Optimizing Sparsity over Lattices and Semigroups

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Abstract. Motivated by problems in optimization we study the sparsity of the solutions to systems of linear Diophantine equations and linear integer programs, i.e., the number of non-zero entries of a solution, which is often referred to as the $\ell_0$-norm. Our main results are improved bounds on the $\ell_0$-norm of sparse solutions to systems $Ax = b$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $x$ is either a general integer vector (lattice case) or a non-negative integer vector (semigroup case). In the lattice case and certain scenarios of the semigroup case, we give polynomial time algorithms for computing solutions with $\ell_0$-norm satisfying the obtained bounds.

1 Introduction

This paper discusses the problem of finding sparse solutions to systems of linear Diophantine equations and integer linear programs. We investigate the $\ell_0$-norm $\|x\|_0 := |\{i : x_i \neq 0\}|$, a function widely used in the theory of compressed sensing \cite{6,9}, which measures the sparsity of a given vector $x = (x_1, \ldots, x_n) \top \in \mathbb{R}^n$ (it is clear that the $\ell_0$-norm is actually not a norm).

Sparsity is a topic of interest in several areas of optimization. The $\ell_0$-norm minimization problem over reals is central in the theory of the classical compressed sensing, where a linear programming relaxation provides a guaranteed approximation \cite{8,9}. Support minimization for solutions to Diophantine equations is relevant for the theory of compressed sensing for discrete-valued signals \cite{11,12,17}. There is still little understanding of discrete signals in the compressed sensing paradigm, despite the fact that there are many applications in which the signal is known to have discrete-valued entries, for instance, in wireless communication \cite{22} and the theory of error-correcting codes \cite{7}. Sparsity was also investigated in integer optimization \cite{1,10,20}, where many combinatorial optimization problems have useful interpretations as sparse semigroup problems. For example, the edge-coloring problem can be seen as a problem in the semigroup generated by matchings of the graph \cite{18}. Our results provide natural out-of-the-box sparsity bounds for problems with linear constraints and integer variables in a general form.
1.1 Lattices: sparse solutions of linear Diophantine systems

Each integer matrix $A \in \mathbb{Z}^{m \times n}$ determines the lattice $\mathcal{L}(A) := \{Ax : x \in \mathbb{Z}^n\}$ generated by the columns of $A$. By an easy reduction via row transformations, we may assume without loss of generality that the rank of $A$ is $m$.

Let $[n] := \{1, \ldots, n\}$ and let $\binom{[n]}{m}$ be the set of all $m$-element subsets of $[n]$. For $\gamma \subseteq [n]$, consider the $m \times |\gamma|$ submatrix $A_\gamma$ of $A$ with columns indexed by $\gamma$. One can easily prove that the determinant of $\mathcal{L}(A)$ is equal to

$$\gcd(A) := \gcd \left\{ \det(A_\gamma) : \gamma \in \binom{[n]}{m} \right\}.$$

Since $\mathcal{L}(A_\gamma)$ is the lattice spanned by the columns of $A$ indexed by $\gamma$, it is a sublattice of $\mathcal{L}(A)$. We first deal with a natural question: Can the description of a given lattice $\mathcal{L}(A)$ in terms of $A$ be made sparser by passing from $A$ to $A_\gamma$, with $\gamma$ having a smaller cardinality than $n$ and satisfying $\mathcal{L}(A) = \mathcal{L}(A_\gamma)$? That is, we want to discard some of the columns of $A$ and generate $\mathcal{L}(A)$ by $|\gamma|$ columns with $|\gamma|$ being possibly small.

For stating our results, we need several number-theoretic functions. Given $z \in \mathbb{Z}_{>0}$, consider the prime factorization $z = p_1^{s_1} \cdots p_k^{s_k}$ with pairwise distinct prime factors $p_1, \ldots, p_k$ and their multiplicities $s_1, \ldots, s_k \in \mathbb{Z}_{>0}$. Then the number of prime factors $\sum_{i=1}^k s_i$ counting the multiplicities is denoted by $\Omega(z)$. Furthermore, we introduce $\Omega_m(z) := \sum_{i=1}^k \min\{s_i, m\}$. That is, by introducing $m$ we set a threshold to account for multiplicities. In the case $m = 1$ we thus have $\omega(z) := \Omega_1(z) = k$, which is the number of prime factors in $z$, not taking the multiplicities into account. The functions $\Omega$ and $\omega$ are called prime $\Omega$-function and prime $\omega$-function, respectively, in number theory [15]. We call $\Omega_m$ the truncated prime $\Omega$-function.

**Theorem 1** Let $A \in \mathbb{Z}^{m \times n}$, with $m \leq n$, and let $\tau \in \binom{[n]}{m}$ be such that the matrix $A_\tau$ is non-singular. Then the equality $\mathcal{L}(A) = \mathcal{L}(A_\tau)$ holds for some $\gamma$ satisfying $\tau \subseteq \gamma \subseteq [n]$ and

$$|\gamma| \leq m + \Omega_m (|\det(A_\tau)|/\gcd(A)). \quad (1)$$

Given $A$ and $\tau$, the set $\gamma$ can be computed in polynomial time.

