Fulde–Ferrell–Larkin–Ovchinnikov State in Perpendicular Magnetic Fields in Strongly Pauli-Limited Quasi-Two-Dimensional Superconductors

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We examine the Fermi-surface effect called the nesting effect for the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) state in strongly Pauli-limited quasi-two-dimensional superconductors, focusing on the effect of three-dimensional factors, such as interlayer electron transfer, interlayer pairing, and off-plane magnetic fields including those perpendicular to the most conductive layers (hereinafter called the perpendicular fields). It is known that the nesting effect for the FFLO state can be strong in quasi-low-dimensional systems in which the orbital pair-breaking effect is suppressed by applying the magnetic field parallel to the layers. Hence, it has sometimes been suggested that it may not work for perpendicular fields. We illustrate that, contrary to this view, the nesting effect can strongly stabilize the FFLO state for perpendicular fields as well as for parallel fields when $t_z$ is small so that the Fermi surfaces are open in the $k_z$-direction, where $t_z$ denotes the interlayer transfer energy. In particular, the nesting effect in perpendicular fields can be strong in interlayer states. For example, in systems with cylindrical Fermi surfaces warped by $t_z \neq 0$, interlayer states with $\Delta_{qz} \propto \sin k_z$ exhibit $\mu_s H_c^z \approx 1.65A_{1qz}$ for perpendicular fields, which is much larger than typical values for parallel fields, such as $\mu_s H_c = \Delta_{qz}$ of the s-wave state and $\mu_s H_c \approx 1.28A_{1qz}$ of the d-wave state in cylindrical systems with $t_z = 0$. Here, $\mu_s$ and $H_c$ are the electron magnetic moment and upper critical field of the FFLO state at $T = 0$, respectively, and $\Delta_{qz} \equiv 2\Delta_{qz}e^{|1/t_z|}$. We discuss the possible relevance of the nesting effect to the high-field superconducting phases in perpendicular fields observed in the compounds CeCoIn$_5$ and FeSe, which are candidates for the FFLO state.

1. Introduction

The Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) state has been examined in quasi-two-dimensional systems for candidate compounds, such as some organic and heavy fermion superconductors. In many cases in theoretical studies, two-dimensional models are useful as effective models of quasi-two-dimensional systems; however, depending on the problem under examination, the three-dimensionality of the systems due to interlayer electron transfer must be treated explicitly, for example, when interlayer pairing and/or off-plane magnetic fields play essential roles.

Since the FFLO state is a superconducting state induced by Cooper pairs with a finite center-of-mass momentum $q$, (hereinafter called the FFLO vector), it significantly depends on the Fermi-surface structure, for example, because of the direction of $q$ relative to the anisotropic Fermi surface. At the same time, unless the orbital pair-breaking effect is too weak or too strong, the direction of $q$ is locked in the direction of the magnetic field $H$; i.e., $q \parallel H$. From this point of view, it is physically interesting that all the candidates discovered thus far are quasi-low-dimensional, which can be attributed to the following two reasons: (i) the suppression of the orbital pair-breaking effect when $H$ is parallel to the most conductive layers (hereinafter called the ab-plane) and (ii) the Fermi-surface nesting effect for the FFLO state.

For two of the strongest candidate compounds, CeCoIn$_5$ and FeSe, reason (i) would not apply, because the conduction electrons have large effective masses, which result in large Maki parameters; hence, $H$ would not need to be parallel to the ab-plane for the emergence of the FFLO state. In fact, high-field superconducting phases, which can be considered to be the FFLO state, were observed in these compounds when $H \parallel c$ as well as when $H \perp c$, where $a$, $b$, and $c$ denote the lattice vectors in the directions of the crystal $a$-, $b$-, and $c$-axes.

The Fermi-surface nesting effect for the FFLO state mentioned in reason (ii) is an effect analogous to the nesting effect for spin- and charge-density waves (SDW and CDW). In quasi-low-dimensional systems, when nesting instabilities such as SDW and CDW instabilities are suppressed by sufficient distortion of the Fermi surfaces, the highly anisotropic Fermi-surface structures help stabilize the FFLO state. In the absence of the orbital effect for the FFLO state, one can examine the effect by considering the overlap of one of the Fermi surfaces of the up- and down-spin electrons and another that is shifted by $q$. In a simple two-dimensional model in which the interlayer electron transfer is neglected, the Fermi surfaces can touch on a vertical line for $H \perp c$ because $q \parallel H$ as mentioned above, whereas for $H \parallel c$, they cannot touch each other. Hence, it may be thought that the nesting effect does not work when $H \parallel c$ in quasi-two-dimensional systems. For example, in CeCoIn$_5$ and FeSe, the high-field phases for $H \parallel c$ may appear to be inconsistent with reason (ii) as well as with reason (i). However, as shown in the following, the nesting effect can work even when $H \parallel c$, in the presence of the warp of the Fermi surfaces in the direction of $c$.

