SOME COMPUTATIONS IN EQUIVARIANT COBORDISM IN RELATION TO MILNOR MANIFOLDS

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Abstract. Let \( \mathcal{N}_* \) be the unoriented cobordism algebra, let \( G = (\mathbb{Z}_2)^n \) and let \( Z_*(G) \) denote the equivariant cobordism algebra of \( G \)-manifolds with finite stationary point sets. Let \( \epsilon_* : Z_*(G) \rightarrow \mathcal{N}_* \) be the homomorphism which forgets the \( G \)-action. We use Milnor manifolds (degree 1 hypersurfaces in \( \mathbb{RP}^m \times \mathbb{RP}^n \)) to construct non-trivial elements in \( Z_*(G) \) in degrees up to \( 2^n - 5 \). Moreover, in most cases these elements can be arranged to be in \( \text{Ker}(\epsilon_*) \).

1. Introduction

The unoriented cobordism algebra \( \mathcal{N}_* \) of smooth closed manifolds is well understood. For \( G = (\mathbb{Z}_2)^n \), let \( Z_*(G) \) denote the equivariant cobordism algebra of smooth closed manifolds with smooth \( G \)-actions with finite stationary point sets. For \( G = (\mathbb{Z}_2)^n, n \leq 2 \), the algebras are completely determined \([\mathbf{1}]\). For instance, when \( n = 2 \), it is well-known that \( Z_*(G) \) is isomorphic to the polynomial algebra with one generator in degree 2. In fact, the equivariant cobordism algebra is generated by the cobordism class \([\mathbb{RP}^2, \phi]\), where the action \( \phi \) of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) on \( \mathbb{RP}^2 \) is given by generators \( T_1, T_2 \) of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) as follows:

\[
T_1([x, y, z]) = [-x, y, z], \quad T_2([x, y, z]) = [x, -y, z].
\]

For \( n > 2 \), the structure of the equivariant cobordism algebra is not known. In order to understand the structure of \( Z_*(G) \), for \( n > 2 \), it is important to have complete knowledge about its indecomposable elements. In \( \mathbf{3} \), the authors considered this question and proved a sufficient criterion to determine indecomposable elements. It was shown that flag manifolds with suitable actions provide a large supply of indecomposable elements in \( Z_*(\mathbb{Z}_2)^n) \) for \( d \leq n \). In the present paper, we consider Milnor manifolds \( \mathcal{H}(m, n) \) with suitable actions of \( G = (\mathbb{Z}_2)^n \), with finite stationary points and provide indecomposable elements of \( Z_*(\mathbb{Z}_2)^n) \) in dimensions up to \( 2^n - 5 \). Finally, we prove that \( 1 \leq d \leq 2^{k-i+1} - 5 \), then there are at least \( i \) linearly independent, indecomposable elements in \( Z_*(\mathbb{Z}_2)^k) \) (Theorem \[\mathbf{1.4}\] Theorem \[\mathbf{1.5}\]). In fact the choices can be so arranged that the corresponding Milnor manifolds \( \mathcal{H}(m, n) \) bound non-equivariantly (Remark \[\mathbf{1.6}\]).

The paper is organized as follows. In Section 2 we fix notations and recall necessary background. In Section 3 we consider Milnor manifolds \( \mathcal{H}(m, n), m \leq n \) with suitable actions of \( (\mathbb{Z}_2)^n \) with finite stationary point sets and show that the equivariant cobordism class \([\mathbb{Z}_2)^n, \mathcal{H}(m, n)] \) is non trivial if \( n \geq 3, m < n \). In Section 4 we prove the indecomposability of \([\mathbb{Z}_2)^k, \mathcal{H}(m, n)] \) in certain cases to conclude the results mentioned above.

Date: October 24, 2013.

