FULLY NONLINEAR EQUATIONS OF KRYLOV TYPE ON RIEMANNIAN MANIFOLDS WITH NEGATIVE CURVATURE

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Abstract. In this paper, we consider fully nonlinear equations of Krylov type on Riemannian manifolds with negative curvature which naturally arise in conformal geometry. Moreover, we prove the a priori estimates for solutions to these equations and establish the existence results. Our results can be viewed as an extension of previous results given by Gursky-Viaclovsky and Li-Sheng.

1. Introduction

Let \((M, g_0)\) be a smooth closed Riemannian manifold of dimension \(n \geq 3\) and \([g_0]\) denote the conformal class of \(g_0\) on \(M\), the well-known \(\sigma_k\)-Yamabe problem is of finding a metric \(g \in [g_0]\) satisfies the following equation on \(M\)

\[
\sigma_k(A_g) = \text{constant},
\]

where

\[
A_g = \frac{1}{n-2} \left( \Ric_g - \frac{R_g}{2(n-1)} g \right)
\]

is the Schouten tensor of \(g\), \(\Ric_g\) and \(R_g\) are the Ricci and scalar curvatures of \(g\) respectively, we denote by \(\sigma_k(\lambda)\) the \(k\)-th elementary symmetric polynomial

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n,
\]

for \(1 \leq k \leq n\) and we set \(\sigma_0(\lambda) = 1\), and \(\sigma_k(A_g)\) means that \(k\)-th elementary symmetric polynomial \(\sigma_k\) is applied to the eigenvalues of \(g^{-1} \cdot A_g\).

For \(k = 1\), the equation (1.1) is just the classical Yamabe equation which has been solved by Yamabe [61], Trudinger [59], Aubin [1] and Schoen [52]. The fully nonlinear elliptic equation (1.1) \((k \geq 2)\) has been studied extensively after the pioneering works of Viaclovsky [60, 61, 62]. Under the assumption that the eigenvalues \(\lambda(A_{g_0})\) of the matrix \(g^{-1} \cdot A_{g_0}\) belong to \(\Gamma_k\) with

\[
\Gamma_k = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n | \quad \sigma_j(\lambda) > 0, \quad \forall 1 \leq j \leq k \},
\]

the \(\sigma_k\)-Yamabe equation (1.1) has been solved for either \(k = 2\), or \(k \geq \frac{n}{2}\), or \(M\) being locally conformally flat by the works of Chang-Gursky-Yang [8, 7], Guan-Wang [26].

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Theorem 1.1. Assume that \[(1.3)\]

\[\sigma \]  

We proved the existence of solutions to the equation \[(1.3)\] without the sign requirement for the coefficient function \[\alpha\] in equation \[(1.3)\]. Gursky-Viaclovsky [29] proved that for \[\tau < 1\] and \[\lambda(-A_{g_0}^\tau) \in \Gamma_k\], there exists a unique conformal metric \[g \in [g_0]\] satisfying

\[(1.2)\]

\[\sigma_k(-A_g^\tau) = f(x)\]

for any smooth positive function \[f(x)\] on \[M\]. A parabolic proof was later given by Li-Sheng [44]. Since the equation \[(1.2)\] is not necessarily elliptic for \[\tau > 1\] and the \(C^2\) estimate does not work for the equation \[(1.2)\] with \[\tau = 1\] as noted previously in [61], the restriction \[\tau < 1\] must be made in [29] and [44]. Those works motivated the later research on the equation \[(1.2)\] with \[\tau\] the boundary conditions [28, 18, 19]. See [50] for related research and [57] for the recent progress on noncompact manifolds.

In this paper, we study an extension of the equation \[(1.2)\]. Let \((M, g_0)\) be a smooth closed Riemannian manifold of dimension \(n \geq 3\) and \([g_0]\) denote the conformal class of \(g_0\) on \(M\), we want to find a metric \(g \in [g_0]\) satisfies the following equation on \(M\)

\[(1.3)\]

\[\sigma_k(-A_g^\tau) + \alpha(x)\sigma_{k-1}(-A_g^\tau) = \sum_{l=0}^{k-2} \alpha_l(x)\sigma_l(-A_g^\tau), \quad 3 \leq k \leq n.\]

The following is our main theorem.

**Theorem 1.1.** Assume that \[\tau < 1\] and \[\lambda(-A_{g_0}^\tau) \in \Gamma_k\]. Let \[\alpha_l(x)\] with \(0 \leq l \leq k-2\) and \[\alpha(x)\] be smooth functions on \(M\). Then there exists a conformal metric \(g \in [g_0]\) satisfies equation \[(1.3)\] if \[\alpha_l(x) > 0\] for all \(0 \leq l \leq k-2\) and \(x \in M\).

**Remark 1.2.** We proved the existence of solutions to the equation \[(1.3)\] in Theorem 1.1 without the sign requirement for the coefficient function \[\alpha(x)\] in the equation \[(1.3)\]. This type of the equation was first considered by Guan-Zhang [27].

**Remark 1.3.** In fact, Theorem 1.1 holds for \[\alpha_l(x)\] with \(0 \leq l \leq k-2\) satisfying either \[\alpha_l(x) > 0\] for all \(x \in M\), or \[\alpha_l \equiv 0\], but at least one of \[\alpha_l(x)\] with \(0 \leq l \leq k-2\) is positive on \(M\). Thus, Theorem 1.1 recovers the previous results proved by Gursky-Viaclovsky [29] and Li-Sheng [44].

