The quantum theory and topological features of photon

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In this paper, we have proposed the spinor wave equation of free and non-free photon. On this basis, we have given the spin operators and spin wave functions of photon, and calculated the wave function of photon in vacuum and medium. In addition, we have given the quantum Berry phase and Chern number with the photon wave function, which can be used to study the quantum topological features of photon in one-dimensional period medium.

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1. Introduction

The Dirac equation \cite{1} is the important equation for the relativistic particles of spin $\frac{1}{2}$. Within quantum electrodynamics, the theoretical calculations and experimental results are highly consistent. In addition, the Dirac equation is the basis of the theory of electroweak interactions, quantum chromodynamics and quantum hadrodynamics. The Klein-Gordon equation \cite{2, 3} is the fundamental equations of quantum field theories, describes bosons with spin zero. The present work wants to show that there exists a modification of the Klein-Gordon equation, which includes the relativistic spin effects \cite{4}.

The Dirac equation is a first order differential equation, whereas the Maxwell equations and Klein-Gordon equation correspond to a second order differential equation for the photon field. After Dirac discovered the relativistic equation for a particle with spin $1/2$, much work was done to study spinor and vectors within the Lorentz group theory for any spin particle.

The initial motivation of this paper was the question whether one can find a first order differential equation of photon. The spinors are considered as fundamental physical quantity in quantum field theory \cite{5-6}. For this reason, we have proposed the spinor wave equations of photon. In recent years, various novel topological phenomena addressed in the condensed matter physics\cite{7-9} have been achieved in photonic systems, where different types of topological insulators have all found their counterparts\cite{10}. The realization of classical analogues of topological insulators in artificial crystals has been an emerging research area \cite{11-13}. Photonic topological insulators (PTIs) have been theoretically proposed and experimentally demonstrated in different photonic systems \cite{14-16}. The Berry phase provides a universal framework which relates the robust quantization of physical observables at the boundaries of a non-interacting system to the topological properties of the bulk.

2. Relativistic spinor wave equation of free photon

As is known to all, Dirac equation describes the particle of spin $\frac{1}{2}$ by factorizing Einstein’s dispersion relation, such that the field equation becomes the first order in time derivative \cite{28}. Namely, Dirac factorized the relativistic dispersion relation employing four by four matrices, which is expressed as

$$E^2 - c^2 \vec{p}^2 - m_0^2 c^4 = (E - c \vec{p} \cdot \vec{\alpha} - m_0 c^2 \beta)(E + c \vec{p} \cdot \vec{\alpha} + m_0 c^2 \beta) = 0,$$

thus we get

$$E - c \vec{\alpha} \cdot \vec{p} - m_0 c^2 \beta = 0.$$  

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By canonical quantization Eq. (2), i.e., $E \rightarrow i\hbar \frac{\partial}{\partial t}$, $\vec{p} \rightarrow -i\hbar \nabla$, we can obtain the Dirac spinor wave equation

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = (-i\hbar \vec{\alpha} \cdot \vec{\nabla} + m_0 c^2 \beta) \psi(\vec{r}, t),$$

(3)

where $\vec{\alpha}$ and $\beta$ are Dirac matrices.

With Dirac’s factorization approach, we can obtain the spinor wave equation of free photon. For a photon, its mass $m_0 = 0$, Eq. (2) becomes

$$E - c\vec{\alpha} \cdot \vec{p} = 0.$$  

(4)

By canonical quantization Eq. (4), we obtain the spinor wave equation of photon

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = -ic\hbar \vec{\alpha} \cdot \vec{\nabla} \psi(\vec{r}, t) = H \psi(\vec{r}, t),$$

(5)

where $H = -i\hbar \vec{\alpha} \cdot \vec{\nabla}$ is Hamiltonian operator and $\psi$ is the spinor wave function of photon. For the proper Lorentz group $L_p$, the irreducibility representations of spin $s = 1$ photon are $D^{10}$, $D^{01}$ and $D^{12}$, respectively, and the dimension of irreducibility representations corresponds to three, three and four, respectively. We choose photon’s spinor wave function as the basis vector of three dimension irreducibility representation, i.e.

$$\psi(\vec{r}, t) = \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \\ \psi_3(\vec{r}, t) \end{pmatrix},$$

(6)

and the $\vec{\alpha}$ matrices are denoted by

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \alpha_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \alpha_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(7)

which are Hermitean matrices, $\vec{\alpha}^\dagger = \vec{\alpha}$. The photon’s Hamiltonian operator is also Hermitean $H^\dagger = H$.

