ON THE SPECTRAL FLOW FOR PATHS OF ESSENTIALLY
HYPERBOLIC BOUNDED OPERATORS ON BANACH SPACES

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To my parents and my sister

Abstract. We give a definition of the spectral flow for paths of bounded essentially hyperbolic operators on a Banach space. The spectral flow induces a group homomorphism on the fundamental group of every connected component of the space of essentially hyperbolic operators. We prove that this homomorphism completes the exact homotopy sequence of a Serre fibration. This allows us to characterize its kernel and image and produce examples of spaces where it is not injective or not surjective, unlike what happens for Hilbert spaces. For a large class of paths, namely the essentially splitting, the spectral flow of \( A \) coincides with \(-\text{ind}(F_A)\), the Fredholm index of the differential operator \( F_A(u) = u' - Au \).

INTRODUCTION

The spectral flow appeared first in [7] for a family of elliptic and self-adjoint operators \( A_t \), ascribed to a joint work of M. Atiyah and G. Lusztig. We outline their effective description as “net number of eigenvalues that change sign (from \(-\) to \(+\)” while the parameter family is completing a period” in the definition given by J. Robbin and D. Salamon in [23, Theorem 4.21]: In a neighbourhood \([t_-, t_+]\) of the real line of a point \( t \in \mathbb{R} \) (called “crossing”) such that \( 0 \in \sigma(A_t) \), \( \sigma(A_t) \) can be

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described as a finite family of continuously differentiable curves $\lambda_i: [t_-, t_+] \to \mathbb{R}$ such that $\lambda_i'(t) \neq 0$. The contribution to the spectral flow of a crossing is given by

$$\sum_{\lambda_i} \text{sign}(\lambda_i(t_+)) - \text{sign}(\lambda_i(t_-))$$

and the spectral flow is the sum of these contributions over all the crossings. The spectral flow was used to define a Morse index for 1-periodic Hamiltonian orbits and then define the Floer homology. Such construction requires some differentiability hypotheses and transversality condition, even though such requirements are dense in continuous families. P. Rabier extended their work for unbounded families of operators in Banach spaces in [22]. In [20], J. Phillips simplified their definition as follows: $A_t \in F^{sa}(H)$ was assumed to be a path of Fredholm and self-adjoint operators, on $[0, 1]$. If $U$ is a neighbourhood of the origin such that

1) $\sigma(A_s) \cap \partial U = \emptyset$ for every $s \in [t_-, t_+]$;
2) $\sigma(A_s) \cap U$ is a finite set of eigenvalues,

then the contribution to the spectral flow on the interval $[t_-, t_+]$ is defined as

$$\dim(P(A(t_+); U)) - \dim(P(A(t_-); U)).$$

The spectral flow is the sum over a partition of the unit interval, $J_i$, where a neighbourhood $U_i$ as in 1,2) exists. It is invariant for fixed-endpoints homotopies and defines a group homomorphism on each connected component of $F^{sa}(H)$. If $H$ is separable, there are only three, corresponding to $I$ and $-I$, contractible to a point, and $2P - I$, where $P$ is a projector with infinite-dimensional kernel and image. On the third one the spectral flow

$$sf: \pi_1(F_{sa}^*) \to \mathbb{Z}$$

is a group isomorphism. J. Phillips extended afterwards his definition in [8] to families of unbounded, Fredholm and self-adjoint operators. C. Zhu and Y. Long in [27] extended such definition to bounded, admissible operators on Banach spaces. Given a partition $J_i$ of $[0, 1]$ and spectral sections, that is

$$Q_i: J_i \to \mathcal{P}(\mathcal{L}(E))$$

$$Q_i(t) - P^+(A(t))$$

is compact for $t \in J_i$, the contribution of $J_i$ to the spectral flow is

$$[Q_i(t'_+) - P^+(A(t'_+))] - [Q_i(t'_-) - P^+(A(t'_-))],$$

where $[Q - P]$ denotes the Fredholm index of $P: \text{Range}(Q) \to \text{Range}(P)$. The purposes of this work are essentially three:

1) Simplifying further the definition of spectral flow. Given a path $A_t \in \mathcal{H}(E)$ of essentially hyperbolic operators (which are compact perturbation of hyperbolic operators and correspond to the admissible ones used in [27]), by the homotopy lifting property of the Serre fibration

$$p: \mathcal{P}(\mathcal{L}(E)) \to \mathcal{P}(\mathcal{C}(E)).$$

we prove that there exists a continuous path of projectors on $[0, 1]$ such that $P(t) - P^+(A(t))$ is compact. Thus, we define

$$sf(A) = [P(0) - P^+(A(0))] - [P(1) - P^+(A(1))].$$

Therefore, we do not need to partitionate the unit interval as long as we do not require $P(t)$ to be a spectral projector of $A(t)$. Such definition coincides with the
one in \[27\]. In \[3\] we show that is invariant for fixed-endpoints homotopies and
that \(\text{sf}(A * B) = \text{sf}(A) + \text{sf}(B)\).

II) Studying the homomorphism properties of the spectral flow. We prove that
\(eH(E)\) is homotopically equivalent to \(\mathcal{P}(\mathcal{C}(E))\), the space of projectors of the Calkin
algebra and that the spectral flow completes the exact homotopy sequence of the
fibration above. That allows us to characterize the kernel and the image of the
spectral flow \(\text{sf}_P\) defined on the fundamental group of the connected component of
\(2P - I \in eH(E)\). Precisely,

\[\begin{align*}
\text{h1)} & \text{ an integer } m \text{ belongs to the image of } \text{sf}_P \text{ if and only if there exists a } \\
& \text{projector } Q \text{ connected to } P \text{ by a path in the space of projectors } \mathcal{P}(\mathcal{L}(E)), \text{such that } Q - P \text{ is compact and } [P - Q] = m; \\
\text{h2)} & \text{ im}(p_*) \cong \ker(\text{sf}_P).
\end{align*}\]

The existence of a projector satisfying properties h1,2) highly depends on the struc-
ture of the Banach space \(E\). When \(E\) is Hilbert, J. Phillips proved in \[20\] that for
every projector with infinite-dimensional range and kernel, h1) with \(m = 1\) and h2)
hold. We show that this happens for the spaces \(\ell^p\) and \(\ell^\infty\). For the spaces satisfying
the hypotheses of Proposition \[14\] and Proposition \[16\] such a projector exists. The
question is strongly related with the existence of complemented subspaces isomor-
phic to closed subspaces of co-dimension \(m\). In direct sums of spaces isomorphic to
hyperplanes \(\text{sf}_P\), where \(P\) is a projector over each of the summands, is surjective;
in spaces isomorphic to hypersquares, but not hyperplanes, as the ones constructed
by W. T. Gowers and B. Maurey in \[14\], the image is \(2\mathbb{Z}\) and so on. In a Douady
space (cf. \[11\]), we prove that \(\text{sf}_P\) can be not injective or trivial, even in projectors
of infinite-dimensional range and kernel.

III) Comparing the spectral flow with the Fredholm index of the diffe-
rential operator
\[F_A : W^{1,p}(\mathbb{R}) \to L^p(\mathbb{R}), \quad u \mapsto \left( \frac{d}{dt} - A(t) \right) u,\]
We prove in Theorem 5.6 that for a large class of path, which are essentially hy-
perbolic and essentially splitting (cf. \[2\]), the equality
\[\text{ind}(F_A) = -\text{sf}(A)\]
holds. The equality above applies, for instance, in the special case where \(A\) is
a continuous, compact perturbation of a path of hyperbolic operators with some
boundary conditions (check \[2\] Theorem E]). That confirms the guess of A. Abbon-
dandolo and P. Majer in \S 7 of \[2\] that for these paths the equality above hold.

