Holonomy control operators in classical and quantum completely integrable Hamiltonian systems

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Abstract.
Completely integrable Hamiltonian systems look rather promising for controllability since their first integrals are stable under an internal evolution, and one may hope to find a perturbation of a Hamiltonian which drives the first integrals at will. Action-angle coordinates are most convenient for this purpose. Written with respect to these coordinates, a Hamiltonian and first integrals of a (time-dependent) completely integrable system depend only on action variables. We introduce a suitable perturbation of an internal Hamiltonian by a term containing time-dependent parameters (control fields) so that an evolution of action variables is nothing else than a holonomy displacement along a curve in a parameter space. Therefore, one can determine this evolution in full by an appropriate choice of control fields. Similar holonomy controllability of finite level quantum systems is of special interest in connection with quantum computation. We provide geometric quantization of a time-dependent completely integrable Hamiltonian system in action-angle variables. Its Hamiltonian and first integrals have time-independent countable spectra. The holonomy control operator in this countable level quantum system is constructed.

1 Introduction

A time-dependent Hamiltonian system of $m$ degrees of freedom is called a completely integrable system (henceforth CIS) if there exist $m$ independent first integrals in involution. We show that such a system admits the action-angle coordinates around any regular instantly compact invariant manifold. Written relative to these coordinates, its Hamiltonian and first integrals are functions only of action coordinates. In comparison with perturbations in the KAM theorem, we consider perturbations of a Hamiltonian of a CIS by the term which contains time-dependent parameters. A generic Hamiltonian of a mechanical system with time-dependent parameters includes a term which is linear in momenta and the temporal derivatives of parameter functions [15, 11, 23]. Then the corresponding evolution operator of action variables depends only on a trajectory of the parameter functions in a parameter space. Therefore, it is determined completely by an appropriate choice of these parameter functions (control fields), and plays the role of a holonomy control operator.

At present, holonomy control operators in quantum systems attract special attention in connection with quantum computation, based on the generalization of Berry’s phase by

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means of driving a finite level degenerate eigenstate of a Hamiltonian over the parameter manifold \([9, 20, 28]\). Information is encoded in this degenerate state. Bearing in mind this application, we quantize a time-dependent CIS in question, and construct the quantum holonomy control operator in this system.

An essential simplification is that, given action-angle variables, a time-dependent CIS can be quantized as an autonomous one because its Hamiltonian and first integrals relative to these coordinates are time-independent. Of course, the choice of action-angle coordinates by no means is unique. They are subject to canonical transformations. Therefore, we employ the geometric quantization technique \([8, 25, 26]\), which remains equivalent under canonical transformations, but essentially depends on a choice of polarization \([4, 21]\).

Geometric quantization of an autonomous CIS has been studied with respect to polarization spanned by Hamiltonian vector fields of first integrals \([18]\). The well-known Simms quantization of the harmonic oscillator is also of this type. The problem is that the associated quantum algebra includes functions which are ill defined on the whole momentum phase space, and elements of the carrier space fail to be smooth sections of the quantum bundle. Indeed, written with respect to the action-angle variables, this quantum algebra consists of functions which are affine in angle coordinates.

We choose a different polarization spanned by almost-Hamiltonian vector fields of angle variables. The associated quantum algebra \(\mathcal{A}\) consists of smooth functions which are affine in action variables. This quantization is equivalent to geometric quantization of the cotangent bundle of the torus \(T^m\) with respect to the vertical polarization. This polarization is known to lead to Schrödinger quantization. We show that \(\mathcal{A}\) possesses a set of inequivalent representations in the separable pre-Hilbert space \(\mathbb{C}^\infty(T^m)\) of smooth complex functions on \(T^m\). In particular, the action operators read

\[
\hat{I}_k = -i\partial_k - \lambda_k,
\]

where \(\lambda_k\) are real numbers which specify different representations of \(\mathcal{A}\). By virtue of the multidimensional Fourier theorem, an orthonormal basis for \(\mathbb{C}^\infty(T^m)\) consists of functions

\[
\psi_{(n_r)} = \exp[i(n_r \phi^r)], \quad (n_r) = (n_1, \ldots, n_m) \in \mathbb{Z}^m,
\]

where \(\phi^i\) are cyclic coordinates on \(T^m\). With respect to this basis, the action operators (1) are written as countable diagonal matrices

\[
\hat{I}_k \psi_{(n_r)} = (n_k - \lambda_k) \psi_{(n_r)}.
\]

Given the representation \([1]\), any polynomial Hamiltonian \(\mathcal{H}(I_k)\) of a CIS is uniquely quantized as a Hermitian element \(\hat{\mathcal{H}}(I_k) = \mathcal{H}(\hat{I}_k)\) of the enveloping algebra of \(\mathcal{A}\). It has the time-independent countable spectrum

\[
\hat{\mathcal{H}}(I_k) \psi_{(n_r)} = E_{(n_r)} \psi_{(n_r)}, \quad E_{(n_r)} = \mathcal{H}(n_k - \lambda_k), \quad n_k \in (n_r).
\]

Since \(\hat{I}_k\) are diagonal, one can also quantize Hamiltonians \(\mathcal{H}(I_j)\) which are analytic functions on \(\mathbb{R}^m\).

Quantum CISs look rather promising for quantum computation. Its Hamiltonian depends only on action variables and possesses a time-independent countable spectrum. Moreover, it can be made degenerate at will by appropriate canonical transformations. We construct the holonomy control operator in this countable level quantum system. Note that one used to study controllability of finite level quantum systems \([1, 4, 24]\).
In order to introduce action-angle variables for a time-dependent CIS, we use the fact that a time-dependent Hamiltonian system of \( m \) degrees of freedom can be extended to an autonomous one of \( m + 1 \) degrees of freedom where the time is treated as a dynamic variable \([3, 1, 14]\). The classical theorem \([3, 13]\) on action-angle coordinates around a regular compact invariant manifold can not be applied to this autonomous CIS since its invariant manifolds are never compact because of the time axis. Generalizing the above mentioned theorem, we first prove that there is a system of action-angle coordinates on a open neighbourhood \( U \) of a regular connected invariant manifold \( M \) of an autonomous CIS if Hamiltonian vector fields on \( U \) are complete and the foliation of \( U \) by invariant manifolds is trivial.

If \( M \) is a compact regular invariant manifold, these conditions always hold \([13]\). Afterwards, we show that, if a regular connected invariant manifold of a time-dependent CIS is compact at each instant, it is diffeomorphic to the product of the time axis \( \mathbb{R} \) and an \( m \)-dimensional torus \( T^m \), and admits an open neighbourhood endowed with time-dependent action-angle coordinates \( (I_i; t, \phi^i), i = 1, \ldots, m \), where \( t \) is the Cartesian coordinate on \( \mathbb{R} \) and \( \phi^i \) are cyclic coordinates on \( T^m \).

