Test compatible metrics and 2-branes

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We propose a sufficient condition for a general spherical symmetric static metric to be compatible with classical tests of gravity. A 1-parametric class of such metrics are constructed. The Schwarzschild metric as well as the Yilmaz-Rosen metric are in this class. By computing the scalar curvature we show that the non-Schwarzschild metrics can be interpreted as close 2-branes. All the manifolds endowed the described metrics contain in a class of pseudo-Riemannian manifolds with scalar curvature of a fixed sign.

The question in turn is: What is similar in a long-distance analytical behavior of these two metrics and which other metrics have the same behavior? We work backwards. We do not specify the field equations, but simply postulate a jet-space of solutions which looks promising.

Consider the general spherical symmetric static metric in spatial conformal (isotropic) coordinates

\[ ds^2 = e^{f(r)} dt^2 - e^{g(r)} (dx^2 + dy^2 + dz^2). \]

We are interested in asymptotically flat metrics, thus require the functions \( f \) and \( g \) to have Taylor expansions of the form

\[ f(r) = \frac{a_1}{r} + \frac{a_2}{r^2} + O\left(\frac{1}{r^3}\right), \]
\[ g(r) = \frac{b_1}{r} + \frac{b_2}{r^2} + O\left(\frac{1}{r^3}\right). \]

The zeroth order terms can be vanishing by rescaling the coordinates. Write the Schwarzschild line element, in the same isotropic coordinates

\[ ds^2 = \left(1 - \frac{m}{r}\right)^2 dt^2 - \left(1 + \frac{m}{2r}\right)^4 (dx^2 + dy^2 + dz^2). \]

The Taylor expansions of the functions \( f \) and \( g \) for this metric are

\[ f(r) = 2 \ln \left(1 - \frac{m}{r}\right) = -\frac{2m}{r} + O\left(\frac{1}{r^2}\right), \]
\[ g(r) = 4 \ln \left(1 + \frac{m}{2r}\right) = \frac{2m}{r} - \frac{m^2}{2r^2} + O\left(\frac{1}{r^3}\right), \]

and the coefficients are

\[ a_1 = -2m, \quad b_1 = 2m, \quad a_2 = 0, \quad b_2 = -\frac{m^2}{2}, \quad \ldots \]

The corresponding coefficients for the Yilmaz-Rosen metric \( \mathbb{B} \) are

\[ a_1 = -2m, \quad b_1 = 2m, \quad a_i = b_i = 0 \quad \text{for} \quad i > 1. \]
Comparing (8) and (10) we obtain a sufficient condition for a general static spherical-symmetric asymptotic flat metric in isotropic coordinates (9) to be compatible with the macroscopic classical tests:

\[ a_1 + b_1 = 0, \quad a_2 = 0. \] (9)

In order to describe by the metric (8) the field of a point with arbitrary mass, the actual value of the coefficient \( a_1 \) should, also, be arbitrary. Thus the conditions (8) define a 2-jet space of functions \( f(r) \) and a 1-jet space of functions \( g(r) \). These spaces correspond to the \( \frac{3}{2} \)-order approximation of GR.

As an example of functions containing in these jet-spaces consider a family of metrics

\[ ds^2 = \left(1 + \frac{m}{kr}\right)^k dt^2 - \left(1 + \frac{m}{kr}\right)^{2k} (dr^2 + dy^2 + dz^2), \] (10)

where \( k \) is a dimensionless parameter. It is easy to see that the metric (10) satisfies the conditions (8) for an arbitrary choice of the parameter \( k \).

In order to have analytically correct functions in (10) we require the parameter \( k \) to be integer. Note that this restriction is taken only for simplification. We can also consider the parameter \( k \) as an arbitrary real number, but in this case we have made an analytical redefinition of the metric on the small distances \( r \leq \frac{m}{k} \).

For \( k = 2 \) one obtain, certainly, Schwarzschild metric while in the limits \( k \rightarrow \pm \infty \) (10) approaches Yilmaz-Rosen metric.

In order to obtain the the metric (10) in Schwarzschild coordinates we have to use the new radial coordinate \( r = r(\rho) \), which is implicit defined by the equation

\[ \left(1 + \frac{m}{kr}\right)^k r = \rho. \] (11)

For \( k < 0 \) the metric (10) is singular at a distance

\[ r = \frac{m}{k} \implies \rho = 0 \] (12)

i.e. in the origin of the Schwarzschild coordinates.

