Polyhedrons and Perceptrons Are Functionally Equivalent

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Abstract

Mathematical definitions of polyhedrons and perceptron networks are discussed. The formalization of polyhedrons is done in a rather traditional way. For networks, previously proposed systems are developed. Perceptron networks in disjunctive normal form (DNF) and conjunctive normal forms (CNF) are introduced. The main theme is that single output perceptron neural networks and characteristic functions of polyhedrons are one and the same class of functions. A rigorous formulation and proof that three layers suffice is obtained. The various constructions and results are among several steps required for algorithms that replace incremental and statistical learning with more efficient, direct and exact geometric methods for calculation of perceptron architecture and weights.

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1 Introduction

A perceptron unit is the characteristic function of a linear half space, a perceptron layer is a product of units and a perceptron network is a composable sequence of perceptron layers.

A collection \( H_1, \ldots, H_n \) of half spaces in \( \mathbb{R}^m \) generates a finite Boolean algebra of subsets of \( \mathbb{R}^m \). A polyhedron \( K \) is any member of this algebra.

With well structured mathematical theories of polyhedrons and perceptron networks a natural proof can be given that for any polyhedron \( K \) in \( \mathbb{R}^m \) there exists an \( m \)-input, single output perceptron neural network \( P \) which is functionally equivalent to \( K \). Functionally equivalent means that the characteristic function \( \chi[K] \) is equal to the function \( F[P] \), so

\[
\forall K \exists P \text{ such that } \chi[K] = F[P]
\]

Function \( F[P] \) is the composition of the layers of \( P \), traditional and operationally called a “forward pass”; see section 9. So, from a functional viewpoint all polyhedrons are perceptrons.

If \( K \) is given, calculation of \( P \) requires writing \( K \) in disjunctive normal form (DNF) and, depending on the way \( K \) is specified, at this point intractability may arise. But if polyhedron \( K \) is already expressed in disjunctive normal form, \( K = K_{\text{DNF}}[H; \Delta] \), then calculation of the architecture, weights and transfer functions of the network \( P \) is immediate. Actually, \( P \) can be taken equal to a DNF network, \( P = P_{\text{DNF}}[H; \Delta] \), which can be trivially obtained from scheme \( \Delta \). Schemes are defined in section 10 and DNF polyhedrons in section 18.
The converse is also proved, namely, that all perceptrons are functionally equivalent to polyhedrons. Given an $m$-input, $k$-layer (arbitrary $k \geq 1$), single output perceptron neural network $P$, there exists a polyhedron $K$ in $\mathbb{R}^m$ which is functionally equivalent to $P$

$$\forall P \ \exists K \text{ such that } F[P] = \chi[K]$$

The existence of $K$ implies that —rudely stated— perceptrons cannot do more than polyhedrons. Generally speaking polyhedron $K$ is calculable from the architecture and weights of perceptron network $P$, but at computational level intractability appears. However, intractability disappears if $P$ is a DNF network.

A DNF perceptron network is a three layer network having conjunctive second layer and disjunctive third layer. See definition in section 21 below. In case network $P$ is such DNF perceptron network, $P = P_{\text{DNF}}[H; \Delta]$, there is no intractability to calculate a functionally equivalent polyhedron $K$. In fact, calculation of $K$ as a DNF polyhedron is immediate. That DNF polyhedrons and networks are freely convertible is fortunate. Similarly for their duals, the CNF polyhedrons and CNF networks; see definitions in sections 18 and 23 below.

Our discussion proves in particular that for any perceptron network $P$ there exists a functionally equivalent DNF perceptron network $P_{\text{DNF}}[H; \Delta]$, that is, such that $F[P] = F[P_{\text{DNF}}[H; \Delta]]$. This means that for perceptron networks 3 layers suffice. Again, when passing from general $P$ to more manageable $P_{\text{DNF}}[H; \Delta]$ intractability arises. When dealing with data recognition problems it is possible, and always advisable, stay within the realm of DNF polyhedrons and DNF networks. Or within its dual CNF realm.

Replacing DNF polyhedrons with CNF polyhedrons (=intersections of unions of linear half spaces) produces valid dual statements. And furthermore, all results generalize to $r$-tuples of polyhedrons and $r$-output networks, to be detailedly discussed in [2].

As mentioned, several results are hindered by intractability. It is a fact, however, that direct and efficient calculation of architecture and weights of DNF perceptron networks that perfectly recognize given data —and maintains margins, preset at will up to largest theoretically admissible values— is computationally bland.

## 2 Half spaces

A **linear form** is a non-constant function $f : \mathbb{R}^n \to \mathbb{R}$, $f(y_1, \ldots, y_n) = w_0 + w_1 y_1 + \cdots + w_n y_n$. Numerical specification of $f$ is done by the coefficients $w_i$. For $f$ to be non-constant it is necessary and sufficient that at least one of $w_1, \ldots, w_m$ be non-zero.

