ON THE DIMENSIONS OF OSCILLATOR-LIKE ALGEBRAS
INDUCED BY ORTHOGONAL POLYNOMIALS:
NON-SYMMETRIC CASE

G. HONNOUVO¹ AND K. THIRULOGASANTHAR²

Abstract. There is a generalized oscillator-like algebra associated with every
class of orthogonal polynomials \{Ψₙ(x)\}_{n=0}^{\infty} on the real line, satisfying a four
term non-symmetric recurrence relation \(xΨₙ(x) = bₙΨₙ₊₁(x) + aₙΨₙ(x) + bₙ₋₁Ψₙ₋₁(x), \Psi₀(x) = 1, \ b₋₁ = 0\). This note presents necessary and suffi-
cient conditions on \(aₙ\) and \(bₙ\) for such algebras to be of finite dimension. As
examples, we discuss the dimensions of oscillator-like algebras associated with
Laguerre and Jacobi polynomials.

1. Introduction

The usual harmonic oscillator annihilation, creation and the number operators
are defined respectively as
\[
(1.1) \quad aΨₙ = \sqrt{n}Ψₙ₋₁; n ≥ 1, \quad a†Ψₙ = \sqrt{n + 1}Ψₙ₊₁; n ≥ 0, \quad N = a†a
\]
and \(aΨ₀ = 0\), where \(\{Ψₙ\}_{n=0}^{\infty}\) is an orthonormal basis of the harmonic oscillator
Fock space. In this case
\[
[a, a†] = I, \quad [N, a] = -a, \quad [N, a†] = a, \quad (a†)† = a, \quad N† = N
\]
and the algebra generated by \(\{I, a, a†, N\}\) is the usual Weyl-Heisenberg algebra.
We call this algebra \(A_{WH}\). It is well-known that the dimension of this algebra is
four.

Several generalizations and deformations of the algebra \(A_{WH}\) have been studied
in the literature, for example, \([1, 2, 3, 4, 6, 9, 10, 12]\). In generalizing or deforming
the algebra \(A_{WH}\) we inclined to stay as close as to the commutation relations of
the algebra \(A_{WH}\). In the following we shall provide conditions, in terms of the
coefficients of some recurrence relations satisfied by the Fock basis of generalized
oscillator-like algebras, for such algebras to be of the same dimension as the algebra
\(A_{WH}\).

In particular, in the following we shall discuss the dimensions of generalized
oscillator algebras presented in \([4]\) and the dimensions of a modified version of an
oscillator-like algebra presented in \([4]\).

In a recent paper, \([14]\), we have considered the dimension of generalized oscil-
lator algebras associated with orthogonal polynomials, on the real line, that are
orthogonal with respect to a symmetric probability measure.

Date: September 7, 2015.
1991 Mathematics Subject Classification. Primary 33C45, 33C80, 33D80.
Key words and phrases. Orthogonal polynomials, oscillator-like algebras, deformed oscillator
algebras.
Let $\mathcal{H} = L^2(\mathbb{R}, d\mu)$, where $\mu$ is a probability measure on $\mathbb{R}$ with finite moments
\begin{equation}
\mu_n = \int_{-\infty}^{\infty} x^n d\mu(x).
\end{equation}
These moments uniquely define the real sequences $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$ and the system of orthogonal polynomials $\{\Psi_n(x)\}_{n=0}^\infty$ satisfying the recurrence relation $\Psi_n(x)$
\begin{equation}
x \Psi_n(x) = b_n \Psi_{n+1}(x) + a_n \Psi_n(x) + b_{n-1} \Psi_{n-1}(x), \quad \Psi_0(x) = 1, \quad b_{-1} = 0; \quad n = 0, 1, 2, \cdots.
\end{equation}
The polynomials (normalized) $\{\Psi_n(x)\}_{n=0}^\infty$ form an orthonormal basis for a Fock space associated with a generalized oscillator algebra provided that $b_n$’s and $\mu_n$’s are connected by a specific relation $3$. There are two cases associated with (1.3) $\Psi_n(x)$:
(i) $a_n = 0$, symmetric case
(ii) $a_n \neq 0$, non-symmetric case
The primary aim of this article is to investigate the dimension of an oscillator-like algebra obeying the recurrence relation (1.3). We shall provide necessary and sufficient conditions, in terms of $a_n$ and $b_n$ of (1.3), for such an oscillator-like algebra to be of finite dimension. This result, in a manner, can be viewed as a dimension wise classification for such algebras.
The rest of the article is organized as follows. In subsection 2.1 we briefly discuss the symmetric case. In particular we shall respond to the claims made in [7, 8] about our earlier paper [14]. In section 2.2 we discuss the non-symmetric case and also comment on the results provided in [7, 8] about the oscillator algebra associated with the non-symmetric case. Subsection 2.3 deals with oscillator-like algebras obeying the recurrence relation (1.3). In section 3 we present the main results of this manuscript. That is, we present a necessary and sufficient condition on $a_n$ and $b_n$ of (1.3) for oscillator-like algebras to be of finite dimension. Some examples accommodating our claim are presented in section 4. Section 5 ends the manuscript with a conclusion.

