INTEGRAL POINTS ON CUBIC HYPERSURFACES

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1. Introduction

Let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be an absolutely irreducible cubic polynomial whose homogeneous part is non-degenerate. The primary goal of this paper is to investigate the set of integer solutions to the equation \( g = 0 \). Specifically, we shall try to determine conditions on \( g \) under which we can show that there are infinitely many solutions. An obvious necessary condition for the existence of integer solutions is that the congruence
\[
g(x_1, \ldots, x_n) \equiv 0 \pmod{p^k},
\]
should be soluble for every prime power \( p^k \). We shall henceforth refer to this condition as “the Congruence Condition”.

There is no condition on the size of \( n \) sufficient to ensure that (1.1) is always soluble for non-homogeneous \( g \). In fact, even when the Congruence Condition is satisfied for a polynomial \( g \), and \( n \) is large, this is still not sufficient to ensure the existence of integer solutions to the corresponding equation \( g = 0 \). An illustration of this is provided by the polynomial
\[
g = (2x_1 - 1)(1 + x_1^2 + \cdots + x_n^2) + x_1x_2,
\]
discovered by Watson. Now it can be shown quite easily that \( g \) satisfies the Congruence Condition for \( n \geq 2 \). However, the equation \( g = 0 \) is insoluble in integers, since \( |2x_1 - 1| \geq 1 \) for every \( x_1 \in \mathbb{Z} \), so \( |g| \geq 1 + x_1^2 + x_2^2 - |x_1x_2| > 0 \).

In view of Watson’s example, it will be necessary to introduce an auxiliary condition on the polynomials \( g \) that we are able to handle. Throughout this work we shall write \( g_0 \) for the homogeneous cubic part of a polynomial \( g \in \mathbb{Z}[x_1, \ldots, x_n] \). The condition that we shall work with is phrased in terms of the singular locus of the projective hypersurface \( g_0 = 0 \) in \( \mathbb{P}^{n-1}_Q \), which we denote by \( \text{sing}(g_0) \), a proper projective subvariety of \( \mathbb{P}^{n-1}_Q \). We set
\[
s(g_0) := \dim \text{sing}(g_0)
\]
for its projective dimension. Following the convention that \( s(g_0) = -1 \) if and only if \( g_0 \) is a non-singular cubic form, we see that \( s(g_0) \) is an integer contained in the interval \([-1, n - 2]\). We are now ready to state our main result.

**Theorem 1.** Suppose that \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) is a cubic polynomial that satisfies the Congruence Condition, such that \( g_0 \) is non-degenerate, and having \( n \geq 11 + s(g_0) \).

Then the equation \( g = 0 \) has infinitely many solutions in integers.

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This improves on work of Skinner [15], who has established the same conclusion under the assumption that $n \geq 18 + s(g_0)$. At this point we note that the polynomial (1.2) has homogeneous cubic part $g_0 = 2x_1(x_1^2 + \cdots + x_n^2)$, which defines a reducible cubic hypersurface with singular locus of dimension $s(g_0) = n - 3$. Hence Theorem 1 is not applicable to this particular example.

It is interesting to place Theorem 1 in the context of other work in the literature. This topic has been extensively studied only in the framework of homogeneous $g$. For arbitrary cubic forms the best result available was, until very recently, due to Davenport [1]. This shows that there exists a non-trivial integer solution to the homogeneous equation $g = 0$ as soon as $n \geq 16$. This has now been improved by the second author [6], who has shown that $n \geq 14$ variables are enough. One can do even better when the form under consideration is non-singular, and the second author has shown that $n \geq 10$ variables suffice for such forms [5]. This in turn has been sharpened by Hooley in a series of papers [8, 9, 11], to the extent that when $n \geq 9$ and the Congruence Condition is satisfied, then the homogeneous equation $g = 0$ is soluble in integers provided that the corresponding hypersurface has only finitely many singularities and these are linearly independent double points.

Returning to the topic of arbitrary cubic polynomials $g \in \mathbb{Z}[x_1, \ldots, x_n]$, Davenport and Lewis [2] have also considered the problem of determining when the equation $g = 0$ has an integer solution. Their main results are phrased in terms of the so-called $h$-invariant. Let $g_0$ denote the cubic homogeneous part of $g$, as above. Then the invariant $h = h(g_0)$ is defined to be the least positive integer such that $g_0$ can be written identically as

$$g_0(x_1, \ldots, x_n) = L_1Q_1 + \cdots + L_hQ_h,$$

for linear forms $L_i \in \mathbb{Z}[x_1, \ldots, x_n]$ and quadratic forms $Q_i \in \mathbb{Z}[x_1, \ldots, x_n]$. With this notation in mind, Davenport and Lewis show that the equation $g = 0$ has infinitely many solutions in integers provided that $g$ satisfies the Congruence Condition, and has $h(g_0) \geq 17$. In the course of generalising this work to the setting of arbitrary number fields, Pleasants [14] has improved this lower bound to $h(g_0) \geq 16$. In a series of papers, culminating in [16], Watson has shown that the equation $g = 0$ is soluble in integers provided that $g$ satisfies the Congruence Condition, and has

$$4 \leq h(g_0) \leq n - 3.$$

One may combine this result with [14], to conclude that the equation $g = 0$ is soluble provided that $g$ satisfies the Congruence Condition, and has

$$n \geq 18, \quad h(g_0) \geq 4.$$

Note that one has $h(g_0) = 1$ in (1.2), so that none of these results apply to this particular example.

Outside of the work of Skinner [15], it is not entirely straightforward to compare the relative merits of Theorem 1 with this previous body of work. It is true, however, that the condition on $s(g_0)$ is much easier to check than the condition on $h(g_0)$. Theorem 1 also has something new to say about the case in which $g$ is homogeneous. In this setting one can view it as a bridge
between [5] and [6], giving a new result for cubic forms \( g \) such that
\[
13 \geq n \geq \begin{cases} 
11, & \text{if } s(g) = 0, \\
12, & \text{if } s(g) = 1, \\
13, & \text{if } s(g) = 2.
\end{cases}
\]

It is a natural question whether the approach of Hooley [8, 9, 11] can be adapted to handle polynomials rather than forms. However we shall content ourselves with investigating the extension of the second author’s methods [5], since Hooley’s technique is considerably more delicate. Our strategy will be to prove Theorem 1 for the case in which \( g_0 \) is non-singular, that is to say that \( s(g_0) = -1 \), and subsequently to deduce the general case via a hyperplane slicing argument.

Much of the work in this paper consists of trivial generalizations of the second author’s paper [5]. However there are three new things to be done. Firstly, we have new complete exponential sums to consider, which we shall reduce to Deligne’s results [3], through a technique of Hooley [7]. Secondly, we must reconsider the singular series and the Congruence Condition. Thirdly, we shall show how to treat polynomials for which \( g_0 \) is singular.

