SMOOTH STRUCTURES ON PSEUDOMANIFOLDS WITH ISOLATED CONICAL SINGULARITIES

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Abstract. In this note we introduce the notion of a smooth structure on a conical pseudomanifold \( M \) in terms of \( C^\infty \)-rings of smooth functions on \( M \). For a finitely generated smooth structure \( C^\infty(M) \) we introduce the notion of the Nash tangent bundle, the Zariski tangent bundle, the tangent bundle of \( M \), and the notion of characteristic classes of \( M \). We prove the vanishing of a Nash vector field at a singular point for a special class of Euclidean smooth structures on \( M \). We introduce the notion of a conical symplectic form on \( M \) and show that it is smooth with respect to a Euclidean smooth structure on \( M \). If a conical symplectic structure is also smooth with respect to a compatible Poisson smooth structure \( C^\infty(M) \), we show that its Brylinski-Poisson homology groups coincide with the de Rham homology groups of \( M \). We show nontrivial examples of these smooth conical symplectic-Poisson pseudomanifolds.

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1. INTRODUCTION

Since the second half of the last century the theory of smooth manifolds has been extended from various points of view to a large class of topological spaces admitting singularities, see e.g. [8, 11, 12, 23, 24, 26, 27]. Roughly speaking, a \( C^k \)-structure, \( 1 \leq k \leq \infty \), on a topological space \( M \) is defined by a choice of a subalgebra \( C^k(M) \) of the \( \mathbb{R} \)-algebra \( C^0(X) \) of all
continuous \( \mathbb{R} \)-valued functions on \( M \), which satisfies certain axioms varying in different approaches. Most of efforts have been spent on construction of a convenient category of smooth spaces, which should satisfy good formal properties, see [1] for a survey. Notably, the theory of de Rham cohomology has been extended to a large class of singular spaces, see [24], [27].

In this note we develop the theory of smooth structures on singular spaces in a different direction. We pick a class of topological spaces and ask, if we can provide these spaces with a family of reasonable smooth structures and what is the best smooth structure on a singular space. This question is motivated by the question of finding the best compactification of an open smooth manifold. We are looking not only for an extension of classical theorems on smooth manifolds, but we are also looking for new phenomena on these manifolds, which are caused by presence of nontrivial singularities.

We study in this note pseudomanifolds with isolated conical singularities. Our choice is motivated by the following reasons. Firstly, isolated conical singularities are geometrically the simplest possible, but they already serve to illustrate new phenomena that are typical for the more general situation. Secondly, the theory of smooth structures on singular spaces should include investigations related to different geometric structures compatible with these smooth structures. A closely related field of research has been developed since Cheeger wrote the seminal paper on spectral geometry of Riemannian spaces with isolated conical singularities [5]. We would like to emphasize that Cheeger and other people working on spectral geometry and index theory on singular spaces, e.g. [6], [7], [16], deal with the analysis on the open regular strata \( M^{\text{reg}} \) of a compact singular space \( M \). Although \( M^{\text{reg}} \) is open, for a large class of spaces the compactness of \( M \) forces the most fundamental features of the theory on compact manifolds to continue to hold for \( M^{\text{reg}} \). They did not consider \( M \) as a smooth space.

The plan of our note is as follows. In section 2 we introduce the notion of a pseudomanifold \( M \) with isolated conical singularities, which we will abbreviate as a pseudomanifold w.i.c.s., their cotangent bundles, the notion of smooth functions, smooth differential forms on these spaces and the notion of smooth mappings between these spaces. Known and new examples are given, see Example 2.5, some important properties of these smooth structures are proved, see Lemma 2.4, Proposition 2.12, Corollary 2.13, which are important in later sections. Our approach is close to the approach by Mostow in [24], which is formalized in the theory of \( C^\infty \)-rings as in [23]. Roughly speaking, a smooth structure on a pseudomanifold w.i.c.s. is specified by the canonical smooth structure on its regular stratum and a smooth structure around its singular points, which dictates the way to “compactify” the smooth structure around the singular point, see Definition 2.3. In section 3 we consider finitely generated smooth structures. We introduce different notions of tangent bundles of a smooth pseudomanifold w.i.c.s., which leads to the notion of characteristic classes of a finitely generated smooth pseudomanifold w.i.c.s., see Remark 3.3. We investigate some properties of
related smooth vector fields, see Proposition 3.6, Lemma 3.7. In section 4 we introduce the notion of a conical symplectic form on a pseudomanifold w.i.c.s. $M$. We show that this symplectic form is smooth with respect to a Euclidean smooth structure on $M$, see Corollary 4.6 and it possesses a unique up to homotopy compatible $C^1$-smooth conical Riemannian metric, see Lemma 4.7. We also show that if this conical symplectic form is compatible with a Poisson smooth structure on $M$, the symplectic homology of $M$ is well-defined, see Remark 4.8, Lemma 4.9. If the symplectic form is also smooth with respect to the compatible Poisson smooth structure, we prove that the symplectic homology coincides with the de Rham cohomology of $M$ with reverse grading, see Corollary 4.13. In Remark 4.8 we show non-trivial examples of smooth Poisson structures which are compatible with a conical symplectic structure on $M$. In section 5 we summarize our main results, and pose some questions for further investigations.

2. Pseudomanifolds w.i.c.s. and their smooth structures

In this section we introduce the notion of a pseudomanifold w.i.c.s. $M$, their cotangent bundles, the notion of a smooth structure, smooth differential forms on these spaces and the notion of smooth mappings between these spaces. We provide known and new examples, representing the algebra of smooth functions in terms of generators and relations, see Example 2.5, Lemma 2.14.1, Remark 2.16. We compare our concepts with some existing concepts. We prove some important properties of these smooth structures, see Lemma 2.4, Proposition 2.12, Corollary 2.13, Lemma 2.14.2, and we characterize the removability of a singular point $s \in M$ in terms of the local algebra of smooth functions on a neighborhood $N_s$ of $s$, see Lemma 2.21.

If $L$ is a smooth manifold, the cone over $L$ is the topological space

$$cL := L \times [0, \infty)/L \times \{0\}.$$ 

The image of $L \times \{0\}$ is the singular point of cone $cL$. Let $[z,t]$ denote the image of $(z,t)$ in $cL$ under the projection $\pi : L \times [0, \infty) \to cL$. Let $\rho_{cL} : cL \to [0, \infty)$ be defined by $\rho_{cL}([z,t]) := t$. We call $\rho_{cL}$ the defining function of the cone. For any $\varepsilon > 0$ we denote by $cL(\varepsilon)$ the open subset $\{[z,t] \in cL | t < \varepsilon\}$.

**Definition 2.1.** (cf. [7, Definition 1.1]) A second-countable locally compact Hausdorff topological space $M^m$ is called a pseudomanifold with isolated conical singularity of dimension $m$, if there is a finite set $S$ (or $S_{M^m}$) of isolated singular points $s_i \in M^m$ such that:

1. $M^m \setminus \bigcup_i \{s_i\}$ is an open smooth manifold $M^{reg}$ of dimension $m$.

2. For each singular point $s$ there is an open neighborhood $N_s$ of $s$ together with a homeomorphism $\phi_s : N_s \to cL_s(\varepsilon_s)$, where $L_s$ is a closed smooth manifold, $\varepsilon_s > 0$, and the restriction of $\phi_s$ to $N_s \setminus \{s\}$ is a smooth diffeomorphism on its image.

3. If $s_0, s_1 \in S$, then either $N_{s_0} \cap N_{s_1} = \emptyset$, or $s_0 = s_1$. 


The smooth manifold \( M^{reg} := M^m \setminus S \) is called the regular stratum of \( M^m \), and \( L_s \) (or simply \( L \)) is called the singularity link of a singular point \( s \). The map \( \phi_s : N_s \to cL(\varepsilon_s) \) is called a singular chart (around a singular point \( s \)). We also denote by \( N_s(\varepsilon) \) the preimage \( \phi_s^{-1}(cL(\varepsilon)) \) for \( 0 < \varepsilon \leq \varepsilon_s \).

Let us notice that that we assume the singularity link \( L_s \) to be compact. For the simplicity of exposition we suppose in this note that \( M^{reg} \) and \( L_s \) are orientable and \( M^{reg} \) is connected.

**Example 2.2.** 1. Let \( M \) be a smooth manifold with boundary \( \partial M = L \) which is a disjoint union of \( k \) compact connected components \( L_i \). An easy way to construct a pseudomanifold w.i.c.s. is to glue to \( M \) the closed cone \( \bar{cL} := L \times [0,1]/L \times \{0\} \), or to glue to \( M \) the union of the closed cones \( \bar{cL}_i \) along the boundary \( \partial M = L \times \{1\} = \cup_i (L_i \times \{1\}) \).

2. The quadric \( Q_m = \{ z \in \mathbb{C}^{m+1} | \sum_{i=1}^{m+1} z_i^2 = 0 \} \) with isolated singularity at 0 is a pseudomanifold w.i.c.s.. The quadric \( Q_m \) is a cone over \( L = Q_m \cap S^{2m+1}(\sqrt{2}) \), where \( S^{2m+1}(\sqrt{2}) \) is the sphere of radius \( \sqrt{2} \) in \( \mathbb{C}^{m+1} \). It is easy to see that \( L \) consists of all pairs \( (x, \sqrt{-1}y) \in S^m(1) \times S^m(1) \subset \mathbb{R}^{m+1} \oplus \sqrt{-1} \mathbb{R}^{m+1} = \mathbb{C}^{m+1} \) such that \( \langle x, y \rangle = 0 \). Hence \( L \) is diffeomorphic to the real Stiefel manifold \( V_{2,m+1} = SO(m+1)/SO(m-1) \).

3. Any smooth manifold with \( k \) marked points is a pseudomanifold w.i.c.s. with singular points being the marked points.

Now let us introduce the notion of a smooth structure on a pseudomanifold w.i.c.s. by refining the Mostow’s concept [24 §1]. We denote by \( C^\infty(X^{reg}) \) (resp. \( C^\infty_0(X^{reg}) \)) the space of smooth functions on \( X^{reg} \) (resp. the space of smooth functions with compact support in \( X^{reg} \)). Note that any function \( f \in C^\infty_0(X^{reg}) \) has a unique extension to a continuous function \( j_* f \) on \( X \) by setting \( j_* f(x) := 0 \) if \( x \in X \setminus X^{reg} \). The image \( j_* (C^\infty_0(X^{reg})) \) is a sub-algebra of \( C^\infty(X) \).

**Definition 2.3.** A smooth structure on a pseudomanifold w.i.c.s. \( M \) is a choice of a subalgebra \( C^\infty(M) \) of the algebra \( C^0(M) \) of all real-valued continuous functions on \( M \) satisfying the following three properties.

1. \( C^\infty(M) \) is a germ-defined \( C^\infty \)-ring, i.e. it is the \( C^\infty \)-ring of all sections of a sheaf \( SC^\infty(M) \) of continuous real-valued functions (for each open set \( U \subset M \) there is a collection \( C^\infty(U) \) of continuous real-valued functions on \( U \) such that the rule \( U \mapsto C^\infty(U) \) defines the sheaf \( SC^\infty(M) \), moreover, for any \( n \) if \( f_1, \ldots, f_n \in C^\infty(U) \) and \( g \in C^\infty(\mathbb{R}^n) \), then \( g(f_1, \ldots, f_n) \in C^\infty(U) \). [24 §1].

2. \( C^\infty(M)|_{M^{reg}} \subset C^\infty(M^{reg}) \).

3. \( j_*(C^\infty_0(M^{reg})) \subset C^\infty(M) \).

We refer the reader to [23] for the theory of \( C^\infty \)-algebras.

**Lemma 2.4.** Any smooth structure on a pseudomanifold w.i.c.s. \( M \) satisfies the following partially invertibility. If \( f \in C^\infty(M) \) is nowhere vanishing, then \( 1/f \in C^\infty(M) \).
Proof. Assume that \( f \in C^\infty(M) \) is nowhere vanishing. It suffices to show that locally \( 1/f \) is a smooth function. Since \( f \neq 0 \), shrinking a neighborhood \( U \) of \( x \) if necessary, we can assume that there is an open interval \( (-\varepsilon, \varepsilon) \) which has no intersection with \( f(U) \). Now there exists a smooth function \( \psi : \mathbb{R} \to \mathbb{R} \) such that

\[
\psi_{(U)} = Id,
\]

a) \( \psi_{(U)}(U) \) does not intersect with \( \psi(\mathbb{R}) \).