2010 Mathematics Subject Classification. Primary: 55N22, 55N91; Secondary: 55P91, 55Q91, 55M35.
2. Representation and Cobordism

Let $G$ be a finite group and $M^d$ a smooth, closed manifold of dimension $d$. We will denote by $(G, M^d)$ a smooth action of $G$ on $M$ with finite stationary point set. The action map will be denoted by $\phi: G \times M \to M$. Given such an action, $(G, M^d)$, we say that $M$ bords equivariantly if there is a smooth action of $G$ on a compact $(d+1)$-dimensional manifold $W$, such that the induced $G$-action on the boundary of $W$ is equivariantly diffeomorphic to $(G, M^d)$. Two actions $(G, M_1^d)$ and $(G, M_2^d)$ are said to be equivariantly cobordant if there disjoint union $(G, M_1^d \sqcup M_2^d)$ bords equivariantly. We will denote the equivalence class of $(G, M^d)$, under the relation of equivariant cobordism, by $[G, M^d]$ and the set of equivalence classes of $d$-dimensional equivariantly cobordant, smooth, closed manifolds by $Z_d(G)$. Note that $Z_d(G)$ becomes an abelian group under disjoint action. The operation of cartesian product and diagonal action makes $\sum_{d \geq 0} Z_d(G)$ into a graded commutative algebra, called the equivariant cobordism algebra $Z_*(G)$. Throughout this document we will assume that the group $G$ is a 2-group, i.e., $G = (\mathbb{Z}_2)^n$, for some $n \geq 0$.

Recall from [1] the results relating equivariant cobordism to tangential representations. Let $R_n(G)$ denote the $\mathbb{Z}_2$-vector space whose basis is the set of $n$-dimensional real $G$-representations up to isomorphism. The operation of direct sum makes $R_\ast(G)$ into a graded algebra. The ring $R_\ast(G)$ is a polynomial ring over $\mathbb{Z}_2$ on the set of irreducible real representations of $G$.

For $G = (\mathbb{Z}_2)^n$ we have $R_\ast(G) \cong \mathbb{Z}_2[\hat{G}]$ where $\hat{G} = \text{Hom}(G, \mathbb{Z}_2)$. For $n \in \mathbb{N}$ denote by $\underline{n}$ the subset of $\mathbb{N}$ given by $\{1, 2, \ldots, n\}$. Let $T_1, \ldots, T_n$ denote the generators of $G$. For any subset $S \in \underline{n}$, let $\chi_S \in \hat{G}$ denote the character defined by $\chi_S(T_i) = 1$ if $i \in S$ and $\chi_S(T_i) = 0$ if $i \notin S$. Let $Y_S$ denote the irreducible representation class of $\chi_S$. Then we have, \[
R_\ast(G) \cong \mathbb{Z}_2[Y_S | S \subset \underline{n}].
\]

Consider an action $(G, M^d)$ with finite stationary point set $\{x_1, \ldots, x_k\}$. For each $i$, let $X(x_i)$ denote the tangential representation at $x_i$. Then we have an algebra homomorphism \[
\eta_\ast: Z_\ast(G) \longrightarrow R_\ast(G)
\]
\[
[G, M^d] \mapsto \sum_{i=1}^k X(x_i) \in R_d(G).
\]

Stong ([6]) proved that $\eta_\ast$ is a monomorphism. Note that finite stationary point set condition automatically implies that the image of $\eta$ lies in $\tilde{R}_\ast(G)$, the subalgebra of $R_\ast(G)$ generated by $\{Y_S | S \subset \underline{n}, S \neq \emptyset\}$, where $\emptyset$ denotes the empty set.

For $m \geq 1$, let $B = \mathbb{Z}_2[b_1, \ldots, b_m, \ldots]$ be the graded $\mathbb{Z}_2$-algebra, where $\deg b_i = i$, $i \geq 1$. Often the convention $b_0 = 1$ is used. Let $L$ denote the $B$-algebra of all formal power series in variables $y_1, \ldots, y_n$, where for all $i$, $\deg y_i = 1$. That is, $L = B[[y_1, \ldots, y_n]]$. Let $Q(L)$ denote the quotient field of $L$.

Define a $\mathbb{Z}_2$-algebra homomorphism $\gamma: \tilde{R}_\ast(G) \to Q(L)$ as follows:

\[
\gamma(Y_S) = \frac{1}{\sum_{i \in S} b_i} \sum_{r \geq 0} b_r \left( \sum_{i \in S} y_i \right)^r.
\]

Tom Dieck ([7]) has proved that $\gamma \circ \eta: Z_\ast(G) \to Q(L)$ is injective and $\text{Im}(\gamma \circ \eta) \subset L$.