**Notation.** Throughout our work \( \mathbb{N} \) will denote the set of positive integers. For any \( \alpha \in \mathbb{R} \), we shall follow common convention and write \( e(\alpha) := e^{2\pi i \alpha} \) and \( e_q(\alpha) := e^{2\pi i \alpha/q} \). All order constants will be allowed to depend on the polynomial \( g \).

## 2. The Circle Method

In this section we recall the framework of the Hardy–Littlewood circle method, as it applies to our problem on cubic polynomials. Our approach is based on the version of the circle method due to the second author [5], which incorporates Kloosterman’s method for tracking the precise location of the endpoints in the decomposition of the unit interval into Farey arcs. We have decided to follow [5] as closely as possible, rather than to incorporate some of the improvements introduced by Hooley. We hope readers will appreciate having to familiarize themselves with only one source, rather than two.

We begin by choosing a non-zero real point \( x_0 \) on \( g_0(x) = 0 \), satisfying the additional condition that the matrix of second derivatives of \( g_0 \) should have rank at least \( n - 1 \) at \( x_0 \). The existence of such a point is established as [5, Lemma 5]. We let \( P \) be a large parameter, which we think of as tending to infinity, and we define the weight
\[
w(x) := \exp(-|x - Px_0|^2P_0^{-2}),
\]
where \( P_0 := P(\log P)^{-2} \). Our main task is then to examine the asymptotic behaviour of
\[
N(g; P) := \sum_{\substack{x \in \mathbb{Z}^n \\mid\\ g(x) = 0}} w(x),
\]
as \( P \to \infty \). We define the singular series
\[
\mathcal{S}(g) := \sum_{q=1}^{\infty} q^{-n} \sum_{\substack{a = 1 \\mod q \\mid\\ \gcd(a, q) = 1}} \sum_{x} e_q(\alpha g(x)),
\]
when it converges, and the singular integral
\[ \mathcal{J}(g; P) := \int_{-1}^{1} \int_{\mathbb{R}^n} w(x)e(\lambda g(x))dx\,dz. \]

Then we shall prove the following estimate.

**Theorem 2.** Let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be a cubic polynomial for which \( g_0 \) is non-singular. Assume that \( n \geq 10 \). Then there exists \( \delta > 0 \) such that
\[ N(g; P) = \mathcal{G}(g) \mathcal{J}(g; P) + O(P^{n-3-\delta}). \quad (2.2) \]

We have
\[ P^{n-3}(\log P)^{2-2n} \ll \mathcal{J}(g; P) \ll P^{n-3}(\log P)^{2-2n}. \]
Moreover \( \mathcal{G}(g) > 0 \) providing that the Congruence Condition holds.

As is implicit in the statement of Theorem 2, the singular series \( \mathcal{G}(g) \) is convergent for \( n \geq 10 \). In fact we shall establish absolute convergence under this hypothesis.

Define the cubic exponential sum
\[ S(\alpha) := \sum_{x \in \mathbb{Z}^n} w(x)e(\alpha g(x)), \]
for \( P \geq 2 \). Then \( S(\alpha) \) converges absolutely, and for any \( Q \geq 1 \) we have
\[ N(g; P) = \int_{0}^{1} S(\alpha)d\alpha = \int_{-\frac{1}{1+Q}}^{1+\frac{1}{1+Q}} S(\alpha)d\alpha, \]
where \( N(g; P) \) is given by (2.1). In the form of the circle method developed by the second author [5], one proceeds to break the interval \([-\frac{1}{1+Q}, 1+\frac{1}{1+Q}]\) according to the Farey dissection of order \( Q \). This ultimately yields
\[ N(g; P) = \sum_{q \leq Q} \int_{-\frac{1}{1+Q}}^{\frac{1}{1+Q}} S_0(q; z)dz + O(Q^{-2}E(g; P, Q)), \quad (2.3) \]
for any \( Q \geq 1 \), where
\[ E(g; P, Q) := \sum_{q \leq Q} \sum_{|u| \leq \frac{q}{Q}} \max_{\frac{1}{2} \leq |z| \leq 1} |S_u(q; z)| \frac{1}{1 + |u|}, \]
and
\[ S_u(q; z) := \sum_{a=1}^{q} e_q(au)S(a/q + z). \]

This is [5, Lemma 7]. We shall find that our work is optimised by taking \( Q = P^{3/2} \) in (2.3). As in [5, §4] we shall estimate \( S_u(q; z) \) via an application of the Poisson summation formula. This leads to the expression
\[ S_u(q; z) = q^{-n} \sum_{v \in \mathbb{Z}^n} S_u(q; v)I(z; q^{-1}v), \]
where
\[ S_u(q; v) := \sum_{a=1}^{q} e_q(au) \sum_{y \mod q} e_q(ay(v) - v \cdot y), \]
In this section, we consider the measure of the set of vectors \( x_1 \) for which the constraints (4.7) and (4.8) of [5] hold, that is to say, for which

\[
|x_1 - Px_0| \ll LP_0
\]

and

\[
|z\nabla g(x_1) + \beta| \ll L^7(P^{-1} + |z|^{1/2}P^{1/2}).
\]

Since \( \nabla g(x_1) = \nabla g_0(x_1) + O(P) \), this latter constraint yields

\[
|z\nabla g_0(x_1) + \beta| \ll L^7(P^{-1} + |z|^{1/2}P^{1/2} + |z|P).
\]

We can then proceed exactly as before so as to deduce the following extension of [5, Lemma 8].

**Lemma 1.** For \(|z| \leq 1\) we have

\[
S_u(q; z) = q^{-n} \sum_{\substack{v \in \mathbb{Z}^n \\ |v| \leq V_0}} S_u(q; v)I(z; q^{-1}v) + O(1),
\]

with

\[
V_0 \ll (\log P)^7q(P^{-1} + |z|P^2).
\]

Moreover for \(|z| \leq 1\) we have

\[
I(z; \beta) \ll (\log P)^7n \left(P + \min\{P^n, P^{(3-n)/2}|z|^{(1-n)/2}\}\right).
\]

3. The sum \( S_u(q; v) \) when \( q \) is prime

The sum \( S_u(q; v) \) satisfies the multiplicativity property

\[
S_u(rs; v) = S_{s^2u}(r; rv)S_{r^2u}(s; rv), \quad \text{for } \gcd(r, s) = 1,
\]

where \( \bar{r}, \bar{s} \) are any integers such that \( r\bar{r} + s\bar{s} = 1 \). Note that this reduces to (5.1) in [5] when \( g \) is homogeneous. It therefore suffices to examine the case in which \( q \) is a prime power. In this section we handle prime values of \( q \), using the following lemma, which summarizes the technique developed by Hooley [7].
Lemma 2. Let $F$ and $G$ be polynomials over $\mathbb{Z}$, of degree at most $d$, and let
\[ S := \sum_{x \in \mathbb{F}_p^n : G(x) = 0} e_p(F(x)), \]
for any prime $p$. For each $j \geq 1$ write
\[ N_j(\tau) = \# \{ x \in \mathbb{F}_{p^n} : G(x) = 0, F(x) = \tau \}, \]
and suppose that for each $j$ there is a real number $N(j)$ such that
\[ \sum_{\tau \in \mathbb{F}_{p^n}} |N_j(\tau) - N(j)|^2 \ll_{d,n} p^{kj}, \quad (3.1) \]
where $k$ is an integer independent of $j$. Then
\[ S \ll_{d,n} p^{k/2}. \]

Our second key tool is an extension of the famous result of Deligne [3], due to Hooley [10].

