Paradoxical decompositions and finitary rules

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Abstract

We colour every point $x$ of a probability space $X$ according to the colours of a finite list $x_1, x_2, \ldots, x_k$ of points such that each of the $x_i$, as a function of $x$, is a measure preserving transformation. We ask two questions about a colouring rule: (1) does there exist a finitely additive extension of the probability measure for which the $x_i$ remain measure preserving and also a colouring obeying the rule almost everywhere that is measurable with respect to this extension?, and (2) does there exist any colouring obeying the rule almost everywhere? If the answer to the first question is no and to the second question yes, we say that the colouring rule is paradoxical. A paradoxical colouring rule not only allows for a paradoxical partition of the space, it requires one. We pay special attention to generalizations of the Hausdorff paradox.

Key words: measurable colouring, Cayley graphs, Hausdorff Paradox, paradoxes in measure theory
1 Introduction

H. Lebesgue asked if his measure can be defined on all subsets of the real line $\mathbb{R}$. A negative answer was provided by G. Vitali. Lebesgue measure $\lambda$ on $\mathbb{R}$ is a countably additive measure that is isometry invariant and such that $\lambda([0,1]) = 1$. Therefore, in the light of Vitali’s construction, it is natural to ask if one can relax the condition of countable additivity and demand that such a measure to be defined on all subsets of $\mathbb{R}$. F. Hausdorff showed in [3] that there is no finitely additive rotation-invariant measure $\mu$ defined on all subsets of the unit sphere $S^2$ and such that $\mu(S^2) = 1$. His construction can be easily modified to obtain that there is no isometry invariant finitely additive measure $\mu$ defined on all subsets of $\mathbb{R}^n$ ($n \geq 3$) and such that $\mu([0,1]^n) = 1$ (see [6] for a simple proof). Finally, S. Banach [1] proved that there is a finitely additive measure with required properties in $\mathbb{R}^n$ for $n = 1, 2$ (see [6], Cor. 12.9). Both Vitali’s and Hausdorff’s constructions use Axiom of Choice (AC) for uncountable families of sets. Namely, the authors construct some sets which then satisfy some rules (related to partitions of the space) that violate conditions required by the measures in question.

In the present paper we reverse the above perspective. We consider how to partition a probability space $X$ into subsets $A_1, \ldots, A_k$ according to rules defined with a group or semi-group $G$ acting on the space $X$ in a measure preserving way. We call such rules colouring rules, and we are interested in colouring rules where the only sets that satisfy these rules cannot be measurable with respect to any finitely additive $G$-invariant measure $\mu$ extending the initial probability measure such that $\mu(X) = 1$. Furthermore we are not really interested in such rules that are merely inconsistent, but rather in those for which, using AC, the rule can be satisfied in some way. We call colouring rules paradoxical when they can be satisfied almost everywhere but never in the finitely additive measurable way described above.

Let us analyze closer the Hausdorff’s construction in [3] that the unit sphere $S^2$ can be decomposed, modulo a countable set $D$ (the intersection with the sphere of the set of all axes from the appropriate subgroup of rotations) into the sets $A, B$ and $C$ satisfying the relation

\[ (*) \quad A \cong B \cong C \cong B \cup C, \]

where $\cong$ denotes the congruence of sets. Here the congruences are witnessed by a subgroup of $SO_3(\mathbb{R})$ isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ (and for convenience we
will call this subgroup \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \). With \( \sigma \) generating \( \mathbb{Z}_2 \) and \( \tau \) generating \( \mathbb{Z}_3 \), we have the set of rules \( \sigma(A) = B \cup C \), \( \sigma(B \cup C) = A \), \( \tau(A) = B \), \( \tau(B) = C \), and \( \tau(C) = A \). We call these rules the \textit{Hausdorff colouring rules}. Measurability with respect to any measure (where both \( \sigma \) and \( \tau \) are still measure preserving) would require that both half and one-third of the space must be taken by the set \( A \). And from these three sets, we can also create six sets that partition the whole space, namely \( A \cap \sigma(B) \), \( A \cap \sigma(C) \), \( \tau(A \cap \sigma(B)) \), \( \tau(A \cap \sigma(C)) \), \( \tau^2(A \cap \sigma(B)) \) and \( \tau^2(A \cap \sigma(C)) \), such that by choosing the appropriate rotations these six sets are congruent to two copies of the whole sphere, modulo the countable set \( D \).

Now we perceive the above congruences, using the actions \( \sigma \) and \( \tau \), as two sets of colouring rules, with the colours \( A \), \( B \), and \( C \). We colour any point \( x \) according to the colours of \( \tau^{-1}(x) \) and \( \sigma(x) \). Suppose that \( \tau^{-1}(x) \) is coloured already by \( B \) and \( \sigma(x) \) is coloured already by \( C \); the \( \sigma \) part of the rules say that \( x \) should be coloured \( A \) and the \( \tau \) part of the rules say that \( x \) should be coloured \( C \). How could this conflict be reconciled? One way would be to move further afield to determine the colour of \( x \). For example, one could specify a new colour \( E \) that can exist at only one point in every orbit of \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \), and then colour the rest of the orbit by \( A \), \( B \), and \( C \) according to the group element needed to move to the point in question from the representative coloured \( E \) (as was done in the original construction of the Hausdorff paradox). After doing this, one could then assign the subset \( E \) to the appropriate colour in \( A \), \( B \), or \( C \) as needed. To insure that there is one and only one representative coloured \( E \) in every orbit we could require that \( gx \) in \( E \) for any \( g \in \mathbb{Z}_2 \ast \mathbb{Z}_3 \setminus \{ e \} \) implies that \( x \) cannot be in \( E \) and if \( gx \) is not in \( E \) for any \( g \in \mathbb{Z}_2 \ast \mathbb{Z}_3 \setminus \{ e \} \) then \( x \) must be in \( E \). However for our purposes such a set of rules based on the colour \( E \) is not satisfactory, not only because an extra colour has been added. More importantly such a colouring rule would be infinite in character. The determination of membership in \( A \), \( B \), \( C \), or \( E \) would involve an unbounded number of group elements.

