Scalar field perturbation on six-dimensional ultra-spinning black holes

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We have studied the scalar field perturbations on six-dimensional ultra-spinning black holes. We have numerically calculated the quasinormal modes of rotating black holes. Our results suggest that such perturbations are stable.

I. INTRODUCTION

The production of higher-dimensional black holes in collider [1, 2, 3] is the important prediction arising from the large extra dimension scenario [4, 5] or the warped compactification (RS1) scenario [6], where the fundamental (higher-dimensional) scale of gravitation can be set to the order of TeV. Such phenomena would strongly suggest the presence of the extra dimensions, and we would also obtain the experimental method to seek for the quantum gravity. Hence, it is important to investigate the fundamental properties of higher-dimensional black holes such as stability, which is the subject of the present paper.

There is a variety of the higher-dimensional black objects with nontrivial topology unlike those in four-dimensional space-times where each component of stationary black hole is a topological two-sphere. In fact, higher-dimensional vacuum Einstein equation admits simple solutions called black branes which are direct product of the four-dimensional Schwarzschild black hole and the flat space. We are usually interested in the case where the flat dimensions in the black brane space-time are compactified into say a torus.

Gregory and Laflamme [7] have found that the black branes are unstable under linear gravitational perturbations when the compactification scales are large compared with the Schwarzschild radius. On the other hand, Ishibashi and Kodama [8] have shown that the higher-dimensional Schwarzschild black holes in the spherically symmetric space-times are gravitationally stable as in the case with the four space-time dimensions. An accurate way to compute the quasinormal frequencies of higher-dimensional Schwarzschild black holes has been obtained via 6th order WKB approximation [9].

It is also important to consider the stability of the rotating black holes, since black holes produced in the collider generically have angular momenta and the production cross section is larger for more rapidly rotating black holes [10]. In contrast to the Kerr-bound on four-dimensional rotating black holes, there are black hole solutions with an arbitrary large angular momentum for a fixed mass in higher-dimensional general relativity. Recently, the existence of the effective Kerr-bound on such rapidly rotating black holes in higher dimension has been conjectured by Emparan and Myers [11]. They showed that the geometry of the event horizon of such rapidly rotating black holes in six or higher dimension behaves like the black membranes and argued that such rapidly rotating black holes therefore become unstable.

Although we should analyze the gravitational perturbations of rotating black holes to confirm such instability, any formalism where the perturbative equations are separable is yet unknown. Here we just consider the perturbation of the massless scalar field propagating on the rotating black hole space-times. Even if we restrict ourselves to field perturbation on fixed background space-time, there exists an instability of massless field perturbation on rotating black branes and strings discovered by Cardoso and Lemos [12]. The equation of motion for the massless scalar field perturbation on the rotating black holes is known to be separable in any dimension [13, 14], and authors examined the stability of this system in five-dimension as the simplest case in the previous work [14]. The present paper is the complementary work on the renewed interest in the instability (or stability) of the rapidly rotating black holes in six dimensions.
II. MYERS-PERRY BLACK HOLE AND ITS MEMBRANE LIMIT

The \((4 + n)\)-dimensional Myers-Perry metric with only one nonzero angular momentum is given by

\[
g = -\frac{\Delta - a^2 \sin^2 \vartheta}{\Sigma} dt^2 - \frac{2a(r^2 + a^2 - \Delta)}{\Sigma} \sin^2 \vartheta dt d\varphi + \left(\frac{r^2 + a^2}{\Sigma} - \Delta a^2 \sin^2 \vartheta\right) \sin^2 \vartheta d\varphi^2 + \sum_\Lambda \sin^2 \vartheta d\vartheta^2 + r^2 \cos^2 \vartheta d\Omega_n^2, \tag{1}
\]

where

\[
\Sigma = r^2 + a^2 \cos^2 \vartheta, \tag{2}
\]

\[
\Delta = r^2 + a^2 - \mu r^{1-n}, \tag{3}
\]

and \(d\Omega_n^2\) denotes the standard metric of the unit \(n\)-sphere. This metric describes a rotating black hole in asymptotically flat, vacuum space-time with the mass and angular momentum proportional to \(\mu\) and \(a\), respectively. Hereafter, \(\mu, a > 0\) are assumed. The event horizon is located at \(r = r_H\), such that \(\Delta|_{r=r_H} = 0\), which is homeomorphic to \(S^{n+2}\).

