Research Article

Optimal Decay Rate Estimates of a Nonlinear Viscoelastic Kirchhoff Plate

Baowei Feng1 and Mostafa Zahri2

1Department of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China
2College of Sciences, Department of Mathematics, Research Group MASEP, University of Sharjah, Sharjah, UAE

Correspondence should be addressed to Mostafa Zahri; mzahri@sharjah.ac.ae

Received 2 December 2019; Accepted 14 February 2020; Published 16 March 2020

Copyright © 2020 Baowei Feng and Mostafa Zahri. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is concerned with a nonlinear viscoelastic Kirchhoff plate \( u_{tt} - \sigma \Delta u_{tt} + \Delta^2 u(t) - \int_0^t g(t-s)\Delta^2 u(s)\,ds = \text{div} F(\nabla u(t)), \) in \( \Omega \times \mathbb{R}^+, \) together with simply supported boundary condition

\[
u(t) = \Delta u(t) = 0, \quad \text{on } \partial \Omega \times \mathbb{R}^+, \tag{2}
\]

and initial conditions

\[
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{3}
\]

where \( \Omega \subseteq \mathbb{R}^n \) is a regular and bounded domain; \( \sigma > 0 \) is a constant; and \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a vector field which will be assumed later. The relaxation function \( g(t) \) is a real function. From the physical point of view, systems (1)–(3) are related to classical theory for beams/plates appearing from materials with viscoelastic structures.

For viscoelastic wave equation,

\[
u_{tt} - \sigma \Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = F(u), \tag{4}
\]

the problems are truly overworked. For example, for specific behavior of the relaxation function \( g \), it is shown that the energy decays exponentially (polynomially) if \( g \) decays exponentially (polynomially) (see Cabanillas and Muñoz Rivera[1], Muñoz Rivera et al. [2, 3], Santos [4], Cavalcanti and Oquendo [5], and so on). Messaoudi [6, 7] considered two classes of system (4), by first introducing the assumption \( g'(t) \leq -\xi(t)\), and established general decay results. Han and Wang [8], Liu [9, 10], Messaoudi and Mustafa [11], Mustafa [12], and Park and Park [13] also used this assumption on \( g \) to get general decay of energy for problems related to (4). In [14], Lasiecka and Tataru introduced a more general assumption on \( g \), which satisfies \( g'(t) \leq -H(g(t)) \), where \( H \) is strictly increasing and convex function. There are also so many stability results established by using this condition. We refer the reader to Cavalcanti et al. [15, 16], Lasiecka et al. [17, 18], Mustafa [19], Mustafa and Messaoudi [20], and Xiao and Liang [21]. Very recently, in [22, 23], Mustafa considered two classes of wave equations and proved general and explicit decay.
results of energy by using a new general assumption on \( g \):
\[
g'(t) \leq -\xi(t)H(g(t)).
\]
For viscoelastic plate equation, Rivera et al. [24] studied the following equation:
\[
u_{tt} - \gamma \Delta u_t + \Delta^2 u - \int_0^t g(t - s)\Delta^2 u(s)ds = 0,
\]
(5)
together with initial and dynamical boundary conditions. They proved the energy decays exponentially (resp. polynomially) if the relaxation function \( g \) decays exponentially (resp. polynomially). Alabau-Boussouira et al. [25] considered
\[
u_{tt} + \Delta^2 u - \int_0^t g(t - s)\Delta^2 u(s)ds = f(u),
\]
(6)
together with Dirichlet–Neumann boundary conditions and established exponential and polynomial decay results for sufficiently small initial data. When \( f(u) = 0 \) in (6), Cavalcanti [26] considered the equation subject to nonlinear boundary conditions and established exponential decay of energy by assuming a nonlinear and nonlocal feedback action on the boundary and provided that the relaxation function decays exponentially. In Andrade et al. [27], the authors considered a viscoelastic plate equation with \( p \)-Laplacian:
\[
u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t - s)\Delta u(s)ds - \Delta u_t + f(u) = 0,
\]
(7)
where \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \), and they established an exponential decay of energy under the assumption
\[
g'(t) \leq -k_1 g(t).\]
When \( \Delta_p u \) is replaced by \( \text{div}(|\nabla u|^{p-2}\nabla u) \) in (7), Ferreira and Messaoudi [28] proved a general decay result of energy. Feng [29] established a general decay result of a plate equation with time delay. Recently, Gomes Tavares et al. [30] considered a class of nonlinear plate equations with memory; they proved, by using the methods developed by Lasiecka and Wang [18], the decay rates are expressed in terms of the solution to a given nonlinear dissipative ODE. In [31], the authors proved the well-posedness of solutions to problem (1)–(3) with the case \( \sigma > 0 \) and the case \( \sigma = 0 \). Under the assumptions on \( g(t) \):
\[
g'(t) \leq -\xi(t)g(t), \quad \forall t > 0,
\]
(8)
\[
\xi(t) > 0, \quad \xi'(t) \leq 0, \quad \forall t > 0,
\]
they established the general decay rates of energy of the form
\[
E(t) \leq ce^{-\gamma t}\int_0^t \xi(s)ds.
\]
(9)
If the memory term is infinite, one can find some results on plate equation with history memory in [32–38] and so on.

In this paper, we continue to study (1)–(3), in which we consider \( \sigma = 1 \) for simplicity, with minimal conditions on the \( L^1(0,\infty) \) relaxation function \( g \) (see (12)). We establish explicit and general energy decay results of systems (1)–(3) by using the idea of Mustafa [22, 23] and some properties of convex functions developed in [18, 39]. We point out that the decay results established here are optimal exponential and polynomial rates for \( 1 \leq p < 2 \) when \( G(t) = t^p \), which improved the previous known results for \( 1 \leq p < (3/2) \). Under this level of generality, the decay rates we get are optimal, and our results improve the stability results in previous works. At last, we give some numerical illustrations.

