EULER CHARACTERISTICS OF ALGEBRAIC VARIETIES

SYLVAIN E. CAPPELL, LAURENTIU MAXIM, AND JULIUS L. SHANESON

Abstract. This note studies the behavior of Euler characteristics and of intersection homology Euler characteristics under proper morphisms of algebraic (or analytic) varieties. The methods also yield, for algebraic (or analytic) varieties, formulae comparing these two kinds of Euler characteristics. The main results are direct consequences of the calculus of constructible functions and Grothendieck groups of constructible sheaves. Similar formulae for Hodge theoretic invariants of algebraic varieties under morphisms were announced by the first and third authors in [3, 13].

1. Introduction

We study the behavior of (intersection homology) Euler characteristics under proper morphisms of complex algebraic (or analytic) varieties. We begin by discussing simple formulae for the usual Euler-Poincaré characteristic, then show that similar formulae hold for the intersection homology Euler characteristic, as well as for the corresponding Chern homology classes of MacPherson. The methods used in the present paper also yield formulae expressing the Euler characteristics of usual and intersection homology of an algebraic (or analytic) variety in terms of each other and corresponding invariants of the subvarieties formed by the closures of its singular strata.

The main results of this note are direct applications of the standard calculus of constructible functions and Grothendieck groups of constructible sheaves. Some of the formulae on the intersection homology Euler characteristic were originally proven (cf. [4]) with the aid of the deep BBDG decomposition theorem for the pushforward of an intersection homology complex under a proper morphism (cf. [1, 7]). The functorial approach employed here was suggested by the referee. However, the core calculations used in proving these results are modeled on our original approach based on BBDG.

This note is a first step in an ongoing project that deals with the study of genera of complex algebraic (or analytic) varieties. In the forthcoming papers [5] and [6] we will discuss the behavior of Hodge theoretic genera under proper morphisms, and provide explicit formulae for the pushforward of various characteristic classes. The functorial approach and the language of Grothendieck groups of constructible sheaves used in this paper allow a simple translation of the underlying ideas to the forthcoming papers, where Grothendieck groups of Saito's algebraic mixed Hodge modules will be employed.

Date: March 2, 2022.

The first and third authors partially supported by grants from NSF and DARPA. The second author partially supported by a grant from the NYU Research Challenge Fund.
Unless otherwise specified, all homology and intersection homology groups in this paper are those with rational coefficients. We assume the reader's familiarity with intersection homology, and for some arguments also with (Grothendieck groups of) constructible sheaves and derived categories. However, our results are also explained in the simpler language of constructible functions, which only relies on Euler characteristic information.

Acknowledgements. We are grateful to the anonymous referee for his valuable comments and suggestions regarding the functorial approach used in the present paper.

2. Topological Euler-Poincaré characteristic

For a complex algebraic variety $X$, let $\chi(X)$ denote its topological Euler characteristic. Then $\chi(X)$ equals the compactly supported Euler characteristic, $\chi_c(X)$ (cf. [9], p.141), [14], §6.0.6). The additivity property for the Euler characteristic reads as follows: for $Z$ a Zariski closed subset of $X$, the long exact sequence of the compactly supported cohomology

$$\cdots \to H^i_c(X \setminus Z) \to H^i_c(X) \to H^i_c(Z) \to H^{i+1}_c(X \setminus Z) \to \cdots$$

yields that $\chi_c(X) = \chi_c(Z) + \chi_c(X \setminus Z)$, therefore the same relation holds for $\chi$. The multiplicativity property for fibrations asserts that if $F \to E \to B$ is a locally trivial topological fibration such that the three Euler characteristics $\chi(B)$, $\chi(F)$ and $\chi(E)$ are defined, then $\chi(E) = \chi(B) \cdot \chi(F)$ (e.g., see [8], Corollary 2.5.5). In particular, if $f : X \to Y$ is a proper smooth submersion of smooth manifolds, with $Y$ connected and generic fiber $F$, then:

$$\chi(X) = \chi(Y) \cdot \chi(F).$$

(2.1)

Indeed, by Ehresmann’s theorem, such a map is a locally trivial fibration in the complex topology.

In this note we generalize this multiplicative property of proper smooth submersions in two different directions: first, we study the behavior of the usual Euler characteristic under arbitrary proper maps of possibly singular varieties; second, we replace the usual cohomology by intersection cohomology when dealing with singular varieties, and study the behavior of the intersection homology Euler characteristic under arbitrary proper morphisms. The formulae we obtain here are classically referred to as the stratified multiplicative property for Euler characteristics (cf. [3, 13]).

