Maximum directivity of arbitrary dipole arrays

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Abstract: A fully analytical form, more general than any reported previously, for the optimum current distribution and maximum directivity of an arbitrary array consisting of dipoles parallel with each other, is derived. For an array configuration consisting of several hundred elements, a calculation takes a fraction of a second making many-element superdirective arrays candidates for adaptive antenna applications. Examples for two-dimensional (2D) and 3D arrays are given and some earlier superconductive arrays are revisited. It is also shown in further examples that superconductivity is not incompatible with moderate values of the quality factor.

1 Introduction

The idea of superdirectivity first emerged close to a hundred years ago. Based on Maxwell’s equations Oseen [1] has shown that a radiation system of finite size can produce an arbitrarily sharp beam. The claim was not only counterintuitive, it stood also against everything antenna engineers knew and cherished. It shared the fate of many a radical initiative by lying dormant for a couple of decades. There were subsequently four further papers that played a major role in the acceptance of superdirectivity as something curious, but not unrealisable. Schelkunoff [2] showed that by inserting appropriately placed zeros into the polynomial describing the radiation pattern of linear arrays, it is possible to design antennas having arbitrarily sharp beams. A very influential paper was that of Uzkov [3] who showed that for an endfire array of \( N \) elements the directivity tends to \( N^2 \) as the distance between the elements tends to zero. The emphasis on directivity of general linear arrays is because of Bloch et al. [4] who derived in closed-form the maximum directivity achievable. Limitations on superdirectivity were pointed out by Uzosky and Solymar [5] by introducing the concept of tolerance sensitivity, \( T \), and quality factor, \( Q \), the latter on the basis of a circuit analogy. They presented a mathematical procedure for finding the maximum directivity for given values of \( Q \) and \( T \).

We should also mention here a classical paper by Toraldo di Francia [6] that relates superdirectivity to optical resolution and three further papers by Chu [7], Harrington [8] and Pozar [9] based on the expansion of spherical harmonics. On the experimental front an early, unjustly forgotten, paper by Bacon and Medhurst [10] needs to be quoted. They built a four-element endfire array with elements unequally spaced and with only one element fed. They were able to beat the classical limit (amplitudes equal and phase adjusted so that the fields add up in the axial direction) by five and a half decibel. Feasibility proven one might have expected a sudden upsurge of interest in experimental realisations. That did not occur, the probable reason being that classical arrays worked quite well. Theoretical interest continued however unabated. We shall mention here a few [11–19], but exclude those based on Chebyshev polynomials since the mathematical technique in those papers is quite different. Some notable recent work on directivity maximisation was done by Azavedo [20, 21] and Smierzchalski et al. [22]. Interest in the experimental realisation of superdirective arrays was reignited by Newman and Schrote [23] who designed a four-element array, built it and measured its performance. More recently, practical superdirective arrays consisting of two elements, driven independently or parasitic, were reported in [24–31]. The authors stressed the significance of using electrically small resonant elements that eases the problems of impedance matching. Metamaterial inspired superdirective antennas were reported in [32–37]. Yet another recent approach to superdirectivity was that of Ludwig et al. [38] for whom it was a by-product of their research on near-field subwavelength focusing.

The aim of the present paper is to revisit superdirectivity by finding fully analytical solutions both for the optimum current distribution and for the maximum achievable directivity. The speed with which the calculations can be performed (takes a fraction of a second for several hundred elements) makes it possible to find and modify the current distributions in real-time adaptive antennas. We shall investigate a large number of dipole arrays, both two-dimensional (2D) and 3D cases, both square and hexagonal configurations up to 450 elements and show the ranges in which maximum directivities are well above the values that can be obtained by uniform current distributions. Some previously determined optimisations using various
approximations will be revisited to show the power and generality of our formulation. Although the analytical expressions are valid for the main beam to be in any direction for arbitrary scanning angles, we shall restrict here the treatment to broadside arrays. We shall give examples likely to be of interest to antenna engineers who ever contemplated to beat the classical limit by wide margins.

