OPERS AND THETA FUNCTIONS

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Abstract. We construct natural maps (the Klein and Wirtinger maps) from moduli spaces of vector bundles on an algebraic curve \( X \) to affine spaces, as quotients of the nonabelian theta linear series. We prove a finiteness result for these maps over generalized Kummer varieties (moduli of torus bundles), leading us to conjecture that the maps are finite in general. The conjecture provides canonical explicit coordinates on the moduli space. The finiteness results give low–dimensional parametrizations of Jacobians (in \( \mathbb{P}^{g-3} \) for generic curves), described by \( 2\Theta \) functions or second logarithmic derivatives of theta.

We interpret the Klein and Wirtinger maps in terms of opers on \( X \). Opers are generalizations of projective structures, and can be considered as differential operators, kernel functions or special bundles with connection. The matrix opers (analogues of opers for matrix differential operators) combine the structures of flat vector bundle and projective connection, and map to opers via generalized Hitchin maps. For vector bundles off the theta divisor, the Szegő kernel gives a natural construction of matrix oper. The Wirtinger map from bundles off the theta divisor to the affine space of opers is then defined as the determinant of the Szegő kernel. This generalizes the Wirtinger projective connections associated to theta characteristics, and the associated Klein bidifferentials.

1. Introduction.

Let \( X \) be a compact connected Riemann surface (or equivalently, a connected smooth projective algebraic curve over \( \mathbb{C} \)). Let \( \mathfrak{M}_X(n) \) denote the moduli space of semistable vector bundles over \( X \) of rank \( n \) and Euler characteristic zero (hence of degree \( n(g-1) \)), and \( \mathfrak{M}_X(n) \subset \mathfrak{M}_X \) is the moduli space of vector bundles with fixed determinant \( \Omega^n_X \) (for a fixed theta–characteristic \( \Omega_X \) on \( X \)). Let \( \Theta \subset \mathfrak{M}_X(n) \) denote the canonical theta divisor; its complement \( \mathfrak{M}_X(n) \setminus \Theta \) is an affine variety, parametrizing rank \( n \) vector bundles with vanishing cohomology.

The theory of nonabelian theta functions provides an embedding of the \((n^2-1)(g-1)\)–dimensional affine variety \( \mathfrak{M}_X(n) \setminus \Theta \) into an affine space of dimension \( n^g \). Specifically, by restricting the canonical theta function to the image of the Jacobian \( \text{Jac}_X \) in \( \mathfrak{M}_X(n) \) obtained by translating a given \( E \in \mathfrak{M}_X(n) \setminus \Theta \) by line bundles, we obtain elements of the \( n\Theta \) linear series on the Jacobian.

It is tempting to look for lower–dimensional parametrizations of \( \mathfrak{M}_X(n) \setminus \Theta \) which come closer to giving explicit coordinates on the moduli space. Optimistically, one can hope for a natural finite map of \( \mathfrak{M}_X(n) \setminus \Theta \) to affine space of the same dimension. In this paper we give a construction of such a map to affine space, which we conjecture is finite, and explain its relations to theta functions, projective structures, and differential operators on the Riemann surface.
Our construction assigns special differential operators, or *opers*, on $X$ to a vector bundle $E$ with vanishing cohomologies. We define a map, the *Wirtinger map* $\mathbb{W}$, from $\mathcal{R}_X(n) \setminus \Theta$ to the space of all opers, which is an affine space for the Hitchin base space of $X$. The dimension of the space of opers is same as that of the moduli space (namely, $(n^2 - 1)(g - 1)$). By realizing the opers as *kernel functions* on $X \times X$ we define the *Klein map* $\mathbb{K}$, sending the moduli space to a (somewhat bigger) affine space of bidifferentials. Our main result establishes the finiteness of the Klein map (for all $X$) and the Wirtinger map (for generic $X$) when restricted to the moduli space of torus bundles – the generalized Kummer variety $\mathcal{R}_X(n) \subset \mathcal{N}_X(n)$.

The case $n = 2$ provides new finite parametrizations of $\text{Jac}_X \setminus \Theta$ (factoring through the Kummer $\mathcal{R}_X(2) = \text{Jac}_X / \{L \sim L^*\}$) in affine spaces of dimensions $\binom{g}{2}$ and $3g - 3$ (that is, quadratic and linear in the genus $g$), improving on the parametrization given by the $2\Theta$ linear series (which requires exponential dependence $2^g$ on the genus to embed the Kummer). As a side–note we obtain that the collection of second logarithmic derivatives of the theta function (considered in [Mu1]) suffice to give a (generically) finite parametrization of the Jacobian, and hence of a generic abelian variety. Our proof uses the behavior of the (abelian) Szegő kernel near the theta divisor (in fact near blowups of Brill–Noether loci) to show that the maps are proper, hence finite, on the affine varieties $\mathcal{R}_X(n) \setminus \Theta$ (giving finite extensions of the Gauss map of the theta divisor).

The Klein and Wirtinger maps may be defined either in terms of restrictions of theta functions, or in terms of determinants of nonabelian Szegő kernels. From the point of view of theta functions, the maps appear as certain quotients of the theta linear series, obtained by restricting the theta function first from $\mathcal{R}_X(n)$ to $\text{Jac}_X$, then to $X \times X$ (via the difference map that sends $(x, y) \in X \times X$ to $\mathcal{O}_X(x - y)$) and further to the $n$th order infinitesimal neighborhood of the diagonal. The theta function thereby defines kernel functions, sections on $X \times X$ of certain sheaves of differentials. Such kernel functions, expanded near the diagonal, are naturally interpreted as differential operators acting between different line bundles on $X$. On the $n$th order infinitesimal neighborhood of the diagonal, we obtain monic differential operators with vanishing subprincipal symbol, which we interpret as $\text{SL}_n$–opers on $X$.

Opers (for a reductive group $G$) are special principal bundles with connection, which play a central rôle in integrable systems and representation theory of loop algebras. They were introduced in [BL] in the context of the geometric Langlands program, providing a coordinate–free expression for the connections which appeared first in [DS] as the phase space of the generalized Korteweg–de Vries hierarchies. Opers form an affine space, modeled on the vector space which is the base of Hitchin’s integrable system on the cotangent bundle of the moduli space of bundles. For $G$ a classical group, opers are identified with certain differential operators acting between line bundles on $X$. In the case $G = \text{SL}_2$, opers are identified (after the choice of a theta characteristic $\Omega_X^\dagger$, which we fix) with projective connections (or projective structures) on $X$.

By writing opers in terms of their kernel function, we obtain explicit constructions of opers, generalizing the constructions of projective structures from theta functions.
due to Klein and Wirtinger ([1Y]). This helps clarify some constructions of differential operators on Riemann surfaces with projective structure ([BR]).

Another point of view on the Klein and Wirtinger maps is given by matrix opers and the Szegö kernel. We define matrix opers by applying the oper interpretation of differential operators to matrix differential operators. Matrix opers combine the structures of connections on a vector bundle and oper in a natural way (they play the same rôle for multicomponent soliton equations that opers play for KdV). A special class of matrix opers, the extended connections (combining connections with projective structures) appear in [BS] (implicitly) and [BR] (explicitly) as twisted cotangent spaces to the universal moduli space of vector bundles on Riemann surfaces. In analogy with the Hitchin system, we may apply invariant polynomials to matrix opers, and obtain (scalar) opers. For extended connections, we show the determinant map in fact gives a deformation of the quadratic Hitchin map, which appears in the theory of Virasoro–Kac–Moody algebras and isomonodromic deformation ([BP]).

To every vector bundle \( E \) off of the theta divisor, there is a canonical matrix oper on \( E \), defined by the nonabelian Szegö kernel of Fay ([Fa2, Fa3, BB]). Applying the determinant map to the Szegö kernel we recover the pullback of the theta function, and thus the Klein and Wirtinger maps. This point of view is motivated by conformal field theory, where this map appears from taking correlation functions associated to \( W \)-algebra symmetries of current algebras. We hope to describe this point of view in future work, and expect it to facilitate the precise description of the Wirtinger map and the proof of our finiteness conjecture.

Since we believe the point of view provided by opers is important in understanding the rôle of Klein and Wirtinger maps, we describe their structure in some detail in the first section. However, we recommend readers to first jump ahead to the last two sections (which can be read largely independently) where the maps are described in elementary terms. The paper is organized as follows: in §2 we review the description of differential operators as kernel functions, review some basics of opers, and describe matrix opers, extended connections and their analogue of the Hitchin map. In §3 we introduce the Klein and Wirtinger maps via the Szegö kernel, and prove the finiteness theorem for Kummers, Theorem 3.1.7. Finally in §4 we explain the relation with classical constructions with theta functions, and draw conclusions about \( 2\Theta \) functions and logarithmic derivatives on Jacobians.

### 2. Differential Operators and Kernels

#### 2.1. Notations

Let \( X \) be a compact connected Riemann surface of genus \( g \) (a connected smooth complex projective curve – unless explicitly noted, all constructions will be algebraic). Let \( p_i : X \times X \longrightarrow X, i = 1,2, \) be the projection to the \( i \)-th factor. The diagonal divisor on \( X \times X \) will be denoted by \( \Delta \). The involution on \( X \times X \) given by interchange of factors will be denoted by \( \sigma \), so \( \sigma(x,y) = (y,x) \). Given holomorphic vector bundles \( V \) and \( W \) on \( X \), we denote vector bundles on \( X \times X \) by

\[
V \boxtimes W := p_1^*V \otimes p_2^*W, \quad V \boxtimes W(n\Delta) := p_1^*V \otimes p_2^*W \otimes \mathcal{O}_{X \times X}(n\Delta)
\]

In particular \( p_1^*V = V \boxtimes \mathcal{O} \) and \( p_2^*W = \mathcal{O} \boxtimes W \). For a vector bundle \( V \) on \( X \) we denote by \( V^* = V^* \otimes \Omega_X \) the Serre dual vector bundle, where \( \Omega_X \) is the holomorphic
cotangent bundle of $X$. For a sheaf $W$, we will denote by $\Gamma(W) = H^0(X,W)$ and $h^i(W) = \dim H^i(X,W)$.

Given a holomorphic vector bundle $V$ over a complex manifold $M$, a torsor, or affine bundle, for $V$ over $M$ is a submersion of complex manifolds $\pi: A \to M$ with a simply transitive, holomorphic action of the sheaf of sections of $V$ on the sections of $A$. So the map $A \times_M V \to A \times_M A$ defined by $(a,v) \mapsto (a,a + v)$ is an isomorphism. In particular, for $x \in M$ the fiber $A_x$ is an affine space over the vector space $V_x$.