One can easily see that $\omega(z) \leq \Omega_m(z) \leq \Omega(z) \leq \log_2(z)$ for every $z \in \mathbb{Z}_{>0}$. The estimate using $\log_2(z)$ gives a first impression on the quality of the bound (1). It turns out, however, that $\Omega_m(z)$ is much smaller on the average. Results in number theory [15, §22.10] show that the average values $\frac{1}{z}(\omega(1) + \cdots + \omega(z))$ and $\frac{1}{z}(\Omega(1) + \cdots + \Omega(z))$ are of order $\log \log z$, as $z \to \infty$.

As an immediate consequence of Theorem 1 we obtain

**Corollary 2** Consider the linear Diophantine system

$$Ax = b, \quad x \in \mathbb{Z}^n \quad (2)$$
with \( A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m \) and \( m \leq n \). Let \( \tau \in \binom{[n]}{m} \) be such that the \( m \times m \) matrix \( A_\tau \) is non-singular. If (2) is feasible, then (2) has a solution \( x \) satisfying the sparsity bound
\[
\|x\|_0 \leq m + \Omega_m \left( \frac{\left| \det(A_\tau) \right|}{\gcd(A)} \right).
\]
Under the above assumptions, for given \( A, b \) and \( \tau \), such a sparse solution can be computed in polynomial time.

From the optimization perspective, Corollary 2 deals with the problem
\[
\min \{ \|x\|_0 : Ax = b, \ x \in \mathbb{Z}^n \}
\]
of minimization of the \( \ell_0 \)-norm over the affine lattice \( \{ x \in \mathbb{Z}^n : Ax = b \} \).

### 1.2 Semigroups: sparse solutions in integer programming

Consider next the standard form of the feasibility constraints of integer linear programming
\[
Ax = b, \ x \in \mathbb{Z}^n_{\geq 0}.
\]

For a given matrix \( A \), the set of all \( b \) such that (3) is feasible, is the *semigroup* \( Sg(A) = \{ Ax : x \in \mathbb{Z}^n_{\geq 0} \} \) generated by the columns of \( A \).

If (3) has a solution, i.e., \( b \in Sg(A) \), how sparse can such a solution be? In other words, we are interested in the \( \ell_0 \)-norm minimization problem
\[
\min \{ \|x\|_0 : Ax = b, \ x \in \mathbb{Z}^n_{\geq 0} \}.
\]
It is clear that Problem (4) is NP-hard, because deciding the feasibility of (3) \cite[§ 18.2]{25} or even solving the relaxation of (4) with the condition \( x \in \mathbb{Z}^n_{\geq 0} \) replaced by \( x \in \mathbb{R}^n \) \cite{23} is NP-hard.

Taking the NP-hardness of Problem (4) into account, our aim is to *estimate* the optimal value of (4) under the assumption that this problem is feasible. In \cite[Theorem 1.1 (i)]{2}, \cite[Theorem 1]{1} (see also \cite[Theorem 1]{1}), it was shown that for any \( b \in Sg(A) \), there exists a \( x \in \mathbb{Z}^n \), such that \( Ax = b \) and
\[
\|x\|_0 \leq m + \left\lfloor \log_2 \left( \frac{\sqrt{\det(AA^T)}}{\gcd(A)} \right) \right\rfloor.
\]

In \cite[Theorem 2]{1}, it was shown that Equation (5) cannot be improved significantly, but nevertheless we show here how to improve it in some special cases. As a consequence of Theorem 1 we obtain the following.

**Corollary 3** Let \( A \in \mathbb{Z}^{m \times n} \) be a matrix whose columns positively span \( \mathbb{R}^m \) and let \( b \in \mathbb{Z}^m \). Then \( \mathcal{L}(A) = Sg(A) \). Furthermore, if \( b \in \mathcal{L}(A) \), and \( \tau \in \binom{[n]}{m} \) is a set, for which the matrix \( A_\tau \) is non-singular, then there is a solution \( x \) of
the integer-programming feasibility problem $A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}_m^{n \geq 0}$ that satisfies the sparsity bound
\[
\|\mathbf{x}\|_0 \leq 2m + \Omega_m\left(\frac{|\det(A_{\tau})|}{\gcd(A)}\right). \tag{6}
\]
Under the above assumptions, for given $A$, $\mathbf{b}$ and $\tau$, such a sparse solution $\mathbf{x}$ can be computed in polynomial time.

Note that for a fixed $m$, (6) is usually much tighter than (5), because the function $\Omega_m(z)$ is bounded from above by the logarithmic function $\log_2(z)$ and is much smaller than $\log_2(z)$ on the average. Furthermore, $|\det(A_{\tau})| \leq \sqrt{\det(AA^T)}$ in view of the Cauchy-Binet formula.

