Proximal determination of convex functions

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Abstract

We provide comparison principles for convex functions through its proximal mappings. Consequently, we prove that the norm of the proximal operator determines a convex function up to a constant. A new characterization of Lipschitzianity in terms of the proximal operator is given.

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. By determination of a convex function $f: \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, we mean a result of type “if $f$ satisfies a given condition, then $f$ is uniquely determined up to constant.” The first determination result was proved by J.J. Moreau in Hilbert spaces (see [5, p.287]):

**Theorem 1.1** (Moreau). If $f, g: \Gamma_0(\mathcal{H})$ are two functions such that

$$\text{prox}_f(x) = \text{prox}_g(x)$$

for all $x \in \mathcal{H},$

then $f$ and $g$ differ by a constant.

Moreau used the latter result to prove that the subgradients uniquely determine a convex function, which is known as an integration result. Since then, several integration results appeared for convex and nonconvex functions (see, e.g., [7, 3, 8]). In this paper, by using a recent result on the determination of convex functions [6], we provide a new determination result by showing that the norm of the proximal operator determines a convex function up to a constant (Proposition 4.1 and Theorem 4.1). For this, we establish comparison principles for convex functions through its proximal mapping (Theorem 3.1), which is also used to obtain a new characterization of Lipschitzianity (Proposition 3.2).

The paper is organized as follows. After some preliminaries, in Section 3 we present comparison principles for convex functions in terms of its proximal operators and a new characterization of Lipschitzianity through proximal operators. These principles are the basis of the developments of Section 4, where it is shown that the norm of the proximal operator determines a convex function completely, up to a constant.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. We denote by $\Gamma_0(\mathcal{H})$ the set of all proper, convex and lower semicontinuous functions from $\mathcal{H}$ with values in $\mathbb{R} \cup \{+\infty\}$. For $f \in \Gamma_0(\mathcal{H})$, its Legendre-Fenchel conjugate function $f^*: \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is given by

$$f^*(x^*) = \sup_{v \in \mathcal{H}} \{ \langle x^*, v \rangle - f(v) \}.$$

It is known that $f^* \in \Gamma_0(\mathcal{H})$ and that for every $(x, x^*) \in \mathcal{H} \times \mathcal{H}$, the Legendre-Fenchel inequality holds, that is

$$f(x) + f^*(x^*) \geq \langle x^*, x \rangle.$$

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For a closed set $C$, we denote by $\delta_C$ the indicator function of $C$, that is, $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \not\in C$. It is clear that $\delta_C \in \Gamma_0(\mathcal{H})$ if and only if $C$ is closed and convex. Moreover, $(\delta_C)^* = \sigma_C$, where $\sigma_C$ is the support function of $C$ defined by $\sigma_C(x) = \sup_{y \in C} \langle y, x \rangle$.

For $\lambda > 0$, the Moreau envelope of $f$ of index $\lambda$ is the function $f_\lambda : \mathcal{H} \to \mathbb{R}$ given by

$$f_\lambda(x) := \inf_{y \in \mathcal{H}} \left\{ f(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\}.$$  

The above infimum is attained at a unique point, $\text{prox}_\lambda f(x)$. The mapping $\text{prox}_\lambda f : \mathcal{H} \to \mathcal{H}$ is non-expansive and for $\lambda = 1$ it is called the proximal operator, that is,

$$\text{prox}_f(x) = \arg\min_{y \in \mathcal{H}} \left\{ f(y) + \frac{1}{2} \|x - y\|^2 \right\}.$$  

It is known that $f_\lambda$ is convex, continuously differentiable on $\mathcal{H}$, and its derivative is given by

$$\nabla f_\lambda(x) = \frac{1}{\lambda} (x - \text{prox}_\lambda f(x)) \text{ for all } x \in \mathcal{H} \tag{1}$$

Moreover,

$$(f_\lambda)^*(x) = f^*(x) + \frac{1}{2\lambda} \|x\|^2 \text{ for all } x \in \mathcal{H}. \tag{2}$$

We refer to [1] for more details of Moreau envelope and its applications.

To obtain our results, we need the Moreau decomposition (see [2, p. 280]).

**Proposition 2.1** (Moreau decomposition). If $f \in \Gamma_0(\mathcal{H})$, then

$$\text{prox}_f(x) + \text{prox}_{f^*}(x) = x \quad \text{for all } x \in \mathcal{H}.$$  

We end this section with a comparison principle for convex functions through its gradients (see [3, Theorem 3.1]). This principle is the basis for the determination of convex functions through the norm of (sub)gradients. We refer to [4] for further results in this direction.

**Proposition 2.2.** Let $f, g \in \Gamma_0(\mathcal{H})$ be two Gâteaux differentiable convex functions bounded from below such that

$$\|\nabla f(x)\| \leq \|\nabla g(x)\| \quad \text{for all } x \in \mathcal{H}.$$  

Then, $f - \inf f \leq g - \inf g$.

### 3 Comparison principles

The following result is a comparison principle for convex functions.

