The Spacelike-Characteristic Cauchy Problem of General Relativity in Low Regularity

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Abstract
In this paper we study the spacelike-characteristic Cauchy problem for the Einstein vacuum equations. Given initial data on a maximal spacelike hypersurface \( \Sigma \simeq B_1 \subset \mathbb{R}^3 \) and the outgoing null hypersurface \( \mathcal{H} \) emanating from \( \partial \Sigma \), we prove \textit{a priori} estimates for the resulting future development in terms of low-regularity bounds on the initial data at the level of curvature in \( L^2 \). The proof uses the bounded \( L^2 \) curvature theorem [22], the extension procedure for the constraint equations [12], Cheeger-Gromov theory in low regularity [13], the canonical foliation on null hypersurfaces in low regularity [15] and global elliptic estimates for spacelike maximal hypersurfaces.

1 Introduction

1.1 Einstein Vacuum Equations and the Cauchy Problem of General Relativity

A Lorentzian 4-manifold \((\mathcal{M}, g)\) is called a \textit{vacuum spacetime} if it solves the Einstein vacuum equations

\[
\text{Ric} = 0,
\]

where \( \text{Ric} \) denotes the Ricci tensor of the Lorentzian metric \( g \). The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1.1) is a non-linear geometric coupled system of partial differential equations of order 2 for \( g \). In suitable coordinates, for example so-called \textit{wave coordinates}, it can be shown that (1.1) is
hyperbolic and hence admits an initial value formulation, see for example Chapter 10 in [36] for background on the Cauchy problem of general relativity.

One way of prescribing initial data for the Einstein vacuum equations is by specifying a triplet \((\Sigma, g, k)\) where \((\Sigma, g)\) is a Riemannian 3-manifold and \(k\) is a symmetric 2-tensor on \(\Sigma\) satisfying the constraint equations,

\[
\begin{align*}
R_{\text{scal}} &= |k|^2_g - (\text{tr}_g k)^2, \\
\text{div} k &= d(\text{tr}_g k),
\end{align*}
\]

(1.2)

where \(R_{\text{scal}}\) denotes the scalar curvature of \(g\), \(d\) the exterior derivative on \((\Sigma, g)\) and for a symmetric 2-tensor \(F\) on \(\Sigma\),

\[
|F|^2_g := g^{ad} g^{bc} F_{ab} F_{cd}, \quad \text{tr}_g F := g^{ij} F_{ij}, \quad (\text{div} F)_i := \nabla^j F_{ij}.
\]

Here \(\nabla\) denotes the covariant derivative on \((\Sigma, g)\) and we use, as in the rest of this paper, the Einstein summation convention. In the future development \((M, g)\) of such initial data \((\Sigma, g, k)\), \(\Sigma \subset M\) is a spacelike hypersurface with induced metric \(g\) and second fundamental form \(k\). Hence we say that such initial data is posed on a spacelike hypersurface.

For the purposes of this paper, it suffices to consider initial data posed on maximal spacelike hypersurfaces, that is, satisfying \(\text{tr}_g k = 0\) in addition to (1.2); see also [3]. In this case, we say that \((\Sigma, g, k)\) is maximal initial data, and the constraint equations (1.2) reduce to

\[
\begin{align*}
R_{\text{scal}} &= |k|^2_g, \\
\text{div} k &= 0, \\
\text{tr}_g k &= 0.
\end{align*}
\]

### 1.2 Weak Cosmic Censorship and the Bounded \(L^2\) Curvature Theorem

One of the main open questions in general relativity is the so-called weak cosmic censorship conjecture formulated by Penrose in 1969, see [26].

**Conjecture 1.1** (Weak cosmic censorship conjecture) Generically, all singularities forming in the context of gravitational collapse are covered by black holes.

In the pioneering work [7], Christodoulou proves the weak cosmic censorship conjecture for the Einstein vacuum-scalar field equations in spherical symmetry. In Christodoulou’s proof, a low regularity control of the Einstein equations is essential for analysing the dynamical formation of black holes. This strongly suggests that a crucial step to prove the weak cosmic censorship in the absence of symmetry is to control the Einstein vacuum equations in very low regularity.

We remark that in the \((1 + 1)\)-setting of spherical symmetry, Christodoulou bounds the regularity of initial data in a scale-invariant BV-norm. Outside of spherical symmetry, however, this BV-norm is not suitable anymore and regularity should be measured with respect to \(L^2\)-based spaces; we refer the reader to the introduction of [22].
A breakthrough result in the low regularity control of the Einstein equations in absence of symmetry is the bounded $L^2$ curvature theorem by Klainerman-Rodnianski-Szeftel [22]. Before stating it, we define the volume radius of a Riemannian 3-manifold.

**Definition 1.2** (Volume radius) Let $(\Sigma, g)$ be a Riemannian 3-manifold, and let $r > 0$ be a real number. The *volume radius of $\Sigma$ at scale $r$* is defined by

$$r_{vol}(\Sigma, r) := \inf_{p \in \Sigma} \inf_{0 < r' < r} \frac{\text{vol}_g(B_g(p, r'))}{r'^3},$$

where $B_g(p, r')$ denotes the geodesic ball of radius $r'$ centred at $p \in \Sigma$.

The following theorem is proved in [22], see also the companion papers [31]- [35]. We state a more technical version in Section 3.5, see Theorem 3.11.

**Theorem 1.3** (The bounded $L^2$ curvature theorem, version 1) Let $(\Sigma, g, k)$ be asymptotically flat, maximal initial data for the Einstein vacuum equations such that $\Sigma \simeq \mathbb{R}^3$. Assume further that for some $\varepsilon > 0$,

$$\|\text{Ric}\|_{L^2(\Sigma)} \leq \varepsilon, \quad \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon$$

and $r_{vol}(\Sigma, 1) \geq \frac{1}{2}$,

where $\text{Ric}$ denotes the Ricci tensor of $(\Sigma, g)$. There is a universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the maximal globally hyperbolic future development $(\mathcal{M}, g)$ of the initial data $(\Sigma, g, k)$ contains a foliation $(\Sigma_t)_{0 \leq t \leq 1}$ of maximal spacelike hypersurfaces defined as level sets of a time function $t$ such that $\Sigma_0 = \Sigma$ and for $0 \leq t \leq 1$,

$$\|\text{Ric}\|_{L^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \|k\|_{L^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \inf_{0 \leq t \leq 1} r_{vol}(\Sigma_t, 1) \geq \frac{1}{4},$$

where $\text{Ric}$ denotes the intrinsic Ricci tensor and $k$ the second fundamental form of $\Sigma_t \subset \mathcal{M}$.

**Remarks on Theorem 1.3.**

1. By the finite speed of propagation for the Einstein vacuum equations (1.1), Theorem 1.3 is local in nature, and hence we do not specify here further the asymptotic flatness condition on $(\Sigma, g, k)$, see also Remark 2.3 in [22].

2. Theorem 1.3 is primarily to be understood as a *continuation result for smooth solutions of the Einstein vacuum equations*, see Remark 1.2 in the introduction of [22].

3. The proof of Theorem 1.3 relies crucially on a plane wave representation formula for the wave equation on low regularity spacetimes developed in [31]- [35]. This plane wave representation formula is constructed as a Fourier integral operator which necessitates the assumption $\Sigma \simeq \mathbb{R}^3$. 
However, Christodoulou’s work [7] as well as related results on the formation of trapped surfaces [2, 8, 16, 21] and gravitational impulses [24, 25] consider initial data posed on null hypersurfaces rather than on a spacelike hypersurface as assumed in Theorem 1.3. This motivates the study of the Cauchy problem of general relativity in low regularity with initial data posed on null hypersurfaces.

1.3 The Spacelike-Characteristic Cauchy Problem

In this paper, we consider the spacelike-characteristic Cauchy problem of general relativity, where initial data is posed on

1. a maximal spacelike hypersurface with boundary $\Sigma \simeq \overline{B_1} \subset \mathbb{R}^3$,
2. the outgoing null hypersurface $\mathcal{H}$ emanating from $\partial \Sigma$.

Remark 1.4 Initial data for the Einstein vacuum equations posed on null hypersurfaces must satisfy constraint equations, namely the so-called null structure equations, see, for example, [15]. We do not state them here as they do not play a role in this paper.

Remark 1.5 Initial data for the spacelike-characteristic Cauchy problem of general relativity must satisfy additional algebraic compatibility conditions on $\partial \Sigma$, see for example Section 7.6 in [11]. We do not state them here as they do not play a role in this paper.

Local existence for the spacelike-characteristic Cauchy problem for smooth initial data follows from [4, 29]. The classical work of Bruhat [5] (based on a perturbation argument) implies furthermore the local existence of a foliation by maximal spacelike hypersurfaces. The proof of the next lemma is omitted. We note that such a local existence result can also be formulated in terms of finite regularity.

Lemma 1.6 (Classical local existence of maximal foliation)

Consider smooth initial data for the spacelike-characteristic Cauchy problem on $\Sigma$ and $\mathcal{H}$, and let $(S_v)_{v \geq 1}$ denote a smooth foliation on $\mathcal{H}$ by spacelike 2-spheres such that $S_1 = \partial \Sigma$.
Then there exists a real number $\tau > 0$ such that the future development $(\mathcal{M}, g)$ contains a foliation by smooth spacelike maximal hypersurfaces $(\Sigma_t)_{1 \leq t \leq 1+\tau}$ given as level sets of a smooth time function $t$ such that $\Sigma_1 = \Sigma$ and for each $1 \leq t \leq 1+\tau$,

$$\partial \Sigma_t = S_t.$$ 

Moreover, the foliation $(\Sigma_t)$ can be locally continued in a smooth fashion as long as the foliation $(\Sigma_t)$ and the spacetime remain smooth.

The next is a rough version of our main result, see Theorem 2.20 for a precise statement. We prove low-regularity a priori estimates for the spacelike-characteristic Cauchy problem.

**Theorem 1.7 (Main result, version 1)**

Consider a smooth solution $(\mathcal{M}, g)$ of the spacelike-characteristic Cauchy problem, foliated by maximal spacelike hypersurfaces $(\Sigma_t)_{1 \leq t \leq 2}$ such that $\partial \Sigma_t = S_t$ for $1 \leq t \leq 2$ (see also Lemma 1.6).

Assume that for some real number $\varepsilon > 0$, the initial data $(g, k)$ on the maximal spacelike hypersurface $\Sigma$ satisfies

$$\|\text{Ric}\|_{L^2(\Sigma)} \leq \varepsilon, \quad \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon, \quad r_{\text{vol}}(\Sigma, 1/2) \geq 1/4, \quad \text{vol}_g(\Sigma) \leq 8\pi.$$

Assume further that with respect to the foliation by spacelike 2-spheres $(S_v)_{1 \leq v \leq 2}$ of $\mathcal{H}$, the characteristic initial data on $\mathcal{H}$ satisfies

$$\|\alpha\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} + \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} \leq \varepsilon,$$

where $(\alpha, \beta, \rho, \sigma, \beta)$ on $\mathcal{H}$ denote null components of the Riemann curvature tensor $\text{R}$ of $(\mathcal{M}, g)$, and $\text{tr}_X$ and $\text{tr}_X$ denote the outgoing and incoming null expansions on $\mathcal{H}$, respectively; see Sections 2.2 and 2.6 for definitions.

There is a universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then for $1 \leq t \leq 2$,

$$\|\text{Ric}\|_{L^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \|k\|_{L^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty L^2(\Sigma_t)} \lesssim \varepsilon,$$

$$\inf_{1 \leq t \leq 2} r_{\text{vol}}(\Sigma_t, 1/2) \geq \frac{1}{8}, \quad \text{vol}_g(\Sigma_t) \leq 32\pi.$$

\[\text{Springer}\]
Remarks (1) Theorem 1.7 assumes solely initial data bounds at the level of curvature in $L^2$ and makes no symmetry assumptions. In contrast, in the available literature the Cauchy problem for the Einstein vacuum equations with initial data on null hypersurfaces outside of symmetry is studied under the assumption of

- higher regularity of the full initial data, see, for example, [6, 10, 29],
- higher regularity of specific components of the initial data, see for example [23–25]. For instance, in [24] the null curvature component $\alpha$ is only assumed to be a measure on $\mathcal{H}$ while $\beta, \rho, \sigma$ and $\underline{\beta}$ are assumed to be controlled up to two angular derivatives in $L^2$.

(2) The assumed geometric control (1.3) of the foliation $(S_v)_v \geq 1$ on $\mathcal{H}$ is essential for the regularity of the spacetime. In the authors’ companion paper [15], it is shown that assuming small bounded $L^2$ curvature flux on $\mathcal{H}$ (with respect to the geodesic foliation) and further low-regularity geometry bounds on the initial sphere $S_1 = \Sigma \cap \mathcal{H}$, the so-called canonical foliation $(S_v)$ exists for $1 \leq v \leq 2$ and satisfies the control (1.3).

(3) The assumptions $r_{vol}(\Sigma, 1/2) \geq 1/4$ and $\text{vol}_g(\Sigma) \leq 8\pi$ on $\Sigma$ are only used to invoke the Cheeger-Gromov theory developed in Section 5, see Theorem 4.1.

(4) The proof of Theorem 1.7 uses as black boxes the bounded $L^2$ curvature theorem, see Theorems 1.3 and 3.11, and the extension procedure for the constraint equations, see Theorem 3.9.

(5) The methods developed in this paper and [12]-[15] appear promising for a future study of the characteristic Cauchy problem of general relativity where initial data is posed on two transversally intersecting null hypersurfaces.

(6) The a priori estimates (1.4) of Theorem 1.7 can be combined with the local existence result of Lemma 1.6 and higher regularity estimates (i.e. propagation of higher regularity) in a standard bootstrapping argument to conclude a continuation result for the spacelike-characteristic Cauchy problem analogous to the phrasing of Theorem 1.3 (i.e. that for smooth initial data with sufficiently small low-regularity norm, the time of existence is bounded from below by $T \geq 1$). The necessary higher regularity estimates follow from the ideas of this paper with the methods of the bounded $L^2$-curvature theorem [22] and the stability of Minkowski spacetime [9]. Their proof is omitted.

In the next section, we rigorously define the geometric setup, fully state and give the proof of the main result and provide an overview of the paper.

2 Geometric Setup and Main Result

In this section, we introduce the notation and main equations of this paper, state the precise version of our main result (see Section 2.8) and give its proof (see Section 2.9).

Notation For a real number $r > 1$, let $B_r \subset \mathbb{R}^3$ denote the open ball of radius $r$. Lowercase Latin letters range over $\{1, 2, 3\}$ and uppercase Latin letters over $\{1, 2\}$. 
Greek letters range over \{0, 1, 2, 3\}. We tacitly use the Einstein summation convention. In an inequality, a constant \(C_{\alpha_1, \ldots, \alpha_k}\) depends on the quantities \(\alpha_1, \ldots, \alpha_k\).

### 2.1 The Bel–Robinson Tensor

In this section, we define the Bel-Robinson tensor associated to the Riemann curvature tensor of a vacuum spacetime. We follow the presentation in the introduction and Sections 7 and 8 of [9]. The Bel-Robinson tensor is used in this paper to prove curvature estimates, see Proposition 2.2 below.

**Definition 2.1** (Bel-Robinson tensor) Let \(\mathbf{R}\) be the Riemann curvature tensor of a vacuum spacetime \((\mathcal{M}, g)\). Then the Bel-Robinson tensor \(Q(\mathbf{R})\) of \(\mathbf{R}\) is defined by

\[
Q(\mathbf{R})_{\alpha\beta\gamma\delta} := \mathbf{R}_{\alpha\nu\gamma\mu} \mathbf{R}^{\nu}_{\beta\delta} + \ast\mathbf{R}_{\alpha\nu\gamma\mu} \ast\mathbf{R}^{\nu}_{\beta\delta},
\]

where \(\ast\mathbf{R}\) denotes the left dual

\[
\ast\mathbf{R}_{\alpha\beta\gamma\delta} := \frac{1}{2} \in_{\alpha\beta\mu\nu} \mathbf{R}^{\mu\nu}_{\gamma\delta},
\]

with \(\in\) being the volume form on \((\mathcal{M}, g)\).

The following classical Bel-Robinson estimate follows from Section 8 in [9], see the introduction and Lemma 8.1.1 therein.

**Proposition 2.2** (Classical Bel-Robinson estimate)

Let \((\mathcal{M}, g)\) be the future maximal globally hyperbolic development of spacelike-characteristic initial data on \(\Sigma \cup \mathcal{H}\), where \(\Sigma\) is a compact spacelike hypersurface with boundary and \(\mathcal{H}\) the outgoing null congruence emanating from \(\partial\Sigma\). Let \(\mathbf{R}\) be the Riemann curvature tensor of \((\mathcal{M}, g)\), and let \(\bar{T}\) be a timelike vectorfield on \(\mathcal{M}\). Let \(\bar{\Sigma}\) be a spacelike hypersurface of \(\mathcal{M}\) and denote by \(J^-(\bar{\Sigma})\) the past of \(\bar{\Sigma}\) in \(\mathcal{M}\). Then it holds that

\[
\int_{\bar{\Sigma}} Q(\mathbf{R})_{\bar{T}\bar{T}\bar{T}\bar{T}} \leq \int_{\Sigma \cap J^-(\bar{\Sigma})} Q(\mathbf{R})_{T\bar{T}T\bar{T}} + \int_{\mathcal{H} \cap J^-(\bar{\Sigma})} Q(\mathbf{R})_{L\bar{T}T\bar{T}} - \frac{3}{2} \int_{J^-(\bar{\Sigma})} Q(\mathbf{R})_{\alpha\beta\bar{T}\bar{T}} \pi^{\alpha\beta},
\]

where \(T\) and \(\bar{T}\) denote the future-pointing timelike unit normals to \(\Sigma\) and \(\bar{\Sigma}\), respectively, and \(\pi := \mathcal{L}_{\bar{T}} g\) is the deformation tensor of \(\bar{T}\).

Proposition 2.2 analogously holds for a null hypersurface \(\mathcal{H} \subset \mathcal{M}\) replacing the spacelike hypersurface \(\bar{\Sigma}\).
2.2 Foliations of Null Hypersurfaces

Let \((\mathcal{M}, g)\) be a vacuum spacetime and let \(\mathcal{H}\) be an outgoing null hypersurface emanating from a spacelike 2-sphere \((S_1, g)\). Let moreover \(T\) be a given timelike vectorfield on \(S_1\). In the following we introduce the geometric setup of foliations on \(\mathcal{H}\) following the notations and normalisations of [15] and [17].

**Definition 2.3** (Geodesic foliation on \(\mathcal{H}\))

Let \(L\) be the unique \(\mathcal{H}\)-tangential null vectorfield on \(S_1\) with \(g(L, T) = -1\). Extend \(L\) as null geodesic vectorfield onto \(\mathcal{H}\). Let \(s\) be the affine parameter of \(L\) on \(\mathcal{H}\) defined by

\[ Ls = 1 \text{ on } \mathcal{H}, \quad s|_{S_1} = 1. \]

Denote the level sets of \(s\) by \(S_s\) and the geodesic foliation by \((S_s)\).

**Definition 2.4** (General foliations on \(\mathcal{H}\))

Let \(v\) be a given scalar function on \(\mathcal{H}\) such that \(L v > 0\). We denote the level sets of \(v\) by \(S_v = \{ v = v_0 \} \) and the corresponding foliation by \((S_v)\). We define the null lapse \(\Omega\) of \((S_v)\) on \(\mathcal{H}\) by

\[ \Omega := L v. \]  

(2.2)

**Definition 2.5** (Orthonormal null frame)

Let \((S_v)\) be a foliation on \(\mathcal{H}\). Let \(L\) be the unique null vector field on \(\mathcal{H}\) orthogonal to each \(S_v\) and such that \(g(L, L) = -2\). The pair \((L, L)\) is called a null pair for the foliation \((S_v)\). Let \((e_1, e_2)\) be an orthonormal frame tangential to each \(S_v\). The frame \((L, L, e_1, e_2)\) is called an orthonormal null frame for the foliation \((S_v)\).

Let \((S_v)\) be a foliation on \(\mathcal{H}\) and let \((L, L, e_1, e_2)\) be an orthonormal null frame for \((S_v)\).

- Denote by \(\bar{g}\) and \(\bar{\nabla}\) the induced metric and covariant derivative on \(S_v\),
- For a given \(S_v\)-tangential \(n\)-tensor \(W\), define

\[ \bar{\nabla}_L W_{A_1...A_n} := \Pi_{A_1}^{\beta_1} \cdots \Pi_{A_n}^{\beta_n} D_L W_{\beta_1...\beta_n}, \]

where \(\Pi\) denotes the projection operator onto the tangent space of \(S_v\) and \(D\) is the covariant derivative on \((\mathcal{M}, g)\).
- Let the null connection coefficients be defined by

\[ \chi_{AB} := g(D_A L, e_B), \quad \hat{\chi}_{AB} := g(D_A L, e_B), \quad \zeta_A := \frac{1}{2} g(D_A L, L). \]

Further decompose \(\chi\) and \(\hat{\chi}\) into their trace and tracefree parts,

\[ \text{tr} \chi := g^{AB} \chi_{AB}, \quad \hat{\chi}_{AB} := \chi_{AB} - \frac{1}{2} \text{tr} \chi g_{AB}, \]

\[ \text{tr} \hat{\chi} := g^{AB} \hat{\chi}_{AB}, \quad \hat{\chi}_{AB} := \hat{\chi}_{AB} - \frac{1}{2} \text{tr} \hat{\chi} g_{AB}. \]
We define the null components of the Riemann curvature tensor $R$ of a vacuum spacetime $({\mathcal M}, g)$ as follows,

$$
\begin{align*}
\alpha_{AB} & := R_{ALB}^L, \\
\beta_A & := \frac{1}{2} R_{ALL}^L, \\
\rho & := \frac{1}{4} R_{LLL}^L, \\
\sigma & := \frac{1}{4} r^L_{LLL}, \\
\beta_A & := \frac{1}{2} R_{ALLL}, \\
\alpha_{AB} & := R_{ALBL}.
\end{align*}
$$

(2.3)

For $S_v$-tangent vector fields $X$ define

$$
\text{div} X := \nabla A X A, \\
\text{curl} X := \epsilon^{AB} \nabla A X B,
$$

where $\epsilon^{AB} := \epsilon_{ABL} L$.

Define on $\mathcal{H}$ the positive-definite metric $h^v$ with respect to the foliation $(S_v)$ by

$$
h^v_{\alpha\beta} := g_{\alpha\beta} + \frac{1}{2} (L + L)_{\alpha} (L + L)_{\beta}.
$$

(2.4)

The following Ricci equations hold, see [9],

$$
\begin{align*}
D_L L & = 0, \\
D_A L & = \chi_{AB} e_B - \zeta_A L, \\
D_L e_A & = \nabla L e_A + (-\zeta_A - \nabla A \log \Omega)L, \\
D_A e_B & = \nabla A e_B + \frac{1}{2} \chi_{AB} L + \frac{1}{2} \chi_{AB} L.
\end{align*}
$$

(2.5)

### 2.3 Foliations of Vacuum Spacetimes by Spacelike Maximal Hypersurfaces

Let $t$ be a scalar function on a vacuum spacetime $({\mathcal M}, g)$ whose level sets $\Sigma_t$ constitute a foliation of spacelike maximal hypersurfaces.