Motivated by these experimental and theoretical studies, we examine the effects of three-dimensional factors, such as interlayer electron transfer, interlayer pairing, and off-plane magnetic fields including perpendicular fields. Magnetic fields can cause extreme phenomena, such as vortex states with higher Landau-level indices, through the orbital pair-breaking effect. However, because the orbital effect has been examined in previous studies, in the present study we focus on the nesting effect in the absence of the orbital effect. We also incorporate the possibility of interlayer pairing because, owing to the finite $q$, strong interplay...
between the three-dimensional structures of the gap function and the Fermi surface is expected. In addition, the FFLO state is worth studying for interlayer pairing because of the compound Sr2RuO4, for which both the FFLO state28) and interlayer pairing29,30) can be considered.

For quasi-two-dimensional systems in perpendicular fields, Song and Koshelev proposed a theory of interplay between orbital-quantization effects and the FFLO state, and discussed the FFLO state in FeSe when \( H \parallel c \).31) In the present study, we assume weaker magnetic fields.

In Sect. 2, we briefly review formulas and define the systems and states to be examined. In Sect. 3, we examine systems with Fermi surfaces straight in the \( k_c \)-direction. In Sect. 4, we examine systems with warped Fermi surfaces. We discuss the possible relevance of our results to the high-field phases in the compounds CeCoIn5 and FeSe. The final section summarizes and concludes the paper. We define the \( x \)-, \( y \)-, and \( z \)-axes along the crystal \( a \)-, \( b \)-, and \( c \)-axes. The lattice constants \( a \), \( b \), and \( c \) are absorbed into the definitions of the momentum components \( k_x \), \( k_y \), and \( k_z \). We use unit cells where \( h = k_B = 1 \), and we denote the electron magnetic moment by \( \mu_e = g \mu_B / 2 \). For convenience, we define the functions

\[
f_o(q) = -\int_0^\infty \frac{dx}{\pi} \cos(\xi x) \ln|1 - p \cos x|, \tag{1}
\]

Their explicit forms are shown in Appendix A.

2. Formulas and Model

Formula for the Critical Field — We use the formula32) for the upper critical field at \( T = 0 \)

\[
h_c = \mu_e H_c = \frac{1}{2} \frac{\Delta_0}{\gamma} \max_q \left[ \frac{\epsilon(q)}{\gamma} \right] \tag{2}
\]

with

\[
f_\alpha(q) = -\frac{1}{(\gamma_\alpha(\hat{k}))^2} \left( \gamma_\alpha(\hat{k})^2 \ln|1 - \frac{\mu_e \cdot q}{2 \hbar c}| \right) \tag{3}
\]

for the \( \alpha \)-wave state with the gap function \( \Delta_\alpha = \gamma_\alpha \gamma_{\alpha}(\hat{k}) \), where \( \gamma_\alpha \) denotes a basis function of \( \hat{k} = k / |k| \), and \( \Delta_\alpha = 2 \Delta_0 e^{-1/k_z} \) is a scale of the gap function.33) The average is defined as

\[
\langle g(\hat{k}) \rangle = \int d^2 \hat{k} \frac{\rho(0, \hat{k})}{S_0} \langle N(0) \rangle g(\hat{k}) \tag{4}
\]

for an arbitrary \( g(\hat{k}) \), where \( \rho(\xi, \hat{k}) \) is the angle-dependent density of states, \( N(\xi) \) is the density of states, and \( S_0 \) is an appropriate normalization constant.33) The FFLO vector is the vector \( \alpha \) that gives the highest \( h_c \), in accordance with the variational principle,31) however, only the magnitude \( q \equiv |q| \) is optimized in Eq. (2) because \( q \parallel H \) in the present problem. The formula in Eq. (2) is derived in the weak coupling theory,32,33) where a second-order transition is assumed. If the second-order transition at \( h_c \) is to occur, \( h_c \) must exceed the Pauli limit \( h_P = \mu_e H_P \).36)

The Pauli paramagnetic limit \( h_P \) in anisotropic superconductors is given by the formula32,33)

\[
h_P = \frac{\sqrt{\gamma_\alpha^2 + \Delta_0 \gamma_\alpha}}{\gamma} \frac{\Delta_0}{\sqrt{2}}. \tag{5}
\]

where \( \tilde{\gamma}_\alpha = \exp(\gamma_\alpha^2 \ln \gamma_\alpha / (\gamma_\alpha^2)) \). The inequality

\[
h_P \leq \frac{\Delta_0}{\sqrt{2}} \tag{5}
\]

can be proved in general as shown in Appendix B. The equality sign holds if and only if \( \gamma_\alpha(\hat{k}) \) is constant.

The real upper critical field would be smaller than the value of \( h_c \) given by Eq. (2) owing to negative effects, such as the orbital pair-breaking effect and the fluctuation effect. However, the value of \( h_c \) is useful because a higher \( h_c \) must imply that the free energy of the FFLO state at \( h \equiv \mu_e H \sim \Delta_0 \) is lower. The value of \( h_c \) can be regarded as an index of the strength of the positive effects that stabilize the FFLO state.

