Comments on T-dualities
of Ramond-Ramond Potentials

Masafumi Fukuma$^1$, Takeshi Oota$^2$ and Hirokazu Tanaka$^1$

$^1$ Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

$^2$ Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Tanashi, Tokyo 188-8501, Japan

Abstract

The type IIA/IIB effective actions compactified on $T^d$ are known to be invariant under the T-duality group $SO(d, d; \mathbb{Z})$ although the invariance of the R-R sector is not so direct to see. Inspired by a work of Brace, Morariu and Zumino, we introduce new potentials which are mixture of R-R potentials and the NS-NS 2-form in order to make the invariant structure of R-R sector more transparent. We give a simple proof that if these new potentials transform as a Majorana-Weyl spinor of $SO(d, d; \mathbb{Z})$, the effective actions are indeed invariant under the T-duality group. The argument is made in such a way that it can apply to Kaluza-Klein forms of arbitrary degree. We also demonstrate that these new fields simplify all the expressions including the Chern-Simons term.

*e-mail address: fukuma@yukawa.kyoto-u.ac.jp
† e-mail address: toota@tanashi.kek.jp
‡ e-mail address: hirokazu@yukawa.kyoto-u.ac.jp
1 Introduction

Recent developments in string theory have been based on various kinds of duality symmetries. Among them, the T-duality was found first [1, 2], which changes the size of the compactified space into its inverse in string unit. Although this symmetry was first recognized in the spectra of perturbative strings, it came to be believed that this should hold as an exact symmetry not simply as a perturbative one [3]. Later, at the level of low energy effective action, the T-duality invariance of the type IIA/IIB theory was identified with a part of already known, much larger, and hidden symmetries of type II supergravities [4]–[7]. It was actually conjectured that the duality group of the full string theory can be extended to the U-duality group $E_{d+1}^d ((d+1)\mathbb{Z})$ when compactified on a $d$-dimensional torus [8].

Being a subgroup of the U-duality group, the T-duality group $SO(d, d; \mathbb{Z})$ has a special property: it is the maximum subgroup which consists of the elements that transform NS-NS and R-R fields into themselves, respectively. On the other hand, we sometimes encounter situations where NS-NS and R-R fields are better treated in a separate way. This is often the case when classical black-hole solutions of string theory are considered. Another example may be given by study of classical configurations based on the Born-Infeld action. Thus, it should be useful if one can know in a simple manner how NS-NS and R-R fields transform under the T-duality group, without resorting to embedding the whole structure once into the vast U-duality group.

The T-duality invariance can actually be seen very easily for the NS-NS sector of supergravity action [9]. There the kinetic term of the Kaluza-Klein (KK) scalars $(G_{ij}, B_{ij})$ ($i, j = 1, \ldots, d$) can be written as

$$\mathcal{L}_{NS} = \frac{1}{8} e^{-2\phi} \text{tr} \left( \partial_{\mu} M^{-1} \partial^{\mu} M \right) \quad (1.1)$$

with a $2d \times 2d$ matrix

$$M = (M_{rs}) = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \quad (1.2)$$

$^1$ This $B_{ij}$ will be denoted by $B_{ij}^{(0)}$ in the following sections to notify that this is a scalar for the noncompact $(10 - d)$-dimensional space-time with coordinates $x^\mu$ ($\mu = 0, 1, \ldots, 9 - d$). We will also take the string unit $\alpha' = 1$. 

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and the \((10−d)\)-dimensional dilaton \(\phi\). Thus the kinetic term is manifestly invariant under T-duality transformations \(\Lambda \in O(d, d; \mathbb{Z})\) if the dilaton does not change and \(M = (M_{rs})\) \((r, s = 1, .., 2d)\) transforms as

\[
\overline{M} = (\Lambda^{-1})^T \cdot M \cdot \Lambda^{-1}.
\]  
(1.3)

The KK 1-forms \((G_{\mu i}, B_{\mu i})\) give a vector representation of \(O(d, d; \mathbb{Z})\) and also have an invariant kinetic term [9]. These facts will be reviewed later in more detail.

On the other hand, the invariance of the sector including R-R potentials under the T-duality group \(SO(d, d; \mathbb{Z})\) is not so manifest as that for the NS-NS sector is. There have actually been many works in which T-duality was studied as a subgroup of U-duality group \(E_{d+1}(d+1)\) \((\mathbb{Z})\) [10]. However, in order to write down the action in a manifestly U-invariant form, one needs to make a non-trivial mapping from the original fields to some other fields, which usually makes the T-duality symmetry for the original fields indirect. As for the works based on the T-duality itself, results have been obtained [11] only for Nahm transformations which generate a subgroup of \(O(d, d; \mathbb{Z})\).

By decomposing representations of \(E_{d+1(d+1)}(\mathbb{Z})\) with respect to \(SO(d, d; \mathbb{Z})\), it has been also known that Majorana-Weyl representations of \(SO(d, d; \mathbb{Z})\) should appear in the R-R sector (see, for example, [12]). However, as was discussed in detail for type IIA with \(d = 3\) in [13, 14], the R-R potentials themselves do not give Majorana-Weyl spinors directly. Instead, one needs to combine them with the NS-NS 2-form to get new fields that have such simple transformation properties under \(SO(d, d; \mathbb{Z})\). Although prescription on how to arrange these fields was known for each \(d\) by starting from 11-dimensional supergravity [15], it is rather complicated due to the manner of field redefinitions which strongly depends on the dimensionality. The main aim of this article is to present the prescription of constructing the new fields and to demonstrate the T-duality invariance of the R-R sector with the Chern-Simons term in a simple form. We give a discussion by investigating solely the structure of the effective action of type IIA/IIB strings with all fermionic fields set zero. Inclusion of fermions with analysis of supersymmetry will be discussed elsewhere. This work is inspired by analysis made by Brace, Morariu and Zumino [13, 14].

\footnote{Each of type IIA and type IIB is only invariant under the subgroup \(SO(d, d; \mathbb{Z})\) of \(O(d, d; \mathbb{Z})\), as we will see later.}
The main result can be summarized as follows. First, we introduce new potentials
\[ D_{p+1} = \frac{1}{(p+1)!} D_{\mu_1...\mu_{p+1}} d\xi^\mu_1 \wedge ... \wedge d\xi^\mu_{p+1} \] \( (\hat{\mu}_1, ..., \hat{\mu}_{p+1} = 0, 1, ..., 9) \) which are mixtures of R-R potentials and the NS-NS 2-form as
\[ D_0 \equiv C_0, \quad D_1 \equiv C_1, \]
\[ D_2 \equiv C_2 + \hat{B}_2 \wedge C_0, \quad D_3 \equiv C_3 + \hat{B}_2 \wedge C_1, \]
\[ D_4 \equiv C_4 + \frac{1}{2} \hat{B}_2 \wedge C_2 + \frac{1}{2} \hat{B}_2 \wedge \hat{B}_2 \wedge C_0, \]
where \( C_{p+1} \) is the original \( (p + 1) \)-form R-R potential and \( \hat{B}_2 \) is the NS-NS 2-form in 10 dimensions. We further introduce potentials of higher degree, \( D_{p+1} \) \( (p+1 = 5, ..., 8) \), as their electromagnetic duals. More precisely, we introduce the sum of field strengths
\[ F \equiv e^{-\hat{B}_2} \wedge \sum_{p+1=0}^{8} dD_{p+1} = \sum_{p+2=1}^{9} F_{p+2}, \]
and require the following relations in their equations of motion:
\[ *F_1 = F_9, \quad *F_2 = -F_8, \]
\[ *F_3 = -F_7, \quad *F_4 = F_6, \]
\[ *F_5 = F_5, \quad *F_6 = -F_4, \]
\[ *F_7 = -F_3, \quad *F_8 = F_2, \]
\[ *F_0 = F_1. \]
Note that \( *^2 F_n = (-1)^{n+1} F_n \) in 10-dimensional Minkowski space. The existence of these fields, \( D_5, ..., D_8 \), is allowed by the equations of motion for \( D_0, ..., D_4 \).

Our first claim is that, as far as the equations of motion are concerned, the R-R action with the Chern-Simons term can be rewritten into the following simple form:
\[ S_{R+CS}^{(IIA)} = \frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-g} \sum_{p+2=3, 5, 7, 9} F_{p+2} \wedge *F_{p+2} \]
\[ S_{R+CS}^{(IIB)} = \frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-g} \sum_{p+2=2, 4, 6, 8} F_{p+2} \wedge *F_{p+2} \]
with all the \( D_0, ..., D_8 \) being regarded as independent variables and with \([1.6]\) being the constraints to be imposed after the equations of motion are derived.