**Theorem 3.1.** Let $f, g : \Gamma_0(\mathcal{H})$ be two functions such for some $x_0 \in \text{dom } f \cap \text{dom } g$ and

$$\|\text{prox}_f(x) - x_0\| \leq \|\text{prox}_g(x) - x_0\| \quad \text{for all } x \in \mathcal{H}.$$  

Then, $g - g(x_0) \leq f - f(x_0)$.

**Proof.** By virtue of Legendre-Fenchel inequality, for all $x, u \in \mathcal{H}$

$$f^*(x) + f(u) \geq \langle x, u \rangle \quad \text{and} \quad g^*(x) + g(u) \geq \langle x, u \rangle.$$

Thus, if $x_0 \in \text{dom } f \cap \text{dom } g$, then

$$f^*(x) - \langle x, x_0 \rangle \geq -f(x_0) \quad \text{and} \quad g^*(x) - \langle x, x_0 \rangle \geq -g(x_0),$$

for all $x \in \mathcal{H}$. 

$$2$$
Then, due to (2),

\[ f = (f^* - \langle x_0, \cdot \rangle)_\lambda \quad \text{and} \quad g = (g^* - \langle x_0, \cdot \rangle)_\lambda. \]

Then, \( \tilde{f} \) and \( \tilde{g} \) are \( C^{1,1} \) and bounded from below functions with

\[ \nabla \tilde{f}(x) = x - \text{prox}_{f^* - \langle x_0, \cdot \rangle}(x) \quad \text{and} \quad \nabla \tilde{g}(x) = x - \text{prox}_{g^* - \langle x_0, \cdot \rangle}(x). \]

Moreover, according to Moreau’s decomposition and properties of the proximal operator, for all \( x \in H \)

\[ \nabla \tilde{f}(x) = x - \text{prox}_{f^* - \langle x_0, \cdot \rangle}(x) = \text{prox}_{f^*}(x + x_0) = \text{prox}_{f^*}(x + x_0) - x_0, \]

\[ \nabla \tilde{g}(x) = x - \text{prox}_{g^* - \langle x_0, \cdot \rangle}(x) = \text{prox}_{g^*}(x + x_0) = \text{prox}_{g^*}(x + x_0) - x_0. \]

Therefore, for all \( x \in H \)

\[ \| \nabla \tilde{f}(x) \| \leq \| \nabla \tilde{g}(x) \|. \]

Hence, by virtue of Proposition 2.2,

\[ \tilde{f} \leq \tilde{g} + \inf \tilde{f} - \inf \tilde{g} = \tilde{g} - f(x_0) + g(x_0), \]

where we have used that

\[ \inf \tilde{f} = \inf (f^* - \langle x_0, \cdot \rangle) = -f^{**}(x_0) = -f(x_0) \quad \text{and} \quad \inf \tilde{g} = \inf (g^* - \langle x_0, \cdot \rangle) = -g^{**}(x_0) = -g(x_0). \]

Then, by conjugation, we obtain that

\[ (\tilde{g})^* \leq (\tilde{f})^* - f(x_0) + g(x_0). \]

Then, due to Proposition 3.1

\[ (\tilde{f})^*(x) = (f^* - \langle x_0, \cdot \rangle)^*(x) + \frac{1}{2} \| x \|^2 = f(x + x_0) + \frac{1}{2} \| x \|^2, \]

\[ (\tilde{g})^*(x) = (g^*(x) - \langle x_0, \cdot \rangle)^*(x) + \frac{1}{2} \| x \|^2 = g(x + x_0) + \frac{1}{2} \| x \|^2. \]

Hence,

\[ g(x + x_0) \leq f(x + x_0) - f(x_0) + g(x_0), \]

which ends the proof.

The following proposition provides an example of application of Theorem 3.1

**Proposition 3.1.** Let \( \ell \geq 0 \) and \( g \in \Gamma_0(H) \) such that \( g^* \) is bounded from below and

\[ \| x \| - \ell \leq \| \text{prox}_g(x) \| \quad \text{for all} \ x \in H. \]

Then, \( g - g(0) \leq \ell \| \cdot \|. \) Moreover, if \( \ell \equiv 0, \) then \( g \) is constant.

**Proof.** Indeed, if \( f = \ell \| \cdot \|, \) then

\[ \text{prox}_f(x) = \left( 1 - \frac{\ell}{\max\{\| x \|, \ell\} \right) x, \]

and \( f^* \) is bounded from below with \( \inf f^* = 0. \) Thus, by Theorem 3.1, \( g - g(0) \leq \ell \| \cdot \|. \) Finally, if \( \ell = 0, \) then \( g \) is a constant function (a convex function which is bounded from above is constant).

The following result gives a Lipschitzianity characterization for a convex function.
Proposition 3.2. Let $f : \mathcal{H} \to \mathbb{R}$ be a convex and lower semicontinuous function. Then, $f$ is $\ell$-Lipschitz if and only if

$$||x|| - \ell \leq ||\text{prox}_f(x + y) - y|| \text{ for all } x, y \in \mathcal{H}. \quad (3)$$

Proof. On the one hand, if $f$ is $\ell$-Lipschitz, then for all $x, y$

$$||x|| - \ell \leq ||x + y - \text{prox}_f(x + y)|| + ||\text{prox}_f(x + y) - y|| \leq \ell + ||\text{prox}_f(x + y) - y||,$$

where we have used that $x + y - \text{prox}_f(x + y) \in \partial f(\text{prox}_f(x + y)) \subset \ell \mathbb{B}$.