We take a closer look at the case $m = 1$ of a single equation and tighten the given bounds in this case. That is, we consider the knapsack feasibility problem
\[
\mathbf{a}^\top \mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in \mathbb{Z}_n^{n \geq 0}, \tag{7}
\]
where $\mathbf{a} \in \mathbb{Z}^n$ and $\mathbf{b} \in \mathbb{Z}$. Without loss of generality we can assume that all components of the vector $\mathbf{a}$ are not equal to zero. It follows from (5) that a feasible problem (7) has a solution $\mathbf{x}$ with
\[
\|\mathbf{x}\|_0 \leq 1 + \left\lfloor \log\left(\frac{\|\mathbf{a}\|_2}{\gcd(\mathbf{a})}\right) \right\rfloor. \tag{8}
\]
If all components of $\mathbf{a}$ have the same sign, without loss of generality we can assume $\mathbf{a} \in \mathbb{Z}_0^n$. In this setting, Theorem 1.2 in [2] strengthens the bound (8) by replacing the $\ell_2$-norm of the vector $\mathbf{a}$ with the $\ell_\infty$-norm. It was conjectured in [2, page 247] that a bound $\|\mathbf{x}\|_0 \leq c + \log_2\left(\frac{\|\mathbf{a}\|_\infty}{\gcd(\mathbf{a})}\right)$ with an absolute constant $c$ holds for an arbitrary $\mathbf{a} \in \mathbb{Z}^n$. We obtain the following result, which covers the case that has not been settled so far and yields a confirmation of this conjecture.

**Corollary 4** Let $\mathbf{a} = (a_1, \ldots, a_n)^\top \in (\mathbb{Z} \setminus \{0\})^n$ be a vector that contains both positive and negative components. If the knapsack feasibility problem $\mathbf{a}^\top \mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in \mathbb{Z}_0^n$ has a solution, then there is a solution $\mathbf{x}$ satisfying the sparsity bound
\[
\|\mathbf{x}\|_0 \leq 2 + \min \left\{ \omega\left(\frac{|a_i|}{\gcd(\mathbf{a})}\right) : i \in [n] \right\}. \tag{9}
\]
Under the above assumptions, for given $\mathbf{a}$ and $\mathbf{b}$, such a sparse solution $\mathbf{x}$ can be computed in polynomial time.

Our next contribution is that, given additional structure on $A$, we can improve on [2, Theorem 1.1 (i)], which in turn also gives an improvement on [2, Theorem 1.2]. For $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$, we denote by $\text{cone}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ the convex conic hull of the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Now assume the matrix $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}_m^{m \times n}$ with columns $\mathbf{a}_i$ satisfies the following conditions:

\begin{align*}
\mathbf{a}_1, \ldots, \mathbf{a}_n &\in \mathbb{Z}^m \setminus \{\mathbf{0}\}, \quad \tag{9} \\
\text{cone}(\mathbf{a}_1, \ldots, \mathbf{a}_n) &\text{ is an } m\text{-dimensional pointed cone,} \quad \tag{10} \\
\text{cone}(\mathbf{a}_1) &\text{ is an extreme ray of } \text{cone}(\mathbf{a}_1, \ldots, \mathbf{a}_n). \quad \tag{11}
\end{align*}
Note that the previously best sparsity bound for the general case of the integer-programming feasibility problem is (5). Using the Cauchy-Binet formula, (5) can be written as
\[\|x\|_0 \leq m + \log_2 \frac{\sqrt{\sum_{I \in \binom{[n]}{m}}} \det(A_I)^2}{\gcd(A)}\].

The following theorem improves this bound in the “pointed cone case” by removing a fraction of $m/n$ of terms in the sum under the square root.

**Theorem 5** Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^{m \times n}$ satisfy (9)–(11) and, for $b \in \mathbb{Z}^m$, consider the integer-programming feasibility problem
\[Ax = b, \ x \in \mathbb{Z}^n_{\geq 0}.\] (12)

If (12) is feasible, then there is a feasible solution $x$ satisfying the sparsity bound
\[\|x\|_0 \leq m + \left\lfloor \log_2 \frac{q(A)}{\gcd(A)} \right\rfloor,
\]
where
\[q(A) := \sqrt{\sum_{I \in \binom{[n]}{m}} \det(A_I)^2} \cdot \gcd(A)^{-1} \min_{1 \in I} \{a_1, \ldots, a_n\}^{-1}.\]

We omit the proof of this result due to the page limit for the IPCO proceedings. Instead we focus on the particularly interesting case $m = 1$. In this case, assumption (10) is equivalent to $a \in \mathbb{Z}^n_{>0} \cup \mathbb{Z}^n_{<0}$. Without loss of generality, one can assume $a \in \mathbb{Z}^n_{>0}$.