Systems and States to be Examined — In the following, we consider a three-dimensional structure of the order parameter by assuming

\[
\gamma_\alpha(\hat{k}) = \gamma_\alpha(\varphi, k_c) = \gamma_\alpha(\varphi) \gamma_{\alpha}(k_c) \tag{6}
\]

with \( \alpha = (\alpha_s, \alpha_c) \), where \( \gamma_\alpha(\varphi) \) and \( \gamma_{\alpha}(k_c) \) are basis functions that satisfy

\[
\int_0^{2\pi} \frac{d\varphi}{2\pi} |\gamma_\alpha(\varphi)|^2 = \int_0^{2\pi} \frac{dk_c}{2\pi} |\gamma_{\alpha}(k_c)|^2 = 1. \tag{7}
\]

We define the one-particle (electron or hole) energy

\[
e_i(k) = \epsilon_i(k) - 2t_c \cos k_c - \mu, \tag{8}
\]

where

\[
e_i(\hat{k}) = \frac{k^2}{2m} + \frac{k_c^2}{2m_c} \tag{9}
\]

in the following for conciseness.

The present model can be effective for systems in which the electron (or hole) density is small. For an arbitrary \( \epsilon_i(\hat{k}) \), redefining the momentum coordinate appropriately and, if necessary, making the electron-hole transformation, we can assume that the minimum of \( \epsilon_i(\hat{k}) \) is at \( \hat{k} = (0, 0) \). Expanding \( \epsilon_i(\hat{k}) \) around \( \hat{k} = (0, 0) \) and redefining the \( k_x - k_y - k_z \) axes along the principal axes, we obtain Eq. (7) as an approximate form when the carrier density is small.38)

3. Systems with Straight Fermi Surfaces

Before examining systems with \( t_z \neq 0 \), let us examine systems with \( t_z = 0 \). When \( t_z = 0 \), the Fermi surfaces are straight (elliptic) cylinders. The cylindrical system has been examined in previous studies when the magnetic field is parallel to the layers; however, the study in this section covers wider situations, including an arbitrary direction of \( H \) except for the \( c \)-direction, interlayer pairing as well as intralayer pairing, and systems with effective mass anisotropy \( m_x \neq m_y \). Equation (3) reduces to

\[
f_\alpha(q) = -\int_0^{2\pi} \frac{d\varphi}{2\pi} |\gamma_{\alpha}(\varphi)|^2 \ln|1 - \tilde{q} \cos \varphi|, \tag{10}
\]
where $\tilde{q} = v_F q_1/2h_c$, $v_F = k_F^2/m$, $q_1 = |q_1|$, and $q_1 = (q_z, q_y)$. Here, $\gamma_0(h_c)$ has disappeared from the equation; hence, the argument in this section does not depend on $\alpha_z$.

First, we examine the states in which the symmetry of $\Delta_k$ is s-wave in each layer and arbitrary in the $k_z$-direction. In such states, $\gamma_0(h_c) = 1$, and hence, $f_0(q) = f_0(\tilde{q})$; i.e.,

$$f_0(q) = -\int_0^{2\pi} \frac{d\theta}{2\pi} \ln|1 - \tilde{q} \cos \theta|.
$$

Therefore, we obtain

$$f_0(q) = \begin{cases} -\ln(\tilde{q}/2) & \text{for } \tilde{q} \geq 1, \\ -\ln\left[1 + (1 - \tilde{q}^2)^{1/2}\right] & \text{for } \tilde{q} \leq 1, \end{cases}
$$

which is the same as the equation in the previous paper except for the definition of $\tilde{q}$ and the extended applicability mentioned above. From Eq. (2), it follows that $h_c = \Delta_{00}$ and $\tilde{q} = 1$. This result holds for an arbitrary symmetry in the $k_z$-direction and an arbitrary magnetic-field direction except for $H \parallel c$.

The fact that $h_c = \Delta_{00}/\mu_c$ is much larger than the Pauli limit $H_F \leq V\mu_c h_c$ can be attributed to the nesting effect. As shown in Fig. 1, the Fermi surfaces touch on a line (hereinafter called the nesting line) by a displacing vector $\mathbf{q}_0$ that has $|q_1| = 2h_c/v_F$. The FFLO vector $\mathbf{q}$ obtained above also has the same $q_1$, and the large value of $h_c$ can be attributed to the fact that the Fermi surfaces touch each other.