A **half space** is a subset $H$ of $\mathbb{R}^m$ defined by means of inequalities imposed on $f$. Given $f$ two linear inequalities will be considered, namely the lax inequality, $f \geq 0$, and the strict inequality $f > 0$, which define two corresponding half spaces. These are the **lax half space of $f$** or **closed half space of $f$**, denoted $H[f; \geq]$, and the **strict half space of $f$** or **open half space of $f$**, denoted $H[f; >]$

$$H[f; \geq] = \{ y \in \mathbb{R}^n \mid f(y) \geq 0 \} \quad H[f; >] = \{ y \in \mathbb{R}^n \mid f(y) > 0 \}$$
A half space of $f$ is either $H = H[f; \geq]$ or $H = H[f; >]$. For computational purposes half spaces are specified by the coefficients $w_i$ of the form and one of the inequality symbols $\geq$, or $>$. Taking complements interchanges lax and strict, or equivalently interchanges open and closed
\[ \mathbb{R}^m - H[f; \geq] = H[(-f); >] \quad \mathbb{R}^m - H[f; >] = H[(-f); \geq] \]
To begin providing geometric structure, half spaces will be accommodated in $n$-tuples $H = (H_1, \ldots, H_n)$
In order to specify a subcollection of $H$ it suffices to give an index set $I \subseteq \mathbb{N} = \{1, \ldots, n\}$. When considering multilayer perceptron networks, use will be made of sequences $H^{(1)}, \ldots, H^{(k)}$, with $H^{(i)}$ an $n_i$-tuple of half spaces in $\mathbb{R}^{n_i-1}$. See section 8.

### 3 Cells and cocells

A cell over $H$ is an intersection of some of the half spaces appearing in $H$, and some of their complements. To make this precise, take two subsets $I^1, I^0$ of $\mathbb{N}$, let $\Gamma = (I^1, I^0)$ and define the cell of $\Gamma$ over $H$ as the following intersection of half spaces
\[
C^\ast[H; \Gamma] = \bigcap_{i \in I^1} H_i \cap \bigcap_{i \in I^0} (\mathbb{R}^m - H_i)
\]
If $H$ and $\Gamma$ are explicitly known then $C^\ast[H; \Gamma]$ is computationally bland in the sense that establishing whether or not a numerically given $x \in \mathbb{R}^m$ belongs to the cell is a routine evaluation of linear functions, inequalities and conjunctions, without dreaded complexity issues.
Cells are always convex. They can be bounded or not, and open, closed or neither. In particular, cells are allowed to be non-compact. Half spaces of $H$, and their complementary half spaces, are cells over $H$. For appropriated $H$, the convex hull of finitely many points is a cell. By definition, $d$-dimensional simplexes are convex hulls of $d+1$ affinely independent points, hence simplexes are cells. Simplexes in $\mathbb{R}^m$ of dimension $d < m$ have empty interior, therefore cells can have empty interior. Hyperplanes can be expressed as intersections of two closed half spaces, $H[f; \geq] \cap H[-f; \geq]$. Affine subspaces $A$ are intersections of hyperplanes, hence of half spaces, thus they are cells, with empty interior except when $A = \mathbb{R}^m$.
Dually, the cocell of $\Gamma$ over $H$ is the union
\[
C^\ast[H; \Gamma] = \bigcup_{i \in I^1} H_i \cup \bigcup_{i \in I^0} (\mathbb{R}^m - H_i)
\]
4 Polyhedrons

The *polyhedral algebra generated by the half spaces of* $H$ *is denoted* $\mathcal{A}_*[H]$ *and has a standard existential definition as the smallest collection of subsets of* $\mathbb{R}^m$ *that includes the ambient space* $\mathbb{R}^m$, *and the half spaces of* $H$, *and is closed under intersections and complements.

One can also existentially define the *copolyhedral algebra generated by the half spaces of* $H$ *as the smallest collection* $\mathcal{A}^*[H]$ *containing* $\mathbb{R}^m$, *the half spaces of* $H$, *and closed under unions and complements.

As is well known, these are one and the same algebra, $\mathcal{A}_*[H] = \mathcal{A}^*[H]$, *to be simply denoted* $\mathcal{A}[H]$. *The algebra* $\mathcal{A}[H]$ *is a finite Boolean algebra of subsets of* $\mathbb{R}^m$. *Compare with Halmos [8], Chapter 1. For a more detailed discussion of Boolean algebras see Halmos and Gehring [9].

A *polyhedron over* $H$ *is by definition a member* $K$ *of the polyhedral algebra,* $K \in \mathcal{A}_*[H]$. *Also by definition, a* copolyhedron over $H$ *is any member of the copolyhedral algebra* $\mathcal{A}^*[H]$. *And since the algebras are equal the terms polyhedron and copolyhedron designate the same objects. The definitions do not require, and do not provide, an explicit description of* $K$ *in terms of the* $H_i$s. *Preference for one of the terms polyhedron/copolyhedron, may depend on intention to allude one of the disjunctive/conjunctive normal forms. See section 18 below. It often suffices to mention only polyhedrons.

5 Variety of polyhedrons

Polyhedrons are not required to be convex, nor connected, nor simply connected. Any “higher connectivity” can occur. For a more technical statement we invoke standard Algebraic Topology spellings, not to be conjured elsewhere in this paper. Let $X$ be any finite CW complex of dimension $k$. *There are finite simplicial complexes* $S$, *of same dimension* $k$, *with geometric realization* $|S|$ *which is homotopy equivalent to* $X$; *see [10], Chapter 2, Section 2C, Theorem 2C.5. Then, there is a linear embedding of* $|S|$ *into* $\mathbb{R}^m$ *with* $m = 2k + 1$; *see Spanier, [12], Chapter 3, Theorem 9. The image* $K$ *of the linear embedding is a finite union of simplexes, and because simplexes are cells,* $K$ *is a polyhedron. Hence polyhedrons* $K$ *exist with homotopy types of arbitrary finite CW complexes* $X$. *Perceptron networks* $P$ *exist with “forward pass” function equal to the characteristic function of* $K$, $F[P] = \chi[K]$; *see Corollary 1 below. Existence of networks* $P$ *with characteristic set* $K = F[P]^{-1}(1)$ *equal to a polyhedron with the homotopy type of an arbitrary finite CW complex, reflects the rich non-linear nature of multilayer perceptron neural networks. On the other hand, being unions of convex cells, polyhedrons are conceptually simple and provide fruitful geometric imagery that is the key for the practical solution of data recognition problems.
6 Perceptron units