Fix a theta characteristic $\Omega^\frac{1}{2}_X$ on $X$—in other words a holomorphic line bundle $\Omega^\frac{1}{2}_X$ equipped with an isomorphism $(\Omega^\frac{1}{2}_X)^\otimes 2 \cong \Omega_X$. If $X$ is compact of genus $g$, there are $2^{2g}$ possible (distinct) choices. For any $m \in \mathbb{Z}$, we will denote $(\Omega^\frac{1}{2}_X)^\otimes m$ by $\Omega^\frac{m}{X}$. The constructions below are independent of the choice of $\Omega^\frac{1}{2}_X$ (see Remark 2.3.4).

### 2.2. Kernel functions.

Let $V$, $W$ be vector bundles over the Riemann surface $X$, and $\text{Diff}^n(V,W)$ the sheaf of differential operators over $X$ of order $n$ from $V$ to $W$. Differential operators are a bimodule over the sheaf $\mathcal{O}$ of holomorphic functions, via pre– and post–multiplication. We identify $\mathcal{O}_X$–bimodules with sheaves on $X \times X$ (via the two pullback maps from $\mathcal{O}_X$ to $\mathcal{O}_{X \times X}$). This way (following Grothendieck and Sato) differential operators are identified with “integral kernels” on $X \times X$ (see e.g. [BS, FB]): there is a canonical isomorphism of $\mathcal{O}_X$–bimodules (supported on the divisor $(n + 1)\Delta$)

$$\text{Diff}^n(V,W) = \frac{W \boxtimes V^\vee ((n + 1)\Delta)}{W \boxtimes V^\vee}.$$  

This is a coordinate–free reformulation of the Cauchy integral formula: differential operators of order $n$ on functions may be realized as kernel functions of the form $\psi(z_1, z_2)dz_2$ with pole of order $(n + 1)$ at $z_1 = z_2$, via the assignment

$$f(z) \mapsto \text{Res}_{z_1 = z_2} f(z_1)\psi(z_1, z_2)dz_2.$$  

The resulting differential operator depends only on the polar part of $\psi$, equivalently on the restriction of $\psi$ in $\mathcal{O}_X \boxtimes \mathcal{O}_X((n + 1)\Delta)|_{(n+1)\Delta}$. For example, the de Rham differential $d: \mathcal{O}_X \to \Omega_X$ is given by a holomorphic section $\mu_d$ of $\Omega_X \boxtimes \Omega_X(2\Delta)$ over $2\Delta$. In local coordinates this section is given by $\frac{d z_1 \boxtimes dz_2}{(z_1 - z_2)^2}$, where $z_i$ the pullback of a coordinate function $z$ on $X$ using the projection to the $i$th factor.

#### 2.2.1. Connections as kernels.

Let $\nu$ be an integer, and consider the line bundle

$$\mathcal{M}_\nu = \Omega^\frac{\nu}{X} \boxtimes \Omega^\frac{\nu}{X}(\nu\Delta)$$

over $X \times X$. As we have observed in § 2.2, the de Rham differential $d$ defines a section $\mu_d$ of $\mathcal{M}_2$ over $2\Delta$. In fact, for any $\nu \in \mathbb{Z}$ there is a unique trivialization $\mu_\nu$ of $\mathcal{M}_\nu|_{2\Delta}$ such that

1. $\mathcal{M}_\nu|_{\Delta} \cong \mathcal{O}_X$ is the natural trivialization (defined by adjunction) (in other words, $\mu_{\nu}|_{\Delta} = 1$);

2. the trivialization is symmetric, i.e., respects the identification $\mathcal{M}_\nu \cong \sigma^*\mathcal{M}_\nu$ in the sense that $\sigma^*\mu_\nu = (-1)^\nu \mu_\nu$. 
(Recall that $\sigma : X \times X \to X \times X$ is the interchange of factors.) In particular note that $\mu_d = \mu_2$ and $\mu_\nu = (\mu_1)^\nu$. For a vector bundle $E$, denote by $M_\nu(E)$ the vector bundle

$$M_\nu(E) = E \otimes E^* \otimes M_\nu = (E \otimes \Omega^2_X) \boxtimes (E \otimes \Omega^2_X)(\nu \Delta)$$
on X \times X.$$

Consider the space $\text{Conn}(E)$ of holomorphic connections on $E$. A connection is given, following Grothendieck, by an isomorphism between the two pullbacks $p_1^*E = E \otimes \mathcal{O}$ and $p_2^*E = \mathcal{O} \boxtimes E$ over $2\Delta$ (the first–order infinitesimal neighborhood of the diagonal), which restricts to the identity automorphism of $E$ on the diagonal. In other words, a connection is determined by a section of $M_0(E) = E \otimes E^*$ on $2\Delta$ with “symbol” the identity map $\text{Id}_E$ on the diagonal.

A connection on $E$ may also be described as a first–order differential operator

$$\nabla : E \to E \otimes \Omega_X$$

whose symbol is the identity map $\text{Id}_E$. Thus $\nabla$ gives rise to a section of

$$M_2(E) = (E \otimes \Omega_X) \boxtimes (E^* \otimes \Omega_X)(2\Delta)$$
on $2\Delta$ with biresidue the identity. These two formulations are related by tensoring with the de Rham kernel $\mu_2 = \mu_d$ trivializing $M_2$ on $2\Delta$. Similarly we can identify connections with sections of $M_\nu(E)|_{2\Delta}$ for any $\nu$. Note also that the difference between any two connection kernels is a section of $\Omega_X \otimes \text{End} E$. Thus $\text{Conn}(E)$ is an affine space for the space $H^0(\Omega_X \otimes \text{End} E)$ of endomorphism–valued one–forms, or Higgs fields, on $E$.

Any holomorphic connection on a Riemann surface is flat (since there are no nonzero holomorphic two–forms on $X$.) This means that the identification between nearby fibers of $E$ can be uniquely extended to an isomorphism $p_1^*E \to p_2^*E$ to any order along $\Delta$ (in fact to local trivializations in the analytic topology). Equivalently there is a canonical extension from sections of $E \boxtimes E^*|_{2\Delta}$ which are identity on the diagonal to sections $\kappa_\nu$ on $\nu \Delta$ for any $\nu > 0$, which in terms of a local flat basis of sections $\{e_i\}$ with dual basis $\{e_i^*\}$, is given by

$$\kappa_\nu = \sum e_i \boxtimes e_i^* \in \Gamma((E \boxtimes E^*)|_{\nu \Delta}).$$

In particular we obtain an isomorphism $E \boxtimes E^*|_{n\Delta} \cong p_1^* \text{End} E|_{n\Delta}$. In the language of kernels this map may be described as the composition

$$E \boxtimes E^* \xrightarrow{\otimes \kappa_\nu^t} \text{End} E \boxtimes \text{End} E \xrightarrow{\text{tr}_E \otimes \text{Id}} \mathcal{O} \boxtimes \text{End} E.$$

Here $\kappa_\nu^t = \sigma^* \kappa_\nu \in \Gamma(E^* \boxtimes E|_{n\Delta})$ is the transpose of $\kappa_\nu$, and $\text{tr}_E$ is the trace divided by the rank of $E$.

Note that this extension is nonlinear with respect to the affine structure on $\text{Conn}(E)$: it involves solving the differential equation defining flat sections.

### 2.3. Opers and kernel functions.

We would like to consider monic $n$th order differential operators

$$L = \partial^n_t - q_1 \partial^{n-1}_t - q_2 \partial^{n-2}_t - \cdots - q_{n-1} \partial_t - q_n$$
on a Riemann surface $X$. To make this notion coordinate–independent, we take $L : \mathcal{A} \to \mathcal{A}'$ to be a $n$th order operator between two holomorphic line bundles, whose principal symbol is an isomorphism. Since the symbol is a section of $\text{Hom}(\mathcal{A}, \mathcal{A}' \otimes \Omega^{-n}_X)$, we must have $\mathcal{A}' \cong \mathcal{A} \otimes \Omega^n_X$.

It is convenient to label the differential operator $L$ not by the line bundle $\mathcal{A}$ but by its twist $L = \mathcal{A} \otimes \Omega^{-1/2}_X$:

2.3.1. **Definition.** A $\text{GL}_n$–oper on $X$ consists of the data of a line bundle $\mathcal{L}$ and a monic $n$th order differential operator $L \in \Gamma(\text{Diff}_n(\mathcal{A}, \mathcal{A} \otimes \Omega^{1-n/2}_X)) = \Gamma(\text{Diff}_n(\mathcal{L} \otimes \Omega^{1/2}_X, \mathcal{L} \otimes \Omega^{1+n/2}_X))$ over $X$ where $\mathcal{A} = \mathcal{L} \otimes \Omega^{1-n/2}_X$. The space of all $\text{GL}_n$–opers on $X$ is denoted by $\mathcal{O}p^n$, and opers for given $L$ by $\mathcal{O}p^n(L)$.

2.3.2. It follows from the differential operator–kernel dictionary that $\text{GL}_n$–opers for given $L$ correspond to kernel functions in $\mathcal{M}^{n+1}_n(L)$ on $(n+1)\Delta$, whose restriction to the diagonal is the constant 1 (by the trivialization defined using adjunction). Moreover, note that restricting the kernel function to $2\Delta$ we obtain a section of $\mathcal{M}^{n+1}_n(L)|_{2\Delta}$, which by §2.2.1 defines a connection on $L$. (This is the reason for labeling opers by $L$ rather than by $\mathcal{A}$.) Thus we have a morphism $\mathcal{O}p_n(L) \to \text{Conn}(L)$. In particular, for $L = \mathcal{O}$, we can look for opers which induce the trivial connection on $\mathcal{O}$, so that the associated kernel on $2\Delta$ is the de Rham kernel $\mu_{n+1}$. In terms of differential operators, the induced connection (restriction to $2\Delta$) is determined by the *subprincipal symbol* $q_1$. Thus we are considering differential operators $L$ of the form

$$L = \partial^n_t - q_2\partial^{n-2}_t - \cdots - q_n.$$

(Conversely the vanishing of the subprincipal symbol forces $L$ and the associated connection to be trivial.)

2.3.3. **Definition.** A $\text{SL}_n$–oper on $X$ (for fixed theta characteristic $\Omega^{1/2}_X$) is a monic $n$th order differential operator $L \in \Gamma(\text{Diff}_n(\Omega^{1-n/2}_X, \Omega^{1+n/2}_X))$ with vanishing subprincipal symbol. Equivalently, $L$ is defined by a section of $\mathcal{M}^{n+1}_n(L)|_{2\Delta}$, which by §2.2.1 defines a connection on $L$. (This is the reason for labeling opers by $L$ rather than by $\mathcal{A}$.) Thus we have a morphism $\mathcal{O}p_n(L) \to \text{Conn}(L)$. In particular, for $L = \mathcal{O}$, we can look for opers which induce the trivial connection on $\mathcal{O}$, so that the associated kernel on $2\Delta$ is the de Rham kernel $\mu_{n+1}$. The space of $\text{SL}_n$–opers (for fixed $\Omega^{1/2}_X$) is denoted by $\mathcal{O}p^n_\mathcal{O}$.

