Resolvability on Continuous Alphabets

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Abstract—We characterize the resolvability region for a large class of point-to-point channels with continuous alphabets. In our direct result, we prove not only the existence of good resolvability codebooks, but adapt an approach based on the Chernoff-Hoeffding bound to the continuous case showing that the probability of drawing an unsuitable codebook is doubly exponentially small. For the converse part, we show that our previous elementary result carries over to the continuous case under some mild continuity assumption.

I. INTRODUCTION

Channel resolvability has been established as an important tool in information-theoretic security [3, 11, 13]. In particular, strong secrecy can be derived directly from channel resolvability, the latter, roughly speaking, is defined as the asymptotically smallest rate of a uniform random seed that is needed to generate a channel input for which the channel output well approximates a given target output under some suitable approximation measure. Potential measures that are commonly used in the literature are the Kullback-Leibler divergence and the variational distance. The latter, roughly speaking, is defined as the resolvability. The latter extension of is strong enough to be related to security concepts from cryptography [11].

To the best of our knowledge, Wierer [22] was the first to propose the problem of approximating a given output distribution over a communication channel with as little randomness as possible on the transmitter side. In [22], he used a normalized Kullback-Leibler divergence to measure the deviation between the actual and the target output distribution. Han and Verdú [14] introduced the notion of channel resolvability and formulated a similar problem except that they assumed the variational distance as a metric. Hou and Kramer [16] considered unnormalized Kullback-Leibler divergence to measure the distance between the distributions. The results of [11, 12, 16] show that not only good resolvability codebooks exist but also that the probability of drawing an unsuitable random codebook is doubly exponentially small. Second order results for resolvability rate are presented in [21]. [6] and for MAC in [10]. Nonasymptotic results are obtained in [15].

Converse theorems for arbitrary input and output distribution (without i.i.d. assumption across channel uses) are contained in [14, 23] and [19] for MAC. A converse resolvability result based on Kullback-Leibler divergence is shown in [22] and a simpler argument is given in [17]. As we focus on variational distance, these results do not carry over to our case.

For MAC with finite alphabets and i.i.d. inputs, we have established direct and converse results in [11]. In this work, we extend those results to continuous alphabets in the point-to-point setting.

II. PRELIMINARIES

A channel $W = ([X, F], [Y, G], K)$ is given by an input alphabet $X$ with $\sigma$-algebra $F$, an output alphabet $Y$ with $\sigma$-algebra $G$ and a stochastic kernel $K$ which defines a stochastic transition between the input and output alphabets. I.e., $K$ is a mapping from $X \times G$ to $[0, 1]$ such that $K(., A)$ is measurable for each $A \in G$ and $K(x, .)$ is a probability measure on $(Y, G)$ for each $x \in X$. We assume throughout this paper that the input and output alphabets are Polish with Borel $\sigma$-algebra.

In this work, we focus on the latter one which is strong enough to be related to security concepts from cryptography [11]. The product $\sigma$-algebra $F \otimes G$ is denoted by $\mathcal{F}^\otimes$. The $n$th extension of $K$ is given by $K^{\otimes n}(x^n, \mathcal{X}_n \otimes \mathcal{A}_1)$ := $\prod_{j=1}^n K(x_j, A_j)$. An input distribution $Q_X$ on $(X, F)$ induces a joint distribution $Q_{XY}$ on $(X \times Y, F \otimes G)$ via $Q_{XY}(A_1 \times A_2)$ := $\int_{A_1} K(x, A_2)Q_X(dx)$ for $A_1 \in F$, $A_2 \in G$. The induced output distribution is denoted $Q_Y$. A codebook is a product $\sigma$-algebra $F \otimes G$. Likewise, the product of $n$ copies of $F$ is denoted by $\mathcal{F}^\otimes n$. The $n$th extension of $Q_X$ is given by $Q_X^{\otimes n}(x^n, \mathcal{X}_n \otimes \mathcal{A}_1)$ := $\prod_{j=1}^n Q_X(x_j)$, where the codewords are $C(m) \in \mathcal{X}^n$. We define the input distribution induced by $C$ as $P_{X^n|C}(A) := \exp(-nR) \sum_{m=1}^{\exp(nR)} 1_{C(m) \in A}$. $P_{X^n|C}$ and $K^{\otimes n}$ induce an input-output distribution $P_{X^nY^n|C}$ and output distribution $P_Y^{n|C}$. Any $Q_X$ on $(X, F)$ induces a distribution $P_C$ on the set of possible codebooks by drawing all the components of all the codewords i.i.d. from $X$ according to $Q_X$. Given probability measures $\mu$ and $\nu$ on $(X, F)$, we define the variational distance as $||\mu - \nu||_{TV}$ := sup$_{A \subset X} (\mu(A) - \nu(A))$. We say that $\mu$ is absolutely continuous with respect to $\nu$, in symbols $\mu \ll \nu$, if all $\mu$-null sets are $\nu$-null sets. If $\mu \ll \nu$, the Radon-Nikodym theorem states that there exists a measurable function $d\mu : X \to [0, \infty)$, called the Radon-Nikodym derivative, such that for every $A \in F$, $\mu(A) = \int_A \frac{d\mu}{d\nu}(x)d\nu(x)$. Given a channel and an input distribution, we define for any $x \in X$ and $y \in Y$ the information density of $(x, y)$ as $i(x, y) := \log \frac{dK(x, y)}{d\nu(x)}$. By convention, we say that the information density is $\infty$ on the singular set where $K(x, .)$ is not absolutely continuous with respect to $Q_Y$ and $-\infty$ where the relative density is $0$. Note that if the information density is finite almost everywhere, we can pick versions of the Radon-Nikodym derivatives such that $i(x, y)$ is measurable with respect to $F \otimes G$ [24, Chap. 5, Theorem 4.44]. In this case, we can define the mutual information of $X$ and $Y$ as $I(X; Y) := \mathbb{E}_{Q_{X,Y}} i(x, y)$. The work was supported by the German Research Foundation (DFG) under grant STA864/7-1 and by the German Federal Ministry of Education and Research under grant 16KIS0605.
Due to lack of space, we omit some proofs and we also skip some details in our proofs. For complete and more detailed proofs, we refer the reader to the extended version of this paper.