Let $Y$ be a topological space with a finite partition $\mathcal{V}$ into a disjoint union of finitely many connected subsets $V$ satisfying the frontier condition: “$V \cap \bar{W} \neq \emptyset$ implies that $V \subset \bar{W}$”. (The main examples of such spaces are complex algebraic or compact analytic varieties with a fixed Whitney stratification.) Then $\mathcal{V}$ is partially ordered by “$V \leq W$ if and only if $V \subset \bar{W}$”. Let $F_\mathcal{V}(Y)$ be the abelian group of $\mathcal{V}$-constructible functions on $Y$. This fact is not true outside the category of complex varieties (e.g., if $X$ is an oriented $n$-dimensional topological manifold, then Poincaré duality yields that $\chi_c(X) = (-1)^n \chi(X)$).
$Y$, i.e., of functions $\alpha : Y \to \mathbb{Z}$ such that $\alpha|_V$ is constant for all $V \in \mathcal{V}$. This is a free abelian group with basis $\{1_V|V \in \mathcal{V}\}$, so that

$$\alpha = \sum_{V \in \mathcal{V}} \alpha(V) \cdot 1_V.$$  

Note that $\{1_V|V \in \mathcal{V}\}$ is another basis for $F_{\mathcal{V}}(Y)$, since

$$1_{\bar{V}} = \sum_{W \leq V} 1_W$$

and the matrix $A = (a_{W,V})$, with $a_{W,V} := 1$ for $W \leq V$ and 0 otherwise, is upper triangular with respect to $\leq$, with all diagonal entries equal to 1. Thus $A$ is invertible. The non-zero entries of $A^{-1} = (a'_{W,V})$ can inductively be calculated (e.g., see [16], Prop. 3.6.2) by $a'_{V,V} = 1$ and, for $W < V$,

$$a'_{W,V} = - \sum_{W \leq S < V} a'_{W,S} \cdot a_{S,V}.$$  

This implies the following

**Proposition 2.1.** For each $V \in \mathcal{V}$, define inductively $\hat{1}_V$ by the formula

$$\hat{1}_V = 1_{\bar{V}} - \sum_{W < V} \hat{1}_W.$$  

Then, for any $\alpha \in F_{\mathcal{V}}(Y)$, one has the equality

(2.2) $$\alpha = \sum_{V} \alpha(V) \cdot \hat{1}_V.$$  

**Proof.** As the notation indicates, $\hat{1}_V$ depends only on the space $\bar{V}$ with its induced partition. Then by the above considerations we have

$$\alpha = \sum_{V} \alpha(V) \cdot 1_V = \sum_{W \leq V} \alpha(V) \cdot a'_{W,V} \cdot 1_{\bar{W}},$$

and formula (2.2) follows from the inductive identification (for $V$ fixed):

$$\sum_{W \leq V} a'_{W,V} \cdot 1_{\bar{W}} = 1_{\bar{V}} - \sum_{W \leq S < V} a'_{W,S} \cdot a_{S,V} \cdot 1_{\bar{W}} = \hat{1}_V.$$  

□

**Remark 2.2.** (1) If there is a stratum $S \in \mathcal{V}$ which is dense in $Y$, i.e., $\bar{S} = Y$, or $V \leq S$ for all $V \in \mathcal{V}$, then formula (2.2) can be rewritten as

(2.3) $$\alpha = \alpha(S) \cdot 1_Y + \sum_{V < S} (\alpha(V) - \alpha(S)) \cdot \hat{1}_V.$$
(2) For a group homomorphism \( \phi : F_Y(Y) \to G \) for some abelian group \( G \), one obtains similar descriptions for \( \phi(\alpha) \) in terms of

\[
\hat{\phi}(V) := \phi(1_V) = \phi(1_V) - \sum_{W < V} \phi(1_W).
\]

For the rest of this section we specialize to the complex algebraic (or compact analytic context), with \( Y \) a reduced complex algebraic variety (or a reduced compact complex analytic space), and all \( V \in \mathcal{V} \) locally closed constructible subsets. The group \( F_c(Y) \) of all complex algebraically (resp. analytically) constructible functions is defined as the direct limit of these \( F_c(V) \) (e.g., see [11, 12, 14, 15]):

1. The Euler characteristic with compact support \( \chi_c : F_c(Y) \to \mathbb{Z} \) characterized by \( \chi_c(1_Z) = \chi_c(Z) \) for \( Z \subset Y \) a locally closed constructible subset.
2. The Euler characteristic \( \chi : F_c(Y) \to \mathbb{Z} \) characterized by \( \chi(1_Z) = \chi(Z) \) for \( Z \subset Y \) a closed algebraic (resp. analytic) subset.
3. For \( f : X \to Y \) a proper complex algebraic (resp. analytic) map, the functorial pushdown \( f_* : F_c(X) \to F_c(Y) \) is characterized by \( f_*(1_Z)(y) = \chi(Z \cap \{ f = y \}) \) for \( Z \subset X \) a closed algebraic (resp. analytic) subset.
4. The Chern-MacPherson class transformation \( c_* : F_c(Y) \to H^{BM}_2(Y; \mathbb{Z}) \), which commutes with proper pushdowns, and is uniquely characterized by this property together with the normalization \( c_*(1_M) = c^*(TM) \cap [M] \), for \( M \) a complex algebraic (resp. analytic) manifold.

