The BFKL Pomeron Calculus in zero transverse dimensions: summation of Pomeron loops and generating functional for the multiparticle production processes

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Abstract: In this paper we address two problems in the BFKL Pomeron calculus in zero transverse dimensions: the summation of the Pomeron loops and the calculation of the processes of multiparticle generation. We introduce a new generating functional for these processes and obtain the evolution equation for it. We argue that in the kinematic range given by $1 \ll \ln(1/\alpha_s^2) \ll \alpha_S Y \ll \frac{1}{\alpha_s}$, we can reduce the Pomeron calculus to exchange of non-interacting Pomerons with the renormalized amplitude of their interaction with the target. Therefore, the summation of the Pomeron loops can be performed using Mueller, Patel, Salam and Iancu approximation.

Keywords: BFKL Pomeron, Generating functional, Mean field approach, Exact solution.

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1. Introduction

The problem of Pomeron interaction in zero transverse dimensions have been discussed in the framework of the Reggeon calculus about three decades ago. However, recently, we have seen a revival of the interest to this problem (see and references therein). The very reason for this in our opinion is related to a hope to solve the old problem of finding the high energy asymptotic behaviour of the scattering amplitude in QCD. We hope for a solution not in the mean field approximation, where the solution has been discussed and well understood both analytically and numerically (see and references therein), but in the approach where the so called Pomeron loops should be taken into account and references therein. The problem of taking into account the Pomeron loops can be reduced to the BFKL Pomeron calculus or and to the solution of statistical physics problem: Langevin equation and directed percolation. The last approach is based on the probabilistic interpretation of the Pomeron calculus which also has roots in the past.

The BFKL Pomeron calculus in zero transverse dimensions being an oversimplified model has the same description in terms of the directed percolation as the general approach. Thus solving this model we can gain an experience that will be useful for the solution to a general problem of interaction of the BFKL Pomeron in QCD.

It is well known that the BFKL calculus in zero transverse dimensions can be treated as a system that evolves in imaginary time \( it = Y \) with the Hamiltonian:

\[
H = -\Delta \phi \phi^+ + \lambda \left( \phi \phi^{+2} - \phi^2 \phi^+ \right) \quad \text{and evolution equation for wave function } \frac{d\Psi}{dY} = -H \Psi
\]
where the Pomeron intercept $\Delta \propto \alpha_S$ and the triple Pomeron vertex $\lambda \propto \alpha_S^2$.

In the next section we will discuss the evolution equation for the generating functional that describes the system of Pomerons in terms of probabilities to find ‘wee’ partons (color dipoles [31]). We introduce $\Gamma(1 \to 2) = \Delta$ and $\Gamma(2 \to 1) = \Delta \gamma$ with $\gamma$ is the amplitude for low energy interaction of the colour dipole with the target (target-Pomeron vertex) and $\gamma \propto \alpha_S^2$. The estimates for $\Delta$ and $\lambda$ are given in the leading order of perturbative QCD. We can trust the approach with the Hamiltonian of Eq. (1.1) only in the kinematic region of $Y$ given by the following equation

$$1 \ll \ln(1/\alpha_S^2) \ll \alpha_S Y \ll \frac{1}{\alpha_S}$$

(1.2)

Indeed, the $n$ Pomeron exchanges give contribution which is proportional to $(\gamma e^\Delta Y)^2$ and, therefore we need to sum them in the kinematic region where $\gamma e^\Delta Y \geq 1$ or $\Delta Y \geq \ln(1/\gamma) \approx \ln(1/\alpha_S^2)$. However, we cannot go to ultra high energies since we do not know the higher order corrections to the BFKL kernel and to triple Pomeron vertex. The contribution of the BFKL Pomeron exchange can be written as $\gamma e^{(\Delta + \text{Const} \alpha_S^2) Y}$ and the high order correction term is essential for $\alpha_S Y > 1/\alpha_S$.

In this paper for the kinematic range given by Eq. (1.2) we obtain two results. We introduce a new generating functional which allows us to calculate processes of multiparticle generation since it gives us the probability to have a given number cut and uncut Pomerons (see [32]). We derive the evolution equation for this generating functional both in the mean field approach and in the approach that takes into account the Pomeron loops.

The second result is related to the method of summation of the Pomeron loops. We claim that in kinematic region of Eq. (1.2) the Pomeron interaction given by Hamiltonian of Eq. (1.1) can be reduce to the system of free Pomerons with the renormalized amplitude of dipole-target interaction at low energies. In other words, we can view the evolution of our system of ‘wee’ partons (colour dipoles) as a system of not interacting partons only with emission absorbed in the evolution of the BFKL Pomerons, and all specific features of this system being determined by the low energy amplitude of ‘wee’ parton interaction with the target. Having this in mind we state that Mueller, Patel, Salam and Iancu [33] approach gives the solution to the problem.

The paper is organized as follows. In the next section we introduce the mean field approach and discuss it in the framework of the generating functional. We introduce a new generating functional which gives us a possibility to calculate the probability to find a given number of cut and uncut Pomerons. Therefore, knowing this generating functional we can calculate the cross section with given multiplicities. We derive the evolution equation for this functional. In section three we take into account the Pomeron loops and generalize the evolution equation. In this section we reconsider the problem of summation of all Pomeron loops in the kinematic region of Eq. (1.2) and argue that we can reduce this problem to consideration of a system of non-interacting Pomerons with renormalized vertex of Pomeron-target interaction. Based on this idea we use the Mueller, Patel, Salam and Iancu approach to calculate scattering amplitude at high energies both for elastic and inelastic interactions with different multiplicities of particles in the final state.

