EQUIVARIANT HOMOLOGICAL MIRROR SYMMETRY FOR $\mathbb{C}$ AND $\mathbb{C}P^1$

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Abstract. In this paper we define an equivariant Floer $A_\infty$ algebra for $\mathbb{C}$ and $\mathbb{C}P^1$ by using Cartan model. We then prove an equivariant homological mirror symmetry, i.e. an equivalence between an $A_\infty$ category of equivariant Lagrangian branes and the category of matrix factorizations of Givental’s equivariant Landau-Ginzburg potential function.

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1. Introduction

One form of the homological mirror symmetry for a toric Fano manifold $X$ relates the Fukaya category of $X$ to the dg-category of matrix factorizations of its Landau-Ginzburg mirror $F_X$, e.g. [12]. In this paper we study an equivariant version of this story.

Givental introduced the equivariant Landau-Ginzburg potential $F_X$ mirror to the toric manifold $X$, e.g. [17]. The potential is of the form

$$F_X = f_X - \lambda_1 \log g_1 - \cdots - \lambda_r \log g_r,$$

where $f_X$ is the non-equivariant Landau-Ginzburg mirror potential of $X$ and $\lambda_1, \ldots, \lambda_r$ are equivariant parameters corresponding to a basis of the Lie algebra of the torus $T$ acting on $X$, and $g_1, \ldots, g_r$ are some invertible functions.

Cho-Oh [7] defined a potential function by using Floer theory and showed that it coincides with the (nonequivariant) Landau-Ginzburg potential in the case of compact toric Fano manifolds based on the idea of Hori-Vafa [21], cf. [4] [16]. Kim-Lau-Zheng [25] showed that Givental’s equivariant Landau-Ginzburg mirror is recovered from the $T$-equivariant potential function when $X$ is a semi-projective semi-Fano toric manifold.

The equivariant Fukaya category is yet to be defined, but should consist of the Lagrangians preserved by the group action and of the equivariant Floer $A_\infty$ algebras.

There have been several approaches to the equivariant Floer theory. Zernik [31, 32] defined an equivariant Floer cohomology for possibly nonorientable Lagrangians using the Cartan model, and applied it to the study of the open Gromov-Witten theory of the real projective space inside the complex projective space. Kim-Lau-Zheng mentioned above employed the Morse model to study the disc potential.

In this paper we introduce our version of the equivariant Floer $A_\infty$ algebra using the Cartan model, and prove the following.

**Theorem 1.1** (Section 4, Theorems 5.1 and 5.3). Let $X$ be $\mathbb{C}P^1$ or $\mathbb{C}^n$ with the standard $S^1$ action. We have a cohomologically fully faithful $A_\infty$ functor from $\mathcal{F}_X$ to $\text{Br}(\mathcal{F}_X)$ whose image split-generates the triangulated category $[\text{Br}(\mathcal{F}_X)]$, where $\mathcal{F}_X$ is an equivariant Floer $A_\infty$ category and $\text{Br}(\mathcal{F}_X)$ is the dg-category of matrix factorizations of $F_X$.

For precise statements, see Section 5.

This paper is organized as follows. In Section 2 we review algebraic notions: $g$-differential spaces and its $A_\infty$ version, the Cartan models and the gapped filtered $A_\infty$ categories. In Section 3 we review some Floer theory, and then define an equivariant Floer $A_\infty$ algebra and an $A_\infty$ category of equivariant Lagrangian branes. In Section 4 we define the dg-category of matrix factorizations for Givental type potential functions. In Section 5 we formulate and prove the main theorems. In Appendix A, we compute the dimension of the Jacobian ring of the equivariant Landau-Ginzburg mirror of a semi-projective toric manifold (Theorem 6.7). In Appendix B, we introduce another version of categories of branes for equivariant Landau-Ginzburg potentials.

**Notations.**

- $\Lambda := \{ \sum_{i \geq 0} a_i T^{\lambda_i} | a_i \in \mathbb{C}, \lambda_0 < \lambda_1 < \cdots \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = \infty \}$ the Novikov field over $\mathbb{C}$,
- $\Lambda_0 := \{ \sum_{i \geq 0} a_i T^{\lambda_i} | a \in \Lambda, \lambda_0 \geq 0 \} \subset \Lambda$ the Novikov ring, and
- $\Lambda_+ := \{ \sum_{i \geq 0} a_i T^{\lambda_i} | a \in \Lambda, \lambda_0 > 0 \}$ the maximal ideal of the Novikov ring.

We define the valuation of elements of the Novikov field by its $T$-exponent of the initial term. Namely, $\text{val}(a) := \lambda_0$ for $a = \sum_{i \geq 0} a_i T^{\lambda_i} \in \Lambda^*$ with $a_0 \neq 0$ and set $\text{val}(0) := \infty$. We also use the following notations:

- $\hat{\otimes}_C : \mathbb{Z}/2\mathbb{Z}$-graded completed tensor product over $\mathbb{C}$.
- $|x|$ : the degree of a homogeneous element of a graded module.
- $|x'| := |x| - 1$ : the shifted degree.
Spec : the set of prime ideals with Zariski topology.
Spm : the set of maximal ideals with Zariski or analytic topology.
Sym : the graded symmetric tensor product.
[1] : the degree one shift of a graded module, i.e., for a graded module \( V \), the degree \( k \) part \( (V[1])^k \) is \( V^{k+1} \).

2. Algebraic preliminaries

In this section, let \( (g, [~,~]) \) be a complex Lie algebra (considered as a \( \mathbb{Z} \)-graded Lie algebra concentrated in degree 0).

2.1. Equivariant cohomology of \( g \)-differential spaces. In this subsection, we recall some basics on de Rham models of equivariant cohomology. See, e.g., [18] for more details. Let \( (M, \delta) \) be a cochain complex, i.e., \( M \) is a \( \mathbb{Z} \)-graded vector space over \( \mathbb{C} \) and \( \delta \) is a degree 1 endmorphism of \( M \) with \( \delta^2 = 0 \).

**Definition 2.1.** Let \( (M, \delta) \) be a cochain complex. Suppose that two linear maps \( i \) and \( L \) of degree \(-1\) and \( 0 \) respectively are given

\[
g \otimes M \to M : X \otimes x \mapsto iXx \\
g \otimes M \to M : X \otimes x \mapsto LXx
\]

which satisfy the following \((X, Y \in g)\):

\[
\begin{align*}
\delta LX - LX \delta &= 0, \\
\delta iX + iX \delta &= LX, \\
LXL_Y - LYL_X &= L_{[X,Y]}, \\
iXLY - iYLX &= i_{[X,Y]}, \\
iXiY + iyixX &= 0.
\end{align*}
\]

Then \((M, \delta, L, i)\) is called a \( g \)-differential space (we denote it briefly by \( M \)).

For \( g \)-differential spaces \( M, N \), we easily see that \( M \otimes N \) naturally a \( g \)-differential space. For a \( g \)-differential space \( M \), we set

\[
\begin{align*}
M^0 &:= \{ x \in M \mid L_Xx = 0 \text{ for all } X \in g \}, \\
M_{hor} &:= \{ x \in M \mid i_Xx = 0 \text{ for all } X \in g \}, \\
M_{bas} &:= \{ x \in M \mid L_Xx = 0, i_Xx = 0 \text{ for all } X \in g \}.
\end{align*}
\]

Then \( M_{hor} \) is closed under \( L_X \), and \( M^0 \) and \( M_{bas} \) are closed under \( \delta \). For a \( \mathbb{C} \)-linear subspace \( N \subseteq M \), the intersection \( N \cap M^0 \) is also denoted by \( N^0 \).

Let \( \wedge g \) be the (\( \mathbb{Z} \)-graded) symmetric algebra \( \text{Sym}(g[1]) \), \( \wedge g^\vee \) be \( \text{Sym}(g^\vee[-1]) \), and \( Sg^\vee \) be \( \text{Sym}(g^\vee[-2]) \). Choose a basis \( e_1, e_2, \ldots, e_r \) of \( g \) and let \( e^1, e^2, \ldots, e^r \) be the dual basis of \( g^\vee \). Let \( c_{jk} \) be the structure constants of \( g \), i.e., \( c_{jk} := \langle e^i, [e_j, e_k] \rangle \). There exists a natural non-degenerate bilinear paring \( \langle \cdot, \cdot \rangle : \wedge g^\vee \otimes \wedge g \to \mathbb{C} \). For \( X_1, \ldots, X_k \in g, x_1, \ldots, x_k \in g^\vee \), we have

\[
\langle x^1 \wedge \cdots \wedge x^k, X_1 \wedge \cdots \wedge X_k \rangle = \det((\langle x^i, X_j \rangle)_{i,j=1}^k).
\]

The element \( e^i \in g^\vee \) is also denoted by \( \theta^i \) if it is considered as an element of \( \wedge g^\vee \).

We introduce a structure of a \( g \)-differential space on \( \wedge g^\vee \). We define

\[
\delta : \wedge g^\vee \to \wedge g^\vee
\]

by the formula \( \delta(\theta^i) = -\frac{1}{2} \sum_{j,k} c_{jk} \theta^j \theta^k \). (Note that this \( \delta \) comes from the dual of the Lie bracket.)
For \(X, Y \in \mathfrak{g}\) and \(x \in \mathfrak{g}^\vee\), set
\[
\langle L_X x, Y \rangle := -\langle x, [X, Y] \rangle, \quad i_X x = (X, x).
\]
Extending \(L_X\) and \(i_X\) by the Leibniz rule, we obtain two linear maps
\[
L_X, i_X : \wedge \mathfrak{g}^\vee \to \wedge \mathfrak{g}^\vee.
\]
We see that \((\wedge \mathfrak{g}^\vee, \delta, L, i)\) is a \(\mathfrak{g}\)-differential space. We note that \(\delta\) is also written as \(\frac{1}{2} \sum_i \theta^i \circ L_{e_i}\), where \(\theta^i\) is the left multiplication by \(\theta^i\).

We next introduce the Weil algebra. Set \(W_\mathfrak{g} : = S(d\mathfrak{g}^\vee \otimes \wedge \mathfrak{g}^\vee)\), which is naturally a \(\mathbb{Z}\)-graded commutative algebra. For simplicity, \(1 \otimes e^i \in W_\mathfrak{g}\) is also denoted by \(\theta^i\) and \(e^i \otimes 1 \in W_\mathfrak{g}\) is denoted by \(\overline{e}^i\). Set \(F^i : = \overline{e}^i + \frac{1}{2} \sum_{j,k} c_{jk}^i \theta^j \theta^k\), then \(\theta^1, \ldots, \theta^r, F^1, \ldots, F^r\) also generate \(W_\mathfrak{g}\). We note that \(\theta^i = 1, \overline{e}^i = 2\), and \(|F^i| = 2\). We define
\[
\delta_W (\theta^i) = \overline{e}^i = F^i - \frac{1}{2} \sum_{j,k} c_{jk}^i \theta^j \theta^k \in W_\mathfrak{g}, \quad \delta_W (F^i) = 0,
\]
and extend it to \(\delta_W : W_\mathfrak{g} \to W_\mathfrak{g}\) by the Leibniz rule. From the definition, we easily see that \((W_\mathfrak{g}, \delta_W)\) is acyclic. By using the Jacobi identity, we have
\[
\delta_W (F^i) = \sum_{j,k} c_{jk}^i F^j \theta^k.
\]
Similarly, we define two linear maps \(L, i : \mathfrak{g} \otimes W_\mathfrak{g} \to W_\mathfrak{g}\) by
\[
i_{e_j} (\theta^i) := \delta^i_j, \quad i_{e_j} (F^i) := 0, \quad L_{e_j} (\theta^i) := -\sum_k c_{jk}^i \theta^k, \quad L_{e_j} (F^i) := -\sum_k c_{jk}^i F^k.
\]
Then \((W_\mathfrak{g}, \delta_W, L, i)\) gives a \(\mathfrak{g}\)-differential space, which is called a Weil algebra. By definition, we see that \((W_\mathfrak{g})_{\text{hor}} \cong S_\mathfrak{g}^\vee\), which is freely generated by \(F^1, F^2, \ldots, F^r \in (W_\mathfrak{g})_{\text{hor}}\) and closed under \(L_X\).

Let \(M\) be a \(\mathfrak{g}\)-differential space. Then \(\mathfrak{g} \otimes W_\mathfrak{g}\) is also a \(\mathfrak{g}\)-differential space and its differential is also denoted by \(\delta_W\). Set \(C^*_\mathfrak{g}(M) := (\mathfrak{g} \otimes W_\mathfrak{g})_{\text{bas}}\). This is a \(\mathbb{Z}\)-graded \((\mathfrak{g} \otimes W_\mathfrak{g})_{\text{bas}} \cong (S\mathfrak{g}^\vee)^{\mathfrak{g}}\)-module and \(\delta_W\) is compatible with this module structure.

**Definition 2.2.** The cochain complex \((C^*_\mathfrak{g}(M), \delta_W)\) is called a Weil model and its cohomology is called an equivariant cohomology of \(M\). The equivariant cohomology is an \((S\mathfrak{g}^\vee)^{\mathfrak{g}}\)-module which is denoted by \(H^*_\mathfrak{g}(M)\).

We next define the Cartan model of equivariant cohomology of a \(\mathfrak{g}\)-differential space \(M\). Set \(C^*_\mathfrak{g}(M) := (\mathfrak{g} \otimes (W_\mathfrak{g})_{\text{hor}})^{\mathfrak{g}}\), which is a \(\mathbb{Z}\)-graded \((S\mathfrak{g}^\vee)^{\mathfrak{g}}\)-module. To simplify notation, for \(x \in W_\mathfrak{g}\), the left multiplication by \(x\) is also denoted by \(x\). We define a differential \(\delta_{\text{Car}} : C^*_\mathfrak{g}(M) \to C^*_\mathfrak{g}(M)\) by
\[
\delta_{\text{Car}} := \delta \otimes 1 - \sum_i i_{e_i} \otimes F^i,
\]
which is compatible with the \(\mathbb{Z}\)-graded \((S\mathfrak{g}^\vee)^{\mathfrak{g}}\)-module structure.

**Definition 2.3.** The cochain complex \((C^*_\mathfrak{g}(M), \delta_{\text{Car}})\) is called a Cartan model.

Set \(\gamma := \sum_j i_{e_j} \otimes \theta^j \in \text{End}(\mathfrak{g} \otimes W_\mathfrak{g})\). This is a degree 0 nilpotent operator and we define an automorphism \(\phi := \exp(\gamma)\) (called a Mathai-Quillen morphism). This morphism \(\phi\) is compatible with the \(\mathbb{Z}\)-graded \(W_\mathfrak{g}\)-module structure on \(\mathfrak{g} \otimes W_\mathfrak{g}\).
Theorem 2.4 (See, e.g., [23] Theorem 3.2 and [18] Chapter 4). The image of the Weil model $C^\infty_W(M)$ by $\phi$ is $C^\infty_{\text{Cart}}(M)$ and $\delta_{\text{Cart}} \circ \phi = \phi \circ \delta_V$. Hence $\phi$ gives an isomorphism between the equivariant cohomology $H^*_\mu(M)$ and the cohomology of the Cartan model as $(\mathcal{S}g^\vee)^\#$-modules.

2.2. Preliminaries on gapped filtered $A_\infty$ categories. Let $G$ be a discrete submonoid of $2\mathbb{Z} \times \mathbb{R}_{\geq 0}$. We denote by $\mu: G \to 2\mathbb{Z}$ the first projection and by $\omega: G \to \mathbb{R}_{\geq 0}$ the second projection. Suppose that for each $E \in \mathbb{R}_{\geq 0}$

$$\# \{ \beta \in G \mid \omega(\beta) \leq E \} < \infty.$$ Let $R$ be a $\mathbb{Z}$-graded commutative algebra over $\mathbb{C}$. We first recall the notion of an $A_\infty$ category (over $R$).