The rest of this paper is as follows. In Section 2, we give some assumptions and results. The general decay result of the energy will be established in Section 3. In Section 4, we give some numerical illustrations.

### 2. Assumptions and Results

In the following, for simplicity, we write \( \| \cdot \| \) instead of \( \| \cdot \|_2 \). \( c > 0 \) is used to denote a generic constant. The positive constants \( \lambda_1 \) and \( \lambda_2 \) represent the embedding constants
\[
\lambda_1 \| u \|_2 \leq \| \Delta u \|_2, \quad \lambda_2 \| \nabla u \|_2 \leq \| u \|_2,
\]
(10)
for \( u \in H^2(\Omega) \cap H_0^1(\Omega) \).

For relaxation function \( g \), we assume the following:

- **(A1)** \( g(t) \): \([0, \infty) \longrightarrow \mathbb{R}^+ \) are nonincreasing \( C^1 \) functions satisfying
\[
g(0) > 0,
\]
\[
1 - \int_0^\infty g(s)ds = l > 0.
\]
(11)

In addition, there exists a \( C^1 \) function \( G : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \) satisfying \( G(0) = 0 \). The function \( G(t) \) is linear or it is an increasing strictly convex function of class \( C^1(\mathbb{R}^+) \) on \((0, r)\), \( r \leq g(0) \), such that
\[
g'(t) \leq -\xi(t)G(g(t)), \quad \forall t \geq 0,
\]
(12)
where \( \xi(t) \) is a \( C^1 \) function satisfying
\[
\xi(t) > 0, \quad \xi'(t) \leq 0, \quad \forall t \geq 0.
\]
(13)

With respect to \( E \), we assume that

- **(A2)** \( F : \mathbb{R}^n \longrightarrow \mathbb{R}^n \) is a \( C^1 \)-vector field given by \( F = (F_1, F_2, \ldots, F_n) \) such that
\[
\| \nabla F_j(\omega) \| \leq k_j (1 + |\omega|^{(p_j-1)/2}) \quad \forall \omega \in \mathbb{R}^n,
\]
(14)
where for \( j = 1, 2, \ldots, n \), the constants \( k_j > 0 \) and \( p_j \) satisfy...
Let (11) and (A2) hold. If the initial data satisfies
\[ 0 \leq f(u) \leq F(u) \cdot u, \quad \forall u \in \mathbb{R}^n, \]  
where \( F \) is a conservation field with \( F = \nabla f \) and \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is a real valued function.

Remark 1 (see [31]). Condition (14) implies that there exists a positive constant \( K = K(k_0, p, n), j = 1, 2, \ldots, n, \) such that
\[ |F(u) - F(v)| \leq K \sum_{j=1}^{n} \left( 1 + |u| \left( |(p-1)/2| + |v| \left( |(p-1)/2| \right) \right) |u - v|, \]  
\[ \forall u, v \in \mathbb{R}^n. \]  
(17)

The existence of global solutions has been proved in [31].

Theorem 1. Let (11) and (A2) hold. If the initial data \( (u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega), \) problem (1)–(3) has a unique weak solution satisfying that for any \( T > 0, \)
\[ u(t) \in L^\infty([0, T]; H^2(\Omega) \cap H^1_0(\Omega)), \]  
\[ u_t(t) \in L^\infty([0, T]; H^1_0(\Omega)). \]  
(18)

The total energy of problem (1)–(3) is given by
\[ E(t) = \frac{1}{2} \left\| u_t(t) \right\|^2 + \left\| \nabla u_t(t) \right\|^2 + \left( 1 - \int_0^t g(s) \, ds \right) \left\| \Delta u(t) \right\|^2 + (g \circ \Delta u)(t) \]  
\[ + \int_{\Omega} f(\nabla u)(t) \, dx, \]  
where
\[ (g \circ \nu)(t) = \int_0^t g(t - s) \nu(t) - \nu(s) \, ds. \]  
(19)

Now, we give the stability result of energy to problem (1)–(3).

Theorem 2. Let (A1) and (A2) hold. If \( (u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega), \) then the energy \( E(t) \) satisfies
\[ E(t) \leq k_2 \left( \int_{g^{-1}(r)}^t \xi(s) \, ds \right), \quad \forall t > g^{-1}(r), \]  
where the positive constant \( k_2 > 0, \) and
\[ G_1(t) = \int_t^0 \frac{1}{s G'(s)} \, ds. \]  
(22)

In particular, for \( G(t) = t^p \) in (12), we can get
\[ E(t) \leq \begin{cases} k \exp \left( -k_1 \int_0^t \xi(s) \, ds \right), & \text{if } p = 1, \\ \frac{k}{1 + \int_0^t \xi(s) \, ds} \frac{t^{p-1}}{1 - (p-1)}, & \text{if } 1 < p < 2, \end{cases} \]  
(23)
where \( k, k_1, \) and \( k_2 \) are positive constants.

Remark 2
(1) Since the constant \( \epsilon G_t \) appearing within the estimate (27) depends on \( E(0), \) which leads to the final constant which will depend on the initial data, the constant \( k_1 > 0 \) depends on the size of initial data.

