Littlewood–Paley–Stein operators on Damek–Ricci spaces

Anestis Fotiadis | Effie Papageorgiou

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54.124, Greece

Correspondence
Anestis Fotiadis, Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54.124, Greece.
Email: fotiadisanestis@math.auth.gr

Abstract
We obtain pointwise upper bounds on the derivatives of the heat kernel on Damek–Ricci spaces and we study the $L^p$-boundedness of Littlewood–Paley–Stein operators.

KEYWORDS
Damek–Ricci spaces, gradient estimates, heat kernel, Littlewood–Paley–Stein operator, maximal operator, time derivative

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1 | INTRODUCTION AND STATEMENT OF THE RESULTS

In this article we prove estimates of the derivatives of the heat kernel on a Damek–Ricci space. Applying these estimates we prove the $L^p$-boundedness of Littlewood–Paley–Stein operators.

For the statement of our results we need to introduce some notation. A Damek–Ricci space $S$ is a solvable extension $S = A \ltimes N$ of a Heisenberg type group $N$, equipped with an invariant Riemannian structure. They are named after E. Damek and F. Ricci, who noted that they are harmonic spaces, [8]. As Riemannian manifolds, these solvable Lie groups include all symmetric spaces of the noncompact type and rank one. Most of them are nonsymmetric harmonic manifolds, and provide counterexamples to the Lichnerowicz conjecture.

Let $n$ be the dimension of $S$ and let $-\Delta_S$ denote the associated Laplace–Beltrami operator. The $L^2$ spectrum of the Laplace–Beltrami operator is the half line $[Q^2/4, +\infty)$, where $Q$ is the homogeneous dimension of $N$ (see Section 2 for more details). Then, the heat kernel $h_t$ of $S$ is the fundamental solution of the heat equation $\partial_t h_t = -\Delta_S h_t$. It is a radial convolution kernel on $S$, i.e. $h_t(x, y) = h_t(d(x, y))$. By realizing the group $S$ as the unit ball in the corresponding Lie algebra $\mathfrak{s}$, the heat kernel can be expressed as a radial function $h_t(r)$, where $r$ is the geodesic distance to the origin in the ball model.

One of the main objectives in the present article is to prove the following result.

**Theorem 1.1.** If $S$ is a Damek–Ricci space, then for all $\varepsilon > 0$ and $i \in \mathbb{N}$ there is a constant $c > 0$ such that

$$
\left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq c t^{\frac{n}{2} - i} e^{-(1-\varepsilon)\left( \frac{Q^2}{4} t + \frac{Q}{2} r + \frac{r^2}{4} \right)},
$$

for all $t > 0$ and $r \geq 0$.

For the proof we use an inductive argument and we obtain a decreasing sequence of upper bounds for the derivatives of the heat kernel.
Denote by $H_t = e^{-Δ S_t}$ the heat semigroup on $S$. We denote with $\mathbb{N}$ the set of natural numbers, including 0. Fix $i \in \mathbb{N}$.

Then, for all $\sigma \geq 0$ we consider as in [2, 13], the $\sigma$-maximal operator

$$H_{\sigma, \text{max}}(f) = \sup_{t > 0} e^{\sigma t} \left| \frac{\partial^i}{\partial t^i} H_t f \right|,$$

and the Littlewood–Paley–Stein operator

$$H_{\sigma}(f)(x) = \left( \int_0^\infty e^{2\sigma t} \left( t \left| \frac{\partial^i}{\partial t^i} H_t f(x) \right|^2 + \| \nabla_x H_t f(x) \|^2 \right) \frac{dt}{t} \right)^{1/2}.$$

Next, we apply Theorem 1.1 in order to prove the following result.

**Theorem 1.2.** Suppose that $S$ is a Damek–Ricci space. Then, the operators $H_{\sigma, \text{max}}$ and $H_{\sigma}$ are bounded on $L^p(S)$, $p \in (1, \infty)$, provided that

$$\sigma < Q^2/\rho \rho'.$$  

Note that the aforementioned result coincides with that of [2, pp. 276, 279] when $S$ is a noncompact symmetric space of rank one.

There is a very rich and long literature concerning heat kernel estimates in various geometric contexts. See for example [1, 4, 6, 10], and the references therein. Davies and Mandouvalos in [9] obtained optimal estimates of the heat kernel in hyperbolic spaces and Anker, Damek and Yacoub in [3], obtained estimates of the heat kernel in the case of Damek–Ricci spaces. The time derivative of the heat kernel has been estimated in [15] for hyperbolic spaces, in [13] for some manifolds of exponential volume growth, and in [11] in a more general setting.

The Littlewood–Paley–Stein operator was first introduced and studied by Lohoué [14], in the case of Riemannian manifolds of non-positive curvature. In a variety of geometric settings it has been proved that $H_{\sigma, \text{max}}$ and $H_{\sigma}$ are bounded on $L^p$, $p \in (1, \infty)$, under some conditions on $\sigma$ (see for example [14] and [2]).

Let us now outline the organization of the paper. In Section 2 we recall some preliminaries about Damek–Ricci spaces and the heat kernel. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorem 1.2.

Throughout this article the different constants will always be denoted by the same letter $c$. When their dependence or independence is significant, it will be clearly stated.

## 2 | PRELIMINARIES

### 2.1 | Damek–Ricci spaces

We shall recall some basic facts on Damek–Ricci spaces. For details, see [7, 8]. A Damek–Ricci space $S$ is a solvable Lie group, equipped with a left-invariant metric. More precisely, $S = A \ltimes N$ is a semi-direct product of $A \simeq \mathbb{R}$ with a Heisenberg type Lie group $N$.