In conclusions we summarize the results and discuss open problems.

2. The mean field approximation

2.1 General approach

Our approach to multiparticle production is based on the AGK cutting rules [32]. These rules stem from the unitarity constraint in $s$-channel, namely,

$$2 \text{Im} A^{BFKL}(s, b) = 2 N^{BFKL}(s, b) = G^{BFKL}_{in}(s, b)$$

(2.1)
where $I m A^{B F K L}(s, b) \equiv N^{B F K L}(s, b)$ denotes the imaginary part of the elastic scattering amplitude for dipole-dipole interaction at energy $W = \sqrt{s}$ and at the impact parameter $b$. It is normalized in the way that the total cross section is equal to $\sigma_{t o t} = 2 \int d^2 b N^{B F K L}(s, b)$. $G^{B F K L}_{in}$ is the contribution of all inelastic processes for the BFKL Pomeron and $\sigma_{in} = \int d^2 b G_{in}(s, b)$. Therefore, Eq. (2.1) gives us the structure of the BFKL Pomeron exchange through the inelastic processes and it can be formulated as the statement that the exchange of the BFKL Pomeron is related to the processes of multi-gluon production in a certain kinematics (see Fig. 1). Eq. (2.1) is proven in \cite{23} in QCD. Using it, we can express all processes of multiparticle production in terms of exchange and interactions of the BFKL Pomerons and/or the cut BFKL Pomerons (see Fig. 2).

**Figure 2:** Several examples of the Pomeron diagrams that contribute to the multiparticle production: the diffraction production of the bunch of particles in the region of rapidity $\ln(M^2/m^2) = Y - Y_0$ (Fig. 2-a); the process of multiparticle production with the average multiplicity due to exchange of four Pomerons (Fig. 2-b); and the process of multiparticle production with the multiplicity in four times larger than the average multiplicity.

The AGK cutting rules establish the relation between different processes that stems from BFKL Pomeron diagrams. For example, the simple triple Pomeron diagrams in Fig. 3 leads to three inelastic processes: the diffractive production of the system with mass $\ln(M^2/m^2) = Y - Y_0$ (Fig. 3-A); the multi-gluon production in the entire kinematic region of rapidity $Y - 0$ with the same multiplicity of gluons as in one Pomeron (Fig. 3-B); and the multi-gluon production in the region $Y - 0$ but with the same multiplicity of gluons as in one Pomeron only in the rapidity window $Y - Y_0$ while for the rapidity $Y_0 - 0$ the gluon multiplicity in two times larger than for one Pomeron (Fig. 3-C). The AGK cutting rules \cite{32} say that the cross sections of these three processes are related as

\[
\sigma_A : \sigma_B : \sigma_C = 1 : -4 : 2 \quad (2.2)
\]

At first sight the cross section of the process B is negative but it should be stressed that one Pomeron also contributes to the same process and the resulting contribution is always positive.

**Figure 1:** The diagram for the cut BFKL Pomeron that illustrates Eq. (2.1).
Figure 3: The AGK cutting rules for the triple Pomeron diagram: the diffractive production of the system with mass $\ln(M^2/m_0^2) = Y - Y_0$ (Fig. 3-A); the multi-gluon production in the entire kinematic region of rapidity $Y - 0$ with the same multiplicity of gluons as in one Pomeron (Fig. 3-B); and the multi-gluon production in the region $Y - 0$ but with the same multiplicity of gluons as in one Pomeron only in the rapidity window $Y - Y_0$ while for the rapidity $Y_0 - 0$ the gluon multiplicity in two times larger than for one Pomeron (Fig. 3-C).

Fig. 1 and Fig. 3 as well as Eq. (2.2) allow us to understand the equation for the single diffractive production in the mean field approximation that has been written by Kovchegov and Levin [34] and that has the following form (see Fig. 4 for graphical representation of this equation):

$$\frac{\partial N^D(x_{01}, b, Y, Y_0)}{\partial Y} = \frac{\alpha C_F}{\pi^2} \int_\rho d^2 x_2 \frac{x_{01}^2}{x_{02} x_{12}}$$  \hspace{1cm} (2.3)

$$\left( N^D(x_{02}, b + \frac{1}{2} x_{12}, Y, Y_0) + N^D(x_{12}, b + \frac{1}{2} x_{02}, Y, Y_0) - N^D(x_{01}, b, Y, Y_0) ight)$$

$$+ N^D(x_{02}, b + \frac{1}{2} x_{12}, Y, Y_0) N^D(x_{12}, b + \frac{1}{2} x_{02}, Y, Y_0) \hspace{1cm} \text{Eq. (2.3)(A)}$$

$$- 4 N^D(x_{02}, b + \frac{1}{2} x_{12}, Y, Y_0) N_0(x_{12}, b + \frac{1}{2} x_{02}, Y) \hspace{1cm} \text{Eq. (2.3)(B)}$$

