E_Y[R(J', Y)] - E_Y[R(J, Y)] = \Pi \left[ r_k (p_{k+1} - p_k) + r_{k+1} \left( p_k (1 - p_{k+1}) - p_{k+1}(1 - p_k) \right) \right]
= \Pi \left[ r_k (r_k - r_{k+1})(p_{k+1} - p_k) \right]
\]
which is non-negative, since \( \Pi \geq 0, r_k \geq r_{k+1} \), and \( p_{k+1} \geq p_k \).

2. Next, let \( J = \langle x_{j_1}, x_{j_2}, \ldots, x_{j_k}, \ldots, x_{j_{l'}} \rangle \) be an arbitrary sequence, and let a perturbed sequence \( J'' = \langle x_{j_1}, x_{j_2}, \ldots, x_{j_{k'}}, \ldots, x_{j_{l'}} \rangle \) be obtained from \( J \) just by replacing item \( x_{j_k} \) by \( x_{j_{k'}} \), where \( p(x_{j_{k'}}) \geq p(x_{j_k}) \). Again, we need to show that \( E_Y[R(J'', Y)] \geq E_Y[R(J, Y)] \). This claim immediately follows from the monotonicity property contained in Lemma \( \text{[4]} \), thereby showing that, for any given length \( s \), the best assortment of items
in $J$ is one that contains those corresponding to the $s$ largest marginal probabilities $p(x_j)$. In turn, combined with the previous item, this implies that $J^*$ has necessarily the form $J^*_s = \langle x_{j_1}, x_{j_2}, \ldots, x_{j_s} \rangle$, for some length $s \in \{1, \ldots, b_t\}$, where $x_{j_1}, x_{j_2}, \ldots, x_{j_s}$ are the items associated with the $s$ largest marginal probabilities $p(x_j)$, sorted in non-increasing order.

3. What remains is to maximize over length $s \in \{0, 1, \ldots, b\}$. Notice that there is no guarantee that, viewed as a function of $s$, the quantity $E_Y [R(J^*_s, Y)]$ will have a specific behavior, like unimodality. Hence, we need to try out all allowed values of $s \leq b$, including $s = 0$.

This concludes the proof. \[\square\]

The next lemma will be the basis for our regret analysis.

**Lemma 3.** Let us assume the independence model for outcome $Y$. Then, for given set of actions $A$, and budget $b$, let $J^*$ be the Bayes optimal sequence and $J = \langle x_{j_1}, \ldots, x_{j_s} \rangle$ be the sequence computed by Algorithm 1 on $A$ and $b$, with link function $\sigma$ such that $\sigma'(\Delta) \leq \epsilon$ for all $\Delta \in \mathbb{R}$. Further, let $\Delta_j = u^\top x_j$, and $\hat{\Delta}_j = w^\top x_j$, for all $x_j \in A$, and assume $|\Delta_j - \hat{\Delta}_j| \leq \epsilon_j$ for all $j$ such that $x_j \in A$, where $w$ is the vector used by Algorithm 1 to compute $J$. Then the one-time regret $E_Y [R(J^*, Y)] - E_Y [R(J, Y)]$ can be bounded as follows:

$$E_Y [R(J^*, Y)] - E_Y [R(J, Y)] \leq \left\{ \begin{array}{ll} 4z \sum_{i=1}^s \epsilon_j, & \text{if } J \neq \emptyset \\ \pi |(1 - \sigma(\Delta_j))|, & \text{if } J = \emptyset \\ \end{array} \right. \tag{13}$$

**Proof.** Irrespective of whether $J \neq \emptyset$ or $J^* \neq \emptyset$, we can write

$$E_Y [R(J^*, Y)] - E_Y [R(J, Y)] \leq \hat{E}_Y [R(J^*, Y)] - E_Y [R(J, Y)]$$

(using the first part of Lemma 2 combined with the condition $|\Delta_j - \hat{\Delta}_j| \leq \epsilon_j$)

$$\leq \hat{E}_Y [R(J, Y)] - E_Y [R(J, Y)]$$

(since, by definition of $J$, $\hat{E}_Y [R(J^*, Y)] \leq E_Y [R(J, Y)]$).

Notice that this implies that in the case where our algorithm happens to play $J = \emptyset$ the regret is $\leq 0$.

$$= E(\hat{\Delta}_{j_1} + \epsilon_j, \ldots, \hat{\Delta}_{j_s} + \epsilon_j) - E(\Delta_{j_1}, \ldots, \Delta_{j_s})$$

(where $E(\cdot)$ is defined in (9))

$$\leq E(\Delta_{j_1} + 2\epsilon_j, \ldots, \Delta_{j_s} + 2\epsilon_j) - E(\Delta_{j_1}, \ldots, \Delta_{j_s})$$

(using again the first part of Lemma 2 together with $|\Delta_j - \hat{\Delta}_j| \leq \epsilon_j$).

Now, by the mean-value theorem, we can write

$$E(\Delta_{j_1} + 2\epsilon_j, \ldots, \Delta_{j_s} + 2\epsilon_j) - E(\Delta_{j_1}, \ldots, \Delta_{j_s}) = 2 \sum_{i=1}^s \frac{\partial E(\Delta_{j_1}, \ldots, \Delta_{j_s})}{\partial \Delta_{j_i}} |_{\Delta_{j_i} = \xi_{j_i}, \ldots, \Delta_{j_s} = \xi_{j_s}} \epsilon_j,$$

where $\xi_{i_j} \in (\Delta_{j_i}, \Delta_{j_i} + 2\epsilon_j)$, for $i \in [s]$. The third part of Lemma 2 then allows us to write

$$\frac{\partial E(\xi_{j_1}, \ldots, \xi_{j_s})}{\partial \Delta_{j_i}} \leq 2z (1 - \sigma(\xi_{j_i})) \ldots (1 - \sigma(\xi_{j_{i-1}}))$$

$$\leq 2z (1 - \sigma(\Delta_{j_i})) \ldots (1 - \sigma(\Delta_{j_{i-1}})),$$

the second inequality deriving from the monotonicity of $\sigma(\cdot)$ and the fact that $\xi_{j_i} \in (\Delta_{j_i}, \Delta_{j_i} + 2\epsilon_j)$. Replacing back and summing over $i$ yields the claimed bound. \[\square\]

**Lemma 4.** Consider any item $x_{j_i} \in A$, and the random variable $s_{j_i} \in \{-1, 0, 1\}$ whose value is given in the algorithm’s pseudocode. Also, assume $x_{j_i}$ occurs in the $i$-th position of sequence $J = \langle x_{j_1}, x_{j_2}, \ldots, x_{j_s} \rangle$. Let $c_\sigma$ and $c_{\sigma^2}$ be two positive constants such that, for all $\Delta \in [-D, D]$ we have $|L'(\Delta)| \leq c_\sigma$ and $L''(\Delta) \geq c_{\sigma^2}$. Set $\Delta_{j_i} = u^\top x_{j_i}$. Then, for any $\Delta'_{j_i} \in \mathbb{R}$ we have

$$0 \leq \text{VAR}[L(s_{j_i} \hat{\Delta}_{j_i}) - L(s_{j_i} \Delta_{j_i}) | J] \leq \frac{2(c_\sigma)^2}{c_{\sigma^2}} E[L(s_{j_i} \hat{\Delta}_{j_i}) - L(s_{j_i} \Delta_{j_i}) | J].$$
We can write
\[ \Delta_j = v_j^\top x, \quad p_{j_1} = \sigma(\Delta_j), \quad \Pi_{i-1} = (1 - \sigma(\Delta_j)) \ldots (1 - \sigma(\Delta_{j-1})) . \]

We have
\[ P(s_j = 1 \mid J) = \Pi_{i-1} p_{j_1} , \quad P(s_j = -1 \mid J) = \Pi_{i-1} (1 - p_{j_1}) , \quad P(s_j = 0 \mid J) = 1 - P(s_j = 1 \mid J) - P(s_j = -1 \mid J) . \]

Hence, for all \( \hat{\Delta}_j \in \mathbb{R} \) we have
\[
E[L(s_j, \hat{\Delta}_j)] - L(s_j, \Delta_j) \mid J \]
\[ = \Pi_{i-1} \left( p_{j_1} \left( L(\hat{\Delta}_j) - L(\Delta_j) \right) + (1 - p_{j_1}) \left( L(-\hat{\Delta}_j) - L(-\Delta_j) \right) \right) \]
\[ \geq \Pi_{i-1} \left( p_{j_1} \left( L'(\hat{\Delta}_j)(\hat{\Delta}_j - \Delta_j) + \frac{c_{\sigma'}}{2} (\hat{\Delta}_j - \Delta_j)^2 \right) \right) \]
\[ + (1 - p_{j_1}) \left( L'(-\hat{\Delta}_j)(\hat{\Delta}_j - \Delta_j) + \frac{c_{\sigma'}}{2} (\hat{\Delta}_j - \Delta_j)^2 \right) \]
\[ \text{(using \( L''(\Delta_j) \geq c_{\sigma''} \))} \]
\[ = \Pi_{i-1} \frac{c_{\sigma'}}{2} (\hat{\Delta}_j - \Delta_j)^2 \]
\[ \text{(since \( p_{j_1} = L'(\Delta_j) \) and \( 1 - p_{j_1} = L'(\Delta_j) \)).} \]

Moreover,
\[ \text{VAR}[L(s_j, \hat{\Delta}_j)] - L(s_j, \Delta_j) \mid J \]
\[ \leq \Pi_{i-1} (c_{\sigma})^2 (\hat{\Delta}_j - \Delta_j)^2 \]
\[ \text{(using \( |L'(\Delta_j)| \leq c_{\sigma'} \)).} \]

Piecing together gives the claimed bound.