Let us recall the structure of an $H$-type Lie group. Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra, equipped with an inner product $(\cdot, \cdot)$. Let us denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{v}$ (so that $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$), with $k = \dim \mathfrak{z}$ and $m = \dim \mathfrak{v}$.

Let $J_Z : \mathfrak{v} \to \mathfrak{v}$ be the linear map defined by

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle,$$

for $X, Y \in \mathfrak{v}, Z \in \mathfrak{z}$.

Then $\mathfrak{v}$ is of Heisenberg type if the following condition is satisfied:

$$J^2_Z = -|Z|^2 I \quad \text{for every } Z \in \mathfrak{z}.$$
The corresponding connected Lie group \(N\) is then called of Heisenberg type, and we shall identify \(N\) with the Lie algebra \(n\) via the exponential map:

\[
\mathfrak{b} \times \mathfrak{z} \to N
\]

\[(X, Z) \mapsto \exp(X + Z).\]

Thus, multiplication in \(N \equiv n = \mathfrak{b} \oplus \mathfrak{z}\) will be

\[(X, Z)(X', Z') = \left(X + X', Z + Z' + \frac{1}{2}[X, X']\right).\]

Let \(a\) denote the Lie algebra of \(A\) and \(H\) a vector in \(a\), acting on \(\mathfrak{b}\) with eigenvalue \(1/2\) and \(\mathfrak{z}\) with eigenvalue \(1\); we extend the inner product on \(n\) to the algebra \(\mathfrak{s} = n \oplus a\), by requiring \(n\) and \(a\) to be orthogonal and \(H\) to be a unit vector \([5, p. 977]\). The product in \(S = N \ltimes \mathbb{R}^*_+\) is given by the rule

\[(X, Z, a)(X', Z', a') = \left(X + a^{1/2}X', Z + aZ' + \frac{1}{2}a^{1/2}[X, X'], aa'\right).\]

Then, \(S\) is solvable connected Lie group, with Lie algebra \(\mathfrak{s} = \mathfrak{b} \oplus \mathfrak{z} \oplus \mathbb{R}\) and Lie bracket

\[
[(X, Z, l), (X', Z', l')] = \left(\frac{1}{2}lX' - \frac{1}{2}l'X, lZ' - l'Z + [X, X'], 0\right).
\]

Let us endow \(S\) with the left invariant Riemannian metric which agrees with the inner product on \(\mathfrak{s}\) at the identity and is induced by

\[
\langle (X, Z, l), (X', Z', l') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + ll'
\]

on \(\mathfrak{s}\). Let \(d\) denote the distance induced by this Riemannian structure. The Riemannian manifold \((S, d)\) is then called a Damek–Ricci space.

The associated left-invariant Haar measure on \(S\) is given by

\[
a^{-Q}dXdZ\frac{da}{a},\quad (2.1)
\]

where \(Q = \frac{m}{2} + k\) is the homogeneous dimension of \(N\).

Similarly to the case of hyperbolic spaces, these general groups \(S\) can be realized as the unit ball

\[B(\mathfrak{s}) = \{(X, Z, l) \in \mathfrak{s} : |X|^2 + |Z|^2 + l^2 < 1\}\]

in \(\mathfrak{s}\), through a Cayley type transform:

\[
C : \quad S \to B(\mathfrak{s})
\]

\[
x = (X, Z, a) \iff x' = (X', Z', l'),
\]

where

\[
\begin{align*}
X' &= \left\{\left(1 + a + \frac{1}{4}|X|^2\right)^2 + |Z|^2\right\}^{-1}\left\{\left(1 + a + \frac{1}{4}|X|^2\right) - J_z\right\}X, \\
Z' &= \left\{\left(1 + a + \frac{1}{4}|X|^2\right)^2 + |Z|^2\right\}^{-1}2Z, \\
l' &= \left\{\left(1 + a + \frac{1}{4}|X|^2\right)^2 + |Z|^2\right\}^{-1}\left\{-1 + \left(a + \frac{1}{4}|X|^2\right)^2 + |Z|^2\right\}.
\end{align*}
\]
In the ball model $B(s)$, the geodesics passing through the origin are the diameters. The geodesic distance to the origin is given by

$$r = d(x', 0) = \log \frac{1 + \|x'\|}{1 - \|x'\|}, \quad \rho = \|x'\| = \frac{\tanh r}{2},$$

and the Riemannian volume is

$$dV = 2^n (1 - \rho^2)^{-Q-1} \rho^{n-1} d\rho d\sigma = 2^{m+k} \left( \sinh \frac{r}{2} \right)^{m+k} \left( \cosh \frac{r}{2} \right)^k d\rho d\sigma,$$

where $d\sigma$ denotes the surface measure on the unit sphere $\partial B(s)$ in $s$ and $n = \dim S = m + k + 1$.

### 2.2 The heat kernel on Damek–Ricci spaces

If $\kappa = \kappa(r)$ is a locally integrable radial function and $\ast |\kappa|$ denotes the convolution operator whose kernel is $|\kappa|$, then in [3, Thm. 3.3] it is proved that

$$\| \ast |\kappa| \|_{L^p(S) \to L^p(S)} = \int_S |\kappa(x)| \phi_i \left( \frac{1}{p^2} \right) Q(x),$$

(2.2)

where $\phi_i$ are the elementary spherical functions, $Q$ is the homogeneous dimension of the Heisenberg type Lie group $N$ and $dx$ is the measure in (3). Note that (2.2) is a version of the Kunze–Stein phenomenon [12] which holds true for Damek–Ricci spaces.