2.3.4. **Remark.** The restriction of the bundles $\mathcal{M}_n$ to any neighborhood $k\Delta$ are independent of the choice of theta characteristic $\Omega^{1/2}_X$. This follows from the fact that the ratio of two theta characteristic is a bundle of order two, $\mathcal{L}^2 = \mathcal{O}_X$, and so carries a canonical flat connection (inducing the trivial connection on $\mathcal{O}_X$), which gives rise to a trivialization of $\mathcal{L} \boxtimes \mathcal{L}^*$ on $n\Delta$ for any $n$. This may also be seen from the universal form of the transition functions defining $\mathcal{O}^{1/2}_X \boxtimes \mathcal{O}^{1/2}_X|_{(n+1)\Delta}$ - in fact these transition functions make sense for an arbitrary complex number $\nu$, since the Taylor expansion of
an expression \( \frac{dz_{\nu}^{\nu} \otimes dz_{\nu}^{\nu'}}{(z_{1} - z_{2})^{2\nu}} \) in terms of a new coordinate \( w = w(z) \) has coefficients which are polynomials (with integer coefficients) in \( \nu \). In other words, all of these bundles are attached to natural representations of the group of formal changes of coordinates on \( X \) (see \([\text{FB}, 7.2]\)).

Thus the spaces of opers for different choices of \( \Omega_{X}^{1} \) are all isomorphic. Alternatively, one can define \( \text{PSL}_{n} \)-opers and then identify \( \text{SL}_{n} \)-opers with pairs consisting of a \( \text{PSL}_{n} \)-oper and a theta characteristic \((\text{BD})\).

### 2.3.5. Example.
On \( \mathbb{P}^{1} \), there is a unique \( \text{SL}_{n} \)-oper for every \( n \) (here \( \Omega_{\mathbb{P}^{1}}^{1} = \mathcal{O}_{\mathbb{P}^{1}}(-1) \) is the unique theta characteristic). It is defined by the kernel function

\[
\gamma_{\nu} = \frac{dz_{n+1}^{\nu} \otimes dz_{n+1}^{\nu'}}{(z_{1} - z_{2})^{n+1}}
\]

on \((n + 1)\), where \( z \) is the natural coordinate function on \( \mathbb{C} \subset \mathbb{C} \cup \{\infty\} = \mathbb{P}^{1} \). This \( \gamma_{\nu} \) is a holomorphic section over \( \mathbb{P}^{1} \times \mathbb{P}^{1} \), and it invariant under the diagonal action of \( \text{PSL}_{2} \) on \( \mathbb{P}^{1} \times \mathbb{P}^{1} \).

### 2.3.6. Lemma.
There is a canonical isomorphism \( \mathcal{O}_{\eta_{n}}(\mathcal{L}) = \text{Conn}(\mathcal{L}) \times \mathcal{O}_{\eta_{n}}^{0} \).  

### 2.3.7. Proof.
An oper \( L \in \mathcal{O}_{\eta_{n}}(\mathcal{L}) \) defines a connection on \( \mathcal{L} \) as above. Solving the connection defines a trivialization \( \kappa_{n+1} \) of \( \mathcal{L} \otimes \mathcal{L}^{*} \) on \((n + 1)\Delta \). This trivialization gives an isomorphism \( \mathcal{M}_{n+1}(\mathcal{L}) \cong \mathcal{M}_{n+1} \), which sends \( L \) to an \( \text{SL}_{n} \)-oper \( L' \). The kernel of \( L' \) is explicitly given by \( \kappa_{n}^{-1} \times \) the kernel of \( L \), from which it is obvious that the restriction to \( 2\Delta \) is indeed \( \mu_{n+1} \).

### 2.3.8. Projective Structures.
A projective structure on a Riemann surface \( X \) (see \([\text{Gu}], [\text{De}]\)) is an equivalence class of atlases \( \{U_{\alpha}, \varphi_{\alpha}\}_{\alpha \in I} \) on \( X \), where \( \varphi_{\alpha} \) is a holomorphic embedding of the open set \( U_{\alpha} \) in \( \mathbb{P}^{1} \), so that the transition maps \( \varphi_{\beta} \circ \varphi_{\alpha}^{-1} \) are Möbius (or fractional linear) transformations (elements of \( \text{PSL}_{2} \mathbb{C} \)). The space of projective structures will be denoted \( \mathcal{P}_{\text{proj}} \). A projective structure on \( X \) allows us to pull back any \( \text{PSL}_{2} \)-invariant construction from \( \mathbb{P}^{1} \) to \( X \). In particular, we may pull back the \( \text{SL}_{n} \) opers on \( \mathbb{P}^{1} \) (and their kernel functions \( \gamma_{n+1} \)) to define \( \text{SL}_{n} \)-opers for every \( n \), or equivalently monic differential operators \( D_{n} \) with vanishing subprincipal symbol. The symbol of each such operator is the constant function 1. The operator \( D_{0} \) is the identity automorphism of \( \Omega_{X}^{1} \). The operator \( D_{1} \) is the exterior derivative \( d : \mathcal{O}_{X} \rightarrow \Omega_{X} \). The operator \( D_{2} \in \Gamma(\text{Diff}^{2}(\Omega_{X}^{1}, \Omega_{X}^{2})) \) over \( X \) is the Sturm–Liouville operator, or projective connection, associated with a projective structure. Thus \( D_{2} \) is the differential operator which in projective local coordinates has the form \( \partial_{t}^{2} \).

The projective structure can be recovered from the associated projective connection, setting up a bijection \( \mathcal{P}_{\text{proj}} \cong \mathcal{O}_{\eta_{2}}^{2} \): the projective atlases are defined by the ratios of any two local linearly independent solutions of the Sturm–Liouville operator \( D_{2} \).
2.4. **Opers as connections.** Opers have an interpretation in terms of vector bundles with connection, which also enables the generalization from $\text{GL}_n$ to an arbitrary reductive group. This observation and its current formulation are due to Drinfeld–Sokolov [DS] and Beilinson–Drinfeld [BD], respectively. Recall that the study of the differential operator

\[ L = \partial_t^n - q_1 \partial_t^{n-1} - q_2 \partial_t^{n-2} - \cdots - q_n \]

is equivalent to that of the system of $n$ first–order equations which can be written in terms of the first–order matrix operator

\[
\begin{pmatrix}
q_1 & q_2 & q_3 & \cdots & q_n \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]

Now suppose $L$ is a $\text{GL}_n$–oper on $X$ for the line bundle $A$. It is not hard to see that the above first–order systems patch together to define a connection $\nabla : F \to F \otimes \Omega_X$ on a rank $n$ vector bundle $F$, which carries a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$, with $F_1 \cong A$. The key features of the above matrix system are the appearance of zeros beneath the subdiagonal (Griffiths transversality) and 1s on the subdiagonal (nondegeneracy). Locally, the bundle and flag $(F, F_\bullet)$ admit a unique trivialization so that the connection has the above form. Moreover for the connection to be an $\text{SL}_n$–connection (so that the determinant line bundle and its connection are trivial) the subprincipal symbol $q_1$ must vanish, so that we obtain $\text{SL}_n$–opers.

2.4.1. **Proposition.** ([BD]) $\text{GL}_n$–opers on $X$ correspond canonically to the data of a rank $n$ vector bundle $F$, equipped with a flag

\[ 0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F, \]

and a connection $\nabla$, satisfying

- $\nabla(F_i) \subset F_{i+1} \otimes \Omega_X$.
- The induced maps $F_i/F_{i-1} \to (F_{i+1}/F_i) \otimes \Omega_X$ are isomorphisms for all $i$.

$\text{SL}_n$–opers are $\text{GL}_n$–opers for which the determinant line bundle of the flat vector bundle $(F, \nabla)$ is trivial.

2.4.2. In fact, the transversality condition on the connection is sufficiently rigid to force the underlying vector bundle $F$ to be the $(n-1)$st jet bundle $F \cong J^{n-1}(F/F_{n-1})$, with its canonical filtration.

Note that from the connection point of view, the extension of projective connections to $n$th order operators is simply the operation of inducing an $\text{SL}_n$–oper from a $\text{SL}_2$–oper by taking the associated bundle for the $(n-1)$st symmetric power representation of $\text{SL}_2$ into $\text{SL}_n$. 
2.5. **The Hitchin base.** An important non-obvious feature of opers (for fixed \( L \)) is that they form an affine space over the Hitchin base space of \( X \),

\[
\text{Hitch}_n(X) = \Gamma(\Omega_X) \oplus \Gamma(\Omega_X^2) \oplus \cdots \oplus \Gamma(\Omega_X^n).
\]

This generalizes the statement that projective structures form an affine space over quadratic differentials \( \Gamma(\Omega_X^2) \).

2.5.1. **Proposition.** (BD) There is a canonical isomorphism

\[
\text{Conn}(L) \times \text{Proj} \times \text{Hitch}^2_n \rightarrow \text{Op}_n(L).
\]

2.5.2. **Remarks on proof.** Geometrically, the proposition is an expression of the fact that the tangent bundle to \( \mathbb{P}^{n-1} \) restricted to the rational normal curve splits canonically (i.e., \( \text{PSL}_2 \)-equivariantly) into a sum of line bundles. Namely, the stabilizer in \( \text{PSL}_2 \) of a point on the rational normal curve (which is isomorphic to upper triangular matrices \( B_0 \)) acts on the tangent space at that point through its \( \mathbb{C}^\times \)-quotient. In fact since a \( \text{SL}_n \)-oper naturally gives rise to a projective structure (on restriction to \( 3\Delta \)), the proposition reduces to this fact since any infinitesimal (or complex-local) statement on \( \mathbb{P}^1 \) which is \( \text{PSL}_2 \)-equivariant generalizes to any Riemann surface with projective structure. In particular, given an oper \( F \) induced from a \( \text{SL}_2 \)-oper, one identifies a subbundle \( \mathcal{V} \sim = \bigoplus_1^n \Gamma(\Omega_X^i) \) inside the Higgs fields \( \Gamma(\text{End} F \otimes \Omega_X) \). Addition of sections from \( \mathcal{V} \) acts transitively on oper connections on \( F \) – in particular the action of \( \Gamma(\Omega_X) \) changes the connection, while that of \( \Gamma(\Omega_X^2) \) changes the projective structure.

2.5.3. **Remark.** It follows from the proposition that the dimensions of \( \text{Op}_p \) and \( \text{Op}_p^\circ \) on a compact Riemann surface \( X \) of genus \( g \) are \((g-1)(n^2-1) + g \) and \((g-1)(n^2-1) \) respectively.

