ON P-GROUPS OF MAXIMAL CLASS

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Abstract.

Recall that a $p$-group of order $p^n > p^3$ is of maximal class, if its nilpotency class is $n - 1$. In this paper, we study the $p$-groups of maximal class. Furthermore, we introduce a subgroup of a $p$-group of maximal class called the fundamental subgroup. This group plays a fundamental role in the development of the general theory of $p$-groups of maximal class. As an application, we study some special class of finite $p$-groups of maximal class and exponent $p$.

Keywords: $p$-groups of maximal class, fundamental subgroup, CGZ-group.

1 Introduction

A group of order $p^n > p^3$ and nilpotency class $n - 1$, is said to be of maximal class. These groups have been studied by various authors [6, 10]. But the main reference in the theory of $p$-groups of maximal class is Blackburn’s paper [4]. Other famous references for these $p$-groups are [1, 7, 8]. The $p$-groups of maximal class with an abelian maximal subgroup were completely classified by Wiman in [9]. More recently, it is proved that the finite $p$-group of maximal class can be determined by centralizers of some subgroups. For example, the theorem of Suzuki ([33] Theorem III.14.23), [1] Proposition 1.8) shows that a finite $p$-group $G$ is of maximal class if there is a self centralizing subgroup of order $p^2$, and in [11] Proposition 10.17], it is showed that if $G$ is a finite $p$-group, $B \leq G$ is a nonabelian subgroup of order $p^3$ such that $C_G(B) < B$, then $G$ is of maximal class. Some results about these groups that we present in this paper can be found in [11, 2, 3]. These results play a fundamental role in finite $p$-group theory. Throughout this paper, we use the standard notation, such as in [11].

The paper is organized as follows: In the second section, we recall some preliminaries of nilpotent groups. The third section covers the basic material about $p$-groups of maximal class. In section 4, we introduce a characteristic subgroup of a $p$-group of maximal class called the fundamental subgroup. This group plays a

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fundamental role in the development of the general theory of $p$-groups of maximal class. In section 5, which is the end section of this work, we deal with some special class of finite $p$-groups of maximal class and exponent $p$, namely $CGZ$-groups. More precisely, we prove that any $CGZ$-group of order $p^n$ and exponent $p$ (where $3 \leq n \leq p$) admits a unique characteristic elementary abelian subgroup of index $p$.

2 Preliminaries

Let $G$ be a group and $H$ be a subgroup of $G$. An element $g \in G$ normalizes $H$ if $gHg^{-1} = H$. We call $N_G(H) = \{g \in G | gHg^{-1} = H\}$ the normalizer of $H$ in $G$. An element $g \in G$ centralizes $H$ if $ghg^{-1} = h$ for any $h \in H$. We call $C_G(H) = \{g \in G | ghg^{-1} = h, \forall h \in H\}$ the centralizer of $H$ in $G$. If $H = G$, then $Z(G) = C_G(G)$ is called the center of $G$.

Note that for a subgroup $H$ of $G$, $C_G(H) = G$ if and only if $H \leq Z(G)$. It is easy to see that, for any subgroup $H$, $N_G(H)$ and $C_G(H)$ are subgroups of $G$ with $C_G(H) \leq N_G(H)$, and that if $H \leq G$ then $H \leq N_G(H)$. The N/C-theorem [1] Introduction, Proposition 12 asserts that if $H \leq G$, then the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$. In particular, $G/Z(G)$ is isomorphic to a subgroup of $Aut(G)$.

Let $G$ be a group. Set $Z_0(G) = \{1\}, Z_1(G) = Z(G)$. Suppose that $Z_i(G)$ has been defined for $i \leq k$. Define $Z_{k+1}(G)$ as follows: $Z_{k+1}(G)/Z_k(G) = Z(G/Z_k(G))$. The chain $\{1\} = Z_0(G) \leq Z_1(G) \leq \ldots \leq Z_k(G) \leq \ldots$ is said to be the upper central series of $G$. All members of that series are characteristic in $G$.

