TOWARDS A CLASSIFICATION OF FUSION RULE ALGEBRAS IN RATIONAL CONFORMAL FIELD THEORIES

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Abstract

We review the main topics concerning Fusion Rule Algebras (FRA) of Rational Conformal Field Theories. After an exposition of their general properties, we examine known results on the complete classification for low number of fields (\(\leq 4\)). We then turn our attention to FRA’s generated polynomially by one (real) fundamental field, for which a classification is known. Attempting to generalize this result, we describe some connections between FRA’s and Graph Theory. The possibility to get new results on the subject following this “graph” approach is briefly discussed.

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1 Introduction

Conformal Field Theories (CFT) in two dimensions \cite{1, 2} have raised a lot of interest owing to their connections with critical phenomena and string theories. In particular, the Rational Conformal Field Theories (RCFT’s), endowed with a finite number of primary fields with respect to some chiral algebra, and characterized by rational values of central charge and conformal weights, have been studied extensively. The attempts to describe and classify these theories with the tools of 2d conformal analysis and modular invariance have shown the relevant role played by Fusion Rule Algebras (FRA) \cite{3, 4}, which encode the algebraic properties of primary fields. In some sense FRA’s can be considered as the skeletons of RCFT’s, and lots of results have been obtained in the reconstruction of RCFT’s from the knowledge of their FRA’s \cite{3, 4, 5, 6, 7, 8, 9, 10, 11}.

One of the open problems in the context of RCFT’s is to find their basic underlying principles and structures. So far we may rely essentially on two approaches to tackle this problem.

- In the first one, quantum groups are proposed as the underlying algebraic structure of RCFT’s \cite{12}, and this approach is particularly successful in the case of WZW models, although its extension to general RCFT’s is still unclear. In particular this line of thought, when applied toward any classification program, shows severe limitations due to the fact that the classification of quantum groups themselves is still an open problem.

- The second approach exploits the relationships between RCFT’s and three-dimensional Chern-Simons theories \cite{13}; in particular the Hilbert space associated to a constant time slice with charges in the three dimensional theories is related to the space of conformal blocks of corresponding RCFT’s.

It is quite remarkable that both these approaches meet and fuse together at the level of FRA’s. Hence, any progress in trying to understand, organize and classify FRA’s could be an important first step toward a deeper comprehension not only of RCFT’s, but also of Quantum Groups and Chern-Simons theories.

In the following we will review some known results about FRA’s and will discuss some open issues. Let us anticipate which will be our strategy: FRA’s in RCFT’s must satisfy several stringent constraints coming from different directions: diophantine constraints due to the fact that the structure constants are positive integers, “modular” constraints on the matrix $S$ which diagonalizes the FRA, “duality” constraints on the allowed conformal weights and central charge of the underlying RCFT. The main idea is then to enforce all these constraints together by writing the FRA’s in terms of their 1-dim irreducible representations; these encode, in a non trivial way, all the relevant properties of the FRA’s and have a natural interpretation from the point of view of the quantum group approach, some of them being the quantum dimensions \cite{12, 14} of the irreps of the underlying quantum group.
During the last few years an impressive amount of new results in this context has been presented, and to review all of them would be not only beyond the scope of this contribution but also beyond our forces. Therefore we hope to give at least a flavour of the progress made in this field by reviewing the main results obtained by various groups, trying to emphasize all the connections among different approaches, and striving to be, as far as possible, self contained.

This paper is organized as follows: sect.2 is devoted to a general introduction on FRA’s, in sect.3 we discuss the classification of FRA’s for low number of primary fields, while sect.4 deals with the so called polynomial FRA’s introduced, studied and classified in [8] and later investigated (for the particular case of WZW models) also in [15]. In sect.5 a connection is traced between FRA’s and Graph Theory. Finally sect.6 discusses some open questions and other interesting results.

2 Definitions and general setting

A RCFT is a CFT whose physical Hilbert space decomposes into a finite sum of highest weight irreducible representations (HWR) of the (maximally extended [14]) chiral algebra \( \mathcal{A} \otimes \overline{\mathcal{A}} \)

\[
\mathcal{H} = \bigotimes_{i, \bar{i}} \mathcal{H}_{i} \otimes \overline{\mathcal{H}}_{i} \tag{1}
\]

where here and in the following (if not otherwise stated) the indices of the middle of latin alphabet \( i, j, k, \ldots \) run on a finite set \( X \) labelling HWR’s of \( \mathcal{A} \) at fixed central extension. This set can be put in one to one correspondence with a set of integers \( 0, 1, 2, \ldots, r \) such that \( \mathcal{H}_0 \) denotes the representation whose highest weight state is the vacuum \( |0\rangle \) of the theory. One can consider the so called chiral vertex operators [16, 17], i.e. intertwiners among \( \mathcal{H}_i \),

\[
\Phi_{ij}^{k}(z)_t : \mathcal{H}_i \otimes \mathcal{H}_j \otimes \mathcal{H}_k^* \to \mathbb{C} \tag{2}
\]

where \( \mathcal{H}_i^* \) is the dual space of \( \mathcal{H}_i \). Among \( \mathcal{H}_i, \mathcal{H}_j \) and \( \mathcal{H}_k^* \) there can be in general more that one possible coupling, i.e. more that one chiral vertex operator. The index \( t \) distinguishes among them. Different \( \Phi_{ij}^{k}(z)_t \) for \( t = 1, \ldots, N_{ij}^k \) span a vector space \( V_{ij}^k \) of dimension \( N_{ij}^k \). The numbers \( N_{ij}^k \) are referred to as fusion rules.