- Let $g$ denote the induced metric on $\Sigma_t$ and $\nabla$ its covariant derivative. Let $\Delta$ denote the Laplace-Beltrami operator of $g$.
- Let $e_0 := T$ denote the future-pointing timelike unit normal to $\Sigma_t$, and let $(e_i)_{i=1,2,3}$ be an orthonormal frame tangent to $\Sigma_t$. Define the second fundamental form $k$ of $\Sigma_t$ by

$$
k_{ij} := -g(D_i T, e_j).
$$

Define the foliation lapse $n$ by

$$
n^{-2} := -g(Dt, Dt),
$$
satisfying in particular,

\[ T = -nDt, \quad D_T T = n^{-1} \nabla n, \quad T(t) = n^{-1}. \]  

(2.6)

We remark that the deformation tensor \( \pi := \mathcal{L}_T g \) can be expressed as

\[ \pi_{\alpha\beta} = -2k_{\alpha\beta} - n^{-1}(T_\alpha \nabla_\beta n + T_\beta \nabla_\alpha n) \text{ for } \alpha, \beta = 0, 1, 2, 3, \]  

(2.7)

where, as in the rest of this paper, \( k \) is extended to a tensor on \( \mathcal{M} \) by \( k_{0\mu} = 0 \) for \( \mu = 0, 1, 2, 3 \).

- For two symmetric \( g \)-tracefree 2-tensors \( V \) and \( W \) and a vectorfield \( X \) on \( \Sigma_t \), define

\[
\begin{align*}
\text{div} \, V_i &:= \nabla^j V_{ji}, \\
\text{curl} \, V_{ij} &:= \frac{1}{2} \left( \varepsilon_{ilm} \nabla^l V^m_{ij} + \varepsilon_{jlm} \nabla^l V^m_{ij} \right), \\
(V \times W)_{ij} &:= \varepsilon_{iab} \varepsilon_{jcd} V_{ac} W_{bd} + \frac{1}{3}(V \cdot W)g_{ij}, \\
(V \wedge W)_i &:= \varepsilon_{imn} V_m W_{ln}, \\
(X \wedge V)_{ij} &:= \varepsilon_{imn} X_m V_{nj} + \varepsilon_{jmn} X_m V_{in},
\end{align*}
\]

where \( \varepsilon_{abc} := \varepsilon_{abcT} \).

- We have the following electric-magnetic decomposition with respect to \( T \) for the Riemann curvature tensor \( R \) of a vacuum spacetime,

\[ E_{ab} := R_{aTbT}, \quad H_{ab} := *R_{aTbT}. \]

The 2-tensors \( E \) and \( H \) are \( \Sigma_t \)-tangent, symmetric and \( g \)-tracefree, see Section 7.2 in [9]. By definition of the Bel-Robinson tensor, see Definition 2.1,

\[ |E|^2 + |H|^2 = Q(R)_{TTTT}. \]  

(2.8)

- Define the positive-definite metric \( h' \) on \( \mathcal{M} \) by

\[ h'_{\alpha\beta} := g_{\alpha\beta} + 2T_\alpha T_\beta, \]  

(2.9)

and for \( n \)-tensors \( W \) on \( \Sigma_t \), let

\[ |W|_{h'}^2 := W_{\alpha_1...\alpha_n} W'_{\alpha'_1...\alpha'_n} (h')^{\alpha_1}\alpha'_1 \cdots (h')^{\alpha_n}\alpha'_n. \]  

(2.10)

In particular, it holds by (7.2.1) in [9] and (2.8) that

\[ |R|_{h'}^2 \lesssim Q(R)_{TTTT} = |E|^2 + |H|^2 \lesssim |R|_{h'}^2. \]  

(2.11)
It also holds that
\[ |Q(R)_{\mu_\nu,\lambda,\tau}| \lesssim Q(R)^{TTTT}, \]  
where evaluation is made with respect to an orthonormal frame \((e_\mu)_{\mu=0,1,2,3}^\cdot\).

The Einstein vacuum equations imply the following structure equations of the maximal foliation, see equations (1.0.11a)-(1.0.14d) in [9]. We have the first variation equation,
\[ \mathcal{L}_T g_{ij} = -2k_{ij}, \]
the second variation equation,
\[ D_T k_{ij} = E_{ij} - n^{-1} \nabla_i \nabla_j n - k_{il} k_{lj}, \]  
(2.13a)
the Gauss-Codazzi equations
\[ \text{div } k_i = 0, \]  
(2.13b)
\[ \text{curl } k_{ij} = H_{ij}, \]  
(2.13c)
the maximality of \(\Sigma_t\),
\[ \text{tr}_g k = 0, \]  
(2.13d)
the lapse equation,
\[ \Delta n = n |k|_g^2, \]  
(2.13e)
the traced Gauss equation,
\[ \text{Ric}_{ij} = E_{ij} + k_{ia} k_{aj}, \]  
(2.13f)
and the twice-traced Gauss equation,
\[ R_{scal} = |k|_g^2. \]  
(2.13g)

With respect to a foliation \((\Sigma_t)\) by maximal hypersurfaces, the Bianchi equations of \((\mathcal{M}, g)\) imply in particular the following, see Proposition 7.2.1 in [9],
\[ \text{div } E = k \wedge H, \]
\[ \text{div } H = -k \wedge E. \]  
(2.14)
2.4 Spherical Coordinates on Spacelike Hypersurfaces

Let \((\Sigma, g)\) be a maximal spacelike hypersurface in a vacuum spacetime \((\mathcal{M}, g)\). Assume there exists a real number \(1 \leq t \leq 2\) such that there is a global coordinate chart \(\phi : \overline{B}_t \to \Sigma\). Using the chart \(\phi\), define standard spherical coordinates \((r, \theta^1, \theta^2)\) on \(\Sigma\) with \(r \in [0,t]\). We denote the level sets of \(r\) by \(S_r\), and for two reals \(0 \leq r_1, r_2 \leq t\), let \(A(r_1, r_2)\) denote the coordinate annulus

\[
A(r_1, r_2) := \{ p \in \Sigma : r_1 \leq r(p) \leq r_2 \}.
\]  

(2.15)

Then:

- The metric \(g\) can be expressed in coordinates \((r, \theta^1, \theta^2)\) for \(r > 0\) as

\[
g = a^2 dr^2 + \hat{g}_{AB}(b^A dr + d\theta^A)(b^B dr + d\theta^B),
\]

where

- \(a\) is the foliation lapse,
- \(\hat{g}\) is the induced metric on \(S_r\),
- \(b\) is the \(S_r\)-tangent shift vector.

- Let \(N\) be the outward pointing unit normal to \(S_r\) and let \((e_1, e_2)\) denote an orthonormal frame tangent to \(S_r\). Define the second fundamental form of \(S_r\) for \(r > 0\) by

\[
\Theta_{AB} := g(\nabla_A N, e_B).
\]

We split \(\Theta\) into its trace and tracefree part,

\[
\text{tr}\Theta := \hat{g}^{AB} \Theta_{AB}, \quad \hat{\Theta}_{AB} := \Theta_{AB} - \frac{1}{2} \text{tr}\Theta \hat{g}_{AB},
\]

Further, in coordinates \((r, \theta^1, \theta^2)\) we can express for \(r > 0\),

\[
N = \frac{1}{a} \partial_r - \frac{1}{a} b, \quad \Theta_{AB} = \frac{1}{2a} \partial_r (\hat{g}^A_{\phantom{A}B}) - \frac{1}{2a} (\mathcal{L}_b \hat{g})_{AB},
\]  

(2.16)

where \(\mathcal{L}\) denotes the Lie derivative on \(S_r\). In particular, by (2.16)

\[
\partial_r \sqrt{\det g} = \sqrt{\det \hat{g}} (a \text{tr}\Theta - \text{div} b).
\]

(2.17)

- Let \(\nabla\) and \(\Delta\) denote the induced covariant derivative and Laplace-Beltrami operator on \(S_r\), respectively. We note the relations (see Chapter 3 in [9])

\[
\nabla_N N = -a^{-1} \nabla a, \quad \nabla_A N = \Theta_{AB} e_B, \quad \text{div} N = \text{tr}\Theta.
\]
we decompose the second fundamental form $k$ on $\Sigma$ into $S_r$-tangential tensors as follows,

$$\delta := k_{NN}, \quad \epsilon_A := k_{NA}, \quad \eta_{AB} := k_{AB}.$$ 

We note that $\text{tr}\eta = -\delta$ because $\text{tr}_g k = 0$ on $\Sigma$ by maximality.

With this notation, we can decompose $\nabla k$ as follows (see Sections 3.1 and 4.4 in [9]),

$$\nabla_A k_{BC} = \bar{\nabla}_A \eta_{BC} + \Theta_{AB} \epsilon_C + \Theta_{AC} \epsilon_B,$$

$$\nabla_N k_{NN} = N(\delta) + 2a^{-1} \bar{\nabla} a \cdot \epsilon,$$

$$\nabla_N k_{AB} = \bar{\nabla}_N \eta_{AB} - a^{-1} \bar{\nabla}_A a \epsilon_B - a^{-1} \bar{\nabla}_B a \epsilon_A,$$

$$\nabla_N k_{NA} = \bar{\nabla}_N \epsilon_A + a^{-1} \bar{\nabla}_C a \eta_{CA} - a^{-1} \bar{\nabla}_A a \delta,$$

$$\nabla_{BkNA} = \bar{\nabla}_B \epsilon_A - \eta_{A} \Theta_{CB} + \delta \Theta_{AB},$$

where for an $S_r$-tangential tensor $F$, $\bar{\nabla}_N F$ is defined as the projection of $\nabla_N F$ onto $S_r$.

Further, the Gauss-Codazzi equations (2.13b) and (2.13c) imply the following relations (see Proposition 4.4.3 in [9]),

$$N(\delta) + d\text{v}\epsilon = -\delta \text{tr}\Theta + \eta \cdot \Theta - 2a^{-1} \bar{\nabla} a \cdot \epsilon,$$

$$\bar{\nabla}_N \epsilon_A = \bar{\nabla}_A \delta = -\hat{H}_{N\dot{A}} - 2\Theta_{AC} \epsilon_C - a^{-1} \bar{\nabla}_C a \eta_{CA} + a^{-1} \bar{\nabla}_A a \delta,$$

$$\bar{\nabla}_N \hat{\eta}_{AB} = \frac{1}{2} \text{tr}\Theta \hat{\eta}_{AB} = \frac{1}{2} (\bar{\nabla}_N \epsilon)_{AB} + \frac{3}{2} \delta \hat{\Theta}_{AB} - a^{-1} \bar{\nabla} a (\hat{\Theta} \epsilon)_{AB},$$

$$d\text{v}\epsilon_A + \bar{\nabla}_A \delta = -\hat{H}_{N\dot{A}} + \Theta_{AC} \epsilon_C - \text{tr}\epsilon A,$$

where we decomposed the source term $H$ into $S_r$-tangential tensors

$$H_{NN}, \quad H_{N\dot{A}} := H_{NA}, \quad H_{AB} := H_{AB},$$

and for $S_r$-tangential vectorfields $X$ and $Y$ and symmetric tracefree 2-tensors $F$ and $G$, we denote

$$X_A := \epsilon_{AB} X^B, \quad (X \hat{\otimes} Y)_{AB} := X_A Y_B + X_B Y_A - (X \cdot Y)\hat{g},$$

$$F_{AB} := \epsilon_{AC} F^C_B, \quad F \wedge G := \epsilon^{AB} F_{AC} G^C_B.$$ 

### 2.5 Relations Between Foliations on Vacuum Spacetimes and Null Hypersurfaces

Let $(M, g)$ be a vacuum spacetime. Let $\Sigma \simeq \overline{B_1}$ be a spacelike maximal hypersurface and let $\mathcal{H}$ denote the outgoing null hypersurface emanating from $\partial \Sigma$. Let

- $(S_v)_{1 \leq v \leq 2}$ be a foliation on $\mathcal{H}$ by spacelike 2-spheres such that $S_1 = \partial \Sigma$. Let $(L, \overline{L}, e_1, e_2)$ be an orthonormal null frame of $(S_v)$.
- $(\Sigma_t)_{1 \leq t \leq 2}$ be a foliation on $M$ by maximal spacelike hypersurfaces given as level sets of a time function $\tau$ such that $\Sigma_1 = \Sigma$ and such that for $1 \leq t \leq 2$,

$$\partial \Sigma_t = S_t, \quad \text{i.e. } t = \tau \text{ on } \mathcal{H}.$$ 

(2.21)
Let $T$ denote the unit normal to $\Sigma_t$.

**Definition 2.6** The slope $\nu$ on $\mathcal{H}$ is defined by

$$
\nu := - \mathbf{g}(L, T).
$$

(2.22)

The proof of the next lemma is left to the reader.

**Lemma 2.7** *(Algebraic relations)* On $\mathcal{H}$ it holds that

$$
T = \frac{1}{2} \nu L + \frac{1}{2} \nu^{-1} L, \quad N = \frac{1}{2} \nu L - \frac{1}{2} \nu^{-1} L,
$$

(2.23)

$$
L = \nu^{-1} (T + N), \quad \nu = \nu(T - N),
$$

and further,

$$
\Theta_{AB} = \frac{1}{2} \nu \chi_{AB} - \frac{1}{2} \nu^{-1} \chi_{AB},
$$

$$
\eta_{AB} = - \frac{1}{2} \nu \chi_{AB} - \frac{1}{2} \nu^{-1} \chi_{AB},
$$

$$
\delta = - \text{tr} \eta = \frac{1}{2} \nu \text{tr} \chi + \frac{1}{2} \nu^{-1} \text{tr} \chi.
$$

**Remark 2.8** By Lemma 2.7, the definition of $\nu$ in (2.22) is equivalent to

$$
\nu^{-1} := - \mathbf{g}(L, T).
$$

Thus, by Definition 2.3 it follows that we have the normalisation

$$
\nu = 1 \text{ on } S_1.
$$

**Lemma 2.9** *(Slope equation)* On $\mathcal{H}$ it holds that

$$
\nu^{-1} \nabla_A \nu = - \epsilon_A + \zeta_A.
$$

(2.24)

**Proof** Using (2.5), (2.22) and Lemma 2.7, we have

$$
\nu^{-1} \nabla_A \nu = - \nu^{-1} \nabla_A (\mathbf{g}(L, T))
$$

$$
= - \nu^{-1} \left( \mathbf{g}(\mathbf{D}_A L, T) + \mathbf{g}(L, \mathbf{D}_A T) \right)
$$

$$
= - \nu^{-1} \left( \mathbf{g}(\chi_{AB} e_B + \zeta_A L, T) - \mathbf{g}(L, k_{A j} e_j) \right)
$$

$$
= \zeta_A + \nu^{-1} \epsilon_A \mathbf{g}(L, N)
$$

$$
= \zeta_A - \epsilon_A.
$$

This finishes the proof of (2.24). \qed

The lapse $n$ can be expressed on $\partial \Sigma_t$ as follows.
Lemma 2.10  It holds on $\partial \Sigma_t$ that
\[ n = v^{-1} \Omega^{-1}. \]  \hfill (2.25)

Proof  Indeed, by (2.2) and Lemma 2.7,
\[ \Omega = L(v) = L(t) = v^{-1}T(t) = v^{-1}n^{-1}, \]
where we used (2.21) and that $T(t) = n^{-1}$, see (2.6). \qed

The next lemma follows from Lemma 7.3.1 in [9] and Lemma 2.7.

Lemma 2.11  It holds that on $\mathcal{H}$,
\[ Q(R)_{LTTT} = \frac{1}{4} v^3 |\alpha|^2 + \frac{3}{2} v |\beta|^2 + \frac{3}{2} v^{-1} (\rho^2 + \sigma^2) + \frac{1}{2} v^{-3} |\beta|^2. \]
Moreover, for $\|v - 1\|_{L^\infty(\mathcal{H})}$ sufficiently small,
\[ |Q(R)_{\mu\nu\lambda\varepsilon}| \lesssim Q(R)_{LTTT}. \]  \hfill (2.26)

2.6 Integration and Norms

In this section, we define integration and norms.

Definition 2.12  (Norms on $S$)
Let $(S, g)$ be a Riemannian 2-sphere. Let $F$ be an $S$-tangent tensor on $S$. For real numbers $1 \leq p < \infty$, define
\[ \|F\|_{L^p(S)} := \int_S |F|^p, \]
where the integrals over $S$ are with respect to the metric $g$. Moreover, let
\[ \|F\|_{L^\infty(S)} := \sup_S |F|. \]

Definition 2.13  (Integration on $\mathcal{H}$)
Let $1 < v_0 < \infty$ be a real number. Let $\mathcal{H} \subset \mathcal{M}$ be a null hypersurface foliated by spacelike 2-spheres $(S_v)_{1 \leq v \leq v_0}$. Let $f$ be a scalar function on $\mathcal{H}$. Let
\[ \int_{\mathcal{H}} f := \int_1^{v_0} \left( \int_{S_v} \Omega^{-1} f \right) dv, \]
where the integral over $S_v$ is with respect to the induced metric $g$ and $\Omega := L(v)$ is the null lapse of the foliation $(S_v)$. 

\[ \text{Springer} \]
Definition 2.14 (Norms on $\mathcal{H}$)

Let $(\mathcal{M}, g)$ be a vacuum spacetime and let $\mathcal{H} \subset \mathcal{M}$ be a null hypersurface. For a real number $1 < v_0 < \infty$, let $(S_v)_{1 \leq v \leq v_0}$ be a foliation of spacelike 2-spheres on $\mathcal{H}$ and denote $\mathcal{H}_{v_0} := \mathcal{H} \cap (S_v)_{1 \leq v \leq v_0}$. Let $1 \leq p < \infty$ be a real number and let $F$ be an $S_v$-tangent tensor on $\mathcal{H}$. For integers $m \geq 0$, define

\[
\|F\|_{L^2(\mathcal{H}_{v_0})} := \left( \int_{\mathcal{H}_{v_0}} |F|^2 \right)^{1/2},
\]

\[
\|F\|_{L^\infty_{[1,v_0]} L^p(S_v)} := \sup_{1 \leq v \leq v_0} \|F\|_{L^p(S_v)},
\]

\[
\|F\|_{L^\infty_{[1,v_0]} L^\infty(S_v)} := \sup_{1 \leq v \leq v_0} \|F\|_{L^\infty(S_v)},
\]

\[
\|F\|_{L^\infty_{[1,v_0]} H^{1/2}(S_v)} := \sup_{1 \leq v \leq v_0} \|F\|_{H^{1/2}(S_v)},
\]

where the fractional Sobolev space $H^{1/2}(S_v)$ is defined in Section 3.1. Further, for tensors $W$ on $\mathcal{M}$, define

\[
\|W\|_{L^\infty(\mathcal{H}_{v_0})} := \sup_{1 \leq v \leq v_0} \sup_{S_v} |W|_{h^v},
\]

where $h^v$ denotes the positive-definite metric on $\mathcal{H}$ associated to the foliation $(S_v)$, see (2.4).

Notation For ease of presentation, we omit the index $v_0$ in this paper whenever it is clear what interval we consider. For example, we write $\|\cdot\|_{L^2(\mathcal{H})}$ instead of $\|\cdot\|_{L^2(\mathcal{H}_{v_0})}$ and $\|\cdot\|_{L^\infty_{[1,v_0]} L^p(S_v)}$ instead of $\|\cdot\|_{L^\infty_{[1,v_0]} L^p(S_v)}$.

Definition 2.15 (Norms on $\mathcal{M}$) Let $t_1$ and $t_2$ be two real numbers. Let $(\mathcal{M}, g)$ be a vacuum spacetime foliated by spacelike hypersurfaces $(\Sigma_t)_{t_1 \leq t \leq t_2}$ given as level sets of a time function $t$ on $\mathcal{M}$. For $\Sigma_t$-tangential tensors $F$ define

\[
\|F\|_{L^\infty_{[t_1,t_2]} L^2(\Sigma_t)} := \sup_{t_1 \leq t \leq t_2} \left( \int_{\Sigma_t} |F|^2 \right)^{1/2},
\]

where the integral over $\Sigma_t$ is with respect to the induced metric $g$. Further, for tensors $W$ on $\mathcal{M}$, let

\[
\|W\|_{L^\infty_{[t_1,t_2]} L^2(\Sigma_t)} := \sup_{t_1 \leq t \leq t_2} \left( \int_{\Sigma_t} |W|^2 \right)^{1/2},
\]

\[
\|W\|_{L^\infty(\Sigma_t)} := \sup_{\Sigma_t} |W|_{h^t},
\]

where $h^t$ denotes the positive-definite metric associated to the foliation $(\Sigma_t)$, see (2.10).

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Notation For ease of presentation, we leave away the index \([t_1, t_2]\) in this paper whenever it is clear what interval of integration we consider. For example, we write \(\| \cdot \|_{L^\infty_t L^2(\Sigma_t)}\) instead of \(\| \cdot \|_{L^\infty_{[t_1, t_2]} L^2(\Sigma_t)}\).

2.7 Initial Data Regularity and Norms for the Spacelike-Characteristic Cauchy Problem

In this section, we introduce the notions of regularity and the initial data norms used in our main result.

Weakly regular 2-spheres. First, we have the following definition of weak regularity of 2-spheres, see [15] and [30]. This level of regularity allows to apply the basic calculus tools on 2-spheres of Section 3.

Definition 2.16 (Weakly regular 2-spheres) Let \(1 \leq N < \infty\) be an integer and \(c > 0\) be a real number. A Riemannian 2-sphere \((S, g)\) is a weakly regular sphere with constants \(N, c\) if

- it can be covered by \(N\) coordinate patches,
- there is a partition of unity \(\eta\) adapted to the above coordinate patches,
- there are functions \(0 \leq \tilde{\eta} \leq 1\) which are compactly supported in the coordinate patches and equal to 1 on the support of \(\eta\),
- on each coordinate patch there exists an orthonormal frame \((e_1, e_2)\) such that on each coordinate patch,

\[
c^{-1} \leq \sqrt{\det g} \leq c, \quad c^{-1}|\xi|^2 \leq g_{AB} \xi^A \xi^B \leq c|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2,
\]

\[
|\partial \eta| + |\partial^2 \eta| + |\partial \tilde{\eta}| \leq c, \quad \|\nabla^A_{\partial g}\|_{L^2} + \|\nabla^A e_{A}\|_{L^4} \leq c \text{ for } A = 1, 2,
\]

where here for \(\xi \in \mathbb{R}^2, |\xi|^2 := (\xi^1)^2 + (\xi^2)^2\).

Remark 2.17 In [15] it is shown that for the canonical foliation, the spheres \((S_v)\) are weakly regular 2-spheres with constants \(N, c\) uniformly controlled by the \(L^2\)-curvature flux through \(\mathcal{H}\) and low regularity assumptions on the foliation geometry of the initial sphere \(S_1\). Therefore, for ease of presentation, in this paper we do not explicitly indicate the dependence of estimates on \(N, c\).

Weakly regular 3-balls. The following regularity property is related to the existence of global coordinates, see Theorem 4.1.