**Theorem 6** Let $a = (a_1, \ldots, a_n)^\top \in \mathbb{Z}^n_{>0}$ and $b \in \mathbb{Z}_{\geq 0}$. If the knapsack feasibility problem $a^\top x = b$, $x \in \mathbb{Z}^n_{\geq 0}$ has a solution, there is a solution $x$ satisfying the sparsity bound
\[\|x\|_0 \leq 1 + \left\lfloor \log_2 \frac{\min_{1 \in I} \{a_1, \ldots, a_n\}}{\gcd(a)} \right\rfloor.
\]

When dealing with bounds for sparsity it would be interesting to understand the worst case scenario among all members of the semigroup, which is described by the function
\[\text{ICR}(A) = \max_{b \in \mathcal{S}_{\mathbb{Z}^n}(A)} \min\{\|x\|_0 : Ax = b, \ x \in \mathbb{Z}^n_{\geq 0}\}.\] (13)

We call ICR$(A)$ the integer Carathéodory rank in resemblance to the classical problem of finding the integer Carathéodory number for Hilbert bases [24]. Above results for the problem $Ax = b$, $x \in \mathbb{Z}^n_{\geq 0}$ can be phrased as upper bounds on ICR$(A)$. We are interested in the complexity of computing ICR$(A)$. The first question is: can the integer Carathéodory rank of a matrix $A$ be computed at all? After all, remember that the semigroup has infinitely many elements.
and, despite the fact that ICR($A$) is a finite number, a direct usage of (13) would result into the determination of the sparsest representation $Ax = b$ for all of the infinitely many elements $b$ of $Sg(A)$. It turns out that ICR($A$) is computable, as the inequality ICR($A$) ≤ $k$ can be expressed as the formula $\forall x \in \mathbb{Z}^n_\geq 0 \exists y \in \mathbb{Z}^n_\geq 0 : (Ax = Ay) \wedge (\|y\|_0 \leq k)$ in Presburger arithmetic [14]. Beyond this fact, the complexity status of computing ICR($A$) is largely open, even when $A$ is just one row:

**Problem 7** Given the input $a = (a_1, \ldots, a_n)^\top \in \mathbb{Z}^n$, is it NP-hard to compute ICR($a^\top$)?

The Frobenius number $\max \mathbb{Z}^\geq_0 \setminus Sg(a^\top)$, defined under the assumptions $a \in \mathbb{Z}^\geq_0$ and $\gcd(a) = 1$, is yet another value associated to $Sg(a^\top)$. The Frobenius number can be computed in polynomial time when $n$ is fixed [5,16] but is NP-hard to compute when $n$ is not fixed [21]. It seems that there might be a connection between computing the Frobenius number and ICR($a^\top$).

2 Proofs of Theorem 1 and its consequences

The proof of Theorem 1 relies on the theory of finite Abelian groups. We write Abelian groups additively. An Abelian group $G$ is said to be a direct sum of its finitely many subgroups $G_1, \ldots, G_m$, which is written as $G = \bigoplus_{i=1}^m G_i$, if every element $x \in G$ has a unique representation as $x = x_1 + \cdots + x_m$ with $x_i \in G_i$ for each $i \in [m]$. A primary cyclic group is a non-zero finite cyclic group whose order is a power of a prime number. We use $G/H$ to denote the quotient of $G$ modulo its subgroup $H$.

The fundamental theorem of finite Abelian groups states that every finite Abelian group $G$ has a primary decomposition, which is essentially unique. This means, $G$ is decomposable into a direct sum of its primary cyclic groups and that this decomposition is unique up to automorphisms of $G$. We denote by $\kappa(G)$ the number of direct summands in the primary decomposition of $G$.

For a subset $S$ of a finite Abelian group $G$, we denote by $\langle S \rangle$ the subgroup of $G$ generated by $S$. We call a subset $S$ of $G$ non-redundant if the subgroups $\langle T \rangle$ generated by proper subsets $T$ of $S$ are properly contained in $\langle S \rangle$. In other words, $S$ is non-redundant if $\langle S \setminus \{x\} \rangle$ is a proper subgroup of $\langle S \rangle$ for every $x \in S$. The following result can be found in [13, Lemma A.6].

**Theorem 8** Let $G$ be a finite Abelian group. Then the maximum cardinality of a non-redundant subset $S$ of $G$ is equal to $\kappa(G)$.

We will also need the following lemmas, proved in the Appendix.

**Lemma 1.** Let $G$ be a finite Abelian group representable as a direct sum $G = \bigoplus_{j=1}^m G_j$ of $m \in \mathbb{Z}^\geq_0$ cyclic groups. Then $\kappa(G) \leq \Omega_m(|G|)$.

**Lemma 2.** Let $\Lambda$ be a sublattice of $\mathbb{Z}^m$ of rank $m \in \mathbb{Z}^\geq_0$. Then $G = \mathbb{Z}^m/\Lambda$ is a finite Abelian group of order $\det(\Lambda)$ that can be represented as a direct sum of at most $m$ cyclic groups.
Proof (Theorem 1). Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the columns of $A$. Without loss of generality, let $\tau = [m]$. We use the notation $B := A_\tau$.

