A Note on Black Holes in Asymptotically Lifshitz Spacetime

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Abstract

We investigate several aspects of exact black hole solutions in asymptotically Lifshitz spacetime, which were proposed in 0812.0530. Firstly, we calculate the tidal forces and find that in the near horizon region of such black hole backgrounds, the tidal forces diverge in the near extremal limit. Secondly, we evaluate the Wilson loops in both extremal and finite temperature cases. Finally, we obtain the corresponding shear viscosity and square of the sound speed and find that the ratio of shear viscosity to entropy density takes the universal value $1/4\pi$ in arbitrary dimensions while the square of the speed of sound saturates the conjectured bound $1/3$ in five dimensions.
1 Introduction

By now, the AdS/CFT correspondence [1, 2, 3] is the unique approach which relates strongly coupled field theories to weakly coupled gravity theory. It has been extensively investigated in the past decade and its validity has been widely recognized in the theoretical high-energy physics community. Recently there has been enormous progress on the application of AdS/CFT correspondence, or even the more general gauge/gravity correspondence to physical systems in the real world, such as AdS/QCD and holographic methods for condensed matter physics. Two nice reviews are given by [4, 5].

It is well known that certain questions which are difficult to deal with in the field theory side, become more transparent and more tractable in the gravity side via the AdS/CFT correspondence. In condensed matter physics there are many strongly coupled systems, so it is widely hoped that the AdS/CFT correspondence can provide some useful tools for studying condensed matter physics. Recently interesting gravity models dual to various condensed matter systems have been proposed [6]-[11].

Special attention has been paid to gravity duals of Lifshitz-like fixed points, which is
initially proposed in [12]. One can find critical phenomena with unconventional scaling behavior in many condensed matter systems

\[ t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x, \quad (1.1) \]

where \( z \neq 1 \). A toy model realizing this scaling behavior with \( z = 2 \) is the so-called Lifshitz field theory,

\[ \mathcal{L} = \int d^2x dt ((\partial_t \phi)^2 - \kappa (\nabla^2 \phi)^2). \quad (1.2) \]

The corresponding gravity dual takes the following form [12]

\[ ds^2 = L^2 (-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 dx^2), \quad (1.3) \]

where \( dx^2 = dx_1^2 + \cdots + dx_d^2 \). This metric exhibits the following scale invariance

\[ t \rightarrow \lambda^z t, \quad r \rightarrow \frac{r}{\lambda}, \quad x \rightarrow \lambda x. \quad (1.4) \]

Note that when \( z = 1 \), it turns out to be the usual \( AdS_{d+2} \) spacetime. In four-dimensional spacetime, the corresponding action is a gravity theory with negative cosmological constant, coupled with abelian gauge fields \( A_{(1)}, B_{(2)} \)

\[ S = \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{2} \int (\ast F_{(2)} \wedge F_{(2)} + \ast H_{(3)} \wedge H_{(3)}) - c \int B_{(2)} \wedge F_{(2)}, \quad (1.5) \]

where \( F_{(2)} = dA_{(1)}, H_{(3)} = dB_{(2)} \) and the cosmological constant \( \Lambda = -5/L^2 \).

There are also several generalizations along a similar way. Various anisotropic gravity solutions in general spacetime dimensions with different scaling behavior were discussed in [13]. The aspects of holography in general anisotropic, non-relativistic backgrounds were investigated extensively in [14]. The geometry of Lifshitz spacetime was studied in [15]. Furthermore, the embedding of such anisotropic gravity background with somewhat different scaling behavior into string theory was realized quite recently in [16], where the corresponding black brane configurations were also obtained.