In fact, as already pointed out, in this context we have \( \chi = \chi_c \). Moreover, \( \chi \circ f_* = \chi \), and for \( Y \) compact one gets by functoriality \( \chi(\alpha) = \deg(c_*(\alpha)) \), for any \( \alpha \in F_c(Y) \).

Now let \( f : X \to Y \) be a proper complex algebraic (resp. analytic) map, with \( Y \) as above. Assume \( f_*(\alpha) \in F_Y(Y) \) for a given \( \alpha \in F_c(X) \) (e.g., \( \mathcal{Y} \) and \( \mathcal{V} \) are complex Whitney stratifications of \( X \) and resp. \( Y \), such that \( f \) is a stratified submersion, and \( \alpha \in F_Y(X) \)). Then

\[
f_*(\alpha) = \sum_{V \in \mathcal{V}} f_*(\alpha)(V) \cdot 1_V,
\]

with

\[
f_*(\alpha)(V) = \chi(\alpha|_{F_Y}),
\]

for \( F_Y \) the fiber of \( f \) over a point in \( V \). Assume, moreover, that \( Y \) is irreducible, so there is a dense stratum \( S \in \mathcal{V} \), with \( F := F_S \) a general fiber of \( f \). In terms of \( f_*(\alpha) \), formula (2.3) yields the following:

\[
f_*(\alpha) = \chi(\alpha|_F) \cdot 1_Y + \sum_{V < S} (\chi(\alpha|_{F_Y}) - \chi(\alpha|_F)) \cdot 1_V.
\]

By applying the homomorphism \( \chi \) and resp. \( c_* \) to the equation (2.4), we obtain the following formulae:
Corollary 2.3.
\begin{equation}
\chi(\alpha) = \chi(\alpha|_F) \cdot \chi(Y) + \sum_{V < S} (\chi(\alpha|_{F_V}) - \chi(\alpha|_F)) \cdot \hat{\chi}(\bar{V}).
\end{equation}

(2.5)

\begin{equation}
f_*(c_*(\alpha)) = \chi(\alpha|_F) \cdot c_*(Y) + \sum_{V < S} (\chi(\alpha|_{F_V}) - \chi(\alpha|_F)) \cdot \hat{c}_*(\bar{V}),
\end{equation}

(2.6)

where \(c_*(Y) := c_*(1_Y)\), and similarly for \(\hat{c}_*(\bar{V})\), which by the functoriality of \(c_*\) is regarded as a homology class in the Borel-Moore homology \(H^{BM}_*(Y; \mathbb{Z})\).

By letting \(\alpha = 1_X\) in the formulae (2.5) and resp. (2.6) above, we obtained the stratified multiplicative property for the topological Euler-Poincaré characteristic and resp. for the Chern-MacPherson class:

Proposition 2.4. Let \(f : X \to Y\) be a proper complex algebraic (resp. analytic) map, with \(Y\) irreducible (and compact in the analytic context) and endowed with a complex algebraic (or analytic) Whitney stratification \(\mathcal{V}\). Assume \(f_*(1_X) \in F_*(Y)\). Then:

\begin{equation}
\chi(X) = \chi(F) \cdot \chi(Y) + \sum_{V < S} (\chi(F_V) - \chi(F)) \cdot \hat{\chi}(\bar{V}).
\end{equation}

(2.7)

(2.8)

\begin{equation}
f_*(c_*(X)) = \chi(F) \cdot c_*(Y) + \sum_{V < S} (\chi(F_V) - \chi(F)) \cdot \hat{c}_*(\bar{V}).
\end{equation}

3. Intersection homology Euler characteristics

Let \(Y\) be a topological pseudomanifold (or a locally cone-like stratified space, \([14],\) p.232), with a stratification \(\mathcal{V}\) by finitely many oriented strata of even dimension. By definition, strata of \(\mathcal{V}\) satisfy the frontier condition, and \(\mathcal{V}\) is locally topologically trivial along each stratum \(V\), with fibers the cone on a compact pseudomanifold \(L_{V,Y}\), the “link” of \(V\) in \(Y\). Note that each stratum \(V\), its closure \(\bar{V}\), and in general any locally closed union of strata gets an induced stratification of the same type. Examples are given by a complex algebraic (or analytic) Whitney stratification of a reduced complex algebraic (or compact complex analytic) variety.