(2) Here, \( G_1 \) is strictly convex and decreasing on \( (0, r] \) with \( \lim_{t \to 0} G_1(t) = +\infty. \)

Remark 3. It follows from (A1) that \( \lim_{t \to 1,0,0} g(t) = 0. \) We know that there exists some \( t_1 \geq 0 \) large enough such that
\[ g(t_1) = r \implies g(t) \leq r, \quad \forall t \geq t_1. \]  
(24)

Then, we can get for every \( t \in [0, t_1], \)
\[ 0 < g(t_1) \leq g(t) \leq g(0), \]  
\[ 0 < \xi(t) \leq \xi(t) \leq \xi(0). \]  
(25)

Therefore, there exist positive constants \( a \) and \( b, \)
\[ a \leq \xi(t)G_t(g(t)) \leq b, \]  
(26)
which yields for every \( t \in [0, t_1], \)
\[ g'(t) \leq -\xi(t) G_t(g(t)) \leq -\frac{a}{G(0)} G(0) \leq -\frac{a}{G(0)} g(t). \]  
(27)

We end this section by giving three examples.

Example 1. Let \( g(t) = \alpha e^{-bt}, (b > 0). \) We take \( \alpha > 0 \) satisfying (11). If we choose \( G_t = t, \) then \( g'(t) = -bG_t(g(t)). \) From (23), we find
\[ E(t) \leq k e^{-bk_1 t}. \]  
(28)

Example 2. Let \( g(t) = (a/(1 + t^b)), (b > 1). \) We take \( a > 0 \) satisfying (11). For a fixed positive constant \( p, \) we have \( g'(t) = -pG_t(g(t)) \) with \( G_t = tp, p = ((1 + b)/b) \in (1, 2). \) From (23), we get
\[ E(t) \leq \frac{k}{(1 + t)^p}. \]  
(29)

Example 3. Let \( g(t) = a \exp(-1(1 + t)^b), (0 < b < 1). \) We take \( a > 0 \) satisfying (11). Then, \( g'(t) = -\xi(t) G_t(g(t)) \) with \( G_t = t \) and \( \xi(t) = b(1 + t)b - 1. \) Then, we obtain from (23) that
\[ E(t) \leq k \exp\left(-b(1 + t)^{b-1}\right). \] (30)

### 3. Optimal Decay

To prove Theorem 2, we need some lemmas.

#### 3.1. Technical Lemmas

**Lemma 1.** It holds that for any \( t \geq 0 \),

\[ E'(t) = \frac{1}{2} (g' \cdot \Delta u)(t) - \frac{1}{2} g(t)\|\Delta u(t)\|^2. \] (31)

**Proof.** Multiplying \( L^2(\Omega) \) in equation (1) by \( u_t \) and using integration by parts and boundary conditions, we can get (31).

Let us define the functionals:

\[ \phi(t) = \int_{\Omega} (u_t(t) - \Delta u_t(t)) u_t(t) \, dx, \]
\[ \psi(t) = -\int_{\Omega} (u_t(t) - \Delta u_t(t)) \left( \int_0^t (g(t - s)(u(t) - u(s))ds \right) \, dx. \] (32)

**Lemma 2.** The functional \( \phi(t) \) satisfies for any \( t \geq 0 \),

\[ \phi'(t) \leq \frac{1}{2} \|\Delta u(t)\|^2 + \|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 \]
\[ -\int_{\Omega} f(\nabla u(t)) dx + \frac{C_\alpha}{T} (h \cdot \Delta u)(t), \] (33)

where \( h(t) = a g(t) - g'(t), \)

introduced in [22, 23].

**Proof.** In view of (1), and integration by parts, we can obtain

\[ \phi'(t) = -\|\Delta u(t)\|^2 + \|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 \]
\[ -\int_{\Omega} F(\nabla u(t)) \cdot \nabla u(t) dx \]
\[ + \int_{\Omega} \nabla u(t) \int_0^t g(t - s) \Delta u(s) ds \, dx \]
\[ = -\left(1 - \int_0^t g(s) ds\right) \|\Delta u(t)\|^2 + \|u_t(t)\|^2 \]
\[ + \|\nabla u_t(t)\|^2 - \int_{\Omega} F(\nabla u(t)) \cdot \nabla u(t) dx \]
\[ + \int_{\Omega} \nabla u(t) \int_0^t g(t - s)(\Delta u(s) - \Delta u(t)) ds dx. \] (35)

It follows from (16) that

\[ -\int_{\Omega} F(\nabla u(t)) \cdot \nabla u(t) dx \leq -\int_{\Omega} f(\nabla u) dx. \] (36)

Hölder’s inequality gives us

\[ \int_{\Omega} \left( \int_0^t g(t - s)\Delta u(s) - \Delta u(t) ds \right)^2 \, dx \]
\[ = \int_{\Omega} \left( \int_0^t \frac{g(t - s)}{ag(t - s) - g'(t - s)} a g(t - s) - g'(t - s) \right) |\Delta u(s) - \Delta u(t)| ds \right) \, dx \]
\[ \leq \left( \int_0^t \frac{g^2(s)}{ag(s) - g'(s)} ds \right) \int_\Omega \int_0^t [a g(t - s) - g'(t - s)] |\Delta u(s) - \Delta u(t)|^2 ds \, dx \]
\[ = C_\alpha (h \cdot \Delta u)(t), \] (37)

which, together with Young’s inequality, implies

\[ \int_{\Omega} \Delta u(t) \int_0^t g(t - s)(\Delta u(s) - \Delta u(t)) ds \, dx \]
\[ \leq \frac{1}{2} \|\Delta u(t)\|^2 + \frac{1}{2r} \int_{\Omega} \left( \int_0^t g(t - s)\Delta u(s) - \Delta u(t) ds \right)^2 \, dx \]
\[ \leq \frac{1}{2} \|\Delta u(t)\|^2 + C_\alpha (h \cdot \Delta u)(t). \] (38)
Repeating (36) and (38) in (35), we can get the desired estimate (33).