Second, for \( d \)-dimensional toroidal compactification, we assemble the set of KK scalars
into the form \((D_\alpha)\) with \(2^{d-1}\) entries:

\[
\begin{align*}
\text{IIA:} & \\
d = 1: & (D_\alpha) = (D_1) \\
d = 2: & (D_\alpha) = (D_1, D_2) \\
d = 3: & (D_\alpha) = (D_1, D_2, D_3, D_{123}) \\
d = 4: & (D_\alpha) = (D_1, D_2, D_3, D_4, D_{123}, D_{124}, D_{134}, D_{234}) \\
& \vdots
\end{align*}
\]

\[
\begin{align*}
\text{IIB:} & \\
d = 1: & (D_\alpha) = (D) \\
d = 2: & (D_\alpha) = (D, D_{12}) \\
d = 3: & (D_\alpha) = (D, D_{12}, D_{13}, D_{23}) \\
d = 4: & (D_\alpha) = (D, D_{12}, D_{13}, D_{14}, D_{23}, D_{24}, D_{34}, D_{1234}) \\
& \vdots
\end{align*}
\]

where \(D_\alpha = D_{i_1 \ldots i_{p+1}}\) is the component of \(D_{p+1}\) in the compact directions \(y^{i_1}, \ldots, y^{i_{p+1}}\) \((1 \leq i_1 < \cdots < i_{p+1} \leq d)\). Similarly, we also assemble the set of KK 1-forms \(D_{\mu i_1 \ldots i_p}\) \((\mu = 0, 1, \ldots, 9-d)\):

\[
\begin{align*}
\text{IIA:} & \\
d = 1: & (D_{\mu \alpha}) = (D_{\mu}) \\
d = 2: & (D_{\mu \alpha}) = (D_{\mu}, D_{\mu 12}) \\
d = 3: & (D_{\mu \alpha}) = (D_{\mu}, D_{\mu 12}, D_{\mu 13}, D_{\mu 23}) \quad (1.10) \\
d = 4: & (D_{\mu \alpha}) = (D_{\mu}, D_{\mu 12}, D_{\mu 13}, D_{\mu 14}, D_{\mu 23}, D_{\mu 24}, D_{\mu 34}, D_{\mu 1234}) \\
& \vdots
\end{align*}
\]

\[
\begin{align*}
\text{IIB:} & \\
d = 1: & (D_{\mu \alpha}) = (D_{\mu 1}) \\
d = 2: & (D_{\mu \alpha}) = (D_{\mu 1}, D_{\mu 2}) \\
d = 3: & (D_{\mu \alpha}) = (D_{\mu 1}, D_{\mu 2}, D_{\mu 3}, D_{\mu 123}) \quad (1.11) \\
d = 4: & (D_{\mu \alpha}) = (D_{\mu 1}, D_{\mu 2}, D_{\mu 3}, D_{\mu 4}, D_{\mu 123}, D_{\mu 124}, D_{\mu 134}, D_{\mu 234}) \\
& \vdots
\end{align*}
\]

This assembling may continue to KK forms of higher degree when \(d\) is low enough.

Our second claim is that the dimensionally-reduced action of the R-R sector with the Chern-Simons term can be rewritten for type IIA and IIB, respectively, as

\[
\begin{align*}
L_{\text{R+CS}} &= \frac{1}{4} \partial_\mu D_\alpha S^\alpha_\beta(M) \partial^\beta D_\beta + \frac{1}{16} \partial_\mu D_\nu |_\alpha S^\pm_\alpha_\beta(M) \partial^\mu D^\nu_\beta + \cdots 
\end{align*}
\]

\(^3\) Precise form is given by (1.23) - (1.27).
where $S^\pm_{\alpha\beta}(M) (\alpha, \beta = 1, ..., 2^{d-1})$ is a representation matrix of $M$ in the Majorana-Weyl representation of $SO(d,d; \mathbb{R})$ with chirality $\pm$. The invariance of the action thus now becomes apparent by assuming that both of the $D_\alpha$ and $D_{\mu \alpha}$ transform as Majorana-Weyl spinors:

\begin{align}
\overline{D}_\alpha &= S^\mp_{\alpha\beta}(\Lambda)D_\beta \\
\overline{D}_{\mu \alpha} &= S^\pm_{\alpha\beta}(\Lambda)D_{\mu \beta}.
\end{align}

We will prove the identity (1.12) for arbitrary $d$ including KK forms of arbitrary degree. We simplify the argument with the use of the fermionic oscillator construction of Majorana spinor representation given in \cite{[13], [14]}.

The present paper is organized as follows. In section 2, in order to fix our convention, we first give a brief review on the invariance of the NS-NS sector and then introduce new potentials $D_{p+1}$. In section 3, we explicitly construct the spinor representations of $O(d,d; \mathbb{Z})$ closely following \cite{[13], [14]}, and then rewrite the R-R action plus the Chern-Simons term into a manifestly T-duality invariant form in section 4. Section 5 is devoted to discussions. The existence of the fields $D_5, ..., D_8$ is proved in Appendix, with a demonstration that our new fields $D_{p+1}$ greatly simplify all the expressions including the Chern-Simons term.

## 2 Type IIA/IIB effective actions

The action of ten-dimensional type IIA/IIB supergravity in the string metric can be split into three parts \cite{[10]}:

\[ S = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}}. \]  

The first term is the action for the NS-NS sector:

\[ S_{\text{NS}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \ e^{-2\phi} \left( \tilde{R} + 4 |d\phi|^2_g - \frac{1}{2} |\tilde{H}_3|^2_g \right), \]  

where $x^{\hat{\mu}} (\hat{\mu} = 0, 1, ..., 9)$ are 10-dimensional coordinates, and $\tilde{g}_{\hat{\mu}\hat{\nu}}$, $\tilde{B}_{\hat{\mu}\hat{\nu}}$ and $\tilde{\phi}$ denote the 10-dimensional metric, NS-NS 2-form and dilaton, respectively. The NS-NS field strength is written as $\tilde{H}_3 = dB_2$ with $B_2 = (1/2)\tilde{B}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} \wedge dx^{\hat{\nu}}$. We adopt a rule that the subscript
of a form stands for its degree when it has a definite meaning in 10 dimensions. We also
often consider a sum of forms of various degree like \( \Omega = \sum_K \Omega_K = \sum_K (1/K!) \Omega_{\hat{\mu}_1 \ldots \hat{\mu}_K} dx^{\hat{\mu}_1} \wedge \cdots \wedge dx^{\hat{\mu}_K} \), and for this we introduce the invariant norm as

\[
| \Omega |^2_g \equiv \sum_K \frac{1}{K!} \hat{g}^{\hat{\mu}_1 \hat{\nu}_1} \cdots \hat{g}^{\hat{\mu}_K \hat{\nu}_K} \Omega_{\hat{\mu}_1 \ldots \hat{\mu}_K} \Omega_{\hat{\nu}_1 \ldots \hat{\nu}_K} .
\]

The action for the R-R sector, \( S_R \), can be written for IIA and IIB, respectively, as

\[
S^{\text{IIA}}_R = - \frac{1}{4 \kappa_1^2} \int d^{10}x \sqrt{-\hat{g}} \left( |F_2|_g^2 + |F_4|_g^2 \right) \]
\[
S^{\text{IIB}}_R = - \frac{1}{4 \kappa_1^2} \int d^{10}x \sqrt{-\hat{g}} \left( |F_1|_g^2 + |F_3|_g^2 + \frac{1}{2} |F_5|_g^2 \right) ,
\]

where the R-R field strengths \( F_{p+2} \) are defined from the \((p+1)\)-form R-R potentials \( C_{p+1} = (1/(p+1)!) C_{\hat{\mu}_1 \ldots \hat{\mu}_{p+1}} dx^{\hat{\mu}_1} \wedge \cdots \wedge dx^{\hat{\mu}_{p+1}} \) as

\[
F_1 = dC_0, \quad F_2 = dC_1, \\
F_3 = dC_2 + \hat{H}_3 \wedge C_0, \quad F_4 = dC_3 + \hat{H}_3 \wedge C_1, \\
F_5 = dC_4 + \frac{1}{2} \hat{H}_3 \wedge C_2 - \frac{1}{2} \hat{B}_2 \wedge dC_2 .
\]

The Chern-Simons term \( S_{CS} \) is given by

\[
S^{\text{IIA}}_{CS} = \frac{1}{4 \kappa_1^2} \int \hat{B}_2 \wedge dC_3 \wedge dC_3 \\
S^{\text{IIB}}_{CS} = \frac{1}{4 \kappa_1^2} \int \hat{B}_2 \wedge dC_4 \wedge dC_2 .
\]

We take a convention that NS-NS fields wear the hat (\( \hat{\cdot} \)) in 10 dimensions while R-R fields do not. This is because NS-NS fields generally need to be redefined after toroidal compactification in order to nicely behave as fields living on the noncompact \((10 - d)\)-dimensional space-time (see, for example, (2.7), (2.16), (2.17) and (2.20)).