2. Classes of generalized oscillator and oscillator-like algebras

In this section we shall provide a class of generalized oscillator and oscillator-like algebras based on [4] [12]. In particular we shall respond to the claims made in [11, 8] about our earlier paper [14].

2.1. Symmetric case. Let $\mu$ be a symmetric probability measure on the real line, $\mathbb{R}$. That is, the measure $\mu$ satisfies
\begin{equation}
\int_{-\infty}^{\infty} \mu(dx) = 1, \quad \text{and} \quad \mu_{2k+1} = \int_{-\infty}^{\infty} x^{2k+1} \mu(dx) = 0; \quad k = 0, 1, \cdots
\end{equation}
Let
\begin{equation}
\{b_n\}_{n=0}^\infty, \quad b_n > 0; \quad n = 0, 1, \cdots
\end{equation}
be a positive sequence defined by the algebraic equations system
\begin{equation}
\sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{s=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{m+s} a_{2m-1,n-1} a_{2s-1,n-1} \frac{\mu_{2m-2m-2s+2}}{(b_n^{2s-1})!} = b_{n-1}^2 + b_n^2, \quad n = 0, 1, 2, \cdots.
\end{equation}
where \((b^2_{n-1})! = b^2_0 b^2_1 ... b^2_{n-1}\), the integral part of \(a\) is denoted by \([a]\), and the coefficients \(\alpha_{ij}\) are given by

\[
(2.4) \quad \alpha_{2p-1, n-1} = \sum_{k_1 = 2p-1}^{n-1} b^2_{k_1} \sum_{k_2 = 2p-3}^{k_1-2} b^2_{k_2} ... \sum_{k_p = 1}^{k_{p-1}-2} b^2_{k_p}.
\]

Let us consider a system \(\{\Psi_n(x)\}_{n=0}^\infty\) of polynomials defined by the recurrence relations \((n \geq 0)\):

\[
(2.5) \quad x \Psi_n(x) = b_n \Psi_{n+1}(x) + b_{n-1} \Psi_{n-1}(x), \quad \Psi_0(x) = 1, \quad b_{-1} = 0,
\]

where \(\{b_n\}_{n=0}^\infty\) is a given positive sequence satisfying the relation \((2.3)\). The following theorem was proved in \([4]\).

**Theorem 2.1.** The polynomial system \(\{\Psi_n(x)\}_{n=0}^\infty\) is orthonormal in the Hilbert space \(\mathcal{H}\) if and only if the coefficients \(b_n\) and the moments \(\mu_{2k}\) are connected by relation \((2.9)\).

Let \(\{\Psi_n(x)\}_{n=0}^\infty\) be an orthonormal basis of the Fock space \(\mathcal{H}_s\) which satisfies the recurrence relation \((2.4)\). That is,

\[
\mathcal{H}_s = \text{span}\{\Psi_n(x) \mid n = 0, 1, 2, ...\},
\]

where the bar stands for the closure of the linear span. Define the ladder operators \(a^\dagger_s, a_s\) and the number operator \(N_s\) in the Fock space, \(\mathcal{H}_s\), by the usual formulas:

\[
(2.6) \quad a^\dagger_s \Psi_n(x) = \sqrt{2} b_n \Psi_{n+1}(x), \quad a_s \Psi_n(x) = \sqrt{2} b_{n-1} \Psi_{n-1}(x), \quad N_s \Psi_n(x) = n \Psi_n(x).
\]

It can be readily seen that \((a^\dagger_s)^\dagger = a_s\). The polynomial set \(\{\Psi_n(x)\}_{n=0}^\infty\) is called a canonical polynomial system when it is defined by the recurrence relation \((2.5)\). The canonical polynomial system \(\{\Psi_n(x)\}_{n=0}^\infty\) is uniquely determined by the symmetric probability measure \(\mu\). Now, as usual, let the position operator be

\[
(2.7) \quad Q_s = \frac{a_s + a^\dagger_s}{\sqrt{2}}.
\]

In order to guarantee

\[
(2.8) \quad Q_s \Psi_n(x) = x \Psi_n(x)
\]

the symmetry of the measure is required and the relation \((2.8)\) is essential for the three term recursion relation \((2.4)\). In fact, the relation \((2.8)\) provides the connection between the operators \(a_s, a^\dagger_s\) and the recurrence relation \((2.4)\) \([4]\) \([14]\).