Definition 2.5. A $\mathbb{Z}$-graded unital $A_\infty$ category $(\mathcal{A}, \{m_k^\mathcal{A}\})$ over $R$ consists of the following data (the first three data is simply denoted by $\mathcal{A}$):

- $\text{Ob} \mathcal{A}$: a set of objects,
- $\mathcal{A}(A, B)$: $\mathbb{Z}$-graded modules over $R$ for pairs $(A, B) \in (\text{Ob} \mathcal{A})^2$,
- $1_A$: degree 0 elements of $\mathcal{A}(A, A)$ for $A \in \text{Ob} \mathcal{A}$,
- $R$-module maps $m_k^\mathcal{A}$ ($k \geq 1$) of degree 1 for $(k + 1)$-tuples $(A_0, ..., A_k) \in (\text{Ob} \mathcal{A})^{k+1}$,

$$m_k^\mathcal{A}: \mathcal{A}(A_0, A_1)[1] \otimes \cdots \otimes \mathcal{A}(A_{k-1}, A_k)[1] \to \mathcal{A}(A_0, A_k)[1]$$

such that they satisfy the following relations:

- $A_\infty$ relations:

$$\sum_{k_1 + k_2 = k+1 \atop 1 \leq k_1, k_2 \leq k \atop 0 \leq i < k_1 + k_2 \leq k} (-1)^{|x_i|} m_{k_1}^\mathcal{A}(x_1, ..., x_{i+k_1}, m_{k_2}^\mathcal{A}(x_{i+1}, ..., x_{i+k_2}), x_{i+k_2+1}, ..., x_k) = 0$$

where $x_i$ are homogeneous elements of $\mathcal{A}(A_{i-1}, A_i)$ and $\# = |x_1'| + \cdots + |x_i'|$.

- Unitality:

$$m_2^\mathcal{A}(1_A, x) = (-1)^{|x|} m_2^\mathcal{A}(x, 1_A) = x \text{ if } k = 2,$$

$$m_k^\mathcal{A}(..., 1_A, ...) = 0 \text{ if } k \neq 2.$$ For simplification, a $\mathbb{Z}$-graded unital $A_\infty$ category $(\mathcal{A}, \{m_k^\mathcal{A}\})$ is also denoted by $\mathcal{A}$.

The morphisms $m_k^\mathcal{A}$ naturally induce degree $2 - k$ morphisms

$$\mathcal{A}(A_0, A_1) \otimes \cdots \otimes \mathcal{A}(A_{k-1}, A_k) \to \mathcal{A}(A_0, A_k)$$

which are also denoted by $m_k^\mathcal{A}$.

We next recall the notion of a unital $G$-gapped filtered $A_\infty$ category.

Definition 2.6 (cf. [13]). A unital $G$-gapped filtered $A_\infty$ category $(\mathcal{A}, \{m_k, \beta\})$ consists of the following data:

- $\mathcal{A}$: a unital $\mathbb{Z}$-graded $A_\infty$ category over $R$.
- $R$-module morphisms of degree $1 - \mu(\beta)$ with $m_{k, 0} = m_k$ (especially $m_{0, 0} = 0$) for $A_0, ..., A_k \in \text{Ob} \mathcal{A}, k \in \mathbb{Z}_{\geq 0}, \beta \in G$:

$$m_{k, \beta}: \mathcal{A}(A_0, A_1)[1] \otimes \cdots \otimes \mathcal{A}(A_{k-1}, A_k)[1] \to \mathcal{A}(A_0, A_k)[1],$$

which naturally induce degree $2 - k - \mu(\beta)$ morphisms $m_{k, \beta}$ similar to (1), such that they satisfy the following relations:
For \( A_0, \ldots, A_k \in \text{Ob} \mathcal{A} \) and homogeneous elements \( x_i \in \mathcal{A}(A_{i-1}, A_i) \), the morphisms \( m_{k,\beta} \) satisfy the \( A_\infty \) relations for \((k, \beta) \in \mathbb{Z}_{\geq 0} \times G\):

\[
\sum_{k_1 + k_2 = k+1, \ 0 \leq k_1, k_2, \ 0 \leq k - k_2, \ \beta_1 + \beta_2 = \beta} (-1)^{\#} m_{k_1, \beta_1}(x_1, \ldots, x_{i_1}, m_{k_2, \beta_2}(x_{i_1+1}, \ldots, x_{i_1+k_2}), x_{i_1+k_2+1}, \ldots, x_k) = 0
\]

where \( \# = |x_1'| + \cdots + |x_i'| \).

- The units \( 1_A \) satisfy

\[
m_{k,\beta}(\cdots, 1_A, \cdots) = 0 \quad \text{if} \ (k, \beta) \neq (2, 0), \tag{2}
\]

\[
m_{2,0}(1_A, x) = (-1)^{|x|} m_{2,0}(x, 1_A) = x \quad \text{if} \ (k, \beta) = (2, 0). \tag{3}
\]

We simply say an unital gapped filtered \( A_\infty \) category when we don’t specify \( G \).

To simplify notation \( m_{1,0} \) is also denoted by \( \delta \), then we have \( \delta^2 = 0 \).

**Remark 2.7.** We obtain a \( \mathbb{Z}/2\mathbb{Z} \)-graded unital curved \( A_\infty \) category \((A, \{m_k\})\) over \( \Lambda \) from a gapped filtered \( A_\infty \) category \((\mathcal{A}, \{m_{k,\beta}\})\) over \( \mathbb{C} \) by taking

\[
\text{Ob} A := \text{Ob} \mathcal{A},
\]

\[
A(A, B) := \mathcal{A}(A, B) \otimes_{\mathcal{C}} \Lambda,
\]

\[
1_A := 1_A \in A(A, A),
\]

\[
m_k := \sum_{\beta \in G} \frac{\omega(\beta)}{2\pi} m_{k,\beta}.
\]

### 2.3. \( \mathfrak{g} \)-differential gapped filtered \( A_\infty \) categories

Let \( \mathfrak{g} \) be a complex Lie algebra.

**Definition 2.8.** Let \((\mathcal{A}, \{m_{k,\beta}\})\) be a unital \( G \)-gapped filtered \( A_\infty \) category over \( \mathbb{C} \). Suppose that \( \mathcal{A}(A, B) \) is a \( \mathfrak{g} \)-differential space with the differential \( \delta = m_{1,0} \) for each \( A, B \in \text{Ob} \mathcal{A} \). These data is called a \( \mathfrak{g} \)-differential unital \( G \)-gapped filtered \( A_\infty \) category over \( \mathbb{C} \) if they satisfy the following equation for each \( k \in \mathbb{Z}_{\geq 0} \) and \( \beta \in G \) with \((k, \beta) \neq (1, 0)\):

\[
i_X m_{k,\beta}(x_1, \ldots, x_k) + \sum_{i=1}^{k} (-1)^{|x_1'|+\cdots+|x_{i-1}'|} m_{1,\beta}(x_1, \ldots, i_X x_i, \ldots, x_k) = 0. \tag{4}
\]

Here \( X \in \mathfrak{g} \) and \( x_i \in \mathcal{A}(A_{i-1}, A_i) \) are homogeneous elements.

**Proposition 2.9.** For \( k \in \mathbb{Z}_{\geq 0}, \beta \in G, X \in \mathfrak{g}, \) and homogeneous elements \( x_i \in \mathcal{A}(A_{i-1}, A_i) \) \((i = 1, 2, \ldots, k)\), we have

\[
L_X m_{k,\beta}(x_1, \ldots, x_k) = \sum_{i=1}^{k} m_{k,\beta}(x_1, \ldots, L_X x_i, \ldots, x_k). \tag{5}
\]

**Proof.** For \((k, \beta) = (1, 0)\), this proposition follows from the definition of \( \mathfrak{g} \)-differential space. We assume \((k, \beta) \neq (1, 0)\). Set

\[
\delta(x_1 \otimes \cdots \otimes x_k) := \sum_{i=1}^{k} (-1)^{|x_1'|+\cdots+|x_{i-1}'|} x_1 \otimes \cdots \otimes \delta x_i \otimes \cdots \otimes x_k
\]

\[
i_X(x_1 \otimes \cdots \otimes x_k) := \sum_{i=1}^{k} (-1)^{|x_1'|+\cdots+|x_{i-1}'|} x_1 \otimes \cdots \otimes i_X x_i \otimes \cdots \otimes x_k
\]
We also set
\[
\hat{m}_{k,\beta}^2(x_1 \otimes \cdots \otimes x_k) := \sum (-1)^{|x_1|+\cdots+|x_k|} m_{k,\beta_1} (x_1 \otimes \cdots \otimes m_{k_2,\beta_2} (x_{i+1} \otimes \cdots \otimes x_{i+k_2}) \otimes \cdots \otimes x_k),
\]
where the sum is taken over the set
\[
\{(k_1, k_2, \beta_1, \beta_2, i) \mid k_1+k_2=k+1, \beta_1+\beta_2=\beta, 0 \leq i \leq k-k_2, (k_1, \beta_1) \neq (1,0), (k_2, \beta_2) \neq (0,1)\}.
\]
By definition, we have \(\delta \circ m_{k,\beta} + m_{k,\beta} \circ \delta + \hat{m}_k^2 = 0\). Then we see that
\[
d_i X m_{k,\beta} = -\delta m_{k,\beta} i_X = m_{k,\beta} \delta i_X + \hat{m}_k^2 i_X,
\]
\[
i_X \delta m_{k,\beta} = -i_X m_{k,\beta} \delta - i_X \hat{m}_k^2 = m_{k,\beta} \delta i_X - \hat{m}_k^2 i_X.
\]
Hence we have \(L_X m_{k,\beta} = m_{k,\beta} (\delta i_X + i_X \delta)\). Combined with the equation
\[
(\delta i_X + i_X \delta) (x_1 \otimes \cdots \otimes x_k) = \sum_{i=1}^k x_1 \otimes \cdots \otimes L_X x_i \otimes \cdots \otimes x_k,
\]
we obtain the desired equation. \(\square\)

Since \(i_X \circ m_{2,0} + m_{2,0} \circ (i_X \otimes \text{id} + \text{id} \otimes i_X) = 0\) for \(X \in \mathfrak{g}\), we have
\[
i_X (1_A) = 0, \quad L_X (1_A) = (\delta \circ i_X + i_X \circ \delta) (1_A) = 0. \tag{6}
\]

Let \((\overline{A}, \{m_{k,\beta}\})\) be a \(\mathfrak{g}\)-differential unital G-gapped filtered \(A_{\infty}\) category over \(\mathbb{C}\). Let \(\bullet\) be \(\text{W}\) or \(\text{Car}\). By Equations (5), we have \(1_A \in C^0_\bullet (\overline{A}, A, A)\).

For \(A_0, A_1, \ldots, A_k \in \text{Ob} \overline{A}\) and \((k, \beta) \neq (1,0)\), let \(m_{k,\beta}^\bullet\) be the trivial extension of \(m_{k,\beta}\) to
\[
\begin{align*}
C_\bullet (\overline{A}(A_0, A_1))[1] \otimes \cdots \otimes C^\bullet (\overline{A}(A_{k-1}, A_k))[1],
\end{align*}
\]
\(\text{i.e., for homogeneous elements } x_i \in \overline{A}(A_{i-1}, A_i) \text{ and } f_j \in W \mathfrak{g}\)
\[
m_{k,\beta}^\bullet (x_1 \otimes f_1, \ldots, x_k \otimes f_k) := (-1)^{\sum_{i<j} f_j |x_i|} m_{k,\beta} (x_1, \ldots, x_k) \otimes (f_1 \cdots f_k)
\]
and set \(m_{1,0}^\bullet := \delta_\bullet\).

Then \(C_\bullet (\overline{A})\) consists of the following data:
- a set of objects \(\text{Ob} C_\bullet (\overline{A}) := \text{Ob} \overline{A}\)
- \(\mathbb{Z}\)-graded \((S^0 \mathfrak{g})^\theta\)-modules \(C_\bullet (\overline{A})(A, B) := C^\bullet (\overline{A}(A, B))\) for \(A, B \in \text{Ob} \overline{A}\)
- degree 0 morphisms \(1_A \in C_0 (\overline{A}(A, A))\) for \(A \in \text{Ob} \overline{A}\)
- the homomorphisms \(m_{k,\beta}^\bullet\).

By using Equations (4), (5), and (6), we easily see that \(m_{k,\beta}^\bullet\) give a unital G-gapped filtered \(A_{\infty}\) algebra structure over \((S^0 \mathfrak{g})^\theta\) on \(C_\bullet (\overline{A})\). Moreover the Mathai-Quillen morphism \(\phi\) satisfies
\[
\phi \circ m_{k,\beta}^W = m_{k,\beta}^{\text{Car}} \circ (\phi \otimes \cdots \otimes \phi)
\]
i.e., \(\phi\) gives an \(A_{\infty}\) functor.

Suppose that \(\mathfrak{g}\) is abelian. Let \(\lambda\) be a \(\mathbb{C}\)-algebra homomorphism from \((W \mathfrak{g})_{\text{her}} \cong S^0 \mathfrak{g}^\vee\) to \(A_0\), which is called an equivariant parameter. By evaluating \(m_{k,\beta}^{\text{Car}}\) at \(\lambda\), we will define a \(\mathbb{Z}/2\mathbb{Z}\)-graded unital curved \(A_{\infty}\) category \((A^\theta, \{m_{k,\lambda}^\theta\})\) over \(A\). We define \(\overline{A}^\theta\) by
- a set of objects \(\text{Ob} \overline{A}^\theta := \text{Ob} \overline{A}\)
- \(\mathbb{Z}\)-graded modules \(\overline{A}^\theta(A, B) := \overline{A}(A, B)^\theta\)
- degree 0 elements \(1_A \in \overline{A}^\theta(A, A)\) (recall Equations (6)).
By Equation (4), the restrictions of \( m_{k,\beta} \) to the \( g \)-invariant part give a unital \( G \)-gapped filtered \( A_\infty \) structure \( m_{k,\beta}^g \) on \( \mathcal{A} \). This unital \( G \)-gapped filtered \( A_\infty \) algebra induces a \( \mathbb{Z}/2\mathbb{Z} \)-graded unital curved \( A_\infty \) structure \( m_{k,\beta}^g \) on \( \mathcal{A}^\theta \), where \( \mathcal{A}^\theta(A, B) := \overline{\mathcal{A}}(A, B) \otimes \Lambda \) and \( m_{k,\beta}^g := \sum_{\beta \in G} T^{(\beta)}_{m_{k,\beta}} \). Set

\[
m_k^\lambda := \begin{cases} m_k^g & \text{if } k \neq 1 \\ m_k^g - \sum_{i=1}^r \lambda(F^i) i_{e_i} & \text{if } k = 1.
\end{cases}
\]

By Equations (11) and (5), we see that \( (\mathcal{A}^\theta, \{m_k^\lambda\}) \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded unital curved \( A_\infty \) category over \( \Lambda \). We note that this \( \mathbb{Z}/2\mathbb{Z} \)-graded curved \( A_\infty \) category is induced from a \( (\mathbb{Z}-\text{graded}) \) \( G' \)-gapped filtered \( A_\infty \) category \( (\overline{\mathcal{A}}, \{m_{k,\beta}^g\}) \) over \( \mathbb{C} \) for some \( G' \supseteq G \).

