Crystal Bases for the Quantum Superalgebra

\[ U_q(D(N, 1)), U_q(B(N, 1)) \]

SUZUKI Kenei

Graduate School of Mathematical Sciences, University of Tokyo
Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan

e-mail address: kenei@ms557pur.ms.u-tokyo.ac.jp

Abstract

Let \( V(\lambda) \) be the irreducible lowest weight \( U_q(D(N, 1)) \)-module with lowest weight \( \lambda \). Assume \( \lambda = n_0 \omega_0 - \sum_{i=0}^{N} n_i \omega_i, n_i \in \mathbb{Z}_{\geq 0} \), where \( \omega_0 \) is the fundamental weight corresponding to the unique odd coroot \( h_0 \). \( V(\lambda) \) is called typical if \( n_0 \geq 0 \). In this paper, we construct polarizable crystal bases of \( V(\lambda) \) in the category \( \mathcal{O}_{int} \), which is a class of integrable modules. We also describe the decomposition of the tensor product of typical representations into irreducible ones, using the generalized Littlewood-Richardson rule for \( U_q(D(N)) \).

We also present analogous results for the quantum superalgebra \( U_q(B(N, 1)) \).

keywords: Crystal bases; Quantum superalgebras; \( U_q(D(N, 1)) \); \( U_q(B(N, 1)) \); Tensor products
1 Introduction

The theory of crystal base for quantized Lie algebras initiated by Kashiwara has brought and is still bringing a great deal of fruits in representation theory. It is hence natural to generalize it to the case of Lie superalgebras.

Finite dimensional simple Lie superalgebras \( \mathfrak{g} \) were classified by Kac [5]. He also determined the conditions under which a \( \mathfrak{g} \)-module is finite dimensional. The quantized enveloping algebras for Lie superalgebras are defined by Yamane [11] for finite dimensional contragredient Lie superalgebras, and by Benkart, Kang and Melville [2] for Borcherds superalgebras. Borcherds superalgebras include Kac-Moody superalgebras, which are analogous to Kac-Moody Lie algebras.

The first work on the crystal bases for quantized Lie superalgebras was the one by Musson and Zou [9]. They defined and constructed crystal bases of finite dimensional modules of \( U_q(B(0,N)) = U_q(\mathfrak{osp}(1,2N)) \). The crystal bases for \( U_q(B(0,N)) \) are essentially the same as those for \( U_q(B(N)) \). Among the Lie superalgebras in Kac’s list, \( B(0,N) \) is distinguished because it is the unique one whose finite dimensional representations are completely reducible. \( B(0,N) \) is also a Kac-Moody superalgebra, while other finite dimensional simple Lie superalgebras are not. Jeong [4] generalized these results from the point of view of Kac-Moody superalgebras. He defined crystal bases and showed their existence for the quantized Kac-Moody superalgebras when the modules are integrable.

Benkart, Kang, Kashiwara [3] defined crystal bases for quantum contragredient superalgebras which are not Kac-Moody Lie superalgebras (i.e. the ones containing \( \otimes \) in their Dynkin diagrams). They introduced a category \( \mathcal{O}_{int} \) of \( U_q(\mathfrak{g}) \)-modules for contragredient Lie superalgebra \( \mathfrak{g} \) (see Definition 3.1), and defined the notion of crystal base for modules \( M \) in \( \mathcal{O}_{int} \). The Kashiwara operators for odd roots behave quite differently from the case of Lie algebras and the Kac-Moody Lie superalgebras (see (3.1) and Proposition 3.5). They also showed the existence of polarizable crystal bases of finite dimensional irreducible \( U_q(\mathfrak{gl}(m,n)) \)-modules under some conditions on the highest weight and described them in terms of Young tableaux. Benkart and Kang [1] give a concise review for these results for quantized Lie superalgebras.

A noteworthy feature of \( U_q(\mathfrak{gl}(m,n)) \) is that the vector representation belongs to \( \mathcal{O}_{int} \). However this fails for the other types of classical Lie superalgebras \( B(m,n) \) \( (m \geq 1) \), \( C(n) \), \( D(m,n) \) and \( D(2,1;\alpha) \). Therefore, when one attempts to generalize the results of [3], the first question is to find modules in \( \mathcal{O}_{int} \).

The first generalization to these quantum superalgebras was for \( U_q(D(2,1;\alpha)) \) by Zou [12]. He found highest weight modules in \( \mathcal{O}_{int} \) which are infinite dimensional, and constructed crystal bases of them. In [12], \( \alpha \) is assumed to be an integer satisfying \( \alpha \leq -2 \). The reason he studied infinite dimensional modules is that any finite dimensional \( U_q(D(2,1;\alpha)) \)-module \( M \) does not satisfy the condition (iv) in the definition of \( \mathcal{O}_{int} \), that is

\[
\text{(iv) } \quad \text{For any } \mu \in P, \quad M_\mu \neq 0 \text{ implies } \langle h_0, \mu \rangle \geq 0,
\]

where \( h_0 \) is the unique simple odd coroot of \( U_q(D(2,1;\alpha)) \).

In this paper, we present two results on crystal bases for \( U_q(D(N,1)) \). Let \( V(\lambda) \) be the irreducible lowest weight module with lowest weight

\[
\lambda = n_0\omega_0 - \sum_{i=1}^{N} n_i\omega_i, \quad n_i \in \mathbb{Z}_{\geq 0} \quad \text{for } 0 \leq i \leq N,
\]
where \( \omega_i \) are the fundamental weights. The first result is that \( V(\lambda) \) admits a crystal base \( B(\lambda) \) (Theorem 6.1). The weight (1.2) is said to be typical if \( n_0 \geq 1 \). As the second result, we describe the decomposition of the tensor product of the irreducible representations which have typical lowest weights (Theorem 9.3). Our description heavily relies on the generalized Littlewood-Richardson rule for \( U_q(D(N)) \) by Nakashima [10]. We give similar results also for the algebra \( U_q(B(N,1)) \).

Our work is a kind of generalization of Zou’s work, since \( U_q(D(2,1;\alpha)) = U_q(D(2,1)) \) holds if \( \alpha = 1 \) and there is an isomorphism of algebras between \( U_q(D(2,1;\alpha)) \) and \( U_q(D(2,1;-1-\alpha)) \). However, it should be noted that we have to adopt another approach because a \( U_q(D(2,1;\alpha))- \)module in \( \mathcal{O}_{int} \) does not necessarily belong to \( \mathcal{O}_{int} \) when viewed as a \( U_q(D(2,1;-1-\alpha))- \)module.

As in the case of \( U_q(D(2,1;\alpha)) \), (1.1) fails for any finite dimensional \( U_q(D(N,1))- \)module. In addition, infinite dimensional highest weight modules do not belong to \( \mathcal{O}_{int} \). We first show that the irreducible lowest weight module \( V(-\omega_N) \) with lowest weight \( -\omega_N \) belongs to \( \mathcal{O}_{int} \), and give an explicit construction of a crystal base \( B(-\omega_N) \) (Proposition 5.3). We find that \( B(-\omega_N) \) is indexed by the crystal bases of the spin representations for \( U_q(D(N)) \) (denoted by \( \mathcal{B}_{sp}^\pm \) in this paper) and non-negative integers. Next we show that \( B(-\omega_N) \otimes B(-\omega_N) \) contains \( B(\omega_0), B(-\omega_1), \ldots, B(-\omega_N) \) using the decomposition of \( \mathcal{B}_{sp}^+ \otimes \mathcal{B}_{sp}^+ \). The existence of the crystal base \( B(\lambda) \) for general \( \lambda \) (1.2) follows by taking the tensor product of them.

If a weight \( \lambda' \) is typical, we observe that \( b' \otimes b \in B(\lambda') \otimes B(\lambda) \) is the lowest weight vector for \( U_q(D(N,1)) \) if and only if \( b' \otimes b \in B(\lambda') \otimes B(\lambda) \) is the lowest weight vector for \( U_q(D(N)) \). We use this fact to study the tensor product of typical representations. To get the lowest weight vectors of this tensor product, we decompose \( B(\lambda) \) into copies of crystal bases of \( U_q(D(N)) \) labeled by certain non-negative integers (Proposition 8.4). The typicality of \( B(\lambda) \) and the generalized Littlewood-Richardson rule enable us to find this decomposition. The integer labels are obtained through properties on Young tableaux for \( U_q(D(N)) \) due to Koga [8] (see Proposition 4.4 and Lemma 10.6). Applying the generalized Littlewood-Richardson rule again, we obtain all the lowest weight vectors of \( B(\lambda') \otimes B(\lambda) \). This is Theorem 9.3.

This paper is organized as follows. In Section 2, we give the definition of the quantized Lie superalgebra \( U_q(D(N,1)) \) following Yamane [11] and set up the notation. In Section 3, we review basic facts about the category \( \mathcal{O}_{int} \) and the crystal bases in the context of \( U_q(D(N,1)) \). In Section 4, we recall some properties of the crystal bases for \( U_q(D(N)) \) which we use in this paper. In Section 5, we construct a crystal base of \( V(-\omega_N) \). In Section 6 and Section 7, we state the existence of crystal bases and give a decomposition of \( B(\lambda) \) mentioned above for typical \( \lambda \). In Section 8, we decompose the tensor product of typical representations into irreducible ones in Theorem 9.3. We also present examples for Theorem 9.3 in the case of \( U_q(D(2,1)) \) and \( U_q(D(4,1)) \). The results for \( U_q(B(N,1)) \) are summarized in Section 9.

## 2 Definition of \( U_q(D(N,1)) \)

In this section we fix our notation concerning the quantum universal enveloping algebra for the Lie superalgebra \( \mathfrak{g} = D(N,1) \).

Let \( P = \oplus_{i=0}^N \mathbb{Z} \omega_i \) be a free \( \mathbb{Z} \)-module with basis \( \{ \omega_i \}_{i=0}^N \). Let \( \{ h_i \}_{i=0}^N \) be the dual basis of \( P^* = \text{Hom}_{\mathbb{Z}}(P,\mathbb{Z}) \) (simple coroots) relative to the pairing \( \langle \cdot, \cdot \rangle \). Define simple roots \( \{ \alpha_i \}_{i=0}^N \subset P \) so that \( a_{ij} = \langle h_i, \alpha_j \rangle \) is given by following Cartan matrix \( A = (a_{ij})_{0 \leq i, j \leq N} \).
The associated Dynkin diagram is

![Dynkin Diagram]

We put

\[ l_0 = 1, \quad l_1 = \cdots = l_N = -1, \]  

and introduce a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h}^* = P \otimes \mathbb{Z} \mathbb{C} \) by

\[ l_i \langle h_i, \lambda \rangle = (\alpha_i, \lambda) \]  

for any \( \lambda \in P, \quad 0 \leq i \leq N. \)

**Definition 2.1 (Yamane [11] Theorem 10.5.1).** Let \( U_q'(D(N,1)) \) be the associative algebra over \( \mathbb{Q}(q) \) with 1 generated by \( e_i, f_i, q^h \) \((0 \leq i \leq N, \, h \in P^*)\) with the following relations.

\[ q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \]  

for \( h, h' \in P^* \),

\[ q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{(-h,\alpha_i)} f_i \]  

for \( 0 \leq i \leq N, h \in P^* \),

\[ e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \]  

for \( 0 \leq i, j \leq N \),

\[ \sum_{\nu=0}^{1+|a_{ij}|} (-1)^{\nu} \left[ 1 + \frac{|a_{ij}|}{\nu} \right] e_i^{1+|a_{ij}|-\nu} e_j e_i^\nu = \sum_{\nu=0}^{1+|a_{ij}|} (-1)^{\nu} \left[ 1 + \frac{|a_{ij}|}{\nu} \right] f_i^{1+|a_{ij}|-\nu} f_j f_i^\nu = 0, \]  

for \( 1 \leq i \leq N, 0 \leq j \leq N, i \neq j \),

\[ e_0^2 = 0, \quad f_0^2 = 0, \]

where

\[ t_i = q^{l_i h_i}, \]

\[ p(0) = 1, \quad p(i) = 0 \text{ for } 1 \leq i \leq N, \]
We have
\[ \sigma = ( -1 )^p i e_i, \quad \sigma ( f_i ) = ( -1 )^p ( i ) f_i, \quad \sigma ( q^h ) = q^h \quad ( h \in P ). \]
Then the quantized Lie superalgebra \( U_q ( D ( N, 1 ) ) \) is defined to be
\[ U_q ( D ( N, 1 ) ) = U'_q ( D ( N, 1 ) ) \oplus U''_q ( D ( N, 1 ) ) \sigma \]
with the algebra structure given by \( \sigma^2 = 1 \) and \( \sigma u \sigma = \sigma ( u ) \) for \( u \in U'_q ( D ( N, 1 ) ) \).