When \[\alpha_l \equiv 0\] for \(1 \leq l \leq k-2\) and \[\tau = 0\], the equation \[(1.3)\] was considered by the authors with Guo [9]. In fact, \[(1.3)\] is the equation of Krylov type which has been introduced and studied by Krylov in [40] thirty years ago, and can be seen the extension
of the landmark work [5, 6] on the Hessian equation investigated by Caffarelli-Nirenberg-Spruck. In detail, Krylov studied the general Hessian equation

\[ \sigma_k(D^2 u) + \alpha(x)\sigma_{k-1}(D^2 u) = \sum_{l=0}^{k-2} \alpha_l(x)\sigma_l(D^2 u), \quad x \in \Omega \subset \mathbb{R}^n. \]  

(1.4) In particular, he observed that if \( \alpha(x) \leq 0 \) and \( \alpha_l(x) \geq 0 \) for \( 0 \leq l \leq k - 2 \), the natural admissible cone to make the equation (1.4) elliptic is also the \( \Gamma_k \)-cone which is the same as the \( k \)-Hessian equation case. Recently, Guan-Zhang [27] studied the equation of Krylov type in the problem of prescribing convex combination of area measures [53] (1.5) \( \sigma_k(D^2 u + uI) + \alpha(x)\sigma_{k-1}(D^2 u + uI) = \sum_{l=0}^{k-2} \alpha_l(x)\sigma_l(D^2 u + uI) \) on \( S^n \),

with \( \alpha_l(x) \geq 0 \) for \( 0 \leq l \leq k - 2 \), but without the sign requirement for the coefficient function \( \alpha(x) \). In this case, they observed that the proper admissible set of solutions of the equation (1.5) is \( \Gamma_{k-1} \), not \( \Gamma_k \). Based on this important observation, they also studied Krylov’s equation (1.4) in \( \Gamma_{k-1} \). In fact, such type of the equations with its structure as a combination of elementary symmetric functions in fact arise naturally from many important geometric problems, such as the so-called Fu-Yau equation arising from the study of the Hull-Strominger system in theoretical physics (see Fu-Yau [15, 16] and Phong-Picard-Zhang [49, 50, 51]), the special Lagrangian equations introduced by Harvey-Lawson [31], and so on.

The present paper is built up as follows. In Sect. 2 we start with some background. In Sect. 3, we obtain the a priori estimates. We will prove Theorem 1.1 in Sect. 4.

2. Preliminaries

Let \((M, g_0)\) be a smooth closed Riemannian manifold of dimension \( n \geq 3 \) with Levi-Civita connection \( \nabla \). For later convenience, we first state our conventions on derivative notation. For a \((0, r)\)-tensor field \( V \) on \( M \), its covariant derivative \( \nabla V \) is a \((0, r+1)\)-tensor field whose coordinate expression is denoted by

\[ \nabla V = (V_{k_1 \ldots k_r}). \]

Similarly, the coordinate expression of the second covariant derivative of \( V \) is denoted by

\[ \nabla^2 V = (V_{k_1 \ldots k_r i}), \]

and so on for the higher order covariant derivatives. Under the conformal transformation \( g = \exp(2u)g_0 \), the Ricci curvature of \( g \) is given by the formula (see [29])

\[ -A_g^r = \nabla^2 u + \frac{1 - \tau}{n - 2} \Delta u g_0 + \frac{2 - \tau}{2} |\nabla u|^2 g_0 - du \otimes du - A_{g_0}^r, \]

where (and throughout the paper) \( \Delta u \) and \( \nabla^2 u \) denote the Laplacian and Hessian of \( u \) with respect to the background metric \( g_0 \). Consequently, the proof of Theorem 1.1
Proposition 2.1. which was first observed by Guan-Zhang [27]. The following proposition says the proper admissible set for the solutions of (2.2) is $\Gamma_k$ where

$$u$$ is elliptic and concave about $u$ is $C^\infty(M)$ to the partial differential equation of second order

$$\frac{\sigma_k(U)}{\sigma_{k-1}(U)} - \sum_{l=0}^{k-2} \alpha_l(x) \exp(2(k-l)u) \frac{\sigma_l(U)}{\sigma_{k-1}(U)} = -\alpha(x) \exp(2u),$$

where

$$U = \nabla^2 u + \frac{1-\tau}{n-2} \Delta u g_0 + \frac{2-\tau}{2} |\nabla u|^2 g_0 - du \otimes du - A^{\tau}_{g_0},$$

and $\sigma_k(U)$ means that $k$-th elementary symmetric polynomial $\sigma_k$ is applied to the eigenvalues of $g_0^{-1}.U$. To solve the equation (2.1), we consider a family of equations

$$G(U^t) := \frac{\sigma_k(U^t)}{\sigma_{k-1}(U^t)} - \sum_{l=0}^{k-2} \left( [ (1-t)c + t\alpha_l(x) ] \exp[2(k-l)u] \sigma_l(U^t) \right) \frac{\sigma_k(U^t)}{\sigma_{k-1}(U^t)}$$

(2.2)

$$= -t\alpha(x) \exp(2u),$$

where $t \in [0, 1], c = \frac{\sigma_{e-2}(e)}{\sum_{l=0}^{k-2} \sigma_l(e)}, e = (1, \cdots, 1)$, and

$$U^t = \nabla^2 u + \frac{1-\tau}{n-2} \Delta u g_0 + \frac{2-\tau}{2} |\nabla u|^2 g_0 - du \otimes du - tA^{\tau}_{g_0} + (1-t)g_0.$$ 

Here $\sigma_k(U^t)$ means that $k$-th elementary symmetric polynomial $\sigma_k$ is applied to the eigenvalues of $g_0^{-1}.U^t$.

Now we denote by $\lambda(U^t)$ the eigenvalues of the matrix $g_0^{-1}.U^t$ throughout the paper. The following proposition says the proper admissible set for the solutions of (2.2) is $\Gamma_{k-1}$ which was first observed by Guan-Zhang [27].

