Leavitt vs. $C^*$ pullbacks

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Abstract

We show that certain pullbacks of $*$-algebras equivariant with respect to a compact group action remain pullbacks upon completing to $C^*$-algebras. This unifies a number of results in the literature on graph algebras, showing that pullbacks of Leavitt path algebras lift automatically to pullbacks of the corresponding graph $C^*$-algebras.

Key words: Leavitt path algebra, graph $C^*$-algebra, pullback

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Introduction

The present note was prompted by [2, 6] and the desire to explain the relationship between their respective main results, [2, Theorem, p.2] and [6, Theorem 3.2] respectively. The former puts a pullback structure on a graph $C^*$-algebra, whereas the latter proves the parallel result for the corresponding Leavitt path algebras.

Given the close relationship between the two settings, it would be natural to seek an abstract framework that would allow one to simply morph the Leavitt result into its $C^*$ analogue without having to retrace the proofs. We propose such a framework here.

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1 Preliminaries

1.1 Generalities on equivariant structures

We will work with $G$-equivariant structures on $*$ or $C^*$-algebras (or more generally Banach spaces) for a compact group $G$. For Banach spaces $V$ this simply means a strongly continuous action as automorphisms (isometries for Banach spaces or $C^*$ automorphisms for $C^*$-algebras) in the sense that

$$G 
i g \mapsto gv \in V$$

is a continuous map for each $v \in V$; this is the customary and apparently most appropriate continuity assumption on actions on operator algebras (e.g. [10, §2.2]).
On the other hand, for $\ast$-algebras $A$ a $G$-equivariant structure means a comodule structure

$$A \rightarrow A \otimes O(G),$$

where $O(G)$ is the Hopf algebra of representative functions on $G$: the span of matrix coefficients of finite-dimensional continuous $G$-representations. When $G = S^1$, which will be the case in the concrete applications discussed below, $O(G)$ is nothing but the algebra $\mathbb{C}[t^\pm 1]$ of Laurent polynomials and an equivariant structure is a $\mathbb{Z}$-grading on the $\ast$-algebra $A$.

For a $G$-$C^\ast$-algebra $A$ we have the faithful expectation

$$A \ni a \mapsto \int_G g \triangleright a \ d\mu(g) \in A^G$$
ono

onto the fixed-point subalgebra $A^G \leq A$, where $d\mu$ is the Haar measure on $G$. 

Given a $G$-equivariant inclusion $A \leq A$ of a $\ast$-algebra into a $C^\ast$-algebra we have (see e.g. [4, Proposition 6.1])

**Lemma 1.1** The invariant subalgebra $A^G$ is dense in $\overline{A}^G$. ■

A number of elementary remarks apply to the functor $A \mapsto A^G$ from $G$-$C^\ast$-algebras to $C^\ast$-algebras. First, it is a right adjoint with left adjoint

$$A \mapsto A$$

equipped with the trivial $G$-action.

In particular the fixed-point functor is continuous (i.e. it preserves all limits) [9, §V.5, Theorem 1]. Additionally:

**Lemma 1.2** The fixed-point functor $A \mapsto A^G$ on $G$-$C^\ast$-algebras reflects monomorphisms, in the sense that if $f : A \rightarrow B$ is a morphism of $G$-$C^\ast$-algebras such that

$$f^G : A^G \rightarrow B^G$$

is monic then so is $f$.

**Proof** Indeed, since the expectation $A \rightarrow A^G$ is faithful, the kernel $J$ of $A \rightarrow B$ vanishes if and only if $J^G$ does, which happens by hypothesis. ■

### 1.2 Algebras attached to graphs

We will need some background on graph algebras: Leavitt path algebras as in [1, Definition 1.2.3] and their analytic counterparts, graph $C^\ast$-algebras, defined, say, as in [5, §2]. Briefly:

**Definition 1.3** A graph is a quadruple $E = (E^0, E^1, s, t)$ consisting of sets $E^0$ and $E^1$ of vertices and respectively edges and source and target maps $s, t : E^1 \rightarrow E^0$.

