Janet Bases of Toric Ideals

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Abstract

In this paper we present a version of the general polynomial involutive algorithm for computing Janet bases specialized to toric ideals. The relevant data structures are Janet trees which provide a very fast search for a Janet divisor. We broach also efficiency issues in view of application of the algorithm presented to computation of toric ideals.

1 Introduction

We consider the problem of computing a Janet basis of a toric ideal \( I_A \) in \( \mathbb{K}[x] \equiv \mathbb{K}[x_1, \ldots, x_n] \) generated by binomials of the form [1]

\[
I_A = \{ \ x^u - x^v \ | \ u, v \in \mathbb{N}^n, \ \pi(u) = \pi(v), \ \gcd(x^u, x^v) = 1 \ \}
\]

Here \( u, v \in \mathbb{N}^n \) and \( \pi \) is the semigroup homomorphism

\[
\pi : \mathbb{N}^n \rightarrow \mathbb{Z}^d, \quad u = \{u_1, \ldots, u_n\} \rightarrow u_1a_1 + \cdots + u_na_n
\]

where \( a_i \in \mathbb{Z}^d \ (1 \leq i \leq n) \).

Given a set of binomials generating a toric ideals, the problem of constructing its Gröbner basis is usually (except small problems) rather expensive from the computational point of view [2]. In practice, for this particular problem, one typically deals with a large number \( n \) of variables and their degrees. If \( d \) is the maximal degree of the initial binomials, then the degree of a reduced Gröbner basis is bounded by [3]

\[
2 \cdot \left( \frac{d^2}{2} + d \right)^{2^{n-1}}.
\]
But for all that the reduced Gröbner basis is also binomial since the binomial structure is preserved during the Buchberger algorithm \([4, 5]\). Similarly, the involutive algorithms \([6]\) based on the sequential multiplicative reductions of nonmultiplicative prolongations of the intermediate polynomials preserve the binomial structure. The output involutive basis which is also a Gröbner basis though generally redundant.

Thus, unlike construction of reduced Gröbner bases or involutive bases for general polynomial ideals, the integer arithmetics which may take most of computing time is not important for binomial ideals. In this case a fast search of monomial divisors for performing reductions of \(S\)-polynomials may become crucial in acceleration of computations.

Recently \([7, 8]\) we designed and implemented involutive algorithms specialized to constructing Janet bases of monomial and polynomial ideals. Janet division as well as any other involutive division \([6]\) provides uniqueness of an involutive divisor in a polynomial set with co-prime leading monomials. This allows one to organize a very fast search for a Janet divisor using special data structures for intermediate polynomial sets called Janet trees.

The main goal of this paper is to discuss the issue of practical efficiency in computing Janet bases of toric ideals based on the use of Janet trees. Since one of the most important applications of toric ideals is integer programming we shortly describe this application \([9]\) in the next section.

## 2 Toric Ideals and Integer Programming

Let \(\mathcal{A}\) be a matrix of dimension \(m \times n\) with integer entries and \(b \in \mathbb{Z}^m\), \(c \in \mathbb{Z}^n\) be vectors. The following optimization problem

\[
\min \{ \, c^T \mathbf{x} \mid \mathbf{x} \in \mathbb{N}^n, \, \mathcal{A} \mathbf{x} = \mathbf{b} \, \}
\]

is called a problem of integer programming.

We shall assume that \(c \in \mathbb{N}^n\). If there exists vector \(\mathbf{x}_0\) satisfying \(\mathcal{A} \mathbf{x}_0 = \mathbf{b}\), \(\mathbf{x}_0 \in \mathbb{N}^n\), then the problem of finding minimum of function \(c^T \mathbf{x}\) can be reduced to all kinds of transformation of the initial vector state \(\mathbf{x}_0\) using \(\ker(\mathcal{A})\).