We are interested only in colouring rules that are \textit{finitary} in character, meaning that there are finitely many colours and the colour of a point is determined by finitely many group or semi-group elements. For the rest of this paper we will assume that all colouring rules are finitary. Of special interest are rules that always allow for some assigned colour (unlike the above Hausdorff rules); we say that such colouring rules have \textit{rank one}. Rank one means that no matter how other point are coloured, there is always an acceptable colouring for \( x \). Rank one rules relate measure theoretic paradoxes
to problems of optimisation, especially when that optimisation is limited in scope. A determination of one best colour (or a subset of equivalently optimal colours) according to finite many parameters is within the grasp of an automaton.

Furthermore we allow for colouring rules that depend on the position in the space $X$. For example, let $X$ be $\{0, 1\}^G$ where the group or semi-group $G$ acts on $X$ by the canonical right action: $(g(x))^h = x^{hg}$. There could be two distinct set of rules, depending on whether the coordinate $x^e$ is equal to 0 or 1. We identify a special class of colouring rules: those that do not depend on position, called stationary rules, with the others called non-stationary rules. The distinction of stationary vs non-stationary plays an important role in this paper, as there are spaces for which it is relatively easy to find a paradoxical rank one non-stationary rule, yet it is unknown whether there are rank one stationary paradoxical rules with the same number of colours. As we prefer colouring rules that can be preformed by an automaton, it should be easy to determine which part of the rule should apply. In general, we requires that the change in rules should be measurable with respect to the original probability distribution. But in all example presented here the spaces are Cantor sets and the domains for the constant parts of the rules are cylinder sets.

Ramsey Theory is a weakly related topic. A non-amenable group $G$ could act on two probability spaces $X$ and $Y$ such that every element in $G$ is measure preserving in both spaces. Assume there is a paradoxical colouring rule for the space $X$ using the group $G$ and $Y$ possesses a finitely additive $G$-invariant extension defined on all subsets. The consequence would be that these rules cannot be realised in any way on the space $Y$, though they would not be contradictory in a logical way that is independent of the space. The failure of the colouring rule on part of such a space $Y$ is analogous to the existence of a forbidden structure with Ramsey Theory. The main difference with Ramsey Theory is that the relevant relations between the vertices to be coloured (e.g. with edges or arithmetic progressions) are not locally bounded, while our colouring rules are so bounded.

The above mentioned reversal of orientation, the primacy of the colouring rules, is related to other approaches.

A proper colouring is a colouring of the edges of a graph such that no two vertices connected by an edge are given the same colour. Placed into our context of a group action, the edges of the graph are defined by the
generators of a group acting on the probability space, hence in general orbits isomorphic to the Cayley graph. The primary concern with Borel colouring (the questions raised in [4]); is what is the Borel chromatic number, the least number of colours needed for a proper colouring such that each colour class defines a Borel measurable set. Furthermore there is interest in the $\epsilon$-Borel chromatic number – for every positive $\epsilon$ the least number of colours needed for a proper colouring when the colours define Borel measurable sets and one exempts a measurable set of size $\epsilon$ from the proper colouring requirement. Notice with finitely many generators that proper colouring defines a colouring rule in our context. Given a free action and a common degree to the Cayley graph, unless the number of colours is larger than this common degree, proper colouring defines a colouring rule that is not of rank one. The colouring rule of the Hausdorff paradox is a restriction of the rule requiring only a proper colouring with three colours. It adds only two requirements to the rule of proper colouring, that the sets cycle $A, B, C$ in the direction of $\tau$, rather than in the direction of $\tau^{-1}$, and that $B$ opposite $C$ (with respect to $\sigma$) is not allowed. It is clear that the number of colours plays a central role in proper colouring, that colourings into Borel sets is made much easier when the number of colours increases. Likewise the number of colours plays a central role with paradoxical colouring rule; both in making measurable colouring easier but also in making it easier to transform a higher rank paradoxical colouring rule to one that is rank one and mimicks the former rule.

With a one dimensional compact continuum of colours (a probability simplex determined by two extremal colours) and a Cantor set (see conclusion more of the connection to this paper), Simon and Tomkowicz [5] demonstrated a colouring rule using a semi-group action for which there is no $\epsilon$-Borel colouring for sufficiently small $\epsilon > 0$ (meaning a colouring function that is Borel measurable and obeys the rule in all but a set of measure $\epsilon$), though there is a colouring that uses almost everywhere the two extremal colours of the continuum.

The rest of this paper is organised as follows. The second section introduces the basic definitions and some important initial results. The third section demonstrates some examples of paradoxical colouring rules. The fourth section looks at another example using semi-group action. The fifth section applies inclusion-exclusion as it pertain to probability calculations and a paradoxical colouring rules using the group generated freely by two generators. The concluding section explores open questions and directions for further study.
2 The Basics

Let $X$ be a probability space with $\mathcal{F}$ the sigma algebra on which a probability measure $m$ is defined. Usually $X$ will have a topology and $\mathcal{F}$ will be the induced Borel sets. We say that the relation $R \subseteq X \times X$ is admissible if:

(i) for any $x \in X$ there are only finitely many elements $x_1, \ldots, x_k$ of $X$ such that $(x, x_i) \in R$ for $i = 1, \ldots, k$.

(ii) for almost all $x \in X$, $R$ is irreflexive.

We will call the elements $x_1, \ldots, x_k$ appearing in the (i) the descendants of $x$; and we say that $x_i$ is in the $i$-th position of $x$.