The next lemma helps us define the upper confidence parameters \( c_j \). To this effect, for \( t \in [T] \), let \( d_t(u, w) \) be the Mahalanobis distance between vectors \( u \) and \( w \) as
\[ d_t(u, w) = (u - w)^\top M_{c_t}(u - w) , \]
where \( M_{c_t} \) is the matrix maintained by Algorithm 1 at the \( c_t \)-th update. In order to quantify \( c_j \) in Lemma 3, we introduce a suitable surrogate loss function \( L(\cdot) \) that determines the dynamics of the algorithm (i.e., the proposed update rule being an online Newton step w.r.t. to this loss function), along with its convergence guarantees. In the proof of this lemma (see the appendix) we set \( L(\Delta) = \log(1 + e^{-\Delta}) \). Notice that \( \sigma(\Delta) = -L'(\Delta) \). The lemma follows indeed from somewhat standard arguments, and relies on the exp-concavity of \( L(\cdot) \).

**Lemma 5.** Assume there exists \( D > 0 \) such that \( u^\top x_j \in [-D, D] \) for all \( x_j \in A \). Let \( c_{\sigma} \) and \( c_{\sigma'} \) be two positive constants such that, for all \( \Delta \in [-D, D] \) we have \( 0 < 1 - c_{\sigma} \leq \sigma(\Delta) \leq c_{\sigma} < 1 \) and \( \sigma'(\Delta) \geq c_{\sigma'} \). Then with probability at least \( 1 - \delta \), with \( \delta < 1/e \), we have
\[ d_{c_{\sigma}}(u, w'_{c_{\sigma}'}) \leq bD^2 + \left( \frac{c_{\sigma}}{c_{\sigma'}} \right)^2 d \log \left( 1 + \frac{2}{b} \left( \frac{t c_{\sigma}}{1 - c_{\sigma}} + 4 \log \frac{2(t + 1)}{\delta} \right) \right) + \left( 12 \left( \frac{c_{\sigma}}{c_{\sigma'}} \right)^2 + \frac{36(1 + D)}{c_{\sigma}} \right) \log \frac{2b(t + 4)}{\delta} \]
uniformly over \( c t \in [bT] \), where \( b_t \leq b \) for all \( t \in [T] \).

**Proof.** Given items A, the update rules \( w'_{c_{\sigma} + j-1} \rightarrow w_{c_{\sigma} + j} \rightarrow w'_{c_{\sigma} + j} \) combined with the lower bound \( L''(\Delta) \geq c_{\sigma'} \) allows us to write for all \( t \) (adapted from, e.g., Hazan et al. 2007, Gentile and Orabona 2012)
\[ d_{c_{\sigma}}(u, w_{c_{\sigma}'}) \leq bD^2 + \left( \frac{1}{c_{\sigma'}} \right)^2 \sum_{k=1}^{t} \sum_{j=1}^{s_k} \nabla_{j,k} M_{c_{\sigma}+j-1} \nabla_{j,k} - \frac{2}{c_{\sigma'}} \sum_{k=1}^{t-1} \sum_{j=1}^{s_k} L(s_{j,k} x_{j}^\top w'_{c_{\sigma} + j - 1} - L(s_{j,k} x_{j}^\top w_{c_{\sigma} + j - 1})) , \]

\[ (14) \]
where $c_k = \hat{s}_1 + \hat{s}_2 + \ldots + \hat{s}_{k-1}$.

In particular, notice that the step $w_{c_i+j} \rightarrow w'_{c_i+j}$ is a projection of $w_{c_i+j}$ onto the convex set \( \{w \in \mathbb{R}^d : -D \leq w^\top x_j \leq D \} \) w.r.t. Mahalanobis distance $d_{c_i+j-1}(:, :)$. This projection can be computed in closed form as follows:

$$
 w'_{c_i+j-1} = \begin{cases} 
 w_{c_i+j-1} & \text{if } |w^\top_{c_i+j-1} x_j| \leq D \\
 \frac{w^\top_{c_i+j-1} x_j - D}{x_j^\top M_{c_i+j-1}^{-1} x_j} M_{c_i+j-1}^{-1} x_j & \text{if } w^\top_{c_i+j-1} x_j > D \\
 \frac{w^\top_{c_i+j-1} x_j + D}{x_j^\top M_{c_i+j-1}^{-1} x_j} M_{c_i+j-1}^{-1} x_j & \text{if } w^\top_{c_i+j-1} x_j < -D .
\end{cases}
$$

Further, we lower bound with high probability $\sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} \left( L(s_j,k x_j^\top w'_{c_k+j-1}) - L(s_j,k w^\top x_j) \right) \geq - \left( \frac{(c_\sigma)^2}{c_\sigma^*} + 18L(-D) \right) \log \left( \frac{b(t+4)}{\delta} \right)$

with $b \geq b_t$ for all $t$, holds with probability $\geq 1 - \delta/(bt(t+1))$, the boundedness of the difference sequence following from the fact that $|w^\top x_j| \leq D$ holds by assumption, and $|x_j^\top w'_{c_k+j-1}| \leq D$ holds by the projection steps $w_{c_k+j-1} \rightarrow w'_{c_k+j-1}$. We then upper bound $L(-D)$ by $1 + D$ and exploit a known upper bound:

$$
 \sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} \nabla _{j,k}^\top M_{c_k+j-1}^{-1} \nabla _{j,k} = \sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} \sigma^2 (-s_j,k x_j^\top w'_{c_k+j-1}) |s_j,k| (x_j^\top M_{c_k+j-1}^{-1} x_j)
$$

$$
\leq (c_\sigma)^2 \sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} |s_j,k| (x_j^\top M_{c_k+j-1}^{-1} x_j)
$$

(from the fact that $L'(\Delta) \leq c_\sigma$ for all $\Delta \in [-D,D]$, and $|x_j^\top w'_{c_k+j-1}| \leq D$)

$$
\leq (c_\sigma)^2 d \log \left( 1 + \frac{1}{b} \sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} |s_j,k| \right) \tag{15}
$$

(from a standard inequality, e.g., [Azoury and Warmuth 2001], [Cesa-Bianchi et al. 2005]).

Since $|s_j,k|$ is a Bernoulli random variable which is 1 (that is, the corresponding component of outcome vector $Y_k$ is observed) with (conditional) probability $\Pi_{j-1,k} = \prod _{i=1} ^{j-1} (1 - \sigma(\Delta_{i,k}))$, where

$$
\Delta_{i,k} = u^\top x_i, \quad i = 1, \ldots, \hat{s}_k ,
$$

we can apply again the aforementioned Freedman–like inequality from [Kakade and Tewari 2008] to conclude that

$$
\mathbb{P} \left( \exists t : \sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} |s_j,k| \leq 2 \sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} \Pi_{j-1,k} + 4 \log \left( \frac{t+1}{\delta} \right) \right) \geq 1 - \delta .
$$

In turn, since $\Delta_{i,k} \in [-D,D]$, we have $1 - \sigma(\Delta_{i,k}) \leq c_\sigma$ for all $i$ and $k$, so that $\sum _{i=1} ^{\infty} (c_\sigma)^i = \frac{c_\sigma}{1-c_\sigma}$. After some upper approximations, the above implies

$$
\mathbb{P} \left( \exists t : \sum _{k=1} ^{t-1} \sum _{j=1} ^{\hat{s}_k} |s_j,k| \leq 2(t-1) \frac{c_\sigma}{1-c_\sigma} + 8 \log \left( \frac{t+1}{\delta} \right) \right) \geq 1 - \delta .
$$

We plug it back into (15), then back into (14) and replace $\delta$ by $\delta/2$ to obtain the claimed result. \[\square\]

\[^{10}\text{Here, Lemma 4 is applied with expectations conditioned on past history.}\]
Lemma 6. Let $M$ be a $d \times d$ positive definite matrix whose minimal eigenvalue is $\geq b$, for some $b \in \{1, 2, \ldots, \}$, and $x_1, x_2, \ldots, x_b \in \{x \in \mathbb{R}^d : ||x|| \leq 1\}$. Then

$$\sum_{j=1}^{b} x_j^T M^{-1} x_j \leq e \sum_{j=1}^{b} x_j^T M_j^{-1} x_j ,$$

where $M_j = M + x_1 x_1^T + \ldots x_j x_j^T$, and $e$ is the base of natural logarithms.