Consider formally an \( r + 1 \) dimensional algebra \( \mathcal{F} \) over \( \mathbb{C} \) and a basis \( \Xi = \{ \phi_0, \phi_1, \ldots, \phi_r \} \) of it, such that in \( \Xi \) the structure constants are given exactly by the numbers \( N_{ij}^k \)

\[
\phi_i \phi_j = \sum_{k} N_{ij}^k \phi_k \tag{3}
\]

The algebra \( \mathcal{F} \) is called Fusion Rules Algebra (FRA) associated with the left chiral part of the given RCFT. The vectors \( \phi_i \) are evidently in 1 to 1 correspondence with the HWR \( \mathcal{H}_i \), thus allowing to define a fusion product between HWR’s of \( \mathcal{A} \). The same procedure can be applied to the right chiral algebra \( \overline{\mathcal{A}} \) to define fusion products between HWR \( \overline{\mathcal{H}}_i \) and a right fusion algebra \( \overline{\mathcal{F}} \).
Properties of \( \mathcal{F} \) can be deduced from those of the chiral vertex operators. These latter have been summarized in eqs.(4.17-4.18) of ref. [17]; here we simply mention those facts that justify the following assumptions for \( \mathcal{F} \):

**P1:** \( \mathcal{F} \) is commutative: \( N^k_{ij} = N^k_{ji} \). This follows from the isomorphism (called \( \Omega \) in [17]) between \( V^k_{ij} \) and \( V^k_{ji} \).

**P2:** \( \mathcal{F} \) is associative:

\[
\sum_m N^m_{ij} N^l_{km} = \sum_m N^m_{ik} N^l_{jm}
\]

This follows from existence of the Fusion matrix isomorphism

\[
\mathcal{F} \begin{bmatrix} i & k \\ j & l \end{bmatrix} : \bigoplus_m V^j_{im} \otimes V^m_{kl} \cong \bigoplus_m V^j_{ml} \otimes V^m_{ik}
\]

which is related to the assumption of duality.

**P3:** Identity: there exists in \( \Xi \) a vector \( \phi_0 \) called the identity such that for all \( i, j \in X \):
\( N^0_{0i} = \delta^i_j \). The vector \( \phi_0 \) is the one in 1 to 1 correspondence with the HWR \( \mathcal{H}_0 \).

**P4:** Charge conjugation: there exists a one to one map of \( X \), say \( C : i \mapsto \hat{i} \), such that \( \hat{i} = i \) and \( \mathcal{F} \) is invariant under \( C \), i.e. \( N^k_{ij} = N^k_{\hat{i} \hat{j}} \) and \( N^0_{ij} = \delta_{ij} \). The matrix \( C_{ij} \equiv N^0_{ij} \) uppers and lowers indices in \( \mathcal{F} \), and from P1,P2, \( N_{ijk} \) is totally symmetric in its indices.

**P5:** Integrality of structure constants: \( N^k_{ij} \in \mathbb{N} \). This follows from the trivial fact that the dimension of a vector space like \( V^k_{ij} \) is a non-negative integer.

In the following we shall adopt properties P1,...,P5 as postulates defining (partially) the algebra \( \mathcal{F} \) and study the mathematical properties of such an object.

The vectors \( \phi_i \) are often called fields, to remember that they are closely related to (chiral) primary fields of the underlying RCFT. A field \( \phi_i \) is referred to as the conjugate field to \( \phi_i \). A FRA such that \( \hat{i} = i \) for all \( i \in X \) will be named selfconjugate.

A crucial role in the following is played by the regular representation of \( \mathcal{F} \) which associates to \( \phi_i \) the matrix \( N_i \) of elements \( (N_i)_j^k \equiv N^k_{ij} \). It is a well known result in the mathematical theory of associative, commutative algebras that all the \( N_i \)'s can be simultaneously diagonalized by a suitable matrix \( S \), i.e. all \( N_i \)'s share the same set of \( (r + 1) \) non-null eigenvectors. In particular if \( \mathcal{F} \) is selfconjugate, then all \( N_i \)'s are Hermitian, and \( S \) is orthogonal. The \( i \)-th eigenvalue of \( N_j \) will be denoted \( \lambda_i(j) \). The spectrum of \( \mathcal{F} \), i.e. the set \( \{ \lambda_i(j) \} \), encodes all the information about \( \mathcal{F} \). We recall the following properties:

**T1:** Being solutions of the algebraic equation (of degree \( r + 1 \))

\[
det(N_j - \lambda_i(j)1) = 0,
\]

with integer coefficients and with 1 as coefficient of the highest power, the \( \lambda \)'s are algebraic integers.
The $\lambda$’s provide all the irreps of $\mathcal{F}$; these latter being all one dimensional. In the $l$-th irrep, the role of $\phi_i$ is played by $\lambda_l(i)$, or

$$\lambda_l(i)\lambda_l(j) = \sum_{k=0}^{r} N^k_{ij} \lambda_l(k) \quad (4)$$

As a consequence of (4) $\lambda_l(\hat{i}) = \lambda_l(i)^*$ so that selfconjugate FRA’s have real spectrum.

