SUBEXTENSIONS FOR A PERMUTATION $\text{PSL}_2(q)$-MODULE

ANDREI V. ZAVARNITSINE

Abstract. Using cohomological methods, we prove the existence of a subgroup isomorphic to $\text{SL}_2(q)$, $q \equiv -1 \pmod{4}$, in the permutation module for $\text{PSL}_2(q)$ in characteristic 2 that arises from the action on the projective line. A similar problem for $q \equiv 1 \pmod{4}$ is reduced to the determination of certain first cohomology groups.

Keywords: finite simple groups, permutation module, group cohomology

MSC2010: 20D06, 20J06, 20B25

1. Introduction

We denote by $\mathbb{F}_q$ a finite field of order $q$ and by $\mathbb{Z}_n$ a cyclic group of order $n$.

Let $q$ be an odd prime power and let $G = \text{PSL}_2(q)$. From the Universal embedding [1, Theorem 2.6.A], it follows that the regular wreath product $\mathbb{Z}_2 \wr G$ contains a subgroup isomorphic to $\text{SL}_2(q)$. It is of interest to know if the same is true for a permutation wreath product that is not necessarily regular. In particular, let $\rho$ be the natural permutation representation of $G$ of degree $q + 1$ on the projective line over $\mathbb{F}_q$. The following problem arose in the research [2].

Problem 1. Does the permutation wreath product $\mathbb{Z}_2 \wr \rho G$ contain a subgroup isomorphic to $\text{SL}_2(q)$?

Although stated in purely group-theoretic terms, this problem is cohomologic in nature. In the next section, we reformulate a generalized version of this question as an assertion about a homomorphism between second cohomology groups of group modules. We then apply some basic theory to obtain the following

Theorem 1. If $q \equiv -1 \pmod{4}$ then the answer to Problem 1 is affirmative.

It seems that the case $q \equiv 1 \pmod{4}$ is more complicated and requires some deeper considerations than those presented here. We put forward

Conjecture 1. If $q \equiv 1 \pmod{4}$ then $\text{SL}_2(q)$ is not embedded in $\mathbb{Z}_2 \wr \rho G$.

For small values $q = 5, 9, 13, 17$, Conjecture 1 was confirmed using a computer. In the last section, we show how to reduce this problem to the determination of the first cohomology groups $H^1(G, U_{\pm})$, where $U_{+}$ and $U_{-}$ are the two nontrivial absolutely irreducible $G$-modules in the principal 2-block.

Supported by the Russian Foundation for Basic Research (projects 11–01–00456, 11–01–91158, 12–01–90006, 13–01–00505); by the Federal Target Grant “Scientific and educational personnel of innovation Russia” (contract 14.740.11.0346).
2. Subextensions for group modules

Let $G$ be a group and let $L, M$ be right $G$-modules. Let
\[
0 \to L \to M
\]
and
\[
1 \to M \to E \xrightarrow{\pi} G \to 1
\]
be exact sequences of modules and groups, where the conjugation action of $E$ on $M$ agrees with the $G$-module structure, i.e. $m^e = m \cdot \pi(e)$ for all $m \in M$ and $e \in E$, and we identify $M$ with its image in $E$. Then we call $E$ an extension of $M$ by $G$. It is natural to ask if there is a subgroup $H \leq E$ such that
\[
H \cap M = L, \quad HM = E,
\]
where we implicitly identify $L$ with its image in $M$. A subgroup $H$ with these properties is itself an extension of $L$ by $G$, and will thus be called a subextension of $E$ that corresponds to the embedding (1). The classification of all such subextensions of $E$ (whenever they exist) up to equivalence is also of interest.

Recall that extensions $H_1, H_2$ of $L$ by $G$ are equivalent if there is a homomorphism $\alpha$ that makes the diagram

\[
\begin{array}{ccc}
H_1 & \to & G \\
\downarrow \alpha & & \downarrow \pi \\
L & \to & E \\
\end{array}
\]
commutative. It is known \cite{3} that the equivalence classes of such extensions are in a one-to-one correspondence with (thus are defined by) the elements of the second cohomology group $H^2(G, L)$. Furthermore, the sequence (1) gives rise to a homomorphism
\[
H^2(G, L) \xrightarrow{\phi} H^2(G, M).
\]
The following assertion is nothing more than an interpretation of this homomorphism in group-theoretic terms.

**Lemma 2.** Let $L, M$ be $G$-modules and $E$ an extension as specified above. Let $\bar{\gamma} \in H^2(G, M)$ be the element that defines $E$. Then the set of elements of $H^2(G, L)$ that define the subextensions $H$ of $E$ corresponding to the embedding (1) coincides with $\varphi^{-1}(\bar{\gamma})$, where $\varphi$ is the induced homomorphism (3). In particular, $E$ has such a subextension $H$ if and only if $\bar{\gamma} \in \text{Im} \varphi$.

