Comments on “Design of Asymmetric Shift Operators for Efficient Decentralized Subspace Projection”

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Abstract—This correspondence disproves the main results in the paper “Design of Asymmetric Shift Operators for Efficient Decentralized Subspace Projection” by S. Mollaebrahim and B. Beferull-Lozano. Counterexamples and counterproofs are provided when applicable. A correction is suggested for some of the flaws. However, it does not seem possible to amend most of the flaws since the overall approach based on a Schur decomposition of the shift matrix does not appear to be helpful to solve the desired problem.

INTRODUCTION

This correspondence comments on [1], where the goal is to design shift matrices for graph filters whose output is the result of projecting their input onto a given subspace. The paper relies on a Schur decomposition $S = W(D + Q)W^T$ of the sought shift matrix $S$. The goal is therefore the same as in the earlier paper [2] with the exception that $S$ is required to be symmetric in [2]. The reason for such a requirement in [2] was to obtain suitable necessary and sufficient conditions to characterize the set of feasible shift matrices. The work in [1] claims to solve the problem when that assumption is lifted. The present correspondence shows that this is not the case.

Notation: The notation in [1] is adopted here. Besides, $A^0$ denotes the identity matrix, where $A$ is an arbitrary square matrix. Equation numbers of the form ($N$) refer to [1], whereas ($CN$) refers to the present correspondence.

TECHNICAL CONTENT

Problem formulation. Early in Sec. IV, the problem is formulated as finding asymmetric matrices that are “polynomial” and “topological”. Although not mentioned in [1], these definitions were introduced by [2, Sec. III]. The aforementioned formulation in [1] excludes symmetric matrices, which conflicts with the optimization approach adopted later. The reason is that symmetric matrices are included in the closure of the set of asymmetric matrices and the infimum of a continuous function on a set equals the infimum of that function on the closure of that set. This first issue could be amended by replacing “asymmetric” with “possibly asymmetric”.

Theorem 1, Part 1. Although not mentioned, this result aims at extending [2, Th. 1, C2]. In [1], Theorem 1, Part 1, states that “A necessary condition to have a valid polynomial graph shift matrix $S$ is that $D_1$ and $D_2$ do not share any common value.”, which is obviously incorrectly phrased as these matrices are diagonal and therefore they necessarily share the entry 0, unless they are $1 \times 1$. By looking at the proof, it is clear that what this result aims to say is that the aforementioned two matrices “do not share any common eigenvalue” or, equivalently, “do not share any common diagonal value”.

But even such a (corrected) statement would be false, as proved by the following counterexample. Let

$$
U_1 = [u_1, u_2], \quad U_\perp = [u_3, u_4], \quad U = [U_{\|}, U_\perp],
$$

(C1)

$$
D_1 = D_2 = \text{diag}\{[1,0]\}, \quad Q = 0, \quad W = [u_1, u_3, u_2, u_4],
$$

where $\{u_1\}_i \subset \mathbb{R}^4$ are arbitrary orthonormal vectors. This clearly satisfies that $S = W(D + Q)W^T = U_{\|}U_{\|}^T$ and therefore it is a “polynomial shift” according to Sec. IV, yet it violates the aforementioned “necessary” condition. One can also obtain an asymmetric $S$ violating this condition; set e.g. $(Q)_{2,1}$ equal to 1, which yields $S^2 = U_{\|}U_{\|}^T$.

Since the result is incorrect, the proof is necessarily incorrect. The issue can be found where it introduces the assumption that there exist $c, D$, and $Q$ such that

$$
\sum_{l=0}^{L-1} c_l (D + Q)^l = \begin{bmatrix} 1_r & 0 \\ 0 & 0 \end{bmatrix}.
$$

(C2)

This assumption is not a hypothesis of the theorem and does entail loss of generality. Therefore, this step is logically flawed. It is clear that one cannot introduce arbitrary assumptions in a proof. Otherwise one could prove any statement $s$ and its negation $\neg s$, which would yield a contradiction.

This flaw propagates throughout the paper and invalidates the proposed optimization problems; cf. the discussions below.