The spectrum satisfies the following orthogonality relations:

$$\sum_{k=0}^{r} \lambda_i(k)\lambda_j(k)^* = \frac{1}{\nu_i} \delta_{ij} \quad (5)$$

$$\sum_{k=0}^{r} \nu_k \lambda_k(i)\lambda_k(j)^* = \delta_{ij} \quad (6)$$

where $\nu_i = (\sum_{k=0}^{r} |\lambda_i(k)|^2)^{-1}$. The meaning of the real numbers $\nu_i$’s will be clear later; notice that $0 < \nu_i < 1$ because $\lambda_j(0) = 1$.

As a consequence of eqs.(4,6) we can express the structure constants in terms of the $\lambda$’s as follows

$$N^k_{ij} = \sum_{l=0}^{r} \nu_l \lambda_l(i)\lambda_l(j)\lambda_l(k)^*$$

This clearly shows how $\lambda$’s really encode all the information about $\mathcal{F}$.

The unitary matrix which diagonalizes all $N_l$’s is

$$S^l_j = \sqrt{\nu_j} \xi^j_l \quad (7)$$

where $\nu_j$’s care for the normalization of the $j$-th eigenvector $\xi^j_l$: $N_k\xi^j_l = \lambda_j(k)\xi^j_l$, whose $l$-th component is related to $\lambda$’s by $\xi^j_l = (\xi^j)_l = \lambda_j(l)$.

Notice that different FRA’s may actually describe the same algebraic structure. We say that two FRA’s $\mathcal{F}^{(1)}, \mathcal{F}^{(2)}$ are isomorphic if there is a permutation $\pi : X \rightarrow X$ such that:

- $\phi^{(1)}_{\pi(i)} = \phi^{(2)}_i$ for all $i \in X$
- $\pi(0) = 0$
- $[\pi, C] = 0$
- $N^{(1)}_{\pi(i)\pi(j)} = N^{(2)}_{ij}$ for all $i, j, k \in X$
Therefore FRA’s of a given type are “partitioned” into isomorphism classes, whose characterization is the final task of any classification program.

A very interesting property of FRA’s is encoded by the matrix $S$. It also performs the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ on the characters of the underlying RCFT. This fact is known as Verlinde theorem in RCFT \cite{3, 4, 14}. From our point of view, this amounts to a couple of new postulates to be required on $\mathcal{F}$:

\textbf{P6:} $S^2 = C$. This, due to unitarity of $S$, is equivalent to the requirement that $S$ is symmetric, i.e. $\sqrt{\nu_j}\lambda_j(i) = \sqrt{\nu_i}\lambda_i(j)$, so that $\sqrt{\nu_j} = \sqrt{\nu_0}\lambda_0(j)$ and upon elimination of $\nu$’s:

$$\lambda_0(j)\lambda_j(i) = \lambda_0(i)\lambda_i(j)$$

where $\lambda_0(j) \neq 0$, for all $r \in X$.

\textbf{P7:} $(ST)^3 = C$, where $T_{ij} = \delta_{ij}\exp 2\pi i(\Delta_i - c/24)$. where $\Delta_i$ is the conformal dimension attached to the HWR $\mathcal{H}_i$ (i.e. the eigenvalue of the zero mode of the Virasoro subalgebra of $\mathcal{A}$ for the highest weight vector of $\mathcal{H}_i$) and $c$ is the central charge of the same Virasoro subalgebra.

The first assumption \textbf{P6} amounts to a very selective constraint on possible FRA’s. Many FRA’s satisfying \textbf{P1,...,P5} fail to satisfy this constraint and must be discarded as candidates to build a RCFT. The second \textbf{P7} gives instead a set of equations to be satisfied by the numbers \{c, $\Delta_i$\}. Some facts follow from these two new assumptions:

\textbf{T7:} It is interesting to see that \textbf{P6} and \textbf{P7} can be combined in the following simple expressions \cite{10, 12}:

$$S^i_j = \sum_m S^m_0 N^i_{jm} e^{2\pi i (\Delta_i + \Delta_j - \Delta_m)}$$

$$e^{2\pi i \Delta_j S^j_0} = e^{\pi ic/4} \sum_m e^{2\pi i \Delta_m S^m_0 S^j_m}$$

\textbf{T8:} In any FRA of RCFT the eigenvalues $\lambda_i(j)$ are always linear combinations of roots of unity with integer coefficients. Otherwise stated, $\lambda_i$’s not only are algebraic integers, but they belong to the algebraic integers of some cyclotomic field. This powerful and intriguing theorem has been proven in \cite{10}; the proof makes use of \textbf{P1,...,P6}.