Recall that an element of the equivariant cobordism algebra is said to be indecomposable if it cannot be written as a sum of product of lower dimensional equivariant cobordism classes. G. Mukherjee and P. Sankaran ([8]) gave a sufficient criterion for indecomposability of an element of $Z_\ast(G)$. With notations as above
2.1. Theorem. Let \([G, M^d] \in Z_d(G)\). Suppose for some \(k > d\), either the coefficient of \(b_k\) or the coefficient of \(b_{k-1}b_1\) in \(\gamma \circ \eta([G, M^d])\), is non-zero. Then \([G, M] \in Z_d(G)\) is indecomposable.

3. 2-Group Actions on Milnor Manifolds

The Milnor manifold \(\mathcal{H}(m, n), m \leq n\), is defined to be the submanifold (of dimension \(m+n-1\)) of \(\mathbb{R}^m \times \mathbb{R}^n\), given by

\[
\{(x_0, \ldots, x_m), (y_0, \ldots, y_n) | \sum_{j=0}^m x_jy_j = 0\}.
\]

Tom Dieck [2] has defined actions of 2-groups on \(\mathcal{H}(m, n)\). We consider a special case. Fix \(T_k\) (1 \( \leq k \leq n\)) a set of \(n\) generators on \((\mathbb{Z}_2)^n\). Define an action of \((\mathbb{Z}_2)^n\) on \(\mathcal{H}(m, n)\) by:

\[
T_k([x_0, \ldots, x_m], [y_0, \ldots, y_n]) = \begin{cases} 
([x_0, \ldots, -x_k, \ldots, x_m], [y_0, \ldots, -y_k, \ldots, y_n]) & k \leq m \\
([x_0, \ldots, x_m], [y_0, \ldots, -y_k, \ldots, y_n]) & k > m
\end{cases}
\]

We refer to the above action as \(\phi\). Note that this action has \(n(m+1)\) stationary points

\[
P_{i,j} = ([0, \ldots, 1, \ldots, 0], [0, \ldots, 1, \ldots, 0])
\]

where the first 1 is in the \(i\)-th place and the second 1 is in the \(j\)-th place, \(0 \leq i \leq m\), \(0 \leq j \leq n\) and \(i \neq j\). We will apply Theorem 2.1 to prove that in many cases the induced elements in the equivariant cobordism algebra are indecomposable. In this section we show that the equivariant cobordism class \([\mathbb{Z}_2)^n, \mathcal{H}(m, n)\) is non trivial if \(n \geq 3\), \(m < n\).

Identify the tangent space to \(\mathcal{H}(m, n)\) at the stationary point \(P_{i,j}, T_{P_{i,j}}(\mathcal{H}(m, n))\), with \(\mathbb{R}^m \times \mathbb{R}^n\), with coordinates given by \((x_0, \ldots, x_m, y_0, \ldots, y_j, \ldots, y_n)\). The Milnor manifold with these coordinates is the hypersurface

\[
y_i + \sum_{l \neq i} x_iy_i = 0, \quad j > m
\]

\[
y_i + x_j + \sum_{l \neq i,j} x_iy_l = 0, \quad j \leq m.
\]

Therefore, \(T_{P_{i,j}}(\mathcal{H}(m, n))\) is the subspace of \(\mathbb{R}^m \times \mathbb{R}^n\) determined by the equations

\[
y_i = 0, \quad j > m
\]

\[
y_i + x_j = 0, \quad j \leq m.
\]

Let \(e_0, \ldots, e_m, f_0, \ldots, f_n\) denote the standard basis of \(\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}\). Then a basis of \(T_{P_{i,j}}(\mathcal{H}(m, n))\) may be written as \(\{e_0, \ldots, \hat{e}_i, \ldots, e_m, f_0, \ldots, \hat{f}_j, \ldots, f_n\}\) if \(j > m\), and as

\[
\{e_0, \ldots, \hat{e}_i, \hat{e}_j, \ldots, e_m, f_0, \ldots, \hat{f}_j, \ldots, f_n\} + \{f_i = -e_j\}, \quad j \leq m.
\]