The problem of determining \(\ker(\mathcal{A})\) can be formulated in terms of toric ideals. Indeed, every vector \(\mathbf{u} \in \ker(\mathcal{A})\) may be uniquely represented as \(\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-\) where both \(\mathbf{u}^+\) and \(\mathbf{u}^-\) are nonnegative and have disjoint support. Associate symbol \(v_i\) with the \(i\)-th column of matrix \(\mathcal{A}\). Then the ideal

\[
\mathcal{I}_{\mathcal{A}} = \{ \, \mathbf{v}^{\mathbf{u}^+} - \mathbf{v}^{\mathbf{u}^-} \mid \mathbf{u}^+ - \mathbf{u}^- = \mathbf{u} \in \ker(\mathcal{A}) \, \}
\]

associated with \(\ker(\mathcal{A})\) is toric. Given the initial solution \(\mathbf{x}_0\), the optimal solution can be found as follows \([9]\):

1. Construct a basis of the toric ideal \(\mathcal{I}_{\mathcal{A}}\).
2. Construct a reduced Gröbner basis or an involutive basis of \(\mathcal{I}_{\mathcal{A}}\) with respect to the admissible monomial ordering \(\succ_{\mathbf{c}}\) generated by vector \(\mathbf{c}\).
3. Reduce monomial \(\mathbf{v}^{\mathbf{x}_0}\) modulo the constructed basis to obtain the optimal solution.

Therefore, the reduced Gröbner basis or any involutive basis of the associated toric ideal \(\mathcal{I}_{\mathcal{A}}\) provide an algorithmic tool for solving the problem of integer programming.
3 Janet Bases of Toric Ideals

3.1 Definition of Janet Basis

In our papers [6] the Gröbner bases of special type, called involutive and based on the concept of involutive division were introduced. Given a set of coprime monomials and an involutive division, any monomial may have at most one involutive divisor in the set. This property of the involutive division allows one to design an efficient search for the involutive divisor using the method of separative monomials [10] for a general involutive division or Janet trees [7] for Janet division.

Because of a larger number of variables and unimportance of integer arithmetical operations over coefficients of the binomials, the practical complexity of an algorithm for construction of Gröbner or Janet bases is caused by an enormous number of binomials arising in computation of the basis. A faster search for divisors may accelerates the computation substantially.

By definition of Janet division [6] (which formalizes the pioneering ideas of Janet [11]) induced by the order

\[ x_1 \succ x_2 \succ \ldots \succ x_n \]  

on \( x \), a polynomial set \( F \) is partitioned into the groups labeled by non-negative integers \( d_1, \ldots, d_i \):

\[ [d_1, \ldots, d_i] = \{ f \in F \mid d_j = \deg_j(\text{lm}(f)), \ 1 \leq j \leq i \} \]

where \( \deg_i(u) \) denotes the degree of \( x_i \) in monomial \( u \) and \( \text{lm}(f) \) denotes the leading monomial of \( f \). A variable \( x_i \) is called (Janet) multiplicative for \( f \in F \) if \( i = 1 \) and

\[ \deg_1(\text{lm}(f)) = \max\{ \deg_1(\text{lm}(g)) \mid g \in F \}, \]

or if \( i > 1 \), \( f \in [d_1, \ldots, d_{i-1}] \) and

\[ \deg_i(\text{lm}(f)) = \max\{ \deg_i(\text{lm}(g)) \mid g \in [d_1, \ldots, d_{i-1}] \}. \]

If a variable is not multiplicative for \( f \in F \), it is called (Janet) nonmultiplicative for \( f \). In the latter case we shall write \( x_i \in NM_J(f, F) \). \( u \in \text{lm}(F) \) is a Janet divisor of \( w \in \mathbb{M} \), if \( u \mid w \) and monomial \( w/u \) contains only multiplicative variables for \( u \). In this case we write \( u \mid_J w \).

Let \( \text{lm}(F) = \{ \text{lm}(f) \mid f \in F \} \). Then a polynomial set \( F \) is called Janet autoreduced if each term in every \( f \in F \) has no Janet divisors among \( \text{lm}(F) \setminus \text{lm}(f) \). A polynomial \( h \) is said to be in the Janet normal form modulo \( F \) if every term in \( h \) has no Janet divisors in \( \text{lm}(F) \). In that follows \( NF_J(f, F) \) denotes the Janet normal form \( f \) modulo \( F \).

