Transductive-Inductive Cluster Approximation
Via Multivariate Chebyshev Inequality

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Abstract. Approximating adequate number of clusters in multidimensional data is an open area of research, given a level of compromise made on the quality of acceptable results. The manuscript addresses the issue by formulating a transductive inductive learning algorithm which uses multivariate Chebyshev inequality. Considering clustering problem in imaging, theoretical proofs for a particular level of compromise are derived to show the convergence of the reconstruction error to a finite value with increasing (a) number of unseen examples and (b) the number of clusters, respectively. Upper bounds for these error rates are also proved. Non-parametric estimates of these error from a random sample of sequences empirically point to a stable number of clusters. Lastly, the generalization of algorithm can be applied to multidimensional data sets from different fields.

Keywords: Transductive Inductive Learning, Multivariate Chebyshev Inequality

1 Introduction

The estimation of clusters has been approached either via a batch framework where the entire data set is presented and different initializations of seed points or prototypes tested to find a model of cluster that fits the data like in k-means [9] and fuzzy C-means [2] or an online strategy clusters are approximated as new examples of data are presented one at a time using variational Dirichlet processes [7] and incremental clustering based on randomized algorithms [3]. It is widely known that approximation of adequate number of clusters using a multidimensional data set is an open problem and a variety of solutions have been proposed using Monte Carlo studies [5], Bayesian-Kullback learning scheme in mean squared error setting or gaussian mixture [19], model based approaches [6] and information theory [17], to cite a few.

This work deviates from the general strategy of defining the number of clusters apriori. It defines a level of compromise, tolerance or confidence in the quality of clustering which gives an upper bound on the number of clusters generated. Note that this is not at all similar to defining the number of clusters. It only indicates the level of confidence in the result and the requirement still is to
estimate the adequate number of clusters, which may be way below the bound. The current work focuses on dealing with the issue of approximating the number of clusters in an online paradigm when the confidence level has been specified. In certain aspects it finds similarity with the recent work on conformal learning theory \cite{18} and presents a novel way of finding the approximation of cluster with a degree of confidence.

Conformal learning theory \cite{15}, which has its foundations in employing a transductive-inductive paradigm deals with the idea of estimating the quality of predictions made on the unlabeled example based on the already processed data. Mathematically, given a set of already processed examples \((x_1, y_1), (x_2, y_2), \ldots, (x_{i-1}, y_{i-1})\), the conformal predictors give a point prediction \(\hat{y}\) for the unseen example \(x_i\) with a confidence level of \(\Gamma\). Thus it estimates the confidence in the quality of prediction using the original label \(y_i\) after the prediction has been made and before moving on to the next unlabeled example. These predictions are made on the basis of a non-conformity measure which checks how much the new example is different from a bag of already seen examples. A bag is considered to be a finite sequence \(Z = (z_1, z_2, \ldots, z_{i-1})\) of examples, where \(z_i = (x_i, y_i)\). Then using the idea of exchangeability, it is known from \cite{18}, under a weaker assumption that for every positive integer \(i\), every permutation \(\pi\) of \(\{1, 2, \ldots, i\}\), and every measurable set \(E \subset Z^i\), the probability distribution \(P\{\{z_1, z_2, \ldots, z_i\} \in Z^\infty : (z_1, z_2, \ldots, z_i) \in E\} = P\{\{z_1, z_2, \ldots \} \in Z^\infty : (z_{\pi(1)}, z_{\pi(2)}, \ldots, z_{\pi(i)}) \in E\}\). A prediction for the new example \(x_i\) is made if and only if the frequency (p-value) of exchanging the new example with another example in the bag is above certain value.

