On a new closed formula for the solution of second order linear difference equations and applications

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Abstract

In this paper, we establish a new closed formula for the solution of homogeneous second order linear difference equations with constant coefficients by using only matrix theory. This in turn gives new closed formulas for many sequences of this type such as the Fibonacci and Lucas sequences and many others. Then we present two applications: one deals with finding interesting summation formulas relating the elements of such sequences. The second application is to show how our approach gives rise to a new method for solving systems of first order difference equations. Finally, we introduce the generalized ratio of such a sequence and find a formula for it which can be considered as a generalization of the golden ratio, the silver ratio and many others.

keywords. Recurrence relations, difference equations, Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence, Jacobsthal-Lucas sequence, Horadam sequence, Tchebychev polynomials, Fibonacci polynomials, Lucas polynomials, golden ratio, silver ratio.

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1 Introduction

For any \((n, z) \in \mathbb{Z} \times \mathbb{C}\) where \(\mathbb{Z}\) is the set of integers and \(\mathbb{C}\) is the field of complex numbers, we call a generalized linear second order recurrent sequence any sequence which is given by the following second order linear difference equation:

\[ R_{n+1}(z) = f(z)R_n(z) + g(z)R_{n-1}(z), \quad R_0(z) = h(z), \quad R_1(z) = k(z) \quad (1) \]

where \(f, g, h\) and \(k\) are any complex functions\(^1\). Without loss of generality, we may assume that \(f(z) \neq 0\) and \(g(z) \neq 0\) since otherwise we obtain one trivial case for \(f(z) = g(z) = 0\), and two other cases which are easy to handle: one corresponds to \(f(z) = 0, g(z) \neq 0\) and the other is associated with \(f(z) \neq 0, g(z) = 0\). In addition, for simplicity, we may sometimes use the notation: \(R_n = R_n(z) = R(n, f, g, h, k)\) to refer to the \(n\)th term of such a sequence. Similarly, when there is no confusion, we often will write \(f, g, h\) and \(k\) to denote the functions \(f(z), g(z), h(z)\) and \(k(z)\) respectively. Our intention here is to study this sequence assuming that \(f, g, h\) and \(k\) are any complex functions.

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\(^1\)Note that \(f, g, h\) and \(k\) can also be considered as multi-variables functions, however this obviously will not affect the results.
It is worth noting here that for \( n \geq 1 \), the sequence defined in (1) can be thought of as an extended family of Gibonacci polynomials \( R_n(z) \). Obviously, sequences of this type include the following. For \( n \geq 1 \), the Fibonacci numbers \( F_n \) and the Lucas numbers \( L_n \) are defined by \( F_{n+1} = F_n + F_{n-1}, F_0 = 0, F_1 = 1 \) and \( L_{n+1} = L_n + L_{n-1}, L_0 = 2, L_1 = 1 \) respectively. On the one hand, the Pell numbers satisfy the recurrence relation \( P_n = 2P_{n-1} + P_{n-2} \) with initial conditions \( P_0 = 0, P_1 = 1 \), and the Pell-Lucas numbers are related by the recurrence relation \( Q_n = 2Q_{n-1} + Q_{n-2} \) with \( Q_0 = Q_1 = 2 \). On the other hand, the Jacobsthal numbers \( J_n \) and the Jacobsthal-Lucas numbers \( j_n \) satisfy the recurrence relation \( J_n = J_{n-1} + 2J_{n-2} \) with initial conditions \( J_0 = 0, J_1 = 1 \) and \( j_0 = 2, j_1 = 1 \) respectively. In addition, The Fibonacci polynomials are defined by the recurrence relation \( f_{n+1}(x) = xf_n(x) + f_{n-1}(x) \) with \( f_1(x) = 1, f_2(x) = x \). Similarly, the definition of the Lucas polynomials is given by \( l_n(x) = xl_{n-1}(x) + l_{n-2}(x) \), \( l_0(x) = 1, l_1(x) = x \). Finally, the Tchebychev polynomials of the first kind are given by \( T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \), \( T_0(x) = 1, T_1(x) = x \); see for example [6, 7, 8, 9] and the references therein.

Certainly there is an extensive work in the literature concerning linear second order difference equations and their applications (see for example [3, 7, 8, 10]). However only a few deals with finding a unifying framework under which all these sequences come together under one theory. Our main objective in this paper is to deal with this issue. Note that other authors have dealt with such issue but their approach is different. We particularly mention the work done in [11] where the authors deals with this issue but one major difference between their approach and ours is that in [11], the parameters are taken to be real numbers instead of complex numbers. Beside this difference of the approaches, there is also another difference which lies in taking \( n \) to be any integer of \( \mathbb{Z} \) in our case. A different approach to the very same problem can be also found in [1].

This paper is organized as follows. Section 2 contains the main results where we present a new closed formula solution for the sequence defined by (1) which in turn gives new closed formulas for many sequences of this type such as Fibonacci, Lucas, Pell, Pell-Lucas, Horadam, Jacobsthal and Jacobsthal-Lucas number sequences as well as Tchebychev, Fibonacci and Lucas polynomials. Section 3 deals with applications; in particular, we reveal how the proposed common closed formula can be used to derive summation formulas relating the elements of the sequence \( \{R_n\}_n \). Then, we shall show how our approach can be easily used to solve systems of non-homogeneous first order difference equations. Finally, we introduce the generalized ratio of such a sequence and find a formula for it which can be considered as a generalization of the golden ratio, the silver ratio and many others.

