RANDOM VECTORS IN THE ISOTROPIC POSITION

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ABSTRACT. Let $y$ be a random vector in $\mathbb{R}^n$, satisfying

$$E y \otimes y = id.$$ 

Let $M$ be a natural number and let $y_1, \ldots, y_M$ be independent copies of $y$. We prove that for some absolute constant $C$

$$E \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left( E \|y\|^{\log M} \right)^{1/\log M},$$

provided that the last expression is smaller than 1.

We apply this estimate to obtain a new proof of a result of Bourgain concerning the number of random points needed to bring a convex body into a nearly isotropic position.

1. Introduction

The problem we consider has arisen from a question studied by R. Kannan, L. Lovász and M. Simonovits [K-L-S]. To construct a fast algorithm for calculating the volume of a convex body, they needed to bring it into some symmetric position. More precisely, let $K$ be a convex body in $\mathbb{R}^n$. We shall say that it is in the isotropic position if for any $x \in \mathbb{R}^n$

$$\left( 1 - \varepsilon \right) \cdot \|x\|^2 \leq \frac{1}{\text{vol} (K)} \int_K \langle x, y \rangle^2 \, dy \leq \left( 1 + \varepsilon \right) \cdot \|x\|^2.$$

By $\|\cdot\|$ we denote the standard Euclidean norm.

The notion of isotropic position was extensively studied by V. Milman and A. Pajor [M-P]. Note that our definition is consistent with [K-L-S]. The normalization in [M-P] is slightly different.

If the information about the body $K$ is uncomplete it is impossible to bring it exactly to the isotropic position. So, the definition of the isotropic position has to be modified to allow a small error. We shall say that the body $K$ is in $\varepsilon$-isotropic position if for any $x \in \mathbb{R}^n$

$$(1 - \varepsilon) \cdot \|x\|^2 \leq \frac{1}{\text{vol} (K)} \int_K \langle x, y \rangle^2 \, dy \leq (1 + \varepsilon) \cdot \|x\|^2.$$
Let $\varepsilon > 0$ be given. Consider $M$ random points $y_1, \ldots, y_M$ independently uniformly distributed in $K$ and put

$$T = \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i.$$ 

If $M$ is sufficiently large, than with high probability

$$\left\| T - \frac{1}{\text{vol}(K)} \int_K y \otimes y \right\|$$

will be small, so the body $T^{-1/2}K$ will be in $\varepsilon$-isotropic position. R. Kannan, L. Lovász and M. Simonovits ([K-L-S]) proved that it is enough to take

$$M = c \frac{n^2}{\varepsilon}$$

for some absolute constant $c$. This estimate was significantly improved by J. Bourgain [B]. Using rather delicate geometric considerations he has shown that one can take

$$M = C(\varepsilon)n \log^3 n.$$ 

Since the situation is invariant under a linear transformation, we may assume that the body $K$ is in the isotropic position. Then the result of Bourgain may be reformulated as follows:

**Theorem 0.** [B] Let $K$ be a convex body in $\mathbb{R}^n$ in the isotropic position. Fix $\varepsilon > 0$ and choose independently $M$ random points $x_1, \ldots, x_M \in K$, 

$$M \geq C(\varepsilon)n \log^3 n.$$ 

Then with probability at least $1 - \varepsilon$ for any $x \in \mathbb{R}^n$ one has

$$(1 - \varepsilon) \|x\|^2 \leq \frac{1}{M} \sum_{i=1}^{M} \langle x, y \rangle^2 \leq (1 + \varepsilon) \|x\|^2.$$ 

We shall show that this theorem follows from a general result about random vectors in $\mathbb{R}^n$. Let $y$ be a random vector. Denote by $\mathbb{E} X$ the expectation of a random variable $X$. We say that $y$ is in the isotropic position if

$$\mathbb{E} y \otimes y = \text{id}.$$ 

If $y$ is uniformly distributed in a convex body $K$, then this is equivalent to the fact that $K$ is in the isotropic position.

