CALDERÓN PROBLEM FOR MAXWELL’S EQUATIONS IN THE WAVE GUIDE

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Abstract. For Maxwell’s equations in a wave guide, we prove the global uniqueness in determination of the conductivity, the permeability and the permittivity by partial Dirichlet-to-Neumann map limited to an arbitrary subboundary.

Let \( \hat{\Omega} \) be a cylinder in \( \mathbb{R}^3 \) and let \( i = \sqrt{-1}, x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Let \( \hat{E} = (\hat{E}_1, \hat{E}_2, \hat{E}_3) \) be the electric field, \( \hat{H} = (\hat{H}_1, \hat{H}_2, \hat{H}_3) \) the magnetic field, \( \sigma \) be the conductivity, \( \mu \) the permeability and \( \epsilon \) permittivity and \( \omega \in \mathbb{R}^1, \omega \neq 0 \) be a frequency. Then Maxwell’s equations are given by

\[
\begin{align*}
(0.1) & \quad \text{curl } \hat{E} - i\omega \mu \hat{H} = 0, \quad \text{in } \hat{\Omega}, \\
(0.2) & \quad \text{curl } \hat{H} + i\omega \gamma \hat{E} = 0, \quad \text{in } \hat{\Omega}.
\end{align*}
\]

In this paper, we consider Maxwell’s equation inside a wave guide \( \hat{\Omega} \) (e.g., Jackson [10], Chapter 8). More precisely, let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary \( \partial \Omega \) and \( \hat{\Omega} = \Omega \times (-\infty, \infty) = \{ (x_1, x_2, x_3); (x_1, x_2) \in \Omega, x_3 \in \mathbb{R} \} \). We assume

\[
\hat{E}(x_1, x_2, x_3) = E(x_1, x_2)e^{hx_3}, \quad \hat{H}(x_1, x_2, x_3) = H(x_1, x_2)e^{hx_3}, \quad x_3 \in \mathbb{R}^1,
\]

where \( h \in \mathbb{C} \) is a constant. Then we can rewrite Maxwell’s equations in \( \Omega \):

\[
\begin{align*}
(0.3) & \quad L_{1,\mu,\gamma}(x, D)(E, H) := \left( \begin{array}{c} \partial_{x_2} E_3 - h E_2 \\ -\partial_{x_1} E_3 + h E_1 \\ \partial_{x_1} E_2 - \partial_{x_2} E_1 \end{array} \right) - i\omega \mu \left( \begin{array}{c} H_1 \\ H_2 \\ H_3 \end{array} \right) = 0, \quad \text{in } \Omega, \\
(0.4) & \quad L_{2,\mu,\gamma}(x, D)(E, H) := \left( \begin{array}{c} \partial_{x_2} H_3 - h H_2 \\ -\partial_{x_1} H_3 + h H_1 \\ \partial_{x_1} H_2 - \partial_{x_2} H_1 \end{array} \right) + i\omega \gamma \left( \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \right) = 0, \quad \text{in } \Omega.
\end{align*}
\]

Here and henceforth we set

\[
\gamma = \epsilon + \frac{i\sigma}{\omega}
\]

and

\[
(0.5) \quad L_{\mu,\gamma}(x, D)(E, H) = (L_{1,\mu,\gamma}(x, D)(E, H), L_{2,\mu,\gamma}(x, D)(E, H)).
\]

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By \((\nu_1, \nu_2)\) we denote the outward unit normal vector to \(\partial \Omega\) and we set \(\vec{\nu} = (\nu_1, \nu_2, 0)\). Then we note that
\[
\vec{\nu} \times E = \begin{pmatrix} \nu_2 E_3 \\ -\nu_1 E_3 \\ \nu_1 E_2 - \nu_2 E_1 \end{pmatrix} \quad \text{on } \partial \Omega.
\]

Let \(\tilde{\Gamma}\) be some fixed open subset of \(\partial \Omega\) and \(\Gamma_0 = \partial \Omega \setminus \tilde{\Gamma}\). Consider the following Dirichlet-to-Neumann map
\[
\Lambda_{\mu, \gamma} f = \vec{\nu} \times H = \begin{pmatrix} \nu_2 H_3 \\ -\nu_1 H_3 \\ \nu_1 H_2 - \nu_2 H_1 \end{pmatrix} \quad \text{on } \tilde{\Gamma},
\]
where
\[
L_{\mu, \gamma}(x, D)(E, H) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times E|_{\Gamma_0} = 0, \quad \vec{\nu} \times E|_{\tilde{\Gamma}} = f.
\]

In general for some values of the parameter \(\omega\), the boundary value problem
\[
L_{\mu, \gamma}(x, D)(E, H) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times E|_{\Gamma_0} = 0, \quad \vec{\nu} \times E|_{\tilde{\Gamma}} = f
\]
may not have a solution for some \(f\). By \(D_{\mu, \gamma}\) we denote the set of functions \(f \in W^1_2(\tilde{\Gamma})\) such that there exists at least one solution to (0.8). As for the mathematical theory on the boundary value problem for Maxwell’s equations, we refer for example to Dautray and Lions [3].

In general for some \(f \in D_{\mu, \gamma}\), there exists more than one solutions. In that case as the value of \(\Lambda_{\mu, \gamma} f\), we consider the set of all functions \(\vec{\nu} \times H\) where the pairs \((E, H)\) are the all possible solutions to (0.8). Thus our definition of the Dirichlet-to-Neumann map is different from the classical one, and we have to specify the conception of the equality of the Dirichlet-to-Neumann maps.

**Definition.** We say that the Dirichlet-to-Neumann maps \(\Lambda_{\mu_1, \gamma_1}\) and \(\Lambda_{\mu_2, \gamma_2}\) are equal if \(D_{\mu_1, \gamma_1} \subset D_{\mu_2, \gamma_2}\) and for any pair \((E, H)\) which solves
\[
L_{\mu_1, \gamma_1}(x, D)(E, H) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times E|_{\Gamma_0} = 0, \quad \vec{\nu} \times E|_{\tilde{\Gamma}} = f,
\]
there exists a pair \((\tilde{E}, \tilde{H})\) which solves
\[
L_{\mu_2, \gamma_2}(x, D)(\tilde{E}, \tilde{H}) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times \tilde{E}|_{\Gamma_0} = 0, \quad \vec{\nu} \times \tilde{E}|_{\tilde{\Gamma}} = f
\]
and
\[
\vec{\nu} \times H = \vec{\nu} \times \tilde{H} \quad \text{on } \tilde{\Gamma}.
\]

Then we can state our main result:

**Theorem** We assume that \(h^2 + \omega^2 \gamma_j \mu_j \neq 0\) on \(\overline{\Omega}\), \(j = 1, 2\). Let \(\Omega\) be a simply connected domain, \(\mu_j, \epsilon_j, \sigma_j \in C^5(\Omega)\) for \(j \in \{1, 2\}\) and \(\mu_j, \epsilon_j\) be the positive functions on \(\Omega\). Suppose that \(\Lambda_{\mu_1, \gamma_1} = \Lambda_{\mu_2, \gamma_2}\) and
\[
\mu_1 - \mu_2 = \gamma_1 - \gamma_2 = 0 \quad \text{on } \tilde{\Gamma}.
\]
Then \(\mu_1 = \mu_2, \epsilon_1 = \epsilon_2\) and \(\sigma_1 = \sigma_2\) in \(\Omega\).
Moreover by lim \( \eta \to \infty \) we define \( \Phi \) as (1.1) this Dirichlet-to-Neumann map is also known. From (0.3) and (0.4) we obtain \( (1.5) \). We rewrite equations (1.1) and (1.2) as

\[
\begin{align*}
H_1 &= \frac{1}{i\omega \mu} (\partial_{x_2} E_3 - h E_2), \\
H_2 &= -\frac{1}{i\omega \mu} (\partial_{x_1} E_3 - h E_1), \\
\partial_{x_1} H_2 - \partial_{x_2} H_1 &= -i\omega \gamma E_3 \quad \text{in } \Omega.
\end{align*}
\]

and

\[
\begin{align*}
E_1 &= -\frac{1}{i\omega \gamma} (\partial_{x_2} H_3 - h H_2), \\
E_2 &= \frac{1}{i\omega \gamma} (\partial_{x_1} H_3 - h H_1), \\
\partial_{x_1} E_2 - \partial_{x_2} E_1 &= i\omega \mu H_3 \quad \text{in } \Omega.
\end{align*}
\]

Denote \( E' = (E_1, E_2), H' = (H_1, H_2) \). After we plug into the third equation of (1.1) expressions for \( H_1 \) and \( H_2 \) from the first two equations we obtain

\[
\text{div} \left( \frac{1}{i\omega \mu} \nabla E_3 \right) - h \text{div} \left( \frac{E'}{i\omega \mu} \right) - i\omega \gamma E_3 = 0 \quad \text{in } \Omega.
\]

Similarly, from (1.2) we obtain

\[
\text{div} \left( \frac{1}{i\omega \gamma} \nabla H_3 \right) - h \text{div} \left( \frac{H'}{i\omega \gamma} \right) - i\omega \mu H_3 = 0 \quad \text{in } \Omega.
\]

We rewrite equations (1.1) and (1.2) as

\[
\begin{align*}
\begin{cases}
H_1 + \frac{h E_2}{i\omega \mu} = \frac{1}{i\omega \mu} \partial_{x_2} E_3, \\
H_2 - \frac{h E_1}{i\omega \mu} = -\frac{1}{i\omega \mu} \partial_{x_1} E_3,
\end{cases} \\
\begin{cases}
E_1 - \frac{h E_2}{i\omega \gamma} = -\frac{1}{i\omega \gamma} \partial_{x_2} H_3, \\
E_2 + \frac{h H_3}{i\omega \gamma} = \frac{1}{i\omega \gamma} \partial_{x_1} H_3.
\end{cases}
\end{align*}
\]

1. **Step 1: Reduction of the Maxwell’s equations to the decoupled system of elliptic equations.**

In this section we transform the system (1.3), (1.4) to some second order system of elliptic equation decoupled respect to the principal part. Then we set up some new Dirichlet-to-Neumann associated with this system of elliptic equations and show that if \( \Lambda_{\mu, \gamma} \) is known this Dirichlet-to-Neumann map is also known. From (1.3) and (1.4) we obtain

(1.1) 

\[
\begin{align*}
H_1 &= \frac{1}{i\omega \mu} (\partial_{x_2} E_3 - h E_2), \\
H_2 &= -\frac{1}{i\omega \mu} (\partial_{x_1} E_3 - h E_1), \\
\partial_{x_1} H_2 - \partial_{x_2} H_1 &= -i\omega \gamma E_3 \quad \text{in } \Omega.
\end{align*}
\]

and

(1.2) 

\[
\begin{align*}
E_1 &= -\frac{1}{i\omega \gamma} (\partial_{x_2} H_3 - h H_2), \\
E_2 &= \frac{1}{i\omega \gamma} (\partial_{x_1} H_3 - h H_1), \\
\partial_{x_1} E_2 - \partial_{x_2} E_1 &= i\omega \mu H_3 \quad \text{in } \Omega.
\end{align*}
\]

Denote \( E' = (E_1, E_2), H' = (H_1, H_2) \). After we plug into the third equation of (1.1) expressions for \( H_1 \) and \( H_2 \) from the first two equations we obtain

(1.3) 

\[
\text{div} \left( \frac{1}{i\omega \mu} \nabla E_3 \right) - h \text{div} \left( \frac{E'}{i\omega \mu} \right) - i\omega \gamma E_3 = 0 \quad \text{in } \Omega.
\]

Similarly, from (1.2) we obtain

(1.4) 

\[
\text{div} \left( \frac{1}{i\omega \gamma} \nabla H_3 \right) - h \text{div} \left( \frac{H'}{i\omega \gamma} \right) - i\omega \mu H_3 = 0 \quad \text{in } \Omega.
\]

We rewrite equations (1.1) and (1.2) as

(1.5) 

\[
\begin{align*}
\begin{cases}
H_1 + \frac{h E_2}{i\omega \mu} = \frac{1}{i\omega \mu} \partial_{x_2} E_3, \\
H_2 - \frac{h E_1}{i\omega \mu} = -\frac{1}{i\omega \mu} \partial_{x_1} E_3,
\end{cases} \\
\begin{cases}
E_1 - \frac{h E_2}{i\omega \gamma} = -\frac{1}{i\omega \gamma} \partial_{x_2} H_3, \\
E_2 + \frac{h H_3}{i\omega \gamma} = \frac{1}{i\omega \gamma} \partial_{x_1} H_3.
\end{cases}
\end{align*}
\]

See Caro, Ola and Salo [11] and Ola, Päivärinta and Somersalo [12] for the uniqueness results for Maxwell’s equations in three dimensions. In a special two dimensional case of \( h = 0 \), we refer to Imanuvilov and Yamamoto [9], where we can reduce Maxwell’s equations to the conductivity equation with zeroth order term and apply the uniqueness result in Imanuvilov, Uhlmann and Yamamoto [5] to prove the theorem.
Solving the linear system (1.5) respect to $E_1, E_2, H_1, H_2$ we have

\[
\begin{aligned}
(1 + \frac{h^2}{\omega^2 \gamma \mu})E_2 &= \frac{h}{\omega^2 \gamma \mu} \partial_x E_3 + \frac{1}{i \omega \gamma} \partial_x H_3, \\
(1 + \frac{h^2}{\omega^2 \gamma \mu})H_1 &= \frac{h}{\omega^2 \gamma \mu} \partial_x H_3 + \frac{1}{i \omega \mu} \partial_x E_3, \\
(1 + \frac{h^2}{\omega^2 \gamma \mu})E_1 &= \frac{h}{\omega^2 \gamma \mu} \partial_x E_3 - \frac{1}{i \omega \gamma} \partial_x H_3, \\
(1 + \frac{h^2}{\omega^2 \gamma \mu})H_2 &= \frac{h}{\omega^2 \gamma \mu} \partial_x H_3 - \frac{1}{i \omega \mu} \partial_x E_3.
\end{aligned}
\]

We set $g = (1 + \frac{h^2}{\omega^2 \gamma \mu})$ and $\nabla P = (\frac{\partial P}{\partial x_2}, -\frac{\partial P}{\partial x_1})$.

We can rewrite these equations as

\[
\begin{aligned}
(1.7) & \quad E' = \frac{h}{\omega^2 \gamma \mu} \nabla E_3 - \frac{1}{i \omega \gamma} \nabla H_3, \\
(1.8) & \quad H' = \frac{h}{\omega^2 \gamma \mu} \nabla H_3 + \frac{1}{i \omega \mu g} \nabla E_3,
\end{aligned}
\]

The simple computations imply

\[
\begin{aligned}
\operatorname{div} \left( \frac{E'}{i \omega \mu} \right) &= \operatorname{div} \left( \frac{h^2}{i \omega^3 \gamma \mu^2} \nabla E_3 + \frac{h}{\omega^2 \gamma \mu g} \nabla H_3 \right) = \\
&= \left( \nabla \frac{h^2}{i \omega^3 \gamma \mu^2}, \nabla E_3 \right) + \left( \nabla \frac{h}{\omega^2 \gamma \mu g}, \nabla H_3 \right) + \frac{h^2}{i \omega^3 \gamma \mu^2} \Delta E_3
\end{aligned}
\]

and

\[
\begin{aligned}
\operatorname{div} \left( \frac{H'}{i \omega \gamma} \right) &= \operatorname{div} \left( \frac{h^2}{i \omega^3 \gamma \mu^2} \nabla H_3 - \frac{h}{\omega^2 \gamma \mu g} \nabla E_3 \right) = \\
&= \left( \nabla \frac{h^2}{i \omega^3 \gamma \mu^2}, \nabla H_3 \right) - \left( \nabla \frac{h}{\omega^2 \gamma \mu g}, \nabla E_3 \right) + \frac{h^2}{i \omega^3 \gamma \mu^2} \Delta H_3
\end{aligned}
\]

We substitute the formulae for $\operatorname{div} \left( \frac{E'}{i \omega \mu} \right)$ in (1.3) and formulae for $\operatorname{div} \left( \frac{H'}{i \omega \gamma} \right)$ into (1.4) to obtain:

\[
\begin{aligned}
(1.11) & \quad L_{(1)}(x, D)E_3 = \operatorname{div} \left( \left( \frac{1}{i \omega \mu} - \frac{h^2}{i \omega^3 \gamma \mu^2} \right) \nabla E_3 \right) - \left( \nabla \frac{h}{\omega^2 \gamma \mu g}, \nabla H_3 \right) - i \omega \gamma E_3 = 0 \quad \text{in } \Omega, \\
(1.12) & \quad E_3|_{\Gamma_0} = 0
\end{aligned}
\]

and

\[
\begin{aligned}
(1.13) & \quad L_{(2)}(x, D)H_3 = \operatorname{div} \left( \left( \frac{1}{i \omega \gamma} - \frac{h^2}{i \omega^3 \gamma \mu^2} \right) \nabla H_3 \right) + \left( \nabla \frac{h}{\omega^2 \gamma \mu g}, \nabla E_3 \right) - i \omega \mu H_3 = 0 \quad \text{in } \Omega, \\
(1.14) & \quad \frac{\partial H_3}{\partial \nu}|_{\Gamma_0} = 0.
\end{aligned}
\]
We set \( \rho = (\rho_1, \rho_2, \rho_3) \) where

\[
\rho_1 = \frac{1}{i\omega\mu} - \frac{\omega}{i\omega^3g\gamma\mu^2} = \frac{\omega\gamma}{i(h^2 + \omega^2\gamma\mu)}.
\]

\[
\rho_2 = \frac{1}{i\omega\gamma} - \frac{h^2}{i\omega^3g\gamma^2\mu} = \frac{\omega\mu}{i(h^2 + \omega^2\gamma\mu)}.
\]

\[
\rho_3 = \frac{h}{\omega^2\mu\gamma g}.
\]

Then we write (1.11) - (1.14) as

\[
L(x, D)W = \Delta W + 2A \partial \bar{z} W + 2B \partial \bar{z} W + QW = 0 \text{ in } \Omega, \quad B(x, D)W|_{\Gamma_0} = 0,
\]

where

\[
A = \left( \begin{array}{cc}
\frac{\partial_z \ln \rho_1}{\rho_1} & -\frac{i}{\rho_1} \partial_z \rho_3 \\
\frac{i}{\rho_2} \partial_z \rho_3 & \frac{\partial_z \ln \rho_2}{\rho_2}
\end{array} \right), \quad B = \left( \begin{array}{cc}
\frac{\partial_z \ln \rho_1}{\rho_1} & \frac{i}{\rho_3} \partial_z \rho_3 \\
-\frac{i}{\rho_2} \partial_z \rho_3 & \frac{\partial_z \ln \rho_2}{\rho_2}
\end{array} \right), \quad Q = \left( \begin{array}{cc}
\frac{-i\omega\gamma}{\rho_1} & 0 \\
0 & \frac{-i\omega\mu}{\rho_2}
\end{array} \right),
\]

\( W = (E_3, H_3) \) and \( B(x, D)W = (E_3, \frac{\partial H_3}{\partial \nu}) \).

