Spacetime characterizations of $\Lambda$-vacuum metrics with a null Killing 2-form

Marc Mars$^{1,3}$ and José M M Senovilla$^2$

$^1$Instituto de Física Fundamental y Matemáticas, Universidad de Salamanca Plaza de la Merced s/n, E-37008 Salamanca, Spain
$^2$Física Teórica, Universidad del País Vasco, Apartado 644, E-48080 Bilbao, Spain

E-mail: marc@usal.es and josemm.senovilla@ehu.es

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Abstract

An exhaustive list of four-dimensional $\Lambda$-vacuum spacetimes admitting a Killing vector $\xi$ whose self-dual Killing two-form $\mathcal{F}_{\mu\nu}$ is null is obtained assuming that the self-dual Weyl tensor is proportional to the tensor product of $\mathcal{F}_{\mu\nu}$ by itself. Our analysis complements previous results (Mars 1999 Class Quantum Grav. 16 2507–23; Mars and Senovilla 2015 Ann. H. Poincaré 16 1509–50) concerning the case with non-null $\mathcal{F}_{\mu\nu}$. We analyze both cases with $\Lambda$ vanishing or not. In the latter case we prove that $\Lambda < 0$ must hold necessarily, and we find a characterization of the Einstein spacetimes conformal to pp-waves. In the former case we obtain spacetime characterizations of vacuum plane waves and of the stationary vacuum Brinkmann spacetimes. At the light of the full set of results, old and new, we reformulate the case with non-null $\mathcal{F}_{\mu\nu}$ and $\Lambda = 0$. We finally present a table collecting the results for both null, and non-null, $\mathcal{F}_{\mu\nu}$.

Keywords: characterization, $\Lambda$-vacuum, null two-form, alignment, Killing vector

1. Introduction

Finding properties that locally characterize spacetimes is a relevant problem both in geometry and in physics as it provides us, among other things, with tools to approach uniqueness results under suitable global assumptions and with methods to describe quantitative deviations from a given spacetime. Characterizations involving the vanishing of a tensorial object are particularly relevant in this respect. The classic and certainly most important example is the vanishing of the Riemann tensor that characterizes locally flat metrics. In General Relativity,
the Kerr spacetime and its generalization with non-zero cosmological constant $\Lambda$ and/or NUT parameter (the so-called Kerr–NUT or Kerr–NUT–(A) de Sitter spacetimes) play a relevant role and substantial effort has been devoted to characterizing them locally. A number of results along these lines are known involving various properties, such as for instance separability of the Hamilton–Jacobi equation or, more geometrically, the existence of a closed conformal Killing–Yano tensor in $\Lambda$-vacuum spacetimes (in arbitrary dimensions) [15, 16, 19]. Characterizations within the class of $\Lambda$-vacuum spacetimes of Petrov type D with shear-free and geodesic principal null congruences have been obtained by Ferrando and Sáez [8].

In the case when the spacetime admits a local isometry with Killing generator $\xi$, the Kerr metric was found [20] to be uniquely characterized among stationary and asymptotically flat vacuum spacetimes by the property that the self-dual Weyl tensor and the self-dual Killing two-form satisfy a suitable alignment condition, (22) below. This condition was motivated by previous results of Simon [34] who characterized the Kerr metric in terms of objects defined in the Killing vector quotient space. The condition of asymptotic flatness was used in [20] only to fix the values of two constants and to ensure that the Killing vector was timelike on at least one point. This leads to a local characterization of the Kerr spacetime in terms of the vanishing of a suitable complex tensor (see theorem 1 in [21], which however needs amendment as the condition that $\xi$ is timelike at some point was overlooked; see below for a detailed discussion of this issue). Such characterization has been used successfully [1, 17] to prove a uniqueness theorem of rotating black holes under suitable smallness assumptions (or appropriate boundary conditions at the bifurcation surface) without assuming that the spacetime is analytic. It has also been used to characterize initial data for the Kerr metric in [10] and to define a quality factor measuring the deviation of a given metric admitting a Killing vector with respect to the Kerr metric [9].

This local characterization of Kerr (or of Kerr–NUT, when the conditions on the constants are relaxed [22]) has been extended [27] to the case of $\Lambda$-vacuum spacetimes. The full class of spacetimes satisfying the Einstein field equations with a cosmological constant $\Lambda$ (of any value, including zero) and admitting a Killing vector such that the alignment condition (22) holds was studied in detail by exploiting an interesting underlying conformal submersion that arises naturally. In particular the Kerr–NUT–(A) de Sitter spacetime was identified invariantly within this class. The results in [27] have been used recently [25, 26] to characterize the Kerr–NUT–de Sitter spacetimes (and other ($\Lambda > 0$)-vacuum spacetimes) in terms of their Killing data at past or future null infinity.

All these characterizations of spacetimes in terms of (22) made the assumption that the square $F^2 := F_{\alpha\beta}F^{\alpha\beta}$ of the self-dual Killing form $F_{\alpha\beta}$ (see below for definitions) is non-zero somewhere—inside the set ($Q \neq 0$). Then $F^2$ is automatically non-zero everywhere and the spacetime is of Petrov type D at every point. Given the relevance of the spacetimes that arise under this assumption, it is most natural to ask what happens in the complementary case when $F^2$ is identically zero. This is the problem we analyze in this paper. By doing so we complete the local characterization of all $\Lambda$-vacuum spacetimes admitting a Killing vector and satisfying the alignment condition (22). The spacetimes thus characterized turn out to be, in the case $F^2 = 0$, physically relevant too. This reinforces the idea that the geometric condition (22) is both physically and geometrically meaningful.

The vanishing of $F^2$ means that the self-dual Killing form is null, and this introduces a privileged null direction $k$ defined as the wave vector of $F_{\alpha\beta}$. Our analysis exploits the simultaneous existence of $\xi$ and $k$ in the spacetime and extracts relevant geometric information that plays a crucial role in determining the local form of the metric. Our main
result is summarized in theorem 1 of section 5. The classification splits into two cases depending on whether the cosmological constant vanishes or not. In the first case the spacetime is either locally isometric to the vacuum plane waves (when $\xi$ and $k$ are orthogonal) or to the vacuum stationary Brinkmann spacetimes (when they are not). Since the first class always admits at least a five-dimensional Killing algebra and the second one also admits more than one linearly independent Killing vector, it may (and it does) occur that the intersection of both cases is non-empty. We identify this intersection as the class of vacuum irreducible locally symmetric spacetimes, which depend on a single parameter. The case of non-zero cosmological constant requires $\Lambda < 0$ necessarily and leads uniquely to the so called Siklos spacetime.

Since this paper completes the classification of $\Lambda$-vacuum spacetimes admitting a Killing vector field $\xi$ such that (22) holds, we include in this paper a table that summarizes all these results. Before doing so we show that the case $\{\mathcal{F}^2 \neq 0\}$, which in [27] was split in three disjoint classes, can be rearranged for the case with $\Lambda = 0$ into two disjoint classes: (i) the so-called gen-Kerr–NUT, which corresponds to the Kerr spacetime with NUT charge together with its generalizations from spherical to plane or hyperbolic surfaces, and (ii) the so-called type D vacuum Kundt. The former class is written in a coordinate system that covers all cases at once, including the situation where the Killing $\xi$ is hypersurface orthogonal. This rearrangement in the case $\{\Lambda = 0, \mathcal{F}^2 = 0\}$ is given in theorem 2 and allows us to discuss in detail the omission in the statement of theorem 1 in [21] already mentioned above. Indeed, the type D vacuum Kundt class can be singled out by the property that the Killing vector $\xi$ is orthogonal everywhere to the two null eigendirections of $\mathcal{F}_{ab}$. Thus, when the Killing $\xi$ is timelike at one point the type D vacuum Kundt class gets automatically excluded. Given that [20] dealt with spacetimes where $\xi$ is timelike somewhere, this case did not show up in the analysis. The problem was that the quotation made in theorem 1 of [21] of the results in [20] did not include this assumption. We amend the statement in theorem 3 by including all necessary assumptions and also give an independent proof based on the splitting in two classes presented in theorem 2.

The paper is organized as follows. In section 2 we present our basic assumptions and find a number of identities that will play a role later. In section 3 we study the case with vanishing cosmological constant. After proving in proposition 1 that, with a suitable choice of scale, $k$ is a parallel vector field with constant scalar product with $\xi$, the spacetime is immediately shown to be a Brinkmann space and the theory of Kerr–Schild vector fields [7] allows us to determine the most general form of the Killing vector $\xi$, as well as the field equations that the remaining metric function must satisfy. At this point, it becomes necessary to split the analysis depending on whether the (constant) scalar product between $\xi$ and $k$ vanishes or not, thus leading to the spacetimes identified in theorem 1 when $\Lambda = 0$. In section 4 we show in proposition 2 that $\Lambda < 0$ necessarily and that the horizons of $\xi$ (which is null, hypersurface orthogonal and nowhere zero in this case) have cross sections of constant negative curvature. From here a local coordinate system is constructed in terms of three geometrically defined scalars and a choice of hypersurface transversal to the horizons. The metric in this coordinate system is simple enough so that the alignment condition (22) can be imposed and solved immediately. The paper closes with section 5 where our main result and the two theorems mentioned above are stated and proven. Finally we summarize the full local classification of spacetimes satisfying (22), for any value of $\Lambda$ and any value of $\mathcal{F}^2$, in table 1.
2. Preliminaries

Throughout this paper, a spacetime $(\mathcal{M}, g)$ is a smooth, orientable four-dimensional, connected manifold endowed with a metric $g$ of Lorentzian signature (at least $C^3$). We assume further that the spacetime is oriented and time oriented. The Levi-Civita covariant derivative of $g$ is denoted by $\nabla$ and the volume form by $\eta$. From now on a tensor field will be called smooth iff it is $C^3$.

Our basic assumptions are

(i) $(\mathcal{M}, g)$ admits a Killing vector field $\xi$ with non-identically vanishing self-dual Killing form $F_{\alpha\beta}$, defined by

$$F_{\alpha\beta} := F_{\alpha\beta} + i F^*_{\alpha\beta}, \quad F_{\alpha\beta} = \nabla_\alpha \xi_\beta,$$

where $*$ is the Hodge dual operator. $E_{\alpha\beta}$ is a two-form by the Killing equations and $F$ is self-dual, i.e. it satisfies $F_{\alpha\beta} = -i F^*_{\alpha\beta}$.

(ii) The two-form $F_{\alpha\beta}$ is singular, or null, that is to say, it satisfies

$$F^2 := F_{\alpha\beta} F^{\alpha\beta} = 0.$$  

Any self-dual two-form satisfies the algebraic identities (see e.g. [18])

$$F_{\mu\nu} F^{\nu\rho} = \frac{1}{4} g^{\mu\rho}, \quad F_{\mu\nu} F^{\mu\nu} = 0,$$

where overbar denotes complex conjugation. Hence, condition (ii) can be equivalently stated as

$$F_{\mu\rho} F^{\rho\nu} = 0,$$

so that the self-dual Killing two-form is nilpotent. This entails, in particular, that $F$ and $F^*$ are simple 2-forms.