This manuscript finds its motivation from the foregoing theory of online prediction using transductive-inductive paradigm. The research work applies the concept of coupling the creation of new clusters via transduction and aggregation of examples into these clusters via induction. It finds its similarity with \cite{15} in utilizing the idea of prediction region defined by a certain level of confidence. It presents a simple algorithm that differs significantly from conformal learning in the following aspect: (1) Instead of working with sequences of data that contain labels, it works on unlabeled sequences. (2) Due the first formulation, it becomes imperative to estimate the number of clusters which is not known apriori and the proposed algorithm comes to rescue by employing a Chebyshev inequality. The inequality helps in providing an upper bound on the number of clusters that could be generated on a random sample of sequence. (3) The quality of the prediction in conformal learning is checked based on the p-values generated online. The current algorithms relaxes this restriction in checking the quality online and just estimates the clusters as the data is presented. (4) The foregoing step makes the algorithm a weak learner as it is sequence dependent. To take stock of the problem, a global solution to the adequate number of cluster is approximated by estimating kernel density estimates on a sample of random sequences of a data. Finally, the level of compromise captured by a parameter in the inequality gives an upper bound on the number of clusters generated. In case of clustering in static images, for a particular parameter value, theoretical proofs
show that the reconstruction error converges to a finite value with increasing (a) number of unseen examples and (b) the number of clusters. Empirical kernel density estimates of reconstruction error over a random sample of sequences on toy examples indicate the number of clusters that have high probability of low reconstruction error. *It is not necessary that labeled data are always present to compute the reconstruction error.* In that case the proposed algorithm stops short at density estimation of approximated number of clusters from a random sequence of examples, with a certain degree of confidence.

Another dimension of the proposed work is to use the generalization of multivariate formulation of Chebyshev inequality [1], [10], [11]. It is known that Chebyshev inequality helps in proving the convergence of random sequences of different data. Also the multivariate formulation of the Chebyshev inequality facilitates in providing bounds for multidimensional data which is often afflicted by the curse of dimensionality making it difficult to compute multivariate probabilities. One of the generalizations that exist for multivariate Chebyshev inequality is the consideration of probability content of a multivariate normal random vector to lie in an Euclidean $n$-dimensional ball [14], [13]. This work employs a more conservative approach is the employment of the Euclidian $n$-dimensional ellipsoid which restricts the spread of the probability content [4]. Work by [9] and [16] provide motivation in employment of multivariate Chebyshev inequality.

Efficient implementation and analysis of $k$-means clustering using the multivariate Chebyshev inequality has been shown in [9]. The current work differs from $k$-means in (1) providing an online setting to the problem of clustering (2) estimating the number of clusters for a particular sequence representation of the same data via convergence through ellipsoidal multivariate Chebyshev inequality, given the level of confidence, compromise or tolerance in the quality of results (3) generating global approximations of number of clusters from non-parametric estimates of reconstruction error rates for sample of random sequences representing the same data and (4) not fixing the cluster number apriori. It must be noted that in $k$-means, the solutions may be different for different initializations for a particular value of $k$ but the value of $k$ as such remains fixed. In the proposed work, with a high probability, an estimate is made regarding the minimum number of clusters that can represent the data with low reconstruction error. This outlook broadens the perspective of finding multiple solutions which are upper bounded as well as approximating a particular number of cluster which have similar solutions. This similarity in solutions for a particular number of cluster is attributed to the constraint imposed by the Chebyshev inequality. A key point to be noted is that using increasing levels of compromise or confidence as in conformal learners, the proposed work generates a nested set of solutions. Low confidence or compromise levels generate tight solutions and vice versa. Thus the proposed weak learner provides a set of solutions which are robust over a sample.

This manuscript also extends the work of [16] on employment of multivariate Chebyshev inequality for image representation. Work in [16] presents a hybrid model based on Hilbert space filling curve [8] to traverse through the image.
Since this curve preserve the local information in the neighbourhood of a pixel, it reduces the burden of good image representation in lower dimensions. On the other side, it acts as a constraint on processing the image in a particular fashion. The current work removes this restriction of processing images via the space filling curves by considering any pixel sequence that represents the image under consideration. Again, a single sequence may not be adequate enough for the learner to synthesize the image to a recognizable level. This can be attributed to the fact that in an unsupervised paradigm the number of clusters are not known a priori and also the learner would be sequence dependent. To reiterate, the proposed work addresses the issues of • recognizability, by defining a level of compromise that a user is willing to make via the Chebyshev parameter $C_p$ and • sequence specific solution, by taking random samples of pixel sequences from the same image. The latter helps in estimating a population dependent solution which would be robust and stable synthesis. Regularization of these error over approximated number of clusters for different levels of compromise leads to an adequate number of clusters that synthesize the image with minimal deviation from the original image.

