SHORT TIME EXISTENCE AND SMOOTHNESS OF THE NONLOCAL MEAN CURVATURE FLOW OF GRAPHS

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Abstract. We consider the geometric evolution problem of entire graphs moving by fractional mean curvature. For this, we study the associated nonlocal quasilinear evolution equation satisfied by the family of graph functions. We establish, using an analytic semigroup approach, short time existence, uniqueness and optimal Hölder regularity in time and space of classical solutions of the nonlocal equation, depending on the regularity of the initial graph. The method also yields $C^\infty$—smoothness estimates of the evolving graphs for positive times.

1. Introduction

In the present paper, we study the geometric evolution problem of hypersurfaces moving by fractional mean curvature. More precisely we consider the problem of existence and uniqueness of a family of (sufficiently regular) open subsets $\{E(t)\}_{t>0}$ of $\mathbb{R}^N$ satisfying

$$
\partial_t X_t \cdot \nu(X_t) := -H^\alpha_{E(t)}(X_t), \quad \text{for all } X_t \in \partial E(t) \text{ and } t \in [0, T],
$$

where $\nu$ is the unit normal vector field on $\partial E(t)$ and $H^\alpha_{E(t)}$ is the fractional mean curvature of order $\alpha \in (0, 1)$ at $X_t \in \partial E(t)$. Recall that for a set $E \subset \mathbb{R}^N$ of class $C^{1+\beta}$, with $\beta > \alpha$, the fractional (nonlocal) mean curvature of $\partial E$ is well defined and it is given at a point $x \in \partial E$ by

$$
H^\alpha_E(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{1_{E^c}(y) - 1_E(y)}{|x - y|^{N+\alpha}} dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{1_{E^c}(y) - 1_E(y)}{|x - y|^{N+\alpha}} dy,
$$

where $1_E$ denotes the characteristic function of the set $E$ and $E^c := \mathbb{R}^N \setminus E$. We recall that the notion of fractional (nonlocal) mean curvature of order $\alpha \in (0, 1)$ appeared for the first time in the work of Caffarelli and Souganidis in [8] around 2008. Since that time, the nonlocal mean curvature has been studied extensively in various settings, see e.g. [1, 5, 16, 25] and the references therein.

As a nonlocal counterpart of the classical mean curvature flow (see e.g. [18, 23] and the references therein), the nonlocal mean curvature flow has attracted much interest in recent years, see e.g. [3, 10, 14, 19, 20, 25] and the survey paper [13]. The first paper dealing with the existence and uniqueness of solutions of this flow is [19], where Imbert used the level set formulation of the geometric flow to prove existence and uniqueness of viscosity solutions while the recent papers [10] and [12] deal with regularity results of viscosity solution of the flow starting from Lipschitz graphs. We also mention the paper [25] where the authors proved a comparison principle for the flow (1.1) and found bounds on the maximal existence time and uniqueness of smooth solutions.

Keywords: Nonlocal mean curvature flow; Quasilinear evolution equations; Analytic semigroup theory; Fixed point theorem.
Despite the attention that the nonlocal mean curvature flow has already received, the existence and regularity of classical solutions remained an open problem until the recent work \cite{20}. In this paper, the authors prove short time existence and uniqueness of classical solution to the fractional mean curvature flow, starting from a bounded \(C^{1,1}\) initial set.

In the present paper, we consider the complementary case of short time existence and uniqueness of classical solutions of problem (1.1) when the sets \(E(t)\) are given by subgraphs of functions \(u(t, \cdot) \in C^{1+\beta}_\text{loc}(\mathbb{R}^{N-1})\), where \(\beta > \alpha\). We recall first that if \(E_u := \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : y < u(x)\}\) then, see e.g. \cite{16} Proposition 3.5, we have that

\[
H(u)(x) := H^p_{E_u}(x, u(x)) = P.V. \int_{\mathbb{R}^{N-1}} \frac{G(p_u(x, y))}{|x - y|^{N-1+\alpha}} \, dy, \quad (1.3)
\]

where

\[
p_u(x, y) = \frac{u(y) - u(x)}{|x - y|}, \quad G(p) := -\int_{-p}^{p} \frac{d\tau}{(1 + \tau^2)^{\frac{3+\alpha}{2}}}. \quad (1.4)
\]

We point out that, due to the boundedness of the function \(G\), the expression \(H(u)(x)\) is well-defined if \(u : \mathbb{R}^{N-1} \to \mathbb{R}\) is measurable and of class \(C^{1+\beta}\) for some \(\beta > \alpha\) in a neighborhood of \(x\). With this expression of the fractional mean curvature, we can derive the quasilinear evolution problem corresponding to (1.1) when \(E_u(t) := \{(x(t), y(t)) \in \mathbb{R}^{N-1} \times \mathbb{R} : y(t) < u(t, x(t))\}\), with \(u : [0, T] \times \mathbb{R}^{N-1} \to \mathbb{R}\). Indeed, recall that the unit interior normal vector field on \(\partial E_u(t)\) is given by \(\nu(X_t) := (-\nabla u(t, x(t)), 1)\). Therefore for every \(X_t \in \partial E_u(t) = \{(x(t), u(t, x(t))) : x(t) \in \mathbb{R}^{N-1}\}\), we have that

\[
\partial_t X_t = \left( \dot{x}(t), \dot{x}(t) \cdot \nabla u(t, x(t)) + \partial_t u(t, x(t)) \right).
\]

Hence

\[
\partial_t X_t \cdot \nu(X_t) = -\frac{\dot{x}(t) \cdot \nabla u(t, x(t))}{\sqrt{1 + |\nabla u(t, x(t))|^2}} + \frac{\dot{x}(t) \cdot \nabla u(t, x(t))}{\sqrt{1 + |\nabla u(t, x(t))|^2}} + \frac{\partial_t u(t, x(t))}{\sqrt{1 + |\nabla u(t, x(t))|^2}}
\]

\[
= \frac{\partial_t u(t, x(t))}{\sqrt{1 + |\nabla u(t, x(t))|^2}}. \quad (1.5)
\]

Therefore, from (1.5) and (1.1), the evolution of \(u\) is given by the flow associated to the quasilinear evolution equation

\[
\partial_t u = -\sqrt{1 + |\nabla u|^2} H(u), \quad t \in (0, T], \quad u(0) = u_0. \quad (1.6)
\]

Our first main result is the following.

**Theorem 1.1.** Let \(\nu > 0\), \(\beta \in (\alpha, 1)\), \(\rho \in (0, \frac{1}{1+\alpha})\) and \(\gamma_\rho := \beta + \rho(1 + \alpha)\). Then for all \(u_0 \in C^{1+\gamma_\rho}_{\text{loc}}(\mathbb{R}^{N-1})\), with \(\|\nabla u_0\|_{C^{\alpha}_\rho(\mathbb{R}^{N-1})} \leq \nu\), there exist \(T, C_0 > 0\) only depending on \(\rho, \alpha, \beta, \gamma, N\) and \(\nu\) such that the problem

\[
\begin{cases}
\partial_t u + \sqrt{1 + |\nabla u|^2} H(u) = 0 & \text{in } [0, T] \times \mathbb{R}^{N-1} \\
\partial_u u = u_0 & \text{in } \mathbb{R}^{N-1}
\end{cases}
\]

(1.7)

admits a unique solution \(u \in C^\rho([0, T], C^{1+\beta}_{\text{loc}}(\mathbb{R}^{N-1})) \cap C^{1+\rho}([0, T], C^{\beta-\alpha}_{\text{loc}}(\mathbb{R}^{N-1}))\) satisfying

\[
\|u - u_0\|_{C^\rho([0, T], C^{1+\beta}_{\text{loc}}(\mathbb{R}^{N-1}) \cap C^{1+\rho}([0, T], C^{\beta-\alpha}_{\text{loc}}(\mathbb{R}^{N-1})))} \leq C_0.
\]

If, in addition, \(\nabla u_0 \in C^{1+\gamma_\rho}(\mathbb{R}^{N-1})\) then for all \(\beta' \in (\alpha, \beta)\) there exists \(C > 0\) only depending on \(\rho, \alpha, \beta, \gamma, N, \nu, T\) and \(\beta'\) such that

\[
\|\nabla u\|_{C^\rho([0, T], C^{1+\beta'}(\mathbb{R}^{N-1}))} \leq C \|\nabla u_0\|_{C^{1+\gamma_\rho}(\mathbb{R}^{N-1})}.
\]

(1.9)
The following result complements Theorem 1.1 by providing, under the same assumptions, smoothness estimates for positive times.

**Theorem 1.2.** Under the assumptions of Theorem 1.1, we have $u(t, \cdot) \in C^\infty(\mathbb{R}^{N-1})$ for every $t \in (0, T]$. Moreover, for every $\beta' \in (\alpha, \beta)$, $\rho \in (0, \frac{1}{1+\alpha}]$ and for all $k \in \mathbb{N} \setminus \{0\}$, there exists $C_k > 0$ only depending on $\rho, \alpha, \beta, \gamma, N, \nu, \beta', T$ and $k$ such that

$$
\| t^k \nabla u \|_{C^\rho([0,T],C^{k+\beta'}(\mathbb{R}^{N-1}))} \leq C_k. \quad (1.10)
$$

Related to Theorem 1.2, we also mention [12], where the authors state that solutions to the fractional mean flow of the graph, starting from a Lipschitz graph, are $C^\infty$-smooth for positive times provided $H(u_0) \in L^\infty(\mathbb{R}^{N-1})$. However, [12] does not provide enough details that confirm this statement. Indeed, the authors apply a priori H"older regularity estimates from [26, 27] to the linearization of the first equation of (1.7). However the linearization of the weighted nonlocal mean curvature operator $u \mapsto \sqrt{1 + |\nabla u|^2} H(u)$ does not fall in the class of nonlocal operators considered in [26, 27].

Our next result provides universal estimates of the Lipschitz norm and the mean curvature of the evolving graph in terms of the initial data.

**Theorem 1.3.** Under the assumptions of Theorem 1.1, we have

$$
\| \nabla u \|_{L^\infty((0,T) \times \mathbb{R}^{N-1})} \leq \| \nabla u_0 \|_{L^\infty(\mathbb{R}^{N-1})}
$$

and

$$
\| \partial_t u \|_{L^\infty((0,T) \times \mathbb{R}^{N-1})} \leq \| \sqrt{1 + |\nabla u_0|^2} H(u_0) \|_{L^\infty(\mathbb{R}^{N-1})}.
$$

If moreover $u_0 \in L^\infty(\mathbb{R}^{N-1})$, then $\| u \|_{L^\infty((0,T) \times \mathbb{R}^{N-1})} \leq \| u_0 \|_{L^\infty(\mathbb{R}^{N-1})}$.

As we shall see below, the existence in Theorem 1.1 is a consequence of a more general existence result concerning problem (1.7). Indeed, we define first the Banach space

$$
C^\theta_0(\mathbb{R}^{N-1}) = \frac{C^\infty_c(\mathbb{R}^{N-1})}{\{ f \in \mathbb{R} \}} \quad \text{for } \theta \in \mathbb{R} _+ \setminus \mathbb{N},
$$

endowed with the norm of $C^\theta(\mathbb{R}^{N-1})$. We then write (1.7) as

$$
\begin{cases}
\partial_t u + \mathcal{L}_0 u = F(u) & \text{in } [0, T) \times \mathbb{R}^{N-1} \\
u(0) & = u_0 & \text{in } \mathbb{R}^{N-1},
\end{cases} \quad (1.11)
$$

where $\mathcal{L}_0 := D\mathcal{H}(u_0) : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$ is the derivative of the weighted mean curvature operator

$$
u(u) := \sqrt{1 + |\nabla u|^2} H(u) \quad (1.12)
$$

at the initial condition $u_0$ and the nonlinear map $u \mapsto F(u) = -H(u) + \mathcal{L}_0 u$ has the property that $F(u) - F(u_0)$ has superlinear growth in $u - u_0$. Our strategy is now to show that $\mathcal{L}_0$ generates a strongly continuous analytic semigroup on $C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$ and to apply a fixed point argument. However, we need precise estimates, as we want all the regularity estimated by the H"older norms of $\nabla u_0$.

The main result from which we partly derive Theorem 1.1 is the following

**Theorem 1.4.** Let $\beta \in (\alpha, 1)$, $\nu > 0$, $\rho \in (0, \frac{1}{1+\alpha})$ and $\gamma_\rho := \beta + \rho(1+\alpha)$. Then, there exists $T > 0$, depending only on $\rho, \alpha, \beta, N$ and $\nu$, such that for all $u_0 \in C^{1+\beta}_0(\mathbb{R}^{N-1})$, with

$$
\| \nabla u_0 \|_{C^{\gamma_\rho}(\mathbb{R}^{N-1})} \leq \nu,
$$
the initial value problem
\[
\begin{aligned}
\partial_t u + \sqrt{1+|\nabla u|^2} H(u) &= 0 & \text{in } [0,T] \times \mathbb{R}^{N-1} \\
u(0) &= u_0 & \text{in } \mathbb{R}^{N-1}
\end{aligned}
\] (1.13)

admits a unique solution \( u \in C^{1+\rho}([0,T], C^{\beta-\alpha}(\mathbb{R}^{N-1})) \cap C^\rho([0,T], C^{1+\beta(\mathbb{R}^{N-1}))}. Moreover, there exists \( C_0 > 0 \), depending only on \( \rho, \alpha, \beta, N \) and \( \nu \), such that
\[
\|u - u_0\|_{C^{1+\rho}([0,T], C^{\beta-\alpha}(\mathbb{R}^{N-1})) \cap C^\rho([0,T], C^{1+\beta}(\mathbb{R}^{N-1}))} \leq C_0.
\] (1.14)

We note that, thanks to the regularity of \( E_u(t) \) proved in Theorem 1.4, the uniqueness of the flow \((E_u(t))_{t \in [0,T]}\) also follows from [25], where the authors proved that \((1.1)\) admits a unique classical solution. On the other hand, the proof of our Theorem 1.4 yields existence and uniqueness simultaneously.

Recall that, provided the linear operator \( \mathcal{L}_0 := D\mathcal{H}(u_0) : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1}) \) generates an analytic semigroup, to get optimal \( \mathcal{C} \)-positivity of the principle yields further interesting qualitative properties of the flow such as preservation of continuous analytic semigroup generated by a general class of nonlocal operators. Section 4 deals with the regularity properties satisfied by the weighted nonlocal mean curvature operator \( \mathcal{H}(u) \) and its linearization. Finally, in Section 5 we prove our main results.

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2. Notation and preliminary estimates

For \( m \in \mathbb{N} \), we denote by \( C^m_b(\mathbb{R}^{N-1}) \) the space of \( m \)-times bounded continuously differentiable functions endowed with the norm
\[
\|u\|_{C^m_b} = \sum_{k=0}^{m} \|D^k u\|_{L^\infty} := \sum_{k=0}^{m} \sup_{x \in \mathbb{R}^{N-1}} |D^k u(x)|. 
\]
For \( 0 < \gamma < 1 \), we consider the space of Hölder continuous functions
\[
C^\gamma(\mathbb{R}^{N-1}) = \left\{ u \in C_b(\mathbb{R}^{N-1}) : [u]_{C^\gamma} := \sup_{x,y \in \mathbb{R}^{N-1}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma} < \infty \right\} 
\]
endowed with the norm
\[
u \mapsto \|u\|_{C^\gamma} = \|u\|_{L^\infty} + [u]_{C^\gamma}.
\]
More generally, for \( m \in \mathbb{N} \) and \( 0 < \gamma < 1 \), we let
\[
C^{m+\gamma}(\mathbb{R}^{N-1}) = \left\{ u \in C^m_b(\mathbb{R}^{N-1}) : D^m u \in C^\gamma(\mathbb{R}^{N-1}) \right\},
\]
endowed with the norm
\[
u \mapsto \|u\|_{C^{m+\gamma}} = \|D^m u\|_{C^\gamma} + \|D^m u\|_{C^\gamma}.
\]
For late use, we recall the interpolation inequality that, for all \( \gamma \in (0, 1) \), \( \gamma' \in [0, \gamma) \) and for all \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
\|\nabla v\|_{C^{\gamma'}} \leq \varepsilon \|\nabla v\|_{C^\gamma} + C_\varepsilon \|v\|_{L^\infty} \quad \text{for all } v \in C^{1,\gamma}(\mathbb{R}^{N-1}).
\]
Next, we define
\[
C^{m+\gamma}_0(\mathbb{R}^{N-1}) := \frac{C^{\infty}(\mathbb{R}^{N-1})}{\|C^{m+\gamma}\}}
\]
endowed with the \( C^{m+\gamma} \)-norm. We then recall the following result.