III. RESOLVABILITY REGION

Given a channel \( W = ((X, F), (Y, G), K) \) and an output distribution \( Q_Y \), a rate \( R \in [0, \infty) \) is called achievable if there is a sequence \( (C_n)_{n \geq 1} \) of codebooks with strictly increasing block lengths \( n \) and rate \( R \) such that \( \lim_{n \to \infty} \| P_{Y^n|C_n} - Q_Y^n \|_TV = 0 \). The resolvability region \( S_{W,Q_Y} \) is the closure of the set of all achievable rates.

**Theorem 3.** Let \( W = ((X, F), (Y, G), K) \) be a channel such that \( X \) is compact and for each \( A \subseteq Y \), \( x \mapsto K(x, A) \) is a continuous mapping. Let \( Q_Y \) be an output distribution. Define

\[
G(Q_Y) := \{ Q_X : Q_X induces Q_Y through W, I_{Q_X,Y}(X;Y) < \infty \}.
\]

Then

\[
S_{W,Q_Y} = \left\{ R \in \mathbb{R} : R \geq \inf_{Q_X \in G(Q_Y)} I_{Q_X,Y}(X;Y) \right\}.
\]

The inclusion \( \subseteq \) is a direct consequence of Theorem 4 in Section V which is a variation of Theorem 2. Theorem 2 even states that not only there exists a sequence of codebooks witnessing that the rate is achievable, but also that the probability of randomly drawing a “bad” codebook vanishes absolutely exponentially with increasing block length. The inclusion \( \supseteq \) is a consequence of Theorem 5 proven in Section Y.

IV. DIRECT RESULTS

The main results of this section is the following theorem.

**Theorem 2.** Given a channel \( W = ((X, F), (Y, G), K) \), an input distribution \( Q_X \) such that the moment-generating function \( \mathbb{E}_{Q_X,Y} \exp(t \cdot i(X, Y)) \) of the information density exists and is finite for some \( t > 0 \), and \( R > I(X;Y) \), there exist \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) such that for large enough block lengths \( n \), the randomized codebook distributions of block length \( n \) and rate \( R \) satisfy

\[
P_{P_Y|C} \left( \| P_{Y^n|C} - Q_Y^n \|_TV > \exp(-\gamma_1 n) \right) \leq \exp(\exp(-\gamma_2 n)). \tag{1}
\]

With a slight refinement of the proof, we can also establish the following second-order result.

**Theorem 3.** Given a channel \( W = ((X, F), (Y, G), K) \), an input distribution \( Q_X \) such that the information density \( i(X, Y) \) has finite central second moment \( V \) and finite absolute third moment \( \rho \), \( \xi > 0 \) and \( c > 1 \), suppose the rate \( R \) depends on \( n \) in the following way:

\[
R = I(X;Y) + \sqrt{\frac{V}{\rho}} Q^{-1}(\xi) + c \log n.
\tag{2}
\]

where \( Q := 1 - \Phi \) with \( \Phi \) the distribution function of the standard normal density. Then, for any \( d \in (0, c - 1) \), we have

\[
P_{P_Y|C} \left( \| P_{Y^n|C} - Q_Y^n \|_TV > \mu \left( 1 + \frac{1}{n^{d/2}} + \frac{1}{\sqrt{n}} \right) \right) \leq \exp \left( - \frac{1}{3} \frac{\mu}{n} \exp(nR) \right) + \sqrt{3 \pi} \exp \left( \frac{3}{4} \exp \left( - \frac{n}{2} c d - 1 \right) \right), \tag{3}
\]

where

\[
\mu := Q \left( Q^{-1}(\xi) + d \log \sqrt{n} \frac{V}{\rho} + \frac{\rho}{\sqrt{V} \sqrt{n}} \right) \tag{4}
\]

tends to \( \xi \) for \( n \to \infty \).

