Majorana-Hubbard Ladders

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Models of interacting Majorana modes may be realized in vortex lattices in superconducting films in contact with topological insulators and may be tuned to the strong interaction regime by adjusting the chemical potential. Extending the results on one- and two-dimensional Majorana-Hubbard models, here we study two- and four-leg ladders using both field theory and the density-matrix renormalization group (DMRG) methods, finding a phase diagram largely consistent with that proposed for the two-dimensional model on the square lattice.

I. INTRODUCTION

Majorana zero modes (MZM) are expected to emerge in various superconducting systems [1-4]. There has been significant experimental progress in their realizations recently [5-9]. In some of the proposed realizations, multiple MZMs with single-particle coupling can arise [2-10], motivating a in-depth study of the many-body phases of matter emerging from MZM building blocks [12-14].

One particular system is of considerable interest, as it gives access to the strongly interacting limit in a tunable way: A thin superconducting film in contact with a topological insulator in the presence of a transverse magnetic field is predicted to have a Majorana mode localized in every vortex core [2]. The low-energy effective Hamiltonian contains short-range hopping and interaction terms. By tuning the chemical potential to zero with respect to the Dirac point of the topological insulator surface, the hopping term can be made to vanish due to an extra chiral symmetry, which changes the topological classification from $Z_2$ to $Z$. Consequently, tuning the chemical potential to small values results in small hopping parameters and brings the system into the strong coupling regime. The simplest one-dimensional case was studied [16, 17] using a combination of field theory and density-matrix renormalization group (DMRG) methods. It was shown to have a rich phase diagram which includes a supersymmetric tricritical Ising phase transition and strong-coupling phases where the Majorana modes combine in pairs to form ordinary complex (“Dirac”) fermions, spontaneously breaking various discrete symmetries of the model. Other one-dimensional models exhibit a similar phase diagram with the phase transition at weaker interaction strengths [21, 25].

The two-dimensional (2D) square lattice version of this model, with Hamiltonian,

$$H = \sum_{m,n} \{i t_{m,n} [(-1)^m \gamma_m + \gamma_n + 1] + g \gamma_m \gamma_n \gamma_m + 1, n \gamma_m + 1, n + 1 \gamma_m + 1\}$$

(1.1)

was studied in [26] where similar strong coupling phases were argued to occur using mean field theory, field theory and renormalization group methods. The infinite coupling limit was studied in [27] using quantum Monte Carlo and an exact mapping into the “Compass” spin model, obtaining results largely consistent with [26]. When the hopping term is included a Monte Carlo sign problem exists. The best numerical method to use is therefore probably DMRG on ladders. Here we study the case of two- and four-leg ladders. We show that the two-leg case maps into a well-understood model, the XXZ $S = 1/2$ chain. The four-leg case exhibits novel behavior which we study with a combination of field theory and DMRG methods. We again find that Majorana modes combine to form occupied/empty Dirac modes at strong coupling, breaking discrete symmetries, in both two- and four-leg cases.

In Sec. II we review results on the 2D case. In Sec. III we study the two-leg ladder, with periodic boundary conditions (PBC), showing that it has a hidden $U(1)$ symmetry and can be mapped into the XXZ $S = 1/2$ spin chain model. It has a massless phase at weak coupling with broken symmetry states occurring at sufficiently strong coupling of either sign, corresponding to Majoranas on rungs forming Dirac fermions, consistent with our predictions in the 2D case. In Sec. IV we show that the weak coupling limit for a ladder with an even number of legs and PBC is equivalent to the two-leg case, with the corresponding massless phase. We then study the strong coupling limit of the four-leg ladder with PBC, showing that Majorana pairing occurs, largely consistent with our predictions in D=2. In Sec. V, we study the two and four-leg ladder with open boundary conditions in the $y$-direction, which lead to quite different behavior. In Sec. VI we include second neighbor interactions for the two-leg case. In Sec. VII, we map the four-leg Majorana ladder into a two-leg spin ladder and check that the strong coupling limit discussed in Sec. IV is recovered. In Sec. VIII we analyze analytically the phase transition that occurs in the four-leg ladder for $g > 0$. In Sec. IX we present numerical results on the phase diagram of the four-leg ladder. Sec. X contains conclusions.
II. REVIEW OF 2D CASE

Due to the alternating sign of the horizontal hopping term in Eq. (1.1) (stemming from the flux quantization condition in the underlying vortex lattice [28]), the unit cell contains two sites and it convenient to introduce e/o labels for even and odd rows:

\[ \gamma_{2j} \equiv \gamma_{2j}^e, \quad \gamma_{2j+1} \equiv \gamma_{2j+1}^o. \]  \tag{2.1}

(Here we use slightly different notation than in [26] to simplify some formulae.) For a chain of 2W rows of length L we Fourier transform:

\[ \gamma_e \vec{k} \equiv \frac{1}{\sqrt{2WL}} \sum_{m,n} e^{i(mk_x+2nk_y)} \gamma_{m,2n}^e, \quad \gamma_o \vec{k} \equiv \frac{1}{\sqrt{2WL}} \sum_{m,n} e^{i(mk_x+(2n+1)k_y)} \gamma_{m,2n+1}^o. \]  \tag{2.2}

The hopping term in \( H \) then becomes:

\[ H_0 = -4t \sum_{k_x>0,k_y} \left[ (\gamma_{k}^{e/0})^+ \gamma_{k}^{e/0} \sin k_x + (\gamma_{k}^{e/0})^+ \gamma_{k}^{e/0} \sin k_y \right]. \]  \tag{2.3}

Here we have used the fact that \( \gamma_{-\vec{k}}^{e/o} = \gamma_{\vec{k}}^{e/o} \), to restrict the Brillouin zone to \( 0 \leq k_x < \pi, -\pi/2 \leq k_y < \pi/2 \). The energy bands are:

\[ E_{\pm} = \pm 4t \sqrt{\sin^2 k_x + \sin^2 k_y}. \]  \tag{2.4}

The low energy Hamiltonian corresponds to two two-component relativistic Majorana fermions at the two “Dirac points” \((0,0)\) and \((\pi,0)\) which can be combined into a single relativistic Dirac fermion, \( \psi \). The interaction term becomes:

\[ H_{int} = 32g(\bar{\psi}\psi)^2 \]  \tag{2.5}

which is an irrelevant interaction in the relativistic (2 + 1)-dimensional field theory, leading to a massless phase for sufficiently weak coupling. We predicted in Ref. [26] that, at a critical positive coupling, \( g_c \), there is a transition into a phase with pairs of neighboring Majoranas forming Dirac fermions. At a mean field level these Dirac energy levels are either filled or empty as indicated in Fig. (1); unfilled circles correspond to empty states. In addition to these ground states two others occur, rotated by \( \pi/2 \) with Dirac fermions forming on horizontal links. For large enough negative \( g \), \( g < g_c \) a symmetry-breaking phase occurs with the Dirac fermions levels alternating filled and empty as indicated in Fig. (1). As shown in Fig. (1), the strongly coupled ordered phase is four-fold (eight-fold) degenerate for large positive (negative) \( g \).

III. TWO-LEG LADDER

In this case the model can be converted into a particle number conserving Dirac model by defining:

\[ c_m \equiv \frac{\gamma_{m,0} + i(-1)^m \gamma_{m,1}}{2}. \]  \tag{3.1}

We combine Majoranas on each vertical link to make Diracs. Thus

\[ \gamma_{m,0} = c_m + c_m^\dagger \]
\[ \gamma_{m,1} = (-1)^m i(c_m^\dagger - c_m) \]  \tag{3.2}

The horizontal hopping term become:

\[ H_0 = 2it \sum_m [c_m^\dagger c_{m+1} - c_{m+1}^\dagger c_m]. \]  \tag{3.3}

The vertical hopping term vanishes with periodic boundary conditions in the y direction since \( \gamma_{m,0} \gamma_{m,1} + \gamma_{m,1} \gamma_{m,0} = 0 \). The interaction term becomes:

\[ H_{int} = 2g(2c_m^\dagger c_m - 1)(2c_{m+1}^\dagger c_{m+1} - 1). \]  \tag{3.4}
Figure 1: (a) The symmetry-breaking pattern for two of the four strong-coupling ground states of the two-dimensional Majorana-Hubbard model for $g > 0$ on the square lattice. The other two states can be obtained by a $\pi/2$ rotation. (b) Four of the eight symmetry-breaking patterns of the mean-field strong-coupling ground states predicted for $g < 0$. The other four states can be obtained by a $\pi/2$ rotation. Blue circles are the Majorana modes, a bond between them indicates the combination of the MZMs into a Dirac fermion. The larger circle on the bond represent the occupation of the Dirac mode.