2 Main Results

We shall start by introducing some notation. The determinant of a square matrix \( X \) will be denoted by \( |X| \). For our purposes, we next introduce the following four matrices which will be used throughout this paper and they constitute the essential foundations of our main results. Define

\[
A = \begin{bmatrix}
    2gh + f^2h - fk & 2kg - fhg \\
    2k - fh & 2gh + fk
\end{bmatrix}, \quad
B = \begin{bmatrix}
    0 & g \\
    1 & f
\end{bmatrix}.
\]
\[ C = \begin{bmatrix} 2g & fg \\ f & f^2 + 2g \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -f & 2g \\ 2 & f \end{bmatrix} \]

where \( g, h \) and \( k \) are as above. Now a simple check shows that

\[ AB = BA = \begin{bmatrix} 2kg - fhg & 2g^2 h + gfk \\ 2gh + fk & 2gk + gfh + f^2 k \end{bmatrix}, \]

\[ AC = CA = \begin{bmatrix} gf^2 h + 4g^2 h & f^2 gk + 4g^2 k \\ 4gk + f^2 k & 4gfk + gf^2 h + 4g^2 h + f^3 k \end{bmatrix}, \]

and

\[ AD = DA = \begin{bmatrix} -f^3 h - 4gf h + f^2 k + 4gk & gf^2 h + 4g^2 h \\ f^2 h + 4gh & 4gk + f^2 k \end{bmatrix}. \]

Similarly one can easily check that \( BC = CB, BD = BD, \) and \( DC = CD \) which means that \( A, B, C \) and \( D \) are pairwise commuting. As a result, we have the following useful lemma that connects the matrices \( A, B, C \) and \( D \) with the sequence \( \{R_n\}_n \).

**Lemma 2.1** Let the matrices \( A, B, C \) and \( D \) be defined as above and let \( R_n \) be the general term of the sequence defined by (1). Then for any integer \( n \), we have

\[ AB^n = CR_n + DgR_{n-1}. \quad (2) \]

**Proof.** We first do the proof for the case \( n \geq 1 \) by induction. Clearly (2) is true for \( n = 1 \) since \( AB = CR_1 + DgR_0 \) as an inspection shows that

\[ \begin{bmatrix} 2gh + f^2 h - fk & 2kg - fhg \\ 2k - fh & 2gh + fk \end{bmatrix} \begin{bmatrix} 0 & g \\ 1 & f \end{bmatrix} = \begin{bmatrix} 2g & fg \\ f & f^2 + 2g \end{bmatrix} k + \begin{bmatrix} -f & 2g \\ 2 & f \end{bmatrix} gh. \]

Now suppose that (2) is true for \( n \), then we have

\[ \begin{bmatrix} 2gh + f^2 h - fk & 2kg - fhg \\ 2k - fh & 2gh + fk \end{bmatrix} \begin{bmatrix} 0 & g \\ 1 & f \end{bmatrix}^n = \begin{bmatrix} 2g & fg \\ f & f^2 + 2g \end{bmatrix} R_n + \begin{bmatrix} -f & 2g \\ 2 & f \end{bmatrix} gR_{n-1}. \]

Multiplying both sides of this last equation to the right by \( B = \begin{bmatrix} 0 & g \\ 1 & f \end{bmatrix} \), we obtain the
following equality:

\[ AB^{n+1} = (CR_n + DgR_{n-1}) B \]

\[ = \begin{bmatrix} fgR_n + 2g^2 R_{n-1} & g(2g + f^2)R_n + fg^2 R_{n-1} \\ (f^2 + 2g)R_n + fgR_{n-1} & f(f^2 + 3g)R_n + g(2g + f^2)R_{n-1} \end{bmatrix} \]

\[ = \begin{bmatrix} 2g & f \\ f & f^2 + 2g \end{bmatrix} (fR_n + gR_{n-1}) + \begin{bmatrix} -f & 2g \\ 2 & f \end{bmatrix} gR_n \]

\[ = CR_{n+1} + DgR_n. \]

Next, for the case \( n = 0 \), it is easy to see that the sequence defined by (1) becomes \( R_{-1} = g^{-1}(k - fh) \). Then a simple check now shows that (2) is also valid for \( n = 0 \). Finally, the proof for the case \( n \leq -1 \), can easily be done by induction in a similar manner as in the first case, and the proof is complete.

It should be noted here that equation (2) of the preceding lemma involves only \( R_n \) and \( R_{n-1} \) so that it has an interesting consequence which can be stated as follows. Solving a second order linear homogeneous difference equation that has constant coefficients and which is coupled with initial conditions, is essentially equivalent to solving a non-homogeneous linear first order difference equation.

Next note that a simple check shows that the following determinant formulas are valid:

\[ |A| = (f^2 + 4g) \left( gh^2 - k^2 + fhk \right), \]

and \( |CR_n + DgR_{n-1}| = g \left( f^2 + 4g \right) \left( R_{n-1}^2 - gR_{n-1}^2 - fR_nR_{n-1} \right) \). As a conclusion of this, we have the following lemma which also appears as Theorem 5 in [11] (for \( z \) real) albeit arrived at by different means.

**Lemma 2.2** Let \( R_n, f, g, h \) and \( k \) be as given above. If \( f^2 + 4g \neq 0 \) then we have:

\[ R_n^2 - gR_{n-1}^2 - fR_nR_{n-1} = (-gh^2 + k^2 - fhk)(-g)^{n-1}, \quad (3) \]

for all integers \( n \).