There is still an important issue to be discussed: one has to require analytic closure of the spaces of conformal blocks on Riemann surfaces of any genus and any number of punctures. To find a complete formulation to this requirement would be equivalent to solve (and classify) the whole set of RCFT’s. Here we shall be concerned with a more modest constraint that can be deduced from a differential equation approach to analytic closure of the space of conformal blocks of the 4-point function on the sphere \cite{5} and of partition function on the torus \cite{6}. For an almost equivalent constraint (although less powerful, as it does not give lower bounds on $\Delta_i$’s and $c$) see \cite{7}:
P8: Conformal dimensions $\Delta_i$ and central charge $c$ must satisfy the following set of equations:

\[
\sum_m \left( N^m_{ij} N^l_{km} + N^m_{ik} N^l_{jm} + N^m_{il} N^m_{kj} \right) \Delta_m - N^l_{ijk} (\Delta_i + \Delta_j + \Delta_k + \Delta_l) = N^l_{ijk} (N^l_{ijk} - 1) - R^l_{ijk}
\]

where $N^l_{ijk} = \sum_m N^m_{ij} N^l_{km}$ and $R^l_{ijk}$ are non-negative integers, and

\[
c = \frac{24}{n} \sum_{i=0}^{n-1} \Delta_i - 2(n - 1) + 4l
\]  

(11)

where $n$ is the number of independent characters in the partition function on the torus and $l = 0, 2, 3, \ldots$. These equations can be deduced \[5, 6\] as Fuchs conditions on the differential equations to be satisfied by conformal blocks.

What is surprising is that the equations of P8 are in general incompatible with those of P7 (in the form (9, 10)). Only when compatibility holds, the FRA is acceptable as a candidate for RCFT. The experience of low $r$ results \[8, 19\] shows that this compatibility between P7 and P8 is an extremely selective requirement (see for examples how it works in the classification of all FRA’s for $r = 1$ in ref. \[19\] or for $r = 2$ in ref. \[8\]).

Once all these constraints are imposed, the number of FRA’s reduces sensibly and one can hope, encouraged by results for low $r$ (see sect.3), to get a classification of all possible structures obeying P1,...,P8. Although this problem of classification seems at present to be a formidable one, some progress has been made. This will be the subject of next sections.

Moreover, the classification of all FRA’s is only a first step in the classification program of RCFT’s. Once the FRA is given, one is faced with the reconstruction problem, i.e. how to find, classify and solve the RCFT’s sharing the given FRA. Some steps in this direction have been done too (see for example \[3, 4, 20\]). Here we recall only a well known result on how to glue left and right FRA’s to get a reasonable theory: it has been proven \[14\] that, if the chiral algebra $A$ is maximally extended, the right FRA must be isomorphic to the left one, and right labelling of fields can differ from left ones only by an automorphism of the FRA itself.

Let us conclude this introductory section with some remarks on the concept of quantum dimension which arises in the context of the quantum group approach to RCFT’s and has a natural interpretation in the language of FRA; it was introduced by Dijkgraaf, E.Verlinde and H.Verlinde \[14\] as a (regularized) definition of dimension for the unitary HWR’s $H_i$ of $A$. The recipe is to divide the character $\chi_i(\tau)$ of the irrep by the character of the identity ($H_0$) irrep and then to take the limit $\tau \to 0$ (the rationale behind this definition is that if the Hilbert space were finite dimensional, then $\chi_i(0)$ would exactly count the number of states):

\[
d_i \equiv \lim_{\tau \to 0} \frac{\chi_i(\tau)}{\chi_0(\tau)}.
\]  

(12)
We can compute \( d_i \) using the modular transformation \( S \):
\[
 d_i = \lim_{\tau \to 0} \frac{\sum_j S^j_i \chi_j(-1/\tau)}{\sum_k S^k_0 \chi_k(-1/\tau)},
\]
(13)
then, since in unitary theories \( \Delta_i \geq 0 \) and \( \Delta_i = 0 \) only for the identity field, we get
\[
 d_i = \frac{S^i_0}{S^0_0} = \lambda_0(i)
\]
(14)
which are the eigenvalues of the corresponding FRA. Remarkably enough, in the case of a WZW model built on a group \( G \), this quantum dimension exactly coincides with a suitable definition (using the so called Markov trace) of dimension of the highest weight irreducible representation of the quantum deformation of the same group \( G \) (see for instance the last of ref.s [12]).

3 Classification of FRA’s with low \( r \)

Solving the Diophantine equations introduced in the previous section is in general a difficult task. Similarly it is usually very difficult to identify and reconstruct all the allowed RCFT’s given a consistent FRA. Notwithstanding this, there are some simple, but non trivial, cases in which this can be done. In particular it is possible to classify completely all the FRA’s with one field plus the identity \( (r = 1) \) and with two fields plus the identity \( (r = 2) \); but the complexity grows exponentially as the number of operators increases and seem to forbid much progress in this direction. Some numerical results have been obtained for \( r = 3 \).

