Mixing, demixing, and structure formation in a binary dipolar Bose-Einstein condensate

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We study static properties of disk-shaped binary dipolar Bose-Einstein condensates of 168Er-164Dy and 52Cr-164Dy mixtures under the action of inter- and intra-species contact and dipolar interactions and demonstrate the effect of dipolar interaction using the mean-field approach. Throughout this study we use realistic values of inter- and intra-species dipolar interactions and the intra-species scattering lengths and consider the inter-species scattering length as a parameter. The stability of the binary mixture is illustrated through phase plots involving number of atoms of the species. The binary system always becomes unstable as the number of atoms increases beyond a certain limit. As the inter-species scattering length increases corresponding to more repulsion, an overlapping mixed state of the two species changes to a separated demixed configuration. During transition from a mixed to a demixed configuration as the inter-species scattering length is increased for parameters just below the stability line, the binary condensate shows special structures in density in the form of red-blood-cell-like biconcave and Saturn-ring-like shapes, which are direct manifestations of dipolar interaction.

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I. INTRODUCTION

After the experimental realization [1–5] of a dipolar Bose-Einstein condensate (BEC) of 52Cr atoms with magnetic moments, there has been renewed interest in the study of static and dynamic properties of such a condensate in the pursuit of novel and interesting properties and features emerging as a consequence of anisotropic long-range dipolar interaction. The atomic interaction in a dilute BEC of alkali-metal atoms is taken as a S-wave short-range (delta-function) potential. However, the anisotropic long-range dipolar interaction is nonlocal in nature acting in all partial waves. More recently, BEC of 164Dy [6,7] and 168Er [8] atoms with larger dipole moments are available for experimental studies, and polar molecules with much larger (electric) dipole moment are being considered [9] for BEC experiments. Among the novel features of a BEC with anisotropic dipolar interaction, one can mention the peculiar shape and stability properties of a stationary state [10], red-blood-cell-like biconcave shape in density due to radial and angular roton-like excitations [11], anisotropic D-wave collapse [12], formation of anisotropic soliton, vortex soliton [13] and vortex lattice [14], anisotropic sound and shock wave propagation [15], and anisotropic Landau critical velocity [10] among others. Distinct stable checkerboard, stripe, and star configurations in dipolar BECs have been identified in a two-dimensional (2D) optical lattice as stable Mott insulator [17] as well as superfluid soliton [15] states. A new possibility of studying universal properties of dipolar BECs for large scattering length has been suggested [19]. Many of these features have only been predicted theoretically, although some of them have already been experimentally confirmed. To enhance the effect of the anisotropic dipolar interaction in most of these theoretical studies, the contact interaction has been set equal to zero. In 52Cr atom, this could be necessary as the strength of the repulsive contact interaction is much stronger than that of the dipolar interaction. However, dipolar BEC of atoms such as 164Dy with much larger dipolar interaction can show effects of anisotropic nonlocal dipolar interaction without switching off the contact interaction.

Now with available dipolar BECs of different species of atoms, there is the possibility to investigate a binary mixture with two dipolar BECs of two types of dipolar interactions acting on each species, e.g., intra- and inter-species, superposed on intra- and inter-species contact interactions, thus creating a much richer platform to study the effect of dipolar interaction. Previously, there have been studies of nondipolar binary boson-boson [20,21], boson-fermion [22] and fermion-fermion [23] as well as dipolar-nondipolar binary boson-boson [24] mixtures. For a binary system without dipolar interactions it is possible to have either a mixed or demixed phase. In this paper, we consider binary dipolar BECs composed of 164Dy and 52Cr atoms as well as 164Dy and 168Er atoms. With all the dipole moments aligned along the z axis, the net dipolar interaction is attractive in the cigar shape along z axis and repulsive in the disk shape confined in the x–y plane, as parallel dipoles in a chain along z axis attract and those confined in the x–y plane repel each other. Consequently, the binary dipolar BEC with dipoles polarized along z axis is more stable in the disk shape confined in the x–y plane compared to the cigar shape along the z axis vulnerable to collapse and we shall consider only such a disk-shaped binary dipolar BEC in

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\[\text{We study static properties of disk-shaped binary dipolar Bose-Einstein condensates of } 168\text{Er-}164\text{Dy and } 52\text{Cr-}164\text{Dy mixtures under the action of inter- and intra-species contact and dipolar interactions and demonstrate the effect of dipolar interaction using the mean-field approach. Throughout this study we use realistic values of inter- and intra-species dipolar interactions and the intra-species scattering lengths and consider the inter-species scattering length as a parameter. The stability of the binary mixture is illustrated through phase plots involving number of atoms of the species. The binary system always becomes unstable as the number of atoms increases beyond a certain limit. As the inter-species scattering length increases corresponding to more repulsion, an overlapping mixed state of the two species changes to a separated demixed configuration. During transition from a mixed to a demixed configuration as the inter-species scattering length is increased for parameters just below the stability line, the binary condensate shows special structures in density in the form of red-blood-cell-like biconcave and Saturn-ring-like shapes, which are direct manifestations of dipolar interaction.}\]

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this study. Even such a disk-shaped binary dipolar BEC is found to be unstable due to collapse instability for dipolar interaction above a critical value controlled by the number of atoms. This is consistent with a similar conclusion that a single-component dipolar BEC for any trap anisotropy collapses above a critical value of dipolar interaction.