Proposition 2.1. Assume $\tau < 1$, then the operator

$$G(U^t) := \frac{\sigma_k(U^t)}{\sigma_{k-1}(U^t)} - \sum_{l=0}^{k-2} \left( [ (1-t)c + t\alpha_l(x) ] \exp[2(k-l)u] \sigma_l(U^t) \right) \frac{\sigma_k(U^t)}{\sigma_{k-1}(U^t)}$$

is elliptic and concave about $u$ if $\lambda(U^t) \in \Gamma_{k-1}$, and $\alpha_l(x) \in C^\infty(M)$ is nonnegative for $0 \leq l \leq k-2$.

Proof. The proof is almost the same to that of Proposition 2.2 in [27], so we omit it. \qed

3. The a priori estimates

3.1. $C^0$ estimate. We begin with an important property of $\sigma_k$.

Lemma 3.1. Let $A$ and $B$ be symmetric $n \times n$ matrices and $0 \leq l < k \leq n$.

1. Assume that $A$ is positive semi-definite, $B \in \Gamma_{k-1}$, and $A + B \in \Gamma_{k-1}$. Then, we have

$$\frac{\sigma_k}{\sigma_{k-1}}(A + B) \geq \frac{\sigma_k}{\sigma_{k-1}}(B)$$

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and
\[
\left(\frac{\sigma_{k-1}}{\sigma_{l}}\right)^{\frac{1}{k-1}}(A + B) \geq \left(\frac{\sigma_{k-1}}{\sigma_{l}}\right)^{\frac{1}{k-1}}(B).
\]

(2) Assume that $A$ is negative semi-definite, $B \in \Gamma_{k-1}$, and $A + B \in \Gamma_{k-1}$. Then, we have
\[
\frac{\sigma_{k}}{\sigma_{k-1}}(A + B) \leq \frac{\sigma_{k}}{\sigma_{k-1}}(B)
\]
and
\[
\left(\frac{\sigma_{k-1}}{\sigma_{l}}\right)^{\frac{1}{k-1}}(A + B) \leq \left(\frac{\sigma_{k-1}}{\sigma_{l}}\right)^{\frac{1}{k-1}}(B).
\]

Proof. Since (2) can be easily proved by applying (1) for matrices $-A$ and $A + B$, it is sufficient to prove (1). We know from the concavity of $\sigma_{k}$ for $A \in \Gamma_{k-1}$, which implies in view of the positive semi-definite of $A$
\[
\frac{\sigma_{k}}{\sigma_{k-1}}(A + B) \geq \frac{\sigma_{k}}{\sigma_{k-1}}(A) + \frac{\sigma_{k}}{\sigma_{k-1}}(B)
\]
for $A \in \Gamma_{k-1}$ and $B \in \Gamma_{k-1}$, we have
\[
\frac{\sigma_{k}}{\sigma_{k-1}}(A + B) \geq \frac{\sigma_{k}}{\sigma_{k-1}}(B).
\]

So, we complete the proof of the first inequality in (1). The second inequality in (1) can be proved similarly if we notice that
\[
\left[\frac{\sigma_{k-1}}{\sigma_{l}}\right]^{\frac{1}{k-1}}\text{ is concave in } \Gamma_{k-1} \text{ for } 0 \leq l < k - 1
\]
(see Chapter XV in [47]).

With the help of Lemma 3.1, $C^0$ estimate can be obtained consequently.

Lemma 3.2. Assume that $\tau < 1$ and $\lambda(-A_{x_0}^{l}) \in \Gamma_{k}$. Let $\alpha_{l}(x)$ be a positive smooth function on $M$ for all $0 \leq l \leq k - 2$ and $\alpha(x)$ be a smooth function on $M$. Suppose $u$ is a smooth solution of (2.2) with $\lambda(U^{t}) \in \Gamma_{k-1}$. Then there exists a constant $C$ depending on $\tau$, $g_{0}$, $\|\alpha\|_{C^{0}(M)}$, $\|\alpha_{l}\|_{C^{0}(M)}$ and $\inf_{M} \alpha_{l}$ with $0 \leq l \leq k - 2$, such that
\[
\sup_{M} |u| \leq C.
\]

Proof. Suppose the maximum point of $u$ is attained at $x_{1}$. Thus $\nabla^{2}u(x_{1})$ is negative semi-definite and $\nabla u(x_{1}) = 0$ which implies
\[
\nabla^{2}u + \frac{1 - \tau}{n - 2} \Delta u g_{0} + \frac{2 - \tau}{2} \nabla u |u| g_{0} - du \otimes du
\]
is negative semi-definite at $x_{1}$ if $\tau < 1$. Thus, we arrive at $x_{1}$ from Lemma 3.1
\[
\frac{\sigma_{k}(B_{g_{0}})}{\sigma_{k-1}(B_{g_{0}})} \geq \frac{\sigma_{k}(U^{t})}{\sigma_{k-1}(U^{t})},
\]
and
\[
\frac{\sigma_{l}(B_{g_{0}})}{\sigma_{k-1}(B_{g_{0}})} \leq \frac{\sigma_{l}(U^{t})}{\sigma_{k-1}(U^{t})}.
\]
for $0 \leq l < k - 1$, where $B_{g_0} = -tA_{g_0}^* + (1-t)g_0$. Plugging the inequalities (3.2) and (3.3) into the equation (2.2) arrives at $x_1$

$$C + C \exp (2u(x_1)) \geq C \sum_{l=0}^{k-2} \exp [2(k-l)u(x_1)].$$

Thus,

$$\sup_{x \in M} u(x) \leq C.$$ 

Similarly, we have at one minimum point $x_2$ of $u$

(3.4) $$\frac{\sigma_k(B_{g_0})}{\sigma_{k-1}(B_{g_0})} - C \exp (2u(x_2)) \leq C \sum_{l=0}^{k-2} \exp [2(k-l)u(x_2)].$$

Since $\lambda(-A_{g_0}^*) \in \Gamma_k$, we can obtain $\frac{\sigma_k(B_{g_0})}{\sigma_{k-1}(B_{g_0})} > 0$. Thus, we can conclude from (3.4)

$$\inf_{x \in M} u(x) \geq -C.$$ 

So, the proof is complete. \(\square\)

**Remark 3.3.** In fact, to get an upper bound of $u$, we only need $\lambda(-A_{g_0}^*) \in \Gamma_{k-1}$. However, $\lambda(-A_{g_0}^*) \in \Gamma_k$ is necessary to show that $u$ is bounded from below.

