Characterization of the quantumness of unsteerable tripartite correlations

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Quantumness for a bipartite unsteerable quantum correlation is operationally characterized by the notion of super-unsteerability. Super-unsteerability refers to the requirement of a larger dimension of the random variable that the steering party has to preshare with the party to be steered in the classical simulation protocol to generate an unsteerable correlation than the local Hilbert space dimension of the quantum states (reproducing the given unsteerable correlation) at the steering party’s side. In the present study, this concept of super-unsteerability has been generalized by defining the notion of super-bi-unsteerability for tripartite correlations, which is unsteerable across a bipartite cut. Specific example of super-bi-unsteerability for tripartite correlations has been presented. This study provides a tool to characterize the quantumness of tripartite quantum correlations which are unsteerable across a bipartite cut.

I. INTRODUCTION

Quantum composite systems exhibit several nonclassical features such as entanglement [1], Einstein-Podolsky-Rosen (EPR) steering [2–4] and Bell nonlocality [5–7]. In the Bell scenario, local quantum measurements on certain spatially separated system leads to nonlocal correlations which cannot be explained by local hidden variable (LHV) theory [5]. However, it is well-known that quantum mechanics (QM) is not maximally nonlocal as there are post-quantum correlations, obeying the no-signalling (NS) principle, which are more nonlocal than QM. Popescu-Rohrlich (PR) box [8] is one of these correlations. Nonlocality in QM is limited by the Tsirelson bound [9].

Motivated by the seminal argument by Einstein, Podolsky and Rosen (EPR) [2] demonstrating the incompleteness of QM, Schrodinger introduced the concept of ‘quantum steering’ [10]. The task of quantum steering [3, 4] is to prepare different ensembles at one part of a bipartite system by performing local quantum measurements on another part of the bipartite system in such a way that these ensembles cannot be explained by a local hidden state (LHS) model. In other words, quantum correlations, which are steerable, cannot be reproduced by local hidden variable-local hidden state (LHV-LHS) model. In recent years, studies related to quantum steering have been acquiring considerable interest, as witnessed by a wide range of studies [11–19]. Bell-nonlocal states form a subset of the steerable states which also form a subset of the entangled states [3, 20]. However, unlike quantum nonlocality and entanglement, the task of quantum steering is inherently asymmetric [21]. In this case, the outcome statistics of one subsystem (which is being steered) is due to valid QM measurements on a valid QM state. On the other hand, there is no such constraint for the other subsystem. Quantum steering has also applications in semi device independent scenario where the party, which is being steered, has trust on his/her quantum device but the other party’s device is untrusted. Secure quantum key distribution (QKD) using quantum steering has been demonstrated [22], where one party cannot trust his/her devices.

Recently, it has been demonstrated that certain quantum information tasks may become advantageous even using separable states if they have quantum discord [23–25], which is a generalized measure of quantum correlations. This motivated the study of nonclassicality going beyond nonlocality. Certain separable states which have quantumness may improve quantum protocols if the shared randomness between the parties is finite [26]. This provides an operational meaning of the measures of quantumness such as quantum discord. In the context of classical simulation of local entangled states, Bowles et. al. [27] have shown that the statistics of all local entangled states can be simulated by using only finite shared randomness and they defined a measure which is the minimal dimension of that shared classical randomness. On the other hand, all the previous works have used unbounded shared randomness to simulate a given local entangled state. In Ref. [28], the minimal dimension of the shared classical randomness required to simulate any local correlation in a given Bell scenario have been demonstrated. Motivated by this, an interesting feature of certain local boxes, called superlocality, has been defined as follows: there exist certain local boxes which can be simulated by quantum systems of local dimension lower than the minimum dimension of the
shared classical randomness needed to simulate them. This implies that superlocality refers to the dimensional advantage in simulating certain local boxes by using quantum systems. In particular, it has been shown \cite{28, 29} that entanglement enables superlocality, however, superlocality occurs even for separable states. Recently it has been pointed out \cite{30} that superlocality cannot occur for arbitrary separable states, in particular, the separable states which are a classical-quantum state \cite{31} or its permutation can never lead to superlocality. One important point to be stressed here is that the bipartite quantum states which are not a classical-quantum state must have quantumness as quantified by quantum discord. Recently, Generalizing the concept of superlocality, the notion of super-correlation \cite{32} has been defined as follows: the requirement for a larger dimension of the preshared randomness to simulate the correlations than that of the quantum states that generate them. In particular, the quantumness of certain unsteerable correlations has been pointed out by the notion of super-unsteerability \cite{32}, the requirement for a larger dimension of the classical variable that the steering party has to preshare with the party to be steered for simulating the unsteerable correlation than the local Hilbert space dimension of the quantum system (reproducing the given unsteerable correlation) at the steering party’s side (i.e., at the untrusted party’s side).

The extension of the Bell-type scenario to more than two parties was first presented in the seminal work by Greenberger, Horne, and Zeilinger \cite{33}. Certain interesting features of nonlocality in tripartite scenario have been established \cite{34–37}. Genuine tripartite quantum discord has been defined to quantify the quantumness shared among all three subsystems of the tripartite quantum state \cite{38–40}. Recently, it has been demonstrated that the limited dimensional quantum simulation of certain local tripartite correlations must require genuine tripartite quantum discord states. To study genuine nonclassicality of these correlations, two quantities called, Svetlichny strength and Mermin strength has been defined in the context of tripartite no-signaling boxes \cite{41}.

In case of multipartite systems, earlier studies have established that certain nonlocal measures may indeed be amplified by the addition of system dimensions \cite{42–45}. Multipartite quantum entanglement displays complicated structures, which can be broadly classified according to whether entanglement is shared among all subsystems of a given multipartite system or not. In this context, the notion of genuine multipartite nonlocality has been introduced and Bell-type inequalities have been derived to detect it \cite{34}. Genuine multipartite quantum nonlocality can be quantified by classical communication models, where the $n$ parties are grouped into $m$ disjoint groups; within each group, the parties can freely communicate with each other, but are not allowed to do the same between distinct groups \cite{46}. The minimal amount of communication between these disjoint groups required to reproduce a given nonlocal correlation determines the extent of multipartite quantum nonlocality of that correlation. Recently, the operational characterization of genuine nonclassicality of local multipartite correlations has been presented and the notion of superlocality has been generalised in the context of local multipartite correlations \cite{47}.

Against the above backdrop, the motivation of the present study is to generalize the notion of super-unsteerability in the tripartite scenario to analyze the resource requirement for simulating the tripartite bi-unsteerable correlations (which are unsteerable across some particular bipartitions) in the context of a given steering scenario. In particular, we show that quantumness is necessary to reproduce certain tripartite bi-unsteerable correlations in the scenario where the dimension of the resource reproducing the correlations is restricted. We demonstrate that there are certain tripartite bi-unsteerable correlations whose simulation with LHV-LHS model requires preshared randomness with dimension exceeding the local Hilbert space dimension of the quantum system (reproducing the given bi-unsteerable correlation) at the untrusted party’s side. This is termed as "super-bi-unsteerability". It provides a tool to give an operational characterization of the quantumness of certain bi-unsteerable correlations in the tripartite scenario.

The plan of the paper is as follows. In Section II, the basic notions of NS polytope and the fundamental ideas of quantum steering in tripartite scenario has been presented. Our purpose is to decompose the given NS correlation in terms of convex combinations of extremal boxes of NS polytope which leads to a LHV-LHS decomposition of the given correlation. In Section III, we demonstrate the formal definition of super-bi-unsteerability, which is followed by Section IV presenting specific examples of super-bi-unsteerability. In Section V, quantumness of certain bi-unsteerable tripartite correlations captured by super-bi-unsteerability has been discussed. Finally, in the concluding Section VI, we elaborate a bit on the significance of the results obtained.

## II. FRAMEWORK

### A. No-signalling Polytope

Let us consider the quantum correlations arising from the following tripartite Bell scenario. Suppose, three spatially separated parties (say, Alice, Bob and Charlie) share a quantum mechanical system $\rho_{ABC} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, where $\mathcal{H}_k$ denotes Hilbert space of $k$th party. In this scenario, a correlation between the outcomes is described by the set of conditional probability distributions $P(abc|A,B,C)$, where $A$, $B$, and $C$ denote the inputs (measurement choices) and $a$, $b$ and $c$ denote the outputs (measurement outcomes) of Alice, Bob and Charlie respectively (with $x,y,z,a,b,c \in \{0,1\}$). Suppose $M^a_A$, $M^b_B$, and $M^c_C$, denote the measurement operators of Alice, Bob, and Charlie, respectively. Then any conditional probability distribution can be
expressed in quantum mechanics through the Born’s rule as follows:

\[ P(\text{abc}|A_b|B_b|C_c) = \text{Tr} \left( \rho_{ABC} M_{A_b}^a \otimes M_{B_b}^1 \otimes M_{C_c}^c \right). \]  

(1)

The set of no-signaling (NS) boxes with two binary inputs and two binary outputs forms a convex polytope \( \mathcal{N} \) in a 26 dimensional space [48], which includes the set of quantum correlations \( Q \) as a proper subset. Any box belonging to this polytope can be fully specified by 6 singlepartite, 12 bipartite and 8 tripartite expectations,

\[
P(\text{abc}|A_b|B_b|C_c) = \frac{1}{8} \left[ 1 + (-1)^a \langle A \rangle_x + (-1)^b \langle B \rangle_y + (-1)^c \langle C \rangle_z + (-1)^{\alpha\beta} \langle A \times B \rangle_{\alpha\beta} + (-1)^{\alpha\gamma} \langle A \times C \rangle_{\alpha\gamma} + (-1)^{\beta\gamma} \langle B \times C \rangle_{\beta\gamma} \right. \\
\left. + (-1)^{\alpha\beta\gamma} \langle A \times B \times C \rangle_{\alpha\beta\gamma} \right],
\]

where \( \langle A \rangle = \sum_a (-1)^a P(a|A) \), \( \langle A \times B \rangle = \sum_a \sum_b (-1)^{\alpha\beta} P(ab|A \times B) \) and \( \langle A \times B \times C \rangle = \sum_a \sum_b \sum_c (-1)^{\alpha\beta\gamma} P(abc|A \times B \times C) \), \( \oplus \) denotes modulo sum 2. The set of boxes that can be simulated by a fully LHV model are of the form,

\[
P(\text{abc}|A_b|B_b|C_c) = \sum_{d=0}^{d-1} p_d P_d(a|A) P_d(b|B) P_d(c|C),
\]