We introduce a notion of bounding cochains for unital \( g \)-differential gapped filtered \( A_\infty \) categories. Let \( (\mathcal{A}, \{m_{k,\beta}\}) \) be a unital \( g \)-differential \( G \)-gapped filtered \( A_\infty \) category over \( \mathbb{C} \) and \( b_{+,i} \) are odd elements of \( \mathcal{A}(A_i, A_i) \otimes \Lambda_+ \) \((A_0, \ldots , A_k) \in \text{Ob} \mathcal{A}\). We recall that the following operators

\[
m_k^{b_+}(x_1, \ldots , x_k) := \sum_{l_0 + \cdots + l_k = l} m_{k+i}(b_{+,0}, \ldots , b_{+,0}, x_1, b_{+,1}, \ldots , b_{+,1}, \ldots , x_k, b_{+,k}, \ldots , b_{+,k})
\]

also satisfy the \( A_\infty \) relations and unitality with units \( 1_{A_0}, \ldots , 1_{A_k} \) (cf. [13, Proposition 1.20]). Suppose that

\[
b_{+,i} \in \mathcal{A}(A_i, A_i) \otimes \Lambda_+
\]

and there exist \( c_0, \ldots , c_k \in \text{Hom}_\mathbb{C}(g, \Lambda_+) \) such that

\[
i_X(b_{+,0}) = c_0(X) \cdot 1_{A_0}, \ldots , i_X(b_{+,k}) = c_k(X) \cdot 1_{A_k}
\]

for all \( X \in g \) and \( i = 0, 1, \ldots , k \), where \( \mathcal{A}(A_i, A_i) \) denotes the degree one part of \( \mathcal{A}(A_i, A_i) \). If \( c_0 = c_1 = \cdots = c_k \), then, by using Equations (2) and (3), we easily see that the operators \( m_k^{b_+} \) also satisfy the Equation (4) (for some monoid \( G' \)). Hence, by choosing a finite collection \( \{(A_i, b_{+,i})\} \) \((i = 0, 1, \ldots , k) \) as objects of \( \mathcal{A}^{bc} \) and putting

\[
\mathcal{A}^{bc}((A_i, b_{+,i}),(A_j, b_{+,j})) = \begin{cases} \mathcal{A}(A_i, A_j) & \text{if } c_i = c_j \\ 0 & \text{if } c_i \neq c_j.
\end{cases}
\]

we obtain a unital \( g \)-differential gapped filtered \( A_\infty \) category \( (\mathcal{A}^{bc}, \{m_{k,\beta}^{b_+}\}) \).

**Definition 2.10.** Let \( (\mathcal{A}, \{m_{k,\beta}\}) \) be a unital \( g \)-differential \( G \)-gapped filtered \( A_\infty \) category over \( \mathbb{C} \). Let \( A \in \text{Ob} \mathcal{A} \) and \( b_+ \) be an element of \( \mathcal{A}(A, A) \otimes \Lambda_+ \) with \( \delta b_+ = 0 \). Suppose that \( i_X(b_+) \in \Lambda_+ \cdot 1_A \) for all \( X \in g \). The element \( b_+ \) is called a bounding cochain if \( m_0^{b_+}(1) \in \Lambda_+ \cdot 1_A \).

Choose an equivariant parameter \( \lambda \) and suppose that \( b_{+,i} \) are bounding cochains. Then we have \( L_X b_{+,i} = 0 \) and we can deform \( (\mathcal{A}^{bc}, \{m_{k,\beta}^{b_+}\}) \) by the same way as Equation (7). Thus we obtain a unital gapped filtered \( A_\infty \) category \( (\mathcal{A}^{\theta}, \{m_{k,\beta}^{b_+,\lambda}\}) \) (for some monoid \( G' \)). Let \( (\mathcal{A}^\theta, \{m_{k,\beta}^{b_+,\lambda}\}) \) be the associated curved \( A_\infty \) category. By the definition, the curvature term \( m_0^{b_+,\lambda}(1) \in \mathcal{A}^\theta(A_i, A_i) \) is

\[
\sum_{k=0}^{\infty} m_k^\theta(b_{+,i}, \ldots , b_{+,i}) - \sum_{j=1}^r \lambda(F^j) i_{e_j}(b_{+,i}).
\]
Since \( b_{+,i} \) is a bounding cochain, we have

\[ m^{b_+,\lambda}_0(1) \in \Lambda_+ \cdot 1_{A_+}. \]

Finally, we obtain a \( \mathbb{Z}/2\mathbb{Z} \)-graded (uncurved) unital \( A_\infty \) category \( (\mathcal{A}_\rho, \{ m^{b_+,\lambda}_k \}) \) over \( \Lambda \) by setting

\[
\mathcal{A}_\rho^g((A_i, b_{+,i}), (A_j, b_{+,j})) = \begin{cases} 
\mathcal{A}_\rho((A_i, b_{+,i}), (A_j, b_{+,j})) & \text{if } m^{b_+,\lambda}_0(1) = m^{b_+,\lambda}_0(1) \\
0 & \text{if } m^{b_+,\lambda}_0(1) \neq m^{b_+,\lambda}_0(1)
\end{cases}
\]

and

\[
m^{b_+,\lambda}_k(x_1, x_2, \ldots, x_k) = \begin{cases} 
m^{b_+,\lambda}_k(x_1, x_2, \ldots, x_k) & \text{if } k \geq 1 \text{ and } x_1 \neq 0, \ldots, x_k \neq 0 \\
0 & \text{if } k = 0 \text{ or } x_i = 0 \text{ for some } i.
\end{cases}
\]

Here \( x_i \) are homogeneous elements of morphism spaces of \( \mathcal{A}_\rho^g \).

3. Equivariant Floer theory

In this section we shall formulate a version of the equivariant Floer theory using the de Rham model. We first review some basics on the equivariant cohomology to fix conventions.

3.1. Equivariant cohomology. Let \( G \) be a compact connected Lie group, \( \text{Lie}(G) \) be its Lie algebra and \( g := \text{Lie}(G) \otimes \mathbb{C} \) be its complexification. We take a basis \( e_1, \ldots, e_r \) of \( \text{Lie}(G) \), regarded also as the basis of \( g \) over \( \mathbb{C} \) and its dual basis \( e^1, \ldots, e^r \) of \( g^* \). Then \( G \) acts on \( g^* \) via \((g \cdot f)(X) := f(g^{-1} \cdot X)\) where \( g \in G \) and \( X \in g \).

Let \( M \) be an \( n \)-dimensional \( G \)-manifold, i.e. \( G \) acts smoothly on \( M \) from the left and let \( \rho \) be a \( \mathbb{C} \)-local system on \( M \). For \( X \in g \), \( X = \Gamma(TM) \) is defined by \( \frac{d}{dt} \exp(tX) \cdot p \) for \( p \in M \).

**Definition 3.1.** Let \( \Omega^*(M; \rho) \) be the de Rham complex with coefficients in \( \rho \) with the differential locally defined as \( d(\alpha \otimes s) := d\alpha \otimes s \), where \( \alpha \) is a complex valued form and \( s \) is a flat section of \( \rho \). We simply denote by \( \Omega^*(M) \) the de Rham complex of \( \mathbb{C} \)-valued differential forms with the trivial local system of rank 1.

Let \( i_X : \Omega^*(M; \rho) \to \Omega^{*-1}(M; \rho) \) be the interior product locally defined by \( i_X(\alpha \otimes s) := i_X(\alpha) \otimes s \) for \( X \in g \), and let \( L_X : \Omega^*(M; \rho) \to \Omega^*(M; \rho) \) be the Lie derivative \( L_X := di_X + i_X d \). Then the quadruple \((\Omega^*(M; \rho), \delta, L, i)\) forms a \( g \)-differential space, where \( \delta \) is the differential, \( L_X := L_X \) and \( i_X := i_X \).

The equivariant cohomology of \( M \) is defined to be the cohomology of the Weil model of this \( g \)-differential space: \( H^*(G) := H^*(\Omega^*(M; \rho), \delta_W) \). See Atiyah-Bott [1, Theorem 4.13] for the relationship between the equivariant cohomologies defined by the Weil model and by the homotopy quotient \( EG \rtimes G \).

Let \( M \) be a compact manifold with corners. There are several different formulations of manifolds with corners. In this paper we use the formulation by Joyce [23]. We call \( M \) a G-manifold with corners if it is a manifold with corners equipped with a smooth \( G \)-action. When \( M \) is oriented, \( \partial M \) can be equipped with an orientation so that the orientation of the boundary of the sides of the following coincide: \( T_x M = \mathbb{R}v + T_x \partial M \) where \( x' \in \partial M \), \( x = i(x') \) via \( i : \partial M \to M \) and \( v \) is an outward vector at \( x \) (cf. [23, Convention 7.2]). The smooth differential \( r \)-forms on \( M \) with coefficients in \( \rho \) are defined to be the smooth sections \( M \to \bigwedge^r T^* M \otimes \rho \). We denote by \( \eta |_{\partial M} \) the pull-back of a form \( \eta \) on \( M \) via \( i : \partial M \to M \).

Let \( M \) and \( N \) be compact \( G \)-manifolds with corners, \( f : M \to N \) be a smooth \( G \)-equivariant map and \( \rho \) be a \( \mathbb{C} \)-local system on \( N \). Note that \( f^*(i_X \eta) = i_{f^*X}^* f^* \eta \), \( f^*(L_X \eta) = L_{f^*X} f^* \eta \) hold and therefore \( f \) induces a homomorphism of complexes \( \Omega^*(N; \rho)^G \to \Omega^*(M; f^* \rho)^G \) which we also denote by \( f^* \) abusing notation.

Assume further that \( f \) is submersive (i.e. \( f \) restricted to any stratum is submersive, [23, Definition 3.2 (iv)]) and that \( M \) and \( N \) are both oriented. We orient the fibers of \( f \) such that the orientation of \( T_{f(p)} N \oplus T_p \text{fib} M \) coincides with that of \( T_pM \), where \( T_p \text{fib} M \) is the tangent space of the fiber \( f^{-1}(f(p)) \subseteq M \) at \( p \in M \) of \( f \).
Then the integration along the fiber \( f^* \eta \) of the form \( \eta \) on \( M \) with coefficients in \( f^* \rho \) is defined as a form on \( N \) with coefficients in \( \rho \). This is characterized by that the formula

\[
\int_N \omega \wedge f^* \eta = \int_M f^* \omega \wedge \eta
\]

holds for any \( \omega \in \Omega^*(N; \rho^\vee) \) (note that \( \omega \wedge f^* \eta \in \Omega^*(N; \rho^\vee \otimes \rho) \cong \Omega^*(N), f^* \omega \wedge \eta \in \Omega^*(M; f^* \rho^\vee \otimes f^* \rho) \cong \Omega^*(M) \)).

**Lemma 3.2.** \( i_X (f^* \eta) = f^* (i_X \eta) \).

The proof is based on the local calculation.

This lemma implies that \( L_X f^* \eta = (di_X + i_X d)(f^* \eta) = f^*(L_X \eta) \). Therefore \( f \) induces \( \mathbb{C} \)-linear maps between the spaces of \( g \)-invariant forms \( f^*: \Omega^*(M; f^* \rho)^g \to \Omega^* - \text{dim } M + \text{dim } N (\rho)^g \). The restriction \( f|_{\partial M}: \partial M \to M \) is also submersive when \( f \) is submersive, and the integrations along the fiber both for \( f \) and for \( f|_{\partial M} \) are defined. Then the following holds.

**Lemma 3.3.**

1. The Stokes formula holds: \( df \eta = f^* d\eta + (-1)^{|\eta| + \text{dim } M} (f|_{\partial M})^* \eta |_{\partial M} \).
2. Let \( L \) be another compact oriented manifold with corners and \( g: N \to L \) be a smooth submersive map. Then the integration along the fiber is compatible with composition, i.e., \( (g \circ f)^* \eta = g^* \circ f^* \eta \).
3. \( f^* (f^* \omega \wedge \eta) = \omega \wedge f^* \eta \).

Let \( M, N \) be compact oriented manifolds with corners, \( L \) be a closed oriented manifold with a \( \mathbb{C} \)-local system \( \rho \) and \( f: M \to L, g: N \to L \) be submersions. Recall that we can form the fiber product

\[
\begin{array}{ccc}
N \times M & \xrightarrow{t} & M \\
L & \xleftarrow{s} & \end{array}
\]

so that \( s \) and \( t \) are smooth submersions. The fiber product \( N \times M \) is orientable and we define the orientation of the main stratum as follows: decompose the tangent spaces of \( N \) and \( M \) at interior points as \( T_N = \text{ker } dg \oplus g^* TL \) and \( TM = f^* TL \oplus \text{ker } df \), and define the orientation of the fiber product by the decomposition \( T(N \times M) = t^* \text{ker } dg \oplus t^* g^* TL \oplus s^* \text{ker } df \). Then we have the following formula.

**Lemma 3.4.** For \( \eta \in \Omega^*(M; f^* \rho) \),

\[
g^* \circ f^* \eta = t^* \circ s^* \eta.
\]

### 3.2. Floer theory

In this subsection we review some basics on the Floer theory.

Let \((M, \omega)\) be a compact symplectic manifold with \( \text{dim } M = 2n \) and \( L \) be a compact oriented spin Lagrangian submanifold in \( M \). Choose an almost complex structure \( J \) compatible with \( \omega \). For \( k \in \mathbb{Z}_{\geq 0} \) and \( \beta \in H_2(M; \mathbb{Z}) \), we denote by \( u: (D, \partial D; z_0, ..., z_k) \to (M, L) \) a \( J \)-holomorphic map \( u: D \to M \) such that \( u(\partial D) \subset L, u_*([D, \partial D]) = \beta \) with \((k+1)\) boundary marked points \( z_0, ..., z_k \in \partial D \) which have counterclockwise order. We say two such maps \( u: (D, \partial D; z_0, ..., z_k) \to (M, L), u': (D, \partial D; z'_0, ..., z'_k) \to (M, L) \) are equivalent if there exists a biholomorphic map \( \phi: D \to D \) such that \( u' = u \circ \phi \) and \( \phi(z'_i) = z_i \), and denote
by $u \sim u'$. We then define the moduli space of $(k+1)$-pointed $J$-holomorphic disks bounded by $L$ to be the set of equivalence classes

$$\mathcal{M}_{k+1,\beta}(L) := \{ u : (D, \partial D; z_0, \ldots, z_k) \to (M, L) \} / \sim$$

and denote by $\overline{\mathcal{M}}_{k+1,\beta}(L)$ its compactification consisting of stable maps when $\beta \neq 0$ or $k \geq 2$.

**Proposition 3.5** ([14 Proposition 7.1.1]). The moduli space $\overline{\mathcal{M}}_{k+1,\beta}(L)$ has a Kuranishi structure with an orientation of dimension $(n + k + \mu(\beta) - 2)$ where $\mu(\beta)$ denotes the Maslov index of the class $\beta$, and we have an isomorphism of the spaces with Kuranishi structures with orientations

$$\partial \overline{\mathcal{M}}_{k+1,\beta}(L) \cong \prod_{k_1 + k_2 = k+1, \beta_1 + \beta_2 = \beta} (-1)^{k_1} \overline{\mathcal{M}}_{k_1,\beta_1}(L)_{ev_1} \times ev_0 \overline{\mathcal{M}}_{k_2,\beta_2}(L)$$

where $z_1 := k_1 k_2 + ik_2 + i + n$ and the sum is taken over $(k, \beta)$ satisfying $\beta \neq 0$ or $k \geq 2$.

**Remark 3.6.** The orientation of the moduli space is determined by the orientation and the spin structure of $L$ [15 Chapter 8].

**Definition 3.7** (cf. [13 Section 7]). We can define the following operators.

$$m_{k,\beta}(x_1, \ldots, x_k) := (-1)^{\mu(x_1 + \cdots + x_k)} \ev_0 (\ev_1 \times \cdots \times \ev_k)^*(x_1 \times \cdots \times x_k),$$

for $\beta \neq 0$ or $k \geq 2$, and

$$m_{1,0}(x_1) := dx_1$$

for $(k, \beta) = (1, 0)$, where $x_1, \ldots, x_k \in \Omega^*(L)$ and $\Omega_2 := \sum_{i=1}^k i|x_i| + 1$.

**Theorem 3.8** ([14 Theorem 7.1]). $(\Omega^*(L), \{m_{k,\beta}\})$ forms a unital $G$-gapped filtered $A_\infty$ algebra for some $G$. The constant function $1 \in \Omega^0(L)$ gives the unit.

The pushforward $\ev_0$, in the right hand side of (10) is defined using the CF perturbation. It can however be calculated using ordinary pullback and pushforward under the following assumption.

**Assumption 3.9.** All the moduli spaces concerned in the definition of the $A_\infty$ structure are manifolds with corners, and the evaluation maps are submersions.