On the other hand, assume that (3) holds and fix $y \in \mathcal{H}$. Let us consider the functions $h := f(\cdot + y)$ and $g = \ell \cdot ||\cdot||$. Then, for all $x \in \mathcal{H}$

$$\text{prox}_h(x) = \text{prox}_f(x + y) - y \quad \text{and} \quad \text{prox}_g(x) = \left(1 - \frac{\ell}{\max\{||x||, \ell\}}\right)x.$$

Moreover, since $\text{dom}(f) = \mathcal{H}$, $h^*$ is bounded from below and $\inf h^* = -f(y)$. Therefore, for all $x \in \mathcal{H}$

$$||\text{prox}_g(x)|| \leq ||x|| - \ell \leq ||\text{prox}_f(x + y) - y|| = ||\text{prox}_h(x)||. \quad (4)$$

By virtue of Theorem 3.1, we obtain that

$$f(x + y) \leq \ell ||x|| + \inf g^* - \inf h^*.$$

Finally, since $\inf g^* = 0$ and $\inf h^* = -f(y)$, we get that

$$f(x + y) \leq f(y) + \ell ||x||,$$

which implies that $f$ is $\ell$-Lipschitz.

\[ \square \]

4 Determination of convex functions

Since then, several integration results appeared In this section, we present the main finding of the paper; that is, the norm of the proximal operator determines a convex function up to a constant. The following two results extends Theorem 1.1.

Proposition 4.1. Let $f, g : \Gamma_0(\mathcal{H})$ be two functions such that for some $x_0 \in \text{dom } f \cap \text{dom } g$

$$||\text{prox}_f(x) - x_0|| = ||\text{prox}_g(x) - x_0|| \text{ for all } x \in \mathcal{H}.$$  

Then, $f - f(x_0) = g - g(x_0)$.

The following result summarizes several determination principles for convex functions.

Theorem 4.1. Let $f, g : \Gamma_0(\mathcal{H})$ be two functions such that $f^*$ and $g^*$ are bounded from below. Then, the following assertions are equivalent:

(i) For all $x \in \mathcal{H}$, $||\text{prox}_f(x)|| = ||\text{prox}_g(x)||$.
(ii) For all $x \in \mathcal{H}$, $f(x) = g(x) + \inf f^* - \inf g^*$.
(iii) For all $x \in \mathcal{H}$, $\partial f(x)^\circ = \partial g(x)^\circ$, where $\partial f(x)^\circ = \text{Proj}_{\partial f(x)}(0)$ and $\partial g(x)^\circ = \text{Proj}_{\partial g(x)}(0)$.
(iv) For all $x \in \mathcal{H}$, $\partial f(x) = \partial g(x)$.
(v) For all $x \in \mathcal{H}$, $\text{prox}_f(x) = \text{prox}_g(x)$.  

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Proof. (i)⇒(ii) follows from Theorem 3.1 (ii)⇒(iii) is trivial. (iii)⇒(iv) follows from [4] Corollaire 2.2. (iv)⇒(v) follows from the formula $\text{prox}_f(x) = (I + \partial f)^{-1}(x)$. Finally, (v)⇒(i) is trivial.

The following example shows that the hypotheses for the implication (i)⇒(ii) are sharp.

Example 4.1. Let us consider $f = \delta_{\{x_1\}}$ and $g = \delta_{\{x_2\}}$, where $x_1 \neq x_2$ and $\|x_1\| = \|x_2\|$. Then, for all $x \in H$

$$\|\text{prox}_f(x)\| = \|x_1\| = \|x_2\| = \|\text{prox}_g(x)\|.$$  

However, $f^* = \langle x_1, \cdot \rangle$ and $g^* = \langle x_2, \cdot \rangle$ are not bounded from below.

Theorem 4.1 allow us to obtain the following characterization of support functions.

Corollary 4.1. Let $C$ be a nonempty, closed and convex set containing 0. Then, $f \in \Gamma_0(H)$ satisfies

$$\|\text{prox}_f(x)\| = d_C(x)$$  \hspace{1cm} (5)

if and only if $f$ is the support of $C$ up to a constant.

Proof. Indeed, on the one hand, if $f$ is the support of $C$ up to a constant, then $\text{prox}_f = \text{prox}_{\sigma_C}$, which implies (5). On the other hand, if (5) holds, then $f^*$ is bounded from below. Moreover, by Proposition 2.1

$$\|\text{prox}_f(x)\| = d_C(x) = \|x - \text{proj}_{C}(x)\| = \|\text{prox}_{\sigma_C}(x)\|$$  \hspace{1cm} for all $x \in H$,

where we have used that $(\delta_C)^* = \sigma_C$. Therefore, by Theorem 4.1 $f$ is the support of $C$ up to a constant.

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