STABLE ISOMORPHISM AND STRONG MORITA EQUIVALENCE OF OPERATOR ALGEBRAS

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ABSTRACT. We introduce a Morita type equivalence: two operator algebras $A$ and $B$ are called strongly $\Delta$-equivalent if they have completely isometric representations $\alpha$ and $\beta$ respectively and there exists a ternary ring of operators $M$ such that $\alpha(A)$ (resp. $\beta(B)$) is equal to the norm closure of the linear span of the set $M^*\beta(B)M$, (resp. $M\alpha(A)M^*$). We study the properties of this equivalence. We prove that if two operator algebras $A$ and $B$, possessing countable approximate identities, are strongly $\Delta$-equivalent, then the operator algebras $A \otimes K$ and $B \otimes K$ are isomorphic. Here $K$ is the set of compact operators on an infinite dimensional separable Hilbert space and $\otimes$ is the spatial tensor product. Conversely, if $A \otimes K$ and $B \otimes K$ are isomorphic and $A, B$ possess contractive approximate identities then $A$ and $B$ are strongly $\Delta$-equivalent.

1. Introduction

An operator algebra $A$ is both an operator space and a Banach algebra for which there exists a Hilbert space $H$ and a completely isometric homomorphism $\alpha: A \to B(H)$, where $B(H)$ is the set of bounded operators acting on $H$. If this algebra is a dual space and the map $\alpha$ is weak* continuous, it is called a dual operator algebra. The topic of non-selfadjoint operator algebras, studied initially by Kadison, Singer, Ringrose and Arveson, has been motivational for the theory of operator spaces.

Rieffel introduced the notion of strong Morita equivalence of $C^*$ algebras and since then many articles have been devoted to this topic. In [6], Brown, Green and Rieffel proved that two $C^*$ algebras with countable approximate identities are strongly Morita equivalent if and only if they are strongly stably isomorphic. Blecher, Muhly and Paulsen introduced another concept of strong Morita equivalence for operator algebras, [4]. We call this equivalence "BMP-strong Morita equivalence". In that article they gave an example showing that BMP-strong Morita equivalence does not induce a stable isomorphism between the operator algebras even if they possess an identity element of norm 1.

In the present article we construct a Morita-type equivalence of operator algebras (strong $\Delta$-equivalence) and prove that if two operator algebras with
countable approximate identities are strongly $\Delta$–equivalent then they are strongly stably isomorphic. Conversely, if they are strongly stably isomorphic and they possess contractive approximate identities, then they are strongly $\Delta$–equivalent.

A fundamental tool in our theory is the concept of a ternary ring of operators (TRO). A subspace $M$ of the set $B(H, K)$ of bounded operators from the Hilbert space $H$ to a Hilbert space $K$ is called a TRO if $MM^*M \subseteq M$. In the Morita theory of $C^*$–algebras, a TRO is an equivalence bimodule. In the case of $\Delta$–equivalence, the equivalence bimodules are “generated” by TROs.

In [9], the notion of weak TRO equivalence was defined and its properties were studied in [10, 11] and [13]. It is important that the weak TRO equivalence of dual operator algebras is related to the notion of weak stable isomorphism. We recall some definitions and results from the above papers:

**Definition 1.1.** Suppose $A$ and $B$ are weak* closed algebras acting on the Hilbert spaces $H$ and $K$ respectively. We say that $A$ and $B$ are weakly TRO equivalent if there exists a TRO $M \subseteq B(H, K)$ such that

$$A = [M^*BM]^{w*} \quad \text{and} \quad B = [MAM^*]^{w*}.$$ 

**Definition 1.2.** Suppose $A$ and $B$ are dual operator algebras. We say that $A$ and $B$ are weakly $\Delta$-equivalent if they have completely isometric normal representations $\alpha$ and $\beta$, respectively, such that $\alpha(A)$ and $\beta(B)$ are weakly TRO equivalent.

If two dual operator algebras are weakly $\Delta$-equivalent, then they are weakly Morita equivalent in the sense of [2, 15]. The converse does not hold, [10, 11, 12].

**Theorem 1.1.** [13] Two dual operator algebras $A$ and $B$ are weakly $\Delta$-equivalent iff there exists a cardinal $I$ such that the dual operator algebras $A \otimes^\sigma B(l^2(I))$ and $B \otimes^\sigma B(l^2(I))$ are isomorphic as dual operator algebras. Here $\otimes^\sigma$ is the normal spatial tensor product.

A similar theorem for dual operator spaces is the main result of [14].

We also need notions from [4]: Suppose that $A$ is an operator algebra and $U$ (resp. $V$) is a right (resp. left) operator module over $A$. We also assume that $\Omega$ is the norm closure of the linear span of the set

$$\{(ua) \otimes v - u \otimes (av) : u \in U, \ v \in V, \ a \in A\}$$

in the Haagerup tensor product $U \otimes^h V$. Then the $A$–balanced Haagerup tensor product of $U$ and $V$ is the quotient $U \otimes^h_A V = (U \otimes^h V)/\Omega$. The last operator space has the property to linearize the completely bounded bilinear maps $\phi : U \times V \to B(H)$, where $H$ is a Hilbert space, which are $A$–balanced:

$$\phi(u, av) = \phi(ua, v) \ \forall \ u \in U, \ v \in V, \ a \in A.$$
Definition 1.3. [4] Let $A$ and $B$ be operator algebras. We call them BMP-strongly Morita equivalent if there exist an operator $B - A$ bimodule $U$ and an operator $A - B$ bimodule $V$ such that the space $V \otimes_B U$ (resp. $U \otimes_A V$) is completely isometrically isomorphic as an operator $A$ bimodule (resp. $B$ bimodule) with $A$ (resp. $B$).

In this paper we introduce the notion of strong TRO equivalence and of strong $\Delta$-equivalence:

Definition 1.4. Suppose $A$ and $B$ are norm closed algebras acting on the Hilbert spaces $H$ and $K$ respectively. We call them strongly TRO equivalent if there exists a TRO $M \subset B(H, K)$ such that $A = [M^* BM]^{-\|\cdot\|}$ and $B = [MAM^*]^{-\|\cdot\|}$.

Definition 1.5. Suppose $A$ and $B$ are operator algebras. We call them strongly $\Delta$-equivalent if they have completely isometric representations $\alpha$ and $\beta$ respectively such that $\alpha(A)$ and $\beta(B)$ are strongly TRO equivalent.

In Section 2, we study some properties of Definitions 1.4 and 1.5 and we prove that both strong TRO equivalence and strong $\Delta$-equivalence are equivalence relations. We also prove that strong $\Delta$-equivalence is stronger than the BMP-strong Morita equivalence. (In Section 3 we will see that strong $\Delta$-equivalence is strictly stronger than BMP-strong Morita equivalence). In Section 2 we also prove that two $C\ast$-algebras are strongly Morita equivalent in the sense of Rieffel [18] iff they are strongly $\Delta$-equivalent.

