Welschinger invariant and enumeration of real rational curves

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Abstract

Welschinger’s invariant bounds from below the number of real rational curves through a given generic collection of real points in the real projective plane. We estimate this invariant using Mikhalkin’s approach which deals with a corresponding count of tropical curves. In particular, our estimate implies that, for any positive integer \( d \), there exists a real rational curve of degree \( d \) through any collection of \( 3d - 1 \) real points in the projective plane, and, moreover, asymptotically in the logarithmic scale at least one third of the complex plane rational curves through a generic point collection are real. We also obtain similar results for curves on other toric Del Pezzo surfaces.

1 Introduction

In contrast to spectacular achievements in complex enumeration geometry, real enumerative geometry remained in its almost embryonic state, with Sottile’s \([14]\) and Eremenko-Gabrielov’s \([3]\) results on the real Schubert calculus as the only serious exception. One of the difficulties in real enumerative geometry is the lack of invariants: the number of real objects usually varies along the parameter space. One of the simplest known examples \([1]\) showing this phenomenon is the following: while over \( \mathbb{C} \) the number of rational plane cubics through given 8 generic points is 12, the number of real ones can take three different values, 8, 10, and 12, depending on the positions of the 8 real points. One of the main questions in real enumerative geometry is that of the lower bounds on the number of real solutions (an upper
bound provided by the number of complex solutions gives rise to another interesting question, that is of the sharpness of such a bound, see the discussion of "total reality" in [13] and further references therein).

In the present note we focus on the enumeration of real rational curves passing through fixed real points on an algebraic surface, a real counterpart of the complex Gromov-Witten theory, and, in particular, on the following question: *whether through any generic $3d-1$ points in the real plane always passes a real rational curve of degree $d$?* (The respective number of complex rational curves [6] is even for every $d \geq 3$, so the existence of required real curves does not immediately follow from the computation in the complex case.)

Till recently, the answer to the above question was not known even for degree 4. The situation has radically changed after the discovery by J.-Y. Welschinger [16, 17] of a way of attributing weights to real solutions making the number of solutions counted with the weights to be independent of the configuration of points. Since the weights take values $\pm 1$ exclusively, the absolute value of Welschinger’s invariant $W_d$ immediately provides a lower bound on the number $R_d$ of real solutions: $R_d \geq |W_d|$. The next natural question arises: *how non-trivial is Welschinger’s bound?*

In the present note we show that Welschinger’s bound implies the following statement.

**Theorem 1.1** For any integer $d \geq 1$, through any $3d-1$ generic points in $\mathbb{R}P^2$ there can be traced at least $d!/2$ real rational curves of degree $d$.

As a corollary, we obtain an affirmative answer to the aforementioned question: there always exists at least one real rational curve going through the given real points (such an existence statement extends from generic point data to an arbitrary one at least if reducible rational curves are taken into consideration as well).

To establish the non-triviality of the Welschinger invariant and to estimate it, we apply another key ingredient, Mikhalkin’s approach to counting nodal curves passing through specific configurations of points [9]. We also point out that the Welschinger invariant does not generalize directly even to the case of elliptic curves (section 3), thereby leaving open the enumeration problem of real plane curves of positive genera.

Note that the expression given in Theorem 1.1 is not the exact value of the Welschinger invariant. Using the same methods one can perform explicit calculations for, say, small $d$. For example, in this way one gets $W_4 = 240$ and $W_5 = 18264$. 

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Remark 1.2 Comparing with the formulae in [6], one can see that the logarithm of the lower bound in Theorem 1.1 is asymptotically equal to $\frac{1}{3}\log N_{\mathbb{P}^2,d}^C$, where $N_{\mathbb{P}^2,d}^C$ is the number of complex rational plane curves of degree $d$ passing through 3$d$ − 1 generic points. (In fact, log $N_{\mathbb{P}^2,d}^C \sim 3d \log d$, as follows from the inequalities $(3d-4)! \cdot 54^{-d} \leq N_{\mathbb{P}^2,d}^C \leq (3d - 5)!$ which, in turn, easily follow from Kontsevich’s recurrence formula [6].)

We treat in a similar way enumerative problems for real rational curves on other toric Del Pezzo surfaces equipped with their usual real structures, and obtain the following statement as a corollary of Welschinger’s bound.

Denote by $Q$ the hyperboloid $\mathbb{C}P^1 \times \mathbb{C}P^1$, and by $P_k$, $k = 1, 2, 3$ the complex projective plane with $k$ blown up generic real points. For $P_k$, let $L$ be the pull back of a generic straight line, and $E_1, ..., E_k$ the exceptional divisors.

Theorem 1.3 Let $\Sigma$ be $Q$, $P_1$, $P_2$ or $P_3$, and $D$ a very ample divisor on $\Sigma$. Then through any $c_1(\Sigma) \cdot D$ − 1 generic real points in $\Sigma$ there can be traced at least $\rho$ real rational curves belonging to the linear system $|D|$, where

$$\rho = \begin{cases} \max\{d_1, d_2\}, & \text{if } \Sigma = Q, \ D \text{ is of bi-degree } (d_1, d_2), \\ \frac{1}{2} d! / d_1!, & \text{if } \Sigma = P_1, \ D \sim dL - d_1E_1, \ d > d_1 > 0, \\ (d - d_2)! / d_1!, & \text{if } \Sigma = P_2, \ D \sim dL - d_1E_1 - d_2E_2, \\ & \quad d_1 + d_2 < d, \text{ and } d_1 \geq d_2 > 0, \\ (d - d_2 - d_3)! / d_1!, & \text{if } \Sigma = P_3, \ D \sim dL - d_1E_1 - d_2E_2 - d_3E_3, \\ & \quad d_1 + d_2 + d_3 \leq d, \text{ and } d_1 \geq d_2 \geq d_3 > 0, \\ (d_1!) / (d - d_2 - d_3)!, & \text{if } \Sigma = P_3, \ D \sim dL - d_1E_1 - d_2E_2 - d_3E_3, \\ & \quad d_1 + d_2 + d_3 > d, \text{ and } d_1 \geq d_2 \geq d_3 > 0. \end{cases}$$