For \(n \geq 2\), \(\Delta\) always has a positive root for arbitrary \(a\), namely regular black hole solutions exist with arbitrarily large angular momentum per unit mass.

In the limit where \(a \to \infty\), the coordinate radius of the horizon is approximated by \(r_H \approx (\mu/a^2)^{1/(n-1)}\). Hence, the horizon radius \(r_H\) shrinks to zero when one take the limit where \(a \to \infty\) with fixed \(\mu\). To avoid a vanishing horizon radius, Emparan and Myers introduced the black membrane limit where \(a \to \infty\) and \(\mu \to \infty\) with fixed \(\mu := \mu/a^2\). In this limit, the Myers-Perry metric behaves like a black membrane metric near the pole \(\vartheta = 0\):

\[
g = \left(1 - \frac{\hat{\mu}}{r^{n-1}}\right) dt^2 + \frac{dr^2}{1 - \hat{\mu}/r^{n-1}} + r^2 d\Omega_n^2 + d\varsigma^2 + \varsigma^2 d\varphi^2, \tag{4}
\]

where the new coordinate is defined by \(\varsigma := a \sin \vartheta\). In this paper, we are concerned with the dimensionless Kerr parameter \(a_* := a/r_H\). The limit where \(a_* \to \infty\) with fixed \(r_H\) is consistent to the membrane limit.

III. SCALAR FIELD EQUATIONS

Here we consider the perturbation of the massless scalar field on the background Myers-Perry metric \([13]\). We put the scalar field configuration as

\[
\phi = e^{i\omega t + im\varphi} R(r) S(\vartheta) Y(\Omega), \tag{5}
\]

where \(Y(\Omega)\) is the hyperspherical harmonics on the \(n\)-sphere with the eigenvalue \(j(j+n-1), (j = 0, 1, 2, \cdots)\). Then we obtain the field equations with separated variables:

\[
\frac{1}{\sin \vartheta \cos \vartheta} \left(\frac{d}{d\vartheta} \sin \vartheta \cos \vartheta \frac{dS}{d\vartheta}\right) + \left[\omega^2 a^2 \cos^2 \vartheta - m^2 \csc^2 \vartheta - j(j+n-1) \sec^2 \vartheta + A\right] S = 0, \tag{6}
\]

and

\[
\frac{1}{r^n} \frac{d}{dr} \left(r^n \frac{dR}{dr}\right) + \left\{\frac{\omega(r^2 + a^2) - ma}{\Delta} - \frac{j(j+n-1)a^2}{r^2} - \lambda\right\} R = 0, \tag{7}
\]

where \(\lambda := A - 2m \omega a + \omega^2 a^2\).

We will solve the eigenvalue problem under the quasinormal boundary condition:

\[
R \sim \begin{cases} 
(r-r_H)^{\sigma} & (r \to r_H), \\
-r^{-(n+2)/2} e^{-i\omega r} & (r \to +\infty),
\end{cases} \tag{8}
\]

where

\[
\sigma := \frac{(r_H^2 + a^2) \omega - ma}{(n-1)(r_H^2 + a^2) + 2r_H^2}. \tag{9}
\]
When \( a = 0 \), the eigenfunctions for the angular equation (6) are analytically given in terms of the hypergeometric functions:

\[
S_{\ell jm} = (\sin \vartheta)^{|m|} (\cos \vartheta)^{j} F \left(-\ell, \ell + |m| + \frac{n+1}{2}, j + \frac{n+1}{2}; \cos^{2} \vartheta \right),
\]

with the eigenvalues

\[
A_{\ell jm} = (j + |m| + 2\ell)(j + |m| + 2\ell + 1), \quad (\ell = 0, 1, 2, \cdots).
\]

Then \( e^{-im\varphi} S_{\ell jm}(\vartheta)Y(\Omega) \) behaves as the hyperspherical harmonics on \( S^{n+2} \), which belongs to the irreducible representation of the rotational group \( SO(n+3) \). Especially for \( n = 2 \), the hyperspherical function on \( S^{4} \) with the eigenvalue \( L(L+3) \) belongs to the representation characterized by the Dynkin index \([ L, 0 ]\) and has \((L+1)(L+2)(2L+3)/6\)-fold degeneracy. When \( a \neq 0 \), \( SO(5) \) symmetry of the space-time breaks down to \( SO(2) \times SO(3) \) and the degenerate mode partially split into the several modes with \((2j+1)\)-fold degeneracies, where \( j \) and \( m \) are constrained by \( L = j + |m| + 2\ell \), \((\ell = 0, 1, 2, \cdots)\).