3.2. **Gradient estimate.** For the convenience, we will denote by

$$G_k(U^t) = \frac{\sigma_k(U^t)}{\sigma_{k-1}(U^t)}, \quad G_l(U^t) = -\frac{\sigma_l(U^t)}{\sigma_{k-1}(U^t)} \quad \text{for } 0 \leq l \leq k - 2,$$

and

$$\beta_l(x, u, t) = [(1-t)c + t\alpha_l(x)] \exp[2(k-l)u] \quad \text{for } 0 \leq l \leq k - 2.$$

We further denote by

$$G^{ij} = \frac{\partial G}{\partial U_{ij}^t}, \quad G^{ij}_k = \frac{\partial G_k}{\partial U_{ij}^t}, \quad G^{ij}_l = \frac{\partial G_l}{\partial U_{ij}^t} \quad \text{for } 0 \leq l \leq k - 2,$$

and

$$G^{ij,rs}_k = \frac{\partial^2 G_k}{\partial U_{ij}^t \partial U_{rs}^t}, \quad G^{ij,rs}_l = \frac{\partial^2 G_l}{\partial U_{ij}^t \partial U_{rs}^t} \quad \text{for } 0 \leq l \leq k - 2.$$

**Lemma 3.4.** Assume $\alpha_l(x) > 0$ for all $0 \leq l \leq k - 2$ and $x \in M$, and $u$ is a smooth solution of (2.2) with $\lambda(U^t) \in \Gamma_{k-1}$, then we have

(3.5) $$0 < \frac{\sigma_l(U^t)}{\sigma_{k-1}(U^t)} \leq C, \quad 0 \leq l \leq k - 2,$$

where the constant $C$ depends only on $n, k, \sup M u$ and $\inf M \alpha_l$ with $0 \leq l \leq k - 2$. 
Proof. Firstly, if \( \frac{\sigma_k}{\sigma_{k-1}} \leq 1 \), then we get from the equation (2.2)
\[
\beta_l \frac{\sigma_l}{\sigma_{l-1}} \leq \frac{\sigma_k}{\sigma_{k-1}} + t\alpha(x) \exp(2u) \leq 1 + C, \quad 0 \leq l \leq k - 2.
\]
Thus,
\[
\frac{\sigma_l}{\sigma_{l-1}} \leq \frac{1 + C}{\inf M \beta_l}, \quad 0 \leq l \leq k - 2.
\]
Secondly, if \( \frac{\sigma_k}{\sigma_{k-1}} > 1 \), i.e. \( \frac{\sigma_{k-1}}{\sigma_k} < 1 \). We can get for \( 0 \leq l \leq k - 2 \) by the Newton-MacLaurin inequality [58, 48]
\[
\frac{\sigma_l}{\sigma_{l-1}} \leq \left( C_n^{k-1-l} C_n^l \right) \left( \frac{\sigma_{k-1}}{\sigma_k} \right)^{k-1-l} \leq \left( C_n^{k-1-l} C_n^l \right) \leq C(n, k).
\]
So, the result follows. \( \square \)

Lemma 3.5. Assume \( u \) is a smooth solution of (2.2) with \( \lambda(U^t) \in \Gamma_{k-1} \) and \( \alpha_l(x) > 0 \) with \( 0 \leq l \leq k - 2 \), then we have

(3.6) \[
G^{ij} U^t_{ij} \geq -t\alpha(x) \exp(2u),
\]

(3.7) \[
G^{ij}(g_0)_{ij} \geq \frac{n - k + 1}{k},
\]

(3.8) \[
G^{ij} U^t_{ijp} + \sum_{i=0}^{k-2} \left[ \beta_l(x, u, t) \right]_p G_l = -\left[ t\alpha(x) \exp(2u) \right]_p,
\]

and

(3.9) \[
G^{ij} U^t_{ijpp} \geq -\sum_{l=0}^{k-2} \left[ \frac{1}{1 + \frac{1}{k-1-l}} \frac{([\beta_l(x, u, t)]_p)^2}{\beta_l} \right] G_l
\]

\[
- \sum_{l=0}^{k-2} \left[ \beta_l(x, u, t) \right]_{pp} G_l - \left[ t\alpha(x) \exp(2u) \right]_{pp}.
\]

Proof. (1) By direct calculation, we have
\[
G^{ij} U^t_{ij} = G_k U^t_{ij} + \sum_{l=0}^{k-2} \beta_l(x, u, t) \sum_{i,j} G^{ij} U^t_{ij}
\]
\[
= G_k + \sum_{l=0}^{k-2} (l - k + 1) \beta_l(x, u, t) G_l
\]
\[
\geq G = -t\alpha(x) \exp(2u).
\]

The first inequality follows consequently.
(2) See page 12 in [27] for the proof of the second inequality. So we just outline the proof by the following simple calculation

\[ G^{ij}(g_0)_{ij} = G^{ij}_k(g_0)_{ij} + \sum_{l=0}^{k-1} \beta_l G^{ij}_l(g_0)_{ij} \geq G^{ij}_k(g_0)_{ij} \geq n - k + 1 \frac{k}{k}, \]

where we get the last inequality from the following inequality

\[ \sum_{i=1}^{n} \frac{\partial (\sigma_k \sigma_{k-1})}{\partial \lambda_i} \geq n - k + 1 \frac{k}{k} \]

for \( \lambda \in \Gamma_{k-1} \) (see Lemma 2.2.19 in [17]).