Let $R \subseteq X \times X$ be an admissible relation. Let $A = \{A_1, \ldots, A_n\}$ be a finite set of colours. We say that $F : X \times A^k \to 2^A \setminus \{\emptyset\}$ is a colouring rule of rank one on $X$ if

(1) for every choice of of an element $b$ in $A^k$ the $F(x, b)$ is a $\mathcal{F}$ measurable function in $x$, and

(2) all descendants $x_1, \ldots, x_k$ are measure preserving transformations as functions of $x$, with respect to $\mathcal{F}$ and $m$.

The colouring rule is stationary if $F(x, b)$ is determined only by $b$. The colouring rule is continuous if the probability space $X$ has a topology, $\mathcal{F}$ are the Borel sets, and for every choice of of an element $b$ in $A^k$ the $F(x, b)$ is a continuous function of $x$. All of our examples are continuous, with the space $X$ a Cantor set, and the different applications of the rule determined by cylinder sets. A colouring rule of rank $n$ is a collection of $n$ colouring rules of rank one such that they cannot be expressed as a collection of fewer colouring rules of rank one.

A colouring $c : X \to A = \{A_1, \ldots, A_n\}$ satisfies a colouring rule $F$ of rank one at $x \in X$ if almost everywhere (with respect to $m$) it follows that $c(x) \subseteq F(x, c(x_1), \ldots c(x_n))$. It satisfies a colouring rule of higher rank if it satisfies all the correspondences $F$ defining the colouring rules of rank one that make up that colouring rule of higher rank.

We say that a rule on $X$ is non-contradictory if it is satisfied by some colouring. It is paradoxical if it is non-contradictory and for every finitely additive extension $\mu$ of $m$ for which the descendants are still measure pre-
serving there is no colouring satisfying the rule almost everywhere such that the sets defined by the colours are measurable with respect to $\mu$.

## 3 First Examples

We need to establish that there are paradoxical rules of rank one. The first example is that of a rank one non-stationary rule with five colours on a space that may fail to have any paradoxical stationary rule of rank one using the same number of colours. The second and third examples are that of a stationary rank one paradoxical rules which mimick the Hausdorff rules and paradox. With the second example this is done by doubling the space and with the third example by squaring the number of colours. We are not so interested in the second and third examples because they represent a kind of cutting of a Gordian knot. As stated in the conclusion, we would like to find some way to differentiate between these two examples and the others.

**Example 1:** Let $G$ be $\mathbb{Z}_2 \ast \mathbb{Z}_5$, with $\mathbb{Z}_2$ generated by $\sigma$ and $\mathbb{Z}_5$ generated by $\tau$. The space $X$ is $\{0, 1\}^G$ and there are five colours, $A_1, A_2, A_3, A_4, A_5$. All arithmetic is modulo 5 and we present the rules with some simplifying abuse of notation. The rule is deterministic, meaning it is a function. We assume that $\tau^{-1}(x)$ is coloured $A_i$. If either $x^e = 1$, $\tau^{-1}(x)$ is coloured $A_5$ or $\sigma(x)$ is coloured $A_1$ then $x$ is coloured $A_{i+1}$. Otherwise, if none of these three conditions apply, $x$ is coloured $A_i$.

**Theorem 1:** The colouring rule of Example 1 is paradoxical.

**Proof:** If the colouring is well defined and follows the colouring rule, there are only two possibilities: every $\tau$ cycle $(x, \tau(x), \tau^2(x), \tau^3(x), \tau^4(x))$ is all of the same colour (using a single colour $A_i$ for some $i$) or it contains all five colours. This allows us to make two independent calculations concerning the proportion of the set coloured $A_1$ (a proportion we assume exists under the assumption of finitely additive measurability and $G$ invariance).

First we estimate from below and outside the $\tau$ cycles the probability of the set coloured $A_1$. In every $\tau$ cycle, there are a certain number of points whose $e$ coordinates are 0. If just one of the $e$ coordinates is 1, it is necessary for the cycle to include all five colours. For each possibility for the number of
points whose $e$ coordinate is 0, the probabilities of these cycles are based on $\frac{1}{2^{5}} = \frac{1}{32}$ and the binomial expansion. We consider the points opposite to the cycle, namely the points \{\(\sigma(x), \sigma \circ \tau(x), \sigma \circ \tau^{2}(x), \sigma \circ \tau^{3}(x), \sigma \circ \tau^{4}(x)\}\}, and see how many of them must be coloured \(A_{1}\). These calculations are based on \(\frac{1}{160}\), with 160 = 32 \cdot 5. The probability of either one or no such points in the cycle with 0 as its $e$-coordinate is $1 + \frac{5}{32}$, and in that case there need not be any points opposite the cycle in \(A_{1}\) (as the one place $x$ in the cycle where $x^{e} = 0$ could be the one place that is coloured \(A_{5}\)). Following the same logic, if there are exactly $k$ points in the cycle whose $e$ coordinate is 0, with $2 \leq k \leq 4$, then there must be at least $k - 1$ points opposite the cycle coloured by \(A_{1}\). If $k = 5$ then of course the entire cycle may be coloured with just one colour, not requiring any opposite point to be in \(A_{1}\). The proportion of the space taken up by the set \(A_{1}\) must be at least $\frac{1}{5}\left(\binom{5}{2} + 2 \cdot \binom{5}{3} + 3 \cdot \binom{5}{4}\right) = \frac{10}{160} + \frac{20}{160} + \frac{15}{160} = \frac{9}{32}$.

Second we estimate from above and inside a \(\tau\) cycle the probability of the set coloured \(A_{1}\). The probability of a \(\tau\) cycle having the same colour throughout cannot exceed $\frac{1}{32}$, the probability that all members of the cycle have 0 for the $e$ coordinate. Otherwise at most one-fifth of the cycle can be in \(A_{1}\). This implies that the proportion of the set coloured \(A_{1}\) in the space cannot exceed $\frac{1}{32} + \frac{1}{5} \cdot \frac{31}{32} = \frac{9}{40}$. Therefore the proportion, if it exists, must be below \(\frac{9}{40}\) and above \(\frac{9}{32}\), a contradiction.