3. The spin operators of photon

In this section, we shall prove that the selection of $\vec{\alpha}$ matrices in Eq. (7) is reasonable, and the Eqs. (5), (6) and (7) are the spinor wave equation of free photon, i.e., they are corresponding to the spinor wave equation of spin $s = 1$ and mass $m_0 = 0$.

The equation (5) can be written as

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = c(\vec{p} \cdot \vec{\alpha}) = H \psi(\vec{r}, t),$$

(8)

where $H = c\vec{p} \cdot \vec{\alpha}$. The orbital angular momentum of photon satisfies

$$\frac{d}{dt} L_x = \frac{1}{i\hbar} [L_x, H] = c(\alpha_y p_z - \alpha_z p_y) = c(\vec{\alpha} \times \vec{p})_x,$$

(9)

so

$$[\vec{L}, H] = i\hbar c(\vec{\alpha} \times \vec{p}).$$

(10)

The Eq. (10) is shown that the orbital angular momentum of photon is not conservation, but the total angular momentum of photon should be conservative. Thus, photon should have an intrinsic angular momentum, i.e., spin angular momentum $\vec{s}$, and the total angular momentum of photon $\vec{J}$ is

$$\vec{J} = \vec{L} + \vec{s},$$

(11)
Comparing with the both sides of Eq. (14), we obtain the commutation relations
\[ [s, H] = -[\vec{J}, H] = -i\hbar c(\vec{a} \times \vec{p}), \]
where the spin component \( s_x \) satisfies
\[ [s_x, H] = [s_x, c\vec{a} \cdot \vec{p}] = -i\hbar c(\vec{a} \times \vec{p})_x = i\hbar c(\alpha_x p_y - \alpha_y p_x), \]
i.e.,
\[ [s_x, \alpha_x p_x + \alpha_y p_y + \alpha_z p_z] = c[s_x, \alpha_x p_x + c[s_x, \alpha_y] p_y + c[s_x, \alpha_z] p_z = i\hbar c(\alpha_x p_y - \alpha_y p_x). \]
Comparing with the both sides of Eq. (14), we obtain the commutation relations
\[ [s_x, \alpha_x] = 0, [s_x, \alpha_y] = i\hbar \alpha_z, [s_x, \alpha_z] = -i\hbar \alpha_y. \]
Similarly, it is obtained
\[ [s_y, \alpha_y] = 0, [s_y, \alpha_x] = -i\hbar \alpha_z, [s_y, \alpha_z] = i\hbar \alpha_x, \]
and
\[ [s_z, \alpha_z] = 0, [s_z, \alpha_x] = i\hbar \alpha_y, [s_z, \alpha_y] = -i\hbar \alpha_x. \]
According to commutation relations in Eqs. (15), (16) and (17) and using Eq. (7), we can calculate the spin matrices \( \vec{s} \) of photon, they are
\[ s_x = \begin{pmatrix} a & 0 & 0 \\ 0 & a & -i\hbar \\ 0 & i\hbar & a \end{pmatrix}, s_y = \begin{pmatrix} b & 0 & i\hbar \\ 0 & b & 0 \\ -i\hbar & 0 & b \end{pmatrix}, s_z = \begin{pmatrix} c & -i\hbar & 0 \\ i\hbar & c & 0 \\ 0 & 0 & c \end{pmatrix}, \]
where \( a, b, c \) are to be determined by their eigenvalues.

