The mechanical first law of black hole spacetimes with a cosmological constant and its application to the Schwarzschild–de Sitter spacetime

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Abstract

The mechanical first law (MFL) of black hole spacetimes is a geometrical relation which relates variations of the mass parameter and horizon area. While it is well known that the MFL of an asymptotic flat black hole is equivalent to its thermodynamical first law, however we do not know the detail of the MFL of black hole spacetimes with a cosmological constant which possess a black hole and cosmological event horizons. This paper aims to formulate an MFL of the two-horizon spacetimes. For this purpose, we try to include the effects of two horizons in the MFL. To do so, we make use of the Iyer–Wald formalism and extend it to regard the mass parameter and the cosmological constant as two independent variables which make it possible to treat the two horizons on the same footing. Our extended Iyer–Wald formalism preserves the existence of the conserved Noether current and its associated Noether charge, and gives an abstract form of the MFL of black hole spacetimes with a cosmological constant. Then, as a representative application of this formalism, we derive the MFL of the Schwarzschild–de Sitter (SdS) spacetime. Our MFL of the SdS spacetime relates the variations of three quantities: the mass parameter, the total area of the two horizons and the volume enclosed by the two horizons. If our MFL is regarded as a thermodynamical first law of the SdS spacetime, it offers a thermodynamically consistent description of the SdS black hole evaporation process: the mass decreases while the volume and the entropy increase. In our suggestion, a generalized second law is not needed to ensure the second law of SdS thermodynamics for its evaporation process.

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1. Introduction

Black hole thermodynamics has already been well established for spacetimes with a single black hole event horizon [1]. However, there is no consistent thermodynamical formulation for black hole spacetimes with a cosmological constant $\Lambda$ which possess a black hole event horizon (BEH) and a cosmological event horizon (CEH). For example, in the Schwarzschild–de Sitter (SdS) spacetime, the Hawking temperature of the BEH is higher than that of the CEH [2]. This temperature difference causes difficulties in formulating SdS thermodynamics, since the BEH (CEH) seems to be in some non-equilibrium state under the influence of Hawking radiation of different temperatures by the CEH (BEH). In general, a similar difficulty also arises for any two-horizon spacetime with $\Lambda$.

However, we may be able to search for the thermodynamical first law of two-horizon spacetimes with $\Lambda$ through the mechanical first law (MFL). The MFL is a geometrical relation which relates the variations of the mass parameter, horizon area and other supplemental quantities. For example, the MFL of the Schwarzschild black hole is equivalent to its thermodynamical first law and formulated by using the mass parameter as the only variable which describes the effect of a single horizon on the MFL. This fact of the Schwarzschild black hole leads us to expect that the MFL of two-horizon spacetimes with $\Lambda$ can be a candidate of the thermodynamical first law of the spacetimes and that the existence of two horizons requires two independent variables in the resultant MFL of the spacetimes to include the effects of two horizons. Then we adopt the following working hypothesis to search for the MFL of two-horizon spacetimes with $\Lambda$.

**Working hypothesis (two independent variables).** Generally for spacetimes with a cosmological constant which possess the BEH and CEH, the mass parameter $M$ and the cosmological constant $\Lambda$ are regarded as two independent variables in the MFL of the spacetimes.

Indeed, it will be shown in section 3 that the MFL of the SdS spacetime becomes mathematically inconsistent if $\Lambda$ is not an independent variable. When one considers a non-variable $\Lambda$ as a physical situation, it is obtained by setting the variation of $\Lambda$ zero ($\delta \Lambda = 0$) in the MFL after constructing it by regarding $\Lambda$ as an independent variable. In such a case, the variable $\Lambda$ is interpreted as a ‘working variable’ to obtain the MFL of two-horizon spacetimes with $\Lambda$.

On the other hand, for the SdS spacetime, the search for an MFL with and without regarding $\Lambda$ as an independent variable has already been tried in some existing works [3, 4]. These works make use of some conserved quantities defined by some integrals (e.g. Hamiltonian, action integral and so on). One can obtain some candidates of the MFL or the thermodynamical first law of the SdS spacetime by regarding the integral quantities as variables in the first law (see [4] for example). However, when one uses the integral quantities, there arises a problem of the choice of the integration constant (e.g. the so-called boundary counter-term or subtraction term to eliminate some divergent term in the Hamiltonian or action integral).

Then, in order to derive the MFL of two-horizon spacetimes with $\Lambda$, we make use of the Iyer–Wald formalism which is free from the problem of the integration constant [5]. We extend the original Iyer–Wald formalism to make it so general to be applicable to any spacetime with $\Lambda$ under the working hypothesis of two independent variables. The extended Iyer–Wald formalism preserves the existence of the conserved Noether current and its associated Noether

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4 Some proposals for thermodynamics of the BEH and CEH are already given for a case with some special matter fields and for an extreme case with a magnetic/electric charge [11]. These examples are artificial to vanish the temperature difference of the BEH and CEH. But in this paper, we consider a more general case which is not extremal and does not depend on artificial matter fields.
charge, and enables us to observe how the resultant MFL is related to the Noether charge which is locally constructed from the metric (free from the problem of the integration constant).

The general extension of the Iyer–Wald formalism to include the variable $\Lambda$ gives simply an abstract form of the MFL of two-horizon spacetimes with $\Lambda$, since the metric is not concretely specified. Then, as a representative application of our extended Iyer–Wald formalism, we derive the MFL of the SdS spacetime by this formalism. Moreover, we examine what the MFL of the SdS spacetime implies for SdS black hole evaporation if the MFL is regarded as a thermodynamical first law.

Here let us note that Sekiwa [4] has already treated $\Lambda$ as an independent variable for the SdS spacetime. (In [3], $\Lambda$ is not an independent variable but a complete constant.) In [4], the variable $\Lambda$ is concluded from the mathematical consistency of the MFL of the SdS spacetime, although there remains an uncertainty of an integration constant in the resultant MFL given in [4]. However, we will show in this paper that the extended Iyer–Wald formalism gives the same MFL as Sekiwa [4] in a straightforward way without the problem of an integration constant. Furthermore, as an advantage of our procedure, the extended Iyer–Wald formalism enables us to introduce naturally a state variable which represents the size of the two-horizon system. Then, consequently, we will show that the new state variable of the system size leads natural definitions of effective temperature and pressure of a two-horizon system. These state variables are suitable to describing the SdS black hole evaporation process. Such good state variables are not given in [4] and the evaporation process is also not described in [4].