The Hopf algebra structure on \( U_q ( D ( N, 1 ) ) \) is given as follows.
The comultiplication \( \Delta \), the antipode \( S \), and the counit \( \varepsilon \) are defined by
\[
\begin{align*}
\Delta ( \sigma ) &= \sigma \otimes \sigma, & S ( \sigma ) &= \sigma, & \varepsilon ( \sigma ) &= 1, \\
\Delta ( q^h ) &= q^h \otimes q^h, & S ( q^h ) &= q^{-h}, & \varepsilon ( q^h ) &= 1, \\
\Delta ( e_i ) &= e_i \otimes t_i^{-1} + \sigma^v ( i ) \otimes e_i, & S ( e_i ) &= -\sigma^v ( i ) e_i t_i, & \varepsilon ( e_i ) &= 0, \\
\Delta ( f_i ) &= f_i \otimes 1 + \sigma^v ( i ) t_i \otimes f_i, & S ( f_i ) &= -\sigma^v ( i ) t_i^{-1} f_i, & \varepsilon ( f_i ) &= 0.
\end{align*}
\]

We define an orthogonal basis \( \{ \delta, \varepsilon_1, \ldots, \varepsilon_N \} \) of \( \mathfrak{h}^* \) by
\[ \alpha_0 = \delta - \varepsilon_1, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N, \quad \alpha_N = \varepsilon_{N-1} + \varepsilon_N. \]
We have
\[ ( \delta, \delta ) = 1, \quad ( \varepsilon_i, \varepsilon_i ) = -1 \quad \text{for } 1 \leq i \leq N, \]
\[ \omega_0 = \delta, \]
\[ \omega_i = -\delta + \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \leq i \leq N - 2, \]
\[ \omega_{N-1} = \frac{1}{2} ( -\delta + \varepsilon_1 + \cdots + \varepsilon_{N-1} - \varepsilon_N ), \]
\[ \omega_N = \frac{1}{2} ( -\delta + \varepsilon_1 + \cdots + \varepsilon_{N-1} + \varepsilon_N ). \]
The set of even and odd roots are \( \Delta_i = \Delta_i^+ \cup ( -\Delta_i^+ ) \) \( ( i = 0, 1 ) \), where
\[ \Delta_0^+ = \{ \varepsilon_i \pm \varepsilon_j, 2 \delta \}_{1 \leq i < j \leq N}, \quad \Delta_1^+ = \{ \pm \varepsilon_i + \delta \}_{1 \leq i \leq N}. \]
We put \( \overline{\Delta_1} = \Delta_1 \) and
\[ \rho_i = \sum_{\beta \in \Delta_i^+} \beta \quad \text{for } i = 0, 1, \quad \rho = \rho_0 - \rho_1. \]

We denote by \( V ( \lambda ) \) the irreducible lowest weight module with lowest weight \( \lambda \).

**Definition 2.2 (Kac [6] Theorem 1).** For \( \lambda \in P \), \( V ( \lambda ) \) is a typical representation if
\[ ( \lambda - \rho, \beta ) \neq 0 \text{ for any } \beta \in \overline{\Delta_1}. \]
In this case, \( \lambda \) is called a typical weight. A weight which is not typical is called an atypical weight.

In this paper we will study \( V ( \lambda ) \) where \( \lambda = n_0 \omega_0 - \sum_{i=1}^N n_i \omega_i \) with \( n_i \in \mathbb{Z}_{\geq 0} \) for \( 0 \leq i \leq N \). It is typical if and only if \( n_0 \geq 1 \).
3 Category $\mathcal{O}_{\text{int}}$ and Crystal Base

**Definition 3.1** ([3] Definition 2.2). $\mathcal{O}_{\text{int}}$ is the category whose objects are $U_q(D(N,1))$-modules $M$ and whose morphisms are $U_q(D(N,1))$-linear homomorphisms satisfying the following conditions:

(i) $M$ has a weight decomposition $M = \bigoplus_{\lambda \in \mathbb{P}} M_{\lambda}$, where $M_{\lambda} = \{ u \in M; q^h u = q^{\langle h, \lambda \rangle} u \text{ for any } h \in \mathbb{P}^* \}$,

(ii) $\dim M_{\lambda} < \infty$ for any $\lambda \in \mathbb{P}$,

(iii) For $1 \leq i \leq N$, $M$ is locally $U_q(D(N,1))_i$-finite, that is, $\dim U_q(D(N,1))_i u < \infty$ for any $u \in M$,

(iv) For any $\lambda \in \mathbb{P}$, $M_{\lambda} \neq 0$ implies $\langle h_0, \lambda \rangle \geq 0$,

(v) $e_0 M_{\lambda} = f_0 M_{\lambda} = 0$ for any $\lambda \in \mathbb{P}$ such that $\langle h_0, \lambda \rangle = 0$.

We define the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ on $M \in \mathcal{O}_{\text{int}}$. For any $u \in M_{\lambda}$, and $i = 1, \ldots, N$, let

$$u = \sum_{k \geq 0, -\langle h_i, \lambda \rangle} f_i^{(k)} u_k$$

be the unique expression with $e_i u_k = 0$ for each $k$. We define

$$\tilde{e}_i u = \sum_k f_i^{(k-1)} u_k,$$

$$\tilde{f}_i u = \sum_k f_i^{(k+1)} u_k.$$

For $i = 0$, define

$$\tilde{e}_0 u = q_0^{-1} t_0 e_0 u,$$  \hspace{1cm} (3.1)

$$\tilde{f}_0 u = f_0 u.$$  \hspace{1cm} (3.2)

We set

$$A = \left\{ \frac{f}{g}; f, g \in \mathbb{Q}[q], g(0) \neq 0 \right\}.$$

**Definition 3.2** ([3] Definition 2.3, 2.4). Let $M$ be a $U_q(D(N,1))$-module in the category $\mathcal{O}_{\text{int}}$. A crystal base of $M$ is a pair $(L, B)$ such that

(L1) $L$ is a free $A$-submodule satisfying $M = L \otimes_A \mathbb{Q}(q)$,

(L2) $\sigma L = L$ and $L$ has a weight decomposition $L = \bigoplus_{\lambda \in \mathbb{P}} L_{\lambda}$ with $L_{\lambda} = L \cap M_{\lambda}$,

(L3) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for $0 \leq i \leq N$,

(B1) $B$ is a subset of $L/qL$ such that $\sigma b = \pm b$ for any $b \in B$, and $B = \bigcup_{\lambda \in \mathbb{P}} B_{\lambda}$ with $B_{\lambda} = B \cap (L_{\lambda}/qL_{\lambda})$. 

(B2) $B$ is a pseudo-base of $L/qL$, that is, $B = B' \cup (-B')$ for a $\mathbb{Q}$ basis $B'$ of $L/qL$.

(B3) $\bar{e}_i B \subset B \cup \{ 0 \}$ and $\bar{f}_i B \subset B \cup \{ 0 \}$ for $0 \leq i \leq N$.

(B4) For any $b, b' \in B$, and $0 \leq i \leq N$, $b = \bar{f}_i b'$ if and only if $\bar{e}_i b = b'$.

We often denote $(L, B)$ by $B$.

Let $\eta$ be the anti-automorphism of $U_q(D(N,1))$ defined by

$$
\eta(\sigma) = \sigma, \quad \eta(q^h) = q^h, \quad \eta(e_i) = q_i f_i t_i^{-1}, \quad \eta(f_i) = q_i^{-1} t_i e_i.
$$

A symmetric bilinear form $(\cdot, \cdot)$ on a $U_q(D(N,1))$-module $M$ is called polarization if $(au, v) = (u, \eta(a)v)$ holds for any $u, v \in M$ and $a \in U_q(D(N,1))$.

**Definition 3.3.** A crystal base $(L, B)$ for a $U_q(D(N,1))$-module $M$ is polarizable if there exists a polarization $(\cdot, \cdot)$ of $M$ such that $(L, L) \subset A$, and with respect to the induced $\mathbb{Q}$-valued symmetric bilinear form $(\cdot, \cdot)_0$ on $L/qL$,

$$(b, b')_0 = \begin{cases} 
\pm 1 & \text{if } b' = \pm b, \\
0 & \text{otherwise}
\end{cases}
$$

holds for any $b, b' \in B$.

**Remark 3.4.** Theorem 8 of [5] implies that condition (iv) of Definition 3.1 fails for all non-trivial irreducible finite dimensional $U_q(D(N,1))$-modules. For this reason, we treat infinite dimensional lowest weight modules.

For $b \in B$, $0 \leq i \leq N$, we define

$$
\varepsilon_i(b) = \max\{ n \in \mathbb{Z}_{\geq 0}; \bar{e}_i^n b \neq 0 \}, \quad \varphi_i(b) = \max\{ n \in \mathbb{Z}_{\geq 0}; \bar{f}_i^n b \neq 0 \}.
$$

We write $\text{wt}(b) = \lambda$ for $b \in B_\lambda$.

**Proposition 3.5 ([3] Proposition 2.8).** Let $(L_i, B_i)$ be polarizable crystal bases of $U_q(D(N,1))$-modules $M_i \in \mathcal{O}_{\text{int}}$, $i = 1, 2$. Then $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a polarizable crystal base of $M_1 \otimes M_2$, and the actions of $\bar{e}_i$ and $\bar{f}_i$ are given as follows.

(1) $i = 1, 2, \ldots, N$

$$
\bar{e}_i(b_1 \otimes b_2) = \begin{cases} 
b_1 \otimes \bar{e}_i(b_2) & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2), \\
\bar{e}_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2),
\end{cases}
$$

(3.3)

$$
\bar{f}_i(b_1 \otimes b_2) = \begin{cases} 
b_1 \otimes \bar{f}_i(b_2) & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2), \\
\bar{f}_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2),
\end{cases}
$$

(3.4)
(2) \( i = 0 \)

\[
\widehat{e}_0(b_1 \otimes b_2) = \begin{cases} 
\sigma b_1 \otimes \widehat{e}_0(b_2) & \text{if } \langle h_0, \text{wt}(b_1) \rangle = 0, \\
\widehat{e}_0(b_1) \otimes b_2 & \text{if } \langle h_0, \text{wt}(b_1) \rangle > 0,
\end{cases}
\]  

(3.5)

\[
\widehat{f}_0(b_1 \otimes b_2) = \begin{cases} 
\sigma b_1 \otimes \widehat{f}_0(b_2) & \text{if } \langle h_0, \text{wt}(b_1) \rangle = 0, \\
\widehat{f}_0(b_1) \otimes b_2 & \text{if } \langle h_0, \text{wt}(b_1) \rangle > 0.
\end{cases}
\]  

(3.6)

Note that in (3.3)-(3.4) the inequality signs are opposite to those for ordinary Lie algebra case. This is due to the negative sign of \( l_i \).