After toroidal compactification on \( T^d \), there will appear various KK forms both from the NS-NS and the R-R sectors. We first review the NS-NS case, closely following [4].

**NS-NS sector:**

We parametrize the 10-dimensional metric as

\[
ds^2 \equiv \hat{g}_{\hat{\mu} \hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} = g_{\mu \nu} dx^\mu dx^\nu + G_{ij} (dy^i + A_i^\mu dx^\mu)(dy^j + A_j^\nu dx^\nu) .
\]
Here the 10-dimensional coordinates are decomposed as \((x^\hat{\mu}) = (x^\mu, y^i)\) \((\mu = 0, 1, ..., 9-d; \ i = 1, 2, ..., d)\), and we assume that all the fields depend only on the noncompact coordinates \(x^\mu\). With this parametrization, the kinetic term for potentials will take a complicated form since the KK 1-form
\[
A^{(1)}_i \equiv A^i_\mu dx^\mu
\]
will appear when contracting the indices in the compact directions. To simplify this, we follow the prescription of [9] which we found can be restated as follows. First, given a sum of forms \(\Omega = \sum_K \Omega_K\), we decompose it as
\[
\Omega = \sum_q \sum_n \frac{1}{n!} \Omega^{(q)}_{i_1...i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n},
\]
where the superscript \((q)\) indicates that \(\Omega^{(q)}_{i_1...i_n}\) is a \(q\)-form for noncompact indices:
\[
\Omega^{(q)}_{i_1...i_n} = \frac{1}{q!} \Omega_{\mu_1...\mu_q i_1...i_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}.
\]
Second, we introduce a new form \(\Omega'\) by replacing \(dy^i\) in \(\Omega\) with \(dy^i - A^{(1)}_i\), and reorganize it as in (2.9):
\[
\Omega' \equiv \Omega \bigg|_{dy^i \rightarrow dy^i - A^{(1)}_i} = \sum_q \sum_n \frac{1}{n!} \Omega^{(q)}_{i_1...i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n}.
\]
Then the kinetic term can be expressed in such a way that all the indices are contracted only with \(g^{\mu\nu}\) and \(G^{ij}\):
\[
|\Omega|^2_g = |\Omega'|^2_{g,G} \equiv \sum_q \sum_n \left| \Omega^{(q)}_{n} \right|^2_{g,G},
\]
where we have defined
\[
\left| \Omega^{(q)}_{n} \right|^2_{g,G} \equiv \frac{1}{n!} G^{i_1j_1} \cdots G^{i_nj_n} \frac{1}{q!} g^{\mu_1\nu_1} \cdots g^{\mu_q\nu_q} \Omega^{(q)}_{\mu_1...\mu_q i_1...i_n} \Omega'_{\nu_1...\nu_q j_1...j_n}.
\]
For example, the NS-NS field strength \(\tilde{H}\) is rewritten as
\[
\begin{align*}
\tilde{H}^{(1)}_{ij} &= dB^{(0)}_{ij} \\
\tilde{H}^{(2)}_i &= dB^{(1)}_i - B^{(0)}_{ij} dA^{(1)} j \\
\tilde{H}^{(3)} &= dB^{(2)} - \frac{1}{2} \left( B^{(1)}_i dA^{(1)} i + dB^{(1)}_i A^{(1)} i \right),
\end{align*}
\]
where we have introduced

\[
B_{ij}^{(0)} \equiv \hat{B}_{ij}^{(0)} \\
B_i^{(1)} \equiv \hat{B}_i^{(1)} + \hat{B}_{ij}^{(0)} A^{(1)j} \\
B^{(2)} \equiv \hat{B}^{(2)} - \frac{1}{2} \hat{B}_i^{(1)} A^{(1)i}.
\] (2.15)

Conversely, we have

\[
\hat{B}_{ij}^{(0)} \equiv B_{ij}^{(0)} \\
\hat{B}_i^{(1)} \equiv B_i^{(1)} - B_{ij}^{(0)} A^{(1)j} \\
\hat{B}^{(2)} \equiv B^{(2)} + \frac{1}{2} B_i^{(1)} A^{(1)i} + \frac{1}{2} B_{ij}^{(0)} A^{(1)i} A^{(1)j},
\] (2.16)

which give the original \(\tilde{B}_2\) as

\[
\tilde{B}_2 = \frac{1}{2} \hat{B}_{ij}^{(0)} dy^i \wedge dy^j + \hat{B}_i^{(1)} dy^i + \hat{B}^{(2)}
\] (2.17)

\[
= \frac{1}{2} B_{ij}^{(0)} (dy^i + A^{(1)i})(dy^j + A^{(1)j}) + B_i^{(1)} (dy^i + A^{(1)i}) + B^{(2)} - \frac{1}{2} B_i^{(1)} A^{(1)i}.
\]

Then the NS-NS part of the action will be rewritten [9] as

\[
S_{NS} = \frac{1}{2\kappa_{10-d}^2} \int d^{10-d}x \sqrt{-g} L_{NS},
\] (2.18)

where

\[
\frac{1}{2\kappa_{10-d}^2} = \frac{1}{2\kappa_{10}^2} \int d^d y.
\] (2.19)

By introducing the \((10 - d)\)-dimensional dilaton \(\phi\) as

\[
e^{-2\phi} \equiv e^{-2\hat{\phi}} \sqrt{G},
\] (2.20)

the factor \(L_{NS} = L_1 + L_2 + L_3 + L_4\) is given by

\[
L_1 = e^{-2\phi} [R + 4 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi] \\
L_2 = \frac{1}{8} e^{-2\phi} g^{\mu\nu} \text{tr} \left( \partial_\mu M^{-1} \partial_\nu M \right) \\
L_3 = -\frac{1}{4} e^{-2\phi} g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} F_{\mu_1 \mu_2}^{r r} M_{rs} F_{\nu_1 \nu_2}^{s s} \\
L_4 = -\frac{1}{12} e^{-2\phi} g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} g^{\mu_3 \nu_3} \tilde{H}^{t \mu_1 \mu_2 \mu_3} \tilde{H}_{t \nu_1 \nu_2 \nu_3},
\] (2.21)
where

\[
M = (M_{rs}) \equiv \begin{pmatrix} G^{-1} & -G^{-1}B^{(0)} \\ B^{(0)}G^{-1} & G - B^{(0)}G^{-1}B^{(0)} \end{pmatrix}, \quad (B^{(0)} \equiv (B_{ij}^{(0)})),
\]

\[
\frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \equiv \begin{pmatrix} dB_{i}^{(1)} \\ dA_{(1) i} \end{pmatrix} \quad (r, s = 1, \ldots, 2d; \, i, j = 1, \ldots, d).
\]

This form of action makes manifest its invariance under the T-duality group \(O(d, d; \mathbb{Z})\) provided that the fields transform as

\[
\Lambda^{-1} T \cdot M \cdot \Lambda^{-1}, \quad \begin{pmatrix} B_{i}^{(1)} \\ A_{(1) i} \end{pmatrix} = \Lambda \begin{pmatrix} B_{i}^{(1)} \\ A_{(1) i} \end{pmatrix}, \quad B^{(2)} = B^{(2)},
\]

for \( \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d, d; \mathbb{Z}) \) satisfying \( \Lambda^T J \Lambda = J \) with \( J = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \). The first transformation rule is equivalent to \( E = (aE + b)(cE + d)^{-1} \) for \( E_{ij} = G_{ij} + B_{ij}^{(0)} \). 

