ON FEYNMAN GRAPHS, MATROIDS, AND GKZ-SYSTEMS

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ABSTRACT. We show in several important cases that the $A$-hypergeometric system attached to a Feynman diagram in Lee–Pomeransky form, obtained by viewing the momenta and the nonzero masses as indeterminates, has a normal underlying semigroup. This continues a quest initiated by Klausen, and studied by Helmer and Tellander. In the process we identify several relevant matroids related to the situation and explore their relationships.

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1. Introduction

Throughout, $G$ is a graph with edge set $E := E_G$ and vertex set $V := V_G$. Denote by $T_G$ its set of $i$-forests, so $F \in T_G$ whenever it is circuit-free and the graph on the set of vertices of $G$ with the set of edges of $F$ has exactly $(i - 1)$ more connected components than $G$ does. The nomenclature comes from the fact that a $i$-forest in a connected graph has $i$ connected components. If $G$ is connected, a 1-forest is often called a spanning tree.

In the theory of Feynman integrals, edges correspond to particles, and vertices to particle interactions. Some of the vertices are labelled as “external”; the set of external vertices is denoted $V_{\text{Ext}}$. An external vertex connects to an external edge (that is not part of $G$) and these external edges represent the externally measurable in- and output particles that interact according to the graph.

Throughout we consider a mass function

$$m : E \rightarrow \mathbb{R}_{\geq 0},$$

and denote by $m_e$ the mass of the particle corresponding to edge $e$. As a matter of general notation, we call mass the edges $e$ with $m_e \neq 0$; the other edges are massless.

There is a momentum function $p$ on the external vertices of $G$, with values in the 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ with indefinite “norm" $p^2 = [(p_0, p_1, p_2, p_3)]^2 := p_0^2 - (p_1^2 + p_2^2 + p_3^2)$. Momentum conservation dictates that the momenta of the external particles must sum to zero. We will assume (see Hypothesis \[19\] below) that the momenta do not satisfy any other constraints. In particular, when measurements of experiment are taken, the momenta can be seen as generic (subject to summing to zero); this setup fits most QFTs.

No generality on the Feynman diagram is lost if one assumes that the underlying graph $G$ be connected, since disconnected graphs describe separate particle interactions. Slightly more generally, one may assume that the graph have no cut vertex: the removal of any single vertex of $G$ should not increase the number of connected components. This property is in the Feynman context referred to as (1VI), short for “one vertex irreducible”; see for example [Sch18]. Physically, the presence of a cut vertex means that the particle interaction can be interpreted as a two-stage process with independent parts.

A bridge is an edge whose removal increases the number of connected components. In the presence of bridges, as well as when the graph has edges linking some vertex to itself, the corresponding Feynman amplitude factors into amplitudes from simpler graphs. In physics, a connected graph without any edges linking a vertex to itself, and without bridges is called (1PI), short for “one particle irreducible”. It implies in particular that no edge is part of every 1-forest.

**Definition 1.1.** We will say that the graph $G$ is strongly 1-irreducible, abbreviated as (s1I) if it is both one particle irreducible and one vertex irreducible. Equivalently, such graphs are connected, and have no bridges, cut vertices or edges that link a vertex to itself.

Mathematically, the (sI) property is “the graphical (or, equivalently, the co-graphical) matroid to $G$ is connected”, see Subsection 2.3 below.

The graph $G$ induces several interesting functions on

$$\mathbb{R}^E := \bigoplus_{e \in E} \mathbb{R} \cdot e_e,$$

that lie inside the polynomial ring $\mathbb{C}[x_E]$ on variables $x_E := \{x_e \mid e \in E\}$ indexed by $E$ the dual graph polynomial

$$\mathcal{U} := \sum_{T \in T_G} (x^E / x^T),$$

where here and elsewhere, $x^S := \prod_{e \in S} x_e$ for any $S \subseteq E$, and more generally $x^a := \prod_{e \in E} x_1^{a_1}$ for $a \in \mathbb{Z}^E$.

\[1\] We will typically use $E$ and reserve $E_G$ for cases where extra clarity is needed, for example when several graphs are around.
Given a set of external momenta, a second polynomial can be derived from \( G \), namely
\[
F_0 := - \sum_{F \in T^2 \cap G} |p(F)|^2 \left( \frac{x^E}{x^F} \right).
\]
Here, \( p(F) \) is the sum of the momenta of the external vertices of \( G \) that belong to one of the two components \( F^2 \).\(^2\) Compare the introduction of \([\text{TH21}]\).

Many QFT techniques take recourse to Wick rotation, the coordinate transformation that multiplies the momentum coordinate \( p_0 \) by \( \sqrt{-1} \). We shall write \( F_0^W \) for the result of Wick rotation on \( F_0 \). The effect is that the Minkowski norm turns into the Euclidean norm, but it also moves the study of Feynman amplitudes to the complex domain. For certain purposes, such as considering families of Feynman type integrals in the spirit discussed below, this is no actual disadvantage.

In contrast to the momenta, there is no genericity assumption on the masses, and in particular they can be zero. One then defines
\[
F := U \cdot \left( \sum_{e \in G} m_e^2 x_e \right) + F_0^W.
\]
In the theory of Feynman integrals, in Lee–Pomeransky form, the function
\[
G_m := U + F = U \cdot (1 + \sum_{e \in G} m_e^2 x_e) + F_0^W
\]
and its integrals are relevant, see \([\text{Kla20, TH21}]\).

**Remark 1.2.** (1) The Lee–Pomeransky formalism assumes that the underlying graph \( G \) is of type (s1I). As noted, if a particle interaction is modeled by a graph that is not (s1I) then one can decompose the situation into subproblems whose graph is in fact (s1I).

(2) The Lee–Pomeransky form of the Feynman integral assumes Wick rotation. This means that one must allow for complex components in the momenta, which then raises the possibility of cancellation of coefficients in the sum \( F \), resulting in the possible disappearance of certain monomials. For degree reasons no cancellation can occur between terms of \( U \) and terms of \( F \).

\( \diamond \)

In order to avoid the pathologies mentioned the previous remark, we shall make the following assumptions.

**Hypothesis 1.3 (Feynman Hypotheses).** Throughout, we shall assume that

(1) the underlying graph \( G \) is (s1I) and has at least one edge (hence actually at least two);
(2) the values of the momenta are sufficiently generic, so that
   (a) in the sum \( U \cdot \left( \sum_{e \in G} m_e^2 x_e \right) + F_0^W \) no cancellation of terms occurs, and
   (b) no proper subset of \( V_{\text{Ext}} \) has zero momentum sum.
(3) At least one 2-forest term appears in \( G_m \).

\( \diamond \)

**Remark 1.4.** (1) Hypothesis 1.3(1) can be postulated since Feynman amplitudes to graphs that fail this condition can be decomposed into amplitudes that come from graphs that satisfy the condition.

(2) Hypothesis 1.3(2) is sometimes assumed without the requisite advertisement. It is always in force when the external momenta are in the Euclidean region. Moreover, for the purpose of studying Feynman integrals as a family (for example, via GKZ-systems), momenta can be viewed as generic (subject to the external momentum sum being zero), and then Hypothesis 1.3(2) holds as well.

\( ^2 \)Since the momenta sum is zero, both 2-forest components give the same coefficient.
(3) If Hypothesis 1.3.(3) is violated, the problem is trivialized to \( G_m = U \) in which case it is known that the semigroup spanned by its support vectors is normal. \([\text{TH21}])\).

\[\Box\]

Treating the nonzero masses and momenta as indeterminates, one arrives at a differentiable family of integrals. One method to study Feynman integrals is by computing differential equations that govern this family, and then solving them with a power series Ansatz. After that, one may consider the specialization of certain variables to special values, or one can investigate geometric behavior (such as monodromy) of the family.

Let

\[
A_m := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{pmatrix}
\]

be the matrix the columns of which are given by the lifted exponents \( a_i \) of the monomials \( x^{a_i} \) appearing in \( G_m \); here and elsewhere we call \((1, a) \in \mathbb{Z} \times \mathbb{Z}^E\) the lift of \( a \in \mathbb{Z}^E \). More generally, let \( A \) be any integer \((1 + |E|) \times n\) matrix. We shall refer to the group of integer linear combinations of the columns of \( A \),

\[
\mathbb{Z}A := \{ \sum m_i a_i \mid m_i \in \mathbb{Z} \}
\]

as the lattice of \( A \). In conjunction with any choice of a complex parameter vector \( \beta \in \mathbb{C} \times \mathbb{C}^E \), such matrix \( A \) induces a GKZ-system (or also called \( A \)-hypergeometric system) \( H_A(\beta) \) of linear partial differential equations in \( n \) new variables \( y_1, \ldots, y_n \), as we explain in the next section. It is known for \( A = A_m \) that a suitable choice of the parameter \( \beta \) causes the \( A_m \)-hypergeometric system \( H_{A_m}(\beta) \) to have among its solutions the family of Feynman integrals to the graph \( G \); see \([\text{Kla20, TH21}]\) for a down-to-earth discussion on this.

In the construction of the hypergeometric system \( H_A(\beta) \) enters a certain toric ideal \( I_A \) in the polynomial ring \( R_A = \mathbb{C}[\partial] \) in the partial differentiation operators \( \partial_1 := \frac{\partial}{\partial y_1}, \ldots, \partial_n := \frac{\partial}{\partial y_n} \); it is induced by the monomial map from \( \mathbb{C}^* \times (\mathbb{C}^*)^E \) to \( \mathbb{C}^n \) encoded in \( A \). Let \( I_A \) be the ideal of \( R_A \) describing the closure of the image of \( \mathbb{C}^* \times (\mathbb{C}^*)^E \) in \( \mathbb{C}^n \). If the quotient

\[
S_A := \mathbb{C}[NA] \simeq R_A/I_A
\]

enjoys a certain algebraic property known as Cohen–Macaulay, then various desirable simplifications regarding the solutions of \( H_A(\beta) \) occur. As is discussed in \([\text{Kla20}]\), of practical value in the theory of Feynman integrals are: suitable initial ideals of \( H_A(\beta) \) become computable in elementary fashion without need to look at Gröbner bases, and classical combinatorial recipes for manufacturing solutions become much simpler, see \( [\text{SST00}] \) for background on hypergeometric differential equations.

The Cohen–Macaulayness of \( S_A \) is implied by, but by no means equivalent to, the condition that the semigroup \( NA \subseteq \mathbb{R} \times \mathbb{R}^E \) be saturated, which means that the intersection of the non-negative rational cone \( \mathbb{R}_{\geq 0}N \) spanned by the columns of \( A \) over the origin with the lattice \( \mathbb{Z}A \) contains no other lattice points than those in \( NA \); see \( [\text{SST00, MMW05}] \) for more details on Cohen–Macaulayness in this context. Saturatedness is an arithmetic condition that involves study of the interior points of the dilations of the polytope spanned by the columns of \( A \).