2.1. Null eigenvector

Assuming that (2) holds at a given point $p \in \mathcal{M}$ where $F$ is non-zero it follows the existence of a unique real null 1-form $k|_p \in \Lambda^1_p$ such that

$$k \wedge F|_p = 0,$$

which is also equivalent to $k^\nu F_{\nu\rho}|_p = 0$. This implies that $F|_p$ takes the simple form $F|_p = k \wedge V|_p$ for some non-zero complex vector $V|_p = V|_p + i Y|_p$ satisfying $g(k|_p, V|_p) = 0$ and $g(V|_p, V|_p) = 0$ (i.e. $g(X|_p, Y|_p) = 0$, $g(X|_p, X|_p) = g(Y|_p, Y|_p)$ $> 0$). These vectors are defined uniquely up to scaling and shift transformations, given by $k|_p = ak|_p$, $V|_p = a^{-1}(V|_p + b k|_p)$, with $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{C}$. The null direction $(k|_p)$ is algebraically characterized as the unique real direction lying in the kernel of $F|_{\alpha\beta}$, and it is usually called the wave vector, or the null eigenvector, of $F$.

Assume now that $F$ is null and nowhere zero on an open neighborhood $\mathcal{U}$, then the decomposition

$$F_{\alpha\beta} = k_\alpha V_\beta - k_\beta V_\alpha$$

holds on $\mathcal{U}$ for smooth vector fields $k$ and $V$ [18]. We show this explicitly for completeness. From time orientability there exists a unit, future directed timelike vector field $u$ on $\mathcal{U}$. The scaling and shift freedom at $p \in \mathcal{U}$ can be uniquely fixed by requiring $g(u|_p, \bar{k}|_p) = -1$ and
$g(u_\mu, \bar{V}_\mu) = 0$. $\bar{V}$ is smooth on $\mathcal{U}$ because $\mathcal{F}_{\alpha\beta} = \bar{V}_\alpha$ and the left hand-side is smooth. $\bar{k}$ is also smooth on $\mathcal{U}$ because it is uniquely characterized by $g(\bar{k}, \bar{k}) = 0$, $g(\bar{k}, \bar{V}) = 0$, $g(\bar{k}, u) = -1$. Now, the most general pair of vector fields $\{k, V\}$ for which (3) holds is given by $k' = a^{-1} \bar{k}$ and $V = a \bar{V} - (b/a) \bar{k}$ where $a, b$ are smooth functions on $\mathcal{U}$ with $b$ complex and $a$ nowhere vanishing. By taking $a^{-1}$ as the positive square root of $g(\bar{V}, \bar{V})$ (which is positive everywhere) and defining $m = a \bar{V}$ we conclude that any nowhere zero self-dual, null two-form on an open set admits the following decomposition in terms of smooth $k'$ and $m$: $\mathcal{F}_{\alpha\beta} = k'_\alpha m_\beta - m_\alpha k'_\beta$, $g(k', k') = g(k', m) = g(m, m) = 0$, $g(m, m) = 1$. (4)

The complex null vector field $m$ is defined up to $m \rightarrow m + B k'$, with $B$ an arbitrary smooth complex function on $\mathcal{U}$.

2.2. Ernst one-form and ‘energy–momentum’ of $F$

The Ernst one-form is the complex one-form defined by

$$\chi_\alpha = 2 \xi^\alpha \mathcal{F}_{\alpha\beta}$$

and it is automatically orthogonal to $\xi$, i.e. $\xi^\beta \chi_\beta = 0$. In the null case it also satisfies the identity

$$\chi^\mu \mathcal{F}_{\mu\nu} = 0$$

as a direct consequence of (2). In fact $\chi$ can be explicitly written in terms of $k$ and $m$ as

$$\chi_\mu = 2(\xi^\mu k'_\nu) m_\nu - 2(\xi^\nu m_\nu) k'_\mu.$$ (7)

Let $N = - g(\xi, \xi)$ be minus the square norm of the Killing. The real part of the Ernst form is directly linked to $N$ by

$$\chi_\mu + \chi_\mu = 4 \xi^\mu F_{\mu\nu} = - 4 \xi^\mu \nabla_\mu \xi_\nu = 2 \nabla_\mu N$$

so that we can always write

$$\chi_\mu = \nabla_\mu N + i \omega_\mu$$ (8)

for some real one-form $\omega$ called the twist of $\xi$ and which vanishes if and only if $\xi_\alpha$ is a hypersurface orthogonal one-form. We will also use the symmetric tensor

$$t_{\mu\nu} = \frac{1}{2} \mathcal{F}_{\mu\rho} \mathcal{F}^{\rho\nu} = \frac{1}{2} k'_\mu k'_\nu$$

which is nothing but the ‘energy–momentum tensor’ of the 2-form $F_{\mu\nu}$. $\xi$ being a Killing vector, it follows that $\xi_\mu \mathcal{F}_{\alpha\beta} = 0$ and thus $\xi_\mu t_{\alpha\beta} = 0$, so that (9) implies

$$\xi_\mu k'_\mu = 0.$$ (10)

2.3. $\Lambda$-vacuum

Throughout this paper we will assume that the space–time satisfies Einstein’s field equations for vacuum with a (possibly vanishing) cosmological constant $\Lambda$, that is to say

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta},$$ (11)

where $R_{\alpha\beta}$ is the Ricci tensor (our sign conventions follow e.g. [36]). Such spacetimes will be called $\Lambda$-vacuum. In the specific case that $\Lambda = 0$ we will simply say vacuum or, also, Ricci-
flat. The Riemann tensor of \( \Lambda \)-vacuum spacetimes decomposes as
\[
R_{\alpha \beta \lambda \mu} = C_{\alpha \beta \lambda \mu} + \frac{\Lambda}{3} (g_{\alpha \lambda} g_{\beta \mu} - g_{\alpha \mu} g_{\beta \lambda}),
\]
where \( C_{\alpha \beta \lambda \mu} \) is the Weyl tensor. We shall denote the (right) self-dual of \( C_{\alpha \beta \lambda \mu} \) by
\[
C_{\alpha \beta \lambda \mu} = C_{\alpha \beta \lambda \mu} + i C^{*}_{\alpha \beta \lambda \mu}, \quad C^{*}_{\alpha \beta \lambda \mu} \coloneqq \frac{1}{2} \eta_{\alpha \rho \sigma \lambda} C_{\rho \sigma \beta \mu}.
\]
The algebraic properties of the Weyl tensor (i.e. vanishing trace and first Bianchi identity) imply that \( C_{\alpha \beta \lambda \mu} \) can be defined equivalently using the left self-dual of \( C_{\alpha \beta \lambda \mu} \) (with obvious notations) and that the following properties hold
\[
C^{\alpha}_{\beta \mu \lambda} = 0, \quad C_{\alpha [\beta \mu \lambda]} = 0,
\]
where square brackets denote antisymmetrization. The second Bianchi identities can be rewritten as
\[
\nabla^\alpha C_{\alpha \beta \lambda \mu} = 0
\]
and the standard identity for Killing vectors \( \nabla_\beta \nabla_\lambda \xi_\mu = \xi^\alpha R_{\alpha \beta \lambda \mu} \) becomes, in the language of complex self-dual objects
\[
\nabla_\mu F_{\alpha \beta} = \xi^\rho \left( C_{\rho \mu \alpha \beta} + \frac{4\Lambda}{3} F_{\rho \mu \alpha \beta} \right),
\]
where
\[
I_{\rho \mu \alpha \beta} = \frac{1}{4} (g_{\rho \mu} g_{\alpha \beta} - g_{\rho \alpha} g_{\mu \beta} + i \eta_{\rho \mu \alpha \beta})
\]
is the canonical metric in the space of complex self-dual 2-forms, i.e. symmetric by pairs \( I_{\rho \mu \alpha \beta} = I_{\mu \rho \alpha \beta} \) and satisfying \( I_{\rho \mu \alpha \beta} W^{\rho \alpha \beta} = W_{\rho \mu \alpha \beta} \) for any self-dual 2-form \( W \). Note that the trace of \( I \) is
\[
I^{\alpha}_{\rho \mu \alpha \beta} = \frac{3}{4} g_{\rho \beta}.
\]
Taking the covariant derivative in \( (5) \) one finds
\[
\nabla_\mu \chi_\beta = 2 (\nabla_\mu \xi^\alpha) F_{\alpha \beta} + 2 \xi^\alpha \nabla_\mu F_{\alpha \beta}
\]
\[
= (F^{\alpha}_{\mu} + \overline{F}_{\mu}^\alpha) F_{\alpha \beta} + 2 \xi^\alpha \xi^\nu \left( C_{\rho \mu \alpha \beta} + \frac{4\Lambda}{3} F_{\rho \mu \alpha \beta} \right)
\]
\[
= - k'_\mu k'_\beta + 2 \xi^\alpha \xi^\nu C_{\rho \mu \alpha \beta} - \frac{2\Lambda}{3} (N g_{\rho \beta} + \xi_\rho \xi_\beta),
\]
where in the second equality we used \( (15) \) and in the third equality we used the nilpotency \( (2) \) and expression \( (9) \). Hence, the Ernst one-form in \( \Lambda \)-vacuum spacetimes is closed (see e.g. [22])
\[
\nabla_\mu \chi_\beta = 0.
\]
Immediate consequences of \( (15) \), combined with \( (13) \) and \( (16) \), are
\[
\nabla_\mu F_{\alpha \beta} = i \frac{\Lambda}{3} \xi^\rho \eta_{\rho \mu \alpha \beta}, \quad \nabla^\alpha F_{\alpha \beta} = - \Lambda \xi_\beta.
\]
while taking the derivative of the null condition (1) and using (15) gives

\[ 2\xi^\nu \mathcal{F}^{\alpha\beta} + \frac{4\Lambda}{3} \chi^\mu = 0. \]  

(20)

As a preliminary result to be used later, we have

**Lemma 1.** If the self-dual Killing 2-form \( \mathcal{F} \) vanishes on a non-empty open neighborhood \( \mathcal{U} \subset \mathcal{M} \) of a \( \Lambda \)-vacuum spacetime, then \( \Lambda = 0 \) necessarily.

**Proof.** This follows immediately from (19).

### 2.4. Alignment of \( \mathcal{C} \) and \( \mathcal{F} \)

The self-dual \( \mathcal{C}_{\alpha\beta\mu} \) defines an eigenvalue problem acting on self-dual 2-forms. The content of this problem leads to the important Petrov classification of the Weyl tensor [35]. In the case under consideration, the space–time has a distinguished self-dual 2-form, the Killing 2-form \( \mathcal{F}_{\mu\nu} \). It seems natural to ask what are the implications of \( \mathcal{F}_{\mu\nu} \) being an eigen-2-form of \( \mathcal{C}_{\alpha\beta\mu} \)

\[ \mathcal{C}_{\alpha\beta\mu} \mathcal{F}^{\alpha\beta} \propto \mathcal{F}_{\mu\nu} \]

(21)

and this was the essential assumption in the several unique characterizations of the Kerr and other relative spacetimes given in [20–23, 27].

The previous relation does not restrict the Petrov type of the space–time in general. However, a particular simple form where that condition is achieved is by assuming (recall (1))

\[ \mathcal{C}_{\alpha\beta\mu} = Q \left( \mathcal{F}_{\alpha\beta} \mathcal{F}_{\mu\nu} - \frac{1}{3} \mathcal{F}^2 \mathcal{T}_{\alpha\beta\mu} \right) \]

(22)

\[ = Q \mathcal{F}_{\alpha\beta} \mathcal{F}_{\mu\nu} \]

(23)

which is actually the starting point in [20, 23, 25–27]. We will assume this condition herein, even though we will make occasional comments and derive some results for the more general case above. If (23) holds, then at \( p \in \mathcal{M} \)

\[ k^{\alpha} \mathcal{C}_{\alpha\beta\mu} = 0 \]

showing that the real null eigenvector of \( \mathcal{F}_{\mu\nu} \) is the unique multiple principal null direction of the Weyl tensor so that the Petrov type is N if \( Q \mid_p \neq 0 \), or type 0 if \( Q \mid_p = 0 \).