Recall that the configuration space of a time-dependent mechanical system is a fibre bundle \( Q \to \mathbb{R} \) over the time axis \( \mathbb{R} \) equipped with the bundle coordinates \( (t, q^k), k = 1, \ldots, m \). The corresponding momentum phase space is the vertical cotangent bundle \( V^*Q \) of \( Q \to \mathbb{R} \) endowed with holonomic coordinates \( (t, q^j, p_k = \dot{q}_k) \) \([13, 22]\). The cotangent bundle \( T^*Q \), coordinated by \( (q^\lambda, p_\lambda) = (t, q^k, p_0, p_k) \), plays a role of the homogeneous momentum phase space. It is provided with the canonical Liouville form \( \Xi = p_\lambda dq^\lambda \), the canonical symplectic form \( \Omega = dp_\lambda \wedge dq^\lambda \), and the corresponding Poisson bracket

\[ \{f, f'\}_T = \partial_\lambda f \partial_\lambda f' - \partial_\lambda f \partial_\lambda f', \quad f, f' \in C^\infty(T^*Q). \]  

There is the one-dimensional trivial affine bundle

\[ \zeta : T^*Q \to V^*Q. \]  

Given its global section \( h \), one can equp \( T^*Q \) with the global fibre coordinate \( r = p_0 - h \). The fibre bundle \([6]\) provides the vertical cotangent bundle \( V^*Q \) with the canonical Poisson structure \( \{\cdot, \cdot\}_V \) such that

\[ \zeta^*\{f, f'\}_V = \{\zeta^*f, \zeta^*f'\}_T, \quad \forall f, f' \in C^\infty(V^*Q), \]  

\[ \{f, f'\}_V = \partial^k f \partial_k f' - \partial_k f \partial_k f', \]  

A Hamiltonian of time-dependent mechanics is defined as a global section

\[ h : V^*Q \to T^*Q, \quad p_0 \circ h = -H(t, q^j, p_j), \]  

of the affine bundle \( \zeta \) \([3, 13, 22]\). It yields the pull-back Hamiltonian form

\[ H = h^*\Xi = p_k dq^k - H dt \]  

on \( V^*Q \). There exists a unique vector field \( \gamma_H \) on \( V^*Q \) such that

\[ \gamma_H | dt = 1, \quad \gamma_H | dH = 0, \]  

\[ \gamma_H = \partial_t + \partial^k H \partial_k - \partial_k H \partial^k. \]
Its trajectories obey the Hamilton equation

\[
\dot{q}^k = \partial^k \mathcal{H}, \quad \dot{p}_k = -\partial_k \mathcal{H}. \tag{11}
\]

A first integral of the Hamilton equation (11) is defined as a smooth real function \( F \) on \( V^*Q \) whose Lie derivative

\[
\mathbf{L}_{\gamma_H} F = \gamma_H | dF = \partial_t F + \{ \mathcal{H}, F \}_V
\]

along the vector field \( \gamma_H \) vanishes, i.e., \( F \) is constant on trajectories of \( \gamma_H \). A time-dependent Hamiltonian system \((V^*Q, H)\) is said to be completely integrable if the Hamilton equation (11) admits \( m \) first integrals \( F_k \) which are in involution with respect to the Poisson bracket \( \{ \cdot, \cdot \}_V \) (8), and whose differentials \( dF_k \) are linearly independent almost everywhere (i.e., the set of points where this condition fails is nowhere dense). One can associate to this CIS an autonomous CIS on \( T^*Q \) as follows.

Given a Hamiltonian \( h \) (9), it is readily observed that

\[
\mathcal{H}^* = \partial_t | (\Xi - \zeta^* h^* \Xi) = p_0 + \mathcal{H}
\]

is a function on \( T^*Q \). Let us regard \( \mathcal{H}^* \) as a Hamiltonian of an autonomous Hamiltonian system on the symplectic manifold \((T^*Q, \Omega)\) \([17]\). Its Hamiltonian vector field

\[
\gamma_T = \partial_t - \partial_t \mathcal{H} \partial^0 + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k
\]

is projected onto the vector field \( \gamma_H \) on \( V^*Q \) so that

\[
\zeta^* (\mathbf{L}_{\gamma_H} f) = \{ \mathcal{H}^*, \zeta^* f \}_T, \quad \forall f \in C^\infty(V^*Q).
\]

An immediate consequence of this relation is the following.

**Proposition 1.** (i) Given a time-dependent CIS \((H; F_k)\) on \( V^*Q \), the Hamiltonian system \((\mathcal{H}^*, \zeta^* F_k)\) on \( T^*Q \) is a CIS. (ii) Let \( N \) be a connected regular invariant manifold of \((H; F_k)\). Then \( h(N) \subset T^*Q \) is a connected regular invariant manifold of the autonomous CIS \((\mathcal{H}^*, \zeta^* F_k)\).

Hereafter, we assume that the vector field \( \gamma_H \) is complete. In this case, the Hamilton equation (11) admits a unique global solution through each point of the momentum phase space \( V^*Q \), and trajectories of \( \gamma_H \) define a trivial bundle \( V^*Q \to V^*_tQ \) over any fibre \( V^*_tQ \) of \( V^*Q \to \mathbb{R} \). Without loss of generality, we choose the fibre \( i_0 : V^*_0Q \to V^*Q \) at \( t = 0 \). Since \( N \) is an invariant manifold, the fibration

\[
\xi : V^*Q \to V^*_0Q
\]

also yields the fibration of \( N \) onto \( N_0 = N \cap V^*_0Q \) such that \( N \cong \mathbb{R} \times N_0 \) is a trivial bundle.
3 Time-dependent action-angle coordinates

Let us introduce the action-angle coordinates around an invariant manifold $N$ of a time-dependent CIS on $V^*Q$ by use of the action-angle coordinates around the invariant manifold $h(N)$ of the autonomous CIS on $T^*Q$ in Proposition 1. Since $N$ and, consequently, $h(N)$ are non-compact, we refer to the following.

**Proposition 2.** Let $M$ be a connected invariant manifold of an autonomous CIS $\{F_\lambda\}$, $\lambda = 1, \ldots, n$, on a symplectic manifold $(Z, \Omega_Z)$. Let $U$ be an open neighbourhood of $M$ such that: (i) the differentials $dF_\lambda$ are independent everywhere in $U$, (ii) the Hamiltonian vector fields $\vartheta_\lambda$ of the first integrals $F_\lambda$ on $U$ are complete, and (iii) the submersion $\times F_\lambda : U \to \mathbb{R}^n$ is a trivial bundle of invariant manifolds over a domain $V' \subset \mathbb{R}^n$. Then $U$ is isomorphic to the symplectic annulus

$$W' = V' \times (\mathbb{R}^{n-m} \times T^m),$$

provided with the action-angle coordinates

$$(I_1, \ldots, I_n; x^1, \ldots, x^{n-m}; \phi^1, \ldots, \phi^m)$$

such that the symplectic form on $W'$ reads

$$\Omega_Z = dI_a \wedge dx^a + dI_i \wedge d\phi^i,$$

and the first integrals $F_\lambda$ depend only on the action coordinates $I_a$.