The factor of 2 in $H_{int}$ arises because there are 2 interaction terms for each $m$ due to the periodic boundary conditions in the $y$-direction; these both have the same sign. This is the standard spinless Dirac fermion model with nearest neighbour interactions: attractive for $g > 0$. (We may eliminate the factor of $i$ from the hopping term by the transformation:

$$c_m \rightarrow i^m c_m$$

which shifts the wave-vector by $\pi/2$. This transformation has no effect on the interaction term.) After a Jordan-Wigner transformation the Hamiltonian becomes:

$$H = \sum_m \left[ t (\sigma^1_m \sigma^1_{m+1} + \sigma^2_m \sigma^2_{m+1}) + 2g \sigma^3_m \sigma^3_{m+1} \right],$$

(3.6)

the well-known XXZ model. For $|g| < 1/2$ a Luttinger liquid phase occurs. At $g = \pm 1/2$ transitions occur into ordered phases. Noting that

$$i\gamma_m,0 \gamma_{m,1} = (-1)^m(2c_m^\dagger c_m - 1) = (-1)^m \sigma^3_m,$$

(3.7)

we see that for $g > 1/2$ the mean field phase of Fig. (1a) occurs and for $g < -1/2$ the mean field phase of Fig. (1b) occurs. [The transformation of Eq. (3.2) switches ferromagnetic with anti-ferromagnetic phases.] It is interesting to note that, for the two-leg ladder, only the vertical order occurs, not the horizontal order. Furthermore, due to the absence of the vertical hopping term, filled or empty Dirac levels are degenerate, leading to a total of two ground states for $g > 0$ as well as $g < 0$. It is also interesting to note that at infinite $g/t$, corresponding to $t = 0$, the case analyzed in [27], there is an infinite set of operators $\sigma^3_m$, commuting with the Hamiltonian. This corresponds to a trivial case of the infinite number of “intermediate symmetries” of the compass model. As expected in the 2D model [27] at any finite temperature the broken symmetry is restored.
IV. FOUR-LEG LADDER

A. Weak coupling

We first discuss the noninteracting model for any even number of legs with PBC. There are now $W$ values of $k_y$ for a ladder with $2W$ legs. We see from Eq. (2.4) that only the $k_y = 0$ mode is gapless, with

$$E_{\pm} = \pm 4t \sin k_z.$$  \hspace{1cm} (4.1)

This follows from Eq. (2.3) due to the absence of any even-odd coupling for $k_y = 0$. So, the low energy theory contains only the $k_y = 0$ mode. Fourier transforming with respect to $y$ only and keeping only $k_y = 0$, we recover precisely the two-leg ladder model. Thus we see that, for any $W$ and small enough $|g|$, we recover the massless Luttinger liquid phase discussed above. However, we may expect that the transitions to gapped phases will occur at different values of $g$ and the universality classes of the phase transitions to gapped phases to be different than in the two-leg case.

B. Strong coupling

We now consider the four-leg ladder in the limit $t = 0$ for finite $g$; we later add an infinitesimal $t$. We combine pairs of Majoranas on vertical rungs into Dirac fermions as

$$c_{m,1} = (\gamma_{m,0} + i\gamma_{m,1})/2$$
$$c_{m,2} = (\gamma_{m,2} + i\gamma_{m,3})/2,$$  \hspace{1cm} (4.2)

implying

$$\begin{align*}
\gamma_{m,0} &= c_{m,1} + c_{m,1}^\dagger \\
\gamma_{m,1} &= i(c_{m,1}^\dagger - c_{m,1}) \\
\gamma_{m,2} &= c_{m,2} + c_{m,2}^\dagger \\
\gamma_{m,3} &= i(c_{m,2}^\dagger - c_{m,2}) \\
&
\end{align*}$$

in Eq. (4.3). The Hamiltonian can be written as

$$H = -g \sum_{m} \sum_{j=0}^{3} (i\gamma_{m,j} \gamma_{m,j+1})(i\gamma_{m+1,j} \gamma_{m+1,j+1})$$  \hspace{1cm} (4.4)

Noting that all terms in $i\gamma_{m,j} \gamma_{m,j+1}$ either preserve the fermion number on the $m^{th}$ rung or change it by 2, we see that fermion number is conserved mod 2 on each rung. Equivalently, fermion parity is conserved on each rung. We may simplify the $i\gamma_{m,j} \gamma_{m,j+1}$ operators depending on which parity sector we are in. It is convenient to identify the 2 states on each rung of given fermion parity with spin up and down states for a spin-1/2 variable.

*Even fermion parity:

We identify

$$|\downarrow\rangle = |0\rangle$$
$$|\uparrow\rangle = c_{1}^\dagger c_{2}^\dagger |0\rangle.$$  \hspace{1cm} (4.5)

This gives:

$$\begin{align*}
i\gamma_{m,0} \gamma_{m,1} &= i\gamma_{m,2} \gamma_{m,3} = \sigma_{m}^3 \\
i\gamma_{m,1} \gamma_{m,2} &= -i\gamma_{m,3} \gamma_{m,0} = \sigma_{m}^i
\end{align*}$$  \hspace{1cm} (4.6)
In the second line, we used the fact that the first term in these operators can be dropped in the even fermion parity sector and the fact that \( c_1^\dagger c_2^\dagger |\downarrow\rangle = |\uparrow\rangle \).

**Odd fermion parity:**

We now identify

\[
|\downarrow\rangle = c_1^\dagger |0\rangle \\
|\uparrow\rangle = c_2^\dagger |0\rangle.
\]  

(4.7)

This gives:

\[
-i\gamma_{m,0}\gamma_{m,1} = i\gamma_{m,2}\gamma_{m,3} = \sigma_m^3 \\
i\gamma_{m,1}\gamma_{m,2} = i\gamma_{m,3}\gamma_{m,0} = -\sigma_m^1.
\]  

(4.8)

In the second line we now used the fact that the second term in these operators can be dropped. Assuming the same fermion parity on every rung,

\[
H = -2g \sum_m (\sigma_m^3 \sigma_{m+1}^3 + \sigma_m^1 \sigma_{m+1}^1)
\]  

(4.9)

If any neighbouring pair of rungs have opposite fermion parity, that term in the Hamiltonian vanishes, due to the opposite signs in the formulas for \( i\gamma_{m,j}\gamma_{m,j+1} \). Thus the ground state clearly has the same fermion parity on every site. This spin model has no broken symmetries and is gapless. However, there is actually a 2-fold ground state degeneracy - even or odd fermion parity. Again there is an infinite number of conserved quantities, the fermion parity on every rung, as in the compass model. Unlike the two-leg case, the four-leg model is massless at \( t = 0 \). This massless behavior is consistent with that of the compass model.