**Definition 2.1.** For elements $x, y \in G$, their commutator $x^{-1}y^{-1}xy$ is written as $[x, y]$. If $X, Y \subseteq G$, then $[X, Y]$ is the subgroup generated by all commutators $[x, y]$ with $x \in X, y \in Y$.

The lower central series $G = K_1(G) \geq K_1(G) \geq K_2(G) \geq \ldots$ of $G$ is defined as follows: $K_1(G) = G, K_{i+1}(G) = [K_i(G), G], i > 0$. All members of that series are characteristic in $G$. We have $K_i(G)/K_{i+1}(G) = Z(G/K_i(G))$. If $H \leq G$, then $K_i(H) \leq K_i(G)$ for all $i$.

Since $[y, x] = [x, y]^{-1}$, we have $[Y, X] = [X, Y]$. We write $[G, G] = G'$, the subgroup $G'$ is called the commutator (or derived subgroup) of $G$. We also write $G^0 = G, G' = G^1$. Then the subgroup $G^{i+1} = [G^i, G^i]$ is called the $(i+1)$th derived subgroup of $G$. $i \geq 0$. The chain $G = G^0 \geq G^1 \geq \ldots \geq G^n \geq \ldots$ is called the derived series of $G$. All members of this series are characteristic in $G$ and all factors $G^i/G^{i+1}$ are abelian. The group $G$ is said to be solvable if $G^n = 1$ for some $n$.

**Definition 2.2.** A group $G$ is said to be nilpotent if the upper central series of $G$ contains $G$. In other words, $G$ is nilpotent of class $c$, if $Z_c(G) = G$ but $Z_{c-1}(G) < G$, we write $c = cl(G)$. In particular, the class of the identity group is 0 and the class of a nonidentity abelian group is 1.

**Lemma 2.1.** The following are equivalent:

1. $G$ is nilpotent of class $c$.

2. $G = K_1(G) > K_2(G) > K_3(G) > \ldots > K_{c+1}(G) = \langle 1 \rangle$. 

3. \( \langle 1 \rangle = Z_0(G) < Z_1(G) < \ldots < Z_{c-1}(G) < Z_c(G) = G. \)

**Theorem 2.1.** Let \( G \) be a \( p \)-group of order \( p^m \geq p^2 \). Then:

1. \( G \) is nilpotent of class at most \( m - 1 \).
2. If \( G \) has nilpotency class \( c \) then \( |G : Z_{c-1}(G)| \geq p^2 \).
3. The maximal subgroups of \( G \) are normal and of index \( p \).
4. \( |G : G'| \geq p^2 \).

**Corollary 2.1.** Let \( G \) be a \( p \)-group and let \( N \) be a normal subgroup of \( G \) of index \( p^i \geq p^2 \). Then \( K_i(G) \leq N \).

**Proof.** The group \( G/N \) has order \( p^i \geq p^2 \). It follows from part (1) of the Theorem that \( G/N \) has class \( \leq i - 1 \) and consequently \( K_i(G/N) = \langle 1 \rangle \). Since \( K_i(G/N) = K_i(G)/N/N \), this proves that \( K_i(G) \leq N \).

## 3 \( p \)-groups of maximal class

Recall that a group of order \( p^m \) is of maximal class, if \( cl(G) = m - 1 > 1 \). Of course, any group of order \( p^2 \) and any nonabelian \( p \)-group of order \( p^3 \) have maximal class. If \( G \) is a group of maximal class and of order \( p^m \), then the lower and upper central series of \( G \) are:

\[
G = K_1(G) > K_2(G) > K_3(G) > \ldots > K_m(G) = \langle 1 \rangle,
\]

and

\[
\langle 1 \rangle = Z_0(G) < Z_1(G) < \ldots < Z_{m-2}(G) < Z_{m-1}(G) = G.
\]

The two series are the same, and all members of these series are characteristic in \( G \). The sections are all of order \( p \), except the first which has order \( p^2 \) and is not cyclic. Clearly \( G/K_i(G) \) is also of maximal class for \( 4 \leq i \leq m - 1 \).