Thus the current work provides a new perspective in approximation of cluster number at a particular confidence level. To test the propositions made, the problem of clustering in images is taken into account. Generalizations of the algorithm can be made and applied to different fields involving multidimensional data sets in an online setting. Let $I$ be an RGB image. A pixel in $I$ is an example $x_i$ with $N$ dimensions (here $N = 3$). It is assumed that examples appear randomly without repetition for the proposed unsupervised learner. Note that when the sample space (here the image $I$) has finite number of examples in it (here $M$ pixels), then the total number of unique sequences is $M!$. When $M$ is large, $M! \to \infty$. Currently, the algorithm works on a subset of unique sequences sampled from $M!$ sequences. The probability of a sequence to occur is equally likely (in this case $1/M$). RGB images from the Berkeley Segmentation Benchmark (BSB) [12] have been taken into consideration for the current study.

2 Transductive-Inductive Learning Algorithm

Given that the examples $(z_i = x_i)$ in a sequence appear randomly, the challenge is to (1) learn the association of a particular example to existing clusters or (2) create a new cluster, based on the information provided by already processed examples. The current algorithm handles the two issues via (1) evaluation of a nonconformity measure defined by multivariate Chebyshev’s inequality formulation and (2) 1-Nearest Neighbour (NN) transductive learning, respectively. The multivariate formulation of the generalized Chebyshev inequality [4] is applied to a new example using a single Chebyshev parameter. This inequality tests the deviation of the new example from the mean of a cluster of examples and gives a lower probabilistic bound on whether the example belongs to the cluster under investigation. If the new random example passes the test, then it is associated with the cluster and the mean and covariance matrix for the cluster is recom-


Algorithm 1 Unsupervised Learner

1: procedure Unsupervised Learner(img, C_p)
2:   [nrows, ncols] ← size(img)
3:   M ← nrows × ncols ▷ Total no. of unseen examples
4:   pt_idx ← 0 ▷ Number of examples encountered
5:   pt_idx ← {1, 2, ..., M} ▷ Total no. of indices of unseen examples
6: Initialize Variables
7:   cluster_cnt ← 0 ▷ Number of clusters
8:   CumErr_val ← 0 ▷ Cumulative value
9:   Err1 ← [] ▷ Error rate as no. of examples increase
10:  Err2 ← [] ▷ Error rate as no. of clusters increase
11: while Card(pt_idx) examples remain unprocessed do ▷ pt_idx ⊂ {1, 2, ..., M}
12:   Choose a random example x_i s.t. i ∈ pt_idx
13:   pt_cnt ← pt_cnt + 1
14:   Update pt_idx i.e pt_idx ← pt_idx − {i}
15:   CRITERION ← []
16:   Err_val ← 0
17:   ∀q clusters were q ∈ {0, 1, 2, ..., cluster_cnt} ▷ x means all examples in cluster q
18:   Err_val ← ∑k=1(x_k − E_q(x))^2 ▷ x means all examples in cluster q
19:   CumErr_val ← CumErr_val + Err_val
20:   Compute D ← (x_i − E_q(x))^T Σ_q^−1 (x_i − E_q(x)) ▷ Σ_q is Chebyshev parameter
21:   CRITERION ← [CRITERION; D_q, q]
22:   If more than one cluster that associates to x_i, i.e length(CRITERION) ≥ 1
23:     Associate x_i to selected cluster q with minimum D_q
24:     Err1 ← [Err1, Err_val/Cnt]
25:   If x_i is not associated with any cluster, i.e sum(FOUND) == 0
26:     Err2 ← [Err2, Err_val/Cnt]
27:     cluster_cnt ← cluster_cnt + 1
28:   Using 1-NN find x_j closest to x_i s.t. j ∈ pt_idx
29:   Update pt_idx i.e pt_idx ← pt_idx − {j}
30:   Form a new cluster {x_i, x_j}
31: end while
32: end procedure

