Globally hyperbolic moment model of arbitrary order for three-dimensional special relativistic Boltzmann equation

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Abstract

This paper extends the model reduction method by the operator projection to the three-dimensional special relativistic Boltzmann equation. The derivation of arbitrary order moment system is built on our careful study of infinite families of the complicate Grad type orthogonal polynomials depending on a parameter and the real spherical harmonics. We derive the recurrence relations of the polynomials, calculate their derivatives with respect to the independent variable and parameter respectively, and study their zeros. The recurrence relations and partial derivatives of the real spherical harmonics are also given. It is proved that our moment system is globally hyperbolic, and linearly stable. Moreover, the Lorentz covariance is also studied in the 1D space.

Key words: Moment method, hyperbolicity, relativistic Boltzmann equation, model reduction, operator projection.

1 Introduction

The relativistic kinetic theory of gases has been widely applied in the astrophysics and cosmology [11,23]. Different from a non-relativistic monatomic gas, a relativistic gas has a

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bulk viscosity. It has called much attention to a number of applications of this theory: the effect of neutrino viscosity on the evolution of the universe and the study of galaxy formation, neutron stars, and controlled thermonuclear fusion etc., and the interest has grown in recent years as experimentalists are now able to make reliable measurements on physical systems where relativistic effects are no longer negligible. However, the relativistic kinetic theory has been sparsely used to model phenomenological matter in comparison to fluid models. In the non-relativistic case, the kinetic theory has been intensively studied as a mathematical subject during several decades, and also played an important role from an engineering point of view, see e.g. [10,12]. One could determine the distribution function and transport coefficients of gases from the Boltzmann equation, however it was not so easy. An approximate solution of the integro-differential equation was obtained by Hilbert from a power series expansion of a parameter (being proportional to the mean free path). The transport coefficients were independently calculated by Chapman and Enskog for gases whose molecules interacted according to any kind of spherically symmetric potential function. Another method is to expand the distribution function in terms of tensorial Hermite polynomials and introduce the balance equations corresponding to higher order moments of the distribution function [21,22]. The Chapman-Enskog method assumes that in the hydrodynamic regime the distribution function can be expressed as a function of the hydrodynamic variables and their gradients, and has been extended to the relativistic cases, see e.g. [14,19,20,24,25], but unfortunately, there exists difficulty to derive the relativistic hydrodynamic equations from the kinetic theory [13]. The moment method is also generalized to the relativistic cases, see e.g. [2,5,29,30,31,35,39]. Combining the Chapman-Enskog method with the moment method has been attempted in order to state the influence of the Knudsen number [13,32]. Recently, some latest progresses were gotten on the Grad moment method in the non-relativistic case. A regularization was presented for the 1D Grad moment system to achieve global hyperbolicity [6]. It was based on the observation that the characteristic polynomial of the flux Jacobian of the Grad’s moment system did not depend on the intermediate moments, and further extended to the multi-dimensional case [7,8]. By replacing the exact integration with some quadrature rule, the quadrature based projection methods could derive hyperbolic PDE systems for the Boltzmann equation [33,34]. In the 1D case, it is similar to the regularization in [6]. Those contributions led to well understanding the hyperbolicity of the Grad moment systems. Based on the operator projection, a general framework of model reduction technique was recently presented [18], in which the time and space derivatives in the kinetic equation were synchronously projected into a finite-dimensional weighted polynomial space, and most of the existing moment systems mentioned above might be regotten.

There exists a huge difficulty in deriving the relativistic hyperbolic moment system of higher order since the family of Grad type orthogonal polynomials can not be found easily. Several attempts have been conducted to construct them analogous to the Hermite polynomials, see e.g [3,23], where the weighted polynomial space was spanned by the products of the weighted function, Grad type orthogonal polynomials, and irreducible tensors. Their application can be found in [13,32,40]. Unfortunately, there seems no explicit expression of the moment systems if the order of the moment system is larger than 3, and the hyperbolicity of existing general moment systems is not proved, even for the
second order moment system (e.g. the general Israel and Stewart system). For a special case with heat conduction and no viscosity, Hiscock and Lindblom proved that the Israel and Stewart moment system was globally hyperbolic and linearly stable in the Landau frame, but converse in the Eckart frame. In the general case, they only proved that the Israel and Stewart moment system was hyperbolic near the equilibrium. The readers are referred to [24,27,28]. It is easy to show by the approach [26,27] that the Israel and Stewart moment system is not globally hyperbolic in the Landau frame too if the viscosity exists. There does not exist any result on the hyperbolicity or loss of hyperbolicity of (existing) general higher-order moment systems for the relativistic kinetic equation. Such proof is very difficult and challenging. The loss of hyperbolicity may cause the solution blow-up when the distribution is far away from the equilibrium state. Even for the non-relativistic case, increasing the number of moments seems not to avoid such blow-up [9]. Recently, globally hyperbolic moment system of arbitrary order was derived for the 1D special relativistic Boltzmann equation [36] with the aid of the model reduction method by the operator projection [18]. The key is to choose the weight function and define the weighted polynomial spaces and their basis as well as the projection operator.

The aim of this paper is to derive globally hyperbolic moment system of arbitrary order for the three-dimensional special relativistic Boltzmann equation. Unlike the 1D case [36], the basis of the weighted polynomial space are very difficult to be obtained, because the irreducible tensors [23] are linearly dependent, and it is difficult to find their maximally linearly independent set. Our contribution are that the irreducible tensors are replaced with the real spherical harmonics in order to find the basis of weighted polynomial spaces and the properties of infinite families of the complicate Grad type orthogonal polynomials depending on a parameter and the real spherical harmonics are carefully studied. The paper is organized as follows. Section 2 introduces the special relativistic Boltzmann equation and some macroscopic quantities defined via the kinetic theory. Section 3 gives infinite families of orthogonal polynomials dependent on a parameter, and studies their properties: recurrence relations, derivative relations with respect to the variable and the parameter, and zeros. Section 4 introduces the real spherical harmonics and their properties. Section 5 derives the moment system of the special relativistic Boltzmann equation and discusses its limitation and corresponding quasi-1D cases. Section 6 studies the properties of our moment system: the hyperbolicity, linear stability, and Lorentz covariance. Section 7 concludes the paper.

2 Preliminaries and notations

In the special relativistic kinetic theory of gases [11], a microscopic gas particle of rest mass $m$ is characterized by the 4 space-time coordinates $(x^\alpha) = (x^0, \mathbf{x})$ and momentum 4-vector $(p^\alpha) = (p^0, \mathbf{p})$, where $x^0 = ct$, $c$ denotes the speed of light in vacuum, and $t$ and $\mathbf{x}$ are the time and 3D spatial coordinates, respectively. Besides the contravariant notation (e.g. $p^\alpha$), the covariant notation such as $p_\alpha$ will also be used and is related to the contravariant by $p_\alpha = g_{\alpha\beta}p^\beta$ or $p^\alpha = g^{\alpha\beta}p_\beta$, where $(g^{\alpha\beta})$ is chosen as the Minkowski space-time metric.
tensor, \((g^{\alpha \beta}) = \text{diag}\{1, -I_3\}\), \(I_3\) is the unit matrix of size 3, \((g_{\alpha \beta})\) is the inverse of \((g^{\alpha \beta})\), and the Einstein summation convention over repeated indices has been used. For a free relativistic particle, one has the relativistic energy-momentum relation (aka “on-shell” or “mass-shell” condition) \(E^2 - \mathbf{p}^2c^2 = m^2c^4\). If putting \(p^0 = c^{-1}E = \sqrt{\mathbf{p}^2 + m^2c^2}\), then the “mass-shell” condition may be rewritten as \(p^\alpha p_\alpha = m^2c^2\).

The special relativistic Boltzmann equation provides a statistical description of a gas of relativistic particles that are interacting through binary collisions in Minkowski space [11]. For a single gas it reads

\[
p^\alpha \frac{\partial f}{\partial x^\alpha} = Q(f, f),
\]

where \(f = f(x, p, t)\) denotes one-particle distribution function, \(Q(f, f)\) is the collision term depending on the product of the distribution functions of two particles at collision, e.g.

\[
Q(f, f) = \int_{\mathbb{R}^3} \int_{S_+^4} (f'_s f' - f_f) B d\Omega \frac{d^3 p_s}{p^0_s},
\]

here the distributions \(f\) and \(f_s\) depend on the momenta before a collision, while \(f'\) and \(f'_s\) rely on the momenta after the collision, \(d\Omega\) denotes the element of the solid angle, the collision kernel is given by \(B = \sigma \sqrt{(p^2_s p^0_s)^2 - m^2c^2}\) for a single non degenerate gas (e.g. electron gas), and \(\sigma\) denotes the differential cross section of collision. The collision term satisfies

\[
\int_{\mathbb{R}^3} Q(f, f) \frac{d^3 p}{p^0} = 0, \quad \int_{\mathbb{R}^3} p^\alpha Q(f, f) \frac{d^3 p}{p^0} = 0,
\]

so that 1 and \(p^\alpha\) are called collision invariants. Moreover, (2.1) satisfies the entropy dissipation relation (in the sense of classical statistics)

\[
\int_{\mathbb{R}^3} Q(f, f) \ln(f) \frac{d^3 p}{p^0} \leq 0,
\]

where the equal sign corresponds to the local thermodynamic equilibrium.

The macroscopic description of gas can be represented by the first and second moments of the distribution function \(f\), namely, the partial particle 4-flow \(N^\alpha\) and the partial energy-momentum tensor \(T^{\alpha \beta}\), defined respectively by

\[
N^\alpha = c \int_{\mathbb{R}^3} p^\alpha f \frac{d^3 p}{p^0}, \quad T^{\alpha \beta} = c \int_{\mathbb{R}^3} p^\alpha p^\beta f \frac{d^3 p}{p^0},
\]

which have the Landau-Lifshitz decompositions

\[
N^\alpha = nU^\alpha + n^\alpha, \quad T^{\alpha \beta} = c^{-2} \varepsilon U^\alpha U^\beta - \Delta^{\alpha \beta} (P_0 + \Pi) + \pi^{\alpha \beta},
\]

where \((U^\alpha) = (\gamma(u)c, \gamma(u)u)\) denotes the macroscopic velocity 4-vector of gas, \(\gamma(u) = (1 - c^{-2}|u|^2)^{-\frac{1}{2}}\) is the Lorentz factor, \(\Delta^{\alpha \beta}\) is defined by

\[
\Delta^{\alpha \beta} := g^{\alpha \beta} - c^{-2} U^\alpha U^\beta,
\]
which is a symmetric projector onto the 3D subspace and orthogonal to $U^\alpha$, i.e. $\Delta^{\alpha\beta} U_\beta = 0$. Here, the number density $n$, the particle-diffusion current $n^\alpha$, the energy density $\varepsilon$, the shear-stress tensor $\pi^{\alpha\beta}$, and the sum of thermodynamic pressure $P_0$ and bulk viscous pressure $\Pi$ are related to the distribution $f$ by

\begin{align}
  n := c^{-2} U_\alpha N^\alpha &= c^{-1} \int_{\mathbb{R}^3} E f \frac{d^3 p}{p^0}, \\
  n^\alpha := \Delta_\beta^{\alpha\beta} N^\beta &= c \int_{\mathbb{R}^3} p^{<\alpha>} f \frac{d^3 p}{p^0}, \\
  \varepsilon := c^{-2} U_\alpha U_\beta T^{\alpha\beta} &= c^{-1} \int_{\mathbb{R}^3} E^2 f \frac{d^3 p}{p^0}, \\
  \pi^{\alpha\beta} := \Delta_{\mu\nu}^{\alpha\beta} T^{\mu\nu} &= c \int_{\mathbb{R}^3} p^{<\alpha} p^{>\beta} f \frac{d^3 p}{p^0}, \\
  P_0 + \Pi := -\frac{1}{3} \Delta_{\alpha\beta}^{\alpha\beta} T^{\alpha\beta} &= \frac{1}{3c} \int_{\mathbb{R}^3} (E^2 - m^2 c^4) f \frac{d^3 p}{p^0},
\end{align}

(2.4)

where $E := U_\alpha p^\alpha$, $p^{<\alpha>} := \Delta_\beta^{\alpha\beta} p^\beta$ and $p^{<\alpha} p^{>\beta} := \Delta_{\mu\nu}^{\alpha\beta} p^\mu p^\nu$ are the first and second order irreducible tensors, respectively, and

$$
\Delta_{\mu\nu}^{\alpha\beta} := \frac{1}{2} \left( \Delta_\mu^{\alpha\beta} \Delta_\nu^{\gamma\delta} + \Delta_\nu^{\alpha\beta} \Delta_\mu^{\gamma\delta} - \frac{2}{3} \Delta_{\mu\nu}^{\gamma\delta} \Delta^{\alpha\beta} \right).
$$

At the local thermodynamic equilibrium, $n^\alpha$, $\Pi$, and $\pi^{\alpha\beta}$ will be zero. Multiplying (2.1) by 1 and $p^\alpha$ respectively, integrating them over $\mathbb{R}^3$ with respect to $p$, and using (2.2) can give the following conservation laws

$$
\partial_\alpha N^\alpha = 0, \quad \partial_\alpha T^{\alpha\beta} = 0.
$$

(2.5)

The present work chooses the macroscopic velocity 4-vector $U^\alpha$ as the velocity of the energy transport (the Landau-Lifshitz frame [37])

$$
U_\alpha T^{\alpha\beta} = \varepsilon U^\alpha,
$$

(2.6)

i.e.

$$
\Delta_\beta^{\alpha\beta} T^{\beta\gamma} U_\gamma = c \int_{\mathbb{R}^3} E p^{<\alpha>} f \frac{d^3 p}{p^0} = 0,
$$

(2.7)

and considers the Anderson-Witting model [4]

$$
Q(f, f) = -\frac{U_\alpha p^\alpha}{\tau c^2} (f - f^{(0)}),
$$

(2.8)

where $f^{(0)} = f^{(0)}(x, p, t)$ denotes the distribution function at the local thermodynamic equilibrium, and the relaxation time $\tau$ may rely on $n$ and $\theta$ and be defined by $\tau = \frac{1}{n n_0 d \bar{g}}$, $d$ denotes the diameter of gas particles, and $\bar{g}$ is proportional to the mean relative speed $\bar{\xi}$ between two particles, e.g. $\bar{g} = \sqrt{2} \bar{\xi}$ or $\bar{\xi}$ [11]. The local-equilibrium distribution $f^{(0)}$ can be explicitly given by the Maxwell-Jüttner distribution

$$
f^{(0)} = n g^{(0)}, \quad g^{(0)} = \frac{\zeta}{4 \pi m^3 c^3 K_2(\zeta)} \exp\left(-m^{-1} c^{-2} \zeta E\right),
$$

(2.9)
which can completely determine the number density \( n \) and energy density \( \varepsilon \) by

\[
n = n_0 := c^{-1} \int_{\mathbb{R}} E f^{(0)} \frac{d^3 p}{p^0}, \quad \varepsilon = \varepsilon_0 := c^{-1} \int_{\mathbb{R}} \frac{E^2 f^{(0)} d^3 p}{p^0} = nmc^2 \left( G(\zeta) - \zeta^{-1} \right),
\]

where \( G(\zeta) := K_2^{-1}(\zeta)K_3(\zeta), \quad \zeta = (k_B T)^{-1}(mc^2) \) is the ratio between the particle rest energy \( mc^2 \) and the thermal energy of the gas \( k_B T \), \( k_B \) denotes the Boltzmann constant, \( T \) is the thermodynamic temperature, and \( K_\nu(\zeta) \) denotes the modified Bessel function of the second kind, defined by

\[
K_\nu(\zeta) = \int_0^\infty \cosh(\nu \vartheta) \exp(-\zeta \cosh \vartheta) d\vartheta,
\]

satisfying the recurrence relation

\[
K_{\nu+1}(\zeta) = K_{\nu-1}(\zeta) + 2\nu \zeta^{-1} K_\nu(\zeta).
\]

The particles behave as non-relativistic (resp. ultra-relativistic) for \( \zeta \gg 1 \) (resp. \( \zeta \ll 1 \)).

It is possible to determine the other macroscopic quantities from the knowledge of \( f^{(0)} \) by

\[
n_0^\alpha := c \int_{\mathbb{R}^3} p^{<\alpha>} f^{(0)} \frac{d^3 p}{p^0} = 0, \quad P_0 := \frac{1}{3c} \int_{\mathbb{R}^3} (E^2 - m^2 c^4) f^{(0)} d^3 p = nk_B T = nmc^2 \zeta^{-1},
\]

and rewrite the conservation laws (2.5) into the following form

\[
\frac{\partial (nU^\alpha)}{\partial x^\alpha} = 0, \quad \frac{\partial \left( c^{-2} nhU^\alpha U^\beta - g^{\alpha\beta} P_0 \right)}{\partial x^\beta} = 0,
\]

where \( h := n^{-1}(\varepsilon + P_0) = mc^2 G(\zeta) \) denotes the specific enthalpy. It means that the special relativistic hydrodynamic equations (2.14) can be derived from the special relativistic Boltzmann equation (2.1) when \( f = f^{(0)} \). How to find the reduced model equations to describe the states with \( f \neq f^{(0)} \)? This paper attempts to answer it and derive its globally hyperbolic moment model of arbitrary order by extending the moment method by operator projection [18] and [36] to (2.1), see Section 5.

Before ending this section, let us discuss calculation of the physically admissible macroscopic states \( \{n, u, \theta = \zeta^{-1} : n > 0, |u| < c, \theta > 0\} \) from a given nonnegative distribution \( f(x, p, t) \), which is not identically zero.

**Theorem 2.1** For the nonnegative distribution \( f(x, p, t) \), which is not identically zero, the density current \( N^\alpha \) and energy-momentum tensor \( T^{\alpha\beta} \) calculated by (2.3) satisfy the properties:

(i) \( T^{\alpha\beta} \) is positive definite, and the matrix-pair \( (T^{\alpha\beta}, g^{\alpha\beta}) \) has an unique positive generalized eigenvalue \( \varepsilon \) larger than \( nmc^2 \) and corresponding generalized eigenvector \( U_\alpha \) satisfying \( U_0 = \sqrt{U_1^2 + U_2^2 + U_3^2 + c^2} \), i.e., \( T^{\alpha\beta} U_\beta = \varepsilon g^{\alpha\beta} U_\beta \).
(ii) The macroscopic velocity vector \( u := -c(U_0^{-1}U_1, U_0^{-1}U_2, U_0^{-1}U_3)^T \) is bounded by the speed of light, that is, \( |u| < c \).

(iii) The number density \( n := c^{-2}U_\alpha N^\alpha \) is positive.

(iii) The equation

\[
G(\theta^{-1}) - \theta = n^{-1}m^{-1}c^{-2}\varepsilon, \tag{2.15}
\]

has an unique positive solution for \( \theta \in (0, +\infty) \).

**Proof**

(i) For any nonzero vector \( X = (x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4 \) and the given nonnegative distribution \( f(x, p, t) \), which is not identically zero, using (2.3) gives

\[
X^T (T^{\alpha\beta}) X = cX^T \left( \int_{\mathbb{R}^3} p^\alpha p^\beta f \frac{d^3p}{p^0} \right) X = c \int_{\mathbb{R}^3} x_\alpha p^\alpha p^\beta x_\beta f \frac{d^3p}{p^0} = c \int_{\mathbb{R}^3} (x_\alpha p^\alpha)^2 f \frac{d^3p}{p^0} > 0.
\]

It means that \( T^{\alpha\beta} \) is positive definite.