Under the assumption (23) equations (15) and (20) become, after using the definition of Ernst one-form (5)

\[ \nabla_{\mu} \mathcal{F}_{\alpha\beta} = \frac{1}{2} Q_{\mu} \mathcal{F}_{\alpha\beta} + \frac{4}{3} \Lambda \xi^\nu \mathcal{F}_{\nu\rho\alpha\beta}, \]

(24)

\[ \Lambda \chi^\mu = 0. \]

(25)

The vector field \( \xi \) being Killing, it follows that \( \xi_{\mu} \mathcal{F}_{\alpha\beta} = 0 \) and \( \xi_{\mu} \mathcal{C}_{\alpha\beta\mu} = 0 \) and thus from (23)

\[ \xi_{\mu} Q = \xi^\nu \nabla_{\nu} Q = 0. \]

(26)

We want to elaborate the second Bianchi identity (14) under the alignment condition. We start with the following identity
where in the last equality we have expanded \( \delta_{\rho\mu}^{\lambda\sigma} = 3! \delta_{\rho\mu}^{\lambda\sigma} \delta_{\rho\mu}^{\lambda\sigma} \) and have used the definition (5). We note that the identity (27) is valid for any self-dual two-form and any vector field \( \xi \) provided \( \chi \) is defined as in (5). The second ingredient is the following identity which only holds for null two-forms as it relies on the expression (4)

\[
\mathcal{F}_{\alpha\beta\gamma} = \mathcal{F}_{\alpha\beta}(\nabla_\gamma k^\alpha) m_\nu + k^\alpha \nabla_\gamma m_\nu - (\nabla_\gamma m^\alpha) k^\alpha_\nu - m^\alpha \nabla_\gamma k^\alpha_\nu \\
= (k^\alpha_\nu m_\beta - k^\beta_\nu m_\alpha)((\nabla_\gamma k^\alpha) m_\nu - (\nabla_\gamma m^\alpha) k^\alpha_\nu) \\
= (m_\alpha \xi^\alpha_\nu - k^\alpha_\nu m_\alpha) m^\alpha \nabla_\gamma k^\alpha_\nu \\
= - \mathcal{F}_{\beta\alpha\gamma} m^\alpha \nabla_\gamma k^\alpha_\nu.
\]

where in the second equality we used that \( k^\alpha \) and \( m^\alpha \) lie in the kernel of \( \mathcal{F} \) and in the second one we used the orthogonality properties of \( \xi^\alpha \) and \( \chi_\nu \).

The second Bianchi identity (14) becomes

\[
\nabla_\mu \mathcal{C}_{3 \mu \nu} = (\nabla_\mu \mathcal{Q} \mathcal{F}_{\alpha \beta} - \Lambda Q \mathcal{F}_{\alpha \beta})(m_\alpha \nabla_\gamma \mathcal{F}_{\beta} - m^\alpha \nabla_\gamma \mathcal{F}_{\beta} - \nabla_\mu \mathcal{F}_{\alpha \beta} + \nabla_\gamma \mathcal{F}_{\alpha \beta}) \\
= (\nabla_\mu \mathcal{Q} \mathcal{F}_{\alpha \beta} - 2 \Lambda Q \mathcal{F}_{\alpha \beta})(m_\alpha \nabla_\gamma \mathcal{F}_{\beta} - m^\alpha \nabla_\gamma \mathcal{F}_{\beta} - \nabla_\mu \mathcal{F}_{\alpha \beta} + \nabla_\gamma \mathcal{F}_{\alpha \beta}) = 0,
\]

where in the first equality we used (19) and in the second we inserted identities (27) and (28) and used the fact that \( \Lambda \chi_\alpha = 0 \), (see (25)).

To conclude the section we find an identity for the derivative of the energy–momentum tensor \( t_{\mu \nu} \) (9). Direct substitution of (24) yields

\[
\nabla_\mu t_{\gamma \alpha} = \frac{1}{2} \nabla_\mu (\mathcal{F}_{\alpha \beta} \mathcal{F}_{\beta}^{\gamma}) = \frac{1}{2} (\mathcal{Q} \chi_\mu + \nabla_\alpha \mathcal{Q}) t_{\gamma \alpha} + \frac{2}{3} \Lambda \xi^\alpha (\mathcal{I}_{\sigma \mu \alpha \beta} \mathcal{F}_{\gamma}^{\beta} + \mathcal{I}_{\sigma \gamma \alpha \beta} \mathcal{F}_{\beta}^{\gamma}).
\]

To elaborate this further we need an expression for the second term II. We first note the identity

\[
\mathcal{X}_{\alpha \beta} \mathcal{Y}^{\gamma} = \mathcal{X}_{\gamma} \mathcal{Y}^{\beta} = 0
\]

which is valid for any pair of self-dual two-forms \( \mathcal{X}, \mathcal{Y} \) (this identity is a direct consequence of the even more fundamental identity \( X_{\alpha \beta} Y^{\gamma} = X^{\gamma} \mathcal{Y}_{\alpha \beta} = \frac{1}{2} \eta_{\gamma \alpha \beta} Y_{\alpha \beta} \) which holds for any two 2-forms [18]). Applying this to \( \mathcal{I}_{\sigma \nu \alpha \beta} \) (in the second pair of indices) and \( \mathcal{F}_{\gamma}^{\beta} \) gives

\[
II = \mathcal{I}_{\sigma \nu \alpha \beta} \mathcal{F}_{\gamma}^{\beta} + \mathcal{I}_{\sigma \gamma \alpha \beta} \mathcal{F}_{\beta}^{\gamma} = \mathcal{I}_{\sigma \nu \alpha \beta} \mathcal{F}_{\gamma}^{\beta} + \mathcal{I}_{\sigma \gamma \alpha \beta} \mathcal{F}_{\beta}^{\gamma} \\
= \frac{1}{4} (g_{\gamma \alpha} (\mathcal{F}_{\sigma \mu} + \mathcal{F}_{\nu \alpha}) - g_{\gamma \alpha} (\mathcal{F}_{\sigma \mu} + \mathcal{F}_{\nu \alpha})) + \frac{1}{4} \eta_{\gamma \alpha \beta} (\mathcal{F}_{\alpha}^{\beta} - \mathcal{F}_{\alpha}^{\beta}) \\
= \frac{1}{2} (g_{\gamma \alpha} F_{\sigma \mu} - g_{\gamma \alpha} F_{\nu \alpha} - g_{\gamma \alpha} F_{\sigma \nu} - g_{\gamma \alpha} F_{\sigma \nu} - g_{\gamma \alpha} F_{\alpha}^{\sigma} - g_{\gamma \alpha} F_{\sigma}^{\alpha}),
\]

where in the third equality we inserted the explicit expression for \( \mathcal{I}_{\sigma \gamma \alpha \beta} \) and in the last one we used \( \mathcal{F}_{\alpha}^{\beta} - \mathcal{F}_{\alpha}^{\beta} = -2i F_{\alpha}^{\beta} = -i \eta_{\alpha \beta} F^{\alpha \beta} \) and have expanded the product of volume forms, as in (27). Identity (30) is valid for any self-dual two-form \( \mathcal{F}_{\alpha \beta} = F_{\alpha \beta} + i F_{\alpha \beta}^{*} \). Contracting II with \( \xi^\alpha \) and using that \( \Lambda \chi_\alpha = 0 \) we finally arrive at
\[
\nabla_{\mu}a_{\gamma} = \frac{1}{2}(Q\chi_{\mu} + \tilde{\Theta}\tau_{\mu})a_{\gamma} + \lambda \frac{1}{3}(\xi_{\gamma}F_{\mu} + \xi_{\mu}F_{\gamma}).
\]  

(31)

It follows from e.g. (25) that the cases \( \Lambda = 0 \) and \( \Lambda \neq 0 \) are different from each other, and thus we treat them separately.

3. The case \( \Lambda = 0 \)

From lemma 1 we know that the self-dual Killing 2-form \( \mathcal{F} \) can vanish on an open neighborhood only if \( \Lambda = 0 \). Thus, we first consider this possibility now. In this case, the main assumption (23) implies that the Weyl tensor vanishes, so the spacetime has a vanishing Riemann tensor and is locally flat. The alignment condition (23) is satisfied by any of the ten linearly independent Killing vectors (by simply choosing \( Q = 0 \)) and the subclass for which \( \mathcal{F} = 0 \) is the four-dimensional space of covariantly constant vector fields (i.e. the translations). This is rather an exceptional situation and, as we will see, is included naturally as a limit in the solutions of the generic case with \( \mathcal{F} \neq 0 \), which we consider next.

The next proposition goes a long way in identifying the spacetimes contained in this class.

**Proposition 1.** Let \((\mathcal{M}, g)\) be a vacuum spacetime with a Killing vector \( \xi \) such that the self-dual Killing form \( \mathcal{F}_{\mu} \) is null and nowhere zero and the self-dual Weyl tensor takes the form (23). Then \((\mathcal{M}, g)\) admits a smooth, null and parallel local vector field \( k \). Moreover, \( \xi \) and \( k \) commute and \( g(\xi, k) \) is constant. In the case that this constant is non-zero, \( k \) is defined globally and can be chosen so that \( g(\xi, k) = -1 \) (in particular, \( \xi \) vanishes nowhere on \( \mathcal{M} \)).

**Proof.** From (9) and (31) with \( \Lambda = 0 \) we have

\[
2\nabla_{\lambda}k'_{\mu}k'_{\nu} = \frac{1}{2}(Q\chi_{\lambda} + \tilde{\Theta}\tau_{\lambda})k'_{\mu}k'_{\nu}
\]

which is equivalent to

\[
\nabla_{\lambda}k'_{\mu} = \frac{1}{4}(Q\chi_{\lambda} + \tilde{\Theta}\tau_{\lambda})k'_{\mu}
\]

(32)

meaning that the vector field \( k' \) is recurrent, and in particular \( m^\nu\nabla_{\nu}k'_{\mu} = 0 \) so that the Bianchi identity (29) simplifies to \( \mathcal{F}_{\mu} \nabla^{\nu}Q = 0 \). Thus \( \nabla_{\alpha}Q = Ck'_{\alpha} + Bm_{\alpha} \) which together with (26) implies

\[
C\xi^\alpha k'_{\alpha} + B\xi^\alpha m_{\alpha} = 0.
\]

(33)

A (local) vector field in the direction of \( k' \) will then be parallel if the proportionality one-form \( Q\chi_{\lambda} + \tilde{\Theta}\tau_{\lambda} \) is closed. This is the case as follows from the following calculation. Using that \( \chi \) is closed (18)

\[
\text{d}(Q\chi) = \text{d}Q \wedge \chi = 2(Ck' + Bm) \wedge [(\xi^\alpha k'_{\alpha})m - (\xi^\alpha m_{\alpha})k']
\]

\[
= 2(C\xi^\alpha k'_{\alpha} + B\xi^\alpha m_{\alpha})k' \wedge m = 0,
\]

where we have used (7) in the second equality and (33) in the last one. It follows that locally there exists a real function \( w \) (defined up to an additive constant) such that \( Q\chi_{\lambda} + \tilde{\Theta}\tau_{\lambda} = 4\nabla_{\lambda}w \) and (32) reads

\[
\nabla_{\lambda}(e^{-w}k'_{\mu}) = 0
\]
so that \( k := e^{-w} k' \) is a covariantly constant null vector field. Taking into account that \( \xi^\mu \nabla_\mu w = 0 \) and (10)

\[
\xi \cdot k = 0
\]

(34)

so that \( \xi \) and \( k \) commute. Moreover the scalar product \( \xi^k \) is constant because

\[
\nabla_\mu (\xi^k \xi_\mu) = (\nabla_\mu \xi_\mu) k^\rho = \text{Re}(\mathcal{F}_{\alpha\beta}) k^\rho = 0
\]

because \( k \) lies in the kernel of \( \mathcal{F} \). Thus \( g(\xi, k) \) is either zero everywhere on \( \mathcal{M} \) or else vanishes nowhere.