(3) Differentiating the equation (2.2) arrives

\[ G^{ij} U^{t}_{ijp} + \sum_{l=0}^{k-2} [\beta_l(x, u, t)]_p G_l = -[t \alpha \exp (2u)]_p. \]

So the third equality follows.

(4) Differentiating the equation (2.2) twice gives

\[
\begin{align*}
G^{ij} U^{t}_{ijp} + G^{ij,rs}_{k} U^{t}_{ijp} U^{t}_{rsp} + \sum_{l=0}^{k-2} \beta_l(x, u, t) G^{ij,rs}_l U^{t}_{ijp} U^{t}_{rsp} \\
+ 2 \sum_{l=0}^{k-2} [\beta_l(x, u, t)]_p G^{ij}_l U^{t}_{ijp} + \sum_{l=0}^{k-2} [\beta_l(x, u, t)]_{pp} G_l \\
= -[t \alpha \exp (2u)]_{pp}.
\end{align*}
\]

Then using the concavity of \( G_k \) in \( \Gamma_{k-1} \) (see [34]), we deduce that \( G^{ij,rs}_k U_{ijp} U_{rsp} \leq 0 \). Hence,

\[
\begin{align*}
G^{ij} U^{t}_{ijpp} &\geq - \sum_{l=0}^{k-2} \beta_l(x, u, t) G^{ij,rs}_l U^{t}_{ijp} U^{t}_{rsp} - 2 \sum_{l=0}^{k-2} [\beta_l(x, u, t)]_p G^{ij}_l U^{t}_{ijp} \\
&\quad - \sum_{l=0}^{k-2} [\beta_l(x, u, t)]_{pp} G_l - [t \alpha \exp (2u)]_{pp}.
\end{align*}
\]

Moreover, using the concavity of \( \left[ \frac{\sigma_l}{\sigma_{l-1}} \right]^{\frac{1}{k-1-l}} \) in \( \Gamma_{k-1} \) for \( 0 \leq l \leq k - 2 \) (see also (3.10) in [27] or Chapter XV in [47]), we obtain for \( 0 \leq l \leq k - 2 \)

\[
\begin{align*}
-G^{ij,rs}_l U_{ijp} U_{rsp} &\geq -(1 + \frac{1}{k-1-l}) G_l^{-1} G^{ij}_l G^{rs}_l U_{ijp} U_{rsp}.
\end{align*}
\]
By virtue of (3.11), it yields
\[
\sum_{l=1}^{k-2} \beta_l G_l^{ij,rs} U_{ijp} U_{rsp} + 2 \sum_{l=0}^{k-2} [\beta_l]_p G_l^{ij} U_{ijp}^t \leq \sum_{l=0}^{k-2} \beta_l (1 + \frac{1}{k-1-l}) G_l^{-1} (G_l^{ij} U_{ijp}^t)^2 + 2 \sum_{l=0}^{k-2} [\beta_l]_p G_l^{ij} U_{ijp}^t \leq \sum_{l=0}^{k-2} \frac{1}{1 + \frac{l}{k-1-l}} [\beta_l]_p^2 G_l.
\]

Plugging the inequality above into (3.10), we arrive
\[
G^{ij} U_{ijpp} \geq - \sum_{l=0}^{k-2} \frac{1}{1 + \frac{l}{k-1-l}} [\beta_l]_p G_l - \sum_{l=0}^{k-2} [\beta_l(x,u,t)]_{pp} G_l - [t \alpha \exp(2u)]_{pp}.
\]

So, we complete the proof the last inequality. □

At last, we recall Lemma 4 in [52] or Lemma 4.2 in [29] as follows.

**Lemma 3.6.** Assume that \( s_1 < s < s_2 \). Then we may choose constants \( c_1, c_2, \) and \( p \) depending only on \( s_1 \) and \( s_2 \) so that \( \gamma(s) = c_1(c_2 + s)^p \) satisfies
\[
\gamma'(s) > 0
\]
and
\[
\gamma''(s) - \gamma'(s)^2 > \gamma'(s).
\]

Now, we begin to prove the gradient estimate.

**Lemma 3.7.** Let \( \tau < 1, \alpha_l(x) \) be a positive smooth function on \( M \) for all \( 0 \leq l \leq k-2 \) and \( \alpha(x) \) be a smooth function on \( M \). Assume \( u \) is a solution of (2.2) with \( \lambda(U^t) \in \Gamma_{k-1}. \) Then there exists a constant \( C, \) depending on \( \tau, \) \( g_0, \) \( ||\alpha_l||_{C^2(M)}, ||u||_{C^0(M)}, \inf_M \alpha_l, \) and \( ||\alpha_l||_{C^2(M)} \) with \( 0 \leq l \leq k-2 \) such that
\[
\sup_M |\nabla u| \leq C.
\]

**Proof.** Consider the auxiliary function
\[
Q = (1 + \frac{|\nabla u|^2}{2}) e^{\gamma(u)},
\]
where \( \gamma(u) = c_1(c_2 + u)^p \) is the function in Lemma 3.6. Assume that max \( M \) \( Q \) is attained at a point \( \tilde{x}. \) After an appropriate choice of the normal frame at \( \tilde{x}, \) we may assume that
$U^t_{ij}(x)$ is diagonal at this point. Hence $G^{ij}$ is diagonal at $\tilde{x}$. Differentiating $Q$ at the point $\tilde{x}$ twice, we obtain

\begin{equation}
Q_i(\tilde{x}) = e^{\gamma(u)} \left(1 + \frac{|\nabla u|^2}{2}\right)\gamma' u_i + \sum_l u_l u_{li} = 0,
\end{equation}

and

\begin{equation}
0 \geq Q_{ij}(\tilde{x}) = e^{\gamma(u)} \left(1 + \frac{|\nabla u|^2}{2}\right)(\gamma')^2 u_i u_j + \gamma' u_{ij} + \gamma'' u_i u_j + \sum_l \left(2\gamma' u_l u_{ij} u_i + u_{ij} u_l + u_l u_{ij}\right).
\end{equation}