In Section 3 we will prove that strong $\Delta$-equivalence is the appropriate context for the strong stable isomorphism of operator algebras. Actually, generalising the results of [6], we will prove that if two operator algebras $A$ and $B$ with countable approximate identities are strongly $\Delta$-equivalent, then they are strongly stably isomorphic. This means that the algebras $A \otimes K$ and $B \otimes K$, where $K$ is the algebra of compact operators acting on an infinite dimensional separable Hilbert space and $\otimes$ is the spatial tensor product, are completely isometrically isomorphic through an algebraic homomorphism. Conversely, if $A \otimes K$ and $B \otimes K$ are isomorphic and $A$ and $B$ possess contractive approximate identities, then $A$ and $B$ are strongly $\Delta$-equivalent.

Throughout this paper, we will use the following lemma, which can be deduced from the proof of Theorem 6.1 of [4].

Lemma 1.2. Suppose $M$ is a norm closed TRO. Then there exist nets $(u_t)_t$, $(f_\lambda)_\lambda$ where

$$u_t = \sum_{i=1}^{l_t} (m_i^t)^* m_i^t, \quad f_\lambda = \sum_{i=1}^{k_\lambda} n_i^\lambda (n_i^\lambda)^*$$

and

$$\{m_i^t, n_j^\lambda : 1 \leq i \leq l_t, 1 \leq j \leq k_\lambda\} \subset M$$
such that
\[ \|u_t\| \leq 1, \|f_\lambda\| \leq 1, \quad \forall t, \lambda \]
and such that
\[ \|\cdot\| - \lim_{t \to \infty} u_tm^* = m^*, \quad \|\cdot\| - \lim_{\lambda \to \infty} f_\lambda m = m \quad \forall m \in M. \]

A representation of an operator algebra \( A \) is a completely contractive homomorphism \( \alpha : A \to B(H) \) where \( H \) is a Hilbert space such that \( \alpha(A) \) acts nondegenerately on \( H \). In case \( A \) is a dual operator algebra, we call \( \alpha \) a normal representation of \( A \) if it is weakly* continuous. The reader can use the books [3, 8, 16, 17] for the notions and theorems of operator space theory which appear in this present paper. If \( X \) is a vector space, \( M_{m,n}(X) \) denotes the set of \( m \times n \) matrices with entries in \( X \) and we write \( M_n(X) \) for \( M_{n,n}(X) \), \( C_n(X) \) for \( M_{n,1}(X) \), and \( R_n(X) \) for \( M_{1,n}(X) \).

2. Strong TRO equivalence and strong \( \Delta \)-equivalence

**Theorem 2.1.** Strong TRO equivalence is an equivalence relation.

*Proof.* If \( A \) is an operator algebra acting on the Hilbert space \( H \), then
\[ A = M^*AM = MAM^* \]
where \( M \) is the TRO \( CI_H \). So it suffices to prove the transitivity of strong TRO equivalence.

Suppose \( A, B, \) and \( C \) are operator algebras acting on the Hilbert spaces \( H, K, \) and \( L \), respectively, such that there exist TROs \( M \subset B(H, K) \) and \( N \subset B(K, L) \) satisfying
\[ A = [M^*BM]^{-\|\cdot\|}, \quad B = [MAM^*]^{-\|\cdot\|} = [N^*CN]^{-\|\cdot\|}, \quad C = [NBN^*]^{-\|\cdot\|}. \]

We have to show that \( A \) and \( C \) are strongly TRO equivalent.

Let \( D \) be the \( C^* \)-algebra generated by the set \( M^* \cup N^*N \). Put
\[ T = [NDM]^{-\|\cdot\|} \subset B(H, L). \]

We shall show that \( T \) is a TRO implementing the TRO equivalence of \( A \) and \( C \). Firstly, we see that \( T \) is a TRO: Observe
\[ NDM^*DN^*NDM \subset NDM \subset T. \]
Thus, \( TT^*T \subset T \). Now we have that
\[ TAT^* \subset [NDMAM^*DN^*]^{-\|\cdot\|} \subset [NDBDN^*]^{-\|\cdot\|}. \]
Since
\[ MM^*B \subset B, \quad N^*NB \subset B, \quad BMM^* \subset B, \quad BN^*N \subset B, \]
and \( D \) is generated by \( MM^* \cup N^*N \), we have
\[ DBD \subset B. \]
Thus
\[ TAT^* \subset [NBN^*]^{-\|\cdot\|} \subset C. \]
On the other hand,
\[ C = [NBN^*]^{-\|\cdot\|} = [NN^*NBNN^*N]^{-\|\cdot\|} \subset [NDBDN^*]^{-\|\cdot\|} = [NDBDN^*]^{-\|\cdot\|} = [TAT^*]^{-\|\cdot\|}. \]
We have proved
\[ C = [TAT^*]^{-\|\cdot\|}. \]
Similarly, we can prove that
\[ A = [T^*CT]^{-\|\cdot\|}. \]
The proof is complete.

**Theorem 2.2.** Suppose \( A \) and \( B \) are \( C^* \)-algebras. Then \( A \) and \( B \) are strongly \( \Delta \)-equivalent iff they are strongly Morita equivalent in the sense of Rieffel.

**Proof.** Suppose that \( A \) and \( B \) are strongly Morita equivalent \( C^* \)-algebras in the sense of Rieffel. Then there exist faithful \(*\)-homomorphisms \( \alpha \) of \( A \) and \( \beta \) of \( B \) to \( B(H) \) and \( B(K) \), respectively, where \( H \) and \( K \) are Hilbert spaces, and a TRO \( M \subset B(H,K) \) such that
\[ \alpha(A) = [M^*M]^{-\|\cdot\|}, \quad \beta(B) = [MM^*]^{-\|\cdot\|}. \]
Obviously
\[ \beta(B) = [M\alpha(A)M^*]^{-\|\cdot\|}, \quad \alpha(A) = [M^*\beta(B)M]^{-\|\cdot\|}. \]
For the converse, suppose that \( A \) and \( B \) are \( C^* \)-algebras of operators and that there exists a TRO \( M \) such that
\[ A = [M^*BM]^{-\|\cdot\|}, \quad B = [MAM^*]^{-\|\cdot\|}. \]
Let \( N = [BM]^{-\|\cdot\|}. \) We have \( NN^*N \subset [BMM^*BM]^{-\|\cdot\|}. \) Since \( MM^*M \subset M \), we have \( MM^*B \subset B \) and thus
\[ NN^*N \subset [BM]^{-\|\cdot\|} = N. \]
So \( N \) is a TRO. We now see that
\[ [N^*N]^{-\|\cdot\|} = [M^*BBM]^{-\|\cdot\|} = [M^*BM]^{-\|\cdot\|} = A, \]
\[ [NN^*]^{-\|\cdot\|} = [BMM^*B]^{-\|\cdot\|}. \]
Since \( M = [MM^*M]^{-\|\cdot\|}, \) we have
\[ B = [MM^*MAM^*]^{-\|\cdot\|} = [MM^*B]^{-\|\cdot\|}. \]
So
\[ [NN^*]^{-\|\cdot\|} = [BB]^{-\|\cdot\|} = B. \]
Similarly we can prove

\[ A = [N^*N]^{-\|\cdot\|}. \]

\[ \square \]