Next, we examine the states in which the symmetry of $\Delta_k$ is d-wave in each layer and arbitrary in the $k_z$-direction. For such states, we adopt $\gamma_0(\alpha_z) = \sqrt{2}\cos(2\alpha_z)$ as the principal in-plane basis function. As many authors have reported, $h_c$ depends on the direction of $H$, where $H = (H_x, H_z)$. Let $q_0$ denote the nesting vector that is parallel to $H$ and has $q_1 = 2h_c/v_F$. When $H \parallel a$, because $\Delta_k$ is maximum on the nesting line, the FFLO vector $\mathbf{q}$ is equal to $\mathbf{q}_0$, as in the s-wave state. By contrast, when $H \parallel [1,1,0]$, because $\Delta_k$ vanishes on the nesting line, we obtain $\mathbf{q} \neq \mathbf{q}_0$. This is shown by an explicit calculation as outlined below. It can be easily verified that

$$f_0(q) = \begin{cases} f_0(q) + f_1(q) & \text{for } H \parallel a, \\ f_0(q) - f_1(q) & \text{for } H \parallel [1,1,0], \end{cases}
$$

and the functions $f_0$ and $f_1$ can be obtained as shown in Appendix A. This result in $h_c$ and $\tilde{q}$ as summarized in Table I. The value $\tilde{q} = 1$ for $H \parallel a$ implies that the Fermi surfaces touch on a line. By contrast, the value $\tilde{q} \approx 1.210 > 1$ for $H \parallel [1,1,0]$ implies that the Fermi surfaces are intersected by two vertical lines. Because of the nesting effect for the FFLO state, $h_c$ for $H \parallel a$ is approximately 30% larger than that for $H \parallel [1,1,0]$.

### Table I. Results for the states with a symmetry that is d-wave in each layer and arbitrary in the $k_z$-direction, when $t_c = 0$ and $H \parallel c$.

| $H_x$ | $H_z$ || $1$, $1$, $0$ |
|-------|-------|----------------|
| Max. of $f_0$ | $\ln 2 + \frac{1}{2}$ | $\frac{1}{2} \ln(\sqrt{3} + 1) + \frac{\sqrt{3} - 1}{4}$ |
| $\tilde{q}$ | $1$ | $\sqrt{2}/(\sqrt{3} - 1)^{1/2} = 1.210$ |
| Fermi surfaces | Touch on a line | Intersected by two lines |
| $h_c/\Delta_{00}$ | $c^{1/4} = 1.284$ | $1/2 \left(\sqrt{3} + 1\right)^{1/2} e^{-\pi/8} = 0.992$ |

### 4. Systems with Warped Fermi Surfaces

In this section, we examine systems with $t_c \neq 0$ in perpendicular fields, i.e., $H \parallel c$, for which $q = (0,0,q)$. We assume that $t_c$ is sufficiently small so that the Fermi surfaces are open in the $k_z$-direction. For the states with $\alpha = (\alpha_1, \alpha_z)$, we obtain

$$f_0(q) = -\int_0^{2\pi} \frac{dk_z}{2\pi} \left|\gamma_0(\alpha_z)\right|^2 \ln|1 - \tilde{q} \sin k_z|.
$$

which is mathematically equivalent to Eq. (10); however, $\cos \theta$ in Eq. (10) originates from the relative angle $\theta$ between $q_1$ and $k_1$, whereas $\sin k_z$ in Eq. (14) originates from the variation of the Fermi velocity in the $k_z$-direction. The same calculation leads to exactly the same equation as Eq. (11) except for the definition of $\tilde{q}$, and hence, we obtain $h_c = \Delta_{00}$ and $\tilde{q} = 1$. Interestingly, the results coincide for the completely different systems, as summarized in Table II.

Although the exact coincidence in $h_c$ is only a consequence of the simplifications of the models, which lead to the mathematical similarity of Eqs. (10) and (14), it is physically significant that $h_c$ is of the same order in the two cases. The physical reason is interpreted using Fig. 2. The split Fermi surfaces shown in Fig. 2(a) touch on a circle (the red dotted curve, hereinafter called the nesting curve) when one of the Fermi surfaces is shifted by $q \parallel c$ as shown in Fig. 2(b). The nesting curve is a full circle (or ellipse) because all the cross sections are circular (or elliptic). In general, when the shapes of the cross sections perpendicular to the $k_z$-axis are the same, the Fermi surfaces touch in a similar manner when one of
them is shifted by the nesting vector $q_0$ ($\parallel e$) appropriate for the Fermi-surface geometry. For this mechanism, the nesting curve is not necessarily closed like a full circle. When parts of the cross sections are similar, the similar effect can work. The quasi-low-dimensionality plays an essential role in the present nesting effect even for perpendicular fields. In Fig. 2(b), the smaller Fermi surface is completely inside the larger Fermi surface. This implies that on the nesting curve, the Fermi surfaces touch but do not cross. The open structure of the Fermi surfaces in the $k_z$-direction favors this behavior because of the presence of the inflection points at $k_z = \pm \pi/2$. In contrast, when $t_z$ is large, the Fermi surfaces are closed and round near the $k_z$-axis, and analogously with the spherical system, $h_c$ is lower than $\Delta_{s0}$. Hence, for the present mechanism of the nesting effect in perpendicular fields, $t_z$ must be sufficiently small. The upper limit of $t_z$ below which the Fermi surfaces are open and touch on a curve decreases as the carrier density decreases.