IV. NUMERICAL COMPUTATION

We use the continued fraction method to determine the resonant frequency \( \omega \) and the separation constant \( A \). We assume the following series expansion for \( S(t) \) and converges at any time.

\[
S = (\sin \vartheta)^{|m|} (\cos \vartheta)^{j} \sum_{k=0}^{\infty} a_{k} (\cos^{2} \vartheta)^{k},
\]

which automatically satisfies the regular boundary conditions at \( \vartheta = 0, \pi/2 \) and converges at any time. Substituting this expansion into the angular equation (6), we obtain the three-term recurrence relations:

\[
\tilde{\alpha}_{k}a_{k} + \tilde{\beta}_{k}a_{k-1} + \tilde{\gamma}_{k}a_{k-2} = 0, \quad (k = 1, 2, \cdots),
\]

where \( a_{-1} = 0 \) and the coefficients are given by

\[
\tilde{\alpha}_{k} = -8k(2j + 2k + n - 1),
\]

\[
\tilde{\beta}_{k} = 2[(j + |m| + 2k - 2)(j + |m| + 2k + n - 1) - A],
\]

\[
\tilde{\gamma}_{k} = -a^{2}_{*} \omega^{2}. \]

Here \( a_{*} := a/r_{H} \) and \( \omega_{*} := \omega r_{H} \) are the dimensionless quantities. The limit where \( a_{*} \rightarrow \infty \) corresponds to the membrane limit as mentioned above.

We expand the radial function \( R \) as

\[
R = e^{-i\omega r/r_{H}} \left( \frac{r - r_{H}}{r_{H}} \right)^{i\sigma} \left( \frac{r + r_{H}}{r_{H}} \right)^{-(n+2)/2 - i\sigma} \sum_{k=0}^{\infty} b_{k} \left( \frac{r - r_{H}}{r + r_{H}} \right)^{k},
\]

which automatically satisfies the quasinormal boundary conditions and converges at any time. When \( n = 2 \), substituting Eq. (17) into the radial equation (7), we obtain the eight-term recurrence relations

\[
\alpha_{k}b_{k} + \beta_{k}b_{k-1} + \gamma_{k}b_{k-2} + \delta_{k}b_{k-3} + \epsilon_{k}b_{k-4} + \zeta_{k}b_{k-5} + \eta_{k}b_{k-6} + \theta_{k}b_{k-7} = 0, \quad (k = 1, 2, \cdots),
\]

where \( b_{-1} = b_{-2} = \cdots = 0 \) and the coefficients are given by

\[
\alpha_{k} = -(a^{2}_{*} + 1)(a^{2}_{*} + 3)^{2}k(k + 2i\sigma),
\]

\[
\beta_{k} = (a^{2}_{*} + 3)[-2(5a^{4}_{*} + 22a^{2}_{*} + 33)a^{2} - 4ma_{*}(a^{2}_{*} + 5)\sigma, + (a^{2}_{*} + 1)[(5k^{2} - 7k + 4)a^{2} + 3k^{2} - 3k + 6 + 2j(j + 1)a^{2}_{*} + 2\lambda)],
\]

\[
+ i(a^{3}_{*} + 3)[((14k - 9)a^{4}_{*} + (40k - 22)a^{2}_{*} + 42k - 21)\sigma + 2ma_{*}(a^{2}_{*} + 3)(2k - 1)],
\]

\[
\gamma_{k} = -(27a^{6}_{*} + 117a^{4}_{*} + 89a^{2}_{*} - 129)a^{2} + 4ma_{*}(a^{2}_{*} + 3)(5a^{2}_{*} + 1)\sigma + 4m^{2}a^{2}_{*}(a^{2}_{*} + 5)
\]

\[
- (a^{2}_{*} + 1)[9k^{2} - 26k + 24]a^{4}_{*} + (14k^{2} - 42k + 40)a^{2}_{*} - 3k^{2} - 24 + 4j(j + 1)a^{2}_{*}(2a^{2}_{*} + 3) - 12\lambda]
\]

