Non-hyperelliptic curves of genus three over finite fields of characteristic two

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Abstract

Let $k$ be a finite field of even characteristic. We obtain in this paper a complete classification, up to $k$-isomorphism, of non singular quartic plane curves defined over $k$. We find explicit rational normal models and we give closed formulas for the total number of $k$-isomorphism classes. We deduce from these computations the number of $k$-rational points of the different strata by the Newton polygon of the non hyperelliptic locus $\mathcal{M}_{3}^{\text{nh}}$ of the moduli space $\mathcal{M}_{3}$ of curves of genus 3. By adding to these computations the results of [NS04], [SZ02] on the hyperelliptic locus we obtain a complete picture of these strata for $\mathcal{M}_{3}$.

Introduction

The study of non singular plane quartics has a long history, going back to XIX-th century geometers, and it is still today a fruitful area of research. These genus 3 curves are the first example of non hyperelliptic curves and their geometry is completely different: classification and invariants of hyperelliptic curves rely on a binary theory given by abscissas of Weierstrass points; in the case of quartics, we deal with bitangents and we obtain a ternary theory. Over $\mathbb{C}$, in spite of recent progress, the description of the moduli space $\mathcal{M}_{3}$ of genus 3 curves is not complete: from a geometric point of view, the works of L. Caporaso and E. Sernesi [CS03] and D. Lehavi [Leh02] show that the old construction of Riemann of a quartic from one of its Aronhold systems (certain subsets of seven bitangents) give in fact an explicit finite morphism between an open subset of $\mathbb{P}^{6}$ and $\mathcal{M}_{3}^{\text{nh}}$, the non hyperelliptic locus of $\mathcal{M}_{3}$. From the

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point of view of the theory of invariants, one tries to understand the graded algebra $A$ of invariants for the natural action of $\text{SL}(3, \mathbb{C})$ on the vector space of homogeneous polynomials of degree 4 in three variables. Dixmier in [Dix87] constructed a homogeneous system of parameters for $A$, of degrees 3, 6, 9, 12, 15, 18 and 27, but the precise number of generators of $A$ is not known. At last, Katsylo in [Kat92] proved that $\mathcal{M}_3$ is rational.

Can we precise the situation over other fields? If $k$ is a field of characteristic 2, we may consider a stratification of $\mathcal{M}_3$ by the 2-rank of the Jacobian. One can show that quartics with 2-rank 3 (resp. 2, 1, 0) have 7 (resp. 4, 2, 1) bitangents. For each case, C.T.C Wall [Wal95] found a family of plane quartics containing all $\bar{k}$-isomorphy classes, together with a group acting fully by isomorphisms on the family.

In this paper we obtain, for $k = \mathbb{F}_q$ a finite field of characteristic 2, a complete classification up to $k$-isomorphism of non singular quartics defined over $k$. Note that, even for $k = \mathbb{F}_2$, this had not been completely fulfilled (cf. [Duu01] and the references quoted there). By applying descent theory to each of the four Wall families, we obtain for each value of $n = 7, 4, 2, 1$ an explicit description of the set of $k$-isomorphy classes of curves with $n$ bitangents in the form $\Pi_i(G_i/\mathcal{N}_i^{(n)})$, where $\mathcal{N}_i^{(n)}$ are different families of rational normal models and each $G_i$ is a finite group acting fully by $k$-isomorphisms on $\mathcal{N}_i^{(n)}$. Actually, we use the notation $O$ (for ordinary) as an alternative to $\mathcal{N}^{(7)}$ and $S$ (for supersingular) as an alternative to $\mathcal{N}_0^{(1)}$. Altogether, we obtain thirteen families of rational normal models, which are denoted by

$$O_1, O_2, O_3, O_4, O_{7,0}, O_{7,1}; \mathcal{N}_1^{(4)}, \mathcal{N}_1^{(4)}, \mathcal{N}_3^{(4)}, \mathcal{N}_3^{(2)}, \mathcal{N}_0^{(2)}, \mathcal{N}_1^{(1)}, S.$$  

We compute also the $k$-automorphism group of each curve $C$ in $\mathcal{N}_i^{(n)}$, which is given by the stabilizer of $C$ under the action of $G_i$. Moreover, we obtain closed formulas for the number $|G_i/\mathcal{N}_i^{(n)}|$ of $k$-isomorphy classes in each family $\mathcal{N}_i^{(n)}$.

This is carried out in Section 1 for the ordinary case, which serves as a prototype, and in Section 2 for the non-ordinary cases. In Section 1 we precise also the structure of the ordinary stratum of $\mathcal{M}_3^{\text{nh}}$. We show that it is isomorphic to an explicit open set of an affine variety whose coordinate ring $R$ is the ring of invariants of conics under the natural action of $\text{PGL}_3(\mathbb{F}_2)$ on $\mathbb{P}^2(\bar{k})$. A complete description of $R$ is obtained in [MR04].

Finally, in Section 3 we deduce from these computations the number of $k$-rational points of the strata by the Newton polygon of $\mathcal{M}_3^{\text{nh}}$. In all cases the Newton polygon is determined by the 2-rank, except for the curves with 2-rank zero, whose Newton polygon has either two sides with slopes 1/3, 2/3 (type 1/3) or one side with slope 1/2 (supersingular case). By adding to these computations the results of [NS04], [SZ02] on the hyperelliptic locus $\mathcal{M}_3^h$ we obtain a complete picture of these strata for $\mathcal{M}_3$. The results are:
1 Ordinary curves

Let $k = \mathbb{F}_q$ be a finite field of characteristic 2 and let $\bar{k}$ be a fixed algebraic closure of $k$. We denote by $Q$ the set of quadratic forms:

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx,$$

(1)

with coefficients in $\bar{k}$, such that:

$$abc \neq 0, \quad a + b + d \neq 0, \quad b + c + e \neq 0, \quad a + c + f \neq 0, \quad a + b + c + d + e + f \neq 1.$$

To any $Q \in Q$ we can associate the non-singular quartic plane curve $C_Q$ determined by the equation:

$$C_Q: \quad Q(x, y, z)^2 = xyz(x + y + z) \quad \text{(shortly written as } Q^2 = xyz(x + y + z))$$

All these curves have the same set of bitangents. It is the Fano plane $B_0$ formed by the seven lines of $\mathbb{P}^2$ defined over $\mathbb{F}_2$. If we identify the lines of $\mathbb{P}^2$ with linear forms in $x, y, z$ up to multiplication by a non-zero constant, we can write:

$$B_0 := \{x, y, z, x + y + z, x + y, y + z, x + z\}.$$

Let us denote by $C_Q$ the family of all these curves $C_Q$. This family contains all $\bar{k}$-isomorphy classes of ordinary non-hyperelliptic curves of genus three [Wal95], [Rit03].

**Proposition 1.1.** Let $C$ be a non-singular quartic plane curve defined over $\bar{k}$. The following conditions are equivalent:

1. The Jacobian variety $J_C$ of $C$ is ordinary.
2. $C$ has seven bitangents.
3. The set $\mathcal{B}$ of bitangents of $C$ is a Fano plane; that is,

$$\mathcal{B} = \{\ell_1, \ell_2, \ell_3, \ell_1 + \ell_2 + \ell_3, \ell_1 + \ell_2, \ell_2 + \ell_3, \ell_1 + \ell_3\},$$

(2)

for some linear forms $\ell_1, \ell_2, \ell_3$. 

\[ \text{ordinary} \quad \begin{array}{cccc}
2\text{-rank one} & 2\text{-rank two} & \text{type } 1/3 & \text{supersingular} \\
M_3^{\text{nh}} & q^6 - q^5 + 1 & q^5 - q^4 & q^4 - q^3 & q^3 - q^2 & q^2 \\
M_3^{\text{h}} & q^5 - q^4 & q^4 - 2q^3 + q^2 & 2(q^3 - q^2) & q^2 & 0 \\
M_3 & q^6 - q^4 + 1 & q^5 - 2q^3 + q^2 & q^4 + q^3 - 2q^2 & q^3 & q^2 \\
\end{array} \]
4. The set of bitangents of $C$ is $\text{PGL}_3(\bar{k})$-equivalent to the Fano plane $\mathcal{B}_0$.

5. $C$ is isomorphic to some curve $C_Q$, with $Q \in Q$.

For any isomorphism $\phi: C \to C_Q$, given by three linear forms
\[ \phi(x, y, z) = (\ell_1(x, y, z), \ell_2(x, y, z), \ell_3(x, y, z)), \]
the set of bitangents of $C$ is the Fano plane generated by $\ell_1, \ell_2, \ell_3$ as in (2). Hence,

Lemma 1.2. Let $k \subseteq K \subseteq \bar{k}$ be a finite extension of $k$. An ordinary curve $C$ has all its bitangents defined over $K$ iff $C$ is $K$-isomorphic to $C_Q$ for some $Q \in Q$ with coefficients in $K$. \hfill $\square$

If $C$ is defined over $k$, the minimum field of definition of all bitangents of $C$ has degree 1, 2, 3, 4 or 7 over $k$ [Rit03].

In this section we describe the $k$-isomorphy classes of ordinary non-singular quartics by applying descent theory to the family $\mathcal{C}_Q$ (paragraph 1.2). We find explicit formulas for the number of curves and we exhibit rational normal models for them (paragraphs 1.3 and 1.4). Before of that, we recall some generalities, we study the action of the subgroup of $\text{PGL}_3(\bar{k})$ that preserves the family $\mathcal{C}_Q$ and we describe the ring of invariants for ordinary curves (paragraph 1.1).

1.1 Action of $\text{PGL}_3(\mathbb{F}_2)$ on the curves $C_Q$ and invariants for ordinary curves

We have a natural action of $\text{PGL}_3(\bar{k})$ on the right on the set of homogeneous polynomials $F(x, y, z)$ in three variables, up to multiplication by a non-zero constant:
\[ F^\gamma(x, y, z) = F(\ell_1(x, y, z), \ell_2(x, y, z), \ell_3(x, y, z)), \quad \forall \gamma \in \text{PGL}_3(\bar{k}), \]
where $\ell_1, \ell_2, \ell_3$ are the linear forms having as coefficients the entries of the rows of any representative of $\gamma$ in $\text{GL}_3(\bar{k})$. If we think $\gamma$ as an automorphism of $\mathbb{P}^2$ and $V(F) \subseteq \mathbb{P}^2$ is the subvariety of zeros of $F$, we have $\gamma(V(F)) = V(F^{\gamma^{-1}})$.

Definition. Let $\mathcal{N}$ be a family of smooth projective curves defined over a field $K \subseteq \bar{k}$ and let $G$ be a group acting on the left on $\mathcal{N}$. We say that $G$ acts by $K$-isomorphisms if there are distinguished $K$-isomorphisms $g_C: C \sim \rightarrow g(C)$ for all pairs $C \in \mathcal{N}$, $g \in G$, satisfying:
\[ (g'g)_C = g'_{g(C)} \circ g_C, \quad \forall C \in \mathcal{N}, \forall g, g' \in G. \]
We say that $G$ acts fully by $K$-isomorphisms if moreover:

$\text{Isom}_K(C, C') = \{g_C \mid g \in G, g(C) = C'\}, \quad \forall C, C' \in \mathcal{N}$.

In particular, in this latter case $\text{Aut}_K(C)$ can be identified with the stabilizer of $C$ under the action of $G$. In the case $K = \bar{k}$ we say simply that $G$ acts (fully) by isomorphisms.

If $C$ is a non-singular plane quartic, the inclusion $C \subseteq \mathbb{P}^2$ coincides with the canonical embedding; hence, the group $\text{PGL}_3(\bar{k}) = \text{Aut}(\mathbb{P}^2)$ acts fully by isomorphisms on the family of all non-singular quartics, with $g_C = g_{|C}$ for all $g \in \text{PGL}_3(\bar{k})$.

Let us denote by $\Gamma := \text{Aut}_{\mathbb{F}_2}(\mathbb{P}^2) = \text{PGL}_3(\mathbb{F}_2) = \text{GL}_3(\mathbb{F}_2)$ the subgroup of $\text{PGL}_3(\bar{k})$ of those automorphisms that leave the Fano plane $\mathcal{B}_0$ invariant as a set. The group $\Gamma$ acts fully by isomorphisms on the family $C_Q$. In fact, if we apply to the curve $C_Q$ an automorphism $\gamma \in \Gamma$ we obtain $\gamma(C_Q) = C_{Q'}$ for some $Q' \in \mathcal{Q}$. On the other hand, any isomorphism $C_Q \sim C_{Q'}$ between two curves in $\mathcal{Q}$ sends the set of bitangents of $C$ into the set of bitangents of $C'$; hence, it is given by an element in $\Gamma$.