**Theorem 2.3.** Suppose \( A \) and \( B \) are strongly TRO equivalent operator algebras acting on the Hilbert spaces \( H \) and \( K \), respectively. Then their diagonals

\[ \Delta(A) = A \cap A^*, \quad \Delta(B) = B \cap B^* \]

are strongly TRO equivalent.

**Proof.** There exists a TRO \( M \subset B(H, K) \) such that

\[ A = [M^*BM]^{-\|\cdot\|}, \quad B = [MAM^*]^{-\|\cdot\|}. \]

Since \( \Delta(A) \) and \( \Delta(B) \) are \( C^* \)-algebras, we have

\[ M^*\Delta(B)M \subset \Delta(A), \quad M\Delta(A)M^* \subset \Delta(B). \]

Suppose that \( b \in \Delta(B) \). Let \( (f_\lambda) \) be the net from Lemma 1.2. We have

\[ \| \cdot \| - \lim_{\lambda} f_\lambda m = m \quad \forall \ m \in M. \]

Since \( B = [MAM^*]^{-\|\cdot\|}, \) we have

\[ \| \cdot \| - \lim_{\lambda} f_\lambda b = b. \]

Also, since

\[ \| \cdot \| - \lim_{\lambda} m^* f_{\lambda'}^* = m^* \quad \forall \ m \in M, \]

we have

\[ \| \cdot \| - \lim_{\lambda'} cf_{\lambda'}^* = c \quad \forall \ c \in B. \]

So

\[ \| \cdot \| - \lim_{\lambda'} f_\lambda b f_{\lambda'}^* = f_\lambda b. \]

But

\[ f_\lambda b f_{\lambda'}^* \in [MM^*\Delta(B)MM^*]^{-\|\cdot\|} \subset [M\Delta(A)M^*]^{-\|\cdot\|}. \]

Thus \( b \in [M\Delta(A)M^*]^{-\|\cdot\|}. \) We have proved \( \Delta(B) = [M\Delta(A)M^*]^{-\|\cdot\|}. \) Similarly we can prove \( \Delta(A) = [M^*\Delta(B)M]^{-\|\cdot\|}. \)

\[ \square \]

**Corollary 2.4.** Suppose \( A \) and \( B \) are operator algebras which are strongly \( \Delta \)-equivalent. Then their diagonals \( \Delta(A) = A \cap A^*, \quad \Delta(B) = B \cap B^* \) are strongly \( \Delta \)-equivalent.

We remark that the diagonal \( \Delta(A) \) of an operator algebra \( A \) can be trivial. Thus, such an algebra can’t be strongly \( \Delta \)-equivalent with an algebra \( B \) whose diagonal is nontrivial.

**Theorem 2.5.** Suppose that \( A \) and \( B \) are strongly \( \Delta \)-equivalent operator algebras with contractive approximate identities. Then \( A \) and \( B \) are BMP-strongly Morita equivalent.
Proof. Let $H$ and $K$ be Hilbert spaces such that $A \subset B(H)$ and $B \subset B(K)$.
Assume that there exists a norm closed TRO $D \subset B(H, K)$ such that
\[ A = [D^*BD]^{-\|\cdot\|}, \quad B = [DAD^*]^{-\|\cdot\|}. \]
Set
\[ U = [BD]^{-\|\cdot\|} \text{ and } V = [D^*B]^{-\|\cdot\|}. \]
Since $BDD^* \subset B$, we have
\[ BDD^*BD \subset BD \subset U \Rightarrow UA \subset U. \]
So $U$ is a $B - A$ bimodule. Similarly, we can prove that $V$ is an $A - B$ bimodule.
Define the $C^*$ algebra $E = [DD^*]^{-\|\cdot\|}$. Observe that $EBE \subset B$. The completely contractive bilinear map
\[ D^* \times B \to V : (\delta, u) \to \delta u \]
induces a completely contractive $E$-module map
\[ \theta : D^* \otimes_E h^* B \to V : \delta \otimes_E b \to \delta b. \]
We are going to prove that $\theta$ is isometric and so onto $V$. It suffices to prove that if $\delta \in R_k(D^*), b \in C_k(B)$ then
\[ \| \delta \otimes_E b \| \leq \| \delta b \|. \]
By Lemma 1.2 there exists a net
\[ u_t = \sum_{i=1}^{k_t} (m^t_i)^* m^t_i \]
such that $m^t_i \in D$ and $\| u_t \| \leq 1$ for all $i, t$ and such that $\| \cdot \| - \lim_t u_t \delta = \delta$. Thus for $\epsilon > 0$, there exists $t$ such that
\[ \| \delta \otimes_E b \| - \epsilon \leq \| (u_t \delta) \otimes_E b \|. \]
Suppose that
\[ \delta_t = ((m^t_1)^*, ..., (m^t_{k_t})^*). \]
We have
\[ \| (u_t \delta) \otimes_E b \| = \| (\delta_t \delta^*_t) \otimes_E b \|. \]
Since $m^t_i \delta \in R_k(E)$ for all $i$ and $t$ we have
\[ \| \delta \otimes_E b \| - \epsilon \leq \| \delta_t \otimes_E (\delta^*_t \delta b) \| \leq \| \delta_t \| \| \delta^*_t \delta b \| \leq \| \delta b \|. \]
Since $\epsilon$ is arbitrary we conclude that
\[ \| \delta \otimes_E b \| \leq \| \delta b \|. \]
The proof of the fact $\theta$ is completely isometric is similar. Thus, the operator spaces $V$ and $D^* \otimes^h_E B$ are completely isometrically isomorphic: $V \cong D^* \otimes^h_E B$. Similarly we can prove that

$$U \cong B \otimes^h_E D, \quad A \cong D^* \otimes^h_E U.$$ 

Now we have

$$A \cong D^* \otimes^h_E U \cong D^* \otimes^h_E (B \otimes^h_E D) \cong D^* \otimes^h_E (B \otimes^h_E B) \otimes^h_E D \cong (D^* \otimes^h_E B) \otimes^h_E (B \otimes^h_E D) \cong V \otimes^h_B U.$$ 

Analogously we can prove that $B \cong U \otimes^h_A V$. 