Let \( (L, B) \) be a crystal base for a \( U_q(D(N, 1)) \)-module in \( O_{int} \). We define

\[
LW(B) = \{ b \in B ; \widehat{f}_i(b) = 0 \text{ for } 0 \leq i \leq N \},
\]

\[
\overline{LW}(B) = \{ b \in B ; \widehat{f}_i(b) = 0 \text{ for } 1 \leq i \leq N \}.
\]

**Lemma 3.6.** Let \( (L_i, B_i) \) be as in Proposition 3.5. Assume \( LW(B_1) \neq \emptyset \) for \( i = 1, 2 \). Then the element of \( LW(B_1 \otimes B_2) \) is of the form \( u_1 \otimes v \) with \( u_1 \in LW(B_1) \).

**Proof.** Assume that \( u \otimes v \in LW(B_1 \otimes B_2) \) with \( u \notin LW(B_1) \). We have two cases;

Case 1: \( \widehat{f}_i(u) \neq 0 \) for some \( 1 \leq i \leq N \).

Since \( 0 = \widehat{f}_i(u \otimes v) = u \otimes \widehat{f}_i(v) \), (3.4) implies \( \varepsilon_i(u) < \varphi_i(v) \). Hence \( \widehat{f}_i(v) \neq 0 \). This is a contradiction.

Case 2: \( \widehat{f}_0u = 0 \), \( \widehat{f}_i u = 0 \) for \( 1 \leq i \leq N \).

Since \( 0 = \widehat{f}_0(u \otimes v) = u \otimes \widehat{f}_0(v) \), (3.6) implies \( \langle h_0, \text{wt}(u) \rangle = 0 \). This contradicts Definition 3.1(v).

\n
In [12], the algebra \( U_q(D(2, 1; \alpha)) \) with \( \alpha \leq -2 \) is considered. Lemma 3.6 fails in this case since the \( l_i \) have both positive and negative signs.

## 4 Results on crystal bases for \( U_q(D(N)) \)-modules

The even part of \( U_q'(D(N, 1)) \) is the eigenspace of \( \sigma \) with eigenvalue +1, denoted by \( U_q'(D(N, 1)_0) \). In our case it is given by \( U_q(D(N)) \otimes U_q(C(1)) \), where \( U_q(D(N)) \) is the subalgebra with generators \( e_i, f_i, q^{h_i} \) \( (1 \leq i \leq N) \), and \( U_q(C(1)) \simeq U_q(s_{\mathfrak{sl}_2}) \) is the one generated by \( E, F, q^H \), where \( H = 2(h_0 - h_1 - \cdots - h_{N-2}) - h_{N-1} - h_N \), \( E \) and \( F \) are the elements corresponding to the root \( 2(\alpha_0 + \cdots + \alpha_{N-2}) + \alpha_{N-1} + \alpha_N \). In this section we recall known properties of crystal bases for \( U_q(D(N)) \).

We denote the irreducible lowest weight module of \( U_q(D(N)) \) with lowest weight \( \Lambda \) and its crystal base by \( \overline{V}(\Lambda) \) and \( \overline{B}(\Lambda) \) respectively, and refer to the crystal base as a \( U_q(D(N)) \)-crystal for short. Let \( \{ \Lambda_i \}_{i=1}^N \) be the fundamental weights of \( U_q(D(N)) \). We denote \( \overline{B}(-\Lambda_{N-1}) \) by \( \overline{B}_{sp} \), and \( \overline{B}(-\Lambda_N) \) by \( \overline{B}_{sp} \).

The crystal bases of spin representations are realized as

\[
\overline{B}_{sp}^\pm \simeq \{ b = (i_1, \ldots, i_N); i_1, \ldots, i_N = \pm, \text{ and } - \text{ appears an even number of times} \},
\]
$\overline{B}_{sp} \cong \{ b = (i_1, \ldots, i_N); i_1, \ldots, i_N = \pm \text{ and } - \text{ appears an odd number of times} \}$,
with the lowest weight vectors $(+, \ldots, +)$ and $(+, \ldots, +, -)$ respectively. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ read

\begin{align*}
\tilde{f}_l(i_1, i_2, \ldots, i_N) &= \begin{cases} (i_1, \ldots, l, l+1, \ldots, i_N) & \text{if } i_l = -, i_{l+1} = +, \\
0 & \text{otherwise}, \end{cases} \\
\tilde{e}_l(i_1, i_2, \ldots, i_N) &= \begin{cases} (i_1, \ldots, l, l+1, \ldots, i_N) & \text{if } i_l = +, i_{l+1} = -, \\
0 & \text{otherwise}, \end{cases}
\end{align*}

for $1 \leq l \leq N - 1$ and,

\begin{align*}
\tilde{f}_N(i_1, i_2, \ldots, i_N) &= \begin{cases} (i_1, \ldots, N-1, N) & \text{if } i_{N-1} = -, i_N = -, \\
0 & \text{otherwise}, \end{cases} \\
\tilde{e}_N(i_1, i_2, \ldots, i_N) &= \begin{cases} (i_1, \ldots, N-1, N) & \text{if } i_{N-1} = +, i_N = +, \\
0 & \text{otherwise}. \end{cases}
\end{align*}

Set

\[ \Xi_0 = 0, \quad \Xi_i = \Lambda_i \quad (1 \leq i \leq N - 2), \quad \Xi_{N-1} = \Lambda_{N-1} + \Lambda_N, \quad \Xi_N' = 2\Lambda_{N-1}, \quad \Xi_N = 2\Lambda_N. \]

**Proposition 4.1 (Nakashima [10]).** We have the decomposition of crystals

\[ \overline{B}^+_sp \otimes \overline{B}^+_sp = \bigoplus_{0 \leq k \leq N \mod 2} \overline{B}(-\Xi_k), \]

\[ \overline{B}^+_sp \otimes \overline{B}^-_sp = \bigoplus_{0 \leq k \leq N-1 \mod 2} \overline{B}(-\Xi_k). \]

For $0 \leq i \leq N$, the lowest weight vector corresponding to the connected component $\overline{B}(-\Xi_i)$ is

\[(+, \ldots, +) \otimes \underbrace{+}_{i}, +, \ldots, +, - , \ldots, -. \]

Each connected component $\overline{B}(-\Xi_k)$ can be given an explicit characterization. For that purpose it is convenient to use an alternative description of $\overline{B}_{sp}^\pm$ in terms of semi-standard Young tableaux [7]. Consider the set of letters $S = \{1, \ldots, N, -N, \ldots, -1\}$. We introduce an ordering $\prec$ on $S$ by

\[ 1 < 2 < \cdots < N-1 < \frac{N}{N} < \frac{-N-1}{N} < \cdots < \frac{-2}{N} < \frac{-1}{N}. \]
Then, there is an isomorphism of crystals

\[
\mathcal{B}_{sp}^+ \sqcup \mathcal{B}_{sp}^- \cong \left\{ \begin{array}{l}
\begin{array}{c}
\begin{array}{c}
 a_1 \\
 \vdots \\
 a_N
\end{array}
\end{array} \\
(1) \ a_1, \ldots, a_N \in S \\
(2) \ a_1 \prec \cdots \prec a_N \\
(3) \ a \text{ and } \overline{a} \text{ do not appear simultaneously}
\end{array}
\right\}.
\] (4.4)

In this description, \(a\) corresponds to the \(a\)-th + and \(\overline{a}\) corresponds to the \(a\)-th – in the former description.

We introduce notations of Young tableaux for convenience.

**Notation 4.2.**
1. A Young tableau \(t(a_1, \ldots, a_N)\) is denoted by \(t_{a_1} \cdots t_{a_N}\).
2. A skew Young tableau \(t(a_1, \ldots, a_N; b_1, \ldots, b_N)\) is denoted by

\[
\begin{array}{c}
\begin{array}{c}
 a_1 \\
 \vdots \\
 a_N
\end{array} \\
\begin{array}{c}
 b_1 \cdots b_N
\end{array}
\end{array}
\]

is denoted by \(t(a_1, \ldots, a_N; b_1, \ldots, b_N)\).

Note that in this skew Young tableau, the number of rows which has two boxes is \(k\).

**Definition 4.3.** A skew Young tableau \(t(a_1, \ldots, a_N; b_1, \ldots, b_N)\) is semi-standard if

1. \(a_1, \ldots, a_N\) satisfy (1),(2) and (3) in (4.4),
2. \(b_1, \ldots, b_N\) satisfy (1),(2) and (3) in (4.4),
3. \(b_r \leq a_{N-k+r}\) holds for \(1 \leq r \leq k\).

**Proposition 4.4 (Koga[8]).** (1) Assume \(u \otimes v = t(a_1, \ldots, a_N) \otimes t(b_1, \ldots, b_N) \in \mathcal{B}_{sp}^+ \otimes \mathcal{B}_{sp}^+\). Then we have

(1A) For \(0 \leq k \leq N-2, \ k \equiv N \mod 2\),

\[
u \otimes v \in \mathcal{B}(-\Xi_k) \iff \begin{array}{l}
t(a_1, \ldots, a_{N-k}; a_{N-k+1}, \ldots, a_N; b_1, \ldots, b_k; b_{k+1}, \ldots, b_N) \text{ is semi-standard and} \\
t(a_1, \ldots, a_{N-k}; a_{N-k+1}, \ldots, a_N; b_1, \ldots, b_k; b_{k+1}, \ldots, b_N) \text{ is not semi-standard,}
\end{array}
\]

(1B)

\[
u \otimes v \in \mathcal{B}(-\Xi_N) \iff t(; a_1, \ldots, a_N; b_1, \ldots, b_N) \text{ is semi-standard.}
\]
(2) Assume \( u \otimes v = t(a_1, \ldots, a_N) \otimes t(b_1, \ldots, b_N) \in B_{sp}^+ \otimes B_{sp}^-. \) Then we have

\[(2A) \quad \text{For } 0 \leq k \leq N - 3, k \equiv N - 1 \mod 2
\]

\[u \otimes v \in B(-\Xi_k) \iff t(a_1, \ldots, a_{N-k}; a_{N-k+1}, \ldots, a_N|b_1, \ldots, b_k; b_{k+1}, \ldots, b_N) \text{ is semi-standard and}
\]

\[t(a_1, \ldots, a_{N-k-2}; a_{N-k-1}, \ldots, a_N|b_1, \ldots, b_{k+2}; b_{k+3}, \ldots, b_N) \text{ is not semi-standard,}
\]

\[(2B) \quad u \otimes v \in B(-\Xi_{N-1}) \iff t(a_1; a_2, \ldots, a_{N-1}; b_N) \text{ is semi-standard.}
\]

5 Crystal Base of \( V(-\omega_N) \)

We describe an analogue of spin representations for \( U_q(D(N, 1)) \) using the \( U_q(D(N)) \)-crystals.