Let us explicit the action on the left of $\Gamma$ on $\mathcal{Q}$, that reflects the natural action on the curves $C_Q$:

$$\gamma(Q) := Q^{\gamma^{-1}} + H_{\gamma^{-1}}, \quad \forall \gamma \in \Gamma, \forall Q \in \mathcal{Q},$$

where, for any $\gamma$ with rows $\ell_1, \ell_2, \ell_3$, $H_\gamma$ is the quadratic form determined by:

$$\ell_1 \ell_2 \ell_3 (\ell_1 + \ell_2 + \ell_3) = xyz(x + y + z) + H^2_\gamma.$$

The fact that this is a well-defined action of $\Gamma$ on $\mathcal{Q}$ translates into the 1-cocycle condition: $H_{\gamma \rho} = (H_\gamma)^{\rho} + H_\rho$. Note that $H_\gamma = 0$ precisely when $\gamma$ permutes the four lines $x, y, z, x + y + z$.

With this notation, we have $\gamma(C_Q) = C_{\gamma(Q)}$ and

$$\text{Isom}(C_Q, C_{Q'}) = \{\rho \in \Gamma \mid \rho(Q) = Q'\}, \quad \forall Q, Q' \in \mathcal{Q}.$$

Througouht the paper we shall abuse of notation and denote a quadratic form (1) simply by $Q = (a, b, c, d, e, f)$.

We want to study now the invariants of ordinary quartics. We know that every ordinary quartic is $\bar{k}$-isomorphic to one of the form $C_Q$ and we want to study invariants of these curves under the action of $\Gamma$. In order to obtain a linear action we consider now models

$$C_Q^{\text{pr}} : \quad Q^2 = g^2 xyz(x + y + z).$$

By considering $(a, b, c, d, e, f, g)$ as homogeneous coordinates, we can identify the set of these quartics with an open subset of $\mathbb{P}^6$. Consider the following generators of $\Gamma$,

$$A, B : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad A(x, y, z) = (x, y, x + z), \quad B(x, y, z) = (y, z, x).$$
Their action on $C_Q^{pr}$ induces a linear representation on $\mathbb{P}^6$ given by the matrices:

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Changing the model, we may simplify the action. The change of coordinates

$$(a, b, c, d, e, f, g) \mapsto (a + g, b + g, c + g, d + g, e + g, f + g, g)$$

corresponds to substituting the models $C_Q$ by models

$$(ax^2 + by^2 + cz^2 + dxy + eyz + fzx)^2 = g^2 C_K,$$

with $C_K = x^4 + y^4 + z^4 + (xy)^2 + (yz)^2 + (zx)^2 + x^2yz + xy^2z + xyz^2$. This curve $C_K$ is a twist of the Klein quartic and so it is invariant by the action of $\Gamma$. Thus, the action of $\Gamma$ restricts to the conic $Q$. Let $R = \mathbb{F}_2[a, b, c, d, e, f]^\Gamma$ be the ring of invariants of conics under the natural action of $\Gamma$ on $\mathbb{P}^2$.

**Proposition 1.3.** The locus of ordinary quartics in $M_3$ is isomorphic to an explicit open subset (given by non-singularity conditions) of the affine variety $\text{Spec}(R)$.

The structure of the ring $R$ is analyzed in more detail in [MR04].

### 1.2 Descent data

Let $\sigma$ be the Frobenius automorphism of $\bar{k}$ relative to $k$: $\sigma a = a^q$, $\forall a \in \bar{k}$.

By Proposition 1.1, the descents to $k$ of all curves in the family $C_Q$ take into account all $k$-isomorphy classes of non-singular quartic plane curves defined over $k$.

Let us briefly recall the basic facts of descent theory for the specific case of curves defined over finite fields. The basic reference is [Wei56]. Given a curve $C$ defined over $\bar{k}$, a family of descent data for $C$ over $k$ is generated by any isomorphism, $\gamma: C \to \sigma C$, such that

$$\sigma^{n-1} \gamma \circ \cdots \circ \sigma \gamma \circ \gamma = 1,$$

(3)
for some $n \geq 1$, which is called the degree of descent. We call the pair $(C, \gamma)$ a descent datum over $k$. To such datum we can associate a curve $C$ defined over $k$ and a $\bar{k}$-isomorphism $\phi: C \to \bar{C}$ such that $\gamma = \sigma \phi \circ \phi^{-1}$; this curve $C$ is unique up to $k$-isomorphism. The degree of descent is the degree of the minimum field of definition of the isomorphism $\phi$. We denote by $\text{dsc}(C, \gamma)$ the class of $k$-isomorphism of $C$. We have:

$$\text{dsc}(C, \gamma) = \text{dsc}(C', \gamma') \iff \exists \rho: C \to C' \text{ such that } \gamma' = \gamma \circ \rho \circ \rho^{-1}. \quad (4)$$

The descent theory of the family $C_Q$ is simplified by the fact that the isomorphisms involved in the descent data are galois invariant. In fact, if $(C_Q, \gamma)$ is a descent datum over $k$, we have necessarily $\gamma \in \Gamma$, since $\sigma(C_Q) = C_{\sigma Q}$. Therefore, condition (3) amounts in our case to $\gamma^n = 1$. Since the elements in $\Gamma$ have order 1, 2, 3, 4 or 7, we get another proof that these are the possible degrees of descent.

We abuse of language and consider our descent data to be pairs $(Q, \gamma)$ instead of $(C_Q, \gamma)$. We denote by $D$ the set of all descent data over $k$ of curves in the family $C_Q$:

$$D = \{(Q, \gamma) \mid \gamma \in \Gamma, Q \in Q, \gamma(Q) = C_Q\}.$$ 

If we denote by $C_k$ the set of $k$-isomorphy classes of non-singular quartic plane curves whose Jacobian is ordinary, we get an onto map $\text{dsc}: D \to C_k$. By (4):

$$\text{dsc}(Q, \gamma) = \text{dsc}(Q', \gamma') \iff \exists \rho \in \Gamma \text{ such that } \rho(Q) = Q' \text{ and } \gamma' = \rho \gamma \rho^{-1}. \quad (5)$$

**Lemma 1.4.** Let $C, C'$ be ordinary curves defined over $k$ which have been obtained by descent from respective pairs $(Q, \gamma)$, $(Q', \gamma')$, with the same $\gamma \in \Gamma$. Then, the sets $B, B'$ of bitangents of $C$ and $C'$ are $\text{PGL}_3(k)$-equivalent.

**Proof.** Let $\phi$, resp. $\phi'$, be $\bar{k}$-isomorphisms $\phi: C \to C_Q$, resp. $\phi': C' \to C_{Q'}$, such that $\sigma \phi \circ \phi^{-1} = \gamma = \sigma \phi' \circ \phi'^{-1}$. We have $\phi(B) = B_0 = \phi'(B')$. Hence, $\phi^{-1} \circ \phi'(B') = B$, and clearly $\phi^{-1} \circ \phi'$ is defined over $k$. \hfill $\square$

**Notation.** We denote by $\Gamma_\gamma := \{\rho \in \Gamma \mid \rho \gamma = \gamma \rho\}$ the centralizer of $\gamma$ in $\Gamma$. Moreover, for any subset $B$ of $\mathbb{P}^2$ and field $K \subseteq \bar{k}$ we denote

$$\text{Aut}_K(B) := \{\rho \in \text{PGL}_3(K) \mid \rho(B) = B\}.$$

For $K = \bar{k}$ we write simply $\text{Aut}(B)$.

**Lemma 1.5.** Let $C$ be an ordinary curve defined over $k$ which has been obtained by descent from $(Q, \gamma)$ and let $\phi: C \to C_Q$ be an isomorphism such that $\sigma \phi \circ \phi^{-1} = \gamma$. Then, for the set of bitangents $B$ of $C$ we have:

$$\text{Aut}_k(B) = \phi^{-1} \circ \Gamma_\gamma \circ \phi.$$
Assume moreover that $C'$ is another ordinary curve such that $\phi(C') \in \mathcal{C}_Q$, say $\phi(C') = C'_Q$. Then,

$$\text{Isom}_k(C, C') = \phi^{-1} \circ \{ \rho \in \Gamma \gamma \mid \rho(Q) = Q' \} \circ \phi.$$  

In particular, $\text{Aut}_k(C) = \phi^{-1} \circ \Gamma_\gamma(Q) \circ \phi$, where $\Gamma_\gamma(Q)$ denotes the stabilizer of $Q$ under the action of $\Gamma_\gamma$.

**Proof.** We have clearly:

$$\text{Aut}(B) = \phi^{-1} \circ \text{Aut}(B_0) \circ \phi = \phi^{-1} \circ \Gamma \circ \phi,$$

$$\text{Isom}(C, C') = \phi^{-1} \circ \text{Isom}(C_Q, C'_Q) \circ \phi = \phi^{-1} \circ \{ \rho \in \Gamma \mid \rho(Q) = Q' \} \circ \phi.$$

Finally, one checks immediately that for any $\rho \in \Gamma$ the automorphism $\phi^{-1} \circ \rho \circ \phi$ is defined over $k$ if and only if $\rho$ commutes with $\gamma$.

For any fixed $\gamma \in \Gamma$, let us denote by $\mathcal{D}_\gamma$ the set of all pairs $(Q, \gamma)$ belonging to $\mathcal{D}$. By Lemma 1.4, the sets of bitangents of all curves obtained by descent of all elements in $\mathcal{D}_\gamma$ lie in the same orbit under the action of $\text{PGL}_3(k)$; we denote by $B_\gamma$ this common orbit.

**Proposition 1.6.** For any $\gamma, \gamma' \in \Gamma$, the following conditions are equivalent:

1. There exists $\rho \in \Gamma$ such that $\gamma' = \rho \gamma \rho^{-1}$.
2. $\text{dsc}(\mathcal{D}_\gamma) = \text{dsc}(\mathcal{D}_{\gamma'})$.
3. $\text{dsc}(\mathcal{D}_\gamma) \cap \text{dsc}(\mathcal{D}_{\gamma'}) \neq \emptyset$.
4. $B_\gamma = B_{\gamma'}$.

**Proof.** 1 implies 2: Since $\rho$ is galois invariant, for any descent datum $(Q, \gamma) \in \mathcal{D}_\gamma$, we see that $(\rho(Q), \gamma')$ is a descent datum in $\mathcal{D}_{\gamma'}$. By (5), $\text{dsc}((Q, \gamma)) = \text{dsc}((\rho(Q), \gamma'))$; hence $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma'}$ and by symmetry these two sets coincide.

2 implies 3 and 3 implies 4 are trivial.

4 implies 1: Let $\phi : C \to C_Q$, resp. $\phi' : C' \to C'_Q$, be isomorphisms such that $\gamma = \sigma \phi \circ \phi^{-1}$, resp. $\gamma' = \sigma \phi' \circ \phi'^{-1}$.

Let $B, B'$ be the respective sets of bitangents of $C, C'$ and assume that there is $\eta \in \text{PGL}_3(k)$ such that $\eta(B) = B'$. Then, $\rho := \phi' \circ \eta \circ \phi^{-1}$ belongs to $\Gamma$, since it leaves $B_0$ invariant. Moreover, $\rho \gamma \rho^{-1} = \sigma \rho \gamma \rho^{-1} = \gamma'$.