Finally, the proof adapts the arguments in [2, Explanation around (12)] and is also related to those in [3] and [4]. However, this is presented without reference to these works.

Theorem 1, Part 1, tries to extend the necessary (and also sufficient) condition in [2, Theorem 1, C2], to asymmetric shift matrices. In [2], this is possible because assuming that $S$ is symmetric allows one to use its eigendecomposition $S = WAW^T$. The columns of $W$ are eigenvectors of $\sum_l c_l S_l$ for all $\{c_l\}_l$, which enables one to group the eigenvalues of $S$ into two groups: those associated with eigenvectors in $\mathcal{R}\{U_{\|}\}$ and those associated with eigenvectors in $\mathcal{R}\{U_\perp\}$. The necessary condition is that the same eigenvalue cannot be in both groups. In contrast, the columns of $W$ in the Schur decomposition $S = W(D + Q)W^T$ adopted in [1] are not generally eigenvectors of $\sum_l c_l S_l$ and, therefore, no such a grouping seems possible. Thus, it is not just the fact that the diagonal entries of $D$ in a Schur decomposition may appear in any order and, therefore, $D_1$ and $D_2$ are not well defined in Theorem 1, Part 1. In turn, the core approach in [1], which relies on a Schur decomposition, seems to help little towards solving the targeted problem.

Theorem 1, Part 2. This result is also incorrect, as the proof is flawed at multiple points. First, matrix $Q$ is assumed upper triangular, which entails loss of generality given that $Q$ is upper quasi-triangular; see sentence after (5). This implies that (8) is not correct and invalidates the rest of the proof.

Second, note that the proof relies on the assumption that $T'$ is full-row rank. Since it has $N_2$ rows, the assumption is rank($T'$) = $N_2$. However, this violates the Cayley-Hamilton Theorem. To see this, recall that this theorem states that every matrix $Z \in \mathbb{R}^{N \times N}$ is a root of its own characteristic polynomial. Let this polynomial be $p(Z) = \sum_{i=0}^N \beta_i Z^i$, where, as we know, $\beta_N = 1$. Then, $p(Z) = 0$ clearly implies

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that \( Z^N = -\sum_{k=0}^{N-1} \beta_k Z^k \). Multiplying both sides by \( Z^k \), for any integer \( k \geq 0 \), shows that every power \( Z^{N+k} \) is a linear combination of the powers of \( Z \) with lower exponents. This implies that the matrix \([\text{vec}[Z^0], \text{vec}[Z^1], \ldots, \text{vec}[Z^N]]\), for an arbitrary \( M \), can never be of rank greater than \( N \).

Matrix \( T' \) in this paper is obtained by removing rows from \( T = [\text{vec}[Z^0], \text{vec}[Z^1], \ldots, \text{vec}[Z^{N-1}]] \), where \( Z = D + Q \). Since \( Z \) is a \( N \times N \) matrix, it follows from the Cayley-Hamilton theorem that \( \text{rank}(T') \leq \text{rank}(T) \leq N \). However, the proof assumes that its rank is \( N_2 = (N^2 + N)/2 - m_1 - m_2 \), which satisfies \( N_2 \geq (N^2 + N)/2 - N = (N^2 - N)/2 \). Thus, except for the trivial case \( N \leq 3 \), this number is greater than the maximum imposed by the Cayley-Hamilton Theorem. This contradiction doubtly voids the proof. Besides, the sufficient condition in Theorem 1, Part 2, is therefore useless since it will never hold for \( N \geq 3 \).

**Centralized algorithm.** The matrix \( S \) solving (11) and (12) is claimed to provide an exact subspace projection; cf. Sec. II and discussions around (11) and (12). However, such a claim is clearly false. The cause of this issue is that the condition provided by Theorem 1, Part 1, is taken as a sufficient condition, whereas what that theorem actually states is that this is a necessary condition. One could however think that the solution actually yields the desired subspace projection despite the fact that the argumentation is incorrect. Unfortunately, this is not the case. To see this, note that given the choices of \( W_{i1}, W_{1i}, \) and \( W \) in the paper, an exact projection is attained only if (7) holds. However, the solution to (11) and (12) will not generally satisfy (7). To see this, note that (7) comprises \( N^2 \) equations and these equations are not generally satisfied by an arbitrarily selected matrix. In other words, one would need to somehow constrain \( S \) in these problems to satisfy (7). However, this is not the case.