**Proposition 3.1.** Let \( G \) be a \( p \)-group of maximal class and order \( p^m \). Then:

1. \( |G : G'| = p^2 \), \( |Z(G)| = p \) and \( |K_i(G) : K_{i+1}(G)| = p \) for \( 2 \leq i \leq m - 1 \).
2. If \( 1 \leq i < m - 1 \), then \( G \) has only one normal subgroup of order \( p^i \). More precisely, if \( N \) is a normal subgroup of \( G \) of index \( p^i \geq p^2 \), then \( N = K_i(G) \).
3. \( G \) has \( p + 1 \) maximal subgroups.

**Proof.** (1) We have that

\[
p^m = |G| = |G : G'| \prod_{i=2}^{m-1} |K_i(G) : K_{i+1}(G)|
\]
Now it suffices to observe that $|G : G'| \geq p^2$, by theorem 2.1 and that $|K_i(G) : K_{i+1}(G)| \geq p$ for $2 \leq i \leq m - 1$.

(2) Let $N$ be any normal subgroup of $G$ of index $p^i$ with $0 \leq i \leq m$. If $i = 0$ or $1$ then $N = K_1(G)$ or $N$ is maximal in $G$. Otherwise $i \geq 2$ and $K_i(G) \leq N$ by Corollary 2.1. Since $|G : K_i(G)| = p^i$, we conclude that $N = K_i(G)$.

(3) As $G/K_2(G)$ has exponent $p$, the Frattini subgroup $\Phi(G) = K_2(G)$. Hence $G/\Phi(G)$ has order $p^2$ and can be regarded as a vector space over $\mathbb{F}_p$ of dimension 2. This vector space has $p + 1$ subspaces of dimension 1 and these correspond to the maximal subgroups of $G$.

\[ \square \]

**Remark 3.1.**

1. (The converse of the proposition 3.1(2)) If $G$ is a noncyclic group of order $p^m > p^2$ containing only one normal subgroup of order $p^i$ for each $1 \leq i < m - 1$, then it is of maximal class [1, Lemma 9.1].

2. A p-group $G$ of order $p^n$ has exactly $n + p - 1$ nontrivial normal subgroups if and only if it is of maximal class [1, Exercise 0.30].

Suppose that a p-group $G$ has only one normal subgroup $T$ of index $\geq p^{i+1}$. If $G/T$ is of maximal class so is $G$ (see [1, Theorem 12.9]). Conversely, let $G$ be a p-group of maximal class. If $N$ is a normal subgroup of $G$ of index $\geq p^2$, then $G/N$ has also maximal class. Indeed, since the class of $G/K_i(G)$ is $i - 1$ whenever $2 \leq i \leq m$, then the result follows immediately from proposition 3.1.

**Proposition 3.2.** Let $G$ be a nonabelian group of order $p^m > p^2$. If $G$ contains only one normal subgroup of index $p^k$ for any $k \in \{2, ..., p + 1\}$, then it is of maximal class.

**Proof.** Obviously, $|G : G'| = p^2$ hence $d(G) = 2$. Assume that $G$ is not of maximal class, then $m > p + 1$. Let $T < G'$ be $G$-invariant of index $p^{p+1}$ in $G$, then $G/T$ is of maximal class. As, by hypothesis, $T$ is the unique normal subgroup of index $p^{p+1}$ in $G$, then $G$ is of maximal class. \[ \square \]

**Proposition 3.3.** (M. Suzuki). Let $G$ be nonabelian p-group. If $A < G$ of order $p^2$ is such that $C_G(A) = A$, then $G$ is of maximal class.

**Proof.** We use induction on $|G|$. Since $p^2 \mid |\text{Aut}(A)|$ then, by N/C-theorem, $N_G(A)$ is nonabelian of order $p^3$. As $A < G$, then $Z(G) < A$ and $|Z(G)| = p$. Obviously, $C_{G/Z(G)}(A/Z(G)) = N_{G/Z(G)}(A/Z(G))$. Since $C_{G/Z(G)}(N_G(A)/Z(G)) \leq C_{G/Z(G)}(A/Z(G)) = N_{G/Z(G)}(A/Z(G))$ is of order $p^2$ then, by induction, $G/Z(G)$ is of maximal class so $G$ is also of maximal class since $|Z(G)| = p$. \[ \square \]