**Proposition 5.1.** The irreducible lowest weight module \( V(-\omega_N) \) with lowest weight \(-\omega_N\) has a basis over \( \mathbb{Q}(q) \)

\[
\{ v(i_1, \ldots, i_N)_{2n} : n \in \mathbb{Z}_{\geq 0}, (i_1, \ldots, i_N) \in B_{sp}^+ \} \cup
\]

\[\{ v(i_1, \ldots, i_N)_{2n+1} : n \in \mathbb{Z}_{\geq 0}, (i_1, \ldots, i_N) \in B_{sp} \}
\]

with the lowest weight vector \( v(+, \ldots, +)_0 \) such that the actions of \( \sigma \) and \( e_i \) read as follows;

\[\sigma v(+, \ldots, +)_0 = v(+, \ldots, +)_0, \quad (5.1)\]

\[e_i(v(i_1, \ldots, i_N)_k) = \begin{cases} v(i'_1, \ldots, i'_N)_k & \text{if } \bar{e}_i(i_1, \ldots, i_N) = (i'_1, \ldots, i'_N) \neq 0 \text{ in } B_{sp}^+, \\
0 & \text{otherwise,} \end{cases} \quad (5.2)\]

for \( 1 \leq i \leq N, \) and

\[e_0(v(i_1, i_2, \ldots, i_N)_k) = \begin{cases} q^{-k}v(+, i_2, \ldots, i_N)_{k+1} & \text{if } i_1 = -, \\
0 & \text{otherwise.} \end{cases} \quad (5.3)\]

**Proof.** Let \( v(+, \ldots, +)_0 \) be the lowest weight vector of \( V(-\omega_N). \)

Claim 1: \( V(-\omega_N) \) is infinite dimensional.

As \( U'_q(D(N, 1)_0) \supset U_q(C_1)\)-module, the weight of \( v(+, \ldots, +)_0 \) is 1. Hence \( V(-\omega_N) \) is infinite dimensional.

Because \(-\omega_N\) is \(-\Lambda_N\) as a weight of \( U_q(D(N)) \), \( v(+, \ldots, +)_0 \) is the lowest weight vector of the spin representation. For \( (i'_1, \ldots, i'_N) \in B_{sp}^+ \), we define \( v(i'_1, \ldots, i'_N)_0 \) by

\[v(i'_1, \ldots, i'_N)_0 = e_i(v(i_1, \ldots, i_N)_0) \quad \text{where } \bar{e}_i(i_1, \ldots, i_N) = (i'_1, \ldots, i'_N). \quad (5.4)\]
In $\mathfrak{p}(-\Lambda_N)$, $\bar{e}_i = e_i$ on each weight vector for $1 \leq i \leq N$. Moreover, if $\bar{e}_i \bar{e}_j = \bar{e}_j \bar{e}_i$ for $1 \leq i \neq j \leq N$ in $B_{sp}^+$, $i$-th node and $j$-th node are not connected in the Dynkin diagram. Hence $e_i e_j = e_j e_i$. These mean that (5.4) is well-defined. (5.2) holds for $k = 0$ by the definition.

Claim 2: $e_0(v(+,\ldots,+)_0) = 0$
Because $f_i(e_0(v(+,\ldots,+)_0)) = 0$ for $0 \leq i \leq N$ by (2.4), $e_0(v(+,\ldots,+)_0)$ is a singular vector if it is not 0. This contradicts the irreducibility of $V(-\omega_N)$.

Claim 3: $e_0(v(+,i_2,\ldots,i_N)_0) = 0$
Let $v(+,i_2,\ldots,i_N)_0 = e_{i_1} \cdots e_{i_p}(v(+,\ldots,+)_0)$. By (5.4), $(+,i_2,\ldots,i_N) = \bar{e}_{i_1} \cdots \bar{e}_{i_p}(+,\ldots,+) +$ in $B_{sp}^+$. Because $\bar{e}_1$ changes $(+,-\ldots)$ into $(-,\ldots)$ in $B_{sp}^+$, it follows that $i_1,\ldots,i_p \in \{2,\ldots,N\}$. By (2.5) and Claim 2, we have
\[
e_0(v(+,i_2,\ldots,i_N)_0) = e_0 e_{i_1} \cdots e_{i_p} v(+,\ldots,+)_0 \\
= e_{i_1} \cdots e_{i_p} e_0 v(+,\ldots,+)_0 \\
= 0.\tag{5.5}
\]

Claim 4: $e_0(v(\ldots,+,\ldots,+,+)_0) \neq 0$
Because $-$ appears an even number of times in $(-,i_2,\ldots,i_N) \in B_{sp}^+$, we may assume
\[
(-,i_2,\ldots,i_N) = \bar{e}_{i_1} \cdots \bar{e}_{i_r}(-,+\ldots,+) \\
\text{with} \quad l_1,\ldots,l_r \in \{2,\ldots,N\}.
\]
This implies
\[
v(-,i_2,\ldots,i_N)_0 = e_{i_1} \cdots e_{i_r} v(-,\ldots,+)_0 \\
\text{with} \quad l_1,\ldots,l_r \in \{2,\ldots,N\}
\]
by (5.4). Hence,
\[
0 \neq e_0(v(-,i_2,\ldots,i_N)_0) = e_0 e_{i_1} \cdots e_{i_r} v(-,\ldots,+)_0 \\
= e_{i_1} \cdots e_{i_r} e_0 v(-,\ldots,+)_0.
\]
We put
\[
v(+,\ldots,+,+)_1 = e_0 v(-,\ldots,+)_0.\tag{5.6}
\]
Since $f_i(v(-,\ldots,+,+)_0) = 0$ holds for $2 \leq i \leq N$, we have
\[
f_i v(+,\ldots,+,+)_1 = f_i e_0 v(-,\ldots,+,+)_0 \\
= 0 \quad \text{for} \quad 1 \leq i \leq N,
\]
where we used (5.5) for $i = 1$. This implies that $v(+,\ldots,+,+)_1$ is the lowest weight vector with lowest weight $-\Lambda_{N-1}$ as $U_q(D(N))$-module. As in the case of $k = 0$, we define $v(i'_1,\ldots,i'_N)_1$ for $(i'_1,\ldots,i'_N) \in B_{sp}^-$ by
\[
v(i'_1,\ldots,i'_N)_1 = e_i(v(i_1,\ldots,i_N)_1), \quad \text{where} \quad \bar{e}_i(i_1,\ldots,i_N) = (i'_1,\ldots,i'_N).
\]
Then (5.2) holds for $k = 1$.

**Claim 5:**

$$e_0(v(-, i_2, \ldots, i_N)_0) = v(+, i_2, \ldots, i_N)_1$$ for any $(-, i_2, \ldots, i_N) \in \overline{B}_{sp}^+$

By the definition of $v(+, i_2, \ldots, i_N)_1$, we have

$$v(+, i_2, \ldots, i_N)_1 = e_{i_1} \cdots e_{i_r} e_0 v(-, +, \ldots, +, -)_0$$

with $i_1, \ldots, i_r \in \{2, \ldots, N\}$

$$= e_0 e_{i_1} \cdots e_{i_r} v(-, +, \ldots, +, -)_0$$

$$= e_0(v(-, i_2, \ldots, i_N)_0).$$

In the cases of $k \geq 1$, we put

$$v(+, \ldots, +, -, -)_{k+1} = q^{-k} e_0(v(-, +, \ldots, +, -)_k)$$

in place of (5.6). Then the rest of the proof is similar. ■

**Remark 5.2.** Suppose $wt(v(i_1, \ldots, i_N)_k) = n_0 \omega_0 - \sum_{i=1}^{N} n_i \omega_i$. Then

$$n_0 = \begin{cases} 
  k & \text{if } i_1 = +, \\
  k + 1 & \text{if } i_1 = -.
\end{cases}$$

This is because among $e_i$’s only $e_1$ changes the value of $n_0$.

**Proposition 5.3.** The irreducible lowest weight module $V(-\omega_N)$ has a polarizable crystal base $(L, B)$ given as follows.

$$L = \bigoplus_{(i_1, \ldots, i_N) \in \overline{B}_{sp}^+, \ n \in \mathbb{Z}_{\geq 0}} Av(i_1, \ldots, i_N)_{2n} \oplus \bigoplus_{(i_1, \ldots, i_N) \in \overline{B}_{sp}^-, \ n \in \mathbb{Z}_{\geq 0}} Av(i_1, \ldots, i_N)_{2n+1}$$

(5.7)

$$B = \{ \pm v(i_1, \ldots, i_N)_{2n} \ mod \ qL ; (i_1, \ldots, i_N) \in \overline{B}_{sp}^+, n \in \mathbb{Z}_{\geq 0} \} \sqcup$$

$$\{ \pm v(i_1, \ldots, i_N)_{2n+1} \ mod \ qL ; (i_1, \ldots, i_N) \in \overline{B}_{sp}^-, n \in \mathbb{Z}_{\geq 0} \}$$

(5.8)

The Kashiwara operators $e_i$ act on $B$ as (we omit mod $qL$)

$$\tilde{e}_i v(i_1, \ldots, i_N)_k = \begin{cases} 
  v(i_1', \ldots, i_N)_k & \text{if } \tilde{e}_i(i_1, \ldots, i_N) = (i_1', \ldots, i_N) \neq 0 \ in \ \overline{B}_{sp}^\pm, \\
  0 & \text{otherwise},
\end{cases}$$

for $1 \leq i \leq N$ and,

$$\tilde{e}_0 v(i_1, i_2, \ldots, i_N)_k = \begin{cases} 
  v(+, i_2, \ldots, i_N)_{k+1} & \text{if } i_1 = -, \\
  0 & \text{otherwise}.
\end{cases}$$

(5.9)

(5.10)

**Proof.** First, we show that $L$ is a crystal lattice.
It suffices to show (L3). We have only to show it in the case of $i = 0$ because the remaining cases are the same as in $U_q(D(N))$. Since $\langle h_0, wt(v(-, i_2, \ldots, i_N)_k) \rangle = k + 1$,

$$
\bar{e}_0 v(-, i_2, \ldots, i_N)_k = q_0^{-1} t_0 e_0 v(-, i_2, \ldots, i_N)_k = q_0^{-1} q_0^{k+1} q^{-k} v(+, i_2, \ldots, i_N)_{k+1} = v(+, i_2, \ldots, i_N)_{k+1} \tag{5.11}
$$

and

$$
\bar{f}_0 v(+, i_2, \ldots, i_N)_{k+1} = f_0 v(+, i_2, \ldots, i_N)_{k+1} = q^k f_0 e_0 v(-, i_2, \ldots, i_N)_k = \frac{q^{2k+2} - 1}{q^2 - 1} v(-, i_2, \ldots, i_N)_k \in L. \tag{5.12}
$$

Next, we show that $(L, B)$ is a crystal base. (B1) and (B2) follow from the definition of $B$. (B3) and (B4) follow from (5.11) and (5.12).

Finally the following symmetric bilinear form on $V(-\omega_N)$ is a polarization.

$$
(v(i_1, \ldots, i_N)_0, v(i_1, \ldots, i_N)_0) = 1,
$$

$$
(v(i_1, \ldots, i_N)_k, v(i_1, \ldots, i_N)_k) = \prod_{j=1}^{k} \frac{q^{2j} - 1}{q^2 - 1} \quad \text{if} \; k \geq 1,
$$

$$
= 0 \quad \text{otherwise}.
$$

Hence, $(L, B)$ is a polarizable crystal base. \[\blacksquare\]

Figure 1 is the crystal graph of $B(-\omega_4)$ of $D(4, 1)$ (we omit $v$ and $\mod qL$). The 8 nodes connected with each other by $i$-arrow $(1 \leq i \leq 4)$ form the crystal graph of $\hat{B}^{\pm}_{sp}$ of $U_q(D(4))$. The rightmost 0-arrow for example changes the first signature from $+$ into $-$ and the integer from 0 into 1. The coefficient of $\omega_0$ increases 1 each time we cross 1-arrow from the lower right to the upper left.
6 Crystal Bases for Fundamental Representations

It is natural to ask which representations admit crystal base. Next theorem, which is one of our main results, is the answer. Our tool is the decomposition of $U_q(D(N))$-crystals.