For notation, let the support \( \text{Supp}(f) \) of a Laurent polynomial \( f = \sum c_a x^a \) be the exponent vectors

\[
\text{Supp}(f) := \{ a \mid c_a \neq 0 \}
\]

of the monomials appearing with nonzero coefficient in \( f \). Denoting the convex hull of a set \( S \subseteq \mathbb{R}^E \) by \( \overline{S} \), the support polytope of \( f \) is \( \text{Supp}(f) \). Let \( P_m \) be the support polytope of \( G_m \). Helmer and Tellander \([\text{TH21}]\) showed in the following two extreme cases that the semigroup of \( A_m \) is saturated:

1. in the massive case where each particle mass is positive,
2. in the massless case where each particle mass is zero (with the additional assumption that every vertex is external).
In both cases, this means that $S_{A_m}$ is Cohen–Macaulay. The tools they use include edge-unimodularity, flag matroid polytopes, Cayley and Minkowski sums, which they used to study IDP properties of polytopes.

In this note, we start with discussing the support vectors of $G_m$ from the point of view of matroid theory. Of course, the support vectors of $U$, interpreted as indicator functions, describe the co-graphical matroid of $G$. We show here that the support vectors of $F_0$ and those of the square-free terms in $U \cdot (\sum_{e \in G} m_e^2 x_e)$ both describe matroids as well. We also show that, quite surprisingly, their union forms a matroid as well. So, for all Feynman graphs the support vectors of the square-free terms of $F$ form a matroid.

We use these matroidal results, and some ideas of [TH21] to show that the semigroup generated by $A_m$ is saturated for (s11) graphs $G$ in the two cases

1. if every 2-forest of $G$ induces a nonzero term in $G_m$ (Theorem 4.3);
2. if $m_e = 0$ for all $e$ (Theorem 4.9);

which generalize the two corresponding cases in [TH21]. In consequence, in these cases $A_m$ defines a hypergeometric system that enjoys the Cohen–Macaulay property.

In the next section we set up the necessary notation, and carefully describe the needed details about hypergeometric systems, as well as graphs, polytopes and matroids. In Section 3, we discuss the advertised matroids, and in Section 4 we state and prove the semigroup results. Under Condition (1) above, this follows from an inspection of the way that the cone over $A_m$ behaves under specialization of a mass to zero. In the massless case we follow the route of [TH21] in the corresponding context. We also provide some partial results towards the general case. In the last section we discuss some examples of the failure of Hypothesis 1.3. For the convenience of the reader, we provide a list of symbols at the end.

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2. Notation and basic concepts

If $e \in E$ then we denote the unit vector of $\mathbb{R}^E$ pointing in $e$-direction by $e_e$, and if $S \subseteq E$ is a collection of edges then we write $v_S$ for the indicator vector of $S$ defined by

$$v_S = \sum_{e \in S} e_e.$$

2.1. Hypergeometric systems. We give here a minimal introduction to $A$-hypergeometric systems invented by Gel’fand, Graev, Kapranov and Zelevinsky in the mid-1980s. For details and literature on them and on parametric integrals that occur as their solutions we refer to the book [SST00, Sec. 5.4], and to the survey [RSSW21].

Take an integer matrix $A \in \mathbb{Z}^{(1+d) \times n}$, and a set of variables $y = y_1, \ldots, y_n$. Denote the partial derivative operators $\partial_i/\partial y_j$ by $\partial_j$ and consider the Weyl algebra $DA$ in variables $y_1, \ldots, y_n$ given as the non-commutative ring $\mathbb{C}[\partial](y)$. The elements of $DA$ can be interpreted as linear differential operators in $y$ with polynomial coefficients.
The matrix $A$ induces a monomial action
\[
(C^*)^{1+d} \times \mathbb{C}^n \to \mathbb{C}^n,
(t, \eta) \mapsto (t^{\eta_1}, \ldots, t^{\eta_n})
\]
of the $(1+d)$-torus on the affine space with coordinates $\partial_1, \ldots, \partial_n$. The usual closure of the orbit of the point $(1, \ldots, 1) \in \mathbb{C}^n$ is also Zariski closed, and defined by the toric ideal $I_A$ generated by the binomials $\square_{u,v} := \partial^u - \partial^v$, running over all $u, v \in \mathbb{N}^n$ with $A \cdot u = A \cdot v$. One may view $I_A$ as a subset of $D_A$ via the embedding of rings $\mathbb{C}[\partial] \hookrightarrow D_A$.

The matrix $A$ also induces $(1+d)$ Euler operators
\[
e_i := \sum_{j=1}^n a_{i,j} y_j \partial_j \in D_A \quad \text{for } 0 \leq i \leq d.
\]
Given a choice of $\beta \in \mathbb{C}^{1+d}$, the hypergeometric ideal to $A$ and $\beta$ is
\[
H_A(\beta) := D_A \cdot (I_A, \{E_i - \beta_i\}_{i=0}^d).
\]
Any left ideal $H = \sum D_A Q_i$ of $D_A$ generated by the operators $\{Q_i\}_i \subseteq D_A$ can be interpreted as a system of linear partial differential equations on a solution function $\phi(y)$, by asking that $Q \cdot (\phi(y)) = 0$ for all $Q \in H$ (or, equivalently, that $Q \cdot (\phi(y)) = 0$ for all $i$). As is explained in [TH21], if one reads the coefficients of $\mathcal{G}_m$ as parameters then the Feynman integrals corresponding to $A_m$ appear as solutions of $H_{A_m}(\beta)$ for the right choice of $\beta$. For the study of Feynman integrals, the entire family is useful; for some purposes even $\beta$ is viewed as a variable.

**Remark 2.1.** A frequent hypothesis in the theory of $A$-hypergeometric systems is that the group $\mathbb{Z}A$ generated by the columns of $A$ agrees with the ambient lattice $\mathbb{Z}^{1+d}$ inside $\mathbb{R}^{1+d}$. The hypothesis is not crucial to the majority of known results, but it usually allows a much simpler formulation. However, the question whether a semigroup ring is normal is only decided by the saturatedness of the semigroup in its own lattice, the group it generates.

### 2.2. Polytopes

A polytope $P$ in $\mathbb{R}^{1+d}$ is a lattice polytope if its vertices belong to the lattice $\mathbb{Z} \times \mathbb{Z}^d$ inside $\mathbb{R}^{1+d}$.

Given two polytopes $P, P'$ in $\mathbb{R}^E$, their Minkowski sum $P + P'$ is the set of points $\{w = v + v' \in \mathbb{R}^E \mid v \in P, v' \in P'\}$. The edges of a Minkowski sum are parallel to edges of the input polytopes. The vertices of a Minkowski sum are always sums of vertices of the input polytopes (although some such sums might be interior points of the sum polytope). In contrast, the set of the lattice points in a Minkowski sum is often not equal to the sum of the sets of lattice points in the two input polytopes.

Let us set
\[
E_m := \{e \in E \mid m(e) \neq 0\} \quad \text{and} \quad E_0 := E \setminus E_m.
\]
Writing $m_e$ for $m(e)$ to ease notation, set
\[
\Sigma_m := \sum_{e \in E_m} m_e^2 x_e \quad \text{and} \quad \Delta_m := \text{Supp}(\Sigma_m);
\]
the latter is the simplex in $\mathbb{R}^E$ spanned by the unit vectors $\{e_e\}_{e \in E_m}$.

We also set
\[
\hat{\Sigma}_m := 1 + \Sigma_m, \quad \text{and} \quad \hat{\Delta}_m := \text{Supp}(\hat{\Sigma}_m).
\]
If we already have a specific mass function $m$ in mind, we write
\[
\begin{align*}
\Sigma_E & := \Sigma_m + \sum_{e \in E_0} x_e, \\
\Delta_E & := \text{Supp}(\Sigma_E), \\
\hat{\Sigma}_E & := 1 + \Sigma_E, \\
\hat{\Delta}_E & := \text{Supp}(\hat{\Sigma}_E).
\end{align*}
\]
According to Hypothesis 2.3 the support polytope of $G_m$ is the same as the polytope spanned by the union $\text{Supp}(U) \cup \text{Supp}(U \cdot \Sigma_m) \cup \{\text{Supp}(p(F)^2 \cdot \mathbf{x}^F) | F \in T_G^2\}$ since in the sum $G_m = U \cdot \Sigma_m + F_0$ no terms are lost due to coefficient cancellations.

2.3. Graphs and their matroids. We generally use the graph and matroid language as it prevails in mathematics. So, for us a loop is an edge that is incident to only one vertex; a circuit is a set of edges whose union in a realization of the graph is homeomorphic to a polygon (in physics this is sometimes called a loop).

In each term of $\mathcal{U}$ and of $\mathcal{F}_0$, each variable appears (by definition) with degree at most one. On the other hand, $U \cdot \Sigma_m$ can have some terms with some variable of degree two (and the other variables of degree one or zero). Such square terms can occur only for massive variables (and if a variable is in fact massive then it will occur in some term with degree two since the corresponding edge cannot not belong to every 1-forest in the (sII) graph $G$).

A matroid $M$ is determined by a distinguished collection $\mathcal{B}_M \subseteq 2^E$ of bases, all of equal cardinality, taken from a fixed ground set $E$. From this angle, the defining property of a matroid is a version of the Exchange Axiom of Steiner from linear algebra: if $B, B'$ are two bases of a matroid, and $e \in B$, then there is $e' \in B'$ such that $(B \setminus \{e\}) \cup \{e'\}$ is again a basis. In fact, there is an equivalent “strong” version where in the same notation the set $(B' \setminus \{e'\}) \cup \{e\}$ can also be arranged to be a basis. The notion of a matroid generalizes the idea of linear independence of sets of vectors, and much of the nomenclature is borrowed from linear algebra. We refer to [Oxl11] for background and all facts that we use about matroids.

For example, matroids have a rank function

$$\text{rk}_M : 2^E \to \mathbb{N},$$

and the bases are precisely the minimal sets (with respect to inclusion) of maximum possible rank in $M$. The rank of a matroid is (by definition) the size of any of its bases (which is indeed a well-defined integer). A loop of a matroid $M$ is an element $e$ for which $\text{rk}_M(\{e\}) = 0$. To each basis $B$ one has an indicator vector $v_B$ in $\{0,1\}^E$ with $v_B(e) = 1$ if and only if $e \in B$; so the entry sum of any $v_B$ is the rank of $M$. The convex hull of the lattice vectors $\{v_B | B \in \mathcal{B}_M\}$ is the matroid polytope of $M$.