**R-R sector with the Chern-Simons term:**

The R-R potentials \( C_{p+1} = (1/(p+1)! \) \( C_{\hat{\mu}_1 \ldots \hat{\mu}_{p+1}} \, dx^{\hat{\mu}_1} \wedge \ldots \wedge dx^{\hat{\mu}_{p+1}} \) also produce KK forms of various degree after toroidal compactification. To simplify all the expressions, we first combine the R-R potentials with the NS-NS 2-form in 10 dimensions as follows:

\[
D_0 \equiv C_0, \quad D_1 \equiv C_1, \\
D_2 \equiv C_2 + \hat{B}_2 \wedge C_0, \quad D_3 \equiv C_3 + \hat{B}_2 \wedge C_1, \\
D_4 \equiv C_4 + \frac{1}{2} \hat{B}_2 \wedge C_2 + \frac{1}{2} \hat{B}_2 \wedge \hat{B}_2 \wedge C_0.
\]

The R-R field strengths are then expressed with these \( D_{p+1} \) as

\[
F_1 = dD_0, \quad F_2 = dD_1, \\
F_3 = dD_2 - \hat{B}_2 \wedge dD_0, \quad F_4 = dD_3 - \hat{B}_2 \wedge dD_1, \\
F_5 = dD_4 - \hat{B}_2 \wedge dD_2 + \frac{1}{2} \hat{B}_2 \wedge \hat{B}_2 \wedge dD_0.
\]

These can be written in a simple form

\[
F = e^{-\hat{B}_2} \wedge dD
\]

if we introduce

\[
D \equiv \sum_{p+1=0}^{4} D_{p+1}, \quad F \equiv \sum_{p+2=1}^{5} F_{p+2}.
\]

\(^4\) For the type IIA, the potentials \( D_1 \) and \( D_3 \) can be found in [14].
The equations of motion for $D_0, \ldots, D_4$ turn out to allow us to introduce extra R-R potentials of higher degree, $D_{p+1}$ $(p+1 = 5, \ldots, 8)$, that preserve the relation (2.26) with

$$D \equiv \sum_{p+1=0}^{8} D_{p+1}, \quad F \equiv \sum_{p+2=1}^{9} F_{p+2},$$

if we introduce the following identification for the field strengths of higher degree:

$${*F}_1 = F_9, \quad {*F}_2 = -F_8,$$

$${*F}_3 = -F_7, \quad {*F}_4 = F_6,$$

$${*F}_5 = F_5, \quad {*F}_6 = -F_4,$$

$${*F}_7 = -F_3, \quad {*F}_8 = F_2,$$

$${*F}_9 = F_1$$

(see Appendix). Interestingly, as far as the equations of motion are concerned, we can in turn regard all the R-R potentials, $D_0, \ldots, D_8$, as independent variables and choose

$$S_{(IIA)}^{R+CS} \equiv -\frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-g} \sum_{p+2=2, 4, 6, 8} |F_{p+2}|_g^2$$

$$S_{(IIB)}^{R+CS} \equiv -\frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-g} \sum_{p+2=1, 3, 5, 7, 9} |F_{p+2}|_g^2$$

as their action functional, with the understanding that the constraints (2.29) are imposed after (and only after) the equations of motion are derived. In fact, one can prove that this system gives the same equations of motion with those from the sum of R-R and Chern-Simons terms $S_R + S_{CS}$, (2.4)–(2.6). We give a proof of this statement in Appendix.

For $d$-dimensional toroidal compactification, we introduce the primed field for $F$ as

$$F' \equiv F|_{dy^{i} \to dy^{i} - A^{(1)i}}.$$  

Then the action for the R-R and Chern-Simons sector will be expressed as

$$S_{R+CS} = \frac{1}{2\kappa_{10-d}^2} \int d^{10-d}x \sqrt{-g} \mathcal{L}_{R+CS}$$

with

$$\mathcal{L}_{R+CS} = -\frac{1}{4} \sqrt{G} |F'|^2_{g,G}.$$  

To show that $\mathcal{L}_{R+CS}$ is invariant under $O(d, d; \mathbb{Z})$ when the set of KK fields coming from $D$ transforms as a Majorana spinor of $O(d, d; \mathbb{Z})$, in the next section we explicitly
construct the spinor representation of $O(d, d; \mathbb{R})$ by using fermionic operators. We mostly follow the convention of [13, 14].

Before concluding this section, we would like to make a comment on the potentials $D_1$ and $D_3$ in the type IIA case. It is well known that the type IIA supergravity can be obtained from the 11-dimensional supergravity [5] by dimensional reduction. A coordinate transformation along the 11-th direction $x^{10} \rightarrow x^{10} + \xi$ becomes a $U(1)$ symmetry in 10-dimensions:

$$\delta \hat{B}_2 = 0, \quad \delta C_1 = d\xi, \quad \delta C_3 = -\hat{B}_2 \wedge d\xi.$$  \hspace{1cm} (2.34)

Thus, these $D$ fields diagonalize the $U(1)$ symmetry: $D_1 \rightarrow D_1 + d\xi$, $D_3 \rightarrow D_3$. These are 10-dimensional analogues of $A'$ fields of [3].

3 Spinor representation of $O(d, d; \mathbb{R})$

We first recall that the group $O(d, d; \mathbb{R})$ consists of $2d \times 2d$ matrices $\Lambda$ satisfying

$$\Lambda^T J \Lambda = J, \quad J = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix},$$  \hspace{1cm} (3.1)

The group $O(d, d; \mathbb{Z})$ is defined as a subgroup that consists of matrices with integer-valued elements. It is known that both are generated by the following three types of matrices [17]:

$$\Lambda_B = \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}, \quad B^T = -B, \quad \text{or} \quad B \in GL(d; \mathbb{R}) \text{ or } GL(d; \mathbb{Z}),$$  \hspace{1cm} (3.2)

$$\Lambda_R = \begin{pmatrix} R^{-1} & 0 \\ 0 & R^T \end{pmatrix}, \quad R \in GL(d; \mathbb{R}) \text{ or } GL(d; \mathbb{Z}),$$  \hspace{1cm} (3.3)

$$\Lambda_i = -\begin{pmatrix} 1 - e_i & -e_i \\ -e_i & 1 - e_i \end{pmatrix}, \quad (e_i)_{jk} = \delta_{ij}\delta_{ik}, \quad (i = 1, \ldots, d).$$  \hspace{1cm} (3.4)

Note that $\det \Lambda_B = \det \Lambda_R = +1$ and $\det \Lambda_i = -1$. Thus one can construct a subgroup $SO(d, d; \mathbb{R})$ or $SO(d, d; \mathbb{Z})$ as such that are generated by $\Lambda_B$, $\Lambda_R$ and $\Lambda_i \Lambda_j$.

The Dirac matrices $\Gamma_r = (\Gamma_{r\alpha\beta})$ with $2^d \times 2^d$ components are introduced as

$$\{\Gamma_r, \Gamma_s\} = 2J_{rs}, \quad (r, s = 1, \ldots, 2d),$$  \hspace{1cm} (3.5)
and the spinor representation \( S(\Lambda) = (S_{\alpha\beta}(\Lambda)) \) is characterized by the property
\[
S(\Lambda) \cdot \Gamma_s \cdot S(\Lambda)^{-1} = \sum_r \Gamma_r \Lambda^r_s.
\] (3.6)

To construct this representation, we introduce fermionic operators \( \psi^\dagger_i \) and \( \psi_i \) with the anti-commutation relations
\[
\{\psi_i, \psi^{\dagger}_j\} = \delta^i_j \mathbf{1}, \quad \{\psi_i, \psi_j\} = 0 = \{\psi^{\dagger}_i, \psi^{\dagger}_j\} \quad (i, j = 1, ..., d). \tag{3.7}
\]