Clearly \( G : \mathbb{R} \to \mathbb{R} \) defined by \( G(x) = \psi(x)^{-1} \) is a smooth function. Note that \( 1/f(y) = G(f(y)) \) for all \( y \in U \). This completes the proof of our claim. \( \square \)

Example 2.5. 1. Let \( \tilde{M} \) be an orientable smooth manifold with a connected orientable boundary \( \partial \tilde{M} = L \) and \( M \) obtained by \( \tilde{M} \) by collapsing \( L \) to a point, see Example \( \ref{2.2}. \) Let \( C^\infty(\tilde{M}) \) be the canonical smooth structure on \( \tilde{M} \). Denote by \( \pi : \tilde{M} \to M \) the surjective continuous map which is 1-1 on \( \tilde{M} \setminus L \) to its image \( M^{\text{reg}} \). We set

\[
C^\infty_w(M) := \{ f \in C^0(M) | \pi^*(f) \in C^\infty(\tilde{M}) \}.
\]

It is easy to see that \( C^\infty_w(M) \) satisfies the conditions in Definition \( \ref{2.3} \). We call \( M \) the canonical resolution of \( M \).

2. Let \( L = S^n \) and \( X \) be the blowup of the point of origin \( 0 \in \mathbb{R}^{n+1} \), i.e. \( X = \{(x,l) \in \mathbb{R}^{n+1} \times \mathbb{R}P^n | x \in l \} \). Let \( \pi : X \to \mathbb{R}^{n+1} = cL \) be the projection on the first factor. We set \( C^\infty_{cL}(cL) := \{ f \in C^0(cL) | \pi^*(f) \in C^\infty(X) \} \). It is easy to see that \( C^\infty_{cL}(cL) \) is a smooth structure according to Definition \( \ref{2.3} \).

3. Let \( L = S^{2n+1} \) and \( X \) be a blowup of the point of origin \( 0 \in \mathbb{C}^{n+1} \) and \( \pi : X \to cL = \mathbb{C}^{n+1} \) be the canonical projection. Using this resolution \( (X \to cL) \) we define another smooth structure \( C^\infty_{cL}(cL) \) on \( cL \) which clearly also satisfies the condition in Definition \( \ref{2.3} \).

4. Let \( M \) be a pseudomanifold w.i.c.s. and \( \tilde{M} \) be a smooth manifold. We call \( \tilde{M} \) a resolution of \( M \) if there exists a continuous surjective map \( \pi : \tilde{M} \to M \) such that the restriction of \( \pi \) to \( \tilde{M} \setminus \pi^{-1}(S_M) \) is a smooth diffeomorphism on its image. Using the same construction as in examples above we define a resolvable smooth structure \( C^\infty_M(M) \) on \( M \). We observe that there are many non-diffeomorphic resolutions of a given conical pseudomanifold, which lead to different smooth structures on \( M \), e.g. Examples \( \ref{2.5}, \ref{2.6} \).

5. Let \( C^\infty_1(M) \) and \( C^\infty_2(M) \) be smooth structures on \( M \). Then \( C^\infty_1(M) \cap \ C^\infty_2(M) \) is a smooth structure on \( M \).

Remark 2.6. If \( L \) is a standard sphere, then \( cL \) has the standard smooth structure, which appears in a family of natural smooth structures on \( cL \), see Remark \( \ref{2.16}. \). Note that the larger the space \( C^\infty(M) \) is, the smaller is the space of smooth mappings from a smooth manifold \( N \) to \( M \). Thus our choice of a smooth structure on \( M \) should be guided by our desire to have a large space or a small space of smooth mappings from a smooth manifold \( N \) to \( M \).
Definition 2.7. Let $M$ and $N$ be conical pseudomanifolds provided with smooth structures $C^\infty(M)$ and $C^\infty(N)$ respectively. A continuous map $\sigma : M \to N$ is called a smooth map, if $\sigma^* (f) \in C^\infty(M)$ for all $f \in C^\infty(N)$.

Remark 2.8. Denote by $i$ the inclusion $M_{reg} \to M$. Condition (1) in Definition 2.3 implies that $i$ is a smooth map. Since the kernel of the homomorphism $i^* : C^\infty(M) \to C^0(M_{reg})$ is zero, we can regard $C^\infty(M)$ as a subalgebra of $C^\infty(M_{reg})$. In the same way we can regard $C^\infty_0(M_{reg})$ as a subalgebra of $C^\infty(M)$.

The existence of a smooth partition of unity on a smooth space is an important condition for the validity of many theorems in analysis and geometry, for example it is used in the proof of Lemma 4.7 below. In [24] Mostow analyzed several consequences of the existence of a smooth partition of unity. We will show that our smooth structures satisfy the existence of partition of unity.

Lemma 2.9. Let $s \in S$ and let $U$ be a neighborhood of $s$. Then there exists a function $f \in C^\infty(M)$ such that

1. $0 \leq f \leq 1$ on $M$;
2. $f(s) = 1$;
3. $f = 0$ outside $U$.

Proof. Obviously, there exists $\varepsilon > 0$ such that $N_s(\varepsilon) \subseteq U$. We will construct the required function $f$ in several steps using a singular chart $\phi_s : N_s(\varepsilon) \to cL(\varepsilon)$.

In the first step we define an auxiliary smooth function $\chi \in C^\infty_0((0, \varepsilon))$. It is defined in the following way.

$$
\chi(a) = 0 \text{ for } a \in (0, \frac{1}{5}\varepsilon], \quad 0 < \chi(a) < 1 \text{ for } a \in \left(\frac{1}{5}\varepsilon, \frac{2}{5}\varepsilon\right],
$$

$$
\chi(a) = 1 \text{ for } a \in \left[\frac{2}{5}\varepsilon, \frac{3}{5}\varepsilon\right], \quad 0 < \chi(a) < 1 \text{ for } a \in \left(\frac{3}{5}\varepsilon, \frac{4}{5}\varepsilon\right],
$$

$$
\chi(a) = 0 \text{ for } a \in \left[\frac{4}{5}\varepsilon, \varepsilon\right).
$$

In the second step we define a continuous function $\chi_M \in C^0(M)$ by setting

$$
\chi_M(x) := \chi \circ \rho_{cL}(\phi_s(x)) \text{ for } x \in N_s(\varepsilon),
$$

$$
\chi(x) := 0 \text{ for } x \in (M_{reg} \setminus \phi_s^{-1}(L \times (0, \varepsilon))).
$$

Note that $\chi_M$ is a smooth function on $M_{reg}$ with compact support, and consequently an element of $C^\infty(M)$.

In the third step we define a new function $\psi \in C^0(M)$. We set

$$
\psi(x) := 1 \text{ for } x \in (M \setminus \phi_s^{-1}(cL(\varepsilon))) \text{ or } x \in \phi_s^{-1}((L \times (\frac{2}{5}\varepsilon, \varepsilon)))
$$

$$
\psi(x) := \chi_M(x) \text{ for } x \in \phi_s^{-1}(L \times (0, \frac{2}{5}\varepsilon)], \text{ and } \psi(s) := 0.
$$
Let us show that on a neighborhood of any point \( x \in M \) the function \( \psi \) coincides with a function from \( C^\infty(M) \). If \( x \in (M \setminus \psi_s^{-1}(cL(\varepsilon))) \) or \( x \in \psi_s^{-1}(L \times (\frac{1}{2},\varepsilon)) \), then on a neighborhood of \( x \) the function \( \psi \) coincides with the constant function \( 1 \in C^\infty(M) \). If \( x \in \psi_s^{-1}(L \times (0,\varepsilon/2)) \), then on a neighborhood of \( x \) the function \( \psi \) coincides with the function \( \chi \in C^\infty(M) \). Finally on a neighborhood of the point \( s \) the function \( \psi \) coincides with a constant function \( 0 \in C^\infty(M) \). Consequently \( \psi \in C^\infty(M) \), and then also \( f = 1 - \psi \in C^\infty(M) \). This function has all the required properties.

**Lemma 2.10.** For every compact subset \( K \subset M \) and every neighborhood \( U \) of \( K \) there exists a function \( f \in C^\infty(M) \) such that

1. \( f \geq 0 \) on \( M \);
2. \( f > 0 \) on \( K \);
3. \( f = 0 \) outside \( U \).

**Proof.** For each point \( x \in K \) we take its open neighborhood \( V_x \) in such a way that \( V_x \subset V_i \) and we take a function \( f_x \in C^\infty(M) \) described in Lemma 2.9 (note that Lemma 2.9 trivially holds for any regular point \( x \in M^{reg} \)). Finally, we take an open neighborhood \( W_x \subset V_x \) of \( x \) such that \( f_{x,W_x} > \frac{1}{2} \).

Because \( K \) is compact, we can find a finite number of \( x_1, \ldots, x_r \) in \( K \) such that

\[
W_{x_1} \cup \cdots \cup W_{x_r} \supset K.
\]

Now it is sufficient to set \( f = f_{x_1} + \cdots + f_{x_r} \).

**Lemma 2.11.** Let \( \{U_i\}_{i \in I} \) be a locally finite open covering of \( M \). Then there exists a locally finite open covering \( \{V_i\}_{i \in I} \) (with the same index set) such that \( \bar{V}_i \subset U_i \).

**Proof.** The proof is standard.

**Proposition 2.12.** Let \( \{U_i\}_{i \in I} \) be a locally finite open covering of \( M \) such that each \( U_i \) has a compact closure \( \bar{U}_i \). Then there exists a partition of unity \( \{f_i\}_{i \in I} \) subordinate to \( \{U_i\}_{i \in I} \).

**Proof.** Let \( \{V_i\}_{i \in I} \) be the same covering as in Lemma 2.11. Let \( \{W_i\}_{i \in I} \) be an open covering such that \( V_i \subset W_i \subset \bar{W}_i \subset U_i \). According to Lemma 2.10 for every \( i \in I \) there exists a function \( g_i \in C^\infty(M) \) such that

1. \( g_i \geq 0 \) on \( M \);
2. \( g_i > 0 \) on \( \bar{V}_i \);
3. \( g_i = 0 \) outside \( W_i \).

Because \( V_i \subset \text{supp } g_i \subset U_i \) for every \( i \in I \), the sum \( g = \sum_{i \in I} g_i \) is well defined and everywhere positive. Since our algebra \( C^\infty(M) \) is germ-defined, \( g \) belongs to \( C^\infty(M) \), and according to the partial invertibility property in Lemma 2.3 \( 1/g \in C^\infty(M) \). Consequently, defining \( f_i = g_i/g \), we obtain the desired partition of unity.

**Corollary 2.13.** Smooth functions on \( M \) separate points on \( M \).
Proof. Let \( x_1, x_2 \in M \), \( x_1 \neq x_2 \). We take an \( \varepsilon \)-neighborhood \( N_{x_2}(\varepsilon) \) of \( x_2 \) such that \( x_1 \not\in N_{x_2}(\varepsilon) \). Then it suffices to take a function \( f \) from Lemma 2.9 and we have \( f(x_1) = 0 \) and \( f(x_2) = 1 \). \( \square \)

Next we would like to define a notion of a locally smoothly contractible differentiable structure on \( M \). For this purpose we shall have to take a product \( U(x) \times [0, 1] \), where \( U(x) \) is an open neighborhood of \( x \in M \), and endow it with a differentiable structure. Though the product \( U(x) \times [0, 1] \) need not be a pseudomanifold w.\,i.\,c.\,s., we can use the same concept of a smooth structure as Mostow used \cite{24} §3. We say that \( C^\infty(M) \) is locally smoothly contractible, if for any \( x \in M \) there exists an open neighborhood \( U(x) \ni x \) together with a smooth homotopy \( \sigma : U(x) \times [0, 1] \to U(x) \) joining the identity map with the constant map \( U(x) \ni x \to x \) \cite{24} §5. Note that there is a natural smooth structure \( C^\infty(U(x) \times [0, 1]) \) generated by \( C^\infty(U(x)) \) and \( C^\infty([0, 1]) \) \cite{24} §3, more precisely, the sheaf \( SC^\infty(U(x) \times [0, 1]) \) is generated by \( \pi_1(SC^\infty([0, 1])) \) and \( \pi_2(SC^\infty(U(x))) \), where \( \pi_1 \) and \( \pi_2 \) is the projection from \( U(x) \times [0, 1] \) to \([0, 1]\) and \( U(x) \) respectively. In particular, \( \pi_1 \) and \( \pi_2 \) are smooth maps.