Proof. Consider the quantity $x^T M_j^{-1} x$, with $M_0 = M$. We first prove that, for any $x \in \mathbb{R}^d$,

$$x^T M_j^{-1} x \leq \left(1 + \frac{1}{b}\right)^j x^T M_j^{-1} x \tag{16}$$

holds for all $j \in [b]$. By the Sherman-Morrison formula for matrix inversion we have, for an arbitrary $x \in \mathbb{R}^d$, and $j \geq 1$,

$$x^T M_j^{-1} x = x^T (M_{j-1} + x_j x_j^T)^{-1} x$$

$$= x^T M_{j-1}^{-1} x - \frac{(x^T M_{j-1}^{-1} x_j)^2}{1 + x_j^T M_{j-1}^{-1} x_j}$$

$$\geq x^T M_{j-1}^{-1} x - \frac{(x^T M_{j-1}^{-1} x_j)(x_j^T M_{j-1}^{-1} x_j)}{1 + x_j^T M_{j-1}^{-1} x_j} \tag{from the Cauchy-Schwarz inequality}$$

so that

$$x^T M_{j-1}^{-1} x \leq x^T M_j^{-1} x + \frac{(x^T M_{j-1}^{-1} x_j)(x_j^T M_{j-1}^{-1} x_j)}{1 + x_j^T M_{j-1}^{-1} x_j} .$$

Hence, rearranging terms, we can write

$$x^T M_j^{-1} x \leq x^T M_j^{-1} x (1 + x_j^T M_{j-1}^{-1} x_j) \leq x^T M_j^{-1} x \left(1 + \frac{1}{b}\right) ,$$

the second inequality deriving from the assumption $||x_j|| \leq 1$ and the fact that since the smallest eigenvalue of $M$ is at least $b$, so is the smallest eigenvalue of $M_{j-1} \geq M$. Unwrapping this recurrence over $j$ gives (16).

From (16), since $(1 + 1/b)^j \leq e$ when $j \leq b$, we have

$$x^T M_j^{-1} x \leq e x^T M_j^{-1} x .$$

Since this holds for a generic $x$, we instantiate in turn $x$ to $x_1, x_1, \ldots, x_b$, and sum over $j \in [b]$. This yields

$$\sum_{j=1}^{b} x_j^T M_j^{-1} x_j \leq e \sum_{j=1}^{b} x_j^T M_j^{-1} x_j ,$$

as claimed. \( \square \)

Proof of Theorem. Consider matrix $M_{c_1-1}$ in Lemma 5. If $J_r = \langle x_{j_r,1}, \ldots, x_{j_r,s_r} \rangle$, for $r = 1, \ldots, t - 1$, we can write

$$M_{c_1-1} = bI + \sum_{r=1}^{t-1} \sum_{j=1}^{s_r} |s_{j,r}| x_{j_r,j} x_{j_r,j}^T ,$$

where $|s_{j,r}|$ is a Bernoulli random variable which is 1 (that is, the corresponding component of outcome vector $Y_r$ is observed) with probability $\Pi_{j-1,r} = \prod_{i=1}^{r-1} (1 - \sigma(\Delta_{i,r}))$, where

$$\Delta_{i,r} = u^T x_{j_r,i} , \quad i = 1, \ldots, \hat{s}_r .$$
Let
\[ \tilde{M}_{c,t} = bI + \sum_{r=1}^{t-1} \sum_{j=1}^{s_r} \Pi_{j-1,r} x_{j,r,j}^c x_{j,r,j}^c, \]
and consider the matrix martingale difference sequence
\[ |s_{j,r}| x_{j,r,j}^c x_{j,r,j}^c - \Pi_{j-1,r} x_{j,r,j}^c x_{j,r,j}^c, \quad r = 1, \ldots, t-1, \ j = 1, \ldots, s_r. \]

By a standard Freedman-style matrix martingale inequality (e.g., Tropp [2011]) adapted to our scenario we have, for positive constants \( \theta \) and \( \theta' \),
\[
\mathbb{P} \left( \exists t : \lambda_{\text{max}} \left( M_{c,t-1} - \tilde{M}_{c,t-1} \right) \geq \theta, ||M_{c,t-1}|| \leq \theta' \right) \leq d \exp \left( \frac{-\theta^2/2}{\theta' + \theta/3} \right), \tag{17}
\]
where \( \lambda_{\text{max}}(\cdot) \) denotes the algebraically largest eigenvalue of the matrix at argument, and \( || \cdot || \) denotes the spectral norm.

We now proceed according to a standard stratification argument (e.g., Cesa-Bianchi and Gentile [2008]). Setting \( A(x, \delta) = 2 \log \frac{d}{\delta} \) and \( f(A, r) = 2A + \sqrt{Ar} \), we can write
\[
\mathbb{P} \left( \exists t : \lambda_{\text{max}} \left( M_{c,t-1} - \tilde{M}_{c,t-1} \right) \geq f(A(||M_{c,t-1}||, \delta), ||M_{c,t-1}||) \right)
\leq \sum_{r=0}^{\infty} \mathbb{P} \left( \exists t : \lambda_{\text{max}} \left( M_{c,t-1} - \tilde{M}_{c,t-1} \right) \geq f(A(||M_{c,t-1}||, \delta), ||M_{c,t-1}||), 2^r - 1 \leq ||M_{c,t-1}|| \leq 2^{r+1} \right)
\leq \sum_{r=0}^{\infty} d \exp \left( \frac{-f^2(2A(2^{r+1}, \delta), 2^{r+1})/2}{2^{r+1} + f(A(2^{r+1}, \delta), 2^{r+1})/3} \right),
\]
the last inequality deriving from (17).

Since \( f(A, r) \) satisfies \( f^2(A, r) \geq Ar + A + 2/3f(A, r)A \), the exponent in the last exponential is at least \( A(2^{r+1}, \delta)/2 \), implying
\[
\sum_{r=0}^{\infty} \exp \left( -A(2^{r+1}, \delta)/2 \right) = \sum_{r=0}^{\infty} \frac{\delta}{d2^{r+1}} = \delta/d,
\]
which in turn implies
\[
\mathbb{P} \left( \exists t : \lambda_{\text{max}} \left( M_{c,t-1} - \tilde{M}_{c,t-1} \right) \geq f(A(||M_{c,t-1}||, \delta), ||M_{c,t-1}||) \right) \leq \delta.
\]

Plugging back the definitions of \( f(A, r) \) and \( A(x, \delta) \), noticing that, \( ||M_{c,t-1}|| = \lambda_{\text{max}}(M_{c,t-1}) \leq b(t+1) \) (due to the fact that \( ||M_{c,t-1}|| \) is positive definite and \( ||x_{j,r,j}|| \leq 1 \)), and overapproximating gives
\[
\mathbb{P} \left( \exists t : \lambda_{\text{max}} \left( M_{c,t-1} - \tilde{M}_{c,t-1} \right) \geq 4 \log \frac{bd(t+1)}{\delta} + \sqrt{2 \lambda_{\text{max}}(M_{c,t-1}) \log \frac{bd(t+1)}{\delta}} \right) \leq \delta.
\]

Further, we use \( \sqrt{ab} \leq a/2 + b/2 \) with \( a = \lambda_{\text{max}}(\tilde{M}_{c,t-1}) \) and \( b = 2 \log \frac{bd(t+1)}{\delta} \). Rearranging gives
\[
\mathbb{P} \left( \exists t : \frac{1}{2} \lambda_{\text{max}} (\tilde{M}_{c,t-1}) - \lambda_{\text{max}}(M_{c,t-1} - \tilde{M}_{c,t-1}) \leq -5 \log \frac{bd(t+1)}{\delta} \right) \leq \delta
\]
or
\[
\mathbb{P} \left( \forall t : \frac{1}{2} \lambda_{\text{max}} (M_{c,t-1}) - \lambda_{\text{max}}(M_{c,t-1} - \tilde{M}_{c,t-1}) \geq -5 \log \frac{bd(t+1)}{\delta} \right) \geq 1 - \delta.
\]