The centralizer $\Gamma_\gamma$ of any $\gamma \in \Gamma$ operates on $\mathcal{D}_\gamma$ by: $\rho(Q, \gamma) = (\rho(Q), \gamma)$. By (5), two descent data in $\mathcal{D}_\gamma$ are equivalent iff they are in the same orbit under this action. Therefore, we have proved the following:
Theorem 1.7. For any system of representatives $C(\Gamma)$ of conjugacy classes of $\Gamma$, the mapping $dsc$ determines a bijection
\[ dsc: \Pi_{\gamma \in C(\Gamma)} \Gamma \setminus D_\gamma \longrightarrow C_k. \]

All elements of order $1,2,3,4$ of $\Gamma$ lie respectively in one single conjugacy class, whereas the elements of order 7 split into two conjugacy classes, according to their trace being 0 or 1 [Hir79, Section 7.4]. We fix once and for all,
\[ C(\Gamma) = \{1, \gamma_2, \gamma_3, \gamma_4, \gamma_7, 0, \gamma_7, 1\}, \tag{6} \]
where:
\[
\begin{align*}
\gamma_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\gamma_3 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
\gamma_4 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\
\gamma_{7,0} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \\
\gamma_{7,1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

By Theorem 1.7, we get a splitting of $C_k$ into the disjoint union of six subfamilies:
\[ C_k = C_1 \amalg C_2 \amalg C_3 \amalg C_4 \amalg C_{7,0} \amalg C_{7,1}, \]
where the subindex indicates the degree of the descent, which by Lemma 1.2 is the degree of the minimum field of definition of all bitangents. By Proposition 1.6 these families are in one-to-one correspondence with the orbits of Fano planes defined over $k$ (as a set) under the action of $\text{PGL}_3(k)$.

In the next two paragraphs we compute the cardinalities of each of these families and we exhibit rational normal models for them.

### 1.3 Number of ordinary curves

We denote by $k_2, k_3, k_4, k_7$ the respective extension of $k$ in $\bar{k}$, of degree $2,3,4,7$. The action of Frobenius will be denoted sometimes by $(\cdot)': \sigma a = a', \sigma^2 a = a''$, etc.

After Theorem 1.7, in order to compute $C_k$ we need only to compute $|\Gamma \setminus D_\gamma|$ for $\gamma$ running on the system of representatives of conjugacy classes of $\Gamma$ specified in (6).

In general, when we have a finite group $G$ acting on a finite set $X$, the number of orbits can be counted with the well-known formula:
\[ |G \setminus X| = \frac{1}{|G|} \sum_{g \in G} |X(g)| = \sum_{g \in C(G)} \frac{|X(g)|}{|G_g|}, \tag{7} \]
where $X(g) := \{ x \in X \mid g(x) = x \}$ is the set of fixed points of $g$, $C(G)$ is a system of representatives of conjugacy classes of $G$ and $G_g$ is the centralizer of $g$. 

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Theorem 1.8. There are \( q^6 - q^5 + q^4 - 3q^3 + 5q^2 - 6q + 7 \) \( \mathbb{F}_q \)-isomorphism classes of ordinary non-hyperelliptic curves of genus three defined over \( \mathbb{F}_q \). More precisely, according to the minimum field of definition of its bitangents, these curves are distributed in the following way:

| \( C_1 \) | \( \frac{1}{108}(q^6 - 7q^5 + 42q^4 - 140q^3 + 343q^2 - 462q + 328) \) |
|---|---|
| \( C_2 \) | \( \frac{1}{8}(q^6 - 3q^5 + 6q^4 - 12q^3 + 15q^2 - 6q) \) |
| \( C_3 \) | \( \frac{1}{3}(q^6 - q^5 - 2q^3 + 4q^2 - 6q + 7) \) |
| \( C_4 \) | \( \frac{1}{4}(q^6 - q^5 - q^2 - 2q + 4) \) |
| \( C_{7,0} + C_{7,1} \) | \( \frac{1}{7}(q^6 + 6) + \frac{1}{7}(q^6 + 6) \) |

Proof. For commodity, we drop the \( \gamma \) from the pair \((Q, \gamma)\). More precisely, we identify \( D_\gamma = \{ Q \in \mathbb{Q} | \gamma(Q) = \sigma Q \} \), for any \( \gamma \in \Gamma \).

Case 1. For \( \gamma = 1 \) we have \( \Gamma_\gamma = \Gamma \) and \( D_\gamma = \{ Q \in \mathbb{Q} | Q = \sigma Q \} =: \mathbb{Q}_k \) coincides with the set of quadratic forms in \( \mathbb{Q} \) with coefficients in \( k \).

The cardinality of the centralizers of \( \Gamma \) is well-known [Hir79]: \( |\Gamma_\rho| = 168, 8, 3, 4, 7 \), according to the order of \( \rho \) being respectively 1,2,3,4,7. Thus, in order to apply (7) we need only to compute \( |\mathbb{Q}_k(\rho)| \) for all \( \rho \in C(\Gamma) \).

Lemma 1.9. \( |\{(a, b) \in k^* \times k^* | a + b \neq 1\}| = q^2 - 3q + 3 \).

Proof. For \( a = 1 \), resp. \( a \neq 1 \), we have \( q - 1 \), resp. \( q - 2 \) possibilities for \( b \).

Let \( \mathcal{U} = (k^*)^3 \times k^3 \). In order to compute \( |\mathbb{Q}_k| \) we need to count how many elements \((a, b, c, d, e, f) \in \mathcal{U}\) fail to satisfy at least one of the following linear relations:

\[
\begin{align*}
    a + b + d = 0, & \quad b + c + e = 0, & \quad a + c + f = 0, & \quad a + b + c + d + e + f = 1.
\end{align*}
\]

For \( i = 1, 2, 3 \), there are \( (q - 1)^3q^{3-i} \) elements in \( \mathcal{U} \) satisfying exactly \( i \) of these relations, since we can fix any value of \((a, b, c, d, e, f) \in (k^*)^3\) and \( i \) of the values of \( d, e, f \) are determined by the linear relations. By Lemma 1.9, there are \( q^2 - 3q + 3 \) elements in \( \mathcal{U} \) satisfying all four linear equations simultaneously, since we can choose \((a, b) \in (k^*)^2\) with \( a + b \neq 1 \) arbitrarily and then determine \( c, d, e, f \) by the relations.

By the inclusion-exclusion principle, we have

\[
|\mathbb{Q}_k| = |\mathcal{U}| - 4(q - 1)^3q^2 + 6(q - 1)^3q - 4(q - 1)^3 + (q^2 - 3q + 3) = q^6 - 7q^5 + 21q^4 - 35q^3 + 35q^2 - 21q + 7.
\]
On the other hand,
\[ Q_k(\gamma_2) = \{(a, a, c, d, e, e) \in k^6 \mid acd \neq 0, a + c + e \neq 0, c + d \neq 1\}. \]

By Lemma 1.9, \(|Q_k(\gamma_2)| = (q^2 - 3q + 3)(q - 1)^2\), since we can choose \((c, d) \in (k^*)^2\) with \(c + d \neq 1\) and then we have \(q - 1\) possibilities, both for \(a\) and for \(e\). Moreover,
\[ Q_k(\gamma_3) = \{(a, a, a, d, d, d) \in (k^*)^6 \mid a + d \neq 1\} \implies |Q_k(\gamma_3)| = q^2 - 3q + 3, \]
\[ Q_k(\gamma_4) = \{(a, b, a, b, b, b) \in (k^*)^6\} \implies |Q_k(\gamma_4)| = (q - 1)^2, \]
\[ Q_k(\gamma, 0) = Q_k(\gamma, 1) = \{(1, 1, 1, 1, 1, 1)\} \implies |Q_k(\gamma, 0)| = |Q_k(\gamma, 1)| = 1. \]
By (7), we get:
\[ |\Gamma \setminus Q_k| = \frac{1}{168} (q^6 - 7q^5 + 42q^4 - 140q^3 + 343q^2 - 462q + 328). \]

**Case 2.** For \(\gamma = \gamma_2\) we have \(\Gamma_\gamma = \{1, \gamma, \tau, \gamma \tau, \rho, \rho^3, \rho \tau, \tau \rho\} \simeq D_8\), where
\[ \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (8) \]

The center of \(\Gamma_\gamma\) is \(\{1, \gamma\}\) and \(\tau^2 = 1, \rho^2 = \gamma\). The conjugacy classes of \(\Gamma_\gamma\) are:
\[ \Gamma_\gamma = \{1\} \amalg \{\gamma\} \amalg \{\tau, \gamma \tau\} \amalg \{\rho, \rho^3\} \amalg \{\rho \tau, \tau \rho\}. \]

We have now,
\[ D_\gamma = \{(a, a', c, d, e', e) \in (k_2)^6 \mid ac \neq 0, c, d \in k, \]
\[ d \neq \text{Tr}(a), a + c + e \neq 0, c + d + \text{Tr}(a) + \text{Tr}(e) \neq 1\}. \]

Let \(U = \{(a, a', c, d, e', e) \in (k_2)^6 \mid ac \neq 0, c, d \in k\}\), with \(|U| = (q^2 - 1)(q - 1)q^3\).

We want to count how many elements in \(U\) fail to satisfy at least one of the following linear relations:
\[ d = \text{Tr}(a), \quad a + c + e = 0, \quad c + d + \text{Tr}(a) + \text{Tr}(e) = 1. \]

There are \((q^2 - 1)(q - 1)q^2\) elements in \(U\) satisfying \(d = \text{Tr}(a)\) or \(c + d + \text{Tr}(a) + \text{Tr}(e) = 1\), whereas only \((q^2 - 1)(q - 1)q\) elements satisfy \(e = a + c\). There are \((q^2 - 1)(q - 1)\) elements satisfying \(e = a + c\) and one of the other two relations, whereas \((q^2 - 1)(q - 1)q\) elements satisfy \(d = \text{Tr}(a)\) and \(c + \text{Tr}(e) = 1\). Finally, there
are $q^2 - q - 1$ elements satisfying simultaneously the three equations (necessarily $\text{Tr}(a) \neq 1$ and the values of $c, d, e$ are determined by the choice of $a$).

By the inclusion-exclusion principle, we have

$$|\mathcal{D}_\gamma| = |\mathcal{U}| - (q^2 - 1)(q - 1)(2q^2 + q) + (q^2 - 1)(q - 1)(q + 2) - (q^2 - q - 1) = q^6 - 3q^5 + q^4 + 5q^3 - 5q^2 - q + 3.$$  

Now, $\mathcal{D}_\gamma(\gamma) = Q_k(\gamma_2)$ has $(q^2 - 3q + 3)(q - 1)^2$ elements, as we saw in Case 1, whereas,

$$\mathcal{D}_\gamma(\rho) = \{(a, a, c, 1, c, c) \in (k^*)^6\} \implies |\mathcal{D}_\gamma(\rho)| = (q - 1)^2.$$  

$$\mathcal{D}_\gamma(\tau) = \{(a, a', c, d, c, c) \in (k_2)^6 \mid ac \neq 0, c, d \in k, d \neq \text{Tr}(a), c + d + \text{Tr}(a) \neq 1\},$$  

$$\mathcal{D}_\gamma(\rho \tau) = \{(a, a, c, e + e' + 1, e', e) \in (k^*)^4 \times (k_2)^2 \mid \text{Tr}(e) \neq 1, a + c + e \neq 0\}.$$  

We have $|\mathcal{D}_\gamma(\tau)| = (q^2 - 1)(q^2 - q + 3)$, since for any fixed $a \in (k_2)^*$ there are $q^2 - q + 3$ possibilities for the pair $(c, d + \text{Tr}(a))$ (hence for the pair $(c, d)$) by Lemma 1.9. Moreover, $|\mathcal{D}_\gamma(\rho \tau)| = (q^2 - q - 1)(q - 1)^2$ since for any fixed pair $(a, c) \in (k^*)^2$ there are $q^2 - q - 1$ possible values for $e$. We get, finally, by (7):

$$|\Gamma_\gamma \setminus \mathcal{D}_\gamma| = \frac{1}{8} (q^6 - 3q^5 + 6q^4 - 12q^3 + 15q^2 - 6q).$$

**Case 3.** For $\gamma = \gamma_3$ we have $\Gamma_\gamma = \{1, \gamma, \gamma^2\} \simeq C_3$ and

$$\mathcal{D}_\gamma = \{(a, a', a'', d, d', d'') \in (k_3)^6 \mid a \neq 0, a + a' + d \neq 0, \text{Tr}(a) + \text{Tr}(d) \neq 1\}.$$  