**Proof.** In accordance with the well-known theorem [3], the invariant manifold $M$ is diffeomorphic to the product $\mathbb{R}^{n-m} \times T^m$, which is the group space of the quotient $G = \mathbb{R}^n/\mathbb{Z}^m$ of the group $\mathbb{R}^n$ generated by Hamiltonian vector fields $\vartheta_\lambda$ of first integrals $F_\lambda$ on $M$. Namely, $M$ is provided with the group space coordinates $(y^\lambda) = (s^a, \varphi^i)$ where $\varphi^i$ are linear functions of parameters $s^a$ along integral curves of the Hamiltonian vector fields $\vartheta_\lambda$ on $U$. Let $(J_\lambda)$ be coordinates on $V'$ which are values of first integrals $F_\lambda$. Let us choose a trivialization of the fibre bundle $U \to V$ seen as a principal bundle with the structure group $G$. We fix its global section $\chi$. Since parameters $s^a$ are given up to a shift, let us provide each fibre $M_J$, $J \in V$, with the group space coordinates $(y^\lambda)$ centred at the point $\chi(J)$. Then $(J_\lambda; y^\lambda)$ are bundle coordinates on the annulus $W'$ ([4]). The rest of the proof in Appendix A reduces to transformation of the coordinates $(J_\lambda; y^\lambda)$ to the desired coordinates

$$I_a = J_a, \quad I_i(J_j), \quad x^a = s^a + S^a(J_\lambda), \quad \phi^i = \varphi^i + S^i(J_\lambda, s^b).$$

(17)

Of course, the action-angle coordinates ([4]) by no means are unique. For instance, let $\mathcal{F}_a$ be $(n - m)$ arbitrary smooth functions on $\mathbb{R}^m$. Let us consider the canonical coordinate transformation

$$I_a' = I_a - \mathcal{F}_a(I_j), \quad I_k' = I_k, \quad x'^a = x^a, \quad \phi^i = \phi^i + x^a \partial_a \mathcal{F}_a(I_j).$$

(18)

Then $(I_a', I_k'; x'^a, \phi'^i)$ are action-angle coordinates on the symplectic annulus which differs from $W'$ ([4]) in a trivialization.
Let us apply Proposition 2 to the CISs in Proposition 1.

**Proposition 3.** Let \( N \) be a connected regular invariant manifold of a time-dependent CIS on \( V^*Q \), and let the image \( N_0 \) of its projection \( \xi \) (14) be compact. Then the invariant manifold \( h(N) \) of the associated autonomous CIS on \( T^*Q \) has an open neighbourhood \( U \) obeying the condition of Proposition 2 (see Appendix B).

In accordance with Proposition 2, the open neighbourhood \( U \) of the invariant manifold \( h(N) \) of the autonomous CIS in Proposition 3 is isomorphic to the symplectic annulus

\[
W' = V' \times (\mathbb{R} \times T^m)
\]

provided with the action-angle coordinates \((I_0, \ldots, I_m; t, \phi^1, \ldots, \phi^m)\) such that the symplectic form on \( W' \) reads

\[
\Omega = dI_0 \wedge dt + dI_k \wedge d\phi^k.
\]

By the construction in Proposition 2, \( I_0 = J_0 = \mathcal{H}^*(12) \) and the corresponding angle coordinate is \( x^0 = t \), while the first integrals \( J_k = \zeta^*F_k \) depend only on the action coordinates \( I_i \).

Since the action coordinates \( I_i \) are independent of the coordinate \( J_0 \), the symplectic annulus \( W' \) (19) inherits the fibration (14) which reads

\[
\zeta : W' \ni (I_0, I_i; t, \phi^i) \rightarrow (I_i, t, \phi^i) \in W = \mathbb{R} \times T^m \times V.
\]

By the relation similar to (16), the product \( W \) (20) is provided with the Poisson structure

\[
\{f, f'\}_W = \partial_i f \partial_i f' - \partial_i f \partial_i f', \quad f, f' \in C^\infty(W).
\]

Therefore, one can regard \( W \) as the momentum phase space of the time-dependent CIS in question around an invariant manifold \( N \).

It is readily observed that the Hamiltonian vector field \( \gamma_T \) of the autonomous Hamiltonian \( \mathcal{H}^* = I_0 = \mathcal{H}^* \) is \( \gamma_T = \partial_t \), and so is its projection \( \gamma_H \) (11) on \( W \). Consequently, the Hamilton equation (11) of a time-dependent CIS with respect to the action-angle coordinates take the form \( \dot{I}_i = 0, \dot{\phi}^i = 0 \). Hence, \((I_i; t, \phi^i)\) are the initial data coordinates. One can introduce such coordinates as follows. Given the fibration \( \xi \) (14), let us provide \( N_0 \times V \subset V_0^*Q \) in Proposition 3 with the action-angle coordinates \((T_i; \phi)\) for the CIS \( \{i^*_0F_k\} \) on the symplectic leaf \( V_0^*Q \). Then, it is readily observed that \((T_i; t, \phi)\) are time-dependent action-angle coordinates on \( W \) (20) such that the Hamiltonian \( \mathcal{H}(T_j) \) of a time-dependent CIS relative to these coordinates vanishes, i.e., \( \mathcal{H}^* = \mathcal{T}_0 \). Using the canonical transformations (18), one can obtain different time-dependent action-angle coordinates. In particular, given a smooth function \( \mathcal{H} \) on \( \mathbb{R}^m \), one can endow \( W \) with the action-angle coordinates

\[
I_0 = \mathcal{T}_0 - \mathcal{H}(T_j), \quad I_i = T_i, \quad \phi^i = \overline{\phi}^i + t \partial^i \mathcal{H}(T_j)
\]

such that \( \mathcal{H}(I_i) \) is a Hamiltonian of time-dependent CIS on \( W \).
4 The classical control operator

A generic momentum phase space of a Hamiltonian system with time-dependent parameters is a composite fibre bundle \( \Pi \to \Sigma \to \mathbb{R} \), where \( \Pi \to \Sigma \) is a symplectic bundle and \( \Sigma \to \mathbb{R} \) is a parameter bundle whose sections are parameter functions \([11, 15, 23, 27]\). Here, we assume that all bundles are trivial and, moreover, their trivializations hold fixed. Then the momentum phase space of a Hamiltonian system with time-dependent parameters on the Poisson manifold \( W(20) \) is the product

\[
\Pi = S \times W = V \times (\mathbb{R} \times S \times T^m) \to \mathbb{R} \times S \to \mathbb{R},
\]

equipped with the coordinates \((I_k; t, \sigma^\alpha, \varphi^k)\). It is convenient to suppose for a time that parameters are also dynamic variables. The momentum phase space of such a system is \( \Pi' = T^*S \times W \), coordinated by \((I_k; t, \sigma^\alpha, p_\alpha, \varphi^k)\). The dynamics of a time-dependent mechanical system on the momentum phase space \( \Pi' \) is characterized by a Hamiltonian form

\[
H_\Sigma = p_\alpha d\sigma^\alpha + I_k d\varphi^k - \mathcal{H}_\Sigma(t, \sigma^\beta, p_\beta, I_j, \varphi^j) dt
\]

\[
\mathcal{H}_\Sigma = p_\alpha \Gamma^\alpha_t + I_k (\Lambda^k_t + \Gamma^\alpha_t \Lambda^k_\alpha) + \tilde{\mathcal{H}},
\]

where \( \Gamma = (\Gamma^\alpha_t) \) is a connection on the parameter bundle \( \mathbb{R} \times S \to \mathbb{R} \) and \( \Lambda = (\Lambda^k_t, \Lambda^k_\alpha) \) is a connection on \( \mathbb{R} \times S \times T^m \to \mathbb{R} \times S \) \([11, 13, 23]\).