puted. In case there exists more than one cluster which qualify for association, then the cluster with lowest deviation to the new example is picked up for association. It is also possible to assign the new example to a random chosen cluster from the selected clusters to induce noise and then check for the approximations on the number of clusters. This has not been considered in the current work for the time being. In case of failure to find any association, the algorithm employs 1-NN transductive algorithm to find a closest neighbour of the current example under processing. This neighbour together with the current example forms a new cluster.

Several important implications arise due to the usage of a probabilistic inequality measure as a nonconformal measure. These will be elucidated in detail in the later sections. An important point to consider here is the usage of 1-NN
algorithm to create a new cluster. Even though it is known that 1-NN suffers from the problem of the curse of dimensionality, for problems with small dimensions, it can be employed for transductive learning. The aim of the proposed work is not to address the curse of dimensionality issue. Also, note that in the general supervised conformal learning algorithm, a prediction has to be made before the next random example is processed. This is not the case in the current unsupervised framework of the conformal learning algorithm. In case the current random example fails to associate with any of the existing clusters, under the constraint yielded by the Chebyshev parameter, the NN helps in finding the closest example (in feature space) from the remaining unprocessed sample data set, to form a new cluster. Thus the formation of a new cluster depends on the strictness of the Chebyshev parameter $C_p$. The procedure for unsupervised conformal learning is presented in algorithm 1. It does not strictly follow the idea of finding confidence on the prediction as labels are not present to be tested against. The goal here is to reconstruct the clusters from a single pixel sequence such that they represent the image. The quality of the reconstruction is taken up later on when a random sample of pixel sequences are used to estimate the probability density of the reconstruction error rates. Note that in the algorithm, $E_q(x)$ represents the mean of the examples $x$ in the $q^{th}$ cluster and $\Sigma_p$ is the covariance matrix of $ND$ feature examples of the $q^{th}$ cluster.

3 Theoretical Perspective

Application of the multivariate Chebyshev inequality that yields a probabilistic bound enforces certain important implications with regard to the clusters that are generated. For the purpose of elucidation of the algorithm, the starfish image is taken from [12].

3.1 Multivariate Chebyshev Inequality

Let $X$ be a stochastic variable in $N$ dimensions with a mean $E[X]$. Further, $\Sigma$ be the covariance matrix of all observations, each containing $N$ features and $C_p \in \mathbb{R}$, then the multivariate Chebyshev Inequality in [4] states that:

\[
\mathbb{P}\{ (X - E[X])^T \Sigma^{-1} (X - E[X]) \geq C_p \} \leq \frac{N}{C_p}
\]

\[
\mathbb{P}\{ (X - E[X])^T \Sigma^{-1} (X - E[X]) < C_p \} \geq 1 - \frac{N}{C_p}
\]

(1)

i.e. the probability of the spread of the value of $X$ around the sample mean $E[X]$ being greater than $C_p$, is less than $N/C_p$. There is a minor variation for the univariate case stating that the probability of the spread of the value of $x$ around the mean $\mu$ being greater than $C_p\sigma$ is less than $1/C_p^2$. Apart from the minor difference, both formulations convey the same message about the probabilistic bound imposed when a random vector or number $X$ lies outside the mean of the sample by a value of $C_p$. 

Fig. 1. A random sequence of Starfish Image segmented via unsupervised conformal learning algorithm. \((C_p, \text{NoClust}, \text{TRErr})\) represent the tuple containing the Chebyshev parameter \((C_p)\), number of clusters generated (NoClust) while using \(C_p\) and the total reconstruction error of the generated image from the original image (TRErr)(a) (3, 1034, 17.746), (b) (5, 271, 36.32), (c) (7, 159, 54.71), (d) (9, 45, 40.591), (e) (11, 31, 62.606), (f) (13, 29, 66.061), (g) (15, 33, 65.424), (h) (17, 24, 64.98).