Indeed the equation \( E_3|_{\Gamma_0} = 0 \) follows from (0.6) and (0.8). Let us show that the boundary condition (1.14) holds true. By (0.6), (0.8), (1.2) and the boundary condition \( \vec{\nu} \times E|_{\Gamma_0} = 0 \) we have on \( \Gamma_0 \):

\[
0 = \nu_1 E_2 - \nu_2 E_1 = \nu_1 \frac{1}{i\omega\gamma}(\partial_x H_3 - hH_1) + \nu_2 \frac{1}{i\omega\gamma}(\partial_x H_3 - hH_2) = \frac{1}{i\omega\gamma}(\partial_\nu H_3 - h(\vec{\nu}, H')).
\]

From (1.1) we have

\[
(\vec{\nu}, H') = \frac{1}{i\omega\mu}\partial_\nu E_3 - h\nu_1 E_2 + h\nu_2 E_1 = 0 \text{ on } \Gamma_0.
\]

Hence (1.12) and (0.6), (0.8) imply (1.14).

We set \( \mathcal{R}(x, D)W = (\partial_\nu W_1, W_2) \).

**Proposition 1.1.** Let the traces of the functions \( \gamma \) and \( \mu \) are given on \( \tilde{\Gamma} \). If the Dirichlet-to-Neumann map (0.7) is known, then the following Dirichlet-to-Neumann map is known:

\[
f \to \mathcal{R}(x, D)W|_{\tilde{\Gamma}}, \quad L(x, D)W = 0 \text{ in } \Omega, \quad \mathcal{B}(x, D)W = f, \quad \text{supp } f \subset \tilde{\Gamma}.
\]

**Proof.** Let \( W = (W_1, W_2) \) be a smooth function such that \( L(x, D)W = 0 \) in \( \Omega \). We set \( E_3 = W_1 \) and \( H_3 = W_2 \). Then we introduce functions \( H_1, H_2, E_1, E_2 \) by formulae (1.6). These formulae are equivalent to (1.7) and (1.8). Then the formulae (1.9) and (1.10) holds true. Hence we may write the equations \( L(1)(x, D)W = 0 \) in form (1.3) and equation \( L(2)(x, D)W = 0 \) in form (1.4). By (1.6) we have first two formulae in (1.1) and (1.2). Finally from first two formulae in (1.2) and (1.4) we obtain the third formula in (1.1). And from first two formulae in (1.1) and (1.3) we obtain the third formula in (1.2). Let \( f = (f_1, f_2) \) be a smooth function, \( \text{supp } f \subset \tilde{\Gamma} \) such that there exists a solution to the boundary value problem

\[
L(x, D)W = 0 \text{ in } \Omega, \quad \mathcal{B}(x, D)W = f.
\]
We set \( f = (\nu_2 f_1, -\nu_1 f_1, -\frac{1}{i\omega^\gamma} f_2 + \frac{h}{\omega^2 \gamma^\mu \partial \tau f_1}) \). Then if the functions \( H \) and \( E \) are defined as above we have \( \nu \times E|_{\Omega} = f \) and \( \text{supp } f \subset \tilde{\Gamma} \). Therefore \( \Lambda_{\mu, \gamma} f \) is given. The formula (0.7) implies that \( H_3|_{\Gamma} = W_2 \) is given and \( (\nu_1 H_2 - \nu_2 H_1)|_{\tilde{\Gamma}} \) is given. Using the formula

\[
(\nu_1 H_2 - \nu_2 H_1) = -\frac{1}{i\omega^\mu} \partial_{\nu} E_3 + \frac{h}{i\omega^\mu} (\tilde{\nu}, E') = -\frac{1}{i\omega^\mu} \partial_{\nu} E_3 + \frac{h}{\omega^2 \gamma^\mu} \partial_{\tau} H_3 + \frac{h}{\omega^2 \gamma^\mu} (\nu_1 H_2 - \nu_2 H_1) \quad \text{on } \tilde{\Gamma}
\]

we obtain that \( \partial_{\nu} E_3 = \partial_{\nu} W_1 \) is given on \( \tilde{\Gamma} \). \( \blacksquare \)

2. **Step 2: Construction of the operators \( P_B \) and \( T_B \).**

Let \( B \) be a \( 2 \times 2 \) matrix with elements from \( C^{5+\alpha}(\Omega) \) with \( \alpha \in (0, 1) \) and \( \tilde{x} \) be some fixed point from \( \Omega \). By Proposition 9 of [7] for the equation

\[
(2\partial_{\tau} + B)u = 0 \quad \text{in } \Omega,
\]

we can construct solutions \( U_{0,k} \) such that

\[
U_{0,k}(\tilde{x}) = \tilde{c}_k, \quad \forall k \in \{1, 2\}.
\]

Consider the matrix

\[
\Pi(x) = (U_{0,1}(x), U_{0,2}(x)).
\]

Then

\[
\left( \partial_{\tau} + \frac{1}{2} \text{tr} B \right) \det \Pi = 0 \quad \text{in } \Omega.
\]

Hence there exists a holomorphic function \( q(z) \) such that \( \det \Pi = q(z) e^{-\frac{1}{2} \partial_{\tau}^{-1} (\text{tr} B)} \) (see [13]). By \( Q \) we denote the set of zeros of the function \( q \) on \( \overline{\Omega} : Q = \{z \in \overline{\Omega} : q(z) = 0\} \). Obviously \( \text{card } Q < \infty \). By \( \kappa \) we denote the highest order of zeros of the function \( q \) on \( \overline{\Omega} \).

Using Proposition 9 of [7] we construct solutions \( \tilde{U}_{0,k} \) to problem (2.1) such that

\[
\tilde{U}_{0,k}(x) = \tilde{c}_k \quad k \in \{1, 2\} \quad \forall x \in Q.
\]

Set \( \tilde{\Pi}(x) = (\tilde{U}_{0,1}, \tilde{U}_{0,2}) \). Then there exists a holomorphic function \( \tilde{q} \) such that \( \det \tilde{\Pi} = \tilde{q}(z) e^{-\frac{1}{2} \partial_{\tau}^{-1} (\text{tr} B)} \). Let \( \tilde{Q} = \{z \in \overline{\Omega} : \tilde{q}(z) = 0\} \) and \( \tilde{\kappa} \) the highest order of zeros of the function \( \tilde{q} \).

By \( \tilde{U}_{0,k}(x) = \tilde{c}_k \) for \( x \in Q \), we see that

\[
\tilde{Q} \cap Q = \emptyset.
\]

Therefore there exists a holomorphic function \( r(z) \) such that

\[
r|_{Q} = 0 \quad \text{and} \quad (1 - r(z))|_{\tilde{Q}} = 0
\]

and the orders of zeros of the function \( r \) on \( Q \) and the function \( 1 - r(z) \) on \( \tilde{Q} \) are greater than or equal to the \( \max\{\kappa, \tilde{\kappa}\} \).

We set

\[
P_B f = \frac{1}{2} \Pi \tilde{\alpha}_x^{-1}(\Pi^{-1} r f) + \frac{1}{2} \tilde{\Pi} \tilde{\alpha}_x^{-1}(\tilde{\Pi}^{-1}(1 - r f)).
\]
Then
\[ P_B^* f = -\frac{1}{2} r (\Pi^{-1})^* \partial_x^{-1} (\Pi^* f) - \frac{1}{2} (1 - r) (\Pi^{-1})^* \partial_y^{-1} (\Pi^* f). \]

We have

**Proposition 2.1.** The linear operators \( P_B, P_B^* \in \mathcal{L}(L^2(\Omega), W^1_2(\Omega)) \) solve the differential equations
\[ (2.3) \quad (-2\partial_x + B^*) P_B^* g = g, \quad (2\partial_x + B) P_B g = g \text{ in } \Omega. \]

**Proof.** Since \( \partial_x \Pi = -\frac{1}{2} B \Pi \) and \( \partial_x \tilde{\Pi} = -\frac{1}{2} B \tilde{\Pi} \), short computations imply
\[
\partial_x P_B f = \partial_x \left\{ \frac{1}{2} \Pi \partial_x^{-1} (\Pi^{-1} r f) + \frac{1}{2} \tilde{\Pi} \partial_x^{-1} (\tilde{\Pi}^{-1} (1 - r) f) \right\}
\]
\[
= \frac{1}{2} (\partial_x \Pi) \partial_x^{-1} (\Pi^{-1} r f) + \frac{1}{2} (\partial_x \tilde{\Pi}) \partial_x^{-1} (\tilde{\Pi}^{-1} (1 - r) f)
\]
\[
+ \frac{1}{2} \Pi (\Pi^{-1} r f) + \frac{1}{2} \tilde{\Pi} (\tilde{\Pi}^{-1} (1 - r) f)
\]
\[
= -\frac{1}{4} B \Pi \partial_x^{-1} (\Pi^{-1} r f) - \frac{1}{4} B \tilde{\Pi} \partial_x^{-1} (\tilde{\Pi}^{-1} (1 - r) f)
\]
\[
+ \frac{1}{2} r f + \frac{1}{2} (1 - r) f = -\frac{1}{2} B P_B f + \frac{1}{2} f.
\]

Hence the second equality in \((2.3)\) is proved. In order to prove the first one observe that since \( \Pi \Pi^{-1} = E \) on \( \overline{\Omega} \setminus \mathcal{Q} \). The differentiation of this identity gives
\[
0 = \partial_x (\Pi \Pi^{-1}) = \partial_x \Pi \Pi^{-1} + \Pi \partial_x \Pi^{-1} = -\frac{1}{2} B \Pi \Pi^{-1} + \Pi \partial_x \Pi^{-1}.
\]

This equality can be written as \( \Pi \partial_x \Pi^{-1} = \frac{1}{2} B \). Multiplying both sides of this equality by \( \Pi^{-1} \) we have
\[
\partial_x \Pi^{-1} = \frac{1}{2} \Pi^{-1} B \text{ on } \overline{\Omega} \setminus \mathcal{Q}.
\]

Next we take the adjoint for the left- and the right-hand sides of the above equality:
\[
(\partial_x \Pi^{-1})^* = \partial_x (\Pi^{-1})^* = \left( \frac{1}{2} \Pi^{-1} B \right)^* = \frac{1}{2} B^* (\Pi^{-1})^* \text{ on } \overline{\Omega} \setminus \mathcal{Q}.
\]

Observing that \( (\Pi^{-1})^* = (\Pi^*)^{-1} \), we obtain
\[ (2.4) \quad \partial_x (\Pi^*)^{-1} = \frac{1}{2} B^* (\Pi^*)^{-1} \text{ on } \overline{\Omega} \setminus \mathcal{Q}. \]

Similarly we obtain
\[ (2.5) \quad \partial_x (\tilde{\Pi}^*)^{-1} = \frac{1}{2} B^* (\tilde{\Pi}^*)^{-1} \text{ on } \overline{\Omega} \setminus \tilde{\mathcal{Q}}. \]

Denote by \( \Pi_{ij} \) the cofactor of the \( i, j \)-element of the matrix \( \Pi \) and by \( \tilde{\Pi}_{ij} \) the cofactor of the \( i, j \)-element of the matrix \( \tilde{\Pi} \). Setting
\[
\Gamma = e^{\frac{1}{2} \partial_x^{-1} (\mathrm{tr} B)} \begin{pmatrix} \Pi_{11} & \Pi_{21} \\ \Pi_{12} & \Pi_{22} \end{pmatrix}, \quad \tilde{\Gamma} = e^{\frac{1}{2} \partial_x^{-1} (\mathrm{tr} B)} \begin{pmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{21} \\ \tilde{\Pi}_{12} & \tilde{\Pi}_{22} \end{pmatrix},
\]
we can write the matrices \((\Pi^*)^{-1}, (\tilde{\Pi}^*)^{-1}\) as
\[
(\Pi^*)^{-1} = \frac{1}{q(z)} \Gamma^* \quad \text{on} \quad \overline{\Omega} \setminus \mathcal{Q}, \quad (\tilde{\Pi}^*)^{-1} = \frac{1}{q(z)} \tilde{\Gamma}^* \quad \text{on} \quad \overline{\Omega} \setminus \tilde{\mathcal{Q}}.
\]
Then (2.4) and (2.5) imply
\[
\partial_\tau (\Gamma^*)^{-1} - \frac{1}{2} B^* (\Gamma^*)^{-1} = \partial_\tau (\tilde{\Gamma}^*)^{-1} - \frac{1}{2} B^* (\tilde{\Gamma}^*)^{-1} = 0 \quad \text{on} \quad \overline{\Omega}.
\]
Since \(r/q\) and \((1 - r)/\tilde{q}\) are smooth functions, the above equalities yield
\[
(2.6) \quad \partial_\tau (r(z)(\Pi^*)^{-1}) = \frac{r(z)}{2} B^* (\Pi^*)^{-1}, \quad \partial_\tau ((1 - r(z))(\tilde{\Pi}^*)^{-1}) = \frac{1 - r(z)}{2} B^* (\tilde{\Pi}^*)^{-1} \quad \text{in} \quad \Omega.
\]
Using (2.6), we compute
\[
\partial_\tau P_B^* f = -\partial_\tau \{ \frac{1}{2} r(z)(\Pi^*)^{-1} \partial_\tau (\Pi^* f) + \frac{1}{2} (1 - r(z))(\tilde{\Pi}^*)^{-1} \partial_\tau (\tilde{\Pi}^* f) \}
\]
\[
= -\frac{1}{2} \partial_\tau (r(z)(\Pi^*)^{-1}) \partial_\tau (\Pi^* f) - \frac{1}{2} \partial_\tau ((1 - r(z))(\tilde{\Pi}^*)^{-1}) \partial_\tau (\tilde{\Pi}^* f)
\]
\[
- \frac{1}{2} r(z)(\Pi^*)^{-1} \Pi^* f - \frac{1}{2} (1 - r(z))(\tilde{\Pi}^*)^{-1} \tilde{\Pi}^* f
\]
\[
= -\frac{1}{4} r(z) B^* (\Pi^*)^{-1} \partial_\tau (\Pi^* f) - \frac{1}{4} (1 - r(z)) B^* (\tilde{\Pi}^*)^{-1} \partial_\tau (\tilde{\Pi}^* f)
\]
\[
- \frac{1}{2} r(z) f - \frac{1}{2} (1 - r(z)) f = \frac{1}{2} B^* P_B^* f - \frac{1}{2} f.
\]
The proof of Proposition 2.1 is complete. \(\blacksquare\)

In a similar way we construct matrices \(\Pi_0, \tilde{\Pi}_0\), an antiholomorphic function \(r_0(\overline{z})\) and operators
\[
(2.7) \quad T_B f = \frac{1}{2} \Pi_0 \partial_z^{-1} (\Pi_0^{-1} r_0(\overline{z}) f) + \frac{1}{2} \tilde{\Pi}_0 \partial_z^{-1} (\tilde{\Pi}_0^{-1} (1 - r_0(\overline{z})) f)
\]
and
\[
(2.8) \quad T_B^* f = -\frac{1}{2} r_0(\overline{z})(\Pi_0^{-1})^* \partial_z^{-1} (\Pi_0^* f) - \frac{1}{2} (1 - r_0(\overline{z}))(\tilde{\Pi}_0^{-1})^* \partial_z^{-1} (\tilde{\Pi}_0^* f).
\]

For any matrix \(B \in C^{\delta + \alpha}(\overline{\Omega}), \alpha \in (0, 1)\), the linear operators \(T_B\) and \(T_B^*\) solve the differential equation
\[
(2.9) \quad (2\partial_z + B) T_B g = g \quad \text{in} \quad \Omega; \quad (-2\partial_z + B^*) T_B^* g = g \quad \text{in} \quad \Omega.
\]

Next we introduce two operators
\[
(2.10) \quad \mathcal{R}_{\tau, B} g = e^{\tau(\Phi - \Phi)} T_B (e^{\tau(\Phi - \Phi)} g), \quad \mathcal{R}_{\tau, B} g = e^{\tau(\Phi - \Phi)} T_B (e^{\tau(\Phi - \Phi)} g).
\]
3. Step 3: Construction of complex geometric optics solutions.

In this step, we will construct two complex geometric optics solutions \( u_1 \) and \( v \) respectively for operators \( L_1(x, D) \) and \( L_2(x, D) \).

As the phase function for such a solution we consider a holomorphic function \( \Phi(z) \) such that
\[
\Phi(z) = \varphi(x_1, x_2) + i \psi(x_1, x_2)
\]
with real-valued \( \varphi \) and \( \psi \). For some \( \alpha \in (0, 1) \) the function \( \Phi \) belongs to \( C^{6+\alpha}(\overline{\Omega}) \). Moreover
\[
(3.1) \quad \partial_z \Phi = 0 \text{ in } \Omega, \quad \text{Im } \Phi|_{\Gamma_0} = 0.
\]

Denote by \( \mathcal{H} \) the set of all the critical points of the function \( \Phi \):
\[
\mathcal{H} = \{ z \in \overline{\Omega}; \Phi'(z) = 0 \}.
\]
Assume that \( \Phi \) has no critical points on \( \overline{\Gamma} \), and that all critical points are nondegenerate:
\[
(3.2) \quad \mathcal{H} \cap \partial \Omega = \emptyset, \quad \Phi''(z) \neq 0, \quad \forall z \in \mathcal{H}.
\]

The following proposition asserts the convergence and was proved in [5].

**Proposition 3.1.** Let \( \bar{x} \) be an arbitrary point in the simply connected domain \( \Omega \). There exists a sequence of functions \( \{ \Phi_\epsilon \}_{\epsilon \in (0,1)} \) satisfying (3.1), (3.2) and there exists a sequence \( \{ \bar{x}_\epsilon \}, \epsilon \in (0,1) \) such that
\[
(3.3) \quad \bar{x}_\epsilon \in \mathcal{H}_\epsilon = \{ z \in \overline{\Omega}; \Phi'(z) = 0 \}, \quad \bar{x}_\epsilon \to \bar{x} \text{ as } \epsilon \to +0.
\]
and
\[
(3.4) \quad \text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \notin \{ \text{Im } \Phi_\epsilon(x); x \in \mathcal{H}_\epsilon \setminus \{ \bar{x}_\epsilon \} \} \quad \text{and} \quad \text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \neq 0.
\]

Let the function \( \Phi \) satisfy (3.1), (3.2) and \( \bar{x} \) be some point from \( \mathcal{H} \). Without loss of generality, we may assume that \( \overline{\Gamma} \) is an arc with the endpoints \( x_\pm \).