A non-measurable colouring satisfying the above rule is essentially the same as that for the Hausdorff paradox: all \(\tau\) cycles go through all five colours and all but one of the five points opposite to any cycle opposite are coloured \(A_{1}\) (and the $e$ coordinates are ignored).

**Example 2:** Let \(G = \mathbb{Z}_{2} \ast \mathbb{Z}_{3}\) and \(X = \{R, S\} \times \{0, 1\}\) with half probability given to both \(\{R\} \times \{0, 1\}\) and \(\{S\} \times \{0, 1\}\). We define \(\rho\) to be the measure preserving involution that switches between \((R, x)\) and \((S, x)\) for every \(x \in \{0, 1\}\). As before, \(\mathbb{Z}_{2}\) is generated by \(\sigma\) and \(\mathbb{Z}_{3}\) is generated by \(\tau\). We use the three colours \(A, B, C\), the same from the Hausdorff rule. The new rule is as follows. It assigns to every \((S, y)\) the same colour as that of \((R, y)\). The rules for colouring the points of the form \((R, y)\) are based on the Hausdorff rules and are very easy to describe. We have described above the rank two rules defining the Hausdorff paradox, determining the colour of \(y\) according to the colours of \(\sigma(y)\) and \(\tau^{-1}(y)\). If the colour of \((S, y)\) follows the Hausdorff rule with respect to \((R, \sigma y)\) and \((R, \tau^{-1}y)\) (after dropping the \(R\) and \(S\)), then colour \((R, y)\) the same colour as that of \((S, y)\). Otherwise colour \((R, y)\) differently than \((S, y)\), with either other way valid. In any colouring
satisfying the rule the points \((R, y)\) and \((S, y)\) are given the same colour, implying that the Hausdorff rules are being followed on both copies of \(\{0, 1\}^G\).

**Example 3:** Return to the uncopied space \(X = \{0, 1\}^{\mathbb{Z}_2 \times \mathbb{Z}_3}\). By using nine colours, namely the pairs \((A, A), (A, B), \ldots\), we can create an analogous rule to that of Example 2, where the second colour of \(x\) is the copy of first colour of \(\sigma x\), and the first colour of \(x\) agrees with the second colour of \(\sigma x\) if and only if the second colour of \(\sigma x\) is the correct colour for \(x\) when following the Hausdorff rule with respect to the first colours of \(\sigma x\) and \(\tau^{-1}(x)\). This shows that there is a stationary rank one paradoxical colouring rule that recreates the Hausdorff paradox after dropping the second colour.

4 A colouring rule that is paradoxical using a semi-group

**Example 4:** Let \(G = \mathbb{N}_0 * \mathbb{Z}_2\) be the free product with generators \(T\) and \(\sigma\), respectively, where \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\), meaning that only non-negative powers of \(T\) are used. There are two colours, \(A\) and \(B\). Let \(X = \{0, 1\}^G\) be the Cantor space with the canonical measure \(m\). Consider the following colouring rule \(Q\) involving two colours \(A\) and \(B\); (opposite to means using the other colour):

\[
\begin{align*}
(i) & \text{ if } Tx \text{ is coloured } A \text{ then colour } x \text{ opposite to the colour of } \sigma x; \\
(ii) & \text{ if } Tx \text{ is coloured } B \text{ and } x^e = 0 \text{ then colour } x \text{ the same as colour of } \sigma x; \\
(iii) & \text{ if } Tx \text{ is coloured } B \text{ and } x^e = 1 \text{ then colour } x \text{ opposite to the colour of } \sigma x;
\end{align*}
\]

It seems that this colouring rule \(Q\) must be more than paradoxical, namely contradictory. Every pair \(x, y \in X\) determine four pairs of points \(z\) and \(\sigma x\) such that \(Tz = x, T\sigma z = y\), namely the four possibilities of \(z^e = 0, 1\) combined with \((\sigma z)^e = 0, 1\). Furthermore, from the measure preserving properties of \(T\) and \(\sigma\) the distribution \(m\) on \(X\) is determined independently by those four choices of pairs and a choice of \(x\) and \(y\) in \(X \times X\) in the \(T\) and \(T\sigma\) positions, according to the product measure \(m \times m\). If both \(x\) and \(y\) are coloured \(B\), then half of the four pairs of possibilities of \(z\) and \(\sigma z\) cannot
be coloured at all, since if $z^e = 0$ and $(\sigma z)^e = 1$ (or vice versa) there is no way to colour one or the other of these two points. Also, if $x$ is coloured $B$ and $y$ is coloured $A$ then either the $z$ satisfying $z^e = 0$ or its twin $\sigma z$ cannot be coloured. Hence from the measure preserving property of $T$ and $\sigma$, any positive measure $r$ given to the subset coloured $B$ results in at least $\frac{r}{4}$ of the space being uncolourable according to the rule. On the other hand, from the probability of at least $\frac{1}{2}$ that $(\sigma x)^e$ is equal to 1, the subset coloured $A$ cannot exceed a size of $\frac{5}{6}$, meaning that the colouring rule cannot hold in at least $\frac{1}{24}$ of the space. Clearly a measurable colouring is not possible by any interpretation of measurability. But even without assuming that the sets coloured $A$ and $B$ are measurable in any way, it seems that if they are to be coloured, then it must be in a measurable way. If the colouring rule holds almost everywhere relative to the Borel measure (with or without measurability of any kind to the colouring), from the above logic the whole set $B$ presents a problem to the colouring of $T^{-1}(B)$– therefore $B$ should be contained in a set of Borel measure zero. But this means that $B$ is measurable and of measure zero with respect to the completion of the Borel measure, a contradiction.