Now consider turning on an infinitesimal hopping term. The hopping terms in the vertical direction on the \( m^{th} \) rung become:

\[
2t\sigma_m^3 \quad \text{(even fermion parity)} \\
-2\sigma_m^1 \quad \text{(odd fermion parity)}.
\]  

(4.10)

On the other hand, hopping terms in the horizontal direction change the fermion number on each rung by \( \pm 1 \), reversing the fermion parity. This gives a state in which two of the neighboring bonds have opposite fermion parity. Acting with the hopping term on the \( m \leftrightarrow (m+1) \) bond gives opposite fermion parity on the \( (m-1) \leftrightarrow m \) and \( (m+1) \leftrightarrow (m+2) \) bonds. This is a high energy state so its effects are \( O(t^2/g) \). The leading-order Hamiltonian then becomes

\[
H = \sum_m [\ -2g(\sigma_m^3 \sigma_{m+1}^3 + \sigma_m^1 \sigma_{m+1}^1) + 2t\sigma_m^3] \quad \text{(even fermion parity)}
\]

\[
= \sum_m [\ -2g(\sigma_m^3 \sigma_{m+1}^3 + \sigma_m^1 \sigma_{m+1}^1) - 2t\sigma_m^1] \quad \text{(odd fermion parity)}.
\]  

(4.11)

For \( g > 0 \), \( \langle \sigma_m^3 \rangle < 0 \) for even fermion parity and \( \langle \sigma_m^1 \rangle > 0 \) for odd fermion parity. For even fermion parity, we see from Eq. (4.10), that \( \langle \sigma_m^3 \rangle < 0 \) corresponds to the two Dirac fermion levels formed on vertical bonds 0 – 1 and 2 – 3, being empty. For odd fermion parity, we see from Eq. (4.11) that \( \langle \sigma_m^1 \rangle > 0 \) corresponds to the two Dirac fermion levels formed on vertical bonds 1 – 2 and 3 – 0 being empty. These correspond precisely to the two mean field 2D ground states of Fig. (1). Again, we see that the ladder geometry favors Dirac fermions forming on vertical, not horizontal links. So, the number of ground states is 2, not 4. It is also interesting to note that \( \langle \sigma_m^3 \rangle \) is presumably zero in the even fermion parity sector. Thus \( \langle i\gamma_{m,1}\gamma_{m,2} \rangle = \langle i\gamma_{m,3}\gamma_{m,0} \rangle = 0 \). This is consistent with the mean field picture of \( \gamma_{m,0} \) and \( \gamma_{m,1} \) combining to form a Dirac fermion and also \( \gamma_{m,2} \) and \( \gamma_{m,3} \). \( \gamma_{m,1} \) and \( \gamma_{m,2} \) are not entangled so their product has zero expectation value and similarly for the \( \gamma_{m,3}\gamma_{m,0} \) product. Similarly, in the odd fermion parity sector \( \langle \sigma_m^1 \rangle = 0 \) corresponding to \( \gamma_0 - \gamma_1 \) and \( \gamma_2 - \gamma_3 \) not being entangled.

For \( g < 0 \), we expect that a uniform in-plane magnetic field applied to an antiferromagnet leads to in-plane antiferromagnetic order, with the antiferromagnetic order parameter perpendicular to the field. Thus we expect

\[
\langle \sigma_m^1 \rangle \propto (-1)^m \quad \text{(even fermion parity)}
\]

\[
\langle \sigma_m^3 \rangle \propto (-1)^m \quad \text{(odd fermion parity)}.
\]  

(4.12)

We see from Eqs. (4.10) and (4.11), that this again corresponds to forming Dirac levels on vertical bonds, with the levels alternating filled and empty along each row. However, unlike our mean field prediction, the two levels on each rung
have opposite filling, as illustrated in Fig. (2). This can be simply understood in the odd sector. The total number of electrons on each rung is 1 in the odd sector, so if the $0-1$ level has filling $<1/2$ then the $2-3$ level must have filling $>1/2$ and vice versa. Similarly, in the even sector, from Eq. (4.3),

$$i\gamma_{m,1}\gamma_{m,2} + i\gamma_{m,3}\gamma_{m,0} = -2\left(c_{m,1}^\dagger c_{m,2} + c_{m,2}^\dagger c_{m,1}\right),$$

and this annihilates both empty and doubly occupied states. Thus, if we form Dirac modes on sites $1-2$ and $3-0$,

$$c_{m,1}' = (\gamma_{m,1} + i\gamma_{m,2})/2, \quad c_{m,2}' = (\gamma_{m,3} + i\gamma_{m,0})/2,$$

then

$$i\gamma_{m,1}\gamma_{m,2} + i\gamma_{m,3}\gamma_{m,0} = 2\left(c_{m,1}'c_{m,1}' + c_{m,2}'c_{m,2}' - 1\right) = 0.$$  

The total occupancy must be 1 so again if the Dirac mode on 1 2 has filling $<1/2$ the Dirac model on 3 0 must have filling $>1/2$ and vice versa.

Now let us consider the nature of the phase transition that occurs at $t = 0$, into a gapped phase. It is convenient to rotate the spins to obtain a more standard model:

$$H = \sum_m [2\gamma_{m,1}\sigma_{m,1}^1 + \sigma_{m,2}\sigma_{m,1}^1] + 2t(-1)^m\sigma_{m,1}^1 \quad (g > 0)$$

$$= \sum_m [2\gamma_{m,1}\sigma_{m,1}^1 + \sigma_{m,2}\sigma_{m,1}^1] + 2t\sigma_{m,2}^1 \quad (g < 0).$$

In both cases we obtain the antiferromagnetic XY model with an in-plane staggered field for $g > 0$ and an in-plane uniform field for $g < 0$. Bosonization gives:

$$\sigma_{m,1}^1 = \cos(\sqrt{3}\phi)[(-1)^mA + B\cos\sqrt{3}\pi\phi].$$

For $g > 0$ the staggered field is a relevant operator of dimension $d = 1/4$. We expect it to produce a mass gap scaling as

$$m \propto t^{1/(2-d)} = t^{4/7}.$$  

Naively, we might expect that since the uniform field has dimension $5/4$ it will produce a mass scaling with exponent $1/(2-d) = 4/3$. However, this argument can be seen to be wrong. We expect the gapped phase to correspond to antiferromagnetic order in the 2 direction. This implies $\phi$ is pinned. But if $\phi$ is pinned, $\theta$ fluctuates strongly so this uniform field term becomes irrelevant. However, at second order in $t$ we should generate a relevant $\cos\sqrt{3}\pi\phi$ term of dimension 1. This should lead to pinning of $\phi$. In fact, this corresponds to anisotropic exchange, $J^y > J^z$, also favouring antiferromagnetic order. Actually the XY model with $J^x \neq J^y$ is exactly solvable by Jordan-Wigner transformation and certainly has a gap. Once we add a uniform field there is no symmetry forbidding anisotropic exchange so this looks reasonable. This suggests that the relevant coupling constant is $\propto t^2$ and of dimension 1, leading to a gap $\propto t^2$, rather than $|t|^{4/3}$. Including $g$, on dimensional grounds, the gap scales as $|t|^{4/7}g^{3/7}$ for $g \to +\infty$ and $t^2/|g|$ for $g \to -\infty$.

We were able to verify the above prediction numerically for $g/t \to +\infty$ by computing the energy gap $\Delta E$ with antiperiodic boundary conditions with DMRG for various system sizes ($N = 40,..80$) and extrapolating them to the thermodynamic limit. We fixed $g = 1$ and used several small values of $t = 0.01 \ldots 0.05$. Plotting $\ln(\Delta E)$ as a function of $\ln(t)$, as shown in Fig. (3) we find a very good linear fit with slope 0.58 that is very close to $4/7 \approx 0.57$.  

Figure 2: 1 of the 4 ground states occurring in the 4-leg ladder for $g \ll -t$.  

\[ \text{Figure 2: 1 of the 4 ground states occurring in the 4-leg ladder for } g \ll -t. \]
Figure 3: A linear fit of ln(ΔE) as a function of ln(t) for g = 1 and small t supports a t^{4/7} scaling.

**V. OPEN BOUNDARY CONDITIONS IN y-DIRECTION**

Open boundary conditions eliminate the massless $k_y = 0$ mode in the non-interacting model. In the limit $W \to \infty$, the 2D model, we expect this to be unimportant. But for finite width ladders it has a large effect.