Let $H$ be a half space. The **perceptron unit of $H$, $p[H]$**, is its characteristic function

$$p[H] = \chi[H]$$

Let $f$ be a linear form. Define then the **lax perceptron unit of $f$, denoted $p[f; \geq]$**, and the **strict perceptron unit of $f$, denoted $p[f; >]$**, as the characteristic functions of the respective half spaces

$$p[f; \geq] = \chi[H[f; \geq]] \quad p[f; >] = \chi[H[f; >]]$$

For a given $n$-tuple of half spaces $H = (H_1, \ldots, H_n)$ a **perceptron unit of $H$** is a characteristic function of a component half space $H_i$ of $H$

$$p[H_i] = \chi[H_i] : \mathbb{R}^m \to \mathbb{B} = \{1, 0\}$$

Since $H_i = \chi[H_i]^{-1}(1)$ and $\mathbb{R}^m - H_i = \chi[H_i]^{-1}(0)$, both $H_i$ and $\mathbb{R}^m - H_i$ can be specified by means of $p[H_i]$.

7 Perceptron layers

The **perceptron layer of $H$** is the product of all the perceptron units of $H$

$$p[H] = (p[H_1], \ldots, p[H_n]) : \mathbb{R}^m \to \mathbb{B}^n$$

If $I = \{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$ is an index set, the **perceptron layer of $I$ over $H$** is

$$p[H; I] = (p[H_{i_1}], \ldots, p[H_{i_t}]) : \mathbb{R}^m \to \mathbb{B}^t$$

As defined, units and layers are functions, bit valued in the first case, binary vector valued in the second. Since $\mathbb{B} \subseteq \mathbb{R}$ and $\mathbb{B}^n \subseteq \mathbb{R}^n$, the codomains of unit $p[H_i]$ and of layer $p[H]$ can be enlarged to make them real valued and real vector valued functions, $p[H_i] : \mathbb{R}^m \to \mathbb{R}$ and $p[H] : \mathbb{R}^m \to \mathbb{R}^n$ respectively.

8 Perceptron networks

Consider a sequence $f_i : X_{i-1} \to Y_i$, $i = 1, \ldots, k$, of functions between sets. The sequence is **composable** if $Y_i \subseteq X_i$, $i = 1, \ldots, k - 1$. In such case there is a well defined composition $f = f_k \circ \cdots \circ f_1 : X_0 \to Y_k$.

A **$k$-layer perceptron network** is a composable sequence $P = (p^{(1)}, \ldots, p^{(k)})$ of perceptron layers. The **diagram of $P$** is

$$\mathbb{R}^{n_0} \xrightarrow{p^{(1)}} \mathbb{R}^{n_1} \xrightarrow{p^{(2)}} \cdots \xrightarrow{\cdots} \mathbb{R}^{n_{k-1}} \xrightarrow{p^{(k)}} \mathbb{B}^{n_k}$$
The architecture of $P$ is the sequence of dimensions
\[ n_0 \cdots n_k \]
Intentionally, this definition of architecture does not keep track of which weights have zero value.

The network is over $H$ if the first layer is $p[H]$, $p^{(1)} = p[H]$. And is a single output network if $n_k = 1$. The collection of perceptron networks over $H$ will be denoted $\mathcal{P}[H]$. All $k$-layer networks, with any $k \geq 1$, are included in $\mathcal{P}[H]$.

## 9 Network functions

For $k \geq 2$ perceptron networks are not functions, but composition of their layers are functions. The function of $P$, or perceptron map, usually referred to as “forward pass of $P$”, is the composition
\[ F[P] = p^{(k)} \circ p^{(k-1)} \circ \cdots \circ p^{(1)} : \mathbb{R}^{n_0} \to \mathbb{B} \]

A terminological remark: Let $X$ be any set. The characteristic function of a subset $A \subseteq X$ is the bit valued function $\chi[A] : X \to \mathbb{B}$. And the characteristic set of a bit valued function $f : X \to \mathbb{B}$ is, by definition, $\chi[f] = f^{-1}(1)$. Note that symbol $\chi$ has duplicity: It is an operator from subsets to bit valued functions. And is also an operator from bit valued functions to subsets. Formalities would require to write something like $\chi^F : \text{Subsets} \to \text{Functions}$ and $\chi^S : \text{Functions} \to \text{Subsets}$. But there is scant margin for confusion if we simply state that $f = \chi[\chi[f]]$ and $A = \chi[\chi[A]]$.

In particular since $F[P] : \mathbb{R}^{n_0} \to \mathbb{B}$ is bit valued, it is the characteristic function of its characteristic set: $F[P] = \chi[K]$, with $K = F[P]^{-1}(1)$. We now prove that $K$ is a polyhedron.

