Supersymmetric Exact Sequence, Heat Kernel and Super KdV Hierarchy

S. Andrea*, A. Restuccia**, A. Sotomayor***

January 6, 2022

*Departamento de Matemáticas,
**Departamento de Física
Universidad Simón Bolívar
***Departamento de Ciencias Básicas
Unexpo, Luis Caballero Mejías
e-mail: sandrea@usb.ve, arestu@usb.ve, asotomay@zeus.unexpo.edu.ve

Abstract

We introduce the free $N = 1$ supersymmetric derivation ring and prove the existence of an exact sequence of supersymmetric rings and linear transformations. We apply necessary and sufficient conditions arising from this exact supersymmetric sequence to obtain the essential relations between conserved quantities, gradients and the $N = 1$ super KdV hierarchy. We combine this algebraic approach with an analytic analysis of the super heat operator. We obtain the explicit expression for the Green’s function of the super heat operator in terms of a series expansion.
and discuss its properties. The expansion is convergent under the assumption of bounded bosonic and fermionic potentials. We show that the asymptotic expansion when \( t \to 0^+ \) of the Green’s function for the super heat operator evaluated over its diagonal generates all the members of the \( N = 1 \) super KdV hierarchy.

1 Introduction

The analysis of supersymmetric quantum problems has been recently considered in several relevant physical contexts. At very high energies one way of studying the \( M \)-theory, which has been proposed as a theory of unification of all known interactions in nature, is through Matrix models describing supersymmetric quantum problems. The supersymmetry is one of the relevant ingredients of these models. In particular the presence of supersymmetry may change completely the spectrum of the quantum hamiltonian. The bosonic hamiltonian in the case of the \( D = 11 \) supermembrane, with Minkowski target space, has a discrete spectrum while its supersymmetric extension has a continuous spectrum. Moreover in that case the Green’s function does not admit a representation in terms of the Feynman path integral since the potential is not bounded from below in some directions in spite of the fact that the SUSY hamiltonian is nonnegative. These systems are very closely related to certain supersymmetric integrable systems. These models in themselves are a great help in understanding integrable systems. The \( N = 1, 2 \) supersymmetric extensions of the KdV equations were found several years ago [11 2 3 4], while the \( N = 3 \) and \( N = 4 \) have been considered more recently in [5 6 7]. The bi-hamiltonian structure of the super KdV equations was studied in [8 9]. For a review of \( N = 1 \) and \( N = 2 \) super KdV equations see [10]. Extensions of the supersymmetric model have been proposed in [11].
In the first part of this work we focus on the algebraic structure of $N = 1$ supersymmetric models. We introduce the “Free SUSY derivation ring on a single fermionic generator” which contains a parity automorphism, a canonical superderivation and a supersymmetric gradient operator. We established an exact sequence of SUSY rings and linear transformations. Several of these relations were already known in the literature [2, 12]. This is a general algebraic construction valid for one dimensional supersymmetric models. Using necessary and sufficient conditions arising from the exact sequence we obtain the essential relations between conserved quantities, gradients and the $N = 1$ SKdV hierarchy. In the second part of this work we use the exact sequence together with an analytic analysis of the super heat kernel to show that the asymptotic expansion when $t \to 0^+$ of the Green’s function for the super heat operator, evaluated over its diagonal, generates all the left hand members of the SKdV hierarchy. Using the symmetry properties of that Green’s function we obtain an iterative procedure to obtain all the gradients and members of the SKdV hierarchy. The algebraic approach arising from the exact SUSY sequence together with the analytic approach arising from the analysis of the Green’s function of the superheat operator combine to give a satisfactory description of the SKdV hierarchy.

2 The Free Supersymmetric Derivation Ring on a single Fermionic Generator

2.1 Definitions

A ring $\mathcal{A}$ is associative but not necessarily commutative. It is “oriented” when there is given a ring automorphism $P : \mathcal{A} \to \mathcal{A}$ satisfying $P^2 = I$, as well as $P(f + g) = Pf + Pg$
and $P(fg) = (Pf)(Pg)$. An element $f$ of $A$ is “oriented” when $Pf = \pm f$, $f$ being called “bosonic” when $Pf = f$ and “fermionic” when $Pf = -f$. The oriented ring $(A, P)$ is said to be “commutative” when

$$fg = gf(-1)^{\sigma\tau}$$

wherever $f, g \in A$ satisfy $Pf = (-1)^{\sigma}f$, $Pg = (-1)^{\tau}g$. In this situation the commutation formula

$$uf = f(P^\sigma u)$$

holds for all $u \in A$, given that $Pf = (-1)^{\sigma}f$; this follows from $u = u_0 + u_1 = \text{boson} + \text{fermion}$. A linear map $D : A \to A$ is called an “ordinary derivation” when

$$DP = PD$$

$$D(uv) = (Du)v + u(Dv)$$

for all $u, v \in A$, and is called a “superderivation” when

$$DP = -PD$$

$$D(uv) = (Du)v + (Pu)(Dv).$$

The two definitions may be combined by saying that $\Pi(D) = (-1)^{\delta}$ when

$$DP = (-1)^{\delta}PD$$

$$D(uv) = (Du)v + (P^\delta u)Dv.$$ 

Then, if $(A, P)$ is commutative and $f$ satisfies $Pf = (-1)^{\sigma}f$, the product $fD$ will also be a derivation, with

$$\Pi(fD) = (-1)^{\sigma + \delta}.$$

Thus a fermionic $f$ times a bosonic $D$ will be a fermionic $fD$, and similarly for the other three cases.
2.2 Construction of the ring $\mathcal{A}$

We may begin with the commutative ring $\mathcal{B}$ of all polynomials in the commuting letters $b_2, b_4, b_6,...$ The ordinary derivations $\frac{\partial}{\partial b_n}: \mathcal{B} \rightarrow \mathcal{B}$ all commute, as $n$ runs through even positive integers. Then $\mathcal{A}$ is the supersymmetric extension of $\mathcal{B}$, constructed as follows.