In the latter case we can choose the additive constant in \( w \) (i.e. scale \( k \) by a constant) so that \( g(\xi, k) = -1 \) everywhere. It follows that \( k \) is defined not only locally on \( \mathcal{M} \) but also globally. To see this, assume that \( g(\xi, k') = 0 \) somewhere. Let \( \mathcal{M}^k \) be the set of points where \( \xi \) is not zero. Given that \( k' \) is smooth, nowhere zero and globally defined on \( \mathcal{M} \), a smooth vector field \( k \) (we use the same notation because \( a \) \( \text{p} \)osteriori this vector field is the same as before) can be uniquely defined on \( \mathcal{M}^k \) by the conditions of being proportional to \( k' \) and satisfying \( g(\xi, k) = -1 \). The property of being recurrent is invariant under scaling, so \( k \) is also recurrent and we have \( \nabla_\mu k_\beta = W_\alpha k_\beta \) for some smooth one-form \( W_\alpha \). But then

\[
0 = \nabla_\mu (\xi^k \xi_\rho) = \text{Re}(\mathcal{F}_{\alpha\beta}) k^\rho + \xi^\rho W_\alpha k_\rho = -W_\alpha,
\]

where again we used \( \mathcal{F}_{\alpha\beta} k^\rho = 0 \). Thus \( k \) is in fact parallel and smooth on \( \mathcal{M}^k \). It must extend smoothly to the closure \( \overline{\mathcal{M}}^k \) (consider e.g. geodesics \( \gamma \) starting at \( p \in \partial \mathcal{M}^k \) into \( \mathcal{M}^k \) and use the property that \( g(\gamma', k) \) is constant along the geodesic) and hence to all of \( \mathcal{M} \). Since \( g(\xi, k) \) also extends smoothly to all of \( \mathcal{M} \) (and takes the value \(-1\)), we in fact conclude that \( \xi \) had no fixed points on \( \mathcal{M} \) after all. \( \square \)

We can now find explicitly the spacetimes satisfying the hypotheses of proposition 1. Spacetimes with a covariantly constant null vector field \( k \) are called Brinkmann spaces [4]. Schimming [31] has shown that Brinkmann spaces admit local coordinates \( \{ v, u, x, y \} \) near any point \( p \in \mathcal{M} \) such that

\[
k = \partial_v, \quad k = -du
\]

and the metric takes the form

\[
dx^2 = -2dvdu + 2Hdx^2 + dx^2 + dy^2,
\]

where \( H(u, x, y) \) is a function not dependent on \( v \) (as \( k \) is, in particular, a Killing vector). These spacetimes are vacuum if and only if \( H \) solves the elliptic PDE

\[
H_{xx} + H_{yy} = 0.
\]

The local coordinates are not fully fixed and the form (36) together with (35) are kept under several coordinate changes, see e.g. [3, 35], in particular under the local transformation

\[
u = u', \quad x = x' + p_1(u), \quad y = y' + p_2(u), \quad v = v' + \dot{p}_1 x' + \dot{p}_2 y' + s(u),
\]

where \( p_1(u) \) and \( p_2(u) \) are functions of only \( u \), dots stand for derivative with respect to \( u \), and the new function \( H'(u', x', y') \) is

\[
H' = H + \dot{p}_1 x' + \dot{p}_2 y' + \dot{s} + \frac{1}{2}(\dot{p}_1^2 + \dot{p}_2^2).
\]

It follows that any terms linear on \( x \) and \( y \) (with \( u \)-dependent coefficients) appearing in \( H \) can be removed by such a transformation, while keeping the vacuum condition (37). We will use this freedom presently.
We want to identify which subclass of Brinkmann spacetimes are included in the spacetime satisfying the hypotheses of proposition 1. Our strategy is to identify the null self-dual two-forms $\Omega_{ab}$ associated to a Killing vector $\xi$ in the coordinates where (36) holds. Then, we will also require that the alignment condition (23) holds.

To that end, observe that the metric (36) takes a Kerr–Schild form [35]
\begin{equation}
g = \eta + 2Hk \otimes k,
\end{equation}
where $\eta$ is the flat Minkowski metric. Hence, any Killing vector $\xi$ of the metric with the property (34) satisfies
\begin{equation}
\xi \cdot g = \xi \cdot \eta + 2\xi(H) \cdot k \otimes k = 0 \implies \xi \cdot \eta = -2\xi(H) \cdot k \otimes k
\end{equation}
and these are called Kerr–Schild vector fields [7] of the Minkowski spacetime relative to the parallel null direction $k$. The general solution for such Kerr–Schild vector fields was found in [7] and, taking into account (34) they read (adapting the notation)
\begin{equation}
\xi = -g(\xi, k)\partial_a + (ax + by + c(u))\partial_u + (a(u) + \alpha y)\partial_x + (b(u) - \alpha x)\partial_y,
\end{equation}
for arbitrary real functions $a(u), b(u), c(u)$ and constant $\alpha \in \mathbb{R}$, and with
\begin{equation}
\xi(H) = \dot{c} + \dot{a}x + \dot{b}y.
\end{equation}

To check which of these vector fields produce a null Killing 2-form, we first note that
\begin{equation}
\xi = g(\xi, k)dv - (\dot{a}x + \dot{b}y + c(u) + 2Hg(\xi, k)du + (a(u) + \alpha y)dx + (b(u) - \alpha x)dy
\end{equation}
so that
\begin{equation}
d\xi = 2du \wedge [(\dot{a} + g(\xi, k)H_{ax})dx + (\dot{b} + g(\xi, k)H_{ay})dy] + 2\alpha dy \wedge dx
\end{equation}
and the 2-form $d\xi$ is null, with $k$ as null eigenvector, if and only if $\alpha = 0$, which we assume henceforth. Therefore
\begin{equation}
\xi = -g(\xi, k)\partial_a + (ax + by + c(u))\partial_u + a(u)\partial_x + b(u)\partial_y,
\end{equation}
the PDE (40) becomes
\begin{equation}
\xi(H) = -g(\xi, k)H_{ax} + a(u)H_{ax} + b(u)H_{ay} = \dot{c} + \dot{a}x + \dot{b}y
\end{equation}
and the Killing null 2-form $F = d\xi/2$ and its dual are (the basis $\{\partial_a, \partial_u, \partial_x, \partial_y\}$ is taken as positively oriented)
\begin{equation}
F = du \wedge [(\dot{a} + g(\xi, k)H_{ax})dx + (\dot{b} + g(\xi, k)H_{ay})dy],
\end{equation}
\begin{equation}
F^* = du \wedge [-(\dot{b} + g(\xi, k)H_{ay})dx + (\dot{a} + g(\xi, k)H_{ax})dy].
\end{equation}
Thus, we have
\begin{equation}
\mathcal{F} = (\dot{a} - \dot{b} + g(\xi, k)(H_{ax} - iH_{ay}))du \wedge (dx + idy).
\end{equation}
Given the simple form of the metric (36), it is a matter of straightforward calculation to compute the self-dual Weyl tensor $C_{\alpha\beta\mu\nu}$, which turns out to be
\begin{equation}
C_{\alpha\beta\mu\nu} = -\frac{1}{2}(H_{\alpha\beta} - H_{\beta\alpha})\mathcal{V}_{\alpha\beta}^{\nu}_{\mu\nu},
\end{equation}
where $\mathcal{V}$ is the self-dual two form
\begin{equation}
\mathcal{V} = k \wedge (dx + idy) = -du \wedge (dx + idy).
\end{equation}
Thus, our main assumption, the alignment condition (23), is clearly satisfied for (41) as follows from (44) and (43).
Now it becomes necessary to split the analysis into two cases, namely when 
\( g(\xi, k) = -1 \) and when \( g(\xi, k) = 0 \).

In the case \( x = g_k, 0 \), equation (42) implies after derivation with respect to \( x, y \) and taking (37) into account

\[ a(u)H_{xx} + b(u)H_{xy} = \ddot{a}, \quad a(u)H_{xy} - b(u)H_{xx} = \ddot{b} \]

so that \( H_{xx} \) and \( H_{xy} \) depend only on \( u \) (note that \( a(u) \) and \( b(u) \) cannot vanish simultaneously on an open interval because \( F \) is assumed to be non-zero, see (43) with \( g(\xi, k) = 0 \)). It follows that there exist five (real) functions \( A(u), B(u), s_1(u), s_2(u), s_3(u) \) such that

\[
H = \frac{1}{2}A(u)(x^2 - y^2) + B(u)xy + s_1(u)x + s_2(u)y + s_3(u) \tag{45}
\]

and the equation (42) requires

\[
\ddot{a} = Aa + Bb, \quad \ddot{b} = Ba - Ab, \tag{46}
\]
\[
\dot{c} = s_1a + s_2b. \tag{47}
\]

However, as mentioned after equation (38), all linear terms in \( x \) and \( y \) with arbitrary \( u \)-dependent coefficients can be removed in (45) by means of the transformation (38), and thus we can assume, without loss of generality, that \( s_1 = s_2 = s_3 = 0 \) which in particular implies, from (47), that \( c = c_0 \) is a constant\(^4\). The general solution of the system (46) has four integration constants. We have thus found that the most general local form of a spacetime satisfying the hypotheses of proposition 1 with \( g(\xi, k) = 0 \) can be written as

\[
dx^2 = -2dudv + dx^2 + dy^2 + (A(u)(x^2 - y^2))du^2, \tag{48}
\]
\[
\xi = (ax + by + c_0)\partial_x + a\partial_y + b\partial_y, \tag{49}
\]

where \( c_0 \in \mathbb{R} \) and \( \{a(u), b(u)\} \) is any solution of (46). If \( A(u) = B(u) = 0 \) on some non-empty set \( \mathcal{U} \subset \mathcal{M} \) then \( (\mathcal{U}, g) \) is obviously locally flat leading to the exceptional case mentioned at the beginning of this section. If, on the other hand, \( A(u) \) and \( B(u) \) do not vanish simultaneously on any open set \( \mathcal{U}' \), these metrics are called vacuum plane waves\(^2, 35\), they admit the 5 parameter family of Killing vectors (49). The alignment condition (23) holds for any of these Killing vectors. The Killing form is (see (43))

\[
\mathcal{F}_{\alpha\beta} = (-\dot{a} + ib)\nabla_{\alpha\beta}
\]

and the proportionality function \( Q \) is, from (44)

\[
Q = -\frac{A - iB}{(\dot{a} - ib)^2}.
\]

The set of points where \( \mathcal{F}_{\alpha\beta} = 0 \) (i.e. those with \( \dot{a} = \dot{b} = 0 \)), which were excluded by the hypothesis of \( F \) being non-zero, can be attached to the spacetime at the cost of \( Q \) being non-smooth there. However, this is merely an artifact of the proportionality function between the Killing form \( \mathcal{F}_{\alpha\beta} \) and the two-form \( \nabla_{\alpha\beta} \) becoming zero. The extended spacetime obviously remains smooth also at those points.

