WAVEFRONT SETS OF CONVOLUTIONS OF DISTRIBUTIONS WITH (WEIGHTED) LINE INTEGRAL DISTRIBUTIONS

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Excluding pathological cases of curves $\gamma \in C^\infty ((-\varepsilon, \infty), \mathbb{R}^n)$, for some $\varepsilon > 0$, the following line integral:

$$\langle T_{\gamma}, \phi \rangle = \int_0^\infty \phi (\gamma(t)) \|\gamma'(t)\| \, dt, \quad \phi \in \mathcal{S} (\mathbb{R}^n),$$

defines a distribution. Moreover, if we replace $\|\gamma'(t)\|$ with any bounded positive weight function $v \in C^\infty ((-\varepsilon, \infty))$ for some $\varepsilon > 0$, the following also defines a distribution:

$$\langle T_{\gamma, v}, \phi \rangle = \int_0^\infty \phi (\gamma(t)) v(t) \, dt, \quad \phi \in \mathcal{S} (\mathbb{R}^n),$$

provided that either $\|\gamma'(t)\|$ is bounded away from zero, or $v$ decays sufficiently fast so that the above integral converges for any $\phi \in \mathcal{S} (\mathbb{R}^n)$.

The convolution of distributions $u_1$ and $u_2$, once of which has compact support, is defined in Hörmander[?], as the unique distribution $u = u_1 \ast u_2$ satisfying:

$$u_1 \ast (u_2 \ast \phi) = u \ast \phi, \quad \phi \in C^\infty_0 (\mathbb{R}^n).$$

As such, $w \ast T_{\gamma}$ and $w \ast T_{\gamma, v}$ are well-defined as distributions whenever $w \in \mathcal{E}' (\mathbb{R}^n)$. This will give rise to formally defining the notations

$$w \ast T_{\gamma} (\xi) = \int_0^\infty w (\xi - \gamma(t)) \|\gamma'(t)\| \, dt,$$
$$w \ast T_{\gamma, v} (\xi) = \int_0^\infty w (\xi - \gamma(t)) v(t) \, dt,$$

both of which will agree with the usual notion of an integral converging for almost every $\xi \in \mathbb{R}^n$ whenever $w \in \mathcal{L}^1 (\mathbb{R}^n)$. We will explore such convolutions, as well as their wavefront sets, particularly exploring how the convolution scatters the singularities of $w$. However, we will require a more direct formulation of such convolutions than the definition of convolution given in Hörmander provides for.

To avoid pathological cases, we will focus on choices of $w$ and $\gamma$ for which given any $\xi$, $\xi - \gamma(t)$ lies outside the support of $w$ for $t$ sufficiently large.

1. The Distributional Directional Antiderivative

We may begin by extending the idea of directional antiderivatives to compactly-supported distributions. In particular, given $\tilde{\nu} \in S^{n-1}$, we want to focus on the directional antiderivatives of the form:

$$\mathcal{I}_\tilde{\nu} f (\tilde{\nu} + s \tilde{\nu}) = \int_{-\infty}^s f (\tilde{\nu} + t \tilde{\nu}) \, dt, \quad \tilde{\nu} \in \tilde{\nu}^\perp, s \in \mathbb{R}.$$
Let $\psi$ be defined as defined above in terms of $\psi_1$. We then define the distributional directional antiderivative 

\begin{equation}
I_{\nu} \phi (\mathbf{u} + t\mathbf{v}) = \int_{-\infty}^{t} \phi (\mathbf{u} + s\mathbf{v}) - X_{\nu} \phi \otimes \psi_0 (\mathbf{u} + s\mathbf{v}) \, ds, \quad \mathbf{u} \in \mathbf{v}^\perp, \ t \in \mathbb{R},
\end{equation}

where $X_{\nu} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbf{v}^\perp)$ denotes the x-ray transform restricted to the direction $\mathbf{v}$, and the tensor product $X_{\nu} \phi \otimes \psi_0$ is interpreted as:

$$X_{\nu} \phi \otimes \psi_0 (\mathbf{u} + s\mathbf{v}) = X_{\nu} (\mathbf{u}) \psi_0 (s), \quad \mathbf{u} \in \mathbf{v}^\perp, \ s \in \mathbb{R}.$$

We then define the distributional directional antiderivative by

$$\langle I_{\nu} w, \phi \rangle = - \langle w, I_{\nu} \phi \rangle.$$

Since we defined $I_{\nu}$ in a way that depends on an arbitrary choice of $\psi_0$, we will want to verify that a different choice of $\psi_0$ will not alter $I_{\nu}$.

**Proposition 1.2.** While $I_{\nu} \phi$ depends on choice of $\psi_0$, $I_{\nu} w$ does not, so long as $\text{supp} \psi_0 \subseteq (-\infty, t_{\min})$.

**Proof.** Let $\psi_0, \psi_1 \in C_0^\infty (\mathbb{R})$ both have support in $(-\infty, t_{\min})$, and take $I_{\nu}^0$ and $I_{\nu}^1$ as defined above in terms of $\psi_0$ and $\psi_1$, respectively. Then for $\phi \in \mathcal{S}(\mathbb{R}^n)$ we observe that:

$$I_{\nu}^0 \phi (\mathbf{u} + t\mathbf{v}) - I_{\nu}^1 \phi (\mathbf{u} + t\mathbf{v})$$

$$= \int_{-\infty}^{t} \phi (\mathbf{u} + s\mathbf{v}) - X_{\nu} \phi \otimes \psi_0 (\mathbf{u} + s\mathbf{v}) \, ds$$

$$- \int_{-\infty}^{t} \phi (\mathbf{u} + s\mathbf{v}) - X_{\nu} \phi \otimes \psi_1 (\mathbf{u} + s\mathbf{v}) \, ds$$

$$= X_{\nu} \phi (\mathbf{u}) \int_{-\infty}^{t} (\psi_0 (s) - \psi_1 (s)) \, ds.$$