This paper is organized as follows. Section 2 is devoted to the extension of the Iyer–Wald formalism to include the variable $\Lambda$ and obtain an abstract form of the MFL of any two-horizon spacetimes with $\Lambda$. Section 3 is for the application of an extended Iyer–Wald formalism to the SdS spacetime and obtain the MFL of the SdS spacetime. In that section, the state variables of the system size, effective temperature and pressure are given, and the mathematical consistency of the MFL with variable $\Lambda$ is then also examined. In section 4, we discuss what is implied about SdS black hole evaporation if our MFL of the SdS spacetime is regarded as a thermodynamical first law, and give a comment about the non-equilibrium nature due to the difference of Hawking temperatures of the BEH and CEH.

Throughout this paper, we use the Planck units, $c = \hbar = G = k_B = 1$.

2. Iyer–Wald formalism with variable $\Lambda$

For simplicity, we consider an empty spacetime in $n$ dimensions without matter fields and start with the diffeomorphism-invariant Lagrangian $n$-form $L(g, \Lambda) = L(g, \Lambda)\varepsilon$, where $g$ denotes the metric $g_{\mu\nu}$, $\varepsilon$ is the volume $n$-form and $L(g, \Lambda)$ is the Lagrangian scalar density. The inclusion of matter fields is very straightforward in the following discussions. In the following section, $L$ is specified to be the ordinary Einstein–Hilbert form in four dimensions:

$$L = \frac{1}{16\pi} (R - 2\Lambda)\varepsilon,$$

(2.1)

where $R$ is the Ricci scalar and $\Lambda$ is the variable cosmological constant. But in this section, $L(g, \Lambda)$ is an arbitrary Lagrangian of the metric and the variable $\Lambda$. The first variation of this Lagrangian is expressed as

$$\delta L = E_g \delta g + d\Theta(g, \delta g) + E_\Lambda \delta \Lambda,$$

(2.2)

where $\delta g$ is the abbreviation of metric variation $\delta g_{\mu\nu}$, $\Theta$ is the $(n-1)$-form called the symplectic potential which corresponds to the boundary term in the variational principle of the action
integral, \( E_g = E_\varepsilon \) gives the Einstein equation of the metric by \( E_g = 0 \) and \( E_\Lambda = E_\Lambda \varepsilon \) is the variation of \( L \) with respect to the working variable \( \Lambda \):

\[
E_\Lambda = \frac{\partial L}{\partial \varepsilon}.
\] (2.3)

For \( L \) in equation (2.1), we get \( E_\Lambda = -(1/8\pi)\varepsilon \) which will be used in the following section.

Here, we have to emphasize the following two remarks. The first one is on the so-called on-shell condition, which restricts the metric (and the matter field if it exists) to be the solution of equation of motion. When the on-shell condition in the presence of metric variation is going to be later required in this section, then the variation \( \delta g_{\mu \nu} \) is to be understood as a solution to the ‘extended’ linearized Einstein equation due to the variable \( \Lambda \):

\[
\delta G_{\mu \nu} + \Lambda \delta g_{\mu \nu} + g_{\mu \nu} \delta \Lambda = 0,
\] (2.4)

where \( G_{\mu \nu} \) is the Einstein tensor and \( g_{\mu \nu} \) is the ‘unperturbed’ metric given by the Einstein equation \( E_g = 0 \) \(^5\). (The right-hand side would not be zero but would be the variation \( \delta T_{\mu \nu} \) of the stress-energy tensor if matter fields exist.) The third term \( g_{\mu \nu} \delta \Lambda \) does not appear for the ordinary linearized equation with completely constant \( \Lambda \), but now it appears due to the variable \( \Lambda \). Under the working hypothesis of two independent variables, we can regard \( \Lambda \) not as the kin of universal constants such as Newton’s constant but as the kin of constants of motion such as mass parameter \( M \) which is a variable in asymptotic flat black hole thermodynamics.

The second remark we should emphasize here is on the variational principle. Exactly speaking, the variational principle gives the equations of motion of dynamical variables via the vanishing variation of action integral \( I \) with respect to the dynamical variable \( \phi(x) \) which generally depends on the spacetime points, \( \delta I / \delta \phi(x) = 0 \), where \( x \) represents the spacetime dependence. However, the ‘working’ variable \( \Lambda \) is not regarded as any dynamical variable and has no spacetime dependence. \( \Lambda \) is simply a working variable to ensure the mathematical consistency of the resultant MFL of two-horizon spacetimes. Hence, even if \( E_\Lambda \) is set zero, it can never be interpreted as any equation of motion of a dynamical variable. This means that the on-shell condition requires only the Einstein equation \( E_g = 0 \) (and the extended linearized equation (2.4) under the presence of variation \( \delta g_{\mu \nu} \)), and \( E_\Lambda \) is non-vanishing \( (E_\Lambda \neq 0) \). Now it is recognized that the physical principle we rely on is the ‘extended’ variational principle in which the variations are taken with respect to not only the dynamical variable \( g_{\mu \nu} \) (and matter field if it exists) but also the working variable \( \Lambda \), while the equations of motion are given by the variation of the Lagrangian with respect to dynamical variables.

As a by-product of the above two remarks, it will be shown below that, in our extended Iyer–Wald formalism, the conserved Noether current and its associated Noether charge are defined in the same way as in the original Iyer–Wald formalism. Our ‘extension’ of the Iyer–Wald formalism has three meanings: (1) the extended Iyer–Wald formalism includes the variable \( \Lambda \), (2) \( \Lambda \) is not a dynamical variable but simply a working variable which means that \( E_\Lambda \) does not give any equation of motion and (3) the conserved Noether current is obtained with the same definition as in the original Iyer–Wald formalism.

Then let us proceed to the extension of the Iyer–Wald formalism. Equation (2.2) is the starting point. The extended Iyer–Wald formalism with variable \( \Lambda \) differs from the original formalism on the following two points. One of them is a manifest point expressed by the third term on the right-hand side of equation (2.2). This term does not arise in the original Iyer–Wald formalism, but arises in our extended formalism by the variation \( \delta \Lambda \) in, for example, the second term on the right-hand side of equation (2.1). Another point is a subtle point included in the metric variation \( \delta g_{\mu \nu} \). In our extended formalism, the variation \( \delta g_{\mu \nu} \) also gives rise to

\(^5\) This \( g_{\mu \nu} \) can also be regarded as a ‘background’ metric of the perturbation \( \delta g_{\mu \nu} \).
the variation $\delta \Lambda$ if the concrete form of the metric depends on $\Lambda$ as for SdS, de Sitter and anti-de Sitter spacetimes.