A Janet autoreduced set \( F \) is called a Janet basis if

\[ (\forall f \in F) \ (\forall x \in NM_J(f, F) \ [ NF_J(f \cdot x, F) = 0 ] \). \]  

A Janet basis \( G \) is called minimal if for any other Janet basis \( F \) of the same ideal the inclusion \( \text{lm}(G) \subseteq \text{lm}(F) \) holds. If both \( G \) and \( F \) are monic this inclusion implies \( G \subseteq F \). A Janet basis is a Gröbner one, though generally not reduced. However, similarly to a reduced Gröbner basis, a monic minimal Janet basis is uniquely defined by an ideal and a monomial order. In that follows we deal with minimal Janet bases only and omit the word ”minimal”.

3
3.2 Janet Trees and Search for Janet Divisor

Consider now a binary Janet tree \([7]\) whose structure reflects the above partition of elements in \(U\) into the groups which sorted in the degrees of variables within every group. Before description of the general structure of Janet trees we explain it in terms of the concrete example \([7]\)

\[
U = \{x^2y, xz, y^2, yz, z^2\}, \quad (x \succ y \succ z)
\]

and portray it in the form of Janet tree as shown below. In doing so, the monomials in set \(U\) are assigned to the leaves of the tree. The monomial with increased by one degree of the current variable is assigned to the left child whereas the right child points at the next variable with respect to chosen ordering. In contrast to Janet tree presented in paper \([7]\), the below tree takes into account sparseness of monomials that is inherent in toric ideals. The related information is given in pairs of integers placed in brackets where the first element represents the number of current variable and the second one represents its degree.

![Janet Tree](image)

Consider now the structure of Janet tree of the general form as a set \(JT := \cup\{\nu\}\) of internal nodes and leaves which corresponds to a nonempty binomial set. To every element \(\nu\) of the tree we shall assign the set of five elements \(\nu = \{v, d, nd, nv, nb\}\) with the following structure:

- \(\text{var}(\nu) = v\) is the index of the current variable
- \(\text{dg}(\nu) = d\) is the degree of the current variable
- \(\text{ndg}(\nu) = nd\) is the pointer to the next node in degree
- \(\text{nvr}(\nu) = nv\) is the pointer to the next node in variable
- \(\text{bnm}(\nu) = bn\) is the pointer to binomial

In the absence of a child we shall assign the value \(\text{nil}\) to the corresponding pointer. Wherever it does not lead to misunderstanding we shall identify the pointers \(nd\) and \(nv\) with the nodes they point out. To the root of \(JT\) we assign \(\nu_0\) with \(\text{var}(\nu_0) = 1\) in accordance with labeling \([1]\) and \(\text{dg}(\nu_0) = 0\).
The internal nodes and leaves of tree $JT$ are characterized by the states:

**Internal node:** $((nv \neq \text{nil} \land v < \text{var}(nv)) \lor (nd \neq \text{nil} \land d < \text{dg}(nd)) \land bn = \text{nil})$

**Leaf:** $nv = \text{nil} \land nd = \text{nil} \land bn \neq \text{nil} \land d = \text{dg(\text{lm}(bn))}$. 

For a fast search for Janet divisor in the given tree one can use the following algorithm **J-divisor** which is an adaptation to the above structure of Janet tree of the algorithm described in [7].

**Algorithm: J-divisor($JT$, $w$)**

```plaintext
Input: $JT$, a Janet tree; $w$, monomial
Output: $bn$, a binomial such that $\text{lm}(bn) \mid_J w$, or $\text{nil}$, otherwise
1: $\nu := \nu_0$
2: while $\text{deg}_{\text{var}(\nu)}(w) \geq \text{dg}(\nu)$ do
3: while $\text{ndg}(\nu)$ and $\text{deg}_{\text{var}(\text{ndg}(\nu))}(w) \geq \text{dg}(\text{ndg}(\nu))$ do
4: $\nu := \text{ndg}(\nu)$
5: od
6: if $nvr(\nu)$ then
7: $\nu := \text{nvr}(\nu)$
8: elif $\text{ndg}(\nu)$ then
9: return $\text{nil}$
10: else
11: return $\text{bmn}(\nu)$
12: fi
13: od
14: return $\text{nil}$
```

Apparently, the next theorem formulated and proved in [7] is valid for the adapted algorithm as well.