In order to prove these theorems, given a codebook \( C \), we write the variational distance as

\[
\| P_{Y^n|C} - Q_Y^n \|_TV = \sup_{A \in G^n} \left( P_{Y^n|C}(A) - Q_Y^n(A) \right)
\]

\[
= \sup_{A \in G^n} \int_A \left( \frac{dP_{Y^n|C}}{dQ_Y^n}(y^n) - 1 \right) Q_Y^n(dy^n)
\]

\[
= \mathbb{E}_{Q_Y^n} \left[ \frac{dP_{Y^n|C}}{dQ_Y^n}(y^n) - 1 \right]^+ \tag{5}
\]

Note that throughout the proofs, we only consider codebooks \( C \) for which \( P_{Y^n|C} \) is absolutely continuous with respect to \( Q_Y^n \). We can do this because \( K^{\otimes n}(x^n, \cdot) \) is absolutely continuous with respect to \( Q_Y^n \) for almost every \( x^n \), and so the probability of drawing a codebook for which \( P_{Y^n|C} \) is not absolutely continuous with respect to \( Q_Y^n \) is 0. The existence of the other Radon-Nikodym derivatives that appear in this proof can be assumed by a similar argument.

We define the typical set

\[
T_{\varepsilon} := \left\{ (x^n, y^n) : \frac{1}{n} i(x^n, y^n) \leq I(X;Y) + \varepsilon \right\} \tag{6}
\]

and split \( P_{Y^n|C} \) into two measures

\[
P_{1,c}(A) := \exp(-nR) \sum_{m=1}^{\exp(nR)} K^{\otimes n} (C(m), A \cap \{ y^n : (C(m), y^n) \in T_{\varepsilon} \}) \tag{7}
\]

\[
P_{2,c}(A) := \exp(-nR) \sum_{m=1}^{\exp(nR)} K^{\otimes n} (C(m), A \cap \{ y^n : (C(m), y^n) \notin T_{\varepsilon} \}). \tag{8}
\]

We observe \( P_{Y^n|C} = P_{1,c} + P_{2,c} \), which allows us to split (5) into a typical and an atypical part

\[
\| P_{Y^n|C} - Q_Y^n \|_TV = \mathbb{E}_{Q_Y^n} \left[ \frac{dP_{1,c}}{dQ_Y^n}(y^n) + \frac{dP_{2,c}}{dQ_Y^n}(y^n) - 1 \right]^+ \leq \mathbb{E}_{Q_Y^n} \left[ \frac{dP_{1,c}}{dQ_Y^n}(y^n) - 1 \right]^+ + P_{2,c}(Y^n). \tag{9}
\]
We next state and prove two lemmas that we will use as tools to bound the typical and atypical parts of this term separately.

**Lemma 1** (Bound for atypical terms). Suppose $Q_{X^n,Y^n}(\mathcal{X}^n \times \mathcal{Y}^n \setminus \mathcal{T}_e) \leq \mu$ and $\delta > 0$. Then

$$P_c(\mathcal{P}_2,\mathcal{C}(Y^n) > \mu(1 + \delta)) \leq \exp\left(-\frac{1}{3} \delta^2 \mu \exp(nR)\right).$$

**Proof.** Observe $\mathbb{E}_{P_c}(\mathcal{P}_2,\mathcal{C}(Y^n)) = Q_{X^n,Y^n}(\mathcal{X}^n \times \mathcal{Y}^n \setminus \mathcal{T}_e) \leq \mu$ and bound

$$P_c(\mathcal{P}_2,\mathcal{C}(Y^n) > \mu(1 + \delta)) = P_c(\exp(nR)P_c,\mathcal{P}_2,\mathcal{C}(Y^n) > \mu \exp(nR)(1 + \delta))$$

$$= P_c\left(\sum_{m=1}^{\infty} K_{\mathcal{C}}^\circ n(C(m),\{y^n : (C(m), y^n) \not\in \mathcal{T}_e\}) \geq \mu \exp(nR)(1 + \delta)\right)$$

$$\leq \exp\left(-\frac{1}{3} \delta^2 \mu \exp(nR)\right).$$

The inequality follows from the Chernoff-Hoeffding bound [8, Ex. 1.1] by noting that we sum probabilities (i.e. values in $\mathcal{Y}$) by noting that we sum probabilities (i.e. values in $\mathcal{Y}$).