(3)

which form a fully local polytope [49, 50] denoted \( \mathcal{L} \). Here \( \lambda \) denotes shared classical randomness/local hidden variable (LHV) which occurs with probability \( p_\lambda \). For a given fully local box, the form (3) determines a classical simulation protocol with dimension \( d_1 \) [28]. The extremal boxes of \( \mathcal{L} \) are 64 fully local vertices which are fully deterministic boxes given by,

\[
P^{\alpha\beta\gamma\epsilon\zeta\eta}_{D}(\text{abc}|A_b|B_b|C_c) = \begin{cases} 1, & a = \alpha x \oplus \beta \\ b = \gamma y \oplus \epsilon \\ c = \zeta z \oplus \eta \\ 0, & \text{otherwise} \end{cases}
\]

(4)

Here, \( \alpha, \beta, \gamma, \epsilon, \zeta, \eta \in [0,1] \). The above boxes can be written as the product of deterministic distributions corresponding to Alice and Bob-Charlie, i.e.,

\[
P^{\alpha\beta\gamma\epsilon\zeta\eta}_{D}(\text{abc}|A_b|B_b|C_c) = P^{\alpha\beta}_{D}(a|A) P^{\epsilon\zeta\eta}_{D}(bc|B \times C),
\]

(5)

and

\[
P^{\alpha\beta\gamma\epsilon\zeta\eta}_{D}(bc|B_b|C_c) = \begin{cases} 1, & b = \gamma y \oplus \epsilon \\ c = \zeta z \oplus \eta \\ 0, & \text{otherwise} \end{cases}
\]

(6)

which can also be written as the product of deterministic distributions corresponding to Bob and Charlie, i.e.,

\[
P^{\alpha\beta\gamma\epsilon\zeta\eta}_{D}(bc|B_b|C_c) = P^{\alpha\beta\gamma\epsilon}_{D}(b|B) P^{\epsilon\zeta\eta}_{D}(c|C),
\]

(7)

and

\[
P^{\alpha\beta\gamma\epsilon\zeta\eta}_{D}(c|C_c) = \begin{cases} 1, & c = \zeta z \oplus \eta \\ 0, & \text{otherwise} \end{cases}
\]

(8)

Hence, one can write,

\[
P^{\alpha\beta\gamma\epsilon\zeta\eta}_{D}(abc|A_b|B_b|C_c) = P^{\alpha\beta\gamma\epsilon}_{D}(a|A) P^{\beta\gamma\epsilon\zeta}_{D}(b|B) P^{\epsilon\zeta\eta}_{D}(c|C).
\]

The set of local boxes and quantum boxes satisfy \( \mathcal{L} \subset Q \subset \mathcal{N} \). Boxes lying outside \( \mathcal{L} \) are called nonlocal boxes and they cannot be written as a convex mixture of the local deterministic boxes alone.

Nonlocal boxes can be classified into two categories:

i) genuinely three-way nonlocal

ii) two-way local boxes.

A nonlocal box is genuinely three-way nonlocal if and only if (iff) it cannot be written in the two-way local form [51] given by,

\[
P(\text{abc}|A_b|B_b|C_c) = p_1 \sum_{\lambda} P^{\lambda}_{A} P^{\lambda|B}_{B} P^{\lambda|C}_{C} + p_2 \sum_{\lambda} q_{AB} P^{\lambda|C}_{A} + p_3 \sum_{\lambda} r_{ABC}^\lambda P^{\lambda|BC}_{A},
\]

(9)
where, \( P_{AB}^{ABC} = P_{ABC}(ab|A_{1}B_{1}) \) \( \mathcal{P}_1(c|C_{2}) \), and, \( P_{ABC}^{AB} = P_{ABC}(a|A_{1}) \) \( \mathcal{P}_3(c|C_{2}) \) are similarly defined. Each bipartite distribution in this decomposition can have arbitrary nonlocality consistent with the NS principle. The set of boxes that admit a decomposition as in Eq. (9) again forms a convex polytope, which is called two-way local polytope denoted by \( L_2 \). The extremal boxes of this polytope are the 64 local vertices and 48 two-way local vertices. There are 16 two-way local vertices in which a PR-box [8] is shared between \( A \) and \( B \),

\[
P_{12}^{Sv}(abc|A_{1}B_{1}C_{2}) = \begin{cases} \frac{1}{2}, & a \oplus b = x \cdot y \oplus ax \oplus b y \oplus y \\ 0, & \text{otherwise} \end{cases}
\]

(10)

and

\[
P_{D}^{Sv}(c|C_{2}) = \begin{cases} 1, & c = \gamma z \oplus \epsilon \\ 0, & \text{otherwise} \end{cases}
\]

(11)

Though in the above the two-way local vertices, \( P_{13}^{Sv} \) and \( P_{23}^{Sv} \), in which a PR-box is shared by \( AC \) and \( BC \), respectively, are similarly defined. The extremal boxes in Eq. (10) can be written in the factorized form, \( P_{12}^{Sv}(abc|A_{1}B_{1}C_{2}) = P_{PR}^{Sv}(ab|A_{1}B_{2})P_{D}^{Sv}(c|C_{2}) \), where \( P_{PR}^{Sv}(ab|A_{1}B_{2}) \) are the 8 PR-boxes given by,

\[
P_{PR}^{Sv}(ab|A_{1}B_{2}) = \begin{cases} \frac{1}{2}, & \text{if } a \oplus b = x y \oplus ax \oplus by \oplus y \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
P_{D}^{Sv}(c|C_{2}) = \begin{cases} 1, & c = \gamma z \oplus \epsilon \\ 0, & \text{otherwise} \end{cases}
\]

are one of the classes of facet inequalities of the two-way local polytope. The violation of a Svetlichny inequality implies one of the forms of genuine nonlocality. The following extremal three-way nonlocal boxes:

\[
P_{Sv}^{Sv}(abc|A_{1}B_{1}C_{2}) = \begin{cases} \frac{1}{4}, & a \oplus b \oplus c = x \cdot y \oplus x \cdot z \oplus y \cdot z \oplus ax \oplus by \oplus y \cdot \gamma \oplus \epsilon \\ 0, & \text{otherwise} \end{cases}
\]

(13)

which violate a Svetlichny inequality to its algebraic maximum are called Svetlichny boxes. Mermin inequalities [35] are one of the classes of facet inequalities of the fully local polytope [52, 53]. One of the Mermin inequalities is given by,

\[
\langle A_{0}B_{0}C_{0} \rangle - \langle A_{0}B_{1}C_{1} \rangle - \langle A_{1}B_{0}C_{1} \rangle - \langle A_{1}B_{1}C_{0} \rangle \leq 2,
\]

(14)

and the other 15 Mermin inequalities can be obtained from the above inequality by local reversible operations (LRO), which are analogous to local unitary operations in quantum theory and include local relabeling of the inputs and outputs (conditionally on the input). Mermin inequalities detect certain nonlocal boxes which are two-way local. Quantum correlations that violate a Mermin inequality to its algebraic maximum demonstrate Greenberger–Horne–Zeilinger (GHZ) paradox [33] and are called Mermin boxes.

If a nonlocal box, which violates a Svetlichny inequality, is decomposed in the context of NS polytope, then it necessarily has a Svetlichny-box fraction in the decomposition. In other words, the box can be decomposed as a convex mixture of an irreducible Svetlichny-box and a Svetlichny-local box \( P_{Sv,L} \) [2],

\[
P = P_{Sv}^{Sv} + (1 - P_{Sv})P_{Sv,L}.
\]

Svetlichny strength [41] is the maximal Svetlichny-box fraction of a Svetlichny-nonlocal box obtained by maximizing the Svetlichny-box fraction \( P_{Sv} \) over all possible decompositions such that the Svetlichny-local box \( P_{Sv,L} \) in the decomposition does not have the Svetlichny box \( P_{Sv}^{Sv} \) fraction excessively. Similarly, Mermin strength is defined as the maximal Mermin-box weight of a correlation that violates a Mermin inequality [41]. Unlike the nonlocal cost [54], Svetlichny strength and/or Mermin strength can also be nonzero for certain local correlations.
B. Quantum Bi-Steering in tripartite scenario

Let us generalize the concept of quantum steering (in bipartite scenario) as introduced by Wiseman et al. [3, 4] to tripartite scenario [55, 56]. Suppose, three spatially separated parties (say, Alice, Bob and Charlie) share a quantum mechanical system $\rho_{ABC} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Let us assume that the sets of observables in the Hilbert space of Alice, Bob and Charlie’s system are denoted by $\mathcal{D}_A$, $\mathcal{D}_B$ and $\mathcal{D}_C$, respectively. $A_\lambda$ denotes an element of $\mathcal{D}_A$, and the set of outcomes are labeled by $a \in \mathcal{L}(A)$. Similarly, $B_\lambda$ denotes an element of $\mathcal{D}_B$, and the set of outcomes are labeled by $b \in \mathcal{L}(B)$; $C_\lambda$ denotes an element of $\mathcal{D}_C$, and the set of outcomes are labeled by $c \in \mathcal{L}(C)$. The joint state $\rho_{ABC}$ of the system shared between Alice, Bob and Charlie is bi-steerable from Alice to Bob-Charlie iff for all $a \in \mathcal{L}(A)$, $b \in \mathcal{L}(B)$, $c \in \mathcal{L}(C)$, $A_\lambda \in \mathcal{D}_A$, $B_\lambda \in \mathcal{D}_B$, $C_\lambda \in \mathcal{D}_C$, the joint probability distributions cannot be written in the form:

$$P_{\rho_{ABC}}(abc|A_\lambda B_\lambda C_\lambda) = \sum_{\lambda} r_i P_{\lambda}(a|A_\lambda) P(b|B_\lambda, C_\lambda, \rho_{BC}^\lambda),$$

(16)

where, $\sum_i r_i = 1$. $P_\lambda(a|A_\lambda)$ denotes an arbitrary probability distribution (deterministic/non-deterministic boxes) arising from local hidden variable (LHV) $\lambda$ (which occurs with probability $r_i$) and $P(b|B_\lambda, C_\lambda, \rho_{BC}^\lambda)$ denotes the quantum probability of obtaining the outcomes $b$ and $c$, when measurements $B_\lambda$ and $C_\lambda$ are performed by Bob and Charlie, respectively, on the bipartite local hidden state (LHS) $\rho_{BC}^\lambda$ shared between Bob and Charlie. The probability distribution $P(b|B_\lambda, C_\lambda, \rho_{BC}^\lambda)$ can demonstrate quantum nonlocality, or EPR-steering (from Bob to Charlie, or from Charlie to Bob, or both), or locality, or unsteerability. Hence, the state $\rho_{ABC}$ will be called bi-steerable from Alice to Bob-Charlie iff it does not have the aforementioned LHV-LHS model. Similarly one can obtain the necessary and sufficient condition of bi-steerability of the state $\rho_{ABC}$ from Bob to Alice-Charlie, or from Charlie to Alice-Bob. Note that the bi-unsteerable states form a subset of the two-way local correlations, as the bipartite distributions in the two-way local correlations are NS box and the bipartite distributions in the bi-unsteerable correlations are quantum correlations.