A vertex $v \in E^0$ is regular if $s^{-1}(v)$ is finite and non-empty (we also say that $v$ is a finite emitter and is not a sink).

With this in place, recall [1, Definition 1.2.3]:

**Definition 1.4** Given a graph $E$, the Leavitt path algebra $L_k(E)$ over a unital commutative ring $k$ is the $k$-algebra equipped with an anti-multiplicative involution $\ast$ defined by generators $v = v^*$ for $v \in E^0$ and $e, e^*$ for $e \in E^1$ subject to relations
\[
\bullet vv' = \delta_{v,v'};
\]
\[
\bullet s(e)e = er(e) = e;
\]
\[
\bullet e^*e' = \delta_{e,e'}r(e);
\]
\[
\sum_{e \in s^{-1}(v)} ee^* = v \text{ for all regular vertices } v \in E^0.
\]

While in this generality the involution ‘*’ is assumed to act as the identity on \(k\), for \(k = \mathbb{C}\) one usually works with a modified definition whereby ‘*’ is complex conjugation on \(\mathbb{C}\). This makes the corresponding algebra (which we will then denote simply by \(L(E)\)) into a complex \(\ast\)-algebra.

On the other hand (see [5, §2] or [1, §5.2]):

**Definition 1.5** The graph C*-algebra \(C^*_E\) is the universal C*-algebra generated by \(v = v^*\), \(v \in E^0\) and \(e,e^*\) for \(e \in E^1\) subject to the same relations as in Definition 1.4, together with the additional conditions

\[
e e^* \leq s(e), \forall e \in E^1.
\]

The additional requirement in Definition 1.5 is not needed if the graph \(E\) is **row-finite**, i.e. if every vertex emits finitely many (possibly zero) edges. All graphs considered in [6, 2, 7] referred to below are row-finite.

**Remark 1.6** It turns out that \(C^*_E\) is nothing but the C*-envelope of \(L(E)\), i.e. the obvious map \(L(E) \to C^*_E\) is an initial object in the category of \(\ast\)-morphisms from \(L(E)\) into C*-algebras [1, §5.2].

Graph algebras (Leavitt of C*) admit \(S^1\)-actions in this sense (called the *gauge actions* in the literature [11, Chapter 2] or [1, §2.1]): \(z \in S^1\) simply scales every edge generator \(e \in E^1\) by \(z\) in the C* case, while the corresponding grading on the corresponding Leavitt path algebra assigns degree 1 to each \(e\) and degree \(-1\) to each \(e^*\).

## 2 Graph algebra pullbacks

Let \(A \leq P_G(\overline{A}) \leq \overline{A}\) be a dense inclusion of a \(G\)-\(\ast\)-algebra into a \(G\)-C*-algebra as in Section 1. Our first remark is

**Lemma 2.1** If the invariant subalgebra \(A^G\) admits a unique C* norm then the norm of \(\overline{A}\) is the unique \(G\)-invariant C*-norm on \(A\).

**Proof** Consider a surjection \(f : \overline{A} \to B\) of \(G\)-\(\ast\)-algebras, faithful on \(A\). Its restriction \(\overline{A}^G \to B^G\) is monic by the unique-norm assumption and hence \(f\) is monic by Lemma 1.2. Since \(f\) was also onto, we are done. ■

**Corollary 2.2** If the invariant subalgebra \(A^G\) is AF then the norm of \(\overline{A}\) is the only \(G\)-invariant C*-norm on \(A\).

**Proof** Indeed, AF \(\ast\)-subalgebras of C*-algebras have unique C*-norms, so Lemma 2.1 applies. ■
We consider a diagram

\[
\begin{array}{ccc}
A & \overset{\ell}{\longrightarrow} & C \\
& \swarrow \downarrow r \searrow & \\
D & \longrightarrow & B
\end{array}
\]  

(1)

in the category of $G$-$*$-algebras.