To formulate an abstract form of the MFL with the variable $\Lambda$, let us follow the same procedure of the original Iyer–Wald formalism [5]. If we introduce an arbitrary vector field $\xi^\mu$, which is not a dynamical variable in $L$, and consider the variation given by the Lie derivative $\delta = L_\xi$ along $\xi^\mu$, then we get from equation (2.2)

$$d[\Theta(g, L_\xi g) - \xi \cdot L] = -E_g L_\xi g - E_\Lambda L_\xi \Lambda,$$

(2.5)

where $\xi \cdot L \equiv \xi^\mu L_{\mu\nu_1\ldots\nu_{n-1}}$ and the relation $L_\xi A = \xi \cdot dA + d(\xi \cdot A)$ of the Lie and exterior derivatives of a form $A$ is used. The $(n-1)$-form on the left-hand side, $\Theta(g, L_\xi g) - \xi \cdot L =: J_\xi$, is called the Noether current. Here note that, since $\Lambda$ has no spacetime dependence, $L_\xi \Lambda \equiv 0$ holds. Therefore, when the on-shell condition $E_g = 0$ is required, the Noether current is closed $dJ_\xi = 0$ which guarantees the local existence of the $(n-2)$-form $Q_\xi$ called the Noether charge:

$$dQ_\xi := J_\xi = \Theta(g, L_\xi g) - \xi \cdot L.$$

(2.6)

This $Q_\xi$ is locally constructed from the on-shell metric which satisfies the Einstein equation. Note that the conservation of the Noether current $dJ_\xi = 0$ under the on-shell condition corresponds to the Noether theorem, and the Noether charge $Q_\xi$ is the conserved charge of $J_\xi$ associated with the symmetry generator $\xi$. The definition of $Q_\xi$ in equation (2.6) is the same as that in the original Iyer–Wald formalism. Here, it should also be emphasized that the existence condition of $Q_\xi$ is the on-shell condition $E_g = 0$ for the metric $g_{\mu\nu}$ appearing in equation (2.6). This condition is the same as that in the original Iyer–Wald formalism. Hence we find that, in our extended Iyer–Wald formalism, the conserved Noether current and its associated Noether charge are defined in the same way as in the original Iyer–Wald formalism.

Next, since the vector $\xi^\mu$ (not the 1-form $\xi_\mu$) is not a dynamical variable, its variation does not exist ($\delta \xi^\mu \equiv 0$) under the variation of dynamical and working variables. Then, from the variation of $J_\xi$, the following relation is obtained:

$$\delta J_\xi = \delta \Theta(g, L_\xi g) - \xi \cdot \delta L$$

$$= \omega(g, \delta g, L_\xi g) + d[\xi \cdot \Theta(g, \delta g)] - \xi \cdot E_\Lambda \delta \Lambda,$$

(2.7)

where $\omega$ is defined as $\omega(g, \delta g, L_\xi g) := \delta \Theta(g, L_\xi g) - L_\xi \Theta(g, L_\xi g)$ and the on-shell condition $E_g = 0$ is required. Here we must note that, as explained in the second paragraph of this section, the on-shell condition under the presence of metric variation $\delta g_{\mu\nu}$ denotes that the unperturbed metric $g_{\mu\nu}$ and the variation $\delta g_{\mu\nu}$ appearing in equation (2.7) satisfy, respectively, the Einstein equation $E_g = 0$ and the extended linearized Einstein equation (2.4). Furthermore, as noted in [5], the $(n-1)$-form $\omega$ vanishes ($\omega = 0$) when $\xi$ is the generator of a symmetry of all dynamical variables in $L$, i.e. $L_\xi g_{\mu\nu} = 0$. For stationary spacetimes, the timelike Killing vector can be regarded as the symmetry generator $\xi$. Hence, at least for the stationary case, we get

$$\xi \cdot E_\Lambda \delta \Lambda + \delta J_\xi - d[\xi \cdot \Theta(g, \delta g)] = 0.$$

(2.8)

Here, the Noether charge defined in equation (2.6) gives $\delta J_\xi = d(\delta Q_\xi)$. Therefore, by integrating equation (2.8) on a hypersurface $\Sigma$ and applying the Stokes theorem, we obtain

$$\int_{\Sigma} \xi \cdot E_\Lambda \delta \Lambda + \int_{\partial \Sigma} [\delta Q_\xi - \xi \cdot \Theta(g, \delta g)] = 0.$$

(2.9)

The first term is due to variable $\Lambda$. There are three conditions to hold equation (2.9): (1) the Einstein equation $E_g = 0$ for the unperturbed metric $g_{\mu\nu}$, (2) the extended linearized Einstein
equation (2.4) of the metric variation $\delta g_{\mu\nu}$ and (3) the existence of the symmetry generator $\xi$ to give $\xi_\mu g = 0$.

In the original Iyer–Wald formalism, the first term in equation (2.9) disappears. Wald et al [5] show that, when the original formalism is applied to asymptotic flat black holes, equation (2.9) without the first term reduces to the MFL of these black holes. Therefore, we can regard equation (2.9) as the primitive MFL of any spacetime with $\Lambda$ in any dimensions.\(^6\)

3. Mechanical first law of the SdS spacetime

3.1. Preparations

Before deriving the MFL of the SdS spacetime from equation (2.9), we summarize the SdS metric and the MFL of asymptotic flat black holes derived from the original Iyer–Wald formalism. For the first, the line element of the SdS spacetime is

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(3.1)

where

$$f(r) := 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2.$$  

(3.2)

The equation $f(r) = 0$ has two positive roots and one negative root for the parameter range

$$0 < 9M^2 \Lambda < 1.$$  

(3.3)

Throughout this paper, our discussion is restricted in this range. This parameter range guarantees the existence of the BEH and CEH. The radii of BEH $r_b$ and that of CEH $r_c$ are given by, respectively, a smaller positive root and a larger positive root of $f(r) = 0$. Then the parameter range (3.3) gives the relations of these radii:

$$2M < r_b < 3M < \frac{1}{\sqrt{\Lambda}} < r_c < \frac{3}{\sqrt{\Lambda}}.$$  

(3.4)

The equations $f(r_b) = 0$ and $f(r_c) = 0$ are rearranged to

$$M = \frac{r_br_c(r_b + r_c)}{2(r_b^2 + r_br_c + r_c^2)}, \quad \Lambda = \frac{3}{r_b^2 + r_br_c + r_c^2}.$$  

(3.5)

Since the SdS spacetime is static, each event horizon possesses a bifurcation sphere, on which the timelike Killing vector $\xi := \partial_t$ vanishes. The surface gravity of BEH $\kappa_b$ and that of CEH $\kappa_c$ are defined by the following equations evaluated at the horizons,

$$\xi^\mu \nabla_\nu \xi^\nu|_{\text{BEH}} = \kappa_b \xi^\mu|_{\text{BEH}}, \quad \kappa_c \xi^\mu|_{\text{CEH}} = \xi^\mu \nabla_\nu \xi^\nu|_{\text{CEH}}.$$  