Using the photon spin matrices \( s_x, s_y, s_z \), their eigenvalues should be \( \pm \hbar \). For the \( s_x \) eigenvalue problem, we have
\[ \begin{pmatrix} a & 0 & 0 \\ 0 & a & -i\hbar \\ 0 & i\hbar & a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \]
and its characteristic equation is
\[ \begin{vmatrix} a - \lambda_1 & 0 & 0 \\ 0 & a - \lambda_1 & -i\hbar \\ 0 & i\hbar & a - \lambda_1 \end{vmatrix} = 0, \]
i.e.,
\[ (a - \lambda_1)[(a - \lambda_1)^2 - \hbar^2] = 0. \]
In order to get the eigenvalues \( \lambda_1 = \pm \hbar \), we should set \( a = 0 \). In the similar method, we also have \( b = 0 \) and \( c = 0 \). Finally, we may obtain the spin matrices of photon in Eq. (18) and after calculation these matrices square are
\[ \vec{s}^2 = s_x^2 + s_y^2 + s_z^2 = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hbar^2 = s(s + 1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ (a - \lambda_1)[(a - \lambda_1)^2 - \hbar^2] = 0. \]
i.e., $s = 1$, the spin matrices Eq. (18) with $a = 0$, $b = 0$ and $c = 0$ are photon’s spin matrices. So Eqs. (5), (6) and (7) are just the spinor wave equation of free photon. According to Eq. (7), we find

$$s_x = \hbar \alpha_x, \quad s_y = \hbar \alpha_y, \quad s_z = \hbar \alpha_z.$$ (23)

### 5. The probability conservation equation of photon

In the following section, we shall give the probability density and probability conservation equation of photon.

The hermitian conjugate of Eq. (5) is

$$-i \hbar \frac{\partial \psi^+}{\partial t} = i \hbar c \nabla \psi^+ \cdot \vec{\alpha},$$ (24)

right multiplying Eq. (24) by $\psi$

$$-i \hbar \frac{\partial \psi^+}{\partial t} \psi = i \hbar \nabla \psi^+ \cdot \vec{\alpha} \psi,$$ (25)

and left multiplying Eq. (5) by $\psi^+$

$$i \hbar \psi^+ \frac{\partial \psi}{\partial t} = -i \hbar c \psi^+ \vec{\alpha} \cdot \nabla \psi,$$ (26)

we get

$$i \hbar (\psi^+ \frac{\partial \psi}{\partial t} + \frac{\partial \psi^+}{\partial t} \psi) + i \hbar c \psi^+ \vec{\alpha} \cdot \nabla \psi + i \hbar \nabla \psi^+ \cdot \vec{\alpha} \psi = 0,$$ (27)

or

$$\frac{1}{c} \frac{\partial}{\partial t} (\psi^+ \psi) + \psi^+ \vec{\alpha} \cdot (\nabla \psi) + (\nabla \psi^+) \cdot \vec{\alpha} \psi = 0,$$ (28)

and obtain the probability conservation equation of photon

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0.$$ (29)

Where $\rho = \psi^+ \psi$ and $\vec{J} = c \psi^+ \vec{\alpha} \psi$ are the probability and probability current density of photon, respectively, which are expressed by the spinor wave functions $\psi$ and $\psi^+$ of photon, and the photon probability density $\rho \geq 0$.

When photon is incident to a uniform medium of finite volume, there are incident, reflection and transmission photon in the medium, we can define the quantum transmission coefficient $t$ and reflection coefficient $r$, they are

$$t = \left| \frac{J_D}{J_I} \right| = \left| \frac{\psi_D^+(x) \alpha_x \psi_D(x)}{\psi_I^+(x) \alpha_x \psi_I(x)} \right|,$$ (30)

and

$$r = \left| \frac{J_R}{J_I} \right| = \left| \frac{\psi_R^+(x) \alpha_x \psi_R(x)}{\psi_I^+(x) \alpha_x \psi_I(x)} \right|.$$ (31)

Where $\psi_D^+(x)$, $\psi_R^+(x)$ and $\psi_I^+(x)$ are the incident, reflection and transmission wave functions of photon in the medium.
6. The plane wave solution and helicity of free photon

For the spin $\frac{1}{2}$ Dirac particle, there are the plane wave solutions corresponding to positive energy and negative energy. For the photon, there are also the plane wave solutions corresponding to positive energy and negative energy. Based on the above discussion, we have the spinor equation of free photon

$$i\hbar \frac{\partial}{\partial t} \psi = H \psi,$$

where

$$H = c\vec{\alpha} \cdot \vec{p}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \quad \text{(33)}$$