We study the limits of stability of binary dipolar disk-shaped BECs formed of $^{168}\text{Er}$ and $^{164}\text{Dy}$ atoms and of $^{52}\text{Cr}$ and $^{164}\text{Dy}$ atoms and present the results in terms of stability phase plots which should aid in the experimental preparation and study of binary dipolar BECs. Of all the atomic interactions, the inter- and intra-species dipolar interactions controlled by the known dipole moments of the two species will be considered known. The contact interactions governed by the approximately known intra-species scattering lengths will also be taken as fixed parameters in this study. We then study the stability of the binary mixture by varying the number of atoms in each species and the yet unknown inter-species scattering length $a_{12}$. For small values of $a_{12}$, the dipolar binary BEC prefers a mixed configuration and with the augmentation of the inter-species repulsion the system moves into a demixed configuration. During this process of mixing-demixing, distinct biconcave red-blood-cell-like structures are found in the densities of the two components just below the line of stability of the binary dipolar BEC for a certain value of inter-species scattering length. This configuration corresponds to saddle-point structures in the 2D densities of the two components in the $x - z$ plane. It was demonstrated that the dipolar interaction corresponds to a saddle structure in 2D responsible for the saddle structure in density in the present study on binary dipolar BEC. The biconcave structure is a consequence of roton instability near the stability line due to dipolar interaction and was studied in detail in single-component dipolar BECs. The Saturn-ring-like density distribution was never observed in a single-component dipolar BEC. To enhance the dipolar effect, in these previous studies the repulsive contact interaction was set to zero, whereas in the present study we set all the intra- and inter-species repulsive contact interactions to their large experimental values and yet obtain these special structures because of the larger values of dipole interactions appropriate for $^{164}\text{Dy}$ and $^{168}\text{Er}$ atoms.

In Sec. II we present the mean-field model for the binary dipolar BEC interacting via inter- and intra-species contact and dipolar interactions in an axially-symmetric confinement. Details of numerics together with an exposition of our numerical results are included in Sec. III. After the application of our model to a simplified binary dipolar BEC without any contact interaction, we present the results for the binary $^{168}\text{Er},^{164}\text{Dy}$ and $^{52}\text{Cr},^{164}\text{Dy}$ mixtures. First, we obtain the stability phase plots of the binary dipolar BECs. Then we demonstrate mixing, demixing, and structure formation in densities in both cases. Close to the stability line, we find Saturn-ring- and red-blood-cell-like density distributions in the components — only possible in the presence of dipolar interaction. Finally, in Sec. III we present a brief summary of our findings.

II. MEAN-FIELD MODEL FOR THE BINARY DIPOLAR BEC

We consider a general two-species dipolar BEC with inter- and intra-species dipolar and contact interactions with the mass, number, magnetic moment, scattering length denoted by $m_i, N_i, \mu_i, a_i, i = 1, 2$, respectively. The angular frequencies for the axially-symmetric trap along $x$, $y$ and $z$ directions are taken as $\omega_{y}^{(i)} = \omega_{y}^{(i)} = \omega_i$ and $\omega_z = \lambda \omega_i$. The intra- and inter-species interactions for two atoms at positions $r_1$ and $r_2$ are taken as

$$V_i(\mathbf{R}) = \frac{\mu_0 \mu_i^2}{4\pi} \frac{1 - 3 \cos^2 \theta}{|\mathbf{R}|^3} + \frac{4\pi \hbar^2 a_i}{m_i} \delta(\mathbf{R}),$$  \hspace{1cm} (1)$$

$$V_{12}(\mathbf{R}) = \frac{\mu_0 \mu_1 \mu_2}{4\pi} \frac{1 - 3 \cos^2 \theta}{|\mathbf{R}|^3} + \frac{2\pi \hbar^2 a_{12}}{m_R} \delta(\mathbf{R}),$$  \hspace{1cm} (2)$$

where $\mathbf{R} = r_1 - r_2$, $\mu_0$ is the permeability of free space, $\theta$ is the angle made by the vector $\mathbf{R}$ with the polarization $z$ direction, and $m_R = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the two species of atoms. With these interactions, the coupled Gross-Pitaevskii (GP) equations for the binary dipolar BEC can be written as

$$i\hbar \frac{\partial \phi_i(r,t)}{\partial t} = \left[ - \frac{\hbar^2}{2m_i} \nabla^2 + \frac{3}{2} m_1 \omega_i^2 (\rho^2 + \lambda_x^2 z^2) + N_1 \frac{\mu_0 \mu_i^2}{4\pi} \int V_{dd}(\mathbf{R})|\phi_1(r',t)|^2 d\mathbf{r}' + \frac{2\pi \hbar^2}{m_R} a_{12} N_2 |\phi_2(r,t)|^2 + \frac{4\pi \hbar^2}{m_1} a_{11} N_1 |\phi_1(r,t)|^2 \right] \phi_i(r,t),$$  \hspace{1cm} (3)$$

$$i\hbar \frac{\partial \phi_2(r,t)}{\partial t} = \left[ - \frac{\hbar^2}{2m_2} \nabla^2 + \frac{3}{2} m_2 \omega_2^2 (\rho^2 + \lambda_x^2 z^2) + N_2 \frac{\mu_0 \mu_2^2}{4\pi} \int V_{dd}(\mathbf{R})|\phi_2(r',t)|^2 d\mathbf{r}' + \frac{2\pi \hbar^2}{m_R} a_{12} N_1 |\phi_1(r,t)|^2 + \frac{4\pi \hbar^2}{m_2} a_{22} N_2 |\phi_2(r,t)|^2 + N_1 \frac{\mu_0 \mu_1 \mu_2}{4\pi} \int V_{dd}(\mathbf{R})|\phi_1(r',t)|^2 d\mathbf{r}' \right] \phi_2(r,t),$$  \hspace{1cm} (4)$$

$$V_{dd}(\mathbf{R}) = \frac{1 - 3 \cos^2 \theta}{|\mathbf{R}|^3},$$  \hspace{1cm} (5)$$

with $\rho^2 = x^2 + y^2$.