In fact, the Welschinger invariant counts real $J$-holomorphic curves on real symplectic rational surfaces equipped with a generic almost complex structure. A literal use of this invariant for an integrable structure on a surface is possible if for this surface there is no difference between Gromov-Witten invariants and the count of irreducible rational curves. The latter condition is verified for Del Pezzo surfaces. It is no longer the case for a surface containing an exceptional divisor $E$ with $E^2 \leq -2$. On the other hand, Mikhalkin’s approach to counting nodal curves applies only to toric surfaces. That is why we restrict ourselves to the case of toric Del Pezzo surfaces $\mathbb{C}P^2$, $Q$, and $P_k$, $k = 1, 2, 3$. The next natural question would be on existence of a real rational curve passing through a generic collection of real points on a non-singular toric surface, and in particular, on a geometrically ruled surface $\Sigma_n$ with $n \geq 2$. However, to solve this question one needs additional tools.
Due to the conformal uniqueness of symplectic structures on $\mathbb{C}P^2$ and certain deformation invariance of Welschinger numbers (see [16, 17]), Theorem 1.1 extends to the corresponding symplectic settings. In particular, by means of such a generalization of Theorem 1.1 one can extend the Hilbert-type inequalities for real plane algebraic curves (see, for example, [1]) to the case of real pseudo-holomorphc curves on the real symplectic projective plane. Define a partial order on the set of connected components of a smooth curve in $\mathbb{R}P^2$ in the following way: if a component $C$ is contained in the disc bounded by a component $C'$, then $C'$ dominates $C$.

**Proposition 1.4** For any nonsingular real pseudo-holomorphic curve of degree $d$ in the real symplectic projective plane and any integer $s \geq 1$, the total length of $3s - 1$ disjoint linearly ordered chains of real components of the curve does not exceed $ds/2$ if no element of one chain dominates all the elements the other chains.

Note that in the integrable case there is a proof of such a statement (see [1]) which is based on the possibility to trace a connected real cubic (of genus 0) through 8 points and a connected real quartic (of genus 3) through 13 points. A similar proof could be proposed in the symplectic category. For that, however, in addition to tracing a real rational pseudo-holomorphic curve of degree 3 through 8 points (one of particular Welschinger’s results, [16, 17]), we should know that for a generic almost-complex structure the (one-dimensional) family of pseudo-holomorphic curves of degree 4 passing through generic 13 points contains an odd number of singular elements, which seems to be still an open question.

## 2 Tropical calculation of the Welschinger invariant

Let $\Sigma$ be $\mathbb{C}P^2$, $Q$, $P_1$, $P_2$ or $P_3$ equipped with its standard real structure, and $D$ a very ample divisor on $\Sigma$. The linear system $|D|$ is generated, with respect to suitable affine coordinates, by monomials $x^iy^j$, where $(i, j)$ ranges over all the integer points (i.e., points having integer coordinates) of a convex lattice polygon $\Delta$ of the following form. If $\Sigma = \mathbb{C}P^2$ and $D \sim d[\mathbb{C}P^1]$, then $\Delta$ is the triangle with vertices $(0, 0)$, $(d, 0)$, $(0, d)$. If $\Sigma = Q$ and $D$ is of bi-degree $(d_1, d_2)$, then $\Delta$ is the rectangle with vertices $(0, 0)$, $(d_1, 0)$, $(d_1, d_2)$, $(0, d_2)$. If $\Sigma = P_k$, $k = 1, 2, 3$, and $D \sim dL - \sum_{i=1}^{k} d_iE_i$, then $\Delta$ is respectively the quadrangle with vertices $(0, 0)$, $(d - d_1, 0)$, $(d - d_1, d_1)$, $(0, d)$, or the pentagon with vertices $(0, 0)$, $(d - d_1, 0)$, $(d - d_1, d_1)$, $(d_2, d - d_2)$, $(0, d - d_2)$, or the hexagon with vertices $(d_3, 0)$, $(d - d_1, 0)$, $(d - d_1, d_1)$, $(d_2, d - d_2)$, $(0, d - d_2)$.
Figure 1: Polygons associated with \( \mathbb{C}P^2, Q, P_1, P_2, \) and \( P_3 \)

(0, \( d_3 \)) (see Figure 1). Let \( r(\Delta) \) be the number of integer points on the boundary of \( \Delta \) diminished by 1, and \( \delta(\Delta) \) be the number of interior integer points of \( \Delta \). Note that \( r(\Delta) = c_1(\Sigma) \cdot D - 1 \) and \( \delta(\Delta) \) is the genus of nonsingular representatives of \( |D| \). As well known, the number of curves of genus \( 0 \leq g \leq \delta(\Delta) \) in \( |D| \) passing through \( c_1(\Sigma) \cdot D - 1 + g \) generic points is finite.

A. Welschinger numbers. For any integer \( 0 \leq g \leq \delta(\Delta) \) and any set \( w \) of \( c_1(\Sigma) \cdot D - 1 + g \) generic distinct real points in \( \Sigma \), let \( N_{\Sigma,D,g}^{\mathbb{R}}(w) \) be the number of irreducible real nodal curves of genus \( g \) in \( |D| \) passing through all the points of \( w \), and let \( N_{\Sigma,D,g}^{\text{even}}(w) \) (resp., \( N_{\Sigma,D,g}^{\text{odd}}(w) \)) be the number of real irreducible nodal curves of genus \( g \) in \( |D| \) passing through all the points of \( w \) and having even (resp., odd) number of solitary nodes (i.e., double points locally given by \( x^2 + y^2 = 0 \)). Define the Welschinger number as \( W_{\Sigma,D,g}(w) = N_{\Sigma,D,g}^{\text{even}}(w) - N_{\Sigma,D,g}^{\text{odd}}(w) \).

Theorem 2.1 (J.-Y. Welschinger, see [16, 17]). If \( g = 0 \), then \( W_{\Sigma,D,g}(w) \) does not depend on the choice of the (generic) set \( w \).