(3.6)

In general, the numerical value of surface gravity depends on the normalization of $\xi$. With the normalization $\xi := \partial_t$, we get

$$\kappa_b = \frac{1}{2} \frac{df(r)}{dr} \bigg|_{r=r_b} = \frac{1 - \Lambda r_b^2}{2r_b}, \quad \kappa_c = \frac{1}{2} \frac{df(r)}{dr} \bigg|_{r=r_c} = \frac{1 - \Lambda r_c^2}{2r_c}.$$  

(3.7)

Then, from equation (3.4), the following relations hold:

$$\kappa_b > 0, \quad \kappa_c < 0, \quad \kappa_b > |\kappa_c|.$$  

(3.8)

\(^6\) The primitive MFL (2.9) seems applicable to spacetimes without the CEH like asymptotic anti-de Sitter black holes. When equation (2.9) is applied to such black holes, the two degrees of freedom expressed by $M$ and $\Lambda$ may describe two independent effects in the MFL; one of them is due to the BEH and another is due to the ‘wall’ given by the infinitely large gravitational potential barrier of the anti-de Sitter metric.
Next, we summarize the MFL of asymptotic flat black holes derived from the original Iyer–Wald formalism [5]. The first term in equation (2.9) does not exist in the original formalism. When the original formalism is applied to asymptotic flat black holes and \( \Sigma_1 \) connects the bifurcation sphere \( B \) of the horizon and the spatial infinity \( \infty \), then equation (2.9) without the first term is rearranged to

\[
\int_\infty [\delta Q_\xi - \xi \cdot \Theta(g, \delta g)] - \int_B \delta Q_\xi = 0, \tag{3.9}
\]

where \( \xi := \partial_t \) with the usual static time \( t \) and \( \xi = 0 \) on \( B \) is used. As shown in [5], this reduces to the MFL of asymptotic flat black holes:

\[
\delta M = \frac{\kappa_{\text{BH}}}{8\pi} \delta A_{\text{BH}} + \text{work terms}, \tag{3.10}
\]

where \( M \) is the ADM mass of the black hole, \( A_{\text{BH}} \) is the area of \( B \), \( \kappa_{\text{BH}} \) is the surface gravity of the horizon with an appropriate normalization of the Killing vector, the ‘work terms’ are given by electromagnetic fields and angular momentum for Reissner–Nordström and Kerr black holes [6]. (This MFL in equation (3.10) is a differential form of the so-called Smarr formula [7].) It should be emphasized that the first term (integration on \( \infty \)) on the left-hand side of equation (3.9) gives the variation of the ADM mass \( \delta M \) and the ‘work terms’ in equation (3.10), and the second term (integration on \( B \)) in equation (3.9) gives the ‘heat term’ \((\kappa_{\text{BH}}/8\pi)\delta A_{\text{BH}}\). In asymptotic flat black hole thermodynamics, the MFL in equation (3.10) is equivalent to the thermodynamical first law of asymptotic flat black holes in which \( M \) is regarded as the internal energy and \( A_{\text{BH}}/4 \) is regarded as the entropy (the so-called entropy–area law) due to the Hawking temperature \( \kappa_{\text{BH}}/2\pi \) [1].

### 3.2. Application of the extended Iyer–Wald formalism to the SdS spacetime

Now we apply equation (2.9) to the SdS spacetime to obtain the appropriately expressed MFL of the SdS spacetime. Here, the meaning of ‘appropriate expression’ is that the mass parameter \( M \) of the SdS spacetime is expressed as a function of two independent quantities which seem to correspond to the entropy and the state variable of the system size when the MFL is regarded as a thermodynamical first law. The quantity corresponding to entropy gives the ‘heat term’ in the MFL and the quantity corresponding to the system size gives the ‘work term’ in the MFL. In the thermodynamical first law of ordinary laboratory systems, the heat term is the product of temperature and variation of entropy and the work term is the product of pressure and variation of the system size.

Concerning the quantity \( S \) which seems to correspond to the entropy of the SdS spacetime, we refer simply to the entropy–area law of asymptotic flat black holes and define \( S \) by the total spatial area of the BEH and CEH:

\[
S := \pi r_b^2 + \pi r_c^2. \tag{3.11}
\]

Here note that the horizon radii \( r_b \) and \( r_c \) are regarded as two independent variables due to the working hypothesis of two independent variables, and that the total horizon area includes manifestly the effects of the BEH and CEH through these two variables \( r_b \) and \( r_c \). When this definition (3.11) is adopted, the variation \( \delta S \) gives the ‘heat term’ in the appropriate MFL of the SdS spacetime.

Next, we have to define the system size \( V \) whose variation \( \delta V \) composes the ‘work term’ in the appropriate MFL of the SdS spacetime. Here, for the system size of the SdS spacetime, it seems reasonable to consider the three-dimensional volume of a spacelike hypersurface connecting the BEH and CEH as the object of thermodynamical interest, since the Hawking radiations of two horizons coexist there. As such a hypersurface, let us consider \( \Sigma_{\text{bd}} \) which
connects the bifurcation spheres of the BEH and CEH at $t = \text{constant}$. Furthermore, let us refer to the first term in equation (2.9). In that term, the integral $\int_{\Sigma_{\text{bif}}} \xi \cdot E_A$ appears as a natural quantity of the three-dimensional volume. Then concerning the quantity $V$ which seems to correspond to the system size of the SdS spacetime, we adopt the following definition:

$$V := \int_{\Sigma_{\text{bif}}} \xi \cdot E_A = \frac{4\pi}{3}(r_c^3 - r_b^3),$$

(3.12)

where $\xi := \partial_t$.

Definitions (3.11) and (3.12) are equivalent to searching for the MFL of the SdS spacetime which is expressed as

$$\delta M = T_{\text{eff}} \delta S - p_{\text{eff}} \delta V,$$

(3.13)

where the concrete forms of coefficients $T_{\text{eff}}$ and $p_{\text{eff}}$ will be obtained below. These coefficients $T_{\text{eff}}$ and $p_{\text{eff}}$ are simply the partial derivatives of $M$. If the MFL in equation (3.13) is regarded as a thermodynamical first law of the SdS spacetime, then $T_{\text{eff}}$ and $p_{\text{eff}}$ are interpreted as, respectively, the effective temperature and pressure. Therefore, because the temperature and pressure are positive definite for ordinary laboratory systems, it is natural to require that the appropriately expressed MFL should satisfy the following requirement:

$$T_{\text{eff}} > 0, \quad p_{\text{eff}} > 0,$$

(3.14)

If the MFL in equation (3.13) is regarded as a thermodynamical first law of the SdS spacetime, then the positivity of $T_{\text{eff}}$ and $p_{\text{eff}}$ makes it plausible to call $T_{\text{eff}}$ and $p_{\text{eff}}$, respectively, the effective temperature and pressure. A comment on $T_{\text{eff}}$ in relation to the non-equilibrium nature of the SdS spacetime will be given in section 4.