The set $\mathcal{U} := k_3^* \times k_3$ has $(q^3 - 1)q^3$ elements. There are $q^3 - 1$ elements $(a, d) \in \mathcal{U}$ satisfying $d = a + a'$, $(q^3 - 1)q^2$ elements satisfying $\text{Tr}(d) = 1 + \text{Tr}(a)$ and $q^2$ elements satisfying both conditions (necessarily $\text{Tr}(a) = 1$). Hence,

$$|\mathcal{D}_\gamma| = |\mathcal{U}| - (q^3 - 1)(q^2 + 1) + q^2 = q^6 - q^5 - 2q^3 + 2q^2 + 1.$$  

For $i = 1, 2$, $\mathcal{D}_\gamma(\gamma^i) = Q_k(\gamma_3)$ has $q^2 - 3q + 3$ elements, as we saw in Case 1. We get:

$$|\Gamma_\gamma \setminus \mathcal{D}_\gamma| = \frac{1}{3} (q^6 - q^5 - 2q^3 + 4q^2 - 6q + 7).$$

**Case 4.** For $\gamma = \gamma_4$ we have $\Gamma_\gamma = \{1, \gamma, \gamma^2, \gamma^3\} \simeq C_4$ and

$$\mathcal{D}_\gamma = \{(c', b, c, b+c'+c'', b+c+c''', b'+c+c') \mid c \in k_4^*, b \in k_2^*, \text{Tr}_{k_4/k}(c) + \text{Tr}_{k_2/k}(b) \neq 1\}.$$  

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There are $q^3$ elements $c \in k_4^*$ with $\text{Tr}_{k_4/k}(c) = 1$; for each of them, we can choose $q^2 - q$ elements $b \in k_2^*$ with $\text{Tr}_{k_2/k}(b) \neq 0$. There are $q^4 - q^3 - 1$ elements $c \in k_4^*$ with $\text{Tr}(c)_{k_4/k} \neq 1$; for each of them, we can choose $q^2 - q - 1$ elements $b \in k_2^*$ with $\text{Tr}_{k_2/k}(b) \neq 1 + \text{Tr}_{k_4/k}(c)$. Altogether, we have

$$|D_\gamma| = q^3(q^2 - q) + (q^4 - q^3 - 1)(q^2 - q - 1) = q^6 - q^5 - q^4 + q^3 - q^2 + q + 1.$$  

For $i = 1, 3$, $D_\gamma(\gamma^i) = Q_k(\gamma^4)$ has $(q - 1)^2$ elements, as we saw in Case 1, whereas,

$$D_\gamma(\gamma^2) = \{(c', b, c + c', b + c + c', b' + c + c') \mid b, c \in k_2^*, \, \text{Tr}_{k_2/k}(b) \neq 1\} \implies |D_\gamma(\gamma^2)| = (q^2 - 1)(q^2 - q - 1) = q^4 - q^3 - 2q^2 + q + 1.$$  

We get:

$$|\Gamma_\gamma \backslash D_\gamma| = \frac{1}{4} \left(q^6 - q^5 - q^2 - 2q + 4\right).$$  

Cases $(7,0)$, $(7,1)$. For $\gamma = \gamma_{7,0}$ we have $\Gamma_\gamma = \langle \gamma \rangle \simeq C_7$ and

$$D_\gamma = \{(b'', b, b' + b'' + b'''', b + b' + b'' + b''' + b''') \mid b \in k_7^*, \, \text{Tr}(b) = 1\}.$$  

Now, $|D_\gamma| = q^6$, $D_\gamma(\gamma^i) = \{(1,1,1,1,1,1)\}$ for $i = 1, \ldots , 6$, and

$$|\Gamma_\gamma \backslash D_\gamma| = \frac{1}{7} \left(q^6 + 6\right).$$

For $\gamma = \gamma_{7,1}$ we obtain a completely analogous result.

**1.4 Rational normal models**

In paragraph 1.2 we obtained a partition:

$$C_k = C_1 \amalg C_2 \amalg C_3 \amalg C_4 \amalg C_{7,0} \amalg C_{7,1},$$

of the $k$-isomorphy classes of ordinary plane quartics, according to the minimum field of definition of their set of bitangents. In this section we exhibit rational models for each of the six families.

By Theorem 1.7, each family is identified to $\Gamma_\gamma \backslash D_\gamma$ for some $\gamma \in C(\Gamma)$; moreover, by Proposition 1.6 it corresponds to an orbit of Fano planes defined over $k$ (as a set) under the action of $\text{PGL}_3(k)$. We shall choose a Fano plane $B$ in each orbit and consider the family $\mathcal{O}$ of normal models:

$$N_Q: \quad Q^2 = \ell_1 \ell_2 \ell_3 (\ell_1 + \ell_2 + \ell_3),$$

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where \( \ell_1, \ell_2, \ell_3 \) are three fixed non-concurrent lines of \( B \) and \( Q \) is a quadratic form such that \( N_Q \) is defined over \( k \) and non-singular. The nonsingularity condition amounts to \( N_Q \) not passing through any of the seven points of the Fano plane \( B \).

In all cases, the automorphism \( \phi \) of \( \mathbb{P}^2 \) given by:

\[
\phi(x, y, z) = (\ell_1(x, y, z), \ell_2(x, y, z), \ell_3(x, y, z)),
\]

satisfies \( \sigma \phi \circ \phi^{-1} = \gamma \) and the mapping

\[
\mathcal{O} \rightarrow \mathcal{D}_\gamma, \quad N_Q \mapsto (\phi(N_Q), \gamma)
\]
establishes a bijection between \( \mathcal{O} \) and \( \mathcal{D}_\gamma \). Thus, the models \( N_Q \) faithfully represent dsc(\( D_\gamma \)). Moreover, by Lemma 1.5 the group \( \text{Aut}_k(B) = \phi^{-1} \circ \Gamma \circ \phi \) acts fully by \( k \)-isomorphisms on the family \( \mathcal{O} \).

We shall choose the first three lines of \( B \) in such a way that \( \phi \) will be represented by a symmetric matrix. Then,

\[
\phi^{-1} \circ \gamma \circ \phi = \phi^{-1} \circ \sigma \phi = \ell_\gamma,
\]

and \( \text{Aut}_k(B) = \Gamma_\gamma \). As in section 1.1, the action of \( \Gamma_\gamma \) on the family of the curves \( N_Q \) can be interpreted in terms of an action on the corresponding set of quadratic forms \( Q \). This action has again the shape: \( \rho(Q) = Q^\rho + H_{\rho^{-1}} \), where the term \( H_{\rho^{-1}} \) depends on how \( \rho \) permutes the seven lines of \( B \). More precisely, \( H_\rho \) is defined by:

\[
\ell_1^\rho \ell_2^\rho \ell_3^\rho (\ell_1 + \ell_2 + \ell_3)^\rho = \ell_1 \ell_2 \ell_3 (\ell_1 + \ell_2 + \ell_3) + H_\rho^2.
\]

The explicit description of this action allows us to determine the \( k \)-isomorphy classes contained in \( \mathcal{O} \) and to compute the \( k \)-automorphism group of each curve \( N_Q \).

Any nonsingular quartic plane curve \( C \) is \( k \)-isomorphic to one of our models. By computing its set of bitangents we know to what family it corresponds; any \( k \)-automorphism of \( \mathbb{P}^2 \) taking the set of bitangents of \( C \) to \( B \) will furnish a \( k \)-isomorphism of the curve \( C \) with one of the normal models.

**Normal models \( \mathcal{O}_1 \)**

We choose \( \mathcal{B} = \mathcal{B}_0 = \{ x, y, z, x + y + z, x + y, y + z, x + z \} \), with \( \text{Aut}_k(B) = \Gamma \). The family \( \mathcal{O}_1 \) gathers the curves \( N_Q := C_Q \) with \( Q \) defined over \( k \):

\[
N_Q = C_Q: \quad Q^2 = xyz(x + y + z), \quad Q \in \mathcal{Q}_k.
\]

The action of \( \Gamma \) on \( \mathcal{Q}_k \) is the ordinary action that we introduced in section 1.1. Two curves \( C_Q, C_Q' \) in this family are \( k \)-isomorphic iff they are \( \bar{k} \)-isomorphic iff \( Q' = \rho(Q) \) for some \( \rho \in \Gamma \). The different possibilities for the group \( \text{Aut}_k(C_Q) = \text{Aut}(C_Q) \) in terms of \( Q \) can be found in [Wal95].
Normal models $\mathcal{O}_2$

We fix as a generator of $k_2/k$ an element $u \in k_2 \setminus k$, with equation $u^2 + u = r$, for certain $r \not\in AS(k) := \{x + x^2 \mid x \in k\}$. We choose $\ell(x, y, z) = ux + u'y$ and

$$B = \{\ell, \ell', z, x + y + z, x + y, \ell' + z, \ell + z\},$$

$$Q_2 := \{Q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fxy | a, b, c, d, e, f \in k, \ c \neq 0, a + b + d \neq 0, a + b + c + d + e + f \neq 1, Q(u, u', 0) \neq 0, Q(u, u', 1) \neq 0\}.$$

We consider the family $\mathcal{O}_2$ of normal models:

$$N_Q: \quad Q^2 = \ell \ell' z(x + y + z) = (rx^2 + ry^2 + xy)z(x + y + z), \quad Q \in Q_2.$$ 

Since $\gamma := \gamma_2$ is symmetric, we have

$$\text{Aut}_k(B) = \Gamma_\gamma = \Gamma = \{1, \gamma, \tau, \gamma \tau, \rho, \rho \gamma, \rho \tau, \tau \rho\} \simeq D_8,$$

where $\tau, \rho$ are given in (8). The orbit of $Q = (a, b, c, d, e, f)$ under the action of $\Gamma$ is

$$\begin{array}{ccccccc}
 a & b & c & d & e & f \\
 b & a & c & d & e & f \\
 a + c + f & b + c + e & c & d + e + f & e & f \\
 b + c + e & a + c + f & c & d + e + f & f & e \\
 b + c + e & a + c + f & \Sigma + 1 & d + e + f & d + e + 1 & d + f + 1 \\
 a + c + f & b + c + e & \Sigma + 1 & d + e + f & d + f + 1 & d + e + 1 \\
 b & a & \Sigma + 1 & d & d + f + 1 & d + e + 1 \\
 a & b & \Sigma + 1 & d & d + e + 1 & d + f + 1 \\
\end{array}$$

where $\Sigma = a + b + c + d + e + f$. Hence,

$$\text{Aut}_k(N_Q) = \begin{cases}
\Gamma_\gamma \simeq D_8, & \text{if } a = b, d = 1, c = e = f, \\
\{1, \gamma, \tau \rho, \rho \tau\} \simeq C_2 \times C_2, & \text{if } a = b, d = 1, e = f \neq c, \\
\{1, \gamma, \tau, \gamma \tau\} \simeq C_2 \times C_2, & \text{if } a = b, d \neq 1, c = e = f, \\
\langle \tau \rho \rangle \simeq C_2, & \text{if } a = b + e + f, d = 1, e \neq f, \\
\langle \rho \tau \rangle \simeq C_2, & \text{if } a = b, d + e + f = 1, e \neq f, \\
\langle \tau \rangle \simeq C_2, & \text{if } c = e = f, a \neq b, \\
\langle \gamma \tau \rangle \simeq C_2, & \text{if } e = f = a + b + c, a \neq b, \\
\langle \gamma \rangle \simeq C_2, & \text{if } a = b, e = f \neq c, d \neq 1, \\
\{1\}, & \text{otherwise.}
\end{cases}$$
Normal models $\mathcal{O}_3$

We fix as a generator of $k_3/k$ an element $v \in k_3 \setminus k$, with equation $v^3 + v^2 = s$. By [CNP02, Lemma 7], we have $v^3 = (v + 1)^{-1}$, $v^\prime = v^2((t + 1)v + 1)^{-1}$, where $t \in k$ satisfies $t^2 + t + 1 = s^{-1}$. We take $\ell(x, y, z) = vx + v'y + v''z$ and

$$\mathcal{B} = \{\ell, \ell', \ell'', x + y + z, \ell + \ell', \ell + \ell''\},$$

$$\mathcal{Q}_3 := \{Q = (a, b, c, d, e, f) \in k^6 \mid Q(v, v', v'') \neq 0, Q(v + 1, v', v'' + 1) \neq 0, Q(1, 1, 1) \neq 1\}.$$