3.2 Association to Clusters

Once a cluster is initialized (say with \(x_i\) and \(x_j\)), the size of the cluster depends on the number of examples getting associated with it. The multivariate formalism
of the Chebyshev inequality controls the degree of uniformity of feature values of examples that constitute the cluster. The association of the example to a cluster happens as follows: Let the new random example (say \(x_t\)) be considered for checking the association to a cluster. If the spread of example \(x_t\) from \(E_q(x)\) (the mean of the \(q^{th}\) cluster \(\{x_i, x_j\}\)), factored by the covariance matrix \(\Sigma_q\), is below \(C_p\), then \(x_t\) is considered as a part of the cluster. Using Chebyshev inequality, it boils down to:

\[
P\left\{\left(x_t - E_q(x)\right)^T \Sigma_q^{-1} \left(x_t - E_q(x)\right) \geq C_p \right\} \leq \frac{N}{C_p} \\
N_{C_p} \geq 1 - \frac{N}{C_p}
\]

(2)

Satisfaction of this criterion suggests a possible cluster to which \(x_t\) could be associated. This test is conducted for all the existing clusters. If there are more than one cluster to which \(x_t\) can be associated, then the cluster which shows the minimum deviation from the new random point is chosen. Once the cluster is chosen, its size is extended to by one more example i.e. \(x_t\). The cluster now constitutes \(\{x, x_j, x_t\}\). If no association is found at all, a new cluster is initialized and the process repeats until all unseen examples have been processed. The satisfaction of the inequality gives a lower probabilistic bound on size of cluster by a value of \(1 - (N/C_p)\), if the second version of the Chebyshev formula is under consideration. Thus the size of the clusters grow under a probabilistic constraint in a homogeneous manner. For a highly inhomogeneous image, a cluster size may be very restricted or small due to big deviation of pixel intensities from the cluster it is being tested with.

Once the pixels have been assigned to respective decompositions, all pixels in a single decomposition are assigned the average value of intensities of pixels that constitute the decomposition. Thus is done under the assumption that decomposed clusters will be homogeneous in nature with the degree of homogeneity controlled by \(C_p\). Figure 1 shows the results of clustering for varying values of \(C_p\) for the starfish image from [12].

3.3 Implications

In [?] various implications have been proposed for using multivariate Chebyshev inequality for image representation using space filling curve. In order to extend on their work, a few implications are reiterated for further development. The inequality being a criterion, the probability associated with the same gives a belief based bound on the satisfaction of the criterion. In order to proceed, first a definition of Decomposition is needed.

**Definition 1.** Let \(D\) be a decomposition which contains a set of points \(x\) with a mean of \(E_q(x)\). The set expands by testing a new point \(x_t\) via the Chebyshev inequality \(P\left\{\left(x_t - E_q(x)\right)^T \Sigma_q^{-1} \left(x_t - E_q(x)\right) < C_p \right\} \geq 1 - \frac{N}{C_p}\).
The decomposition may include the point $x_t$ depending on the outcome of the criterion. A point to be noted is that, if the new point $x_t$ belongs to $D$, then $D$ can be represented as $(x_t - \mathbf{E}_q(x))^T \Sigma_q^{-1} (x_t - \mathbf{E}_q(x))$.

**Lemma 1.** Decompositions $\mathcal{D}$ are bounded by lower probability bound of $1 - (N/C_p)$.

**Lemma 2.** The value of $C_p$ reduces the size of the sample from $\mathcal{M}$ to an upper bound of $\mathcal{M}/C_p$ probabilistically with a lower bound of $1 - (N/C_p)$. Here $\mathcal{M}$ is the number of examples in the image.

**Lemma 3.** As $C_p \to \infty$ the lower probability bound drops to zero, implying large number of small decompositions $\mathcal{D}$ can be achieved. Vice versa for $C_p \to \infty$.