To compare the dipolar and contact interactions, the intra- and inter-species dipolar interactions will be expressed in terms the length scales $a_{dd}^{(i)}$ and $a_{dd}^{(12)}$, respectively, defined by

$$\frac{\mu_0 \mu_i^2}{4\pi} = \frac{3\hbar^2}{m_i} a_{dd}^{(i)}, \hspace{1cm} \frac{\mu_0 \mu_1 \mu_2}{4\pi} = \frac{3\hbar^2}{m_1} a_{dd}^{(12)}.$$  \hspace{1cm} (6)
For the intra-species dipolar scale $a_{dd}^{(i)}$, the mass of the corresponding species $m_i$ has been used to define the scale. To define the inter-species dipolar scale $a_{dd}^{(12)}$, we have used the mass $m_1$ of the first species. We express the strengths of the dipolar interactions in Eqs. (3) and (4) by these length scales and transform these equations into the following dimensionless form:

$$i \frac{\partial \phi_1(r, t)}{\partial t} = \left[ -\frac{\nabla^2}{2} + \frac{1}{2} (\rho^2 + \lambda_1^2 z^2) \right] \phi_1(r, t) + g_1 |\phi_1|^2 \phi_1(r, t) + g_{12} |\phi_2|^2 \phi_1(r, t) + \int V_{dd}(R)|\phi_1(r', t)|^2 dr' \phi_1(r, t),$$

$$i \frac{\partial \phi_2(r, t)}{\partial t} = \left[ -\frac{\nabla^2}{2} + \frac{1}{2} (\rho^2 + \lambda_2^2 z^2) \right] \phi_2(r, t) + g_2 |\phi_2|^2 \phi_2(r, t) + g_{12} |\phi_1|^2 \phi_2(r, t) + \int V_{dd}(R)|\phi_2(r', t)|^2 dr' \phi_2(r, t),$$

where $m_{12} = m_1/m_2$, $m_\omega = \omega^2/(m_{12} \omega_1^2)$, $g_1 = 4\pi a_{dd} N_1$, $g_{12} = 3N a_{dd}$, $g_{dd} = 2\pi m_{12} N_1/m_R$, $g_{21} = 2\pi m_{12} N_1/m_R$, $g_{dd} = 3N a_{dd}$, $g_{dd} = 3N a_{dd}$. In Eqs. (7) and (8), length is expressed in units of oscillator length for the first species $l_0 = \sqrt{\hbar/m_\omega}$, energy in units of oscillator energy $h\omega_1$, density $|\phi_i|^2$ in units of $l_0^{-3}$, and time in units of $t_0 = \omega_1^{-1}$.

### III. NUMERICAL RESULTS

For the binary dipolar BEC we solve Eqs. (7) and (8) numerically after discretization [27]. The divergence of the dipolar term at short distances has been handled by treating this term in momentum ($k$) space. The dipolar integral in (Fourier) momentum space is tackled by the following convolution integral [12, 28]:

$$\int dr' V_{dd}(R)n(r') = \int \frac{dk}{(2\pi)^3} e^{-ikr} V_{dd}(k)n(k),$$

with $n(r) = |\phi(r)|^2$. The Fourier transformation (FT) is defined by

$$A(k) = \int dr B(r)e^{-ikr},$$

$$B(r) = \frac{1}{(2\pi)^3} \int dk A(k)e^{-ikr}.$$

The FT $V_{dd}(k)$ of the dipolar potential is known analytically [28]:

$$V_{dd}(k) = \frac{4\pi}{3} \left( \frac{3k^2}{k^2} - 1 \right).$$

![Graph](image_url)

**FIG. 1:** (Color online) The maximum allowed value of net intra-species dipolar nonlinearity $\zeta_{cr} (\equiv 3N a_{dd})$ versus fraction of atoms of the first component $\eta (\equiv N_1/N)$ for a binary dipolar BEC of oppositely polarized dipolar gases in a spherically symmetric trap compared with the findings of Góral and Santos [28].

The FT $n(k)$ of density is calculated numerically by a fast FT (FFT) routine. The inverse FT is also evaluated numerically by the FFT routine. The whole procedure is performed in three-dimensional (3D) Cartesian coordinate system irrespective of the underlying trap symmetry.

#### A. Same species of atoms in isotropic trap

To test the model and the numerical routine, we consider a simple binary dipolar BEC of same species of atoms without contact interaction in a spherically-symmetric trap. However, the two components are considered to be oppositely polarized along $z$ and $-z$ directions, respectively. In this model, first considered by Góral and Santos [28], $m_i = m$, $\omega_i = \omega$, $a_{dd}^{(i)} = -a_{dd}^{(12)} = a_{dd}$, $a_i = 1$.

Consequently, in Eqs. (7) and (8), $g_1 = g_{12} = 0$, $g_{dd}^{(1)} = 3N a_{dd}$, $g_{dd}^{(12)} = -3N a_{dd}$, and $g_{dd}^{(21)} = -3N a_{dd}$. Using $\eta = N_1/N$, $\zeta = 3a_{dd}N/N = N_1 + N_2$ we can write Eqs. (7) and (8) as in Ref. [28]:

$$i \frac{\partial \phi_1}{\partial t} = \left[ -\frac{\nabla^2}{2} + \frac{\rho^2}{2} + \zeta \left( \eta U_1(r) - (1 - \eta)U_2(r) \right) \right] \phi_1,$$

$$i \frac{\partial \phi_2}{\partial t} = \left[ -\frac{\nabla^2}{2} + \frac{\rho^2}{2} + \zeta \left( (1 - \eta)U_2(r) - \eta U_1(r) \right) \right] \phi_2,$$

$$U_i(r) = \int V_{dd}(R) |\phi_i(r')|^2 dr', \quad i = 1, 2.$$
dipolar condensate of same species of atoms, but of oppositely polarized dipolar moments, in a spherically symmetric trap is shown in Fig. 1 in agreement with Góral and Santos [23]. Independent of the composition of the system governed by the parameter η, the system is unstable beyond a critical value of the total number of atoms \( N \).