2.6. **Shifted opers.** The projection from \( \text{Op}_p \) to \( \text{Proj} \) may be described conveniently using kernels. Given an \( \text{SL}_n \)-oper with kernel \( s \in \Gamma(M_{n+1}(n+1)\Delta) \), its restriction to \( 3\Delta \) defines an element of the space

\[
\text{Proj}(k) = \{ s \in \Gamma(M_{n+1}|3\Delta) \mid s|_{2\Delta} = \mu_{n+1} \}.
\]

These “shifted” projective kernels are however naturally identified with projective structures. Note that the difference between any two sections of \( \text{Proj}(k) \) vanishes on \( 2\Delta \), and so may be identified with a section of \( M_k(-2\Delta)|_\Delta \cong \Omega_X^{\otimes 2} \), that is, a quadratic differential on \( X \). It follows that \( \text{Proj}(k) \) is a torsor for the quadratic differentials \( \Gamma(\Omega_X^{\otimes 2}) \) on \( X \). Recall that we may rescale the torsor structure on a fixed affine bundle by any scalar \( \lambda \in \mathbb{C}^\times \), keeping the manifold \( \pi : A \rightarrow M \) the same but making \( v \in V \) act by \( \lambda \cdot v \).

2.6.1. **Lemma.** The spaces \( \text{Proj}(k) \) for \( k \neq 0 \) are all isomorphic (with rescaled torsor structure over quadratic differentials).
2.6.2. Proof. The $k$-th power map $\rho \mapsto \rho^{\otimes k}|_{3\Delta}$ identifies the sheaves $\mathcal{P}roj(1)$ and $\mathcal{P}roj(k)$. On affine structures, this has the effect of rescaling by $k$:

$$(\rho + q)^{\otimes k}|_{3\Delta} = (\rho^{\otimes k} + k \cdot q)|_{3\Delta}$$

(since all higher terms vanish on $3\Delta$). Note that multiplication by $k$ sends $\mathcal{P}roj(1)$ isomorphically to sections of $\mathcal{M}_1|_{3\Delta}$ whose restriction to $2\Delta$ is $k\mu_1$, and has the same effect on torsor structures.

2.6.3. It follows that the restriction of an $\text{SL}_n$-oper to $3\Delta$ may be naturally identified with a projective structure on $X$. (For $\text{GL}_n$-opers, we must twist by the connection on $L$ given by the restriction to $2\Delta$.)

One naturally encounters other kernel realizations of the spaces of opers:

2.6.4. Definition. The space of $k$-shifted opers on $L$ is defined to be the space

$$\mathcal{O}p_n(L)(k) = \{ s \in \Gamma(M_k(L)|_{(n+1)\Delta}) \mid s|_{\Delta} = 1 \}.$$ 

2.6.5. (Note that for $k > n$ shifted opers form a quotient of $\mathcal{O}p_{k-1}$ by differential operators of order $k-n-2$.) The restriction of a shifted oper to $3\Delta$ defines an element of $\mathcal{P}roj(k)$, hence a projective structure on $X$. It follows that we may identify all shifted opers canonically with honest opers. Explicitly, this is done by tensoring with the kernels $\gamma_{n+1-k}$ obtained from the projective structure by (2.3.1), which trivializes $\mathcal{M}_{n+1-k}$ to any order near the diagonal.

2.7. Projective kernels and projective connections. Many projective connections arising in Riemann surface theory arise naturally from projective kernels, or global bidifferentials of the second kind, with biresidue one:

2.7.1. Definition.

1. A $\text{SL}_n$-oper kernel on $X$ is a global section $s \in H^0(X \times X, \mathcal{M}_n)$ with $s|_{2\Delta} = \mu_n$. The space of oper kernels is denoted by $\mathcal{K}ern_n$.

2. A projective kernel on $X$ is a symmetric $\text{SL}_2$-oper kernel. In other words, a bidifferential

$$\omega \in H^0(X \times X, \Omega_X \boxtimes \Omega_X(2\Delta))^{\text{sym}}$$

with biresidue one on $X \times X$. The space of projective kernels is denoted by $\mathcal{K}ern_2^{\text{sym}} \subset \mathcal{K}ern_2$.

By $H^0(X \times X, \Omega_X \boxtimes \Omega_X(2\Delta))^{\text{sym}}$ we mean section invariant under the involution $\sigma$.

2.7.2. The difference between two projective kernels is a holomorphic symmetric bidifferential on $X$, so that $\mathcal{K}ern_2^{\text{sym}}$ is an affine space for $\text{Sym}^2 H^0(X, \Omega)$, of dimension $\binom{g}{2}$. Restriction to $3\Delta$ defines a map $\mathcal{K}ern_2^{\text{sym}} \to \mathcal{P}roj_2$, which is surjective for $X$ non-hyperelliptic.

A key rôle of the spaces $\mathcal{P}roj$ and $\mathcal{K}ern_2^{\text{sym}}$ is in relation to moduli spaces. Namely, $H^0(X, \Omega_X^{\otimes 2})$ is the cotangent space to the moduli space of curves (or Teichmüller space) at $X$, and $\text{Sym}^2 H^0(X, \Omega_X)$ is the cotangent space to the moduli of abelian varieties (or Siegel upper half space) at the Jacobian $\text{Jac}_X$ of $X$. The spaces $\mathcal{P}roj$ and $\mathcal{K}ern_2^{\text{sym}}$ are naturally identified as the fibers, at $X$ and $\text{Jac}_X$ respectively, of the space of connections.
on a natural (Hodge or theta) line bundles on the respective moduli spaces (in other words the fibers of appropriate twisted cotangent bundles). (See [11, 31].)

An important example of a projective kernel is the Bergman kernel \( \omega_B \). Let \( \omega_i \) \((i = 1, \ldots, g)\) be the normalized basis of holomorphic differentials on \( X \), with respect to a normalized homology basis \( A_1, B_1 \) and \( \frac{\partial}{\partial z} \) the dual basis of vector fields on the Jacobian. The Bergman kernel is characterized by having vanishing periods, and the forms \( \omega_i \) as its \( B \)-periods.

2.8. Matrix opers. In this section we describe a matrix version of opers. Thus we consider \( n \)th order differential operators with matrix coefficients,

\[
L = \partial^n_t - q_1\partial^{n-1}_t - q_2\partial^{n-2}_t - \cdots - q_n
\]

where the \( q_{n+1} \) are now \( k \) by \( k \) matrices, acting on \( \mathbb{C}^r \). Let \( L : E_1 \to E_2 \) be a \( n \)th order differential operator between vector bundles \( E_1, E_2 \) on \( X \), whose symbol is an isomorphism, so that \( E_2 \cong E_1 \otimes \Omega^n_X \).

2.8.1. Definition. A \( n \)th order matrix oper on \( E \) is an \( n \)th order differential operator \( L \in \Gamma(Diff^n(E \otimes \Omega^{1-n}_X, E \otimes \Omega^{1+n}_X)) \) over \( X \) with principal symbol the identity \( \text{Id}_E \). The space of matrix opers on \( E \) is denoted by \( \text{MOp}_n[E] \).

2.8.2. We may follow the same procedure as in § 2.4 to describe \( n \)th order matrix opers \( L \) by first order matrix systems, now of rank \( nk \). The resulting connections were called coupled connections in [11]. We have the following statement (see [11] for more details):

2.8.3. Proposition. Let \( E \) be a vector bundle. There is a natural identification between matrix opers \( L : E \to E \otimes \Omega^n_X \) of order \( n \), and vector bundles \( F \) equipped with a filtration \( 0 = F_0 \subset F_1 \subset \cdots \subset F_n = F \), with \( F_1 \cong E \otimes \Omega^{1-n}_X \), and a connection \( \nabla : F \to F \otimes \Omega_X \) satisfying the two conditions

- \( \nabla : F_\nu \to F_{\nu+1} \otimes \Omega_X \) (Griffiths transversality), and
- the homomorphism \( F_\nu/F_{\nu-1} \to F_{\nu+1}/F_\nu \otimes \Omega_X \) induced by \( \nabla \) is an isomorphism for all \( \nu \).

2.8.4. Proof. Recall that a \( n \)th order operator \( L \in \Gamma(Diff^n(E_1, E_1 \otimes \Omega^n_X)) \) over \( X \) is a homomorphism from \( J^n(E_1) \) to \( E_1 \otimes \Omega^n_X \). This is equivalent to a splitting of the jet sequence

\[
0 \to E_1 \otimes \Omega^n_X \to J^n(E_1) \to J^{n-1}(E_1) \to 0,
\]

and thus to a lift \( J^{n-1}(E_1) \) to \( J^n(E_1) \). However there is a natural homomorphism \( J^n(E_1) \to J^1(J^{n-1}(E_1)) \) for any bundle. Thus we have constructed a lifting from \( J^{n-1}(E_1) \) to its sheaf of 1-jets, in other words a connection on \( J^{n-1}(E_1) \). The strict Griffiths transversality with respect to the natural connection on \( J^{n-1}(E) \) follows automatically.

In the reverse direction, given a filtered vector bundle \( F \) and a connection \( \nabla \) on \( F \) as above, consider the homomorphism

\[
\psi_k : F \to J^k(F) \to J^k(F/F_{n-1})
\]
where the first arrow is the flat extension map given by the connection and the second is the projection. The transversality condition ensures that $\psi_{n-1}$ is an isomorphism. Therefore, $\psi_n \circ \psi_{n-1}^{-1} : J^{n-1}(F/F_{n-1}) \to J^n(F/F_{n-1})$ gives a splitting of the jet sequence as above, in other words, a differential operator as desired.

2.8.5. Developing maps. A geometric description of coupled connections $\nabla$ as in Proposition 2.8.3, generalizing the description of projective structures via period maps, is given in [Bi]. Namely consider the Grassmannian bundle $G_k(F)$ of $k$–dimensional subspaces of $F$. This inherits a connection from $\nabla$ and a section from $F_1$, which is nowhere flat. It follows that on simply connected opens (or on the universal cover of $X$) we obtain period maps to $G_k(C^n)$ using the connection to trivialize $G_k(F)$ and the section to map. These period maps satisfy natural nondegeneracy conditions. Conversely such nondegenerate period maps with transitions coming from the action of $GL_{nk}$ on $G_k$ give rise to coupled connections.

2.8.6. Decomposition of matrix opers. Matrix opers of order $n$ on $E$ correspond to kernel functions in $M_{n+1}(E)|(n+1)\Delta$, that is a section of $M_{n+1}(E)$ over $(n+1)\Delta$, whose restriction to the diagonal is

$$Id_E \in \text{End} E \cong M_{n+1}(E)|_\Delta.$$ 

For example, if $E = \mathcal{L}$ is a line bundle, then matrix opers and $GL_n$–opers for $\mathcal{L}$ are the same. It follows that by restriction to $2\Delta$, a matrix oper defines a section of $M_{n+1}(E)|_{2\Delta}$ with residue $Id_E$ and thus a flat connection on $E$ (§2.2.1). So there is a canonical projection $MOp_n(E) \to \text{Conn}(E)$.

The induced connection on $E$ allows us to identify $M_{n+1}(E)$ with $M_{n+1} \otimes p^*_1 \text{End} E$ to any order near $\Delta$. Thus, if $p \in \mathbb{C}[\mathfrak{gl}_n]^{GL_n}$ is an invariant polynomial on matrices (i.e., a coefficient of the characteristic polynomial) we obtain a map

$$p_* : MOp_n(E) \longrightarrow \mathcal{O}p_n^\circ$$

by applying $p$ to $\text{End} E$ and identifying the resulting shifted oper with an oper.