Figure 4: Different Pomeron cuts contributing to the cross section of diffractive dissociation which lead to different terms on the right hand side in Eq. (2.3).
\[ +2 N_0(x_02, b + \frac{1}{2}x_{12}, Y) N_0(x_{12}, b + \frac{1}{2}x_02, Y) \]  
\text{Eq. (2.3)(C)}

with the initial condition given by

\[ N^D(x_\perp, b, Y = Y_0, Y_0) = N_0^2(x_\perp, b, Y_0). \]  
\text{(2.4)}

where \( N_0 \) is the solution of the Balitsky-Kovchegov equation \[13\] and \( N^D(x, b; Y; Y_0) \) is the diffraction dissociation of the colourless dipole with size \( x \) at impact parameter \( b \) into the system of gluon with the rapidity gap larger than \( Y_0 \) at energy \( Y \). At first sight, Eq. (2.3) contradicts the AGK relations given by Eq. (2.2) (see Fig. 3), but this contradiction can be easily resolved if we take into account coefficient 2 in Eq. (2.1) (see [34] for more details as well as for a proof based directly on the dipole approach to high energy scattering).

Despite a simple structure of Eq. (2.3), which is only a little bit more complicated than the Balitsky-Kovchegov equation, as far as we know there exists no analytical solution to this equation and there is the only attempt to solve it numerically \[37\]. However, this equation has a simple solution in the toy model (see \[34\] ) which we are going to discuss.

### 2.2 The BFKL Pomeron calculus in zero transverse dimensions: general approach

The mean field approximation looks extremely simple in the BFKL Pomeron calculus in zero transverse dimensions (the toy model \[31, 35, 36\]). Indeed, in the toy model, in which there is no dependence on the sizes of interacting dipoles, the generating functional degenerates to the generating function that has the form

\[ Z_0(u|Y) = \sum_{n=0}^{\infty} P_n(Y) u^n \]  
\text{(2.5)}

where \( P_n(Y) \) is the probability to find \( n \)-dipoles (or/and \( n \)-Pomerons) at rapidity \( Y \). For the probabilities \( P_n(Y) \) we can write Markov chain, namely \[31, 35, 36\]

\[ \frac{dP_n(Y)}{dY} = -\Gamma(1 \to 2) n P_n(Y) + \Gamma(1 \to 2) (n-1) P_{n-1}(Y) \]  
\text{(2.6)}

Eq. (2.6) has a simple structure: for the process of dipole splitting we see two terms. The first one with the negative sign describes a decrease of probability \( P_n \) due to the process of splitting of dipoles. The second term with positive sign is responsible for the increase of the probability due to the same processes of dipole interactions.

Eq. (2.6) can be re-written in the elegant form of the master equation for \( Z_0 \), namely,

\[ \frac{\partial Z_0(u|Y)}{\partial Y} = -\Gamma(1 \to 2) u (1-u) \frac{\partial Z_0(u|Y)}{\partial u} \]  
\text{(2.7)}

where \( \Gamma(1 \to 2) \propto \bar{\alpha}_S \) in QCD. The initial and boundary conditions look as follows

- initial condition: \( Z_0(u|Y = 0) = u \);
- boundary condition: \( Z_0(u = 1|Y) = 1 \).

(2.8)

In Eq. (2.8) the initial condition means that we are studying the evolution of one dipole, while the boundary condition follows from the normalization of the sum of probabilities. With this initial condition the linear differential equation Eq. (2.7) can be written as non-linear one

\[ \frac{\partial Z_0(u|Y)}{\partial Y} = -\Gamma(1 \to 2) Z_0(u|Y) + \Gamma(1 \to 2) Z_0^2(u|Y) \]  
\text{(2.9)}
Introducing scattering amplitude
\[ N_0(\gamma|Y) = ImA_{el} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n Z_0(u|Y)}{\partial u^n}|_{u=1} \gamma^n \] (2.10)
with \( \gamma \) being a scattering amplitude of interaction of single dipole with the target, we can find Eq. (2.7) for the amplitude [39].

Using initial condition \( N_0(\gamma|Y = 0) = \gamma \) one can rewrite Eq. (2.11) as non-linear equation for the amplitude
\[ \frac{\partial N_0(\gamma|Y)}{\partial Y} = \Gamma(1 \rightarrow 2)(\gamma - \gamma^2) \frac{\partial N_0(\gamma|Y)}{\partial \gamma} \] (2.12)
Eq. (2.12) is easy to solve in this model and the solution has been found in [34, 38, 3]. It was noticed in [13] that one can get Eq. (2.12) directly from Eq. (2.9) by substitution \( N_0(\gamma|Y) = 1 - Z_0(1 - \gamma|Y) \). Using this fact we can go back to Eq. (2.7) and write it as
\[ \frac{\partial Z(1 - \gamma|Y)}{\partial Y} = \Gamma(1 \rightarrow 2) \left( \gamma \frac{\partial Z(1 - \gamma|Y)}{\partial \gamma} - \gamma^2 \frac{\partial Z(1 - \gamma|Y)}{\partial \gamma} \right) \] (2.13)

