The Fermionic Signature Operator in the Exterior Schwarzschild Geometry

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Abstract. The structure of the solution space of the Dirac equation in the exterior Schwarzschild geometry is analyzed. Representing the space-time inner product for families of solutions with variable mass parameter in terms of the respective scalar products, a so-called mass decomposition is derived. This mass decomposition consists of a single mass integral involving the fermionic signature operator as well as a double integral which takes into account the flux of Dirac currents across the event horizon. The spectrum of the fermionic signature operator is computed. The corresponding generalized fermionic projector states are analyzed.

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1. Introduction

The fermionic signature operator introduced in [12,13] gives a general setting for spectral geometry in Lorentzian signature [9] and is useful for constructing quasi-free Dirac states in globally hyperbolic space-times [4,10]. In the present paper, the fermionic signature operator is constructed for the first time in a black hole geometry, namely the exterior Schwarzschild geometry. The event horizon makes it necessary to modify the constructions considerably. In order to explain these modifications, we briefly recall the general idea and the basic construction in [12,13]. (The necessary preliminaries will be provided in Sect. 2 below.) On solutions $\psi_m, \phi_m$ of the Dirac equation of mass $m$ in a globally hyperbolic space-time $(\mathcal{M}, g)$, one has two inner products: One is the scalar product obtained by integrating the polarized probability density over a Cauchy surface $N$,

$$
(\psi_m | \phi_m)_m := 2\pi \int_N \langle \psi_m | \gamma^j \nu_j \phi_m \rangle_x \, d\mu_N(x) \quad (1.1)
$$

(where $\nu$ is the future-directed normal), whereas the other is obtained by integrating the pointwise inner product of Dirac wave functions over all of space-time,

$$
<\psi | \phi> := \int_{\mathcal{M}} <\psi | \phi> \, d\mu_{\mathcal{M}}. \quad (1.2)
$$

The scalar product (1.1) endows the solution space of the Dirac equation with the structure of a Hilbert space $(\mathcal{H}_m, (\cdot | \cdot)_m)$. In non-technical terms, the fermionic signature operator arises when representing the space-time inner product $<\cdot | \cdot>$ with respect to the scalar product $(\cdot | \cdot)_m$. In space-times of finite lifetime [12], this method can be implemented directly by demanding that the relation

$$
<\psi_m | \phi_m> = (\psi_m | S_m \phi_m)_m \quad (1.3)
$$

should hold for all $\psi_m, \phi_m \in \mathcal{H}_m$. This uniquely defines $S_m$ as a symmetric bounded operator on $\mathcal{H}_m$. In space-times of infinite lifetime [13], the relation (1.3) in general is not sensible (except in specific situations like the Rindler space-time [11]), simply because the time integration in (1.2) may diverge for Dirac solutions. The way out is to make use of mass oscillations in the following sense. Instead of analyzing solutions for a fixed mass $m$, one considers families $(\psi_m)_m \in I$ of Dirac solutions for a mass parameter $m$ which varies in an interval $I := (m_L, m_R)$ with $0 \notin I$. Integrating over the mass parameter,

$$
p\psi := \int_I \psi_m \, dm,
$$

we obtain a superposition of waves oscillating with different frequencies. Intuitively speaking, this leads to destructive interference, giving rise to the desired decay of the Dirac wave functions for large times. This makes it possible to replace (1.3) by the condition

$$
<p\psi | p\phi> = \int_I (\psi_m | S_m \phi_m)_m \, dm, \quad (1.4)
$$
to be satisfied for all families of solutions \((\psi_m)_{m \in I}\) and \((\phi_m)_{m \in I}\) which lie in a suitably chosen dense subspace \(\mathcal{H}^\infty \subset \mathcal{H}\) of the Hilbert space of families of solutions (for details see Sect. 2.1). The space \(\mathcal{H}^\infty\) is referred to as the \textit{domain} for the mass oscillations. This construction gives for every \(m \in I\) a uniquely defined bounded linear operator \(S_m\) on \(\mathcal{H}_m\). The conditions needed for the construction to work are subsumed in various notions of \textit{mass oscillation properties}. For details, we refer the interested reader to the general construction in [13] and to the applications in [10,14].

In order to put our results into context, let us say a few general words on the fermionic signature operator and its significance. The fermionic signature operator \(S_m\) is a symmetric operator on the Hilbert space \(H_m\). Since its construction involves the Dirac wave functions, it clearly contains information on the solutions of the Dirac equation. However, since only the inner products \((.|.)_m\) and \(<.|.>\) of these wave functions are used, the Dirac wave functions do not enter pointwise, but merely integrated over space and spacetime. The construction is covariant and does not depend on observers nor on the choice of Cauchy surfaces. Moreover, as is made precise in [15], the fermionic signature operator respects all space-time symmetries. To summarize, the fermionic signature operator encodes specific information on the global behavior of the Dirac wave functions in space-time. Apart from being of independent interest as a novel geometric operator, the fermionic signature operator has two main applications. The first application is that it provides a setting for spectral geometry with Lorentzian signature. This has been explored for two-dimensional space-times in [9]. As the second application, it gives a new method for constructing quasi-free states of the quantized Dirac field. This is based on the observation that spectral subspaces of \(S_m\) are distinguished subspaces of \(\mathcal{H}_m\) characterized purely geometrically independent of observers. These distinguished subspaces can be used to construct distinguished quasi-free Dirac states, referred to as the \textit{fermionic projector state} or generalized fermionic projector states (see [4,12] and [2,10,11,13,14]).

In an exterior black hole geometry, the main complication is that part of the Dirac wave may cross the event horizon and disappear in the black hole. As a consequence, the mass oscillation properties no longer hold, and a representation of the form (1.4) no longer exists. Instead, based on the integral representation of the Dirac propagator [8], we derive a so-called \textit{mass decomposition} of the form (for details see Theorem 3.3)

\[
<p\psi|p\phi> = \int_I (\psi_m | S_m \phi_m)_m \, dm
\]  

\[
+ \frac{i}{\pi} \int_I dm \int_I dm' \frac{PP}{m - m'} \mathfrak{B}(\psi_m, \phi_{m'}),
\]

where \(\mathfrak{B}(\psi_m, \phi_{m'})\) is a smooth function in \(m\) and \(m'\) (and \(PP\) denotes the principal value). The above equation holds for all families \((\psi_m)_{m \in I}\) and \((\phi_m)_{m \in I}\) in a conveniently chosen domain \(\mathcal{H}^\infty\) (for details see Definition 3.1). We point out that (1.6) gives a contribution for pairs of solutions \(\psi_m\) and \(\phi_{m'}\) of the
Dirac equation with different masses $m \neq m'$. This contribution can be associated to the flux of a corresponding “current” $J^k(x) = \langle \psi_m | \gamma^j \phi_{m'} \rangle_x$ through the event horizon of the black hole. (This connection is explained in Sect. 3.3 and worked out in Sect. 5 using the so-called fermionic flux operator.)

The contribution (1.5) to the mass decomposition again uniquely defines for every $m \in I$ a fermionic signature operator $S_m$. We analyze the properties of this operator. Our results are summarized as follows. First of all, the fermionic signature operator is a bounded self-adjoint operator on $H_m$ with $\|S_m\| \leq 2$. It respects the symmetries of space-time, meaning that it has a joint spectral decomposition with the angular momentum operator $A$ and the Dirac Hamiltonian $H$ of the form

$$S_m = \sum_{k,n} \int_{-\infty}^{\infty} S_{kn}(\omega) F_{k,n} dE_\omega,$$

(1.7)

where $F$ and $E$ are the spectral measures of the operators $A$ and $H$, respectively; i.e.,

$$A = \sum_{k,n} \lambda_n F_{k,n} \quad \text{and} \quad H = \int_{-\infty}^{\infty} \omega dE_\omega.$$

(1.8)

(Here $k \in \mathbb{Z} + \frac{1}{2}$ is the azimuthal eigenvalue and $n \in \mathbb{Z}$ labels the eigenvalues of the spin-weighted angular operator $A$; for details see Sect. 2.3 below.) For clarity, we note that the spectral decomposition (1.7) already follows abstractly from the fact that the space-time symmetries can be described by local groups of isomorphisms of the spinor bundle [15]. Here, we obtain this representation with a computational approach, which has the great advantage that we get detailed information on the eigenvalues of the operators $S_{kn}(\omega)$ in (1.7) (for details see Theorem 4.1):

(i) The operators $S^{kn}(\omega)$ vanish if $\omega \in [-m, m]$.

(ii) In the range $\omega \in \mathbb{R} \setminus [-m, m]$, the operator $S^{kn}(\omega)$ is

$$S^{kn}(\omega) \begin{cases} \text{positive definite} & \text{if } \omega > m \\ \text{negative definite} & \text{if } \omega < -m. \end{cases}$$

(1.9)

Its eigenvalues are given by

$$\mu_{\pm}(\omega) = \epsilon(\omega) \pm \sqrt{\frac{\|f_{\infty, m, 1}^{kn}\|_{C^2}^2 - 1}{\|f_{\infty, m, 1}^{kn}\|_{C^2}^2 + 1}},$$

(1.10)

where $\epsilon$ is the sign function and $f_{\infty, m, 1}^{kn}$ are the transmission coefficients of the radial ODE (for details see Sect. 2.4).

These results show that the fermionic signature operator contains surprisingly rich information on the black hole geometry and on properties of the Dirac solutions: According to (i), the kernel of $S_m$ consists of all Dirac solutions which necessarily “fall into” the black hole because their kinetic energy is not large enough for the wave to propagate to the asymptotic end. According to (1.9), the positive and negative spectral subspaces of $S_m$ yield the frequency splitting for an observer in a rest frame at infinity. Finally, the formula (1.10)
shows that the gravitational force acting on the Dirac wave functions has an interesting influence on the spectrum of $S_m$.

We also analyze the corresponding fermionic projector state. It is obtained by applying Araki’s construction in [1] to the projection operator onto the negative spectral subspace of the fermionic signature operator (for details see [10, Section 6]). In view of (1.9), we obtain the following result:

**Corollary 1.1.** The pure quasi-free fermionic projector state obtained from the fermionic signature operator coincides with the Hadamard state which is obtained by frequency splitting for the observer in a rest frame at infinity.

Having non-trivial eigenvalues (1.10), one obtains many other quasi-free states by applying Araki’s construction to the positive operators $W(S_m)$ with $W$ a nonnegative Borel function. However, at present the physical significance of these so-called generalized fermionic projector states is unclear. These states are in general not Hadamard (for details see Sect. 6).

The paper is organized as follows. In Sect. 2, we give the necessary background on the Dirac equation in globally hyperbolic space-times and in the exterior Schwarzschild geometry. The main point is to specialize the integral representation of the Dirac operator in the Kerr geometry which was derived and analyzed in [6,8] to the exterior Schwarzschild geometry. We closely follow the procedure in these papers and use a similar notation. In Sect. 3, the mass decomposition (1.5) and (1.6) is derived. In Sect. 4, the fermionic signature operator $S_m$ is computed and analyzed. In Sect. 5, we define and analyze the fermionic flux operator $B_m$ which describes the flux of Dirac currents through the event horizon. Finally, Sect. 6 is devoted to the resulting quasi-free quantum states.