Using polar coordinates on $S$, [3, p. 656],

$$\phi_i \left( \frac{1}{p^2} \right) Q(r) \approx \begin{cases} e^{-\frac{Qr}{p}}, & \text{if } 1 \leq p < 2, \\ (1 + r) e^{-\frac{Qr}{2}}, & \text{if } p = 2. \end{cases}$$

(2.3)

Denote by $h_t$ the heat kernel on the Damek–Ricci space $S$. Then, $h_t$ is a radial right-convolution kernel on $S$:

$$h_t(x, y) = h_t(d(x, y)).$$

Then, the following estimate holds:

$$h_t(r) \approx t^{-3/2} (1 + r) \left( 1 + \frac{1 + r}{t} \right) \left( 1 + \frac{Q^2}{4} - \frac{Q^2 - r^2}{4t} \right),$$

(2.4)

for $t > 0$ and $r \geq 0$ (see [3, p. 664] for details).

Consequently, (2.4) implies the upper bound

$$h_t(r) \leq c t^{-3/2} (1 + t) \left( 1 + r \right) \left( 1 + \frac{Q^2}{4} - \frac{Q^2 - r^2}{4t} \right).$$

(2.5)

Also, in [3, p. 669] the authors derive the following gradient estimate:

$$\| \nabla h_t(r) \| \approx t^{-3/2} r \left( 1 + \frac{1 + r}{t} \right) \left( 1 + \frac{Q^2}{4} - \frac{Q^2 - r^2}{4t} \right).$$

(2.6)
which implies
\[ \|\nabla h_t(r)\| \leq c t^{-\frac{n+2}{2}} (1 + t)^{-\frac{n-1}{2}} (1 + r)^{-\frac{n+1}{2}} e^{-\frac{c^2}{4} (t + \frac{r}{2} + \frac{r^2}{4t})}. \] 

Grigoryan in [11] derived Gaussian upper bounds for all time derivatives of the heat kernel, under some assumptions on the on-diagonal upper bound for \( h_t \) on an arbitrary complete non-compact Riemannian manifold \( M \). More precisely, he proves that if there exists an increasing continuous function \( f(t) > 0 \), \( t > 0 \), such that
\[ h_t(x, x) \leq \frac{1}{f(t)}, \text{ for all } t > 0 \text{ and } x \in M, \]
then,
\[ \left| \frac{\partial^i h_t}{\partial t^i} (x, y) \right| \leq \frac{1}{\sqrt{f(t)} f_2(t)}, \text{ for all } i \in \mathbb{N}, t > 0, x, y \in M, \] 
(2.8)
where the sequence of functions \( f_i = f_i(t) \), is defined by
\[ f_0(t) = f(t) \] and \( f_i(t) = \int_0^t f_{i-1}(s) \, ds, i \geq 1. \)

3 \quad PROOF OF THEOREM 1.1

In this section we give the proof of Theorem 1.1. More precisely we shall prove the following estimate: for all \( \epsilon > 0 \) and \( i \in \mathbb{N} \), there is a \( c > 0 \) such that
\[ \left| \frac{\partial^i h_t}{\partial t^i} (r) \right| \leq c t^{-\frac{n}{2}} e^{-\frac{(1-\epsilon)(O^2 + \frac{r^2}{4t})}{2}}. \] 
(3.1)
for all \( r \geq 0 \) and all \( t > 0 \).

For the proof of (3.1) we need several lemmata. The following lemma is technical but important for the proof of Theorem 1.1. It provides a method to obtain estimates for the first derivative of a function, given some upper bounds on the function and its second derivative.

**Lemma 3.1.** Let
\[ \alpha > \beta, \quad D \geq D_\alpha, \quad B \geq B_\alpha, \quad C \geq C_\alpha, \] 
and assume that for fixed \( r \geq 0 \), the function \( f_r : (0, +\infty) \to \mathbb{R} \) satisfies
\[ \left| f_r(t) \right| \leq c t^{-\alpha} (1 + t)^{\beta} (1 + r)^{\gamma} e^{-D t - B r - C r^2/(4t)}, \] 
(3.3)
and
\[ \left| \frac{d^2 f_r}{dt^2} (t) \right| \leq c t^{-\alpha - 2} (1 + t)^{\beta} (1 + r)^{\gamma} e^{-D t - B r - C r^2/(4t)}. \] 
(3.4)
Then, for all \( \epsilon \in (0, 1) \), there is a constant \( c > 0 \), that does not depend on \( r, t \), such that for all \( r \geq 0 \),
\[ \left| \frac{d f_r}{dt} (t) \right| \leq c t^{-\alpha - 1} (1 + t)^{\beta} (1 + r)^{\gamma} e^{-((D + D_\alpha) t/2 + (B + B_\alpha) r/2 + (C + C_\alpha) \epsilon^2 t^2)} \]
where \( \lambda_\epsilon = \frac{1 - \epsilon}{1 + \epsilon} \).
Proof. By applying twice the mean value theorem, one can prove that

\[ \left| \frac{df_r}{dt}(t) \right| \leq \frac{1}{\delta} \left( |f_r(t)| + |f_r(t+\delta)| \right) + \delta \sup_{\tau \in (t+t+\delta)} \left| \frac{d^2 f_r}{dt^2}(\tau) \right|, \text{ for all } \delta > 0. \] (3.5)

Note that \( t \mapsto t^{-\alpha}(1 + t)^{\beta} \) is a decreasing function of \( t \), since \( \alpha > \beta \), therefore

\[ (t + \delta)^{-\alpha}(1 + t + \delta)^{\beta} \leq t^{-\alpha}(1 + t)^{\beta}. \]