2.3 The generating functional for the multiparticle production: definition and linear evolution equation

Here we would like to develop a method that will allow us not only to find the cross sections of the diffractive production but also to consider all processes of multi-particle production at high energy. Having this goal in mind, we propose a generalization of the generating functional given by Eq. (2.5), namely, we introduce a new generating functional \( Z(u, v|Y) \) as follows
\[ Z(u, v|Y) = \sum_{n=0, m=0}^{\infty} P(n, m|Y) u^n v^m \] (2.14)
where \( P(n, m|Y) \) is a probability to find \( n \) uncut Pomerons and \( m \) cut Pomerons. Directly from Eq. (2.14) and from the fact that \( P(n, m|Y) \) is a probability we find the first boundary condition
\[ Z(u = 1, v = 1|Y) = 1 \] (2.15)
To find the second boundary condition we can use the full form of the \( s \)-channel unitarity constraint. Assuming that the scattering amplitude is pure imaginary at high energy, this constraint looks as follows
\[ 2 N(s, b) = |N(s, b)|^2 + G_{in} \] (2.16)
where the first term in the l.h.s. is the elastic term with no cut Pomerons, while the second is the total contribution of the inelastic processes (in other words, sum over all cut Pomerons). Using functionals \( Z_0(u|Y) \) and \( Z(u, v|Y) \) we can calculate the left and right hand sides of Eq. (2.16), namely,
\[ N_0 = 1 - Z_0(1 - \gamma|Y) \quad \text{(see [13])} \] (2.17)
\[ |N(s, b)|^2 + G_{in} = 2 (1 - Z(1 - \gamma, 1 - \gamma_{in}|Y)) \] (2.18)
where \( \gamma = N(s = 0, b) \) is the imaginary part of the scattering amplitude of a dipole with the target at low energies, while \( \gamma_{in} \) is the inelastic contribution \((|N(s, b)|^2 + G_{in})\) for interaction of a dipole with the target at low
energy. Generally speaking both these amplitudes are arbitrary and have to be calculated from non-perturbative QCD, however, assuming the low energies are not very low and we can apply the relation of Eq. (2.1) we see that $2 \gamma = \gamma_{in}$. Using this relation we can re-write Eq. (2.17) and Eq. (2.18) in the form

$$Z_0(u|Y) = Z(u, u|Y)$$ (2.19)

The initial condition depends on what we want to calculate. This is the main advantage of the generating functional that it allows us to calculate everything. For example, the cross section of single diffraction integrated over all produced masses ($\sigma_{sd}$) we can find just calculating $Z(u, v = 0|Y - Y_0)$ for the initial condition in the form

$$Z(u, v|Y = Y_0) = v$$ (2.20)

The cross section is equal to

$$\sigma_{sd}(\gamma(Y_0)|Y - Y_0) = 1 - Z(u = 1 - \gamma, v = 0|Y - Y_0)$$ (2.21)

The main idea of this paper is to introduce cut Pomeron being split into three different states. By analogy with Eq. (2.13) we can relate to each process correspondent term of differential equation

$$\mathcal{P} \to \mathcal{P} + \mathcal{P} \sim \gamma_{in}^2 \frac{\partial Z}{\partial \gamma_{in}};$$ (2.22)

$$\mathcal{P} \to \mathcal{P} + P \sim \gamma_{in} \frac{\partial Z}{\partial \gamma_{in}};$$ (2.23)

$$\mathcal{P} \to P + P \sim \gamma_{in}^2 \frac{\partial Z}{\partial \gamma_{in}};$$ (2.24)

where the notation $\mathcal{P}$ and $P$ stand for cut and uncut Pomeron respectively and only second order terms in $\gamma (\gamma_{in})$ are shown. The next step it to use AGK cutting rules to write the resulting evolution equation

$$\frac{\partial Z}{\partial Y} = \Gamma(1 \to 2)(\gamma - \gamma^2) \frac{\partial Z}{\partial \gamma} + \Gamma(1 \to 2)(2\gamma^2 - 4\gamma_{in} + \gamma_{in}^2) \frac{\partial Z}{\partial \gamma_{in}}$$ (2.25)

It can be easily seen that the second term reproduces the first term for $2 \gamma = \gamma_{in}$. We note that $u = 1 - \gamma$ and $v = 1 - \gamma_{in}$ and thus Eq. (2.22) for $u$ and $v$ reads as

$$\frac{\partial Z}{\partial Y} = -\Gamma(1 \to 2)u(1 - u) \frac{\partial Z}{\partial u} - \Gamma(1 \to 2)(2u^2 - 4uv + v^2 + v) \frac{\partial Z}{\partial v}$$ (2.26)

