Non-uniqueness in conformal formulations of the Einstein constraints

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Abstract
Standard methods in nonlinear analysis are used to indicate that there exists a parabolic branching of solutions of the Lichnerowicz–York equation with an unscaled source. We also apply these methods to the extended conformal thin sandwich formulation and, by assuming that the linearized system develops a kernel solution for sufficiently large initial data, we reproduce the parabolic solution curves for the conformal factor, lapse and shift found numerically by Pfeiffer and York.

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1. Introduction

With the onset of gravitational wave astronomy approaching, it is of crucial importance to have constructed suitably realistic theoretical and numerical models of the spacetime structure of gravitational wave sources. This demands that we study the initial value problem for Einstein’s gravitational field equations. These form a complicated quasi-linear system of partial differential equations. They naturally split into six evolution equations and four constraint equations on the initial data. It is the constraint equations that this work will focus on.

This work was initiated to try to explain the intriguing non-uniqueness results found numerically by Pfeiffer and York in [1]. They found a parabolic curve of regular solutions for each of the five unknowns in the extended conformal thin sandwich formulation of the Einstein constraints. This system has enjoyed much support amongst numerical relativists but their results show the existence of two regular solutions for the entire range of wave amplitude considered. We present a simple local derivation of these non-uniqueness results.

We first review the important features of the conformal method for solving the constraints. The four constraint equations of general relativity are

\[ R(\tilde{g}) - \tilde{K}^{ij} \tilde{K}_{ij} + \tilde{K}^2 = 16\pi \rho \]  

(1)
\[ \nabla_i (\bar{K}^{ij} - \bar{g}^{ij} \bar{K}) = 8\pi J^j \]  

(2)

where \( \bar{K}^{ij} \) is the extrinsic curvature of the spacelike initial data slice and \( \bar{K} = \text{tr}_g \bar{K}^{ij} \) and \( \rho \) and \( J^j \) must satisfy conservation equations. In the various conformal formulations an initial 3-metric \( \bar{g}_{ij} \) is chosen which is conformally related to the physical solution of the constraints \( \bar{g}_{ij} :\)

\[ \bar{g}_{ij} = \phi^4 g_{ij}. \]  

(3)

We decompose \( \bar{K}^{ij} \) as \( \bar{K}^{ij} = \bar{A}^{ij} + \frac{1}{3} \bar{g}^{ij} \bar{K} \) and define the tracefree conformal extrinsic curvature tensor \( A^{ij} \) according to

\[ \bar{A}^{ij} = \phi^{-10} A^{ij} \]  

(4)

which leads to

\[ \bar{A}^{ij} \bar{A}^{ij} = \phi^{-12} A^{ij} A^{ij}, \]  

(5)

where barred objects are defined with respect to the physical metric \( \bar{g} \) and \( \bar{K} = K \). This property is used in the transformation of the Hamiltonian constraint (1). Together with the transformation for the scalar curvature given by

\[ R(\bar{g}) \phi^5 = R(g) \phi - 8\nabla^2 \phi \]  

(6)

it yields the Lichnerowicz–York (LY) equation for \( \phi \):

\[ \nabla^2 \phi - \frac{R(g) \phi}{8} = -\frac{1}{8} A_{ij} A^{ij} \phi^7 + \frac{K^2}{12} \phi^5 - 2\pi \rho \phi^5. \]  

(7)

Under the conformal transformations (3)–(4) a symmetric tracefree tensor satisfies

\[ \nabla_i \bar{A}^{ij} = \phi^{-10} \nabla_i A^{ij}, \]  

(8)

which simplifies the momentum constraint (2),

\[ \nabla_i A^{ij} - \frac{2}{3} \phi^6 \nabla^i K = 8\pi \phi^{10} J^j. \]  

(9)

We see immediately that the vacuum (i.e. \( \rho = J^j = 0 \)) constraint equations decouple if we have constant mean curvature, \( K = \text{const} \). The original conformal transverse traceless (CTT) formulation by York and his later conformal thin sandwich (CTS) formulation have identical existence and uniqueness properties (see [2]). They differ in their construction of the tensor \( A^{ij} \). The CTT method relies upon tensor splittings to construct a TT tensor. The CTS formulation is much simpler and bypasses these complications. There \( A^{ij} \) takes the form

\[ A^{ij} = \frac{\mathbb{L} \beta^{ij} - U^{ij}}{2N}, \]  

(10)

where \( U^{ij} \) is the tracefree part of the time derivative of the conformal metric, \( N \) is the conformal lapse related to the physical lapse \( \bar{N} \) by \( \bar{N} = N \phi^6 \), \( \beta^i \) has a natural interpretation as the shift vector for the spatial slice, and \( \mathbb{L} \) is the conformal killing operator defined as \( \mathbb{L} X^{ij} = \nabla^i X^j + \nabla^j X^i - \frac{2}{3} g^{ij} \nabla_k X^k \). The initial data for the four equation system (7), (9) is \( (g_{ij}, U_{ij}, K, N) \).

The extended conformal thin sandwich (XCTS) system [1, 3] extends the CTS system by allowing the slicing condition to be propagated (\( \dot{K} \) is now initial data so that the evolution equation for \( K \) becomes a constraint equation). This is highly desirable since it is natural to choose \( \dot{K} = 0 \) at quasi-equilibrium (i.e. when \( U^{ij} = 0 \), for example, while it is unclear what choice for the conformal lapse in the standard CTS formulation should be chosen for this situation. However, this condition couples the lapse fixing equation \( \dot{K} \) to the four constraint
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The initial data are now in the Lagrangian form $(g_{ij}, U_{ij}; K, \dot{K})$. The five vacuum XCTS equations are

$$\nabla^2 \phi - \frac{R(g)\phi}{8} = -\frac{a(\beta)\phi^7}{32\chi^2} + \frac{K^2}{12}\phi^5 \quad (11)$$

$$\nabla^2 \chi - \frac{R(g)\chi}{8} = \frac{7a(\beta)\phi^5}{32\chi} + \frac{5K^2}{12}\phi^4 - \phi^5(\dot{K} - \beta^i \nabla_i K) \quad (12)$$

$$\nabla_i \left( \frac{\nabla_{\beta}^i - U_{ij}}{2N} \right) - \frac{2\phi^6}{3} \nabla^j K = 0 \quad (13)$$

where $a(\beta) = (\nabla_{\beta}^i - U_{ij})(\nabla_{\beta}^j - U_{ij})$ and $\chi = N\phi^7$. This extended system shares the conformal covariance properties of the original system by construction since the new equation it couples to is the physical lapse fixing equation. In the analysis that follows we restrict our attention to asymptotically flat initial data.