We call \( W_{\Sigma,D,0}(w) \) the Welschinger invariant and denote it by \( W_{\Sigma,D} \). Clearly, \( N_{\Sigma,D,0}^{\mathbb{R}}(w) \geq |W_{\Sigma,D}| \). This Welschinger bound together with an explicit estimate for \( W_{\Sigma,D} \), obtained below, implies Theorems 1.1, 1.3. To get the estimate we use the methods of tropical algebraic geometry [9, 13].

B. Correspondence theorems. Let \( A \) be a finite collection of integer points in \( \mathbb{R}^2 \), and \( \nu : A \to \mathbb{R} \) a function. Consider the function \( \hat{\nu} : \mathbb{R}^2 \to \mathbb{R} \) defined by \( \hat{\nu}(x,y) = \max_{(i,j) \in A} \{ ix + jy - \nu(i,j) \} \). The function \( \hat{\nu} \) is called the Legendre transform of \( \nu \). Note that \( \hat{\nu} \) is a piecewise-linear convex function. Consider the corner locus \( \Pi \subset \mathbb{R}^2 \) of \( \hat{\nu} \), i.e., the set where \( \hat{\nu} \) is not smooth. The set \( \Pi \) has a natural structure of one-dimensional complex, and any edge of \( \Pi \) is a segment or a ray. Denote by \( \Delta(A) \) the convex hull of \( A \). Any connected component of the complement of \( \Pi \) corresponds to an integer point of \( \Delta(A) \). Thus, the edges of \( \Pi \)
are naturally equipped with positive integer weights. Namely, the weight of an edge separating two connected components of the complement of \( \Pi \) is the (integer) length of the segment joining the corresponding two integer points of \( \Delta(A) \). The resulting weighted complex is called the *tropical curve* associated with the pair \((A, \nu)\); cf. [8] and [15]. The polygon \( \Delta(A) \) is called a *Newton polygon* of this tropical curve.

Let \( K \) be the field of Puiseux series with complex coefficients equipped with its standard non-Archimedian valuation \( \text{val} : K^* \to \mathbb{R} \). According to Kapranov’s theorem [5], the underlying set \( \Pi \) of the tropical curve associated with \((A, \nu)\) is the closure in \( \mathbb{R}^2 \) of the non-Archimedian amoeba (see [5] and [7]) of a curve in \((K^*)^2\) defined by a polynomial \( f(z, w) = \sum_{(i, j) \in A} c_{i,j} z^i w^j \) such that for any point \((i, j)\) in \( A \) one has \( \text{val}(c_{i,j}) = \nu(i, j) \). Clearly, such a polynomial \( f \) determines the pair \((A, \nu)\), and thus, the associated tropical curve.

The function \( \nu \) defines a subdivision of \( \Delta(A) \) into convex polygons in the following way. Consider the overgraph \( \Gamma_{\nu} \) of \( \nu \), i.e., the convex hull of the set \( \{(i, j, k) \in \mathbb{R}^3 : (i, j) \in A, k \geq \nu(i, j)\} \). The polyhedron \( \Gamma_{\nu} \) is naturally projected onto \( \Delta(A) \). The faces of \( \Gamma_{\nu} \) which project injectively, define a subdivision of \( \Delta(A) \). Denote this subdivision by \( S(A, \nu) \). Let \( T \) be the tropical curve associated with \((A, \nu)\). Note that \( T \) does not determine uniquely the pair \((A, \nu)\) (and even the polygon \( \Delta(A) \)). However, once the polygon \( \Delta(A) \) is fixed, the tropical curve \( T \) determines uniquely the subdivision \( S(A, \nu) \). For any tropical curve \( T \) with Newton polygon \( \Delta \), denote by \( D_{\Delta}(T) \) the subdivision of \( \Delta \) determined by \( T \).

A tropical curve is called *irreducible* if it cannot be presented as a union of two proper tropical subcurves. A tropical curve \( T \) with Newton polygon \( \Delta \) is called *nodal* if the subdivision \( D_{\Delta}(T) \) of \( \Delta \) verifies the following properties.

- any polygon of \( D_{\Delta}(T) \) is either triangle or a parallelogram,
- any integer point on the boundary of \( \Delta \) is a vertex of \( D_{\Delta}(T) \).

In this case, the subdivision \( D_{\Delta}(T) \) is also called nodal. Let \( T \) be a nodal tropical curve with Newton polygon \( \Delta \). The *rank* of \( T \) is the difference diminished by 1 between the number of vertices of \( D_{\Delta}(T) \) and the number of parallelograms in \( D_{\Delta}(T) \). The *multiplicity* \( \mu(T) \) of \( T \) (and the multiplicity \( \mu(D_{\Delta}(T)) \) of \( D_{\Delta}(T) \)) is the product of areas of all the triangles in \( D_{\Delta}(T) \) (we normalize the area in such a way that the area of a triangle whose only integer points are the vertices is equal to 1).

Let \( n \) be a natural number, \( u \) a generic set of \( n \) points in \( \mathbb{R}^2 \). Consider the collection \( C(u) \) of nodal tropical curves of rank \( n \) and of Newton polygon \( \Delta \) which
pass through all the points of $u$, and denote by $T_n^C(u)$ the number of curves in $C(u)$ counted with their multiplicities.

**Theorem 2.2** (G. Mikhalkin, see [9] and [13]). Let $u$ be a generic set of $n = r(\Delta) + g$ points in $\mathbb{R}^2$, where $0 \leq g \leq \delta(\Delta)$ is an integer. Then $T_n^C(u)$ is equal to the number of complex curves in the linear system $|D|$ which pass through a fixed generic collection of $n$ points in $\Sigma$ and have $\delta(\Delta) - g$ nodes.

A nodal tropical curve $T$ with Newton polygon $\Delta$ and the corresponding subdivision $\mathcal{D}_{\Delta}(T)$ of $\Delta$ are called *odd*, if each triangle in $\mathcal{D}_{\Delta}(T)$ has an odd area. A nodal tropical curve $T$ with Newton polygon $\Delta$ and the subdivision $\mathcal{D}_{\Delta}(T)$ are called *positive* (resp., *negative*) if the sum of the numbers of interior integer points over all the triangles of $\mathcal{D}_{\Delta}(T)$ is even (resp, odd). Denote by $T_n^+(u)$ (resp., $T_n^-(u)$) the number of odd positive (resp., negative) irreducible curves in $C(u)$ (counted without multiplicities).