We remark that our work deals with bounded operators. The differential operator
\(F_A\) arises naturally from the linearization of a vector field \(\xi \in C^1(E, E)\) on a
solution \(v'(t) = \xi(v(t))\), such that the endpoints are zeroes of \(\xi\) and \(A(t) = D\xi(v(t))\).
If \(F_A\) is Fredholm and surjective (and the zeroes are hyperbolic), then the set
\[W_\xi(p, q) = \{ v : \mathbb{R} \to E : v'(t) = \xi(v(t)), v(-\infty) = p, v(+\infty) = q \}\]
is a sub-manifold of dimension \(\text{ind}(F_A)\). That constitutes a landmark for the study
of the Morse theory on Banach manifolds, as in \[3\]. A proof of this can be found
in \[2\] \S 8. If \(A\) fulfills the hypotheses of Theorem 5.6 \(\text{sf}(A)\) determines \(\text{ind}(F_A)\).

The spectral flow also provides naturally an index for 1-periodic solutions of
\(u'(t) = X(u(t))\). If \(DX(u_t)\) is essentially hyperbolic, then \(\text{index}_X(u) = \text{sf}(DX(u_t))\).
Thus, in order to define a Morse complex, the question whether there are loops with non-trivial spectral flow becomes relevant.

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1. Preliminaries

We recall basics definitions, results on spectral theory and Fredholm operators. Our main references are [24, Ch. X] and [15, IV.4,5].

1.1. Spectral theory. A Banach algebra is the given of an algebra $A$ with unit, 1, over the real or complex field and a norm $\| \cdot \|$ such that
- $(A, \| \cdot \|)$ is a Banach space;
- $\|xy\| \leq \|x\| \cdot \|y\|$ for every $x, y \in A$.

we denote with $G(A)$ the set of invertible elements of the algebra; it is an open subset of $A$. Given $x \in A$, the subset of the field

$\sigma(x) = \{ \lambda \in F : x - \lambda \cdot 1 \notin G(A) \}$

is called spectrum of $x$. We recall some properties of the spectrum.

Proposition 1.1. For every $x \in A$, $t \in F$ and $\Omega \subset F$ open subset,

(i) if $\sigma(x)$ is non-empty, is closed and bounded;
(ii) $\sigma(x + t) = \sigma(x) + t$, $\sigma(tx) = t\sigma(x)$;
(iii) there exists $\delta > 0$ such that $\sigma(y) \subset \Omega$ for every $y \in B(x, \delta)$;
(iv) if $A$ is complex, then $\sigma(x)$ is non-empty;
(v) if $f : A \to B$ is an algebras homomorphism such that $f(1) = 1$, then $\sigma(f(x)) \subseteq \sigma(x)$.

Given a real algebra, we can consider the complex algebra associated to it $A_C = A \otimes \mathbb{C}$. We have an inclusion of algebras

$A \hookrightarrow A_C, \ x \mapsto x \otimes 1$

and $\sigma(x) \subseteq \sigma(x \otimes 1)$. Hereafter, we will take $\sigma(x \otimes 1)$ as spectrum of $x$.

Definition 1.2. A finite family of closed curves in the complex plane, $\Gamma = \{ c_i : 1 \leq i \leq n \}$ is said simple if, for every $z \notin \bigcup_{i=1}^n \text{im}c_i$,

$\text{ind}_\Gamma(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\zeta}{\zeta - z} \in \{0, 1\}.$

we denote with $\Omega_0$ and $\Omega_1$ the subsets of the complex space such that $\text{ind}_\Gamma(z)$ is zero or 1, respectively.

Definition 1.3. An element $p \in A$ is said projector if $p^2 = p$. We can associate to it the sub-algebra $A(p) = \{ pxp : x \in A \}$, with the unit $p$. We denote with $\sigma_p$ the spectrum of the elements of $A(p)$.

If $xp = px$, we have the following property

$\sigma(x) = \sigma_p(px) \cup \sigma_{1-p}((1-p)x(1-p)).$

Theorem 1.4. Let $x \in A$, $\Sigma \subset \sigma(x)$ open and closed in $\sigma(x)$. Then, there exists a projector, called spectral projector, that we denote with $P(x; \Sigma)$ such that

(i) $\sigma_p(PxP) = \Sigma$;
(ii) $Px = xP$;
(iii) \( I - P = P(x; \Sigma^c) \);
(iv) given an algebras homomorphism, \( f : \mathcal{A} \rightarrow \mathcal{B} \), \( f(P(x; \Sigma)) = P(f(x); \Sigma) \).

Given \( \Gamma \) as above such that \( \Sigma \subset \Omega_1 \), then
\[
P(x; \Sigma) = P_\Gamma(x) := \frac{1}{2\pi i} \int_\Gamma (x - z)^{-1} dz.
\]

On the open subset \( \mathcal{A}(\Gamma) = \{ x \in \mathcal{A} : \sigma(x) \cap \Gamma = \emptyset \} \), \( P_\Gamma(x) \) is a continuous map.

For more details check [24, Theorem 10.27] or [15, Theorem 6.17].

1.2. Spaces of projectors. Given a Banach space \( E \), we denote with \( L(E) \) the space of bounded operators on \( E \) and with \( L_c(E) \) the closed linear subspace of compact operators. We denote the quotient space with \( C(E) \). It inherits a structure of Banach algebra and it is called Calkin algebra. The quotient projection \( p : L(E) \rightarrow C(E), A \mapsto A + L_c(E) \) is an algebras homomorphism.

**Definition 1.5.** Given a Banach algebra \( \mathcal{A} \), we define the following subsets
(i) \( P(\mathcal{A}) = \{ p \in \mathcal{A} : p^2 = p \} \), projectors;
(ii) \( Q(\mathcal{A}) = \{ q \in \mathcal{A} : q^2 = 1 \} \), square roots of the unit;
(iii) \( H(\mathcal{A}) = \{ x \in \mathcal{A} : \sigma(x) \cap i \mathbb{R} = \emptyset \} \), hyperbolic elements.

**Properties and remarks.** \( \mathcal{P} \) and \( \mathcal{Q} \) are closed subsets, locally arc-wise connected and analytic sub-manifold. A proof of this can be found in [4] after REMARK 1.4. They are diffeomorphic to each other through the diffeomorphism \( p \mapsto 2p - 1 \). From (iii) of Proposition 1.1, \( H(\mathcal{A}) \) is open.

**Theorem 1.6.** Given two projectors \( p, q \) such that \( \| p - q \| < 1 \) or are in the same connected component of \( P(\mathcal{A}) \), there exists \( u \in G_1(\mathcal{A}) \) such that \( up = qu \).

By \( G_1(\mathcal{A}) \) we denote the connected component of \( G(\mathcal{A}) \) of the unit. For the proof and details we refer to [4, Proposition 1.2]. The theorem above has two consequences

\begin{itemize}
\item c1) \( P(\mathcal{A}) \) is locally arc-wise connected;
\item c2) when \( \mathcal{A} = L(E) \), two projectors in the same connected component have isomorphic ranges and kernels.
\end{itemize}

The quotient projection \( p \) restricts to the subset of projectors and roots of the unit
\[
\begin{align*}
\mathcal{P}(p) : P(L(E)) &\rightarrow P(C(E)), \quad P \mapsto P + L_c(E) \\
\mathcal{Q}(p) : Q(L(E)) &\rightarrow Q(C(E)), \quad Q \mapsto Q + L_c(E).
\end{align*}
\]

**Definition 1.7.** A continuous map \( p : E \rightarrow B \) has the homotopy lifting property w.r.t. a topological space \( X \) if, given continuous maps
\[
h : X \times [0, 1] \rightarrow B, \quad f : X \times \{0\} \rightarrow E,
\]
there exists \( H : X \times [0, 1] \rightarrow E \) such that \( H(x, 0) = f(x, 0) \) and \( p \circ H = h \). If the homotopy lifting property holds w.r.t. \( [0, 1]^n \) for every \( n \geq 0 \), then \( p \), is called Serre fibration.