The non-uniqueness results in [1] should not be confused with the non-uniqueness that results from choosing trivial initial data in the standard CTT/CTS formulations. In this case we need only solve (7) which is simply $\nabla^2 \phi = 0$. We get a unique regular solution (flat space $\phi = 1$) or any number of singular moment of time symmetry Schwarzschild solutions by linearity. However, Pfeiffer and York found two regular solutions for each nonzero wave amplitude considered and their results suggest that the limiting case of trivial initial data is identical to the CTT/CTS scenario (as it should be given that the two systems are essentially equivalent for trivial initial data). We only consider regular solutions in this work.

In the following section we review the basic uniqueness properties of the maximal $(K = 0)$ CTT/CTS formulation of the constraints. In section 3, we focus on the LY equation with unscaled sources. We assume that a (critical) solution exists whose linearization has a kernel and show using Lyapunov–Schmidt theory a parabolic branching of solutions. In section 4, we consider the non-uniqueness results for the XCTS system. We show that with the assumption of a 1D kernel, the full XCTS system exhibits a parabolic branching in all five variables, exactly as was found numerically in [1]. We conclude with a discussion of the possible implications of these results for evolutions.

2. The linear system and bifurcations

It is important to understand why the branching features described in [1] and outlined below do not occur in the maximal CTT/CTS formulation. It is known that the LY equation admits unique solutions [4] away from trivial initial data. For the maximal (i.e. decoupled) CTT/CTS systems the existence and uniqueness of solutions of the LY equation is determined by a variational inequality for the initial data. A theorem by Cantor and Brill in [5], later corrected by Maxwell in [6], states that on an asymptotically Euclidean manifold there exists a positive solution of the LY equation if and only if $\forall f \in C^\infty_0$ the following inequality holds

$$\inf_{f \in C^\infty_0} \int \left( |\nabla f|^2 + Rf^2 \right) dv > 0 \quad (14)$$

where $R$ is the scalar curvature of the conformal metric and the volume element and inner product are with respect to this metric. No such theorem is known for the XCTS system.

In nonlinear systems the existence of a bifurcating branch of solutions is indicated by the non-invertibility of the linearized system. Thus in order to explain the non-uniqueness of solutions in [1] we need to find a background solution whose linearization is not invertible.
Given a nonlinear scalar equation $\triangle u = f(x, u)$, such as (7) with $K = 0$, our first task is to check if it is linearization stable; see [8]. If $f_u \leq 0$ then the linearization has the ‘wrong sign’ for use of the maximum principle to show local uniqueness of solutions. To illustrate this we recall the following result proven in [9].

On a suitably smooth asymptotically flat background the operator

$$(\nabla^2 - f) : H_{k,0} \rightarrow H_{k-2,0}$$

where $f \geq 0$ and $f \in H^{3,0}$ with $\delta_0 < -2$ (so that $f$ is continuous and falls off faster than $r^{-2}$), these function spaces are defined in the appendix) is an isomorphism if $-1 < \delta < 0$ and $2 \leq k \leq 4$. If we linearize the maximal CTT/CTS vacuum LY equation about a solution it reduces to this isomorphic form, so that the implicit function theorem gives us complete neighbourhoods of nearby solutions ('linearization stability'). This provides a qualitative explanation for the uniqueness results found by Pfeiffer and York for the maximal 4-equation CTS system (see figure 1 in [1]).

In the following section, we consider the unscaled source model studied by York in [10] which is not linearization stable, its linearization has the ‘wrong sign’ for application of the maximum principle so that a kernel solution may exist for sufficiently large initial data (the function $f$ in (15) is negative). We demonstrate the existence of a parabolic solution curve similar to those found in [1] in a small neighbourhood of such a critical solution.

3. Non-uniqueness for unscaled sources

In [10] York considered the case of moment of time symmetry conformally flat initial data with positive energy density $\rho$ (a more general analysis of unscaled sources including existence proofs may be found in [11]). Then the LY equation (7) reads

$$\nabla^2 \phi + 2\pi \rho \phi^5 = 0$$

(16)

where $\phi \rightarrow 1$ at spatial infinity. He noted that this equation is not linearization stable because the linearization has the ‘wrong sign’ for use of the maximum principle to prove local uniqueness.

We now study the form of this non-uniqueness using Lyapunov–Schmidt (LS) theory. When a linear operator $B$ has a kernel we are no longer able to invert the equation $BV = h(x)$, with $h(x)$ given, unless $\int V^* h = 0$ where $V^*$ is the kernel of the adjoint problem i.e. $V^* \in \text{Ker} B^* = \text{coker} B$. The LS theory may be regarded as a local extension of these ideas to nonlinear problems. In this method, we remove the kernel of the linearization from the domain and project the source terms, now nonlinear combinations of the unknown $V$, onto the image of the linearization. In this way a bijective operator $\hat{B}$ is defined in (22). The implicit function theorem gives a unique solution to the mapping between the modified spaces from which we reconstruct a solution to the original problem from a Taylor expansion of a real valued function (the LS equation).

We briefly review the LS theory here following [12] closely (see also [13]) whilst keeping in mind the immediate application to equation (16). We work in weighted Sobolev spaces (defined in the appendix) to ensure that the integrals below are finite, because the Fredholm properties of mappings between these spaces are well known (see [9, 19]) and because the implicit function theorem is easily defined for such spaces. We write the nonlinear equation in functional form $F(X, \lambda) = 0$ where $F$ is a smooth function of the unknown $X$ and a parameter $\lambda$. $F$ defines a map between Hilbert spaces (defined in the appendix) $F : X \times \mathbb{R} \rightarrow Y$. To apply the implicit function theorem $F$ must be at least $C^1$ and this follows from the multiplication properties of functions in these spaces. For an example of the implicit function theorem
applied to the CTS formalism with $K \neq 0$ between these function spaces see \[11\]. Once a solution to the restricted problem is obtained via the implicit function theorem the remainder of our analysis is formal.