Bearing in mind that \( \sigma^\alpha \) are parameters, one should choose the Hamiltonian \( \mathcal{H}_\Sigma \) to be affine in momenta \( p_\alpha \). Furthermore, in order to describe a Hamiltonian system with a fixed parameter function \( \sigma^\alpha = \xi^\alpha(t) \), one defines the connection \( \Gamma \) such that

\[
\nabla^\Gamma \xi = 0,
\]

\[
\Gamma^\alpha_t(t, \xi^\beta(t)) = \partial_t \xi^\alpha.
\]

Then the pull-back

\[
H_\xi = \xi^*H_\Sigma = I_k d\varphi^k - (I_k [\Lambda^k_t(t, \xi^\beta, \varphi^j) + \Lambda^k_\alpha(t, \xi^\beta, \varphi^j) \partial_t \xi^\alpha] + \tilde{\mathcal{H}}(t, \xi^\beta, I_j, \varphi^j)) dt
\]

is a Hamiltonian form on the Poisson manifold \( W(20) \). Let us put

\[
\tilde{\mathcal{H}} = \mathcal{H} - I_k \Lambda^k_t,
\]

where \( \mathcal{H}(I) \) is a Hamiltonian of the original CIS on \( W(20) \). Then the Hamiltonian form

\[
H_\xi = I_k d\varphi^k - \mathcal{H}_\xi dt = I_k d\varphi^k - [I_k \Lambda^k_t(t, \xi^\beta, \varphi^j) \partial_t \xi^\alpha + \mathcal{H}(I_j)] dt
\]

(23)

describes a perturbed time-dependent CIS on the Poisson manifold \( W(20) \). The corresponding Hamilton equation reads

\[
\partial_t I_k = -\partial_k \Lambda^j_\alpha I_j \partial_t \xi^\alpha, \quad \partial_t \varphi^k = \partial^k \mathcal{H} + \Lambda^k_\alpha \partial_t \xi^\alpha.
\]

(24)

In order to make the term

\[
\Delta = I_k \Lambda^k_\alpha \partial_t \xi^\alpha
\]

(25)

in the perturbed Hamiltonian \( H_\xi \) \([23]\) a control operator, let us assume that the coefficients \( \Lambda^k_\alpha \) of the connection \( \Lambda \) are independent of time. Then, in view of the trivialization (24),
its part \((\Lambda^k_{\alpha})\) can be seen as a connection on the fibre bundle \(S \times T^m \to S\). Let us choose the initial data action-angle variables. Then the internal Hamiltonian \(\mathcal{H}\) of the original CIS vanishes, and the Hamilton equation \((24)\) takes the form

\[
\begin{align*}
\partial_t I_i &= -I_k \partial_i \Lambda^k_{\alpha} \partial_t \xi^\alpha, \\
\partial_t \phi^i &= \Lambda^i_{\alpha} \partial_t \xi^\alpha.
\end{align*}
\]

(26)

\begin{align*}
\text{(27)}
\end{align*}

This is the control equation as follows.

Any smooth complex function on the product \(\mathbb{R} \times T^m\) is represented by a multidimensional Fourier series of functions \(\psi_{(n_r)}\) \((2)\) on \(T^m\) with coefficients the smooth functions on \(\mathbb{R}\). Let us rewrite the equation \((27)\) as the countable system of equations

\[
\partial_t \psi_{(n_r)} = i \psi_{(n_r)} \partial_t \phi^i = i \psi_{(n_r)} n_i \Lambda^i_{\alpha}(\xi^\beta, \psi_{(m_r)}) \partial_t \xi^\alpha, \quad n_i \in (n_r),
\]

for functions \(\psi_{(n_r)}\) \((2)\). Let

\[
\Lambda^i_{\alpha} = \sum_{(m_r)} \Lambda^i_{\alpha(m_r)}(\xi^\beta) \psi_{(m_r)}
\]

(28)

be the Fourier series for \(\Lambda^k_{\alpha}\). Since \(\psi_{(n_r)} \psi_{(m_r)} = \psi_{(n_r+m_r)}\), we obtain a countable system of linear ordinary differential equations

\[
\partial_t \psi_{(n_r)} = \sum_{(n_q)} [iM^{(k_r)}_{\alpha(n_q)}(\xi^\beta)] \partial_t \xi^\alpha \psi_{(k_q)},
\]

(29)

\[
M^{(k_r)}_{\alpha(n_q)} = \sum_{(m_r+k_r = n_r)} n_i \Lambda^i_{\alpha(m_r)},
\]

(30)

with time-dependent coefficients \([iM^{(k_r)}_{\alpha(n_q)}(\xi^\beta)] \partial_t \xi^\alpha\]. Its solution with the initial data \(\psi^t(0)\) can be written as the formal time-ordered exponential

\[
\psi_{(n_r)}(t) = U(t)^{(k_r)}_{(n_r)} \psi_{(k_r)}(0),
\]

\[
U(t) = \exp \left[ i \int_0^t \hat{M}_{\alpha}(\xi^\beta(t')) \partial_t \xi^\alpha dt' \right] = \exp \left[ i \int_{\xi([0,t])} \hat{M}_{\alpha}(\sigma^\beta) d\sigma^\alpha \right],
\]

(31)

where \(\hat{M}_{\alpha}\) denotes the matrix with time-dependent entries \(\{30\}\) \([12, 13]\). A glance at the evolution operator \(U(t)\) \((31)\) shows that solutions of the equations \((28)\) are functions \(\psi_{(n_r)}(\xi(t))\) of a point \(\xi(t)\) of the curve \(\xi : \mathbb{R} \to S\) in the parameter space \(S\).