2 Derivation of optimum current and maximum directivity

Our starting point is a 3D irregular array of parallel dipoles located at positions \( r_1 \ldots r_N \) within a cube of linear dimension \( L \) (Fig. 1). A reference point inside the cube is given by the vector \( r_0 \) arbitrary in general, but if the array has a centre of symmetry it may be convenient to choose that. The current in the \( n \)th dipole is denoted by \( I_n \). We shall initially choose a spherical coordinate system with its \( z \)-axis in the direction the dipoles are pointing: the elevation and azimuthal angles \( \theta \) and \( \phi \). Next we shall find the electric field due to the array at a distance \( R \gg L \) in the far-field where all the rays emanating from the dipoles may be regarded to be parallel. Characterising the direction in which we wish to determine the field by the unit vector

\[
i_R = \sin \theta \cos \phi i_x + \sin \theta \sin \phi i_y + \cos \theta i_z
\]

where \( i_x, i_y, \) and \( i_z \) are unit vectors in the directions of the \( x, y \) and \( z \) axes, the phase difference between the rays from the \( n \)th dipole and from the reference point may be obtained as

\[
\Delta \Phi_n = \beta (r_m - r_0) \cdot i_R
\]

where \( \beta = 2\pi \lambda \) is the propagation coefficient and \( \lambda \) is the wavelength. The field in the \( i_R \) direction is given by the well-known expression (see e.g. [5])

\[
E = C \sin \theta \sum_{m=1}^{N} I_m e^{-j\beta(r_m-r_0)k_z}
\]

where \( \sin \theta \) is the radiation pattern of the dipoles and \( C \) is a constant. The power density pointing in the \( i_R \) direction in the far-field is then given by

\[
P_d = \frac{|E|^2}{2Z_0} = \frac{C^2 \sin^2 \theta}{2Z_0} \left[ \sum_{m=1}^{N} I_m e^{-j\beta(r_m-r_0)k_z} \right] \left[ \sum_{n=1}^{N} I_n^* e^{j\beta(r_n-r_0)k_z} \right]
\]

where \( Z_0 \) is the free space impedance.

We shall adopt here the notations used in [5] that neither obviates the need for introducing row and column matrices nor it necessary to use complex conjugates and transposed matrices. They are all in the definitions and the result is much neater expressions as will be seen below. The inner product of two \( N \)-dimensional vectors \( a \) and \( b \) is defined as

\[
ab = \sum_{m=1}^{N} a_m b_m^*
\]

The product of a matrix \( c \) and an \( N \)-dimensional vector \( b \) yields the \( N \)-dimensional vector \( a \) defined as

\[
a_m = c_{mn} b_n^*
\]

The electric field (3) may then be written in the more condensed form as

\[
E = CIV
\]

where \( I \) and \( V \) are \( N \)-dimensional vectors

\[
I = \begin{bmatrix} I_1 & \cdots & I_m & \cdots & I_N \end{bmatrix}
\]

and

\[
V = \sin \theta [e^{j\beta(r_1-r_0)k_z} \cdots e^{j\beta(r_m-r_0)k_z} \cdots e^{j\beta(r_N-r_0)k_z}]
\]

and \( IV \) is their inner product. The power density (4) may be written in the form

\[
P_d = \frac{C^2}{2Z_0} (IV)(IV)^* = \frac{C^2}{2Z_0} |B|I
\]

where the matrix \( B \) is defined by the outer product

\[
B = V \circ V
\]

The average power density over the solid angle \( 4\pi \) is given by

\[
\langle P_d \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi P_d \sin \theta \, d\theta \, d\phi
\]