Thanks of (2.6), the matrix-pair \( (T^{\alpha\beta}, g^{\alpha\beta}) \) has an unique positive generalized eigenvalue \( \varepsilon \), which satisfies

\[
U_\alpha T^{\alpha\beta} U_\beta = \varepsilon U_\alpha g^{\alpha\beta} U_\beta, \quad U_\alpha \neq 0.
\]

Because \( T^{\alpha\beta} \) is positive definite, the left hand side is larger than zero and thus one has \( U_0^2 > U_1^2 + U_2^2 + U_3^2 \) and \( U_0 = \sqrt{U_1^2 + U_2^2 + U_3^2 + c^2} \) via multiplying \( (U_\alpha) \) by a scaling constant \( c(U_0^2 - U_1^2 - U_2^2 - U_3^2)^{-1/2} \). As a result, the macroscopic velocity vector \( u \) can be calculated by \( u = -c(U_0^{-1}U_1, U_0^{-1}U_2, U_0^{-1}U_3)^T \) and satisfies

\[
|u| = cU_0^{-1}\sqrt{U_1^2 + U_2^2 + U_3^2} < c. \tag{2.16}
\]

(ii) Using the Cauchy-Schwarz inequality gives

\[
E - mc^2 = \left( \sum_{i=1}^{3} U_i^2 + c^2 \sum_{i=1}^{3} p_i^2 + m^2 c^2 - \sum_{i=1}^{3} U_i p_i \right) - mc^2 > 0,
\]

which implies

\[
n = c^{-2}U_\alpha N^\alpha = c^{-1} \int_{\mathbb{R}^3} Ef \frac{d^3p}{p^0} > 0.
\]

(iii) Because \( E > mc^2 \) in (ii), one has

\[
\varepsilon - nm^2 = c^{-2}U_\alpha T^{\alpha\beta} U_\beta - mU_\alpha N^\alpha = c^{-1} \left( \int_{\mathbb{R}^3} E^2 f \frac{d^3p}{p^0} - mc^2 \int_{\mathbb{R}^3} E f \frac{d^3p}{p^0} \right) = c^{-1} \int_{\mathbb{R}^3} E(E - mc^2) f \frac{d^3p}{p^0} > 0.
\]

On the other hand, it holds

\[
\lim_{\theta \to 0} \left( G(\theta^{-1}) - \theta \right) = 1, \quad \lim_{\theta \to +\infty} \left( G(\theta^{-1}) - \theta \right) = \lim_{\theta \to +\infty} 3\theta = +\infty,
\]

7
and

\[ \frac{\partial (G(\theta^{-1}) - \theta)}{\partial \theta} = -\theta^{-2} \left( G(\theta^{-1})^2 - 5G(\theta^{-1})\theta + \theta^2 - 1 \right) =: \tilde{\psi}(G(\theta^{-1}), \theta). \]

Because

\[ 0 < mc \int_{\mathbb{R}^3} \frac{E - mc^2 f^{(0)} d^3p}{E + mc^2} = -n\theta \left( (3\theta + 2)G(\theta^{-1}) - 2(6\theta^2 + 4\theta + 1) \right), \]
\[ 0 < c^{-1} \int_{\mathbb{R}^3} (E - mc^2)^2 f^{(0)} d^3p \frac{d^3p}{p^0} = nmc^2 (2G(\theta^{-1}) - 5\theta - 2), \]

one gets

\[ \frac{5}{2}\theta + 1 < G(\theta^{-1}) < \frac{2(6\theta^2 + 4\theta + 1)}{3\theta + 2}. \]

Hence one has

\[ \tilde{\psi}(G(\theta^{-1}), \theta) > \tilde{\psi} \left( \frac{2(6\theta^2 + 4\theta + 1)}{3\theta + 2}, \theta \right) > 3(3\theta + 2)^{-2}(9\theta^2 + 12\theta + 1) > 0, \]

i.e.

\[ \frac{\partial (G(\theta^{-1}) - \theta)}{\partial \theta} > 0, \]

which implies that \( G(\theta^{-1}) - \theta \) is a strictly monotonic function of \( \theta \) in the interval \((0, +\infty)\).

Thus (2.15) has an unique solution in the interval \((0, +\infty)\). The proof is completed.

Furthermore, the following conclusion holds.

**Theorem 2.2** Under the assumptions of Theorem 2.1, the bulk viscous pressure \( \Pi \) satisfies

\[ \Pi > -nmc^2\theta. \]

**Proof** Using Theorem 2.1 yields that \( \{n, u, \theta\} \) satisfy

\[ n > 0, \quad |u| < c, \quad \theta > 0. \quad (2.17) \]

Combining them with the last equations in (2.4) and (2.13) gives

\[ \Pi = -\frac{1}{3} \int_{\mathbb{R}^3} \Delta_{\alpha\beta} p^\alpha p^\beta f \frac{d^3p}{p^0} - nmc^2\theta = -\frac{1}{3c} \int_{\mathbb{R}^3} (E^2 - m^2 c^4) f \frac{d^3p}{p^0} - nmc^2\theta > -nmc^2\theta. \]

The proof is completed.

**Remark 2.1** Theorem 2.1 provides a recovery procedure of the admissible primitive variables \( \rho, u, \) and \( \theta \) from the nonnegative distribution \( f(x, p, t) \) or the given density current \( N^\alpha \) and energy-momentum tensor \( T^{\alpha\beta} \) satisfying

\[ U_\alpha N^\alpha > 0, \text{ and } T^{\alpha\beta} \text{ is positive definite.} \quad (2.18) \]

It is useful in the derivation of the moment system as well as the numerical scheme.
Remark 2.2 If setting
\[ x = L \hat{x}, \quad p = c \hat{p}, \quad p^0 = c \hat{p}^0, \quad t = \frac{L}{c} \hat{t}, \quad g = c \hat{g}, \quad f = \frac{n_0}{c^3} \hat{f}, \]
where \( L, \ n_0, \) and \( \theta_0 = mc^2/k_B \) are the macroscopic characteristic length, the reference number density and temperature, respectively, then the relativistic Boltzmann equation (2.1) with the Anderson-Witting model (2.8) can be non-dimensionalized as follows

\[
\hat{p}^\alpha \frac{\partial \hat{f}}{\partial \hat{x}^\alpha} = \frac{\hat{n}}{K_n} \hat{U}_\alpha \hat{p}^\alpha \left( \hat{f}^{(0)} - \hat{f} \right),
\]
(2.19)

where \( K_n = \frac{\lambda}{L} = \frac{n_0c}{L} = \frac{1}{n_0 L \pi d^2} \) denotes the Knudsen number. If considering \( \tilde{\tau} := \frac{K_n}{\hat{n}} \) as a new “relaxation time”, then in (2.19) the collision term is the same as that of the non-relativistic Bhatnagar-Gross-Krook model, and the notations \( \tilde{\tau}, \ \hat{x}, \ \hat{t}, \ \hat{f}, \ \hat{p}, \ \hat{p}^0, \ \hat{n} \) can still be replaced with \( \tau, \ x, \ t, \ f, \ p, \ p^0, \ n \) respectively.

3 Families of orthogonal polynomials

This section introduces (infinite) families of Grad type orthogonal polynomials dependent on a parameter \( \zeta \), and studies their properties. The polynomials are the same as those in [3], but different from those in [36]. Their properties are not discussed in the literature but are crucial for deriving globally hyperbolic moment model of arbitrary order for the three-dimensional special relativistic Boltzmann equation.

If considering
\[
\omega^{(\ell)}(x; \zeta) = \frac{\zeta (x^2 - 1)^{\ell + \frac{1}{2}}}{(2\ell + 1)K_2(\zeta)} \exp (-\zeta x), \ \ell \in \mathbb{N},
\]
(3.1)
as the weight functions in the interval \([1, +\infty)\), where \( \zeta \in \mathbb{R}^+ \) denotes a parameter, and \( \mathbb{N} \) is the set of the non-negative integers, then the inner product with respect to \( \omega^{(\ell)}(x; \zeta) \) may be introduced as follows
\[
(f, g)_{\omega^{(\ell)}} := \int_{1}^{+\infty} f(x)g(x)\omega^{(\ell)}(x; \zeta)dx, \quad f, g \in L^2_{\omega^{(\ell)}}[1, +\infty), \ \ell \in \mathbb{N},
\]
where \( L^2_{\omega^{(\ell)}}[1, +\infty) := \left\{ f \mid \int_{1}^{+\infty} f(x)^2\omega^{(\ell)}(x; \zeta)dx < +\infty \right\} \). It is worth noting that the choice of the weight function \( \omega^{(\ell)}(x; \zeta) \) is dependent on the equilibrium distribution \( f^{(0)}(x, p, t) \) in (2.9).

Let \( \{P_k^{(\ell)}(x; \zeta)\}, \ \ell \in \mathbb{N}, \) be infinite families of standard orthogonal polynomials with respect to the weight function \( \omega^{(\ell)}(x; \zeta) \) in the interval \([1, +\infty)\), i.e.
\[
(P_i^{(\ell)}, P_k^{(\ell)})_{\omega^{(\ell)}} = \delta_{i,k}, \quad \ell \in \mathbb{N},
\]
(3.2)
where the degree of \( P_k^{(\ell)}(x; \zeta) \) is equal to \( k \), \( \delta_{i,k} \) denotes the Kronecker delta function, which is equal to 1 if \( i = k \), and 0 otherwise. Obviously, \( \{ P_k^{(\ell)}(x; \zeta) \} \) satisfies
\[
(P_k^{(\ell)}, x^i)_{\omega(\ell)} = 0, \quad i \leq k - 1,
\]
and
\[
Q(x; \zeta) = \sum_{i=0}^{k} \left( Q(x; \zeta), P_k^{(\ell)} \right)_{\omega(\ell)} P_k^{(\ell)}(x; \zeta),
\]
for any polynomial \( Q(x; \zeta) \) of degree \( \leq k \) in \( L^2_{\omega(\ell)}[1, +\infty) \).

The orthogonal polynomials \( \{ P_k^{(\ell)}(x; \zeta) \} \) may be obtained by using the Gram-Schmidt process. For example, several orthogonal polynomials of lower degree are given as follows
\[
P_0^{(0)}(x; \zeta) = \frac{1}{\sqrt{G(\zeta) - 4\zeta^{-1}}},
\]
\[
P_1^{(0)}(x; \zeta) = \frac{\sqrt{G(\zeta) - 4\zeta^{-1}}}{\sqrt{G(\zeta)^2 - 5\zeta^{-1}G(\zeta) + 4\zeta^{-2} - 1}} \left( x - \frac{1}{G(\zeta) - 4\zeta^{-1}} \right),
\]
\[
P_2^{(0)}(x; \zeta) = \sqrt{3}\frac{\zeta\sqrt{G(\zeta)^2 - 5\zeta^{-1}G(\zeta) + 4\zeta^{-2} - 1}}{\sqrt{2G(\zeta)^3 - 13\zeta^{-1}G(\zeta)^2 - 2G(\zeta) + 20\zeta^{-2}G(\zeta) + 3\zeta^{-1}}}
\times \left( x^2 - 3\frac{G(\zeta)^2 - 4\zeta^{-1}G(\zeta) - 1}{\zeta (G(\zeta)^2 - 5\zeta^{-1}G(\zeta) + 4\zeta^{-2} - 1)}x - \frac{G(\zeta)^2 - 5\zeta^{-1}G(\zeta) + \zeta^{-2} - 1}{G(\zeta)^2 - 5\zeta^{-1}G(\zeta) + 4\zeta^{-2} - 1} \right),
\]
\[
P_0^{(1)}(x; \zeta) = \sqrt{\zeta},
\]
\[
P_1^{(1)}(x; \zeta) = \frac{\sqrt{\zeta}}{\sqrt{\left( -G(\zeta)^2 - 5\zeta^{-1}G(\zeta) + 1 \right)}} \left( x - G(\zeta) \right),
\]
\[
P_0^{(2)}(x; \zeta) = \frac{\zeta}{\sqrt{3G(\zeta)}},
\]
and plotted in Fig. 3.1 with respect to \( x \) and \( \zeta \). Obviously, the coefficients in those orthogonal polynomials are so irregular that it is quite complicate to study the properties of \( \{ P_k^{(\ell)}(x; \zeta) \} \). Let \( c_k^{(\ell)} \) be the leading coefficient of \( P_k^{(\ell)}(x; \zeta) \), and without loss of generality, assume \( c_k^{(\ell)} > 0 \), \( \ell \in \mathbb{N} \). Because the polynomial \( P_k^{(\ell)}(x; \zeta) \) has exactly \( n \) real simple zeros in the interval \( (1, +\infty) \), denoted by \( \{ x_{i,k}^{(\ell)} \}_{i=1}^{k} \) in an increasing order, the polynomial \( P_k^{(\ell)}(x; \zeta) \) can be rewritten as follows
\[
P_k^{(\ell)}(x; \zeta) = c_k^{(\ell)} \prod_{i=1}^{k} (x - x_{i,k}^{(\ell)}), \quad \ell \in \mathbb{N}.
\]
In the following, we will derive the recurrence relations of \( \{ P_n^{(\ell)}(x; \zeta) \} \), calculate their derivatives with respect to \( x \) and \( \zeta \), respectively, and study the properties of zeros and coefficient matrices in the recurrence relations.

10
3.1 Recurrence relations

This section presents the recurrence relations for the orthogonal polynomials \( \{ P_\ell^k(x; \zeta) \} \), \( \ell \in \mathbb{N} \), the recurrence relations between \( \{ P_\ell^k(x; \zeta) \} \) and \( \{ P_{\ell-1}^k(x; \zeta) \} \), \( \ell \in \mathbb{N}^+ = \mathbb{N}/\{0\} \), and the specific forms of the coefficients in those recurrence relations.

Using the three-term recurrence relation and the existence theorem of zeros of general orthogonal polynomials in Theorems 3.1 and 3.2 of [38] gives the following conclusion.

**Theorem 3.1** For \( \ell \in \mathbb{N} \), a three-term recurrence relation for the orthogonal polynomials \( \{ P_\ell^k(x; \zeta) \} \) can be given by

\[
x P_\ell^k = a_{\ell-1}^k P_{\ell-1}^k + b_\ell^k P_\ell^k + a_\ell P_{\ell+1}^k,
\]

or in the matrix-vector form

\[
x P_\ell^k = J_\ell^k P_\ell^k + a_\ell^k P_{\ell+1}^k e_{\ell+1}, \quad P_\ell^k := (P_0^\ell, \ldots, P_\ell^\ell)^T,
\]

where both coefficients

\[
a_\ell^k := \left( x P_\ell^k, P_{\ell+1}^k \right)_{\omega^\ell}, \quad b_\ell^k := \left( x P_\ell^k, P_\ell^k \right)_{\omega^\ell} = \sum_{i=1}^{k+1} x_{i,k+1} - \sum_{i=1}^k x_{i,k},
\]

Fig. 3.1. The polynomials \( P_\ell^k(x, \zeta) \) given in (3.5).
are positive, \( e_{k+1} \) is the last column of the identity matrix of order \((k+1)\), and

\[
J_k^{(\ell)} := \begin{pmatrix}
    b_0^{(\ell)} & a_0^{(\ell)} & 0 \\
    a_0^{(\ell)} & b_1^{(\ell)} & a_1^{(\ell)} \\
    \vdots & \vdots & \ddots \\
    a_{k-2}^{(\ell)} & b_{k-1}^{(\ell)} & a_{k-1}^{(\ell)} \\
    0 & a_{k-1}^{(\ell)} & b_k^{(\ell)}
\end{pmatrix} \in \mathbb{R}^{(k+1)\times(k+1)},
\]

which is a symmetric positive definite tridiagonal matrix with the spectral radius larger than 1.

Besides those, the recurrence relations between \( \{P_k^{(\ell)}(x; \zeta)\} \) and \( \{P_k^{(\ell-1)}(x; \zeta)\} \) can also be obtained.

**Theorem 3.2** For \( \ell \in \mathbb{N}^+ \), it holds:

(i) The three-term recurrence relations between \( \{P_k^{(\ell)}(x; \zeta)\} \) and \( \{P_k^{(\ell-1)}(x; \zeta)\} \) can be given by

\[
\frac{2\ell - 1}{2\ell + 1}(x^2 - 1)P_k^{(\ell)} = p_k^{(\ell)}P_k^{(\ell-1)} + q_k^{(\ell)}P_k^{(\ell-1)} + r_k^{(\ell)}P_{k+1}^{(\ell-1)},
\]

or in the matrix-vector form

\[
\frac{2\ell - 1}{2\ell + 1}(x^2 - 1)P_k^{(\ell)} = J_k^{(\ell)}P_k^{(\ell-1)} + r_k^{(\ell)}P_{k+1}^{(\ell-1)} e_{k+1},
\]

where

\[
p_k^{(\ell)} := \frac{c_k^{(\ell-1)}}{c_k^{(\ell)}}, \quad r_k^{(\ell)} := \frac{2\ell - 1}{2\ell + 1} \frac{c_k^{(\ell-1)}}{c_k^{(\ell)}},
\]

\[
q_k^{(\ell)} := \frac{2\ell - 1}{2\ell + 1} \frac{c_k^{(\ell)}}{c_{k+1}^{(\ell-1)}} \left( \sum_{i=1}^{k+2} x_{i,k+2}^{(\ell-1)} - \sum_{i=1}^{k} x_{i,k}^{(\ell)} \right) = \frac{c_k^{(\ell-1)} c_{k+1}^{(\ell)}}{c_k^{(\ell)}} \sum_{i=1}^{k} (x_{i,k+1}^{(\ell-1)} - x_{i,k}^{(\ell-1)}),
\]

and

\[
J_k^{(\ell)} := \begin{pmatrix}
    p_0^{(\ell)} & q_0^{(\ell)} & r_1^{(\ell)} & 0 & 0 & \cdots & 0 \\
    0 & p_1^{(\ell)} & q_1^{(\ell)} & r_2^{(\ell)} & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & 0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix} \in \mathbb{R}^{(k+1)\times(k+2)}.
\]

(ii) The two-term recurrence relations between \( \{P_k^{(\ell-1)}(x; \zeta)\} \) and \( \{P_k^{(\ell)}(x; \zeta)\} \) can be
derived as follows
\[
\frac{2\ell - 1}{2\ell + 1} (x^2 - 1) P_k^{(\ell)} = \tilde{p}_k^{(\ell)} (x + \tilde{q}_k^{(\ell)}) P_{k+1}^{(\ell-1)} + \tilde{r}_k^{(\ell)} P_k^{(\ell-1)},
\]
(3.15)
\[
P_{k+1}^{(\ell-1)} = \frac{2\ell - 1}{2\ell + 1} \frac{1}{P_k^{(\ell)}} (x - \tilde{q}_k^{(\ell)}) P_k^{(\ell)} - \frac{a_k^{(\ell)}}{a_k^{(\ell-1)}} \tilde{r}_k^{(\ell)} P_{k-1}^{(\ell)},
\]
(3.16)
where
\[
\tilde{p}_k^{(\ell)} := \frac{2\ell - 1}{2\ell + 1} c_k^{(\ell)} \sum_{i=1}^{k+1} x_{i,k+1} - \sum_{i=1}^{k} x_{i,k}, \quad \tilde{q}_k^{(\ell)} := \sum_{i=1}^{k+1} x_{i,k+1} - \sum_{i=1}^{k} x_{i,k}, \quad \tilde{r}_k^{(\ell)} := p_k^{(\ell)} \left( 1 - \frac{2\ell + 1}{2\ell - 1} (\tilde{p}_k^{(\ell)})^2 \right).
\]
(3.17)

**Proof** (i) For \( i \leq k + 2 \), taking the inner product with respect to \( \omega^{(\ell-1)} \) between the polynomials \( P_i^{(\ell-1)}(x; \zeta) \) and \( (x^2 - 1) P_k^{(\ell)}(x; \zeta) \) gives
\[
\frac{2\ell - 1}{2\ell + 1} \left( (x^2 - 1) P_k^{(\ell)} , P_{k+1}^{(\ell-1)} \right)_{\omega^{(\ell-1)}} = \frac{2\ell - 1}{2\ell + 1} \left( c_k^{(\ell)} x^{k+2} , P_{k+2}^{(\ell-1)} \right)_{\omega^{(\ell-1)}} = r_{k+1},
\]
\[
\frac{2\ell - 1}{2\ell + 1} \sum_{i=1}^{k+1} x_{i,k+1} - \sum_{i=1}^{k} x_{i,k} = \left( x^{k+1} + \sum_{i=1}^{k+1} x_{i,k+1} - \sum_{i=1}^{k} x_{i,k} \right) = x_{i,k+1} x^{k+1} = q_{k+1},
\]
\[
= \sum_{i=1}^{k+1} x_{i,k+1} - \sum_{i=1}^{k} x_{i,k} = \sum_{i=1}^{k+1} \left( x_{i,k+1} - x_{i,k} \right), = \sum_{i=1}^{k+1} \left( x_{i,k+1} - x_{i,k} \right) = q_{i,k},
\]
\[
= \frac{2\ell - 1}{2\ell + 1} \left( (x^2 - 1) P_k^{(\ell)} , P_{k+1}^{(\ell-1)} \right)_{\omega^{(\ell-1)}} = \left( P_k^{(\ell)} , P_k^{(\ell)} \right)_{\omega^{(\ell-1)}} = \left( P_k^{(\ell)} , P_k^{(\ell-1)} \right)_{\omega^{(\ell-1)}} = 0, \quad i \leq k - 1,
\]
Substituting them into (3.4) gives (3.10).