**Proof.** Taking determinants of both sides of (2) we obtain \( |AB^n| = |CR_n + DgR_{n-1}| \), or

\[ \left| \begin{bmatrix} 2gh + f^2h - fh \\ 2k - fh \end{bmatrix} \begin{bmatrix} 0 & g \\ 1 & f \end{bmatrix} \right|^n = \left| \begin{bmatrix} 2g & fg \\ f & f^2 + 2g \end{bmatrix} R_n + \begin{bmatrix} -f & 2g \\ 2 & f \end{bmatrix} gR_{n-1} \right|. \]

Using the fact that \( |XY| = |X||Y| \) for any square matrices of the same size, we get

\[ (f^2 + 4g) \left( gh^2 - k^2 + fhk \right)(-g)^n = g \left( f^2 + 4g \right) \left( R_{n-1}^2 - gR_{n-1}^2 - fR_nR_{n-1} \right). \]

Therefore (3) is valid. ■

Now we are in a position to present one of the main results of this paper.
Theorem 2.3 For any integer \( n \) and for any complex functions \( f \neq 0 \) and \( g \neq 0 \), the general term \( R_n \) of the recurrence \( R_{n+1} = fR_n + gR_{n-1} \), with seeds \( R_0 = h, R_1 = k \), satisfies one of the following:

1) If \( f^2 + 4g \neq 0 \) and \( gh^2 - k^2 + fhk \neq 0 \), then

\[
R_n^2 = \frac{(-g)^n(gh^2 - k^2 + fhk)}{(f^2 + 4g)} [M + (-g)^n B^{-2n}] 
\]

where

\[
M = \begin{bmatrix}
\frac{gh^2 + f^2h^2 + k^2 - 2fhk}{gh^2 - k^2 + fhk} & \frac{2k - fh}{gh^2 - k^2 + fhk} \\
\frac{2k - fh}{gh^2 - k^2 + fhk} & \frac{gh^2 + k^2}{gh^2 - k^2 + fhk}
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
0 & g \\
1 & f
\end{bmatrix}.
\]

2) If \( f^2 + 4g = 0 \) then we obtain

\[
R_n = \frac{n(2k - fh) + fh}{2n} f^{n-1}.
\]

3) If \( f^2 + 4g \neq 0 \) and \( gh^2 - k^2 + fhk = 0 \), then we obtain the following closed formula:

\[
R_n = hkh^n.
\]

Proof. Case 1: First note that \( -gB^{-2} = \begin{bmatrix}
-\frac{1}{g} (g + f^2) & f \\
\frac{f}{g} & -1
\end{bmatrix} \), and also as one can easily check, \( M \) and \( B \) commute. Therefore, the right-hand side of (4) can be rewritten as

\[
\text{RHS} = \frac{(-g)^n(gh^2 - k^2 + fhk)}{(f^2 + 4g)} |B^{-n}| |MB^n + (-g)^n B^{-n}|
\]

\[
= \frac{gh^2 - k^2 + fhk}{(f^2 + 4g)} |B^{-n}| |MB^n + (-g)^n B^{-n}|
\]

Now from equality (2), we know that \( B^n = A^{-1}(CR_n + DrR_n) \). Finding \( A^{-1} \) and multiplying, we obtain

\[
B^n = \begin{bmatrix}
\frac{gR_n - kR_{n-1}}{gh^2 - k^2 + fhk} & \frac{-kR_n + fhR_n + ghR_{n-1}}{gh^2 - k^2 + fhk} \\
\frac{-kR_n + fhR_n + ghR_{n-1}}{gh^2 - k^2 + fhk} & \frac{ghR_n - fR_n + f^2hR_n - ghR_{n-1} + fhR_{n-1}}{gh^2 - k^2 + fhk}
\end{bmatrix}.
\]

As a result, we get

\[
MB^n = \begin{bmatrix}
\frac{gR_n + kR_{n-1} - fhR_{n-1}}{gh^2 - k^2 + fhk} & \frac{kR_n + gR_{n-1}}{gh^2 - k^2 + fhk} \\
\frac{kR_n + gR_{n-1}}{gh^2 - k^2 + fhk} & \frac{ghR_n + fR_n + ghR_{n-1}}{gh^2 - k^2 + fhk}
\end{bmatrix}.
\]

\[
= \frac{1}{gh^2 - k^2 + fhk} \begin{bmatrix}
g(hR_n + kR_{n-1} - fhR_{n-1}) & g(kR_n + ghR_{n-1}) \\
kR_n + ghR_{n-1} & ghR_n + fR_n + ghR_{n-1}
\end{bmatrix}.
\]
Similarly, we can write

\[ (-g)^n B^{-n} = (g)^n (CR_n + DhR_{n-1})^{-1} A \]

\[ = (g)^n \begin{pmatrix}
\frac{ghR_n - fR_{n+1} + f^2 hR_n - ghR_{n-1} + fghR_{n-1}}{g(R_n^2 - R_{n-1}^2 + fR_n - R_{n-1})} & -kR_n + fhR_n + ghR_{n-1} \\
-tR_n + fhR_n + ghR_{n-1} & g \end{pmatrix} \]

\[ = \frac{g}{(gh^2 - k^2 + fhk)} \begin{pmatrix}
\frac{(g - fh)(R_n - (gh - fh)R_{n-1})}{R_n^2 - R_{n-1}^2 + fhR_n - ghR_{n-1}} & (k - fh)R_n - ghR_{n-1} \\
-\frac{kR_n - fhR_n - ghR_{n-1}}{R_n^2 - R_{n-1}^2 + fhR_n - ghR_{n-1}} & hR_n - kR_{n-1}
\end{pmatrix}. \]

Substituting these two expressions of \( B^n \) and \((-g)^n B^{-n}\), we obtain an expression that involves a determinant of sum of two matrices. More explicitly,

\[ \text{RHS} = \frac{(gh^2 - k^2 + fhk)}{(f^2 + 4g)(gh^2 - k^2 + fhk)} \begin{pmatrix}
gr \begin{pmatrix}
R_n \left(fh + 2gh - fh\right) & gR_n \left(2k - fh\right) \\
R_n \left(2k - fh\right) & R_n \left(2gh + fh\right)
\end{pmatrix}
\end{pmatrix} +
\]

After arranging terms and simplifying by \( gh^2 - k^2 + fhk \), we obtain

\[ \text{RHS} = \frac{1}{(f^2 + 4g)(gh^2 - k^2 + fhk)} \begin{pmatrix}
R_n \left(f^2h + 2gh - fh\right) & gR_n \left(2k - fh\right) \\
R_n \left(2k - fh\right) & R_n \left(2gh + fh\right)
\end{pmatrix} \]

\[ = R_n^2. \]

This completes the proof of the first case.