III. DEFINITIONS OF SUPER-BI-UNSTEERABILITY OF TRIPARTITE BOXES

For a given bipartite or n-partite box, let $d_i$ denotes the minimal dimension of the shared classical randomness. Before we define super-bi-unsteerability for bi-unsteerable tripartite boxes, let us recapitulate the notion of super-unsteerability [32] for unsteerable bipartite boxes.

**Definition 1:** Suppose two spatially separated party (say, Alice and Bob) share a bipartite quantum state $\rho_{AB}$ in $\mathbb{C}^{d^A} \otimes \mathbb{C}^{d^B}$ which produces a correlation box $P(ab|A_\lambda B_\lambda)$ (unsteerable from Alice to Bob). Then, super-unsteerability holds iff there is no decomposition of the form:

$$P(ab|A_\lambda B_\lambda) = \sum_{\lambda=0}^{d_1-1} r_i P_\lambda(a|A_\lambda) P(b|B_\lambda, \rho_{B}^\lambda),$$

(17)

where $d_1 \leq d^A$. Here, $P_\lambda(a|A_\lambda)$ denotes an arbitrary probability distribution (deterministic/non-deterministic boxes) arising from local hidden variable (LHV) $\lambda$ and $P(b|B_\lambda, \rho_{B}^\lambda)$ are the quantum probability of obtaining the outcome $b$, when measurement $B_\lambda$ is performed by Bob on LHS $\rho_{B}^\lambda$ in $\mathbb{C}^{d^B}$.

We now define super-bi-unsteerability for the bi-unsteerable tripartite boxes.

**Definition 2:** Suppose three spatially separated party (say, Alice, Bob and Charlie) share a tripartite quantum state $\rho'_{ABC}$ in $\mathbb{C}^{d^A} \otimes \mathbb{C}^{d^B} \otimes \mathbb{C}^{d^C}$ which produces a correlation box $P(abc|xyz)$ (bi-unsteerable from Alice to Bob-Charlie). Then, super-bi-unsteerability holds (in the bipartite cut Alice and Bob-Charlie) iff there is no decomposition of the form:

$$P(abc|A_\lambda B_\lambda C_\lambda) = \sum_{\lambda=0}^{d_1-1} r_i P_\lambda(a|A_\lambda) P(bc|B_\lambda C_\lambda, \rho_{BC}^\lambda),$$

(18)

where $d_1 \leq d^A$. Here, $P_\lambda(a|A_\lambda)$ denotes an arbitrary probability distribution (deterministic/non-deterministic boxes) arising from local hidden variable (LHV) $\lambda$ and $P(bc|B_\lambda C_\lambda, \rho_{BC}^\lambda)$ are the quantum probability of obtaining the outcomes $b$ and $c$, when measurements $B_\lambda$ and $C_\lambda$ are performed by Bob and Charlie, respectively, on the bipartite LHS $\rho_{BC}^\lambda$ in $\mathbb{C}^{d^B} \otimes \mathbb{C}^{d^C}$. $P(bc|B_\lambda C_\lambda, \rho_{BC}^\lambda)$ may demonstrate quantum nonlocality or EPR-steering.
The definitions of super-bi-unsteerability in the bipartite cut Bob and Alice-Charlie, or in the bipartite cut Charlie and Alice-Bob are similar to the above definition.

In the following Section, we are going to study some specific examples of super-bi-unsteerability.

IV. SPECIFIC EXAMPLES OF SUPER-BI-UNSTEERABILITY

We consider quantum correlations that belong to the noisy Mermin family defined as

\[ P_{MF}^V(abc|A_xB_yC_z) = \frac{1 + (-1)^{x_0y_0z_0} \delta_{x_0y_0z_0}}{8} V, \]

where \( 0 < V \leq 1 \). The above box is two-way local, but not fully local for \( V > \frac{1}{2} \) as it violates the Mermin inequality (given in Eq. (14)) in this range, and for \( V \leq \frac{1}{2} \), it is fully local as in this range the correlation does not violate any Bell inequality. Note that for any \( V > 0 \), the quantum simulation of the Mermin family by using a \( 2 \otimes 2 \otimes 2 \) quantum state or using a \( 3 \otimes 2 \otimes 2 \) quantum state necessarily requires genuine quantum discord [38, 39] in the state. Because, the Mermin family has nonzero Mermin strength for any \( V > 0 \) [41, 47]. We now give example of simulating the noisy Mermin family by using a quantum state which has quantumness. Consider, the three spatially seperated party (say, Alice, Bob and Charlie) share the following \( 2 \otimes 2 \otimes 2 \) GHZ state:

\[ \rho_1 = V|GHZ\rangle\langle GHZ| + (1 - V)\frac{T_2}{2} \otimes \frac{T_2}{2} \otimes \frac{T_2}{2}, \]

where \( |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \); \( 0 < V \leq 1 \); \(|0\rangle \) and \(|1\rangle \) are the eigenstates of operator \( \sigma_z \) corresponding to eigenvalues \(+1\) and \(-1\) respectively; \( T_2 \) is the \( 2 \otimes 2 \) identity matrix. Then the noisy Mermin family can be reproduced if Alice, Bob and Charlie perform projective qubit measurement corresponding to the operators: \( A_0 = \sigma_y, A_1 = -\sigma_x; B_0 = \sigma_y, B_1 = -\sigma_x; C_0 = \sigma_y, C_1 = -\sigma_x \) respectively. Hence, noisy Mermin family can be simulated with \( 2 \otimes 2 \otimes 2 \) quantum states.

A. Simulating noisy Mermin family with LHV at Alice’s side and LHS at Bob-Charlie’s side

The correlation belonging to noisy Mermin family can be written as

\[ P_{MF}^V(abc|A_xB_yC_z) = \sum_{a,d=0}^{3} r_a P_d(a|A_x) P(bc|B_yC_z; \rho_{bc}^d), \]

where \( r_0 = r_1 = r_2 = r_3 = \frac{1}{8} \), and

\[ P_0(a|A_x) = P_0^D, P_1(a|A_x) = P_1^D, P_2(a|A_x) = P_2^D, P_3(a|A_x) = P_3^D. \]

where,

\[ P_D(a|A_x) = \begin{cases} 1, & a = \alpha x \oplus \beta \\ 0, & \text{otherwise.} \end{cases} \]

Now,

\[
P_{bc}^D = \begin{pmatrix}
00 & 01 & 10 & 11 \\
01 & \frac{1+V}{4} & \frac{1-V}{4} & \frac{1-V}{4} \\
10 & \frac{1-V}{4} & \frac{1-V}{4} & \frac{1+V}{4} \\
11 & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4}
\end{pmatrix},
\]

where each row and column corresponds to a fixed measurement settings \((yz)\) and a fixed outcome \((bc)\) respectively. Throughout the paper we will follow the same convention.

This joint probability distribution at Bob and Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators \( B_0 = \sigma_x, B_1 = \sigma_y \); and \( C_0 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y), C_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) \) on the state given by

\[ \rho_{bc}^0 = |\psi_0\rangle\langle \psi_0|, \]

where, \( |\psi_0\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle \) \((0 \leq \theta \leq \frac{\pi}{2})\) with \( \sin 2\theta = \sqrt{2V} \); \(|0\rangle\) and \(|1\rangle\) are the eigenstates of \( \sigma_z \) corresponding to the eigenvalues \(+1\) and \(-1\) respectively.
"A case of noisy Mermin family, all the marginal probability distributions of Alice, Bob and Charlie are maximally mixed: having dimension 3 and with different deterministic distributions at Alice’s side for 0 < V ≤ \frac{1}{\sqrt{2}}. Before proceeding, we want to mention that in this case of noisy Mermin family, all the marginal probability distributions of Alice, Bob and Charlie are maximally mixed:

\[
P(bc|B_y, C_z, \rho_{bc}^1) = \begin{pmatrix} \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \\ \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \\ \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \\ \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \end{pmatrix},
\]

This joint probability distribution at Bob and Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators \( B_0 = \sigma_x, B_1 = \sigma_z \); and \( C_0 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y), C_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) \) on the state given by

\[
\rho_{bc}^1 = |\psi_1\rangle\langle\psi_1|,
\]

where, |\psi_1\rangle = cos\theta|00\rangle - sin\theta|11\rangle (0 ≤ \theta ≤ \frac{\pi}{4}) with sin2\theta = \sqrt{2V}.

\[
P(bc|B_y, C_z, \rho_{bc}^2) = \begin{pmatrix} \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} \\ \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} \\ \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} \\ \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} & \frac{1-V}{4} \end{pmatrix},
\]

This joint probability distribution at Bob and Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators \( B_0 = \sigma_x, B_1 = \sigma_z \); and \( C_0 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y), C_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) \) on the state given by

\[
\rho_{bc}^2 = |\psi_2\rangle\langle\psi_2|,
\]

where, |\psi_2\rangle = (-i) cos\theta|00\rangle + sin\theta|11\rangle (0 ≤ \theta ≤ \frac{\pi}{4}) with sin2\theta = \sqrt{2V}.

\[
P(bc|B_y, C_z, \rho_{bc}^3) = \begin{pmatrix} \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \\ \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \\ \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \\ \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} & \frac{1+V}{4} \end{pmatrix},
\]

This joint probability distribution at Bob and Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators \( B_0 = \sigma_x, B_1 = \sigma_z \); and \( C_0 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y), C_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) \) on the state given by

\[
\rho_{bc}^3 = |\psi_3\rangle\langle\psi_3|,
\]

where, |\psi_3\rangle = i cos\theta|00\rangle + sin\theta|11\rangle (0 ≤ \theta ≤ \frac{\pi}{4}) with sin2\theta = \sqrt{2V}.