# 2. Preliminaries

## 2.1. The Dirac Equation in Globally Hyperbolic Space-Times

We recall the setting in [12,13], restricting attention to four-dimensional space-times. Thus, we let $(\mathcal{M}, g)$ be a smooth, globally hyperbolic Lorentzian spin manifold of dimension four. For the signature of the metric, we use the convention $(+,−,−,−)$. We denote the corresponding spinor bundle by $S\mathcal{M}$. Its fibers $S_x\mathcal{M}$ are endowed with an inner product $\langle .,. \rangle_x$ of signature $(2,2)$, referred to as the spin scalar product. Clifford multiplication is described by a mapping $\gamma$ which satisfies the anti-commutation relations,

$$\gamma : T_x\mathcal{M} \to \text{L}(S_x\mathcal{M}) \quad \text{with} \quad \gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g(u,v)I_{S_x\mathcal{M}}.$$ 

We write Clifford multiplication in components with the Dirac matrices $\gamma^j$. The metric connections on the tangent bundle and the spinor bundle are denoted by $\nabla$. The sections of the spinor bundle are also referred to as wave functions. We denote the smooth sections of the spinor bundle by $C^\infty(\mathcal{M},S\mathcal{M})$. Similarly, $C^\infty_0(\mathcal{M},S\mathcal{M})$ denotes the smooth sections with compact support. On the wave functions, one has the Lorentz invariant inner product

$$\langle .,. \rangle : C^\infty(\mathcal{M},S\mathcal{M}) \times C^\infty_0(\mathcal{M},S\mathcal{M}) \to \mathbb{C},$$
\[ \langle \psi | \phi \rangle = \int_\mathcal{M} \langle \psi | \phi \rangle_x \, d\mu_\mathcal{M}. \tag{2.1} \]

The Dirac operator \( \mathcal{D} \) in a gravitational field is defined by
\[ \mathcal{D} := i \gamma^j \nabla_j : C^\infty (\mathcal{M}, S\mathcal{M}) \to C^\infty (\mathcal{M}, S\mathcal{M}). \]

For a given real parameter \( m \in \mathbb{R} \) (the “mass”), the Dirac equation reads
\[ (\mathcal{D} - m) \psi_m = 0. \]

For clarity, we always denote solutions of the Dirac equation by a subscript \( m \).

The assumption of global hyperbolicity yields the existence of a smooth foliation by Cauchy surfaces. Given smooth initial data on a Cauchy surface \( \mathcal{N} \), the Dirac equation has a unique global smooth solution. We mainly consider solutions in the class \( C^\infty_\text{sc} (\mathcal{M}, S\mathcal{M}) \) of smooth sections with spatially compact support. On such solutions, one has the scalar product
\[ \langle \psi_m | \phi_m \rangle_m = 2\pi \int_\mathcal{N} \langle \psi_m | \nu^j \gamma_j \phi_m \rangle_x \, d\mu_\mathcal{N}(x), \tag{2.2} \]
where \( \nu \) is the future-directed normal on \( \mathcal{N} \). (Due to current conservation, the scalar product is in fact independent of the choice of \( \mathcal{N} \); for details see [12, Section 2].) Forming the completion gives the Hilbert space \( (\mathcal{H}_m, (\cdot | \cdot)_m) \).

We shall also work with the Hilbert space of families of solutions of the Hilbert space defined as follows. We consider the mass parameter in a bounded open interval, \( m \in I := (m_L, m_R) \) with \( 0 \notin I \). For a given Cauchy surface \( \mathcal{N} \), we consider a function \( \psi_{\mathcal{N}}(x, m) \in S_x \mathcal{M} \) with \( x \in \mathcal{N} \) and \( m \in I \). We assume that this wave function is smooth and has compact support in both variables, \( \psi_{\mathcal{N}} \in C^\infty_0 (\mathcal{N} \times I, S\mathcal{M}) \). For every \( m \in I \), we let \( \psi(., m) \) be the solution of the Cauchy problem for initial data \( \psi_{\mathcal{N}}(., m) \),
\[ (\mathcal{D} - m) \psi(x, m) = 0, \quad \psi(x, m) = \psi_{\mathcal{N}}(x, m) \quad \forall \ x \in \mathcal{N}. \tag{2.3} \]

Since the solution of the Cauchy problem is smooth and depends smoothly on parameters, we know that \( \psi \in C^\infty (\mathcal{M} \times I, S\mathcal{M}) \). Moreover, due to finite propagation speed, \( \psi(., m) \) has spatially compact support. Finally, the solution is clearly compactly supported in the mass parameter \( m \). We summarize these properties by writing
\[ \psi \in C^\infty_{\text{sc}, 0} (\mathcal{M} \times I, S\mathcal{M}), \tag{2.4} \]
where \( C^\infty_{\text{sc}, 0} (\mathcal{M} \times I, S\mathcal{M}) \) denotes the smooth wave functions with spatially compact support which are also compactly supported in \( I \). We often denote the dependence on \( m \) by a subscript, \( \psi_m(x) := \psi(x, m) \). Then for any fixed \( m \), we can take the scalar product (2.2). On families of solutions \( \psi, \phi \in C^\infty_{\text{sc}, 0} (\mathcal{M} \times I, S\mathcal{M}) \) of (2.3), we introduce a scalar product by integrating over the mass parameter,
\[ \langle \psi | \phi \rangle := \int_I \langle \psi_m | \phi_m \rangle_m \, dm \tag{2.5} \]
(where \( dm \) is the Lebesgue measure). Forming the completion gives the Hilbert space \( (\mathcal{H}, (\cdot | \cdot)) \). It consists of measurable functions \( \psi(x, m) \) such that for almost all \( m \in I \), the function \( \psi(., m) \) is a weak solution of the Dirac equation which is square integrable over any Cauchy surface. Moreover, this spatial integral
is integrable over \( m \in I \), so that the scalar product (2.5) is well defined. We denote the norm on \( \mathcal{H} \) by \( \| \| \). The Hilbert space \( \mathcal{H} \) can be regarded as the direct integral of the Hilbert spaces \( \mathcal{H}_m \), sometimes denoted alternatively by

\[
\mathcal{H} = L^2(I, \mathcal{H}_m; dm) = \int_I \mathcal{H}_m \, dm.
\]

Our procedure clarifies that the space \( C^{\infty}_{sc,0}(\mathcal{M} \times I, S\mathcal{M}) \) is dense in \( \mathcal{H} \).

2.2. The Dirac Equation in the Exterior Schwarzschild Geometry

In Schwarzschild coordinates, the line element of the Schwarzschild geometry takes the form

\[
ds^2 = g_{jk} \, dx^j \, dx^k = \frac{\Delta}{r^2} \, dt^2 - \frac{r^2}{\Delta} \, dr^2 - r^2 \, d\vartheta^2 - r^2 \sin^2 \vartheta \, d\varphi^2,
\]

where

\[
\Delta := r^2 - 2Mr,
\]

and \( M > 0 \) is the mass of the black hole. The zero \( r_1 := 2M \) of \( \Delta \) defines the event horizon. We here restrict attention to the exterior region outside the event horizon. Thus, the coordinates \((t, r, \vartheta, \varphi)\) are in the range

\[-\infty < t < \infty, \quad r_1 < r < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi.\]

The exterior region is globally hyperbolic. The surfaces of constant coordinate time \( t \) form a foliation by Cauchy surfaces.

In [6,8], the Dirac equation is computed in the Kerr geometry and the solution of the Cauchy problem is expressed in terms of the radial and angular ODEs arising in the separation of variables. In the remainder of the preliminaries, we recall a few steps of the construction, specialized to the exterior Schwarzschild geometry. We choose the pseudo-orthonormal frame

\[
u_0 = -\frac{r}{\sqrt{\Delta}} \frac{\partial}{\partial t}, \quad u_1 = \frac{1}{r} \frac{\partial}{\partial \vartheta}, \quad u_2 = \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi}, \quad u_3 = \frac{\sqrt{\Delta}}{r} \frac{\partial}{\partial r}.
\]

For the Dirac operator, we make the ansatz

\[
D = iG^j \partial_j + B.
\]

In order to satisfy the anti-commutation relations

\[
g^{jk}(x) \mathbb{1}_{S\mathcal{M}} = \frac{1}{2} \left\{ G^j(x), G^k(x) \right\},
\]

we choose \( G^j(x) = u_a^j(x) \gamma^a \), where \( \gamma^a \) are the usual Dirac matrices in the Weyl representation. More precisely, we set

\[
G^t(x) = -\frac{r}{\sqrt{\Delta}} \gamma^0, \quad G^\vartheta(x) = \frac{1}{r} \gamma^1, \quad G^\varphi(x) = \frac{1}{r \sin \vartheta} \gamma^2, \quad G^r(x) = \frac{\sqrt{\Delta}}{r} \gamma^3,
\]

where

\[
\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \tilde{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}
\]
(and $\bar{\sigma}$ are the Pauli matrices). In order to arrange that these matrices are symmetric with respect to the spin scalar product $\langle \cdot | \cdot \rangle_x$ in (1.1) and (1.2), we choose

$$\langle \psi | \phi \rangle_x := - \left\langle \psi, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \phi \right\rangle_{\mathbb{C}^4}. \tag{2.6}$$

(Here the minus sign is a good convention because then the inner product $\langle \cdot | G^t \cdot \rangle_x$ is positive definite.) The corresponding zero-order term $B$ in the Dirac operator is given by (see also the general method for diagonal metrics in [5, Proposition 9.1],)

$$B = \frac{i}{2\sqrt{|\det g|}} \partial_j \left( \sqrt{|\det g|} G^j \right) = \frac{i}{2r^2 \sin \vartheta} \partial_j \left( r^2 \sin \vartheta G^j \right)$$

$$= \frac{i}{2 \sin \vartheta} \partial_\vartheta \left( \sin \vartheta G^\vartheta \right) + \frac{i}{2r^2} \partial_r \left( r^2 G^r \right) = \frac{i \cot \vartheta}{2r} \gamma^1 + \frac{i \partial_r (r \sqrt{\Delta})}{2r^2} \gamma^3.$$

The resulting Dirac operator takes the form

$$D = \begin{pmatrix} 0 & 0 & \alpha_+ & \beta_+ \\ 0 & 0 & \beta_- & \alpha_- \\ -\beta_- & \alpha_+ & 0 & 0 \\ \alpha_- & -\beta_+ & 0 & 0 \end{pmatrix}$$

with

$$\beta_\pm = \frac{i}{r} \left( \partial_\vartheta + \frac{\cot \vartheta}{2} \right) \pm \frac{1}{r \sin \vartheta} \partial_\varphi$$

$$\alpha_\pm = -\frac{ir}{\sqrt{\Delta}} \frac{\partial}{\partial t} \pm \frac{\sqrt{\Delta}}{r} \left( i \frac{\partial}{\partial r} + i \frac{r - M}{2\Delta} + \frac{i}{2r} \right).$$

### 2.3. Separation of the Dirac Equation

In preparation, we let $S(r)$ and $\Gamma(r)$ be the diagonal matrices

$$S = \Delta^{\frac{1}{2}} \sqrt{r} \mathbf{1}_{\mathbb{C}^4}, \quad \Gamma = -ir \diag (1, -1, -1, 1).$$