**Theorem.** Let $d$ be the maximal total degree of the leading monomials of binomials in $n$ variables which constitute the finite set $U$. Then the complexity bound of the algorithm J – divisor and the binary search algorithm is given by

$$t_{J\text{-divisor}} = O(d + n),$$
$$t_{\text{BinarySearch}} = O(n((d + n) \log(d + n) - n \log(n) - d \log(d))).$$

Thus, the complexity bound for the search of Janet divisor is $O(n + d)$ where $n$ is the number of variables and $d$ is the maximal degree of the leading monomials in the binomial basis. Since this bound is even lower than that for the binary search algorithm, one can expect that the involutive completion of binomial ideals may be faster than the reduced Gröbner basis completion.
3.3 Algorithms for Binomial Janet Bases

Given the generating binomial set $F$ of a toric ideal $\mathcal{I}_A$, the following algorithm BinomialJanetBasis which is a special form of the general polynomial algorithm \[6, 8\] constructs a Janet basis of $\mathcal{I}_A$.

**Algorithm: BinomialJanetBasis($F, \prec$)**

**Input:** $F \in \mathbb{R} \setminus \{0\}$, a finite binomial set; $\prec$, an admissible ordering

**Output:** $G$, a Janet basis of the ideal generated by $F$

1. choose $f \in F$ with the lowest $\text{lm}(f)$ w.r.t. $\prec$
2. $T := \{f, \text{lm}(f), \emptyset\}$
3. $Q := \{ \{q, \text{lm}(q), \emptyset\} \mid q \in F \setminus \{f\} \}$
4. $Q := \text{JanetReduce}(Q, T)$
5. while $Q \neq \emptyset$
6. choose $p \in Q$ with the lowest $\text{lm}(\text{bin}(p))$ w.r.t. $\prec$
7. $Q := Q \setminus \{p\}$
8. if $\text{lm}(\text{bin}(p)) = \text{anc}(p)$ then
   9. for all $\{ r \in T \mid \text{lm}(\text{bin}(r)) \prec \text{lm}(\text{bin}(p)) \}$ do
      10. $Q := Q \cup \{r\}$; $T := T \setminus \{r\}$
   11. od
12. $p := \text{NF}_J(\text{bin}(p), T)$
13. fi
14. $T := T \cup \{p\}$
15. for all $q \in T$ and $x \in \text{NM}_J(\text{bin}(q), T) \setminus \text{nmp}(q)$ do
16. $Q := Q \cup \{ \{\text{bin}(q) \cdot x, \text{anc}(q), \emptyset\} \}$
17. $\text{nmp}(q) := \text{nmp}(q) \cap \text{NM}_J(\text{bin}(q), T) \cup \{x\}$
18. od
19. $Q := \text{JanetReduce}(Q, T)$
20. od
21. return $G := \{ \text{bin}(f) \mid f \in T \}$

As well as in \[8\] to apply the involutive criteria and avoid repeated prolongations we shall endow with every binomial $f \in F$ the triple structure

$$p = \{ f, u, \text{vars} \}$$

such that

- $\text{bin}(p) = f$ is binomial itself,
- $\text{anc}(b) = u$ is the leading monomial of a binomial ancestor of $f$ in $F$
- $\text{nmp}(p) = \text{vars}$ is a (possible empty) subset of variables.

Here the *ancestor* of $f$ is a polynomial $g \in F$ with $u = \text{lm}(g)$ and such that $u \mid \text{lm}(p)$. Moreover, if $\deg(u) < \deg(\text{lm}(p))$, then every variable occurring in the monomial $\text{lm}(p)/u$
is nonmultiplicative for \( g \). Besides, for the ancestor \( g \) the equality \( \text{anc}(g) = \text{lm}(g) \) must hold. These conditions mean that polynomial \( p \) was obtained in the course of the below algorithm \textbf{BinomialJanetBasis} from \( g \) by a sequence of nonmultiplicative prolongations. This tracking of the history in the algorithm allows one to use the involutive analogues of Buchberger’s criteria to avoid unnecessary reductions.