Recall from Section 2 the notation \(Y_S\) of the irreducible representation of \((\mathbb{Z}_2)^n\) of induced by \(S\) (the corresponding homomorphism \(Y_S: (\mathbb{Z}_2)^{n+1} \rightarrow \mathbb{Z}_2 \cong \{\pm 1\}\) satisfies \(Y_S(T_i) = \text{e}^{-1}\) if \(i \in S\) and \(1\) otherwise). In the notation above observe that the action of \((\mathbb{Z}_2)^n\) on \(R\{e_i\} \subset T_{P_{i,j}}(\mathcal{H}(m, n))\) is given by \(Y_{(i)}\) if \(l > 0\) and \(Y_{(i)}\) if \(l = 0\). We make similar computations to deduce

\[
\eta_\ast([\mathbb{Z}_2)^n, \mathcal{H}(m, n)] = \prod_{i=1}^m Y_{(i)} \left( \sum_{j=1}^n \prod_{k=1, k \neq j}^n Y_{(k,j)} \right) + \sum_{i=1}^m Y_{(i)} \prod_{k=1, k \neq i}^n Y_{(k,i)} \left( \prod_{l=1, l \neq i}^n Y_{(l)} + \sum_{j=1, j \neq i}^n Y_{(j)} \prod_{l=1, l \neq i, j}^n Y_{(l,j)} \right)
\]
Note that in the above expression the coefficient of $Y_{(1)} \cdots Y_{(n)}$ cannot be cancelled if $m \geq 3$ and $m = 1$. For $m = 2$ the term $Y_{(1)} Y_{(l)} \prod_{j=1,j \neq l}^{n} Y_{(j,n)} \prod_{k=1,k \neq l,j}^{n} Y_{(k,l)}$ cannot be cancelled if $i < m$, $l > m$. Therefore $\eta_{*}([\mathbb{Z}_{2}^{*}, H(m,n)])$ cannot be zero if $m < n$. Hence we obtain,

3.1. Proposition. $([\mathbb{Z}_{2}]^{*}, H(m,n)) \neq 0$ in $Z_{m+n-1}(\mathbb{Z}_{2}^{n})$ for $m < n$.

Next we consider actions of $(\mathbb{Z}_{2})^{r}$ on $H(m,n)$ by pulling back the above action via group homomorphisms $(\mathbb{Z}_{2})^{r} \to (\mathbb{Z}_{2})^{n}$. To get suitable formulae we represent such a homomorphism $\Lambda$ as $\Lambda_{S_{1}, \ldots, S_{n}}: (\mathbb{Z}_{2})^{r} \to (\mathbb{Z}_{2})^{n}$

$$T_{i} \mapsto \prod_{i \in S_{i}} T_{j}$$

We call the pullback action of $\phi$ on $H(m,n)$ $\phi_{S_{1}, \ldots, S_{n}}$. In order to obtain an element of $Z_{m+n-1}(\mathbb{Z}_{2}^{r})$ we need a condition when such an action will have finite stationary points. This is the content of the following proposition.

3.2. Proposition. If $S_{1}, \ldots, S_{n}$ are distinct non-empty subsets of $\mathbb{Z}$ then the stationary points of $\phi_{S_{1}, \ldots, S_{n}}$ are precisely $P_{i,j}$ $0 \leq i \leq m$, $0 \leq j \leq n$, $i \neq j$.

Proof. When $S_{i}$ are all distinct and non-empty, for any two coordinates there is a $T_{i}$ which multiplies as 1 on one and by -1 on the other. It follows that a point $([x_{0}, \cdots, x_{m}], [y_{0}, \cdots, y_{n}])$ can be fixed only if only one $x_{i}$ is non-zero and only one $y_{j}$ is non-zero.