Substituting this solution into the equation \((24)\), we obtain the system of \(l\) ordinary linear differential equations with time-dependent coefficients:

\[
\partial_t I_i = -[L^k_{\alpha i}(\xi^\beta(t), \psi_{(n_r)}(\xi(t))) \partial_t \xi^\alpha] I_k, \quad L^k_{\alpha i} = \partial_t \Lambda^k_{\alpha},
\]

Its solution is given by the time-ordered exponential

\[
I_i(t) = U(t)^k_i I_k(0),
\]

\[
U(t) = \exp \left[ - \int_0^t \hat{L}_{\alpha}(\xi^\beta(t), \psi_{(n_r)}(\xi(t))) \partial_t \xi^\alpha dt' \right] = \exp \left[ - \int_{\xi([0,t])} \hat{L}_{\alpha}(\sigma^\beta, \psi_{(n_r)}(\sigma)) d\sigma^\alpha \right],
\]

\[
T \exp \left[ - \int_{\xi([0,t])} \hat{L}_{\alpha}(\sigma^\beta, \psi_{(n_r)}(\sigma)) d\sigma^\alpha \right],
\]

\[
8
\]
where $\hat{L}_\alpha$ denotes the matrix with time-dependent entities $L^k_{\alpha i}$. This solution is a function of a point $\xi(t)$ of the curve $\xi : \mathbb{R} \to S$ in the parameter space $S$.

It follows that, if the holonomy group of the connection $(\Lambda^k_{\alpha})$ on the fibre bundle $S \times T^m \to S$ is the whole group $GL(m, \mathbb{R})$, one can obtain any desired trajectory of action variables $I_i$ and, consequently, of first integrals $F_i$ by an appropriate choice of control functions $\xi^\alpha(t)$.

## 5 Quantum completely integrable systems

In order to quantize a time-dependent CIS on the Poisson manifold $(W, \{,\}_W)$, one may follow the general procedure of instantwise geometric quantization of time-dependent Hamiltonian systems in [10]. As was mentioned above, it can however be quantized as an autonomous CIS on the symplectic annulus

$$P = V \times T^m,$$

equipped with fixed action-angle coordinates $(I, \phi^i)$ and provided with the symplectic form

$$\Omega_P = dI_i \wedge d\phi^i. \quad (32)$$

In accordance with the standard geometric quantization procedure [25, 26], because the symplectic form $\Omega_P$ (32) is exact, the prequantum bundle is defined as a trivial complex line bundle $C$ over $P$. Since the action-angle coordinates are canonical for the symplectic form (32), the prequantum bundle $C$ need no metaplectic correction. Let its trivialization

$$C \cong P \times \mathbb{C} \quad (33)$$

hold fixed. Any other trivialization leads to equivalent quantization of $P$. Given the associated bundle coordinates $(I_k; \phi^k, c)$, $c \in \mathbb{C}$, on $C$ (33), one can treat its sections as smooth complex functions on $P$.

The Konstant–Souriau prequantization formula associates to each smooth real function $f \in C^\infty(P)$ on $P$ the first order differential operator

$$\hat{f} = -i\nabla_{\vartheta_f} + f \quad (34)$$

on sections of $C$, where $\vartheta_f = \partial^k f \partial_k - \partial_k f \partial^k$ is the Hamiltonian vector field of $f$ and $\nabla$ is the covariant differential with respect to a suitable $U(1)$-principal connection on $C$. This connection preserves the Hermitian metric $g(c, c') = c\overline{c'}$ on $C$, and its curvature form obeys the prequantization condition $R = i\Omega_P$. This connection reads

$$A = A_0 + icI_k d\phi^k \otimes \partial_c, \quad (35)$$

where $A_0$ is a flat $U(1)$-principal connection on $C \to P$. The equivalence classes of flat principal connections on $C$ are indexed by the set $\mathbb{R}^m/\mathbb{Z}^m$ of homomorphisms of the de Rham cohomology group

$$H^1(P) = H^1(T^m) = \mathbb{R}^m$$
of $P$ to the cycle group $U(1)$ \footnote{3}. We choose their representatives of the form

$$A_0[(\lambda_k)] = dI_k \otimes \partial^k + d\phi^k \otimes (\partial_k + i\lambda_k c\partial_c), \quad \lambda_k \in [0, 1).$$

Then the connection (35) up to gauge transformations reads

$$A[(\lambda_k)] = dI_k \otimes \partial^k + d\phi^k \otimes (\partial_k + i(I_k + \lambda_k)c\partial_c).$$

(36)

For the sake of simplicity, we will assume that the numbers $\lambda_k$ in the expression (36) belong to $\mathbb{R}$, but will bear in mind that connections $A[(\lambda_k)]$ and $A[(\lambda'_k)]$ with $\lambda_k - \lambda'_k \in \mathbb{Z}$ are gauge conjugated. Given a connection (36), the prequantization operators (34) read

$$\hat{f} = -i\partial f + (f - (I_k + \lambda_k)\partial^k f).$$

(37)

Let us choose the above mentioned angle polarization $V\pi$ which is the vertical tangent bundle of the fibration $\pi : P \rightarrow T^m$, and is spanned by the vectors $\partial^k$. It is readily observed that the corresponding quantum algebra $A \subset C^\infty(P)$ consists of affine functions

$$f = a^k(\phi^j)I_k + b(\phi^j)$$

(38)

of action coordinates $I_k$. The carrier space of its representation by operators (37) is defined as the space $E$ of sections $\rho$ of the prequantum bundle $C$ of compact support which obey the condition $\nabla_\varphi \rho = 0$ for any Hamiltonian vector field $\varphi$ subordinate to the distribution $V\pi$. This condition reads

$$\partial_k f \partial^k \rho = 0, \quad \forall f \in C^\infty(T^m).$$

It follows that elements of $E$ are independent of action variables and, consequently, fail to be of compact support, unless $\rho = 0$. This well-known problem of Schrödinger geometric quantization is solved as follows \footnote{3, 10}.

Let us fix a slice $i_T : T^m \rightarrow T^m \times V$. Let $C_T = i_T^*C$ be the pull-back of the prequantum bundle $C$ (33) over the torus $T^m$. It is a trivial complex line bundle $C_T = T^m \times \mathbb{C}$ provided with the pull-back Hermitian metric $g(c, c') = c\bar{c}'$. Its sections are smooth complex functions on $T^m$. Let

$$\overline{A} = i_T^*A = d\phi^k \otimes (\partial_k + i(I_k + \lambda_k)c\partial_c)$$

be the pull-back of the connection $A$ (33) onto $C_T$. Let $\mathcal{D}$ be a metalinear bundle of complex half-forms on the torus $T^m$. It admits the canonical lift of any vector field $\tau$ on $T^m$, and the corresponding Lie derivative of its sections reads

$$L_\tau = \tau^k \partial_k + \frac{1}{2} \partial_k \tau^k.$$ 

Let us consider the tensor product

$$Y = C_T \otimes \mathcal{D} \rightarrow T^m.$$ 

(39)

Since the Hamiltonian vector fields

$$\varphi_f = a^k \partial_k - (I_r \partial_k a^r + \partial_k b)\partial^k$$
of functions $f$ (38) are projectable onto $T^m$, one can associate to each element $f$ of the quantum algebra $\mathcal{A}$ the first order differential operator

$$\hat{f} = (-i\nabla_{\pi\vartheta} + f) \otimes \text{Id} + \text{Id} \otimes L_{\pi\vartheta} = -ia^k \partial_k - \frac{i}{2} \partial_k a^k - a^k \lambda_k + b$$

(40)
on sections of $Y$. A direct computation shows that the operators (40) obey the Dirac condition

$$[\hat{f}, \hat{f}'] = -i\{\hat{f}, \hat{f}'\}.$$

Sections $\rho_T$ of the quantum bundle $Y \to T^m$ (39) constitute a pre-Hilbert space $E_T$ with respect to the non-degenerate Hermitian form

$$\langle \rho_T | \rho'_T \rangle = \left(\frac{1}{2\pi}\right)^m \int_{T^m} \rho_T \rho'_T,$$

$\rho_T, \rho'_T \in E_T$.