For $t$ below or above both supports of $\psi_0$ and $\psi_1$, this integral is zero. In particular, the support of $I_{\nu}^0 \phi - I_{\nu}^1 \phi$ is contained inside $\mathbf{v}^\perp + (-\infty, t_{\min}) \mathbf{v}$. Hence:

$$\langle I_{\nu}^0 w, \phi \rangle - \langle I_{\nu}^1 w, \phi \rangle = - \langle w, I_{\nu}^0 \phi \rangle + \langle w, I_{\nu}^1 \phi \rangle$$

$$- \langle w, I_{\nu}^0 \phi - I_{\nu}^1 \phi \rangle$$

$$= 0.$$

We now wish to verify that $I_{\nu}$ acts on functions in $L^1 (\mathbb{R}^n)$ satisfying the support condition in the desired manner.
**Proposition 1.3** (Distributional Anti-partial derivative of $L^1$ functions). If $f \in L^1(\mathbb{R}^n)$ satisfies the support condition \[1.1\] then $I_\varphi f$ is in fact a function given by

$$I_\varphi f (\vec{u} + s\vec{v}) = \int_{-\infty}^{s} f (\vec{u} + t\vec{v}) \, dt.$$  

**Proof.** Let $t_{\min} = \min_{x \in \text{supp} f} \vec{x} \cdot \vec{v}$, and choose $\psi_0$ as described in \[1.1\]. Observe:

$$\langle I_\varphi f, \phi \rangle = -\langle f, I_\varphi \phi \rangle = -\int_{\mathbb{R}^n} f (\vec{x}) I_\varphi \phi (\vec{x}) \, d\vec{x} = -\int_{\mathbb{R}^n} \int_{\mathbb{R}} f (\vec{u} + t\vec{v}) I_\varphi \phi (\vec{u} + t\vec{v}) \, dt \, d\vec{u}$$

$$= -\int_{\mathbb{R}^n} \int_{\mathbb{R}} f (\vec{u} + t\vec{v}) \left( \phi (\vec{u} + s\vec{v}) - X_{\varphi} \phi \otimes \psi_1 (\vec{u} + s\vec{v}) \right) \, ds \, dt \, d\vec{u}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{-\infty}^{\infty} f (\vec{u} + t\vec{v}) \left( \phi (\vec{u} + s\vec{v}) - X_{\varphi} \phi \otimes \psi_1 (\vec{u} + s\vec{v}) \right) \, ds \, dt \, d\vec{u}$$

The latter integral vanishes for all $s$, since:

$$\int_{-\infty}^{s} f (\vec{u} + t\vec{v}) \, dt = 0, \quad s < t_{\min},$$

and:

$$\psi_0 (s) = 0, \quad s > t_{\min}.$$  

Hence:

$$\langle I_\varphi f, \phi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{-\infty}^{s} f (\vec{u} + t\vec{v}) \, dt \, \phi (\vec{u} + s\vec{v}) \, ds \, d\vec{u}. \tag*{\blacksquare}$$

**Proposition 1.4.** For $w \in \mathcal{E}'(\mathbb{R}^n)$:

$$\mathcal{D}_\varphi I_\varphi w = w, \quad I_\varphi \mathcal{D}_\varphi w = w.$$  

**Proof.** We first observe that for $\phi \in \mathcal{S}(\mathbb{R}^n)$, $X_{\varphi} \mathcal{D}_\varphi \phi = 0$, and so:

$$I_\varphi \mathcal{D}_\varphi \phi (\vec{u} + t\vec{v}) = \int_{-\infty}^{t} \phi (\vec{u} + s\vec{v}) \, ds = \phi (\vec{u} + t\vec{v}).$$

Then:

$$\langle \mathcal{D}_\varphi I_\varphi w, \phi \rangle = -\langle I_\varphi w, \mathcal{D}_\varphi \phi \rangle = \langle w, \mathcal{D}_\varphi \phi \rangle = \langle w, \phi \rangle.$$  

On the other hand:

$$\mathcal{D}_\varphi I_\varphi \phi (\vec{u} + t\vec{v}) = \phi (\vec{u} + s\vec{v}) - X_{\varphi} \phi \otimes \psi_0 (\vec{u} + t\vec{v}),$$
and since $X_\varphi \hat{\otimes} \psi_0$ is supported away from the support of $w$, a similar computation also yields:

$$\mathcal{I}_\varphi D_\varphi w = w.$$  \hfill $\square$

**Proposition 1.5.** Let $U \subseteq \mathbb{R}^n$ be open and assume $U$ is invariant under translation in the direction $-\bar{v}$. That is, $U - t\bar{v} \subseteq U$ for $t \geq 0$. If $w_1, w_2 \in E'(\mathbb{R}^n)$, and are equal on $U$, then $\mathcal{I}_\varphi w_1 = \mathcal{I}_\varphi w_2$ are equal on $U$.

**Proof.** Let $\phi$ be supported in $U$. Then if we inspect:

$$\mathcal{I}_\varphi \phi (\bar{u} + t\bar{v}) = \int_{-\infty}^{t} \phi (\bar{u} + s\bar{v}) - X_\varphi \otimes \psi_0 (\bar{u} + s\bar{v}) \, ds,$$

Then for $\bar{u} + t\bar{v} \notin U$, we must have $\bar{u} + \tau\bar{v} \notin U$ for $\tau \geq t$ and so:

$$\mathcal{I}_\varphi \phi (\bar{u} + t\bar{v}) = \int_{-\infty}^{\infty} \phi (\bar{u} + s\bar{v}) \, ds - X_\varphi \phi (\bar{u}) \int_{-\infty}^{t} \psi_0 (s) \, ds$$

$$= X_\varphi \phi (\bar{u}) \int_{t}^{\infty} \psi_0 (s) \, ds.$$  

In particular, $\mathcal{I}_\varphi \phi (\bar{u} + t\bar{v}) = 0$ when $\bar{u} + t\bar{v} \notin U$ and $t \geq t_{\text{min}}$. In particular:

$$\text{supp} \mathcal{I}_\varphi \phi \subseteq U \cup V, \quad V = \{\bar{u} + t\bar{v} : t < t_{\text{min}}\}.$$  

