COMMENTS ON "NEW GENERATING RELATIONS FOR PRODUCTS OF TWO LAGUERRE POLYNOMIALS"

XIAOXIA WANG A,∗, ARJUN K. RATHIE B

A Department of Mathematics, Shanghai University, Shanghai, 200444, P. R. China;

b Department of Mathematics, School of Mathematical and Physical Sciences, Central University of Kerala,
Riverside Transit Campus, Padennakkad P. O. Nileshwar Kasaragod-671 328, Kerala State, India.

Abstract. By utilizing a two-dimensional extension of a very general series transform given by Bailey, Exton [Indian J. pure appl. Math. 24 (6) (1993), 401-408] deduced a very general double generating relation of a product of a pair of Laguerre polynomials and obtained a number of useful relations with elementary functions, Bessel functions, Hermite polynomials and single series expansions of pairs of Laguerre polynomials. Unfortunately, some of the results given by Exon contain errors and thus this is the aim of this short note to provide the corrected form of these results.

2000 AMS Subject Classification: 33C20; 33C05; 33B20.

Keywords: Generating relations; Laguerre polynomial; Bessel function; Hermite polynomial.

1. Introduction

We recall with the definition of generalized hypergeometric function [4] with p numerator and q denominator parameters by

\[ _pF_q\left[ \begin{array}{c} \alpha_1, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \\ z \end{array} \right] = \frac{\Gamma(\alpha_1 + \cdots + \beta_q + n)}{\Gamma(\beta_1 + \cdots + \beta_q + n)} \]

where \((\alpha, n)\) denotes the well known Pochhammer symbol (or the shifted factorial, since \((1, n) = n!\)) defined for complex number \(a\) by

\[ (\alpha, n) = \begin{cases} \alpha(\alpha + 1) \cdots (\alpha + n - 1), & n \in \mathbb{N}; \\
1, & n = 0. \end{cases} \]

Using the fundamental function relation \(\Gamma(a + 1) = a\Gamma(a)\), \((\alpha, n)\) can be written in the form

\[ (\alpha, n) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, \quad (n \in \mathbb{N} \cup \{0\}), \]

where \(\Gamma\) is the well known Gamma function. For more detail about convergence of this function, we refer to [4].

On the other hand, we recall the double hypergeometric function which is defined and introduced by Kampé de Fériet and subsequently abbreviated by Bunchall and Chaundy [6]. Here, we present a slightly modified notation given by Srivastava and Panda [7, p.423, Eq.(26)] as follows.

∗ Corresponding author.
E-mail addresses: xiaoxiawang@shu.edu.cn (X. Wang), akrathie@rediffmail.com (A. K. Rathie).
For more detail about the convergence of this function, we refer to \[7\].

The Laguerre polynomials have been researched in various branches of pure and applied mathematics \[1, 5\], which can be expressed by the confluent hypergeometric function as

\[
L_n^{(a)}(x) = \frac{(a + 1, n)}{n!} \, _1F_1 \left[ -n \atop a + 1 \right] x.
\] (1.5)

In 1974, Exton \[3\] obtained the well known Bailey’s transform in two dimension in the following theorem.

**Theorem 1.** If

\[
\beta_{m,n} = \sum_{p=0}^{m} \sum_{q=0}^{n} \alpha_{p,q} \mu_{m-p,n-q} \nu_{m+p,n+q}
\] (1.6)

and

\[
\gamma_{m,n} = \sum_{p=m}^{\infty} \sum_{q=n}^{\infty} \delta_{p,q} \mu_{p-m,q-n} \nu_{p+m,q+n},
\] (1.7)

then

\[
\sum_{m,n=0}^{\infty} \alpha_{m,n} \gamma_{m,n} = \sum_{m,n=0}^{\infty} \beta_{m,n} \delta_{m,n}.
\] (1.8)

Here, it is understood that \(\alpha, \delta, \mu\) and \(\nu\) are functions of \(p\) and \(q\) only and all the series concerned are either convergent or terminating.