Denote by \( C^\infty_{L \times \{0\}}(L \times [0, \infty)) \) the subalgebra in \( C^\infty((L \times [0, \infty)) \) consisting of functions taking constant values along \( L \times \{0\} \). Clearly \( C^\infty_{L \times \{0\}}(L \times [0, \infty)) \) is isomorphic (as \( \mathbb{R} \)-algebra) to \( C^\infty_w(cL) \).

**Lemma 2.14.** 1. A function \( f(x, t) \in C^\infty(L \times [0, \infty)) \) belongs to \( C^\infty_{L \times \{0\}}(L \times [0, \infty)) \) if and only if \( f \) can be written as \( f(x, t) = t \cdot g(x, t) + c \), where \( g \in C^\infty(L \times [0, \infty)) \) and \( c \in \mathbb{R} \).

2. Let \( C^\infty_e(cL) \subset C^\infty_w(cL) \) be the subalgebra consisting of all functions \( f \) on which can be written as \( f([x, t]) = g(tf_1(x), \ldots, tf_k(x)) \) for some \( g \in C^\infty(\mathbb{R}^k) \) and \( f_i \in C^\infty(L) \). Then \( C^\infty_e(cL) \) is a locally smoothly contractible smooth structure on \( cL \).

**Proof.** 1) The “if” assertion in the first statement is obvious. Let us prove the “only if” assertion. For any \( f \in C^\infty(L \times [0, \infty)) \) we have

\[
 f(x, t) = f(x, 0) + \int_0^1 \frac{df(x, tr)}{dr} dr = f(x, 0) + t \int_0^1 \frac{df(x, tr)}{d(tr)} dr.
\]

Clearly \( \int_0^1 \frac{df(x, tr)}{d(tr)} dr \in C^\infty(L \times [0, \infty)) \). This proves the first statement.

2) It is easy to see that \( C^\infty_e(cL) \) satisfies the first condition in Definition 2.3. We observe that \( C^\infty_e(cL) \) also satisfies the second condition of Definition 2.3, i.e. \( f_*([x, t], \lambda) \ni [x, \lambda t] \) for any \( f \in C^\infty_0(cL) \), since this assertion is a consequence of Remark 2.16.4 below. To prove the second statement of Lemma 2.14 it suffices to show that the map

\[
 F : cL(1) \times [0, 1] \to cL, \ ([x, t], \lambda) \mapsto [x, \lambda t]
\]

is a smooth map. Equivalently we have to show that any function \( F^*(f) \), \( f \in C^\infty_e(cL(1)) \), belongs to the germ-defined \( C^\infty \)-ring \( C^\infty(cL(1) \times [0, 1]) \) generated by \( C^\infty_e(cL(1)) \) and \( C^\infty([0, 1]) \). Repeating the previous argument,
we can write \( f([x, t]) = g(t f_1(x), \ldots, t f_k(x)) \), where \( f_i \in C^\infty(L) \) and \( g \in C^\infty(\mathbb{R}^k) \). Clearly \((F^*(f))(x, t), \lambda) = g(\lambda t f_1(x), \ldots, \lambda t f_k(x))\) can be written as a function \( G(\lambda, t f_1(x), \ldots, t f_k(x)) \), hence it belongs to \( C^\infty(cL(1) \times [0, 1]) \).

\[ \square \]

Now we show a geometric way to construct a nice locally smoothly contractible smooth structure on a conical pseudomanifold \( M \).

**Definition 2.15.** A Euclidean smooth structure on a pseudomanifold w.i.c.s. \( M \) is defined by a smooth embedding \( I_s : L_s \to S^l(1) \subset \mathbb{R}^{l+1} \) and a trivialization \( \phi_s : \mathcal{N}_s \to cL_s \) for each \( s \in S_M \) as follows. Let \( \tilde{I}_s \) denote the induced embedding of \( cL_s \to \mathbb{R}^{l+1} \). A continuous function on \( M \) is called smooth, if it is smooth on \( M^{reg} \) and its restriction to \( \mathcal{N}_s \) is a pull back of a smooth function on \( \mathbb{R}^{l+1} \) via \( \tilde{I}_s \circ \phi_s \) for all \( s \).

By composing an embedding \( I_s \) with an isometric embedding \( g_{i,l+k} : S^l(1) \to S^{l+k}(1) \subset \mathbb{R}^{l+k+1} \) we get another embedding \( I_{s,k} : L_s \to S^{l+k}(1) \). Denote by \( \tilde{I}_{s,k} \) the induced embedding \( cL_s \to \mathbb{R}^{l+k+1} \). It is easy to see that the smooth structures defined by \( I_s \) and \( I_{s,k} \) are equivalent. This motivates us to give the following concept.

Two smooth embeddings \( I_1^s : L_s \to S^{k_1}(1) \subset \mathbb{R}^{k_1+1} \) and \( I_2^s : L_s \to S^{k_2}(1) \subset \mathbb{R}^{k_2+1} \) are called Euclidean equivalent, if there exists a diffeomorphism \( \Theta : \mathbb{R}^{k_1+k_2+2} \to \mathbb{R}^{k_1+k_2+2} \) such that \( \Theta \circ \tilde{I}_{s,k_1}^s = \tilde{I}_{s,k_2}^s \). Two Euclidean smooth structures are called Euclidean equivalent, if the corresponding embeddings \( I_s \) are Euclidean equivalent. The embedding \( \tilde{I}_s \circ \phi_s \) is called a smooth chart around singular point \( s \in S_M \).

**Remark 2.16.** 1. A pseudomanifold w.i.c.s. may have more than one Euclidean smooth structure. For example, conical pseudomanifolds \( cL(z = \frac{1}{2}) \) and \( cL(z = 0) \) are not isomorphic, where \( L(z = \theta) \) is the circle in \( S^2(1) \subset \mathbb{R}^3(x, y, z) \) defined by the equation \( z = \theta \in (-1, 1) \). This is proved by observing that the function \( f(x, y, z) = (x^2 + y^2 + z^2)^{1/2} \) is smooth on \( cL(z = \frac{1}{2}) \) but it is not smooth on \( cL(z = 0) \). Using the diffeomorphism \( T_\alpha : \mathbb{R}^3 \to \mathbb{R}^3, z \mapsto \alpha z, \alpha \neq 0 \), we conclude that all \( cL(z = \alpha) \) are diffeomorphic, if \( 0 < |\alpha| < 1 \). Note that the “smallest” smooth structure on \( cS^1 \) is the isolated smooth structure \( cL(z = 0) \).

2. Clearly any Euclidean smooth structure is locally smoothly contractible, since the homotopy \( cL \times [0, 1] \to cL : ([x, t], \lambda) \mapsto [x, \lambda t] \) is a smooth map.

3. In the next section, see Proposition 3.5 we will show that for any fixed \( L \) there are infinitely many non-equivalent Euclidean structures on \( L \).

4. Suppose that \( L \) is compact and \( C^\infty(cL) \) is a Euclidean smooth structure on \( cL \). Let \( I_s : L \to \mathbb{R}^k \) is defined by \( k \) smooth functions \( f_i \in C^\infty(L), i = 1, \ldots, k \). Then \( \tilde{f}_i(t, x) := tf_i(x) \) are generators of the associated Euclidean smooth structure \( C^\infty(cL) \). Thus \( C^\infty(cL) \) is a subalgebra of the algebra \( C^\infty_c(cL) \).
We say that $C^\infty(M)$ is \textit{finitely generated}, if there is a finite number of functions $f_1, \ldots, f_k \in C^\infty(M)$ such that any $g \in C^\infty(M)$ is of form $g := \hat{g}(f_1, \ldots, f_k)$ for some $\hat{g} \in C^\infty(\mathbb{R}^k)$. Functions $f_1, \ldots, f_k$ are called \textit{generators} of $C^\infty(M)$. Remark 2.164 asserts that a Euclidean smooth structure is finitely generated.

**Proposition 2.17.** Suppose that $M$ and $N$ are pseudomanifolds \textit{w.i.c.s.} provided with a finitely generated smooth structure. A continuous map $\sigma : M \to N$ is smooth, if and only if for each $x \in M$ there exist a smooth chart $\phi_x : U(x) \to \mathbb{R}^n$, a smooth chart $\phi_{\sigma(x)} : U(\sigma(x)) \to \mathbb{R}^m$, and a smooth map $\tilde{\sigma} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\phi_{\sigma(x)} \circ \sigma = \tilde{\sigma} \circ \phi_x$. Consequently, two Euclidean smooth structures on a pseudomanifold \textit{w.i.c.s.} $M$ are Euclidean equivalent, if and only if they are equivalent.

\textbf{Proof.} 1) The first assertion of Proposition 2.17 is a special case of Proposition 1.3.8 in [29], see also [23] Proposition 1.5 for an equivalent formulation. For the convenience of the reader we give a proof of this assertion, which is similar to the proof in the case of smooth manifolds. The "if" part is clear, so we will prove the "only" part. Let $y_1, \ldots, y_m$ be coordinate functions on $\mathbb{R}^m$. By our assumption, $y_k(\phi_{\sigma(x)} \circ \sigma)$ is a smooth function on $U(x)$, hence there exist smooth functions $f_k$ on $\mathbb{R}^n$ such that $f_k(\phi_x) = y(\phi_{\sigma(x)} \circ \sigma)$ for $k = 1, m$. Now we define a smooth map $\tilde{\sigma} : \mathbb{R}^n \to \mathbb{R}^m$ by setting

$$\tilde{\sigma}(x) = (f_1(x), \ldots, f_k(x)).$$

Clearly $\tilde{\sigma}$ satisfies the condition of our Proposition 2.17.1.

2) Let us prove the "only if" part of second assertion of Proposition 2.17. Assume that two Euclidean smooth structures $C^\infty_1(M)$ and $C^\infty_2(M)$ are Euclidean equivalent. Using the existence of a smooth partition of unity, see Lemma 2.12 and finiteness of $S_M$, it is easy to see that $C^\infty_1(M)$ and $C^\infty_2(M)$ are equivalent, i.e. there exists a homeomorphism $\sigma : M \to M$ such that $\sigma^*(C^\infty_1(M)) = C^\infty_2(M)$. (The proof is similar to the proof for the case of smooth manifolds $M, N$.)

Now we will prove the "if" part of the second assertion, i.e. we assume that there exists a homeomorphism $\sigma : M \to M$ such that $\sigma^*(C^\infty_1(M)) = C^\infty_2(M)$. Let $\{(I_i : L_{s_i} \to S_{l_i} \subset \mathbb{R}^{l_i+1}, \phi_{s_i} : N_{s_i} \to cL_{s_i}) | s_i \in S_M \}$ be embeddings defining $C^\infty_1(M)$. Then $\{(I_{s_i}, \sigma \circ \phi_{s_i}) | s_i \in S_M \}$ are embeddings defining $C^\infty_2(M)$. This proves that $C^\infty_1(M)$ and $C^\infty_2(M)$ are Euclidean equivalent. \hfill $\Box$

Next we introduce the notion of the cotangent bundle of a stratified space $X$, which is similar to the notions introduced in [26], [29] B.1. Note that the germs of smooth functions $C^\infty_x(X)$ is a local $\mathbb{R}$-algebra with the unique maximal ideal $m_x$ consisting of functions vanishing at $x$. Set $T^*_x(X) := m_x/m^2_x$. Since the following exact sequence

\begin{equation}
0 \to m_x \to C^\infty_x \xrightarrow{\partial} \mathbb{R} \to 0
\end{equation}
split, where \( j \) is the evaluation map: \( j(f_x) = f_x(x) \) for any \( f_x \in C_x^{\infty} \), the space \( T^*_x X \) can be identified with the space of Kähler differentials of \( C_x^{\infty}(X) \).