**Lemma 2** (Bound for typical terms). Let $\delta, \lambda > 0$. Then

$$P_c\left(\mathbb{E}_{Q_{Y^n}} \left[ \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) - 1 \right] > \delta\right) \leq \sqrt{\frac{\pi}{3}} \exp\left(\frac{3}{4} \exp(-n(R - I(X;Y) - \varepsilon)) \lambda^2\right) \cdot \exp\left(-\frac{1}{2} n(R - I(X;Y) - \varepsilon) \right) \lambda \exp(-\delta \lambda).$$

**Proof.** An application of the Chernoff bound yields

$$\hat{P} \leq P_c\left(\mathbb{E}_{Q_{Y^n}} \exp(\lambda E_{Q_{Y^n}} f(\mathcal{C}, y^n))\right) \exp(-\delta \lambda).$$

We can then prove [10] by successive applications of Jensen’s inequality and Fubini’s theorem.

**Proof of Lemma 2.** We begin by examining parts of the term in [10] for fixed, but arbitrary $\mathcal{C}$ and $y^n$ and rewrite

$$\exp(n(R - I(X;Y) - \varepsilon)) \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n)$$

$$= \sum_{m=1}^{\infty} \exp(n(-I(X;Y) - \varepsilon)) \frac{dK_{\mathcal{C}}^\circ n(C(m),\cdot)}{dQ_{Y^n}}(y^n) \cdot 1_{[C(m),y^n) \in \mathcal{T}_e}.$$ 

Now, we observe that the indicator function bounds the relative density to be at most $\exp(n(I(X;Y) + \varepsilon))$ and thus every term in the sum to range within $[0,1]$ and that furthermore

$$\mathbb{E}_{P_c}\left(\exp(n(R - I(X;Y) - \varepsilon)) \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) \right) \leq \exp(n(R - I(X;Y) - \varepsilon)) \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) \right)$$

$$\mathop{\sum_{m=1}^{\infty}} \mathbb{E}_{P_c}\left(\frac{dK_{\mathcal{C}}^\circ n(C(m),\cdot)}{dQ_{Y^n}}(y^n) \right)$$

$$= \exp(n(R - I(X;Y) - \varepsilon)) \mathop{\sum_{m=1}^{\infty}} \mathbb{E}_{P_c}\left(\frac{dK_{\mathcal{C}}^\circ n(C(m),\cdot)}{dQ_{Y^n}}(y^n) \right).$$

We then use these observations to yield, for any $\xi > 0$,

$$P_c\left(\exp\left(\lambda \left[ \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) - 1 \right] \right) > \exp(\lambda \xi)\right)$$

$$= P_c\left(\frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) > 1 + \xi\right)$$

$$= P_c\left(\exp(n(R - I(X;Y) - \varepsilon)) \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) \right)$$

$$> (1 + \xi) (\exp(n(R - I(X;Y) - \varepsilon)))$$

$$\leq \exp\left(-\frac{\xi^2}{3} \exp(n(R - I(X;Y) - \varepsilon))\right).$$

where [12] holds because the two measured events are equal and [13] follows by the Chernoff-Hoeffding bound [8, Ex. 1.1].

We will in the following use the substitutions

$$r := \exp(n(R - I(X;Y) - \varepsilon))$$

$$a := \exp(\lambda \xi)$$

$$b := \frac{\log(a)}{\lambda} = \sqrt{\frac{2r}{3}} - \sqrt{\frac{3}{2r}} \lambda.$$ (16)

Since we will be using [16] for integration by substitution, we note that it implies

$$\frac{d}{db} a = \exp\left(b \sqrt{\frac{3}{2r}} + \lambda^2 \frac{3}{2r}\right) \lambda \sqrt{\frac{3}{2r}}.$$ (17)

We bound

$$\mathbb{E}_{P_c}\left(\exp\left(\lambda \left[ \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) - 1 \right] \right)\right)$$

$$= \int_0^\infty P_c\left(\exp\left(\lambda \left[ \frac{dP_{1,\mathcal{C}}}{dQ_{Y^n}}(y^n) - 1 \right] \right) > a\right) da$$ (18)

$$\leq \int_0^\infty \exp\left(-\frac{(\log a)^2}{3\lambda^2 r}\right) da$$ (19)
\[ = \int_{-\infty}^{\infty} \exp \left( -b^2 \lambda^2 \frac{3}{2r} + 2b \lambda \left( \frac{\lambda}{2} \right)^{\frac{3}{2r}} + \lambda^4 \left( \frac{3}{2r} \right)^2 \right) \cdot \exp \left( \frac{3}{2r} db \right) \]

where (18) is e.g. by [2] Eq. 21.9. (19) follows by substituting (13) as well as (14) and (15), and (20) follows by the density, and get (10).

