The Midpoint Upwind Scheme on the Bakhvalov-Shishkin Mesh for Parabolic Singularly Perturbed Problems

Quan Zheng\textsuperscript{1,a}, Yi-Yang Wang\textsuperscript{2}

\textsuperscript{1}College of Science, North China University of Technology, Beijing, China
\textsuperscript{2}College of Science, North China University of Technology, Beijing, China
\textsuperscript{a>Email: zhengq@ncut.edu.cn}

Abstract: In this paper, the parabolic singularly perturbed convection-diffusion problem is discretized by using the backward Euler method in time and the midpoint upwind scheme on the Bakhvalov-Shishkin mesh in space. This method is shown to be first-order convergent in time and space. Finally numerical experiments confirm the theoretical results.

1. Introduction
Consider the parabolic singularly perturbed convection-diffusion problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - \varepsilon \Delta u &= f(x,t), (x,t) \in \Omega, \\
u(x,0) &= s(x), 0 \leq x \leq 1, \\
u(0,t) &= u(1,t) = 0, 0 < t \leq T,
\end{align*}
\]
where $L\varepsilon u(x,t) = -\varepsilon u_{xx}(x,t) + \alpha(x) u_x(x,t) + b(x,t) u(x,t)$, $\varepsilon$ is a positive parameter, $0 < \varepsilon << 1$, $\alpha(x) > 0$ and $b=b(x,t) \geq 0$ on $\overline{\Omega}$.

In the case of $b(x,t)=b(x)$, many authors discussed the finite difference methods to solve the singularly perturbed parabolic convection-diffusion problem (see [1-3]). In time direction, these papers used the backward Euler method, and in space direction, they used the simple upwind scheme, the midpoint upwind scheme and the hybrid numerical method respectively on the Shishkin mesh. In the case of $b=b(x,t)$, Clavero, Gracia and Stynes[4] discretized the time variable by using the backward Euler method, and the spatial variable by using the mixed difference scheme. Comparing [5] and [6], the convergence effect on Bakhvalov-Shishkin mesh is better than that on S mesh in the two-point BVP. Therefore, this paper uses the midpoint upwind scheme to discretize the spatial variable on the B-S mesh.

When all data of (1) satisfies some corner compatibility conditions[7], (1) has a unique solution with a boundary layer at $x = 1$ (see [7]). [4] proved that when the data of the problem (1) satisfies the third-order compatibility condition, the solution of (1) satisfies the bound
\[
\left| \partial^k \partial_t u(x,t) / \partial x^m \right| \leq C(1 + e^{-k e^{-a(1-x)/\varepsilon}}), \text{ for } (x,t) \in \Omega \text{ and } k + 2m \leq 6.
\]

2. The numerical method
2.1. Mesh generation in time and space
The interval $[0, T]$ is divided into $M$ subintervals, set $t_j = jT/M$ and $\tau = t_j - t_{j-1}$ for $j = 0, 1, \ldots, M$. 

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI. Published under licence by IOP Publishing Ltd.
Choose 1-\(\sigma\) as a turning point, and divide the space interval [0,1] into intervals [0,1-\(\sigma\)] and [1-\(\sigma\),1]. Here, \(\sigma = \text{min}\{1/2,(4\varepsilon/\alpha)\ln N\}\), and suppose that \((4\varepsilon/\alpha)\ln N \leq 1/2\). The B-S mesh is

\[
x_i = \begin{cases} 
(2(1-\sigma)/N)i, & 0 \leq i \leq N/2, \\
1 + 4\varepsilon/\alpha \ln[1-2(1-1/N)(1-i/N)], & N/2 \leq i \leq N.
\end{cases}
\]

(3)

Lemma 2.1. ([4]) \(h_i = x_i - x_{i-1} = 2(1-\sigma)/N \in [N^{-1},2N^{-1}]\) and \(h_{N/2+i} < 8\varepsilon/\alpha \leq CN^{-1}, \ i = 1,2,\ldots, N/2\).

\[
\frac{\partial_i U_i}{\partial x_i} + L_{x,x}^{M,N} U_i = f_i^{1/2}, \quad 0 < i < N, 0 \leq j < M, \\
U_i^0 = \eta(x_i), \ i = 0,1,\ldots,N, \\
U_0^j = U_0^j, \ j = 0, 1,\ldots,M,
\]