We define the hermitian conjugation as
\[
(\psi^\dagger_i) = \psi^{\dagger}_i, \tag{3.8}
\]
and introduce the vacuum \( |0\rangle \) such that \( \psi_i |0\rangle = 0 \quad (i = 1, ..., d) \) and \( \langle 0 | 0 \rangle = 1 \). Then the \( 2^d \)-dimensional fermion Fock space is spanned by the vectors
\[
|\alpha\rangle = \psi^{i_1\dagger} \cdots \psi^{i_n\dagger} |0\rangle \quad (n = 0, ..., d), \tag{3.9}
\]
where \( \alpha \) is a multi-index \( \alpha = (i_1, ..., i_n) \quad (i_1 < \cdots < i_n) \), and the Dirac matrices can be introduced with respect to this as follows:
\[
\psi^{\dagger}_i |\beta\rangle = \sum_\alpha |\alpha\rangle \frac{1}{\sqrt{2}} (\Gamma_i)_{\alpha\beta},
\]
\[
\psi_i |\beta\rangle = \sum_\alpha |\alpha\rangle \frac{1}{\sqrt{2}} (\Gamma_{d+i})_{\alpha\beta}. \tag{3.10}
\]

Thus, if we can always introduce an operator \( \Lambda \) to any element
\[
\Lambda = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in O(d, d; \mathbb{R}) \tag{3.11}
\]
such that
\[
(\Lambda \psi^{\dagger}_i \Lambda^{-1}, \Lambda \psi_j \Lambda^{-1}) = (\psi^{\dagger}_i a_{ij} + \psi_{ij} c, \psi^{\dagger}_i b_{ij} + \psi d_{ij}) = (\psi^{\dagger}_i, \psi_i) \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.12}
\]
then, introducing the matrix \( S_{\alpha\beta}(\Lambda) \) by \( \Lambda |\beta\rangle = \sum_\alpha |\alpha\rangle S_{\alpha\beta}(\Lambda) \), we can establish the relation (3.6). For this, it is enough to construct the operators that correspond to the elements given in (3.2)–(3.4), and it is easy to show that the followings are solutions [13, 14]:
\[
\Lambda_B = e^{-B} \equiv \exp \left( -\frac{1}{2} B_{ij} \psi^{\dagger}_i \psi^{\dagger}_j \right),
\]
\[
\Lambda_R = (\det R)^{1/2} \exp \left( -\psi^{\dagger}_i A_{ij} \psi_j \right) \quad \left( R = (R_i^j) = \exp \left( A_i^j \right) \right),
\]
\[
\Lambda_i = \psi_i + \psi^{\dagger}_i \quad (i = 1, ..., d). \tag{3.13}
\]
Notice that all of these operators give real-valued matrix elements, so that the resulting representation is automatically Majorana. Note also that the $\Lambda_i$’s do not give a faithful representation so that there are always ambiguities in their orderings.

In order to construct Weyl representations, we define a matrix

$$\Gamma_{2d+1} \equiv \frac{1}{2^d} \prod_{i=1}^{d} (\Gamma_i + \Gamma_{d+i})(\Gamma_i - \Gamma_{d+i}) \quad (3.14)$$

which satisfies $\{\Gamma_{2d+1}, \Gamma_r\} = 0 \quad (r = 1, \ldots, 2d)$. By looking at the correspondence (3.10), one can easily see that $\Gamma_{2d+1}$ corresponds to $(-1)^{N_F}$ with $N_F = \sum_i \psi_i^\dagger \psi_i$. Thus, the projection to the subspace with $(-1)^{N_F} = 1$ leads to a Majorana-Weyl representation $(2^d-1)_s$ and the other one with $(-1)^{N_F} = -1$ to $(2^d-1)_c$. Note that $\Lambda_i$ is a linear function of fermions and thus changes the chirality. Therefore, in order for an operator to preserve the chirality it must correspond to an element in $SO(d, d; \mathbb{R})$.

We further introduce an operator $J$ that corresponds to $J = (J_{rs})$:

$$J = i^{d(d-1)/2} \Lambda_1 \cdots \Lambda_d, \quad (3.15)$$

where the phase factor is chosen such that $J^2 = 1$. One can actually prove that

$$J \psi_i^\dagger J = \psi_i, \quad J \psi_i J = \psi_i^\dagger. \quad (3.16)$$

It is easy to check that for all these $\Lambda$’s in (3.13) (and thus for all elements in $O(d, d; \mathbb{R})$), their transposes $\Lambda^T = J \cdot \Lambda^{-1} \cdot J$ are mapped to $\Lambda^\dagger$:

$$\Lambda^\dagger = J \Lambda^{-1} J. \quad (3.17)$$

In particular, we have

$$\Lambda^\dagger_B = e^{-B^T} = \exp \left( \frac{1}{2} B_{ij} \psi_i \psi_j \right), \quad \Lambda^T_B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \quad (3.18)$$

Note also that the normalization of the operators (3.13) is correctly chosen such that they satisfy the condition (3.17).

We finally make a comment that this operator $J$ is essentially the charge conjugation operator. In fact, the operators defined by

$$C^\pm \equiv \Lambda_1^\pm \cdots \Lambda_d^\pm \quad (3.19)$$
with

$$\Lambda_i^\pm \equiv \psi_i^\dagger \pm \psi_i$$

(3.20)

can be easily seen to satisfy

$$C^\pm (C^\pm)^\dagger = 1, \quad (C^\pm)^2 = (-1)^{d(d+1)/2} 1,$$

(3.21)

$$C^\pm \psi_i^\dagger (C^\pm)^{-1} = \mp (-1)^d \psi_i, \quad C^\pm \psi_i (C^\pm)^{-1} = \mp (-1)^d \psi_i^\dagger.$$

This implies that the matrices $C^\pm = (C^\pm_{\alpha\beta})$ defined by $C^\pm |\beta\rangle = |\alpha\rangle C^\pm_{\alpha\beta}$ satisfy the condition for the charge conjugation of $SO(d, d)$ [18]:

$$C^\pm (C^\pm)^\dagger = 1, \quad (C^\pm)^T = (-1)^{d(d+1)/2} C^\pm,$$

(3.22)

$$C^\pm \Gamma_r (C^\pm)^{-1} = \mp (-1)^d (\Gamma_r)^T.$$

4 R-R potentials and T-duality

In this section, we show that the R-R action plus the Chern-Simons term after toroidal compactification on $T^d$, (2.32)–(2.33), is actually invariant under $SO(d, d; \mathbb{Z})$ if a set of our R-R fields transform as a Majorana-Weyl spinor.

We first introduce a one-to-one correspondence between the set of forms and the space of creation operators by replacing the differential in the compact direction $dy^i$ with the fermion creation operator $\psi_i^\dagger$ as

$$\Omega = \sum_{n} \frac{1}{n!} \Omega_{i_1...i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n} = \sum_q \sum_{n} \frac{1}{n!} \Omega^{(q)}_{i_1...i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n}$$

(4.1)

to

$$\Omega \equiv \sum_{n} \frac{1}{n!} \Omega_{i_1...i_n} \psi_{i_1}^\dagger \cdots \psi_{i_n}^\dagger = \sum_q \sum_{n} \frac{1}{n!} \Omega^{(q)}_{i_1...i_n} \psi_{i_1}^\dagger \cdots \psi_{i_n}^\dagger.$$

(4.2)

This actually gives an isomorphism as algebra. We also extend our rule such that $\Omega^{(q)}_{i_1...i_n}$ has $N_F = q$, and thus it will anticommute with all the fermionic operators when $q$ is odd. We define a state corresponding to $\Omega$ as

$$|\Omega\rangle \equiv |\Omega| 0\rangle.$$

(4.3)

Recall that the superscript $(q)$ indicates that $\Omega^{(q)}_{i_1...i_n}$ is a $q$-form for noncompact indices (see [2.10]).
Note that the following holds for any two forms $\Omega$ and $\Xi$:

$$\Omega | \Xi \rangle = | \Omega \wedge \Xi \rangle.$$  \hspace{1cm} (4.4)