To state the main result of this section we need

**Lemma 2.3** Let $\pi : M \to N$ be a $*$-algebra surjection embedding in its $C^*$ completion $\overline{\pi} : \overline{M} \to \overline{M}$. Then, $\ker \pi$ is dense in $\ker \overline{\pi}$.

**Proof** First, note that the closure of $J := \ker \pi$ in $\overline{M}$ is a $C^*$ ideal contained in $\ker \overline{\pi}$, so we have a factorization

\[
\begin{array}{ccc}
\overline{M} & \overset{\cdot}{\longrightarrow} & \overline{M}/J \\
\downarrow \pi & & \downarrow \pi \\
N & & N
\end{array}
\]

The fact that the $\cdot$ map in the above diagram is an isomorphism follows from an examination of the (Hilbert space) representations of the two $C^*$-algebras: those of $\overline{M}/J$ are precisely the $*$-representations of $M$ vanishing on $J = \ker \pi$, and hence precisely the $*$-representations of $N$ (and hence of $\overline{M}$).

As a consequence, we have the following simple remark.

**Lemma 2.4** Let $\pi : M \to N$ be a $*$-algebra surjection embedding into its $C^*$ completion $\overline{\pi}$. Then, for every $y \in N$ of norm $\|y\| < \varepsilon$ in the completion $\overline{N}$ there is

\[
x \in \pi^{-1}(y), \quad \|x\| < \varepsilon \quad \text{in} \quad \overline{M}.
\]

**Proof** On the one hand we can find

\[
\overline{\pi} \in (\pi)^{-1}(y) \subset \overline{M}
\]

of norm $< \varepsilon$ simply because $\overline{\pi}$ is a $C^*$-algebra surjection. On the other hand, the surjectivity of $\pi$ ensures the existence of some $x' \in \pi^{-1}(y) \subset M$, perhaps not satisfying the norm constraint. But then $\overline{\pi} - x' \in \ker \overline{\pi}$, meaning by Lemma 2.3 that it is arbitrarily approximable with elements $x - x' \in \ker \pi$. If the latter is sufficiently close to $\overline{\pi} - x'$ then $x \in A$ will be sufficiently close to $\overline{\pi}$ to achieve $\|x\| < \varepsilon$.

We are now ready for the main statement.

**Theorem 2.5** Assume that

- (1) is a pullback;
- $r$ is onto;
- $C^G$ is AF;
- (1) embeds in its $C^*$ completion.
Then, the $C^*$-completed diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ell} & \mathcal{C} & \xrightarrow{r} & \mathcal{B} \\
\mathcal{D} & \xleftarrow{\ell} & & \xleftarrow{r} & \\
\end{array}
\]

(2)

is a pullback of $G$-$C^*$-algebras.

**Proof** Consider the $C^*$-pullback

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ell} & \mathcal{P} & \xleftarrow{\ell} & \mathcal{D} \\
\mathcal{P} & \xleftarrow{r} & \mathcal{B} & \xrightarrow{r} & \\
\end{array}
\]

(3)

By Corollary 2.2 the canonical map $\mathcal{C} \to \mathcal{P}$ is one-to-one, because $\mathcal{C}$ admits a unique $G$-invariant $C^*$ norm, which must be the universal one inherited from $\mathcal{C}$.

On the other hand, the surjectivity of $\mathcal{C} \to \mathcal{P}$ follows from that of $r$. To see this, consider a pair of elements $a \in \mathcal{A}$, $b \in \mathcal{B}$ with equal images in $\mathcal{D}$ through $\ell$ and $r$ in (3) respectively. The pair forms an element $(a, b) \in \mathcal{P}$, by the very definition of a pullback. We want to argue that $(a, b)$ is arbitrarily approximable in the sup norm on $\mathcal{A} \times \mathcal{B}$ by elements $(a_i, b_i)$ in the pullback $\mathcal{C}$, i.e. such that $\ell(a_i) = r(b_i)$.