Every $v_B$ is a vertex of the matroid polytope, since it is even a vertex of the polytope spanned by all integer vectors that have only 0/1 entries and entry sum $\text{rk}(M)$. Indeed, among such integer vectors, $v_B$ realizes the unique maximum of the linear function that takes dot product against $v_B$.

A matroid is Boolean if $E$ itself is a basis (and then the only one). More generally, the Strong Exchange Axiom implies that the edges of the matroid polytope are precisely those that link (indicator vectors of) bases that agree in all but two positions. In particular, edges of the matroid polytope are parallel to the vectors $e_e - e_{e'}$ [GGMS87].

A circuit of a matroid is a set that is not contained in any basis, and minimal (with respect to inclusion) in this regard. Loops are circuits. An independent set is one that contains no circuit; independent sets are exactly those subsets of $E$ on which the rank function agrees with the cardinality function, and they can also be described as the sets that are subsets of bases. Bases are maximal independent sets, and proper subsets of circuits are independent.

If $G$ is a graph, the collection $T_G^1$ of 1-forests of $G$ forms the set of bases for a matroid $M_G^1$ on the underlying set $E$ of edges. Circuits of the graph are then circuits of $M_G^1$, and (graph-theoretic) loops correspond to (matroid-theoretic) loops. Matroids that arise this way are called graphic.

For a set of edges $S$ from $G$ (which we read as a subgraph of $G$ on the same vertex set $V_G$) we call their span the collection of all edges of $G$ that connect vertices of $G$ that belong to the same connected component in the subgraph $S$. In other words, the vertex partitions of $V_G$ by sets of connected components of $S$ and span ($S$) are the same, and span ($S$) is the largest subgraph of $G$ in this regard. Then $\text{rk}(S) = \text{rk}(\text{span}(S))$ is the difference of the number of components of $S$ (as graph on the vertex set of $G$) and $|V_G|$. The rank
function can also be interpreted as the size of the largest circuit-free subset, and span in a general matroid is the largest superset with the same rank as the given set.

The set of complements \( \{E \setminus T \mid T \in \mathcal{T}_1^G\} \) forms the set of bases for another matroid \( M_{G,1}^{1,\perp} \) on \( E \) that turns out to be dual to \( M_G \) in a suitable sense. For this *cographic* matroid \( M_{G,1}^{1,\perp} \), a loop is an edge that is part of every 1-forest of \( G \). Its removal thus disconnects the graph and such edge cannot occur in a (s1I) Feynman diagram. So, for an (s1I) graph, neither the graphic nor the cographic matroid has loops.

Similarly, the set of 2-forests \( \mathcal{T}_2^G \), as well as the set of their complements, form matroids that we denote \( M_2^G \) and \( M_{2,1}^{1,\perp} \) respectively.

Any matroid can be written as a matroid sum of simple matroids; a matroid is *simple* if it is impossible to write the set of bases \( B_M \) as the set of unions of the bases of two submatroids on disjoint subsets \( E_1, E_2 \) of \( E \). A graph is (s1I) if and only if its graphic and cographic matroid are simple.

Let \( x = \{x_e \mid e \in E\} \) be a set of indeterminates that are in correspondence with the elements of the ground set of \( M \). There is an induced matroid basis polynomial

\[
\Phi_M = \sum_{B \in B_M} x^B \in \mathbb{C}[x]
\]

with very interesting combinatorial properties. The polynomial \( U \) is the matroid basis polynomial \( \Phi_{M_{G,1}^{1,\perp}} \) of \( M_{G,1}^{1,\perp} \), and the induced polytope

\[
P_G^{1,\perp} := \text{Supp}(U)
\]

is the matroid polytope to \( M_{G,1}^{1,\perp} \). On the other hand, \( \Delta_E \) is the matroid polytope to the cographic matroid on \( E \) corresponding to a polygon with \(|E|\) edges (or to the graphic matroid to the graph on \(|E|\) edges with only two vertices and no loops; these are called banana or sunset graphs).

If \( M \) is any matroid on the set \( E \), then the semigroup

\[
\{ v_b \mid B \in B_M \}
\]

is saturated in its own lattice, by [Whi77, Thms. 1, 2].

For any pointed (i.e., no invertibles except for the neutral element) sub-semigroup \( S \) of a free Abelian group of finite rank, the semigroup ring \( \mathbb{C}[S] \) is normal if and only if \( S \) is saturated in the group generated by \( S \) according to [Hoc72], and this happens if and only if the semigroup of its lifts is saturated in the ambient lattice. All such semigroup rings are toric, and therefore their normality implies Cohen–Macaulayness.

By a *unimodular matrix* we mean here a matrix with integer entries whose maximal minors are all in the set \( \{-1, 0, 1\} \).

In [WH21] it is shown that the semigroup generated by the lifts of the support vectors of \( G_m \) is normal provided that either a) all masses are nonzero, or b) all masses are zero and every vertex is an external vertex.

### 3. Matroids in Feynman Theory

Recall that we assume that \( G \) satisfies the conditions in Hypothesis 1.3, and that \( \mathcal{T}_G^{1} \) and \( \mathcal{T}_G^{1,\perp} \) denote the collections of spanning trees and 2-forests of \( G \) respectively.

By Hypothesis 1.3, the monomials appearing in \( G_m \) are exactly those appearing \( U \), plus those appearing in either \( U \cdot \Sigma_m \) or \( F_0 \). The squarefree ones in these last two polynomials are indexed, respectively, by a massive edge in a spanning tree for \( G \), or a 2-forest with non-vanishing moment coefficient. In this section we investigate the matroidal properties of these two sets. They form the tools for the main results in the next section.

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3A more general class of polynomials arises from realizations of matroids, see for example [BEK06, Pat11, DSW21, DPSW21].
In order to simplify the discussion we introduce some language.

**Notation 3.1.** If \( G', G'' \) are subgraphs of \( G \) then if \( e \in E_G \) is an edge we say it *links* \( G' \) to \( G'' \) if it involves one vertex from \( G' \) and one vertex from \( G'' \). We further say that \( e \) is *supported on* \( G' \) if both vertices of \( e \) are vertices of \( G' \). This does not require that \( e \) be an edge of \( G' \).

\[\diamond\]

### 3.1. Momentous 2-forests.

**Definition 3.2.** A 2-forest \( F \in \mathcal{T}_2^2 \) is *momentum-free* if the momentum coefficient \(|p(F)|^2\) of \( x^{E \setminus F} \) in \( F_0 \) is zero. We denote the set of momentum-free 2-forests of \( G \) by \( \mathcal{T}^2_{G,0} \).

We call the elements of the complementary set

\[\mathcal{T}^2_{G,\#} := \mathcal{T}^2_G \setminus \mathcal{T}^2_{G,0}\]

the *momentous 2-forests*.

Note that, by Hypothesis 1.3, a 2-forest \( F \in \mathcal{T}^2_{G,\#} \) is a 3-forest \( F = F_1 \cup F_2 \) with connected components \( F_1, F_2 \) if either \( V_{\text{Ext}} \subseteq F_1 \) or \( V_{\text{Ext}} \subseteq F_2 \).

\[\diamond\]

As a very special example of a momentum-free 2-forest, let \( v \) be an interior vertex and let \( F \) be a spanning tree for the graph obtained by deleting \( v \) and all incident edges from \( G \). Then \( F \cup \{v\} \) is a 2-forest for \( G \) that lies in \( \mathcal{T}^2_{G,0} \). More extremely, if \( G \) were permitted to have only one external vertex, no momentous 2-forest would exist at all, and \( F_0 \) would be zero altogether.

**Lemma 3.3.** The set \( \mathcal{T}^2_{G,\#} \) is the set of bases of a matroid on the edge set \( E \) of \( G \).

**Proof.** If \(|V_{\text{Ext}}| = 1\), there are no momentous 2-forest, so there is nothing to show. So we assume that at least two external vertices exist.

If \( \mathcal{T}^2_{G,\#} \) is non-empty, we need to show that the set of momentous 2-forests satisfies the matroid basis exchange axiom. So, choose \( F \in \mathcal{T}^2_{G,\#} \), and suppose \( F' \) is an arbitrary second 2-forest. Choose \( e \in F \); then \( F \setminus \{e\} \) is a 3-forest \( F_1 \sqcup F_2 \sqcup F_3 \) of \( G \), where the \( F_i \) are the connected components of \( F \setminus \{e\} \).

Since the full collection \( \mathcal{T}^2_2 \) of 2-forests forms the set of bases of a matroid, some edges of \( F' \), when added to \( F \setminus \{e\} \), produce again a 2-forest. These are precisely those edges of \( F' \) that link \( F_i \) to \( F_j \), for \( i \neq j \) in \( \{1, 2, 3\} \).

Since \( F \) is in \( \mathcal{T}^2_{G,\#} \), the external vertices do not lie entirely inside one of the components of \( F \), and even less do they lie entirely inside a connected component \( F_i \) of \( F \setminus \{e\} \). Thus, after possibly relabeling, both \( F_1 \) and \( F_2 \), and possibly also \( F_3 \), will contain an external vertex. If \( F_3 \) does in fact contain an external vertex, then adding any edge \( f \in F' \) to \( F_1 \sqcup F_2 \sqcup F_3 \) will leave the external vertices split between at least two different connected components. Combined with the previous paragraph and Hypothesis 1.3(3) we can dispose of the case when \( F_3 \) also contains an external vertex.

Now suppose \( F_3 \) does not contain an external vertex, so \( V_{\text{Ext}} \) is in the disjoint union \( F_1 \sqcup F_2 \). If \( F' \) contains an edge \( f \) that links \( F_3 \) either to \( F_1 \) or to \( F_2 \), we are done, since then \( (F \setminus \{e\}) \cup \{f\} \) is a 2-forest in \( \mathcal{T}^2_{G,\#} \). So consider the possibility that \( F' \) has no such edge; then no edge of \( F' \) links \( F_3 \) to \( F_1 \sqcup F_2 \). This disconnection shows that the 2-forest \( F' \) has one connected component that uses the vertices of \( F_3 \), and one component that uses the vertices of \( F_1 \sqcup F_2 \). But then \( F' \) has \( V_{\text{Ext}} \) inside one of its components and thus can’t be in \( \mathcal{T}^2_{G,\#} \). The lemma follows.

\[\square\]

**Definition 3.4.** We denote the matroid of the previous lemma by \( M^2_{G,\#} \).

Recall that a matroid \( M' \) is a quotient of the matroid \( M \) if (they are matroids on the same ground set and) any circuit in \( M \) is a union of circuits in \( M' \).
Lemma 3.5. $M_{G,≠}^2$ is a quotient of $M_{G}^1$.

Proof. The graphic matroid $M_{G}^1$ of $G$ has as circuits the circuits of $G$. Suppose $C$ is one such circuit; it cannot be independent in $M_{G,≠}^2$ since it cannot be contained in any 2-forest. We will show that it is the union of circuits in $M_{G,≠}^2$.