**Proof.** Let $H$ be a required subextension of $E$. Choose a transversal $\tau : G \to H$ of $L$ in $H$. Then, for all $g_1, g_2 \in G$, we have $\tau(g_1)\tau(g_2) = \tau(g_1g_2)\beta(g_1, g_2)$ for a 2-cocycle $\beta \in Z^2(G, L)$ and the element $\bar{\beta} - \bar{\beta} + B^2(G, L)$ of $H^2(G, L)$ defines $H$. Let $\gamma$ be the composition of $\beta$ with the embedding (1). Then $\gamma \in Z^2(G, M)$ arises from the same transversal $\tau$ (composed with the embedding $H \to E$), hence $\gamma + B^2(G, M)$ is the element of $H^2(G, M)$ that defines $E$ which is $\bar{\gamma}$. Therefore, $\varphi(\bar{\beta}) = \bar{\gamma}$.

Conversely, let $\varphi(\bar{\beta}) = \bar{\gamma}$ for some $\bar{\beta} \in H^2(G, L)$. Then there is a representative 2-cocycle $\gamma \in Z^2(G, M)$ whose values lie in $L$ and which, when viewed as a map
$G \times G \to L$, is a 2-cocycle $\beta \in Z^2(G, L)$ representative for $\bar{\beta}$. Now $E$ can be identified with the set of pairs $(g, m)$ with $g \in G$, $m \in M$ subject to the multiplication 

$$(g_1, m_1)(g_2, m_2) = (g_1g_2, m_1 \cdot g_2 + m_2 + \beta(g_1, g_2))$$

and, if we set $H = \{(g, m) \mid g \in G, m \in L\}$, then $H$ is clearly a subextension of $E$ defined by $\bar{\beta}$. \hfill \Box

It is known that the zero element of $H^2(G, M)$ defines the split extension (which fact is also a particular case of Lemma 2 with $L = 0$). Therefore, we have

**Corollary 3.** Let $L, M$ be $G$-modules as above and let $E$ be the split extension of $M$ by $G$. Then the subextensions of $E$ that correspond to the embedding $[1]$ are defined by the elements of $\text{Ker } \varphi$, where $\varphi$ is the induced homomorphism $[3]$.  

3. Notation and auxiliary results

Basic facts of homological algebra can be found in [3, 5]. For abelian groups $A$ and $B$, we denote $\text{Hom}(A, B) = \text{Hom}_2(A, B)$ and $\text{Ext}(A, B) = \text{Ext}_2(A, B)$.

**Lemma 4** (The Universal Coefficient Theorem for Cohomology, [3, Theorem 3]). For all $i \geq 1$ and every trivial $G$-module $A$,

$$H^i(G, A) \cong \text{Hom}(H_i(G, \mathbb{Z}), A) \oplus \text{Ext}(H_{i-1}(G, \mathbb{Z}), A).$$

**Lemma 5** (Shapiro’s lemma, [5, §6.3]). Let $H \leq G$ with $|G : H|$ finite. If $V$ is an $H$-module and $i \geq 0$ then $H^i(G, V^G) \cong H^i(H, V)$, where $V^G$ is the induced $G$-module.

Given a group $G$, we denote by $M(G)$ the Schur multiplier of $G$. If $A$ is a finite abelian group and $p$ a prime then $A_{(p)}$ denotes the $p$-primary component of $A$.

**Lemma 6.** [4, Theorem 25.1] Let $G$ be a finite group, $p$ a prime, and let $P \in \text{Syl}_p(G)$. Then $M(G)_{(p)}$ is isomorphic to a subgroup of $M(P)$.

4. Projective action of $\text{PSL}_2(q)$

We denote $G = \text{PSL}_2(q)$ for $q$ odd. Let $\mathcal{P}$ be the projective line over $\mathbb{F}_q$ and let $V$ be the permutation $\mathbb{F}_2G$-module that corresponds to the natural action of $G$ on $\mathcal{P}$. The sum of the basis vectors of $V$, which are permuted by $G$, spans a $1$-dimensional submodule $I$, and we have the exact sequence

$$(4) \quad 0 \to I \to V \to W \to 0,$$

where $W \cong V/I$. The following result clarifies the composition structure of the module $V$. Let $k = \mathbb{F}_2$ be the algebraic closure of $\mathbb{F}_2$.

**Lemma 7** ([6, Lemma 1.6]). In the notation above, $I$ is the unique minimal submodule of $V$ and $W$ has a unique maximal submodule $U$ such that

$$(5) \quad 0 \to U \to W \to I \to 0$$

is a nonsplit short exact sequence. Moreover, $U \otimes k = U_+ \oplus U_-$, where $U_+$ and $U_-$ are the two nontrivial absolutely irreducible $kG$-modules in the principal $2$-block of $G$.