Then, the transformed wave function

$$\Psi = S \psi \tag{2.7}$$

satisfies the Dirac equation

$$\Gamma S (D - m) S^{-1} \Psi = 0. \tag{2.8}$$

Moreover,

$$\Gamma S (D - m) S^{-1} = \mathcal{R} + \mathcal{A}$$

with

$$\mathcal{R} = \begin{pmatrix} imr & 0 & \sqrt{\Delta} D_+ & 0 \\ 0 & -imr & 0 & \sqrt{\Delta} D_- \\ \sqrt{\Delta} D_- & 0 & -imr & 0 \\ 0 & \sqrt{\Delta} D_+ & 0 & imr \end{pmatrix}$$
\[ A = \begin{pmatrix}
0 & 0 & 0 & L_+ \\
0 & 0 & -L_- & 0 \\
0 & L_+ & 0 & 0 \\
-L_- & 0 & 0 & 0
\end{pmatrix} \]

and

\[ D_\pm = \frac{\partial}{\partial r} \pm \frac{r^2}{\Delta} \frac{\partial}{\partial t}, \quad L_\pm = \frac{\partial}{\partial \vartheta} + \frac{\cot \vartheta}{2} \pm \frac{i}{\sin \vartheta} \frac{\partial}{\partial \varphi}. \]

For the separation of the Dirac equation, we first employ for \( \Psi \) the ansatz

\[ \Psi(t, r, \vartheta, \varphi) = e^{-i\omega t} e^{-ik\varphi} \Phi(r, \vartheta), \quad \text{with } \omega \in \mathbb{R}, \ k \in \mathbb{Z} + \frac{1}{2}. \tag{2.9} \]

Next, for the function \( \Phi \) we make the ansatz

\[ \Phi(r, \vartheta) = \begin{pmatrix}
X_-(r) Y_- (\vartheta) \\
X_+(r) Y_+ (\vartheta) \\
X_+(r) Y_- (\vartheta) \\
X_-(r) Y_+ (\vartheta)
\end{pmatrix}, \tag{2.10} \]

composed of radial functions \( X_\pm(r) \) and angular functions \( Y_\pm(\vartheta) \). By substituting (2.9) and (2.10) into the transformed Dirac equation (2.8), we obtain the eigenvalue problems

\[ R \Psi = \lambda \Psi, \quad A \Psi = -\lambda \Psi, \]

under which the Dirac equation (2.8) decouples into the system of ODEs

\[ \begin{pmatrix}
\sqrt{\Delta} D_+ \\
-i mr - \lambda
\end{pmatrix}
\begin{pmatrix}
X_+ \\
Y_+
\end{pmatrix} = 0, \tag{2.11}
\]

\[ \begin{pmatrix}
L_+ \\
\lambda
\end{pmatrix}
\begin{pmatrix}
X_-\\
Y_-
\end{pmatrix} = 0. \tag{2.12}
\]

with

\[ D_\pm = \frac{\partial}{\partial r} \pm \frac{i}{\Delta} \frac{r^2}{\Delta} \frac{\partial}{\partial \varphi}, \quad L_\pm = \frac{\partial}{\partial \vartheta} + \frac{\cot \vartheta}{2} \pm \frac{k}{\sin \vartheta}. \tag{2.13} \]

2.4. Fundamental Solutions and Their Asymptotics

The angular equation (2.12) with \( L_\pm \) according to (2.13) does not involve \( \omega \). For any \( k \in \mathbb{Z} + \frac{1}{2} \), it can be regarded as an eigenvalue equation for the angular function \( Y \). This eigenvalue equation can be solved by the so-called spin-weighted spherical harmonics (for details see [16]). We thus obtain an orthonormal eigenvector basis \( Y_{kn} \) with \( n \in \mathbb{Z} \) in the Hilbert space \( L^2((-1, 1), d\cos \vartheta)^2 \), i.e.,

\[ \langle e^{-ik\varphi} Y_{kn}(\vartheta), e^{-ik'\varphi} Y_{k'n'}(\vartheta) \rangle_{L^2(S^2)^2} = \delta_{k,k'} \delta_{n,n'}. \tag{2.14} \]

We denote the corresponding eigenvalues by \( \lambda_{kn} \).

Choosing the separation constant \( \lambda = \lambda_{kn} \) as one of these eigenvalues, the radial ODE (2.11) can be written as

\[ \left[ \frac{d}{du} + i \omega \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right] X = \frac{\sqrt{\Delta}}{r^2} \begin{pmatrix} 0 & \lambda - i mr \\ \lambda + i mr & 0 \end{pmatrix} X, \tag{2.15} \]
where for convenience we transformed the radial variable to the Regge–Wheeler coordinate $u \in \mathbb{R}$ defined by
\[
\frac{du}{dr} = \frac{r^2}{\Delta}.
\]
The limit $u \to -\infty$ describes the event horizon, whereas in the limit $u \to \infty$ one reaches spatial infinity. The asymptotics of the solutions of the radial ODE have been worked out in [8, Lemmas 3.1 and 3.5]:

**Lemma 2.1.** Every solution $X$ of (2.15) is asymptotically as $u \to -\infty$ of the form
\[
X(u) = \left( e^{-i\omega u} f_0^+ + e^{i\omega u} f_0^- \right) + R_0(u)
\]
with $f_0^\pm \in \mathbb{C}$ and an exponentially decaying error term, i.e.,
\[
|R_0| \leq c e^{du}
\]
with a constant $c, d > 0$ which can be chosen locally uniformly in $\omega$ and $|f_0^\pm|$.

**Lemma 2.2.** In the case $|\omega| < m$, the ODE (2.15) has one fundamental solution which decays exponentially as $u \to \infty$ and one fundamental solution which increases exponentially in this limit.

In the case $|\omega| > m$, on the other hand, every solution $X$ of (2.15) is asymptotically as $u \to \infty$ of the form
\[
X(u) = A \left( e^{-i\Phi^\omega_m(u)} f_\infty^+ + e^{i\Phi^\omega_m(u)} f_\infty^- \right) + R_\infty(u)
\]
with $f_\infty^\pm \in \mathbb{C}$ and
\[
\Phi^\omega_m = \epsilon(\omega) \left( \sqrt{\omega^2 - m^2} u + \frac{Mm^2}{\sqrt{\omega^2 - m^2}} \log u \right)
\]
\[
A = \begin{pmatrix} \cosh \Theta & \sinh \Theta \\ \sinh \Theta & \cosh \Theta \end{pmatrix}, \quad \Theta = \frac{1}{4} \log \left| \frac{\omega - m}{\omega + m} \right|
\]
\[
|R_\infty| \leq \frac{C}{u},
\]
where the constant $C > 0$ can be chosen locally uniformly in $\omega$ and $f_\infty^\pm$.

Based on these results, we choose fundamental solutions of the Dirac equation
\[
\Psi_{k,\omega n}^{m, a} \quad \text{with } k \in \mathbb{Z}, \frac{1}{2}, \omega \in \mathbb{R}, n \in \mathbb{Z} \text{ and } a = 1, 2
\]
as follows. We always use the separation ansatz (2.9) and (2.10) with $Y$ chosen as the angular eigenfunction $Y_{kn}$ and the radial eigenfunction $X$ as a solution of the radial ODE (2.15) with $\lambda = \lambda_{kn}$. In the case $|\omega| < m$, the solution for $a = 1$ decays exponentially at infinity, whereas the solution for $a = 2$ increases exponentially. We normalize the solutions such that they have norm one at the event horizon
\[
\|f_{0, m, a}^{k, \omega n}\|_{C^2} = 1 \quad \text{for } a = 1, 2
\]
(where $f_{\omega n}^{k,0,a} \in \mathbb{C}^2$ is the vector with components denoted by $\pm$). In the case $|\omega| > m$, on the other hand, we choose the fundamental solutions with the asymptotics near the event horizon

$$f_{0, m, 1}^{k,\omega n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_{0, m, 2}^{k,\omega n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.21)$$

The asymptotics of these solutions at spatial infinity are described in (2.16) by coefficients $f_{\omega}^\pm$. For clarity, we combine these so-called transmission coefficients to a vector denoted by $f_{\omega, m, a}^{k,\omega n} \in \mathbb{C}^2$.

### 2.5. An Integral Representation of the Dirac Propagator

The dynamics of Dirac waves in a black hole geometry can be analyzed with different methods. Scattering theory as developed in [3,17,18] gives information on the behavior of the waves near the event horizon and near infinity. Moreover, it connects the asymptotic waves by scattering operators. Another approach is integral representations for the Dirac propagator which were derived in [6,8] and used to study decay rates in [7]. The advantage of the integral representations is that they give finer information on the solutions of the Cauchy problem. However, the integral representations apply only in special geometries where the Dirac equation is separable into ordinary differential equations (ODEs), namely in the Kerr–Newman black hole geometry and special cases thereof like the Kerr, Reissner–Nordström and Schwarzschild black holes. Here, we shall work with the integral representations because this makes it possible to work out the mass decomposition explicitly in terms of the fundamental solutions of the radial ODE and the corresponding transmission coefficients.

More precisely, an integral representation of the Dirac propagator was derived in [8, Theorem 3.6]. Here, we write this representation as follows. First, it was shown that every smooth and spatially compact solution of the Dirac equation $\psi_m \in C^\infty_0(\mathcal{M}, S\mathcal{M})$ can be represented as

$$\psi_m(t, r, \vartheta, \varphi) = \sum_{k, n} \int_{\mathbb{R}} d\omega \ e^{-i\omega t} \sum_{a=1}^{2} \hat{\psi}_{m, a}^{k, n}(\omega) \Psi_{m, a}^{k, \omega n}(r, \vartheta, \varphi) \quad (2.22)$$

with complex-valued functions $\hat{\psi}_{m, a}^{k, n}(\omega)$, where we sum over $k \in \mathbb{Z} + \frac{1}{2}$ and $n \in \mathbb{Z}$. Here, $\Psi_{m, a}^{k, \omega n}$ are the fundamental solutions of the coupled ODEs introduced in (2.20). Moreover, in the case $|\omega| < m$ only those fundamental solutions appear which decay at spatial infinity. We implement this fact by always choosing

$$\hat{\psi}_{m, 2}^{k, n}(\omega) = 0 \quad \text{for all } \omega \in (-m, m) \text{ and all } k, n. \quad (2.23)$$

Second, the integral representation in [8, Theorem 3.6] also gives explicit formulas for the functions $\hat{\psi}_{m, a}^{k, n}$ in (2.22) in terms of the initial data $\Psi_0 \in C^\infty_0(N, S\mathcal{M})$ (where $N$ is the hypersurface $\{t = 0\}$). Indeed,

$$\hat{\psi}_{m, a}^{k, n}(\omega) = \frac{1}{2\pi^2} \sum_{b=1}^{2} \hat{t}_{ab}^{k, \omega n}(\Psi_{m, b}^{k, \omega n} | \psi_m)_{m} \big|_{t=0} \quad (2.24)$$
where $\langle . | . \rangle_m$ is again the scalar product (2.2), and the coefficients $t_{ab}^{knm}$ can be expressed explicitly in terms of the transmission coefficients. (For the prefactor $1/(2\pi^2)$, one must keep in mind that the scalar product (2.2) involves a factor $2\pi$.)

3. Derivation of the Mass Decomposition

For the analysis of mass oscillations, we consider a variable mass parameter $m$ in an interval $I = (m_L, m_L)$ with $0 \not\in I$. Following the general procedure in [13], we want to choose the domain $H^\infty$ as a dense subspace of $H$ spanned by families of Dirac solutions which are “nice” in the sense that it is easy to handle technical issues like verifying differentiability, integrability and interchangeability of limits and integrals. Here, it is most convenient to make the following choice:

**Definition 3.1.** The domain $H^\infty \subset H$ is chosen as the space of all Dirac solutions of the form (2.22) and (2.23) which satisfy the following conditions:

(i) The functions $\hat{\psi}_{m,a}^{kn} (\omega)$ in (2.22) vanish identically for almost all $k \in \mathbb{Z} + \frac{1}{2}$ and $n \in \mathbb{Z}$.