**Lemma 3.** Assume that (A1) and (A2) hold, then the functional $\psi(t)$ satisfies for any $\delta > 0$,

$$
\psi'(t) \leq -\left( \int_0^t g(s) ds - \delta \right) \|u_t(t)\|^2 \\
- \left( \int_0^t g(s) ds - \delta \right) \|\nabla u_t(t)\|^2 + \delta \|\Delta u(t)\|^2 + \frac{C_\alpha + 1}{\delta} (h \cdot \Delta u(t)).
$$

**Proof.** Taking the derivative of $\psi(t)$, using equation (1) and integration by parts, we can derive that

$$
\psi'(t) = \left( 1 - \int_0^t g(s) ds \right) \int_\Omega \Delta u(t) \int_0^t g(t-s) (\Delta u(t) - \Delta u(s)) ds dx \\
+ \int_\Omega \left( \int_0^t g(t-s) (\Delta u(t) - \Delta u(s)) ds \right)^2 dx
$$

$$
= I_1 + I_2 + I_3 + I_4
$$

By Young’s inequality and (37), we shall see below, for any $\delta > 0$,

$$
I_1 \leq \frac{\delta}{2} \|\Delta u(t)\|^2 + \frac{C_\alpha}{\delta} (h \cdot \Delta u)(t),
$$

$$
I_2 \leq C_\alpha (h \cdot \Delta u)(t),
$$

$$
I_3 = \int_\Omega u_t(t) \int_0^t h(t-s) (u(t) - u(s)) ds dx - \int_\Omega u_t(t) \int_0^t a g(t-s) (u(t) - u(s)) ds dx
$$

$$
\leq \|u_t(t)\|^2 \frac{1}{2\delta} \int_\Omega \left( \int_0^t h(t-s) (u(t) - u(s)) ds \right)^2 dx
$$

$$
+ \frac{a^2}{2\delta} \int_\Omega \left( \int_0^t g(t-s) (u(t) - u(s)) ds \right)^2 dx
$$

$$
\leq \|u_t(t)\|^2 + \frac{\int_0^t h(s) ds}{2\delta} (h \cdot u)(t) + \frac{a^2 C_\alpha}{2\delta} (h \cdot u)(t)
$$

$$
\leq \|u_t(t)\|^2 + \frac{C_\alpha (h \cdot \Delta u)(t)}{\delta}.
$$
Similarly, we can get for any $\delta > 0$,

$$I_5 \leq \delta \|\nabla u(t)\|^2 + \frac{c(C_a + 1)}{\delta} (h \cdot \Delta u(t)).$$  \hfill (42)

With respect to $I_3$, we have

$$|I_3| \leq \int_0^I g(t - s) \left( \int_\Omega |F(\nabla u(t))| |\nabla u(t) - \nabla u(s)| dx \right) ds.$$  \hfill (43)

Following the same method as in [31], we can get

$$\left( I_3^2 \right) \leq \mu_1 + \mu_2 [E(0)]^{(p - 1)/2},$$  \hfill (45)

where $\mu_1$ and $\mu_2$ are two positive constants and

$$I_3 \leq \int_0^I g(t - s) \|\Delta u(t)\|^2 \left[ \frac{K}{\lambda_2} \sum_{j=1}^n \mu_p \left( |\Omega| \left( (p_j - 1)/(p_j + 1) \right) + \|\nabla u(t)\|_{H^{p_j + 1}} \right) \right] \|\Delta u(t) - \Delta u(s)\| ds.$$  \hfill (47)

By using (A2), we derive that

$$p = \max\{p_1, \ldots, p_n\}, \text{ if } E(0) \geq 1,$$

$$p = \min\{p_1, \ldots, p_n\}, \text{ if } E(0) < 1.$$  \hfill (46)

In view of Young and Hölder’s inequalities and (37), we have for any $\delta > 0$,

$$\theta(t) = \sigma(0) \|\Delta u(t)\|^2 - \int_0^I \int_\Omega g(t - s) |\Delta u(s)|^2 dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

By using Young and Hölder’s inequalities, we will show that

$$\theta(t) = \sigma(0) \|\Delta u(t)\|^2 - \int_0^I \int_\Omega g(t - s) |\Delta u(s)|^2 dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

By using Young and Hölder’s inequalities, we will show that

$$\theta(t) = \sigma(0) \|\Delta u(t)\|^2 - \int_0^I \int_\Omega g(t - s) |\Delta u(s)|^2 dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$

$$\leq -2 \int_\Omega \int_0^t g(t - s) (\Delta u(s) - \Delta u(t)) dx ds + \sigma(t) \|\Delta u(t)\|^2$$
Since \( \sigma(t) \leq \sigma(0) = 1 - l \) and \( \int_0^t g(s)ds \leq 1 - l \), we can obtain (49).

Our argument in the following is based on the choice of a suitable Lyapunov function \( \mathcal{L}(t) \) by

\[
\mathcal{L}(t) = NE(t) + N_1 \psi(t) + N_2 \psi(t),
\]

where \( N, N_1, \) and \( N_2 \) are positive constants. Clearly, for \( N \) large, there exist \( \beta_1 > 0 \) and \( \beta_2 > 0 \) such that

\[
\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t).
\]

**Lemma 5.** It holds that for any \( t \geq t_1 \),

\[
\mathcal{L}'(t) \leq -4(1-l)\|\Delta u(t)\|^2 - \|u_t(t)\|^2 - \|\nabla u_t(t)\|^2
- \int_\Omega f(\nabla u(t))dx + \frac{1}{4} (g * \Delta u)(t).
\]

