In this work, we show the robustness of uberholography and its associated quantum error correcting code against the breakdown of entanglement wedge in the presence of highly entropic mixed states in the bulk. We show that for Cantor-set-like erasure in the boundary in $AdS_3/CFT_2$, the code distance is independent of the mixed-state entropy in the bulk in the $m \to \infty$ limit. We also show that for a Sierpinski triangle shaped boundary subregion with fractal boundary erasures in $AdS_4/CFT_3$, bulk reconstruction is possible in the presence of highly entropic mixed states in the bulk in the large $m$ regime.

I. INTRODUCTION

The study of quantum error correction has proven quite fruitful in the context of AdS/CFT, from the resolution of the commutator puzzle [1] to the formalization of bulk reconstruction [2][3][4][5] to ideas on subregion duality [6]. For recent reviews on holography and quantum error correction, see [7][8].

Increasingly, there is also interest in how holographic studies of quantum error correction can influence quantum error correction as applied in quantum computing. There are some known limitations of existing non-holographic quantum error correcting codes such as the surface code, particularly in terms of their robustness against fractal noise [9]. In a previous work, we demonstrated the robustness of the holographic code and presumably codes derived thereupon to such types of noise using entanglement wedge-based arguments.

In the present work, we seek to expand upon this study by studying robustness properties against fractal noise in the presence of a black hole background in AdS/CFT, or equivalently at states of finite temperature in the boundary CFT. The work will be divided as follows. In sections 2 and 3 we will give a brief review of state-specific bulk-reconstruction and uberholography, as developed in [10] and [11], respectively. In section 4, we will generalize this study to higher dimensions, in the context of a black hole background, and demonstrate that the robustness property continues to hold. Finally, we will give some concluding comments in section 5.

II. STATE-SPECIFIC RECONSTRUCTION

Recently it has been shown in [10] that there is a macroscopic breakdown of the entanglement wedge when the von Neumann entropy of an object in the bulk cannot be ignored. In particular, this can occur when the bulk state is a highly mixed state. While one way to get past this is to purify those highly mixed states into e.g Bell pairs, the existence of this breakdown nevertheless establishes the fact that the entanglement wedge is state-specific in general. An object that was defined in the study of such breakdows is the reconstruction wedge, defined by [10] as the following:

Definition: The reconstruction wedge of a boundary region $A$ is the intersection of all entanglement wedges of $A$ for every state in the code subspace, pure or mixed. It is the region of spacetime within which bulk operators are guaranteed to be reconstructible from the boundary in a state-independent manner.

Recently, it has also become clear that the entropy of the boundary at higher orders is no longer given by the RT surface area, but a more generalized entropy is associated with the quantum extremal surface, which is also defined in a state-dependent way [12].

The key idea is to relate the bulk entropy with the boundary entropy while taking into account the entropy of the bulk state. For a boundary subregion $B$ corresponding to a bulk subregion $b$ with a bulk state $\rho$ with von Neumann entropy $S_b(\rho)$ the entanglement entropy $S(B)_\rho$ is given by

$$S(B)_\rho = \frac{A_B(b)}{4G} + S_b(\rho)$$

where $G$ is Newton’s constant and $A_B(b)$ is the area of the bulk surface bounding $b$.

III. UBERHOLOGRAPHY IN $AdS_3/CFT_2$

The scheme of uberholography in $AdS_3/CFT_2$ was first proposed in [11] and was extended to $AdS_4/CFT_3$ in [14]. More literature related to uberholography can be found in [15][16][17].
The idea of **uberholography** is as follows: Given a boundary subregion R, one punches a hole H from it leaving disjoint boundary subregions $R_1$, $R_2$ and hole $H$ as shown in figures [1] and [2] with their respective entanglement wedges. The hole $H$ is punched such that

$$|R_1| = |R_2| = \left(\frac{r}{2}\right)|R|, \quad |H| = (1 - r)|R|.$$  \hfill (2)

For the same boundary subregion R, there are two candidate surfaces that might be the minimal surface, giving rise to two different candidate entanglement wedges. Bulk reconstruction is possible when the RT surface (in this dimension, a geodesic) corresponding to the candidate surfaces that might be the minimal surface, giving rise to two different candidate entanglement wedges. For the same boundary subregion R, there are two candidate surfaces that might be the minimal surface, giving rise to two different candidate entanglement wedges. Bulk reconstruction is possible when the RT surface (in this dimension, a geodesic) corresponding to the candidate surfaces that might be the minimal surface, giving rise to two different candidate entanglement wedges.