Proof of Theorem 2. In order to bound the atypical term in the sum (29), note first that for any \( \alpha > 1 \),

\[ Q_{X^n,Y^n} = dP_{X^n,Y^n} \]

\[ = dP_{X^n,Y^n} \bigg| \{ (x^n, y^n) : \sum_i |x_i - y_i| / n > I(X;Y) + \epsilon \} \bigg] \]

\[ = dP_{X^n,Y^n} \bigg| \{ (x^n, y^n) : \exp ((\alpha - 1)n (I(X;Y) + \epsilon)) \} \bigg] \]

\[ \leq \int_{X^n \times Y^n} \exp ((\alpha - 1)n (I(X;Y) + \epsilon)) \cdot \exp (-n(\alpha - 1)(I(X;Y) + \epsilon)) \cdot Q_{X,Y}(d(x^n, y^n)) \]

\[ = \exp (-n(\alpha - 1)(I(X;Y) + \epsilon) - D_\alpha (Q_{X,Y} || Q_{X^n,Y^n})) \]

\[ \leq \exp (-n(\beta_1)), \] (24)

where (22) follows by applying Markov’s inequality and (24) as long as

\[ \beta_1 \leq (\alpha - 1)(I(X;Y) + \epsilon) - D_\alpha (Q_{X,Y} || Q_{X^n,Y^n}). \] (25)

Note that since the moment-generating function \( \mathbb{E}_{Q_{X,Y}} \exp(t \cdot i(X)) \) exists and is finite for some \( t > 0 \), there is some \( \alpha' > 1 \) such that \( D_{\alpha'} (Q_{X,Y} || Q_{X^n,Y^n}) \) is finite, and thus \( D_\alpha (Q_{X,Y} || Q_{X^n,Y^n}) \) is finite and continuous in \( \alpha \) for \( \alpha \leq \alpha' \). Since \( D_\alpha (Q_{X,Y} || Q_{X^n,Y^n}) \rightarrow I(X;Y) \) for \( \alpha \rightarrow 1 \), we can choose \( \alpha > 1 \), but sufficiently close to 1 such that the bound on \( \beta_1 \) is positive.

We can now apply Lemma 1 with \( \mu := \exp(-n\beta_1) \) and \( \delta := 1 \) and get

\[ P_C \left( \frac{dP_{X^n,Y^n}}{dP_{X^n,Y^n}}(y^n) \right) > 2 \exp(-n\beta_1) \]

\[ \leq \exp \left( \frac{1}{3} \exp(n(R - \beta_1)) \right). \] (26)

To bound the typical term in (9), we apply Lemma 2 with \( \lambda := \exp(n\beta_2) \) and \( \delta := \exp(-n\beta_1) \), which yields

\[ P_C \left( \int_{Y^n} \left[ \frac{dP_{X^n,Y^n}}{dQ_{Y^n}}(y^n) - 1 \right]^+ \right) \]

\[ \leq \sqrt{3 \pi} \exp \left( \frac{3}{4} \exp(-n(R - I(X;Y) - \epsilon - 2\beta_2)) \right) \cdot \exp \left( -\frac{1}{2}n(R - I(X;Y) - \epsilon - 2\beta_2) \right) \]

\[ \cdot \exp \left( -n(\beta_2 - \beta_1) \right). \] (27)

We are now ready to put everything together: Considering (26) and (27), an application of the union bound yields

\[ P_C \left( \| Y^n - p_{Y^n} \|_{TV} > \frac{3}{4} \exp(-n\beta_1) \right) \]

\[ \leq \exp \left( -\frac{1}{3} \exp(n(R - \beta_1)) \right) \]

\[ + \sqrt{3 \pi} \exp \left( \frac{3}{4} \exp(-n(R - I(X;Y) - \epsilon - 2\beta_2)) \right) \cdot \exp \left( -\frac{1}{2}n(R - I(X;Y) - \epsilon - 2\beta_2) \right) \]

\[ \cdot \exp \left( -n(\beta_2 - \beta_1) \right). \] (28)

We choose \( \epsilon < R - I(X;Y) \), then \( \beta_1 < (R - I(X;Y) - \epsilon)/2 \) small enough to satisfy (25), then \( \beta_2 \) such that \( \beta_1 < \beta_2 < (R - I(X;Y) - \epsilon)/2 \), and finally we choose \( \gamma_1 < \beta_1 \) and \( \gamma_2 < \min(R - \beta_1, \beta_2 - \beta_1) \). With these choices, (28) implies (11) for all sufficiently large \( n \), thereby concluding the proof. \( \square \)

The existence of the moment-generating function is only needed to ensure the doubly exponential convergence in (11). In fact, modifying the preceding proof slightly, we can also establish the following variation of Theorem 2.

Theorem 4. Given a channel \( \mathcal{W} = ((\mathcal{X}, \mathcal{F}), (\mathcal{Y}, \mathcal{G}), K) \), an input distribution \( Q_X \) such that \( I(X;Y) \) exists and is finite, there exist \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) such that for large enough block lengths \( n \), the randomized codebook distributions of block length \( n \) and rate \( R \) satisfy

\[ \lim_{n \rightarrow \infty} P_C \left( \| Y^n - p_{Y^n} \|_{TV} > \delta \right) = 0 \] (29)