Indeed showing that this colouring rule $Q$ is paradoxical (e.g. also satisfiable) is different from the previous examples. Rather than first showing that the rule cannot be satisfied in any measurable way and then showing satisfaction of the rule by some colouring, we reverse the process and show first the satisfaction. In order to do this, we must use only part of the set $X$ and redefine the Borel measure to apply only to this subset. In this special subset of $X$, the set coloured $B$ will never be in the image of $T$, hence the problem presented above will be avoided.

We define the functions $\alpha : X \to X$ and $\beta : X \to X$ by $\alpha(x)^e + x^e = 1$ (modulo 2), $\beta(x)^\sigma + x^\sigma = 1$ (modulo 2), and otherwise $\alpha(x)^g = x^g$ for all $g \neq e$ and $\beta(x)^g = x^g$ for all $g \neq \sigma$. In other words, $\alpha$ switches the $e$ coordinate and $\beta$ switches the $\sigma$ coordinate. Both $\alpha$ and $\beta$ are measure preserving transformations. We include $\alpha(x)$ and $\beta(x)$ among the descendents of $x$, although they will play no role in the colouring of $x$. We extend the semigroup $G$ to $\overline{G}$, the semigroup generated by $G$ and $\alpha$ and $\beta$. From the above analysis, for every pair of subsets $D, E \subseteq X$, $\alpha$ and $\beta$ map the set $T^{-1}(D) \cap \sigma T^{-1}(E)$ back to itself.

We use the same approach that has been applied to Wiener processes and the restriction to the paths that are continuous. With this approach, $\Omega$ is a topological space with a regular Borel probability measure and $C$ is a non-
measurable subset of outer measure one. The subset $C$ inherits the original structure of the Borel measure through intersection, so that any Borel set $B$ of $\Omega$ defines a measurable subset $B \cap C$ of $C$ whose measure $m(B \cap C)$ is the same as $m(B)$ (see [2] for more details).

We will show first that there is a colouring following the rule $Q$ on a subset $X'$ of outer measure 1. Except for the outer measure, the method is similar to that used in Simon and Tomkowicz [5]. The measure preserving property of the $T, \sigma, \alpha$ and $\beta$ remain after such a restriction to the set $X'$.

We say that a set $S(x) \subseteq X$ is the semigroup orbit of $x$ if it has the form 
\[ \{ y \in X : y = U(x) \text{ and } U \text{ is a semigroup word in } T, \sigma \} \].
A subset $A$ of $X$ is pyramidic if for every $x \in A$ the semigroup orbit of $x$ is also in $A$. The link $R(y)$ of a point $y$ is $\{ U^{-1}(y) | U \text{ is an element of } G \}$. The semi-group orbit of a point is a countable set, while its link is an uncountable set. However due to the measure preserving property of all elements of $G$ and that there are countably many members of $G$, all links are sets of measure zero (following from the atomless character of the probability space).

We define the outer measure $m^*$ of $Y \subseteq X$ by 
\[ \inf \{ m(L) : Y \subseteq L : L \text{ is an open set of } X \} \).

Assuming the continuum hypothesis or weaker Martin’s axiom we have the following theorem:

**Theorem 2:** There exists a pyramidic subset $X'$ of outer measure one and a colouring $c$ of $X'$ that satisfies the rule $Q$ such that $T(X') \subseteq A$ and the set of points coloured $A$ has outer measure 1.

**Proof:** We will be proceeding by transfinite induction. First we order all uncountable compact subsets of $\{0, 1\}^G$ with positive measure $m$ into a transfinite sequence $\{K_\alpha\}$. Since $\{0, 1\}^G$ is a separable space and the open sets have a countable base, there are continuum many compact sets of $\{0, 1\}^G$. Pick now a point $x_0 \in K_0$ and colour it $A$. Then consider the semigroup orbit $S_0$ of $x_0$ and colour it in the way that $TV(x_0)$ is always coloured $A$, where $V$ is a semigroup word in $\sigma$ and $T$. Clearly, the colouring of $S_0$ satisfies the rule $Q$.

Suppose now that we have coloured the semigroup orbits $S_0, ..., S_\beta$ for some ordinal $\beta < c$. Clearly, the set $T_\beta = \bigcup_{i \leq \beta} S_i$ has the cardinality $m$ less
then the continuum \( c \) (this follows from the fact that any cardinal number is the least ordinal number with the given cardinality).

Now, given a point \( y \in T_\beta \), we consider the set of links \( R(y) \) and then the set \( W_\beta = \bigcup_{y \in T_\beta} R(y) \). As the links are sets of measure zero, the set \( W_\beta = \bigcup_{y \in T_\beta} R(y) \) is also a set of measure zero. Now take a point \( x_{\beta+1} \) in \( K_{\beta+1} \setminus W_\beta \). Clearly, for any word \( U \in G \) we have \( U(x_{\beta+1}) \notin T_\beta \). Hence we colour \( x_{\beta+1} \) in \( A \) and we colour the remaining points in the semigroup of \( x_{\beta+1} \) in the way to satisfy the rule \( Q \). Therefore for any compact set \( K_\alpha \) there is a point \( x_\alpha \in K_\alpha \) coloured \( A \). If \( A \) failed to have outer measure one, there would have to be a cover of \( A \) by open sets whose measures add up to strictly less than 1, hence there would have to be a compact set of measure strictly greater than 0 with an empty intersection with \( A \), something we have excluded through the construction. Finally we define \( X' \) as the union of the sets coloured \( A \) and \( B \). \( \square \)

Still there does seem to be something wrong with the above theorem. With \( T \) measure preserving, one could think that \( X' = T^{-1}T(X') \) should be the same measure as that of \( T(X') \), which is the subset coloured \( A \). This would suggest, after an abuse of notation, that \( B = \sigma(A) \) is also of measure one, a contradiction. But we never imply in the context of intersection with \( X' \) that \( T(X') \) is a measurable set, and indeed the measurability of \( X' \) does not imply the measurability of \( T(X') \) (as it is an uncountable to one function).