The set \( \text{vars} \) contains those nonmultiplicative variables that have been already used in the algorithm for construction of nonmultiplicative prolongations. This set serves to prevent repeated prolongations.

In order to provide minimality of the output Janet basis we separate the whole polynomial data into two subsets which are contained in sets \( T \) and \( Q \). Set \( T \) is a part of the intermediate binomial basis. Another part of the intermediate basis is contained in set \( Q \) together with all the nonmultiplicative prolongations of polynomials in \( T \) which must be examined in accordance to the above definition of Janet bases. In so doing, after every insertion of a new element \( p \) in \( T \) all elements \( r \in T \) such that \( \text{lm}(\text{bin}(r)) \succ \text{lm}(\text{bin}(p)) \) are moved from \( T \) to \( Q \) as the for-loop 6-11 in algorithm \textbf{BinomialJanetBasis} does. Such a displacement guaranties that the output basis is minimal.

It should also be noted that for any triple \( p \in T \) the set \( \text{vars} \) must always be a subset of the set of nonmultiplicative variables for \( \text{bin}(p) \)

\[
\text{vars} \subseteq \text{NM}_J(\text{bin}(p), T). \tag{3}
\]

In the description of algorithm \textbf{JanetBinomialBases} we use the contractions:

\[
\text{NM}_J(\text{bin}(p), T) \equiv \text{NM}_J(\text{bin}(p), \{\text{bin}(f) \mid f \in T\}),
\]
\[
\text{NF}_J(\text{bin}(p), T) \equiv \text{NF}_J(\text{bin}(p), \{\text{bin}(f) \mid f \in T\}),
\]

The insertion of a new polynomial in \( T \) may generate new nonmultiplicative prolongations of elements in \( T \) which are added to \( Q \) in line 16. To avoid repeated prolongations the set \( \text{nmp}(q) \) of Janet nonmultiplicative variables for \( g \) has been used to construct its prolongations is enlarged with \( x \) in line 17. The intersection placed in this line preserves the condition (3).

The subalgorithms \textbf{JanetReduce} and \textbf{NF}_J perform Janet reduction of polynomials in \( Q \) modulo polynomials in \( T \) and presented below. In addition to reductions in lines 4 and 19, the Janet normal form computation is placed in line 12. This is because the replacement of elements from \( T \) to \( Q \) may lead to the tail reducibility of the binomial in \( p \). Such a reducibility may be caused by converting of some nonmultiplicative variables for binomials in \( T \) into multiplicative due to the replacement.

In subalgorithm \textbf{JanetReduce} computation of the Janet normal form \( h \) is done in line 6 for every binomial \( \text{bin}(p) \) in \( T \). If \( h \) is nonzero, then line 8 checks if \( \text{lm}(\text{bin}(p)) \) was subjected by reduction. If the reduction took place \( \text{lm}(h) \) cannot be multiple of any monomial in the set \( \{\text{lm}(\text{bin}(g)) \mid g \in T\} \) \[6\]. Therefore, one has to insert the triple with \( h \) in the output set \( Q \) as shown in line 9 as \( h \) cannot have ancestors among polynomials in \( T \) and one must also examine all nonmultiplicative prolongations of \( h \). If \( \text{lm}(\text{bin}(p)) \) is Janet irreducible modulo \( \{\text{lm}(\text{bin}(g)) \mid g \in T\} \), then the triple \( \{h, \text{anc}(p), \text{nmp}(p)\} \) is added to \( Q \) in line 11.
Algorithm: JanetReduce(Q, T)

**Input:** Q and T, sets of triples

**Output:** Q whose polynomials are Janet head reduced modulo T

1: \( S := Q \)
2: \( Q := \emptyset \)
3: while \( S \neq \emptyset \) do
4: \( \text{choose } p \in S \)
5: \( S := S \setminus \{p\} \)
6: \( h := \text{NF}_J(p, T) \)
7: if \( h \neq 0 \) then
8: \( \text{if } \text{lm}(\text{bin}(p)) \neq \text{lm}(h) \text{ then} \)
9: \( Q := Q \cup \{h, \text{lm}(h), \emptyset\} \)
10: else
11: \( Q := Q \cup \{h, \text{anc}(p), \text{nmp}(p)\} \)
12: fi
13: fi
14: od
15: return \( Q \)