Lemma 3. Let $q = p^j$ and let $H(x_1, \ldots, x_m)$ be a form of degree $d$ defined over $\mathbb{F}_q$. Assume that $p \nmid d$, and write $s \geq -1$ for the dimension of the singular locus of $H = 0$ in $\mathbb{P}^{m-1}(\mathbb{F}_q)$. Then
\[ \# \{ x \in \mathbb{F}_q^m : H(x) = 0 \} = q^m - O_{d,m}(q^{(m+1+s)/2}). \]

Note that if $H$ is not absolutely irreducible we interpret $s$ as the dimension of the variety $\nabla H(x) = 0$, so that $s \geq m - 3$. The result is therefore trivial for such $H$. Indeed it remains true even if $H$ vanishes identically, since then $s = m - 1$.

Our basic estimate for $S_u(p; v)$ is now provided by the following result.

Lemma 4. Let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be a cubic polynomial, and let $p$ be a prime. Suppose that $g_0$ is non-singular modulo $p$, and that $p \nmid u$. Then there is a constant $C(n)$ such that
\[ |S_u(p; v)| \leq C(n)p^{(n+1)/2}. \]

A more general result has been given recently by Katz [13], but we shall give a shorter self-contained treatment, based on Lemma 2.

Lemma 4 is trivial for $p \leq 3$, so we shall assume that $p \geq 5$. For the proof we set
\[ F(a, b, x) = ub + ag(x) - v \cdot x, \quad G(a, b, x) = ab - 1, \]
so that $S = S_u(p; v)$ in the notation of Lemma 2. Then
\[ N_j(\tau) = \# \{ (b, x) \in \mathbb{F}_{q^n}^2 : b^2 u + g(x) - bv \cdot x - b\tau = 0, b \neq 0 \}, \]
where $q = p^j$. We convert this into a question about projective varieties by defining $\bar{g}(z, x) := z^3 g(z^{-1}x)$ and
\[ H_{\tau}(b, z, x) := ub^2 z + \bar{g}(z, x) - bzv \cdot x - bz^2 \tau, \]
so that
\[ N_j(\tau) = \frac{1}{q-1} \# \{ (b, z, x) \in \mathbb{F}_{q^n}^3 : H_{\tau}(b, z, x) = 0, bz \neq 0 \}. \]
Solutions with \( z = 0 \) have \( \tilde{g}(0, \mathbf{x}) = g_0(\mathbf{x}) = 0 \). According to Lemma 3 one has \( g_0(\mathbf{x}) = 0 \) for \( q^{n-1} + O_n(q^{n/2}) \) values of \( \mathbf{x} \), whence

\[
\#\{(b, z, x) \in \mathbb{F}_q^{n+2} : H_{\tau}(b, z, x) = 0, \; b \neq 0, \; z = 0\} = (q - 1)q^{n-1} + O_n(q^{(n+2)/2}).
\]

When \( b = 0 \) the equation \( H_{\tau} = 0 \) reduces to \( \tilde{g}(z, \mathbf{x}) = 0 \). Let \( V \) denote, temporarily, the singular locus of the variety defined by \( \tilde{g}(z, \mathbf{x}) = 0 \), and let \( H \) denote the hyperplane given by \( z = 0 \). Any points \( (0, \mathbf{x}) \) on \( V \cap H \) would satisfy \( \nabla g_0(\mathbf{x}) = 0 \). Since \( g_0 \) is assumed to be non-singular modulo \( p \) we conclude that \( V \cap H \) is empty. Thus \( V \) has dimension at most zero. An application of Lemma 3 now shows that

\[
\#\{(b, z, x) \in \mathbb{F}_q^{n+2} : H_{\tau}(b, z, x) = 0, \; b = 0\} = q^n + O_n(q^{(n+2)/2}).
\]

Finally, a third application of Lemma 3 yields

\[
\#\{(b, z, x) \in \mathbb{F}_q^{n+2} : H_{\tau}(b, z, x) = 0\} = q^{n+1} + O_n(q^{(n+3+s)/2}),
\]

where \( s \) is the dimension of the singular locus of the variety \( W_{\tau} \subseteq \mathbb{F}_q^{n+1} \) defined by \( H_{\tau}(b, z, x) = 0 \).

On combining our various results we conclude that

\[
N_j(\tau) = (q - 1)q^{n-1} + O_n(q^{(n+1+s)/2}).
\]

We claim that \( s = -1 \) for all but \( O_n(1) \) values of \( \tau \) in the algebraic closure \( \overline{\mathbb{F}_q} \), and that \( s = 0 \) for the remaining values. If we take \( N(j) = (p^j - 1)p^{n+1-j} \) in Lemma 2 we will then obtain the required estimate \( (3.1) \) with \( k = n + 1 \), whence \( S = S_n(p; \nu) = O_n(p^{(n+1)/2}) \) as required.