Now, |\psi_3\rangle = i cos\theta|00\rangle + sin\theta|11\rangle (0 ≤ \theta ≤ \frac{\pi}{4}), which implies that \( V ≤ \frac{1}{\sqrt{2}} \). Hence, one can state that the noisy Mermin family can be expressed with a LHV-LHS decomposition (21) having hidden variables of dimension 4 in the range \( 0 < V ≤ \frac{1}{\sqrt{2}} \). The noisy Mermin family for \( V ≤ \frac{1}{\sqrt{2}} \), therefore, is bi-unsteerable from Alice to Bob-Charlie. Each joint probability distribution at Bob-Charlie’s side \( P(bc|B_y, C_z, \rho_{bc}^\lambda) \) (\( \lambda = 0, 1, 2, 3 \)) produced from the LHS demonstrates EPR-steering when \( \frac{1}{2} < V ≤ \frac{1}{\sqrt{2}} \) (if the two measurement settings of the party which is being steered are mutually unbiased), because in this range, each of the \( P(bc|B_y, C_z, \rho_{bc}^\lambda) \) violates the analogous Clauser-Horne-Shimony-Holt inequality for steering [57]. Each joint probability distribution at Bob-Charlie’s side \( P(bc|B_y, C_z, \rho_{bc}^\lambda) \) produced from the LHS demonstrates super-unsteerability when \( 0 < V ≤ \frac{1}{2} \) (for detailed calculations, see the Appendix A).

Hence, the decomposition (21) represents a LHV-LHS decomposition of the bi-unsteerable noisy Mermin box with different deterministic distributions at Alice’s side for \( 0 < V ≤ \frac{1}{\sqrt{2}} \).

**Theorem 1.** The LHV-LHS decomposition of bi-unsteerable noisy Mermin box cannot be realized with hidden variables having dimension 3 for \( V > \frac{1}{\sqrt{2}} \).

**Proof.** Let us try to generate a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variables having dimension 3 and with different deterministic distributions at Alice’s side. Before proceeding, we want to mention that in case of noisy Mermin family, all the marginal probability distributions of Alice, Bob and Charlie are maximally mixed:

\[
P(a|A_x) = P(b|B_y) = P(c|C_z) = \frac{1}{2}, a, b, c, x, y, z.
\]
Consider that the noisy Mermin family can be decomposed in the following way:

\[ P_{MF}^V(abc|A_x B_y C_z) = \sum_{\lambda=0}^{2} r_\lambda P_\lambda(a|A_x) \sum_{\lambda=0}^{2} r_\lambda P_\lambda(b|B_y, C_z, \rho_{BC}^\lambda), \quad (29) \]

Here, \( r_0 = u, r_1 = v, r_2 = w \) (0 < \( u < 1, 0 < v < 1, 0 < w < 1 \), \( u + v + w = 1 \)). Since Alice’s strategy is deterministic one, the three probability distributions \( P_0(a|A_x), P_1(a|A_x) \) and \( P_2(a|A_x) \) must be equal to any three among \( P_{D0}^{A}, P_{D1}^{A}, P_{D2}^{A} \). But any such combination will not satisfy the marginal probabilities \( P(a|A_x) \) for Alice. So it is impossible to generate a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variables having dimension 3 and with different deterministic distributions at Alice’s side.

Let us try to generate a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variables having dimension 3 and with different non-deterministic distributions at Alice’s side. We note that the noisy Mermin family is fully local for \( V \leq \frac{1}{3} \) and it is two-way local, but not fully local for \( \frac{1}{3} < V \leq 1 \). Hence from any decomposition of the noisy Mermin family in terms of fully deterministic boxes or two-way local vertices, one may construct a LHV-LHS model of the bi-unsteerable noisy Mermin family as in Eq.(21) with different deterministic distributions at Alice’s side, which does not require hidden variables of dimension more than 4 since there are only 4 possible different deterministic distributions given by Eq.(5) at Alice’s side. Hence, a LHV-LHS model of the bi-unsteerable noisy Mermin family with hidden variables of dimension 3 can also be achieved by constructing a LHV-LHS model of the bi-unsteerable noisy Mermin family with hidden variables of dimension 4 with different deterministic distributions at Alice’s side followed by taking equal joint probability distributions (having quantum realisations) at Bob-Charlie’s side as common and making the corresponding distributions at Alice’s side non-deterministic.

If the hidden variable dimension in the LHV-LHS decomposition of bi-unsteerable noisy Mermin family can be reduced from 4 to 3, then noisy Mermin family can be decomposed in the following way:

\[ P_{MF}^V(abc|A_x B_y C_z) = \sum_{\lambda=0}^{3} r_\lambda P_\lambda(a|A_x) \sum_{\lambda=0}^{3} r_\lambda P_\lambda(b|B_y, C_z, \rho_{BC}^\lambda), \quad (30) \]

where \( P_\lambda(a|A_x) \) are different deterministic distributions and any two of the four joint probability distributions \( P(b|B_y, C_z, \rho_{BC}^\lambda) \) are equal to each other; 0 < \( r_\lambda < 1 \) for \( \lambda = 0, 1, 2, 3 \); \( \sum_{\lambda=0}^{3} r_\lambda = 1 \). Then taking equal joint probability distributions \( P(b|B_y, C_z, \rho_{BC}^\lambda) \) at Bob-Charlie’s side as common and making corresponding distribution at Alice’s side non-deterministic will reduce the dimension of the hidden variable from 4 to 3. For example, let us consider

\[ P(b|B_y, C_z, \rho_{BC}^0) = P(b|B_y, C_z, \rho_{BC}^2). \quad (31) \]

Now in order to satisfy Alice’s marginal given by Eq. (28), one must take \( r_0 = r_1 = r_2 = r_3 = \frac{1}{2} \). Hence, the decomposition (36) can be written as,

\[ P_{MF}^V(abc|A_x B_y C_z) = q_0 \mathbb{P}_0(a|A_x) P(b|B_y, C_z, \rho_{BC}^0) + \frac{1}{4} P_1(a|A_x) P(b|B_y, C_z, \rho_{BC}^1) + \frac{1}{4} P_2(a|A_x) P(b|B_y, C_z, \rho_{BC}^3), \quad (32) \]

where,

\[ \mathbb{P}_0(a|A_x) = \frac{P_0(a|A_x) + P_2(a|A_x)}{2}, \quad (33) \]

which is a non-deterministic distribution at Alice’s side, and

\[ q_0 = \frac{1}{2}. \quad (34) \]

The decomposition (32) represents a LHV-LHS model of noisy Mermin family having different deterministic/non-deterministic distributions at Alice’s side with the dimension of the hidden variable being 3. Now in this protocol, if all the tripartite distributions \( P_{MF}^V(abc|A_x B_y C_z) \) are reproduced, quantum realisations of all the joint probability distributions \( P(b|B_y, C_z, \rho_{BC}^\lambda) \) are not possible for \( V > \frac{1}{\sqrt{3}} \) (for detailed calculations, see the Appendix B).

There are the following other cases in which the dimension of the hidden variable in the LHV-LHS decomposition of the bi-unsteerable noisy Mermin family can be reduced from 4 to 3:

\begin{itemize}
  \item[i)] \( P(b|B_y, C_z, \rho_{BC}^0) = P(b|B_y, C_z, \rho_{BC}^1); \)
  \item[ii)] \( P(b|B_y, C_z, \rho_{BC}^0) = P(b|B_y, C_z, \rho_{BC}^3); \)
  \item[iii)] \( P(b|B_y, C_z, \rho_{BC}^1) = P(b|B_y, C_z, \rho_{BC}^3); \)
  \item[iv)] \( P(b|B_y, C_z, \rho_{BC}^1) = P(b|B_y, C_z, \rho_{BC}^3); \)
  \item[v)] \( P(b|B_y, C_z, \rho_{BC}^2) = P(b|B_y, C_z, \rho_{BC}^3); \)
\end{itemize}
Now in cases i) and ii), it can be shown that all the tripartite distributions $P^V_{MF}(abc|A,B,C)$ is reproduced iff $V = 0$. On the other hand, in cases iii) and iv), following similar procedure adopted in Appendix B it can be shown that if all the tripartite distributions $P^V_{MF}(abc|A,B,C)$ are reproduced, quantum realisations of all the joint probability distributions $P(bc|B_y,C_z,\rho_{BC}^{3})$ are not possible for $V > \sqrt{\frac{2}{3}}$.

Hence, one can conclude that the LHV-LHS decomposition of bi-unsteerable noisy Mermin box cannot be realized with hidden variables having dimension 3 for $V > \sqrt{\frac{2}{3}}$ with deterministic/non-deterministic distributions at Alice’s side.

**Theorem 2.** The LHV-LHS decomposition of bi-unsteerable noisy Mermin box cannot be realized with hidden variables having dimension 2 or 1 for $V > 0$.

Now, let us try to generate a LHV-LHS decomposition of the noisy Mermin family with hidden variables of dimension 2 having different deterministic distributions at Alice’s side. In this case the noisy Mermin family can be decomposed in the following way:

$$P^V_{MF}(abc|A,B,C) = \sum_{l=0}^{1} r_l P_l(a|A_l)P(bc|B_y,C_z,\rho_{BC}^{3}).$$

Here, $r_0 = u, r_1 = v (0 < u < 1, 0 < v < 1, u + v = 1)$. Since Alice’s strategies are deterministic, the two probability distributions $P_0(a|A_y)$ and $P_1(a|A_y)$ must be equal to any two among $P_{00}^{D}, P_{01}^{D}, P_{10}^{D}$ and $P_{11}^{D}$. In order to satisfy the marginal probabilities for Alice, the only two possible choices of $P_0(a|A_y)$ and $P_1(a|A_y)$ are:

1) $P_{00}^{D}$ and $P_{01}^{D}$ with $u = v = \frac{1}{2}$.
2) $P_{10}^{D}$ and $P_{11}^{D}$ with $u = v = \frac{1}{2}$.