Definition 2.18 (Weakly regular 3-balls) Let \(r > 0\) and \(0 < C_{\text{ball}} < 1/2\) be two real numbers. A Riemannian 3-manifold with boundary \((\Sigma, g)\) is a weakly regular ball of radius \(r\) with constant \(C_{\text{ball}}\) if there is a global coordinate chart \(\phi : \overline{B_r} \to \Sigma\) such that on \(\overline{B_r}\),

\[
(1 - C_{\text{ball}})|\xi|^2 \leq g_{ij} \xi^i \xi^j \leq (1 + C_{\text{ball}})|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2,
\]

\[
\|\partial g_{ij}\|_{L^2(B_r)} + \|\partial^2 g_{ij}\|_{L^2(B_r)} \leq C_{\text{ball}}.
\]
Initial data norms. Consider initial data for the spacelike-characteristic Cauchy problem of general relativity on the spacelike hypersurface $\Sigma \simeq B_1$ and the outgoing null hypersurface $\mathcal{H}$ emanating from $\partial \Sigma$. Let further $(S_\nu)_{1 \leq \nu \leq 2}$ denote the canonical foliation on $\mathcal{H}$ with $S_1 = \partial \Sigma$. Define

$$
\mathcal{O}_\Sigma := \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)},
$$

$$
\mathcal{R}_\Sigma := \|\text{Ric}\|_{L^2(\Sigma)},
$$

$$
\mathcal{O}_\mathcal{H} := \left\| \text{tr} \chi - \frac{2}{v} \right\|_{L^\infty L^2(S_\nu)} + \left\| \text{tr} \chi + \frac{2}{v} \right\|_{L^\infty L^\infty(S_\nu)} + \left\| \nabla \text{tr} \chi \right\|_{L^\infty L^2(S_\nu)}
$$

$$
+ \left\| \nabla \Omega \right\|_{L^\infty L^2(S_\nu)} + \left\| \text{tr} \chi \right\|_{L^\infty L^4(S_\nu)} + \left\| \frac{\chi}{2} \right\|_{L^\infty L^4(S_\nu)} + \left\| \chi \right\|_{L^\infty H^{1/2}(S_\nu)},
$$

$$
\mathcal{R}_\mathcal{H} := \|\alpha\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} + \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \left\| -\frac{\chi}{2} \right\|_{L^2(\mathcal{H})}.
$$

Here $H^{1/2}(S_\nu)$ is an $L^2$-based fractional Sobolev space on $S_\nu$ bounding 1/2 derivatives, see Definition 3.2.

Remark 2.19 In [15] it is shown that for the canonical foliation, the norm $\mathcal{O}_\mathcal{H}$ can be bounded by the $L^2$ curvature flux $\mathcal{R}_\mathcal{H}$ and low regularity bounds on the geometry of the initial sphere $S_1 = \Sigma \cap \mathcal{H}$, see the introduction of [15].

2.8 Main Result

The following is the main result of this paper. We prove a priori estimates for the spacelike-characteristic Cauchy problem at $L^2$-curvature regularity.

Theorem 2.20 (Main theorem, version 2) Consider a smooth solution $(\mathcal{M}, g)$ of the spacelike-characteristic Cauchy problem, foliated by maximal spacelike hypersurfaces $(\Sigma_t)_{1 \leq t \leq 2}$ satisfying $\partial \Sigma_t = S_t$ for $1 \leq t \leq 2$ (see also Lemma 1.6).

$$
\Sigma = \Sigma_1
\quad
\mathcal{H}
\quad
\Sigma_2
\quad
(S_\nu)_{1 \leq \nu \leq 2}
$$

Assume that the 2-spheres $(S_\nu)_{1 \leq \nu \leq 2} \subset \mathcal{H}$ are uniformly weakly regular with constants $N, c$, and that for some real number $\varepsilon > 0$,

$$
\mathcal{O}_\Sigma + \mathcal{R}_\Sigma + \mathcal{O}_\mathcal{H} + \mathcal{R}_\mathcal{H} \leq \varepsilon, \quad 1/4 \leq r_{\text{vol}}(\Sigma, 1/2) \leq 8, \quad 2\pi \leq \text{vol}_g(\Sigma) \leq 8\pi.
$$

Moreover, let $0 < C_{\text{ball}} < 1/2$ be a real number. There is a universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then it holds that for $1 \leq t \leq 2$,

$$
\Sigma_t \text{ is a weakly regular ball of radius } 1/t \text{ with constant } C_{\text{ball}}.
$$
and moreover,
\[
\begin{align*}
\|\text{Ric}\|_{L^\infty_t L^2(S_t)} + \|\mathbf{R}\|_{L^\infty_t L^2(S_t)} &\lesssim \varepsilon, \\
\|k\|_{L^\infty_t L^2(S_t)} + \|\nabla k\|_{L^\infty_t L^2(S_t)} + \|\mathbf{D}_T k\|_{L^\infty_t L^2(S_t)} &\lesssim \varepsilon, \\
\|n - 1\|_{L^\infty_t L^\infty(S_t)} + \|\nabla n\|_{L^\infty_t L^2(S_t)} + \|\nabla^2 n\|_{L^\infty_t L^2(S_t)} &\lesssim \varepsilon,
\end{align*}
\] (2.28)

Remarks on Theorem 2.20.

(1) In the proof of Theorem 2.20, we derive estimates for the curvature tensor \( \mathbf{R} \) by using the Bel-Robinson tensor \( Q(\mathbf{R}) \), see Proposition 2.2, which in turn requires a trilinear estimate for the corresponding error term. It is due to this trilinear estimate that we need to invoke the bounded \( L^2 \) curvature theorem (Theorem 3.11).

(2) By Theorem 2.20, each hypersurface \( \Sigma_t \) is a weakly regular ball of radius \( t \) with constant \( C_{\text{ball}} \) which means that there are global coordinates on \( \Sigma_t \) such that \( \partial^2 g_{ij} \in L^2(\Sigma_t) \). However, because these global coordinates are constructed by Cheeger-Gromov theory on \( \Sigma_t \) (see Theorem 4.1), we have no control on the regularity of the coordinate components \( g_{ij} \) in the \( t \)-direction.

(3) In [15] it is shown that for the so-called canonical foliation, the weak regularity of the 2-spheres \( S_v \) and the norm \( \mathcal{O}^\mathcal{H} \) can be bounded by the \( L^2 \) curvature flux \( \mathcal{R}^\mathcal{H} \) and low-regularity bounds on the geometry of the initial sphere \( S_1 = \Sigma \cap \mathcal{H} \).

(4) The regularity assumptions (2.27) (successfully established for the canonical foliation in [15]) are crucial for the proof of Theorem 2.20. However, the precise choice of foliation is not relevant for our estimates, i.e. any foliation with similar regularity bounds can be used to prove Theorem 2.20.

2.9 Proof of the Main Result

The proof of Theorem 2.20 is based on a standard bootstrapping argument. Let \( T \in [1, 2] \) be defined as
\[
T := \sup_{t^* \in [1, 2]} \left\{ (2.28) \text{ holds holds on } 1 \leq t \leq t^* \right\}.
\]

We conclude that \( T = 2 \) for \( \varepsilon > 0 \) sufficiently small by a continuity argument with the following three standard steps:

(1) It holds that \( T > 1 \). Indeed, this follows by the smoothness of \( (\mathcal{M}, g) \) and the continuous dependence of (2.28).

(2) Assuming that a set of bootstrap assumptions holds up to some \( 1 < t_0^* < 2 \), we show that we can improve them for \( \varepsilon > 0 \) sufficiently small.

(3) Similarly as in (1), we deduce that \( T > t_0^* \).

For the rest of this paper, we are concerned with (2). The following proposition is proved in Section 4 and forms the core of this paper.
Proposition 2.21 (Improvement of bootstrap assumptions) Consider a smooth solution \((\mathcal{M}, g)\) of the spacelike-characteristic Cauchy problem, foliated by maximal spacelike hypersurfaces \((\Sigma_t)_{1 \leq t \leq 2}\) satisfying \(\partial \Sigma_t = S_t\) for \(1 \leq t \leq 2\) (see also Lemma 1.6).

Assume that the 2-spheres \((S_v)_{1 \leq v \leq 2} \subset \mathcal{H}\) are uniformly weakly regular with constants \(N, c\), and that for some real number \(\varepsilon > 0\),

\[
O^\Sigma + R^\Sigma + O^\mathcal{H} + R^\mathcal{H} \leq \varepsilon, \quad 1/4 \leq r_{vol}(\Sigma, 1/2) \leq 8, \quad 2\pi \leq vol_g(\Sigma) \leq 8\pi.
\]

Moreover, assume that for some fixed large real number \(D > 0\), the following bootstrap assumptions hold for \(1 \leq t \leq t_0^*\),

\[
\|\text{Ric}\|_{L^\infty_t L^2(\Sigma_t)} + \|\text{R}\|_{L^\infty_t L^2(\Sigma_t)} \leq D\varepsilon, \\
\|k\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty_t L^2(\Sigma_t)} + \|k\|_{L^\infty_t L^2(S_t)} \leq D\varepsilon, \\
\|v - 1\|_{L^\infty_t L^\infty(S_t)} + \|\nabla v\|_{L^\infty_t H^{1/2}(S_t)} \leq D\varepsilon, \\
\frac{1}{8} \leq r_{vol}(\Sigma_t, 1/2) \leq 16, \\
\frac{\pi}{2} \leq vol_g(\Sigma_t) \leq 32\pi.
\]

Let further \(0 < C_{\text{ball}} < 1/2\) be a real number. There is a real number \(\varepsilon_0 > 0\) such that if \(0 < \varepsilon < \varepsilon_0\), then for \(1 \leq t \leq t_0^*\),

\(\Sigma_t\) is a weakly regular ball of radius \(t\) with constant \(C_{\text{ball}}\),

and

\[
\|\text{Ric}\|_{L^\infty_t L^2(\Sigma_t)} + \|\text{R}\|_{L^\infty_t L^2(\Sigma_t)} \leq D'\varepsilon, \\
\|k\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty_t L^2(\Sigma_t)} + \|k\|_{L^\infty_t L^2(S_t)} + \|\text{DT} k\|_{L^\infty_t L^2(\Sigma_t)} \leq D'\varepsilon, \\
\|v - 1\|_{L^\infty_t L^\infty(S_t)} + \|\nabla v\|_{L^\infty_t H^{1/2}(S_t)} \leq D'\varepsilon, \\
\|n - 1\|_{L^\infty_t L^\infty(\Sigma_t)} + \|\nabla n\|_{L^\infty_t L^2(\Sigma_t)} + \|\nabla^2 n\|_{L^\infty_t L^2(\Sigma_t)} \leq D'\varepsilon, \\
\frac{1}{8} < r_{vol}(\Sigma_t, 1/2) < 16, \\
\frac{\pi}{2} < vol_g(\Sigma_t) < 32\pi,
\]

for a constant \(0 < D' < D\).
2.10 Organisation of the Paper

The paper is organised as follows.

- In Section 3, we recapitulate calculus inequalities and prerequisites.
- In Section 4, we prove Proposition 2.21.
- In Section 5, we prove the existence of global coordinates by Cheeger-Gromov theory (see Theorem 4.1), which is applied in Section 4.3 to prove that $\Sigma_t$ are weakly regular 3-balls.
- In Appendix A, we prove a global elliptic estimate for the second fundamental form of a maximal spacelike hypersurface with boundary.
- In Appendix B, we prove the calculus inequalities of Lemmas 3.5 and 3.6.
- In Appendix C, we prove a Riemannian rigidity result, see Lemma 5.4, used in the proof of existence of global coordinates in Section 5.
- In Appendix 1, we prove comparison estimates between two maximal foliations on a vacuum spacetime used in Section 4.

3 Calculus Inequalities and Prerequisites

3.1 Calculus on Weakly Regular Riemannian 2-Spheres

In this section, we recapitulate calculus prerequisites on weakly regular Riemannian 2-spheres $(S, g)$. The following lemma is proved in [30].

Lemma 3.1 (Sobolev inequalities)

Let $(S, g)$ be a weakly regular Riemannian 2-sphere with constants $N, c$. Let $1 \leq p < \infty$ be a real number. Then it holds that for each tensor $F$ on $S$,

$$
\|F\|_{L^p(S)} \lesssim \|\nabla F\|_{L^2(S)} + \|F\|_{L^2(S)},
$$

$$
\|F\|_{L^\infty(S)} \lesssim \|\nabla F\|_{L^4(S)} + \|F\|_{L^2(S)},
$$

where the constants depend on $p, N, c$ and $N, c$, respectively.

We introduce the following fractional Sobolev spaces.

Definition 3.2 (Fractional Sobolev spaces) Let $(S, g)$ be a Riemannian 2-sphere and let $-\infty < s < \infty$ be a real number. For tensors $F$ on $S$, define the norm

$$
\|F\|_{H^s(S)} := \|(1 - \Delta)^{s/2} F\|_{L^2(S)},
$$

where the fractional Laplace-Beltrami operator is defined by standard spectral decomposition, see [30].

The following properties of fractional Sobolev spaces are well-known, see for example Section 2 and Theorem 3.6 in [30] and Section 3 and Appendix B in [15].
Lemma 3.3 (Properties of fractional Sobolev spaces) Let $(S, g)$ be a weakly regular Riemannian 2-sphere with constants $N, c$. Let $F, F_1$ and $F_2$ be tensors on $S$. Then it holds that

$$\|F\|_{L^4(S)} \lesssim \|F\|_{H^{1/2}(S)},$$
$$\|\nabla F\|_{H^{-1/2}(S)} \lesssim \|F\|_{H^{1/2}(S)},$$
$$\|F\|_{H^{1/2}(S)} \lesssim \|F\|_{H^1(S)},$$
$$\|F_1 F_2\|_{H^{1/2}(S)} \lesssim (\|F_1\|_{L^\infty(S)} + \|\nabla F_1\|_{L^2(S)}) \|F_2\|_{H^{1/2}(S)},$$

where the constants depend on $N, c$.

3.2 Calculus on Weakly Regular Balls

In this section, we recall low regularity calculus prerequisites on weakly regular balls (see Definition 2.18). The following Sobolev inequalities are well-known, see, for example, Section 3.5 in [33].

Lemma 3.4 (Sobolev inequalities on $\Sigma$) Let $1 \leq r \leq 2$ and $0 < C_{\text{ball}} < 1/2$ be real numbers, and let $(\Sigma, g)$ be a weakly regular ball of radius $r$ with constant $C_{\text{ball}}$. Then for each tensor $F$ on $\Sigma$,

$$\|F\|_{L^\infty(\Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)} + \|\nabla^2 F\|_{L^2(\Sigma)},$$
$$\|F\|_{L^\infty(\Sigma)} \lesssim \|F\|_{L^4(\Sigma)} + \|\nabla F\|_{L^4(\Sigma)},$$
$$\|F\|_{L^6(\Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}.$$

The following trace estimates are well-known, see, for example, [1] and [30]. For completeness, a proof is provided in Appendix B.

Lemma 3.5 (Trace estimates onto $\partial \Sigma$) Let $1 \leq r \leq 2$ and $0 < C_{\text{ball}} < 1/2$ be real numbers, and let $(\Sigma, g)$ be a weakly regular ball of radius $r$ with constant $C_{\text{ball}}$. Then for each tensor $F$ on $\Sigma$, it holds that

$$\|F\|_{L^4(\partial \Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)},$$
$$\|F\|_{H^{1/2}(\partial \Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}.$$

(3.1)

The following lemma allows to estimate the $L^2$-norm of a tensor by the $L^2$-norm of its covariant derivative and its boundary value; see Appendix B for a proof.

Lemma 3.6 Let $1 \leq r \leq 2$ and $0 < C_{\text{ball}} < 1/2$ be two real numbers. Let $(\Sigma, g)$ be a weakly regular ball of radius $r$ with constant $C_{\text{ball}}$, and let $F$ be a tensor on $\Sigma$. Then,

$$\|F\|_{L^2(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)} + \|F\|_{L^2(\partial \Sigma)}.$$
For real numbers $1 \leq r \leq 2$ and $0 < C_{\text{ball}} < 1/2$, let $(\Sigma, g)$ be a weakly regular ball of radius $r$ with constant $C_{\text{ball}}$. In Section 2.4, we defined spherical coordinates $(r, \theta^1, \theta^2)$ on $\Sigma$ and expressed the metric as follows,

$$g = a^2 r^2 + \mathcal{g}_{AB} (b^A dr + d\theta^A) (b^B dr + d\theta^B).$$

Moreover, we defined for real numbers $1 \leq r_1 \leq r_2 \leq 2$ the annulus $A(r_1, r_2)$ in (2.15).

The following lemma shows that in spherical coordinates, the metric components are estimated by the constant $C_{\text{ball}}$; see Lemma 2.22 in [12] for a proof.

**Lemma 3.7** (Estimates for metric components in spherical coordinates) Let $1 \leq r \leq 2$ and $0 < C_{\text{ball}} < 1/2$ be two real numbers. Let $(\Sigma, g)$ be a weakly regular ball of radius $r$ with constant $C_{\text{ball}}$. Then

$$\|a - 1\|_{H^2(A(r/2, r))} + \|b\|_{H^2(A(r/2, r))} + \|\mathcal{g}_{AB} - \gamma_{AB}\|_{H^2(A(r/2, r))} \lesssim C_{\text{ball}},$$

$$\left\|\operatorname{tr} \Theta - \frac{2}{r'}\right\|_{H^1(A(r/2, r))} + \|\Theta\|_{H^1(A(r/2, r))} + \|\nabla a\|_{H^1(A(r/2, r))} \lesssim C_{\text{ball}},$$

where $\gamma_{AB}$ denotes the standard round metric on $S_r$.

### 3.3 Global Elliptic Estimates for the Laplace–Beltrami Operator on $\Sigma$

We apply the following global elliptic estimates to the maximal lapse equation (2.13e). The proof follows from a generalisation of the estimates proved in Appendix B in [13].

**Proposition 3.8** Let $1 \leq r \leq 2$ and $0 < C_{\text{ball}} < 1/2$ be two real numbers. Let $(\Sigma, g)$ be a weakly regular ball of radius $r$ with constant $C_{\text{ball}}$. Then for any scalar function $f$ on $\Sigma$,

$$\sum_{0 \leq m \leq 2} \|\nabla^m f\|_{L^2(\Sigma)} \lesssim \|\Delta f\|_{L^2(\Sigma)} + \|\nabla f\|_{H^{1/2}(\partial \Sigma)} + \|f\|_{L^2(\partial \Sigma)}.$$

### 3.4 An Extension Procedure for the Constraint Equations

The following result of [12] is used as a black box in this paper.

**Theorem 3.9** (Extension procedure for the constraints, [12]) Let $1 \leq r \leq 2$ be a real number. Let $(\bar{g}, \bar{k})$ be maximal initial data for the Einstein equations on $B_r \subset \mathbb{R}^3$. There exists a universal constant $\varepsilon > 0$ such that if

$$\sum_{0 \leq m \leq 2} \|\partial^m (\bar{g}_{ij} - e_{ij})\|_{L^2(B_r)} + \sum_{0 \leq m \leq 1} \|\partial^m \bar{k}_{ij}\|_{L^2(B_r)} < \varepsilon,$$

where $e_{ij}$ denotes the standard Euclidean metric in Cartesian coordinates, then $(\bar{g}, \bar{k})$ can be smoothly extended to maximal initial data $(g, k)$ on $\mathbb{R}^3$ with

$$(g, k)|_{B_r} = (\bar{g}, \bar{k}),$$
such that
\[
\| g_{ij} - e_{ij} \|_{H^2_{-1/2}(\mathbb{R}^3)} + \| k_{ij} \|_{H^1_{-3/2}(\mathbb{R}^3)} \lesssim \sum_{0 \leq m \leq 2} \| \partial^m (\bar{g}_{ij} - e_{ij}) \|_{L^2(B_r)} \\
+ \sum_{0 \leq m \leq 1} \| \partial^m \bar{k}_{ij} \|_{L^2(B_r)},
\]
where \( H^2_{-1/2}(\mathbb{R}^3) \) and \( H^1_{-3/2}(\mathbb{R}^3) \) are weighted Sobolev spaces bounding 2 and 1 coordinate derivatives, respectively, and measuring asymptotic flatness, see [12]. In particular, the constructed maximal initial data \((g, k)\) on \( \mathbb{R}^3 \) satisfies
\[
rvol(\mathbb{R}^3, 1/2) > 1/4
\]
and
\[
\| \text{Ric} \|_{L^2(\mathbb{R}^3)} + \| k \|_{L^2(\mathbb{R}^3)} + \| \nabla k \|_{L^2(\mathbb{R}^3)} \lesssim \sum_{0 \leq m \leq 2} \| \partial^m (\bar{g}_{ij} - e_{ij}) \|_{L^2(B_r)} \\
+ \sum_{0 \leq m \leq 1} \| \partial^m \bar{k}_{ij} \|_{L^2(B_r)}.
\]

**Remark 3.10** In [12], Theorem 3.9 is proved for maximal initial data \((\bar{g}, \bar{k})\) given on the unit ball \( B_1 \). However, it is straightforward to generalise that result to maximal initial data given on \( B_r \) for \( 1 \leq r \leq 2 \), as stated in Theorem 3.9 above.

### 3.5 The Bounded \( L^2 \) Curvature Theorem

The following theorem is a paraphrasing of Theorems 2.4 and 2.5 in [31] and Theorem 2.18 in [33], and used as a black box in this paper.

**Theorem 3.11** (The bounded \( L^2 \) curvature theorem, version 2) Let \( (\Sigma, g, k) \) be asymptotically flat maximal initial data for the Einstein vacuum equations such that \( \Sigma \simeq \mathbb{R}^3 \). Assume moreover that for some \( \varepsilon > 0 \),
\[
\| \text{Ric} \|_{L^2(\Sigma)} \leq \varepsilon, \quad \| k \|_{L^2(\Sigma)} + \| \nabla k \|_{L^2(\Sigma)} \leq \varepsilon, \quad rvol(\Sigma, 1/2) > 1/4.
\]
Then there is a universal constant \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \), then the maximal globally hyperbolic future development \((\mathcal{M}, g)\) of the initial data \((\Sigma, g, k)\) contains a foliation of maximal spacelike hypersurfaces \( (\Sigma_t)_{0 \leq t \leq 1} \) with \( \Sigma_0 = \Sigma \) such that on each \( \Sigma_t \),
\[
\| \text{Ric} \|_{L^\infty_t L^2(\Sigma_t)} + \| R \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \\
\| k \|_{L^\infty_t L^2(\Sigma_t)} + \| \nabla k \|_{L^\infty_t L^2(\Sigma_t)} + \| D_T k \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \\
\| n - 1 \|_{L^\infty(\mathcal{M})} + \| \nabla n \|_{L^\infty(\mathcal{M})} + \| \nabla^2 n \|_{L^\infty_t L^2(\Sigma_t)} + \| \nabla T(n) \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon,
\]
\[
rvol(\Sigma_t, 1/2) \geq 1/8.
\]
Moreover, for each \( \omega \in \mathbb{S}^2 \), there is a foliation \( (\mathcal{H}_{\omega u})_{\omega u} \in \mathcal{M} \) by weakly regular (see remarks below) null hyperplanes \( \mathcal{H}_{\omega u} \) given as level sets of an optical function \( \omega u \) such that

\[
\sup_{\omega \in \mathbb{S}^2} \| \mathbf{R} \cdot L \|_{L^\infty_{\omega u} L^2(\mathcal{H}_{\omega u})} \lesssim \varepsilon,
\]

where \( L \) denotes the \( \mathcal{H}_{\omega u} \)-tangential null vectorfield with \( g(T, L) = -1 \). Here

\[
\| \mathbf{R} \cdot L \|_{L^\infty_{\omega u} L^2(\mathcal{H}_{\omega u})} := \| \alpha \|_{L^\infty_{\omega u} L^2(\mathcal{H}_{\omega u})} + \| \beta \|_{L^\infty_{\omega u} L^2(\mathcal{H}_{\omega u})} + \| \rho \|_{L^\infty_{\omega u} L^2(\mathcal{H}_{\omega u})} + \| \sigma \|_{L^\infty_{\omega u} L^2(\mathcal{H}_{\omega u})},
\]

where \( \alpha, \beta, \rho, \sigma, \beta \) are the \( \mathcal{H}_{\omega u} \cap \Sigma_t \)-tangential tensors defined by \( (2.3) \) with \( L := 2T - L \). In addition, the following trilinear estimate holds,

\[
\left| \int_{\mathcal{M}} Q(R)_{ijTTk}^{ij} \right| \lesssim \varepsilon \| \mathbf{R} \|_{L^\infty_{\omega u} L^2(\Sigma_t)} + \varepsilon \| \mathbf{R} \|_{L^2(\mathcal{M})} \sup_{\omega \in \mathbb{S}^2} \| \mathbf{R} \cdot L \|_{L^\infty_{\omega u} L^2(\mathcal{H}_{\omega u})}. \tag{3.3}
\]

Remarks

1. Theorem 1.3 is the small data version of the bounded \( L^2 \) curvature theorem. A corresponding large data version is obtained in [22] by a rescaling procedure.
2. We refer to Definition 5.3 in [22] for a definition of weakly regular null hypersurfaces. For the purposes of this paper, it suffices to note that weak regularity is sufficient for an application of Stokes’ theorem as in Proposition 2.2.
3. For \( \omega \in \mathbb{S}^2 \) the optical function \( \omega u \) is initialized on \( \Sigma \simeq \mathbb{R}^3 \) such that, for \( x \in \mathbb{R}^3 \),

\[
\omega u(x) \sim x \cdot \omega \quad \text{as} \quad |x| \to \infty;
\]

we refer to the introduction of [31] (see condition C1a) for more details on the construction of \( \omega u \).