(ii) Taking the inner product with respect to \( \omega^{(\ell)} \) between \( P_{k+1}^{(\ell-1)}(x; \zeta) \) and \( P_i^{(\ell)}(x; \zeta) \) with
i \leq k + 1

\left( P^{(\ell-1)}_{k+1}, P^{(\ell)}_{k+1} \right)_{\omega(t)} = \left( (x^{k+1}, P^{(\ell)}_{k+1}) \right)_{\omega(t)} = p_{k+1} \left( P^{(\ell)}_{k+1}, P^{(\ell)}_{k+1} \right)_{\omega(t)} = p^{(\ell)}_{k+1},
\left( P^{(\ell-1)}_{k+1}, P^{(\ell)}_{k} \right)_{\omega(t)} = \frac{2\ell - 1}{2\ell + 1} \left( p^{(\ell-1)}_{k+1}, (x^2 - 1)^{P^{(\ell)}_{k}} \right)_{\omega(t)} = q^{(\ell)}_{k},
\left( P^{(\ell-1)}_{k+1}, P^{(\ell)}_{k-1} \right)_{\omega(t)} = \frac{2\ell - 1}{2\ell + 1} \left( P^{(\ell-1)}_{k+1}, (x^2 - 1)^{P^{(\ell)}_{k-1}} \right)_{\omega(t)} = r^{(\ell)}_{k},
\left( P^{(\ell-1)}_{k+1}, P^{(\ell)}_{k} \right)_{\omega(t)} = \frac{2\ell - 1}{2\ell + 1} \left( P^{(\ell-1)}_{k+1}, (x^2 - 1)^{P^{(\ell)}_{k}} \right)_{\omega(t)} = 0, \quad i \leq k - 2.

Similarly, substituting them into (3.4) gives (3.11).

(iii) If using (3.7) to eliminate \( P^{(\ell-1)}_{k+2} \) and \( P^{(\ell-1)}_{k+1} \) in (3.10) and (3.11) respectively, then one yields

\[ \frac{2\ell - 1}{2\ell + 1} (x^2 - 1)^{P^{(\ell)}_{k}} = \tilde{p}^{(\ell)}_{k} (x + \tilde{q}^{(\ell)}_{k}) P^{(\ell-1)}_{k+1} + \tilde{r}^{(\ell)}_{k} P^{(\ell-1)}_{k}, \]

\[ P^{(\ell-1)}_{k+1} = \frac{2\ell - 1}{2\ell + 1} \tilde{p}^{(\ell)}_{k} (x - \tilde{\zeta}^{(\ell)}_{k}) P^{(\ell-1)}_{k} - \tilde{a}^{(\ell-1)}_{k+1} \tilde{\zeta}^{(\ell)}_{k} P^{(\ell)}_{k-1}, \]

with

\[ \tilde{p}^{(\ell)}_{k} = \frac{a^{(\ell-1)}_{k+1}}{a^{(\ell-1)}_{k+1}} = \frac{2\ell - 1 \tilde{c}^{(\ell-1)}_{k+1}}{2\ell + 1 \tilde{c}^{(\ell-1)}_{k+1}} = \frac{2\ell - 1 \tilde{a}^{(\ell-1)}_{k+1}}{2\ell + 1 \tilde{P}^{(\ell)}_{k+1}} = \tilde{p}_{k}, \]

\[ \tilde{q}^{(\ell)}_{k} = \frac{1}{\tilde{p}^{(\ell)}_{k}} \tilde{q}^{(\ell)}_{k} = b^{(\ell-1)}_{k+1} = \sum_{i=1}^{k+1} x^{(\ell-1)}_{i,k+1} - \sum_{i=1}^{k} x^{(\ell)}_{i,k} = b^{(\ell)}_{k} - \frac{2\ell + 1}{2\ell - 1} \tilde{p}^{(\ell)}_{k} \tilde{q}^{(\ell)}_{k} = \tilde{q}^{(\ell)}_{k}, \]

\[ \tilde{r}^{(\ell)}_{k} = P^{(\ell)}_{k} - \tilde{p}^{(\ell)}_{k} \tilde{a}^{(\ell-1)}_{k+1} = P^{(\ell)}_{k} \left( 1 - \frac{2\ell + 1}{2\ell - 1} \tilde{p}^{(\ell)}_{k} \tilde{q}^{(\ell)}_{k} \tilde{r}^{(\ell)}_{k} \right) = \frac{a^{(\ell-1)}_{k+1}}{a^{(\ell-1)}_{k+1}} \left( -\tilde{r}^{(\ell)}_{k} + \frac{2\ell - 1}{2\ell + 1} \tilde{P}^{(\ell)}_{k} \tilde{a}^{(\ell-1)}_{k+1} \right) = \tilde{r}^{(\ell)}_{k}. \]

The proof is completed.

3.2 Partial derivatives

This section calculates the derivatives of the polynomial \( P^{(\ell)}_{k}(x; \zeta) \) with respect to \( x \) and \( \zeta, \ell \in \mathbb{N}. \)

**Theorem 3.3** For \( \ell \in \mathbb{N} \), the first-order derivative of the polynomial \( P^{(\ell)}_{k+1}(x; \zeta) \) with respect to the parameter \( \zeta \) satisfies

\[ \frac{\partial P^{(\ell)}_{k+1}}{\partial \zeta} = a^{(\ell)}_{k} P^{(\ell)}_{k} - \frac{1}{2} \left( G(\zeta) - \zeta^{-1} - b^{(\ell)}_{k+1} \right) P^{(\ell)}_{k+1}. \tag{3.18} \]

**Proof** With the aid of definition (2.11) and recurrence relation (2.12) of the second kind
modified Bessel functions, one has

\[
\frac{\partial}{\partial \zeta} \omega^{(\ell)}(x; \zeta) = \frac{K_2(\zeta) + K_1(\zeta) + 2\zeta^{-1}K_2(\zeta) - 2xK_2(\zeta)}{2K_2(\zeta)} \left( \frac{\zeta}{(2\ell + 1)K_2(\zeta)} (x^2 - 1)^{\ell - \frac{1}{2}} \exp(-\zeta x) \right) = \left(G(\zeta) - \zeta^{-1} - x\right) \omega^{(\ell)}(x; \zeta).
\]

Taking the partial derivative of both sides of identities

\[
\left( P_{k+1}^{(\ell)}, P_{i}^{(\ell)} \right)_{\omega^{(\ell)}} = \delta_{k+1,i}, \ i = 0, \ldots, k + 1,
\]

with respect to \( \zeta \) and using (3.9) gives

\[
\frac{\partial}{\partial \zeta} \left( P_{k+1}^{(\ell)}, P_{k+1}^{(\ell)} \right)_{\omega^{(\ell)}} = 2 \left( \frac{\partial}{\partial \zeta} P_{k+1}^{(\ell)}, P_{k+1}^{(\ell)} \right)_{\omega^{(\ell)}} + \left( G(\zeta) - \zeta^{-1} \right) \left( P_{k+1}^{(\ell)}, P_{k+1}^{(\ell)} \right)_{\omega^{(\ell)}} - \left( x P_{k+1}^{(\ell)}, P_{k+1}^{(\ell)} \right)_{\omega^{(\ell)}}
\]

\[
= 2 \left( \frac{\partial}{\partial \zeta} P_{k+1}^{(\ell)}, P_{k+1}^{(\ell)} \right)_{\omega^{(\ell)}} + \left( G(\zeta) - \zeta^{-1} - b_{k+1}^{(\ell)} \right) = 0,
\]

\[
\frac{\partial}{\partial \zeta} \left( P_{k+1}^{(\ell)}, P_{k}^{(\ell)} \right)_{\omega^{(\ell)}} = \left( \frac{\partial}{\partial \zeta} P_{k+1}^{(\ell)}, P_{k}^{(\ell)} \right)_{\omega^{(\ell)}} + \left( P_{k+1}^{(\ell)}, \frac{\partial}{\partial \zeta} P_{k}^{(\ell)} \right)_{\omega^{(\ell)}} + \left( G(\zeta) - \zeta^{-1} \right) \left( P_{k+1}^{(\ell)}, P_{k}^{(\ell)} \right)_{\omega^{(\ell)}} - \left( x P_{k+1}^{(\ell)}, P_{k}^{(\ell)} \right)_{\omega^{(\ell)}}
\]

\[
= \left( \frac{\partial}{\partial \zeta} P_{k+1}^{(\ell)}, P_{k}^{(\ell)} \right)_{\omega^{(\ell)}} = 0, \ i \leq k - 1.
\]

Thus one has

\[
\left( \frac{\partial}{\partial \zeta} P_{k+1}^{(\ell)}, P_{k+1}^{(\ell)} \right)_{\omega^{(\ell)}} = -\frac{1}{2} \left( G(\zeta) - \zeta^{-1} - b_{k+1}^{(\ell)} \right),
\]

\[
\left( \frac{\partial}{\partial \zeta} P_{k+1}^{(\ell)}, P_{k}^{(\ell)} \right)_{\omega^{(\ell)}} = a_{k}^{(\ell)}, \ \left( \frac{\partial}{\partial \zeta} P_{k+1}^{(\ell)}, P_{i}^{(\ell)} \right)_{\omega^{(\ell)}} = 0, \ i \leq k - 1.
\]

Because \( \frac{\partial P_{k+1}^{(\ell)}(x; \zeta)}{\partial \zeta} \) is a polynomial and its degree is not larger than \( k + 1 \), using (3.4) gives (3.18). The proof is completed.

**Theorem 3.4** For \( \ell \in \mathbb{N}^+ \), the first-order derivatives of the polynomials \( \{P_{k}^{(\ell)}(x; \zeta)\} \) with respect to the independent variable \( x \) satisfy

\[
\frac{\partial P_{k+1}^{(\ell-1)}(x)}{\partial x} = \frac{2\ell - 1}{2\ell + 1} \frac{k + 1}{P_{k}^{(\ell)}} P_{k}^{(\ell)} + \zeta r^{(\ell)}_{k} P_{k-1}^{(\ell)}, \quad (3.19)
\]
\[
\frac{2\ell - 1}{2\ell + 1} \left( x^2 - 1 \right) \frac{\partial P_k^{(t)}}{\partial x} + (2\ell - 1) x P_k^{(t)} = (k + 2\ell + 1) P_k^{(t)} P_{k+1}^{(t-1)} + \zeta P_k^{(t)} P_{k}^{(t-1)}. \tag{3.20}
\]

**Proof** Similar to the proof of Theorem 3.3, one has

\[
\frac{\partial}{\partial x} \omega^{(t)}(x; \zeta) = (2\ell - 1) x \omega^{(t-1)}(x; \zeta) - \zeta \omega^{(t)}(x; \zeta).
\]

Because the degrees of polynomials \( \frac{\partial P_{k}^{(t-1)}}{\partial x} \) and \( \frac{2\ell - 1}{2\ell + 1} \left( x^2 - 1 \right) \frac{\partial P_k^{(t)}}{\partial x} + (2\ell - 1) x P_k^{(t)} \) are not larger than \( n \) and \( k + 1 \), respectively, and

\[
limit_{x \to +\infty} P_i^{(t-1)}(x; \zeta) P_j^{(t)}(x; \zeta) \omega^{(t)}(x; \zeta) = 0, \quad \limit_{x \to -\infty} P_i^{(t-1)}(x; \zeta) P_j^{(t)}(x; \zeta) \omega^{(t)}(x; \zeta) = 0, \quad \forall i, j \in \mathbb{K},
\]

one can calculate the expansion coefficients in (3.4) as follows

\[
\left( \frac{\partial}{\partial x} P_{k+1}^{(t-1)}, P_k^{(t)} \right)_{\omega^{(t)}} = \left( (k + 1)c_{k+1}^{(t-1)} x, P_k^{(t)} \right)_{\omega^{(t)}} = \frac{2\ell - 1}{2\ell + 1} \frac{1}{\tilde{P}_k^{(t)}} \left( P_k^{(t)}, P_k^{(t)} \right)_{\omega^{(t)}} = \frac{2\ell - 1}{2\ell + 1} \frac{1}{\tilde{P}_k^{(t)}},
\]

\[
\left( \frac{\partial}{\partial x} P_{k+1}^{(t-1)}, P_k^{(t)} \right)_{\omega^{(t)}} = \int_{1}^{+\infty} \frac{\partial}{\partial x} \left( P_{k+1}^{(t-1)} P_k^{(t-1)} \right)_{\omega^{(t)}} dx - \frac{2\ell - 1}{2\ell + 1} \left( P_{k+1}^{(t-1)}, x^2 - 1 \right) \frac{\partial}{\partial x} P_{k-1}^{(t)}_{\omega^{(t-1)}} - (2\ell - 1) \left( P_{k+1}^{(t-1)}, x P_k^{(t)} \right)_{\omega^{(t-1)}},
\]

\[
\left( \frac{\partial}{\partial x} P_{k+1}^{(t-1)}, P_i^{(t)} \right)_{\omega^{(t)}} = \int_{1}^{+\infty} \frac{\partial}{\partial x} \left( P_{k+1}^{(t-1)} P_i^{(t)} \right)_{\omega^{(t)}} dx - \frac{2\ell - 1}{2\ell + 1} \left( P_{k+1}^{(t-1)}, x^2 - 1 \right) \frac{\partial}{\partial x} P_{i}^{(t)}_{\omega^{(t-1)}} - (2\ell - 1) \left( P_{k+1}^{(t-1)}, x P_i^{(t)} \right)_{\omega^{(t-1)}},
\]

and

\[
\left( \frac{2\ell - 1}{2\ell + 1} \left( x^2 - 1 \right) \frac{\partial P_k^{(t)}}{\partial x} + (2\ell - 1) x P_k^{(t)} \right)_{\omega^{(t-1)}} = \frac{2\ell - 1}{2\ell + 1} \left( (k + 2\ell + 1) c_k^{(t)} x, P_{k+1}^{(t-1)} \right)_{\omega^{(t-1)}},
\]

\[
\left( \frac{2\ell - 1}{2\ell + 1} \left( x^2 - 1 \right) \frac{\partial P_k^{(t)}}{\partial x} + (2\ell - 1) x P_k^{(t)} \right)_{\omega^{(t-1)}} = \frac{2\ell - 1}{2\ell + 1} \left( (k + 2\ell + 1) c_k^{(t)} x, P_{k+1}^{(t-1)} \right)_{\omega^{(t-1)}},
\]

\[
\left( \frac{\partial}{\partial x} P_k^{(t-1)}, P_{k+1}^{(t-1)} \right)_{\omega^{(t-1)}} = \left( \frac{\partial}{\partial x} P_k^{(t-1)}, P_{k+1}^{(t-1)} \right)_{\omega^{(t-1)}} = \left( \frac{\partial}{\partial x} P_k^{(t-1)}, P_{k+1}^{(t-1)} \right)_{\omega^{(t-1)}} = \left( \frac{\partial}{\partial x} P_k^{(t-1)}, P_{k+1}^{(t-1)} \right)_{\omega^{(t-1)}} = \left( \frac{\partial}{\partial x} P_k^{(t-1)}, P_{k+1}^{(t-1)} \right)_{\omega^{(t-1)}}.
\]

The proof is completed.\]
3.3 Zeros

Using the separation theorem of zeros of general orthogonal polynomials [38] gives the following conclusion on the orthogonal polynomials \( \{P_k^{(\ell)}(x; \zeta)\} \).

**Theorem 3.5** For \( \ell \in \mathbb{N} \), the zeros \( \{x_{i,k}^{(\ell)}\}_{i=1}^{k} \) of \( P_k^{(\ell)}(x; \zeta) \) and \( \{x_{i,k+1}^{(\ell)}\}_{i=1}^{k+1} \) of \( P_{k+1}^{(\ell)}(x; \zeta) \) satisfy the separation property

\[
1 < x_{1,k+1}^{(\ell)} < x_{1,k}^{(\ell)} < x_{2,k+1}^{(\ell)} < \cdots < x_{k,k}^{(\ell)} < x_{k+1,k+1}^{(\ell)}.
\]

There is still another important separation property for the zeros of the orthogonal polynomials \( \{P_k^{(\ell)}(x; \zeta), \ell \in \mathbb{N}\} \).

**Theorem 3.6** The \( k \) zeros \( \{x_{i,k}^{(\ell-1)}\}_{i=1}^{k} \) of \( P_k^{(\ell)} \) and \( (k + 1) \) zeros of \( \{x_{i,k+1}^{(\ell-1)}\}_{i=1}^{k+1} \) of \( P_{k+1}^{(\ell-1)} \) satisfy

\[
1 < x_{1,k+1}^{(\ell-1)} < x_{1,k}^{(\ell-1)} < x_{2,k+1}^{(\ell-1)} < \cdots < x_{k,k}^{(\ell-1)} < x_{k+1,k+1}^{(\ell-1)}.
\]

**Proof** Substituting \( \{x_{i,k+1}^{(\ell-1)}\}_{i=1}^{k+1} \) into (3.15) gives

\[
\frac{2\ell - 1}{2\ell + 1} \left( (x_{i,k+1}^{(\ell-1)})^2 - 1 \right) P_k^{(\ell)}(x_{i,k+1}^{(\ell-1)}; \zeta) = r_k^{(\ell)} P_k^{(\ell-1)}(x_{i,k+1}^{(\ell-1)}; \zeta).
\]

which implies that \( r_k^{(\ell)} \neq 0 \). In fact, if assuming \( r_k^{(\ell)} = 0 \), then the above identity and the fact that \( (x_{i,k+1}^{(\ell-1)})^2 - 1 > 0 \) imply \( P_k^{(\ell)}(x_{i,k+1}^{(\ell-1)}; \zeta) = 0 \), which contradicts with \( P_k^{(\ell)} \) being a polynomial of degree \( k \).

Using Theorem 3.5 gives

\[
\text{sign} \left( P_k^{(\ell)}(x_{i,k+1}^{(\ell-1)}; \zeta) P_k^{(\ell)}(x_{i,k+1,k+1}^{(\ell-1)}; \zeta) \right) = \text{sign} \left( P_k^{(\ell-1)}(x_{i,k+1}^{(\ell-1)}; \zeta) P_k^{(\ell-1)}(x_{i,k+1,k+1}^{(\ell-1)}; \zeta) \right) < 0.
\]

Thus the number of zero of the polynomial \( P_k^{(\ell)} \) is not less than one in each subinterval \( (x_{i,k+1}^{(\ell-1)}, x_{i+1,k+1}^{(\ell-1)}) \). The proof is completed.

According to Theorems 3.5 and 3.6, one can know the sign of the coefficients of the recurrence relations in Theorem 3.2.

**Corollary 3.1** All quantities \( p_k, q_k, r_k \) in (3.14) and \( \bar{p}_k, \bar{q}_k, \bar{r}_k \) in (3.17) are positive.

**Proof** It is obvious that

\[
p_k^{(\ell)} = \frac{c_k^{(\ell)-1}}{c_k^{(\ell)}} > 0, \quad r_k^{(\ell)} = \frac{2\ell - 1}{2\ell + 1} \frac{c_k^{(\ell)-1}}{c_k^{(\ell)-1}} > 0, \quad \bar{p}_k^{(\ell)} = \frac{2\ell - 1}{2\ell + 1} \frac{c_k^{(\ell)-1}}{c_k^{(\ell)-1}} > 0.
\]
Comparing the coefficients of the term of order \( n \) gives

Using Corollary 3.1, \( \tilde{q}_k^{(\ell)} \) gives

Taking partial derivative of \( \tilde{q}_k^{(\ell)} \) gives

Proof

Using Theorems 3.1 and 3.6 gives

Comparing the coefficients of the term of order \( n \) at two sides of (3.15) gives

where \( x_{0,k} = 0 \). The proof is completed.

Using Theorem 3.6 gives \( \tilde{r}_k^{(\ell)} > 0 \). The proof is completed.

Using Corollary 3.1, \( \tilde{r}_k^{(\ell)} = p_k^{(\ell)} (1 - \frac{2r+1}{2r+2} (\bar{q}_k^{(\ell)})^2) \), and \( \bar{p}_k^{(\ell)} = \frac{2r+1}{2r+2} (c_{k+1}^{(\ell)})^{-1} c_k^{(\ell)} \) can give the following corollary.

**Corollary 3.2** The leading coefficient of \( P_{k+1}^{(\ell)} \) is \( \sqrt{\frac{2r+1}{2r+2}} \) times larger than that of \( P_k^{(\ell)} \), i.e. \( c_{k+1}^{(\ell)} > \sqrt{\frac{2r+1}{2r+2}} c_k^{(\ell)} \).

According to Theorems 3.3 and 3.5, one has the following further conclusion.