Let $\mathbb{M} = \{1, 3, 5, ...\}$ be the set of all positive odd numbers, and let $2^\mathbb{M} = \{\phi, 1, 3, 13, ...\}$ be the collection of all finite subsets of $\mathbb{M}$, including the empty set $\phi$. Then $\mathcal{A}$ is to consist of all finitely supported functions $f: 2^\mathbb{M} \rightarrow \mathcal{B}$. The product of two elements $f$ and $g$, evaluated at a finite subset $E \subset \mathbb{M}$, is defined to be

$$(fg)(E) = \sum_{A \cup B = E} f(A)g(B)\varepsilon(A, B).$$

Here the function $\varepsilon: 2^\mathbb{M} \times 2^\mathbb{M} \rightarrow \{-1, 0, 1\}$ is defined to be

$$\varepsilon(A, B) = \prod_{a \in A} \prod_{b \in B} \varepsilon(a, b),$$

where $\varepsilon: \mathbb{M} \times \mathbb{M} \rightarrow \{-1, 0, 1\}$ is defined by

$$\varepsilon(a, b) = \begin{cases} 1 & a < b \\ 0 & a = b \\ -1 & a > b \end{cases}$$

If the number of elements in $E$ is $|E|$, then the above sum has $2^{|E|}$ terms, since $\varepsilon(A, B) \neq 0$ only occurs when $A$ and $B$ are disjoint. The parity automorphism $P: \mathcal{A} \rightarrow \mathcal{A}$ is given by $(Pf)(E) = f(E)(-1)^{|E|}$.

The easy formula $\varepsilon(A, B) = \varepsilon(B, A)(-1)^{|A||B|}$ makes it clear that $(\mathcal{A}, P)$ is commutative.

When $B, C \subset \mathbb{M}$ are disjoint, one has $\varepsilon(A, B \cup C) = \varepsilon(A, B)\varepsilon(A, C)$. This shows that the product operation is associative, the value of $(fg)h = f(gh)$ on $E \subset \mathbb{M}$ being given
by
\[
\sum_{A \cup B \cup C = E} f(A)g(B)h(C)\varepsilon(A, B)\varepsilon(A, C)\varepsilon(B, C).
\]

This completes the construction of the oriented ring \((A, P)\). The generating elements \(a_1, a_2, a_3, \ldots \in A\) are now to be identified.

The inclusion \(B \subset A\) is realized by associating each \(b \in B\) with that function \(2^M \to B\) which sends the empty set \(\phi\) to \(b\), and everything else to zero. Thus, for \(m\) even, \(a_m(\phi) = b_m\) and \(a_m(E) = 0\) for all nonempty \(E \subset M\).

When \(p\) is an odd positive integer, \(a_p : 2^M \to B\) is defined by \(a_p(p) = 1 \in B\) and \(a_p(E) = 0\) for all other finite subsets of \(M\).

This gives us \(\{a_1, a_2, a_3, \ldots\} \subset A\).

Evidently \(P a_n = (-1)^n a_n\). Further, if \(1 \leq p_1 < p_2 < \ldots < p_n\) are odd, the product \(a_{p_1}a_{p_2} \cdots a_{p_n} \in A\) takes the value +1 on the subset \(\{p_1, p_2, \ldots, p_n\} \subset M\) and zero everywhere else. Therefore every element of \(A\) may be written as a finite polynomial in the elements \(a_1, a_2, a_3, \ldots\).

2.3 Derivations of \(A\)

The fundamental superderivation of \(A\) is defined by
\[
D = a_2 \frac{\partial}{\partial a_1} + a_3 \frac{\partial}{\partial a_2} + a_4 \frac{\partial}{\partial a_3} + \cdots
\]
It sends \(a_1 \to a_2 \to a_3 \to \cdots\), exchanging bosons and fermions. This suggests that \((A, P, D)\) is in some sense the natural model of an oriented superderivation ring generated by a single fermion \(a_1\).

The square of a superderivation is an ordinary derivation. Thus \(D^2\) restricted to \(B \subset A\) sends \(a_2 \to a_4 \to a_6 \cdots\), and the pair \((B, D^2)\) can be called the free bosonic derivation ring on a single generator.
The inclusions $DA_n \subset A_n$ which follow from $DE = ED$ will be used later in the proofs of the exact sequence.

2.4 The Supersymmetric Gradient Operator

Given $h \in A$ we ask whether $f \subset A$ exists with $h = Df$. A linear operator $M : A \to A$ with $MD = 0$ would give at least a necessary condition.

In analogy with the bosonic case, for which it is known the gradient operator $M$, it can be found that the Susy $M$ operator has the expression

$$M = \frac{\partial}{\partial a_1} + D \frac{\partial}{\partial a_2} - D^2 \frac{\partial}{\partial a_3} - D^3 \frac{\partial}{\partial a_4} + D^4 \frac{\partial}{\partial a_5} + \cdots$$

It is a linear operator sending $A$ into itself, and we have seen that the equation $Mh = 0$ is a necessary condition for the existence of $f \in A$ with $Df = h$.

Later it will be shown that the condition $Mh = 0$ is also sufficient. (With respect to parity we note that $PM = -MP$).

2.5 Operators and Adjoints

Given $(A, P, D)$ as constructed: a “differential operator” is a linear map of $A$ into itself having the form

$$L = \sum_{n=0}^{N} l_n D^n$$

with coefficients $l_n \in A$. $L$ is the identically zero map $A \to A$ if and only if all the coefficients are zero.

Then $O_p A$ is defined to be the set of all differential operators. It is an associative ring: the composition of two operators has the same form, because $D_n(fI)$ can be expanded as
a finite sum \( \sum_{r=0}^{n} g_r D^r \), by using the defining property of \( D \) on products of elements of \( \mathcal{A} \).

Composing \( L \) with the parity automorphism \( P : \mathcal{A} \to \mathcal{A} \) we find that

\[
PLP = \sum_{0}^{N} (P l_n)(-D)^n.
\]

Therefore \( O_p \mathcal{A} \) is also an oriented ring, its parity automorphism given by \( L \to PLP \). As before, \( L \) can be called “oriented” if \( PL = \pm LP \), “fermionic” in one case and “bosonic” in the other.

We now ask how to integrate by parts in \( \mathcal{A} \). Suppose \( u, v \in \mathcal{A} \) are oriented elements with \( Pu = u(-1)^\alpha, Pv = v(-1)^\beta \). Starting with \( D \in O_p \), we compute

\[
D(uv) = (Du)v + (-1)^\alpha u(Dv)
\]

\[
= (Du)v + (-1)^\alpha (Dv)u(-1)^{\alpha(\beta+1)}
\]

\[
= (Du)v + (Dv)u(-1)^{\alpha\beta}.
\]

This can be written \((Du)v \equiv (-Dv)u(-1)^{\alpha\beta}\), if the congruence notation \( f \equiv g \) in \( \mathcal{A} \) is defined to mean that \( f - g = Dh \) for some \( h \in \mathcal{A} \).