Simplify notation letting $m = n_0, n = n_1, H = (H_1, \ldots, H_n), x_i = \chi[H_i]$ and $p^{(1)} = p[H]$. Write $F[P] = g \circ \chi[H]$ with $g = p^{(k)} \circ p^{(k-1)} \circ \cdots \circ p^{(2)}$. Then each $b = (b_1, \ldots, b_n) \in \mathbb{B}^n = \mathbb{B}^{n_1}$ has inverse image by first layer equal to
\[ \chi[H]^{-1}(b) = \bigcap_{i=1}^{n} \chi_i^{-1}(b_i) \]

But $\chi_i^{-1}(1) = H_i$ and $\chi_i^{-1}(0) = \mathbb{R}^m - H_i$. Hence $\chi[H]^{-1}(b)$ is a cell over $H$. Taking $B = g^{-1}(1)$ we have
\[ F[P]^{-1}(1) = \bigcup_{b \in B} \chi[H]^{-1}(b) \]

Thus, the characteristic set of $F[P]$ is a union of cells, hence is a polyhedron. The reasoning only required elementary notions about sets and functions. But the result is crucial for perceptron networks, hence it will be stated as
Theorem 1. The function of a single output network \( P \) over \( H \) is the characteristic function of some polyhedron over \( H \)

\[
P \in \mathbb{P}[H] \Rightarrow \exists K \in \mathbb{A}[H] \text{ such that } F[P] = \chi[K]
\]

Proof: Done.

10 Indexes

To handle DNF/CNF polyhedrons and networks over an \( n \)-tuple \( H \) of half spaces, index sets will be used. Recall that acronym DNF stands for for disjunctive normal form and CNF for conjunctive normal form.

An index set \( I \) is over \( n \) if \( I \subseteq \{1, \ldots, n\} \). When not empty, set \( I \) can always be written as \( I = \{i_1, \ldots, i_t\} \) with \( 1 \leq i_1 < \cdots < i_t \leq n \). The collection of index sets over \( n \) is the power set \( 2^n \) of \( n \), which is partially ordered under inclusion. There is also a total order on \( 2^n \), the lexicographic order, where for \( I' = \{i'_1, \ldots, i'_r\} \) we put \( I' < I \) if \( i'_j = i_j \) for \( j = 1, \ldots, k - 1 \), and \( i'_k < i_k \). In consequence for any collection of index sets, \( I = \{I_1, \ldots, I_q\} \), it can be assumed that \( I_1 < \cdots < I_q \).

An index pair over \( n \) is a pair \( \Gamma = (I_1, I_0) \) with components that are index sets over \( n \). The index pair is consistent if \( I_1 \cap I_0 = \emptyset \). As explained above, index pairs will allow to specify cells over \( H \). The collection of index pairs is \( 2^n \times 2^n \). Here, with \( \Gamma' = (I'^1, I'^0) \), the lexicographic order is \( \Gamma' \prec \Gamma \) whenever \( I'^1 < I^1 \), or if \( I'^1 = I^1 \) and \( I'^0 < I^0 \).

An index pair collection of multiplicity \( q \) over \( n \) of \( \Gamma \) is a set \( \Gamma \) with \( q \) elements such that each element \( i \) an index pair over \( n \)

\[
\Gamma = \{\Gamma_1, \ldots, \Gamma_q\} = \{(I^1_1, I^0_1), \ldots, (I^1_q, I^0_q)\}
\]

It can be assumed that \( \Gamma_1 < \cdots < \Gamma_q \). The collection of all such \( \Gamma \) constitute a set denoted \( G^q[n] \),

\[
G^q[n] = (2^n \times 2^n) \times \cdots \times (2^n \times 2^n)
\]

A scheme over \( n \) is a pair \( \Delta = (\Gamma, J) \) with \( \Gamma \in G^q[n] \) and \( J \subseteq \{1, \ldots, q\} \). Schemes \( \Delta \) will be used to specify DNF and CNF polyhedrons over \( H \). They will also define DNF and CNF networks.

11 Adders

For \( I = \{i_1, \ldots, i_t\} \subseteq n \) define the adder of \( I \) as the linear form \( \text{Ad}[I] : \mathbb{R}^n \to \mathbb{R} \) given by

\[
\text{Ad}[I](y) = \sum_{i \in I} y_i = y_{i_1} + \cdots + y_{i_t}
\]
For binary vectors $b = (b_1, \ldots, b_n) \in \mathbb{B}^n$ we have

\[
\begin{align*}
\text{Ad}[I](b) &= 0 \iff \forall i \in I \ b_i = 0 \\
\text{Ad}[I](b) &= |I| \iff \forall i \in I \ b_i = 1 \\
0 < \text{Ad}[I](b) < |I| &\iff \text{otherwise}
\end{align*}
\]

Here $|I|$ is the number of elements of $I$.