Taking partial derivatives, we see that singular points on \( W_{\tau} \) satisfy

\[
2ubz - zv \cdot \mathbf{x} - z^2\tau = 0, \quad (3.2)
\]

\[
ub^2 + \frac{\partial \tilde{g}}{\partial z}(z, \mathbf{x}) - bv \cdot \mathbf{x} - 2bz\tau = 0 \quad (3.3)
\]

and

\[
\frac{\partial \tilde{g}}{\partial x_i}(z, \mathbf{x}) - bzh_i = 0, \quad (1 \leq i \leq n). \quad (3.4)
\]

If \( z = 0 \) then \( (3.4) \) reduces to \( \nabla g_0(\mathbf{x}) = 0 \), whence \( \mathbf{x} = 0 \), since \( g_0 \) is non-singular modulo \( p \). We then have \( b = 0 \) from \( (3.3) \). Thus there can be no singular points with \( z = 0 \), so that \( (3.2) \) may be replaced by

\[
2ub - v \cdot \mathbf{x} - z\tau = 0. \quad (3.5)
\]

We now eliminate \( \tau \) from \( (3.3) \) and \( (3.5) \) to produce

\[
-3ub^2 + \frac{\partial \tilde{g}}{\partial z}(z, \mathbf{x}) + bv \cdot \mathbf{x} = 0. \quad (3.6)
\]

It follows that all singular points on \( W_{\tau} \), regardless of the value of \( \tau \), lie on the variety \( U \), say, given by \( (3.4) \) and \( (3.6) \). We proceed to examine the intersection \( I \), say, of \( U \) with the hyperplane \( z = 0 \). For points on \( I \) the equation \( (3.4) \) reduces to \( \nabla g_0(\mathbf{x}) = 0 \), whence \( \mathbf{x} = 0 \). Since \( p \nmid u \) it then follows from \( (3.6) \) that \( b = 0 \), so that \( I \) is empty, as a subset of \( \mathbb{F}_q^{n+1} \). We therefore conclude that \( U \) has at most dimension zero, and therefore contains \( O_n(1) \) points \( (b, z, \mathbf{x}) \). Finally we conclude that the various varieties \( W_{\tau} = 0 \) have between them at most \( O_n(1) \) singular points. Since one has \( z \neq 0 \) for
any singular point, as noted above, any singular point determines exactly one corresponding value of $\tau$, via (3.5). It therefore follows that there are $O(n(1))$ values of $\tau$ for which $W_\tau$ is singular, and that if $W_\tau$ is singular then it has $s = 0$. This establishes the claim above, and thereby completes the proof of Lemma 4.

For the case in which $p | u$ the situation is more complicated. We consider the projective variety defined by $g_0(x) = 0$. Then the dual variety is a hypersurface, defined by an equation $g^*(x) = 0$, say. We now have the following estimate.

**Lemma 5.** Let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be a cubic polynomial, and let $p$ be a prime. Suppose that $g_0$ is non-singular modulo $p$. Then there is a constant $C(n)$ such that

$$\left| S_0(p; v) \right| \leq C(n)p^{(n+1)/2}(p, g^*(v))^{1/2}.$$

As before this is trivial for $p \leq 3$. When $p \nmid g^*(v)$ we apply the estimate

$$\sum_{x \pmod{p}} e_p(f(x)) \ll_d n^{1/2},$$

of Deligne [3], which applies to any polynomial $f$ over $\mathbb{F}_p$ of degree $d$, in $n$ variables, whose homogeneous part is non-singular modulo $p$. Taking $f(x) = ag(x) - v.x$ we see that

$$\sum_{x \pmod{p}} e_p(ag(x) - v.x) \ll n^{1/2}$$

for $p \nmid a$. Summing over $a$ yields a satisfactory bound when $p \nmid g^*(v)$.

For the general case we begin by observing that $p \nmid v$, since $p \nmid g^*(v)$. It follows that

$$\sum_{x \pmod{p}} e_p(v.x) = 0,$$

whence

$$S_0(p; v) = \sum_{a,x \pmod{p}} e_p(ag(x) - v.x) = p \sum_{x \pmod{p}} e_p(-v.x) = pS,$$

say. It is possible to handle this by an application of Hooley’s method. However a more general result due to Katz [12] is already available. To put our sum into the correct form for Katz’ estimate, we define $\bar{g}(z, y) = z^3g(z^{-1}y)$ and substitute $x \equiv z^{-1}y \pmod{p}$. If we then let $z$ run over the residue classes coprime to $p$ we find that

$$S = \frac{1}{p-1} \sum_{z,y} e_p(-z^{-1}v.y)$$

where $z, y$ run over solutions of $\bar{g}(z, y) \equiv 0 \pmod{p}$ with $z \neq 0 \pmod{p}$. Thus we have

$$S = \sum_{(z, y) \in V} e_p(-z^{-1}v.y),$$

where $V$ is the projective variety over $\mathbb{F}_p$ given by $\bar{g} = 0$ and $z \neq 0$. For this type of sum Katz [12] shows that $S \ll_n p^{m/2}$, where $m = n - 1$ is the dimension of $V$ in projective space, under the conditions that $\bar{g}$ is absolutely
irreducible over $\mathbb{F}_p$, and that the variety $\tilde{g}(z, y) = v \cdot y = z = 0$ is smooth and of dimension $m - 2 = n - 3$ in $\mathbb{P}^n(\mathbb{F}_p)$. Since $\tilde{g}(z, y) = v \cdot y = z = 0$ implies $g_0(y) = v \cdot y = 0$, this second condition follows from our assumption that $p \nmid g^*(v)$. Moreover $g_0$ is absolutely irreducible, since it is non-singular, and the absolute irreducibility of $\tilde{g}$ follows. This completes our treatment of Lemma 5.

4. THE SUM $S_u(q; v)$ WHEN $q$ IS SQUARE-FULL

When $q$ is square-full we follow the analysis of [5, §6], with only minor modifications. The sum $S_{k, h}$ becomes

$$S_{k, h} = \sum_{j \pmod{q_2}} e_{q_2}(k, j + \frac{s}{2}j^T M(h)j),$$

where $M(h)$ is the matrix of second derivatives of $g(h)$. Similarly $N(q_3; h)$ becomes

$$\tilde{N}(q_3; h) = \# \{j \pmod{q_3} : q_3 \mid 1_6 M(h)j \}. $$

We now have the following analogue of [5, Lemma 4].

Lemma 6. Let

$$\tilde{N}(q) := \# \{h, j \pmod{q} : q \mid 1_6 M(h)j \}. $$

Then there is a constant $A$ such that

$$\tilde{N}(q) \leq A^{\omega(q)} q^n$$

for every square-free $q$.

To prove this we observe that

$$M(h)j = M_0(h)j + M_1j,$$

where $M_0(h)$ is the matrix of second derivatives of $g_0$, and $M_1$ is the matrix of second derivatives of the quadratic part of $g$. It follows that

$$\tilde{N}(q) = \sum_{j \pmod{q}} \# \{h \pmod{q} : q \mid 1_6 (M_0(h)j + M_1j) \}$$

$$= \sum_{j \pmod{q}} \# \{h \pmod{q} : q \mid 1_6 (M_0(j)h + M_1j) \}. $$

However if $M_0$ is a square matrix and $c$ is a constant vector we claim that

$$\# \{h \pmod{q} : q \mid M_0 h + c \} \leq \# \{h \pmod{q} : q \mid M_0 h \}.$$

It will then follow that

$$\tilde{N}(q) \leq \sum_{j \pmod{q}} \# \{h \pmod{q} : q \mid 1_6 M_0(j)h \}$$

$$= \# \{j, h \pmod{q} : q \mid 1_6 M_0(j)h \}.$$

The lemma therefore follows from [5, Lemma 4], since the final expression is just $N(q)$ for the non-singular form $g_0$.