Case 2: \((5)\) can be easily obtained by a direct substitution. If, in addition, \( gh^2 - k^2 + fhk = 0 \) then solving for \( k \) we obtain \( k = \frac{fh}{2} \). Substituting in \((5)\), then the proof can be easily completed.

Case 3: Clearly, \((3)\) implies that

\[ R_n^2 - gR_{n-1} - fR_nR_{n-1} = (-gh^2 + k^2 - fhk)(-g)^{n-1} = 0. \]

From which we can easily get \( R_n = \frac{f \pm \sqrt{f^2 + 4g}R_{n-1}}{2} \), with \( R_0 = h \). Hence \( R_n = h k^n \).

An immediate consequence of the preceding theorem is the following corollary which gives new closed formulas for the sequences that were introduced earlier.

**Corollary 2.4** The following new closed formulas hold.

1. If \( f = x, g = 1, h = 0 \) and \( k = 1 \), we get the Fibonacci polynomials:

\[ f_n(x) = \sqrt{\frac{(-1)^{n+1}}{4 + x^2}} \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix} + \begin{pmatrix}
-1 + x^2 & x \\
x & -1
\end{pmatrix}^n. \]

2. For \( f = 1, g = 1, h = 0 \) and \( k = 1 \), then the Fibonacci numbers are given by:

\[ F_n = \sqrt{\frac{(-1)^{n+1}}{5}} \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix} + \begin{pmatrix}
-2 & 1 \\
1 & -1
\end{pmatrix}^n. \]
(3) If \( f = x, g = 1, h = 2, \) and \( k = x \) we get the Lucas polynomials:

\[
L_n(x) = \sqrt{(-1)^n \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{c} -1 + x^2 \\ x \end{array} \right]}.
\]

(4) For \( f = 1, g = 1, h = 2 \) and \( k = 1 \), the Lucas numbers are given by:

\[
L_n = \sqrt{(-1)^n \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} -2 & 1 \\ 1 & -1 \end{array} \right]}.
\]

(5) If \( f = 1, g = 2x, h = 1 \) and \( k = 1 \), we get the Jacobsthal polynomials:

\[
J_n(x) = \sqrt{(-1)^n \left( \frac{(2x)^n+1}{1 + 8x} \right) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} -\frac{1}{2x} - 1 & 1 \\ \frac{1}{2x} & -1 \end{array} \right]}.
\]

(6) If \( f = 1, g = 2, h = 0 \) and \( k = 1 \), we get the Jacobsthal numbers:

\[
J_n = \sqrt{\frac{-2)^n}{-9} \left[ \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right] + \left[ \begin{array}{cc} -3/2 & 1 \\ 1/2 & -1 \end{array} \right]}.
\]

(7) For \( f = 1, g = 2x, h = 2 \) and \( k = 1 \), we obtain the Jacobsthal-Lucas polynomials:

\[
j_n(x) = \sqrt{(-2)^n \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} -\frac{1}{2x} - 1 & 1 \\ \frac{1}{2x} & -1 \end{array} \right]}.
\]

(8) For \( f = 1, g = 2, h = 2 \) and \( k = 1 \), we obtain the Jacobsthal-Lucas numbers:

\[
j_n = \sqrt{\frac{-2)^n}{-9} \left[ \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right] + \left[ \begin{array}{cc} -3/2 & 1 \\ 1/2 & -1 \end{array} \right]}.
\]

(9) If \( f = 2, g = 1, h = 2 \) and \( k = 2 \), we get the Pell-Lucas numbers:

\[
Q_n = \sqrt{\left( -1 \right)^n \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} -5 & 2 \\ 2 & -1 \end{array} \right]}.
\]

(10) For \( f = 2, g = 1, h = 0 \) and \( k = 1 \), we obtain the Pell numbers:

\[
P_n = \sqrt{\frac{\left( -1 \right)^{n+1}}{8} \left[ \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right] + \left[ \begin{array}{cc} -5 & 2 \\ 2 & -1 \end{array} \right]}.
\]

(11) For \( f = 2x, g = -1, h = 1 \) and \( k = x \), we obtain the Tchebychev polynomial of the first kind:

\[
T_n(x) = \sqrt{\frac{1}{4} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} 4x^2 - 1 & 2x \\ -2x & -1 \end{array} \right]}.
\]

**Remark 1** It should be noted here that new closed formulas for Fermat polynomials \( F_n(x) \), Fermat-Lucas polynomials \( FL_n(x) \) and Horadam sequence \( W_n(a, b, p, q) \) (see, for example, Wolfram MathWorld for their definitions) as well as the sequences of Oresme numbers \( O(n) \) and that of \( k \)-Oresme numbers \( Ok(n) \) (see [2]) can also be found using our approach, since we have the following:
\item $F_n(x) = R(n, 3x, -2, 0, 1)$ and $FL_n(x) = R(n, 3x, -2, 2, 3x)$,
\item $W_n(a, b, p, q) = R(a, b, p, q)$,
\item $O(n) = R(n, 1, -\frac{1}{4}, 0, \frac{1}{2})$ and $Ok(n) = R(n, 1, -\frac{1}{4\pi}, 0, \frac{1}{\pi})$.