Our analysis is perturbative and we will assume in this section and the next that there exists a critical solution $X_c$ corresponding to parameter value $\lambda = \lambda_c$. By this we mean that when we linearize $F$ about $(X_c, \lambda_c)$ we find that the linear operator $B := D_{X_c} F(X_c, \lambda_c)$ has a kernel i.e. there is a nonzero function $V_0$ such that $BV_0 = 0$. We require that $B$ be Fredholm of index zero (i.e. \( \dim \ker B = \dim \text{coker } B \), see the appendix) and focus our attention on the case when the kernel of $B$ is 1D. We perturb $\lambda_c$ by $\lambda = \lambda_c - \epsilon$ and look for solutions to $F(X, \lambda) = 0$ of the form $X = X_c + V$ where $V$ is a small perturbation.

Clearly $F = 0$ is equivalent to $BV = R(V, \epsilon)$ (17) where $R(V, \epsilon) = BV - F(X, \lambda)$. For equation (16) $B$ is self-adjoint and (for functions with falloff satisfying $\delta \in (-1, 0)$, see the appendix) we have $V_0 = V_0^*$ where $V_0^*$ spans the kernel of $B^*$, the formal adjoint of $B$. Given $V_0 \in H_{k, \delta}$ we define $z \in C^\infty_0$, for simplicity, so that $\int V_0 z = 1$. We then split the domain of $B$ as

$$X \subset H_{k, \delta} \ni V = \xi V_0 + u.$$ (18)

If we define the parameter $\xi = \int Vz \, dv$ then $\int uz \, dv = 0$. Thus $X$ is split parallel and perpendicular to $z$ by this choice of parameter. We take $k \geq 4$ so that if $V \in H_{k, \delta}$ then we also have $V \in C^2$ (see the appendix).

Similarly we split the range of $B$ as

$$Y \subset H_{k-2, \delta-2} \ni R = dz + w.$$ (19)

and taking $d = \int RV_0 \, dv$ means $\int wV_0 \, dv = 0$ and so the range is split parallel and perpendicular to $V_0$.

This splitting allows us to define the following projection operators:

$$P : X \rightarrow X, \quad PX = \xi V_0$$ (20)

$$Q : Y \rightarrow Y, \quad QY = dz.$$ (21)

We now apply separately the projections $1 - Q$ and $Q$ to (17) to obtain

$$\hat{B}u = R(\xi V_0 + u, \epsilon)$$ (22)

$$0 = QR(\xi V_0 + u, \epsilon),$$ (23)

where the operator $\hat{B} : (1 - P)X \rightarrow (1 - Q)Y$ is now bijective. This means that the linearization of (22), $\hat{B}u = 0$, only has the zero solution. The implicit function theorem then implies that there exists a unique small solution $u = u(\xi, \epsilon) \in C^2$ to (22) with $u(0, 0) = 0$ and $u_\xi(0, 0) = 0$ since there are no terms in (22) linear in $\xi$.

We now substitute this result into (23) which yields the LS equation (see [12])

$$QR(\xi V_0 + u(\xi V_0, \epsilon), \epsilon) = 0.$$ (24)

With $u(\xi V_0, \epsilon)$ a known function given by the implicit function theorem, this should be viewed as a real valued equation. It gives a relation between the parameters $\xi = \xi(\epsilon)$.

Since $Q$ is a projection operator we may write the LS equation as

$$d(\xi, \epsilon) = \int_{\mathbb{R}^3} R(\xi V_0 + u(\xi V_0, \epsilon), \epsilon) V_0 \, dv = 0.$$ (25)

i.e. $d = 0$ in (19).
At this stage, from the splitting (18), we have a solution to (17) of the form

\[ V = \xi(\epsilon)V_0 + u(\xi, \epsilon) \]

with the particular form of \( \xi(\epsilon) \) determined by (25). We note that for each value of the perturbation \( \epsilon \), \( \xi(\epsilon) \) traces a curve along which the splittings above are valid. If \( D_\epsilon F(\phi_c, \lambda_c) \epsilon \neq 0 \) for some \( \epsilon \) then we know that the set of zeros of \( d(\epsilon), d^{-1}(0) \), form a smooth submanifold (see [13]).

We now proceed with a formal argument to determine \( \xi(\epsilon) \). Expanding the known \( C^2 \) function \( u \) as a Taylor series in the dependent parameter \( \xi \) and the independent parameter \( \epsilon \) we have

\[ u(\xi, \epsilon) = \epsilon u_\epsilon(0, 0) + O(2) := \epsilon u^* + O(2) \] (27)

since \( u_\xi(0, 0) \neq 0 \) is incompatible with (22) due to the absence of linear terms in \( V \) in \( R \) (where subscripts denote partial derivatives and \( O(2) \) denotes second-order terms in \( \xi, \epsilon \)).

Now our solution to (17) is

\[ V = \xi(\epsilon)V_0 + \epsilon u^* + O(2). \] (28)

The LS equation may now be written in the form

\[ L_{20} + \xi \epsilon L_{11} + \epsilon L_{01} + \cdots = 0 \] (29)

where \( L_{mn} = \int V_0 R_{mn} dv \) and \( R_{mn} \) denotes the \( m \)th-order term in \( \xi \) and the \( n \)th-order term in \( \epsilon \) resulting from substitution of the solution (28) into \( R(V, \epsilon) \). This fixes \( \xi = \xi(\epsilon) \).

In [12] Vainberg and Trenogin prove that the small solutions (i.e. where \( \xi(\epsilon) \to 0 \) as \( \epsilon \to 0 \)) of the LS equation are in 1–1 correspondence with the small solutions of \( F(X, \lambda) = 0 \).

We now show that branching of the type found for the XCTS system in [1] is actually a generic property of solutions of the standard LY equation with an unscaled source (this was shown recently in the special case of the constant density star in [20]). We examine the local behaviour of the solutions to this equation at a critical point of the linearization as the unscaled source term is varied. We multiply \( \rho \) in (16) by a positive parameter \( \lambda \) and seek a continuous family of solutions to the LY equation:

\[ \nabla^2 \phi + 2\pi \rho \lambda \phi^5 = 0, \quad \phi \to 1 \quad \text{as} \quad r \to \infty \] (30)

on a fixed flat background. If \( \rho \) is a suitably smooth compactly supported function and we seek solutions \( \phi \) such that \( \phi - 1 \in H_{k, \delta} \) where \( \delta \in (-1, 0) \) then the integral relations above will be finite.