In the two leg case, open boundary conditions lead to a vertical hopping term. We see from Eq. (3.7) that this changes the spin Hamiltonian to
\begin{equation}
H = \sum_m [t(\sigma^1_m \sigma^1_{m+1} + \sigma^2_m \sigma^2_{m+1}) + g\sigma^3_m \sigma^3_{m+1} + t(-1)^m \sigma^3_m],
\end{equation}
adding a staggered field in the 3-direction which can produce a gap, even for $g = 0$. This gap is present for all $g \geq 0$. However, for $g < 0$, where the exchange term is ferromagnetic this staggered field leads to frustration. At large enough $g < 0$ ferromagnetic order occurs. There is an intermediate range of $g < 0$ where a massless Luttinger liquid phase occurs.\[29, 30\]

In the 4-leg case, the gapless phase at small $g$ is again eliminated. The gapless phase at $t = 0$ is also eliminated. This follows because the plaquette interaction involving rows 3 and 0 doesn’t appear so the Hamiltonian becomes
\begin{equation}
H = -g \sum_m [2\sigma^3_m \sigma^3_{m+1} + \sigma^1_m \sigma^1_{m+1}].
\end{equation}
This model should be gapped with order in the 3 direction, ferromagnetic for $g > 0$ and antiferromagnetic for $g < 0$. Again this corresponds to Dirac fermion formation on vertical bonds but now only occurring on 0 − 1 and 2 − 3 bonds. A small hopping term adds a perturbation:
\begin{equation}
\delta H = t \sum_m (2\sigma^3_m - \sigma^3_m).
\end{equation}
For $g > 0$, this leads to a unique ground state with the Dirac levels empty. For $g < 0$ it doesn’t change the two ground states.

**VI. INCLUDING SECOND NEIGHBOUR HOPPING FOR 2 LEGS**

With periodic boundary conditions, the Hamiltonian has the additional term
\begin{equation}
H_2 = 2it_2 \sum_m [\gamma_{m,0} \gamma_{m+1,1} + \gamma_{m+1,0} \gamma_{m,1}].
\end{equation}
(With open boundary conditions, $2t_2 \to t_2$.) In terms of Dirac fermions this becomes
\begin{equation}
H_2 = 4t_2 \sum_m (-1)^m (c_{m+1} c_m + h.c.).
\end{equation}
Making the transformation of Eq. (3.5), this becomes
\begin{equation}
H_2 = 4t_2i \sum_m (c_{m+1} c_m - h.c.).
\end{equation}
Making the Jordan-Wigner transformation, this becomes:
\[ H_2 = -2t_2 \sum_m (\sigma_m^1 \sigma_{m+1}^2 + \sigma_m^2 \sigma_{m+1}^1). \] (6.4)

Making a rotation of the spin variables in the 1\,-\,2 plane on every second site we obtain:
\[ H = \sum_m [(t + 2t_2)\sigma_m^1 \sigma_{m+1}^1 + (t - 2t_2)\sigma_m^2 \sigma_{m+1}^2 + 2g\sigma_m^3 \sigma_{m+1}^3]. \] (6.5)

Note that while the \( t_2 \) term breaks time-reversal in the 2D model it doesn’t for the 2-leg ladder with periodic boundary conditions due to the absence of the vertical hopping term. We may define time-reversal as
\[ \gamma_{m,n} \rightarrow (-1)^m \gamma_{m,n}, \quad i \rightarrow -i. \] (6.6)

Choosing \( t, t_2 > 0 \), we get antiferromagnetic order in the 1 direction for sufficiently small \( g \). This is a gapped phase. Although there is a spontaneously broken symmetry in the spin model, with an order parameter of the form \( \sigma_m^1 \cos \theta + \sigma_m^2 \sin \theta \), this becomes a non-local operator in the fermion model after the Jordan-Wigner transformation and there is no spontaneously broken symmetry in the fermion model. This is also evident from the fact that, for \( g = 0 \) it corresponds to a simple free fermion model with hopping and pairing terms. This model is equivalent to the Kitaev wire model[4], for which it is known that the topological degeneracy maps onto the symmetry breaking one under the Jordan-Wigner transformation. For sufficiently large positive \( g \) the direction of the antiferromagnetic order should switch to the 3 direction corresponding to spontaneous breaking of the \( \sigma_m^1 \rightarrow -\sigma_m^1 \) symmetry and the type of Majorana order discussed above.

**VII. SPIN LADDER MAPPING**

We first rewrite the Hamiltonian in terms of momentum 0 and \( \pi \) channels, with periodic boundary conditions. For \( g = 0 \), the momentum 0 channel is gapless and the momentum \( \pi \) channel is gapped. While the 2 channels are decoupled in the non-interacting model, the interaction term couples them together (while also introducing interactions within each channel). It is then convenient to attempt to integrate out the gapped \( \pi \)-channel to get an effective Hamiltonian for the gapless 0-channel in order to study the phase transitions.

The momentum 0 and \( \pi \) channels are defined by:
\[
\begin{align*}
\gamma_{m,0}^e &:= \frac{1}{\sqrt{2}}(\gamma_{m,0} + \gamma_{m,2}), \quad \gamma_{m,0}^o := \frac{1}{\sqrt{2}}(\gamma_{m,1} + \gamma_{m,3}) \\
\gamma_{m,\pi}^e &:= \frac{1}{\sqrt{2}}(\gamma_{m,0} - \gamma_{m,2}), \quad \gamma_{m,\pi}^o := \frac{1}{\sqrt{2}}(\gamma_{m,1} - \gamma_{m,3}).
\end{align*}
\] (7.1)

Solving:
\[
\begin{align*}
\gamma_{m,0} &= \frac{1}{\sqrt{2}}(\gamma_{m,0}^e + \gamma_{m,\pi}^e), \quad \gamma_{m,1} = \frac{1}{\sqrt{2}}(\gamma_{m,0}^o + \gamma_{m,\pi}^o) \\
\gamma_{m,2} &= \frac{1}{\sqrt{2}}(\gamma_{m,0}^e - \gamma_{m,\pi}^e), \quad \gamma_{m,3} = \frac{1}{\sqrt{2}}(\gamma_{m,0}^o - \gamma_{m,\pi}^o).
\end{align*}
\] (7.2)

The vertical hopping term contains
\[
\sum_{n=0}^{3} \gamma_{m,n} \gamma_{m,n+1} = \frac{1}{2} [(\gamma_{m,0}^e + \gamma_{m,\pi}^e)(\gamma_{m,0}^o + \gamma_{m,\pi}^o) + (\gamma_{m,0}^o + \gamma_{m,\pi}^o)(\gamma_{m,0}^e - \gamma_{m,\pi}^e) + (\gamma_{m,0}^e - \gamma_{m,\pi}^e)(\gamma_{m,0}^o - \gamma_{m,\pi}^o)] = 2\gamma_{m,\pi}^e \gamma_{m,\pi}^o.
\] (7.3)

This term produces a gap in the \( \pi \)-channel only. The horizontal hopping term contains:
\[
\begin{align*}
\gamma_{m,0}^e \gamma_{m,1}^e + \gamma_{m,1}^e \gamma_{m,2}^e + \gamma_{m,2}^e \gamma_{m,3}^e - \gamma_{m,3}^e \gamma_{m,4}^e \\
= \frac{1}{2} [(\gamma_{m,0}^e + \gamma_{m,\pi}^e)(\gamma_{m,0}^e + \gamma_{m,\pi}^e) - (\gamma_{m,0}^o + \gamma_{m,\pi}^o)(\gamma_{m,0}^o + \gamma_{m,\pi}^o)] \\
+ (\gamma_{m,0}^e - \gamma_{m,\pi}^e)(\gamma_{m,0}^e - \gamma_{m,\pi}^e) - (\gamma_{m,0}^o - \gamma_{m,\pi}^o)(\gamma_{m,0}^o - \gamma_{m,\pi}^o)]
= \gamma_{m,0}^e \gamma_{m,1}^e + \gamma_{m,1}^e \gamma_{m,2}^e + \gamma_{m,2}^e \gamma_{m,3}^e - \gamma_{m,3}^e \gamma_{m,4}^e.
\end{align*}
\] (7.4)
Thus the total hopping term is:

$$H_0 = it \sum_m [\gamma_{m,0}^{e} \gamma_{m+1,0}^{e} + \gamma_{m,\pi}^{e} \gamma_{m+1,\pi}^{e} - \gamma_{m,0}^{o} \gamma_{m+1,0}^{o} - \gamma_{m,\pi}^{o} \gamma_{m+1,\pi}^{o} + 2 \gamma_{m,\pi}^{e} \gamma_{m,0}^{o}].$$