**Corollary 3.3** The zeros \( \{x_{i,k}^{(\ell)}\}_{i=1}^{k} \) of \( P_k^{(\ell)} \) strictly decrease with respect to \( \zeta \), i.e.

\[
\frac{\partial x_{i,k}^{(\ell)}}{\partial \zeta} < 0.
\]

**Proof** Taking partial derivative of \( P_k^{(\ell)}(x_{i,k}^{(\ell)}; \zeta) \) with respect to \( \zeta \) and using Theorem 3.3 gives

\[
\frac{\partial x_{i,k}^{(\ell)}}{\partial \zeta} = - \left( \frac{\partial P_k^{(\ell)}(x_{i,k}^{(\ell)}; \zeta)}{\partial x} \right)^{-1} \left( \frac{\partial P_k^{(\ell)}(x_{i,k}^{(\ell)}; \zeta)}{\partial \zeta} \right) = -a_{k-1}^{(\ell)} \left( \frac{\partial P_k^{(\ell)}(x_{i,k}^{(\ell)}; \zeta)}{\partial x} \right)^{-1} P_{k-1}^{(\ell)}(x_{i,k}^{(\ell)}; \zeta).
\]
Due to Theorem 3.5, one has
\[
\text{sign}(P_{k-1}^{(\ell)}(x_{i,k}^{(\ell)}, \zeta)) = (-1)^{k+i} = \text{sign} \left( \frac{\partial P_k^{(\ell)}}{\partial x}(x_{i,k}^{(\ell)}, \zeta) \right),
\]
Combining them completes the proof.

4 Some properties of real spherical harmonics

The real spherical harmonics are known as tesseral spherical harmonics [41]. The harmonics with \( m > 0 \) are said to be of cosine type, and those with \( m < 0 \) are of sine type. The reason for that can be seen by writing the spherical harmonic functions in terms of the associated Legendre polynomials
\[
Y_{\ell,m}(y, \phi) = \begin{cases} 
\sqrt{2} \hat{P}_m^{\ell}(y) \sin(|m|\phi), & m < 0, \\
\hat{P}_m^{\ell}(y), & m = 0, \\
\sqrt{2} \hat{P}_m^{\ell}(y) \cos(m\phi), & m > 0,
\end{cases}
\]
where
\[
\hat{P}_m^{\ell}(y) = \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} \hat{P}_m^{\ell}(y), \quad |m| \leq \ell, \quad \ell \in \mathbb{N},
\]
and \( \{\hat{P}_m^{\ell}(y), |m| \leq \ell, \ell \in \mathbb{N}\} \) are the associated Legendre polynomials defined by
\[
\hat{P}_m^{\ell}(y) = (-1)^m (1 - y^2)^{\frac{m}{2}} \frac{1}{2^{\ell+1} \ell!} \frac{d^{\ell+m}}{dy^{\ell+m}}(y^2 - 1)^{\ell}. 
\]
Moreover, it holds that \( \hat{P}_m^{\ell}(y) = 0 \) for \( |m| > \ell \); \( \hat{P}_m^{-m}(y) = (-1)^m \hat{P}_m^{\ell}(y) \), and \( \hat{P}_m^{m}(y) = 0 \).
Thus the following orthogonality properties hold
\[
\int_{-1}^{1} \int_{0}^{2\pi} Y_{\ell,m} Y_{\ell',m'} d\phi dy = \frac{4\pi}{2\ell + 1} \delta_{\ell,\ell'} \delta_{m,m'}.
\]

4.1 Recurrence relations and Partial derivatives

**Lemma 4.1 ([1])** The associated Legendre polynomials \( \{\hat{P}_m^{\ell}(y)\} \) have the following properties:
\[
y\hat{P}_m^{\ell} = \frac{1}{2\ell + 1} \left( (\ell - m + 1)\hat{P}_{\ell+1}^m + (\ell + m)\hat{P}_{\ell-1}^m \right),
\]
\[
y(1 - y^2)^{-\frac{1}{2}} \hat{P}_m^{\ell} = -\frac{1}{2m} \left( \hat{P}_{\ell+1}^{m+1} + (\ell + m)(\ell - m + 1)\hat{P}_{\ell-1}^{m-1} \right),
\]
Theorem 4.1  (1) The real spherical harmonics satisfy the recurrence relations

\[(1 - y^2)^{-\frac{1}{2}} \hat{P}_\ell^m = -\frac{1}{2\ell + 1} \left( \hat{P}_{\ell-1}^{m+1} + (\ell + m - 1)(\ell + m)\hat{P}_{\ell-1}^{m-1} \right), \quad (4.6)\]

\[(1 - y^2)^{\frac{1}{2}} \hat{P}_\ell^m = \frac{1}{2\ell + 1} \left( (\ell - m + 1)(\ell - m + 2)\hat{P}_{\ell+1}^{m-1} - (\ell + m - 1)(\ell + m)\hat{P}_{\ell-1}^{m-1} \right), \quad (4.7)\]

\[(1 - y^2)^{\frac{1}{2}} \hat{P}_\ell^{m+1} = \frac{1}{2\ell + 1} \left( -\hat{P}_{\ell+1}^{m+1} + \hat{P}_{\ell-1}^{m+1} \right), \quad (4.8)\]

\[(1 - y^2)^{\frac{1}{2}} \hat{P}_\ell^{m-1} = (\ell - m + 1)\hat{P}_{\ell+1}^{m-1} - (\ell + m + 1)y\hat{P}_\ell^m, \quad (4.9)\]

\[(y^2 - 1) \frac{d}{dy} \hat{P}_\ell^m = \ell y\hat{P}_\ell^m - (\ell + m)\hat{P}_{\ell-1}^m, \quad (4.10)\]

\[(y^2 - 1) \frac{d}{dy} \hat{P}_\ell^{m+1} = \sqrt{1 - y^2}\hat{P}_{\ell+1}^{m+1} + my\hat{P}_\ell^m. \quad (4.11)\]

Using Lemma 4.1 can give the following recurrence relations and partial derivatives of the real spherical harmonics.

**Theorem 4.1**  (1) The real spherical harmonics satisfy the recurrence relations

\[I_0 Y_{\ell,m} = \frac{1}{2\ell + 1} \left( h_{\ell+1,m}Y_{\ell+1,m} + h_{\ell,m}Y_{\ell-1,m} \right), \quad (4.12)\]

\[I_1 Y_{\ell,m} = \frac{1}{2(2\ell + 1)} \left( \hat{s}_m(\hat{h}_{\ell+1,-m}Y_{\ell+1,m} - \hat{h}_{\ell-1,-m}Y_{\ell-1,m}) \right. \]

\[- \hat{s}_m(\hat{h}_{\ell+1,m}Y_{\ell+1,m+1} - \hat{h}_{\ell-1,-m}Y_{\ell-1,m+1}) \left. \right), \quad (4.13)\]

\[I_2 Y_{\ell,m} = \frac{1}{2(2\ell + 1)} \left( s_m(\hat{h}_{\ell-1,-m}Y_{\ell-1,m-1} - \hat{h}_{\ell+1,m+1}Y_{\ell+1,m-1}) \right. \]

\[+ \hat{s}_m(\hat{h}_{\ell-1,-m}Y_{\ell-1,m-1} - \hat{h}_{\ell+1,-m+1}Y_{\ell+1,m-1+1}) \right), \quad (4.14)\]

where \(I_0(y, \phi) := y, I_1(y, \phi) := \sqrt{1 - y^2}\cos \phi, I_2(y, \phi) := \sqrt{1 - y^2}\sin \phi, h_{\ell,m} := \sqrt{(\ell + m)(\ell - m)}, \hat{h}_{\ell-1,m} := \sqrt{(\ell + m - 1)(\ell + m)}, \)

\[s_m := \text{sign}(m)\sqrt{\delta_m - 1 + \delta_{m,0}} + 1, \quad \hat{s}_m := \text{sign}(m)(1 - \delta_m)(1 - \delta_{m,1}), \]

\[\tilde{s}_m := \text{sign}(m)(1 - \delta_{m,-1})\sqrt{\delta_m + 1}, \quad \hat{s}_m := \text{sign}(m)(1 - \delta_{m,0})\sqrt{\delta_{m,1} + 1}, \]

with

\[\text{sign}(m) = \begin{cases} 1, & m \geq 0, \\ -1, & m < 0. \end{cases}\]

(2) The partial derivatives of real spherical harmonics satisfy

\[I_0 \frac{d}{dy} Y_{\ell,m} - \ell I_0 Y_{\ell,m} = -h_{\ell,m}Y_{\ell-1,m}, \quad (4.15)\]

\[I_0 \frac{d}{dy} Y_{\ell,m} + (\ell + 1)I_0 Y_{\ell,m} = h_{\ell+1,m}Y_{\ell+1,m}, \quad (4.16)\]
\[ I_1 \frac{d}{dy} Y_{\ell,m} - \ell I_1 Y_{\ell,m} - m \hat{I}_1 Y_{\ell,m} = \frac{1}{2} (\hat{s}_m \hat{h}_{\ell-1,m} Y_{\ell-1,m-1} - \hat{s}_m \hat{h}_{\ell-1,-m} Y_{\ell-1,m+1}), \]  
(4.17)

\[ I_2 \frac{d}{dy} Y_{\ell,m} - \ell I_2 Y_{\ell,m} - m \hat{I}_2 Y_{\ell,m} = -\frac{1}{2} (\hat{s}_m \hat{h}_{\ell-1,m} Y_{\ell-1,m+1} + s_m \hat{h}_{\ell-1,-m} Y_{\ell-1,m-1}), \]  
(4.18)

\[ I_1 \frac{d}{dy} Y_{\ell,m} + (\ell + 1)I_1 Y_{\ell,m} - m \hat{I}_1 Y_{\ell,m} = \frac{1}{2} (\hat{s}_m \hat{h}_{\ell+1,m} Y_{\ell+1,m-1} - \hat{s}_m \hat{h}_{\ell+1,-m} Y_{\ell+1,m+1}), \]  
(4.19)

\[ I_2 \frac{d}{dy} Y_{\ell,m} + (\ell + 1)I_2 Y_{\ell,m} - m \hat{I}_2 Y_{\ell,m} = -\frac{1}{2} (\hat{s}_m \hat{h}_{\ell+1,m} Y_{\ell+1,m+1} + s_m \hat{h}_{\ell+1,-m} Y_{\ell+1,m-1}), \]  
(4.20)

\[ I_1 \frac{d}{dy} Y_{\ell,m} - m \hat{I}_3 Y_{\ell,-m} = \frac{1}{2} (\hat{s}_m \hat{h}_{\ell,m} Y_{\ell,m-1} - \hat{s}_m \hat{h}_{\ell,-m} Y_{\ell,m+1}), \]  
(4.21)

\[ I_2 \frac{d}{dy} Y_{\ell,m} - m \hat{I}_4 Y_{\ell,-m} = -\frac{1}{2} (\hat{s}_m \hat{h}_{\ell,m} Y_{\ell,m+1} + s_m \hat{h}_{\ell,-m} Y_{\ell,m-1}). \]  
(4.22)

Where

\[ \hat{I}_0(y, \phi) := y^2 - 1, \quad \hat{I}_1(y, \phi) := y \sqrt{1 - y^2} \cos \phi, \]
\[ \hat{I}_2(y, \phi) := y \sqrt{1 - y^2} \sin \phi, \quad \hat{I}_3(y, \phi) := (1 - y^2)^{-\frac{1}{2}} \sin \phi, \]
\[ \hat{I}_4(y, \phi) := -(1 - y^2)^{-\frac{1}{2}} \cos \phi, \quad \hat{h}_{\ell,m} := \sqrt{(\ell + m)(\ell - m + 1)}. \]

**Proof** Only the case of \( m > 1 \) is taken as an example because other cases can be obtained similarly.

(1) Thanks to (4.4), the identity (4.12) is obvious. Due to (4.7) and (4.8), one has

\[ I_1 \hat{P}^m_{\ell} \cos((m+1)\phi) = \frac{1}{2} \sqrt{1 - y^2} \hat{P}^m_{\ell} \cos((m+1)\phi) + \frac{1}{2} \sqrt{1 - y^2} \hat{P}^m_{\ell} \cos((m-1)\phi) \]
\[ = \frac{1}{2\ell + 1} (-\hat{P}^m_{\ell+1} + \hat{P}^m_{\ell-1}) \cos((m+1)\phi) \]
\[ + \frac{1}{2\ell + 1} ((\ell - m + 1)(\ell - m + 2) \hat{P}^m_{\ell+1} - (\ell + m - 1)(\ell + m) \hat{P}^m_{\ell-1}) \cos((m-1)\phi). \]

Thus (4.13) and (4.14) can be gotten.

(2) Due to (4.10), one has

\[ \hat{I}_0 \frac{d}{dy} \hat{P}^m_{\ell} - \ell \hat{I}_0 \hat{P}^m_{\ell} = -(\ell + m) \hat{P}^m_{\ell-1}, \]

which deduces (4.15). Combining it with (4.12) gives (4.16).
(3) Due to (4.5), (4.9) and (4.10), one has
\[
\left( \hat{I}_1 \frac{d}{dy} \hat{\rho}^m - \ell I_1 \hat{\rho}^m \right) \cos(m \phi) - m \hat{I}_1 \hat{\rho}^m \sin(m \phi)
\]
\[
= \sqrt{1 - y^2} \left( \ell y^2 \hat{\rho}^m - (\ell + m)y \hat{\rho}^m - \ell (y^2 - 1) \hat{\rho}^m \right) \cos \phi \cos(m \phi) - m (1 - y^2)^{-\frac{1}{2}} \hat{\rho}^m \sin \phi \sin(m \phi)
\]
\[
= - (1 - y^2)^{-\frac{1}{2}} \left( \left( \ell y^2 \hat{\rho}^m - (\ell + m)y \hat{\rho}^m - \ell (y^2 - 1) \hat{\rho}^m \right) \cos \phi \cos(m \phi) + m \hat{\rho}^m \sin \phi \sin(m \phi) \right)
\]
\[
= - m (1 - y^2)^{-\frac{1}{2}} \hat{\rho}^m \cos((m - 1) \phi) - \hat{\rho}^m \hat{\rho}^m \cos(m \phi)
\]
\[
= \frac{1}{2} \left( \hat{\rho}^m + (\ell + m - 1)(\ell + m) \hat{\rho}^m \right) \cos((m - 1) \phi) - \frac{1}{2} \hat{\rho}^m \hat{\rho}^m \cos((m + 1) \phi).
\]
Thus (4.17) and (4.18) can be gotten. Combining them with (4.13) and (4.14) obtains (4.19) and (4.20).

(4) Due to (4.5) and (4.11), one has
\[
I_1 \frac{d}{dy} \hat{\rho}^m \cos(m \phi) - m \hat{I}_3 \hat{\rho}^m \sin(m \phi)
\]
\[
= \sqrt{1 - y^2} \left( \sqrt{1 - y^2} \hat{\rho}^m + my \hat{\rho}^m \right) \cos \phi \cos(m \phi) - my (1 - y^2)^{-\frac{1}{2}} \hat{\rho}^m \sin \phi \sin(m \phi)
\]
\[
= - my (1 - y^2)^{-\frac{1}{2}} \hat{\rho}^m \cos((m - 1) \phi) - \hat{\rho}^m \hat{\rho}^m \cos(m \phi)
\]
\[
= \frac{1}{2} \left( \hat{\rho}^m + (\ell + m)(\ell + m - 1) \hat{\rho}^m \right) \cos((m - 1) \phi) - \frac{1}{2} \hat{\rho}^m \hat{\rho}^m \cos((m + 1) \phi)
\]
\[
= \frac{1}{2} (\ell + m - 1)(\ell + m) \hat{\rho}^m \cos((m - 1) \phi) - \frac{1}{2} \hat{\rho}^m \hat{\rho}^m \cos((m + 1) \phi).
\]
Thus (4.21) and (4.22) are gotten. The proof is completed.

5 Moment method by operator projection

This section begins to extend the moment method by operator projection [18,36] to the special relativistic Boltzmann equation (2.1) and derive its arbitrary order hyperbolic moment model. For the sake of convenience, units in which both the speed of light c and rest mass m of particle are equal to one will be used hereafter.

Similar to [3], the momentum \( p^\alpha \) at a point is decomposed as
\[
p^\alpha = U^\alpha E + \sqrt{E^2 - 1} l^\alpha, \tag{5.1}
\]
where \( l^\alpha \) is an unit spacelike vector orthogonal to \( U^\alpha \), i.e.
\[
l^\alpha l_\alpha = -1, \quad l^\alpha U_\alpha = 0.
\]
We introduce an orthogonal tetrad $n^\alpha_i$ ($i = 1, 2, 3$) orthogonal to $U^\alpha$ so that

$$U_\alpha n^\alpha_i = 0, \quad g_{\alpha\beta} n^\alpha_i n^\beta_j = -\delta_{i,j}.$$  

Using the Lorentz transformation [17] to the local rest frame where $(U^\alpha) = (1, 0, 0, 0)$, and taking $n^\alpha_i = \delta_{i,\alpha}$ ($i = 1, 2, 3$) can obtain

$$n^0_i = U^i, \quad n^\hat{i}_i = ((U^0)^2 - 1)^{-\frac{1}{2}} U^i U^j (U^0 - 1) + \delta_{i,j}, \quad i, j = 1, 2, 3. \quad (5.2)$$

Thus $l^\alpha$ can be expressed as

$$l^\alpha = \sin \xi \cos \phi n^\alpha_1 + \sin \xi \sin \phi n^\alpha_2 + \cos \xi n^\alpha_3,$$

where

$$(\sin \xi \cos \phi, \sin \xi \sin \phi, \cos \xi) := -(E^2 - 1)^{-\frac{1}{2}} (n^\alpha_1 p_\alpha, n^\alpha_2 p_\alpha, n^\alpha_3 p_\alpha),$$

and $\xi \in [0, \pi]$, $\phi \in [0, 2\pi)$. It is easy to prove that it is well-defined, that is to say, the above vector is an unit vector. If denoting $y := \cos \xi$, then one has

$$(\sqrt{1 - y^2} \cos \phi, \sqrt{1 - y^2} \sin \phi, y) := -(E^2 - 1)^{-\frac{1}{2}} (n^\alpha_1 p_\alpha, n^\alpha_2 p_\alpha, n^\alpha_3 p_\alpha). \quad (5.3)$$

5.1 Weighted polynomial space

In order to use the moment method by the operator projection to derive the hyperbolic moment model of the kinetic equation, one should define weighted polynomial spaces and norms as well as the projection operator. Thanks to the equilibrium distribution $f^{(0)}$ in (2.9), the weight function is chosen as $g^{(0)}$, which will be replaced with the new notation $g^{(0)}_{[u,\theta]}$, considering the dependence of $g^{(0)}$ on the macroscopic fluid velocity $u$ and $\theta = k_B T/mc^2 = \zeta^{-1}$, that is

$$g^{(0)}_{[u,\theta]} = \frac{\zeta}{4\pi K_2(\zeta)} \exp \left( -\frac{E}{\theta} \right), \quad E = U_\alpha p^\alpha. \quad (5.4)$$

Associated with the weight function $g^{(0)}_{[u,\theta]}$, our weighted polynomial space is defined by

$$\mathbb{H}^{(0)}_{[u,\theta]} := \text{span} \left\{ p^{\mu_1} p^{\mu_2} \cdots p^{\mu_\ell} g^{(0)}_{[u,\theta]} : \mu_i = 0, 1, 2, 3, \quad \ell \in \mathbb{N} \right\},$$

which is an infinite-dimensional linear space equipped with the inner product

$$\langle f, g \rangle_{g^{(0)}_{[u,\theta]}} := \int_{\mathbb{R}^3} \frac{1}{g^{(0)}_{[u,\theta]}} f(p) g(p) \frac{d^3 p}{p^0}, \quad f, g \in \mathbb{H}^{(0)}_{[u,\theta]}.$$
Similarly, for a finite positive integer $M \in \mathbb{N}$, a finite-dimensional weighted polynomial space can be defined by

$$
\mathbb{H}^{(0)}_{M} := \text{span} \left\{ p^{\mu_1} p^{\mu_2} \cdots p^{\mu_{\ell}} g^{(0)}_{[u, \theta]} : \mu_i = 0, 1, 2, 3, \quad \ell = 0, 1, \ldots, M \right\},
$$

which is a closed subspace of $\mathbb{H}^{(0)}_{M}$, obviously.