Each $Q \in \mathcal{Q}_3$ provides a normal model:

$$N_Q: \quad Q^2 = \ell\ell''(x + y + z) = (x + y + z) \cdot (s(x^3 + y^3 + z^3) + xyz + st(xy^2 + x^2z + yz^2) + s(t + 1)(xz^2 + x^2y + y^2z)).$$

Let us denote $\gamma := \gamma_3$. Since $\gamma' = \gamma^{-1}$, we have $\text{Aut}_k(\mathcal{B}) = \Gamma_\gamma = \{1, \gamma, \gamma^2\}$. Since $\gamma$ permutes the four bitangents $\ell, \ell', \ell'', x + y + z$, the action of $\Gamma_\gamma$ on the normal models is given by $\rho(Q) = Q^{\gamma^{-1}}$. Thus, the orbit of $Q = (a, b, c, d, e, f)$ is the cyclic orbit generated by $\gamma(Q) = (b, c, a, e, f, d)$ and all $k$-automorphism groups are trivial except for:

$$\text{Aut}_k(N_Q) = \Gamma_\gamma \simeq C_3, \quad \text{if } a = b = c, d = e = f.$$

Normal models $\mathcal{O}_4$

We fix as a generator of $k_4/k$ an element $w \in k_4 \setminus k_2$, with equation $w^4 + (t + t^2)w^2 + t^2w = 1$, where $t$ is any element in $k$ such that $t^{-1} \notin \text{AS}(k)$ [NS04, Proposition 6]. For instance, we can choose $t = 1$ if $q$ is not a square. It is easy to check that $w + w'' = t$ and $\alpha := w + w'$ satisfies the equation $\alpha^2 + t\alpha = t$. We take,

$$\mathcal{B} = \{\ell, \ell', \ell'', \ell + \ell', \ell + \ell'', \ell + \ell', \ell + \ell''\},$$

where $\ell(x, y, z) = wx + w'y + w''z$. Note that $\ell + \ell' + \ell'' = \ell''$ since $w$ has null trace. The linear forms $\ell + \ell', \ell + \ell''$ are defined over $k_2$ and conjugate, whereas $\ell + \ell'' = t(x + y + z)$ is defined over $k$.

The seven points of the Fano plane $\mathcal{B}$ are $(w + t + 1, \alpha, w + \alpha + 1)$ and its three conjugates, $(\alpha, t, \alpha')$ and its conjugate, and $(1, 0, 1)$. Hence, we consider

$$\mathcal{Q}_4 := \{Q = (a, b, c, d, e, f) \in k^6 \mid Q(w + t + 1, \alpha, w + \alpha + 1) \neq 0, Q(\alpha, t, \alpha') \neq 0, Q(1, 0, 1) \neq t^2\}.$$

We get as normal models:

$$N_Q: \quad Q^2 = \ell\ell''\ell''', \quad Q \in \mathcal{Q}_4.$$
For $\gamma := \gamma_4$ we have $\text{Aut}_k(B) = \Gamma_{\gamma} = \{1, \gamma, \gamma^2, \gamma^3\}$. Since $\gamma$ permutes the four bitangents $\ell, \ell', \ell''$, the orbit of $Q = (a, b, c, d, e, f)$ is the cyclic orbit generated by $\gamma(Q) = Q^{\gamma^{-1}} = (a + b + c + d + e + f, a, b, d + f, d, d + e)$ and

$$\text{Aut}_k(N_Q) = \begin{cases} \langle \gamma \rangle \simeq C_4, & \text{if } a = b = c, f = 0, d = e, \\ \langle \gamma^2 \rangle \simeq C_2, & \text{if } a = c, d + e + f = 0, (f \neq 0 \text{ or } b \neq c), \\ \{1\}, & \text{otherwise.} \end{cases}$$

Normal models $O_{7,0}$ and $O_{7,1}$

We are interested in elements $\zeta \in k_7 \setminus k$ such that

$$\{0\} \cup \{\zeta, \zeta', \zeta'', \zeta'''', \zeta''''', \zeta'''', \zeta''''\}$$

is an additive subgroup of $k_7$. If $f(x) \in k[x]$ is the monic minimal polynomial of $\zeta$ over $k$, this condition is equivalent to $xf(x)$ being an additive polynomial; hence, it is equivalent to $f(x) = x^7 + ax^5 + bx + c$, for some $a, b, c \in k$. Let us denote by $S \subseteq k[x]$ the set of all septic irreducible polynomials of this type. Among these elements $\zeta$ we can distinguish two cases:

$$\zeta + \zeta'' = \zeta', \quad \zeta + \zeta''' = \zeta''',$$

which we denote respectively as “case 0” and “case 1”. It is easy to check that there are no other possibilities. In terms of the minimal polynomial we distinguish the two cases according to $f(x)$ dividing $x^3 + x^2 + x$ (case 0) or $x^3 + x^2 + x$ (case 1). Thus, our set $S$ splits as the disjoint union $S = S_0 \cup S_1$ of two subsets gathering the irreducible polynomials of each type. We want to ensure that these subsets $S_0, S_1$ are non-empty. Actually, we have:

**Lemma 1.10.** $(x^3 + x^2 + x)(x^3 + x^2 + x) = x^2 \prod_{f(x) \in S} f(x)$.

In particular, $|S_0| = |S_1| = (q^3 - 1)/7$.

**Proof.** The change of variables, $\tau^n := x^n, n \geq 0$, establishes an isomorphism between the ring of $k$-linear polynomials in $x$, with coefficients in $k$ (with composition as the product operation) and the ring $k[\tau]$ of polynomials in $\tau$. Thus, the identity:

$$\tau^3 + 1 = (\tau^3 + \tau + 1)(\tau^3 + \tau^2 + 1)(\tau + 1),$$

implies that there exists a $k$-linear polynomial $P(x)$ such that $x^3 + x = P(x^3 + x^2 + x)$, and similarly for $x^2 + x + x$. In particular, these two polynomials divide $x^3 + x$, so that all their irreducible factors, apart from $x$, have degree seven. \qed

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Let us deal now simultaneously with the cases $C_{7,0}$ and $C_{7,1}$. Let $ζ ∈ k_7$ be a root of any fixed polynomial in $S_0$ (resp. $S_1$). For instance, if $7 ∤ [k: F_2]$ we can take $ζ$ to be a root of $x^7 + x + 1$ (resp. $x^7 + x^3 + 1$). We take,

$$B = \{\ell, \ell', \ell'', \ell^v, \ell^v\}$$

where $\ell(x, y, z) = ζx + ζ' y + ζ'' z$. We have,

$$\ell'' = \ell + \ell', \ell^v = \ell + \ell'', \ell^v = \ell + \ell', \ell^v = \ell + \ell', \ell^v = \ell + \ell', \ell^v = \ell + \ell'$$

respectively in the cases $(7, 0)$, $(7, 1)$. The rational models will be of the type:

$$N_Q: \quad Q^2 = ℓℓ'' ℓv, \quad \text{resp. } Q^2 = ℓℓ'' ℓv.$$ 

These curves are defined over $k$ if and only if $Q + Q' = ℓℓ''$. The seven points of the Fano plane $B$ are conjugate, hence, if $P$ is the intersection point of $ℓ$ and $ℓ'$, the set of quadratic forms $Q$ such that $N_Q$ is defined over $k$ and non-singular is:

$$Q_7 := \{Q = (a, b, c, d, e, f) ∈ (k_7)^6 \mid Q + Q' = ℓℓ'', Q(P) ≠ 0\}.$$ 

For $γ := γ_{7,0}$ (resp. $γ := γ_{7,1}$), we have $\text{Aut}_k(B) = Γ_γ = \langle t_γ \rangle$. One checks easily that $H_γ = ℓℓ''$, so that

$$t_γ^{-1}(Q) = Q_γ + ℓℓ'' = (Q_γ)'.$$ 

Thus, $t_γ(Q) = (Q_γ^{-1})'$ and the orbit of $Q = (a, b, c, d, e, f)$ is the cyclic orbit generated respectively by:

$$t_γ(Q) = (a' + c' + f', a', b', f', d', d' + e'), \quad t_γ(Q) = (b' + c' + e', a', b', d' + f', d', e').$$

We have $\text{Aut}_k(N_Q) = \{1\}$ for all $N_Q$ except for one, which is a twist of the Klein quartic (this was seen in paragraph 1.3 too). This special model $N_Q$ corresponds to:

$$Q = ℓ^2 + (ℓ')^2 + (ℓ'')^2 + ℓℓ' + ℓℓ'' + ℓℓ'',$$

and it has $\text{Aut}(N_Q) = \langle t_γ \rangle ≃ C_7$.

2 Non-ordinary curves

As in the ordinary case, our starting point is a family of models representing all $\bar{k}$-isomorphy classes of non-ordinary curves [Wal95] [Rit03].
Proposition 2.1. Let $C \subseteq \mathbb{P}^2$ be a non-ordinary, non-singular quartic plane curve defined over $\bar{k}$. Then $C$ satisfies one of the following equivalent conditions:

1. The Jacobian $J_C$ of $C$ has 2-rank 2, resp. 1, resp. 0.
2. $C$ has 4, resp. 2, resp. 1 bitangents.
3. $C$ is isomorphic to a curve $C_Q$ with equation:
   \[ Q^2 = xyz(y + z); \quad \text{resp.} \quad Q^2 = xy(y^2 + xz); \quad \text{resp.} \quad Q^2 = x(y^3 + x^2 z), \]
   where $Q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ is a quadratic form satisfying:
   \[ abc \neq 0, \quad b + c + e \neq 0; \quad \text{resp.} \quad ac \neq 0; \quad \text{resp.} \quad c \neq 0. \]

In this section we find several families $\mathcal{N}$ of rational normal models containing disjoint sets of $k$-isomorphism classes, such that every non-ordinary quartic is $k$-isomorphic to some normal model. For each family $\mathcal{N}$ we find a finite group $G$ acting fully by $k$-isomorphisms on $\mathcal{N}$; this allows us to explicitly determine the $k$-isomorphy classes and to compute the $k$-automorphism group of each curve. Also, we obtain the number $|G \setminus \mathcal{N}|$ of $k$-isomorphism classes contained in $\mathcal{N}$ by applying (7).

As before, we abuse of language and denote the quadratic forms (1) in $x, y, z$ simply by $Q = (a, b, c, d, e, f)$.

2.1 Curves with four bitangents

Let $\mathcal{Q}$ be the set of all quadratic forms $Q = (a, b, c, d, e, f)$ defined over $\bar{k}$ such that $abc \neq 0$, $b + c + e \neq 0$. Let $C_Q$ be the family of all curves:

\[ C_Q : \quad Q(x, y, z)^2 = xyz(y + z), \quad Q \in \mathcal{Q}. \]

The set of bitangents of these curves is $\mathcal{B} = \{x, y, z, y + z\}$. As in the ordinary case, the group $\text{Aut}(\mathcal{B})$ acts fully by isomorphisms on the family $C_Q$; in fact, the elements of $\text{Aut}(\mathcal{B})$ are the only transformations that preserve the quartic $xyz(y + z)$ modulo squares. This group is isomorphic to $S_3 \times \bar{k}^*$ [Wal95], where the subgroup isomorphic to $S_3$ contains the automorphisms permuting $y, z, y + z$:

\[ \{1, \gamma_r(x, y, z) = (x, y, y + z), \gamma_{r_2}(x, y, z) = (x, z, y), \gamma_{r_3}(x, y, z) = (x, y + z, z), \gamma_\rho(x, y, z) = (x, z, y + z), \gamma_{\rho^2}(x, y, z) = (x, y + z, y)\}, \]

and the subgroup isomorphic to $\bar{k}^*$ consists of the transformations:

\[ \gamma_t(x, y, z) = (t^3 x, t^{-1} y, t^{-1} z), \quad t \in \bar{k}^*. \]
Every element of $\text{Aut}(B)$ can be written in a unique way as $\gamma_{\beta,t} := \gamma_t \circ \gamma_\beta = \gamma_\beta \circ \gamma_t$, with $\beta \in S_3$, $t \in \bar{k}^*$. Moreover,

$$
\gamma_{\beta',t'} \circ \gamma_{\beta,t} = \gamma_{\beta',t'} \circ \gamma_{\beta,t}, \quad \forall \beta, \beta' \in S_3, \forall t, t' \in \bar{k}^*.
$$

For any $\beta \in S_3$ let us denote by $\mathcal{D}_\beta$ the set of all descent data over $k$ of curves of $\mathcal{C}_Q$ generated by $\gamma_\beta$ (cf. paragraph 1.2):

$$
\mathcal{D}_\beta = \{(C, \gamma_\beta) \mid C \in \mathcal{C}_Q, \gamma_\beta(C) = \sigma C\}.
$$

**Proposition 2.2.**

$$
\text{dsc}(\mathcal{C}_Q) = \text{dsc}(\mathcal{D}_1) \amalg \text{dsc}(\mathcal{D}_r) \amalg \text{dsc}(\mathcal{D}_\rho).
$$

Moreover, two descent data in the same family $\mathcal{D}_\beta$ are equivalent iff they are in the same orbit under the action of the centralizer of $\gamma_\beta$ in $\text{Aut}_k(B) \simeq S_3 \times \bar{k}^*$.