\[
- i[(30k - 42)a^{6}_{*} + (106k - 152)a^{4}_{*} + (58k - 114)a^{2}_{*} - 114k + 108]\sigma + 4ma_{*}(a^{2}_{*} + 3)(3k - 4)a^{2}_{*} - 3(k - 1)],
\]
\[ \delta_k = -(3a_\ast^2 - 1)(5a_\ast^4 + 6a_\ast^2 - 15)a^2 - 8ma_\ast(2a_\ast^2 + 3)(a_\ast^2 - 3)a - 4m^2 a_\ast^2(a_\ast^2 - 7) \\
+ (a_\ast^2 + 1)(5k^2 - 25k + 36)a_\ast^4 - 2(k - 1)(3k - 2)a_\ast^2 - 3k + 3k + 24 + 4j(j + 1)a_\ast^2(3a_\ast^2 + 2) - 4(a_\ast^2 - 2)\lambda \\
+ i\{(18k - 45)a_\ast^6 + (6k - 55)a_\ast^4 - (42k + 25)a_\ast^2 + 66k - 141| \sigma + 4ma_\ast[(2k - 5)a_\ast^4 + (-4k + 5)a_\ast^2 + 6k - 12]\} \]

\[ \epsilon_k = -2(9a_\ast^2 + 25)(a_\ast^2 + 1)^2a^2 - 8ma_\ast(2a_\ast^4 + 3a_\ast^2 - 3)a - 4m^2 a_\ast^2(a_\ast^2 - 3) \\
+ (a_\ast^2 + 1)(5k^2 - 20k + 16)a_\ast^4 + (14k^2 - 60k + 64)a_\ast^2 + 5k^2 - 24k + 40 - 4j(j + 1)a_\ast^2(2a_\ast^2 + 1) + 4\lambda \\
+ i\{(18k - 36)a_\ast^6 + (78k - 164)a_\ast^4 + (110k - 252)a_\ast^2 + 82k - 204| \sigma + 4ma_\ast[(2k - 4)a_\ast^4 + (4k - 9)a_\ast^2 + 6k - 15]\} \]

\[ \zeta_k = (a_\ast^2 + 1)(4(6a_\ast^4 + 15a_\ast^2 + 5)\sigma^2 + 20ma_\ast(a_\ast^2 + 1)\sigma + 4m^2 a_\ast^2 \\
- (k - 3)(9k - 28)a_\ast^4 - (10k^2 - 60k + 86)a_\ast^2 - (k - 2)(5k + 19) + 2(a_\ast^2 + 1)[j(j + 1)a_\ast^2 + \lambda] \\
- i(a_\ast^2 + 1)\{(30k - 93)a_\ast^4 + (52k - 160)a_\ast^2 - 2k + 13| \sigma + 2ma_\ast[(6k - 19)a_\ast^2 - 2k + 7]\}, \]

\[ \eta_k = (a_\ast^2 + 1)^2\{-(9a_\ast^2 + 13)\sigma^2 - 4ma_\ast\sigma + (k - 4)[(5k - 18)a_\ast^2 + k - 4] \\
+ 2i(a_\ast^2 + 1)^2\{(7k - 27)a_\ast^2 + 7k - 28| \sigma + 2ma_\ast(k - 4)\}, \]

\[ \theta_k = (a_\ast^2 + 1)^3| \sigma^2 - (k - 4)(k - 5) - i(2k - 9)| \sigma \].

After we repeatedly use the Gaussian elimination as written in the Appendix, the eight-term recurrence relations are reduced to the three-term ones

\[ a_k^{m+1}b_k + b_k^{m+1}a_k = 0, \quad (k = 1, 2, \ldots). \]