**Theorem 2.3** (cf. [9] and [13]). Let $u$ be a generic set of $n = r(\Delta)$ points in $\mathbb{R}^2$. Then $W_{\Sigma,D} = T_n^+(u) - T_n^-(u)$.

**Theorem 2.3** immediately follows from Proposition 5.1 and Lemma 2.6 in [13].

**C. Lattice paths.** Denote by $p$ the highest point of $\Delta$ on the vertical axis, and by $q$ the most right point of $\Delta$ on the horizontal axis. The points $p$ and $q$ divide the boundary of $\Delta$ in two parts. Denote the upper part of the boundary by $\partial \Delta_+$, and the lower one by $\partial \Delta_-$. Fix a linear function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ which is injective on the integer points of $\Delta$, and such that the restriction of $\lambda$ on $\Delta$ takes its minimum at $p$ and takes its maximum at $q$.

Let $l$ be a natural number. A path $\gamma : [0, l] \to \Delta$ is called $\lambda$-*admissible* if

- $\gamma(0) = p$ and $\gamma(l) = q$,
- the composition $\lambda \circ \gamma$ is injective,
- for any integer $0 \leq i \leq l - 1$ the point $\gamma(i)$ is integer, and $\gamma([i, i + 1])$ is a segment.

The number $l$ is called the *length* of $\gamma$. A $\lambda$-admissible path $\gamma$ divides $\Delta$ in two parts: the part $\Delta_+(\gamma)$ bounded by $\gamma$ and $\partial \Delta_+$ and the part $\Delta_-(\gamma)$ bounded by $\gamma$ and $\partial \Delta_-$. Define an operation of *compression* of $\Delta_+(\gamma)$ in the following way. Let $j$ be the smallest positive integer $1 \leq j \leq l - 1$ such that $\gamma(j)$ is the vertex of
a generic set \( u \) with the angle less than \( \pi \) (a compression of \( \Delta_+ (\gamma) \) is defined only if such an integer \( j \) does exist). A compression of \( \Delta_+ (\gamma) \) is \( \Delta_+ (\gamma') \), where \( \gamma' \) is either the path defined by \( \gamma'(i) = \gamma(i) \) for \( i < j \) and \( \gamma'(i) = \gamma(i+1) \) for \( i \geq j \), or the path defined by \( \gamma'(i) = \gamma(i) \) for \( i \neq j \) and \( \gamma'(j) = \gamma(j-1) + \gamma(j+1) - \gamma(j) \) (if \( \gamma(j-1) + \gamma(j-1) + \gamma(j) \in \Delta \)). Note that \( \gamma' \) is also a \( \lambda \)-admissible path. A sequence of compressions started with \( \Delta_+ (\gamma) \) and ended with a path whose image coincides with \( \partial \Delta_+ \) defines a subdivision of \( \Delta_+ (\gamma) \) which is called compressing. A compression and a compressing subdivision of \( \Delta_- (\gamma) \) is defined in the completely similar way. A pair \( (\mathcal{S}_+, \mathcal{S}_-) ) \), where \( \mathcal{S}_\pm (\gamma) \) is a compressing subdivision of \( \Delta_\pm (\gamma) \), produces a subdivision of \( \Delta \). Denote by \( \mathcal{N}_\lambda (\gamma) \) the collection of nodal subdivisions of \( \Delta \) which can be obtained in this way starting with \( \gamma \).

**Theorem 2.4** (G. Mikhalkin, see [9]). Let \( 0 \leq g \leq \delta (\Delta) \) be an integer. There exists a generic set \( u \) of \( n = r (\Delta) + g \) points in \( \mathbb{R}^2 \) such that the map \( \mathcal{D}_\Delta \), associating to a nodal tropical curve \( T \) with Newton polygon \( \Delta \) the corresponding subdivision \( \mathcal{D}_\Delta (T) \) of \( \Delta \) (see subsection B), establishes a 1-to-1 correspondence between the set \( \mathcal{C} (u) \) and the disjoint union \( \Pi_\gamma \mathcal{N}_\lambda (\gamma) \), where \( \gamma \) runs over all the \( \lambda \)-admissible paths in \( \Delta \) of length \( n \). In particular, \( \mathcal{T}^C_n (u) = \sum_\gamma \sum_{\mathcal{S} \in \mathcal{N}_\lambda (\gamma)} \mu (\mathcal{S} ) \), where \( \mu (\mathcal{S} ) \) is the multiplicity of \( \mathcal{S} \).

Take a set \( u \) with the properties described in Theorem 2.4 and denote by \( \mathcal{N}_\lambda^+ (\gamma) \) (resp., \( \mathcal{N}_\lambda^- (\gamma) \)) the number of odd positive (resp., negative) subdivisions in \( \mathcal{N}_\lambda (\gamma) \) which are the images of irreducible tropical curves under the bijection \( \mathcal{D}_\Delta \mid_{\mathcal{C} (u)} : \mathcal{C} (u) \rightarrow \Pi_\gamma \mathcal{N}_\lambda (\gamma) \). The following statement is an immediate corollary of Theorems 2.3 and 2.4.

**Proposition 2.5** The Welschinger number \( W_{\Sigma, D} \) is equal to \( \sum_\gamma (\mathcal{N}_\lambda^+ (\gamma) - \mathcal{N}_\lambda^- (\gamma)) \), where \( \gamma \) runs over all the \( \lambda \)-admissible paths in \( \Delta \) of length \( r (\Delta) \).

**D. Positivity of the Welschinger invariant.** Let \( \lambda^0 : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a linear function defined by \( \lambda^0 (i, j) = i - \varepsilon j \), where \( \varepsilon \) is sufficiently small positive irrational number (so that \( \lambda^0 \) defines a kind of a lexicographical order on the integer points of the polygon \( \Delta \)).