However, although we can study the zero-temperature Lifshitz spacetime, it is difficult to obtain exact black hole solutions in the context of the original action (1.5). The black hole in asymptotically Lifshitz spacetime was constructed in [17] using numerical methods, while Lifshitz topological black holes were obtained in [18]. It should be pointed out that an exact solution of topological Lifshitz black holes was obtained in [18] for certain particular case.
When discussing the aspects of holography in more general anisotropic, non-relativistic backgrounds in [14], an exact solution with finite temperature was obtained by making use of a different action. It can be seen that by performing some coordinate transformation, the black hole solution is asymptotically Lifshitz-like. In this note we discuss several aspects of this exact solution. In section 2 we rewrite the black hole solution in a more transparent way. Then in section 3 we calculate the tidal forces and it turns out that such tidal forces become divergent in the near horizon region, while the horizon area remains large. In this sense, this type of Lifshitz black hole is “naked” [19]. In the next section we evaluate the Wilson loops in this asymptotically Lifshitz black hole background, the results agree with previous examples in the extremal limit and the finite temperature cases are calculated numerically. We discuss the hydrodynamic properties in section 5. It can be shown that the ratio of shear viscosity to entropy density is $1/4\pi$ in arbitrary dimensions and the square of the speed of sound is $1/3$ in five dimensional spacetime, both of which saturate the well-known bounds. A summary and discussion on further directions are given in the final section.

2 The black hole solution

In this section we review the asymptotic Lifshitz solutions proposed in [14], including the extremal solution and black hole solution. We will rewrite the solutions in a more transparent way.

Consider the following action in $(d+2)$-dimensional spacetime

$$S = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{4} e^{\lambda \phi} F_{\mu \nu} F^{\mu \nu} \right],$$

(2.1)

where $\Lambda$ is the cosmological constant and the matter fields are a massless scalar and an abelian gauge field. The equations of motion can be written as follows:

$$\partial_{\mu} (\sqrt{-g} e^{\lambda \phi} F^{\mu \nu}) = 0,$$

(2.2)

$$\partial_{\mu} (\sqrt{-g} \partial^{\mu} \phi) - \frac{\lambda}{4} \sqrt{-g} e^{\lambda \phi} F_{\mu \nu} F^{\mu \nu} = 0,$$

(2.3)

$$R_{\mu \nu} = \frac{2}{d} \Lambda g_{\mu \nu} + \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{2} e^{\lambda \phi} F_{\mu \rho} F^{\rho}_{\nu} - \frac{1}{4d} g_{\mu \nu} e^{\lambda \phi} F_{\rho \sigma} F^{\rho \sigma}.$$  

(2.4)
We make the following ansatz for the metric

\[ ds^2 = L^2 \left[ -r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{d} dx_i^2 \right], \tag{2.5} \]

where \( z \geq 1 \) and the only non-vanishing component of the field strength is \( F_{rt} \).

We can obtain the following expression for \( F_{rt} \) by solving (2.2)

\[ F_{rt} = q e^{-\lambda \phi} r^{z-d-1}, \tag{2.6} \]

where \( q \) is a constant which can be related to the charge of the black hole. Furthermore, solving the \( tt \) and \( rr \) components of (2.4) we can arrive at

\[ \partial_r \phi \partial_r \phi = \frac{2(z - 1) d}{r^2}. \tag{2.7} \]

When \( z = 1 \), it can be easily seen that the solution is \( \phi = \phi_0 = \text{const} \). The full solution can be obtained by solving the remaining equations of motion

\[ ds^2 = L^2 \left[ -r^{2} dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{d} dx_i^2 \right], \]

\[ \phi = \text{const}, \quad F_{rt} = 0, \quad \Lambda = -\frac{d(d+1)}{2L^2}. \tag{2.8} \]

It is simply the AdS solution in Poincaré coordinates. It also admits black hole solution with \( f(r) = 1 - r^{d+1}/r^{d+1} \) and other fields remaining the same as the AdS solution.