We prove the following

**Theorem 1.** Let $y \in \mathbb{R}^n$ be a random vector in the isotropic position. Let $M$ be a natural number and let $y_1, \ldots, y_M$ be independent copies of $y$. Then

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - \text{id} \right\| \leq C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left( \mathbb{E} \|y\|^\log M \right)^{1/\log M},$$

for some absolute constant $C$.
provided that the last expression is smaller than 1.

Here and later $C, c, \ldots$ denote absolute constants whose values may vary from line to line.

Remark. Taking the trace of (1.1) we obtain that $\mathbb{E} \|y\|^2 = n$, so to make the right hand side of (1.2) smaller than 1, we have to assume that $M \geq cn \log n$.

Using Theorem 1 we prove a better estimate of the length of approximate John’s decompositions [R1] and thus improve the results about approximating a convex body by another one having a small number of contact points, obtained in [R2]. Estimating the moment of the norm of random vector in a convex body, we obtain a different proof of Theorem 0 which gives also a better estimate.

2. Main results.

The proof of Theorem 1 consists of two steps. First we introduce a Bernoulli random process and estimate the expectation of the norm in (1.2) by the expectation of its supremum. Then we construct a majorizing measure to obtain a bound for the latest.

The first step is relatively standard. Let $\varepsilon_1, \ldots, \varepsilon_M$ be independent Bernoulli variables taking values 1, $-1$ with probability $1/2$ and let $y_1, \ldots, y_M, \bar{y}_1, \ldots, \bar{y}_M$ be independent copies of $y$. Denote $E_y, E_{\varepsilon}$ the expectation according to $y$ and $\varepsilon$ respectively. Since $y_i \otimes y_i - \bar{y}_i \otimes \bar{y}_i$ is a symmetric random variable, we have

$$
\mathbb{E}_y \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq \mathbb{E}_y \mathbb{E}_{\bar{y}} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - \frac{1}{M} \sum_{i=1}^{M} \bar{y}_i \otimes \bar{y}_i \right\| = \mathbb{E}_y \mathbb{E}_{\varepsilon} \mathbb{E}_{\bar{y}} \left\| \frac{1}{M} \sum_{i=1}^{M} \varepsilon_i (y_i \otimes y_i - \bar{y}_i \otimes \bar{y}_i) \right\| \leq 2 \mathbb{E}_y \mathbb{E}_{\varepsilon} \left\| \frac{1}{M} \sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i \right\|.
$$

To estimate the last expectation, we need the following Lemma, which generalizes Lemma 1 [R3].

Lemma. Let $y_1, \ldots, y_M$ be vectors in $\mathbb{R}^n$ and let $\varepsilon_1, \ldots, \varepsilon_M$ be independent Bernoulli variables taking values 1, $-1$ with probability $1/2$. Then

$$
\mathbb{E} \left\| \sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i \right\| \leq C \sqrt{\log M} \cdot \max_{i=1, \ldots, M} \|y_i\| \cdot \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\|^{1/2}.
$$

We postpone the proof of the Lemma to the next section.

Applying the Lemma, we get

$$
\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq C \cdot \sqrt{\log M} \cdot \left( \max_{i=1, \ldots, M} \|y_i\|^2 \right)^{1/2} \cdot \left( \mathbb{E} \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\| \right)^{1/2}. \tag{2.1}
$$
We have
\[
\left( \mathbb{E} \max_{i=1,\ldots,M} \| y_i \|^2 \right)^{1/2} \leq \left( \mathbb{E} \left( \sum_{i=1}^{M} \| y_i \|^2 \log M \right)^{2/\log M} \right)^{1/2} \leq \sqrt{M} M^{1/\log M} \cdot \left( \mathbb{E} \| y \|^2 \log M \right)^{1/\log M}.
\]

Thus, denoting
\[
D = \mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\|,
\]
we obtain by (2.1)
\[
D \leq C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left( \mathbb{E} \| y \|^2 \log M \right)^{1/\log M} \cdot (D + 1)^{1/2}.
\]

If
\[
C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left( \mathbb{E} \| y \|^2 \log M \right)^{1/\log M} \leq 1,
\]
we get
\[
D \leq 2C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left( \mathbb{E} \| y \|^2 \log M \right)^{1/\log M},
\]
which completes the proof of Theorem 1.