**Proposition 2.6** For any \( \lambda^0 \)-admissible path \( \gamma \) in \( \Delta \), the number \( \mathcal{N}_{\lambda^0}^- (\gamma) \) equals 0.

**Proof.** Let \( \gamma \) be a \( \lambda^0 \)-admissible path in \( \Delta \), and \( \mathcal{S} \) a subdivision in the collection \( \mathcal{N}_{\lambda^0} (\gamma) \). The subdivision \( \mathcal{S} \) does not have an edge with the endpoints \( (i_1, j_1) \) and \( (i_2, j_2) \) such that \( |i_1 - i_2| > 1 \); otherwise, at least one integer point on the boundary
Proof of Theorems 1.1 and 1.3. We consider only the cases of $\mathbb{C}P^2$ and $P_3$, since the constructions presented below can be easily adapted to the remaining cases.

If $\Sigma = P_3$, we can assume that $d_1 + d_2 + d_3 \leq d$. Indeed, we have $|dL - d_1 E_1 - d_2 E_2 - d_3 E_3| = |d'L - d'_1 E'_1 - d'_2 E'_2 - d'_3 E'_3|$, where $d' = 2d - d_1 - d_2 - d_3$, $d'_i = d - d_j - d_k$, $E'_i = E_j - E_k$ for all $\{i, j, k\} = \{1, 2, 3\}$. It remains to notice that $d'_1 + d'_2 + d'_3 < d'$ as far as $d_1 + d_2 + d_3 > 0$, and $(d_1!)/(d - d_2 - d_3)! = (d' - d'_2 - d'_3)!/(d'_1!)$.

Take the $\lambda^0$-admissible lattice path $\gamma$ of length $r(\Delta)$ passing successively through $r(\Delta) + 1$ integer points of $\Delta$ described below.

- If $\Sigma = \mathbb{C}P^2$, then the points are $(0, d - j)$, where $j = 0, \ldots, d$; $(i, 1)$, $(i, 0)$, where $i = 1, \ldots, d - 1$; and $(d, 0)$ (see Figure 2(a)).
- If $\Sigma = P_3$, then the points are $(0, d - d_2 - j)$, where $j = 0, \ldots, d - d_2 - d_3$; $(i, d_3 - i + 1)$, $(i, d_3 - i)$, where $i = 1, \ldots, d_3$; $(i, 1)$, $(i, 0)$, where $i = d_3 + 1, \ldots, d - d_1 - 1$; $(d - d_1, d_1 - j)$, where $j = 0, \ldots, d_1$ (see Figure 2(b)).

Then we choose the following subdivisions of $\Delta$ constructed along the procedure of subsection C starting with $\gamma$.

If $\Sigma = \mathbb{C}P^2$, in the domain $\{(\xi, \eta) \in \Delta \ : \ 0 \leq \xi \leq 1, \ \eta \geq \xi\}$, to which we attribute the number $i = 0$, and in each strip $\{(\xi, \eta) \in \Delta \ : \ i \leq \xi \leq i + 1, \ \eta \geq 1\}$, $1 \leq i \leq d - 3$, we pack arbitrarily $d - i - 2$ parallelograms of (normalized) area 2.
The remaining part of $\Delta$ is uniquely covered by triangles of area 1 in order to obtain a pair $(S_+(\gamma), S_-(\gamma))$ of compressing subdivisions.

If $\Sigma = P_3$, in the domain

$$\{(\xi, \eta) \in \Delta : \xi \leq d_2 + d_3 - 1, \eta \geq \min\{d_3, \max\{d_3 + 1 - \xi, 1\}\} \}$$

we pack parallelograms of area 2 in a way compatible with the compressing procedure (see Figure 2(b)). Then in each strip

$$\{(\xi, \eta) \in \Delta : d_2 + d_3 - 1 + i \leq \xi \leq d_2 + d_3 + i, \eta \geq 1 \},$$

$i = 0, ... , d - d_1 - d_2 - d_3 - 1$, we pack arbitrarily $d - d_2 - d_3 - i - 1$ parallelograms of area 2. The remaining part of $\Delta$ is uniquely covered by triangles of area 1 in order to obtain a pair $(S_+(\gamma), S_-(\gamma))$ of compressing subdivisions.

We illustrate the construction in Figure 2, where the shadowed domain represents the common part of the described subdivisions.

All the chosen subdivisions of $\Delta$ are nodal and odd, and they correspond to irreducible tropical curves. The number of subdivisions under consideration is equal respectively to $d!/2$ for $CP^2$, and to $(d - d_2 - d_3)!/d_1!$ for $P_3$. Thus, according to Propositions 2.5 and 2.6 the same numbers are lower bounds for the Welschinger invariant.

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\square
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3 Non-invariance of the Welschinger number for elliptic curves

Theorem 3.1 For any $d \geq 4$, there exist generic collections $w'$ and $w''$ of 3$d$ distinct points in $\mathbb{R}P^2$ such that $W_{CP^2,d,1}(w') \neq W_{CP^2,d,1}(w'').$

Proof. As is known (see, for example, [11]), for any generic set $z$ of $3d - 1$ points in $CP^2$, the plane nodal irreducible elliptic curves of degree $d$, which pass through each of the points $z_1, ... , z_{3d-1}$ in $z$ and are nonsingular at these points, form a smooth quasi-projective one-dimensional variety $V = V(z) \subset |O_{CP^2}(d)| \simeq CP^{d(d+3)/2}$; the Zariski (projective) tangent space to $V$ at $C \in V$ is the one-dimensional linear system $\Lambda(\text{Sing}(C), z)$ of curves of degree $d$ which pass through $\text{Sing}(C)$ and $z$.