Now that we have the above isomorphism, we can introduce the operator corresponding to $F$ in (2.26):

$$F = e^{-\hat{B}_2} dD. \hspace{1cm} (4.5)$$

Since $F_{p+2}$ are even (odd) forms for type IIA (IIB), we have $(-1)^{N_p} | F \rangle = + | F \rangle$ for type IIA and $= - | F \rangle$ for type IIB. This implies that each state has a definite chirality and thus forms a Majorana-Weyl representation of $SO(d, d; \mathbb{Z})$. Noticing that the replacement $dy^i \rightarrow dy^i - A^{(1)} i$ as in (2.31) is equivalent to the operation

$$\psi_i \rightarrow e^{\psi_i A^{(1)} i} \psi_i e^{-\psi_i A^{(1)} i} = \psi_i - A^{(1)} i,$$  \hspace{1cm} (4.6)

we can simply express the operator corresponding to $F'$ as

$$F' = e^{\psi_i A^{(1)} i} F e^{-\psi_i A^{(1)} i}, \hspace{1cm} (4.7)$$

and thus the corresponding state can be written as

$$| F' \rangle = F' | 0 \rangle = e^{\psi_i A^{(1)} i} F | 0 \rangle$$

$$= e^{\psi_i A^{(1)} i} e^{-\hat{B}_2} | dD \rangle = e^{\psi_i A^{(1)} i} e^{-\hat{B}_2} e^{-\psi_i A^{(1)} i} \cdot e^{\psi_i A^{(1)} i} | dD \rangle. \hspace{1cm} (4.8)$$

Here one can use (2.17) to show that

$$e^{\psi_i A^{(1)} i} \hat{B}_2 e^{-\psi_i A^{(1)} i} = \frac{1}{2} B^{(1)}_{ij} \psi^i \psi^j + B^{(1)}_i \psi^i + B^{(2)} - \frac{1}{2} B^{(1)}_i A^{(1)} i. \hspace{1cm} (4.9)$$

Therefore we have

$$| F' \rangle = e^{-B^{(0)}} e^{-B^{(2)}} e^{(1/2)B^{(1)}_i A^{(1)} i} e^{\psi^i B^{(1)}_i} e^{\psi_i A^{(1)} i} | dD \rangle$$

$$= e^{-B^{(0)}} e^{-B^{(2)}} e^V | dD \rangle, \hspace{1cm} (4.10)$$

where

$$B^{(0)} \equiv \frac{1}{2} B^{(0)}_{ij} \psi^i \psi^j$$

$$V \equiv \psi^i B^{(1)}_i + \psi_i A^{(1)} i. \hspace{1cm} (4.11)$$
Since \((B_i^{(1)}, A_i^{(1)})^T\) transforms as a vector of \(O(d, d; Z)\), one can see that \(V\) transforms as

\[
\nabla = \Lambda V \Lambda^{-1}
\]

(4.12)

for \(\Lambda \in SO(d, d; Z)\). In fact,

\[
\nabla = (\psi^\dagger, \psi) \begin{pmatrix} B^{(1)} \\ A^{(1)} \end{pmatrix} = (\psi^\dagger, \psi) \Lambda \begin{pmatrix} B^{(1)} \\ A^{(1)} \end{pmatrix} \\
= \Lambda (\psi^\dagger, \psi) \Lambda^{-1} \begin{pmatrix} B^{(1)} \\ A^{(1)} \end{pmatrix} = \Lambda V \Lambda^{-1}.
\]

(4.13)

On the other hand, if we make a block-wise Gauss decomposition of \(M\) as

\[
M = \begin{pmatrix} 1_d & 0 \\ B^{(0)} & 1_d \end{pmatrix} \cdot \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \cdot \begin{pmatrix} 1_d & -B^{(0)} \\ 0 & 1_d \end{pmatrix}
\]

\[
= \Lambda_{B^{(0)}}^T \Lambda G \Lambda_{B^{(0)}},
\]

(4.14)

then the corresponding operator \(M\) will be written as

\[
M = e^{-B^{(0)\dagger}} \Lambda_G e^{-B^{(0)}}
\]

(4.15)

with

\[
\Lambda_G \equiv \sqrt{G} e^{-\psi^\dagger h_i^j \psi_j} \quad \left( (G_{ij}) = e^{(h_i^j)} \right).
\]

(4.16)

This operator \(\Lambda_G\) has a special property. In fact, suppose that for a given state

\[
|\Omega\rangle = \sum_q \sum_n \frac{1}{n!} \Omega^{(q)}_{i_1\ldots i_n} \psi_i^{i_1\dagger} \cdots \psi_i^{i_n\dagger} |0\rangle,
\]

(4.17)

we introduce its hermitian conjugate as

\[
\langle \Omega | = \sum_q \sum_n \frac{1}{n!} \langle 0 | \psi_i^{i_n} \cdots \psi_i^{i_1} \ast_{10-d} \Omega^{(q)}_{i_1\ldots i_n}.
\]

(4.18)

where \(\ast_{10-d}\) is the Hodge-star in the noncompact \((10-d)\) dimensions. Then the following identity holds:

\[
d^{10-d}x \sqrt{-g} \sqrt{G} |\Omega|_{g,G}^2 = - \langle \Omega | \Lambda_G | \Omega \rangle.
\]

(4.19)
In fact, using
\[ \Lambda_G | \Omega \rangle = \sum_q \sum_n \frac{\sqrt{G}}{n!} (e^{-h})_{i_1}^{j_1} \cdots (e^{-h})_{i_n}^{j_n} \Omega^{(q)}_{j_1 \cdots j_n} \psi^{i_1 \dagger} \cdots \psi^{i_n \dagger} | 0 \rangle \]
\[ = \sum_q \sum_n \frac{\sqrt{G}}{n!} G_{i_1 j_1} \cdots G_{i_n j_n} \Omega^{(q)}_{j_1 \cdots j_n} \psi^{i_1 \dagger} \cdots \psi^{i_n \dagger} | 0 \rangle, \]  
(4.20)
we can show
\[ \langle \Omega | \Lambda_G | \Omega \rangle = \sum_q \sum_n \frac{\sqrt{G}}{n!} (\ast_{10-d} \Omega)_{i_1 \cdots i_n} \wedge \Omega^{(q)}_{j_1 \cdots j_n} G_{i_1 j_1} \cdots G_{i_n j_n} \]  
\[ = - d^{10-d} x \sqrt{-g} \sqrt{G} | \Omega |_{g,G}^2. \]  
(4.21)
Setting \( \Omega = F' \) in (4.19), we have
\[ d^{10-d} x \sqrt{-g} \sqrt{G} | F' |_{g,G}^2 = - \langle F' | \Lambda_G | F' \rangle. \]  
(4.22)
Since this \( F' \) is written as in (4.10), the R-R action with the Chern-Simons term is expressed as
\[ S_{R+CS} = - \frac{1}{8 \kappa_{10-d}^2} \int d^{10-d} x \sqrt{-g} \sqrt{G} | F' |_{g,G}^2 \]
\[ = \frac{1}{8 \kappa_{10-d}^2} \int_{10-d} \langle F' | \Lambda_G | F' \rangle \]  
\[ = \frac{1}{8 \kappa_{10-d}^2} \int_{10-d} \langle K | M | K \rangle \]  
(4.23)
with
\[ | K \rangle = \exp \left( -B^{(2)} \right) \exp \left( V \right) | dD \rangle. \]  
(4.24)
This can also be written as
\[ S_{R+CS} = \frac{1}{8 \kappa_{10-d}^2} \int_{10-d} \mathcal{S}_{\alpha \beta} (M) K_{\alpha} \wedge \ast_{10-d} K_{\beta}; \]  
(4.25)
where \( K_{\alpha} \) is a sum of forms in noncompact directions:
\[ K_{\alpha} = e^{-B^{(2)}} \wedge \left( e^{(1/\sqrt{2}) \Gamma_{\alpha} V^{\alpha}} \right)_{\alpha \beta} \wedge dD_{\beta} \]  
(4.26)
with
\[ B^{(2)} = \frac{1}{2} B_{\mu \nu} dx^\mu \wedge dx^\nu \quad \text{(see (2.15))} \]
\[ V^\alpha = \begin{pmatrix} B_{\mu z} dx^\mu \\ A_{i} dx^\mu \end{pmatrix} \]  
(4.27)
\[ D_{\alpha} = \sum_q \frac{1}{q!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} D_{\mu_1 \cdots \mu_q \alpha}. \]
Since $M$, $V$ and $B^{(2)}$ transform as

$$\overline{M} = (\Lambda^{-1})^\dagger M \Lambda^{-1}, \quad \overline{V} = \Lambda V \Lambda^{-1}, \quad \overline{B^{(2)}} = B^{(2)},$$

we see that the action is invariant under the whole T-duality group $SO(d, d; \mathbb{Z})$ if the $D = (D_\alpha)$ transforms as a Majorana-Weyl spinor:

$$|\mathcal{D}\rangle = \Lambda |D\rangle.$$

Furthermore, if we expand $D$ with respect to noncompact indices as

$$D = \sum_{q} \frac{1}{q!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} D_{\mu_1...\mu_q}$$

with

$$D_{\mu_1...\mu_q} = \sum_{n} \frac{1}{n!} D_{\mu_1...\mu_q i_1...i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n},$$

then each coefficient $D_{\mu_1...\mu_q}$ will also transform as a Majorana spinor

$$|\mathcal{D}_{\mu_1...\mu_q}\rangle = \Lambda |D_{\mu_1...\mu_q}\rangle,$$

or equivalently,

$$\mathcal{D}_{\mu_1...\mu_q} = \sum_\beta S_{\alpha\beta}(\Lambda) D_{\mu_1...\mu_q \beta}.$$

with multi-indices $\alpha = (i_1, ..., i_n)$ ($i_1 < \cdots < i_n$; $n = 0, ..., d$). Since $D_{\mu_1...\mu_q i_1...i_n}$ vanishes if $q + n =$ even (odd) for type IIA (IIB), it has a definite chirality. This implies that $D_{\mu_1...\mu_q} = (D_{\mu_1...\mu_q \alpha})$ transforms as a Majorana-Weyl spinor for each set of noncompact indices $(\mu_1, ..., \mu_q)$.

