SINGULAR MINIMAL TRANSLATION GRAPHS IN EUCLIDEAN SPACES

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ABSTRACT. In this paper, we consider the problem of finding the hypersurface $M^n$ in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ that satisfies an equation of mean curvature type, called singular minimal hypersurface equation. Such an equation physically characterizes the hypersurfaces in the upper halfspace $\mathbb{R}^{n+1}_+(\mathbf{u})$ with lowest gravity center, for a fixed unit vector $\mathbf{u} \in \mathbb{R}^{n+1}$. We first state that a singular minimal cylinder $M^n$ in $\mathbb{R}^{n+1}$ is either a hyperplane or an $\alpha$-catenary cylinder. It is also shown that this result remains true when $M^n$ is a translation hypersurface and $\mathbf{u}$ a horizontal vector. As a further application, we prove that a singular minimal translation graph in $\mathbb{R}^3$ of the form $z = f(x) + g(y + cx) + c$, $c \in \mathbb{R} - \{0\}$, with respect to a certain horizontal vector $\mathbf{u}$ is either a plane or an $\alpha$-catenary cylinder.

1. INTRODUCTION

Let the pair $(\mathbb{R}^3, g)$ denote the Euclidean 3-space and $\mathbf{u}$ a fixed unit vector in $\mathbb{R}^3$. Given a smooth immersion $\sigma$ of an oriented surface $M^2$ into the halfspace $\mathbb{R}^3_+(\mathbf{u}) = \{q \in \mathbb{R}^3 : g(q, \mathbf{u}) > 0\}$.

Let $\xi$ and $H$ be the Gauss map and the mean curvature of $\sigma$, respectively (see [2]). Then, for some real constant $\alpha$, the potential $\alpha$-energy of $\sigma$ in the direction of $\mathbf{u}$ can be introduced by (see [4], [20]-[22])

$$E(\sigma) = \int_{M^2} g(q, \mathbf{u})^\alpha dM^2,$$

where $dM^2$ denotes the measure on $M^2$ with respect to the induced metric tensor from the Euclidean metric $g$ in $\mathbb{R}^3$ and $q = \sigma(p), p \in M^2$.

Denoting $\Sigma : M^2 \times (-\theta, \theta) \to \mathbb{R}^3_+(\mathbf{u})$ a compactly supported variation of $\sigma$ with variation vector field $\zeta$, the first variation of $E$ then becomes

$$E'(0) = -\int_{M^2} (2Hg(\sigma, \mathbf{u}) - \alpha g(\xi, \mathbf{u})) g(\xi, \zeta)^{\alpha-1} dM^2.$$

Taking $\sigma$ as a critical point of $E$, it then follows

$$2H = \frac{\alpha g(\xi, \mathbf{u})}{g(\sigma, \mathbf{u})},$$

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which we call singular minimal (SM) surface equation, see [5]. In the cited paper Dierkes calls a surface $M^2$ a SM surface or $\alpha$–minimal surface if its mean curvature $H$ fulfills Eq. (1.2).

Eq. (1.2) is clearly of mean curvature type and an extension of the classic minimal surface equation ([17] p. 17), i.e. the situation $\alpha = 0$. Notice also that Eq. (1.2) is the Euler equation (see [27] p. 33) of the variational integral given in Eq. (1.1).

Let $\gamma = \gamma (s)$ be a curve in $\mathbb{R}^2$ and $u \in \mathbb{R}^2$ a fixed unit vector. For the curve $\gamma$ one dimensional case of Eq. (1.2) writes

$$\kappa (s) = \frac{g(N(\gamma (s)) u)}{g(\gamma (s), u)},$$

where $\kappa$ and $N$ are the curvature and unit principal normal vector field of $\gamma$. Hereinafter the curve $\gamma$ that the curvature $\kappa$ satisfies Eq. (1.3) is referred to as $\alpha$–catenary [5]. Up to a change of coordinates, we can take $u = (0, 1)$ and $\gamma$ as the graph of $y = f(s)$ . In that case Eq. (1.3) writes

$$\frac{f''}{1 + f'^2} = \frac{\alpha}{f}. $$

Let the $y$–axis in $\mathbb{R}^2$ denote the direction of the gravity. What to solve Eq. (1.4) in case $\alpha = 1$ is physically to find the curve $\gamma$ in the upper halfplane $y > 0$ with the lowest gravity center [23]. The solution of Eq. (1.4) is then the catenary

$$ f(s) = \frac{1}{\lambda} \cosh(\lambda s + \mu), \lambda, \mu \in \mathbb{R}, \lambda \neq 0. $$