(ii) For all $k \in \mathbb{Z} + \frac{1}{2}$, $n \in \mathbb{Z}$ and $a \in \{1, 2\}$, the functions $\hat{\psi}_{m,a}^{kn} (\omega)$ are smooth and compactly supported in $\omega$ and $m$. Moreover, they are supported away from $\omega = \pm m$, i.e.,

$$\text{supp} \hat{\psi}_{m,a}^{kn} (\cdot) \subset \{ (\omega, m) \in \mathbb{R} \times I \text{ with } \omega \neq \pm m \}. \quad (3.1)$$

We clarify this definition with two remarks. We first note that, by approximation, the integral representation (2.22) can be extended to wave functions which do not have spatially compact support. In particular, choosing $\hat{\psi}_{m,a}^{kn} (\omega)$ according to Definition 3.1 (i) and (ii), for all $m \in I$ one gets a well-defined Dirac solution in $H^m$. The second remark concerns the condition (3.1). This condition states that the waves in $H^\infty$ vanish identically for frequencies in a neighborhood of $\omega = \pm m$. In other words, we disregard Dirac solutions which asymptotically at infinity have zero momentum. This assumption is a major technical simplification. It is justified by the fact that $H^\infty$ is dense in $H$, as can be verified in various ways: One method is to work again with the integral representation (2.22) and to approximate the functions $\hat{\psi}_{m,a}^{kn}$ coming from a solution $\psi \in C^\infty(\mathcal{M}, S\mathcal{M})$ by functions which vanish in a neighborhood of $\omega = \pm m$. More abstractly, the denseness of $H^\infty$ follows from the fact that the Dirac Hamiltonian has no point spectrum. Using methods of scattering theory, this follows from the asymptotic completeness results in [17,18].

3.1. Integral Representation of the Scalar Product

After performing the transformation (2.7), the scalar product (2.2) with the spin scalar product according to (2.6) takes for all $\psi, \phi \in H^\infty$ the form

$$\langle \psi_m | \phi_m \rangle_m = 2\pi \int_{r_1}^{r_2} dr \int_{-1}^{1} d\cos \vartheta \int_{0}^{2\pi} d\varphi \frac{r^2}{\Delta} \langle \Psi_m, \Phi_m \rangle_{C^4}.$$
Employing (2.22) as well as the separation ansatz (2.10), one can use the orthogonality of the angular eigenfunctions (2.14) to obtain
\[
(\psi_m | \phi_m)_m = 4\pi \sum_{k,n} \int_{r_1}^{\infty} dr \int_{r_1}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{i(\omega-\omega')t} \\
\times \sum_{a,a'=1}^2 \overline{\hat{\psi}^{kn}_{m,a}(\omega)} \hat{\phi}^{kn}_{m,a}(\omega') \left\langle X^{k\omega n}_{m,a}(r), X^{k\omega'n}_{m,a'}(r) \right\rangle_{C^2},
\]
where the functions \(\hat{\phi}^{kn}_{m,a}\) and \(\hat{\psi}^{kn}_{m,a}\) are defined by (2.24). (Note that a factor of two arises because a four-spinor involves \(X\) twice.) This representation has the disadvantage that, due to the factor \(e^{i(\omega-\omega')t}\), current conservation is not apparent. Therefore, the representations derived in the following lemma are more useful. (These representations were first used in [7, Section 9].)

**Lemma 3.2.** For any Dirac solutions \(\psi_m, \phi_m \in C^\infty_\text{sc}(\mathcal{M}, S\mathcal{M})\), their scalar product can be written in the alternative forms
\[
(\psi_m | \phi_m)_m = 2\pi \sum_{k,n} \int_{\mathbb{R}} \int_{\mathbb{R}}^2 (T^{kn}(\omega))^{-1} \overline{\hat{\psi}^{kn}_{m,a}(\omega)} \hat{\phi}^{kn}_{m,b}(\omega) d\omega
\]
\[
= \frac{1}{2\pi^2} \sum_{k,n} \sum_{a,b=1}^2 \int_{\mathbb{R}}^2 t^{k\omega n}_{ab} (\psi_m | \Psi^{k\omega n}_{m,a})_m (\Psi^{k\omega n}_{m,b} | \phi_m)_m |_{t=0} d\omega,
\]
where \(T^{kn}(\omega)\) is the \(2 \times 2\)-matrix with entries \(t^{k\omega n}_{ab}\).

**Proof.** Combining (2.22) and (2.24) and setting \(t = 0\), we obtain
\[
\psi_m |_{t=0}(r, \theta, \varphi) = \frac{1}{2\pi^2} \sum_{k,n} \sum_{a,b=1}^2 \int_{\mathbb{R}}^2 t^{k\omega n}_{ab} (\psi_m | \Psi^{k\omega n}_{m,a})_m (\Psi^{k\omega n}_{m,b} | \psi_m)_m |_{t=0} d\omega.
\]
Taking the scalar product with another Dirac solution \(\phi_m\) gives (3.3). Again applying (2.24) gives (3.2). \(\square\)

### 3.2. Mass Decomposition of the Space-Time Inner Product

After the transformation (2.7), the space-time inner product (2.1) (with the spin scalar product according to (2.6)) becomes
\[
<\psi | \phi> = - \int_{-\infty}^{\infty} dt \int_{\mathbb{R}}^1 dr \int_{-1}^{1} d\cos \theta \int_{0}^{2\pi} d\varphi \frac{r}{\sqrt{\Delta}} \left\langle \Psi, \left(0 \ \ 1 \right) \Phi \right\rangle_{C^4}.
\]
Applying (2.22) as well as the separation ansatz (2.10), one can again use the orthogonality of the angular eigenfunctions (2.14) as well as Plancherel’s theorem to obtain for all \(\psi, \phi \in \mathcal{H}^\infty\)
\[
<\psi | p \phi> = -2\sum_{k,n} \int_{-\infty}^{\infty} dt \int_{r_1}^{\infty} dr \frac{r}{\sqrt{\Delta}} \int_I dm \int_{I'} dm' \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{i(\omega-\omega')t} \\
\times \sum_{a,a'=1}^2 \overline{\hat{\psi}^{kn}_{m,a}(\omega)} \hat{\phi}^{kn}_{m,a'}(\omega') \left\langle X^{k\omega n}_{m,a}(r), X^{k\omega'n}_{m,a'}(r) \right\rangle_{C^2}.
\]
\[= -4\pi \sum_{k,n} \int_{r_1}^{\infty} \frac{dr}{r} \int_I dm \int_{-\infty}^{\infty} d\omega \]
\[\times \sum_{a,a'=1}^{2} \hat{\psi}_{m,a}(\omega) \hat{\phi}_{m',a'}(\omega) \left\langle X_{k\omega}^n(r), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_{k\omega',a'}^n(r) \right\rangle_{C^2}. \]

(Note that again a factor of two arises because a four-spinor involves \(X\) twice.) Since in this formula as well as in the formulas of Lemma 3.2 we get a pairing only between wave functions with the same angular momentum, in what follows we may restrict attention to a single angular momentum mode. Consequently, from now on we always leave out the sums over \(k\) and \(n\) and omit the indices \(k\) and \(n\).

The \(r\) integration in (3.4) can be carried out, giving the following result.

**Theorem 3.3** (mass decomposition of the space-time inner product). Restricting attention to one angular momentum mode, for all \(\psi, \phi \in H^\infty\) the following identity holds,

\[
\langle p\psi | p\phi \rangle = 4\pi^2 \int_I dm \int_{\mathbb{R} \setminus [-m,m]} \epsilon(\omega) \, d\omega \sum_{a,a'=1}^{2} \hat{\psi}_{m,a}(\omega) \hat{\phi}_{m,a}(\omega) \left\langle f_{\infty}^{\omega}, m, a; f_{\infty}^{\omega}, m, a' \right\rangle_{C^2}
\[- 4\pi i \int_I dm \int_{I} dm' \frac{PP}{m - m'} \int_{-\infty}^{\infty} d\omega \]
\[\times \sum_{a,a'=1}^{2} \hat{\psi}_{m,a}(\omega) \hat{\phi}_{m',a'}(\omega) \left\langle f_{\omega}^{\omega}, m, a; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f_{\omega}^{\omega}, m', a' \right\rangle_{C^2}, \]

where \(\hat{\psi}_{m,a}\) and \(\hat{\phi}_{m',a'}\) are the functions in the representation (2.22) (which always satisfy the condition (2.23)). Moreover, \(f_{\omega}^{\omega}, m, a\) and \(f_{\infty}^{\omega}, m, a\) describe the asymptotics of the radial fundamental solutions near the event horizon and at infinity (see Lemmas 2.1 and 2.2).

The remainder of this section is devoted to the proof of this theorem. Its physical significance will be explained afterward in Sect. 3.3. We begin with preparatory lemmas.

**Lemma 3.4.** In the range \(|\omega| > m\), the transmission coefficients \(f_{\infty}^{\omega}, m, a\) satisfy the relations

\[
|f_{\infty}^{\omega}, m, a|_2^2 - |f_{\infty}^{\omega}, m, a|_2^2 = \begin{cases} 1 & \text{if } a = 1 \\ -1 & \text{if } a = 2 \end{cases}. \quad (3.5)
\]
\[
\|f_{\infty}^{\omega}, m, 1\|_{C^2} = \|f_{\infty}^{\omega}, m, 2\|_{C^2} \geq 1. \quad (3.6)
\]

**Proof.** The form of the matrices in the ODE (2.15) implies that (see [8, proof of Lemma 3.3]),

\[
\frac{d}{du} (|X_+|^2 - |X_-|^2) = 0. \quad (3.7)
\]

Hence, in view of the asymptotics near the event horizon (2.21), it follows that

\[
|X_+(u)|^2 - |X_-(u)|^2 = \begin{cases} 1 & \text{if } a = 1 \\ -1 & \text{if } a = 2 \end{cases} \quad \text{for all } u \in \mathbb{R}.
\]
Using the asymptotics at infinity of Lemma 2.2, we conclude that
\[
\langle Af_\infty^{\omega}, m, a \mid A f_\infty^{\omega}, m, a \rangle_{C^2} = \begin{cases} 
1 & \text{if } a = 1 \\
-1 & \text{if } a = 2.
\end{cases}
\]
It is useful to regard the bilinear form on the left as an indefinite inner product generated by the matrix \(\text{diag}(1, -1)\). By direct computation, one verifies that the matrix \(A\) in (2.18) is unitary with respect to this inner product. This gives (3.5).

In order to derive (3.6), we make use of the fact that the identity (3.7) holds for any linear combination of our fundamental solutions. This implies that the transmission coefficients are pseudo-orthonormal with respect to the indefinite inner product generated by the matrix \(\text{diag}(1, -1)\), i.e.,
\[
\langle f_\infty^{\omega}, m, a \mid f_\infty^{\omega}, m, b \rangle_{C^2} = \delta_{ab} \times \begin{cases} 
1 & \text{if } a = 1 \\
-1 & \text{if } a = 2.
\end{cases}
\]
As a consequence, the transmission coefficients can be parametrized as
\[
f_\infty^{\omega}, m, 1 = \left( e^{i\beta} \cosh \vartheta \right),
\quad f_\infty^{\omega}, m, 2 = e^{i\delta} \left( e^{i\gamma} \cosh \vartheta \right),
\]
with four real parameters \(\vartheta, \beta, \gamma, \delta\). Indeed, the left equation is a general parametrization of a vector satisfying the normalization (3.5). The right equation, on the other hand, parametrizes a general vector which again satisfies the normalization (3.5) and is orthonormal to the first vector with respect to the inner product in (3.8).