In the sequel of this section we are going to prove that if $A$ and $B$ are operator algebras with contractive approximate identities (cai’s) and are strongly $\Delta$-equivalent, then for every completely isometric representation $\alpha$ of $A$, there exists a completely isometric representation $\beta$ of $B$ such that $\alpha(A)$ and $\beta(B)$ are strongly TRO equivalent. We may assume that $A \subset B(R)$ and $B \subset B(L)$ for $R$ and $L$ some Hilbert spaces, and that there exists a norm closed TRO $M \subset B(R, L)$ such that

$$A = [M^* BM]^{-\|\cdot\|}, \quad B = [MAM^*]^{-\|\cdot\|}.$$ 

Let

$$Y = [MA]^{-\|\cdot\|} \quad \text{and} \quad X = [AM^*]^{-\|\cdot\|}.$$ 

We can easily see that

$$Y = [BM]^{-\|\cdot\|}, \quad \text{and} \quad X = [M^* B]^{-\|\cdot\|},$$

thus

$$BYA \subset Y, \quad AXB \subset X.$$ 

By Theorem 2.5 and its proof, the algebra $A$ (resp. $B$) is completely isometrically isomorphic as an $A$-bimodule (resp. a $B$-bimodule) with the space $X \otimes^h_A Y$ (resp. $Y \otimes^h_A X$). We assume that $\alpha : A \rightarrow B(H)$ is a completely isometric representation such that $\alpha(A)(H) = H$. We define the space $K = Y \otimes^h_A H$, which is the underlying Hilbert space of a representation of $B$, Theorem 3.10 in [1], through the following completely contractive map:

$$\beta : B \rightarrow B(K), \quad \beta(b)(y \otimes_A h) = (by) \otimes_A h.$$ 

We are going to prove that $\beta$ is a complete isometry and that the algebras $\alpha(A)$ and $\beta(B)$ are strongly TRO equivalent.

Lemma 2.6. Let $(f_{\lambda})$ be the net from Lemma 1.2. Let

$$\theta_{\lambda} : K \rightarrow C_{k_{\lambda}}(H)$$
be the map defined by
\[ \theta_\lambda(y \otimes_A h) = (\alpha((n_\lambda^1)^* y)(h), ..., \alpha((n_\lambda^k)^* y)(h))^t. \]
If \( \langle \cdot, \cdot \rangle_K \) is the inner product of \( K \), then
\[ \langle u, v \rangle_K = \lim_\lambda \langle \theta_\lambda(u), \theta_\lambda(v) \rangle_{C_\lambda(H)} \quad \forall \; u, v \in K. \]

The proof of the above lemma can be deduced by arguments similar to those in the proof of [4, Theorem 3.10].

**Lemma 2.7.** For every \( a, b \in A, c \in [M^*M]^{-\|\|} \), and \( h, \xi \in H \), we have
\[ \langle \alpha(a)(h), \alpha(cb)(\xi) \rangle = \langle \alpha(c^*a)(h), \alpha(b)(\xi) \rangle. \]

**Proof.** We denote the \( C^* \)-algebra \( C = [M^*M]^{-\|\|} \) and by \( \mathcal{M}_l(A) \) the left multiplier algebra of \( A \). Put
\[ \sigma : C \times A \rightarrow A, \quad \sigma(c, a) = ca. \]
Since \( A = [CA]^{-\|\|} \) if \( (c_t) \) is a cai for \( C \), we have
\[ \lim_t \sigma(c_t, a) = \lim_t c_t a = a \quad \forall \; a \in A. \]
So \( \sigma \) is an oplication in the sense of Theorem 4.6.2 in [3]. Therefore, by that theorem, there exists a \(*\)-homomorphism
\[ \hat{\theta} : C \rightarrow \mathcal{M}_l(A) \cap \mathcal{M}_l(A)^*, \quad \hat{\theta}(c)(a) = \sigma(c, a) = ca. \]
Let \( \Omega \) be the algebra
\[ \{ T \in B(H) : T\alpha(A) \subset \alpha(A) \}. \]
By Theorem 2.6.2 in [3], there exists a completely isometric homomorphism
\[ \rho : \Omega \rightarrow \mathcal{M}_l(A) : \quad \rho(T)(a) = \alpha^{-1}(T\alpha(a)). \]
Put
\[ \theta = \rho^{-1} \circ \hat{\theta} : C \rightarrow \Omega. \]
Since
\[ \hat{\theta}(c)(a) = ca \quad \forall \; a \in A \Rightarrow \rho(\theta(c))(a) = ca \quad \forall \; a \in A. \]
So
\[ \alpha(ca) = \alpha(\rho(\theta(c)))(a)) = \theta(c)\alpha(a) \quad \forall \; c \in C, \; a \in A. \]
Since \( \theta \) is a \(*\)-homomorphism,
\[ \langle \alpha(a)(h), \alpha(cb)(\xi) \rangle = \langle \alpha(a)(h), \theta(c)\alpha(b)(\xi) \rangle = \langle \theta(c^*a)(h), \alpha(b)(\xi) \rangle. \]

**Lemma 2.8.** The map \( \phi : Y \rightarrow B(H, K) \) given by \( \phi(y)(h) = y \otimes_A h \) is a complete isometry.
Proof. Clearly $\phi$ is a completely contractive map. It suffices to prove that
\[ \|y\| \leq \|\phi(y)\| \]
for arbitrary $y \in M_n(Y)$ and $n \in \mathbb{N}$.

Since $Y = [MA]^{-1}$, we need to show $\|y\| \leq \|\phi(y)\|$ for $y = (y_{ij}) \in M_n(Y)$, where $y_{ij} = m_{ij}a_{ij}$ with $m_{ij} \in R_k(M)$, $a_{ij} \in C_k(A)$ and $k \in \mathbb{N}$. There exist $s \in \mathbb{N}$, $m_i \in R_s(M)$, and $a_j \in C_s(A)$ such that $y_{ij} = m_ia_j$ for $1 \leq i, j \leq n$. For example, if
\[
\begin{pmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{pmatrix} = \begin{pmatrix}
m_{11}a_{11} & m_{12}a_{12} \\
m_{21}a_{21} & m_{22}a_{22}
\end{pmatrix},
\]
then $y_{ij} = m_ia_j$ for the rows $m_1 = (m_{11}, 0, m_{12}, 0)$, $m_2 = (0, m_{21}, 0, m_{22})$ and the columns $a_1 = (a_{11}, a_{21}, 0, 0)^t$, $a_2 = (0, 0, a_{12}, a_{22})^t$.