If $M_{G,≠}^2$ is the trivial matroid, each singleton is a circuit, and the lamme follows. So, we can assume that $M_{G,≠}^2$ is not trivial.

For the moment assume that $C$ contains at least one, but not every, external vertex. Let $e$ be any edge of $C$. As $C \setminus \{e\}$ is independent in $M_{G}^1$, we can embed it into a spanning tree $T$ for $G$. Then let $v$ be an external vertex not in $C$. Since the set $C \setminus \{e\}$ is connected and $T$ is a tree, there is a unique shortest path in $T$ that connects $v$ with $C \setminus \{e\}$. Remove one of the edges $f$ in this shortest path to obtain from $T$ a 2-forest $F$ in which $v$ and $C \setminus \{e\}$ lie in different components. Note that by construction $f$ is not in $C$. It follows that $F$ is a basis in $M_{G,≠}^2$ and so $C \setminus \{e\} \subseteq F$ is independent in $M_{G,≠}^2$. Since this is so for any $e \in C$, $C$ is a circuit in $M_{G,≠}^2$.

Now suppose $C$ contains no external vertex. Again, remove an arbitrary edge $e \in C$ and embed the resulting $C \setminus \{e\}$ into a spanning tree $T$ for $G$. Choose any two external vertices $v, v'$. Within $T$ there is a unique minimal path from $v$ to $v'$. Since neither vertex is in $C$, there is at least one edge $f$ in this minimal path that does not belong to $C$. Remove $f$ from $T$ to arrive at a 2-forest containing $C \setminus \{e\}$. It is momentous by Hypothesis 1.3(3) since the external vertices are not all in one component. It follows that removing any edge from $C$ makes it independent in $M_{G,≠}^2$ and thus $C$ is a circuit in $M_{G,≠}^2$.

Finally, suppose $C$ contains all $\ell \geq 2$ external vertices. Denote the vertices of $C$ by $v_1, \ldots, v_\ell$, written in such a way that $(v_j, v_{j+1})$ are the edges of the circuit (with the understanding that $v_{\ell+1} = v_1$). Let $1 \leq i_1 < \ldots < i_\ell \leq \ell$ be the labels that correspond to the $\ell = |V_{\text{Ext}}|$ external vertices. Let $C_k$ be the result of removing from $C$ the edges $(v_{i_k}, v_{i_k+1}), \ldots, (v_{i_{k+1}} - 1, v_{i_{k+1}})$ that lie in the chosen orientation of $C$ between the external vertices $v_{i_k}$ and $v_{i_{k+1}}$. (Again, we agree that $v_{i_{\ell+1}} = v_1$). Then in $M_{G}^1$, these sets $C_k$ are independent, but in $M_{G,≠}^2$, they are still dependent since they contain all external vertices. We claim that $C_k$ is in fact a circuit in $M_{G,≠}^2$. Indeed, for any edge $e \in C_k$, the graph $C_k \setminus \{e\}$ has two connected components and $V_{\text{Ext}}$ is not contained in either: one component contains $v_{i_k}$ and the other contains $v_{i_{k+1}}$. Thus, $C_k \setminus \{e\}$ can be completed to a 2-forest such that neither of its components contains $V_{\text{Ext}}$ and is hence independent in $M_{G,≠}^2$. To finish the proof, observe that $C$ is covered by the various $C_k$.

\[\Box\]

3.2. Massive truncations.

Definition 3.6. A 2-forest $F$ that can be written as $T \setminus \{e\}$ for a spanning tree $T$ and a massive edge $e$ is called a massive truncation (of $T$ by $e$). We denote by $T_{G,m.t.}^2$ the collection of massive truncations. ♦

The massively truncated 2-forests are those that label nonzero squarefree terms in $F_{0}^W$.

Lemma 3.7. The set of massively truncated 2-forests forms the bases of a matroid on the edge set $E$ of $G$.

Proof. We need to show that the set of massively truncated 2-forests, if non-empty, satisfies the Exchange Axiom.

Let $F, F'$ be massively truncated 2-forests and choose massive edges $e, e'$ such that $T = F \cup \{e\}$ and $T' = F' \cup \{e'\}$ are spanning trees. Let $f \in F$ and consider the 3-forest $F \setminus \{f\}$ with connected components $F_1, F_2a, F_2b$ where $F_1$ is one component of $F$ and $F_2a \cup F_2b \cup \{f\}$ is the other. We need to show that for a suitable $g \in F'$, the set $(F \setminus \{f\}) \cup \{g\}$ is a massively truncated 2-forest.
Now suppose $E$ is a 2-forest. Moreover, the edge $e = T \setminus F$ from above links $F_1$ to either $F_{2a}$ or $F_{2b}$; without loss of generality we can and do assume that $e$ links in fact $F_1$ to $F_{2a}$.

If some edge $g$ of $F'$ links a vertex of $F_{2a}$ to a vertex of $F_{2b}$, then $(F \setminus \{f\}) \cup \{g\}$ is a 2-forest on the same connected components as $F$ and thus can be completed by the massive edge $e$ to a spanning tree. Similarly, if any edge $g$ of $F'$ links $F_1$ to $F_{2a}$, then $(F \setminus \{f\}) \cup \{g\}$ is a 2-forest in which $F_{2a}$ is a connected component and again the 2-forest $(F' \setminus \{f\}) \cup \{g\}$ can be completed by the massive edge $e$ to a spanning tree. So, assume from now on that $F'$ has no edges from $F_{2a}$ to $F_{2b}$, and no edges from $F_1$ to $F_{2b}$.

In that case, the vertices of $F_{2b}$ must be exactly the vertices in one of the two components of the 2-forest $F'$ and therefore the other component of $F'$ uses exactly the vertices of $F_1 \cup F_{2a}$. In particular there is guaranteed to be an edge $g$ in $F'$ from a vertex of $F_1$ to a vertex of $F_{2a}$. Note that $(F \setminus \{f\}) \cup \{g\}$ is then a 2-forest. Now recall that $F' = F' \setminus \{e'\}$ is a massive truncation. Clearly, $e'$ must connect the two components of $F'$ and so links $F_{2b}$ to either $F_1$ or $F_{2a}$. In that case, $(F \setminus \{f\}) \cup \{g\}$ is a massive truncation by $e'$.

**Definition 3.8.** We denote the matroid of massively truncated 2-forests of $G$ from Lemma 3.7 by $M_{G,m}^2$. ◦

We show next that the matroid of massively truncated 2-forests is also quotient of $\mathcal{T}_G^1$, but we use a different strategy than for the momentous 2-forests.

**Definition 3.9.** Suppose $M$ is a matroid on the set $E$ and $E' \subseteq E$. Define $B_{E'}$ to be the set of subsets $B$ of $E$ that have the property that there is some $e' \in E' \setminus B$ such that $B \cup \{e'\}$ is a basis in $M$.

By Lemma 3.11 below, the sets in $B_{E'}$ are the bases of a matroid that we denote $M_{E'}$. Note that when $E'$ has rank zero (so $E'$ contains only loops) then $M_{E'}$ is the trivial matroid. ◦

**Lemma 3.10.** The set $B_{E'}$ is the set of bases of a matroid.

**Proof.** Let $B, B' \in B_{E'}$ and choose $f \in B$. Let $e \in E' \setminus B$ and $e' \in E' \setminus B'$ be such that $B \cup \{e\}, B' \cup \{e'\}$ are bases for $M$. Then for some element $g$ of $B' \cup \{e'\}$ the Exchange Axiom in $M$ guarantees that $((B \cup \{e\}) \setminus \{f\}) \cup \{g\}$ is a basis for $M$. Since $f \neq e \neq g$, $(B \setminus \{f\}) \cup \{g\}$ is the new basis for $M_{E'}$ that we want. ◇

**Remark 3.11.** Note that the independent sets of $M_{E'}$ are those contained in a basis of $M_{E'}$ and therefore are the subsets of $E$ that can be augmented to an independent set in $M$ by an element of $E'$. In particular, if $e' \in E'$ is a loop, then $M_{E'} = M_{E' \setminus \{e'\}}$. ◦

**Lemma 3.12.** The matroid $M_{E'}$ is a quotient of the matroid $M$.

**Proof.** We shall assume that $E'$ has positive rank. Let $C$ be a circuit of $M$; in particular, $|C| = \text{rk}(C) + 1$.

Suppose first, that the span of $C$ does not contain all of $E'$, and choose $e' \in E' \setminus C$. Select $c \in C$. Then $|(C \setminus \{c\}) \cup \{e'\}| = |C| = \text{rk}(C) + 1 = \text{rk}(C \cup \{e'\}) = \text{rk}((C \setminus \{c\}) \cup \{e'\})$. It follows that $(C \setminus \{c\}) \cup \{e'\}$ is independent in $M$ and hence $C \setminus \{c\}$ is independent in $M_{E'}$. Thus, in this case $C$ is a circuit in $M_{E'}$.

Now suppose $E' \subseteq \text{span}(C)$. Let $C'$ be a minimal subset of $C$ such that $\text{span}(C')$ contains $E'$. Note that $C'$ is dependent in $M_{E'}$. Take $e' \in C'$; then $\text{span}(C' \setminus \{e'\})$ does not contain $E'$ and so there is $e \in E'$ such that $\text{rk}(C' \setminus \{e'\}) < \text{rk}((C' \setminus \{e'\}) \cup \{e\})$. Since $C' \setminus \{c\}$ is independent in $M$, so that rank agrees with cardinality, the same is true for $(C' \setminus \{c\}) \cup \{e'\}$ and so this set must be independent in $M$. It follows that the $M_{E'}$-dependent set $C'$ is covered by the $M_{E'}$-independent sets $C' \setminus \{c'\}$ and thus a $M_{E'}$-circuit.

Next, consider $C \setminus \{c\}$ for $c' \in C'$. Since $C$ is an $M$-circuit, span$(C \setminus \{c\}) = \text{span}(C) \supseteq E'$ and it follows that $C \setminus \{c\}$ is $M_{E'}$-dependent. For any $c \neq c'$ in $C$, we have $\text{rk}((C \setminus \{c, c'\}) \cup E') > \text{rk}(C \setminus \{c, c'\}) = |C| - 2$. (Indeed, a flat that contains $C \setminus \{c\}$ and $E'$ must contain $C'$, and so any flat that contains $(C \setminus \{c, c'\}) \cup E'$ must contain $(C \setminus \{c\})$, of rank $|C| - 1$). It follows, that for a suitable element $e' \in E' \setminus (C \setminus \{c, c'\})$, $(C \setminus \{c, c'\}) \cup \{e'\}$ is $M$-independent. It follows that $C \setminus \{c, c'\}$ is $M_{E'}$-independent, and so $C \setminus \{c\}$ is a $M_{E'}$-circuit.
The union of all circuits discussed covers $C$. 