We turn now to the applications of Theorem 1. Applying Theorem 1 to the question of Kannan, Lovász and Simonovits, we obtain the following

**Corollary 2.1.** Let \( \varepsilon > 0 \) and let \( K \) be an \( n \)-dimensional convex body in the isotropic position. Let
\[
M \geq C \cdot \frac{n}{\varepsilon^2} \cdot \log^2 \frac{n}{\varepsilon^2}
\]
and let \( y_1, \ldots, y_M \) be independent random vectors uniformly distributed in \( K \). Then
\[
\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq \varepsilon.
\]

**Proof.** It follows from a result of S. Alesker [A], that
\[
\mathbb{E} \exp \left( \frac{\| y \|^2}{c \cdot n} \right) \leq 2
\]
for some absolute constant \( c \). Then
\[
\mathbb{E} \| y \|^2 \log M \leq \left( \mathbb{E} e^{\| y \|^2/c \cdot n} \right)^{1/2} \cdot \left( \mathbb{E} \left( \| y \|^2 \log M \cdot e^{-\| y \|^2/c \cdot n} \right) \right)^{1/2} \leq \sqrt{2} \cdot \left( \max_{t \geq 0} t^{\log M} \cdot e^{-t/c \cdot n} \right)^{1/2} \leq (C \cdot n \cdot \log M)^{\log M / 2}.
\]

Corollary 2.1 follows from this estimate and Theorem 1. \( \square \)

By a Lemma of Borell [M-S, Appendix III], most of the volume of a convex body in the isotropic position is concentrated within the Euclidean ball of radius \( c \sqrt{n} \). So, it might be of interest to consider a random vector uniformly distributed in the intersection of a convex body and such a ball. In this case the previous estimate may be improved as follows.
Corollary 2.2. Let $\varepsilon, R > 0$ and let $K$ be an $n$-dimensional convex body in the isotropic position. Suppose that $R \geq c \sqrt{\log 1/\varepsilon}$ and let

$$M \geq C_0 \cdot \frac{R^2 \cdot n}{\varepsilon^2} \cdot \log \frac{R^2 \cdot n}{\varepsilon^2}$$

and let $y_1, \ldots, y_M$ be independent random vectors uniformly distributed in $K \cap R\sqrt{n} \cdot B_n^2$. Then

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - \text{id} \right\| \leq \varepsilon.$$

Proof. Denote $a = R \cdot \sqrt{n}$ and let $z$ be a random vector uniformly distributed in $K \cap aB_n^2$. Then for $x \in B_n^2$

$$\mathbb{E} \langle z, x \rangle^2 = \frac{\text{vol}(K)}{\text{vol}(K \cap aB_n^2)} \cdot \left( \frac{1}{\text{vol}(K)} \int_{K} \langle y, x \rangle^2 \, dy - \frac{1}{\text{vol}(K)} \int_{K} \langle y, x \rangle^2 \cdot 1_{\{u : \|u\| \geq a\}}(y) \, dy \right).$$

By a result of S. Alesker [A] and Khinchine type inequality [M-P], we have

$$\frac{\text{vol}(K)}{\text{vol}(K \cap aB_n^2)} \leq 1 + e^{-ca^2/n} \leq 1 + \frac{\varepsilon}{4}$$

and

$$\frac{1}{\text{vol}(K)} \int_{K} \langle y, x \rangle^2 \cdot 1_{\{u : \|u\| \geq a\}}(y) \, dy \leq \left( \frac{1}{\text{vol}(K)} \int_{K} \langle y, x \rangle^4 \, dy \right)^{1/2} \cdot \left( \frac{1}{\text{vol}(K)} \int_{K} 1_{\{u : \|u\| \geq a\}}(y) \, dy \right)^{1/2} \leq Ce^{-ca^2/2n} \leq \frac{\varepsilon}{4}.$$