In the parametrization (3.9), the relations in (3.6) are verified immediately by a short computation. \(\square\)

**Lemma 3.5.** For all \(\psi \in \mathcal{H}^{\infty}\) and all \(\omega \in \mathbb{R}\),
\[
\int_I \, dm \, \sum_{a=1}^{2} \hat{\psi}_{m,a}(\omega) \, X_{m,a}^{\omega}(r) = O\left(\frac{1}{r}\right).
\]

**Proof.** We first point out that, due to the support assumption (3.1), the function \(\hat{\psi}_{m,a}(\omega)\) vanishes in a neighborhood of \(m = \pm \omega\). This has the technical advantage that the transformation functions in Lemma 2.2 are smooth in \(m\). Moreover, we may treat the cases \(|\omega| < m\) and \(|\omega| > m\) separately.

In the case \(|\omega| < m\), the fundamental solution \(X_{m,1}^{\omega}\) decays exponentially, giving the desired decay for large \(r\). In the remaining case \(|\omega| > m\), we may clearly disregard the error term \(R_\infty\) in (2.16). The remaining summand in (2.16) is smooth and involves the oscillatory factors \(e^{\pm i \Phi_{m,a}(u)}\). Using that the Fourier transform of a smooth function has rapid decay, the resulting term decays even rapidly in \(r\). \(\square\)

In the next lemma, the Dirac operator is “integrated by parts” in the space-time inner product. As in [13, Section 3.1], we denote the operator of multiplication with the mass parameter by \(T\),
\[
T : \mathcal{H}^{\infty} \to \mathcal{H}^{\infty}, \quad (T \psi)_m = m \psi_m.
\]
Lemma 3.6. For all $\psi, \phi \in \mathcal{H}^\infty$,
\[
<p\psi | pT\phi> - <pT\psi | p\phi> = 4\pi i \int_I dm \int_{I'} dm' \\
\times \int_{-\infty}^{\infty} \sum_{a, a' = 1}^2 \hat{\psi}_{m,a}(\omega) \hat{\phi}_{m',a'}(\omega) \left\langle f_{0, m,a}^\omega, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f_{0, m',a'}^\omega \right\rangle_{C^2} d\omega.
\]
(3.10)

Proof. We again restrict attention to a single angular momentum mode. We write the radial equation (2.15) as
\[
\mathcal{D}X = mX
\]
with the “radial Dirac operator”
\[
\mathcal{D} = i\sqrt{\Delta} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dr} + \frac{\lambda}{r} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{\omega r}{\sqrt{\Delta}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Since all the matrices in this equation are symmetric with respect to the “separated spin scalar product”
\[
\langle . , . \rangle := -\left\langle . , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \right\rangle_{C^2},
\]
the radial Dirac operator is symmetric with respect to the inner product $<.|.>$. This means that on the left side of (3.10) our task is to compute the boundary terms at infinity and on the horizon, i.e., using (3.4)
\[
<p\psi | pT\phi> - <pT\psi | p\phi> = -4\pi i \left( \int_I dm \int_{I'} dm' \int_{-\infty}^{\infty} d\omega \\
\times \sum_{a, a' = 1}^2 \hat{\psi}_{m,a}(\omega) \hat{\phi}_{m',a'}(\omega) \left\langle X_{m,a}(r), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X_{m',a'}(r) \right\rangle_{C^2} \right\vert_{r=\infty}^{r=r_1}.
\]
The boundary terms at spatial infinity vanish in view of Lemma 3.5. On the event horizon, on the other hand, we can use the asymptotics in Lemma 2.1, giving the result. □

We next compute the singular contribution to the space-time inner product at $m = m'$.

Lemma 3.7. For all $\psi, \phi \in \mathcal{H}^\infty$,
\[
<p\psi | p\phi> = 4\pi^2 \int_I dm \int_{\mathbb{R}\setminus[-m,m]} \epsilon(\omega) d\omega \sum_{a, a' = 1}^2 \hat{\psi}_{m,a}(\omega) \hat{\phi}_{m,a'}(\omega) \left\langle f_{\infty, m,a}^\omega, f_{\infty, m,a'}^\omega \right\rangle_{C^2}
\]
(3.11)
\[
-4\pi i \int_I dm \int_{I'} dm' \frac{PP}{m - m'} \\
\times \int_{\mathbb{R}\setminus[-m,m]} \left( \hat{\psi}_{m,1}(\omega) \hat{\phi}_{m',1}(\omega) - \hat{\psi}_{m,2}(\omega) \hat{\phi}_{m',2}(\omega) \right) d\omega
\]
(3.12)
\[ + \int_I \, dm \int_I \, dm' \, h(m, m') \]  

(3.13)

with a bounded function \( h \in L^\infty(I \times I) \).

**Proof.** Considering again one angular momentum mode, (3.4) becomes

\[
\langle p \psi \mid p \phi \rangle = -4\pi \int_{r_1}^{\infty} \frac{dr}{\sqrt{\Delta}} \int_I \, dm \int_I \, dm' \int_{-\infty}^{\infty} \, d\omega \times \sum_{a, a' = 1}^{2} \hat{\psi}_{m, a}(\omega) \hat{\phi}_{m', a}(\omega) \left\langle X_{m, a}(r), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\rangle \left\langle X_{m, a'}^\omega(r), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\rangle e^{-r}.
\]

(3.14)

According to Lemma 2.1 and the fact that the weight \( r/\sqrt{\Delta} \) is integrable, the \( r \) integral converges near the event horizon for any fixed \( \omega, m \) and \( m' \), uniformly in these parameters. Near spatial infinity, on the other hand, the plane wave asymptotics of Lemma 2.2 show that the \( r \) integral is well defined only if the integrals are performed in the order given in (3.14). However, we want to analyze the \( r \) integral first. To this end, we insert a convergence-generating factor \( e^{-\varepsilon r} \), making it possible to interchange the integrals. (In the end, we shall take the limit \( \varepsilon \searrow 0 \) in the distributional sense.) We thus obtain

\[
\langle p \psi \mid p \phi \rangle = -4\pi \lim_{\varepsilon \searrow 0} \int_I \, dm \int_I \, dm' \int_{-\infty}^{\infty} \, d\omega \sum_{a, a' = 1}^{2} \hat{\psi}_{m, a}(\omega) \hat{\phi}_{m', a}(\omega) \times \int_{r_1}^{\infty} \frac{r}{\sqrt{\Delta}} \left\langle X_{m, a}^\omega(r), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) X_{m', a'}^\omega(r) \right\rangle e^{-r} \, dr.
\]

(3.15)

In the case \( |\omega| < m \), the fundamental solution decays exponentially at infinity, so that the resulting contribution can be absorbed into the function \( h(m, m') \).

In the remaining case \( |\omega| > m \), the term involving the error term \( R_{\infty} \) squared is integrable over \( r \) and can again be absorbed into the function \( h(m, m') \). The term involving one exponential factor and one error term \( R_{\infty} \) is not integrable in the Lebesgue sense, but it exists as an improper Riemann integral, giving a bounded function. Hence, it can also be absorbed into the function \( h(m, m') \). Therefore, it remains to consider the \( r \) integral for the leading term in (2.16). Transforming for convenience to the Regge–Wheeler coordinate, we obtain

\[
R := \int_{r_1}^{\infty} \frac{r}{\sqrt{\Delta}} \left\langle X_{m, a}^\omega(r), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) X_{m, a'}^\omega(r) \right\rangle e^{-r} \, dr
\]

\[
= \int_0^{\infty} \left\langle A_m^\omega \left( e^{-i\Phi_m^\omega(u)} e^{\Phi_m^\omega(u)} f_{\infty, m, a}^\omega(u) \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) A_{m'}^\omega \left( e^{-i\Phi_m^\omega(u)} e^{\Phi_m^\omega(u)} f_{\infty, m', a'}^\omega(u) \right) \right\rangle e^{-u} \, du
\]

\[+ h(m, m')
\]

(here it suffices to integrate over \( [0, \infty) \) with the integration measure \( du \) because the error can again be absorbed into the function \( h(m, m') \)). Using the formula for the phase \( \Phi_m^\omega \) in (2.17), one finds that the contributions involving the sum of the phases \( \pm (\Phi_m^\omega(u) + \Phi_m^\omega(u)) \) are finite due to the oscillations
even in the limit $\varepsilon \searrow 0$ and can thus be absorbed into the function $h(m, m')$. We thus obtain

$$
R = \int_0^\infty \left( (A_m^\omega)^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_m^\omega \right)^\dagger \frac{f_m^{\omega+}, m, a f_m^{\omega+}, m', a'}{i\varepsilon} e^{i\Phi_m^\omega(u) - i\Phi_m^{\omega'}, (u) - \varepsilon u} \, du
$$

$$
+ \int_0^\infty \left( (A_m^\omega)^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_m^{\omega'} \right)^\dagger \frac{f_m^{\omega-}, m, a f_m^{\omega-}, m', a'}{i\varepsilon} e^{i\Phi_m^\omega(u) + i\Phi_m^{\omega'}, (u) - \varepsilon u} \, du
$$

$$
+ h(m, m')
$$

with a new function $h \in L^\infty(I \times I)$. Now a straightforward computation using the explicit form of the matrix $A$ in (2.18) and the phase in (2.17) yields

$$
R = -\varepsilon(\omega) \frac{m}{\omega^2 - m^2} \left( \frac{f_m^{\omega+}, m, a f_m^{\omega+}, m', a'}{i(\omega - m + \varepsilon)} \right) + h(m, m')
$$

(again with a new function $h \in L^\infty(I \times I)$). Linearizing near $m = m'$ gives the distributional equation

$$
\lim_{\varepsilon \searrow 0} R = -\lim_{\varepsilon \searrow 0} \left( \frac{f_m^{\omega+}, m, a f_m^{\omega+}, m', a'}{i(m - m')} - \varepsilon(\omega) \varepsilon \right) + h(m, m')
$$

$$
= -\pi \varepsilon(\omega) \delta(m - m') \left( f_m^{\omega+}, m, a f_m^{\omega+}, m', a' + f_m^{\omega-}, m, a f_m^{\omega-}, m', a' \right)
$$

$$
+ i \frac{PP}{m - m'} \left( f_m^{\omega+}, m, a f_m^{\omega+}, m', a' - f_m^{\omega-}, m, a f_m^{\omega-}, m', a' \right) + h(m, m')
$$

(3.16)

The combination of transmission coefficients in (3.18) can be simplified using the relation (3.5) to obtain

$$
\frac{f_m^{\omega+}, m, a f_m^{\omega+}, m', a'}{f_m^{\omega-}, m, a f_m^{\omega-}, m', a'} = \Theta(m - m') + \delta_{a, a'} \times \begin{cases} 1 & \text{if } a = 1 \\ -1 & \text{if } a = 2 \end{cases}
$$

Using these formulas in (3.15) gives the result.