Denote by $\zeta : \tilde{C} \to C$ the normalization of $C \in V$, by $\tilde{z} = \{\tilde{z}_1, ... , \tilde{z}_{3d-1}\}$ and $\tilde{S}$ the pull-back of $z$ and $\text{Sing}(C)$, and by $\mathcal{N}$ the normal line bundle of the immersion $\phi : \tilde{C} \to C \to CP^2$. According to the adjunction formula, the bundle $\mathcal{N}$ is of degree
3d and is induced by \( \mathcal{O}_{\mathbb{C}P^2}(3) \), and, as follows from Riemann-Roch formula, there is one and only one point \( \tilde{\omega}_{C, z} \in \tilde{C} \) such that \( \tilde{z}_1 + \cdots + \tilde{z}_{3d-1} + \tilde{\omega}_{C, z} = c_1(N) \in \text{Pic } \tilde{C} \). On the other hand, for any \( C^{(1)} \neq C \) from \( \Lambda(\text{Sing}(C), z) \) one has

\[
\phi^* C^{(1)} = \tilde{z}_1 + \cdots + \tilde{z}_{3d-1} + \tilde{\omega}_{C, z} + \sum_{s \in \mathcal{S}} s, \tag{1}
\]

and any such curve \( C^{(1)} \) of degree \( d \) together with \( C \) generates \( \Lambda(\text{Sing}(C), z) \).

If \( z_1, \ldots, z_{3d-1} \) are real, the variety \( V \) is real, and for any real \( C \in V \) all the above objects, including \( \tilde{C} \) and \( \tilde{\omega}_{C, z} \), are real as well.

**Claim A.** For a generic \( z \) the map \( C \in V \mapsto \omega_C = \phi(\tilde{\omega}_{C, z}) \in \mathbb{C}P^2 \) is not constant.

Denote by \( \bar{V} \) the closure of \( V \), by \( V \) the closure of \( \mathcal{V} = \{(C, p) \in V \times \mathbb{C}P^2 : p \in C \} \), and by \( \pi : \bar{V} \times \mathbb{C}P^2 \to \mathbb{C}P^2 \) the canonical projection. Here, we deal with an immersed surface \( \mathcal{V} \), and, according to the above characterization of \( \Lambda(\text{Sing}(C), z) \), the image \( \theta \) of the map \( C \in V \mapsto \omega_C \in \mathbb{C}P^2 \) is the critical value set of \( \pi|_\mathcal{V} \), whereas \( \Theta = \{(C, \omega_{C, z}) : C \in V \} \subset \mathcal{V} \) is its critical point set.

**Claim B.** For a generic \( z \) and a generic \( \omega_C \in \theta \), the fibre \( \pi^{-1}(\omega_C) \) meets \( \bar{V} \) only inside \( V \), and, in addition, all the points \( (C', \omega_C) \in \pi^{-1}(\omega_C) \) distinct from \( (C, \omega_C) \) are regular.

Hence the set \( \pi^{-1}(z) \cap V \) may bifurcate only in a neighborhood of \( (C, \omega_C) \) as \( z \) ranges over a neighborhood of \( \omega_C \) in \( \mathbb{R}P^2 \).

Let \( \omega \) be a smooth point of \( \theta \) and let \( \omega = \omega_C \) for some \( C \in V \). The order \( m \) of ramification of \( \pi \) at \( (C, \omega) \) is given by the variation formula

\[
m = m(\tilde{C}(t) \cdot \Theta(t))\tilde{\omega}(C, z) = (C(t) \cdot \theta(t))\omega(C(t)) = 1 + \left( \frac{d}{dt} C(t) \cdot C(t) \right)\omega(C(t)),
\]

where \( t \) is a uniformization parameter of \( \theta \) at \( \omega \). On the other hand, Bézout’s theorem implies (cf. \( \square \)) that \( \left( \frac{d}{dt} C(t) \cdot C(t) \right)\omega(C(t)) = 1 \). Since \( \pi|_\mathcal{V} \) has a double folding along the germ of \( \theta \) at \( \omega \), the set \( \pi^{-1}(z) \cap \mathbb{R}V \) (where \( \mathbb{R}V \) is the real part of \( \mathcal{V} \)) either losess or obtains two elements when \( z \) varies in \( \mathbb{R}P^2 \) along a line through \( \omega_C \) transverse to the real part \( \mathbb{R}C \) of \( C \). Both the elements of \( \pi^{-1}(z) \cap \mathbb{R}V \) which appear (disappear) in this bifurcation, are real nodal elliptic curves with the same number of solitary nodes as \( C \), and hence the Welschinger number changes. Theorem \( \square \) is proven. (The proofs of Claims A and B are found in Appendix.) \( \square \)
Remark 3.2 The above bifurcation can be characterized as a jump of \( \dim \Lambda(\text{Sing}(C), z_1, \ldots, z_{3d}) \) from 0 to 1, where \( \Lambda(\text{Sing}(C), z_1, \ldots, z_{3d}) \) is the linear system of curves of degree \( d \), passing through the singular locus \( \text{Sing}(S) \) of a nodal elliptic curve \( C \) of degree \( d \) and through some points \( z_1, \ldots, z_{3d} \in C \setminus \text{Sing}(C) \). This cannot happen in the case of rational curves, since, by Riemann-Roch theorem, \( \dim \Lambda(\text{Sing}(C), z_1, \ldots, z_{3d-1}) = 0 \) for any rational nodal curve \( C \) of degree \( d \) and any distinct points \( z_1, \ldots, z_{3d-1} \in C \setminus \text{Sing}(C) \).

4 A remark on counting real plane curves with one node

To count curves with one node, one has to consider linear pencils. In the case of plane curves of degree \( d \), the computation of the Euler characteristic of \( \mathbb{R}P^2 \) by means of a pencil with \( d^2 \) real base points as done in [1] for \( d = 3 \), gives \( d^2 - 1 \) as the lower bound for the number of real nodal curves in such a pencil. The upper bound \( 3(d-1)^2 \) is given by the number of complex nodal curves in the pencil. To show that the bounds are sharp, consider a pencil generated by \( \prod_{\alpha=1}^{d}(x-x_\alpha) \) and \( \prod_{\alpha=1}^{d}(y-y_\alpha) \). The number of real nodal curves in this pencil is \( (d-1)^2 \). If we perturb \( \prod_{\alpha=1}^{d}(x-x_\alpha) \) and \( \prod_{\alpha=1}^{d}(y-y_\alpha) \) into unions of generic lines and then take a close generic pencil, the number of real nodal curves becomes \( 3(d-1)^2 \). If we perturb \( \prod_{\alpha=1}^{d}(x-x_\alpha) \) and \( \prod_{\alpha=1}^{d}(y-y_\alpha) \) by means of affine polynomials without minima and maxima (see [12]), we obtain \( (d-1)^2 + 2(d-1) = d^2 - 1 \) real nodal curves in the pencil.