**Proposition 2.1.** Let \( \beta, \gamma \in (0, 1) \), with \( \gamma > \beta \). Let \( u \in C^{\gamma}(\mathbb{R}^{N-1}) \) be such that \( \lim_{R \to \infty} \|f\|_{C^\beta(\mathbb{R}^{N-1} \setminus B_R)} = 0 \). Then \( f \in C^{\beta}_0(\mathbb{R}^{N-1}) \).

We need some more notation related to space of time dependent functions. Let \( \mathcal{X} \) be a Banach space. For \( T > 0 \), we consider the Banach space \( L^\infty([0, T], \mathcal{X}) \), consisting of bounded functions \( u : [0, T] \to \mathcal{X} \). The spaces \( L^\infty([0, T], \mathcal{X}) \) is endowed with the norm
\[
\|u\|_{L^\infty([0, T], \mathcal{X})} = \sup_{t \in [0, T]} \|u(t)\|_{\mathcal{X}}.
\]
For \( \mu \in (0, 1) \), we define the Banach space
\[
C^{1+\mu}([0, T], \mathcal{X}) = \left\{ u \in C^1([0, T], \mathcal{X}) : [\partial_t u]_{C^\mu([0, T], \mathcal{X})} = \sup_{0 \leq s < t \leq T} \frac{\|\partial_t u(t) - \partial_t u(s)\|_{\mathcal{X}}}{|t - s|^\mu} < +\infty \right\},
\]
endowed with the norm \( \|u\|_{C^{1+\mu}([0, T], \mathcal{X})} = \|u\|_{L^\infty([0, T], \mathcal{X})} + [\partial_t u]_{C^\mu([0, T], \mathcal{X})}. \)
2.1. Estimates in polar coordinates. In our study of linearizations of the nonlocal mean curvature operator, we will have to consider integral operators whose associated kernels have anisotropic singularities which can be resolved partially in polar coordinates. For this, the following basic estimates will be of key importance.

Lemma 2.2. Let $\gamma \in (\alpha, 1)$, $v \in C^{1+\gamma}(\mathbb{R}^{N-1})$ be such that $\nabla v \in L^\infty(\mathbb{R}^{N-1})$. Let $\mu \in C^{\gamma-\alpha}(\mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2})$ and $\nu \in C^{\gamma}(\mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2})$ with $\nu(\cdot, 0, \cdot) = 0$. We define

$$I_\epsilon v(x) := \int_{S^{N-2}} \int_0^\infty \delta_\epsilon v(x, r, \theta)\mu(x, r, \theta)r^{-2-\alpha} drd\theta$$

and

$$I_\delta v(x) := \int_{S^{N-2}} \int_0^\infty \delta_\delta v(x, r, \theta)\nu(x, r, \theta)r^{-2-\alpha} drd\theta$$

for $x \in \mathbb{R}^{N-1}$, where

$$\delta_\epsilon v(x, r, \theta) := \frac{1}{2}(2v(x) - v(x + r\theta) - v(x - r\theta)) \quad \text{and} \quad \delta_\delta v(x, r, \theta) := v(x) - v(x - r\theta).$$

Then the functions $I_\epsilon v$ and $I_\delta v$ satisfy the following estimates with constants $C$ independent of $\mu, \nu$ and $v$.

(i) We have

$$\|I_\delta v\|_{C^{\gamma-\alpha}} \leq C\|\nabla v\|_{L^\infty} \|\nu\|_{C^{\gamma}(\mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2})}$$

and if $\nabla v \in C^\gamma(\mathbb{R}^{N-1})$, then

$$\|I_\epsilon v\|_{C^{\gamma-\alpha}} \leq C\|\nabla v\|_{C^\gamma(\mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2})}.$$ (2.5)

(ii) If $\nabla v \in C^\gamma(\mathbb{R}^{N-1})$ and $\nabla v$ is compactly supported, then

$$\lim_{R \to \infty} \|I_\epsilon v\|_{C^{\gamma-\alpha}(\mathbb{R}^{N-1} \setminus B_R)} = \lim_{R \to \infty} \|I_\delta v\|_{C^{\gamma-\alpha}(\mathbb{R}^{N-1} \setminus B_R)} = 0.$$ (2.6)

(iii) If $v \in C^{1+\beta}_0(\mathbb{R}^{N-1})$, with $\beta \in (\alpha, \gamma)$, then $I_\epsilon v, I_\delta v \in C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$.

Proof. We assume for simplicity that $\|\nu\|_{C^{\gamma}(\mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2})} \leq 1$. We first observe that

$$|\nu(x, r, \theta)| \leq \min(1, r^\gamma), \quad |\delta_\delta v(x, r, \theta)| = r \left| \int_0^1 \nabla v(x - rt\theta) \cdot \theta dt \right| \leq r\|\nabla v\|_{L^\infty}$$

and for $x, x' \in \mathbb{R}^{N-1}$

$$|\nu(x, r, \theta) - \nu(x', r, \theta)| \leq \min(|x - x'|^\gamma, r^\gamma),$$

and

$$|\delta_\delta v(x, r, \theta) - \delta_\delta v(x', r, \theta)| \leq 2\|\nabla v\|_{L^\infty} \min(|x - x'|, r).$$

Then, by (2.8), we have

$$\sup_{x \in \mathbb{R}^{N-1}} \int_{S^{N-2}} \int_0^\infty |\delta_\delta v(x, r, \theta)\nu(x, r, \theta)|r^{-2-\alpha} drd\theta$$

$$\leq C\|\nabla v\|_{L^\infty} \int_0^\infty \min(1, r^\gamma)r^{-1-\alpha} dr \leq C\|\nabla v\|_{L^\infty}.$$ (2.9)

We write

$$I_\delta v(x) - I_\delta v(z) = \int_{S^{N-2}} \int_0^\infty r^{-2-\alpha}\delta_\delta v(x, r, \theta)[\nu(x, r, \theta) - \nu(z, r, \theta)] drd\theta$$

$$+ \int_{S^{N-2}} \int_0^\infty r^{-2-\alpha}[\delta_\delta v(x, r, \theta) - \delta_\delta v(z, r, \theta)]\nu(x, r, \theta) drd\theta.$$
From this, (2.18), (2.9) and (2.10), we get
\[ |I_0v(x) - I_0v(z)| \leq C\|\nabla v\|_{L^\infty} \int_0^\infty r^{-1-\alpha} \min\{r^\gamma, |x-z|^\gamma\} \, dr \]
\[ + C\|\nabla v\|_{L^\infty} \int_0^\infty r^{\gamma-2-\alpha} \min\{|x-z|, r\} \, dr \leq C\|\nabla v\|_{L^\infty} |x-z|^{\gamma-\alpha}. \]
Therefore
\[ |I_0v|_{C^{\gamma-\alpha}} \leq C\|\nabla v\|_{L^\infty}. \] (2.12)
The inequalities (2.11) and (2.12) imply (2.5).

For simplicity, we assume that \( \|\mu\|_{C^{\gamma}(\mathbb{R}^{N-1} \times [0,\infty) \times S^{N-2})} \leq 1 \). We write
\[ \delta_\varepsilon v(x, r, \theta) = \frac{r}{2} \int_0^1 \nabla v(x - rt\theta) - \nabla v(x + rt\theta) \, dt. \] (2.13)
Hence, we have
\[ |\delta_\varepsilon v(x, r, \theta)| \leq r \min(1,r^\gamma)\|\nabla v\|_{C^\gamma} \] (2.14)
and for \( x, x' \in \mathbb{R}^{N-1} \)
\[ |\mu(x, r, \theta) - \mu(x', r, \theta)| \leq \min(|x - x'|^\gamma, 1) \] (2.15)
and
\[ |\delta_\varepsilon v(x, r, \theta) - \delta_\varepsilon v(x', r, \theta)| \leq r\|\nabla v\|_{C^\gamma} \min(|x - x'|^\gamma, r^\gamma). \] (2.16)

We now estimate, using (2.14),
\[ \|I_\varepsilon v\|_{L^\infty} = \sup_{x \in \mathbb{R}^{N-1}} \int_{S^{N-2}} \int_0^\infty |\delta_\varepsilon v(x, r, \theta)\mu(x, r, \theta)| r^{-2-\alpha} \, dr \, d\theta \]
\[ \leq C\|\nabla v\|_{C^\gamma} \int_0^\infty \min(1,r^\gamma) r^{-1-\alpha} \, dr \leq C\|\nabla v\|_{C^\gamma}. \] (2.17)
We have
\[ I_\varepsilon v(x) - I_\varepsilon v(z) = \int_{S^{N-2}} \int_0^\infty r^{-2-\alpha} \delta_\varepsilon v(x, r, \theta)\mu(x, r, \theta) - \delta_\varepsilon v(z, r, \theta)\mu(z, r, \theta) \, dr \, d\theta \]
\[ + \int_{S^{N-2}} \int_0^\infty r^{-2-\alpha} \delta_\varepsilon v(x, r, \theta)\mu(x, r, \theta) - \delta_\varepsilon v(x, r, \theta)\mu(x, r, \theta) \, dr \, d\theta. \]

From this, (2.8), (2.9) and (2.10), we get
\[ |I_\varepsilon v(x) - I_\varepsilon v(z)| \leq C\|\nabla v\|_{C^\gamma} \min\{1, |x-z|^\gamma\} \int_0^\infty r^{-1-\alpha} \min\{1,r^\gamma\} \, dr \]
\[ + C\|\nabla v\|_{C^\gamma} \int_0^\infty r^{-1-\alpha} \min\{|x-z|^\gamma, r^\gamma\} \, dr \leq C\|\nabla v\|_{C^\gamma} |x-z|^{\gamma-\alpha}. \]
This and (2.17) give (2.6). The proof of (i) is thus complete.

For (ii), we assume that \( \text{Supp} \nabla v \subset B_{R'}, \) for some \( R' > 1. \) We let \( R > 2R' \) and we split
\[ I_\varepsilon v(x) = \int_{S^{N-2}} \int_0^R \delta_\varepsilon v(x, r, \theta)\mu(x, r, \theta) r^{-2-\alpha} \, dr \, d\theta + \int_{S^{N-2}} \int_R^\infty \delta_\varepsilon v(x, r, \theta)\mu(x, r, \theta) r^{-2-\alpha} \, dr \, d\theta \]
\[ =: I^1_\varepsilon (x) + I^2_\varepsilon (x). \]

By (2.13),
\[ I^1_\varepsilon (x) = 0 \quad \text{for } |x| \geq R' + R. \] (2.18)
In addition, for $|x| > R$, by (2.13),

$$|I^2_e(x)| \leq \|\nabla v\|_{L^\infty} \int_0^1 \int_R^{|x|+R'} r^{-1-\alpha} dr dt \leq C \|\nabla v\|_{L^\infty} \left(R^{-\alpha} + (|x| + R')^{-\alpha}\right), \quad (2.19)$$

so that

$$\lim_{R \to \infty} \|I_e\|_{L^\infty} = 0. \quad (2.20)$$

Now, for $h \in \mathbb{R}^{N-1}$,

$$|I^2_e(x + h) - I^2_e(x)| \leq |h|^{\gamma-\alpha} \|\nabla v\|_{C^{\gamma-\alpha}} \int_0^1 \int_R^{|x|+|h|+R'} r^{-1-\alpha} dr dt \leq C |h|^{\gamma-\alpha} \|\nabla v\|_{C^{\gamma-\alpha}} (R^{-\alpha} + (|x| + |h| + R')^{-\alpha}).$$

Combining this with (2.20) and (2.18), we deduce that $\lim_{R \to \infty} \|I_e v\|_{C^{\gamma-\alpha}(\mathbb{R}^{N-1} \setminus B_R)} = 0$. By similar argument we have $\lim_{R \to \infty} \|I_0 v\|_{C^{\gamma-\alpha}(\mathbb{R}^{N-1} \setminus B_R)} = 0$, so that (2.7) holds.

For (iii), we pick $v \in C^{0+\gamma}(\mathbb{R}^{N-1})$. Then there exists $v_n \in C^{\infty}_c(\mathbb{R}^{N-1})$ such that $v_n \to v$ in $C^{1+\gamma}(\mathbb{R}^{N-1})$. Thanks to (i), we have $I_e v_n, I_0 v_n \in C^{\gamma-\alpha}_0(\mathbb{R}^{N-1})$. Now, by (ii) and Proposition 2.1, we get $I_e v_n, I_0 v_n \in C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$. Once again by (i), we have $I_e v_n \to I_e v$ and $I_0 v_n \to I_0 v$ in $C^{\beta-\alpha}(\mathbb{R}^{N-1})$. This then yields the conclusion, since $C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$ is closed in $C^{\beta-\alpha}(\mathbb{R}^{N-1})$. \hfill \Box

### 2.2. A nonlocal comparison principle in the entire space.

In this subsection we provide a comparison principle for a class of nonlocal evolution equations relying on kernel assumptions in polar coordinates, which will be used later on.