When \( z \neq 1 \), from (2.7) we can obtain

\[ \phi = \pm \sqrt{2(z - 1)d} \log r, \tag{2.9} \]

where we have taken the integration constant to be zero without loss of generality. Similarly, we can summarize the extremal solution as follows

\[ ds^2 = L^2 (-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{d} dx_i^2), \]

\[ F_{rt} = q e^{-\lambda \phi} r^{z-d-1}, \quad e^{\lambda \phi} = r^{\lambda \sqrt{2(z-1)d}}, \]

\[ \lambda^2 = \frac{2d}{z-1}, \quad q^2 = 2L^2 (z-1)(z+d), \quad \Lambda = -\frac{(z+d-1)(z+d)}{2L^2}. \tag{2.10} \]
It is just the Lifshitz spacetime with non-trivial dilaton and gauge fields. It should be pointed out that the finite temperature generalization
\[ ds^2 = L^2(-r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{d} dx_i^2), \quad f(r) = 1 - \frac{r^{z+d}_{+}}{r^{z+d}}, \] (2.11)
is also a solution to the equations of motion with the same field configuration. Thus the finite temperature solution is an asymptotically Lifshitz black hole.

Now let us focus on the asymptotically Lifshitz black hole solution. The temperature is
\[ T_H = \frac{(z + d)r^z_+}{4\pi}, \] (2.12)
and the black hole entropy is
\[ S_{BH} = \frac{V_d}{4G_{d+2}} L^d r^d_+, \] (2.13)
where \( V_d \) denotes the volume of the \( d \) dimensional spatial coordinates. One can rewrite the entropy as a function of temperature
\[ S_{BH} = \frac{V_d L^d}{4G_{d+2}} \left( \frac{4\pi}{z + d} \right)^\frac{z}{2} T_H^\frac{d}{2}, \] (2.14)
which exhibits the expected behavior of an anisotropic scale invariant theory.

The thermodynamic quantities can be obtained via the Euclidean path integral method, which were calculated explicitly in [14]. Here we shall not dwell on the details but only list some useful results. Consider the following Euclidean action
\[ I_E = -\frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} [R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda \phi} F_{\mu\nu} F^{\mu\nu}] - \frac{1}{8\pi G_{d+2}} \int d^{d+1}x \sqrt{h} K, \] (2.15)
where the second term is the Gibbons-Hawking boundary term. After substituting the background configuration, the Euclidean action turns out to be
\[ I_E = -\frac{r^{z+d}_{+} L^d V_d \beta_H}{16\pi G_{d+2}}, \] (2.16)
where \( \beta_H = 1/T_H \). We can calculate the other thermodynamic quantities in a standard way as soon as we obtain the Euclidean action. For example, the mass of the black hole is
\[ M = \frac{r^{z+d}_{+} d L^d V_d}{16\pi G_{d+2}}, \] (2.17)
and the charge of the black hole is given by

\[ Q = \frac{1}{32\pi G_{d+2}} \int e^{\lambda \Phi}(\ast F) = \frac{q L^d V_d}{32\pi G_{d+2}}. \]  

(2.18)

From (2.16) we can see that there is no interesting phase structure for this asymptotically Lifshitz black hole, as the Euclidean action is always negative. This can also be seen from the heat capacity

\[ C = \frac{dM}{dT} = \frac{\partial M/\partial r_+}{\partial T/\partial r_+}. \]  

(2.19)

Using (2.12) and (2.17), we can obtain

\[ C = \frac{d V_d r^d_+}{4 \pi G_{d+2}}, \]  

(2.20)

which shows that the black hole is always thermodynamically stable.

\section{3 Tidal forces}

In this section we will calculate the tidal forces of the Lifshitz black hole, following [19]. It has been shown that there exist a class of black holes whose horizon area is large and all curvature invariants are small near the horizon, while any object falling in experiences large tidal forces outside the horizon. As the region of large tidal forces is visible to distant observers, such black holes are called "naked".