\(^4\) The explicit coordinate change (38) that achieves this has \( \{p_1, p_2, s\} \) solving the system of ODEs

\[
\dot{p}_1 = Ap_1 + Bp_2 + s_1, \quad \dot{p}_2 = Bp_1 + Ap_2 + s_2, \quad 2s = 2s_1 + Ap_1 + Bp_2 + s_1^2 + p_1^2 + p_2^2.
\]
The case $g(\xi, k) = -1$ is simpler. Consider the coordinate transformation
\[ p_1, p_2, s \] satisfying
\[ \dot{p}_1 = a, \quad \dot{p}_2 = b, \quad \frac{d}{ds}(s - p_1 \dot{p}_1 - p_2 \dot{p}_2) = \dot{c} - a \dot{p}_1 - b \dot{p}_2. \]

The Killing vector $\xi$ in (41) is simply $\xi = \partial_u$ in the new coordinates. It follows that we can assume without loss of generality $\xi = \partial_u$ (i.e. $a = b = c = 0$) in the expressions above. The PDE (42) states simply that $H(x, y)$ is independent of $u$. Therefore, these are precisely the stationary vacuum Brinkmann spacetimes (or the stationary vacuum pp-waves) considered in [14].

This metric admits in general a 2-parameter family of Killing vectors generated by the parallel $k$ and by $\xi$. The length of the 2-form $k \wedge \xi$ is constant.

Note that $H$ being a solution of the Laplace equation (37), $H$ is the real part of some complex function $\sigma$ holomorphic in the variable $\zeta = x + iy$. We also have $\mathcal{F}_{\alpha \beta} = (H_x - iH_y) V_{\alpha \beta} = \sigma \zeta \overline{V}_{\alpha \beta}$.

The alignment condition (44) can be rewritten as

\[ C_{\alpha \beta \mu \nu} = -\sigma \zeta \zeta V_{\alpha \beta} \overline{V}_{\mu \nu} \]

so that, since we are assuming $\mathcal{F} \equiv 0$, the function $Q$ reads

\[ Q = \frac{\sigma \zeta \zeta}{(\sigma \zeta)^2}. \]

As in the previous case, the set of points where $\mathcal{F}_{\alpha \beta} = 0$, which are given by the zeros of the holomorphic function $\sigma$, were excluded by assumption, can be attached to the spacetime at the cost of making $Q$ non-smooth there, but keeping the extended spacetime fully smooth also at those points.

Remark: It must be noticed that there is a non-empty intersection of the two families of spacetimes identified in this section. They are given by the analytic function $\sigma = \beta \zeta^2$ with $\beta \in \mathbb{C}$, $2\beta = A - iB$ —where now $A, B$ are constants. In this case there is a 6-parameter family of Killing vectors satisfying the alignment (23). These metrics are precisely the irreducible locally symmetric vacuum spacetimes, that is, they satisfy the condition $\nabla_a R_{a\beta \mu \nu} = 0$, and were identified in [6], see also [3]. For later reference, we write down explicitly the metric and the Killing vectors. Define constants $\kappa > 0$ and $\alpha$ by $A = \kappa^2 \cos(2\alpha)$ and $B = \kappa^2 \sin(2\alpha)$. Then, the coordinate change $x' = x \cos \alpha + y \sin \alpha$, $y' = -x \sin \alpha + y \cos \alpha$ transforms the metric (after dropping the primes) into (48) with $A = \kappa^2 > 0$ and $B = 0$. The solution of (46) (with $A = \kappa^2, B = 0$) is

\[ a(u) = c_1 e^{au} + c_2 e^{-au}, \quad b(u) = c_3 \cos(\kappa u) + c_4 \sin(\kappa u), \]

where $c_1, c_2, c_3, c_4$ are arbitrary constants. Thus, the metric and Killing vectors are

\[ ds^2 = -2udv + dx^2 + dy^2 + \kappa^2(x^2 - y^2)du^2, \]
\[ \xi = (c_3 + \kappa x(c_1 e^{au} - c_2 e^{-au}) + \kappa y(-c_3 \sin(\kappa u) + c_4 \cos(\kappa u))) \partial_v + (c_1 e^{au} + c_2 e^{-au}) \partial_u + (c_3 \cos(\kappa u) + c_4 \sin(\kappa u)) \partial_r + c_5 \partial_u. \]

The alignment condition is satisfied by any non-trivial Killing vector in (53). If $c_5 = 0$ we are in the case $g(\xi, k) \equiv 0$ while $c_5 = 0$ corresponds to $g(\xi, k) = 0$. The proportionality factor
\( Q \) in (23) for the general Killing (53) takes the form (with \( a(u), b(u) \) as in (51))

\[
Q = \frac{-\kappa^2}{(a - ib - c_2\kappa^2(x + iy))^2}.
\]

4. The case \( \Lambda \neq 0 \)

In the \( \Lambda \)-vacuum case we start with the following result

**Proposition 2.** Let \((\mathcal{M}, g)\) be a \( \Lambda \)-vacuum \( (\Lambda \neq 0) \) spacetime with a Killing vector \( \xi \) such that the self-dual Killing form \( \mathcal{F}_{\alpha\beta} \) is null and such that the self-dual Weyl tensor takes the form (23). Then \( \Lambda < 0 \), both \( \mathcal{F}_{\alpha\beta} \) and \( \xi \) vanish nowhere and \( \xi \) is null, hypersurface orthogonal and pointing along the principal null direction of \( \mathcal{F}_{\alpha\beta} \). Moreover, the quotient metric on each null hypersurface orthogonal to \( \xi \) is of negative constant curvature \( R^{(3)} = \frac{2\Lambda}{3} \).

**Proof.** Consider the set \( \mathcal{U} \) where \( \mathcal{F}_{\alpha\beta} \) is not zero. To start with, \( \mathcal{U} \) cannot be empty, as otherwise \( \mathcal{F} \) would vanish on \( \mathcal{M} \) which is impossible for \( \Lambda \neq 0 \) according to lemma 1. From (25), on \( \mathcal{U} \) we have \( \chi_{\mu} = 0 = \xi^\nu \mathcal{F}_{\nu\mu} \), so that \( \xi \) must be proportional to the unique principal null direction \( \kappa \) of \( \mathcal{F}_{\alpha\beta} \). Thus

\[
\xi_\alpha = L \kappa^\alpha,
\]

for some function \( L \). Note that \( \xi \) is automatically null, so \( N = 0 \). \( L \) is in fact a constant directly related to \( \Lambda \) as follows from (17) with \( \chi_{\mu} = 0 \) and \( N = 0 \)

\[
0 = -k_\mu k^\nu - \frac{2}{3} \Lambda \xi_\mu \xi_\nu = -k_\mu k^\nu \left( 1 + \frac{2}{3} \Lambda L^2 \right)
\]

leading to

\[
\Lambda = -\frac{3}{2L^2} < 0.
\]

From (9) we have \( 2t_{\mu\nu} = 2 \mathcal{F}_{\nu\rho} \mathcal{F}_{\mu}^{\rho} = -\frac{2\Lambda}{3} \xi_\mu \xi_\nu \). This shows that \( \partial \mathcal{U} \) is empty as otherwise, at \( p \in \partial \mathcal{U} \), we would have both \( \mathcal{F}_{\alpha\beta} |_p = 0 \) and \( \xi_\alpha |_p = 0 \) which cannot happen for non-identically zero Killing vectors. Thus, \( \mathcal{U} = \mathcal{M} \) and neither \( \mathcal{F}_{\alpha\beta} \) nor \( \xi \) vanish anywhere.

Equation (31) becomes, on using (9)

\[
k_\mu \left( \nabla_\mu k_\nu - \frac{2\Lambda}{3} LF_{\nu\mu} \right) + k_\nu \left( \nabla_\nu k_\gamma - \frac{2\Lambda}{3} LF_{\gamma\nu} \right) = 0
\]

so that

\[
\nabla_\mu k_\nu = \frac{2\Lambda}{3} LF_{\nu\mu} = -\frac{1}{2L} [k_\mu (m_\mu + m_\nu) - k_\nu (m_\mu + m_\alpha)].
\]

This expression implies, in particular, that \( k_\mu \) (and hence \( \kappa_\mu \)) is hypersurface orthogonal. Thus, by the Fröbenius theorem, through each point \( p \in \mathcal{M} \) there passes a unique, maximal, injectively immersed null hypersurface \( \mathcal{H}_p \) orthogonal to \( \kappa \). We call such hypersurfaces ‘horizons’. From (56) we know that any spacelike section \( S \) of \( \mathcal{H}_p \) has a vanishing second fundamental form along \( k^\nu \), which also implies that the mean curvature vector of \( S \) in \( \mathcal{M} \) is null. This, together with the fact that \( k^\mu C_{\alpha\beta\gamma\delta} = 0 \) and using the standard form of the Gauss equation for \( S \) (see e.g. formula (9) in [32], the formula previous to (7) in [24], or equation...
(16) in [13]) leads to
\[ K_S = \frac{R^{(2)}}{2} = \frac{\Lambda}{3}, \]
where \( K_S \) represents the Gaussian curvature of \( S \) and \( R^{(2)} \) its scalar curvature.

We want to use the information provided in the proof of this proposition to determine the local form of the spacetime metric. To that aim we use that \( k' \) is integrable and nowhere zero and introduce two smooth functions \( f \) and \( u \) such that
\[ k' = -f \, du. \tag{57} \]
Since neither \( f \) nor \( du \) vanish anywhere we can, without loss of generality, assume that \( f > 0 \). Let us find the PDE’s that this function satisfies. Combining the exterior differential of (57) \( d k' = f^{-1} \, df \wedge k' \) with expression (56) yields
\[ \frac{1}{L} (m_\alpha + m_\alpha) = -\frac{1}{f} \, \nabla^\alpha f + h_1 k'_\alpha \tag{58} \]
for some smooth function \( h_1 \). Note that this expression implies in particular that
\[ \nabla^\alpha \nabla^\alpha f = \frac{2f^2}{L^2}. \tag{59} \]
Inserting (57) and (58) into (56) it follows
\[ \nabla_\mu \nabla_\nu u = -\frac{1}{2f} (\nabla_\mu u \nabla_\nu f + \nabla_\nu u \nabla_\mu f). \]
We can evaluate the integrability conditions of this equation. Using the fact that
\[ C^\rho_{\alpha \mu \beta} \nabla_\rho u = -\frac{1}{f} C^\rho_{\alpha \mu \beta} k'_\rho = -\frac{1}{f} \text{Re} (C^\rho_{\alpha \mu \beta} k'_\rho) = 0, \]
the integrability condition turns out to be
\[ \nabla_\mu \left( \frac{1}{f} \nabla_\rho \nabla_\alpha f - \frac{1}{2f} \nabla_\mu f \nabla_\alpha f - \frac{1}{L^2} \delta_{\rho \mu} \right) = 0 \]
which is equivalent to
\[ \nabla_\mu \nabla_\alpha f - \frac{1}{2f} \nabla_\mu f \nabla_\alpha f - \frac{1}{L^2} \delta_{\rho \mu} = h_2 \nabla_\mu u \nabla_\alpha u \tag{60} \]
for some smooth function \( h_2 \). At this point, it is convenient to define a smooth positive function \( x \) by \( f := \frac{2}{x^2} \). In terms of \( x \), the norm condition (59) and the Hessian equation (60) become
\[ \nabla_\mu x \nabla^\mu x = \frac{x^2}{2L^2}, \tag{61} \]
\[ \nabla_\mu \nabla_\alpha x - \frac{2}{x} \nabla_\mu x \nabla_\alpha x + \frac{x}{2L^2} \delta_{\rho \mu} = -\frac{h_2 x^3}{4L} \nabla_\mu u \nabla_\alpha u \tag{62} \]
and we also have, from (58)
\[ k'^\alpha \nabla_\alpha x = 0. \tag{63} \]
We now construct a local coordinate system near any point \( p \in M \). Consider a hypersurface \( \Sigma \) passing through \( p \) and transversal to the horizon \( \mathcal{H}_{u(p)} = \{ u = u(p) \} \). Consider an open neighborhood \( U_p \) of \( p \) and define \( I_u \subset \mathbb{R} \) as the set of values of \( u \) such that the horizons \( \mathcal{H}_u \) of constant \( u \) intersect \( U_p \). Restricting \( U_p \) if necessary we can fulfill the following three conditions:

- \( I_u \) is connected,
- \( \Sigma \) is transverse to \( \mathcal{H}_u \) for all \( u \in I_u \),
- the spacelike surface \( S_u = \Sigma \cap \mathcal{H}_u \) is non-empty and connected for all \( u \in I_u \).