**Theorem 6.1.** The irreducible lowest weight module $V(\lambda)$ with the lowest weight

$$\lambda = n_0\omega_0 - \sum_{i=1}^{N} n_i\omega_i, \quad n_i \in \mathbb{Z}_{\geq 0} \quad \text{for } 0 \leq i \leq N$$

(6.1)

admits a polarizable crystal base.

**Proof.** We prove this when $N$ is even. The proof for odd $N$ is similar.

We show that

$$LW(B(-\omega_N) \otimes B(-\omega_N)) = \left\{ \begin{array}{c}
v(+, \ldots, +)_0 \otimes v(-, \ldots, -)_{2k} \\ v(+, \ldots, +)_0 \otimes v(+, \ldots, +,-, \ldots,-)_0 \\ v(+, \ldots, +)_0 \otimes v(+, \ldots, +)_0 \\ 2i \\ 1 \leq i \leq \frac{N-2}{2}
\end{array} \right\}$$

(6.2)

The element in the left hand side of (6.2) is $v(+, \ldots, +)_0 \otimes b$ for some $b \in B(-\omega_N)$ by Lemma 3.6. By (5.9) and (4.3) in Proposition 4.1,

$$\tilde{f}_i (v(+, \ldots, +)_0 \otimes b) = 0 \quad \text{for } 1 \leq i \leq N$$

$$\iff b = v(-, \ldots, -)_{2k} \text{ or } v(+, \ldots, +,-, \ldots,-)_l \quad \text{for some } k, l \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq N.$$
If \( l \geq 1 \), and \( 1 \leq j \leq N \),

\[
\tilde{f}_0(v(+, \ldots, +)_0 \otimes v(+, \ldots, +, -j, \ldots, -)) = v(+, \ldots, +)_{j-1} \otimes \tilde{f}_0(v(+, \ldots, +, -j, \ldots, -))
\]

\[
\neq 0.
\]

Hence \( l = 0 \) for this case and (6.2) follows.

As a consequence, we have

\[
B(-\omega_N) \otimes B(-\omega_N) = B(-2\omega_N) \oplus \bigoplus_{j=1}^{N-2} B(-\omega_{2j}) + \bigoplus_{k \in \mathbb{Z}_{\geq 0}} B((2k+1)\omega_0).
\]

Similarly, we have

\[
B(-\omega_N) \otimes B(-\omega_{N-1}) = B(-\omega_N - \omega_{N-1}) \oplus \bigoplus_{j=0}^{N-4} B(-\omega_{2j+1}) + \bigoplus_{k \in \mathbb{Z}_{\geq 0}} B((2k+2)\omega_0).
\]

In particular, there are polarizable crystal bases with lowest weights \( \omega_0, -\omega_1, \ldots, -\omega_N \). Together with Proposition 3.5, we obtain the desired statement.

Corollary 6.2. Let \( \lambda \) and \( \lambda' \) be

\[
\lambda = n_0\omega_0 - \sum_{i=1}^{N} n_i\omega_i, \quad \lambda' = n'_0\omega_0 - \sum_{i=1}^{N} n'_i\omega_i, \quad n_i, n'_i \in \mathbb{Z}_{\geq 0} \quad \text{for } 0 \leq i \leq N.
\]

Then the tensor product \( V(\lambda') \otimes V(\lambda) \) is completely reducible.

Proof. This is a direct consequence of Proposition 3.5.

7 Properties of \( B(\omega_0) \)

Next we treat the tensor product of modules with weights as in Corollary 6.2. Especially we are interested in the case when modules are typical because of the following lemma.

Lemma 7.1. Let \( \lambda \) and \( \lambda' \) be as in Corollary 6.2, and \( u_{\lambda'} \) be the lowest weight vector of \( B(\lambda') \). If \( \lambda' \) is typical, then

\[
u_{\lambda'} \otimes u \in LW(B(\lambda') \otimes B(\lambda)) \iff u_{\lambda'} \otimes u \in LW(B(\lambda') \otimes B(\lambda))
\]

holds for \( u \in B(\lambda) \).

Proof. Assume \( u_{\lambda'} \otimes u \in LW(B(\lambda') \otimes B(\lambda)) \). By (3.6), we have \( \tilde{f}_0(u_{\lambda'} \otimes u) = \tilde{f}_0(u_{\lambda'}) \otimes u = 0 \). Hence \( u_{\lambda'} \otimes u \in LW(B(\lambda') \otimes B(\lambda)) \).

\( \blacksquare \)
We further restrict $B(\lambda)$ in Lemma 7.1 also to be typical because the crystal base of typical representations have nice properties which will be stated in Proposition 8.2. We first study the structure of $B(\omega_0)$ because it plays an important role to investigate typical representations. $B(\omega_0)$ can be realized by the embedding in the proof of Theorem 6.1, that is $B(\omega_0) \hookrightarrow B(-\omega_N) \otimes B(-\omega_N)$ when $N$ is even, $B(\omega_0) \hookrightarrow B(-\omega_N) \otimes B(-\omega_{N-1})$ when $N$ is odd.

Roughly speaking, the proof of Theorem 6.1 shows that $B(-\omega_i)$ is the union of infinitely many $\overline{B}(-\Xi_k)$’s, where each $\overline{B}(-\Xi_k)$ is connected with others by $e_0$ and $f_0$. We describe more precisely this situation that a $U_q(D(N))$-crystal is contained in a $U_q(D(N,1))$-crystal.

**Definition 7.2.** We assume the following conditions.

1. $\lambda = n_0\omega_0 - \sum_{i=1}^{N} n_i\omega_i$, $n_j \in \mathbb{Z}_{\geq 0}$ for $0 \leq j \leq N$,
2. $\Lambda = -\sum_{i=1}^{N} l_i\Lambda_i$, $l_i \in \mathbb{Z}_{\geq 0}$ for all $i$,
3. $b \in B(\lambda)$ satisfies $\text{wt}(b) = l_0\omega_0 - \sum_{i=1}^{N} l_i\omega_i$, $l_0 \in \mathbb{Z}_{\geq 0}$,
4. $\overline{f}_i b = 0$ for $1 \leq i \leq N$.

Then we define a $U_q(D(N))$-crystal in $B(\lambda)$ by

$$\overline{B}(\Lambda; l_0) = \{e_{i_1}^\prime \cdots e_{i_p}^\prime (b); 1 \leq i_1, \ldots, i_p \leq N, p \geq 0 \} - \{0\}.$$

**Notation 7.3.** In order to relate a weight of $U_q(D(N))$ with that of $U_q(D(N,1))$, we fix some notations. Let $n_0, \ldots, n_N$ be non-negative integers.

1. For $\lambda = n_0\omega_0 - \sum_{i=1}^{N} n_i\omega_i$, we define a dominant integral weight of $U_q(D(N))$ by

$$\lambda_{\text{cr}} = -\sum_{i=1}^{N} n_i\Lambda_i.$$ 

2. For $\Lambda = -\sum_{i=1}^{N} n_i\Lambda_i$, we define a weight of $U_q(D(N,1))$ by

$$\Lambda_{\text{su}} = -\sum_{i=1}^{N} n_i\omega_i.$$ 

Note that

$$\text{wt}(LW(\overline{B}(\Lambda; l_0))) = \Lambda_{\text{su}} + l_0\omega_0 \quad \text{as } U_q(D(N,1))\text{-crystal.}$$

We now determine the places where 0-arrows exist in the above tensor products.

**Definition 7.4.** We define

$$\overline{B}(\Lambda; l) \xrightarrow{0} \overline{B}(\Lambda'; l')$$

by the condition

for any $b \otimes b' \in \overline{B}(\Lambda; l) \subset B(-\omega_N) \otimes B(-\omega_N)$ or $B(-\omega_N) \otimes B(-\omega_{N-1})$,

$$\overline{f}_0(b \otimes b') = \sigma b \otimes \overline{f}_0 b' \neq 0 \implies \overline{f}_0(b \otimes b') \in \overline{B}(\Lambda'; l'),$$

and

$$\overline{B}(\Lambda; l) \xrightarrow{0} \overline{B}(\Lambda'; l')$$

by the condition

for any $b \otimes b' \in \overline{B}(\Lambda; l) \subset B(-\omega_N) \otimes B(-\omega_N)$ or $B(-\omega_N) \otimes B(-\omega_{N-1})$,

$$\overline{f}_0(b \otimes b') = (\overline{f}_0 b) \otimes b' \neq 0 \implies \overline{f}_0(b \otimes b') \in \overline{B}(\Lambda'; l').$$
Lemma 7.5. In $B(-\omega_N) \otimes B(-\omega_N)$ and $B(-\omega_N) \otimes B(-\omega_{N-1})$, we have

(1) for $k = 2, 3, \ldots, N$
\[
B(-\Xi_k; l) \xrightarrow{0}{R} B(-\Xi_{k-1}; l - 1),
\]

(2)
\[
B(-\Xi_1; l) \xrightarrow{0}{R} B(-\Xi_0; l),
\]

(3) for $k = 2, \ldots, N - 1$
\[
B(-\Xi_{k-1}; l) \xrightarrow{0}{L} B(-\Xi_k; l - 1),
\]

(4)
\[
B(-\Xi_0; l) \xrightarrow{0}{L} B(-\Xi_1; l - 2),
\]

(5)
\[
B(-\Xi_N; l) \xrightarrow{0}{L} B(-\Xi_{N-1}; l - 1),
\]

(6)
\[
B(-\Xi_{N-1}; l) \xrightarrow{0}{L} B(-\Xi_N'; l - 1).
\]

Proof. We prove this when $N$ is even. We denote $u \otimes w = v(i_1, \ldots, i_N)_p \otimes v(j_1, \ldots, j_N)_{p'}$ by $t(a_1, \ldots, a_N)_p \otimes t(b_1, \ldots, b_N)_{p'}$. We also assume $p + p' = l$.

Let us show (1) with $1 \leq k \leq N - 2$. Assume $u \otimes w \in B(-\Xi_k; l)$ satisfies $\tilde{f}_0(u \otimes w) = \sigma u \otimes \tilde{f}_0 w \neq 0$. Let us denote $\tilde{f}_0 w = t(b'_1, \ldots, b'_N)_{p'}$. Because $\tilde{f}_0$ changes $(+, \ldots, +)_{p'}$ into $(-, \ldots, -)_{p'-1}$, $p_1 = p' - 1$, $b_1 = 1$, $b'_l = b_{l+1}$ $(1 \leq l \leq N - 1)$, and $b'_N = 1$.