Together with Proposition 2.5.1, this gives a very simple description of matrix opers. Let

$$\text{Hitch}^n_1(E)^\circ = \bigoplus_{i=2}^{n} \Gamma(\text{End}^i E \otimes \Omega_X^{\leq i}),$$

the space of traceless $\text{End} E$–valued polydifferentials.

2.8.7. Proposition. There is a canonical isomorphism

$$\text{Conn}(E) \times \mathcal{O}p_n^\circ \times \text{Hitch}^n_1(E)^\circ \longrightarrow MOp_n(E).$$

2.8.8. Proof. We describe the isomorphism in the languages of kernels and of coupled connections.

We define the map $\text{Conn}(E) \times \mathcal{O}p_n^\circ \to MOp_n(E)$ using the tensor decomposition of $M_n(E)$ by taking the tensor product of sections. It follows that the decomposition (Proposition 2.5.1) of sections of $M_{n+1}$ gives rise to a direct sum decomposition of sections of this tensor product. We identify $\mathcal{O}p_n^\circ$ with the scalar endomorphisms, thereby
obtaining the proposition. The projection back to $\text{SL}_n$-opers is given by the induced map $\text{tr}_* \text{ for } p(A) = \text{tr}(A)/\text{rk}(E)$ above.

Viewing an oper through the corresponding flat bundle $(F, F_\bullet, \nabla)$, where $F_\bullet$ is a filtration of subbundles of $F$, we may take the tensor product of vector bundles $E \otimes F$, with its induced filtration and connection. The result is a coupled connection, which we consider as a matrix oper on $E$. Inside the space of Higgs fields on $E \otimes F$ compatible with the filtration we find the tensor product of $\text{End}_E$ with the space $V$ of Higgs fields from Proposition $\text{2.5.1}$, so we can modify the coupled connection by $\text{End}_E$-valued polydifferentials. Again one checks this gives a bijective parametrization of coupled connections.

To see the compatibility of the constructions, note that a vector bundle with connection is canonically trivialized to any order near a point $x \in X$, up to a constant matrix (the change of trivialization of its fiber at $x$). Hence the compatibility reduces to the (equivariant) compatibility in the case of the trivial bundle, which is obvious.

2.8.9. The determinant. The determinant map for matrix opers may also be described directly, without solving the associated connection. Let $s$ be a section of $(E_1 \boxtimes E_2) \otimes L$, where $E_1, E_2$ are vector bundles on $X$ of the same rank $k$ and $L$ is a line bundle on $X \times X$. Then we may define the determinant section $\text{det } s = \wedge^k s$ of $(\text{det } E_1 \boxtimes \text{det } E_2) \otimes L^k$ (e.g. consider $s$ as a homomorphism from $p_1^* E_1$ to $p_2^* E_2 \otimes L$ of rank $k$ vector bundles and take its determinant).

If $s \in E \boxtimes E^*|\Delta$ is a connection on $E$, then $\text{det } s \in \Gamma(\text{det } E \boxtimes \text{det } E^*|\Delta)$ is the determinant connection on $\text{det } E$. More generally, the determinant defines a canonical map

$$\det: \text{MOp}_n(E) \rightarrow \text{O}_n(\text{det } E)$$

Namely, the determinant of $s \in \Gamma(M_n(E)|k\Delta)$ defines a section

$$\text{det } s \in \Gamma(M_{n, \text{rk } E}(\text{det } E)|k\Delta)$$

which is the identity on the diagonal, i.e., a shifted oper, and which we identify with an (unshifted) oper as in § 2.6. There is a commutative diagram

$$\begin{array}{ccc}
\text{MOp}_n(E) & \rightarrow & \text{Conn}(E) \times \Gamma(M_n \otimes p_1^* \text{End } E) \\
\downarrow & & \downarrow \\
\text{O}_n(\text{det } E) & \rightarrow & \text{Conn}(\text{det } E) \times \text{O}_n
\end{array}$$

where the horizontal arrows are given by trivializing $E, \text{det } E$ using the connection, and the vertical arrows are the determinant maps on kernels and on endomorphisms. This identifies the determinant map for matrix opers above with the determinant of the associated kernel.

2.9. Extended Connections. The splitting in Proposition $\text{2.8.7}$ picks out a particularly interesting subspace $\text{Conn}(E) \times \text{Proj}$ of matrix opers on $E$, the extended connections on $E$. In fact extended connections most naturally appear as a quotient of matrix opers. Their rôle is as an affine space for the cotangent space of the moduli of the pair $(X, E)$. As such they do not split as a product: the splitting in Proposition $\text{2.8.7}$ is nonlinear (since it involves solving the connection to some order), and in fact a deformation the quadratic part of the Hitchin map.
2.9.1. Definition. The space \( \mathcal{E}x\mathcal{C}onn_{n+1}(E) \) of extended connections on \( E \) is the space of monic sections of the quotient of \( M_{n+1}(E)\lvert_{3\Delta} \) by the subsheaf of sections vanishing on \( 2\Delta \) and with vanishing trace on \( 3\Delta \).

2.9.2. Here trace refers to the composition

\[
M_{n+1}(E)(-2\Delta) \rightarrow \Omega_X^{\otimes 2} \otimes \text{End } E \rightarrow \Omega_X^2,
\]

and monic sections are sections restricting to \( \text{Id} \) on the diagonal. Thus we have modified \( M_{n+1}(E)\lvert_{3\Delta} \) by forgetting all but the trace of the lowest–order term.

It follows that restriction to \( 2\Delta \) makes \( \mathcal{E}x\mathcal{C}onn_{n+1}(E) \) an affine bundle for quadratic differentials \( H^0(X, \Omega_X^2) \) over \( \mathcal{C}onn(E) \). Consider the space of extended Higgs fields

\[
\mathcal{E}x\mathcal{H}iggs(E) = \{ s \in \Gamma(M_{n+1}(E)\lvert_{2\Delta}) \mid s\lvert_{\Delta} = 0 \} / \Gamma(M_{n+1}(E)\lvert_{2\Delta}).
\]

Note that this space \( \mathcal{E}x\mathcal{H}iggs(E) \) is independent of \( n+1 \) since \( M_{n+1}\lvert_{2\Delta} \) is canonically trivialized. The space of extended connections is clearly a torsor over extended Higgs fields. The importance of the latter is as the cotangent space at \( (X, E) \) to the moduli of pairs of Riemann surfaces and vector bundles. They form an extension

\[
0 \rightarrow H^0(X, \Omega_X^2) \xrightarrow{\mathcal{E}x\mathcal{H}iggs(E)} \mathcal{E}x\mathcal{H}iggs(E) \rightarrow H^0(X, \text{End } E \otimes \Omega_X) \rightarrow 0
\]

of Higgs fields on \( E \) by quadratic differentials. It is proven in [BH] that the torsors \( \mathcal{E}x\mathcal{C}onn_{n+1}(E) \) over \( \mathcal{E}x\mathcal{H}iggs(E) \) for varying \( X, E \) form a twisted cotangent bundle over the moduli space: in particular there is an isomorphism

\[
\mathcal{E}x\mathcal{C}onn_1 \cong \mathcal{C}onn(\Theta)
\]

with the affine bundle of connections on the theta line bundle over the moduli space.

2.9.3. Lemma. For every \( n \in \mathbb{Z} \), the map

\[
\mathcal{C}onn(E) \times \mathcal{P}roj_{n+1} \rightarrow \Gamma(M_{n+1}(E)\lvert_{3\Delta}) \rightarrow \Gamma(M_{n+1}\lvert_{3\Delta} / M_{n+1}(E)^\circ)
\]

defines an isomorphism \( \mathcal{C}onn \times \mathcal{P}roj \rightarrow \mathcal{E}x\mathcal{C}onn_{n+1}(E) \) and thereby lifts the latter to \( \mathcal{M}\mathcal{O}p_2(E)(n+1) \) (and hence \( \mathcal{M}\mathcal{O}p_n \) for every \( n \)).

2.9.4. The deformed quadratic Hitchin map. The projection

\[
\mathcal{E}x\mathcal{C}onn_{n+1}(E) \rightarrow \mathcal{P}roj
\]

of extended connections back to projective structures may be described in several ways. Following § 2.8.6, it is given by sending \( M_{n+1}(E)\lvert_{3\Delta} \rightarrow M_{n+1}\lvert_{3\Delta} \otimes \text{End } E \) via the connection and thence to \( \mathcal{P}roj(n+1) \) via trace of sections. Alternatively, it can be deduced from the determinant map

\[
\det : M_{n+1}(E)\lvert_{3\Delta} \rightarrow M_{k(n+1)}(\det E)\lvert_{3\Delta}.
\]

We identify the resulting \( \text{GL}_2 \)--oper with an element of \( \mathcal{P}roj(k(n+1)) \) by tensoring with the connection of \( \det E \) as in Lemma 2.3.6, and thence with an element of \( \mathcal{P}roj(n+1) \) by Lemma 2.6.1 (This agrees with the trace map since we are restricting to \( 3\Delta \), thereby keeping only the leading term of the determinant.)
Another description of the projection is given by taking the trace of the square of the kernel. More precisely, for \( s \in \Gamma(M_{n+1}(E)|_{3\Delta}) \), its transpose \( s^t = \sigma^*s \in M_{n+1}(E^*)|_{3\Delta} \), so that the tensor product lives in
\[
s \otimes s^t \in (\text{End } E \boxtimes \text{End } E) \otimes M_{2(n+1)}
\]
over \( 3\Delta \). We apply trace to both factors, obtaining
\[
S(s) = \text{tr}_E \boxtimes \text{tr}_E(s \otimes s^t) \in \Gamma(M_{2(n+1)}|_{3\Delta})
\]
which is monic if \( s \) is. To compare this with the other constructions, suppose \( \rho \in M_{n+1}|_{3\Delta} \) is a projective structure, \( \nabla \) is a connection and \( \kappa \in E \boxtimes E^*|_{3\Delta} \) is the corresponding kernel function giving the isomorphism \( p_2^*E \to p_1^*E \). Note that
\[
(Id \otimes \text{tr}_E)(\kappa \otimes \kappa^t) = \text{Id}_E \boxtimes 1 \in \text{End } E \boxtimes 0,
\]
simply expressing the fact that \( \kappa^t \) is the flat kernel for the inverse map \( p_1^*E \to p_2^*E \).

It follows that
\[
S(\rho \otimes \kappa) = \rho,
\]
so that \( S \) is indeed the projection back on projective structures.