The description in terms of generating function becomes clear for Markov chain. We use the definition of generating function given by Eq. (2.14) to find the differential equation for probabilities

$$\frac{\partial P(n, m|Y)}{\partial (\Gamma(1 \to 2) Y)} =$$

$$(P \to P + P) \quad - n P(n, m|Y) + (n - 1) P(n - 1, m|Y)$$ (2.27)

$$(P \to P + P) \quad + m P(n, m|Y) - (m - 1) P(n, m - 1|Y)$$ (2.28)

$$(P \to P + P) \quad - 4m P(n, m|Y) + 4m P(n - 1, m|Y)$$ (2.29)

$$(P \to P + P) \quad + 2m P(n, m|Y) - 2(m + 1) P(n - 2, m + 1|Y)$$ (2.30)

where each line describes specific process of Pomeron splitting discussed above. It is instructive to compare this equation to Eq. (2.6). Each of Eq. (2.27) - Eq. (2.30) consists of two terms: one describes to increase of probability...
to find $n$-Pomerons due to decay of one Pomeron to two and one is responsible for the decrease of this probability since one of $n$ Pomerons can decay. In all equations, except Eq. (2.28) and Eq. (2.30), the increase leads to the plus sign and a decrease to the minus sign. However, in Eq. (2.28) and Eq. (2.30) the signs are opposite in accordance with the AGK cutting rules (see Fig. 3). In this case we have to say that decay $P \rightarrow P + P$ and $P \rightarrow P + P$ have negative amplitudes. It should be stressed that in terms of the amplitude (see Eq. (2.25)) we obtain positive arbitrary function of variables $\xi$ since one of the signs and a decrease to the minus sign. However, in Eq. (2.28) and Eq. (2.30) the signs are opposite in accordance with the AGK cutting rules (see Fig. 3).

Equations Eq. (2.27) - Eq. (2.30) give clear probabilistic interpretation of all Pomeron splitting processes under discussion.

It turns out that Eq. (2.26), being a typical Liouville equation, has the solutions that depend on two variables $\xi_1 = \Gamma(1 \rightarrow 2)Y + \ln \frac{u}{1-u}$ and $\xi_2 = \Gamma(1 \rightarrow 2)Y + \ln \frac{2u-v}{2u-v}$. The general solution of this equation is given by an arbitrary function of variables $\xi_1$ and $\xi_2$, which in our case can be written as sum of two functions, namely,

$$Z = F_1 \left\{ \Gamma(1 \rightarrow 2)Y + \ln \frac{u}{1-u} \right\} + F_2 \left\{ \Gamma(1 \rightarrow 2)Y + \ln \frac{2u-v}{1-(2u-v)} \right\}$$  \hspace{1cm} (2.31)

For our initial condition Eq. (2.20) the solution reads

$$Z = \frac{2ue^{-\Gamma(1 \rightarrow 2)Y}}{1 + u(e^{-\Gamma(1 \rightarrow 2)Y} - 1)} - \frac{(2u-v)e^{-\Gamma(1 \rightarrow 2)Y}}{1 + (2u-v)(e^{-\Gamma(1 \rightarrow 2)Y} - 1)}$$  \hspace{1cm} (2.32)

One can easily see that this solution satisfies both initial Eq. (2.20) and boundary Eq. (2.15) conditions.

2.4 The generating functional for the multiparticle production: non-linear equation

Using our initial condition Eq. (2.20) we can rewrite linear differential equation Eq. (2.26) as non-linear one. We use the fact mentioned above, namely, that Eq. (2.26) is differential equation of two variable $\xi_1$ and $\xi_2$ and, thus, has no separate dependence on $Y$. This means that the differential equation Eq. (2.26) written at some rapidity $Y$ keeps the same form for any rapidity. We pick initial rapidity $Y = 0$ and substitute generating function given by Eq. (2.20) into Eq. (2.26)

$$\frac{\partial Z}{\partial Y} = 0 - \Gamma(1 \rightarrow 2)(2u^2 - 4uv + v^2 + v)$$  \hspace{1cm} (2.33)

Now use initial condition from Eq. (2.8) for generating function with no cut Pomerons. In terms of $Z_0$ and $Z$ Eq. (2.33) reads

$$\frac{\partial Z}{\partial Y} = -\Gamma(1 \rightarrow 2)(2Z_0^2 - 4Z_0Z + Z^2 + Z)$$  \hspace{1cm} (2.34)

We identify scattering amplitude and diffractive cross section with $N_0(\gamma|Y) = 1 - Z_0(1 - \gamma|Y)$ and $N(\gamma, \gamma_{in}|Y) = 1 - Z(1 - \gamma, 1 - \gamma_{in}|Y)$, respectively. For $N_0(\gamma|Y)$ and $N(\gamma, \gamma_{in}|Y)$ Eq. (2.34) reads

$$\frac{\partial N(\gamma, \gamma_{in}|Y)}{\partial Y} = -\Gamma(1 \rightarrow 2) \left\{ 2N_0^2(\gamma|Y) - 4N_0(\gamma|Y)N(\gamma, \gamma_{in}|Y) + N^2(\gamma, \gamma_{in}|Y) \right\}$$  \hspace{1cm} (2.35)

This equation has the same form as the equation for the diffraction production obtained by Kovchegov and Levin (see Eq. 2.3), but it is written for a general functional and describes not only diffractive production but also the processes of particle production with any value of multiplicity. The fact that Eq. (2.3) has a simple generalization on the general case of QCD gives us a hope to generalize this equation.