Fix $h_1, ..., h_n \in H$. We can see that
\[
\|\phi(y)(h_1, ..., h_n)^t\| = \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} y_{ik} \otimes A \ h_k \right\|_K^2.
\]
We recall the maps $\theta_\lambda$ from Lemma 2.6. We have
\[
\|\phi(y)(h_1, ..., h_n)^t\| = \lim_{\lambda} \sum_{i=1}^{n} \left\langle \theta_\lambda(\sum_{k=1}^{n} y_{ik} \otimes A \ h_k), \theta_\lambda(\sum_{l=1}^{n} y_{il} \otimes A \ h_l) \right\rangle =
\]
\[
\lim_{\lambda} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left\langle \theta_\lambda(m_ia_k \otimes A \ h_k), \theta_\lambda(m_ia_l \otimes A \ h_l) \right\rangle =
\]
\[
\lim_{\lambda} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{j=1}^{k} \left\langle \alpha((n_j^\lambda)^*m_ia_k)(h_k), \alpha((n_j^\lambda)^*m_ia_l)(h_l) \right\rangle.
\]
By Lemma 2.7, we have
\[
\|\phi(y)(h_1, ..., h_n)^t\| = \lim_{\lambda} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{j=1}^{k} \left\langle \alpha(m_i^*n_j^\lambda m_ia_k)(h_k), \alpha(a_l)(h_l) \right\rangle =
\]
\[
\sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left\langle \alpha(m_i^*m_ia_k)(h_k), \alpha(a_l)(h_l) \right\rangle.
\]
Again by Lemma 2.7, we have
\[ \| \phi(y)(h_1, ..., h_n)^t \|^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left\langle \alpha((m_s^* m_i)^{1/2} a_k)(h_k), \alpha((m_s^* m_i)^{1/2} a_l)(h_l) \right\rangle = \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} \alpha((m_s^* m_i)^{1/2} a_k)(h_k) \right\|^2 = \left\| \alpha((m_s^* m_i)^{1/2} a_k)(h_1, ..., h_n)^t \right\|^2. \]
Taking the supremum over all \((h_1, ..., h_n)^t\) with \(\|(h_1, ..., h_n)^t\| \leq 1\), we obtain
\[ \| \phi(y) \|^2 = \| \alpha((m_s^* m_i)^{1/2} a_k)(i,k) \|^2. \]
Since \(\alpha\) is a complete isometry,
\[ \| \phi(y) \|^2 = \| ((m_s^* m_i)^{1/2} a_k)(i,k) \|^2 = \left\| \sum_{k=1}^{n} \alpha_i^* m_s^* m_k a_j \right\| \left\| \sum_{k=1}^{n} y_k^* y_j \right\| = \| y^* y \| = \| y \|^2. \]
The proof is complete. \(\square\)

**Lemma 2.9.** If \(b \in M_n(B)\) and \(n \in \mathbb{N}\), then
\[ \| b \| = \sup_{y \in Ball(M_n,k(Y)), k \in \mathbb{N}} \| by \|. \]

**Proof.** Suppose that \(b = (b_{ij})\). Since \(B\) is completely isometrically isomorphic as a \(B\) bimodule to \(Y \otimes_A Y\), there exist nets \((y_k)_k, (x_k)_k\), where
\[ y_k \in Ball(R_{n_k}(Y)), x_k \in Ball(C_{n_k}(X)) \]
such that
\[ b_{ij} = \| \cdot \| - \lim_k b_{ij} y_k x_k, \]
for all \(i, j\), Lemma 2.9 in [4]. So for any \(\epsilon > 0\), there exists a \(k\) such that
\[ \| b \| - \epsilon < \| (b_{ij} y_k x_k)_{i,j} \| = \| (b_{ij})_{ij}(y_k \oplus \ldots \oplus y_k)(x_k \oplus \ldots \oplus x_k) \| \leq \| by \|, \]
where \(y = (y_k \oplus \ldots \oplus y_k)\). Since \(\epsilon\) was arbitrary, the proof is complete. \(\square\)

**Lemma 2.10.** The map \(\beta\) is a complete isometry.

**Proof.** Fix \(b \in M_n(B)\) for some \(n \in \mathbb{N}\). By Lemmas 2.8 and 2.9, we have
\[ \| b \| = \sup_{y \in Ball(M_n,k(Y)), k \in \mathbb{N}} \| by \| = \sup_{y \in Ball(M_n,k(Y)), k \in \mathbb{N}} \| \phi(by) \| = \sup_{y \in Ball(M_n,k(Y)), k \in \mathbb{N}} \sup_{\|(h_1, \ldots, h_k)^t\| \leq 1} \| \phi(by)(h_1, ..., h_k)^t \|. \]
We can see that
\[ \phi(by)(h_1, ..., h_k)^t = \beta(b)(y \otimes_A (h_1, ..., h_k)^t). \]
So
\[ \|\phi(by)(h_1, ..., h_k)^t\| \leq \|\beta(b)\| \]
for all \( y \in Ball(M_{n,k}(Y)), h = (h_1, ..., h_k)^t \) with \( \|h\| \leq 1. \)
Thus \( \|b\| \leq \|\beta(b)\|. \)

Fix \( a \in A \) and \( h \in H \). If \((a_t)_t\) is a cai for \( A \) and \( m \in M \), then
\[ \|ma \otimes_A h\| = \lim_t \|ma_t a \otimes_A h\| = \lim_t \|ma_t \otimes_A \alpha(a)(h)\|. \]
So for any \( \epsilon > 0 \), there exists \( t \) such that
\[ \|ma \otimes_A h\| - \epsilon \leq \|ma_t \otimes_A \alpha(a)(h)\| \leq \|m\|\|\alpha(a)(h)\|. \]
Since \( \epsilon \) was arbitrary, we have
\[ \|ma \otimes_A h\| \leq \|m\|\|\alpha(a)(h)\|. \]
So we can define a map
\[ \alpha(A)(H) \rightarrow K : \alpha(a)(h) \rightarrow ma \otimes_A h \]
since this map is bounded and \( H = \alpha(A)(H) \) extends to
\[ \mu(m) : H \rightarrow K, \mu(m)(\alpha(a)(h)) = ma \otimes_A h. \]
We are going to prove that \( N = \overline{\mu(M)}^{\|\cdot\|} \) is a TRO implementing a TRO equivalence between \( \alpha(A) \) and \( \beta(B) \).