The Kähler derivation \( d : C_x^{\infty}(X) \to T^*_x X \) is defined as follows:

\[
(2.2) \quad d(f_x) = (f_x - j^{-1}(f_x(x))) + m_x^2,
\]

where \( j^{-1} : \mathbb{R} \to C_x^{\infty} \) is the left inverse of \( j \), see e.g. \cite{19} Chapter 10, or \cite{29} Proposition B.1.2. We call \( T^*_x X \) the cotangent space of \( X \) at \( x \). Its dual space \( T^*_x X := \text{Hom}(T_x^* X, \mathbb{R}) \) is called the Zariski tangent space of \( X \) at \( x \). The union \( \bigcup_{x \in X} T^*_x X \) is called the cotangent bundle of \( X \). The union \( \bigcup_{x \in X} T^*_x X \) is also called the Zariski tangent bundle of \( X \).

Let us denote by \( \Omega^1_x(X) \) the \( C_x^{\infty}(X) \)-module \( C_x^{\infty}(X) \otimes \mathbb{R} m_x/m_x^2 \). We called \( \Omega^1_x(X) \) the germs of 1-forms at \( x \). Set \( \Omega^k_x(X) := C_x^{\infty}(X) \otimes \mathbb{R} \Lambda^k(m_x/m_x^2) \). Then \( \otimes \Omega^k_x(X) \) is an exterior algebra with the following wedge product

\[
(2.3) \quad (f \otimes_R d g_1 \wedge \cdots \wedge d g_k) \wedge (f' \otimes_R d g_{k+1} \wedge \cdots \wedge d g_l) = (f \cdot f') \otimes_R d g_1 \wedge \cdots \wedge d g_l,
\]

where \( f, f' \in C_x^{\infty} \) and \( d g_i \in T^*_x M \).

Note that the Kähler derivation \( d : C_x^{\infty}(X) \to \Omega^0_x(X) \) extends to the unique derivation \( d : \Omega^k_x(X) \to \Omega^{k+1}_x(X) \) satisfying the Leibniz property. Namely we set

\[
d(f \otimes 1) = 1 \otimes df,
\]

\[
d((f \otimes \alpha) \wedge (g \otimes \beta)) = d(f \otimes \alpha) \wedge g \otimes \beta + (-1)^{\deg \alpha} f \otimes \alpha \wedge d(g \otimes \beta).
\]

**Definition 2.18.** (cf. \cite{21} §2) A section \( \alpha : X \to \Lambda^k T^* x(X) \) is called a smooth differential \( k \)-form, if for each \( x \in X \) there exists \( U(x) \subset X \) such that \( \alpha(x) \) can be represented as \( \sum_{i_0 \cdots i_k} f_{i_0} df_{i_1} \wedge \cdots \wedge df_{i_k} \), for some \( f_{i_0}, \ldots, f_{i_k} \in C_x^{\infty}(X) \).

Denote by \( \Omega(X) = \otimes_k \Omega^k(X) \) the space of all smooth differential forms on \( X \). We identify the germ at \( x \) of a \( k \)-form \( \sum_{i_0 \cdots i_k} f_{i_0} df_{i_1} \wedge \cdots \wedge df_{i_k} \) with element \( \sum_{i_0 \cdots i_k} f_{i_0} \otimes d f_{i_1} \wedge \cdots \wedge df_{i_k} \in \Omega^k_x(X) \). Clearly the Kähler derivation \( d \) extends to a map also denoted by \( d \) mapping \( \Omega(X) \) to \( \Omega(x) \).

**Remark 2.19.** Let \( i^*(\Omega(X)) \) be the restriction of \( \Omega(X) \) to \( X^{\text{reg}} \). By Remark \ref{2.8} the kernel \( i^* : \Omega(X) \to \Omega(X^{\text{reg}}) \) is zero. Roughly speaking, we can regard \( \Omega(X) \) as a subspace in \( \Omega(X^{\text{reg}}) \).

**Lemma 2.20.** Let \( f : M \to N \) be a smooth map between pseudomanifolds w.i.c.s.. Then there is a natural map \( f^* : T^* N \to T^* M \) such that \( f^* (\alpha) \) is a smooth, if \( \alpha \) is smooth.

**Proof.** Let \( f^*(C^{\infty}_{f(N)}(N)) \) be the germs of smooth functions in \( f^*(C^{\infty}(N)) \) at \( x \). This defines a map \( : f^*(\Omega^0_{f(N)}(N)) \to \Omega^0_{f(N)}M \). Denote by \( n_{f(x)} \) the maximal ideal in \( C^{\infty}_{f(x)}(N) \) (resp. in \( C^{\infty}_{x}(M) \)). Clearly \( f^*(n_{f(x)}) \subset m_x \).

This induces a map \( f^* : T^*_{f(N)}N \to T^*_{f(N)}M \). Since \( f^*(C^{\infty}(N)) \subset C^{\infty}(M) \), the pull back \( f^*(\alpha) \) is also a smooth differential form, if \( \alpha \) is smooth. This proves Lemma \ref{2.20}.

\[\square\]
The following Lemma characterizes the singularity of a smooth structure $C^\infty(cL)$. For any pseudomanifold w.i.c.s. $M$ denote by $\text{rk}(C^\infty(M))$ the minimal number of the generators of $C^\infty(M)$.

**Lemma 2.21.** A Euclidean smooth structure $C^\infty(cL)$ has no singularity, if and only if $\text{rk}(C^\infty(cL)) = \text{rk}(C^\infty(L)) = \dim(L) + 1$.

**Proof.** Assume that $C^\infty(cL)$ has no singularity, so there is a local diffeomorphism $f : cL(1) \to B^{l+1} \subset \mathbb{R}^{l+1}$, where $B^{l+1}$ is a ball in $\mathbb{R}^{l+1}$. Observe that $f$ sends $L$ to $\partial B^{l+1}$, so we get the “only if” assertion of Lemma 2.21. Now let us prove the “if” assertion. The condition $\text{rk}(C^\infty(L)) = \dim(L) + 1$ holds, if and only if $L$ can be embedded in $\mathbb{R}^{l+1}$ as a hypersurface, where $l = \dim L$. Since $(C^\infty(cL))$ is a Euclidean smooth structure, the cone $cL$ is a star-shaped domain in $\mathbb{R}^{l+1}$. So the smooth structure on $cL$ induced by the embedding $cL \to \mathbb{R}^{l+1}$ is a smooth structure without singularity. \hfill \Box

### 3. Tangent bundles and vector fields on a pseudomanifold w.i.c.s. with a finitely generated smooth structure

In this section we study only finitely generated smooth structures, so we omit the adjective “finitely generated”, if no misunderstanding can occur. We introduce the notion of the Nash tangent bundle of a smooth pseudomanifold w.i.c.s. $M$, the notion of the Zariski tangent bundle of $M$, and the notion of the tangent bundle of $M$, as well as different notions of a smooth vector field on $M$. We introduce the notion of characteristic classes of $M$, see Remark 3.5. Nash vector fields are related with infinitesimal diffeomorphisms of $M$, see Proposition 3.6.2. Zariski vector fields are related with derivations of smooth functions on $M$. Their relation has been analyzed in Lemma 3.7 and Remark 3.8. Using the invariance of the tangent cone and the cotangent space at singular points on $M$, we prove the existence of infinitely many Euclidean smooth structures on any conical pseudomanifold $M$, see Proposition 3.5. We find a sufficient condition for the vanishing of a Nash vector field at singular points, see Proposition 3.6.1.

Let $M^m$ be a pseudomanifold w.i.c.s.. Since $C^\infty(M)$ is finitely generated, there is a smooth embedding $F : M \to \mathbb{R}^{l+1}$ such that $F^*(C^\infty(\mathbb{R}^{l+1})) = C^\infty(M)$. Denote by $\text{Gr}_m(\mathbb{R}^{l+1})$ the set of oriented $m$-planes in $\mathbb{R}^{l+1}$. The embedding $F$ induces the gaussian map $\tilde{F} : M^\text{reg} \to \mathbb{R}^{l+1} \times \text{Gr}_m(\mathbb{R}^{l+1})$ sending a point $x$ to the pair $(x, (T_xM^\text{reg}))$. Denote by $\hat{M}^m$ the closure of the image $\tilde{F}(M^\text{reg})$ in $\mathbb{R}^{l+1} \times \text{Gr}_m(\mathbb{R}^{l+1})$. We called $\hat{M}^m$ the Nash blowup of $M^m$. We define the projection $\pi : \hat{M}^m \to M^m$ by setting $\pi(x,v) := x$.

We note that the fiber $\pi^{-1}(s), s \in S_M$, is a closed set in $\text{Gr}_m(\mathbb{R}^{l+1})$. Hence $\hat{M}^m$ is compact, if $M^m$ is compact.

We define the Nash tangent cone $\hat{T}_xM^m$ at a point $x \in M$ by setting $\hat{T}_xM^m := \{v \in \mathbb{R}^{l+1} \mid v \in \pi^{-1}(x)\}$. If $x$ is a regular point we have $\hat{T}_xM^m = T_xM^m$. The union $\hat{T}M^m := \cup_{x \in M^m} \hat{T}_xM^m$ is called the Nash tangent bundle of $M^m$. The Nash tangent bundle carries a natural topology, since $\hat{T}M$ is a
locally closed subset in $\mathbb{R}^{l+1} \times \text{Gr}_m(\mathbb{R}^{l+1})$. Clearly the inclusion $TM^{reg} \to \hat{T} M^{m}$ is a continuous map with respect to this topology. Let $\pi : \hat{T} M^{m} \to M^{m}$ denote the natural projection. Then $\pi$ is a continuous map.

Now we want to introduce the notion of a Nash smooth vector field on $M$. For this purpose we will provide $\hat{T} M$ with a smooth structure, that is a choice $\mathbb{R}$-subalgebra of “smooth functions” in $C^0(\hat{T} M)$ in the following way. The Nash tangent bundle $\hat{T} M^{m}$ has a natural smooth structure defined using the induced embedding of $\hat{T} M^{m}$ into the product $\mathbb{R}^{l+1} \times \mathbb{R}^{l+1} : (x,v) \mapsto (F(x),v)$ in the following way. Note that if $K$ is a subset of a space $M$ with a smooth structure $C^\infty(M)$ then we define a continuous function $f$ on $K$ to be smooth ($f \in C^\infty(K)$) if $f$ is the restriction of $\hat{f} \in C^\infty(U(K))$ to $K$, where $U(K)$ is an open neighborhood of $K$ in $M$, see [23, p.16]. If $K$ is locally closed and $M$ is finitely generated, then $C^\infty(M)$ is finitely generated. If $K$ is closed then $C^\infty(K) = C^\infty(M)_{|K}$; [23, p. 20]. By Proposition 2.17 the projection $\pi : \hat{T} M^{m} \to M^{m}$ is a smooth map, since it is the restriction of the smooth projection $\mathbb{R}^{l+1} \times \mathbb{R}^{l+1} \to \mathbb{R}^{l+1}$. A smooth section $V : M \to \hat{T} M$ is called a smooth Nash vector field on $M$. By Proposition 2.17 a section $V$ is a smooth vector field, if and only if $F_1 \circ V : M \to T\mathbb{R}^{l+1} = \mathbb{R}^{l+1} \times \mathbb{R}^{l+1}$ is a smooth map, where $F_1 : \hat{T} M \to T\mathbb{R}^{l+1}$ is the inclusion.

Example 3.1. We consider the smooth pseudomanifold w.i.c.s. $cL(z = \frac{1}{2})$ in Remark 2.10.1. It is easy to that the Nash blowup of $cL(z = \frac{1}{2})$ is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$. The Nash tangent space $T_{OC}L(z = \frac{1}{2})$ is the cone over $\mathbb{R}^2 \setminus (B^2)$, where $B^2$ is the open disk on $\mathbb{R}^2$ whose boundary $\partial B^2$ is $S^1(z = \frac{1}{2})$.

Lemma 3.2. 1. The homeomorphism type of the Nash blowup $\hat{M}^{m}$ of a smooth pseudomanifold w.i.c.s. $M^{m}$ does not depend on the choice of a smooth embedding $F : M^{m} \to \mathbb{R}^{l+1}$.

2. Let $f : M \to N$ be a smooth map. Then the differential map $Df : TM^{reg} \to TN^{reg}$ extends naturally to a smooth map $Df : \hat{T} M \to \hat{T} N$.