Now, it can be easily checked that none of these two possible choices will satisfy all the tripartite joint probability distributions $P^i_{MF}(abc|A,B,C)$ for $V > 0$ (for detailed calculations, see the Appendix C). It is, therefore, impossible to generate a LHV-LHS decomposition of the noisy Mermin family with hidden variables of dimension 2 having different deterministic distributions at Alice’s side.

Now, let us try to generate a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variables having dimension 2 and with different non-deterministic distributions at Alice’s side. As noted earlier, this can also be achieved by constructing a LHV-LHS model of the bi-unsteerable noisy Mermin family with hidden variables of dimension 4 or 3 having different deterministic distributions at Alice’s side followed by taking equal joint probability distributions (having quantum realisations) at Bob-Charlie’s side as common and making the corresponding distributions at Alice’s side non-deterministic.

It has already been shown that it is impossible to generate a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variables having dimension 3 and with different deterministic distributions at Alice’s side. Hence, there is no scope to reduce the hidden variable dimension from 3 to 2 in the LHV-LHS decomposition of bi-unsteerable noisy Mermin family.

Now, if the hidden variable dimension in the LHV-LHS decomposition of bi-unsteerable noisy Mermin family can be reduced from 4 to 2, then noisy Mermin family can be decomposed in the following way:

$$P^V_{MF}(abc|A,B,C) = \sum_{l=0}^{3} r_l P_l(a|A_l)P(bc|B_y,C_z,\rho_{BC}^{3}),$$

where $P_l(a|A_l)$ are different deterministic distributions; and either any three of the four joint probability distributions $P(bc|B_y,C_z,\rho_{BC}^{3})$ are equal to each other or there exists two sets each containing two equal joint probability distributions $P(bc|B_y,C_z,\rho_{BC}^{3})$:

$0 < r_1 < 1$ for $\lambda = 0, 1, 2, 3$; $\sum_{l=0}^{3} r_l = 1$. Then, as described earlier, taking equal joint probability distributions $P(bc|B_y,C_z,\rho_{BC}^{3})$ at Bob-Charlie’s side as common and making corresponding distribution at Alice’s side non-deterministic will reduce the dimension of the hidden variable from 4 to 2.

There are the following seven cases in which the dimension of the hidden variable in the LHV-LHS decomposition of bi-unsteerable noisy Mermin family can be reduced from 4 to 2:

- $P(bc|B_y,C_z,\rho_{BC}^{3}) = P(bc|B_y,C_z,\rho_{BC}^{3})$ as well as $P(bc|B_y,C_z,\rho_{BC}^{3}) = P(bc|B_y,C_z,\rho_{BC}^{3})$;
- $P(bc|B_y,C_z,\rho_{BC}^{3}) = P(bc|B_y,C_z,\rho_{BC}^{3})$ as well as $P(bc|B_y,C_z,\rho_{BC}^{3}) = P(bc|B_y,C_z,\rho_{BC}^{3})$;
- $P(bc|B_y,C_z,\rho_{BC}^{3}) = P(bc|B_y,C_z,\rho_{BC}^{3})$ as well as $P(bc|B_y,C_z,\rho_{BC}^{3}) = P(bc|B_y,C_z,\rho_{BC}^{3})$;
Now in any of these possible cases, considering arbitrary joint probability distributions \( P(bc|B_x,C_z;P_{BC}^1) \) at Bob-Charlie’s side (without considering any constraint), it can be shown that all the tripartite distributions \( P_{MF}(abc|A_x,B_y,C_z) \) are not reproduced simultaneously for \( V > 0 \). Hence, this also holds when the boxes \( P_{SF}(bc|y) \) satisfy NS principle as well as have quantum realizations.

It can be checked that the noisy Mermin box is nonproduct across all three bipartite cuts for any \( V > 0 \). It is, therefore, impossible to generate a LHV-LHS decomposition of the bi-unsteerable noisy Mermin box (0 < \( V \leq \frac{1}{2\sqrt{2}} \)) with hidden variables having dimension 1.

Hence, one can conclude that the LHV-LHS decomposition of bi-unsteerable noisy Mermin box cannot be realized with hidden variables having dimension 2 or 1 for \( V > 0 \).

Theorem 2 implies the following.

**Corollary 1.** The bi-unsteerable noisy Mermin family demonstrates super-bi-unsteerability (in the bipartite cut Alice and Bob-Charlie) for \( 0 < V \leq \frac{1}{2\sqrt{2}} \).

**Proof.** The bi-unsteerable noisy Mermin family \((0 < V \leq \frac{1}{2\sqrt{2}})\) can be reproduced by appropriate measurements on the quantum state in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) (given by Eq.\((20)\)). On the other hand, we have shown that the bi-unsteerable noisy Mermin family \((0 < V \leq \frac{1}{2\sqrt{2}})\) can be simulated with LHV at Alice’s side and LHS at Bob-Charlie’s side with the minimum dimension of the hidden variable being greater than 2. The bi-unsteerable noisy Mermin family \((0 < V \leq \frac{1}{2\sqrt{2}})\), therefore, demonstrates super-bi-unsteerability across the bipartite cut \( A|BC \).

In a similar way, it can be shown that the bi-unsteerable noisy Mermin family \((0 < V \leq \frac{1}{2\sqrt{2}})\) demonstrates super-bi-unsteerability across any bipartite cut.

Now, we consider quantum correlations that belong to the noisy Svetlichny family defined as

\[
P_{SF}^V(abc|A_x,B_y,C_z) = \frac{2 + (-1)^{xyz} \sqrt{2V}}{16}
\]

where \(0 < V \leq 1\). Since the noisy Svetlichny family has nonzero Svetlichny strength for any \( V > 0 \), the quantum simulation of these correlations by using a 2 \( \otimes \) 2 \( \otimes \) 2 quantum state or using a 3 \( \otimes \) 2 \( \otimes \) 2 quantum state necessarily requires genuine quantum discord \([38, 39]\) in the state \([41, 47]\). Following the similar argument presented earlier in case noisy Mermin family, it can be stated that the bi-unsteerable noisy Svetlichny family demonstrates super-bi-unsteerability (in all the three possible bipartite cut) for \(0 < V \leq \frac{1}{2\sqrt{2}}\) \([56]\).

V. QUANTUMNESS OF TRIPARTITE CORRELATIONS AS CAPTURED BY “SUPER-BI-UNSTEERABILITY”

Note that the dimension of the hidden variable needed to simulate the LHV-LHS model of the bi-unsteerable noisy Mermin family in the range \( V > \frac{1}{2\sqrt{2}} \) must be greater than 3. On the other hand, that in the range \( V > 0 \) must be greater than 2. Hence, the bi-unsteerable noisy Mermin family certifies quantumness of the 2 \( \otimes \) 2 \( \otimes \) 2 dimensional resource reproducing it in the range \(0 < V \leq \frac{1}{2\sqrt{2}}\). For example, the bi-unsteerable noisy Mermin family in the range \(0 < V \leq \frac{1}{2\sqrt{2}}\) characterizes the quantumness of the state \((20)\). The bi-unsteerable noisy Mermin family in the range \(\frac{1}{2\sqrt{2}} < V \leq \frac{1}{\sqrt{2}}\) also certifies quantumness of the 3 \( \otimes \) 2 \( \otimes \) 2 dimensional resource reproducing it. For example, consider that the three spatially separated party (say, Alice, Bob and Charlie) share the following 3 \( \otimes \) 2 \( \otimes \) 2 quantum state:

\[
\rho_2 = V|\text{GHZ}\rangle\langle\text{GHZ}| + (1 - V)|2\rangle\langle 2| \otimes \frac{I_2}{2} \otimes \frac{I_2}{2}
\]

where \(|\text{GHZ}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle); 0 < V \leq 1; |0\rangle, |1\rangle \) and \(|2\rangle\) form an orthonormal basis in the Hilbert space in \(C^3\); \(|0\rangle\) and \(|1\rangle\) form an orthonormal basis in the Hilbert space in \(C^2\); \(I_2 = |0\rangle\langle 0| + |1\rangle\langle 1|\). If Alice, Bob and Charlie perform appropriate measurements on the state given in Eq.\((38)\), the noisy Mermin family can also be reproduced (for detailed calculations, see the Appendix D). Hence, the bi-unsteerable noisy Mermin family in the range \(\frac{1}{2\sqrt{2}} < V \leq \frac{1}{\sqrt{2}}\) characterizes the quantumness the 3 \( \otimes \) 2 \( \otimes \) 2 state \((38)\).

The notion of genuine tripartite quantum discord has been defined in a tripartite quantum state to capture the genuine quantumness of separable states \([38]\). Note that, for \( V > 0 \), the simulation of the noisy Mermin family by using a 2 \( \otimes \) 2 \( \otimes \) 2 quantum state or using a 3 \( \otimes \) 2 \( \otimes \) 2 quantum state requires genuine quantumness in the form of genuine quantum discord as they have nonzero Mermin strength \([41, 47]\). Hence, the super-bi-unsteerable states given by Eqs.\((20, 38)\) (which reproduce the super-bi-unsteerable noisy Mermin family in the range \(0 < V \leq \frac{1}{2\sqrt{2}}\)) have non-zero genuine quantum discord.
Genuine tripartite quantum discord becomes zero iff there exists a bipartite cut of the tripartite system such that no quantum correlation exist between the two parts [39]. It is well-known that a bipartite quantum state has no (Alice to Bob) quantum discord iff it can be written in the classical-quantum (CQ) state form, $\rho_{CQ} = \sum_i p_i |i\rangle^A \langle i| \otimes \rho_i^B$ [58].

The tripartite classical-quantum state is defined as follows.

**Definition 3:** A fully separable tripartite state has a classical-quantum state form with respect to the bipartite cut $A$ versus $BC$ if it can be decomposed as

$$\rho_{CQ}^{ABC} = \sum_i p_i |i\rangle^A \langle i| \otimes \rho_i^B \otimes \rho_i^C,$$

where $|i\rangle^A \langle i|$ is some orthonormal basis of Alice’s Hilbert space $\mathcal{H}_A$.