Note that the composition

$$(\mathbb{Z}_{2})^{r} \xrightarrow{\Lambda_{S_{1}, \ldots, S_{n}}} (\mathbb{Z}_{2})^{n} \xrightarrow{Y_{(i)}} \mathbb{Z}_{2}$$

is $Y_{S_{i}}$ ($1 \leq i \leq n$). The composition

$$(\mathbb{Z}_{2})^{r} \xrightarrow{\Lambda_{S_{1}, \ldots, S_{n}}} (\mathbb{Z}_{2})^{n} \xrightarrow{Y_{(i,j)}} \mathbb{Z}_{2}$$

is $Y_{S_{i} \Delta S_{j}}$ where $S_{i} \Delta S_{j}$ equals the symmetric difference of the sets $S_{i}$ and $S_{j}$ ($S_{i} \Delta S_{j} = (S_{i} - S_{j}) \cup (S_{j} - S_{i})$). It follows that

$$\eta_{*}([\mathbb{Z}_{2}^{*}, H(m,n), \phi_{S_{1}, \ldots, S_{n}}]) = \prod_{i=1}^{m} Y_{S_{i}} \left( \sum_{j=1}^{n} \prod_{k=1,k \neq j}^{n} Y_{S_{j} \Delta S_{j}} \right) + \sum_{i=1}^{m} Y_{S_{i}} \prod_{k=1,k \neq i}^{n} Y_{S_{i} \Delta S_{i}} \left( \prod_{l=1,l \neq i}^{n} Y_{S_{l}} + \sum_{j=1,j \neq i}^{n} Y_{S_{j}} \prod_{l=1,l \neq j}^{n} Y_{S_{j} \Delta S_{j}} \right).$$

It is not hard to put conditions on $S_{i}$ for which the above expression is not zero. As an example if all the sets $S_{i}$ and $S_{k} \Delta S_{l}$ are different. Then the argument for Proposition 3.1 yields

$$[[\mathbb{Z}_{2}^{*}, H(m,n), \phi_{S_{1}, \ldots, S_{n}}] \neq 0.$$

4. Indecomposability of Certain Classes

In this section, we use Milnor manifolds to construct indecomposable elements in the equivariant cobordism algebra $Z_{*}(\mathbb{Z}_{2}^{*})$ in dimensions up to $2^{n} - 5$. We begin with the lemma

4.1. Lemma. With $\mathbb{Z}_{2}$ coefficients, we have the following:
(i) \[ \sum_i \frac{1}{\prod_{j \neq i} (y_j - y_i)} = 0, \]

(ii) \[ \frac{1}{y_1 \cdots y_n} + \sum_{i=1}^n \frac{1}{y_i \prod_{j \neq i} (y_n + y_j)} = 0, \]

(iii) \[ \sum_{j=1}^n \frac{y_i^k}{\prod_{j \neq i} (y_n + y_j)} = \begin{cases} 1 & k = n \\ 0 & k < n \end{cases}. \]

Proof. The proof is by induction.

Let \( \sum_{j \neq i} (y_j - y_i) =: p(y_1, \cdots, y_n) \). Then

\[ \frac{p(y_1, \cdots, y_n)}{y_1 + y_{n+1}} = \frac{p(y_1, \cdots, y_{n+1})}{y_1 + y_{n+1}}. \]

Applying induction hypothesis to \( p(y_1, \cdots, y_n) \) and \( p(y_2, \cdots, y_{n+1}) \), we get \( p(y_1, \cdots, y_{n+1}) = 0 \). Hence we have (i).

The proof of (ii) is similar.

To prove (iii), define \( q_k(y_1, \cdots, y_n) := \sum_{j=1}^n \prod_{j \neq i} (y_{n+1} + y_j) \). We have

\[ q_k(y_1, \cdots, y_{n+1}) = q_k(y_1, \cdots, y_n) + q_k(y_2, \cdots, y_{n+1}). \]

Then, once again, the required statement follows by induction. \( \square \)

Now consider the action of \((\mathbb{Z}_2)^n\) on \( \mathcal{H}(m, n) \), where \( m \leq n \), given by \( \phi \). Recall that this action has \((m + 1)n\) fixed points, \( P_{i,j} \), where \( i \neq j \). The sum of the tangential representations at these points is given by

\[ \eta_*[(\mathbb{Z}_2)^n, \mathcal{H}(m, n)] = \prod_{i=1}^m Y_{(i)} \left( \sum_{j=1}^n \prod_{k=1, k \neq j}^n Y_{(k,j)} \right) + \sum_{i=1}^m \prod_{k=1, k \neq i}^m Y_{(k,i)} \left( \prod_{l=1, l \neq i}^n Y_{(l)} \right) + \sum_{j=1}^n \prod_{j \neq i}^n Y_{(j)} \left( \prod_{l=1, l \neq j}^n Y_{(l,j)} \right). \]  