Proof. 1. Let $F_1 : M^{m} \to \mathbb{R}^{l+1}$ and $F_2 : M^{m} \to \mathbb{R}^{l+1}$ be two smooth embeddings. Let $s \in S_M$. The maps $F_1 \circ F_2^{-1} : F_2(M) \to F_1(M)$ and $F_2 \circ F_1^{-1} : F_1(M) \to F_2(M)$ are smooth maps, hence the argument in the proof of Proposition 2.17 yields that, there are smooth maps $\sigma_{12} : \mathbb{R}^{l+1} \to \mathbb{R}^{l+1}$, $\sigma_{21} : \mathbb{R}^{l+1} \to \mathbb{R}^{l+1}$ such that $(F_2 \circ F_1^{-1})_{|U(s)} = (\sigma_{12})_{|F_1(U(s))}$ and $(F_1 \circ F_2^{-1})_{|U(s)} = (\sigma_{21})_{|F_2(U(s))}$ for some small neighborhood $U(s)$ of $s$. The smooth maps $\sigma_{12}$ and $\sigma_{21}$ lift to smooth maps $\tilde{\sigma}_{12} : \mathbb{R}^{l+1} \times \text{Gr}_m(\mathbb{R}^{l+1}) \to \mathbb{R}^{l+1} \times \text{Gr}_m(\mathbb{R}^{l+1})$ and $\tilde{\sigma}_{21} : \mathbb{R}^{l+1} \times \text{Gr}_m(\mathbb{R}^{l+1}) \to \mathbb{R}^{l+1} \times \text{Gr}_m(\mathbb{R}^{l+1})$.

These maps induce a map $h_1 : \tilde{F}_1(U(s)) \to \tilde{F}_2(U(s))$ and a map $h_2 : \tilde{F}_2(U(s)) \to \tilde{F}_1(U(s))$ such that $h_1 \circ h_2 = Id_{|\tilde{F}_2(U(s))}$ and $h_2 \circ h_1 = Id_{|\tilde{F}_1(U(s))}$. Hence $h_1$ and $h_2$ are homeomorphisms. This proves the first assertion of Lemma 3.2.

2. The second assertion follows directly from the construction of $\hat{T} M$. □
Remark 3.3. We can imitate the Mather construction of characteristic classes using the Nash blowup \[22, \S 2\]. Let $T\hat{M}$ denote the restriction of the tautological bundle $V^m$ of the Grassmanian $Gr_m(\mathbb{R}^{l+1})$ to $\hat{M}$ (more precisely $T\hat{M} = (i \circ \pi)^*V^m$, where $\pi: \mathbb{R}^{l+1} \times Gr_m(\mathbb{R}^{l+1}) \to Gr_m(\mathbb{R}^{l+1})$ is the projection, and $i: M \to \mathbb{R}^{l+1} \times Gr_m(\mathbb{R}^{l+1})$ is the embedding). Then we set

$$\text{char}(M) := \pi_*(\text{Dual}(\text{char}(T\hat{M}))),$$

where Dual denotes the Poincaré duality map defined by capping with the fundamental homology class. It is easy to see that this definition is well-defined and it satisfies functorial properties of characteristic classes.

We define the tangent cone $T_s M$ as the subset in $\hat{T}_s M$ consisting of vectors of the form $\gamma(0)$, where $\gamma(r): [0, \infty) \to M$ is a smooth curve (ray) such that $\gamma(0) = x$. Clearly the tangent cone $T_s M$ at a regular point coincides with the tangent space $T_{\hat{M}}M^{reg}$. The tangent bundle $TM$ is defined as the union $\cup_{x \in M} T_s M$. It is a closed subset of $\hat{T}M$, hence it has the natural induced smooth structure, see the explanation before Example 3.4.

Example 3.4. Let $\gamma(r) = [\alpha(r), \beta(r)]$ be a smooth curve (interval) on $cL$ with $\alpha(0) \in L$ and $\beta(0) = 0$. We provide $cL$ with a Euclidean smooth structure using the natural embedding of $cL \to \mathbb{R}^{l+1}$ as a cone over smooth submanifold $L \subset S^l(1) \subset \mathbb{R}^{l+1}$ which sends $[x, t]$ to $x, t \in \mathbb{R}^{l+1}$, here $t \in \mathbb{R}$ acts on $\mathbb{R}^{l+1}$ by multiplication. By Proposition 2.17, $\gamma(r)$ is smooth iff $\alpha(r) \cdot \beta(r)$ is a smooth curve in $\mathbb{R}^{l+1}$. Since $\beta(r) = 0$, we get $\dot{\gamma}(0) = \beta'(0)\alpha(0) \in \mathbb{R}^{l+1}$. Thus $T_s cL = \cup_{x \in L} \{\partial_t(x)\}_\mathbb{R}$.

We define the degree of flatness of the tangent cone $T_s M$ as the number of connected components of the subset $T_s M := \{v \in T_s M | v \neq 0 \}$ of flat tangent vectors $v$. Clearly the collection of degrees of flatness of the tangent cones at singular points $s \in S_M$ is a diffeomorphism invariant of $M$. Using this we will prove the following

Proposition 3.5. For any pseudomanifold w.i.c.s. $M$ there exist infinitely many Euclidean smooth structures on $M$.

Proof. It suffices to show that there is a smooth structure on $cL$ with a given degree of flatness. First we embed $L \to S^l(1) \cap \{x \in \mathbb{R}^{l+1} | x_{l+1} = 1/2\}$, so that the degree of flatness of $T_s cL$ is zero. Now pick $k$ points $x_1, \ldots, x_k \in L \subset \mathbb{R}^{l+1}$. Clearly $-x_1, \ldots, -x_k \in S^l(1)$. It is easy to construct a new embedding $L \to S^{l+1}$ such that $x_i, -x_i \in L$ for all $i = 1, k$. Moreover, a careful construction of this new embedding can be made so that $L \cap (-L) = \{x_1, -x_1, \ldots, x_k, -x_k\}$. This completes the proof of Proposition 3.5. \[\square\]

Let $f : N \to M$ be a smooth map between pseudomanifolds w.i.c.s. Denote by $f^*(\hat{T}M)$ the fiber product (the pullback) of $f$ and $\pi: \hat{T}M \to M$: $f^*(\hat{T}M) = \{(x, v) \in N \times TM | f(x) = \pi(y)\}$. A section $s : N \to f^*(\hat{T}M)$ is called smooth, if the decomposition $i \circ s$ is a smooth map $N \to \hat{T}M$, where
The orthogonal projection of $\partial x$ on $L$ is denoted by $\tilde{\partial}_x$. Assume that all singular points of $M$ are non-trivial, i.e. $N$ is not diffeomorphic to a ball for all $s \in S_M$. Then any diffeomorphism $\psi_t$ of $M$ which is isotopic to the identity must leave $S_M$ fixed. By Proposition 2.17, $\frac{d}{dt}|_{t=0}\psi_t$ is a smooth Nash vector field $V$ on $M$, which vanishes at $S_M$.

A singular point $s$ is called trivial, if $L_s$ is the standard sphere and $cL_s$ is diffeomorphic to $\mathbb{R}^{l+1}$. Otherwise $s$ is called a nontrivial singular point.

**Proposition 3.6.** Let $M$ be a compact pseudomanifold w.i.c.s. provided with a Euclidean smooth structure, and $V$ a smooth Nash vector field on $M$.

1. If a singular point $s \in S_M$ is nontrivial, then $V(s) = 0$.

2. If $V(s) = 0$ for all $s \in S_M$, there exists a one-parameter group of smooth diffeomorphisms $\psi_t$ on $M$ such that $\frac{d}{dt}|_{t=0}\psi_t(x) = V(x)$ and $\psi_0 = Id$.

**Proof.** 1) Let $V$ be a smooth Nash vector field on a compact pseudomanifold w.i.c.s. $M$. By using a smooth partition of unity it suffices to consider the case that $\text{sp}(V) \subset N_s$ for some $s \in S_M$. Fix an embedding $I_s : L_s \to S^l(1) \subset \mathbb{R}^{l+1}$. By Lemma 3.2.2 we can assume that $V$ is a vector field on $(cL_s) \subset \mathbb{R}^{l+1}$. Suppose that $V(s) \neq 0$. By a linear transformation of $\mathbb{R}^{l+1}$ we can assume that $V(s) = \partial x_1$. Since $T_xcL_s = T_{\lambda x}cL_s$ for all $x \in cL_{s_{reg}}$ and for all $\lambda > 0$, using the compactness of the Grassmanian $Gr_m(\mathbb{R}^{l+1})$, $m = \dim M$, we conclude that $\partial x_1$ belongs to $T_xcL_s$ for all $x \in L_s$. We note that $T_xcL = T_xL \oplus \langle \partial x \rangle_{\mathbb{R}}$ for any $x \in L$. Let us denote by $\tilde{V}$ the projection of $\partial x_1$ to $TL$ with respect to the above decomposition. Then $\tilde{V}$ is a smooth vector field on $L$. We write

$$\tilde{V}(x) = \partial x_1 + \lambda(x)\partial_l(x).$$

Let us denote by $|.|$ the norm defined by the Euclidean metric on $\mathbb{R}^{l+1}$. Then $|\tilde{V}(x)| \leq |\partial x_1| = 1$. Hence $|\lambda(x)| \leq 1$. Denote by $\tilde{x}_1$ the restriction of the coordinate function $x_1$ to $L$. Since $|x_1(x)| \leq 1$ we get $\tilde{V}(\tilde{x}_1)(x) = 1 + \tilde{x}_1(x)\lambda(x) \geq 0$. Furthermore, $\tilde{V}(\tilde{x}_1)(x) = 0$ only if $\tilde{x}_1(x)\lambda(x) = -1$, hence $\tilde{x}_1(x) = \pm 1 = -\lambda(x)$. Hence the differential $d\tilde{x}_1$ vanishes at maximal two points on $L$, which are the south pole and the north pole of $S^l(1)$. Now we show that $L$ is a totally geodesic sphere in $S^l(1)$. Denote by $W_1$ the orthogonal projection of $\partial x_1$ to $S^l(1)$. Clearly for all $x \in L$ we have $\tilde{V}(x) = W_1(x)$, where $\tilde{V}$ is defined above. Hence the integral curves of $\tilde{V}$ on $L$ coincide the integral curves of $W_1$, if they have a common point. Note the integral curve of $W_1$ coincides with a geodesic after reparametrization. (At a point $x \in S^l$ we intersect $S^l$ with the plane $\mathbb{R}^2$ spanned on $\partial_l(x), \partial x_1$. Clearly the integral curve of $W_1$ through $x$ lies on this intersection.) Hence $L_s$ is totally geodesic. Thus $s$ is a removable singularity.
2) Let $F : M \to \mathbb{R}^{l+1}$ be a smooth embedding, i.e. $F^*(C^\infty(\mathbb{R}^{l+1})) = C^\infty(M)$. We will show that there exists a smooth vector field $\tilde{V}$ with compact support on $\mathbb{R}^{l+1}$ such that the restriction of $\tilde{V}$ to $F(M)$ coincides with the vector field $F_*(V)$.

Since $V$ is a smooth map from $M$ to $\hat{T}M$, the argument in the proof of Lemma [2.17] yields that there exists a smooth map $\sigma : \mathbb{R}^{l+1} \to T\mathbb{R}^{l+1} \supset \hat{T}M$ such that for all $x \in \mathbb{R}^{l+1}$, $|V(X)| = |\sigma|_{F(M)} = F_*(V)$. Using a cut off function we can assume that $\sigma$ has a compact support in $\mathbb{R}^{l+1}$, since $M$ is compact.

Now we set $\tilde{V}(x) := (x, \pi_2 \circ \sigma)$ for $x \in \mathbb{R}^{l+1}$, where $\pi_2 : T\mathbb{R}^{l+1} = \mathbb{R}^{l+1} \oplus \mathbb{R}_2^{l+1} \to \mathbb{R}_2^{l+1}$ is the projection onto the second summand. Clearly $\tilde{V}$ is a smooth vector field on $\mathbb{R}^{l+1}$ such that $\tilde{V}|_{F(M)} = F_*(V)$.