Let \(\text{Sh}_\mathcal{V}(Y)\) be the category of \(\mathcal{V}\)-constructible sheaves of rational vector spaces, i.e., sheaves \(\mathcal{F}\) with the property that for all \(V \in \mathcal{V}\) the restriction \(\mathcal{F}|_V\) is a locally constant sheaf of \(\mathbb{Q}\)-vector spaces, with finite dimensional stalks. Denote by \(D^b_\mathcal{V}(Y)\) the corresponding derived category of bounded complexes with \(\mathcal{V}\)-constructible cohomology sheaves (compare \([2, 11, 14]\)). Then one has an equality of Grothendieck groups (e.g. compare \([11],\) p. 77, \([14],\) Lemma 3.3.1)

\[ K_0(\text{Sh}_\mathcal{V}(Y)) = K_0(D^b_\mathcal{V}(Y)) \]

obtained by identifying the class of a complex with the alternating sum of the classes of its cohomology sheaves. Moreover one has a canonical group epimorphism

\[ \chi_Y : K_0(D^b_\mathcal{V}(Y)) \to F_*(Y) \]
defined by taking stalkwise the Euler characteristic. Note that $\chi_Y$ is not injective in general (e.g., see [5], p.98), except for when all strata $V \in \mathcal{V}$ are simply-connected, e.g. for $Y = \{pt\}$, in which case we use the shorter notion $K_0(pt)$. So $K_0(pt)$ is just the Grothendieck group of finite dimensional $\mathbb{Q}$-vector spaces, and it is a commutative ring with respect to tensor product, with unit $\mathbb{Q}_{pt}$. Moreover, there is an isomorphism $K_0(pt) \cong \mathbb{Z}$ induced by the Euler characteristic homomorphism. $K_0(D^b_Y(Y))$ becomes a unitary $K_0(pt)$-module, with the multiplication defined by the exterior product:

$$K_0(D^b_Y(Y)) \times K_0(pt) \to K_0(D^b_{Y \times \{pt\}}(Y \times \{pt\})) = K_0(D^b_Y(Y)),$$

and the Euler characteristic homomorphisms $\chi_Y$ and $\chi$ are compatible with this structure (more generally $\chi_Y$ commutes with exterior products).

Important examples of $\mathcal{V}$-constructible complexes are provided by the intersection cohomology complexes $IC_Y$ of the closures of the strata $V \in \mathcal{V}$, extended by 0 to all of $Y$ (cf. [1, 2, 10]). These are selfdual with respect to Verdier duality (and become important in the context of perverse sheaves and mixed Hodge modules, as in our forthcoming papers [5] and [6]). The normalization axiom for $IC_Y$ (in the conventions of [1]) yields that $IC_Y|_V = \mathbb{Q}_V[\dim(V)]$, with $\dim(V) := \dim_{\mathbb{R}}(V)/2$ (the complex dimension in the complex algebraic/analytic context). Since we work in Grothendieck groups, in order to avoid signs in our calculations, we will use the normalization condition of [2], that is, we work with $IC'_Y := IC_Y[-\dim(V)]$, whose hypercohomology is exactly the intersection cohomology of $V$.

Let us fix for each $W \in \mathcal{V}$ a point $w \in W$ with inclusion $i_w : \{w\} \hookrightarrow Y$. Then

$$(3.1) \quad i^*_w[IC'_Y] = [i^*_wIC'_W] = [\mathbb{Q}_{pt}] \in K_0(w) = K_0(pt),$$

and $i^*_w[IC'_Y] \neq [0] \in K_0(pt)$ only if $W \leq V$. If we let

$$(3.2) \quad ic_Y := \chi_Y(IC'_Y) \in F_Y(Y)$$

be the corresponding constructible function, then

$$(3.3) \quad supp(ic_Y) = \bar{V} \quad \text{and} \quad ic_Y|_V = 1_V.$$ 

Note that $ic_Y(w)$ does not depend on the choice of $w \in W$, and this is also the case for $i_w^*[IC_Y] \in K_0(pt)$. In fact, since for any $j \in \mathbb{Z}$,

$$\mathcal{H}^j(i^*_wIC'_Y) \simeq IH^j(c^0L_{W,V})$$

with $c^0L_{W,V}$ the open cone on the link $L_{W,V}$ of $W$ in $\bar{V}$ for $W \leq V$ (cf. [2], p.30, Prop.4.2), we have that