Thus (3.5) and the estimates (3.3) and (3.4) imply that

\[ \left| \frac{df_r}{dt}(t) \right| \leq c \frac{1}{\delta} t^{-\alpha}(1 + t)^{\beta}(1 + r)^{\gamma} e^{-(D_{\tau} - B_{\tau} - C_{\tau} r^2)/4(t+\delta)} + c \delta t^{-\alpha-2}(1 + t)^{\beta}(1 + r)^{\gamma} e^{-D_{\tau} t - B_{\tau} r - C_{\tau} r^2/4(t+\delta)}. \]

Choose now

\[ \delta = \varepsilon \, t e^{-(D - D^*) t/2 - (B - B^*) r/2 - (C - C^*) r^2/8t}, \]

in order to balance the exponential terms. Thus,

\[ \left| \frac{df_r}{dt}(t) \right| \leq c \left( \varepsilon \left( 1 \right)^{\beta}(1 + r)^{\gamma} e^{-(D + D^*) r/2 - (B + B^*) r/2} \right. \]
\[ \left. \times \left( \frac{1}{\varepsilon} e^{(C - C^*) r^2/8t) - C_{\tau} r^2/4(t+\delta)} + \varepsilon e^{-(C - C^*) r^2/8t) - C_{\tau} r^2/4(t+\delta)} \right) \right). \] (3.6)

From (3.2) it follows that \( \delta \leq \varepsilon \, t \). Thus

\[ \frac{1}{2t} - \frac{1}{t + \delta} \leq - \frac{1 - \varepsilon}{2t(1 + \varepsilon)} = - \frac{\lambda_{\varepsilon}}{2t}. \]

Consequently,

\[ \frac{C - C_{\tau} r^2}{2} - C \frac{r^2}{4(t + \delta)} \leq - \frac{r^2 C_{\tau} + C_{\tau}}{2}, \]

and similarly

\[ \frac{C - C_{\tau} r^2}{2} + \frac{C_{\tau} r^2}{4(t + \delta)} \geq \frac{r^2 C_{\tau} \lambda_{\varepsilon} + C}{4t} \frac{r^2}{2}. \] (3.7)

Thus, from (3.6), (3.7) and (3.8) it follows that

\[ \left| \frac{df_r}{dt}(t) \right| \leq c t^{-\alpha-1}(1 + t)^{\beta}(1 + r)^{\gamma} e^{-(D_{\tau} + D^*) t/2 + (B_{\tau} + B^*) r/2 + (C_{\tau} + C_{\tau}) r^2/8t). \] \( \square \)

We shall now apply estimate (2.8) in the case of a Damek–Ricci space. The following lemma provides an initial estimate for all the derivatives of the heat kernel.

**Lemma 3.2.** Suppose that \( S \) is a Damek–Ricci space. For all \( i \in \mathbb{N} \) there is a constant \( c > 0 \) such that

\[ \left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq c t^{-\alpha-1}(1 + t)^{\frac{r^2}{2}}, \text{ for all } t > 0, r \geq 0. \] (3.9)
Proof. We note that according to (2.5),

$$h_t(r) \leq c t^{-\frac{n}{2}}(1 + t)^{-\frac{n-3}{2}}$$

for all $t > 0, r \geq 0$.

Thus, we can apply (2.8) with

$$f(t) = t^{-\frac{n}{2}}(1 + t)^{-\frac{n-3}{2}}.$$

Consider the sequence of functions $f_i$, which is defined by

$$f_0(t) = f(t) \quad \text{and} \quad f_i(t) = \int_0^t f_{i-1}(s) \, ds, i \geq 1.$$

Note that $f$ is an increasing function, since $\frac{n}{2} > \frac{n-3}{2}$, thus we can invoke (2.8). By an induction argument we get that

$$f_i(t) \geq t^{-\frac{n}{2}+i}(1 + t)^{-\frac{n-3}{2}}. \quad (3.10)$$

Then, by (2.8) and (3.10) we get that

$$\left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq \frac{1}{\sqrt{f(t)f_2(t)}} \leq c t^{-\frac{n-i}{2}}(1 + t)^{-\frac{n-3}{2}}. \quad (3.11)$$

Note that the derivative estimates in Lemma 4 do not involve the distance $r$. Given the estimates (2.5), and (3.9), we can apply Lemma 3.1 and find improved upper bounds for the $i$-th derivative, for every $i \geq 1$. In this way, we can refine the upper bound for the $i$-th derivative given by Lemma 4. We can apply an inductive argument in order to obtain $\ell \in \mathbb{N}$ successive upper bounds for the $i$-th derivative, for every $i \geq 1$.