(7.5)

Defining Dirac modes:

$$c_{m,0} = \frac{\gamma_{m,0}^{e} + i \gamma_{m,0}^{o}}{2}, \quad c_{m,\pi} = \frac{\gamma_{m,\pi}^{e} + i \gamma_{m,\pi}^{o}}{2}$$

(7.6)

and using

$$\gamma_{m}^{e} \gamma_{m+1}^{e} - \gamma_{m}^{o} \gamma_{m+1}^{o} = (c_{m}^{+} + c_{m})(c_{m+1}^{+} + c_{m+1}) + (c_{m}^{+} - c_{m})(c_{m+1}^{+} - c_{m+1}) = 2(c_{m}c_{m+1} - c_{m+1} c_{m}^{+}),$$

(7.7)

$$H_0 = 2t \sum_m [i(c_{m,0} c_{m+1,0} - c_{m+1,0}^{+} c_{m,0}^{+}) + i(c_{m,\pi} c_{m+1,\pi} - c_{m+1,\pi}^{+} c_{m,\pi}^{+}) + 2(2c_{m,\pi} c_{m,\pi} - 1)].$$

(7.8)

To get this into a more standard form, we make the transformation:

$$c_{2m,k} \rightarrow (-1)^m c_{2m,k},$$

$$c_{2m+1,k} \rightarrow -i(-1)^m e_{2m+1,k}$$

(7.9)

for both $k = 0$ and $\pi$ modes, giving

$$H_0 = 2t \sum_m [-\gamma_{m,0}^{e} \gamma_{m+1,0}^{e} + \gamma_{m,\pi}^{e} \gamma_{m+1,\pi}^{e} + h.c. - i(-1)^m \gamma_{m+1,\pi}^{e}].$$

(7.10)

We see that the $k = 0$ mode is in a gapless XY phase while the $k = \pi$ mode is gapped due to a staggered field in the spin representation.

Now let’s include the interaction terms:

$$H_{int} = g \sum_m [\gamma_{m,0}^{e} \gamma_{m+1,0}^{e} \gamma_{m+1,1}^{e} \gamma_{m,1}^{e} + \gamma_{m,1}^{e} \gamma_{m+1,1}^{e} \gamma_{m+1,2}^{e} \gamma_{m,2}^{e} + \gamma_{m,2}^{e} \gamma_{m+1,2}^{e} + \gamma_{m,3}^{e} \gamma_{m+1,3}^{e} + \gamma_{m,3}^{e} \gamma_{m+1,3}^{e} \gamma_{m,0}^{e} \gamma_{m,0}^{e}].$$

(7.11)

Next, we transform the $c$ operators as in Eq. (7.9), giving

$$H_{int} = g \sum_m [(2c_{m,0}^{+} c_{m,0}^{+} - 1)(2c_{m,0}^{+} c_{m,0}^{+} - 1) + (2c_{m,\pi}^{+} c_{m,\pi}^{+} - 1)(2c_{m,\pi}^{+} c_{m,\pi}^{+} - 1)$$

$$- (c_{m,0}^{+} + c_{m,0}^{+})(c_{m+1,0}^{+} - c_{m+1,0}^{+})(c_{m+1,1}^{+} - c_{m+1,1}^{+})(c_{m+1,2}^{+} - c_{m+1,2}^{+})$$

$$- (c_{m,\pi}^{+} + c_{m,\pi}^{+})(c_{m+1,\pi}^{+} - c_{m+1,\pi}^{+})(c_{m+1,1}^{+} - c_{m+1,1}^{+})(c_{m+1,2}^{+} - c_{m+1,2}^{+})].$$

(7.12)

Ignoring inter-leg coupling this is just 2 copies of the spinless fermion model with nearest neighbour interactions and a staggered potential for the $\pi$ sector. The $0$ sector remains gapless for $|g| < t$. 


Next we make a Jordan-Wigner transformation:

$$c_{m,0} = \left( \prod_{m' < m} \sigma_{m',0}^x \sigma_{m',\pi}^z \right) \sigma_{m,0}^-$$

$$c_{m,\pi} = \left( \prod_{m' < m} \sigma_{m',0}^x \sigma_{m',\pi}^z \right) \sigma_{m,0}^z$$

$$2c_{m,k}^\dagger c_{m,k} - 1 = \sigma_{m,k}^z$$  \hspace{1cm} (7.13)

for \(k = 0\) and \(\pi\). The hopping terms become:

$$H_0 = t \sum_m \left[ \sigma_{m,0}^+ \sigma_{m,\pi}^x \sigma_{m+1,0}^- + \sigma_{m,0}^- \sigma_{m,\pi}^x \sigma_{m+1,0}^+ + \sigma_{m,\pi}^+ \sigma_{m+1,0}^x \sigma_{m+1,\pi}^- + \sigma_{m,\pi}^- \sigma_{m+1,0}^x \sigma_{m+1,\pi}^+ + 2(-1)^m \sigma_{m,\pi}^z \right].$$  \hspace{1cm} (7.14)

To transform the interaction term note that

$$(c_{m,0}^\dagger + c_{m,\pi})(c_{m,\pi}^\dagger - c_{m,\pi})(c_{m+1,0}^\dagger + c_{m+1,\pi})(c_{m+1,\pi}^\dagger - c_{m+1,\pi})$$

$$\rightarrow (\sigma_{m,0}^- + \sigma_{m,\pi}^-) \sigma_{m,0}^+ (\sigma_{m,\pi}^+ - \sigma_{m,\pi}^-) (\sigma_{m+1,0}^- + \sigma_{m+1,\pi}^-) (\sigma_{m+1,\pi}^+ - \sigma_{m+1,\pi}^-)$$

$$= (-\sigma_{m,0}^- + \sigma_{m,\pi}^-) (\sigma_{m,\pi}^+ - \sigma_{m,\pi}^-) (\sigma_{m+1,0}^- - \sigma_{m+1,\pi}^-) (\sigma_{m+1,\pi}^+ - \sigma_{m+1,\pi}^-)$$

$$= \sigma_{m,0}^z \sigma_{m,\pi}^y \sigma_{m+1,0}^y \sigma_{m+1,\pi}^y.$$  \hspace{1cm} (7.15)

$$H_{int} = g \sum_m \left[ \sigma_{m,0}^z \sigma_{m,\pi}^z \sigma_{m+1,0}^- + \sigma_{m,\pi}^z \sigma_{m+1,\pi}^- - \sigma_{m,0}^y \sigma_{m,\pi}^y \sigma_{m+1,0}^y \sigma_{m+1,\pi}^y - \sigma_{m,0}^x \sigma_{m,\pi}^x \sigma_{m+1,\pi}^x \sigma_{m+1,\pi}^y \right].$$  \hspace{1cm} (7.16)

We effectively get a 2-leg spin ladder with the legs corresponding to \(k = 0\) and \(k = \pi\) modes. We have an unusual inter-leg 4-spin coupling which breaks the \(U(1)\) symmetry of the decoupled legs. Ignoring \(H_0\), we can find the exact ground state of \(H_{int}\) as we saw earlier in the Majorana basis. Here we confirm that we can also solve it exactly in the spin basis. The rung fermion parity is

$$i\gamma_1 \gamma_2 \gamma_3 = i\gamma_0^x \gamma_0^y + i\gamma_0^x \gamma_0^y \rightarrow \sigma_{m,0}^0 + \sigma_{m,\pi}^z.$$  \hspace{1cm} (7.17)