$$i^*_w[IC'_Y] = [IH^*(c^0L_{W,V})] \in K_0(pt).$$

In terms of constructible functions, this gives

$$(3.4) \quad ic_Y(w) = I(\chi(c^0L_{W,V}) := \chi([IH^*(c^0L_{W,V})]).$$

In particular, $\{ic_Y|V \in \mathcal{V}\}$ is another distinguished basis of $F_Y(Y)$ since, by $(3.3)$, the transition matrix to the basis $\{1_V\}$ is upper triangular with respect to $\leq$, with all diagonal entries equal to 1. Moreover, by $(3.1)$ the $K_0(pt)$-submodule $\langle [IC'_Y] \rangle$ of $K_0(D^b_Y(Y))$
generated by the elements \([IC'_V]\) \((V \in V)\) is in fact freely generated by them, and the restriction
\[(3.5) \quad \chi_Y : \langle [IC'_V]\rangle \to F_Y(Y)\]
is an isomorphism.

The main technical result of this section is the following

**Theorem 3.1.** Assume \(Y\) has an open dense stratum \(S \in V\) so that \(V \leq S\) for all \(V\). For each \(V \in V \setminus \{S\}\) define inductively
\[(3.6) \quad \hat{IC}(\hat{V}) := [IC'_V] - \sum_{W < V} \hat{IC}(\hat{W}) \cdot \iota^*_w[IC'_V] \in K_0(D^b_Y(V)),\]
and similarly
\[(3.7) \quad \hat{c}(\hat{V}) := ic_V - \sum_{W < V} \hat{c}(\hat{W}) \cdot I\chi(c^\circ L_{W,Y}) \in F_Y(Y)\]
so that \(\chi_Y(\hat{IC}(\hat{V})) = \hat{c}(\hat{V}).\) As the notation suggests, \(\hat{IC}(\hat{V})\) and \(\hat{c}(\hat{V})\) depend only on the stratified space \(\hat{V}\) with its induced stratification.

1. Assume \([\mathcal{F}] \in K_0(D^b_Y(V))\) is an element of the \(K_0(\text{pt})\)-submodule \(\langle [IC'_V] \rangle\). Then
\[(3.8) \quad [\mathcal{F}] = [IC'_V] \cdot i^*_s[\mathcal{F}] + \sum_{V < S} \hat{IC}(\hat{V}) \cdot (i^*_v[\mathcal{F}] - i^*_s[\mathcal{F}] \cdot i^*_v[IC'_V]) \in K_0(D^b_Y(V)).\]

2. For any \(V\)-constructible function \(\alpha \in F_Y(Y)\), one has the equality
\[(3.9) \quad \alpha = \alpha(s) \cdot ic_Y + \sum_{V < S} (\alpha(v) - \alpha(s) \cdot I\chi(c^\circ L_{V,Y})) \cdot \hat{c}(\hat{V}).\]

**Proof.** Note that the equation \((3.9)\) of the second part of the theorem is a direct consequence of formula \((3.8)\) from the first part. Indeed, by \((3.5)\) we can first represent any \(\alpha \in F_Y(Y)\) as \(\alpha = \chi_Y([\mathcal{F}])\), for some \([\mathcal{F}] \in \langle [IC'_V] \rangle\). Then, assuming \((3.8)\) holds for this choice of \([\mathcal{F}]\), we can apply \(\chi_Y\) to this equation and obtain \((3.9)\).

In order to prove formula \((3.8)\), consider
\[(3.10) \quad [\mathcal{F}] = \sum_{V} [IC'_V] \cdot L(V),\]
for some \(L(V) \in K_0(\text{pt})\). The aim is to identify these coefficients \(L(V)\). Since \(S\) is an open stratum, by applying \(i^*_s\) to \((3.10)\) we obtain:
\[i^*_s[\mathcal{F}] = L(S) \in K_0(s) = K_0(\text{pt}).\]

Next fix a stratum \(W \neq S\), and apply \(i^*_w\) to \((3.10)\). Recall that \(i^*_w[IC'_W] = [\mathbb{Q}_{pt}] \in K_0(w) = K_0(\text{pt})\), and \(i^*_w[IC'_V] \neq [0] \in K_0(\text{pt})\) only if \(W \leq V\). We obtain
\[(3.11) \quad i^*_w[\mathcal{F}] = L(W) + \sum_{W < V} i^*_w[IC'_V] \cdot L(V) \in K_0(w) = K_0(\text{pt}).\]
Since $S$ is dense, we have that $W < S$, so the stratum $S$ appears in the summation on the right hand side of (3.11). Therefore

\[(3.12) \quad i^*_w[F] - i^*_w[IC'_Y] \cdot i^*[F] = L(W) + \sum_{W < V < S} i^*_w[IC'_V] \cdot L(V) \in K_0(w) = K_0(pt).\]