**Corollary 3.1.** A p-group $G$ is of maximal class if and only if $G$ has an element with centralizer of order $p^2$.

**Proposition 3.4.** Let $G$ be a p-group, $B \leq G$ nonabelian of order $p^3$ and $C_G(B) < B$. Then $G$ is of maximal class.
Proof. Assume that $|G| \geq p^4$ and the proposition has been proved for groups of order $< |G|$. It is known that a Sylow $p$-subgroup of $\text{Aut}(B)$ is nonabelian of order $p^3$. Now, $C_G(B) = Z(B) = Z(G)$. Therefore, by N/C-Theorem, $N_G(B)/Z(G)$ is nonabelian of order $p^3$. If $x \in G - C_G(B)$ centralizes $N_G(B)/Z(G)$, then $x$ normalizes $B$ so $x \in N_G(B)$, a contradiction. Thus, $C_G(N_G(B)/Z(G)) < N_G(B)/Z(G)$ so, by induction, $G/Z(G)$ is of maximal class. Since $|Z(G)| = p$, we are done.

\[ \square \]

Proposition 3.5. Let $G$ be a $p$-group. If $G$ has a subgroup $H$ such that $N_G(H)$ is of maximal class, then it is of maximal class.

**Proof.** Assume that $|N_G(H)| > p^3$ (otherwise, $C_G(H) = H$ and $G$ is of maximal class, by Proposition 3.1). We use induction on $|G|$. One may assume that $N_G(H) < G$, then $H$ is not characteristic in $N_G(H)$ so by Proposition 3.1 we have $|N_G(H) : H| = p$ hence $|H| > p$. As $|Z(N_G(H))| = p$ and $Z(G) < N_G(H)$, we get $Z(G) = Z(N_G(H))$ so $|Z(G)| = p$ and $Z(G) < H$. Then $N_{G/Z(G)}(H/Z(G)) = N_G(H)/Z(G)$ is of maximal class, so $G/Z(G)$ is also of maximal class by induction. Since $|Z(G)| = p$, then $G$ is of maximal class.

\[ \square \]

Lemma 3.1. Let $G$ be a $p$-group and let $N \trianglelefteq G$ be of order $> p$. Suppose that $G/N$ of order $> p$ has cyclic center. If $R/N \not\trianglelefteq G/N$ is of order $p$ in $G/N$, then $R$ is not of maximal class.

**Proof.** Let $T$ be a $G$-invariant subgroup of index $p^2$ in $N$. Then $R \leq C_G(N/T)$ so $R/T$ is abelian of order $p^3$, and we conclude that $R$ is not of maximal class.

\[ \square \]

Proposition 3.6. Let $A < G$ be of order $> p$. If all subgroups of $G$ containing $A$ as a subgroup of index $p$ are of maximal class, then $G$ is also of maximal class.

**Proof.** Set $N = N_G(A)$. In view of proposition 3.5 and hypothesis, one may assume that $|N : A| > p$ (otherwise, there is nothing to prove). Let $D < A$ be $N$-invariant of index $p^2$ ($D$ exists since $|A| > p$). Set $C = C_N(A/D)$, then $C > A$. Let $F/A \leq C/A$ be of order $p$, then $F$ is not of maximal class, a contradiction.

Let $H < G$ be of index $> p^k$, $k > 1$. If all subgroups of $G$ of order $p^k | H|$, containing $H$, are of maximal class, then $G$ is also of maximal class. Indeed, let $H < M < G$, where $|M : H| = p^k - 1$. Then all subgroups of $G$ containing $M$ as a subgroup of index $p$, are of maximal class. Now the result follows from Proposition 3.6.

Proposition 3.7. Let $G$ be a $p$-group of maximal class and order $p^m$, $p > 2$, $m > 3$, and let $N \trianglelefteq G$ be of index $p^3$. Then $\exp(G/N) = p$.