B. Different species of atoms in disk trap

**Binary \(^{168}\text{Er}-^{164}\text{Dy}\) mixture:** We consider two different binary mixtures of dipolar BECs. First, we consider the \(^{168}\text{Er}-^{164}\text{Dy}\) mixture. In this case we take BEC number 1 as \(^{168}\text{Er}\) and BEC number 2 as \(^{164}\text{Dy}\) with parameters \( \mu_1 = 7\mu_B, \mu_2 = 10\mu_B, a_{(1)}^{(1)} = 66a_0, a_{(2)}^{(12)} = 94a_0, a_{(2)}^{(2)} = 131a_0 \) with \( a_0 \) the Bohr radius and \( \mu_B \) the Bohr magneton. Without accurate experimental estimates [7] of the intra-species scattering lengths, we use \( a_1 = 110a_0 \). The angular frequencies for the axial trap for \(^{164}\text{Dy}\) are taken as \( \omega_2 = 2\pi \times 243 \text{ Hz} \), \( \lambda_2 = \sqrt{10} \approx 3.1623 \), compared with experimental frequencies [2] \( \{f_x, f_y, f_z\} = \{205, 195, 760\} \text{ Hz} \) with trap aspect ratio \( \lambda = 3.8. \) We take \( m_\omega = (m_2\omega_2^2)/(m_1\omega_1^2) = 1 \), corresponding to \( \omega_1 = 2\pi \times 240 \text{ Hz} \), \( \lambda_1 = \sqrt{10} \). The unit of length in this study is \( l_0 = \sqrt{\hbar/(m_1\omega_1)} \approx 0.5 \mu \text{m} \) and the unit of time \( t_0 = \omega_1^{-1} \approx 0.663 \text{ ms} \).

**Binary \(^{52}\text{Cr}-^{164}\text{Dy}\) mixture:** In this case, taking the condensate number 1 as \(^{52}\text{Cr}\) and the one number 2 as \(^{164}\text{Dy}\), the parameters are: \( \mu_1 = 6\mu_B, \mu_2 = 10\mu_B, a_{(1)}^{(1)} = 15a_0, a_{(2)}^{(2)} = 131a_0, a_{(12)}^{(12)} = 25a_0 \). We take \( a_1 = 110a_0 \) close to their experimental estimates [4] [5] [7]. The angular frequencies for the axial trap for \(^{52}\text{Cr}\) are taken as \( \omega_1 = 2\pi \times 195 \text{ Hz} \), \( \lambda_1 = \sqrt{10} \approx 3.1623 \), compared with experimental frequencies [2] \( \{f_x, f_y, f_z\} = \{150, 150, \lambda f_z\} \) Hz with trap aspect ratio \( \lambda < 10 \). We take \( m_\omega = (m_2\omega_2^2)/(m_1\omega_1^2) = 1 \), corresponding to \( \omega_1 = 2\pi \times 110 \text{ Hz} \), \( \lambda_1 = \sqrt{10} \). The unit of length in this calculation is \( l_0 = \sqrt{\hbar/(m_1\omega_1)} \approx 1 \mu \text{m} \) and the unit of time \( t_0 = \omega_1^{-1} \approx 0.816 \text{ ms} \).

C. Stability phase plot

By solving the coupled set of GP equations [7] and [8] for the binary mixture of dipolar BECs with the above parameters we find that the system becomes unstable above a certain number of atoms with the total dipolar interaction beyond a limiting value in agreement with a similar conclusion [25] in a single-component dipolar BEC. To perform a systematic study of the stability of the binary dipolar BEC, we solve Eqs. [7] and [8] with the above parameters, fixing the number of atoms in the first species (\(^{52}\text{Cr\ or \ ^{168}\text{Er}}\) and searching for the critical number of atoms (\( N_{cr} \)) of second species \(^{164}\text{Dy}\) beyond which the system becomes unstable for different values of inter-species scattering length \( a_{12} \). If the binary mixture has a smaller number of atoms of the first species \((^{52}\text{Cr or } ^{168}\text{Er})\), it can accommodate a larger number of atoms of the second species \((^{164}\text{Dy})\) while the limiting value of the total dipolar interaction is reached maintaining all other parameters fixed.

We present the results of our study in stability phase plots in Fig. 2. In this figure we show the critical number of \(^{164}\text{Dy}\) atoms \( N_{cr}(\text{Dy}) \) versus \( a_{12} \) for \( N_{cr}(\text{Dy}) = 1000 \) and \( N_{cr}(\text{Dy}) = 10000 \) \(^{52}\text{Cr\ or \ ^{168}\text{Er}}\) atoms in binary \(^{52}\text{Cr}-^{164}\text{Dy}\) and \(^{168}\text{Er}-^{164}\text{Dy}\) mixtures, respectively. The system is stable below the lines of Fig. 2 and unstable above. Stability phase plots of Fig. 2 could be relevant for planning future experiments on binary dipolar BEC. The system may have distinct structure in density in the shaded regions in this figure. This region is similar to the darker region in Fig. 1 of Ref. [25], where biconcave shape in density of a single-component dipolar BEC appeared due to roton instability. The biconcave shape in density in a disk-shaped dipolar BEC emerges because of dipolar repulsion in the plane of the disk. Due to this repulsion the atoms come to the peripheral region of the disk and a region of low density appears in the center. Similar low-density central region may appear in a rotating BEC due to centrifugal repulsion, as has been found in a binary nondipolar BEC in a rotating trap [21]. We shall see below that bicon-
cave and Saturn-ring-like shapes in density of the components of the binary dipolar BEC may appear in the shaded region in Fig. 2. The system can accommodate a larger number of atoms for large positive values of $a_{12}$ responsible for large contact repulsion which stabilize the binary dipolar BEC. As $a_{12}$ reduces, the contact repulsion reduces and the system becomes more vulnerable to collapse for larger number of atoms due to dipolar interaction and hence can accommodate only a small number of $^{168}$Er atoms as can be seen in Fig. 2.