**Proposition 1.8.** The maps \( \mathcal{P}(p) \) and \( \mathcal{Q}(p) \) are surjective Serre fibrations.
In general, every surjective algebra homomorphism induces a Serre fibration, for a proof check \cite[Theorem 2.4]{[10]}. The surjectivity of the restriction follows, for instance, \cite[Proposition 4.1]{[4]}. In fact, \( P(p) \) and \( Q(p) \) are locally trivial fiber bundles, refer for instance \cite[Proposition 1.3]{[4]} or \cite[Theorem 4.2]{[12]}.

1.3. **Fredholm operators and relative dimension.** Let \( T \in \mathcal{L}(E, F) \) be a bounded operator. If the image of \( T \) is a closed subspace, we have two Banach spaces associated to it, namely \( \ker(T) \) and \( E/\text{Range}(T) = \text{coker}(T) \).

**Definition 1.9.** An operator as above is said **semi-Fredholm** if either \( \ker(T) \) or \( \text{coker}(T) \) are finite-dimensional spaces. If both have finite dimension, \( T \) is called **Fredholm** and the integer

\[
\text{ind}(T) = \dim \ker(T) - \dim \text{coker}(T)
\]

is the **Fredholm index**. If one between the two spaces have infinite dimension, the index is defined to be \(+\infty\) or \(-\infty\) whether \( \ker(T) \) or \( \text{coker}(T) \) have infinite dimension.

we denote with \( \text{Fred}(E, F) \) and \( \text{Fred}(E) \) the set of Fredholm operator in \( \mathcal{L}(E, F) \) and \( \mathcal{L}(E) \), respectively; \( \text{Fred}_k(E, F) \) is the set of Fredholm operators of index \( k \). Let \( T \in \text{Fred}(E, F) \), \( S \in \mathcal{L}(F, G) \) and \( K \in \mathcal{L}_c(E, F) \). We have

a) \( \text{Fred}_k(E, F) \subseteq \mathcal{L}(E, F) \) is an open subset;

b) \( T + K \in \text{Fred}(E, F) \) and \( \text{ind}(T + K) = \text{ind}(T) \);

c) \( S \circ T \in \text{Fred}(E, G) \) and \( \text{ind}(S \circ T) = \text{ind}(S) + \text{ind}(T) \);

d) given \( B \in \mathcal{L}(E, F) \), there exists \( \varepsilon > 0 \) such that the maps \( \dim \ker(T + \lambda B) \) and \( \dim \text{coker}(T + \lambda B) \) are constant on \( B(0, \varepsilon) \setminus \{0\} \);

e) \( T \in \text{Fred}(E, F) \) if and only if there exists \( S \in \mathcal{L}(F, E) \) such that \( T \circ S - I \in \mathcal{L}_c(E) \) and \( S \circ T - I \in \mathcal{L}_c(E) \).

These results hold true for unbounded operators too. Check \cite[IV.4.5]{[15]}.

**Definition 1.10.** A pair of closed subspaces \((X, Y)\) is **semi-Fredholm** if and only if their sum is closed and either \( X \cap Y \) or \( E/(X + Y) \) has a finite dimension. If both have finite dimension, then it is defined the **Fredholm index** of the pair \((X, Y)\) and

\[
\text{ind}(X, Y) = \dim X \cap Y - \text{codim} X + Y.
\]

Otherwise, the index is \(+\infty\) or \(-\infty\).

Two projectors \( P, Q \) are compact perturbation of each other if \( P - Q \in \mathcal{L}_c(E) \). In this case, the restriction of \( Q \) to \( \text{Range}(P) \) is in \( \text{Fred}(\text{Range}(P), \text{Range}(Q)) \). The **relative dimension**, between \( P \) and \( Q \) is defined as

\[
[P - Q] := \text{ind}(Q \circ \text{Range}(P) \to \text{Range}(Q)).
\]

Such definition is meant to generalize the dimension for finite-dimensional spaces to Banach spaces. The notation above is used by C. Zhu and Y. Long in \cite{[27]}, Corresponding definitions are known in Hilbert setting, considered by A. Abbondandolo and P. Majer in \cite{[1]} (see also \cite[Remark 4.9]{[9]}). A definition for pairs of closed subspaces \((X, Y)\), not necessarily complemented, can be found in \cite[Definition 5.8]{[12]}.

**Theorem 1.11.** Given pairs of projectors \((P, Q)\) and \((Q, R)\) with compact difference, we have

(i) if \( \text{Range}(P), \text{Range}(Q) \) have finite dimension, then

\[
[Q - P] = \dim \text{Range}(P) - \dim \text{Range}(Q);
\]
(ii) \(|P - R| = |P - Q| + |Q - R|\);
(iii) on the subset \(|\{(P, Q) \in \mathcal{P}(\mathcal{L}(E)) \times \mathcal{P}(\mathcal{L}(E)) : P - Q \in \mathcal{L}_c(E)\}\), the map \(|P - Q|\) is continuous;
(iv) \(|P - Q| = [(I - Q) - (I - P)]\);
(v) \((\text{Range}(P), \ker(Q))\) is a Fredholm pair and \(\text{ind}(\text{Range}(P), \ker(Q)) = |Q - P|\).

Property (iii) follows from stability results for the index of semi-Fredholm pairs, check [15, Theorem 4.30] and [12, Theorem 3.3]. For a proof of (iv) and (v) check [27, Lemma 2.3] and [12, Proposition 5.3], respectively; (i) follows from [12, Theorem 3.3] and the special case when \(PQ = Q = PQ\).

2. ESSENTIALLY HYPERBOLIC OPERATORS

We recall that a bounded operator \(A \in \mathcal{L}(E)\) - or more generally an element of a Banach algebra \(A\) - is called hyperbolic if its spectrum does not meet the imaginary axis. We denote with \(\text{GL}(E)\) the group of invertible operators on \(E\) and with \(\text{GL}_f(E)\) the connected component of the identity operator.

**Definition 2.1.** An operator \(A\) is called essentially hyperbolic if \(A + \mathcal{L}_c(E)\) is a hyperbolic element in \(\mathcal{L}(E)\).

The spectrum of \(A + \mathcal{L}_c(E)\) is the essential spectrum, denoted with \(\sigma_e(A) = \{\lambda : A - \lambda \notin \text{Fred}(E)\}\). Thus, an operator \(A \in \mathcal{L}(E)\) is essentially hyperbolic if and only if its essential spectrum does not meet the imaginary axis. A consequence of this property is the following:

**Lemma 2.2.** Let \(D\) be the set of isolated points of \(\sigma(A)\) and let \(\partial \sigma(A)\) be the set of boundary points of \(\sigma(A)\). Then \(\partial \sigma(A) \setminus D\) is a subset of \(\sigma_e(A)\).

**Proof.** Let \(\lambda \in \partial \sigma(A) \setminus D\) and suppose that \(\lambda \notin \sigma_e(A)\), thus \(A - \lambda \notin \text{Fred}(E)\). Let \(\varepsilon > 0\) as in d) of §1.3, with \(B = I\). Since \(\lambda \in \partial \sigma(A)\), there exists \(w \in B(\lambda, \varepsilon) \setminus \{\lambda\}\) such that \(A - w\) is invertible. Thus, for every \(z \in B(\lambda, \varepsilon) \setminus \{\lambda\}\), \(A - z\) is invertible, hence \(\lambda\) is isolated in \(\sigma(A)\). \(\square\)

We need a well-known fact about the topology of the complex plane:

**Proposition 2.3.** A closed proper subset of the complex plane with a discrete boundary is discrete.

By Lemma 2.2 \(\sigma(A) \cap i\mathbb{R}\) is discrete and, by the proposition above, is discrete. Thus, it is a finite set, by (i) of Proposition 1.4.

**Proposition 2.4.** Each of the points of \(\sigma(A) \cap i\mathbb{R}\) is an eigenvalue of finite algebraic multiplicity.