This implies that we can inductively calculate $L(V)$ in terms of

\[L'(W) := i^*_w[F] - i^*_w[IC'_Y] \cdot i^*[F].\]

Indeed, (3.12) can be rewritten as

\[(3.13) \quad L'(W) = \sum_{W \leq V < S} i^*_w[IC'_V] \cdot L(V) \in K_0(pt),\]

and the matrix $A = (a_{W,V})$, with $a_{W,V} := i^*_w[IC'_V] \in K_0(pt)$ for $W, V \in \mathcal{V} \setminus \{S\}$, is upper-triangular with respect to $\leq$, with ones on the diagonal. So $A$ can be inverted. The non-zero coefficients of $A^{-1} = (a'_{W,V})$ can inductively be calculated by $a'_{W,V} = 1$ and

\[(3.14) \quad a'_{W,V} = - \sum_{W \leq T < V} a'_{W,T} \cdot a_{T,V}\]

for $W < V$. Then (3.10) becomes

\[(3.15) \quad [F] = [IC'_Y] \cdot i^*[F] + \sum_{W < S} [IC'_W] \cdot L(W) = [IC'_Y] \cdot i^*[F] + \sum_{W \leq V < S} [IC'_W] \cdot a'_{W,V} \cdot L'(V).\]

The result follows by the inductive identification (for $V < S$ fixed):

\[\sum_{W \leq V} [IC'_W] \cdot a'_{W,V} = [IC'_Y] - \sum_{W \leq T < V} [IC'_W] \cdot a'_{W,T} \cdot a_{T,V} = \hat{IC}(V).\]

Remark 3.2. In this paper, we only make use of the equation (3.9), and this could be proven directly by working in $F_V(Y)$, following the same arguments as above. However, the formula of equation (3.8) is particularly important since in the complex algebraic context it extends to the framework of Grothendieck groups of algebraic mixed Hodge modules that will be used in our forthcoming paper [5]. Of course, the technical condition used in proving formula (3.8) is not generally satisfied, but it holds under the assumption of trivial monodromy along all strata $V \in \mathcal{V}$ (e.g., if all strata $V$ are simply-connected). For more details, see [5].

For the remaining part of this section, we will specialize to the complex algebraic (or compact complex analytic) context, that is, $Y$ is a reduced complex algebraic variety (or a reduced compact complex analytic space), with a complex algebraic (resp. analytic) Whitney stratification $\mathcal{V}$. In this setting, let $f : X \to Y$ be a proper complex algebraic (or analytic) map. Assume $f_*(\alpha) \in F_V(Y)$ for a given $\alpha \in F_*(X)$, e.g., we choose $\mathcal{V}'$
Euler characteristics

and $\mathcal{V}$ complex Whitney stratifications of $X$ and resp. $Y$ such that $f$ is a stratified submersion, and $\alpha \in F_Y(X)$. Then

$$f_\ast(\alpha) = \sum_{V \in \mathcal{V}} f_\ast(\alpha)(V) \cdot 1_V,$$

with

$$f_\ast(\alpha)(V) = \chi(\alpha|_{F_Y}),$$

for $F_Y$ the fiber of $f$ over a point in $V$. Assume, moreover, that $Y$ is irreducible, so there is a dense stratum $S \in \mathcal{V}$ with $F := F_S$ a general fiber of $f$. In terms of $f_\ast(\alpha)$, the equation (3.9) of Theorem 3.1 becomes

$$f_\ast(\alpha)(v) = \chi(\alpha|_{F_Y}) + \sum_{V < S} (\chi(\alpha|_{F_Y}) - \chi(\alpha|_{F_Y}) \cdot I\chi(c^\circ L_{V,Y})) \cdot \tilde{c}(V) \in F_Y(Y).$$

(3.16)

By applying the group homomorphism $\chi$ and resp. $c_\ast$ to the equation (3.16), we obtain the following (recall $\chi \circ f_\ast = \chi$ and $c_\ast \circ f_\ast = f_\ast \circ c_\ast$ for $f$ proper):

**Corollary 3.3.**

$$\chi(\alpha) = \chi(\alpha|_{F_Y}) \cdot I\chi(Y) + \sum_{V < S} (\chi(\alpha|_{F_Y}) - \chi(\alpha|_{F_Y}) \cdot I\chi(c^\circ L_{V,Y})) \cdot \tilde{I}\chi(V).$$