Suppose that \( m \in M, y_i \in Y, \) and \( h_i \in H, i = 1, ..., k; \) and let \((u_t)_t\) be the net in Lemma 1.2. We have
\[ \|m\| \left\| \sum_{i=1}^k y_i \otimes_A h_i \right\| \geq \left\| \sum_{i=1}^k u_i m^* y_i \otimes_A h_i \right\| = \left\| \sum_{i=1}^k \alpha(u_i m^* y_i)(h_i) \right\|. \]
Since \( m^* = \|\cdot\| - \lim_t u_t m^* \), we have
\[ \|m\| \left\| \sum_{i=1}^k y_i \otimes_A h_i \right\| \geq \left\| \sum_{i=1}^k \alpha(m^* y_i)(h) \right\|. \]
Thus we can define a bounded map
\[ \nu(m^*) : K \rightarrow H, \ y \otimes_A h \rightarrow \alpha(m^* y)(h). \]
We are going to prove that \( \mu(m) \) is the adjoint of \( \nu(m^*) \).

Lemma 2.11.
\[ \nu(m^*) = \mu(m)^* \ \forall \ m \in M. \]
Theorem 2.12. Suppose that

\[ \mu(m)(\alpha(a)(h)), rb \otimes_A \xi = (ma \otimes_A h, rb \otimes_A \xi) = \lim_{\lambda} (\theta_{\lambda}(ma \otimes a h), \theta_{\lambda}(rb \otimes A \xi)) = \lim_{\lambda} \langle (\alpha((n_{1}^{\lambda})^*ma)(h), \ldots, \alpha((n_{k}^{\lambda})^*ma)(h))^t, \ldots, \alpha((n_{k}^{\lambda})^*rb)(\xi), \ldots, \alpha((n_{k}^{\lambda})^*rb)(\xi))^t \rangle = \lim_{\lambda} \sum_{j=1}^{k_{\lambda}} \langle \alpha((n_{j}^{\lambda})^*ma)(h), \alpha((n_{j}^{\lambda})^*rb)(\xi) \rangle . \]

By Lemma 2.7,

\[ \langle \mu(m)(\alpha(a)(h)), rb \otimes_A \xi \rangle = \lim_{\lambda} \sum_{j=1}^{k_{\lambda}} \langle \alpha(r^*f_{\lambda}ma)(h), \alpha(b)(\xi) \rangle = \lim_{\lambda} \langle \alpha(r^*f_{\lambda}ma)(h), \alpha(b)(\xi) \rangle = \langle \alpha(a)(h), \alpha(m^*rb)(\xi) \rangle = \langle \alpha(a)(h), \nu(m^*)(rb \otimes A \xi) \rangle . \]

Since \( \alpha(A)(H) \) is dense in \( H \) and \( Y = [MA]^{-\|\|} \), the proof is complete. \( \square \)

Theorem 2.12. Suppose that \( A \) and \( B \) are operator algebras with contractive approximate identities which are strongly \( \Delta \)-equivalent. Then for every completely isometric representation \( \alpha \) of \( A \), there exists a completely isometric representation \( \beta \) of \( B \) such that \( \alpha(A) \) and \( \beta(B) \) are strongly TRO equivalent.

Proof. We assume that \( A, B, \) and \( M \) are as above. We also recall the maps \( \alpha, \beta, \mu, \) and \( \nu \). By Lemma 2.10, \( \beta \) is a complete isometry. If \( N = \mu(M) \), we are going to prove that \( N \) is a TRO and

\[ \alpha(A) = [N^* \beta(B)N]^{-\|\|}, \quad \beta(B) = [N \alpha(A)N^*]^{-\|\|}. \]

If \( m_1, m_2, m_3 \in M, a \in A, \) and \( h \in H \), we have

\[ \mu(m_3)\mu(m_2)^*\mu(m_1)(\alpha(a)(h)) = \mu(m_3)\mu(m_2)^*(ma \otimes h) = \mu(m_3)(\alpha(m_2^*ma)(h)) = m_3m_2^*m_1a \otimes_A h = \mu(m_1m_2^*m_3)(\alpha(a)(h)). \]

So

\[ \mu(m_3)\mu(m_2)^*\mu(m_1) = \mu(m_3m_2^*m_1) \in \mu(M) \subset N. \]

Thus

\[ NN^*N \subset N. \]

If \( m_1, m_2 \in M, b \in B, a \in A, \) and \( h \in H \), we have

\[ \mu(m_2)^*\beta(b)\mu(m_1)(\alpha(a)(h)) = \nu(m_2^*b)(ma \otimes h) = \nu(m_2^*)(bm_1a \otimes h) = \alpha(m_2^*bm_1a)(h) = \alpha(m_2^*bm_1a)(h). \]

So

\[ \mu(m_2)^*\beta(b)\mu(m_1) = \alpha(m_2^*bm_1). \]
Since \( \alpha \) and \( \beta \) are completely isometric maps and \( A = [M^*BM]^{-\parallel \cdot \parallel} \), then
\[
\alpha(A) = [N^*\beta(B)N]^{-\parallel \cdot \parallel}.
\]
If additionally \( y \in Y \), then
\[
\mu(m_2)\alpha(a)\mu(m_1)^*(y \otimes_A h) = \mu(m_2)\alpha(a)\nu(m_1^*)y \otimes_A h = \mu(m_2)(\alpha(am_1^*y)(h)) = m_2am_1^*y \otimes_A h = \beta(m_2am_1^*)(y \otimes_A h).
\]
Thus
\[
\mu(m_2)\alpha(a)\mu(m_1)^* = \beta(m_2am_1^*).
\]
Since
\[
B = [MAM^*]^{-\parallel \cdot \parallel} \Rightarrow \beta(B) = [N\alpha(A)N^*]^{-\parallel \cdot \parallel}.
\]
The proof is complete. \( \square \)

**Corollary 2.13.** Strong \( \Delta \)-equivalence is an equivalence relation of operator algebras with contractive approximate identities.

**Proof.** We need to prove its transitivity. Suppose that \( A, B, \) and \( C \) are operator algebras with contractive approximate identities and that \( A \) and \( B \) (resp. \( B \) and \( C \)) are strongly \( \Delta \)-equivalent. By Definition 1.5, there exist completely isometric representations \( \alpha \) of \( A \) and \( \beta \) of \( B \) such that \( \alpha(A) \) and \( \beta(B) \) are strongly TRO equivalent. By Theorem 2.12, there exists a completely isometric representation \( \gamma \) of \( C \) such that the algebras \( \beta(B) \) and \( \gamma(C) \) are strongly TRO equivalent. By Theorem 2.1, the algebras \( \alpha(A) \) and \( \gamma(C) \) are strongly TRO equivalent. \( \square \)

### 3. Stable isomorphisms of operator algebras

If \( X \) is an operator space, \( M_\infty(X) \) denotes the operator space of \( \infty \times \infty \) matrices with entries in \( X \), whose finite submatrices have uniformly bounded norm. Let \( M_\infty^{fin}(X) \) denote the subspace of finitely supported matrices and write \( K_\infty(X) \) for its norm closure in \( M_\infty(X) \). We can see that \( K_\infty(X) \) is isomorphic as an operator space with \( X \otimes K \), where \( \otimes \) is the spatial tensor product and \( K \) is the algebra of compact operators acting on an infinite dimensional separable Hilbert space.