Recall the metric

\[ ds^2 = L^2 (-r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{d} dx_i^2), \quad f(r) = 1 - \frac{r_+^{z+d}}{r^{z+d}}, \]  

(3.1)

and the vielbein in the static frame is given as

\[ (e_0)_\mu = -r r f(r)^{1/2} \partial_\mu t, \quad (e_1)_\mu = L r^{-1} f(r)^{-1/2} \partial_\mu r, \quad (e_i)_\mu = L r \partial_\mu x_i. \]  

(3.2)

Consider timelike geodesics in the above background, with proper time \( \tau \) and tangent vector \( u^\mu = dx^\mu/d\tau \). The constants of motion can be written as follows

\[ E = L^2 r^{z-d} f(r) \dot{t}, \quad p_i = L^2 r^2 \dot{x}_i, \]  

(3.3)
where an overdot denotes $d/d\tau$. For simplicity, we just consider radial geodesics, i.e. $p_i = 0$. We can arrive at the following expression due to the normalization condition $u^\mu u_\mu = -1$

$$\dot{r}^2 = \frac{E^2}{L^4 r^{2z-2}} - \frac{r^2}{L^2} f(r). \quad (3.4)$$

The parallel-propagated orthonormal frame $(e_0')_\mu = u_\mu$ can be obtained by a boost of the original static frame

$$(e_0')_\mu = u_\mu = -E \partial_\mu t + \frac{\dot{r} L^2}{r^2 f(r)} \partial_\mu r$$

$$\equiv \cosh \alpha (e_0)_\mu + \sinh \alpha (e_1)_\mu$$

$$(e_1')_\mu = \sinh \alpha (e_0)_\mu + \cosh \alpha (e_1)_\mu, \quad (3.5)$$

where $\cosh \alpha = E[L^2 r^{2z} f(r)]^{-1/2}$ and the other components remain invariant. It can be seen that the boost parameter $\alpha$ diverges at the horizon.

The components of the Riemann curvature in the boosted frame can be calculated by working out the components in the static frame first and then performing some transformations. However, there is another simple route to calculate such quantities which has a more direct physical meaning [19]. We will calculate $R_{0'i0'k}$ which correspond to tidal forces in the transverse directions. Consider a class of radial infalling geodesics whose tangent vector is $u^\mu$ and the deviation vectors are $\eta^i = \partial/\partial x_i$, we have

$$u^\nu \nabla_\nu \eta^\sigma = u^\nu \Gamma^\sigma_{\nu\rho} \eta^\rho = \frac{\dot{H}}{H} \eta^\sigma, \quad (3.6)$$

where $H = L r$. Thus the geodesic deviation equation gives

$$R_{\mu\nu\rho\sigma} u^\mu \eta^\nu u^\rho = -u^\mu \nabla_\mu (u^\nu \nabla_\nu \eta^\sigma) = -\frac{\dot{H}}{H} \eta^\sigma. \quad (3.7)$$

Therefore

$$R_{0'i0'k} = R_{\mu\nu\rho\sigma} u^\mu (e_i)^\nu u^\rho (e_i)_\sigma = -\frac{\dot{H}}{H}$$

$$= \frac{(z - 1) E^2}{L^4 r^{2z}} + \frac{1}{L^2} [1 + \frac{(z + d - 2) r^z + d}{2 r^z + d}]. \quad (3.8)$$

The enhancement of the curvature in the geodesic frame leads to the term proportional to $E^2$ in the above expression. It can be seen that if we take the conserved quantity $E$ to be very large, the tidal force can be made arbitrarily large. Conversely, we can also
make the tidal force very small. Thus in order to avoid such ambiguities, we assume that the conserved quantity \( E \) is chosen to be order one. It is sufficient to keep the term proportional to \( E^2 \) only, as such term represents the difference between the static frame and the boosted frame. Then the tidal force in the near horizon region is given by

\[
R_{0'0'i} = \frac{(z - 1)E^2}{L^4r^2_+}. \tag{3.9}
\]

It can be easily seen that the tidal force vanishes in \( z = 1 \) case then we will consider two different near-extremal limits with the assumption \( z > 1 \) in the following.

- \( r_+ << 1 \) with fixed mass.