More generally, \( L \) and \( L^* \in O_p \mathcal{A} \) may be said to be “mutually adjoint” if

\[
(Lu)v \equiv (L^*v)u(-1)^{\alpha\beta}
\]

for all oriented \( u, v \) as above. Thus, \( D^* = -D \), while a zeroth order operator \( l_0I \) is its own adjoint.

The uniqueness of the adjoint operator is argued as follows: if \( L = 0 \) then every \( h = L^*v \) satisfies \( hu \equiv 0 \) for all \( u \in \mathcal{A} \). However, it can be shown that for any nonzero \( h \in \mathcal{A} \) there exist \( u \) such that \( hu \) cannot be of the form \( Df \) for any \( f \in \mathcal{A} \).
Consequently, $L = 0$ in $\mathcal{O}_p\mathcal{A}$ implies that all $L^*v = 0$ in $\mathcal{A}$, and hence $L^* = 0$ in $\mathcal{O}_p\mathcal{A}$. This shows that any $L \in \mathcal{O}_p\mathcal{A}$ can have at most one adjoint $L^* \in \mathcal{O}_p\mathcal{A}$, and furthermore that $(L^*)^* = L$.

The commutation of the constructions $L \to PLP$ and $L \to L^*$ is shown by applying $P$ to the congruence

$$(LPu)Pv \equiv (L^*Pv)Pu(-1)^{\alpha\beta}.$$ 

Thus if $L$ has an adjoint then so does $PLP$, and

$$(PLP)^* = PL^*P.$$

The existence of adjoints for all differential operators must now be shown.

**Proposition** Suppose $K, L \in \mathcal{O}_p\mathcal{A}$ have adjoints, and that $PKP = K(-1)^{\kappa}$, $PLP = L(-1)^{\lambda}$. Then $KL$ has an adjoint, and it is given by

$$(KL)^* = L^*K^*(-1)^{\kappa\lambda}.$$ 

The proposition generalizes immediately to finite products of operators $L_1L_2 \cdots L_m$ in which each $L_k$ has an adjoint and is oriented with $PL_kP = L_k(-1)^{\lambda_k}$. Then

$$(L_1L_2 \cdots L_m)^* = (L_m^*L_{m-1}^* \cdots L_1^*)(-1)^{\mu},$$

$$\mu = \sum_{1 \leq i < j \leq m} \lambda_i\lambda_j.$$ 

Thus $lD^k$, when $Pl = \pm l$, has an adjoint $\pm D^k(lI)$, the sign depending on $k$ and the parity of $l$. This proves that every $L \in \mathcal{O}_p\mathcal{A}$ possesses a unique adjoint $L^* \in \mathcal{O}_p\mathcal{A}$, the bijection $L \leftrightarrow L^*$ satisfying $(L^*)^* = L$. 

9
We conclude by computing the adjoint of $D^p l D^q$, $p + q = m$. If $pl = -l$ then all the $m + 1$ exponents are $+1$ and $\mu = \frac{m(m+1)}{2}$.

Then

\[
(D^p l D^q)^* = (-D)^q l (-D)^p (-1)^{\frac{m(m+1)}{2}}
= D^q l D^p (-1)^{m+\frac{m(m+1)}{2}}
= D^q (P^m l) D^p (-1)^{\frac{m(m+1)}{2}}.
\]

On the other hand, if $pl = +l$ then all but one of the exponents $\lambda_1, ..., \lambda_{m+1}$ is $+1$, the remaining exponent being zero. Then

\[
(D^p l D^q)^* = (-D)^q l (-D)^p (-1)^{\frac{m(m-1)}{2}}
= D^q l D^p (-1)^{\frac{m(m+1)}{2}}
= D^q (P^m l) D^p (-1)^{\frac{m(m+1)}{2}}.
\]

But every $l \in A$ is uniquely the sum of a boson and a fermion. Therefore, for any $l \in A$ and any nonnegative integers $p$ and $q$, the adjoint of $D^p l D^q \in O_p A$ is given by

\[
(D^p l D^q)^* = D^q (P^m l) D^p (-1)^{\frac{m(m+1)}{2}},
\]

where $p + q = m$.

### 2.6 Frechet derivative

The construction of the Frechet derivative operator gives a linear map $A \to A$, sending $f \to L_f$. Given $f(a_1, a_2, ..., a_n)$ an element of $A$, the action of $L_f$ on a fermionic element $v \in A, P v = -v$, may be defined by

\[
L_f v = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} f(a_1 + \epsilon v, a_2 + \epsilon D v, ..., a_n + \epsilon D^{n-1} v).
\]
The coefficients of $L_f$ are obtained as follows. If $q$ is odd and $f = ga_q h$ with $g$ and $h$ independent of $a_q$, we have $\frac{\partial}{\partial a_q} f = (Pg)h$, while $g(D^{q-1}v)h = g(Ph)(D^{q-1}v)$ since $D^{q-1}v$ is fermionic. Therefore $D^{q-1}v$ is multiplied on the left by $(P \frac{\partial}{\partial a_q} f)$. If $m$ is even then $\frac{\partial}{\partial a_m}$ is an ordinary derivation and $D^{m-1}v$ is bosonic. In this case there is no anticommutation, and $D^{m-1}v$ is multiplied on the left by $\frac{\partial f}{\partial a_m}$. Combining these two cases we obtain the general formula

$$L_f = \sum_{n=1}^{\infty} (P^n \frac{\partial f}{\partial a_n}) D^{n-1},$$

which gives the Frechet derivative operator $L_f \in O_p A$ for any $f \in A$.

Applying $L_f$ to the generating element $a_1 \in A$ we get

$$L_f a_1 = \sum_{n=1}^{\infty} (P^n \frac{\partial f}{\partial a_n}) a_n.$$

But $h a_n = a_n (P^n h)$ for all $h \in A$. Hence $L_f a_1 = Ef$, connecting $L_f$ to the Euler operator $E$.