Now, observing that
\[
\lambda_{\text{max}} (M_{c,t-1}) - \lambda_{\text{max}}(2M_{c,t-1} - 2\tilde{M}_{c,t-1}) \leq \lambda_{\text{max}} (2M_{c,t-1} - \tilde{M}_{c,t-1})
\]
the above implies
\[ P \left( \forall t \lambda_{\max} \left( M_{t-1} - \frac{1}{2} \bar{M}_{t-1} \right) \geq -5 \log \frac{bd(t+1)}{\delta} \right) \geq 1 - \delta , \]
which can be rewritten as
\[ P \left( \forall v \in \mathbb{R}^d : v^T \left( M_{t-1} - \frac{1}{2} \bar{M}_{t-1} \right) v \geq -5 \log \frac{bd(t+1)}{\delta} \right) \geq 1 - \delta . \]
If we define
\[ \bar{d}_{t-1}(u, w) = (u - w)^T \bar{M}_{t-1} (u - w) \]
the above inequality allows us to conclude that
\[ d_{t-1}(u, w) \geq \frac{1}{2} \bar{d}_{t-1}(u, w) - 20D^2 \log \frac{bd(t+1)}{\delta} \]
holds with probability at least \( 1 - \delta \), uniformly over all \( u, w \in \mathbb{R}^d \) such that \( ||u - w|| \leq 2D \) and all rounds \( t \). Hence, combining with Lemma 5 and upper bounding \( t \) by \( T \),
\[ \bar{d}_{t-1}(u, w_{t'}) \leq \alpha(b, d, T, 2\delta) \]
where
\[ \alpha(b, d, T, 2\delta) = 2bD^2 + \left( \frac{c_\sigma}{c_{\sigma'}} \right)^2 \frac{d}{2} \left( 1 + \frac{2}{b} \left( \frac{T}{c_\sigma} - \log c_\sigma \right) \right) \]
\[ + 2 \left( 12 \left( \frac{c_\sigma}{c_{\sigma'}} \right)^2 + \frac{36}{c_\sigma} + \frac{1}{\delta} \right) \log \frac{b(T+4)}{\delta} + 20D^2 \log \frac{bd(T+1)}{\delta} \]
with probability at least \( 1 - 2\delta \).
Then Cauchy-Schwarz inequality allows us to write, for all \( x \in \mathbb{R}^d \),
\[ (u^T x - x^T w_{t'})^2 \leq x^T \bar{M}_{t-1} x \bar{d}_{t-1}(u, w_{t'}) \leq \left( x^T \bar{M}_{t-1} x \right) \alpha(b, d, T, 2\delta) . \]
We are therefore in a position to apply Lemma 3 with \( J \) therein set to \( J_t = \langle x_{j_1, t}, \ldots, x_{j_{s_t}, t} \rangle \) and \( \epsilon_j \)
set to \( \epsilon_{j_1, j} = \sqrt{x^T J_{j_1, j} M_{t-1} x_{j_1, j}} \alpha(b, d, T, 2\delta) \), for \( j = 1, \ldots, s_t \). Thus we can write
\[ \sum_{t=1}^T \mathbb{E}_{Y_t} [R(J^*_t, Y_t)] - \mathbb{E}_{Y_t} [R(J_t, Y_t)] \leq 4\sqrt{\alpha(b, d, T, 2\delta)} \sum_{t=1}^T \sum_{j=1}^{s_t} \left( x^T J_{j_1, j} M_{t-1} x_{j_1, j} \right) \Pi_{j_1-1, t} . \] (18)
Now, for each round \( t \), consider the quantity
\[ \sum_{j=1}^{s_t} \left( x^T J_{j_1, j} M_{t-1} x_{j_1, j} \right) \Pi_{j_1-1, t} \]
Noticing that \( \bar{M}_0 = bI \), we invoke Lemma 6 with \( x_j \) therein set to \( x^T J_{j_1, j} \sqrt{\Pi_{j_1-1, t}} \) and write
\[ \sum_{j=1}^{s_t} \left( x^T J_{j_1, j} M_{t-1} x_{j_1, j} \right) \Pi_{j_1-1, t} \leq \delta \sum_{j=1}^{s_t} \left( x^T J_{j_1, j} M_{t-1} x_{j_1, j} \right) \Pi_{j_1-1, t} \] (19)
where
\[ \bar{M}_{t-1+j} = \bar{M}_{t-1} + \sum_{i=1}^{j} x^T J_{j_1, j} \Pi_{j_1-1, t} , \]
with $\Pi_{0,t} = 1$. Thus, for each $t$,

$$
\sum_{j=1}^{\tilde{s}_t} \sqrt{\sum_{j=1}^{\tilde{s}_t} \left( \frac{E_{t}^{j} Y^{\sqrt{T}}}{M_{c_{t-1}x_{j}}} \right)} \Pi_{j-1,t} = \sum_{j=1}^{\tilde{s}_t} \sqrt{\sum_{j=1}^{\tilde{s}_t} \left( \frac{E_{t}^{j} Y^{\sqrt{T}}}{M_{c_{t-1}x_{j}}} \right)} \Pi_{j-1,t}
$$

(from the Cauchy-Schwarz inequality)

$$
\leq \sum_{j=1}^{\tilde{s}_t} \left( \frac{e c_{t}}{1 - c_{t}} \right) \Pi_{j-1,t}
$$

(again from the Cauchy-Schwarz inequality)

$$
\leq 4z \alpha(b, d, T, 2\delta) d \log \left( 1 + \frac{1}{b} \sum_{t=1}^{T} \sum_{j=1}^{\tilde{s}_t} \Pi_{j-1,t} \right)
$$

(from a standard inequality, e.g., [2001], [2005], [2011])

$$
\leq 4z \alpha(b, d, T, 2\delta) d \log (1 + T)
$$

Since the above holds with probability $\geq 1 - 2\delta$, we replace $\delta$ by $\delta/2$ in $\alpha(b, d, T, 2\delta)$ so as to obtain the claimed result.

\[ \square \]

## B Algorithm for the Case of Dependent Outcomes

For completeness, we give in Algorithm 3 the pseudocode of the greedy algorithm used in our experiments. All in all, the algorithm performs the same updates as Algorithm 1 but applied to the coverage difference vectors $\hat{c}(x_{j,k} \mid x_{j,1}, \ldots, x_{j,k-1})$ instead of the original feature vectors $x_{j,k}$. Moreover, Algorithm 3 replaces the computation of $J_{l}$ by mimicking GREEDY, described in Section 4.

In the pseudocode of Algorithm 3, we define

$$
\alpha(b, d', T, \delta) = 2bD^{2} + \left( \frac{c_{t}}{c_{t'}} \right)^{2} d' \log \left( 1 + \frac{2}{b} \frac{T c_{t}}{1 - c_{t}} + 4 \log \frac{4(T + 1)}{\delta} \right) + 2 \left( \frac{2b(T + 4)}{\delta} \right) + 20D^{2} \log \frac{2bd'(T + 1)}{\delta}
$$
Algorithm 2: 

Input: Confidence level $\delta \in [0, 1]$, width parameter $D > 0$, maximal budget parameter $b > 0$; 
Init: $M_0 = bI \in \mathbb{R}^{d' \times d'}$, $w_1 = 0 \in \mathbb{R}^{d'}$, $c_1 = 1$

For $t = 1, 2, \ldots, T$:

1. Get:
   - Set of actions $A_t = \{x_{1,t}, \ldots, x_{|A_t|,t}\} \subseteq \{x \in \mathbb{R}^{d'} : ||x|| \leq 1\}$,
   - budget $b_t \leq b$;

2. Compute $J_t$:
   - For $k = 1, \ldots, \min\{b_t, |A_t|\}$
     
     \[
     x_{j_t,k}^* = \arg\max_{x \in A_t \setminus \{x_{j_t,1}, \ldots, x_{j_t,k-1}\}} \sigma\left(\tilde{c}(x \mid x_{j_t,1}, \ldots, x_{j_t,k-1})^\top w_{c_t} + \epsilon_t(x \mid x_{j_t,1}, \ldots, x_{j_t,k-1})\right),
     \]
     
     where \(\epsilon_t^2(x \mid x_{j_t,1}, \ldots, x_{j_t,k-1}) = \tilde{c}(x \mid x_{j_t,1}, \ldots, x_{j_t,k-1})^\top M_{c_t-1}^{-1} \tilde{c}(x \mid x_{j_t,1}, \ldots, x_{j_t,k-1})\).