Recalling (2.17), we can arrive at the following equation

\[
M_{\text{fixed}} \rightarrow L \sim r^{-(z+d)/d}. \tag{3.10}
\]

Then the tidal force becomes

\[
R_{0'0'i} = (z - 1)E^2r^{4+z/d-2}_+. \tag{3.11}
\]

The horizon area satisfies

\[
A \propto L^d r^d_+ \sim r^{-z^d}_+. \tag{3.12}
\]

So \( r_+ << 1 \) makes the horizon area large. However, when \( d = 2 \), the tidal force is \((z - 1)E^2r^4_+\), that is, the tidal force also turns out to be very small. In this limit the black holes are not “naked”. When \( d > 2 \), the requirement that the tidal force is large gives

\[
4 + 4 \frac{z}{d} - 2z < 0 \quad \rightarrow \quad z > \frac{4d}{2d - 4} = 2 + \frac{8}{2d - 4}. \tag{3.13}
\]

It can be seen that the \( z = 2 \) case can never lead to “naked” black holes.

- \( r_+ << 1 \) without fixing the mass.

Here the requirement that the horizon area is large gives

\[
A \propto L^d r^d_+ >> 1 \quad \rightarrow \quad Lr_+ >> 1. \tag{3.14}
\]

The tidal force turns out to be

\[
R_{0'0'i} \propto \frac{1}{(Lr_+)^4r^{2z-4}_+}. \tag{3.15}
\]

Thus it is possible to have “naked” black holes only in the \( z > 2 \) cases.
In this section we study Wilson loops for asymptotically Lifshitz black holes. The Wilson loops describe the behavior of quarks by hanging strings from the boundary where the quarks locate at the ends of the strings. Although it is quite difficult to embed Lifshitz spacetime into string theory, the calculations presented here can provide some qualitative information. Consider rectangular Wilson loops in Euclidean spacetime, the dynamics is described by the Nambu-Goto action

\[ S = - \int d\sigma \sqrt{\det h_{ab}}, \quad h_{ab} = g_{\mu\nu} \partial_a X^\mu(\tau, \sigma) \partial_b X^\nu(\tau, \sigma), \] (4.1)

where \( X^\mu(\tau, \sigma) \) denote the string coordinates and \( \tau, \sigma \) parametrize the string worldsheet.

In the following we will focus on five-dimensional asymptotic Lifshitz black holes, whose metric is

\[ ds^2 = L^2[-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2f(r)} + r^2(dx_1^2 + dx_2^2 + dx_3^2)].f(r) = 1 - \frac{r_+^{z+3}}{r^{z+3}}. \] (4.2)

Then we can obtain the equations of motion

\[ \left( \frac{r^{2z+2}f(r)x'}{\sqrt{r^{2z-2}r'^2 + r^{2z+2}f(r)x'^2}} \right)' = 0, \]

\[ \left( \frac{r^{2z-2}r'}{\sqrt{r^{2z-2}r'^2 + r^{2z+2}f(r)x'^2}} \right)' = \frac{1}{2} \frac{(2z - 2)r^{2z-3}r'^2 + \partial_r (r^{2z+2}f(r))x'^2}{\sqrt{r^{2z-2}r'^2 + r^{2z+2}f(r)x'^2}}, \] (4.3)

where the prime stands for derivative with respect to \( \sigma \).

One possible static configuration is a pair of straight macroscopic strings which are stretched between \( r = \infty \) and \( r = r_+ \). The corresponding total energy is

\[ E_0 = 2L^2 \int_{r_+}^{\infty} r^{z-1} dr. \] (4.4)