**Lemma 3.3.** Suppose that $S$ is a Damek–Ricci space. Let us fix $\varepsilon \in (0, 1)$ and set $\lambda_\varepsilon = \frac{1-\varepsilon}{1+\varepsilon}$. For every $i, \ell \in \mathbb{N}$, consider the sequences $\beta_\varepsilon^i, \gamma_\varepsilon^i$ that satisfy the iteration formulas

$$\beta_\varepsilon^i = \frac{1}{2} \left( \beta_{\varepsilon}^{i-1} + \beta_{\varepsilon}^{i+1} \right),$$

$$\gamma_\varepsilon^i = \frac{1}{2} \left( \lambda_\varepsilon \gamma_{\varepsilon}^{i-1} + \gamma_{\varepsilon}^{i+1} \right), \quad (3.12)$$

and the initial conditions

$$\beta_\varepsilon^0 = 0, \gamma_\varepsilon^0 = 0, \text{ for all } i \geq 1, \beta_\varepsilon^0 = 1, \gamma_\varepsilon^0 = 1, \text{ for all } \ell \geq 0. \quad (3.13)$$

Then, there is a constant $c > 0$

$$\left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq c t^{-\frac{n-i}{2}}(1 + t)^{-\frac{n-3}{2}}(1 + r)^{-\frac{n-3}{2}} e^{-\beta_\varepsilon^i \left( \frac{\varepsilon^2 + \varepsilon r}{\varepsilon^2 + \varepsilon} \right)} e^{-\gamma_\varepsilon^i \frac{r^2}{\varepsilon^2}}, \quad (3.14)$$

for all $t > 0$ and $r \geq 0$, where the constant $c$ depends on $\varepsilon, i, \ell$.

Proof. For every $\ell \in \mathbb{N}$ consider the following statement.

$L(\ell)$: for all $i \in \mathbb{N}$, $\frac{\partial^i h_t}{\partial t^i}(r)$ satisfies the estimate (3.14) and the constants $\beta_\varepsilon^i, \gamma_\varepsilon^i$, appearing in (3.14), satisfy the iteration formulas (3.12) and the initial conditions (3.13).
We shall then prove by induction, that \( L(\ell) \) holds for every \( \ell \in \mathbb{N} \).

For \( \ell = 0 \) we have to prove that for all \( i \in \mathbb{N} \), \( \frac{\partial^n h_t}{\partial t^i}(r) \) satisfies the estimate (3.14) and that the constants \( \beta_i^0, \gamma_i^0 \geq 0 \) satisfy \( \beta_0^i = \gamma_0^i = 0 \), for all \( i \geq 1 \), and \( \beta_0^0 = \gamma_0^0 = 1 \). Indeed, from Lemma 3.2 we get that for \( i \geq 1 \)

\[
\left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq c t^{-\frac{n-1}{2}}(1 + t)^{\frac{n-3}{2}} \left( 1 + r \right)^{\frac{n-1}{2}} e^{-\frac{n-1}{2}}, \quad \text{for all } t > 0, r \geq 0. \tag{3.15}
\]

But, \( 1 \leq (1 + r)^{\frac{n-1}{2}} \). Thus

\[
\left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq c t^{-\frac{n-1}{2}}(1 + t)^{\frac{n-3}{2}} (1 + r)^{\frac{n-1}{2}}, \quad \text{for all } t > 0, r \geq 0,
\]

i.e. (3.14) holds true for all \( i \geq 1 \), with \( \beta_0^i = \gamma_0^i = 0 \). Furthermore, from the estimates of the heat kernel in (2.5), we obtain that

\[
|h_t(r)| \leq c t^{-\frac{n-1}{2}}(1 + t)^{\frac{n-3}{2}} (1 + r)^{\frac{n-1}{2}} e^{-\frac{n-1}{2}}, \quad \text{for all } t > 0, r \geq 0.
\]

Thus (3.14) holds true also for \( i = 0 \) and \( \beta_0^0 = \gamma_0^0 = 1 \). Therefore the statement \( L(0) \) holds true.

Let us assume now that \( L(\ell - 1) \) holds true. Thus, for all \( i \in \mathbb{N} \), there are constants \( c, \beta_{i-1}, \gamma_{i-1} \geq 0 \) such that \( \frac{\partial^i h_t}{\partial t^i}(r) \) satisfies the estimate (3.14).

We shall prove that \( L(\ell) \) holds true. Indeed, from the estimates of the heat kernel in (2.5), we have

\[
|h_t(r)| \leq c t^{-\frac{n-1}{2}}(1 + t)^{\frac{n-3}{2}} (1 + r)^{\frac{n-1}{2}} e^{-\frac{n-1}{2}}, \quad \text{for all } t > 0, r \geq 0.
\]

Thus (3.14) holds true for \( i = 0 \) with \( \beta_0^0 = \gamma_0^0 = 1 \). For \( i \geq 1 \), consider the function

\[ f_r(t) = \frac{\partial^{i-1} h_t}{\partial t^{i-1}}(r). \]

From the validity of \( L(\ell - 1) \), we get that for \( i - 1 \) and \( i + 1 \) we have

\[
|f_r(t)| = \left| \frac{\partial^{i-1} h_t}{\partial t^{i-1}}(r) \right| \leq c t^{-\alpha}(1 + t)^{\beta} (1 + r)^{\gamma} e^{-D t - B \frac{r^2}{2} - C \frac{r^2}{4}},
\]

\[
\left| \frac{d^2 f_r}{dt^2}(t) \right| = \left| \frac{\partial^{i+1} h_t}{\partial t^{i+1}}(r) \right| \leq c t^{-\alpha-2}(1 + t)^{\beta} (1 + r)^{\gamma} e^{-D_* t - B_* \frac{r^2}{2} - C_* \frac{r^2}{4}},
\]

with

\[
\alpha = \frac{n}{2} + i - 1, \quad \beta = \frac{n-3}{2}, \quad \gamma = \frac{n-1}{2},
\]

\[
D = B = \beta_{i-1}^{i-1}, \quad C = \gamma_{i-1}^{i-1} \text{ and } D_* = B_* = \beta_{i-1}^{i+1}, \quad C_* = \gamma_{i-1}^{i+1}.
\]

