An effective fluid description, for a brane world model in five dimensions, is discussed for both signs of the brane tension. We found several cosmological scenarios where the effective equation differs widely from the bare equation of state. For universes with negative brane tension, with a bare fluid satisfying the strong energy condition, the effective fluid can cross the barrier $\omega_{\text{eff}} = -1$.

I. INTRODUCTION

Consideration of dimensionality of the universe greater than four has had a long history passed from the original idea of Kaluza-Klein [1] to modern ideas of string theory [2, 3]. In particular the Randall-Sundrum scenario has acquired a great attention in the last decade [4, 5]. From the cosmological point of view, brane world offers a novel approach to our understanding of the evolution of the universe. The most spectacular consequence of this scenario is the modification of the Friedmann equation. In those models, for instance in five dimensions, matter is confined to a four dimensional brane, while gravity can be propagated in the bulk, i.e., gravity is the only field that can feel the extra dimension. From the perspective of string theory [6], brane world cosmology has been a big challenge for modern cosmology. For a comprehensible review on BW cosmology see Ref. [7]. For example, consequences of a chaotic inflationary universe scenario in a BW model was described [8], where it was found that the slow-roll approximation is enhanced by the modification of the Friedman equation. In the Einstein approach the DE appears as an exotic fluid in the energy-momentum tensor, so the field equations [10] becomes

$$G_{\mu\nu} = 8\pi G N (T^{\text{matter}}_{\mu\nu} + T^{\text{DE}}_{\mu\nu}),$$  \hspace{1cm} (1)

where $G_{\mu\nu}$ is the 4$D$ Einstein tensor, $T^{\text{matter}}_{\mu\nu}$ is the stress tensor for matter and $T^{\text{DE}}_{\mu\nu}$ is the stress tensor for dark energy, a new exotic component with a negative pressure. This description can be modified in such way that this exotic component arises naturally from modifications in the geometric sector of the field equations,

$$G_{\mu\nu} - \mathcal{K}_{\mu\nu} = 8\pi G N T^{\text{matter}}_{\mu\nu},$$  \hspace{1cm} (2)

where $\mathcal{K}_{\mu\nu}$ denotes a tensor that arise from the extrinsic curvature, due to the embedding of our brane universe in the 5$D$ bulk (for a review see Ref. [14] and references therein). Using projection techniques [13] it was found the effective Friedmann equation onto the brane, which can be written as follows,

$$3H^2 = 8\pi G N f(\rho),$$  \hspace{1cm} (3)

where the function $f(\rho)$ encoded all geometric modification of the field equations. In this framework exotic matter is not necessary to explain the late acceleration of the universe and the coincidence problem. In the context of this framework, the evolution of universes filled with a perfect fluid has been investigated in many works, see for example Refs. [7, 8, 11, 12].

The aim of this article is discuss the behavior of the brane world models in five dimensions for a positive and negative brane tension, in terms of an bare and effective fluid description.

The plan of the paper is as follows: In Sec. II we specify the effective four dimensional cosmological equations from Randall-Sundrum model. In Sec III we give the principal equations corresponding an effective fluid description for the cosmological evolution. We show the behavior of the found scenarios for a positive and negative brane tension. In Sec IV we discuss the cosmological scenarios in terms of the bare and effective description.
II. RANDALL-SUNDRUM COSMOLOGICAL SCENARIO

For an homogeneous and isotropic flat 4-brane, described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the field equations are

\[ 3H^2 = \rho \left( 1 \pm \frac{\rho}{2\lambda} \right), \]  
(4)

where the positive and negative sign are related to positive and negative brane tension. The matter content sector satisfy the conservation equation given by

\[ \dot{\rho} + 3H (\rho + p) = 0. \]  
(5)

Deriving Eq. (4) with respect to the cosmological time and using Eq. (5) we obtain

\[ \ddot{a} = -\frac{1}{6} \left( \rho + 3p \right) \left[ 1 \pm \frac{\rho}{\lambda} \right] \mp \frac{\rho^2}{6\lambda}. \]  
(6)

In what follows we consider a barotropic the equation of state \( p = \omega \rho \) for the bare fluid.