Now let $\tilde{\psi}_\tau$ be the smooth diffeomorphisms on $\mathbb{R}^{l+1}$ generated by $\tilde{V}$. We will show that $\tilde{\psi}_\tau(x) \in M$ for all $\tau$ and for all $x \in M$. Note that $\tilde{V}(s) = V(s) = 0$. Hence there is a fixed point of the flow $\tilde{\psi}_\tau(s)$ for all $\tau > 0$. Next we note that since $M$ is compact, there exists a positive number $\varepsilon$ such that for all $x \in M$ we have $\tilde{\psi}_\tau(x) \in M$, if $0 \leq \tau \leq \varepsilon$. Clearly the restriction of $\tilde{\psi}_\tau$ to $M$ provides us with the required one-parameter family of diffeomorphisms. This proves the second assertion. \hfill \Box

Let us define the Zariski tangent cone $T^Z_x M$ at a point $x$ in a smooth conical pseudomanifold $M$ by setting $T^Z_x M := \text{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R})$. The universal property of the Kähler differentials implies that $T^Z_x M$ can be identified with the space of all derivations of $C^\infty_x(M)$ with values in $\mathbb{R}$, see [19, 26.C], [29, B.1.2]. If $x$ is a regular point of $M$, then $T^Z_x M = T_x M = \hat{T}_x M$.

Now we compare the Nash tangent cone and the Zariski tangent cone at a given singular point $s \in S_M$. Without loss of generality we can assume that $M = cL \subset \mathbb{R}^{l+1}$. Let $V \in \hat{T}_s cL$. We set for $f \in C^\infty_s(cL)$,

$$V(f)_s := V(\tilde{f})_s,$$

where $\tilde{f} \in C^\infty_x(\mathbb{R}^n)$ such that the restriction of $\tilde{f}$ to $cL$ is $f$. By the definition of the Nash tangent cone there exists a sequence $x_n \in cL^{reg}$ such that $V(s) = \lim_{x_n \to s} V(x_n)$, where $V(x_n) \in T_{x_n} cL^{reg} \subset \mathbb{R}^{l+1} \times \mathbb{R}^{l+1}$. Then

$$V(\tilde{f})_s = \lim_{x_n \to s} V(\tilde{f})_{x_n} = \lim_{x_n \to s} V(\tilde{f}|_{T_{x_n}^{reg}})_{x_n} = \lim_{x_n \to s} V(f)_{x_n}.$$

Thus the above expression $V(f)_s$ does not depend on the choice of $\tilde{f}$. This defines a map $i : \hat{T}_s M \to T^Z_s M$.

**Lemma 3.7.** 1. Let $\dim \mathfrak{m}_s/\mathfrak{m}_s^2 = k$. Then there exist a neighborhood $\mathcal{N}_s(\varepsilon)$ and a smooth embedding $\psi_s : \mathcal{N}_s(\varepsilon) \to \mathbb{R}^k$, i.e. $\psi^* (C^\infty(\mathbb{R}^k)) = C^\infty(\mathcal{N}_s)$.

2. Assume that the smooth structure on $M$ is Euclidean. Then the Zariski tangent cone is generated by the Nash tangent cone, i.e. any element in $T^Z_s M$ is a linear combination of elements in $i(\hat{T}_s M)$.

**Proof.** 1) The first assertion is a special case of [29, Proposition 1.3.10]. For the convenience of the reader we sketch here the proof of this assertion.
Assume the opposite, i.e. there is a smooth embedding \( N_s \to \mathbb{R}^l \), where \( l \geq k + 1 \) is the minimal number of the dimension \( \mathbb{R}^l \), where such a smooth embedding is realizable. Choose a neighborhood \( N_s(\varepsilon) \) and \( k \) functions \( f_1, \cdots, f_k \in C^\infty(\mathcal{N}_s(\varepsilon)) \) such that \( df_i(s) \) form a basis in \( \mathfrak{m}_s/\mathfrak{m}^2_s \). Let \( \bar{f}_i \) be the extension of \( f_i \) to a smooth functions on \( \mathbb{R}^l \), whose existence follows from Proposition \( \ref{prop:2.17} \). Denote by \( I \) the ideal of smooth functions on \( \mathbb{R}^l \) vanishing on \( \mathcal{N}_s(\varepsilon) \). We choose \( f_{k+1}, \cdots, f_l \in I \) such that \( d\bar{f}_1, \cdots, d\bar{f}_l \) form a basis in \( T^*_s(\mathbb{R}^l) = \mathfrak{m}_s/\mathfrak{m}^2_s \). Its follows that \( f_1, \cdots, f_k : \mathcal{N}_s(\varepsilon) \to \mathbb{R}^k \) is a smooth embedding. This proves the first assertion.

2) It suffices to prove this assertion for \( M = cL \subset \mathbb{R}^{l+1} \). We first show that there exists a smooth embedding \( L \to S^{k-1} = S^l \cap \mathbb{R}^k \) such that \( \langle \hat{T}_s cL \rangle = \mathbb{R}^k \subset \mathbb{R}^{l+1} = T_s(\mathbb{R}^{l+1}) \), where \( \langle \hat{T}_s cL \rangle \) is the linear span of \( \hat{T}_s cL \) in \( T_s(\mathbb{R}^{l+1}) = \mathbb{R}^{l+1} \). Let us denote the linear span \( \langle \hat{T}_s cL \rangle \) in \( \mathbb{R}^{l+1} \) by \( \mathbb{R}^k \). Let \( \alpha_i, i = 1, n - k \), be 1-forms on \( \mathbb{R}^{l+1} \) annihilating the subspace \( \mathbb{R}^k \). Since \( \alpha_i \) annihilates any tangent vector in \( TL_s \), it follows that \( L_s \subset \mathbb{R}^k \). Hence \( L \subset S^{k-1} = S^l \cap \mathbb{R}^k \). Since \( \alpha_i \) also annihilates radial vector field \( \partial_l(x) \) we concludes that \( cL_s \subset \mathbb{R}^k \).

It follows that the map \( i : \langle \hat{T}_s cL \rangle = \mathbb{R}^k \to T^*_s cL \) is surjective, since \( T^*_s \mathbb{R}^k \to T^*_s cL \) is a surjective map. Hence Lemma \( \ref{lem:3.7} \) follows.

A Zariski vector field on \( M \) is a section of the Zariski tangent bundle \( T^Z M \). The projection \( \pi : T^Z M \to M \) provides \( T^Z M \) with a topology under which \( \pi \) is continuous, and all sections \( M \to T^Z M \) are continuous. Regarding a Zariski vector field as a linear function on the space of differential 1-forms, we introduce the notion of a smooth Zariski vector field as follows. A Zariski vector field \( V \) is called of class \( C^k \), \( 1 \leq k \leq \infty \), if \( V(\alpha) \) is a \( C^k \)-function, for any smooth differential 1-form \( \alpha \) on \( M \). Equivalently, if \( V(df) \) is a \( C^k \)-function, for any \( f \in C^\infty(M) \).

**Remark 3.8.** Clearly, a smooth Nash vector field is also a smooth Zariski vector field.

## 4. Symplectic pseudomaniolds w.i.c.s.

In this section we introduce the notion of a conical symplectic form on a pseudomaniifold \( M \) w.i.c.s. We provide many known examples of symplectic pseudomaniolds w.i.c.s., see Example \( \ref{ex:4.3} \). We prove that any conical symplectic form is smooth with respect to some Euclidean smooth structure \( C^\infty(M) \), see Corollary \( \ref{cor:4.6} \). In particular it is smooth with respect to the smooth structures \( C^\infty(M) \subset C^\infty(M) \). We also show the existence and uniqueness up to homotopy of a \( C^1 \)-conical Riemannian metric compatible with given conical smooth symplectic structure, see Lemma \( \ref{lem:4.7} \). We compare our concept with some existing concepts, see Remark \( \ref{rem:4.8} \). Finally we show that the Brylinski-Poisson homology can be defined on a symplectic pseudomaniold w.i.c.s. \( M \), if the conical symplectic form is compatible with a Poisson smooth structure. Moreover, its Brylinski-Poisson homology
groups are isomorphic to the deRham cohomology groups of $M$ with the reverse grading if the conical symplectic form is also smooth with respect to the compatible Poisson smooth structure, see Lemma 4.9 and Corollary 4.13. We show non-trivial examples of these symplectic-Poisson smooth pseudomanifolds w.i.c.s., see Remark 4.8.

**Definition 4.1.** A pseudomanifold w.i.c.s. $M^{2n}$ is called conical symplectic, if $M^{reg}$ is provided with a symplectic form $\omega$ and for each singular point $s \in S_M$ there exists a singular chart $(N_s, \phi_s, cL_s(\varepsilon_s))$ such that the restriction of $\omega$ to $N_s$ has the form $\phi_s^*\tilde{\omega}$ with

$$\tilde{\omega}([z, t]) = t^2 \tilde{\omega}(z) + t dt \wedge \alpha(z),$$

where $\tilde{\omega} \in \Omega^2(L_s)$ and $\alpha \in \Omega^1(L_s)$.

Sometimes we also denote by $\omega(\alpha)$ the symplectic form defined by $\tilde{\omega}$. We call $\omega(\alpha)$ a conical symplectic form.

**Remark 4.2.** 1. Taking into account $d\omega = 0$, formula (4.1) implies that $d\alpha = 2\tilde{\omega}$. Hence $\alpha$ defines a contact structure on $L_s$, since $\tilde{\omega}^n = t^{2n-1} \tilde{\omega}^{n-1} dt \wedge \alpha \neq 0$. Thus we can write $\tilde{\omega} = \frac{1}{2} d(t^2 \alpha)$.

2. Let $V$ be the radial vector field on $cL_s$ such that $V(z, t) = t \partial_v$. Then we have

$$l^2 \cdot \alpha = V(z, t)|\tilde{\omega},$$

$$\mathcal{L}_V(\tilde{\omega}) = d(l^2 \alpha) = l^2 d\alpha + 2tdt \wedge \alpha = 2\tilde{\omega}.$$  

It follows that $M \setminus (\cup_{s \in S_M} N_s)$ is a symplectic manifold with concave boundary. (Recall that a boundary $\partial M$ of a symplectic manifold $(M, \omega)$ is called concave, if there exists a vector field $X$ defined near $\partial M$ and pointing inwards such that $\mathcal{L}_X \omega = \omega$, see e.g. [20] or [9].) Such a vector field $X$ is called a Liouville vector field.

**Example 4.3.** 1. Let $\alpha_0$ be the restriction of 1-form $\sum_{i=1}^{k+1} (x_i dy_i - y_i dx_i)$ on $\mathbb{R}^{2k+2}$ to the sphere $S^{2k+1}(1) \subset \mathbb{R}^{2k+2}$. Then the standard symplectic form $\omega_0 = \sum_{i=1}^{k+1} dx_i \wedge dy_i$ on $\mathbb{R}^{2k+2}$ can be written as in formula (4.1) with $L = S^{2k+1}(1)$. Hence, a symplectic manifold with $m$ marked points $s_i, i = \overline{1, m}$, is conical symplectic.

2. Let $G$ be a finite group of $U(n)$ acting freely on $S^{2n-1} \subset (\mathbb{R}^{2n}, \omega, J)$. Then the quotient $\mathbb{R}^{2n}/G$ is a conical symplectic manifold $cL$ with isolated singularity at $0$, where $L = S^{2n-1}(1)/G$. Using (1.2) we observe that the contact form $\alpha$ on $S^{2n-1}$ is invariant under the action of $G$, since $G$ preserves $\tilde{\omega} = \omega_0$, the vector field $V(z, t)$ and $\tilde{\omega} = (\omega_0)|_{S^{2n-1}}$.

3. Let $H := \{z \in \mathbb{C}^{n+1} | Q(z) = 0\}$ is a hypersurface in $\mathbb{C}^{n+1}$, where $Q(z)$ is a homogeneous polynomial such that the projectivization $P(H) := \{z \in \mathbb{C}P^n | Q(z) = 0\}$ is a nonsingular hypersurface. Then $(H, (\omega_0)|_H)$ is a symplectic cone $cL$, whose base $L \subset S^{2n+1}$ is a $S^1$-fibration over $P(H)$.
equipped with the standard contact form $\alpha = (\alpha_0)_L$. A particular case with $H = Q_3$ has been considered in [4].

4. A slightly different example is the closure $\mathcal{O}_{\min}$ of a smallest non-zero nilpotent orbit $\mathcal{O}_{\min}$ of the adjoint action on a simple complex Lie algebra $g$ [2, 2.6], [28]. The regular stratum $\mathcal{O}_{\min}^{reg} = \mathcal{O}_{\min}$ is provided with the Kirillov-Kostant symplectic form, which can be checked easily that it is a conical symplectic form, see also [15] Example 3.6. Clearly $\mathcal{O}_{\min}$ is a complex cone over the smooth variety $\mathcal{O}_{\min}/\mathbb{C}^* \subset P(g)$.