**Corollary:** The colouring rule \( Q \) applied to \( X' \) is paradoxical, given that \( \alpha \) and \( \beta \) are descendants and the finitely additive extensions are \( G \) invariant.

**Proof:** Having defined \( X' \) as above, we have a colouring into \( A \) and \( B \) that satisfies the colouring rule. Now remove the initial partition of \( X' \) into the two colours \( A \) and \( B \) and consider any colouring according to the rule \( Q \) (not necessarily in the same way as we used to construct \( X' \)). We assume that this new colouring is measurable with respect to a finitely additive measure \( \mu \) which is \( G \) invariant and according to this new colouring we call \( \overline{A} \) the subset coloured \( A \) and \( \overline{B} \) the subset coloured \( B \). With both \( \alpha \) and \( \beta \) mapping \( T^{-1}(\overline{B}) \cap \sigma T^{-1}(\overline{B}) \) to itself and \( T^{-1}(\overline{B}) \cap \sigma T^{-1}(\overline{A}) \) to itself, from the \( \alpha \) and \( \beta \) invariance we must conclude (for both sets) that the four possibilities for the \( e \) and \( \sigma \) coordinates are equally likely (according to \( \mu \)). Following the same arguments as above, at least \( \frac{1}{16} \) of the space cannot not be coloured.
according to the rule $Q$, a contradiction. \hfill $\square$

**Note:** An alternative approach would be to construct $X'$ using orbits and the links defined according to larger group $G$. In the proof of Theorem 2, one could try to make the set coloured $A$ equal to the image $T(X')$.

## 5 Inclusion-Exclusion principle

In this section we will present an application of the Inclusion-Exclusion principle to paradoxical rules. Let $\mu$ be any finitely additive measure. The inclusion-exclusion rule for finitely many sets $A_1, \ldots, A_k$ can be phrased as:

$$\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{j=1}^{k} (-1)^{j+1} \left( \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq k} \mu(A_{i_1} \cap \ldots \cap A_{i_j}) \right).$$

Consider now some sets $A_1, A_2$ and $A_3$ contained in $X$ and such that $\mu(A_1) + \mu(A_2) + \mu(A_3) = 1$. Then, by the Inclusion-Exclusion principle and the fact that $\mu(X) = 1$, we get that

$$\mu(X) - \mu(A_1 \cup A_2 \cup A_3) = \mu(A_1 \cap A_2) + \mu(A_2 \cap A_3) + \mu(A_1 \cap A_3) - \mu(A_1 \cap A_2 \cap A_3) = 0.$$

Clearly, the same equality works, mutatis mutandis, for any $k$ subsets $A_1, \ldots, A_k$ of $X$ such that $\mu(A_1) + \ldots + \mu(A_k) = 1$. Of particular interest is when $B_1, \ldots, B_k$ partition a probability space and $A_i = \sigma_i B_i$ for some measure preserving transformations $\sigma_1, \ldots, \sigma_k$. If we can show that $\mu(A_i \cap A_j) = 0$ for all $i \neq j$ then any argument that $X \setminus (A_1 \cup \cdots \cup A_k)$ should have positive measure would be a contradiction to the measurability assumption.

The following example demonstrates a non-stationary paradoxical rule with four colours for the space $\{0, 1\}^F$ using the inclusion-exclusion principle. As $F_2$ is isomorphic to a subgroup of $\mathbb{Z}_2 \ast \mathbb{Z}_3$, this example shows also that there is a paradoxical colouring rule for $\{0, 1\}^{\mathbb{Z}_2 \ast \mathbb{Z}_3}$ with four colours. With $\sigma$ the generator of $\mathbb{Z}_2$ and $\tau$ the generator of $\mathbb{Z}_3$, this follows from the fact that $\tau \sigma^2 \sigma$ and $\tau^2 \sigma \tau \sigma$ generate of a subgroup of $\mathbb{Z}_2 \ast \mathbb{Z}_3$ isomorphic to $F_2$.

**Example 5:** With $X = \{a, b\}^F$, initially there will be two colours, $P$ and $N$ for positive and negative. Before specifying the rule determining the
colouring, let $P_i$ be the subset coloured positive such that the $e$ coordinate is $i = a, b$, and define $N_i$ in the same way. Let $T_a$ and $T_b$ be the two generators of $\mathbb{F}_2$. If we can create a rank one rule such that the four sets $T_a(N_a)$, $T_a^{-1}(P_a)$, $T_b(N_b)$, $T_b^{-1}(P_b)$ are mutually disjoint almost everywhere, then automatically the rule must be paradoxical. This follows from the fact that the set of $x$ such that $T_a(x)^e = b$, $T_a^{-1}(x)^e = b$, $T_b(x)^e = a$, $T_b^{-1}(x)^e = a$ is $\frac{1}{16}$ of the whole space, yet by the inclusion-exclusion principle the measure of this set must be 0.

To define the rule, we will create four colours by splitting the initial two colours $P$ and $N$ into four colours, $P^u$, $P^c$, $N^u$ and $N^c$, the $C$ or $U$ for whether the vertex is “crowded” or “uncrowded”. Given any particular colouring, we place an arrow from every $x$ to one of its four neighbours. If $x^e = a$, we place an arrow from $x$ to $T_a(x)$ if $x$ is coloured either $P^u$ or $P^c$ and otherwise place an arrow from $x$ to $T_a^{-1}(x)$. Likewise if $x^e = b$, we place an arrow from $x$ to $T_b(x)$ if $x$ is coloured $P^c$ or $P^U$ and otherwise place an arrow from $x$ to $T_b^{-1}(x)$. Now in defining the rule, we have to decide for every $x \in X$ whether to colour it $P$ or $N$ and whether to colour it crowded $C$ or uncrowded $U$. The rule is relatively simple. If $x^e = d$ with $d \in \{a, b\}$ and $T_d(x)$ is uncrowded and $T_d^{-1}(x)$ is crowded then colour $x$ with $P$. Likewise of $x^e = d$ and $T_d(x)$ is crowded and $T_d^{-1}(x)$ is uncrowded, colour $x$ with $N$. Otherwise if the appropriate adjacent vertices are both uncrowded or crowded then colour $x$ however one wants. A vertex is coloured “crowded” or $C$ if there are two or more arrows pointed toward it and it is coloured “uncrowded” or $U$ if there is one or no arrows pointed toward it.

