Towards linear phononics and nonlocality tests in ion traps

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We explore the possibility to manipulate ‘massive’, i.e. motional, degrees of freedom of trapped ions. In particular, we demonstrate that, if local control of the trapping frequencies is achieved, one can reproduce the full toolbox of linear optics on radial modes. Furthermore, assuming only global control of the trapping potential, we show that unprecedented degrees of continuous variable entanglement can be obtained and that nonlocality tests with massive degrees of freedom can be carried out.

The last decade saw a boom in the development of experimental capabilities available for quantum information processing. The ability to manipulate the information of discrete variables encoded in polarisation, spin and atomic degrees of freedom has by now reached very high standards. On the other hand, the control of continuous variable degrees of freedom is still almost exclusive to light fields in quantum optics. Even though quantum optical systems rely on well established elementary quantum decision generation in such systems is strongly limited by the efficiency of parametric processes in nonlinear crystals; moreover, ‘static’ optical degrees of freedom – i.e. light resonating in cavities – are seriously affected by losses and decoherence over the typical dynamical time scales.

Here, we discuss in detail the possibility of controlling the radial motion of trapped ions, described by continuous variable quantum degrees of freedom which we will refer to as radial modes [1, 2]. We highlight the remarkable potential of such modes in view of the refined technology that has been developed in ion traps. In this experimental setting, we demonstrate that any linear optical operation can be obtained for radial modes of trapped ions by controlling the individual radial trapping frequencies. Furthermore we show that, even if only global control of the trapping potential is possible, such systems would outperform optical modes in both achievable degrees of entanglement and decoherence rates. Finally, as an application, we consider the violation of non locality tests with radial modes, and show it to be achievable with current technology, demonstrating the potential of such setups not only for information processing but also as probes of fundamental physics.

The trap – We shall consider the radial modes of n ions of mass m and charge ze in a linear Paul trap [4]. Let \( X_j \) and \( \hat{P}_j \) be the position and momentum operators associated to the radial degree of freedom of the j-th ion, which is trapped in the radial direction with angular frequency \( \omega_j \). In the following, the longitudinal trapping frequency \( \nu_l \) will be the unit of frequency and will set the unit of length as well (equal to \( \sqrt{\hbar/e^2/(4\pi\epsilon_0 m \nu_l^2)} \), where \( \epsilon_0 \) is the dielectric constant); also, we shall set \( \hbar = 1 \). The Coulomb interaction affects the local radial oscillation frequencies: for convenience, let us then define the ‘effective’ local radial frequencies \( \nu_j = \sqrt{\omega_j^2 - \sum_{i \neq j} 1/|u_j - u_i|^3} \), \( \{u_j\} \) being the equilibrium positions of the ions in the length unit set by the longitudinal frequency \( \omega_0 \). Rescaling the canonical operators according to \( \hat{x}_j = \hat{x}_j / \sqrt{m \nu_j} \), \( \hat{p}_j = \hat{p}_j / \sqrt{m \nu_j} \), and grouping them together in a vector of operators \( \hat{R} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n)^T \), allows one to express the global Hamiltonian of the system in the harmonic approximation as the following quadratic form

\[
\hat{H} = \frac{1}{2} \hat{R}^T \begin{pmatrix} \kappa & 0 \\ 0 & \nu \end{pmatrix} \hat{R},
\]

where \( \nu \) is a diagonal matrix: \( \nu = \text{diag}(\nu_1, \ldots, \nu_n) \), while the potential matrix \( \kappa \) has diagonal entries \( \kappa_{jj} = \nu_j \) and off-diagonal entries \( \kappa_{jk} = 1/(\sqrt{m \nu_j} |u_j - u_k|^3) \) for \( j \neq k \). Let us also recall that the canonical commutation relations can be expressed as \( [\hat{R}_j, \hat{R}_k] = i\Omega_{jk} \), where the \( 2n \times 2n \) matrix \( \Omega \) has entries \( \Omega_{jk} = \delta_{n,k-j} - \delta_{n,j-k} \) for \( 1 \leq j, k \leq 2n \), and that, in quantum optics, Gaussian states are defined as states with Gaussian characteristic function: a Gaussian state \( \rho \) is completely determined by the ‘covariance matrix’ (CM) \( \sigma \), with entries \( \sigma_{jk} \equiv \text{Tr} \left( \{\hat{R}_j, \hat{R}_k\}\rho \right)/2 - \text{Tr}[\hat{R}_j \rho] \text{Tr}[\hat{R}_k \rho] \) and by the vector of first moments \( \bar{R} \), with components \( R_j \equiv \text{Tr}[\hat{R}_j \rho] \), in terms of the vector of canonical operators \( \hat{R} \).