Using the knowledge of the Schur multiplier of $G$ we can determine $H^2(G, I)$.

**Lemma 8.** Let $q$ be an odd prime power. For $\text{PSL}_2(q)$ acting trivially on $\mathbb{Z}_2$, we have

$$H^2(\text{PSL}_2(q), \mathbb{Z}_2) \cong \mathbb{Z}_2.$$
Proof. Applying Lemma 4 for the trivial action of \( G = \text{PSL}_2(q) \) on \( \mathbb{Z}_2 \), we have

\[
H^2(G, \mathbb{Z}_2) \cong \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z}_2) \oplus \text{Ext}(H_1(G, \mathbb{Z}), \mathbb{Z}_2).
\]

Since \( H_2(G, \mathbb{Z}) = M(G) \), according to [4, Theorem 25.7], we have

\[
H_2(G, \mathbb{Z}) = \begin{cases} 
\mathbb{Z}_2, & q \neq 9, \\
\mathbb{Z}_6, & q = 9.
\end{cases}
\]

It follows that \( \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z}_2) \cong \mathbb{Z}_2 \). Since the first integral homology group \( H_1(G, \mathbb{Z}) \) is isomorphic to the abelianization \( G/G' \), we have

\[
H_1(G, \mathbb{Z}) = \begin{cases} 
0, & q \neq 3, \\
\mathbb{Z}_3, & q = 3.
\end{cases}
\]

Therefore, we always have \( \text{Ext}(H_1(G, \mathbb{Z}), \mathbb{Z}_2) = 0 \), and the claim follows. \( \square \)

We now determine the group \( H^2(G, V) \).

**Lemma 9.** Let \( V \) be the above-defined permutation module. Then we have

\[
H^2(G, V) \cong \begin{cases} 
0, & q \equiv -1 \pmod{4}, \\
\mathbb{Z}_2, & q \equiv 1 \pmod{4}.
\end{cases}
\]

**Proof.** Since the action of \( G \) on \( \mathcal{P} \) is transitive, we have \( V \cong T^G \), where \( T \) is the principal \( \mathbb{F}_2H \)-module for a point stabilizer \( H \leq G \). By Lemma 5, \( H^2(G, V) \cong H^2(H, T) \). We have \( H \cong \mathbb{F}_q \ltimes \mathbb{Z}_{(q-1)/2} \), a Frobenius group. If \( q \equiv -1 \pmod{4} \), the order \( |H| \) is odd. By the Schur-Zassenhaus theorem, every extension of a 2-group by \( H \) splits, which yields \( H^2(H, T) = 0 \). Suppose that \( q \equiv 1 \pmod{4} \). Let \( P \in \text{Syl}_2(H) \). Lemma 6 implies that \( H_2(H, \mathbb{Z})(2) \) is a subgroup of \( H_2(P, \mathbb{Z}) \) which is 0, since cyclic groups have trivial Schur multiplier. Therefore,

\[
\text{Hom}(H_2(H, \mathbb{Z}), \mathbb{Z}_2) = \text{Hom}(H_2(H, \mathbb{Z})(2), \mathbb{Z}_2) = 0.
\]

Note also that \( H_1(H, \mathbb{Z}) = \mathbb{Z}_{(q-1)/2} \). Now, \( H \) acts trivially on \( T \cong \mathbb{Z}_2 \), so we can use again the universal coefficient formula (6) to obtain

\[
H^2(H, T) = \text{Ext}(\mathbb{Z}_{(q-1)/2}, \mathbb{Z}_2) \cong \mathbb{Z}_2,
\]

since \( (q-1)/2 \) is even by assumption. This completes the proof. \( \square \)

5. **Proof of Theorem 1**

Given a permutation representation \( \rho \) of \( G = \text{PSL}_2(q) \) as in the introduction, observe that the permutation wreath product \( E = \mathbb{Z}_2 \wr_\rho G \) is isomorphic to the split extension of the permutation \( \mathbb{F}_2G \)-module \( V \) by \( G \). Since \( \text{SL}_2(q) \) is an extension of the principal \( \mathbb{F}_2G \) module \( I \) by \( G \), and \( I \) is a unique minimal submodule of \( V \) by Lemma 7, it follows that Problem 1 is equivalent to the question of whether \( \text{SL}_2(q) \) is a subextension of \( E \) that corresponds to the embedding \( I \rightarrow V \). Corollary 3 implies that such subextensions are defined by the elements of \( \text{Ker} \varphi \), where \( \varphi \) is the induced homomorphism

\[
H^2(G, I) \xrightarrow{\varphi} H^2(G, V).
\]