**Proposition 2.3.** Let $\alpha \in (0,1)$, $\beta \in (\alpha,1)$, $T > 0$, $V \in C([0,T], C_b(\mathbb{R}^{N-1}))$ and let $\mu \in C([0,T], C^\beta(\mathbb{R}^{N-1} \times [0,\infty) \times S^{N-2}))$ with $\mu > 0$ and $\mu(t,x,0,\theta) = \mu(t,x,0,-\theta)$ Moreover, let $\beta > \alpha$, and let $u \in L^\infty([0,T], C^{\beta}(\mathbb{R}^{N-1}))$ satisfy

$$u(\cdot,x) \in C^1([0,T]) \quad \text{for every } x \in \mathbb{R}^{N-1}, \quad u(t,\cdot) \in C^{1+\beta}_{loc}(\mathbb{R}^{N-1}) \quad \text{for every } t \in [0,T]$$

and

$$\partial_t u(t,x) + P.V. \int_{\mathbb{R}^{N-1}} \frac{u(t,x) - u(t,x+y)}{|y|^{N+\alpha}} \mu(t,x,|y|,y/|y|) dy + V(t,x) \cdot \nabla u(t,x) \leq 0 \quad (2.21)$$

for $(t,x) \in [0,T] \times \mathbb{R}^{N-1}$. Then \( \sup_{[0,T] \times \mathbb{R}^{N-1}} u = \sup_{x \in \mathbb{R}^{N-1}} u(0,x). \)

**Proof.** We consider $\eta \in C^\infty_c(B_2)$ with $0 \leq \eta \leq 1$ on $\mathbb{R}^{N-1}$ and $\eta = 1$ on $B_1$. We define, for $\varepsilon,R > 0,$

$$v(t,x) = \eta_R(x)u(t,x) - \varepsilon t,$$

where $\eta_R(x) = \eta(x/R)$. We also define, for $w \in C^{1+\beta}_{loc}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ and $x \in \mathbb{R}^{N-1},$

$$\mathcal{L}(t)w(x) = P.V. \int_{\mathbb{R}^{N-1}} \frac{w(x) - w(x+y)}{|y|^{N+\alpha}} \mu(t,x,|y|,y/|y|) dy.$$
We then have

\[
\begin{align*}
\mathcal{L}(t)(\eta_Ru(t))(x) &= \eta_R(x)\mathcal{L}(t)(u(t))(x) \\
+ u(t, x)P.V. &\int_{\mathbb{R}^N} \frac{\eta_R(x) - \eta_R(x + y)}{|y|^{N+\alpha}} \mu(t, x, |y|, |y|) \, dy \\
+ P.V. &\int_{\mathbb{R}^N} \frac{\eta_R(x) - \eta_R(x + y)}{|y|^{N+\alpha}} (u(t, x + y) - u(t, x)) \mu(t, x, |y|, |y|) \, dy.
\end{align*}
\]

From this, we get

\[
\begin{align*}
\mathcal{L}(t)(\eta_Ru(t))(x) - \eta_R(x)P.V. &\int_{\mathbb{R}^N} \frac{u(t, x) - u(t, x + y)}{|y|^{N+\alpha}} \mu(t, x, |y|, |y|) \, dy \\
= &\frac{u(t, x)}{2}P.V. \int_{\mathbb{R}^N} \frac{2\eta_R(x) - \eta_R(x - y) - \eta_R(x + y)}{|y|^{N+\alpha}} \mu(t, x, 0, |y|) \, dy \\
+ u(t, x)P.V. &\int_{\mathbb{R}^N} \frac{\eta_R(x) - \eta_R(x + y)}{|y|^{N+\alpha}} [\mu(t, x, |y|, |y|) - \mu(t, x, 0, |y|)] \, dy \\
+ P.V. &\int_{\mathbb{R}^N} \frac{\eta_R(x) - \eta_R(x + y)}{|y|^{N+\alpha}} (u(t, x + y) - u(t, x)) \mu(t, x, |y|, |y|) \, dy.
\end{align*}
\]

Hence, by Lemma 2.2 we get

\[
\|\mathcal{L}(t)(\eta_Ru(t)) - \eta_R\mathcal{L}(t)u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C\|\nabla\eta_R\|_{C^1_0}\|u(t)\|_{C^\beta} \leq \frac{C}{R}\|u(t)\|_{C^\beta}
\]

for \( t \in [0, T] \). From this and (2.21), we then obtain

\[
\partial_t v + \mathcal{L}(t)v + V \cdot \nabla v \leq -\varepsilon + F_R \quad \text{in } [0, T] \times \mathbb{R}^N,
\]

with

\[
\|F_R\|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq \frac{C\|u\|_{L^\infty((0,T)\times\mathbb{R}^N)}^\beta}{R}(1 + \|V\|_{L^\infty((0,T)\times\mathbb{R}^N)}).
\]

We claim that

\[
\max_{[0,T] \times \mathbb{R}^N} v = \max_{x \in \mathbb{R}^N} v(0, x).
\]

Indeed, let \((t_0, x_0) \in [0, T] \times \mathbb{R}^N\) be such that \(v(t_0, x_0) = \max_{[0,T] \times \mathbb{R}^N} v\). Suppose that \(t_0 > 0\). The maximality property then implies that \(\nabla_x v(t_0, x_0) = 0\) and also \(L(t_0)v(x_0) \geq 0\), since \(\mu \geq 0\) by assumption. By (2.22) we thus have

\[
0 \leq \mathcal{L}(t_0)v(x_0) \leq -\varepsilon + \|F_R\|_{L^\infty((0,T)\times\mathbb{R}^N)}
\]

which is not possible if \(R\) is large enough, thanks to (2.23). Therefore \(t_0 = 0\) and thus (2.24) holds as claimed.

Letting now \(R \to \infty\) and then \(\varepsilon \to 0\) in (2.24), we get the result. \(\square\)

3. Analytic semigroups, their generators, intermediate spaces and associated semilinear evolution equations

We begin by introducing some notions regarding function spaces and analytic semigroups (see e.g. [22 Chapter 2]). For normed vector spaces \(\mathcal{X}, \mathcal{X}'\) we let \(\mathcal{L}(\mathcal{X}, \mathcal{X}')\) denote the space of continuous linear operators \(\mathcal{X} \to \mathcal{X}'\), endowed with the usual operator norm. As usual, we also set \(\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})\).

A strongly continuous analytic semigroup on a Banach space \(\mathcal{X}\) is a family of operators \(\{T(t)\}_{t \geq 0} \subset \mathcal{L}(\mathcal{X})\) with the following properties:

(i) \(T(0) = \text{id}, T(t + s) = T(t)T(s)\) for all \(t, s \geq 0\).
(ii) The function \((0, \infty) \to \mathcal{L}(\mathcal{X}), t \mapsto T(t)\) is analytic.

(iii) The function \([0, \infty) \to \mathcal{X}, t \mapsto T(t)u\) is continuous for every \(u \in \mathcal{X}\).

The generator of such a semigroup is given as an (unbounded) linear operator \(B_0 : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}\) by
\[
B_0 u := \lim_{t \to 0^+} \frac{T(t)u - u}{t}, \quad u \in \mathcal{Y},
\]
where the domain \(\mathcal{Y} \subset \mathcal{X} of \(B_0\) is given as the subspace of \(u \in \mathcal{X}\) for which this limit exists.

We recall that an operator \(B_0 : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}\) generates a strongly continuous analytic
semigroup in this sense if and only if \(B_0\) is sectorial (see e.g. [22, Def. 2.0.1] for a definition)
and its domain \(\mathcal{Y}\) is dense in \(\mathcal{X}\). In such a case, the operator \(B_0\) is also closed, which means
that \(\mathcal{Y}\) is a Banach space with the graph norm \(u \mapsto \|u\|_{\mathcal{X}} + \|B_0 u\|_{\mathcal{X}}\) on \(\mathcal{Y}\). As a consequence,
by the open mapping theorem, the graph norm on \(\mathcal{Y}\) is equivalent to any other given norm
\(\| \cdot \|_{\mathcal{Y}}\) on \(\mathcal{Y}\) for which \((\mathcal{Y}, \| \cdot \|_{\mathcal{Y}})\) is a Banach space.

In the following, if \(B_0 : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}\) generates a strongly continuous analytic semigroup,
we shall denote this semigroup by
\[
t \mapsto e^{B_0 t} \in \mathcal{L}(\mathcal{X}), \quad t \geq 0.
\]
As noted in [22, Proposition 2.1.1], for all \(k \in \mathbb{N}\), there exists \(M_k > 0\), such that
\[
\|t^k B_0^k e^{B_0 t}\|_{\mathcal{L}(\mathcal{Y})} \leq M_k, \quad \text{for all } t \in (0, 1]. \tag{3.1}
\]
In order to obtain optimal regularity estimates in time, it is convenient to introduce, for \(\rho \in (0, 1)\), the intermediate space
\[
\mathcal{D}_{B_0}(\rho, \infty) = \{ f \in \mathcal{Y} : [f]_{\mathcal{D}_{B_0}(\rho, \infty)} = \sup_{0 < t \leq 1} \|t^{1-\rho} B_0 e^{B_0 t} f\|_{\mathcal{Y}} < \infty\}, \tag{3.2}
\]
endowed with the norm \(\|f\|_{\mathcal{D}_{B_0}(\rho, \infty)} = \|f\|_{\mathcal{Y}} + [f]_{\mathcal{D}_{B_0}(\rho, \infty)}\).

We then have the following result taken from [22, Theorem 4.3.1 (iii)].

**Theorem 3.1.** Let \(B_0 : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}\) be the generator of a strongly continuous analytic
semigroup. Let \(T > 0, \rho \in (0, 1), u_0 \in \mathcal{X}\), and let \(f \in C^\rho([0, T], \mathcal{Y})\) be such that \(f(0) - B_0 u_0 \in \mathcal{D}_{B_0}(\rho, \infty)\). Then the problem
\[
\begin{aligned}
\left\{
\begin{array}{ll}
  u'(t) + B_0 u(t) &= f(t), \quad t \in (0, T] \\
  u(0) &= u_0
\end{array}
\right.
\end{aligned}
\]
amits a unique solution \(u \in C^\rho([0, T], \mathcal{X}) \cap C^{1+\rho}([0, T], \mathcal{Y})\). Moreover, there exists \(C_T = C(T, \rho, M_0, M_1, M_2) > 0\) such that
\[
\|B_0(u - u_0)\|_{C^\rho([0, T], \mathcal{Y})} + \|u - u_0\|_{C^{1+\rho}([0, T], \mathcal{Y})} \leq C_T \left(\|f - B_0 u_0\|_{C^\rho([0, T], \mathcal{Y})} + \|f(0) - B_0 u_0\|_{\mathcal{D}_{B_0}(\rho, \infty)} \right) \tag{3.3}
\]
and \(C_T \leq C_{T_0}\) for all \(T \leq T_0\).

**Proof.** The existence, uniqueness and (3.3) follow from [22, Theorem 4.3.1]. Moreover, as explained in the beginning of Section 4.1 in [22], the constant \(C_T\) is increasing in \(T\). \(\square\)

**Remark 3.2.** Let \(B_0 : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}\) be the generator of a strongly continuous analytic
semigroup and \(\sigma \in (0, 1)\). Then, as noted in [22, Proposition 2.2.2], the intermediate space
\(\mathcal{D}_{B_0}(\rho, \infty)\) does not depend on the operator \(B_0\) itself, as it coincides with a real interpolation
space between the spaces \(\mathcal{Y}\) and \(\mathcal{X}\) (with equivalence of respective norms). We shall use this fact in the following where we consider a Hölder space setting.
3.1. **Intermediate spaces in a Hölder space setting.** In our application to the nonlocal mean curvature flow, we will need to consider the special case where \( \mathcal{Y} = C^{1+\beta}(\mathbb{R}^{N-1}) \) and \( \mathcal{X} = C^{\beta-\gamma}(\mathbb{R}^{N-1}) \) for values \( \gamma \in (0,1) \). Our next result provides a characterization of the intermediate space \( \mathcal{D}_{B_0}(\rho,\infty) \) defined in (3.2) in this particular case. As noted in Remark [3.2] this space does not depend on the particular choice of a generator \( B_0 : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X} \) of a strongly continuous analytic semigroup.

**Proposition 3.3.** Let \( \gamma \in (-1,1) \), \( \beta \in (\sigma,1+\sigma) \) with \( \beta \notin \mathbb{N} \). Let \( B_0 : C^{1+\beta}(\mathbb{R}^{N-1}) \subset C^{\beta-\gamma}(\mathbb{R}^{N-1}) \to C^{\beta-\gamma}(\mathbb{R}^{N-1}) \) be any infinitesimal generator of a strongly continuous analytic semigroup on \( C^{\beta-\gamma}(\mathbb{R}^{N-1}) \). Let \( \rho \in (0,\min(1,1/\sigma)) \) and put \( \gamma \rho := \beta + \rho(1 + \sigma) \). Then \( C^{\beta-\gamma}(\mathbb{R}^{N-1}) \cap C^{\gamma\rho-\sigma}(\mathbb{R}^{N-1}) \subset \mathcal{D}_{B_0}(\rho,\infty) \) and there exists \( c = c(\sigma,\beta,\rho) > 0 \) such that

\[
\|f\|_{\mathcal{D}_{B_0}(\rho,\infty)} \leq c\|f\|_{C^{\gamma\rho-\sigma}} \quad \text{for all } f \in C^{\beta-\gamma}(\mathbb{R}^{N-1}) \cap C^{\gamma\rho-\sigma}(\mathbb{R}^{N-1}).
\]

If moreover \( \gamma \rho - \sigma \notin \mathbb{N} \), then \( C^{\beta-\gamma}(\mathbb{R}^{N-1}) \cap C^{\gamma\rho-\sigma}(\mathbb{R}^{N-1}) = \mathcal{D}_{B_0}(\rho,\infty) \), with equivalence of their respective norms.

**Proof.** By [2, Corollary 2.17], for \( \sigma \in (-1,1) \), the standard fractional Laplacian \( (-\Delta)^{1/2} : C^{1+\beta}(\mathbb{R}^{N-1}) \to C^{\beta-\gamma}(\mathbb{R}^{N-1}) \) is a generator of a strongly continuous analytic semigroup, and therefore

\[
\mathcal{D}_{B_0}(\rho,\infty) = \mathcal{D}(-\Delta)^{1/2}(\rho,\infty),
\]

with equivalence of their respective norms, see Remark [3.2].

For simplicity, we write \( \gamma = \gamma \rho \). Letting \( f \in C^{\beta-\gamma}(\mathbb{R}^{N-1}) \cap C^{\gamma\rho-\sigma}(\mathbb{R}^{N-1}) \), we then have

\[
e^{-t(-\Delta)^{1/2}} f(x) = K(\cdot,t) * f(x) = \int_{\mathbb{R}^{N-1}} K(x-y,t)f(y)dy,
\]

where \( K \) is the heat kernel of \( (-\Delta)^{1/2} \). It is known, see e.g. [6], that

\[
K(x,t) = t^{-\frac{N-1}{1+\sigma}} P(t^{-\frac{1}{1+\sigma}} x),
\]

for some radially symmetric function \( P \in C^\infty(\mathbb{R}^{N-1}) \), with

\[
|D^k P(y)| \leq \frac{C(k,N,\sigma)}{1 + |y|^{N+\sigma+k}}.
\]

From this, we can apply [17, Lemma 2.1] to get

\[
|(-\Delta)^{1/2} P(z)| \leq \frac{C}{1 + |z|^{N+\sigma}}.
\]

We have, using that \( \int_{\mathbb{R}^{N-1}} (\Delta)^{1/2} P(z) dz = 0 \) and a change of variables,

\[
H_t(x) := t (-\Delta)^{1/2} e^{-t(-\Delta)^{1/2}} f(x) = t \int_{\mathbb{R}^{N-1}} K(x,y,t) f(y) dy
\]

\[
= t^{-\frac{N-1}{1+\sigma}} \int_{\mathbb{R}^{N-1}} (\Delta)^{1/2} P(t^{-\frac{1}{1+\sigma}} (x-y)) f(y) dy
\]

\[
= \int_{\mathbb{R}^{N-1}} P(z) f(x-t^{-\frac{1}{1+\sigma}} z) dz = \int_{\mathbb{R}^{N-1}} (\Delta)^{1/2} P(z) [f(x-t^{-\frac{1}{1+\sigma}} z) - f(x)] dz.
\]

Now using that \( (-\Delta)^{1/2} P \) is even, we conclude that

\[
H_t(x) = \frac{1}{2} \int_{\mathbb{R}^{N-1}} (\Delta)^{1/2} P(z) [f(x-t^{-\frac{1}{1+\sigma}} z) + f(x+t^{-\frac{1}{1+\sigma}} z) - 2 f(x)] dz.
\]
Suppose that \( \gamma - \beta \leq 1 \). We then deduce, from that
\[
|H_t(x)| \leq t^{\frac{\alpha}{1 + \sigma}} C[f]_{C^{\gamma - \beta}} \int_{\mathbb{R}^{N-1}} |z|^\gamma |z|^{-\beta} \left( -\Delta \right)^{\frac{1 + \sigma}{4}} P(z) \, dz.
\] (3.9)