One can verify that \( \alpha > \beta, B \geq B_* \) and \( C \geq C_* \). Thus, by Lemma 3.1 applied for the function \( f_r(t) \), it follows that

\[
\left| \frac{df_r}{dt}(t) \right| = \left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq c t^{-\frac{n-1}{2}}(1 + t)^{\frac{n-3}{2}} (1 + r)^{\frac{n-1}{2}} e^{-\beta_i^0 \left( \frac{Q^2}{4} t + \frac{Q}{2} r + \frac{r^2}{4} \right)} e^{-\gamma_i^0 \frac{r^2}{4}},
\]

for all \( i \geq 1 \), where \( \beta_i^0 \) and \( \gamma_i^0 \) satisfy (3.12). Thus, the statement \( L(\ell) \) is valid and the proof of the lemma is complete. \( \square \)
Remark 3.4. The constant $c = c(i, \ell, \epsilon)$ in relation (3.14) of Lemma 3.3 depends on $i, \ell$ and $\epsilon$ and it increases to infinity (when either $i \to \infty$ or $\ell \to \infty$ or $\epsilon \to 0$), but we only need the fact that it is finite for fixed $i, \ell, \epsilon$.

In the following lemma, we shall prove by induction that the exponential coefficients $\beta^i_\ell, \gamma^i_\ell$ of Lemma 5 are convergent sequences of $\ell$. Using this fact, we shall show that these coefficients, after a sufficiently large number of iterations and $\epsilon$ sufficiently close to zero, can get arbitrarily close to 1.

**Lemma 3.5.** For any $i \in \mathbb{N}$,

$$\lim_{\ell \to \infty} \gamma^i_\ell = \left(1 - \sqrt{1 - \lambda \epsilon}\right)^i \quad \text{and} \quad \lim_{\ell \to \infty} \beta^i_\ell = 1.$$  \hspace{1cm} (3.16)

**Proof.** We shall deal only with $\gamma^i_\ell$. The proof that $\lim_{\ell \to \infty} \beta^i_\ell = 1$ is similar, and we shall omit it.

**Claim 1.** For every $\ell \in \mathbb{N}$ consider the following statement $L(\ell)$: for all $i \in \mathbb{N}$,

$$\gamma^i_\ell \leq 1.$$  \hspace{1cm} (3.17)

We shall prove by induction that $L(\ell)$ is valid for all $\ell \in \mathbb{N}$.

For $\ell = 0$ we have to prove that for all $i \in \mathbb{N}$, we have that $\gamma^0_0 \leq 1$. Indeed, this is a consequence of the initial conditions $\gamma^0_0 = 0$ and $\gamma^0_0 = 1$. Thus $L(0)$ holds true.

Let us assume now that $L(\ell - 1)$ holds true. Thus, for all $i \in \mathbb{N}$, we have that $\gamma^i_{\ell - 1} \leq 1$.

We shall prove that $L(\ell)$ holds true. Recall that by the induction assumption, for all $i \in \mathbb{N}$, for $i - 1$ and $i + 1$ we have that $\gamma^{i-1}_{\ell-1} \leq 1$ and $\gamma^{i+1}_{\ell-1} \leq 1$. Thus, from (3.12) it follows that

$$\gamma^i_\ell = \frac{\lambda \epsilon}{2} \gamma^{i-1}_{\ell-1} + \frac{1}{2} \gamma^{i+1}_{\ell-1} \leq \frac{\lambda \epsilon}{2} + \frac{1}{2} \leq 1,$$

thus the statement $L(\ell)$ is valid and this completes the proof of Claim 1.

**Claim 2.** For every $\ell \in \mathbb{N}$ consider the following statement $L(\ell)$: for all $i \in \mathbb{N}$,

$$\gamma^i_\ell \leq \gamma^i_{\ell+1}.$$  \hspace{1cm} (3.18)

We shall prove that $L(\ell)$ is valid for all $\ell \in \mathbb{N}$. We proceed once again by induction in $\ell \in \mathbb{N}$.

For $\ell = 0$ we have to prove that for all $i \in \mathbb{N}$, $\gamma^i_0 = 0 \leq \gamma^i_1$. Indeed, from (3.13) it follows that $\gamma^i_0 = 0 \leq \gamma^i_1$, for all $i > 0$ and $\gamma^0_0 = 1 = \gamma^0_1$. Therefore the statement $L(0)$ holds true.

Let us assume now that $L(\ell - 1)$ holds true, i.e. that for all $i \in \mathbb{N}$, $\gamma^i_{\ell - 1} \leq \gamma^i_{\ell - 1}$. We shall prove that $L(\ell)$ holds true, i.e. that for all $i \in \mathbb{N}$, $\gamma^i_\ell \leq \gamma^i_{\ell + 1}$. Recall that by (3.12) we have

$$\gamma^i_\ell = \frac{1}{2} \left( \lambda \epsilon \gamma^{i-1}_{\ell-1} + \gamma^{i+1}_{\ell-1} \right).$$  \hspace{1cm} (3.19)

Then by the induction assumption for $i - 1$ and $i + 1$ we have that $\gamma^{i-1}_{\ell-1} \leq \gamma^{i-1}_{\ell}$ and $\gamma^{i+1}_{\ell-1} \leq \gamma^{i+1}_{\ell}$. Hence, from (3.19) we get that

$$\gamma^i_\ell \leq \frac{1}{2} \left( \lambda \epsilon \gamma^{i-1}_{\ell} + \gamma^{i+1}_{\ell} \right) = \gamma^i_{\ell + 1}.$$

Thus the statement $L(\ell)$ is valid and the proof of Claim 2 is complete.