We now choose a partition of unity $\rho_U$ and $\rho_V$ for $U$ and $V$ so that $\rho_U + \rho_V = 1$ on $U \cup V$, supp $\rho_U \subseteq U$, and supp $\rho_V \subseteq V$. Then

$$\langle \mathcal{I}_\varphi w_1, \phi \rangle = - \langle w_1, \mathcal{I}_\varphi \phi \rangle$$

$$= - \langle w_1, \rho_U \mathcal{I}_\varphi \phi \rangle - \langle w_1, \rho_V \mathcal{I}_\varphi \phi \rangle$$

$$= - \langle w_2, \rho_U \mathcal{I}_\varphi \phi \rangle - \langle w_2, \rho_V \mathcal{I}_\varphi \phi \rangle$$

$$= - \langle w_2, \mathcal{I}_\varphi \phi \rangle$$

$$= \langle \mathcal{I}_\varphi w_2, \phi \rangle.  \hfill \square$$

**Proposition 1.6.** Let $t_0 > t_{\text{max}} = \max_{\bar{x} \in \text{supp} \, \bar{w}} \bar{x} \cdot \bar{v}$, then define $w^*$ by:

$$(w^*, \phi) = \langle w, \phi (\bar{x} + t\bar{v}) - \phi (\bar{x} + (2t_0 - t) \bar{v}) \rangle.$$  

Then $w^*$ and $\mathcal{I}_\varphi w^*$ are distribution with odd and even symmetries across the hyperplane \{ $t = t_0$ \}, and furthermore, $\mathcal{I}_\varphi w^*$ is compactly supported, and is equal to $\mathcal{I}_\varphi w$ on \{ $t < t_0$ \}.

**Proof.** We observe

$$\langle w^*, \phi (\bar{u} + (2t_0 - t) \bar{v}) \rangle = \langle w, \phi (\bar{u} + (2t_0 - t) \bar{v}) - \phi (\bar{u} + t\bar{v}) \rangle$$

$$= - \langle w, \phi (\bar{u} + t\bar{v}) - \phi (\bar{u} + (2t_0 - t) \bar{v}) \rangle$$

$$= - \langle w^*, \phi (\bar{u} + t\bar{v}) \rangle.$$
This implies that if $\phi$ has even symmetry across $\{t = t_0\}$, then $\langle w^*, \phi \rangle = 0$. Then

$$
\langle \mathcal{I}_\phi w^*, \phi (\bar{u} + (2t_0 - t) \bar{v}) \rangle = -\langle w^*, \mathcal{I}_\phi \{ \phi (\bar{u} + (2t_0 - t) \bar{v}) \} \rangle
$$

$$
= -\langle w^*, \int_{-\infty}^{t} \phi (\bar{u} + (2t_0 - s) \bar{v}) - \mathcal{X}_\phi \phi \otimes \psi_0 (\bar{u} + t \bar{v}) \rangle \ ds \rangle
$$

$$
= -\langle w^*, \int_{2t_0 - t}^{\infty} \phi (\bar{u} + s \bar{v}) - \mathcal{X}_\phi \phi \otimes \psi_0 (\bar{u} + (2t_0 - s) \bar{v}) \rangle \ ds \rangle
$$

$$
= \langle w^*, \int_{-\infty}^{t} \phi (\bar{u} + s \bar{v}) - \mathcal{X}_\phi \phi \otimes \psi_0 (\bar{u} + (2t_0 - s) \bar{v}) \rangle \ ds \rangle
$$

$$
= -\langle w^*, \mathcal{I}_\phi \phi \rangle + \langle w^*, \mathcal{X}_\phi \phi (\bar{u}) \int_{-\infty}^{t} \psi_0 (s) - \psi_0 (2t_0 - s) \rangle \ ds \rangle.
$$

We then observe that $\int_{-\infty}^{t} \psi_0 (s) - \psi_0 (2t_0 - s) \ ds$ has even symmetry across $\{t = t_0\}$, and so:

$$
\langle \mathcal{I}_\phi w^*, \phi (\bar{x} + (2t_0 - t) \bar{v}) \rangle = \langle \mathcal{I}_\phi w^*, \phi \rangle.
$$

The symmetry easily implies that $\mathcal{I}_\phi w^*$ must be compactly-supported. Furthermore, it is clear that $w = w^*$ on $\{t < t_0\}$, so we have $\mathcal{I}_\phi w = \mathcal{I}_\phi w^*$ on $\{t < t_0\}$.

We will now establish a relationship between $WF(w)$ and $WF(\mathcal{I}_\phi w)$. But first, we start with the following theorem:

**Theorem 1.7** (Microlocal property[7]). If $P$ is a differential operator of order $m$ with $\mathcal{C}^\infty$ coefficients on a manifold $X$, then:

$$
WF(u) \subseteq \text{Char} \ P \cup WF(Pu), \quad u \in \mathcal{G}'(X),
$$

where the characteristic set $\text{Char} \ P$ is defined by

$$
\text{Char} \ P = \left\{ \left( \bar{x}, \bar{\xi} \right) \in T^* (X) \left| P_m \left( \bar{x}, \bar{\xi} \right) = 0 \right\} = \mathbb{R}^n \times \mathbb{v}^\perp,
$$

and $P_m$ is the principal symbol of $P$.

In the case that $P$ is merely a directional derivative, i.e., $P = \mathcal{D}_\xi$, then $P_m \left( \bar{x}, \bar{\xi} \right) = i\bar{\xi} \cdot \bar{v}$, and so

$$
\text{Char} \ P = \mathbb{R}^n \times \mathbb{v}^\perp.
$$

While[7] implies that $\mathcal{I}_\phi$ will extend the wavefront set of a distribution by at most $\mathbb{R}^n \times \left( \mathbb{v}^\perp \setminus \bar{0} \right)$, the following result states that $\mathcal{I}_\phi w$ will not contain an element $(\bar{x}, \bar{\eta}_0) \in \mathbb{R}^n \times \mathbb{v}^\perp$ in its wavefront set if $WF(w)$ omits $\mathbb{R}^n \times \{\bar{\eta}_0\}$ altogether.