The other possible configuration is a macroscopic U-shape string whose each end is connected to the quark and anti-quark at the boundary. In the static gauge \( \sigma = x \), the equations of motion turn out to be

\[ \left( \frac{r^{2z+2}f(r)}{\sqrt{r^{2z-2}r'^2 + r^{2z+2}f(r)}} \right)' = 0, \]

\[ \left( \frac{r^{2z-2}r'}{\sqrt{r^{2z-2}r'^2 + r^{2z+2}f(r)}} \right)' = \frac{1}{2} \frac{(2z - 2)r^{2z-3}r'^2 + \partial_r (r^{2z+2}f(r))}{\sqrt{r^{2z-2}r'^2 + r^{2z+2}f(r)}}. \] (4.5)
We can arrive at the following result by extremizing the action
\[
\frac{f(r)r^{2z+2}}{\sqrt{r^{2z-2}r'^2 + f(r)r^{2z+2}}} = \text{const} = f_{\text{min}}^{1/2}\frac{r_{\text{min}}}{r_{\text{min}}},
\] (4.6)
where \(r_{\text{min}}\) is \(r\) coordinate of the string tip which is the closest to the horizon and \(f_{\text{min}} = f(r)|_{r=r_{\text{min}}}\). Note that \(\partial r/\partial x = 0\) at \(r = r_{\text{min}}\). From the above expression we can rewrite \(x\) as a function of \(r\)
\[
x = \int_{r_{\text{min}}}^{r} \frac{1}{r^{2}f(r)^{1/2}\sqrt{\left(\frac{r}{r_{\text{min}}}\right)^{2z+2}\left(\frac{f}{f_{\text{min}}}\right)} - 1} dr.
\] (4.7)
Thus the boundary distance between the endpoints of the string is given by
\[
\ell = 2\int_{r_{\text{min}}}^{\infty} \frac{1}{r^{2}f(r)^{1/2}\sqrt{\left(\frac{r}{r_{\text{min}}}\right)^{2z+2}\left(\frac{f}{f_{\text{min}}}\right)} - 1} dr.
\] (4.8)
The total energy of the U-shape string with inter-quark separation \(\ell\) is
\[
E = L^2 \int_{-\ell}^{\ell} dx \sqrt{r^{2z-2}r'^2 + r^{2z+2}f(r)}
\]
\[
= 2L^2 \int dr \frac{r^{4}f(r)^{1/2}}{\sqrt{r^6f(r) - r_{\text{min}}^6f_{\text{min}}}}.
\] (4.9)
Finally, the heavy quark potential is given by
\[
V = E - E_0,
\] (4.10)
where we have subtracted the contribution of two straight strings.

In the extremal background, i.e. \(r_+ = 0\), it can be seen that these results agree with those given in [17] and these expressions reduce to those of [20] and [21] when \(z = 1\). In the finite temperature case [22], the integration can be worked out analytically by making use of the elliptic integral. Unfortunately, here we cannot obtain analytical results thus we have to evaluate the integrals numerically. For simplicity we just consider the five-dimensional case with \(z = 2\), that is, \(f = 1 - \frac{r^5}{r^5}\). The expressions for the boundary distance \(\ell\) and the potential energy can be rewritten as follows
\[
\ell = \frac{2}{r_+} \sqrt{a^6 - a} \int_{a}^{\infty} \frac{\sqrt{y}}{\sqrt{(y^5 - 1)}[(y^6 - a^6) - (y - a)]} dy,
\] (4.11)
Figure 4.1: The boundary distance between the endpoints of a string $\ell$ as a function of $a$, with $r_+ = 1$.

and

$$V = 2r_+^2 L^2 \left[ \int_a^\infty dy \left( \frac{y^{3/2} \sqrt{y^5 - 1}}{\sqrt{(y^6 - a^6) - (y - a)}} - y - \frac{1}{2} (a^2 - 1) \right) \right],$$  

(4.12)

where we have introduced $y \equiv r/r_+$ and $a \equiv r_{\text{min}}/r_+$. Note that the parameter $a$ should be larger than 1, that is, the string always stays outside the horizon.