### 12 Conjunctive forms

Let $I \subseteq \{1, \ldots, n\}$. The original conjunctive linear form of $I$ is the linear form $C_n[I] : \mathbb{R}^n \to \mathbb{R}$ defined as

\[
C_n[I] = \text{Ad}[I] - |I| + \frac{1}{2}
\]

On binary vectors $b \in \mathbb{B}^n$ the values of $C_n[I]$ are half integers, are never a whole integer, and by inspection we conclude that

\[
\begin{align*}
C_n[I](b) &= \frac{1}{2} \iff \forall i \in I \ b_i = 1 \\
C_n[I](b) &\leq -\frac{1}{2} \iff \exists i \in I \ b_i = 0
\end{align*}
\]

The complementary conjunctive linear form of $I$ is

\[
\overline{C_n[I]} = -\text{Ad}[I] + \frac{1}{2}
\]

Hence for binary vectors

\[
\begin{align*}
\overline{C_n[I]}(b) &= \frac{1}{2} \iff \forall i \in I \ b_i = 0 \\
\overline{C_n[I]}(b) &\leq -\frac{1}{2} \iff \exists i \in I \ b_i = 1
\end{align*}
\]

Consider a consistent pair $\Gamma = (I^1, I^0)$ of index sets over $n$. The conjunctive linear form of $\Gamma$ is

\[
C_n[\Gamma] = \text{Ad}[I^1] - |I^1| + \frac{1}{2} - \text{Ad}[I^0]
\]

Note that, for $I \subseteq \mathbb{n}$, $C_n[I] = C_n[(I, \emptyset)]$ and $\overline{C_n[I]} = C_n[(\emptyset, I)]$.

On binary vectors $b \in \mathbb{B}^n$ form $C_n[\Gamma]$ satisfies

\[
\begin{align*}
C_n[\Gamma](b) &= \frac{1}{2} \iff \forall i \in I^1 \ b_i = 1 \text{ and } \forall i \in I^0 \ b_i = 0 \\
C_n[\Gamma](b) &\leq -\frac{1}{2} \iff \exists i \in I^1 \ b_i = 0 \ \text{ or } \exists i \in I^0 \ b_i = 1
\end{align*}
\]
13 Disjunctive forms

Define the original disjunctive linear form of $I$ as the function $D_s[I] : \mathbb{R}^n \to \mathbb{R}$ given by

$$D_s[I] = A_d[ I ] - \frac{1}{2}$$

which for binary vectors $b \in \mathbb{B}^n$ has values

$$D_s[I](b) \geq \frac{1}{2} \Leftrightarrow \exists i \in I \ b_i = 1$$
$$D_s[I](b) = -\frac{1}{2} \Leftrightarrow \forall i \in I \ b_i = 0$$

Define also the complementary disjunctive linear form of $I$ by the expression

$$D_{s}[\overline{I}] = -A_d[ I ] + |I| - \frac{1}{2}$$

 Evaluated on binary vectors this gives

$$D_{s}[\overline{I}](b) = \frac{1}{2} \Leftrightarrow \forall i \in I^1 \ b_i = 1$$
$$D_{s}[\overline{I}](b) \leq -\frac{1}{2} \Leftrightarrow \exists i \in I \ b_i = 0$$

Consider again a consistent pair $\Gamma = (I^1, I^0)$ of index sets over $\mathbb{B}$. The disjunctive linear form of $\Gamma$ is $D_s[\Gamma] : \mathbb{R}^n \to \mathbb{R}$ given by

$$D_s[\Gamma] = A_d[ I^1 ] - A_d[ I^0 ] + |I^0| - \frac{1}{2}$$

Note that, whenever $I \subseteq \mathbb{B}$, $D_s[I] = D_s[(I, \emptyset)]$ and $D_s[I] = D_s[(\emptyset, I)]$. Values on binary vectors $b \in \mathbb{B}^n$ are given by

$$D_s[\Gamma](b) = -\frac{1}{2} \Leftrightarrow \forall i \in I^1 \ b_i = 0 \text{ and } \forall i \in I^0 \ b_i = 1$$
$$D_s[\Gamma](b) \geq \frac{1}{2} \Leftrightarrow \exists i \in I^1 \ b_i = 1 \text{ or } \exists i \in I^0 \ b_i = 0$$

14 Conjunctive units

Perceptron units of linear forms $f$ were defined in section 6. For conjunctive linear forms, $f = \text{Cn}[\Gamma]$, we obtain the conjunctive lax perceptron unit of $\Gamma$ and the conjunctive strict perceptron unit of $\Gamma$ defined as

$$p[\text{Cn}[\Gamma]; \geq] \quad p[\text{Cn}[\Gamma]; >]$$
These are bit valued functions defined on $\mathbb{R}^n$. In the case of conjunctive and disjunctive forms, both lax and strict units will serve our goals equally well. For the sake of definiteness “conjunctive perceptron unit” will refer to the lax unit, to be simply denoted $\Gamma^\cap$, thus $\Gamma^\cap = p[Cn[\Gamma]; \geq]$.

For binary vectors we then have

$$
\begin{align*}
\Gamma^\cap(b) = 1 & \iff \forall i \in I^1 b_i = 1 \text{ and } \forall i \in I^0 b_i = 0 \\
\Gamma^\cap(b) = 0 & \iff \exists i \in I^1 b_i = 0 \text{ or } \exists i \in I^0 b_i = 1
\end{align*}
$$

Therefore the characteristic function of cell $C_*[\mathbf{H}; \Gamma]$ is

$$
\chi[C_*[\mathbf{H}; \Gamma]] = \Gamma^\cap \circ p[\mathbf{H}]
$$

### 15 Conjunctive layers

Let $\Gamma = \{\Gamma_1, \ldots, \Gamma_q\}$ be a collection of index set pairs over $n$, $\Gamma_j = (I^1_j, I^0_j)$. The **conjunctive perceptron layer of $\Gamma$**, denoted $\Gamma^\cap : \mathbb{R}^n \rightarrow \mathbb{B}^q$, is the product of the conjunctive units of its member index set pairs

$$
\Gamma^\cap = (\Gamma^\cap_1, \ldots, \Gamma^\cap_q)
$$

The value of $\Gamma^\cap$ on a binary vector $b \in \mathbb{B}^n$ has bit components $\Gamma^\cap_1(b), \ldots, \Gamma^\cap_q(b)$. In consequence the $q$-tuple of characteristic functions of the cells is the composition of the perceptron layer of $\mathbf{H}$ with the conjunctive perceptron layer of $\Gamma$

$$
(\chi[C_*[\mathbf{H}; \Gamma_1]], \ldots, \chi[C_*[\mathbf{H}; \Gamma_q]]) = \Gamma^\cap \circ p[\mathbf{H}]
$$