Consider the following operator:
\[
L_1(x, D) = 4\partial_z \partial_{\overline{z}} + 2A_1 \partial_z + 2B_1 \partial_{\overline{z}} + Q_1
\]
\[
= (2\partial_z + B_1)(2\partial_{\overline{z}} + A_1) + Q_1(1) = (2\partial_z + A_1)(2\partial_z + B_1) + Q_1(2). \quad (3.5)
\]

Here
\[
Q_1(1) = -2\partial_z A_1 - B_1 A_1 + Q_1, \quad Q_2(1) = -2\partial_{\overline{z}} B_1 - A_1 B_1 + Q_1.
\]

Let \( U_0 = (U_{0,1}, U_{0,2}), \tilde{U}_0 = (\tilde{U}_{0,1}, \tilde{U}_{0,2}) \in C^{6+\alpha}(\overline{\Omega}) \) be a nontrivial solution to the boundary value problem:
\[
K(x, D)(U_0, \tilde{U}_0) = (2\partial_z U_0 + A_1 U_0, 2\partial_{\overline{z}} \tilde{U}_0 + B_1 \tilde{U}_0) = 0 \quad \text{in } \Omega, \quad (3.6)
\]
\[
U_0 + I \tilde{U}_0 = 0 \quad \text{on } \Gamma_0, \quad (3.7)
\]
where
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We have
Proposition 3.2. Let $A_1, B_1 \in C^{5+\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, $\vec{r}_{0,k}, \ldots, \vec{r}_{2,k} \in \mathbb{C}^3$ be arbitrary vectors and $x_1, \ldots, x_k$ be mutually distinct arbitrary points from the domain $\Omega$. There exists a solution $(U_0, \tilde{U}_0) \in C^{6+\alpha}(\Omega)$ to problem (3.6), (3.7) such that

$$\partial^j U_0(x_k) = \vec{r}_{j,k} \quad \forall k \in \{1, \ldots, 5\},$$

(3.8)

$$\lim_{x \to x_\pm} \frac{|U_0(x)|}{|x - x_\pm|^{98}} = \lim_{x \to x_\pm} \frac{|\tilde{U}_0(x)|}{|x - x_\pm|^{98}} = 0.$$ 

(3.9)

In (3.9), the number 98 has no special sense. As exponent we can choose sufficiently large $m \in \mathbb{N}$ and henceforth we choose such a number 98.

**Proof.** Let us fix a point $\vec{x}$ from $\mathcal{H} \setminus \{\vec{x}\}$. By Proposition 4.2 of [6] there exists a holomorphic function $a(z) \in C^7(\Omega)$ such that $Im\,a|_{\Gamma_0} = 0$, $a(\vec{x}) = 1$ and $a$ vanishes at each point of the set $\{x_\pm\} \cup \mathcal{H} \setminus \{\vec{x}\}$. Let $(U_{0,0}, \tilde{U}_{0,0}) \in C^{6+\alpha}(\Omega)$ be a solution to problem (3.6) such that $U_{0,0}(\vec{x}) = \vec{z}$. Since $(U_{0,0}, \tilde{U}_{0,0}) = (a^{100}U_{0,0}, \tilde{a}^{100}\tilde{U}_{0,0})$ solves equations (3.6) and satisfies (3.7), the proof of the proposition is completed. \hfill \blacksquare

Now we start the construction of complex geometric optics solution. Let the pair $(U_0, \tilde{U}_0)$ be defined by Proposition 3.2. Short computations and (3.5) yield

$$L_1(x, D)(U_0 e^{r\Phi}) = Q_1(1)U_0 e^{r\Phi}, \quad L_1(x, D)(\tilde{U}_0 e^{r\Phi}) = Q_1(2)\tilde{U}_0 e^{r\Phi}.$$ 

(3.10)

Let $e_1, e_2$ be smooth functions such that

$$\text{supp} \, e_1 \subset \subset \text{supp} \, e = 1, \quad e_1 + e_2 = 1 \text{ on } \Omega,$$ 

(3.11)

and $e_1$ vanishes in a neighborhood of $\partial \Omega$ and $e_2$ vanishes in a neighborhood of the set $\mathcal{H}$.

For any positive $\epsilon$ denote $G_\epsilon = \{x \in \Omega; \text{dist} (\text{supp} \, e_1, x) > \epsilon\}. We have

Proposition 3.3. Let $G_\epsilon \cap \text{supp} e = \emptyset$, $B, q \in C^{5+\alpha}(\Omega)$ for some positive $\alpha \in (0, 1)$ and $\vec{q} \in W^1_p(\Omega)$ for some $p > 2$. Suppose that $q|_{\mathcal{H}} = \vec{q}|_{\mathcal{H}} = 0$. There exist functions $m_{\pm, \vec{x}} \in C^2(\overline{\mathcal{G}}_\epsilon), \vec{x} \in \mathcal{H}$ independent of $\tau$ such that the asymptotic formulae hold true:

$$\lim_{\tau \to +\infty} \frac{\tau}{|\vec{x}|} e^{r\Phi} \left( \sum_{\vec{x} \in \mathcal{H}} \frac{m_{\pm, \vec{x}} e^{2i\tau\psi(\vec{x})}}{\tau^2} + o_{C^2(\mathcal{G}_\epsilon)}\left(\frac{1}{\tau^2}\right) \right) = 0,$$ 

(3.12)

$$\lim_{\tau \to +\infty} \frac{\tau}{|\vec{x}|} e^{-r\Phi} \left( \sum_{\vec{x} \in \mathcal{H}} \frac{m_{\mp, \vec{x}} e^{-2i\tau\psi(\vec{x})}}{\tau^2} + o_{C^2(\mathcal{G}_\epsilon)}\left(\frac{1}{\tau^2}\right) \right) = 0.$$ 

(3.13)

Denote

$$q_1 = P_{A_1}(Q_1(1)U_0) - M_1, \quad q_2 = T_{B_1}(Q_1(2)\tilde{U}_0) - M_2 \in C^{5+\alpha}(\Omega),$$

(3.14)

where the functions $M_1 \in \text{Ker}(2\partial_\tau + A_1)$ and $M_2 \in \text{Ker}(2\partial_\tau + B_1)$ are taken such that

$$q_1(x) = q_2(x) = 0, \quad \forall x \in \mathcal{H}.$$ 

(3.15)

Moreover by (3.9) we can assume that

$$\lim_{x \to x_\pm} \frac{|q_1(x)|}{|x - x_\pm|^{98}} = \lim_{x \to x_\pm} \frac{|q_2(x)|}{|x - x_\pm|^{98}} = 0.$$ 

(3.16)
Next we introduce the functions $U_{-1}, \tilde{U}_{-1}$ as solutions to the following boundary value problems:

(3.17) \[ \mathcal{K}(x, D)(U_{-1}, \tilde{U}_{-1}) = 0 \quad \text{in } \Omega, \quad (U_{-1} + \mathbf{I}\tilde{U}_{-1})|_{r_0} = \mathbf{I}(\frac{q_1}{2\Phi'} + \frac{q_2}{2\Phi'}) + (0, -\frac{1}{\partial_{\nu}\psi}(U_0 + \tilde{U}_0), \bar{e}_2)). \]

We set $p_1 = -Q_1(\frac{e_1\xi}{2\Phi'} - U_{-1}) + L_1(x, D)(\frac{e_2\mathbf{I}q_2}{2\Phi'})$, $p_2 = -Q_1(1)(\frac{e_1\xi}{2\Phi'} - \tilde{U}_{-1}) + L_1(x, D)(\frac{e_2\mathbf{I}q_2}{2\Phi'})$, $\tilde{q}_1 = T_{h_1}p_1 - \tilde{M}_1, \tilde{q}_1 = P_{A_1}p_1 - \tilde{M}_1$, where $\tilde{M}_1 \in Ker(2\partial_{\tau} + A_1)$ and $\tilde{M}_2 \in Ker(2\partial_{\tau} + B_1)$ are taken such that

(3.18) \[ \tilde{q}_1(x) = \tilde{q}_2(x) = 0, \quad \forall x \in \mathcal{H}. \]

By Proposition 3.3 there exist functions $m_{\pm, \bar{x}} \in C^2(\mathcal{G}_x)$ such that

(3.19) \[ \tilde{R}_{\tau, A_1}(e_1(q_1 + \frac{\tilde{q}_1}{\tau}))|_{\mathcal{G}_x} = e^{\tau(\Phi - \varphi)} \left( \sum_{\bar{x} \in \mathcal{H}} \frac{m_{\pm, \bar{x}} e^{2\mathbf{i}\mathbf{v}(\bar{x})}}{\tau^2} + o_{\mathcal{W}_2}(\mathcal{G}_x) \left( \frac{1}{\tau^2} \right) \right) \quad \text{as } |\tau| \to +\infty \]
and

(3.20) \[ \tilde{R}_{\tau, B_1}(e_1(q_1 + \frac{\tilde{q}_1}{\tau}))|_{\mathcal{G}_x} = e^{\tau(\Phi - \varphi)} \left( \sum_{\bar{x} \in \mathcal{H}} \frac{m_{\pm, \bar{x}} e^{-2\mathbf{i}\mathbf{v}(\bar{x})}}{\tau^2} + o_{\mathcal{W}_2}(\mathcal{G}_x) \left( \frac{1}{\tau^2} \right) \right) \quad \text{as } |\tau| \to +\infty. \]

For any $\bar{x} \in \mathcal{H}$ we introduce the functions, $a_{\pm, \bar{x}}, b_{\pm, \bar{x}} \in C^2(\mathcal{G}_x)$ as solutions to the boundary value problem

(3.21) \[ \mathcal{K}(x, D)(a_{\pm, \bar{x}}, b_{\pm, \bar{x}}) = 0 \quad \text{in } \Omega, \quad (a_{\pm, \bar{x}} + \mathbf{I}b_{\pm, \bar{x}})|_{r_0} = \pm m_{\pm, \bar{x}}. \]

Since by (3.18) the functions $\frac{\bar{q}_1}{2\Phi'}, \frac{\bar{q}_2}{2\Phi'}$ belong to the space $W^1_2(\partial\Omega)$ there exists a solution $(U_{-2}, \tilde{U}_{-2}) \in W^1_2(\Omega)$ to the boundary value problem

(3.22) \[ \mathcal{K}(x, D)(U_{-2}, \tilde{U}_{-2}) = 0 \quad \text{in } \Omega, \]
\[ (U_{-2} + \mathbf{I}\tilde{U}_{-2})|_{r_0} = \left( (\frac{\bar{q}_1}{2\Phi'} + \frac{\bar{q}_2}{2\Phi'}, \bar{e}_1), (-\frac{\bar{q}_1}{2\Phi'} + \frac{\bar{q}_2}{2\Phi'}, \bar{e}_2) \right) \]
\[ -\frac{1}{i\partial_{\nu}\psi}(0, \partial_{\nu}(U_{-1} - \frac{e_2q_1}{2\Phi'} + \tilde{U}_{-1} - \frac{e_2q_2}{2\Phi'}, \bar{e}_2)). \]

We introduce the functions $U_{0, \tau}, \tilde{U}_{0, \tau} \in W^1_2(\Omega)$ by

(3.23) \[ U_{0, \tau} = U_0 + \frac{U_{-1} - e_2q_1/2\Phi'}{\tau} + \frac{1}{\tau^2} \left( \sum_{\bar{x} \in \mathcal{H}} \left( e^{2\mathbf{i}\mathbf{v}(\bar{x})} a_{+, \bar{x}} + e^{-2\mathbf{i}\mathbf{v}(\bar{x})} a_{-, \bar{x}} \right) + U_{-2} - \frac{\tilde{q}_1e_2}{2\Phi'} \right) \]
and

(3.24) \[ \tilde{U}_{0, \tau} = \tilde{U}_0 + \frac{\tilde{U}_{-1} - e_2q_2/2\Phi'}{\tau} + \frac{1}{\tau^2} \left( \sum_{\bar{x} \in \mathcal{H}} \left( e^{2\mathbf{i}\mathbf{v}(\bar{x})} b_{+, \bar{x}} + e^{-2\mathbf{i}\mathbf{v}(\bar{x})} b_{-, \bar{x}} \right) + \tilde{U}_{-2} - \frac{\tilde{q}_2e_2}{2\Phi'} \right). \]
Simple computations and Proposition 8 of [7] imply for any $p \in (1, \infty)$ the asymptotic formula:

\[
\begin{align*}
L_1(x, D)(-e^{r\Phi} \tilde{R}_{\tau, B_1}(e_1(q_1 + \tilde{q}_1/\tau)) &- \frac{e_2(q_1 + \tilde{q}_1/\tau)e^{r\Phi}}{2\tau \Phi'} - e^{r\Phi} \tilde{R}_{\tau, A_1}(e_1(q_2 + \tilde{q}_2/\tau)) \notag \\
&- \frac{e_2(q_2 + \tilde{q}_2/\tau)e^{r\Phi}}{2\tau \Phi'} = -L_1(x, D)(e^{r\Phi} \tilde{R}_{\tau, B_1}(e_1(q_1 + \tilde{q}_1/\tau)) + \frac{e_2(q_1 + \tilde{q}_1/\tau)e^{r\Phi}}{2\tau \Phi'}) \\
&- L_1(x, D)(e^{r\Phi} \tilde{R}_{\tau, A_1}(e_1(q_2 + \tilde{q}_2/\tau)) + \frac{e_2(q_2 + \tilde{q}_2/\tau)e^{r\Phi}}{2\tau \Phi'}) \\
&= -Q_1(2)e^{r\Phi} \tilde{R}_{\tau, B_1}(e_1(q_1 + \tilde{q}_1/\tau)) - Q_1(1)e^{r\Phi} \tilde{R}_{\tau, A_1}(e_1(q_2 + \tilde{q}_2/\tau)) \\
&- e^{r\Phi} L_1(x, D)(\frac{e_2(q_1 + \tilde{q}_1/\tau)}{2\tau \Phi'}) - e^{r\Phi} L_1(x, D)(\frac{e_2(q_2 + \tilde{q}_2/\tau)}{2\tau \Phi'}) \\
&- Q_1(2)\tilde{U}_0 e^{r\Phi} - Q_1(1)U_0 e^{r\Phi} + \frac{1}{\tau}(Q_1(2)(\frac{e_1 q_1}{2\Phi'} - U_1) + L_1(x, D)(\frac{e_2 q_1}{2\Phi'}))e^{r\Phi} \\
&+ \frac{1}{\tau}(Q_1(1)(\frac{e_2 q_2}{2\Phi'} - \tilde{U}_1) + L_1(x, D)(\frac{e_2 q_2}{2\Phi'}))e^{r\Phi} = -\frac{1}{\tau} Q_1(2)U_{-1} e^{r\Phi} - \frac{1}{\tau} Q_1(1)\tilde{U}_{-1} e^{r\Phi} \\
&- Q_1(2)\tilde{U}_0 e^{r\Phi} - Q_1(1)U_0 e^{r\Phi} + e^{r\varphi} o_{L^p(\Omega)}(\frac{1}{\tau}).
\end{align*}
\]

(3.25)

We set

\[ U = U_{0, \tau} e^{r\Phi} + U_{0, \tau} e^{-r\Phi} - e^{r\Phi} \tilde{R}_{\tau, B_1}(e_1(q_1 + \tilde{q}_1/\tau)) - e^{r\Phi} \tilde{R}_{\tau, A_1}(e_1(q_2 + \tilde{q}_2/\tau)). \]

Using (3.25) we prove the following proposition.

**Proposition 3.4.** For any $p > 1$, we have the asymptotic formula:

\[
L_1(x, D)U = e^{r\varphi} o_{L^p(\Omega)}(\frac{1}{\tau}),
\]

(3.26)

\[
B(x, D)U|_{\Gamma_0} = e^{r\varphi} \left( \frac{o_{W^2_2(I_{\Gamma_0})}(\frac{1}{\tau})}{o_{W^2_2(I_{\Gamma_0})}(\frac{1}{\tau})} \right).
\]

(3.27)

**Proof.** By (3.1), (3.19)-(3.21) and (3.17)-(3.24), we have

\[
(U_{0, \tau} e^{r\Phi} + \tilde{U}_{0, \tau} e^{r\Phi} - e^{r\Phi} \tilde{R}_{\tau, B_1}(e_1(q_1 + \tilde{q}_1/\tau)) - e^{r\Phi} \tilde{R}_{\tau, A_1}(e_1(q_2 + \tilde{q}_2/\tau), \tilde{e}_1))|_{\Gamma_0}
\]

\[
= (U_{0, \tau} e^{r\varphi} + \tilde{U}_{0, \tau} e^{r\varphi} - e^{r\varphi} \tilde{R}_{\tau, B_1}(e_1(q_1 + \tilde{q}_1/\tau)) - e^{r\varphi} \tilde{R}_{\tau, A_1}(e_1(q_2 + \tilde{q}_2/\tau), \tilde{e}_1))|_{\Gamma_0}
\]

\[
e^{r\varphi}(U_0 + \frac{U_{-1} - e_2 q_1/2\Phi'}{\tau} + \frac{1}{\tau^2} \sum_{\bar{x} \in H} (\frac{e^{2ir\psi(\bar{x})}}{\Phi'} a_{\bar{x}, \bar{x}} + \frac{e^{2ir\psi(\bar{x})}}{\Phi'} a_{\bar{x}, \bar{x}}) + U_{-2} - \frac{\tilde{q}_1 e_2}{2\Phi'})
\]

\[
+ \tilde{U}_0 + \frac{\tilde{U}_{-1} - e_2 q_2/2\Phi'}{\tau} + \frac{1}{\tau^2} \sum_{\bar{x} \in H} (\frac{e^{2ir\psi(\bar{x})}}{\Phi'} b_{\bar{x}, \bar{x}} + \frac{e^{2ir\psi(\bar{x})}}{\Phi'} b_{\bar{x}, \bar{x}}) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2\Phi'})
\]

\[
= e^{r\varphi}(\frac{1}{\tau^2} \sum_{\bar{x} \in H} (\frac{e^{2ir\psi(\bar{x})}}{\Phi'} d_{\bar{x}, \bar{x}} + \frac{e^{2ir\psi(\bar{x})}}{\Phi'} d_{\bar{x}, \bar{x}}) + e^{2ir\psi(\bar{x})} e_{\bar{x}, \bar{x}} + e^{2ir\psi(\bar{x})} e_{\bar{x}, \bar{x}})
\]

\[
= e^{r\varphi}(\frac{1}{\tau^2} \sum_{\bar{x} \in H} (\frac{e^{2ir\psi(\bar{x})}}{\Phi'} d_{\bar{x}, \bar{x}} + \frac{e^{2ir\psi(\bar{x})}}{\Phi'} d_{\bar{x}, \bar{x}}) + e^{2ir\psi(\bar{x})} e_{\bar{x}, \bar{x}} + e^{2ir\psi(\bar{x})} e_{\bar{x}, \bar{x}})
\]