For \( \lambda = 0 \) the only regular solution satisfying the boundary conditions is \( \phi \equiv 1 \). As we increase \( \lambda \) we find that the linearized homogenous equation

\[ \nabla^2 V_0 + 10 \pi \rho \lambda \phi^4 V_0 = 0 \] (31)

has only the trivial solution \( V_0 \equiv 0 \). The implicit function theorem then tells us that (30) has a smooth sequence of solutions, \( \phi(\lambda) \), with \( \phi(0) = 1 \) and the maximum principle tells us that \( \phi(\lambda) \geq 1 \). We now assume that there exists a critical solution \( \phi_c \) at \( \lambda = \lambda_c \), whose linearization has a kernel \( V_0 \), corresponding to the lowest eigenstate of the Schrödinger-type equation (31). We know that this eigenstate is unique up to scaling and has no nodes, so we may take \( V_0 > 0 \).

We now expand (30) about this critical value where the linearization, (31), has a kernel. Taking

\[ \phi := \phi_c + V \text{ with } \phi_c > 0 \text{ and } \lambda := \lambda_c - \epsilon \] we find

\[ \nabla^2 (\phi_c + V) + 2\pi \rho (\lambda - \epsilon) (\phi_c + V)^5 = 0 \] (32)
which gives

\[ BV = (\nabla^2 + 10 \pi \lambda_c \phi^5_{\epsilon} \rho) V \]

\[ = 2 \pi \epsilon \rho \phi^5_{\epsilon} - 20 \pi \lambda_c \rho \phi^3_{\epsilon} V^2 + 10 \pi \rho \phi^5_{\epsilon} \epsilon V + 7 \text{ other terms.} \]  

(33)

The LS equation (25) for this problem is

\[ \int_{\mathbb{R}^3} V_{0} \left( 2 \pi \rho \phi^5_{\epsilon} + 10 \pi \rho \phi^3_{\epsilon} \epsilon V - 20 \pi \lambda_c \rho \phi^3_{\epsilon} V^2 + \cdots \right) \, dv = 0. \]

(34)

If we substitute the solution \( V \) in the form (28) into (35) we obtain the coefficients \( L_{mn} \):

\[ L_{01} = \int_{\mathbb{R}^3} V_0 (2 \pi \rho \phi^5_{\epsilon}) \, dv > 0 \]

(36)

\[ L_{11} = \int_{\mathbb{R}^3} V_0 (10 \pi \rho \phi^3_{\epsilon} \nat V_0 - 40 \pi \lambda_c \rho \phi^3_{\epsilon} V_0 u^* \nat) \, dv \]

(37)

\[ L_{20} = \int_{\mathbb{R}^3} V_0 (-20 \pi \lambda_c \rho \phi^3_{\epsilon} V_0^2) \, dv < 0, \]

(38)

and so on.

By choosing \( \xi \) and \( \epsilon \) small enough we may truncate the LS branching equation (29) at any order. In particular we can write

\[ \xi^2 L_{20} + \xi \epsilon |L_{11}| + \epsilon L_{01} \approx 0. \]

(39)

Solving the quadratic equation for \( \xi \) we find

\[ \xi = \pm \left( \frac{L_{01}}{|L_{20}|} \right)^{1/2} + o(\sqrt{\epsilon}), \]

(40)

where \( o(\sqrt{\epsilon}) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). This tells us that to lowest order and provided \( L_{01} \neq 0 \) that we may ignore the contribution of \( L_{11} \).

Therefore, in a small neighbourhood of the critical solution \( \phi_c \) we find that as the parameter \( \epsilon \) is varied that the conformal factor traces a parabola in the solution space:

\[ \phi = \phi_c \pm \left( \frac{L_{01}}{|L_{20}|} \right)^{1/2} V_0 + O(\sqrt{\epsilon}). \]

(41)

If \( L_{01} = 0 \) then \( 2 \pi \rho \phi^5_{\epsilon} \in \text{Image}(B) \) and a qualitatively different situation arises. The LS equation then tells us that there is more than one branch of solutions passing through \( \lambda = \lambda_c \) provided not all \( L_{ij} \neq 0 \). In this case the zeros of the bifurcation equation (35) do not form a smooth submanifold. Note that this is not the behaviour observed in [1] where only a single smooth (parabolic) curve of solutions is found. This phenomenon was observed recently in a numerical study of non-unique solutions to (16) corresponding to a constant density star [14]. Further details can be found in [13].

As noted above, specializing (16) to the constant density star, upper and lower branches of solutions were found in [20] and an explicit form for the critical and kernel solutions were given. Generalizing from \( \rho = \text{const} \) and compactly supported to just \( \rho > 0 \) and compactly supported, only the lower branch of solutions was given and a parabolic approach to the critical solution was found in [20]. The existence of an upper branch was conjectured and is easily determined using the LS methods outlined above which give the required \( \pm \sqrt{\epsilon} \) behaviour in a small neighbourhood of the critical point.
In [10] York notes that by specifying a conformal energy density \( \hat{\rho} \) related to the physical energy density by \( \rho = \hat{\rho} \phi^{-s} \) where \( s > 5 \), we transform the linearization of (30) into an isomorphic form yielding unique solutions. Therefore non-uniqueness results from a poor choice of conformal scaling in this case. We argue in the next section that the coupling of the \( K \) evolution equation to the four constraint equations in the XCTS system similarly introduces an undesirable scaling of the variables leading to non-uniqueness as above.

4. The XCTS system

As in the case of unscaled sources above, to apply the LS method we must first assume the existence of a critical solution about which the linearization of this system has a (vector) kernel.

We note here that the XCTS system arises from the variation of an action given by

\[
I(u) = \int_M \left( \nabla_X \nabla \phi + \frac{R \phi X}{8} + \frac{\alpha(\beta) \phi^7}{32 \chi} + \frac{K^2}{12} \chi \phi^5 - \frac{\phi^6}{6} (K - \beta^i \nabla_i K) \right) \, dv. \tag{42}
\]

We may also include scaled versions of the energy density and the current density. This allows us to examine the existence and uniqueness properties of the XCTS system from a variational perspective.

First we investigate the necessary conditions for the functional (42) to have a minimum. We must have that the functional is bounded below on an appropriately defined domain of admissible functions \( \Pi \) which will be some Sobolev space \( H^{k,\delta} \) subject to the constraints \( N > 0, \phi > 0 \).