It was stated that the image can be reconstructed from pixel sequences at a certain level of compromise. From lemma 2 it can be seen that $C_p$ reduces the sample size while inducing a certain amount of error due to loss of information via averaging. This reduction in sample size indicates the level of compromise at which the image is to processed. This reduction in sample size or level of compromise is directly related to the construction of probabilistically bounded decompositions also. Since the decompositions are generated via the usage of $C_p$ in equation 1, the belief of their existence in terms of a lower probability bound (from lemma 1) suggests a confidence in the amount of error incurred in reconstruction of the image. For a particular pixel, this reconstruction error can be computed by squaring the difference between the value of the intensity in the original image and the intensity value assigned after clustering. Since a somewhat homogeneous decomposition is bounded probabilistically, the reconstruction error of pixels that constitute it are also bounded probabilistically. Thus for all decompositions, the summation of reconstruction errors for all pixels is bounded. The bound indicates the confidence in the generated reconstruction error. Also, by lemma 2 since the number of decompositions or clusters is upper bounded, the total reconstruction error is also upper bounded. It now remains to be proven that for a particular level of compromise, the error rates converge as the number of processed examples and the number of clusters increase.

In algorithm 1 three error rates are computed as the random sequence of examples get processed. For each original pixel $x_i \in \mathcal{R}^N$ in the image, let $x_i^R$ be the intensity value assigned after clustering. Then the reconstruction error for pixel $x_i$ is $||x_i - x_i^R||_2$. Since a pixel is assigned to a particular decomposition $\mathcal{D}_q$, it gets a value of the mean of the all pixels that constitute the decomposition $\mathcal{D}_q$. Thus the reconstruction error for a pixel turns out to be $||x_i - \mathbf{E}_q(x)||_2$. For each cluster $q$, the reconstruction error is $\text{Err}_{\mathcal{D}_q} = \sum_{i=1}^{n} ||x_i - \mathbf{E}_q(x)||_2$. Note that the error also indicates how much the examples deviate from the mean of their respective cluster. As new examples are processed based on the information present from the previous examples, the total error computed at after processing the first $p_{\text{cnt}}$ examples in a random sequence is $\text{Err}_{\text{val}} = \sum_{\text{cluster cnt}} \text{Err}_{\mathcal{D}_q}$. The error rate for these $p_{\text{cnt}}$ examples is $\text{Err}_1 = \text{Err}_{\text{val}}/p_{\text{cnt}}$. Finally, an error rate is computed that captures how the deviation of the examples from
their respective cluster means happen, after the formation of a new cluster. This error is denoted by \( Err_2 \). The formula for \( Err_2 \) is the same as \( Err_1 \) but with a minute change in conception. The \( Err_{val} \) are divided by the total number of point processed after the formation of every new cluster.

**Theorem 1.** Let \( Z_i \) be a random sequence that represents the entire image \( I \). If \( Z_i \) is decomposed into clusters via the Chebyshev Inequality using the unsupervised learner, then the reconstruction error rate \( Err_1 \) converges asymptotically with a probabilistically lower bound or confidence level of \( 1 - N/C_p \) or greater.

**Proof.** It is known that the total reconstruction error after \( pt_{cnt} \) examples have been processed, is \( Err_{val} = \sum_{q=1}^{\text{cluster\_cnt}} Err_{D_q} \). And the error rate is \( Err_1 = Err_{val}/pt_{cnt} \). It is also known from equation 1 that an example is associated to a particular decomposition \( D_q \) if it satisfies the constraint \((x_t - E_q(x))^T \Sigma_q^{-1} (x_t - E_q(x)) \leq C_p \). Since \( C_p \) defines level of compromise on the image via lemma 2 and the decompositions \( D_q \) is almost homogeneous, all examples that constitute a decomposition have similar attribute values. Due to this similarity between the attribute values, the non-diagonal elements of the covariance matrix in the inequality above approach to zero. Thus, \( \Sigma_q^{-1} \approx det|\Sigma_q^{-1}|I \), were \( I \) is the identity matrix. The inequality then equates to:

\[
(x_t - E_q(x))^T det|\Sigma_q^{-1}|I(x_t - E_q(x)) \leq C_p
\]

\[
det|\Sigma_q^{-1}|(x_t - E_q(x))^T I(x_t - E_q(x)) \leq C_p
\]

\[
(x_t - E_q(x))^T I(x_t - E_q(x)) \leq \frac{C_p}{det|\Sigma_q^{-1}|}
\]
\[ ||x_t - E_q(x)||_2 \leq \frac{C_p}{\det(|\Sigma_q^{-1}|)} \]  