For the given three-term recurrence relations, the successive coefficients are obtained in two ways, the finite and infinite continued-fraction representations:

\[ \frac{a_k}{a_{k+1}} = \frac{-\tilde{\beta}_k}{\beta_k} \frac{1}{\beta_k} \frac{-\gamma_2}{\gamma_1} \frac{-\gamma_3}{\gamma_2} \frac{-\gamma_4}{\gamma_3} \ldots, \]

\[ a_k = \frac{-\tilde{\beta}_k}{\beta_k} \frac{1}{\beta_k} \frac{-\gamma_2}{\gamma_1} \frac{-\gamma_3}{\gamma_2} \frac{-\gamma_4}{\gamma_3} \ldots, \]

where \((1/x+y/z)\) is an abbreviation for \(1/(x+y/z)\). Similarly for Eq. (21), we obtain

\[ \frac{b_k}{b_{k+1}} = \frac{-\tilde{\beta}_k^{m+1}}{\beta_k^{m+1}} \frac{1}{\beta_k^{m+1}} \frac{-\gamma_2^{m+1}}{\gamma_1^{m+1}} \frac{-\gamma_3^{m+1}}{\gamma_2^{m+1}} \frac{-\gamma_4^{m+1}}{\gamma_3^{m+1}} \ldots, \]

\[ b_k = \frac{-\tilde{\beta}_k^{m+1}}{\beta_k^{m+1}} \frac{1}{\beta_k^{m+1}} \frac{-\gamma_2^{m+1}}{\gamma_1^{m+1}} \frac{-\gamma_3^{m+1}}{\gamma_2^{m+1}} \frac{-\gamma_4^{m+1}}{\gamma_3^{m+1}} \ldots, \]

For fixed \(k\) and \(k'\), we obtain nonlinear algebraic equations (28) and (29) for two unknown complex numbers \(A\) and \(\lambda\). We use a standard nonlinear root search algorithm provided by the MINPACK subroutine package to evaluate the numerical solutions of these equations. The equations for different sets of \((k,k')\) are used as numerical check.

When the perturbation become unstable under the membrane limit, we expect that the behavior of such instability resembles one of the Gregory-Laflamme instability. Since they found that the \(s\) wave is unstable, we consider the \(j = 0\) modes. The critical value of the wavelength of the Gregory-Laflamme instability is of the order of \(r_H\). In the coordinate \(\zeta\) along the rotation plane, the perturbation has the wave length \(\zeta \sim a\theta \sim a_r r_H / \ell\) for \(\ell\)-th mode. Hence, we expect that the possible instabilities occur for \(a_r > \ell\).

Typical examples of quasinormal modes of the six-dimensional Myers-Perry black holes are plotted in Figs. 1 and 4. Here the real and imaginary parts of the resonant frequencies \(\omega_\ast\) as functions of the parameter \(a_\ast / \sqrt{1 + a_\ast^2}\) for various values of \(m\) and \(\ell\) are plotted. Although the Kerr parameter \(a_\ast\) is taken to have a sufficiently large value, each mode does not seem to have negative imaginary part in any case.

V. SUMMARY AND DISCUSSION

We investigated the massless scalar field equation in the six-dimensional Myers-Perry black hole background. Under the quasinormal boundary condition, we have searched for the resonant modes.
In the non-rotating case \((a_*=0)\), the quasinormal modes with same \(L(=j+|m|+2\ell)\) are degenerate. Continuously varying the rotational parameter \(a_*\), these modes split into several modes characterized by \((j,m,\ell)\). The sequences of quasinormal modes we obtained have positive imaginary parts, thus they do not show any evidence for instability.

Recently, Cardoso, Siopsis and Yoshida investigated the stability of the same system. Several quasinormal modes within our present work are also calculated by them in different way to expand the radial function and their results are consistent to our present results. They also conclude that the stability of scalar field perturbations on Myers-Perry black holes are suggested.

**Acknowledgments**

The authors would like to thank G. w. Kang, and H. Kodama for useful discussions and comments. Y. M. is supported by a Grant-in-Aid for the 21st Century COE “Center for Diversity and Universality in Physics”.

**APPENDIX A: GAUSSIAN ELIMINATIONS**

The eight-term recurrence relations are reduced to the seven-term ones

\[ \alpha'_k b_k + \beta'_k b_{k-1} + \gamma'_k b_{k-2} + \delta'_k b_{k-3} + \epsilon'_k b_{k-4} + \zeta'_k b_{k-5} + \eta'_k b_{k-6} = 0, \quad (k = 1, 2, \cdots), \tag{A1} \]

via the Gaussian elimination:

\[
\begin{align*}
\alpha'_1 &= \alpha_1, & \beta'_1 &= \beta_1, \\
\alpha'_2 &= \alpha_2, & \beta'_2 &= \beta_2, & \gamma'_2 &= \gamma_2, \\
\alpha'_3 &= \alpha_3, & \beta'_3 &= \beta_3, & \gamma'_3 &= \gamma_3, & \delta'_3 &= \delta_3, \\
\alpha'_4 &= \alpha_4, & \beta'_4 &= \beta_4, & \gamma'_4 &= \gamma_4, & \delta'_4 &= \delta_4, & \epsilon'_4 &= \epsilon_4, \\
\alpha'_5 &= \alpha_5, & \beta'_5 &= \beta_5, & \gamma'_5 &= \gamma_5, & \delta'_5 &= \delta_5, & \epsilon'_5 &= \epsilon_5, & \zeta'_5 &= \zeta_5, \\
\alpha'_6 &= \alpha_6, & \beta'_6 &= \beta_6, & \gamma'_6 &= \gamma_6, & \delta'_6 &= \delta_6, & \epsilon'_6 &= \epsilon_6, & \zeta'_6 &= \zeta_6, & \eta'_6 &= \eta_6, \\
\alpha'_k &= \alpha_k, & \beta'_k &= \beta_k - \alpha_{k-1}\theta_k / \eta'_{k-1}, & \gamma'_k &= \gamma_k - \beta'_{k-1}\theta_k / \eta'_{k-1}, & \delta'_k &= \delta_k - \gamma'_{k-1}\theta_k / \eta'_{k-1}, & \epsilon'_k &= \epsilon_k - \delta'_{k-1}\theta_k / \eta'_{k-1} - \zeta'_k, & \eta'_k &= \eta_k - \zeta'_{k-1}\theta_k / \eta'_{k-1}, \quad (k = 7, 8, \cdots). \tag{A2}
\end{align*}
\]

The similar procedures

\[
\begin{align*}
\alpha''_1 &= \alpha'_1, & \beta''_1 &= \beta'_1, \\
\alpha''_2 &= \alpha'_2, & \beta''_2 &= \beta'_2, & \gamma''_2 &= \gamma'_2, \\
\alpha''_3 &= \alpha'_3, & \beta''_3 &= \beta'_3, & \gamma''_3 &= \gamma'_3, & \delta''_3 &= \delta'_3, \\
\alpha''_4 &= \alpha'_4, & \beta''_4 &= \beta'_4, & \gamma''_4 &= \gamma'_4, & \delta''_4 &= \delta'_4, & \epsilon''_4 &= \epsilon'_4, \\
\alpha''_5 &= \alpha'_5, & \beta''_5 &= \beta'_5, & \gamma''_5 &= \gamma'_5, & \delta''_5 &= \delta'_5, & \epsilon''_5 &= \epsilon'_5, & \zeta''_5 &= \zeta'_5, \\
\alpha''_6 &= \alpha'_6, & \beta''_6 &= \beta'_6 - \alpha_{k-1}\eta'_{k-1}, & \gamma''_6 &= \gamma_k - \beta'_{k-1}\eta'_{k-1}, & \delta''_6 &= \delta_k - \gamma'_{k-1}\eta'_{k-1}, & \epsilon''_6 &= \epsilon_k - \delta'_{k-1}\eta'_{k-1} - \zeta'_k - \eta'_{k-1}, & \zeta''_6 &= \zeta_k - \epsilon_{k-1}\eta'_{k-1} + \eta'_k, & \eta''_6 &= \eta_k - \zeta'_{k-1}\eta'_{k-1} - \eta'_k, \quad (k = 6, 7, \cdots), \tag{A3}
\end{align*}
\]