Proof. Let \( g_1 = \int_0^{t_1} g(s) ds > 0 \). Combining (31), (33), and (39), taking \( \delta = 1/(2N_2) \), and noting \( g' = ag - h \), we can infer that for any \( t \geq t_1 \),

\[
\mathcal{L}'(t) \leq -\left( \frac{N_1}{2} - 1 \right) \|\Delta u(t)\|^2
- \left( N_2 g_1 - \frac{1}{2} N_1 \right) \|u_t(t)\|^2
- \left( N_2 g_1 - \frac{1}{2} N_1 \right) \|\nabla u_t(t)\|^2 + \frac{\alpha}{2} N (g * \Delta u)(t)
- N_1 \int_\Omega f(\nabla u(t))dx
- \left[ \frac{N}{2} - 2cN_2^2 - C_a \left( \frac{N_1}{4} + 2cN_2^2 \right) (h \cdot \Delta u)(t) \right].
\]

(55)

Firstly, we take \( N_1 \) large enough so that

\[
N_1 > 1, \quad \frac{N_1}{2} - \frac{1}{2} > 4(1-l),
\]

and then we choose \( N_1 \) so large that

\[
N_2 g_1 - \frac{1}{2} N_1 > 1.
\]

(56)

Since \( ((ag^2(s))/(2ag(s) - g'(s))) < g(s) \), using the Lebesgue dominated convergence theorem, we can get

\[
\lim_{a \to 0} \alpha C_a = \lim_{a \to 0} \int_0^\infty \frac{ag^2(s)}{ag(s) - g'(s)} ds = 0.
\]

(58)

Hence, there exist some \( \alpha_0 (0 < \alpha_0 < 1) \) such that if \( \alpha < \alpha_0 \), then

\[
\alpha C_a < \frac{1}{8((N_1/2) + 2cN_2^2)}
\]

(59)

At last, for any fixed \( N_1 \) and \( N_2 \), we choose \( N \) large enough and choose \( \alpha \) satisfying

\[
\frac{1}{4} N - 2cN_2^2 > 0,
\]

\[
\alpha = \frac{1}{2N} < \alpha_0,
\]

and then

\[
\frac{N}{2} - 2cN_2^2 - C_a \left( \frac{N_1}{4} + 2cN_2^2 \right) > 0.
\]

(61)

This ends the proof.

3.2. Proof of Theorem 2. Taking into account (27) and (31), we find that for any \( t \geq t_1 \),

\[
\int_0^{t_1} g(s) \int_\Omega (|u_t(t) - \Delta u(t - s)|^2 dx ds
\]

\[
\leq \frac{g(0)}{a} \int_0^{t_1} g'(s) \int_\Omega (|u_t(t) - \Delta u(t - s)|^2 dx ds \leq -cE'(t),
\]

(62)

which, along with (54), gives us for some constant \( m > 0 \) and for all \( t \geq t_1 \),

\[
\mathcal{L}'(t) \leq -mE(t) + c(g * \Delta u)(t)
\]

\[
\leq -mE(t) - cE'(t)
\]

\[
+ c \int_{t_1}^t g(s) \int_\Omega (|u_t(t) - \Delta u(t - s)|^2 dx ds.
\]

Define \( F(t) = \mathcal{L}(t) + cE(t) \sim E(t) \). Then, we find from (63),

\[
F'(t) \leq -mE(t) + c \int_{t_1}^t g(s) \int_\Omega (|u_t(t) - \Delta u(t - s)|^2 dx ds.
\]

(64)

We consider the following two cases.

**Case 1.** The particular case \( G(t) = tp \).

(1) \( p = 1 \).

Multiplying (64) by \( \xi(t) \) and using (31) and (A2)-(A3), we have

\[
\xi(t)F'(t) \leq -m\xi(t)E(t) - cE'(t), \quad \forall t \geq t_1.
\]

(65)

Since \( \xi(t) \) is a nonincreasing continuous function and \( \xi'(t) \leq 0 \) for a.e. \( t \), then

\[
(\xi F + cE)'(t) \leq \xi(t)F'(t) + cE'(t) \leq -m\xi(t)E(t), \quad \text{a.e. } t \geq t_1.
\]

(66)

In view of \( \xi F + cE \sim E \), we obtain that there exist two positive constants \( c_1, c_2 > 0 \),
\[ E(t) = c_1 e^{-c_0 t} \int e^{\xi(t)s} ds. \quad (67) \]

(II) \( 1 < p < 2 \).

Define \( \mathcal{E}(t) \) by
\[ \mathcal{E}(t) = \mathcal{E}(t) + \theta(t). \quad (68) \]

It follows from (49) and (54) that \( \mathcal{E}(t) \geq 0 \) and there exists a positive constant \( \beta \) such that for any \( t \geq t_1 \),
\[ \mathcal{E}'(t) \leq -\beta \left( \| u(t) \|^2 + \| \Delta u(t) \|^2 + \| \nabla u(t) \|^2 \right) + \int_\Omega f(\nabla u(t)) dx - \frac{1}{4} (g \cdot \Delta u)(t). \quad (69) \]

Then, there exists a certain constant \( \beta_1 > 0 \),
\[ \mathcal{E}'(t) \leq -\beta_1 E(t), \quad \forall \ t \geq t_1. \quad (70) \]

This gives us
\[ \beta_1 \int_{t_1}^t E(s) ds \leq \mathcal{E}(t_1) - \mathcal{E}(t) \leq \mathcal{E}(t_1). \quad (71) \]

Hence,
\[ \int_0^\infty E(s) ds < \infty. \quad (72) \]

Define
\[ I(t) = \int_0^\infty \int_\Omega |\nabla (t) - \nabla (t-s)|^2 dx ds, \quad (73) \]

and we know that
\[ I(t) \leq c \int_0^t E(s) ds. \quad (74) \]