\footnote{The difference in overall minus sign corresponds to different definitions of rapidity variable moving in opposite direction.}
2.5 The generating functional for the multiparticle production: consistency with the AGK cutting rules

We want to check the consistency of our solution Eq. (2.32) with the AGK cutting rules in an explicit way. To do this we calculate cross section of a process \( \sigma^{(k)} \) with a given multiplicity \( k \) from both generating function given by Eq. (2.32) and directly from the AGK cutting rules, and compare them. We define cross section with multiplicity \( k \) as

\[
\sigma^{(k)} = \frac{1}{k!} \left| \frac{\partial^k N(\gamma, \gamma_{in}|Y)}{\partial \gamma_{in}^k} \right|_{\gamma_{in}=0} \gamma_{in}^k
\]  

(2.36)

where \( N(\gamma, \gamma_{in}|Y) = 1 - Z(1 - \gamma, 1 - \gamma_{in}|Y) \).

As an example we pick multiplicity to be that of one cut Pomeron, i.e. \( k = 1 \) for any number of uncut Pomerons. From Eq. (2.36) with \( k = 1 \) we get

\[
\sigma^{(1)} = \frac{\gamma_{in} e^{\Gamma(1\to2)Y}}{(1 + 2\gamma e^{\Gamma(1\to2)Y} - 1)^2}
\]  

(2.37)

On the other hand we can use coefficients for multiple Pomeron exchange from the AGK cutting rules [32]. In this case cross section for multiplicity of \( k \) cut Pomerons reads

\[
\sigma^{(k)} = \sum_{n=0}^{\infty} (-1)^{n-k} C_n(2\gamma)^n e^{n\Gamma(1\to2)Y}
\]  

(2.38)

where \( e^{\Gamma(1\to2)Y} \) stands for Pomeron propagator.

For \( k = 1 \) we sum over \( n \) in Eq. (2.38) and obtain

\[
\sigma^{(1)} = \frac{2\gamma e^{\Gamma(1\to2)Y}}{(1 + 2\gamma e^{\Gamma(1\to2)Y} - 1)^2}
\]  

(2.39)

In high energy limit \( \gamma_{in} = 2\gamma \) and \( e^{\Gamma(1\to2)Y} - 1 \simeq e^{\Gamma(1\to2)Y} \), thus Eq. (2.37) reproduces Eq. (2.39). It can be easily shown that this holds for any value of \( k \).

We can further compare cross sections obtained from generating function and direct summation of fan diagrams using AGK rules [32]. In our approach this corresponds to

\[
\sigma_{sd} = N(\gamma, \gamma_{in} = 0|Y) \quad \sigma_{in} = N(\gamma, \gamma_{in} = 2\gamma_{in}|Y)
\]  

(2.40)

where \( N(\gamma, \gamma_{in}|Y) = 1 - Z(1 - \gamma, 1 - \gamma_{in}|Y) \). The resulting expressions are identical and given by

\[
\sigma_{sd} = \frac{2\gamma^2 e^{\Gamma(1\to2)Y} (e^{\Gamma(1\to2)Y} - 1)}{(1 + \gamma (e^{\Gamma(1\to2)Y} - 1))(1 + 2\gamma (e^{\Gamma(1\to2)Y} - 1))}
\]

\[
\sigma_{el} = \frac{2\gamma e^{\Gamma(1\to2)Y}}{1 + 2\gamma (e^{\Gamma(1\to2)Y} - 1)}
\]  

(2.41)

\[
\sigma_{tot} = \frac{2\gamma e^{\Gamma(1\to2)Y}}{1 + \gamma (e^{\Gamma(1\to2)Y} - 1)}
\]  

(2.42)
3. Pomeron loops

3.1 Evolution equation with loops

Now we want to account for contributions of Pomeron loops. This problem has already been solved for a case with no cut Pomerons \[36\]. The master equation for no cut Pomerons is given by

\[
\frac{\partial Z}{\partial Y} = -\Gamma(1 \rightarrow 2)u(1 - u)\frac{\partial Z}{\partial u} + \frac{1}{2}\Gamma(2 \rightarrow 1)(u - w^2)\frac{\partial^2 Z}{\partial u^2}
\] (3.1)

Unfortunately, one cannot generalize Eq. (2.26) for diffractive processes by just adding second order derivative terms by analogy with Eq. (3.1). The reason for that is because this type of equation would include diagrams that does not exist. For example, diagrams of the type \(P \rightarrow P + P \rightarrow P\) where cut Pomeron splits to two uncut Pomerons with further merging to uncut Pomeron are forbidden. To resolve this problem we introduce two separate variables \(w\) and \(\bar{w}\) for uncut Pomeron being in amplitude or conjugate amplitude, respectively. Naturally, these two subsets of Pomerons evolve separately till the cut Pomeron is introduced, alternatively, their evolutions mix only via cut Pomeron. Using our previous discussions we can readily write this new type of evolution equation based on Eq. (2.26)

\[
\frac{\partial Z}{\partial Y} = -\Gamma(1 \rightarrow 2)\{w(1 - w)\frac{\partial Z}{\partial w} - \bar{w}(1 - \bar{w})\frac{\partial Z}{\partial \bar{w}}\} - \Gamma(1 \rightarrow 2)(2\bar{w}w - 2\bar{w}v - 2\bar{w}v + v^2 + v)\frac{\partial Z}{\partial v} + \frac{1}{2}\Gamma(2 \rightarrow 1)\{(w - w^2)\frac{\partial^2 Z}{\partial w^2} + (\bar{w} - \bar{w}^2)\frac{\partial^2 Z}{\partial \bar{w}^2}\} - \frac{1}{2}\Gamma(2 \rightarrow 1)\{2(v - \bar{w}w)\frac{\partial^2 Z}{\partial w \partial \bar{w}} + 2(v - \bar{w}v)\frac{\partial^2 Z}{\partial \bar{w} \partial v} + (v - v^2)\frac{\partial^2 Z}{\partial v^2}\} Z
\] (3.2)