Suppose that \( X \) and \( Y \) are operator spaces. We call them strongly stably isomorphic if \( K_\infty(X) \) and \( K_\infty(Y) \) are isomorphic as operator spaces. In this section we are going to generalise, to the setting of nonselfadjoint operator algebras, the following very important theorem from [6]:

**Theorem 3.1.** Two \( C^* \)-algebras which possess countable approximate identities are strongly Morita equivalent iff they are strongly stably isomorphic.

Our generalisation states:
Theorem 3.2. If two operator algebras which possess countable approximate identities are strongly \( \Delta \)-equivalent then they are strongly stably isomorphic. Conversely, if two operator algebras which possess contractive approximate identities are strongly stably isomorphic then they are strongly \( \Delta \)-equivalent.

The one direction of the proof is a consequence of the results of Section 2. We use Corollary 2.13: suppose \( A \) and \( B \) are operator algebras with contractive approximate identities such that \( K_\infty(A) \) and \( K_\infty(B) \) are isomorphic as operator spaces. (We recall that \( \mathcal{C}^* \) algebras have contractive approximate identities). We may assume that \( A \) acts on the Hilbert space \( H \) and \( B \) acts on \( L \). We can see that

\[
K_\infty(A) = [M^*AM]^{-\|\cdot\|}, \quad A = MK_\infty(A)M^*,
\]

where \( M \) is the norm closure of finitely supported rows with scalar entries. Thus \( A \) and \( K_\infty(A) \) are strongly TRO equivalent. Since also \( K_\infty(B) \) and \( B \) are strongly TRO equivalent and \( K_\infty(A) \) and \( K_\infty(B) \) are isomorphic, we conclude that \( A \) and \( B \) are strongly \( \Delta \)-equivalent. For this direction we didn’t use the hypothesis of the existence of a countable approximate identity. For the converse, we use this assumption. Examples in [6] show that the hypothesis that the \( \mathcal{C}^* \) algebras have countable approximate units (equivalently, strictly positive elements) is not superfluous in the strong stable isomorphism theorem.

For the proof of Theorem 3.2, we fix operator algebras \( A \) and \( B \) acting on the Hilbert spaces \( H \) and \( K \), respectively, such that \( A(H) \) (resp. \( B(K) \)) is dense in \( H \) (resp. \( K \)) and which possess countable approximate identities. We also assume that there exists a norm closed TRO \( M \subset B(H,K) \) such that

\[
A = [M^*BM]^{-\|\cdot\|}, \quad B = [MAM^*]^{-\|\cdot\|}.
\]

We are going to prove that \( K_\infty(A) \) and \( K_\infty(B) \) are isomorphic as operator spaces. We define the spaces

\[
Y = [MA]^{-\|\cdot\|} = [BM]^{-\|\cdot\|}, \quad X = [AM^*]^{-\|\cdot\|} = [M^*B]^{-\|\cdot\|}.
\]

Also observe that

\[
A = [M^*MAM^*M]^{-\|\cdot\|}.
\]

We define the \( \mathcal{C}^* \)-algebra

\[
D = \prod_{i=1}^k a_i^*b_i, \quad a_i, b_i \in A, \ k \in \mathbb{N}^{-\|\cdot\|}.
\]

Lemma 3.3. There exists an element \( a_0 \in D \) such that \( D = Da_0^{-\|\cdot\|} \).
Proof. It suffices to prove that $D$ has a strictly positive element. Suppose that $(e_n)_{n \in \mathbb{N}}$ is an approximate identity for $A$. Define

$$a_0 = \sum_{n=1}^{\infty} \frac{e_n^* e_n}{\|e_n\|^2 2^n}$$

and fix a state $\phi$ of $D$. We are going to prove that $\phi(a_0) > 0$. If, on the contrary, $\phi(a_0) = 0$, then $\phi(e_n^* e_n) = 0$ for all $n$. Fix an arbitrary $d \in D$ and $a, b \in Ball(A)$. Since $a^* b e_n \in A^* A A \subset D$, we have

$$|\phi(d a^* b e_n)|^2 \leq \phi(d d^*) \phi(e_n^* b a^* b e_n).$$

But

$$0 \leq e_n^* b a^* b e_n \leq e_n^* e_n.$$

Thus

$$\phi(e_n^* b a^* b e_n) = 0 \Rightarrow \phi(d a^* b e_n) = 0 \ \forall \ n.$$ 

The sequence $(b e_n)_n$ converges to $b$. We conclude that $\phi(d a^* b) = 0$ for all $d \in D$, $a, b \in A$, which implies $\phi(\Pi_{i=1}^k a_i^* b_i) = 0$ for all $a_1, ..., a_k, b_1, ..., b_k \in A$, $k \in \mathbb{N}$. It follows that $\phi = 0$. This contradiction completes the proof. \qed

Lemma 3.4. There exists a sequence $(m_i)_{i \in \mathbb{N}} \subset M$ such that

$$\left\| \sum_{i=1}^{k} m_i^* m_i \right\| \leq 1, \ \forall \ k \ \in \mathbb{N}$$

and

$$\| \cdot \| - \lim_k a \sum_{i=1}^{k} m_i^* m_i = a \ \forall \ a \ \in A.$$

Proof. Using the above lemma and the fact

$$M^* M D M^* M \subset D$$

we can prove as in Lemma 2.3 of [5] that there exists a sequence $(m_i)_{i \in \mathbb{N}} \subset M$ such that

$$\left\| \sum_{i=1}^{k} m_i^* m_i \right\| \leq 1, \ \forall \ k \ \in \mathbb{N}$$

and

$$\| \cdot \| - \lim_k d \sum_{i=1}^{k} m_i^* m_i = d \ \forall \ d \ \in \ D.$$
Fix $a \in A$ and suppose that $a = u|a|$ is the polar decomposition of $a$. Since $|a| = (a^*a)^{1/2}$ and $A^*A \subset D$, we have $|a| \in D$. Thus,

$$
\| \cdot \| - \lim_k |a| \sum_{i=1}^k m_i^* m_i = |a| \Rightarrow \| \cdot \| - \lim_k u|a| \sum_{i=1}^k m_i^* m_i = u|a| \Rightarrow \\
\| \cdot \| - \lim_k a \sum_{i=1}^k m_i^* m_i = a.
$$

\qed

We will use the following notation.

If $Z$ is a norm closed subspace of $B(L, R)$, where $L$ and $R$ are Hilbert spaces, we denote by $R_\infty(Z)$ the subspace of $B(L_\infty, R)$ containing all operators of the form $(z_1, z_2, \ldots)$ such that $z_i \in Z, \forall i$ and such that the sequence $(\sum_{i=1}^n z_i z_i^*)_n$ converges in norm.