**Corollary 3.13.** The matroid $M_{G,\text{m.t.}}^2$ is a quotient of $M_G^1$.

*Proof.* In the previous lemma, take $M = M_G^1$ and $E'$ to be the massive edges. Then the definition of $M_{G,\text{m.t.}}^2$ matches that of $(M_G^1)/E'$. □

**Remark 3.14.** The term “quotient matroid” comes historically from the following quotient construction of vector spaces. Suppose $R = \{r_e\}_{e \in E} \subseteq \mathbb{R}^n$ is a collection of vectors. It is often referred to a representation of a matroid. Indeed, define a matroid $M$ by selecting as bases of the matroid the maximal independent subsets of $R$. That these form indeed a matroid follows from the Exchange Property in linear algebra. The rank function corresponds to span dimension and independence is the same on both sides.

It is well-known that not all matroids are representable. In fact, most are not.

Suppose further that $E' \subseteq E$ is a distinguished subset. Let $\rho$ be a general linear combination of the elements of $R' := \{r_e\}_{e \in E'}$ and consider the quotient set $R/\rho$ in the quotient space $R'/\mathbb{R}\cdot \rho$. Let $M/\rho$ be the matroid defined by the representation $R/\rho$. Then:

1. The bases $B_{M/\rho}$ of $M/\rho$ are precisely the sets $\{e_1, \ldots, e_{\text{rk}(M)-1}\}$ for which there exists $e \in E'$ such that the collection $\{r_{e_1}, \ldots, r_{e_{\text{rk}(M)-1}}, r_e\}$ is a basis for $M$. In particular, the rank of $M/\rho$ is $\text{rk}(M) - 1$.
2. The matroid $M/\rho$ is a quotient of $M$.

It is customary, if $E' = E$, to say that $M/\rho$ is the truncation of $M$.

We do not know whether the matroid of momentous $2$-forests allows a quotient construction by truncations, but it seems unlikely. □

### 3.3. 2-forests of $G_m$.

Astoundingly, the union of the two matroids $M_{G,\#}^2$ and $M_{G,\text{m.t.}}^2$ is also a matroid, as we show next.

**Proposition 3.15.** The set of $2$-forests in the Feynman graph $G$ that arises as the union of the momentous $2$-forests and the massively truncated $2$-forests forms the set of bases of a matroid.

*Proof.* Let $F, F'$ be in $M_{G,\text{Feyn}}^2$. We need to show the validity of the simple Steiner exchange axiom. Since $M_{G,\text{m.t.}}^2$ and $M_{G,\#}^2$ are matroids by Lemmas 3.3 and 3.3, it suffices to consider the two cases listed below.

**Case 1:** $F$ is momentous and $F'$ is massively truncated. Let $e \in F$ be any edge; then $F \setminus \{e\}$ is a $3$-forest, with components denoted $F_1, F_{2a}, F_{2b}$ where $e$ links $F_{2a}$ to $F_{2b}$. Since the set of all $2$-forests is in fact a matroid, there is at least one edge $g \in F'$ such that $(F \setminus \{e\}) \cup \{g\}$ is a $2$-forest. If this is a momentous $2$-forest we are done with this case. So, in the sequel we assume that no edge of $F'$ combines with $(F \setminus \{e\})$ to a momentous $2$-forest.

Let $g \in F'$ form a $2$-forest $(F \setminus \{e\}) \cup \{g\}$. Then, since $F$ itself is momentous and $(F \setminus \{e\}) \cup \{g\}$ contains no circuits, such $g$ cannot link $F_{2a}$ to $F_{2b}$ and so will connect either $F_1$ to $F_{2a}$ or $F_1$ to $F_{2b}$. Depending on the case, the implication would be that the external vertices are either completely contained in $F_1 \cup F_{2a}$ or in $F_{2b}$, or in $F_1 \cup F_{2b}$ or in $F_{2a}$. In other words, the external vertices are either contained completely in $F_1 \cup F_{2a}$ or in $F_1 \cup F_{2b}$. Without loss of generality, let us assume they are all inside $F_1 \cup F_{2a}$ and so none is in $F_{2b}$.

Note that momentousness of $F$ implies that some external vertices are in $F_1$ and some in $F_{2b}$. In particular then, the edge $g$ from the start of this paragraph that creates the non-momentous $2$-forest $(F \setminus \{e\}) \cup \{g\}$ connects a vertex of $F_1$ to a vertex of $F_{2b}$.

It follows that if for no edge $g \in F'$ the set $(F \setminus \{e\}) \cup \{g\}$ is a momentous $2$-forest, then all edges of $F'$ are either supported on one of $F_1, F_{2a}, F_{2b}$, or they must connect $F_1$ to $F_{2b}$. That means that all edges of $F'$ are supported either on $F_{2a}$, or on $F_1 \cup F_{2b}$, implying that the vertex sets of $F_{2a}$ and $F_1 \cup F_{2b}$ are the same as the vertex sets of the two components of $F'$. 

Now recall that $F'$ is massively truncated, and let $f$ be a massive edge such that $F' \cup \{f\}$ is a spanning tree. By the previous paragraph, $f$ must link a vertex of $F_1 \cup F_{2b}$ to one of $F_{2a}$. It follows that $(F \setminus \{e\}) \cup \{g\}$ is massively truncated via $f$.

**Case 2: $F$ is massively truncated and $F'$ is momentous.** Fix an edge $e \in F$, and a massive edge $f$ such that $F' \cup \{f\}$ is a spanning tree. Then $F \setminus \{e\}$ has three components $F_1, F_{2a}, F_{2b}$ with $e$ linking a vertex from $F_{2a}$ to one from $F_{2b}$, and $f$ linking $F_1$ to either $F_{2a}$ or $F_{2b}$. Without loss of generality, assume the latter case.

Since 2-forests form a matroid, at least one edge $g$ of $F'$ turns $F \setminus \{e\}$ into a 2-forest. Suppose all edges $g$ of $F'$ are either supported on one of $F_1, F_{2a}$ or $F_{2b}$, or make it impossible to certify $(F \setminus \{e\}) \cup \{g\}$ as massively truncated via $f$ (i.e., $(F \setminus \{e\}) \cup \{g\} \cup \{f\}$ contains a circuit). Then all edges of $F'$ are either supported on one of $\{F_1, F_{2a}, F_{2b}\}$, or link $F_1$ to $F_{2b}$. Note that therefore an edge $g \in F'$ linking $F_1$ to $F_{2b}$ must exist, as else $F'$ should have more than two components. Since $F'$ has exactly two components, these must be supported on $F_1 \cup F_{2b}$ and $F_{2a}$ respectively. Since $F'$ is momentous, $F_{2a}$ contains some but not all external vertices. Then with the edge $g \in F'$ that links a vertex from $F_1$ to one of $F_{2b}$, we find that $(F \setminus \{e\}) \cup \{g\}$ is momentous, finishing the second case and the proof. □

**Definition 3.16.** We denote the matroid from the previous lemma by $\mathcal{M}_{G, \text{Feyn}}^2$.

4. **Main Theorems**

4.1. **All 2-forests present.** We recall a result from [1121] that will be used in the proof below.

**Theorem 4.1.** In the massive case, the semigroup spanned by the lifts of $\text{Supp}(\mathcal{G}_m)$ is normal. □

In the massive case, the momenta are inconsequential since the terms involving $\tilde{\Sigma}_m \cdot \mathcal{U}$ alone ensure that the support of $\mathcal{G}_m$ is as large as it can possibly be for any mass and any moment function—keeping in mind Hypothesis 1.3. We shall prove here that the conclusion continues to hold as long as every 2-forest of $G$ contributes to the support of $\mathcal{G}$; it is immaterial which terms with squares appear.

For this, recall Equations (2.2.1), (2.2.2) and set

$$\mathcal{G}_E := \mathcal{U} \cdot \tilde{\Sigma}_E + \mathcal{F}_0^W.$$  

By the genericity hypothesis on the momenta, all momomials that appear in $\mathcal{G}_m$ also appear in $\mathcal{G}_E$.

**Remark 4.2.** An idea that will be used repeatedly is the obvious observation:

$$(1\text{-forest complement) } \cup \text{ (element outside the 1-forest)} = (2\text{-forest complement).}$$

By the (s11) condition, any given edge $e$ is not a loop, and hence contained in a 1-forest $T$. Removing $e$ arrives at a 2-forest $F = T \setminus \{e\}$, those to $x^{E \setminus T}$ and to $x^{E \setminus F}$. If $F$ labels a nonzero term in $\mathcal{G}_m$ then $A_m$ contains two columns whose difference is $e_\mu$. It follows that $\mathbb{Z} A_m$ contains $\mathbb{Z} E$, and therefore also $\mathbb{Z} \times \mathbb{Z}^d$. Thus, when all 2-forests are present (and in most other cases), the lattice of $A_m$ agrees with the ambient lattice. □

In the massive case, the supports of $\mathcal{G}_E$ and $\mathcal{G}_m$ agree, and hence the semigroup generated by $(1, \text{Supp}(\mathcal{G}_E))$ is saturated in $\mathbb{Z} \times \mathbb{Z}^d = \mathbb{Z} A_m$ by the Helmer–Tellander result. Our strategy will be to show that as long as all 2-forests of $G$ contribute to the support of $\mathcal{G}_m$, then the semigroup to the lifts of $\text{Supp}(\mathcal{G}_m)$ can be obtained from the semigroup to the lifts of $\text{Supp}(\mathcal{G}_E)$ by intersecting with suitable halfspaces of $\mathbb{C} \times \mathbb{C}^E$. The point is that halfspaces contain saturated semigroups, and intersections of saturated semigroups are saturated.

Let us denote by $\mu_\epsilon : \mathbb{C}^E \rightarrow \mathbb{C}$ the $\epsilon$-th coordinate function on $\mathbb{C}^E$; on $\mathbb{C} \times \mathbb{C}^E$ we include the coordinate function $\mu_0$ on the first factor into the notation.
We can now prove the following generalization of [TH21, Thm. 1.1, part 1]:

**Theorem 4.3.** Let \( G \) be a (s1I) Feynman graph with mass function \( m: E \to \mathbb{R}_{\geq 0} \) satisfying Hypothesis 4.1.1. If \( M^2_G = M^2_{G, \text{Ferm.}} \) or equivalently if every 2-forest of \( G \) contributes to \( \text{Supp}(G) \), then the semigroup \( \mathbb{N}A_m \) is saturated and thus the semigroup ring \( \mathbb{K}[\mathbb{N}A_m] \) is normal and Cohen–Macaulay for all fields \( \mathbb{K} \).