5 Discussion

In the present article, we have given a simple proof that if the R-R potentials $C_{p+1}$ are combined with the NS-NS 2-form as in (2.24), then their KK forms transform as Majorana-Weyl spinors under the T-duality group $SO(d, d; \mathbb{Z})$ in order to make the action invariant.

\footnote{To be more precise, the following discussion holds only when $q + d \leq 10$.}
There should be various applications once transformation rules are obtained explicitly for the whole T-duality group. One will be to establish relations among various classical solutions of type IIA/IIB supergravities by using the full T-duality group together with the S-duality of type IIB. The work in this direction is in progress and will be reported elsewhere [19].

We finally make a comment on the dilaton dependence in R-R potentials, assuming the case $\hat{B}_2 = 0$ in which there is no distinction between the original R-R potential $C_{p+1}$ and our potential $D_{p+1}$. Usually we expect that another field strength defined by $\tilde{F}_{p+2} = e^{\hat{\phi}} dC_{p+1}$ corresponds to an R-R vertex operator of NSR strings in a flat background. To see this in our formulation, we first recall that we have introduced the $(10 - d)$-dimensional dilaton $\phi$ as a singlet of $O(d, d; \mathbb{Z})$. This implies that the 10-dimensional dilaton $\hat{\phi}$ should transform as $e^{\hat{\phi}} \propto G^{1/4}$. On the other hand, we could have further decomposed the operator $\Lambda_G$ as $\Lambda_G = \Lambda_E^\dagger \Lambda_E$ where $E = (E_{ia}) (i, a = 1, \ldots, d)$ is a vielbein for $G$, $G = E E^T$. Then one might say that the state $\Lambda_E |dC\rangle$ corresponds to an R-R vertex in a flat background. Thus, noticing that the operator $\Lambda_E$ will carry the factor $G^{1/4}$, we expect that $e^{\hat{\phi}} dC$ will transform as in a flat case.

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**Note added**

After the first version of the present paper was put on the bulletin board, some related works appeared [20, 21], which also investigate the T-duality transformation of R-R fields from a different point of view.
Appendix: “Self-dual” formulation of type II effective actions

In this appendix, we prove that the original R-R action plus the Chern-Simon term, (2.4)–(2.6):

\[
S_{\text{R}}^{(\text{IIA})} + S_{\text{CS}}^{(\text{IIA})} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left( |F_2|^2_g + |F_4|^2_g \right) \\
+ \frac{1}{4\kappa_{10}^2} \int \hat{B}_2 \wedge dC_3 \wedge dC_3
\]

(A.1)

\[
S_{\text{R}}^{(\text{IIB})} + S_{\text{CS}}^{(\text{IIB})} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left( |F_1|^2_g + |F_3|^2_g + \frac{1}{2} |F_5|^2_g \right) \\
+ \frac{1}{4\kappa_{10}^2} \int \hat{B}_2 \wedge dC_4 \wedge dC_2
\]

with \( C_1, C_3 \) (or \( D_1, D_3 \)) and \( C_0, C_2, C_4 \) (or \( D_0, D_2, D_4 \)) being independent variables, respectively, is equivalent to the new action (2.30):

\[
S_{\text{R+CS}}^{(\text{IIA})} \equiv \frac{1}{8\kappa_{10}^2} \int \sum_{p+2=2,4,6,8} F_{p+2} \wedge \ast F_{p+2} = -\frac{1}{8\kappa_{10}^2} \int \sum_{p+2=2,4,6,8} |F_{p+2}|_g^2
\]

(A.2)

\[
S_{\text{R+CS}}^{(\text{IIB})} \equiv \frac{1}{8\kappa_{10}^2} \int \sum_{p+2=1,3,5,7,9} F_{p+2} \wedge \ast F_{p+2} = -\frac{1}{8\kappa_{10}^2} \int \sum_{p+2=1,3,5,7,9} |F_{p+2}|_g^2
\]

with \( D_1, D_3, D_5, D_7 \) and \( D_0, D_2, D_4, D_6, D_8 \) being independent variables, in the sense that both give the same equations of motion when the constraints (2.29):

\[
\ast F_1 = F_9, \quad \ast F_2 = -F_8, \\
\ast F_3 = -F_7, \quad \ast F_4 = F_6, \\
\ast F_5 = F_5, \quad \ast F_6 = -F_4, \\
\ast F_7 = -F_3, \quad \ast F_8 = F_2, \\
\ast F_0 = F_1
\]

(A.3)

are imposed on the extra variables, \( D_5, \ldots, D_8 \), after the equations of motion are derived from (A.2). Here their field strengths are defined by

\[
F \equiv \sum_{p+2=1}^9 F_{p+2} \equiv e^{-\hat{B}_2} \wedge dD
\]

(A.4)

with

\[
D \equiv \sum_{p+1=0}^8 D_{p+1}
\]

(A.5)
and the 10-dimensional Hodge-star * is defined by
\[
* \left( dx^\mu_1 \wedge \cdots \wedge dx^\mu_n \right) = \frac{1}{(10 - n)!} \frac{1}{\sqrt{-g}} \epsilon^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_{10-n}} \hat{g}_{\nu_1 \lambda_1} \cdots \hat{g}_{\nu_{10-n} \lambda_{10-n}} d x^\lambda_1 \wedge \cdots \wedge d x^{\lambda_{10-n}}
\]
with $\epsilon^{01 \cdots 9} = +1$. Note that $K$-forms satisfy $*^2 \Omega_K = (-1)^{K+1} \Omega_K$ and
\[
d^{10} x \sqrt{-g} | \Omega_K |^2 = d^{10} x \sqrt{-g} \frac{1}{K!} \hat{g}^{\mu_1 \nu_1} \cdots \hat{g}^{\mu_K \nu_K} \Omega_{\mu_1 \cdots \mu_K} \Omega_{\nu_1 \cdots \nu_K}
\]
(A.7)
in 10-dimensional Minkowski space.

First, we note that the original action (A.1) can be written as
\[
S_{\text{R}}^{(\text{IIA})} + S_{\text{CS}}^{(\text{IIA})} = \frac{1}{4 \kappa_1^2} \int \left( F_2 \wedge * F_2 + F_4 \wedge * F_4 + \hat{B}_2 F_4^2 + \hat{B}_2^2 F_4 F_2 + \frac{1}{3} \hat{B}_2^3 F_2^2 \right)
\]
and
\[
S_{\text{R}}^{(\text{IIB})} + S_{\text{CS}}^{(\text{IIB})} = \frac{1}{4 \kappa_1^2} \int \left( F_1 \wedge * F_1 + F_3 \wedge * F_3 + \frac{1}{2} F_5 \wedge * F_5 \right.
\]
\[
+ \hat{B}_2 F_5 F_3 + \frac{1}{2} \hat{B}_2^2 F_5 F_1 + \frac{1}{6} \hat{B}_2^3 F_3 F_1 \right)
\]
(A.8)

Combined with the NS-NS action $S_{\text{NS}}$, (2.2), the equations of motion are thus