III. EFFECTIVE FLUID DESCRIPTION ON THE BRANE

Since we are interested in an effective description of the modified Friedmann equations obtained from the Randall-Sundrum models, we define the dimensionless variable, \( x = \rho/2\lambda \). This allow us to rewrite Eq. (4) in the standard form

\[ 3H^2 = \rho_{\text{eff}}, \]  
(7)

where the effective density is given by

\[ \rho_{\text{eff}} = 2\lambda x \left( 1 \pm x \right). \]  
(8)

The quadratic term in the \( x \) variable describes brane world correction on the cosmological equations. Taking the derivative of Eq. (7) with respect to cosmological time we obtain

\[ \dot{H} = -\frac{1}{2} \left[ \rho_{\text{eff}} + p_{\text{eff}} \right], \]  
(9)

where the effective pressure is given by

\[ p_{\text{eff}} = p \left( 1 \pm 2x \right) \pm 2\lambda x^2. \]  
(10)

In this sense, the effective state equation is given by

\[ \omega_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{1}{1 \pm x} \left[ \omega \left( 1 \pm 2x \right) \pm x \right], \]  
(11)

Meanwhile, the effective fluid satisfy the usual equation of conservation

\[ \dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + p_{\text{eff}}) = 0. \]  
(12)

The acceleration in terms of the variable \( x \) is given by

\[ \frac{\ddot{a}}{a} = -\frac{\lambda}{3} \left[ (1 + 3\omega)x \left( 1 \pm 2x \right) \pm 2x^2 \right]. \]  
(13)

This effective scenario allows us to describe the Randall-Sundrum cosmology in the standard way and we can fix or constraint the cosmological parameters. In the following we are discussing this effective scenarios and we particularize in the sign of the brane tension.
A. SCENARIOS FOR A POSITIVE BRANE TENSION

• Accelerated universes with an inflexion point

In the case of a brane with positive tension the modified Friedmann becomes,

\[ 3H^2 = 2\lambda x (1 + x). \]  

The acceleration is given in this case by

\[ \frac{\ddot{a}}{a} = -\frac{\lambda}{3} \left[(1 + 3\omega) x^2(1 + 2x) + 2x^2 \right]. \]  

If we search for an scenarios where exist an inflexion point between phases of acceleration and deceleration, it is necessary to impose the existence of an inflexion point, \( x_t \), where \( \ddot{a}(x_t) = 0 \). This point is given by

\[ x_t = -\frac{1}{2} \left(1 + \frac{3\omega}{1 + 3\omega} \right). \]

Since we must have \( x_t > 0 \), the equation of state of the bare matter is constrained to the range \(-\frac{2}{3} < \omega < -\frac{1}{3}\). Note that this implies that the inflexion point exists the strong energy condition is violated, i.e., \( 1 + 3\omega < 0 \).

The acceleration given in Eq. (15) can be rewritten in the form

\[ \frac{\ddot{a}}{a} = -\frac{2\lambda}{3} (2 + 3\omega)(x - x_t)x. \]  

Using Eq. (15) with a constant \( \omega \) we obtain a decreasing energy density of the bare fluid as the universe expands, then in early times the universe begins in a decelerated phase, reach the point of inflexion and then begin to accelerate. This scenarios resembles the future evolution of the universe at late times, as we observe today. In order to discuss the how behaves effective cosmological fluid we rewrite the effective equation of state, given by Eq. (11), in terms of the inflexion point, yielding

\[ 1 + 3\omega_{\text{eff}} = 2 \left(\frac{2 + 3\omega}{1 + x} \right) (x - x_t). \]  

Note that independent of the \( \omega \), at the inflexion point, \( x_t \), the effective fluids is \( \omega_{\text{eff}} = -1/3 \).