4 Proof of Proposition 2.21

In this section we prove Proposition 2.21. Let \( \varepsilon > 0 \) and \( 1 < t_0^* < 2 \) be real numbers. Assume that the 2-spheres \( (S_v)_{1 \leq v \leq 2} \subset \mathcal{H} \) are uniformly weakly regular with constants \( N, c, \) and that

\[
O^\Sigma + R^\Sigma + O^\mathcal{H} + R^\mathcal{H} \leq \varepsilon, \quad 1/4 \leq v_{\mathrm{vol}}(\Sigma, 1/2) \leq 8, \quad 2\pi \leq v_{\mathrm{vol}}(\Sigma) \leq 8\pi, \tag{4.1}
\]
and that, for a large fixed constant $D > 0$ and $1 \leq t \leq t_0^*$,

\[
\|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\text{R}\|_{L_t^\infty L^2(\Sigma_t)} \leq D\varepsilon,
\]

\[
\|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(\Sigma_t)} \leq D\varepsilon,
\]

\[
\|\nu - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(\Sigma_t)} \leq D\varepsilon,
\]

(4.2)

In the following, we prove that for $\varepsilon > 0$ sufficiently small, for $1 \leq t \leq t_0^*$,

\[
\|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\text{R}\|_{L_t^\infty L^2(\Sigma_t)} \leq D'\varepsilon,
\]

\[
\|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(\Sigma_t)} \leq D'\varepsilon,
\]

\[
\|\nu - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(\Sigma_t)} \leq D'\varepsilon,
\]

(4.3)

for a constant $0 < D' < D$. In addition, we show that

\[
\|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} \leq D'\varepsilon,
\]

\[
\|D T k\|_{L_t^\infty L^2(\Sigma_t)} \leq D'\varepsilon.
\]

(4.4)

**Notation**

1. Pick $1 \leq t^* \leq t_0^*$. In the following, we prove (4.3) and (4.4) on $\Sigma_t^*$. As $t^*$ was chosen arbitrarily, this implies (4.3) and (4.4) for $1 \leq t \leq t_0^*$.

2. In the following estimates, the universal constant in $\lesssim$ is independent of $D > 0$. Moreover, $D > 0$ is chosen sufficiently large such that, for example, $C \sqrt{D} < D$, where $C > 0$ is the universal constant in $\lesssim$.

### 4.1 Overview of the Proof

In the following, we outline the main steps of the proof of Proposition 2.21. An important tool is the following theorem about the existence of global coordinates which is applied in the proof of Proposition 2.21 to each hypersurface $\Sigma_t$. Its proof is given in Section 5 and based on the Cheeger-Gromov theory of manifold convergence [13].

**Theorem 4.1** *(Existence of global regular coordinates)*

Let $(M, g)$ be a compact Riemannian 3-manifold with boundary such that $M \simeq \overline{B_1} \subset \mathbb{R}^3$. Assume that for real numbers $1 \leq t \leq 2$, $\varepsilon > 0$ and $0 < V < \infty$,

\[
\|\text{Ric}\|_{L^2(M)} \leq \varepsilon,
\]

\[
\|\text{tr} \Theta - \frac{2}{t}\|_{L^4(\partial M)}
\]

\[
+ \|\hat{\Theta}\|_{L^4(\partial M)} \leq \varepsilon,
\]

\[
r_{\text{vol}}(M, 1/2) \geq 1/4, \quad \text{vol}_g(M) \leq V
\]

\[\square\]
where $\Theta$ denotes the second fundamental form of $\partial M \subset M$. Then for every real number $0 < C_{\text{ball}} < 1/2$ there is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then

$$(M, g) \text{ is a weakly regular ball of radius } t \text{ with constant } C_{\text{ball}},$$

that is, there is a global coordinate chart $\phi : \overline{B_t} \to M$ such that

$$(1 - C_{\text{ball}}) |\xi| \leq g_{ij} \xi^i \xi^j \leq (1 + C_{\text{ball}}) |\xi|^2 \text{ for all } \xi \in \mathbb{R}^2,$$

$$\sum_{0 \leq m \leq 2} \|\partial^m (g_{ij} - e_{ij})\|_{L^2(B_t)} \lesssim C_{\text{ball}}.$$

We are now in position to give an overview of the proof of Proposition 2.21. As noted above, it suffices to improve the bootstrap assumptions on $\Sigma_{t^*}$ for a fixed real number

$$1 \leq t^* \leq t_0^*.$$

(1) Let $0 < C_{\text{ball}} < 1/2$ be a real number to be determined below. By the bootstrap assumptions (4.2) together with Theorem 4.1, we deduce that for $\varepsilon > 0$ sufficiently small, $\Sigma_{t^*}$ is a weakly regular ball of radius $t^*$ with constant $C_{\text{ball}}$. For $C_{\text{ball}} > 0$ sufficiently small, this directly improves the bootstrap assumptions on $\text{vol}_{g} (\Sigma_{t^*})$ and $r_{\text{vol}} (\Sigma_{t^*}, 1/2)$, see Section 4.3.

(2) For $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small, the extension procedure for the constraint equations (Theorem 3.9) can be applied to $\Sigma_{t^*}$. This yields an extension of the maximal initial data $(\Sigma_{t^*}, g, k)$ to an asymptotically flat maximal initial data set of size bounded by $C_{\text{ball}}$, see Section 4.5.

(3) For $C_{\text{ball}} > 0$ sufficiently small, we can subsequently apply backwards the bounded $L^2$ curvature theorem (Theorem 3.11) to the above extended initial data set. This yields a foliation of the past of $\Sigma_{t^*}$ in $\mathcal{M}$ by maximal hypersurfaces $(\tilde{\Sigma}_t)_{0 \leq t \leq t^*}$ which satisfies in particular $\tilde{\nabla} n \in L^\infty (\mathcal{M}_{t^*})$ and admits a trilinear estimate, see Section 4.5.

(4) Using the $\tilde{\Sigma}_t$-foliation, we apply the Bel-Robinson estimate for $\mathbf{R}$ (see Proposition 2.2) to estimate the curvature flux through $\Sigma_{t^*}$ by the curvature fluxes through $\mathcal{H}$ and $\Sigma$. As the latter are bounded by the initial data norms (see Remark 4.2 below), this improves the bootstrap assumption on the curvature flux on $\Sigma_{t^*}$. It is in this estimate that the control of $\tilde{\nabla} n \in L^\infty (\mathcal{M}_{t^*})$ and the trilinear estimate for the $(\tilde{\Sigma}_t)_{0 \leq t \leq t^*}$-foliation are essential, see Section 4.6.

(5) The second fundamental form $k$ on $\Sigma_{t^*}$ satisfies a Hodge system. By applying global elliptic estimates (Corollary A.3), we improve the bootstrap assumptions on $\nabla k \in L^2 (\Sigma_{t^*})$. Here we use that the source terms in the Hodge system are curvature terms which were improved in the previous step. Moreover, here it is crucial to analyse the boundary integrals appearing in the global elliptic estimates for $k$. Indeed, they admit a special structure which allows to split them up into two parts: One part which has the right sign to control the slope $\nu$ between $\mathcal{H}$ and $\Sigma_{t^*}$ and the value of $k$ on $\partial \Sigma$, and a second part which can be estimated by the initial data norm on $\mathcal{H}$. See Section 4.7 for details.
(6) The bootstrap assumptions for \( \nu \) on \( \partial \Sigma_t^* \) are fully improved by using the slope equation (2.24) together with the previously improved bounds for \( k \) and the initial data norms on \( \mathcal{H} \), see Section 4.10.

(7) The foliation lapse \( n \) of the foliation \( \Sigma_t^* \) is improved by global elliptic estimates applied to the maximal lapse equation. Here we use that \( k \) on \( \Sigma_t^* \) and the boundary value \( n = \Omega^{-1} \nu^{-1} \) on \( \partial \Sigma_t^* \) are improved in the previous steps, see (4.47). The additional estimate for \( D_T k \) on \( \Sigma_t^* \) follows from the second variation equation (2.13a), see Corollary 4.11.

**Remark 4.2** To bound the curvature fluxes through \( \mathcal{H} \) and \( \Sigma \) by the initial data norms on \( \mathcal{H} \) and \( \Sigma \), one needs a comparison argument between the two maximal foliations \((/\Sigma_t^*)_{1 \leq t \leq t^*}\) and \((/\Sigma_0^*)_{0 \leq \tilde{t} \leq t^*}\), see Lemma 4.6. This comparison requires the control of \( n \in L^\infty(\mathcal{M}_t^*) \), and hence in the proof of Proposition 2.21 we first bound \( n - 1 \) of size \( D_\epsilon \) by using the smallness assumption (4.1) and the bootstrap assumptions (4.2), before improving the curvature estimates on \( \Sigma_t^* \).

### 4.2 First Consequences of the Bootstrap Assumptions

We first remark that by (4.1) and (4.2),

\[
\left\| \text{tr} \chi - \frac{2}{t} \right\|_{L_t^\infty L^\infty(\Sigma_t)} + \left\| \text{tr} \chi + \frac{2}{t} \right\|_{L_t^\infty L^\infty(\Sigma_t)} \lesssim \epsilon, \\
\left\| \hat{\chi} \right\|_{L_t^\infty L^4(\Sigma_t)} + \left\| \hat{\chi} \right\|_{L_t^\infty L^4(\Sigma_t)} \lesssim \epsilon, \\
\left\| \nu - 1 \nu_{L_t^\infty L^\infty(\Sigma_t)} \lesssim D_\epsilon. \quad (4.5)
\]

Using that by Lemma 2.7

\[
\text{tr} \Theta - \frac{2}{t} = \frac{1}{2} \nu \text{tr} \chi - \frac{1}{2} \nu^{-1} \text{tr} \chi - \frac{2}{t} \\
= \frac{1}{2} (\nu - 1) \text{tr} \chi + \frac{1}{2} \left( \text{tr} \chi - \frac{2}{t} \right) - \frac{1}{2} (\nu - 1) \text{tr} \chi - \frac{1}{2} \left( \text{tr} \chi + \frac{2}{t} \right),
\]

we therefore have by (4.5) for \( \epsilon > 0 \) sufficiently small that

\[
\left\| \text{tr} \Theta - \frac{2}{t} \right\|_{L_t^\infty L^\infty(\Sigma_t)} \lesssim D_\epsilon. \quad (4.6)
\]

Moreover, using that by Lemma 2.7

\[
\hat{\Theta} = \frac{1}{2} \nu \hat{\chi} - \frac{1}{2} \nu^{-1} \hat{\chi},
\]

we get by (4.5) for \( \epsilon > 0 \) sufficiently small that

\[
\left\| \hat{\Theta} \right\|_{L_t^\infty L^4(\Sigma_t)} \lesssim \epsilon. \quad (4.7)
\]
4.3 Weakly Regular Ball Property of $\Sigma_t$

From (4.2), (4.6) and (4.7), we have for $\varepsilon > 0$ sufficiently small that

$$
\| \text{Ric} \|_{L^2(\Sigma_t)} \lesssim D \varepsilon,
$$

$$
\left\| \text{tr}\Theta - \frac{2}{t^*} \right\|_{L^4(S_t^*)} + \left\| \hat{\Theta} \right\|_{L^4(S_t^*)} \lesssim D \varepsilon,
$$

$$
r_{vol}(\Sigma_t^*, 1/2) \geq 1/4,
$$

$$
\text{vol}_g(\Sigma_t^*) \leq 32 \pi,
$$

where we used that by assumption, the spheres $(S_v) \subset \mathcal{H}$ are weakly regular with constants $N, c$, and hence their area is well-controlled and inclusions such as $L^\infty(S_v) \hookrightarrow L^4(S_v)$ hold.

Let $0 < C_{\text{ball}} < 1/2$ be a real number to be determined below. By Theorem 4.1, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $\Sigma_t^*$ is a weakly regular ball of radius $t^*$ with constant $C_{\text{ball}}$, that is, there is a global coordinate chart $\phi : B_{t^*} \to \Sigma_t^*$ such that

$$
(1 - C_{\text{ball}}) |\xi|^2 \leq g_{ij} \xi^i \xi^j \leq (1 + C_{\text{ball}}) |\xi|^2 \text{ for all } \xi \in \mathbb{R}^2,
$$

$$
\sum_{0 \leq m \leq 2} \| \partial^m (g_{ij} - e_{ij}) \|_{L^2(B_{t^*})} \lesssim C_{\text{ball}}.
$$

(4.8)

For $C_{\text{ball}} > 0$ sufficiently small, it follows from (4.8) that

$$
1/4 < r_{vol}(\Sigma_t^*, 1/2) < 8, \quad \pi/2 < \text{vol}_g(\Sigma_t^*) < 32 \pi,
$$

which improves the bootstrap assumptions on $r_{vol}(\Sigma_t^*, 1/2)$ and $\text{vol}_g(\Sigma_t^*)$ in (4.2).

**Remark 4.3** In this paper it generally holds that $C_{\text{ball}} \gg \varepsilon$, see Theorem 4.1. In particular, demanding $C_{\text{ball}} > 0$ to be sufficiently small thus stipulates that $\varepsilon > 0$ be sufficiently small. For ease of presentation, this is tacitly acknowledged in the following.

**Remark 4.4** In particular, the above shows that the calculus results of Section 3.2 for weakly regular balls hold on $\Sigma_t$.

4.4 Estimates for the Lapse $n$ on $\Sigma_t$

The lapse function $n$ is by (2.13e) and (2.25) a solution to the following elliptic boundary value problem,

$$
\Delta n = n|k|^2_g \quad \text{on } \Sigma_t^*,
$$

$$
n = \nu^{-1} \Omega^{-1} \quad \text{on } \partial \Sigma_t^*.
$$

(4.9)
In this section, we use (4.1), (4.2) and global elliptic estimates to prove that for \( \varepsilon > 0 \) sufficiently small,
\[
\| n - 1 \|_{L^\infty(\Sigma^\ast)} + \| \nabla n \|_{L^2(\Sigma^\ast)} + \| \nabla^2 n \|_{L^2(\Sigma^\ast)} \lesssim D \varepsilon. \tag{4.10}
\]

**Remark 4.5** In accordance with the continuity argument of the proof of the main result, we do not have any bootstrap assumptions on \( n \) in (4.2).

On the one hand, by (4.1), (4.2) and Lemma 3.3, the boundary value \( n = n^{-1} \Omega^{-1} \) satisfies for \( \varepsilon > 0 \) sufficiently small
\[
\| n - 1 \|_{L^\infty(\partial \Sigma^\ast)} \lesssim \| \nabla n \|_{L^2(\Sigma^\ast)} + \| \Omega - 1 \|_{L^\infty(\partial \Sigma^\ast)} \lesssim D \varepsilon,
\]
\[
\| \nabla n \|_{L^2(\partial \Sigma^\ast)} \lesssim \| \nabla v \|_{L^2(\partial \Sigma^\ast)} + \| \nabla \Omega \|_{L^2(\partial \Sigma^\ast)} \lesssim D \varepsilon,
\]
as well as
\[
\| \nabla n \|_{H^{1/2}(\partial \Sigma^\ast)} = \left\| \frac{1}{\nu^2} \nabla v + \frac{1}{\nu^2} \nabla \Omega \right\|_{H^{1/2}(\partial \Sigma^\ast)} \lesssim \left( \left\| \nabla \left( \frac{1}{\nu^2} \right) \right\|_{L^2(\partial \Sigma^\ast)} + \left\| \frac{1}{\nu^2} \right\|_{L^\infty(\partial \Sigma^\ast)} + \left\| \nabla \left( \frac{1}{\nu^2} \right) \right\|_{L^2(\partial \Sigma^\ast)} + \left\| \nabla \nu \right\|_{H^{1/2}(\partial \Sigma^\ast)} + \left\| \nabla \Omega \right\|_{H^{1/2}(\partial \Sigma^\ast)} \right) \lesssim \| \nabla \nu \|_{H^{1/2}(\partial \Sigma^\ast)} + \| \nabla \Omega \|_{H^{1/2}(\partial \Sigma^\ast)} \lesssim D \varepsilon.
\]

On the other hand, by (4.2), (4.8), (4.9) and Lemma 3.4, we have that
\[
\| \Delta n \|_{L^2(\Sigma^\ast)} = \| n |k|^2 \|_{L^2(\Sigma^\ast)} \lesssim \| n \|_{L^6(\Sigma^\ast)} \| k \|_{L^6(\Sigma^\ast)} \lesssim (1 + \| n - 1 \|_{L^2(\Sigma^\ast)} + \| \nabla n \|_{L^2(\Sigma^\ast)}) \left( \| k \|_{L^2(\Sigma^\ast)} + \| \nabla k \|_{L^2(\Sigma^\ast)} \right)^2 \lesssim (1 + \| n - 1 \|_{L^2(\Sigma^\ast)} + \| \nabla n \|_{L^2(\Sigma^\ast)})(D \varepsilon)^2.
\]

By the elliptic estimates of Proposition 3.8, we hence get that for \( C_{ball} > 0 \) and \( \varepsilon > 0 \) sufficiently small,
\[
\| n - 1 \|_{L^2(\Sigma^\ast)} + \| \nabla n \|_{L^2(\Sigma^\ast)} + \| \nabla^2 n \|_{L^2(\Sigma^\ast)} \lesssim (\| \Delta n \|_{L^2(\Sigma^\ast)} + \| \nabla n \|_{L^2(\Sigma^\ast)} + \| n \|_{L^2(\Sigma^\ast)}) \lesssim \left( \| n - 1 \|_{L^2(\Sigma^\ast)} + \| \nabla n \|_{L^2(\Sigma^\ast)} \right)(D \varepsilon)^2 + D \varepsilon.
\]
For \( \varepsilon > 0 \) sufficiently small, we can absorb the first term on the right-hand side into the left-hand side to get that

\[
\| n - 1 \|_{L^2(\Sigma^*)}^2 + \| \nabla n \|_{L^2(\Sigma^*)}^2 + \| \nabla^2 n \|_{L^2(\Sigma^*)}^2 \lesssim D \varepsilon.
\]

The estimate (4.10) follows then by Lemma 3.4.

### 4.5 Construction of a Background Foliation of the Past of \( \mathcal{M}_{t^*} \)

Let \( \mathcal{M}_{t^*} \) denote the past of \( \Sigma_{t^*} \) in \( \mathcal{M} \). In this section, we apply backwards the bounded \( L^2 \) curvature theorem on \( \Sigma_{t^*} \) to construct a background foliation of \( \mathcal{M}_{t^*} \).

First, on the one hand, we have by (4.8) that

\[
(1 - C_{\text{ball}}) |\xi|^2 \leq g_{ij} \xi^i \xi^j \leq (1 + C_{\text{ball}}) |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2,
\]

\[
\sum_{0 \leq m \leq 2} \| \partial^m (g_{ij} - e_{ij}) \|_{L^2(B_{t^*})} \lesssim C_{\text{ball}}.
\]  

(4.11)

On the other hand, by (4.2),

\[
\| k \|_{L^2(\Sigma^*)} + \| \nabla k \|_{L^2(\Sigma^*)} \leq D \varepsilon.
\]  

(4.12)

For \( \varepsilon > 0 \) sufficiently small, (4.11) and (4.12) imply by standard product estimates that

\[
\sum_{0 \leq m \leq 1} \| \partial^m k_{ij} \|_{L^2(B_{t^*})} \lesssim C_{\text{ball}}.
\]

In particular, with (4.11), it holds that

\[
\sum_{0 \leq m \leq 2} \| \partial^m (g_{ij} - e_{ij}) \|_{L^2(B_{t^*})} + \sum_{0 \leq m \leq 1} \| \partial^m k_{ij} \|_{L^2(B_{t^*})} \lesssim C_{\text{ball}}.
\]  

(4.13)

Second, using (4.13) and assuming \( C_{\text{ball}} > 0 \) is sufficiently small, we can apply the extension procedure of Theorem 3.9 to extend \( (g, k) \) from \( \Sigma_{t^*} = \phi(B_{t^*}) \) to an asymptotically flat maximal initial data \( (g', k') \) on \( \mathbb{R}^3 \) satisfying

\[
r_{\text{vol}}(\mathbb{R}^3, 1/2) > 1/4,
\]

\[
\| \text{Ric}' \|_{L^2(\mathbb{R}^3)} + \| k' \|_{L^2(\mathbb{R}^3)} + \| \nabla k' \|_{L^2(\mathbb{R}^3)} \lesssim C_{\text{ball}}.
\]  

(4.14)

Here \( r_{\text{vol}}(\mathbb{R}^3, 1/2) \), Ric' and \( \nabla' \) denote the volume radius, the Ricci curvature and the covariant derivative with respect to \( g' \) on \( \mathbb{R}^3 \).

Third, by (4.14) the maximal initial data \( (\mathbb{R}^3, g', k') \) satisfies for \( C_{\text{ball}} > 0 \) sufficiently small the assumptions of the bounded \( L^2 \) curvature theorem (see Theorem 3.11). Thus, Theorem 3.11 yields the following; see also Fig. 1 below.
Fig. 1 Applying the bounded $L^2$-curvature theorem, the backwards development $(\tilde{\mathcal{M}}, \tilde{g})$ of the extended initial data $(\mathbb{R}^3, g', k')$ is foliated by maximal spacelike hypersurfaces $\tilde{\Sigma}_t$ which, in particular, foliate $\mathcal{M}_{t^*}$.