### 3 APPLICATIONS

In this section, we present three applications; one deals with finding summation relations that the elements $R_i$ of sequence (1) satisfy. The second one shows how our results can be used to solve systems of first order difference equations. Finally, the last application is concerned with introducing the so-called \textit{generalized ratio} of such a sequence and then finding a formula for it which can be considered as a generalization of the golden ratio, the silver ratio and many others.

#### 3.1 Summation relations for the $R_i$

Although the next result is a particular case of Theorem 8 in [11] where the \textit{generating function} of the sequence $R_n$ (for $z$ real) is given and is used in its proof, however we will include it here since our proof relies only on the definition of sequence (1) and might be of independent interest.

**Theorem 3.1** For any integers $n$ and $i$, the elements $R_n$ of sequence (1) satisfy the following:

$$(f + g - 1) \sum_{i=0}^{n} R_i = R_{n+1} + gR_n + (f - 1)h - k.$$

**Proof.** Recalling that for all integers $i$, we have $R_{i+1} = fR_i + gR_{i-1}$. Taking summation of both sides to obtain $\sum_{i=1}^{n} R_{i+1} = f \sum_{i=1}^{n} R_i + g \sum_{i=1}^{n} R_{i-1}$ which implies that $\sum_{i=2}^{n+1} R_{i-1} = f \sum_{i=2}^{n} R_i + g \sum_{i=1}^{n} R_{i-1}$. However this in turn implies that $R_{n+1} + R_n + \sum_{i=3}^{n} R_{i-1} = f(k + R_n + \sum_{i=3}^{n} R_{i-1}) + g(h + k + \sum_{i=3}^{n} R_{i-1})$. Thus $(f + g - 1)(\sum_{i=0}^{n} R_i) = (-h - k + gR_n + R_{n+1} + fh)$, and the proof is complete. \qed

Another more interesting relations among the $R_i$ are given in the following.

**Theorem 3.2** For any integer $n$ and for any complex functions $f, g, h$ and $k$, the following two equations concerning the $R_i$ are valid:

$$(f^2 - g^2 + 2g - 1) \sum_{i=1}^{n} R_i^2 = (R_{n+1}^2 - k^2) - g^2(R_n^2 - h^2) + 2Sg,$$

$$f(f^2 - g^2 + 2g - 1) \sum_{i=1}^{n} R_iR_{i-1} = (1 - g)(R_{n+1}^2 - k^2) + g(g - 1 + f^2)(R_n^2 - h^2) + S(1 - f^2 - g^2)$$

where $S = \begin{cases} (k^2 - gh^2 - fkh) \frac{1 - (-g)^n}{1 + g} & \text{if } g \neq -1 \\ n(k^2 - gh^2 - kht) & \text{if } g = -1. \end{cases}$
Proof. It is easy to see that

\[ \sum_{i=1}^{n} R_{i+1}^2 = \sum_{i=1}^{n} (f R_i + g R_{i-1})^2 = f^2 \sum_{i=1}^{n} R_i^2 + g^2 \sum_{i=1}^{n} R_{i-1}^2 + 2fg \sum_{i=1}^{n} R_i R_{i-1}. \]  

(6)

Now if we let \( x_n = \sum_{i=1}^{n} R_i^2 \) and \( y_n = \sum_{i=1}^{n} R_i R_{i-1} \), then clearly we can write \( \sum_{i=1}^{n} R_i^2 = x_n + k^2 - R_{n+1}^2 \) and \( \sum_{i=1}^{n} R_i R_{i-1} = x_n + h^2 + k^2 - R_n^2 - R_{n+1}^2 \). Substituting these in (6) and then rearranging the terms, we get

\[ x_n(1 - f^2 - g^2) - 2fgy_n = f^2(k^2 - R_{n+1}^2) + g^2(h^2 + k^2 - R_n^2 - R_{n+1}^2). \]  

(7)

Making use of (3) of Lemma 2.2 in (6), we get

\[ \sum_{i=1}^{n} R_i^2 - g \sum_{i=1}^{n} R_{i-1}^2 - f \sum_{i=1}^{n} R_i R_{i-1} = \sum_{i=1}^{n} (k^2 - gh^2 - fhk) (-g)^{i-1}. \]

After arranging terms, we obtain

\[ (1 - g)x_n - fy_n = S - (k^2 - R_{n+1}^2) + g(h^2 + k^2 - R_n^2 - R_{n+1}^2), \]  

(8)

where

\[ S = \sum_{i=1}^{n} (k^2 - gh^2 - fhk) (-g)^{i-1} = \begin{cases} (k^2 - gh^2 - fhk) \frac{1 - (-g)^n}{1 + g} & \text{if } g \neq -1 \\ n(k^2 - gh^2 - kht) & \text{if } g = -1. \end{cases} \]

Solving the system (7) and (8) for \( x_n \) and \( y_n \), we then obtain two equations where the first is in \( x_n \) and is given by

\[ (f - g + 1)(f + g - 1)x_n = (f^2 + 2g - g^2)(R_{n+1}^2 - k^2) - g^2(R_n^2 - h^2) + 2Sg, \]

and the second equation involves only \( y_n \) and is given by

\[ f(f - g + 1)(f + g - 1)y_n = (1 - g)(R_{n+1}^2 - k^2) + g(g - 1 + f^2)(R_n^2 - h^2) + S(1 - f^2 - g^2). \]

This completes the proof. \( \blacksquare \)

Another interesting relation that involves products of \( R_i \) and somewhat is more general than (3) is given in the next theorem which is similar to Theorem 7 presented in [11] for the case \( r = n \) and \( z \) real but we include it here as it serves to illustrate the methods used in this paper.