\[
- \tilde{R}_{\tau, B_1}(e_1(q_1 + \tilde{q}_1/\tau)) - \tilde{R}_{\tau, A_1}(e_1(q_2 + \tilde{q}_2/\tau)), \tilde{e}_1)|_{\Gamma_0} = e^{r\varphi} o_{W^2_2(I_{\Gamma_0})}(\frac{1}{\tau^2}).
\]
The short computations imply
\[
\mathcal{I} = \partial_\nu \left( U_{0, r} e^{\tau \phi} + \tilde{U}_{0, r} e^{\tau \tilde{\phi}} - e^{\tau \phi} \tilde{R}_{\tau, B_1} (e_1 (q_1 + \tilde{q}_1 / \tau)) - e^{\tau \tilde{\phi}} \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau), \tilde{e}_2) \right) |_{\Gamma_0} \\
= (i \tau \partial_\nu \psi U_{0, r} e^{\tau \phi} - i \tau \partial_\nu \tilde{U}_{0, r} e^{\tau \tilde{\phi}} + \partial_\nu \tilde{U}_{0, r} e^{\tau \phi} + \partial_\nu \tilde{U}_{0, r} e^{\tau \tilde{\phi}} + e^{\tau \phi} i \tau \partial_\nu \psi \tilde{R}_{\tau, B_1} (e_1 (q_1 + \tilde{q}_1 / \tau)) + i \tau \partial_\nu \psi e^{\tau \phi} \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau)) \\
- e^{\tau \phi} \partial_\nu \tilde{R}_{\tau, B_1} (e_1 (q_1 + \tilde{q}_1 / \tau)) - e^{\tau \tilde{\phi}} \partial_\nu \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau), \tilde{e}_2) |_{\Gamma_0}
\]
\[
= (e^{\tau \phi} i \tau \partial_\nu \psi (U_0 + \frac{U_{-1} - e_2 q_1 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \phi (\bar{x})} a_{+ \bar{x}} + e^{-2i \tau \phi (\bar{x})} a_{- \bar{x}}) + U_{-2} - \frac{\tilde{q}_1 e_2}{2 \Phi'} \right)) \\
- e^{\tau \phi} i \tau \partial_\nu \psi (\tilde{U}_0 + \frac{\tilde{U}_{-1} - e_2 q_2 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \tilde{\phi} (\bar{x})} b_{+ \bar{x}} + e^{-2i \tau \tilde{\phi} (\bar{x})} b_{- \bar{x}}) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)) + (e^{\tau \phi} \partial_\nu (U_0 + \frac{U_{-1} - e_2 q_1 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \phi (\bar{x})} a_{+ \bar{x}} + e^{-2i \tau \phi (\bar{x})} a_{- \bar{x}}) + U_{-2} - \frac{\tilde{q}_1 e_2}{2 \Phi'} \right)) \\
+ e^{\tau \phi} \partial_\nu (\tilde{U}_0 + \frac{\tilde{U}_{-1} - e_2 q_2 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \tilde{\phi} (\bar{x})} b_{+ \bar{x}} + e^{-2i \tau \tilde{\phi} (\bar{x})} b_{- \bar{x}}) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)) \\
- e^{\tau \phi} i \tau \partial_\nu \psi \tilde{R}_{\tau, B_1} (e_1 (q_1 + \tilde{q}_1 / \tau)) + i \tau \partial_\nu \psi e^{\tau \phi} \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau)) - e^{\tau \phi} \partial_\nu \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau), \tilde{e}_2) |_{\Gamma_0}.
\]

By (3.6) and (3.17) we obtain
\[
\mathcal{I} = (e^{\tau \phi} i \tau \partial_\nu \psi (\frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \phi (\bar{x})} a_{+ \bar{x}} + e^{-2i \tau \phi (\bar{x})} a_{- \bar{x}}) + U_{-2} - \frac{\tilde{q}_1 e_2}{2 \Phi'} \right)) \\
- e^{\tau \phi} i \tau \partial_\nu \psi (\frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \tilde{\phi} (\bar{x})} b_{+ \bar{x}} + e^{-2i \tau \tilde{\phi} (\bar{x})} b_{- \bar{x}}) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)) + (e^{\tau \phi} \partial_\nu (U_0 + \frac{U_{-1} - e_2 q_1 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \phi (\bar{x})} a_{+ \bar{x}} + e^{-2i \tau \phi (\bar{x})} a_{- \bar{x}}) + U_{-2} - \frac{\tilde{q}_1 e_2}{2 \Phi'} \right)) \\
+ e^{\tau \phi} \partial_\nu (\tilde{U}_0 + \frac{\tilde{U}_{-1} - e_2 q_2 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \tilde{\phi} (\bar{x})} b_{+ \bar{x}} + e^{-2i \tau \tilde{\phi} (\bar{x})} b_{- \bar{x}}) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)) \\
- e^{\tau \phi} i \tau \partial_\nu \psi \tilde{R}_{\tau, B_1} (e_1 (q_1 + \tilde{q}_1 / \tau)) + i \tau \partial_\nu \psi e^{\tau \phi} \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau)) - e^{\tau \phi} \partial_\nu \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau), \tilde{e}_2) |_{\Gamma_0}.
\]

Using Proposition 3.33 the formulæ (3.19) and (3.20) we obtain
\[
\mathcal{I} = (e^{\tau \phi} i \tau \partial_\nu \psi (\frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \phi (\bar{x})} a_{+ \bar{x}} + e^{-2i \tau \phi (\bar{x})} a_{- \bar{x}}) + U_{-2} - \frac{\tilde{q}_1 e_2}{2 \Phi'} \right)) \\
- e^{\tau \phi} i \tau \partial_\nu \psi (\frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \tilde{\phi} (\bar{x})} b_{+ \bar{x}} + e^{-2i \tau \tilde{\phi} (\bar{x})} b_{- \bar{x}}) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)) + (e^{\tau \phi} \partial_\nu (U_0 + \frac{U_{-1} - e_2 q_1 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \phi (\bar{x})} a_{+ \bar{x}} + e^{-2i \tau \phi (\bar{x})} a_{- \bar{x}}) + U_{-2} - \frac{\tilde{q}_1 e_2}{2 \Phi'} \right)) \\
+ e^{\tau \phi} \partial_\nu (\tilde{U}_0 + \frac{\tilde{U}_{-1} - e_2 q_2 / 2 \Phi'}{\tau}) + \frac{1}{\tau^2} \left( \sum_{x \in \mathcal{H}} (e^{2i \tau \tilde{\phi} (\bar{x})} b_{+ \bar{x}} + e^{-2i \tau \tilde{\phi} (\bar{x})} b_{- \bar{x}}) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)) \\
- e^{\tau \phi} i \tau \partial_\nu \psi \tilde{R}_{\tau, B_1} (e_1 (q_1 + \tilde{q}_1 / \tau)) + i \tau \partial_\nu \psi e^{\tau \phi} \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau)) - e^{\tau \phi} \partial_\nu \tilde{R}_{\tau, A_1} (e_1 (q_2 + \tilde{q}_2 / \tau), \tilde{e}_2) |_{\Gamma_0}.
\]
+ e^{\tau \varphi} \partial_\nu \left( \tilde{U}_{-1} - e_2 \frac{q_2}{2 \Phi'} \right) \frac{1}{\tau} + \frac{1}{\tau^2} \sum_{\tilde{x} \in \cal{H}} \left( e^{2i\tau \psi(x)} b_{+,\tilde{x}} + e^{-2i\tau \psi(x)} b_{-,\tilde{x}} \right) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)

- e^{\tau \varphi} i \partial_\nu \psi \sum_{\tilde{x} \in \cal{H}} \frac{m_{+,\tilde{x}} e^{2i\tau \psi(x)}}{\tau} + e^{\tau \varphi} i \partial_\nu \psi \sum_{\tilde{x} \in \cal{H}} \frac{m_{-,\tilde{x}} e^{-2i\tau \psi(x)}}{\tau}

- e^{\tau \varphi} \sum_{\tilde{x} \in \cal{H}} \frac{\partial_\nu m_{+,\tilde{x}} e^{2i\tau \psi(x)}}{\tau^2} - e^{\tau \varphi} \sum_{\tilde{x} \in \cal{H}} \frac{\partial_\nu m_{-,\tilde{x}} e^{-2i\tau \psi(x)}}{\tau^2}, e_2 \big|_{\Gamma_0} + e^{\tau \varphi} \partial_{W^2_\Gamma(\Gamma_0)} \left( \frac{1}{\tau} \right).

Using (3.22) write down \( I \) as

\[ I = \left( e^{\tau \varphi} i \tau \partial_\nu \psi \left( \frac{1}{\tau^2} \sum_{\tilde{x} \in \cal{H}} \left( e^{2i\tau \psi(x)} a_{+,\tilde{x}} + e^{-2i\tau \psi(x)} a_{-,\tilde{x}} \right) \right) + e^{\tau \varphi} \partial_\nu \left( \frac{1}{\tau^2} \sum_{\tilde{x} \in \cal{H}} \left( e^{2i\tau \psi(x)} b_{+,\tilde{x}} + e^{-2i\tau \psi(x)} b_{+,\tilde{x}} \right) \right) \right)

+ e^{\tau \varphi} \partial_\nu \left( \frac{1}{\tau^2} \sum_{\tilde{x} \in \cal{H}} \left( e^{2i\tau \psi(x)} b_{+,\tilde{x}} + e^{-2i\tau \psi(x)} b_{-,\tilde{x}} \right) + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right)

- e^{\tau \varphi} i \partial_\nu \psi \sum_{\tilde{x} \in \cal{H}} \frac{m_{+,\tilde{x}} e^{2i\tau \psi(x)}}{\tau} + e^{\tau \varphi} i \partial_\nu \psi \sum_{\tilde{x} \in \cal{H}} \frac{m_{-,\tilde{x}} e^{-2i\tau \psi(x)}}{\tau}

- e^{\tau \varphi} \sum_{\tilde{x} \in \cal{H}} \frac{\partial_\nu m_{+,\tilde{x}} e^{2i\tau \psi(x)}}{\tau^2} - e^{\tau \varphi} \sum_{\tilde{x} \in \cal{H}} \frac{\partial_\nu m_{-,\tilde{x}} e^{-2i\tau \psi(x)}}{\tau^2}, e_2 \big|_{\Gamma_0} + e^{\tau \varphi} \partial_{W^2_\Gamma(\Gamma_0)} \left( \frac{1}{\tau} \right).

Finally, applying (3.21) we have

\[ I = \left( e^{\tau \varphi} \partial_\nu \left( \frac{1}{\tau^2} \sum_{\tilde{x} \in \cal{H}} \left( e^{2i\tau \psi(x)} a_{+,\tilde{x}} + e^{-2i\tau \psi(x)} a_{-,\tilde{x}} \right) + U_{-2} - \frac{\tilde{q}_1 e_2}{2 \Phi'} \right), e_2 \right) \]

+ e^{\tau \varphi} \partial_\nu \left( \frac{1}{\tau^2} \sum_{\tilde{x} \in \cal{H}} \left( e^{2i\tau \psi(x)} b_{+,\tilde{x}} + e^{-2i\tau \psi(x)} b_{-,\tilde{x}} + \tilde{U}_{-2} - \frac{\tilde{q}_2 e_2}{2 \Phi'} \right), e_2 \right) + e^{\tau \varphi} \partial_{W^2_\Gamma(\Gamma_0)} \left( \frac{1}{\tau} \right) = e^{\tau \varphi} \partial_{W^2_\Gamma(\Gamma_0)} \left( \frac{1}{\tau} \right).

The proof of equality (3.27) is complete now.

Similarly to (3.10) we obtain

\[ L_1(x, D)(U_0, e^{\varphi}) + \tilde{U}_0, e^{\varphi} + e^{2q_1 + \tilde{q}_1/\tau} \frac{e^{\varphi}}{2 \Phi'} \left( \frac{e_2(q_1 + \tilde{q}_1/\tau)}{2 \Phi'} \right) + e^{2q_2 + \tilde{q}_2/\tau} \frac{e^{\varphi}}{2 \Phi'} \left( \frac{e_2(q_2 + \tilde{q}_2/\tau)}{2 \Phi'} \right) \]

(3.28) \[ = Q_1(1)(U_0, e^{\varphi} + \frac{e^{2q_1 + \tilde{q}_1/\tau}}{2 \Phi'} \frac{e^{\varphi}}{2 \Phi'}) + Q_1(2)(\tilde{U}_0, e^{\varphi} + \frac{e^{2q_2 + \tilde{q}_2/\tau}}{2 \Phi'} \frac{e^{\varphi}}{2 \Phi'}). \]

By (3.28) and (3.26), we obtain (3.20).}

Consider the boundary value problem

\[ L(x, D)u = \Delta u + 2A \partial_2 u + 2B \partial_2 u + Qu = f \quad \text{in} \quad \Omega, \quad B(x, D)u|_{\partial \Omega} = 0. \]

Then we prove a Carleman estimate with boundary terms whose weight function is degenerate.
Proposition 3.5. Suppose that \( \Phi = \varphi + i \psi \) satisfies (3.1), (3.2), the coefficients of the operator \( L \) matrices \( A, B, Q \) belong to \( L^\infty(\Omega) \). Then there exist \( \tau_0 \) and \( C \), independent of \( u \) and \( \tau \), such that
\[
|\tau||ue^{\tau\varphi}|^2_{L^2(\Omega)} + ||ue^{\tau\varphi}|^2_{W^1_2(\Omega)} + \tau^2\left|\frac{\partial \Phi}{\partial z}\right|^2_{L^2(\Omega)} \leq C(||(L(x, D)u)e^{\tau\varphi}|^2_{L^2(\Omega)} + |\tau|\int_{\Gamma}(|\nabla u|^2 + \tau^2 u^2)e^{2\tau\varphi}d\sigma)
\]
for all \( |\tau| > \tau_0 \) and all \( u \in W^2_2(\Omega), B(x, D)u|_{\Gamma_0} = 0 \).

For the scalar equation, the estimate is proved in [11] and [6] for the case of the Dirichlet and Neumann boundary conditions respectively. In order to prove this estimate for the system, it is sufficient to apply the scalar estimate to each equation in the system and take advantage of the second large parameter in order to absorb the right-hand side.

Using estimate (3.29), we obtain

Proposition 3.6. There exists a constant \( \tau_0 \) such that for \( |\tau| \geq \tau_0 \) and any \( f \in L^2(\Omega) \), there exists a solution to the boundary value problem
\[
L(x, D + i\tau \nabla \varphi)u = f \quad \text{in } \Omega, \quad B(x, D)u|_{\Gamma_0} = 0
\]
such that
\[
\|u\|_{W^2_2^{1,\tau}(\Omega)} \leq C\|f\|_{L^2(\Omega)}.
\]
Moreover if \( f/\Phi' \in L^2(\Omega) \), then for any \( |\tau| \geq \tau_0 \) there exists a solution to the boundary value problem (3.30) such that
\[
\|u\|_{W^2_2^{1,\tau}(\Omega)} \leq C\|f/\Phi'\|_{L^2(\Omega)}.
\]
The constants \( C \) in (3.31) and (3.32) are independent of \( \tau \).

We set \( \mathcal{O}_\epsilon = \{ x \in \Omega; \text{dist}(x, \partial \Omega) \leq \epsilon \} \). In order to construct the last term in complex geometric optics solution, we need the following proposition:

Proposition 3.7. Let \( A, B \in C^{5+\alpha}(\overline{\Omega}) \) and \( Q \in C^{4+\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \), \( f \in L^p(\Omega) \) for some \( p > 2 \), \( \text{dist}(\Gamma_0, \text{supp } f) > 0 \), \( q \in W^1_2(\Gamma_0) \), and \( \epsilon \) be a small positive number such that \( \overline{\mathcal{O}_\epsilon} \cap \mathcal{H} = \emptyset \). Then there exists \( C \) independent of \( \tau \) and \( \tau_0 \) such that for all \( |\tau| > \tau_0 \), there exists a solution \( w \in W^3_2(\Omega) \) to the boundary value problem
\[
L(x, D)w = fe^{\tau\varphi} \quad \text{in } \Omega, \quad B(x, D)w|_{\Gamma_0} = qe^{\tau\varphi}/\tau
\]
such that
\[
\sqrt{|\tau|}\|we^{-\tau\varphi}|^2_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}(||e^{-\tau\varphi}|^2_{L^2(\Omega)} + ||we^{-\tau\varphi}|^2_{H^1,\tau(\mathcal{O}_\epsilon)} \leq C(||f||_{L^p(\Omega)} + ||q||_{W^1_2(\Gamma_0)}).
\]

Proof. First let us assume that \( f \) is identically equal to zero. Let \((d, \tilde{d}) \in W^1_2(\Omega) \times W^1_2(\overline{\Omega})\) satisfy
\[
\mathcal{K}(x, D)(d, \tilde{d}) = 0 \quad \text{in } \Omega, \quad (d + 1\tilde{d})|_{\Gamma_0} = q.
\]
For existence of such a solution see e.g. [14]. By (3.10) and (3.35), we have
\[ L(x, D)(\frac{d}{\tau}e^{\tau\Phi} + \frac{\bar{d}}{\tau}e^{\tau\overline{\Phi}}) = \frac{1}{\tau}(Q - 2\partial_\tau A - BA)e^{\tau\Phi} + \frac{1}{\tau}(Q - 2\partial_\tau B - AB)e^{\tau\overline{\Phi}}. \]

By Proposition 3.6, there exists a solution \( w \) to the boundary value problem
\[ L(x, D)\bar{w} = -\frac{1}{\tau}(Q - 2\partial_\tau A - BA)e^{\tau\Phi} - \frac{1}{\tau}(Q - 2\partial_\tau B - AB)e^{\tau\overline{\Phi}}, \quad \mathcal{B}(x, D)\bar{w}|_{\Gamma_0} = 0 \]
such that there exists a constant \( C > 0 \) such that
\[ \|\bar{w}e^{-\tau\Phi}\|_{H^1,\tau(\Omega)} \leq C \frac{\|L - \tau\partial_\tau L\|_{\mathcal{B}^\ast\mathcal{B}}}{\sqrt{|\tau|}} \|\mathcal{B}(e^{\tau\Phi})\|_{L^2(\Omega)} \]
for all large \( \tau > 0 \).

Then the function \( \bar{w} \) is a solution to (3.33) which satisfies (3.34) if \( f \equiv 0 \).

If \( f \) is not identically equal zero without the loss of generality we may assume that \( q \equiv 0 \).

Then we consider the function \( \bar{w} = \tilde{e}e^{\tau\Phi}\tilde{R}_\tau B(e_1q_0) \), where \( \tilde{e} \in C_0^\infty(\Omega) \), \( \tilde{e}|_{\text{supp} e_1} = 1 \) and \( q_0 = P_A f - M \), where a function \( M \in C^5(\Omega) \) belongs to \( \text{Ker}(2\partial_\tau + B) \) and chosen such that \( q_0|_{\mathcal{H}} = 0 \). Then \( L(x, D)\bar{w} = (Q - 2\partial_\tau B - AB)e^{\tau\Phi} + \tilde{e}e_1f e^{\tau\Phi} + 2\tilde{e}e^{\tau\Phi}(2\partial_\tau + A)\tilde{R}_\tau B(e_1q_0) \). Since, by Proposition 8 of [17], the function \( \tilde{f}(\tau, \cdot) = e^{\tau\Phi}L(x, D)\bar{w} - f \) can be represented as a sum of two functions, where the first one equal to zero in a neighborhood of \( \mathcal{H} \) and is bounded uniformly in \( \tau \) in \( L^2(\Omega) \) norm, the second one is \( O_{L^2(\Omega)}(\frac{1}{\tau}) \). Applying Proposition 3.6 to the boundary value problem
\[ L(x, D)U_\ast \equiv \tilde{f}e^{\tau\Phi} \text{ in } \Omega, \text{ } \mathcal{B}(x, D)U_\ast|_{\Gamma_0} = 0, \]
we construct a solution such that
\[ \|U_\ast e^{-\tau\Phi}\|_{W_{1,\tau}^1(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \]
The function \( U^\ast - \bar{w} \) solves the boundary value problem (3.33) and satisfies estimate (3.34).