D. Mixing, demixing, and structure formation

Next we study in detail the density of the two species of atoms in the binary dipolar BEC to look for mixing (overlapping phases of two components), demixing (separated phases) and distinct structure formation in the shaded regions of the phase plots shown in Fig. 2. For small $a_{12}$ the inter-atomic contact repulsion between the two species is small and one has a mixed configuration. As $a_{12}$ is increased the inter-species contact repulsion increases and for a large enough value of $a_{12}$ one can have demixing of the two species. In the demixed configuration one of the species occupy the central region and the other the external region. To illustrate this phenomenon, we plot in Fig. 3 the reduced 2D density in the radial direction $|\Phi(x,y=0)|^2 = \int dz |\phi(x,y=0,z)|^2$ versus $x$ under different situations in case of the binary $^{168}$Er-$^{164}$Dy mixture. The mixing-demixing is illustrated in Figs. 3 (a), (b), and (c) for 10000 $^{168}$Er and 20000 $^{164}$Dy atoms, respectively, for $a_{12} = 110a_0$, 125$a_0$, and 130$a_0$. In Fig. 3 (a) we have a mixed (overlapping) configuration and with the increase of inter-species contact repulsion, in Fig. 3 (c) we have a demixed (separated) configuration of the two species. In Fig. 3 (b), for $a_{12} = 125a_0$ we have a partially demixed configuration. In the demixed configuration in Fig. 3 (c), 20000 $^{164}$Dy atoms have a larger contribution to energy of the system and the $^{164}$Dy atoms expel the $^{168}$Er atoms from the central region due to inter-species repulsion. The opposite panorama is also possible for a larger number of $^{168}$Er atoms, while the $^{168}$Er atoms can expel the $^{164}$Dy atoms from the central region. Such a situation is illustrated in Fig. 3 (d) with 10000 $^{168}$Er atoms and 500 $^{164}$Dy atoms for $a_{12} = 160a_0$. In this case $^{168}$Er atoms have a larger contribution to the energy of the system and expel $^{164}$Dy atoms from the center.

A careful examination of the passage from the partially mixed configuration of Fig. 3 (b) to the demixed configuration of Fig. 3 (c) reveal interesting feature. To study this, we consider the 3D profile of the density $|\phi(x,y,z)|^2$ of the two species for the cases depicted in Figs. 3 (b) and (c). In Figs. 4 (a) and (b) we plot the 3D contours of $^{168}$Er and $^{164}$Dy BEC profiles for the parameters of Fig. 3 (b). The same for the parameters of Fig. 3 (c) are illustrated in Figs. 4 (c) and (d). The density $|\phi(x,y,z)|^2$ on the surface of these plots is 0.0005/$l_0^3$. With $l_0 = 0.5 \mu m$, and $N(\text{Er}) = 10000$, and $N(\text{Dy}) = 20000$ this leads to atom densities on contour of $40 \mu m^{-3}$ for $^{168}$Er and of $80 \mu m^{-3}$ for $^{164}$Dy. The maxima of atom density in the interior of the BECs of Figs. 3 (a) – (d) are, respectively, 800 $\mu m^{-3}$, 1160 $\mu m^{-3}$, 460 $\mu m^{-3}$, and 1000 $\mu m^{-3}$. In Figs. 4 (a) and
the \(^{168}\text{Er}\) and \(^{164}\text{Dy}\) BECs have Saturn-ring-like and red-blood-cell-like biconcave profiles, respectively. The biconcave profile is a manifestation of the inter-species dipolar interaction: the net dipolar repulsion in \(^{164}\text{Dy}\) in disk shape drives the dysprosium atoms to peripheral region thus creating the biconcave shape as in the single-component dipolar BEC. In the single-component case the biconcave profile near the stability limit has been related to roton instability \(^{25}\). The Saturn-ring-like \(^{168}\text{Er}\) profile in Fig. 3(a) is a manifestation of the inter-species dipolar interaction: the biconcave density distribution of the \(^{164}\text{Dy}\) species with a density minimum at center, due to inter-species repulsion, expels a part of the \(^{168}\text{Er}\) BEC to the central region of lower density and another part to the peripheral region, also of lower density, thus creating the Saturn-ring-like profile. Such a profile is not possible in a single-component dipolar BEC, or in a binary BEC without dipolar interaction. With further increase of inter-species contact repulsion, for \(a_{12} = 130\alpha_0\), the contact repulsions dominate over the dipolar interactions and hence plays a major role thus creating a simpler situation of demixing shown in Fig. 3(c) and Figs. 4(c) and (d), where the net inter-species repulsion is so strong that all \(^{168}\text{Er}\) atoms are driven to the peripheral region in the form of a ring with the dominant \(^{164}\text{Dy}\) atoms occupying the central region. The ring-like structure in Fig. 4(a) as opposed to a shell-like demixed configuration is a consequence of the dipolar interaction: a pure inter-species contact repulsion would have led to a hollow shell-like configuration for \(^{168}\text{Er}\) surrounding a compact disk-shaped \(^{164}\text{Dy}\) core. The dipolar interaction is attractive in the \(z\) direction and this transforms the shell-like configuration into a ring. Such transition from a shell-like configuration to a ring with the increase of dipolar interaction has been demonstrated in a single-component dipolar BEC in a shell-like trap \(^{29}\).