   - Let $\hat{J}_{t,s} = \langle x_{j_t,1}, \ldots, x_{j_t,s}\rangle$ for any $s \leq b_t$;

   - Set $\bar{s}_t = \arg \max_{s=0,1,\ldots,b_t} E_{Y_t}[R(\hat{J}_{t,s}, Y_t)]$, with

     \[
     \tilde{\Delta}_{j_t,k,t}^2 = \tilde{c}(x_{j_t,k} \mid x_{j_t,1}, \ldots, x_{j_t,k-1})^\top w_{c_t}
     \]

     \[
     \tilde{c}^2_{j_t,k,t} = \tilde{c}(x_{j_t,k} \mid x_{j_t,1}, \ldots, x_{j_t,k-1})^\top M_{c_t-1}^{-1} \tilde{c}(x_{j_t,k} \mid x_{j_t,1}, \ldots, x_{j_t,k-1}) \alpha(b, d', T, \delta)
     \]

     \[
     E_{Y_t}[R(\hat{J}_{t,s}, Y_t)] = \begin{cases} 
     E(\tilde{\Delta}_{j_t,t,t}^2 + \epsilon_t^{s, j_t,t} + \epsilon_t^{\bar{s}_t, j_t,t}) & \text{if } s \geq 1 \\
     \ell_{0,t} & \text{otherwise},
     \end{cases}
     \]

     where function $E(\cdot, \cdot, \cdot)$ is as in (7) in Lemma 3 with $p(\cdot)$ therein replaced by $\sigma(\cdot)$;

   - Finally, $J_t = \hat{J}_{t,\bar{s}_t}$;

3. Observe feedback

   $Y_t \downarrow J_t = \begin{cases} 
   (y_{1,j_t}, y_{2,j_t}, \ldots, y_{b_t,j_t}) & \text{if } \bar{s}_t' \leq \bar{s}_t \quad \text{or} \\
   (y_{1,j_t}, y_{1,j_t}, \ldots, y_{1,j_t}) & \text{if } \bar{s}_t' = 0
   \end{cases}$

4. For $k = 1, \ldots, \bar{s}_t$ (in the order of occurrence of items in $J_t$) update:

   \[
   w_{c_t+k} = w_{c_t+k-1} + \frac{1}{c_{s_t}} M_{c_t+k-1}^{-1} \nabla_{k,t},
   \]

   where

   \[
   s_{k,t} = \begin{cases} 
   1 & \text{If } y_{t,k} \text{ is observed and } y_{t,k} = 1 \\
   -1 & \text{If } y_{t,k} \text{ is observed and } y_{t,k} = 0 \\
   0 & \text{If } y_{t,k} \text{ is not observed},
   \end{cases}
   \]

   and \(\nabla_{k,t} = \sigma(-s_{k,t} \tilde{\Delta}_{k,t}) s_{k,t} \tilde{c}(x_{j_t,k} \mid x_{j_t,1}, \ldots, x_{j_t,k-1})\), where

   \[
   \tilde{\Delta}_{k,t} = \tilde{c}(x_{j_t,k} \mid x_{j_t,1}, \ldots, x_{j_t,k-1})^\top w_{c_t+k-1}
   \]

   with

   \[
   w_{c_t+k-1} = \arg \min_w \text{ for } -D \leq w^\top \tilde{c}(x_{j_t,1}, \ldots, x_{j_t,k-1}) \leq D \\
   \frac{d_{c_t+j-2}(w, w_{c_t+k-1})}{1 + \exp(w)}
   \]

5. $c_{t+1} \leftarrow c_t + \bar{s}_t$.

**Algorithm 2:** The contextual bandit algorithm in the dependent case. Here the link function $\sigma(\cdot)$ is $\sigma(x) = \frac{\exp(x)}{1 + \exp(x)}$.

Below we give the derivation for the approximation ratio claimed in the main body of the paper.

**Lemma 7.** Fix $s \in \{0, 1, \ldots, b\}$. Let $J^* = \langle x_{j_1}, \ldots, x_{j_s}\rangle$ be the Bayes optimal sequence under model (5) with unknown vector $u$. Let $(x_{j_1}', x_{j_2}', \ldots)$ be the order of items according to Eq. (7) and the unknown vector $u$ and $J' = \langle x_{j_1}', x_{j_2}', \ldots\rangle$ be the sequence taking first $s$ elements. Suppose
Then let $E\|u\|_1\leq \frac{\sqrt{T}(z-(1-1/e)c_{r'})}{6(z^2-(1-1/e)c_{r'})}$. Moreover, let the reward and loss sequences satisfy

$$s(r_s - \ell_s) \max \left\{ \frac{1}{s}, 1 - \frac{s - 1}{2} c_{r'} \right\} \left( 1 - \left( 1 - \frac{1}{e} \right) \frac{c_{r'}}{z} \right) + 3 \ell_s \left( 1 - \max \left\{ \frac{1}{s}, 1 - \frac{s - 1}{2} c_{r'} \right\} \left( 1 - \frac{1}{e} \right) \frac{c_{r'}(r_s - \ell_s)}{z(r_1 - \ell_s)} \right) \geq 0.$$

Let

$$\gamma(s) = \begin{cases} \max\left\{ \frac{1}{s}, 1 - \frac{s - 1}{2} c_{r'} \right\} (1 - \frac{1}{e}) \frac{c_{r'}(r_s - \ell_s)}{z(r_1 - \ell_s)}, & \text{if } s \geq 2, \\ 1, & \text{if } s = 0, 1. \end{cases}$$

Then

$$\mathbb{E}_Y[R(J', Y)] \geq \gamma(s) \mathbb{E}_Y[R(J', Y)].$$

**Proof.** It is immediate to see the conclusion holds for $s = 0, 1$. Now assume $s \geq 2$. Let $J = \langle x_{j_1}, \ldots, x_{j_s} \rangle$ be any sequence of length $s$. Then, setting for brevity $a = 2/\sqrt{d'}$ and $a' = -1/\sqrt{d'}(1, \ldots, 1)^T$, we can write

$$\mathbb{E}_Y[R(J, Y)] = E(\Delta_{j_1}, \Delta_{j_1,j_2}, \ldots, \Delta_{j_1,j_2,\ldots,j_s})$$

$$= r_1p(\Delta_{j_1}) + r_2p(\Delta_{j_1,j_2})(1 - p(\Delta_{j_1})) + \cdots + r_sp(\Delta_{j_1,\ldots,j_s}) \prod_{i=1}^{s-1}(1 - p(\Delta_{j_1,\ldots,j_i}))$$

$$+ \ell_s \prod_{i=1}^{s}(1 - p(\Delta_{j_1,\ldots,j_i}))$$

$$= (r_1 - \ell_s)p(\Delta_{j_1}) + (r_2 - \ell_s)p(\Delta_{j_1,j_2})(1 - p(\Delta_{j_1})) + \cdots$$

$$+ (r_s - \ell_s)p(\Delta_{j_1,\ldots,j_s}) \prod_{i=1}^{s-1}(1 - p(\Delta_{j_1,\ldots,j_i})) + \ell_s$$

$$\leq (r_1 - \ell_s)(1 - \prod_{i=1}^{s}(1 - p(\Delta_{j_1,\ldots,j_i}))) + \ell_s$$

$$\leq (r_1 - \ell_s) \sum_{i=1}^{s} p(\Delta_{j_1,\ldots,j_i}) + \ell_s$$

$$= (r_1 - \ell_s) \sum_{i=1}^{s} \sigma(a \cdot c'(x_{j_1}, x_{j_1}, \ldots, x_{j_{i-1}})^T u + a'^T u) + \ell_s$$

$$\leq (r_1 - \ell_s) \sum_{i=1}^{s} \left( \sigma(0) + z \cdot a \cdot c'(x_{j_1}, x_{j_1}, \ldots, x_{j_{i-1}})^T u + c_{r'}a'^T u \right) + \ell_s$$

$$\leq (r_1 - \ell_s) \sum_{i=1}^{s} \left( \sigma(0) + z \cdot a \cdot c'(x_{j_1}, x_{j_1}, \ldots, x_{j_{i-1}})^T u + c_{r'}a'^T u \right) + \ell_s$$

$$= (r_1 - \ell_s)(s/2 + s c_{r'} a'^T u + z \cdot a \cdot \langle c'(\{x_{j_1}, \ldots, x_{j_s}\}), u \rangle) + \ell_s$$

where the fourth and third lines from last are both from the properties of the $\sigma$ function. Also

$$\mathbb{E}_Y[R(J, Y)]$$

11 Notice that, since $z \equiv 1$ and $c_{r'} = e^{-D}/(1 + e^{-D})^2$, this requirement is essentially equivalent to something like $\|u\|_1 = O(\sqrt{T})$.

12 For example, this requirement holds when $r_s \geq 5|\ell_s|$ for all $s \geq 1$, and $c_{r'} \leq \frac{1}{2(1 - 1/e)^2}$. 

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where the second inequality is by Lemma 1 of Hiranandani et al. [2020], and the fourth and fifth lines
is by the definition of $\Delta_j$.

The next lemma is the dependent outcome counterpart to Lemma 3.

The second inequality is based on the fact that the selection of $\Delta_j$ is the vector used by Algorithm 2 to compute $J$.