Eq. (7.16) preserves rung parity since \(\sigma_{m,0}^x \sigma_{m,\pi}^y \sigma_{m,\pi}^y \sigma_{m,\pi}^x\) changes the the value of

$$\sigma_{m,0}^x \equiv \sigma_{m,\pi}^x + \sigma_{m,\pi}^x.$$  \hspace{1cm} (7.18)

by either 0 or ±4. \(\sigma_{m}^z\) takes the values ±2 in one sector and 0 in the other. In the ±2 sector we may replace:

$$\sigma_{m,0}^0 \sigma_{m,\pi}^z \rightarrow (\sigma_{m,\pi}^0 + h.c.), \quad \sigma_{m,0}^y \sigma_{m,\pi}^y \rightarrow -(\sigma_{m,\pi}^0 + h.c.).$$  \hspace{1cm} (7.19)

And in the 0 sector we may replace:

$$\sigma_{m,\pi}^x \sigma_{m,\pi}^x \rightarrow (\sigma_{m,\pi}^0 + h.c.), \quad \sigma_{m,\pi}^y \sigma_{m,\pi}^y \rightarrow (\sigma_{m,\pi}^0 + h.c.).$$  \hspace{1cm} (7.20)

If two neighboring sites are in opposite sectors the corresponding term in the Hamiltonian vanishes. For the

$$\sigma_{m,0}^z \sigma_{m+1,0}^z + \sigma_{m,\pi}^z \sigma_{m+1,\pi}^z$$  \hspace{1cm} (7.21)

terms this follows because, for example, if \(m\) is in the 0 sector and \(m + 1\) is in the ±2 sector then \(\sigma_{m,0}^z = -\sigma_{m,\pi}^z\) and \(\sigma_{m+1,0}^z = \sigma_{m+1,\pi}^z\). For the XYYY* terms this follows because

$$\sigma_{m,0}^x \sigma_{m,\pi}^x \sigma_{m+1,0}^x \sigma_{m+1,\pi}^x + \sigma_{m,0}^y \sigma_{m,\pi}^y \sigma_{m+1,0}^y \sigma_{m+1,\pi}^y$$

$$\rightarrow (\sigma_{m,0}^+ \sigma_{m,\pi}^- + h.c.) (\sigma_{m+1,0}^0 \sigma_{m+1,\pi}^0 + h.c.) - (\sigma_{m,0}^+ \sigma_{m,\pi}^0 + h.c.) (\sigma_{m+1,0}^0 \sigma_{m+1,\pi}^0 + h.c.) = 0.$$  \hspace{1cm} (7.22)

Thus we may assume that each rung is in the same fermion parity sector. There are then only 2 states on each rung. In the 0 sector we may label them:

$$|\uparrow\rangle \equiv |\uparrow,\downarrow\rangle, \quad |\downarrow\rangle \equiv |\downarrow,\uparrow\rangle,$$  \hspace{1cm} (7.23)

and in the ±2 sector:

$$|\uparrow\rangle \equiv |\uparrow,\uparrow\rangle, \quad |\downarrow\rangle \equiv |\downarrow,\downarrow\rangle.$$  \hspace{1cm} (7.24)
Here the first arrow refers to the 0 leg and second arrow to the π leg. In the ±2 sector, for example, we may replace

\[ \sigma_{m,0}^y \sigma_{m,\pi}^y \rightarrow -\sigma_{m,0}^x \sigma_{m,\pi}^x + h.c. \rightarrow -\sigma_m^+ - \sigma_m^- = -\sigma_m^z \]  

(7.25)

Similarly, we may replace

\[ \sigma_{m,0}^z \rightarrow \sigma_m \]  

(7.26)

Thus:

\[ H_{int} \rightarrow -2g \sum_m [\sigma_m^x \sigma_{m+1}^x + \sigma_{m-1}^x \sigma_{m}^x] \]  

(7.27)

Now consider adding a small hopping term. The horizontal hopping term changes the fermion parity on a pair of channels so it can be ignored in lowest order perturbation theory. The vertical hopping term becomes:

\[ H_0 \approx \pm t \sum_m (-1)^m \sigma_m^z \]  

(7.28)

where the + factor occurs in the ±2 sector and the − factor in the 0 sector. Thus we have reproduced the results of Sec. IV in the spin basis.

**VIII. KOSTERLITZ-THOULESS TRANSITION**

Here we argue that the transition at \( g = g_{KT} \approx 0.8 \) is of Kosterlitz-Thouless type. We first consider the sum of the non-interacting Hamiltonian in Eq. (7.14) and the interacting Hamiltonian in (7.16), taking into account only the intra-channel interactions. Then, at least for sufficiently small \( g/t \), we see that the \( k = 0 \) sector is in a gapless Luttinger liquid phase corresponding to the XXZ model while the \( k = \pi \) sector is gapped. Now let’s include the inter-channel interactions and integrate out the gapped \( k = \pi \) sector. At first order in \( g \) we can just replace \( \sigma_m^x \sigma_{m+1}^x \) and \( \sigma_m^y \sigma_{m+1}^y \) by their expectation values. This simply renormalizes the XY couplings \( \propto t \) in the \( k = 0 \) sector and does not break the \( U(1) \) symmetry. At next order in \( g \) we generate interactions of the form:

\[ \delta H = \sum_{m,m'} [g_{m-m'}(\sigma_{m,0}^x \sigma_{m+1,0}^x + \sigma_{m,0}^y \sigma_{m+1,0}^y + \sigma_{m,0}^x \sigma_{m+1,0}^x + \sigma_{m,0}^y \sigma_{m+1,0}^y) + \lambda_{m-m'} \sigma_{m,0}^x \sigma_{m+1,0}^x \sigma_{m+1,0}^y h.c.] \]  

(8.1)

where the \( g \) and \( \lambda \) couplings drop off exponentially with distance. These do break the \( U(1) \) symmetry. The \( U(1) \) breaking part is

\[ \delta H' = \sum_{m,m'} (2g_{m-m'} - \lambda_{m-m'})(\sigma_{m,0}^+ \sigma_{m+1,0}^+ \sigma_{m+1,0}^+ + h.c.) \]  

(8.2)

To study its effect we can bosonize the spin model (or the fermion model). The most relevant term comes from the staggered part of the spin operators:

\[ \sigma_{m,0}^+ \propto (-1)^m e^{i\pi/\sqrt{K}}. \]  

(8.3)

Here \( K = 1 \) for the XY model. \( K \) decreases to \( 1/2 \) at the antiferromagnetic Heisenberg model and increases to \( \infty \) at the ferromagnetic Heisenberg model where a Lifshitz transition occurs. Thus we get a bosonized symmetry breaking interaction:

\[ \delta H \propto \int dx \cos[4\sqrt{\pi/\sqrt{K}} \theta(x)] \]  