**Theorem 1.8** (Main Result). Let $w \in \mathcal{E}'(\mathbb{R}^n)$, and $\bar{\eta}_0 \in \mathbb{v}^\perp$. If:

$$
WF(w) \cap (\mathbb{R}^n \times \{\bar{\eta}_0\}) = \emptyset,
$$

then:

$$
WF(\mathcal{I}_\phi w) \cap (\mathbb{R}^n \times \{\bar{\eta}_0\}) = \emptyset.
$$
Proof. Let \( w^* \) be the distribution extending \( w \), having odd symmetry across a plane \( \{ t = t_0 \} \), with \( t_0 \) large enough so that \( w = w^* \) on \( \{ t < t_0 \} \). Then:
\[
WF (w^*) \cap (\mathbb{R}^n \times \{ \vec{\eta}_0 \}) = \emptyset,
\]
and it will suffice to show that:
\[
WF (\mathcal{I}_Q w^*) \cap (\mathbb{R}^n \times \{ \vec{\eta}_0 \}) = \emptyset.
\]
Indeed, since \( \mathcal{I}_Q w^* \) and \( w^* \) are compactly-supported distribution, their Fourier transforms exist as entire functions, and:
\[
i \tau \hat{\mathcal{I}_Q w^*} (\vec{\eta} + \tau \vec{v}) = \hat{w^*} (\vec{\eta} + \tau \vec{v}), \quad \vec{\eta} \in \vec{v}^\perp, \tau \in \mathbb{R}.
\]
Because of the odd symmetry of \( w^* \) across the plane \( \{ t = t_0 \} \), \( \hat{w^*} \) vanishes on \( \vec{v}^\perp \), and also:
\[
\hat{\mathcal{I}_Q w^*} (\vec{\eta} + \tau \vec{v}) = \begin{cases} \hat{w^*} (\vec{\eta} + \tau \vec{v}), & \text{if } \tau \neq 0, \\ -i \mathcal{D}_Q \hat{w^*} (\vec{\eta}), & \text{if } \tau = 0,
\end{cases}
\]
for \( \vec{\eta} \in \vec{v}^\perp \) and \( \tau \in \mathbb{R} \). The case \( \tau = 0 \) comes from an application of l'Hôpital's rule. However, \( -i \mathcal{D}_Q \hat{w^*} = -i \hat{w^*} \), and:
\[
WF (-i \hat{w^*}) \subseteq WF (w^*),
\]
so we can expect \( -i \mathcal{D}_Q \hat{w^*} (\vec{\xi}) \) to decay rapidly in an open conic neighborhood \( \Gamma \) of \( \vec{\eta}_0 \). For our following argument, we will require that \( \Gamma \) be chosen to be convex in \( \tau \), i.e., if \( \vec{\eta} + \tau \vec{v} \in \Gamma \) for \( k = 1, 2 \), and \( \tau_1 < \tau_2 \), then \( \vec{\eta} + \tau \vec{v} \in \Gamma \) for \( \tau_1 \leq \tau \leq \tau_2 \). In particular, we can set:
\[
\Gamma = \left\{ \sigma \vec{\eta}_0 + \vec{\xi} + \tau \vec{v} : \sigma, \tau > 0, \vec{\xi} \in \vec{v}^\perp \cap \vec{\eta}_0^\perp, \max \{ \| \vec{\xi} \|, \| \tau \| \} < \varepsilon \sigma \right\},
\]
for some \( \varepsilon > 0 \) sufficiently small. Then for each \( N \geq 0 \), we can choose \( C_N \) so that:
\[
\left| \mathcal{D}_Q \hat{w^*} (\vec{\eta} + \tau \vec{v}) \right| \leq C_N \left( 1 + \| \vec{\eta} \|^2 + \tau^2 \right)^{-N/2}, \quad \vec{\eta} \in \vec{v}^\perp, \tau \in \mathbb{R}, \vec{\eta} + \tau \vec{v} \in \Gamma.
\]
This bound on the derivative then gives us the following bound on \( \hat{w^*} \):
\[
\left| \hat{w^*} (\vec{\eta} + \tau \vec{v}) \right| \leq C_N \left( 1 + \| \vec{\eta} \|^2 + \tau^2 \right)^{-N/2} |\tau|, \quad \vec{\eta} \in \vec{v}^\perp, \tau \in \mathbb{R}, \vec{\eta} + \tau \vec{v} \in \Gamma.
\]
It then follows immediately that:
\[
\left| \hat{\mathcal{I}_Q w^*} (\vec{\eta} + \tau \vec{v}) \right| \leq C_N \left( 1 + \| \vec{\eta} \|^2 + \tau^2 \right)^{-N/2}, \quad \vec{\eta} \in \vec{v}^\perp, \tau \in \mathbb{R}, \vec{\eta} + \tau \vec{v} \in \Gamma.
\]
As this is true for arbitrary \( N \geq 0 \), this proves that:
\[
WF (\mathcal{I}_Q w^*) \cap (\mathbb{R}^n \times \{ \vec{\eta}_0 \}) = \emptyset.
\]
Since \( \mathcal{I}_Q w = \mathcal{I}_Q w^* \) on \( \{ t < t_0 \} \), we can say that:
\[
WF (\mathcal{I}_Q w) \cap (\{ t < t_0 \} \times \{ \vec{\eta}_0 \}) = \emptyset,
\]
and then let \( t_0 \to \infty \) to obtain:
\[
WF (\mathcal{I}_Q w) \cap (\mathbb{R}^n \times \{ \vec{\eta}_0 \}) = \emptyset.
\]
The above result implies that the only way $I_{\vec{v}}w$ can include an element $(\vec{x}_0, \vec{\eta}_0) \in \mathbb{R}^n \times \vec{v}^\perp$ in its wavefront set is if $w$ itself already contains some element of $\mathbb{R}^n \times \{\vec{\eta}_0\}$. The following result further refines the previous result by describing a necessary condition on $\vec{x}_0$ in order for $I_{\vec{v}}w$ to contain $(\vec{x}_0, \vec{\eta}_0)$ in its wavefront set. Intuition tells us that $(\vec{x}_0 - t\vec{v}, \vec{\eta}_0)$ must already belong to the wavefront set of $w$ for some $t \geq 0$. 