If also \( \gamma - \beta > 1 \) then since
\[
||f(x - t^{\frac{1}{1 + \sigma}} z) + f(x + t^{\frac{1}{1 + \sigma}} z) - 2f(x)|| \leq 2|\nabla f|_{C^{\gamma - \beta - 1} t^{\frac{1}{1 + \sigma}} |z|} \min(1, (t^{\frac{1}{1 + \sigma}} |z|)^{\gamma - \beta - 1}),
\]
we still have (3.9). As a consequence
\[
\|H_t\|_{L^\infty} \leq C t^{\rho} \|f\|_{C^{\gamma - \beta}} \quad \text{for all } \rho \in (0, \min(1, \frac{1}{1 + \sigma})).
\] (3.10)

To proceed, we start with the case \( \gamma - \sigma \leq 1 \). Then for \( x \neq x' \in \mathbb{R}^{N-1} \), by (3.7) and the fact that \( 0 < \gamma - \beta = \rho(1 + \sigma) < 1 + \sigma \), we have
\[
|H_t(x) - H_t(x')| \leq C[f]_{C^{\gamma - \sigma}} \int_{\mathbb{R}^{N-1}} |(-\Delta)^{\frac{1 + \sigma}{2}} P(z)| \min(|x - x'|, t^{\frac{1}{1 + \sigma}} |z|)^{\gamma - \sigma} \, dz
\]
\[
\leq C[f]_{C^{\gamma - \sigma}} t^{\frac{1}{1 + \sigma}} \int_{|z| \leq t^{\frac{1}{1 + \sigma}} |x - x'|} |z|^\gamma |z|^{-\beta} \left( -\Delta \right)^{\frac{1 + \sigma}{4}} P(z) \, dz
\]
\[
+ C[f]_{C^{\gamma - \sigma}} |x - x'|^{\gamma - \sigma} \int_{|z| \geq t^{\frac{1}{1 + \sigma}} |x - x'|} |(-\Delta)^{\frac{1 + \sigma}{2}} P(z)| \, dz
\]
\[
\leq C[f]_{C^{\gamma - \sigma}} t^{\frac{1}{1 + \sigma}} t^{\frac{\gamma - \beta}{1 + \sigma}} |x - x'|^{\gamma - \sigma} \int_{\mathbb{R}^{N-1}} |z|^\gamma |z|^{-\beta} \left( -\Delta \right)^{\frac{1 + \sigma}{4}} P(z) \, dz
\]
\[
+ C[f]_{C^{\gamma - \sigma}} |x - x'|^{\gamma - \sigma} \int_{|z| \geq t^{\frac{1}{1 + \sigma}} |x - x'|} \frac{1}{1 + |z|^{N-1 + (\gamma - \beta)}} \, dz
\]
\[
\leq C[f]_{C^{\gamma - \sigma}} \left( t^{\frac{\gamma - \beta}{1 + \sigma}} |x - x'|^{\gamma - \sigma} + |x - x'|^{\gamma - \sigma} (t^{\frac{1}{1 + \sigma}} |x - x'|)^{(\gamma - \beta)} \right)
\]
\[
\leq C[f]_{C^{\gamma - \sigma}} t^{\frac{\gamma - \beta}{1 + \sigma}} |x - x'|^{\gamma - \sigma}.
\]

This clearly implies that
\[
\sup_{t \in (0,1)} t^{-\rho} |H_t|_{C^{\beta - \sigma}} \leq C[f]_{C^{\gamma - \sigma}(\mathbb{R}^{N-1})}.
\]

As a consequence, using also (3.8), we have
\[
\sup_{t \in (0,1)} t^{-\rho} \|H_t\|_{C^{\beta - \sigma}} \leq C \|f\|_{C^{\gamma - \sigma}} \quad \text{for } \gamma - \sigma \leq 1.
\] (3.11)

We now assume that \( \gamma - \sigma > 1 \) and we observe that from the upper bounds of \( \rho \) and \( \beta \) we have \( \gamma - \sigma \leq 2 \). Therefore (recalling (3.8)) using that
\[
||f(x - t^{\frac{1}{1 + \sigma}} z) + f(x + t^{\frac{1}{1 + \sigma}} z) - 2f(x)|| \leq 2|\nabla f|_{C^{\gamma - \sigma - 1} t^{\frac{1}{1 + \sigma}} |z|} \min(|x - x'|, (t^{\frac{1}{1 + \sigma}} |z|)^{\gamma - \sigma - 1})
\]
and the same argument as above, we get
\[
\sup_{t \in (0,1)} t^{-\rho} |H_t|_{C^{\beta - \sigma}} \leq C[\nabla f]_{C^{\gamma - \sigma - 1}}.
\]

From this, (3.10) and (3.11), we conclude that for all \( \rho \in (0, \min(1, \frac{1}{1 + \sigma})) \),
\[
\|f\|_{(-\Delta)^{\frac{1 + \sigma}{4}}(\rho, \infty)} = \|f\|_{C^{\beta - \sigma}} + \sup_{t \in (0,1)} t^{-\rho} \|H_t\|_{C^{\beta - \sigma}} \leq C \|f\|_{C^{\gamma - \sigma}}.
\]
Thanks to \((3.5)\) and \((3.4)\), we obtain
\[
\|f\|_{D_{B_0}(\rho,\infty)} \leq C\|f\|_{C^{\gamma-\sigma}}.
\]
Therefore \(C_{\beta-\sigma}(\mathbb{R}^{N-1}) \cap C_{\gamma-\sigma}(\mathbb{R}^{N-1}) \subset D_{B_0}(\rho,\infty)\) and \((3.4)\) holds.

Next, by \([22\text{, Proposition 2.2.2}]\) and \([22\text{, Corollary 1.2.18}]\) we have that \(D_{B_0}(\rho,\infty)\) is continuously embedded in \(C_{\beta-\sigma}(\mathbb{R}^{N-1})\), provided \(\gamma - \sigma \not\in \mathbb{N}\). Since, by definition \(D_{B_0}(\rho,\infty) \subset C_{0-\sigma}(\mathbb{R}^{N-1})\), we get the desired result.

\(\square\)

**Remark 3.4.** We point out that Proposition 3.3 still holds when \(\beta \in \{0,1\}\) and \(\gamma - \sigma = 1\), provided that, for \(k \in \{1,2\}\), the spaces \(C^k(\mathbb{R}^{N-1})\) and \(C_0^k(\mathbb{R}^{N-1})\) are, respectively, replaced with the Hölder-Zygmund space \(C^k(\mathbb{R}^{N-1})\) and the space \(C_0^k(\mathbb{R}^{N-1}) = C^\infty(\mathbb{R}^{N-1})\)\(|\|c_k\|\). Recall that the space \(C^1(\mathbb{R}^{N-1})\) is defined by
\[
C^1(\mathbb{R}^{N-1}) := \left\{ u \in C_0(\mathbb{R}^{N-1}) : [u]_{C^1} := \sup_{x,y \in \mathbb{R}^{N-1}, x \neq y} \frac{|u(x) - 2u(\frac{x+y}{2}) + u(y)|}{|x-y|} < \infty \right\}
\]
and \(C^2(\mathbb{R}^{N-1})\) is given by the set of \(u \in C^1(\mathbb{R}^{N-1})\) such that \(\partial_i u \in C^1(\mathbb{R}^{N-1})\) for \(i = 1, \ldots, N-1\).

### 3.2. A class of nonlocal operators generating strongly continuous analytic semigroups

In this section, we consider a class of nonlocal operators which we prove to generate strongly continuous analytic semigroups.

For fixed \(\gamma \in (\alpha, 1)\), we consider linear nonlocal operator \(L_K : C^{1+\gamma}(\mathbb{R}^{N-1}) \to C^{\gamma-\alpha}(\mathbb{R}^{N-1})\) given by
\[
L_Ku(x) = P.V. \int_{\mathbb{R}^{N-1}} \frac{u(x) - u(y)}{|x-y|^{N+\alpha}} K(x,y) dy,
\]
where \(K : (\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}) \setminus \{(x,x) : x \in \mathbb{R}^{N-1}\} \to \mathbb{R}\) is a measurable function satisfying the following assumptions.

**Assumptions 3.5.**

(i) \(K(x,y) = K(y,x)\) for all \(x,y \in \mathbb{R}^{N-1}, x \neq y\).
(ii) \(\frac{1}{\kappa} \leq K(x,y) \leq \kappa\), for some \(\kappa > 1\).
(iii) there exists \(A_K \in C^\gamma(\mathbb{R}^{N-1} \times [0,\infty) \times S^{N-2})\), with \(A_K(x,0,0) = A_K(x,0,-\theta)\) such that \(A_K(x,r,\theta) = K(x,x - r\theta)\) for all \(r > 0\) and \(\|A_K\|_{C^\gamma(\mathbb{R}^{N-1} \times [0,\infty) \times S^{N-2})} \leq \kappa\).

In order to apply the estimates in Lemma 2.2 it will be useful to decompose \(L_K\) in two parts, writing
\[
L_Ku(x) = \int_{S^{N-2}} \int_0^\infty \frac{\delta_u(x,r,\theta)}{r^{2+\alpha}} A_K(x,r,\theta) dr d\theta + \int_{S^{N-2}} \int_0^\infty \frac{\delta_u(x,r,\theta)}{r^{2+\alpha}} A_{\nu}^K(x,r,\theta) dr d\theta,
\]
for \(u \in C^{1+\gamma}(\mathbb{R}^{N-1})\) with
\[
A_K^\nu(x,r,\theta) = A_K(x,r,\theta) + A_K(x,r,-\theta), \quad A_{\nu}^K(x,r,\theta) = A_K(x,r,\theta) - A_K(x,r,-\theta)
\]
and \(\delta_u, \delta_u\) defined as in \((2.4)\). We now define the set
\[
O^\nu_{\text{loc}} := \{ u \in C^{1+\gamma}_{\text{loc}}(\mathbb{R}^{N-1}) : \|\nabla u\|_{C^\gamma} \leq \nu\},
\]
for \(\nu > 0\), and we state the following estimates.
Lemma 3.6.  
(i) Let $\gamma > \alpha$. Then there exist $C, C' > 0$ depending only on $N, \alpha, \gamma, \kappa$ such that
\[
\|L_Ku\|_{C^{\gamma-\alpha}} \leq C'\|\nabla u\|_{C^\gamma} \quad \text{for all } u \in \mathcal{O}_{\nu}^\gamma
\] (3.16)
and
\[
\|\nabla u\|_{C^\gamma} \leq C(\|L_Ku\|_{C^{\gamma-\alpha}} + \|\nabla u\|_{L^\infty}) \quad \text{for all } u \in \mathcal{O}_{\nu}^\gamma.
\] (3.17)
(ii) If $\beta \in (\alpha, \gamma)$, then there exist $L_Ku \in C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$ for all $u \in C^{1+\beta}_0(\mathbb{R}^{N-1}) \cap C^\gamma_{\nu}$. 

Proof. (i) It is clear from Lemma 2.2 and (3.14) that (3.16) holds.

Now we let $f(x) = L_Ku(x)$, since $K(\cdot + x_0, \cdot + x_0)$ satisfies also Assumptions 3.5 for all $x_0 \in \mathbb{R}^{N-1}$, by [13, Theorem 1.3(ii), Theorem 1.4(iii)], we get
\[
\|\nabla u\|_{C^\gamma(B_1(x_0))} = \|\nabla(u - u(x_0))\|_{C^\gamma(B_1(x_0))},
\]
where $C$ may change value from one line to another. Since $x_0$ is arbitrary, (3.17) follows.

(ii) By Lemma 2.2 and (3.11) if $u \in C^{1+\beta}_0(\mathbb{R}^{N-1})$, then $L_Ku \in C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$.

Combining Lemma 3.6 and a result from [2], we now exhibit a useful class of operators generating strongly continuous analytic semigroups in Hölder spaces.

Proposition 3.7. Let $\gamma \in (\beta, 1)$ and $b \in C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$ be such that
\[
\frac{1}{\kappa} \leq |b(x)| \leq \kappa \quad \text{for all } x \in \mathbb{R}^{N-1}.
\]
Let $L : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$ be a bounded linear operator satisfying for all $\varepsilon > 0$, there exists $c_\varepsilon$ such that
\[
\|Lu\|_{C^{\beta-\alpha}} \leq \varepsilon\|L_Ku\|_{C^{\beta-\alpha}} + c_\varepsilon\|u\|_{C^{\beta-\alpha}} \quad \text{for all } u \in C^{1+\beta}_0(\mathbb{R}^{N-1}).
\] (3.18)
Suppose that for all $k \in \mathbb{N}$,
\[
\sup_{\theta \in S^{N-2}} \|\nabla^k_\theta A_K^\gamma(\cdot, 0, \theta)\|_{C^\gamma} \leq C_k.
\] (3.19)
Then, the operator
\[
b(x)L_K + L : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1}),
\] (3.20)
with domain $C^{1+\beta}_0(\mathbb{R}^{N-1})$, generates a strongly continuous analytic semigroup on $C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$.

Proof. Recalling (3.14), we write
\[
b(x)L_K := L_1 + L_2,
\] (3.21)
where

\[ L_1 u(x) = b(x) \left( P.V. \int_{\mathbb{R}^N} \frac{(u(x) - u(x + y))}{|y|^{N+\alpha}} A^c_K(x, 0, y/|y|) \, dy \right) = \frac{b(x)}{2} \int_{\mathbb{R}^N} \frac{(2u(x) - u(x + y) - u(x - y))}{|y|^{N+\alpha}} A^c_K(x, 0, y/|y|) \, dy \]

and

\[ L_2 u(x) = \int_{\mathbb{R}^N} \frac{(u(x) - u(x + y))}{|y|^{N+\alpha}} b(x) [A^c_K(x, y, y/|y|) - A^c_K(x, 0, y/|y|)] \, dy. \]

By Lemma 2.2, the operator \( L_1 : C_0^{1+\beta}(\mathbb{R}^{N-1}) \to C_0^{\beta-\alpha}(\mathbb{R}^{N-1}) \) is bounded.

Letting \( k(x, y) := b(x)A^c_K(x, 0, y/|y|)|y|^{-\alpha} \) and \( \chi \in C_c^\infty(\mathbb{R}^{N-1}) \) be nonnegative, radially symmetric and satisfy \( \chi = 1 \) for \( |y| \leq 2 \). One can split \( k = k_1 + k_2 \), where \( k_1(x, y) = k(x, y)\chi(y) \) and \( k_2(x, y) = k(x, y)(1 - \chi(y)) \). Then, it is easy to see that \( k_1 \) is zero for \( |y| \geq 2 \) and \( k_2 \) is zero for \( |y| \leq 1 \). In addition, using also (3.19), for all \( n \in \mathbb{N}^{N-1} \) with \( |n| \leq N \), there exists \( C > 0, c > 0 \) only depending on \( N, \alpha, \gamma, n \) and \( \kappa \) such that

\[
\begin{cases}
\|g^\xi_k(\cdot, y)\|_{C^\gamma} \leq C|y|^{-\alpha - |n|}, & 0 < |y| \leq 2, \\
k_1(x, y) \leq C|y|^{-\alpha}, & 0 < |y| \leq 1, x \in \mathbb{R}^{N-1}, \\
k_2(\cdot, y) \|_{C^\gamma} \leq C|y|^{-\alpha'}, & 0 < |y| \leq 1, \quad \text{for } 0 \leq \alpha' < \alpha < 1 \\
\int_{|y| \geq 1} \|k_2(\cdot, y)\|_{C^\gamma} \, dy < \infty, \\
\lim_{|y| \to \infty} \|k_2(\cdot, y)\|_{C^\gamma} = 0.
\end{cases}
\]  

(3.22)

In view of this, we can apply [2] Corollary 2.17, to deduce that, for \( \beta \in (\alpha, \gamma) \), the operator \( L_1 : C_0^{1+\beta}(\mathbb{R}^{N-1}) \to C_0^{\beta-\alpha}(\mathbb{R}^{N-1}) \) generates a strongly continuous analytic semigroup.