This description of the determinant map for extended connections presents it as a deformation of the quadratic Hitchin map. Namely let
\[
\mathcal{E}x\text{Conn}^{\lambda}_{n+1}(E) = \{ s \in \Gamma(M_{n+1}(E)|_{3\Delta}) \mid s|_{\Delta} = \lambda \text{Id}\} / \Gamma(M_{n+1}(E)^0|_{3\Delta})
\]
be the family deforming extended connections to extended Higgs fields.

2.9.5. Proposition. The determinant map \( \mathcal{E}x\text{Conn}_{n+1}(E) \to \text{Proj}(n+1) \) deforms to a map
\[
\mathcal{E}x\text{Conn}^{\lambda}_{n+1}(E) \to \text{Proj}(\lambda(n+1))
\]
(for \( \lambda \in \mathbb{C} \)), which for \( \lambda = 0 \) factors through the quadratic Hitchin map
\[
\mathcal{E}x\text{Higgs}(E) \to \Gamma(\Omega_X \otimes \text{End } E) \to \Gamma(\Omega_X^2) = \text{Proj}(0),
\]
sending \( \eta \in \Gamma(\Omega_X \otimes \text{End } E) \) to \( \text{tr}_E(\eta^2) \).

2.9.6. Proof. If \( s|_{\Delta} = \lambda \text{Id}_E \) then \( S(s)|_{2\Delta} = \lambda^2 \mu_{2(n+1)} \) (by symmetry with respect to transposition of factors). For \( \lambda \neq 0 \) the space of such kernels is isomorphic (by rescaling and taking square–root, Lemma 2.6.1) with projective structures. In fact the resulting map \( \mathcal{E}x\text{Conn}^{\lambda}_{n+1}(E) \to \text{Proj}(\lambda(n+1)) \) is a morphism of torsors for quadratic differentials (the square root \( \text{Proj}(2\lambda(n+1)) \to \text{Proj}(\lambda(n+1)) \) compensates for the quadratic expression \( s \otimes s^t \)). This map clearly descends to \( \mathcal{E}x\text{Conn}^{\lambda}_{n+1}(E) \). On the other hand, for \( \lambda = 0 \), we obtain a quadratic differential, realized as a section of \( M_{n+1}|_{3\Delta} \) vanishing on \( 2\Delta \). This quadratic differential depends only on the Higgs field \( \eta \) underlying the extended Higgs field \( s \), and equals \( \text{tr}_E(\eta^2) \) (the first trace squares \( \eta \) by contracting indices, while the other trace takes trace of the resulting matrix).
3. The Klein and Wirtinger maps.

Let \( \mathcal{M}_X(n) \) denote the moduli space of semistable vector bundles over \( X \) of rank \( n \) and Euler characteristic 0. It is known that \( \mathcal{M}_X(n) \) is an irreducible normal projective variety of dimension \( (g-1)(n^2-1)+g \). In particular, \( \mathcal{M}_X(1) = \text{Pic}^g \), the moduli of degree \( g-1 \) line bundles. Let \( \mathcal{M}_X(n)_0 \) denote the moduli space of semistable vector bundles of rank \( n \) and degree 0. The chosen theta characteristic \( \Omega^\frac{1}{2} \) gives an isomorphism

\[
\mathcal{M}_X(n) \to \mathcal{M}_X(n)_0, \quad E \mapsto E_0 = E \otimes \Omega^{-\frac{1}{2}}_X
\]

(since tensoring by a line bundle preserves semistability).

The determinant map \( E \mapsto \det E \) sends \( \mathcal{M}_X(n) \) to \( \text{Pic}_X^{n(g-1)} \). We may identify a closed subvariety

\[
\mathfrak{M}_X(n) = \det^{-1}(\{\Omega^{\frac{n}{2}}\}) \subset \mathcal{M}_X(n)
\]

which is isomorphic, via \( E \mapsto E_0 = E \otimes \Omega^{-\frac{1}{2}}_X \), to the moduli of semistable \( \text{SL}_n \)-bundles. The dimension of \( \mathfrak{M}_X(n) \) is \( (g-1)(n^2-1) \).

The subvariety

\[
\Theta := \{V \in \mathfrak{M}_X(n) \mid H^0(X, V) \neq 0\}
\]

is a (reduced) divisor, the generalized theta divisor, that gives the ample generator of the Picard group \( \text{Pic}(\mathfrak{M}_X(n)) \). Note that for any \( E \) in \( \mathfrak{M}_X(n) \), we have \( h^0(E) = h^1(E) \). The condition \( h^0(E) = h^1(E) = 0 \) also guarantees that \( E \) is semistable. Indeed, if a subbundle \( F \) of \( E \) contradicts the semistability condition of \( E \), then the Riemann–Roch theorem ensures that \( h^0(F) > 0 \), thus contradicting the condition that \( h^0(E) = 0 \). The smooth locus of the theta divisor \( \Theta \) is precisely the subvariety \( \Theta^o \) of vector bundles \( E \) with \( h^0(E) = h^1(E) = 1 \).

Let \( \mathfrak{K}_X(n) \subset \mathfrak{M}_X(n) \) denote the subvariety consisting of vector bundles, which are isomorphic to a direct sum of line bundles. Thus for \( n = 2 \), \( \mathfrak{K}_X(2) \) consists of vector bundles of the form \( \mathcal{L} \oplus \mathcal{L}^\vee \cong \mathcal{L}^\vee \oplus \mathcal{L} \), so that \( \mathfrak{K}_X(2) \) is isomorphic to the Kummer variety \( \mathfrak{K}_X(2) = \text{Pic}^{2-1} \mathbb{P}(\mathcal{L} \sim \mathcal{L}) \).

3.1. The Szegő kernel. For \( E \in \mathfrak{M}_X(n) \), with \( E_0 = E \otimes \Omega^{-\frac{1}{2}}_X \in \mathfrak{M}_X(n)_0 \), denote by \( \mathcal{M}(E) \) the sheaf

\[
\mathcal{M}(E) = \mathcal{M}_1(E_0) \cong E \boxtimes E^\vee(\Delta).
\]

(By Remark 2.3.4 \( \mathcal{M}(E)_{|\Delta} \) is independent of \( \Omega^{\frac{1}{2}}_X \).) Let \( \mathcal{M}(E)^o \) denote the subsheaf

\[
\mathcal{M}(E)^o = \{s \in \mathcal{M}(E) : s|_{\Delta} = \lambda \text{Id}_E (\lambda \in \mathbb{C})\}.
\]

When \( E \in \mathfrak{M}_X(n) \setminus \Theta \), there is a canonical kernel function associated to \( E \), the non-abelian Szegő kernel of Fay (see also (BR)). In particular we will use the following characterization of the Szegő kernel:
3.1.1. Proposition. (BB)  
(1) If \( h^0(E) = h^1(E) = 0 \), then \( H^0(X \times X, \mathcal{M}(E)^\circ) = C \cdot s_E \), where \( s_E \), the Szegö kernel of \( E \), is the unique section with \( s_E|_\Delta = \text{Id}_E \).

(2) Otherwise, the inclusion  
\[ H^0(X, E) \otimes H^0(X, E^\vee) \cong H^0(X \times X, E \boxtimes E^\vee) \hookrightarrow H^0(X \times X, \mathcal{M}(E)^\circ) \]

is an isomorphism. In other words, all global sections of \( \mathcal{M}(E)^\circ \) vanish on \( \Delta \).

3.1.2. Thus \( s_E|_{k\Delta} \in \mathcal{MO}_k(E_0)(1) \) is a canonical (shifted) matrix oper on \( E_0 \) (§ 2.8). The proposition follows from Serre duality and the long exact sequence of cohomologies of \( E \boxtimes E^\vee \) with poles along the diagonal.

3.1.3. Definition.  
(1) The Wirtinger oper associated to a bundle \( E \in \mathcal{M}_X(n) \setminus \Theta \) is the GL\(_n\)-oper  
\[ \text{det} s_E \in \Gamma(\mathcal{M}_n(\text{det} E_0)). \]

Restricting to \( k\Delta \) defines a (shifted) GL\(_k\)-oper for the line bundle \( \text{det} E_0 \). (We will identify shifted opers with opers, using § 2.6.)

(2) The Klein oper kernel associated to a bundle \( E \in \mathcal{M}_X(n) \setminus \Theta \) is the kernel  
\[ \text{det} s_E \in H^0(X \times X, \mathcal{M}_n(\text{det} E)). \]

The resulting map  
\[ \mathcal{K} : \mathcal{N}_X(n) \setminus \Theta \rightarrow \text{Kern}_n \]

is the Klein map (of rank \( n \)).

3.1.4. Note that the dimensions of \( \mathcal{M}_X(n) \) and \( \mathcal{O}_p_n \) agree, as do those of \( \mathcal{N}_X(n) \) and \( \mathcal{O}_p^\circ_n \). Thus if we knew \( \mathcal{W} \) to be a finite map, it would give a canonical system of étale coordinates on an open subvariety of the moduli space. This leads us to conjecture:

3.1.5. Conjecture.  
(1) The Klein map is finite onto its image for all \( X \).

(2) The Wirtinger map is finite for generic \( X \).

3.1.6. We will prove the conjecture in the case of torus bundles, i.e., along \( \mathcal{R}_X(n) \subset \mathcal{M}_X(n) \). We first describe the Szegö kernel and its determinant for torus bundles. Suppose \( E \cong L_1 \oplus L_2 \oplus \cdots \oplus L_n \). Then \( E \in \mathcal{M}_n(X) \setminus \Theta \) if and only if each \( L_i \in \text{Pic}^0_X \setminus \Theta \). Moreover in this case \( s_E = s_{L_1} \oplus \cdots \oplus s_{L_n} \), and \( \text{det} s_E = \bigotimes_{i=1}^n s_{L_i} \). If \( E \in \mathcal{M}_X(n) \) then we have in addition  
\[ \bigotimes_{i=1}^n L_i = \Omega^2_X. \]

For example, if \( n = 2 \), \( E = L \oplus L^\vee \) and  
\[ s_E = s_L s_{L^\vee}. \]

Recall the Petri map  
\[ H^0(X, L) \otimes H^0(X, L^\vee) \rightarrow H^0(X, \Omega) \]
obtained by tensoring of sections [ACGH, p. 127]. Under the K"unneth isomorphism
\[ H^0(X \times X, \mathcal{L} \boxtimes \mathcal{L}^\vee) = H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}^\vee), \]
the Petri map is identified with the restriction to the diagonal
\[ H^0(X \times X, \mathcal{L} \boxtimes \mathcal{L}^\vee) \longrightarrow \Gamma(\mathcal{L} \boxtimes \mathcal{L}^\vee|_\Delta) = H^0(X, \Omega). \]
Thus injectivity of the Petri map implies that global sections of \( \mathcal{L} \boxtimes \mathcal{L}^\vee \) are determined by their restriction to the diagonal. The curve \( X \) is called \textit{Brill-Noether general} if the Petri map is injective for every line bundle \( \mathcal{L} \). By the Petri conjecture (Lazarsfeld’s Theorem), this condition is satisfied by a generic curve of genus \( g \).