Finally, although not mentioned in [1], the technique to deal with constraint (11f) therein was proposed in [2].

**Decentralized algorithm.** In Equation (13), \( \Psi \) is not defined. After (13), matrix \( \Delta_n \) is improperly defined and \( \epsilon_1 \) is not defined. This and other undefined symbols limits the reader’s ability to understand and verify the contents of Sec. V.

After some guessing regarding the symbol definitions, it is easy to see that the algorithm proposed in Sec. V need not converge to the solution of the targeted problem. To this end, note that \( a_n \) in (13d) is a function of \( D^{(l)} \) with \( l \neq n \), where \( D^{(l)} \) is an optimization variable of the \( l \)-th node. However, \( a_n \) is treated as a constant by the algorithm: this would need to be fixed by introducing \( N - 1 \) constraints similar to (14e) but for \( D^{(l)} \), \( l \neq n \), and by introducing \( N - 1 \) additional update equations in (28). This clearly means that the proposed algorithm is not guaranteed to converge to the solution of (12) and, in fact, it is not even clear whether the algorithm will converge at all.

The fact that this algorithm is wrong is further corroborated by the simulations, where the centralized and decentralized algorithms yield different results, something that would not be possible if the algorithm was correctly derived and implemented, given that no approximation is introduced.

Finally, this algorithm is not truly decentralized since all the nodes need to know the subspace of interest (cf. \( W \) shows up in (28)) and the topology of the graph (cf. \( \Psi \) shows up in (28)). Thus, all nodes have global information and can solve the problem on their own without interacting with others. One could argue that interaction with other nodes could decrease the computational complexity. Unfortunately, this is not the case: such a decentralized approach would have the same complexity order (yet probably substantially more operations in absolute terms) as its centralized counterpart, cf. Sec. VIII-D, and it would introduce additional overhead in terms of time, communication, and power consumption.

**Theorem 2.** Although not mentioned, this theorem is based on Theorem 2 and Corollary 1 in [2]; see also the explanation afterwards. In fact, Appendix C in [1] is an adaptation of [2] Appendix C, including the same steps and equations up to small modifications. Besides, the proof in [1] is flawed. To see that, note that (31) is wrong since the matrices in the set \( \widehat{S}_{pp} \) are not necessarily polynomial shift matrices as per the definition; cf Sec. IV. A simple counterexample is the zero matrix, which is in \( \widehat{S}_{pp} \) but is not a “polynomial shift”. This clearly voids the proof and Theorem 2.

**Equation (21c).** Since \( d \) is undefined, the problem (21) cannot be understood.

**Proposition 1.** This is also incorrect. The proof suffers from the same issues as the proof of Theorem 1; see above.

**Simulation study.** It is remarkable that the simulation experiments do not compare with the algorithms proposed in [2], given that the contribution is based on the benefits of allowing for asymmetric shift matrices. This could have been easily done by running an experiment with directed graphs (upon applying [2] on a symmetric adjacency matrix that results from removing the necessary edges) and another experiment with undirected graphs, thereby assessing the benefits of the proposed scheme relative to the state of the art. Instead, the proposed methods are compared against algorithms not designed for subspace projection.

I. Conclusions

This short correspondence has disproved Theorem 1, Theorem 2, and Proposition 1 in [1] by means of counterexamples and counterproofs. The proposed centralized algorithm, which aims at solving (12) was shown to be flawed. Similarly, the proposed decentralized algorithm is also incorrectly derived. Besides, it was pointed out which core ideas were obtained from [2] without acknowledgment. Whenever possible, a correction was proposed. However, for the most part, the adopted approach based on a Schur decomposition seems intractable and amendments are most likely impossible.

References

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