Using Propositions 3.7 and 3.4, we construct the last term \( u_{-1} \) in complex geometric optics solution which satisfies the equation
\[ L_1(x, D)(e^{\tau\Phi}u_{-1}) = -L_1(x, D)U, \text{ } \mathcal{B}(x, D)(e^{\tau\Phi}u_{-1}) = -\mathcal{B}(x, D)U \text{ on } \Gamma_0 \]
and satisfies the estimate
\[ \sqrt{|\tau|} \|u_{-1}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}} \|
abla u_{-1}\|_{L^2(\Omega)} + \|u_{-1}\|_{W_{1,\tau}^1(\Omega)} = o(\frac{1}{\tau}) \text{ as } \tau \to +\infty. \]

Finally we obtain a complex geometric optics solution in the form:
\[ u_1(x) = U_0,\tau e^{\tau\Phi} + \tilde{U}_0,\tau e^{\tau\overline{\Phi}} - e^{\tau\Phi}\tilde{R}_{\tau, B_1}(q_1 + \bar{q}_1/\tau) - e^{\tau\overline{\Phi}}\tilde{R}_{\tau, A_1}(q_2 + \bar{q}_2/\tau) + e^{\tau\Phi}u_{-1}. \]

Obviously
\[ L_1(x, D)u_1 = 0 \text{ in } \Omega, \text{ } \mathcal{B}(x, D)u_1|_{\Gamma_0} = 0. \]

Consider the operator
\[ L_2(x, D)^\ast = 4\partial_\tau \partial_\tau - 2A_2^\ast\partial_\tau - 2B_2^\ast\partial_\tau - Q_2^\ast - 2\partial_\tau A_2^\ast - 2\partial_\tau B_2^\ast \]
\[ = (2\partial_\tau - A_2^\ast)(2\partial_\tau - B_2^\ast) + Q_1(2) = (2\partial_\tau - B_2^\ast)(2\partial_\tau - A_2^\ast) + Q_2(2). \]
Here
\[ Q_1(2) = Q_2^* - 2\partial_2 A_2^* + A_2^* B_2^*, \quad Q_2(2) = Q_2^* - 2\partial_2 B_2^* + B_2^* A_2^*. \]
Similarly we construct the complex geometric optics solutions to the operator \( L_2(x, D)^* \). Let \((V_0, \tilde{V}_0) \in C^{6+\alpha}(\Omega) \times C^{6+\alpha}(\overline{\Omega})\) be a solutions to the following boundary value problem:
\[
(3.40) \quad \mathcal{M}(x, D)(V_0, \tilde{V}_0) = ((2\partial_2 - A_2^*)V_0, (2\partial_2 - B_2^*)\tilde{V}_0) = 0 \quad \text{in } \Omega, \quad (V_0 + i\tilde{V}_0)|_{\Gamma_0} = 0,
\]
such that
\[
(3.41) \quad \lim_{x \to x_\pm} \frac{|V_0(x)|}{|x - x_\pm|^{1/2}} = \lim_{x \to x_\pm} \frac{|\tilde{V}_0(x)|}{|x - x_\pm|^{1/2}} = 0.
\]
Such a pair \((V_0, \tilde{V}_0)\) exists due to Proposition 3.2. Observe that
\[
L_2(x, D)^*(\tilde{V}_0 e^{-\tau\Phi}) = Q_1(2)\tilde{V}_0 e^{-\tau\Phi} \quad \text{in } \Omega, \quad L_2(x, D)^*(V_0 e^{-\tau\Phi}) = Q_2(2)V_0 e^{-\tau\Phi} \quad \text{in } \Omega.
\]
We set
\[
(3.42) \quad q_3 = P_{-B_2^*}(Q_1(2)\tilde{V}_0) - M_3, \quad q_4 = T_{-A_2^*}(Q_2(2)V_0) - M_4,
\]
where \(M_3 \in \text{Ker}(2\partial_2 - B_2^*)\) and \(M_4 \in \text{Ker}(2\partial_2 - A_2^*)\) are chosen such that
\[
(3.43) \quad q_3(x) = q_4(x) = 0, \quad \forall x \in \mathcal{H} \quad \text{and} \quad \lim_{x \to x_\pm} \frac{|q_j(x)|}{|x - x_\pm|^{1/2}} = 0, \quad j \in \{3, 4\}.
\]
By (3.43) the functions \(\frac{q_3}{2\Phi}, \frac{q_4}{2\Phi}\) belong to the space \(C^2(\Gamma_0)\). Therefore we can introduce the functions \(V_{-1}, \tilde{V}_{-1}\) as a solutions to the following boundary value problem:
\[
(3.44) \quad \mathcal{M}(x, D)(V_{-1}, \tilde{V}_{-1}) = 0 \quad \text{in } \Omega, \quad (V_{-1} + i\tilde{V}_{-1})|_{\Gamma_0} = -(\mathcal{I} \frac{q_3}{2\Phi'} + \frac{q_4}{2\Phi'} - (0, \partial_\nu(V_0 + \tilde{V}_0)))/i\partial_\nu\psi).
\]
Let
\[
p_3 = Q_1(2)(\frac{e_1q_3}{2\Phi'} + \tilde{V}_{-1}) + L_2(x, D)^*(\frac{q_3 e_2}{2\Phi'}), \quad p_4 = Q_2(2)(\frac{e_1q_4}{2\Phi'} + V_{-1}) + L_2(x, D)^*(\frac{q_4 e_2}{2\Phi'})
\]
and
\[
\tilde{q}_3 = (P_{-B_2^*}p_3 - \tilde{M}_3), \quad \tilde{q}_4 = (T_{-A_2^*}p_3 - \tilde{M}_4),
\]
where \(\tilde{M}_3 \in \text{Ker}(2\partial_2 - B_2^*), \tilde{M}_4 \in \text{Ker}(2\partial_2 - A_2^*)\), and \((\tilde{q}_3, \tilde{q}_4)\) are chosen such that
\[
(3.45) \quad \tilde{q}_3(x) = \tilde{q}_4(x) = 0, \quad \forall x \in \mathcal{H}.
\]
The following asymptotic formula holds true:

**Proposition 3.8.** Let \(\mathcal{G}_r \cap \text{supp } \psi = \emptyset\). There exist smooth functions \(\tilde{m}_{\pm, \tilde{x}} \in C^2(\mathcal{G}_r), \tilde{x} \in \mathcal{H},\) independent of \(\tau\) such that
\[
(3.46) \quad \tilde{R}_{-\tau, -A_2^*}(e_1(q_3 + \tilde{q}_3/\tau))|_{\mathcal{G}_r} = \sum_{\tilde{x} \in \mathcal{H}} \frac{\tilde{m}_{\pm, \tilde{x}} e^{2i\tau(\psi - \psi(\tilde{x}))}}{\tau^2} + e^{2i\tau\psi} a_{W_2}(\partial_\theta)(\frac{1}{\tau^2}) \quad \text{as } |\tau| \to +\infty
\]
and
\[
(3.47) \quad R_{-\tau, -B_2^*}(e_1(q_4 + \tilde{q}_4/\tau))|_{\mathcal{G}_r} = \sum_{\tilde{x} \in \mathcal{H}} \frac{\tilde{m}_{-\tau, \tilde{x}} e^{-2i\tau(\psi - \psi(\tilde{x}))}}{\tau^2} + e^{-2i\tau\psi} a_{W_2}(\partial_\theta)(\frac{1}{\tau^2}) \quad \text{as } |\tau| \to +\infty.
\]
Using the functions \( \tilde{m}_{\pm, \bar{x}} \) we introduce functions \( \tilde{a}_{\pm, \bar{x}}, \tilde{b}_{\pm, \bar{x}} \in C^2(\Omega) \) which solve the boundary value problem

\[
\mathcal{M}(x, D)(\tilde{a}_{\pm, \bar{x}}, \tilde{b}_{\pm, \bar{x}}) = 0 \quad \text{in} \quad \Omega, \quad (\tilde{a}_{\pm, \bar{x}} + \tilde{b}_{\pm, \bar{x}})|_{\Gamma_0} = \pm (-\tilde{m}_{\pm, \bar{x}}) \quad \forall \bar{x} \in \mathcal{H}.
\]

By (3.45), there exists a pair \( (V_{-}, \tilde{V}_{-}) \in W_2^1(\Omega) \times W_2^1(\Omega) \) which solves the boundary value problem

\[
\mathcal{M}(x, D)(V_{-}, \tilde{V}_{-}) = 0 \quad \text{in} \quad \Omega,
\]

\[
(V_{-} + \tilde{V}_{-})|_{\Gamma_0} = -(\frac{\tilde{q}_3}{2\Phi} + \frac{\tilde{q}_4}{2\Phi}) - (0, \tilde{e}_2, \partial_\nu(\tilde{V}_{-} + \frac{e_2\tilde{q}_3}{2\Phi'} + V_{-} + \frac{e_2\tilde{q}_4}{2\Phi'})) / i\partial_\nu\psi).
\]

We introduce functions \( V_{0, \tau}, \tilde{V}_{0, \tau} \) by formulas

\[
\tilde{V}_{0, \tau} = \tilde{V}_0 + \frac{V_{-} + e_2\tilde{q}_4/2\Phi'}{\tau} + \frac{1}{\tau^2} \left( \sum_{\tilde{x} \in \mathcal{H}} (e^{2i\tau\psi(\tilde{x})}\tilde{b}_{+, \tilde{x}} + e^{-2i\tau\psi(\tilde{x})}\tilde{b}_{-, \tilde{x}}) + \tilde{V}_{-} + e_2\tilde{q}_3/2\Phi' \right)
\]

and

\[
V_{0, \tau} = V_0 + \frac{V_{-} + e_2\tilde{q}_4/2\Phi'}{\tau} + \frac{1}{\tau^2} \left( \sum_{\tilde{x} \in \mathcal{H}} (e^{2i\tau\psi(\tilde{x})}\tilde{a}_{+, \tilde{x}} + e^{-2i\tau\psi(\tilde{x})}\tilde{a}_{-, \tilde{x}}) + V_{-} + e_2\tilde{q}_4/2\Phi' \right).
\]

By (3.43) and (3.45), the functions \( V_{0, \tau}, \tilde{V}_{0, \tau} \) belong to \( W_2^1(\Omega) \). After short computations, for any \( p \in (1, +\infty) \) we have

\[
\begin{align*}
L_2(x, D)^* \left( -e^{-\tau\Phi}\tilde{R}_{-\tau, -A_2}(e_1(q_3 + \frac{\tilde{q}_3}{\tau})) + e^{-\tau\Phi}e_2(q_3 + \frac{\tilde{q}_3}{\tau}) + \frac{e^{-\tau\Phi}e_2(q_4 + \frac{\tilde{q}_4}{\tau})}{2\tau\Phi'} \right) \\
= -e^{-\tau\Phi}Q_2(2)\tilde{R}_{-\tau, -A_2}(e_1(q_3 + \frac{\tilde{q}_3}{\tau})) + e^{-\tau\Phi}L_2(x, D)^* \left( \frac{e_2(q_3 + \frac{\tilde{q}_3}{\tau})}{2\tau\Phi'} \right) \\
- e^{-\tau\Phi}Q_1(2)\tilde{R}_{-\tau, -B_2}^{(2)}(e_1(q_4 + \frac{\tilde{q}_4}{\tau})) + e^{-\tau\Phi}L_2(x, D)^* \left( \frac{e_2(q_4 + \frac{\tilde{q}_4}{\tau})}{2\tau\Phi'} \right) \\
- e^{-\tau\Phi}Q_2(2)\frac{e_1q_3}{2\tau\Phi'} - e^{-\tau\Phi}Q_1(2)\frac{e_1q_4}{2\tau\Phi'} - e^{-\tau\Phi}L_2(x, D)^* \left( \frac{e_2q_3}{2\tau\Phi'} \right) \\
- e^{-\tau\Phi}L_2(x, D)^* \left( \frac{e_2q_4}{2\tau\Phi'} \right) \\
- Q_1(2)(\tilde{V}_{-} + \frac{V_{-} - \tilde{V}_{-}}{\tau}) e^{-\tau\Phi} - Q_2(2)(V_{-} + \frac{V_{-} - \tilde{V}_{-}}{\tau}) e^{-\tau\Phi} - e^{-\tau\Phi}o_{L_p(\Omega)}(\frac{1}{r}).
\end{align*}
\]

\[(53) \quad = -Q_1(2)(\tilde{V}_{-} + \frac{V_{-} - \tilde{V}_{-}}{\tau}) e^{-\tau\Phi} - Q_2(2)(V_{-} + \frac{V_{-} - \tilde{V}_{-}}{\tau}) e^{-\tau\Phi} + e^{-\tau\Phi}o_{L_p(\Omega)}(\frac{1}{r}).\]

Setting \( V_* = V_{0, \tau} e^{-\tau\Phi} + \tilde{V}_{0, \tau} e^{-\tau\Phi} - e^{-\tau\Phi}\tilde{R}_{-\tau, -A_2}(e_1(q_3 + \frac{\tilde{q}_3}{\tau})) - e^{-\tau\Phi}R_{-\tau, -B_2}^{(2)}(e_1(q_4 + \frac{\tilde{q}_4}{\tau})) \) for any \( p \in (1, \infty) \), we obtain that
The first equality in (3.54) follows from (3.53) and the second one can be obtained by argument similar to one used in the proof of Proposition 3.4.

Using (3.54) and Proposition 3.7, we construct the last term \( v_{-1} \) in complex geometric optics solution which solves the boundary value problem

\[
L_2(x, D)^*v_{-1} = L_2(x, D)^*V^* \text{ in } \Omega, \quad B(x, D)v_{-1}|_{\Gamma_0} = -B(x, D)V^*,
\]

and we obtain

\[
\sqrt{|\tau|}\|v_{-1}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla v_{-1}\|_{L^2(\Omega)} + \|v_{-1}\|_{W^{1,\tau}_2(\Gamma_0)} = o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty.
\]

Finally we have a complex geometric optics solution for the Schrödinger operator \( L_2(x, D)^* \) in a form:

\[
v = V_{1,\tau}e^{-\tau\Phi} + \bar{V}_{1,\tau}e^{-\tau\Phi} - e^{-\tau\Phi}\mathcal{R}_{-\tau,-A_2}(e_1(\frac{q_3}{\tau^2})) - e^{-\tau\Phi}\mathcal{R}_{-\tau,-B_2}(e_1(\frac{q_4}{\tau^2})) + v_{-1}e^{-\tau\Phi}.
\]

By (3.57), (3.54) and (3.55), we have

\[
L_2(x, D)^*v = 0 \text{ in } \Omega, \quad B(x, D)v|_{\Gamma_0} = 0.
\]

4. Step 4: Asymptotic

We introduce the following functionals

\[
\mathfrak{F}_\tau u = \sum_{\tilde{x} \in \mathcal{H}} \frac{\pi}{2} \left( \frac{u(\tilde{x})}{\tau} - \frac{\partial_{\tilde{z}}^2 u(\tilde{x})}{2\Phi'(\tilde{x})\tau^2} + \frac{\partial_{\bar{z}}^2 u(\tilde{x})}{2\Phi'(\tilde{x})\tau^2} \right) .
\]

and

\[
\mathfrak{J}_\tau r = \int_{\partial\Omega} r \left( \frac{\nu_1 - i\nu_2}{2\tau\Phi'} \right) e^{\tau(\Phi - \bar{\Phi})} d\sigma - \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2)}{\Phi'} \partial_{\nu} \left( \frac{r}{2\tau^2\Phi'} \right) e^{\tau(\Phi - \bar{\Phi})} d\sigma.
\]

Using these notations and the fact that \( \Phi \) is the harmonic function we rewrite the classical result of theorem 7.7.5 of [4] as

**Proposition 4.1.** Let \( \Phi(z) \) satisfies (3.1), (3.2) and \( u \in C^{5+\alpha}(\Omega), \alpha \in (0,1) \) be some function. Then the following asymptotic formula is true:

\[
\int_{\Omega} u e^{\tau(\Phi - \bar{\Phi})} dx = \mathfrak{F}_\tau u + \mathfrak{J}_\tau u + o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty.
\]

Denote

\[
\mathcal{H}(x, \partial_x, \partial_{\bar{x}}) = 2A\partial_x + 2B\partial_{\bar{x}} + Q,
\]

where \( A(x), B(x) \) and \( Q(x) \) are some \( 2 \times 2 \) matrices. We have
Proposition 4.2. Suppose that for any $U_{0,\tau}, \bar{U}_{0,\tau}$ given by (3.23), (3.24) and $V_{0,\tau}, \bar{V}_{0,\tau}$ given by (3.31), (3.32) with any function $\Phi$ which satisfies (3.1), (3.2) we have

$$J_\tau = \int_\Omega (H(x, \partial_z, \partial_{\bar{z}})U_{0,\tau}, V_{0,\tau})dx = o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty$$