Without loss of generality assuming \( t_1 \) so large that \( I(t_1) > 0 \), then
\[ 0 < I(t_1) \leq I(t) < \infty, \quad \forall \ t \geq t_1. \quad (75) \]

Using Jensen's inequality and (12), we can derive from (64) that for some constant \( q > 0 \),
\[ F'(t) \leq -q E(t) + \frac{c I(t)}{I(t)} \int_{t_1}^t (g^\nu(s))^{1/p} \int_\Omega |\Delta u(t) - \Delta u(t-s)|^2 dx ds \]
\[ \leq -q E(t) + \frac{c I(t)}{I(t)} \left( \int_{t_1}^t g^\nu(s) \int_\Omega |\Delta u(t) - \Delta u(t-s)|^2 dx ds \right)^{1/p} \]
\[ \leq -q E(t) + \frac{c I(t)}{I(t)} \left( \int_{t_1}^t \frac{g^\nu(s)}{\xi(s)} \int_\Omega |\Delta u(t) - \Delta u(t-s)|^2 dx ds \right)^{1/p} \]
\[ \leq -q E(t) + \frac{c I(t)}{I(t)} \left( \int_{t_1}^t \frac{g^\nu(s)}{\xi(s)} \int_\Omega |\Delta u(t) - \Delta u(t-s)|^2 dx ds \right)^{1/p} \]
\[ \leq -q E(t) + \frac{c I(t)}{I(t)} \left[ \int_{t_1}^t (g^\nu(s)) \int_\Omega |\Delta u(t) - \Delta u(t-s)|^2 dx ds \right]^{1/p} \]
\[ \leq -q E(t) + \frac{c I(t)}{I(t)} [-E'(t)]^{1/p}. \quad (76) \]

We multiply (76) by \( E^{-1}(t) \) and use (31) to deduce
\[ (FE^{-1})'(t) \leq F'(t) E^{-1}(t) \]
\[ \leq -q E(t) + c \left[ \left( \frac{E}{\xi(t)} \right) \right]^{1/p} E^{-1}(t). \quad (77) \]

By Young's inequality, we have for any \( \epsilon_1 > 0 \),
\[ (FE^{-1})'(t) \leq -q E(t) + \epsilon_1 E(t) + \frac{c}{\epsilon_1} \left[ \frac{E}{\xi(t)} \right] \quad (78) \]

Taking \( \epsilon_1 < (1/2)q \), we conclude

\[ \left( FE^{-1} \right)'(t) \leq -q E(t) + c \frac{E(t)}{\xi(t)}. \quad (79) \]

Define \( \mathcal{F}(t) = \xi E^{-1} + c E - E. \) Multiplying (79) by \( \xi(t) \), we have
\[ \mathcal{F}'(t) \leq -q \xi(t) E(t). \quad (80) \]

Then, there exists a certain constant \( q_0 > 0 \) such that
\[ \mathcal{F}'(t) \leq -q_0 \xi(t) \mathcal{F}(t), \quad (81) \]

from which we obtain
\[ E(t) \leq c_1 \left( 1 + \int_0^t \xi(s) ds \right)^{-1/(p-1)}, \]  

(82)

where \( c_1 \) is a positive constant.

Combining (I) and (II), we can get (35).

**Case 2. The general case.**

Define \( I(t) \) by

\[ I(t) = q \int_{t_1}^t \int_\Omega |\Delta(t) - \Delta(u(t-s))|^2 dx ds. \]  

(83)

It follows from (72) that we can choose a constant \( 0 < q < 1 \) so that for all \( t \geq t_1 \),

\[ I(t) < 1. \]  

(84)

Without loss of generality, we assume that \( I(t) > 0 \) for all \( t \geq t_1 \); otherwise, \( (64) \) implies an exponential decay. In addition, we define \( \lambda(t) \) by

\[ \lambda(t) = -\int_{t_1}^t g(s) \int_\Omega |\Delta u(t) - \Delta(u(t-s))|^2 dx ds. \]  

(85)

Clearly, \( \lambda(t) \leq -cE'(t) \). Since \( G(t) \) is strictly convex on \((0, r) \) and \( G(0) = 0 \), then

\[ G(\nu x) \leq \nu G(x), \]  

(86)

provided \( 0 \leq \nu \leq 1 \) and \( x \in (0, r) \). By using (12), (84), and Jensen’s inequality, we can obtain

\[ \lambda(t) = \frac{1}{qI(t)} \int_{t_1}^t I(t)(-g'(s)) \int_\Omega q|\Delta u(t) - \Delta(u(t-s))|^2 dx ds \]

\[ \geq \frac{1}{qI(t)} \int_{t_1}^t I(t)\xi(s)G(g(s)) \int_\Omega q|\Delta u(t) - \Delta(u(t-s))|^2 dx ds \]

\[ \geq \frac{\xi(t)}{qI(t)} \int_{t_1}^t G(I(t))g(s) \int_\Omega q|\Delta u(t) - \Delta(u(t-s))|^2 dx ds \]

\[ \geq \frac{\xi(t)}{q} G \left( \frac{1}{qI(t)} \int_{t_1}^t I(t)g(s) \int_\Omega q|\Delta u(t) - \Delta(u(t-s))|^2 dx ds \right) \]

\[ = \frac{\xi(t)}{q} G \left( q \int_{t_1}^t g(s) \int_\Omega |\Delta u(t) - \Delta(u(t-s))|^2 dx ds \right) \]

\[ = \frac{\xi(t)}{q} \int_{t_1}^t g(s) \int_\Omega |\Delta u(t) - \Delta(u(t-s))|^2 dx ds. \]  

(87)