From Figs. 3 we find that for a binary dipolar BEC composed of 10000 \(^{168}\text{Er}\) atoms and 20000 \(^{164}\text{Dy}\) atoms the transition from a mixed to a demixed configuration happens around \(a_{12} = 120\alpha_0\). We verified that similar transition also takes place in a binary dipolar BEC for 10000 \(^{168}\text{Er}\) atoms and a reduced number of \(^{164}\text{Dy}\) atoms also for \(a_{12} \approx 120\alpha_0\). Nevertheless, the biconcave structure in the density of \(^{164}\text{Dy}\) is obtained for a small shaded region below the stability line in Fig. 2(a) with large number of \(^{164}\text{Dy}\) atoms. As in the single component case \(^{29}\), the binary dipolar BEC possessing biconcave shape in density in the \(^{164}\text{Dy}\) BEC in the shaded region in Fig. 2(a) is usually transient and instability appears after crossing the stability lines in Fig. 2. In the case of a single-component dipolar BEC a conveniently chosen initial density could be necessary for obtaining the biconcave structure in density. In imaginary-time propagation the initial density should either be chosen as a Gaussian with widths slightly larger than the actual widths of the desired state. The present structures were obtained by using imaginary time propagation of the coupled mean-field equations with Gaussian profiles for initial densities. We started with the Gaussian profiles of the linear oscillator states and increased the dipolar and nondipolar nonlinearities in small steps during numerical simulation until the final nonlinearities are reached. In this fashion the structure in density in the shaded regions of phase plots in Fig. 2 are obtained.

To gain further insight into the density distributions in Figs. 3(a) – (d) we plot in Figs. 5(a) – (d) contour plots of asymmetric 2D density \(\psi(x, y, z)^2 = \int dy |\psi(x, y, z)|^2\) of the binary \(^{168}\text{Er}^{164}\text{Dy}\) mixture for parameters corresponding to Figs. 4(a) – (d), respectively. The double and single saddle structures in density in the \(x - z\) plane in Figs. 5(a) and (b) are manifestations of the dipolar interaction. In these plots a minimum in density along \(x\) direction coincides with a maximum in density in \(z\) direction thus creating a saddle point, which appears as a clear manifestation of the saddle-shaped dipolar interaction \(^{3}\). Such a density distribution is not possible in the absence of dipolar interaction and to the best of our knowledge has not been demonstrated even in the single-component dipolar BEC. In Fig. 3(a), for \(a_{12} = 125\alpha_0\), we have two saddle points on the \(x\) axis in the density of \(^{168}\text{Er}\) and one saddle point in the density of \(^{164}\text{Dy}\). Similarly, for \(a_{12} = 130\alpha_0\), we have a single saddle point only in the density of \(^{168}\text{Er}\) and none in \(^{164}\text{Dy}\). These saddle points have curvatures consistent with saddle-shaped
dipolar interaction $\mathbf{3}$ with negative curvature along the direction of magnetization ($z$ axis) and positive curvature orthogonal to it ($x$ axis).

In Fig. 6 we plot the chemical potential $\mu$ and root mean square (rms) sizes $\langle z \rangle$, and $\langle \rho \rangle$ versus inter-species scattering length $a_{12}$, for (a) $N(\text{Er}) = N(\text{Dy}) = 10000$; (b) $N(\text{Er}) = 10000$, $N(\text{Dy}) = 100000$; (c) $N(\text{Er}) = 1000$, $N(\text{Dy}) = 10000$. It is found that the chemical potential and rms sizes are slowly varying functions of inter-species scattering length. As expected, the chemical potential of the first species increases more rapidly with $a_{12}$ when there are more atoms in the second species and vice versa.

With this study of the binary $^{168}\text{Er}-^{164}\text{Dy}$ BEC, we also considered a detailed investigation of the binary $^{52}\text{Cr}-^{164}\text{Dy}$ BEC. Similar mixing, demixing, and structure formation are also found in this second case, although the quantitative results and estimates are different. We only report here the studies on the interesting structure formation in the binary $^{52}\text{Cr}-^{164}\text{Dy}$ BEC. Indeed we find that one can have the condensate with biconcave shape in this case. In Figs. 7 (a) and (b) we plot the linear density in the $x - y$ plane along the $x$ direction $|\Phi(x, y = 0)|^2$, as in Fig. 3, for a binary mixture of 10000 $^{52}\text{Cr}$ atoms and 60000 $^{164}\text{Dy}$ atoms for $a_{12} = 70a_0$ and 80$a_0$, respectively. Figure 7 (a) illustrates a case similar to Fig. 5 (b) denoting a transition from a mixed state to a demixed state. For $a_{12} < 60a_0$, we have a fully mixed configuration of the two condensates as in Fig. 3 (a). For $a_{12} = 70a_0$ a biconcave shape has appeared in the density of $^{52}\text{Cr}$. For $a_{12} > 750a_0$, a complete demixed configuration appears as illustrated in Fig. 7 (b) for $a = 80a_0$, where the $^{52}\text{Cr}$ atoms are expelled from the central region occupied by only $^{164}\text{Dy}$ atoms. In Figs. 7 (c) and (d) we show the 3D contour of the density $|\phi(x, y, z)|^2$ of $^{52}\text{Cr}$ and $^{164}\text{Dy}$ BECs corresponding to the parameters of Fig. 7 (a) with the density $0.0005l_0^{-3}$ on contour. This corresponds to number densities on the contour of $5\mu m^{-3}$ and $30\mu m^{-3}$ in $^{52}\text{Cr}$ and $^{164}\text{Dy}$, respectively, for $l_0 = 1\mu m$. The $^{52}\text{Cr}$ BEC shows a pronounced biconcave shape which is a direct manifestation of the dipolar interaction. However, we could not find a biconcave shape in the density of the $^{164}\text{Dy}$ BEC, as in the case of a binary dipolar $^{168}\text{Er}-^{164}\text{Dy}$ mixture, possibly because the inter-species dipole interaction in the present system is much smaller in strength compared to the same in the binary $^{168}\text{Er}-^{164}\text{Dy}$ system so as to create the biconcave shape in density in the presence of dominating large contact repulsions acting on $^{164}\text{Dy}$. The much larger dipole...
interactions in $^{164}$Dy might be responsible for the biconcave shape in density in Fig. 4 (b). Even in the single-component case, the biconcave profile in density appears for certain values of the trap aspect ratio $\frac{80}{0}$, and it