5. Any symplectic manifold $(M, \omega)$ with concave boundary $\partial M$ extends to a conical symplectic manifold with one singular point by attaching to $M$ a symplectic closed cone $\partial M$ as follows. Define a symplectic form on $\partial M$ by (4.1), see also Remark 4.2.2. Then we glue $\partial M$ with $(M, \omega)$ using Darboux’s theorem, which states that a symplectic neighborhood $(U(\partial M), \omega|_{U(\partial M)})$ of $\partial M$ is symplectomorphic to $(\partial M \times (1 - \varepsilon, 1 + \varepsilon), \omega(\alpha))$, where $\omega(\alpha)$ is defined by (4.1), see e.g. exercise 3.36 in [21].

6. Any contact 3-manifold $(M^3, \xi)$ is a concave boundary of some symplectic manifold $(M^4, \omega)$ [9, Theorem 1.3]. Attaching a symplectic cone to $(M^4, \omega)$ as above we get a conical symplectic pseudomanifold.

Assume that $L_s$ is a singularity link of a conical symplectic pseudomanifold w.r.t. $\alpha$ and $\alpha$ is defined by (4.1).

**Lemma 4.4.** There exists a smooth embedding $I_s : (L_s, \alpha) \to (S^{2k+1}(1), \alpha_0)$ such that $I_s^*(\alpha_0) = \alpha$, if $N \geq n(n-1)$, where $n-1 = \dim L_s$. This embedding gives rise to a symplectic embedding $I_s : (cL_s^{reg}, \omega(\alpha)) \to (\mathbb{R}^{2k+2}, \omega_0)$.

**Proof.** Let $(L^n, \alpha)$ be a smooth $n$-dimensional manifold equipped with a 1-form $\alpha$. Using the Nash trick we can find an open covering $A_i$ on $L^n$:

$$L^n = \bigcup_{i=0}^n A_i$$

such that each $A_i$ is the union of disjoint open balls $D_{i,j}$, $j = 1, \ldots, J(i)$ on $M^n$. Pick a simplicial decomposition of $L^n$ and construct $A_i$ by the induction on $i$. Let $D_{0,j}$ be a small coordinate neighborhood of the $j$-th vertex. We may assume that they are mutually disjoint. Set $A_0 = \bigcup_{j=1}^{f(0)} D_{0,j}$. Suppose that $A_0, \ldots, A_i$ are defined. Let $D_{i+1,j}$ be a small coordinate neighborhood, which contains $S^{i+1}_j \setminus \bigcup_{\ell=0}^i A_\ell$, where $S^{i+1}_j$ is the $j$-th $i+1$-dimensional simplex. We may assume that they are mutually disjoint. Set $A_{i+1} = \bigcup_{j=1}^{f(i+1)} D_{i+1,j}$. Hence we obtain desired open sets $A_0, \ldots, A_n$.

Let $\{\rho_i\}$ be a partition of unity on $L^n$ subordinate to the covering $\{A_i\}$. We write $\alpha(x) = \sum_{i=0}^n \rho_i(x) \cdot \alpha$. Clearly the form $\alpha_i = \rho_i(x) \cdot \alpha$ has support on $A_i$.

Let $\gamma_n := \sum_{j=1}^n y^j dz^j$ be a smooth 1-form on $\mathbb{R}^{2n}(y^j, z^j)$. Let us recall

**Proposition 4.5.** [14, Proposition A.3] There is an embedding $f^i : A_i \to (\mathbb{R}^{2n}(y^i, z^i), \gamma_n)$ such that $f^i_*(\gamma_n) = \alpha_i$. Moreover $f^i$ can be chosen such that the image $f^i(A_i)$ lies in arbitrary (small) neighborhood of the origin $0 \in \mathbb{R}^{2n}$. 

Now we construct an embedding $\tilde{f} : M \to \mathbb{R}^{2n(n+1)}$ by setting $\tilde{f} := (f_0, \cdots, f_n)$. Clearly $f^*(\gamma_{n(n+1)}) = \alpha$.

By Proposition 4.5 for any arbitrary small neighborhood $O_\varepsilon(0)$ of the origin 0 of $\mathbb{R}^{2n(n+1)}$ there exists a smooth embedding $f : L^n \to O_\varepsilon(0) \subset \mathbb{R}^{2n(n+1)}$ such that $f^*(\sum_{k=1}^{n+1} \sum_{j=1}^{n} x_k^j \frac{dy_j}{y_k}) = \alpha$. Here $(x_k^j, y_k)$ are coordinates on $\mathbb{R}^{2n(n+1)}$. Now let $\alpha_1 := dz + \sum_{k=1}^{n+1} \sum_{j=1}^{2n} x_k^j \frac{dy_j}{y_k}$ be a contact form on $\mathbb{R}^{2n(n+1)+1}$. Let $\tilde{O}_\varepsilon(0) \subset O_\varepsilon(0)$ be a small neighborhood of 0 in $\mathbb{R}^{2n(n+1)+1}$ such that there exists a diffeomorphism $\psi : \tilde{O}_\varepsilon(0) \to U \subset S^{2n(n+1)+1}$ such that $\psi^*(\alpha_0) = \alpha_1$. The existence of $\tilde{O}_\varepsilon(0)$ together with $\psi$ follows from the Darboux theorem for contact manifolds (see e.g. [10]). This completes the proof of the first assertion. The second assertion follows from the first one, using Example 4.3.1. This completes the proof of Lemma 4.4.

From Lemma 4.4 we get immediately that any conical symplectic pseudomanifold w.i.c.s. admits a smooth structure which is compatible with the given conical symplectic form, i.e. the symplectic form on a singular chart $N_s$ is induced by the smooth embedding $I_s : cL_s \to \mathbb{R}^2$, defined in Lemma 4.4. Let us consider one such compatible smooth structure on a conical symplectic manifold. We get the following consequence of Lemma 2.17.

**Corollary 4.6.** 1. Any conical symplectic structure is smooth with respect to some Euclidean smooth structure $C^\infty(M)$. In particular any conical symplectic structures is smooth with respect to the smooth structures $C^\omega_e(M)$, $C^\omega_w(M)$.

2. Suppose that $N$ is a smooth manifold and $h : N \to (M, \omega)$ is a smooth map with respect to the compatible smooth structure $C^\infty(M)$. Then $h^*(\omega)$ is a smooth differential form on $N$.

A conical Riemannian metric $g$ on a pseudomanifold w.i.c.s. $M$ is a Riemannian metric on $M^{reg}$ such that for all $s \in S_M$ the restriction of $g$ to a conical neighborhood $N_s$ has the form $dt^2 + t^2 g|_{L_s}$, see e.g. [10], [7, 6.3.4]. Recall that a Riemannian metric $g$ on $M^{reg}$ is called compatible with a symplectic form $\omega$, if there exists an almost complex structure $J$ on $M^{reg}$ such that $g(X, Y) = \omega(X, JY)$ is a Riemannian metric on $M^{reg}$. If the resulted metric $g$ is conical, we call $J$ a conical compatible almost complex structure. Now let $\omega$ be a conical symplectic form defined by (1.1). Denote by $R$ the Reeb field on the contact manifold $(L_s, \alpha)$. Let $J$ be a conical almost complex structure on $M$ compatible with $\omega$. Since $g(\partial_t, TL_s) = 0$, we get $J\partial_t \in TL_s$. Furthermore, using $\omega(J\partial_t, \ker \omega|_{L_s}) = 0$ and $\omega(\partial_t, R/t) = 1$, we obtain $J(t\partial_t) = R$. Thus any conical Riemannian metric on $(M, \omega)$ compatible with $\omega$ has the form $g = dt^2 + t^2 (d\alpha^2 + g|_{\ker \alpha})$.

**Lemma 4.7.** Any conical symplectic pseudomanifold w.i.c.s. $(M^m, \omega)$ admits a compatible conical Riemannian metric $g$, which is unique up to homotopy. Let $C^\infty(M)$ be a smooth structure constructed in Corollary 4.6. The compatible conical Riemannian metrics are also smooth with respect
to $C^\infty(M)$ except at the singular points, where they are $C^1$-smooth. The compatible conical Riemannian metrics are smooth w.r.t. $C^\infty_e(M)$, $C^\infty_w(M)$.

Proof. Let us consider the fiber bundle $\mathcal{M}(M^{reg}, \omega) \to M^{reg}$ whose fiber $\mathcal{M}(x)$ consists of all Riemannian metrics compatible with symplectic form $\omega(x)$. It is well-known that $\mathcal{M}(x) = Sp(2m)/U(m)$ is contractible. Now let us consider the subspace $\mathcal{M}_{cone}(M^{reg}, \omega) \subset \mathcal{M}(M^{reg}, \omega)$ consisting of conical Riemannian metrics. The fiber $\mathcal{M}_{cone}(y)$ for $y = [x, t] \in \mathcal{N}_s$ consists of Riemannian metrics of the form $dt^2 + t^2(\alpha^2 + g^0_{\ker \alpha})$, see above. This fiber is isomorphic to the space $Sp(2m - 2)/U(m - 1)$, so it is contractible. Let us take a section $s : \cup_{s \in SM} L_s \to \cup_{s \in SM} \mathcal{M}_{cone}(M^{reg}, \omega)|_{L_s}$. This section extends to a smooth section of $\mathcal{M}(M^{reg} \setminus \cup_{s \in SM, s \neq s} \mathcal{N}_s, \omega)$. It also extends smoothly on $\cup_{s \in SM} \mathcal{N}_s(s)$ by setting $g(y = [x, t]) = dt^2 + t^2(\alpha^2 + g^0_{\ker \alpha})$ for $y \in \mathcal{N}_s$. Using a smooth partition of unity we get the existence of a compatible conical Riemannian metric on $M^{reg}$ by gluing these local sections. The uniqueness up to homotopy follows from the fact that the restriction of two sections $g_1$ and $g_2$ of $\mathcal{M}_{cone}(M^{reg}, \omega)$ to $\cup_{s \in SM} L_s$ are homotopic over $\cup_{s \in SM} L_s$, furthermore this homotopy can be extended to a homotopy by sections of $\mathcal{M}_{cone}(M^{reg}, \omega)$ joining $g_1$ and $g_2$ using smooth partitions of unity. This proves the uniqueness up to homotopy.

Let us prove the second assertion of Lemma 4.7. Choose an embedding $I_s : L_s \to S^{2l+1}$ satisfying the condition of Lemma 4.4. Let $g$ denote the restriction of a compatible conical metric $\hat{g}$ on $M^{reg}$ to $L_s$. We note that there exists a metric $\hat{g}$ on $S^{2l+1}$, which is compatible with $\alpha_0$, i.e. $\hat{g}(R, R) = 1$, $\hat{g}(R, \ker \alpha_0) = 0$ and the restriction of $\hat{g}$ to $\ker \alpha_0$ is compatible with $\alpha_0$, and moreover, the restriction of $\hat{g}$ to $I_s(L_s)$ coincides with the induced metric $(I_s^{-1})^*g$, (note that $I_s^{-1}$ is defined only on the image of $I_s$). Denote by $g_0$ the Euclidean metric on $\mathbb{R}^{2l+2}$. Note that $g_0$ can be written as $dt^2 + t^2(\alpha_0^2 + g_0_{\ker \alpha_0})$. Set $\hat{g} := dt^2 + t^2(\alpha_0^2 + g_{\ker \alpha_0})$. We claim that $\hat{g}$ is a $C^\infty$-metric on $\mathbb{R}^{2l+2} \setminus \{0\}$ and it is $C^1$-smooth at $0 \in \mathbb{R}^{2l+2}$. Substituting $t^2 = x_1^2 + \cdots + x_{2l+2}$ we reduce the proof of this assertion to verifying that the function $f(x) := \partial_{x_1}[(x_1^2 + \cdots + x_{2l+2})(\hat{g} - g_0)(x/|x|)]$ is a continuous function on $\mathbb{R}^{2l+2}$. Note that $(\hat{g} - g_0)$ is a smooth quadratic form on $S^{2l+1}$, so its restriction to any great circle $S^1 \subset S^{2l+1}$ is smooth. Thus we can reduce this smoothness problem to the case when $l = 0$, where the validity of our claim follows by using the identity $(\arctan x)' = \frac{1}{1+x^2}$ and expressing the coordinates $(x_1, y_1)$ in terms of polar coordinates $(r, \theta)$. This proves the second assertion of Lemma 4.7.