(1) $\mathcal{M}_{t^*}$ is foliated by spacelike maximal hypersurfaces $(\tilde{\Sigma}_t)_{0 \leq \tilde{t} \leq t^*}$ given as level sets of a time function $\tilde{t}$ with $\Sigma_{t^*} = \tilde{\Sigma}_{t^*}$ and satisfying for $0 \leq \tilde{t} \leq t^*$,

$$
\|\tilde{\text{Ric}}\|_{L^2_t \Sigma_t} \lesssim C_{\text{ball}}, \quad \|\tilde{k}\|_{L^2_t \Sigma_t} + \|\tilde{\nabla}\tilde{k}\|_{L^2_t \Sigma_t} \lesssim C_{\text{ball}},
$$

$$
\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \lesssim C_{\text{ball}}, \quad \|\tilde{n} - 1\|_{L^\infty_t L^\infty(\Sigma_t)} + \|\tilde{\nabla}\tilde{n}\|_{L^\infty_t L^\infty(\Sigma_t)} \lesssim C_{\text{ball}}.
$$

where $\tilde{\text{Ric}}, \tilde{k}$ and $\tilde{\nabla}$ denote the induced Ricci curvature, the second fundamental form and the induced covariant derivative on $\tilde{\Sigma}_t$, respectively, and $\tilde{n}$ denotes the lapse of the foliation $(\tilde{\Sigma}_t)_{0 \leq \tilde{t} \leq t^*}$.

(2) For each $\omega \in S^2$, the spacetime portion $\mathcal{M}_{t^*}$ is foliated by a family of null hyperplanes $(\mathcal{H}_{\omega u})_{\omega u \in \mathbb{R}}$ given as level sets of an optical function $\omega u$ satisfying

$$
\sup_{\omega \in S^2} \|\mathbf{R} \cdot \tilde{L}\|_{L^\infty_t L^2(\mathcal{H}_{\omega u})} \lesssim C_{\text{ball}},
$$

where $\tilde{L}$ is the unique $\mathcal{H}_{\omega u}$-tangent null vectorfield with $g(\tilde{L}, \tilde{T}) = -1$ and $\tilde{T}$ denotes the future-pointing time-like unit normal to $\tilde{\Sigma}_t$.

(3) Define the angle $\tilde{\nu}$ between $T$ and $\tilde{T}$ by

$$
\tilde{\nu} := -g(T, \tilde{T}).
$$

The proof of the next lemma is provided in Appendix 1.

**Lemma 4.6 (Comparison of maximal foliations on $\mathcal{M}_{t^*}$)** For $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small, it holds that

$$
\|\tilde{\nu} - 1\|_{L^\infty_t L^{\infty}(\Sigma_t)} \lesssim C_{\text{ball}}, \quad \|\tilde{k}\|_{L^\infty_t L^4(\Sigma_t)} \lesssim C_{\text{ball}},
$$

where $\tilde{k}$ denotes the second fundamental form of $\tilde{\Sigma}_t$.

**4.6 Estimate for the Curvature Tensor on $\Sigma_t$**

In this section, we prove that

$$
\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon.
$$
The spacetime region $\mathcal{M}_{t^*}$

Using that $\Sigma_{t^*} = \bar{\Sigma}_{t^*}$ by construction, see Section 4.5, (4.18) implies in particular that

$$\|R\|_{L^2(\Sigma_{t^*})} \lesssim \varepsilon. \quad (4.19)$$

We turn to the proof of (4.18). In the following we prove the stronger estimate

$$\|R\|_{L^2(\Sigma_{t^*})} + \sup_{\omega \in S^2} \|R \cdot L_{\omega}\|_{L^2(H_{\omega u})} \lesssim \varepsilon^2, \quad (4.20)$$

where from Section 4.5 we recall that for each $\omega \in S^2$, $(H_{\omega u})_{u \in \mathbb{R}}$ is a family of null hyperplanes foliating the spacetime region $\mathcal{M}_{t^*}$, where $\mathcal{M}_{t^*}$ is defined as the past of $\Sigma_{t^*}$ in $\mathcal{M}$.

The proof of (4.20) follows by classical Bel-Robinson estimates (that is, Stokes’ theorem applied to the Bel-Robinson tensor $Q(R)$, see Definition 2.1 and (2.1), over spacetime regions, see Figs. 2 and 3), and trilinear estimates for the error term.

We start by a Bel-Robinson estimate for $Q(R)$ with $\bar{T}$ as multiplier fields over the spacetime region $\mathcal{M}_{\bar{t}}$, defined as past of $\bar{\Sigma}_{\bar{t}}$ in $\mathcal{M}$ (for some fixed $\bar{t} \leq t^*$), see Fig. 2. Proposition 2.2 yields

$$\|R\|_{L^2(\Sigma_{\bar{t}})} \lesssim \int_{\mathcal{M}_{\bar{t}}} Q(R)_{\bar{T}T\bar{T}T} + \int_{\mathcal{H}} Q(R)_{\bar{T}\bar{T}T\bar{T}} + \left| \int_{\mathcal{M}_{\bar{t}}} Q(R)_{\alpha\beta} T_{\alpha\beta} \tilde{\pi}^{\alpha\beta} \right| =: \mathcal{E}, \quad (4.21)$$

where we used (2.11) and Lemma 2.11. The error term $\mathcal{E}$ on the right-hand side of (4.21) is bounded as follows. By (2.7) the components of $\tilde{\pi} := \hat{L}_T g$ are

$$\tilde{\pi}_{\bar{T}\bar{T}} = 0, \quad \tilde{\pi}_{\bar{T}j} = \tilde{n}^{-1} \nabla_j \tilde{n}, \quad \tilde{\pi}_{ab} = -2 \tilde{k}_{ab}.$$
Hence, by (4.15),

\[ \mathcal{E} = \left| \int_{\mathcal{M}_t} Q(R)_{\alpha\beta} \tilde{T}_{ij} \tilde{T}^{\alpha\beta} \right| \]

\[ \lesssim \left| \int_{\mathcal{M}_t} Q(R)_{ab} \tilde{T} \tilde{k}^{ab} \right| + \left| \int_{\mathcal{M}_t} \tilde{n}^{-1} Q(R)_{\bar{j}j} \tilde{T} \tilde{\nabla} j \tilde{n} \right| \]

\[ \lesssim \left| \int_{\mathcal{M}_t} Q(R)_{ab} \tilde{T} \tilde{k}^{ab} \right| + \left| \int_{\mathcal{M}_t} \tilde{n}^{-1} Q(R)_{\bar{j}j} \tilde{T} \tilde{\nabla} j \tilde{n} \right| \]

\[ \lesssim \left| \int_{\mathcal{M}_t} Q(R)_{ab} \tilde{T} \tilde{k}^{ab} \right| + C_{\text{ball}} \left\| R \right\|_{L^\infty_t L^2_t(\Sigma_t)}^2 \]

(4.22)

where, as in the following, in the second term on the right-hand side the supremum in the norm \( \left\| \cdot \right\|_{L^\infty_t L^2_t(\Sigma_t)} \) is taken over \( \tilde{T} \leq t^* \).

The first term on the right-hand side of (4.22) is controlled as follows,

\[ \left| \int_{\mathcal{M}_t} Q(R)_{ab} \tilde{T} \tilde{k}^{ab} \right| \lesssim C_{\text{ball}} \left\| R \right\|_{L^\infty_t L^2_t(\Sigma_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \left\| R \cdot \tilde{L} \right\|_{L^\infty_t L^2_t(\mathcal{H}_{\omega_u})}^2. \]

(4.23)

Indeed, (4.23) is a straightforward localisation of the trilinear estimate (3.3) of Theorem 3.11. In [22] it is shown that \( \tilde{k} \) admits the following special structure (see (3.16) and Lemma 6.5 in [22])

\[ -k_{ij} = (A_i)_0 j, \quad A = \text{curl}(B) + E, \]

where the good term \( E \) is bounded by

\[ \| E \|_{L^2_t L^\infty_t(\Sigma_t)} \lesssim (C_{\text{ball}})^2. \]
Thus we get that, see also Section 11.2 in [22],

\[
\left| \int_{\mathcal{M}_t} Q(R)_{ab} \tilde{\tilde{T}}^a T^b \right| \leq \left| \int_{\mathcal{M}_t} Q(R)_{,b} \tilde{\tilde{T}}^a (\text{curl}(B))^b \right| + \left| \int_{\mathcal{M}_t} Q(R)_{,b} \tilde{\tilde{T}}^a E^b \right| \tag{4.24}
\]

\[
\lesssim \int_{\mathcal{M}_t} Q(R)_{,b} \tilde{\tilde{T}}^a (\text{curl}(B))^b + (C_{\text{ball}})^2 \|R\|_{L^\infty_t L^2(\tilde{\Sigma}_t)}^2.
\]

The first term of the right-hand side of (4.24) is bounded by using the background foliation control of Section 4.5 and a localized application of the trilinear estimates in Section 11.2 in [22]. Indeed, a direct inspection shows that the crucial estimates on pages 188 and 189 in [22] (see, in particular, the important structure (11.11)) also go through in our localized setting. We subsequently get that

\[
\int_{\mathcal{M}_t} Q(R)_{,b} \tilde{\tilde{T}}^a (\text{curl}(B))^b \lesssim C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R(R \cdot \tilde{\tilde{T}})\|_{L_{\tilde{\tilde{T}}\omega_u}^1(\tilde{\mathcal{H}}_{\omega_u})}
\]

\[
\lesssim C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \left( \|R\|_{L^2(\mathcal{M}_{t,*})} \cdot \|R \cdot \tilde{\tilde{T}}\|_{L_{\tilde{\tilde{T}}\omega_u}^\infty L^2(\tilde{\mathcal{H}}_{\omega_u})} \right)
\]

\[
\lesssim C_{\text{ball}} \|R\|_{L^2(\mathcal{M}_{t,*})}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{\tilde{T}}\|_{L_{\tilde{\tilde{T}}\omega_u}^\infty L^2(\tilde{\mathcal{H}}_{\omega_u})}^2
\]

\[
\lesssim C_{\text{ball}} \|R\|_{L^\infty_t L^2(\tilde{\Sigma}_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{\tilde{T}}\|_{L_{\tilde{\tilde{T}}\omega_u}^\infty L^2(\tilde{\mathcal{H}}_{\omega_u})}^2.
\]

Plugging (4.25) into (4.24) finishes the proof of (4.23). Plugging (4.22) and (4.23) into (4.21) then leads to

\[
\|R\|_{L^2(\tilde{\Sigma}_t)}^2 \lesssim \int_{\Sigma_1} Q(R)_{\tilde{\tilde{T}}\tilde{\tilde{T}}T} + \int_{\tilde{\mathcal{H}}} Q(R)_{\tilde{\tilde{T}}\tilde{\tilde{T}}}L
\]

\[
+ C_{\text{ball}} \|R\|_{L^\infty_t L^2(\tilde{\Sigma}_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{\tilde{T}}\|_{L_{\tilde{\tilde{T}}\omega_u}^\infty L^2(\tilde{\mathcal{H}}_{\omega_u})}^2.
\]

Taking the supremum over \( \tilde{\tau} \leq t^* \) of (4.26), we get that

\[
\|R\|_{L^\infty_t L^2(\tilde{\Sigma}_t)}^2 \lesssim \int_{\Sigma_1} Q(R)_{\tilde{\tilde{T}}\tilde{\tilde{T}}T} + \int_{\tilde{\mathcal{H}}} Q(R)_{\tilde{\tilde{T}}\tilde{\tilde{T}}}L
\]

\[
+ C_{\text{ball}} \|R\|_{L^\infty_t L^2(\tilde{\Sigma}_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{\tilde{T}}\|_{L_{\tilde{\tilde{T}}\omega_u}^\infty L^2(\tilde{\mathcal{H}}_{\omega_u})}^2.
\]

This finishes the Bel-Robinson estimate with \( \tilde{T} \) as multiplier fields over the spacetime region \( \mathcal{M}_t \).
Fig. 3 The spacetime region $\mathcal{M}_{\omega_u}$

Next, we apply a Bel-Robinson estimate with $\tilde{T}$ as multiplier fields over the spacetime region $\mathcal{M}_{\omega_u}$, defined to be the past of $\mathcal{H}_{\omega_u}$ in $\mathcal{M}_{t^*}$, for $\omega \in \mathbb{S}^2$ and $\omega_u \in \mathbb{R}$, see Fig. 3.

By Proposition 2.2 and the same argument as above, this leads to the estimate

$$
\|R \cdot \tilde{L}\|_{L^2(\mathcal{H}_{\omega_u})} \lesssim \int_{\Sigma_1} Q(R)_{\tilde{T} \tilde{T} \tilde{T} \tilde{T}} + \int_{\mathcal{H}} Q(R)_{\tilde{T} \tilde{T} \tilde{T} \tilde{T} L} + C_{\text{ball}} \|R\|_{L^2_{\tilde{t}} L^2(\tilde{\Sigma}_t)} + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{L}\|_{L^2_{\omega_u} L^2(\mathcal{H}_{\omega_u})}.
$$

Taking the supremum of (4.28) over $\omega_u \in \mathbb{R}$ and $\omega \in \mathbb{S}^2$, we get that

$$
\sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{L}\|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})} \lesssim \int_{\tilde{\Sigma}_t} Q(R)_{\tilde{T} \tilde{T} \tilde{T} \tilde{T}} + \int_{\mathcal{H}} Q(R)_{\tilde{T} \tilde{T} \tilde{T} \tilde{T} L} + C_{\text{ball}} \|R\|_{L^\infty_{\tilde{t}} L^2(\tilde{\Sigma}_t)} + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{L}\|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})}.
$$

Combining (4.27) and (4.29), we get that

$$
\|R\|_{L^\infty_{\tilde{t}} L^2(\tilde{\Sigma}_t)} + \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{L}\|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})} \lesssim \int_{\tilde{\Sigma}_t} Q(R)_{\tilde{T} \tilde{T} \tilde{T} \tilde{T}} + \int_{\mathcal{H}} Q(R)_{\tilde{T} \tilde{T} \tilde{T} \tilde{T} L} + C_{\text{ball}} \|R\|_{L^\infty_{\tilde{t}} L^2(\tilde{\Sigma}_t)} + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|R \cdot \tilde{L}\|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})}.
$$

$\square$ Springer
For \( C_{\text{ball}} > 0 \) sufficiently small, we can absorb the third and fourth term on the right-hand side of (4.30) into the left, and get that

\[
\| R \|_{L^\infty_t L^2(\Sigma_1)}^2 + \sup_{\omega \in S^2} \| R \cdot \tilde{\omega} \|_{L^\infty_t L^2_\omega (\mathcal{H}_{\omega u})}^2 \lesssim \int_{\Sigma_1} Q(R) \tilde{\omega} \tilde{\omega} \tilde{\omega} \tilde{\omega} \cdot \mathcal{H} \cdot \tilde{\omega} \tilde{\omega} \tilde{\omega} \tilde{\omega} \cdot \mathcal{H} := I_1
\]

\[
\lesssim \int_{\Sigma_1} Q_{TTTT} + \int_{\Sigma_1} Q(R) \tilde{\omega} \tilde{\omega} \tilde{\omega} \tilde{\omega} \cdot \mathcal{H} := I_2
\]

(4.31)

It remains to control the boundary integrals \( I_1 \) and \( I_2 \) on the right-hand side of (4.31). In the following we simply denote \( Q(R) \) by \( Q \).

**Control of \( I_1 \).** Let \((e_i)_{i=1,2,3}\) be an orthonormal frame of \( \Sigma_1 \). Decompose \( \tilde{T} \) with respect to this frame into

\[
\tilde{T} = \tilde{v} T + C^1 e_1 + C^2 e_2 + C^3 e_3,
\]

(4.32)

and denote \( C^0 := \tilde{v} - 1 \) and \( e_0 := T \). By (4.17), we have that

\[
\| C^0 \|_{L^\infty(\Sigma_1)} \lesssim C_{\text{ball}}.
\]

(4.33)

Using that \( T \) and \( \tilde{T} \) are unit timelike vectors, that \((e_1, e_2, e_3)\) is an orthonormal frame tangent to \( \Sigma_1 \) and (4.33), we have that

\[
-1 = g(\tilde{T}, \tilde{T}) = -\tilde{v}^2 + \sum_{i=1}^3 |C^i|^2,
\]

so that on \( \Sigma_1 \),

\[
|C^i| \leq \sqrt{\tilde{v}^2 - 1} = \sqrt{C^0 \tilde{v} + 1} \lesssim \sqrt{C_{\text{ball}}} \quad \text{for } i = 1, 2, 3.
\]

(4.34)

Using (2.11), (2.12), (4.1), (4.32), (4.33) and (4.34), we get that

\[
I_1 := \int_{\Sigma_1} Q_{TTTT} + \int_{\Sigma_1} \sum_{\mu} C^\mu Q_{\mu TTT} + \int_{\Sigma_1} \sum_{\nu} C^\nu C^\nu Q_{\mu \nu TT} + \int_{\Sigma_1} \sum_{\mu, \nu, \lambda} C^\mu C^\nu C^\lambda Q_{\mu \nu \lambda T}
\]

\[
\lesssim \left( 1 + \sqrt{C_{\text{ball}}} + \sqrt{C_{\text{ball}}^2} + \sqrt{C_{\text{ball}}^3} \right) \int_{\Sigma_1} Q_{TTTT}
\]

\[
\lesssim \| R \|_{L^2(\Sigma_1)}^2
\]

\[
\lesssim \varepsilon^2.
\]

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Control of $I_2$. Let $(N, e_1, e_2)$ be a local frame on $\mathcal{H}$ such that $(e_1, e_2)$ is an orthonormal frame tangent to $\partial \Sigma_t$ and $N$ is tangent to $\Sigma_t$ and normal to $\partial \Sigma_t$. Decompose $\tilde{T}$ into

$$\tilde{T} = \tilde{v}T + C^1 e_1 + C^2 e_2 + C^3 N,$$

and denote $C^0 := \tilde{v} - 1$ and $e_0 = T$. Using (4.17), we have that

$$\|C^0\|_{L^\infty(\Sigma_t)} \lesssim C_{\text{ball}}. \tag{4.36}$$

Using that $T$ and $\tilde{T}$ are timelike unit, that $(N, e_1, e_2)$ is an orthonormal frame and (4.36), it follows that

$$\|C^i\|_{L^\infty(\mathcal{H})} \lesssim \sqrt{C_{\text{ball}}} \quad \text{for } i = 1, 2, 3. \tag{4.37}$$

By Lemma 2.11, (2.26), (4.1), (4.17), (4.35), (4.36) and (4.37), we get that for $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small,

$$I_2 := \int_{\mathcal{H}} Q_{TTTL}$$

$$= \int_{\mathcal{H}} QTTTL + \int_{\mathcal{H}} C^{i\mu} Q_{\mu TTL} + \int_{\mathcal{H}} C^{i\mu} C^{i\nu} Q_{\mu \nu TL} + \int_{\mathcal{H}} C^{i\mu} C^{i\nu} C^{i\lambda} Q_{\mu \nu \lambda L}$$

$$\lesssim \left(1 + \sqrt{C_{\text{ball}}} + \sqrt{C_{\text{ball}}}^2 + \sqrt{C_{\text{ball}}}^3\right) \int_{\mathcal{H}} QTTTL$$

$$\lesssim \|\tilde{v}\|_{L^3(\mathcal{H})}^3 \|\alpha\|_{L^2(\mathcal{H})}^2 + \|\tilde{v}\|_{L^\infty(\mathcal{H})} \|\beta\|_{L^2(\mathcal{H})}^2$$

$$+ \|\tilde{v}^{-1}\|_{L^\infty(\mathcal{H})} \left(\|\rho\|_{L^2(\mathcal{H})}^2 + \|\sigma\|_{L^2(\mathcal{H})}^2\right) + \|\tilde{v}^{-3}\|_{L^\infty(\mathcal{H})} \|\beta\|_{L^2(\mathcal{H})}^2$$

$$\lesssim \|\alpha\|_{L^2(\mathcal{H})}^2 + \|\beta\|_{L^2(\mathcal{H})}^2 + \|\rho\|_{L^2(\mathcal{H})}^2 + \|\sigma\|_{L^2(\mathcal{H})}^2 + \|\beta\|_{L^2(\mathcal{H})}^2 \lesssim \varepsilon^2.$$

To summarise, plugging the above control of $I_1$ and $I_2$ into (4.31), we get that for $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small,

$$\|R\|_{L^\infty_i L^2(\Sigma_t)}^2 + \sup_{\omega \in S^2} \|R \cdot \tilde{L}\|_{L^\infty_{i\omega} L^2(\mathcal{H}^i_{\omega^i})} \lesssim \varepsilon^2.$$

This finishes the proof of (4.20), and thus of (4.18) and subsequently (4.19).

### 4.7 Elliptic Estimates for the Second Fundamental form $k$ on $\Sigma_t$

In this section we prove the following proposition to improve the bootstrap assumption (4.2) for $k$. 

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Proposition 4.7 (Global elliptic estimate for \( k \)) It holds that
\[
\| \nabla k \|^2_{L^2(\Sigma_{t^*})} + \| k \|^2_{L^2(\Sigma_{t^*})} + \| k \|^4_{L^4(\Sigma_{t^*})} + \| \nabla v \|^2_{L^2(\partial \Sigma_{t^*})} \lesssim (\sqrt{D}\epsilon)^2.
\]

**Proof** Using (2.13b), (2.13c), (2.13d) and Corollary A.3, we have the following well-known classical global elliptic estimate for \( k \),
\[
\int_{\Sigma_{t^*}} |\nabla k|^2 + \frac{1}{4} |k|^4 - \int_{\partial \Sigma_{t^*}} \nabla_a k_b N k^b a \lesssim \int_{\Sigma_{t^*}} |R|^2_{h'},
\]
(4.38)
where \( N \) denotes the outward-pointing unit normal to \( \partial \Sigma_{t^*} \subset \Sigma_{t^*} \).