By Lemma 2.2 and (2.2), for all \( \varepsilon > 0 \), there exists \( C_2(\varepsilon) > 0 \) such that

\[ \|L_2 u\|_{C^{\beta-\alpha}} \leq \varepsilon \|\nabla u\|_{C_0^\gamma} \leq C (\varepsilon \|u\|_{C^{1+\beta}} + C_2(\varepsilon) \|u\|_{L^\infty}). \]

Combining this information with Lemma 3.6 (i), we get

\[ \|L_2 u\|_{C^{\beta-\alpha}} \leq C (\varepsilon \|(L_1 + L_2) u\|_{C^{\beta-\alpha}} + C_2(\varepsilon) \|u\|_{L^\infty}). \]

with a possibly different constant \( C \) not depending on \( \varepsilon \). The above two estimates give

\[ \|L_2 u\|_{C^{\beta-\alpha}} \leq C (\varepsilon \|L_1 u\|_{C^{\beta-\alpha}} + C_2(\varepsilon) \|u\|_{L^\infty}). \]

From this and (3.18), we have

\[ \|(L_2 + L) u\|_{C^{\beta-\alpha}} \leq \varepsilon \|L_1 u\|_{C^{\beta-\alpha}} + C(\varepsilon) \|u\|_{C^{\beta-\alpha}} \quad \text{for all } u \in C_0^{1+\beta}(\mathbb{R}^{N-1}). \]

Now applying [24] Theorem 2.1, we deduce that \( b(x)L_K + L = L_1 + L_2 + L \) is a generator of a strongly continuous analytic semigroup.

We shall now derive the following existence and uniqueness result from Theorem 3.1 Proposition 3.3 and Proposition 3.7

**Theorem 3.8.** Let \( 0 < \alpha < \beta < 1 \) and \( B_0 := b(x)L_K + L : C_0^{1+\beta}(\mathbb{R}^{N-1}) \to C_0^{\beta-\alpha}(\mathbb{R}^{N-1}) \) be given by (3.20). Let \( T > 0 \) and \( \rho \in (0, \frac{1}{1+\alpha}). \) Let \( u_0 \in C_0^{1+\beta}(\mathbb{R}^{N-1}) \) and \( f \in \)
\( C^\beta([0, T], C^{\beta-\alpha}_0(\mathbb{R}^{N-1})) \) be such that \( f(0) - B_0u_0 \in C^\gamma(\mathbb{R}^{N-1}) \) with \( \gamma = \beta + \rho(1 + \alpha) \). Then the problem

\[
\begin{aligned}
&u'(t) + B_0u(t) = f(t), \\
u(0) = u_0
\end{aligned}
\tag{3.23}
\]

admits a unique solution \( u \in C^\beta([0, T], C^{1+\beta}_0) \) \( \cap C^{1+\rho}([0, T], C^{\beta-\alpha}_0) \). Moreover, there exists \( C_T = C(T, \rho, \kappa, N, \alpha, \gamma, \beta) > 0 \) such that

\[
\|u - u_0\|_{C^\rho([0, T], C^{\beta-\alpha})} + \|u - u_0\|_{C^{1+\rho}([0, T], C^{\beta-\alpha})}
\leq C_T \left( \|f - B_0u_0\|_{C^\rho([0, T], C^{\beta-\alpha})} + \|f(0) - B_0u_0\|_{C^{\gamma-\alpha}} \right).
\tag{3.24}
\]

In addition, \( C_T \leq C_{\overline{T}} \) for all \( T \leq \overline{T} \).

**Proof.** From Theorem 3.1 and Proposition 3.7, we get a unique solution \( u \in C^\rho([0, T], C^{1+\beta}_0) \) \( \cap C^{1+\rho}([0, T], C^{\beta-\alpha}_0) \) to (3.23). Moreover,

\[
\|B_0(u - u_0)\|_{C^\rho([0, T], C^{\beta-\alpha})} + \|u - u_0\|_{C^{1+\rho}([0, T], C^{\beta-\alpha})}
\leq C_T \left( \|f - B_0u_0\|_{C^\rho([0, T], C^{\beta-\alpha})} + \|f(0) - B_0u_0\|_{D_{B_0}(\rho, \infty)} \right).
\]

By Proposition 3.3, we have \( \|f(0) - B_0u_0\|_{D_{B_0}(\rho, \infty)} \leq C'\|f(0) - B_0u_0\|_{C^{\gamma-\alpha}} \). On the other hand, since \( B_0 \) is closed, we obtain

\[
\|B_0\|_{C^{\beta-\alpha}} + \|v\|_{C^{\beta-\alpha}} \geq C (\|v\|_{C^{1+\beta}} + \|v\|_{C^{\beta-\alpha}}), \quad \forall v \in C^{1+\beta}(\mathbb{R}^{N-1}).
\]

From this and (3.24), we get the result. \( \square \)

## 4. Regularity property of the nonlocal mean curvature operator

By changes of variables and the fact that \( G(p) = -\int_{\rho}^{p} \frac{d\tau}{(1 + \tau^2)^{\frac{N+\alpha}{2}}} \) is odd, for \( w \in C^{1,\gamma}_{loc}(\mathbb{R}^{N-1}) \), with \( \gamma > \alpha \), we have

\[
H(w)(x) = P.V. \int_{\mathbb{R}^{N-1}} \frac{G(p_w(x, y))}{|x-y|^{N+\alpha}} dy
= \frac{P.V.}{2} \int_{\mathbb{R}^{N-1}} \frac{G(p_w(x, x-z)) + G(p_w(x, x+z))}{|z|^{N+\alpha}} dz
= \frac{P.V.}{2} \int_{\mathbb{R}^{N-1}} \frac{G(p_w(x, x-z)) - G(-p_w(x, x+z))}{|z|^{N+\alpha}} dz.
\]

Therefore, by the fundamental theorem of calculus and the fact that \( G'(p) = -2(1+p^2)^{-\frac{N+\alpha}{2}} \) is even, we get

\[
H(w)(x) = \int_{\mathbb{R}^{N-1}} \frac{2w(x) - w(x-z) - w(x+z)}{|z|^{N+\alpha}} \mathcal{K}_w(x, z) dz,
\tag{4.1}
\]

where

\[
\mathcal{K}_w(x, z) = \int_0^1 \left( 1 + (\tau p_w(x, x-z) - (1 - \tau)p_w(x, x+z) \right)^{\frac{N+\alpha}{2}} d\tau.
\tag{4.2}
\]

For the remainder of this section, it will be convenient to write

\[
H(w)(x) = \int_{S^{N-2}} \int_0^\infty \frac{\delta_{w}(x, r, \theta)}{r^{2+\alpha}} \mathcal{A}_w(x, r, \theta) dr d\theta,
\tag{4.3}
\]
where \( \delta_e w(x, r, \theta) = 2w(x) - w(x - r\theta) - w(x + r\theta) \), for all \((x, r, \theta) \in \mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2}\) and

\[
\mathcal{A}_w : \mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2} \to \mathbb{R}
\]

\[
(x, r, \theta) \mapsto \mathcal{A}_w(x, r, \theta) = K_w(x, r, \theta), \quad \mathcal{A}_w(x, 0, \theta) = \left(1 + (\nabla w(x) \cdot \theta)^2\right)^{-\frac{N+\alpha}{2}}. \tag{4.4}
\]

**Lemma 4.1.** Let \( k \in \mathbb{N} \) and \( 0 < \gamma < 1 \). Then there exists \( C = C(k, N, \alpha, \gamma) > 0 \) such that for all \( u, w_1, \ldots, w_k \in C^{1+\gamma}(\mathbb{R}^{N-1}) \), we have

\[
\|\partial^k_u \mathcal{A}_u[w_1, \ldots, w_k]\|_{C^\gamma(\mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2})} \leq C (1 + \|\nabla u\|_{C^\gamma}) \prod_{i=1}^k \|\nabla w_i\|_{C^\gamma}.
\]

**Proof.** This follows from the definition of \( \mathcal{A} \). \( \square \)

We have the following lemma.

**Lemma 4.2.** Let \( \gamma \in (\alpha, 1) \). Then the map \( H : C^{1+\gamma}(\mathbb{R}^{N-1}) \to C^{\gamma-\alpha}(\mathbb{R}^{N-1}) \) is of class \( C^\infty \) and for all \( k \in \mathbb{N}, u, w_1, \ldots, w_k \in C^{1+\gamma}(\mathbb{R}^{N-1}) \),

\[
\|D^k H(u)[w_1, \ldots, w_k]\|_{C^{\gamma-\alpha}} \leq C (1 + \|\nabla u\|_{C^\gamma}) \prod_{i=1}^k \|\nabla w_i\|_{C^\gamma}. \tag{4.5}
\]

with \( C > 0 \) only depending on \( k, N, \alpha, \gamma \). Moreover \( H : C^{1+\gamma}_0(\mathbb{R}^{N-1}) \to C^{\gamma-\alpha}_0(\mathbb{R}^{N-1}) \) is of class \( C^\infty \) and (4.5) holds for all \( u, w_1, \ldots, w_k \in C^{1+\gamma}_0(\mathbb{R}^{N-1}) \).

**Proof.** For all \( k \in \mathbb{N}, w_0, w_1, \ldots, w_k \in C^{1+\gamma}(\mathbb{R}^{N-1}) \), we define

\[
\mathcal{B}^\gamma_k(u)[w_0; w_1, \ldots, w_k](x) = \int_{S^{N-2}} \int_0^\infty \delta_e w_0(x, r, \theta) \partial^k_u \mathcal{A}_u[w_1, \ldots, w_k](x, r, \theta) r^{-2\alpha} dr d\theta.
\]

We now prove that \( H \) is of class \( C^\infty \) and for all \( k \in \mathbb{N},

\[
D^k H(u)[w_1, \ldots, w_k] = \mathcal{B}^\gamma_k(u)[w_0; w_1, \ldots, w_k] + \sum_{j=1}^k \mathcal{B}^\gamma_{k-1}(u)[w_j; w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_k]. \tag{4.6}
\]

For this, it is enough to prove that \( u \mapsto \mathcal{B}^\gamma_k(u)[w_0; w_1, \ldots, w_k] \) is differentiable. By the linearity of the map \( u \mapsto \delta_e u \) it is also enough to prove the differentiability of \( u \mapsto \mathcal{B}^\gamma_k(u)[w_0; w_1, \ldots, w_k] \) with

\[
\|D_u \mathcal{B}^\gamma_k(u)[w_0; w_1, \ldots, w_k]\|_{C^{\gamma-\alpha}} \leq C (1 + \|\nabla u\|_{C^\gamma}) \prod_{i=0}^k \|\nabla w_i\|_{C^\gamma}. \tag{4.7}
\]
Let $v \in C^{1+\gamma}(\mathbb{R}^{N-1})$, with $\|\nabla v\|_{C^\gamma} \leq 1$. We have

$$\mathcal{B}_k^e(u + v)[w_0; w_1, \ldots, w_k](x) - \mathcal{B}_k^e(u)[w_0; w_1, \ldots, w_k](x) - \int_{S^{N-2}} \int_0^\infty \delta_e w_0(x, r, \theta) \partial^{k+1}_u \mathcal{A}_u^e[w_1, \ldots, w_k, v](x, r, \theta) r^{-2-\alpha} \, dr \, d\theta$$

$$= \int_{S^{N-2}} \int_0^\infty r^{-2-\alpha} \delta_e w_0(x, r, \theta) \times \int_1^\infty \left( \partial^{k+1}_u \mathcal{A}_u^e[z_1, \ldots, z_k, v](x, r, \theta) - \partial^{k+1}_u \mathcal{A}_u^e[z_1, \ldots, z_k, v](x, r, \theta) \right) \, \rho \, dr \, d\theta$$

$$= \int_{S^{N-2}} \int_0^\infty r^{-2-\alpha} \delta_e w_0(x, r, \theta) \int_1^\infty \rho \int_0^1 \partial^{k+2}_u \mathcal{A}_u^e[z_1, \ldots, z_k, v](x, r, \theta) \, d\rho \, dr \, d\theta$$

Now by Lemma 2.2 and Lemma 4.1, we get

$$\left\| \mathcal{B}_k^e(u + v)[w_0; w_1, \ldots, w_k] - \mathcal{B}_k^e(u)[w_0; w_1, \ldots, w_k] - \int_{S^{N-2}} \int_0^\infty r^{-2-\alpha} \delta_e w_0(x, r, \theta) \partial^{k+1}_u \mathcal{A}_u^e[w_1, \ldots, w_k, v](x, r, \theta) \, dr \, d\theta \right\|_{C^{\gamma-\alpha}}$$

$$\leq C \left( 1 + \|\nabla u\|_{C^\gamma} \right) \prod_{i=0}^k \|\nabla w_i\|_{C^\gamma} \|\nabla v\|_{C^{\gamma-\alpha}}^2.$$

From this, we deduce that $u \mapsto \mathcal{B}_k^e(u)[w_0; w_1, \ldots, w_k]$ is differentiable and (4.7) holds.

Now the fact that $H : C^{1+\gamma}(\mathbb{R}^{N-1}) \to C^{\gamma-\alpha}(\mathbb{R}^{N-1})$ is of class $C^\infty$, follows easily by induction, thanks to (4.6) and the estimates on $\mathcal{B}_k^e$. Moreover the estimate on the derivative $H$ is an immediate consequence of those of $\mathcal{B}_k^e$.

In view of (4.6), Lemma 2.2 and Lemma 4.1, the fact that $H : C^{1+\gamma}(\mathbb{R}^{N-1}) \to C^{\gamma-\alpha}(\mathbb{R}^{N-1})$ is of class $C^\infty$ follows similarly as above. In fact one can simply replace in the above argument $C^{1+\gamma}(\mathbb{R}^{N-1})$ with $C_0^{1+\gamma}(\mathbb{R}^{N-1})$ and $C^{\gamma-\alpha}(\mathbb{R}^{N-1})$ with $C_0^{\gamma-\alpha}(\mathbb{R}^{N-1})$. □

We compute next the explicit expression of $DH$.

**Lemma 4.3.** Let $u \in C^{1+\gamma}_{\text{loc}}(\mathbb{R}^{N-1})$, for some $\gamma \in (\alpha, 1)$, with $\nabla u \in C^\gamma(\mathbb{R}^{N-1})$. Then for all $w \in C^{1+\gamma}(\mathbb{R}^{N-1})$

$$DH(u)[w](x) = -P.V. \int_{\mathbb{R}^{N-1}} \frac{w(x) - w(y)}{|x - y|^{N+\alpha}} \mathcal{G}'(p_u(x, y)) \, dy$$

(4.8)

and

$$\|DH(u)[w]\|_{C^{\gamma-\alpha}} \leq C \left( 1 + \|\nabla u\|_{C^\gamma} \right) \|\nabla w\|_{C^\gamma},$$

where $\mathcal{G}'(p) = -2(1 + p^2)^{-\frac{N+\alpha}{2}}$. Here $C > 0$, depends only on $N, \alpha, \beta$ and $\gamma$. If moreover $\gamma > \beta$ and $w \in C_0^{1+\beta}(\mathbb{R}^{N-1})$, then $DH(u)[w] \in C_0^{\beta-\alpha}(\mathbb{R}^{N-1})$.