Since $\frac{\partial H}{\partial t} = 0$ and $[\vec{p}, H] = 0$, the photon energy $E$ and momentum $\vec{p}$ are conserved quantity, they have common eigenstate, namely,

$$\psi_{E,\vec{p}}(\vec{r}, t) = u(\vec{p})e^{i(\vec{p} \cdot \vec{r} - Et) / \hbar}, \quad \text{(34)}$$

where

$$u(\vec{p}) = \begin{pmatrix} u_1(\vec{p}) \\ u_2(\vec{p}) \\ u_3(\vec{p}) \end{pmatrix}. \quad \text{(35)}$$

Substituting Eqs. (34) and (35) into (32) yields

$$c\vec{\alpha} \cdot \vec{p} \quad u(\vec{p}) = E u(\vec{p}), \quad \text{(36)}$$

i.e.,

$$\begin{pmatrix} 0 & -icp_z & icp_y \\ icp_z & 0 & -icp_x \\ -icp_y & icp_x & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \quad \text{(37)}$$

Expanding Eq. (37), we get

$$Eu_1 + icp_z u_2 - icp_y u_3 = 0, \quad \text{(38)}$$

$$icp_z u_1 - Eu_2 - icp_x u_3 = 0, \quad \text{(39)}$$

$$icp_y u_1 - icp_x u_2 + Eu_3 = 0. \quad \text{(40)}$$

by the sufficient necessary condition of nonzero solution of $u_1$, $u_2$, and $u_3$, we get

$$\begin{vmatrix} E & icp_z & -icp_y \\ icp_z & -E & -icp_x \\ icp_y & -icp_x & E \end{vmatrix} = 0, \quad \text{(41)}$$

with the eigenvalues $E$ being

$$E_1 = +c|\vec{p}|, \quad E_2 = -c|\vec{p}|. \quad \text{(42)}$$

From Eqs. (39) and (40), we have

$$icp_z p_y u_1 - Ep_y u_2 - icp_x p_y u_3 = 0, \quad \text{(43)}$$

$$icp_y p_z u_1 - icp_x p_z u_2 + Ep_z u_3 = 0. \quad \text{(44)}$$
taking the difference of Eqs. (43) and (44), it is obtained that

\[(E p_y - ic p_z p_z) u_2 + (E p_z + ic p_x p_y) u_3 = 0,\]  

\[(45)\]

or

\[\frac{u_2}{u_3} = \frac{E p_z + ic p_x p_y}{E p_y - ic p_z p_z}.\]  

\[(46)\]

Substituting Eq. (47) into Eq. (38), there is

\[u_1 u_3 = \frac{ic(p_y^2 + p_z^2)}{E p_y - ic p_z p_z}.\]  

\[(47)\]

According to Eqs. (46) and (47), there is

\[\frac{u_1}{u_2} = \frac{-ic(p_y^2 + p_z^2)}{E p_z + ic p_x p_y},\]  

\[(48)\]

thus the \(u(p)\) can be written as

\[u(p) = N \begin{pmatrix} \frac{ic(p_y^2 + p_z^2)}{E p_y - ic p_z p_z} \\ \frac{-(E p_z + ic p_x p_y)}{E p_y - ic p_z p_z} \end{pmatrix},\]  

\[(49)\]

where \(N\) is normalization constant. Using \(u^+(p)u(p) = 1\), we can obtain

\[2E^2 N^2 (p_y^2 + p_z^2) = 1,\]  

\[(50)\]

and the normalization constant is

\[N = \sqrt{\frac{1}{2E^2 (p_y^2 + p_z^2)}}.\]  

\[(51)\]

Therefore,

\[u(p) = \sqrt{\frac{1}{2E^2 (p_y^2 + p_z^2)}} \begin{pmatrix} ic(p_y^2 + p_z^2) \\ -(E p_z + ic p_x p_y) \end{pmatrix},\]  

\[(52)\]

and

\[\psi_{E,p}(\vec{r},t) = \sqrt{\frac{1}{2E^2 (p_y^2 + p_z^2)}} \begin{pmatrix} ic(p_y^2 + p_z^2) \\ -(E p_z + ic p_x p_y) \end{pmatrix} e^{i(\vec{p} \cdot \vec{r} - Et)/\hbar},\]  

\[(53)\]

which is just the plane wave solution of free photon. Substituting \(E_1 = +c|\vec{p}|\) and \(E_2 = -c|\vec{p}|\) into Eq. (53), we can obtain the photon plane wave solutions of positive energy and negative energy.