The appropriately expressed MFL of the SdS spacetime should be obtained by using definitions (3.11) and (3.12) under the requirement (3.14) by regarding $M$ and $\Lambda$ as two independent variables. To obtain the appropriate MFL, we must be careful to carry out the integrations in equation (2.9). According to the asymptotic flat case summarized in the previous subsection, we expect that, when the second term (surface term) in equation (2.9) is evaluated on the bifurcation sphere in the SdS spacetime, it should give the ‘heat term’ $T_{\text{eff}} \delta S$ in the MFL of the SdS spacetime (3.13). But when the second term in equation (2.9) is evaluated on some sphere which is not the bifurcation sphere, it should contribute to the mass variation $\delta M$ in equation (3.13), as for the ADM mass in an asymptotic flat case (3.10). The first term (three-volume term) in equation (2.9) should contribute to the ‘work term’ $-p_{\text{eff}} \delta V$ in equation (3.13). Then, since there seems no natural sphere except for the bifurcation spheres of horizons, one problem arises: how to prepare a sphere to get the mass variation in the MFL of the SdS spacetime. Hence, in order to extract the mass variation from equation (2.9), we adopt a strategy composed of three steps as follows.

*Step 1.* Consider two hypersurfaces, $\Sigma_b$ and $\Sigma_c$, at $t = \text{constant}$. Here $\Sigma_b$ covers the region $r_b < r < r_{\text{cut}}$ and $\Sigma_c$ covers $r_{\text{cut}} < r < r_c$, where $r_{\text{cut}}$ is a working variable which is introduced to extract the mass variation $\delta M$ and must not appear in the resultant MFL of the SdS spacetime.

*Step 2.* Carry out the integrations in equation (2.9) for the hypersurface $\Sigma_b$. Do the same for $\Sigma_c$.

*Step 3.* Combine these integrated equations to introduce the variations of $M$, $S$ and $V$ into the appropriately expressed MFL of the SdS spacetime (3.13).

Note that the working variable $r_{\text{cut}}$ is introduced simply to carry out the integrations in equation (2.9) as a ‘piecewise’ integration. If the resultant MFL obtained in step 3 depends on
where the indices $a$, $\Theta_1$ potential determined by the normal directions to $\delta \kappa_b \nabla B_b = \nabla \epsilon$ satisfies which is given in equation (3.1), and the timelike Killing vector $\dot{t}$ is regarded as the symmetry generator $\xi := \partial_t$.

Equation (2.9) with $\Sigma_b$ in the SdS spacetime reads as

$$- \frac{1}{8\pi} \int_{\Sigma_b} \xi \cdot \epsilon \delta \Lambda - \int_{B_b} \delta \mathbf{Q}_\xi + \int_{B_{cut}} [\delta \mathbf{Q}_\xi - \xi \cdot \Theta(g, \delta g)] = 0,$$

(3.15)

where $\xi = 0$ at $B_b$ is used in the second term and the signs of the second and third terms are determined by the normal directions to $\Sigma_b$ at $B_b$ and $B_{cut}$ which are, respectively, inward and outward pointing along the 'r-axis'. The first term on the left-hand side of equation (3.15) is a simple three-dimensional volume integral in $\Sigma_b$:

$$- \frac{1}{8\pi} \int_{\Sigma_b} \xi \cdot \epsilon \delta \Lambda = - \frac{1}{6} \epsilon_{cut} \delta \Lambda + \frac{1}{6} \epsilon_b \delta \Lambda.$$  

(3.16)

For the second term on the left-hand side of equation (3.15), note that the symplectic potential $\Theta$ includes the $(n-1)$-volume form as its factor by definition (2.2), and consequently the Noether charge $\mathbf{Q}_\xi$ includes the $(n-2)$-volume form as its factor by definition (2.6). $(n = 4$ for the SdS spacetime.) This implies that the integral $\int \mathbf{Q}_\xi$ and the variation $\delta$ of the Noether charge are commutative. $\int_{B_b} \delta \mathbf{Q}_\xi = \delta(\int_{B_b} \mathbf{Q}_\xi)$. On the other hand, we can find for the Lagrangian in equation (2.1) with the variable $\Lambda$ that the Noether charge $\mathbf{Q}_\xi$ takes the same form as the ordinary general relativity without $\Lambda$ [5].

$$\mathbf{Q}_\xi = - \frac{1}{16\pi} \epsilon_{\mu \nu \alpha \beta} \nabla^a \xi^\beta.$$  

(3.17)

Then, referring to the proof of theorem 6.1 in the original Iyer–Wald formalism [5] which holds even with the variable $\Lambda$, we get

$$\delta \left[ \int_{B_b} (\mathbf{Q}_\xi)_{ab} \right] = - \frac{\kappa_b}{16\pi} \delta \left( \int_{B_b} \epsilon_{\alpha \beta} \nabla^\alpha \xi^\beta \right),$$

(3.18)

where the indices $a$ and $b$ are the abstract indices [8], $\epsilon_{\alpha \beta}$ is the bi-normal 2-form to $B_b$ which satisfies $\nabla_\mu \epsilon_{\alpha \beta} = \kappa_b \epsilon_{\mu \nu}$ on $B_b$ and the vanishing variation of the surface gravity at $B_b$ ($\delta \kappa_b |_{B_b} = 0$) is used7. Hence, we obtain for the second term in equation (3.15)

$$\int_{B_b} \delta \mathbf{Q}_\xi = \frac{\kappa_b}{16\pi} \delta \left( 2 \int_{B_b} d^2 \Omega_b \right) = \frac{\kappa_b}{2\pi} \delta (\pi r_b^2),$$

(3.19)

where $d^2 \Omega_b := r_b^2 \sin \theta \, d\theta \, d\varphi$ is the area element on $B_b$.