(4.1)

We have shown in the previous section that \([(\mathbb{Z}_2)^n, \mathcal{H}(m, n), \phi]\) does not bound. Now, we check the indecomposability of the cobordism class \([(\mathbb{Z}_2)^n, \mathcal{H}(m, n), \phi]\) in \( Z_*(\mathbb{Z}_2^n) \) by applying

\[ \gamma: \tilde{R}_*(\mathbb{Z}_2^n) \rightarrow Q(L) \] to \( \eta_*[(\mathbb{Z}_2)^n, \mathcal{H}(m, n), \phi] \).

Recall that

\[ \gamma(Y) = \frac{1}{\sum_{i \in S} y_i} \sum_{r \geq 0} \left( \sum_{i \in S} y_i \right)^r. \]

4.2. Example. \((m = 1)\) We have

\[ \eta_*[(\mathbb{Z}_2)^n, \mathcal{H}(1, n)] = Y_{(1)} \left( \sum_{j=1}^n \prod_{k=1, k \neq j}^n Y_{(k,j)} \right) + Y_{(1)} \left( \prod_{l=2}^n Y_{(l)} \right) + \sum_{j=2}^n Y_{(j)} \left( \prod_{l=2, l \neq j}^n Y_{(l,j)} \right). \]
We calculate the coefficient of $b_m$ in $\gamma(\eta_*([Z_2]^n), \mathcal{H}(1,n))$ for $m \gg 0$ is a power of 2. For such an expression $\Delta$ we use the notation $b_m(\Delta)$ for the coefficient of $b_m$ in $\Delta$. Then

$$b_m(\gamma(\eta_*([Z_2]^n), \mathcal{H}(1,n))) = \frac{1}{y_1} b_m(\Delta_1) + y_1^{m-1} b_0(\Delta_1) + \frac{1}{y_1} b_m(\Delta_2) + y_1^{m-1} b_m(\Delta_2)$$

where

$$\Delta_1 = \gamma(\prod_{i=1}^{n} Y_{i(j)})$$

and

$$\Delta_2 = \gamma(\prod_{i=1}^{n} Y_{i(l)} + \sum_{j=2}^{n} Y_{j} \prod_{l=2, l \neq j}^{n} Y_{l(i,j)}).$$

Note from Lemma 4.1 it follows that $b_0(\Delta_1) = 0$ and $b_0(\Delta_2) = 0$ and since $m$ is a power of 2 with $Z_2$ coefficients we have

$$b_m(\Delta_1) = \sum_{j=1}^{n} \frac{y_j^n}{y_j^{k=1, k \neq j}(y_k + y_j)}$$

and

$$b_m(\Delta_2) = \frac{y_j^n}{y_j^{k=2, l \neq j}(y_l + y_j)}.$$
We simplify the expression for $A$, using Lemma 4.1 and that $k \leq l < n$ to get

$$A = \sum_{i=1}^{l} \left( \sum_{j=1}^{n} y_{i}^{N+k} \right) y_{i+1} \cdots y_{m} \left( \frac{\prod_{i=1}^{n} y_{i}^{N+k}}{y_{i} \prod_{j \neq i} y_{j+k}} \right).$$

Similarly for $B$ using Lemma 4.1 (iii), we have

$$B = \sum_{i=1}^{m} \frac{1}{y_{i} \prod_{j \neq i} y_{j+k}} \left[ \sum_{i \neq j} y_{i}^{N+k} \sum_{j=1}^{n} y_{j-1} \right] y_{i+k} \cdots y_{m} \frac{\prod_{i=1}^{n} y_{i}^{N+k}}{y_{i} \prod_{j \neq i} y_{j+k}} \left( \frac{\prod_{i=1}^{n} y_{i}^{N+k}}{y_{i} \prod_{j \neq i} y_{j+k}} \right).$$