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where

$$\alpha = \frac{\log 2}{\log 2/r_C} = \frac{1}{\log_2(\sqrt{2} + 1)} = 0.786,$$  \hfill (4)

where 'a' is the short-distance cutoff and $r_C$ is the critical value of $r$ for the continuous phase to be favoured over the discontinuous one. We will drop the subscript C from now on for convenience and just write $r$.

### A. Reconstruction Wedge in $AdS_3/CFT_2$

Having said this, there are two questions that we will address in this section. First, does uberholography survive state-dependent reconstruction, i.e., can we still perform bulk reconstruction with fractal erasures in the boundary, when the bulk has a mixed state with large von Neumann entropy? Second, even if the scheme of reconstructing bulk information with fractal erasures on the boundary survives, to reconstruct the bulk information, how does the support on the boundary change, i.e., does it increase with the introduction of a maximally mixed state in the bulk? We find that the answer to the first question is yes, and while the intuitive answer to the second question is "yes", the answer turns out to be "no" to leading order.

Consider a state in the bulk with entropy $S_b$. At the first iteration, for the continuous phase to dominate over the discontinuous one, the following inequality must be satisfied:

$$|\chi_{R_1}| + |\chi_{R_2}| \geq |\chi_R| + |\chi_H| + 4GS_b$$  \hfill (5)

hinting an upper bound on $S_b$ for the error correction scheme, or a scheme for bulk reconstruction, of deep interior operators to exist. This perceived upper bound is given by

$$4GS_b \leq 2L \log \left(\frac{(r/2)^2}{(1 - r)}\right)$$  \hfill (6)

Re-arranging, the critical condition for $r$ is given by

$$\frac{(\frac{r}{2})^2}{(1 - r)} = \frac{4GS_b}{2L}.$$  \hfill (7)

Setting $S_b = 0$, we recover the condition in [11]

$$\frac{(\frac{r}{2})^2}{(1 - r)} = 1$$  \hfill (8)

As one goes to arbitrarily higher levels of iteration, the same condition [8] holds when $S_b = 0$.

Apparently it may look like from (6) that at level $1$, $S_b$ is bounded above by a finite value and as the bulk entropy becomes larger than this value, the connected phase gets dominated by the disconnected one, implying that information in the bulk can no longer be reconstructed. But

\[\text{Level 0 indicates no holes in the boundary, Level 1 refers to punching out holes once, Level 2 twice and so on...}\]
it is not the case, as we will show here. For simplicity, let \( x = e^{4G S_b/2L} \), which gives us the inequality

\[
r^2 + 4rx - 4x \geq 0
\]  

(9)

Solving for the critical case of equality, we get

\[
r = 2x \left( \sqrt{1 + \frac{1}{x}} - 1 \right)
\]  

(10)

Setting \( x = 1( S_b = 0) \), we recover the value of \( r \) as in (8).

But it is clear that \( 0 < r < 1 \) for any value of \( S_b \), making bulk reconstruction possible theoretically. However, as we increase \( S_b \), \( r \) approaches 1 (see fig 3).

With a bulk state having entropy \( S_b \), we arrive at the following inequality in place of (6):

\[
4G S_b \leq 2L(2^m - 1) \log \left( \frac{(r/2)^2}{(1-r)} \right)
\]  

(12)

In the \( m \to \infty \) limit, bulk reconstructability holds for any finite non-zero bulk entropy \( S_b \), i.e, the connected phase always dominates over the disconnected one and the scheme of quantum error correction via uberholography for fractal erasures on the boundary is robust against state-dependent reconstruction. However, the support on the boundary increases. We will now show that this increment is very small (negligible). Inverting equation (12) and considering the critical equality, we get the condition on \( r \)

\[
\left( \frac{r}{2} \right)^2 = e^{4G S_b/2L(2^m - 1)}
\]  

(13)

In the limit \( m \to \infty \), with or without taking \( L \to \infty \), the RHS \( \to 1 \), approaching the value of \( r \) in \( S_b = 0 \) limit [11]

\[
r \to 2(\sqrt{2} - 1)
\]  

(14)

Comparing this result with figure [3] the asymptotic value of \( r \) at a higher level \( m \to \infty \) is not 1, but (14) (see figure 4).

One may notice, that even with a non-zero \( S_b \), one may also recover the limit in [5] by setting \( L \to \infty \). However, we will be showing that for fractal errors in the boundary (at level \( m \to \infty \)), setting \( L \to \infty \) is sufficient but not necessary to approximately recover (8). Punching more and more holes suffice.