(3.17)

$$f_\ast(c_\ast(\alpha)) = \chi(\alpha|_{F_Y}) \cdot Ic_\ast(Y) + \sum_{V < S} (\chi(\alpha|_{F_Y}) - \chi(\alpha|_{F_Y}) \cdot I\chi(c^\circ L_{V,Y})) \cdot \tilde{I}c_\ast(V).$$

(3.18)

Here $I\chi(Y) := \chi(ic_Y) = \chi([I\pi^*(Y; \mathbb{Q})])$ is the intersection homology Euler characteristic of $Y$, and similarly for $\tilde{I}\chi(V)$. Also $Ic_\ast(Y) := c_\ast(ic_Y)$, and similarly for $\tilde{I}c_\ast(V)$, which by functoriality is regarded as a homology class in $H_{2\ast}^{BM}(Y; \mathbb{Z})$, the even degree Borel-Moore homology of $Y$.

By letting $\alpha = 1_X$ in the formulae (3.17) and resp. (3.18) above, we obtain:

**Proposition 3.4.** Let $f : X \to Y$ be a proper complex algebraic (resp. analytic) map, with $Y$ irreducible (and compact in the analytic context) and endowed with a complex algebraic (or analytic) Whitney stratification $\mathcal{V}$. Assume $f_\ast(1_X) \in F_Y(Y)$. Then:

$$\chi(X) = \chi(F) \cdot I\chi(Y) + \sum_{V < S} (\chi(F_V) - \chi(F) \cdot I\chi(c^\circ L_{V,Y})) \cdot \tilde{I}\chi(V).$$

(3.19)

$$f_\ast(c_\ast(X)) = \chi(F) \cdot Ic_\ast(Y) + \sum_{V < S} (\chi(F_V) - \chi(F) \cdot I\chi(c^\circ L_{V,Y})) \cdot \tilde{I}c_\ast(V).$$

(3.20)

In the special case when $f$ is the identity map, the equation (3.19) yields a formula expressing the Euler characteristics of usual and intersection homology of an algebraic (or analytic) variety in terms of each other and corresponding invariants of the subvarieties formed by the closures of its singular strata. Similarly, (3.20) yields in this case a comparison between the corresponding Chern homology classes of MacPherson:
Corollary 3.5. Let \( Y \) be an irreducible complex algebraic (or compact analytic) variety endowed with a complex algebraic (or analytic) Whitney stratification \( \mathcal{V} \). Then in the above notations we have:

\[
\chi(Y) = I\chi(Y) + \sum_{V < S} (1 - I\chi(c^oL_{V,Y})) \cdot \hat{I}\chi(\bar{V}).
\]

\[
c_*(Y) = Ic_*(Y) + \sum_{V < S} (1 - I\chi(c^oL_{V,Y})) \cdot \hat{I}c_*(\bar{V}).
\]

The stratified multiplicative property for the intersection homology Euler characteristic and for the corresponding homology characteristic classes is obtained from (3.17) and resp. (3.18) above in the case when \( \alpha = ic_X \). Indeed, we have:

Proposition 3.6. Let \( f : X \to Y \) be a proper complex algebraic (resp. analytic) map, with \( X \) pure dimensional and \( Y \) irreducible (and compact in the analytic context). Assume \( Y \) is endowed with a complex algebraic (or analytic) Whitney stratification \( \mathcal{V} \) so that \( f^*(ic_X) \in F_{\mathcal{V}}(Y) \). Then:

\[
I\chi(X) = I\chi(F) \cdot I\chi(Y) + \sum_{V < S} (I\chi(f^{-1}(c^oL_{V,Y})) - I\chi(F) \cdot I\chi(c^oL_{V,Y})) \cdot \hat{I}\chi(\bar{V}).
\]

\[
f_*(Ic_*(X)) = I\chi(F) \cdot Ic_*(Y) + \sum_{V < S} (I\chi(f^{-1}(c^oL_{V,Y})) - I\chi(F) \cdot I\chi(c^oL_{V,Y})) \cdot \hat{I}c_*(\bar{V}).
\]

Proof. Based on the above considerations, it suffices to show that

\[
\chi(ic_X|_F) = I\chi(F)
\]

and

\[
\chi(ic_X|_{F_V}) = I\chi(f^{-1}(c^oL_{V,Y})).
\]

Since the general fiber \( F \) of \( f \) is locally normally nonsingular embedded in \( X \), we have a quasi-isomorphism (\[10\], §5.4.1):

\[
IC'_X|_F \simeq IC'_F,
\]

hence an equality \( ic_X|_F = ic_F \), thus proving (3.25).