Linear phononics – In the first part of the paper, we shall assume that the trapping frequencies \( \{\omega_j\} \), and thus \( \{\nu_j\} \), can be controlled locally and changed suddenly. This may be achieved by building small, local radial electrodes, by adding local optical standing waves [7], or in Penning trap arrays [8, 9]. Our first aim here is to show how, in principle, this control allows one to perform any arbitrary ‘linear optical’ operation on the radial modes of the ions, that is any unitary operation under which, in the Heisenberg picture, the vector of operators \( \hat{R} \) transforms linearly: \( \hat{R} \mapsto S \hat{R} \). The matrix \( S \) has to be ‘symplectic’, i.e. \( S^T \Omega S = \Omega \), to preserve the canonical commutation relations. Any symplectic operation \( S \) on a system of many canonical degrees of freedom (‘modes’) can be decomposed into a combination of generic single-mode symplectic transformations and two-mode rotations (‘beam splitters’, in the quantum optical terminology) [10, 11]. It is therefore sufficient for us to establish the possibility of performing these subclasses of operations on our system of \( n \) ions by manipulating the local frequencies. Single qubit operations – In what follows we assume that the original frequencies of the ions are different but commensurate, as given by, say, \( \nu_j = j \nu \), and that \( \nu \) is large enough
so that interaction between ions suppressed \[\text{[12]}\]. Let us then consider the reaction of the system if the frequency of the \(j\)-th ion changes suddenly from \(\nu_j\) to \(\alpha_j \nu_j\), for some real \(\alpha_j\). The Heisenberg equation of motion for \(\hat{x}_j\) and \(\hat{p}_j\) can be immediately integrated in such a case, resulting into a symplectic transformation \(S_j(t)\)

\[
S_j(t) = \left( \begin{array}{cc} \alpha_j^{\frac{1}{2}} & 0 \\ 0 & \alpha_j^{-\frac{1}{2}} \end{array} \right) \left( \begin{array}{cc} c & s \\ -s & c \end{array} \right) \left( \begin{array}{cc} \alpha_j^{\frac{1}{2}} & 0 \\ 0 & \alpha_j^{-\frac{1}{2}} \end{array} \right),
\]

(2)

with \(c \equiv \cos(\nu_j \alpha_j t)\) and \(s \equiv \sin(\nu_j \alpha_j t)\). The first and last factor of this decomposition are ‘squeezing’ operations in the quantum optical terminology, whereas the second factor is known as a ‘phase shift’ (i.e., a rotation in the single-mode phase space). Combinations of squeezings and phase-shifts make up any possible single-mode symplectic operation: we thus need to show that such operations can be implemented individually on any ion of the system in a controllable manner.

**Phase-shift** – To realise a phase-shift operation on the \(k\)-th ion, it is sufficient to change the frequencies of all the other ions in the same way, such that \(\alpha_k = 1\) and \(\alpha_j \neq 1\) for \(j \neq k\). As apparent from Eq. (2), after a time \(t_\alpha = 2\pi/(\nu \alpha)\) one has \(S_j = \mathbb{I}_2\) for \(j \neq k\) (let us recall that \(\nu_j = j \nu\) by assumption), whereas the oscillation of the \(k\)-th ion will have acquired a phase \(\varphi_k = 2 \pi k / \alpha\) (with no squeezing, as \(\alpha_k\) is kept equal to 1). If the frequencies are switched back to the original values after a time \(t_\alpha\), the net effect of the evolution is then analogous to an ‘optical’ phase-shift on the ion \(k\).