**Proof.** Assume that this is false. Let $T$ be a $G$-invariant subgroup of index $p$ in $N$. By hypothesis, $G/N$ has two distinct cyclic subgroups $C/N$ and $Z/N$ of order $p^2$. Then $C/N \cap Z/N = Z(G/T)$ and $G/T$ is not of maximal class, a contradiction.

\[ \square \]
Let $A$ be an abelian subgroup of index $p$ of a nonabelian p-group $G$. By \[1\] lemma 1.1, we have $|G| = p|G'||Z(G)|$. Hence, we have the following proposition.

**Proposition 3.8.** Suppose that a nonabelian p-group $G$ has an abelian subgroup $A$ of index $p$. If $|G : G'| = p^2$, then $G$ is of maximal class.

**Proof.** We proceed by induction on $|G|$. One may assume that $|G| > p^3$. We have $|Z(G)| = \frac{1}{p}|G : G'| = p$, so $Z(G)$ is a unique minimal normal subgroup of $G$. Since $Z(G) < G'$, then we have $|G/Z(G) : G'/Z(G)| = |G : G'| = p^2$. Hence, the quotient group $G/Z(G)$ is of maximal class by induction, and the result follows since $|Z(G)| = p$. □

The previous result also holds if $G$ contains a subgroup of maximal class and index $p$ (see \[1\] Theorem 9.10).

**Proposition 3.9.** Let $G$ be a $p$-group of order $p^4$. Show that $G$ is of maximal class if and only if $|G : G'| = p^2$.

**Proof.** Let $|G : G'| = p^2$. We have to prove that $G$ is of class 3. Assume that $cl(G) = 2$. It follows from \[2\] Lemma 65.1 that $G$ contains a nonabelian subgroup $B$ of order $p^3$. By proposition 3.4, we have $G = BZ(G)$, then $|G : G'| = p^3$, a contradiction. □

**Proposition 3.10.** Let $G$ be a $p$-group of order $p^4$ and exponent $p$. Prove that if $G$ has no nontrivial direct factors then it is of maximal class.

**Proof.** Let $H$ be an $A_1$-subgroup of $G$; then $|H| = p^3$. If $Z(G) \not\leq H$, then $G = H \times C$, where $C < Z(G)$ is of order $p$ such that $C \not\leq H$. Thus, $Z(G) < H$ so $Z(G)$ is of order $p$. Since $C_G(H) = Z(G)$, we get $C_G(H) < H$. Then, by proposition 3.4, $G$ is of maximal class. □

**Proposition 3.11.** If $G$ is of order $p^4$ and exponent $p$. Prove that if $d(G) = 2$ then $G$ is of maximal class.

**Proof.** Obviously, the group $G$ has an abelian subgroup of index $p$. Since $G' = \Phi(G)$ and, by hypothesis, $|G : G'| = p^2$, the result follows from Proposition 3.8. □

4 Fundamental subgroup of a $p$-group of maximal class

**Lemma 4.1.** Let $G$ be a $p$-group of maximal class and order $p^m$, $m > 3$. For each $2 \leq i \leq m - 2$, $M_i = C_G(K_i(G)/K_{i+2}(G))$ is a maximal subgroup of $G$.

**Proof.** Indeed, $K_i(G)/K_{i+2}(G)$ is a noncentral normal subgroup of order $p^2$ in $G/K_{i+2}(G)$. So by $N/C$-theorem, the quotient group $G/M_i$ is isomorphic to a subgroup of $Aut(K_i(G)/K_{i+2}(G))$. But, a $p$-sylow subgroup of $Aut(K_i(G)/K_{i+2}(G))$ has order $p$. Thus, $|G, M_i| = p$. □
The subgroup \(M_2 = C_G(K_2(G)/K_1(G))\) plays distinguished role in what follows; we denote it by \(G_1\) and call the fundamental subgroup of \(G\). In the following, if \(G\) is of maximal class, then \(G_1\) denotes always the fundamental subgroup of \(G\). We shall write \(G_i\) instead of \(K_i(G)\) for all \(i \geq 2\) when there is no possible confusion. We have \(G/G_2\) is elementary abelian of order \(p^2\) and \([G_i, G_{i+1}] = p\) for \(1 \leq i \leq n-1\). Hence \(|G : G_i| = p^i\) for \(1 \leq i \leq n\).