Let us now take the surface $M^2$ as a generalized cylinder in $\mathbb{R}^3$ (see [8] p. 439) i.e. $M^2 = \gamma (s) + tw$, $s, t \in \mathbb{R}$, where $\gamma (s)$ is the so-called base curve, $w \in \mathbb{R}^3$ a fixed unit vector. We call it cylinder, shortly. Lopez [20] proved that a SM cylinder is a $\alpha$–catenary cylinder, namely a cylinder that takes the base curve as a $\alpha$–catenary. More generally, in [19][20], one was proved that a SM translation surface, i.e. a graph of the form $z = f(x) + g(y)$, is a $\alpha$–catenary cylinder, where $\{x, y, z\}$ is the orthogonal coordinate system in $\mathbb{R}^3$.

In this paper, we generalize to higher dimensions the mentioned results in previous paragraph. For this, we concern the following equation

$$nH = \frac{g(\xi, u)}{g(\sigma, u)}, \quad n \geq 2,$$

where $u \in \mathbb{R}^{n+1}$ is a fixed unit vector, $\xi$ and $H$ the Gauss map and the mean curvature of the smooth immersion $\sigma$ of an oriented hypersurface $M^n$ into the halfspace

$$\mathbb{R}^{n+1}_+(u) : \{q \in \mathbb{R}^{n+1} : g(q, u) > 0\}. $$

We call Eq. (1.5) SM hypersurface equation.

Let $\{x_1, ..., x_{n+1}\}$ denote the orthogonal coordinate system in $\mathbb{R}^{n+1}$ and the $x_{n+1}$–axis the direction of the gravity. Then Eq. (1.5) characterizes the hypersurfaces in $\mathbb{R}^{n+1}_+(u)$ with lowest gravity center.

To study Eq. (1.5) we first take a generalized cylinder $M^n$ in $\mathbb{R}^{n+1}$. By a generalized cylinder in $\mathbb{R}^{n+1}$, we mean a hypersurface given in the following form

$$M^n = \{\gamma (s) + t_1 w_1 + ... + t_{n-1} w_{n-1} : t_1, ..., t_{n-1} \in \mathbb{R}, s \in I \subseteq \mathbb{R}\},$$
where $w_1, w_2, ..., w_{n-1}$ are orthonormal vectors in $\mathbb{R}^{n+1}$ and $\gamma$ (so-called base curve) a 2-planar curve lying in $\Gamma = \text{Span} \{w_1, ..., w_{n-1}\}^\perp$. As before, we call it cylinder, shortly. We prove that besides hyperplanes only SM cylinders in $\mathbb{R}^{n+1}$ are the $\alpha-$catenary cylinders.

Afterwards we study SM translation hypersurfaces in $\mathbb{R}^{n+1}$, i.e., the graphs of the form (see [6])

$$x_{n+1} = f_1(x_1) + \ldots + f_n(x_n),$$

where $f_1, ..., f_n$ are smooth functions of single variable. We obtain that besides hyperplanes SM translation hypersurfaces in $\mathbb{R}^{n+1}$ are $\alpha-$catenary cylinders. More geometric details on this class of hypersurfaces can be found in [7], [9]-[11], [13, 14], [18], [24]-[26].

As a further application, we concern SM translation graphs in $\mathbb{R}^3$ of the form

$$z = f(x) + g(y + cx), c \in \mathbb{R} - \{0\}.$$ The study of the surfaces of this kind was initiated by Liu and Yu [15], obtaining explicit equations of minimal ones. These translation graphs, so-called affine translation surfaces, belong to the family of surfaces invariant by a group of translations (see Section 5). For some results and progress on those surfaces, see [12, 16], [28]-[30]. We show that a SM such translation graph in $\mathbb{R}^3$ is either a plane or a $\alpha-$catenary cylinder.

2. Preliminaries

This section is devoted to present a short brief for hypersurfaces in $\mathbb{R}^{n+1}$. Further details can be found in [3].