By Proposition 4.4(1A), $u \otimes w \in \overline{B}(-\Xi_k; l)$ implies
\[
t(a_1, \ldots, a_{N-k}; a_{N-k+1}, \ldots, a_N|1, b_2, \ldots, b_k; b_{k+1}, \ldots, b_N) \quad \text{is semi-standard and}
\]
\[
t(a_1, \ldots, a_{N-k-2}; a_{N-k-1}, \ldots, a_N|1, b_2, \ldots, b_k; b_{k+2}, \ldots, b_N) \quad \text{is not semi-standard.}
\]

It follows that
\[
t(a_1, \ldots, a_{N-k+1}; a_{N-k+2}, \ldots, a_N|b_2, \ldots, b_k; b_{k+1}, \ldots, b_N, 1) \quad \text{is semi-standard and}
\]
\[
t(a_1, \ldots, a_{N-k-1}; a_{N-k}, \ldots, a_N|b_2, \ldots, b_k; b_{k+2}, b_{k+3}, \ldots, b_N, 1) \quad \text{is not semi-standard.}
\]

This means that $\tilde{f}_0(u \otimes w) \in \overline{B}(-\Xi_{k-1}; l')$ for some $l'$. Because the lowest weight vector of
\[
\overline{B}(-\Xi_{k-1}; l') \quad \text{is $(+, \ldots, +)_p \otimes (+, \ldots, +, -, \ldots, -)_{p'-1}$ as $U_q(D(N))$-crystal, we have $l' = p + (p' - 1) = l - 1$ by Remark 5.2.}

The rest of (1) and (2) are similar.

Let us verify (3). Assume $u \otimes w \in \overline{B}(-\Xi_{k-1}; l)$ satisfies $\tilde{f}_0(u \otimes w) = (\tilde{f}_0 u) \otimes w \neq 0$, and $\tilde{f}_0 u = t(a'_1, \ldots, a'_N)_{p_2}$. As in (1), $p_2 = p - 1$, $a_1 = 1$, $a'_l = a_{l+1}$ $(1 \leq l \leq N - 1)$, $a'_N = 1$.
By Proposition 4.4(2A),
\[ t(1, a_2, \ldots, a_{N-k+1}; a_{N-k+2}, \ldots, a_N|b_1, \ldots, b_{k-1}; b_k, \ldots, b_N) \text{ is semi-standard and} \]
\[ t(1, a_2, \ldots, a_{N-k-1}; a_{N-k}, \ldots, a_N|b_1, \ldots, b_{k+1}; b_{k+2}, \ldots, b_N) \text{ is not semi-standard.} \]

It follows that
\[ t(a_2, \ldots, a_{N-k+1}; a_{N-k+2}, \ldots, a_N, \overline{b}_1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots, b_N) \text{ is semi-standard and} \]
\[ t(a_2, \ldots, a_{N-k-1}; a_{N-k}, \ldots, a_N, \overline{b}_1, \ldots, b_{k+1}, b_{k+2}, \ldots, b_N) \text{ is not semi-standard.} \]

This means that \( \tilde{f}_0(u \otimes w) \in \overline{B}(-\Xi_k; l') \). Because the lowest weight vector of \( \overline{B}(-\Xi_k; l') \) is \((+, \ldots, +)_{p-1} \otimes (+, \ldots, +, -, \ldots, -)_{p'} \) as \( U_q(D(N)) \)-crystal, we have \( l' = l - 1 \). The proofs of (4),(5) and (6) are similar.

\[ \text{Rem. 7.6. In Lemma 10.6, (1) and (2) occur only if } p = 0 \text{ by (3.6).} \]

\[ \text{Corollary 7.7. We have} \]
\[ B(\omega_0) \cong \bigoplus_{\nu \in W} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} B(\nu; z^i(\nu) + 2n), \]
where \( W = \{-\Xi_0, \ldots, -\Xi_N, -\Xi'_N\} \). The integers \( i_\nu, z^i(\nu) \) are given as follows.

- For \( \nu = -\Xi_0 \), \( i_\nu = 2, \quad z^1(\nu) = 1, \quad z^2(\nu) = 2N + 1 \)
- For \( \nu = -\Xi_k \quad (1 \leq k \leq N - 1) \), \( i_\nu = 2, \quad z^1(\nu) = k, \quad z^2(\nu) = 2N - k \)
- For \( \nu = -\Xi'_N \text{ or } -\Xi_N \), \( i_\nu = 1, \quad z^1(\nu) = N. \)

\[ \text{Proof. } B(\omega_0) \text{ is the connected component of } B(-\omega_N) \otimes B(-\omega_N) \text{ containing } \overline{B}(0; 1) \text{ when } N \text{ is even, and of } B(-\omega_N) \otimes B(-\omega_{N-1}) \text{ when } N \text{ is odd. This corollary follows from Lemma 10.6 and Rem. 7.6.} \]

\section{Crystal bases of typical representations}

The aim of this section is to obtain a description as in Corollary 7.7 for typical representations. In this section, we assume
\[ \lambda = -\sum_{i=1}^{N} n_i \omega_i, \quad n_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq N. \quad (8.1) \]

We denote the lowest weight vector of \( B(\lambda) \) by \( u_\lambda \).

When we consider typical representations, the coefficient of \( \omega_0 \) does not matter essentially.

\[ \text{Lemma 8.1. Let } \lambda \text{ be as in (8.1) and } n \geq 1. \text{ Then there is a bijection} \]
\[ \varphi : B(\lambda + n \omega_0) \xrightarrow{\sim} B(\lambda + (n+1) \omega_0) \quad (8.2) \]
which commutes with the Kashiwara operators and
\[ \text{wt}(\varphi(b)) = \text{wt}(b) + \omega_0. \quad (8.3) \]
Proof. Note that $B(\lambda + (n + 1)\omega_0)$ is the lowest component of $B(\lambda + n\omega_0) \otimes B(\omega_0)$. We define $\varphi$ by $\varphi(b) = b \otimes u_{\omega_0}$. Lemma 3.6 implies $b \otimes u_{\omega_0} \in B(\lambda + (n + 1)\omega_0)$ for all $b \in B(\lambda + n\omega_0)$. Hence it suffices to show that $\varphi$ is surjective and that $\varphi$ commutes with $\tilde{e}_i$'s and $\tilde{f}_i$'s. To show the surjectivity, we show that for all $i$

\begin{equation}
\tilde{e}_i (\varphi (B(\lambda + n\omega_0))) \subset \varphi (B(\lambda + n\omega_0)),
\end{equation}

\begin{equation}
\tilde{f}_i (\varphi (B(\lambda + n\omega_0))) \subset \varphi (B(\lambda + n\omega_0))
\end{equation}

instead. By Proposition 10.6, $\tilde{e}_iu_{\omega_0} = 0$ for $1 \leq i \leq N$ and by (3.3), it follows that $\tilde{e}_i(b \otimes u_{\omega_0}) = (\tilde{e}_ib) \otimes u_{\omega_0}$ for $1 \leq i \leq N$. Since $\langle h_0, wt(u_{\omega_0}) \rangle > 0$, we have $\tilde{e}_0(b \otimes u_{\omega_0}) = (\tilde{e}_0b) \otimes u_{\omega_0}$. The case of $\tilde{f}_i$'s is similar. Hence (8.4) holds and $\varphi$ commutes with $\tilde{e}_i$'s and $\tilde{f}_i$'s.

In view of the above lemma, we have only to consider typical representations $B(\lambda + \omega_0)$ with $\lambda$ as in (8.1). $B(\lambda + \omega_0)$ is the connected component of $B(\omega_0) \otimes B(\lambda)$ containing $u_{\omega_0} \otimes u_\lambda$. Note that $u_\lambda$ is contained in $\overline{B}(\lambda_{cd};0)$.

The next proposition is one of the favorable properties of typical representations.

Proposition 8.2. Let $\lambda$ be as in (8.1). Then we have

\begin{equation}
B(\lambda + \omega_0) \cong \{ b \otimes u; b \in B(\omega_0), u \in \overline{B}(\lambda_{cd};0) \}.
\end{equation}

Proof. Let $J$ be the right hand side of (8.5). First we claim that $J$ is stable under $\tilde{e}_i$'s and $\tilde{f}_i$'s. Since $\overline{B}(\lambda_{cd};0) \subset B(\lambda)$ is stable under $\tilde{e}_i, \tilde{f}_i$ ($1 \leq i \leq N$), $J$ is stable under $\tilde{e}_i, \tilde{f}_i$ ($1 \leq i \leq N$). Because

\begin{equation}
\tilde{e}_0(b \otimes u) = (\tilde{e}_0b) \otimes u,
\end{equation}

\begin{equation}
\tilde{f}_0(b \otimes u) = (\tilde{f}_0b) \otimes u
\end{equation}

holds for $b \otimes u \in J$, $J$ is stable under all $\tilde{e}_i$ and $\tilde{f}_i$.

Next we show that $LW(J) = \{ u_{\omega_0} \otimes u_\lambda \}$. Assume $LW(J) \ni b \otimes u$ satisfies $b \neq u_{\omega_0}$ and $u \neq u_\lambda$. There exists $0 \leq i \leq N$ such that $\tilde{f}_ib \neq 0$. If $i \neq 0$, then by (3.4),

$$\tilde{f}_i^k(b \otimes u) = (\tilde{f}_ib) \otimes (\tilde{f}_i^{k-1}u)$$

for some $k \geq 1$ and $\tilde{f}_i^{k-1}u \in \overline{B}(\lambda_{cd};0)$ hold. Together with (8.7), we may assume $b = u_{\omega_0}$. If $\tilde{f}_iu \neq 0$ for some $1 \leq i \leq N$,

$$\tilde{f}_i(u_{\omega_0} \otimes u) = u_{\omega_0} \otimes \tilde{f}_iu \neq 0.$$ 

This contradicts to the fact that $b_{\omega_0} \otimes u \in LW(J)$. Hence $LW(J) = \{ u_{\omega_0} \otimes u_\lambda \}$. This means that $J$ is the connected component of $B(\omega_0) \otimes B(\lambda)$ containing $u_{\omega_0} \otimes u_\lambda$.

For $\nu \in W$, $\lambda$ as in Lemma 8.1, the generalized Littlewood-Richardson rule [10] gives the decomposition of the tensor product $\overline{B}(\nu) \otimes \overline{B}(\lambda_{cd})$. In the decomposition

\begin{equation}
\overline{B}(\nu) \otimes \overline{B}(\lambda_{cd}) \twoheadrightarrow \bigoplus_{j=1}^{\nu_1} \overline{B}(\mu^j(\nu, \lambda_{cd})),
\end{equation}

let $w_j$ be the lowest weight vector of $\overline{B}(\mu^j(\nu, \lambda))$, and $u_\nu \otimes u_j$ ($u_j \in \overline{B}(\lambda_{cd})$) be the corresponding vector in the left hand side.

In order to know the weight of $w_j$ as $U_q(D(N,1))$-module, we have to obtain the coefficients of $\omega_0$. 

20
**Definition 8.3.** Assume $N \geq 3$ and $u_j = \tilde{e}_{i_1} \tilde{e}_{i_2} \cdots \tilde{e}_{i_p} u_{\lambda_{cl}}$ $(1 \leq i \leq N, p \in \mathbb{Z}_{\geq 0})$ in (8.8). Then we define a positive integer

$$a(\mu^j(\nu, \lambda_{cl})) = \# \{ i ; l_i = 1 \} . \quad (8.9)$$

See Example 9.2 for the case of $N = 2$.

The decomposition of a typical $U_q(D(N, 1))$-crystal as in Corollary 7.7 is the following proposition.