Thus for any $x \in B_n^2$

$$|\mathbb{E} \langle z, x \rangle^2 - 1| \leq \frac{\varepsilon}{2}.$$

Define a random vector

$$y = (\mathbb{E} z \otimes z)^{-1/2} z.$$

Then $y$ is in the isotropic position and

$$\left( \mathbb{E} \|y\|^{\log M} \right)^{1/\log M} \leq \left( (\mathbb{E} z \otimes z)^{-1/2} \right) \cdot \left( \mathbb{E} \|z\|^{\log M} \right)^{1/\log M} \leq 2a,$$

so

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - \text{id} \right\| \leq C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot 2a \leq \frac{\varepsilon}{2}$$

provided the constant $C_0$ in (2.2) is large enough. Thus,

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} z_i \otimes z_i - \text{id} \right\| \leq \mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - \text{id} \right\| \cdot \|\mathbb{E} z \otimes z\| + \|\mathbb{E} z \otimes z - \text{id}\| \leq \varepsilon.$$
The next application is connected to the approximation of a convex body by another one having a small number of contact points [R2]. Let $K$ be a convex body in $\mathbb{R}^n$ such that the ellipsoid of minimal volume containing it is the standard Euclidean ball $B_2^n$. Then by the theorem of John, there exist $N \leq (n + 3)n/2$ points $z_1, \ldots, z_N \in K$, $\|x_i\| = 1$ and $N$ positive numbers $c_1, \ldots, c_N$ satisfying the following system of equations

\begin{align}
(2.3) \quad id &= \sum_{i=1}^{N} c_i \, z_i \otimes z_i \\
(2.4) \quad 0 &= \sum_{i=1}^{N} c_i \, z_i.
\end{align}

It was shown in [R1] for convex symmetric bodies and in [R2] in the general case, that the identity operator can be approximated by a sum of a smaller number of terms $x_i \otimes x_i$. We derive from Theorem 1 the following corollary, which improves Lemma 3.1 [R2].

**Corollary 2.3.** Let $\varepsilon > 0$ and let $K$ be a convex body in $\mathbb{R}^n$, so that the ellipsoid of minimal volume containing it is $B_2^n$. Then there exist

\begin{equation}
M \leq \frac{C}{\varepsilon^2} \cdot n \cdot \log \frac{n}{\varepsilon}
\end{equation}

contact points $x_1, \ldots, x_M$ and a vector $u$, $\|u\| \leq \frac{C}{\sqrt{M}}$, so that the identity operator in $\mathbb{R}^n$ has the following representation

\begin{equation}
id = \frac{n}{M} \sum_{i=1}^{M} (x_i + u) \otimes (x_i + u) + S,
\end{equation}

where

\begin{equation}
\sum_{i=1}^{M} (x_i + u) = 0
\end{equation}

and

\[ \|S : \ell_2^n \to \ell_2^n\| < \varepsilon. \]

**Proof.** Let (2.3) be a decomposition of the identity operator. Let $y$ be a random vector in $\mathbb{R}^n$, taking values $\sqrt{n}z_i$ with probability $c_i/\sqrt{n}$. Then, by (2.3), $y$ is in the isotropic position. Obviously, for all $1 \leq p < \infty$

\[ \|x\|_p \leq \left( \mathbb{E} \|x\|^p \right)^{1/p} \leq \sqrt{2n}. \]
So, taking $M$ as in (2.5), we obtain that for sufficiently large $C$

\[(2.7) \quad \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \leq \frac{\varepsilon}{2} \]

with probability greater than $3/4$. Since by (2.4), $\mathbb{E} y = 0$ and $\| y \| = \sqrt{n}$, we have

\[(2.8) \quad \left\| \sum_{i=1}^{M} y_i \right\| \leq 2\sqrt{M} \]

with probability greater than $3/4$. Take $y_1, \ldots, y_M$ for which (2.7) and (2.8) hold and put

$$ x_i = \frac{1}{\sqrt{n}} \cdot y_i, \quad u = -\frac{1}{M} \sum_{i=1}^{M} x_i. $$