Let $\sigma: M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of an oriented hypersurface $M^n$. Then the Gauss map $\xi: M^n \to S^n$ maps $M^n$ to the unit hypersphere $S^n$ of $\mathbb{R}^{n+1}$. The differential $d\xi$ of the Gauss map $\xi$ is called Weingarten map. For some vectors $v$ and $w$ tangent to $M^n$ at the point $p \in M^n$, the shape operator $A_p$ is given by

$$\tilde{g}(A_p(v), w) = \tilde{g}(d\xi(v), w),$$

where $\tilde{g}$ is the induced metric tensor on $M^n$ from the Euclidean metric $g$ on $\mathbb{R}^{n+1}$. The second fundamental form $h$ of $\sigma$ is given in terms of the shape operator $A$ by

$$g(\xi, h(v, w)) = \tilde{g}(A_p(v), w).$$

The mean curvature of $\sigma$ at $p$ is defined by

$$H(p) = \frac{1}{n} \text{tr} A_p,$$

where $\text{tr}$ denotes the trace of $A_p$. A hypersurface is called minimal if $H$ vanishes identically.

The following result is well-known (see [11]).

**Proposition 2.1.** For a graph of $\mathbb{R}^{n+1}$ of the form $x_{n+1} = f(x_1, ..., x_n)$, we have

1. the unit normal vector field is

$$\xi = \frac{-1}{\phi} (f_{x_1}, ..., f_{x_n}, -1),$$

where $\phi = \sqrt{1 + \sum_{j=1}^{n} (f_{x_j})^2}$ and $f_{x_j} = \frac{\partial f}{\partial x_j}$;

2. the components of the induced metric tensor (or the first fundamental form) are

$$g_{ij} = \delta_{ij} + f_{x_i} f_{x_j},$$
where $\delta_{ij}$ is Kronocker’s Delta.

3. the components of the second fundamental form are

$$h_{ij} = \frac{f_{x_i}x_j}{\phi},$$

4. the matrix $[a_{ij}]$ of the shape operator is

$$a_{ij} = \sum_l h_{il}g^{lj} = \frac{f_{x_i}x_j}{\phi} - \sum_l \frac{f_{x_i}f_{x_j}}{\phi^3},$$

where $f_{x_i}x_j = \frac{\partial^2 f}{\partial x_i \partial x_j}$, and $[g^{lj}] = [g_{lj}]^{-1}$.

5. the mean curvature $H$ is

$$H = \frac{1}{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( \frac{f_{x_j}}{\phi} \right).$$

3. Generalized Cylinders

Let $w_1, \ldots, w_{n-1}$ be fixed orthonormal vectors in $\mathbb{R}^{n+1}$ and $\gamma = \gamma (s) , s \in I \subseteq \mathbb{R}$ a unit speed curve lying in the 2-plane $\Gamma = \text{Span} \{ w_1, \ldots, w_{n-1} \}$. Consider the cylinder in $\mathbb{R}^{n+1}$ given by Eq. (1.6). Denoting $\times$ the cross product in $\mathbb{R}^{n+1}$ the unit normal vector field $\xi$ of $M^n$ becomes

$$\xi (s) = w_1 \times \ldots \times w_{n-1} \times \gamma' (s) = \sum_{i=1}^{n+1} \det (e_i, w_1, \ldots, w_{n-1}, \gamma' (s)) e_i,$$

where $\{ e_1, \ldots, e_{n+1} \}$ is the standard basis of $\mathbb{R}^{n+1}$ and $\gamma' = \frac{d\gamma}{ds}$. Then, as the components of the fundamental forms of $M^n$, we get $g_{ij} = \delta_{ij}$, $1 \leq i, j \leq n$, and

$$h_{ij} = \begin{cases} \kappa, & \text{for } i = j = n \\ 0, & \text{otherwise} \end{cases}$$

where $\kappa = \kappa (s), s \in I$, denotes the curvature of $\gamma$. The mean curvature $H$ of $M^n$ turns to $H (s) = \frac{\kappa (s)}{n}$ and hence Eq. (1.5) leads to

$$\kappa (s) = \frac{\alpha (w_1 \times \ldots \times w_{n-1} \times \gamma' (s), u)}{\sum_{i=1}^{n-1} \alpha (w_i, u) t_i + \alpha (\gamma (s), u)}$$

or equivalently

$$\begin{align*}
(3.1) \quad & \kappa (s) \sum_{i=1}^{n-1} \alpha (w_i, u) t_i + \kappa (s) \alpha (\gamma (s), u) - \alpha (w_1 \times \ldots \times w_{n-1} \times \gamma' (s), u) = 0.
\end{align*}$$