Since $G^{ij}$ is positive definite by Proposition 2.1 and $Q_{ij}$ is negative definite at $\tilde{x}$, we find

\begin{equation}
0 \geq \sum_i (G^{ii} + \frac{1 - \tau}{n - 2} \sum_p G^{pp} g^{ii}_0) Q_{ii}(\tilde{x})
\end{equation}

\begin{align*}
&\geq (G^{ii} + \frac{1 - \tau}{n - 2} \sum_p G^{pp} g^{ii}_0) \cdot \left(\sum_l u_l u_{il} + (1 + \frac{|\nabla u|^2}{2}) \left(\sum_l (\gamma')^2 + \gamma' u_i + \gamma'' u_{ii}\right) + \sum_l \left(2\gamma' u_l u_{il} u_i + u_{il} u_l\right)\right) \\
&\geq \sum_i (G^{ii} + \frac{1 - \tau}{n - 2} \sum_p G^{pp} g^{ii}_0) \cdot \left(\sum_l u_l u_{il} + (1 + \frac{|\nabla u|^2}{2}) \left(\sum_l (\gamma')^2 + \gamma' u_i + \gamma'' u_{ii}\right) + \sum_l \left(2\gamma' u_l u_{il} u_i + u_{il} u_l\right)\right)
\end{align*}

\begin{equation}
(3.15) \quad -C \sum_i G^{ii} |\nabla u|^2.
\end{equation}

Moreover, recalling the definition of $U^t$, using \(3.6\) and \(3.8\), we obtain at $\tilde{x}$

\begin{align*}
0 \geq \sum_i \sum_l u_l G^{ii} \left(U^t_{ii} - \left[\frac{2 - \tau}{n - 2} |\nabla u|^2 - u_i^2\right]_l\right) \\
+ \gamma' \sum_i G^{ii} \left(U^t_{ii} - \left[\frac{2 - \tau}{n - 2} |\nabla u|^2 - u_i^2\right]\right) \left(1 + \frac{|\nabla u|^2}{2}\right) \\
+ \sum_i G^{ii} \left(1 + \frac{|\nabla u|^2}{2}\right) \left(\gamma' u_i^2 + \gamma' u_{ii}\right) \left(u_i^2 + \frac{(1 - \tau)}{n - 2} |\nabla u|^2\right) \\
+ 2 \sum_i G^{ii} \left(\gamma' \sum_l u_l u_{il} + \frac{(1 - \tau)}{n - 2} \gamma' \sum_{p,l} u_p u_{lp} u_l\right) - C \sum_i G^{ii} \left(1 + |\nabla u|^2\right)
\end{align*}
\[ \geq \sum_i G^{ii} \left( - (2 - \tau) \sum_{p,l} u_l u_p u_{pl} + 2 \sum_l u_l u_{li} \right) \\
+ \gamma^' \left[ \frac{2 - \tau}{2} |\nabla u|^2 + u_i^2 \right] \left[ 1 + \frac{|\nabla u|^2}{2} \right] \\
+ [(\gamma')^2 + \gamma''] \left( 1 + \frac{|\nabla u|^2}{2} \right) \sum_i G^{ii} \left( u_i^2 + \frac{1 - \tau}{n - 2} |\nabla u|^2 \right) \\
+ 2 \sum_i G^{ii} \left( \gamma^{'} \sum_l u_i u_{li} + \frac{1 - \tau}{n - 2} \gamma^{'} \sum_{p,l} u_p u_{lp} u_l \right) \\
\geq -C(|\nabla u|^2 + 1) \sum_i G^{ii} + C(|\nabla u|^2 + 1) \left( \sum_{l=0}^{k-2} G_l - 1 \right). \tag{3.16} \]

From (3.13), we know at \( \tilde{x} \)
\[ \sum_l u_i u_{li} = -\gamma^{'} \left( 1 + \frac{|\nabla u|^2}{2} \right) u_i, \]
which implies at \( \tilde{x} \)
\[ \sum_i G^{ii} \left( - (2 - \tau) \sum_{p,l} u_l u_p u_{pl} + 2 \sum_l u_l u_{li} \right) \\
= \gamma^{'} \left( 1 + \frac{|\nabla u|^2}{2} \right) \sum_i G^{ii} \left( (2 - \tau)|\nabla u|^2 - 2u_i^2 \right), \]
and
\[ \sum_i G^{ii} \left( \gamma^{'} \sum_l u_i u_{li} u_l + \frac{1 - \tau}{n - 2} \gamma^{'} \sum_{p,l} u_p u_{lp} u_l \right) \\
= - (\gamma')^2 \left( 1 + \frac{|\nabla u|^2}{2} \right) \sum_i G^{ii} \left( u_i^2 + \frac{1 - \tau}{n - 2} |\nabla u|^2 \right). \]

Then, plugging the two inequalities above into (3.16), we arrive at \( \tilde{x} \) from Lemma 3.6
\[ 0 \geq \sum_i G^{ii} \left( \gamma^{'} \left[ 1 + \frac{|\nabla u|^2}{2} \right] \left[ \frac{2 - \tau}{2} |\nabla u|^2 - u_i^2 \right] \\
+ \left[ 1 + \frac{|\nabla u|^2}{2} \right] \left( -\gamma'^2 + \gamma'' \right) \left[ u_i^2 + \frac{1 - \tau}{n - 2} |\nabla u|^2 \right] \right) \\
- C(|\nabla u|^2 + 1) \sum_i G^{ii} + C(|\nabla u|^2 + 1) \left( \sum_{l=0}^{k-2} G_l - 1 \right) \]