Then (2.6) is satisfied and

$$ \left\| \frac{n}{M} \sum_{i=1}^{M} (x_i + u) \otimes (x_i + u) - id \right\| \leq \frac{\varepsilon}{2} + \frac{4n}{M} \leq \varepsilon. $$

Substituting Lemma 3.1 [R2] by Corollary 2.3 in the proof of Theorem 1.1 [R2] we obtain the following

**Corollary 2.3.** Let $B$ be a convex body in $\mathbb{R}^n$ and let $\varepsilon > 0$. There exists a convex body $K \subset \mathbb{R}^n$, so that $d(K, B) \leq 1 + \varepsilon$ and the number of contact points of $K$ with the ellipsoid of minimal volume containing it is less than

$$ M(n, \varepsilon) = \frac{C}{\varepsilon^2} \cdot n \cdot \log \frac{n}{\varepsilon}. $$

### 3. Proof of the Lemma

The proof of the Lemma is similar to that of Lemma 1 [R3]. For the reader’s convenience we present here a complete proof.

Without loss of generality, we may assume that

$$ \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\| = 1. $$

Define a random process

$$ V_\varepsilon = \sum_{i=1}^{M} \varepsilon_i \langle x, y \rangle^2 $$
for $x \in B_2^n$. We have to estimate

\begin{equation}
\mathbb{E} \sup_{x \in B_2^n} V_x.
\end{equation}

Note that the process $V_x$ has a subgaussian tail estimate

$$
P\{|V_x - V_{\bar{x}}| > a\} \leq \exp\left(-C \frac{a^2}{d^2(x, \bar{x})}\right),$$

where $C$ is an absolute constant and

$$
\tilde{d}(x, \bar{x}) = \left(\sum_{i=1}^M \left(\langle x, y_i \rangle^2 - \langle \bar{x}, y_i \rangle^2\right)^2\right)^{1/2}.
$$

The function $\tilde{d}$ is not a metric on $B_2^n$, since $\tilde{d}(x, \bar{x}) = 0$ does not imply $x = \bar{x}$. To avoid this obstacle, we shall estimate $\tilde{d}$ by a quasimetric $d$ defined by

$$
d(x, \bar{x}) = \left(\sum_{i=1}^M \langle x - \bar{x}, y_i \rangle^2 \left(\langle x, y_i \rangle^2 + \langle \bar{x}, y_i \rangle^2\right)\right)^{1/2}.
$$

Then for all $x, \bar{x} \in B_2^n$

$$
\tilde{d}(x, \bar{x}) \leq \sqrt{2} \cdot d(x, \bar{x}),
$$

so we may treat $V_x$ as a subgaussian process with the quasimetric $d$. It can be easily shown that $d$ satisfies a generalized triangle inequality

\begin{equation}
d(x, \bar{x}) \leq 4 \cdot (d(x, z) + d(z, \bar{x}))
\end{equation}

for all $x, \bar{x}, z \in B_2^n$.

Denote by $B_\rho(x)$ a ball in the quasimetric $d$ with center $x$ and radius $\rho$. Then for any $x \in B_2^n$ and $\rho > 0$ we have

$$
\text{conv } B_\rho(x) \subset B_{4\rho}(x).
$$

The proof of this fact is the same as that of Lemma 3 [R3], so we shall omit it. To estimate the expression (3.1), we apply the following version of the Majorizing measure theorem.