Subalgorithm \( \text{NF}_J(p, T) \) performs the Janet reduction of a binomial \( g = \text{bin}(p) \) modulo polynomial set in \( T \):

Algorithm: \( \text{NF}_J(f, T) \)

**Input:** \( f = \{\text{bin}(f), \text{anc}(f), \text{nmp}(f)\} \), a triple; \( T \), a set of triples

**Output:** \( h = \text{NF}_J(\text{bin}(f), T) \), the Janet normal form of the binomial in \( f \) modulo binomial set in \( T \)

1: \( G := \{\text{bin}(g) \mid g \in T\} \)
2: \( h := \text{bin}(f) \)
3: if \( \text{lm}(h) \) is Janet reducible modulo \( G \) then
4: \( \text{choose } g \in T \text{ such that } \text{lm}(\text{bin}(g)) \mid_J \text{lm}(h) \)
5: \( \text{if } \text{lm}(h) \neq \text{anc}(f) \text{ and } \text{CriterionI}(f, g) \text{ or CriterionII}(f, g) \text{ then} \)
6: \( \text{return } 0 \)
7: fi
8: else
9: while \( h \neq 0 \text{ and } h \) has a term \( t \) Janet reducible modulo \( G \) do
10: \( \text{choose } q \in G \text{ such that } \text{lm}(q) \mid_J t \)
11: \( h := h - q \cdot t/\text{lm}(q) \)
12: od
13: fi
14: return \( h \)

For the head reducible input binomial \( \text{bin}(f) \) the two criteria are verified in line 5:
• **Criterion I** \((f, g)\) is true iff \(\text{anc}(f) \cdot \text{anc}(g) \mid \text{lm}(\text{bin}(f))\).

• **Criterion II** \((f, g)\) is true iff \(\text{deg}(\text{lcm}(\text{anc}(f) \cdot \text{anc}(g))) < \text{deg}(\text{lm}(\text{bin}(f)))\).

These criteria are the Buchberger criteria \([12]\) adapted to the involutive completion procedure. If any of the two criteria is true, then \(NF(\text{bin}(f), T) = 0\) \([8]\).

It should be noted that the Janet normal form is uniquely defined and, hence, uniquely computed by the above subalgorithm. This uniqueness hold because of the uniqueness of a Janet divisor among the leading terms of binomials in \(T\) at every step of intermediate computations \([9]\).

## 4 Examples

As we emphasized in Sect.3.1, in the course of involutive completion of the initial binomial generators for a toric ideal the reduction can be performed very fast due to the fast search for a Janet divisor. This fast search is provided by the use of the Janet tree structures for intermediate binomial set. Our computer experiments with C/C++ codes implementing polynomial algorithms for Janet bases \([8]\) perfectly strengthen this theoretical fact. In particular this fast reduction in addition to suppressing swell of intermediate integer coefficients results in high computational speed observed for the benchmark collection used for testing Gröbner bases software \([8]\). These benchmarks, however, are not very ”sparse” with respect to degrees of variables occurring in the generating set. By contrast, the generating binomial sets for toric ideals especially for those arising in integer programming problem are usually highly sparse. This may lead to much larger cardinality of a Janet basis than that of the reduced Gröbner basis and thereby annihilate the advantages of involutive reduction.