**Proposition 1.9.** Let $U \subseteq \mathbb{R}^n$ be open and assume $U$ is invariant under translation in the direction $-\vec{v}$, and $\vec{\eta}_0 \in \vec{v}^\perp$. If:

\begin{equation}
WF(w) \cap (U \times \{\vec{\eta}_0\}) = \emptyset,
\end{equation}

then:

\begin{equation}
WF(\mathcal{I}_\vec{v}w) \cap (U \times \{\vec{\eta}_0\}) = \emptyset.
\end{equation}

**Proof.** Let $U^*$ be an open subset of $U$ whose closure is entirely contained in $U$, and is also closed under translation in the direction $-\vec{v}$, and choose a $C^\infty$ function $\psi \geq 0$ supported in $U$ that is equal to 1 on $U^*$. Then $\psi w = w$ on $U^*$, so $\mathcal{I}_\vec{v} (\psi w) = \mathcal{I}_\vec{v} w$, and in fact, we also have $\psi \mathcal{I}_\vec{v} w = \mathcal{I}_\vec{v} w$ on $U^*$. Furthermore, (1.2) implies:

\begin{equation}
WF(\psi w) \cap (\mathbb{R}^n \times \{\vec{\eta}_0\}) = \emptyset,
\end{equation}

and so:

\begin{equation}
WF(\mathcal{I}_\vec{v} (\psi w)) \cap (\mathbb{R}^n \times \{\vec{\eta}_0\}) = \emptyset.
\end{equation}

We then have:

\begin{equation}
WF(\mathcal{I}_\vec{v} w) \cap (U^* \times \{\vec{\eta}_0\}) = WF(\mathcal{I}_\vec{v} (\psi w)) \cap (U^* \times \{\vec{\eta}_0\})
\subseteq WF(\mathcal{I}_\vec{v} (\psi w)) \cap (\mathbb{R}^n \times \{\vec{\eta}_0\})
= \emptyset.
\end{equation}

Since $U^*$ was arbitrary, and for each $\vec{x} \in U$, we can find such a $U^*$ containing $\vec{x}$, we can deduce that:

\begin{equation}
WF(\mathcal{I}_\vec{v}w) \cap (U \times \{\vec{\eta}_0\}) = \emptyset.
\end{equation}

**Corollary 1.10** (Propagation of singularities of the distributional directional antiderivative). Let $w \in \mathcal{E}'(\mathbb{R}^n)$, define:

\begin{equation}
R_\vec{v}(\vec{x}) = \{\vec{x} + t\vec{v} : t \geq 0\}, \quad \vec{x} \in \mathbb{R}^n, \vec{v} \in S^{n-1},
\end{equation}

and let:

\begin{equation}
V_{\vec{\eta}_0} = \bigcup_{(\vec{x}, \vec{\eta}_0) \in WF(w)} R_\vec{v}(\vec{x}), \quad U_{\vec{\eta}_0} = V_{\vec{\eta}_0}^C, \quad \vec{\eta}_0 \in \vec{v}^\perp.
\end{equation}

Then:

\begin{equation}
WF(\mathcal{I}_\vec{v}w) \subseteq WF(w) \cup \bigcup_{\vec{\eta}_0 \in \vec{v}^\perp} (V_{\vec{\eta}_0} \times \{\vec{\eta}_0\})
= WF(w) \cup \{(\vec{x} + t\vec{v}, \vec{\eta}_0) \mid (\vec{x}, \vec{\eta}_0) \in WF(w), \vec{\eta}_0 \perp \vec{v}, t \geq 0\}.
\end{equation}

**Proof.** Since $D_\vec{v} \mathcal{I}_\vec{v} w = w$, we can already narrow down $WF(\mathcal{I}_\vec{v} w)$ to:

\begin{equation}
WF(\mathcal{I}_\vec{v}w) \subseteq WF(w) \cup (\mathbb{R}^n \times \vec{v}^\perp).
\end{equation}

We want to be able to replace $\mathbb{R}^n \times \vec{v}^\perp$ with $\bigcup_{\vec{\eta}_0 \in \vec{v}^\perp} (V_{\vec{\eta}_0} \times \{\vec{\eta}_0\})$.

We next observe that for each $\vec{\eta}_0 \in \vec{v}^\perp$, since $w$ is compactly-supported, $V_{\vec{\eta}_0}$ must be closed. Then $U_{\vec{\eta}_0}$ is an open set that is invariant under translation in the direction of $-\vec{v}$, and:

\begin{equation}
WF(w) \cap (U_{\vec{\eta}_0} \times \{\vec{\eta}_0\}) = \emptyset,
\end{equation}

and so:

\begin{equation}
WF(\mathcal{I}_\vec{v}w) \cap (U_{\vec{\eta}_0} \times \{\vec{\eta}_0\}) = \emptyset.
\end{equation}
Therefore, if \((\xi_0, \eta_0) \in WF(\mathcal{I}_w w)\), but \((\xi_0, \eta_0) \notin WF(w)\), then \([1.5]\) indicates that \(\eta_0 \in \nu^\perp\), and then \([1.6]\) would require that \(\xi \notin U_{\eta_0}\), and so \(\{\xi_0, \xi_0\} \in \mathcal{V}_{\eta_0} \times \nu_0\).

Note that if we lift the restriction of compact support on \(w\), we must instead use:

\[
V_{\eta_0} = \bigcup_{(\xi, \eta) \in WF(w)} R_{\mathcal{I}_w}(\xi).
\]

We now strengthen \([1.9]\) by with the next result. Intuitively, even if \((\xi_0, \eta_0) \in WF(\mathcal{I}_w w)\), if \(WF(\mathcal{I}_w w)\) omits some \((\xi_0 + t_0 \nu, \eta_0)\), for some \(t_0 > 0\), then the only way for \(WF(\mathcal{I}_w w)\) to pick up any more elements of the form \((\xi_0 + t \nu, \eta_0)\) for \(t > t_0\) is for \(WF(w)\) to contain some \((\xi_0 + t_1 \nu, \eta_0)\), for some \(t_1 > t_0\).

**Proposition 1.11.** Let \(w \in \mathcal{E}'(\mathbb{R}^n)\), \(\eta_0 \in \nu^\perp\), and \(U_0\) be a bounded open set, \(t_1 > 0\), and

\[
U_t = U + t \nu, \quad t \in \mathbb{R},
\]

\[
U = \bigcup_{0 \leq t \leq t_1} U_t.
\]

If:

\[
WF(\mathcal{I}_w w) \cap (U_0 \times \{\eta_0\}) = \emptyset,
\]

and:

\[
WF(w) \cap (U \times \{\eta_0\}) = \emptyset,
\]

then:

\[
WF(\mathcal{I}_w w) \cap (U \times \{\eta_0\}) = \emptyset.
\]

**Proof.** We may assume without loss of generality that \(U_0\) is convex, otherwise, we can apply the following argument to every convex open subset of \(U_0\). Let \(U_0^*\) be an open set whose closure is contained in \(U_0\), then define \(U^*\) in much the same way as \(U\). Now let \(\psi \in \mathcal{C}_c^\infty(U)\) be equal to 1 on \(U^*\). Consider the distributional partial derivative:

\[
D_{\mathcal{I}_w} (\psi \cdot \mathcal{I}_w w) = D_{\mathcal{I}_w} \psi \cdot \mathcal{I}_w w + \psi \cdot w.
\]