The boundary distance between the endpoints of a string $\ell$ as a function of $a$ is shown in Fig. 4.1 while the potential energy as a function of the boundary distance $\ell$ is shown in Fig. 4.2. Compared to the results in [22], it can be seen that these functions exhibit similar behavior. The boundary distance between the endpoints of a string has a maximum value $\ell_{\text{max}}$. For a fixed $\ell < \ell_{\text{max}}$, there are two possible U-shape string configurations at two different values of $a$. The energy of the U-shape string is plotted in Fig. 4.2. The configuration with smaller $a$ has a nearly zero potential energy and the configuration with larger $a$ has lower energy. The potential crosses zero at $\ell = \ell_\ast$. The pair of straight strings has lower energy than the U-shape string configuration once $\ell > \ell_\ast$.

5 Hydrodynamics

In this section we discuss the hydrodynamic properties of such asymptotically Lifshitz black holes, including the shear viscosity and the speed of sound. We will see that the
Figure 4.2: The potential energy as a function of the boundary distance $\ell$, with $r_+ = 1$.

ratio of shear viscosity to entropy density is $1/4\pi$ in arbitrary dimensions, which saturates the well known KSS bound [24], while the square of the speed of sound is $1/d$. It should be pointed out that in five-dimensional case, the square of the speed of sound is $1/3$, which also saturates the bound proposed very recently [25], [26].

Firstly let us focus on the shear viscosity $\eta$. We will apply the Kubo formula

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G^R(\omega, \vec{k} = 0),$$

where $G^R$ is the retarded two-point function of the scalar mode of the stress tensor

$$G^R(\omega, \vec{k} = 0) = -i \int d^d x d t e^{i \omega t} \theta(t) < [T_{xy}(t, \vec{x}), T_{xy}(0, 0)].$$

Following the prescription proposed in [27], the linearized Einstein equation for $\phi \equiv h^r_i(r)e^{-i\omega t}$ is the scalar wave function in the same background, due to the $SO(2)$ symmetry of rotations in the $xy$–plane.

Recalling the black hole solution (2.11), we make coordinate transformation $u^2 = r^{z+d}_+/r^{z+d}_-$ for convenience. Then the black hole metric turns out to be

$$ds^2 = L^2 \left[ - \left( \frac{r^{z+d}_+}{u^2} \right)^{\frac{2d}{z+d}} f(u) dt^2 + \frac{4}{(z + d)^2 u^2 f(u)} du^2 + \left( \frac{r^{z+d}_+}{u^2} \right)^{\frac{2d}{z+d}} \sum_{i=1}^{d} dx_i^2 \right], \quad f(u) = 1 - u^2. \tag{5.3}$$

Assuming $\Phi(\omega, u) = \phi(u)e^{-i\omega t}$, the scalar wave equation

$$\Box \Phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) + g^{tt} \partial_t \partial_t \phi$$

\begin{align*}
\frac{d}{dt} \Phi &= i \omega \Phi, \\
\frac{d}{dt} \phi &= i \omega \phi.
\end{align*} \tag{5.4}
\[ u^3 f(u) \partial_u \left( \frac{f(u)}{u} \partial_u \phi \right) + u \frac{4\pi \beta}{\omega^2} \partial^2 \phi = 0, \]  
(5.5)

where \( \beta^{-1} \equiv \frac{1}{4}r_+^2 (z + d) \). Following the standard procedure, we set \( \phi_k = (1 - u)^\alpha \) and require that the most singular terms at \( u = 1 \) cancel as well as the incoming wave boundary condition. These requirements finally fix

\[ \alpha = -\frac{i}{2} \beta \omega. \]  
(5.6)

Next, we choose \( \phi_k(u) = (1 - u)^{-\frac{i}{2} \beta \omega}(1 + \frac{i}{2} \beta \omega F_1(u)) \), then substitute this expression back to the scalar wave function (5.5). Note that in order to calculate the shear viscosity, we just need the perturbation up to the first order of \( \omega \). Furthermore, \( F_1(u) \) should be zero at \( u = 1 \). It can be easily obtained that

\[ \phi_k(u) = (1 - u)^{-\frac{i}{2} \beta \omega}(1 - \frac{i}{2} \beta \omega \ln \frac{1 + u}{2}). \]  
(5.7)