**Claim 3.** For all $i \in \mathbb{N}$,

$$\lim_{\ell \to \infty} \gamma^i_\ell = \left(1 - \sqrt{1 - \lambda \epsilon}\right)^i.$$  \hspace{1cm} (3.20)
Note that by Claim 2, the sequence $\gamma^l_\ell$ is increasing in $\ell$ and by Claim 1, $\gamma^l_\ell$ is bounded above. Thus $\lim_{\ell \to \infty} \gamma^l_\ell$ exists and since $0 \leq \gamma^l_\ell \leq 1$, then
\[
\lim_{\ell \to \infty} \gamma^l_\ell = \gamma_1 \leq 1.
\]

Note that $\gamma^0_\ell = 1$, for all $\ell \in \mathbb{N}$, thus $\gamma_0 = 1$.

Now, taking limits in the iteration formula (3.19) we obtain that
\[
\gamma_i = \frac{1}{2}(\lambda \gamma_{i-1} + \gamma_{i+1}),
\]
thus
\[
\gamma_{i+1} - 2\gamma_i + \lambda \gamma_{i-1} = 0.
\]
This is a homogeneous linear recurrence relation with constant coefficients and the solutions of this equation are given by
\[
\gamma_i = C_1 \rho_1^i + C_2 \rho_2^i, \quad C_1, C_2 \in \mathbb{R},
\]
where $\rho_1, \rho_2$ are the roots of the equation
\[
\rho^2 - 2\rho + \lambda = 0.
\]
Thus, we conclude that
\[
\gamma_i = C_1 \left(1 - \sqrt{1 - \lambda}\right)^i + C_2 \left(1 + \sqrt{1 - \lambda}\right)^i,
\]
for some $C_1, C_2 \in \mathbb{R}$.

Since $0 \leq \gamma_1 \leq 1$, we get $C_2 = 0$, otherwise $\lim_{i \to \infty} \gamma_i = \infty$. Also, since $\gamma_0 = 1$, we get $C_1 = 1$. Thus, from (3.21) for $C_1 = 1, C_2 = 0$, we get (3.20) and the proof is complete.

**End of the proof of Theorem 1.1.** To complete the proof of Theorem 1.1, note that $\lim_{\ell \to 0} \left(1 - \sqrt{1 - \lambda}\right)^i = 1$. Thus, taking $\ell \in \mathbb{N}$ sufficiently large and $\epsilon$ sufficiently close to zero, one has $\gamma^l_\ell \geq 1 - \epsilon$ and $\beta^l_\ell \geq 1 - \epsilon$. Thus, from (3.14) and (3.16) it follows that
\[
\left| \frac{\partial^i h_\ell}{\partial t^i}(r) \right| \leq c t^{-n-1}(1 + t)^{-n-1} e^{-(1-\epsilon)\left(\frac{a^2 t^2 + a r + \frac{a^2}{4}}{\pi^2}\right)}.
\]
Taking now into account that if $a, b > 0$, then there exists a constant $c = c(a, b)$ such that $x^a \leq ce^{bx}$ for all $x > 0$, we conclude that for every $\epsilon > 0$, there exists a constant $c > 0$ such that
\[
\left| \frac{\partial^i h_\ell}{\partial t^i}(x, y) \right| \leq c t^{-n - \frac{a}{2}} e^{-(1-\epsilon)\left(\frac{a^2 t^2 + a r + \frac{a^2}{4}}{\pi^2}\right)},
\]
and the proof of Theorem 1.1 is complete. \qed
4 | PROOF OF THEOREM 1.2

In this section we apply the estimates of the derivatives of the heat kernel. We claim that the operators $H_{\sigma,\text{max}}$ and $H_\sigma$ defined in Section 1 are bounded on $L^p(S)$, $p \in (1, \infty)$, provided that

$$\sigma < Q^2 / pp'. $$

We shall give only the proof for $H_{\sigma,\text{max}}$. The proof for $H_\sigma$ is similar and then omitted. Note that in this case, apart from Theorem 1.1, gradient estimates (2.7) are also required.

We shall consider separately the small time maximal operator

$$H^0_{\sigma,\text{max}}(f)(x) = \sup_{0 < t \leq 1} \left| e^{\sigma t} \left| \frac{\partial^i}{\partial t^i} H_t(f)(x) \right| \right|$$

and the large time maximal operator

$$H^\infty_{\sigma,\text{max}}(f)(x) = \sup_{t \geq 1} \left| e^{\sigma t} \left| \frac{\partial^i}{\partial t^i} H_t(f)(x) \right| \right|.$$

As noted in [2, p. 276], the whole problem comes from the component $H^\infty_{\sigma,\text{max}}$.

Let

$$k^\infty_{\sigma,\text{max}}(x) = \sup_{t \geq 1} \left| e^{\sigma t} \left| \frac{\partial^i}{\partial t^i} h_t(x) \right| \right|.$$  

Then, the component $H^\infty_{\sigma,\text{max}}$ can be handled by estimating

$$H^\infty_{\sigma,\text{max}}(f)(x) \leq |f| * k^\infty_{\sigma,\text{max}}$$

and applying the Kunze–Stein phenomenon (2.2).

To be more precise, Theorem 1.1 implies the following upper bound.

**Lemma 4.1.** For all $\epsilon \in (0, 1)$ there exists $c > 0$ such that

$$|k^\infty_{\sigma,\text{max}}(r)| \leq c e^{-(1-\epsilon)Qr / 2} e^{-(1-\epsilon)\sqrt{Q^2 + x^2 / 1-\epsilon}},$$

for all $r > 0$.