By utilizing the above theorem and (1.6), (1.7) and (1.8) are appropriately specified, Exton \[2\] obtained the following very general double generating function

\[
\sum_{m,n=0}^{\infty} \frac{((d), m+n)s^n}{((g), m+n)m!n!} \, _1F_1 \left[ -m \atop p \right] _1F_1 \left[ -m \atop p' \right] \frac{1}{F_{D,0;0;G,0;0}} \left[ (d) + m + n : - ; - ; x, s \right],
\] (1.9)

and further as a simple consequence of the binomial theorem, the inner double series on the right-hand side of (1.9) is immediately reduce to a single series. Also, if the confluent hypergeometric functions on the left-hand side of (1.9) are replaced by their representations as Laguerre polynomials (of course, changing \(y\) to \(-y\) and \(t\) to \(-t\) and using (1.5)), we arrive at the following result.

\[
\sum_{m,n=0}^{\infty} \frac{((d), m+n)s^n}{((g), m+n)(p, m)(p', n)m!n!} L_{m-1}^{p-1}(y)L_{n-1}^{p'-1}(t)
\]  

\[
eq \sum_{m,n=0}^{\infty} \frac{((d), m+n)(-xy)^n(-st)^n}{((g), m+n)(p, m)(p', n)m!n!} \, _0F_G \left[ (d) + m + n \atop g + m + n \right] \, x + s.
\] (1.10)

This is a two-dimensional very general generating relation of a pair of Laguerre polynomials.
Exton in his paper [2] deduced a number of useful relations with elementary functions, Bessel functions, Hermite polynomials and single series expansions of pairs of Laguerre polynomials by utilizing (1.10). Unfortunately, some of the results given by Exton contain errors.

The remainder part of this short note is organized as follows. In section 2, Exton’s general result (1.9) will be established by another method. In section 3, we will list Exton’s results in corrected form.

2. Another proof of (1.9):

In order to establish (1.9), we proceed as follows. Denoting the right-hand side of (1.9) by $S$, expressing the double hypergeometric function with the help of the definition (1.4), we have

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \frac{((d), m+n)((d)+m+n,u+v)x^{m+n}s^{n+v}y^{m+n}t^n}{((g), m+n)((g)+m+n,u+v)(p,m)(p',n)m!n!u!v!}$$

where, we have applied the elementary relation $(a,m)(a+m,n) = (a,m+n)$. Performing the transformations $u \rightarrow u - m$ and $v \rightarrow v - n$ on the above identity and then applying the transformation $(m-n)! = (-1)^n m!/(-m,n)$, we get

$$S = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((d), u+v)x^{u+s}y^{m+n}t^n}{((g), u+v)(p,m)(p',n)(u-m)!(v-n)!m!n!}$$

$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-u,m)(-v,n)(-y)^{m+n}(-t)^n}{(p,m)(p',n)m!n!u!v!}$

$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-u,m)(-v,n)(-y)^{m+n}(-t)^n}{(p,m)(p',n)m!n!u!v!}$

$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-u,m)(-v,n)(-y)^{m+n}(-t)^n}{(p,m)(p',n)m!n!u!v!}$

$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-u,m)(-v,n)(-y)^{m+n}(-t)^n}{(p,m)(p',n)m!n!u!v!}$

$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-u,m)(-v,n)(-y)^{m+n}(-t)^n}{(p,m)(p',n)m!n!u!v!}$

$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-u,m)(-v,n)(-y)^{m+n}(-t)^n}{(p,m)(p',n)m!n!u!v!}$

Let $u \rightarrow m$ and $v \rightarrow n$, the above identity arrives at the left-hand side of (1.9). This completes the proof of (1.9).

3. Exton’s results in corrected form

In [2], we find that some of the results are not correct. In this section, we will present the corrected form of these results by the method applied by Exton with specializing the parameters in the main transformation (1.10).