The adjoint operator to $L_f$ may be written down using the results of the previous section:

$$L_f^* = \sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} D^{n-1}((P \frac{\partial f}{\partial a_n}) I).$$

Applying the operator parity automorphism $K \rightarrow PKP$ we get

$$PL_f^* P = \sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} (-D)^{n-1}(\frac{\partial f}{\partial a_n}) I.$$

The coefficient of the identity operator is readily accessible since

$$D^{n-1}(\frac{\partial f}{\partial a_n}) I = (D^{n-1} \frac{\partial f}{\partial a_n}) I + (?) D + \cdots$$

After checking the $\pm$ signs we see that this coefficient is just the supersymmetric gradient $Mf$ of $f$: thus

$$L_f^* = (PMf) I + (?) D + \cdots$$
Going back to the equation $Mh = 0$, we see that this condition would imply the existence of $K \in \mathcal{O}_p\mathcal{A}$ satisfying $L_h^* = KD$ and hence also $L_h = \pm DK^*$ if $h$ is oriented. Applying this operator equation to the generating element $a_1 \in \mathcal{A}$, we find that $Eh = Df$ for some $f \in \mathcal{A}$.

Thus $Mh = 0$ implies $h \equiv 0$, at least when $h$ is oriented and $Eh = nh, n > 0$. But the two extra conditions are no obstacle:

**Proposition** Given $h \in \mathcal{A}$, with zero constant term. Then $Mh = 0$ if and only if $h = Df$ for some $f \in \mathcal{A}$.

**Proof** The equation $PM = -MP$ shows that $h \pm Ph$ also has zero gradient. Replacing $h$ by either summand in $h = \text{boson} + \text{fermion}$, we may suppose that $h$ is oriented. The presentation

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$$

by eigenspaces of the Euler operator permits $h$ to be written as $h = h_0 + h_1 + h_2 + \cdots$, in which $h_0 = 0$ by hypothesis. Then, since $MA_n \subset \mathcal{A}_{n-1}$ the condition $Mh = 0$ implies $Mh_n = 0$ for all $n \geq 1$.

By what was said earlier, there exist $f_n$ with $Df_n = Eh_n = nh_n$. Therefore $h = D(f_1 + \frac{1}{2}f_2 + \frac{1}{3}f_3 + \cdots)$ and the proof is complete.

In the next section we will need to know the interaction between the $f \rightarrow L_f$ construction and the parity automorphism $P: \mathcal{A} \rightarrow \mathcal{A}$. The formula $L_{Pf} = -PL_fP$ is easily verified by

$$\left(P^n \frac{\partial}{\partial a_n} Pf\right)D^{n-1} = (-1)^n \left(P^{n+1} \frac{\partial f}{\partial a_n}\right)D^{n-1}$$
\[ L_D f = D L_f. \]

(ii) When \( g = M h \), the Frechet derivative operator of \( g \) is antisymmetric:

\[ L_g^* = -L_g. \]

Evidently the second fact gives a necessary condition for \( g \) to be a gradient. But it is also sufficient:

**Proposition** Given \( g \in A \). The antisymmetry equation \( L_g^* = -L_g \) in \( O_p A \) is necessary and sufficient for the existence of \( h \) with \( M h = g \).

**Proof** Suppose first that \( g \) is oriented and satisfies \( Eg = n g, n \geq 0 \), as well as the hypothesis \( L_g^* = -L_g \). If \( P g = g(-1)^n \), then the Frechet derivative operator of element \( h = a_1 g \) is given by

\[ L_h = (P g) I + \sum_{n=1}^{\infty} \left( P^n (-1)^n a_1 \frac{\partial g}{\partial a_n} \right) D^{n-1} \]

\[ = (P g) I + a_1 L_g. \]
From \( L_{Pg} = -PL_g P \) we see that \( L_g \) has orientation opposite to that of \( g \), that is, \( PL_g P = L_g (-1)^{\nu+1} \). This permits the adjoint of \( L_h \) to be calculated as

\[
L_h^* = (Pg)I + L^*_g (a_1 I)(-1)^{\nu+1}
\]

\[
= (-1)^\nu \{ gI + L_g(a_1 I) \}.
\]

This shows that \((-1)^\nu L_h^* = (g + Eg) + (?) D + \cdots\)

But it was seen before that \( L_h^* = (PMh)I + (?) D + \cdots \) Observing \( PM = -MP \) and \( Ph = h(-1)^{\nu+1} \), we conclude that \( M(a_1 g) = g + Eg = (n + 1)g \) in consequence of three assumptions made at the beginning of this proof. Returning now to the general case, we note that operator adjoints, Frechet derivative operators, and the parity automorphism are interconnected by

\[
(PL_g P)^* = PL^*_g P
\]

\[
= L_{Pg} = -PL_g P.
\]

Thus, if \( g \in \mathcal{A} \) has an antisymmetric Frechet derivative operator then so do \( Pg \) and \( g \pm Pg \). Hence it suffices to treat only the case of oriented \( g \). Expanding \( g = g_0 + g_1 + g_2 + \cdots \) by homogeneous components in \( \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \cdots \), we observe that the coefficients of \( L_{gn} \) and \( L^*_n \) fall within \( \mathcal{A}_{n-1} \). Therefore the antisymmetry of \( L_g \) implies the antisymmetry of all the \( L_{gn} \). From what was said before \( g = Mh \) with \( h = a_1(g_0 + \frac{1}{2}g_1 + \frac{1}{3}g_2 + \cdots) \).

This completes the proof.

2.8 Summary: the Exact Sequence

A ring \( \mathcal{A} \), the “free SUSY derivation ring on a single fermionic generator”, has been constructed. It has a parity automorphism \( P \), a canonical superderivation \( D \), and a
SUSY gradient operator $M$.

Necessary and sufficient conditions have been given for recognizing which elements of $\mathcal{A}$ are derivatives and which are gradients. In terms of the SUSY gradient operator $M$, the Frechet derivative operator $L_g$, and the operator adjoint construction $L \rightarrow L^*$, these conditions are expressed by the following exact sequence of rings and linear transformations:

\[ \mathcal{O}_p \mathcal{A} \leftarrow \mathcal{A} \leftarrow \mathcal{A} \leftarrow \mathcal{A} \leftarrow \mathbb{R} \leftarrow 0 \]

\[ Df \leftarrow f \]

\[ Mh \leftarrow h \]

\[ L_g + L^*_g \leftarrow g \]

The sequence is exact in that the kernals of the outgoing transformations coincide with the images of the incoming transformations.