**Squeezing** – In order to squeeze the state of ion \(k\), one can conversely change only the pertinent frequency, so that \(\alpha_k \neq 1\) and \(\alpha_j = 1\) for \(j \neq k\). Then, after a time period \(t = 2\pi / \nu\), all the other ions will have returned to the initial state, while ion \(k\) will be squeezed and phase-shifted according to Eq. (2). Notice that the phase-shift can always be corrected by applying the strategy described above. Let us remark that the degree of squeezing achieved in Eq. (2) depends crucially on the phase-shift operation, as the two squeezing operations act along orthogonal directions and are the inverse of each other. In the case \(\alpha_k = (\frac{1}{4} + h)/k\) for \(h \in \mathbb{N}\), the phase-shift can be balanced by a counter-rotation of \(\pi/4\) in phase space and the final squeezing operation is a diagonal matrix given by \(\text{diag}(\alpha_k, \alpha_k^{-1})\) \[\text{[13]}\]. Also notice that, by placing the ions inside cavities, the squeezing of the massive degrees of freedom could be transferred to light, so that radial modes could act as an effective source of squeezing (and potentially even entanglement) for optical systems as well.

**Beam-splitters** – Let us now turn to ‘beam-splitting’ operations\(\text{[eq\[2\]}}\) between any two radial modes. To this aim, it is sufficient to bring two modes (hereafter labeled by \(j\) and \(k\)) to the same frequency \(\nu = \nu_j = \nu_k\), so that the Coulomb interaction between them is no longer suppressed. Switching to an interaction picture, one has the following interaction Hamiltonian between the two modes:

\[
\kappa_{jk} \hat{x}_j(t) \hat{x}_k(t) = \kappa_{jk} (a_j e^{-i\omega t} + a_j^\dagger e^{i\omega t})(a_k e^{-i\omega t} + a_k^\dagger e^{i\omega t}),
\]

where the ladder operators are defined as \(\hat{x}_j = (a_j + a_j^\dagger)\). If the frequency \(\nu\) is sufficiently large the rotating wave approximation applies to yield \(\kappa_{jk} (a_k^\dagger a_j + a_j^\dagger a_k)\). This Hamiltonian realises exactly the desired beam splitter-like evolution, resulting into a symplectic transformation which mixes \(\hat{x}_j\) with \(\hat{x}_k\) and \(\hat{p}_j\) with \(\hat{p}_k\) (rotating such pairs equally, by the angle \(\kappa_{jk} t\)). For instance, a \(50:50\) beam splitter is achieved after a time \(t = \pi/(4\kappa_{jk})\). Since the interaction requires a change of the local frequencies it includes automatically in it a local operation, which may however be corrected before or after the ‘beam-splitting’ procedure.

Summing up, we have shown that any symplectic (i.e., “linear optical”) operation, including squeezing, can be implemented for radial modes of trapped ions by a proper tuning of the frequencies of the microtraps. Displacement operations on individual ions, which shift the operators \(\hat{R}_j\) by a real number, can also be implemented in microtraps by shifting the radial equilibrium position of the ion, or as in \[\text{[15]}\]. Because the free evolution rotates the state of the radial modes in phase space (see Eq. (2) for \(\alpha_j = 1\)), if the operation is carried out at the proper time such a shift can be implemented in any direction of phase space and not only in the positions \(\hat{x}_j\). The unitary operator displacing the canonical operators of mode \(j\) by, respectively, \(x_j\) and \(p_j\) will be denoted by \(D_j(x_j, p_j)\).

These findings show that all the developments based on Gaussian states in the quantum optical scenario, in particular concerning entanglement manipulation \[\text{[3]}\] and information protocols \[\text{[16]}\], could be carried over to radial modes of ion traps if local control is achieved. In fact, linear optical operations, complemented by displacements, correspond to all the unitary transformations that preserve the Gaussian character of the initial state.