\begin{figure}[ht]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{(Color online) Similar to Fig. 1, but the Fermi surfaces are warped and $q \parallel e$. (In (a), the Fermi surface of up-spin electrons is shifted by $q_0$ with $q_0 = h_c/t_z$. For this value of $q_0$, the Fermi surfaces touch on the circle shown by the red dotted curve.}
\end{figure}

Interlayer pairing — In this part, we consider the states induced by interlayer pairing (hereinafter called the interlayer states). For the interlayer pairing between electrons on adjacent layers, $\Delta_k$ is proportional to $\gamma_{cz} = \sqrt{2} \cos k_z$ or $\gamma_{sz} = \sqrt{2} \sin k_z$, where the indices $\alpha_z = cz$ and $sz$ are defined. When $t_z \neq 0$ and $H \parallel e$, the nesting curve is at $k_z = \pi/2$ as shown in Fig. 2(b), the nesting effect does not work for the states with $\Delta_k \propto \cos k_z$, whereas it significantly enhances $h_c$ for the states with $\Delta_k \propto \sin k_z$. This behavior is shown in the following.

For the interlayer states, we obtain

$$f_0(q) = \begin{cases} f_0(\bar{q}) + f_2(\bar{q}) & \text{for } \alpha_z = sz \\
\frac{f_0(\bar{q}) - f_2(\bar{q})}{2} & \text{for } \alpha_z = cz,
\end{cases} \quad (15)$$

where the functions $f_0$ and $f_2$ are given in Appendix A. The results for $h_c$ and $\bar{q}$ are summarized in Table III. Equation (15) and Table III are quite similar to Eq. (12) and Table I for the $d$-wave state in parallel fields. The argument that applies the relation between the value of $\bar{q}$ and the nesting for the $d$-wave state also applies to that for the present interlayer states. As shown in Table III, the upper critical field $h_c = e^{1/2}\Delta_{s0}$ for the $sz$-wave states is much larger than $h_c = \Delta_{s0}/\sqrt{2}$ for the $cz$-wave states, although both values are larger than $h_p$ because of Eq. (5). Their ratio, $e^{1/2}/(1/\sqrt{2}) \approx 2.33$, is much larger than the corresponding ratio of $h_c(1.284/0.992 \approx 1.29)$ for the $d$-wave state in parallel fields shown in Table I.

\begin{table}[ht]
\centering
\caption{Results for interlayer states with an arbitrary in-plane symmetry $\alpha_z$ when $t_z \neq 0$ and $H \parallel e$.}
\begin{tabular}{|c|c|c|c|}
\hline
$\alpha_z$ & sz & cz &
\hline
Nodes & $k_z = 0, \pm \pi$ & $k_z = \pm \pi/2$ &
\hline
Max. of $f_0$ & $\ln 2 + \frac{1}{2}$ & $\frac{1}{2} \ln 2$ &
\hline
$\bar{q}$ & 1 & $\sqrt{2} \approx 1.414$ &
\hline
Fermi surfaces & Touch on a circle & Intersected by two circles &
\hline
$\frac{h_c}{\Delta_{s0}}$ & $e^{1/2} \approx 1.649$ & $\frac{1}{\sqrt{2}} \approx 0.707$ &
\hline
\end{tabular}
\end{table}

Compounds CeCoIn$_5$ and FeSe — The present mechanism may explain the existence of the high-field superconducting phases for $H \parallel e$ in the compounds CeCoIn$_5$ and FeSe, which are considered to be the FFLO state. At least, as illustrated above, the perpendicular direction ($H \parallel e$) is not necessarily disadvantageous to the nesting effect for the FFLO state in quasi-low-dimensional systems. For FeSe, a small carrier density and nearly cylindrical Fermi surfaces are compatible with the present model in Eqs. (6) or (7). For this compound, the specific feature $\Delta \sim t_c$ should be incorporated in future research. In CeCoIn$_5$, a first-principles calculation suggests that some of the Fermi surfaces are cylindrical but corrugated. The present theory may be applicable to those Fermi surfaces. For accurate prediction of the FFLO state, extremely accurate information on the Fermi-surface structure would be required, and analysis incorporating realistic shapes of Fermi surfaces is a future research direction.

5. Summary and Conclusion

We examined the FFLO state in quasi-two-dimensional superconductors in magnetic fields perpendicular to the ab-plane. The nesting effect for the FFLO state can enhance $h_c$ for perpendicular fields in systems with cylindrical Fermi surfaces warped by $t_z \neq 0$. For intralayer states, the nesting effect in perpendicular fields is as strong as that for the s-wave
state in parallel fields (Table II). For the interlayer states, the nesting effect in perpendicular fields can be more pronounced because of the $k_c$-dependence in $\Delta_k$. In fact, it was shown that states with $\Delta_k \propto \sin k_c$ exhibit $h_c/\Delta_0 \approx 1.649$.

In conclusion, the nesting effect for the FFLO state can be significant for perpendicular fields as well as for parallel fields. For the nesting effect in perpendicular fields, the value of $t_s$ must be sufficiently small, and in particular, quasi-low-dimensional systems with open Fermi surfaces favor the present mechanism. The upper limit of $t_s$ decreases as the carrier density decreases.