Let us proceed with the calculation of the third term on the left-hand side of equation (3.15). For the first we calculate the integral of the Noether charge $\int_{B_b} \mathbf{Q}_\xi$, where $B_b$...
is a two-sphere of an arbitrary radius \( r \). Note that the \( \theta \)-\( \varphi \) component of \( Q_\xi \) in equation (3.17) contributes to the integral
\[
Q_\xi = -\frac{1}{16\pi} \epsilon_{\theta\varphi ab} \nabla^a \xi^b = -\frac{1}{16\pi} \frac{\partial g_{\mu\nu}}{\partial r} \xi^\mu \sqrt{-\det(g_{\mu\nu})} \, d\theta \wedge d\varphi.
\]
(3.20)
This gives
\[
\int_{B_r} Q_\xi = -\frac{1}{16\pi} \int_{B_r} d^2\Omega_r \frac{\partial g_{\mu\nu}}{\partial r} \xi^\mu = \frac{1}{16\pi} \frac{df(r)}{dr} 4\pi r^2,
\]
(3.21)
where \( d^2\Omega_r := r^2 \sin \theta \, d\theta \, d\varphi \) is the area element on \( B_r \) and the definitions \( \xi := \partial_\theta \) and \( g_{\mu\nu} = -f(r) \) in equation (3.2) are used. Then, using the relation \( \int_{B_r} Q_\xi = \delta(\int_{B_r} Q_\xi) \), the first integral in the third term in equation (3.15) becomes
\[
\int_{B_{cut}} \delta Q_\xi = \delta \int_{B_{cut}} Q_\xi = \delta \left( \frac{r_{cut}^3 f' (r_{cut})}{4} \right) = \frac{r_{cut}^3}{4} \delta f'(r_{cut}),
\]
(3.22)
where \( f' := df/dr \) and equation (3.21) is used in the second equality. Next, the explicit form of the symplectic potential \( \Theta \) for the Lagrangian (2.1) with the variable \( \Lambda \) is
\[
\Theta_{\lambda\mu
u} = -\frac{1}{16\pi} \epsilon_{\lambda\mu\nu ab} g^{ab} g^{\alpha\beta} [\nabla_{\alpha}(\delta g_{\beta\alpha}) - \nabla_{\beta}(\delta g_{\alpha\beta})].
\]
(3.23)
This is the same form as the ordinary general relativity without \( \Lambda \) [5]. Here, under the variations of \( \delta M \) and \( \delta \Lambda \), the metric variation \( \delta g_{\mu\nu} \) is given by equation (3.1):
\[
\delta (ds^2) = \delta g_{\mu\nu} \, dx^\mu \, dx^\nu = -\delta f(r) \, dr^2 + \frac{-\delta f(r)}{f^2(r)} \, dr^2.
\]
(3.24)
where the spherical part vanishes since that part does not depend on \( M \) and \( \Lambda \), and
\[
\delta f(r) = -\frac{2}{r} \delta M - \frac{r^2}{3} \delta \Lambda.
\]
(3.25)
It should be emphasized here that the metric variation (3.24) is a static solution to the extended linearized Einstein equation (2.4) and satisfies the condition to ensure equation (2.9). Furthermore note that, for the integrand \( \xi^\mu \Theta_{\mu\nu} \) in the integral \( \int_{B_{cut}} \xi \cdot \Theta \), the indices \( \mu \) and \( \nu \) denote the tangential components to \( B_{cut} \) (\( \theta \)-\( \varphi \) component), \( \xi^\mu \Theta_{a\theta\varphi} \). Then we obtain
\[
\int_{B_{cut}} \xi \cdot \Theta = -\frac{1}{16\pi} \int_{B_{cut}} d^2\Omega_{cut} \frac{\partial g_{ab} g^{ab}}{\partial r} \left[ \nabla_{\alpha}(\delta g_{\beta\alpha}) - \nabla_{\beta}(\delta g_{\alpha\beta}) \right]
\]
\[
= -\frac{1}{16\pi} \int_{B_{cut}} d^2\Omega_{cut} \left[ -\delta f'(r) - \frac{2}{r} \delta f(r) \right]
\]
\[
= \frac{r_{cut}^2}{4} \left[ \delta f'(r_{cut}) + \frac{2}{r_{cut}} \delta f(r_{cut}) \right],
\]
(3.26)
where \( d^2\Omega_{cut} := r_{cut}^2 \, \sin \theta \, d\theta \, d\varphi \) is the area element on \( B_{cut} \) and \( \xi := \partial_\theta \) and \( \epsilon_{\mu\nu\theta} = \epsilon_{r\theta\nu} \) are used in the first equality. Hence, the third term in equation (3.15) becomes
\[
\int_{B_{cut}} [\delta Q_\xi - \xi \cdot \Theta(g, \delta g)] = \delta M + \frac{1}{6} r_{cut}^3 \delta \Lambda.
\]
(3.27)
Then, collecting equations (3.16), (3.19) and (3.27), we obtain from equation (3.15)
\[
\delta M = \frac{k_\Lambda}{2\pi} \delta \left( \pi r_{cut}^2 \right) - \frac{1}{6} r_{cut}^3 \delta \Lambda.
\]
(3.28)
It should be noted that the working variable \( r_{cut} \) does not appear in this equation.
Next turn our calculation to equation (2.9) with the hypersurface $\Sigma_c$, which reads as
\[
\frac{1}{8\pi} \int_\Sigma \xi \cdot \epsilon \delta \Lambda + \int_{B_c} \delta Q_\xi = \int_{B_{cut}} [\delta Q_\xi - \xi \cdot \Theta(g, \delta g)] = 0,
\]
where $\xi = 0$ at $B_c$ is used in the second term and the signs of the second and third terms are determined by the normal directions to $\Sigma_c$ at $B_c$ and $B_{cut}$ which are, respectively, outward and inward pointing along the ‘$r$-axis’. Then, following the same calculations to obtain equation (3.28), we obtain from equation (3.29)
\[
\delta M = \frac{\kappa_c}{2\pi} \delta (\pi r_c^2) - \frac{1}{6} r_c^3 \delta \Lambda.
\]
(3.30)
It should be noted that the working variable $r_{cut}$ does not appear in this equation.

So far we have carried out step 2 to obtain equations (3.28) and (3.30). Before proceeding to step 3 of our strategy, let us comment on Sekiwa [4]: as mentioned in section 1, Sekiwa [4] has already discussed the MFL of the SdS spacetime with variable $\Lambda$ using some integral quantities as state variables and obtained two MFLs separately for BEH and CEH. These MFLs in [4] are the same as our equations (3.28) and (3.30). Although the derivation of MFLs in [4] includes the problem of the choice of the integration constant, our derivation based on the extended Iyer–Wald formalism is free from such a problem. Furthermore, the integral quantity used in [4] requires to place a boundary (corresponding to $B_{cut}$ in our step 2) at the CEH in deriving equation (3.28) and at the BEH in deriving equation (3.30). If we interpret the boundary as the position of an observer, it seems not to be physically acceptable to place the observer at the event horizon. However, our surface $B_{cut}$ may be appropriate as a candidate of the position of the observer who measures thermodynamical quantities of a two-horizon system.