**Theorem.** Let $(T, d)$ be a quasimetric space. Let $(X_t)_{t \in T}$ be a collection of mean 0 random variables with the subgaussian tail estimate

$$
P\{|X_t - X_{\bar{t}}| > a\} \leq \exp\left(-C \frac{a^2}{d^2(t, \bar{t})}\right),$$

for all $a > 0$. Let $r > 1$ and let $k_0$ be a natural number so that the diameter of $T$ is less than $r^{-k_0}$. Let $\{\varphi_k\}_{k=k_0}^\infty$ be a sequence of functions from $T$ to $\mathbb{R}^+$, uniformly bounded by a constant depending only on $r$. Assume that there exists $\sigma > 0$ so that for any $k$ the functions $\varphi_k$ satisfy the following condition:
for any \( s \in T \) and for any points \( t_1, \ldots, t_N \in B_{r-k}(s) \) with mutual distances at least \( r^{-k-1} \) one has

\[
\max_{j=1, \ldots, N} \varphi_{k+2}(t_j) \geq \varphi_k(s) + \sigma \cdot r^{-k} \cdot \sqrt{\log N}.
\]

Then

\[
\mathbb{E} \sup_{t \in T} X_t \leq C(r) \cdot \sigma^{-1}.
\]

This Theorem is a combination of the majorizing measure theorem of Fernique [L-T] and the general majorizing measure construction of Talagrand (Theorems 2.1 and 2.2 [T1] or Theorems 4.2, 4.3 and Proposition 4.4 [T2]).

Let

\[
Q = \max_{i=1, \ldots, M} \| y_i \|.
\]

Let \( r \) be a natural number and let \( k_0 \) and \( k_1 \) be the largest numbers so that

\[
r^{-k_0} \geq Q, \quad r^{-k_1} \geq \frac{Q}{\sqrt{n}}.
\]

Then \( k_1 - k_0 \leq (2 \log r)^{-1} \log n \). Define now the functions \( \varphi_k : B_2^n \to \mathbb{R} \) by

\[
\varphi_k(x) = \min \{ \| u \|^2 \mid u \in \text{conv} B_{8r-k}(x) \} + \frac{k - k_0}{\log M}, \quad \text{if } k = k_0, \ldots, k_1,
\]

\[
\varphi_k(w) = 1 + \frac{1}{2 \log r} + \sum_{l=k_1}^{k} r^{-l} \cdot \frac{\sqrt{n \cdot \log(1 + 4Q r^l)}}{Q \sqrt{\log M}}, \quad \text{if } k > k_1.
\]

The functions \( \varphi_k \) form a nonnegative nondecreasing sequence bounded by a constant depending only on \( r \). Indeed, for \( k \leq k_1 \),

\[
\varphi_k(x) \leq 1 + \frac{1}{2 \log r} \cdot \frac{\log n}{\log M}.
\]

For \( k > k_1 \) we have

\[
\varphi_k(w) \leq 1 + \frac{1}{2 \log r} + \sum_{l=k_1}^{\infty} r^{-l} \cdot \frac{\sqrt{n \cdot \log(1 + 4Q r^l)}}{Q \sqrt{\log M}} \leq
\]

\[
1 + \frac{1}{2 \log r} + c(r) \cdot r^{-k_1} \cdot \sqrt{n} \cdot \frac{\sqrt{\log(1 + 4Q r^{k_1})}}{Q \sqrt{\log M}} \leq C(r).
\]

Now let \( x_1, \ldots, x_N \in B_{r-k}(x) \) and suppose that

\[
d(x_i, x_j) > r^{-k-1}
\]

for any \( i \neq j \). We have to prove that

\[
\max_{i=1, \ldots, N} \varphi_{k+2}(x_i) \geq \varphi_k(x) + \frac{C}{\sigma} \cdot \sqrt{\log N}.
\]
Note that for \( x, \bar{x} \in B^n_2 \)

\[
d(x, \bar{x}) \leq \sqrt{2} \frac{|\langle x - \bar{x}, y_i \rangle|}{\sqrt{\sum_{i=1}^{M} (\langle x, y_i \rangle)^2 + (\langle \bar{x}, y_i \rangle)^2}} \leq \sqrt{2} \cdot \max_{i=1, \ldots, M} |\langle x - \bar{x}, y_i \rangle| \cdot \left(\sum_{i=1}^{M} |\langle x, y_i \rangle| \cdot (\|x\|^2 + \|\bar{x}\|^2)\right)^{1/2} \leq 2 \cdot \max_{i=1, \ldots, M} |\langle x - \bar{x}, y_i \rangle|.
\]