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Appendix A: Explicit forms of $f_n(p)$

The integral in Eq. (1) can be carried out explicitly. The results for $n = 0$ are

$$f_0(p) = \begin{cases} -\ln \frac{p}{2} & \text{for } p \geq 1, \\ -\ln \frac{1 + \sqrt{1 - p^2}}{2} & \text{for } p \leq 1, \end{cases}$$

and those for an integer $n \neq 0$ are

$$f_n(p) = \begin{cases} \frac{1}{n} \cos(n \arccos \frac{1}{p}) & \text{for } p \geq 1, \\ \frac{1}{n} \left( \frac{|p|}{1 + \sqrt{1 - p^2}} \right)^n & \text{for } p \leq 1. \end{cases}$$

The functions $f_2(p)$ and $f_3(p)$ can be expressed as

$$f_2(p) = -\frac{1}{2} + \frac{1}{p^2}, \quad f_3(p) = \frac{1}{4} - \frac{2}{p^2} + \frac{2}{p^4}$$

for $p \geq 1$.

Appendix B: Proof of Eq. (5)

For an arbitrary function $g(x)$ and an arbitrary average $\langle \cdot \cdot \cdot \rangle$ over an arbitrary variable $x$, it can be proved that if $\langle g \rangle = 1$, $\langle g \ln g \rangle \geq 0$, (B·1)

where the equality sign holds for $g(x) = 1$. Applying Eq. (B·1) to the average defined in Eq. (4) and $g = \gamma_a^2/(\gamma_0^2)$, we obtain $\sqrt{\gamma_a^2} \leq \gamma_0$, which leads to Eq. (5).

Proof of Eq. (B·1) — It is sufficient to prove this for a simple average such as

$$\langle g \rangle = \frac{1}{n} \sum_{k=1}^{n} g_k$$

with an arbitrary positive integer $n$. In fact, an arbitrary probability function $p(x)$ can be realized by a sufficiently dense distribution of $x_k$ ($k = 1, 2, \cdots, n$) on the $x$-axis as

$$\int dx p(x)g(x) \approx \frac{1}{n} \sum_{k=1}^{n} g_k,$$

where $g_k = g(x_k)$. For the average defined by Eq. (B·2), the inequality in Eq. (B·1) is easily proved with mathematical induction as follows. For $n = 2$, defining $x$ with $g_1 = 1 + x$ and $g_2 = 1 - x$, $f(x) = \langle g \ln g \rangle$ satisfies $f'(x) \geq 0$ and $f(0) = 0$. Hence, $f(x) \geq 0$. When Eq. (B·1) is satisfied for $n$, Eq. (B·1) is satisfied for $n + 1$. In fact, assuming $g_{n+1} \leq 1$ without loss of generality,

$$\hat{g}_k = \frac{n g_k}{n + 1 - g_{n+1}}$$

satisfies

$$\frac{1}{n} \sum_{k=1}^{n} \hat{g}_k \ln \hat{g}_k \geq 0$$

because of the induction hypothesis. Hence,

$$\frac{1}{n + 1} \sum_{k=1}^{n+1} g_k \ln g_k \geq F(g_{n+1}) \geq 0$$

with

$$F(x) \equiv (n + 1 - x) \ln \frac{n + 1 - x}{n} + x \ln x.$$

The last inequality holds because $F(1) = 0$ and $F'(x) \leq 0$ for $x \leq 1$. It is evident that the equality sign holds when $g_k = 1$ for all integers $k$.

Appendix C: Effective Mass Anisotropy

It follows from the definitions

$$\tilde{k}_\mu = \sqrt{\frac{m}{m_{\mu}}} k_\mu,$$

and $m = \sqrt{m_|| m_\perp}$ that, $dk_\mu dk_\nu = d\tilde{k}_\mu d\tilde{k}_\nu$ and

$$e_{\tilde{k}_\mu}^\parallel = \tilde{k}_\mu^2 / 2m,$$

where $\tilde{k}_\mu$ and $\phi$ are defined by

$$(\tilde{k}_x, \tilde{k}_y) = (\tilde{k}_0 \cos \phi, \tilde{k}_0 \sin \phi).$$

We also obtain $\rho = m / 2\pi$ and

$$\tilde{\rho} = \frac{\tilde{e}_{\tilde{k}_\mu}^\parallel}{\tilde{g}_0} = \frac{d\tilde{\phi} dk_\mu}{2\pi}. $$

The FFLO vector $\mathbf{q}$ is also transformed as $\tilde{q}_\mu = \sqrt{m_\mu} q_\mu$ and

$$\mathbf{q} \cdot \tilde{q} = \frac{q_0}{2\hbar_c} \cos \tilde{q}_0 + \frac{t_0 q_x}{2\hbar_c} \sin k_x,$$

where $(\tilde{q}_0, \tilde{q}_x) = (\tilde{q}_0 \cos \tilde{q}_0 \tilde{q}_x \sin \tilde{q}_0)$, and $\tilde{q} = \tilde{\phi} - \tilde{q}_0$. Hence, all the equations for $m_\parallel \neq m_\perp$ have exactly the same form as those for $m_\parallel = m_\perp = m$.