Proof of Theorem 3.3. The formula comes about by combining the results of the previous two lemmas: While Lemma 3.6 determines the contribution for $m \neq m'$, Lemma 3.7 tells us about the singular behavior at $m = m'$.

In order to compute the contribution for $m \neq m'$, we assume that the mass supports of $\hat{\omega}_{\cdot, a}(\omega)$ and $\hat{\omega}_{\cdot, a'}(\omega)$ are disjoint. Then, according to (3.4),

$$
\langle \mathbf{p}\psi | \mathbf{p}T\phi \rangle - \langle \mathbf{p}T\psi | \mathbf{p}\phi \rangle = -4\pi \int_{r_1}^\infty dr \frac{r}{\sqrt{\Delta}} \int I \, dm \int I' \, dm' \int_{-\infty}^\infty d\omega
$$

$$
\times \sum_{a, a'=1}^2 (m' - m) \mathbf{\hat{w}}_{a, a'}(\omega) \mathbf{\hat{w}}_{a', a}(\omega) \left\langle X_{m, a}^{k\omega n}(r), \begin{pmatrix} 0 \\ 1 \end{pmatrix} X_{m', a'}^{k\omega' n}(r) \right\rangle_{C^2}.
$$
Since \( \hat{\psi}_{m,a} \) and \( \hat{\phi}_{m',a'} \) may be multiplied by arbitrary test functions in \( m \) and \( m' \), respectively, comparing again with (3.4) and the formula of Lemma 3.6, we obtain

\[
\langle p \psi | p \phi \rangle = -4\pi i \int_I \int_I \frac{1}{m - m'} \sum_{a,a'=1}^2 \bar{\psi}_{m,a}(\omega) \hat{\phi}_{m',a'}(\omega) \left\langle f^{\omega}_{0,m,a}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f^{\omega}_{0,m',a'} \right\rangle \, d\omega.
\]

We have thus derived the contribution for \( m \neq m' \) in the formula of the theorem. Moreover, the summands (3.11) and (3.12) also appear in the above formula.

It remains to show that in the case \( |\omega| < m \), the integrals involving the principal part can be combined with the function \( h \) in (3.13). In other words, our task is to show that the function

\[
\chi_{[-m,m]}(\omega) \frac{1}{m - m'} \bar{\psi}_{m,1}(\omega) \hat{\phi}_{m',1}(\omega) \left\langle f^{\omega}_{0,m,1}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f^{\omega}_{0,m',1} \right\rangle_{C^2}
\]

is bounded (meaning that the corresponding integrals are well defined even in the Lebesgue sense without taking a principal value). To this end, we note that in the case \( |\omega| < m \), the fundamental solution \( X^{\omega}_{m,1}(r) \) tends to zero at spatial infinity. Therefore, the differential equation (3.7) implies that

\[
\left\langle f^{\omega}_{0,m,1}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f^{\omega}_{0,m,1} \right\rangle_{C^2} = 0.
\]

As a consequence, the corresponding inner product in (3.19) is of the order \( O(m - m') \), giving the result. \( \Box \)

### 3.3. Interpretation of the Mass Decomposition

We now explain how the different terms in the mass decomposition of Theorem 3.3 come about and discuss their significance. The structure of the mass decomposition becomes clearer if we write it in the form (1.5) and (1.6), where we set

\[
(\psi_m | S_m \phi_m)_m := 4\pi^2 \int_{\mathbb{R} \setminus [-m,m]} \epsilon(\omega) \, d\omega \sum_{a,a'=1}^2 \bar{\psi}_{m,a}(\omega) \hat{\phi}_{m,a'}(\omega) \left\langle f^{\omega}_{\infty,m,a}, f^{\omega}_{\infty,m,a'} \right\rangle_{C^2},
\]

and \( \mathfrak{B}(\psi_m, \phi_m) \) stands for the integrands of the principal value integrals in Theorem 3.3. The relation (3.21) will serve as the definition of the fermionic signature operator \( S_m \) (see Sect. 4); for the moment, the left side of (3.21) merely is a convenient abbreviation.

The term (1.5) involves a single mass integral. Intuitively speaking, this term can be understood from the fact that the mass oscillations give a factor \( \delta(m - m') \), making it possible to carry out one of the mass integrals. Consequently, the contribution (1.5) tells us about the behavior of the Dirac wave functions in the asymptotic end for large times, as shown in Fig. 1 in a conformal diagram.
In this diagram, a spatially compact solution is shown. (The gray region denotes the support.) Clearly, only the diamond up to the horizons $\mathcal{H}^{\pm}$ belongs to our space-time. The mass oscillations come into play at lightlike and time-like infinity. The term (1.6), on the other hand, arises in our analysis as the boundary terms when integrating the Dirac operator by parts (see the proof of Lemma 3.6). With this in mind, these contributions can be understood as boundary terms on the event horizon (as shown in Fig. 1). Usually, such boundary terms describe the flux of Dirac currents. In our setting, the situation is a bit more involved because the masses of the wave functions $\psi_m$ and $\phi_{m'}$ are in general different. But at least in the limit when the masses coincide,

$$\lim_{m' \to m} \mathcal{B}(\psi_m, \phi_{m'}) ,$$

the integrand of the double mass integral goes over to the probability flux of the Dirac current through the event horizon. This can be made more precise by introducing the so-called fermionic flux operator similar to (3.21) by (for details see Sect. 5),

$$(\psi_m | \mathcal{B}_m \phi_m)_m := \lim_{m' \to m} \mathcal{B}(\psi_m, \phi_{m'}) . \quad (3.22)$$

An interesting point is that the boundary terms and the terms arising from the mass oscillations are not independent of each other, but they come from a joint pole structure (as one sees best in (3.16)). This phenomenon can be understood from the unitarity of the time evolution, which gives connections between the behavior of the wave on the event horizon and in the asymptotically flat end.

We finally remark that a similar connection between boundary terms and double mass integrals involving a principal value is found in cosmological de Sitter space-time [2].
4. The Fermionic Signature Operator

The next question is what information on the solution space $H_m$ for fixed $m$ can be extracted from the mass decomposition of Theorem 3.3. One method is to analyze the integrand in the first line; this will be done in this section. Another method is to analyze the integrand of the double integrals in the limiting case $m' \to m$; this will be explained in Sect. 5 below.

The main result of this section is to show that (3.21) uniquely defines the fermionic signature operator $S_m$ as an operator with the following properties:

**Theorem 4.1.** Restricting attention to one angular momentum mode, for all $\psi_m \in H_m$ the fermionic signature operator defined by (3.21) has the alternative representations

$$S_m \psi_m = \frac{1}{2\pi^2} \int_{\mathbb{R}\setminus[-m,m]} \frac{\epsilon(\omega)}{\|f_\infty, m, 1\|_{C_2}^2 + 1} \left( \sum_{a=1}^{2} \Psi^\omega_{m,a} (\Psi^\omega_{m,a} | \psi_m)_m \right) d\omega \quad (4.1)$$

$$(S_m \psi_m)_{m,a}(\omega) = \frac{\epsilon(\omega) \chi_{\mathbb{R}\setminus[-m,m]}(\omega)}{\|f_\infty, m, 1\|_{C_2}^2 + 1} \sum_{b=1}^{2} (T(\omega)^{-1})^{ab} \hat{\psi}_{m,b}(\omega). \quad (4.2)$$

The fermionic signature operator is a bounded symmetric operator on $H_m$ with

$$\|S_m\| \leq 2. \quad (4.3)$$

Moreover, it commutes with the Dirac Hamiltonian $H$. It has the spectral representation

$$S_m = \int_{\mathbb{R}\setminus[-m,m]} S_m(\omega) \, dE_\omega, \quad (4.4)$$

where $E$ is the spectral measure of the Hamiltonian (see (1.8)) with operators $S_m(\omega)$ having the eigenvalues (1.10).

The remainder of this section is devoted to the proof of this theorem. We begin with the representation (3.2),

$$(\psi_m | S_m \phi_m)_m = 2\pi^2 \int_{\mathbb{R}} d\omega \sum_{a,b=1}^{2} (T(\omega)^{-1})^{ab} \hat{\psi}_{m,a}(\omega) (S_m \phi_m)_{m,b}(\omega).$$

Comparing with (3.21) gives

$$\frac{1}{2} \int_{\mathbb{R}} d\omega \sum_{a,b=1}^{2} (T(\omega)^{-1})^{ab} \hat{\psi}_{m,a}(\omega) (S_m \phi_m)_{m,b}(\omega)$$

$$= \int_{\mathbb{R}\setminus[-m,m]} \epsilon(\omega) \, d\omega \sum_{a,a'=1}^{2} \hat{\psi}_{m,a}(\omega) \phi_{m,a'}(\omega) \langle f_\infty, m, a, f_\infty, m, a' \rangle_{C_2}$$

$$= \int_{\mathbb{R}\setminus[-m,m]} \epsilon(\omega) \, d\omega \sum_{a,b,c,a'=1}^{2} (T(\omega)^{-1})^{ab} \hat{\psi}_{m,a}(\omega) \left( t_{bc} \langle f_\infty, m, c, f_\infty, m, a' \rangle_{C_2} \phi_{m,a'}(\omega) \right).$$

Hence,

$$(S_m \phi_m)_{m,b}(\omega) = 2 \epsilon(\omega) \chi_{\mathbb{R}\setminus[-m,m]}(\omega) \sum_{c,a'=1}^{2} t_{bc} \langle f_\infty, m, c, f_\infty, m, a' \rangle_{C_2} \hat{\phi}_{m,a'}(\omega).$$
Using (2.22) and (2.24), we obtain
\[
(8_m\phi_m)(r, \vartheta, \varphi) = \int_{\mathbb{R}} \left(2 \sum_{b=1}^{2} (8_m\phi_m)(r, \vartheta, \varphi) \right) \Psi_{m,b}(r, \vartheta, \varphi) \, d\omega
\]
\[
= 2 \int_{\mathbb{R}[-m, m]} \epsilon(\omega) \, d\omega \sum_{b,c,a' = 1}^{2} t_{bc}^{\omega} \left\langle f_{\infty}^m, m, c, f_{\infty}^m, m, a' \right\rangle_{C^2} \hat{\phi}_{m,a'}(\omega) \Psi_{m,b}(r, \vartheta, \varphi)
\]
\[
= \frac{1}{\pi^2} \int_{\mathbb{R}[-m, m]} \epsilon(\omega) \, d\omega \sum_{b,c,a' = 1}^{2} \frac{t_{bc}^{\omega}}{2} \left\langle f_{\infty}^m, m, c, f_{\infty}^m, m, a' \right\rangle_{C^2} \frac{t_{a'd}^{\omega} (\Psi_{m,d}(\phi_m)_m \Psi_{m,b}(r, \vartheta, \varphi))}{(4.5)}
\]

We conclude that
\[
8_m = \frac{1}{\pi^2} \int_{\mathbb{R}[m, -m]} \epsilon(\omega) \, d\omega \sum_{a,b,c,d = 1}^{2} \Psi_{m,a} \frac{t_{ab}^{\omega} \left\langle f_{\infty}^m, m, b, f_{\infty}^m, m, c \right\rangle_{C^2}}{\pi^2} \frac{t_{cd}^{\omega} (\Psi_{m,d}(\phi_m)_m \Psi_{m,b}(r, \vartheta, \varphi))}{(4.5)}
\]

The combination of transmission coefficients and matrix elements \(t_{ab}^{\omega}\) appearing here is computed in the next lemma.