The last assertion of Lemma 4.7 follows from the Nash embedding theorem which asserts that any Riemannian manifold admits an isometric embedding into sphere $S^N(1) \subset \mathbb{R}^{N+1}$, if $N$ is large enough, thus the conical compatible Riemannian metric is smooth with respect to this “new” embedding, and taking into account the fact that $C^\infty_e(M)$ contains any subalgebra $C^\infty(M)$ associated with some Euclidean smooth structure on $M$. □
Remark 4.8. Let us compare our definition of a conical symplectic structure with the definition of a symplectic structure on stratified manifolds given by Sjamaar and Lerman in [26], Definition 1.12. In that paper they define a symplectic structure on a stratified manifold \( M \) to be an algebra of smooth functions \( C^\infty(M) \) equipped with a Poisson bracket such that the restriction of \( C^\infty(M) \) to each smooth strata \( S \) of \( M \) is an algebra of smooth functions on \( S \) with the induced symplectic structure. Let us denote by \( G_{\omega_0} \) the following linear bivector

\[
G_{\omega_0} = \partial y_1 \wedge \partial x_1 + \cdots + \partial y_n \wedge \partial x_n.
\]

It is known that \( G_{\omega_0} \) does not depend on the choice of symplectic coordinates \((x_i, y_i)\) on \( \mathbb{R}^{2n} \) [3, §1.1]. Clearly \( \omega \) defines a Poisson structure on \( C^\infty(M) \) by the formula

\[
\{f, g\}_\omega := G_\omega(df \wedge dg),
\]

if and only if \( G(\omega) \) extends to a smooth section of \( \Lambda^2 T^*Z(M) \) (i.e. \( i(G_\omega) \) sends smooth differential forms to smooth differential forms, cf. the definition of a smooth Zariski vector field at the end of §3). A smooth structure \( C^\infty(M) \) equipped with such a smooth section \( G(\omega) \) is called a compatible Poisson smooth structure. Examples of conical symplectic manifolds with a compatible smooth Poisson structure are the quotient \((M, \omega) / G\) with isolated singularity where \( G \) is a compact subgroup of \( \text{Sym}(M, \omega) \), and certain singular symplectic reductions [26], [17], see also a detailed explanation in [15, Example 3.4]. Another example of a compatible smooth Poisson structure on a conical symplectic manifold is a resolvable smooth structure on the closure \( \bar{O}_{\text{min}} \) of an even minimal nilpotent orbit \( O_{\text{min}} \) in complex semisimple Lie algebras, see Remarks 4.3.3 above. A detailed explanation is given in [15, Example 3.6].

We end this section by introducing the notion of the symplectic homology (also called Brylinski-Poisson homology) on a conical pseudomanifold with a compatible smooth Poisson structure \( C^\infty(M) \). Let \( \Omega^p(M^m) \) be the space of all smooth differential \( p \)-forms on \( M \). Then \( \Omega(M) = \bigoplus_{p=0}^m \Omega^p(M^m) \). By Remark 2.19 \( i^*(\Omega(M)) \cong \Omega(M) \) is a subalgebra in \( \Omega(M^{reg}) \).

We consider the canonical complex

\[
\rightarrow \Omega^{n+1}(M) \xrightarrow{\delta} \Omega^n(M) \rightarrow ...
\]

where \( \delta \) is a linear operator defined as follows. Let \( \alpha \in \Omega(M) \) and \( \alpha = \sum f_i df^i_1 \wedge \cdots \wedge df^i_p \) be a local representation of \( \alpha \) as in Definition 2.18. Then we set [13, §3, Lemma 1.2.1]

\[
\delta(f_0 df_1 \wedge \cdots \wedge df_n) = \sum_{i=1}^n (-1)^{i+1} \{f_0, f_i\}_\omega df^i_1 \wedge \cdots \wedge \widehat{df}^i_i \wedge \cdots \wedge df^i_n
\]

\[
+ \sum_{1 \leq i < j \leq n} f_0 d\{f_i, f_j\}_\omega \wedge df^i_1 \wedge \cdots \wedge \widehat{df}^i_i \wedge \cdots \wedge df^i_j \wedge \cdots \wedge df_n.
\]

Lemma 4.9. 1) We have \( \delta = i(G_\omega) \circ d - d \circ i(G_\omega) \). In particular \( \delta \) is well-defined.

2) \( \delta^2 = 0 \).
Proof. 1) The first assertion of Lemma 4.9 has been proved for case of a smooth Poisson manifold $M^{reg}$ by Brylinski in [3] Lemma 1.2.1. Since both $\delta$ and $i(G_\omega) \circ d - d \circ i(G_\omega)$ are local operators, moreover they preserve the subspace $i^*(\Omega(M)) \subset \Omega(M^{reg})$, Lemma 4.9.1 follows from [3] Lemma 1.2.1.

2) To prove the second assertion we note that $\delta^2(\alpha)(x) = 0$ for all $x \in M^{reg}$, since $\delta$ is local operator by the first assertion. Hence $\delta^2(\alpha)(x) = 0$ for all $x \in M$. \hfill \Box

We denote by $*_\omega$ the symplectic star operator

$$*_\omega : \Lambda^p(\mathbb{R}^{2n}) \to \Lambda^{2n-p}(\mathbb{R}^{2n})$$

satisfying

$$\beta \wedge *_\omega \alpha = G^k(\beta, \alpha) vol, \text{ where } vol = \omega^n/n!.$$ 

Now let us consider a conical symplectic neighborhood $(M^{2n}, \omega)$ with a compatible Poisson smooth structure. Operator $*_\omega : \Lambda^p(T^*_x M^{reg}) \to \Lambda^{2n-p}(T^*_x M^{reg})$ extends to a linear operator $*_\omega : \Omega^p(M^{reg}) \to \Omega^{2n-p}(M^{reg})$. In particular, we have $*_\omega(i^*(\Omega^p(A))) \subset \Omega^{2n-p}(M^{reg})$.

Proposition 4.10. Suppose that a conical symplectic form $\omega$ on $M^{2n}$ is compatible with a smooth Poisson structure $C^\infty(M^{2n})$. If $\omega$ is also smooth w.r.t. $C^\infty(M^{2n})$ then $*_\omega(i^*(\Omega^k(M^{2n}))) = i^*(\Omega^{2n-k}(M^{2n}))$.

Proof. We set

$$\Omega_A(M^{2n}) := \{ \gamma \in \Omega(M^{2n}) | *_\omega i^*(\gamma) \in i^*(\Omega(M^{2n})) \}.$$ 

To prove Proposition 4.10 it suffices to show that $\Omega_A(M^{2n}) = \Omega(M^{2n})$. Note that the $C^\infty(M^{2n})$-module $\Omega^{2n}(M^{2n})$ is generated by $\omega^n$ since $\omega^n$ is smooth with respect to $C^\infty(M^{2n})$ and $C^\infty(M^{reg})$-module $\Omega^{2n}(M^{reg})$ is generated by $\omega^n$. Furthermore, $*_\omega(i^*f) = i^*(f) i^*(\omega^n)$ for any $f \in C^\infty(M^{2n})$. This proves $*_\omega(i^*(\Omega^k(M^{2n}))) = i^*(\Omega^{2n-k}(M^{2n}))$. In particular $\Omega^0(M^{2n}) \subset \Omega_A(X^{2n})$, and $\Omega^{2n}(X^{2n}) \subset \Omega_A(X^{2n})$.

Lemma 4.11. We have

$$*_\omega(i^*(\Omega_A(M^{2n}))) = i^*(\Omega_A(M^{2n})).$$

Proof of Lemma 4.11. Let $\gamma \in \Omega_A(M^{2n})$. By definition $*_\omega(i^*\gamma) = \beta \in i^*(\Omega(M^{2n}))$. Using the identity $*_\omega \beta = Id$, we get $*_\omega \beta = i^*\gamma$. It follows $\beta \in i^*(\Omega_A(M^{2n}))$. This proves $*_\omega(i^*(\Omega_A(M^{2n}))) \subset i^*(\Omega_A(M^{2n}))$. Taking into account $*_\omega^2 = Id$, this proves Lemma 4.11. \hfill \Box

Lemma 4.12. 1. $\Omega_A(M^{2n})$ is a $C^\infty(M)$-module.

2. $d(\Omega_A(M^{2n})) \subset \Omega_A(M^{2n})$.

Proof of Lemma 4.12. 1. The first assertion follows from the identity $*_\omega(i^*f(x) \phi(x)) = i^*(f(x)) \cdot *_\omega i^*(\phi(x))$ for $x \in M^{reg}$, $f \in C^\infty(M)$, $\phi \in \Omega^k(M^{2n})$, and using the fact that $\Omega(M^{2n})$ is a $C^\infty(M^{2n})$-module.

2. To prove the second assertion it suffices to show that for any $\gamma \in \Omega_A(M^{2n})$ we have $*_\omega(i^*(d\gamma)) \in \Omega(M^{2n})$. Using Lemma 4.11, we can write
\[ i^*\gamma = *_{\omega} \beta \text{ for some } \beta \in i^*(\Omega_A(M^{2n})). \] Since \( \beta \in \Omega(M^{reg}) \), we can apply the identity \( \delta \beta = (-1)^{deg \beta + 1} *_{\omega} d *_{\omega} \) \[ \text{[3, Theorem 2.2.1]}, \] which implies \[ i^* \omega \beta = (-1)^{deg \beta + 1} \delta \beta \in i^*(\Omega(M^{2n})), \] since \( i^* \circ \delta = \delta \circ i^* \). Hence \( d\gamma \in \Omega_A(M^{2n}) \). This proves the second assertion of Lemma 4.12. \[ \square \]

Let us complete the proof of Proposition 4.10. Since \( \Omega^1(M^{2n}) \) is a \( C^\infty(M^{2n}) \)-module whose generators are differentials \( df \), \( f \in C^\infty(M^{2n}) \), using Lemma 4.12 we obtain that \( \Omega^1(M^{2n}) \subset \Omega_A(M^{2n}) \). Inductively, we observe that \( \Omega^k(M^{2n}) \) is a \( C^\infty(M^{2n}) \)-module whose generators are the \( k \)-forms \( d(\phi(x)) \), where \( \phi(x) \in \Omega^{k-1}(M^{2n}) \). By Lemma 4.12 \( \Omega^k(M^{2n}) \subset \Omega_A(M^{2n}) \), if \( \Omega^{k-1}(M^{2n}) \subset \Omega_A(M^{2n}) \). This completes the proof of Proposition 4.10. \[ \square \]

From Proposition 4.10 we get immediately

**Corollary 4.13.** Suppose \((M, \omega, C^\infty(M))\) is a smooth conical symplectic pseudomanifold satisfying the conditions in Proposition 4.10. The symplectic homology of the complex \((\Omega(M), \delta)\) is isomorphic to the deRham cohomology with reverse grading: \( H_k(\Omega(M), \delta) = H^{m-k}(\Omega, d) \). It is equal to the singular cohomology \( H^{m-k}(M, \mathbb{R}) \), if the smooth structure is locally smoothly contractible.

We like to mention that a theory of De Rham cohomology for symplectic quotients has been considered by Sjamaar in [25].

**5. Concluding remarks**

(1) We have introduced the notion of smooth structures with many good properties on conical pseudomanifolds. Some of our results has been extended to a larger class of singular spaces, see [15]. Our concept of smooth structures and smooth symplectic structures comprises many known examples in algebraic geometry and in the orbifold theory.

(2) It would be interesting to investigate, when a smooth structure \( C^\infty_M M \) given by a resolution of \( M \) is finitely generated.

(3) It would be interesting to find a sufficient condition for the nonvanishing of characteristic classes of a smooth conical pseudomanifold \((M, C^\infty(M))\).

(4) It would be interesting to find a necessary or sufficient condition for a conical symplectic manifold to admit a compatible Poisson smooth structure.

(5) It would be interesting to develop a Hodge theory for a compact smooth conical Riemannian pseudomanifolds and compare these results with those developed by Cheeger in [6].

(6) It would be interesting to find sufficient conditions for developing a Gromov-Witten theory on smooth compact conical symplectic manifolds, which may lead to new invariants for symplectic manifolds with concave boundary.
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