**Proof.** From the estimates (1.1) of the derivative $\frac{\partial^i}{\partial t^i} h_t$, provided by Theorem 1.1, we get

$$|k^\infty_{\sigma}(r)| \leq c \sup_{t \geq 1} e^{\sigma t} \left| \frac{\partial^i}{\partial t^i} h_t(r) \right| \leq c \sup_{t \geq 1} e^{\sigma t} e^{-(1-\epsilon)\left( \frac{Q^2}{4} t + \frac{Qr}{2} + \frac{\epsilon^2}{4t} \right)}$$

$$\leq c \sup_{t \geq 1} e^{\sigma t} e^{-(1-\epsilon)\left( \frac{Q^2}{4} t + \frac{Qr}{2} + \left( \frac{Q^2}{4} - \frac{\epsilon}{1-\epsilon} \right) t + \frac{\epsilon^2}{4t} \right)}$$

$$\leq c e^{-(1-\epsilon)\frac{Qr}{2}} e^{-(1-\epsilon)\sqrt{Q^2 + x^2 / 1-\epsilon}}.$$  

□
Next, by (4.1) and (2.2) we find
\[
\|H_{\sigma, \text{max}}^\infty\|_{L^p(S) \rightarrow L^p(S)} \leq \int_S dx \ |k_{\sigma, \text{max}}^\infty(x)| \phi_1(\frac{1}{p} - \frac{1}{2}) Q(x).
\] (4.2)

Then, taking into account (2.3), we conclude that the integral condition (4.2) becomes
\[
\left\{ \begin{array}{ll}
\int_0^1 dr \ r^{n-1} + \int_1^{+\infty} dr \ e^{\frac{Q}{r}} k_{\sigma, \text{max}}^\infty(r) < +\infty, & \text{if } 1 \leq p < 2, \\
\int_0^1 dr \ r^{n-1} + \int_1^{+\infty} dr \ (1 + r) e^{\frac{Q}{r}} k_{\sigma, \text{max}}^\infty(r) < +\infty, & \text{if } p = 2,
\end{array} \right.
\] (4.3)

[3, p. 656]. We shall show that the integrals in (4.3) converge. Using the estimates of \(k_{\sigma, \text{max}}^\infty\) obtained in Lemma 4.1 and (4.3), we get
\[
\int_S dr \ k_{\sigma, \text{max}}^\infty(x) \phi_1(\frac{1}{p} - \frac{1}{2}) Q(x)
\]
\[
\leq c \int_0^1 dr \ r^{n-1} e^{-(1-2\epsilon)\frac{Q}{r}} \sqrt{\frac{Q^2}{4} - \frac{\sigma}{1-2\epsilon}} + \int_1^{+\infty} dr \ e^{\frac{Q}{r}} e^{-(1-2\epsilon)\frac{Q}{r}} \sqrt{\frac{Q^2}{4} - \frac{\sigma}{1-2\epsilon}},
\] (4.4)

for all \(p \in (1, 2)\), where we used that \((1 + r)e^{-(1-\epsilon)\frac{Q}{r}} \leq c e^{-(1-2\epsilon)\frac{Q}{r}}\).

The first integral is finite, while the second integral converges, provided that
\[
\frac{Q}{p} - (1 - 2\epsilon) \frac{Q}{2} - (1 - 2\epsilon)\sqrt{\frac{Q^2}{4} - \frac{\sigma}{1 - 2\epsilon}} < 0.
\] (4.5)

Choosing \(\epsilon\) small enough, it follows from (4.5) that the integral in (4.4) converges when
\[
\sigma < \frac{Q^2}{pp'}.
\] (4.6)

Thus, by duality, \(H_{\sigma, \text{max}}^\infty\) is bounded on \(L^p(S)\), \(p \in (1, \infty)\), if (4.6) holds true.

Next, it is left to show that the component \(H_{\sigma, \text{max}}^0\) is bounded on \(L^p(S)\), \(p \in (1, \infty)\). We split the operator \(H_{\sigma, \text{max}}^0\) into two parts
\[
H_{\sigma, \text{max}}^{0,0}(f)(x) = \sup_{0 < t \leq 1} e^{\sigma t} t^{\frac{1}{2}} \left| \frac{\partial}{\partial t} f * \psi h_t(x) \right|
\]
and
\[
H_{\sigma, \text{max}}^{0,\infty}(f)(x) = \sup_{0 < t \leq 1} e^{\sigma t} t^{\frac{1}{2}} \left| \frac{\partial}{\partial t} f * (1 - \psi) h_t(x) \right|,
\]
using a smooth cutoff function \(\psi \in C_c^\infty(S)\) with \(\psi \equiv 1\) near the origin. Then we observe that the second term \(H_{\sigma, \text{max}}^{0,\infty}\) can be handled like \(H_{\sigma, \text{max}}^\infty\) and the first term \(H_{\sigma, \text{max}}^{0,0}(f)(x)\) can be handled as in the Euclidean case (see for example [2, p. 278], [3, p. 670]).

Remark 4.2. In a similar way one can study the boundedness of the operators (1.2) and (1.3) related to the Poisson operator \(P_t = e^{-t(-\Delta_\sigma)^{1/2}}\), as well as the boundedness of the Riesz transform \(R = \nabla (-\Delta_\sigma)^{-1/2}\), already studied in [3]. However, the estimates obtained in the present paper are not sharp enough to deal with weak \(L^1\) boundedness.
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ORCID
Anestis Fotiadis https://orcid.org/0000-0002-6338-7965
Effie Papageorgiou https://orcid.org/0000-0001-8475-3226

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