Then

$$J_\tau = \frac{1}{\tau} \sum_{k=-1}^{1} \tau^k J_k + \frac{1}{\tau} ((J_+ + \sum_{\bar{z} \in \mathcal{H}} I_+,\Phi(\bar{z}) + K_+)e^{2\tau \psi(\bar{z})} + (J_- + \sum_{\bar{z} \in \mathcal{H}} I_-,\Phi(\bar{z}) + K_-)e^{-2\tau \psi(\bar{z})})$$

$$+ \int_\Omega ((\nu_1 - i\nu_2)(AU_0e^{\tau\Phi}, V_0e^{-\tau\Phi}) + (\nu_1 + i\nu_2)(B\bar{U}_0e^{\tau\Phi}, \bar{V}_0e^{-\tau\Phi}))d\sigma$$

$$+ \mathcal{J}_\tau(q_1, T^{*}_{B_1}B^*V_{0,\tau} - A^*V_{0,\tau} + 2T^{*}_{B_1}\partial_z B^*V_{0,\tau} + T^{*}_{B_1}(B^*(A_2^*V_{0,\tau} - 2\tau\Phi'V_{0,\tau}))$$

$$- 2\mathcal{J}(P_{-A_{2}}^*(A(\partial_z U_{0,\tau} + \tau\Phi'U_{0,\tau})), B\partial_z U_{0,\tau}, q_4) + o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty,$$

where

$$J_+ = \frac{\pi}{2}((-2\partial_z AU_0, V_0) - (AU_0, B^*_2V_0) - (2B A_1U_0, V_0) - \frac{1}{2}(Q_1(1)U_0, T^{*}_{B_1}(B^*V_0))$$

$$- (Q_1(2)\bar{U}_0, P_{-A_{2}}^*(AU_0))) - (QU_0, V_0),$$

$$J_- = \frac{\pi}{2}((-2\partial_z \bar{U}_0, V_0) - (2\partial_{\bar{z}} B\bar{U}_0, \bar{V}_0) - (2B\bar{U}_0, \partial_{\bar{z}} V_0)$$

$$- ((\partial_z \bar{q}_1, P_{A_1}^*(A^*V_0)) + (\partial_{\bar{z}} \bar{q}_2, T_{-B_2}^*(B\bar{U}_0))) + (QU_0, V_0),$$

$$I_+,\Phi(x) = \sum_{\bar{z} \in \mathcal{H}} \int_{\partial\Omega} \left\{ (\nu_1 - i\nu_2)((2Bb_{+,\bar{z}}\Phi', V_0) + (2\Phi'U_0, \bar{a}_{+,\bar{z}}))$$

$$+ (\nu_1 + i\nu_2)((2a_{+,\bar{z}}\Phi', \bar{V}_0) + (2\Phi'U_0, \bar{b}_{+,\bar{z}})) \right\} d\sigma,$$

$$I_-,\Phi(x) = \sum_{\bar{z} \in \mathcal{H}} \int_{\partial\Omega} \left\{ (\nu_1 - i\nu_2)((2Bb_{+,\bar{z}}\Phi', V_0) + (2\Phi'U_0, \bar{a}_{+,\bar{z}}))$$

$$+ (\nu_1 + i\nu_2)((2a_{+,\bar{z}}\Phi', \bar{V}_0) + (2\Phi'U_0, \bar{b}_{+,\bar{z}})) \right\} d\sigma,$$

$$K_+ = \tau \mathcal{K}_\tau(q_1, T^{*}_{B_1}B^*V_{0,\tau} - A^*V_{0,\tau} + 2T^{*}_{B_1}\partial_z B^*V_{0,\tau} + T^{*}_{B_1}(B^*(A_2^*V_{0,\tau} - 2\tau\Phi'V_{0,\tau}))$$

$$- 2\tau \mathcal{K}(P_{-A_{2}}^*(A(\partial_z U_{0,\tau} + \tau\Phi'U_{0,\tau})), B\partial_z U_{0,\tau}, q_4).$$

Proof. Denote

$$U_1 = -\mathcal{R}_{\tau, B_1}(e_1(q_1 + \bar{q}_1/\tau)), \quad \bar{U}_1 = -\mathcal{R}_{\tau, A_1}(e_1(q_2 + \bar{q}_2/\tau)),$$

$$\bar{V}_1 = -\mathcal{R}_{-\tau, -A_2}(e_1(q_3 + \bar{q}_3/\tau)), \quad V_1 = -\mathcal{R}_{-\tau, -B_2}(e_1(q_4 + \bar{q}_4/\tau)).$$
Integrating by parts and using Proposition 4.1, we obtain

\[ M_1 = \int_\Omega (2A\partial_z(U_{0,\tau}e^{r\Phi}) + 2B\partial_z(U_{0,\tau}e^{r\Phi}), V_{0,\tau}e^{-r\Phi})dx = \]

\[ \int_\Omega ((-2\partial_z AU_{0,\tau}e^{r\Phi}, V_{0,\tau}e^{-r\Phi}) - (2AU_{0,\tau}e^{r\Phi}, \partial_z V_{0,\tau}e^{-r\Phi}) + (2B\partial_z U_{0,\tau}e^{r\Phi}, V_{0,\tau}e^{-r\Phi}))dx \]

\[ + \int_{\partial\Omega} (\nu_1 - i\nu_2)(AU_{0,\tau}e^{r\Phi}, V_{0,\tau}e^{-r\Phi})d\sigma = \]

\[ \mathcal{F}_\tau(-2\partial_z AU_0, V_0) - (2AU_0, \partial_z V_0) + (2B\partial_z U_0, V_0) \]

\[ + \mathcal{J}_\tau(-2\partial_z AU_{0,\tau}, V_{0,\tau}) - (2AU_{0,\tau}, \partial_z V_{0,\tau}) + (2B\partial_z U_{0,\tau}, V_{0,\tau}) \]

\[ + \int_{\Gamma} (\nu_1 - i\nu_2)(AU_0, V_0)e^{r(\Phi - \tilde{\Phi})}d\sigma + \kappa_{0,0} + \frac{\kappa_{0,1}}{\tau} + o\left(\frac{1}{\tau}\right), \]

where \( \kappa_{0,j} \) are some constants independent of \( \tau \).

Integrating by parts we obtain that there exist constants \( \kappa_{1,j} \) independent of \( \tau \) such that

\[ \int_\Omega (2A\partial_z(\tilde{U}_{0,\tau}e^{r\Phi}) + 2B\partial_z(\tilde{U}_{0,\tau}e^{r\Phi}), V_{0,\tau}e^{-r\Phi})dx = \]

\[ (2\partial_z AU_{0,\tau}, V_{0,\tau})_{L^2(\Omega)} + (2B(\partial_z \tilde{U}_0, \tau + r\tilde{\Phi}'\tilde{U}_0)_{L^2(\Omega)} = \]

\[ \tau\kappa_{1,1} + \kappa_{1,0} + \frac{\kappa_{1,-1}}{\tau} + \frac{1}{\tau} \sum_{\tilde{x} \in \mathcal{H}} (e^{2ir\psi(\tilde{x})}(2Bb_{+\tilde{x}}\tilde{\Phi}', V_0)_{L^2(\Omega)} + e^{-2ir\psi(\tilde{x})}(2Bb_{-\tilde{x}}\tilde{\Phi}', V_0)_{L^2(\Omega)}) \]

\[ + \frac{1}{\tau} \sum_{\tilde{x} \in \mathcal{H}} (e^{2ir\psi(\tilde{x})}(2B\tilde{\Phi}'U_0, \tilde{a}_{+\tilde{x}})_{L^2(\Omega)} + e^{-2ir\psi(\tilde{x})}(2B\tilde{\Phi}'U_0, \tilde{a}_{-\tilde{x}})_{L^2(\Omega)}) + o\left(\frac{1}{\tau}\right). \]

Since for any \( \tilde{x} \) from \( \mathcal{H} \)

\[ (2B\tilde{\Phi}'U_0, \tilde{a}_{\pm\tilde{x}}) = 4\partial_z(\tilde{\Phi}'U_0, \tilde{a}_{\pm\tilde{x}}), \quad \text{and} \quad (2Bb_{\pm\tilde{x}}\tilde{\Phi}', V_0) = 4\partial_z(b_{\pm\tilde{x}}\tilde{\Phi}', V_0) \quad \text{in} \ \Omega \]

from (4.11) we have

\[ M_2 = \int_\Omega (2A\partial_z(\tilde{U}_{0,\tau}e^{r\Phi}) + 2B\partial_z(\tilde{U}_{0,\tau}e^{r\Phi}), V_{0,\tau}e^{-r\Phi})dx = \]

\[ \tau\kappa_{1,1} + \kappa_{1,0} + \frac{\kappa_{1,-1}}{\tau} + \int_{\partial\Omega} \sum_{\tilde{x} \in \mathcal{H}} \frac{(\nu_1 - i\nu_2)}{\tau} (e^{2ir\psi(\tilde{x})}(2Bb_{+\tilde{x}}\tilde{\Phi}', V_0) + e^{-2ir\psi(\tilde{x})}(2Bb_{-\tilde{x}}\tilde{\Phi}', V_0))d\sigma \]

\[ + \int_{\partial\Omega} \sum_{\tilde{x} \in \mathcal{H}} \frac{(\nu_1 - i\nu_2)}{\tau} (e^{2ir\psi(\tilde{x})}(2\tilde{\Phi}'U_0, \tilde{a}_{+\tilde{x}}) + e^{-2ir\psi(\tilde{x})}(2\tilde{\Phi}'U_0, \tilde{a}_{-\tilde{x}}))d\sigma + o\left(\frac{1}{\tau}\right). \]
Integrating by parts we obtain that there exist constants $\kappa_{2,j}$ independent of $\tau$ such that

$$
\int_\Omega (2A\partial_\bar{z}(U_{0,\tau}e^{\tau\Phi}) + 2B\partial_\bar{z}(U_{0,\tau}e^{\tau\Phi}), \bar{V}_{0,\tau}e^{-\tau\Phi})dx =
\int_\Omega (2A\partial_\bar{z}(U_{0,\tau} + \tau\Phi'U_{0,\tau}) + 2B\partial_\bar{z}U_{0,\tau}, \bar{V}_{0,\tau})_{L^2(\Omega)} =
\tau\kappa_{2,1} + \kappa_{1,0} + \frac{\kappa_{2,1}}{\tau} + \int_{\Omega} \frac{1}{\tau} \sum_{\tilde{x} \in \mathcal{H}} (e^{2i\tau\psi(\tilde{x})} (2Aa_+\tilde{x}\Phi', \bar{V}_{0,\tau})_{L^2(\Omega)} + e^{-2i\tau\psi(\tilde{x})} (2Aa_-\tilde{x}\Phi', \bar{V}_{0,\tau})_{L^2(\Omega)})
$$

(4.13) \quad + \frac{1}{\tau} \sum_{\tilde{x} \in \mathcal{H}} (e^{2i\tau\psi(\tilde{x})} (2A\Phi'\bar{U}_{0,\tau}, \tilde{b}_+, \bar{z})_{L^2(\Omega)} + e^{-2i\tau\psi(\tilde{x})} (2A\Phi'\bar{U}_{0,\tau}, \tilde{b}_-, \bar{z})_{L^2(\Omega)}) + o\left(\frac{1}{\tau}\right).

Since for any $\tilde{x}$ from $\mathcal{H}$

$$(2Aa_+\tilde{x}\Phi', \bar{V}_{0,\tau}) = 4\partial_\bar{z}(a_+\tilde{x}\Phi', \bar{V}_{0,\tau}) \quad \text{and} \quad (2A\Phi'\bar{U}_{0,\tau}, \tilde{b}_+, \bar{z}) = 4\partial_\bar{z}(\Phi'\bar{U}_{0,\tau}, \tilde{b}_+, \bar{z}) \quad \text{in} \ \Omega$$

we obtain from (4.13)

$$
\mathcal{M}_3 = \int_\Omega (2A\partial_\bar{z}(U_{0,\tau}e^{\tau\Phi}) + 2B\partial_\bar{z}(U_{0,\tau}e^{\tau\Phi}), \bar{V}_{0,\tau}e^{-\tau\Phi})dx =
\tau\kappa_{2,1} + \kappa_{1,0} + \frac{\kappa_{2,1}}{\tau} + \int_{\partial\Omega} (\nu_1 + i\nu_2) \frac{1}{\tau} \sum_{\tilde{x} \in \mathcal{H}} (e^{2i\tau\psi(\tilde{x})} (2a_+\tilde{x}\Phi', \bar{V}_{0,\tau}) + e^{-2i\tau\psi(\tilde{x})} (2a_-\tilde{x}\Phi', \bar{V}_{0,\tau}))d\sigma
$$

(4.14) \quad + \int_{\partial\Omega} (\nu_1 + i\nu_2) \frac{1}{\tau} \sum_{\tilde{x} \in \mathcal{H}} (e^{2i\tau\psi(\tilde{x})} (2\Phi'\bar{U}_{0,\tau}, \tilde{b}_+, \bar{z}) + e^{-2i\tau\psi(\tilde{x})} (2\Phi'\bar{U}_{0,\tau}, \tilde{b}_-, \bar{z}))d\sigma + o\left(\frac{1}{\tau}\right).

Integrating by parts, using (3.6) and Proposition 4.1 we obtain that there exists some constants $\kappa_{3,j}$ independent of $\tau$ such that

$$
\mathcal{M}_4 = \int_\Omega (2A\partial_\bar{z}((\bar{U}_{0,\tau}e^{\tau\Phi}) + 2B\partial_\bar{z}(\bar{U}_{0,\tau}e^{\tau\Phi}), \bar{V}_{0,\tau}e^{-\tau\Phi})dx =
\int_\Omega ((2A\partial_\bar{z}\bar{U}_{0,\tau}e^{\tau\Phi}, \bar{V}_{0,\tau}e^{-\tau\Phi}) - (2\partial_\bar{z}B\bar{U}_{0,\tau}e^{\tau\Phi}, \bar{V}_{0,\tau}e^{-\tau\Phi}) - (2B\bar{U}_{0,\tau}e^{\tau\Phi}, \partial_\bar{z}\bar{V}_{0,\tau}e^{-\tau\Phi}))dx
$$

$$
\quad + \int_{\partial\Omega} (\nu_1 + i\nu_2)(B\bar{U}_{0,\tau}e^{\tau\Phi}, \bar{V}_{0,\tau}e^{-\tau\Phi})d\sigma =
\bar{F}_{-\tau}((2A\partial_\bar{z}\bar{U}_{0,\tau}, \bar{V}_{0,\tau}) - (2\partial_\bar{z}B\bar{U}_{0,\tau}, \bar{V}_{0,\tau}) - (2B\bar{U}_{0,\tau}, \partial_\bar{z}\bar{V}_{0,\tau}))
$$

$$
\quad + \bar{J}_{-\tau}(((2A\partial_\bar{z}\bar{U}_{0,\tau}, \bar{V}_{0,\tau}) - (2\partial_\bar{z}B\bar{U}_{0,\tau}, \bar{V}_{0,\tau}) - (2B\bar{U}_{0,\tau}, \partial_\bar{z}\bar{V}_{0,\tau}))
$$

(4.15) \quad + \int_{\Gamma} (\nu_1 + i\nu_2)(B\bar{U}_{0}e^{\tau\Phi}, \bar{V}_{0}e^{-\tau\Phi})d\sigma + \kappa_{3,1} + \frac{\kappa_{3,1}}{\tau} + o\left(\frac{1}{\tau}\right).

Integrating by parts and using Proposition 4.1 we obtain
Integrating by parts and using Proposition 4.1 we have

\[ M_5 = \int_{\Omega} (2A \partial_z(U_1 e^{\tau \Phi}) + 2B \partial_z(U_1 e^{\tau \Phi}), V_{0,\tau} e^{-\tau \Phi}) dx = \]
\[ \int_{\Omega} (A(-B_1 U_1 - q_1) e^{\tau \Phi} - 2\partial_z B(U_1 e^{\tau \Phi}), V_{0,\tau} e^{-\tau \Phi}) dx + \]
\[ \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_0) e^{\tau(\Phi - \Phi')} d\sigma - (2BU_1, \partial_z(V_0,\tau) e^{\tau(\Phi - \Phi')})_{L^2(\Omega)} = \]
\[ \int_{\Omega} (A(B_1T_B(e^{\tau(\Phi - \Phi')} q_1) - q_1) e^{\tau(\Phi - \Phi')}, V_{0,\tau}) + 2\partial_z B(T_B(e^{\tau(\Phi - \Phi')} q_1), V_{0,\tau}) dx + \]
\[ (BT_B(e^{\tau(\Phi - \Phi')} q_1), A^* V_{0,\tau} - 2\tau \Phi' V_{0,\tau})_{L^2(\Omega)} + \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_{0,\tau}) e^{\tau(\Phi - \Phi')} d\sigma = \]
\[ \tilde{\mathbf{f}}_\tau(q_1, T_B^* B_1^* A^* V_0 - A^* V_0 + 2T_B^* (\partial_z B^* V_0) + T_B^* (B^*(A^* V_0 - 2\tau \Phi' V_0))) + \]
\[ + \mathbf{j}_\tau(q_1, T_B^* B_1^* A^* V_{0,\tau} - A^* V_{0,\tau} + 2T_B^* (\partial_z B^* V_{0,\tau}) + T_B^* (B^*(A^* V_{0,\tau} - 2\tau \Phi' V_{0,\tau}))) + \]
\[ + \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_{0,\tau}) e^{\tau(\Phi - \Phi')} d\sigma + o\left(\frac{1}{\tau}\right). \]

After integration by parts we have

\[ \int_{\Omega} (2A \partial_z(U_1 e^{\tau \Phi}) + 2B \partial_z(U_1 e^{\tau \Phi}), \tilde{V}_{0,\tau} e^{-\tau \Phi}) dx = \]
\[ \int_{\Omega} (A(-B_1 U_1 - q_1) - 2\partial_z BU_1, \tilde{V}_{0,\tau}) dx + \]
\[ (2BU_1, \partial_z \tilde{V}_0)_{L^2(\Omega)} + \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, \tilde{V}_{0,\tau}) d\sigma. \]

Using (4.8) and Proposition 8 of [7] we obtain that

\[ M_6 = \int_{\Omega} (2A \partial_z(U_1 e^{\tau \Phi}) + 2B \partial_z(U_1 e^{\tau \Phi}), \tilde{V}_{0,\tau} e^{-\tau \Phi}) dx = - \int_{\Omega} (Aq_1, \tilde{V}_{0,\tau}) dx + o\left(\frac{1}{\tau^2}\right) \text{ as } \tau \to +\infty. \]