As for \( k > 0 \) the metric (8) is singular on a sphere

\[ r = \frac{m}{k} \implies \rho = \frac{2^k}{k}. \] (13)

Note that the physical radius of singular sphere \( \rho \) increases very fast with growth of the parameter \( k \). In order to clarify the nature of the singularities compute the scalar curvature of the metric (10)

\[ R = \frac{2 - k}{k} \left(\frac{m^2}{r^4} \cdot \frac{1 + \frac{m}{r}}{1 - \frac{m}{r}}\right)^2 \left(1 + \frac{m}{kr}\right)^{-2k} (1 + \frac{m}{kr})^{-2k}. \] (14)

Note that the expression in the brackets are positive thus the sign of the scalar curvature depends only on the value of the parameter \( k \). Thus all the manifolds endowed the metric (10) contain in a class of pseudo-Riemannian manifolds with a scalar curvature of a fixed sign.

The scalar curvature is zero only in the case of Schwarzschild metric - \( k = 2 \).

Consider the different regions for the values of the parameter \( k \):

1) \( k > 2 \)

The scalar curvature negative in every final point on the manifold. Near the singular value of coordinates \( r = \frac{m}{k} \) the scalar curvature is singular \( R \rightarrow -\infty \). Thus this coordinate singularity is physical. The scalar curvature inside of the spherical envelope decreases very fast. This singularity can be interpreted as a rigid sphere - close 2-brane.

![FIG. 1. The scalar curvature (14) plotted as a function \( R/m^2 \) of a radial distance \( r/m \) for \( k = 3 \).](image)

For \( k \rightarrow \infty \) we obtain the expression for the scalar curvature of Yilmaz-Rosen metric (4)

\[ R = -2 \frac{m^2}{r^4} e^{-2\frac{m}{r}}. \] (15)

2) \( k = 1 \)

The scalar curvature is positive. We have the metric in isotropic coordinates

\[ ds^2 = \left(1 + \frac{m}{r}\right)^2 dt^2 - \left(1 + \frac{m}{r}\right)^2 (dr^2 + dy^2 + dz^2). \] (16)

From the relation (11) we obtain the transform to the Schwarzschild radial coordinate

\[ \rho = r + m. \]

The metric in these coordinates takes the form

\[ ds^2 = \left(1 - 2 \frac{m}{\rho}\right) dt^2 - \left(1 - \frac{m}{\rho}\right) d\rho^2 - \rho^2 d\Omega^2. \] (17)

The scalar curvature of the metric (16) is
$$R = \frac{m^2}{r^4} \cdot \frac{1 + (1 - \frac{m}{r})^2}{(1 - \frac{m}{r})^2 (1 + \frac{m}{r})^4}. \quad (18)$$

This expression is positive in every point of the manifold. It is singular for $r = m$ or equivalently for $\rho = 2m$. Thus the coordinate singularity for $\rho = 2m$ is physical.

FIG. 2. The scalar curvature $R$ plotted as a function $R/m^2$ of $r/m$ for $k = 1$.

This singularity can be also interpreted as a rigid sphere - close 2-brane.

3) $k \leq -1$

The scalar curvature is negative. Consider for instance $k = -1$. The metric takes the form

$$ds^2 = \left(\frac{1 - \frac{m}{r}}{1 + \frac{m}{r}}\right) dt^2 - \left(1 - \frac{m}{r}\right)^{-2} (dx^2 + dy^2 + dz^2). \quad (19)$$

The metric is singular at the Schwarzschild radius $r = m$.

The scalar curvature takes a form

$$R = -3 \frac{m^2}{r^4} \cdot \frac{1 + (1 + \frac{m}{r})^2}{(1 + \frac{m}{r})^2}. \quad (20)$$

This value is regular for every $r$ (except of the origin).

FIG. 3. The scalar curvature (20) plotted as a function $R/m^2$ of $r/m$ for $k = 1$.

In order to describe by the metric (19) a black hole one should locate it’s surface at a distance $r = m$ where $g_{00} = 0$. In fact this surface cannot be reached by any material object. The proper radial distance from the surface $r = m$ to a point $r_0 > m$ is

$$l = \int_m^{r_0} \frac{dr}{(1 - \frac{m}{r})^2} \to \infty$$

The proper time for a radial null geodesic is also infinite

$$T = \int_m^{r_0} \frac{(1 + \frac{m}{r})^2}{(1 - \frac{m}{r})^2} \frac{dr}{r} \to \infty.$$ 

Thus the surface of a star cannot never reach the Schwarzschild radius and a realistic physical system can be modeled by the metric (19) only for a distance $r > m$. The behavior of the metric is similar to the spherical-symmetric solution in the gravity model of Lee and Lightman. [8].

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