**Proof.** We consider the linear operator $M$ defined as

$$Mw(x) := -P.V. \int_{\mathbb{R}^{N-1}} \frac{w(x) - w(y)}{|x - y|^{N+\alpha}} \mathcal{G}'(p_u(x, y)) \, dy.$$
We then have
\[ H(u + w)(x) - H(u)(x) - Mw(x) = P.V. \int_{\mathbb{R}^{N-1}} \left[ G'(p_{u+\tau w}(x,y)) - G'(p_u(x,y)) \right] w(x,y) \, d\tau \, dx. \]

We then get
\[ \Gamma(x) := H(u + w)(x) - H(u)(x) - Mw(x) = P.V. \int_{\mathbb{R}^{N-1}} \frac{(p_w(x,x-y))^2 \mu(x,|y|/|y|)}{|y|^{N-1+\alpha}} \, dx, \]

where
\[ \mu(x,r,\theta) := 2 \int_0^1 \tau \int_0^1 G''(p_{u+\tau w}(x,x-\tau \theta)) \, d\tau \, d\varphi \]

and
\[ \mu(x,0,\theta) := 2 \int_0^1 \tau \int_0^1 G''(\nabla u(x) \cdot \theta + \tau \varrho \nabla w(x) \cdot \theta) \, d\tau \, d\varphi. \]

Using polar coordinates and recalling (2.4), we can write
\[ \Gamma(x) := P.V. \int_{\mathbb{S}^{N-2}} \int_0^1 r^{-2-\alpha} \left( w(x) - w(x-r\varrho) \right) \int_0^1 \nabla w(x-r\tau \theta) \cdot \theta \, d\tau \mu(x,r,\theta) \, dr \, d\varrho \]

where we used that \( \mu(x,0,-\theta) = -\mu(x,0,\theta) \), from the oddness of \( G'' \). Since
\[ \|\mu\|_{C^\gamma([0,\infty) \times S^{N-2})} \leq C(1 + \|\nabla w\|_{C^\gamma} + \|\nabla u\|_{C^\gamma}), \]

we can thus apply Lemma 2.2(i) to deduce that
\[ \|\Gamma\|_{C^{\gamma-\alpha}} \leq C(1 + \|\nabla w\|_{C^\gamma} + \|\nabla u\|_{C^\gamma})\|\nabla w\|_{C^\gamma}^2. \]

Recalling (4.9), this completes the proof of the first statement of the lemma. The second statement follows from Lemma 2.2(iii). \( \square \)

We deduce the following important result for the weighted fractional mean curvature operator \( \mathcal{H} \) defined in (1.12).

**Corollary 4.4.** The map \( \mathcal{H} : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1}) \) is of class \( C^\infty \). Moreover, letting \( u_0 \in C^{1+\gamma}_0(\mathbb{R}^{N-1}) \) for some \( \gamma \in (\beta,1) \), with \( \nabla u_0 \in C^\gamma(\mathbb{R}^{N-1}) \), then the map
\[ F : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1}), \quad F(u) = D\mathcal{H}(u_0)[u] - \mathcal{H}(u) \]

is of class \( C^\infty \) and for all \( k \in \mathbb{N} \),
\[ \|D^k F(u)\|_{C^{\beta-\alpha}} \leq C \left( 1 + \|\nabla u\|_{C^\beta} + \|\nabla u_0\|_{C^\beta} \right)^C, \]

with \( C = C(k,N,\alpha,\beta) > 1 \).

**Proof.** Recall that \( \mathcal{H}(u) = Q(u) H(u) \) with \( Q(u) = \sqrt{1 + \|\nabla u\|^2} \) and we observe that \( Q \in C^\infty(C^{1+\beta}_0(\mathbb{R}^{N-1}),C^{\beta-\alpha}_0(\mathbb{R}^{N-1})) \) and \( DQ(u_0) \in C^\infty(C^{1+\beta}_0(\mathbb{R}^{N-1}),C^{\beta-\alpha}_0(\mathbb{R}^{N-1})) \). The conclusion then follows from Lemma 4.2 and Lemma 4.3. \( \square \)
4.1. Regularity properties and linearization of the weighted fractional mean curvature operator. We recall that
\[ H(u) := Q(u)H(u), \quad Q(u) := \sqrt{1 + |\nabla u(x)|^2}. \]
Our aim is to prove that, provided \( u_0 \in C^{1+\gamma}_{loc}(\mathbb{R}^{N-1}) \) with \( \gamma < (\beta, 1) \) and \( \nabla u_0 \in C^\gamma_{loc}(\mathbb{R}^{N-1}) \), we have that \( D\mathcal{H}(u_0) : C^{1+\gamma}(\mathbb{R}^{N-1}) \to C^\gamma(\mathbb{R}^{N-1}) \) generates a strongly continuous analytic semigroup. We observe that for all \( w \in C^{1+\beta}(\mathbb{R}^{N-1}) \),
\[ D\mathcal{H}(u_0)[w] := Q(u_0)\bar{L}_1 w + \bar{L}_2 w, \quad (4.10) \]
where
\[ \bar{L}_1 w(x) := -\text{P.V.} \int_{\mathbb{R}^{N-1}} \frac{w(x) - w(y)}{|x-y|^{N+\alpha}} G'(p_{u_0}(x,y)) \, dy \]
\[ = \text{P.V.} \int_{\mathbb{R}^{N-1}} \frac{w(x) - w(y)}{|x-y|^{N+\alpha}} (1 + (p_{u_0}(x,y))^2)^{\frac{N+\alpha}{2}} \, dy \quad (4.11) \]
and
\[ \bar{L}_2 w(x) := H(u_0)(x)DQ(u_0)[w](x) = H(u_0)(x)\frac{\nabla u_0(x) \cdot \nabla w(x)}{Q(u_0)(x)}. \]
For the following, we recall the set
\[ \mathcal{O}_\nu := \{ u \in C^{1+\gamma}_{loc}(\mathbb{R}^{N-1}) : \| \nabla u \|_{C^\gamma} \leq \nu \}, \quad (4.12) \]
for \( \nu > 0 \). We have the following result.

**Lemma 4.5.** Let \( \gamma \in (\alpha, 1) \). Then the nonlocal mean curvature operator \( H \) satisfies the following properties.

(i) There exists \( C' = C'(N, \alpha, \gamma) > 0 \) such that for all \( u_0 \in \mathcal{O}_\nu \),
\[ \| \mathcal{H}(u_0) \|_{C^{\gamma-\alpha}} + \| H(u_0) \|_{C^{\gamma-\alpha}} \leq C' \| \nabla u_0 \|_{C^\gamma}. \quad (4.13) \]
(ii) If \( u \in C^{1+\gamma}_{loc}(\mathbb{R}^{N-1}) \) then \( \mathcal{H}(u), H(u) \in C^{\gamma-\alpha}_{loc}(\mathbb{R}^{N-1}) \).
(iii) There exists \( C = C(N, \alpha, \gamma, \nu) \) such that if \( u \in \mathcal{O}_{\nu}^{\gamma-\alpha} \cap C^{1+\beta}_{loc}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1}) \), for some \( \beta > \alpha \), and satisfies \( \mathcal{H}(u) \in C^{\gamma-\alpha}(\mathbb{R}^{N-1}) \), then \( u \in C^{\gamma+\alpha}(\mathbb{R}^{N-1}) \) and
\[ C \| \nabla u \|_{C^\gamma} \leq \| \mathcal{H}(u) \|_{C^{\gamma-\alpha}} + \| u \|_{L^\infty}. \quad (4.14) \]

**Proof.** In view of (4.3) and since \( \mathcal{H}(u_0) = \sqrt{1 + |\nabla u_0|^2}H(u_0) \), we can apply Lemma 2.2 to get (i) and (ii).

For (iii), we let \( f := \mathcal{H}(u) \) and recalling (1.1), we then have
\[ \int_{\mathbb{R}^{N-1}} \frac{2u(x) - u(x-y) - u(x+y)}{|y|^{N+\alpha}} \mathcal{K}_u(x,y) \, dy = f(x) \quad \text{for all } x \in \mathbb{R}^{N-1}, \]
where \( \mathcal{K}_u(x,y) = \sqrt{1 + |\nabla u(x)|^2}\mathcal{K}_u(x,y) \). Since \( u \in \mathcal{O}_{\nu}^{\gamma-\alpha} \), we have
\[ \sup_{x,y \in \mathbb{R}^{N-1}} |\mathcal{K}_u(x+h,y) - \mathcal{K}_u(x,y)| \leq C(\nu, N, \alpha, \gamma) |h|^{\gamma-\alpha} \quad \text{for all } h \in \mathbb{R}^{N-1}. \]
In addition \( (1 + 4u^2)^{\frac{\gamma-\alpha}{2}} \leq \mathcal{K}_u(x,y) \leq 1 \), for all \( x, y \in \mathbb{R}^{N-1} \). Therefore, applying [4, Theorem 1.2], we obtain
\[ C \| \nabla u \|_{C^\gamma} \leq \| f \|_{C^{\gamma-\alpha}} + \| u \|_{L^\infty}, \]
and the proof is complete. \( \square \)

We now have the following result.
Lemma 4.6. Let \( w, v \in \mathcal{O}_0^\gamma \), for some \( \gamma \in (\alpha, 1) \). Then the linear operator

\[
B[w, v] : C^{1+\gamma}(\mathbb{R}^{N-1}) \to C^{\gamma-\alpha}(\mathbb{R}^{N-1}), \quad B[w, v]u := \int_0^1 D\mathcal{H}(qw + (1 - \varrho)v)[u]d\varrho, \tag{4.15}
\]

satisfies, for all \( u \in C^{1+\gamma}_{loc}(\mathbb{R}^{N-1}) \), with \( \nabla u \in C^\gamma(\mathbb{R}^{N-1}) \),

\[
\|B[w, v]u\|_{C^{\gamma-\alpha}} \leq C\|\nabla u\|_{C^\gamma}, \tag{4.16}
\]

and

\[
C'\|\nabla u\|_{C^\gamma} \leq \|B[w, v]u\|_{C^{\gamma-\alpha}} + \|\nabla u\|_{L^\infty}, \tag{4.17}
\]

for some constants \( C, C' \) depending only on \( \alpha, \gamma, \alpha \) and \( v \).

Moreover for all \( \beta \in (\alpha, \gamma) \), the operator \( B := B[w, v] \), with domain \( C^{1+\beta}_0(\mathbb{R}^{N-1}) \), is an infinitesimal generator of a strongly continuous analytic semigroup \( \{e^{tB} : t \geq 0\} \) on \( C^{\beta-\alpha}_0(\mathbb{R}^{N-1}) \).

In addition, there exists \( m = m(N, \alpha, \beta, \gamma, \nu) \in \mathbb{R} \) such that for \( k \in \mathbb{N} \), there exists \( M_k = M_k(N, \alpha, \beta, \gamma, \nu, m) > 0 \), such that

\[
\|t^k B^ke^{tB}\|_{\mathcal{L}(C^{\beta-\alpha}_0(\mathbb{R}^{N-1}))} \leq M_ke^{mt}, \quad \text{for all } t > 0. \tag{4.18}
\]

Proof. Thanks to (4.10), we can write

\[
B[w, v] = Q(w)B_1 + B_2, \tag{4.19}
\]

where \( Q(w) = \sqrt{1 + |\nabla w|^2} \) and

\[
B_1u = -\int_{\mathbb{R}^{N-1}} \frac{u(x) - u(y)}{|x - y|^{N+\alpha}} \int_{-1}^1 \mathcal{G}'(qp_w(x, y) + (1 - \varrho)p_u(x, y))d\varrho dy,
\]

\[
B_2u := H(v) \int_0^1 DQ(qw + (1 - \varrho)v)[u]d\varrho.
\]

Recall that \( \mathcal{G}'(p) = -(1 + p^2)^{-\frac{N+\alpha}{2}} \) and

\[
p_u(x, y) = -\frac{U(x) - U(y)}{|x - y|} = -\int_0^1 \nabla U(\tau(x - y) + y) : \frac{x - y}{|x - y|} d\tau.
\]

Therefore since \( p_u(x, x - r\vartheta) = -\int_0^1 \nabla U(x - r\vartheta) : \vartheta d\vartheta \), we see that \( B_1 \) satisfies the properties in Assumptions 3.5 with \( K(x, y) := -\int_{-1}^1 \mathcal{G}'(qp_w(x, y) + (1 - \varrho)p_u(x, y))d\varrho \) and \( \kappa \), depending only on \( N, \alpha, \gamma, \beta \) and \( \nu \). Applying Lemma 2.2, we obtain (4.16). On the other hand by Lemma 3.6 we have

\[
C'\|\nabla u\|_{C^\gamma} \leq \|B_1u\|_{C^{\gamma-\alpha}} + \|\nabla u\|_{L^\infty}. \tag{4.20}
\]

From this and (2.2), we get

\[
\|B_2u\|_{C^{\gamma-\alpha}} \leq \varepsilon\|B_1u\|_{C^{\gamma-\alpha}} + c_\varepsilon\|\nabla u\|_{L^\infty}. \tag{4.21}
\]

Combining this with (4.20), we obtain (4.17).

We now let \( \beta \in (\alpha, \gamma) \) and \( u \in C^{1+\beta}_0(\mathbb{R}^{N-1}) \) so that \( B_2u \in C^{\beta-\alpha}_0(\mathbb{R}^{N-1}) \). Then (2.2) and (4.21) imply that

\[
\|B_2u\|_{C^{\beta-\alpha}} \leq \varepsilon\|B_1u\|_{C^{\beta-\alpha}} + c_\varepsilon\|u\|_{C^{\beta-\alpha}}.
\]

Moreover,

\[
A^\nu_K(x, 0, \vartheta) = -2\int_{-1}^1 (1 + (\varrho \nabla w(x) \cdot \vartheta + (1 - \varrho) \nabla v(x) \cdot \vartheta)^2)^{-\frac{N+\alpha}{2}} d\varrho.
\]
and clearly satisfies $\sup_{\theta \in S^{N-2}} \| \nabla_{\theta}^k A_k(\cdot, 0, \theta) \|_{C^\gamma} \leq C_k(N, \alpha, \gamma, \nu)$. We can thus apply Lemma 3.7 to deduce that $B[w, v] : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$ generates a strongly continuous analytic semigroup. Since $B[w, v]$ is sectorial, (4.18) follows from [22 Proposition 2.1.1].

As a consequence of the above lemma, we have the following result.

**Corollary 4.7.** Let $\beta \in (\alpha, 1)$ and $u_0 \in \mathcal{O}_\gamma$, for some $\gamma \in (\beta, 1)$. Then the linear operator

$$L_0 : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^{\beta-\alpha}_0(\mathbb{R}^{N-1}), \quad L_0 u := D\mathcal{H}(u_0)[u], \quad (4.22)$$

is an infinitesimal generator of a strongly continuous analytic semigroup $\{e^{tL_0} : t \geq 0\}$ on $C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$.

Moreover, for $T > 0$, $\rho \in (0, \frac{1}{1+\alpha})$, $u_0 \in C^{1+\beta}_0(\mathbb{R}^{N-1})$ and $f \in C^\rho([0, T], C^{\beta-\alpha}_0(\mathbb{R}^{N-1}))$ such that $f(0) - L_0 u_0 \in C^{\gamma-\alpha}(\mathbb{R}^{N-1})$, with $\gamma = \beta + \rho(1+\alpha)$, the initial value problem

$$\begin{align*}
&u(t) + L_0 u(t) = f(t), \quad t \in (0, T] \\
u(0) & = u_0
\end{align*}$$

admits a unique solution $u \in C^\rho([0, T], C^{1+\beta}_0(\mathbb{R}^{N-1})) \cap C^{1+\rho}(\mathbb{R}^{N-1})$, with $C_T = C(T, \rho, \nu, N, \alpha, \gamma, \beta) > 0$ such that

$$\|u - u_0\|_{C^\rho([0, T]; C^{1+\beta}_0(\mathbb{R}^{N-1}))} + \|u - u_0\|_{C^{1+\rho}(\mathbb{R}^{N-1})} \leq C_T \left( \|f - L_0 u_0\|_{C^\rho([0, T]; C^{\beta-\alpha}_0(\mathbb{R}^{N-1}))} \right),$$

with, $C_T \leq C_{T_0}$ for all $T \leq T_0$.