Integrating by parts and using Proposition 4.1 we have

\[ M_7 = \int_{\Omega} (2A \partial_z(U_{0,\tau} e^{\tau \Phi}) + 2B \partial_z(U_{0,\tau} e^{\tau \Phi}), V_1 e^{-\tau \Phi}) dx = \]
\[ 2 \int_{\Omega} (A(\partial_z U_0 + \tau \Phi' U_{0,\tau}) e^{\tau \Phi} + B \partial_z U_{0,\tau} e^{\tau \Phi}, V_1 e^{-\tau \Phi}) dx = \]
\[ -2 \int_{\Omega} (P^{*}_{-A_2}(A(\partial_z U_0 + \tau \Phi' U_0) + B \partial_z U_{0,\tau}), q_4 e^{\tau(\Phi - \Phi')}) dx = \]
\[ -2 \tilde{\mathbf{f}}_\tau(P^{*}_{-A_2}(A(\partial_z U_0 + \tau \Phi' U_0) + B \partial_z U_{0,\tau}), q_4) + \]
\[ -2 \mathbf{j}_\tau(P^{*}_{-A_2}(A(\partial_z U_{0,\tau} + \tau \Phi' U_{0,\tau}) + B \partial_z U_{0,\tau}), q_4) + o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty. \]
Integrating by parts and using Proposition 8 of [7] we have
\begin{align}
\mathcal{M}_8 &= \int_{\Omega} (2A\partial_z(U_0,\tau e^{\tau\Phi}) + 2B\partial_z(U_0,\tau e^{-\tau\Phi}), \tilde{V}_1 e^{-\tau\Phi}) dx = \\
& \int_{\Omega} ((-2\partial_zAU_0 + B\partial_zU_0, \tilde{V}_1) - (AU_0, -B_1^z\tilde{V}_1 - q_3)) dx \\
& + \int_{\partial\Omega} (\nu_1 - i\nu_2)(AU_0, \tilde{V}_1) d\sigma = -\int_{\Omega} (AU_0, q_3) dx + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty
\end{align}

and
\begin{align}
\mathcal{M}_9 &= \int_{\Omega} (2A\partial_z(\bar{U}_1 e^{\tau\Phi}) + 2B\partial_z(\bar{U}_1 e^{-\tau\Phi}), V_{0,\tau} e^{-\tau\Phi}) dx = \\
& \int_{\Omega} ((\bar{U}_1, -\partial_z(2A^*V_{0,\tau})) + (B(-A_1\bar{U}_1 - q_2), V_{0,\tau})] dx \\
& + \int_{\partial\Omega} (\nu_1 - i\nu_2)(A\bar{U}_1, V_{0,\tau}) e^{\tau(\Phi - \Phi)} d\sigma = \\
& \int_{\Omega} (q_2, P_{A_1}^*(2\partial_z(A^*\bar{V}_{0,\tau}) - 2\tau\Phi' A^*\bar{V}_0) - B^*\bar{V}_0 - P_{A_1}^*(A_1^* B^* \bar{V}_0)) e^{\tau(\Phi - \Phi)} dx \\
& + \int_{\partial\Omega} (\nu_1 - i\nu_2)(A\bar{U}_1, \bar{V}_0) e^{\tau(\Phi - \Phi)} d\sigma = \\
& \tilde{\mathcal{E}}_{-\tau}(q_2, P_{A_1}^*(2\partial_z(A^*\bar{V}_0) - \tau\Phi' 2A^*\bar{V}_0) - B^*\bar{V}_0 + P_{A_1}^*(A_1^* B^* \bar{V}_0)) \\
& + \tilde{\mathcal{E}}_{-\tau}(q_2, P_{A_1}^*(2\partial_z(A^*\bar{V}_{0,\tau}) - \tau\Phi' 2A^*\bar{V}_{0,\tau}) - B^*\bar{V}_{0,\tau} + P_{A_1}^*(A_1^* B^* \bar{V}_{0,\tau})) + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.
\end{align}

By (3.14) and Proposition 4.1 we obtain
\begin{align}
\mathcal{M}_{10} &= \int_{\Omega} (2A\partial_z(\bar{U}_1 e^{\tau\Phi}) + 2B\partial_z(\bar{U}_1 e^{-\tau\Phi}), \bar{V}_{0,\tau} e^{-\tau\Phi}) dx = \\
& \int_{\Omega} ((\bar{U}_1, -\partial_z(2A^*\bar{V}_{0,\tau}) + \tau\Phi' 2A^*\bar{V}_0) + (B(-A_1\bar{U}_1 - q_2), \bar{V}_{0,\tau}) e^{\tau(\Phi - \Phi)} dx + \\
& \int_{\partial\Omega} (\nu_1 - i\nu_2)(A\bar{U}_1, \bar{V}_{0,\tau}) e^{\tau(\Phi - \Phi)} d\sigma = \\
& \tilde{\mathcal{E}}_{-\tau}(q_2, P_{A_1}^*(2\partial_z(A^*\bar{V}_0) - \Phi' 2A^*\bar{V}_0) - B^*\bar{V}_0 + P_{A_1}^*(A_1^* B^* \bar{V}_0)) \\
& + \tilde{\mathcal{E}}_{-\tau}(q_2, P_{A_1}^*(2\partial_z(A^*\bar{V}_{0,\tau}) - \tau\Phi' 2A^*\bar{V}_{0,\tau}) - B^*\bar{V}_{0,\tau} + P_{A_1}^*(A_1^* B^* \bar{V}_{0,\tau})) + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.
\end{align}

By (3.14) and Proposition 4.1 we obtain
\begin{align}
\mathcal{M}_{11} &= \int_{\Omega} (2A\partial_z(\bar{U}_{0,\tau} e^{\tau\Phi}) + 2B\partial_z(\bar{U}_{0,\tau} e^{-\tau\Phi}), \bar{V}_1 e^{-\tau\Phi}) dx = \\
& \int_{\Omega} ((2A\partial_z\bar{U}_{0,\tau} + 2B(\partial_z\bar{U}_{0,\tau} + \tau\bar{\Phi}'\bar{U}_{0,\tau}), \bar{V}_1) e^{\tau(\Phi - \Phi)} dx = \\
& -\int_{\Omega} (q_3, T_{-B_2^*}^*(2A\partial_z\bar{U}_{0,\tau} e^{\tau\Phi} + 2B(\partial_z\bar{U}_{0,\tau} + \tau\bar{\Phi}'\bar{U}_{0,\tau})) e^{\tau(\Phi - \Phi)} dx = \\
& -\tilde{\mathcal{E}}_{-\tau}(q_3, T_{-B_2^*}^*(2A\partial_z\bar{U}_0 + 2B(\partial_z\bar{U}_0 + \tau\bar{\Phi}'\bar{U}_0))) \\
& - \tilde{\mathcal{E}}_{-\tau}(q_3, T_{-B_2^*}^*(2A\partial_z\bar{U}_{0,\tau} + 2B(\partial_z\bar{U}_{0,\tau} + \tau\bar{\Phi}'\bar{U}_{0,\tau}))) + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.
\end{align}
By Proposition 4.1 we there exist constants $\kappa_{4,j}$ independent of $\tau$ such that

$$\mathcal{M}_{12} = \int_{\Omega} (U_{0,\tau} e^{r\phi} + \tilde{U}_{0,\tau} e^{r\phi}, V_{0,\tau} e^{-r\phi} + \tilde{V}_{0,\tau} e^{-r\phi}) dx =$$

$$\kappa_{4,0} + \kappa_{4,-1}/\tau + \frac{\pi}{2\tau} ((Q U_{0}, V_{0})(\tilde{x}) e^{2i\tau\psi(\tilde{x})} + (Q \tilde{U}_{0}, \tilde{V}_{0})(\tilde{x}) e^{-2i\tau\psi(\tilde{x})}) + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.$$ 

Since $J_\tau = \sum_{k=1}^{12} \mathcal{M}_k$ the proof of the proposition is complete. ■

**Proposition 4.2.** Under assumptions of Proposition 4.1, the following equality holds true

$$T_{B_1}(\mathcal{B}^* \Phi V_0) = P_{-A_1}^*(\mathcal{A} \Phi V_0) = P_{A_1}^*(\mathcal{B} \Phi V_0) = T_{-B_2}^*(\mathcal{A}^* \Phi \tilde{U}_0) \quad \text{on} \quad \tilde{\Gamma}$$

and

$$T_{B_1}(\mathcal{B}^* V_0) = P_{-A_1}^*(\mathcal{A} V_0) = P_{A_1}^*(\mathcal{B} V_0) = T_{-B_2}^*(\mathcal{A}^* \tilde{U}_0) \quad \text{on} \quad \tilde{\Gamma}.$$

**Proof.** We construct the function $r$ with domain $\partial \Omega$ in the following way. Let us parameterize the boundary of simply connected domain $\Omega$ clockwise. We remind that without the loss of generality we one can assume that $\tilde{\Gamma}$ is an arc with the endpoints $x_-$ and $x_+$. Let $x^*$ be an arbitrary point from $\tilde{\Gamma}$. On $\Gamma_0$ we set the function $r$ be equal zero. From $x_-$ till $x^*$ this is the strictly increasing function and form $x^*$ to $x_+$ the function $r$ is strictly decreasing. At $x^*$ the function $r$ has nondegenerate critical point and there are no critical points on $(x_-, x_+)$ except of $x^*$. Finally, at point $x_-$ we assume that the right tangential derivative of order five of the function $r$ is not equal to zero and at point $x_+$ we assume that the left tangential derivative of order five of the function $r$ is not equal to zero. Let $\psi$ be a harmonic function in $\tilde{\Omega}$ such that $\psi|_{\partial \Omega} = r$. Since domain $\Omega$ is simply connected there exists a smooth harmonic function $\tilde{\varphi}$ such that the function $\Phi = \tilde{\varphi} + i\psi$ is holomorphic in the domain $\Omega$. Moreover, after possible small perturbation of the function $r$ with support near point where $r$ is strictly increasing we may assume that

$$\partial_{\nu} \tilde{\varphi}(x_+) \neq 0 \quad \text{and} \quad \partial_{\nu} \tilde{\varphi}(x^*) \neq 0.$$ 

Then the function $\Phi$ does not have a critical points on $\tilde{\Omega}$. Consider the complex geometric optics solution constructed with the phase function $\tilde{\Phi} = \tilde{\varphi} + i\psi$ instead of $\Phi$ and the pair $(\tilde{\varphi}' U_0, \tilde{\varphi}' \tilde{U}_0)$ instead of $(U_0, \tilde{U}_0)$. By Proposition 4.2, the equality (4.2) holds true. Since there are no critical points of the function $\tilde{\Phi}$ on $\tilde{\Omega}$ the terms $I_\pm, K_\pm, J_\pm$ are equal to zero. Let $e \in C^5(\partial \Omega)$ be a function with support concentrated near points $x_\pm$ and equal to one in some neighborhood of points $x_\pm$. By (3.9), (3.11) and the fact that the left or the right tangential derivative of order five of the function $\psi$ at points $x_\pm$ is not equal zero using the stationary phase argument we have

$$\int_{\partial \Omega} e((\nu_1 - i\nu_2) \mathcal{A} U_0, V_0) e^{r(\Phi - \tilde{\Phi})} d\sigma = c_1 + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.$$ 

and

$$\int_{\partial \Omega} e((\nu_1 + i\nu_2) \mathcal{B} U_0, V_0) e^{r(\Phi - \tilde{\Phi})} d\sigma = c_2 + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.$$
By (3.16), (3.43) and the fact that left or right tangential derivative of order five of the function $\psi$ at points $x_\pm$ is not equal zero using using theorem 7.7.1 of [4] we obtain

$$
(4.28) \quad \int_{\partial \Omega} e((q_1, T_{B_1}^*(B^*\Phi'V_0)) - (q_4, P_{-A_2}^*(A\Phi'V_0)) e^{\tau(\Phi - \bar{\Phi})} d\sigma = c_3 + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
$$

and

$$
(4.29) \quad \int_{\partial \Omega} e((q_2, P_{A_1}(B\Phi'\bar{V}_0)) - (q_3, T_{-B_2}^*(A^*\bar{\Phi}U_0)) e^{\tau(\Phi - \bar{\Phi})} d\sigma = c_4 + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
$$

Then using (4.26)-(4.29) and using Theorem 7.7.1 of [4], we have

$$
(4.30) \quad (\nu_1 - i\nu_2)A U_0, V_0)(x^*) - 2(q_1, T_{B_1}^*(B^*\Phi'V_0))(x^*) - 2(q_4, P_{-A_2}^*(A\Phi'V_0))(x^*) = 0
$$

and

$$
(4.31) \quad (\nu_1 + i\nu_2)B U_0, V_0)(x^*) - 2(q_2, P_{A_1}(B\Phi'\bar{V}_0))(x^*) - 2(q_3, T_{-B_2}^*(A^*\bar{\Phi}U_0))(x^*) = 0.
$$

By Proposition 3.2 for any an arbitrary vectors $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ we can choose function $q_j$ in such a way, that $q_j(x^*) = \bar{z}_j$. Therefore (4.32) and (4.33) imply (4.24). In order to prove (4.25) we consider the complex geometric optics solution constructed with the phase function $\Phi$ instead of $\Phi$ and the pair $(\frac{1}{\nu_1}U_0, \frac{1}{\nu_1}U_0)$ instead of $(U_0, \bar{U}_0)$. By Proposition 4.2 the equality (4.2) holds true. Since there are no critical points of the function $\tilde{\Phi}$ on $\Omega$ the terms $I_\pm, K_\pm, J_\pm$ are equal to zero. Then applying the theorem 7.7.5 of [4] we have

$$
(4.32) \quad (\nu_1 - i\nu_2)A U_0, V_0)(x^*) - 2(q_1, T_{B_1}^*(B^*\Phi'V_0))(x^*) - 2(q_4, P_{-A_2}^*(A\Phi'V_0))(x^*) = 0
$$

and

$$
(4.33) \quad (\nu_1 + i\nu_2)B U_0, V_0)(x^*) - 2(q_2, P_{A_1}(B\Phi'\bar{V}_0))(x^*) - 2(q_3, T_{-B_2}^*(A^*\bar{\Phi}U_0))(x^*) = 0.
$$

By Proposition 3.2 for any an arbitrary vectors $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ we can choose function $q_j$ in such a way, that $q_j(x^*) = \bar{z}_j$. Therefore (4.32) and (4.33) imply (4.25).

Using Proposition 4.3 we prove the following:

**Proposition 4.4.** Under assumptions of Proposition 4.2 the following equality holds true

$$
(4.34) \quad P_{-A_2}^*(A U_0 \Phi') = \Phi' P_{-A_2}^*(A U_0), \quad T_{B_1}(B^*\Phi'V_0) = \bar{\Phi} T_{B_1}(B^*V_0)
$$

$$
(4.35) \quad P_{A_1}^*(A^*\bar{V}_0 \Phi') = \Phi' P_{A_1}^*(A^*\bar{V}_0), \quad T_{-B_2}^*(B\Phi'\bar{U}_0) = \bar{\Phi} T_{-B_2}^*(B^*\bar{U}_0).
$$

**Proof.** We prove the first equality in (4.34). The proof of the remaining three equalities is the same. By (4.24) and (4.25)

$$
P_{-A_2}^*(A U_0 \Phi') = \Phi' P_{-A_2}^*(A U_0) = 0 \quad \text{on } \bar{\Gamma}.
$$

We set $r_1 = P_{-A_2}^*(A U_0 \Phi')$ and $r_2 = \Phi' P_{-A_2}^*(A U_0)$. Then functions $r_j$ satisfy

$$
-2\partial_j r_j - A_2^* r_j = A U_0 \Phi' \quad \text{in } \Omega, \quad r_j|_{\bar{\Gamma}} = 0 \quad j \in \{1, 2\}.
$$
By the uniqueness of the Cauchy problem for the $\partial_{\xi}$ operator we have that $r_1 = r_2$. The proof of Proposition 4.4 is complete. ■

We use Proposition 4.4 to simplify the formula (4.2).

**Proposition 4.5.** Under conditions of Proposition 4.2 we have

\begin{equation}
(4.36)
-(2\partial_{\xi}A U_0, V_0)(\bar{x}) - (2A U_0, \partial_2 V_0)(\bar{x}) + (2B \partial_{\xi} U_0, V_0)(\bar{x})
- ((Q_1(1) U_0, T_{B_1}(B^* V_0)) + (Q(2) V_0, P_{-A_2^*}^* (A U_0)))(\bar{x}) + (Q_0 U_0, V_0)(\bar{x}) + I_+ \phi(\bar{x}) = 0 \quad \text{in } \Omega,
\end{equation}

where function $\Phi$ satisfies (5.2) and

\begin{equation}
(4.37)
\text{Im} \Phi(\bar{x}) \notin \{ \text{Im} \Phi(x); x \in H \setminus \{ \bar{x} \} \}.
\end{equation}

**Proof.** By Proposition 4.2 equality (4.2) holds true. Thanks to (4.37) and Proposition 4.3 we can write it as

\begin{equation}
(4.38)
(J_+ + I_+ + K_+)(\bar{x}) - \pi(\mathcal{L}(q_1, T_{B_1}(B^* \Phi^* V_0)) + \mathcal{L}(q_4, P_{-A_2}^* (A \Phi^* U_0)) + I_+ \phi(\bar{x}) = 0,
\end{equation}

where $\mathcal{L} u = -\frac{\partial^2 u(\bar{x})}{\partial \Phi^* (\bar{x})} + \frac{\partial^2 u(\bar{x})}{2\Phi^* (\bar{x})}$. Let us transform the function $K_+$, which is given by (4.7). By Propositions 4.1 and 4.4 we obtain

\begin{equation}
(4.39)
\frac{3}{2\tau}(q_1, T_{B_1}(B^* V_0))(\bar{x}) = -\frac{\pi}{2\tau}(Q_1(1) U_0, T_{B_1}(B^* V_0))(\bar{x})
\end{equation}

and

\begin{equation}
(4.40)
-2\frac{3}{2\tau}(P_{-A_2}^* (A(\partial_{\xi} U_0 + \tau \Phi^* U_0)) + B \partial_{\xi} U_{0, \tau}, q_4) = -2\frac{3}{2\tau}(P_{-A_2}^* (A \tau \Phi^* U_0)), q_4 =
-2\frac{3}{2\tau}(\tau \Phi^* (P_{-A_2}^* (A U_0)), q_4) = \frac{\pi}{2\tau}(P_{-A_2}^* (A U_0), 2\partial_2 q_4)(\bar{x}) = \frac{\pi}{2\tau}(P_{-A_2}^* (A U_0), Q_2(2) V_0)(\bar{x}).
\end{equation}

By (4.39) and (4.40)

\begin{equation}
(4.41)\quad K_+ = -\frac{\pi}{2\tau}(Q_1(1) U_0, T_{B_1}(B^* V_0))(\bar{x}) + \frac{\pi}{2\tau}(P_{-A_2}^* (A U_0), Q_2(2) V_0)(\bar{x}).
\end{equation}

Substituting into equality (4.38) the right hand side of formula (4.41) we obtain (4.36). The proof of the proposition is complete. ■

5. **Step 5: End of the proof.**

Suppose that for the operators $L_{\mu_1, \gamma_1}(x, D)$ and $L_{\mu_2, \gamma_2}(x, D)$ given by (0.5) and the Dirichlet-to-Neumann maps $\Lambda_{\mu_2, \gamma_2}$ given by (0.7) are the same. Then, by Proposition 1.1 the Dirichlet-to-Neumann maps for the operators $L_1(x, D)$ and $L_2(x, D)$ given by formula (1.19) are the same. Let $u_1$ be the complex geometric optics solution given by (3.38) constructed for the
operator $L_1(x, D)$. There exists a function $u_2$ be a solution to the following boundary value problem:

$$L_2(x, D)u_2 = 0 \quad \text{in } \Omega, \quad \mathcal{B}(x, D)(u_1 - u_2)|_{\partial \Omega} = 0, \quad \mathcal{R}(x, D)(u_1 - u_2) = 0 \quad \text{on } \tilde{\Gamma}.$$ 

Let $\eta$ be a function such that

\begin{equation}
\eta \in C_0^\infty(\Omega), \quad \eta|_\partial \Omega = 1,
\end{equation}

where $\Omega$ is some open set such that $\mathcal{H} \subset \Omega \subset \Omega$. The operator $L_1(x, s, D) = e^{sn}L_1(x, D)e^{-sn}$ has the same Dirichlet-to-Neumann map as the operator $L_1(x, D)$. Then the function $\tilde{u}_1 = e^{sn}u_1$ solves the boundary value problem

\begin{equation}
L_1(x, s, D)\tilde{u}_1 = 0 \quad \text{in } \Omega, \quad \mathcal{B}(x, D)(\tilde{u}_1 - u_2)|_{\partial \Omega} = 0, \quad \mathcal{R}(x, D)(\tilde{u}_1 - u_2) = 0 \quad \text{on } \tilde{\Gamma}.
\end{equation}