The tripartite quantum states which have the classical-quantum state form given above do not have nonzero genuine quantum discord since subsystem $A$ is always classically correlated with $B$ and $C$ subsystems. Now, Consider tripartite boxes arising from three-qubit classical-quantum states which have the form as given in Eq. (39) with $i = 0, 1$. The correlations obtained from this state can manifestly be simulated by presharing classical random variable $\lambda$ of dimension 2. Hence, the states given by Eq. (39) represent a family of states that do not demonstrate super-bi-unsteerability across the bipartite cut $A$ versus $BC$. This implies that for any three-qubit state which do not have genuine quantumness, there exists a bipartite cut in which it is not super-bi-unsteerable. One can, therefore, conclude that genuine nonclassicality of bi-unsteerable correlations (produced from three-qubit states) [41] is necessary for implying super-bi-unsteerability across every bipartite cut.

**VI. DISCUSSION AND CONCLUSIONS**

In the present work we have introduced the notion of super-bi-unsteerability by showing that there are certain bi-unsteerable correlations whose simulation with LHV-LHS model requires preshared randomness with dimension higher than the local Hilbert space dimension of the quantum systems (reproducing the given bi-unsteerable correlations) at the untrusted party’s side. The super-bi-unsteerability of the noisy Mermin family has been demonstrated in the present study.

In Ref. [30], the authors have shown that the nonclassicality of a family of bipartite local correlations in the Bell-CHSH scenario can be characterized by superlocality. Extending this approach, it has been shown that the nonclassicality in the related steering scenario can also be pointed out by the notion of super-unsteerability [32] of certain bipartite unsteerable correlations. The notion of superlocality of bipartite local correlations has also been generalized to demonstrate superlocality of multipartite boxes [47]. Motivated by this, in the present paper, we generalize the concept of super-unsteerability in the tripartite scenario and define the notion of “super-bi-unsteerability” in the context of tripartite bi-unsteerable correlations.

Before concluding, we note that nonlocality or steerability of any correlation in QM or in any convex operational theory can be characterized by the non-zero communication cost that must be supplemented with pre-shared randomness in order to simulate the correlations. The question of an analogous operational characterization of quantumness of bi-unsteerable tripartite correlations has been addressed here, and associated with super-bi-unsteerability.

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Appendix A: Demonstrating super-unsteerability of each joint probability distribution at Bob-Charlie’s side $P(bc|B_xC_x,r'_{bc})$ produced from the LHS of the LHV-LHS decomposition of noisy Mermin family when $0 < V \leq \frac{3}{2}$.

The correlation belonging to noisy Mermin family can be written as

$$P_{MF}^V(abc|A_xB_xC_x) = \sum_{j=0}^{3} r_j P_j(a|A_x)P(bc|B_xC_x,r'_{bc}).$$

(A1)

where $r_0 = r_1 = r_2 = r_3 = \frac{1}{4}$, and

$$P_0(a|A_x) = \rho_D^{00}, \quad P_1(a|A_x) = \rho_D^{01}, \quad P_2(a|A_x) = \rho_D^{10}, \quad P_3(a|A_x) = \rho_D^{11}.$$
where each row and column corresponds to a fixed measurement \((yz)\) and a fixed outcome \((bc)\) respectively. This correlation can be written as

\[
P(bc|B, C, \rho_{BC}^0) = \sum_{j=0}^{3} q_{j}^0 \rho^0_{A}(b|B_j) P^{0}(c|C, \rho_{C}^0),
\]

where \(q_{0}^0 = q_{1}^0 = q_{2}^0 = q_{3}^0 = \frac{1}{4}\), and \(P_{0}^{0}(b|B_0) = P_{D}^{0}, P_{1}^{0}(b|B_1) = P_{D}, P_{2}^{0}(b|B_2) = P_{D}^{10}, P_{3}^{0}(b|B_3) = P_{D}^{11}\).

Now,

\[
P^{1}(c|C, \rho_{C}^0) = \left( \begin{array}{cc} 0 & 1 \\ \frac{1+2\sqrt{2}}{2} & \frac{1-2\sqrt{2}}{2} \end{array} \right),
\]

where each row and column corresponds to a fixed measurement \((z)\) and a fixed outcome \((c)\) respectively. Now, \(0 \leq P^{0}(c|C, \rho_{C}^0) \leq 1 \forall C, z\), which implies that \(0 < V \leq \frac{1}{2}\).

This probability distribution at Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators: \(C_0 = \sigma_z, C_1 = \sigma_x\) on the state given by

\[
|\psi_C^0\rangle = \sqrt{\frac{1+2V}{2}} |0\rangle + e^{i\phi_0} \sqrt{\frac{1-2V}{2}} |1\rangle,
\]

where, \(\phi_0 = \frac{\pi}{4}\); \(|0\rangle\) and \(|1\rangle\) are the eigenstates of \(\sigma_z\) corresponding to the eigenvalues +1 and −1 respectively.

\[
P^{0}(c|C, \rho_{C}^0) = \left( \begin{array}{cc} \frac{1-2V}{2} & \frac{1+2V}{2} \\ \frac{1+2V}{2} & \frac{1-2V}{2} \end{array} \right).
\]

Now, \(0 \leq P^{0}(c|C, \rho_{C}^0) \leq 1 \forall C, z\), which implies that \(0 < V \leq \frac{1}{2}\).

This probability distribution at Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators: \(C_0 = \sigma_z, C_1 = \sigma_x\) on the state given by

\[
|\psi_C^1\rangle = \sqrt{\frac{1-2V}{2}} |0\rangle + e^{i\phi_1} \sqrt{\frac{1+2V}{2}} |1\rangle,
\]

where, \(\phi_1 = \frac{\pi}{2}\).

\[
P^{0}(c|C, \rho_{C}^0) = \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right).
\]

Now, \(0 \leq P^{0}(c|C, \rho_{C}^0) \leq 1 \forall C, z\), which implies that \(0 < V \leq \frac{1}{2}\).

This probability distribution at Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators: \(C_0 = \sigma_z, C_1 = \sigma_x\) on the state given by

\[
|\psi_C^2\rangle = \sqrt{\frac{1}{2}} |0\rangle + e^{i\phi_2} \sqrt{\frac{1}{2}} |1\rangle,
\]

where, \(\phi_2 = \cos^{-1}(2V)\).

\[
P^{0}(c|C, \rho_{C}^0) = \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right).
\]

Now, \(0 \leq P^{0}(c|C, \rho_{C}^0) \leq 1 \forall C, z\), which implies that \(0 < V \leq \frac{1}{2}\).

This probability distribution at Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators: \(C_0 = \sigma_z, C_1 = \sigma_x\) on the state given by

\[
|\psi_C^3\rangle = \sqrt{\frac{1}{2}} |0\rangle + e^{i\phi_3} \sqrt{\frac{1}{2}} |1\rangle,
\]

where, \(\phi_3 = \cos^{-1}(2V)\).

\[
P^{0}(c|C, \rho_{C}^0) = \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right).
\]
Now, \(0 \leq P^0(c|C_2, \rho^0_C) \leq 1 \forall c, z\), which implies that \(0 < V \leq \frac{3}{2}\).

This probability distribution at Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators: \(C_0 = \sigma_z, C_1 = \sigma_x\) on the state given by

\[
|\psi^3_C\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\phi} |1\rangle),
\]

where, \(\phi_3 = \cos^{-1}(-2V)\).

Hence, one can state that \(P(bc|B_2, C_2, \rho^0_{BC})\) can be expressed with a LHV-LHS decomposition having hidden variables of dimension 4 with different deterministic distributions at Bob’s side in the range \(0 < V \leq \frac{3}{2}\).

Now, let us try to generate a LHV-LHS decomposition of \(P(bc|B_2, C_2, \rho^0_{BC})\) with hidden variables having dimension 2 and with different deterministic distributions at Bob’s side. Before proceeding, we want to mention that in case of \(P(bc|B_2, C_2, \rho^0_{BC})\), all the marginal probability distributions of Bob and Charlie are maximally mixed:

\[
P(b|B_2, \rho^0_{BC}) = P(c|C_2, \rho^0_C) = \frac{1}{2} |db, c, y, z\rangle
\]

Now, in this case the unsteerable box \(P(bc|B_2, C_2, \rho^0_{BC})\) can be decomposed in the following way:

\[
P(bc|B_2, C_2, \rho^0_{BC}) = \sum_{j=0}^{1} q^0_{A} P^j(b|B_2)P^j(c|C_2, \rho^0_C).
\]

Here, \(q^0_0 = e, q^0_1 = f (0 < e < 1, 0 < f < 1, e + f = 1)\). Since Bob’s strategy is deterministic one, the two probability distributions \(P^0(b|B_2)\) and \(P^1(b|B_2)\) must be equal to any two among \(P^0_D, P^1_D, P^{10}_D\) and \(P^{11}_D\). In order to satisfy the marginal probabilities for Bob \(P(b|B_2, \rho^0_{BC})\), the only two possible choices of \(P^0(b|B_2)\) and \(P^1(b|B_2)\) are:

1) \(P^0_D\) and \(P^{11}_D\) with \(e = f = \frac{1}{2}\)

2) \(P^{10}_D\) and \(P^{11}_D\) with \(e = f = \frac{1}{2}\).

Now, it can be easily checked that none of these two possible choices will satisfy all the joint probability distributions \(P(bc|B_2, C_2, \rho^0_{BC})\) simultaneously. It is therefore, impossible to generate a LHV-LHS decomposition of \(P(bc|B_2, C_2, \rho^0_{BC})\) with hidden variables having dimension 2 and with different deterministic distributions at Bob’s side.

Now, we will show that it is impossible to generate a LHV-LHS decomposition of \(P(bc|B_2, C_2, \rho^0_{BC})\) with hidden variables having dimension 2 and with deterministic or non-deterministic distributions at Bob’s side. Before proceeding we note that from any decomposition of the unsteerable (as well as local) box \(P(bc|B_2, C_2, \rho^0_{BC}) (0 < V \leq \frac{3}{2})\) in terms of deterministic boxes (6), one can derive a LHV-LHS model with different deterministic distributions at Bob’s side, which does not require Bob to preshare the hidden variable of dimension more than 4 [28] since there are only 4 possible different deterministic distributions given by Eq. (7) at Bob’s side. Hence, a LHV-LHS model with hidden variable of dimension 2 of the unsteerable box \(P(bc|B_2, C_2, \rho^0_{BC}) (0 < V \leq \frac{3}{2})\) can be achieved by constructing a LHV-LHS model of the unsteerable box \(P(bc|B_2, C_2, \rho^0_{BC}) (0 < V \leq \frac{3}{2})\) with hidden variable of dimension 3 or 4 with different deterministic distributions at Bob’s side followed by taking equal probability distributions at Charlie’s side as common and making the corresponding distributions at Bob’s side non-deterministic.