Then it is readily observed that $\hat{f}$ (40) are Hermitian operators in $E_T$. In particular, the action operators take the form (1).

Of course, the above quantization depends on the choice of a connection $A[(\lambda_k)]$ (36) and a metalinear bundle $\mathcal{D}$. The latter need not be trivial. If $\mathcal{D}$ is trivial, sections of the quantum bundle $Y \to T^m$ (39) obey the transformation rule

$$\rho_T(\phi^k + 2\pi) = \rho_T(\phi^k)$$

for all indices $k$. They are naturally complex smooth functions on $T^m$. In this case, $E_T$ is the above mentioned pre-Hilbert space $\mathbb{C}^\infty(T^m)$ of complex smooth functions on $T^m$ whose basis consists of functions (2). The action operators $\hat{I}$ (1) with respect to this basis are represented by countable diagonal matrices (3), while functions $a(\phi)$ are decomposed into the pull-back functions $\psi_{(n_r)}$ which act on $\mathbb{C}^\infty(T^m)$ by multiplications

$$\psi_{(n_r)} \psi_{(n'_r)} = \psi_{(n_r)} \psi_{(n'_r)} = \psi_{(n_r+nn'_r)}.$$  

(41)

If $\mathcal{D}$ is a non-trivial metalinear bundle, sections of the quantum bundle $Y \to T^m$ (39) obey the transformation rule

$$\rho_T(\phi^j + 2\pi) = -\rho_T(\phi^j)$$

(42)

for some indices $j$. In this case, the orthonormal basis of the pre-Hilbert space $E_T$ can be represented by double-valued complex functions

$$\psi_{(n_i, n_j)} = \exp\{i(n_i \phi^i + (n_j + \frac{1}{2}) \phi^j)\}.$$  

(43)

on $T^m$. They are eigenvectors

$$\hat{I}_i \psi_{(n_i, n_j)} = (n_i - \lambda_i) \psi_{(n_i, n_j)}, \quad \hat{I}_j \psi_{(n_i, n_j)} = (n_j - \lambda_j + \frac{1}{2}) \psi_{(n_i, n_j)}$$

of the operators $\hat{I}_k$ (1), and the functions $a(\phi)$ act on the basis (39) by the above law (14). It follows that the representation of $\mathcal{A}$ determined by the connection $A[(\lambda_k)]$ (36)
in the space of sections (12) of a non-trivial quantum bundle $Y$ (39) is equivalent to its representation determined by the connection $A[(\lambda_i, \lambda_j - \frac{1}{2})]$ in the space $\mathbb{C}^\infty(T^m)$ of smooth complex functions on $T^m$.

Therefore, one can restrict the study of representations of the quantum algebra $A$ to its representations in $\mathbb{C}^\infty(T^m)$ associated to different connections (36). These representations are inequivalent, unless $\lambda_k - \lambda'_k \in \mathbb{Z}$ for all indices $k$.

Now, in order to quantize the Poisson manifold $(W, \{\cdot, \cdot\})$, one can simply replace functions on $T^m$ with those on $\mathbb{R} \times T^m$ [10, 25]. Let us choose the angle polarization of $W$ spanned by the vectors $\partial^k$. The corresponding quantum algebra $A_W \subset C^\infty(W)$ consists of affine functions
\[ f = a^k(t, \phi^j)I_k + b(t, \phi^j) \] (44)
of action coordinates $I_k$, represented by the operators (40) in the space $C^\infty(\mathbb{R} \times T^m)$ of smooth complex functions on $\mathbb{R} \times T^m$. This space is provided with the structure of the pre-Hilbert $C^\infty(\mathbb{R})$-module with respect to the non-degenerate $C^\infty(\mathbb{R})$-bilinear form
\[ \langle \psi | \psi' \rangle = \left( \frac{1}{2\pi} \right)^m \int_{T^m} \overline{\psi} \psi', \quad \psi, \psi' \in C^\infty(\mathbb{R} \times T^m). \]

Its basis consists of the pull-backs onto $\mathbb{R} \times T^m$ of the functions $\psi_{(n_r)}$ (2).

Since the Poisson structure (21) defines no dynamics on the momentum phase space $W$ (21), we should quantize the homogeneous momentum phase space $W'$ (19) in order to describe evolution of a quantum time-dependent CIS. Following the general scheme in [10, 11], one can provide the relevant geometric quantization of the symplectic annulus $(W', \Omega')$. The corresponding quantum algebra $A_{W'} \subset C^\infty(W')$ consists of affine functions
\[ f = a^\lambda(t, \phi^j)I_\lambda + b(t, \phi^j) \] of action coordinates $I_\lambda$. It suffices to consider its subalgebra consisting of the elements $f$ and $I_0 + f$ for all $f \in A_{W'}$ (44). They are represented by the operators $\hat{f}$ (44) and $\hat{I}_0 = -i\partial_t$ in the pre-Hilbert module $C^\infty(\mathbb{R} \times T^m)$. If a Hamiltonian $\mathcal{H}(I_j)$ of the time-dependent CIS is a polynomial (or analytic) function in action variables, the Hamiltonian $\mathcal{H}^*$ of the associated autonomous CIS is quantized as
\[ \hat{\mathcal{H}}^* = -i\partial_t + \mathcal{H}(\hat{I}_j). \]

Then we obtain the Schrödinger equation
\[ \hat{\mathcal{H}}^* \psi = -i\partial_t \psi + \mathcal{H}(-i\partial_k - \lambda_k)\psi = 0, \quad \psi \in C^\infty(\mathbb{R} \times T^m). \]

Its solutions are the series
\[ \psi = \sum_{(n_r)} B_{(n_r)} \exp[-iE_{(n_r)}t] \psi_{(n_r)}, \quad B_{(n_r)} \in \mathbb{C}, \]
where $E_{(n_r)}$ are the eigenvalues (1) of the Hamiltonian $\hat{\mathcal{H}}$. 