**Proof.** Since $B[u_0, u_0] = D\mathcal{H}(u_0)$, the result follows from Lemma 4.6 (4.19) and Theorem 3.8. \hfill \square

**Remark 4.8.** Let $\rho \in (0, \frac{1}{1+\alpha})$ and define $\gamma_\rho = \beta + \rho(1+\alpha)$ and suppose that $\gamma_\rho - \alpha \neq 1$. Then by Proposition 3.3, if $B := B[w, v]$ is given by (4.15), then thanks to (4.16), $Bu \in \mathcal{D}_B(\rho, \infty)$ as soon as $\nabla u \in C^\rho(\mathbb{R}^{N-1}) \cap C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$.

As a consequence, if $u_0 \in C^{1+\beta}_0(\mathbb{R}^{N-1})$, for some $\beta > \alpha$, then by (4.15), $\mathcal{H}(u_0) \in \mathcal{D}_{L_0}(\rho, \infty)$ if and only if $\nabla u_0 \in C^\rho(\mathbb{R}^{N-1}) \cap C^{\beta-\alpha}_0(\mathbb{R}^{N-1})$, where $L_0 := D\mathcal{H}(u_0)$.

Our next result shows that the operator $B[w, v]$ (defined in (4.15)) satisfies the maximum principle.

**Lemma 4.9.** Let $\gamma > \alpha$ and $w, v \in C([0, T], \mathcal{O}_\gamma)$. Then for all $u \in \mathcal{O}_\gamma$,

$$B[w(t), v(t)]u(x) = P.V. \int_{\mathbb{R}^{N-1}} \frac{u(x) - u(x+y)}{|y|^{N+\alpha}} \mu(t, x, |y|, |y|) \, dy + V(t, x) \cdot \nabla u(x),$$

where, for $(x, r, \theta) \in \mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2}$ we have $p_w(x, x-r \theta) = - \int_0^1 \nabla w(x-r \tau \theta) \cdot \theta \, d\tau$,

$$\mu(t, x, r, \theta) := - \int_{-1}^1 \mathcal{G}'(q \mu(t)) x - r\theta + (1-q)p(t) x - r\theta)Q(qw(t) + (1-q)v(t))(x) \, dq,$$

and

$$V(t, x) := \int_{-1}^1 q \frac{\nabla w(t) + (1-q) \nabla v(t)}{Q(qw(t) + (1-q)v(t))(x)} H(qw(t) + (1-q)v(t))(x) \, dq.$$
Moreover \( \mu(t, x, 0, \theta) = \mu(t, x, 0, -\theta) \) and \( \mu \in C([0, T], C^\gamma(\mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2})) \) and \( V \in C_b([0, T] \times \mathbb{R}^{N-1}) \).

**Proof.** By the fundamental theorem of calculus, we can write \( p_w(x, x - r\theta) = \frac{w(x - r\theta) - w(x)}{r} - \int_0^1 \nabla w(x - r\tau\theta) \cdot \theta \, d\tau \). Using (4.19), we get the expression of \( B[w(t), v(t)] \). Moreover from the evenness of \( G' \) we have \( \mu(t, x, 0, \theta) = \mu(t, x, 0, -\theta) \). The regularity of \( V \) is a consequence of (4.13). \( \square \)

As a consequence, of Lemma 4.9 and Proposition 2.3 we have the following result.

**Corollary 4.10.** Under the assumptions of Lemma 4.9, let \( u \in L^\infty([0, T], C^\beta(\mathbb{R}^{N-1})) \), with \( u(\cdot, x) \in C^1([0, T]) \) and \( u(t, \cdot) \in C^{1+\beta}(\mathbb{R}^{N-1}) \), satisfy

\[
\partial_t u + B[u(t), v(t)]u \leq 0 \quad \text{in } [0, T] \times \mathbb{R}^{N-1}.
\]

Then \( \sup_{[0,T] \times \mathbb{R}^{N-1}} u = \sup_{x \in \mathbb{R}^{N-1}} u(0, x) \).

## 5. Proof of the main results

For \( 0 < \rho < \frac{1}{1+\alpha} \), \( \beta \in (\alpha, 1) \) and \( T > 0 \), we define for the following Banach space

\[
E_T = C^{\rho}([0, T], C^{1+\beta}_0(\mathbb{R}^{N-1})) \cap C^{1+\rho}([0, T], C^{\beta-\alpha}_0(\mathbb{R}^{N-1}))
\]

(5.1)

endowed with the norm \( \| \cdot \|_{E_T} = \| \cdot \|_{C^{\rho}([0, T], C^{1+\beta}_0(\mathbb{R}^{N-1}))} + \| \cdot \|_{C^{1+\rho}([0, T], C^{\beta-\alpha}_0(\mathbb{R}^{N-1}))} \).

**Lemma 5.1.** Let \( u_0 \in C^{1+\gamma}_0(\mathbb{R}^{N-1}) \cap C^{1+\beta}_0(\mathbb{R}^{N-1}) \), for some \( \gamma > \beta \), with \( \nabla u_0 \in C^{\gamma}(\mathbb{R}^{N-1}) \). For \( R > 0 \), we define

\[
E_{T,R} := \left\{ u \in E_T : u(0) = u_0, \| \nabla u - \nabla u_0 \|_{C^{\rho}([0, T], C^{\beta}_0(\mathbb{R}^{N-1}))} \leq R \right\}.
\]

(5.2)

Then, there exists \( C = C(N, \alpha, \beta) > 1 \) with the property that for every \( u, v \in E_{T,R} \) we have

\[
\| F(u) - F(v) \|_{C^{\rho}([0, T], C^{\beta-\alpha})} \leq CR(1 + R + \| \nabla u_0 \|_{C^{\beta}}) T^\rho (1 + T^\rho) \| u - v \|_{E_T},
\]

where \( F : C^{1+\beta}_0(\mathbb{R}^{N-1}) \to C^0_0(\mathbb{R}^{N-1}) \) is given by \( F(u) = -H(u) + D\mathcal{H}(u_0)u \).

**Proof.** Let \( T, R > 0, u, v \in E_{T,R} \) and \( w = u - v \). Then we have

\[
F(u) - F(v) = -\int_0^1 (D\mathcal{H}(\lambda u + (1-\lambda)v) - D\mathcal{H}(u_0))[w] \, d\lambda
\]

\[
= -\int_0^1 \lambda \int_0^1 D^2\mathcal{H}(\tau(\lambda u + (1-\lambda)v) + (1-\tau)u_0)[\lambda(u-u_0) + (1-\lambda)(v-u_0), w] \, d\tau d\lambda.
\]

Next, for \( t \in [0, T], \lambda, \tau \in [0, 1] \) we define \( L_{\lambda, \tau}(t) \in C^0_0(\mathbb{R}^{N-1}) \) by

\[
L_{\lambda, \tau}(t) := D^2\mathcal{H}(\tau(\lambda u(t) + (1-\lambda)v(t)) + (1-\tau)u_0).
\]

We observe that, by Corollary 4.4 for all \( s, t \in [0, T] \),

\[
\| L_{\lambda, \tau}(t) - L_{\lambda, \tau}(s) \|_{C^{\beta-\alpha}} \leq CR(1 + R + \| \nabla u_0 \|_{C^{\beta}}) |s-t|^\rho
\]

(5.3)

and

\[
\| L_{\lambda, \tau}(t) \|_{C^{\beta-\alpha}} \leq CR(1 + R + \| \nabla u_0 \|_{C^{\beta}})^{\gamma}. \]

(5.4)
We write

\[
[F(u) - F(v)](t) - [F(u) - F(v)](s) = \int_0^1 \lambda \int_0^1 \left( L_{\lambda,t}(s)[\lambda(u(s) - u_0) + (1 - \lambda)(v(s) - u_0), w(s)]
- L_{\lambda,t}(t)[\lambda(u(t) - u_0) + (1 - \lambda)(v(t) - u_0), w(t)] \right) dt \, d\lambda.
\]

Using (5.3), (5.4) and the fact that \( u, v \in E_{T,R} \), we then get

\[
\| [F(u) - F(v)](t) - [F(u) - F(v)](s) \|_{C^\beta - \alpha} \leq CR(1 + R + \| \nabla u_0 \|_{C^\beta}) C \left( t^\rho + s^\rho \right) |t - s|^{\rho} \| w \|_{E_T}. (5.5)
\]

Using the fact that \([F(u) - F(v)](0) = 0, w \in E_T\) and taking \( s = 0 \) in (5.5), we get

\[
\| F(u) - F(v) \|_{L^\infty([0,T] C_0^{\beta-\alpha})} \leq CR(1 + R + \| \nabla u_0 \|_{C^\beta}) C T^{2\rho} \| w \|_{E_T}. (5.6)
\]

The result follows from (5.5) and (5.6).

\[\square\]

**Proof of Theorem 1.4.** Let \( u_0 \in C_0^{1+\beta}(\mathbb{R}^{N-1}) \cap C_{loc}^{\beta+\rho(1+\alpha)}(\mathbb{R}^{N-1}) \) with \( \| \nabla u_0 \|_{C^{\beta+\rho(1+\alpha)}} \leq \nu \).

Then by Corollary 4.7, the operator \( L_0 := D \mathcal{H}(u_0) \) is a generator of strongly continuous analytic semigroup. Now, to solve problem (1.13), we rewrite it as

\[
\begin{cases}
\partial_t u + L_0 u = F(u) & \text{in } [0,T] \times \mathbb{R}^{N-1} \\
u(0) = u_0 & \text{in } \mathbb{R}^{N-1}
\end{cases}
(5.7)
\]

and use a fixed point argument, with \( F(u) := -\mathcal{H}(u) + D \mathcal{H}(u_0)[u] = -\mathcal{H}(u) + L_0 u \).

Let \( u \in E_T \) and recall (5.1) for the definition of \( E_T \). Note that by (4.16) and Lemma 4.5, \( F(u)(0) - L_0 u(0) = -\mathcal{H}(u_0) \in C_0^{\beta-\alpha}(\mathbb{R}^{N-1}) \cap C_{loc}^{\beta+\rho(1+\alpha)-\alpha}(\mathbb{R}^{N-1}) \). Hence by Corollary 4.7, there exists a unique function \( \Phi(u) \in E_T \) satisfying

\[
\begin{cases}
\partial_t \Phi(u) + L_0 \Phi(u) = F(u) & \text{in } [0,T] \times \mathbb{R}^{N-1} \\
\Phi(u)(0) = u_0 & \text{in } \mathbb{R}^{N-1}.
\end{cases}
(5.8)
\]

We define

\[ \mathcal{E}_{T,R} := \{ u \in E_T : u(0) = u_0, \| u - u_0 \|_{E_T} \leq R \} \].

Clearly a fixed point of the map \( \Phi : E_T \to E_T \) in the set \( \mathcal{E}_{T,R} \) will be a solution to (1.13).

We claim that, provided \( \| \nabla u_0 \|_{C^{\beta+\rho(1+\alpha)}}(\mathbb{R}^{N-1}) \leq \nu \), there exist a large constant \( R > 0 \) and a small constant \( T > 0 \), both depending only on \( N, \alpha, \beta, \rho, \gamma \) and \( \nu \), such that

\begin{enumerate}
\item \( \Phi(\mathcal{E}_{T,R}) \subset \mathcal{E}_{T,R} \).
\item \( \Phi \) is a contraction on \( \mathcal{E}_{T,R} \).
\end{enumerate}
To prove (i), we use Corollary 4.7 and Lemma 5.1 to get \( C = C(\rho, N, \alpha, \beta, \gamma, \nu) > 1 \) such that for all \( T \in (0, 1) \),
\[
\| \Phi(u) - u_0 \|_{E_T} \leq C \left( \| F(u) - \mathcal{L}_0 u_0 \|_{C^\rho([0,T], C^{\beta-\alpha})} + \| \mathcal{H}(u_0) \|_{C^{\beta+\rho(1+\alpha)-\alpha}} \right)
\leq C \left( \| F(u) - F(u_0) \|_{C^\rho([0,T], C^{\beta-\alpha})} + \| \nabla u_0 \|_{C^{\beta+\rho(1+\alpha)}} \right)
\leq C \left( T^\rho(1 + T^\rho) R(1 + R + \| \nabla u_0 \|_{C^\beta}) C + \| \nabla u_0 \|_{C^{\beta+\rho(1+\alpha)}} \right).
\]
We can thus let \( R = 2\nu C \) and choose \( T > 0 \), so that \( \Phi(u) \in \mathcal{E}_{T,R} \).

To prove (ii), we let \( u, v \in \mathcal{E}_{T,R} \). Then \( W := \Phi(u) - \Phi(v) \) satisfies
\[
\begin{align*}
\partial_t W + \mathcal{L}_0 W &= F(u) - F(v) & \text{in } [0, T] \times \mathbb{R}^{N-1} \\
W(0) &= 0 & \text{in } \mathbb{R}^{N-1}.
\end{align*}
\]
Hence by Corollary 4.7 and Lemma 5.1 there exists \( C = C(\rho, N, \alpha, \beta, \gamma, \nu) > 1 \) such that for \( T \in (0, 1) \),
\[
\| \Phi(u) - \Phi(v) \|_{E_T} \leq C \| F(u) - F(v) \|_{C^\rho([0,T], C^{\beta-\alpha})} \leq C T^\rho R(1 + R + \| \nabla u_0 \|_{C^\beta}) C \| u - v \|_{E_T}.
\]
Now decreasing \( T \), if necessary, we obtain \( \| \Phi(u) - \Phi(v) \|_{E_T} \leq \frac{1}{2} \| u - v \|_{E_T} \), which is (ii). This
ends the proof of the claim.

We can thus apply the Banach fixed point theorem on \( \mathcal{E}_{T,R} \) to get a unique fixed point \( u \in \mathcal{E}_{T,R} \) of \( \Phi \). The proof of the theorem is thus finished.

Theorem 1.1 and Theorem 1.2 are consequences of the following result which deals with bounded functions.

**Theorem 5.2.** Under the assumptions of Theorem 1.4, we have the following properties.

(i) For all \( \beta' \in (\alpha, \beta), \rho \in (0, \frac{\alpha}{1+\alpha}], k \in \mathbb{N} \) and \( t \in (0, T) \), we have \( u(t) \in C^{k+1+\beta'}(\mathbb{R}^{N-1}) \)
and there exists \( C_k > 0 \) only depending on \( k, \rho, \alpha, \beta, \gamma, N, \nu, \beta' \) and \( T \) such that
\[
\| t^k \nabla u \|_{C^\rho([0,T], C^{1+\beta'})} \leq C_k, \tag{5.9}
\]

(ii) If, in addition, \( \nabla u_0 \in C^{1+\gamma}(\mathbb{R}^{N-1}) \) for \( \rho \in (0, \frac{1}{1+\alpha}) \), then there exists \( C > 0 \) only depending on \( \rho, \alpha, \beta, \gamma, N, \nu, T \) and \( \beta' \) such that
\[
\| \nabla u \|_{C^\rho([0,T], C^{1+\beta'})} \leq C \| \nabla u_0 \|_{C^{1+\gamma}}, \tag{5.10}
\]
where \( \gamma_\rho = \beta + \rho(1 + \alpha) \).