### 16 Disjunctive units

The disjunctive linear form $f = Ds[\Gamma] : \mathbb{R}^n \rightarrow \mathbb{R}$ defines a **disjunctive lax perceptron unit of $\Gamma$** and a **disjunctive strict perceptron unit of $\Gamma$**

$$
\begin{align*}
p[\mathbf{H} \geq] : \mathbb{R}^n & \rightarrow \mathbb{B} \\
p[\mathbf{H} >] : \mathbb{R}^n & \rightarrow \mathbb{B}
\end{align*}
$$

Again for definiteness, $\Gamma^\cup$ will denote a lax unit, $\Gamma^\cup = p[\mathbf{H} \geq]$.

The values of $\Gamma^\cup$ on binary vectors are

$$
\begin{align*}
\Gamma^\cup(b) = 1 & \iff \exists i \in I^1 b_i = 1 \text{ or } \exists i \in I^0 b_i = 0 \\
\Gamma^\cup(b) = 0 & \iff \forall i \in I^1 b_i = 0 \text{ and } \forall i \in I^0 b_i = 1
\end{align*}
$$

Hence the characteristic function of the cocell $C^*[\mathbf{H}; \Gamma]$ is

$$
\chi[C^*[\mathbf{H}; \Gamma]] = \Gamma^\cup \circ p[\mathbf{H}]
$$
17 Disjunctive layers

As before, let $\Gamma = \{\Gamma_1, \ldots, \Gamma_q\}$ be a $q$-tuple of index set pairs over $n$. The \textit{disjunctive perceptron layer of} $\Gamma$, denoted $\Gamma^\cup : \mathbb{R}^n \rightarrow \mathbb{B}^q$, is the product of respective disjunctive units

$$\Gamma^\cup = (\Gamma_1^\cup, \ldots, \Gamma_q^\cup)$$

Layer $\Gamma^\cup$ evaluated on a binary vector $b \in \mathbb{B}^n$ is a binary vector with components $\Gamma_1^\cup(b), \ldots, \Gamma_q^\cup(b)$. Dually to the case of cells, the $q$-tuple of characteristic functions of the cocells is equal to the composition of the layer of $H$ with the disjunctive perceptron layer of $\Gamma$

$$(\chi[H; \Gamma_1], \ldots, \chi[H; \Gamma_q]) = \Gamma^\cup \circ p[H]$$

18 DNF and CNF polyhedrons

Let $\Delta = (\Gamma, J)$ be a scheme over $n$. By definition the \textit{DNF polyhedron of} $\Delta$ \textit{over} $H$ is the union of the cells specified by index pairs $\Gamma_j$ with $j \in J$

$$K_{\text{DNF}}[H; \Delta] = \bigcup_{j \in J} C_*[H; \Gamma_j]$$

Polyhedron $K_{\text{DNF}}[H; \Delta]$ is specified by a of collection $H$ of half spaces; by a list $\Gamma = \{\Gamma_1, \ldots, \Gamma_q\}$ of index pairs over $n$ that define cells over $H$; and by an index set $J$ over $q$ that tells which of the cells to include in the union. So defined, DNF polyhedrons are polyhedrons, $K_{\text{DNF}}[H; \Delta] \in \mathcal{A}[H]$, endowed with an explicit description.

Define the \textit{CNF polyhedron of} $\Delta$ \textit{over} $H$ as

$$K_{\text{CNF}}[H; \Delta] = \bigcap_{j \in J} C^*[H; \Gamma_j]$$

The specification of $K_{\text{CNF}}[H; \Delta]$ consists of the collection $H$ of half spaces; of a list $\Gamma = \{\Gamma_1, \ldots, \Gamma_q\}$ of index pairs over $n$ with each pair defining a cocell; and of a collection of cocells to be intersected, indicated by the index set $J$ over $q$. This notion is dual of DNF polyhedron. Note that “CNF copolyhedron” could have been used as a consistent name for what has been called CNF polyhedron.

When a DNF polyhedron is given, some half spaces of $H$ could eventually be “mute” in the sense that they will appear in none of the cells. And some cells may also turn out be mute since they may be left out of the polyhedron. This seems wasteful. But note that different cells are made from different half spaces, and different polyhedrons are made from different cells. Thus, when considering a polyhedron, what for one cell is a mute half space may be needed for another cell. If several polyhedrons are simultaneously discussed, what are mute cells for one of these may be needed for the others. On the other hand and for efficiency, half spaces and cells with participation in more than one object need only appear once. These comments also apply to the CNF case.
19 DNF and CNF polyhedral algebras

Consider $K \in \mathcal{A}[H]$. A DNF presentation of $K$ is a scheme $\Delta$ such that $K = K_{\text{DNF}}[H;\Delta]$. The DNF polyhedral algebra of $H$, denoted $\mathcal{A}_{\text{DNF}}[H]$, is the class of polyhedrons that have some DNF presentation.