**Proof.** Let \(\lambda \in \sigma(A) \cap i\mathbb{R}\). We infer that \(A - \lambda \notin \text{Fred}_0(E)\). In fact, by a,d) of §1.3, there exists a neighbourhood \(\lambda \in V\) such that

\[ A - z \in \text{Fred}_0(E), \quad z \in V \quad \text{dim ker}(A - z) \equiv n, \quad \text{dim coker}(A - z) \equiv m, \quad z \in V \setminus \{\lambda\}, \]

where \(k = \text{ind}(A - \lambda)\). Since \(\lambda\) is isolated, there exists \(z' \in V \setminus \{\lambda\}\) such that \(A - z'\) is invertible, hence \(m = n = 0\) and \(k = 0\). Thus, \(\lambda\) is an eigenvalue and isolated. These two conditions, by [15, Theorem 5.10,5.28], imply that \(\lambda\) has finite multiplicity, that is the range of \(P(A; \lambda)\) is a finite-dimensional space. \(\square\)
Theorem 1.4 provides us with projectors $P_i = P(A; \lambda_i)$ for every $\lambda_i \in \sigma(A) \cap i\mathbb{R}$, and $P = P(A; \sigma(A) \cap \{\text{re } z \neq 0\})$. We can write

$$A = \left( A P + \sum_{i=1}^{n} P_i \right) + (A - I) \sum_{i=1}^{n} P_i. \tag{2}$$

According to (i,ii) of Theorem 1.4, the term in the brackets is hyperbolic and the last term has finite rank. Thus, we have proved that an essentially hyperbolic operator is a compact perturbation of a hyperbolic one. Conversely, a compact perturbation of a hyperbolic operator is essentially hyperbolic. In fact, let $H, K$ be a hyperbolic and a compact operator, respectively. By b) of §1.3 $\sigma_c(H + K) = \sigma_c(H) \subseteq \sigma(H)$. Since $H$ is hyperbolic, the latter does not meet the imaginary axis, so does $\sigma_c(H)$.

Thus, we have proved the following

**Theorem 2.5.** An operator is essentially hyperbolic if and only if it is a compact perturbation of a hyperbolic one.

We denote with $e\mathcal{H}(E)$ the set of essentially hyperbolic operators endowed with the topology induced by the operator norm. By applying (iii) of Proposition 1.1 to the algebra $\mathcal{C}(E)$, it can be checked that $e\mathcal{H}(E) \subset \mathcal{C}(E)$ is an open subset, thus locally arc-wise connected.

Let $p$ be the quotient projection $\mathcal{L}(E) \to \mathcal{C}(E)$, and let $s$ be a continuous global section of $p$, i.e. $p \circ s$ is the identity map. Since $\mathcal{L}_c(E)$ is contractible, classical results on continuous selections ensure the existence of such a map, refer [4, Proposition A.1] for a proof and references.

**Proposition 2.6.** The space of essentially hyperbolic operators is homeomorphic to the product $\mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E)$.

**Proof.** Since $e\mathcal{H}(E)$ is defined as $p^{-1}(\mathcal{H}(\mathcal{C}))$, $s(\mathcal{H}(\mathcal{C})) \subset e\mathcal{H}(E)$. We define the continuous maps

$$e\mathcal{H}(E) \to \mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E), \quad A \mapsto (p(A), A - s(p(A))),$$

$$\mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E) \to e\mathcal{H}(E), \quad (x, K) \mapsto s(x) + K.$$ 

It is easy to check that their compositions in both orders are the identity. \hfill $\square$

**Definition 2.7.** Given $x \in \mathcal{A}$ such that $\sigma^+(x) = \sigma(x) \cap \{\text{re } z > 0\}$ disconnects $\sigma(x)$, we denote with $p^+(x)$ the projector $P(x; \{\text{re } z > 0\})$. Similarly, we define $p^-(x)$.

**Proposition 2.8.** The map $p^+: \mathcal{H}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ defines a homotopy equivalence, a homotopy inverse being the map $j: \mathcal{P}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$, $j(p) = 2p - 1$.

**Proof.** Given $x \in \mathcal{H}(\mathcal{A})$, $\sigma^+(x)$ is a spectral subset. There is a continuous, positively oriented and simple curve $c$, such that $\text{im}(c)$ is the boundary of a rectangle $Q = (0, a) \times (-b, b)$ and $\sigma^+(x) \subset Q$. Thus,

$$\sigma(x) \subset (\mathbb{R} \times [-b, b])^c \cap \text{im}(c)^c := \Omega.$$

By (iii) of Proposition 1.4 there exists $\delta > 0$ such that, if $d(y, x) < \delta$, then $y \in \mathcal{H}(\mathcal{A})$ and $\sigma(y) \subset \Omega$. Thus $\Omega \cap \sigma(y) = \sigma^+(y)$ and $p^+(y) = P_c(y)$ which is continuous by Theorem 1.4. Thus, $p^+$ is continuous in a neighbourhood of $x$, namely $B(x, \delta)$,
By the lifting properties of $L$ there exists a path of projectors $P$.

By Theorem 1.6, there exists a path of projectors $P$.

Given $p$, $j(p)$ is a square root of the unit, thus $\sigma(j(p)) \subseteq \{−1, 1\}$, hence is hyperbolic. Given $\zeta \in \mathbb{C} \setminus \{−1, 1\}$, we have

$$(j(p)−\zeta)^{-1} = \frac{\zeta}{1−\zeta^2} + \frac{j(p)}{1−\zeta^2} = \frac{1}{2} \left( \frac{-1}{\zeta+1} + \frac{1}{1−\zeta} \right) − \frac{1}{2} \left( \frac{-1}{\zeta+1} − \frac{1}{1−\zeta} \right) j(p).$$

If we integrate both sides over a curve surrounding the compact set $\{1\}$ in $\mathbb{C} \setminus \{−1\}$, we obtain

$$2\pi i \cdot p^+(j(p)) = 2\pi i \left( \frac{1}{2} + \frac{1}{2}(2p−1) \right) = 2\pi i \cdot p.$$ 

Thus, $p^+ \circ j$ is the identity map on $\mathcal{H}(A)$. In order to prove that $j \circ p^+$ is homotopically equivalent to the identity on $\mathcal{P}(A)$, we define

$$H(t,x) = (((1−t)x + t)p^+(x) + ((1−t)x − t)p^−(x)).$$

By (1), we have

$$(3) \quad \sigma(H(t,x)) = \sigma_{p^+(x)}((1−t)x p^+(x) + t p^+(x)) \cup \sigma_{p^−(x)}((1−t)x p^−(x) − t p^−(x)) = \{(1−t)\sigma^+(x) + t\} \cup \{(1−t)\sigma^−(x) − t\}.$$ 

Since the subsets of the complex plane $\{\text{re}z > 0\}$ and $\{\text{re}z < 0\}$ are convex, the sets in the second line of (3) do not meet the imaginary axis, thus $H(t,x)$ is hyperbolic. Moreover, $H(0,\cdot)$ is the identity map and $H(1,x) = j \circ p^+(x)$.

Since $\mathcal{L}_c(E)$ is a vector space, thus it is contractible to a point, the projection onto the first factor in $\mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E)$ is a homotopy equivalence. Together with the last two propositions we have proved the following

**Corollary 2.9.** The map $\Psi : \mathcal{eH}(E) \to \mathcal{P}(\mathcal{C}(E))$, $A \mapsto p^+(A + \mathcal{L}_c(E))$ is a homotopy equivalence.