The polarization vector acts for the photon as the "spin part" of the wave function, the polarization of the photon is in a certain relationship to photon’s helicity. For a given momentum photon, it has two different polarizations, which may be taken to be two mutually perpendicular linear polarizations, and the two circular polarizations having opposite directions of rotation, i.e., the right-hand and left-hand circular polarizations. By the helicity of photon, we can obtain the two different polarizations of photon. The helicity is defined as the projection of spin in the momentum direction, i.e.

\[h = \frac{\vec{\alpha} \cdot \vec{p}}{|\vec{p}|},\]  

\[(54)\]
with Eq. (33), the helicity becomes

$$ h = \frac{H}{c|\vec{p}|} \quad (55) $$

Since the eigenvalues of $H$ are $+c|\vec{p}|$ and $-c|\vec{p}|$, the eigenvalues of $h$ are $+1$ and $-1$, which corresponds to a right-handed and left-handed, transverse, circularly polarized helicity state of photon.

6. The spin wave functions of photon

In the section 3, we have given the spin operator of photon, we can calculate the spin wave functions of photon. From Eqs. (18) and (22), we find that $\vec{s}^2$ commutes with $s_x$, $s_y$ and $s_z$, respectively. Thus we can calculate the common eigenstates of $\vec{s}^2$ and $s_z$, expressed as

$$ \vec{s}^2 \chi_{\mu} = 2\hbar^2 \chi_{\mu}, \quad (56) $$

$$ s_z \chi_{\mu} = \mu \hbar \chi_{\mu}, \quad (57) $$

where $(\chi_{\mu})^T = (\varphi_1, \varphi_2, \varphi_3)$ is their common eigenstate. Considering Eq. (38), we easily rewritten as

$$ \begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \mu \hbar \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \quad (58) $$

and according to its characteristic equation, the eigenvalues $\mu$ are

$$ \mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = -1, \quad (59) $$

Substituting $\mu_1 = 0$ into Eq. (58), we get

$$ \begin{cases} -i\varphi_2 = 0 \\ i\varphi_1 = 0 \end{cases}, \quad (60) $$

i.e.,

$$ \varphi_1 = \varphi_2 = 0, \varphi_3 \neq 0, \quad (61) $$

and the normalization spin wave function is

$$ \chi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (62) $$

Similarly, by substituting $\mu_2 = 1$ and $\mu_3 = -1$ into Eq. (58), respectively, the corresponding normalization spin wave function is

$$ \chi_1 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad (63) $$

and

$$ \chi_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}. \quad (64) $$
These spin wave functions satisfy the normalization condition
\[ \sum_{\alpha} \chi_\mu^*(\alpha)\chi_{\mu'}(\alpha) = \delta_{\mu\mu'}. \] (65)

By the spin wave functions of single-photon, we can obtain the spin wave functions of two-photon and multiphoton, and can further give the spin entanglement states of two-photon and multiphoton, which can be used in quantum information.

7. The spinor wave equation of non-free photon

In the sections above, we have given the spinor wave equation of free photon, i.e., the photon is in the vacuum. When photon is in the medium, it becomes non-free photon. Next, we shall give the spinor wave equation of non-free photon.

For the non-free particle, the Einstein’s dispersion relation is
\[ (E - V)^2 = c^2\vec{p}^2 + m_0^2c^4. \] (66)

Factorizing Eq. (66), we obtain
\[ (E - V)^2 - c^2\vec{p}^2 - m_0^2c^4 = (E - V - c\vec{p} \cdot \vec{\alpha} - m_0c^2\beta)(E - V + c\vec{p} \cdot \vec{\alpha} + m_0c^2\beta) = 0. \] (67)

For a photon, \( m_0 = 0 \), Eq. (67) becomes
\[ (E - V - c\vec{p} \cdot \vec{\alpha})(E - V + c\vec{p} \cdot \vec{\alpha}) = 0, \] (68)
or
\[ (E - V - c\vec{p} \cdot \vec{\alpha}) = 0. \] (69)

According to canonical quantization, Eq. (69) turns into
\[ i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = -ich\vec{\alpha} \cdot \vec{\nabla}\psi(\vec{r},t) + V\psi(\vec{r},t). \] (70)