Unlike the one dimensional case [36], the basis of $\mathbb{H}^{(0)}_{M}$ cannot be obtained easily. People usually use the following weighted polynomials [13, 23]

$$
g^{(0)}_{[u, \theta]} P_{k}^{(\ell)}(E; \theta^{-1}) p^{<\mu_1 \cdots \mu_{\ell}>} \tag{5.5}
$$
to span the spaces $\mathbb{H}^{(0)}_{M}$ and $\mathbb{H}^{(0)}_{M}$, where $p^{<\mu_1 \cdots \mu_{\ell}>} := \Delta_{\mu_1, \ldots, \mu_\ell}^p R_{\mu_1 \cdots \mu_\ell}$ denotes the $\ell$th order irreducible tensor, here

$$
\Delta_{\beta} := g_{\beta} - \frac{1}{\ell^2} U_{\alpha} U_{\beta}, \quad \Delta_{\mu \nu} := \frac{1}{2} \left( \Delta_{\mu} \Delta_{\nu} + \Delta_{\nu} \Delta_{\mu} - 2 \frac{2}{3} \Delta_{\mu \nu} \Delta_{\beta \beta} \right),
$$

and the higher order ones could be found in [23] and are symmetric and traceless. Moreover, the irreducible tensors satisfy

$$
\int_{\mathbb{R}^3} p^{<\mu_1 \cdots \mu_m>} p_{\nu_1 \cdots \nu_k>} F(E) \frac{d^3 p}{p^0} = \frac{(-1)^m \delta_{mk} \Delta_{\nu_1, \ldots, \nu_m}}{(2m + 1)!!} \Delta_{\mu_1, \ldots, \mu_m} \int F(E) (E^2 - 1)^{\ell} \frac{d^3 p}{p^0},
$$

for any function $F(E)$ of $E$. Unfortunately, the weighted polynomials in (5.5) are linearly dependent. It is obvious for the case of $\ell = 1$, so they cannot be the basis of $\mathbb{H}^{(0)}_{M}$ or $\mathbb{H}^{(0)}_{M}$. In the one dimensional case, one can simply delete $p^{<0>}$ to obtain the basis, because the irreducible tensors are zeros for $\ell \geq 2$. But in the three dimensional case, it is complicate. This work uses the real spherical harmonics to replace the irreducible tensors and presents the basis of $\mathbb{H}^{(0)}_{M}$ or $\mathbb{H}^{(0)}_{M}$.

Thanks to Theorem 2.2, for all physically admissible $u$ and $\theta$ satisfying $|u| < 1$ and $\theta > 0$, introduce two notations

$$
\mathcal{P}_\infty[u, \theta] := (\tilde{P}_{0,0}^{(0)}[u, \theta], \ldots, \tilde{P}_{M,0}^{(0)}[u, \theta], \tilde{P}_{M-1,0}^{(0)}[u, \theta], \tilde{P}_{M-1,0}^{(0)}[u, \theta]),
$$

$$
\mathcal{P}_{M}[u, \theta] := (\tilde{P}_{0,0}^{(0)}[u, \theta], \tilde{P}_{0,0}^{(0)}[u, \theta], \tilde{P}_{0,0}^{(0)}[u, \theta], \tilde{P}_{0,0}^{(0)}[u, \theta], \tilde{P}_{0,0}^{(0)}[u, \theta], \ldots, \tilde{P}_{M,0}^{(0)}[u, \theta], \ldots, \tilde{P}_{M,0}^{(0)}[u, \theta], \ldots, \tilde{P}_{M,0}^{(0)}[u, \theta]),
$$

$$
\tilde{P}_{k,m}^{(\ell)}[u, \theta] := g^{(0)}_{[u, \theta]} P^{(\ell)}_{k} (E; \theta^{-1}) Y_{\ell,m}, \quad \tilde{Y}_{\ell,m} := (E^2 - 1)^{\ell} Y_{\ell,m}(y, \phi), \quad |m| \leq \ell, \ell \in \mathbb{N},
$$

and the length of the vector (5.7) is equal to

$$
\sum_{\ell=0}^{M} (2\ell + 1)(M + 1 - \ell) = (M + 1) \left( \frac{M(2M + 1)}{6} + 1 \right) =: N_M.
$$
Lemma 5.1 The set of all components of $\mathcal{P}_\infty[u, \theta]$ (resp. $\mathcal{P}_M[u, \theta]$) form a standard orthogonal basis of $\mathbb{H}^{(0)}_{\ell, M}[u, \theta]$ (resp. $\mathbb{H}^{(0)}_{\ell, M}[u, \theta]$).

Proof (i) Using (4.1) and (4.2) gives

$$\hat{Y}_{\ell,m} = (E^2 - 1)^{1/2} \kappa_{\ell,m} (1 - y^2)^{3/2} \frac{d^{m}}{dy^m} \hat{P}_{\ell}(y) \begin{cases} \sin(|m|\phi), & m < 0, \\ \cos(m\phi), & m \geq 0, \end{cases}$$

so each component of $\mathcal{P}_\infty[u, \theta]$ (resp. $\mathcal{P}_M[u, \theta]$) holds. One has to show that $\mathcal{P}_\infty[u, \theta]$ (resp. $\mathcal{P}_M[u, \theta]$) belongs to $\mathbb{H}^{(0)}_{\ell, M}[u, \theta]$ (resp. $\mathbb{H}^{(0)}_{\ell, M}[u, \theta]$).

(ii) The mathematical induction is used to prove that any element in the space $\mathbb{H}^{(0)}_{\ell, M}[u, \theta]$ (resp. $\mathbb{H}^{(0)}_{\ell, M}[u, \theta]$) can be written into a linear combination of vectors in $\mathcal{P}_\infty[u, \theta]$ (resp. $\mathcal{P}_M[u, \theta]$). For $M = 1$, it is clear to have the linear combination

$$p^\alpha g^{(0)}_{[u, \theta]} = \left(U^\alpha E + \sqrt{E^2 - 1}\left(\sqrt{1 - y^2 \cos \phi n_1^\alpha + \sqrt{1 - y^2 \sin \phi n_2^\alpha + y n_3^\alpha}}\right)\right) g^{(0)}_{[u, \theta]}$$

where the decomposition of the particle velocity vector (5.1) has been used.

Assume that the linear combination

$$p^{\mu_1}p^{\mu_2} \cdots p^{\mu_M} g^{(0)}_{[u, \theta]} = \sum_{\ell=0}^{M} \sum_{\mu_{-\ell}}^{M-\ell} c_{\ell,m}^{\mu_1 \cdots \mu_M} \tilde{P}_{\ell,m}^{(\mu_1 \cdots \mu_M)}, \quad \mu_{-\ell} = 0, 1, 2, 3, \quad c_{\ell,m}^{\mu_1 \cdots \mu_M} \in \mathbb{R},$$

holds. One has to show that $p^{\mu_1}p^{\mu_2} \cdots p^{\mu_{M+1}} g^{(0)}_{[u, \theta]}$ may be written into a linear combination
of components of $\mathcal{P}_{M+1}[u, \theta]$. Because

$$p^{\mu_1}p^{\mu_2} \cdots p^{\mu_{M+1}}g^{(0)}_{[u, \theta]}$$

$$= \left( \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M-\ell} \sum_{i,m,\ell} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}} \tilde{P}_{i,m}^{(\ell)} \right) \left( U^{M+1} E + \sqrt{E^2 - 1}(\sqrt{1 - y^2} \cos \phi n_{1}^{M+1} + \sqrt{1 - y^2} \sin \phi n_{2}^{M+1} + y n_{3}^{M+1}) \right)$$

$$= \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M-\ell} \sum_{i,m,\ell} \left\{ U^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}} E \tilde{P}_{i,m}^{(\ell)} + n_{1}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}} \sqrt{E^2 - 1} \tilde{P}_{i,m}^{(\ell)} + n_{2}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}} \sqrt{E^2 - 1} \tilde{P}_{i,m}^{(\ell)} \right\}$$

$$= g^{(0)}_{[u, \theta]} \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M-\ell} \sum_{i,m,\ell} \left\{ U^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}} E P_{i}^{(\ell)} \tilde{Y}_{\ell,m}^{(\ell)} \right. \right.$$  

$$\left. + \frac{n_{1}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}}}{2\ell + 1} \left( h_{\ell+1,m} P_{i}^{(\ell)} \tilde{Y}_{\ell+1,m}^{(\ell)} + h_{\ell,m} (E^2 - 1) P_{i}^{(\ell)} \tilde{Y}_{\ell-1,m}^{(\ell)} \right) \right.$$  

$$\left. + \frac{n_{2}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}}}{2\ell + 1} \left( h_{\ell+1,m} Y_{\ell+1,m}^{(\ell)} + h_{\ell-1,m} (E^2 - 1) Y_{\ell-1,m}^{(\ell)} \right) \right.$$  

$$\left. + \frac{n_{3}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}}}{2\ell + 1} \left( h_{\ell+1,m} \tilde{Y}_{\ell+1,m}^{(\ell)} + h_{\ell-1,m} \tilde{Y}_{\ell-1,m}^{(\ell)} \right) \right),$$

where (4.12)-(4.14) have been used.

By using the three-term recurrence relations (3.7), (3.10), and (3.11) for the orthogonal polynomials $\{P_{k}^{(\ell)}(x; \zeta), \ell \in \mathbb{N}\}$, one has

$$p^{\mu_1}p^{\mu_2} \cdots p^{\mu_{M+1}}g^{(0)}_{[u, \theta]}$$

$$= \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M-\ell} \sum_{i,m,\ell} \left\{ U^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}} \left( a_{i-1}^{(\ell)} \tilde{P}_{i-1,m}^{(\ell)} + b_{i}^{(\ell)} \tilde{P}_{i,m}^{(\ell)} + c_{i}^{(\ell)} \tilde{P}_{i+1,m}^{(\ell)} \right) \right. \right.$$  

$$\left. + \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M-\ell} \sum_{i,m,\ell} \frac{n_{1}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}}}{2\ell + 1} \left( h_{\ell+1,m} \tilde{P}_{i-1,m}^{(\ell+1)} + h_{\ell-1,m} \tilde{P}_{i-1,m}^{(\ell+1)} \right) \right.$$  

$$\left. + \frac{n_{2}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}}}{2\ell + 1} \left( h_{\ell+1,m} P_{i-1,m}^{(\ell)} + h_{\ell-1,m} P_{i-1,m}^{(\ell)} \right) \right.$$  

$$\left. + \frac{n_{3}^{M+1} c_{i,m,\ell}^{\mu_1,\cdots,\mu_{M+1}}}{2\ell + 1} \left( h_{\ell+1,m} Q_{i-1,m}^{(\ell)} + h_{\ell-1,m} Q_{i-1,m}^{(\ell)} \right) \right),$$

where (4.12)-(4.14) have been used.
\[
\begin{align*}
&- \frac{1}{2\ell+1} \hat{s}_m \hat{h}_{\ell+1,m} \left( r_{i-1}^{(\ell+1)} \hat{P}_i^{(\ell+1)} + q_i^{(\ell+1)} \hat{P}_{i-1,m+1} + p_i^{(\ell+1)} \hat{P}_{i,m+1} \right) \\
&+ \frac{1}{2\ell-1} \hat{s}_m \hat{h}_{\ell-1,m} \left( p_i^{(\ell)} \hat{P}_{i,m+1} + q_i^{(\ell)} \hat{P}_{i+1,m+1} + r_{i+1}^{(\ell)} \hat{P}_{i+2,m+1} \right) \\
&- \frac{1}{2\ell-1} \hat{s}_m \hat{h}_{\ell-1,m} \left( p_i^{(\ell)} \hat{P}_{i-1,m+1} + q_i^{(\ell)} \hat{P}_{i-1,m+1} + r_{i+1}^{(\ell)} \hat{P}_{i+2,m+1} \right)
\end{align*}
\]

\[
+ \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M} \frac{n^2}{2} \sum_{i,\ell,\mu \in \mathbb{M}^M, \mu_1, \ldots, \mu_M} \hat{P}_{i,m}^{(\ell)}
\]

(iii) Because of (5.1), one has

\[
\frac{d^3 p}{p^0} = \left| \det \left( \frac{\partial (p_1, p_2, p_3)}{\partial (E, y, \phi)} \right) \right| d\phi dy dE = \sqrt{E^2 - 1} d\phi dy dE,
\]

where

\[
\frac{\partial (p_1, p_2, p_3)}{\partial (E, y, \phi)} = \begin{pmatrix}
\left( t^1 + \frac{E}{\sqrt{E^2-1}} \right), & \left( -\frac{E}{\sqrt{1-y^2}} \left( \cos \phi_1^1 + \sin \phi_1^1 \right) + n_3^1 \right), & \left( \sqrt{E^2-1} \left( \sqrt{1-y^2} \left( -\sin \phi_1^1 + \cos \phi_1^1 \right) \right) \right) \\
\left( t^2 + \frac{E}{\sqrt{E^2-1}} \right), & \left( -\frac{E}{\sqrt{1-y^2}} \left( \cos \phi_1^2 + \sin \phi_1^2 \right) + n_3^2 \right), & \left( \sqrt{E^2-1} \left( \sqrt{1-y^2} \left( -\sin \phi_1^2 + \cos \phi_1^2 \right) \right) \right) \\
\left( t^3 + \frac{E}{\sqrt{E^2-1}} \right), & \left( -\frac{E}{\sqrt{1-y^2}} \left( \cos \phi_1^3 + \sin \phi_1^3 \right) + n_3^3 \right), & \left( \sqrt{E^2-1} \left( \sqrt{1-y^2} \left( -\sin \phi_1^3 + \cos \phi_1^3 \right) \right) \right)
\end{pmatrix}
\]

is the Jacobi matrix.

Using (3.2) gives

\[
< \hat{P}_{i,m}^{(\ell)}, \hat{P}_{j,m'}^{(\ell')} >_{g^{(0)}_{\mathbf{u}, \partial}} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} P_i^{(\ell)} P_j^{(\ell')} (E^2 - 1)^{\ell + \ell'} \frac{\epsilon_{i,\ell,\ell'}^{\ell+\ell'}}{2} Y_{m,m'} \bigg|_{\mathbf{u}, \partial} \left[ g^{(0)}_{\mathbf{u}, \partial} \right] \frac{d^3 p}{p^0}
\]

\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} P_i^{(\ell)} P_j^{(\ell')} (E^2 - 1)^{\ell + \ell'} \frac{\epsilon_{i,\ell,\ell'}^{\ell+\ell'}}{2} g^{(0)}_{\mathbf{u}, \partial} Y_{m,m'} d\phi dy dE d\phi dy dE
\]

\[
= \frac{4\pi}{2\ell+1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} P_i^{(\ell)} P_j^{(\ell')} (E^2 - 1)^{\ell + \ell'} \frac{\epsilon_{i,\ell,\ell'}^{\ell+\ell'}}{2} g^{(0)}_{\mathbf{u}, \partial} dE d\phi dy dE
\]

\[
= \left( P_i^{(\ell)}, P_j^{(\ell')} \right)_{\omega^{(\ell)}} \delta_{\ell,\ell'} \delta_{m,m'} = \delta_{i,j} \delta_{\ell,\ell'} \delta_{m,m'}, \quad \ell \in \mathbb{N}, |m| \leq \ell.
\]

Combining (i) and (ii) with (iii) completes the proof.
Since $\mathbb{H}_M^{(0)}$ is a subspace of $\mathbb{H}_N^{(0)}$ when $M < N < +\infty$, there exists a matrix $P_{M,N} \in \mathbb{R}^{(M)_N \times (N)_N}$ with full row rank such that $P_{M}[u,\theta] = P_{M,N}P_{N}[u,\theta]$, where

$$P_{M,N} := \text{diag}\{I_{N,M},O_{N,M,N-N+1}\}.$$ 

Using the properties of the orthogonal polynomials $\{P^{(\ell)}_n(x;\zeta)\,\ell = 0, 1, n \geq 0\}$ in Section 3 can further give calculation of the partial derivatives and recurrence relations of the basis functions $\{f^{(\ell)}_k[u,\theta]\}$.

**Lemma 5.2 (Derivative relations)** The partial derivatives of basis functions can be calculated by

$$\frac{\partial f^{(\ell)}_{k,m}[u,\theta]}{\partial s} = -\frac{\partial \theta}{\partial s} s^2 \left( \frac{1}{2} \left( G(\zeta) - \zeta^{-1} - b^{(\ell)}_k \right) \tilde{f}^{(\ell)}_{k,m}[u,\theta] - a^{(\ell)}_k \tilde{f}^{(\ell)}_{k+1,m}[u,\theta] \right)$$

$$- \frac{\partial u_i}{\partial s} U^0 g^{(0)}_{i[u,\theta]} \left[ \left( \frac{2\ell + 1}{2\ell + 3} \tilde{p}^{(\ell+1)}_k - \zeta q^{(\ell+1)}_{k-1} \right) \right.$$  

$$\left. \cdot \left( n^i_3 h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k-1,m} + n^i_1 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k-1,m} - \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k-1,m} \right) \right) \right) \right.$$  

$$- n^i_2 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k-1,m} + s_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k-1,m} \right) \right)$$  

$$- \zeta \tilde{p}^{(\ell+1)}_k \left( n^i_3 h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k,m} + n^i_1 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k,m} - \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k,m} \right) \right) \right)$$  

$$- n^i_2 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k,m} + s_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k,m} \right) \right)$$  

$$- \frac{2\ell + 1}{2\ell - 1} \left( (k + 2\ell + 1) \tilde{p}^{(\ell)}_k - \zeta q^{(\ell)}_k \right)$$  

$$\times \left( n^i_3 h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k,m} - n^i_1 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+1,m} - \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+1,m} \right) \right) \right)$$  

$$+ n^i_2 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+1,m} + s_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+1,m} \right) \right)$$  

$$+ \frac{2\ell + 1}{2\ell - 1} \zeta \tilde{p}^{(\ell)}_k \left( n^i_3 h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+2,m} - n^i_1 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+2,m} - \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+2,m} \right) \right) \right)$$  

$$+ n^i_2 \left( \frac{1}{2} \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+2,m} + s_m h_{\ell+1,m} \tilde{p}^{(\ell+1)}_{k+2,m} \right) \right)$$  

$$+ \frac{\partial u_i}{\partial s} U^0 (U^0 + 1)^{-1} \left[ -m(U^2 \delta_{1,i} - U^1 \delta_{2,i}) \tilde{p}^{(\ell)}_k \right.$$  

$$- \frac{1}{2} \left( \delta_{1,i} U^3 - \delta_{3,i} \right) \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell)}_{k,m} - \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell)}_{k,m} \right)$$  

$$- \frac{1}{2} \left( \delta_{2,i} U^3 - \delta_{3,i} \right) \left( \tilde{s}_m h_{\ell+1,m} \tilde{p}^{(\ell)}_{k,m} + s_m h_{\ell+1,m} \tilde{p}^{(\ell)}_{k,m} \right) \right].$$

for $s = x^\alpha$. It indicates that $\frac{\partial f^{(\ell)}_{k,m}[u,\theta]}{\partial s} \in H_{k+\ell+1}$.