Provided such a domain has been defined, we proceed to the calculation of the first and second variations of \( I \). If we define \( J(s) = I(u + sv) \), where \( v \in \Pi \), a stationary value of \( I \) is given by \( J'(0) = 0 \) where \( ' \) denotes differentiation with respect to \( s \). A simple calculation reveals that the XCTS system corresponds to a stationary point of this functional.

It is straightforward to show that the second variation of (42) is not of definite sign. The following inequality is used in [15] as the basic assumption on the integrand \( F \) to apply direct methods in the calculus of variations;

\[
F_{P^i_{\alpha \beta}}(x, u, \nabla u) \xi^i_{\alpha} \xi^j_{\beta} > 0 \tag{43}
\]

for all rank one matrices \( \xi^i_{\alpha} \) where \( i = 1–5, \alpha = 1–3 \) and \( P^i_{\alpha} \) denote the usual derivatives in the Euler–Lagrange equations. This condition, called ‘strong ellipticity’ in [15], is the requirement that \( F(x, u, p) \) be convex with respect to \( p \) (\( p = \nabla u \in \mathbb{R}^{5 \times 3} \)). The XCTS Lagrangian (42) fails this criterion; it is convex in \( \nabla \beta \) but, due to the mixed term \( \nabla_X \nabla \phi \), \( F \) is not convex in \( \nabla \phi \) or in \( \nabla X \).

This lack of convexity means we cannot expect stationary points, if they exist, to be unique. An analogous situation occurs frequently in nonlinear elasticity where non-convex functionals are necessary in order to model buckling equilibrium configurations of materials which are known physically to be non-unique, i.e. an input stress can lead to numerous buckled states. This argument and the analogy to the unscaled source equation (16) lends support to our assumption below that the linearized XCTS system develops a kernel for sufficiently large initial data.

4.1. Non-uniqueness in the XCTS system

In this section, we assume that the XCTS system (11)–(13) has a critical solution \( \vec{X}_c \), and that the linearization about this solution has a one-dimensional kernel \( \vec{V}_0 \). To apply the LS method
we must also check that the formally adjoint system has a kernel of equal dimension. We know from [9] that for sufficiently smooth initial data with very general falloff conditions at spatial infinity that the kernel of this linearized system is finite dimensional and that it has a closed range. We show in the appendix that the linear system is actually Fredholm with an index of zero (so that $\dim \text{Ker} = \dim \text{CoKer}$) between suitably defined Sobolev spaces. Assuming now that the kernel is one dimensional, we have satisfied the requirements to implement the LS theory outlined above.

We introduce a parameter $\lambda$ to allow us to continuously vary the conformal initial data $g_{ij}$ and $U^{ij}$ in the conformal background (in [1] the parameter $A$ was used which corresponded to the amplitude of a Teukolsky wave in the conformal background.) The 1-parameter family of initial data considered in [1] is

$$g_{ij} = \delta_{ij} + \lambda \hat{h}_{ij} \quad (44)$$

$$U_{ij} = \lambda \hat{h}_{ij}, \quad (45)$$

where $\hat{h}_{ij}$ is the tracefree part of the metric perturbation (these tensors are given explicitly in [1]). This corresponds to a gravitational wave perturbation of flat space with a Gaussian wave profile. The perturbation $\hat{h}_{ij}$ decays exponentially with distance so the metric is asymptotically flat. (Our construction is valid for systems with much weaker power-law falloff.) Furthermore, due to the fast falloff, we know that this conformal metric will not contribute to theADM energy of the physical solution.

In (11)–(13) we denote the dependence of the initial data on $\lambda$ by

$$\nabla \mapsto \nabla_{\lambda}, \quad U^{ij} \mapsto \lambda U^{ij}, \quad R \mapsto R(\lambda).$$

When $\lambda = 0$ the XCTS equations (11)–(13) decouple due to the non-existence of conformal killing vectors that vanish at infinity (where we have the constraint on the conformal lapse that $N > 0$), and we obtain flat space as the unique regular solution. Applying the implicit function theorem then gives a local curve of solutions parameterized by $\lambda$. We assume that there exists a critical solution $\vec{X}_c = (\phi_c, \chi_c, \beta^i_c)$ occurring at $\lambda = \lambda_c$. At this point we apply the LS method to continue the curve through $\lambda_c$.

Below we will, as in the previous example, perturb the system about this critical solution. It is convenient to absorb the critical shift vector into $U^{ij}$ by a gauge transformation. We perturb the XCTS system (11)–(13) at the critical solution $\vec{X}_c = (\phi_c, \chi_c, 0, 0, 0)$ according to

$$\lambda = \lambda_c - \epsilon, \quad \phi = \phi_c + \phi_1, \quad \chi = \chi_c + \chi_1, \quad \beta^i = 0 + \beta^i_1.$$

We expand the background scalar curvature in a Taylor series about $\lambda_c$ so that

$$R(\lambda) = R(\lambda_c) - \epsilon R'(\lambda_c) + \cdots. \quad (46)$$

The linear terms in the expansion give the following inhomogenous system

$$\Delta \phi_1 - \frac{1}{8} R(\lambda_c) \phi_1 + \frac{\phi_1^6}{32 \chi_c^2} \left( -2 U \cdot \nabla \beta_1 - \left( \frac{\chi_1}{\chi_c} - 7 \frac{\phi_1}{\phi_c} \right) U \cdot U \right) = \epsilon \frac{R'(\lambda_c)}{8} \phi_c + \epsilon \Gamma$$

$$\Delta \chi_1 - \frac{1}{8} R(\lambda_c) \chi_1 + 7 \frac{\phi_1^6}{32 \chi_c^2} \left( 2 U \cdot \nabla \beta_1 - \left( 6 \frac{\phi_1}{\phi_c} - \frac{\chi_1}{\chi_c} \right) U \cdot U \right) = \epsilon \frac{R'(\lambda_c)}{8} \phi_c + \epsilon \Gamma$$

$$\nabla_i \left( \frac{\phi_1^7 \chi_c}{\chi_c^2} \beta_1 - \frac{\phi_1^7}{\chi_c} U^{ij} \left( \frac{\chi_1}{\phi_c} - \frac{\chi_1}{\chi_c} \right) \right) = \epsilon \Gamma^j$$