**Proposition 2.10.** Given a connected component $\mathcal{D} \subset \mathcal{eH}(E)$, there exists $P \in \mathcal{P}(\mathcal{C}(E))$ such that $2P − I \in \mathcal{D}$. Moreover, $A,B \in \mathcal{D}$ if and only if there exists $T \in GL_1(E)$ such that

$$TP^+(A)T^{-1} − P^+(B) \in \mathcal{L}_c(E).$$

**Proof.** Given $A \in \mathcal{D}$, by (2) there exists a hyperbolic operator $H$ such that $A−H \in \mathcal{L}_c(E)$. Using the convex combination $t \mapsto H + t(A−H)$, we obtain $H \in \mathcal{D}$. By Proposition 2.8, $2P^+(H) − I \in \mathcal{D}$. Conversely, if $A,B \in \mathcal{D}$, there exists a path $\alpha : [0,1] \to \mathcal{P}(\mathcal{C}(E))$ such that $\alpha(0) = \Psi(A)$ and $\alpha(1) = \Psi(B)$. By Proposition 1.8, there exists a path of projectors $P$ such that $P(0) = P^+(A)$ and $p(P(t)) = \alpha(t)$. By Theorem 1.6, there exists $T \in GL_1(E)$ such that

$$TP^+(A)T^{-1} = P(1).$$

By the lifting properties of $P$, $p(P(1)) = \Psi(B) = p^+(B + \mathcal{L}_c(E))$; the second equality follows from the definition $\Psi$. By (iv) of Proposition 1.4, $p^+(B + \mathcal{L}_c(E)) = P^+(B) + \mathcal{L}_c(E)$, hence $P(1) − P^+(B) \in \mathcal{L}_c(E)$. □
3. The spectral flow

Let $A: [0, 1] \to e\mathcal{H}(E)$ be a continuous path. By Proposition 1.8 there exists a continuous path of projectors $P$, such that $p(P(t)) = P^+(A(t))$. We have an integer associated to it

$$\text{sf}(A; P) := [P(0) - P^+(A(0))] - [P(1) - P^+(A(1))].$$

Moreover, given another path $Q$ such that $p(Q(t)) = p(P(t))$, by (iii) of Theorem 1.11 we have

$$\text{sf}(A; Q) = [Q(0) - P^+(A(0))] - [Q(1) - P^+(A(1))]$$

$$= [Q(0) - P(0)] + [P(0) - P^+(A(0))]$$

$$- [Q(1) - P(1)] - [P(1) - P^+(A(1))] = \text{sf}(A; P).$$

**Definition 3.1.** Given $A$ as above, we define the spectral flow as the integer $\text{sf}(A; P)$ where $P$ is any of the paths of projectors such that $p(P(t)) = P^+(A(t))$. We denote it by $\text{sf}(A)$.

**Proposition 3.2.** The spectral flow satisfies the following properties:

(i) It is well behaved with respect to the catenation of paths;

(ii) the spectral flow of a constant path or a path in $\mathcal{H}(\mathcal{L}(E))$ is zero;

(iii) it is invariant for free-endpoints homotopies in $\mathcal{H}(\mathcal{L}(E))$ and for fixed-endpoints homotopies in $e\mathcal{H}(E)$;

(iv) if $A_i \in C([0, 1], e\mathcal{H}(E_i))$ for $1 \leq i \leq n$, then $\text{sf}({\oplus}_{i=1}^n A_i) = \sum_{i=1}^n \text{sf}(A_i)$;

(v) if $E$ is an $n$-dimensional linear space then, for every integer $-n \leq k \leq n$, there is a path such that $\text{sf}(A) = k$;

(vi) if $E$ has infinite dimension, then, for every $k$ there is $A$ such that $\text{sf}(A) = k$.

**Proof.** (i). Let $A, B$ be two paths such that $A(1) = B(0)$. We can choose paths of projectors $P$ and $Q$ such that $p(P) = \Psi(A)$ and $p(Q) = \Psi(B)$, with $Q(0) = P(1)$. Denote by $C$ and $R$ the catenation of $A, B$ and $P, Q$ respectively. Then,

$$\text{sf}(A * B) = [R(0) - P^+(C(0))] - [R(1) - P^+(C(1))]$$

$$= [P(0) - P^+(A(0))] - [Q(1) - P^+(B(1))] = [P(0) - P^+(A(0))]$$

$$- [P(1) - P^+(A(1))] + [Q(0) - P^+(B(0))] - [Q(1) - P^+(B(1))]$$

$$= \text{sf}(A) + \text{sf}(B).$$

(ii). If $A$ is constant, the path $P$ can be chosen to be constant. Hence, the spectral flow is zero. If $A$ is hyperbolic, $P^+(A(t))$ is continuous and can be chosen as lifting path of $\Psi(A)$. Hence,

$$\text{sf}(A) = [P^+(A(0)) - P^+(A(0))] - [P^+(A(1)) - P^+(A(1))] = 0.$$

(iii). Let $H: I \times I \to e\mathcal{H}(E)$ be a continuous map. There exists $P: I \times I \to \mathcal{P}(E)$ such that

$$P(t, s) - P^+(H(t, s)) \in \mathcal{L}_{\circ}(E), \text{ for every } t, s.$$

Let $H(\cdot, 0) = A$ and $H(\cdot, 1) = B$. We have

$$\text{sf}(A) = [P(0, 0) - P^+(H(0, 0))] - [P(1, 0) - P^+(H(1, 0))].$$
SPECTRAL FLOW IN BANACH SPACES

For $i = 0, 1$ and every $s$, the operator $P(i, s) - P^+(H(i, s))$ is compact. The right summand is constant or continuous, whether the homotopy has fixed endpoints in $e\mathcal{H}(E)$ or laying in $\mathcal{H}(\mathcal{L}(E))$. In both cases

$$[P(i, s) - P^+(H(i, s))] = k_i$$

for every $0 \leq s \leq 1$ and $i = 0, 1$. Thus, $\text{sf}(A) = k_0 - k_1 = \text{sf}(B)$.

(iv). Let $P_i$ be continuous paths of projectors such that $P_i(t) - P^+(A_i(t)) \in \mathcal{L}_c(E_i)$.

$$\text{sf}(\oplus_{i=1}^n A_i) = [\oplus_{i=1}^n P_i(0) - \oplus_{i=1}^n P^+(A_i(0))] - [\oplus_{i=1}^n P_i(1) - \oplus_{i=1}^n P^+(A_i(1))]$$

$$= \sum_{i=1}^n [P_i(0) - P^+(A_i(0))] - [P_i(1) - P^+(A_i(1))] = \sum_{i=1}^n \text{sf}(A_i).$$

(v). Given $0 \leq k \leq n$, the spectral flow of

$$A(t) = (2t - 1)I_k \oplus I_{n-k}$$

can be computed using $P(t) \equiv I$. Since $P^+(A(1)) = I$ and $P^+(A(0)) = 0 \oplus I_{n-k}$, we have $\text{sf}(A) = k$ and $\text{sf}(A(-)) = -k$.

(vi). Given $k \in \mathbb{Z}$, let $E = X^k \oplus R_k$ where $X^k$ is a closed subspace and $\dim(R_k) = k$. Thus, the spectral flow of $A(t) = I_{X^k} \oplus (2t - 1)I_{R_k}$ can be computed with $P(t) \equiv I$ and $\text{sf}(A) = k$ and $\text{sf}(A(-)) = -k$.

3.1. spectral sections. The definition of spectral flow we used corresponds to the one given by C. Zhu and Y. Long in [27] for paths of admissible operators, refer [27, Definition 2.3], which are essentially hyperbolic. We briefly recall their definition

**Definition 3.3.** A s-section, for a path of projectors $Q$ on $J \subset [0, 1]$ is a continuous path $P$ such that $P(t) - Q(t) \in \mathcal{L}_c(E)$.

The authors show in Corollary 2.1 that there exists a partition of the unit interval $\{J_k\}_{k=1}^n$ and a continuous path $Q_k(\cdot)$ on $J_k$ such that $Q_k(t)$ is a spectral projector of $A(t)$ and is an s-section for $P^+(A(\cdot))$ on $J_k$. Then, they define

$$\text{sf}(A) = \sum_{k=1}^n \text{sf}(A_k; Q_k)$$

where $A_k$ is the restriction of $A$ to $J_k$. Using Proposition 1.8 we showed that an s-section of the path $P^+(A(\cdot))$ on $[0, 1]$ exists, even though the projectors need not be spectral. That allows us to simplify the definition of spectral flow, provide simpler proofs of well-known properties, such as the homotopy invariance, from the original proof in [27, Proposition 2.2] or in [20]. It also makes easy to produce examples, as we will show in the next section.