**Proof** For $s = t$ and $x$, it is clear to have

$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial u_i}{\partial s} U^0 (U^i U^a + \delta_{i,a}), \quad \frac{\partial U_a}{\partial s} U^\alpha = 0, \quad \frac{\partial n^\alpha_i}{\partial s} n^\beta_i g_{\alpha \beta} = 0, \quad \frac{\partial n^\alpha_i}{\partial s} U_a + \frac{\partial U_a}{\partial s} n^\alpha_i = 0.$$
Thus one has
\[
\frac{\partial E}{\partial s} = \frac{\partial u_i}{\partial s} U^0 (U^i U_\alpha - \delta_{i, \alpha}) p^\alpha = \frac{\partial u_i}{\partial s} U^0 \sqrt{E^2 - 1} \left(n_1^i I_1 + n_2^i I_2 + n_3^i I_0\right),
\]
\[
\frac{\partial y}{\partial s} = -y E (E^2 - 1)^{-1} \frac{\partial E}{\partial s} - \frac{\partial n_3^i}{\partial s} p_a (E^2 - 1)^{-\frac{1}{2}}
\]
\[
= \frac{\partial u_i}{\partial s} U^0 E (E^2 - 1)^{-\frac{1}{2}} \left(n_1^i \hat{I}_1 + n_2^i \hat{I}_2 + n_3^i \hat{I}_0\right) - \frac{\partial n_3^i}{\partial s} g_{\alpha \beta} (n_1^\beta \hat{I}_1 + n_2^\beta \hat{I}_2)
\]
\[
= \frac{\partial u_i}{\partial s} U^0 \left(\frac{E}{\sqrt{E^2 - 1}} \left(n_1^i \hat{I}_1 + n_2^i \hat{I}_2 + n_3^i \hat{I}_0\right) + \frac{1}{U^0 + 1} \frac{\partial}{\partial s} g_{\alpha \beta} (n_1^\beta \hat{I}_1 + n_2^\beta \hat{I}_2)\right),
\]
\[
\frac{\partial \phi}{\partial s} = -\left(\frac{\partial n_3^i}{\partial s} p_a \frac{\cos \phi}{\sqrt{1 - y^2 \sqrt{E^2 - 1}}} - \frac{\partial n_1^\alpha}{\partial s} p_a \frac{\sin \phi}{\sqrt{1 - y^2 \sqrt{E^2 - 1}}}\right)
\]
\[
= \frac{\partial u_i}{\partial s} U^0 \left(\frac{E}{\sqrt{E^2 - 1}} \left(n_1^i \hat{I}_1 + n_2^i \hat{I}_2\right)\right) + g_{\alpha \beta} \left(\frac{\partial n_3^i}{\partial s} (n_3^\beta \hat{I}_4 - n_1^\beta) + \frac{\partial n_1^\alpha}{\partial s} n_3^\beta \hat{I}_3\right)
\]
\[
= \frac{\partial u_i}{\partial s} U^0 \left(\frac{E}{\sqrt{E^2 - 1}} \left(n_1^i \hat{I}_1 + n_2^i \hat{I}_2\right) + \frac{U^2 \delta_{1,i} - U^1 \delta_{2,i} + (\delta_{1,i} U^3 - \delta_{3,i} U^1) \hat{I}_3 + (\delta_{2,i} U^3 - \delta_{3,i} U^2) \hat{I}_4}{U^0 + 1}\right).
\]

Using those above identities and (5.4) gives
\[
\frac{\partial g_0}{\partial s} = -\left(\frac{\partial \theta}{\partial s} \frac{\zeta^2}{\sqrt{E^2 - 1}} + \zeta \frac{\partial E}{\partial s}\right) g_0.
\]

Using Lemma 4.1 gives
\[
\frac{\partial Y_{\ell,m}}{\partial s} = \frac{\partial y}{\partial s} \frac{d}{dy} Y_{\ell,m} (E^2 - 1)^{\frac{1}{2}} - m Y_{\ell,-m} (E^2 - 1)^{\frac{1}{2}} \frac{\partial \phi}{\partial s} + \ell E Y_{\ell,m} (E^2 - 1)^{\frac{\ell - 1}{2}} \frac{\partial E}{\partial s},
\]
\[
= \frac{\partial u_i}{\partial s} U^0 E (E^2 - 1)^{\frac{\ell - 1}{2}} n_3^i \left(\hat{I}_0 \frac{d}{dy} Y_{\ell,m} - \ell I_0 Y_{\ell,m}\right)
\]
\[
+ \frac{\partial u_i}{\partial s} U^0 E (E^2 - 1)^{\frac{\ell - 1}{2}} n_1^i \left(\hat{I}_1 \frac{d}{dy} Y_{\ell,m} - \ell I_1 Y_{\ell,m} - m \hat{I}_1 Y_{\ell,-m}\right)
\]
\[
+ \frac{\partial u_i}{\partial s} U^0 E (E^2 - 1)^{\frac{\ell - 1}{2}} n_2^i \left(\hat{I}_2 \frac{d}{dy} Y_{\ell,m} - \ell I_2 Y_{\ell,m} - m \hat{I}_2 Y_{\ell,-m}\right)
\]
\[
+ \frac{\partial u_i}{\partial s} U^0 \delta_{1,i} U^3 - \delta_{3,i} U^1 \left(\frac{E^2 - 1}{U^0 + 1}\right)^{\frac{\ell - 1}{2}} \left(I_1 \frac{d}{dy} Y_{\ell,m} - m \hat{I}_3 Y_{\ell,m}\right)
\]
\[
+ \frac{\partial u_i}{\partial s} U^0 \delta_{2,i} U^3 - \delta_{3,i} U^2 \left(\frac{E^2 - 1}{U^0 + 1}\right)^{\frac{\ell - 1}{2}} \left(I_2 \frac{d}{dy} Y_{\ell,m} - m \hat{I}_4 Y_{\ell,m}\right)
\]
\[
- m \frac{\partial u_i}{\partial s} U^0 n_3^i h_{\ell,m} E Y_{\ell,-1,m}
\]
\[
= - \frac{\partial u_i}{\partial s} U^0 n_3^i h_{\ell,m} Y_{\ell,-1,m}
\]
\[
+ \frac{\partial u_i}{\partial s} U^0 n_1^i E \left(\tilde{s}_m \hat{I}_{\ell-1,m} Y_{\ell-1,m} - \tilde{s}_m \hat{I}_{\ell-1,-m} Y_{\ell-1,m+1}\right)
\]
\[
- \frac{\partial u_i}{\partial s} U^0 n_2^i E \left(\tilde{s}_m \hat{I}_{\ell-1,m} Y_{\ell-1,-m+1} + s_m \hat{I}_{\ell-1,-m} \hat{I}_{\ell-1,-m+1}\right)
\]
\[ + \frac{1}{2} \frac{\partial u_i}{\partial s} U_0 \delta_{1,i} U^3 - \delta_{3,i} U^1 \left( \hat{s}_m \hat{h}_{t,m} \hat{Y}_{t,m-1} - \hat{s}_m \hat{h}_{t,-m} \hat{Y}_{t,m+1} \right) \]
\[ - \frac{1}{2} \frac{\partial u_i}{\partial s} U_0 \delta_{2,i} U^3 - \delta_{3,i} U^2 \left( \hat{s}_m \hat{h}_{t,m} \hat{Y}_{t,-m+1} + s_m \hat{h}_{t,-m} \hat{Y}_{t,-m-1} \right) \]
\[ - m \frac{\partial u_i}{\partial s} U_0^2 \delta_{1,i} - U_0^1 \delta_{2,i} \hat{Y}_{t,m}, \]

and

\[
\frac{\partial P^{(\ell)}_k}{\partial s} = - \zeta^2 \frac{\partial P^{(\ell)}_k}{\partial \zeta} \frac{\partial \theta}{\partial s} + \frac{\partial P^{(\ell)}_k}{\partial E} \frac{\partial E}{\partial s} \\
= - \zeta^2 \left( a_{k-1}^{(\ell)} P_{k-1}^{(\ell)} - \frac{1}{2} \left( G(\zeta) - \zeta^{-1} - b_k^{(\ell)} \right) P_k^{(\ell)} \right) \frac{\partial \theta}{\partial s} \\
+ \left( \frac{2\ell + 1}{2\ell + 3} P_{k-1}^{(\ell+1)} + \zeta_{k-1}^{(\ell+1)} P_{k-2}^{(\ell+1)} \right) \frac{\partial E}{\partial s} \\
= - \zeta^2 \left( a_{k-1}^{(\ell)} P_{k-1}^{(\ell)} - \frac{1}{2} \left( G(\zeta) - \zeta^{-1} - b_k^{(\ell)} \right) P_k^{(\ell)} \right) \frac{\partial \theta}{\partial s} \\
+ \left( (E^2 - 1)^{-1} \left( \frac{2\ell + 1}{2\ell + 1} (k + 2\ell + 1) P_k^{(\ell-1)} + \zeta_{k}^{(\ell)} P_{k-1}^{(\ell-1)} \right) - (2\ell + 1) E P_k^{(\ell)} \right) \frac{\partial E}{\partial s}. \]

Combining (3.7) and (3.19) with (3.20) yields

\[
\frac{\partial P^{(\ell)}_k}{\partial s} \gamma_{\ell,m}^{(\ell)} = - \frac{\partial \theta}{\partial s} \zeta^2 \left( \frac{1}{2} \left( G(\zeta) - \zeta^{-1} - b_k^{(\ell)} \right) \tilde{P}_{k,m}^{(\ell)} \left[ \theta, \theta \right] - a_{k}^{(\ell)} \tilde{P}_{k+1,m}^{(\ell)} \left[ \theta, \theta \right] \right) \\
+ \frac{\partial E}{\partial s} \gamma_{\ell,m}^{(\ell)} (E^2 - 1)^{-1} \left( \frac{2\ell + 1}{2\ell + 1} \left( (k + 2\ell + 1) P_k^{(\ell-1)} - \zeta q_k^{(\ell)} \right) P_{k+1}^{(\ell-1)} - \zeta_{k+1}^{(\ell)} P_{k+2}^{(\ell-1)} \right) \\
- (2\ell + 1) E P_k^{(\ell)} \tilde{Y}_{\ell,m} \\
= - \frac{\partial \theta}{\partial s} \zeta^2 \left( \left( G(\zeta) - \zeta^{-1} - b_k^{(\ell)} \right) \tilde{P}_{k,m}^{(\ell)} \left[ \theta, \theta \right] - a_{k}^{(\ell)} \tilde{P}_{k+1,m}^{(\ell)} \left[ \theta, \theta \right] \right) \\
+ \frac{\partial E}{\partial s} \gamma_{\ell,m}^{(\ell)} \left( \frac{2\ell + 1}{2\ell + 3} P_{k-1}^{(\ell+1)} - \zeta q_{k-1}^{(\ell)} \right) P_{k-1}^{(\ell+1)} - \zeta_{k}^{(\ell)} P_{k}^{(\ell+1)} \tilde{Y}_{\ell,m}. \]

The derivation rule of compound function gives

\[
\frac{\partial \tilde{P}_{k,m}^{(\ell)} \left[ \theta, \theta \right]}{\partial s} = \frac{\partial P^{(\ell)}_k}{\partial s} \gamma_{\ell,m}^{(\ell)} \tilde{Y}_{\ell,m} + \frac{\partial \tilde{Y}_{\ell,m}}{\partial s} P_{k}^{(\ell)} \gamma_{\ell,m}^{(\ell)}. \]

Combining the above and using Lemma 4.1 again can complete the proof.

\[ \]

**Lemma 5.3 (Recurrence relations)** The basis functions \( \{ \tilde{P}_{k,m}^{(\ell)} \left[ \theta, \theta \right], \ell, m, k \in \mathbb{N}, |m| \leq \ell, k \leq M - \ell \} \) satisfy the following recurrence relations

\[
p^\alpha P_M^{(\ell)} \left[ \theta, \theta \right] = M^\alpha P_M^{(\ell)} \left[ \theta, \theta \right] + U^\alpha P_M^{(\ell)} e_0^0 + n_1^\alpha P_M^{(\ell)} e_1^1 + n_2^\alpha P_M^{(\ell)} e_2^2 + + n_3^\alpha P_M^{(\ell)} e_3^3, \quad (5.8) \]
where

\[ M_\alpha^M := n_1^\alpha P_M^1 A_M^1 (P_M^p)^T + n_2^\alpha P_M^2 A_M^2 (P_M^p)^T + n_3^\alpha P_M^3 A_M^3 (P_M^p)^T + U^\alpha P_M^0 A_M^0 (P_M^p)^T, \]

in which \( P_M^p \) is a permutation matrix satisfying

\[ P_M^p \hat{P}_M[u, \theta] = P_M[u, \theta], \quad P_M^p (P_M^p)^T = (P_M^p)^T P_M^p = I, \]

with

\[ \hat{P}_M[u, \theta] := (\hat{P}_{0,0}^{(0)}, \ldots, \hat{P}_{M,0}^{(0)}, \hat{P}_{0,-1}^{(1)}, \ldots, \hat{P}_{M-1,-1}^{(1)}, \ldots, \hat{P}_{0,1}^{(1)}, \ldots, \hat{P}_{0,-M}^{(M)}, \ldots, \hat{P}_{0,M}^{(M)})^T. \]

Moreover, \( A_M^\alpha \) is a partitioned matrix such as

\[ A_M^\alpha = \begin{pmatrix}
A_M^\alpha[[0,0],[0,0]] & A_M^\alpha[[0,0],[-1,1]] & A_M^\alpha[[0,0],[0,1]] & \cdots & A_M^\alpha[[0,0],[-M,M]] & \cdots & A_M^\alpha[[-1,1],[0,0]] \\
A_M^\alpha[[-1,1],[0,0]] & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
& \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
A_M^\alpha[[M,M],[0,0]] & \cdots & \cdots & \cdots & \cdots & \cdots & A_M^\alpha[[M,M],[M,M]]
\end{pmatrix}, \]

where

\[ A_M^0[(m, \ell), (m, \ell)] = J_M^{(\ell)}_{M-\ell}, \]

and

\[ A_M^1[(m, \ell - 1), (m - 1, \ell)] = \frac{(J_M^{(\ell)}_{M-\ell}^T \tilde{s}_m \tilde{h}_{\ell,m} \tilde{h}_{-\ell,m})}{2(\ell - 1)}, \quad A_M^1[(m, \ell - 1), (m + 1, \ell)] = \frac{(J_M^{(\ell)}_{M-\ell}^T \tilde{s}_m \tilde{h}_{\ell,m} \tilde{h}_{-\ell,m})}{2(\ell - 1)}, \]

\[ A_M^2[(m, \ell - 1), (m - 1, \ell)] = \frac{(J_M^{(\ell)}_{M-\ell}^T \tilde{s}_m \tilde{h}_{\ell-1,m} \tilde{h}_{-\ell-1,m})}{2(\ell - 1)}, \quad A_M^2[(m, \ell - 1), (m + 1, \ell - 1)] = \frac{(J_M^{(\ell)}_{M-\ell}^T \tilde{s}_m \tilde{h}_{\ell-1,m} \tilde{h}_{-\ell-1,m})}{2(\ell - 1)}, \]

\[ A_M^3[(m, \ell - 1), (m, \ell)] = \frac{(J_M^{(\ell)}_{M-\ell}^T \tilde{s}_m \tilde{h}_{\ell,m} \tilde{h}_{\ell,m})}{2(\ell - 1)}, \quad A_M^3[(m, \ell - 1), (m, \ell - 1)] = \frac{(J_M^{(\ell)}_{M-\ell}^T \tilde{s}_m \tilde{h}_{\ell,m} \tilde{h}_{\ell,m})}{2(\ell - 1)}, \]

with other blocks are zero matrices.

The vector \( e_M^\alpha \) is partitioned as follows

\[ e_M^\alpha = \left( e_M^\alpha[0,0]^T, \ldots, e_M^\alpha[-M,M]^T, \ldots, e_M^\alpha[M,M]^T \right)^T, \]

where

\[ e_M^0[(m, \ell)] = a_{\alpha}^{(\ell)}_{M-\ell} \hat{s}_m \tilde{h}_{\ell,m} \tilde{h}_{\ell,m}, \]

\[ e_M^1[(m, \ell)] = \frac{p_{\alpha}^{(\ell)}_{\ell+1} (\hat{s}_m \tilde{h}_{\ell+1,m} \tilde{h}_{\ell+1,m} - \tilde{s}_m \tilde{h}_{\ell+1,m} \tilde{h}_{\ell+1,m})}{2(\ell + 1)} e_M^{\ell+1}. \]
\[
\begin{align*}
&+ r_{M-\ell+1}(\tilde{s}_m \tilde{h}_{\ell-1,m} \tilde{P}_{\ell-1}^{(\ell-1)} e_{M-\ell+1}, \\
\mathcal{E}_M^2(m, \ell) &= - \frac{p_{\ell+1}^{(\ell+1)}(\tilde{s}_m \tilde{h}_{\ell+1,m} \tilde{P}_{\ell+1}^{(\ell+1)} e_{M-\ell+1}, e_{M-\ell+1}, \\
&+ r_{M-\ell+1}(\tilde{s}_m \tilde{h}_{\ell-1,m} \tilde{P}_{\ell-1}^{(\ell-1)} e_{M-\ell+1}, e_{M-\ell+1}, \\
&+ r_{M-\ell+1}(\tilde{s}_m \tilde{h}_{\ell+1,m} \tilde{P}_{\ell+1}^{(\ell+1)} e_{M-\ell+1}, e_{M-\ell+1}, \\
\mathcal{E}_M^3(m, \ell) &= \frac{p_{\ell+1}^{(\ell+1)}(\tilde{s}_m \tilde{h}_{\ell+1,m} \tilde{P}_{\ell+1}^{(\ell+1)} e_{M-\ell+1}, e_{M-\ell+1}, \\
&\text{and } e_{M-\ell+1} \text{ is the last column of the identity matrix of order } (M-\ell+1).
\end{align*}
\]

**Proof** Using the three-term recurrence relations (3.8), (3.12), (3.13), (4.12)-(4.14), and (5.1) can complete the proof.

For a finite integer \( M \geq 1 \), define an operator \( \Pi_M[u, \theta] : \mathbb{H}_M^{(0)[0]} \rightarrow \mathbb{H}_M^{(0)[0]} \) by

\[
\Pi_M[u, \theta] f := \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M-\ell} f_{i,m}^{(\ell)} \tilde{P}_{i,m}^{(\ell)} [u, \theta],
\]

(5.11)

or in a compact form

\[
\Pi_M[u, \theta] f = [\mathcal{P}_M[u, \theta], f_M]_M,
\]

(5.12)

where

\[
\begin{align*}
&f_{i,m}^{(\ell)} = \langle f_{i,m}^{(\ell)} [u, \theta], \rangle > g_{[0][0]}, \\
&|m| \leq \ell, \quad i \leq M-\ell,
\end{align*}
\]

(5.13)

\[
\begin{align*}
f_M &= (f_{0,0}^{(0)}, f_{1,0}^{(1)}, f_{0,-1}^{(1)}, f_{0,0}^{(1)}, f_{1,1}^{(1)}, \ldots, f_{M,0}^{(1)} f_{M-1,-1}^{(1)}, f_{M-1,0}^{(1)}, \ldots, f_{0,-M}^{(M)}, \ldots, f_{0,0}^{(M)})^T.
\end{align*}
\]

(5.14)

and the symbol \([., .]_M\) denotes the common inner product of two \( N_M \)-dimensional vectors.

**Lemma 5.4** The operator \( \Pi_M[u, \theta] \) is linear bounded and projection operator in sense that

(i) \( \Pi_M[u, \theta] f \in \mathbb{H}_M^{[0][0]} \) for all \( f \in \mathbb{H}_M^{[0][0]} \),

(ii) \( \Pi_M[u, \theta] f = f \) for all \( f \in \mathbb{H}_M^{[0][0]} \).

**Proof** It is obvious that \( \Pi_M[u, \theta] \) is a linear bounded operator and \( \Pi_M[u, \theta] f \in \mathbb{H}_M^{[0][0]} \) for all \( f \in \mathbb{H}_M^{[0][0]} \).

For each \( f \in \mathbb{H}_M^{[0][0]} \), besides (5.11), one has by using Lemma 5.1

\[
f = \sum_{\ell=0}^{M} \sum_{m=-\ell}^{M-\ell} f_{i,m}^{(\ell)} \tilde{P}_{i,m}^{(\ell)} [u, \theta] .
\]

32
Taking respectively the inner product with $\tilde{P}_{i,m}[u, \theta]$ from both sides of the last equation gives

$$f_{i,m}^{(\ell)} = <f, \tilde{P}_{i,m}[u, \theta]>_{g_{[u,\theta]}} = \tilde{f}_{i,m}^{(\ell)}.$$ 

The proof is completed.

Remark 5.1 The so-called Grad type expansion is to expand the distribution function $f(x, p, t)$ in the weighted polynomial space $\mathbb{H}_g^{(0)}[u, \theta]$ as follows

$$f(x, p, t) = [\mathcal{P}_\infty[u, \theta], f_\infty],$$

where the symbol $[\cdot, \cdot]_\infty$ denotes the common inner product of two infinite-dimensional vectors, and $f_\infty = (f_{0,0}^{(0)}, \cdots, f_{M,0}^{(0)}, f_{M-1,0}^{(1)}, f_{M-1,1}^{(1)}, \cdots, f_{0,-M}^{(M)}, \cdots, f_{0,M}^{(M)}, \cdots)^T$.