From (3.13), we know at \( \tilde{x} \)
\[ \sum_i G^{ii} \left( - \gamma^{'} \sum_l u_i u_{li} \right) = \gamma^{'} \left( 1 + \frac{|\nabla u|^2}{2} \right) \sum_i G^{ii} \left( (2 - \tau)|\nabla u|^2 - 2u_i^2 \right), \]
and
\[ \sum_i G^{ii} \left( \gamma^{'} \sum_l u_i u_{li} + \frac{1 - \tau}{n - 2} \gamma^{'} \sum_{p,l} u_p u_{lp} u_l \right) \\
= - (\gamma')^2 \left( 1 + \frac{|\nabla u|^2}{2} \right) \sum_i G^{ii} \left( u_i^2 + \frac{1 - \tau}{n - 2} |\nabla u|^2 \right). \]
\[
\geq \gamma' \left( \frac{1 - \tau}{n - 2} + \frac{2 - \tau}{2} \right) |\nabla u|^2 \left( 1 + \frac{|\nabla u|^2}{2} \right) \sum_i G^{ii}
\]
(3.17)

\[-C(|\nabla u|^2 + 1) \sum_i G^{ii} + C(|\nabla u|^2 + 1) \left( \sum_{l=0}^{k-2} G_l - 1 \right).\]

Since \( \tau < 1 \), we have \( \frac{1 - \tau}{n - 2} + \frac{2 - \tau}{2} > 0 \). Thus, in view of (3.5) and \( \gamma' > 0 \), we know the first term in the right of the inequality (3.17) dominates. Then, absorbing lower order terms results in

\[ C \geq |\nabla u|^2. \]

So, the gradient estimate is immediate. \( \square \)

3.3. \( C^2 \) estimate.

**Lemma 3.8.** Let \( \tau < 1 \), \( \alpha_l(x) \) be a positive smooth function on \( M \) for all \( 0 \leq l \leq k - 2 \) and \( \alpha(x) \) be a smooth function on \( M \). Assume \( u \) is a solution of (2.2) with \( \lambda(U^t) \in \Gamma_{k-1} \). Then there exists a constant \( C \), depending on \( \tau \), \( g_0 \), \( ||\alpha||_{C^2(M)} \), \( ||u||_{C^1(M)} \), \( \inf_M \alpha_l \), and \( ||\alpha||_{C^2(M)} \) with \( 0 \leq l \leq k - 2 \) such that

(3.18) \[ \sup_M |\nabla^2 u| \leq C. \]

**Proof.** Since \( \lambda(U^t) \in \Gamma_2 \), we have

\[ |U_{ij}^t| \leq CtrU^t. \]

Therefore,

(3.19) \[ |u_{ij}| \leq C(\Delta u + 1). \]

So we only estimate \( \Delta u \). Thus, we take the auxiliary function

\[ H(x) = (\Delta u + \mu|\nabla u|^2), \]

where \( \mu \) is a positive constant which will be chosen later. Assume \( x_0 \) is the maximum point of \( H \). After an appropriate choice of the normal frame at \( x_0 \), we further assume \( U^t_{ij} \) and hence \( G^{ij} \) is diagonal at the point \( x_0 \). Then we have at \( x_0 \),

(3.20) \[ H_i(x_0) = \sum_k (u_{kki} + 2\mu u_k u_{ki}) = 0, \]

and

(3.21) \[ H_{ii}(x_0) = \sum_k (u_{kki} + 2\mu u_k u_{kii} + 2\mu u_{ki}^2) \leq 0. \]
From the positivity of $G^{ij}$ and (3.21), we arrive at $x_0$

\[
0 \geq \sum_i \left( G^{ii} + \frac{1 - \tau}{n - 2} \sum_p G^{pp} g^{ij}_0 \right) H_{ii}(x)
\]

\[
\geq \sum_i \left( G^{ii} + \frac{1 - \tau}{n - 2} \sum_p G^{pp} g^{ij}_0 \right) \sum_k \left( u_{kkii} + 2\mu u_k u_{kii} + 2\mu u^2_{ki} \right)
\]

\[
\geq \sum_i \left( G^{ii} + \frac{1 - \tau}{n - 2} \sum_p G^{pp} g^{ij}_0 \right) \sum_k \left( u_{ii} + 2\mu u_k u_{ii} + 2\mu u^2_{ki} - C\Delta u \right),
\]

where we use Ricci identity to get the last inequality. In view of (3.19), we may assume $\Delta u$ is large enough. Thus it follows from the definition of $U^t$ and (3.20) that

\[
0 \geq \sum_i G^{ii} \sum_p \left( U^t_{ip} + (u^2_i)_{pp} - \left[ \frac{2 - \tau}{2} |\nabla u|^2 \right]_{pp} \right)
\]

\[
+ 2\mu u_p \left( U^t_{ip} + (u^2_i)_p - \left[ \frac{2 - \tau}{2} |\nabla u|^2 \right]_p \right) + 2\mu u^2_{pi} + \frac{2\mu(1 - \tau)}{n - 2} \sum_l u^2_{lp}
\]

\[-C \sum_i G^{ii} \Delta u
\]

\[
\geq \sum_i G^{ii} \left( U^t_{ip} + 2u_i u_{ip} - (2 - \tau) \sum_l u_{il} u_{lp} \right) + 2\mu u^2_{pi} + \frac{2\mu(1 - \tau)}{n - 2} \sum_l u^2_{lp}
\]

\[-C \sum_i G^{ii} (1 + \Delta u)
\]

\[
\geq \left( \frac{2\mu(1 - \tau)}{n - 2} - 2 + \tau \right) (\Delta u)^2 - C\Delta u - C \right) \sum_i G^{ii}
\]

\[
+ \sum_{i, p} G^{ii} U^t_{ip} + 2\mu \sum_{i, p} u_p G^{ii} U^t_{ip}.
\]