The operator \(N_s\) is self-adjoint in the Fock space. Therefore for any Borel function \(B\), through the spectral theorem \([11]\), one can define the operator \(B(N_s)\) in

\[
(2.9) \quad B(N_s) \Psi_n(x) = b^2_n \Psi_n(x), \quad \text{and} \quad B(N_s + I_s) \Psi_n(x) = b^2_n \Psi_n(x); \quad n \geq 0,
\]

where \(I_s\) is the identity operator on \(\mathcal{H}_s\). The following result is proved in \([4, 5]\):

**Theorem 2.2.** The operators \(a_s, a^\dagger_s\) and \(N_s\) obey the following commutation relations

\[
(2.10) \quad [a_s, a^\dagger_s] = 2 (B(N_s + I_s) - B(N_s)), \quad [N_s, a^\dagger_s] = a^\dagger_s, \quad [N_s, a_s] = -a_s.
\]
Definition 2.3. An algebra $A_n$ is called a generalized oscillator algebra corresponding to the orthonormal system $\{\Psi_n(x)\}_{n=0}^{\infty}$, which satisfies (2.5), if $A_n$ is generated by the operators $a_n^\dagger$, $a_n$, $N_n$, and $I_n$ and by their commutators. The operators should also satisfy the relations (2.6) and (2.10).

At this point we like to emphasize a word about the above definition. The algebra $A_n$ consists the operators $a_n^\dagger$, $a_n$, $N_n$, and $I_n$ and their repeated commutators only. Further, the above oscillator algebra arises only in the symmetric case [4, 14]. Further, the algebra $A_n$ may be considered as a generalization of the algebra $A_{WH}$. Regarding the dimension of the algebra $A_n$ we have proved the following result in [14].

Theorem 2.4. The generalized oscillator algebra $A_n$ is of finite dimension if and only if

\begin{equation}
(2.11) \quad b_n^2 = a_0 + a_1 n + a_2 n^2, \quad b_{-1} = 0, \quad a_0, a_1, a_2 \in \mathbb{R}
\end{equation}

and in this case the dimension of the algebra is four.

Remark 2.5. At this point we like to respond to the comments made in [7] and [8] about our earlier paper [14]. In [7] and [8] the authors claimed that the sufficient part of the Theorem (2.4) is incorrect. They indicated that for the sufficient part to be true, in addition to (2.11), the coefficients $a_0, a_1$ and $a_2$ of (2.11) must also satisfy the relation

\begin{equation}
(2.12) \quad a_1 = a_0 + a_2
\end{equation}

However, the relation (2.12) is indeed included in Theorem (2.4). It can be easily seen that in the recurrence relation (2.6) we have $b_{-1} = 0$ and in (2.11) if $b_{-1} = 0$ then we obtain $a_1 = a_0 + a_2$. In this regard, Theorem (2.4) is correct in its own form.

2.2. Non-symmetric case. For a symmetric probability measure a non-symmetric recurrence relation can be transformed to a symmetric one. For details we refer the reader to Section 5 in [14]. Also the following is extracted from [14] as needed here.

Let $\mu$ be a probability but not necessarily a symmetric measure on $\mathbb{R}$. Let $A_{n-s} = L^2(\mathbb{R}, \mu)$ and

\begin{equation}
(2.13) \quad \mu_0 = 1, \quad \mu_k = \int_{-\infty}^{\infty} x^k \, d\mu(x); \quad k = 1, 2, \ldots.
\end{equation}

Let the real sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be solutions of the system

\begin{equation}
(2.14) \quad \begin{cases} 
A_{k,n} = b_n A_{k-1,n+1} + a_n A_{k-1,n} + b_{n-1} A_{k-1,n-1}; & n \geq 0, b_{-1} = 0 \\
A_{0,0} = 1, \quad A_{k,0} = \mu_k, \quad A_{0,k} = 0; & k \geq 0.
\end{cases}
\end{equation}

There is a unique solution to the system (2.14) with respect to the variables $(a_n, b_n, A_{k,n})$; $n \geq 0, k \geq 0$.

If sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ are given, then one can define the canonical polynomial system by the recurrence relation

\begin{equation}
(2.15) \quad x \Psi_n(x) = b_n \Psi_{n+1}(x) + a_n \Psi_n(x) + b_{n-1} \Psi_{n-1}(x); \quad n \geq 0, b_{-1} = 0, \Psi_0(x) = 1.
\end{equation}

Theorem 2.6. Let $\{\Psi_n(x)\}_{n=0}^{\infty}$ be a real polynomial system defined by (2.14) and let $\mu$ be a probability measure on $\mathbb{R}$. The system of polynomials $\{\Psi_n(x)\}_{n=0}^{\infty}$ is orthonormal with respect to the measure $\mu$ on $\mathbb{R}$ if and only if the coefficients...
\[ \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \] involved in (2.17) are the solutions of the system (2.14), where \( \mu_k \) are defined by (2.13).