12
6 The quantum control operator

In comparison with the classical control operator in Section 4, we will construct the quantum control operator which preserves the eigenvalues of a Hamiltonian, and acts in its degenerate eigenspaces. For instance, let us choose action-angle coordinates such that a Hamiltonian $H$ of a CIS equals $I_1$, and it is independent of other action variables $I_a$ $(a, b, c = 2, \ldots, m)$. It is quantized as $\hat{H} = \hat{I_1}$. Its eigenvalues are countably degenerate. Let us consider the perturbed Hamiltonian $\mathcal{H}_\xi$ (23) where the perturbation term $\Delta$ (25) depends only on the action-angle coordinates with the indices $a, b, c = 2, \ldots, m$, i.e.,

$$\mathcal{H}_\xi = I_1 + \Lambda^a_\beta(\xi^\mu, \phi^b)\partial_t \xi^\beta I_a.$$  

The perturbation term

$$\Delta = \Lambda^a_\beta(\xi^\mu, \phi^b)\partial_t \xi^\beta I_a$$

of this Hamiltonian is an element of the quantum algebra $\mathcal{A}_W$, and is quantized by the operator

$$\hat{\Delta} = -i\Lambda^a_\beta\partial_t \xi^\beta \partial_a - \frac{i}{2}\partial_a(\Lambda^a_\beta)\partial_t \xi^\beta - \lambda_a \Lambda^a_\beta \partial_t \xi^\beta = \hat{\Delta}_\beta \partial_t \xi^\beta,$$

$$\hat{\Delta}_{\beta(n_1, n_a)} = \sum_{(m_a + k_a = n_a)} [(k_a + \frac{1}{2}m_a - \lambda_a)\Lambda^a_\beta(m_a)(\xi^\alpha)],$$

where the Fourier series decomposition (28) is used.

Since the operators $\hat{\Delta}$ and $\hat{\mathcal{H}}$ mutually commute, the corresponding quantum evolution operator reduces to the product

$$T \exp \left[ -i \int_0^t \hat{\mathcal{H}} dt' \right] = U_1(t) \circ U_2(t) = T \exp \left[ -i \int_0^t \hat{\Delta} dt' \right] \circ T \exp \left[ -i \int_0^t \hat{\Delta} dt' \right]. \quad (45)$$

The first factor in this product is the dynamic evolution operator of the quantum CIS. It reads

$$U_1(t)\psi_{(n_1, n_a)} = \exp[-i(n_1 - \lambda_1)t]\psi_{(n_1, n_a)}. \quad (46)$$

Its eigenvalues are countably degenerate. Recall that the operator (43) acts in the pre-Hilbert module $\mathbb{C}^\infty(\mathbb{R} \times T^m)$. Its eigenvalues are smooth complex functions on $\mathbb{R}$, and its eigenspaces are $\mathbb{C}^\infty(\mathbb{R})$-submodules of $\mathbb{C}^\infty(\mathbb{R} \times T^m)$ of countable rank.

The second factor in the product (45) is

$$U_2(t) = T \exp \left[ -i \int_0^t \hat{\Delta}_\beta(\xi^\alpha(t'))\partial_t \xi^\beta dt' \right] = T \exp \left[ -i \int_{\xi(t)}^{\xi(t')} \hat{\Delta}_\beta(\sigma^\alpha) d\sigma^\beta \right]. \quad (47)$$

It acts as a matrix of countable rank in the eigenspaces of the internal Hamiltonian $\hat{\mathcal{H}}$. Its eigenspace corresponding to the eigenvalue $(n_j - \lambda_j)$ is the pre-Hilbert $\mathbb{C}^\infty(\mathbb{R})$-submodule of $\mathbb{C}^\infty(\mathbb{R} \times T^m)$ whose orthonormal basis is made up by functions $\psi_{(n_1, n_a)}$ for all collections of integers $(n_a)$. 13
A glance at the expression (47) shows that, in fact, the operator \( U_2(t) \) depends on the curve \( \xi([0,1]) \subset S \) in the parameter space \( S \). One can treat it as an operator of parallel displacement with respect to a connection in the \( C^\infty(\Sigma) \)-module of smooth complex functions on \( \Sigma \times T^m \) along the curve \( \xi \) [11, 13, 17]. For instance, if \( \xi([0,1]) \) is a loop in \( S \), the operator \( U_2(47) \) is the geometric Berry factor. In this case, one can think of \( U_2 \) as being a holonomy control operator.

It should be emphasized that, in comparison with operators usually studied [1, 2, 9, 20, 24, 28], the operator (47) acts in a countable level quantum system. Of course, the problem arises if such a system admits complete controllability and if the holonomy control operator (47) can provide this controllability. This problem will be studied elsewhere.

7 Appendix A

Let us complete the proof of 2. Since \( M_J \) are Lagrangian manifolds, the symplectic form \( \Omega_Z \) on \( W' \) is given relative to the bundle coordinates \( (J_\lambda; y^\lambda) \) by the expression

\[
\Omega_Z = \Omega^{\alpha\beta} dJ_\alpha \wedge dJ_\beta + \Omega^i_\lambda dJ_\lambda \wedge dy^i. \tag{48}
\]

By the very definition of coordinates \( (y^\lambda) \), the Hamiltonian vector fields \( \vartheta_\lambda \) of first integrals take the coordinate form

\[
\vartheta_\alpha = \partial_\alpha + \vartheta^i_\alpha (J_\lambda) \partial_i, \quad \vartheta_i = \vartheta^k_i (J_\lambda) \partial_k. \tag{49}
\]

The Hamiltonian vector fields \( \vartheta_\lambda \) obey the relations

\[
\vartheta_\lambda ] \Omega_Z = -dJ_\lambda, \quad \Omega^\alpha_\beta \vartheta^\beta_\lambda = \delta^\alpha_\lambda. \tag{50}
\]

It follows that \( \Omega^\alpha_\beta \) is a non-degenerate matrix and \( \vartheta^\alpha_\lambda = (\Omega^{-1})^\alpha_\lambda \), i.e., the functions \( \Omega^\alpha_\beta \) depend only on coordinates \( J_\lambda \). A substitution of (44) into (50) results in the equalities

\[
\Omega^a_b = \delta^a_b, \quad \vartheta^a_\lambda \Omega^i_\lambda = 0, \quad \vartheta^i_k \Omega^a_k = 0. \tag{51}
\]

The first of the equalities (52) shows that the matrix \( \Omega^a_k \) is non-degenerate, and so is the matrix \( \vartheta^i_k \). Then the second one gives \( \Omega^a_k = 0 \).

By virtue of the well-known Künneth formula for the de Rham cohomology of a product of manifolds, the closed form \( \Omega_Z \) (48) on \( W' \) (45) is exact, i.e., \( \Omega_Z = d\Xi \) where \( \Xi \) reads

\[
\Xi = \Xi^\alpha (J_\lambda, y^\lambda) dJ_\alpha + \Xi_i (J_\lambda) d\varphi^i + \partial_\alpha \Phi (J_\lambda, y^\lambda) dy^\alpha,
\]

where \( \Phi \) is a function on \( W' \). Taken up to an exact form, \( \Xi \) is brought into the form

\[
\Xi = \Xi'^\alpha (J_\lambda, y^\lambda) dJ_\alpha + \Xi_i (J_\lambda) d\varphi^i. \tag{53}
\]

Owing to the fact that components of \( d\Xi = \Omega_Z \) are independent of \( y^\lambda \) and obey the equalities (51) – (52), we obtain the following.
(i) \( \Omega^a_i = -\partial_i \Xi^a + \partial^a \Xi_i = 0 \). It follows that \( \partial_i \Xi^a \) is independent of \( \varphi^i \), i.e., \( \Xi^a \) is affine in \( \varphi^i \) and, consequently, is independent of \( \varphi^i \) since \( \varphi^i \) are cyclic coordinate. Hence, \( \partial^a \Xi_i = 0 \), i.e., \( \Xi_i \) is a function only of coordinates \( J_j \).