A CNF presentation of $K$ is a scheme $\Delta$ such that $K = K_{\text{CNF}}[H;\Delta]$. The DNF polyhedral algebra of $H$, denoted $\mathcal{A}_{\text{DNF}}[H]$, is the class of polyhedrons that have some DNF presentation.

That $\mathcal{A}_{\text{DNF}}[H]$ and $\mathcal{A}_{\text{CNF}}[H]$ (by definition subsets of $\mathcal{A}[H]$) are in fact Boolean algebras requires proof. Schemes for unions, intersections and complements have to be calculated in terms of initially given schemes. We now state formally

**Proposition 1.** $\mathcal{A}_{\text{DNF}}[H]$ and $\mathcal{A}_{\text{CNF}}[H]$ are Boolean Algebras

**Proof:** Elementary. For details see [1].

20 Equality of algebras

The algebra $\mathcal{A}[H]$ of subsets of $\mathbb{R}^m$ was defined as the Boolean algebra generated by the half spaces of $H$.

Let $\Gamma = (\{i\}, \emptyset)$, $\Gamma_1 = \Gamma$, $\Gamma = \{\Gamma_1\}$ and $J = \{1\}$, then scheme $\Delta = (\Gamma, J)$ is a DNF presentation over $H$ of the half space $H_i$, that is, $H_i = K_{\text{DNF}}[H;\Delta]$. Therefore the half space $H_i$ belongs to the DNF polyhedral algebra of $H$, $H_i \in \mathcal{A}_{\text{DNF}}[H]$. Also, the same scheme is a CNF presentation over $H$ of $H_i$, $H_i = K_{\text{CNF}}[H;\Delta]$, and $H_i \in \mathcal{A}_{\text{CNF}}[H]$.

Note that taking $\Gamma = (\emptyset, \{i\})$, we similarly obtain a scheme $\overline{\Delta}$ for the complementary half spaces, $\mathbb{R}^m - H_i = K_{\text{DNF}}[H;\overline{\Delta}]$ and $\mathbb{R}^m - H_i = K_{\text{CNF}}[H;\overline{\Delta}]$.

Proposition 1 implies now that $\mathcal{A}_{\text{DNF}}[H] = \mathcal{A}[H]$ and $\mathcal{A}_{\text{CNF}}[H] = \mathcal{A}[H]$. Therefore

**Theorem 2.** The three Boolean polyhedral algebras are equal

$\mathcal{A}_{\text{DNF}}[H] = \mathcal{A}[H] = \mathcal{A}_{\text{CNF}}[H]$

**Proof:** Done.

21 DNF perceptron networks

Let $\Delta = (\Gamma, J)$ be a scheme over $n$. Define the DNF perceptron network of $\Delta$ over $H$ as the three layer, single output perceptron network, denoted $P_{\text{DNF}}[H;\Delta]$, which has first layer $p[H]$, second layer $\Gamma^\cap$ and third layer $J^\cup$, so that $P_{\text{DNF}}[H;\Delta] = (p[H], \Gamma^\cap, J^\cup)$. This network has diagram

$\mathbb{R}^m \xrightarrow{p[H]} \mathbb{R}^n \xrightarrow{\Gamma^\cap} \mathbb{R}^q \xrightarrow{J^\cup} \mathbb{B}$

Here $\Gamma^\cap$ is the conjunctive layer of $\Gamma$ defined in section 15, and layer $J^\cup$ is the disjunctive unit of $J$ described in section 16.
22 DNF network function

Consider a DNF network $P_{\text{DNF}}[H; \Delta]$ and its network function

$$F[P_{\text{DNF}}[H; \Delta]] = J^\cup \circ \Gamma^\cap \circ p[H]$$

From section 15 we know that composition of the first two layers is the product of the characteristic functions of the cells. And from section 16 we conclude that further composition with the disjunctive unit of $J$ gives the characteristic function of the DNF polyhedron

$$J^\cup \circ \Gamma^\cap \circ p[H] = \lambda[K_{\text{DNF}}[H; \Delta]] : \mathbb{R}^m \rightarrow \mathbb{B}$$

This proves, for any $n$-tuple $H$ of half spaces and for any scheme $\Delta$ over $n$, the following

**Theorem 3.** The DNF polyhedron and the DNF perceptron network of scheme $\Delta$ over $H$ are functionally equivalent

$$F[P_{\text{DNF}}[H; \Delta]] = \lambda[K_{\text{DNF}}[H; \Delta]]$$

**Proof:** Done.

Let $K \in A[H]$ be an arbitrary polyhedron over $H$. Theorem 2 proves that some scheme $\Delta$ exists such that $K = K_{\text{DNF}}[H; \Delta]$. Theorem 3 gives $\lambda[K] = \lambda[K_{\text{DNF}}[H; \Delta]] = P_{\text{DNF}}[H; \Delta]$ and we reach