• Accelerated universes without an inflexion point

Other possible cosmological evolutions not include a inflexion point, which occurs whenever \( \omega > -1/3 \), or \( \omega < -2/3 \), driven an universe which evolves always with \( \ddot{a} < 0 \). Eq.(11) indicates that the effective fluid also has a positive pressure. In general, the effective fluid has higher pressure that the bare one. Since at early times we can assume \( x \sim 1 \), we obtain that \( \omega_{\text{eff}} \sim (3\omega + 1)/2 \). If, for example, the bare matter is radiation, the effective fluid is stiff matter at early times. At late times we see that \( \omega_{\text{eff}} \sim \omega \). Therefore, in general the effective fluid is diluting with the cosmic evolution. An universe with radiation as bare matter expand with deceleration and the effective matter evolves from stiff matter to radiation. An scenario derived in [9], for an holographic cosmology, leads to a FLRW universe with equation of state \( p = \rho \).

For \( \omega \leq -2/3 \), which includes phantom matter, evolves always with an positive acceleration. From Eq.(11) is clear to see that if \( \omega = -1 \), corresponding a cosmological constant in the bare matter, we obtain also \( \omega_{\text{eff}} = -1 \), for the effective fluid. Note that even for bare phantom matter, i. e., \( \omega < -1 \), the effective fluid never becomes a phantom fluid.

It is interesting to mention that if we have a phantom bare matter, the universe could begin with a low density, which is increasing as the cosmic times evolves.

B. SCENARIOS FOR A NEGATIVE BRANE TENSION

• Accelerated universes with an inflexion point

For this case the Friedmann equation is it is interesting to note that, this kind of modified Friedmann equation can be derived also from effective loop quantum cosmology [15]

\[ 3H^2 = 2\lambda x (1 - x), \]  

\[ x \sim 1, \]  

\[ \frac{\ddot{a}}{a} = -\frac{\lambda}{3} \left[(1 + 3\omega) x^2(1 - 2x) + 2x^2 \right]. \]  

If we search for an scenarios where exist an inflexion point between phases of acceleration and deceleration, it is necessary to impose the existence of an inflexion point, \( x_t \), where \( \ddot{a}(x_t) = 0 \). This point is given by

\[ x_t = \frac{1}{2} \left(1 + \frac{3\omega}{1 + 3\omega} \right). \]

Since we must have \( x_t < 0 \), the equation of state of the bare matter is constrained to the range \(\frac{2}{3} < \omega < \frac{1}{3}\). Note that this implies that the inflexion point exists the strong energy condition is violated, i.e., \( 1 + 3\omega > 0 \).

The acceleration given in Eq. (15) can be rewritten in the form

\[ \frac{\ddot{a}}{a} = -\frac{2\lambda}{3} (2 + 3\omega)(x - x_t)x. \]  

Using Eq. (15) with a constant \( \omega \) we obtain a decreasing energy density of the bare fluid as the universe expands, then in early times the universe begins in a decelerated phase, reach the point of inflexion and then begin to accelerate. This scenarios resembles the future evolution of the universe at late times, as we observe today. In order to discuss the how behaves effective cosmological fluid we rewrite the effective equation of state, given by Eq. (11), in terms of the inflexion point, yielding

\[ 1 + 3\omega_{\text{eff}} = 2 \left(\frac{2 + 3\omega}{1 - x} \right) (x - x_t). \]  

Note that independent of the \( \omega \), at the inflexion point, \( x_t \), the effective fluids is \( \omega_{\text{eff}} = -1/3 \).
where we have now the constraint $0 < x < 1$. Let us consider first scenarios where exist a inflexion point in the acceleration. Then in this cases the acceleration is given by

$$\frac{\ddot{a}}{a}(x) = -\frac{2\lambda}{3} (2 + 3\omega) x (x_t - x),$$

where the inflexion point, $x_t$, is given by

$$x_t = \frac{1}{2} \left( 1 + \frac{3\omega}{2 + 3\omega} \right).$$

The condition $x_t > 0$ implies the constraint $\omega > -1/3$ or $\omega < -2/3$. If we consider the first constraint then the strong energy condition $\rho + 3p = (1 + 3\omega) \rho > 0$ is satisfied by the bare matter. In order to discuss the effective behavior we shall to use the effective equation of state given, in this case, by