It remains to prove the claim above. If there is no vector $h$ with $q \mid M_0 h + c$ the result is trivial. Otherwise let $h_0$ be any such vector. Then $q \mid M_0 h + c$ if
and only if \( q \mid M_0(h - h_0) \), and the required bound follows. This completes the proof of Lemma 6.

We may now continue with the analysis as in [5, §6], finding, in the notation of [5, page 242], that

\[
|S_1|^2 \leq \tilde{N}(q_3)|S_0| \ll \tilde{N}(q_3)q_1^n T(a),
\]
say, where

\[
S_0 = \sum_{h_3 \pmod{q_4}} e_{q_4}(q_3 a \nabla g_0(h_3) + sa^TM_1h_3) \sum_{h_2 \pmod{q_1}} e_{q_1}(sa^TM_0(h_3)h_2)
\]
with

\[
T(a) = \#\{h_3 \pmod{q_5} : q_5 \mid M_0(a)h_3\}.
\]

Everything now proceeds as before, leading to the following variant of [5, Lemma 14].

**Lemma 7.** There is a positive constant \( A \), such that for any integer vector \( \mathbf{v}_0 \) we have

\[
\sum_{|v - \mathbf{v}_0| \leq V} |S_1(v; \mathbf{v})| \leq A^{\omega(q)}(\log(q + 1))^{2n}q^{n/2 + 1}(V^n + q^{n/3}),
\]
uniformly in \( \mathbf{v}_0 \), whenever \( q \) is square-full.

The effect of introducing \( \mathbf{v}_0 \) into the analysis of [5, §6] is to modify the sum \( S_2 \) which occurs there. However the same estimate for \( S_2 \) still holds, and the proof goes through as before.

We shall also want to consider the sum in Lemma 7 with \( u = 0 \) and with \( \mathbf{v} \) restricted by the condition \( g^*(\mathbf{v}) = 0 \). Here we follow the analysis in [5, §7], with \( h \nabla F(j) \) replaced by \( \frac{1}{2}j^TM(h)j \) throughout, so that \( N(q_3; h) \) becomes \( \tilde{N}(q_3; h) \). With these trivial changes everything goes through as before, up to the treatment of the sum \( S(q) \) defined in [5, (7.3)]. Let

\[
q_1 = \prod_{p^n \mid q} p^{u/2}, \quad q_2 = \prod_{p^n \mid q} p, \quad \text{and} \quad q_4 = \prod_{2 \mid a, u \geq 13} p^n.
\]

In the present setting we will only be able to establish that if \( q \) is square-full then

\[
S(q) \ll q_1^{n-1+\varepsilon} q_2^{1/2} q_4^{1/2}.
\]

when \( q \) is square-full. The corresponding bound in [5] is somewhat sharper, in that the factor \( q_4^{1/2} \) is absent. We shall prove (4.1) in a moment, but first we show how it suffices for our purposes. Inserting (4.1) into [5, (7.3)], an application of [5, Lemma 15] reveals that

\[
\sum_{|v| \leq V \atop g^*(v)=0} |S_1(v; \mathbf{v})| \ll q_1^{2n+1+\varepsilon} q_2^{(n+3)/2} q_4^{1/2} (1 + V q_1^{-1})^{n-3/2} \log(V + 1)
\]

whenever \( q \) is square-full. In order to establish the analogue of [5, Lemma 16] we are now left with a parallel calculation to the three lines at the bottom of [5, page 245]. Note that \( q_1 \leq q_2 \leq q_1 \) and \( q = q_1 q_2 \). Hence

\[
q_1^{2n+1} q_2^{(n+3)/2} q_4^{1/2} \leq q_1^{2n+1} q_2^{n/2+2} q_4^{1/2} \leq q^{n+1/2}.
\]
Similarly, we have
\[ q_1^{n+5/2} q_2^{(n+3)/2} q_4^{1/2} \leq (q_1^2 q_2)^{(2n+5)/4} q_2^{1/4} q_4^{1/2}. \]
Thus it remains to confirm that \( q_2^{1/4} q_4^{1/2} \leq q^{1/12} \), which it suffices to verify at each prime power \( q = p^e \). This is trivial when \( e \) is even since then \( q_2 = q_4 = 1 \). When \( e = 2f + 1 \) is odd we have
\[ q_2^{1/4} q_4^{1/2} = \begin{cases} p^{1/4}, & \text{if } f \leq 5, \\ p^{3/4}, & \text{if } f \geq 6, \end{cases} \]
which is always at most \( p^{(2f+1)/12} \). Assuming the validity of (4.1), this therefore yields the following result, corresponding to [5, Lemma 16].

**Lemma 8.** We have
\[ \sum_{|v| \leq V} |S_0(q; v)| \ll q^e \left( q^{n+1/2} + V^{n-3/2} q^{n/2+4/3} \right) \log(V + 1), \]
whenever \( q \) is square-full.

It remains to establish (4.1). Let \( v_0 \in \mathbb{Z}^n \) be an arbitrary vector. Taking \( V = q^{1/3} \), it follows from an application of Lemma 7 that
\[ |S_0(q; v_0)| \leq \sum_{|v-v_0| \leq V} |S_0(q; v)| \ll q^{5n/6 + 1 + \varepsilon}, \]
when \( q \) is square-full. Furthermore, the implied constant in this estimate does not depend on \( v_0 \). Arguing as in [5] one easily checks that \( S(q) \) is multiplicative in \( q \), whence it suffices to estimate \( S(p^e) \) for \( e \geq 2 \). When \( e \) is even, so that \( q_3 = 1 \), the argument based on exponential sums in [5, page 245] goes through with no changes. This yields
\[ S(p^{2f}) = S_0(p^f) \ll p^{f(n-1)}, \]
for any \( f \geq 1 \). Indeed, once combined with Lemma 4, the bound in (4.2) gives \( S_0(p^g; 0) \ll p^{g(5n/6 + 1 + \varepsilon)} \) for any \( g \geq 1 \). This argument also takes care of the finitely many primes \( p \) for which \( p \mid 6 \) or for which the reduction of \( g_0 \) modulo \( p \) is singular, since in these cases we have \( \tilde{N}(p; h) \ll 1 \).