(3)

Thus, if \( x_i = x_t \) was the last example to be associated to a decomposition, the reconstruction error \( ||x_i - E_q(x)||_2 \) for that example would be upper bounded by \( \frac{C_p}{\det(|\Sigma_q^{-1}|)} \). Consequently, the total error after processing \( pt_{cntr} \) examples is also upper bounded, i.e.

\[
Err_{val} = \sum_{q=1}^{\text{cluster}_{cntr}} Err_D_q
\]

= \sum_{q=1}^{\text{cluster}_{cntr}} \sum_{i=1}^{n} ||x_i - E_q(x)||_2

\overset{R.A.}{=} \sum_{q=1}^{\text{cluster}_{cntr}} \sum_{i=1}^{n} \frac{C_p}{\det(|\Sigma_q^{-1}|)}

\overset{R.A.}{=} \sum_{q=1}^{\text{cluster}_{cntr}} \sum_{i=1}^{n} \frac{C_p}{\det(|\Sigma_q^{-1}|)}

(4)

Thus the error rate \( Err_1 = \frac{Err_{val}}{pt_{cntr}} \) is also upper bounded. Different decompositions may have different \( \Sigma_q^{-1} \), but in the worst case scenario, if the decomposition with the lowest covariance is substituted for every other decompositions, then the upper bound on the error is

\[
\sum_{q=1}^{\text{cluster}_{cntr}} \sum_{i=1}^{n} \frac{C_p}{\det(|\Sigma_{\text{lowest}}^{-1}|) \times pt_{cntr}}
\]

which equates to \( \frac{C_p}{\det(|\Sigma_{\text{lowest}}^{-1}|)} \).

It is important to note that this error rate converges to a finite value asymptotically as the number of processed examples increases. This is because initially when the learner has not seen enough examples to learn and solidify the knowledge in terms of a stable mean and variance of decompositions, the error rate \( Err_1 \) increases as new examples are presented. This is attributed to the fact that new clusters are formed more often in the initial stages, due to lack of prior knowledge. After a certain time, when large number of examples have been encountered to help solidify the knowledge or stabilize the decompositions, then addition of further examples does not increment the error. This stability of clusters is checked via the multivariate formulation of the Chebyshev inequality in equation \( \overset{2}{\text{2}} \). The stability also casues the error rate \( Err_1 \) to stabilize and thus indicate its convergence in a bounded manner with a probabilistic confidence level. Thus for any value of \( pt_{cntr} \), there exists an upper bound on reconstruction error, which stabilizes as \( pt_{cntr} \) increases.

For \( C_p = 7 \), the image (c) in figure \( \overset{1}{\text{1}} \) shows the clustered image that is generated using the unsupervised conformal learning algorithm. Pixels in a cluster of the generated image have the mean of the cluster as their intensity value or the label. This holds for all the clusters in the generated image. The total number of clusters generated for a particular random sequence was 159. The error rate \( Err_1 \) is depicted in figure \( \overset{2}{\text{2}} \).
Fig. 3. Error rate $Err_2$ for a particular sequence with increasing number of clusters and $C_p = 7$.

**Theorem 2.** Let $Z_i$ be a random sequence that represents the entire image $I$. If $Z_i$ is decomposed into clusters via the Chebyshev Inequality using the unsupervised learner, then the reconstruction error rate $Err_2$ converges asymptotically with a probabilistically lower bound or confidence level of $1 - \frac{N}{C_p}$ or greater.

**Proof.** The error rate $Err_2$ is the computation of error after each new cluster is formed. The upper bound on $Err_2$ as the number of clusters or decompositions increase follows a proof similar to one presented in theorem 1. □

Again for the same $C_p = 7$, the image (c) in figure 1, the error rate $Err_2$ is depicted in figure 3. Intuitively, it can be seen that both the reconstruction error rates converge to an approximately similar value.