Although equations (3.28) and (3.30) are regarded as MFLs of the SdS spacetime in [4], our aim is to propose a single formula (3.13) as the appropriately expressed MFL of the SdS spacetime. The procedure to obtain the appropriate MFL is step 3 of our strategy. Then, let us carry out step 3. Combining equations (3.28) and (3.30) together with definition (3.11), we get
\[
\delta S = 2\pi \left( \frac{1}{\kappa_b} + \frac{1}{\kappa_c} \right) \delta M + \frac{\pi}{3} \left( \frac{r_b^3}{\kappa_b} + \frac{r_c^3}{\kappa_c} \right) \delta \Lambda.
\]
(3.31)
On the other hand, we get from equation (3.5)
\[
\delta r_b = \frac{2\delta M + (r_b^3/3)\delta \Lambda}{1 - \kappa_b r_b^2}, \quad \delta r_c = \frac{2\delta M + (r_c^3/3)\delta \Lambda}{1 - \kappa_c r_c^2}.
\]
(3.32)
These variations of horizon radii together with definition (3.12) give the volume variation
\[
\delta V = 4\pi \left( \frac{r_c}{\kappa_c} - \frac{r_b}{\kappa_b} \right) \delta M + \frac{2\pi}{3} \left( \frac{r_c^4}{\kappa_c} - \frac{r_b^4}{\kappa_b} \right) \delta \Lambda.
\]
(3.33)
Then, substituting this $\delta V$ into equation (3.31), we obtain the appropriate MFL of the SdS spacetime (3.13),
\[
\delta M = T_{eff} \delta S - p_{eff} \delta V,
\]
(3.34)
where the coefficients are
\[
T_{eff} = \frac{r_b^4}{(r_c + r_b)(r_c^2 - r_b^2)} \frac{r_c^4}{(r_c + r_b)(r_c^2 - r_b^2)} \frac{\kappa_b}{2\pi} + \frac{r_c^4}{(r_c + r_b)(r_c^2 - r_b^2)} \frac{\kappa_c}{2\pi} = \frac{1}{4\pi r_c} \frac{x^4 + x^3 - 2x^2 + x + 1}{x(x + 1)(x^2 + x + 1)},
\]
(3.35)
\[ p_{\text{eff}} = \frac{1}{8\pi r_{+}^{2}} \frac{(1-x)(x^4 + 3x^3 + 3x^2 + 3x + 1)}{x(x+1)(x^2 + x + 1)^2}, \] (3.36)

where \( x := r_{h}/r_{+} \) and \( 0 < x < 1 \). We find that \( T_{\text{eff}} \) and \( p_{\text{eff}} \) are positive definite, \( T_{\text{eff}} > 0 \) and \( p_{\text{eff}} > 0 \) and the requirement (3.14) is satisfied. Hence, the MFL in equation (3.34) with the coefficients given above is the appropriately expressed MFL of the SdS spacetime.

Here note that although we refer to the entropy–area law of asymptotic flat black hole thermodynamics as a motivation to adopt definition (3.11), it does not mean to assume \( S \) in equation (3.11) to be the physical entropy of the SdS spacetime. It is a future task to resolve the issue whether this \( S \) is really a physical entropy in SdS thermodynamics. Even if \( S \) is not a physical entropy, our MFL in equation (3.34) suggests the mass formula which relates mass parameter \( M \) and total horizon area \( S \). (The mass formula of asymptotic flat black holes is the Smarr formula [7].)

Our derivation of the appropriately expressed MFL (3.34) is based on the extended Iyer–Wald formalism. However, since the relation among \( M, S \) and \( V \) is definitely determined by the metric (3.1), relation (3.34) must be obtained from \( S \) and \( V \) defined in equations (3.11) and (3.12) without using the extended Iyer–Wald formalism. Indeed, Sekiwa [4] has already obtained equations (3.28) and (3.30) (whose combination gives equation (3.34)) by using some integral quantities as state variables. The advantages of using the extended Iyer–Wald formalism are the following three points: (1) it is free from the problem of the integration constant as mentioned in section 1, (2) it offers naturally the definition of \( V \) in equation (3.12) and (3) it gives the positive-definite effective temperature \( T_{\text{eff}} \) and pressure \( p_{\text{eff}} \), which, together with \( V \), are suitable to describe the SdS black hole evaporation process as discussed in the following section.

Finally in this section, for the completeness of our discussion, let us consider the case that \( \Lambda \) is not regarded as an independent variable. In this case, one obtains a relation, \( \delta S = 2\pi(1/\kappa_{h} + 1/\kappa_{c})\delta M \), from definition (3.11). By rearranging this relation appropriately and introducing \( V \) defined in equation (3.12), one can formally obtain the MFL in equation (3.34) with the same coefficients in equations (3.35) and (3.36). However in this case, we find relations \( \partial M(S, V)/\partial S = (dS/dM)^{-1} \neq T_{\text{eff}} \) and \( -\partial M/\partial V = -(dV/dM)^{-1} \neq -p_{\text{eff}} \). These contradict the definition of coefficients \( T_{\text{eff}} \) and \( p_{\text{eff}} \) in equation (3.34). By the reductive absurdity, this fact indicates the necessity of ‘variable \( \Lambda \)’ for a mathematically consistent MFL of the SdS spacetime of the form in equation (3.34). Here note that the same claim, the necessity of variable \( \Lambda \), is already suggested in [4], although the MFL in [4] is given in a different expression from equation (3.34) and includes the uncertainty of the integration constant of conserved quantities as mentioned in section 1. The necessity of variable \( \Lambda \) seems to be a universal fact for a mathematically consistent MFL of the SdS spacetime.

4. Discussions

Let us discuss what is implied by our appropriate MFL in equation (3.34) if it is regarded as a thermodynamical first law of the SdS spacetime. As an interesting process, we treat the SdS black hole evaporation process at constant \( \Lambda \).

Before considering SdS black hole evaporation, let us recall the so-called generalized second law in the evaporation process of the Schwarzschild black hole. In Schwarzschild thermodynamics, the thermodynamical first law is given by equation (3.10) by setting the ‘work terms’ zero, \( \delta M = T_{\text{BH}}\delta S_{\text{BH}} \), where \( T_{\text{BH}} := \kappa_{\text{BH}}/2\pi = 1/8\pi M \) is the Hawking temperature and \( S_{\text{BH}} := A_{\text{BH}}/4 = \pi(2M)^2 \) is the black hole entropy. When the Schwarzschild black hole evaporates, the mass energy \( M \) decreases and consequently \( S_{\text{BH}} \) decreases. Here,
note that the evaporation of an isolated black hole is an irreversible adiabatic process. Hence, if the total entropy of the whole system is given by $S_{BH}$, the decrease of $S_{BH}$ contradicts the second law of thermodynamics which requires the increase of total entropy for irreversible adiabatic processes. Then the generalized second law claims that the total entropy of the black hole and matter fields of Hawking radiation increases for the evaporation process of isolated black holes [9, 10]. Therefore, at least for the evaporation of asymptotic flat black holes, the generalized second law is necessary to hold the validity of thermodynamical formulation of black holes.