If we take partial derivative of Eq. (3.1) with respect to $t_i$, $i = 1, \ldots, n - 1$, we find

$$\kappa (s) g (w_i, u) = 0$$

and

$$\kappa (s) g (\gamma (s), u) - \alpha (w_1 \times \ldots \times w_{n-1} \times \gamma' (s), u) = 0.$$

Assume, in Eq. (3.2), $\kappa = 0$ on $I$, identically. Then $\gamma$ is a straight-line and Eq. (3.3) follows that $M^n$ is a hyperplane parallel to $u$ because $\{ w_1, \ldots, w_{n-1}, \gamma' (s), u \}$
are linearly dependent at every \( s \in I \). Otherwise, i.e. \( \kappa \neq 0 \), then \( g(w_i, u) = 0 \) and we get that \( u \) is parallel to \( \Gamma \). Hence Eq. (3.3) can be rewritten as

\[
\kappa(s) = \alpha \frac{g(w_1 \times \ldots \times w_{n-1} \times \gamma'(s), u)}{g(\gamma(s), u)} = \alpha \frac{g(N(s), u)}{g(\gamma(s), u)},
\]

where \( N(s) \) denotes the principal unit normal vector field to \( \gamma \) at \( s \in I \). Eq. (3.4) implies that \( \gamma \) is a \( \alpha \)-catenary lying in \( \Gamma \) and \( M^n \) a \( \alpha \)-catenary cylinder.

Summing up, we have proved the following.

**Theorem 3.1.** The only SM cylinders in \( \mathbb{R}^{n+1} \) are either the hyperplanes parallel to \( u \) or the \( \alpha \)-catenary cylinders whose the rulings are orthogonal to \( u \).

### 4. Translation Hypersurfaces

Let \( \{x_1, \ldots, x_{n+1}\} \) be the orthogonal coordinate system in \( \mathbb{R}^{n+1} \). A translation hypersurface \( M^n \) in \( \mathbb{R}^{n+1} \) can be described as the sum of \( n \) curves \( \gamma_1, \ldots, \gamma_n \), so-called translating curves. Then \( M^n \) parameterizes locally as

\[
\sigma(x_1, \ldots, x_n) = \gamma_1(x_1) + \ldots + \gamma_n(x_n).
\]

If the translating curves \( \gamma_1, \ldots, \gamma_n \) lie in orthogonal 2-planes mutually then after a change of coordinates \( M^n \) becomes the graph of the form

\[
x_{n+1} = f_1(x_1) + \ldots + f_n(x_n),
\]

where \( f_1, \ldots, f_n \) are smooth functions of single variable. We mean this graph by a translation hypersurface throughout the section.

By Proposition 2.1 the Gauss map \( \xi \) and the mean curvature \( H \) of \( M^n \) are

\[
\xi = \frac{(-f_1', \ldots, -f_n', 1)}{\left[1 + \sum_{i=1}^n (f_i')^2\right]^{1/2}}
\]

and

\[
H = \frac{\sum_{i=1}^n \left(1 + \sum_{j \neq i}^n (f_j')^2\right) f_i''}{n \left[1 + \sum_{i=1}^n (f_i')^2\right]^{3/2}},
\]

where \( f_i' = \frac{df_i}{dx_i} \) and \( f_i'' = \frac{d^2f_i}{dx_i^2} \), \( i = 1, \ldots, n \). We have the following result:

**Theorem 4.1.** Let \( M^n \) be a SM translation hypersurface in \( \mathbb{R}^{n+1} \) with respect to a horizontal vector \( u \). Then it is either a hyperplane parallel to \( u \) or a \( \alpha \)-catenary cylinder whose the rulings are horizontal straight-lines orthogonal to \( u \).