Notably, even non Gaussian states can be engineered in this setup with relative ease, either by entering the nonlinear regime of the Coulomb interactions or by exploiting the internal degrees of freedom of the ions. The latter also allows for Gaussian and non Gaussian measurements on individual ions: the tomography of trapped ions, corresponding to homodyning, was proposed in \[\text{[13, 17, 18]}\] and partially realized in \[\text{[19]}\], while local number states and parity could be measured using the scheme suggested and realized for cavity QED in \[\text{[20]}\]. Quite remarkably, such a scheme would allow one to measure parity on a single copy of the state and run of the apparatus. In the remainder of the paper, in order to demonstrate the potential of Gaussian states of radial modes in experimentally accessible settings, we will consider entanglement generation and nonlocality tests requiring only global control of the trapping potential.

**Entanglement generation** – The specific Gaussian situation we shall address starts off from the ground state \(\varrho_0\) of Hamiltonian \[\text{[1]}\] – with all frequencies being equal, i.e. \(\omega_j = \omega_i\) for \(1 \leq j \leq n\) – as the initial state (which can be well approximated in the laboratory by cooling the system to its ground state \[\text{[21]}\]). Next, the frequency is changed to \(\omega_f\), so that the state \(\varrho_0\) will not be stationary anymore under the modified Hamiltonian. For large \(\omega_i\), the initial state \(\varrho\) contains very little entanglement but entanglement builds up during
FIG. 1: Entanglement between first and last ions of the chain as a function of time for initial frequency $\omega_i = 100$ MHz, final frequency $\omega_f = 2$ MHz, temperature $T = 294^\circ K$ (corresponding to $N = 2 \times 10^7$ thermal phonons) and different couplings to the environment $\gamma$: thicker curves refer to ions 1 and 2 for $n = 2$, while thinner curves refer to ions 1 and 3 for $n = 3$. The curves for $\gamma = 0$ and $\gamma = 10^{-6}$ Hz are very close: decoherence is almost negligible for such heating rates.

the subsequent evolution (see [22] for an analogous scheme in chains of nanomechanical oscillators). Entanglement may be quantified by the logarithmic negativity $E_N = \log_2 \| \tilde{\rho} \|_1$, where $\| \tilde{\rho} \|_1$ stands for the trace norm of the ‘partially transposed’ density matrix of the considered system (with this definition, a Bell pair has $E_N = 1$), which is computable for Gaussian states [23]. The ground state of Hamiltonian (1) is just a Gaussian state with a block diagonal CM $\sigma_{\gamma} = (\sigma_{\gamma} \otimes 1)/2$, where $\sigma_{\gamma} = \nu^{1/2}(\nu^{1/2})^\dagger - 1/2\nu^{1/2}$, and vanishing first moments. After the change of potential, resulting into the new quadratic Hamiltonian $H' = R^T H' R$ (where the ‘Hamiltonian matrix’ $H'$ is implicitly defined), the evolved state after a time $t$ is a Gaussian state with CM given by $\sigma_t = S_t \sigma_{\gamma} S^T_t$, for $S_t \equiv \exp(\Omega H't)$. Furthermore, we have taken into account decoherence in an environment of phonons with temperature $T$ and ‘loss rate’ $\gamma$, as the master equation under such conditions admits Gaussian solutions with CM $e^{-\gamma t/2} (1 - e^{-\gamma t}) S_t \sigma_{\gamma} S^T_t$, where $\sigma_{\infty} \equiv \text{diag}(\frac{1}{2} + N_j, \frac{1}{2} + N_j)$ and $N_j = 1/(e^{\nu_j/T} - 1)$ is the number of thermal phonons at frequency $\nu_j$ (setting $k_B = 1$) [24].