5.2 Derivation of the moment model

Based on the weighted polynomial spaces $\mathbb{H}_g^{(0)}[u, \theta]$ and $\mathbb{H}_M^{(0)}[u, \theta]$ in Section 5.1 and the projection operator $\Pi_M[u, \theta]$ defined in (5.11), the moment method by the operator projection [18,36] may be implemented for the 3D special relativistic Boltzmann equation (2.1). In view of the fact that the variables $\{n, u, \theta, \Pi\}$ are several physical quantities of practical interest and the first three are required in calculating the equilibrium distribution $f^{(0)}$, the $N_M$-dimensional vector

$$W_M = (n, u, \theta, \Pi, \tilde{f}_{0,-1}^{(1)}, \tilde{f}_{0,0}^{(1)}, \tilde{f}_{0,1}^{(1)}, \cdots, f_{M,0}^{(0)}, f_{M-1,-1}^{(1)}, f_{M-1,0}^{(1)}, f_{M-1,1}^{(1)}, \cdots, f_{0,-M}^{(M)}, \cdots, f_{0,M}^{(M)}, \cdots)^T,$$

will be considered as the dependent variable vector, instead of $f_M$ defined in (5.14), where $\tilde{f}_{0,m}^{(1)} := f_{0,m}^{(1)}(c_0^{(1)})^{-1}$ satisfying

$$n^\alpha = -n_1^\alpha \tilde{f}_{0,1}^{(1)} + n_2^\alpha \tilde{f}_{0,-1}^{(1)} + n_3^\alpha \tilde{f}_{0,0}^{(1)}.$$ 

Thanks to (2.7) and (2.10), the relations between $W_M$ and $f_M$ is

$$f_M = D^{W}_M W_M,$$ 

where the square matrix $D^{W}_M$ depends on $\theta$ and is of the following explicit form

$$D^{W}_1 = \begin{pmatrix} (c_0^{(0)})^{-1} \\ O_{4 \times 4} \end{pmatrix}, \quad D^{W}_2 = \begin{pmatrix} D^{12}_1 & D^{12}_{5 \times 4} \\ D^{22}_{4 \times 4} & I_{5 \times 5} \end{pmatrix},$$

33
where

\[
D_{5\times 4}^{12} = \begin{pmatrix}
-3c_0^{(0)} & 0 & 0 & 0 \\
-3\zeta c_1^{(0)}x_{1,1}^{(0)} & 0 & 0 & 0 \\
0 & c_0^{(1)} & 0 & 0 \\
0 & 0 & c_0^{(1)} & 0 \\
0 & 0 & 0 & c_0^{(1)} \\
\end{pmatrix}, \quad D_{4\times 4}^{22} = \begin{pmatrix}
-3c_2^{(0)}x_{1,2}^{(0)} & 0 & 0 & 0 \\
0 & -c_1^{(1)}x_{1,1}^{(1)} & 0 & 0 \\
0 & 0 & -c_1^{(1)}x_{1,1}^{(1)} & 0 \\
0 & 0 & 0 & -c_1^{(1)}x_{1,1}^{(1)} \\
\end{pmatrix},
\]

and \(D_M^W = \text{diag}\{D_2^W, I_{N_M-N_2}\}\) for \(M \geq 3\). Referring to the schematic diagram shown in Fig. 5.1, the arbitrary order moment system for the Boltzmann equation (2.1) may be derived by the operator projection as follows:

**Step 1 (Projection 1):** Projecting the distribution function \(f\) into space \(H_g^{[0]}\) by the operator \(\Pi_M[u, \theta]\) defined in (5.12).

**Step 2:** Calculating the partial derivatives in time and space provides

\[
\frac{\partial \Pi_M[u, \theta]f}{\partial s} = \left[ \frac{\partial \mathcal{P}_M[u, \theta]}{\partial s}, f_M \right]_M + \left[ \mathcal{P}_M[u, \theta], \frac{\partial f_M}{\partial s} \right]_M
\]

\[
= \left[ C_{M+1}P_{M,M+1}^T\mathcal{P}_M[u, \theta], P_{M,M+1}^Tf_M \right]_{M+1} + \left[ \mathcal{P}_M[u, \theta], \frac{\partial f_M}{\partial s} \right]_M, \tag{5.16}
\]

for \(s = x^\alpha\), where \(C_{M+1}\) is a square matrix of order \(N_{M+1}\) and directly derived with the aid of the derivative relations of the basis functions in Lemma 5.2.

**Step 3 (Projection 2):** Projecting the partial derivatives in (5.16) into the space \(H_M^{[0]}\) gives

\[
\Pi_M[u, \theta]\frac{\partial \Pi_M[u, \theta]f}{\partial s} = \left[ \mathcal{P}_M[u, \theta], C_M^Tf_M \right]_M + \left[ \mathcal{P}_M[u, \theta], \frac{\partial f_M}{\partial s} \right]_M
\]
\[
\mathbf{D}_M = \begin{pmatrix}
\mathbf{D}_2 & \mathbf{0} \\
0 & \mathbf{I}_{N_M-N_2}
\end{pmatrix}, \quad M \geq 3, \quad \mathbf{D}_2 = \begin{pmatrix}
\mathbf{D}^{11}_{5 \times 5} & \mathbf{D}^{12}_{5 \times 4} & \mathbf{0} \\
\mathbf{D}^{21}_{4 \times 5} & \mathbf{D}^{22}_{4 \times 4} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}_{5 \times 5}
\end{pmatrix},
\]

and

\[
\mathbf{D}^{11}_{5 \times 5} = \begin{pmatrix}
(c_0^{(0)})^{-1} & 0 & 0 & 0 & -n\zeta^2 (c_1^{(0)})^{-2} (c_0^{(0)})^{-1} \\
0 & 0 & 0 & 0 & n\zeta^2 (c_1^{(0)})^{-1} \\
0 & -nU_0 c_1^{(1)} & -nU_0 c_2^{(1)} & -nU_0 c_3^{(1)} & 0 \\
0 & nU_0 c_1^{(1)} & nU_0 c_2^{(1)} & nU_0 c_3^{(1)} & 0 \\
0 & -nU_0 c_1^{(1)} & -nU_0 c_2^{(1)} & -nU_0 c_3^{(1)} & 0
\end{pmatrix},
\]

\[
\mathbf{D}^{21}_{4 \times 5} = \begin{pmatrix}
0 & -c_2^{(0)} j_{0,0,1}(x_1,2,x_2,2,2) & c_2^{(0)} j_{1,0,1}(x_1,2,x_2,2,2) & -c_2^{(0)} j_{0,1,1}(x_1,2,x_2,2,2) & 0 \\
0 & -U_0 c_1^{(1)} & -U_0 c_2^{(1)} & -U_0 c_3^{(1)} & 0 \\
0 & U_0 c_1^{(1)} & U_0 c_2^{(1)} & U_0 c_3^{(1)} & 0 \\
0 & -U_0 c_1^{(1)} & -U_0 c_2^{(1)} & -U_0 c_3^{(1)} & 0
\end{pmatrix}.\]

**Step 4:** Multiplying (5.17) by the particle momentum \(p^\alpha\) yields

\[
p^\alpha \Pi_M[u, \theta] \frac{\partial \Pi_M[u, \theta]}{\partial x^\alpha} := [p^\alpha \mathcal{P}_M[u, \theta], \mathbf{D}_M \frac{\partial W_M}{\partial x^\alpha}]_M
\]

\[
= [M^\alpha_{M+1} \mathcal{P}^T_{M,M+1} \mathcal{P}_M[u, \theta], \mathcal{P}^T_{M,M+1} \mathbf{D}_M \frac{\partial W_M}{\partial x^\alpha}]_{M+1}. \tag{5.19}
\]
Step 5 (Projection 3): Projecting (5.19) into the space $\Pi_M^{(0)}[u, \theta]$ gives
\[\Pi_M[u, \theta]\left( p^0\Pi_M[u, \theta]\frac{\partial\Pi_M[u, \theta]f}{\partial x^\alpha}\right) = [\mathcal{P}_M[u, \theta], M^\alpha_M D_M \frac{\partial W_M}{\partial x^\alpha}]_M. \quad (5.20)\]

Step 6: Substituting them into the special relativistic Boltzmann equation (2.1) derives the abstract form of the moment system
\[\Pi_M[u, \theta]\left( p^0\Pi_M[u, \theta]\left( \frac{\partial\Pi_M[u, \theta]f}{\partial x^\alpha}\right)\right) = \Pi_M[u, \theta]Q(\Pi_M[u, \theta]f, \Pi_M[u, \theta]f), \quad (5.21)\]
and then matching the coefficients in front of the basis functions $\tilde{\mathcal{P}}_{k,m}^{(\ell)}[u, \theta]$ leads to an “explicit” matrix-vector form of the moment system
\[B^\alpha_M \frac{\partial W_M}{\partial x^\alpha} = S(W_M), \quad (5.22)\]
which consists of $N_M$ equations, where $B^\alpha_M = M^\alpha_M D_M$. For a general collision term $Q(f, f)$, it is difficult to get an explicit expression of the source term $S(W_M)$ in (5.22).

For the Anderson-Witting model (2.8), the right-hand side of (5.21) becomes
\[\frac{1}{\tau}\Pi_M[u, \theta]Q(\Pi_M[u, \theta]f, \Pi_M[u, \theta]f) = -\frac{1}{\tau}\Pi_M[u, \theta]E\Pi_M[u, \theta](f - f^{(0)})
\]
\[= -\frac{1}{\tau}\Pi_M[u, \theta][\mathcal{P}^p_{M+1} A^0_{M+1}(\mathcal{P}^p_{M+1})^T \mathcal{P}^T_{M,M+1} \mathcal{P}_M[u, \theta], \mathcal{P}^T_{M,M+1}(f_M - f^{(0)}_M)]_M + 1
\]
\[= -\frac{1}{\tau}[\mathcal{P}_M[u, \theta], \mathcal{P}^p_M A^0_M (\mathcal{P}^p_M)^T \tilde{D}^W_M W_M]_M, \]
which implies that the source term $S(W_M)$ can be explicitly given by
\[S(W_M) = -\frac{1}{\tau} \mathcal{P}^p_M A^0_M (\mathcal{P}^p_M)^T \tilde{D}^W_M W_M, \quad (5.23)\]
where $f^{(0)}_M = \left(n \sqrt{G(\zeta)} - 4\zeta^{-1}, 0, \cdots, 0 \right)^T$, and the matrix $\tilde{D}^W_M$ is the same as $D^W_M$ except that the component of the upper left corner is zero. It is worth noting that the first five components of $S(W_M)$ are zero due to (2.7) and (2.10).

Remark 5.2 When $M = 1$, it is obvious to show $\Pi_M[u, \theta]f = f^{(0)}$ by the definition of $D^W_M$ such that the moment system (5.22) gives the macroscopic RHD equations (2.14).

5.3 The limited case

This section discusses two important special cases: the non-relativistic limit (small temperature and velocities) for a cool gas and the ultra-relativistic limit (zero rest mass of the particles).
5.3.1 The non-relativistic limit

In the non-relativistic limit, one has that \( \zeta \to +\infty \), \( K_\nu(\zeta) \to \sqrt{\frac{2\nu}{\pi}} \exp(-\zeta) \), \( uc^{-1} \to 0 \), and \( pc^{-1} \to 0 \). Thus it holds

\[
E = \sqrt{c^2 + u^2 \sqrt{m^2c^2 + p^2}} - u \cdot p = (mc + \frac{p^2}{2mc})(c + \frac{u^2}{2c}) - u \cdot p \\
\to mc^2 + \frac{muc^2}{2} + \frac{p^2}{2m} - u \cdot p = mc^2 + \frac{m|\xi - u|^2}{2},
\]

where \( \xi = pm^{-1} \), and the Maxwell-Jüttner distribution in (2.9) reduces to

\[
f^{(0)} = ng^{(0)} , \quad g^{(0)} = \frac{1}{(2\pi mkT)^{\frac{3}{2}}} \exp \left( -\frac{m|\xi - u|^2}{2kT} \right),
\]

which means that the weight function in (5.4) reduces to

\[
g_{[u,\theta]}^{(0)} = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp \left( -\frac{|\xi - u|^2}{2\theta} \right).
\]

Corresponding to such limited weight function \( g_{[u,\theta]}^{(0)} \), the previous standard orthogonal bases of the weighted polynomial spaces and the moment system will respectively reduce to those in [18] for the non-relativistic case.

5.3.2 The ultra-relativistic limit

In the ultra-relativistic limit \( (m \to 0) \), one has \( K_2(\zeta) \to \frac{2}{\zeta^2} \), \( p^0 \to p \), and the Maxwell-Jüttner distribution in (2.9) reduces to

\[
f^{(0)} = ng^{(0)} , \quad g^{(0)} = \frac{c^3}{8\pi k^3 T^3} \exp \left( -\frac{U_0p^0}{kT} \right),
\]

so that the weight function in (5.4) tends to

\[
g_{[u,\theta]}^{(0)} = \frac{1}{8\pi \theta^3} \exp \left( -\frac{E}{\theta} \right).
\]

For this weight function \( g_{[u,\theta]}^{(0)} \), the previous standard orthogonal bases of the weighted polynomial spaces become the followings.

**Theorem 5.1** The basis of the weighted polynomial space \( H_{M}^{g^{(0)}_{[u,\theta]}} \) in (5.6) reduces to

\[
\mathcal{P}_\infty[u,\theta] := (\tilde{P}^{(0)}_{0,0}[u,\theta], \cdots, \tilde{P}^{(0)}_{M,0}[u,\theta], \tilde{P}^{(1)}_{M-1,1}[u,\theta], \cdots, \tilde{P}^{(1)}_{M-1,0}[u,\theta]), \\
\tilde{P}^{(1)}_{M-1,1}[u,\theta], \cdots, \tilde{P}^{(M)}_{M-1,0}, \cdots, \tilde{P}^{(M)}_{0,0}, \cdots, \tilde{P}^{(0)}_{M,0}[u,\theta]), \\
\mathcal{P}_{M}[u,\theta] := (\tilde{P}^{(0)}_{0,0}[u,\theta], \tilde{P}^{(0)}_{1,0}[u,\theta], \tilde{P}^{(1)}_{0,1}[u,\theta], \tilde{P}^{(1)}_{0,0}[u,\theta], \tilde{P}^{(1)}_{1,1}[u,\theta], \cdots, \tilde{P}^{(0)}_{M,0}[u,\theta]),
\]

37
Due to the ultra-relativistic limit of (5.4), the weight function in (3.1) reduces to

\[
\omega^{(\ell)}(x; \zeta) = \frac{\zeta^3 x^{2\ell+1}}{(2\ell+1)} \exp(-\zeta x), \ell \in \mathbb{N}.
\]

If using the notation \( \tilde{x} = \zeta x \), then one has

\[
P^{(\ell)}_k(x, \zeta) = C_{\ell, k} \bar{P}^{(2\ell+1)}_k(\tilde{x}), \quad x > 0,
\]

where \( \{ \bar{P}^{(2\ell+1)}_k \} \) are the generalized Laguerre polynomials with the parameter \( 2\ell + 1 \), satisfying

\[
\int_0^\infty \tilde{x}^{2\ell+1} \exp(-\tilde{x}) \bar{P}^{(2\ell+1)}_k(\tilde{x}) \bar{P}^{(2\ell+1)}_m(\tilde{x}) d\tilde{x} = \frac{\Gamma(k+2\ell+2)}{k!} \delta_{k,m},
\]

and

\[
C_{\ell, k} = \frac{(2\ell+1)k!}{\zeta^2 \Gamma(k+2\ell+2)}.
\]

Thus, it holds

\[
\int_0^\infty P^{(\ell)}_k(x, \zeta) P^{(\ell)}_m(x, \zeta) \omega^{(\ell)}(x; \zeta) dx = \delta_{k,m}.
\]

The proof is completed.

Based on the above discussion, one can derive the moment system in the case of ultra-relativistic limit by the procedure in Sec. 5.2.

### 5.4 Quasi-1D case

In practice, only one spatial coordinate \( x \in \mathbb{R} \) may be considered for some problems, e.g. the shock tube or spherical symmetric problems, but the particle momentum should still be 3D, i.e. \( p \in \mathbb{R}^3 \).

Take \( x = x_3 \in \mathbb{R} \) as an example, and assume that the distribution \( f(x, p, t) \) is symmetric in \( p^1 \) and \( p^2 \) directions, and constant in \( x_1 \) and \( x_2 \), that is, \( f(x, p, t) = f(x_3, p, t), f(x, -p^1, p^2, p^3, t) = f(x, p^1, p^2, p^3, t), f(x, p^1, -p^2, p^3, t) = f(x, p^1, p^2, p^3, t) \). The energy-momentum tensor becomes

\[
T^{\alpha\beta} = \begin{pmatrix}
\int_{\mathbb{R}^3} p^0 p^0 f \frac{d^3p}{p^\beta} & 0 & 0 & \int_{\mathbb{R}^3} p^0 p^3 f \frac{d^3p}{p^\beta} \\
0 & \int_{\mathbb{R}^3} p^1 p^1 f \frac{d^3p}{p^\beta} & 0 & 0 \\
0 & 0 & \int_{\mathbb{R}^3} p^2 p^2 f \frac{d^3p}{p^\beta} & 0 \\
\int_{\mathbb{R}^3} p^0 p^3 f \frac{d^3p}{p^\beta} & 0 & 0 & \int_{\mathbb{R}^3} p^3 p^3 f \frac{d^3p}{p^\beta}
\end{pmatrix},
\]
thus one has $U^1 = U^2 = 0$, thanks to Theorem 2.1.

According to (5.2) and (5.3), one gets

$$n_1^a = (0, 1, 0, 0), \quad n_2^a = (0, 0, 1, 0), \quad n_3^a = (U^3, 0, 0, U^0),$$

$$y = (E^2 - 1)^{-1/2}(U^0p^3 - U^3p^0),$$

$$\cos \phi = \frac{p^1}{\sqrt{1 - y^2}}, \quad \sin \phi = \frac{p^2}{\sqrt{1 - y^2}},$$

$$p^0 = U^0E + \sqrt{E^2 - 1}U^3y, \quad p^1 = \sqrt{E^2 - 1}y \cos \phi, \quad p^2 = \sqrt{E^2 - 1}y \sin \phi.$$  

Using Lemma 5.2 yields

$$\frac{\partial \tilde{P}_{k,m}(u, \theta)}{\partial \theta} = -\frac{\partial \theta}{\partial s} \left( \frac{1}{2} G(\zeta) - \zeta^{-1} - b_k^{(0)} \right) \tilde{P}_{k,m} - a_k^{(0)} \tilde{P}_{k+1,m},$$

$$- \frac{\partial u_3}{\partial s} U_3 \left[ \left( 2\ell + 1 \right) \zeta^{(\ell+1)} - \zeta_k^{(\ell+1)} \right] n_3 h_{\ell+1,m} \tilde{P}_{k,m} - \zeta_k^{(\ell+1)} n_3 h_{\ell+1,m} \tilde{P}_{k,m} + \frac{2\ell + 1}{2\ell - 1} \zeta^{(\ell)} \frac{1}{2} \tilde{P}_{k+1,m} + \frac{2\ell + 1}{2\ell - 1} \zeta^{(\ell)} \frac{1}{2} \tilde{P}_{k+2,m},$$

and $d\phi = 0$. Further using Lemmas 5.3 and (5.18) derives

$$\dot{M}_M^0 = U^0 P_M^0 A_M^0 (P_M^0)^T + U^3 P_M^0 A_M^3 (P_M^0)^T, \quad \dot{M}_M^3 = U^3 P_M^0 A_M^0 (P_M^0)^T + U^0 P_M^0 A_M^3 (P_M^0)^T,$$

and the second to fourth columns of $D_M$ have only one nonzero component in each row, where the column for $u^3$ only corresponds to the row for $\tilde{F}_{k,0}$. On the other hand, the physical quantities of practical interest $(n, u_3, \theta, \Pi, n^3)$ can be expressed as $(f_0^{(0)}, f_1^{(1)}, f_0^{(0)}, f_2^{(0)})$ due to (5.15), and the expansion of $f$ can be reduced as follows

$$\tilde{M}_M[u, \theta] = \sum_{\ell=0}^{M} \sum_{i=0}^{M-\ell} \tilde{f}_{i,0}^{(\ell)} \tilde{F}_{i,0}(u, \theta), \quad \dot{W}_M = (n, u, \theta, \check{\Pi}, f_0^{(0)}, f_1^{(1)}, f_0^{(0)}, f_1^{(0)}, f_0^{(0)}, f_1^{(0)}, f_0^{(0)})^T,$$

where $u = u^3$. Thus if eliminating corresponding rows and columns of $D_M$, $M^\alpha$ and $S(W_M)$ and then denoting them by the notations $\dot{D}_M$, $\dot{M}^\alpha$, and $\dot{S}(\dot{W}_M)$, then the moment system (5.22) becomes

$$\dot{\dot{B}}_M^0 \frac{\partial \dot{W}_M}{\partial t} + \dot{\dot{B}}_M^3 \frac{\partial \dot{W}_M}{\partial x} = \dot{S}(\dot{W}_M),$$

where $\dot{\dot{B}}_M^\alpha = \dot{\dot{M}}^\alpha \dot{D}_M$ and $\dot{N}_M = \frac{(M+1)(M+2)}{2}$.

6 Properties of the moment system

This section studies some properties of moment system (5.21) or (5.22).
6.1 \textit{Hyperbolicity}

In order to prove the hyperbolicity of the moment system (5.22), one has to verify that $B_M^0$ to be invertible and $B_M := (B_M^0)^{-1} \sum_{i=1}^{3} \tilde{n}_i M^i_M$ to be real diagonalizable for a given unit vector $\tilde{n} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$. In the following, assume that the first five components of $W_M$ satisfy three inequalities in (2.17).