Let \( S_{u,v} \) be the set of surfaces obtained by Lie dragging \( S_u \) along the (affinely parametrized) null generator with tangent \( \xi \) of \( \mathcal{H}_u \) and satisfying \( S_{u,v=0} = S_u \). Restricting \( U_p \) further we can assume that \( \nu \) takes values on an open interval \( I_v \) containing zero and that all \( S_{u,v}, \forall v \in I_v \) are diffeomorphic to \( S_u \). Moreover we can assume all \( S_u, u \in I_u \) to be simply connected.

Consider the function \( x_S := x_{|S_u} \) restricted to the surface \( S_{u,v} \) and endow \( S_{u,v} \) with the induced metric \( h \) and the corresponding Levi-Civita covariant derivative, which we denote by \( D \). The pull-back of the Hessian of a function \( F \) and the Hessian of the restriction \( F_S \) on an embedded submanifold \( S \) with embedding \( \varphi \) are related by

\[
\text{Hess}^D F_S = \varphi^* (\text{Hess} F) - g (\mathbb{I}, \text{grad} F)
\]

where \( \mathbb{I} \) is the shape tensor—also called second fundamental form vector—of \( S \) in \( M \) (see e.g. [28]). In order to apply this to \( x_S \) we observe that the shape tensor of \( S_{u,v} \) satisfies \( g (\mathbb{I}, k') = 0 \), because \( S_{u,v} \) lies in a horizon, which as seen in the proof of proposition 2 are null hypersurfaces with identically vanishing second fundamental form. This means that \( \mathbb{I} \) is parallel to \( k' \) and we have \( g (\mathbb{I}, \text{grad} x) = 0 \) as a consequence of (63). Pulling-back (62) onto \( S_{u,v} \) by means of (64) we thus find

\[
D_A D_B x_S = -\frac{2}{x_S} (D_A x_S)(D_B x_S) + \frac{x_S}{2L^2} h_{AB} = 0,
\]

where \( h_{AB} \) is the induced metric on \( S_{u,v} \). Moreover, from (61) and (63) the square norm of \( x_S \) satisfies

\[
(D_A x_S)(D^A x_S) = \frac{x_S^2}{2L^2}.
\]

The trace of (65) is therefore

\[
D_A D^A x_S = 0
\]

so that \( x_S \) is a harmonic function with \( dx_S \) nowhere vanishing. Its Hodge dual \( \ast dx_S \) in \( S_{u,v} \) is then a closed one-form and simply connectedness of \( S_{u,v} \) implies the existence of a function \( y_S \) satisfying \( \ast dx_S = dy_S \). By construction \( dx_S \) and \( dy_S \) are mutually orthogonal and have the same norm, which implies that the metric \( h_{AB} \) takes the form

\[
h = \frac{2L^2}{x_S} (dx_S^2 + dy_S^2).
\]

This metric has constant negative curvature equal to \(-1/(2L^2)\) in accordance with proposition 2. The pair \( \{ x_S, y_S \} \) is a coordinate system of \( S_{u,v} \). The function \( y_S \) is defined up to an additive constant on each \( S_{u,v} \). We can fix partially this constant by selecting, on each \( \mathcal{H}_u \) one null generator \( \gamma (v) \) and imposing \( y_S |_{\gamma (v)} = y_0 \) for some fixed constant \( y_0 \).
By construction the set of coordinates which assigns to each point \( q \in U_p \) the values \( \{u, v\} \) of the surface \( S_{u,v} \) containing \( q \) and the values \( \{x_\gamma, y_\gamma\} \) of its coordinates within \( S_{u,v} \) is a coordinate system of \( U_p \). We will denote this coordinate system by \( \{v, u, x, y\} \) (we can use \( x \) because by construction this coordinate agrees with the function \( x \) introduced before). We already know that \( k'^{\alpha}\nabla_\alpha u = 0 \) from (57), and we also have (63) and \( k'^{\alpha}\nabla_\alpha v = 1/L \), the latter as a consequence of \( v \) being the affine parameter associated to \( \xi = Lk' \). Moreover \( k'^{\alpha}\nabla_\alpha y = 0 \), as we show next. We work within a fixed \( \mathcal{H}_{uv} \). Let \( W_1, W_2 \) be vectors tangent to \( S_{u,v} \) satisfying \([k', W_1] = [k', W_2] = 0\). \( k' \) being an isometry implies that

\[
0 = \mathcal{L}_{k'}(g(W_1, W_2)) = \mathcal{L}_{k'}(h(W_1, W_2)) = \mathcal{L}_{k'}\left(\frac{2L^2}{x^2}(dx(W_1)dx(W_2) + dy(W_1)dy(W_2))\right)
\]

\[
= \frac{2L^2}{x^2}d(\mathcal{L}_{k'}y)(W_1)d(\mathcal{L}_{k'}y)(W_2)
\]

where in the last equality we used (63) (i.e. \( \mathcal{L}_{k'}x = 0 \)) and the fact that the Lie derivative commutes with the exterior differential. Since this holds for any \( W_1, W_2 \) tangent to \( S_{u,v} \), the only possibility is \( d(k'(y)) = dy \), i.e. that there exists a function \( G(v) \) on \( \mathcal{H}_{uv} \) such that \( k'(y) = G(v) \). Since \( k'(y) \) vanishes on a null generator \( \gamma(v) \), it must be \( G(v) = 0 \), and hence \( k'(y) = 0 \). As the argument applies to each \( \mathcal{H}_{uv} \), we conclude \( k'^{\alpha}\nabla_\alpha y = 0 \), as claimed.

In summary, the vector field \( k' \) takes the form

\[
k' = \frac{1}{L}\partial_t
\]

in the local coordinates \( \{v, u, x, y\} \). Taking (57) into account and the definition of \( x \) we derive the metric coefficients \( g_{\alpha\beta} = -\frac{2L^2}{x^2}\delta_{\alpha\beta} \) and the spacetime metric must take the local form

\[
g = \frac{2L^2}{x^2}(-2dudv + dx^2 + dy^2 + 2C_1dudx + 2C_2dudy + C_3du^2)
\]

(66)

for some functions \( C_1, C_2, C_3 \) depending on \( u, v, x, y \) (because \( \xi = \partial_x \) is a Killing vector). The construction of the coordinates described above was based on the choice of a transversal hypersurface \( \Sigma \), which in this coordinates becomes \( \Sigma = \{v = 0\} \). The freedom in choosing \( \Sigma \) is reflected in the coordinate freedom \( v = v' + s(u, x, y) \) which leaves the form of the metric invariant and transforms the functions \( C_1, C_2 \) and \( C_3 \) to

\[
C_1' = C_1 - s_x, \quad C_2' = C_2 - s_y, \quad C_3' = C_3 - 2s_u.
\]

Given the simple form of the metric we can now impose directly the conditions of \((\mathcal{M}, g)\) being \( \Lambda \)-vacuum and satisfying the alignment condition (23). Computing the Ricci tensor of (66) one finds

\[
R_{ux} - \Lambda g_{ux} = \frac{1}{2}\partial_x(\partial_x C_2 - \partial_y C_1) = 0,
\]

\[
R_{uy} - \Lambda g_{uy} = -\frac{x^2}{2}\partial_x[x^{-2}(\partial_x C_2 - \partial_y C_1)] = 0.
\]

The first equation implies that \( \partial_x C_2 - \partial_y C_1 = q(x, u) \), which inserted in the second yields \( \partial_x(x^{-2}q) = 0 \), i.e. \( q = -x^2r(u) \), for some smooth function \( r(u) \). The functions \( \tilde{C}_1 \) and \( \tilde{C}_2 \) defined by \( \tilde{C}_1 := C_1 - r(u)x^2 \), \( \tilde{C}_2 := C_2 \) satisfy then

\[
\partial_x \tilde{C}_2 - \partial_y \tilde{C}_1 = \partial_x C_2 - \partial_y C_1 + r(u)x^2 = 0
\]
so that there exists a function \( s(u, x, y) \) such that \( s_x = \hat{C}_1 \) and \( s_y = \hat{C}_2 \). Applying the coordinate change \( v = v' + s \) the form of the metric simplifies to (we drop the primes in \( v \) and set \( M := C_1' \))

\[
g = \frac{2L^2}{x^2} (-2du dv + dx^2 + dy^2 + 2r(u)x^2ydu dx + M du^2).
\]  

(67)

The \( \Lambda \)-vacuum field equations are satisfied if and only if \( M \) solves the linear inhomogeneous PDE

\[
M_{xx} + M_{yy} - \frac{2}{x} M_x - r^2 x^4 = 0.
\]

A particular solution of this equation is \( \frac{1}{18} r^2(u) x^6 \), so that the general solution is

\[
M = H + \frac{1}{18} r^2(u) x^6
\]

(68)

with \( H \) any solution of the homogeneous equation

\[
H_{xx} + H_{yy} - \frac{2}{x} H_x = 0.
\]

(69)

To solve this equation (see [30, 33]), perform the following change of dependent variable

\[
H = x^2 \partial_x [\hat{h}(u, x, y)/x]
\]

(70)

which transforms (69) into

\[
x (\hat{h}_{xxt} + \hat{h}_{yxy}) - \hat{h}_{xxt} - \hat{h}_{yxy} = x (\hat{h}_{xxt} + \hat{h}_{yxy})_x = (\hat{h}_{xxt} + \hat{h}_{yxy}) = 0
\]

whose general solution reads \( \hat{h}_{xxt} + \hat{h}_{yxy} = 0 \) for some smooth arbitrary function \( f \) independent of \( x \). This provides the general solution for \( \hat{h} \) as follows

\[
\hat{h} = x g(u, y) + h(u, x, y)
\]

where \( g(u, y) \) is arbitrary and

\[
h_{xx} + h_{yy} = 0.
\]

(71)

Introducing this into (70) one checks that

\[
H = x^2 (g(u, y) + h(u, x, y)/x)_x = x^2 (h(u, x, y)/x)_x
\]

so that \( g(u, y) \) does not contribute to the function \( H \) and the general solution of (69) is given by

\[
H = x^2 (h/x)_x
\]

(72)

with \( h(u, x, y) \) any solution of (71).