Fig. 1 shows that robust entanglement, up to about 6 ebits of logarithmic negativity, between two ions can be created in such a setup. A complete analysis of entanglement and information propagation through chains of ions will be detailed in [6]. Here, let us just point out that such degrees of entanglement are by far out of experimental reach for quantum optics (where, to the best of our knowledge, $E_N \simeq 1.6$ is the maximum value so far reported after state reconstruction [23]). The coupling to the bath of $\gamma = 10^{-6}$ Hz (best value considered in the plot) is realistic in view of the recently observed heating rates in ion traps [24, 27]. The plot shows that this estimate is extremely encouraging, especially if compared to the state of the art for quantum optical cavities and resonators, where loss rates still significantly limit performances. Multiparticle entanglement, as well as entanglement between non-neighboring ions, can also be created with global control as, when the frequency is switched from $\omega_i$ to $\omega_f$, all the $n$ ions in the trap start interacting with each other. For instance, for three ions, $\omega_i = 20$ MHz and $\omega_f = 2$ MHz, the initially completely separable state evolves into a ‘fully inseparable’ Gaussian state (inseparable under any bipartition of the modes); let us denote the CM of this state, after an evolution time $t = 5\nu^{-1}_f = 5\mu$s, by $\sigma_3$.

Nonlocality test — Such multiparticle entanglement can be put to use to test quantum nonlocality with massive particles. Taking advantage of the possibility of performing parity measurements (in a single shot) and displacements, we will analyse the violation of the Bell-Klyshko inequalities [28] on the three ions state with CM $\sigma_3$, by the displaced parity test as introduced in [29]. In this instance, the family of (non-Gaussian) local, bounded, dichotomic observables is given by $\Pi_j(x, p_j) = \hat{D_j}(x, p_j)\hat{D_j}(x, p_j)$, where $\hat{D_j}$ and $\hat{n}_j$ are the displacement and number of phonons operators of ion $j$. The three observers, pertaining to the three ions, randomly apply two different displacements $[\hat{D}_1(x_1, p_1), \hat{D}_2(x_2, p_2)]$ on their ions and then measure parity locally. The expectation value of the operator $\Pi(R) \equiv \Pi_1(x_1, p_1) \otimes \Pi_2(x_2, p_2) \otimes \Pi_3(x_3, p_3)$ is proportional to the Wigner function $W(R)$ of the composite system evaluated in the point $R = (x_1, x_2, x_3, p_1, p_2, p_3)^T$. $\Pi(R) = (2/\pi)^3 W(R)$ [30]. Such a function is immediately determined for the Gaussian state under consideration:

$$W(R) = \frac{e^{-\frac{1}{2}R^T \sigma_3^{-1} R}}{\pi^3 \sqrt{\text{Det} \sigma_3}}. \quad (3)$$

The Bell-Klyshko inequality finally reads: $B_3 \equiv \sum \left| W(x_1, x_2, x_3, p_1, p_2, p_3') + W(x_1, x_2', x_3, p_1, p_2', p_3) + W(x_1, x_2, x_3', p_1', p_2, p_3') - W(x_1, x_2', x_3', p_1', p_2', p_3') \right| \leq 2$. Quantum mechanics allows for $2 \leq B_3 \leq 4$. As epitomised by Fig. 2 regions in the space of displacements where the violation of the inequality is substantial and remarkably stable can be found. Therefore, this preliminary study opens up very promising perspectives concerning the violation of Bell inequalities with massive degrees of freedom, which would be a major, not yet probed, testing ground for fundamental quantum mechanics [31]. A detailed analysis of the impact of imperfections and noise on such nonlocality tests will be presented in [3].

Conclusions — We have demonstrated how the local control of the trapping frequencies would allow one to reproduce any linear optical manipulation on radial modes of trapped ions. We also indicated that phonon detection and homodyne detection as well as the implementation of non-Gaussian operations is possible in this setting. Next, we have emphasized that, even restricting to global control, such manipulations enjoy a high efficiency in entanglement generation and low decoherence rates, along with the possibility of implementing number and parity measurements with current technology. The experimental pursuit of the programme outlined in this paper
thus holds considerable promise, concerning both technological developments, such as the storage and manipulation of quantum information, and fundamental physical aspects, as in the nonlocality test for massive degrees of freedom here discussed.

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