Since \(D_w \psi\) vanishes on \(U^*\), we have \(D_{\mathcal{I}_w} (\psi \cdot \mathcal{I}_w w) = w\) on \(U^*\), and since:

\[
WF(w) \cap (U^* \times \{\eta_0\}) \subseteq WF(w) \cap (U \times \{\eta_0\}) = \emptyset,
\]

we have:

\[
WF(D_{\mathcal{I}_w}(\psi \cdot \mathcal{I}_w w)) \cap (U^* \times \{\eta_0\}) = \emptyset.
\]

Furthermore, since:

\[
WF(\mathcal{I}_w w) \cap (U_0 \times \{\eta_0\}) = \emptyset,
\]

it must also follow that:

\[
WF(D_{\mathcal{I}_w}(\psi \cdot \mathcal{I}_w w)) \cap (U_0 \times \{\eta_0\}) = \emptyset,
\]

and so:

\[
WF(D_{\mathcal{I}_w}(\psi \cdot \mathcal{I}_w w)) \cap (U_0 \cup U^* \times \{\eta_0\}) = \emptyset.
\]

We can replace \(U_0 \cup U^*\) with \(\bigcup_{t \leq 0} U_t \cup U^*\), as that introduces no points that are inside the support of \(\psi\) (hence the requirement that \(U_0\) be convex), and is also invariant under translation in the direction \(-\nu\), and so we again apply \([1.5]\) to obtain:

\[
WF(\psi \cdot \mathcal{I}_w w) \cap ((U_0 \cup U^*) \times \{\eta_0\}) = \emptyset.
\]
In particular,
\[ \text{WF} (I_\varphi w) \cap (U^* \times \{ \tilde{t}_0 \}) = \emptyset, \]
and since \( U_0^* \) was arbitrary, we can replace \( U^* \) with \( U \):
\[ \text{WF} (I_\varphi w) \cap (U \times \{ \tilde{t}_0 \}) = \emptyset. \]

We now wish to extend the distributional directional antiderivative further by replacing the support condition \(1.11\) with an even weaker condition, that there exists a \( t_{\text{min}} \in C^\infty (\overline{v^+}) \) such that
\[ \text{supp } w \subseteq \{ \tilde{u} + t\tilde{v} \mid \tilde{u} \in \overline{v^+}, t > t_{\text{min}} (\tilde{u}) \}. \]

Notice this includes the previous support condition by considering the case that \( t_{\text{min}} \) is a constant. If we let \( \chi (\tilde{x}) = \tilde{x} - t_{\text{min}} (\tilde{u}) \tilde{v} \), then the pullback \( \chi^* w \) has the support condition \( \inf_{\tilde{x} \in \text{supp } \chi^* w} \tilde{x} \cdot \tilde{v} > 0 \), and so we can define \( I_\varphi w \) by conjugating \( I_\varphi \) with the pullback map \( \chi^* \). We will want to be sure that this does not change \( I_\varphi w \), however.

**Proposition 1.12.** Let \( w \) satisfy \(1.1\) and define \( \chi (\tilde{x}) = \tilde{x} - t_{\text{min}} \tilde{v} \). Then \( \chi^{-*} I_\varphi \chi^* w = I_\varphi w \).

**Proof.** It is important to note that \( \chi I_\varphi \chi^* = I_\varphi \). For some \( \psi_0 \in C_0^\infty (\mathbb{R}^-) \), we have
\[
\langle \chi^{-*} I_\varphi \chi^* w, \phi \rangle = -\langle \chi^* w, I_\varphi \chi^* \phi \rangle = -\langle \chi^* w, \int_{-\infty}^t \chi^* \phi (\tilde{u} + s\tilde{v}) - \chi_\varphi (\chi^* \phi) \otimes \psi_0 (\tilde{u} + s\tilde{v}) \ ds \rangle \\
= -\langle \chi^* w, \int_{-\infty}^t \phi (\tilde{u} + (s - t_{\text{min}}) \tilde{v}) - \chi_\varphi \phi \otimes \psi_0 (\tilde{u} + s\tilde{v}) \ ds \rangle \\
= -\langle \chi^* w, \int_{-\infty}^{t-t_{\text{min}}} \phi (\tilde{u} + s\tilde{v}) - \chi_\varphi \phi \otimes \psi_0 (\tilde{u} + (s + t_{\text{min}}) \tilde{v}) \ ds \rangle \\
= -\langle \chi^* w, \int_{-\infty}^{t-t_{\text{min}}} \phi (\tilde{u} + s\tilde{v}) - \chi_\varphi \phi \otimes \chi^{-*} \psi_0 (\tilde{u} + s\tilde{v}) \ ds \rangle \\
= -\langle \chi^* w, I_\varphi \phi (\tilde{u} + (t - t_{\text{min}}) \tilde{v}) \rangle \\
= -\langle w, \chi^{-*} \{ I_\varphi \phi (\tilde{u} + (t - t_{\text{min}}) \tilde{v}) \} \rangle \\
= -\langle w, I_\varphi \phi (\tilde{u} + t\tilde{v}) \rangle \\
= \langle I_\varphi w, \phi \rangle,
\]
where \( \chi^{-*} \psi_0 (t) = \psi_0 (t + t_{\text{min}}) \).

Thus, we may define \( I_\varphi w \) when \( w \) satisfies the weaker support condition as follows:

**Definition 1.13.** Let \( w \in \mathcal{D}' (\mathbb{R}^n) \) have the support condition \(1.7\). Define
\[ I_\varphi w = \chi^{-*} I_\varphi \chi^* w, \]
where \( \chi (\tilde{x}) = \tilde{x} - t_{\text{min}} (\tilde{u}) \tilde{v} \).

We can also verify that this definition is independent of the choice of \( \chi \) so long as \(1.7\) is satisfied. We will omit this proof as it would proceed in a fashion similar to the above computation.