Setting $u = \tilde{u}_1 - u_2$, $A_s = A_1 - 2s\partial_2\eta - A_2$, $A = A_0$, $B_s = B_1 - 2s\partial_2\eta - B_2$, $B = B_0$ and $Q_s = Q_1 - Q_2 - s^2|\nabla \eta|^2 + s\Delta \eta + 2sA_2\partial_2\eta + 2sB_2\partial_2\eta$, $Q = Q_0$ we have

\begin{equation}
L_2(x, D)u + 2A_s\partial_2u_1 + 2B_s\partial_\tau u_1 + Q_s\tilde{u}_1 = 0 \quad \text{in } \Omega
\end{equation}

and

\begin{equation}
\mathcal{B}(x, D)u|_{\partial \Omega} = 0, \quad \mathcal{R}(x, D)u|_{\tilde{\Gamma}} = 0.
\end{equation}

Let $v$ be a function given by (3.57). Taking the scalar product of (5.3) with $v$ in $L^2(\Omega)$ and using (3.58) and (5.4), we obtain

\begin{equation}
0 = \mathcal{G}(\tilde{u}_1, v) = \int_\Omega (2A_s\partial_2\tilde{u}_1 + 2B_s\partial_\tau\tilde{u}_1 + Q_s\tilde{u}_1, v)dx.
\end{equation}

**Proposition 5.1.** Let $\tilde{u}_1 = e^{sn}u_1$, where $u_1$ is given by (3.37) and $v$ is given by (3.57). Then the following asymptotics holds true

$$\mathcal{G}(\tilde{u}_1, v) = \int_\Omega (2A_s\partial_2(e^{sn}U) + 2B_s\partial_\tau(e^{sn}U) + Q_s(e^{sn}U), V^*)dx + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.$$ 

**Proof.** Since the form $\mathcal{G}(\cdot, \cdot)$ is bilinear in order to prove the statement of this proposition it suffices to show that

\begin{equation}
\mathcal{G}(e^{\tau \varphi}u_{-1}, V^*) = \mathcal{G}(e^{\tau \varphi}u_{-1}, e^{-\tau \varphi}v_{-1}) = \mathcal{G}(e^{\tau \varphi}u, e^{-\tau \varphi}v) = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
\end{equation}

Obviously, by (3.37) and (3.56), we see that

\begin{equation}
\mathcal{G}(e^{\tau \varphi}u_{-1}, e^{-\tau \varphi}v_{-1}) = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
\end{equation}

Let $\chi \in C_0^\infty(\Omega)$ satisfy $\chi|_{\partial \Omega} = 1$. By (3.37), we have

\begin{equation}
\mathcal{G}(e^{\tau \varphi}u_{-1}, V^*) = \mathcal{G}(e^{\tau \varphi}u_{-1}, \chi V^*) + o\left(\frac{1}{\tau}\right)
\end{equation}

\begin{align*}
= & \int_\Omega (2A_s\partial_2(e^{\tau \varphi}u_{-1}) + 2B_s\partial_\tau(e^{\tau \varphi}u_{-1}), \chi V^*)dx + o\left(\frac{1}{\tau}\right)
\end{align*}

\begin{equation}
= \int_\Omega (2A_s\partial_2(e^{\tau \varphi}u_{-1}), \chi \tilde{V}_0e^{-\tau \varphi}) + (2B_s\partial_\tau(e^{\tau \varphi}u_{-1}), \chi V_0e^{-\tau \varphi})dx + o\left(\frac{1}{\tau}\right).
\end{equation}
Let functions \(w_1, w_5\) solve the equations \((-\partial_z + (B_1^*-2s\partial_z\eta))w_1 = 2\mathcal{A}_s^*V_0\) and \((-\partial_z + (A_1^*-2s\partial_z\eta))w_5 = 2B_s^*V_0\).

Taking the scalar product of equation \((3.36)\) and the function \(\chi(w_5e^{\tau\Phi} + w_4e^{\tau\bar{\Phi}})\), after integration by parts we obtain

\[
\int_\Omega ((2\partial_z(e^{\tau\varphi}\bar{u}_1) + (A_1 + 2s\partial_z\eta)(e^{\tau\varphi}\bar{u}_1), 2\mathcal{A}_s^*V_0e^{-\tau\Phi})
\]

\[
+ (2\partial_z(e^{\tau\varphi}\bar{u}_1) + (B_1 - 2s\partial_z\eta)(e^{\tau\varphi}\bar{u}_1), 2B_s^*V_0e^{-\tau\Phi}))dx = o\left(\frac{1}{r}\right).
\]

By \((5.8)\) and \((5.9)\), we obtain the first equality in \((5.6)\). The proof of the second equality in \((5.6)\) is the same and involves the estimate \((5.50)\).

Thanks to Proposition \(5.1\) the statements of Propositions 4.2 - 4.4 hold true.

**Proposition 5.2.** Let sequence of function \(\Phi_\varepsilon\) given by Proposition \(3.7\) For the functions \(I_{\pm,\Phi_\varepsilon}\) given by \((4.5)\) and \((4.6)\) we have

\[
I_{\pm,\Phi_\varepsilon}(\tilde{x}_\varepsilon) \equiv 0.
\]

**Proof.** We prove this statement for the function \(I_{+,\Phi_\varepsilon}\). The proof for the function \(I_{-,\Phi_\varepsilon}\) is the same. By Proposition \(4.5\) equality \((4.36)\) holds true. Next we observe that

\[
T^*_{B_1}(\mathcal{B}^*V_0) = V_0 + p_1, \quad p_1 \in \text{Ker} T^*_{B_1}
\]

and

\[
P^*_{-A_s^*}(AU_0) = U_0 + p_2, \quad p_2 \in \text{Ker} P^*_{-A_s^*}.
\]

**Proof**. We prove this statement for the function \(I_{+,\Phi_\varepsilon}\). The proof for the function \(I_{-,\Phi_\varepsilon}\) is the same. By Proposition \(4.3\) equality \((4.36)\) holds true. Next we observe that

\[
T^*_{B_1}(\mathcal{B}^*V_0) = V_0 + p_1, \quad p_1 \in \text{Ker} T^*_{B_1}
\]

and

\[
P^*_{-A_s^*}(AU_0) = U_0 + p_2, \quad p_2 \in \text{Ker} P^*_{-A_s^*}.
\]

where functions \(p_1\) and \(p_2\) are given by \((5.11)\) and \((5.12)\) respectively. The direct computations imply that

\[
(-\partial_z + B_s^*)(e^{\text{sn}}(V_0 + p_1)) = e^{\text{sn}}(\mathcal{B}_s^*V_0) \quad \text{and} \quad (-\partial_z + A_2)(e^{\text{sn}}(U_0 + p_2)) = e^{\text{sn}}(A_sU_0).
\]

By Proposition \(4.3\)

\[
T^*_{B_1}(\mathcal{B}^*V_0) = P^*_{-A_s^*}(AU_0) = 0 \quad \text{on} \quad \tilde{\Gamma}.
\]
On the other hand by (5.13) we have
\[ e^{sn}(U_0 + p_1) = e^{sn}(U_0 + p_2) = 0 \quad \text{on } \bar{\Gamma}. \]

The by uniqueness of solution for the Cauchy problem for \partial_x \text{ operator we have (5.15) and (5.16). Using (5.18) and (5.17) we rewrite equation (4.3) as}

\[ -(2\partial_x \mathcal{A}U_0, V_0) - (2\mathcal{A}U_0, \partial_x V_0)e^s + (2\partial_x \mathcal{A}U_0, V_0)e^s \]
\[ -e^s((Q_1(1)U_0, T_{B_1}^*(\mathcal{B}^*V_0)) + e^s(Q(2)V_0, P_{-A_2}^*(\mathcal{A}U_0))) + e^s(QU_0, V_0) + I_{+, \varphi}(\tilde{x}) = 0. \]

This imply (5.10). □

Using (5.10) and the fact that by (3.4) for any \( x \) from \( \Omega \) exists a sequence of \( x_\epsilon \) converging to \( x \) we rewrite the equation (2.2) as

\[ -(2\partial_x \mathcal{A}U_0, V_0) - (2\mathcal{A}U_0, \partial_x V_0) + (2\mathcal{B}\partial_x U_0, V_0) \]
\[ -(Q_1(1)U_0, T_{B_1}^*(\mathcal{B}^*V_0)) + (Q(2)V_0, P_{-A_2}^*(\mathcal{A}U_0))) + (QU_0, V_0) = 0 \quad \text{in } \Omega. \]

By Proposition 3.2 for each point \( \tilde{x} \) from \( \Omega \) one can construct such a function \( U_0, V_0 \) satisfying (3.7), (3.9), (3.10), (3.11) such that

\[ U_0^{(k)}(\tilde{x}) = \tilde{e}_k, \quad V_0^{(\ell)}(\tilde{x}) = \tilde{e}_\ell \quad k, \ell \in \{1, 2\}. \]

Then for each \( \tilde{x} \) there exists positive \( \delta(\tilde{x}) \) such that the matrices \( \{ U_0^{(j)} \} \) and \( \{ V_0^{(j)} \} \) are invertible for any \( x \in B(\tilde{x}, \delta(x)) \). From the covering of \( \Omega \) by such a balls we take the finite subcovering \( \Omega \subset \cup_{k=1}^{N} B(x_k, \delta_k) \). Then from (5.20) we have the differential inequality

\[ |\partial_x \mathcal{A}ij| \leq C(x) \left( \sum_{k=1}^{N} |T_{B_1}^*(\mathcal{B}^*V_0(k))| + |P_{-A_2}^*(\mathcal{A}U_0(k))| + |A| + |B| + |Q| \right) \quad \text{in } \Omega, \quad \forall i, j \in \{1, 2\}. \]

We set \( \rho = (\rho_1, \rho_2, \rho_3) \) where the function \( \rho_i \) are defined by (1.15). Let point \( \hat{x} \) belongs to \( \text{supp}\rho \). Using (1.17) we rewrite (5.21) as

\[ |\Delta \rho| \leq C(x) \left( \sum_{k=1}^{N} |T_{B_1}^*(\mathcal{B}^*V_0(k))| + |P_{-A_2}^*(\mathcal{A}U_0(k))| + |\nabla \rho| + |\rho| \right) \quad \text{in } \Omega. \]

Let \( \tilde{\gamma} \) be a curve, without self-intersections which pass through the point \( \hat{x} \) and couple points \( x_1, x_2 \) from \( \bar{\Gamma} \) in such a way that the set \( \tilde{\gamma} \cap \partial \Omega \setminus \{x_1, x_2\} \) is empty. Denote by \( \Omega_1 \) a domain bounded by \( \tilde{\gamma} \) and and part of \( \partial \Omega \) located between points \( x_1 \) and \( x_2 \). Then we set \( \Omega_{1, \epsilon} = \{ x; \text{dist}(\Omega_1, x) < \epsilon \} \). Let \( \phi_0 \) be a function such that

\[ \nabla \phi_0(x) \neq 0 \quad \text{in } \Omega_1, \quad \partial_\nu \phi_0 |_{\tilde{\gamma}} \leq \alpha' < 0, \quad \phi_0 |_{\tilde{\gamma}} = 0. \]

where \( \nabla \phi_0(x) \neq 0 \) in \( \Omega_1 \), \( \partial_\nu \phi_0 |_{\tilde{\gamma}} \leq \alpha' < 0 \), \( \phi_0 |_{\tilde{\gamma}} = 0 \).

where \( \nabla \phi_0(x) \neq 0 \) is the outward normal derivative to \( \Omega_1 \) and \( \mu_\epsilon \) be a function such that

\[ \mu_\epsilon \in C^\infty_0(\Omega_{1, \epsilon}), \quad \mu_\epsilon = 1 \quad \text{in } \Omega_1. \]

We set \( \rho_\epsilon = \mu_\epsilon \rho \). From (5.22) we have

\[ |\Delta \rho_\epsilon| \leq C(x) \left( \sum_{k=1}^{N} |\mu_\epsilon T_{B_1}^*(\mathcal{B}^*V_0(k))| + |\mu_\epsilon P_{-A_2}^*(\mathcal{A}U_0(k))| + |\mu_\epsilon (|\nabla \rho| + |\rho|) + |[\mu_\epsilon, \Delta \rho]| \right) \quad \text{in } \Omega_{1, \epsilon}, \quad \rho_\epsilon |_{\partial \Omega_{1, \epsilon}} = 0. \]
Set $\psi_0 = e^{\lambda \psi_0}$ with positive $\lambda$ sufficiently large. Applying the Carleman estimate to the above inequality we have

$$
\int_{\Omega_1,\epsilon} e^{2\tau \psi_0} (\tau |\nabla \rho_\epsilon|^2 + \tau^3 |\rho_\epsilon|^2) dx \leq C \int_{\Omega_1,\epsilon} \left( \sum_{k=1}^N |\mu_\epsilon T_{B_1}^* (B^* V_0(k))|^2 \right) dx
$$

$$
+ |\mu_\epsilon P_{-A_2}^* (AU_0(k))|^2 + \mu_\epsilon^2 (|\nabla \rho| + |\rho|^2) + ||\mu_\epsilon, \Delta|\rho|^2)e^{2\tau \psi_0} dx \quad \forall \tau \geq \tau_0.
$$

(5.24)

By the Carleman estimate for the operator $\partial_2$ there exist $C$ and $\tau_0$ independent of $\tau$ such that

$$
\int_{\Omega_1,\epsilon} |\mu_\epsilon T_{B_1}^* (B^* V_0(k))|^2 e^{2\tau \psi_0} dx \leq C \int_{\Omega_1,\epsilon} \left( ||\mu_\epsilon, \partial_2 T_{B_1}^* (B^* V_0(k))|^2 + |B^* V_0(k)|^2 \right) e^{2\tau \psi_0} dx
$$

and

$$
\int_{\Omega_1,\epsilon} |\mu_\epsilon P_{-A_2}^* (AU_0(k))|^2 e^{2\tau \psi_0} dx \leq C \int_{\Omega_1,\epsilon} \left( ||\mu_\epsilon, \partial_2 P_{-A_2}^* (AU_0(k))|^2 + |AU_0(k)|^2 \right) e^{2\tau \psi_0} dx \quad \forall \tau \geq \tau_0
$$

for all $\tau \geq \tau_0$. Combining (5.24), (5.25) and (5.26) we obtain

$$
\int_{\Omega_1,\epsilon} e^{2\tau \psi_0} (\tau |\nabla \rho_\epsilon|^2 + \tau^3 |\rho_\epsilon|^2) dx \leq C \int_{\Omega_1,\epsilon} \left( \sum_{k=1}^N |\mu_\epsilon, \partial_2 P_{-A_2}^* (AU_0(k))|^2 \right) dx
$$

$$
+ |\mu_\epsilon (AU_0(k))|^2 + ||\mu_\epsilon, \partial_2 T_{B_1}^* (B^* V_0(k))|^2 + |\mu_\epsilon B^* V_0(k)|^2 + ||\mu_\epsilon, \Delta|\rho|^2)e^{2\tau \psi_0} dx
$$

$$
\leq C \int_{\Omega_1,\epsilon} \left( \sum_{k=1}^N |\mu_\epsilon, \partial_2 P_{-A_2}^* (AU_0(k))|^2 + |\nabla \rho_\epsilon|^2 + |\rho_\epsilon|^2 \right)
$$

$$
+ ||\mu_\epsilon, \partial_2 T_{B_1}^* (B^* V_0(k))|^2 + ||\mu_\epsilon, \Delta|\rho|^2)e^{2\tau \psi_0} dx \quad \forall \tau \geq \tau_0.
$$

(5.27)

For all sufficiently large $\tau$ the term $\int_{\Omega_1,\epsilon} (|\nabla \rho_\epsilon|^2 + |\rho_\epsilon|^2)e^{2\tau \psi_0} dx$ absorbed by the integral on the left hand side. Moreover, thanks to the choice of the function $\mu_\epsilon$, we have supports of coefficients for the operators $[\mu_\epsilon, \partial_2], [\mu_\epsilon, \partial_2]$ and $[\mu_\epsilon, \Delta]$ are located in the domain $\Omega_{1,\epsilon} \setminus \Omega_{1,\tilde{\tau}}$.

$$
\int_{\Omega_{1,\epsilon}} e^{2\tau \psi_0} (\tau |\nabla \rho_\epsilon|^2 + \tau^3 |\rho_\epsilon|^2) dx \leq C \int_{\Omega_{1,\epsilon} \setminus (\Omega_{1,\tilde{\tau}} \cup \Omega_1)} \left( \sum_{k=1}^N |\mu_\epsilon, \partial_2 P_{-A_2}^* (AU_0(k))|^2 \right) dx
$$

$$
+ ||\mu_\epsilon, \partial_2 |\psi_0(k)|^2 + ||\mu_\epsilon, \Delta|\rho|^2)e^{2\tau \psi_0} dx \quad \forall \tau \geq \tau_1.
$$

(5.28)

By (5.23) for all sufficiently small positive $\epsilon$ there exists a positive constant $\alpha < 1$ such that

$$
\tilde{\psi}_0(x) < \alpha \quad \text{on} \quad \Omega_{1,\epsilon} \setminus (\Omega_{1,\tilde{\tau}} \cup \Omega_1).
$$

(5.29)

Since $\tilde{x} \in \text{supp } \rho \cap \tilde{\gamma}$ and thanks to the fact $\partial_x \phi_0|\tilde{\gamma} \leq \alpha' < 0$ there exists $\kappa > 0$ such that

$$
\kappa \tau \leq \int_{\Omega_{1,\epsilon}} e^{2\tau \psi_0} (\tau |\nabla \rho_\epsilon|^2 + \tau^3 |\rho_\epsilon|^2) dx \quad \forall \tau \geq \tau_1.
$$

(5.30)
By (5.29) we can estimate the right hand side of the inequality (5.28) as

\[
\int_{\Omega_{1,\epsilon}\setminus(\Omega_{1,\epsilon}\cup\Omega_{1})} \left( \sum_{k=1}^{N} \left| \mu_{\epsilon}, \partial_{2} \right| P_{=A_{2}}^{*} (AU_{0}(k)) \right|^{2} + \left| \mu_{\epsilon}, \partial_{2} \right| T_{B_{1}}^{*} (B^{*}V_{0}(k)) \right|^{2} + \left| \mu_{\epsilon}, \Delta \right| \rho \right|^{2} e^{2r\psi_{0}} dx \leq C e^{\alpha \tau} \quad \forall \tau \geq \tau_{1}.
\]

Using (5.30) and (5.31) in (5.28) we obtain

\[
\kappa e^{\tau} \leq C e^{\alpha \tau} \quad \forall \tau \geq \tau_{1}.
\]

Since \( \alpha < 1 \) we arrived to the contradiction. ■

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