**Remark 3.10.** The condition that the resulting form in (10) has nonnegative degree is $\deg x_1 + \cdots + \deg x_k + 2 - \mu(\beta) - k \geq 0$. Therefore only holomorphic disks with Maslov index $\mu(\beta) \leq 2$ contribute the $A_\infty$ structure when $n = 1$. All the holomorphic disks bounded by a moment fiber in a toric manifold are classified by Cho-Oh [7]. In Section 5 we only consider the cases $M = \mathbb{C}P^1$ and $\mathbb{C}$, and in these cases the disk with Maslov index less than or equal to 2 is either a constant disk (then the Maslov index is 0) or a disk with Maslov index 2. Since there is no nonconstant holomorphic sphere with $c_1 \leq 0$, the moduli space $\overline{\mathcal{M}}_{k+1,\beta}(L)$ compactified with stable disks becomes a compact manifold with corners when $\mu(\beta) \leq 2$, and the evaluation maps are submersions. Hence Assumption 3.9 is satisfied.

We also have the following formula.

**Proposition 3.11** (Divisor axiom. [5 Proposition 6.3], see also [10 Proof of Lemma 11.8]). Let $\theta \in \Omega^1(L)$ such that $d\theta = 0$, then

$$\sum_{i=1}^k m_{k,\beta}(x_1, \ldots, x_{i-1}, \theta, x_i, \ldots, x_{k-1}) = (\partial \beta, \theta)m_{k-1,\beta}(x_1, \ldots, x_{k-1})$$

if $\beta \neq 0$ or $k \geq 2$. 
3.3. Equivariant Floer theory. We now proceed to the construction of an equivariant Floer $A_\infty$ algebra. Let $M$ be a compact smooth $G$-manifold with a $G$-invariant symplectic structure $\omega$ and a $G$-invariant $\omega$-compatible almost complex structure, $L$ be its compact oriented spin Lagrangian submanifold preserved by the $G$-action. From now on we assume Assumption 3.9. We first see the following.

**Proposition 3.12.** Under Assumption 3.9 we have the following formulae for $k \geq 2$ or $\beta \neq 0$.

\[ i_X m_{k,\beta}(x_1, \ldots, x_k) + \sum_{i=1}^{k} (-1)^{|x_i|+\cdots+|x_{i-1}|} m_{k,\beta}(x_1, \ldots, i_X x_i, \ldots, x_k) = 0 \]

**Proof.**

\[
i_X m_{k,\beta}(x_1, \ldots, x_k) = (-1)^{2^k} ev_0(\omega x_1 \times \cdots \times \omega x_k) (i_X(x_1 \times \cdots x_k))
\]

\[
= \sum_{i=1}^{k} (-1)^{2^k+|x_i|+\cdots+|x_{i-1}|} ev_0(\omega x_1 \times \cdots \times \omega x_k) (x_1 \times \cdots \times i_X x_i \times \cdots x_k)
\]

\[
= \sum_{i=1}^{k} (-1)^{2^k+|x_i|+\cdots+|x_{i-1}|+2^{i-1}} m_{k,\beta}(x_1, \ldots, i_X x_i, \ldots, x_k)
\]

\[
= - \sum_{i=1}^{k} (-1)^{|x_1|+\cdots+|x_{i-1}|} m_{k,\beta}(x_1, \ldots, i_X x_i, \ldots, x_k).
\]

\[\square\]

**Corollary 3.13.** $\Omega^*(L)$ with the Floer $A_\infty$ structure $\{m_{k,\beta}\}$ becomes a unital $g$-differential gapped $A_\infty$ algebra over $\mathbb{C}$ in the sense of Definition 2.8.

As discussed in Section 2 its Cartan model $(\Omega^*_{\mathcal{C}^*}(L), \{m_{k,\beta}^{\mathcal{C}^*}\})$ becomes a unital gapped filtered $A_\infty$ algebra.

The last step is to define the $\mathbb{Z}/2\mathbb{Z}$-graded filtered $A_\infty$ algebra over the Novikov field by substituting equivariant parameters. From now on we assume that $G$ is a compact torus of dimension $r$.

**Definition 3.14.** Let $\lambda: S g \to \Lambda_0$ be an equivariant parameter. We call $(\Omega^*_{\mathcal{C}^*}(L), \{m_{k,\beta}^{\mathcal{C}^*}\})$ the equivariant Floer $A_\infty$ algebra of $L$, where $\otimes_{\mathcal{C}}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-graded completed tensor. This is a unital $\mathbb{Z}/2\mathbb{Z}$-graded gapped filtered $A_\infty$ algebra over $\Lambda$ with the unit $1 \in \mathbb{C} \subseteq \Omega^0(L)^g$.

Lastly we introduce the deformation of an equivariant Floer $A_\infty$ algebra by a bounding cocycle, slightly generalizing the construction of Section 2 (see also [16] Section 4).

**Definition 3.15.** Take $b = b_0 + b_+$ with $b_0 \in \Omega^1(L; \mathbb{C})$ and $b_+ \in \Omega^1(L; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda_+$ satisfying $db = 0$, and set $\rho: \pi_1(L) \to \mathbb{C}^*$ to be $\rho(\gamma) := e^{(b_0, \gamma)}$ for $\gamma \in \pi_1(L)$. We define the operators $m_{k,\lambda}^{b}$ deformed by $b$ by the following formula:

\[
m_{k,\lambda}^{b}(x_1, \ldots, x_k) := \sum_{\beta} \rho(\beta) T^{\omega(\beta)/2\pi} \sum_{l_0 + \cdots + l_k = l} m_{k+l,\lambda}(b_+, \ldots, x_1, \ldots, x_k, b_+, \ldots, b_+)
\]

\[
= \begin{cases} 
0 & k \neq 0, 1 \\
\sum_{j=1}^{r} \lambda(F^j) i_{\epsilon_j}(x_1) & k = 1 \\
\sum_{j=1}^{r} \lambda(F^j) i_{\epsilon_j}(b) & k = 0
\end{cases}
\]

(12)

We call $b = b_0 + b_+$ a bounding cocycle of $L$ if it satisfies the following:

\[i_X b = i_X b_0 + i_X b_+ \in \Lambda_0 \cdot 1, \quad m_{0,\lambda}^{b}(1) \in \Lambda_0 \cdot 1.\]

(13)
The second equation is called the weak Maurer‐Cartan equation.

If \( b = b_0 + b_+ \) is a bounding cochain, \( \{ m^{b,\lambda}_k \} \) gives a structure of a unital gapped filtered \( A_\infty \) algebra on \( \Omega^*(L)\otimes_{\mathbb{C}} \Lambda \) which satisfies \( m^{b,\lambda}_1 \circ m^{b,\lambda}_1 = 0 \), i.e. we can define its cohomology with respect to the differential \( m^{b,\lambda}_1 \).

Using \( \square \) the curvature of this deformed \( A_\infty \) structure is calculated as
\[
\text{curv}(1) = \sum_{\beta \neq 0} T\omega(\beta)/2\pi e^{(b,\partial\beta)}m^{b,\lambda}_0(1) - \sum_{j=1}^r \lambda(F^j)i_{e_j}(b).
\]

Take \( \rho \) a \( \mathbb{C} \)-local system of rank 1 on \( L \) which is expressed as \( \rho(\gamma) = e^{(b_0,\gamma)} \) for all \( \gamma \in \pi_1(L) \) with some closed 1-form \( b_0 \in \Omega^1(L;\mathbb{C}) \), and \( b_+ \in \Omega^1(L;\mathbb{C})\otimes \Lambda_+ \), such that \( b = b_0 + b_+ \) is a bounding cochain. Then \( b_0 \) can be regarded as a choice of branch associated to \( \rho \) which affects the curvature term of the \( A_\infty \) structure only. Therefore we may regard \( \rho \) instead of \( b_0 \) as part of the relevant data consisting of an object of a Fukaya category.

**Definition 3.16.** We call the triple \( \mathcal{L} = (L,\rho,b_+) \) an equivariant Lagrangian brane, where \( \rho \) is a \( \mathbb{C} \)-local system of rank 1 on \( L \) such that there exists a closed 1-form \( b_0 \in \Omega^1(L;\mathbb{C}) \) satisfying \( \rho(\gamma) = e^{(b_0,\gamma)} \) for all \( \gamma \in \pi_1(L) \) and \( b := b_0 + b_+ \) is a bounding cochain in the sense of Definition 3.15.

As we only use equivariant Lagrangian branes, we sometimes call them Lagrangian branes for short hereafter. We also call \( L \) the underlying Lagrangian submanifold of \( \mathcal{L} \).

### 3.4 Category of equivariant Lagrangian branes.

Let \( G \) be a compact connected Lie group and \( \mathfrak{g} := \text{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C} \) be its complexified Lie algebra, \( M \) be a compact \( G \)-manifold equipped with a \( G \)-invariant symplectic form \( \omega \) and a \( G \)-invariant \( \omega \)-compatible almost complex structure. In this section we first introduce the \( \mathfrak{g} \)-differential gapped filtered \( A_\infty \) category of equivariant Lagrangian branes and then deform it with bounded cochains to get a Fukaya \( A_\infty \) category over \( \Lambda \).

Take a finite collection of pairs of compact oriented spin Lagrangian submanifolds preserved by the \( G \)-action and a \( \mathbb{C} \)-local system of rank 1 on it \( \mathcal{L} = \{ (L,\rho) \} \), such that (i) each \( L \) satisfies Assumption 3.9 (ii) any pair \( L \) and \( L' \) either coincide or do not intersect, and (iii) each \( \rho \) can be expressed as \( \rho(\gamma) = e^{(b_0,\gamma)} \) for some closed 1-form \( b_0 \in \Omega^1(L;\mathbb{C}) \). We first construct a unital \( \mathfrak{g} \)-differential gapped filtered \( A_\infty \) category \( \mathcal{F}_L \) with the set of objects \( \mathbb{L} \) as follows.

Set
\[
\mathcal{F}_L((L,\rho),(L',\rho')) := \begin{cases} 
\Omega^*(L;\text{Hom}(\rho,\rho')) & \text{if } L = L', \\
0 & \text{otherwise}
\end{cases}
\]

Then this \( \mathcal{F}_L((L,\rho),(L',\rho')) \) is a \( \mathfrak{g} \)-differential space with the operators \( L \) and \( i \).

Take a sequence of \( (k+1) \) objects \( (L_0,\rho_0),\ldots,(L_k,\rho_k) \) such that \( L_0 = \cdots = L_k =: L \). Recall the evaluation maps from the compactified moduli space of holomorphic disks
\[
L^k \xrightarrow{ev_1 \times \cdots \times ev_k} \mathcal{M}_{k,\beta}(L) \xrightarrow{ev_0} L.
\]

Denote \( \rho_{j-1,j} := \text{Hom}(\rho_{j-1},\rho_j) \) and \( \rho_{0,k} := \text{Hom}(\rho_0,\rho_k) \). Consider 1-parameter families of evaluation maps \( ev_{j,t} \) along the \( j \)-th boundary arc \( \partial_j \) connecting the \( j \)-th marked point to the \( (j+1) \)-th marked point for \( 0 \leq j \leq k \) and \( \partial_k \) connecting the \( k \)-th marked point to the 0-th marked point, and define \( P_j \) to be the bundle isomorphism \( ev_{j+1}^* \rho_{j-1,j'} \rightarrow ev_{j+1}^* \rho_{j'-1,j'} \) between the pull back of local systems along \( ev_{j,t} \) obtained by parallel transport (where \( ev_{k+1} := ev_0 \) ). \( P_j \)'s are independent of the choices of \( ev_{j,t} \), and we define
\[
\text{ev}_{j,t} \circ (P_k \circ P_{k-1}(s_{k-1}) \circ \cdots \circ (P_k \circ \cdots \circ P_1)(s_1))
\]
for $\beta \neq 0$ or $k \geq 2$ where $# = \sum_{i=1}^{k} i|x_{i}|' + 1$, and

$$m_{1,0}(x_{1}) := dx_{1}$$

for $x_{1} = \alpha_{1} \otimes s_{1} \in \Omega^{*}(L; \mathcal{H}om(\rho_{0}, \rho_{1})), \ldots, x_{k} = \alpha_{k} \otimes s_{k} \in \Omega^{*}(L; \mathcal{H}om(\rho_{k-1}, \rho_{k})).$

It is easy to see that these $m_{k,\beta}$'s satisfy $A_{\infty}$ relations by carrying out the proof of Theorem 4.8 with local systems. They also satisfy the compatibility with the interior product,

$$i_{X}m_{k,\beta}(x_{1}, \ldots, x_{k}) + \sum_{i=1}^{k} (-1)^{i|x_{1}|'+\cdots+|x_{i-1}|'}m_{k,\beta}(x_{1}, \ldots, i_{X}x_{i}, \ldots, x_{k}) = 0.$$

These operators satisfy gapping conditions and the quadruple $(\mathcal{F}_{L}, \{m_{k,\beta}\}, L, i)$ forms a $g$-differential gapped filtered $A_{\infty}$ category.

Next, take a finite set of equivariant Lagrangian branes $\{L_{\alpha} = (L, \rho_{\alpha}, b_{+,\alpha})\}$ with $(L, \rho_{\alpha}) \in \mathcal{L}$ and choose for each $L_{\alpha}$ a closed 1-form $b_{0,\alpha} \in \Omega^{1}(L; \mathbb{C})$ of the local system $\rho_{\alpha}$ such that $\rho_{\alpha}(\gamma) = e^{(b_{0,\alpha}, \gamma)}$ and that $b_{\alpha} := b_{0,\alpha} + b_{+,\alpha}$ is a bounding cochain. We denote by $\mathcal{L}$ the set of such pairs $(L, b_{\alpha})$.

We're going to construct an uncurved $A_{\infty}$ category $\mathcal{F}_{\mathcal{L}}$ with the set of objects $\mathcal{L}$. For each $(L, b_{\alpha})$ we define its curvature $c_{\alpha} \in \Lambda_{+}$ by $m_{0}^{\beta,\alpha}(1) = c_{\alpha} \cdot 1_{L_{\alpha}}$.

**Definition 3.17.** $\mathcal{F}_{\mathcal{L}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded uncurved $A_{\infty}$ category over $\Lambda$ with $Ob \mathcal{F}_{\mathcal{L}} = \mathcal{L}$,

$$\mathcal{F}_{\mathcal{L}}((L, b_{\alpha}), (L', b_{\alpha'})) := \left\{\begin{array}{ll}
\mathcal{T}_{L}(L, \rho_{\alpha}, (L', \rho_{\alpha'}))^0 \otimes \Lambda & \text{if } L = L', i_{X}(b_{\alpha}) = i_{X}(b_{\alpha'}) \\
0 & \text{for all } X \in g \text{ and } c_{\alpha} = c_{\alpha'}
\end{array}\right.$$

with operators $m_{0} := 0$ and

$$m_{k}(x_{1}, \ldots, x_{k}) := m_{k}^{b}(x_{1}, \ldots, x_{k}) - \left\{\begin{array}{ll}
0 & k \neq 0, 1 \\
\sum_{j=1}^{r} \lambda(F_{j})i_{e_{j}}(x_{1}) & k = 1
\end{array}\right.$$

for $k \geq 1$, where

$$m_{k}^{b}(x_{1}, \ldots, x_{k}) := \sum_{\beta} T^{\omega(\beta)/2\pi} \sum_{l_{0} + \cdots + l_{t} = t} m_{k+l_{0}, \beta} \underbrace{b_{+,\alpha}, \ldots, b_{+,\alpha}}_{l_{0}}, x_{1}, \ldots, x_{k}, \underbrace{b_{+,k}, \ldots, b_{+,k}}_{l_{k}}$$

whenever the spaces of morphisms concerned are nonzero.

Note that the $m_{k,\beta}$'s appeared in the above are the operators defined in this subsection. It is easy to check that $m_{k}$'s satisfy the $A_{\infty}$ relations.

4. Matrix factorizations

4.1. Preliminaries on categories. In §4.1 we recall some basic definitions on derived categories. See [10], §5 for more details.

Let $k$ be an algebraically closed field of characteristic zero. For a triangulated category $\mathcal{T}$, the idempotent completion of $\mathcal{T}$ is denoted by $\hat{\mathcal{T}}$ and $\mathcal{T}$ is said to be idempotent complete if $\mathcal{T}$ is naturally equivalent to $\hat{\mathcal{T}}$ (see, e.g., [2]).