(4)

where

\[
\tilde{\delta}_i U_i^j = (\delta_i U_i^j + \delta U_{i-1}^j)/2, \quad \delta_i U_i^j = (U_i^{j+1} - U_i^{j-1})/\tau, \quad f_i^{(1/2)} = f(x_{i-1/2},t_j), \quad D_x U_i^j = (U_i^{j+1} - U_i^{j-1})/h_i, \\
\delta_i^2 U_i^j = 2(D_x^2 U_i^j - D_x U_i^{j+1/2})/(h_{i+1} + h_i), \quad h_{i+1} = h(x_{i+1/2},t_j).
\]

That is

\[
L_{x,x}^{M,N} U_i^j = \left[-\frac{2\varepsilon}{(h_{i+1} + h_i)h_i} - \frac{a_{i+1/2} + b_{i+1/2}^j}{2}U_i^j + \frac{2\varepsilon}{(h_{i+1} + h_i)h_i} + \frac{2\varepsilon}{h_i} + \frac{a_{i+1/2} + b_{i+1/2}^j}{2}U_i^j - \frac{2\varepsilon}{(h_{i+1} + h_i)h_i}U_i^{j+1/2}\right]U_i^j.
\]

(5)

3. Error estimate

Lemma 3.1. Assuming that the fully discrete scheme (3) satisfy

\[
-2\varepsilon(h_{i+1} + h_i)h_i - a_{i+1/2}/h_i + b_{i+1/2}^j/2 \leq 0.
\]

Then, the matrix associated with (3) is an \(M\)-matrix for all time level \(t_j\).

Proof.

\[
\frac{\tilde{\delta}_i U_i^j + L_{x,x}^{M,N} U_i^j}{\delta_i U_i^j + \delta U_{i-1}^j} = \left[-\frac{2\varepsilon}{(h_{i+1} + h_i)h_i} - \frac{a_{i+1/2} + b_{i+1/2}^j}{2}U_i^j + \frac{2\varepsilon}{(h_{i+1} + h_i)h_i} + \frac{2\varepsilon}{h_i} + \frac{a_{i+1/2} + b_{i+1/2}^j}{2}U_i^j - \frac{2\varepsilon}{(h_{i+1} + h_i)h_i}U_i^{j+1/2}\right]U_i^j.
\]

(6)

Decompose \(\gamma_i = \gamma_i^\mu + \mu_i\), it is the truncation error of (1). Define \(\lambda_i^\mu = L_{x,x}^{M,N} u_i^\mu - L_{x,x}^{M,N} u_i^\mu\), \(\mu_i = \tilde{\delta}_i u_i^\mu - (u_i)^\mu\), then the truncation error of the scheme is

\[
[\tilde{\delta}_i \gamma^\mu + L_{x,x}^{M,N} \gamma^\mu] = \lambda_i^\mu + \mu_i.
\]

(7)

For \(\omega_i^\mu\), we have a parabolic problem:
Lemma 3.2. Under the condition (5), \( \nu_j^i \) satisfies
\[
\left| \nu_j^i \right| \leq CN_j N^{-1}.
\] (9)

\( \nu_j^i \) is the truncation error of a convection-diffusion two-point BVP, and (9) has been solved (see [5]).

Lemma 3.3. Assuming that (5) is valid, then there exists a constant \( C \) such that
\[
\max_i \left| 1 - \| \omega_i^j \|_{\infty, d} \right| \leq C(\tau + N^{-1}).
\] (10)

Proof. According to Lemma 2.1, problem (8) satisfies a discrete maximum principle, then
\[
\max_i \left| 1 - \| \omega_i^j \|_{\infty, d} \right| \leq C(\tau + N^{-1}) \|
\] (11)

Since \( a(x) \) is only related to \( x \), we calculate (7) and obtain \( \delta_i \nu \) satisfying
\[
\begin{cases}
L^{i,N}_{x} (\delta_i \nu)^j_{i} = \delta_i \lambda_i^j - (\delta_i b) \nu^{i-1}_j, & i = 1, ..., N - 1, \\
(\delta_i \nu)^j_{0} = (\delta_i \nu)^j_{N} = 0.
\end{cases}
\] (12)

Decompose \( \delta_i \nu \) as \( \delta_i \nu = \phi + \psi \), where for each fixed \( j \) one has
\[
\begin{cases}
L^{i,N}_{x} \phi^j = \delta_i \lambda_i^j, & i = 1, ..., N - 1, \\
\phi^0 = \phi^N = 0,
\end{cases}
\] (13)

\[
\begin{cases}
L^{i,N}_{x} \psi^j = -(\delta_i b) \nu^{i-1}_j, & i = 1, ..., N - 1, \\
\psi^0 = \psi^N = 0.
\end{cases}
\] (14)