Here, \( \overline{G} \) is an extension of \( G \), which is strictly convex and strictly increasing \( C^2 \) function on \((0, \infty) \). We can get from (87) that

\[ \int_{t_1}^t g(s) \int_\Omega |\Delta u(t) - \Delta(u(t-s))|^2 dx ds \leq \frac{1}{q} \overline{G}^{-1} \left( \frac{q\lambda(t)}{\xi(t)} \right). \]  

(88)

which, together with (64), implies for any \( t \geq t_1 \),

\[ F'(t) \leq -mE(t) + c\overline{G}^{-1} \left( \frac{q\lambda(t)}{\xi(t)} \right). \]  

(89)

Now, for \( \varepsilon < r \), we define the function \( \mathcal{K}_1(t) \) by

\[ \mathcal{K}_1(t) = \overline{G} \left( \frac{e_0}{E(0)} \right) F(t) + E(t), \]  

(90)

which is equivalent to \( E(t) \). In view of \( E'(t) \leq 0 \), \( \overline{G}' > 0 \), and \( \overline{G}' > 0 \), and using (89), we conclude that

\[ \mathcal{K}_1'(t) = \varepsilon_0 \frac{E'(t)}{E(0)} \]  

\[ \leq -mE(t) \overline{G} \left( \frac{e_0}{E(0)} \right) \]  

\[ + c\overline{G}^{-1} \left( \frac{q\lambda(t)}{\xi(t)} \right) \overline{G} \left( \frac{E(t)}{e_0} \right) - cE'(t). \]  

(91)

As in Arnold [40], we define the conjugate function of \( \overline{G} \) by \( \overline{G}^* \), which satisfies

\[ M_1M_2 \leq \overline{G}^* (M_1) + \overline{G} (M_2). \]  

(92)

For \( M_1 = \overline{G}' \left( \frac{e_0}{E(0)} \right) \) and \( M_2 = \overline{G}^{-1} \left( \frac{q\lambda(t)}{\xi(t)} \right) \), and using (91), we get

\[ \mathcal{K}_1'(t) \leq -mE(t) \overline{G} \left( \frac{E(t)}{e_0} \right) \]  

\[ + c\overline{G}^* \left( \frac{e_0}{E(0)} \right) \]  

\[ \leq -mE(t) \overline{G} \left( \frac{E(t)}{E(0)} \right) + c\overline{G}^{-1} \left( \frac{q\lambda(t)}{\xi(t)} \right) \overline{G} \left( \frac{E(t)}{e_0} \right) \]  

\[ \leq -m \xi(t) E(t) G' \left( \frac{E(t)}{e_0} \right) + c\xi(t) E(t) \overline{G} \left( \frac{E(t)}{e_0} \right) \]  

\[ + c\xi(t) E(t) G' \left( \frac{E(t)}{e_0} \right) \]  

\[ + c\xi(t) E(t) \overline{G} \left( \frac{E(t)}{e_0} \right) - cE'(t). \]  

(93)

Multiplying (93) by \( \xi(t) \), we see that

\[ \xi(t) \mathcal{K}_1'(t) \leq -m \xi(t) E(t) G' \left( \frac{E(t)}{e_0} \right) \]  

\[ + c\xi(t) E(t) G' \left( \frac{E(t)}{e_0} \right) \]  

\[ \leq -m \xi(t) E(t) G' \left( \frac{E(t)}{e_0} \right) \]  

\[ + c\xi(t) E(t) G' \left( \frac{E(t)}{e_0} \right) - cE'(t). \]  

(94)
where we used the fact that as $\varepsilon_0 (E(t)/E(0)) < r$, $G'(\varepsilon_0 (E(t)/E(0))) = G'(\varepsilon_0 (E(t)/E(0)))$.

Define the functional $\mathcal{K}_2(t)$ by

$$\mathcal{K}_2(t) = \xi(t)\mathcal{K}_1(t) + cE(t).$$

It is easy to obtain that $\mathcal{K}_2(t) \sim E(t)$, i.e., there exist two positive constants $\beta_3$ and $\beta_4$ such that

$$\beta_3 \mathcal{K}_2(t) \leq E(t) \leq \beta_4 \mathcal{K}_2(t).$$

(96)

We choose suitable $\varepsilon_0$ to get from (94) that for a certain constant $k > 0$,

$$\mathcal{K}_2(t) \leq -k\xi(t)\frac{E(t)}{E(0)}G'(\frac{E(t)}{E(0)}) = -k\xi(t)G_2\left(\frac{E(t)}{E(0)}\right),$$

(97)

where $G_2(t) = tG'(\varepsilon_0 t)$. Denote $R(t) = ((\beta_3 \mathcal{K}_2(t))/E(0))$. It follows from (96) that

$$R(t) \sim E(t).$$

(98)

Since $G_2(t) = G'(\varepsilon_0 t) + \varepsilon_0 tG''(\varepsilon_0 t)$, then, using the strict convexity of $G$ on $[0, r]$, we know that $G_2(t), G_2(t) > 0$ on $[0, 1]$. By (97), we get for some $k_1 > 0$ and for all $t \geq t_1$,

$$R'(t) \leq -k_1\xi(t)G_2(R(t)).$$

(99)

Integrating (99) over $(t_1, t)$, we have

$$\int_{t_1}^{t} -R'(s) \, ds \geq k_1 \int_{t_1}^{t} \xi(s) \, ds \Rightarrow \int_{t_1}^{t} \frac{1}{\xi(s)G_2'(s)} \, ds$$

$$\geq k_1 \int_{t_1}^{t} \xi(s) \, ds,$$

(100)

which, noting $G_1$, defined by

$$G_1(t) = \int_{t}^{R} \frac{1}{sG_2'(s)} \, ds,$$

(101)

is strictly decreasing on $(0, r]$ and $\lim_{t \to \infty} G_1(t) = +\infty$, we get

Figure 1: Test 1: damping behavior using the exponential function $g_1(t)$. 
\[ R(t) \leq \frac{1}{\xi_0} G_1^{-1} \left( k_1 \int_{t_1}^t \xi(s) ds \right). \]  

(102)

Then, (21) follows from (98) and (102). The proof is done.