In this section, a differential $\mathbb{Z}/2\mathbb{Z}$-graded category over $k$ is briefly called a dg-category. Let $T$ be a dg-category. The $k$-linear category $[T]$ is defined by taking even cohomology $H^{0}$ of $T$. Let $T^{op}$-mod be the $k$-linear category of right $T$-modules, i.e., a category of dg-functors from $T^{op}$ to the dg-category of $\mathbb{Z}/2\mathbb{Z}$-graded complexes over $k$. We denote by $D(T^{op})$ the localization of $T^{op}$-mod with respect to the set of objectwise quasi-isomorphisms. Then $D(T^{op})$ is an idempotent complete triangulated category which admits arbitrary coproducts. By the Yoneda embedding, $[T]$ is considered as a full subcategory of $D(T^{op})$. Let $[\hat{T}_{pe}]$
be the full subcategory of compact objects in $D(T^{\text{op}})$. Then $[\mathcal{T}_{pc}]$ is the smallest triangulated subcategory of $D(T^{\text{op}})$ containing $[T]$ and closed under direct summands.

**Remark 4.1.** For a dg-category $T$, an associated $\mathbb{Z}/2\mathbb{Z}$-graded $A_\infty$ category $T_\infty$ is defined as follows: The set of objects of $T_\infty$ is the same as $T$. For objects $X$ and $Y$, the morphism space $T_\infty(X, Y) = T(Y, X)$ where $T(Y, X)$ is the morphism space of the dg category $T$. The $A_\infty$ structure $\{m_k\}$ are defined by

$$m_1(x_1) := dx_1, \quad m_2(x_1, x_2) := (-1)^{|x_1|}x_1 \cdot x_2, \quad m_k = 0 \quad (k \geq 3).$$

4.2. **Preliminaries on matrix factorizations.** Let $R$ be a commutative regular $k$-algebra with finite Krull dimension $n$. Take $w \in R \setminus k$. We define a matrix factorization of $w$ by the pair $(P, d_P)$, where $P = P^0 \oplus P^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded finitely generated projective $R$-module and $d_P \in \text{End}^{\text{odd}}(P)$ is an $R$-linear morphism of odd degree with $d_P^2 = w \cdot \text{id}_P$. Then $d_P$ consists of $\varphi \in \text{Hom}_R(P^1, P^0)$ and $\psi \in \text{Hom}_R(P^0, P^1)$ with $\varphi \circ \psi = w \cdot \text{id}_{P_0}$, $\psi \circ \varphi = w \cdot \text{id}_{P_1}$. For matrix factorizations $(P, d_P)$ and $(P', d_{P'})$, the $\mathbb{Z}/2\mathbb{Z}$-graded module of $R$-linear morphisms from $P$ to $P'$ with a differential

$$d(f) := d_{P'} \circ f - (-1)^{|f|}f \circ d_P$$

is denoted by $\text{MF}(P, P')$. Here $f$ is a homogeneous $R$-linear morphism and $|f| \in \mathbb{Z}/2\mathbb{Z}$ is the degree of $f$. These data define a dg-category $\text{MF}(w)$, where compositions of morphisms are naturally defined. Then $[\text{MF}(w)]$ is a triangulated category.

Set $S := R/w$. Let $D^b(S)$ be the derived category of complexes of $S$-modules with finitely generated total cohomology. A complex of $S$-modules is called perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective $S$-modules. We denote by $D^b_{\text{perf}}(S)$ the subcategory of perfect complexes in $D^b(S)$. Then $D^b_{\text{perf}}(S)$ is a thick subcategory of $D^b(S)$ and the Verdier quotient $D^b(S)/D^b_{\text{perf}}(S)$ is denoted by $D^b(S)$, which is called a stabilized derived category of $S$. In some references, a stabilized derived category is also called a triangulated category of singularity. There exists a triangulated functor

$$\text{cok} : [\text{MF}(w)] \to D^b(S),$$

(14)

which sends a matrix factorization $(P, d_P)$ to $\text{cok}(\varphi)$. Moreover, this functor gives an equivalence of triangulated categories.

Let $L$ be a finitely generated $S$-module. Then $L$ is naturally considered as an object of $D^b(S)$. A matrix factorization $L^\text{stab}$ is defined by $\text{cok}(L^\text{stab}) = L$, which is called a stabilization of $L$.

Let $f_1, f_2, \ldots, f_m$ be a regular sequence in $R$ and let $I \neq R$ be the ideal generated by $f_1, f_2, \ldots, f_m$. We assume that $I$ contains $w$. Take $w_1, w_2, \ldots, w_m \in R$ with $w = w_1f_1 + w_2f_2 + \cdots + w_m f_m$. Let $V \cong R^m$ be the free $R$-module of rank $m$ with a basis $e_1, e_2, \ldots, e_m$ and let $e_1^*, e_2^*, \ldots, e_m^*$ be the dual basis. The contraction by $e_i^*$ is denoted by $\iota_i \in \text{End}(V)$. We define $s_0, s_1 \in \text{End}(V)$ by

$$s_0 := f_1\iota_1 + f_2\iota_2 + \cdots + f_m\iota_m,$$

$$s_1 := w_1e_1 \wedge + w_2e_2 \wedge + \cdots + w_m e_m \wedge.$$

Then we easily see that the $\mathbb{Z}/2\mathbb{Z}$-graded $R$-module $V$ equipped with the odd degree morphism $s_0 + s_1$ is a matrix factorization of $w$. By [10 COROLLARY 2.7], this matrix factorization is a stabilization of the $S$-module $R/I$. If $R$ is a local ring with the maximal ideal $\mathfrak{m}$ and $I = \mathfrak{m}$, then this stabilization is denoted by $k^\text{stab}$.

Let $\text{Crit}(w)$ be the critical locus of $w$, i.e., the scheme-theoretic zero locus of $dw$. Set

$$\text{Sing}(S) := \text{Crit}(w) \cap \text{Spec}(S),$$

which is the singular locus of $\text{Spec}(S)$. If $\text{Crit}(w)$ (resp. $\text{Sing}(S)$) is zero-dimensional, then we say that $w$ (resp. $S$) has isolated singularities.
Let \( m \) be a maximal ideal of \( R \) and let \( \hat{R}_m \) be the completion of the local ring \( R_m \) with respect to the \( m \)-adic topology. The element of \( \hat{R}_m \) corresponding to \( w \) is denoted by \( \hat{w}_m \). By \cite[THEOREM 5.7]{10}, it follows that \( \text{MF}(\hat{w}_m) \) is idempotent complete if \( w \in m \) and \( \hat{w}_m \) has isolated singularities. There exists a restriction functor from \( \text{MF}(w) \) to \( \text{MF}(\hat{w}_m) \). By the equivalence \cite[Theorem 2.10]{14} (see also \cite[Proposition 3.4]{28}), we see that these restriction functors give an equivalence between triangulated categories

\[
\text{MF}(w) \cong \prod_{m \in \text{Sing}(S)} \text{MF}(\hat{w}_m)
\]

if \( S \) has isolated singularities. By \cite[THEOREM 5.2, COROLLARY 5.3 and THEOREM 5.7]{10}, we have the following:

**Theorem 4.2.** Suppose that \((R, m)\) is a local \( k\)-algebra and \( w \in m \) has isolated singularities. Then \( k^{\text{stab}} \) split-generates \( \text{MF}(\hat{w}_m) \), i.e., the smallest triangulated subcategory of \( \text{MF}(\hat{w}_m) \) containing \( k^{\text{stab}} \) and closed under direct summand is \( \text{MF}(\hat{w}_m) \). Set \( A := \text{MF}(k^{\text{stab}}, k^{\text{stab}}) \), which is considered as a dg-category with one object. Then, the Yoneda embedding gives an equivalence between triangulated categories

\[
\text{MF}(\hat{w}_m) \cong [\hat{A}_{\text{pe}}].
\]

Let \( W := \text{Hom}_k(m/m^2, k) \) be the Zariski tangent space of \((R, m)\). A Hessian matrix of \( w \) gives a quadratic form on \( W \) and let \( Cl(w) \) be the corresponding Clifford algebra. If \( m \) is a non-degenerate critical point of \( w \), i.e., Hessian is non-degenerate, then we have a natural inclusion \( Cl(-w) \subseteq \text{MF}(k^{\text{stab}}, k^{\text{stab}}) \) which gives a quasi-isomorphism (see, e.g., \cite[§5.5]{10}).

We define a dg-category of branes \( \text{Br}(w) \) by

\[
\text{Br}(w) := \prod_{c \in k} \text{MF}(w - c),
\]

then \( \text{Br}(w) \) is a triangulated category. Moreover, if \( w \) has isolated singularities, then we have

\[
[\text{Br}(w)] \cong \prod_{m \in \text{Crit}(w)} [\text{MF}(\hat{w}_m)].
\]

### 4.3. Categories of branes for Givental type potential functions.

Let \( R \) be a commutative regular \( k \)-algebra with finite Krull dimension \( n \). Choose \( f, g_1, \ldots, g_r \in R \) and \( \lambda_1, \lambda_2, \ldots, \lambda_r \in k \). Assume that \( g_1, \ldots, g_r \) are invertible. For abbreviation, these data is denoted by \( f = -\lambda_1 \log g_1 - \cdots - \lambda_r \log g_r \) or simply by \( F \), which is called a Givental type potential function. Although \( F \) is not well-defined as a single-valued function, its differential can be defined as:

\[
dF = df - \lambda_1 \frac{dg_1}{g_1} - \cdots - \lambda_r \frac{dg_r}{g_r}.
\]

We define \( \text{Crit}(F) \) as the scheme theoretic zero-locus of \( dF \). We assume that \( F \) has isolated singularities, i.e., \( \text{Crit}(F) \) is zero-dimensional.

Let \( m \in \text{Spm}(R) \) be a maximal ideal of \( R \) and \( \hat{R}_m, \hat{m} \) be the completion with respect to the \( \hat{m} \)-adic topology, where \( \hat{m} \) is the maximal ideal. For \( h \in \hat{R}_m \), the corresponding element in \( \hat{R}_m \) is denoted by \( \hat{h}_m \). For an invertible element \( h \in \hat{R}_m \), we define \( \log h \in \hat{m} \) by

\[
\log h = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \left( \frac{h - h(0)}{h(0)} \right)^k,
\]

where \( h(0) \neq 0 \) is the value of \( h \) at \( \hat{m} \). Then \( \hat{F}_m \) at \( m \in \text{Spm}(R) \) can be defined as

\[
\hat{F}_m := \hat{f}_m - \hat{f}_m(0) - \lambda_1 \log \hat{g}_{1,m} - \cdots - \lambda_r \log \hat{g}_{r,m} \in \hat{m}.
\]
Definition 4.3. The dg-category of matrix factorizations of the Givental type potential $F$ is defined to be

$$\Br(F) := \prod_{m \in \Crit(F)} \MF(\hat{F}_m).$$

The idempotent complete triangulated category $[\Br(F)]$ is also called the category of matrix factorizations of $F$.

For $m \in \Crit(F)$, the matrix factorization $(R/m)_{\text{stab}} \in \Br(F)$ (or $[\Br(F)]$) is simply denoted by $e_m^{\text{stab}}$. Then these matrix factorizations split generate $[\Br(F)]$.

5. Equivariant Homological Mirror Symmetry

In this section we study the cases $M = \mathbb{C}P^{1}$ and $\mathbb{C}$. As explained in the introduction, we consider the equivariant Floer $A_{\infty}$ algebras of moment fiber Lagrangians of $M$, following the study in the non-equivariant case by Fukaya-Oh-Ohta-Ono [10].

5.1. The case of $\mathbb{C}P^{1}$. Let $\mathbb{C}P^{1}$ be equipped with the $S^1 = U(1)$-action $[z_0 : z_1] \mapsto [z_0 : \zeta z_1]$ $(\zeta \in U(1))$, an $S^1$-invariant symplectic form $\omega$ with $\omega(\mathbb{C}P^{1}) = 2\pi$. Let $\mu : \mathbb{C}P^{1} \to [0, 1]$ be the associated moment map and let $L_u := \mu^{-1}(u) \subset \mathbb{C}P^{1}$ be the moment fiber Lagrangian over an interior point $u \in (0, 1)$. By identifying $L_u$ with $S^1$ via the $S^1$-action, we choose an orientation of $L_u$ such that it is compatible with that of $S^1$. We choose the standard spin structure of $L_u$ ([3 Section 8]).

Let $\mathfrak{g} := \text{Lie}(S^1) \otimes \mathbb{C}$. Take a basis $e_1$ of $\mathfrak{g}$ consistent with the orientation of $S^1$ and an integral basis $e^1 \in \Omega^1(L_u)^{\mathbb{Z}} \cong H^1(L_u)$, such that $e^1$ coincides with the dual basis $e_1 \in \mathfrak{g}^\vee$ via the identification $S^1 \to L_u$ given by the $S^1$-action.

As we saw in Remark [4,10] only constant disks and holomorphic disks with Maslov index 2 contribute the equivariant Floer $A_{\infty}$ algebras of $L_u$. There are two holomorphic disks of Maslov index 2 up to automorphisms: the one which projects onto $[0, u] \subset [0, 1]$ via $\mu$ and the other one which projects onto $[u, 1] \subset [0, 1]$, see [7]. We denote their relative homology classes by $\beta_1, \beta_2 \in H_2(\mathbb{C}P^{1}, L_u; \mathbb{Z})$ respectively. Then $(e^1, \partial \beta_1) = 1$, $\omega(\beta_1) = 2\pi u$, $\omega(\beta_2) = 2\pi(1 - u)$ and $\partial \beta_2 = -\partial \beta_1$.

Take a closed $S^1$-invariant 1-form $b = b_0 + b_+$ where $b_0 \in \Omega^1(L_u)$, $b_+ \in \Omega^1(L_u) \otimes \Lambda_+$ and put

$$e_0 := e^{(b_0, \partial \beta_1)} \in \mathbb{C}, \quad 1 + c_+ := e^{(b_+, \partial \beta_1)} \in \Lambda_+. \quad (15)$$

Take $\lambda \in \Lambda_0$ and take an equivariant parameter $\lambda : S\mathfrak{g}^\vee \cong \mathbb{C}[e^1] \to \Lambda_0$ which sends $e^1$ to $\lambda$. Recall that the curvature term of the equivariant Floer $A_{\infty}$ algebra evaluated at $\lambda$ and deformed by $b$ is given by

$$m_0^{b, \lambda}(1) = T^{\omega(b_1)/2\pi} e^{(b_0, \partial \beta_1)} + T^{\omega(b_2)/2\pi} e^{(b_0, \partial \beta_2)} - \lambda e_1(b) = T^u c_0 (1 + c_+) + T^{1-u} c_0^{-1} (1 + c_+)^{-1} - \lambda(b, e_1).$$

The differential is calculated as

$$m_1^{b, \lambda}(e^1) = \sum_{\beta} T^{\omega(\beta)/2\pi} e^{(b_0, \partial \beta)} \sum_{l_0 + l_1 = l} m_{l_0 + l_1}^{\beta, \lambda} (b_+, e^1) e^{(b_0, \partial \beta)} - \lambda e_1(e^1) = T^u c_0 (1 + c_+) - T^{1-u} c_0^{-1} (1 + c_+)^{-1} - \lambda$$

by using the divisor axiom. Givental’s potential function for $\mathbb{C}P^{1}$

$$F = T^u x + T^{1-u} x^{-1} - \lambda \log x = X + \frac{T}{X} - \lambda \log X + \text{const}.$$
is recovered from the curvature term $m_{0}^{b,\lambda}(1)$ by introducing the variables $x := c_{0}(1 + c_{+})$ and $X := T^{u}x$. The condition that the differential $m_{1}^{b,\lambda}$ vanishes coincides with the equation for the critical points

$$\partial F = X - T/X - \lambda = 0,$$

where $\partial := X \frac{\partial}{\partial X}$.  

In the following we assume $\lambda \neq \pm 2\sqrt{-1}T^{\frac{1}{2}}$ so that (16) does not have a double root, which implies the equivariant Landau-Ginzburg mirror potential $F$ does not have a degenerate critical point. The case $\lambda = \pm 2\sqrt{-1}T^{\frac{1}{2}}$ can be treated similarly, see Remark 5.2.