Define a norm in \( \mathbb{R}^n \) by

\[
\|x\|_Y = \max_{i=1, \ldots, M} |\langle x, y_i \rangle|.
\]

If \( k \geq k_1 - 2 \) then (3.4) follows from a simple entropy estimate. Indeed, we have

\[
N \leq N(B^n_2, d, r^{-k-1}) \leq N(B^n_2, \|\cdot\|_Y, \frac{1}{2} r^{-k-1}) \leq N(B^n_2, \|\cdot\|, \frac{1}{2Q} r^{-k-1}) \leq (1 + 4Q \cdot r^{k+1})^n,
\]

since \( \|\cdot\|_Y \leq Q \|\cdot\| \). Suppose now that \( k < k_1 - 2 \). For \( j = 1, \ldots, N \) denote \( z_j \) the point of \( \text{conv} B_{8^{r-k-2}}(x_j) \) for which the minimum of \( \|z\|^2 \) is attained and denote \( u \) the similar point of \( B_{8^{r-k}}(x) \). Put

\[
\theta = \max_{j=1, \ldots, N} \|z_j\|^2 - \|u\|^2.
\]

We have to show that

\[
(3.5) \quad r^{-k} \cdot \left( c \cdot Q \cdot \sqrt{\log M} \right)^{-1} \cdot \sqrt{\log N} \leq \max_{j=1, \ldots, N} \varphi_{k+2}(x_j) - \varphi_k(x) = \theta + \frac{2}{\log M}.
\]

Since \( d(x_i, x_j) \geq r^{-k-1} \), it follows from (3.2) that

\[
d(z_i, z_j) \geq \frac{1}{2} r^{-k-1},
\]

provided \( r \) is sufficiently large. From the other side,

\[
d(x, z_j) \leq 4(d(x, x_j) + d(x_j, z_j)) \leq 8r^{-k}.
\]

Since \( \frac{z_j + u}{2} \in \text{conv} B_{8^{r-k}}(x) \), and \( \|u\| \leq \|z_j\| \), we have

\[
\left\| \frac{z_j - u}{2} \right\|^2 = \frac{1}{2} \|z_j\|^2 + \frac{1}{2} \|u\|^2 - \left\| \frac{z_j + u}{2} \right\|^2 \leq \|z_j\|^2 - \left\| \frac{z_j + u}{2} \right\|^2 \leq \|z_j\|^2 - \|u\|^2,
\]

so,

\[
(3.6) \quad \left\| \frac{z_j - u}{2} \right\| \leq 2r^k.
\]
Thus, $N$ is bounded by the $\frac{1}{2}r^{-k-1}$-entropy of the set $K = u + 2\sqrt{\theta}B^n_2$ in the quasimetric $d$. To estimate this entropy we partition the set $K$ into $S$ disjoint subsets having diameter less than $\delta = \frac{1}{16}r^{-k-1}\theta^{-1/2}$ in the $\|\cdot\|_Y$ metric.

Let $g$ be a Gaussian vector in $\mathbb{R}^n$, normalized by $\mathbb{E}\|g\|^2 = n$. Denote by $N(B, \Delta, \varepsilon)$ the $\varepsilon$-entropy of the set $B$ in the metric $\Delta$. By dual Sudakov minoration [L-T] we have

$$\sqrt{\log S} \leq \sqrt{\log N(2\sqrt{\theta}B^n_2, \|\cdot\|_Y, \delta)} \leq \frac{c}{\delta} \cdot 2\sqrt{\theta} \cdot \mathbb{E}\|g\|_Y \leq C \cdot r^k \theta \cdot \mathbb{E}\max_i |\langle g, y_i \rangle| \leq C \cdot r^k \theta \cdot Q \cdot \sqrt{\log M}.$$  

(3.7)

If $S \geq \sqrt{N}$, we are done, because in this case (3.7) implies (3.5). Suppose that $S \leq \sqrt{N}$. Then there exists an element of the partition containing at least $\sqrt{N}$ points $z_j$. Let $J \subset \{1, \ldots, N\}$ be the set of the indices of these points. We have

(3.8)

$$\|z_j - z_l\|_Y \leq \frac{1}{16}r^{-k-1} \cdot \theta^{-1/2}$$

for all $j, l \in J$.