Then, for given set of actions $A$, and budget $b$, let $J^*$ be the Bayes optimal sequence and $J = (x_{j_1}, \ldots, x_{j_s})$ be the sequence computed by Algorithm 2 on $A$ and $b$, with link function $\epsilon$ such that $\sigma'(\Delta) \leq z$ for all $\Delta \in \mathbb{R}$. Further, let $\Delta_{j_1, \ldots, j_k} = u^T \hat{\epsilon}(x_{j_k} \mid x_{j_1}, \ldots, x_{j_{k-1}})$, and $\hat{\Delta}_{j_1, \ldots, j_k} = w^T \hat{\epsilon}(x_{j_k} \mid x_{j_1}, \ldots, x_{j_{k-1}})$, for all conditional vectors computed from $A$, and assume $|\Delta_{j_1, \ldots, j_k} - \hat{\Delta}_{j_1, \ldots, j_k}| \leq \epsilon_{j_1, \ldots, j_k}$ for all $j$ sequence, where $w$ is the vector used by Algorithm 2 to compute $J$. Suppose $\Delta_{j_1, \ldots, j_k} + 2x_{j_1, \ldots, j_k} \in [-D, D]$.

The next lemma is the dependent outcome counterpart to Lemma 3.

**Lemma 8.** Let us assume the dependent model \ref{model} for outcome vector $Y$. Then, for given set of actions $A$, and budget $b$, let $J^*$ be the Bayes optimal sequence and $J = (x_{j_1}, \ldots, x_{j_s})$ be the sequence computed by Algorithm 2 on $A$ and $b$, with link function $\epsilon$ such that $\sigma'(\Delta) \leq z$ for all $\Delta \in \mathbb{R}$. Further, let $\Delta_{j_1, \ldots, j_k} = u^T \hat{\epsilon}(x_{j_k} \mid x_{j_1}, \ldots, x_{j_{k-1}})$, and $\hat{\Delta}_{j_1, \ldots, j_k} = w^T \hat{\epsilon}(x_{j_k} \mid x_{j_1}, \ldots, x_{j_{k-1}})$, for all conditional vectors computed from $A$, and assume $|\Delta_{j_1, \ldots, j_k} - \hat{\Delta}_{j_1, \ldots, j_k}| \leq \epsilon_{j_1, \ldots, j_k}$ for all $j$ sequence, where $w$ is the vector used by Algorithm 2 to compute $J$. Suppose $\Delta_{j_1, \ldots, j_k} + 2x_{j_1, \ldots, j_k} \in [-D, D]$.

This requirement is controllable since $\epsilon_{j_1, \ldots, j_k}$ is reasonably small after $O(\log T)$ rounds.
Yet, it is worth stressing that, despite the two regret bounds \(\xi\) and \(E\), only applies to non-contextual bandit scenarios. Inconvenient dependence on \(1/p\) as alluded to at the end of Section 3, but their comprehensive analysis only applies to non-contextual bandit scenarios.

**Proof.** Irrespective of whether \(J \neq \emptyset\) or \(J^* \neq \emptyset\), we can write

\[
\mathbb{E}[\gamma(s^*_t)R(J^*, Y)] - \mathbb{E}[R(J, Y)] \\
\leq \mathbb{E}[\gamma(s^*_t)R(J^*, Y)] - \mathbb{E}[R(J, Y)] \\
\leq \mathbb{E}[\gamma(J^*, Y)] - \mathbb{E}[R(J, Y)] \\
= E(\hat{\Delta}_{j_1} + \epsilon_{j_1}, \hat{\Delta}_{j_1,j_2} + \epsilon_{j_1,j_2}, \ldots, \hat{\Delta}_{j_1,j_2,\ldots,j_s} + \epsilon_{j_1,j_2,\ldots,j_s}) - E(\Delta_{j_1}, \Delta_{j_1,j_2}, \ldots, \Delta_{j_1,j_2,\ldots,j_s})
\]

where \(\hat{\Delta}_{j_1}\) is defined as in Algorithm 1 by using \(\hat{\Delta}_{j_1,k_k + \epsilon_{j_1,j_k}}\). Here \(J^*\) is computed similarly in Lemma 2, but under \(\hat{\Delta}_{j_1,k_k + \epsilon_{j_1,j_k}}\) and length \(s^*_t\). The \(\gamma(s^*_t)\)-approximation still holds according to Lemma 2. The list \(J^*\) is just \(\hat{J}_s^t\) in Algorithm 2 and is no better than \(J\) under \(\hat{E}_Y\) according to the computation of \(s\).

Similar to the proof of Lemma 3 by the mean-value theorem, we can write

\[
E(\Delta_{j_1} + 2\epsilon_{j_1}, \Delta_{j_1,j_2} + 2\epsilon_{j_1,j_2}, \ldots, \Delta_{j_1,j_2,\ldots,j_s} + 2\epsilon_{j_1,j_2,\ldots,j_s}) - E(\Delta_{j_1}, \Delta_{j_1,j_2}, \ldots, \Delta_{j_1,j_2,\ldots,j_s})
\]

\[
= 2\sum_{i=1}^{s} \frac{\partial E(\Delta_{j_1}, \Delta_{j_1,j_2}, \ldots, \Delta_{j_1,j_2,\ldots,j_s})}{\partial \Delta_{j_1,j_2,\ldots,j_s}} |_{\Delta_{j_1} = \epsilon_{j_1}, \ldots, \Delta_{j_1,j_s} = \epsilon_{j_1,j_s}}
\]

where \(\epsilon_{j_1,j_s} \in (\Delta_{j_1,j_s}, \Delta_{j_1,j_s} + 2\epsilon_{j_1,j_s})\). The second inequality deriving from the monotonicity of \(\sigma(\cdot)\) and the fact that \(\epsilon_{j_1,j_s} \in (\Delta_{j_1,j_s}, \Delta_{j_1,j_s} + 2\epsilon_{j_1,j_s})\). Replacing back, and summing over \(i\) yields the claimed bound.

Based on this lemma, we combine with the corresponding remaining parts in the proof for the independent case. This gives us a scaled regret bound which coincides with the one for the dependent case.

Yet, it is worth stressing that, despite the two regret bounds look alike, the two underlying notions of regret are widely different, both because we have now a scaled regret, and because of the different assumptions on the process generating the outcomes as compared to the independent case.

### C Further Related Work

[Kveton et al. 2015b] studies a variant of cascading bandits where the feedback stops when a 0 outcome is observed, as opposed to a 1 outcome of the standard cascading bandit model. This reward is equivalent to a Boolean AND function on the sequence, and the available sequences are defined by combinatorial constraints of the problem. [Zhou et al. 2018] also studies a variant of cascading bandits where each arm has an extra (unknown) cost when displayed. The length of the recommended sequences can also change, but in their setting this is due to the trade-off between the attractiveness and the cost of an item, while in our setting this is due to the trade-off between attractiveness of items and both reward and loss values. The combinatorial semi-bandit setting with probabilistically triggered arms [ZZ et al. 2018] is a generalization of the cascading bandit setting that also encompasses, for instance, influence maximization problems. The authors are able to remove the inconvenient dependence on \(1/p\) alluded to at the end of Section 3, but their comprehensive analysis only applies to non-contextual bandit scenarios.
Besides cascading bandits, relevant works investigate bandits with submodular reward functions to account for diversity in the item assortment (e.g., Yue and Guestrin [2011], Takemori et al. [2020]). In particular, Takemori et al. [2020] show a regret bound of the form $\sqrt{bT}$ in a submodular bandits scenario with rewards on items similar to our setting, yet relying on a feedback which is more informative than ours. For instance, in the independent case, their setting is equivalent to a (constrained) combinatorial bandits scenario with semi-bandit feedback with linear rewards.

Regarding the generative model for outcome vectors, following previous work [Li and Zhang, 2018], we assumed the probability that an item is successful is ruled by a generalized linear model (GLM). Such a model is more convenient than a purely linear model, since the sigmoidal link function would always map values to $\{0, 1\}$ which we need here to encode probabilities and compute the Bayes optimal sequence. The bandit problem under GLM assumptions is first studied in Filippi et al. [2010], whose regret bound can be improved by the finer self-concordant analysis of Faury et al. [2020]. The online Newton step analysis presented here is inspired by the GLM-based bandit analysis contained in Gentile and Orabona [2012]. See also Zhang et al. [2016] for similar results. Li et al. [2017] gives an optimal solution for this model up to a constant coefficient.

Finally, the update method that deals with long sequences in our paper also often appears in the study of bandit algorithms with delayed feedback. There is indeed some kind of similarity between a cascading model and a delayed feedback model in bandits: both share the need for a bandit algorithm to deal with signals that are received somehow later than the time the algorithm commits to actions. Relevant works in bandits with delayed feedback include Dudik et al. [2011], Joulani et al. [2013], Cesa-Bianchi et al. [2019], Pike-Burke et al. [2018], Zhou et al. [2019], Arya and Yang [2020]. Yet, we are not aware of a way to reduce the delayed bandit model to the cascading bandit model, or vice versa.