1) P. Fulde and R. A. Ferrell, Phys. Rev. 135, A550 (1964).
2) A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 47, 1136 (1964); translation: Sov. Phys. JETP, 20, 762 (1965).
3) R. Casalbuoni and G. Nardulli, Rev. Mod. Phys. 76, 263 (2004).
4) Y. Matsuda and H. Shimahara, J. Phys. Soc. Jpn. 76, 051005 (2007).
5) H. Shimahara, in The Physics of Organic Superconductors and Conductors, ed. A.G. Lebed (Springer, Berlin, 2008), p. 687.
6) J. Wosnitza, Ann. Phys. (Berlin) 530, 1700282 (2018).
7) J. Singleton, J. A. Symington, M. S. Num, A. Ardavan, M. Karmoo, and P. Day, J. Phys. Condens. Matter 12, L641 (2000).
8) H. A. Radovan, N. A. Fortune, T. P. Murphy, S. T. Hannahs, E. C. Palm, S. W. Tozer, and D. Hall, Nature 425, 51 (2003).
9) A. Bianchi, R. Movshovich, C. Capan, P. G. Pagliuso, and J. L. Sarrao, Phys. Rev. Lett. 91, 187004 (2003).
10) S. Yonezawa, S. Kusaba, Y. Maeno, P. Auban-Senzier, C. Pasquier, K. Bechgaard, and D. Jerome, Phys. Rev. Lett. 100, 117002 (2008).
11) S. Kasahara, Y. Sato, S. Liciardiello, M. Čušo, S. Arsenijević, T. Ottenbro, T. Tominaga, J. Boker, I. Ermen, T. Shibachiu, J. Wosnitza, N.E. Hussey, and Y. Matsuda, Phys. Rev. Lett. 124, 107001 (2020).

12) Here, the term “two-dimensional models” does not mean purely two-dimensional models, but those in which weak three-dimensional interactions that stabilize the long-range order are implicitly taken into account by applying mean-field approximations.

13) Below the upper critical field, FFLO states with more than two q’s can be stable, especially in quasi-low-dimensional systems. However, because the second-order transition point can be examined by considering a single q, we do not consider multiple q states in the present study.

14) L. W. Gruenberg and L. Gunther, Phys. Rev. Lett. 16, 996 (1966).

15) H. Shimahara, Phys. Rev. B 80, 214512 (2009).

16) When the orbital pairing effect is sufficiently strong (w > 1.8), the vortex and the FFLO oscillation coexist. In the coexistence state, q || H as examined in Ref. 14. However, when the orbital effect is very weak, vortex states with higher Landau-level indices n occur, and the oscillation of the order parameter can have components perpendicular to H.\(^\text{15,15}\) The oscillation is reduced to the FFLO oscillation of the pure FFLO state in the limit n → ∞. Hence, when \(\alpha \approx 1\), practically speaking, the FFLO vector has components perpendicular to H. See also Refs. 26 and 27.

17) States with \(\Delta(q) = \cos(q - \varphi)\) have two FFLO vectors, i.e., \(\pm q\), and hence, \(q \parallel H\); however, for the same reason as that mentioned in Ref. 13, we consider a single q.

18) H. Shimahara and D. Rainer, J. Phys. Soc. Jpn. 66, 3591 (1997); see also H. Shimahara, J. Supercond., 12, 469 (1999).

19) This effect is called the nesting effect, in analogy with the nesting effect for the SDW and CDW.

20) Details of the nesting effect were explained in previous papers, such as Refs. 21, 32, 35, and 40–42.

21) H. Shimahara, Phys. Rev. B 50, 12760 (1994).

22) S. Kasahara, T. Watashige, T. Hanaguri, Y. Kohsaka, T. Yamashita, Y. Shimoyama, Y. Mizukami, R. Endo, H. Ikeda, K. Aoyama, T. Terashima, S. Uji, T. Wolf, H. V. Löhlensyen, T. Shibachiu, and Y. Matsuda, Proc. Natl. Acad. Sci. U.S.A. 111, 16309 (2014).

23) T. Hanaguri, K. Iwaya, Y. Kohsaka, T. Machida, T. Watashige, S. Kasahara, T. Shibachi, and Y. Matsuda, Sci. Adv. 4, eaar6419 (2018).

24) T. Shibachiu, T. Hanaguri, and Y. Matsuda, J. Phys. Soc. Jpn. 89, 102002 (2020).

25) K. Kumagai, M. Saitoh, T. Oyazui, Y. Furukawa, S. Takashima, M. Nohara, H. Takagi, and Y. Matsuda, Phys. Rev. Lett. 97, 227002 (2006).