First, approximate $(a, b)$ arbitrarily well with $(a_i, b'_i) \in \mathcal{A} \times \mathcal{B}$ disregarding for the moment the coincidence of their images through $\ell$ and $r$ respectively. We have

$$\|\ell(a_i) - r(b'_i)\| < \varepsilon.$$  

Lemma 2.4 ensures the existence of some $\gamma_i \in \mathcal{B}$ with

- $\|\gamma_i\| < \varepsilon$
- $r(\gamma_i) = r(b'_i) - \ell(a_i)$.

Finally, set $b_i = b'_i - \gamma_i$. This element will

- be close to the original element $b$;
- have image $\ell(a_i)$ through $r$.

In short, $(a_i, b_i) \in \mathcal{C}$ will be a good approximation for $(a, b) \in \mathcal{P}$.  

The surjectivity assumption is essential in Theorem 2.5, as the following remark shows.

**Example 2.6** In (1), we take $D = C(S^1)$, which will hence be its own $C^*$ closure. $A$ and $B$ will be dense $*$-subalgebras thereof such that

- $\mathcal{A} = \mathcal{B} = D$;
- $A \cap B$ consists of precisely the scalars.
To achieve all of this, first let $A$ be the $\ast$-algebra generated by the standard unitary
$$\text{id} : S^1 \to S^1 \subset \mathbb{C}.$$
We indeed have $\mathfrak{A} = C$, since the latter is the universal $C^*$-algebra generated by a unitary.

On the other hand, for $B$ we take the “twist” of $A$ by a sufficiently general self-homeomorphism $\varphi$ of the circle $S^1$; in other words,
$$B = \{ f \circ \varphi \mid f \in A \}.$$
“Sufficiently general” here might mean, for instance, that the set
$$\{ z \in S^1 \mid \varphi \text{ is not differentiable at } z \}$$
is dense in the unit circle. To see that this will ensure $A \cap B = \mathbb{C}$, note that every non-constant trigonometric polynomial $f \in A$ will map some interval arc $J \subseteq S^1$ diffeomorphically onto its image. But then, denoting by
$$\psi : f(J) \to J$$
the inverse of that diffeomorphism, $\psi \circ f \circ \varphi |_{J}$ equals $\varphi$ on $J$. Since the latter has points of non-differentiability in $J$, the non-constant $f \circ \varphi \in B$ cannot possibly be equal to any element of $A$.

We now apply the preceding discussion to the setup of [2, 6]. Recall that the authors of said papers work with graphs $Q' \subset Q''$ and maps

$$
\begin{align*}
L(Q') & \xrightarrow{\delta} L(Q) & L(Q') \otimes k[t,t^{-1}] \xleftarrow{\pi \otimes \text{id}} L(Q'') \otimes k[t,t^{-1}]
\end{align*}
$$

where $L(-)$ denotes the Leavitt path algebra construction, $k$ denotes a ground field, and $\pi$ is onto. There is an analogous picture for graph $C^*$-algebras, whereupon $k = \mathbb{C}$; this is the case of interest here.

The two papers work in the Leavitt path algebra and $C^*$ setting ([6, 2] respectively), proving parallel results: while [6, Theorem 3.2] argues that a certain diagram

$$
\begin{align*}
L(Q') & \xrightarrow{\delta} L(Q) & L(Q'') \otimes k[t,t^{-1}] \xleftarrow{\pi \otimes \text{id}} L(Q'') \otimes k[t,t^{-1}]
\end{align*}
$$

is a pullback of graded $\ast$-algebras, the analytic analogue [2, Theorem, p.2] shows that the $C^*$ graph algebra version

$$
\begin{align*}
C^*(Q') & \xrightarrow{\delta} C^*(Q) & C^*(Q'') \otimes C(S^1) \xleftarrow{\pi \otimes \text{id}} C^*(Q'') \otimes C(S^1)
\end{align*}
$$

is a pullback of $C^*$-algebras equipped with actions by the circle group $S^1$ (with $C(-)$ denoting the algebra of continuous functions).