(ii) \( \Omega^k_i = -\partial_i \Xi^k + \partial^k \Xi_i \). Similarly to item (i), one shows that \( \Xi^k \) is independent of \( \varphi^i \) and \( \Omega^k_i = \partial^k \Xi_i \), i.e., \( \partial^k \Xi_i \) is a non-degenerate matrix.

(iii) \( \Omega^a_i = -\partial_i \Xi^a = \delta^a_i \). Hence, \( \Xi^a = -s^a + D^a(J_\lambda) \).

(iv) \( \Omega^k_i = -\partial_i \Xi^k \), i.e., \( \Xi^k \) is affine in \( s^a \).

In view of items (i) – (iv), the Liouville form \( \Xi \) reads

\[
\Xi = x^a dJ_a + [D^i(J_\lambda) + B^a_i(J_\lambda)s^a]dI_i + \Xi_i(J_j)d\varphi^i,
\]

where we put

\[
x^a = -\Xi^a = s^a - D^a(J_\lambda).
\]

Since the matrix \( \partial^k \Xi_i \) is non-degenerate, one can introduce new coordinates \( I_i = \Xi_i(J_j) \), \( I_a = J_a \). Then we have

\[
\Xi = -x^a dI_a + [D^i(I_\lambda) + B^a_i(I_\lambda)s^a]dI_i + I_id\varphi^i.
\]

Finally, put

\[
\phi^i = \varphi^i - [D^i(I_\lambda) + B^a_i(I_\lambda)s^a]
\]

in order to obtain the desired action-angle coordinates (17). These are bundle coordinates on \( U \to V' \) where the coordinate shifts \( (54) \) – \( (57) \) correspond to a choice of another

8 Appendix B

In order to prove Proposition 3, we first show that functions \( i_0^a F_k \) make up a CIS on the symplectic leaf \( (V_0^* Q, \Omega_0) \) and \( N_0 \) is its invariant manifold without critical points (i.e., where first integrals fail to be independent). Clearly, the functions \( i_0^a F_k \) are in involution, and \( N_0 \) is their connected invariant manifold. Let us show that the set of critical points of \( \{ i_0^a F_k \} \) is nowhere dense in \( V_0^* Q \) and \( N_0 \) has none of these points. Let \( V_0^* Q \) be equipped with some coordinates \( (q^k, p_k) \). Then the trivial bundle \( \xi \) is provided with the bundle coordinates \( (t, q^k, p_k) \) which play a role of the initial data coordinates on the momentum phase space \( V^* Q \). Written with respect to these coordinates, the first integrals \( F_k \) become time-independent. It follows that

\[
dF_k(y) = di_0^a F_k(\xi(y))
\]

for any point \( y \in V^* Q \). In particular, if \( y_0 \in V_0^* Q \) is a critical point of \( \{ i_0^a F_k \} \), then the trajectory \( \xi^{-1}(y_0) \) is a critical set for the first integrals \( \{ F_k \} \). The desired statement at once follows from this result.

Since \( N_0 \) is compact and regular, there is an open neighbourhood of \( N_0 \) in \( V_0^* Q \) isomorphic to \( V \times N_0 \) where \( V \subset \mathbb{R}^m \) is a domain, and \( \{ v \} \times N_0, v \in V \), are also invariant manifolds in \( V_0^* Q \). Then

\[
W'' = \xi^{-1}(V \times N_0) \cong V \times N
\]

is an open neighbourhood in \( V^* Q \) of the invariant manifold \( N \) foliated by invariant manifolds \( \xi^{-1}(\{ v \} \times N_0), v \in V \), of the time-dependent CIS on \( V^* Q \). By virtue of the equality (59),
the first integrals \( \{F_k\} \) have no critical points in \( W'' \). For any real number \( r \in (-\varepsilon, \varepsilon) \), let us consider a section

\[
h_r : V^*Q \to T^*Q, \quad p_0 \circ h_r = -\mathcal{H}(t, q^j, p_j) + r,
\]

of the affine bundle \( \zeta (6) \). Then the images \( h_r(W'') \) of \( W'' \) make up an open neighbourhood \( U \) of \( h(N) \) in \( T^*Q \). Because \( \zeta (U) = W'' \), the pull-backs \( \zeta^* F_k \) of first integrals \( F_k \) are free from critical points in \( U \), and so is the function \( \mathcal{H}^* (12) \). Since the coordinate \( r = p_0 - h \) provides a trivialization of the affine bundle \( \zeta \), the open neighbourhood \( U \) of \( h(N) \) is diffeomorphic to the product

\[
(-\varepsilon, \varepsilon) \times h(W'') \cong (-\varepsilon, \varepsilon) \times V \times h(N)
\]

which is a trivialization of the fibration

\[
\mathcal{H}^* \times (\times \zeta^* F_k) : U \to (-\varepsilon, \varepsilon) \times V.
\]

It remains to prove that the Hamiltonian vector fields of \( \mathcal{H}^* \) and \( \zeta^* F_k \) on \( U \) are complete. It is readily observed that the Hamiltonian vector field \( \gamma_T (13) \) of \( \mathcal{H}^* \) is tangent to the manifolds \( h_r(W'') \), and is the image \( \gamma_T = Th_r \circ \gamma_H \circ \zeta \) of the vector field \( \gamma_H (11) \). The latter is complete on \( W'' \), and so is \( \gamma_T \) on \( U \). Similarly, the Hamiltonian vector field

\[
\gamma_k = -\partial_i F_k \partial^0 + \partial^i F_k \partial_i - \partial_i F_k \partial^i
\]

of the function \( \zeta^* F_k \) on \( T^*Q \) with respect to the Poisson bracket \( \{,\}_T (5) \) is tangent to the manifolds \( h_r(W'') \), and is the image \( \gamma_k = Th_r \circ \partial_k \circ \zeta \) of the Hamiltonian vector field \( \partial_k \) of the first integral \( F_k \) on \( W'' \) with respect to the Poisson bracket \( \{,\}_V (8) \). The vector fields \( \partial_k \) on \( W'' \) are vertical relative to the fibration \( W'' \to \mathbb{R} \), and are tangent to compact manifolds. Therefore, they are complete, and so are the vector fields \( \gamma_k \) on \( U \). Thus, \( U \) is the desired open neighbourhood of the invariant manifold \( h(N) \).

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