For \( q \equiv -1 \pmod{4} \), we have \( H^2(G, V) = 0 \) by Lemma 9. Thus, \( \text{Ker} \varphi = H^2(G, I) \cong \mathbb{Z}_2 \) by Lemma 8. The nonidentity element of \( H^2(G, I) \) defines the unique nonsplit extension \( \text{SL}_2(q) \) of \( I \) by \( G \), which is therefore a subextension of \( E \). The proof is complete.
6. Reduction in the case $q \equiv 1 \pmod{4}$

The above argument does not clarify what $\ker \varphi$ is if $q \equiv 1 \pmod{4}$, because in this case $H^2(G, V) \cong \mathbb{Z}_2$ by Lemma [9]. We will consider the long sequence

$$(8) \quad H^1(G, I) \to H^1(G, V) \to H^1(G, W) \xrightarrow{\delta} H^2(G, I) \xrightarrow{\phi} H^2(G, V)$$

induced by (4). Since this sequence is exact, it follows that $\text{Im} \, \delta = \ker \varphi$, and we might as well study the connecting homomorphism $\delta$.

**Lemma 10.** In the above notation, we have

- (i) $H^1(G, I) = 0$;
- (ii) $H^1(G, V) \cong \begin{cases} 0, & q \equiv -1 \pmod{4}, \\ \mathbb{Z}_2, & q \equiv 1 \pmod{4}. \end{cases}$

**Proof.** (i) This follows from the interpretation of $|H^1(G, I)|$ as the number of conjugacy classes of complements to $I$ in $I \rtimes G$, which is clearly 1, or from Lemma [4] for $i = 1$.

(ii) We again use the fact that $V \cong T^G$ as in the proof of Lemma [3], where $T$ is the principal $\mathbb{F}_2$-$H$-module for the Borel subgroup $H \leq G$. We have $H^1(G, V) \cong H^1(H, T)$ by Lemma [5]. Assume that $q \equiv -1 \pmod{4}$. (Although we have covered this case in the previous sections, we still consider it for the sake of completeness.) We have that $|H|$ is odd and $|H^1(H, T)| = 0$ by the Schur-Zassenhaus theorem. Let $q \equiv 1 \pmod{4}$. By Lemma [4],

$$H^1(H, T) \cong \text{Hom}(H_1(H, \mathbb{Z}), T) \oplus \text{Ext}(H_0(H, \mathbb{Z}), T),$$

where the second summand is zero as $H_0(H, \mathbb{Z}) \cong \mathbb{Z}$ and $\text{Ext}(\mathbb{Z}, T) = 0$, since $\mathbb{Z}$ is projective. Now, $H_1(H, \mathbb{Z}) \cong \mathbb{Z}_{(q-1)/2}$ and $\text{Hom}(\mathbb{Z}_{(q-1)/2}, \mathbb{Z}_2) = \mathbb{Z}_2$, since $(q-1)/2$ is even by assumption. The claim follows.

Lemma [10] implies that $\text{Im} \, \delta \cong H^1(G, W)/H^1(G, V)$ and so it remains to determine $H^1(G, W)$. To this end, we consider the long exact sequence

$$(9) \quad H^0(G, W) \to H^0(G, I) \to H^1(G, U) \to H^1(G, W) \to H^1(G, I)$$

induced by (5). We have $H^1(G, I) = 0$ by Lemma [10] and $H^0(G, W) = 0$ by Lemma [7]. Therefore,

$$H^1(G, W) \cong H^1(G, U)/H^0(G, I),$$

and, since $H^0(G, I) = \mathbb{Z}_2$ is known, in view of Lemma [7], it remains to determine the groups $H^1(G, U_{\pm})$ for the absolutely irreducible modules $U_{\pm}$. This will settle the case $q \equiv 1 \pmod{4}$.

Acknowledgement. The author is thankful to Prof. D. O. Revin for drawing attention to Problem [11] and discussing the content of this paper.

**References**

[1] J. D. Dixon, B. Mortimer, Permutation groups. Graduate Texts in Mathematics, 163. Springer-Verlag, New York, 1996, 346 p.
[2] N. V. Maslova, D. O. Revin, On composition factors of a finite group that is minimal with respect to the prime spectrum. (in preparation)
[3] K. W. Gruenberg, Cohomological topics in group theory, Lecture Notes in Mathematics, Vol. 143, Springer-Verlag, Berlin–New York, 1970, 275 p.
[4] B. Huppert, Endliche Gruppen. I, Die Grundlehren der Mathematischen Wissenschaften, Band 134 Springer-Verlag, Berlin-New York, 1967, 793 p.
[5] Ch. A. Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994, 450 p.
[6] V. P. Burichenko, Extensions of abelian 2-groups by means of $L_2(q)$ with irreducible action. Algebra and Logic 39 (2000), no. 3, 160-183.

Andrei V. Zavarnitsine, Group Theory Lab., Sobolev Institute of Mathematics, 4, Koptyug av., 630090, Novosibirsk, Russia,
E-mail address: zav@math.nsc.ru