Considering that the potential energy of photon in medium is \([29]\)
\[ V = \hbar\omega(1 - n), \] (71)
where \( n \) is the medium refractive index. Substituting Eq. (71) into (70), the spiron equation of photon in medium is expressed as
\[ i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = -ich\vec{\alpha} \cdot \vec{\nabla}\psi(\vec{r},t) + \hbar\omega(1 - n)\psi(\vec{r},t). \] (72)

Using the method of separation variable
\[ \psi(\vec{r},t) = \psi(\vec{r})f(t), \] (73)
Eq. (72) becomes
\[ f(t) = f_0e^{-\frac{\hbar\omega}{c}\vec{r} \cdot \vec{\alpha} \cdot \vec{\nabla} + \hbar(1 - n)\vec{r} \cdot \vec{\alpha} \cdot \vec{\nabla} - \frac{\hbar\omega}{c}\vec{r} \cdot \vec{\alpha} \cdot \vec{\nabla} + \hbar(1 - n)\vec{r} \cdot \vec{\alpha} \cdot \vec{\nabla} - \frac{\hbar\omega}{c}\vec{r} \cdot \vec{\alpha} \cdot \vec{\nabla} + \hbar(1 - n)\vec{r} \cdot \vec{\alpha} \cdot \vec{\nabla}}}, \] (74)
\[ [-ich\vec{\alpha} \cdot \vec{\nabla} + \hbar\omega(1 - n)]\psi(\vec{r}) = E\psi(\vec{r}), \] (75)
where \( E \) is the total energy of photon in medium, and \( n \) is the refractive index of medium. The Eqs. (72) and (75) are the spiron wave equations of time-dependent and time-independent of photon in the medium, which can be used to study the quantum property of photon in medium.
8. The plane wave solution of photon in medium

For the spin $\frac{1}{2}$ Dirac electron, there are the plane wave solutions corresponding to positive energy and negative energy. For the photon, there are also the plane wave solutions corresponding to positive energy and negative energy. The Eq. (72) Hamiltonian operator and spinor wave function are

$$H = c \vec{\alpha} \cdot \vec{p} + \hbar \omega (1 - n), \quad \psi(\vec{r}, t) = \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \\ \psi_3(\vec{r}, t) \end{pmatrix}, \quad (76)$$

since $\frac{\partial H}{\partial t} = 0$ and $[\vec{p}, H] = 0$, the photon energy $E$ and momentum $\vec{p}$ are conserved quantity, they have common eigenstate, namely,

$$\psi(\vec{r}, t) = u(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)/\hbar} = u(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad (77)$$

where $\vec{k} = \vec{p}/\hbar$, $\omega = E/\hbar$ and

$$\psi(\vec{r}) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} e^{i\vec{k} \cdot \vec{r}} = u(\vec{k}) e^{i\vec{k} \cdot \vec{r}}, \quad (78)$$

substituting Eqs. (78) into (72) yields

$$ich \alpha \cdot \vec{\nabla} \psi(\vec{r}) + (E - \hbar \omega (1 - n)) \psi(\vec{r}) = 0, \quad (79)$$

substituting Eqs. (7) and (78) into (79), there are

$$E - (1 - n) \hbar \omega)u_1 + ich_2 u_2 - ich_3 u_3 = 0, \quad (80)$$

$$-ich_2 u_1 + (E - (1 - n) \hbar \omega)u_2 + ich_3 u_3 = 0, \quad (81)$$

and

$$ich_2 u_1 - ich_2 u_2 + (E - (1 - n) \hbar \omega)u_3 = 0, \quad (82)$$

the necessary and sufficient conditions for a non-zero solution of Eqs. (80)-(82) is

$$\begin{vmatrix} E - (1 - n) \hbar \omega & ich_2 & -ich_3 \\ -ich_2 & E - (1 - n) \hbar \omega & ich_1 \\ ich_3 & -ich_1 & E - (1 - n) \hbar \omega \end{vmatrix} = 0, \quad (83)$$

spreading Eq. (83), we obtain

$$E_1 = \hbar \omega (1 - n), \quad (84)$$

$$E_2 = \hbar \omega (1 - 2n), \quad (85)$$

and

$$E_3 = \hbar \omega, \quad (86)$$

since $E_1 < 0$ and $E_2 < 0$, the solution $E_3 = \hbar \omega$ is reasonable. Substituting $E_3$ into (80)-(82), we get

$$\omega n u_1 + ick_z u_2 - ick_y u_3 = 0, \quad (87)$$
\[ i c k_z u_1 - \omega n u_2 - i c k_z u_3 = 0, \quad (88) \]

and

\[ i c k_y u_1 - i c k_x u_2 + \omega n u_3 = 0. \quad (89) \]