**Proof.** Without lose of generality we can take \( u \) as \( (1, 0, \ldots, 0) \). Eqs. (1.5), (4.1) and (4.2) then follow

\[
\sum_{i=1}^n \left(1 + \sum_{i \neq j}^n (f_j')^2\right) f_i'' = \frac{-\alpha f_1'}{x_1} \left(1 + \sum_{i=1}^n (f_i')^2\right).
\]

We have to take \( \alpha f_1' \neq 0 \) in Eq. (4.3) because \( M^n \) is minimal otherwise, which we are not interested in. Let us assume that \( f_1'' = 0 \), or equivalently \( f_1' = \text{const.} \neq 0 \), in Eq. (4.3). Then the partial derivative of Eq. (4.3) with respect to \( x_1 \) leads to
α = 0. This is the case we already ignore and hereinafter \( f''_k \neq 0 \) is assumed. Next taking partial derivative of Eq. (4.3) with respect to \( x_k, k \neq 1 \), gives

\[
2f'_{k} \sum_{k \neq i}^{n} f''_{i} + \left( 1 + \sum_{k \neq i}^{n} (f'_{i})^2 \right) f''_{k} = \frac{-2\alpha f'_{1}}{x_{1}} f'_{k} f''_{k}.
\]

It can be seen that \( f''_{k} = 0, k = 2, 3, ..., n \), is a solution of Eq. (4.4). This means that \( f_k (x_k) = \lambda_k x_k + \mu_k, \lambda_k, \mu_k \in \mathbb{R} \), and \( M^n \) is a cylinder that can be written as

\[
\sigma (x_1, x_2, ..., x_n) = \left( x_1, 0, ..., 0, f_1 (x_1) + \sum_{i=2}^{n} \mu_k \right) + x_2 (0, 1, ..., 0, \lambda_2) + ... + x_n (0, 0, ..., 1, \lambda_n)
\]

Due to Theorem 3.1, we obtain the statement in the hypothesis of the theorem. In order to finish the proof of the theorem, it is needed to show that Eq. (4.4) has no other solution than \( f''_k = 0 \). Assuming now \( f''_k \neq 0 \) and dividing (4.4) with \( 2f'_{k} f''_{k} \), we have

\[
\sum_{k \neq i}^{n} f''_{i} + \left( 1 + \sum_{k \neq i}^{n} (f'_{i})^2 \right) \frac{f''_{k}}{2f'_{k} f''_{k}} = \frac{-\alpha f'_{1}}{x_{1}}.
\]

Taking partial derivative of Eq. (4.5) with respect to \( x_k, k \neq 1 \), gives

\[
\left( 1 + \sum_{k \neq i}^{n} (f'_{i})^2 \right) \left( \frac{f''_{k}}{2f'_{k} f''_{k}} \right)' = 0,
\]

or equivalently,

\[
f''_{k} = 2\nu_k f'_{k} f''_{k},
\]

for some constant \( \nu_k \). We distinguish two cases:

- \( \nu_k = 0, i.e. \ f''_{k} = \lambda_{k}, \) for some nonzero constant \( \lambda_{k}, k = 2, 3, ..., n \). After substituting this into Eq. (4.3), one can be rewritten as

\[
G_1 (x_1) + G_2 (x_1) (f''_{2})^2 + G_3 (x_1) (f''_{3})^2 + ... + G_n (x_1) (f''_{n})^2 = 0,
\]

where

\[
G_1 (x_1) = f''_{1} + \frac{\alpha f'_{1}}{x_1} + \frac{\alpha (f'_{1})^3}{x_1} + \left[ (f'_{1})^2 + 1 \right] \sum_{i=2}^{n} \lambda_i
\]

\[
G_2 (x_2) = f''_{2} + \frac{\alpha f'_{2}}{x_2} + \sum_{i=3}^{n} \lambda_i
\]

\[
G_3 (x_3) = f''_{3} + \frac{\alpha f'_{3}}{x_3} + \sum_{i \neq 2}^{n} \lambda_i
\]

\[
G_n (x_n) = f''_{n} + \frac{\alpha f'_{n}}{x_n} + \sum_{i=2}^{n-1} \lambda_i.
\]

Because \( f''_{k} \neq 0, k = 2, 3, ..., n \), taking partial derivative of (4.7) with respect to \( x_k \) gives that the functions \( G_1, ..., G_n \) are all zero. If we substract second equality in Eq. (4.8) from third one then we find \( \lambda_2 = \lambda_3 \). Analogously if we substract third equality in Eq. (4.8) from fourth one then we find \( \lambda_3 = \lambda_4 \). Hence by proceeding this for other equalities in Eq. (4.8) we
obtain $\lambda_2 = \lambda_3 = \ldots = \lambda_n$. Put $\tilde{\lambda} = \lambda_k$, $k = 2, 3, \ldots, n$. The following can be obtained by some equality in Eq. (4.8) (other than the first one)

\begin{equation}
(4.9)\quad f''_1 + \frac{\alpha f'_1}{x_1} = \tilde{\lambda}(2 - n).
\end{equation}