**Proof.** Assume that $\gamma_{\beta,t} : C_Q \overset{\sim}{\rightarrow} C_{\sigma Q}$ generates descent data over $k$ of degree $n$ for the curve $C_Q$. Clearly:

$$
\sigma^{-1} \gamma_{\beta,t} \circ \cdots \circ \gamma_{\beta,t} \circ \gamma_{\beta,t} = 1 \implies \sigma^{-1} t \cdots \sigma t = 1.
$$

By Hilbert’s 90th theorem, there exists $s \in \bar{k}^*$ such that $t = s/\sigma s$. In particular, $\sigma_{\gamma_{1,s}} \circ \gamma_{\beta,t} \circ \gamma_{1,s}^{-1} = \gamma_{\beta,1}$. Thus, by (4), our original descent datum $(C_Q, \gamma_{\beta,t})$ is equivalent to some descent datum having $t = 1$.

Once restricted to descent data with $t = 1$, the situation is completely analogous to that of paragraph 1.2. The assertion of the proposition derives from results analogous to Proposition 1.6 and Theorem 1.7. \qed

Let $k_2, k_3$ denote respectively the quadratic and cubic extension of $k$ in $\bar{k}$. We fix $u \in k_2 \setminus k$ with equation $u^2 + u = r$, for some $r \notin \text{AS}(k)$. Also, we fix $v \in k_3 \setminus k$ with equation $v^3 + v = s$ for adequate $s \in k^*$. By [CNP02, Lemma 7] we have $v' = s^{-1}v^2 + tv$, $v'' = s^{-1}v^2 + (t + 1)v$, for certain $t \in \bar{k}$ satisfying $t^2 + t + 1 = s^{-1}$.

By Proposition 2.2 there is still an action of $k^*$ on all descents of the family $\mathcal{C}_Q$, given by the isomorphisms $\gamma_{1,t}$, $t \in k^*$. Thus, we can reduce the family of normal models by assuming that the coefficient $a$ in the quadratic forms runs on a fixed system of representatives of $k^*/(k^*)^3$. So, we take the following sets of quadratic forms:

$$
\bar{\mathcal{Q}}_1 := \{(a, b, c, d, e, f) \in k^6 \mid abc \neq 0, b + c + e \neq 0\},
$$

$$
\bar{\mathcal{Q}}_2 := \{(a, b, c, d, e, f) \in k^6 \mid a \neq 0, (b, e) \neq (cr, c)\},
$$

$$
\bar{\mathcal{Q}}_3 := \{(a, b, c, d, e, f) \in k^6 \mid a \neq 0, (b, c, e) \neq (0, 0, 0)\},
$$

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and subsets $Q_i := \{ Q \in \hat{Q}_i \mid a \in k^*/(k^*)^3 \}$. Consider the families of rational normal models:

\[ \tilde{N}_i^{(4)} := \{ N_Q \mid Q \in \hat{Q}_i \}, \quad N_i^{(4)} := \{ N_Q \mid Q \in Q_i \}, \quad i = 1, 2, 3, \]

where $N_Q$ is respectively:

- $N_Q: \quad Q^2 = xyz(y + z)$,
- $N_Q: \quad Q^2 = xy\ell\ell' = xy(ry^2 + yz + z^2)$, $\ell(x, y, z) = uy + z$,
- $N_Q: \quad Q^2 = x\ell\ell\ell'' = x(y^3 + ty^2z + (t + 1)yz^2 + z^3)$, $\ell(x, y, z) = y + v'v^{-1}z$.

The isomorphism:

\[ \phi(x, y, z) = (x, y, z), \quad \text{resp.} \quad (x, y, \ell(x, y, z)) \quad \text{resp.} \quad (s^{-1}x, v\ell(x, y, z), v'v'(x, y, z)) \]

sets the families of models $\tilde{N}_i^{(4)}$, $i = 1, 2, 3$ in respective bijection with the descent data $D_\beta$, for $\beta = 1, \tau, \rho$. By Proposition 2.2 and a result analogous to Lemma 1.5, the group $\phi^{-1} \circ ((S_3)_\beta \times k^*) \circ \phi$ acts fully by $k$-isomorphisms on the corresponding family $\tilde{N}_i^{(4)}$. Hence, the group $G = \phi^{-1} \circ ((S_3)_\beta \times \mu_3(k)) \circ \phi$ acts fully by $k$-isomorphisms on the corresponding family $N_i^{(4)}$. Here and in the sequel, $\mu_n(k)$ denotes the group of $n$-th roots of unity that are contained in $k$.

By Proposition 2.2, every non-singular quartic $C$ over $k$ with four bitangents is $k$-isomorphic to a curve in one of the three families $N_i^{(4)}$. The minimum field of definition of the set of bitangents tells us to which family belongs the $k$-isomorphism class of $C$. Any transformation in $\text{PGL}_3(k)$ sending the set of four bitangents of $C$ to the set of four lines that has been chosen for the family $N_i^{(4)}$, will take $C$ into one of the normal models $N_Q$.

We describe now in each case the $k$-isomorphism classes and the $k$-automorphism group. In the sequel we denote $N := |\mu_3(k)| = |k^*/(k^*)^3|$; that is, $N = 3, 1$, according to $q$ being a square or not.

**Normal models $N_1^{(4)}$**

Since $\phi = 1$, we have $G = \{ \gamma_{\beta,t} \mid \beta \in S_3, t^3 = 1 \} \simeq S_3 \times \mu_3(k)$. For any given $Q = (a, b, c, d, e, f) \in Q_1$ and $t \in \mu_3(k)$, we have

\[ \gamma_{1,t}(a, b, c, d, e, f) = (a, t^2b, t^2c, td, t^2e, tf), \]

which has no fixed points if $t \neq 1$. On the other hand, the orbit of $Q$ under the action of $S_3$ is:

- $(a, b, c, d, e, f) \quad (1)$
- $(a, c, b, f, e, d) \quad (\gamma_{\tau_2})$
- $(a, B, b, d + f, e, d) \quad (\gamma_{\rho})$

and subsets $Q_i := \{ Q \in \hat{Q}_i \mid a \in k^*/(k^*)^3 \}$. Consider the families of rational normal models:

\[ \tilde{N}_i^{(4)} := \{ N_Q \mid Q \in \hat{Q}_i \}, \quad N_i^{(4)} := \{ N_Q \mid Q \in Q_i \}, \quad i = 1, 2, 3, \]

where $N_Q$ is respectively:

- $N_Q: \quad Q^2 = xyz(y + z)$,
- $N_Q: \quad Q^2 = xy\ell\ell' = xy(ry^2 + yz + z^2)$, $\ell(x, y, z) = uy + z$,
- $N_Q: \quad Q^2 = x\ell\ell\ell'' = x(y^3 + ty^2z + (t + 1)yz^2 + z^3)$, $\ell(x, y, z) = y + v'v^{-1}z$.

The isomorphism:

\[ \phi(x, y, z) = (x, y, z), \quad \text{resp.} \quad (x, y, \ell(x, y, z)) \quad \text{resp.} \quad (s^{-1}x, v\ell(x, y, z), v'v'(x, y, z)) \]

sets the families of models $\tilde{N}_i^{(4)}$, $i = 1, 2, 3$ in respective bijection with the descent data $D_\beta$, for $\beta = 1, \tau, \rho$. By Proposition 2.2 and a result analogous to Lemma 1.5, the group $\phi^{-1} \circ ((S_3)_\beta \times k^*) \circ \phi$ acts fully by $k$-isomorphisms on the corresponding family $\tilde{N}_i^{(4)}$. Hence, the group $G = \phi^{-1} \circ ((S_3)_\beta \times \mu_3(k)) \circ \phi$ acts fully by $k$-isomorphisms on the corresponding family $N_i^{(4)}$. Here and in the sequel, $\mu_n(k)$ denotes the group of $n$-th roots of unity that are contained in $k$.

By Proposition 2.2, every non-singular quartic $C$ over $k$ with four bitangents is $k$-isomorphic to a curve in one of the three families $N_i^{(4)}$. The minimum field of definition of the set of bitangents tells us to which family belongs the $k$-isomorphism class of $C$. Any transformation in $\text{PGL}_3(k)$ sending the set of four bitangents of $C$ to the set of four lines that has been chosen for the family $N_i^{(4)}$, will take $C$ into one of the normal models $N_Q$.

We describe now in each case the $k$-isomorphism classes and the $k$-automorphism group. In the sequel we denote $N := |\mu_3(k)| = |k^*/(k^*)^3|$; that is, $N = 3, 1$, according to $q$ being a square or not.

**Normal models $N_1^{(4)}$**

Since $\phi = 1$, we have $G = \{ \gamma_{\beta,t} \mid \beta \in S_3, t^3 = 1 \} \simeq S_3 \times \mu_3(k)$. For any given $Q = (a, b, c, d, e, f) \in Q_1$ and $t \in \mu_3(k)$, we have

\[ \gamma_{1,t}(a, b, c, d, e, f) = (a, t^2b, t^2c, td, t^2e, tf), \]

which has no fixed points if $t \neq 1$. On the other hand, the orbit of $Q$ under the action of $S_3$ is:

\[ (a, b, c, d, e, f) \quad (1) \quad (a, B, c, d + f, e, f) \quad (\gamma_7) \quad (a, c, B, f, e, d + f) \quad (\gamma_{\rho^3}) \]
\[ (a, c, b, f, e, d) \quad (\gamma_{\tau_2}) \quad (a, b, B, d, e, d + f) \quad (\gamma_{\tau_1}) \quad (a, c, B, f, e, d + f) \quad (\gamma_{\rho^3}) \]
where we have denoted \( B := b + c + e \). This leads immediately to:

\[
\text{Aut}_k(N_Q) = \begin{cases} 
\{ \gamma_{3,1} \} \simeq S_3, & \text{if } b = c = e, d = f = 0, \\
\langle \tau \rangle \simeq C_2, & \text{if } c = e, f = 0 \neq d, \\
\langle \tau_2 \rangle \simeq C_2, & \text{if } b = c, d = f \neq 0, \\
\langle \tau_3 \rangle \simeq C_2, & \text{if } b = e, d = 0 \neq f, \\
\{ 1 \}, & \text{otherwise.}
\end{cases}
\]

Finally, we have \(|Q_1| = N(q - 1)^3 q^2\) and

\[
Q_1(\gamma_\tau) = \{(a, b, c, d, c, 0) \in k^6 \mid a \in k^*/(k^*)^3, bc \neq 0\} \implies |Q_1(\tau)| = N(q - 1)^2 q, \\
Q_1(\gamma_\rho) = \{(a, c, c, 0, c, 0) \in k^6 \mid a \in k^*/(k^*)^3, c \neq 0\}, \implies |Q_1(\gamma_\rho)| = N(q - 1).
\]

We get from (7):

\[
|G \backslash N^{(4)}_1| = \frac{1}{6N} \left( N(q - 1)^3 q^2 + 3N(q - 1)^2 q + 2N(q - 1) \right) = \frac{1}{6} \left( q^5 - 3q^4 + 6q^3 - 7q^2 + 5q - 2 \right).
\]