D Further Experimental Results

This section contains details on our experimental setting and results that have been omitted from the main paper.

D.1 Dataset Preprocessing

We report here the pre-processing steps we followed for the Million Songs, Yelp, and MNIST datasets.

- **Million Songs**: The Million Songs Dataset (MSD) is a repository of audio features and metadata of a million contemporary pop songs. We consider the Echo Nest Taste Profile Subset of MSD that contains the play-counts of some of these songs by real users. We pick 100,000 users that have played the highest number of songs and 50,000 songs with the highest number of users. We sample 10,000 songs at random and calculate the singular value decomposition (SVD) of the corresponding $100,000 \times 10,000$ ratings matrix into 10 principal components. The projection matrices from the SVD are used to compute embeddings of dimension $d = 10$ for the remaining 40,000 songs for training the bandit algorithms. The embeddings are normalized to unit $L_2$-norm and the dataset is shuffled randomly. In every round of bandit learning, the algorithm is presented with a non-overlapping chunk of movies as arms ($A_t$). The chunk size is 100. The rate of success of an arm is decided by the mean rating received by the corresponding movie in the dataset. This mean rating is normalized by first re-centering through its median value in the dataset, and then converting to a probability by passing through a sigmoidal function. As mentioned in Section 2.2, for the dependent algorithm the 40,000 SVD-projected $d$-dimensional vectors have been used to compute coverage vectors through a Gaussian Mixture Model (GMM) with $d'$ centroids.

- **Yelp**: The Yelp Dataset Challenge is a library of restaurants (and related businesses) and their reviews from customers. We pick 200,000 users that have reviewed the highest number of businesses and 50,000 businesses with the highest number of reviews. We sample 10,000 businesses at random and calculate the singular value decomposition (SVD) of the corresponding $200,000 \times 10,000$ ratings matrix into 10 principal components. The projection matrices from the SVD are used to compute embeddings of dimension $d = 10$ for the remaining 40,000 businesses for training the bandit algorithms. The embeddings are normalized to unit $L_2$-norm and the dataset is shuffled randomly. In every round of
bandit learning, the algorithm is presented with a non-overlapping chunk of movies as arms \((A_t)\). The chunk size is 100. The rate of success of an arm is decided by the mean rating received by the corresponding movie in the dataset. This mean rating is normalized by first re-centering through its median value in the dataset, and then converting to a probability by passing through a sigmoidal function. As mentioned in Section 2.2 for the dependent algorithm the 40,000 SVD-projected \(d\)-dimensional vectors have been used to compute coverage vectors through a GMM with \(d'\) centroids.

- **MNIST:** The MNIST dataset consists of 60,000 training samples and 10,000 test samples. We draw 19,800 samples at random from the training split for constructing a \(d = 10\)-dimensional embedding space using Principal Component Analysis (PCA) and combine the remaining training samples with the test samples and randomly shuffle it to create a dataset of 50,200 samples for training the bandit algorithm. As mentioned in Section 2.2 for the dependent algorithm the 50,200 SVD-projected 10-dimensional vectors are used to compute coverage vectors through a GMM with \(d'\) centroids. All observed vectors (embeddings and coverage vectors) are scaled to unit \(L_2\)-norm.

MNIST has 10 output classes. For each of these output classes, we define a sub-task that considers that class as the "pivot-class". At every round of bandit learning, we present the agent with a non-overlapping chunk of examples as arms. The agent observes success only if it chooses an arm whose output class matches the pivot class. We choose the pivot class at the beginning of each experiment and keep it constant throughout.

### D.2 Metric

We report the algorithms’ performance in terms of their Normalized Cumulative Reward (NCR) obtained over all rounds of training. If a given dataset has \(T\) chunks then each algorithm is trained for exactly \(T\) rounds. The Cumulative Reward (CR) obtained by an algorithm is normalized with respect to the CR accrued by the random policy and the maximum possible CR over \(T\) rounds with budget \(b_t\) in each round to obtain NCR as:

\[ NCR = \frac{CR_{\text{algorithm}} - CR_{\text{random}}}{CR_{\text{max}} - CR_{\text{random}}} . \]

NCR is meant to capture the fractional advantage in CR over the random policy Rand. This normalization is needed especially when the random policy shows good performance, for example, in the vanilla scenario with large \(b_t\).

### D.3 Results

Our NCR results are reported in Tables [1-13]. Notice that, by construction, the NCR of Rand is always 0.0. Also observe that for large \(b_t\), the vanilla scenario makes all algorithms essentially indistinguishable, and when \(b_t = 50\) or \(b_t = 100\) also Rand performs as well as all other algorithms. This is not the case for the exponential scenario. In a few cases, the tables reflect negative entries (specifically for Eps), which simply means that the algorithm happened to perform worse than Rand.

From these experiments, the following trends emerge.

1. In a vanilla scenario that emphasizes early success \((b_t\) small), the baseline algorithms (Eps, C-UCB1, GL-CDCM) are rarely the winner. In most cases, the winner is either the proposed independent (Ind) or dependent (Dep) algorithms. On the other hand, as the budget \(b_t\) grows the algorithms tend to be indistinguishable. This has to be expected, as when \(b_t\) is large even the random policy (Rand) becomes competitive in the vanilla scenario, and NCRs, by their very definition, tend to be zero.

2. In the exponential scenario, Dep generally outperforms Ind, with the exception of a few cases in the MNIST dataset (specifically on pivot classes 3, 4, 5, 8, and 9). For these tasks, Dep is dramatically underperforming, probably due to the latent space construction, which does not offer a convenient representation – see Section D.4.

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Table 1: Comparison of normalized cumulative reward on the Million Songs Dataset for two different reward/loss scenarios – Vanilla and Exponential. “Rand” refers to the random policy, “Eps” is the $\epsilon$-greedy version of our Algorithm 1, “C-UCB1” is the cascading bandit algorithm of Zong et al. [2016], while “GL-CDCM” is the one from Liu et al. [2018b]. Moreover, “Ind” and “Dep” are abbreviations for the Independent (Algorithm 1) and Dependent (Algorithm 2) algorithms proposed in this paper. Notice that the exponential scenario does not include the baselines “Eps”, “C-UCB1” and “GL-CDCM” since those baselines are defined to work only in the vanilla scenario. For each of the two scenarios and each value of $b_t$, we emphasize in bold the best performance.

|    | Vanilla | Exponential |
|----|---------|-------------|
| $b_t$ | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
| 1  | 0.00 | 0.27 | 0.29 | 0.28 | 0.32 | 0.51 | 0.00 | 0.32 | 0.48 |
| 5  | 0.00 | 0.40 | 0.50 | 0.90 | 0.90 | 0.90 | 0.00 | 0.35 | 0.56 |
| 10 | 0.00 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.00 | 0.38 | 0.57 |
| 50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.38 | 0.50 |
| 100| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.34 | 0.54 |

Table 2: Same as in Table 1 for the Yelp dataset.

|    | Vanilla | Exponential |
|----|---------|-------------|
| $b_t$ | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
| 1  | 0.00 | 0.17 | 0.06 | 0.16 | 0.16 | 0.25 | 0.00 | 0.18 | 0.26 |
| 5  | 0.00 | 0.33 | 0.33 | 0.67 | 0.67 | 0.80 | 0.00 | 0.17 | 0.31 |
| 10 | 0.00 | 0.35 | 0.35 | 0.35 | 0.35 | 0.35 | 0.00 | 0.23 | 0.28 |
| 50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.21 | 0.26 |
| 100| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.18 | 0.29 |

Table 3: Same as in Table 1 for the Movielens dataset.

|    | Vanilla | Exponential |
|----|---------|-------------|
| $b_t$ | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
| 1  | 0.00 | 0.26 | 0.17 | 0.35 | 0.40 | 0.34 | 0.00 | 0.35 | 0.39 |
| 5  | 0.00 | 0.23 | 0.51 | 0.78 | 0.62 | 0.84 | 0.00 | 0.36 | 0.43 |
| 10 | 0.00 | 0.38 | 0.38 | 0.38 | 0.38 | 0.38 | 0.00 | 0.35 | 0.38 |
| 50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.40 | 0.39 |
| 100| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.32 | 0.38 |

Table 4: Comparison of normalized cumulative reward on the MNIST Dataset with pivot-class 0 for two different reward/loss scenarios – Vanilla and Exponential. “Rand” refers to the random policy, “Eps” is the $\epsilon$-greedy version of our Algorithm 1, “C-UCB1” is the cascading bandit algorithm of Zong et al. [2016], while “GL-CDCM” is the one from Liu et al. [2018b]. Moreover, “Ind” and “Dep” are abbreviations for the Independent (Algorithm 1) and Dependent (Algorithm 2) algorithms proposed in this paper. Notice that the exponential scenario does not include the baselines “Eps”, “C-UCB1” and “GL-CDCM” since those baselines are defined to work only in the vanilla scenario. For each of the two scenarios and each value of $b_t$, we emphasize in bold the best performance.