We also wish to show \(1.10\) also applies to this extension of \( I_\varphi \).
Proof. Notice that
\[ \mathcal{D} \chi = \begin{bmatrix} I_{\mathfrak{v}^\perp} & 0 \\ D_{t_{\min}} & 1 \end{bmatrix}, \]
which implies that \( \mathcal{D} \chi^T \) and \( \mathcal{D} \chi^{-T} \) both fix \( \mathfrak{v}^\perp \). Thus,

\[
WF(\mathcal{I}_w) = WF(\chi^* \mathcal{I}_w) = \chi^{-*} (WF(\chi^* w) \cup \{(\bar{x} + t\mathfrak{v}, \bar{\eta}_0) \mid \bar{x}, \bar{\eta}_0 \in WF(\chi^* w), \bar{\eta}_0 \perp \mathfrak{v}, t \geq 0\})
\]

\[
= WF(w) \cup \{(\chi^{-1}(\bar{x} + t\mathfrak{v}), \bar{\eta}_0) \mid \chi^{-1}(\bar{x}), \bar{\eta}_0 \in WF(w), \bar{\eta}_0 \perp \mathfrak{v}, t \geq 0\}
\]

\[
= WF(w) \cup \{(\bar{x} + t\mathfrak{v}, \bar{\eta}_0) \mid (\bar{x}, \bar{\eta}_0) \in WF(w), \bar{\eta}_0 \perp \mathfrak{v}, t \geq 0\}.
\]

\[ CW \]

Definition 1.14. We will use the more familiar notation:
\[
\int_0^\infty w(\bar{x} - t\mathfrak{v}) \, dt
\]
to refer to \( \mathcal{I}_w \).

2. General line integrals

Now that we have given meaning to the integral \((1.8)\) we now wish to give meaning to the following integral:
\[
\int_0^\infty w(\bar{x} - \tilde{\gamma}(t)) v(t) \, dt,
\]
given a \( \tilde{\gamma} \in C^\infty((-\varepsilon, \infty); \mathbb{R}^n) \) and positive-valued \( v \in C^\infty((-\varepsilon, \infty), \mathbb{R}) \), for some \( \varepsilon > 0 \), with \( \tilde{\gamma}(0) = \bar{0} \), and \( \tilde{\gamma}'(t) \neq 0 \) for all \( t > -\varepsilon \). It will also be necessary to impose a support condition that \( w(\bar{x} - \gamma(t)) \) has bounded support in the variable \( t \) to avoid a pathological choice of \( \tilde{\gamma} \), e.g., a choice of \( \tilde{\gamma} \), that given some \( w \in L^1(\mathbb{R}^n) \), the above integral may fail to converge for \( \bar{x} \) in some open set.

We observe that in the case that \( w \in L^1(\mathbb{R}^n) \), the above integral can be interpreted as
\[
\int_0^\infty w(\bar{x} - \tilde{\gamma}(y + t)) \, dt \bigg|_{y=0}.
\]

The notation \( w(\bar{x} - \tilde{\gamma}(y)) \) refers to pulling back \( w \) by the map \( \chi(\bar{x}, y) = \bar{x} - \tilde{\gamma}(y) \).

We then compute the distributional antiderivative in the direction \((\bar{0}, -1)\), the direction corresponding to the negative \( y \)-axis. This antiderivative is then pulled back by the map
\[
\psi_0(\bar{x}) = (\bar{x}, 0).
\]

For ease of notation, we will specialize to the case \( \tilde{\gamma}(0) = \bar{0} \). A result for the general case can be achieved via translations.
**Definition 2.1.** Let \( w \) be a distribution in \( \mathbb{R}^n \), and \( \dot{\gamma} \in C^\infty ((-\varepsilon, \infty) ; \mathbb{R}^n) \) a curve for some \( \varepsilon > 0 \), with \( \dot{\gamma}(0) = \dot{0} \) and \( \dot{\gamma}'(t) \neq 0 \), and assume the pullback
\[
\chi^* w = w(\overline{x} - \dot{\gamma}(y))
\]
has support bounded in \( y \). Then the integral:
\[
\int_0^\infty w(\overline{x} - \dot{\gamma}(t)) \, dt,
\]
is defined as:
\[
\int_0^\infty w(\overline{x} - \dot{\gamma}(y + t)) \, dt \bigg|_{y=0}.
\]
That is,
\[
\int_0^\infty w(\overline{x} - \dot{\gamma}(t)) \, dt = \psi_0^* T(\delta_{-1}) (\chi^* w).
\]
If additionally, \( \nu \in C^\infty ((-\varepsilon, \infty)) \) is a positive-valued weight function, we can then define:
\[
(2.2) \quad \int_0^\infty w(\overline{x} - \dot{\gamma}(t)) \nu(t) \, dt = \int_0^\infty w(\overline{x} - \dot{\gamma}(y + t)) \nu(y + t) \, dt \bigg|_{y=0} = \psi_0^* T(\delta_{-1}) (\chi^* w \nu).
\]

We then extend 1.10 to a more general result for the integral 2.1

**Theorem 2.2.** If \( w \in \mathcal{E}'(\mathbb{R}^n) \) is a distribution such that the support in \( y \) of \( w(\overline{x} - \dot{\gamma}(y)) \) is bounded, then:
\[
WF \left\{ \int_0^\infty w(\overline{x} - \dot{\gamma}(t)) \nu(t) \, dt \right\} \subseteq WF(w),
\]
\[
\subseteq \left\{ (\overline{x}, \xi) \mid \exists t \geq 0 : \xi \perp \gamma'(t) \land (\overline{x} - \dot{\gamma}(t), \dot{\xi}) \in WF(w) \right\}.
\]

**Proof.** We may, without loss of generality, set \( \nu = 1 \), as doing so will not alter the wavefront sets involved in this proof. With:
\[
\chi(\overline{x}, y) = \overline{x} - \dot{\gamma}(y).
\]
We observe that:
\[
\mathcal{D}_\chi(\overline{x}, y) = \left[ I_n \quad \gamma'(y) \right], \quad \mathcal{D}_\chi(\overline{x}, y)^T = \left[ I_n \ \gamma'(y)^T \right],
\]
and so \( \ker \mathcal{D}_\chi(\overline{x}, y)^T \) is trivial, indicating that the pullback \( w(\overline{x} - \dot{\gamma}(y)) \) is indeed well-defined, and:
\[
WF \{ w(\overline{x} - \dot{\gamma}(y)) \} \subseteq \chi^* WF(w),
\]

\[
(2.3) \quad \chi^* WF(w) = \left\{ (\overline{x}, y), \left( \dot{\xi}, \gamma'(y) \cdot \dot{\xi} \right) : (\overline{x} - \dot{\gamma}(y), \dot{\xi}) \in WF(w) \right\}.
\]