Exton’s result (3.3) should be read as

$$\sum_{m,n=0}^{\infty} \frac{((d), m+n)(-1)^n x^{m+n} y^{m+n}}{((g), m+n)(p,m)(p',n)} L^{(p-1)} _m (y)L^{(p'-1)}_n (-y) = D_{+2} F_{G+3} \begin{pmatrix} (d), (p+p'-1)/2, (p+p')/2; & (g), p, p', p+p'-1; & -4xy \end{pmatrix};$$
Exton’s result (3.8) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(p', m + n)(p + p' - 1, m + n)(-1)^n x^{m+n}}{(p + p' - 1)/2, m + n)(p + p')/2, m + n)(p, m)(p', n)
L_{m}^{(p-1)}(y)L_{n}^{(p'-1)}(-y) = \Gamma(p)(2\sqrt(xy))^{1-p} J_{p-1}(4\sqrt(xy));
\]
Exton’s result (3.11) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(p, m + n)(p', m + n)(-1)^n x^{m+n}}{(p + p', m + n)(p, m)(p', n)} L_{m}^{(p-1)}(-y)L_{n}^{(p'-1)}(y)
= \Gamma((p + p')/2)e^{2xy}(xy)^{1-p/2-p'/2}I_{p/2+p'/2-1}(2xy);
\]
Exton’s result (3.12) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(p, m + n)(p', m + n)(-1)^n x^{m+n}}{(p, m)(p', n)} L_{m}^{(p-1)}(-y)L_{n}^{(1-p)}(y) = (1 - 4xy)^{-1/2};
\]
In fact, Exton’s result (3.13) which is obtained by setting \( p' = 2 - p \) in the above identity actually should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(p, m + n)(2-p, m + n)(-1)^n x^{m+n}}{(p, m)(2-p, n)} L_{m}^{(p-1)}(-y)L_{n}^{(1-p)}(y) = (1 - 4xy)^{-1/2};
\]
Exton’s result (4.3) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(p, m + n)(2p-1, m + n)(-1)^n x^{m+n}}{(p, m)(p, n)} L_{m}^{(p-1)}(y)L_{n}^{(p-1)}(y) = {\frac{\Gamma(p)}{\Gamma(2-p)}} = \Gamma(p)(xy)^{1-p} J_{p-1}(2xy);
\]
Exton’s result (4.5) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(p, m + n)(2p-1, m + n)(-1)^n x^{m+n}}{(p, m)(p, n)} L_{m}^{(p-1)}(y)L_{n}^{(p-1)}(y) = (1 + 4x^2y^2)^{1/2-p};
\]
Exton’s result (5.3) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(1/2, m + n)(1/2, m + n)(-1)^m 2^{-m-2n} x^{m+n}}{(m + n)!(1/2, m)(1/2, n)m! n!} H_{2m}(iy^{1/2})H_{2n}(y^{1/2}) = \exp(4xy);
\]
Exton’s result (5.4) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(3/2, m + n)(2, m + n)(-1)^m 2^{-m-2n} x^{m+n}}{(m + n)!(3/2, m)(3/2, n)m! n!} H_{2m+1}(iy^{1/2})H_{2n+1}(y^{1/2}) = iy \exp(4xy);
\]
Exton’s result (5.5) should be read as
\[
\sum_{m,n=0}^{\infty} \frac{(3/2, m + n)(-1)^m 2^{-2m-2n} x^{m+n}}{(1/2, m)(3/2, n)m! n!} H_{2m}(iy^{1/2})H_{2n+1}(y^{1/2}) = y^{1/2}\exp(4xy);
Exton’s result (5.6) should be read as
\[\sum_{m,n=0}^{\infty} \frac{(p', m + n)(p' - 1/2, m + n)(-1)^m n x^{m+n} 2^{-2m}}{((2p' - 1)/4, m + n)((2p' + 1)/4, m + n)(1/2, m)(p', n)m!}H_{2m}(y^{1/2})L_n^{(p'-1)}(-y) = \cos(4x^{1/2}y^{1/2});\]

Exton’s result (5.7) should be read as
\[\sum_{m,n=0}^{\infty} \frac{(1/2, m + n)(-1)^m n x^{m+n} 2^{-2m-2n}}{(1/2, m)(1/2, n)m! n!}H_{2m}(y^{1/2})H_{2n}(y^{1/2}) = \cos(2xy);\]

Exton’s result (5.8) should be read as
\[\sum_{m,n=0}^{\infty} \frac{(3/2, m + n)(-1)^m n x^{m+n+1} 2^{-1-2m-2n}}{(3/2, m)(3/2, n)m! n!}H_{2m+1}(y^{1/2})H_{2n+1}(y^{1/2}) = \sin(2xy);\]

Exton’s result (6.2) should be read as
\[\sum_{m=0}^{q} \frac{(-1)^m}{(p,m)(p',q-m)} L_{m}^{(p-1)}(-y) L_n^{(p'-1)}(y) = \frac{((p + p' - 1)/2, q)((p + p')/2, q)(-4q^q)}{(p,q)(p',q)(p + p' - 1, q)q!};\]

Acknowledgement

X. Wang acknowledges support of National Natural Science Foundation of China (Grant No. 11201291), and Natural Science Foundation of Shanghai, China (Grant No. 12ZR1443800).

References

[1] A. Erdélyi, *Hybergeometric functions of two variables*, Acta Math. 83(1950), 131–164.
[2] H. Exton, *New generating relations for products of two Laguerre polynomials*, Indian J. pure appl. Math. 24(6)(1993), 401-408.
[3] H. Exton, *A theorem on the transformation of Kampé de Fériet functions*, Janabha, sect A, 4(1974), 141-144.
[4] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960, Reprinted by Chelsea Publishing, Bronx, New York, 1971.
[5] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.
[6] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester); Wiley, New York, Chichester, Brisbane and Toronto, 1985.
[7] H. M. Srivastava and R. Panda, *An integral representation for the product of two Jacobi polynomials*, J. London Math. Soc. 12(2)(1976), 419–425.