Theorem 3.3 For any integers \( n \) and \( i \), the elements \( R_n \) of sequence (1) satisfy the following:

\[ (k^2 - fh - gh^2)R_{i+n} + (f^2h - fh^2)R_i R_n + \\
h^2R_{i-1} R_{n-1} + (fh - k)g(R_n R_{i-1} + R_i R_{n-1}) = 0. \]  

(9)
Now system (11) can be transformed into the following homogeneous system:

\( (CR_n + DgR_{n-1})(CR_i + Dg_i) = A(CR_{n+i} + DgR_{n+i-1}) \).

After substitution, we obtain following two equations:

\[ g(4g + f^2)(R_i R_n - hR_{n+i} - kR_{n+i-1} + gR_{n-1} R_n - fhR_{n+i-1}) = 0 \]
\[ g(4g + f^2)( -kR_{n+i} + gR_n R_{n-1} + gR_i R_{n-1} + fR_i R_n - ghR_{n+i-1}) = 0. \]

Since \( 4g + f^2 \neq 0 \), then we conclude the following two equations:

\[ gh (R_i R_n - hR_{n+i} + gR_{n-1} R_n + (fh - k)R_{n+i-1}) = 0, \]
\[ (fh - k)( -kR_{n+i} + gR_n R_{n-1} + gR_i R_{n-1} + fR_i R_n - ghR_{n+i-1}) = 0. \]

Finding the value of \((fh - k)R_{n+i-1}\) from the first equation and substituting it in the second equation, we obtain (9).

### 3.2 Applications II: On difference equations

In this section, we shall show how our approach can be used as a tool to solve systems of first order difference equations (see for example [4]). But first we start with the following remark which is concerned with the non-homogeneous case for which non-homogeneous extensions of the Fibonacci recurrence that has been studied by many authors (see, for example, [5] and the references therein) is a special case.

**Remark 2** It is worth noting that if \( 1 - f - g \neq 0 \), and if \( l \) is any complex function, then recurrences of the form \( R_{n+1} = fR_n + gR_{n-1} + l \) can also be solved by using our approach, since these can easily be transformed into the following second order linear recurrence

\[ G_{n+1} = fG_n + gG_{n-1} \]

where \( G_n \) is any complex function, \( f \) and \( g \) are the coefficients, and \( l \) is the constant term.

The general form of a system of first order difference equations is given by:

\[ X_{n+1} = aX_n + bY_n + p \]
\[ Y_{n+1} = cX_n + dY_n + q \]  

where \( X_0 = \alpha \) and \( Y_0 = \beta \) are the initial conditions. Then obviously, we can rewrite

\[ X_{n+2} = aX_{n+1} + bcX_n + bdY_n + bq + p = aX_{n+1} + bcX_n + dX_{n+1} - adX_n - pd + bq + p. \]

Similarly, we obtain

\[ Y_{n+2} = (a + d)Y_{n+1} + (bc - ad)Y_n + pc - aq + q. \]

From (10), we obtain the following system:

\[ X_{n+2} = (a + d)X_{n+1} + (bc - ad)X_n - pd + bq + p, \]
\[ Y_{n+2} = (a + d)Y_{n+1} + (bc - ad)Y_n + pc - aq + q, \]

\[ Y_0 = \beta, \quad Y_1 = \alpha \]

Now system (11) can be transformed into the following homogeneous system:

\[ R_{n+2} = (a + d)R_{n+1} + (bc - ad)R_n, \]
\[ R_1 = \alpha \]
\[ G_{n+2} = (a + d)G_{n+1} + (bc - ad)G_n, \]
\[ G_1 = \alpha \]

Proof. Since \( A \) and \( B \) commute, then for any integers \( i \) and \( n \), we obtain the matrix equation \( AB^n AB^i = A^2 B^{n+i} \). By (1), this last equation can be rewritten as:

\[ (CR_n + DgR_{n-1})(CR_i + Dg_i) = A(CR_{n+i} + DgR_{n+i-1}) \]
with \( R_n = X_n - \frac{pd + bg + p}{(a+d) - (bc - ad)} \) and \( G_n = Y_n - \frac{pc^2 + 2gh + p}{(a+d) - (bc - ad)} \). From Theorem 2.3, the solution of this last homogeneous system can be easily determined.

### 3.3 The generalized ratio

In this subsection, we introduce the generalized ratio which is defined by \( \lim_{n \to \infty} \frac{R_{n+s}}{R_{n+t}} \) where \( s \) and \( t \) are any integers. The objective here is to find a formula for this generalized ratio in the case when it is convergent. First we start with the following lemma.

**Lemma 3.4** For any integer \( n \) and for any complex functions \( f, g, h \) and \( k \), the elements \( R_n \) of sequence (1) satisfy the equation

\[
R_n = \frac{fR_{n-1} \pm \sqrt{(f^2 + 4g)R_{n-1}^2 - 4(fh - k^2 + gh^2)(-g)^{n-1}}}{2},
\]

where the choice of (+) or (-) sign in the formula is uniquely determined by the seeds \( R_1 = k \) and \( R_0 = h \).

**Proof.** Recall that (3) is given by

\[
R_n^2 - gR_{n-1}^2 - fR_nR_{n-1} = (k^2 - gh^2 - fhk)(-g)^{n-1}.
\]

Solving for \( R_n \), we obtain

\[
R_n = \frac{fR_{n-1} \pm \sqrt{(f^2 + 4g)R_{n-1}^2 - 4(fh - k^2 + gh^2)(-g)^{n-1}}}{2}.
\]

In what follows, we shall assume that the seeds \( R_1 = k \) and \( R_0 = h \) are chosen so that

\[
R_n = \frac{fR_{n-1} + \sqrt{(f^2 + 4g)R_{n-1}^2 - 4(fh - k^2 + gh^2)(-g)^{n-1}}}{2}.
\]

(12)

Note that for the other case, a similar analysis can be applied.