**Corollary 1.** For any polyhedron $K \in A[H]$ there exists a functionally equivalent DNF perceptron network $P_{\text{DNF}}[H; \Delta]$

$$\lambda[K] = F[P_{\text{DNF}}[H; \Delta]]$$

**Proof:** Done.

23 CNF perceptron networks

Dually to section 21, the **CNF perceptron network of $\Delta$ over $H$** is defined as the three layer, single output perceptron network $P_{\text{CNF}}[H; \Delta]$ with first layer $p[H]$, second layer $\Gamma^\cup$, and third layer equal to $J^\cap$, respectively defined in sections 7, 17 and 14. In symbols, $P_{\text{CNF}}[H; \Delta] = (p[H]; \Gamma^\cup, J^\cap)$. The diagram of this perceptron network is

$$\mathbb{R}^m \xrightarrow{p[H]} \mathbb{R}^n \xrightarrow{\Gamma^\cup} \mathbb{R}^q \xrightarrow{J^\cap} \mathbb{B}$$

24 CNF network function

Let $P_{\text{CNF}}[H; \Delta]$ be a CNF network. Its function is

$$F[P_{\text{CNF}}[H; \Delta]] = J^\cap \circ \Gamma^\cup \circ p[H]$$
According to section 17, composition of the first two layers is the product of the characteristic functions of the cocells. Section 14 implies then that composition with the third layer is equal to the characteristic function of the CNF polyhedron

\[ J^c \circ \Gamma^j \circ p[H] = \chi(K_{\text{CNF}}[H; \Delta]) : \mathbb{R}^m \rightarrow \mathbb{B} \]

Thus, the following dual of Theorem 3 has been proved

**Theorem 4.** The CNF polyhedron and the CNF perceptron network of scheme \( \Delta \) over \( H \) are functionally equivalent.

\[ F[P_{\text{CNF}}[H; \Delta]] = \chi(K_{\text{CNF}}[H; \Delta]) \]

**Proof:** Done.

Theorem 2 implies that for any polyhedron \( K \in \mathcal{A}[H] \) there exists CNF presentation \( \Delta \) of \( K \), \( K = K_{\text{DNF}}[H; \Delta] \). The dual of Corollary 1 is

**Corollary 2.** For any polyhedron \( K \in \mathcal{A}[H] \) there exists a functionally equivalent CNF perceptron network \( P_{\text{CNF}}[H; \Delta] \)

\[ \chi(K) = F[P_{\text{CNF}}[H; \Delta]] \]

**Proof:** Done.

### 25 DNF and CNF functional equivalence

Let \( P \) be any \( m \)-input, \( k \)-layer, single output perceptron network with first layer \( H \). Theorems 2, 3 and 4 allow to conclude the following

**Theorem 5.** There are schemes \( \Delta_* \) and \( \Delta^* \) such that

\[ F[P] = F[P_{\text{DNF}}[H; \Delta_*]] = \chi(K_{\text{DNF}}[H; \Delta_*]) = F[P_{\text{CNF}}[H; \Delta^*]] = \chi(K_{\text{CNF}}[H; \Delta^*]) \]

**Proof:** Done.

### 26 Three layers suffice

Because DNF—as well as CNF—perceptron networks have three layers, taking \( P^{(3)} = P_{\text{DNF}}[H; \Delta_*] \) as immediate consequence of the previous theorem we obtain

**Corollary 3.** For any \( m \)-input, \( k \)-layer, single output perceptron network \( P \) with first layer \( H \) there exists a functionally equivalent 3-layer network \( P^{(3)} \) over \( H \)

\[ F[P] = F[P^{(3)}] \]

The interpretation is that “for perceptrons three layers suffice”, in the precise sense that any function from \( \mathbb{R}^m \) to \( \mathbb{B} \) realizable by a \( k \)-layer perceptron network, can also be realized by a network having three layers, and such that for both networks the first layer is the same. See Crespin [6].
27 Conclusions

Along the paper polyhedrons and perceptron neural networks have been compared. The context has been one of formal definitions, propositions, theorems and proofs, all within contemporary standards of mathematical rigor. Results were very basic and are natural consequences of definitions. The viewpoint may contribute to establish foundations for a mathematical theory of perceptron neural networks. It is now clear that polyhedrons and perceptron neural networks are functionally the same, \( P = \text{PNN} \). So what?

Perceptron networks are often used for pattern recognition. We prefer to talk about data recognition. Data are finite subsets of \( \mathbb{R}^m \). If data sets are given —non-empty and mutually disjoint— DNF polyhedrons can be calculated that are adapted to the data, including specification of margins, or distances to “decision boundaries”. Geometry makes possible exquisite adjustments of polyhedrons to data. Conversion of DNF polyhedrons to DNF perceptron networks is immediate, resulting in networks that perfectly recognize the data. Such DNF networks have controllable, ample and flexible generalization capabilities, up to maximum theoretical limits. The methodology has already been software tested. It is considerably simpler and more efficient than backpropagation or support vector machines. The DNF polyhedrons are easy to calculate, and amenable to rule extraction. How to pass from data to polyhedrons will be explained in forthcoming papers.

Direct calculation of DNF polyhedrons provides DNF perceptron networks and competes against learning paradigms. Backpropagation or other types of incremental learning are bypassed. DNF perceptron networks are geometrically gestated and, as in some myths, born with knowledge. The gestation process is brief and efficient. If it is the case that streams of new data keep coming, permanent online gestation would keep the network updated.

That polyhedrons and perceptrons are equivalent is a recurrent theme in neural network literature. The earliest reference known to the present author is the 1987 article \([11]\) of Lippmann, but older papers may exist. Our own line of development has been circulating in \([4]-[7]\), which papers are available at

http://www.matematica.ciens.ucv.ve/dcrespin/Pub/

and also from

http://ucv.academia.edu/DanielCrespin

Oteyeva, Caracas
Tuesday, November 05, 2013.

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