**Lemma 4.2.** For all \(|\omega| > m,\)
\[
\sum_{b,c=1}^{2} t_{ab}^{\omega} \left\langle f_{\infty}^m, m, b, f_{\infty}^m, m, c \right\rangle_{C^2} t_{cd}^{\omega} = \frac{\delta_{ad}}{2 (1 + \|f_{\infty}^m, m, a\|_{C^2})^2}.
\]

**Proof.** We make use of the explicit formulas for the coefficients \(t_{ab}^{\omega}\) as derived in [8, Theorem 3.6] and [7, Lemma 6.1]. We first recall these results and formulate them in a way most convenient for us. In [8], the integral representation is obtained by first analyzing the system with Dirichlet boundary conditions at \(u_2 \in \mathbb{R}\) and then taking the limit \(u_2 \to \infty\). Considering a linear combination of the radial fundamental solutions
\[
X(u) = c_1 X_1(u) + c_2 X_2(u),
\]
the Dirichlet boundary conditions take the form \(X_+(u_2) = X_-(u_2)\). Evaluating these conditions asymptotically as \(u \to \infty\) with the help of (2.16) and keeping in mind that the normalization at the event horizon implies that \(|c_1|^2 + |c_2|^2 = 1\), one finds
\[
c_1 = \frac{t_1}{\sqrt{|t_1|^2 + |t_2|^2}}, \quad c_2 = \frac{t_2}{\sqrt{|t_1|^2 + |t_2|^2}}
\]
with
\[
t_1(\alpha) = f_{\infty}^+ e^{-i\alpha} - f_{\infty}^- e^{i\alpha}, \quad t_2(\alpha) = -f_{\infty}^+ e^{-i\alpha} + f_{\infty}^- e^{i\alpha}\]
and \(\alpha = \Phi_m^\omega(u)\). The coefficients \(t_{ab}^{\omega}\) are obtained by taking the product \(c_a c_b\) and integrating over \(\alpha\) (see [8, eq. (3.46)]),
\[
t_{ab}^{\omega} = \frac{1}{2\pi} \int_0^{2\pi} \frac{t_a t_b}{|t_1|^2 + |t_2|^2} \, d\alpha.
\]
In [7], more detailed formulas for \(t_{ab}^{\omega}\) were derived using specific properties of the radial equation which become most apparent in the reformulation as the so-called planar equation (see [7, Section 4]). For our purposes, it suffices
and is more convenient to incorporate the additional properties of the radial
equation by employing the methods and results of Lemma 3.4.

Again in the parametrization (3.9), the scalar products in (4.6) become
\[\langle f_\infty^\omega, m, 1, f_\infty^\omega, m, 1 \rangle_{C^2} = \langle f_\infty^\omega, m, 2, f_\infty^\omega, m, 2 \rangle_{C^2} = \cosh(2\theta) \]  
\[\langle f_\infty^\omega, m, 1, f_\infty^\omega, m, 2 \rangle_{C^2} = e^{i\delta} \sinh(2\theta) .\]  

Moreover, using this parametrization in (4.7), a short computation shows that

\[|t_1|^2 = |t_2|^2 .\]  
Applying this relation in (4.8), we immediately find

\[t_\omega^{11} = t_\omega^{22} = \frac{1}{2} \]  

and

\[t_\omega^{12} = \frac{1}{4\pi} \int_0^{2\pi} \frac{t_1 t_2}{|t_2|^2} d\alpha = \frac{1}{4\pi} \int_0^{2\pi} \frac{t_1}{t_2} d\alpha .\]  

Introducing \(z = e^{2i\alpha}\) as the new integration variable, we obtain the contour integral

\[t_\omega^{12} = \frac{1}{2\pi} \oint_{\partial B_1(0)} \frac{f_\infty^+ - f_\infty^-}{f_\infty^+ + f_\infty^-} \left( -\frac{i}{2} \frac{dz}{z} \right) .\]  
The integrand has poles at \(z = 0\) and

\[z = \frac{f_\infty^+}{f_\infty^-} \]  

Since the last pole lies outside our integration contour, we only need to take
into account the contour at \(z = 0\). We thus obtain

\[t_\omega^{12} = -\frac{1}{2} \frac{f_\infty^+}{f_\infty^-} = -\frac{1}{2} e^{i\delta} \tanh \theta .\]  

Finally, the coefficient \(t_\omega^{21}\) is obtained by complex conjugation,

\[t_\omega^{21} = t_\omega^{12} = -\frac{1}{2} e^{-i\delta} \tanh \theta .\]  
Combining the identities (4.10), (4.11) and (4.12) with (4.9), a straight-
forward computation yields

\[\sum_{b,c=1}^{2} t_{ab} \langle f_\infty^\omega, m, b, f_\infty^\omega, m, c \rangle_{C^2} t_{cd}^\omega = \frac{\delta_{ad}}{4} \frac{1}{\cosh^2 \theta} .\]  

Rewriting the factor \(\cosh^2 \theta\) as the absolute square of the vectors in (3.9),

\[\cosh^2 \theta = \frac{1}{2} \left( \cosh^2 \theta + \sinh^2 \theta + 1 \right) = \frac{1}{2} \left( \|f_\infty^\omega, m, a\|_{C^2}^2 + 1 \right) \]  

concludes the proof. \(\square\)
Proof of Theorem 4.1. The representation (4.1) follows immediately by using the identity of Lemma 4.2 in (4.5). Applying (2.22) and (2.24) gives (4.2).

According to (4.2), the fermionic signature operator is a multiplication operator in $\omega$. This implies that it commutes with the Hamiltonian and can be represented in the form (4.4). Moreover, the eigenvalues $\mu_{\pm}$ in (1.10) are the eigenvalues of the matrix in (4.2). In order to compute them, we again work in the parametrization (3.9). Then,

$$T(\omega) = \frac{1}{2} \begin{pmatrix} -e^{-i\delta} \tanh \vartheta & -e^{i\delta} \tanh \vartheta \\ -e^{-i\delta} \tanh \vartheta & 1 \end{pmatrix}.$$

This matrix has the eigenvalues

$$\nu_{\pm} = \frac{1}{2} (1 \mp \tanh \vartheta).$$

Thus, the matrix in (4.2) has the eigenvalues

$$\mu_{\pm} = \frac{\epsilon(\omega)}{2 \cosh^2 \vartheta} \frac{1}{\nu_{\pm}} = \epsilon(\omega) \pm \tanh \vartheta.$$

Finally, we express the hyperbolic tangent in terms of the norm of the vectors in (3.9),

$$\tanh^2 \vartheta = \frac{\cosh^2 \vartheta + \sinh^2 \vartheta - 1}{\cosh^2 \vartheta + \sinh^2 \vartheta + 1} = \frac{\|f_{\omega,m,a}\|_{C^2}^2 - 1}{\|f_{\omega,m,a}\|_{C^2}^2 + 1}.$$

This concludes the proof. 

5. The Fermionic Flux Operator

In this section, we shall analyze how one can extract information on the solution space $H_m$ for fixed $m$ from the double integral in Theorem 3.3. For convenience, we again write this double integral in the form (1.6). In Sect. 3.3, we already mentioned the method of representing the integrand $\mathcal{B}(\psi_m, \phi_{m'})$ in the limit $m' \to m$ in terms of the scalar product on $H_m$, giving rise to the so-called fermionic flux operator $\mathcal{B}_m$ in (3.22). Before entering the details of this construction, we point out that this operator is the only operator on $H_m$ which can be constructed from (1.6). Indeed, a more general idea would be to expand to higher order in the masses before taking the limit $m' \to m$,

$$\lim_{m' \to m} \frac{dp}{dm'} \frac{dq}{dm''} \mathcal{B}(\psi_m, \phi_{m'}) \quad \text{with } p + q > 0. \quad (5.1)$$

However, these bilinear forms depend on how the solutions $\phi_m, \psi_m \in H_m$ are extended to families of solutions described by the mass parameter in $I$. For this reason, the bilinear forms (5.1) do not give rise to well-defined operators on $H_m$. 
Theorem 5.1. Restricting attention to one angular momentum mode, for all \( \psi_m \in \mathcal{H}_m^\infty \) the fermionic flux operator defined by (3.22) has the alternative representations

\[
\mathcal{B}_m \psi_m = -\frac{1}{2\pi^2} \int_{\mathbb{R} \setminus [-m,m]} \frac{1}{\|f^\infty_{\infty, m,1}\|_2^2 + 1} \left( \sum_{a=1}^{2} s_a \Psi_{m,a}^\omega (\Psi_{m,a}^\omega | \psi_m)_m \right) d\omega
\]

(5.2)

\[
(\mathcal{B}_m \psi_m)_{m,a}(\omega) = -\frac{\chi_{\mathbb{R} \setminus [-m,m]}}{\|f^\omega_{\infty, m,1}\|_2^2 + 1} \sum_{b=1}^{2} s_a \left( T(\omega)^{-1} \right)_{ab} \hat{\psi}_m,b(\omega),
\]

(5.3)

where

\[
s_1 = 1 \quad \text{and} \quad s_2 = -1.
\]

(5.4)

The fermionic flux operator is a bounded symmetric operator on \( \mathcal{H}_m \) with \( \|\mathcal{B}_m\| \leq 1 \).

It commutes with the Dirac Hamiltonian \( H \). It has the spectral representation

\[
\mathcal{B}_m = \int_{\mathbb{R} \setminus [-m,m]} \mathcal{B}_m(\omega) \, dE_\omega,
\]

where \( E \) is the spectral measure of the Hamiltonian (see (1.8)) with operators \( \mathcal{B}_m(\omega) \) having the eigenvalues

\[
\nu_{\pm}(\omega) = \pm \frac{2}{\|f^\infty_{\infty, m,1}\|_2^2 + 1}.
\]

Comparing with Theorem 4.1, one sees that the spectral decompositions of the fermionic flux operator and the fermionic signature operator are quite different. Indeed, the sign of the eigenvalues of \( \mathcal{B}_m \) does not depend on the sign of \( \omega \). Instead, the negative spectral subspace of \( \mathcal{B}_m \) describes the Dirac waves which “enter the black hole,” whereas the positive spectral subspace corresponds to Dirac waves which “emerge from the white hole.”

Proof of Theorem 5.1. Comparing the formula in Theorem 3.3 with (1.5) and (1.6), we obtain

\[
(\psi_m | \mathcal{B}_m \phi_m)_m = -4\pi^2 \int_{-\infty}^{\infty} \sum_{a,a'=1}^{2} \bar{\psi}_{m,a}(\omega) \hat{\phi}_{m,a'}(\omega) \left( \begin{array}{c} f^\omega_{0, m,1} \\ f^\omega_{0, m,2} \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ -1 \end{array} \right) d\omega.
\]

The inner product on the right can be further simplified: In the case \( |\omega| < m \), according to (2.23) we only need to consider the contributions for \( a = a' = 1 \). These contributions vanish according to (3.20). Therefore, we do not get a contribution if \( |\omega| < m \).