Proof. The statement (11) is proven in (28), using as ingredients (9), (26) and (27), and (9) and (27) do not require that the moment-generating function of the information density exists and moreover, (27) can be weakened to

\[ \lim_{n \rightarrow \infty} P_C \left( \int_{Y^n} \left[ \frac{dP_{X^n,Y^n}}{dQ_{Y^n}}(y^n) - 1 \right]^+ Q_{Y^n}(dy^n) > \frac{\delta}{2} \right) = 0 \]

for any \( \delta > 0 \). In order to find a suitable replacement for (26), we consider

\[ \mathbb{E}_{P_C} (P_{2,C}(Y^n)) = Q_{X^n,Y^n}(\mathcal{X}^n \times \mathcal{Y}^n \setminus \mathcal{T}_e) = \sum_{i=1}^{n} i(x_i, y_i) - I(X;Y) > \epsilon \]
and note that by the law of large numbers, it vanishes for any \( \varepsilon > 0 \) as \( n \) tends to infinity. So by Markov’s inequality

\[
P_C \left( P_{2,C}(Y^n) > \frac{\delta}{2} \right) \leq \frac{2EP_C(P_{2,C}(Y^n))}{\delta}
\]

also vanishes for any \( \delta > 0 \). Thus, applying the union bound in a way analogous to (28), we can derive (29).

**Proof of Theorem 3** We consider the typical set as defined in (6), with

\[
\varepsilon := \sqrt{\frac{V}{n}} Q^{-1}(\xi) + d \log \frac{n}{V}.
\]

In preparation for bounding the atypical term in (6), we observe

\[
Q_{X^n,Y^n}(X^n \times Y^n \backslash T_\varepsilon) = Q_{X^n,Y^n} \left( \left\{ (x^n, y^n) : i(x^n, y^n)/n > I(X;Y) + \varepsilon \right\} \right)
\]

\[
= Q_{X^n,Y^n} \left( \left\{ (x^n, y^n) : \sum_{k=1}^{n} \left( i(x_k, y_k) - I(X;Y) \right) \sqrt{\frac{n}{V}} \right\} \right)
\]

\[
> Q^{-1}(\xi) + d \log \frac{n}{V n^\varepsilon}
\]

(31)

where (31) follows by substituting (30) and (32) by the Berry–Esseen Theorem. Knowing this, we apply Lemma 1 with \( \delta := 1/\sqrt{n} \) and get

\[
P_C \left( P_{2,C}(Y^n) > \mu(1 + 1/\sqrt{n}) \right) \leq \exp \left( -\frac{1}{3} \mu n \exp(nR) \right).
\]

In order to get a bound for the typical part of (6), we apply Lemma 2 with \( \lambda := \exp(n/2(R - I(X;Y) - \varepsilon)) \) and \( \delta := 1/\sqrt{n} \), which yields

\[
P_C \left( \int_{Y^n} \left[ dP_{1,c}(y^n) - 1 \right]^+ Q_{Y^n}(dy^n) > 1/\sqrt{n} \right)
\]

\[
\leq \sqrt{3} \pi \exp \left( \frac{3}{4} \right)
\]

\[
\cdot \exp \left( -\frac{1}{\sqrt{n}} \exp \left( \frac{1}{2} n(R - I(X;Y) - \varepsilon) \right) \right)
\]

\[
= \sqrt{3} \pi \exp \left( \frac{3}{4} \right) \exp \left( -\frac{1}{\sqrt{n}} \exp \left( \frac{1}{2} (c - d) \log n \right) \right)
\]

\[
= \sqrt{3} \pi \exp \left( \frac{3}{4} \right) \exp \left( -n^\varepsilon (c - d - 1) \right),
\]

(35)

where (35) is the application of Lemma 2 and (36) follows by substituting (35) and (30).

Finally, we arrive at (3) by combining (33) and (37) using the union bound.

**V. CONVERSE RESULT**

The main result of this section is Theorem 5, the converse result for resolvability of continuous channels. We first prove Lemma 4 a version of the theorem in which only a finite output alphabet is considered, and then show how the statement can be generalized to continuous alphabets by looking at a sequence of discrete approximations of the channel.

**Theorem 5.** Let \((\mathcal{X}, \mathcal{F}, (\mathcal{Y}, \mathcal{G}), K)\) be a channel such that \(\mathcal{X}\) is compact and for each \(A \subseteq \mathcal{Y}\), \(x \mapsto K(x,A)\) is a continuous mapping. Let \(Q_Y\) be an output distribution and \((C_t)_{t \geq 1}\) a sequence of codebooks with strictly increasing block lengths \(n_t\) and fixed rate \(R\) such that \(\mu = \|P_{Y^n|C_t} - Q_{Y^n}\|_{TV} = \delta_t \leq 1/4\) with \(\delta_t \to 0\). Then there is a joint probability measure \(Q_{X,Y}\) with marginal \(Q_Y\) for \(Y\) such that \(Q_X\) induces \(Q_Y\) through \(K\) and \(I(Q_{X,Y} (X);Y) \leq R\).