3.1 Exact classification for \( r = 1 \)

In this case the general form of the algebras satisfying \( P_1, ..., P_5 \) is
\[
 \phi_1 \phi_1 = \phi_0 + n \phi_1, \quad n \in \mathbb{N}
\]
All these algebras also satisfy \( P_6, P_7 \) (in the form of eq.(9)), gives some constraints on \( c \) and \( \Delta_1 \) that are compatible with those coming from \( P_8 \) only for \( n = 0, 1 \). Hence the full classification of FRA’s with \( r = 1 \) is given by the list of tab.1 In tab.2 some corresponding RCFT’s are identified. Introducing a notation which will be clear in the following we call them \( A_1 \) and \( B_1 \) algebras. It is interesting to notice that all the allowed theories are algebraic.

3.2 Exact classification for \( r = 2 \)

In this case there are only three algebras which satisfy all the constraints [9]. They are listed in tab.3 Some identified RCFT’s, with the corresponding conformal weights
Table 1: The set of FRA’s, central charges and conformal weights compatible with RCFT’s with one operator plus the identity. \( m = 0, 2, 3, \ldots \) and \( l = 0, 1, 2, \ldots \).

| Algebra       | \( c \) | \( \Delta \) |
|---------------|---------|------------|
| \( \phi_1\phi_1 = \phi_0 \) | \( 1 + 4m \) | \( \frac{1}{4} + l \) |
|               | \( 7 + 4m \) | \( \frac{3}{4} + l \) |
| \( \phi_1\phi_1 = \phi_0 + \phi_1 \) | \( \frac{2}{5} + 4m \) | \( \frac{1}{5} + l \) |
|               | \( \frac{14}{5} + 4m \) | \( \frac{2}{5} + l \) |
|               | \( -\frac{22}{5} + 4m \) | \( \frac{1}{5} + l \) |
|               | \( \frac{26}{5} + 4m \) | \( \frac{3}{5} + l \) |

Table 2: Some RCFT’s with one field plus the identity.

| Algebra       | \( c \) | \( \Delta \) | Model       |
|---------------|---------|------------|-------------|
| \( \phi_1\phi_1 = \phi_0 \) | 1       | \( \frac{1}{4} \) | SU(2) \( k=1 \) WZW |
|               | 7       | \( \frac{3}{4} \) | \( E_7 \) \( k=1 \) WZW |
| \( \phi_1\phi_1 = \phi_0 + \phi_1 \) | \( \frac{14}{5} \) | \( \frac{2}{5} \) | \( G_2 \) \( k=1 \) WZW |
|               | \( \frac{26}{5} \) | \( \frac{3}{5} \) | \( F_4 \) \( k=1 \) WZW |
|               | \( -\frac{22}{5} \) | \( -\frac{1}{5} \) | \( (5,2) \) Virasoro |
Table 3: The set of FRA’s, central charges and conformal weights compatible with unitary RCFT’s with two fields plus the identity. \( m = 0, 2, 3, \ldots \) and \( l, n = 0, 1, 2, \ldots \).

| Algebra                  | \( c \) | \( \Delta_1 \) | \( \Delta_2 \) |
|--------------------------|--------|----------------|----------------|
| \( \phi_1 \phi_1 = \phi_2 \) | \( 2 + 8m \) | \( \frac{1}{3} + l \) | \( \frac{1}{3} + l \) |
| \( \phi_2 \phi_2 = \phi_1 \) | \( 6 + 8m \) | \( \frac{2}{3} + l \) | \( \frac{2}{3} + l \) |
| \( \phi_1 \phi_2 = \phi_0 \) | \( \frac{20}{7} + 8m \) | \( \frac{1}{7} + l \) | \( \frac{5}{7} + n \) |
| \( \phi_1 \phi_1 = \phi_0 + \phi_2 \) | \( \frac{36}{7} + 8m \) | \( \frac{6}{7} + l \) | \( \frac{2}{7} + n \) |
| \( \phi_2 \phi_2 = \phi_0 + \phi_1 + \phi_2 \) | \( \frac{1}{2} + m \) | \( \frac{1}{16} + \frac{l}{8} \) | \( \frac{1}{2} + n \) |

Table 4: Some identified models for \( r = 2 \).

| Algebra                  | \( c \) | \( \Delta_1 \) | \( \Delta_2 \) | Model                  |
|--------------------------|--------|----------------|----------------|------------------------|
| \( \phi_1 \phi_1 = \phi_2 \) | \( 2 \) | \( \frac{1}{3} \) | \( \frac{1}{3} \) | SU(3) \( k = 1 \) WZW |
| \( \phi_2 \phi_2 = \phi_1 \) | \( 6 \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) | \( E_6 \) \( k = 1 \) WZW |
| \( \phi_1 \phi_2 = \phi_0 \) | \( \frac{-68}{7} \) | \( \frac{-2}{7} \) | \( \frac{-3}{7} \) | (7,2) Virasoro          |
| \( \phi_1 \phi_1 = \phi_0 + \phi_2 \) | \( \frac{1}{2} \) | \( \frac{1}{16} \) | \( \frac{1}{2} \) | Ising                  |
| \( \phi_2 \phi_2 = \phi_0 \) | \( \frac{3}{2} \) | \( \frac{3}{16} \) | \( \frac{1}{2} \) | SU(2) \( k = 2 \) WZW |
| \( \phi_1 \phi_2 = \phi_1 \) | \( \frac{2n+1}{2} \) | \( \frac{2n+1}{16} \) | \( \frac{1}{2} \) | SO\((2n+1)\) \( k = 1 \) WZW |
|                           | \( \frac{31}{2} \) | \( \frac{15}{16} \) | \( \frac{3}{2} \) | \( E_8 \) \( k = 2 \) WZW |