In the remaining case \( |\omega| > m \), we can use the identity (2.21). We thus obtain

\[
(\psi_m | \mathcal{B}_m \phi_m)_m = -4\pi^2 \int_{\mathbb{R} \setminus [-m,m]} \left( \bar{\psi}_{m,1}(\omega) \hat{\phi}_{m,1}(\omega) - \bar{\psi}_{m,2}(\omega) \hat{\phi}_{m,2}(\omega) \right) d\omega.
\]
Introducing the coefficients $s_a$ by (5.4), we can write this formula as

$$(\psi_m | B_m \phi_m)_m = -4\pi^2 \int_{\mathbb{R} \setminus [-m,m]} \sum_{a=1}^{2} s_a \overline{\psi_{m,a}(\omega)} \hat{\phi}_{m,a}(\omega) \, d\omega$$

$$= -4\pi^2 \int_{\mathbb{R} \setminus [-m,m]} \sum_{a,b,c=1}^{2} (T(\omega)^{-1})_{ab} \overline{\psi_{m,a}(\omega)} t_{bc}^\omega s_c \hat{\phi}_{m,c}(\omega) \, d\omega.$$  

Comparing this formula with (3.2) gives

$$(\hat{B}_m \phi_m)_m,b = -2 \chi_{\mathbb{R} \setminus [-m,m]}(\omega) \sum_{c=1}^{2} t_{bc}^\omega s_c \hat{\phi}_{m,c}(\omega).$$

Using (2.22) and (2.24), we obtain

$$(B_m \phi_m)(r, \vartheta, \varphi) = \int_{\mathbb{R}} \sum_{b=1}^{2} (\hat{B}_m \phi_m)_m,b(\omega) \Psi_m^{\omega}(r, \vartheta, \varphi) \, d\omega$$

$$= -2 \int_{\mathbb{R} \setminus [-m,m]} d\omega \sum_{b_c=1}^{2} t_{bc}^\omega s_c \hat{\phi}_{m,c}(\omega) \Psi_m^{\omega}(r, \vartheta, \varphi)$$

$$= -\frac{1}{\pi^2} \int_{\mathbb{R} \setminus [-m,m]} d\omega \sum_{b_c,d=1}^{2} t_{bc}^\omega s_c t_{cd}^\omega (\Psi_m^{\omega_d} | \phi_m)_m \Psi_m^{\omega}(r, \vartheta, \varphi).$$

We conclude that

$$B_m = -\frac{1}{\pi^2} \int_{\mathbb{R} \setminus [-m,m]} d\omega \sum_{a,b,c=1}^{2} \Psi_m^{\omega_a} t_{ab}^\omega s_b t_{bc}^\omega (\Psi_m^{\omega_c} | \cdot)_m.$$ 

A short computation using the explicit formulas for $t_{ab}^\omega$ as given in (4.10), (4.11) and (4.12) yields

$$\sum_{b=1}^{2} t_{ab}^\omega s_b t_{bc}^\omega = \frac{s_a \delta_{ac}}{4 \cosh^2 \vartheta}.$$ 

Using again (4.13), we obtain (5.2). Applying (2.22) and (2.24) gives (5.3).

The spectrum of the operator $B_m$ is computed similar as in the proof of Theorem 4.1 by diagonalizing the operator in (5.3). \(\square\)

### 6. Generalized Fermionic Projector States

We briefly recall the construction of quasi-free Dirac states as worked out in [10]. According to (4.3), for any $m \in I$ the fermionic signature operator is a bounded symmetric operator on $\mathcal{H}_m$. The fermionic projector $P$ is introduced as the operator (for details see [12, Section 3] and [13, Section 4.2])

$$P = -\chi_{(-\infty,0)}(S_m) k_m : C^\infty(\mathcal{M}, S\mathcal{M}) \to \mathcal{H}_m,$$  

(6.1)
where \( k_m \) is the \textit{causal fundamental solution} defined as the difference of the advanced and retarded Green's operators, 
\[
k_m := \frac{1}{2\pi i} (s^\vee_m - s^\wedge_m) : C_0^\infty(\mathcal{M}, S\mathcal{M}) \to H^\infty_m.
\]

The fermionic projector \( P \) can be written as an integral operator involving a uniquely determined distributional kernel \( P \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}) \), i.e., (for details see [12, Section 3.5])
\[
<\phi|P\psi> = P(\overline{\phi} \otimes \psi) \quad \text{for all } \phi, \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M}).
\]

Araki's construction in [1] yields for any nonnegative operator \( W \) on \( H_m \) a unique quasi-free Dirac state with the property that the two-point distribution coincides with the integral kernel of the operator \(-W k_m\). Applying this construction to the projection operator \( \chi(\mathcal{S}_m) \) gives the so-called \textit{fermionic projector state} (for details see [10, Section 6]).

According to Theorem 4.1, the negative spectral subspace of the fermionic signature operator coincides with the negative spectral subspace of the Hamiltonian; more precisely (see also (1.9))
\[
\chi(-\infty, 0)(\mathcal{S}_m) = \chi(-\infty, m)(H).
\]

We thus reproduce the frequency splitting for the observer in a rest frame at infinity. Clearly, this state is Hadamard. This gives the result of Corollary 1.1.

Applying Araki’s construction to the operator \( W = W(\mathcal{S}_m) \) with \( W \), a nonnegative Borel function gives the so-called \textit{generalized fermionic projector state}. The corresponding two-point distribution is the integral kernel of the operator
\[
P_W = -W(\mathcal{S}_m) k_m : C_0^\infty(\mathcal{M}, S\mathcal{M}) \to H_m.
\]

In ultrastatic space-times, where the operator \( \mathcal{S}_m \) only has the eigenvalues \( \pm 1 \), working with \( W(\mathcal{S}_m) \) does not give anything new. This is why the generalized fermionic projector state was first considered in Rindler space-time [11, Section 11]. (The notion “generalized fermionic projector state” was introduced in [15, Section 2.5].)

The basic question is whether the generalized fermionic projector state is a Hadamard state. We now explain why, for generic \( W \), the generalized fermionic projector cannot be expected to be a Hadamard state. To this end, recall that a state is Hadamard if it realizes the frequency splitting up to smooth contributions. This means in particular that, asymptotically as \( \omega \to \pm \infty \), the state should reproduce the frequency splitting (meaning that \( \lim_{\omega \to \pm \infty} \mu_s = \pm 1 \) for all \( s \in \{+, -\} \)). This condition can be analyzed by looking at the radial equation (2.15). Indeed, as \( \omega \to \pm \infty \), the potential on the right of this equation has little effect on the solutions (as could be made precise for example with a WKB analysis), meaning that the solutions go over asymptotically to plane waves. As a consequence, the norm \( \|f^{k_\omega n}_{m,1}\|_{C^2} \) in (1.10) tends to one. We conclude that
\[
\lim_{\omega \to \pm \infty} \mu_\pm(\omega) = \epsilon(\omega),
\]
implying that, as desired, for large frequencies we recover frequency splitting. This argument has the caveat that in order to obtain a Hadamard state, we must recover frequency splitting uniformly in the angular eigenvalue $\lambda$. But this uniformity does not hold for the following reason: Suppose we are given $\omega$ and a compact interval $[u_0, u_1]$. Since the matrix on the right of (2.15) is Hermitian, by choosing $\lambda$ sufficiently large, we can arrange that the fundamental solutions of the radial equation are exponentially increasing or decreasing on the interval $[u_0, u_1]$. This means that, for any fixed $\omega$, the norm $\|f_{\infty, m, 1}^{k\omega n}\|_{C^2}$ in (1.10) can be made arbitrarily large by increasing $\lambda$. Therefore, except in the case when $W$ is constant on the intervals $[-2, 0)$ and $(0, 2]$ (in which case we get merely a linear combination of the fermionic projector (6.1) and the operator $k_m$), the generalized fermionic projector does not reproduce frequency splitting for large $\omega$. As a consequence, for generic $W$, the generalized fermionic projector state will not be a Hadamard state.

Clearly, this argument leaves the possibility that one gets a Hadamard state for specific choices of the function $W$. If this is the case, the next question would be what this state means physically. We leave these questions as open problems for the future.

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References

[1] Araki, H.: On quasifree states of CAR and Bogoliubov automorphisms. Publ. Res. Inst. Math. Sci. 6, 385–442 (1970/71)
[2] Dappiaggi, C., Finster, F., Murro, S., Radici, E.: The fermionic signature operator in De Sitter space-time, arXiv:1902.09144 [math-ph] (2019)
[3] Daudé, T.: Time-dependent scattering theory for charged Dirac fields on a Reissner–Nordström black hole. J. Math. Phys. 51(10), 102504–57 (2010)
[4] Fewster, C.J., Lang, B.: Pure quasifree states of the Dirac field from the fermionic projector. Class. Quantum Grav. 32(9), 095001–30 (2015). arXiv:1408.1645 [math-ph]
[5] Finster, F., Kamran, N.: Spinors on singular spaces and the topology of causal fermion systems. Mem. Am. Math. Soc. 259(1251), v+83 (2019). arXiv:1403.7885 [math-ph]
[6] Finster, F., Kamran, N., Smoller, J., Yau, S.-T.: Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry. Commun. Pure Appl. Math. 53(7), 902–929 (2000). gr-qc/9905047
[7] Finster, F., Kamran, N., Smoller, J., Yau, S.-T.: Decay rates and probability estimates for massive Dirac particles in the Kerr–Newman black hole geometry, Commun. Math. Phys. 230(2), 201–244 (2002). arXiv:gr-qc/0107094
[8] Finster, F., Kamran, N., Smoller, J., Yau, S.-T.: The long-time dynamics of Dirac particles in the Kerr–Newman black hole geometry. Adv. Theor. Math. Phys. 7(1), 25–52 (2003). arXiv:gr-qc/0005088
[9] Finster, F., Müller, O.: Lorentzian spectral geometry for globally hyperbolic surfaces. Adv. Theor. Math. Phys. 20(4), 751–820 (2016). arXiv:1411.3578 [math-ph]
[10] Finster, F., Murro, S., Röken, C.: The fermionic projector in a time-dependent external potential: mass oscillation property and Hadamard states. J. Math. Phys. 57(7), 072303 (2016). arXiv:1501.05522 [math-ph]
[11] Finster, F., Murro, S., Röken, C.: The fermionic signature operator and quantum states in Rindler space-time. J. Math. Anal. Appl. 454(1), 385–411 (2017). arXiv:1606.03882 [math-ph]
[12] Finster, F., Reintjes, M.: A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I—space-times of finite lifetime. Adv. Theor. Math. Phys. 19(4), 761–803 (2015). arXiv:1301.5420 [math-ph]
[13] Finster, F., Reintjes, M.: A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II—space-times of infinite lifetime. Adv. Theor. Math. Phys. 20(5), 1007–1048 (2016). arXiv:1312.7209 [math-ph]
[14] Finster, F., Reintjes, M.: The fermionic signature operator and Hadamard states in the presence of a plane electromagnetic wave. Ann. Henri Poincaré 18(5), 1671–1701 (2017). arXiv:1609.04516 [math-ph]
[15] Finster, F., Reintjes, M.: The fermionic signature operator and space-time symmetries. Adv. Theor. Math. Phys. (2019). arXiv:1708.09643 [math-ph]
[16] Goldberg, J.N., Macfarlane, A.J., Newman, E.T., Rohrlich, F., Sudarshan, E.C.G.: Spin-s spherical harmonics and ∂. J. Math. Phys. 8, 2155–2161 (1967)
[17] Melnyk, F.: Scattering on Reissner-Nordstrøm metric for massive charged spin 1/2 fields. Ann. Henri Poincaré 4(5), 813–846 (2003)
[18] Nicolas, J.-P.: Scattering of linear Dirac fields by a spherically symmetric black hole. Ann. Inst. H. Poincaré Phys. Théor. 62(2), 145–179 (1995)

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