**Lemma 4.** Let \((\mathcal{X}, \mathcal{F}, (\mathcal{Y}, \mathcal{G}), K)\) be a channel such that \(\mathcal{X}\) is compact, \((\mathcal{Y}, \mathcal{G})\) is a finite discrete space and \(x \mapsto K(x,\cdot)\) is a continuous mapping from \(\mathcal{X}\) to the probability measures on \(\mathcal{Y}\). Let \(Q_Y\) be an output distribution, and \((C_t)_{t \geq 1}\) be a sequence of codebooks with strictly increasing block lengths \(n_t\) and fixed rate \(R\) such that

\[
\|P_{Y^n|C_t} - Q_{Y^n}\|_{TV} = \delta_t \leq 1/4
\]

(38)

with \(\delta_t \to 0\). Define

\[
Q_X^{(t)} := \frac{1}{n} \sum_{j=1}^{n_t} P_{X|j} \cdot C_t
\]

(39)

and \(Q_{X,Y}^{(t)}\) induced by \(Q_X^{(t)}\) through \(K\). Then there is a strictly increasing sequence \((\ell_t)_{t \geq 1}\) such that \(Q_{X,Y}^{(\ell_t)}\) converges weakly to some \(Q_{X,Y}\); the marginal \(Q_X\) induces the marginal \(Q_Y\) through \(K\) and \(I(Q_{X,Y} (X);Y) \leq R\).

**Proof of Lemma 4.** Since \(\mathcal{X}\) is compact, the space of measures \(\mu\) on \((\mathcal{X}, \mathcal{F})\) such that \(\mu(\mathcal{X}) \leq 1\) endowed with the weak topology is compact [11 Corollary 31.3]. Therefore, \((Q_X^{(t)})_{t \geq 1}\) must have a convergent subsequence, or, put differently, there is a strictly increasing sequence \((\ell_t)_{t \geq 1}\) such that \((Q_{X,Y}^{(\ell_t)})_{t \geq 1}\) converges weakly to some \(Q_{X,Y}\). By Theorem A.5.9, \(Q_{X,Y}^{(\ell_t)}\) converges weakly to some \(Q_{X,Y}\) and the marginal \(Q_X\) induces the marginal \(Q_Y\) through \(K\). We note

\[
\|P_{Y|C_t} - Q_Y\|_{TV}
\]

\[
= \frac{1}{n} \sum_{y \in Y} |P_{Y|C_t}(\{y\}) - Q_Y(\{y\})|
\]

\[
= \frac{1}{n} \sum_{y \in Y} \sum_{y^{n_t} \in \mathcal{Y}^{n_t}} \left( P_{Y^n|C_t}(\{y^{n_t}\}) - Q_{Y^n}(\{y^{n_t}\}) \right)
\]

\[
\leq \frac{1}{n} \sum_{y^{n_t} \in \mathcal{Y}^{n_t}} \left| P_{Y^n|C_t}(\{y^{n_t}\}) - Q_{Y^n}(\{y^{n_t}\}) \right|
\]

\[
= \|P_{Y^n|C_t} - Q_{Y^n}\|_{TV}
\]
and therefore
\[ \left\| Q_{Y}^{(t)} - Q_{Y} \right\|_{TV} = \frac{1}{n} \sum_{j=1}^{n} \left\| P_{Y_j|c_{t}} - Q_{Y} \right\|_{TV} \leq \frac{1}{n} \sum_{j=1}^{n} \left\| P_{Y_j|c_{t}} - Q_{Y^n} \right\|_{TV} \leq \left\| P_{Y^n|c_{t}} - Q_{Y^n} \right\|_{TV}. \] (40)

So \( Q_{Y}^{(t)} \) converges to \( Q_{Y} \) in total variation and thus, in particular, weakly. Moreover, we have \( Q_{Y} = Q_{Y^n} \) because marginalization is a continuous operation under the weak topology, so we can write \( Q_{X,Y} \) instead of \( Q_{X,Y}^{(t)} \). We further observe
\[ n_t R > \sum_{x \in X, y \in Y} P_{X^n,Y^n}(x^n, y^n) \log \frac{K(x^n, y^n)}{P_{Y^n|c_t}(y^n)} \]
\[ = \sum_{i=1}^{n} \sum_{x \in X, y \in Y} P_{X,Y|c_t}(x, y) \log K(x, y) + H_{P_{X^n,Y^n|c_t}}(Y^n) \]
\[ = \sum_{x \in X} \sum_{y \in Y} Q_{X,Y}^{(t)}(x, y) \log \frac{Q_{X,Y}^{(t)}(x, y)}{Q_{Y}^{(t)}(y)} + n_t H_{P_{X^n,Y^n|c_t}}(Y^n) \]
\[ = n_t I_{Q_{X,Y}^{(t)}}(X;Y) - n_t H_{Q_{Y}^{(t)}}(Y) + H_{P_{X^n,Y^n|c_t}}(Y^n) \]
\[ \geq n_t I_{Q_{X,Y}^{*}}(X;Y) \geq \frac{1}{2} \frac{\delta_t}{\delta_t} \frac{2}{2\sqrt{Q(Y)}} \]
\[ \geq n_t \left( I_{Q_{X,Y}^{*}}(X;Y) + \frac{1}{2} \frac{\delta_t}{\delta_t} \frac{2}{2\sqrt{Q(Y)}} \right), \] (41)

where (41) is an application of Lemma 2.7 taking into consideration (33) and (40). Thus, in particular, because the mutual information is lower semicontinuous in the weak topology [18, Theorem 1], we can conclude
\[ I_{Q_{X,Y}}(X;Y) \leq \liminf_{\ell \to \infty} I_{Q_{X,Y}^{(t)}}(X;Y) \leq R. \]