We now wish to show that:
\[
WF \{ w(\overline{x} - \dot{\gamma}(y)) \} = \chi^* WF(w).
\]
Indeed, let \( y_0 > -\varepsilon \), and define:
\[
\psi_{y_0}(\overline{x}) = \left[ \overline{x} + \gamma(y_0) \right]_{y_0}.
\]
Then:
\[ D_{\psi_{y_0}}(\mathbf{x}) = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \quad D_{\psi_{y_0}}(\mathbf{x})^T = [I_n \ 0], \]
and so:
\[ \ker D_{\psi_{y_0}}(\mathbf{x})^T = \left\{ (\mathbf{0}, \eta) : \eta \in \mathbb{R} \right\}. \]
Therefore, the set of normals for \( \psi_{y_0} \) satisfies:
\[ N_{\psi_{y_0}} \subseteq (\mathbb{R}^n \times (-\infty, \infty)) \times \left\{ (\mathbf{0}, \eta) : \eta \in \mathbb{R} \right\}, \]
and so \( N_{\psi_{y_0}} \cap WF \{ w(\mathbf{x} - \gamma(y)) \} = \emptyset \). Thus, the pullback \( \psi_{y_0}^* \{ w(\mathbf{x} - \gamma(y)) \} \) is well-defined, and is in fact equal to \( w \). Then:
\[ WF(w) = WF(\psi_{y_0}^* \{ w(\mathbf{x} - \gamma(y)) \}) \subseteq \psi_{y_0}^*WF \{ w(\mathbf{x} - \gamma(y)) \}, \]
where:
\[ \psi_{y_0}^*WF \{ w(\mathbf{x} - \gamma(y)) \} \]
\[ = \left\{ (\mathbf{x}, \xi) : (\mathbf{x} + \gamma(y_0), y_0), (\xi, \eta) \right\} \in WF \{ w(\mathbf{x} - \gamma(y)) \}. \]
Thus, if we chose \( (\mathbf{x}, y_0), (\xi', \gamma'(y_0) \cdot \xi') \) \( \in \chi^*WF(w) \), this choice was based on choosing \( (\mathbf{x} - \gamma(y_0), \xi) \) \( \in WF(w) \). We then have by (2.4) that \( (\mathbf{x} - \gamma(y_0), \xi) \) \( \in \psi_{y_0}^*WF \{ w(\mathbf{x} - \gamma(y)) \} \). That is,
\[ \left( (\mathbf{x}, y_0), (\xi', \eta) \right) = \left( (\mathbf{x} - \gamma(y_0) + \gamma(y_0), y_0), (\xi', \eta) \right) \in WF \{ w(\mathbf{x} - \gamma(y)) \}, \]
for some \( \eta \). Of course, one only needs to choose \( \eta = \gamma'(y_0) \cdot \xi' \), and this will yield the desired set inclusion.

Next, we can use (2.5) to describe \( WF \left\{ \int_0^\infty w(\mathbf{x} - \gamma(y + t)) \, dt \right\} \) as follows:
\[ WF \left\{ \int_0^\infty w(\mathbf{x} - \gamma(y + t)) \, dt \right\} \triangleleft WF \{ w(\mathbf{x} - \gamma(y)) \}
\[ \subseteq \bigcup_{(\xi, \eta) \perp (\xi', -1)} \left\{ (\mathbf{x}, y - t), (\xi', \eta) \right\} : \]
\[ \left( (\mathbf{x}, y), (\xi', \eta) \right) \in WF \{ w(\mathbf{x} - \gamma(y)) \}, t \geq 0 \}
\[ = \bigcup_{\xi \in \mathbb{R}^n} \left\{ \left( (\mathbf{x}, y - t), (\xi', 0) \right) : \left( (\mathbf{x}, y), (\xi', 0) \right) \in \chi^*WF(w), t \geq 0 \right\} \]
\[ = \bigcup_{\xi \in \mathbb{R}^n} \left\{ \left( (\mathbf{x}, y - t), (\xi', 0) \right) : \left( \mathbf{x} - \gamma(y), \xi \right) \in WF(w), \xi \perp \gamma(y), t \geq 0 \right\}. \]
Finally,
\[
WF \left\{ \int_0^\infty w(\vec{x} - \vec{\gamma}(y + t)) \, dt \right|_{y=0} \right\} \\
\subseteq \chi_0^*WF \{w(\vec{x} - \vec{\gamma}(y))\} \cup \chi_0^* \left\{ \left( (\vec{x}, y - t), (\vec{\xi}, 0) \right) : \xi \in \vec{\gamma}'(y) \perp, (\vec{x} - \vec{\gamma}(y) , \vec{\xi}) \in WF (w) , t \geq 0 \right\} \\
= WF (w) \cup \\
\left\{ (\vec{x}, \vec{\xi}) \mid \exists t \geq 0 : (\vec{x} - \vec{\gamma}(t) , \vec{\xi}) \in WF (w) , \vec{\xi} \in \vec{\gamma}'(t) \perp \right\}.
\]

It should be noted that the integral 2.1 reduces to the distributional anti-partial derivative developed in the previous section when \( \vec{\gamma} \) parametrizes a ray:
\[
\vec{\gamma}(t) = t \vec{v}, \quad t \geq 0, \vec{v} \in S^{n-1}.
\]

Furthermore, the result obtained in 2.2 in this case is:
\[
WF \left\{ \int_0^\infty w(\vec{x} - \vec{\gamma}(t)) v(t) \, dt \right\} \setminus WF (w) \\
\subseteq \left\{ (\vec{x}, \vec{\xi}) \mid \exists t \geq 0 : \vec{\xi} \perp \vec{\gamma}'(t) \& (\vec{x} - \vec{\gamma}(t) , \vec{\xi}) \in WF (w) \right\} \\
= \left\{ (\vec{x}, \vec{\xi}) \mid \exists t \geq 0 : \vec{\xi} \perp \vec{v} \& (\vec{x} - t \vec{v}, \vec{\xi}) \in WF (w) \right\},
\]

which is an equivalent formulation to 1.4.