The self-dual Killing form \( \mathcal{F}_{\alpha\beta} \) associated to \( \xi = \partial_x \) of the metric (67) is

\[
\mathcal{F}_{\alpha\beta} = k'_{\alpha} m_{\beta} - k'_{\beta} m_{\alpha}
\]

with \( m = \frac{2}{x} (dx + idy) \) in agreement with (58). In particular \( \mathcal{F}_{uu} = 0 \). Computing the self-dual Weyl tensor of (67) one finds

\[
C_{uuuy} = \frac{L^2 r(u)}{x}
\]

hence the alignment condition (23) forces \( r(u) = 0 \). This restriction turns out to be sufficient for the validity of (23), with the function \( Q \) taking the value
where $H$ is given by (72). In conclusion, the most general solution of $\Lambda$-vacuum spacetimes with a Killing vector $\xi$ with null-self-dual Killing form and satisfying the alignment condition (23) can be written in local form as

$$g = \frac{2L^2}{x^2} (-2dudv + dx^2 + dy^2 + x^2(h/x)dud^2), \quad h_{xx} + h_{yy} = 0, \quad \xi = \partial_v,$$

(73)

with $h$ any $u$-dependent solution of the Laplace equation (71). Therefore, $h$ is the real part of any arbitrary $u$-dependent holomorphic function $\sigma(u, x + iy)$. This spacetime is known as the Siklos wave solutions [33] and corresponds also to the class (IV)$_0$ in [29], and they happen to be the only non-trivial Einstein spaces conformal to non-flat pp-waves [33], see also [12, 30]. It is noteworthy that anti-de Sitter (AdS) spacetime is included in the metric (73) for the case with $h$ the real part of the function $\sigma = (A(u) - iB(u))\zeta^2$, giving

$$h = A(u)(x^2 - y^2) + 2B(u)xy \quad \implies \quad H = A(u)(x^2 + y^2).$$

This case has $Q = 0$, and the metric being conformally flat, it is a portion of AdS. Observe that the metric without the conformal factor $2L^2/x^2$ describes the electromagnetic plane waves [35], which are known to be conformally flat.

**Remark:** Along the way, we have found another class of $\Lambda$-vacuum spacetimes given by (67) with (68) together with (71) and (72) and a non-vanishing function $r(u)$. This solution does not satisfy the special alignment condition (23) but, as one can easily check, it does satisfy the more general alignment condition (21). The Petrov type of this metric is III and was identified previously in [11] (case 3, formula (20) there).

5. Main theorems and concluding remarks

We finish the paper with a theorem that collects our main results herein and with a discussion that summarizes all the results concerning characterizations of spacetimes subject to our main assumption (22).

**Theorem 1.** Let $(\mathcal{M}, g)$ be a $\Lambda$-vacuum $C^3$ spacetime admitting a Killing vector $\xi$ with null self-dual Killing form $\mathcal{F}_{\alpha\beta}$. Assume that the Weyl tensor satisfies the alignment condition (23) where $Q$ is a $C^1$ function except possibly at the boundary of the set $\mathcal{M}^Q = \{ \mathcal{F}_{\alpha\beta} = 0 \}$.

If $\mathcal{M}^Q = \mathcal{M}$, then the spacetime is locally isometric to the Minkowski flat spacetime and $\Lambda = 0$.

If $\mathcal{M}^Q = \emptyset$ then $\Lambda \leq 0$ and furthermore

- if $\Lambda < 0$, then $\mathcal{M}^Q = \emptyset$ and the spacetime is locally isometric to the Siklos wave spacetimes (73)—including AdS as a particular case.
- if $\Lambda = 0$ the spacetime is either locally isometric to the vacuum plane waves (48) (when $\xi$ is orthogonal to the wave vector of $\mathcal{F}_{\alpha\beta}$), or to the vacuum and stationary Brinkmann spacetimes (50) (when $\xi$ is not orthogonal to the wave vector of $\mathcal{F}_{\alpha\beta}$). The intersection of both cases is non-empty, corresponds to the case with $A$ and $B$ constant in the vacuum plane waves and leads to the irreducible Lorentzian locally symmetric vacuum spacetimes (52).
As discussed in the introduction, one of the motivations of this work was to complete the classification of $\Lambda$-vacuum spacetimes admitting a Killing vector and satisfying the alignment condition (22). When $\mathcal{F}^2 = 0$ this was developed in [20, 22] in the case of $\Lambda = 0$ and in [27] when $\Lambda \neq 0$. In this paper we have considered the remaining case $\mathcal{F}^2 = 0$, so it makes sense to summarize all these results here. Given that the entire class satisfying (22) has now been fully identified we have now a clearer perspective on the results, and thus it is worth to reconsider and re-organize some of them. This is what we do next.

The classification in the case $\Lambda \neq 0$ and $\mathcal{F}^2 \neq 0$ discussed in [27] leads to three disjoint classes labeled (A), (B.i) and (B.ii) in theorem 4 of [27], and the Kerr–NUT–(A) de Sitter spacetime (which includes the non-rotating Schwarzschild–NUT–(A) de Sitter case) is located partly in the class (B.i) and partly in the class (B.ii). Something similar happens in the vacuum case, but since the situation is simpler these two classes can be reorganized so that they remain disjoint and one of them contains the whole class of Kerr–NUT metrics (and their plane and hyperbolic generalizations). The following theorem provides this splitting in the vacuum $\mathcal{F}^2 = 0$ case.

**Theorem 2.** Let $(\mathcal{M}, g)$ be a smooth vacuum spacetime admitting a Killing vector $\xi$ and satisfying the alignment condition (22) where $\mathcal{F}_{\alpha\beta}$ is the self-dual Killing form of $\xi$. Assume that $\mathcal{F}^2 = \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ is not identically zero. Then the spacetime is locally isometric to an element of the following two mutually exclusive classes:

- **gen-Kerr–NUT spacetime.** This class depends on two discrete constants $\sigma = \{-1, 0, 1\}$ and $\delta = \{-1, 0, 1\}$ and, away from fixed points of Killing vectors of $(\mathcal{M}, g)$, there exist coordinates $(u, r, \theta, \varphi)$ where $\xi = \partial_u$ and the metric is

  \[ ds^2 = -\frac{\Delta(r)}{\rho^2} [du + G(\theta)(aG(\theta) + 2\ell)\,d\varphi]^2 + \rho^2 (d\theta^2 + G_{,\varphi}^2 d\varphi^2) + 2(\rho^{2} - aG_{,\varphi}^2 d\varphi)(du + G(\theta)(aG(\theta) + 2\ell)\,d\varphi), \]  

  \[ \Delta(r) = \sigma(r^2 - \ell^2) - 2mr - 2a\alpha \frac{\lambda}{\rho}, \quad \Theta(\theta) = a^2 G(\theta)(\sigma G(\theta) - 2\alpha), \]

  \[ \rho^2 = r^2 + (aG(\theta) + \ell)^2, \]

  \[ G_{,\varphi} = \delta + 2\alpha G(\theta) - \sigma G^2(\theta), \quad G_{,\theta} \text{ not identically zero} \]

  where $m$, $a$, $\ell$, $\alpha$ are arbitrary constants which, without loss of generality, can be chosen to satisfy $\alpha = 0$ if $\sigma = 0$ and $\ell = 0$ if $\{\sigma = 0, a \neq 0\}$. Moreover, $\delta = 1$ whenever $\{\sigma = 1, \alpha = 0\}$ or $\{\sigma = 0, \alpha = 0\}$.

- **Type D vacuum Kundt.** Away from fixed points of $\xi$, there exists coordinates $(t, u, r, x)$, where $\xi = \partial_u$ and the metric is

  \[ ds^2 = (r^2 + \ell^2)(-dt^2 + \Sigma_{\ell}^2\,dx^2) + \frac{1}{W} dr^2 + W (dt + 2\ell \Sigma_{\ell}^2(t)\,dx)^2, \]

  \[ W = \frac{\sigma(r^2 - \ell^2) - 2mr}{r^2 + \ell^2}, \quad \ell, m \in \mathbb{R}, \quad \Sigma_{\ell m} = \sigma \Sigma_{\ell \ell}, \]

  where $\sigma = \{+1, 0, -1\}$.

  **Remark:** The name ‘gen-Kerr–NUT spacetime’ is motivated by the fact that the two-dimensional space at constant $r$ and $u$ is conformal to a Riemannian surface of constant curvature $\sigma$. When $\sigma = 1$, this class is the standard Kerr–NUT spacetime with mass $m$, specific angular momentum $a$ and NUT charge $\ell$. The cases with $\sigma = 0$ (resp. $\sigma = -1$) are
analogous in the plane (resp. hyperbolic) cases. This notation follows the one used in figure 16.2 in [12].

**Proof.** Setting $\Lambda = 0$ in theorem 4 in [27] we find that case (A) is impossible. The local metric in case (B.i) is (away from fixed points of Killing vectors of $(\mathcal{M}, g)$)

$$\text{dx}^2 = -N (dv - Z^2 dx)^2 + 2 (dy + Vdx)(dv - Z^2 dx) + (y^2 + Z^2) \left( \frac{dz^2}{V} + V dx^2 \right). \quad (76)$$

$$\xi = \partial_v, \quad N = c - \frac{b_1 v + b_2 Z}{y^2 + Z^2}, \quad V = k + b_2 Z - cZ^2, \quad (77)$$

where $k, b_1, b_2, c$ are constants with the property that $V(Z) > 0$ in some non-empty interval where $Z$ takes values. Define $s > 0$ and $m$ by

$$c = \sigma s^2, \quad b_1 = 2ms^3$$

when $c = 0$, $s$ can be fixed to any non-zero value) and introduce three constants $\ell, \alpha, a$, with $a > 0$ as a solution of the under-determined system

$$\begin{align*}
 b_2 &= 2s^3(\sigma \ell + a\alpha) \\
 k &= s^4(-\sigma\ell^2 - 2a\alpha \ell + \delta a^2)
\end{align*} \quad (78)$$

To show that this system is compatible note that

- when $\sigma = \pm 1$ we can choose $\{2\ell = \sigma b_2 s^{-3}, \alpha = 0\}$ providing $4\delta a^2 = 4ks^{-4} + \sigma b_2^2 s^{-2}$ (and $a > 0$ arbitrary if, in addition, $4ks^{-4} + \sigma b_2^2 s^{-2} = 0$ requiring $\delta = 0$),
- and when $\sigma = 0$ we can choose $\{\ell = 0, 2\alpha = b_2 a^{-1} s^{-3}\}$ where $a$ is given by $\delta a^2 = ks^{-4}$ (or $a > 0$ arbitrary if $k = 0$ which requires $\delta = 0$).

Nevertheless all the expressions below hold for any solution of (78). Consider the coordinate change $Z = s(aG(\theta) + \ell)$ in terms of which the polynomial $V(Z)$ becomes

$$V(Z(\theta)) = s^4 a^2 (\delta + 2\alpha G(\theta) - \sigma G^2(\theta)) > 0$$

(this shows in particular that $\delta = 1$ if $\{\sigma = 1, \alpha = 0\}$ or if $\{\sigma = 0, \alpha = 0\}$, as claimed) and fix $G(\theta)$ by requiring that $dZ^2/V = s^{-2} d\theta^2$, that is to say, as any solution of the ODE

$$G_{,\theta}^2 = \delta + 2\alpha G(\theta) - \sigma G^2(\theta), \quad G_{,\theta} \text{ not identically zero}. \quad (79)$$

The condition $G(\theta)$ not constant is necessary for $Z(\theta)$ to define a coordinate change. Note that, in these circumstances $G(\theta)$ also solves

$$G_{,\theta\theta} = \alpha - \sigma G(\theta). \quad (80)$$

With the additional coordinate changes

$$y = sr, \quad v = \frac{1}{s} \left( u - \frac{\ell^2}{\alpha} \phi \right), \quad x = -\frac{1}{\alpha s^3} \phi$$

a straightforward calculation brings the metric (76) into the form (74) (note that in this process the Killing vector field $\xi$ needs to be rescaled appropriately).