**Proof of Theorem 5** Pick an increasing sequence \((G_k)_{k \geq 1}\) of finite algebras on \( \mathcal{C} \) such that \( \bigcup_{k \geq 1} G_k \) generates \( \mathcal{G} \). Recursively construct sequences \((\ell_{k}^{(t)})_{k \geq 1}\) for each \( k \geq 0 \). Set \( \ell_{k}^{(0)} := i \). In order to construct \((\ell_{k}^{(t)})_{k \geq 1}\) for \( k > 0 \), first define a discrete finite alphabet \( \mathcal{Y}_k := \{ y \in \mathcal{Y} : y \ \text{is an atom of} \ G_k \} \) and note that any probability measure \( \mu \) on \( \mathcal{Y} \) induces a probability measure \( \mu^{(k)} \) on \( \mathcal{Y}_k \). We use the notation \( \mu^{(k)} \) and conversely, \( \mu^{(k)} \) can be seen as a probability measure on \( (\mathcal{Y}, G_k) \) by assigning to any set in \( G_k \) the sum of the probabilities of all contained atoms, or, put equivalently, \( \mu^{(k)} \) can be seen as the restriction of \( \mu \) to \( G_k \).

So for each \( x \in \mathcal{X} \), \( K(x, \cdot) \) induces a probability measure \( K^{(k)}(x, \cdot) \) on \( \mathcal{Y}_k \) and thus the stochastic kernel \( K \) and thereby a channel \((\mathcal{X} \times \mathcal{F}, (\mathcal{Y}_k, P_{Y_k}(\mathcal{Y}_k))) \), \( x \mapsto K^{(k)}(x, \cdot) \) is a continuous map to the space of probability vectors on \( \mathcal{Y}_k \), because \( x \mapsto K(x, y) \) is continuous for each \( y \in \mathcal{Y}_k \). Furthermore, \( Q_{Y}^{(k)} \) induces \( Q_{Y^{(k)}} \) in the same way.

A codebook \( C \) is also a codebook for the new channel, and we note that also \( P_{Y^{(k)}|c_t} \) is induced by \( P_{Y^n|c_t} \) in the same way.

So for any \( A \subseteq \mathcal{Y}_k \), we have \( P_{Y^{(k)}|c_t}(A) = P_{Y^n|c_t}(A) \) and \( Q_{Y^{(k)}}(A) = Q_{Y^n}(A) \). Thus,
\[ \left\| P_{Y^{(k)}|c_t} - Q_{Y^{(k)}} \right\|_{TV} \leq \sup_{A \subseteq \mathcal{Y}_k} \left( P_{Y^{(k)}|c_t}(A) - Q_{Y}^{(k)}(A) \right) \leq \left\| P_{Y^n|c_t} - Q_{Y^n} \right\|_{TV}. \]

We can therefore apply Lemma 2 to the codebook sequence \((C_{i}^{(t-1)}\),\(i \geq 1\)\) to obtain a subsequence \((C_{i}^{(t)})_{i \geq 1}\) of \((C_{i}^{(k-1)})_{i \geq 1}\) such that \( Q_{X,Y}^{(t)} \) converges weakly to some \( Q_{X,Y} \) such that \( Q_{Y}^{(k)} \) induces \( Q_{Y} \) through \( K^{(k)} \). Because of the subsequence construction, all the \( Q_{X,Y}^{(k)} \) are compatible with each other and seeing them as measures on \( (\mathcal{X} \times \mathcal{Y}, \mathcal{F} \otimes G) \), we can define a probability measure \( Q_{X,Y}^{(k)} \) on \( (\mathcal{X} \times \mathcal{Y}, \mathcal{F} \otimes G) \) as the unique extension [2] Theorem 3.1 of \( \bigcup_{i \geq 1} Q_{X,Y}^{(k)} \) to the \( \sigma \)-algebra \( \mathcal{F} \otimes G \). Since also the marginals are compatible (the marginals for \( X \) are even identical), we can apply [13] Corollary 5.2.3 and obtain \( I_{Q_{X,Y}^{(k)}}(X;Y) \to I_{Q_{X,Y}}(X;Y) \). Since we know from the statement of Lemma 4 that for all \( k \), \( I_{Q_{X,Y}^{(k)}}(X;Y) \leq R \), it follows that \( I_{Q_{X,Y}}(X;Y) \leq R \), completing the proof of the theorem. \( \square \)

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