Let \( \{\Psi_n(x)\}_{n=0}^{\infty} \) be an orthonormal system satisfying the four term recurrence relation (2.15) and let

\[ \mathcal{H}_{n-s} = \text{span}\{\Psi_n(x) \mid n = 0, 1, 2, \ldots\}. \]

On \( \mathcal{H}_{n-s} \) define the operators \( Q_{n-s} \) and \( P_{n-s} \) as

\[
Q_{n-s}\Psi_n(x) = b_{n-1}\Psi_{n-1}(x) + a_n\Psi_n(x) + b_n\Psi_{n+1}(x) \\
P_{n-s}\Psi_n(x) = i[b_{n-1}(x) - b_n\Psi_{n}(x)] + a_n\Psi_n(x); \quad n \geq 0.
\]

Let

\[ (2.16) \quad \hat{Q}_{n-s} = \text{Re}(Q_{n-s} - P_{n-s}), \quad \hat{P}_{n-s} = -i\text{Im}(Q_{n-s} - P_{n-s}). \]

Define the ladder operators as

\[ (2.17) \quad \hat{a}_{n-s} = \frac{\hat{Q}_{n-s} - i\hat{P}_{n-s}}{\sqrt{2}}, \quad \hat{a}_{n-s}^\dagger = \frac{\hat{Q}_{n-s} + i\hat{P}_{n-s}}{\sqrt{2}}. \]

Also take \( N_{n-s} = N_s \) and \( I_{n-s} \) to be the identity operator on \( \mathcal{H}_{n-s} \). Then for the operators \( \hat{a}_{n-s}, \hat{a}_{n-s}^\dagger, N_{n-s} \) and \( I_{n-s} \) the formulas (2.6) and Theorem (2.2) are valid. Let the algebra generated in this case be \( \mathcal{A}_{n-s} \). In the non-symmetric case the position operator, \( \hat{Q}_{n-s} \), does not have to be an operator of the multiplication by an independent variable. However, since the orthonormal system \( \{\Psi_n(x)\}_{n=0}^{\infty} \) satisfies the recurrence relation (2.15), by the definition of \( Q_{n-s} \), the operator \( Q_{n-s} \) is an operator of multiplication by an independent variable.

**Remark 2.7.** Since the sets of operators

\[ \{a_s, a_s^\dagger, N_s, I_s\} \quad \text{and} \quad \{\hat{a}_{n-s}, \hat{a}_{n-s}^\dagger, N_{n-s}, I_{n-s}\} \]

are defined by the same relations (2.13) and satisfy the same commutation relations (Theorem 2.4), in [7, 8] the authors claimed that the algebras \( \mathcal{A}_s \) and \( \mathcal{A}_{n-s} \) coincide and therefore Theorem (2.4) is valid for the algebra \( \mathcal{A}_{n-s} \) as well. According to the definition of the algebra \( \mathcal{A}_{n-s} \) given in [7, 8] the authors claim is true. However, since the \( a_n \)'s of the recurrence relation (2.16) are absent from Eq. (2.6) and \( \hat{Q}_{n-s} \) is not necessarily be an operator of multiplication by an independent variable, the connection between the operators \( \hat{a}_{n-s} \) and \( \hat{a}_{n-s}^\dagger \) and the recurrence relation (2.15) is ambiguous. This suggest us, for the non-symmetric case, to look for a better alternative to Theorem (2.4).

### 2.3. Oscillator-like algebras.

A particular kind of deformation to the creation operator of the algebra \( \mathcal{A}_{WH} \) is proposed in [1] and then used, for example, in [2, 10, 11]. In [1] the authors deformed the creation operator as

\[ (2.18) \quad a_\lambda^\dagger = a^\dagger + \lambda I; \quad \lambda \in \mathbb{R}, \]

where \( \lambda \) is a real continuous parameter, without changing the annihilation operator. Even after the deformation the commutation relations of the Weyl-Heisenberg algebra remain unchanged, that is,

\[ (2.19) \quad [a, a_\lambda^\dagger] = I, \quad [N_\lambda, a] = -a, \quad [N_\lambda, a_\lambda^\dagger] = a_\lambda^\dagger, \]

where \( N_\lambda = a_\lambda^\dagger a \). However, \( (a_\lambda^\dagger)^\dagger \neq a \) and \( N_\lambda^\dagger \neq N_\lambda \). In [11, 12] the authors called the algebra generated by \( \{a, a_\lambda^\dagger, N_\lambda, I\} \) an oscillator-like algebra. In the following we
propose a different, however similar to (2.18), deformation to the creation operator of the symmetric case and obtain an oscillator-like algebra. In fact, in (2.18) we replace the real parameter \(\lambda\) by the \(a_n\)'s of the recurrence relation (2.15) and the operator \(a\) by the operator \(a_s\).