4. Numerical Tests

In this section, we present various tests in order to illustrate our theoretical results proved in Theorem 2. We solve problem (1) using the nonlinear Lax–Wendroff method in time and space in the space-time domain \([0, 1] \times [0, 5]\). Moreover, for all partial derivatives of problem (1), we used a second-order discretization in time and space, and we consider the vector field below

\[ F(u) = \left( \frac{\partial u}{\partial x} \right)^2, \text{ where } \text{div}(F(u)) = 2 \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial u}{\partial x}. \]  

(103)

For the following values of the parameter \(\sigma = 0.1, 1, 5\), we simulate six tests of the decay of the energy (19) (for similar constructions, we refer to [41, 42]).

Test 1: in the first three tests, we present the decay case using the exponential function \(g_1(t) = e^{-2t}\), the vector field (103), and the parameters \(\sigma = 0.1, 1, 5\) (Test 1.1, 1.2, and 1.3).

Test 2: in the second three numerical tests, we examine the energy decay (19) using the polynomial function \(g_2(t) = \frac{1}{1 + t}\), the vector field (103), and the parameters \(\sigma = 0.1, 1, 5\) (Test 1.1, 1.2, and 1.3).

In order to ensure the scheme stability, we use \(\Delta t = 0.0005 < dx = 0.005\) satisfying the stability of the Courant–Friedrichs–Lewy (CFL) inequality, where \(\Delta t\) represents the time step and \(dx\) represents the spatial step. The spatial interval \([0, 1]\) is subdivided into 200 subintervals, where the temporal interval \([0, 5]\) is deduced from the stability condition above. We run our code for 10000 time steps using the following initial conditions:

\[ u(x, 0) = x(1 - x); \]  

\[ u_t(x, 0) = x(1 - x) \text{ in } [0, 1]. \]  

(104)
In Figure 1, we show the results of the first three tests, namely, Test 1.1 for $\sigma = 0.1$, Test 1.2 for $\sigma = 5$, and Test 1.3 for $\sigma = 5$. We present the cross section cuts at $x = 0.25$, $x = 0.5$, and $x = 0.75$. The damping behavior is demonstrated for all experiments. Moreover, it should be stressed that for larger $\sigma$, the pseudoperiod decreases within decaying envelope. But for smaller $\sigma$ (tending to 0), the pseudoperiod increases within same decaying envelope. Under similar initial and boundary conditions, we present in Figure 2 the results obtained for the Test 2.1 for $\sigma = 0.1$, Test 2.2 for $\sigma = 5$, and Test 2.3 for $\sigma = 5$. In Figure 3 we can clearly compare the energy decay obtained in Test 1 and in Test 2. We remarked that the energy decay is not affected by the choice of the memory functions $g_1(t) = e^{-2t}$ and $g_2(t) = 1/(1 + t)$.

Finally, it should be stressed that solving problem (1) using linear vector field functional $F$ leads to similar damping behavior of the waves and similar decay results, either for the choice of the type of the function $g_i$, $i = 1, 2$ or the positive parameter $\sigma$.

5. Conclusion

In this paper we investigate a nonlinear Kirchhoff viscoelastic plate. Under suitable assumptions on relaxation function, we establish a more general decay result of energy, by introducing suitable energy and perturbed Lyapunov functionals. The decay results established here are optimal exponential and polynomial rates for $1 \leq p < 2$ when $G(t) = t^p$, which improved the previous known results for $1 \leq p < (3/2)$. Under this level of generality, the decay rates we get are optimal, and our results improve the stability results in previous works. At last, we give some numerical illustrations and related comparisons.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (no. 11701465).

References

[1] E. L. Cabanillas and J. E. Muñoz Rivera, "Decay rates of solutions of an anisotropic inhomogeneous n-dimensional viscoelastic equation with polynomial decaying kernels," Communications in Mathematical Physics, vol. 177, no. 3, pp. 583–602, 1996.
[2] J. E. Muñoz Rivera, "Asymptotic behaviour in linear viscoelasticity," Quarterly of Applied Mathematics, vol. 52, no. 4, pp. 628–648, 1994.
[3] J. E. Muñoz Rivera and A. Peres Salvatierra, "Asymptotic behaviour of the energy in partially viscoelastic materials," Quarterly of Applied Mathematics, vol. 59, no. 3, pp. 557–578, 2001.
[4] M. L. Santos, "Asymptotic behavior of solutions to wave equations with a memory conditions at the boundary," Electronic Journal of Differential Equations, vol. 73, pp. 1–11, 2001.
[5] M. M. Cavalcanti and H. P. Oquendo, "Frictional versus viscoelastic damping in a semilinear wave equation," SIAM Journal on Control and Optimization, vol. 42, no. 4, pp. 1310–1324, 2003.
[6] S. A. Messaoudi, "General decay of solutions of a viscoelastic equation," Journal of Mathematical Analysis and Applications, vol. 341, no. 2, pp. 1457–1467, 2008.
system, to appear in Math,” *Mathematical Methods in the Applied Sciences*, vol. 43, no. 5, pp. 2626–2645, 2019.

[42] J. H. Hassan, S. A. Messaoudi, and M. Zahri, *Existence and New General Decay Results for a Viscoelastic-type Timoshenko System*, to appear in *Zeitschrift für Analysis und ihre Anwendungen*, 2019.