Turning to the case of odd \( e \geq 2 \), we suppose that \( e = 2f + 1 \) and that \( g_0 \) is non-singular modulo \( p \), with \( p \nmid 6 \). Thus \( q_1 = p^f \) and \( q_3 = p \), and it follows that
\[ S(p^{f}) = \sum_{k (\text{mod } p) \atop p \nmid g(k)} \tilde{N}(p; k)^{1/2} M(p^f; k), \]
where
\[ M(p^f; k) = \# \{ h (\text{mod } p^f) : p \mid h - k, \ p^f \mid g(h) \}. \]
When \( p \nmid \nabla g(k) \) a straightforward argument based on Hensel’s lemma reveals that \( M(p^f; k) \ll p^{(f-1)(n-1)} \). Thus the overall contribution from such \( k \) is
\[ \ll p^{(f-1)(n-1)} \sum_{k (\text{mod } p) \atop p \nmid g(k)} \tilde{N}(p; k)^{1/2} \ll p^{f(n-1) + 1/2}, \]
by an application of Cauchy’s inequality and Lemma 6. To handle the contribution from the remaining $k$, we observe that there can only be $O(1)$ values of $k$ modulo $p$ for which $p \mid g(k)$ and $p \mid \nabla g(k)$. This follows from the fact that the corresponding projectivised variety has dimension 0, as we saw in our analysis of $V$ in the proof of Lemma 4. Taking $\tilde{N}(p; k)^{1/2} \leq p^{n/2}$, and incorporating our work above, we deduce that

$$S(p^e) \ll p^{f(n-1)+1/2} + p^{n/2} \max_k M(p^f; k)$$

for $e = 2f + 1$, where the maximum is taken over all $k$ modulo $p$ such that $p \mid g(k)$ and $p \mid \nabla g(k)$. We will show that

$$\max_k M(p^f; k) \ll p^{f(n-1+\varepsilon)+5-n+\theta_p(e)}, \quad (4.3)$$

where $\theta_p(e) = 1$ if $e = 2f + 1$ with $f \geq 6$, and $\theta_p(e) = 0$ otherwise. Once inserted into our bound for $S(p^e)$ this implies that

$$S(p^e) \ll p^{f(n-1+\varepsilon)}(p^{1/2} + p^{5-n/2+\theta_p(e)}) \ll p^{f(n-1+\varepsilon)+1/2+\theta_p(e)/2},$$

since $n \geq 10$. In view of the fact that $\theta_p(e) = 0$ unless $e = 2f + 1 \geq 13$, this is therefore enough to complete the proof of (4.1).

We will use exponential sums to estimate $M(p^f) = M(p^f; k)$. Thus we find that

$$M(p^f) = \frac{1}{p^{f+n}} \sum_{s \pmod{p^f}} \sum_{j \pmod{p}} \sum_{h \pmod{p^f}} e((sg(h)/p^f) + j(h-k)/p)$$

$$= \frac{1}{p^{f+n}} \sum_{0 \leq g \leq f} \sum_{t \pmod{p^g}} \sum_{j \pmod{p}} \sum_{h \pmod{p^f}} e_{p^g}(tg(h) + p^{g-1}j(h-k)),$$

on splitting $s$ according to the value of the highest common factor $p^{f-g}$ of $s$ with $p^f$. Fix a choice of $\ell$, with $1 \leq \ell \leq f$. Let us write $M_1(p^f)$ for the contribution to $M(p^f)$ from values of $g \leq \ell$, and $M_2(p^f)$ for the corresponding contribution from values of $g > \ell$. Beginning with small values of $g$, we reverse the process above to deduce that

$$M_1(p^f) = \frac{1}{p^{f}} \sum_{0 \leq g \leq \ell} \sum_{t \pmod{p^g}} \sum_{h \pmod{p^f}} e_{p^g}(tg(h))$$

$$= \frac{1}{p^{f}} \sum_{0 \leq g \leq \ell} p^{fn} (p^{(1-n)g} M(p^g) - p^{(1-n)(g-1)} M(p^{g-1}))$$

$$= p^{f(n-1)} p^{(1-n)} M(p^f).$$

Here we have followed the convention that $M(p^{g-1}) = 0$ when $g = 0$. Employing the crude upper bound $M(p^f) \ll p^{(f-1)n}$, we deduce that

$$M_1(p^f) \ll p^{f(n-1)+\ell-n}. \quad (4.4)$$
To produce a bound for $M_2(p^f)$, we apply (4.2) to deduce that
\[ \sum_{t \pmod{p^g}} \sum_{h \pmod{p^f}} e^{p^g} (t g(h) + p^{g-1} j, h) = p^{(f-g)n} S_0(p^g, -p^{g-1} j) \]
\[ \ll p^{fn+g(1-n/6+\varepsilon)} \]
for each $g$ and $j$. Hence
\[ M_2(p^f) \ll \frac{1}{p^{f+n}} \sum_{\ell < g \leq f \pmod{p}} \sum_{\ell < g \leq f} p^{fn+g(1-n/6+\varepsilon)} \]
\[ \ll p^{fn+g(1-n/6+\varepsilon)} \]
\[ \ll p^{fn+g(1-n/6+\varepsilon)} - \ell(n/6-1). \]
When $f \geq 6$, we take $\ell = 6$ and combine the above estimate with (4.4) to conclude that
\[ M(p^f) = M_1(p^f) + M_2(p^f) \ll p^{(n-1+\varepsilon)+6-n} = p^{f(n-1+\varepsilon)+5-n+\delta_p(\varepsilon)}. \]
This is satisfactory for (4.3). Alternatively, when $f \leq 5$ we apply (4.4) with $\ell = f$ to deduce that
\[ M(p^f) = M_1(p^f) \ll p^{(n-1+\varepsilon)+5-n}. \]
This too is satisfactory for (4.3), and so completes its proof.

5. Proof of Theorem 2

We now estimate the various terms in (2.3), just as in [5, §§8, 9 and 10], finding that (2.2) holds with
\[ \mathcal{S}(g) = \sum_{q=1}^{\infty} q^{-n} S_0(q; 0), \]
\[ \mathcal{J}(g; P) = \int_{-1}^{1} I(z; 0) dz, \]
and any fixed $\delta < 1/18$. Note that the only difference arises from the additional term $P(\log P)^{7n}$ appearing in the estimate for $I(z; \beta)$ in Lemma 1. It is a trivial matter to check that this term makes a satisfactory overall contribution to $N(g; P)$. It therefore remains to show that $\mathcal{S}(g)$ is strictly positive, and that
\[ P^{n-3} (\log P)^{2-2n} \ll \mathcal{J}(g; P) \ll P^{n-3} (\log P)^{2-2n}. \quad (5.1) \]

For the first task it suffices, as usual, to show that $g(x) = 0$ has a non-singular $p$-adic integer zero for every prime $p$. This problem is discussed by Davenport and Lewis [2, §2], where it is shown that if the Congruence Condition holds, then non-singular $p$-adic solutions exist, except possibly in Case 3b of [2, §2]. In the excluded case there is a non-singular matrix $M$ and a positive integer $r \leq 4$ such that
\[ g(Mx) = x_1 R_1(x_1, \ldots, x_n) + \cdots + x_r R_r(x_1, \ldots, x_n) + R(x_1, \ldots, x_r) \]
for certain quadratic forms \(R_1, \ldots, R_r\) and \(R\). If \(g\) has such a representation then

\[ g_0(Mx) = x_1R_1(x_1, \ldots, x_n) + \cdots + x_RR_r(x_1, \ldots, x_n), \]

whence \(\nabla g_0(Mx)\) vanishes whenever \(x_1, \ldots, x_r\) and \(R_1, \ldots, R_r\) all vanish. This produces a subset of the singular locus having projective dimension at least \(n - 1 - 2r \geq n - 9\). Hence if \(s(g_0) < n - 9\), and in particular if \(s(g_0) = -1\) and \(n \geq 10\), then there will be non-singular \(p\)-adic points whenever the Congruence Condition holds.