Let \(\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty\) be the sequence of real numbers appearing in the four term recurrence relation (2.15) and \(\{\Psi_n(x)\}_{n=0}^\infty\) be an orthonormal polynomials system satisfying the recurrence relation (2.15). Let

\[
\mathcal{H} = \text{span}(\Psi_n(x) | n = 0, 1, 2, \cdots).
\]

In \(\mathcal{H}\), define the operator \(D\) by

\[
(2.20) \quad D\Psi_n = \sqrt{2}a_n\Psi_n, \quad n = 0, 1, ...
\]

Also on \(\mathcal{H}\) define the operators

\[
(2.21) \quad A = a_s, \quad A^\dagger = a_s^\dagger + D, \quad N = N_s
\]

and \(I\), the identity operator on \(\mathcal{H}\). Then their actions take the form

\[
(2.22) \quad A^\dagger\Psi_n = \sqrt{2}b_n\Psi_{n+1} + \sqrt{2}a_n\Psi_n, \quad A\Psi_n = \sqrt{2}b_{n-1}\Psi_{n-1}.
\]

We also have

\[
(2.23) \quad [A, A^\dagger] = 2(B(N + I) - B(N)) + 2Af(N), \quad [N, A] = -A, \quad [N, A^\dagger] = a_s^\dagger,
\]

where \(f(N)\) is a function of the self-adjoint operator \(N\) acting as

\[
f(N)\Psi_n(x) = \sqrt{2}(a_n - a_{n-1})\Psi_n(x).
\]

Once again \((A^\dagger)^\dagger \neq A\). Let \(A\) be the oscillator-like algebra generated by \(\{I, A, A^\dagger, N\}\). Now, as usual, let the position operator be

\[
(2.24) \quad Q = \frac{A + A^\dagger}{\sqrt{2}}.
\]

**Proposition 2.8.** The operator \(Q\) in (2.24) is an operator of the multiplication by an independent variable. That is,

\[
(2.25) \quad Q\Psi_n(x) = x\Psi_n(x).
\]

**Proof.** Since the sequences \(\{a_n\}\), \(\{b_n\}\) and the normalized polynomials system \(\{\Psi_n(x)\}\) satisfy the four term recurrence relation (2.15), we have

\[
Q\Psi_n(x) = \frac{1}{\sqrt{2}}(A + A^\dagger)\Psi_n(x)
\]

\[
= \frac{1}{\sqrt{2}}(\sqrt{2}b_{n-1}\Psi_{n-1}(x) + \sqrt{2}a_n\Psi_n(x) + \sqrt{2}b_n\Psi_{n+1}(x))
\]

\[
= b_{n-1}\Psi_{n-1}(x) + a_n\Psi_n(x) + b_n\Psi_{n+1}(x)
\]

\[
= x\Psi_n(x).
\]

\[\square\]

In this regard, unlike the algebra \(A_{n-5}\), \(A\) is the oscillator-like algebra most closely associated with the orthogonal polynomials system satisfying the four term recurrence relation (2.15).
3. Main results

In this section we shall provide a necessary and sufficient conditions on \( a_n \) and \( b_n \) of the four term recurrence relation (2.15) for the oscillator-like algebra, \( \mathcal{A} \), to be of finite dimension. The following theorem is the main result of this manuscript.

**Theorem 3.1.** The generalized oscillator-like algebra \( \mathcal{A} \) is of finite dimension if and only if

\[
(3.1) \quad b_n^2 = \alpha_2 n^2 + \alpha_1 n + \alpha_0 \quad \text{and} \quad a_n = \beta_1 n + \beta_0, \quad \text{with} \quad b_{-1} = 0,
\]

where \( \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1 \in \mathbb{R} \).

As a corollary we state the following result.

**Corollary 3.2.** If the oscillator-like algebra \( \mathcal{A} \) is of finite dimension, then the dimension of \( \mathcal{A} \) is four.

3.1. **Proof of Theorem 3.1**

We execute the proof in three steps. In step-1 we prove that if \( \dim(\mathcal{A}) < \infty \) then \( b_n \) is of second degree in \( n \). In step-2 we show that if \( \dim(\mathcal{A}) < \infty \) then \( a_n \) is of degree one in \( n \). In step-3 we prove the converse of the theorem.