By \( \text{div} \, k = 0 \), see (2.13b), the boundary term on the left-hand side of (4.38) can be rewritten as
\[
- \int_{\partial \Sigma_{t^*}} \nabla_a k_b N k^b a = - \int_{\partial \Sigma_{t^*}} \left( \nabla_N k_b N k^b N + \nabla_C k_b N k^b C \right)
= - \int_{\partial \Sigma_{t^*}} \left( - \nabla_C k_b C k^b N + \nabla_C k_N N k^N C + \nabla_C k_{A N} k^{A C} \right)
= \int_{\partial \Sigma_{t^*}} \nabla_C k_{A C} k^A N + \nabla_C k_{C} k^N N - \nabla_C k_{N N} k^N C - \nabla_C k_{A N} k^{A C},
\]
(4.39)
where summation over \( A, C = 1, 2 \) indicates taking the trace with respect to a local orthonormal frame \((e_1, e_2)\) on \( \partial \Sigma_{t^*} \). Using the boundary decomposition of Section 2.4 for \( \nabla k \) on \( \partial \Sigma_{t^*} \), that is,
\[
\nabla_C k_{A C} = \text{div} \delta \eta_A + \Theta_{A B} \epsilon^B + \text{tr} \Theta \epsilon_A, \quad \nabla_C k_{N C} = \text{div} \epsilon - \eta \cdot \Theta + \delta \text{tr} \Theta,
\n\nabla_C k_{N N} = \nabla_C \delta - 2 \Theta_{C B} \epsilon^B, \quad \nabla_C k_{A N} = \nabla_C \epsilon_A - \eta_{A B} \Theta_{B C} + \delta \Theta_{A C},
\]
where \( \delta = k_{N N}, \epsilon_A = k_{N A} \) and \( \eta_{A B} = k_{A B} \), and integrating by parts, we get that (4.39) becomes
\[
- \int_{\partial \Sigma_{t^*}} \nabla_a k_b N k^b a = \int_{\partial \Sigma_{t^*}} 2(\text{div} \eta) \epsilon_A e^A - 2 \epsilon \cdot \nabla \delta + 3 \Theta_{A B} \epsilon^A e^B + \text{tr} \Theta |\epsilon|^2
+ \int_{\partial \Sigma_{t^*}} \delta^2 \text{tr} \Theta - 2 \delta \eta_{A B} \Theta^{A B} + \eta_{A B} \Theta_{B C} \eta_C^A
= \int_{\partial \Sigma_{t^*}} -4 \epsilon \cdot \nabla \delta - 2 H_N^A \epsilon^A + 5 \Theta_{A B} \epsilon^A e^B - \text{tr} \Theta |\epsilon|^2.
\[ + \int_{\partial \Sigma^*} \delta^2 \text{tr} \Theta - 2 \delta \eta_{AB} \Theta^A_B + \eta_{AB} \Theta^B_C \eta^A_C, \quad (4.40) \]

where we used the following Gauss-Codazzi equation in the last equality, see (2.18),

\[ \text{div} \eta_A = - \nabla_A \delta - *H^A_N + (\Theta \cdot \epsilon)_A - \text{tr} \Theta \epsilon_A. \]

We can re-arrange (4.40) as

\[ - \int_{\partial \Sigma^*} \nabla_a k_{bN} k^{ba} = - \int_{\partial \Sigma^*} 4 \epsilon \cdot \nabla \delta + 2 \epsilon^{AB} H_{BN} \epsilon_A \]

\[ + \int_{\partial \Sigma^*} \left( \text{tr} \Theta - \frac{2}{t^*} \right) \left( \frac{3}{2} |\epsilon|^2 + 2 |\delta|^2 + \frac{1}{2} |\eta|^2 \right) \]

\[ + \int_{\partial \Sigma^*} 5 \Theta_{AB} \epsilon^A \epsilon^B - 2 \delta \Theta_{AB} \eta^{AB} + \eta_{AC} \Theta^C_B \eta^A_B \]

\[ + \int_{\partial \Sigma^*} \frac{3}{t^*} |\epsilon|^2 + \frac{4}{t^*} |\delta|^2 + \frac{1}{t^*} |\eta|^2, \quad (4.41) \]

where we used that by definition, see (2.19) and (2.20),

\[ *H^A_N \epsilon^A = \epsilon^{AB} H_{BN} \epsilon_A. \]

Since \( \partial \Sigma^* \subset \mathcal{H} \), we can use Lemmas 2.7 and 2.9, that is, the relations

\[ \epsilon_A = - \nu^{-1} \nabla_A \nu + \zeta_A, \]

\[ \delta = \frac{1}{2} \nu \text{tr} \chi + \frac{1}{2} \nu^{-1} \text{tr} \chi, \]

\[ \nabla \delta = \nabla \left( \frac{1}{2} \nu \text{tr} \chi + \frac{1}{2} \nu^{-1} \text{tr} \chi \right) \]

\[ = \left( \frac{1}{t^*} \left( 1 + \frac{1}{\nu^2} \right) + \frac{1}{2} \left( \nu \text{tr} \chi - \frac{2}{t^*} \right) - \frac{1}{2} \frac{1}{\nu^2} \left( \nu \text{tr} \chi + \frac{2}{t^*} \right) \right) \nabla \nu \]

\[ = : F(\nu, \text{tr} \chi, \text{tr} \chi) \]

\[ + \frac{1}{2} \nu \nabla \text{tr} \chi + \frac{1}{2} \nu^{-1} \nabla \text{tr} \chi, \quad (4.42) \]
to rewrite the first term on the right-hand side of (4.41) as follows,

$$- \int_{\partial \Sigma_*} 4 \epsilon \cdot \nabla \delta = - \int_{\partial \Sigma_*} 4 \zeta \cdot \nabla \delta + \int_{\partial \Sigma_*} 4 \nu^{-1} \nabla \nu \cdot \nabla \delta$$

$$= - \int_{\partial \Sigma_*} 4 \zeta \cdot \nabla \delta + \int_{\partial \Sigma_*} 4 \nu^{-1} F(\nu, \text{tr} \chi, \text{tr} \chi) |\nabla \nu|^2$$

$$+ \int_{\partial \Sigma_*} 2 \left( \nabla \nu \cdot \nabla \text{tr} \chi + \nu^{-2} \nabla \nu \cdot \nabla \text{tr} \chi \right).$$

(4.43)

By (4.1) and (4.2), it holds for $\epsilon > 0$ sufficiently small that on $\partial \Sigma_*$,

$$\nu^{-1} F(\nu, \text{tr} \chi, \text{tr} \chi) \geq \frac{1}{8}.$$

Hence, for $\epsilon > 0$ sufficiently small, (4.43) yields

$$- \int_{\partial \Sigma_*} 4 \epsilon \cdot \nabla \delta \geq \int_{\partial \Sigma_*} \frac{1}{2} |\nabla \nu|^2 + \int_{\partial \Sigma_*} 2 \left( \nabla \nu \cdot \nabla \text{tr} \chi + \nu^{-2} \nabla \nu \cdot \nabla \text{tr} \chi \right) - \int_{\partial \Sigma_*} 4 \zeta \cdot \nabla \delta.$$

(4.44)

Plugging (4.41) and (4.44) into (4.38), we get that for $\epsilon > 0$ sufficiently small,

$$\int_{\Sigma_*} |\nabla k|^2 + \frac{1}{4} |k|^4 + \int_{\partial \Sigma_*} |k|^2 + \int_{\partial \Sigma_*} |\nabla \nu|^2$$

$$\subseteq \int_{\Sigma_*} \bar{R}_{\text{h}t}^2 + \int_{\partial \Sigma_*} \varepsilon^{AB} H_{AN} \epsilon_B + \int_{\partial \Sigma_*} \left( \text{tr} \Theta - \frac{2}{t^*} \right) |k|^2$$

$$+ \int_{\partial \Sigma_*} \hat{\Theta}_{AB} \epsilon^A \epsilon^B + \int_{\partial \Sigma_*} \delta \hat{\Theta}_{AB} \eta^{AB} + \int_{\partial \Sigma_*} \eta_{AC} \hat{\Theta}^{CB} \eta^A_B$$

$$+ \int_{\partial \Sigma_*} \zeta \cdot \nabla \delta + \int_{\partial \Sigma_*} \nabla \nu \cdot \nabla \text{tr} \chi + \int_{\partial \Sigma_*} \nu^{-2} \nabla \nu \nabla \text{tr} \chi.$$

(4.45)

In the following, we control the terms $I_1$-$I_9$ on the right-hand side of (4.45).
Control of $I_1$. From (4.19), we have

$$I_1 := \int_{\Sigma_t^{*}} |\mathbf{R}|^2_{h^{*}} \lesssim \varepsilon^2. \quad (4.46a)$$

Control of $I_2$. In the following we use, based on the control (4.8), the spherical coordinates $(r, \theta^1, \theta^2)$ on $\overline{B}_{t^{*}} = \phi^{-1}(\Sigma_t^{*})$ as defined in Section 2.4. Let $\psi : \overline{B}_{t^{*}} \to [0, 1]$ be a smooth radial cut-off function such that $\psi(x) = 1$ for $|x| \geq 3t^{*}/4$ and $\psi(x) = 0$ for $|x| \leq t^{*}/2$, where $|\cdot|$ denotes the Euclidean norm on $\overline{B}_{t^{*}}$. Then by the fundamental theorem of calculus, (4.2), Lemma 3.7 and the fact that due to (2.11) and (4.19),

$$\|E\|_{L^2(\Sigma_t^{*})} + \|H\|_{L^2(\Sigma_t^{*})} \lesssim \|\mathbf{R}\|_{L^2(\Sigma_t^{*})} \lesssim \varepsilon, \quad (4.46b)$$

we have that

$$\int_{\partial \Sigma_t^{*}} \varepsilon^{AB} H_{AN} \epsilon_B^{B} = \int_{t^{*}/2}^{t^{*}} \partial_r \left( \int_{\Sigma_r} \varepsilon^{AB} H_{AN} \epsilon_B^{B} \right) dr$$

$$\lesssim \int_{t^{*}/2}^{t^{*}} \psi \partial_r \left( \int_{\Sigma_r} \varepsilon^{AB} H_{AN} \epsilon_B^{B} \right) dr + \|H\|_{L^2(\Sigma_t^{*})} \|k\|_{L^2(\Sigma_t^{*})}$$

$$\lesssim \int_{t^{*}/2}^{t^{*}} \psi \left( \partial_r \left( \varepsilon^{AB} H_{AN} \epsilon_B^{B} \right) + \varepsilon^{AB} H_{AN} \epsilon_B^{B} (a \text{ tr} \Theta - \text{ div}b) \right) dr$$

$$+ \varepsilon(D\varepsilon)$$

$$\lesssim \int_{t^{*}/2}^{t^{*}} \psi \partial_r \left( \varepsilon^{AB} H_{AN} \epsilon_B^{B} \right) dr + \varepsilon(D\varepsilon); \quad (4.46c)$$

where we used (2.17). Using that $\partial_r = a N + b$ and $\varepsilon^{AB} = \varepsilon^{ABN}$, we can express the integrand on the right-hand side of (4.46c) as

$$\partial_r \left( \varepsilon^{AB} H_{AN} \epsilon_B^{B} \right) = \partial_r \left( \varepsilon^{ijN} H_{iNKJN} \right)$$

$$= a \nabla_N (\varepsilon^{ijN} H_{iNKJN}) + b \left( \varepsilon^{AB} H_{AN} \epsilon_B^{B} \right)$$

$$= a \nabla_N (\varepsilon^{ijN} H_{iNKJN}) + b^{\text{c}} \nabla_c \left( \varepsilon^{AB} H_{AN} \epsilon_B^{B} \right). \quad (4.46d)$$

Further, using that $\nabla_N N = -a^{-1} \nabla a$ and (2.14), that is,

$$\nabla_N H_{AN} = -\text{ div} H_A - \Theta_{AC} H_N^C - \text{ tr} \Theta H_{AN} - (k \wedge E)_A,$$
where $\mathcal{H}$ denotes the projection of $H$ onto $S_r$, the first term on the right-hand side of (4.46d) can be expressed as follows,

$$a \nabla_N (\epsilon^{iN} H_{iN} k_{jN}) = a \epsilon^{ijl} (-a^{-1} \nabla_j a) H_{iN} k_{jN} + a \epsilon^{ijN}$$

$$= a \epsilon^{ijl} (-a^{-1} \nabla_j a) H_{iN} k_{jN} + a \epsilon^{ijN}$$

$$\left( \nabla_N H_{iN} - a^{-1} \nabla^C a H_{iC}) k_{jN} + a \epsilon^{ijN} H_{iN} (\nabla_N k_{jN} - a^{-1} \nabla^C a k_{jC}) \right)$$

$$= a \epsilon^{ijl} (-a^{-1} \nabla_j a) H_{iN} k_{jN} + a \epsilon^{ijN}$$

$$\left( \nabla_N H_{AN} - a^{-1} \nabla^C a H_{AC}) \epsilon_B \right)$$

$$+ a \epsilon^{AB} H_{AN} (\nabla_N k_{BN} - a^{-1} \nabla^C a k_{BC})$$

$$= a \epsilon^{ijl} (-a^{-1} \nabla_j a) H_{iN} k_{jN}$$

$$+ a \epsilon^{AB} (-\text{div} H - \Theta_{AC} H_{N}^C - \text{tr} \Theta H_{NA}$$

$$- (k \wedge E)_A - a^{-1} \nabla^C a H_{AC}) \epsilon_B$$

$$+ a \epsilon^{AB} H_{AN} (\nabla_N k_{BN} - a^{-1} \nabla^C a k_{BC})$$.

Plugging (4.46d) and the above into (4.46c) and integrating by parts on $S_r$ the terms $-a \epsilon^{AB} \text{div} \mathcal{H}_A \epsilon_B$ and $b^c \nabla_c (\epsilon^{AB} H_{AN} \epsilon_B)$, we get that

$$\int_{\Sigma_t}^{t^*} \epsilon^{AB} H_{AN} \epsilon_B \lesssim \int_{t^*/2}^{t^*} \psi \left( \int_{S_r} \epsilon^{ijl} (-a^{-1} \nabla_j a) H_{iN} k_{jN} - \text{div} b (\epsilon^{AB} H_{AN} \epsilon_B) \right) dr$$

$$+ \int_{t^*/2}^{t^*} \psi \left( \int_{S_r} \epsilon^{AB} \nabla^C (a \epsilon_B) H_{AC} \right) dr$$

$$+ \int_{t^*/2}^{t^*} \psi \left( \int_{S_r} a \epsilon^{AB} (-\Theta_{AC} H_{N}^C - \text{tr} \Theta H_{NA}$$

$$- (k \wedge E)_A - a^{-1} \nabla^C a H_{AC}) \epsilon_B \right) dr$$

$$+ \int_{t^*/2}^{t^*} \psi \left( \int_{S_r} a \epsilon^{AB} H_{AN} (\nabla_N k_{BN} - a^{-1} \nabla^C a k_{BC}) \right) dr + \varepsilon D\varepsilon.$$
By Lemmas 3.4 and 3.7, (4.2), (4.8) and (4.46b), the right-hand side can be estimated as

\[ I_2 := \left| \int_{\partial \Sigma^*_{t=}} \Phi_{AB} \epsilon B \right| \lesssim (1 + C_{\text{ball}}) (\|H\|_{L^2(\Sigma^*)} + \|E\|_{L^2(\Sigma^*)}) \, D\epsilon \]

\[ \lesssim (1 + C_{\text{ball}}) \epsilon \, D\epsilon \]

\[ \lesssim (\sqrt{D\epsilon})^2. \quad (4.46e) \]

**Control of \( I_3 \).** Using (4.2) and (4.6), we have

\[ I_3 := \left| \int_{\partial \Sigma^*_{t=}} \left( \frac{\text{tr} \Theta - \frac{2}{t^*}}{t^*} \right) |k|^2 \right| \leq \left\| \text{tr} \Theta - \frac{2}{t^*} \right\|_{L^\infty(\partial \Sigma^*)} \|k\|^2_{L^2(\partial \Sigma^*)} \]

\[ \lesssim (D\epsilon)^3. \quad (4.46f) \]

**Control of \( I_4, I_5 \) and \( I_6 \).** By Lemma 3.5 and (4.7), we have

\[ I_4 := \left| \int_{\partial \Sigma^*_{t=}} \mathcal{G}_{AB} \epsilon B \right| \lesssim \left\| \mathcal{G} \right\|_{L^4(\partial \Sigma^*)} \left( \left\| \epsilon \right\|_{L^2(\partial \Sigma^*)} + \left\| \nabla \epsilon \right\|_{L^2(\Sigma^*)} \right)^2 \]

\[ \lesssim \epsilon (D\epsilon)^2. \quad (4.46g) \]

The terms \( I_5 \) and \( I_6 \) are bounded similarly as

\[ I_5 + I_6 \lesssim \epsilon (D\epsilon)^2. \quad (4.46h) \]

**Control of \( I_7, I_8 \) and \( I_9 \).** By (4.1), (4.2) and (4.42), we have

\[ I_7 := \left| \int_{\partial \Sigma^*_{t=}} \xi \cdot \nabla \delta \right| \]

\[ \lesssim \left( 1 + \left\| \nu - 1 \right\|_{L^\infty(\partial \Sigma^*)} + \left\| \text{tr} \chi \right\|_{L^\infty(\partial \Sigma^*)} \right) \left( \left\| \nabla \text{tr} \chi \right\|_{L^\infty(\partial \Sigma^*)} + \left\| \nabla \text{tr} \chi \right\|_{L^\infty(\partial \Sigma^*)} \right) \]
\[
\cdot \| \xi \|_{L^2(\partial \Sigma_\ast)} \left( \| \nabla v \|_{L^2(\partial \Sigma_\ast)} + \| \nabla \text{tr} \chi \|_{L^2(\partial \Sigma_\ast)} + \| \nabla \text{tr} \chi \|_{L^2(\partial \Sigma_\ast)} \right) \\
\lesssim \varepsilon (D \varepsilon) + \varepsilon^2.
\] (4.46i)

The terms \( I_8 \) and \( I_9 \) are bounded similarly by (4.1) and (4.2),
\[
I_8 + I_9 \lesssim (\sqrt{D} \varepsilon)^2.
\] (4.46j)

Plugging (4.46a)-(4.46j) into (4.45), we get that for \( C_{\text{ball}} > 0 \) and \( \varepsilon > 0 \) sufficiently small,
\[
\int_{\Sigma_\ast} |\nabla k|^2 + \int_{\Sigma_\ast} |k|^4 + \int_{\partial \Sigma_\ast} |k|^2 + \int_{\partial \Sigma_\ast} |\nabla v|^2 \lesssim (\sqrt{D} \varepsilon)^2.
\]

In particular, this implies by Lemma 3.6 that
\[
\| k \|_{L^2(\Sigma_\ast)} \lesssim \| \nabla k \|_{L^2(\Sigma_\ast)} + \| k \|_{L^2(\partial \Sigma_\ast)} \\
\lesssim \sqrt{D} \varepsilon.
\]

Summarising the above, we have that
\[
\int_{\Sigma_\ast} \left( |\nabla k|^2 + |k|^2 + |k|^4 \right) + \int_{\partial \Sigma_\ast} \left( |k|^2 + |\nabla v|^2 \right) \lesssim (\sqrt{D} \varepsilon)^2.
\]

This finishes the proof of Proposition 4.7. \( \square \)

The estimate (4.19) and Proposition 4.7 imply by (2.11) and (2.13f) the improvement of \( \text{Ric} \),
\[
\| \text{Ric} \|_{L^2(\Sigma_\ast)} \lesssim \varepsilon.
\]

Moreover, Proposition 4.7 and Lemma 3.5 yield the following corollary.

**Corollary 4.8** It holds that
\[
\| k \|_{H^{1/2}(\partial \Sigma_\ast)} + \| k \|_{L^4(\partial \Sigma_\ast)} \lesssim \sqrt{D} \varepsilon.
\]

**4.8 Estimates for the Slope \( v \) on \( \partial \Sigma_\ast \)**

We first prove the next lemma.

**Lemma 4.9** It holds that
\[
\| v - 1 \|_{L^2(\partial \Sigma_\ast)} \lesssim \sqrt{D} \varepsilon.
\]
\textbf{Proof} Recall from Lemma 2.7 that on $\partial \Sigma_{t^*}$,
\[
\delta = \frac{1}{2} \nu \text{tr} \chi + \frac{1}{2} \nu^{-1} \text{tr} \chi.
\]
This can be rearranged as
\[
\delta = \frac{1}{2} \nu \left( \text{tr} \chi - \frac{2}{t^*} \right) + \frac{1}{2} \nu^{-1} \left( \text{tr} \chi + \frac{2}{t^*} \right) + \frac{1}{\nu t^*} (\nu + 1)(\nu - 1),
\]
which leads to
\[
v - 1 = \frac{\nu t^*}{\nu + 1} \left( \delta - \frac{1}{2} \nu \left( \text{tr} \chi - \frac{2}{t^*} \right) - \frac{1}{2} \nu^{-1} \left( \text{tr} \chi + \frac{2}{t^*} \right) \right).
\]
Consequently, using (4.1) and Proposition 4.7, we can estimate for $\varepsilon > 0$ sufficiently small,
\[
\|v - 1\|_{L^2(\partial \Sigma_{t^*})} \lesssim (1 + D\varepsilon) \left( \|\delta\|_{L^2(\partial \Sigma_{t^*})} + \left\| \frac{\text{tr} \chi - 2}{t^*} \right\|_{L^2(\partial \Sigma_{t^*})} + \left\| \frac{\text{tr} \chi + 2}{t^*} \right\|_{L^2(\partial \Sigma_{t^*})} \right)
\lesssim \sqrt{D\varepsilon}.
\]
This finishes the proof of Lemma 4.9. \hfill \Box

Moreover, we have the following.

**Lemma 4.10** It holds that
\[
\|\nabla v\|_{L^4(\partial \Sigma_{t^*})} + \|v - 1\|_{L^\infty(\partial \Sigma_{t^*})} + \|\nabla v\|_{H^{1/2}(\partial \Sigma_{t^*})} \lesssim \sqrt{D\varepsilon}.
\]

\textbf{Proof} Indeed, by Lemma 2.9, (4.1) and Corollary 4.8, we have
\[
\|\nabla v\|_{L^4(\partial \Sigma_{t^*})} \lesssim \|\epsilon\|_{L^4(\partial \Sigma_{t^*})} + \|\xi\|_{L^4(\partial \Sigma_{t^*})} \lesssim \sqrt{D\varepsilon} + \varepsilon.
\]
Consequently, by Lemmas 3.1 and 4.9, we have
\[
\|v - 1\|_{L^\infty(\partial \Sigma_{t^*})} \lesssim \|\nabla(v - 1)\|_{L^4(\partial \Sigma_{t^*})} + \|v - 1\|_{L^2(\partial \Sigma_{t^*})} \lesssim \sqrt{D\varepsilon}.
\]

By the above and Lemmas 2.9, 3.3 and 3.5, (4.1) and Proposition 4.7,
\[
\|\nabla v\|_{H^{1/2}(\partial \Sigma_{t^*})} = \|v \left( v^{-1} \nabla v \right)\|_{H^{1/2}(\partial \Sigma_{t^*})} \lesssim \left( \|v\|_{L^\infty(\partial \Sigma_{t^*})} + \|\nabla v\|_{L^2(\partial \Sigma_{t^*})} \right) \left( \|\xi\|_{H^{1/2}(\partial \Sigma_{t^*})} + \|\epsilon\|_{H^{1/2}(\partial \Sigma_{t^*})} \right).
\]
\[
\lesssim \left( 1 + \|v - 1\|_{L^\infty(\partial \Sigma^*)} + \|\nabla v\|_{L^2(\partial \Sigma^*)} \right) \\
\left( \|\zeta\|_{H^{1/2}(\partial \Sigma^*)} + \|\epsilon\|_{L^2(\Sigma^*)} + \|\nabla \epsilon\|_{L^2(\Sigma^*)} \right) \\
\lesssim \sqrt{D\epsilon}.
\]

This finishes the proof of Lemma 4.10. \hfill \Box

This finishes the improvement of the bootstrap assumptions (4.2).

At this point we can reapply the estimates of Section 4.4 for \( n \) to get
\[
\|n - 1\|_{L^\infty(\Sigma^*)} + \|\nabla n\|_{L^2(\Sigma^*)} + \|\nabla^2 n\|_{L^2(\Sigma^*)} \lesssim \sqrt{D\epsilon}. \tag{4.47}
\]

As a consequence of the above, we can prove the following additional bound.