where terms arising from variation of the connection are given the generic symbol $\Gamma$. Due to the form of the conformal initial data we are considering we know that these connection terms decay exponentially with distance.
The XCTS system (11)–(13) will be referred to here as \( F(X, \lambda) = 0 \). As before, \( F \) describes a mapping between Hilbert spaces \( F: X \times \mathbb{R} \to Y \). We assume the existence of a 1D kernel \( V_0 \) and define the dual object \( Z \in \mathcal{C}_0^\infty \) so that \( \int_{\mathbb{R}^3} V_0 Z \, dv = 1 \) where the inner product denotes multiplication of a 5-component row vector with a 5-component column vector and the volume element is taken with respect to the conformal metric \( g_{ij} \). Then we may split the domain \( X \) as five copies of that defined in the scalar case (18) and similarly the range corresponds to five copies of (19). Likewise our relation for \( \xi \) in the scalar case, namely \( \xi = \int V \, \tilde Z \, dv \), becomes

\[
\xi = \int_{\mathbb{R}^3} \tilde V \cdot \tilde Z \, dv. \tag{47}
\]

The critical solution satisfies \( F(\tilde X_c, \lambda_c) = 0 \) and \( \tilde V_0 \) satisfies

\[
B \tilde V_0 = D_X F(\tilde X_c, \lambda_c) \tilde V_0 = 0.
\]

The LS machinery developed for the scalar equation (16) naturally generalizes to systems of elliptic equations. As before we have that \( F = 0 \) is equivalent to

\[
B \tilde V_0 := D_X F(\tilde X_c, \lambda_c) \tilde V_0 = \tilde R (V, \epsilon). \tag{48}
\]

Following the same procedure as (22)–(23), restricting the domain and range of \( B \) by applying the projections \( 1 - Q \) and \( Q \) to \( \tilde R \) gives us

\[
\tilde B \tilde u = \tilde R (\xi \tilde V_0 + \tilde u, \epsilon) \tag{49}
\]

\[
Q \tilde R (\xi \tilde V_0 + \tilde u, \epsilon) = 0 \tag{50}
\]

where \( Q \) denotes the projection onto \( \tilde Z \) as before. The implicit function theorem is also valid for nonlinear systems (see, e.g., [11]). The linearization of (49) now reads \( \tilde B \tilde u = 0 \). Since \( \tilde B \) is now an isomorphism, we obtain a unique small solution \( \tilde u(\xi, \epsilon) \) to (49) with \( \tilde u(0, 0) = \tilde u(0, 0) = 0 \). We substitute this into (50) to obtain the orthogonality relation that defines the curve \( \xi = \xi(\epsilon) \). It is worth pointing out that, since the linear system is Fredholm with index zero and the kernel is assumed to be one dimensional, we still have just one equation, (50), for one unknown \( \xi \) (though the numerical value of \( \xi \) now depends on all five unknowns).

At this point we have a (vector) solution of the form

\[
\tilde V = \xi(\epsilon) \tilde V_0 + \tilde u(\xi, \epsilon)
\]

\[
= \xi(\epsilon) \tilde V_0 + \epsilon \tilde u^* + O(2), \tag{51}
\]

just as before (see (28)). Equation (50) reads

\[
0 = \int_{\mathbb{R}^3} \tilde R_0 \, (\tilde R) \, dv = \int \left( V_0^\phi, V_0^\sigma, V_0^\gamma, V_0^\epsilon, V_0^\delta \right) \, dv \tag{52}
\]

\[
where R^\phi \text{ corresponds to the first order in } \epsilon \text{ background term, } D_\epsilon F(X_c, \lambda_c) \epsilon, \text{ and nonlinear terms arising on the RHS of the Hamiltonian constraint. Note that in the appendix we have shown that}
\]

\[
(V_0^\phi, V_0^\sigma, \frac{1}{8} V_0^\gamma, \frac{1}{8} V_0^\epsilon, \frac{1}{8} V_0^\delta) = (V_0^\phi, V_0^\sigma, V_0^\gamma, V_0^\epsilon, V_0^\delta).
\]
Likewise the LS coefficients are given by

\[ L_{ij} = \int_{\mathbb{R}^3} \vec{V}_0^* \cdot \vec{R}_{ij} \, dv, \]  

where \( \vec{R}_{ij} \) denotes the \( i \)th-order term in \( \xi \) and the \( j \)th-order term in \( \epsilon \) resulting from substitution of the solution (51) into \( \vec{R} \).

The source terms on the RHS of (49) are of the form

\[ (\vec{R}) = \begin{pmatrix} R^\phi \\ R^\chi \\ R^\beta_1 \\ R^\beta_2 \\ R^\beta_3 \\ R^\beta_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{8} \epsilon R'(\lambda_c) + \epsilon \Gamma + T_1(\phi, \chi, L_\beta) + \cdots \\ -\frac{1}{8} \epsilon R'(\lambda_c) + \epsilon \Gamma + T_2(\phi, \chi, L_\beta) + \cdots \\ \epsilon \Gamma + T_3(\phi, \chi, L_\beta) + \cdots \\ \epsilon \Gamma + T_4(\phi, \chi, L_\beta) + \cdots \\ \epsilon \Gamma + T_5(\phi, \chi, L_\beta) + \cdots \end{pmatrix} \]  

(55)

where \( T_i \) represents the quadratic combinations of all variables arising from the perturbation. From the form of the solution (51) we see that the \( T_i \) terms yield terms quadratic in \( \xi \) and \( \epsilon \) and mixed terms proportional to \( \xi \epsilon \).

Clearly

\[ L_{01} = \int_{\mathbb{R}^3} \left( V_0^\phi + V_0^\chi \right) \frac{1}{8} R'(\lambda_c) - V_0 \cdot \Gamma \right) \, dv. \]  

(56)

In the scalar model (16) we worked on a fixed flat background so that the connection was not varied. We also knew that the first eigenfunction of the linearized equation had no nodes. When dealing with a coupled system of equations we lose this property. To obtain the results of [1] we must assume that \( L_{01} \neq 0 \), which means that only a single curve of solutions passes through the critical point (\( d^{-1}(0) \) is a smooth submanifold).