**Proposition 8.4.** Let $\lambda$ be as in (8.1), $\mu^j(\nu, \lambda_{cl})$ be as in (8.8). Then we have

$$B(\lambda + \omega_0) \cong \bigoplus_{\nu \in W} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{i=1}^{\lambda_{nu, \lambda}} \bigoplus_{j=1}^{\lambda_{nu, \lambda}} \mathcal{B}(\mu^j(\nu, \lambda_{cl}); z^i(\nu) + a(\mu^j(\nu, \lambda_{cl})) + 2n) . \quad (8.10)$$

**Proof.** By Proposition 7.7 and Proposition 8.2,

$$B(\lambda + \omega_0) \cong B(\omega_0) \otimes \mathcal{B}(\lambda_{cl}; 0) \cong \bigoplus_{\nu \in W} \bigoplus_{i=1}^{\lambda_{nu, \lambda}} \bigoplus_{j=1}^{\lambda_{nu, \lambda}} \mathcal{B}(\nu; z^i(\nu) + 2n) \otimes \mathcal{B}(\lambda_{cl}; 0)$$

holds. Because $\tilde{e}_1$ makes the coefficient of $\omega_0$ increase by 1, we get

$$B(\lambda + \omega_0) \cong \bigoplus_{\nu \in W} \bigoplus_{i=1}^{\lambda_{nu, \lambda}} \bigoplus_{j=1}^{\lambda_{nu, \lambda}} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{B}(\mu^j(\nu, \lambda_{cl}); z^i(\nu) + a(\mu^j(\nu, \lambda_{cl})) + 2n) . \quad \blacksquare$$

## 9 Tensor Products of typical representations

Assume $\lambda' = -\sum_{i=1}^{N} n'_i \omega_i$, $n'_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq N$ and $\mu^j(\nu, \lambda_{cl})$ as in (8.8). We also assume

$$\mathcal{B}(\lambda_{cl})_{u_{\lambda'}} \otimes \mathcal{B}(\mu^j(\nu, \lambda_{cl})) \cong \bigoplus_{k=1}^{k_{\lambda', \nu, \lambda}} \mathcal{B}(\mu^j_k(\lambda_{cl}', \nu, \lambda_{cl})) \quad (9.1)$$

holds as in (8.8).

We state the main theorem.

**Theorem 9.1 (Main Theorem).** Assume

$$\lambda = -\sum_{i=1}^{N} n_i \omega_i , \quad \lambda' = -\sum_{i=1}^{N} n'_i \omega_i , \quad n_i, n'_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq N . \quad (9.2)$$

Then we obtain

$$B(\lambda' + \omega_0) \otimes B(\lambda + \omega_0)$$

$$= \bigoplus_{\nu \in W} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{i=1}^{\lambda_{nu, \lambda}} \bigoplus_{j=1}^{\lambda_{nu, \lambda}} \bigoplus_{k=1}^{k_{\lambda', \nu, \lambda}} \mathcal{B}(\mu^j_k(\lambda_{cl}', \nu, \lambda_{cl})) + \sum \{ z^i(\nu) + a(\mu^j(\nu, \lambda_{cl})) + a(\mu^j_k(\lambda_{cl}', \nu, \lambda_{cl})) + 2n + 1 \} \omega_0 \quad (9.3)$$

21
Proof. By Lemma 3.6, it suffices to consider \( u_{\lambda' + \omega_0} \otimes u \), \( u \in B(\lambda + \omega_0) \).

Assume \( u \in B(\mu_j(\nu, \lambda_{cl}); z_i^j(\nu) + a(\mu_j(\nu, \lambda_{cl})) + 2n) \). We have

\[
\begin{align*}
&u_{\lambda' + \omega_0} \otimes u \in LW(B(\lambda' + \omega_0) \otimes B(\lambda + \omega_0)) \\
\iff &u_{\lambda' + \omega_0} \otimes u \in LW \left\{ B(\lambda'_{cl}; 1) \otimes B(\mu^j(\nu, \lambda_{cl}); z^i(\nu) + a(\mu^j(\nu, \lambda_{cl})) + 2n) \right\} \\
\iff &u_{\lambda' + \omega_0} \otimes u \in LW \left\{ B \left( \mu_k^j(\lambda'_{cl}, \nu, \lambda_{cl}); z^i(\nu) + a(\mu^j(\nu, \lambda_{cl})) + a(\mu_k^j(\lambda'_{cl}, \nu, \lambda_{cl})) + 2n + 1 \right) \right\}
\end{align*}
\]

for some \( k \).

Here, the first equality follows from Lemma 7.1 and Proposition 8.4, the second from (9.1). Because \( wt(LW(B(\Lambda; p\omega_0))) = \Lambda_{su} + p\omega_0 \) as \( U_q(D(N, 1)) \)-crystal, we have (9.3).

\[\blacksquare\]

Example 9.2. : \( U_q(D(2,1)) \)

The even part is \((U_q(sl_2) \otimes U_q(sl_2)) \otimes U_q(C(1))\). Let \( \lambda \) and \( \lambda' \) be as in (9.2) and let \( \Lambda_1 \) and \( \Lambda_2 \) be the fundamental weights of first two \( U_q(sl_2) \). Because not only the first but also the second node are connected with 0-th node in the Dynkin diagram, we modify (8.9) into

\[
a(\mu^j(\nu, \lambda_{cl})) = \sharp \{ i; l_i = 1 \text{ or } 2 \}.
\]

Note that

\[
B(\lambda'_{cl}) \otimes B(\lambda_{cl}) = \bigoplus_{j=0}^{\min(n_1, n'_1)} \bigoplus_{k=0}^{\min(n_2, n'_2)} B(- (n_1 + n'_1 - 2j) \Lambda_1 - (n_2 + n'_2 - 2k) \Lambda_2)
\]

by the Clebsch-Gordan formula, and

\[
a(- (n_1 + n'_1 - 2j) \Lambda_1 - (n_2 + n'_2 - 2k) \Lambda_2) = j + k.
\]

We assume \(|n_1 - n'_1| \geq 2\) and \(|n_2 - n'_2| \geq 2\) to make the description simple.
Applying the Clebsch-Gordan formula again, we obtain

\[ B(\lambda + \omega_0) \otimes B(\lambda + \omega_0) \]
\[ \min_{(n_1, n_1')} \otimes \min_{(n_2, n_2')} \]
\[ \oplus_{j=0}^{\Xi_0} \oplus_{k=0}^{\Xi_1} \oplus_{n \in \mathbb{Z}_{\geq 0}}^{\Xi_2} \]
\[ B ((n_1 + n_1') - 2j) \lambda_1 - (n_2 + n_2' - 2k) \omega_2 + (j + k + 2 + 2n) \omega_0) \]
\[ \oplus B ((n_1 + n_1') - 2j) \lambda_1 - (n_2 + n_2' - 2k) \omega_2 + (j + k + 6 + 2n) \omega_0) \]
\[ \oplus B ((n_1 + n_1') - 2j + 1) \lambda_1 - (n_2 + n_2' - 2k + 1) \omega_2 + (j + k + 4 + 2n) \omega_0) \]
\[ \oplus B ((n_1 + n_1') - 2j + 1) \lambda_1 - (n_2 + n_2' - 2k - 1) \omega_2 + (j + k + 5 + 2n) \omega_0) \]
\[ \oplus B ((n_1 + n_1') - 2j + 2) \lambda_1 - (n_2 + n_2' - 2k) \omega_2 + (j + k + 3 + 2n) \omega_0) \]
\[ \oplus B ((n_1 + n_1') - 2j + 0) \lambda_1 - (n_2 + n_2' - 2k) \omega_2 + (j + k + 4 + 2n) \omega_0) \]
\[ \oplus B ((n_1 + n_1') - 2j - 2) \lambda_1 - (n_2 + n_2' - 2k) \omega_2 + (j + k + 5 + 2n) \omega_0) \]
\[ \oplus B ((n_1 + n_1') - 2j - 1) \lambda_1 - (n_2 + n_2' - 2k - 1) \omega_2 + (j + k + 6 + 2n) \omega_0) \]

Example 9.3. : \( B(-\omega_4 + \omega_0) \otimes B(\omega_0) \) for \( U_q(D(4,1)) \)

W for \( U_q(D(4,1)) \) is

\[ W = \{ \Xi_0 = 0, -\Xi_1 = -\Lambda_1, -\Xi_2 = -\Lambda_2, -\Xi_3 = -\Lambda_3 - \Lambda_4, -\Xi_4' = -2\Lambda_3, -\Xi_4 = -2\Lambda_4 \}. \]

In \( U_q(D(4)) \), we have

\[ \mathcal{B}(-\Lambda_4) \otimes \mathcal{B}(-\Xi_1) = \mathcal{B}(-\Lambda_1 - \Lambda_4), \quad a(-\Lambda_1 - \Lambda_4) = 0, \quad a(-\Lambda_3) = 1, \]
\[ \mathcal{B}(-\Lambda_4) \otimes \mathcal{B}(-\Xi_2) = \mathcal{B}(-\Lambda_2 - \Lambda_4), \quad a(-\Lambda_2 - \Lambda_4) = 0, \quad a(-\Lambda_1 - \Lambda_3) = 0, \quad a(-\Lambda_3) = 1, \]
\[ \mathcal{B}(-\Lambda_4) \otimes \mathcal{B}(-\Xi_3) = \mathcal{B}(-\Lambda_3 - 2\Lambda_4), \quad a(-\Lambda_3 - 2\Lambda_4) = 0, \quad a(-\Lambda_1 - \Lambda_3) = 0, \quad a(-\Lambda_1 - \Lambda_4) = 0, \quad a(-\Lambda_3) = 1, \]
\[ \mathcal{B}(-\Lambda_4) \otimes \mathcal{B}(-\Xi_4') = \mathcal{B}(-2\Lambda_3 - \Lambda_4), \quad a(-2\Lambda_3 - \Lambda_4) = 0, \quad a(-\Lambda_1 - \Lambda_3) = 0, \quad a(-\Lambda_1 - \Lambda_4) = 0, \quad a(-\Lambda_3) = 1, \]
\[ \mathcal{B}(-\Lambda_4) \otimes \mathcal{B}(-\Xi_4) = \mathcal{B}(-3\Lambda_4), \quad a(-3\Lambda_4) = 0, \quad a(-\Lambda_1 - \Lambda_4) = 0, \quad a(-\Lambda_3) = 1. \]
It follows that

\[
B(-\omega_4 + \omega_0) \otimes B(\omega_0) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} B(-\omega_4 + (2 + 2n)\omega_0) \oplus B(-\omega_4 + (10 + 2n)\omega_0) \\
\oplus B(-\omega_1 - \omega_4 + (2 + 2n)\omega_0) \oplus B(-\omega_1 - \omega_4 + (8 + 2n)\omega_0) \\
\oplus B(-\omega_3 + (3 + 2n)\omega_0) \oplus B(-\omega_3 + (9 + 2n)\omega_0) \\
\oplus B(-\omega_2 - \omega_4 + (3 + 2n)\omega_0) \oplus B(-\omega_2 - \omega_4 + (7 + 2n)\omega_0) \\
\oplus B(-\omega_1 - \omega_3 + (3 + 2n)\omega_0) \oplus B(-\omega_1 - \omega_3 + (7 + 2n)\omega_0) \\
\oplus B(-\omega_4 + (4 + 2n)\omega_0) \oplus B(-\omega_4 + (8 + 2n)\omega_0) \\
\oplus B(-\omega_3 - 2\omega_4 + (4 + 2n)\omega_0) \oplus B(-\omega_3 - 2\omega_4 + (6 + 2n)\omega_0) \\
\oplus B(-\omega_2 - \omega_3 + (4 + 2n)\omega_0) \oplus B(-\omega_2 - \omega_3 + (6 + 2n)\omega_0) \\
\oplus B(-\omega_1 - \omega_4 + (4 + 2n)\omega_0) \oplus B(-\omega_1 - \omega_4 + (6 + 2n)\omega_0) \\
\oplus B(-\omega_3 + (5 + 2n)\omega_0) \oplus B(-\omega_3 + (7 + 2n)\omega_0) \\
\oplus B(-2\omega_3 - \omega_4 + (5 + 2n)\omega_0) \oplus B(-\omega_1 - \omega_3 + (5 + 2n)\omega_0) \\
\oplus B(-3\omega_4 + (5 + 2n)\omega_0) \oplus B(-\omega_2 - \omega_4 + (5 + 2n)\omega_0) \oplus B(-\omega_4 + (6 + 2n)\omega_0).
\]

10 Results for $U_q(B(N, 1))$

The results in previous sections carry over to the case of $U_q(B(N, 1))$. Since the proofs are entirely similar, we only state the results.