4. SPECTRAL FLOW AS GROUP HOMOMORPHISM

By (ii) and (i) of Proposition 3.2 the spectral flow determines a $\mathbb{Z}$-valued group homomorphism on the fundamental group of each connected component of $e\mathcal{H}(E)$. Given a projector $P$, we denote with $\text{sf}_P$ the spectral flow on fundamental group of the connected component of $2P - I$.

The fiber of $p: \mathcal{P}(\mathcal{L}(E)) \to \mathcal{P}(\mathcal{C}(E))$ over a point of the base space, $P + \mathcal{L}_c(E)$ is the set

$$\mathcal{P}_c(E; P) = \{Q \in \mathcal{P}(\mathcal{L}(E)) : Q - P \in \mathcal{L}_c(E)\}$$

$$i: \mathcal{P}_c(E; P) \to \mathcal{P}(\mathcal{L}(E)).$$
**Proposition 4.1.** For every projector $P$, the connected components of $\mathcal{P}_c(E; P)$ correspond to $\mathbb{Z}$ through the bijection $Q \mapsto [P - Q]$. Moreover, if both the range and the kernel have infinite dimension, $\pi_1(\mathcal{P}_c(E; P)) \cong \mathbb{Z}_2$.

It follows from [12, Theorem 6.3] or [11] and from [12, Theorem 7.3]. Since $p$ is a Serre fibration, the sequence of homomorphisms

$$\pi_1(\mathcal{P}_c(E; P), P) \xrightarrow{i_*} \pi_1(\mathcal{L}(E), P) \xrightarrow{p_*} \pi_1(\mathcal{P}(C(E)), P + \mathcal{L}_c(E))$$

is exact. The homotopy equivalence $\Psi$ defined in Corollary 2.9 determines a group isomorphism

$$\Psi_* : \pi_1(e\mathcal{H}(E), 2P - I) \to \pi_1(\mathcal{P}(C(E)), P + \mathcal{L}_c(E)).$$

We set $\varphi_P = \text{sf}_P \circ (\Psi_*)^{-1}$.

**Theorem 4.2.** The sequence of homomorphisms

$$\pi_1(\mathcal{P}(\mathcal{L}(E)), P) \xrightarrow{p_*} \pi_1(\mathcal{P}(C(E)), P + \mathcal{L}_c(E)) \xrightarrow{\varphi_P} \mathbb{Z},$$

is exact.

**Proof.** We prove that $\ker(\varphi_P) \subseteq \text{im}(p_*)$. Let $a$ be a loop in $\mathcal{P}(C(E))$ at the base point $P + \mathcal{L}_c(E)$ and $A_t \in e\mathcal{H}(E)$ such that $\Psi(A(t)) = a(t)$. Thus, there exists a path of projectors $P_t$ such that $P(0) = P$ and $p(P(t)) = P^+(A(t))$. Hence,

$$\varphi_P(a) = \text{sf}_P(A) = [P(0) - P^+(A(0))] - [P(1) - P^+(A(1))] = [P(0) - P(1)] = 0;$$

the third inequality follows from (ii) of Theorem 1.11. By Proposition 4.1 there exists a continuous path $Q$ such that

$$Q(0) = P(1), \quad Q(1) = P, \quad Q(t) - P(t) \in \mathcal{L}_c(E).$$

Thus, $p_*(P*Q) = a$. We prove that $\text{im}(p_*) \subseteq \ker(\varphi_P)$. Given a loop $P_t \in \mathcal{P}_c(E; P)$,

$$\varphi_P(i_*(P)) = [P(0) - P(1)] = 0$$

because $P(0) = P(1)$. \hfill \Box

**Corollary 4.3.** Given $P \in \mathcal{P}(\mathcal{L}(E))$, we have the following properties of the spectral flow $\text{sf}_P$:

h1) $m \in \text{im}(\text{sf}_P)$ if and only if there exists a projector $Q$ in the same connected component of $P$ such that $Q - P$ is compact and $[Q - P] = m$;

h2) $\text{im}(p_*) \cong \ker(\text{sf}_P)$.

The isomorphism classes of the kernel and the image of $\text{sf}_P$ depend only on the conjugacy class of $P + \mathcal{L}_c(E)$ in $\mathcal{P}(C(E))$. We show that in many cases we can find a projector $P$ such that $\text{sf}_P$ is an isomorphism.

**Lemma 4.4.** Let $E$ be a Banach space, and $X, Y \subseteq E$ closed subspaces such that $X \cong Y$ and $X \oplus Y = E$. Two projectors $P_X, P_Y$ with ranges $X$ and $Y$ respectively, are connected by a continuous path on $\mathcal{P}(\mathcal{L}(E))$.

A proof of this can be found in [21, §9] or in [18].

**Proposition 4.5.** Let $X, Y \subseteq E$ be as above. Suppose that $X$ is isomorphic to its closed subspaces of co-dimension $m$. Let $P$ be the projector onto $X$ with kernel $Y$. Then $P$ satisfies the condition h1) with the integer $m$. 
Proof. Let $X^m, R_m \subset X$ be closed subspaces such that $\dim(R_m) = m$ and $X^m \cong X$. We have the decomposition and isomorphism

$$E = R_m \oplus X^m \oplus Y, \ X^m \cong Y.$$  

By applying Lemma 4.4 with $E = X^m \oplus Y$, we obtain that $P_{X^m}$ is connected to $P_Y$. By applying it a second time to $E$ and subspaces $X = R_m \oplus X^m$ and $Y$, we obtain that $P_X$ is connected to $P_Y$. Hence, $P_X$ is connected to $P_{X^m}$. □

We required $E$ to be isomorphic to a product of a space $X$ with itself, but it suffices that $E$ has a complemented subspace $F$ fulfilling the requirements of Proposition 4.3. In fact, if $A_t \in \mathcal{H}(F)$ is such that $\text{sf}(A_t) = m$, then $\text{sf}(I \oplus A_t) = m$, by (iv) of Proposition 4.2.

Proposition 4.6. Given $P \in \mathcal{L}(\mathcal{P}(E))$, the map $\pi: GL(E) \to \mathcal{P}(\mathcal{L}(E)), \ T \mapsto \text{TPT}^{-1}$ defines a principal bundle with fiber $GL(X) \times GL(Y)$, where $X = \text{Range}(P)$ and $Y = \ker(P)$.

A proof of this can be found in [10, 11].

Corollary 4.7. If $GL(E)$ is simply-connected and $GL(X), GL(Y)$ are connected, then $\text{sf}_P$ is injective.

Proof. Since a locally trivial bundle is a Serre fibration, we have a long exact sequence of homomorphisms which ends

$$\pi_1(GL(E), I) \overset{\pi}{\longrightarrow} \pi_1(\mathcal{P}(\mathcal{L}(E)), P) \overset{\Delta}{\longrightarrow} \pi_0(GL(X) \times GL(Y), I).$$

Thus, if $GL(E)$ is simply connected and $GL(X)$ and $GL(Y)$ are connected, the middle group is trivial, hence in (b) $p_*$ is the trivial map, thus $\phi_P$ is injective and $\text{sf}_P$ is injective. □

Then, we have sufficient conditions to a Banach space in order to have at least one projector $P$ such that $\text{sf}_P$ is an isomorphism.

Theorem 4.8. Let $E = X \oplus X$ such that $X$ is isomorphic to its hyperplanes and $GL(E)$ is simply-connected and $GL(X)$ is connected. Then $\text{sf}_{P_X}$ is an isomorphism.

Proof. That $\text{sf}_{P_X}$ is surjective follows from Proposition 4.3. From the corollary above, $\text{sf}_{P_X}$ is also injective. □

A special case of the theorem above is when $E \cong E \times E$, it is isomorphic to a hyperplane and has a contractible general linear group. This, in fact, is the case of the most common infinite-dimensional spaces as separable Hilbert spaces, $c_0, \ell^p$ with $p \geq 1$, and $L^p(\Omega, \mu)$ with $p > 1$, and $C(K, F)$ for large classes of compact spaces $K$ and Banach spaces $F$, and many others. For a richer list, check Theorem 2 of [26] and [16, 19, 18]. Sequence spaces $\ell^p, \ell^\infty$ and $c_0$ are also prime (see [3, Theorem 2.2.4] and [17]) that is they are isomorphic to their complemented, infinite-dimensional subspaces. Thus, for every projector $P$ such that $\text{Range}(P)$ and $\ker(P)$ have infinite dimension $\text{sf}_P$ is an isomorphism.