Substituting (11) into (13) and defining the directivity as power density in the maximum direction divided by the
average power density, we find the directivity in the form
\[
D = \frac{IB_{\text{max}}I}{J\alpha I}
\]
where \(B_{\text{max}}\) is the value of \(B\) when \(I_0\) is the direction of the desired maximum radiation and the Hermitian matrix \(A\) is defined as
\[
A = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\pi} V \cdot V \sin \theta \, d\theta \, d\varphi
\]
We wish to find the current distribution \(I\) that maximises \(D\). For linear arrays, this problem was solved first by Bloch et al. [4]. The solution in compact form was given by Uzsoy and Solymar [5] as
\[
I_{\text{opt}} = A^{-1} V_{\text{max}}
\]
and the maximum available directivity is
\[
D_{\text{max}} = V_{\text{max}} I_{\text{opt}}
\]
It is necessary to evaluate the integral of (15) for a general configuration of dipoles. Using (15) and the definition of the outer product, the general form of the \(mn\) element of matrix \(A\) is
\[
A_{mn} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} B_{mn} \sin \theta \, d\theta \, d\varphi
\]
where
\[
B_{mn} = \sin^2 \theta_e^{0} \sin (r_m - r_n) I_k
\]
The scalar product in the exponent may be expressed as
\[
(r_m - r_n) \cdot I_k = d_{mn} \cos \theta
\]
where
\[
d_{mn} = |r_m - r_n|
\]
is the distance between elements \(m\) and \(n\). To evaluate the integral in (18), we need to do the integration in terms of the new spherical coordinate system whose axis is in the direction of the \(r_m - r_n\) vector, shown as \(x'\) in Fig. 2. Without loss of generality, the coordinate systems \(xyz\) and \(x'y'z'\) are chosen in such a way that the dipoles \(m\) and \(n\) are in the \(xz\)-plane and also in the \(x'z'\)-plane. The relationship between the two coordinate systems, rotated by an angle \(\alpha\) in the \(xz\)-plane, may be expressed by
\[
z = z' \cos \alpha + x' \sin \alpha
\]
and
\[
z = r \cos \theta, x' = r \cos \theta', z' = r \sin \theta' \sin \varphi'
\]
From (22) and (23), we find the required relationship (i.e. expressing \(\sin^2 \theta\) in terms of \(\theta'\) and \(\varphi'\)) as follows (see (24))
\[
\sin^2 \theta = 1 - \sin^2 \theta' \cos^2 \varphi' \cos^2 \alpha - \cos^2 \theta' \sin^2 \alpha - \frac{1}{2} \sin (2\theta') \sin (2\alpha) \cos \varphi'
\]
One of the recent studies on planar arrays is that of Azevedo [20]. For a square planar array of 81 elements with inter-element distance of half-wavelength, he finds the maximum directivity (before imposing any conditions on the sidelobes) to be 23.6 dB corresponding to a factor of 229. Our curve for the same 81 element array has already been shown in Fig. 3. At a value of \( u_0 = \pi \), the directivity is 127. Considering that Azevedo made some approximations, these figures are in reasonable agreement. It needs to be noted that Azevedo calculated the directivity by ignoring radiation in the reverse direction, so his figure is bound to be larger by a factor of 2. It is of course well known that broadside planar arrays radiate equally in both directions and nobody would design an antenna to do that. Therefore if we want to avoid backward radiation and at the same time increase directivity, we may put an identical second layer behind the planar array. We shall return to such configurations in Section 4 when discussing the quality factor.

### 3.2 Square against hexagon

A question that was asked some time ago [41] was as follows: what is the optimum configuration of a planar array with the number of isotropic elements given without restraint on position, and aimed at maximum directivity in the broadside direction. The answer was a hexagonal array with neighbouring elements making up equilateral triangles with the optimum distance monotonically increasing with the number of layers of the hexagonal array. We can therefore ask now the question whether hexagonal dipole arrays would outperform square dipole arrays. Taking the example of 169 elements, we shall show in Fig. 4a that there is not much difference between the performances of these two configurations. The absolute maximum of the directivity for a square 13 × 13 array is 773 for a lower value of \( u_0 = 5.76 \) and for a hexagonal array with eight layers it is 813 for a higher value of \( u_0 = 6.27 \). The same trend holds for any number of elements, see e.g. Fig. 4b with the results for a square 20 × 20 array and for a hexagonal array with 12 layers (a total of 397 elements).

### 3.3 Circular array

Our next example is a circular array (see inset to Fig. 5) of 30 dipoles. The maximum directivity as a function of array diameter was calculated for this array by Cheng and Chen.
They started their analysis with integral equations derived earlier for half-wave dipoles of finite radius which they converted, in order to escape numerical problems, into matrix equations that could be solved without inversion. Their curve as a function of $L/\lambda$, $L$ being the diameter of the array, is shown in Fig. 5a. Our curve for short dipoles, based on (16) and (25), is shown in Fig. 5b. The two curves are remarkably similar. The treatment of Cheng and Chen is more general in the sense that they consider half-wave dipoles of finite radius, whereas we have elementary dipoles. The remarkable similarity shows that as far as directivity is concerned what matters is the radiation pattern that is very similar for a half-wave dipole of finite radius and for an elementary dipole. That is the reason for such detailed agreement between the shapes of the two curves. It should also be emphasised that Cheng and Chen had to resort to a rather complicated analysis starting with an integral equation, whereas a result, the same detailed shape, can be obtained from a simple analytical form introduced in the present paper. We should also mention here a recent publication by Ma et al. [42] concerned with optimisation of circular arrays using acoustic radiators.