### 3 The SUSY heat operator

The ordinary heat equation with potential $u(x)$ and temperature function $f(x,t)$ is

\[ L_u f = 0, \quad L_u = \frac{\partial}{\partial t} - \Delta + u(x). \]

Its Green’s function evaluated at a field point $p = (x,t)$ and source point $q = (x',0)$ may be written as $G(p,q) = G_t(x,x')$. For fixed $x'$ and variable $x$ and $t$ it satisfies the above heat equation with initial value

\[ \lim_{t \downarrow 0} G_t(x,x') = \delta(x - x'). \]
When the potential is zero the Green’s function is

\[ g_t(x - x') \equiv g(x - x', t) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{(x - x')^2}{4t} \right). \]

It has the basic properties

\[
\begin{align*}
  i) & \quad g_t(x - x') > 0 \\
  ii) & \quad \int_{\mathbb{R}^n} g_t(x - x') dx = 1 \\
  iii) & \quad g_{t+s}(x - x') = \int_{\mathbb{R}^n} g_s(y - x')g_t(x - y) dy.
\end{align*}
\]

These properties allow the definition of the conditional Wiener measure, which may be used to express the Green’s function \( G_t(x, x') \) of the operator \( L_u \) by the Feynman-Kac formula \([13]\), when the potential \( u(x) \) is real and bounded from below.

For bounded potentials the Green’s function admits an asymptotic expansion

\[ G_t(x, x') = g(x - x', t) \sum_{n=0}^{\infty} \frac{1}{n!} a_n(x, x') t^n \]

in which the coefficients \( a_n(x, x') \) are determined recursively by \( a_0(x, x') = 1 \),

\[ (n + (x - x')\partial_x) a_n(x, x') = (\partial_x^2 + u(x)) a_{n-1}(x, x'). \]

On the diagonal \( x = x' \), one has

\[ a_n(x, x) = g_n(u(x), u'(x), \ldots), \]

a finite polynomial in the potential function \( u(x) \) and its derivatives.

Then the equations of the KdV hierarchy \([14]\), for unknown functions \( w(x, t) \), are

\[ w_t = \frac{\partial}{\partial x} g_n(w, w_x, w_{xx}, \ldots) \]
We now present a supersymmetric extension of this construction. The potential and the temperature function now have their values in an exterior algebra $\Lambda$, also called a Grassmann algebra.

If anticommuting generators of $\Lambda$ are written as $\theta, \theta_2, \theta_3, \ldots, \theta_m$ then every element of $\Lambda$ has a unique presentation

$$\Phi = \xi + \theta u$$

where $\xi$ and $u$ are in the subalgebra of $\Lambda$ generated by $\theta_2, \theta_3, \ldots, \theta_m$. Then, defining $\partial_\theta \Phi = u$, we obtain a superderivation $\partial_\theta : \Lambda \to \Lambda$, that is,

$$\partial_\theta (\Phi_1 \Phi_2) = (\partial_\theta \Phi_1) \Phi_2 + \Phi_1 (\partial_\theta \Phi_2)$$

where $\Phi \to \bar{\Phi}$ is the parity automorphism of $\Lambda$.

The operator $D = \partial_\theta + \theta \partial_x$ then acts on “superfields”, that is, on differentiable functions $\mathbb{R} \to \Lambda$. Using $\Phi_1$ and $\Phi_2$ to designate superfields, one can check

$$D (\Phi_1 \Phi_2) = (D \Phi_1) \Phi_2 + \Phi_1 (D \Phi_2),$$

$$D^2 = \partial_x.$$ 

It will be assumed that the potential $\Phi$ is fermionic: $\bar{\Phi} = -\Phi$ meaning that $\bar{\xi} = -\xi$ and $\bar{u} = u$.

If we take the dimension of $x$ to be 1:

$$[x] = 1,$$

17
then one must have
\[ [\theta] = \frac{1}{2} \]
\[ [D] = -\frac{1}{2}, \]
consequently
\[ [u] = -2, \]
\[ [\Phi] = [\xi] = -\frac{3}{2}. \]
The most general supersymmetric extension of \( L_u \), assuming positive powers of \( D \), becomes then
\[ L = \frac{\partial}{\partial t} - (D^4 - D\Phi + \lambda\Phi D) \]
where \( \lambda \) is a constant, dimensionless parameter.

When the superpotential \( \Phi \) is zero, it reduces to the heat operator, while if \( \xi = 0 \) and \( \theta = 0 \) it reduces to \( L_u \).

The parameter \( \lambda \) already appeared in the analysis of Mathieu \cite{1} for all supersymmetric extensions of the KdV equation. The case \( \lambda = 1 \) was related to the integrable supersymmetric extension of the KdV equation. We will consider in what follows \( \lambda = 1 \).

There are two supersymmetric extensions of \( \delta(x - x') \).
\( \delta(x - x')\delta(\theta - \theta') \) and \( \delta(x - x' - \theta\theta') = \delta(x - x') - \theta\theta'\delta'(x - x') \) are both invariants under the supersymmetric transformations
\[ x \rightarrow x + \theta\eta \quad , \theta \rightarrow \theta + \eta \]
\[ x' \rightarrow x' + \theta'\eta \quad , \theta' \rightarrow \theta' + \eta. \quad (2) \]

We may then consider two Green’s functions according to each possible initial conditions. We will denote the corresponding Green’s function by \( K_t(x, x', \theta, \theta') \) and \( G_t(x, x', \theta, \theta') \) respectively.
The Green’s function for the potential \( \Phi \), as a function of the source point \( q = (x', 0) \) and field point \( p = (x, t) \), is to be a function \( K_t(x, x', \theta, \theta') \), \( G_t(x, x', \theta, \theta') \) having values in \( \Lambda \) and satisfying \( LK_t = 0, LG_t = 0 \) when \( t > 0 \), while \( \lim_{t \to 0} K_t = \delta(x - x')\delta(\theta - \theta') \), \( \lim_{t \to 0} G_t = \delta(x - x') - \theta'\delta'(x - x') \).

\( K_t \) and \( G_t \) are related by

\[-D'K_t = G_t,\]

in which

\[D' = \frac{\partial}{\partial \theta'} + \theta' \frac{\partial}{\partial x'}\]

is the superderivative with respect to \((x', \theta')\).