(8.4)

of scaling dimension \( d = 4/K \). This follows because in the fermion model this corresponds to \( \psi_L \partial_x \psi_L \psi_R \partial_x \psi_R + h.c. \) of dimension 4 at the free fermion point where \( K = 1 \). For antiferromagnetic interactions, corresponding to \( g > 0 \) this just becomes more irrelevant. But for ferromagnetic interactions, corresponding to \( g < 0 \) it becomes more relevant. For \( g > 0 \) we might expect the usual Kosterlitz-Thouless transition driven by the Umklapp term \( \psi_L^\dagger \partial_x \psi_L^\dagger \psi_R \partial_x \psi_R + h.c. \propto \cos(4\sqrt{\pi K} \theta) \) which becomes relevant at the Heisenberg point, \( K = 1/2 \).
The left-moving single fermion operator bosonizes as
\[ \psi_L \propto e^{i\sqrt{4\pi} \phi_L} \] (8.5)
at \( g = 0 \) where \( K = 1 \). Writing
\[ \phi = \phi_L + \phi_R, \quad \theta = \phi_L - \phi_R, \] (8.6)
this becomes
\[ \psi_L \propto e^{i\sqrt{\pi}(\phi + \theta)}. \] (8.7)
For general \( K \) this becomes:
\[ \psi_L \propto e^{i\sqrt{\pi}(\phi + \theta)}/2 \] (8.8)
of dimension
\[ d = (1/4)(K + 1/K). \] (8.9)
This gives \( d = 1/2 \) for free fermions, \( K = 1 \) and \( d = 5/8 \) at the KT point, \( K = 1/2 \). The equal time Green’s function for the fermion decays with exponent 2d: 1 for free fermions and 5/4 at the KT point. The central charge is \( c = 1 \) along the entire critical line including at the KT point.
For \( g > g_{KT} \) we expect antiferromagnetic order in the \( z \) direction. The order parameter is
\[ (-1)^m \langle \sigma_{m,0}^z \rangle = (-1)^m(2c_{m,0}^\dagger c_{m,0} - 1) \rightarrow (2c_{m,0}^\dagger c_{m,0} - 1) = i\gamma_{m,0}^c \gamma_{m,0}^o \]
\[ = i/2\{\langle \gamma_{m,0} \gamma_{m,1} \rangle - \langle \gamma_{m,1} \gamma_{m,2} \rangle + \langle \gamma_{m,2} \gamma_{m,3} \rangle - \langle \gamma_{m,3} \gamma_{m,4} \rangle \}. \] (8.10)
This corresponds to the order of Fig. (2). Conversely
\[ (-1)^m \langle \sigma_{m,\pi}^z \rangle = i/2 \sum_{n=0}^{3} \langle \gamma_{m,n} \gamma_{m,n+1} \rangle \] (8.11)
as we see from Eq. (7.3). This does not break any symmetries.

IX. NUMERICAL PHASE DIAGRAM OF THE FOUR-LEG LADDER

As discussed above, we expect gapped phases with certain symmetry breaking patterns at the two strongly interacting regimes \( g/t \to \pm \infty \). The noninteracting point is a critical phase with central charge \( c = 1 \), described by a noninteracting Luttinger liquid. Motivated by the results of the single chain, we expect transitions between possibly several critical phases before reaching the broken-symmetry gapped phases. On the positive \( g \) side, however, our theoretical predictions support a single Kosterlitz-Thouless transition from the Luttinger liquid phase. In the chain, we had one (supersymmetric) transition on the positive \( g \) and two transitions on the negative \( g \)-side. Our numerical studies suggest a similar phase diagram with one transition on the positive \( g \) and three transitions on the negative \( g \) side. The first transition is a Lifshitz transition to a critical phase with central charge \( c = 3/2 \). It appears that there is a second transition to another critical phase with central charge \( c = 2 \) upon increasing the negative interaction strength, then a transition to a critical phase with \( c = 1 \), and finally a transition to the gapped phase at very strong negative interactions. Our estimate of these phase transition are shown in Fig. 4.

We combined several numerical diagnostics in determining the phase diagram: a direct calculation of the entanglement entropy, which provides the central charges of the critical phases, calculation of the two-point functions of Majorana operators, which decay as a power-law in critical phases, and direct extrapolation of the spectral gaps.

A. Entanglement entropy and the central charge

Our primary tool in determining the phase diagram is based on a calculation of the the entanglement entropy between two parts of the ladder using DMRG. We can directly compute the central charge of the system if it is
Figure 4: The schematic phase diagram of the four-leg ladder with periodic boundary conditions in the $y$ direction. The interaction strength $|g_3|$ is expected to be very large, but we have not been able to approximate it due to the large correlation length of the gapped phase.

![Phase Diagram](image)

Figure 5: Top: An example of the entanglement entropy for a system of length $L = 140$ and $g = -10$. Bottom: Averaging the entanglement entropy for two nearest integers $\ell$ and $\ell + 1$ (assigning the average to a middle point $x = \ell - 1/2$), effectively eliminates the oscillatory subleading terms, allowing us to extract the central charge $c$ from the slope of the resulting linear curve.

![Entanglement Entropy](image)

in a gapless phase. Up to subleading terms, which happen to be oscillatory in the present model, we expect the entanglement entropy between a subsystem of length $\ell$ and the remaining subsystem of length $L - \ell$ for a (quasi)-one-dimensional system of length $L$ to be given by

$$S(\ell) = \frac{c}{6} \ln \left[ \sin \left( \frac{\pi \ell}{L} \right) \right],$$

with open boundary conditions in the $x$ direction. The ground-state entanglement entropy can be measured directly with DRMG and does conform to the predicted form above in several regions of the phase diagram upon canceling out the subleading oscillatory terms using nearest-neighbor averaging, as shown, e.g., in Fig. 5 for a system of length $L = 140$ and $g = -10$.

Using this approach, we can determine the central charge as a function of the interaction strength. We also analyzed the goodness of the fits to the CFT results. A bad fit indicates that we likely have a gapped phase and the $c$ value obtained from fitting $S(\ell)$ should not be trusted. This occurs in the vicinity of several of the phase transitions, e.g., the Lifshitz transition, as well as the gapped phase at large $g$. Our results for the central charge are shown in Fig. 9.

The full phase diagram can be inferred from these plots of the numerically extracted central charge. We know that $g \to \infty$ is a gapped phase and $g = 0$ is a gapless Luttinger-liquid phase with $c = 1$. Both of these predictions are confirmed by the numerics. A plateau with $c = 1$ is clearly observed around $g = 0$. Upon increasing $g$, the central charge seems to dip at around $g \approx 0.7$, increase to around $c \approx 1.25$ and then drop to zero. This may indicate a multicritical point at $g \approx 2.0$ with central charge $c = 1.25$. However, other diagnostics do not support this picture. It appears that the bumps in the measured central charge is within the gapped phase. We will see this explicitly by calculating the expectation values of the fermionic two-point functions and the extrapolation of the energy gaps. We conclude that for positive $g$ a single transition occurs at around $g \approx 0.8$. On theoretical grounds, we expect this transition to belong to the KT universality class, and the calculations of the Green’s function support this prediction.

On the negative $g$ side, there is strong evidence for a phase transition at around $g \approx -1$. We have strong evidence from the extrapolation of the spectral gap that this transition occurs and is a Lifshitz transition. Stronger interaction
strengths give rise to other phases and transitions. Two robust plateaus with central charge $c = 2$ and $c = 1$ are clearly visible. The values $g_1$ and $g_2$ of $g$, for which these plateaus begin, drift with system size.

To determine and estimated phase diagram, we used a linear extrapolation to estimate the values of $g_1$ and $g_2$ in the thermodynamic limit as shown in Fig. 7. Our extrapolation of these values strongly suggests that there is a finite phase between the Lifshitz transition and the $c = 2$ phase. Although the behavior of the numerically estimated central charge is rather chaotic in this region, it appears that this intermediate phase may have $c = 3/2$. In fact at a Lifshitz transition, we expect a species of low-energy fermions to appear and the smallest change can be the appearance of one low-energy Majorana. We leave an in-depth study of the nature of these phases for negative $g$ to future publications.

Theoretically, we also know that $g \to \infty$ is a gapped phase. In our numerical studies, we were not able to see direct evidence of the gap, which suggests a small gap over a small range of $|t/g|$.