The theoretical proofs and the lemmas suggest that, for a given level of compromise $C_p$ there exists an upper bound on the reconstruction error as well as the number of clusters. But this reconstruction error and the number of clusters is dependent on a pixel sequence presented to the learner. Does this mean that for a particular level of compromise one may find values of reconstruction error and number of clusters that may never converge to a finite value, when a random sample of pixel sequences that represent an image are processed by the learner? Or in a more simplified way, is it possible to find a reconstruction error and the number of clusters at a particular level of compromise that best represents the image? This points to the problem of whether an image can be reconstructed at a particular level of compromise where there is a high probability of finding a low reconstruction error and the number of clusters, from a sample of sequences.
The existence of such a probability value would require the knowledge of the probability distribution of the reconstruction error over increasing (1) number of examples and (2) number of clusters generated. In this work, kernel density estimation (KDE) is used to estimate the probability distribution of the reconstruction error $Err_1$ and $Err_2$. To investigate into the quality of the solution obtained, the error rates were generated for different random sequences and a KDE was evaluated on the observations. The density estimate empirically point to the least error rates with high probability. It was found that the error rates $Err_1$, $Err_2$ and the number of clusters, all converge to a particular value, for a given image.
For $C_p = 10$, the probability density estimates were generated using the density estimates on error rates and the number of clusters obtained on 1000 random sequences of the same image. It was found that the error rates $Err_1$, $Err_2$ and the number of clusters converge to 33.1762, 35.9339 and 38, respectively. Figure 4 shows the graphs for the same. It can be seen from graphs (a) and (b) in figure 4 that both $Err_1$ and $Err_2$ converge nearly to the similar values.

It can been noted that with increasing value of the parameter $C_p$, the bound on the decomposition expands which further leads to generation of lower number of clusters required to reconstruct the image. Thus it can be expected that at lower levels of compromise, the reconstruction error (via KDE) is low but the number of clusters (via KDE) is very high and vice versa. Figure 5 shows the behaviour of these reconstruction error and number of clusters generated as the level of compromise increases. High reconstruction error does not necessarily mean that the representation of the image is bad. It only suggests the granularity of reconstruction obtained. Thus the reconstruction of the image can yield finer details at low level of compromise and point to segmentations at high level of compromise. Regularization over the level of compromise and the number of clusters would lead to a reconstruction which has low reconstruction error as well as adequate number of decompositions that represent an image properly.

There are a few points that need to be remembered when applying such an online learning paradigm. The reconstructed results come near to original image only at a level of imposed compromise. As the size of dataset or the image increases, the time consumed and the number of computations involved for processing also increases. To start with, the learner would perform well in clean images than on noisy images. Adaptations need to be made for processing noisy images or the pre-processing would be a necessary step before application of such an algorithm. Other inequalities can also be taken into account for multivariate information online. It would be tough to compare the algorithm with other
powerful clustering algorithms as the proposed work presents a weak learner and provides a general solution with no tight bounds on the quality of clustering.

Nevertheless, the current work contributes to estimation of cluster number in an unsupervised paradigm using transductive-inductive learning strategy. It can be said that for a fixed Chebyshev parameter, in a bootstrapped sequence sampling environment without replacement, the unsupervised learner converges to a finite error rate along with the a finite number of clusters. The result in terms of clustering and the error rates may not be the most optimal (where the meaning depends on the goal of optimization), but it does give an affirmative clue that image decomposition is robust and convergent.

4 Conclusion

A simple transductive-inductive learning strategy for unsupervised learning paradigm is presented with the usage of multivariate Chebyshev inequality. Theoretical proofs of convergence in number of clusters for a particular level of compromise show (1) stability of result over a sequence and (2) robustness of probabilistically estimated approximation of cluster number over a random sample of sequences, representing the same multidimensional data. Lastly, upper bounds generated on the number of clusters point to a limited search space.

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