The next possibly nonzero terms are \( L_{11} \) and \( L_{20} \). If \( L_{01} \neq 0 \) and \( L_{20} \neq 0 \) then we may truncate our series at a quadratic order and ignore the contribution of \( L_{11} \). The complexity of the combinations of the quadratic terms in (55) do not yield easily to analysis. We prefer to emphasize that it is extremely unlikely that our choice of initial data could lead to the cancellation of all terms at this order. Furthermore, since the kernel solution is removed by the LS method we know that the operator \( \hat{B} \) is an isomorphism which implies that all the \( L_{ij} \) are finite for small \( \xi \) and \( \epsilon \). We conclude that

\[ \xi^2 L_{20} + \epsilon L_{01} \approx 0. \]  

(57)

In principle the numbers \( L_{01}, L_{20} \) can be determined for the choice of initial data that yields the kernel solution \( \vec{V}_0 \) (cf (36)). However, by choosing the sign of \( \epsilon \) so that \( \frac{L_{01} \epsilon}{L_{20}} < 0 \) we know that there is a parabolic branching of all five variables

\[ \vec{X} \approx \vec{X}_c \pm \left( \frac{|L_{01}\epsilon|}{|L_{20}|} \right)^{1/2} \vec{V}_0. \]  

(58)

(cf equation (7) in [1]). The parabola in [1] corresponds to the case where \( \frac{L_{01}}{L_{20}} < 0 \) so that \( \epsilon \) must be positive (i.e. there are no solutions with \( \lambda > \lambda_c \)).

To recap, we list the assumptions that were made to derive (58). The first assumption was that for sufficiently large \( \lambda \) the curve of exact solutions (found numerically in [1]) reaches a critical solution (an exact solution whose linearization has a 1D kernel \( \vec{V}_0 \)) and that none of the components of \( \vec{V}_0 \) vanishes identically. If a component of \( \vec{V}_0 \) was identically zero then the lowest order term in expansion (51) of this component would not vary like \( \pm \sqrt{\epsilon} \) in the vicinity of \( \lambda_c \), contrary to the results in [1].
Secondly, we assumed that $L_{01} \neq 0$ and $L_{20} \neq 0$. The very general form of the integral (56) and the corresponding integral for $L_{20}$ suggests that this is by far the likeliest outcome. This guarantees a parabolic curve such as (58). The solution curves found numerically by Pfeiffer and York correspond to $\frac{L_{01}}{L_{20}} < 0$.

5. Implications for evolutions

We now consider the likely implications of these results for evolutions of the constraints. Recall that the Bianchi identities tell us that if the constraints are satisfied on the initial slice then they are satisfied on all slices of the foliation. However in numerical evolution schemes constraint violation off the initial data hypersurface remains a serious problem. To control constraint errors one may choose to solve the constraints on each slice of the foliation (constrained evolution).

The Bianchi identities also make some of the Einstein equations redundant. Given a solution of the constraints ($\bar{g} = \phi^4 g, K$), we may choose to evolve the conformal factor according to the evolution equation for the metric. For example, evolving initial data from the XCTS formulation naturally gives

$$ (\partial_t - \beta^i \partial_i) \log \phi = \frac{1}{6} (-N \phi^6 K + \nabla_i \beta^i) $$

where N is the conformal lapse and derivatives are with respect to the conformal metric. Whereas (59) selects just one solution of the constraints to evolve, the constrained evolution scheme could possibly lead to a solution jumping between branches during an evolution.

From equation (59) we see that if we choose maximal slicing with zero shift then $\dot{\phi} = 0$ and so a constrained evolution of these data should yield the initial conformal factor throughout the evolution. This provides a possible test of whether a constrained evolution has caused $\phi$ to stray from the correct branch.

The example (16) serves to illustrate the need to choose an appropriate scaling of the extrinsic curvature. The standard scaling of the LY formulation in (4) not only simplifies the momentum constraint but also removes the linearization instability. Many axisymmetric evolution schemes (see [16, 17]) do not scale the momentum. This need not cause problems on the initial slice if moment of time symmetry data is chosen. However, constrained evolution of these initial data could prove problematic as later time slices are susceptible to linearization instability and non-uniqueness as noted in [18]. Also, the conformal scaling in the Hamiltonian constraint in the BSSN formulation is analogous to (16). This is not necessarily a problem if an initial data set is transformed into this formalism and then evolved using a free evolution such as (59).

6. Conclusion

In [1] the solution curves for initial data in CTS and XCTS were compared. As expected the CTS system had unique solutions for $\lambda < \lambda_{crit}$ and no solutions for $\lambda > \lambda_{crit}$. With identical initial data $(g_{ij}, U_{ij})$, but with $K = 0$ replacing $N = 1$ the solution curve for the XCTS system is parabolic and turns back upon itself at the (much smaller) critical wave amplitude. We have shown that this is consistent with the system developing a kernel solution of its linearization and being dominated by a quadratic nonlinearity there.

It is worth emphasizing that all the non-uniqueness results in this work are local in nature, i.e. the multiplicity of solutions is confined to a small neighbourhood of the critical solution. We have argued that the parabolic branching behaviour in [1] is the simplest example of
branching phenomena that can arise in nonlinear elliptic PDEs. However, the global nature of the parabolic branches in [1] is surprising.

The non-uniqueness properties described above are generic in the sense that they do not depend on the form of the (Fredholm) linearized operator $B$ nor on the form of the forcing terms. If $B$ has a 1D kernel and there are quadratic nonlinearities then a parabolic solution curve is to be expected (provided $L_{01}, L_{20} \neq 0$). With higher order nonlinearities or with multiple parameters $\lambda_i$ more complicated solution curves arise. It is therefore natural to expect non-uniqueness (in a neighbourhood of the critical solution) in any nonlinear elliptic system with a non-trivial kernel.

Whilst our results using LS theory mirror the numerical results found in [1], it is important to remember that this non-uniqueness is removed when the wave amplitude $\lambda$ picks up a spatial dependence when we map from the conformal to the physical space. In the original paper it was noted that in the physical space the wave amplitude was not $\lambda$ but rather $\phi^4 \lambda$, and that using instead $\lambda \max \phi^4$ as our wave amplitude parameter leads to a solution curve with no turn-around point, solutions are seen to be unique for this ‘parameter’ (see figure 4 in [1]). We may draw a parallel to the scalar equation (16) for which York noted that choosing an appropriate conformal scaling for $\rho$ leads to linearization stability and unique solutions but at the considerable cost of losing control over a physical quantity, the energy density $\rho$.