### B. Majorana Green’s function and the KT transition

DMRG allows us to also compute the ground-state expectation values of various operators. In this section, we focus on

\[ G(x) = i\langle \gamma_{m,n}\gamma_{m,n+x} \rangle. \]  

As we have periodic boundary condition in the $y$ direction, the Green’s function is independent of $m$. With periodic and antiperiodic boundary conditions in the $x$ direction, it is also independent of $n$. We have found that antiperiodic boundary conditions suppress the subleading terms and allow us to extract the universal behavior of $G(x)$ from the numerics more conveniently. If the system is in a Luttinger-liquid phase with Luttinger parameter $K$, we expect

\[ G(x) \sim x^{(K+1)/2}. \]  

In a finite system, we can replace $x$ with $\frac{L}{2} \sin \pi x/L$. In Fig. 8 we show the behavior of $G(x)$ on the positive $g$ side. For small $g$, we observe the expected power-law decay, with an increasing exponent as we increase $g$. The behavior
transitions to a faster decay for larger $g$ around $g = 0.8$ with $K \approx 1/2$. This is consistent with a KT transition, as analyzed in Sec. VIII. The bump in the central charge around $g = 2$ seems to be an artifact as there is evidence from the Green’s function that we have already entered a gapped phase at this value of $g$.

A curious feature of the central charge data is that for $0.8 < g < 2$, the entanglement entropy looks similar to that of a critical phase and a robust peak is observed at around $g = 2$. The behavior of the correlation functions are, however, suggestive of a gapped phase. It is possible that a large correlation length in finite systems gives the illusion of criticality in the behavior of the entanglement entropy. To investigate this issue further, we directly calculated the energy gap $\Delta E$ (with antiperiodic boundary conditions and in the even fermion parity sector).

The results indicate that the gap $g = 0.8$ likely extrapolates to a finite value in the thermodynamic limit (we used a fit to a second-order polynomial of $1/L$). For a slightly larger $g = 1$, there is strong evidence of a finite energy gap. This suggests that the critical behavior of the entanglement entropy is most likely an artifact.

C. Lifshitz transition and the Luttinger-liquid velocity

We now focus on the negative $g$ side. The first transition out of the Luttinger liquid phase is easier to understand. We claim this transition is a Lifshitz transition analogous to the Majorana chain. A direct evidence is provided by extracting the velocity in the Luttinger-liquid phase and observing that it extrapolated to zero, signalling the emergence of a dynamical exponent $z > 1$ at the transition. In the Luttinger-liquid phase the energy gaps scale as

Figure 8: The numerically computed Green’s function for positive $g$ for a system of length $L = 80$. For $g = 0$ and small values of $g$, the expected power law is observed. When the exponent reaches $\frac{1}{2}(K + 1/K) \approx 5/4$ at around $g \approx 0.8$, which corresponds $K = 1/2$, the behavior of the correlation function shift to a decay faster than power law.

Figure 9: The energy gap $\Delta E$ seems to extrapolate to a finite value for $g = 0.8$. For a slightly larger $g = 1$ there is clear evidence of an energy gap.
Figure 10: The energy gaps in the even fermion parity sector as a function of $1/L$ for $g$ in the LL phase.

Figure 11: The behavior of the velocity as a function of $g$ signaling a Lifshitz transition at around $g = -0.98$, where the velocity vanishes, in agreement with the central charge results.

1/ $L$. In particular we define the velocity as the coefficient $v$ in

$$\Delta E = E_{1}^{\text{even}} - E_{0}^{\text{even}} = 2\pi xv/L,$$

where $\Delta E$ is defined as the gap from the ground state to the first excited state in the even fermion parity sector, and $x$ is a universal constant of order unity (the scaling dimension of the operator corresponding to this energy level). As shown, in Fig. [10] this behavior is confirmed in the numerics, allowing us to extract the coefficient $\tilde{v} = 2\pi xv$ from a linear fit.

Extracting $\tilde{v} \propto v$ and plotting it as a function of $g$ allows us to identify the location of the Lifshitz transition, see Fig. [11]. We note that the simple linear dependence disappears as we move past the transition, and an intricate dependence on system size appears similar to the $c = 3/2$ phase of the chain.

X. CONCLUSIONS

We have studied the 2-leg and 4-leg Majorana-Hubbard model. The behavior is largely consistent with our previous mean field predictions for the 2D case. For pbc a massless phase occurs at sufficiently weak coupling of either sign. As $|g|$ increases, transitions occur to broken symmetry phases with neighboring Majorana fermions combining to form Dirac fermion levels which tend to be empty or filled. While in the 2D case these can occur on vertical or horizontal bonds, for ladders they only occur on vertical bonds, reducing the number of ground states. We also found, in the 4-leg case at large negative $g/t$, that the dimer order has a larger unit cell than predicted by our naive mean field theory, with the Dirac levels alternating filled and empty in both horizontal and vertical directions.
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[1] A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001).
[2] L. Fu and C. L. Kane, Physical review letters 100, 096407 (2008).
[3] R. M. Lutchyn, J. D. Sau, and S. D. Sarma, Physical review letters 105, 077001 (2010).
[4] Y. Oreg, G. Refael, and F. von Oppen, Physical review letters 105, 177002 (2010).
[5] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science 318, 766 (2007).
[6] H. Zhang, C.-X. Liu, X.-L. Qi, X. Dai, Z. Fang, and S.-C. Zhang, Nature physics 5, 438 (2009).
[7] Y. Xia, D. Qian, D. Hsieh, L. Wray, A. Pal, H. Lin, A. Bansil, D. Grauer, Y. Hor, R. Cava, et al., Nature Physics 5, 398 (2009).
[8] V. Mourik, K. Zuo, S. M. Frolov, S. Plissard, E. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
[9] R. Lutchyn, E. Bakkers, L. Kouwenhoven, P. Krogstrup, C. Marcus, and Y. Oreg, Nature Reviews Materials p. 1 (2018).
[10] J. Zhou, Y.-J. Wu, R.-W. Li, J. He, and S.-P. Kou, EPL (Europhysics Letters) 102, 47005 (2013).
[11] R. R. Biswas, Physical review letters 111, 136401 (2013).
[12] C.-K. Chiu, D. Pikulin, and M. Franz, Physical Review B 91, 165402 (2015).
[13] T. Liu and M. Franz, Phys. Rev. B 92, 134519 (2015).
[14] D. Pikulin, C.-K. Chiu, X. Zhu, and M. Franz, Physical Review B 92, 075438 (2015).
[15] M. Hermanns and S. Trebst, Phys. Rev. B 89, 235102 (2014).
[16] A. Rahmani, X. Zhu, M. Franz, and I. Affleck, Physical review letters 115, 166401 (2015).
[17] A. Rahmani, X. Zhu, M. Franz, and I. Affleck, Phys. Rev. B 92, 235123 (2015).
[18] A. Milsted, L. Seabra, I. Fulga, C. Beenakker, and E. Cobanera, Physical Review B 92, 085139 (2015).
[19] W. Witczak-Krempa and J. Maciejko, Physical review letters 116, 100402 (2016).
[20] B. Ware, J. H. Son, M. Cheng, R. V. Mishmash, J. Alicea, and B. Bauer, Physical Review B 94, 115127 (2016).
[21] S. Gangadharaiah, B. Braunecker, P. Simon, and D. Loss, Physical review letters 107, 036801 (2011).
[22] A. Lobos, R. Lutchyn, and S. Das Sarma, Physical review letters 109, 146403 (2012).
[23] C. Li and M. Franz, Phys. Rev. B 98, 115123 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.98.115123.
[24] N. Sannomiya and H. Katsura, arXiv:1712.01148 (2017).
[25] E. O’Brien and P. Fendley, Phys. Rev. Lett. 120, 206403 (2018).
[26] I. Affleck, A. Rahmani, and D. Pikulin, Phys. Rev. B 96, 125121 (2017).
[27] Y. Kamiya, A. Furusaki, Y. Teo, and G.-W. Chern, arxiv 1711.03632 (2017).
[28] A. Stern, Ann. Phys. 323, 204 (2008).
[29] F. C. Alcaraz and A. L. Malvezzi, Journal of Physics A: Mathematical and General 28, 1521 (1995).
[30] K. Okamoto and K. Nomura, Journal of Physics A: Mathematical and General 29, 2279 (1996).