We associate an equivariant Lagrangian brane to a solution $X$ of (16) satisfying $0 < \text{val}(X) < 1$ as follows. Take the moment fiber $L := Lu$ where $u := \text{val}(X)$, take $c_{0} \in \mathbb{C}^*$ and $c_{+} \in \Lambda^+$ such that $X = T^{u}c_{0}(1 + c_{+})$. The positive valuation part of the bounding cochain $\partial_{+}$ is given by $\partial_{+} = \log(1 + c_{+})e^{1}$ where $\log(1 + c_{+}) := \sum_{k \geq 1}(-1)^{k-1} \frac{c_{+}^{k}}{k}$. The leading term $b_{0} = b_{0}e^{1}$ is taken such that $e^{b_{0}} = c_{0}$, and gives a local system $\rho$ on $Lu$ with the monodromy $c_{0}$ ($\rho$ is independent of the choice of $b_{0}$). Then $b = b_{0} + b_{+}$ gives a bounding cochain since $e^{1}$ is an invariant form, and we obtain an equivariant Lagrangian brane $(Lu, \rho, b_{+})$. This construction gives a one-to-one correspondence between the solutions of (16) with $\text{val}(X) \in (0, 1)$ and Lagrangian branes (with standard spin structures) whose equivariant Floer $A_{\infty}$ algebra is not quasi-isomorphic to $0$. This is compatible with (15).

Next we study the solutions of (16). It has two solutions $X_{1}, X_{2}$. Then

1. if $\text{val}(\lambda) \geq \frac{1}{2}$, both solutions have valuation $u_{1} = u_{2} = \frac{1}{2}$, i.e. we consider two Lagrangian branes with the same underlying Lagrangian submanifold $L_{\lambda}$ but with different bounding cochains, and

2. if $0 \leq \text{val}(\lambda) < \frac{1}{2}$, the solutions have valuations $u_{1} = \text{val}(\lambda)$ and $u_{2} = 1 - \text{val}(\lambda)$ respectively and therefore we have two disjoint underlying Lagrangians $Lu_{1}$ and $Lu_{2}$. We only consider the case $\text{val}(\lambda) > 0$ because the Lagrangians collapse to points when $\text{val}(\lambda) = 0$.

Let us denote the corresponding maximal ideals by $m_{1}, m_{2} \in \text{Crit} F$ and the corresponding bounding cochains by $b_{i} = b_{0,i} + b_{+,i}$ ($i = 1, 2$). Since $m_{1}^{b_{+,\lambda}} = 0$, 

$$H(\Omega^{*}(Lu_{i})^{\otimes C} \Lambda, m_{1}^{b_{+,\lambda}}) \cong H^{*}(Lu_{i}) \otimes \Lambda$$

as $\mathbb{Z}/2\mathbb{Z}$-graded $\Lambda$-vector spaces.

Next we see the multiplicative structure. $m_{2}^{b_{+,\lambda}}$ of the equivariant Floer $A_{\infty}$ algebra is calculated as:

$$2m_{2}^{b_{+,\lambda}}(e^{1}, e^{1}) = m_{2}^{b_{+,\lambda}}(e^{1}, e^{1}) + m_{2}^{b_{+,\lambda}}(e^{1}, e^{1})$$

$$= \sum_{\beta} T^{\omega(\beta)/2\pi} e^{(b_{0,i}, \beta)} \sum_{l_{0} + l_{1} + l_{2} = l} \left( m_{l_{0}+2,\beta}(b_{+,i}, ..., b_{+,i}, e^{1}, b_{+,i}, ..., b_{+,i}, e^{1}, b_{+,i}, ..., b_{+,i}) + m_{l_{0}+2,\beta}(b_{+,i}, ..., b_{+,i}, e^{1}, b_{+,i}, ..., b_{+,i}, e^{1}, b_{+,i}, ..., b_{+,i}) \right)$$

$$= \sum_{\beta} T^{\omega(\beta)/2\pi} e^{(b_{0,i}, \beta)} \langle b_{+,i}, \beta \rangle \sum_{l_{0} + l_{1} = l} m_{l_{0}+1,\beta}(b_{+,i}, ..., b_{+,i}, e^{1}, b_{+,i}, ..., b_{+,i})$$

$$= T^{u}c_{0}(1 + c_{+}) + T^{1-u}c_{0}^{-1}(1 + c_{+})^{-1}$$

$$= X + TX^{-1}$$
by applying the divisor axiom twice. Hence we have $2m_2^{b_i, \lambda}(\epsilon^i, \epsilon^1) = \partial^2 F(m_i)$. This is nonzero since we assumed $\lambda \neq \pm 2\sqrt{-1}T^\pm$.

Let $H(\Omega^*(L_{u_i})^g \otimes \Lambda, \Lambda_2^{b_i, \lambda})$ be the cohomology algebra associated with the equivariant Floer $A_\infty$ algebra $(\Omega^*(L_{u_i})^g \otimes \Lambda, \{m_2^{b_i, \lambda}\})$, with the product defined as

$$[x] \cdot [y] := (-1)^{|x|}[m_2^{b_i, \lambda}(x, y)].$$

Then we have an isomorphism of $\Lambda$-algebras

$$H(\Omega^*(L_{u_i})^g \otimes \Lambda, \Lambda_2^{b_i, \lambda}) \to Cl(\hat{F}_{m_i})$$

by sending $[\epsilon^i]$ to the generator of the Clifford algebra. By Sheridan’s intrinsic formality for the Clifford algebras \cite{Hoch}, we can lift it to a quasi-isomorphism of $A_\infty$ algebras

$$(\Omega^*(L_{u_i})^g \otimes \Lambda, \{m_2^{b_i, \lambda}\}) \to Cl(\hat{F}_{m_i})$$

(17)

where $Cl(\hat{F}_{m_i})$ is defined in Remark \[\ref{rem:clifford}]

Now let $\mathcal{L}$ be the set of pairs $\{(L_{u_i}, b_1), (L_{u_j}, b_2)\}$ and let $\mathcal{F}_\mathcal{L}$ be the uncurved $A_\infty$ category over $\Lambda$ as defined in Definition \[\ref{def:uncurved}\]. Since $i \neq j$, we have $\mathcal{F}_\mathcal{L}((L_{u_i}, b_1), (L_{u_j}, b_2)) = 0$ if $i \neq j$. Then by \[\ref{alg}\] and by Dyckerhoff’s result reviewed in Section 4.3, we have an objectwise $A_\infty$ functor

$$\mathcal{F}_\mathcal{L}((L_{u_i}, b_1), (L_{u_i}, b_1)) \to Cl(\hat{F}_{m_i}) \subseteq MF(\Lambda_{m_i}^{\text{stab}}, \Lambda_{m_i}^{\text{stab}}) \infty,$$

where $MF$ denotes the dg-category of matrix factorizations of $F$. Therefore we have the following.

**Theorem 5.1.** (Equivariant homological mirror symmetry for $CP^1$). We have a cohomologically fully-faithful $A_\infty$ functor for $\text{val}(\lambda) > 0$ and $\lambda \neq \pm 2\sqrt{-1}T^\pm$

$$\phi: \mathcal{F}_\mathcal{L} \to \text{Br}(F)$$

by sending $(L_{u_i}, b_1)$ to $\Lambda_{m_i}^{\text{stab}}$, whose image split-generates $[\text{Br}(F)]$.

**Remark 5.2.** (Degenerate cases). Suppose that $\lambda = \pm 2\sqrt{-1}T^\pm$. Then the potential function $F$ has a unique degenerate critical point $m$ with valuation $\frac{1}{2}$. Let $(L_{u_i}, \rho, b = b_0 + b_+)$ be the equivariant Lagrangian brane with a bounding cochain corresponding to $m$. By using the divisor axiom, we easily see that

$$k! \cdot m_2^{b_i, \lambda}(\epsilon^i, \epsilon^1, \ldots, \epsilon^1) = \partial^k F(m).$$

(18)

We note that all structure constants of the equivariant Floer algebra are determined by the above equation \[\ref{alg}\]. On the other hand, by \[\ref{alg}\] Theorem 5.8, this unital $A_\infty$ algebra is quasi-isomorphic to the $A_\infty$ algebra $MF(\Lambda_{m_i}^{\text{stab}}, \Lambda_{m_i}^{\text{stab}}) \infty$.

This means that $\mathcal{F}_\mathcal{L}$ and $\text{Br}(F)$ are Morita equivalent.

5.2. **The case of $\mathbb{C}$**. Let $\mathbb{C}$ be equipped with the $S^1 = U(1)$-action $z \mapsto \zeta z$ ($\zeta \in U(1)$), the standard symplectic form $\omega = dx \wedge dy$ ($z = x + iy$) and the standard complex structure. Let $\mu: \mathbb{C} \to \mathbb{R}_{\geq 0}$: $z \mapsto \frac{1}{2}|z|^2$ be the moment map associated to the $S^1$-action, and let $L_u := \mu^{-1}(u)$ be the moment fiber Lagrangian brane over $u \in \mathbb{R}_{>0}$. By identifying $L_u$ with $S^1$ via the $S^1$-action, we choose an orientation of $L_u$ such that it is compatible with that of $S^1$. We choose the standard spin structure of $L_u$ (\[\ref{ss}\] Section 8)).

Let $\mathfrak{g} := \text{Lie}(S^1) \otimes \mathbb{C}$. Take a basis $\epsilon_1$ of $\mathfrak{g}$ consistent with the orientation of $S^1$ and an integral basis $\epsilon^i \in \Omega^1(L_u)^g \cong H^1(L_u)$, such that $\epsilon^i$ coincides with the dual basis $\epsilon^1 \in \mathfrak{g}^\vee$ via the identification $S^1 \to L_u$ given by the $S^1$-action.

Although $\mathbb{C}$ is noncompact, the arguments in Section \[\ref{sec:noncompact}\] applies to this case as well, because all the holomorphic and stable disks bounded by $L_u$ are contained in a compact set thanks to the maximum principle. There is a unique holomorphic disk up to automorphisms of Maslov index 2 bounded by $L_u$,
which projects onto \([0, u] \subset \mathbb{R}_{\geq 0}\) and whose relative homology class we denote by \(\beta \in H_2(\mathbb{C}P^1, L_u; \mathbb{Z}).\) Then \((e^1, \partial \beta) = 1\) and \(\omega(\beta) = 2\pi u.

Take a closed \(S^1\)-invariant 1-form \(b = b_0 + b_+\) where \(b_0 \in \Omega^1(L_u)\) and \(b_+ \in \Omega^1(L_u) \otimes \Lambda_+\), and put

\[
c_0 := e^{(b_0, \partial \beta)} \in \mathbb{C}, \quad 1 + c_+ := e^{(b_+, \partial \beta)} \in \Lambda_+.
\]

Take \(\lambda \in \Lambda_0\) and take an equivariant parameter \(\lambda: S^1 \cong \mathbb{C}[e^1] \to \Lambda_0\) which sends \(e^1\) to \(\lambda\). The curvature term and the differential of the equivariant \(A_\infty\) algebra are

\[
m_0^{b, \lambda}(1) = T^{\omega(\beta)/2\pi} e^{(b, \partial \beta)} - \lambda i_{e^1}(b) = T^u c_0(1 + c_+) - \lambda(b, e^1),
\]

\[
m_1^{b, \lambda}(e^1) = T^{\omega(\beta)/2\pi} e^{(b_0, \partial \beta)} \sum_{t_0 + t_1 = l} m_{t_0 + 1, \beta}(b_+, \ldots, b_+, e^1, b_+, \ldots, b_+) - \lambda i_{e^1}(e^1) = T^u c_0(1 + c_+) - \lambda.
\]

Givental’s potential function for \(\mathbb{C}\)

\[
F = T^u x - \lambda \log x = X - \lambda \log X + \text{const.}
\]

is recovered by introducing the variables \(x := c_0(1 + c_+)\) and \(X := T^u x\), and the differential \(m_1^{b, \lambda}\) vanishes when the derivative \(\partial F = X - \lambda\) of \(F\) equals zero.

Suppose \(\lambda \neq 0\). Then \(F\) has a unique nondegenerate critical point \(m \in \text{Crit} F\). If \(\text{val}(\lambda) > 0\), \(m\) corresponds to \((L_u, b)\) with nonvanishing equivariant Floer cohomology, where \(u = \text{val}(\lambda), L_u\) is the moment fiber over \(u\), and \(b = b_0 + b_+\) is as in the case of \(\mathbb{C}P^1\).

Then

\[
2m_2^{b, \lambda}(e^1, e^1) = T^u c_0(1 + c_+) = X = \partial^2 F \neq 0.
\]

Let \(\mathcal{F}_L\) be the uncurved \(A_\infty\) category with a single object \(\mathcal{L} = \{(L_u, b = b_0 + b_+)\}\). Then we have the following as in the case of \(\mathbb{C}P^1\).

**Theorem 5.3** (Homological mirror symmetry for \(\mathbb{C}\)). Suppose \(\lambda \neq 0\) and \(\text{val}(\lambda) > 0\). Then we have a cohomologically fully faithful \(A_\infty\) functor

\[
\phi: \mathcal{F}_L \to Br(F)_\infty = MF(F_m)_\infty
\]

by sending \((L_u, b)\) to \(\Lambda_m^{\text{stab}}\), whose image split-generates \([Br(F)]\).

6. Appendix A

In this Appendix, we will compute the dimension of the Jacobian ring associated with a Landau-Ginzburg mirror of a smooth semi-projective toric variety (see Theorem 6.7).

6.1. Preliminaries on polyhedrons. In §6.1 we introduce some notions of polyhedron, which is used throughout this appendix. We mainly follow [9].

Let \(N \cong \mathbb{Z}^n\) be a free abelian group of rank \(n\) and \(M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z})\) be the dual lattice. For a commutative ring \(R\), set \(N_R := N \otimes \mathbb{Z} R, M_R := M \otimes \mathbb{Z} R\). For a convex polyhedral cone \(\sigma \in N_R\), the dual cone \(\sigma^\vee\) is defined by

\[
\sigma^\vee = \{u \in M_R \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.
\]

Note that a convex polyhedral cone is full-dimensional if and only if its dual cone is strictly convex. For convex polyhedral cones \(\tau, \sigma \subseteq N_R\), we write \(\tau \prec \sigma\) if \(\tau\) is a face of \(\sigma\).
Let $\Sigma$ be a fan in $N_\mathbb{R}$ and $\varphi$ be a strictly convex support function on $\Sigma$ (see [9] Definitions 3.1.2, 4.2.11, 6.1.12]). In this appendix, we assume that

the support $|\Sigma|$ is a full-dimensional rational polyhedral cone. (20)

Let $v_1, v_2, \ldots, v_m \in N$ be the set of integral generators of rays of $\Sigma$.

Let $P$ be a polyhedron in $M_\mathbb{R}$ determined by $\Sigma$ and $\varphi$, i.e.,

$$P = \{ u \in M_\mathbb{R} \mid \ell_i(u) := \langle u, v_i \rangle + \varphi(v_i) \geq 0, \quad i = 1, 2, \ldots, m \}.$$

We call integral affine functions $\ell_i$ ($i = 1, 2, \ldots, m$) the defining functions of $P$. For $v \in |\Sigma|$, we note that

$$\min_{u \in P} \langle u, v \rangle = -\varphi(v).$$

Let $U_P$ be the recession cone of $P$, i.e.,

$$U_P := \{ u \in M_\mathbb{R} \mid \langle u, v_i \rangle \geq 0, \quad i = 1, 2, \ldots, m \}.$$

We note that

$$U_P = |\Sigma| = \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \cdots + \mathbb{R}_{\geq 0}v_m.$$

Since $|\Sigma|$ is a full-dimensional rational polyhedral cone, the polyhedron $P$ is pointed, i.e., $U_P$ is strictly convex.