Concerning case (B.ii) with $\epsilon = 1$ in theorem 4 of [27], the spacetime metric (away from fixed points of $\xi$) is
where $h_+$ is a positive definite metric of constant curvature $\kappa$ and volume form $\eta_+$ and $\kappa, \beta, n$ are real constants such that $W(y)$ is positive in some interval where $y$ takes values. Define $s > 0$, $\ell$ and $m$ by

\[
\kappa = \sigma s^2, \quad \beta = -\ell s, \quad m = -2ms^3.
\]

Being $h_+$ of constant curvature $\sigma s^2$ it can be written in local form as

\[
h_+ = \frac{1}{s^2}(d\theta^2 + G_{\varphi}\varphi^2),
\]

where $G(\theta)$ is a non-constant function satisfying the equation $G_{,\theta\theta} = -\sigma G_{,\theta}$. In fact, we can assume without loss of generality that $G(\theta)$ satisfies (79) (and hence also (80)). Equation (82) for $\hat{w}$ can be integrated explicitly as $\hat{w} = -2s^{-1}G(\theta)d\varphi + df$, where $f$ is any function of $\{\theta, \varphi\}$. Performing the coordinate change

\[
y = sr, \quad v = \frac{1}{s}u + f,
\]

the metric (81) becomes

\[
ds^2 = \frac{\sigma(r^2 - \ell^2)}{r^2} - 2mrdu + 2(2G(\theta)d\varphi)^2 + 2dr(du + 2G(\theta)d\varphi)
\]

\[+(r^2 + \ell^2)(d\theta^2 + G_{\varphi}\varphi^2),
\]

which corresponds to the gen-Kerr–NUT class with vanishing $a$.

Finally, case (B.ii) with $\epsilon = -1$ in theorem 4 of [27] corresponds to the metric

\[
ds^2 = W^{-1}dy^2 + W(dv - \hat{w})^2 + (\beta^2 + y^2)h_-
\]

\[
W = (\beta^2 + y^2)^{-1}( -\kappa(\beta^2 - y^2) + my), \quad \hat{w} = 2\beta\eta_-, \quad \xi = \partial_y,
\]

where $h_-$ is a Lorentzian metric of constant curvature $\kappa$ and volume form $\eta_-$. and, as before, $\kappa, \beta, n$ are real constants such that $W(y)$ is positive in some interval where $y$ takes values. We proceed similarly. Define $s$ by $\kappa = \sigma s^2$ and choose local coordinates $\{t, x\}$ so that

\[
h_- = s^{-2}(-dt^2 + \Sigma_{ij}(t)dx^2),
\]

The condition that this metric has constant curvature $\sigma s^2$ is $\Sigma_{ij} = \sigma \Sigma_{ij}$. Define $\ell, m$ by $\beta = -s\ell$ and $n = -2ms^3$. The equation for $\hat{w}$ can be integrated as $\hat{w} = -2s^{-1}Hdx + f(t, x)$. The coordinate change $\{v = s^{-1}u + df, y = sr\}$ brings the metric into the from (75).

**Remark.** As shown in lemma 4 in [20], spacetimes satisfying the hypotheses of theorem 2 admit an exact Ernst one-form $\chi$ associated to $\xi$, which defines the Ernst potential $\chi$ by $\chi = d\chi$. Moreover, selecting (partially) the free additive constant in $\chi$ so that $\text{Re}(\chi) = -g(\xi, \xi)$, there exist complex constants $c$ and $A$ such that

\[
Q = \frac{-6}{c - \chi}, \quad F^2 = A(c - \chi)^2.
\]

The constants $c$ and $A$ play an important role in characterizing locally the Kerr metric among vacuum spacetimes admitting a Killing vector $\xi$ such that the alignment condition (23) holds and $F^2 \equiv 0$ somewhere. Such a purely local characterization was first given in theorem 1.
in [21], where it was claimed that the Kerr spacetime can be characterized by the condition \( \text{Re}(c) > 0 \) and \( A \) real and negative. Unfortunately, theorem 1 in [21] is incorrect as stated because, as we show below, the type D vacuum Kundt spacetime also admits a particular case with \( \text{Re}(c) > 0 \) and \( A < 0 \). The reason why this possibility was missed in theorem 1 in [21] was that the arguments in [20] (on which the validity of theorem 1 was based) made the implicit assumption that the Killing vector \( \xi \) is not everywhere orthogonal to the plane generated by the two real eigenvectors \( \{l, k\} \) of the self-dual Killing form \( \mathcal{F}_{\alpha\beta} \) (this condition is automatically satisfied if \( \xi \) is timelike somewhere, which was the case of interest in [20]).

This assumption needs to be added to the conditions on \( c \) and \( A \) in the local characterization of Kerr in order to make the result correct. We provide here a corrected version of the theorem and give its proof.

**Theorem 3** (Corrected local characterization of the Kerr metric). Let \( (\mathcal{M}, g) \) be a vacuum spacetime admitting a Killing vector \( \xi \) such that the alignment condition (23) holds with \( \mathcal{F}^2 \neq 0 \) on at least one point. Then there exist complex constants \( c, A \) such that

\[
Q = -\frac{6}{c - \chi}, \quad \mathcal{F}^2 = A(c - \chi)^4,
\]

where \( \chi \) is the Ernst potential (i.e., a function satisfying \( \nabla_\beta \chi = 2\xi^\alpha \mathcal{F}_{\alpha\beta} \) and which necessarily exists globally on \( (\mathcal{M}, g) \)). Fix partially the free additive constant in \( \chi \) so that \( \text{Re}(\chi) = -g(\xi, \xi) \). Then \( (\mathcal{M}, g) \) is locally isometric to a Kerr spacetime if and only if the following two conditions are satisfied

i. \( \text{Re}(c) > 0 \) and \( A \) is real and negative.

ii. There is at least one point \( q \in \mathcal{M} \) where \( \mathcal{F}^2|_q \neq 0 \) such that \( \xi|_q \) is not orthogonal to the 2-plane generated by the two real null eigenvectors \( \{l_q, k_q\} \) of \( \mathcal{F}_{\alpha\beta}|_q \).

**Proof.** As discussed above, the ‘if’ part follows from the arguments in [20]. For the ‘only if’ part we could simply compute the vectors \( \{k, l\} \) and the constants \( c \) and \( A \) in the Kerr spacetime and show that both (i) and (ii) hold. However, for completeness we compute these objects for the full class of spacetimes contained in theorem 2 and give a direct and independent proof of the theorem.

For the gen-Kerr–NUT spacetime (74) with the given choice of \( \xi = \partial_u \), a direct calculation shows that the Ernst potential satisfying \( \text{Re}(\chi) = -g(\xi, \xi) \) is

\[
\chi = \frac{\Delta(r) + \Theta(\theta) + 2[(a\alpha + \sigma\ell)r - m(a\Theta(\theta) + \ell)]}{\rho^2} + i\omega_0,
\]

where \( \omega_0 \in \mathbb{R} \) is any constant. Moreover, \( c \) and \( A \) are given by

\[
c = \sigma + i\omega_0, \quad A = \frac{-1}{4(m - i(a\alpha + \sigma\ell))^2}.
\]

The null eigendirection of \( \mathcal{F}_{\alpha\beta} \) with eigenvalue \( (c - \chi)^2/[4(m - i(a\alpha + \sigma\ell))] \) is generated by \( k = \partial_u \). Hence \( g(\xi, k) = g(\partial_u, \partial_r) = 1 \) and \( \xi \) is not orthogonal to the 2-plane generated by \( \{l, k\} \).

For the type D Kundt metric, the Ernst potential, the complex constants \( c \) and \( A \), and the real null eigenvectors \( \{l, k\} \) of \( \mathcal{F}_{\alpha\beta} \) are
\[
\chi = -\frac{\sigma (r^2 - \ell^2) + 2mr + 2i(\sigma - m)}{r^2 + \ell^2} + i\omega_0,
\]
\[c = -\sigma + i\omega_0, \quad A = \frac{1}{4(m + i\sigma \ell)^2},\]
\[l = dt + \Sigma dx \quad \text{with eigenvalue} \quad -\frac{i(\sigma + \chi)^2}{4(m + i\sigma \ell)},\]
\[k = dt - \Sigma dx \quad \text{with eigenvalue} \quad \frac{i(\sigma + \chi)^2}{4(m + i\sigma \ell)}.
\]

The eigenvectors are given in terms of their metrically associated one-forms \{l, k\} so that it is obvious that \(\xi = \partial_\nu\) is everywhere orthogonal to the two-plane generated by \{l, k\}.

The Kerr spacetime corresponds precisely to the subclass with \(\sigma = +1\) and NUT parameter \(\ell = 0\) of the gen-Kerr–NUT class. As \(\sigma = 1\) from theorem 2 we know that we can set \(\alpha = 0\) without loss of generality and then \(A\) is real and negative. However, the subclass with \(\{m = 0, \ell = 0, \sigma = -1\}\) in the type D vacuum Kundt class also has the properties that Re\(c\) > 0 and \(A < 0\) and hence satisfies hypothesis (i) in the theorem. From the properties of the null eigenvectors \{l, k\} of these spacetimes it is clear that item (ii) selects uniquely the Kerr class, as claimed.

\[\Box\]

**Table 1.** The column \(\mathcal{F}^2 = 0\) is the content of theorem 1 and we have emphasized that (a) AdS is included in the Siklos class and (b) that the vacuum plane waves and the vacuum stationary Brinkmann classes are not disjoint by including (in parenthesis) their intersection, namely the irreducible locally symmetric vacuum spacetimes.

| \(\mathcal{F}^2 = 0\) | \(\mathcal{F}^2 \neq 0\) |
|-----------------|-----------------|
| \(\Lambda \neq 0\) | \(\Lambda > 0\) | \(\Lambda < 0\) | Siklos (AdS). |
| \(\Lambda = 0\)  | — Vacuum plane waves. |
| —— Vacuum plane waves. | — Pure LRS (Schwarzschild–(A) de Sitter, Taub–NUT–(A) de Sitter, ...) |
| —— Taub–NUT–(A) de Sitter, Siklos (AdS). |
| —— Type D Kundt. |
| —— Product metrics (uncharged Bertotti–Robinson, Nariai). |
| —— dS and AdS. |

The cell \(\mathcal{F} \neq 0, \Lambda = 0\) is the content of theorem 2. The cell \(\mathcal{F} = 0, \Lambda = 0\) was proven in theorem 4 in [27]. The Plebański case corresponds to case (B.i) in that theorem, while its case (B.ii) with \(\epsilon = 1\) corresponds to pure LRS metrics: these are metrics admitting a four-dimensional local isometry group acting on three-dimensional hypersurfaces. These spacetimes were classified by Cahen and Defrise [5] and contain many special subcases, some of which are explicitly listed in parenthesis. The type D Kundt metrics correspond to case (B.ii) with \(\epsilon = -1\) in theorem 4 of [27], and they contain both cases with \(\Lambda\) vanishing or not. The Kerr–NUT–(A) de Sitter spacetime belongs to the Plebański class as long as the rotation parameter \(a\) is non-zero. The case of vanishing \(a\) belongs to the pure LRS case. Finally the 'product metrics' correspond to case (A) in [27] for which the spacetime metric is locally the product of a Lorentzian two-dimensional metric times a Riemannian two-dimensional metric, both of them of constant curvature \(\Lambda\). As a final observation, it is noteworthy that de Sitter, Minkowski and anti-de Sitter spacetimes are included here, but while AdS and Minkowski have Killing vector fields with \(\mathcal{F}^2\) both vanishing or not, dS does not admit any Killing vector with null Killing 2-form.
For completeness, we write down the second null eigendirection of $\mathcal{F}_{\alpha\beta}$. It has eigenvalue $-(c - \chi)^2/(4(m - i(\alpha + \sigma\ell)))$ and it is generated by

$$l = \partial_\alpha + \frac{a^2\delta + \Delta(r)}{2(r^2 + \ell^2)} \partial_\ell + \frac{a}{r^2 + \ell^2} \partial_r.$$ 

We conclude the paper with table 1 which summarizes the results in this paper and those in [20, 21, 27].

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