The Cartan matrix for $U_q(B(N, 1))$ is

\[
A = ((h_i, \alpha_i)_{i,j=0}^N = (a_{ij})_{i,j=0}^N = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 2 & -1 \\ 0 & -1 & 2 & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & 2 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ \cdots & 0 & -1 & 2 & -1 \\ \cdots & 0 & -2 & 2 \end{pmatrix}. \quad (10.1)
\]

The associated Dynkin diagram is

\[\begin{array}{cccccccc}
\otimes & \bullet & \bullet & \cdots & \cdots & \cdots & \bullet & \bullet \\
0 & 1 & 2 & \cdots & \cdots & \cdots & N-1 & N
\end{array}\]

We put

\[l_0 = 2, \quad l_1 = \cdots = l_{N-1} = -2, \quad l_N = -1.\]

$U_q(B(N, 1))$ is defined as in Definition 2.1, wherein $(a_{ij})$ is replaced by the Cartan matrix (10.1).
\{\delta, \varepsilon_1, \ldots, \varepsilon_N\}, \Delta_0, \Delta_1, \Sigma_1 \text{ for } U_q(B(N, 1)) \text{ is given by}

\[
\alpha_0 = \delta - \varepsilon_1, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N, \quad \alpha_N = \varepsilon_N,
\]

\[
\Delta_0 = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_i, \pm 2 \delta \}, \quad \Delta_1 = \{ \pm \varepsilon_i \pm \delta, \pm \delta \}, \quad \Sigma_1 = \{ \pm \varepsilon_i \pm \delta \}.
\]

Its even part \(U_q(B(N, 1)_0)\) is given by \(U_q(B(N)) \otimes U_q(C(1))\), where \(U_q(B(N))\) is the subalgebra with generators \(e_i, f_i, q^{h_i} (1 \leq i \leq N)\), and \(U_q(C(1)) \simeq U_q(\mathfrak{sl}_2)\) is the one generated by \(E, F, q^H\), where \(H = 2(h_0 - h_1 - \cdots - h_{N-1}) - h_N, E\) and \(F\) are the elements corresponding to the root \(2(\alpha_0 + \cdots + \alpha_{N-2}) + \alpha_N\).

We state properties of \(U_q(B(N))\) corresponding to those of \(U_q(D(N))\) in Section 4. \(U_q(B(N))\) has one spin representation \(\overline{B}_{sp} = \overline{B}(-\Lambda_N)\) whose crystal base is realized as

\[
\overline{B}_{sp} = \{ b = (i_1, \ldots, i_N); i_1, \ldots, i_N = \pm \}
\]

with the lowest weight vector \((+, \ldots, +)\). The actions of \(\overline{e}_i\) and \(\overline{f}_i\) are

\[
\overline{f}_i(i_1, i_2, \ldots, i_N) = \begin{cases} (i_1, \ldots, i_{l-1}, i_{l+1} = +, - \ldots, i_N) & \text{if } i_i = -, i_{i+1} = +, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\overline{e}_i(i_1, i_2, \ldots, i_N) = \begin{cases} (i_1, \ldots, i_{l-1}, i_{l+1} = +, - \ldots, i_N) & \text{if } i_i = +, i_{i+1} = -, \\ 0 & \text{otherwise}, \end{cases}
\]

for \(1 \leq l \leq N - 1\), and

\[
\overline{f}_N(i_1, i_2, \ldots, i_N) = \begin{cases} (i_1, \ldots, i_N = \pm) & \text{if } i_N = -, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\overline{e}_N(i_1, i_2, \ldots, i_N) = \begin{cases} (i_1, \ldots, i_N = \pm) & \text{if } i_N = +, \\ 0 & \text{otherwise}. \end{cases}
\]

We also describe \(\overline{B}_{sp}\) in terms of Young tableaux as we do in the case of \(U_q(D(N))\). We denote

\[
\Xi_0 = 0, \quad \Xi_i = \Lambda_i \ (1 \leq i \leq N - 1), \quad \Xi_N = 2\Lambda_N.
\]

**Proposition 10.1.** We have

\[
\overline{B}_{sp} \otimes \overline{B}_{sp} = \bigoplus_{0 \leq k \leq N} \overline{B}(-\Xi_k).
\]

For \(0 \leq i \leq N\), the lowest weight vector corresponding to the connected component \(\overline{B}(-\Xi_i)\) is

\[
(\ +, \ldots, +) \otimes (\ +, \ldots, +, -, \ldots, -).
\]

**Proposition 10.2 (Koga[8]).** (1) Assume \(u \otimes v = t(a_1, \ldots, a_N) \otimes t(b_1, \ldots, b_N) \in \overline{B}_{sp} \otimes \overline{B}_{sp}\).

Then we have
(1A) For $0 \leq k \leq N - 1$,

\[
\begin{align*}
\{u \otimes v \in B(-\Xi_k) \iff t(a_1, \ldots, a_{N-k}; a_{N-k+1}, \ldots, a_N|b_1, \ldots, b_k; b_{k+1}, \ldots, b_N) \text{ is semi-standard and} \\
&
\{u \otimes v \in B(-\Xi_{k-1}) \iff t(a_1, \ldots, a_{N-k-1}; a_{N-k}, \ldots, a_N|b_1, \ldots, b_{k+1}; b_{k+2}, \ldots, b_N) \text{ is not semi-standard,}
\end{align*}
\]

(1B)

\[
\begin{align*}
\{u \otimes v \in B(-\Xi_N) \iff t(; a_1, \ldots, a_N|b_1, \ldots, b_N) \text{ is semi-standard.}
\end{align*}
\]

Proposition 10.3. The irreducible lowest weight module $V(-\omega_N)$ with lowest weight $-\omega_N$ has a basis over $\mathbb{Q}(q)$

\[
\{v(i_1, \ldots, i_N)_k; k \in \mathbb{Z}_{\geq 0}, (i_1, \ldots, i_N) \in B_{sp}\}
\]

with the lowest weight vector $v(+, \ldots, +)_0$ such that the actions of $\sigma$ and $e_i$ are:

\[
\begin{align*}
\sigma v(+, \ldots, +)_0 &= v(+, \ldots, +)_0, \\
e_i(v(i_1, \ldots, i_N)_k) &= \begin{cases} 
 v(i'_1, \ldots, i'_N)_k & \text{if } \bar{e}_i(v(i_1, \ldots, i_N)) = (i'_1, \ldots, i'_N) \neq 0 \text{ in } B_{sp}, \\
 0 & \text{otherwise,}
\end{cases}
\end{align*}
\]

for $1 \leq i \leq N$, and

\[
\begin{align*}
e_0(v(i_1, i_2, \ldots, i_N)_k) &= \begin{cases} 
 q^{-k}v(+, i_2, \ldots, i_N)_{k+1} & \text{if } i_1 = -, \\
 0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Proposition 10.4. The irreducible lowest weight module $V(-\omega_N)$ has the polarizable crystal base $(L, B)$:

\[
L = \bigoplus_{(i_1, \ldots, i_N) \in B_{sp}, k \in \mathbb{Z}_{\geq 0}} Av(i_1, \ldots, i_N)_k,
\]

\[
B = \{\pm v(i_1, \ldots, i_N)_k \mod qL; (i_1, \ldots, i_N) \in B_{sp}, k \in \mathbb{Z}_{\geq 0}\}.
\]

The Kashiwara operators on $B$ is given by (we omit $\mod qL$)

\[
\begin{align*}
\bar{e}_i v(i_1, \ldots, i_N)_k &= \begin{cases} 
 v(i'_1, \ldots, i'_N)_k & \text{if } \bar{e}_i(v(i_1, \ldots, i_N)) = (i'_1, \ldots, i'_N) \neq 0 \text{ in } B_{sp}, \\
 0 & \text{otherwise,}
\end{cases}
\end{align*}
\]

for $1 \leq i \leq N$, and

\[
\begin{align*}
\bar{e}_0 v(i_1, i_2, \ldots, i_N)_k &= \begin{cases} 
v(+, i_2, \ldots, i_N)_{k+1} & \text{if } i_1 = -, \\
 0 & \text{otherwise.}
\end{cases}
\end{align*}
\]
Because
\[
B(-\omega_N) \otimes B(-\omega_N) = B(-2\omega_N) \oplus \bigoplus_{j=1}^{N-1} B(-\omega_j) \oplus \bigoplus_{k \in \mathbb{Z}_{\geq 0}} B((k+1)\omega_0)
\]
holds, we have the theorem corresponding to Theorem 6.1.

**Theorem 10.5.** The irreducible lowest weight module \(V(\lambda)\) with the lowest weight
\[
\lambda = n_0\omega_0 - \sum_{i=1}^{N} n_i\omega_i, \quad n_i \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad 0 \leq i \leq N
\]
admits a polarizable crystal base.

The 0-arrows in \(B(-\omega_N) \otimes B(-\omega_N)\) can be described as follows.

**Lemma 10.6.** In \(B(-\omega_N) \otimes B(-\omega_N)\), we have

1. for \(k = 2, 3, \ldots, N\)
   \[
   \overline{B}(-\Xi_k; l) \xrightarrow{0} \overline{B}(-\Xi_{k-1}; l - 1),
   \]
2.
   \[
   \overline{B}(-\Xi_1; l) \xrightarrow{0} \overline{B}(-\Xi_0; l),
   \]
3. for \(k = 2, \ldots, N\)
   \[
   \overline{B}(-\Xi_{k-1}; l) \xrightarrow{0} \overline{B}(-\Xi_k; l - 1),
   \]
4. \[
   \overline{B}(-\Xi_0; l) \xrightarrow{0} \overline{B}(-\Xi_1; l - 2),
   \]
5. \[
   \overline{B}(-\Xi_N; l) \xrightarrow{0} \overline{B}(-\Xi_N; l - 1).
   \]

The results corresponding to Corollary 7.7 is the next proposition.

**Proposition 10.7.** We have
\[
B(\omega_0) = \bigoplus_{\nu \in W} \bigoplus_{i \nu = 1}^{i \nu} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \overline{B}(\nu; z^i(\nu) + 2n),
\]
where \(W = \{-\Xi_0, \ldots, -\Xi_N\}\). \(z^i(\nu)\) are given as follows;

- for \(\nu = -\Xi_0\), \(i_\nu = 2, \quad z^1(\nu) = 1, \quad z^2(\nu) = 2N + 2, \)
- for \(\nu = -\Xi_k \quad (1 \leq k \leq N)\), \(i_\nu = 2, \quad z^1(\nu) = k, \quad z^2(\nu) = 2N - k + 1.\)
The results in Section 7 and Section 8 hold for $U_q(B(N, 1))$.

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