**Proof.** That the second statement follows from the first is contained in [Hoc72].

Comparing the terms in \( \mathcal{G}_m \) and \( \mathcal{G}_E \) in light of our assumptions, \( \mathcal{G}_m \) arises from \( \mathcal{G}_E \) by canceling in \( \tilde{\Sigma}_E \) all terms that are divided by the square of a massless variable, and no others. In other words, the monomials \( x^a \) in \( \text{Supp}(\mathcal{G}_m) \) are precisely those in \( \text{Supp}(\mathcal{G}_E) \) whose lifted exponent \((1, a)\) satisfies \( \mu_0((1, a)) \geq \mu_e((1, a)) \) for all massless \( e \in E \).

Let \( A_E \) denote any matrix whose columns are the lifted support exponents of \( \mathcal{G}_E \); in particular, we could order \( A_E \) in such a way that \( A_m \) becomes a submatrix. For elements \((k, a)\) in \( \mathbb{N}A_E \) or \( \mathbb{N}A_m \), we call \( k = \mu_0((k, a)) \) their *degree*. We have noted above that, as subsets of \( \mathbb{Z} \times \mathbb{Z}^E \),

\[
A_m = A_E \cap \bigcap_{m_e = 0} H_e
\]

where

\[
H_e := \{ \alpha \in \mathbb{R} \times \mathbb{R}^E \mid \mu_0(\alpha) \geq \mu_e(\alpha) \}
\]

is the half-space on which \( \mu_0 - \mu_e \) is non-negative. It follows also that

\[
\mathbb{N}A_m \subseteq (\mathbb{N}A_E) \cap \bigcap_{m_e = 0} H_e,
\]

and the remainder of the proof is devoted to showing that this is an equality, which would show that \( \mathbb{N}A_m \) is the intersection of saturated semigroups, hence saturated itself.

Take any lattice element \((k, a)\) in the cone \( \mathbb{R}_{\geq 0}A_m \) of degree \( k \). Since \( \mathbb{N}A_E \) is saturated according to Corollary 4.1.1 one has \( (\mathbb{R}_{\geq 0}A_E) \cap (\mathbb{Z} \times \mathbb{Z}^E) = \mathbb{N}A_E \). Since \( (\mathbb{R}_{\geq 0}A_m) \subseteq (\mathbb{R}_{\geq 0}A_E) \), one can write

\[(4.1.1) \quad (k, a) = (1, a_1) + \ldots + (1, a_k)\]

where each \((1, a_i)\) is a column of \( A_E \).

Note that \((k, a) \in (\mathbb{R}_{\geq 0}A_m) \subseteq H_e\) for all massless \( e \in E_0 \). We will show that, given \( e \in E_0 \), the condition \((k, a) \in H_e\) implies that one can rewrite the sum \((4.1.1)\) in such a way that the following exchange rules hold:

- the new sum only uses summands that are columns of \( A_E \);
- the number of summands is unchanged;
- each summand lies in \( H_e \),

and that, moreover, it can be arranged that

- if all summands were originally in \( \bigcap_{e' \in E'} H_{e'} \) for some set \( E' \subseteq E \), then this holds after the rewriting for the larger set \( E' \cup \{e\} \).

Establishing this rewriting forms the main part of the proof. Indeed, given such rewriting result, fix a massless edge \( e \in E_0 \). Our exchange rules above allow to change the sum in \((4.1.1)\) into one where each support vector is in \( H_e \). Since no exchange operation introduces square terms that were not there before, we can treat \((4.1.1)\) one \( e \in E_0 \) at the time and arrive at a sum as in \((4.1.1)\) in which every term is in \( H_e \) for each \( e \in E_0 \). But that implies that we have written \( a \) as a sum of \( k \) exponent vectors that appear in \( \mathcal{G}_m \), implying that \( \mathbb{N}A_m \) is saturated.

Before we engage in the rewriting, note that for \( e \in E \), the monomials \( x^{a_j} \) appearing in \( \mathcal{G}_E = \tilde{\Sigma}_E \cdot \mathcal{U} + \mathcal{F}_0 \) fall into three categories, depending on whether \( \mu_e(a_j) \) is 0, 1, or 2. Alternatively, they are classified by the value of \( (\mu_0 - \mu_e)((1, a_j)) \in \{-1, 0, 1\} \). Those with \( \mu_e(a_j) = 0 \) fall themselves into two classes:
(1) squarefree monomials without \( x_e \) from \( \mathcal{F}_0 \) or from \( U \cdot \Sigma_E \);
(2) monomials from \( U \cdot \Sigma_E \) that contain a square but not \( x_e \).

Now suppose that the sum decomposition (4.1.1) involves an element \((1, a_i)\) that is not in the positive real cone of \( A_m \) and therefore satisfies \( \mu_e(a_i) = 2 \) for some (necessarily unique) \( e \) with \( n_e = 0 \). In particular, \( a_i \) does then not appear in \( \text{Supp}(U) \) and so we will have \(|a_i| = r + 1\).

Since \( a_i \) is a support vector of \( G_E \) with \( \mu_e(a_i) = 2 \), \( x^{a_i} \) appears in \( U \cdot \Sigma_E \) and so

\[
\begin{align*}
(4.1.2) \quad x^{a_i} = x^{E \setminus T} x_e \quad \text{with } T \in \mathcal{T}_1 \text{ and } e \not\in T.
\end{align*}
\]

Since \((\mu_e - \mu_0)((1, a_i)) > 0 \) but \((\mu_e - \mu_0)((k, a)) \leq 0 \) there must appear a semigroup element \((1, a_j)\) in (4.1.1)

with \((\mu_e - \mu_0)((1, a_j)) < 0 \); choose one such. It must be of one of the types (1), (2) or (3) above.

Case 1: Suppose \( a_j \) is of type (1); then \( x^{a_j} = x^{E \setminus F} \) for some 2-forest \( F \in \mathcal{T}_2 \) with \( e \in F \).

The union \( T \cup \{e\} \) has exactly one circuit \( C \), \( C \) contains \( e \), and \( F \setminus \{e\} \) is a 3-forest. Since \( C \) is a circuit, \( C \setminus \{e\} \) has the same span as \( C \), and so \( \text{span}(C \setminus \{e\}) \cup (F \setminus \{e\}) = \text{span}(C \cup (F \setminus \{e\})) = \text{span}(C \cup F) \), which contains the 2-forest \( F \). Thus, there is a suitable edge \( f \in C \setminus \{e\} = C \cap T \) that combines with the 3-forest \( F \setminus \{e\} \) to a set of rank greater than \( rk(F \setminus \{e\}) \). For such \( f \), \( (F \setminus \{e\}) \cup \{f\} \) is therefore a 2-forest.

However, so is \( T \setminus \{f\} \), and so by the assumptions of the theorem the monomials \( x^{a_i} := x^{E \setminus (T \setminus \{f\})} \) and \( x^{a_j} := x^{E \setminus (F \setminus \{e\}) \cup \{f\}} \) appear in \( \mathcal{F}_0 \). Moreover, their product is \( x^{a_i} x^{a_j} = x^{E \setminus T} x^{E \setminus F} x_e = x^{a_i} x^{a_j} \) and so \((1, a_i) + (1, a_j) = (1, a_i') + (1, a_j') \) in \( \mathcal{N} \mathcal{A}_E \). We can thus replace \( a_i \) by \( a_i' \) and \( a_j \) by \( a_j' \) while preserving (A.1) as a sum in \( \mathcal{N} \mathcal{A}_E \). Note that the replacement terms have no square terms and so no new terms with squares in any variable have been introduced while the overall number of square terms has in fact decreased.

Case 3: Next consider type (3), where \( a_i \) is a support vector of a term in \( \Sigma_E \cdot U \) with \( \mu_f(a_j) = 2 \) for some \( f \in E \), while \( \mu_e(a_j) = 0 \). Thus, (we still have \( a_i \) as in (4.1.1) and) \( x^{a_i} = x_f x^{E \setminus S} \) for some 1-forest \( S \) of \( G \) that does not involve \( f \) (since else \( x_f \) is linear in \( x_f x^{E \setminus S} \)) but does involve \( e \) (so that \( x_e \) does not appear in \( x_f x^{E \setminus S} \)).

Then \( T \cup \{e\} \) contains a unique circuit \( C \supset e \), and the span of \( (C \setminus \{e\}) \cup (S \setminus \{e\}) \) contains \( \text{span}(C \cup (S \setminus \{e\})) = \text{span}(C \cup S) ≥ \text{span}(S) = E \). It follows that some element \( g \in (C \setminus \{e\}) \subset C \cap T \) different from \( e \) turns the 2-forest \( S \setminus \{e\} \) back into a 1-forest. As removal of \( g \) from \( T \cup \{e\} \) breaks the unique circuit \( C \) in \( T \cup \{e\} \), \( (T \cup \{e\}) \setminus \{g\} \) is a 1-forest. Then, \( (x_e x^{E \setminus T}) (x_f x^{E \setminus S}) = (x_e x^{E \setminus (T \cup \{g\})}) (x_f x^{E \setminus (S \cup \{g\}) \setminus \{e\}}) \). In (4.1.1), replace \((1, a_i) + (1, a_j)\) by the sum of \((1, E \setminus (T \setminus \{g\})) = (1, a_i + e_g - e_e) \) and \((1, E \setminus (S \cup \{g\} \setminus \{e\}))) + (0, e_f) = (1, a_j + e_e - e_g) \). Both new terms are lifts of support vectors of \( G_E \), both are in \( H_e \), and the only square factor in either one is \( x^2_f \) in the second one, inherited from \( a_j \).

This finishes the rewriting claim, and as explained above proves the theorem.

\[ \square \]

It is natural to ask under what conditions we have the equality \( M^2_G = M^2_{G,Feyn} \); we address this question next.

**Definition 4.4.** A path \( v_0, v_1, \ldots, v_t \) of vertices in \( G \) (with \( \{v_i, v_{i+1}\} \) adjacent for all \( 0 \leq i < t \)) is called **massive** if all edges \( \{v_i, v_{i+1}\} \) are massive.

**Theorem 4.5.** In an (sI) graph \( G \), the equality \( M^2_G = M^2_{G,Feyn} \) holds if and only if every vertex of \( G \) permits a massive path to an external vertex of \( G \).

**Proof.** By Hypothesis (1), all momentous 2-forests label a nonzero term in \( \mathcal{G}_m \). Thus, assume that the 2-forest \( F \) is not momentous, so that one of the two components of \( F \) contain all external vertices.

Then \( F \) will cause a nonzero term in \( \mathcal{G}_m \) precisely if it is a massive truncation. In other words, if and only if there is a massive edge \( e \) such that \( F \cup \{e\} \) is a 1-forest.
Since one of the components of \(F\) contains all external vertices, the failure of such a massive edge \(e\) to exist implies that the vertices in the other component of \(F\) cannot be linked to \(V_{\text{Ext}}\) by a massive path.