We then having the following result in the direction of the finiteness conjecture:

3.1.7. \textit{Theorem.}

(1) The Klein map for Kummers \( K: \mathfrak{R}_X(n) \setminus \Theta \to \mathcal{Kern}_n \) is finite onto its image for all \( X \).

(2) The Wirtinger map for Kummers \( W: \mathfrak{R}_X(n) \setminus \Theta \to \mathcal{Op}_n \) is finite onto its image for Brill-Noether general \( X \).

3.1.8. \textit{Proof of (1).} Consider the subvariety of \( (\text{Pic}^{g-1}_X)^n \) of line bundles \((\mathcal{L}_1, \ldots, \mathcal{L}_n)\) with \( \bigotimes_{i=1}^n \mathcal{L}_i \cong \Omega^n_X \). (We identify this with \( (\text{Pic}^{g-1}_X)^{n-1} \) through the first \( n-1 \) \( \mathcal{L}_i \).)

For (1), it clearly suffices to show that the map from \( (\text{Pic}^{g-1}_X)^{n-1} \) to \( \mathcal{Kern}_n \) given by
\[ (\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}) \mapsto \bigotimes_{i=1}^n \mathfrak{S}_{\mathcal{L}_i} \]
is finite. To do so we consider \( \mathcal{Kern}_n \) as a subvariety of \( \mathbb{P}H^0(X \times X, \mathcal{M}_n) \) (contained in the affine open of sections with nonzero trace on the diagonal), and complete \( K \) to a morphism
\[ K: (\mathbb{P}^{g-1})^{n-1} \longrightarrow \mathbb{P}H^0(X \times X, \mathcal{M}_n) \]
from a partial resolution of the singular locus of \( \Theta \). Here \( \mathbb{P}^{g-1} \to \text{Pic}^{g-1}_X \) is a projective morphism, which is an isomorphism off the singular part of the theta divisor. (In fact \( \mathbb{P}^{g-1} \) will be the union, for \( X \) Brill-Noether general, of the projectivized conormal bundles to the Brill-Noether loci \( W^{g-1,i} \subset \text{Pic}^{g-1}_X \).) Hence the extended map is automatically proper, and a closer examination shows it remains proper when restricted to \( (\text{Pic}^{g-1}_X \setminus \Theta)^{n-1} \), and hence finite.

We construct \( \mathbb{P}^{g-1} \) as the moduli of pairs \((\mathcal{L}, s)\) consisting of a line bundle \( \mathcal{L} \in \text{Pic}^{g-1}_X \) and a nonzero section \( s \) of \( M(\mathcal{L}) \), up to scalar (i.e., a divisor in the complete linear series \(|M(\mathcal{L})| \) on \( X \times X \)). This is a projective variety mapping to \( \text{Pic}^{g-1}_X \), with the fibers the projective spaces \( \mathbb{P}H^0(X \times X, \mathcal{L} \boxtimes \mathcal{L}^\vee(\Delta)) \). The construction of this projective variety follows from that of the Hilbert scheme of divisors, of the same degree as \( M(\mathcal{L}) \), on the surface \( X \times X \). This Hilbert scheme fibers over the Picard group of \( X \times X \), and we pull it back to \( \text{Pic}^{g-1}_X \) over the morphism \( \text{Pic}^{g-1}_X \to \text{Pic}(X \times X) \) sending \( \mathcal{L} \) to \( M(\mathcal{L}) \).

It follows from Proposition 3.1.3 that over \( \text{Pic}^{g-1}_X \setminus \Theta \) the projection \( \mathbb{P}^{g-1} \to \text{Pic}^{g-1}_X \) is an isomorphism, since the Szegő kernel is the unique section of \( M(\mathcal{L}) \) up to scalars. In fact, the morphism remains an isomorphism on the smooth locus of \( \Theta \), since for
$h^0(\mathcal{L}) = 1$ we have $h^0(M(\mathcal{L})) = h^0(\mathcal{L})h^0(\mathcal{L}^\vee) = 1$. Since by Proposition 3.1.1 every section of $M(\mathcal{L})$ for $\mathcal{L} \in \Theta$ defines a section of $\mathcal{L}$ and one of $\mathcal{L}^\vee$, it follows that the inverse image in $\mathbb{P}^g$ over $\Theta$ (for the projection of $\mathbb{P}^g$ to $\text{Pic}^g_X$) is given by

$$\mathbb{P}^g|_{\Theta} \cong \text{Sym}^g X \times_{\text{Pic}^g_X} \iota^* \text{Sym}^g X,$$

where $\iota : \mathcal{L} \to \mathcal{L}^\vee$ — in other words, the inverse image is the space of pairs of divisors for $\mathcal{L}$ and $\mathcal{L}^\vee$. (Thus $\mathbb{P}^g$ restricts, for $X$ Brill–Noether general, to the union of blowups of the Brill–Noether loci in $\text{Pic}^g_X$.)

We now extend the morphism $\mathbb{K}$ from $(\text{Pic}^g_X)^n$ to $\mathbb{P}^g_n$, the inverse image of $(\text{Pic}^g_X)^n$ in $(\mathbb{P}^g)^n$, i.e., $\mathbb{P}^g_n$ parametrizes $(\mathcal{L}_1, s_1; \ldots; \mathcal{L}_n, s_n)$ where the $\mathcal{L}_i$ add up to $\Omega^2_X$. To such a tuple we assign the line $[\bigotimes_{i=1}^n s_i]$ in

$$\bigotimes_{i=1}^n (\pi_{X \times X})_* M(\mathcal{L}_i) = (\pi_{X \times X})_* M(\Omega^2_X),$$

where $s_i$ are the tautological sections of $\mathcal{L}_i$ given by the $i$th point in $\mathbb{P}^g$ (taken up to scalar). The right hand side is the vector space $H^0(X \times X, M_n)$, independently of the $\mathcal{L}_i$; so we have constructed the desired extension

$$\mathbb{K} : \mathbb{P}^g_n \longrightarrow \mathbb{P}H^0(X \times X, M_n).$$

The completed morphism $\mathbb{K}$ is a morphism of projective varieties, hence proper. We claim its restriction to $(\text{Pic}^g_X \setminus \Theta)^n$ is also proper. Let

$$\mathbb{P}H^0(X \times X, M_n) \subset \mathbb{P}H^0(X \times X, M_n)$$

denote the hyperplane of sections vanishing on the diagonal. By Proposition 3.1.1, for $\mathcal{L} \in \Theta$, all sections of $M(\mathcal{L})$ automatically vanish on the diagonal, while for $\mathcal{L} \in \text{Pic}^g_X \setminus \Theta$ all nonzero sections give nonzero constant functions on the diagonal. Hence the preimage of the complement of this hyperplane is precisely $\text{Pic}^g_X \setminus \Theta$. We obtain that the morphism $\mathbb{K}$ from the affine variety $(\text{Pic}^g_X \setminus \Theta)^n$ is proper, hence finite.

3.1.9. Proof of (2). We embed the affine space $\mathbb{O}p_n^\circ$ in the projective space

$$\overline{\mathbb{O}p_n^\circ} = \mathbb{P}\Gamma(M_n|_{(n+1)\Delta}).$$

Thus $\mathbb{W}$ gives rise to a map

$$\mathbb{W} : (\text{Pic}^g_X \setminus \Theta)^n \longrightarrow \overline{\mathbb{O}p_n^\circ}.$$

In order to prove finiteness of $\mathbb{W}$, we would like to extend it to $\mathbb{P}^g$, whenever possible.

Let $\mathcal{L} \in \Theta$. Then by Proposition 3.1.1, global sections of $\mathcal{L} \boxtimes \mathcal{L}^\vee(\Delta)$ vanish on $\Delta$. If the Petri map of $\mathcal{L}$ is injective, however, such sections are determined by their restriction to $2\Delta$. So we take $X$ to be Brill–Noether general. It follows that for a collection of nonzero sections $s_i \in H^0(X \times X, M(\mathcal{L}_i))$, the restriction $[\bigotimes_{i=1}^n s_i]|_{(n+1)\Delta}$ is also nonzero. Thus the $s_i$ define a point in $\overline{\mathbb{O}p_n^\circ}$, and we have completed $\mathbb{W}$ to a map

$$\mathbb{W} : (\mathbb{P}^g)^n \longrightarrow \overline{\mathbb{O}p_n^\circ}.$$
Again the inverse image of the hyperplane of sections vanishing on the diagonal is precisely the inverse image of the theta divisor, so the map remains proper off $\Theta$, implying finiteness as before.

4. Relations with theta functions.

4.1. The theta linear series. The Klein and Wirtinger maps have natural interpretations as quotients of the theta linear series on $\mathcal{M}_X(n)$ and $\mathfrak{M}_X(n)$. For $E \in \mathcal{M}_X(n)$, consider the sequence of maps

$$((n+1)\Delta) \hookrightarrow X \times X \xrightarrow{\delta} \text{Jac}_X \xrightarrow{\tau_E} \mathcal{M}_X(n).$$

Here

$$\tau_E : \text{Jac}_X := \text{Pic}^0(X) \rightarrow \mathcal{M}_X(n), \quad \tau_E(\mathcal{L}) = E \otimes \mathcal{L}$$

is the translation map, $\delta(x,y) = y - x$ and the composition $\tau_E \circ \delta$ is the difference map

$$\delta_E : X \times X \rightarrow \mathcal{M}_X(n), \quad \delta_E(x,y) = E(y - x).$$

It is well–known that for $E \in \mathfrak{M}_X(n)$, the pullback of nonabelian theta functions

$$\tau^*_E[\mathcal{O}\mathcal{M}_X(n)(\Theta)] = \mathcal{O}_{\text{Jac}_X}(n\Theta)$$

are weight $n$ abelian theta functions. Moreover the resulting map

$$\tau^* : \mathfrak{M}_X(n) \rightarrow \mathbb{P}H^0(\text{Jac}_X, \mathcal{O}_{\text{Jac}_X}(n\Theta))$$

is an embedding (see [Be]). (Note that we have fixed a theta characteristic $\Omega^3_X$, which allows us to principally polarize the Jacobian and pass from line bundles $\mathcal{L}$ of degree $n(g-1)$ to $\mathcal{L}_0$ of degree 0.) Pulling back further to $X \times X$ or $(n+1)\Delta$, we obtain sections of the pullback $\delta^*_E[\mathcal{O}\mathcal{M}_X(n)(\Theta)] = \mathcal{M}_n \otimes \Theta|_E$, the tensor of the line bundle $\mathcal{M}_n$ by the complex line $\Theta|_E$, the fiber of $\Theta$. (See e.g. [BB].)

It follows that we have a sequence of pullback maps

$$H^0(\text{Jac}_X, \mathcal{O}_{\text{Jac}_X}(n\Theta)) \rightarrow H^0(X \times X, \mathcal{M}_n) \rightarrow \Gamma(\mathcal{M}_n|_{(n+1)\Delta}),$$

and consequently rational maps on the corresponding projective spaces. Composing these with $\tau^*$ we obtain rational maps from $\mathfrak{M}_X(n)$ (if the image of $\tau^*$ is not contained in the kernels of the projections).