Then, proceed to the discussion of SdS black hole evaporation at constant $\Lambda$. When the SdS black hole evaporates, it seems reasonable to consider the mass parameter decreases, $\delta M < 0$. Here, for any process at constant $\Lambda$, our MFL in equation (3.34) and $\delta V$ in equation (3.33) are rearranged to $\delta S = 2\pi (1/k_b + 1/k_c) \delta M$ and $\delta V = 4\pi (r_c/k_c - r_b/k_b) \delta M$. Then, we get $\delta S > 0$ and $\delta V > 0$ due to $\delta M < 0$ and equation (3.8). The expansion of volume $\delta V > 0$ is a reasonable result if $p_{\text{eff}}$ is interpreted as a pressure. The increase of entropy $\delta S > 0$ denotes that the generalized second law is not needed to ensure the second law of SdS thermodynamics for its evaporation process at constant $\Lambda$. Hence, our appropriate MFL in equation (3.34) suggests a thermodynamically consistent description of SdS black hole evaporation at constant $\Lambda$.

Finally, let us comment on the coefficient $T_{\text{eff}}$ in equation (3.35). This $T_{\text{eff}}$ is regarded as a temperature if our MFL in equation (3.34) is regarded as a thermodynamical first law. Then one may wonder about what the physical meaning of the temperature is, since the SdS spacetime is essentially a non-equilibrium system due to the difference of Hawking temperatures of the BEH and CEH [2]. Here let us point out that there is a long history of research on two-temperature non-equilibrium systems, and there are many proposals on the definition of non-equilibrium temperature of non-equilibrium systems (see, for example, references of works in [10]). No commonly accepted definition of non-equilibrium temperature exists at present. Therefore, from the point of view of non-equilibrium thermodynamics, the suggestion of defining a non-equilibrium temperature is meaningful at present. If the SdS thermodynamics is formulated by regarding our MFL in equation (3.34) as a thermodynamical first law, then our $T_{\text{eff}}$ in equation (3.35), which can be expressed as a linear combination of the surface gravities of the BEH and CEH, may be understood as one suggestion of an effective temperature of a non-equilibrium system. At present, the physical meaning of $T_{\text{eff}}$ is an open issue which requires further researches on the non-equilibrium nature of the SdS spacetime. The research on SdS thermodynamics provides the stage for the intersection of gravitational physics and non-equilibrium physics.

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References

[1] Bekenstein J D 1973 Black holes and entropy Phys. Rev. D 7 2333
Bekenstein J D 1974 Generalized second law of thermodynamics in black-hole physics Phys. Rev. D 9 3292
Hawking S W 1971 Gravitational radiation from colliding black holes Phys. Rev. Lett. 26 1344
Hawking S W 1975 Particle creation by black holes Commun. Math. Phys. 43 199
Hawking S W and Page D N 1983 Thermodynamics of black holes in anti-de Sitter space Commun. Math. Phys. 87 577
Israel W 1986 Third law of black-hole dynamics: a formulation and proof Phys. Rev. Lett. 57 397
York J W Jr 1986 Black-hole thermodynamics and the Euclidean Einstein action Phys. Rev. D 33 2002
Braden H W, Brown J D, Whiting B F and York J W Jr 1990 Charged black hole in a grand canonical ensemble Phys. Rev. D 42 3376
David J, Martinez E A and York J W Jr 1991 Complex Kerr–Newman geometry and black-hole thermodynamics Phys. Rev. Lett. 66 2281
[2] Gibbons G W and Hawking S W 1977 Cosmological event horizons, thermodynamics, and particle creation Phys. Rev. D 15 2738
[3] Abbot L F and Deser S 1982 Stability of gravity with a cosmological constant Nucl. Phys. B 195 76
Balasubramanian V, de Boer J and Minic D 2002 Mass, entropy and holography in asymptotically de Sitter spaces Phys. Rev. D 65 123508
Ghezelbash A M and Mann R B 2002 Action, mass and entropy of Schwarzschild–de Sitter black holes and the de Sitter/CFT correspondence J. High Energy Phys. JHEP01(2002)005
Gomberoff A and Teitelboim C 2003 De Sitter black holes with either of the two horizons as a boundary Phys. Rev. D 67 104024
Corichi A and Gomberoff A 2004 Black holes in de Sitter space: masses, energies and entropy bounds Phys. Rev. D 69 064016
[4] Sekiwa Y 2006 Thermodynamics of de Sitter black holes: thermal cosmological constant Phys. Rev. D 73 084009
[5] Wald R M 1993 Black hole entropy is the Noether charge Phys. Rev. D 48 R3427
Iyer V and Wald R M 1994 Some properties of the Noether charge and a proposal for dynamical black hole entropy Phys. Rev. D 50 846
[6] Bardeen J M, Carter B and Hawking S W 1973 The four laws of black hole mechanics Commun. Math. Phys. 31 161
[7] Smarr L 1973 Mass formula for Kerr black holes Phys. Rev. Lett. 30 71
[8] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[9] Unruh W G and Wald R M 1982 Acceleration radiation and the generalized second law of thermodynamics Phys. Rev. D 25 942
Unruh W G and Wald R M 1983 Entropy bounds, acceleration radiation, and the generalized second law Phys. Rev. D 27 2271
Frolov V P and Page D N 1993 Proof of generalized second law for quasistationary semiclassical black holes Phys. Rev. Lett. 71 3902
Flanagan E E, Marolf D and Wald R M 2000 Proof of classical version of the Bousso entropy bound and of the generalized second law Phys. Rev. D 62 084035
[10] Saida H 2005 Two-temperature steady state thermodynamics for a radiation field Physica A 356 481
Saida H 2006 The generalized second law and the black hole evaporation in an empty space as a nonequilibrium process Class. Quantum Grav. 23 6227
Saida H 2008 Black hole evaporation as a nonequilibrium process Classical and Quantum Gravity Research ed M N Christiansen and T K Rasmussen (New York: Nova Science) chapter 8 (arXiv:0811.1622 [gr-qc])
[11] Hayward G 1990 Euclidean action and the thermodynamics of manifolds without boundary Phys. Rev. D 41 3248
Hawking S W and Ross S F 1995 Duality between electric and magnetic black holes Phys. Rev. D 52 5865