We now have:

**Proposition 2.7** [6, Theorem 3.2] implies [2, Theorem, p.2].
Theorem 2.5 for $G = S^1$ and the fact that the $S^1$-invariant subalgebras of Leavitt path algebras are AF.

The pattern recurs in [7]: a pushout graph diagram

\[
\begin{array}{c}
F_1 \\ E \\ F_2 \\
\end{array}
\begin{array}{c}
F_1 \\ F_1 \cap F_2 \\ F_2 \\
\end{array}
\]

results via [7, Theorem 3.1] in a pullback diagram in the category of $S^1$-$C^*$-algebras

\[
\begin{array}{c}
C^*(F_1) \\ C^*(E) \\ C^*(F_2) \\
\end{array}
\begin{array}{c}
C^*(F_1) \\ C^*(F_1 \cap F_2) \\ C^*(F_2) \\
\end{array}
\] (7)

consisting of surjections only. It is then observed in [7, Remark 3.2] that a parallel proof would dispatch the Leavitt path algebra version (with all instances of $C^*(-)$ replaced by the corresponding $L(-)$). In that context, the analogue of Proposition 2.7 is

Proposition 2.8 [7, Theorem 3.1] follows from its Leavitt path algebra analogue.

On the other hand, the results of [3] involve non-equivariant pullbacks of graph algebras. The general principle at work is the same however:

Proposition 2.9 [3, Theorem 3.3] follows from its Leavitt path algebra analogue.

Proof This time around the compact group $G$ will be trivial. The graph algebra at the top of the diagram, corresponding to $C$ in (1), is attached to a graph assumed to have no loops. It follows that the Leavitt path algebra is AF (e.g. [8, Theorem 2.4]) and the conclusion once more follows from Theorem 2.5.

References

[1] Gene Abrams, Pere Ara, and Mercedes Siles Molina. Leavitt path algebras, volume 2191 of Lecture Notes in Mathematics. Springer, London, 2017.

[2] F. Arici, F. D'Andrea, P. M. Hajac, and M. Tobolski. An equivariant pullback structure of trimmable graph C*-algebras. ArXiv e-prints, December 2017.

[3] Alexandru Chirvasitu, Piotr M. Hajac, and Mariusz Tobolski. Non-surjective pullbacks of graph C*-algebras from non-injective pushouts of graphs, 2019. arXiv:1907.10260.

[4] Alexandru Chirvasitu, Benjamin Passer, and Mariusz Tobolski. Equivariant dimensions of graph C*-algebras, 2019. arXiv:1907.10010.

[5] D. Drinen and M. Tomforde. The C*-algebras of arbitrary graphs. Rocky Mountain J. Math., 35(1):105–135, 2005.

[6] P. M. Hajac, A. Kaygun, and M. Tobolski. A graded pullback structure of Leavitt path algebras of trimmable graphs. ArXiv e-prints, March 2018.
[7] Piotr M. Hajac, Sarah Reznikoff, and Mariusz Tobolski. Pullbacks of graph C*-algebras from admissible pushouts of graphs, 2018. arXiv:1811.00100.

[8] Alex Kumjian, David Pask, and Iain Raeburn. Cuntz-Krieger algebras of directed graphs. *Pacific J. Math.*, 184(1):161–174, 1998.

[9] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[10] N. Christopher Phillips. *Equivariant K-theory and freeness of group actions on C*-algebras*, volume 1274 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[11] Iain Raeburn. *Graph algebras*, volume 103 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005.

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