Let us try to produce a LHV-LHS decomposition of \(P(bc|B_2, C_2, \rho^0_{BC})\) with hidden variables having dimension 3 and with different deterministic distributions at Bob’s side. In this case the unsteerable box \(P(bc|B_2, C_2, \rho^0_{BC})\) can be decomposed in the following way:

\[
P(bc|B_2, C_2, \rho^0_{BC}) = \sum_{j=0}^{2} q^0_{A} P^j(b|B_2)P^j(c|C_2, \rho^0_C).
\]

Here, \(q^0_0 = e, q^0_1 = f, q^0_2 = g (0 < e < 1, 0 < f < 1, 0 < g < 1, e + f + g = 1)\). Since Bob’s strategy is deterministic one, the three probability distributions \(P^0(b|B_2), P^1(b|B_2)\) and \(P_2(b|B_2)\) must be equal to any three among \(P^0_D, P^1_D, P^{10}_D\) and \(P^{11}_D\). But any such combination will not satisfy the marginal probabilities \(P(b|B_2, \rho^0_{BC})\) for Bob. So it is impossible to generate a LHV-LHS decomposition of \(P(bc|B_2, C_2, \rho^0_{BC})\) with hidden variables having dimension 3 and with different deterministic distributions at Bob’s side.

Therefore, in order to simulate the LHV-LHS decomposition of \(P(bc|B_2, C_2, \rho^0_{BC})\) with different deterministic distributions at Bob’s side, Bob has to share the hidden variables of dimension 4.

Suppose the unsteerable box \(P(bc|B_2, C_2, \rho^0_{BC})\) can be decomposed in the following way:

\[
P(bc|B_2, C_2, \rho^0_{BC}) = \sum_{j=0}^{3} q^0_{A} P^j(b|B_2)P^j(c|C_2, \rho^0_C),
\]

(A15)
where $P^0_l(b|B_x)$ are different deterministic distributions and either any three of the four probability distributions $P^0_l(c|C_z, \rho^l_{BC})$ are equal to each other, or there exists two sets each containing two equal probability distributions $P^0_l(c|C_z, \rho^l_{BC})$, $0 < q^l_1 < 1$ for $l = 0, 1, 2, 3$; $\sum_{l=0}^3 q^l_1 = 1$. Then taking equal probability distributions $P^0_l(c|C_z, \rho^l_{BC})$ at Charlie’s side as common and making corresponding distribution at Bob’s side non-deterministic will reduce the dimension of the hidden variable from 4 to 2.

Now in order to satisfy Bob’s marginal given by Eq. (A12), one must take $q^l_0 = q^l_1 = q^l_2 = q^l_3 = \frac{1}{2}$. It can be easily checked that for all possible cases, in which the hidden variable dimension in the LHV-LHS decomposition (A15) can be reduced from 4 to 2, all the joint probability distributions $P(bc|B_y, C_z, \rho^0_{BC})$ are not satisfied simultaneously for $V > 0$. This can be checked considering arbitrary probability distributions $P^0_l(c|C_z, \rho^l_{BC})$ at Charlie’s side (without considering any constraint). Hence, this also follows when the probability distributions $P^0_l(c|C_z, \rho^l_{BC})$ at Charlie’s side has quantum realizations. It is, therefore, impossible to reduce the dimension from 4 to 2 in the LHV-LHS decomposition (A15) of $P(bc|B_y, C_z, \rho^0_{BC})$.

It can be checked that the joint probability distribution $P(bc|B_y, C_z, \rho^0_{BC})$ is nonproduct. It is, therefore, impossible to generate a LHV-LHS decomposition of the joint probability distribution $P(bc|B_y, C_z, \rho^0_{BC})$ with hidden variables having dimension 1.

Hence, one can conclude that the LHV-LHS decomposition of $P(bc|B_y, C_z, \rho^0_{BC})$ cannot be realized with hidden variables having dimension 2 or 1.

Now, as stated before, the joint probability distribution $P(bc|B_y, C_z, \rho^0_{BC})$ at Bob and Charlie’s side can be reproduced by performing the projective qubit measurements of the observables corresponding to the operators $B_0 = \sigma_x$, $B_1 = \sigma_y$, and $C_0 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y)$, $C_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y)$ on the $2 \otimes 2$ quantum state given by

$$|\psi_0\rangle = \cos\theta |00\rangle + \sin\theta |11\rangle,$$

(A16)

$0 \leq \theta \leq \frac{\pi}{4}$ and $\sin 2\theta = \sqrt{2}V; |0\rangle$ and $|1\rangle$ are the eigenstates of $\sigma_z$ corresponding to the eigenvalues $+1$ and $-1$ respectively.

We have shown that the unsteerable box $P(bc|B_y, C_z, \rho^0_{BC}) (0 < V \leq \frac{1}{2})$ can be simulated with LHV-LHS model with the minimum dimension of the hidden variable being greater than 2. On the other hand, $P(bc|B_y, C_z, \rho^0_{BC})$ can be simulated by appropriate measurement on $2 \otimes 2$ quantum system. Hence, one can state that the unsteerable box $P(bc|B_y, C_z, \rho^0_{BC})$ demonstrates super-unsteerability for $0 < V \leq \frac{1}{2}$.

In a similar way as described above, it can be shown that $P(bc|B_y, C_z, \rho^1_{BC})$, $P(bc|B_y, C_z, \rho^2_{BC})$ and $P(bc|B_y, C_z, \rho^3_{BC})$ also demonstrate super-unsteerability for $0 < V \leq \frac{1}{2}$.

Appendix B: Reducing the dimension of the hidden variable in the LHV-LHS decomposition of the bi-unsteerable noisy Mermin family from 4 to 3

Consider that the noisy Mermin family can be decomposed in the following way:

$$P^V_{MF}(abc|A_xB_yC_z) = \sum_{l=0}^3 r_l P_3(a|A_x) P(bc|B_y, C_z, \rho^l_{BC}),$$

(B1)

where without any loss of generality let us assume that $P_0(a|A_x) = P^0_{D_1}$, $P_1(a|A_x) = P^{11}_{D_1}$, $P_2(a|A_x) = P^{10}_{D_1}$, and $P_3(a|A_x) = P^{00}_{D_1}$; and also assume that $P(bc|B_y, C_z, \rho^0_{BC}) = P(bc|B_y, C_z, \rho^0_{BC})$. Now in order to satisfy Alice’s marginal given by Eq. (28), one must take $r_0 = r_1 = r_2 = r_3 = \frac{1}{4}$. Hence, the decomposition (B1) can be written as,

$$P^V_{MF}(abc|A_xB_yC_z) = q_0 \mathbb{P}_0(a|A_x) P(bc|B_y, C_z, \rho^0_{BC}) + \frac{1}{4} P_3(a|A_x) P(bc|B_y, C_z, \rho^0_{BC}),$$

(B2)

where,

$$\mathbb{P}_0(a|A_x) = \frac{P_0(a|A_x) + P_3(a|A_x)}{2},$$

(B3)

which is a non-deterministic distribution at Alice’s side, and

$$q_0 = \frac{1}{2}.$$

(B4)

The decomposition (B2) represents a LHV-LHS model of noisy Mermin family having different deterministic/non-deterministic distributions at Alice’s side with the dimension of the hidden variable being 3.
Now equating left hand side of Eq. (B2) with its right hand side, we obtain the following unique solutions for the joint probability distributions at Bob-Charlie’s side,

\[ P(bc|B_0, C_0, \rho_{BC}^0) = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix} \]

(B5)

Now equating left hand side of Eq. (B2) with its right hand side, we obtain the following unique solutions for the joint probability distributions at Bob-Charlie’s side,

\[ P(bc|B_1, C_1, \rho_{BC}^1) = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix} \]

(B6)

and

\[ P(bc|B_2, C_2, \rho_{BC}^2) = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix} \]

(B7)

Now from the the necessary and sufficient condition for bipartite correlations to have quantum realisations [59], it can be shown that the bipartite correlations (B5), (B6) and (B7) will have quantum realisations iff \( V \leq \frac{\sqrt{3}}{9} \). Hence, the decomposition (B1) is not a LHV-LHS decomposition of noisy Mermin family for \( V > \frac{\sqrt{3}}{9} \). Hence, in this case the dimension of the hidden variable in the LHV-LHS decomposition (B1) of the bi-unsteerable noisy Mermin family cannot be reduced from 4 to 3 for \( V > \frac{\sqrt{3}}{9} \).

**Appendix C: Demonstrating impossibility to have a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variable of dimension 2 having different deterministic distributions at Alice’s side**

Let us try to generate a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variables having dimension 2 having different deterministic distributions at Alice’s side. In this case the bi-unsteerable noisy Mermin family can be decomposed in the following way:

\[ P(MF)(abc|A_0, B_0, C_0) = \sum_{j=0}^{1} r_{aj} P(\{a\}|\{A\}) P(bc|B_0, C_0, \rho_{BC}^j). \]

(C1)

Here, \( r_0 = u \), \( r_1 = v \) \((0 < u < 1, 0 < v < 1, u + v = 1)\). Since Alice’s strategies are deterministic, the two probability distributions \( P_0(\{a\}|\{A\}) \) and \( P_1(\{a\}|\{A\}) \) must be equal to any two among \( P_0^0 \), \( P_0^1 \), \( P_0^2 \) and \( P_0^3 \). In order to satisfy the marginal probabilities for Alice, the only two possible choices of \( P_0(\{a\}|\{A\}) \) and \( P_1(\{a\}|\{A\}) \) are:

1. \( P_0^0 \) and \( P_0^2 \) with \( u = v = \frac{1}{2} \).
2. \( P_0^0 \) and \( P_0^3 \) with \( u = v = \frac{1}{4} \).