where the first and second lines are generalization of Eq. (2.26) for variables \(w\) and \(\bar{w}\), the third line corresponds to Eq. (3.1) and the last line a little bit more explanation. We have to introduce a new generating function

\[
Z(w, \bar{w}, v|Y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}P(k, l, m|Y)w^k \bar{w}^l v^m
\] (3.3)

where \(P(k, l, m|Y)\) stands for probability to find \(k\) uncut Pomerons in the amplitude, \(l\) uncut Pomerons in the conjugate amplitude and \(m\) cut Pomerons at some rapidity \(Y\). For the last term of Eq. (3.2) we write Markov chain
in a similar manner we did it for Eq. (2.27)-Eq. (2.30), namely,

\[
\frac{\partial P(k,l,m|Y)}{\partial Y} = \Gamma(1 \to 2) \{ (k-1)P(k-1,l,m|Y) - kP(k,l,m|Y) \} \tag{3.4}
\]

\[
(P \to P + P) = + \Gamma(1 \to 2) \{ (k-1)P(k-1,l,m|Y) - lP(k,l,m|Y) \}
\]

\[
(\bar{P} \to \bar{P} + \bar{P}) = + \Gamma(1 \to 2) \{ (l-1)P(k,l-1,m|Y) - MP(k,l,m|Y) \}
\]

\[
(\bar{P} \to \bar{P} + P) = - \Gamma(1 \to 2) \{ (m-1)P(k,l,m-1|Y) - mP(k,l,m|Y) \}
\]

\[
(P \to P + \bar{P}) = + 2 \Gamma(1 \to 2) \{ mp(k-1,l,m|Y) - mP(k,l,m|Y) \}
\]

\[
(\bar{P} \to \bar{P} + \bar{P}) = + 2 \Gamma(1 \to 2) \{ mp(k,l-1,m|Y) - mP(k,l,m|Y) \}
\]

\[
(P \to P + \bar{P}) = - 2 \Gamma(1 \to 2) \{ (m+1)P(k-1,l-1,m+1|Y) - mP(k,l,m|Y) \}
\]

\[
(P + P \to P) = + \frac{1}{2} \Gamma(2 \to 1) \{ (k+1)P(k+1,l,m|Y) - kP(k,l,m|Y) \}
\]

\[
(\bar{P} + \bar{P} \to \bar{P}) = + \frac{1}{2} \Gamma(2 \to 1) \{ (l+1)P(k,l+1,m|Y) - lP(k,l,m|Y) \}
\]

\[
(P + P \to P) = - \frac{1}{2} \Gamma(2 \to 1) \{ (m+1)P(k,l,m+1|Y) - mP(k,l,m|Y) \}
\]

\[
(\bar{P} + \bar{P} \to \bar{P}) = - \frac{1}{2} \Gamma(2 \to 1) \{ (k+1)mP(k+1,l,m|Y) - kmP(k,l,m|Y) \}
\]

\[
(P + \bar{P} \to P) = - \frac{1}{2} \Gamma(2 \to 1) \{ (l+1)mP(k,l+1,m|Y) - lmP(k,l,m|Y) \}
\]

\[
(P + \bar{P} \to \bar{P}) = - \frac{1}{2} \Gamma(2 \to 1) \{ (k+1)(l+1)P(k+1,l+1,m|Y) - klP(k,l,m|Y) \}
\]

where \( \tilde{P} \) denotes Pomeron in conjugate amplitude and factor of \( \frac{1}{2} \) accounts for a fact that the Pomerons are identical in this approach. Each line in Eq. (3.4) as in Eq. (2.27)-Eq. (2.30) has a clear probabilistic interpretation. We multiply Eq. (3.3) by \( w^k\bar{w}^lv^m \), sum over all \( k, l \) and \( m \), and using definition Eq. (3.3) obtain Eq. (3.2).

Even in our simple model with no coordinate dependence Eq. (3.2) is too much complicated and its solution is still to be found. But before the solution is found we can see that putting \( w = \bar{w} = u \), i.e. making no difference between uncut Pomerons, first two lines of Eq. (3.3) correctly reproduce Eq. (2.27). Moreover, we can check it further and using initial condition \( Z(w,\bar{w},v|Y = 0) = v \) we can perform iterations

\[
\frac{\partial Z_1}{\partial Y} = -\Gamma(1 \to 2)(2w\bar{w} - 2wv - 2\bar{w}v + v^2 + v) \tag{3.5}
\]

giving

\[
Z_1 = -\Gamma(1 \to 2)(2w\bar{w} - 2wv - 2\bar{w}v + v^2 + v)Y \tag{3.6}
\]