Substituting (4.9) into the first equality in Eq. (4.8) leads to

\begin{equation}
(4.10)\quad \tilde{\lambda}(n - 1) (f'_1)^2 + \alpha \left( \frac{f'_1}{x_1} \right)^3 + \tilde{\lambda} = 0.
\end{equation}

By taking derivative of Eq. (4.10) with respect to $x_1$ and then dividing $x_1$ we derive

\begin{equation}
(4.11)\quad f''_1 \left[ 2\tilde{\lambda}(n - 1) \frac{f'_1}{x_1} + 3\alpha \left( \frac{f'_1}{x_1} \right)^2 \right] - \left( \frac{f'_1}{x_1} \right)^3 = 0.
\end{equation}

From Eq. (4.9) we get $f''_1 = \tilde{\lambda}(2 - n) - \frac{\alpha f'_1}{x_1}$ and considering this into Eq. (4.11) yields a polynomial equation of $(f'_1)$, in which the leading coefficient is $-3\alpha^2 - 1$. This leads to a contradiction.

- $\nu_k \neq 0$. Hence Eq. (4.5) reduces to

\begin{equation}
(4.12)\quad \sum_{k \neq i}^n f''_i + \nu_k \left( 1 + \sum_{k \neq i}^n (f'_i)^2 \right) = -\frac{\alpha f'_1}{x_1}.
\end{equation}

Taking partial derivative of Eq. (4.12) with respect to $x_1$, $1 \neq l \neq k$, yields

\begin{equation}
(4.13)\quad f'''_l + 2\nu_k f'_i f''_l = 0.
\end{equation}

Because Eq. (4.6) hold for $k = 2, 3, \ldots, n$, we have $f'''_l = 2\nu_l f'_i f''_l$. By substituting this into Eq. (4.13), we obtain

\begin{equation}
(4.14)\quad \nu_k + \nu_l = 0.
\end{equation}

On the other hand, integrating Eq. (4.6) gives

\begin{equation}
(4.15)\quad f''_k = \nu_k (f'_k)^2 + \mu_k.
\end{equation}

Substituting Eqs. (4.14) and (4.15) into Eq. (4.12) leads to

\begin{equation}
(4.16)\quad f''_1 = - \left[ \nu_k (f'_1)^2 + \frac{\alpha f'_1}{x_1} + \varepsilon \right],
\end{equation}

where $\varepsilon = \nu_k + \sum_{k \neq i=2}^n \mu_i$. After substituting Eqs. (4.15) and (4.16) into Eq. (4.3) we can rearrange it as

\begin{equation}
(4.17)\quad \sum_{i=2}^n \left[ (\nu_i - \nu_k) (f'_i)^2 + \mu_i \right] - \nu_k \left( f'_1 \right)^2 + \sum_{i=2}^n \left( 1 + \sum_{j \neq i=2}^n (f'_j)^2 \right) \left( \nu_i (f'_j)^2 + \mu_i \right) + \varepsilon \left( 1 + \sum_{i=2}^n (f'_j)^2 \right) = -\frac{\alpha (f'_1)^3}{x_1}.
\end{equation}

The partial derivatives of Eq. (4.17) with respect to $x_1$ and $x_l$, $1 \neq l \neq k$ lead to

\begin{equation}
(4.18)\quad \nu_k - \nu_l = 0.
\end{equation}

Comparing Eqs. (4.14) and (4.18) contradicts with $\nu_k \neq 0$. 
Let $M^2$ be a translation graph in $\mathbb{R}^3$ of the form

$$z = f(x) + g(y + cx), \ c \in \mathbb{R} \setminus \{0\},$$

for some smooth functions $f$ and $g$. By the change of parameters $\tilde{x} = x$ and $\tilde{y} = y + cx$, we can choose a local parameterization on $M^2$ as

$$\sigma : I \times J \subset \mathbb{R}^2 \to \mathbb{R}^3$$

and

$$\sigma (\tilde{x}, \tilde{y}) = (\tilde{x}, \tilde{y} - c\tilde{x}, f(\tilde{x}) + g(\tilde{y})).$$