is also possible that for other values of trap aspect ratio the biconcave profile might appear in the density of $^{164}$Dy in the present binary $^{52}$Cr-$^{164}$Dy mixture. However, the dipolar interaction leaves its signature in a different fashion as can be seen in Figs. 7 (e) and (f), where we plot the same densities of Figs. 7 (c) and (d) but now with the density $0.0025 l_0^{-3}$ on the contour. This corresponds to number densities on the contour of 25 $\mu m^{-3}$ and 150 $\mu m^{-3}$ in $^{52}$Cr and $^{164}$Dy, respectively, for $l_0 = 1 \mu m$. The Figs. 7 (e) and (f) thus show the structure in the interior of the BECs illustrated in Figs. 7 (c) and (d). Along the radial direction in the $x$-$y$ plane, we find in Figs. 7 (e) and (f) that the maximum of density of one component is accompanied by the minimum of density of the other component due to interspecies contact repulsion and the density of one component may have several local maxima. The global maxima in densities of Figs. 8 (c) and (f) are $0.0064 l_0^{-3}$ and $0.013 l_0^{-3}$, respectively, corresponding to number densities of 64 $\mu m^{-3}$ and 780 $\mu m^{-3}$ for $^{52}$Cr and $^{164}$Dy.

Finally, we study the distinct structure in densities of binary dipolar BECs with a smaller number $^{166}$Er atoms in the shaded region of Fig. 2 (b). In particular we consider the partially mixed configuration with $a_{12} = 110 a_0$, $N(Er) = 1000$, $N(Dy) = 50000$. In Fig. 8 (a), we illustrate the 2D radial density along $x$ axis $|\Phi(x,y=0)|^2 = \int dz |\phi(x,y=0,z)|^2$ of this binary $^{166}$Er-
\(^{164}\)Dy BEC. In Figs. 8(b) and (c) the contour plots of the corresponding 3D densities of \(^{168}\)Er and \(^{164}\)Dy are shown with a cut-off density 0.0025\(\frac{l}{a^3}\) on the contour. The maximum densities inside the \(^{168}\)Er and \(^{164}\)Dy BECs are 0.014\(\frac{l}{a^3}\) and 0.0038\(\frac{l}{a^3}\), respectively. In this case a biconcave shape has appeared in the density profile of \(^{168}\)Er BEC in Fig. 8(b). A Saturn-anel-like profile in the density of \(^{164}\)Dy BEC is shown in Fig. 8(c). The central high-density region of \(^{164}\)Dy BEC has expelled the \(^{168}\)Er BEC from the central region, thus creating a biconcave shape in density. If we compare Figs. 8(a) and (b) with Figs. 8(b) and (c), we find that the roles of \(^{164}\)Dy and \(^{168}\)Er BECs have interchanged. In Fig. 8(a), the \(^{168}\)Er BEC has a Saturn-anel-like profile, whereas, in Fig. 8(c), the \(^{164}\)Dy BEC has similar profile. These profiles are a consequence of dipolar interaction.

IV. SUMMARY

Using a mean-field description we studied the static properties of binary disk-shaped dipolar BECs \(^{168}\)Er-\(^{164}\)Dy and \(^{52}\)Cr-\(^{164}\)Dy to search for the effect of inter-and intra-species dipolar interactions employing realistic values of inter- and intra-species dipolar interactions as of intra-species scattering lengths. The yet unknown inter-species scattering length is considered as a variable parameter. The binary system is found to be stable for number of atoms below a critical value. The stability domain for the system is illustrated in convenient phase plots involving number of atoms of the species and the inter-species scattering length \(a_{12}\). For small values of \(a_{12}\) a mixed state (with overlapping phase) of the two species emerges and this state transforms into a demixed state (with separated phase) for larger \(a_{12}\). Just below the stability line, in the region of transition from the mixed to a demixed configuration distinct structures in 3D densities may appear in the form of the Saturn-ring-like shape or red-blood-cell-like biconcave shape. Such structures are a direct manifestations of the dipolar interaction. Similar biconcave shape in the density of a single-component dipolar BEC was found and related to roton instability, which is presumably also responsible for these structures in the binary dipolar BECs. After this investigation was finished we came to know about another recent work on binary dipolar BEC in 2D [30].