6.2. Preliminaries on polyhedral subdomains. We define a $\Lambda$-algebra $\Lambda\langle U_P \rangle$ by

$$\Lambda\langle U_P \rangle := \left\{ \sum_{v \in N \cap |\Sigma|} c_v y^v \mid c_v \in \Lambda, \lim_{|v| \to \infty} (\langle u, v \rangle + \text{val}(c_v)) = \infty \text{ for all } u \in P \right\},$$

where $|v|$ is a standard Euclidean norm on $N_\mathbb{R}$. For $f = \sum c_v y^v \in \Lambda\langle U_P \rangle$, set

$$|f| := \exp\left( -\inf_{v \in N \cap U_P} (\langle u, v \rangle + \text{val}(c_v)) \right).$$

We note that $\inf_{u \in P} (\langle u, v \rangle + \text{val}(c_v)) = \text{val}(c_v) - \varphi(v)$ for $v \in |\Sigma| \cap N$. Then $| \cdot |$ gives a complete non-archimedean norm on $\Lambda\langle U_P \rangle$ and the pair $(\Lambda\langle U_P \rangle, | \cdot |)$ is a $\Lambda$-Banach algebra ([29, Remark 6.6]). Moreover, we know that $(\Lambda\langle U_P \rangle, | \cdot |)$ is a $\Lambda$-affinoid algebra ([29, Proposition 6.9]).

By construction, the affinoid algebra $\Lambda\langle U_P \rangle$ contains the monoid ring $\Lambda[U_P^* \cap N]$, which induces an inclusion

$$\text{Spm}(\Lambda\langle U_P \rangle) \subseteq \text{Spm}(\Lambda[U_P^* \cap N]),$$

where we denote by $\text{Spm}$ the maximal spectrums (see [29, Proposition 6.9]).

Let $\mathfrak{m} \in \text{Spm}(\Lambda\langle U_P \rangle)$ and $\mathfrak{m} := \mathfrak{m} \cap \Lambda[U_P^* \cap N]$ be the corresponding maximal ideal of $\Lambda[U_P^* \cap N]$. Then we have an isomorphism between complete local rings

$$\Lambda\langle U_P \rangle_{\mathfrak{m}} \cong \Lambda[U_P^* \cap N]_{\mathfrak{m}}.$$

(See the proof of [29, Proposition 6.9]. See also [9, Proposition 7.3.2.3 and 8, Lemma 5.1.2,].) We note that $\Lambda[U_P^* \cap N]_{\mathfrak{m}}$ is a Cohen-Macaulay ring of Krull dimension $n$ (see, e.g., [20, Theorem 1], [11, §13.1]). Hence $\Lambda\langle U_P \rangle_{\mathfrak{m}}$ is a Cohen-Macaulay ring ([29, Proposition 6.9]) of Krull dimension $n$. Moreover, $\Lambda\langle U_P \rangle_{\mathfrak{m}}$ is regular if and only if $\Lambda[U_P^* \cap N]_{\mathfrak{m}}$ is regular.

For a $\Lambda$-Banach algebra $A$, set

$$A_0 := \{ f \in A \mid |f| \leq 1 \}, \quad A_+ := \{ f \in A \mid |f| < 1 \}, \quad \overline{A} := A_0/A_+.$$

Then $A_0$ is a $\Lambda$-Banach algebra, $A_+$ is an ideal of $A_0$, and $\overline{A}$ is a $\mathbb{C}$-algebra. For $v \in |\Sigma| \cap N$, let $y^v \in \overline{\Lambda\langle U_P \rangle}$ be the residue class of $T^{\varphi(v)} y^v \in \Lambda\langle U_P \rangle_0$. 


For $c = \sum_{i=0}^{\infty} c_i T^{\lambda_i} \in \Lambda \setminus \{0\}$, the leading term coefficient $L(c) \in \mathbb{C}^*$ is defined by the coefficient of $T^{\text{val}(c)}$ and set $\text{Sp}(c) := \{\lambda_i \mid c_i \neq 0\} \subset \mathbb{R}$. For $f = \sum_{v \in |\Sigma| \cap N} c_v y^v \in \Lambda(U_P)$, set $\text{Sp}(f) := \bigcup_{v \in |\Sigma| \cap N} \text{Sp}(T^{-\varphi(v)} c_v)$. Note that $\text{Sp}(c)$ and $\text{Sp}(f)$ are discrete subsets of $\mathbb{R}$.

**Proposition 6.1.** We have
\[
\Lambda(U_P) \cong \bigoplus_{v \in |\Sigma| \cap N} \mathbb{C} y^v
\]
as $\mathbb{C}$-vector spaces. The ring structure of $\Lambda(U_P)$ is given by
\[
\overline{y^v} \cdot \overline{y^{v'}} = \begin{cases} 
\overline{y^{v+v'}} & \text{if } v, v' \in \sigma \text{ for some } \sigma \in \Sigma \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** The isomorphism $\Lambda(U_P) \cong \bigoplus_{v \in |\Sigma| \cap N} \mathbb{C} y^v$ easily follows from the definition. Since $\varphi$ is strictly convex, we have the desired conclusion on the ring structure. \(\square\)

Let $\text{Sym}(M_\Lambda)$ be the symmetric algebra of the $\Lambda$-vector space $M_\Lambda$ and $\Lambda(M)$ be the completion of the polynomial ring $\text{Sym}(M_\Lambda)$ with respect to the Gauss norm so that $\Lambda(M)$ is a Tate algebra. Then we have $\Lambda(M) \cong \text{Sym}(M_\Lambda)$. We see that $\text{Spm}(\Lambda(M))$ is naturally isomorphic to $N_{A_\Lambda}$.

For $f = \sum c_v y^v \in \Lambda(U_P)$ and $m \in M$ we define a derivative $\partial_m f \in \Lambda(U_P)$ by $\partial_m f := \sum c_v (m, v) y^v$. We note that $\text{Sp}(\Lambda(U_P)_0, \Lambda(U_P)_+)$ are closed under $\partial_m$ and $\partial_m$ also acts on $\Lambda(U_P)$. We define a $\Lambda$-algebra homomorphism $\psi_f : \Lambda(M) \to \Lambda(U_P)$ by $\psi_f(m) := \partial_m f$. If $f \in \Lambda(U_P)_0$, then we have $\psi_f(\Lambda(M)_0) \subseteq \Lambda(U_P)_0$ and $\psi_f(\Lambda(M)_+) \subseteq \Lambda(U_P)_+$. Hence $\psi_f$ induces a $\mathbb{C}$-algebra morphism $\overline{\psi_f} : \text{Sym}(M) \to \overline{\Lambda(U_P)}$.

**Proposition 6.2.** Let $f \in \Lambda(U_P)_0$. Suppose that $\overline{\Lambda(U_P)}$ is finite over $\text{Sym}(M)$. Then $\Lambda(U_P)_0$ is finite over $\Lambda(M)_0$.

**Proof.** Take $v_1, v_2, \ldots, v_r \in |\Sigma| \cap N$ such that $\overline{y^{v_1}}, \overline{y^{v_2}}, \ldots, \overline{y^{v_r}}$ generate $\overline{\Lambda(U_P)}$ as a $\text{Sym}(M)$-module. Take $g = \sum T^{\varphi(v)} c'_v y^v \in \Lambda(U_P)_0$. Let $G$ be the discrete submonoid of $\mathbb{R}_{\geq 0}$ generated by
\[
\text{Sp}(g) \cup \bigcup_{k \geq 1} \text{Sp}(T^{\varphi(v_i)} f^k y^{v_i}).
\]
We enumerate $G$ by
\[
G = \{c_0, c_1, c_2, \ldots\}, \quad 0 = c_0 < c_1 < c_2 < \cdots.
\]
Let $g = \sum_{\text{val}(c'_v) = 0} L(c'_v) \overline{y^v} \in \overline{\Lambda(U_P)}$ be the residue class of $g$. Choose $\phi_{1,0}, \ldots, \phi_{r,0} \in \text{Sym}(M)$ such that $g = \sum_{i=1}^r \phi_{i,0} \overline{y^{v_i}}$. Set $g_0 = g$ and $g_1 = g_0 - \sum_{i=1}^r T^{\varphi(v_i)} \phi_{i,0} \overline{y^{v_i}}$, where $\phi_{i,0}$ are considered as elements of $\Lambda(M)_0$. Then we have $\text{Sp}(g_1) \subseteq G \setminus \{c_0\}$. Inductively, we can construct
\[
g_{k+1} = \sum_{i=1}^r T^{c_i + \varphi(v_i)} \phi_{i,k} \overline{y^{v_i}}, \quad \text{Sp}(g_{k+1}) \subseteq G \setminus \{c_0, c_1, \ldots, c_k\}.
\]
with
\[
g_{k+1} = g_k - \sum_{i=1}^r T^{c_i + \varphi(v_i)} \phi_{i,k} \overline{y^{v_i}}, \quad \text{Sp}(g_{k+1}) \subseteq G \setminus \{c_0, c_1, \ldots, c_k\}.
\]
Set $\phi_i := \sum_{j=1}^\infty T^{\varphi(v_j)} \phi_{i,j} \in \Lambda(M)_0$, then we have $g = \sum_{i=1}^r \phi_i T^{\varphi(v_i)} y^{v_i}$. This implies that $T^{\varphi(v_i)} y^{v_i} (i = 1, 2, \ldots, r)$ generate $\Lambda(U_P)_0$ as a $\Lambda(M)_0$-module. \(\square\)

**Corollary 6.3.** Under the same assumptions of Proposition 6.2. $\Lambda(U_P)$ is flat over $\Lambda(M)$.
Proof. Let $m$ be a maximal ideal of $\Lambda(U_P)$ and set $m' := \psi^{-1}_f(m)$, then $m'$ is a maximal ideal of $\Lambda(M)$. By Proposition 6.2, we easily see that $\Lambda(U_P)$ is finite over $\Lambda(M)$. Combining with the incomparability (e.g., [11] Corollary 4.18, Proposition 9.2), the fiber $\Lambda(U_P)/m\Lambda(U_P)$ over $m'$ has Krull dimension 0. Since $\Lambda(U_P)_m$ is a Cohen-Macaulay local ring of Krull dimension $n$ and $\Lambda(M)_{m'}$ is a regular local ring of Krull dimension $n$, we see that $\Lambda(U_P)_m$ is flat over $\Lambda(M)_{m'}$ (e.g., [11] Theorem 18.16). Thus $\Lambda(U_P)$ is flat over $\Lambda(M)$. □

In the last of §6.2 we introduce a logarithmic Jacobian ring.

Definition 6.4. Let $f \in \Lambda(U_P)_0$, $\lambda \in N_{\Lambda_0}$, and $m_\lambda$ be the maximal ideal of $\Lambda(M)$ corresponding to $\lambda$. Set $\partial_m f_\lambda := \partial_m f - \langle m, \lambda \rangle$. A logarithmic Jacobian ideal $I_{f, \lambda}$ is defined by

$$I_{f, \lambda} := \langle \partial_m f_\lambda \mid m \in M \rangle,$$

and a logarithmic Jacobian ring $J_{f}(\lambda)$ is defined by $\Lambda(U_P)/I_{f, \lambda}$. We note that $I_{f, \lambda}$ is the ideal generated by $\psi_f(m_\lambda)$ and $\Spec(J_{f}(\lambda))$ is the fiber over $\lambda$.

6.3. Preliminaries on tropicalizations. Let $\mathbb{R}$ be the additive monoid $\mathbb{R} \cup \{\infty\}$ and let $\sigma \subseteq N_\mathbb{R}$ be a rational full-dimensional convex polyhedral cone. We define $M_\mathbb{R}(\sigma^\vee)$ by the space of monoid morphisms

$$M_\mathbb{R}(\sigma^\vee) := \Hom_{\mathbb{R}_{\geq 0}}(\sigma, \mathbb{R})$$

respecting multiplication by $\mathbb{R}_{\geq 0}$. Since $\sigma$ is rational, we easily see that

$$\Hom_{\mathbb{R}_{\geq 0}}(\sigma, \mathbb{R}) \cong \Hom(\sigma \cap N, \mathbb{R}),$$

where $\Hom$ is the space of monoid morphisms. Let $\tau$ be a face of $\sigma$, $u \in M_\mathbb{R}/\tau^\perp$ and $v \in \sigma$. We define $\iota(u) \in M_\mathbb{R}(\sigma^\vee)$ by

$$\langle \iota(u), v \rangle := \begin{cases} (u, v) & \text{if } v \in \tau \\ \infty & \text{otherwise.} \end{cases}$$

Then this correspondence gives an isomorphism

$$\iota : \prod_{\tau < \sigma} M_\mathbb{R}/\tau^\perp \xrightarrow{\cong} M_\mathbb{R}(\sigma^\vee)$$

(see [24] Proposition 3.4). Since $\sigma$ is full-dimensional, $M_\mathbb{R}$ is naturally contained in $M_\mathbb{R}(\sigma^\vee)$. For $m \in \Spm(\Lambda[\sigma \cap N])$, by using $\Spm(\Lambda[\sigma \cap N]) \cong \Hom(\sigma \cap N, \Lambda)$, we denote by $\phi_m$ the corresponding element of $\Hom(\sigma \cap N, \Lambda)$. We define a tropicalization morphism

$$\trop : \Spm(\Lambda[\sigma \cap N]) \to M_\mathbb{R}(\sigma^\vee)$$

by $\trop(m) := \val \circ \phi_m \in M_\mathbb{R}(\sigma^\vee)$.

Recall that $P \subseteq M_\mathbb{R}$ is a polyhedron determined by $\Sigma$ and $\varphi$ with a strictly convex recession cone $U_P$. We define $\overline{P} \subseteq M_\mathbb{R}(U_P)$ as follows: Let $\tau$ be a face of $U_P^\circ$ and $u \in M_\mathbb{R}/\tau^\perp$. Then $\iota(u) \in \overline{P}$ if and only if

$$\ell_\tau(u) = \langle u, v_i \rangle + \varphi(v_i) \geq 0 \text{ for all } v_i \in \tau.$$

Set

$$U_P := \{ m \in \Spm(\Lambda[\overline{U_P^\circ} \cap N]) \mid \trop(m) \in \overline{P} \},$$

then we see that $\Spm(\Lambda(U_P)) = U_P \subseteq \Spm(\Lambda[\overline{U_P^\circ} \cap N])$ (24 Proposition 6.9]). The restriction of the tropicalization morphism $\trop$ to $U_P$ is also denoted by $\trop$.

Definition 6.5. Let $a$ be an ideal of $\Lambda(U_P)$ and set $V(a) := \Spm(\Lambda(U_P)/a) \subseteq U_P$. Then the image $\trop(V(a)) \subseteq \overline{P}$ is called a tropicalization of $V(a)$ and denoted by $\trop(a)$. If $a$ is generated by $f \in \Lambda(U_P)$, $V(a)$ is also denoted by $V(f)$ and its tropicalization is denoted by $\trop(f)$. 
For $c \neq 0 \in \Lambda$, recall that the leading term $L(c) \in \mathbb{C}^*$ is the coefficient of $T^\mathrm{val}(c)$. For $f = \sum c_v y^v \in \Lambda(U_P), \tau \prec U_P^\lambda$, and $u \in (M_R/\tau^\perp) \cap \overline{\mathcal{P}}$, set
\[
m := \min_{v \in \tau \cap N} \left(\langle u, v \rangle + \mathrm{val}(c_v)\right).
\]
We define the initial term $\text{in}_u(f)$ by
\[
\text{in}_u(f) := \sum_{v \in \tau \cap N} L(c_v) y^v \in \mathbb{C}[\tau \cap N].
\]
By [29] Lemma 8.4, we have
\[
\text{Trop}(f) = \{ u \in \overline{\mathcal{P}} \mid \text{in}_u(f) \text{ is not a monomial} \}. \tag{21}
\]

**Remark 6.6.** It is possible that $\text{in}_u(f) = 0$. If $\text{in}_u(f) = 0$, then $\text{in}_u(f)$ is not considered as a monomial.