For $j = 1, \ldots, M$ denote

$$I_j = \{i \in \{1, \ldots, M\} \mid |\langle z_j, y_i \rangle| \geq 2|\langle u, y_i \rangle|\}.$$

Then (3.6) imlies that

$$\sum_{i \in I_j} \langle z_j, y_i \rangle^2 \leq 2 \sum_{i \in I_j} \langle z_j - u, y_i \rangle^2 + 2 \sum_{i \in I_j} \langle u, y_i \rangle^2 \leq 8\theta + \frac{1}{2} \sum_{i \in I_j} \langle z_j, y_i \rangle^2,$$

so,

(3.9)

$$\sum_{i \in I_j} \langle z_j, y_i \rangle^2 \leq 16\theta.$$

Since $d(z_j, z_l) \geq \frac{1}{2}r^{-k-1}$, we have

$$\left(\frac{1}{2}r^{-k-1}\right)^2 \leq \sum_{i=1}^M \langle z_j - z_l, y_i \rangle^2 \cdot \left(\langle z_j, y_i \rangle^2 + \langle z_l, y_i \rangle^2\right) \leq \sum_{i=1}^M \langle z_j - z_l, y_i \rangle^2 \cdot 4\langle u, y_i \rangle^2 + \max_{i \in I_j} \langle z_j - z_l, y_i \rangle^2 \cdot \sum_{i \in I_j} \langle z_j, y_i \rangle^2.$$ \hspace{1cm}

Combining (3.8) and (3.9) we get that the last expression is bounded by

$$2 \cdot 16\theta \cdot \left(\frac{\theta^{-1/2}}{8}r^{-k-1}\right)^2 + 4 \sum_{i \in I_j} \langle z_j - z_l, y_i \rangle^2 \cdot \langle u, y_i \rangle^2.$$
Define a norm \( \| \cdot \|_E \) by
\[
\| x \|_E = \left( \sum_{i=1}^{M} \langle x, y_i \rangle^2 \cdot \langle u, y_i \rangle^2 \right)^{1/2}.
\]
Then, for all \( j, l \in J, j \neq l \) we have
\[
\| z_j - z_l \|_E \geq \frac{1}{8} r^{-k-1}.
\]
Applying again dual Sudakov minoration, we obtain
\[
\sqrt{\log |J|} \leq \sqrt{\log N (2\sqrt{\theta} B_2^n, \| \cdot \|_E, \frac{1}{8} r^{-k-1})} \leq c r^k \cdot 2\sqrt{\theta} \cdot \| g \|_E \leq \]
\[
c r^k \cdot 2\sqrt{\theta} \left( \mathbb{E} \sum_{i=1}^{M} \langle g, y_i \rangle^2 \cdot \langle u, y_i \rangle^2 \right)^{1/2} \leq \]
\[
C r^k \cdot 2\sqrt{\theta} \cdot \max_{i=1, \ldots, M} \| y_i \| \cdot \left\| \sum_{i=1}^{M} y_i \otimes y_i u \right\| \leq C r^k \cdot 2\sqrt{\theta} \cdot Q.
\]
Since for all \( \theta > 0 \)
\[
2\sqrt{\theta} \leq \sqrt{\log M} \cdot \theta + \frac{1}{\sqrt{\log M}},
\]
we get
\[
\sqrt{\log N} \leq 2\sqrt{\log |J|} \leq C \cdot Q \cdot r^k \cdot \sqrt{\log M} \cdot \left( \theta + \frac{1}{\log M} \right),
\]
so (3.5) is satisfied.

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