**IIA**

\[
0 = d \left( * F_4 + \hat{B}_2 F_4 + \frac{1}{2} \hat{B}_2^2 F_2 \right)
\]
\[
0 = d \left( - * F_2 + \hat{B}_2 * F_4 + \frac{1}{2} \hat{B}_2^2 F_4 + \frac{1}{6} \hat{B}_2^3 F_2 \right)
\]
\[
0 = d \left( e^{-2 \hat{\phi}} * \hat{B}_2 \right) + F_2 * F_4 - \frac{1}{2} F_4^2
\]

**IIB**

\[
0 = d \left( * F_5 + \hat{B}_2 F_3 + \frac{1}{2} \hat{B}_2^2 F_1 \right)
\]
\[
0 = d \left( - * F_3 + \hat{B}_2 * F_5 + \frac{1}{2} \hat{B}_2^2 F_3 + \frac{1}{6} \hat{B}_2^3 F_1 \right)
\]
\[
0 = d \left( * F_1 - \hat{B}_2 * F_3 + \frac{1}{4} \hat{B}_2^2 (F_5 + * F_5) + \frac{1}{6} \hat{B}_2^3 F_3 + \frac{1}{24} \hat{B}_2^4 F_1 \right)
\]
\[
0 = d \left( e^{-2 \hat{\phi}} * \hat{B}_2 \right) + F_1 * F_3 + \frac{1}{2} F_3 F_5 + \frac{1}{2} F_3 * F_5,
\]
as well as the Einstein equation with the energy-momentum tensor of R-R fields:

\[
T_{\hat{\mu} \hat{\nu}}^{(\text{R})} = \begin{cases} 
E_{\hat{\mu} \hat{\nu}} (F_2) + E_{\hat{\mu} \hat{\nu}} (F_4) & \text{(IIA)} \\
E_{\hat{\mu} \hat{\nu}} (F_1) + E_{\hat{\mu} \hat{\nu}} (F_3) + \frac{1}{2} E_{\hat{\mu} \hat{\nu}} (F_5) & \text{(IIB)}
\end{cases}
\]
where $\mathcal{E}_{\hat{\mu}\hat{\nu}}$ is defined for an $n$-form $F_n = (1/n!) F_{\mu_1...\mu_n} dx^{\mu_1} \wedge ... \wedge dx^{\mu_n}$ as

$$
\mathcal{E}_{\hat{\mu}\hat{\nu}}(F_n) \equiv \frac{1}{(n-1)!} F_{\hat{\mu}\hat{\mu}_1...\hat{\mu}_{n-1}} F_{\hat{\nu} \hat{\mu}_1...\hat{\mu}_{n-1}} - \frac{1}{2} \hat{g}_{\hat{\mu}\hat{\nu}} |F_n|_g^2. \quad (A.12)
$$

Equations (A.9) and (A.10) imply that $F_1, ..., F_5$ can be expressed in the following form with integration “constants” $D_{p+1}$ ($p + 1 \geq 5$):

**IIA**

$$
\ast F_2 = - \left( e^{-\hat{B}_2} \wedge dD \right)_8 \equiv -F_8 \quad (A.13)
$$

$$
\ast F_4 = \left( e^{-\hat{B}_2} \wedge dD \right)_6 \equiv F_6
$$

**IIB**

$$
\ast F_1 = \left( e^{-\hat{B}_2} \wedge dD \right)_9 \equiv F_9
$$

$$
\ast F_3 = - \left( e^{-\hat{B}_2} \wedge dD \right)_7 \equiv -F_7 \quad (A.14)
$$

$$
\ast F_5 = \left( e^{-\hat{B}_2} \wedge dD \right)_5 \equiv F_5.
$$

For example, the first equation of (A.9) is solved as

$$
\ast F_4 + \hat{B}_2 F_4 + \frac{1}{2} \hat{B}_2^2 F_2 = dD_5 \quad (A.15)
$$

with some 5-form $D_5$. Then $\ast F_4$ is written as

$$
\ast F_4 = dD_5 - \hat{B}_2 F_4 - \frac{1}{2} \hat{B}_2^2 F_2
$$

$$
= dD_5 - \hat{B}_2 (dD_3 - \hat{B}_2 dD_1) - \frac{1}{2} \hat{B}_2^2 dD_1 \quad (A.16)
$$

$$
= dD_5 - \hat{B}_2 dD_3 + \frac{1}{2} \hat{B}_2^2 dD_1
$$

$$
= \left( e^{-\hat{B}_2} \wedge dD \right)_6
$$

$$
\equiv F_6.
$$

Now we restart the argument in the reverse order, and this time we treat all the fields $D_{p+1}$ ($p+1 = 0, ..., 8$) as independent variables with field strengths (A.4), and adopt (A.2) plus $S_{NS}$ as their action functional. The variation of the action with respect to these fields can be easily found to be

23
IIA

\[ 0 = d(*F_8) \]
\[ 0 = d(-*F_6 + \hat{B}_2*F_8) \]
\[ 0 = d\left(*F_4 - \hat{B}_2*F_6 + \frac{1}{2}\hat{B}_2^2*F_8\right) \]  
\[ 0 = d\left(-*F_2 + \hat{B}_2*F_4 - \frac{1}{2}\hat{B}_2^2*F_6 + \frac{1}{6}\hat{B}_2^3*F_8\right), \]  
(A.17)

IIB

\[ 0 = d(*F_9) \]
\[ 0 = d(-*F_7 + \hat{B}_2*F_9) \]
\[ 0 = d\left(*F_5 - \hat{B}_2*F_7 + \frac{1}{2}\hat{B}_2^2*F_9\right) \]  
\[ 0 = d\left(-*F_3 + \hat{B}_2*F_5 - \frac{1}{2}\hat{B}_2^2*F_7 + \frac{1}{6}\hat{B}_2^3*F_9\right) \]
\[ 0 = d\left(*F_1 - \hat{B}_2*F_3 + \frac{1}{2}\hat{B}_2^2*F_5 - \frac{1}{6}\hat{B}_2^3*F_7 + \frac{1}{24}\hat{B}_2^4*F_9\right). \]  
(A.18)

These are nothing but the set of the Bianchi identities and the equations of motion for the original fields \(D_0, \ldots, D_4\) if we identify \(*F_{p+2} = \pm F_{8-p}\) as in (A.3). Furthermore, the variation with respect to \(\hat{B}_2\) gives

IIA

\[ 0 = d\left(e^{-2\phi} * \hat{B}_2\right) + \frac{1}{2} F_2 * F_4 + \frac{1}{2} F_4 * F_6 + \frac{1}{2} F_6 * F_8 \]  
(A.19)

IIB

\[ 0 = d\left(e^{-2\phi} * \hat{B}_2\right) + \frac{1}{2} F_1 * F_3 + \frac{1}{2} F_3 * F_5 + \frac{1}{2} F_5 * F_7 + \frac{1}{2} F_7 * F_9, \]  
(A.20)

which equal the last equations of (A.9) and (A.10), respectively, after the identification (A.3) is made.

The Einstein equation will be accompanied by the new energy-momentum tensor for R-R fields:

\[ T^{(R)}_{\mu\nu} = \begin{cases} 
\frac{1}{2} \sum_{n=2,4,6,8} \mathcal{E}_{\mu\nu}(F_n) & \text{(IIA)} \\
\frac{1}{2} \sum_{n=3,5,7,9} \mathcal{E}_{\mu\nu}(F_n) & \text{(IIB)} 
\end{cases} \]  
(A.21)
This agrees with the previous one (A.11) since the following identity holds for the dual field $\tilde{F}_{10-n} \equiv \star F_n$:

$$\mathcal{E}_{\mu\nu}(\tilde{F}_{10-n}) = \mathcal{E}_{\mu\nu}(F_n),$$

(A.22)

which can be easily proved by using

$$\frac{1}{(9-n)!} \tilde{F}_{\mu_1 \ldots \mu_9 \cdot \nu} \tilde{F}_{\nu}^{\cdot \lambda_1 \ldots \lambda_9 \cdot n} = \frac{1}{(n-1)!} F_{\mu_1 \lambda_1 \ldots \lambda_{n-1}} F_{\nu}^{\cdot \lambda_1 \ldots \lambda_{n-1}} - \tilde{g}_{\mu\nu} \left| F_n \right|^2 \hat{g},$$

(A.23)

Since the equivalence for the variation with respect to the dilaton $\hat{\phi}$ is obvious, we have completed the proof of the equivalence between the two actions (A.1) and (A.2).

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