Conversely, suppose that some vertex \(v\) cannot be linked to \(V_{\text{Ext}}\) by a massive path. We now delete from \(G\) all massive edges and call the result \(G'\). Then \(v\) belongs to a connected component \(U\) of \(G'\) that does not include any external vertex. Take any 2-forest for \(G\) that has one connected component supported in \(U\), and the other on \(G \setminus U\). By our choices, this 2-forest is neither massively truncated nor momentous and hence does not contribute to \(G_m\).

\[\Box\]

### 4.2. The general massless case.

In [TH21], Helmer and Tellander proved that if every vertex of \(G\) is an external vertex, then the semigroup \(\mathbb{N}A_m\) is normal for the mass function that is identically zero. The advantage of the condition on \(V_{\text{Ext}}\) is that it places us in a special case of Theorem 4.8 above, and guarantees that \(G_m\) involves a term from every 2-forest, \(M^2_G = M^2_{G,\text{Feyn}}\). As it turns out, this condition can be completely removed: we now use our results from Section 3 to dispose of the general massless case.

We need to review edge-unimodularity and IDP properties of polytopes.

**Definition 4.6.** An integer matrix is **unimodular** if all maximal minors are in the set \{-1, 0, 1\}. A lattice polytope \(P\) is **edge-unimodular** if there is an integer unimodular matrix \(M\) such that all edges of \(P\) are parallel to columns of \(M\).

**Definition 4.7.** A lattice polytope \(P \subseteq \mathbb{Z}^d\) is said to have the **IDP property** or to be **normal** if the intersection \((kP) \cap \mathbb{Z}^d\) agrees with the sum \(((k-1)P \cap \mathbb{Z}^d) + (P \cap \mathbb{Z}^d)\) for all \(k \in 1 + \mathbb{N}\).

The benefit of the IDP property to the present context is that it is equivalent to the equation
\[
\mathbb{N}((1, P) \cap (\mathbb{Z} \times \mathbb{Z}^d)) = \mathbb{R}_{\geq 0}((1, P)) \cap (\mathbb{Z} \times \mathbb{Z}^d).
\]

In other words, a polytope is IDP if and only if the semigroup generated by the lattice points in its lift is saturated in \(\mathbb{Z} \times \mathbb{Z}^d\).

The following result is due to Howard.

**Theorem 4.8 (How07b Thm. 4.5).** Suppose that \(A \in \mathbb{Z}^{d \times n}\) is a unimodular matrix, and that \(P\) and \(Q\) are lattice polytopes with edges parallel to columns of \(A\). Then, \((P \cap \mathbb{Z}^d) + (Q \cap \mathbb{Z}^d) = (P + Q) \cap \mathbb{Z}^d\).

In fact, the theorem is stated in a much more constrained context (inside a lattice of weights of a Lie algebra) and in a more opaque way, but the proof works in the generality stated here (which is also the version Howard states in [How07a Thm. 1]). As Howard points out, this implies that if \(P\) is a lattice polytope with edges parallel to the columns of a unimodular matrix, then \(P\) is IDP and in consequence the semigroup generated by the lattice points in the lifted polytope \((1, P)\) inside \(\mathbb{Z} \times \mathbb{Z}^d\) is saturated.

**Theorem 4.9.** Let \(G\) be a \((s|I)\) Feynman graph with mass function \(m: E \to \mathbb{R}_{\geq 0}\) that is identically zero: \(m_e = 0\) for all \(e\). Then the semigroup \(A_m\) is saturated and thus the semigroup ring \(\mathbb{K}[\mathbb{N}A_m]\) is normal and Cohen–Macaulay for all fields \(\mathbb{K}\).

**Proof.** The proof follows the one from [TH21], with appropriate modifications.

By our assumptions on \(m\), \(G_m = U + F_0\). Since the momentous 2-forests \(T_{G,\neq}\) form the set of bases of a matroid, the support vectors of \(F_0\) (the complements of the elements of \(T_{G,\neq}\) in \(E\)) are the indicator vectors of the bases of the dual matroid \(M^{2,\bot}_{G,\neq}\) on the edge set \(E\). By [GGMSY7], the support polytopes \(P^2_{G,\neq}\) of \(F_0\) and \(P^2_G\) of \(U\) have their edges within the set of vectors \(\{e_e - e_{e'}\}_{e,e' \in E}\). The matrix with these vectors as columns is unimodular, so the support polytopes of \(F_0\) and \(U\) are edge-unimodular and in particular IDP.

Since edge directions are invariant under scaling, we have for all dilations that \((k \cdot P^2_{G,\neq} + \ell \cdot P^2_G) \cap \mathbb{Z}^d = (k \cdot P^2_{G,\neq} \cap \mathbb{Z}^d) + (\ell \cdot P^2_G \cap \mathbb{Z}^d)\). Recall that the Cayley sum of the lattice polytopes \(P\) and \(Q\) is the convex
hull of \((\{0\} \times P) \cup (\{1\} \times Q)\) in \(\mathbb{R}^{1+d}\). With the IDP properties of \(P_{\mathcal{G}}\#\) and \(P_{\mathcal{G}}\), this implies by a theorem of Tsuchiya that the Cayley sum of \(P_{\mathcal{G}}\#\) and \(P_{\mathcal{G}}\) has the IDP property, [Tsu19] Thm 0.4.

Since the entry sums of the vertices of \(P_{\mathcal{G}}\#\) and \(P_{\mathcal{G}}\) differ by one, an integer coordinate change shows that the Cayley sum of \(P_{\mathcal{G}}\#\) and \(P_{\mathcal{G}}\) can be identified with the convex hull of the union of \(P_{\mathcal{G}}\#\) and \(P_{\mathcal{G}}\) in \(\mathbb{R}^{1+d}\), when embedded into \(\mathbb{R}^{1+d}\) by a constant function. It follows that the union of \(P_{\mathcal{G}}\#\) and \(P_{\mathcal{G}}\), which is the support polytope of \(\mathcal{G}_m\), has the IDP property. So, the semigroup generated by the lattice points in the lift \((1, P)\) of this support polytope \(P\) is saturated.

Both polytopes \(P_{\mathcal{G}}\#\) and \(P_{\mathcal{G}}\) are matroid polytopes, so they have no interior points. They sit in parallel hyperplanes of distance one. Thus, the lattice points in their union are precisely the lattice points of the two polytopes, which are their vertices. Since the vertices are (by definition) support vectors of terms in \(\mathcal{G}_m\), the semigroup generated by lifted support vectors is saturated.

\[\square\] 4.3. Approaching the general case.

**Proposition 4.10.** For all masses and for generic momenta, the support vectors of \(\mathcal{F}_m + \mathcal{F}_0\) are exactly the lattice points inside the support polytope of \(\mathcal{F}_m + \mathcal{F}_0\). In other words, the difference of semigroups \(\mathbb{N}A_m \setminus \mathbb{N}A_m\) has no elements of degree 1.

**Proof.** Suppose \(a = \sum \alpha_i a_i\) is a lattice point in the support polytope of \(\mathcal{F}_m + \mathcal{F}_0\) that can be written as a linear combination of support vectors of \(\mathcal{F}_m + \mathcal{F}_0\) with \(\sum \alpha_i = 1\). We need to show that \(a\) is a support vector itself.

Each \(a_i\) is the support vector of a monomial \(x^{E \setminus T} \cdot x_f\) for some spanning tree \(T\) that does not contain the edge \(f\), or of \(x^{E \setminus F}\) where \(F\) is a momentous 2-forest. In any event, the entries of \(a_i\) are in \(\{0, 1, 2\}\). It follows that the same is true for every entry of \(a\).

If \(a_i\) has a zero entry for some edge \(e\), then this must also be the case for all \(a_i\) with nonzero \(\alpha_i\) in the linear combination. For such \(a_i\), the corresponding tree \(T\) or 2-forest \(F\) must contain \(e\) (and \(e \notin T\) in the tree case). Note that spanning trees and 2-forests of \(G\) that contain a fixed edge \(e\) are in bijection with the spanning trees and 2-forests of the graph \(G/e\) obtained from \(G\) by contracting the edge \(e\); the correspondence linking the spanning tree (resp. 2-forest) \(S \ni e\) of \(G\) to the spanning tree (resp. 2-forest) \(S \setminus \{e\}\) of \(G/e\). Moreover, \(F\) being momentous for \(G/e\) is equivalent to \(F \setminus \{e\}\) being momentous for \(G/e\). It follows that we can replace \(G\) by \(G/e\), and \(a\) and each \(a_i\) by \(a - e_i\) and \(a_i - e_i\) and consider this a computation about \(G/e\). By induction, the claim is already shown for \(G/e\), so the case of a zero entry in \(a\) follows.

If \(a\) has an entry 2 for some edge \(e\), the same is true for every \(a_i\) appearing with nonzero \(\alpha_i\) in the linear combination. This forces each nonzero term to be of the type \(x^{E \setminus T} \cdot x_e\) with \(e \notin T\), and \(e\) must be massive. In particular, \(e\) cannot be a bridge for \(G\) and we can polynomially factor \(x^{E \setminus T} \cdot x_e = x^{(E \setminus \{e\}) \setminus T} \cdot x_e^2\). Any \(T\) appearing here is also a spanning tree for \(G_{<e}\), the graph obtained from \(G\) by deleting \(e\). Then \(\sum \alpha_i a_i = (\sum \alpha_i a_i - e_i) + 2a_e\) and both summands are lattice points. Recall that the set of spanning trees in \(G_{<e}\) is a matroid, and the set of complements is the dual matroid. But matroid polytopes have no interior points, and so \((\sum \alpha_i a_i - e_i)\) is one of the vertices of the matroid polytope of \(M_{1,1}^{G_{<e}}\). In particular, it must agree with one of the terms \(a_i - e_i\), and it follows that if \(a\) has an entry value of 2, then every lattice point in the support polytope is in fact a support vector of \(\mathcal{F}_m + \mathcal{F}_0\).

We are left to deal with the case where no entry is 0 and no entry is 2; thus, all entries are 1. Note that the entry sum of each \(a_i\), and this also of \(a\), is always \(|E \setminus T| + 1\) for any spanning tree \(T\). But if all entries of \(a\) are equal to 1, the entry sum is also equal to \(E\). So, spanning trees must have size 1, which means that (apart for isolated points that make no difference to our purposes) \(G\) must be a banana graph.