Welschinger noticed that his number is not invariant for curves with one node if \( d \geq 4 \). The argument of Welschinger is the following. A pencil of plane curves of degree \( d \) is defined by \( \frac{1}{2}d(d+3) - 1 \) fixed points, and it has \( \frac{1}{2}d(d-3) + 1 \) extra base points. From the above computation of the Euler characteristic, it follows that, in this case, the Welschinger number is equal to \( d^2 - 1 - 2r \), where \( r \) is the number of pairs of conjugated imaginary points among the base ones. It remains to consider pencils with \( r \) varying from 0 to \([d(d-3)/4 + 1/2]\) and to choose each time all the defining \( \frac{1}{2}d(d+3) - 1 \) points real.

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Appendix

Proof of Claim A. Since the non-constancy of $\omega_C$ is an open condition, it is sufficient to verify it for a suitable special configuration of points. Let us choose ten generic points $z_1, \ldots, z_{10}$, trace a (unique) cubic $C_3$ through $z_1, \ldots, z_9$, and pick a rational nodal curve $R_{d-3}$ of degree $d - 3$, passing through $z_{10}$ and intersecting transversely $C_3$ at $3(d - 3)$ points, which all are nonsingular for $R_{d-3}$. Denote by $z_1', \ldots, z_{3d-1}', p_1, p_2$ the latter intersection points. Consider the elements $C \in V = V(z_1, \ldots, z_{10}, z_1', \ldots, z_{3d-1}')$ which are close to $C_3 R_{d-3}$, which have nodes near $p_1$, near each of $z_j'$ ($j = 11, \ldots, 3d - 1$), and near $\text{Sing}(R_{d-3})$, and for which the branch passing through $z_j'$ ($j = 11, \ldots, 3d - 1$) approaches $R_{d-3}$ as $C$ tends to $C_3 R_{d-3}$. By the standard transversality arguments, they smooth off the node at $p_2$ and form, together with $C_3 R_{d-3}$ a smooth (germ of) one-dimensional variety $\tilde{V}$ (one can easily verify the first order conditions which ensure that the linear system $\Lambda(p_2)$ of curves of degree $d$ through $p_2$, the linear systems of curves of degree $d$ meeting $R_{d-3}$ at $z_j'$ with multiplicity $\geq 2$ and the linear system $\Lambda(z_1, \ldots, z_{10}, \text{Sing}(R_{d-3}))$ intersect transversely all together, cf. [10] and [11]).

From now on we argue to contradiction. Since nodal elliptic curves of given degree form an irreducible variety (see [4]), for Claim A it suffices to treat the case when $\omega_C$ is a constant point $\omega$ for all $C \in \tilde{V}\{C_3 R_{d-3}\}$. Then $\omega$ is the intersection point of $C_3 R_{d-3}\{\{z_1, \ldots, z_{10}, z_1', \ldots, z_{3d-1}', p_1\} \cup \text{Sing}(R_{d-3})\}$ with a curve $C' = \frac{d}{dt}C(t) \in \Lambda(z_1, \ldots, z_{10}, z_1', \ldots, z_{3d-1}', p_1) \{C_3 R_{d-3}\}$, where $C(t)$ is a parametrization of $\tilde{V}$. On the other hand, $C \in \tilde{V}\{C_3 R_{d-3}\}$ meets $C_3$ at $z_1, \ldots, z_9, z_1', \ldots, z_{3d-1}'$ and some
point $p_1'$ in a neighborhood of $p_1$. Thus,

$$[z_1 + \ldots + z_9 + z_9' + \ldots + z_{3d-1}' + p_1' + w] = [z_1 + \ldots + z_9 + z_9' + \ldots + z_{3d-1}' + p_1 + w] \in \text{Pic}(C_3),$$

and it remains to check that $p_1' \neq p_1$ under suitable choice of $z_1, \ldots, z_9$.

Suppose further that all $C \in \bar{V}$ pass through $p_1$. Then $C' = \frac{d}{dt}C(t)$ belongs either to the space $T'$ of curves of degree $d$ crossing $C_3$ at $p_1$ with multiplicity $\geq 2$, or to the space $T''$ of curves of degree $d$ crossing $R_{d-3}$ at $p_1$ with multiplicity $\geq 2$. However, the intersection of $T'$ or $T''$ with the linear system $\Lambda'$ of curves of degree $d$ passing through $z_1, \ldots, z_{10}, \text{Sing}(R_{d-3})$ and crossing $R_{d-3}$ at each of $z_9', \ldots, z_{3d-1}'$ with multiplicity $\geq 2$, is transversal and zero-dimensional, since contains only $C_3R_{d-3}$. For example, a curve $H$ of degree $d$ from $T' \cap \Lambda'$ crosses $C_3$ at $z_1, \ldots, z_9, z_9', \ldots, z_{3d-1}'$ and twice at $p_1$, but

$$z_1 + \ldots + z_9 + z_9' + \ldots + z_{3d-1}' + 2p_1$$

can not be cut on $C_3$ by a curve of degree $d$, at least if we move slightly one of the points $z_1, \ldots, z_9$ along $C_3$. Hence $H$ splits off $C_3$. In turn the remaining component of degree $d - 3$ coincides with $R_{d-3}$, since it meets $R_{d-3}$ at $z_{10}, z_9', \ldots, z_{3d-1}, \text{Sing}(R_{d-3})$ with the total multiplicity $\geq 3d - 10 + (d - 4)(d - 5) = (d - 3)^2 + 1$. \hfill \Box