**Lemma 3.5** Let \( n \) be any integer and \( f, g, h \) and \( k \), be any complex functions with \( k^2 - gh^2 - fhk \neq 0 \). If \( \lim_{n \to \infty} \frac{R_{n+1}}{R_n} = L \), then it holds that

1. \( L^2 - fL - g = 0 \).
2. \( \lim_{n \to \infty} \frac{R_n^n}{R_n} = 0 \).

**Proof.** Starting with \( R_{n+1} = fR_n + gR_{n-1} \) and dividing by \( R_n \), and then taking limit as \( n \) tends to infinity, we obtain \( \lim_{n \to \infty} \frac{R_{n+1}}{R_n} = f + g \lim_{n \to \infty} \frac{R_{n-1}}{R_n} \). This implies that \( L = f + g \frac{1}{L} \) and the proof of the first part is complete.

For the second part, dividing equality (3) by \( R_{n-1}^2 \), then clearly we conclude the following equation \( \frac{R_{n}^2}{R_{n-1}^2} - g - f \frac{R_{n}}{R_{n-1}} = (k^2 - gh^2 - fhk)(-g)^{n-1} \). Taking limits of both sides, we obtain

\[
\lim_{n \to \infty} \frac{R_{n}^2}{R_{n-1}^2} - g - f \lim_{n \to \infty} \frac{R_{n}}{R_{n-1}} = (k^2 - gh^2 - fhk) \lim_{n \to \infty} (-g)^{n-1}.
\]

But this implies that \( L^2 - fL - g = (k^2 - gh^2 - fhk) \lim_{n \to \infty} (-g)^{n-1} \). Thus by the first part, the proof is complete.

As a consequence, we have the following theorem.
Theorem 3.6 Suppose that \( R_n = \frac{f R_{n-1} + \sqrt{(f^2 + 4g)R_{n-1}^2 - 4(fh - k^2 + gh^2)(-g)^{n-1}}}{2} \). If \( \lim_{n \to \infty} \frac{R_n}{R_{n-1}} = L \), then
\[
L = \frac{f + \sqrt{f^2 + 4g}}{2}.
\]

Proof. It is easy to see that
\[
L = \lim_{n \to \infty} \frac{R_n}{R_{n-1}} = \lim_{n \to \infty} \frac{f R_{n-1} + \sqrt{(f^2 + 4g)R_{n-1}^2 - 4(fh - k^2 + gh^2)(-g)^{n-1}}}{2R_{n-1}}
\]
\[
= \frac{f + \sqrt{(f^2 + 4g) - 4(fh - k^2 + gh^2) \lim_{n \to \infty} \frac{(-g)^{n-1}}{R_{n-1}^2}}}{2},
\]
and then by the preceding lemma the proof is complete. \( \blacksquare \)

As a conclusion, we obtain the ratios of the following famous sequences.

Corollary 3.7

- For \( f = 1, g = 1 \), we obtain the golden ratio \( \lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \lim_{n \to \infty} \frac{L_n}{L_{n-1}} = \frac{1 + \sqrt{5}}{2} \).
- For \( f = 2, g = 1 \), we obtain the silver ratio \( \lim_{n \to \infty} \frac{F_n}{F_{n-1}} = 1 + \sqrt{2} \).
- For \( f = x, g = 1 \), we obtain the Fibonacci and Lucas polynomials ratio \( \lim_{n \to \infty} \frac{f_n(x)}{f_{n-1}(x)} = \lim_{n \to \infty} \frac{L_n}{L_{n-1}(x)} = \frac{x + \sqrt{x^2 + 4}}{2} \).
- For \( f = 2x, g = -1 \), we obtain the ratio of the Tchebychev polynomial of the first kind \( \lim_{n \to \infty} \frac{T_n(x)}{T_{n-1}(x)} = x + \sqrt{x^2 - 1} \).
- For \( f = 1, g = 2 \), we obtain the Jacobsthal (and similarly the Jacobsthal-Lucas) sequence ratio \( \lim_{n \to \infty} \frac{J_n}{J_{n-1}} = 2 \).

In what follows, we shall assume that \( \lim_{n \to \infty} \frac{R_n}{R_{n-1}} \) exists. Our next goal is to show the same for the generalized ratio \( \lim_{n \to \infty} \frac{R_{n+s}}{R_{n+t}} \) where \( s \) and \( t \) are any integers. First, noticing that
\[
\lim_{n \to \infty} \frac{R_{n+s}}{R_n} = \lim_{n \to \infty} \frac{R_{n+s+t} + gR_n}{R_n} = \frac{1}{2} \left( 2g + f^2 + f \sqrt{4g + f^2} \right)
\]
then we have the following.