From Eq. (87) to (89), we obtain the ratio

\[ \frac{u_2}{u_3} = \frac{ick_z k_y + \omega n k_z}{ick_z k_z - \omega n k_y}, \quad (90) \]

and

\[ \frac{u_1}{u_2} = -\frac{i c (k_y^2 + k_z^2)}{ick_x k_y + \omega n k_z}. \quad (91) \]

and the \( u(\vec{k}) \) spinor is

\[ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = A \begin{pmatrix} -i c (k_y^2 + k_z^2) \\ ick_x k_y + \omega n k_z \\ ick_x k_z - \omega n k_y \end{pmatrix}, \quad (92) \]

its normalization form is

\[ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{ck \sqrt{2(k_y^2 + k_z^2)}} \begin{pmatrix} -i c k_y^2 \\ ick_x k_y + \omega n k_z \\ ick_x k_z - \omega n k_y \end{pmatrix}. \quad (93) \]

In the medium of refractive index \( n \), when the photon propagates in the \( x - y \) plane, the wave vector \( \vec{k} = k_x \vec{i} + k_y \vec{j} \), where \( k_x = k \cos \theta = \frac{\omega}{c} n \cos \theta \), \( k_y = k \sin \theta = \frac{\omega}{c} n \sin \theta \) and \( \theta \) is the angle between \( \vec{k} \) and \( x \) axis, the Eq. (93) becomes

\[ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{\sqrt{2 \omega n \cdot k_y}} \begin{pmatrix} -i c k_y^2 \\ ick_x k_y \\ -\omega n k_y \end{pmatrix}. \quad (94) \]

In the vacuum, the medium refractive index \( n = 1 \), the Eq. (94) becomes

\[ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{\sqrt{2 \omega}} \begin{pmatrix} -i c k_y \\ ick_x \\ -\omega \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \sin \theta \\ i \cos \theta \\ -1 \end{pmatrix}, \quad (95) \]

the plane wave solution of photon in the vacuum or uniform medium is

\[ \psi(\vec{r}, t) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} e^{i(\vec{k} \cdot \vec{r} - \omega t)}. \quad (96) \]

9. The Lagrangean density of photon spinor equation

Before giving the second quantization of the field of a photon particle, we must deepen our understanding of the formal properties of the spinor equation of photon. We start by casting the formalism of the spinor equation of photon into Hamiltonian form, because this is necessary for canonical quantization. For the photon spinor equation (4), we may give its Lagrangean density. In the natural unit, \( \hbar = c = 1 \), the Eq. (5) can be written as

\[ \left( \partial_t + \vec{\alpha} \cdot \vec{\nabla} \right) \psi = 0, \quad (97) \]
or
\[ \beta^\mu \partial_\mu \psi = 0, \]  
(98)

where \( \beta^\mu = (I, \vec{a}) \) and \( I \) is a \( 3 \times 3 \) unit matrix. The adjoint of Eq. (98) is
\[ \partial_\psi \psi^+ + \vec{\nabla} \psi^+ \cdot \vec{a} = 0, \]  
(99)
i.e.,
\[ \partial_\mu \psi^+ \beta^\mu = 0, \]  
(100)
the Lagrangean density of photon can be taken as
\[ \mathcal{L} = \psi^+ \beta^\mu \partial_\mu \psi, \]  
(101)
Using Eq. (101), we have
\[ \frac{\partial \mathcal{L}}{\partial \psi} = 0, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = \psi^+ \beta^\mu, \]  
(102)
and
\[ \frac{\partial \mathcal{L}}{\partial \psi^+} = \beta^\mu \partial_\mu \psi, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^+} = 0. \]  
(103)
Substituting Eqs. (102) and (103) into Lagrangean equations
\[ \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = 0, \]  
(104)
and
\[ \frac{\partial \mathcal{L}}{\partial \psi^+} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^+} = 0, \]  
(105)
we can obtain the spinor photon wave equation (98) and its adjoint equation (100).