Reduction to the case $\gcd(A) = 1$. For a non-singular square matrix $M$, the columns of $M^{-1}A$ are representations of the columns of $A$ in the basis of columns of $M$. In particular, for a matrix $M$ whose columns form a basis of $\mathcal{L}(A)$, the matrix $M^{-1}A$ is integral and the $m \times m$ minors of $M^{-1}A$ are the respective $m \times m$ minors of $A$ divided by $\det(M) = \gcd(A)$. Thus, replacing $A$ by $M^{-1}A$, we pass from $\mathcal{L}(A)$ to $\mathcal{L}(M^{-1}A) = \{M^{-1}z : z \in \mathcal{L}(A)\}$, which corresponds to a change of a coordinate system in $\mathbb{R}^m$ and ensures that $\gcd(A) = 1$.

Sparsity bound (1). The matrix $B$ gives rise to the lattice $A := \mathcal{L}(B)$ of rank $m$, while $A$ determines the finite Abelian group $\mathbb{Z}^m/\Lambda$.

Consider the canonical homomorphism $\phi : \mathbb{Z}^m \to \mathbb{Z}^m/\Lambda$, sending an element of $\mathbb{Z}^m$ to its coset modulo $\Lambda$. Since $\gcd(A) = 1$, we have $\mathcal{L}(A) = \mathbb{Z}^m$, which implies $\langle T \rangle = \mathbb{Z}^m/\Lambda$ for $T := \{\phi(\mathbf{a}_{m+1}), \ldots, \phi(\mathbf{a}_n)\}$. For every non-redundant subset $S$ of $T$, we have

$$
|S| \leq \kappa(\mathbb{Z}^m/\Lambda) \leq \Omega_m(\det(A_\tau))
$$

by Theorem 8 (by Lemmas 1 and 2).

Fixing a set $I \subseteq \{m+1, \ldots, n\}$ that satisfies $|I| = |S|$ and $S = \{\phi(\mathbf{a}_i) : i \in I\}$, we reformulate $\langle S \rangle = \mathbb{Z}^m/\Lambda$ as $\mathbb{Z}^m = \mathcal{L}(A_I) + \Lambda = \mathcal{L}(A_I) + \mathcal{L}(A_\tau) = \mathcal{L}(A_{I \cup \tau})$.

Thus, (1) holds for $\gamma = I \cup \tau$.

Construction of $\gamma$ in polynomial time. The matrix $M$ used in the reduction to the case $\gcd(A) = 1$ can be constructed in polynomial time: one can obtain $M$ from the Hermite Normal Form of $A$ (with respect to the column transformations) by discarding zero columns. For the determination of $\gamma$, the set $I$ that defines the non-redundant subset $S = \{\phi(\mathbf{a}_i) : i \in I\}$ of $\mathbb{Z}^m/\Lambda$ needs to be determined. Start with $I = \{m+1, \ldots, n\}$ and iteratively check if some of the elements $\phi(\mathbf{a}_i) \in \mathbb{Z}^m/\Lambda$, where $i \in I$, is in the group generated by the remaining elements. Suppose $j \in I$ and we want to check if $\phi(\mathbf{a}_j)$ is in the group generated by all $\phi(\mathbf{a}_i)$ with $i \in I \setminus \{j\}$. Since $A = \mathcal{L}(A_\tau)$, this is equivalent to checking $a_j \in \mathcal{L}(A_{I \setminus \{j\} \cup \tau})$ and is thus reduced to solving a system of linear Diophantine equations with the left-hand side matrix $A_{I \setminus \{j\} \cup \tau}$ and the right-hand side vector $a_j$. Thus, carrying the above procedure for every $j \in I$ and removing $j$ from $I$ whenever $a_j \in \mathcal{L}(A_{I \setminus \{j\} \cup \tau})$, we eventually arrive at a set $I$ that determines a non-redundant subset $S$ of $\mathbb{Z}^m/\Lambda$. This is done by solving at most $n - m$ linear Diophantine systems in total, where the matrix of each system is a sub-matrix of $A$ and the right-hand vector of the system is a column of $A$.

Remark 1 (Optimality of the bounds). For a given $\Delta \in \mathbb{Z}_{\geq 2}$ let us consider matrices $A \in \mathbb{Z}^{m \times n}$ with $\Delta = \lvert \det(A_\tau) \rvert / \gcd(A)$. We construct a matrix $\mathbf{A}$ that shows the optimality of the bound (1). As in the proof of Theorem 1, we assume $\tau = [m]$ and use the notation $B := A_\tau$. Consider the prime factorization $\Delta = p_1^{\nu_1} \cdots p_s^{\nu_s}$. We will fix the matrix $B$ to be a diagonal matrix with diagonal entries $d_1, \ldots, d_m \in \mathbb{Z}_{\geq 0}$ so that $\det(B) = d_1 \cdots d_m = \Delta$.

The diagonal entries are defined by distributing the prime factors of $\Delta$ among the diagonal entries of $B$. If the multiplicity $n_i$ of the prime $p_i$ is less than $m$,
we introduce \( p_i \) as a factor of multiplicity 1 in \( n_i \) of the \( m \) diagonal entries of \( B \). If the multiplicity \( n_i \) is at least \( m \), we are able distribute the factors \( p_i \) among all of the diagonal entries of \( B \) so that each diagonal entry contains the factor \( p_i \) with multiplicity at least 1.