Consider the example taken from \([13]\)

\[
\mathcal{I}_A = \{ \ x_0x_1x_2x_3x_4 - 1, x_2^{20}x_3^5 - x_1^{14}x_4^20, x_1^{39} - x_2^{25}x_3^{14} \ \}
\]

Our C++ package \([8]\) generates the degree-reverse-lexicographical Janet basis of \(\mathcal{I}_A\) with 7769 binomials whose sorting with respect to the ordering chosen gives

\[
\{ \ x_0x_1^2x_3x_4^{281} - x_1x_2^{280}, x_0x_2^{61}x_3x_4^{221} - x_1x_2^{279}, x_0x_4^2x_3x_4^{281} - x_2^{280}, \ldots, x_0x_1x_2x_3x_4 - 1 \ \}
\]

where we explicitly show only three highest ranking binomials and the lowest one. The computing time on a Pentium III 700 Mhz based PC running under RedHat Linux 6.2 is 6 seconds that is noticeably larger than the running time for direct computation of the reduced Gröbner basis which contains 19 binomials only:

\[
\{ \ x_0x_1x_2x_3x_4^{281} - x_2^{280}, x_2^{281} - x_1x_4^{280}, x_0x_3^2x_4^{221} - x_1x_2^{218}, x_1^2x_2^{219} - x_3x_4^{220}, \\
x_0x_3^3x_4^{161} - x_1x_2^{156}, x_2^{157} - x_3^2x_4^{160}, x_0x_3x_4^{41} - x_1^7x_2^{94}, x_1^8x_2^{95} - x_3^3x_4^{100}, \\
x_0x_4^4x_4^{61} - x_2^{61}, x_2^6x_3^{63} - x_1^3x_4^60, x_0x_3^5x_4^{41} - x_1^10x_2^{32}, x_1^{11}x_2^{33} - x_3^4x_4^{40}, \\
x_0x_2^{26}x_3^{15}x_4 - x_1^{38}, x_1^{39} - x_2^{25}x_3^{14}, x_0x_1^{15}x_4^{21} - x_2^{28}x_3^{4}, x_2^{29}x_3^{5} - x_1^{14}x_4^{20}, \\
x_0x_3^{10}x_4^{21} - x_1^{24}x_2^3, x_1^{25}x_2^4 - x_3^9x_4^{20}, x_0x_1x_2x_3x_4 - 1 \ \}
\]

Accordingly, such a computer algebra system as *Singular* \([17]\) needs much less than 1 second to compute this Gröbner basis on the same computer.
Having ascertained this drawback of the involutive method with respect to the Gröbner basis one in computing toric ideals we designed another algorithmic approach to computing Gröbner bases \[14\]. This approach preserves the Janet-like tree structure and uniqueness of a divisor though underlying division is not involutive since it does not satisfy the axioms in \[6\]. On the other hand the resulting bases unlike Janet bases are often reduced as Gröbner bases and their cardinality is always less or equal to the cardinality of Janet bases. For toric ideals the new bases are much more compact then Janet bases. We have not implemented yet the new algorithm and so we demonstrate the compactness of its output in comparison with algorithm BinomialJanetBasis by the following simple example taken from \[2\]:

\[
I_A = \{ x^7 - y^2 z, x^4 w - y^3, x^3 y - z w \}.
\]

The reduced Gröbner basis and Janet basis of this toric ideal for the degree-reverse-lexicographic order induced by \(x \succ y \succ z \succ w\) are

\[
\{ x^7 - y^2 z, x^4 w - y^3, x^3 y - z w, y^4 - x z w^2 \}
\]

and

\[
\{ x^7 - y^2 z, x^6 y - x^3 z w, x^6 w - x^2 y^3, x^5 y - x^2 z w, x^2 y^4 - x^3 z w^2, x^5 w - x y^3, \\
x^4 y - x z w, x^2 z w^2 - x y^4, x^4 w - y^3, x^3 y - z w, y^4 - x z w^2 \}
\]

respectively. Their cardinalities are 4 and 11. The new basis contains 5 elements

\[
\{ x^7 - y^2 z, x^4 y - x z w, x^4 w - y^3, x^3 y - z w, y^4 - x z w^2 \}
\]

and contains only single extra element in comparison with the reduced Gröbner basis.

It should be noted that there are also a number of other efficient algorithms computing Gröbner bases of toric ideals (see, for example, \[2\ \[15\ \[16\]) which are differ greatly from just completion of a generating binomial set to a Gröbner basis. After implementation of our new algorithm we are planning to run the underlying code for collection of large examples given in \[1\ \[2\ and other references.

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