\textbf{Lemma 6.1} \textit{The matrix $D_M$ is invertible for $M \geq 1$.}

\textbf{Proof} It is obvious that for $M = 1$, the matrix $D_M$ is invertible because $\det(D_M) = n^4 \zeta^2 (c_0^{(1)})^3 (c_0^{(0)} c_1^{(0)} (U^0 + 1))^{-1} ((U^0)^2 + U^0 + 1) (U^0)^3 < 0$. For $M \geq 2$, according to the form of $D_M$ in Section 5.2, one has

$$\det(D_M) = \det(D_2) = 3 \zeta^3 c_2^{(0)} c_1^{(1)} (x_{1,2}^{(0)} + x_{2,2}^{(0)}) (\xi G(\zeta) + \Pi) n c_1^{(0)} c_0^{(0)} (U^0)^6.$$  

Using Lemma 2.2 gives

$$\det(D_M) > \zeta^3 c_2^{(0)} c_1^{(1)} (x_{1,2}^{(0)} + x_{2,2}^{(0)}) n^2 (G(\zeta) - \zeta^{-1}) c_1^{(0)} c_0^{(0)} (U^0)^6 > 0.$$  

The proof is completed.  

\textbf{Theorem 6.1 (Hyperbolicity)} \textit{The moment system (5.22) is strictly hyperbolic and the spectral radius of $B_M$ is less than one.}

\textbf{Proof} Due to Lemma 6.1, $D_M$ is invertible. According to (5.9), one knows that

$$M_M^0 = \left< p^\alpha P_M[u, \theta], P_M[u, \theta]^T \right>_{g(0)[u, \theta]}.$$  

It is obvious that $M_M^0$ is symmetric, and for a given $N_M$-dimensional nonzero vector $q$,

$$q^T M_M^0 q = \left< p^\alpha q^T P_M[u, \theta], P_M[u, \theta]^T [q]_{g(0)[u, \theta]} \right> = \left< p^\alpha q^T P_M[u, \theta], q^T P_M[u, \theta] \right>_{g(0)[u, \theta]} > 0,$$

thus $M_M^0$ is positive definite. Therefore $B_M^0$ is invertible.

Furthermore, one has

$$B_M = D_M^{-1} (M_M^0)^{-1} \sum_{i=1}^{3} \tilde{n}_i M^i_M D_M$$

$$= \left( (M_M^0)^{\frac{1}{2}} D_M \right)^{-1} \left( M_M^0 \right)^{-\frac{1}{2}} \sum_{i=1}^{3} \tilde{n}_i M^i_M (M_M^0)^{-\frac{1}{2}} \left( (M_M^0)^{\frac{1}{2}} D_M \right).$$

Thus the matrix $B_M$ is real diagonalizable since $(M_M^0)^{-\frac{1}{2}} \sum_{i=1}^{3} \tilde{n}_i M^i_M (M_M^0)^{-\frac{1}{2}}$ is symmet-
ric, and the spectral radius of $B_M$ is equal to that of $(M_M^0)^{-1} \sum_{i=1}^{3} \tilde{n}_i M_M^i$ and satisfies

$$\lambda M_M^0 - \sum_{i=1}^{3} \tilde{n}_i M_M^i = <(\lambda p^0 - \sum_{i=1}^{3} p^i)P_M[u, \theta], P_M[u, \theta]^T>_{g^{(0)}[u, \theta]},$$

for $|\lambda| > 1$. Moreover, for a given $N_M$-dimensional nonzero vector $q$, one has

$$q^T \left( \lambda M_M^0 - \sum_{i=1}^{3} \tilde{n}_i M_M^i \right) q = < p^0 q^T P_M[u, \theta], P_M[u, \theta]^T q >_{g^{(0)}[u, \theta]}$$

$$= \left< (\lambda p^0 - \sum_{i=1}^{3} p^i)q^T P_M[u, \theta], q^T P_M[u, \theta] \right>_{g^{(0)}[u, \theta]} \begin{cases} > 0, & \lambda > 1, \\ < 0, & \lambda < -1, \end{cases}$$

thus the matrix

$$\lambda M_M^0 - \sum_{i=1}^{3} \tilde{n}_i M_M^i$$

is positive (resp. negative) definite for $\lambda > 1$ (resp. $\lambda < -1$). Therefore the spectral radius of $(M_M^0)^{-1} \sum_{i=1}^{3} \tilde{n}_i M_M^i$ is less than one, and the proof is completed.

6.2 Linear stability

It is obvious that the moment system (5.22) has the local equilibrium solution $W_M^{(0)} = (n_0, u_0, \theta_0, 0, \cdots, 0)^T$, where $n_0$, $u_0$, and $\theta_0$ satisfy three inequalities in (2.17). Similar to the procedure in [15,36], the moment system (5.22)–(5.23) is linearized at $W_M^{(0)}$. If assuming that $W_M = W_M^{(0)}(1 + \tilde{W}_M)$ and each component of $\tilde{W}_M$ is small, then the linearized moment system becomes

$$B_M^\alpha \frac{\partial \tilde{W}_M}{\partial x^\alpha} = Q_M \tilde{W}_M, \quad (6.1)$$

where

$$Q_M = -\frac{1}{\tau} U_M M_M^0 \tilde{D}_M^W.$$

Assume that the solution $\tilde{W}_M$ is of the plane wave form

$$\tilde{W}_M = \tilde{W}_M \exp(i(\omega t - kx)),$$

where $i$ is the imaginary unit, $\tilde{W}_M$ is the nonzero amplitude, and $\omega$ and $k$ denote the frequency and wave number, respectively. Substituting the above plane wave into (6.1) gives

$$\left(i\omega B_M^0 - i k_j B_M^j - Q_M \right) \tilde{W}_M^{(0)} \tilde{W}_M = 0.$$

Because the amplitude $\tilde{W}_M$ is nonzero, the above coefficient matrix is singular, i.e.

$$\det \left(i\omega B_M^0 - i k_j B_M^j - Q_M \right) \tilde{W}_M^{(0)} = 0,$$

(6.2)
which implies the dispersion relation between $\omega$ and $k$.

**Theorem 6.2** The moment system (5.22) with the source term (5.23) is linearly stable in time at the local equilibrium, that is, the linearized moment system (6.1) is stable in the sense that $\text{Im}(\omega(k)) \geq 0$ for each $k \in \mathbb{R}^3$.

**Proof** Because the matrix $D_M$ in (5.18) at $W_M = W_M^{(0)}$ can be reformed as follows

$$D_M = \begin{pmatrix} D_{11}^{5 \times 5} & D_{12}^{5 \times 4} & O \\ O & D_{22}^{4 \times 4} & O \\ O & O & I_{N_M - 9} \end{pmatrix},$$

and its inverse is given by

$$D_M^{-1} = \begin{pmatrix} \left( D_{11}^{5 \times 5} \right)^{-1} - \left( D_{11}^{5 \times 5} \right)^{-1} D_{5 \times 4}^{12} \left( D_{4 \times 4}^{22} \right)^{-1} & O \\ O & \left( D_{4 \times 4}^{22} \right)^{-1} & O \\ O & O & I_{N_M - 9} \end{pmatrix},$$

as well as

$$\tilde{D}_M^W = \begin{pmatrix} O & D_{5 \times 4}^{12} & O \\ O & D_{4 \times 4}^{12} & O \\ O & O & I_{N_M - 9} \end{pmatrix},$$

the product of $\tilde{D}_M^W$ and $D_M^{-1}$ is of the form

$$\tilde{D}_M^W D_M^{-1} = \begin{pmatrix} O_{5 \times 5} & D_{5 \times 4}^{12} \left( D_{4 \times 4}^{22} \right)^{-1} O_{5 \times (N_M - 9)} \\ O_{4 \times 5} & I_{4} & O_{4 \times (N_M - 9)} \\ O_{(N_M - 9) \times 3} & O_{(N_M - 9) \times 2} & I_{N_M - 9} \end{pmatrix},$$

where $D_{11}^{5 \times 5}$ is the $5 \times 5$ subblock of the $9 \times 9$ upper left subblock of $D_2$ in the upper left corner, $D_{5 \times 4}^{12}$ denotes the $5 \times 4$ subblock of the $9 \times 9$ upper left subblock of $D_2$ in the upper right corner, and $D_{4 \times 4}^{22}$ is $4 \times 4$ subblock of the $9 \times 9$ upper left subblock of $D_2$ in the bottom right corner. It is obvious that each eigenvalue of $-\tilde{D}_M^W D_M^{-1}$ is non-positive, so does the matrix

$$Q_M := -\frac{1}{\tau} \left( U_\alpha M_M^\alpha \right)^{\frac{1}{2}} \tilde{D}_M^W D_M^{-1} \left( U_\alpha M_M^\alpha \right)^{-\frac{1}{2}}.$$
The matrix $U^\alpha M^\alpha_M$ can be written as follows

\[
\begin{pmatrix}
M_{5 \times 5}^{11} & M_{5 \times 4}^{12} & O_{5 \times (N_M-9)} \\
(M_{5 \times 4}^{12})^T & M_{4 \times 4}^{22} & M_{4 \times (N_M-9)}^{23} \\
O_{(N_M-9) \times 5} & (M_{4 \times (N_M-9)}^{23})^T & M_{(N_M-9) \times (N_M-9)}^{33}
\end{pmatrix},
\]

where $M_{5 \times 5}^{11}$ is the $5 \times 5$ subblock of $P_2^p A_2^0 (P_2^p)^T$ in the upper left corner, $M_{5 \times 4}^{12}$ denotes the $5 \times 4$ subblock of $P_2^p A_2^0 (P_2^p)^T$ in the upper right corner, and $M_{4 \times 4}^{23}$ is $4 \times 4$ subblock of $P_2^p A_2^0 (P_2^p)^T$ in the bottom right corner, the rest subblocks form the $(N_M-9) \times (N_M-9)$ bottom right corner of $P_2^p A_2^0 (P_2^p)^T$.

Thus one has

\[
M_D := (M_{5 \times 4}^{12})^T D_{5 \times 4}^{12} \left( D_{4 \times 4}^{22} \right)^{-1} = - \left( D_{5 \times 4}^{12} \left( D_{4 \times 4}^{22} \right)^{-1} \right)^T M_{5 \times 5}^{11} \left( D_{5 \times 4}^{12} \left( D_{4 \times 4}^{22} \right)^{-1} \right),
\]

which is symmetric because $M_{5 \times 5}^{11} D_{5 \times 4}^{12} \left( D_{4 \times 4}^{22} \right)^{-1} + M_{5 \times 4}^{12} = O_{5 \times 4}$.

On the other hand, because the first five components of $S(W_M)$ are zero, all elements in the first five rows and the first five columns of the matrix

\[
Q_M = -\frac{1}{\tau} U^\alpha M^\alpha_M \tilde{D}_M W^\dagger M^{-1}_M,
\]

are zero, and the matrix $Q_M$ is of form

\[
Q_M = -\frac{1}{\tau} \begin{pmatrix}
O_{5 \times 5} & O_{5 \times 4} & O_{5 \times (N_M-9)} \\
O_{4 \times 5} & M_{4 \times 4}^{22} + M_D & M_{4 \times (N_M-9)}^{23} \\
O_{(N_M-9) \times 5} & (M_{4 \times (N_M-9)}^{23})^T & M_{(N_M-9) \times (N_M-9)}^{33}
\end{pmatrix}.
\]

Hence the matrix $Q_M$ is symmetric. It is obvious that $Q_M$ is congruent with $\hat{Q}_M$, so it is negative semi-definite.

Because both matrices $D_M$ and $D_M^0$ are invertible and $M_M^0$ is positive definite, $Q_M$ is equivalent to

\[
\det \left( i \omega I - i M_{M} - \hat{Q}_M \right) = 0,
\]

where

\[
\hat{Q}_M := \left( M^0_M \right)^{-\frac{1}{2}} Q_M \left( M^0_M \right)^{-\frac{1}{2}},
\]

and

\[
M_M := \left( M^0_M \right)^{-\frac{1}{2}} k_i M^i_M \left( M^0_M \right)^{-\frac{1}{2}}.
\]

It means that the matrix $\hat{Q}_M$ is congruent with $Q_M$ and negative semi-definite, and $M_M$ is symmetric. Using Lemmas 1 and 2 in [15] completes the proof.
In physics, the Lorentz covariance is a key property of space-time following from the special theory of relativity, see e.g. [17]. This section studies the Lorentz covariance of the quasi 1D moment system in Section 5.4, where \( x = x_3 \).

Some Lorentz covariant quantities are first pointed out below.

**Lemma 6.2** (i) Each component of \( \dot{D}_M d\dot{W}_M \) is Lorentz invariant, where \( d\dot{W}_M \) denotes the total differential of \( \dot{W}_M \) and \( \dot{D}_M := \text{diag}\{1, U^0 n_3, 1, \cdots, 1\} \). (ii) The matrices \( \dot{A}_M^0 \), \( \dot{A}_M^3 \) and source term \( S(\dot{W}_M) \) defined in (5.23) are Lorentz invariant.

**Proof** (i) Under the given Lorentz boost (\( x \) direction)

\[
t' = \gamma(v)(t - vx), \quad x' = \gamma(v)(x - vt), \quad \gamma(v) = (1 - v^2)^{-\frac{1}{2}},
\]

where \( v \) is the relative velocity between frames in the \( x \)-direction, one has

\[
(p^0)' = \gamma(v)(p^0 - p^3 v), \quad (p^3)' = \gamma(v)(p^3 - p^0 v),
\]

\[
(U^0)' = \gamma(v)(U^0 - U^3 v), \quad (U^3)' = \gamma(v)(U^3 - U^0 v).
\]

Thus one further gets

\[
E' = (U^0)'(p^0)' - (U^3)'(p^3)' = U^0 p^0 - U^3 p^3 = E,
\]

and

\[
y' = ((E^2 - 1)^{-\frac{1}{2}}(U^0 p^3 - U^3 p^0))' = (E^2 - 1)^{-\frac{1}{2}}(((U^0)'(p^3)' - (U^3)'(p^0)')
\]

\[
= (E^2 - 1)^{-\frac{1}{2}}(U^0 p^3 - U^3 p^0) = y,
\]

\[
\left( \frac{d^3 p}{p^0} \right)' = \frac{(d^3 p _{p^3} d^2 p^1)'}{(p^0)'} = \left( \frac{1 - (p^0)^{-1}p^3 v)d^3 p^0 d^2 p^1}{p^0 - p^3 v} \right) = \frac{d^3 p}{p^0}.
\]

Combining them with (5.13) gives that each component of \( f_M \) is Lorentz invariant, such that the last \( (N_M - 3) \) components of \( \dot{W}_M \) are also Lorentz invariant.

From (2.13), it is not difficult to prove that \( n \) and \( \theta \) are Lorentz invariant.

Moreover, one has

\[
\left( \frac{du^0 n_3^3}{(U^0)'} \right)' = \frac{d(U^3)'}{(U^0)'} = \frac{dU^3 - dU^0 v}{U^0 - U^3 v} = \frac{dU^3 - (U^0)^{-1} U^3 dU^3 v}{U^0 - U^3 v} = \frac{dU^3}{U^0} = \frac{du^0 n_3^3}{(U^0)'}.
\]

Using the above results completes the proof of the first part.

(ii) Because \( \dot{A}_M^0 \) and \( \dot{A}_M^3 \) only depend on \( \theta \), they are Lorentz invariant. The source term
\( \dot{S}(\dot{W}_M) \) in (5.23) can be rewritten into
\[
\dot{S}(\dot{W}_M) = -\frac{1}{\tau} \dot{\rho}_M \dot{A}_M (\dot{P}_M)^T (f_M - f_M^{(0)}),
\]
which has been expressed in terms of the Lorentz covariant quantities. In fact, the general source term \( \dot{S}(\dot{W}_M) \) in the moment system (5.22) is also Lorentz invariant. The proof is completed.

**Theorem 6.3 (Lorentz covariance)** The moment system (5.26) with the source term (5.23) is Lorentz covariant.

**Proof** From the 3rd step in Section 5.2 and Lemma 6.2, one knows that \( \dot{D}_M = \dot{D}_M (\dot{D}_M^u)^{-1} \) can be expressed in terms of the Lorentz covariant quantities, so it is Lorentz invariant. Because
\[
(\dot{M}_M^0)' = \gamma(v)(U^3 - U^0 v) \dot{P}_M \dot{A}_M (\dot{P}_M)^T + \gamma(v)(U^0 - U^3 v) \dot{P}_M \dot{A}_M (\dot{P}_M)^T,
\]
\[
(\dot{M}_M^3)' = \gamma(v)(U^0 - U^3 v) \dot{P}_M \dot{A}_M (\dot{P}_M)^T + \gamma(v)(U^3 - U^0 v) \dot{P}_M \dot{A}_M (\dot{P}_M)^T,
\]
and
\[
\left( \frac{\partial}{\partial t} \right)' = \gamma(v) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right), \quad \left( \frac{\partial}{\partial x} \right)' = \gamma(v) \left( \frac{\partial}{\partial x} + v \frac{\partial}{\partial t} \right),
\]
on one has
\[
\left( \dot{D}_M^u \frac{\partial \dot{W}_M}{\partial t} \right)' = \text{diag} \left\{ 1, (U^0 n_3)', 1, \ldots, 1 \right\} \gamma(v) \left( \frac{\partial(n, (U^3)', \theta, \Pi, f_{0,0}^{(1)}, f_{0,0}^{(2)}, f_{0,0}^{(0)}, \ldots, f_{0,0}^{(M)})^T}{\partial t} 
+ v \frac{\partial(n, (U^3)', \theta, \Pi, f_{0,0}^{(1)}, f_{0,0}^{(2)}, f_{0,0}^{(0)}, \ldots, f_{0,0}^{(M)})^T}{\partial x} \right)
= \text{diag} \left\{ 1, ((U^0)^{-1})', 1, \ldots, 1 \right\} \gamma(v) \left( \frac{\partial(n, (U^3)', \theta, \Pi, f_{0,0}^{(1)}, f_{0,0}^{(2)}, f_{0,0}^{(0)}, \ldots, f_{0,0}^{(M)})^T}{\partial t} 
+ v \frac{\partial(n, (U^3)', \theta, \Pi, f_{0,0}^{(1)}, f_{0,0}^{(2)}, f_{0,0}^{(0)}, \ldots, f_{0,0}^{(M)})^T}{\partial x} \right)
= \dot{D}_M^u \gamma(v) \left( \frac{\partial \dot{W}_M}{\partial t} + v \frac{\partial \dot{W}_M}{\partial x} \right),
\]
where the last equal sign is derived by following the proof of Lemma 6.2. Similarly, one has
\[
\left( \dot{D}_M^u \frac{\partial \dot{W}_M}{\partial x} \right)' = \dot{D}_M^u \gamma(v) \left( \frac{\partial \dot{W}_M}{\partial x} + v \frac{\partial \dot{W}_M}{\partial t} \right).
\]
Thus one yields
\[
\left( B_M^0 \frac{\partial \dot{W}_M}{\partial t} + B_M^3 \frac{\partial \dot{W}_M}{\partial x} \right)',
\]

45
\[
(\mathbf{M}_M^0)' \left( \frac{\partial \mathbf{W}_M}{\partial t} \right)' + (\mathbf{M}_M^3)' \left( \frac{\partial \mathbf{W}_M}{\partial x} \right)' \\
= \left( (U^3)' \mathbf{P}_M^p \mathbf{A}_M^3 (\mathbf{P}_M^p)^T \right) \frac{\partial \mathbf{W}_M}{\partial t} + \left( (U^3)' \mathbf{P}_M^p \mathbf{A}_M^0 (\mathbf{P}_M^p)^T \right) \frac{\partial \mathbf{W}_M}{\partial x} \\
+ \left( (U^0)' \mathbf{P}_M^p \mathbf{A}_M^3 (\mathbf{P}_M^p)^T \right) \frac{\partial \mathbf{W}_M}{\partial t} + \left( (U^0)' \mathbf{P}_M^p \mathbf{A}_M^0 (\mathbf{P}_M^p)^T \right) \frac{\partial \mathbf{W}_M}{\partial x} \\
= B_0 \frac{\partial \mathbf{W}_M}{\partial t} + B_3 \frac{\partial \mathbf{W}_M}{\partial x}.
\]

Combining it with Lemma 6.2 completes the proof.

7 Conclusions

The paper derived the arbitrary order globally hyperbolic moment system of the three-dimensional special relativistic Boltzmann equation for the first time and studied the hyperbolicity and linear stability of the moment system, and Lorentz covariance of the quasi-1D moment system. The technique was the model reduction by the operator projection [18] and [36]. The key contributions were the real spherical harmonics and families of the Grad type orthogonal polynomials were used to establish the bases of the weighted polynomial spaces and the careful study on their recurrence relations and derivatives as well as the zeros of the Grad type orthogonal polynomials were given. It is interesting to develop robust, high order accurate numerical schemes for the moment system and find other basis for the derivation of moment system with some good property, e.g. non-negativity.

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