In case of the first choice, let us assume that \( P_0(\{a\}|\{A\}) = P_0^0 \), \( P_1(\{a\}|\{A\}) = P_0^2 \), \( P(bc|B_0, C_0, \rho_{BC}^0) \) and \( P(bc|B_0, C_0, \rho_{BC}^1) \) are given by,

\[ P(bc|B_0, C_0, \rho_{BC}^0) := \begin{pmatrix}
u_{11} & u_{12} & u_{13} & u_{14} \\
u_{21} & u_{22} & u_{23} & u_{24} \\
u_{31} & u_{32} & u_{33} & u_{34} \\
u_{41} & u_{42} & u_{43} & u_{44} \\
\end{pmatrix} \]

where \( 0 \leq u_{ij} \leq 1 \forall i, j \), and \( \sum_{j} u_{ij} = 1 \forall i \), and let us assume that \( P(bc|B_0, C_0, \rho_{BC}^0) \) can be reproduced by performing appropriate quantum measurements on quantum state \( \rho_{BC}^0 \), and
where $0 \leq w_{ij} \leq 1 \forall i, j,$ and $\sum_j w_{ij} = 1 \forall i$, and let us assume that $P(bc|B_y, C_z, \rho_{BC}^1)$ can be reproduced by performing appropriate quantum measurements on quantum state $\rho_{BC}^1$.

Now, with this choice, the box $P_{MF}^V(abc|A_0B_yC_z)$ given by the model (C1) has

$P_{MF}^V(abc|A_0B_yC_z) = P_{MF}^V(abc|A_1B_yC_z),$

which is not true for the noisy Mermin family as given in Eq. (19) with $V > 0$. Because in case of noisy Mermin family given by Eq. (19),

$P_{MF}^V(abc|A_0B_yC_z) = \frac{1 + (-1)^{\epsilon b \oplus e \oplus y \otimes z \oplus \delta_{ab\oplus1\otimes C_z}V}}{8},$

and

$P_{MF}^V(abc|A_1B_yC_z) = \frac{1 + (-1)^{\epsilon b \oplus e \oplus y \otimes z \oplus \delta_{ab\oplus1\otimes C_z}V}}{8}.$

Hence, in this case, though the marginal probabilities for Alice are satisfied, all the tripartite joint probability distributions $P_{MF}^V(abc|A_1B_yC_z)$ are not satisfied simultaneously for $V > 0$.

Similarly, in case of the first choice, if we assume that $P_0(a|A_y) = P_{D}^{00}$, $P_1(a|A_y) = P_{D}^{01}$, then the marginal probabilities for Alice are satisfied, but all the tripartite joint probability distributions $P_{MF}^V(abc|A_1B_yC_z)$ are not satisfied simultaneously for $V > 0$.

Now, in case of the second choice, let us assume that $P_0(a|A_y) = P_{D}^{10}$, $P_1(a|A_y) = P_{D}^{11}$, $P(bc|B_y, C_z, \rho_{BC}^0)$ and $P(bc|B_y, C_z, \rho_{BC}^1)$ are given by,

$P(bc|B_y, C_z, \rho_{BC}^0) = \left\{ \begin{array}{cccc} u_1' & u_2' & u_3' & u_4' \\ u_1'' & u_2'' & u_3'' & u_4'' \\ u_1''' & u_2''' & u_3''' & u_4''' \\ u_1'''' & u_2'''' & u_3'''' & u_4'''' \end{array} \right\},$

$\begin{array}{c}
\rho_{BC}^0 = \left\{ \begin{array}{cccc}
\begin{array}{cccc}
\rho_{BB} & \rho_{BC} & \rho_{BD} & 0 \\
\rho_{BC} & \rho_{CC} & \rho_{CD} & 0 \\
\rho_{BD} & \rho_{CD} & \rho_{DD} & 0 \\
0 & 0 & 0 & \rho_{EE} \\
\end{array}
\end{array} \right\} \\
\end{array}$

where $0 \leq u_{ij}' \leq 1 \forall i, j,$ and $\sum_j u_{ij}' = 1 \forall i$, and let us assume that $P(bc|B_y, C_z, \rho_{BC}^0)$ can be reproduced by performing appropriate quantum measurements on quantum state $\rho_{BC}^0$; and

$P(bc|B_y, C_z, \rho_{BC}^1) = \left\{ \begin{array}{cccc} w_1' & w_2' & w_3' & w_4' \\ w_1'' & w_2'' & w_3'' & w_4'' \\ w_1''' & w_2''' & w_3''' & w_4''' \\ w_1'''' & w_2'''' & w_3'''' & w_4'''' \end{array} \right\},$

$\begin{array}{c}
\rho_{BC}^1 = \left\{ \begin{array}{cccc}
\begin{array}{cccc}
\rho_{BB} & \rho_{BC} & \rho_{BD} & 0 \\
\rho_{BC} & \rho_{CC} & \rho_{CD} & 0 \\
\rho_{BD} & \rho_{CD} & \rho_{DD} & 0 \\
0 & 0 & 0 & \rho_{EE} \\
\end{array}
\end{array} \right\} \\
\end{array}$

where $0 \leq w_{ij}' \leq 1 \forall i, j,$ and $\sum_j w_{ij}' = 1 \forall i$. 

\[P(bc|B_y, C_z, \rho_{BC}^1) = \left\{ \begin{array}{cccc} w_1' & w_2' & w_3' & w_4' \\ w_1'' & w_2'' & w_3'' & w_4'' \\ w_1''' & w_2''' & w_3''' & w_4''' \\ w_1'''' & w_2'''' & w_3'''' & w_4'''' \end{array} \right\}.

\[\begin{array}{c}
\rho_{BC}^1 = \left\{ \begin{array}{cccc}
\begin{array}{cccc}
\rho_{BB} & \rho_{BC} & \rho_{BD} & 0 \\
\rho_{BC} & \rho_{CC} & \rho_{CD} & 0 \\
\rho_{BD} & \rho_{CD} & \rho_{DD} & 0 \\
0 & 0 & 0 & \rho_{EE} \\
\end{array}
\end{array} \right\} \\
\end{array}\]
where \( 0 \leq w_{ij} \leq 1 \forall i, j, \) and \( \sum_i w_{ij} = 1 \forall i, \) and let us assume that \( P(bc|B_j, C_i; \rho_{BC}^1) \) can be reproduced by performing appropriate quantum measurements on quantum state \( \rho_{BC}^1. \)

Now, with this choice, the box \( P_{MF}^V(abc|A_kB_jC_i) \) given by the model (C1) has,

\[
P_{MF}^V = \begin{pmatrix}
  000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
  0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
  0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
  0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
  0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
  0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
  0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
  0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\
\end{pmatrix},
\]

(C3)

From Eq. (C3), it can be seen that

\[
P_{MF}^V(abc|A_kB_jC_i) = P_{MF}^V(\bar{a}bc|A_kB_jC_i),
\]

where \( \bar{a} = a \oplus 1. \) The above equation is not true for the noisy Mermin family as given in Eq.(19) with \( V > 0. \) Because in case of noisy Mermin family given by Eq.(19),

\[
P_{MF}^V(abc|A_kB_jC_i) = \frac{1 + (-1)^{\sum_i b_i c_i \delta_{y \oplus 1, z}} V}{8},
\]

and

\[
P_{MF}^V(\bar{a}bc|A_kB_jC_i) = \frac{1 + (-1)^{\sum_i b_i c_i \delta_{y \oplus 1, z}} V}{8}.
\]

Hence, in this case, though the marginal probabilities for Alice are satisfied, all the tripartite joint probability distributions \( P_{MF}^V(abc|A_kB_jC_i) \) are not satisfied simultaneously for \( V > 0. \)

Similarly, in case of the second choice, if we assume that \( P_0(a|A_k) = P_{11}^0, P_1(a|A_k) = P_{10}^1 \) then the marginal probabilities for Alice are satisfied, but all the tripartite joint probability distributions \( P_{MF}^V(abc|A_kB_jC_i) \) are not satisfied simultaneously for \( V > 0. \)

It is, therefore, impossible to have a LHV-LHS decomposition of the bi-unsteerable noisy Mermin family with hidden variable of dimension 2 having different deterministic distributions at Alice’s side.

**Appendix D: Reproducing noisy Mermin box using 3 \( \otimes \) 2 \( \otimes \) 2 quantum system**

Consider, the three spatially separated party (say, Alice, Bob and Charlie) share the following 3 \( \otimes \) 2 \( \otimes \) 2 quantum state:

\[
\rho_2 = V |GHZ\rangle\langle GHZ| + (1 - V)|2\rangle\langle 2| \otimes \frac{I_2}{2} \otimes \frac{I_2}{2},
\]

(D1)

where \( |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle); \) \( 0 < V \leq 1; |0\rangle, |1\rangle \) and \( |2\rangle \) form an orthonormal basis in the Hilbert space in \( C^3; |0\rangle \) and \( |1\rangle \) form an orthonormal basis in the Hilbert space in \( C^2 \) (they are eigenvectors of the operator \( \sigma_z \)); \( I_2 = |0\rangle\langle 0| + |1\rangle\langle 1|. \) Now consider the following two dichotomic POVM \( E_i^1 \equiv |E_i^1(i = 0, 1)| \sum_i E_i^1 = I, 0 < E_i^1 \leq I \) and \( E^2 \equiv |E_j^2(j = 0, 1)| \sum_j E_j^2 = I, 0 < E_j^2 \leq I, \)

where

\[
E_0 = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
\end{pmatrix},
\]

and let us assume that the corresponding outcome is 0,

\[
E_1 = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
\end{pmatrix},
\]

and let us assume that the corresponding outcome is 1.
On the other hand,

\[
E_0^2 = \begin{pmatrix}
\frac{1}{2} & -\frac{i}{2} & 0 \\
-\frac{i}{2} & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\]

and let us assume that the corresponding outcome is 0,

\[
E_1^2 = \begin{pmatrix}
\frac{1}{2} & \frac{i}{2} & 0 \\
\frac{i}{2} & -\frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\]

and let us assume that the corresponding outcome is 1.

Here, matrix form of \(E_0^1, E_1^1, E_0^2\) and \(E_1^2\) are written in the basis \(|0\rangle, |1\rangle, |2\rangle\). Now if Alice performs the POVMs corresponding to \(A_0 = E_0^1\) and \(A_1 = E_1^2\); Bob performs the projective qubit measurements corresponding to the operators: \(B_0 = \sigma_y\) and \(B_1 = -\sigma_x\); and Charlie performs the projective qubit measurements corresponding to the operators: \(C_0 = \sigma_y\) and \(C_1 = -\sigma_x\), then the noisy Mermin family can be reproduced.