Then using (3.22) and (3.9), we deduce that

\[
(3.22) \quad 0 \geq \left( \frac{2\mu(1 - \tau)}{n - 2} - (2 - \tau) \right) (\Delta u)^2 - C\Delta u - C \right) \sum_i G^{ii}
\]

\[
(3.23) \quad + C \left( \sum_{l=0}^{k-2} G_l + 1 \right) (\Delta u + 1).
\]

Since $\tau < 1$, we may choose $\mu$ large to dominate the $(2 - \tau)$ term (this is the point where the assumption $\tau < 1$ is crucial). Choosing $\mu > \frac{(2 - \tau)(n - 2)}{2(1 - \tau)}$ and using the inequality (3.5),
we conclude at \(x_0\) from the inequality above
\[
C \geq |\Delta u|^2.
\]
So, we complete the proof. \(\square\)

3.4. Proof of Theorem 1.1

In this section, we use the degree theory for nonlinear elliptic equation developed in [45] to prove Theorem 1.1. After establishing the a priori estimates Lemma 3.2, Lemma 3.7, Lemma 3.8, we know that the equation (2.2) is uniformly elliptic if we notice (3.5) for the case \(l = 0\)
\[
\sigma_{k-1}(U^t) \geq C > 0.
\]
From Evans-Krylov estimates [14, 39] and Schauder estimates, we have
\[
|u|_{C^{4,\delta}(M)} \leq C
\]
for any solution \(u\) to the equation (2.2) with \(\lambda(U^t) \in \Gamma_{k-1}\), where \(0 < \delta < 1\). Recalling the equation (2.2)
\[
G(U^t) := \frac{\sigma_k(U^t)}{\sigma_{k-1}(U^t)} - \frac{\sum_{l=0}^{k-2} \left((1-t)c + t\alpha_l(x)\right) \exp[2(k-l)u]\sigma_l(U^t)}{\sigma_{k-1}(U^t)} = -t\alpha \exp(2u),
\]
where \(t \in [0,1], c = (1, \cdots, 1)\),
\[
c = \frac{\sigma_k(e)}{\sum_{l=0}^{k-2} \sigma_l(e)},
\]
and
\[
U^t = \nabla^2 u + \frac{1-\tau}{n-2} \Delta u g_0 + \frac{2-\tau}{2} |\nabla u|^2 g_0 - du \otimes du - tA^T \tau g_0 + (1-t)g_0.
\]
Then we consider a family of the mappings for \(t \in [0,1]\)
\[
F(.,t) : C^{4,\delta}_0(M) \to C^{2,\delta}(M),
\]
which is defined by
\[
F(u;t) := G(U^t) + t\alpha \exp(2u),
\]
where
\[
C^{4,\delta}_0(M) = \{u \in C^{4,\delta}(M) : \lambda(U^t) \in \Gamma_{k-1}\}
\]
is an open subset of \(C^{4,\delta}(M)\). Let
\[
\mathcal{O}_R = \{u \in C^{4,\delta}_0(M) : |u|_{C^{4,\delta}(M)} < R\},
\]
which clearly is also an open subset of \(C^{4,\delta}(M)\). Moreover, if \(R\) is sufficiently large, \(F(u;t) = 0\) has no solution on \(\partial\mathcal{O}_R\) by (3.24) and the a priori estimate established in
Therefore the degree \( \deg(F(., t), \mathcal{O}_R; 0) \) is well-defined for \( 0 \leq t \leq 1 \). Using the homotopic invariance of the degree, we have

\[
\deg(F(., 1), \mathcal{O}_R, 0) = \deg(F(., 0), \mathcal{O}_R, 0).
\]

When \( t = 0 \), (2.2) becomes

\[
(3.27) \quad \sigma_k(U^0) - c \sum_{l=0}^{k-2} \exp[2(k - l)u] \sigma_l(U^0) = 0
\]

with

\[
U^0 = \nabla^2 u + \frac{1 - \tau}{n - 2} \Delta ug_0 + \frac{2 - \tau}{2} |\nabla u|^2 g_0 - du \otimes du + g_0.
\]

Lemma 3.9. \( u = 0 \) is the unique solution for (3.27).

Proof. Assume \( x \) and \( y \) are the maximum and minimum points of \( u \) respectively. Then we obtain by (3.27),

\[
\sigma_k(e) \leq c \sum_{l=0}^{k-2} \exp[2(k - l)u(x)] \sigma_l(e),
\]

which implies by the definition (3.24) of \( c \)

\[
u(x) \geq 1.
\]

Similarly, we have

\[
\sigma_k(e) \geq c \sum_{l=0}^{k-2} \exp[2(k - l)u(x)] \sigma_l(e),
\]

which implies

\[
u(y) \leq 1.
\]

Thus \( u \equiv 0 \). \( \square \)

Lemma 3.9 shows that \( u = 0 \) is the unique solution to the equation (2.2) for \( t = 0 \). Let \( u(x, s) \) be the variation of \( u = 0 \) such that \( u'_s = \varphi \) at \( s = 0 \). Then

\[
\delta_\varphi F(0; 0) = a_{ij} \varphi_{ij} + 1st \ derivatives \ in \ \varphi - c \sum_{l=0}^{k-2} 2(k - l) \sigma_l(e) \sigma_{k-1}(e) \varphi,
\]

where \( a_{ij} \) is a positive definite matrix and \( \delta F(0; 0) \) is the linearized operator of \( F \) at \( u = 0 \). Clearly, \( \delta F(0; 0) \) is an invertible operator. Therefore,

\[
\deg(F(., 1), \mathcal{O}_R; 0) = \deg(F(., 0), \mathcal{O}_R, 0) = \pm 1.
\]

So, we obtain a solution at \( t = 1 \). This completes the proof of Theorem 1.1.
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