The group \( \mathbb{Z}^m / \Lambda = \mathbb{Z}^m / \mathcal{L}(B) \) is a direct sum of \( m \) cyclic groups \( G_1, \ldots, G_m \) of orders \( d_1, \ldots, d_m \), respectively. By the Chinese Remainder Theorem, these cyclic groups can be further decomposed into the direct sum of primary cyclic groups. By our construction, the prime factor \( p_i \) of the multiplicity \( n_i < m \) generates a cyclic direct summand of order \( p_i \) in \( n_i \) of the subgroups \( G_1, \ldots, G_m \). If \( n_i \geq m \), then each of the groups \( G_1, \ldots, G_m \) has a direct summand, which is a non-trivial cyclic group whose order is a power of \( p_i \). Summarizing, we see that the decomposition of \( \mathbb{Z}^m / \Lambda \) into primary cyclic groups contains \( n_i \) summands of order \( p_i \), when \( n_i < m \), and \( m \) summands, whose order is a power of \( p_i \), when \( n_i \geq m \). The total number of summands is thus \( \sum_{i=1}^n \min\{m, n_i\} = \Omega_m(\Delta) \).

Now, fix \( n = m + \Omega_m(\Delta) \) and choose columns \( a_{m+1}, \ldots, a_n \) so that \( \phi(a_{m+1}), \ldots, \phi(a_n) \) generate all direct summands in the decomposition of \( \mathbb{Z}^m / \Lambda \) into primary cyclic groups. With this choice, \( \phi(a_{m+1}), \ldots, \phi(a_n) \) generate \( \mathbb{Z}^m / \Lambda \), which means that \( \mathcal{L}(A) = \mathbb{Z}^m \) and implies \( \gcd(A) = 1 \). On the other hand, any proper subset \( \{\phi(a_{m+1}), \ldots, \phi(a_n)\} \) generates a proper subgroup of \( \mathbb{Z}^m / \Lambda \), as some of the direct summands in the decomposition of \( \mathbb{Z}^m / \Lambda \) into primary cyclic groups will be missing. This means \( \mathcal{L}(A_{|m\cup I}) \subseteq \mathbb{Z}^m \) for every \( I \subseteq \{m+1, \ldots, n\} \).

Proof (Corollary 2). Feasibility of (2) can be expressed as \( b \in \mathcal{L}(A) \). Choose \( \gamma \) from the assertion of Theorem 1. One has \( b \in \mathcal{L}(A) = \mathcal{L}(A_\gamma) \) and so there exists a solution \( x \) of (2) whose support is a subset of \( \gamma \). This sparse solution \( x \) can be computed by solving the Diophantine system with the left-hand side matrix \( A_\gamma \) and the right-hand side vector \( b \).

Proof (Corollary 3). Assume that the Diophantine system \( Ax = b, x \in \mathbb{Z}^n \) has a solution. It suffices to show that, in this case, the integer-programming feasibility problem \( Ax = b, x \in \mathbb{Z}_{\geq 0}^n \) has a solution, too, and that one can find a solution of the desired sparsity to the integer-programming feasibility problem in polynomial time.

One can determine \( \gamma \) as in Theorem 1 in polynomial time. Using \( \gamma \), we can determine a solution \( x^* = (x_1^*, \ldots, x_n^*)^\top \in \mathbb{Z}^n \) of the Diophantine system \( Ax = b, x \in \mathbb{Z}^n \) satisfying \( x_i^* = 0 \) for \( i \in [n] \setminus \gamma \) in polynomial time, as described in the proof of Corollary 2.

Let \( a_1, \ldots, a_m \) be the columns of \( A \). Since the matrix \( A_\tau \) is non-singular, the \( m \) vectors \( a_i \), where \( i \in \tau \), together with the vector \( v = -\sum_{i \in \tau} a_i \) positively span \( \mathbb{R}^n \). Since all columns of \( A \) positive span \( \mathbb{R}^n \), the conic version of the Carathéodory theorem implies the existence of a set \( \beta \subseteq [m] \) with \( |\beta| \leq m \), such that \( v \) is in the conic hull of \( \{a_i : i \in \beta\} \). Consequently, the set \( \{a_i : i \in \beta \cup \tau\} \) and by this also the larger set \( \{a_i : i \in \beta \cup \gamma\} \) positively span \( \mathbb{R}^m \). Let \( I = \beta \cup \gamma \). By construction, \( |I| \leq |\beta| + |\gamma| \leq m + |\gamma| \).