Eq. (5.1) follows that it can be written as a sum of two planar curves, i.e.

$$M^2 = \gamma (\tilde{x}) + \eta (\tilde{y}),$$

where

$$\gamma : I \subset \mathbb{R} \to \mathbb{R}^3, \ \gamma (\tilde{x}) = (\tilde{x}, -c\tilde{x}, f(\tilde{x}))$$

and

$$\eta : J \subset \mathbb{R} \to \mathbb{R}^3, \ \eta (\tilde{y}) = (0, \tilde{y}, g(\tilde{y})).$$

Notice that $\gamma$ and $\eta$ lie in the non-orthogonal planes to each other. Hence $M^2$ turns to an extension of classical translation surface.

Let us put $f' = \frac{df}{d\tilde{x}}$ and $g' = \frac{dg}{d\tilde{y}}$, etc. Thereby we have

**Theorem 5.1.** Let $M^2$ be a translation graph in $\mathbb{R}^3$ of the form $z = f(x) + g(y + cx), \ c \neq 0$. If $M^2$ is a SM surface with respect to the horizontal vector $u = (1, 0, 0)$ then it is either a plane parallel to $u$ or a $\alpha$-catenary cylinder whose the rulings are horizontal straight-lines orthogonal to $u$.

**Proof.** If $M^2$ is a SM surface then Eq. (1.2) writes

$$\frac{[1 + c^2 + (f')^2]}{1 + f'' + \left(\frac{c + f'g'}{f' + cg'}\right)^2} \frac{g''}{\tilde{x}} = -\alpha \frac{f' + cg'}{\tilde{x}}.$$

We distinguish several cases:

- $f'' = 0$. Hence Eq. (5.2) reduces to

$$\frac{1 + (g')^2}{1 + (f' + cg')^2 + (g')^2} \frac{g''}{\tilde{x}} = -\alpha \frac{f' + cg'}{\tilde{x}},$$

where $f' = f_0$ is some constant. Because $\alpha \neq 0$, the partial derivative of Eq. (5.3) with respect to $\tilde{x}$ gives

$$f_0 + cg' = 0.$$

Eq. (5.4) implies $g'' = 0$. Putting $g' = g_0$, for some nonzero constant $g_0$, we conclude

$$f_0 + cg_0 = 0,$$

which yields that

$$z(x, y) = f_0x + g_0(y + cx) + \mu = g_0y + \mu,$$

for some constant $\mu$. This leads $M^2$ to a plane parallel to $u$. 

• $f'' \neq 0$ and $g'' = 0$. Then we have $g(y + cx) = \lambda(cx + y) + \mu$, for some constants $\lambda, \mu$. Therefore $M^2$ turns to a cylinder of the form

$$
\sigma(x, y) = (x, y, f(x) + \lambda(cx + y) + \mu) = (x, 0, f(x) + \lambda cx + \mu) + y(0, 1, \lambda).
$$

Due to [21, Theorem 1], we obtain that $M^2$ is a $\alpha$–catenary cylinder whose the rulings are parallel to the vector $(0, 1, \lambda)$.

• $f''g'' \neq 0$. By taking partial derivative of Eq. (5.2) with respect to $\tilde{y}$, we have

$$
2g'g''f'' + \left[1 + c^2 + (f')^2\right] g''' = \frac{-ac}{\tilde{x}} \left[1 + (f' + cg')^2 + (g')^2\right] g'' - 2\alpha \frac{f' + cg'}{\tilde{x}} \left[cf' + (c^2 + 1)g'\right] g''.
$$

Dividing Eq. (5.5) with $g''$ leads to

$$
2g'f'' + \left[1 + c^2 + (f')^2\right] \frac{g''}{g'} = \frac{-ac}{\tilde{x}} \left[1 + (f' + cg')^2 + (g')^2\right] - 2\alpha \left(\frac{f' + cg'}{\tilde{x}}\right) \left[cf' + (c^2 + 1)g'\right].
$$

Assume now that $\frac{g''}{g'} = \lambda$, for some constant $\lambda$. Then Eq. (5.6) turns to

$$
2\tilde{x}f''g' + \left[1 + c^2 + (f')^2\right] \lambda \tilde{x} = -ac \left[1 + (f' + cg')^2 + (g')^2\right] - 2\alpha \left(\frac{f' + cg'}{\tilde{x}}\right) \left[cf' + (c^2 + 1)g'\right].
$$

If we take partial derivative of Eq. (5.7) with respect to $\tilde{y}$ and then divide it with $g''$, we find

$$
\tilde{x}f'' + \alpha (3c^2 + 1) f' = -3ac (c^2 + 1) g'.
$$

The partial derivative of Eq. (5.8) with respect to $\tilde{y}$ yields a contradiction.