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and charges, are listed in tab.4. In this case we have two selfconjugate algebras: $A_2$ and $B_2$ and one non-selfconjugate which is isomorphic to the $Z_3$ group. While the classification of the allowed conformal weights and charges is complete, a proof of the completeness of the reconstruction of RCFT’s is still lacking. Remarkably enough, also in this case all the allowed theories are algebraic.

3.3 Numerical results for $r = 3$

A preliminary, numerical, analysis of FRA’s with 3 fields plus the identity has been done following the strategy of ref [8]. Besides the direct products of $r = 1$ algebras we have found that only the expected $A_3, B_3$ and $Z_4$ algebras satisfy all the constraints and are good candidates for RCFT’s. But it must be noticed that lots of interesting structures appear if one only imposes associativity and the modular constraints.

4 Polynomial fusion rule algebras

Remarkably enough there is another situation in which the Diophantine system simplifies, namely the case of FRA’s whose fields are generated polynomially in terms of only one fundamental field (these algebras will be called in the following polynomial fusion rule algebras: PFRA). They are quite important in the context of RCFT’s, because they describe all models somehow related with the $SU(2)$ Kac-Moody algebra, hence, among the others, also the minimal models of Virasoro and Supervirasoro algebras and all the $SU(2) \otimes SU(2)/SU(2)$ cosets.

Let us restrict our attention to those FRA’s which are self-conjugate and such that all fields $\phi_i, 2 \leq i \leq r$, are generated polynomially in terms of a “fundamental” field $\phi \equiv \phi_1$ [4, 9]:

$$\phi_i = p_i(\phi)$$

where $p_i$ is a suitable polynomial with real coefficients and $p_0(\phi) = 1, p_1(\phi) = \phi$. Using the idempotent decomposition of the algebra [4], one can easily prove that

$$\lambda_j(i) = p_i(\xi_j)$$

where $\xi_j \equiv \lambda_j(1)$ are real and distinct, else, by eq.(4), $S$ would have two equal columns and would be singular. Inserting this equation into (4) we get

$$\sum_{k=0}^{r} \nu_k p_i(\xi_k)p_j(\xi_k) = \int_a^b dx \sum_{k=0}^{r} \nu_k \delta(x - \xi_k)p_i(x)p_j(x) = \delta_{ij}$$

for a finite interval $[a, b]$ such that $\xi_k \in [a, b], 0 \leq k \leq r$. Hence the polynomials $p_i(x), x \in \mathbb{R}, i = 0, ..., r$, build up an orthonormal set with respect to the positive definite atomic measure $\mu(x) = \sum_k \nu_k \delta(x - \xi_k)$.

Any set of orthogonal polynomials in one real variable satisfies a 3-term recurrence relation, which can be put in the form

$$xp_i(x) = a_{i+1}p_{i+1}(x) + b_ip_i(x) + c_{i-1}p_{i-1}(x)$$

(18)
where \( b_0 = c_{-1} = 0, \ a_1 = c_0 = 1 \) and \( a_{r+1} \neq 0 \) is arbitrary. Owing to (15,16), this relation can be interpreted in terms of fusion rules of the field \( \phi \equiv \phi_1 \) with the other fields

\[
\phi_1 \phi_i = \sum_{k=0}^r N_{i1}^k \phi_k = \delta_{i<r} a_{i+1} \phi_{i+1} + b_i \phi_i + c_{i-1} \phi_{i-1}, \quad 1 \leq i \leq r,
\]

where \( \delta_{i<r} = 1 \) if \( i < r \), \( \delta_{i<r} = 0 \) if \( i \geq r \). Hence the form of the matrix \( N_1 \) must be tridiagonal, symmetric (i.e. \( c_i = a_{i-1} \)) as the fields are all self-conjugate, and all \( c_i \)'s must not vanish:

\[
N_1 = \begin{pmatrix}
0 & 1 & b_1 & c_1 & & & \\
1 & b_1 & c_1 & b_2 & c_2 & & \\
& c_1 & b_2 & b_3 & & & \\
& & & \ddots & & & \\
& & & & \ddots & & c_{r-1} \\
& & & & & c_{r-1} & b_r
\end{pmatrix}
\]

The ensuing class of FRA’s enjoys rather peculiar properties, as we will see. Once the set of \( 2r-1 \) non-negative integer numbers \( c_i, 1 \leq i \leq r-1, \) and \( b_i, 1 \leq i \leq r, \) is given, then we can generate the whole set of orthogonal polynomials, and hence all the \( N_{ij}^k \), thanks to the fact that the matrices \( N_i \), that constitute the regular representation of the FRA, are given by \( N_i = p_i(N_1) \). Combining this fact with the recurrence relation (19), one can get the following set of equations, which automatically encode all the associativity conditions:

\[
c_i N_{i+1,j}^k = c_{i-1} N_{i,j}^{k-1} + \delta_{k<r} c_{k} N_{i,j}^{k+1} + (b_k - b_i) N_{i,j}^k - c_{i-1} N_{i-1,j}^k
\]

where, as \( i \) increases from 1 to \( r-1 \), all choices for \( j, k \) are considered such that: \( i+1 \leq k \leq j \leq r \). Once \( N_p, p \leq i \) are known, this recurrence relation gives the not yet known elements of \( N_{i+1} \).