Calculating \( \delta_i \lambda_i^j \), then
\[
\delta_i \lambda_i^j = (1/2\tau)(\lambda_i^{i+1} - \lambda_i^{i-1}) + (1/2\tau)(\lambda_i^i - \lambda_i^{i-1}) = \left(1/2\tau\right)[L^{i,N}_{x} u^i_{i+1} - L^{i,N}_{x} u^i_{i-1}] - (L u^i_{i+1} - L u^i_{i-1}) \\
+ (1/2\tau)[L^{i,N}_{x} u^i_{i} - L^{i,N}_{x} u^i_{i-1}] - (L u^i_{i} + L u^i_{i-1})].
\]

Set \( \tilde{L} u = -c a u_x + a u_x + a_{i-1/2} D_x U_i^j \), then
\[
\delta_i \lambda_i^j = (1/2\tau) \int_{x_{i-1}}^{x_i} \left[ L^{i,N}_{x} u(x_i, t) + \tilde{L}^{i,N}_{x} u_i(x_i, t) \right] - (\tilde{L} u(x_i, t) + \tilde{L} u_i(x_i, t)) dt.
\] (15)

Using the Peano kernel theorem as in [[9], Lemma 3.3.1], one obtains
\[
| \delta_i \lambda_i^j | \leq C \max_{x_{i-1, x_{i+1}}} \left| u(x, t) \right| dx + C h \max_{x_{i-1, x_{i+1}}} \left| \sum_{x_{i-1, x_{i+1}}} \left| u(x, t) \right| + \left| u_{xxx} \right| \right) dx.
\] (16)

[5] is used to calculate the above equation, and then the corresponding truncation error for a standard two-point BVP is generated. Then, (13) can be analyzed the same as (12). Therefore, we have
\[
| \phi^j_i | \leq CN_j N^{-1}.
\] (17)

For (14), which satisfies a discrete maximum principle since the related matrix is an M-matrix. So
\[
\max_i \psi_i \leq \max_j \left( (\tilde{\alpha}_j, b) \psi^{j-1} \right) \leq C \max_j \psi^{j-1} \leq CN^{j-1}. \quad j = 0, 1, \ldots, M
\]  

(18)

(10) can be obtained from (9), (11) and (18).

Theorem 1. If (1) satisfies (5), then there is a constant \( C \) such that

\[
\max_{i,j} |u(x_i, t_j) - U^j| \leq C(M^{-1} + N^{-1}).
\]  

(19)

Proof. By the Lemma 3.2 and Lemma 3.3, we have

\[
|u(x_i, t_j) - U^j| \leq \lambda_j |\mu^j| \leq C(M^{-1} + N^{-1}).
\]

4. Numerical Results

Consider the singularly perturbed problem 1

\[
\begin{cases}
-\varepsilon u_{xx} + u_t + u = e^{-\varepsilon (-\pi^2 \varepsilon / 4 \cos(\pi / 2)x + (\pi / 2) \cos(\pi / 2)x)}, (x, t) \in (0,1) \times (0,2), \\
u(x,0) = \nu_0(x) = 1 - e^{-(\varepsilon^2 / \varepsilon / 1 - e^{1/\varepsilon} - \cos(\pi / 2)x), 0 \leq x \leq 1,} \\
u(0,t) = \nu(1,t) = 0, 0 \leq t \leq 1,
\end{cases}
\]

where the exact solution is

\[
u(x, t) = e^{-\varepsilon (1 - e^{-(\varepsilon^2 / \varepsilon / 1 - e^{1/\varepsilon} - \cos(\pi / 2)x})}.\

The maximum errors, the numerical convergence orders in the time direction and the numerical convergence orders in the spatial direction respectively are

\[
e^{M,N} = \max_{i,j} |u(x_i, t_j) - U^j|, \quad order_T = \log_2(e^{M,N} / e^{M,N}_{L}), \quad order_S = \log_2(e^{M,N} / e^{M,N}_{L}).
\]

Table 1. The numerical results in problem 1 when \( N=1536. \)

| \( M \) | \( e^{M,N} \) | \( order_T \) |
|---|---|---|
| 12 | 4.843e-02 | 0.4292 |
| 24 | 3.597e-02 | 0.5158 |
| 48 | 2.516e-02 | 0.5941 |
| 96 | 1.667e-02 | 0.6921 |
| 192 | 1.032e-02 | 0.8588 |
| 384 | 5.688e-03 | 1.2669 |