**Normal models \( N_2^{(4)} \)**

Since \( \phi^{-1} \circ \gamma_\tau \circ \phi = \gamma_\tau \), we have \( G = \{ \gamma_{\beta t} \mid \beta \in \langle \tau \rangle, t^3 = 1 \} \simeq C_2 \times \mu_3(k) \). Since the elements of \( G \) leave the quartic \( xy\ell\ell' \) invariant, the action on \( Q_2 \) is the same than before. Hence, all \( k \)-automorphism groups are trivial except for:

\[
\text{Aut}_k(N_Q) = \langle \gamma_\tau \rangle \simeq C_2, \quad \text{if } c = e, f = 0.
\]

Finally, we have \(|Q_2| = N(q - 1)(q^2 - 1)q^2, \ |Q_2(\gamma_\tau)| = N(q - 1)^2 q \ (a \neq 0, b \neq c, r, d \text{ arbitrary}) \) and we get from (7):

\[
|G \backslash N_2^{(4)}| = \frac{1}{2N} \left( Nq(q - 1)^2(q^2 + q + 1) \right) = \frac{1}{2} \left( q^5 - q^4 - q^2 + q \right).
\]

**Normal models \( N_3^{(4)} \)**

Since \( \phi \) is symmetric, we have \( \phi^{-1} \circ \gamma_\rho \circ \phi = \gamma_\rho \). Hence, \( G = \langle \gamma_\rho \rangle \times \mu_3(k) \simeq C_3 \times \mu_3(k) \). Since the elements of \( G \) leave the quartic \( x\ell\ell'' \) invariant, the action on \( Q_3 \) is the one described in the case \( N_1^{(4)} \). Hence, all \( k \)-automorphism groups are trivial except for:

\[
\text{Aut}_k(N_Q) = \langle \gamma_\rho \rangle \simeq C_3, \quad \text{if } b = c = e, d = f = 0.
\]

Finally, we have \(|Q_3| = N(q^3 - 1)q^2, \ |Q_3(\gamma_\rho)| = N(q - 1) \) and we get from (7):

\[
|G \backslash N_3^{(4)}| = \frac{1}{3N} \left( N(q^3 - 1)q^2 + 2N(q - 1) \right) = \frac{1}{3} \left( q^5 - q^2 + 2q - 2 \right).
\]

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2.2 Curves with two bitangents

Let \( \mathcal{C}_Q \) be the family of non-singular quartics:

\[
C_Q: \quad Q^2 = xy(y^2 + xz),
\]

where \( Q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx \) is a quadratic form with coefficients in \( \bar{k} \) satisfying \( ac \neq 0 \).

The subgroup of automorphisms of \( \mathbb{P}^2 \) that preserve the quartic \( xy(y^2 + xz) \) modulo squares consists of the transformations [Wal95]:

\[
\gamma_{t,u}(x, y, z) = (t^3 x, t^{-1} y, t^{-5} (z + uy)), \quad t \in \bar{k}^*, \ u \in \bar{k}.
\]

The composition of two such transformations is given by:

\[
\gamma_{t',u'} \circ \gamma_{t,u} = \gamma_{t't,u+tu'}, \quad \forall t, t' \in \bar{k}^*, \ u, u' \in \bar{k}.
\]

Clearly, this group acts fully by isomorphisms on the family \( \mathcal{C}_Q \).

**Proposition 2.3.** Every non-singular quartic plane curve \( C \subseteq \mathbb{P}^2 \) with two bitangents is \( k \)-isomorphic to a curve \( C_Q \) for some \( Q \) with coefficients in \( k \).

**Proof.** The assertion is true over \( \bar{k} \) by Proposition 2.1. We need only to check that all descent data over \( k \) of these curves \( C_Q \) are trivial. Assume that \( \gamma_{t,u}: C_Q \sim \rightarrow C_{\sigma Q} \) generates descent data of degree \( n \) for the curve \( C_Q \) defined over \( \bar{k} \). Arguing as in the proof of Proposition 2.2 we can assume that \( t = 1 \). Then, taking \( u \in \bar{k} \) such that \( v + e_0 = u \) we have: \( \sigma \gamma_{1,v} \circ \gamma_{1,u} \circ \gamma_{1,v}^{-1} = 1 \), so that all these descent data are trivial. \( \square \)

On the family of curves \( Q^2 = xy(y^2 + xz) \) with \( Q \) defined over \( k \) we can still apply \( k \)-isomorphisms \( \gamma_{t,u} \) with \( t \in k^*, \ u \in k \). We have, \( \gamma_{t,u}(C_Q) = C_{\gamma_{t,u}(Q)} \), where, for \( Q = (a, b, c, d, e, f) \):

\[
\gamma_{t,u}(Q) = (at^{-6}, (b + eu + cu^2)t^2, ct^{10}, (d + \sqrt{u} + fu)t^{-2}, et^6, ft^2). \tag{9}
\]

This allows us to consider a smaller family of curves still containing all \( k \)-isomorphy classes. By choosing a suitable \( t \) we can prefix the class of a modulo cubes. By choosing a suitable \( u \) we can assume that \( d = 0 \) (if \( f = 0 \) or \( df \in \text{AS}(k) \)) or \( df = r_0 \), for some fixed element \( r_0 \in k \setminus \text{AS}(k) \). This leads to the consideration of two families of rational normal models:

\[
\mathcal{Q}_1 := \{ Q = (a, b, c, d_0, e, f) \in k^6 \mid a \in k^*/(k^*)^3, \ c \neq 0, \ d_0 \in \{0, f^{-1}r_0\} \},
\]

\[
\mathcal{Q}_0 := \{ Q = (a, b, c, 0, e, 0) \in k^6 \mid a \in k^*/(k^*)^3, \ c \neq 0 \},
\]

\[
\mathcal{N}^{(2)}_1 := \{ C_Q \mid Q \in \mathcal{Q}_1 \}, \quad \mathcal{N}^{(2)}_0 := \{ C_Q \mid Q \in \mathcal{Q}_0 \}.
\]
Normal models $N^{(2)}_1$

The group $G := \{ \gamma_{t,u} \mid t^3 = 1, u \in \{0, f^{-2}\}\} \simeq \mu_3(k) \times C_2$ acts fully by $k$-isomorphisms on $N^{(2)}_1$. From (9) we have:

$$\gamma_{t,u}(a, b, c, d, e, f) = (a, (b + eu + cu^2)t^2, ct, d_0t^{-2}, e, ft).$$

Therefore, all $k$-automorphisms of the curves in $N^{(2)}_1$ are trivial except for:

$$\text{Aut}_k(N_Q) = \langle \gamma_{1, f^{-2}} \rangle \simeq C_2, \quad \text{if } e = cf^{-2}.$$ 

Clearly, $|N^{(2)}_1| = 2N(q - 1)^2q^2$, the element $\gamma_{1, f^{-2}}$ has $2N(q - 1)^2q$ fixed points and all other transformations in $G$ have no fixed points. By (7):

$$|G \backslash N^{(2)}_1| = \frac{1}{2N} (2N(q - 1)^2q^2 + 2N(q - 1)^2q) = (q - 1)^2q(q + 1).$$

Normal models $N^{(2)}_0$

The group $G := \{ \gamma_{t,0} \mid t^3 = 1\} \simeq \mu_3(k)$ acts fully by $k$-isomorphisms on $N^{(2)}_0$. From (9) we have now:

$$\gamma_{t,0}(a, b, c, d, e, f) = (a, bt^2, ct, d_0, e, ft).$$

Therefore, all $k$-automorphism groups are trivial and

$$|G \backslash N^{(2)}_0| = \frac{1}{|G|} |N^{(2)}_0| = (q - 1)q^2.$$ 

2.3 Curves with one bitangent

Let $C_Q$ be the family of non-singular quartics:

$$C_Q: \quad Q^2 = x(y^3 + x^2z),$$

where $Q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ is a quadratic form with coefficients in $\overline{k}$ satisfying $c \neq 0$.

On $C_Q$ we have a full action by isomorphisms by the subgroup of automorphisms of $\mathbb{P}^2$ that preserve the quartic $x(y^3 + x^2z)$ modulo squares. This group consists of the transformations [Wal95]:

$$\gamma_{t,u,v}(x, y, z) = (t^3x, t^{-1}(y + ux), t^{-9}(z + u^2y + vx)), \quad t \in \overline{k}^*, u, v \in \overline{k}.$$ 

The composition of two such transformations is given by:

$$\gamma_{t',u',v'} \circ \gamma_{t,u,v} = \gamma_{t't, u + t'u, v + u(u')^2t^8 + v' t^{12}}, \quad \forall t, t' \in \overline{k}^*, u, v, u', v' \in \overline{k}.$$
Proposition 2.4. Every non-singular quartic plane curve $C \subseteq \mathbb{P}^2$ with one bitangent is $k$-isomorphic to a curve $C_Q$ for some $Q$ with coefficients in $k$.

Proof. As in Proposition 2.3 we need only to check that all descent data over $k$ of these curves $C_Q$ are trivial.

Assume that $\gamma_{t,u,v} : C_Q \longrightarrow C_{\sigma_Q}$ generates descent data for the curve $C_Q$ defined over $\bar{k}$. By the argument we used in the proof of Proposition 2.3 we can assume that $t = 1$, $u = 0$. Again, these descent data are all trivial, since taking $w \in \bar{k}$ such that $w + \sigma w = v$ we have: $\sigma \gamma_{1,0,w} \circ \gamma_{1,0,v} \circ \gamma_{1,0,w}^{-1} = 1$.

On the family of curves $Q^2 = x(y^3 + x^2z)$ with $Q$ defined over $k$ we can still apply $k$-isomorphisms $\gamma_{t,u,v}$ with $t \in k^*$, $u, v \in k$. We have, $\gamma_{t,u,v}(C_Q) = C_{\gamma_{t,u,v}(Q)}$, where, for $Q = (a, b, c, d, e, f)$:

$$\gamma_{t,u,v}(Q) = ((a + bt^2 + c(v + u^3)^2 + du + (eu + f)(v + u^3) + \sqrt{v})t^{-6},$$

$$b + cu^2 + cu^4)t^2, et^{18}, (d + ev + fu^2 + \sqrt{u})t^{-2}, et^{10}, (eu + f)t^6). \quad (10)$$

This allows us to consider a smaller family of curves still containing all $k$-isomorphy classes. By choosing a suitable $t$ we can prefix the class of $c$ modulo $(k^*)^9$. If $e \neq 0$, we can achieve $d = f = 0$ by choosing suitable $u, v$; if $e = 0$ we obtain $b = 0$ by a suitable choice of $u$. This leads to the consideration of two families of rational normal models. The family $\mathcal{N}_1^{(1)}$ consists of all curves with equation:

$$N_Q: \quad Q^2 = x(y^3 + x^2z), \quad Q = (a, b, c, 0, e, 0), \quad c \in k^*/(k^*)^9, \quad a, b \in k, \quad e \in k^*,$$

and the family $\mathcal{S} := \mathcal{N}_0^{(1)}$ gathers all curves with equation:

$$N_Q: \quad Q^2 = x(y^3 + x^2z), \quad Q = (a, 0, c, d, 0, f), \quad c \in k^*/(k^*)^9, \quad a, d, f \in k.$$

We shall see below that a quartic defined over $k$ is supersingular if and only if it is $k$-isomorphic to a curve in the family $\mathcal{S}$.

From now on, we let $N := |\mu_9(k)| = |k^*/(k^*)^9|$; that is, $N = 9, 3, 1$, according respectively to $q \equiv 1 \pmod{9}$, $q \equiv 4, 7 \pmod{9}$ or $q \not\equiv 1 \pmod{3}$.