We turn now to the singular integral \(I(g; P)\). It follows from Lemma 1 that

\[ \int_{-1}^{1} I(z; 0) \, dz = \int_{-P^{-11/4}}^{P^{-11/4}} I(z; 0) \, dz + O(P^{7(n-3)/8}(\log P)^{7n}). \]

Moreover \(g(x) = g_0(x) + O(|x|^2) + O(1)\), whence

\[ e(zg_0(x)) = e(zg_0(x)) + O(|z||x|^2) + O(|z|). \]

It therefore follows that

\[ I(z; 0) = J(z) + O\left( \int w(x)|z||x|^2 + 1 \, dx \right) = J(z) + O(|z|P^{n+2}), \]

where

\[ J(z) := \int w(x)e(zg_0(x)) \, dx. \]

We now see that

\[ \int_{-1}^{1} I(z; 0) \, dz = \int_{-P^{-11/4}}^{P^{-11/4}} J(z) \, dz + O(P^{7(n-3)/8}(\log P)^{7n}) + O(P^{n-7/2}). \]

However Lemma 1 also applies to \(J(z)\), whence

\[ \int_{-P^{-11/4}}^{P^{-11/4}} J(z) \, dz = \int_{-1}^{1} J(z) \, dz + O(P^{7(n-3)/8}(\log P)^{7n}). \]

Finally, from [5, (10.3) and (10.4)] we see that

\[ P^{n-3}(\log P)^{2-2n} \ll \int_{-1}^{1} J(z) \, dz \ll P^{n-3}(\log P)^{2-2n}. \]

This therefore establishes (5.1), providing that \(n \geq 4\), which thereby completes the proof of Theorem 2.

6. Proof of Theorem 1: Hyperplane Sections

It remains to prove Theorem 1, which will be achieved by induction on \(s = s(g_0)\). The base case will be \(s = -1\), which follows from Theorem 2. For the induction we will find a non-degenerate affine hyperplane section of \(g = 0\) which again satisfies the Congruence Condition, and for which \(s\) is reduced by 1.

We begin by applying Bertini’s Theorem (see Harris [4, Theorem 17.16], for example) to show that, for a generic vector \(a\), the singular locus of the projective hyperplane section \(g_0(x) = a \cdot x = 0\) has dimension \(s - 1\). Similarly by Harris [4, Proposition 18.10], for generic \(a\) the intersection will be non-degenerate. Thus there is a non-zero form \(f\) say, such that the dimension is \(s - 1\), and the intersection is non-degenerate, whenever \(f(a) \neq 0\). Choose \(a\)
to be any primitive integer vector such that \( f(a) \neq 0 \), whence the singular locus of \( g_0(x) = ax = 0 \) will have dimension \( s - 1 \). The affine hyperplane section we seek will then take the form \( a.x = c \) for a suitably chosen integer \( c \). We can find a matrix \( M \in \text{SL}_n(\mathbb{Z}) \) whose first row is \( a \). If we then write \( g \) in terms of \( y := Mx \), by setting \( h(y) = g(M^{-1}y) \), it follows that the singular locus of \( h_0(y) = y_1 = 0 \) will have dimension \( s - 1 \). Thus, irrespective of the value of \( c \), if we put \( h^*(y_2, \ldots, y_n) = h(c, y_2, \ldots, y_n) \) then \( s(h_0^*) = s - 1 \).

It is clear that distinct integer solutions \( (y_2, \ldots, y_n) \) of \( h^*(y_2, \ldots, y_n) = 0 \), for some value of \( c \), produce distinct solutions of \( g(x) = 0 \).

To complete our induction it therefore suffices to show that there is an integer \( c \) for which \( h^* \) satisfies the Congruence Condition. It will be convenient to use the notation \( u = (y_2, \ldots, y_n) \). We begin by proving the following result.

**Lemma 9.** Suppose \( h(y_1, \ldots, y_n) \in \mathbb{Z}[x_1, \ldots, x_n] \) is a cubic polynomial with \( n \geq 4 + s(h_0) \). Then there is an integer \( p(h) \) depending on \( h \) such that for every prime \( p \geq p(h) \) and every integer \( c \) the congruence

\[
h^*(u) \equiv 0 \pmod{p}
\]

has a non-singular solution modulo \( p \).

By Hensel’s lemma, once we have a non-singular solution modulo \( p \) we will have solutions modulo \( p^k \) for every \( k \).

For the proof we define

\[
H^*(t, u) := t^3h^*(t^{-1}u), \quad H_1(u) := h_0^*(u) = H^*(0, u).
\]

It will be important to observe that \( H_1 \) is independent of \( c \), and that, by construction, \( s(H_1) = s(h_0) - 1 \). We proceed to estimate the number \( N \), say, of solutions to the congruence \( h^*(u) \equiv 0 \pmod{p} \). We have \( N = (N_1 - N_2)/(p - 1) \), where \( N_1 \) counts solutions of

\[
H^*(t, u) \equiv 0 \pmod{p}
\]

and \( N_2 \) counts solutions of \( H_1(u) \equiv 0 \pmod{p} \). For a form \( F \) we shall write \( s_p(F) \) to denote the dimension of the singular locus of \( F = 0 \) over \( \mathbb{F}_p \). If \( p \) is sufficiently large then \( s_p(H_1) = s(H_1) \), where “sufficiently large” will be independent of \( c \), since \( H_1 \) is independent of \( c \). Taking hyperplane sections we can change the dimension of the singular locus by at most one, whence \( s_p(H^*) \leq 1 + s_p(H_1) = s(h_0) \) for large enough \( p \). Thus Lemma 3 yields \( N_1 = p^{n-1} + O_n(p^{n-1+s(h_0)})/2 \) and \( N_2 = p^{n-2} + O_n(p^{n-1+s(h_0)})/2 \). Since \( s(h_0) \leq n - 4 \), by the hypothesis for Lemma 9, these bounds are enough to ensure that \( N \gg_n p^{n-2} \) for large enough \( p \).