|    | Vanilla | Exponential |
|----|---------|-------------|
| $b_t$ | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
| 1  | 0.00 | 0.95 | 0.84 | 0.98 | 0.97 | 1.00 | 0.00 | 0.97 | 1.00 |
| 5  | 0.00 | 0.97 | 0.99 | 0.99 | 0.99 | 1.00 | 0.00 | 0.99 | 1.00 |
| 10 | 0.00 | 0.97 | 0.99 | 0.99 | 0.99 | 0.99 | 0.00 | 0.99 | 1.00 |
| 50 | 0.00 | 0.70 | 0.70 | 0.70 | 0.70 | 0.70 | 0.00 | 0.99 | 1.00 |
| 100| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.99 | 1.00 |

Table 5: Same as in Table 4 for MNIST Dataset with pivot-class 1.

|    | Vanilla | Exponential |
|----|---------|-------------|
| $b_t$ | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
| 1  | 0.00 | 0.91 | 0.93 | 0.99 | 0.98 | 1.00 | 0.00 | 0.98 | 1.00 |
| 5  | 0.00 | 0.97 | 0.99 | 0.99 | 0.99 | 1.00 | 0.00 | 0.99 | 1.00 |
| 10 | 0.00 | 0.99 | 0.99 | 0.99 | 0.99 | 0.99 | 0.00 | 0.99 | 1.00 |
| 50 | 0.00 | 0.58 | 0.58 | 0.58 | 0.58 | 0.58 | 0.00 | 0.99 | 1.00 |
| 100| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.99 | 1.00 |
| \(b_2\) | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.94 | 0.90 | 0.98 | 0.99 | 0.99 | 0.00 | 0.99 | 0.99 |
| 5 | 0.00 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 0.00 | 1.00 | 0.99 |
| 10 | 0.00 | 0.97 | 0.99 | 0.99 | 0.99 | 0.99 | 0.00 | 1.00 | 1.00 |
| 50 | 0.00 | 0.74 | 0.74 | 0.74 | 0.74 | 0.74 | 0.00 | 0.99 | 0.99 |
| 100 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.99 | 0.99 |

Table 7: Same as in Table 4 for MNIST Dataset with pivot-class 3.

| \(b_2\) | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.93 | 0.87 | 0.94 | 0.94 | 0.84 | 0.00 | 0.94 | 0.85 |
| 5 | 0.00 | 0.94 | 0.96 | 0.99 | 0.99 | 0.99 | 0.00 | 0.98 | 0.94 |
| 10 | 0.00 | 0.90 | 0.98 | 0.99 | 0.99 | 0.99 | 0.00 | 0.98 | 0.93 |
| 50 | 0.00 | -0.66 | 0.76 | 0.76 | 0.76 | 0.76 | 0.00 | 0.97 | 0.89 |
| 100 | 0.00 | -1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.98 | 0.89 |

Table 8: Same as in Table 4 for MNIST Dataset with pivot-class 4.

| \(b_2\) | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.93 | 0.83 | 0.93 | 0.92 | 0.71 | 0.00 | 0.92 | 0.71 |
| 5 | 0.00 | 0.92 | 0.99 | 0.99 | 0.99 | 0.96 | 0.00 | 0.98 | 0.85 |
| 10 | 0.00 | 0.92 | 0.99 | 0.99 | 0.99 | 0.98 | 0.00 | 0.98 | 0.84 |
| 50 | 0.00 | -0.52 | 0.75 | 0.75 | 0.75 | 0.75 | 0.00 | 0.97 | 0.78 |
| 100 | 0.00 | -1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.97 | 0.79 |

Table 9: Same as in Table 4 for MNIST Dataset with pivot-class 5.

| \(b_2\) | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.84 | 0.85 | 0.88 | 0.89 | 0.68 | 0.00 | 0.89 | 0.69 |
| 5 | 0.00 | 0.84 | 0.96 | 0.98 | 0.97 | 0.96 | 0.00 | 0.95 | 0.82 |
| 10 | 0.00 | 0.75 | 0.97 | 0.97 | 0.98 | 0.97 | 0.00 | 0.94 | 0.81 |
| 50 | 0.00 | -2.97 | 0.82 | 0.82 | 0.82 | 0.82 | 0.00 | 0.93 | 0.73 |
| 100 | 0.00 | -3.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.92 | 0.72 |

Table 10: Same as in Table 4 for MNIST Dataset with pivot-class 6.

| \(b_2\) | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.94 | 0.97 | 0.97 | 0.95 | 1.00 | 0.00 | 0.95 | 1.00 |
| 5 | 0.00 | 0.97 | 1.00 | 0.98 | 0.98 | 1.00 | 0.00 | 0.98 | 1.00 |
| 10 | 0.00 | 0.96 | 0.99 | 0.99 | 0.99 | 0.99 | 0.00 | 0.99 | 1.00 |
| 50 | 0.00 | 0.46 | 0.73 | 0.73 | 0.73 | 0.73 | 0.00 | 0.98 | 1.00 |
| 100 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.98 | 1.00 |

Table 11: Same as in Table 4 for MNIST Dataset with pivot-class 7.
Table 12: Same as in Table 4 for MNIST Dataset with pivot-class 8.

| $b_t$ | Vanilla | | | | | | Exponential | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
| 1 | 0.00 | 0.89 | 0.92 | 0.91 | 0.91 | 0.71 | 0.00 | 0.91 | 0.71 |
| 5 | 0.00 | 0.83 | 0.94 | 0.98 | 0.98 | 0.96 | 0.00 | 0.95 | 0.86 |
| 10 | 0.00 | 0.80 | 0.97 | 0.98 | 0.98 | 0.98 | 0.00 | 0.95 | 0.84 |
| 50 | 0.00 | -1.39 | 0.78 | 0.78 | 0.78 | 0.78 | 0.00 | 0.94 | 0.78 |
| 100 | 0.00 | -2.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.93 | 0.78 |

Table 13: Same as in Table 4 for MNIST Dataset with pivot-class 9.

| $b_t$ | Vanilla | | | | | | Exponential | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Rand | Eps | C-UCB1 | GL-CDCM | Ind | Dep | Rand | Ind | Dep |
| 1 | 0.00 | 0.77 | 0.80 | 0.80 | 0.75 | 0.74 | 0.00 | 0.76 | 0.74 |
| 5 | 0.00 | 0.66 | 0.97 | 0.99 | 0.99 | 0.98 | 0.00 | 0.92 | 0.89 |
| 10 | 0.00 | 0.56 | 0.98 | 0.99 | 0.99 | 0.99 | 0.00 | 0.91 | 0.86 |
| 50 | 0.00 | -4.56 | 0.72 | 0.72 | 0.72 | 0.72 | 0.00 | 0.88 | 0.81 |
| 100 | 0.00 | -3.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.87 | 0.81 |

D.4 Further investigations

In order to further understand the poor performance of Dep in the MNIST classes 3, 4, 5, 8, and 9 (Tables 7–9, 12, and 13, respectively), we conducted a small investigation to see to what extent the latent space representation can be deemed responsible for this performance.

We run on the MNIST dataset the independent algorithm Ind on the same GMM-generated latent space on which we ran Dep, and optimized the number $d'$ of centroids as in the tuning of Dep. We then compared the results to Dep as reported in Tables 4–13.

Figure 3 collects the outcome of this comparison on two relevant scenarios, vanilla with $b_t = 1$ and exponential with $b_t = 10$. In the x-axis of the two plots are the pivot classes 0, ..., 9, on the y-axis are the final CR performances. As one can clearly see from both plots, when Dep performs poorly (classes 3, 4, 5, 8, and 9), it is also the case that the difference in performance between Ind with GMM and Ind without GMM becomes negative, that is to say, those pivot classes for MNIST are the same classes on which one can observe performance degradation when the GMM-latent space representation is added to Ind.

Though a more thorough investigation on the role of the latent space has to be performed, this finding by itself gives a strong support to the claim that it is indeed the GMM-based latent space that hinders the performance of the bandit algorithms in some cases.

![Figure 3](image-url)

**Figure 3:** MNIST dataset: Correlation across pivot classes 0, ..., 9 between the CR performance of Dep (“dependent”) and the difference in CR performance between Ind run on the GMM latent space and Ind run without latent space (“difference”). On the left plot is a vanilla scenario with $b_t = 1$ or the right an exponential scenario with $b_t = 10$. As one can clearly see, on the classes where Dep performs poorly, that is, pivot classes 3, 4, 5, 8, and 9, there is also a substantial degradation in performance for Ind when run on the latent space.