**Proposition 4.1.** Let \(H\) and \(K\) be two \(p\)-groups of maximal class. Let \(\varphi : H \to K\) be an isomorphism, then \(\varphi(H_1) \subseteq K_1\).

**Proof.** The subgroup \(H_1\) is composed of the elements \(x \in H\) such that \([x, H_2] \leq H_4\). Let \(x \in H_1\), since \(\varphi(H_2) = K_2\) and \(\varphi(H_4) = K_4\), it follows that
\[
[\varphi(x), K_2] = [\varphi(x), \varphi(H_2)] = \varphi([x, H_2]) \leq \varphi(H_4) = K_4.
\]
Thus, \(\varphi(x) \in K_1\). As required.

**Corollary 4.1.** Let \(G\) be a \(p\)-group of maximal class. Then, \(G_1\) is a characteristic subgroup of \(G\).

**Proof.** The corollary follows directly from the preceding proposition by taking \(H = K\).

**Remark 4.1.**

1. If \(N\) is a normal subgroup of \(G\) such that \(|G/N| \geq p^4\), it is clear from the definition that \((G/N)_1 = G_1/N\).

2. Let \(G\) be a \(p\)-group of maximal class, \(|G| > p^4\). Let \(M\) be a maximal subgroup of \(G\) and let \(M_1\) be the fundamental subgroup of \(M\). Then \(|G : M_1| = p^2\) and \(M_1 \triangleleft G\) so \(M_1 = \Phi(G) < G_1\), and we get \(M_1 = G_1 \cap M\).

**Lemma 4.2.** [\(\ddagger\) Theorem 9.6(e)] Let \(G\) be a group of maximal class and order \(p^m\), \(p > 2\), \(m > p + 1\). Then the set of all maximal subgroups of \(G\) is
\[
\Gamma_1 = \{M_1 = G_1; M_2; \ldots; M_{p+1}\}
\]
Where \(G_1\) is the fundamental subgroup of \(G\), and the subgroups \(M_2; \ldots; M_{p+1}\) are of maximal class.

**Proposition 4.2.** Let a \(p\)-group \(G\) of maximal class have order \(p^m > p^{p+1}\). If \(H < G\) is of order \(p^p\) and \(H \nleq G_1\), then \(H\) is of maximal class.

**Proof.** We proceed by induction on \(m\). If \(m = p + 2\), the result follows from Lemma 4.2 (indeed, then all members of the set \(\Gamma_1 - G_1\) are of maximal class). Now let \(m > p + 2\) and let \(H \leq M \in \Gamma_1\), then \(M\) is of maximal class (Lemma 4.2). The subgroup \(M_1 = M \cap G_1\) is the fundamental subgroup of \(M\) (Remark 4.1(2)). As \(H \nleq M_1\), then \(H\) is of maximal class, by induction, applied to the pair \(H < M\).
Proposition 4.3. Let $G$ be of maximal class and order $> p^{p+1}$. If $H < G$ is of order $> p^2$, then either $H \leq G_1$ or $H$ is of maximal class.

Proof. Let $|H| = p^k$. The result is known if $k = 3$ [4]. Assuming that $k > 3$, we use induction on $k$. Then all maximal subgroups of $H$ which $\neq H \cap G_1$ are of maximal class, by induction. Then the set $\Gamma_1(H)$ contains exactly $|\Gamma_1(H)| - 1 \neq 0 \mod p^2$ members of maximal class so $H$ is of maximal class, by [1, Theorem 12.12(c)].

Proposition 4.4. [1, Exercise 9.28] Let $G$ be a $p$-group of maximal class and order $> p^{p+1}$. Show the following:

1. $\exp(G_1) = \exp(G)$.
2. If $x \in G$ is of order $\geq p^3$, then $x \in G_1$.

5 CGZ-group of exponent $p$

Let $p$ be a prime number and $G$ be a finite nonabelian $p$-group. The group $G$ is called a CGZ-group if and only if every nonabelian subgroup $H$ of $G$ satisfies $C_G(H) = Z(H)$. In this section, we prove that any CGZ-group of order $p^n$ and exponent $p$ (where $3 \leq n \leq p$) admits a unique characteristic elementary abelian subgroup of index $p$.