$$1 + 3\omega_{\text{eff}} = \frac{2}{1 - x} (2 + 3\omega) (x_t - x).$$

If $\omega > -1/3$ then $2 + 3\omega > 0$, so for $x < x_t$ the universe is decelerating , with $\omega_{\text{eff}} > -1/3$ and for $x > x_t$ is accelerating with $\omega_{\text{eff}} < -1/3$. According to this, whereas $\omega$ satisfied the strong energy condition, the effective fluid can be described by normal matter, quintessence or phantom matter. It is interesting to pointed out here an important different between the standard description and the effective description. It is well-known in the framework of standard cosmology that for universes filled with one fluid, satisfying the strong energy condition, the evolution shall be driven by just one accelerated phase . In the effective description for a brane with negative tension, where the bare matter satisfy, the strong energy condition we found the presence of both phases: accelerated and decelerated.

Since the variable $x$ is decreasing with the cosmic time, at the early times $x \leq x_t$, which indicates that initially we have an accelerated phase and then an decelerated phase when $0 \leq x \leq x_t$. Note that this scenario can occurs with a bare fluid with positive pressure, in other words, with normal matter.

It is direct to verify that

$$\frac{\ddot{a}(x)}{\ddot{a}(1/2)} = 2(1 + 3\omega)x \left( \frac{x}{x_t} - 1 \right),$$

and from the inequalities

$$\ddot{a}(0 < x < x_t) < \ddot{a}(x_t < x < 1/2) < \ddot{a}(1/2) < \ddot{a}(1/2 < x < 1),$$

where $\ddot{a}(0 < x < x_t) < 0$ and all the accelerations in the ranges indicates above are positives. Note that $\ddot{a}/a(1/2) = \lambda/6$ correspond to a the Sitter phase just at the point $x = 1/2$. In brief, the above results indicates that the effective matter behaves initially as normal matter following to quintessence, de Sitter and phantom matter.

An interesting and novel content can be found if we relax the strong energy condition and we consider that the condition $1 + 3\omega < 0$ is satisfied. From the condition $0 < x_t < 1$ we shall investigate the the constraint upon the the equation of state of the mater. If we rewrite Eq. (20) in the following form

$$x_t = \frac{1}{2} \left( \frac{|1 + 3\omega|}{|1 + 3\omega| - 1} \right),$$

we obtain that if $x_t > 0$ then $|1 + 3\omega| > 1$ and the constraint is $\omega < -2/3$. If $x_t < 1$ then $|1 + 3\omega| > 2$, which leads to the condition $\omega < -1$. Since both constraint must be satisfied we conclude that the matter confined on the brane corresponds to a phantom matter. It is easy to check for bare phantom matter, $\frac{1}{2} < x_t < 1$. In this scenario, the bare phantom matter drives an accelerated phase at the beginning, i.e., for $x < x_t$. After the inflexion point, $x > x_t$, the universe end in a decelerated phase. A remarkable point of this phantom scenario is that does not reach a future singularity. This kind of phantom behavior was discussed in Ref. [16] for an universe with arbitrary (non gravitational) interaction between the components of the cosmic fluid. It was found a wide region in the parameter space where the solutions are free of the big rip singularity, suggesting that phantom models without big rip singularity might be preferred by nature. In terms of the effective fluid, the universe is accelerating , with $\omega_{\text{eff}} < -1/3$ and is decelerating with $\omega_{\text{eff}} > -1/3$.

- **Accelerated universes without a inflexion point**

If the strong energy condition is violated, there are models which not include an inflexion point whenever $-1 \leq \omega \leq -1/3$. This indicate that the bare matter is a quintessence fluid. Notice that that the particular case of $\omega = -1$
implies \( \omega_{\text{eff}} = -1 \). In other words, if the bare matter is a cosmological constant, the effective fluids also behaves like cosmological constant during all the cosmic evolution driving an accelerated expansion.