Trivial spectral flow. When $P$ is a projector with a finite-dimensional range or kernel, $P + \mathcal{L}_c(E) = 0$ or 1, thus its connected component in $\mathcal{P}(C(E))$ is $\{0\}$ or $\{1\}$, thus $\text{sf}_P = 0$. This is the case of finite-dimensional spaces. A spaces is said undecomposable if the only projectors are as above. In [13], the authors showed the existence of a infinite-dimensional, undecomposable space.
Non-trivial and not surjective spectral flow. In [13], W. T. Gowers and B. Maurey proved the existence of a space isomorphic to their hypersquares (that is, subspaces of co-dimension 2), but not their hyperplanes. If we denote with $X$ a space with such property and by $P$ the projector onto the first factor in $E = X \oplus X$, then $2 \in \text{im}(sf_P)$ by Proposition [14]. However, if $1 \in \text{im}(sf_P)$, by condition h1) and c2) in §1.2, $X$ would be isomorphic to its hyperplanes.

4.1. The Douady space. We show the existence of a projector $P$ such that $\varphi_P$, and thus $sf_P$, is not injective.

Proposition 4.9. Let $E = X \oplus X$. Given $T \in GL(X)$, there exists a loop $x$ in the space of projectors such that

$$\Delta(x) = \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}.$$ 

Proof. Let $M$ be the operator defined in the line above. Let $U_t \in GL(E)$ be such that $U(1) = M$ and $U(0) = I$. In general, $T \oplus T'$ is connected to $TT' \oplus I$, check [18]. Since $T$ commutes with $P_X$, the path

$$P(t) = U(t)P_XU(t)^{-1}$$

is a loop in $\mathcal{P}(\mathcal{L}(E))$ with base point $P$. We denote with $x$ its homotopy class. The path $U_t$ is a lifting path for $P$. Hence $\Delta(x) = U(1) = M$. $\square$

Let $F$ and $G$ be Banach spaces such that

(i) every bounded map $G \to F$ is compact;

(ii) both $F$ and $G$ are isomorphic to their hyperplanes.

We have the following:

Lemma 4.10. Let $X = F \oplus G$, $F$ and $G$ as above. There exists a continuous, surjective map $j: GL(X) \to \mathbb{Z}$.

Proof. The Lemma follows from a more general result of A. Douady, [11]. We briefly sketch the proof of B. S. Mitjagin in [18]. Let $T \in GL(X)$ be an invertible operator and $S$ be its inverse. We have

$$\begin{pmatrix} I_F & 0 \\ 0 & I_G \end{pmatrix} = TS = \begin{pmatrix} T_{11}S_{11} + T_{12}S_{21} & T_{11}S_{12} + T_{12}S_{22} \\ T_{21}S_{11} + T_{22}S_{21} & T_{22}S_{22} + T_{21}S_{12} \end{pmatrix}.$$ 

A similar equality holds for $ST$. Since $S_{21}$ and $T_{21}$ are compact operators, $T_{11}$ and $T_{22}$ satisfy e) of [13] with $S_{11}$ and $S_{22}$, respectively. Thus, they are Fredholm operators. We define

$$j(T) = \text{ind}(T_{11}).$$

By a) of [13] there exists $\varepsilon > 0$ such that, if $\|T_{11} - T_{11}'\| < \varepsilon$, then $T_{11}'$ is a Fredholm operator and $\text{ind}(T_{11}') = \text{ind}(T_{11})$. This proves the continuity. Moreover, given two invertible operators $T$ and $S$, we have

$$j(TS) = \text{ind}(T)\text{ind}(S_{11}) = \text{ind}(T_{11}S_{11} + T_{12}S_{21}) = \text{ind}(T_{11}S_{11}) = \text{ind}(T_{11}) + \text{ind}(S_{11}),$$

...
where b,c) of \([13]\) have been used. Thus, \(j\) is a group homomorphism. Let \(F^1\) and \(G^1\) be hyperplanes of \(F\) and \(G\), respectively. We define

\[
\sigma: F \to F^1, \quad \tau: G^1 \to G
\]

\[
F = \langle v \rangle \oplus F^1 = F, \quad G = \langle w \rangle \oplus G^1
\]

\[
B: G \to F, \quad tw + y \mapsto tv.
\]

where \(\sigma, \tau\) are isomorphisms, which exist by (i). We define

\[
T(x, y) = (\sigma(x) + B(y), \tau(Py));
\]

\(T\) is invertible and \(\text{ind}(T_{11}) = \text{ind}(\sigma) = 1\). Since \(j\) is a homomorphism, it is surjective.

**Proposition 4.11.** If \(E = X \oplus X\), where \(X\) is a direct sum of two spaces \(F\) and \(G\) fulfilling the two conditions (i) and (ii) above, then \(\text{sf}_{P_X}\) is surjective, but not injective.

**Proof.** From the lemma above, for every \(k \in \mathbb{Z}\), there exists \(T_k \in GL(X)\) such that \(J(T_k) = k\). By Proposition \([1,\,\,1]\) there exists \(x_k \in \pi_1(P_c(E; P), P_X)\) such that \(\Delta(x_k) = T_k\) and \(x_k \neq x_h\) if \(k \neq h\). Then \(\pi_1(P(L(E)), P_X)\) has infinite elements, while \(\pi_1(P_c(E; P), P_X)\) is a finite group, by Proposition \([1,\,\,1]\). Hence, in \([1,\,\,1]\), \(j\) is not surjective, thus \(x_k \notin \text{im}(j_*)\) for infinitely many \(k \in \mathbb{Z}\). Hence,

\[
p_* (x_k) \neq 0, \quad \varphi_p(p_*(x_k)) = 0.
\]

because the sequence \([1]\) is exact; therefore, the kernel of \(\varphi_p\) is not trivial. Since \(E\) and \(X\) fulfill the hypothesis of Proposition \([1,\,\,5]\), \(\text{sf}_{P_X}\) is surjective. \qed

Pairs of Banach spaces with the properties (i) and (ii) are \((\ell^p, \ell^2)\), with \(p > 2\). We refer to \([25]\) Theorem 4.23.

We show that even when \(P\) has infinite-dimensional range and kernel, \(\text{sf}_P\) can be trivial.

**Proposition 4.12.** Let \(X\) be as above. Then, \(\text{sf}_{P_X} = 0\).

**Proof.** Let \(0 \leq m \in \text{im}(\text{sf})\). Then, \(P_F\) is connected to a projector \(Q \in \mathcal{L}(P(X))\) onto a subspace of \(F \oplus \{0\}\) of co-dimension \(m\), \(F^m \oplus \{0\}\). By Theorem \([1,\,\,7]\) there exists a continuous path \(U_t \in GL(X)\) such that

\[
U(0) = I, \quad U(1)P_F = QU(1).
\]

From these relations it follows that \(U(1)_{11}(F) = F^m\), hence \(j(U(1)) = -m\), and \(j(U(0)) = 0\). By the lemma above, \(j(U(t))\) is constant, therefore \(m = 0\). \qed

5. THE FREDHOLM INDEX OF \(F_A\) AND THE SPECTRAL FLOW

**Definition 5.1.** A path \(A: \mathbb{R} \to \mathcal{L}(E)\) of bounded operators is said to be asymptotically hyperbolic if the limits \(A(+\infty)\) and \(A(-\infty)\) exist and are hyperbolic operators.

If \(A\) is also essentially hyperbolic, we can define the spectral flow as follows: Since the set of hyperbolic operators is an open subset of \(\mathcal{L}(E)\), there exists \(\delta > 0\) such that \(A(t)\) is hyperbolic for every \(t \in (-\infty, -\delta] \cup [\delta, +\infty)\). We set

\[
\text{sf}(A) = \text{sf}(A, [-\delta, \delta]).
\]
The definition does not depend on the choice of \( \delta \) by (i,ii) of Proposition 3.2. Let \( P^+(A) \) be the spectral projector \( P(A; \{ \Re z > 0 \}) \).

