Chaotic exit to inflation: the dynamics of pre-inflationary universes

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We show that anisotropic Bianchi type-IX models, with matter and cosmological constant have chaotic dynamics, connected to the presence of a saddle-center in phase space. The topology of cylinders emanating from unstable periodic orbits about the saddle-center provides an invariant characterization of chaos in the models. The model can be thought to describe the early stages of inflation, the way out to inflation being chaotic.

Some important features in inflationary dynamics arise whenever anisotropy is present even in the form of small perturbations. This point has not been emphasized yet in the literature of inflation. The conjunction of the cosmological constant, anisotropy and matter fields implies the existence of a saddle-center $E$ in phase space. As consequence we have a complex dynamics, based on structures as homoclinic cylinders, which emanate from unstable periodic orbits in the neighborhood of $E$. These cylinders will cross each other transversally producing a chaotic dynamics, analogous to the breaking of homoclinic curves in Poincaré homoclinic phenomena. This structure constitutes an invariant characterization of chaos.

We consider anisotropic Bianchi IX cosmological models characterized by two scale factors $A(t)$ and $B(t)$, with dust plus a cosmological constant $\Lambda$. The dynamics of the models is governed by the Hamiltonian

$$
H(A, B, P_A, P_B) = \frac{P_A P_B}{4 B} - \frac{A P_B^2}{8 B^2} + 2 A - \frac{A^3}{2 B^2} - 2 \Lambda A B^2 - E_0 = 0,
$$

(1)

where $P_A$ and $P_B$ are the momenta canonically conjugated to $A$ and $B$, respectively, and $E_0$ is a constant proportional to the total energy of the models. The associated dynamical system has one saddle-center $E$ in the finite region of the phase space with associated energy $E_0 = E_{crit} = \frac{1}{\sqrt{4 \Lambda}}$ and eigenvalues $\lambda_{1,2} = \pm \frac{1}{2 E_{crit}}$, $\lambda_{3,4} = \pm 2 i E_{crit}$. The phase space has two critical points at infinity, corresponding to the de Sitter solution one of them acting as an attractor. The dynamical system generated by (1) admits the invariant manifold $\mathcal{M}$ ($A = B$, $P_A = \frac{P_B}{2}$). Its phase portrait is depicted in Fig. 2, where the curves represent isotropic universes. The critical point $E$ is contained in $\mathcal{M}$, and the separatrices $S$ constitute boundaries between isotropic models that collapse and escape to the de Sitter configuration.

Due to its saddle-center character, the motion in a small neighborhood of the critical point $E$ is separable into hyperbolic and rotational motions, with respective
energies approximately conserved. The rotational motion corresponds to periodic orbits in a linear neighborhood of $E$ (cf. Fig 1(a)). For zero energy of the hyperbolic motion, we have the linear stable $V_s$ and unstable $V_u$ one-dimensional manifolds of Fig. 1(b). The separatrices $S$ in the invariant manifold $\mathcal{M}$ are actually the non-linear extension of $V_s$ and $V_u$. The direct product of a periodic orbit with $V_s$ and $V_u$ generates in the linear neighborhood of $E$ the structure of stable and unstable cylinders, which coalesce into the periodic orbit for large positive and negative times, respectively (Fig. 1(c)). A general orbit which visits the neighborhood of $E$ has an oscillatory approach to the cylinders, the closer as the energy of the hyperbolic motion goes to zero. The outcome of this oscillatory regime will be to collapse or to escape to the de Sitter attractor depending on the energy of the hyperbolic motion.

Away from a linear neighborhood of $E$, the linear approximation is no longer valid. Higher order terms become important for the dynamics, and the non-integrability of the system results in the distortion and twisting of the cylinders. The stable cylinder and the unstable one will cross each other transversally, producing chaotic sets in the phase space; orbits with initial conditions taken on these chaotic sets will be highly sensitive to small perturbations on the initial conditions.

The phase space under consideration is not compact, and we will actually identify chaotic behaviour associated to the possible asymptotic ($t \to \infty$) outcomes of orbits, namely, escape to the de Sitter attractor at infinity or collapse after a burst of initial expansion. We assume $\Lambda = 0.25$, so that the critical point $E$ is characterized by $A = B = 1.0$, $P_A = 0$ and $E_{\text{crit}} = 1.0$. All calculations were made using the package Poincaré. Let $S_0$ be a point belonging to the separatrix ($E_0 = 1.0$), with coordinates $A = B = 0.4$, $P_B = 2 P_A = 1.3576450198$. Around this point, we construct a 4-dim sphere in the phase space with arbitrary small radius $R_0$, for instance $R_0 = 10^{-5}$. The values of $(A, B, P_A, P_B)$ are taken in energy surfaces which have a non-empty intersection with this sphere. Such energy surfaces are those for which the range of energy, $E_0$, about $E_0 = E_{\text{crit}} = 1.0$ is of the order of, or smaller than the radius $R_0$. Physically we are considering expanding cosmological models with small anisotropic perturbations in a pre-inflationary phase.

After several experiments with the above set of initial conditions, we note that, as expected, two possible outcomes arise: collapse, or expansion into the de Sitter configuration. The main result of this paper is to show that, for a determined interval of energy $\delta E^\ast$, this outcome is chaotic. Indeed, according to our numerical work, we find out that, for each sphere of initial conditions, there always exists a non-null interval $\delta E^\ast$ for which orbits either collapse or escape, in an indeterminate outcome. In Fig. 3 this behaviour is showed for spheres of initial conditions with radius $R_0 = 10^{-5}$. An empirical relation between the gap $\delta E^\ast$ and the radius $R_0$ is obtained, $\delta E^\ast \propto R_0^2$. In other words, any infinitesimal fluctuation of a given initial condition in the sphere (for energies in the interval $\delta E^\ast$ can lead to an indeterminate outcome, that is, collapse or escape. This is evidence of chaos, as a consequence of the crossing of stable and unstable cylinders, emanating from unstable periodic orbits about $E$. This topological structure is actually an invariant characterization of chaos. Finally we remark that the above behaviour is not restricted to initial conditions taken in small neighborhoods of points of the separatrix. Generally, any sets of initial conditions taken in an arbitrary neighborhood of the invariant
manifold $\mathcal{M}$, which result in orbits that visit a small neighborhood of $E$, display the above chaotic behaviour.

Figure 1: (a) Periodic orbits of the Hamiltonian in the linear approximation. (b) The linear unstable $V_u$ and stable $V_s$ one-dimensional manifolds. (c) Stable and unstable cylinders manifolds emanating from the periodic orbit $\tau$.

Figure 2: Integral curves on the invariant manifold $A = B$, $P_A = P_B/2$ for $\gamma = 1$ (dust).

Figure 3: Chaotic exit to inflation: outcome of 50 orbits chosen in a sphere of $R = 10^{-5}$ about a point $S_0$ on the separatrix.

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