By the fundamental theorem of tropical geometry [29] Theorem 7.8, we have
\[
\text{Trop}(a) = \bigcap_{f \in a} \text{Trop}(f). \tag{22}
\]

### 6.4. Main theorem.

In §6.3 we assume that the fan $\Sigma$ is smooth. Let
\[
\Lambda(Z_1, Z_2, \ldots, Z_m)
\]
be the Tate algebra over $\Lambda$ with variables $Z_1, Z_2, \ldots, Z_m$ (the affinoid algebra associated with the polyhedron $\mathbb{R}^m_{\geq 0}$). Let
\[
\psi : \Lambda(Z_1, Z_2, \ldots, Z_m) \to \Lambda(U_P)
\]
be the continuous morphism which is defined by $\psi(Z_i) = T^{\sigma(v_i)} y^{v_i}$. Surjectivity of $\psi$ follows from the smoothness of $\Sigma$. Let
\[
\overline{f} = c_1 Z_1 + c_2 Z_2 + \cdots + c_m Z_m \in \Lambda(Z_1, Z_2, \ldots, Z_m),
\]
where $c_i \in \mathbb{C}^*$ and let $F$ be an element of $\Lambda(Z_1, Z_2, \ldots, Z_m)$ with $|F - \overline{f}| < 1$. Set $\overline{f} = \psi(F)$ and $f = \psi(F)$. Take $\lambda \in N_{\Lambda_0}$. The next statement is the main theorem of this appendix.

**Theorem 6.7.** Let $\Sigma$ be a smooth fan such that the support $|\Sigma|$ is a full dimensional rational polyhedral cone, $\lambda \in N_{\Lambda_0}$, and $f$ is as above. Then $\dim_\Lambda J_\lambda(f) = \dim_\mathbb{C} H^\ast(X_\Sigma; \mathbb{C})$.

To prove this theorem, we first show the next proposition.

**Proposition 6.8.** $\dim_\Lambda J_\lambda(f)$ is independent of $\lambda \in N_{\Lambda_0}$.

**Proof.** We recall that $f$ induces a $\mathbb{C}$-algebra morphism $\overline{\psi}_f : \text{Sym}(M_C) \to \overline{\Lambda(U_P)}$. Using this morphism, we consider $\Lambda(U_P)$ as a module over $\text{Sym}(M_C)$. Set $\overline{y}^{v_i} := c_i \overline{y}^{v_i}$. By definition, for $m \in M_C$, we have $\overline{\psi}_f(m) = \sum_{i=1}^m \langle m, v_i \rangle \overline{y}^{v_i}$. Let $X_\Sigma$ be the $n$-dimensional smooth toric variety associated with the fan $\Sigma$. Then the torus $T := N \otimes \mathbb{Z} \mathbb{C}^*$ naturally acts on $X_\Sigma$. We naturally identify the equivariant cohomology $H_T^\ast(\text{pt})$ with $\text{Sym}(M_C)$ as $\mathbb{C}$-algebras. We easily see that $\Lambda(U_P)$ is isomorphic to the equivariant cohomology $H_T^\ast(X_\Sigma; \mathbb{C})$ as $\text{Sym}(M_C)$-algebras (see, e.g., [9] §12.4). Since $H_T^\ast(X_\Sigma; \mathbb{C})$ is a free module over $\text{Sym}(M_C)$ of finite rank (e.g., [22] Proposition 2.1 and [9] §7.2), the $\Lambda(M)$-module $\Lambda(U_P)$ is finite and flat by Proposition 6.2 and Corollary 6.3. Hence $\Lambda(U_P)$ is a finite locally free $\Lambda(M)$-module and dimensions of fibers are independent of $\lambda \in N_{\Lambda_0}$. \hfill \square

We next compute $\dim_\Lambda J_\lambda(f)$ at a specific point $\lambda \in N_{\Lambda_0}$ (Theorem 6.10).

**Lemma 6.9.** Let $m \in M, \tau \prec |\Sigma|$ and $u \in M_R/\tau^\perp$. Then $u \in \text{Trop}(\partial_m f_\lambda)$ if and only if $u \in \text{Trop}(\partial_m \overline{f}_\lambda)$.
Proof. For $G \in \Lambda(Z_1, Z_2, \ldots, Z_m)$, we set
\[
\partial_m G := \sum_{i=1}^{m} \langle m, v_i \rangle Z_i \frac{\partial G}{\partial Z_i}
\]
Then we see that $\partial_m G \in \Lambda(Z_1, Z_2, \ldots, Z_m)$ and $\psi(\partial_m G) = \partial_m \psi(G)$. Choose a monomial $Z^\lambda := Z_1^{b_1} Z_2^{b_2} \cdots Z_m^{b_m}$, where $b_1, b_2, \ldots, b_m \in \mathbb{Z}_{\geq 0}$. Set
\[
v^\circ := b_1 v_1 + b_2 v_2 + \cdots + b_m v_m, \quad \varphi^\circ := b_1 \varphi(v_1) + b_2 \varphi(v_2) + \cdots + b_m \varphi(v_m).
\]
Then we have
\[
\psi(\partial_m Z^\lambda) = (m, v^\circ) T^{\epsilon} y^\epsilon.
\]
Suppose that $v^\circ \in \tau$ and $\partial_m Z^\lambda \neq 0$, which implies $b_i = 0$ for $v_i \notin \tau$. Then there exists $v_j \in \tau$ with $\langle m, v_j \rangle \neq 0$ and $b_j \neq 0$. We see that
\[
\langle u, v^\circ \rangle + \varphi^\circ = \sum_{v_i \in \tau} b_i \ell_i(u) \geq \ell_j(u).
\]
Since $|F - \mathcal{F}| < 1$, the valuation of each coefficient of $F - \mathcal{F}$ is positive. Hence $Z^\lambda$-term of $F - \mathcal{F}$ does not contribute to $\text{Trop}(\partial_m f_\lambda)$. $\square$

Combining this lemma with Equations (21) and (22), it follows that
\[
\text{Trop}(J_\lambda(f)) \subseteq \bigcap_{m \in M} \text{Trop}(\partial_m f_\lambda).
\]
For each maximal cone $\sigma \in \Sigma(n)$, let $e^\sigma_1, e^\sigma_2, \ldots, e^\sigma_n$ be the integral basis of $N$ which generate $\sigma$ and let $f^\sigma_1, f^\sigma_2, \ldots, f^\sigma_m$ be the dual basis. Set
\[
\lambda^\sigma := (f^\sigma_1, \lambda) \in \Lambda_0.
\]
For each positive dimensional cone $\tau \in \Sigma \setminus \{\{0\}\}$ and $\epsilon \in \mathbb{R}_{>0}$, set
\[
I_\tau := \{v_i \mid \tau \text{ and } v_i \text{ do not span any cone in } \Sigma\},
\]
\[
P_\tau := \{u \in P \mid \ell_i(u) \leq \ell_j(u) \text{ for all } v_i \in \tau \text{ and } v_j \in I_\tau\},
\]
\[
V_{\tau, \epsilon} := \{u \in P \mid \ell_i(u) \leq \epsilon \text{ for all } v_i \in \tau\}.
\]
For $\tau \in \Sigma \setminus \{\{0\}\}$ with $I_\tau \neq \emptyset$, we also set
\[
\epsilon_\tau := \inf_{v_i \in I_\tau, \epsilon \in P_\tau} \ell_i(u),
\]
\[
\epsilon_\tau := \min\{\epsilon_\tau \mid \tau \in \Sigma \setminus \{\{0\}\} \text{ with } I_\tau \neq \emptyset\}.
\]
We easily see that $\epsilon_\tau > 0$ and $V_{\tau, \epsilon} \subseteq P_\tau$.

Theorem 6.10. Suppose that $\epsilon_\tau > \text{val}(\lambda^\sigma) > 0$ for all $\sigma \in \Sigma(n)$ and $i$. For $\sigma \in \Sigma(n)$, we define $u^\sigma_\lambda \in \text{int}P$ by $\ell_i(u) = \text{val}(\lambda^\sigma)$ $(v_i \in \sigma)$. Then
\[
\text{Trop}(J_\lambda(f)) = \{u^\sigma_\lambda \mid \sigma \in \Sigma(n)\}
\]
and, for each $\sigma \in \Sigma(n)$, there exists a unique critical point $m^\lambda_\sigma \in \text{Spm}(J_\lambda(f))$ with $\text{trop}(m^\lambda_\sigma) = u^\sigma_\lambda$. Moreover, these critical points $m^\lambda_\sigma$ are non-degenerate.

Proof. Let $u \in \text{Trop}(J_\lambda(f))$. We first show that $u \in P$. Suppose that $u \in M_{\mathbb{R}}/\tau^{+}$ for some proper face $\tau \prec |\Sigma|$. Take $m \neq 0 \in \tau^{+}$ with $\langle m, \lambda \rangle \neq 0$. Then only the constant term $\langle m, \lambda \rangle$ contributes to $\text{int}u(\partial_m f_\lambda)$. This contradicts $u \in \text{Trop}(J_\lambda(f))$.

Suppose that $u \in P \cap \text{Trop}(J_\lambda(f))$. Choose $v_{i_1}$ with $\ell_{i_1}(u) \leq \ell_i(u)$ for all $i = 1, 2, \ldots, m$ and let $\tau_1$ be the cone in $\Sigma$ spanned by $v_{i_1}$. By considering $\text{Trop}(\partial_{f_1} f_\lambda)$ for some $\sigma \in \Sigma(n)$ and $i$ with $\langle f_1^\sigma, v_{i_1} \rangle = 1$, we see that $\ell_{i_1}(u) \leq \text{val}(\lambda^\sigma) < \epsilon_\tau \leq \ell_{i_1}(u)$, which implies $u \in P_{\tau_1}$. Choose $v_{i_2}$ with $\ell_{i_2}(u) \leq \ell_i(u)$ for all
Hence $L$ where the algebra of $h$ at $V$ with $p$ point $x$. We denote by $\hat{f}^\sigma$, $\hat{v}_j = \delta_{i,j}$ (i, j = 1, 2, ..., n). Then we have
\[
\text{trop}(\partial \hat{f}^\sigma, \hat{T}) = \{ u \in P \mid \ell_i(u) = \text{val}(\lambda_i^\sigma_u) \},
\]
which implies that
\[
\text{Trop}(J_\lambda(f)) \subseteq \{ u_\lambda^\sigma \mid \sigma \in \Sigma(n) \}.
\]
On the other hand, by [29] Theorem 11.7, it follows that for each $\sigma \in \Sigma(n)$ there exists a unique critical point $m \in \text{Spm}(J_\lambda(f))$ with $\text{trop}(m) = u_\lambda^\sigma$ and these critical points are non-degenerate.

Combining Proposition [6.8] and Theorem [6.10] we complete the proof of Theorem 6.7.

7. Appendix B

In §7 we introduce a variant of a category of matrix factorizations of Givental type potentials (Definition [7.1]). In this section, we assume that the base field $k$ is equal to $\mathbb{C}$. Let $R, f, g_1, g_2, \ldots, g_r$ be the same as §4.3 i.e., $R$ is a commutative regular $\mathbb{C}$-algebra of Krull dimension $n$, $f \in R$, and $g_1, g_2, \ldots, g_r$ are invertible elements of $R$. Set
\[
T := R/(g_1 - 1, g_2 - 1, \ldots, g_r - 1).
\]
Suppose that $dg_1, dg_2, \ldots, dg_r$ are linearly independent at each point of $\text{Spec}(T) \subseteq \text{Spec}(R)$, which implies that $T$ is also regular. We denote by $f_T$ the restriction of $f$ to $\text{Spec}(T)$. We will give a relationship between $[\text{Br}(f_T)]$ and a matrix factorization category for a Givental type potential function.

Let $X$ be the analytification of $\text{Spec}(R)$, $Y \subseteq X$ be the analytification of $\text{Spec}(T)$, and $C_X^r$ be the $r$-dimensional complex plane with coordinates $x_1, x_2, \ldots, x_r$. For $\epsilon \in \mathbb{R}_{>0}$, set
\[
U := \{ x \in X \mid |g_1(x) - 1| < \epsilon, |g_2(x) - 1| < \epsilon, \ldots, |g_r(x) - 1| < \epsilon \}.
\]
By taking $\epsilon$ enough small, we can choose the branch of $\log g_i$ on $U$ such that $\log g_i = 0$ along $Y$. We define $F^\text{an}$ by
\[
F^\text{an} := f - \lambda_1 \log g_1 - \lambda_1 \log g_2 - \cdots - \lambda_r \log g_r.
\]
This is an element of the ring $O_{U \times C_X^r}$ of holomorphic functions on $U \times C_X^r$. Let $\text{pr}_1$ be the projection from $X \times C_X^r$ to $X$ and $i$ be the inclusion of $X \approx X \times \{ \emptyset \}$ to $X \times C_X^r$. By the method of Lagrange multipliers, for $y \in \text{Crit}(f_T)$, there exists a unique critical point $L(y) \in \text{Crit}(F^\text{an})$ with $\text{pr}_1(L(y)) = y$ and this map $L$ gives an isomorphism $\text{Crit}(f_T) \approx \text{Crit}(F^\text{an})$. For a complex manifold $Z$ with a holomorphic function $h \in O_Z$, a point $p \in Z$, and a coordinate system $z_1, z_2, \ldots, z_n$ near $p$, we define the Jacobian algebra and the Tyurina algebra of $h$ at $p$ by
\[
J(h)_p := \frac{O_{Z,p}}{(\partial_{z_1} h, \partial_{z_2} h, \ldots, \partial_{z_n} h)},
\]
\[
T(h)_p := \frac{O_{Z,p}}{(h, \partial_{z_1} h, \partial_{z_2} h, \ldots, \partial_{z_n} h)},
\]
where $O_{Z,p}$ is the ring of analytic germs of the holomorphic functions at $p$. The formal Taylor expansion of $h$ at $p$ is denoted by $h_p$. This is an element of the formal completion $\hat{O}_{Z,p}$.

For $y \in \text{Crit}(f_T)$, take an open neighborhood $V \subseteq U \subseteq X$ of $y$ and a coordinate system $t_1, t_2, \ldots, t_n$ on $V$ with $t_1 = \log g_1, t_2 = \log g_2, \ldots, t_r = \log g_r$ and $t_1(y) = \cdots = t_n(y) = 0$. Using this coordinate system, we easily see that
\[
J(F^\text{an})_{L(y)} \cong J(f_T)_y.
\]
Hence $L(y)$ is an isolated singular point of $F^\text{an}$ if and only if $y$ is an isolated singular point of $f_T$. For an isolated singular point $y \in \text{Crit}(f_T)$, we will show that
\[
[\text{MF}(f_T, y)] \cong [\text{MF}(F^\text{an}_{L(y)})].
\]
Using the coordinate system $t_1, t_2, \ldots, t_n$ near $y \in \text{Crit}(f_T)$, we define $F_{an}^n \in \mathcal{O}_{V \times \mathbb{C}_y, i(g)}$ by

$$f_T - \lambda_1 t_1 - \lambda_2 t_2 - \cdots - \lambda_r t_r.$$ 

By easy computation, we see that

$$T(F_{an}^n)_{L_y(\lambda)} \cong T(F_{an}^n)_{L_y(\lambda)} \cong T(f_T)_y.$$ 

By a theorem of Mather and Yau [27], we see that

$$\mathcal{O}_{V \times \mathbb{C}_y, i(g)}/F_{an}^n \cong \mathcal{O}_{V \times \mathbb{C}_y, i(g)}/F_{an}^n.$$ 

Combining with [28, Theorem 2.10], we have

$$\text{MF}(\hat{F}_{an}^n) \cong \text{MF}(\hat{F}_{T,i}(g)).$$ 

On the other hand, by the Knörrer periodicity [26, Theorem 3.1], see also [10, §2.1], we have

$$\text{MF}(\hat{F}_{T,i}(g)) \cong \text{MF}(f_T)_y.$$ 

Thus we see that

$$\text{MF}(\hat{f}_T,y) \cong \text{MF}(\hat{F}_{an}^n).$$ 

**Definition 7.1.** Suppose that $f_T$ has isolated singularities. We define a triangulated category of matrix factorizations of $F_{an}$ by

$$\text{Br}^H(F_{an}) := \prod_{p \in \text{Crit}(F_{an})} \text{MF}(F_{an}^n).$$ 

**Proposition 7.2** (cf. [19, Theorem 1.2]). Suppose that $f_T$ has isolated singularities. Then we have

$$\text{Br}^H(F_{an}) \cong [\text{Br}(f_T)].$$ 

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