**Definition 5.2.** An asymptotically hyperbolic path is called *essentially splitting* if the following properties

(i) \( P^+(A(+\infty)) - P^+(A(-\infty)) \) is compact;

(ii) \( [A(t), P^+(A(+\infty))] \) is compact for every \( t \),

holds, where \([A, P] = AP - PA\).

The definition above corresponds to the one given in [2] Theorem 6.3 when the property (i) above is added. Thus, we refer to the paths above as essentially splitting, as well.

**Lemma 5.3.** Let \( A \) be an asymptotically hyperbolic and essentially hyperbolic path. It is also essentially splitting if and only if the projectors of the subset \( \{ P^+(A(t)) : t \in \mathbb{R} \} \) are compact perturbation of each other.

*Proof.* Suppose \( A \) is essentially splitting and consider the restriction on the positive half-line. We denote with \( E^+ \) and \( E^- \) the ranges of \( P^+ \) and \( P^- \), respectively. With respect to the decomposition \( E = E^+ \oplus E^- \) we can write the operators of the path block-wise:

\[
A(t) = \begin{pmatrix} A_+(t) & K_+(t) \\ K_-(t) & A_-(t) \end{pmatrix}
\]

where \( K_+(t) \) and \( K_-(t) \) are compact operators because \( A \) is essentially splitting. Thus

\[
P^+(A(t)) - P^+(A_+(t)) \oplus P^+(A_-(t))
\]

is compact: in fact if two operators have compact difference, then their spectral projectors are compact perturbation of each other. Since \( A(+\infty) \) is hyperbolic, and \( K_+(+\infty), K_-(+\infty) \) are null operators, \( A_+(+\infty) \) is hyperbolic and there exists \( t_+ > 0 \) such that for every \( t \geq t_+ \) the operator \( A_+(t) \) is hyperbolic on \( E^+ \) and

\[
\|P^+(A_+(t)) - P^+(A_+(+\infty))\| < 1.
\]

Since \( \sigma(A_+(+\infty)) \subset \{ \Re z > 0 \} \), then \( P^+(A_+(+\infty)) = I_{E^+}. \) Thus, \( P^+(A_+(t)) = I_{E^+}. \) Hence, the path

\[
A_+: [-s, t_+] \to e\mathcal{H}(E^+), \ \forall s
\]

lies in the connected component of the identity in \( e\mathcal{H}(E^+) \). By Proposition (2.10), all the projectors \( P^+(A_+(t)) \) are compact perturbation of the identity. Similarly, the spectrum of \( A_-(+\infty) \) is contained in the negative complex half-plane, hence \( P^+(A_-(+\infty)) = 0 \) and there exists \( t_- > 0 \) such that, for every \( t \geq t_- \),

\[
P^+(A_-(t)) = 0.
\]

The path \( A_-: [-s, t_-] \to e\mathcal{H}(E^-) \) lies in the connected component of \( 0 \in e\mathcal{H}(E^-) \) for every \( s > 0 \). By Proposition (2.10) all the projectors \( P^+(A_-(t)) \) have finite rank. To recap, for every \( t \in \mathbb{R} \),

\[
P^+(A_+(t)) - I_{E^+} \in \mathcal{L}_e(E^+), \quad P^+(A_-(t)) \in \mathcal{L}_e(E^-),
\]

The direct sum \( I_{E^+} \oplus 0_{E^-} \) is \( P^+(A(+\infty)) \). Hence,

\[
P^+(A_+(t)) \oplus P^+(A_-(t)) - P^+(A(+\infty))
\]
is compact. Together with (7), we conclude. Conversely, suppose that each of the projectors of the set \( \{ P^+(A(t)) : t \in \mathbb{R} \} \) is compact perturbation of each other and let \( a \) be such that \( A(s) \) is hyperbolic for every \( |s| > a \). Thus \( P^+ \) is continuous and

\[
P^+(A(+\infty)) - P^+(A(-\infty)) = \lim_{|s| \to \infty, |s| > a} \left( P^+(A(s)) - P^+(A(-s)) \right).
\]

because \( A \) is asymptotically hyperbolic. Hence, the left member is the limit of a sequence of compact operators. Since \( \mathcal{L}_c(E) \) is a closed subset of \( \mathcal{L}(E) \), we have proved (i). Property (ii) follows from the equality

\[
[A(t), P^+(A(s))] = [A(t), P^+(A(t))] + [A(t), P^+(A(s)) - P^+(A(t))]
\]

where the first summand of the right member is zero and the second one is compact by hypothesis. In particular, the equality holds for \( s > a \), so we finish our proof by taking the limit as \( s \to +\infty \).

\[ \square \]

5.1. The spectral flow and the Fredholm index of \( F_A \). Given a continuous, bounded path \( A_t \in \mathcal{L}(E) \), we denote with \( F_A \) the differential operator

\[
F_A : W^{1,p}(\mathbb{R}, E) \to L^p(\mathbb{R}, E), \quad F_A(u) = \frac{du}{dt} - A(\cdot)u.
\]

**Theorem 5.4.** Let \( A \) be an asymptotically hyperbolic and essentially splitting path. Then \( F_A \) is a Fredholm operator and

\[
\text{ind}(F_A) = [P^-(A(-\infty)) - P^-(A(+\infty))]
\]

A. Abbondandolo and P. Majer proved it in [2, Theorem 6.3] when \( E \) is a Hilbert space and \( p = 2 \). However, the theorem, as much of the content of their work can be generalized to Banach spaces.

**Theorem 5.5.** Let \( A \) be an asymptotically hyperbolic, essentially splitting and essentially hyperbolic path. Then,

\[
\text{sf}(A) = -[P^-(A(-\infty)) - P^-(A(+\infty))]
\]

**Proof.** Let \( \delta > 0 \) as in (6). By Lemma 5.1, the constant path \( P \equiv P^+(A(\delta)) \) is a section for \( P^+(A_t) \) on \([-\delta, \delta] \) in the sense of Definition 5.2. Hence,

\[
\text{sf}(A) = [P^+(A(\delta)) - P^+(A(-\delta))].
\]

Since \( A \) is hyperbolic on \((-\infty - \delta) \cup [\delta, +\infty)\) the path \( P^+(A_t) \) is continuous on this subset. By (iii) of Theorem 1.11

\[
[P^-(A(-\infty)) - P^-(A(+\infty))] = -[P^+(A(\delta)) - P^+(A(-\delta))] = -\text{sf}(A).
\]

\[ \square \]

From Theorem 5.5 and Theorem 5.4 we have the final result

**Theorem 5.6.** If \( A \) is essentially hyperbolic, essentially splitting and asymptotically hyperbolic, then

\[
\text{ind}(F_A) = -\text{sf}(A).
\]

Let \( A(t) = A_0(t) + K(t) \), where \( A_0(t) \) is hyperbolic and \( A_0, A \) are asymptotically hyperbolic and \( K(t) \) is compact and

\[
A_0(t)E^- \subseteq E^-, \quad A_0(t)E^+ \subseteq E^+;
\]

\[
E^- = E^-(A_0(\pm\infty)), \quad E^+ = E^+(A_0(\pm\infty)).
\]
The second line tells us that \( P^+(A_0(+\infty)) = P^+(A_0(-\infty)) \), thus \( P^+(A(+\infty)) = P^+(A(-\infty)) \) is compact. From the first line, it follows that \([A(t), P^+(A(+\infty))]\) is compact. Thus, by Theorem 5.6, we confirm the guess of A. Abbondandolo and P. Majer in [2] §7, that for paths satisfying the hypotheses of Theorem E, corresponding to those listed above, the relation \([5]\) holds.

When \( A \) is not essentially splitting, the authors provided in [2, Example 6,7] counterexamples to the \([5]\).

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