The XCTS system has been very successful in modelling Black holes using the puncture technique and trivial initial data corresponding to quasi-equilibrium i.e. $(U_{ij} = K = \dot{K} = 0)$ for which the system considerably simplifies (see, e.g., [21] where the value of the shift at the ‘puncture’ was specified to prevent it vanishing everywhere).

When we studied the XCTS system in this work we considered only Dirichlet boundary conditions on an asymptotically flat manifold containing gravitational waves. When modelling black holes an alternative to the puncture technique is to excise a region corresponding to an apparent horizon. Much analytical work has been done to guarantee unique solutions satisfying apparent horizon boundary conditions in the standard CTT formalism. A recent numerical study of the generalization of these methods to the XCTS system revealed many difficulties with well-posedness (see [22] and references therein) as should be expected from the results outlined here. These results add weight to our reservations regarding the possible dangers of constrained evolution using ill-posed formulations of the constraints.

When modelling data with $U_{ij} \neq 0$, or when using apparent horizon boundary conditions it is clear that the XCTS system becomes much more complicated and it may prove more beneficial to revert to the standard and much simpler four equation CTS formalism.

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Appendix. Function spaces and the linearized system

We now define the function spaces used in the text. Our analysis has been restricted to asymptotically flat manifolds $(\mathbb{R}^3, \bar{g}_{ij})$ where $\bar{g} - \delta$ satisfies the falloff conditions below.
We use weighted Sobolev spaces of tensors with norm defined as (see [19])

$$
\|W\|_{H^k,\delta(R^3)}^2 = \sum_{m=0}^{k} \int_{R^3} |\nabla^m W|^2 \sigma^{-2(\delta-m)-3} \, dv
$$

where $\sigma = (1 + r^2)^{\frac{1}{2}}$, $r = |x|$ is the Euclidean distance function and the volume form and covariant derivatives are with respect to the Euclidean metric $\delta$.

The conformal metric considered in [1] was a perturbation to flat space consisting of a quadrupole gravitational wave with a Gaussian profile. The metric therefore had an exponential falloff with distance. When we conformally map to a solution of the constraints we know that this background metric will make no contribution to the ADM energy.

Through the conformal method we aim to construct a Riemannian asymptotically flat 3 metric $\bar{g}$ where

$$
\bar{g}_{ij} - \delta_{ij} \in H_{k,\delta}
$$

and $\bar{U}_{ij} \in H_{k-1,\delta-1}$

and we seek solutions $\phi - 1, 1 - \chi, \beta' \in H_{k,\delta}$ where $k \geq 4$ and $\delta \in (-1,0)$. $k$ is the number of times a tensor is weakly differentiable. Note that with $k \geq 4$ the metric is $C^2$ (we lose $3/2$ degrees of differentiability in passing from weak to strong differentiability). If $v \in H_{k,\delta}$ with $k > 3/2$ then $|v(x)| = o(r^\delta)$ as $r$ tends to infinity; see [19].

The linearized XCTS system is a mapping

$$
B : H_{k,\delta} \rightarrow H_{k-2,\delta-2},
$$

and was given after equation (46).

The formally adjoint system is defined for all $u \in H_{k,\delta}$ according to

$$
\langle Bu, v \rangle = \int v Bu = \int u B^* v
$$

where inner products denote a product of 5-component row and column vectors, the volume form is with respect to the background metric and $v \in H_{k-2,\delta-2}$. Thus $B^*$ is a mapping

$$
B^* : H_{k-2,\delta-2} \rightarrow H_{k-4,\delta},
$$

given by

$$
\Delta \hat{\phi}_1 - \frac{1}{8} R(\lambda_c) \hat{\phi}_1 + \frac{\phi_0^6}{32\chi_0} \left( 16 U \cdot L \hat{\beta}_1 + \left( -6 \frac{\hat{\chi}_1}{\phi_0} + \frac{\phi_1}{\chi_0} \right) U \cdot U \right) = 0
$$

$$
\Delta \hat{\chi}_1 - \frac{1}{8} R(\lambda_c) \hat{\chi}_1 + \frac{\phi_0^3}{32\chi_0^2} \left( -16 U \cdot L \hat{\beta}_1 + \left( 7 \frac{\hat{\chi}_1}{\phi_0} - 2 \frac{\hat{\phi}_1}{\chi_0} \right) U \cdot U \right) = 0
$$

$$
\nabla \left( \frac{L \beta^i_j \phi_0^2}{\chi_0} + \frac{\phi_0^2}{8\chi_0} U^{ij} \left( \frac{\phi_1}{\chi_0} - 7 \frac{\hat{\phi}_1}{\phi_0} \right) \right) = 0.
$$

We now outline the Fredholm properties of $B$. Linear elliptic systems on asymptotically Euclidean manifolds of the form $Bu = \sum_{i=0}^{m} a_i D^i u$ were studied in [9]. Clearly the linearized XCTS system is elliptic. Our initial data satisfy hypothesis one of that work regarding smoothness and falloff. They proved that such a system has a finite-dimensional kernel and a closed range. This says that $B$ is semi-Fredholm so that the domain of $B$ splits as $H_{k,\delta} = \ker B + W$ where $W$ is closed and $B$ is injective on $W$. However, this is not good
enough for the application we have in mind. We need $B$ to be Fredholm with an index of zero (so that the dimension of the cokernel is equal to that of the kernel of $B$).

The domain of $B$ is contained in $H_{k,\delta}$ such that $\delta \in (-1, 0)$. This implies that $-1 - \delta \in (-1, 0)$ so that the domain of $B^*$ is the same as that of $B$. From a theorem in [9] we know that each system has a finite-dimensional kernel and a closed range. At first sight it is not clear that the kernels of $B$ and $B^*$ are related. However, inspection of the systems reveals that if $\vec{V}_0 = (\phi_1, \chi_1, \beta_1^i) \in \ker B$ then $\vec{V}_0^* := (\chi_1, \phi_1, \frac{1}{2} \beta_1^i) \in \ker B^*$ and so the systems are identical under this relabelling of variables. Therefore the systems have kernels of equal finite dimension so that $B$ is Fredholm with an index of zero. This is the basic requirement for the use of the LS methods outlined above. If the linearized system is Fredholm of index zero then we know that the LS relation $QR = 0$ gives $n$ equations in $n$ unknowns $\xi_i$ where $n$ is the dimension of the kernel. If $\text{Dim Ker}$ is 1 then we have a single equation for $\xi$ in terms of $\epsilon$.

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