If two operator spaces $Z_1, Z_2$ are completely isometrically isomorphic, we write $Z_1 \sim Z_2$.

We now return to the proof of Theorem 3.2. We shall use a route to the proof of stable isomorphism that is standard in the literature, see for example [3], [4], or [11]. So we leave the details to the reader. Let $A, B, M, X$, and $Y$ be as in the discussion preceding Lemma 3.3 and let $(m_i)_{i \in \mathbb{N}}$ be the sequence in Lemma 3.4. Put

$$
\alpha : Y \to R_\infty(B), a(y) = (ym_i)_i, \quad \beta : R_\infty(B) \to Y, \beta((b_i)_i) = \sum_{i=1}^\infty b_i m_i, \\
P = \alpha \circ \beta : R_\infty(B) \to R_\infty(B).
$$

We can see that $P \circ P = P$. Define $W = \text{Ran}(id - P)$. Since $\beta \circ \alpha = id_Y$, $\alpha$ is completely isometric. We have

$$
Y \cong \alpha(Y) = \text{Ran}P
$$

and

$$
R_\infty(B) = \text{Ran}P \oplus_r \text{Ran}(id - P) \cong Y \oplus W.
$$

As in the proof of [3, Corollary 8.2.6], we have

$$
R_\infty(B) \cong Y \oplus_r R_\infty(B)
$$

which implies

$$
R_\infty(B) \cong R_\infty(Y) \oplus_r R_\infty(B).
$$

By the same arguments we show

$$
R_\infty(Y) \cong R_\infty(B) \oplus_r R_\infty(Y),
$$

thus

$$
R_\infty(B) \cong R_\infty(Y) \Rightarrow \text{K}_\infty(B) \cong \text{K}_\infty(Y).
$$
Similarly we can prove $K_\infty(Y) \cong K_\infty(A)$. Hence $K_\infty(A) \cong K_\infty(B)$. The proof of Theorem 3.2 is complete.

**Theorem 3.5.** BMP-strong Morita equivalence is strictly weaker than strong $\Delta$-equivalence.

**Proof.** There is an example of BMP-strongly Morita equivalent operator algebras with units of norm 1 which are not stably isomorphic (Example 8.2 in [4]). So, by Theorem 3.2 these algebras can not be strongly $\Delta$-equivalent. □

A different proof of the above theorem is implemented by Example 3.7.

**Example 3.6.** Let $A$ and $B$ be nest algebras corresponding to the nests $\mathcal{L}_1$ and $\mathcal{L}_2$, acting on the separable Hilbert spaces $H$ and $K$, respectively. See the appropriate definition in [7]. We assume that $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are the subalgebras of compact operators. The second duals of $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are the algebras $A$ and $B$. Then the following are equivalent:

(i) $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are strongly stably isomorphic.

(ii) $A$ and $B$ are weakly stably isomorphic.

(iii) There exists a $\ast$-isomorphism $\theta : \mathcal{L}_1'' \to \mathcal{L}_2''$ mapping $\mathcal{L}_1$ onto $\mathcal{L}_2$. Here, $\mathcal{L}_i''$ is the double commutant of $\mathcal{L}_i$, $i = 1, 2$.

The equivalence of (ii) and (iii) is implied by Theorems 3.3 in [9] and 3.2 in [11].

We shall prove that (i) implies (ii). We assume that $\mathcal{K}$ is the algebra of compact operators acting on the infinite dimensional separable Hilbert space $R$. Since $\mathcal{K}(A) \otimes \mathcal{K}$ and $\mathcal{K}(B) \otimes \mathcal{K}$ are isomorphic operator algebras, their second duals $A \otimes^\sigma B(R)$ and $B \otimes^\sigma B(R)$ are isomorphic as dual operator algebras. Here $\otimes$ is the spatial tensor product and $\otimes^\sigma$ is the normal spatial tensor product.

We shall prove that (iii) implies (i). We define the TRO

$$M = \{ m \in B(H, K) : mp = \theta(p)m \ \forall \ p \in \mathcal{L}_1 \}.$$ 

By Theorem 3.3 in [9],

$$A = [M^*BM]^{-w}, \quad B = [MAM^*]^{-w}.$$ 

Thus

$$\mathcal{K}(A) \supset M^*\mathcal{K}(B)M, \quad \mathcal{K}(B) \supset MK(A)M^*.$$ 

On the other hand,

$$M^*MK(A)M^*M \subset M^*\mathcal{K}(B)M.$$ 

By Theorem 8.5.23 in [3], there exists a net of integers $(n_i)$ and operators $m_i \in Ball(C_{n_i}(M)) \ \forall \ i$ such that the identity operator of $H$ is the limit of the net $m_i^*m_i$ in the strong operator topology. Thus

$$k = \| \cdot \| - \lim_{i} m_i^*m_i k$$
for every compact operator \( k \in B(H) \). It follows from (3.1) that
\[
\mathcal{K}(A) \subset [M^*\mathcal{K}(B)M]^{-\|\cdot\|} \Rightarrow \mathcal{K}(A) = [M^*\mathcal{K}(B)M]^{-\|\cdot\|}.
\]
Similarly we can prove that
\[
\mathcal{K}(B) = [MK(A)M^*]^{-\|\cdot\|}.
\]
Since \( A \) and \( B \) are nest algebras acting on separable Hilbert spaces, \( \mathcal{K}(A) \) and \( \mathcal{K}(B) \) have countable approximate identities, \([7]\). So by Theorem 3.2, \( \mathcal{K}(A) \) and \( \mathcal{K}(B) \) are strongly stably isomorphic.

**Example 3.7.** In this example we give a different proof of the fact BMP-strong Morita equivalence is strictly weaker than strong \( \Delta \)-equivalence. Let \( \mathcal{L}_1, \mathcal{L}_2 \) be the nests in Example 7.19 in \([7]\), \( A, B \) be the corresponding nest algebras and \( \mathcal{K}(A), \mathcal{K}(B) \) be the subalgebras of compact operators. Since \( \mathcal{L}_1, \mathcal{L}_2 \) are isomorphic nests the algebras \( \mathcal{K}(A), \mathcal{K}(B) \) are BMP-strongly Morita equivalent, \([12]\). If \( \mathcal{K}(A) \) and \( \mathcal{K}(B) \) were strongly \( \Delta \)-equivalent by Example 3.6 the von Neumann algebras \( \mathcal{L}_1'', \mathcal{L}_2'' \) would be isomorphic. But \( \mathcal{L}_1'' \) is a maximal abelian selfadjoint algebra (masa) with a nontrivial continuous part and \( \mathcal{L}_2'' \) is a totally atomic masa. This is a contradiction.

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