Since the vectors \( a_i \), with \( i \in I \) positively span \( \mathbb{R}^m \), there exist a choice of rational coefficients \( \lambda_i > 0 \) (\( i \in I \)) with \( \sum_{i \in I} \lambda_i a_i = 0 \). After rescaling we
can assume \( \lambda_i \in \mathbb{Z}_{>0} \). Define \( \mathbf{x}' = (x'_1, \ldots, x'_n)^\top \in \mathbb{Z}_{\geq 0}^n \) by setting \( x'_i = \lambda_i \) for \( i \in I \) and \( x'_i = 0 \) otherwise. The vector \( \mathbf{x}' \) is a solution of \( A\mathbf{x} = \mathbf{0} \). Choosing \( N \in \mathbb{Z}_{>0} \) large enough, we can ensure that the vector \( \mathbf{x}^* + N\mathbf{x}' \) has non-negative components. Hence, \( \mathbf{x} = \mathbf{x}^* + N\mathbf{x}' \) is a solution of the system \( A\mathbf{x} = \mathbf{b} \), \( \mathbf{x} \in \mathbb{Z}_{\geq 0}^n \) satisfying the desired sparsity estimate. The coefficients \( \lambda_i \) and the number \( N \) can be computed in polynomial time.

**Proof (Corollary 4).** The assertion follows by applying Corollary 3 for \( m = 1 \) and all \( \tau = \{i\} \) with \( i \in [n] \).

### 3 Proof of Theorem 6

**Lemma 3.** Let \( a_1, \ldots, a_t \in \mathbb{Z}_{>0} \), where \( t \in \mathbb{Z}_{>0} \). If \( t > 1 + \log_2(a_1) \), then the system

\[
\begin{align*}
y_1a_1 + \cdots + y_ta_t &= 0, \\
y_1 &\in \mathbb{Z}_{\geq 0}, \quad y_2, \ldots, y_t \in \{-1, 0, 1\}.
\end{align*}
\]

in the unknowns \( y_1, \ldots, y_t \) has a solution that is not identically equal to zero.

**Proof.** The proof is inspired by the approach in [3, § 3.1] (used in a different context) that suggests to reformulate the underlying equation over integers as two strict inequalities and then use Minkowski’s first theorem [4, Ch. VII, Sect. 3] from the geometry of numbers. Consider the convex set \( Y \subseteq \mathbb{R}^t \) defined by \( 2t \) strict linear inequalities

\[
\begin{align*}
-1 &< y_1a_1 + \cdots + y_ta_t < 1, \\
-2 &< y_i < 2 \text{ for all } i \in \{2, \ldots, t\}.
\end{align*}
\]

Clearly, the set \( Y \) is the interior of a hyper-parallelepiped and can also be described as \( Y = \{y \in \mathbb{R}^t : \|My\|_\infty < 1\} \), where \( M \) is the upper triangular matrix

\[
M = \begin{pmatrix}
a_1 & a_2 & \cdots & a_t \\
1/2 & & \cdots & \\
& 1/2 & & \cdots
\end{pmatrix}.
\]

It is easy to see that the \( t \)-dimensional volume \( \text{vol}(Y) \) of \( Y \) is

\[
\text{vol}(Y) = \text{vol}(M^{-1}[-1, 1]^t) = \frac{1}{\det(M)} 2^t = \frac{4^t}{2a_1}.
\]

The assumption \( t > 1 + \log_2(a_1) \) implies that the volume of \( Y \) is strictly larger than \( 2^t \). Thus, by Minkowski’s first theorem, the set \( Y \) contains a non-zero integer vector \( y = (y_1, \ldots, y_t)^\top \in \mathbb{Z}^t \). Without loss of generality we can assume that \( y_1 \geq 0 \) (if the latter is not true, one can replace \( y \) by \(-y\)). The vector \( y \) is a desired solution from the assertion of the lemma. \( \Box \)
Proof (Theorem 6). Without loss of generality we can assume that \( \gcd(a) = 1 \).
In fact, if \( b \) is divisible by \( \gcd(a) \) we can convert \( \mathbf{a}^\top \mathbf{x} = b \) to \( \tilde{\mathbf{a}}^\top \mathbf{x} = \tilde{b} \) with \( \tilde{\mathbf{a}} = \frac{a}{\gcd(a)} \) \( \mathbf{x} = \frac{b}{\gcd(a)} \), and, if \( b \) is not divisible by \( \gcd(a) \), the knapsack feasibility problem \( \mathbf{a}^\top \mathbf{x} = b \), \( \mathbf{x} \in \mathbb{Z}_\geq 0 \) has no solution.

Without loss of generality, let \( \tilde{a}_1 = \min\{a_1, \ldots, a_n\} \). We need to show the existence of solution of the knapsack feasibility problem satisfying \( \|\mathbf{x}\|_0 \leq 1 + \log_2(\tilde{a}_1) \).