**Step-1:** From (2.28) we have

\[
(3.2) \quad [N, A^\dagger] \Psi_n = a_n^\dagger \Psi_n, \quad n = 0, 1, 2, \ldots.
\]

Therefore, \( a_n^\dagger \in \mathcal{A} \) and hence

\[
(3.3) \quad \mathcal{A}_s \subseteq \mathcal{A}.
\]

Thus, if \( \dim(\mathcal{A}) < \infty \), then \( \dim(\mathcal{A}_s) < \infty \). Therefore, according to [14], \( b_n^2 \) must be of the form

\[
(3.4) \quad b_n^2 = \alpha_2 n^2 + \alpha_1 n + \alpha_0.
\]

**Step-2:** Since \( a_n^\dagger \in \mathcal{A} \), then \( D = A^\dagger - a_n^\dagger \in \mathcal{A} \). Define a family of operators as follows:

\[
(3.5) \quad D_1 = [A^\dagger, D], \quad D_2 = [A^\dagger, D_1], \quad \text{and} \quad D_k = [A^\dagger, D_{k-1}], \quad k = 3, 4, \ldots
\]

and

\[
(3.6) \quad d_n^{(1)} = a_n - a_{n-1}, \quad d_n^{(2)} = d_n^{(1)} - d_{n-1}^{(1)} \quad \text{and} \quad d_n^{(k)} = d_n^{(k-1)} - d_{n-1}^{(k-1)}, \quad k = 3, 4, \ldots
\]

Then, by induction, we have

\[
(3.7) \quad D_k \Psi_n = (\sqrt{2})^{k+1} \prod_{i=1}^{k} b_n - d_n^{(k)} \Psi_{n-k}.
\]

We can see that for any \( \Psi_n(x) \), with \( n \geq k \), \( D_k \) is lowering the level of \( \Psi_n(x) \) by \( k \)-stages if there is no \( k \) such that \( d_n^{(k)} = \text{constant} \), for \( n = 0, 1, 2, \ldots \). Therefore, if \( \dim(\mathcal{A}) < \infty \) then there exists \( k \) such that \( d_n^{(k)} = \text{constant} \), for \( n = 0, 1, 2, \ldots \).

Let

\[
(3.8) \quad p = \inf \{ k \mid d_n^{(k)} = \text{constant}, \quad \text{for} \quad n = 0, 1, 2, \ldots \}.
\]

Then it can be easily shown that the \( a_n \) has the form

\[
(3.9) \quad a_n = \sum_{i=0}^{p} \theta_i n^i = \theta_p n^p + \theta_{p-1} n^{p-1} + \ldots + \theta_1 n + \theta_0,
\]
where $\theta_p, \cdots, \theta_0$ are real constants. Hence, $D$ can be seen as

\begin{equation}
D = \sqrt{2} \sum_{i=0}^{p} \theta_i N^i = \sqrt{2} \left( \theta_p N^p + \theta_{p-1} N^{p-1} + \cdots + \theta_1 N + \theta_0 I \right) \in A. \tag{3.10}
\end{equation}

Now let us show that $\dim(A) = \infty$ if $p \geq 2$. Since $D$, $N$, $I \in A$. Eq. (3.10) implies that

\[ \sqrt{2}(\theta_p N^p + \theta_{p-1} N^{p-1} + \cdots + \theta_2 N^2) \in A. \]

By rescaling, we get

\begin{equation}
W_0 = N^p + \gamma_{p-1} N^{p-1} + \cdots + \gamma_1 N^2 \in A, \tag{3.11}
\end{equation}

where $\gamma_i = \frac{\theta_i}{\sqrt{2}}$, for $i = 2, \cdots, p - 1$. On can see that, just by replacing $p$ by $p + 1$, the Eq. (3.11) is the same as the Eq. (3.11) in [14]. Following step by step the calculations of page 8 in [14] and replacing $A^\dagger$ by $a_1^\dagger$ we can arrive at the conclusion that

\begin{equation}
(a_1^\dagger)^{p+m(p-1)} \in A, \quad \text{for every} \quad m = 1, 2, 3, \cdots. \tag{3.12}
\end{equation}

Further, for $p \geq 2$, the operators $(A^\dagger)^{p+m(p-1)}$ are new elements of $A$ for every $m \geq 1$. Therefore $A$ is of infinite dimension. That is, we have arrived at the conclusion that if $\dim(A) < \infty$, then $p < 2$. Thus, from (3.9), $a_n = \theta_1 n + \theta_0$. Hence, if $\dim(A) < \infty$, then $b_n$ and $a_n$ are second and first degree polynomials in $n$ respectively.