\[
\begin{align*}
\alpha''''_1 &= \alpha''_1, & \beta''''_1 &= \beta''_1, \\
\alpha''''_2 &= \alpha''_2, & \beta''''_2 &= \beta''_2, & \gamma''''_2 &= \gamma''_2, \\
\alpha''''_3 &= \alpha''_3, & \beta''''_3 &= \beta''_3, & \gamma''''_3 &= \gamma''_3, & \delta''''_3 &= \delta''_3, \\
\alpha''''_4 &= \alpha''_4, & \beta''''_4 &= \beta''_4, & \gamma''''_4 &= \gamma''_4, & \delta''''_4 &= \delta''_4, & \epsilon''''_4 &= \epsilon''_4, \\
\alpha''''_5 &= \alpha''_5, & \beta''''_5 &= \beta''_5, & \gamma''''_5 &= \gamma''_5, & \delta''''_5 &= \delta''_5, & \epsilon''''_5 &= \epsilon''_5, & \zeta''''_5 &= \zeta''_5, \\
\alpha''''_6 &= \alpha''_6, & \beta''''_6 &= \beta''_6 - \alpha_{k-1}\varepsilon'_{k-1}, & \gamma''''_6 &= \gamma_k - \beta'_{k-1}\varepsilon'_{k-1}, & \delta''''_6 &= \delta_k - \gamma'_{k-1}\varepsilon'_{k-1}, & \epsilon''''_6 &= \epsilon_k - \delta'_{k-1}\varepsilon'_{k-1} - \zeta'_k - \varepsilon'_{k-1}, & \zeta''''_6 &= \zeta_k - \epsilon_{k-1}\varepsilon'_{k-1} + \varepsilon'_k, & \eta''''_6 &= \eta_k - \zeta'_{k-1}\varepsilon'_{k-1} - \varepsilon'_k, \quad (k = 5, 6, \cdots), \tag{A4}
\end{align*}
\]

\[
\begin{align*}
\alpha'''_1 &= \alpha''_1, & \beta'''_1 &= \beta''_1, \\
\alpha'''_2 &= \alpha''_2, & \beta'''_2 &= \beta''_2, & \gamma'''_2 &= \gamma''_2, \\
\alpha'''_3 &= \alpha''_3, & \beta'''_3 &= \beta''_3, & \gamma'''_3 &= \gamma''_3, & \delta'''_3 &= \delta''_3, \\
\alpha'''_4 &= \alpha''_4, & \beta'''_4 &= \beta''_4, & \gamma'''_4 &= \gamma''_4, & \delta'''_4 &= \delta''_4, & \epsilon'''_4 &= \epsilon''_4, \\
\alpha'''_5 &= \alpha''_5, & \beta'''_5 &= \beta''_5, & \gamma'''_5 &= \gamma''_5, & \delta'''_5 &= \delta''_5, & \epsilon'''_5 &= \epsilon''_5, & \zeta'''_5 &= \zeta''_5, \\
\alpha'''_6 &= \alpha''_6, & \beta'''_6 &= \beta''_6 - \alpha_{k-1}\theta_k / \eta'_{k-1}, & \gamma'''_6 &= \gamma_k - \beta'_{k-1}\theta_k / \eta'_{k-1}, & \delta'''_6 &= \delta_k - \gamma'_{k-1}\theta_k / \eta'_{k-1}, & \epsilon'''_6 &= \epsilon_k - \delta'_{k-1}\theta_k / \eta'_{k-1} - \zeta'_k, & \eta'''_6 &= \eta_k - \zeta'_{k-1}\theta_k / \eta'_{k-1}, \quad (k = 4, 5, \cdots), \tag{A5}
\end{align*}
\]
\[ \alpha_1''' = \alpha_1'''', \quad \beta_1''' = \beta_1'''', \]
\[ \alpha_2''' = \alpha_2'''', \quad \beta_2''' = \beta_2'''', \quad \gamma_2''' = \gamma_2'''', \]
\[ \alpha_k''' = \alpha_k'''', \quad \beta_k''' = \beta_k'''', \quad \gamma_k''' = \gamma_k'''', \quad (k = 3, 4, \ldots), \quad (A6) \]

leads to the three-term relations
\[ \alpha_k'''b_k + \beta_k'''b_{k-1} + \gamma_k'''b_{k-2} = 0, \quad (k = 1, 2, \ldots). \quad (A7) \]
FIG. 2: The quasinormal modes for $(j, m, \ell) = (0, \pm 1, 0)$.

FIG. 3: The quasinormal modes for $(j, m, \ell) = (0, \pm 2, 0)$ and $(0, 0, 1)$.

FIG. 4: The quasinormal modes for $(j, m, \ell) = (0, \pm 3, 0)$ and $(0, \pm 1, 1)$. 
FIG. 5: The quasinormal modes for \((j, m, \ell) = (0, \pm 4, 0), (0, \pm 2, 1)\) and \((0, 0, 2)\).

FIG. 6: The quasinormal modes for \((j, m, \ell) = (0, \pm 5, 0), (0, \pm 3, 1)\) and \((0, \pm 1, 2)\).