### 3.4 Spherical array

Our final example is a spherical array (see inset to Fig. 6) where 14 parallel dipoles are placed symmetrically on the surface of a sphere plus one in the centre of the sphere. The directivity as a function of $u_0$ (where $u_0 = 2\pi a/\lambda$ and $a$ is the radius of the sphere) is shown in Fig. 6. This is a rather unusual example of a 3D array, but the basic features are apparently the same. There is a range where there is little difference between superdirective and 'uniform' directivity, but superdirectivity wins strongly for small $u_0$ and wins modestly for large $u_0$.

### 4 Quality factor

We have plotted the maximum directivity in a number of examples. That is of course not the end of the quest for superdirective antennas. It is well known that the greatest problem with superdirectivity is the narrow bandwidth (see e.g. [39, 40]) as indicated by the high value of the array quality factor, $Q$, defined in [5] as

$$Q = \frac{\|H\|}{|A|} \quad (27)$$

How large value of $Q$ is tolerated depends of course on the application. It is beyond the scope of the present paper to investigate, in general, the relationship between $Q$ and superdirectivity. We wish to show however a few cases when superdirectivity is well above classical directivity and nonetheless $Q$ remains quite low. Values of several hundred might be acceptable in narrow band applications, but we shall be here quite conservative and demand that the value of $Q$ should always be below 40.

![Fig. 6](https://example.com/fig6.jpg)

**Fig. 6** Directivity for optimum (solid lines) and uniform (dashed lines) current distribution as a function of the separation of the elements for a spherical array consisting of 15 dipoles shown in inset

![Fig. 7](https://example.com/fig7.jpg)

**Fig. 7** Optimum current distribution of a 3D array of $2 \times M \times M$ elements

- a Schematic representation
- b Maximum directivity as a function of axial separation for $M = 7, 11$ and $15$ for optimum (solid lines) and uniform (dashed lines) current distribution
- c Corresponding quality factor as a function of axial separation for the optimum current distribution

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For our examples, we shall now return to the directivity plots of planar arrays shown in Fig. 3 and take $M=7$, 11 and 15. In each case, we shall first choose the separation of the elements in the plane to yield the highest possible directivity, and then we shall put an identical planar array behind the first one. We shall then find the optimum current distribution for any axial separation $d_x$ between the two planar arrays (see Fig. 7a) and plot the directivity as a function of $u_x = \beta d_x$. These are shown in Fig. 7b together with the directivity curves for uniform current distribution. Plots for each configuration convey the same message. The directivity due to optimum current distribution is always above that for uniform current distribution and by quite a large margin for smaller values of $u_x$. The corresponding quality factor against axial separation for optimum current distributions is plotted in Fig. 7c. It may be seen that the quality factor only weakly depends on the number of elements. Note that the point at which $Q$ reaches 40 is denoted by a small circle on the curves for optimum directivity of Fig. 7b. To the right of that circle $Q$ is always <40 showing the feasibility of large superdirective arrays.

The quality factor being under 40 would also imply that the corresponding current distributions will only gently vary. To show this, we have chosen the array of $450 = 2 \times 15 \times 15$ elements with a normalised axial distance between the two planes of $u_x = 0.121\pi$ denoted by the black circle in Fig. 7b. The current distribution is shown in Fig. 8. In Fig. 8a, contour plots of the current amplitudes are shown that vary between 1 and 1.28. Note that the amplitudes of corresponding elements are the same in both planes. The variation of the amplitude along a diagonal is shown in Fig. 8b, a smoothly varying function. The phase distributions of the current for the front and rear surfaces are shown in Figs. 8c and d. They vary between 0 and $-5^\circ$ in the front plane and between $-160$ and $-165^\circ$ in the rear plane, shown also in the diagonal plot of Fig. 8e.

5 Conclusions

A fully analytical solution, more general than any reported previously, has been obtained for an arbitrary dipole array. A particular merit of the solution is that it can be speedily found (calculation time is a fraction of a second for several hundred elements) that makes it a candidate for real-time adaptive antenna applications. A large number of 2D and 3D examples up to 450 elements, both square and hexagon configurations, have been discussed and the availability of many superdirective solutions have been shown in which the optimum directivity is well above the classical one and still the quality factor is quite low, below 40 in a wide range. It is hoped that the present work will be instrumental in renewing interest in many-element superdirective antennas.

6 References

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Fig. 8 Current distributions of a $2 \times 15 \times 15$ array with $u_0 = 1.854\pi$ and $u_x = 0.121\pi$ yielding $Q = 40$ (as indicated by the black circle in Fig. 7b)

Amplitude (top) and phase (bottom). Surface (left) and diagonal distributions (right)
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