**Theorem 3:** The colouring rule of Example 5 is paradoxical.

**Proof:** We define the degree of a vertex as the number of potential arrows that could be pointed toward this vertex. In other words, if $(T_a(x))^e = a$, $(T_a^{-1}(x))^e = b$, $(T_b(x))^e = b$ and $(T_b^{-1}(x))^e = b$ then the degree of $x$ is 3. If $(T_a(x))^e = b$, $(T_a^{-1}(x))^e = b$, $(T_b(x))^e = a$ and $(T_b^{-1}(x))^e = a$ then the degree of $x$ is 0. As mentioned before, the degree zero vertices take up $\frac{1}{16}$ of the space. Our claim is that with a colouring satisfying the rule almost all vertices are uncrowded. This implies that the colouring cannot be measurable with respect to any finite extension for which the group $\mathbb{F}_2$ is measure preserving, because at least $\frac{1}{16}$ of the vertices cannot have any arrows points toward them. By one accounting there is an arrow exiting every vertex but by another accounting the average number of arrows coming in to vertices must
be no more than $\frac{15}{16}$.

Now let us assume that $x$ is a crowded vertex and see what is necessary to maintain this situation in a colouring satisfying the rule. There must be two distinct vertices $y$ and $z$ such that there is an arrow from $y$ to $x$ and an arrow from $z$ to $x$. Let's focus on just one of them, without loss of generality the $y$. As the colouring rule is satisfied, the existence of an arrow from $y$ to $x$ implies that the vertex opposite to $x$ from $y$ is another crowded vertex, call it $w$. As the arrow is already defined from $y$ to $x$ it means that there are at least two arrows pointed inward to $w$ that do not start at $y$. Letting $v_1$ and $v_2$ be two of those vertices, let $u_1$ and $u_2$ be the vertices for which there could have been an arrow from $v_i$ to $u_i$, however instead the arrow was from the $v_i$ to $w$. We recognise by induction the existence of a chain of backwardly directed arrows, starting at $w$, moving to $u_1$ and $u_2$ and beyond, such that the induced graph is binary, has two branches at every stage. (If there are three such branches in some places, we could reduce to the existence of a binary tree). Each of these vertices of the binary tree has degree at least three. Now let $p$ be the probability of there existing such an infinite chain, the probability relative to the start at a vertex like $y$ moving in the direction away from $x$. As the space is defined homogeneously (that the probabilities for $a$ or $b$ are independent regardless of shift distances) we can calculate $p$ recursively. There are two possibilities, the next vertex $w$ could be of degree three and the chain of backward arrows continues with these two adjacent vertices on the other side of $y$, or $w$ is degree four and the chain continues with at least two of these three vertices on the other side of $y$. In the first case the conditional probability that $w$ is of degree three is $\frac{3}{8}$ (conditioned on the move from $y$ to $w$) and then a continuation of the chain indefinitely happens with probability $p^2$. In the second case, the conditional probability of degree four is $\frac{1}{8}$ and the probability of continuation $3p^2 - 2p^3$ (three choices for the two next vertices minus the possibility, counted twice, that continuation is possible in all three directions). We have the formula $p = \frac{3}{8}p^2 + \frac{1}{8}(3p^2 - 2p^3) = \frac{3p^2 - p^3}{4}$. Factoring out the $p = 0$ solutions, we are left with $p^2 - 3p + 4$, which has no real solutions. We conclude that the stochastic structure of degrees does not allow for crowded points to exist in more than a set of measure zero.

Now we show (using AC) that there does exist a colouring satisfying the rule. For every orbit choose a representative $x$ and label every other vertex in this same orbit as $gx$ by the group element $g$ used to travel from $x$ to $gx$. 
Every group element $g$ has a length, the minimal number of uses of $T_a, T_a^{-1}, T_b$ and $T_b^{-1}$ used to construct $g$ (where the length of the identity is zero). Let the length of a vertex $gx$ in the orbit be the length of $g$. Colour $x$ with either colour. Colour all vertices of length 1 next, then all vertices of length 2, and so on. At every stage of the process, a vertex of length $l$ is adjacent to one vertex of length $l + 1$ and three of length $l - 1$. Therefore from any vertex $y$ of length $l$ one can always point the arrow toward a vertex of length $l + 1$, regardless of the value of $y$. There can be no other arrow toward $y$, as the three other points adjacent to $y$ are all of length $l + 2$.

6 Further study

Question 1: Given any probability space $X$ and a finitely generated measure preserving group $G$ acting on $X$, let $P$ be the colouring rule that requires only that the colouring must be proper. Is there such an $X$ and $G$ such that the rule $P$ is paradoxical?

A colouring rule with $k$ colours is (stationarily) essential if it is paradoxical of rank at least two and there exists no (stationarily) paradoxical rank one finitary rule with $k$ colours whose satisfaction implies the satisfaction of the original rule. A space $X$ is (stationarily) essential with $k$ colours if there is no (stationarily) paradoxical rank one colouring rule with $k$ colours defined on $X$, and yet there is a paradoxical finitary rule with $k$ colours defined on $X$ of higher rank. Intriguing is the difference between what can be accomplished by stationary and non-stationary colouring rules.