To complete our treatment of “large” primes we estimate the number, \( S \) say, of singular solutions to \( h^*(u) \equiv 0 \pmod{p} \). Clearly \( S \leq S_1/(p - 1) \) where \( S_1 \) is the number of solutions to

\[
H^* = \frac{\partial H^*}{\partial u_1} = \ldots = \frac{\partial H^*}{\partial u_{n-1}} = 0
\]

in \( \mathbb{F}_p \). Suppose these equations define a variety \( V \) say in projective space, and consider the variety \( W \) defined by \( \partial H^*/\partial t = 0 \). Clearly \( W \) has codimension at most 1, and \( V \cap W \) is the singular locus of \( H^* \). Thus \( \dim(V \cap W) = \ldots = \ldots = 0 \).
with symmetric coefficients $c_{i,j,k}$, if we deduce that $M$ follows that $N > S$ for large enough $p$, whence $h^{(c)}(u) \equiv 0 \pmod{p}$ has a non-singular solution, as claimed. This completes the proof of Lemma 9.

We now know that $h^{(c)}$ satisfies the Congruence Condition for $p \geq p(h)$ for every integer $c$. To complete our argument we proceed to choose $c$ so that the condition is satisfied for the remaining small primes. Now we saw in §5 that if the Congruence Condition holds for $g$ then there will in fact be non-singular $p$-adic solutions, if $s = s(g_0) < n - 9$. Thus, under the hypotheses of Theorem 1, we may assume that there is a non-singular $p$-adic solution of $g = 0$ for each $p$. It then follows that $h(y) = 0$ has a non-singular $p$-adic solution $y_0$, say. We will need to know that there must be a solution with $\nabla'h(y) \neq 0$, where

$$\nabla'h(y) := \left( \frac{\partial h}{\partial y_1}, \ldots, \frac{\partial h}{\partial y_n} \right).$$

This is the content of the following result.

**Lemma 10.** Suppose $h(y_1, \ldots, y_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$ is a cubic polynomial with $h_0$ absolutely irreducible. If $h(y) = 0$ has a non-singular solution in $\mathbb{Z}_p$ then there is a solution with $\nabla'h(y) \neq 0$.

We argue by contradiction. Suppose that $\nabla'h(y_0) = 0$ for every non-singular $p$-adic solution $h(y_0) = 0$. Let $w$ be a $p$-adic integer vector with $w_1 = 0$. Then if $|w|_p$ is sufficiently small, Hensel’s Lemma shows that there is a non-singular solution $h(y_0 + w + z) = 0$ with $|z|_p \ll |w|_p^2$. We supposedly have $\nabla'h(y_0 + w + z) = 0$. However if $M = M(y_0)$ is the matrix of second derivatives of $h$ at $y_0$, then

$$\nabla h(y_0 + w + z) = \nabla h(y_0) + Mw + O(|w|_p^2).$$

(By this, we mean that one can replace the error term by a vector whose $p$-adic norm is $O(|w|_p^2)$.) Since $\nabla'h(y_0 + w + z) = \nabla'h(y_0) = 0$ we deduce that $(Mw)_i = O(|w|_p^2)$ for $2 \leq i \leq n$, whenever $w_1 = 0$ and $|w|_p$ is small enough. Since $w$ is arbitrary subject to these restrictions it follows that $M_{ij}(y_0) = 0$ whenever $2 \leq i, j \leq n$.

However $y_0$ was an arbitrary non-singular solution of $h(y_0) = 0$. It therefore follows that $M_{ij}(y) = 0$ for $2 \leq i, j \leq n$, for every non-singular solution of $h(y) = 0$. We may therefore repeat our argument. Since $h(y_0 + w + z) = 0$, we deduce that $M_{ij}(y_0 + w + z) = 0$. This time we have

$$M_{ij}(y_0 + w + z) = M_{ij}(y_0) + 6 \sum_{k=1}^n c_{i,j,k}w_k + O(|w|_p^2),$$

if

$$h_0(y) = \sum_{i,j,k=1}^n c_{i,j,k}y_iy_jy_k$$

with symmetric coefficients $c_{i,j,k}$. Arguing as before we deduce that

$$\sum_{k=1}^n c_{i,j,k}w_k = O(|w|_p^2), \quad (2 \leq i, j \leq n),$$
whenever $w_1 = 0$ and $|w_p|$ is small enough. This allows us to conclude that $c_{ijk} = 0$ for $2 \leq i, j, k \leq n$. It therefore follows that $y_1$ divides $h_0(y)$ identically. We have finally reached a contradiction, and the lemma follows.

In our situation, if $h_0$ were reducible, we would have $s(h_0) \geq n - 3$, which is contrary to hypothesis. Lemma 10 therefore implies that for each $p < p(h)$ we can find a $p$-adic integer vector $y^{(p)}$ with $h(y^{(p)}) = 0$ and $\nabla' h(y^{(p)}) \neq 0$. Suppose that the exponent $k(p)$ satisfies $p^{k(p)} \mid \nabla' h(y^{(p)})$ but $p^{k(p)+1} \nmid \nabla' h(y^{(p)})$, and choose a vector $z^{(p)} \in \mathbb{Z}^n$ with $z^{(p)} \equiv y^{(p)} \pmod{p^{2k(p)+1}}$. We define

$$u^{(p)} = (z_2^{(p)}, \ldots, z_n^{(p)}).$$

Finally let $c$ satisfy $c \equiv z_1^{(p)} \pmod{p^{2k(p)+1}}$ for every prime $p < p(h)$. Such a $c$ exists, by the Chinese Remainder Theorem. Then

$$h^{(c)}(u^{(p)}) = h(c, u^{(p)}) \equiv h(z_1^{(p)}, u^{(p)}) \pmod{p^{2k(p)+1}}$$

$$= h(z^{(p)})$$

$$\equiv h(y^{(p)}) \pmod{p^{2k(p)+1}}$$

$$\equiv 0 \pmod{p^{2k(p)+1}},$$

while

$$\nabla h^{(c)}(u^{(p)}) = \nabla h(c, u^{(p)}) \equiv \nabla h(z^{(p)}) \equiv \nabla h(y^{(p)}) \neq 0 \pmod{p^{k(p)+1}}.$$ 

It follows that the vector $u^{(p)}$ can be lifted to a non-singular $p$-adic solution of $h^{(c)}(u) = 0$. This establishes the Congruence Condition for $h^{(c)}$ for every prime $p < p(h)$, thereby completing the proof of Theorem 1.

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