Hence we conclude $\left(\frac{g''}{g'}\right)' \neq 0$. Next taking partial derivative of Eq. (5.6) with respect to $\tilde{y}$ and dividing it with $g''$

$$
f''' + \frac{1}{2} \left[1 + c^2 + (f')^2\right] \left[\left(\frac{g''}{g'}\right)'/g''\right] = \frac{-2ac}{\tilde{x}} \left[cf' + (c^2 + 1)g'\right] - \frac{\alpha (c^2 + 1)}{\tilde{x}} [f' + cg'].
$$

The partial derivative of Eq. (5.9) with respect to $\tilde{y}$ yields

$$
\left[1 + c^2 + (f')^2\right] \left[\left(\frac{g''}{g'}\right)'/g''\right]' = \frac{-3ac (c^2 + 1)}{\tilde{x}} g' + \frac{\lambda_1}{\tilde{x}} g''
$$

which implies that both hand-sides cannot vanish. Hence Eq. (5.10) leads to

$$
1 + c^2 + (f')^2 = \frac{-3ac (c^2 + 1)}{\lambda_1 \tilde{x}}
$$

and

$$
\left[\left(\frac{g''}{g'}\right)'/g''\right]' = \lambda_1 g''
$$

for nonzero constant $\lambda_1$. Integrating of Eq (5.12) gives

$$
g'' = \frac{\lambda_1}{6} (g')^3 + \frac{\lambda_2}{2} (g')^2 + \lambda_3 g' + \lambda_4.
$$
for some constants $\lambda_2, \lambda_3, \lambda_4$. Substituting Eq. (5.13) into Eq. (5.2) gives

\[
\begin{align*}
\left[1 + (g')^2\right] f'' + \left[1 + c^2 + (f')^2\right] \left[\frac{1}{6} (g')^3 + \frac{\lambda_2}{2} (g')^2 + \lambda_3 g' + \lambda_4\right] = \\
= -\alpha \frac{f' + cg'}{x} \left[1 + (f' + cg')^2 + (g')^2\right].
\end{align*}
\]

Eq. (5.14) is a polynomial equation of $g'$ in which the coefficient of the term of degree 1 satisfies

\[
\lambda_3 \left[1 + c^2 + (f')^2\right] + \frac{\alpha c}{x} + \frac{3\alpha c (f')^2}{x} = 0.
\]

Putting Eq. (5.11) into Eq. (5.15) yields

\[
\frac{-3\alpha \lambda_3 c (c^2 + 1)}{\lambda_1} + \alpha c + 3\alpha (f')^2 = 0,
\]

which implies $f'' = 0$. This is not our case.

6. Conclusions

The results in Sections 4 and 5 on singular minimal translation (hyper)surfaces were obtained by taking $u$ as a horizontal vector. It is pointed out that the vector $u$ is parallel to the hyperplane $x_{n+1} = 0$ where the hypersurface is a graph. The investigating of these graphs is still open when $u$ is a vertical vector, that is, $u$ is normal to the hyperplane $x_{n+1} = 0$.

Moreover, the translation graph considering in Section 5 can be directly extended to higher dimensions as

\[
z (x_1, \ldots, x_n) = f_1 (x_1) + \ldots + f_{n-1} (x_{n-1}) + g \left( x_n + \sum_{i=1}^{n-1} c_i x_i \right), \quad n \geq 2,
\]

for smooth functions $f_1, \ldots, f_{n-1}, g$. This is explicitly a generalization in arbitrary dimensions of a classical translation hypersurface. By taking the vector $u$ as a horizontal or a vertical vector, it could be another interesting problem to find a singular minimal translation graph in $\mathbb{R}^{n+1}$ given by Eq. (6.1).

As a final remark these problems can be also considered by taking the vector $u$ as arbitrary.

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