Proposition 5.1. Let $G$ be a $p$-group of order $p^4$ and exponent $p$. If $G$ has no nontrivial direct factors then it is a CGZ-group.

Proof. Indeed, if $G$ is minimal nonabelian then it is clearly a CGZ-group. Else, let $H$ be a proper nonabelian subgroup of $G$, then $|H| = p^3$. Now, let $x \in C_G(H)$. If $x \notin H$ then $G = \langle x \rangle \cdot H$, a contradiction. So, $C_G(H) < H$ and then $C_G(H) = Z(H)$. As required.

Lemma 5.1. Let $G$ be a nonabelian $p$-group and let $N$ be a nonabelian subgroup of $G$. If $G$ is a CGZ-group, then $N$ is a CNZ-group.

Proof. Let $H$ be a nonabelian subgroup of $N$. Since $G$ is a CGZ-group, it follows that $C_N(H) = N \cap C_G(H) = N \cap Z(H) = Z(H)$. So $N$ is a CNZ-group.

Proposition 5.2. [17, Proposition 2.2] Let $G$ be a nonabelian $p$-group of exponent $p$, where $p \geq 3$. Then $G$ is a CGZ-group if and only if $G$ is a finite $p$-group of maximal class, and there is an abelian subgroup of index $p$.

Proposition 5.3. Let $G$ be a $p$-group of order $p^n > p^4$ and exponent $p$. Then $G$ is a CGZ-group if and only if all proper nonabelian subgroups of $G$ have maximal class.

Proof. Let $H$ be a nonabelian subgroup of $G$. If $G$ is a CGZ-group, by Lemma 5.1, $H$ is a CHZ-group, then it is of maximal class, by Proposition 5.2. Conversely, let $H$ be a minimal nonabelian subgroup of $G$, by [3, Lemma 136.2(ii)], we have $|H| = p^3$. Now, if $x \in C_G(H)$ then $K = \langle x \rangle \cdot H$ is a nonabelian subgroup of
G and then it is of maximal class. But \( Z(H) \leq Z(K) \) and \( |Z(H)| = |Z(K)| = p \), so \( x \in Z(H) \). Thus \( C_G(H) \leq H \) and then \( G \) is of maximal class, by Proposition 3.4. Let \( R \) be a normal subgroup of order \( p^2 \). By \( N/C \)-theorem, we have \( |G, C_G(R)| = p \). If \( C_G(R) \) is not abelian, by hypothesis, it is of maximal class and then \( |Z(C_G(R))| = p \). But \( R \leq Z(C_G(R)) \), a contradiction. So \( C_G(R) \) is abelian and maximal in \( G \). Therefore, by Proposition 5.2, \( G \) is a \( CGZ \)-group. 

**Proposition 5.4.** [11, Proposition 2.6] Let \( G \) be a finite \( p \)-group of maximal class. If \( G \) is a \( CGZ \)-group, then \( G' \) is abelian.

**Remark 5.1.** Let \( G \) be a \( CGZ \)-group of order \( p^n \) and exponent \( p \), by Proposition 5.4, \( G' \) is abelian, then \( G_1 = C_G(G_i/G_{i+2}) \) with \( 1 \leq i \leq n - 2 \) (see [8] Theorem III.14.11).

Hence we have the following interesting corollary:

**Corollary 5.1.** Let \( G \) be a \( CGZ \)-group of order \( p^n \) and exponent \( p \), where \( p \geq 3 \). Then \( G \) possesses a unique characteristic elementary abelian subgroup of index \( p \). In this case, we have necessary \( n \leq p \).

**Proof.** Let \( K = C_G(Z_2(G)) \). By proposition 5.3 we easily see that \( K \) is an abelian subgroup of \( G \). By the above, we get \( K = G_1 \), then \( K \) is the unique characteristic elementary abelian subgroup of index \( p \). Since \( G \) has exponent \( p \), it is a regular \( p \)-group (see [11] Theorem 7.1(b)). Therefore, by a theorem of Blackburn ([8] Theorem III.14.21)) or [11] Theorem 9.5], we obtain \( |G| \leq p^p \). 

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