**Proof.** We define \( \tau_h u(t, x) = \frac{u(t-h, x) - u(t,x)}{|h|} \), for \( 0 < |h| < 1 \). From (1.13), we deduce that
\[
\begin{align*}
\partial_t \tau_h u + L_h(t) \tau_h u &= 0 & \text{in } [0, T] \times \mathbb{R}^{N-1} \\
\tau_h u(0) &= \tau_h u_0 & \text{in } \mathbb{R}^{N-1},
\end{align*}
\]
(5.11)
where for all \( a \in \mathbb{R}^{N-1} \), we define
\[
L_a(t)v := \int_0^1 D\mathcal{H}(qu(t, \cdot + a) + (1 - q)u(t))[v] \, dq. \tag{5.12}
\]
We observe, from Lemma 4.1, that
\[
L_a(t)u(x) = P.V. \int_{\mathbb{R}^{N-1}} \frac{u(t,x) - u(t,x + y)}{|y|^{N+\alpha}} \mu(t, x, |y|/|y|) \, dy + V(t, x) \cdot \nabla u(t, x)
\]
where, for \((x, r, \theta) \in \mathbb{R}^{N-1} \times [0, \infty) \times S^{N-2}\) and \(a \in \mathbb{R}^{N-1},\)

\[
\mu(t, x, r, \theta) := - \int_{-1}^{1} G'(\varphi a(t + a)) (x, x - r \theta + (1 - \varphi) p_u(x, x - r \theta)) Q(a u(\cdot + a) + (1 - \varphi) u(x)) d\varphi
\]

and

\[
V(t, x) := \int_{-1}^{1} \frac{\varphi \nabla u(t, x + a) + (1 - \varphi) \nabla u(t, x)}{Q(a u(\cdot + a) + (1 - \varphi) u(t)(\cdot))} H(a u(\cdot + a) + (1 - \varphi) u(t))(x) d\varphi.
\]

We recall that \(G'(p) = -(1 + p^2)^{-\frac{N-2}{2}}\).

We now consider \((5.9)\) by induction. Indeed, since

\[
G_0(t) = 0
\]

we show by induction that for all \(0 < i \leq k\),

\[
\|G_i(t)\|_{L^1(B(0, C_1))} \leq C(1 + t)^{\frac{N-2}{2}}
\]

and

\[
\|G_{i+1}(t)\|_{L^1(B(0, C_1))} \leq C(1 + t)^{\frac{N-2}{2}}
\]

for some \(C_i\) that for all \(i \in \mathbb{N}\),

\[
C^\rho + \rho(1+\alpha) - \alpha \langle \mathbb{R}^{N-1} \rangle \cap C^\rho - \alpha \langle \mathbb{R}^{N-1} \rangle \subset D_{L_i(0)}(\rho, \infty) \quad \text{for all } \rho \in (0, \frac{1}{1+\alpha}).
\]

We now consider \((5.9)\) by induction. Indeed, since \(\rho \leq \frac{\alpha}{1+\alpha}\), we have \(\beta' + \rho(1 + \alpha) - \alpha \leq \beta'\).

We show by induction that for all \(k \in \mathbb{N}\) and \(t \in (0, T]\),

\[
u(t) \in C^{k+1, \beta'}(\mathbb{R}^{N-1})\quad \text{and} \quad \|t^k \nabla u(t)\|_{C^{\rho}(\mathbb{R}^{N-1})} \leq C_k,
\]

for some \(C_k = C_k(N, \alpha, \beta, \beta', \rho, \nu) > 0\).

Clearly \((5.10)\) holds for \(k = 0\) by the statement of the lemma. We assume that \((5.10)\) holds up to order \(k \geq 1\) and we prove the result for \(k + 1\).

Consider \(L_i(t)\) given by \((5.12)\), which we can be written as (recall \((2.4)\) and \((4.10)\))

\[
L_i(t)u(x) = \frac{1}{2} \int_{0}^{\infty} \int_{S^{N-2}} r^{-2-\alpha} \delta u(x, r, \theta) (\mu(t, x, r, \theta) + \mu(t, x, r, \theta)) drd\theta
\]

\[
+ \int_{0}^{\infty} \int_{S^{N-2}} r^{-2-\alpha} \delta u(x, r, \theta) (\mu(t, x, r, \theta) - \mu(t, x, r, \theta)) drd\theta + V(t, x) \cdot \nabla u(t, x),
\]

with \(\mu(t, x, 0, \theta) = \mu(t, x, 0, -\theta)\). From this and Lemma \(2.2\) we can differentiate \((5.11)\) \(k\) times, so that, letting \(U = \sum_{i=1}^{k} \partial_{\tau_i} \tilde{\mu}(\tau_h U)\), we then get

\[
\partial_t U + L_h(t) U = f(t, x) \quad \text{in } \mathbb{R}^{N-1} \times (0, T],
\]

where

\[
f(t, x) := \sum_{S \in S_k} P.V. \int_{\mathbb{R}^{N-1}} \frac{\partial |S|}{\prod_{i \in S} \partial x_i} (\tau_h u(x) - \tau_h u(x - y)) |y|^{N+\alpha} \prod_{i \in S} \partial x_i \mu(t, x, |y|, y/|y|) dy
\]

\[
+ \sum_{S \in S_k} \frac{\partial |S|}{\prod_{i \in S} \partial x_i} (\tau_h u(x) \cdot \tau_h V(t, x)) \prod_{i \in S} \partial x_i
\]

and

\[
\partial_t U + L_h(t) U = f(t, x) \quad \text{in } \mathbb{R}^{N-1} \times (0, T],
\]

where

\[
\partial_t U + L_h(t) U = f(t, x)
\]

and

\[
\partial_t U + L_h(t) U = f(t, x)
\]
and $S_{k-1}$ is the set of subsets of $\{1, \ldots, k-1\}$. By (5.16) and the smoothness of $G'$, for all $S \in S_{k-1}$,
\[
\left| \left| t^{|S|} \frac{\partial^{k-|S|} \mu}{\prod_{i \notin S} \partial x_i} \right| \right|_{C^\rho([0,T], C^{\rho'})} \leq C_{k-|S|}
\]
and in addition by Lemma 4.2
\[
\left| \left| t^{|S|+1} \frac{\partial^{k} \tau_k \nabla u}{\prod_{i \notin S} \partial x_i} \right| \right|_{C^\rho([0,T], C^{\rho'})} \leq C_{|S|+1}, \quad \left| \left| t^{k-|S|} \frac{\partial^{k-|S|} V}{\prod_{i \notin S} \partial x_i} \right| \right|_{C^\rho([0,T], C^{\rho'-\alpha})} \leq C_{k-|S|}.
\]
It then follows, from (5.16) and Lemma 2.2 that $f(t) \in C^0_\alpha(\mathbb{R}^{N-1})$ and
\[
\left| \left| t^{k+1} f \right| \right|_{C^\rho([0,T], C^{\rho'-\alpha})} \leq C C_k, \quad t^{k+1} f |_{t=0} = 0. \tag{5.18}
\]
We then define $v = t^{k+1} \frac{\partial^k (\tau_k u)}{\partial x_1 \ldots \partial x_k}$, so that, by (5.17),
\[
\partial_t v + L_h(t)v = (k+1) t^k \frac{\partial^k (\tau_k u)}{\partial x_1 \ldots \partial x_k} (t, x) + t^{k+1} f(t, x), \quad v(0) = 0 \tag{5.19}
\]
Letting $g^k := t^k \frac{\partial^k (\tau_k u)}{\partial x_1 \ldots \partial x_k}$, then (5.16) implies that $g^k \in C^\rho([0,T], C^{\rho'})$ and in addition, by (5.15) and the choice of $\rho \leq \frac{\alpha}{1+\alpha}$, we find that
\[
\|g^k\|_{C^\rho([0,T], C^{\rho'-\alpha})} + \|g^k(0)\|_{\mathcal{D}_{L_0(0)}(\rho, \infty)} \leq \|g^k\|_{C^\rho([0,T], C^{\rho'})} + C \|g^k(0)\|_{C^{\rho}} \leq C_k. \tag{5.20}
\]
In view of (5.13) and (5.14), we can thus apply [22 Proposition 6.1.3] to the equation (5.19) and use (5.20) together with (5.18) and deduce that
\[
\|v\|_{C^\rho([0,T], C^{1+\rho'})} \leq C \left( \|g^k\|_{C^\rho([0,T], C^{\rho'-\alpha})} + \|g^k(0)\|_{\mathcal{D}_{L_0(0)}(\rho, \infty)} + \|t^{k+1} f\|_{C^\rho([0,T], C^{\rho'-\alpha})} \right) \leq C_{k+1}.
\]
Letting now $h \to 0$, we finally get
\[
\|t^{k+1} \nabla u\|_{C^\rho([0,T], C^{k+1+\rho'})} \leq C_{k+1}.
\]
This completes the proof of (i).

Finally for (ii), we use (5.13) to obtain
\[
\|L_h(0) \tau_k u_0\|_{\mathcal{D}_{L_0(0)}(\rho, \infty)} \leq C \|\nabla \tau_k u_0\|_{C^{\rho'+(1+\alpha)}} \leq C \|\tau_k u_0\|_{C^{\rho}}.
\]
We can thus apply [22 Proposition 6.1.3] to (5.11) to get
\[
\|\tau_k u\|_{C^\rho([0,T], C^{1+\rho'})} \leq C (\|\tau_k u_0\|_{C^{1+\rho}} + \|\nabla \tau_k u_0\|_{C^{\rho}}).
\]
Letting $|h| \to 0$ in the above estimate, we find that
\[
\|\nabla u\|_{C^\rho([0,T], C^{1+\rho'})} \leq C \|\nabla u_0\|_{C^{1+\rho'}}.
\]
That is (5.10). \hfill \Box

**Proof of Theorem 1.1 and Theorem 1.2.** For $n \in \mathbb{N}$, we let $\eta_n \in C^\infty_c(\mathbb{R}^{N-1})$ such that $\eta \equiv 1$ for $|x| \leq n$, $\eta \equiv 0$ for $|x| \geq 2n$ and $|D^k \eta| \leq \frac{1}{n^k}$ on $\mathbb{R}^{N-1}$. It then follows that $\eta_n u_0 \in C^{1+\beta}_0(\mathbb{R}^{N-1})$ thanks to Proposition 2.3 and that
\[
\|\nabla (\eta_n u_0)\|_{C^\rho} \leq 2 \|\nabla u_0\|_{C^\rho} \leq 2 \nu, \tag{5.21}
\]
where $\gamma_\rho = \beta + \rho(1 + \alpha)$. By Theorem 1.4 there exists a unique function $u^n \in E_T$ solving

$$\begin{cases}
\partial_t u^n + \mathcal{H}(u^n) = 0 & \text{in } [0, T] \times \mathbb{R}^{N-1} \\
u^n(0) = \eta_n u_0 & \text{in } \mathbb{R}^{N-1}
\end{cases}$$

and satisfying

$$\|u^n - \eta_n u_0\|_{E_T} \leq C_0(N, \alpha, \beta, \gamma, \rho, \nu),$$

for $\rho' < \rho$ and $\beta' < \beta$, it then follows from Corollary 4.10 that

$$\|u^n\| \to \infty$$

in (5.23), and the Arzelà-Ascoli theorem, which is Theorem 1.2.

Using (5.9), we obtain (1.10) thanks to the Arzelà-Ascoli theorem, which is Theorem 1.2.

Now (1.9) is a consequence of (5.10), by compactness and thus the proof of Theorem 1.1 is thus complete.

Using (5.9), we obtain (1.10) thanks to the Arzelà-Ascoli theorem, which is Theorem 1.2.

Our next result shows that the (global) sign of the initial condition $u_0$ and the one of the initial fractional mean curvature $\mathcal{H}(u_0)$ are preserved. It also completes the proof of Theorem 1.3.

**Lemma 5.3.** Under the assumptions of Theorem 1.4, we have the following results.

(i) $\|\nabla u\|_{L^\infty([0, T] \times \mathbb{R}^{N-1})} \leq \|\nabla u_0\|_{L^\infty}$,

(ii) $\sup_{[0, T] \times \mathbb{R}^{N-1}} \partial_t u = - \inf_{\mathbb{R}^{N-1}} \mathcal{H}(u_0)$ and $\inf_{[0, T] \times \mathbb{R}^{N-1}} \partial_t u = - \sup_{\mathbb{R}^{N-1}} \mathcal{H}(u_0)$.

(iii) If moreover $u_0 \in L^\infty(\mathbb{R}^{N-1})$ then

$$\sup_{[0, T] \times \mathbb{R}^{N-1}} u = \sup_{\mathbb{R}^{N-1}} u_0 \quad \text{and} \quad \inf_{[0, T] \times \mathbb{R}^{N-1}} u = \inf_{\mathbb{R}^{N-1}} u_0$$

**Proof.** We start with (iii). If $u_0 \in L^\infty(\mathbb{R}^{N-1})$ then by (1.3) we have $u \in C^\rho([0, T], C^{1+\beta}) \cap C^{1+\rho}([0, T], C^{\beta-\alpha})$. Now, in view of (4.1), we can apply Proposition 2.3 to $u$ and $-u$ to get (iii).

Next to prove (i), we define $\tau_h u(t, x) = \frac{u(t, x+h) - u(t, x)}{|h|} \in C([0, T], C^{1+\beta})$, for $0 < |h| < 1$. We then have

$$\partial_t \tau_h u + B[u(t, \cdot) + h, u(t)] \tau_h u(t) = 0 \quad \text{in } [0, T] \times \mathbb{R}^{N-1}.$$

It thus follows from Corollary 4.10 that $\|\tau_h u\|_{L^\infty((0, T) \times \mathbb{R}^{N-1})} \leq \|\tau_h u_0\|_{L^\infty}$.

Hence, letting $h \to 0$, we get (i).
Finally, by Theorem 1.2 and Lemma 4.2 we have \( \partial_t u = -\mathcal{H}(u) \in C^\rho([\epsilon, T], C^{k+\beta'-\alpha}) \) for all \( \epsilon \in (0, T) \) and \( k \geq 0 \). Letting \( \tau \in (\epsilon, T) \), we define \( u^\tau(t, x) = \frac{u(t+\tau, x) - u(t, x)}{\tau} \), for \( \epsilon < t \leq T - \tau \).

We thus get \( u^\tau \in C^\rho([\epsilon, T-\tau], C^{1+\rho}) \cap C^{1+\rho}([\epsilon, T-\tau], C^{\beta-\alpha}) \) and
\[
\partial_t u^\tau + B[u(t+\tau), u(t)]u^\tau(t) = 0 \quad \text{on } [\epsilon, T-\tau] \times \mathbb{R}^{N-1}.
\]

Applying Corollary 4.10, we obtain
\[
\sup_{[\epsilon,T]\times\mathbb{R}^{N-1}} u^\tau = \sup_{x \in \mathbb{R}^{N-1}} u^\tau(\epsilon, x) \quad \text{and} \quad \inf_{[\epsilon,T]\times\mathbb{R}^{N-1}} u^\tau = \inf_{x \in \mathbb{R}^{N-1}} u^\tau(\epsilon, x).
\]
Since \( \partial_t u = -\mathcal{H}(u) \in C^\rho([0, T], L^\infty) \), letting first \( \tau \to 0 \) and then \( \epsilon \to 0 \) in the above identities, we get (ii).

\[\square\]

Proof of Theorem 1.3 (completed). It suffices to apply Lemma 5.3.

\[\square\]

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