Question 2: Is there is a non-stationary rank one paradoxical colouring rule on $X = \{0, 1\} \mathbb{Z}_2 \ast \mathbb{Z}_3$ using the canonical measure preserving group action of $\mathbb{Z}_2 \ast \mathbb{Z}_3$ and three colours, but no such stationary paradoxical rank one paradoxical colouring rule with three colours?

We conjecture that the answer to Question 2 is yes. In other words, we conjecture that this space with this group action is stationarily essential but not essential. If indeed it is not stationarily essential, there remains the question of whether the conventional Hausdorff rule of rank two is essential, utilising either a stationary or non-stationary rule involving a larger number of descendants.
**Question 3:** Does there exist a space and a semi-group action that is essential with respect to any number of colours?

Whenever at least one of the descendents is invertible and its inverse is measure preserving (the latter following from the former whenever the space is compact and the measure is Borel) then a colouring rule of any rank with \(k\) colours can be mimicked by a rank one colouring rule with \(k^2\) colours in the same way as with Example 3. But if none of the descendants are invertible, it is plausible that some paradoxical colouring rules are not multi-functional in character no matter how many colours are allowed.

There is something satisfying about Example 5 and unsatisfying about Examples 2 and 3 (with Example 1 somewhere in between). Example 5 employs a stochastic process that seems to push colourings toward the satisfaction of the rule. On the other hand, the existence of colourings witnessing the rules of Examples 2 and 3 seems to be either accidental or contrived. One could perceive satisfaction of a colouring rule to be a kind of fixed colouring, with the colouring rule defining some kind of iterative process that does or doesn’t bring the colourings closer to satisfaction of the rule. Of course if any colouring rule forced almost everywhere (with respect to the original measure \(m\)) an eventually stable colour in finitely many colouring stages (with respect to some initial measurable colouring not satisfying the rule) then there would be measurable colouring solutions and the rule could not be paradoxical.

**Question 4:** Is there a refinement to the definition of a rank one paradoxical rule that identifies a credible force toward its satisfaction?

Suppose one had a colouring rule for \(S^G\) where \(S\) is a finite set and \(G\) is a group, and \(C\) are the finite set of colours. Another structure to consider is \((S \times C)^G\), where we assume a random colouring start to \(S^G\) inherited from the \(C\) coordinate. We could consider how the colouring rule generates iterations of colourings on \((S \times C)^G\).

The *deficiency* of a colouring rule on \(X\) is the infimum of all probabilities \(\rho\) such that there is a \(\mathcal{F}\) measurable subset of size \(1 - \rho\) such that the finitary rule *can* be satisfied on this subset. The paradoxical size of a paradoxical colouring rule on \(X\) is its deficiency. Likewise we define the paradoxical size of a colouring rule as the difference between its deficiency and the supremum on all \(r\) such that there is a satisfaction of the rules on a \(\mathcal{F}\) measurable subset of size \(1 - r\).
Do paradoxical colouring rules of arbitrarily small paradoxical sizes get converted in an organic way to full paradoxical decompositions? With the Hausdorff paradox the ambiguity concerning the size of the set coloured $A$, that it should be both $\frac{2}{3}$ and $\frac{4}{3}$, leads to a statement of “$1 = 2$” for the whole space. More generally Tarski proved that the existence of a finitely additive measure defined on all subsets is equivalent to the existence of a paradox in the conventional sense, that there are disjoint sets $B_1, \ldots, B_l$ of a subset $E$ of positive measure and measure preserving transformations $\sigma_1, \ldots, \sigma_l$ with $\sigma_1(B_1), \ldots, \sigma_l(B_l)$ forming two copies of $E$. If there is a paradoxical colouring rule on $X$ using a group $G$ of measure preserving transformations, we know automatically that $X$ is non-amenable in character and thus has such a paradoxical decomposition without any application of the paradoxical rule.

**Question 5:** Given a colouring $c$ that satisfies a paradoxical colouring rule, are there disjoint subsets $B_1, \ldots, B_l$ of a set $E$ of positive measure creating two copies of $E$ using the measure preserving descendants such that the $B_i$ sets belonging to the smallest sigma algebra containing the original $\mathcal{F}$, the sets defined by the inverse images of the colouring $c$, and the shifts of these sets as so defined by the descendents?

In the definition of a colouring rule, we use that the descendants are measure preserving. This was a natural way to connect colouring rules to measure theoretic paradoxes.

**Question 6:** Is there a more general definition for paradoxical colouring rules (implying that the sets defined by the colours cannot not be measurable by any invariant finitely additive measure) that weakens the measure preserving property of the descendants?

**Question 7:** When the space $X$ to be coloured has a topology, is there any significant distinction between the continuous and the more general non-stationary colouring rules?

Usually optimisation concerns a choice of a point in a finite dimensional convex set. We generalise to a definition of a *probabilistic* rank one colouring rule in the following way. We keep the initial set up with $X$ a probability space and the finitely many descendants, but we change what is a colour. The set of colours is $A = \prod_{i=1}^n \Delta(C_i)$, where each $C_i$ is a finite set. The
restriction on the choice of a colour is to a non-empty subset of $A$, a restriction determined by $x$ and the descendents $p_1, p_2, \ldots, p_k \in A^k$ in a measurable way. The probabilistic colouring rule is normal if these non-empty subsets of $A$ are convex and as a correspondence defined by the $x$ and the $p_1, \ldots, p_k$ is upper-semi-continuous. We define satisfaction of the colouring rule in the same way as before and understand by measurability with respect to a finitely additive extension $\mu$ of $X$ such that the inverse images of any Borel set in $A$ are $\mu$ measurable. As before, a probabilistic colouring rule is paradoxical if it can be satisfied but not by any colouring function which is measurable with respect to any finitely additive extension that maintains the measure preserving property of the descendants.

**Question 8:** Does there exist a normal probabilistic rank one paradoxical colouring rule?

This would be of great importance to applications, as its affirmation would suggest that there may be some very natural optimization problems that are solvable in some way, but not in any way for which one can make any predictions.

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