The Green’s function \( K_t \) may then be expressed as

\[K_t(x, x'; \theta, \theta') = K_t(x - x', \theta - \theta') - \left\langle \tilde{D} \left[ \Phi(\bar{x}, \bar{\theta})K_t(\bar{x} - x', \bar{\theta} - \theta') \right] K_{\tilde{t}-t}(x - \bar{x}, \theta - \bar{\theta}) \right\rangle_{\bar{x}, \bar{\theta}, \tilde{t}} + \]

\[+ \left\langle \tilde{D} \left[ \Phi(\bar{x}, \bar{\theta})K_t(\bar{x} - x', \bar{\theta} - \theta') \right] \tilde{D} \left[ \Phi(\tilde{x}, \tilde{\theta})K_{\tilde{t}-\tilde{t}}(\tilde{x} - \bar{x}, \tilde{\theta} - \bar{\theta}) \right] K_{\tilde{t}-\tilde{t}}(x - \tilde{x}, \theta - \tilde{\theta}) \right\rangle_{\tilde{x}, \tilde{\theta}, \tilde{t}, \tilde{\tilde{t}}} - \ldots\]

where

\[K_t(x - x', \theta - \theta') = g_t(x - x')(\theta - \theta'),\]

\[0 < \tilde{t} < \tilde{\tilde{t}} < \cdots < t,\]

\[\tilde{D} = \tilde{\theta} \frac{\partial}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{\theta}}\]

and

\[\left\langle \cdot \right\rangle_{\tilde{x}, \tilde{\theta}, \tilde{t}} = \int \frac{\partial}{\partial \tilde{\theta}} \left( \cdot \right)|_{\tilde{\theta}=0} d\tilde{x} d\tilde{t}.\]
The integration on $\tilde{x}$ is over $\mathbb{R}$, the integration on the $\tilde{t}$ variable is according to the above ordering while the integration on the odd coordinate means $\frac{\partial}{\partial \tilde{\theta}}_{\tilde{\theta}=0}$ and it must be performed in the order indicated in our formula. We will show that this series expansion is convergent, under the assumption of bounded potentials.

The generic term of the expansion may be constructed from the previous one replacing $\mathcal{K}_{t-i}(x - \tilde{x}, \theta - \tilde{\theta})$ by $\hat{D} \left[ \Phi(\hat{x}, \hat{\theta}) \mathcal{K}_{\tilde{t}-i}(\tilde{x} - \tilde{x}, \tilde{\theta} - \tilde{\tilde{\theta}}) \right] \mathcal{K}_{t-i}(x - \hat{x}, \theta - \hat{\theta})$ and performing an overall integration on $\hat{x}, \hat{\theta}, \hat{t}$ where this point is an intermediate one with $\tilde{\tilde{t}} < \hat{t} < t$.

The explicit expansion for $G_t$ turns out to be, after some calculations,

$$G_t(x, x'; \theta, \theta') = g_t(x - x' - \theta \theta') - \left\langle \Phi(x, \tilde{\theta}) g_{t-i}(\tilde{x} - x' - \tilde{\theta} \theta') g_{t-i}(x - \tilde{x} - \tilde{\theta} \tilde{\theta}) \right\rangle_{\tilde{x}, \tilde{\theta}} +$$

$$+ \left\langle \left\langle \Phi(x, \tilde{\theta}) g_{t-i}(\tilde{x} - x' - \tilde{\theta} \theta') \Phi(\tilde{x}, \tilde{\theta}) g_{t-i}(\tilde{x} - \tilde{x} - \tilde{\theta} \tilde{\theta}) g_{t-i}(x - \tilde{x} - \tilde{\theta} \tilde{\theta}) \right\rangle_{\tilde{x}, \tilde{\theta}} \right\rangle_{\tilde{\tilde{x}}, \tilde{\tilde{\theta}}} - \ldots$$

The generic term of the expansion may be constructed from the previous one replacing $g_{t-i}(x - \tilde{x} - \theta \tilde{\theta})$ by $\Phi(x, \tilde{\theta}) g_{t-i}(\tilde{x} - \tilde{x} - \tilde{\theta} \tilde{\theta}) g_{t-i}(x - \tilde{x} - \tilde{\theta} \tilde{\theta})$ and performing an overall integration on $\hat{x}, \hat{\theta}, \hat{t}$ where this point is an intermediate one with $\tilde{\tilde{t}} < \hat{t} < t$.

In the bosonic limit, when $\xi(x) = 0$ and $\theta = \theta' = 0$, the integration on the odd variables $\tilde{\theta}$ becomes straightforward and the formula reduces to

$$G_t(x, x'; 0, 0)|_{\xi=0} = g_t(x - x') - \left\langle u(\tilde{x}) g_{t-i}(\tilde{x} - x') g_{t-i}(x - \tilde{x}) \right\rangle_{\tilde{x}, \tilde{i}} +$$

$$+ \left\langle \left\langle u(\tilde{x}) g_{t-i}(\tilde{x} - x') u(\tilde{x}) g_{t-i}(\tilde{x} - x') g_{t-i}(x - \tilde{x}) \right\rangle_{\tilde{x}, \tilde{i}} \right\rangle_{\tilde{\tilde{x}}, \tilde{\tilde{t}}} - \ldots$$

This is exactly the Green’s function $G$ for the operator $L_u = \partial_t - \Delta + u(x)$. If we
assume \( u(x) \) to be bounded and continuous:

\[
|u(x)| < M,
\]

it then follows, using the semigroup property for the Green’s function, that

\[
|G_t(x, x')| < e^{tM} g_t(x - x').
\]

It may also be shown that the convergent expansion for \( G_t(x, x') \) is equal to the Feynman-Kac formula

\[
G_t(x, x') = \int dW_t(x, x') \exp \left( - \int_{-\frac{t}{2}}^{+\frac{t}{2}} u(x(s)) ds \right)
\]

where \( dW_t(x, x') \) denotes the Wiener measure for the continuous paths between \( x' \) and \( x \). When \( u(x) \) is real and bounded both formulas are exactly the same. However, the Feynman-Kac formula may be established even when \( u(x) \) is bounded from below and \( \Delta + u(x) \) is (essentially) self adjoint, while the expansion in terms of the potential is valid when \( u \) is bounded without assuming the self adjoint property of the operator.

\( G_t(x, x') \) is positive. This property arises directly from the Feynman-Kac formula.

It also satisfies

\[
G_t(x, x') = G_t(x', x)
\]

under the interchange of the positions of the field point and the source one.