To go over to a Hamiltonian formalism, the momentum canonically conjugate to \( \psi \) is
\[ \pi = \frac{\partial \mathcal{L}}{\partial \psi} = \psi^+, \]  
(106)
the hamiltonian density is
\[ \mathcal{H} = \pi \dot{\psi} - \mathcal{L} = -\psi^+ \vec{a} \cdot \vec{\nabla} \psi. \]  
(107)
The Hamiltonian (101) is suitable for the quantization of photon field.

10. The topological features of photon in periodic potential

The study of topological properties in condensed matter physics emerges from the discovery of quantum hall effect and quantum anomalous hall effect, which have attracted great attentions in recent years [17, 18]. Recent studies have discovered that these topological properties exist also in photonic systems [19, 20] and acoustic systems [21]. In photonic systems, the Zak phase and Chern number can also be obtained from the photonic Bloch state of one-dimensional photonic crystal [22], which are calculated by classic electrical field, they are the classical Zak phase and Chern number. In the following, we shall study the quantum Zak phase
and Chern number when a photon in one-dimensional photonic crystal with the photon wave function. The photon quantum wave equation is

\[ i\hbar \frac{\partial}{\partial t}\psi(x, t) = (-i c \hbar \vec{a} \cdot \vec{\nabla} + V(x))\psi(x, t), \]  

(108)

with the method of separation variable

\[ \psi(x, t) = \psi(x)f(t), \]  

(109)

Eq. (108) becomes

\[ [-i c \hbar \vec{a} \cdot \vec{\nabla} + V(x)]\psi_m(x) = E_m\psi_m(x), \]  

(110)

where \( m \) labels energy band and \( E_m \) is the energy spectrum for the band \( m \). The Eq. (110) describes the quantum peculiarity of Bloch photon in period potential field \( V(x) \), it has the solution of Bloch wave, it is

\[ \psi_m(x) = e^{ikx}u_{mk}(x), \]  

(111)

where \( k \) is Bloch vector, and \( u_{mk}(x) \) is the period function, i.e.,

\[ u_{mk}(x + na) = u_{mk}(x), \]  

(112)

where \( a \) is the period and \( n \) is a integer. In the medium of refractive index \( n(x) \), the photon potential energy is

\[ V(x) = \hbar \omega (1 - n(x)). \]  

(113)

For media A and B, which refractive index are \( n_a, n_b \) and their thickness are \( a, b \), respectively are constituted one-dimensional photonic crystals, the photon potential energy in one-dimensional photonic crystals is the period potential, it is

\[ V(x + (a + b)) = V(x). \]  

(114)

The quantum Berry phase is defined as

\[ \gamma_m = \int_{-\pi/(a+b)}^{\pi/(a+b)} X_m(k)dk, \]  

(115)

where the quantity \( X_m(k) \) is

\[ X_m(k) = \frac{2\pi}{a+b} \int_0^{a+b} u_{mk}^*(x) \frac{\partial u_{mk}(x)}{\partial k} dx. \]  

(116)

The quantum Chern number on a surface enclosing the point is

\[ C_m = \frac{1}{2\pi} \int_S F_m(k) \cdot dS \]  

(117)

Here \( S \) denotes the integration surface, and \( F_m(k) = \nabla \times \langle u_{mk}(x) | i \partial_k | u_{mk}(x) \rangle \) is the Berry curvature of the \( n \)-th band with the wave function \( u_{mk}(x) \), where the band indices \( n \) for the lower, middle, and higher bands are denoted as −, 0, and +, respectively.

From Eqs. (112) to (116), we can calculate the quantum Zak phase of photon in the period potential field, and we can calculate the quantum Chern number with Eq. (117).

11. Conclusions

In this paper, we have proposed the spinor wave equation of free photon, and given the spin operators, spin wave functions and space wave functions. We have calculated photon helicity and found left-handed and right-handed photon. In addition, we have given the Hamiltonian density of free photon, which is suitable for the quantization of photon field. Otherwise we have further given the spinor wave equation of non-free photon, which should be used to study the quantum property of photon in medium. Using the single-photon spin wave function, we can study two-photon or multiple-photon spin wave function, and further give the spin entanglement states of multiple-photon, which should be applied in quantum communication.
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