Moreover one can find an explicit expression for the polynomials \( p_i \) [9].

With these results it is possible, at least in principle, to gain a complete control over the spectrum of \( \mathcal{F} \), and therefore to write the constraint from P6 as a set of algebraic Diophantine equations in \( b \)'s and \( c \)'s only.

In particular, in [9] using the previous results we obtained a complete classification of all the PFRA’s with structure constants less or equal to one.

Since all the \( c_i \)'s must be nonzero, they are fixed to one in this case and we are left with a simpler problem with only \( r \) degrees of freedom. By imposing associativity (i.e. solving eq.(20)) we find 5 infinite series plus an isolated solution; they are completely characterized by the values of the \( b \)'s (\( b_0 = 0 \)):

\[\mathcal{A}_r: b_i = 0, \ i = 1 \ldots r;\]
\[\mathcal{B}_r: b_i = 0, \ i = 1 \ldots r-1, \ b_r = 1;\]
\[\mathcal{B}_r': b_i = 1, \ i = 1 \ldots r;\]
$C_r$: If $r \geq 2$, $b_i = 1$, $i = 1 \cdots r - 1$, $b_r = 0$;

$D_r$: If $r = 2k \geq 4$, $b_{2i} = 0$, $b_{2i-1} = 1$, $i = 1 \cdots k$;

$E_4$: If $r = 4$: $b_1 = 1$, $b_2 = 0$, $b_3 = 1$, $b_4 = 1$.

These solutions are not independent: one can show that the two series $B_r$ and $B'_r$ are isomorphic; but this is the only isomorphism met in this classification. For every one of these algebras one can compute the $S$ matrix which diagonalizes the $N_i$’s and explicitly check that only $A_r$ and $B_r$ fulfil $P6$. They also satisfy compatibility between $P7$ and $P8$, hence they are good candidates for building RCFT’s.

A numerical check of (20) has been done also allowing $N_{ik} > 1$. The results seem to confirm that $A_r$ and $B_r$ are the only consistent series of PFRA’s.

5 FRA’s and Graph Theory

It is interesting to notice that the $N_1$ matrix of the $A_r$ PFRA is the incidence matrix of the Dynkin diagram of the $A_{r+1}$ Lie algebra. and that the $B_r$ series can be obtained as a $Z_2$ folding of the $A_{2r+2}$ diagrams [9].

$A_r : \quad \cdots \quad \downarrow \quad B_r : \quad \cdots \quad \downarrow$

This suggests a possible connection between PFRA’s and Graph Theory. This connection indeed exists and has been explored in a recent work by De Boer and Goeree [10]. Their results are in complete agreement with ours. Moreover it is now clear that the impressive simplifications which occur in PFRA’s and the fact that it is possible to obtain the above classification are indeed signatures of the rich algebraic structure undelying RCFT’s [10]. Otherwise stated, the problem of classification of FRA’s is related, in this approach, to the problem of classifying matrices with non-negative integer entries, a problem addressed in the book of Goodmann, De la Harpe and Jones [22]. A classification of all non-negative integer valued matrices whose highest component of the Perron-Frobenius eigenvector is less than 2 has been obtained and put in 1 to 1 correspondence with diagrams of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ and $A_{2n}/Z_2$. Now, if we check $P6$ for all the FRA’s generated by these diagrams, we discover that only $A_n$ and $A_{2n}/Z_2$ survive the check. In other words, polynomiality of FRA is equivalent to having highest component of Perron-Frobenius less than 2. (See also the related topics in [23]).

Pushing the graph analogy further, one can ask if the FRA’s of other known series of models can be encoded in some graph. For example, the $D_{even}$ series of $SU(2)$ WZW models has FRA’s with graphs:

\[\text{etc...}\]
These graphs can be obtained by fusing the $D_{2n}$ Dynkin diagrams: a quite remarkable fact that relates them to the $A, D, E$ classification of [22].

Other interesting classes of graphs can be obtained. For example $SU(3)$ WZW models (diagonal series) have FRA’s encoded in the series of graphs:

Finally, we give the graphs for the $G_2$ WZW models:

By suitably orbifolding, fusing and tensoring these graphs one should obtain FRA’s for the non-diagonal series of WZW models, as well as those of minimal models of the corresponding $W$-algebras. Work in this direction is in progress.