26) U. Klein, D. Rainer, and H. Shimahara, J. Low Temp. Phys. (USA), 118, 91 (2000).

27) U. Klein, Phys. Rev. B 69, 134518 (2004).

28) In SrRuO\(_3\), the FFLO state may occur at high fields parallel to the ab-plane because the antiparallel-spin pairing is considered to be confirmed.\(^\text{27}\) The horizontal line node observed in Ref. 48 is compatible with Ref. 13, we consider a single q, parallel spins.

29, 30) For example, \(\Delta_0 = \Delta_0\) for the s-wave state with \(\gamma_s = 1\), whereas \(\Delta_0 = \Delta_0\) for the other \(\alpha\). For example, see Ref. 35. The dimensionless coupling constant \(t_{\text{eff}}\) is expressed as \(\lambda_s = g_s N_\alpha(0)\), where \(N_\alpha(0) = N_\alpha(0)/\gamma_s^2\), and \(g_s\) denotes the coupling constant of the pairing interaction for \(\alpha\)-wave pairing, which is limited by the cutoff energy \(\omega_c\).

31) For example, \(S_0 = 4\pi\) when \(q^2 = \sin \vartheta \cos \vartheta_0\), and \(S_0 = 4\pi^2\) when \(q^2 = \sin \vartheta_0\).

32) H. Shimahara, J. Phys. Soc. Jpn. 68, 3069 (1999).

33) The internal field is enhanced by a factor \(\xi/\xi\), whereas \(\xi, \xi\) are the bare susceptibility and the susceptibility renormalized by the repulsive interactions between electrons, respectively. Hence, the FFLO critical field (in terms of the bare external field) is reduced by the factor \(\xi/\xi\) < 1. On the other hand, because \(N_0(0)/2\) = \(H^2/2\), \(H_p\) is reduced by the factor \(\sqrt{\xi/\xi}\). Consequently, \(H_p/\bar{H}_p\) is reduced by the factor \(\sqrt{\xi/\xi}\) < 1.\(^\text{21,49}\) Note that the attractive interactions between electrons that stabilize the superconductivity are effective interactions in which the retardation effect is incorporated. The bare static interactions between electrons on the Fermi surfaces should be repulsive because of the strong Coulomb repulsion, which results in \(\xi > \xi\). Hence, if the real \(H_p\) is to exceed the real \(H_p\) in the presence of the internal-field enhancement, the difference \(H_p - \bar{H}_p (\geq 0)\) in the absence of the internal-field enhancement must be substantial.

34) Equation (6) results in \(\rho(\xi, \bar{H}) = m/2\pi\) and \(\rho(0) = (K/m, 2\sqrt{\pi} \sin k_\parallel)\), and hence,

\[\frac{\nu}{\nu} = \frac{q}{2\kappa} \cos \theta + \frac{q}{2\kappa} \sin k_\parallel\]

where \(q = |K/\mu|\), and \(\theta\) denotes the angle between \(k_\parallel\) and \(q = (q_, q_\parallel)\).

35) For example, the expansion of \(q_{\text{eff}}\) = \(2/(\cos k_\parallel + \cos k_s)\) results in Eq. (7) with \(m = 1/\kappa\) for small electron densities.

36) If we consider the FFLO state in singlet superconductors, the symmetry in the \(k_s\)-direction should be even and odd when the in-plane state is even and odd, respectively. However, the FFLO state can occur in triplet superconductors as long as the pairing occurs between electrons with antiparallel spins.

37) H. Shimahara and K. Moriwake, J. Phys. Soc. Jpn. 71, 1234 (2002); H. Shimahara and S. Hata, Phys. Rev. B 62, 14541 (2000).

38) N. Miyawaki and H. Shimahara, J. Phys. Soc. Jpn. 83, 024703 (2014).

39) K. Itahashi and H. Shimahara, J. Phys. Soc. Jpn. 87, 083701 (2018); ibid. J. Phys. Soc. Jpn. 89, 024708 (2020).

40) See references in Ref. 3–6. For example, see Ref. 18.

41) W. Hasegawa, K. Machida, and M. Ozaki, J. Phys. Soc. Jpn. 69, 336 (2000).

42) H. Shimahara, J. Phys. Soc. Jpn. 89, 093704 (2020).

43) K. W. Song and A. Kosele, Phys. Rev. B 97, 224520 (2018).

44) H. Shimahara, J. Phys. Soc. Jpn. 66, 541 (1997).

45) At \(T = 0\), \(\Delta_0 = \Delta_0\) for the s-wave state with \(\gamma_s = 1\), whereas \(\Delta_0 = \Delta_0\) for the other \(\alpha\). For example, see Ref. 35. The dimensionless coupling constant \(t_{\text{eff}}\) is expressed as \(\lambda_s = g_s N_\alpha(0)\), where \(N_\alpha(0) = N(0)/\gamma_s^2\), and \(g_s\) denotes the coupling constant of the pairing interaction for \(\alpha\)-wave pairing, which is limited by the cutoff energy \(\omega_c\).