**Step-3:** Let us prove that if $p < 2$, then the algebra $A$ is of finite dimension. In this regard, for $n \geq 0$, assume that

\begin{equation}
b_n^2 = \alpha_2 n^2 + \alpha_1 n + \alpha_0 \quad \text{and} \quad a_n = \beta_1 n + \beta_0, \quad \text{with} \quad b_{-1} = 0. \tag{3.13}
\end{equation}

Then from (2.23) we have

\begin{equation}
[N, A^\dagger] \Psi_n = NA^\dagger \Psi_n - A^\dagger N \Psi_n \tag{3.14}
\end{equation}

\begin{align*}
&= \sqrt{2} b_n \Psi_{n+1} \\
&= \sqrt{2} b_n \Psi_{n+1}(x) + \sqrt{2} (\beta_1 n + \beta_0) \Psi_n - \sqrt{2} (\beta_1 n + \beta_0) \Psi_n \\
&= A^\dagger \Psi_n - \sqrt{2} (\beta_1 n + \beta_0) \Psi_n \\
&= A^\dagger \Psi_n - \sqrt{2} \beta_1 N \Psi_n - \sqrt{2} \beta_0 \Psi_n \\
&= \left( A^\dagger - \sqrt{2} \beta_1 N - \sqrt{2} \beta_0 I \right) \Psi_n.
\end{align*}

Therefore,

\begin{equation}
[N, A^\dagger] = A^\dagger - \sqrt{2} \beta_1 N - \sqrt{2} \beta_0 I. \tag{3.15}
\end{equation}

Further

\begin{equation}
[N, A] \Psi_n = \sqrt{2} (n-1) b_{n-1} \Psi_{n-1} - \sqrt{2} n b_{n-1} \Psi_{n-1} \tag{3.16}
\end{equation}

\begin{align*}
&= -\sqrt{2} b_{n-1} \Psi_{n-1} \\
&= -A \Psi_n.
\end{align*}

Therefore,

\begin{equation}
[N, A] = -A. \tag{3.17}
\end{equation}
Also

\[ [A, A^\dagger] \Psi_n = 2 [2 \alpha_2 n + (\alpha_1 - \alpha_2)] \Psi_n + 2 \beta_1 b_{n-1} \Psi_{n-1} \]
\[ = 2 [2 \alpha_2 N + (\alpha_1 - \alpha_2) I] \Psi_n + \sqrt{2} \beta_1 A \Psi_n \]
\[ = \left[ 4 \alpha_2 N + 2 (\alpha_1 - \alpha_2) I + \sqrt{2} \beta_1 A \right] \Psi_n. \]

Therefore,

\[ [A, A^\dagger] = 4 \alpha_2 N + 2 (\alpha_1 - \alpha_2) I + \sqrt{2} \beta_1 A. \]

That is, in this case, all the commutation relations are linear combinations of the operators \( A, A^\dagger, N \) and \( I \). Therefore, the algebra \( \mathcal{A} \) is closed under the bracket \([ \cdot, \cdot ]\). Hence \( \mathcal{A} \) is of finite dimension.

3.2. **Proof of Corollary 3.2.** The proof follows from step-3 of the above proof.

4. **Some examples**

In this section, as examples, we discuss the dimensions of oscillator-like algebras associated with Laguerre and Jacobi polynomials. We borrow the details of these polynomials from [7, 8]. For an enhanced explanation we refer the reader to [7, 8] and the references therein.

4.1. **Laguerre polynomials.** The Laguerre polynomials are defined by

\[ L_n^\alpha(x) = \frac{\alpha + 1}{n!} {}_1F_1(-n, \alpha + 1; x). \]

These polynomials are orthogonal in the Hilbert space \( \mathcal{H}_L = L^2(\mathbb{R}_+, x^\alpha e^{-x} dx) \).

The normalized polynomials take the form

\[ \Psi_n(x) = d_n^{-1} L_n^\alpha(x) \text{ with } d_n = \sqrt{\frac{\Gamma(n + \alpha + 1)}{n!}}; \quad n \geq 0. \]

These normalized polynomials satisfy the non-symmetric recurrence relation (2.15) with

\[ b_n = \sqrt{(n + 1)(n + \alpha + 1)}, \quad a_n = 2n + \alpha + 1. \]

Therefore, according to theorem (3.1), the related oscillator-like algebra, \( \mathcal{A}_L \), is of finite dimension.

4.2. **Jacobi polynomials.** The Jacobi polynomials

\[ P_n^{(\alpha, \beta)}(x) = \frac{\alpha + 1}{n!} {}_2F_1(-n, n + \alpha + \beta; \alpha + 1; \frac{1 - x}{2}) \]

are orthogonal in the Hilbert space \( L^2([-1, 1], (d_0(\alpha, \beta))^{-2}(1 - x)^\alpha (1 + x)^\beta dx) \), where

\[ d_0^2(\alpha, \beta) = 2^{\alpha + \beta + 1} \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}. \]

The normalized polynomials \( \{ \Psi_n(x) \}_{n=0}^\infty \) are defined by the formula

\[ \Psi_n(x) = d_0 d_n^{-1} P_n^{(\alpha, \beta)}(x), \text{ where } d_n = 2^{\alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{\Gamma(n + \alpha + \beta + 1)n!(2n + \alpha + \beta + 1)}; \quad n > 0. \]
Then the non-symmetric recurrence relation \((2.15)\) is satisfied with
\[
a_n = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)},
\]
\[
b_n = 2\sqrt{\frac{(n+1)(n+1+\alpha)(n+1+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}}.
\]

Therefore, according to theorem \((3.1)\), the corresponding oscillator-like algebra, \(A_J\), is of infinite dimension.