Therefore,

$$\gamma(\eta_{r}([\mathbb{Z}_{2}]^{n}, \mathcal{H}(m, n))) = \sum_{i=1}^{l} \sum_{j=1}^{n} y_{i}^{N+k} \left( \frac{\prod_{i=1}^{n} y_{i}^{N+k}}{y_{i} \prod_{j \neq i} y_{j+k}} \right) + \sum_{i=2}^{m} \frac{1}{y_{i} \prod_{j \neq i} y_{j+k}} \left[ y_{i}^{N+k} + \sum_{j=1}^{n} \frac{1}{y_{i} \prod_{j \neq i} y_{j+k}} \right].$$

In the above equation we assemble together powers of $y_{1}$ of degree $\geq N$. Call this coefficient $C_{N}$.

This is given by

$$C_{N} = \frac{\sum_{i=2}^{l} \left( \sum_{j=1}^{n} y_{j-1} \right) \sum_{i=1}^{m} y_{i+k} \cdots y_{m}}{y_{i} \prod_{j=2}^{n} y_{j+k} \prod_{i=1}^{n} y_{i}^{N+k}} + \sum_{i=2}^{m} \frac{1}{y_{i} \prod_{j \neq i} y_{j+k}} \left[ y_{i}^{N+k} + \sum_{j=1}^{n} \frac{1}{y_{i} \prod_{j \neq i} y_{j+k}} \right].$$

Simplifying, using Lemma 4.1 we get

$$C_{N} = \left\{ \begin{array}{ll}
\frac{1}{y_{i} \prod_{j=2}^{n} y_{j+k}} \sum_{i=2}^{m} y_{i+k} \cdots y_{m}, & l-k < m \\
\frac{1}{y_{i} \prod_{j=2}^{n} y_{j+k}}, & l=m, k=0.
\end{array} \right.$$

This proves that the coefficient of $b_{N+m}$ contains the term $\frac{y_{i}}{y_{i} \prod_{j=2}^{n} y_{j+k}}$, which cannot be cancelled. That is, the coefficient of $b_{N+m}$ in $\gamma \circ \eta_{r}([\mathbb{Z}_{2}]^{n}, \mathcal{H}(m, n), \phi)$ is non-zero. Therefore, using Theorem 2.1 the proof of the Proposition is complete.

Now suppose $\psi: (\mathbb{Z}_{2})^{k} \to (\mathbb{Z}_{2})^{n}$ given by $n$ distinct, non-empty subsets $S_{1}, \cdots, S_{n} \subseteq [k]$ such that $S_{1} = \{1\}$ and $S_{2}, \cdots, S_{n} \subseteq \{2, \cdots, k\}$. Then there is an induced action of $(\mathbb{Z}_{2})^{k}$ on $\mathcal{H}(m, n)$, which we will denote by $\psi \circ \phi$. Then, we have the following:

4.4. Theorem. Let $0 < m \leq n-2 \leq 2^{k-1} - 3$, and $\psi: (\mathbb{Z}_{2})^{k} \to (\mathbb{Z}_{2})^{n}$ defined as above. Then the class of the induced action $[(\mathbb{Z}_{2})^{k}, \mathcal{H}(m, n), \psi \circ \phi]$ is indecomposable in $Z_{m+n-1}(\mathbb{Z}_{2})^{k}$.

Therefore, in the equivariant cobordism algebra $Z_{*}(\mathbb{Z}_{2})^{k}$, there exists indecomposable elements in degrees at least up to $2^{k} - 5$.