Eq. (11) indicates that the effective fluid behaves during the cosmic evolution from \( x \sim 1 \), at early times, to \( x \sim 0 \) going from \( \omega_{\text{eff}} \ll w \) to \( \omega_{\text{eff}} \sim w \), respectively. Obviously, in this scenario the universe is always accelerated.

IV. DISCUSSION

In this article we have studied brane world cosmology in term of the effective fluid description on the brane.

We have found two types of scenarios for a positive and negative brane tension: those that show an inflexion point in the acceleration and those that present a continuous acceleration or deceleration during the cosmic evolution. For universes with a positive brane tension, the first scenario exist if the strong energy condition is violated for the bare matter. In this case the universe evolves initially with deceleration and then begins to accelerate at the inflexion point \( x_i \). This scenario resembles the observed evolution of the universe today from supernova data. In the decelerated phase, the effective fluid has the equation of state \( \omega_{\text{eff}} > -1/3 \). Note that this not exclude that at early times the effective matter could be, for example, dust. In the accelerated phase, the effective fluid has the equation of state \( \omega_{\text{eff}} < -1/3 \), which leads to the possibility to have cosmological constant or phantom as effective matter.

In the other scenario, for positive brane tension, there is no inflexion point for the acceleration. For bare matter with \( \omega > -1/3 \), which include bare matter with positive pressure, the effective fluid has an equation of state with higher pressure that bare one. For example, a universe filled with radiation as bare matter expand with deceleration and the effective matter evolves from stiff matter to radiation. For bare matter with \( \omega < -1/3 \), which include phantom matter, the universe evolves always with a positive acceleration. Although, the bare matter could be phantom matter, the effective matter never becomes a phantom fluid. In this case the early universe has low density.

For a negative brane tension, accelerated universes with an inflexion point, occurs for the both cases of a bare matter satisfying and violating the strong energy condition. If \( \omega > -1/3 \), the universe begins accelerating with \( \omega_{\text{eff}} < -1/3 \), and then is decelerating with \( \omega_{\text{eff}} > -1/3 \). According to this, whereas \( \omega \) satisfied the strong energy condition, the effective fluid can be described by normal matter, quintessence or phantom matter. At the effective level, the equation of state presents a crossing of the barrier \( \omega_{\text{eff}} = -1 \).

When the strong energy condition is violated the bare phantom matter drives an accelerated phase at the beginning, and after the inflexion point the universe end in a decelerated phase. A remarkable point of this phantom scenario is that does not reach a future singularity. As it was said before, in terms of the effective fluid, the universe is accelerating, with \( \omega_{\text{eff}} < -1/3 \) and is decelerating with \( \omega_{\text{eff}} > -1/3 \).

There are models which not include an inflexion point whenever \(-1 \leq \omega \leq -1/3\), i.e., when the bare matter is a quintessence fluid. Exist the particular case \( \omega = -1 \), since the effective equation of state is then \( \omega_{\text{eff}} = -1 \). In other words, if the bare matter is a cosmological constant, the effective fluids also behaves like cosmological constant during all the cosmic evolution driving an accelerated expansion. In general, the effective fluid behaves during the cosmic evolution going from \( \omega_{\text{eff}} \ll w \), at early times, to \( \omega_{\text{eff}} \sim w \), lately.

Within the possible scenarios described above, a remarkable evolution is found when the effective description is taken account This correspond to a universe with negative brane tension, with a bare fluid satisfying the strong energy condition, although the effective fluid could be quintessence or phantom. The effective fluid can cross the barrier \( \omega_{\text{eff}} = -1 \). Also it is of interest the case of a universe with radiation as bare matter, expanding decelerated and with the effective matter evolving from stiff matter to radiation. This scenario have been discussed in the framework of holographic cosmology [9].

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