Table 2. The numerical results in problem 1 when \( M=1536. \)

| \( N \) | \( e^{M,N} \) | \( order_S \) |
|---|---|---|
| 10 | 6.735e-02 | 0.7424 |
| 20 | 4.026e-02 | 0.8768 |
| 40 | 2.192e-02 | 1.0061 |
| 80 | 1.092e-02 | 1.1502 |
| 160 | 4.918e-03 | 1.3266 |
| 320 | 1.961e-03 | 1.6416 |

Table 1 shows that the numerical results in problem 1 in time is first-order convergent and Table 2 shows that the numerical results in problem 1 in space is first-order convergent, which verify Theorem 1. It can be seen from Table 3 that the midpoint upwind scheme of B-S mesh for solving the singularly perturbed parabolic problem is more adaptable to the exponential boundary layer than S mesh. It can be seen from Table 4 that the midpoint upwind scheme achieves better convergence effect than the simple upwind scheme.
Table 3. The numerical results in problem 1 when $M=1536$.

| $N$ | $e^{MN}$ | order | $B - S$ mesh $e^{MN}$ | order |
|-----|----------|-------|------------------------|-------|
| 10  | 7.818e-02 | 0.4236 | 6.735e-02 | 0.7424 |
| 20  | 5.829e-02 | 0.6027 | 4.026e-02 | 0.8768 |
| 40  | 3.838e-02 | 0.7123 | 2.192e-02 | 1.0061 |
| 80  | 2.343e-02 | 0.8632 | 1.092e-02 | 1.1502 |
| 160 | 1.288e-02 | 1.0580 | 4.918e-03 | 1.3266 |
| 320 | 6.185e-03 | 1.4594 | 1.961e-03 | 1.6416 |

Table 4. The numerical results in problem 1 when $M=1536$.

| $N$ | Simple $e^{MN}$ | order | Midpoint $e^{MN}$ | order |
|-----|-----------------|-------|------------------|-------|
| 10  | 1.225e-01 | 0.6015 | 6.735e-02 | 0.7424 |
| 20  | 8.071e-02 | 0.6714 | 4.026e-02 | 0.8768 |
| 40  | 5.068e-02 | 0.7280 | 2.192e-02 | 1.0061 |
| 80  | 3.060e-02 | 0.8027 | 1.092e-02 | 1.1502 |
| 160 | 1.754e-02 | 0.9539 | 4.918e-03 | 1.3266 |
| 320 | 9.055e-03 | 1.3543 | 1.961e-03 | 1.6416 |

5. Conclusion
In this paper, the backward Euler method is used in the time direction and the midpoint upwind method is used in the space direction on the B-S mesh. This method is shown to be first order convergent in time and space.

Acknowledgment
The authors thank the support of Natural Natural Science Foundation of China (No. 11471019).

References
[1] Clavero, C., Jorge, J.C., Lisbona, F. (2003) A uniformly convergent scheme on a nonuniform mesh for convection-diffusion parabolic problems. J. Comput. Appl. Math., 154, 415–429.
[2] Mohan, K., Kadalbajoo, A.A. (2011) The midpoint upwind finite difference scheme for time-dependent singularly perturbed convection-diffusion equations on non-uniform mesh. International Journal for Computational Methods in Engineering Science and Mechanics, 12, 150-159.
[3] Mukherjee, K., Natesan, S. (2009) Parameter-uniform hybrid numerical scheme for time-dependent convection-dominated initial-boundary-value problems. Computing, 84, 209-230.
[4] Clavero, C., Gracia, J.L., Stynes, M. (2011) A simpler analysis of a hybrid numerical method for time-dependent convection-diffusion problems. J. Comput. Appl. Math., 235, 5240-5248.
[5] Zheng, Q., Feng, X.-L., Li, X.-Z. (2014) $\varepsilon$-uniform convergence of the midpoint upwind scheme on the Bakhvalov-Shishkin mesh for singularly perturbed problems. J. Comput. Anal. Appl., 17(1), 40-47.
[6] Stynes, M., Roos, H.G. (1997) The midpoint upwind scheme, Appl. Numer. Math., 23, 361-374.
[7] Roos, H.G., Stynes, M., Tobiska, L. (2008) Robust Numerical Methods for Singularly Perturbed Differential Equations (2nd ed.). Berlin Heidelberg: Springer-Verlag.
[8] Kellogg, R.B., Tsan, A. (1978) Analysis of some difference approximations for a singular perturbation problem without turning points, Math. Comput., 32,1025-1039.