Similarly, since \( \chi(ic_X|_{F_V}) = f_*(ic_X)(v) \), for some \( v \in V \), in order to prove (3.26), it suffices to show that

\[
\mathcal{H}^j(Rf_*IC'_X)_v \cong IH^j(f^{-1}(c^oL_{V,Y}); \mathbb{Q}).
\]

Let \( N \) be a normal slice to \( V \) at \( v \) in local analytic coordinates \((Y, v) \mapsto (\mathbb{C}^n, v)\), that is, a germ of a complex manifold \((N, v) \mapsto (\mathbb{C}^n, v)\), intersecting \( V \) transversally only at \( v \), and with \( \dim V + \dim N = n \). Recall that the link \( L_{V,Y} \) of the stratum \( V \) in \( Y \) is defined as

\[
L_{V,Y} := Y \cap N \cap \partial B_r(v),
\]
where $B_r(v)$ is an open ball of (very small) radius $r$ around $v$. Moreover, $Y \cap N \cap B_r(v)$ is isomorphic (in a stratified sense) to the open cone $c^\circ L_{V,Y}$ on the link (2, p. 44). By factoring $i_v$ as the composition:

$$
\{v\} \xrightarrow{\phi} Y \cap N \xrightarrow{\psi} Y
$$

we can now write:

$$
H^j(Rf_*IC'_X) \sim H^j(Y, i_v \circ Rf_*IC'_X) \sim H^j(\psi^* Rf_*IC'_X) \sim H^j(c^\circ L_{V,Y}, Rf_*IC'_X) \sim H^j(f^{-1}(c^\circ L_{V,Y}), IC'_X) \sim (1) \sim IH^j(f^{-1}(c^\circ L_{V,Y}); \mathbb{Q})
$$

where in (1) we used the fact that the inverse image of a normal slice to a stratum of $Y$ in a stratification of $f$ is (locally) normally non-singular embedded in $X$ (this fact is a consequence of first isotopy lemma).

\[ \square \]

References

[1] Beilinson, A. A., Bernstein, J., Deligne, P., *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[2] Borel, A. et. al., *Intersection cohomology*, Progress in Mathematics, vol. 50, Birkhäuser Boston, Boston, MA, 1984.

[3] Cappell, S. E., Shaneson, J. L., *Genera of algebraic varieties and counting of lattice points*, Bull. Amer. Math. Soc. (N.S.) 30 (1994), no. 1, 62–69.

[4] Cappell, S. E., Maxim, L. G., Shaneson, J. L., *Euler characteristics of algebraic varieties*, math.AT/0606654, v2.

[5] Cappell, S. E., Maxim, L., Shaneson, J. L., *Hodge genera of algebraic varieties, I.*, to appear in Comm. in Pure and Applied Math.

[6] Cappell, S. E., Libgober, A., Maxim, L., Shaneson, J. L., *Hodge genera of algebraic varieties, II.*, math.AG/0702380.

[7] de Cataldo, M. A., Migliorini, L., *The Hodge theory of algebraic maps*, Ann. Scient. Ec. Norm. Sup., 4e serie, t.38, 2005, p. 693-750.

[8] Dimca, A., *Sheaves in Topology*, Universitext, Springer-Verlag, Berlin, 2004.

[9] Fulton, W., *Introduction to toric varieties*, Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[10] Goresky, M., MacPherson, M., *Intersection Homology II*, Invent. Math., 71 (1983), 77-129.

[11] Kashiwara, M., Schapira, P., *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, 292. Springer-Verlag, Berlin, Heidelberg, 1990.

[12] MacPherson, R., *Chern classes for singular algebraic varieties*, Ann. of Math. (2) 100 (1974), 423–432.
[13] Shaneson, S., *Characteristic classes, lattice points and Euler-MacLaurin formulae*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 612–624, Birkhäuser, Basel, 1995.

[14] Schürmann, J., *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne, 63. Birkhäuser Verlag, Basel, 2003.

[15] Schürmann, J., Yokura, S., *A survey of characteristic classes of singular spaces*, in “Singularity Theory” (ed. by D. Chéniot et al), Dedicated to Jean Paul Brasselet on his 60th birthday, Proceedings of the 2005 Marseille Singularity School and Conference, World Scientific, 2007, 865-952.

[16] Stanley, R.P., *Enumerative combinatorics, Volume 1*, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, 2002.

S. E. Cappell: Courant Institute, New York University, New York, NY-10012
E-mail address: cappell@cims.nyu.edu

L. Maxim: Courant Institute, New York University, New York, NY-10012
E-mail address: maxim@cims.nyu.edu

J. L. Shaneson: Department of Mathematics, University of Pennsylvania, Philadelphia, PA-19104
E-mail address: shaneson@sas.upenn.edu