Proof. Note that, since $S_{1}, \cdots, S_{n}$ are distinct subsets, we know, by Proposition 2.2, that the induced action $\psi \circ \phi$ has exactly $(m + 1)n$ isolated fixed points, which are identical to the fixed points of $\phi$. Now consider the following commuting diagram:

$$
\begin{array}{ccc}
R_{*}((\mathbb{Z}_{2})^{n}) & \xrightarrow{\psi^{*}} & R_{*}((\mathbb{Z}_{2})^{k}) \\
\gamma \downarrow & & \downarrow \gamma \\
B_{*}(y_{1}, \cdots, y_{n}) & \xrightarrow{\psi^{*}} & B_{*}(y_{1}, \cdots, y_{n})
\end{array}
$$
Then, we have
\[ \psi^*(\eta_*([\mathbb{Z}_2]^n, \mathcal{H}(m, n), \phi)) = \eta_*([\mathbb{Z}_2]^k, \mathcal{H}(m, n), \psi \circ \phi) \]
and
\[ \psi^*(y_i) = y_1, \quad \psi^*(y_j) = \sum_{l \in S_j} y_l, \quad j > 1. \]

Therefore, the term containing \( y_j \) in the coefficient of \( b_{N+m} \) cannot be cancelled and hence, the coefficient of \( b_{N+m} \) itself, in the expression for \( \gamma(\eta_*([\mathbb{Z}_2]^k, \mathcal{H}(m, n), \psi \circ \phi)) \) is non-zero. This implies \( ([\mathbb{Z}_2]^k, \mathcal{H}(m, n), \psi \circ \phi) \) is indecomposable in \(ZA_{m+n-1}((\mathbb{Z}_2)^k)\).

\[ \square \]

We can extend the above argument to give a lower bound on the number of linearly independent elements.

4.5. Theorem. If \( 1 \leq d \leq 2^{k-i+1} - 5 \), then there are at least \( i \) linearly independent, indecomposable elements in \( Z_d((\mathbb{Z}_2)^k) \).

Proof. The proof of Proposition 4.3 yields by symmetry for \( 1 \leq j \leq m \) that the coefficient of \( y_j^N \) in \( b_{N+m}(\eta_*\mathcal{H}(m, n)) \) does not cancel out. For a fixed \( i \) and \( 1 \leq j \leq i \), consider the maps \( \psi_j : (\mathbb{Z}_2)^k \to (\mathbb{Z}_2)^n \), where, for each \( j \), \( \psi_j \) is given by \( n \) distinct, non-empty subsets \( S_1^j, \ldots, S_k^j \subset \mathbb{Z}_2 \), such that \( S_1^j = \{ j \} \) and \( S_2^j, \ldots, S_k^j \subset \{ i+1, \ldots, k \} \) are \( n \) distinct non-empty subsets. Such a choice is possible provided \( n \leq 2^{k-i} - 1 \). Now consider the actions of \( (\mathbb{Z}_2)^k \) on \( \mathcal{H}(m, n) \) for \( m \geq i \) induced by the maps \( \psi_j \).

Note that for the equivariant cobordism classes \( \lambda_j = ([\mathbb{Z}_2]^k, \mathcal{H}(m, n), \psi_j \circ \phi) \), \( 1 \leq j \leq m - 2 \), we have that \( b_{N+m}(\gamma\eta_*\lambda_j) \) contains a non-cancelling term \( y_j^N \) and no other \( y_r \) for \( 1 \leq r \leq i \). It follows that \( \lambda_j \) (\( 1 \leq j \leq i \)) are \( i \) linearly independent, indecomposable elements of \( Z_{m+n-1}((\mathbb{Z}_2)^k) \).

\[ \square \]

4.6. Remark. For equivariant cobordism the image of the homomorphism \( \epsilon_* : Z_*(G) \to N_* \) to the unoriented cobordism algebra has been determined by tom Dieck ([B], also see [D]) as the subalgebra generated by \( \oplus_{i \leq 2n} N_i \). Therefore, it is important to construct (indecomposable) classes in \( Z_*(G) \) in \( Ker(\epsilon_*) \). From [B] we know that \( \mathcal{H}(m, n) \) bounds if \( m = 1 \), both \( m, n \) are odd and in the case \( n = 2^l - 2 \) for \( l \geq 2 \). Using these values we can arrange the indecomposable elements in \( Z_d((\mathbb{Z}_2)^n) \) in Theorem 4.4 to be in \( Ker(\epsilon_*) \) except for \( d = 2^k - 6 \) and the indecomposable elements in Theorem 4.5 to be in \( Ker(\epsilon_*) \) except for \( d = 2^{k-i+1} - 6 \).
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