5. Conclusion

In this paper, we have discussed the dimensions of oscillator-like algebras induced by orthogonal polynomials satisfying a non-symmetric four term recurrence relation. Further, we have also responded to the claims made in \([7, 8]\) about our previous paper \([14]\).

In \([7]\) the authors have presented some remarks about the dimensions of oscillator algebras associated with two dimensional orthogonal polynomials such as the normalized 2D-Hermite polynomials \(H_{n,m}(z, \overline{z})\) which satisfy the three term recurrence relation \([17, 13]\)
\[
zH_{m,n}(z, \overline{z}) = \sqrt{m+1}H_{m+1,n}(z, \overline{z}) + \sqrt{n}H_{m,n-1}(z, \overline{z}).
\]

It may be interesting to consider a detail study of the dimensions of oscillator algebras arising from 2D orthogonal polynomials satisfying three-term and four-term recurrence relations.

Further, there are several deformations to 1D and 2D orthogonal polynomials, for example see \([3, 6, 9, 10, 12, 15]\). The theory developed in \([14, 7, 8]\) or in this manuscript does not directly apply to the deformed algebras associated with these deformed orthogonal polynomials.

References

[1] Beckers, J., Debergh, N., Szafraniec, F.H., A proposal of new sets of squeezed states, Phys. Lett. A. 243 (1998), 256-260.
[2] Beckers, J., Debergh, N., Szafraniec, F.H., Oscillator like Hamiltonians and squeezing, Int. J. Theor. Phys. 39 (2000), 1515-1527.
[3] Biedenharn, L. C., The quantum group \(SU(2)_q\) and a \(q\)-analogue of the boson operators, J. Phys. A. 22 (1989), L873-L878.
[4] Borzov, V. V., Orthogonal Polynomials and Generalized Oscillator Algebras, Integral Transform. Spec. Funct. 12 (2001), 115-138.
[5] Borzov, V. V., Damaskinsky, E. V., Coherent states for a generalized oscillator in a finite dimensional Hilbert space, J. Math. Sci. (N.Y.) 143 (2007), 2738-2753.
[6] Borzov, V. V., Damaskinsky, E. V., Yegorov, S. B., Representations of the deformed oscillator under different choices of generators, J. Math. Sci. (N.Y.) 100 (2000), 2061-2076.
[7] Borzov, V.V., Damaskinsky, E.V., On the dimensions of oscillator algebras, Days on Diffraction (2014), 48-52.
[8] Borzov, V.V., Damaskinsky, E.V., Comment on “On the dimensions of the oscillator algebras induced by orthogonal polynomials” [J. Math. Phys. 55, 093511 (2014)], arXiv:1503.08202.
[9] Burban, I.M., Unified \((q; \alpha, \beta, \gamma, \nu)\)-deformation of one parameter \(q\)-deformed oscillator algebras, J. Phys. A 42 (2009), 065201.
[10] Bukweli-Kyemba, J.D., Hounkonnou, M.N., Quantum deformed algebras: Coherent states and Special functions, [arXiv:1301.0116 [math-ph]].
[11] de Oliveira, C. R., *Intermediate Spectral theory and Quantum Dynamics*, Progress in Mathematical Physics 54, Birkhäuser, Boston (2009).

[12] Floreanini, R., Vinet, L., *q-Orthogonal polynomials and the oscillator quantum groups*, Lecture in Math. Phys. 22 (1991), 45-54.

[13] Ghanmi, A., *A class of generalized complex Hermite polynomials*, J. Math. Anal. and App. 340 (2008), 1395-1406.

[14] Honnouvo, G., Thirulogasanthar, K., *On the dimensions of oscillator algebras induced by orthogonal polynomials*, J. Math. Phys. 55 (2014), 093511.

[15] Ismail, M.E.H., Zhang, R., *On some 2D orthogonal q-polynomials*, arXiv: 1411.5223.

[16] Roknizadeh, R., Tavassoly, M.K., *Representation of coherent and squeezed states in a f-deformed Fock space*, J. Phys. A: Math. Gen., 37 (2004), 5649.

[17] Wünsche, A., *Hermite and Laguerre 2D polynomials*, J. Comput. Appl. Math. 133 (2001) 665-678.

1 Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, Quebec H3A 2K6, Canada
E-mail address: g.honnouvo@yahoo.fr

2Department of Computer Science and Software Engineering, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec, H3G 1M8, Canada
E-mail address: santhar@gmail.com