Suppose \(G\) is a banana graph with \(m\) massive and \(n\) massless edges, and let \(e_1, \ldots, e_m\) be the massive edges, and suppose \(a = \sum \alpha_i a_i\) equals \((1, \ldots, 1)\). For each \(a_i\), the massless components of \(a_i\) add up to at most \(n\) since for massless edges no second power can occur in any term of \(G\). But the massless components of \(a\) add
up to \( n \), and so each \( a_i \) must have the form \((c_{i,1}, \ldots, c_{i,n}, 1, \ldots, 1)\). Now consider the massive part \((-)_m\), the first \( m \) components of each vector in the linear combination. Since \((a)_m = \sum \alpha_i(a_i)_m\) we have reduced the question to the case of a banana graph with only massive edges. However, we already know this to be true not just for massive banana trees but in fact for all graphs with only massive edges.

\[ \Box \]

5. Normality vs Cohen–Macaulayness, and Hypothesis \([13]\)

Let \( A \) be an integer matrix such that its column span equals the lattice (free Abelian group) spanned by the unit column vectors; examples include the matrices \( A = A_m \) collecting the support vectors of \( \mathcal{G}_m \). The semigroup quotient \( \tilde{\mathcal{A}} \) has an associated \emph{saturation}, the semigroup \( \tilde{\mathcal{N}}A \) given by the full collection of lattice points inside the cone \( \mathbb{R}_{\geq 0} A \). Since \( NA \subseteq \tilde{\mathcal{N}}A \) and the latter is a semigroup, one can consider \( \tilde{\mathcal{N}}A \) as a module over \( \mathcal{N}A \) by restricting the semigroup operation \( \tilde{\mathcal{N}}A \times \tilde{\mathcal{N}}A \to \tilde{\mathcal{N}}A \) to \( \mathcal{N}A \times \tilde{\mathcal{N}}A \). The resulting semigroup quotient module \( \tilde{\mathcal{N}}A/\mathcal{N}A \) is a measure of the non-saturatedness of \( \mathcal{N}A \).

On the level of associated semigroup rings, \( \mathcal{S}_A := k[\tilde{\mathcal{N}}A] \) is by Hochster’s work \([\text{Hoc}72]\) a normal Cohen–Macaulay domain, and \( S_A := k[\mathcal{N}A] \) is a subring of \( \mathcal{S}_A \) over which \( \tilde{\mathcal{N}}A \) is a finite integral extension. The quotient \( Q_A := k[\tilde{\mathcal{N}}A]/k[\mathcal{N}A] \) is an \( S_A \)-module.

While \( Q_A \neq 0 \) is a clear indication that \( \mathcal{N}A \) is not saturated, it can easily happen that \( Q_A \neq 0 \) but \( S_A \) is Cohen–Macaulay.

\textbf{Example 5.1.} We consider here the massive bubble, whose underlying graph is the 2-banana graph given as the loopless graph with two vertices (both external) and two edges. The only 2-forest has no edge, and there are two 1-forests. So \( U = x_1 + x_2 \) and \( \Sigma_m = 1 + m_1^2 x_1 + m_2^2 x_2 \). Because of momentum conservation, the two external momenta are opposite to one another, and if \( p^2 \) denotes the norm at either vertex after Wick rotation then \( \mathcal{F}_0 = p^2 x_1 x_2 \). So,

\[
\mathcal{G}_m = (x_1 + x_2) \cdot (1 + m_1^2 x_1 + m_2^2 x_2) + p^2 x_1 x_2
\]

\[= x_1 + x_2 + m_1^2 x_1^2 + m_2^2 x_2^2 + (p^2 + m_1^2 + m_2^2) x_1 x_2\]

after Wick rotation. If \( p^2 + m_1^2 + m_2^2 = 0 \), \( \text{Supp}(\mathcal{G}_m) = \left\{ (\frac{1}{2}), (\frac{1}{2}), (0, 1), (1, 0) \right\} \). The semigroup to the lifted support vectors is not saturated since on one hand we have the lattice equation

\[
\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]

and so 2 times \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) belongs to the semigroup of \( A_m \), while on the other hand

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
\]

belongs to the lattice spanned by \( A_m \). However, since the toric ideal is a hypersurface, it is automatically Cohen–Macaulay.

The semigroup quotient \( Q_A \) consists here of the lattice points

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} + \mathbb{N} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \mathbb{N} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.
\]

\[\Diamond\]

There are certain conditions that \( Q_A \) must satisfy for \( S_A \) to have the chance of being Cohen–Macaulay. One of the easiest to describe concerns the dimension of the \( S_A \)-module \( \mathcal{S}_A/S_A \), or more precisely the dimensions of
its associated primes. Fortunately, all technical algebraic details can be expressed in terms of the semigroup quotient \( Q_A \). Note the following easy observation:

**Lemma 5.2.** The semigroup quotient \( Q_A \) is generated over \( \mathbb{N}A \) by the lifted support vectors of \( G_E \) whose coefficients in \( G_m \) are zero, the terms of \( G_m \) that violate Hypothesis 1.3. We shall denote this set of lattice points by \( V_m \).

If \( Q_A \) contains an element \( a + \mathbb{N}A \) such that the elements of \( a + \mathbb{N}A - \mathbb{N}A \) are contained in a union of (shifted) faces of cone \( \mathbb{R}_{\geq 0}A \) of dimension \( \dim(\mathbb{N}A) - 2 \) or less, then the ring \( S_A \) is not Cohen–Macaulay.

**Proof.** If one adds \( V_m \) to the columns of \( A_m \) one gets generators for the saturation of \( \mathbb{N}A \).

If \( Q_A \) contains an element as described in the lemma, then \( \tilde{S}_A/\tilde{S}_A \) has an associated prime of dimension less than \( \dim(S_A) - 1 \) and thus has depth less than \( \dim(S_A) - 1 \). By standard results on depth, this makes \( \text{depth}(S_A) = \dim(S_A) \) impossible.

In order to get a feeling, consider the following example.

**Example 5.3.** Let \( G \) be the triple sunset graph on two vertices with three edges and no loop, assuming both vertices to be external. Then \( \mathcal{U} = x_1x_2 + x_2x_3 + x_3x_1 \), \( \Sigma_m = 1 + m_1^2x_1 + m_2^2x_2 + m_3^2x_3 \). The 2-forests are empty, so \( F_0 = p^2x_1x_2x_3 \), where \( p^2 \) is the norm of the momentum at either vertex. One computes that in the massive case

\[
A_m = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2
\end{pmatrix}
\]

plus the lift \( a_0 \) of the support vector of \( (p^2 + m_1^2 + m_2^2 + m_3^2) x_1x_2x_3 \) if the coefficient of this term is nonzero.

Let us denote \( a_1, \ldots, a_9 \) the columns of \( A_m \). If \( c_0 \) is nonzero then the semigroup generated by \( \text{Supp}(G_m) \) is saturated by Theorem 5.3, while otherwise \( Q_{A_m} \) is generated by \( V_m = a_0 \).

In any case, one has the identities \( a_0 + a_1 = a_3 + a_4 \in \mathbb{N}A_m \) and \( a_0 + a_4 = a_5 + a_6 \in \mathbb{N}A_m \). It follows from symmetry that \( a_0 + a_i \in \mathbb{N}A_m \) for \( 1 \leq i \leq 9 \) and so \( Q_{A_m} \) is the singleton \( \{a_0\} \). Equivalently, the \( S_A \)-module \( \tilde{S}_A/S_A \) is a 1-dimensional vector space in multi-degree \((1,1,1,1)\).

Application of the long Euler–Koszul homology functor from [MMW05] to the short exact sequence \( S_A \to \tilde{S}_A \to S_A/S_A \) now implies that the GKZ-system attached to \( A_m \) with parameter \( \beta \) has a larger solution space (namely, of dimension \( v + 9 - 1 \)) than all other GKZ-systems attached to \( A_m \) (whose rank is always the volume \( v \) of the convex hull of \( A_m \)). In particular, \( S_A \) is not Cohen–Macaulay.

An alternative way using commutative algebra is to observe that \( \tilde{S}_A/S_{A_m} \) being a finite dimensional vector space (that is, a zero-dimensional module) means that as \( S_{A_m} \)-module it must have depth zero, which then forces \( S_{A_m} \) to have depth one. But as the dimension of \( S_{A_m} \) is equal to the dimension of the lattice spanned by \( A_m \) (namely, 4), \( S_{A_m} \) is far from satisfying the equality \( \dim(S_{A_m}) = \text{depth}(S_{A_m}) \) that determines Cohen–Macaulayness.

In general, if \( \tilde{S}_A/S_A \) contains a submodule of dimension \( k < \dim(S_A) - 1 \) then the depth of \( S_A \) cannot exceed \( k + 1 \) and thus \( S_A \) cannot be Cohen–Macaulay.

**Example 5.4.** Suppose \( S_A \) is the \( \mathbb{K} \)-algebra inside the polynomial ring \( \mathbb{K}[x, y] \) generated by the monomials \( x^i y, x^i y^2, y, \mathbb{K} \) a field. Then \( S_A \) contains all monomials of \( y \)-degree 2 or more, all powers of \( x^2 \), and all monomials \( x^i y \) except for \( i = 1 \). Then \( Q_A = \{(1, 1)\} \cup \{2t + 1, 0\}_{t \in \mathbb{N}} \). It would be reasonable to say that \( Q_A \) is 1-dimensional since it spreads out infinitely far along the line \((*, 0)\). However, \( Q_A \) contains a submodule of dimension zero, generated by the monomial \( xy \), since \( x^2 y \) and \( xy^2 \) are in \( S_A \). It follows that \( S_A \) has depth zero and is not Cohen–Macaulay.
It seems very likely that Cohen–Macaulayness of $S_A$ fails most of the time that Hypothesis 1.3.(2) fails.

6. List of Symbols

- $(G, m, V_{\text{Ext}})$ a Feynman graph with edge set $E$, mass function $m: E \rightarrow \mathbb{R}$ and external vertices $V_{\text{Ext}}$.
- $E_m, E_0 \subseteq E$ the sets of massive and of massless edges.
- $\mathcal{T}_E$ the set of $i$-forests of $G$.
- $\mathcal{M}^i_G$ the matroid whose bases are the $i$-forests of $G$.
- $\mathcal{M}^2_G$ the matroid whose bases are the momentous 2-forests of $G$.
- $\mathcal{M}^2_{G, m, t}$ the matroid whose bases are the massively truncated 2-forests of $G$.
- $\mathcal{M}^2_{G, \text{Feyn}}$ the matroid whose bases label the square-free terms in $\mathcal{G}_m$.
- $\mathcal{U}$ the first Symmankenz polynomial.
- $\mathcal{F}_W$ the sum over $\mathcal{M}^2_{G, \neq}$, weighted with their Wick rotated moments.
- $\mathcal{G}_m = 1 + \mathcal{G}_m$ the Feynman integrand.
- $\mathcal{F}_E = 1 + \mathcal{F}_E$ $\mathcal{G}_E = \mathcal{E}_E + \mathcal{F}_W$ the Feynman integrand.
- $\mathcal{F}_m$ the support polytope of $\mathcal{G}_m$.

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