We will use the following description of the Szegö kernel:

4.1.1. Theorem. ([BB], see also [GH, Po]) $\det s_E = \delta^*_E \theta/\theta(E)$.

4.1.2. Corollary. The Klein and Wirtinger maps

$$K : \mathfrak{M}_X(n) \setminus \Theta \rightarrow \mathbb{P}H^0(X \times X, \mathcal{M}_n)$$
$$W : \mathfrak{M}_X(n) \setminus \Theta \rightarrow \mathbb{P}\Gamma(\mathcal{M}_n|_{(n+1)\Delta})$$

are equal to the composition of the theta linear series $\tau^*$ with the restrictions to $X \times X$ and $(n+1)\Delta$, respectively.
4.2. The linear series |2Θ|. Let us consider the case $n = 2$. (Our reference for 2Θ functions is [2].) The map $\tau^* : \mathfrak{M}_X(n) \to \mathcal{P}\mathcal{H}^0(\text{Jac}_X, O(2\Theta))$ restricts on the Kummer variety $\text{Jac}_X \to \mathfrak{K}_X(2) \subset \mathfrak{M}_X(2)$ to the map

$$\text{Jac}_X \ni e \mapsto \Theta_e + \Theta_{-e}$$

(where $\Theta_e$ denotes the translate of $\Theta$ by $e$). The Riemann quadratic identity and Kummer identification theorem provide a natural isomorphism between this map and the 2Θ linear series

$$|2\Theta|_* : \text{Jac}_X \to \mathcal{P}\mathcal{H}^0(\text{Jac}_X, O(2\Theta))^*$$

which naturally maps to the dual projective space.

By the symmetry properties of 2Θ it follows that the image of $\mathcal{H}^0(\text{Jac}_X, O(2\Theta))$ in $\mathcal{H}^0(X \times X, M_2)$ consists of symmetric bidifferentials. In fact there is a short exact sequence

$$0 \to \Gamma_{00} \to \mathcal{H}^0(\text{Jac}_X, O(2\Theta)) \to \mathcal{H}^0(X \times X, \Omega \boxtimes \Omega(2\Delta)) \to 0,$$

where the kernel $\Gamma_{00}$ can be characterized as the subspace of 2Θ–functions vanishing to fourth order at 0. The right hand side is a vector space of dimension $(g_2 - 3) + 1$. Its projective space $\mathcal{Ker}_{2\text{sym}} \cong \mathbb{P}^{\frac{3}{2}}$ contains as an affine open the space $\mathcal{Ker}_{2\text{sym}}$ of projective kernels. This vector space has a further quotient $\Gamma(\Omega_X \boxtimes \Omega_X(2\Delta)|_{3\Delta}) \boxtimes \Omega(3\Delta)$, obtained by restricting kernels to $3\Delta$. Its projective space $\mathcal{Proj} \cong \mathbb{P}^{3g-3}$ contains as an affine open the space $\mathcal{Proj}$ of projective structures. Note that the image of $\mathbb{K} : \mathfrak{M}_X(2) \setminus \Theta \to \mathcal{Ker}_{2\text{sym}}$ lies in $\mathcal{Ker}_{2\text{sym}}$, while $\mathbb{W}$ defines a map $\mathbb{W} : \mathfrak{M}_X(2) \setminus \Theta \to \mathcal{Proj}$.

We may thus reinterpret the finiteness theorem as follows:

4.2.1. Corollary.

(1) The rational map $\text{Jac}_X \to \mathbb{P}^{\frac{3}{2}}$ defined by the composition of $|2\Theta|_*$ with projection by $\Gamma_{00}$ is a finite morphism on $\text{Jac}_X \setminus \Theta$.

(2) For $X$ generic, the further projection $\text{Jac}_X \to \mathbb{P}^{3g-3}$ remains finite on $\text{Jac}_X \setminus \Theta$.

4.2.2. Formulas. The explicit description of the Szegö kernel for line bundles is

$$s_{\mathcal{L}}(x, y) = \frac{\theta(y - x + \mathcal{L}_0)}{\theta(\mathcal{L}_0)E(x, y)},$$

where $E(x, y)$ is the prime form (this is the rank one case of Theorem 4.1.1). Thus the Klein map on the Kummer $\mathfrak{K}_X(2)$ becomes

$$\mathbb{K}(\mathcal{L} \oplus \mathcal{L}) = s_{\mathcal{L}}s_{\mathcal{L}} = \frac{\theta(y - x + \mathcal{L}_0)\theta(y - x - \mathcal{L}_0)}{\theta(\mathcal{L}_0)^2E(x, y)^2}.$$

The relation to 2Θ is easily seen explicitly. Let

$$\vec{\theta} : \mathbb{C}^g \to \mathcal{P}\mathcal{H}^0(\text{Jac}_X, O(2\Theta))^*$$

defined by

$$\vec{\theta}(e) = \sum_{\alpha, \beta \in \text{Jac}_X[2]} \theta^\alpha \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (e)$$
be the generating vector of the second order theta functions with characteristics. By Riemann’s quadratic identity (|Mu1|), we may rewrite the expression \( \mathbb{K}(\mathcal{L} \oplus \mathcal{L}^\vee) \) of (4.2.2) as follows:

\[
\frac{\theta(y - x + \mathcal{L}_0)\theta(y - x - \mathcal{L}_0)}{\theta(\mathcal{L}_0)^2E(x,y)^2} = \frac{\theta'(y - x) \cdot \mathcal{L}_0}{\theta(\mathcal{L}_0)^2E(x,y)^2}.
\]

4.3. **The Gauss map.** Let \( \Theta^\circ \subset \text{Pic}^{g-1}_X \) denote the smooth part of the theta divisor. The Gauss map for the theta divisor sends

\[
\Theta^\circ : \Theta^\circ \to \mathbb{P}H^0(X, \Omega).
\]

Since \( H^0(X, \mathcal{L}) = \mathbb{C}l \) is one dimensional for \( \mathcal{L} \in \Theta^\circ \), the Petri map for \( \mathcal{L} \),

\[
\mathcal{L} \mapsto l \otimes l^\vee,
\]

also defines a line in \( H^0(X, \Omega) \), which is known to agree with the Gauss line for \( \mathcal{L} \). On the other hand the extension of \( \mathbb{W} \) to \( \Theta^\circ \subset \mathbb{P}^{g-1} \) sends

\[
\mathcal{L} \mapsto (l \otimes l^\vee) \otimes (l^\vee \otimes l)|_\Delta = (l \otimes l^\vee)^{\otimes 2},
\]

which defines a line in \( \mathbb{P}H^0(X, \Omega) \subset \mathbb{P}\text{proj} \). Thus the tensor square of the Gauss map agrees with the morphism \( \mathbb{W} \):

4.3.1. **Corollary.** For a Brill–Noether general curve, the square of the Gauss map

\[
\mathfrak{G}^{\otimes 2} : \Theta^\circ \to \mathbb{P}H^0(X, \Omega^{\otimes 2})
\]

extends to a finite morphism

\[
\mathbb{W} : \text{Pic}^{g-1}_X \setminus \Theta^{\text{sing}} \to \mathbb{P}\text{proj}.
\]

4.3.2. **Remark.** It is interesting to note that this relation of the Klein map to the theta divisor fails completely in higher rank. Namely, for \( E \in \Theta^\circ \) we still have \( H^0(X, E) = \mathbb{C}s \).

It follows that the Higgs field

\[
s \otimes s^\vee \in \text{End} E \otimes \Omega = (E \otimes E^\vee)|_\Delta
\]

is nilpotent. In fact as \( E \) varies over \( \Theta^\circ \) we obtain this way an irreducible component of the global nilpotent cone in the moduli of Higgs bundles. Thus the “Hitchin–Gauss” map, applying characteristic polynomials to this canonical line of Higgs bundles along \( \Theta^\circ \), is identically zero. In particular the determinant \( \det s \otimes s^\vee = 0 \) vanishes identically on \( X \times X \), so we cannot use this to extend the Klein map across the theta divisor.

4.4. **Logarithmic derivatives of theta.** In |Mu1|, Mumford cites three general techniques for constructing meromorphic functions on Jacobians out of theta functions, of which the third is that of taking second logarithmic derivatives. Namely, there is a collection of \( \binom{g}{2} \) meromorphic functions

\[
\frac{\partial^2 \log \theta}{\partial z_i \partial z_j}
\]
on the Jacobian – or more invariantly, a rational map

\[ e \mapsto \sum_{i=1}^{g} \frac{\partial^2 \log \theta}{\partial z_i \partial z_j} (e) \omega_i(x) \omega_j(y). \]

from the Jacobian to holomorphic symmetric bidifferentials on \( X, H^0(X \times X, \Omega \boxtimes \Omega)^{sym} \).

By translating these holomorphic bidifferentials by the Bergman kernel \( \omega_B \) (§ 2.7), we obtain the Klein projective kernels \( \omega_e \in \text{Kern}^{sym}_2 (\mathbb{L}) \):

\[ \omega_e = \omega_B(x, y) + \sum_{i=1}^{g} \frac{\partial^2 \log \theta}{\partial z_i \partial z_j} (e) \omega_i(x) \omega_j(y). \]

Here the point \( e \in \text{Jac}_X \setminus \Theta \). Classically \( e \) is taken to be a two–torsion point, so that \( \omega_e \) is written in terms of theta functions with characteristics. The corresponding projective connections \( \omega_e |_{3\Delta} \in \text{Proj} \) are known (§ 1.3) as the Wirtinger connections.

The relation of these classical kernels with our Klein and Wirtinger maps is provided by the “second corollary to the trisecant identity” of J. Fay (§ 1.1; also [Mu2]):

\[ K(\mathbb{L}) = \frac{\theta(y-x+\mathbb{L}_0)\theta(y-x-\mathbb{L}_0)}{\theta(\mathbb{L}_0)^2 E(x, y)^2} = \omega_B(x, y) + \sum_{i=1}^{g} \frac{\partial^2 \log \theta}{\partial z_i \partial z_j} (\mathbb{L}_0) \omega_i(x) \omega_j(y) = \omega_{\mathbb{L}_0}. \]

4.4.1. Corollary. The second logarithmic derivatives of \( \theta \) provide a finite parametrization of the complement of the theta divisor in the Jacobian in affine space of dimension \( \binom{g}{2} \). Namely, the holomorphic map

\[ \text{Jac}_X \setminus \Theta \longrightarrow H^0(X \times X, \Omega \boxtimes \Omega)^{sym} \]

of (4.4.1) is finite onto its image.

4.4.2. Remark. It also follows from Corollary 4.4.1 that the second logarithmic derivative map is generically finite for generic abelian varieties, since it is finite on the Jacobian locus.

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