We conclude this section on applications of Graph Theory to FRA’s by an apparently unrelated but intriguing result on a systematic analysis of FRA’s with structure constants less or equal to 1, imposing $P_1, \ldots, P_5$ only, and without using constraints from modular invariance or polynomiality (which are trivial to impose once this first level classification has been obtained). The results are summarized in tab.5. The columns correspond to the number of fields (identity included), while the row number $s \geq 0$ counts the pair of conjugate fields. In fact as $C^2$ is the identity permutation on $\{0, 1, \ldots, r\}$ the cycles of $C$ may have length 1,2 only, and non-selfconjugate FRA’s can be partitioned in a natural way according to the number $2s$ of non-selfconjugate fields.

It is quite impressive to notice that, at least in the range considered, the number of FRA’s as a function of the number of operators grows very slowly, almost linearly. In the case of selfconjugate FRA’s, for instance, this has to be compared with the asymptotic behaviour

$$H_r \sim \frac{2^{\binom{r+2}{3}}}{r!}$$

of the number $H_r$ of non-isomorphic hypergraphs which underly these FRA’s; $r = 6$ yields $H_6 \sim 10^{14}$ and associativity cuts it down to 18. The enumeration of associative hypergraphs seems to be a non trivial problem, and it would be rather interesting to expand tab.5 by resorting to supercomputing.

---

1They are one-to-one with all subsets of the set formed by the unordered triples $(iii), (iii), (ijk)$, $i \neq j \neq k$, $i \neq k$, $1 \leq i, j, k \leq r$, and in eq. 21 the factor $(r!)^{-1}$ cares for the asymptotic elimination of isomorphic algebras (see ref [24]); for $r = 5$ eq. 21 is already accurate because it predicts $H_5 \sim 2.86 \times 10^8$ while $H_5 = 287800704$ (from Polya’s counting theorem).
Table 5: Enumeration of non-isomorphic algebras which satisfy $P_1,\ldots,P_5$ and $N^k_{ij} \leq 1$. The entries show the number of non-isomorphic associative algebras which are not direct products of lower order ones; the subscripts specify the number (when different from 0) of additional direct product algebras.

| $s$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $r+1$ | $1$ | $2$ | $3$ | $3$ | $10$ | $14_6$ | $18$ | $14$ |
| $1$ | $1$ | $4$ | $5$ | $9$ | $14$ | $6_2$ | $7$ | $1$ |
| $2$ | $1$ | $6_2$ | $7$ | $1$ |
| $3$ |

### 6 Other results and open issues

Most of the studies on FRA’s have been done on the WZW models, in particular those related to quantum groups [12]. Moreover it is widely believed that suitable cosets (and orbifolds) of WZW models should exhaust all possible RCFT’s. Finally the RCFT’s which are more important from the statistical mechanics point of view are definitely the minimal models of the Virasoro Algebra and the $c = 1$ models, and all of them are well understood in terms of cosets (and orbifolds) of Kac-Moody algebras. Lots of interesting exact results have been produced in this direction.

#### 6.1 FRA’s of WZW models

All the FRA’s for all possible (diagonal) WZW are known, at least in principle, and can be explicitly written in some different but consistently equivalent ways: using the so called depth rule [25] (which was the first to be studied), using the Verlinde formula [3] and the fact that modular transformation of WZW characters are well known [26]; more recently a highly efficient algorithm which mixes both these approaches has been presented [27].

#### 6.2 FRA’s of Coset Models

An interesting problem is that of finding the FRA’s of the coset models of the type:

$$\frac{G_k \times G_l}{G_{k+l}}$$

(22)

were $G$ can be any Kac-Moody algebra and the index labels the level of the algebra. This point was discussed for $G = SU(2)$, $l = 1$ in [4] and for general $G$’s and levels in [28]. If we denote as $G_k$ the FRA of the WZW model (of the principal series) with group $G$ and level $k$, then the FRA of (22) proposed in [28] is $G_k \otimes G_l \otimes G_{k+l}$. 

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However, to be more precise, if we want to write the actual FRA, we have also to take into account the foldings due to the so called conformal grid symmetry [1]. This explains why in [9] we found in the case of Virasoro minimal models \((p, q)\) and \(q\) is odd, then the corresponding algebra is: \(A_{p^{-2}} \otimes B_{q^{-2}}\). Notice that, as a consequence of this, if both \(p\) and \(q\) are odd, we have a non trivial isomorphism between FRA’s, i.e. \(A_{p^{-2}} \otimes B_{q^{-2}}\) and \(B_{p^{-2}} \otimes A_{q^{-2}}\) turn out to be isomorphic. The same folding procedure should also be applied to FRA’s of higher cosets of the form \((22)\).

### 6.3 Relation with \(N = 2\) superconformal theory

An interesting recent result has been obtained by Gepner [15], who has shown the deep relation existing between FRA’s and the operator product algebra of the chiral fields in \(N = 2\) superconformal field theories. A relation which is in principle unexpected (notice that for \(N = 2\) we are dealing with the full operator product algebra, without truncation of any kind), and could probably lead to new results in the wider context of Frobenius Algebras and of its potential deformations.

All these issues show how the structure and classification of FRA’s is a very intriguing problem whose understanding will surely shed more light on the general properties of CFT’s and two dimensional Field Theories in general.

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