Choose a solution \( \mathbf{x} = (x_1, \ldots, x_n)^\top \) of the knapsack feasibility problem with the property that the number of indices \( i \in \{2, \ldots, n\} \) for which \( x_i \neq 0 \) is minimized. Without loss of generality we can assume that, for some \( t \in \{2, \ldots, n\} \) one has \( x_2 > 0, \ldots, x_t > 0, x_{t+1} = \cdots = x_n = 0 \). Lemma 3 implies \( t \leq 1 + \log_2(\tilde{a}_1) \). In fact, if the latter was not true, then a solution \( \mathbf{y} \in \mathbb{R}^t \) of the system in Lemma 3 could be extended to a solution \( \mathbf{y} \in \mathbb{R}^n \) by appending zero components. It is clear that some of the components \( y_2, \ldots, y_t \) are negative, because \( a_2 > 0, \ldots, a_t > 0 \). It then turns out that, for an appropriate choice of \( k \in \mathbb{Z}_\geq 0 \), the vector \( \mathbf{x}' = (x'_1, \ldots, x'_n)^\top = \mathbf{x} + ky \) is a solution of the same knapsack feasibility problem satisfying \( x'_1 \geq 0, \ldots, x'_t \geq 0, x'_{t+1} = \cdots = x'_n = 0 \) and \( x'_i = 0 \) for at least one \( i \in \{2, \ldots, t\} \). Indeed, one can choose \( k \) to be the minimum among all \( a_i \) with \( i \in \{2, \ldots, t\} \) and \( y_i = -1 \).

The existence of \( \mathbf{x}' \) with at most \( t - 1 \) non-zero components \( x'_i \) with \( i \in \{2, \ldots, n\} \) contradicts the choice of \( \mathbf{x} \) and yields the assertion. \( \Box \)

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4 Appendix

Proof (Lemma 1). Consider the prime factorization \( |G| = p_1^{n_1} \cdots p_s^{n_s} \). Then \( |G_j| = p_1^{n_{i,j}} \cdots p_s^{n_{i,j}} \) with \( 0 \leq n_{i,j} \leq n_i \) and, by the Chinese Remainder Theorem, the cyclic group \( G_j \) can be represented as \( G_j = \bigoplus_{i=1}^s G_{i,j} \), where \( G_{i,j} \) is a cyclic group of order \( p_i^{n_{i,j}} \). Consequently, \( G = \bigoplus_{i=1}^s \bigoplus_{j=1}^h G_{i,j} \). This is a decomposition of \( G \) into a direct sum of primary cyclic groups and, possibly, some trivial summands \( G_{i,j} \) equal to \{0\}. We can count the non-trivial direct summands whose order is a power of \( p_i \), for a given \( i \in [s] \). There is at most one summand like this for each of the groups \( G_j \). So, there are at most \( m \) non-trivial summands in the decomposition whose order is a power of \( p_i \). On the other hand, the direct sum of all non-trivial summands whose order is a power of \( p_i \) is a group of order \( p_i^{n_{i,1} + \cdots + n_{i,s}} = p_i^{n_i} \) so that the total number of such summands is not larger than \( n_i \), as every summand contributes the factor at least \( p_i \) to the power \( p_i^{n_i} \). This shows that the total number of non-zero summands in the decomposition of \( G \) is at most \( \sum_{i=1}^s \min\{m, n_i\} = \Omega_m(|G|) \). \( \Box \)

Proof (Lemma 2). The proof relies on the relationship of finite Abelian groups and lattices, see [23, §4.4]. Fix a matrix \( \mathbf{M} \in \mathbb{Z}^{m \times m} \) whose columns form a basis
of $A$. Then $|\det(M)| = \det(A)$. There exist unimodular matrices $U \in \mathbb{Z}^{m \times m}$ and $V \in \mathbb{Z}^{m \times m}$ such that $D := U M V$ is diagonal matrix with positive integer diagonal entries. For example, one can choose $D$ to be the Smith Normal Form of $M$ [23, §4.4]. Let $d_1, \ldots, d_m \in \mathbb{Z}_{>0}$ be the diagonal entries of $D$. Since $U$ and $V$ are unimodular, $d_1 \cdots d_m = \det(D) = \det(A)$.

We introduce the quotient group $G' := \mathbb{Z}^m / \Lambda' = (\mathbb{Z}/d_1 \mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_m \mathbb{Z})$ with respect to the lattice $\Lambda' := \mathcal{L}(D) = (d_1 \mathbb{Z}) \times \cdots \times (d_m \mathbb{Z})$. The order of $G'$ is $d_1 \cdot \cdots \cdot d_m = \det(D) = \det(A)$ and $G'$ is a direct sum of at most $m$ cyclic groups, as every $d_i > 1$ determines a non-trivial direct summand.

To conclude the proof, it suffices to show that $G'$ is isomorphic to $G$. To see this, note that $\Lambda' = \mathcal{L}(D) = \mathcal{L}(UMV) = \mathcal{L}(UM) = \{Uz : z \in \Lambda\}$. Thus, the map $z \mapsto Uz$ is an automorphism of $\mathbb{Z}^m$ and an isomorphism from $\Lambda$ to $\Lambda'$. Thus, $z \mapsto Uz$ induces an isomorphism from the group $G = \mathbb{Z}^m / \Lambda$ to the group $G' = \mathbb{Z}^m / \Lambda'$.

\[ \square \]

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