At level \( m \) of the fractal boundary erasure, the disconnected phase has a RT surface given by:

\[
|\chi R'|_{\text{disconn.}} = 2L \left[ 2^m \log \left( \frac{(\frac{r}{2})^m |R|}{a} \right) \right]
\]  

(11)

while the connected phase has a RT surface

\[
2L \left[ \log \left( \frac{|R|}{a} \right) + \sum_{j=1}^{m} 2^{j-1} \log \left( \frac{(\frac{r}{2})^{j-1}(1-r)|R|}{a} \right) \right]
\]  

FIG. 3. \( r \) vs \( x \) plot at level 1. One can always tune \( 0 < r < 1 \), such that regardless of how large \( S_b \) is, uberholographic error correction is always possible.

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FIG. 4. \( r \) vs \( S_b \) plot for various \( m \). At \( m \to \infty \), the curve is given by the line \( r = 2(\sqrt{2} - 1) \). We have set \( 4G/L = 1 \) for simplicity.

Thus, we showed that that the quantum extremal surface approaches the Ryu-Takayanagi surface in the \( m \to \infty \) limit, indicating a nearly-state-independent bulk reconstruction in uberholography.

This becomes clearer in the context of the construction discussed in [11]. In figure [3] with \( S_b = 0 \), \( \Lambda \) is the logical boundary and \( \Phi \) is the physical boundary. The boundary subregion \( R \) is the remaining boundary after punching out holes. One may interpret the introduction of a bulk state with entropy \( S_b \) as the bulk geodesic moving further away from the logical inner boundary \( \Lambda \) increasing the...
support on the boundary for bulk reconstruction. One may imagine a boundary $\Lambda'$ to be the new "effective" logical boundary with the inclusion of a mixed state in the bulk. This means $r_{in}$ gets modified to some $r_{in}'$ ($r_{in}' > r_{in}$) while $r_{out}$ remains the same. As long as $r_{out}$ is much greater than $r_{in}'$, the bulk reconstruction is possible. This is a reasonable assumption as $r_{out}$ is the boundary of the AdS space. Said another way, we can write the code rate (with $S_b = 0$) to be

$$k/n = e^{(r_{in}-r_{out})/L}. \quad (15)$$

With the introduction of mixed state in the bulk, $r_{in} \rightarrow r_{in}'$, implying that the code rate depreciates with $k \rightarrow k'$ ($k'>k$). As long as $k' << n$, the is a reasonable error correcting scheme.

**FIG. 5.** The inner logical boundary $\Lambda$ is contained inside the entanglement wedge (shaded in blue) of outer boundary subregion $R \subset \Phi$ where $\Phi$ is the outer physical boundary. We keep punching more holes of decreasing size in the boundary such that the remaining boundary region $R_{m,\in}$ (whose measure goes to zero when $m \rightarrow \infty$) has an entanglement wedge that contains the logical boundary. By introducing a bulk mixed-state with large entropy, the volume of the entanglement wedge increases as the support on the outer boundary $\Phi$ increases. The new minimal area at level 1 is now the red line which does not touch the inner boundary $\Lambda$. One may imagine it as touching a larger inner boundary $\Lambda'$.

**FIG. 6.** The Sierpinski Triangle. The fractal is constructed by removing triangular holes (shaded black) of decreasing size from the big triangle.

Consider the boundary Sierpinski triangle at level 0 (without holes) with a side of length $l_0$. At level 1, we divide this triangle into four smaller triangles and make a hole by taking away the triangle in the center. With every iteration of making a hole, the side-length reduces by half. So at level $m$, we have $3^m$ small triangles of equal area each having a side length $l_m = l_0/2^m$ as shown in figure 6. The holes however have different areas, as expected in a fractal.

**IV. THE HOLOGRAPHIC SIERPINSKI CODE IN $AdS_4/CFT_3$**

An example of uberholography in $AdS_4/CFT_3$ with a Sierpinski triangle shaped boundary is constructed in

[14]. This construction is more subtle in the sense that the connected phase and disconnected phase have the same RT-Surface at leading order at every level of hole iteration. Sub-leading order comparisons render the existence of holographic QECC for this geometry. Moreover, unlike the case in $AdS_3/CFT_2$ where we could tune the boundary erasure using a continuous $r(0 < r < 1)$, the erasure is fixed in this case (i.e., taking away one triangle at the center from four triangles). Previous calculations in $AdS_3/CFT_2$ dealt with the critical value of $r$ for which error correction is possible, but one could take any allowed value of $r$. In the case of Sierpinski triangle shaped boundary subregion in $AdS_4/CFT_3$, there is a priori, no reason to take it for granted that this construction is robust against state-dependent reconstruction at sub-leading order.