**Lemma 4.11** It holds that
\[
\|D_T k\|_{L^2(\Sigma^*)} \lesssim \sqrt{D\epsilon}.
\]

**Proof** Indeed, by the second variation equation (2.13a), that is,
\[
D_T k_{ij} = E_{ij} - n^{-1}\nabla_i n \nabla_j n - k_{ij}k^l_{lj},
\]
we get that for \( \epsilon > 0 \) sufficiently small,
\[
\|D_T k\|_{L^2(\Sigma^*)} \lesssim \|E\|_{L^2(\Sigma^*)} + \|\nabla^2 n\|_{L^2(\Sigma^*)} + \left( \|\nabla k\|_{L^2(\Sigma^*)} + \|k\|_{L^2(\Sigma^*)} \right)^2 \]
\[
\lesssim \sqrt{D\epsilon} + (\sqrt{D\epsilon})^2.
\]

This finishes the proof of Lemma 4.11. \hfill \Box

### 5 Existence of Global Coordinates by Cheeger–Gromov Theory

In this section we prove Theorem 4.1 by applying the Cheeger-Gromov theory developed in [13]. Theorem 4.1 is a low-regularity curvature pinching result and its proof is based, like other curvature pinching results (see for example [28]), on a convergence result and a rigidity result, see Lemmas 5.3 and 5.4 below, respectively.

In the following, we first introduce the necessary definitions and prerequisite results before turning to the proof of Theorem 4.1.

**Notation** We denote the diffeomorphism equivalence and the isometry of manifolds by \( \simeq \) and \( \cong \), respectively. Moreover, given a scalar function \( f \) on a subset \( U \) of Euclidean 3-space, let the (coordinate-based) \( H^2 \)-norm of \( f \) over \( U \) be defined as
\[
\|f\|_{H^2(U)} := \sum_{0 \leq m \leq 2} \|\partial^m f\|_{L^2(U)},
\]
where $\partial^m$ denotes a standard tuple of $m$ (Cartesian) coordinate derivatives in $U \subset \mathbb{R}^3$.

**Definition 5.1** ($H^2$-convergence of functions and tensors)

Let $(M, g)$ be a compact Riemannian 3-manifold with boundary. Let $(\varphi_i)$ be a finite number of fixed charts covering $M$. A sequence of functions $(f_n)_{n \in \mathbb{N}}$ on $M$ is said to converge in $H^2$ as $n \to \infty$, if for each $\varphi_i$, the pullbacks $(\varphi_i)^* f_n$ converge in $H^2$ as $n \to \infty$. The convergence of a sequence of tensors on $M$ in $H^2$ is defined similarly.

**Definition 5.2** ($H^2$-convergence of manifolds with boundary) A sequence $(M_n, g_n)$ of compact Riemannian 3-manifolds with boundary is said to converge to a Riemannian manifold with boundary $(M, g)$ in the $H^2$-topology as $n \to \infty$, if for large $n$ there exist diffeomorphisms $\Psi_n : M \to M_n$ such that $(\Psi_n)^* g_n \to g$ in the $H^2$-topology on $M$.

The following convergence result is applied in the proof of Theorem 4.1.

**Lemma 5.3** ($H^2$-convergence) Let $(M_n, g_n)$ be a sequence of smooth compact Riemannian 3-manifolds with boundary such that $M_n \simeq \overline{B_1} \subset \mathbb{R}^3$ and for real numbers $1 \leq t \leq 2$ and $0 < V < \infty$,

\[
\|\text{Ric}_n\|_{L^2(M_n)} \to 0 \text{ as } n \to \infty,
\]

\[
\left\|\text{tr} \Theta_n - \frac{2}{t} \right\|_{L^4(\partial M_n)} + \|\widehat{\Theta}_n\|_{L^4(\partial M_n)} \to 0 \text{ as } n \to \infty,
\]

\[
\|\text{vol}(M_n, 1/2) \| \geq 1/4,
\]

\[
\text{vol}_{g_n}(M_n) \leq V.
\]

Then, there is a smooth compact Riemannian 3-manifold $(M, g)$ with $M \simeq \overline{B_1} \subset \mathbb{R}^3$ such that as $n \to \infty$,

\[
(M_n, g_n) \to (M, g) \text{ in the } H^2\text{-topology},
\]

that is, for large $n$ there are global diffeomorphisms $\Psi_n : M \to M_n$ such that, with respect to a fixed chart on $M$, for $i, j = 1, 2, 3$,

\[
\sum_{0 \leq m \leq 2} \|\partial^m \left((\Psi_n^* g_n)_{ij} - g_{ij}\right)\|_{L^2} \to 0 \text{ as } n \to \infty.
\]

**Proof of Lemma 5.3** The $H^2$-convergence (5.2) of the sequence (5.1) follows directly by the low-regularity pre-compactness result established in [13], see Theorem 4.1 and Corollary 3.11 therein, together with $H^2$-regularity elliptic estimates for boundary harmonic coordinates, see Section 4 in [13]. We note that the limit manifold is flat by the first of (5.1). This finishes the proof of Lemma 5.3. □

In the proof of Theorem 4.1, we also use the following rigidity result. A proof is provided in Appendix C.
Lemma 5.4 (Rigidity result) Let \((M, g)\) be a smooth compact Riemannian 3-manifold with boundary such that \(M \cong \overline{B_t} \subset \mathbb{R}^3\) and for a real number \(1 \leq t \leq 2\),

\[
\text{Ric} = 0 \text{ on } M, \quad \text{tr } \Theta = \frac{2}{t}, \quad \hat{\Theta} = 0 \text{ on } \partial M.
\]

Then

\[(M, g) \cong (\overline{B_t}, e).
\]

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1 The proof is by contradiction: Assume that there does not exist an \(\varepsilon_0 > 0\) sufficiently small such that the conclusion of Theorem 4.1 holds. In other words, assume that there exist real numbers

\[0 < C_{\text{ball}} < 1/2, \quad 1 \leq t \leq 2, \quad 0 < V < \infty,
\]

and a sequence \((M_n, g_n)\) of smooth compact Riemannian 3-manifolds with boundary such that \(M_n \cong \overline{B_t} \subset \mathbb{R}^3\), such that the assumptions of Theorem 4.1 are satisfied with \(\varepsilon_n = \frac{1}{n}\),

\[
\|\text{Ric}_n\|_{L^2(M_n)} \leq \frac{1}{n},
\]

\[
\left\|\text{tr} \Theta_n - \frac{2}{t}\right\|_{L^4(\partial M_n)} + \|\hat{\Theta}_n\|_{L^4(\partial M_n)} \leq \frac{1}{n},
\]

\[
\mathcal{r}_{\text{vol}}(M_n, \frac{1}{2}) \geq \frac{1}{4},
\]

\[
\text{vol}_{g_n}(M_n) \leq V.
\]

and such that there does not exist an integer \(N \geq 1\) and a family of global charts

\[(\varphi_n : \overline{B_t} \to M_n)_{n \geq N}
\]

such that on \(\overline{B_t}\), for \(i, j = 1, 2, 3\) and \(n \geq N\),

\[
\sum_{0 \leq m \leq 2} \|\partial^m (g_n)_{ij} - e_{ij}\|_{L^2(B_t)} \lesssim C_{\text{ball}},
\]

\[
(1 - C_{\text{ball}})|\xi|^2 \leq (g_n)_{ij} \xi^i \xi^j \leq (1 + C_{\text{ball}})|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2,
\]

where we abused notation by writing \(g_n\) instead of \(\Psi_n^* g_n\).

By Lemma 5.3, there is a smooth limit manifold \((M, g)\) such that

\[(M_n, g_n) \to (M, g) \text{ in the } H^2\text{-topology as } n \to \infty,
\]

and the estimate (5.3) holds.
By (5.4) and (5.6), it follows that \((M, g)\) satisfies

\[\text{Ric} = 0 \text{ in } M, \quad \text{tr} \Theta = \frac{2}{t}, \quad \widehat{\Theta} = 0 \text{ on } \partial M,\]

so by Lemma 5.4,

\[(M, g) \cong (\mathcal{B}_t, e),\]  

(5.7)

which trivially admits global smooth coordinates.

Therefore, by (5.3), (5.6) and (5.7), we get that for \(n\) large there are global diffeomorphisms \(\Psi_n : \mathcal{B}_1 \to M_n\) satisfying

\[
\sum_{0 \leq m \leq 2} \| \partial^m \left((g_n)_{ij} - e_{ij}\right) \|_{L^2(\mathcal{B}_t)} \to 0 \text{ as } n \to \infty.
\]  

(5.8)

This yields a contradiction to (5.5), and hence finishes the proof of Theorem 4.1.  

\[\square\]

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**Appendix A. Global Elliptic Estimate for the Second Fundamental Form**

In this section we derive a global elliptic estimate for the second fundamental form \(k\) of a maximal spacelike hypersurface \(\Sigma \simeq \mathcal{B}_1\). We start by deriving an integral identity for Hodge systems on compact Riemannian 3-manifolds with boundary \(\Sigma\), see Lemma A.2 below. This integral identity is a slight generalisation of Section 8 in [9] where non-compact manifolds without boundary are considered.

First recall the following notation.

**Definition A.1** Let \(m \geq 0\) be an integer. For a given totally symmetric \((m + 2)\)-tensor \(F\), define

\[A(F)_{a_1 \ldots a_{m+1}bc} := \nabla_c F_{a_1 \ldots a_{m+1}b} - \nabla_b F_{a_1 \ldots a_{m+1}c}, \quad D(F)_{a_1 \ldots a_{m+1}} := \nabla^c F_{a_1 \ldots a_{m+1}c}.\]

The following integral identity is a straightforward generalisation of Lemma 4.4.1 in [9] to manifolds with boundary. The proof is by integration by parts and left to the reader.

**Lemma A.2** (Fundamental integral identity for Hodge systems) Let \((\Sigma, g)\) be a compact Riemannian 3-manifold with boundary and let \(m \geq 0\) be an integer. Let \(F\) be a totally symmetric \((m + 2)\)-tensor on \(\Sigma\). Then it holds that

\[
\int_{\Sigma} |\nabla F|^2 = \int_{\Sigma} \frac{1}{2} |A(F)|^2 + |D(F)|^2
\]
\begin{equation}
- \int_{\Sigma} \sum_{i=1}^{m+1} \left( R^l_{ab} F^c_{a_1 \ldots a_m+1c} + \text{Ric}^l_{b} F^{a_1 \ldots a_m+1}_{b} \right) F^{a_1 \ldots a_m+1}_{b}
- \int_{\partial \Sigma} F^{a_1 \ldots a_m+1} N D(F)_{a_1 \ldots a_m+1} + \int_{\partial \Sigma} \nabla_b F^{a_1 \ldots a_m+1} F^{a_1 \ldots a_m+1}_{b}.
\end{equation}

We now use Lemma A.2 to derive global elliptic estimates for the second fundamental form \( k \) of a maximal spacelike hypersurface. Let \((\mathcal{M}, g)\) be a vacuum spacetime and let \( \Sigma \cong \overline{B_1} \) be a compact spacelike maximal hypersurface in \( \mathcal{M} \). By (2.13b), (2.13c) and (2.13d), the second fundamental form \( k \) of \( \Sigma \) satisfies the following Hodge system,

\[
\begin{align*}
\text{div}_g k &= 0, \\
\text{curl}_g k &= H, \\
\text{tr}_g k &= 0.
\end{align*}
\]

In the notation of Definition A.1, \( k \) satisfies

\[
A(k)_{iab} = \varepsilon^{m}_{ab} H_{im}, \quad D(k) = 0. \tag{A.1}
\]

Lemma A.2 together with (A.1) yields the following corollary (see Section 8.3 in [20] for the case of manifolds without boundary).

**Corollary A.3** (Fundamental global elliptic estimate for \( k \)) Let \((\mathcal{M}, g)\) be a vacuum spacetime and let \( \Sigma \cong \overline{B_1} \) be a compact spacelike maximal hypersurface in \( \mathcal{M} \). Then it holds that

\[
\int_{\Sigma} |\nabla k|^2 + \frac{1}{4} |k|^4 - \int_{\partial \Sigma} \nabla_a k_b N k_{ba} \lesssim \int_{\Sigma} |\text{R}|^2_{h'},
\]

where \( N \) denotes the outward-pointing unit normal to \( \partial \Sigma \subset \Sigma \), \( T \) denotes the timelike unit normal to \( \Sigma \) and \( h' \) denotes the positive-definite norm on \( \Sigma \) defined with \( T \), see (2.9).

**Proof** By Lemma A.2 with \( F = k \), we have

\[
\int_{\Sigma} |\nabla k|^2 = \int_{\Sigma} \frac{1}{2} |H|^2 - \int_{\Sigma} \left( R^l_{ab} c_{klc} + \text{Ric}^l_{b} k_{al} \right) k^{ab} + \int_{\partial \Sigma} \nabla_b k_{aN} k^{ab}. \tag{A.2}
\]

In dimension \( n = 3 \), the full Riemann curvature tensor is determined by \( \text{Ric} \), yielding

\[
R^l_{abc} k_{lc} k^{ab} = 2\text{Ric}_{ji} k^j i k^l_i - \frac{1}{2} \text{R}_{scal} |k|^2.
\]
Plugging this into (A.2), we get

\[
\int_{\Sigma} |\nabla k|^2 = \int_{\Sigma} \frac{1}{2}|H|^2 - \int_{\Sigma} \left( 3\text{Ric}^l_{\ k_a l} k^{ab} - \frac{1}{2} \text{R}_{\text{scal}} |k|^2 \right) + \int_{\partial \Sigma} \nabla_b k_{aN} k^{ab}. \quad (A.3)
\]

On the one hand, by (2.13f) and (2.13g), we have

\[E_{ij} = \text{Ric}_{ij} - k_i m k^m_j, \quad \text{R}_{\text{scal}} (g) = |k|^2.g.\]

On the other hand, in dimension \(n = 3\), it holds for symmetric tracefree 2-tensors \(F\) that

\[3\text{tr}(F^4) \geq |F|^4.\]

Hence it follows from (A.3) that

\[
\int_{\Sigma} \frac{1}{2}|H|^2 = \int_{\Sigma} |\nabla k|^2 + 3\text{Ric}^s_{\ a} k_{bs} k^{ba} - \frac{1}{2} |k|^4 - \int_{\partial \Sigma} \nabla_a k_{bN} k^{ba} \\
= \int_{\Sigma} |\nabla k|^2 + 3(E^s_a + k^s_{\ m} k^{m}_a) k_{bs} k^{ba} - \frac{1}{2} |k|^4 - \int_{\partial \Sigma} \nabla_a k_{bN} k^{ba} \\
\geq \int_{\Sigma} |\nabla k|^2 + 3E^s_a k_{bs} k^{ba} + \frac{1}{2} |k|^4 - \int_{\partial \Sigma} \nabla_a k_{bN} k^{ba}.
\]

Using that \(|E|^2 + |H|^2 \lesssim |R|^2_{\text{ht}}\) by (2.8) and (2.11), we obtain

\[
\int_{\Sigma} |\nabla k|^2 + \frac{1}{4}|k|^4 - \int_{\partial \Sigma} \nabla_a k_{bN} k^{ba} \lesssim \int_{\Sigma} |R|^2_{\text{ht}}.
\]

This finishes the proof of Corollary A.3. \(\square\)

**Appendix B. Proof of Lemmas 3.5 and 3.6**

In this section We Prove Lemmas 3.5 and 3.6.

**B.1. Proof of Lemma 3.5**

We have to show that on weakly regular balls \((\Sigma, g)\) of radius \(1 \leq r \leq 2\) with constant \(0 < C_{\text{ball}} < 1/2\), it holds that

\[
\|F\|_{L^2(\partial \Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}, \quad (B.1)
\]

\[
\|F\|_{L^4(\partial \Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)},
\]

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and
\[ \|F\|_{H^{1/2}(\partial \Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}. \quad (B.2) \]

First, the estimates (B.1) are straightforward, see, for example, Lemma 3.26 and Corollary 3.27 in [33] for a concise proof.

We turn to the proof of (B.2). On the one hand, by Sections 7.50 to 7.56 in [1], for each coordinate patch \( U \subset \partial B_r \) and smooth open set \( V \subset \overline{B_r} \) with \( U \subset (V \cap \partial B_r) \), it holds that
\[ H^1(V) \hookrightarrow H^{1/2}(U), \]
where \( H^{1/2}(U) \) denotes a local coordinate-defined fractional Sobolev space on \( U \).

On the other hand, if \( g_{ij} \in H^2(B_r) \) then in particular \( g_{AB} \in W^{1,4}(\partial \overline{B_r}) \) in local coordinates. By Proposition 3.2 in [30], this control suffices to compare the coordinate-defined spaces \( H^{1/2}(U) \) with the space \( H^{1/2}(\partial \overline{B_r}) \) defined in Definition 3.2, see also Appendix B of [30]. This finishes the proof of (B.2). This finishes the proof of Lemma 3.5.

**B.2. Proof of Lemma 3.6**

Let \( 1 \leq r_0 \leq 2 \) and \( 0 < C_{\text{ball}} < 1/2 \) be two real numbers. Let \((\Sigma, g)\) be a weakly regular ball of radius \( r_0 \) with constant \( C_{\text{ball}} \). Let \( F \) be a tensor on \( \Sigma \). We have to show that
\[ \|F\|_{L^2(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)} + \|F\|_{L^2(\partial \Sigma)}. \]

Define spherical coordinates \((r, \theta^1, \theta^2)\) with \( r \in [0, r_0] \) and \((\theta^1, \theta^2) \in \mathbb{S}^2 \) on \( \Sigma \simeq \overline{B_{r_0}} \) as in Section 2.4. Let \( \gamma \) and \( d\mu_\gamma \) on \( \mathbb{S}_r \) denote the standard round metric of radius \( r > 0 \) and its volume element, respectively. Denote by \( \gamma^\circ \) and \( d\mu_{\gamma^\circ} \) the standard round metric on the unit sphere and its volume element, respectively. Note that \( \gamma = r^2 \gamma^\circ \). By the fundamental theorem of calculus and using that \( \partial \Sigma = \mathcal{S}_{r_0} \), we have for \( 0 < r \leq r_0 \),
\[
\int_{\mathcal{S}_r} |F|^2_{g} \, d\mu_{\gamma^\circ} \leq \int_{r_0}^{r} \left( \int_{\mathcal{S}_{r'}} |F|^2_{g} \, d\mu_{\gamma^\circ} \right) dr + \int_{\partial \Sigma} |F|^2_{g} \, d\mu_{\gamma^\circ}
\]
\[
= \int_{r_0}^{r} \left( \int_{\mathcal{S}_{r'}} \nabla_{\partial r} F \cdot F \, d\mu_{\gamma^\circ} \right) dr + \int_{\partial \Sigma} |F|^2_{g} \, d\mu_{\gamma^\circ}
\]
\[
\leq \left\| \partial_r |F|_{g} \right\|_{L^\infty(\Sigma)} \left( \int_{r_0}^{r} \int_{\mathcal{S}_{r'}} |\nabla F|^2_{g} \, d\mu_{\gamma^\circ} \, dr \right)^{1/2} \left( \int_{r_0}^{r} \int_{\mathcal{S}_{r'}} |F|^2_{g} \, d\mu_{\gamma^\circ} \, dr \right)^{1/2}
\]
\[
+ \int_{\partial \Sigma} |F|^2_{g} \, d\mu_{\gamma^\circ}.
\]
Using that by definition of the spherical coordinates on $B_{r_0}$ and the weakly regular ball property of $(\Sigma, g)$,

\[ |\partial_r|^2 = g_{rr} = \frac{x^i x^j}{r} g_{ij} \leq (1 + C_{\text{ball}}) \frac{x^i x^j}{r} e_{ij} = 1 + C_{\text{ball}}, \]

we can estimate the right-hand side as follows,

\[
\int_{\Sigma} |F|^2_{g} d\mu_\gamma \lesssim \left( \int_{r_0} r \int_{S} |\nabla F|^2_{g} d\mu_\gamma dr' \right)^{1/2} \left( \int_{r_0} r \int_{S} |F|^2_{g} d\mu_\gamma dr' \right)^{1/2} + \int |F|^2_{g} d\mu_\gamma
\]

\[
\lesssim \frac{1}{r^2} \left( \int_{r_0} r \int_{S} |\nabla F|^2_{g} d\mu_\gamma dr' \right)^{1/2} \left( \int_{r_0} r \int_{S} |F|^2_{g} d\mu_\gamma dr' \right)^{1/2} + \int |F|^2_{g} d\mu_\gamma
\]

\[
\lesssim \frac{1}{r^2} \left( \int_{B_{r_0}} |\nabla F|^2_{e} d\mu_e \right)^{1/2} \left( \int_{B_{r_0}} |F|^2_{g} d\mu_e \right)^{1/2} + \int |F|^2_{g} d\mu_\gamma
\]

\[
\lesssim \frac{1}{r^2} \|\nabla F\|_{L^2(B_{r_0})} \|F\|_{L^2(B_{r_0})} + \int |F|^2_{g} d\mu_\gamma,
\]

where $d\mu_e$ denotes the measure with respect to the standard Euclidean metric $e$ in Cartesian coordinates, and we used in the last inequality that $\Sigma$ is a weakly regular ball to compare the Euclidean integral with the $g$-dependent norm.

Multiplying the above by $r^2$ and using that by Lemmas 3.1, 3.5 and 3.7,

\[
\int_{\partial \Sigma} |F|^2_{g} d\mu_{\partial \Gamma} \lesssim \|F\|_{L^2(\partial \Sigma)}^2,
\]

we get that for $0 < r \leq r_0$,

\[
\int_{\Sigma} |F|^2_{g} d\mu_{\gamma} \lesssim \|\nabla F\|_{L^2(\Sigma)} \|F\|_{L^2(\Sigma)} + \frac{r^2}{r_0} \int_{\partial \Sigma} |F|^2_{g} d\mu_{\partial \Gamma} \lesssim \|\nabla F\|_{L^2(\Sigma)} \|F\|_{L^2(\Sigma)} + \frac{r^2}{r_0} \|F\|_{L^2(\partial \Sigma)}.
\]

By (B.3) and using that $(\Sigma, g)$ is a weakly regular ball of radius $r_0$, we get that
\[ \| F \|_{L^2(\Sigma)}^2 \lesssim \int_{B_{r_0}} |F|^2 d\mu_e \lesssim \int_0^{r_0} \left( \int_{S_r} |F|^2 d\mu_{\gamma} \right) dr \]
\[ \lesssim \| \nabla F \|_{L^2(\Sigma)} \| F \|_{L^2(\Sigma)} + \| F \|_{L^2(\partial \Sigma)}^2, \]
which implies that
\[ \| F \|_{L^2(\Sigma)} \lesssim \| \nabla F \|_{L^2(\Sigma)} + \| F \|_{L^2(\partial \Sigma)}. \]
This finishes the proof of Lemma 3.6.