Normal models $\mathcal{N}_1^{(1)}$

The group $G := \{\gamma_{t,0,0} \mid t^9 = 1\} \cong \mu_9(k)$ acts fully by $k$-isomorphisms on $\mathcal{N}_1^{(1)}$. From (10) we have:

$$\gamma_{t,0,0}(a, b, c, 0, e, 0) = (at^3, bt^2, c, 0, et, 0).$$

Therefore, the $k$-automorphism groups are all trivial and

$$\frac{|G \setminus \mathcal{N}_1^{(1)}|}{N|\mathcal{N}_1^{(1)}|} = \frac{1}{N} = (q-1)q^2.$$
Normal models for supersingular quartics

The group \( G := \{ \gamma_{t,0,v} \mid t^9 = 1, v \in k \} \approx \mu_9(k) \times k \) acts fully by \( k \)-isomorphisms on \( S \). From (10) we have now:

\[
\gamma_{t,0,v}(a, 0, c, d, 0, f) = ((a + cv^2 + fv + \sqrt{v})t^3, 0, c, dt^{-2}, 0, ft^{-3}).
\]

We could reduce the family of normal models by choosing adequate representatives for the coefficient \( a \) modulo this action, but the analysis of the case is easier if we don't do this. For fixed \( c \in k^* \), \( f \in k \), let us consider the \( \mathbb{F}_2 \)-linear homomorphism,

\[
E_{c,f} : k \to k, \quad E_{c,f}(x) = cx^2 + fx + \sqrt{x}.
\]

One checks immediately that:

\[
\text{Aut}_k(N_q) = \begin{cases} 
\ker(E_{c,f}), & \text{if } d \neq 0, \\
\mu_3(k) \times \ker(E_{c,f}), & \text{if } d = 0, f \neq 0, \\
(\mu_3(k) \times \ker(E_{c,0})) \sqcup \{ \gamma_{t,0,v} \in G \mid t^3 \neq 1, v \in E_{c,0}^{-1}(t^3a) \}, & \text{if } d = f = 0,
\end{cases}
\]

For fixed \( c \in k^* \), \( v \in k \), denote

\[
N(v) := |\{ f \in k \mid v \in \ker(E_{c,f}) \}| = \begin{cases} 
q, & \text{if } v = 0, \\
1, & \text{if } v \neq 0,
\end{cases}
\]

which is independent of \( c \). Clearly \( \sum_{v \in k} N(v) = 2q - 1 \). The number of fixed points of any \( \gamma_{t,0,v} \in \Gamma \) is:

\[
\begin{align*}
Nq^2 N(v), & \quad \text{if } t = 1, \\
Nq N(v), & \quad \text{if } t^3 = 1, t \neq 1, \\
N, & \quad \text{if } t^3 \neq 1.
\end{align*}
\]

\[
(a, d \in k, c \in k^*/(k^*)^9) \quad (a = 0, c \in k^*/(k^*)^9) \quad (f = d = 0, a = t^{-3}E_{c,0}(v), c \in k^*/(k^*)^9)
\]

We can apply now (7) to count the number of \( k \)-isomorphy classes:

\[
|G \backslash S| = \frac{1}{N_q} \sum_{t,v} |S(\gamma_{t,0,v})| = \frac{1}{N_q} \sum_{v \in k} N ((q^2 + \delta q)N(v) + \epsilon) = (q + \delta)(2q - 1) + \epsilon,
\]

where \( \delta = |\mu_3(k)| - 1 \) and \( \epsilon = |\mu_9(k)| - |\mu_3(k)| \).

Stratification by the Newton polygon

For an abelian variety \( A \) over a finite field \( \mathbb{F}_q \), let us denote by \( \text{Slp}(A) \) the set of slopes of the Newton polygon (NP) of \( A \) [Oor91]. It is well known that

\[
\text{Slp}(A_1 \times A_2) = \text{Slp}(A_1) \cup \text{Slp}(A_2),
\]

\[
\dim(A) \leq 2 \implies \text{Slp}(A) \subseteq \{0, 1/2, 1\}. \quad (11)
\]
If \( \dim(A) = 3 \), the NP is determined by the 2-rank of \( A \), except for the abelian varieties of 2-rank zero, whose NP has either two sides with slopes 1/3, 2/3 or one side with slope 1/2. We say respectively that \( A \) is “of type 1/3” or “supersingular”. We use the same terminology for a quartic with one bitangent according to the shape of the NP of its jacobian.

**Proposition 2.5.**  
(a) All quartics in the family \( S \) are supersingular.  
(b) All quartics in the family \( N^{(1)}_1 \) are of type 1/3.

**Proof.**  
a) Let \( C = N_Q \) be a quartic in the family \( S \). By taking \( x = 0 \) as the line at infinity we obtain an affine model:

\[
C : \quad cz^4 + fz^2 + z = y^3 + dy^2 + a.
\]

Let \( v \in \bar{k}^* \) be a non-trivial root of the polynomial \( cz^4 + fz^2 + z \). The curve \( C \) admits the involution \( \varphi(x, y, z) = (x, y, z + v) \) and the quotient curve \( C' := C/\varphi \) has genus one. In fact, working with the affine model, the morphism \( C \to C' \) can be seen at the level of function fields as:

\[
k(y, u) \subseteq k(y, z), \quad u := z(z + v).
\]

Since \( cu^2 + v^{-1}u = cz^4 + fz^2 + z \), we can take

\[
cu^2 + v^{-1}u = y^3 + dy^2 + a,
\]

as an affine model for \( C' \) and this is clearly an elliptic curve.

Therefore, the Jacobian of \( C \) admits a non-trivial morphism to an elliptic curve and it cannot be of type 1/3 because an abelian threefold of type 1/3 is absolutely simple by (11).

b) Let \( M_3 \hookrightarrow A_{3,1} \) be the respective moduli spaces of curves of genus three and principally polarized abelian threefolds over \( \mathbb{F}_2 \). The supersingular locus in \( A_{3,1} \) is a closed subset \([\text{Oor91}, (2.4)]\) and T. Katsura and F. Oort have proved that it has dimension 2 and is absolutely irreducible \([\text{KO}, \text{section } 6]\).

Let \( SS \) be the supersingular locus of \( M_3 \) and let \( S \subseteq M_3 \) be the set of images in \( M_3 \) of all curves of the family \( S \) for all finite fields of characteristic 2. By a) we have \( S \subseteq SS \). Since all curves in \( S \) have non-trivial automorphisms, they lie in the singular locus of the moduli space: \( S \subseteq (M_3)_{\text{sing}} \).

By Lemma 3.3 below, we have \( |S(\mathbb{F}_q)| = q^2 \) for all \( q \). Hence, the result of Katsura-Oort implies that \( \overline{S} = SS \). Since the singular locus of a variety is a closed subset, we get \( SS = \overline{S} \subseteq (M_3)_{\text{sing}} \). On the other hand, all curves in \( N^{(1)}_1 \) furnish smooth points of \( M_3 \) since their automorphism groups are trivial; hence, none of these curves is supersingular. \(\square\)
3 Number of curves of genus three

Gathering all computations of section 1 and section 2 we have:

**Theorem 3.1.** There are

\[ q^6 + q^4 - q^3 + 2q^2 + 4 - [4q - 2]_{q \equiv -1 \pmod{3}} + [6]_{q \equiv 1 \pmod{9}} \]

\( \mathbb{F}_q \)-isomorphism classes of non-singular quartic plane curves defined over \( \mathbb{F}_q \). According to the Newton polygon of the Jacobian, these curves are distributed in the following way:

| Type         | Formula                                                                 |
|--------------|-------------------------------------------------------------------------|
| ordinary     | \( q^6 - q^5 + q^4 - 3q^3 + 5q^2 - 6q + 7 \)                            |
| 2-rank two   | \( q^5 - q^4 + q^3 - 2q^2 + 2q - 1 \)                                  |
| 2-rank one   | \( q^4 - 2q^2 + q \)                                                   |
| type 1/3     | \( q^3 - q^2 \)                                                        |
| supersingular| \( 2q^2 - q + [4q - 2]_{q \equiv 1 \pmod{3}} + [6]_{q \equiv 1 \pmod{9}} \) |

The notation \([a]_{\text{condition}}\) indicates that \( a \) has to be added in the formula if "condition" is satisfied.

The Newton polygon of a hyperelliptic curve of genus 3 is completely determined by the number of Weierstrass points. On one hand, for any hyperelliptic curve \( C \) defined over a perfect field of characteristic 2 we have:

\[ \dim_{\mathbb{F}_2}(J_C[2]) = |W| - 1, \]

where \( J_C \) is the Jacobian of \( C \) and \( W \) the set of Weierstrass points. On the other hand, J. Scholten and H.J. Zhu have proved that there are no supersingular hyperelliptic curves of genus 3 in characteristic 2 [SZ02]. Thus, when \( C \) has 2-rank zero it is necessarily of type 1/3.

In [NS04, Table 2] there are formulas for the number of hyperelliptic curves of genus 3 with prescribed ramification divisor. Adding these computations to the results of Theorem 3.1 we get:

**Corollary 3.2.** There are

\[ q^6 + 2q^5 + q^4 + q^3 + q^2 + q + 2 - [4q - 2]_{q \equiv -1 \pmod{3}} + [6]_{q \equiv 1 \pmod{9}} + [12]_{q \equiv 1 \pmod{7}} \]

\( \mathbb{F}_q \)-isomorphism classes of smooth projective curves of genus three defined over \( \mathbb{F}_q \). According to the Newton polygon of the Jacobian, these curves are distributed in the following way:
\[ q^6 + q^5 - q^4 - q^3 + q^2 - 4q + 7 \]

\[ q^5 + q^4 - 3q^3 + q^2 + q - 1 \]

\[ q^4 + 4q^3 - 4q^2 + q - 2 \]

\[ q^3 + q^2 + [12]_{q \equiv 1 \pmod{7}} \]

\[ 2q^2 - q + [4q - 2]_{q \equiv 1 \pmod{4}} + [6]_{q \equiv 1 \pmod{9}} \]

Also, from the computations of sections 1, 2 we can deduce the number of \( k \)-rational points in the moduli space. Let us quote first a result that is an immediate consequence of [VdG-VdV92, 5.1].

**Lemma 3.3.** Let \( \mathcal{C} \) be a finite family of projective smooth curves of genus \( g > 1 \), defined over a finite field \( k \) with the property that every \( k \)-curve of genus \( g \) that is isomorphic to some curve in \( \mathcal{C} \) is \( k \)-isomorphic to some curve in \( \mathcal{C} \). Suppose that \( \mathcal{C} \) is the disjoint union, \( \mathcal{C} = \bigsqcup_i \mathcal{C}_i \), of subfamilies \( \mathcal{C}_i \) admitting a full action by \( k \)-isomorphisms by a finite group \( G_i \). Then, the image of the family \( \mathcal{C} \) in the subset of \( k \)-rational points of the moduli space of curves of genus \( g \) has cardinality \( \sum_i |\mathcal{C}_i|/|G_i| \).

**Proof.** By [VdG-VdV92, 5.1], the contribution of the curves in this family to the moduli space is the weighted sum \( \sum_C |\text{Aut}_k(C)|^{-1} \), for \( C \) running on a system of representatives of \( k \)-isomorphism classes of the curves in \( \mathcal{C} \). If we have one single group \( G \) acting fully by \( k \)-isomorphisms on the family \( \mathcal{C} \), this weighted sum can be computed as:

\[
\sum_{C \in G \cdot \mathcal{C}} |\text{Aut}_k(C)|^{-1} = \sum_{C \in G \cdot \mathcal{C}} |G(C)|^{-1} = \sum_{C \in G \cdot \mathcal{C}} \frac{|G \cdot C|}{|G|} = \frac{|\mathcal{C}|}{|G|},
\]

where \( G(C) \) is the stabilizer of \( C \) and \( G \cdot C \) the orbit of \( C \). In the general case, the contribution to the moduli space would be \( \sum_i \sum_{C \in G_i \cdot \mathcal{C}_i} |\text{Aut}_k(C)|^{-1} \) and the above computation applies to each subfamily \( \mathcal{C}_i \).

This result can be applied to each of the five families obtained by gathering all normal models with the same Newton polygon, to get closed formulas for the cardinalities of the strata by the Newton polygon of the non hyperelliptic locus \( \mathcal{M}_{3}^{\text{nh}} \) of the moduli space. Adding to these computations the results of [NS04, Table 3], [SZ02] concerning the hyperelliptic locus \( \mathcal{M}_3^{\text{h}} \) we get a complete picture for \( \mathcal{M}_3 \):

**Theorem 3.4.** The number of \( \mathbb{F}_q \)-rational points of the strata by the Newton polygon of \( \mathcal{M}_{3}^{\text{nh}} \), \( \mathcal{M}_3^{\text{h}} \) and \( \mathcal{M}_3 \) are given in the following table:
In particular, $|\mathcal{M}_3(\mathbb{F}_q)| = q^6 + q^5 + q^4 - 2q^3 + q^2 + 2q^3 - 2q^2 + q^2$. This result for the total number of rational points of the moduli space was conjectured by A. Granville and B. Brock [BG01] and proved by J. Bergström [Ber01].

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