Theorem 3.8 For any integer \( s \), and for any \( f, g, h \) and \( k \) as above, then
\[
\lim_{n \to \infty} \frac{R(n + s, f, g, h, k)}{R(n, f, g, h, k)} = R(s, f, g, 0, 1) \left( \frac{f + \sqrt{f^2 + 4g}}{2} \right) + R(s - 1, f, g, 0, 1). \quad (13)
\]

Proof. We proceed by induction on \( s \). Clearly (13) is true for \( s = 1 \) as \( R(1, f, g, 0, 1) = 1 \) and \( R(0, f, g, 0, 1) = 0 \). Suppose it is true for \( s \leq i \) that is
\[
\lim_{n \to \infty} \frac{R(n + i, f, g, h, k)}{R(n, f, g, h, k)} = R(i, f, g, 0, 1) \left( \frac{f + \sqrt{f^2 + 4g}}{2} \right) + R(i - 1, f, g, 0, 1).
\]
Then it is easy to see that
\[
\lim_{n \to \infty} \frac{R(n + i + 1, f, g, h, k)}{R(n, f, g, h, k)} = \lim_{n \to \infty} \frac{R(n + i, f, g, h, k) + R(n + i - 1, f, g, h, k)}{R(n, f, g, h, k)} = \\
\lim_{n \to \infty} \frac{R(n + i, f, g, h, k) + R(n + i - 1, f, g, h, k)}{R(n, f, g, h, k)} = R(i, f, g, 0, 1) \left( \frac{f + \sqrt{f^2 + 4g}}{2} \right) + \\
R(i - 1, f, g, 0, 1) + R(i - 1, f, g, 0, 1) \left( \frac{f + \sqrt{f^2 + 4g}}{2} \right) + R(i - 2, f, g, 0, 1) = \\
R(i + 1, f, g, 0, 1) \left( \frac{f + \sqrt{f^2 + 4g}}{2} \right) + R(i, f, g, 0, 1).
\]

This complete the proof of the theorem. ■

As a result, we have the following conclusion.

**Corollary 3.9** For any integers \( s \) and \( t \), and for any \( f, g, h \) and \( k \) as above, then
\[
\lim_{n \to \infty} \frac{R(n + s, f, g, h, k)}{R(n + t, f, g, h, k)} = R(s - t, f, g, 0, 1) \left( \frac{f + \sqrt{f^2 + 4g}}{2} \right) + R(s - t - 1, f, g, 0, 1).
\]

**Proof.** It is easy to see that
\[
\lim_{n \to \infty} \frac{R(n + s, f, g, h, k)}{R(n + t, f, g, h, k)} = \lim_{n \to \infty} \frac{R(n + t + (s - t), f, g, h, k)}{R(n + t, f, g, h, k)} = \\
\lim_{n \to \infty} \frac{R(n + t + (s - t), f, g, h, k)}{R(n, f, g, h, k)} = R(s - t, f, g, 0, 1) \left( \frac{f + \sqrt{f^2 + 4g}}{2} \right) + R(s - t - 1, f, g, 0, 1).
\]

To illustrate the results of the preceding corollary, we take the following two examples.

**Example 3.10** Using the preceding corollary, we know that
\[
\lim_{n \to \infty} \frac{F_{n+20}}{F_n} = R(20, 1, 1, 0, 1) \left( \frac{1 + \sqrt{5}}{2} \right) + R(19, 1, 1, 0, 1).
\]

From Corollary 2.4, \( R(20, 1, 1, 0, 1) = \sqrt{\frac{(-1)^{20+1}}{5} \left[ \begin{array}{c} -2 \\ 1 \end{array} \right]^{20} - \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]} = 6765, \) and
\[
R(19, 1, 1, 0, 1) = \sqrt{\frac{(-1)^{19+1}}{5} \left[ \begin{array}{c} -2 \\ 1 \end{array} \right]^{19} - \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]} = 4181. \text{ Thus } \lim_{n \to \infty} \frac{F_{n+20}}{F_n} = \frac{15 \cdot 127 + 6765}{2} \sqrt{5}.
\]

**Example 3.11** Similarly, in view of the preceding corollary, we have
\[
\lim_{n \to \infty} \frac{T_{n+10}(x)}{T_{n+3}(x)} = R(10 - 3, 2x, -1, 0, 1)(x + \sqrt{x^2 - 1}) + R(10 - 3 - 1, 2x, -1, 0, 1).
\]

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By Corollary 2.4,

\[ R(7, 2x, -1, 0, 1) = \sqrt{\frac{-1}{-4 + 4x^2}} \left[ \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right] + \left[ \begin{array}{cc} 4x^2 - 1 & 2x \\ -2x & -1 \end{array} \right]^7 = \]

\[ (-4x + 4x^2 + 8x^3 - 1) (-4x - 4x^2 + 8x^3 + 1). \]

Similarly,

\[ R(6, 2x, -1, 0, 1) = \sqrt{\frac{-1}{-4 + 4x^2}} \left[ \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right] + \left[ \begin{array}{cc} 4x^2 - 1 & 2x \\ -2x & -1 \end{array} \right]^6 = \]

\[ 2x (2x + 1)(2x - 1)(4x^2 - 3). \]

Finally, we obtain

\[ \lim_{n \to \infty} \frac{T_{n+10}(x)}{T_{n+3}(x)} = x \left( -8x^2 - 48x^4 + 64x^6 + 5 \right) + \]

\[ (-4x + 4x^2 + 8x^3 - 1) (-4x - 4x^2 + 8x^3 + 1) \sqrt{x^2 - 1}. \quad \square \]

We end our discussion here with finding a generating function \( G(x) \) for the sequence \( R_n \).

Let \( G(x) = \sum_{i=0}^{\infty} R_i x^i \). Then

\[ (1 - fx - gx^2)G(x) = R_0 + x(R_1 - fR_0) + \sum_{i=0}^{\infty} (R_{i+2} - fR_{i+1} - gR_i)x^{i+2} = h + x(k - fh). \]

Thus

\[ G(x) = \frac{h + x(k - fh)}{1 - fx - gx^2}. \]

4 Conclusion and Future Work

We presented in Theorem 2.3 a new unifying closed formula for the solution of homogeneous second order linear difference equations. However, it should be stressed that the right-hand side of (4) is an expression that involves computing the determinant of some matrix. Thus (4) remains valid when this matrix is replaced by another which is similar to it. As a result, one can ask the question of determining among all those similar matrices, which one gives formula (4) more advantages regarding complexity and precision.

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