**Appendix C. Proof of Lemma 5.4**

In this section we prove Lemma 5.4. Let \((M, g)\) be a smooth compact Riemannian 3-manifold with boundary such that \(M \simeq B_1 \subset \mathbb{R}^3\) and for a real number \(1 \leq t \leq 2,\)

\[ \text{Ric} = 0 \text{ on } M, \quad \text{tr } \Theta = \frac{2}{t}, \quad \widehat{\Theta} = 0 \text{ on } \partial M. \]  

We have to prove that
\[ (M, g) \simeq (B_t, e). \]

First, by the Gauss equation
\[ 2K = (\text{tr } \Theta)^2 - |\Theta|^2 + R_{\text{scal}} - 2\text{Ric}(N, N), \]

it follows that the Gauss curvature of \((\partial M, g)\) is constant \(K = \frac{1}{t^2} . \) Hence by classical differential geometry, there exist coordinates \((\theta^1, \theta^2)\) on \(\partial M\) such that
\[ g = t^2 \left( (d\theta^1)^2 + \sin^2(\theta^1) (d\theta^2)^2 \right) \text{ on } \partial M. \]  

Second, by the smoothness of \((M, g)\), there exists in an open neighbourhood \(U \subset M\) of \(\partial M \subset M\) a so-called *Gaussian coordinate system* \((r, \theta^1, \theta^2)\), see for example Section 3.3 in [36], which coincide with (C.2) on \(\partial M\) and are such that for some small real number \(\delta > 0,\)

\[ U = \{ r \in (t - \delta, t) \}, \]
\[ U \cap \partial M = \{ r = t \}, \]
\[ \nabla_{\partial_r} \partial_r = 0, \]
\[ \partial_r \mid_{\partial M} \text{ is normal to } \partial M, \]
\[ g(\partial_r, \partial_r) = 1. \]
In such coordinates we can express $g$ as

$$g = dr^2 + g(r, \theta^1, \theta^2)_{AB} d\theta^A d\theta^B,$$

where $g(r, \theta^1, \theta^2)$ denotes the induced metric on the level sets $S_r$ of $r$. In particular, by (C.2) it holds that

$$g(t, \theta^1, \theta^2) = t^2 \left((d\theta^1)^2 + \sin^2(\theta^1)(d\theta^2)^2\right).$$

Third, we claim that the Riemannian manifold $(M, g)$ smoothly extends onto $\mathbb{R}^3 \setminus B_t$ when identifying $\partial M = \partial B_t \subset \mathbb{R}^3$. It suffices to show that the induced metric

$$\tilde{g}_{AB}(r, \theta^1, \theta^2) := \begin{cases} 
\gamma_{AB}(r, \theta^1, \theta^2), & \text{if } r \geq t, \\
g_{AB}(r, \theta^1, \theta^2) & \text{if } t - \delta < r \leq t,
\end{cases}$$

is smooth across $\{r = t\}$. Here $\gamma_{AB}(r, \theta^1, \theta^2)$ is the standard round metric of radius $r$. In the following calculations we denote $g_{AB}(r, \theta^1, \theta^2)$ on $U$ simply by $g_{AB}$.

By (C.2) it follows that $\tilde{g}$ is continuous across $\{r = t\}$. Further, on the one hand, on $\partial B_t \subset \mathbb{R}^3$, it holds that

$$\partial_r \gamma_{AB}|_{r = t} = 2 t \dot{\gamma}_{AB}, \quad \partial_r^2 \gamma_{AB}|_{r = t} = 2 \ddot{\gamma}_{AB}, \quad \partial_r^m \gamma_{AB}|_{r = t} = 0 \text{ for } m \geq 3, \quad (C.3)$$

where $\dot{\gamma}_{AB}$ denotes the metric components of the standard round metric on $S^2$.

On the other hand, by construction of $(r, \theta^1, \theta^2)$ it holds on $U$ that

$$\partial_r g_{AB} = 2g(\nabla_r \partial_A, \partial_B) = 2g(\nabla_A \partial_r, \partial_B). \quad (C.4)$$

By (C.1), (C.2) and (C.4), on $\partial M$ we thus have that

$$\partial_r g_{AB}|_{r = t} = 2 \Theta_{AB} = 2 \dot{\gamma}_{AB}, \quad (C.5)$$

Differentiating (C.4) in $r$ yields

$$\partial_r^2 g_{AB} = 2g(\nabla_r \partial_A, \nabla_r \partial_B) + 2g(\partial_A, \nabla_r \nabla_r \partial_B)$$

$$= 2g(\nabla_A \partial_r, \nabla_B \partial_r) + 2g(\partial_A, \nabla_r \nabla_B \partial_r)$$

$$= 2g(\nabla_A \partial_r, \nabla_B \partial_r) + 2g(\partial_A, Rm(\partial_r, \partial_B) \partial_r) + 2g(\partial_A, \nabla_B \nabla_r \partial_r)$$

$$= 2g(\nabla_A \partial_r, \nabla_B \partial_r). \quad (C.6)$$

where we used that $Rm = 0$ in $(M, g)$ and $\nabla_\partial_r \partial_r = 0$ in Gaussian coordinates in $U$.

From (C.5) and (C.6) and using that $a = 1$, it follows that

$$\partial_r^2 g_{AB}|_{r = t} = 2t^{-2} \ddot{\gamma}_{CD} t \dot{\gamma}_{AC} t \dot{\gamma}_{BD} = 2 \dot{\gamma}_{AB}. \quad (C.7)$$
Differentiating (C.4) further in $r$ shows that

$$\partial_r^m \tilde g_{AB} |_{r=t} = 0 \text{ for } m \geq 3. \quad (C.8)$$

Comparing (C.3) with (C.5), (C.7) and (C.8) shows that all $r$-derivatives of $\tilde g$ agree on $r = t$. Hence $(M, \tilde g)$ smoothly extends as Riemannian manifold onto $(\mathbb{R}^3 \setminus \overline{B_t}, e)$.

The resulting smooth Riemannian 3-manifold is in particular flat, complete and has cubic volume growth of geodesic balls. By Proposition 4.4 in [13] it must therefore be isometric to $(\mathbb{R}^3, e)$. We conclude that by the above construction,

$$(M, \tilde g) \cong (\mathbb{R}^3 \setminus (\mathbb{R}^3 \setminus \overline{B_t}), e) \cong (\overline{B_t}, e),$$

which finishes the proof of Lemma 5.4.

**Appendix D. Comparison Estimates Between Maximal Spacelike Foliations**

In this section we prove Lemma 4.6. By assumption, for real numbers $D > 0$ and $\varepsilon > 0$, $M_{t^*}$ is foliated by maximal spacelike hypersurface $(\Sigma_t)_{1 \leq t \leq t^*}$ given as level sets of a time function $t$ with $\Sigma_1 = \Sigma$ and satisfying

$$\| \mathbf{R} \|_{L^\infty_t L^2(\Sigma_t)} + \| \text{Ric} \|_{L^\infty_t L^2(\Sigma_t)} \leq D\varepsilon,$$

$$\| \nabla k \|_{L^\infty_t L^2(\Sigma_t)} + \| k \|_{L^\infty_t L^2(\Sigma_t)} \leq D\varepsilon,$$

$$\| n - 1 \|_{L^\infty_t L^2(\Sigma_t)} + \| \nabla n \|_{L^\infty_t L^2(\Sigma_t)} + \| \nabla^2 n \|_{L^\infty_t L^2(\Sigma_t)} \lesssim D\varepsilon. \quad (D.1)$$

Let $T$ denote the timelike unit normal to $\Sigma_t$.

Moreover, by assumption there is another foliation on $M_{t^*}$ (constructed by the bounded $L^2$ curvature theorem) of maximal hypersurfaces $(\tilde \Sigma_t)_{0 \leq t \leq t^*}$ given as level sets of a time function $\tilde t$ with $\Sigma_{t^*} = \tilde \Sigma_{t^*}$ and satisfying

$$\| \mathbf{R} \|_{L^\infty_t L^2(\tilde \Sigma_t)} + \| \tilde \text{Ric} \|_{L^\infty_t L^2(\tilde \Sigma_t)} \lesssim C_{\text{ball}},$$

$$\| \tilde \nabla \tilde k \|_{L^\infty_t L^2(\tilde \Sigma_t)} + \| \tilde k \|_{L^\infty_t L^2(\tilde \Sigma_t)} + \| \tilde \Delta \tilde n \|_{L^\infty_t L^2(\tilde \Sigma_t)} \lesssim C_{\text{ball}}, \quad (D.2)$$

$$\| \tilde n - 1 \|_{L^\infty_t L^2(\tilde \Sigma_t)} + \| \tilde \nabla \tilde n \|_{L^\infty_t L^2(\tilde \Sigma_t)} + \| \tilde \nabla^2 \tilde n \|_{L^\infty_t L^2(\tilde \Sigma_t)} \lesssim C_{\text{ball}},$$

where $\tilde \text{Ric}$, $\tilde \nabla$ and $\tilde k$ denote the Ricci curvature, covariant derivative and the second fundamental form on $\tilde \Sigma_t$, respectively. Let $\tilde T$ denote the timelike unit normal to $\tilde \Sigma_{\tilde t}$.

We need to show that for $\varepsilon > 0$ and $C_{\text{ball}} > 0$ sufficiently small, for $1 \leq t \leq t^*$,

$$\| \tilde v - 1 \|_{L^\infty_t L^\infty(\Sigma_t)} \lesssim C_{\text{ball}}, \quad (D.3)$$

$$\| \tilde k \|_{L^\infty_t L^4(\Sigma_t)} \lesssim C_{\text{ball}}, \quad (D.4)$$

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where the angle $\tilde{\nu}$ between $T$ and $\tilde{T}$ is defined as
\[ \tilde{\nu} := -g(T, \tilde{T}). \]  
(D.5)

The proof of (D.3) and (D.4) is based on a standard continuity argument going backwards in $t$ and starting at $\Sigma_{t^*} = \tilde{\Sigma}_{t^*}$ where by construction $\tilde{\nu} = 1$, $k = \tilde{k}$.

In the following, we only discuss the bootstrap assumption and its improvement. 

**Bootstrap assumption.** Let $1 \leq t_0^* < t^*$ be a real. Assume that for a large constant $M > 0$, for $t_0^* \leq t \leq t^*$,
\[ \| \tilde{\nu} - 1 \|_{L^\infty_t L^\infty(\Sigma_t)} \leq MC_{\text{ball}}. \]  
(D.6)

**First consequences of the bootstrap assumption.** Let $(\tilde{e}_i)_{i=1,2,3}$ be an orthonormal frame on $\tilde{\Sigma}_t$ and let $\tilde{e}_0 = \tilde{T}$. Expressing $T$ as
\[ T = \tilde{\nu} \tilde{T} + g(T, \tilde{e}_i) \tilde{e}_i, \]
and using that $g(T, T) = -1$, it follows that
\[ -1 = -|\tilde{\nu}|^2 + \sum_{i=1,2,3} |g(T, \tilde{e}_i)|^2. \]

This implies by the bootstrap assumption (D.6) for $C_{\text{ball}} > 0$ sufficiently small that for $t_0^* \leq t \leq t^*$,
\[ \| g(T, \tilde{e}_i) \|_{L^\infty_t L^\infty(\Sigma_t)} \lesssim \sqrt{MC_{\text{ball}}}. \]  
(D.7)

Let $(e_i)_{i=1,2,3}$ be the orthonormal frame tangent to $\Sigma_t$ constructed by the Gram-Schmidt method applied to the $\Sigma_t$-tangential frame
\[ (\tilde{e}_1 + g(\tilde{e}_1, T)T, \tilde{e}_2 + g(\tilde{e}_2, T)T, \tilde{e}_3 + g(\tilde{e}_3, T)T), \]
and set moreover $e_0 := T$. By the Gram-Schmidt construction and (D.6) and (D.7), we get that for $C_{\text{ball}} > 0$ sufficiently small, for $t_0^* \leq t \leq t^*$,
\[ \sum_{i=1}^3 \left( \| g(e_i, \tilde{e}_i) - 1 \|_{L^\infty_t L^\infty(\Sigma_t)} + \sum_{j \neq i} \| g(\tilde{e}_i, e_j) \|_{L^\infty_t L^\infty(\Sigma_t)} + \| g(\tilde{T}, e_i) \|_{L^\infty_t L^\infty(\Sigma_t)} \right) \lesssim \sqrt{MC_{\text{ball}}}. \]  
(D.8)
Improvement of the bootstrap assumption. First, by definition of \( \tilde{v} \), see (D.5), and using that \( D_T T = n^{-1} \nabla n \),

\[
T(\tilde{v}) = - g(D_T T, \tilde{T}) - g(T, D_T \tilde{T})
\]

\[
= - \left( g(\tilde{T}, e_i)g(T, e_i) + g(T, \tilde{e}_j)g(\tilde{e}_j, D_g(T, \tilde{e}_j)e_j - g(T, \tilde{e}_j)e_j) \right)
\]

\[
= - \left( g(\tilde{T}, e_i)(n^{-1} \nabla e_i n) - g(T, \tilde{e}_j)g(T, \tilde{e}_j)\tilde{k}_{ij} + \tilde{v}g(T, \tilde{e}_i) (\tilde{n}^{-1} \tilde{\nabla}_i \tilde{n}) \right).
\]

(D.9)

Define shift-free coordinates \((t, x^1, x^2, x^3)\) on \( \mathcal{M}_{t^*} \) by transporting \((x^1, x^2, x^3)\) from \( \Sigma_{t^*} \) backwards along \( T \). Then it holds that \( \partial_t \tilde{v} = n^{-1} T \), and we get from (D.9) that

\[
\partial_t \tilde{v} = - n^{-1} \left( g(\tilde{T}, e_i)(n^{-1} \nabla e_i n) - g(T, \tilde{e}_j)g(T, \tilde{e}_j)\tilde{k}_{ij} + \tilde{v}g(T, \tilde{e}_i) (\tilde{n}^{-1} \tilde{\nabla}_i \tilde{n}) \right).
\]

Integrating this equation in \( t \), using that \( \tilde{v}|_{t=t^*} = 1 \) and applying (D.1), (D.2), (D.6), (D.7), (D.8) and Lemma 3.4, we get that for \( \epsilon > 0 \) and \( C_{ball} > 0 \) sufficiently small, for \( t_0^* \leq t \leq t^* \), it holds that

\[
\| \tilde{v} - 1 \| \infty L^4(\Sigma_i) \lesssim \| g(\tilde{T}, e_i) \| L^\infty(\mathcal{M}_{t_0^*, t^*}) \| \nabla e_i n \| L^4(\mathcal{M}_{t_0^*, t^*})
\]

\[
+ \| g(T, \tilde{e}_j) \| L^\infty(\mathcal{M}_{t_0^*, t^*}) \| g(T, \tilde{e}_j) \| L^4(\mathcal{M}_{t_0^*, t^*}) \| \tilde{k}_{ij} \| L^4(\mathcal{M}_{t_0^*, t^*})
\]

\[
+ \| \tilde{v} \| L^\infty(\mathcal{M}_{t_0^*, t^*}) \| g(T, \tilde{e}_i) \| L^\infty(\mathcal{M}_{t_0^*, t^*}) \| \tilde{\nabla}_i \tilde{n} \| L^4(\mathcal{M}_{t_0^*, t^*})
\]

\[
\lesssim \| g(\tilde{T}, e_i) \| L^\infty(\mathcal{M}_{t_0^*, t^*})
\]

\[
\left( \| \nabla n \| L^\infty L^2(\Sigma_i) + \| \nabla^2 n \| L^\infty L^2(\Sigma_i) \right)
\]

\[
+ \| g(T, \tilde{e}_j) \| L^\infty(\mathcal{M}_{t_0^*, t^*}) \| g(T, \tilde{e}_j) \| L^\infty(\mathcal{M}_{t_0^*, t^*})
\]

\[
\left( \| \tilde{k} \| L^\infty L^2(\Sigma_i) + \| \tilde{\nabla} k \| L^\infty L^2(\Sigma_i) \right)
\]

\[
+ \| \tilde{v} \| L^\infty(\mathcal{M}_{t_0^*, t^*}) \| g(T, \tilde{e}_i) \| L^\infty(\mathcal{M}_{t_0^*, t^*})
\]

\[
\left( \| \tilde{\nabla} n \| L^\infty L^2(\Sigma_i) + \| \tilde{\nabla}^2 n \| L^\infty L^2(\Sigma_i) \right)
\]

\[
\lesssim \sqrt{MC_{ball}(D\epsilon)} + \sqrt{MC_{ball}C_{ball}}.
\]

(D.10)

where \( \mathcal{M}_{t_0^*, t^*} \) denotes the spacetime region \( t_0^* \leq t \leq t^* \).

Second, by definition of \( \tilde{v} \) in (D.5), for \( i = 1, 2, 3 \),

\[
\nabla_{e_i} \tilde{v} = - g(D_{e_i} T, \tilde{T}) - g(T, D_{e_i} \tilde{T})
\]

\[
= - \left( g(\tilde{T}, e_j)k_{ij} + g(T, \tilde{e}_j)g(\tilde{e}_j, D_g(\tilde{T}, \tilde{e}_j)e_j - g(\tilde{T}, \tilde{e}_j)e_j) \right)
\]

\[
= - \left( g(\tilde{T}, e_j)k_{ij} - g(T, \tilde{e}_j)g(e_i, \tilde{e}_j)\tilde{k}_{ij} + g(T, \tilde{e}_j)g(e_i, \tilde{e}_j)(\tilde{n}^{-1} \tilde{\nabla}_j \tilde{n}) \right).
\]
By (D.1), (D.2), (D.8) and Lemma 3.4, we have for $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small,

$$
\| \nabla_v \tilde{v} \|_{L_t^\infty L^1(\Sigma_t)} \lesssim \| \mathbf{g}(\tilde{T}, e_j) \|_{L_t^\infty(L^\infty(M_t^{i_0*, \ast}))} \| \tilde{k}_{ij} \|_{L_t^\infty L^4(\Sigma_t)} 
+ \| \mathbf{g}(T, \tilde{e}_j) \|_{L_t^\infty(L^\infty(M_t^{i_0*, \ast}))} \| \tilde{k}_{ij} \|_{L_t^\infty L^4(\Sigma_t)} 
+ \| \mathbf{g}(T, \tilde{e}_j) \|_{L_t^\infty(L^\infty(M_t^{i_0*, \ast}))} \| \tilde{\nabla}_{\tilde{e}_j} \tilde{n} \|_{L_t^\infty L^\infty(\Sigma_t)} 
\lesssim \sqrt{MC_{\text{ball}}} D\varepsilon + \sqrt{MC_{\text{ball}}} \| \tilde{k}_{ij} \|_{L_t^\infty L^4(\Sigma_t)} + (MC_{\text{ball}}) C_{\text{ball}}.
$$

By (D.10), (D.11) and Lemma 3.4, it follows that

$$
\| \tilde{v} - 1 \|_{L_t^\infty(M_t^{i_0*, \ast})} \lesssim \sqrt{MC_{\text{ball}}} (D\varepsilon + \sqrt{MC_{\text{ball}}} C_{\text{ball}}) + \sqrt{MC_{\text{ball}}} \| \tilde{k}_{ij} \|_{L_t^\infty L^4(\Sigma_t)}. \tag{D.12}
$$

To estimate the remaining term $\| \tilde{k}_{ij} \|_{L_t^\infty L^4(\Sigma_t)}$ on the right-hand side of (D.12), we apply the following technical lemma whose proof is postponed to the end of this section.

**Lemma 5.5** (Technical lemma) Under the assumption of (D.1), (D.2), (D.6) for $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small, it holds for every scalar function $f$ on $M_t^*$ that for $t_0^* \leq t \leq t^*$,

$$
\| f \|_{L_t^\infty L^4(\Sigma_t)} \lesssim \| \mathbf{D} f \|_{L_t^{1/4} L^{2}(\tilde{\Sigma}_t)}^{1/4} \| f \|_{L_t^{3/4} L^{6}(\tilde{\Sigma}_t)}^{3/4} + \| f \|_{L^4(\Sigma_t^*)}.
$$

Applying Lemma 5.5 to $f = \tilde{k}_{ij}$, we have for $\varepsilon > 0$ and $C_{\text{ball}} > 0$ sufficiently small, for $t_0^* \leq t \leq t^*$,

$$
\| \tilde{k}_{ij} \|_{L_t^\infty L^4(\Sigma_t)} \lesssim \| \mathbf{D} (\tilde{k}_{ij}) \|_{L_t^{1/4} L^{2}(\tilde{\Sigma}_t)}^{1/4} \| \tilde{k}_{ij} \|_{L_t^{3/4} L^{6}(\tilde{\Sigma}_t)}^{3/4} + \| \tilde{k}_{ij} \|_{L^4(\Sigma_t^*)} 
\lesssim \left( \| \mathbf{D} \tilde{\alpha} \|_{L_t^\infty L^2(\tilde{\Sigma}_t)} + \| \tilde{\alpha} \|_{L_t^\infty L^4(\tilde{\Sigma}_t)} \| \mathbf{A} \|_{L_t^\infty L^4(\tilde{\Sigma}_t)} \right)^{1/4} \| \tilde{k} \|_{L_t^{3/4} L^{6}(\tilde{\Sigma}_t)}^{3/4} 
+ \| k \|_{L^4(\Sigma_t^*)} 
\lesssim C_{\text{ball}}^{1/4} C_{\text{ball}}^{3/4} + D\varepsilon, \tag{D.13}
$$

where $\tilde{\mathbf{A}}$ denotes the connection 1-form defined by

$$
(\tilde{\mathbf{A}}_{\mu})_{\alpha\beta} := \mathbf{g}(\mathbf{D}_{\tilde{e}_{\mu}} \tilde{e}_\beta, \tilde{e}_\alpha), \quad \text{for } \mu, \alpha, \beta = 0, 1, 2, 3,
$$

and we used that the foliation $(\tilde{\Sigma}_t)_{0 \leq t \leq t^*}$ constructed by the bounded $L^2$ curvature theorem (Theorem 3.11, see also [22]) satisfies in addition to (D.2) the following
bound on $0 \leq \tilde{t} \leq t^*$,

$$\|\tilde{A}\|_{L^\infty_t L^4(S_{\tilde{t}})} \lesssim C_{\text{ball}}.$$  

Plugging (D.13) into (D.12), we get that for $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small, for $t_0^* \leq t \leq t^*$,

$$\|\tilde{V} - 1\|_{L^\infty_t L^\infty(S_{\tilde{t}})} \lesssim \sqrt{MC_{\text{ball}}} (D \varepsilon + C_{\text{ball}} + \sqrt{MC_{\text{ball}}} C_{\text{ball}}) \lesssim M'C_{\text{ball}},$$

for a constant $0 < M' < M$. This improves the bootstrap assumption (D.6) and hence finishes the proof of (D.3). The estimate (D.4) follows directly from (D.13). This finishes the proof of Lemma 4.6.

It remains to prove Lemma 5.5. Let $(t, x^1, x^2, x^3)$ denote the shift-free coordinate system constructed above. Using that $\text{tr}k = 0$ on $\Sigma_t$ for $1 \leq t \leq t^*$, (D.1), (D.2) and $\partial_t = n^{-1} T$, we have for $C_{\text{ball}} > 0$ and $\varepsilon > 0$ sufficiently small, on $t_0^* \leq t \leq t^*$,

$$\int_{\Sigma_t} f^4 d\mu_g = \int_{t^*}^t \partial_{t'} \left( \int_{\Sigma_{t'}} f^4 d\mu_g \right) dt' + \int_{\Sigma_t} f^4 d\mu_g$$

$$= 4 \int_{t^*}^t \left( \int_{\Sigma_{t'}} \partial_{t'} f f^3 d\mu_g \right) dt' + \int_{\Sigma_t} f^4 d\mu_g$$

$$\lesssim \int_{M_{t_0^*, t^*}} |Df| h^t |f|^3 + \int_{\Sigma_t} f^4 d\mu_g$$

$$\lesssim \int_{M_{t_0^*, t^*}} |Df| h^t |f|^3 + \int_{\Sigma_t} f^4 d\mu_g$$

$$\lesssim \|Df\|_{L^\infty_t L^2(S_{\tilde{t}})} \|f\|_{L^\infty_t L^6(S_{\tilde{t}})}^3 + \int_{\Sigma_t} f^4 d\mu_g,$$

where we used (D.6) to compare the positive-definite norms $h^t$ and $h'$. This finishes the proof of Lemma 5.5.

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