**Proof of Claim B.** To prove the first part of Claim B, it is sufficient to show that for a generic $z$ no curve $C' \in \bar{V}(z) \setminus V(z)$ has a component in common with $\theta = \theta(z)$. So assume by contrary that, for a generic $z$, some $C' \in \bar{V}(z) \setminus V(z)$ has a component $C'_1 \subset \theta$. Pick a generic point $q \in C'_1$ and a curve $C \in V(z)$ with $\omega_C = q \neq z_1$ (recall that the map $C \in V \mapsto \omega_C$ is not constant). Due to the generality of $z$, we may suppose that at least for small variations of $z$ the variations of $\bar{V} \setminus V$ are flat. Consider a deformation $z_1(t), t \in (\mathbb{C}, 0)$, of $z_1$ along $C'$. Due to the above flatness, $C'_1$ remains a component of the curve $\theta(z(t)), z(t) = \{z_1(t), z_2, \ldots, z_{3d-1}\}$, and $C = C(0)$ deforms in a family $C(t)$ of elliptic curves passing through $z(t)$ and satisfying $\omega_{C(t)} = q$. By construction, $\theta$ is an envelope of the family $V$, and hence $C'_1$ is tangent to each of $C(t), t \in (\mathbb{C}, 0)$, at $q$. Thus,

$$C(t) \cap C = z_2 + \ldots + z_{3d-1} + 2q + D(t) \tag{2}$$

with a divisor $D(t)$ close to $D = \phi(\bar{S})$. The limit $t = 0$ yields the relation

$$[\bar{z}_2 + \ldots + \bar{z}_{3d-1} + 2\bar{q} + \bar{S}] = \phi^*(O_{P^2}(d)) \in \text{Pic}(\bar{C}). \tag{3}$$

On the other hand, (1) implies

$$[\bar{z}_1 + \ldots + \bar{z}_{3d-1} + \bar{w}_C + \bar{S}] = \phi^*(O_{P^2}(d)) \in \text{Pic}(\bar{C}), \tag{4}$$

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which contradicts \([3]\) since \(q = \omega_C\) and \([z_1] \neq [\omega_C] \in \text{Pic}(C)\).

We prove the regularity statement arguing to contradiction as well. Since the nodal elliptic curves of given degree form an irreducible variety (see \([4]\)), it suffices to treat the case when for generic \(\mathbf{z}\) the whole curve \(\Theta \subset \mathcal{V}\) is multiply mapped on \(\theta\) by \(\pi\). Then, for a generic \(C \in \mathcal{V}(\mathbf{z})\) there exists \(\hat{C} \in \mathcal{V}(\mathbf{z}), \hat{C} \neq C\), with a common point \(\omega_C = \omega_{\hat{C}}\) on a nonsingular branch of \(\theta\). Both \(C\) and \(\hat{C}\) are tangent to \(\theta\) at this point and such a configuration must be preserved under small variations of \(\mathbf{z}\).

Note that \(C\) and \(\hat{C}\) intersect transversally at least at one of the points \(z_1, \ldots, z_{3d-1}\). Indeed, otherwise, taking a deformation \(z_1(t), z_2(t) \in C, t \in (\mathbb{C}, 0)\), such that \([z_1(t) + z_2(t)] = [z_1 + z_2] \in \text{Pic}(C)\), we obtain that \(\omega_C\) does not depend on \(t\), and \(\hat{C} = \hat{C}(0)\) varies in a family \(\hat{C}(t)\) of curves passing through \(z_1(t), z_2(t), z_3, \ldots, z_{3d-1}\) and having \(\omega_{\hat{C}(t)} = \omega_C\). However, the tangency assumption would imply that the total intersection of \(\hat{C}\) and \(\hat{C}(t), t \neq 0\), at \(z_3, \ldots, z_{3d-1}, \omega_C\) and in a neighborhood of \(\text{Sing}(C)\) is at least \(6d - 4 + d(d - 3) = d^2 + 3d - 4 > d^2\), which contradicts Bézout’s theorem.

Thus, we can suppose that \(C\) and \(\hat{C}\) are nonsingular and transverse at \(z_1\). The first order infinitesimal equisingular deformations of the curve \(\hat{C}\) with fixed points \(z_2, \ldots, z_{3d-1}\) correspond to the elements of the two-dimensional space \(H^0(\hat{C}, \mathcal{J}_{Z/\hat{C}} \otimes \mathcal{O}_{\mathbb{C}P^2}(d))\), where \(\mathcal{J}_{Z/\hat{C}}\) is the ideal sheaf of the zero-dimensional scheme \(Z \subset \hat{C}\) defined by the maximal ideals at \(z_2, \ldots, z_{3d-1}\) and \(\text{Sing}(\hat{C})\). Pick a section \(s \in H^0(\hat{C}, \mathcal{J}_{Z/\hat{C}} \otimes \mathcal{O}_{\mathbb{C}P^2}(d))\) which does not vanish neither at \(z_1\) nor at \(\omega_C\). Consider a deformation \(\hat{C}(t), t \in (\mathbb{C}, 0)\), \(\hat{C}(0) = \hat{C}\), generated by \(s\), i.e., \(\frac{d}{dt}\hat{C}(t) \bigg|_0 = s\), and keep \(C\) fixed, i.e., put \(C(t) = C\). The transversality of \(C\) and \(\hat{C}\) at \(z_1\) implies that, for small \(t\), there is a unique intersection point \(z_1(t)\) of \(\hat{C}(t)\) and \(C\) near \(z_1\). Put \(\mathbf{z}(t) = \{z_1(t), z_2, \ldots, z_{3d-1}\}\). Since \(s(\omega_C) \neq 0\), we have \(\frac{d}{dt}\omega_{\hat{C}(t), \mathbf{z}(t)} \notin T\theta\) (where \(T\) stands for the tangent space). On the other hand, \(\frac{d}{dt}\omega_{C(t), \mathbf{z}(t)} \in T\theta\).

It means that the branches of \(\theta(\mathbf{z})\) at \(\omega_C = \omega_{\hat{C}}\) split into separate branches in the above variation, which contradicts the initial assumption on the stability of the multiple projection \(\Theta(\mathbf{z}) \to \theta(\mathbf{z})\). \(\Box\)

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