Acknowledgments

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[1] T. Lahaye et al., Rep. Prog. Phys. 72, 126401 (2009).
[2] T. Lahaye et al., Nature (London) 448, 672 (2007); A. Griesmaier et al., Phys. Rev. Lett. 97, 250402 (2006).
[3] J. Stuhler et al., Phys. Rev. Lett. 95, 150406 (2005).
[4] K. Goral, K. Rzazewski, and T. Pfau, Phys. Rev. A 61, 051601 (2000).
[5] T. Koch et al., Nature Phys. 4, 218 (2008).
[6] M. Lu, S. H. Youn, and B. L. Lev, Phys. Rev. Lett. 104, 063001 (2010); J. J. McClelland and J. L. Hanssen, Phys. Rev. Lett. 96, 143005 (2006); S. H. Youn, M. W. Lu, U. Ray, and B. V. Lev, Phys. Rev. A 82, 043425 (2010).
[7] M. Lu, N. Q. Burdick, Seo Ho Youn, and B. L. Lev, Phys. Rev. Lett. 107, 190401 (2011).
[8] K. Aikawa et al., Phys. Rev. Lett. 108, 210401 (2012).
[9] J. Doyle, B. Friedrich, R. V. Krems, and F. and Masnou-Seeuws, Eur. Phys. J. D 31, 149 (2004); J. Deiglmayr et al., Phys. Rev. Lett. 101, 133004 (2008); M. H. de Miranda et al., Nature Phys. 7, 502 (2011).
[10] N. G. Parker, C. Ticknor, A. M. Martin, and D. H. J. O’Dell, Phys. Rev. A 79, 013617 (2009); R. M. Wilson, S. Ronen, and J. L. Bohn, Phys. Rev. A 80, 023614 (2009); N. G. Parker and D. H. J. O’Dell, Phys. Rev. A 78, 041601 (2008); L. Santos, G. V. Shlyapnikov, P. Zoller, and M. Lewenstein, Phys. Rev. Lett. 85, 1791 (2000); C. Ticknor, N. G. Parker, A. Melatos, S. L. Cornish, D. H. J. O’Dell, and A. M. Martin, Phys. Rev. A 78, 061607 (2008); R. M. W. van Bijnen, A. J. Dow, D. H. J. O’Dell, N. G. Parker, and A. M. Martin, Phys. Rev. A 80, 033617 (2009); A. Junginger, J. Main, G. Wunner, and T. Bartsch, Phys. Rev. A 86, 023632 (2012).
[11] L. Santos, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 90, 250403 (2003); R. M. Wilson, S. Ronen, J. L. Bohn, and H. Pu, Phys. Rev. Lett. 100, 245302 (2008); M. Asad-uz-Zaman and D. Blume, Phys. Rev. A 83, 033616 (2011).
[12] T. Lahaye et al., Phys. Rev. Lett. 101, 080401 (2008).
[13] I. Tikhonenkov, B. A. Malomed, and A. Vardi, Phys. Rev. Lett. 100, 090406 (2008); S. K. Adhikari and P. Muruganandam, J. Phys. B 45, 045301 (2012); L. E. Young-S, P. Muruganandam, and S. K. adhikari, J. Phys. B 44, 101001 (2011); P. Muruganandam and S. K. adhikari, J. Phys. B 44, 121001 (2011); P. Köberle, D. Zajec, G. Wunner, and B. A. Malomed, Phys. Rev. A 85, 023630 (2012).
[14] R. M. W. van Bijnen, D. H. J. O’Dell, N. G. Parker, and A. M. Martin, Phys. Rev. Lett. 98, 150401 (2007); R. Kishor Kumar, P. Muruganandam, J. Phys. B 45, 215301 (2012); M. Abad, M. Guilleumas, R. Mayol, M. Pi, and D. M. Jezek, Phys. Rev. A 79, 063622 (2009).
[15] P. Muruganandam and S. K. Adhikari, Phys. Lett. A 376, 480 (2012); C. Krumnow and A. Pelster, Phys. Rev. A 84, 021608 (2011); G. Bismut, B. Laburthe-Tolra, E. Maréchal, P. Pedri, O. Corkin, and L. Vernac, Phys. Rev. Lett. 108, 155302 (2012).
[16] R. M. Wilson, S. Ronen, and J. L. Bohn, Phys. Rev. Lett. 104, 094501 (2010).
[17] B. Capogrosso-Sansone, C. Trefzger, M. Lewenstein, P. Zoller, and G. Pupillo, Phys. Rev. Lett. 104, 125301 (2010).
[18] K. Lakomy, R. Nath, and L. Santos Phys. Rev. A 85, 033618 (2012); S. K. Adhikari and P. Muruganandam, Phys. Lett. A 376, 2200 (2012).
[19] L. E. Young-S, S. K. adhikari, and P. Muruganandam, Phys. Rev. A 85, 033619 (2012).
[20] E. Timmermans, Phys. Rev. Lett. 81, 5718 (1998); A. A. Svidzinsky and S. T. Chui, Phys. Rev. A 68, 013612 (2003); G. Gligorić, A. Maluckov, M. Stepić, L. Hadzievski, and B. A. Malomed, Phys. Rev. A 82, 033624 (2010); P. Kuopanportti, J. A. M. Huhtamäki, and M. Möttönen, Phys. Rev. A 85, 043613 (2012); Kui-Tian Xi, Jinbin Li, and Da-Ning Shi, Phys. Rev. A 84, 013619 (2011).
[21] I. Corro, R. G. Scott, and A. M. Martin, Phys. Rev. A 80, 033609 (2009).
[22] Y. Takeuchi and H. Mori, Phys. Rev. A 72, 063617 (2005); S. K. Adhikari and L. Salasnich, Phys. Rev. A 75, 053603 (2007).
[23] S. K. Adhikari, Phys. Rev. A 73, 043619 (2006).
[24] H. Saito, Y. Kawaguchi, and M. Ueda, Phys. Rev. Lett. 102, 230403 (2009).
[25] S. Ronen, D. C. E. Bortolotti, and J. L. Bohn, Phys. Rev. Lett. 98, 030406 (2007).
[26] S. Yi and L. You, Phys. Rev. A 63, 053607 (2001).
[27] P. Muruganandam and S. K. Adhikari, Comput. Phys. Commun. 180, 1888 (2009); D. Vudragovic, I. Vidanovic, A. Balaz, P. Muruganandam, and S. K. Adhikari, Comput. Phys. Commun. 183, 2021 (2012).
[28] K. Goral and L. Santos, Phys. Rev. A 66, 023613 (2002).
[29] S. K. Adhikari, Phys. Rev. A 85, 053631 (2012); F. Malet, G. M. Kavoulakis, and S. M. Reimann, Phys. Rev. A 84, 043626 (2011).
[30] R. M. Wilson, C. Ticknor, J. L. Bohn, E. Timmermans, Phys. Rev. A 86, 033606 (2012).