A. Reconstruction Wedge of the Sierpinski Triangle

In the case of $AdS_4/CFT_3$, at level 1, the allowed entropy of the bulk mixed state is bounded above by

$$4GS_{b,m=1} \leq -6b\left(\frac{\pi}{3}\right) \log\left(\frac{3l_0}{4a}\right). \quad (16)$$
where 'a' is the distance cutoff and \( b(\theta) \) is a negative quantity defined in [18]. Unlike the case of \( AdS_3/CFT_2 \), here if the bulk state entropy exceeds this number, quantum error correction is not possible. However, we will see at higher levels \( (m \rightarrow \text{large}) \), this bound on the allowed bulk state entropies relaxes and the bulk reconstruction sheds away state-dependency.

As found in [14], after \( m \) iterations, the connected phase has an area

\[
|\chi R'| = |\chi R| + |\chi H_1| + 3|\chi H_2| + 3^2|\chi H_3| + \cdots + 3^{m-1}|\chi H_m|
\]

\[
= |\chi R| + \sum_{j=1}^{m} 3^{j-1} |\chi H_j|
\]

\[
= \left[ \frac{3 l_0}{a} + \sum_{j=1}^{m} 3^{j-1} \frac{3 l_0}{2^{j} a} \right] - 6b \left( \frac{\pi}{3} \right) \log \left( \frac{3 \left( \frac{l_0}{a} \right)}{a} \right) + \sum_{j=1}^{m} 3^{j-1} \log \left( \frac{3 l_0}{2^{j} a} \right).
\]

while the area of the disconnected phase is

\[
|\chi R'| = 3^m \left[ \frac{3 \left( \frac{l_0}{a} \right)}{a} - 6b \left( \frac{\pi}{3} \right) \log \left( \frac{3 \left( \frac{l_0}{a} \right)}{a} \right) \right]
\]

Introducing an entropic state in the bulk with von Neumann entropy \( S_b \), the critical value of \( S_b \) for the continuous phase to dominate is given by

\[
4G S_b = -6b \left( \frac{\pi}{3} \right) \log \left[ \frac{1}{2^{\frac{m}{2}(m+1)}} \left( \frac{3 \left( \frac{l_0}{a} \right)}{a} \right)^{\frac{1}{2}(3^m-1)} \right].
\]

As long as we are in the regime where the length of the side of the triangle at level \( m \) \( (l_m = \frac{L_m}{2^m}) \) is of the order of lattice spacing 'a' (the ratio \( \frac{L_m}{a} = O(1) \)), the RHS of [19] is a very large positive number, which means that for a finite \( S_b \), the connected phase dominates and reconstruction is possible (for a theory with \( a \to 0 \), the RHS of [19] approaches infinity. However, it is worth mentioning that as we make the length of the side of the triangle smaller than this cutoff length 'a', there is a breakdown of the entanglement wedge, even for small \( S_b \). Such a case would mean that the entropy is bounded above by a negative number and is thus unphysical. This is perhaps a relic of the untrustworthiness of theories below cutoffs. As we are talking about the Sierpinski triangle as boundary to \( AdS_4 \), it is reasonable to assume \( l_0 \) is very large and short-distance cutoff 'a' is very small, one can always choose sufficiently large \( m \) to allow for state-independent bulk reconstruction.

V. CONCLUSION

We see that the robustness of the holographic quantum error correcting code against fractal noise persists even in finite temperature states of the CFT, or equivalently in black hole backgrounds. This is a priori surprising, as one might have thought that the black hole horizon size would have served as a finite cut-off to the uberholography generalization to higher dimensions. However, we believe that the horizon-avoiding properties of extremal surface probes prevents this naive concern from occurring. A recent work [17] that appeared at the time of completion of this manuscript discusses the robustness of uberholography in Cantor-set like erasures in higher dimensions. It also discusses the finite temperature effects on uberholography. This work is complementary to our work and tells a similar story in a different language.

Given the continued robustness of uberholographic codes to fractal noise, the logical next step would be to determine the precise features of the holographic quantum error correcting code that permit its robustness against this form of error, and to port a novel code with such features into the realm of quantum computing. We will reserve both of these studies for future work.

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3 To study finite temperature CFT, the work in [17] has deformed the bulk metric to a BTZ blackhole while we continue to work in empty AdS-metric. As noted in equation (8) of [10], when the entanglement wedge boundaries are very far from the blackhole horizon in the bulk, in that limit, the metric near the boundary is near-vacuum. Figure (1) [17] show similar trend to our figure (3). We arrived at a closed-form expression for \( r \) as a function of blackhole entropy \( S_b \) and show that in the empty-AdS approximation, the code distance in \( AdS_3/CFT_2 \) does not change in the \( m \to \infty \) limit.
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