Finally combining the flux factor

\[ F = K \sqrt{-g} g^{uu} \phi_{-k}(u) \partial_u \phi_k(u), \]  
(5.8)

where \( K = -\frac{1}{32\pi G_{d+2}} \) is the coupling constant and the retarded green function

\[ G^R(\omega, \vec{k} = 0) = -2F|_{u \to 0}, \]  
(5.9)

as proposed in [28], we can obtain

\[ \eta = \frac{L^d d^d}{16\pi G_{d+2}}. \]  
(5.10)

Therefore we can arrive at the famous KSS bound

\[ \frac{\eta}{s} = \frac{1}{4\pi}. \]  
(5.11)

For the speed of sound, we first note that the thermodynamic quantities of the black hole should be identified with the quantities in the field theory side as

\[ \{ I_E, M, S_{BH}, T_H \} \leftrightarrow \{ \Omega / T, E, S, T \} \]  
(5.12)

where \( \Omega \) denotes the thermodynamic potential. Recall the results given in Section 2,

\[ I_E = -\frac{r_+^{z+d} L^d V_d \beta_H}{16\pi G_{d+2}}, \quad T_H = \frac{(z + d)r_+^z}{4\pi}, \quad M = \frac{r_+^{z+d} L^d V_d}{16\pi G_{d+2}}. \]  
(5.13)
Then by using the fact that the thermodynamic potential \( \Omega \) is
\[
\Omega = -PV_d,
\]
where \( P \) denotes the pressure, we can obtain
\[
P = \frac{1}{d} \frac{E}{V_d} = \frac{1}{d} \epsilon,
\]
where \( \epsilon \) is the energy density. Thus the speed of sound is given by
\[
c_s^2 = \frac{\partial P}{\partial \epsilon} = \frac{1}{d}.
\]
Note that in five dimensional spacetime, i.e. \( d = 3 \), we have \( c_s^2 = 1/3 \), which saturates the bound conjectured in [25], [26].

## 6 Summary and Discussion

There has been enormous progress on applying the AdS/CFT correspondence, or the more general gauge/gravity correspondence to systems in condensed matter physics. In this note we discuss several aspects of the exact black hole solutions in asymptotically Lifshitz spacetime. We firstly rewrite the solution proposed in [14] in a more convenient way. Then we show that the tidal forces in the near horizon region tend to be infinity in the near-extremal limit, in which sense the black hole is “naked”. We also evaluate the Wilson loops both analytically and numerically in the extremal and finite temperature cases. Finally, we investigate the hydrodynamic properties of the black holes and find that the shear viscosity and the speed of sound both saturate the conjectured bounds.

There are several directions which worth further studying. Firstly, the embedding of the original Lifshitz background (1.1) into string theory is still unknown. However, some Lifshitz backgrounds with different scaling behavior have been realized in string theory in [16], where the configurations were comprised by D3-D7 and D4-D6 branes. It may be expected that we can embed (1.1) into string theory by superposing more different types of D-branes.

Secondly, it has been observed in [29] that the Lifshitz fixed point has ultralocal correlators at finite temperature. Thus it would be interesting to calculation the correlation functions in the black hole background, following the prescription in [28], and compare the results.
with those obtained in the field theory side. Furthermore, it is necessary to build up a systematic holographic renormalization method [30] in such asymptotically Lifshitz background.

Finally, an interesting model of quantum gravity was proposed by Horava quite recently [31]. In $3+1$ dimensions, this theory has a $z=3$ fixed point in the UV and flows to a $z=1$ fixed point in the IR, which is just the classical Einstein-Hilbert gravity theory. Furthermore, it has been found that there exist black hole solutions in Horava-Lifshitz gravity [32]. Thus it is interesting to study the relations between the asymptotically Lifshitz black hole and black holes in Horava-Lifshitz gravity.

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