In the next section we will analize these properties for the supersymmetric extensions we are considering.
4 The SUSY Green’s function and the SKdV hierarchy

The Green’s function $G_t$ depends on $x, \theta$ and $x', \theta'$ and on the components of the superpotential $u$ and $\xi$. In order to analyze the transformation law, under supersymmetry, of $G_t$ we will write explicitly its dependence on $u$ and $\xi$, $G_t(x, x'; \theta, \theta'; u, \xi)$. Under the supersymmetric transformations:

$$
x \rightarrow x + \delta x = x - \eta \theta
$$
$$
\theta \rightarrow \theta + \delta \theta = \theta + \eta
$$
$$
\Phi \rightarrow \Phi + \delta \Phi = \Phi - \eta u(x) + \eta \theta \xi'(x),
$$

the components of $\Phi$ transforms as

$$
u \rightarrow u + \delta u = u - \eta \xi'
$$
$$
\xi \rightarrow \xi + \delta \xi = \xi - \eta u.
$$

$G_t$ is then invariant under these transformations. That is,

$$
G_t(x + \delta x, x' + \delta x'; \theta + \delta \theta, \theta' + \delta \theta'; u + \delta u, \xi + \delta \xi) =  
= G_t(x, x'; \theta, \theta'; u, \xi)
$$

(3)

To show this invariance property of $G_t$ we notice that

$$(\Phi + \delta \Phi)(\tilde{x}, \tilde{\theta}) = \Phi(\tilde{x} - \delta \tilde{x}, \tilde{\theta} - \delta \tilde{\theta}).$$

We then evaluate the left hand member of (3) using the previous expansion formula and perform a change of variable at each intermediate point $\tilde{x}, \tilde{\theta}$:

$$
\tilde{x} \rightarrow \tilde{x}_1 = \tilde{x} - \delta \tilde{x}
$$
$$
\tilde{\theta} \rightarrow \tilde{\theta}_1 = \tilde{\theta} - \delta \tilde{\theta},
$$
with Jacobian equal to 1.

We then use the property that the combination $(\tilde{x} - \hat{x} - \tilde{\theta})$ is invariant under this change of coordinates. We end up with the relation $[3].$

The other symmetry of the Susy Green’s function $G_t$ is

$$G_t(x, x'; \theta, \theta'; u, \xi) = G_t(x, x'; \theta', \theta; u, \xi).$$  \hspace{1cm} (4)

It follows by performing changes of variables on the time arguments at each integrand on each term of the expansion. In terms of the components of $G_t$ it means

$$G_t(x, x'; \theta, \theta') = A_t(x, x') + \theta' B_t(x, x') + \theta C_t(x, x') + \theta \theta' D_t(x, x')$$

$$A_t(x, x') = A_t(x', x)$$

$$B_t(x, x') = C_t(x', x)$$

$$D_t(x, x') = -D_t(x', x).$$

We will now evaluate $G_t$ by performing all integrations on the odd variables. We start evaluating $G_t(x, x'; 0, 0; u, \xi)$. We denote

$$\tilde{x}, \tilde{t} \mapsto \tilde{x}, \tilde{t} = G_{t-\hat{t}}(\tilde{x}, \tilde{x})$$  the bosonic propagator,

$$\tilde{x}, \tilde{t} \mapsto \tilde{x}, \tilde{t} = -\frac{1}{2} \frac{1}{(\tilde{t} - \hat{t})} g_{t-\hat{t}}(\tilde{x} - \hat{x})$$  the fermionic propagator.

An arrow followed by a vertex $\xi(\tilde{x})$ denotes multiplication of the propagator by the vertex and integration on the corresponding coordinates $\tilde{x}, \tilde{t}$.

The Green’s function at $\theta = \theta' = 0$ may then be expressed by
\[ G_t(x, x'; 0, 0; u, \xi) = x', t' \rightarrow x, t + x', t' \rightarrow \xi \rightarrow x, t + \]
\[ + x', t' \rightarrow \xi \rightarrow \xi \rightarrow x, t + \]
\[ + x', t' \rightarrow \xi \rightarrow \xi \rightarrow \xi \rightarrow x, t + \cdots \]

It can be shown that this expansion on the fermionic vertex \( \xi \) is convergent provided \( u(x) \) is bounded and \( \xi(x) \) is bounded in the following sense. It is possible to express the product
\[ \xi(\tilde{x})\xi(\tilde{x}) = \frac{1}{2}(x - \tilde{x})f(\tilde{x}, \tilde{x}) \quad (5) \]
since the left hand member is antisymmetric on \( \tilde{x} \leftrightarrow \tilde{\tilde{x}} \). We assume then
\[ |u(x)| < M, \]
\[ f(\tilde{x}, \tilde{x}) < M^2. \quad (6) \]
The square arises from dimensional arguments. In fact, let us remember that \([\xi] = -\frac{3}{2}\) and \([u] = -2\) and hence \([f]\) must be \(-4\). After replacing \([5]\) in the expression of \( G_t(x, x'; 0, 0; u, \xi) \), the contributions of the fermionic propagator times the fermionic vertices may be worked out in terms of derivatives of \( g_t \). One may then use \([6]\) and the semigroup properties for \( g_t \) to obtain a bound for the series expansion of \( G_t(x, x'; 0, 0; u, \xi) \).

The complete expression for \( G_t(x, x'; \theta, \theta') \), expressed in terms of its value at \( \theta = \theta' = 0 \), is the following

\[ G_t(x, x'; \theta, \theta') = G_t(x, x', 0, 0) - \theta \langle G_{t - \nu}(\tilde{x}, x', 0, 0)\xi(\tilde{x})g_{t - \nu}'(x - \tilde{x}) \rangle_{\tilde{x}, \tilde{t}} + \]
\[ + \theta' \langle \xi(\tilde{x})g_{t - \nu}'(\tilde{x} - x')G_{t - \nu}(x, \tilde{x}, 0, 0) \rangle_{\tilde{x}, \tilde{t}} - \]
\[ - \theta\theta' \langle \xi(\tilde{x})g_{t - \nu}'(\tilde{x} - x')G_{t - \nu}(\tilde{x}, \tilde{x}, 0, 0)\xi(\tilde{x})g_{t - \nu}'(x - \tilde{x}) \rangle_{\tilde{x}, \tilde{t}} \]

Page 24
Just as in the bosonic case, $G_t$ possesses an asymptotic expansion

$$G_t(x, x', \theta, \theta') = g_t(x - x' - \theta \theta') \sum_{k=0}^{\infty} \frac{t^k}{k!} \Gamma_k(x, x')$$

in which each term has the form

$$\Gamma_k(x, x') = A_k(x, x') + \theta B_k(x, x') + \theta' C_k(x, x') + \theta \theta' D_k(x, x').$$

The approximation of $G_t$ by its asymptotic expansion proves that $A_k, B_k, C_k, D_k$ have the same $x, x'$ symmetry as noted before. As before, $\Gamma_k$ is constructed from the potential $\Phi(x) = \xi(x) + \theta u(x)$ by an iterative procedure starting with $\Gamma_0(x, x') = 1$.

Formally equating $x$ with $x'$ and $\theta$ with $\theta'$ we define

$$g_k(x) = A_k(x, x) + 2\theta B(x, x).$$

This must be a polynomial in $\xi(x), u(x)$, and their derivatives. But it turns out to be expressible as a polynomial in $\Phi = \xi + \theta u, D\Phi = u + \theta \partial_x \xi, D^2 \Phi = \partial_x \xi + \theta \partial_x u, \ldots$

These polynomials can be seen as elements of the free supersymmetric derivation ring on a single fermionic generator, but in the notation $\Phi = a_1, D\Phi = a_2, \ldots$

The first few such polynomials are

$$g_2 = -a_2$$

$$g_6 = -a_6 + 3a_2^2 - 2a_1 a_3$$

$$g_{10} = (-a_{10} + 10a_2 a_6 + 5a_1^2 - 10a_2)^3 - (4a_1 a_7 + a_3 a_5 - 15a_1 a_2 a_3).$$

They have been shown to be gradients of conserved quantities of the SKdV equation, whose unknown function $\Omega(x, t)$ is to satisfy

$$\Omega_t = -D^6 \Omega + 3\Omega (D^4 \Omega) + 3 ((D\Omega)(D^2 \Omega)).$$

25
The symmetry of the asymptotic Green’s function permits the derivation, after several calculations, of the recursive algorithm

\[ D^2 g_{n+4} = \left( D^6 + 2a_1 D^3 - 4a_2 D^2 + a_3 D - 2a_4 I \right) g_n - a_1 l_n, \]

\[ D^2 l_n = -a_2 D g_n + a_3 g_n. \]

The members of the super KdV hierarchy are then given by \( M g_n \) with

\[ M = D^5 - 3a_1 D^2 - a_2 D - 2a_3 I. \]

In particular \( M g_2 \) is the super KdV equation of Mathieu [11], the same one appearing above.

One may use necessary and sufficient conditions arising from the exact sequence established in section (2) to show \( g_n \) are the gradients of conserved quantities of the SKdV hierarchy.

## 5 Conclusions

We introduced the free \( N = 1 \) supersymmetric derivation ring. We established the exact sequence of supersymmetric rings and linear transformations:

\[ \mathcal{O}_p \mathcal{A} \leftarrow \mathcal{A} \leftarrow \mathcal{A} \leftarrow \mathcal{A} \leftarrow \mathbb{R} \leftarrow 0 \]

\[ Df \leftarrow f \]

\[ Mh \leftarrow h \]

\[ L_g + L^*_g \leftarrow g \]

Several of these relations were already known in the literature [2, 12].
We used necessary and sufficient conditions arising from this exact sequence to obtain the essential relations between conserved quantities, gradients and the $N = 1$ super KdV hierarchy. We combine these algebraic conditions together with an analytical analysis of the super heat operator

$$L = \frac{\partial}{\partial t} - (D^4 - D\Phi + \Phi D).$$

We found an explicit series expansion for the Green’s function of the Super heat operator and discussed its properties. The expansion is convergent under the assumption of bounded bosonic and fermionic potentials as established in section (4). This analysis may be relevant since there is no rigorous Feynman-Kac formula for the fermionic case.

Finally we show that the asymptotic expansion when $t \to 0^+$ of the Green’s function of $L$, evaluated over its diagonal, generates all the members of the $N = 1$ super KdV hierarchy.

The exact sequence of susy rings and linear transformations may also be constructed with $N > 1$ susy generators, we hope to discuss this extension elsewhere. We expect that the algebraic construction established in the first part of this work may have a natural extension for more general finite-dimensional quantum systems. This would be of great help in understanding, for example, the quantum behavior of the supermembrane and super $D$-brane theories [15, 16, 17].

References

[1] P. Mathieu, J. Math. Phys. 29, 2499 (1988).

[2] Yu. I. Manin and A. O. Radul, Commun. Math. Phys. 98, 65 (1985).

[3] C. A. Laberge and P. Mathieu, Phys. Lett. B 215, 718 (1988).
[4] Z. Popowicz, Phys. Lett. A174, 411 (1993).

[5] C. M. Yung, The $N = 3$ supersymmetric KdV hierarchy, (1993).

[6] S. Krivonos, A. Pashnev and Z. Popowicz, Mod. Phys. Lett. A113, 1435 (1998).

[7] F. Delduc, L. Gallot and E. Ivanov, Phys. Lett. B396, 122 (1997).

[8] W. Oevel and Z. Popowicz, Comm. Math. Phys. 139, 441 (1991).

[9] J. M. Figueroa-O’Farrill, J. Mas and E. Ramos, Rev. Mod. Phys. 3, 479 (1991).

[10] P. Mathieu, “Open problems for the super KdV equations”, math-ph/0005007.

[11] S. Andrea, A. Restuccia, A. Sotomayor, J. Math. Phys. 42, 2625 (2001).

[12] P. Mathieu, Lett. Math. Phys. 16, 199 (1988).

[13] J. Glimm, A. Jaffe, Quantum Physics, Springer-Verlag, New York, 1987.

[14] I. G. Avramidi and R. Schimming, J. Math. Phys. 36(9), 5042 (1995).

[15] M. Garcia del Moral, A. Restuccia, Phys. Rev. D66, 045023 (2002).

[16] L. Boulton, M. Garcia del Moral, I. Martin, A. Restuccia, Class. Quant. Grav. 19, 2951 (2002).

[17] L. Boulton, M. Garcia del Moral, A. Restuccia, hep-th/0211045 to be published in Nuclear Physics B.