At the next step

\[
Z_2 = + \Gamma^2(1 \to 2) \{ w(1-w)(2\bar{w} - 2v) - \bar{w}(1-\bar{w})(2w - 2v) \} \frac{Y^2}{2}
\]

\[
+ \Gamma^2(1 \to 2)(2w\bar{w} - 2wv - 2\bar{w}v + v^2 + v)(-2w + \bar{w}) + v + 1 \frac{Y^2}{2}
\]

\[
- \frac{1}{2} \Gamma(1 \to 2)\Gamma(2 \to 1) \{ 2(v - w\bar{w})(-2) + 2(v - w\bar{w})2 + 2(v - w\bar{w})2 + (v - v^2)(-2) \} \frac{Y^2}{2} \tag{3.7}
\]

It is easy to see from Eq. (3.7) that Eq. (3.2) correctly reproduces the sign and combinatorics coefficient of Pomeron loops in accordance with AGK cutting rules. For example, in the first term in the third line \(-\frac{1}{2}\Gamma(1 \to 2)\Gamma(2 \to 2)\)
\[ \gamma = \gamma + \gamma \]

\[ R \]

\[ \gamma = \gamma + \gamma^2 \]

1) The renormalization procedure in the case of the simplest ‘fan’ diagram.

Figure 5: The renormalization procedure in the case of the simplest ‘fan’ diagram.

1)\(2(v - w\bar{w})(-2)\), the term proportional to \(v\) describes the Pomeron loop of the type \(P \rightarrow P + \bar{P} \rightarrow \bar{P}\). This loop has factor of 2 and brings plus sign as expected. Similarly, the terms proportional to \(v\) in the second and third term of the last line correspond to loops of the type \(P \rightarrow P + P \rightarrow P\) and \(P \rightarrow P + \bar{P} \rightarrow P\), respectively. Each of them brings factor of \(-2\), putting \(P = \bar{P}\) we have a factor of \(-4\) which is in agreement with AGK cutting rules. Thus, we expect Eq. (3.2) properly include Pomeron loops into evolution.

As one can see, the integration over \(y_1\) reduces the diagram in Fig. 5 to two contributions: the exchange of two non-interacting Pomerons and the exchange of one Pomeron with the renormalized vertex: \(\gamma_R = \gamma + \gamma^2\). In Fig. 6...
is shown the Pomeron ‘fan’ diagram of the second order which we have to integrate over two rapidities $y_1$ and $y_2$. The result is

$$A(\text{Fig. 6}) =$$

$$= 2 \Delta^2 \gamma^3 \int_0^Y d y_1 \int_0^{y_1} d y_2 G(Y - y_1) G(y_1 - 0) G(y_2 - 0) G^2(y_2 - 0)$$

$$= 2 \Delta^2 \gamma^3 \int_0^Y d y_1 \int_0^{y_1} d y_2 e^{\Delta(y_1 + y_2)} = 2 \Delta^2 \gamma^3 \left( \frac{1}{2 \Delta^2} e^{3 \Delta Y} - \frac{1}{\Delta^2} e^{2 \Delta Y} + \frac{1}{2 \Delta^2} e^{\Delta Y} \right)$$

Adding the contributions of this diagram and the diagrams of Fig. 5 we obtain

$$A(\text{Fig. 5}) + A(\text{Fig. 6}) =$$

$$= - \gamma^3 e^{3 \Delta Y} - \gamma (\gamma + \gamma^2) e^{2 \Delta Y} + (\gamma + \gamma^2 + \gamma^3) e^{3 \Delta Y} = \gamma^3 e^{3 \Delta Y} - 2 \gamma \gamma^2 e^{2 \Delta Y} + \gamma R e^{\Delta Y}$$

Therefore, one can see that the scattering amplitude can be rewritten as exchanges of the Pomerons without interaction between them but with Pomeron-particle vertex. In the dipole model this vertex has a meaning of the amplitude of two dipole interaction in the Born approximation of perturbative QCD.

These two examples illustrates our main idea: the BFKL Pomeron calculus in zero transverse dimensions can be viewed as the theory of free, non-interacting Pomerons whose interaction with the target has to be renormalized. It is easy to see that in the MFA we can rewrite the master equation (see Eq. (2.7) and Eq. (2.11)) in the form

$$\frac{\partial N_0(\gamma R | Y)}{\partial Y} = \Gamma(1 \to 2) \gamma R \frac{\partial N_0(\gamma R | Y)}{\partial \gamma R}$$

with

$$\gamma R = \frac{\gamma}{1 - \gamma}$$

The way Eq. (3.13) has started to build in perturbation expansion we have shown in Eq. (3.8) and Eq. (3.10).

The general solution of Eq. (3.12) is the system of non-interacting Pomerons and the scattering amplitude can be found in the form

$$N_0(\gamma R | Y) = \sum_{n=1}^{\infty} C_n \gamma^n R^n (Y - 0)$$

Figure 6: The renormalization procedure in the case of the ‘fan’ diagram of the second order.
where coefficients $C_n$ could be found from the initial conditions, namely, from the expression for the low energy amplitude. In particular, the initial condition

$$N_0 (\gamma_R | Y = 0) = \gamma = \gamma_R/(1+\gamma_R)$$

(3.15)

generates $C_n = (-1)^n$ and the solution is

$$N_0 (\gamma_R | Y) = \frac{\gamma_R e^{\Delta Y}}{1 + \gamma_R e^{\Delta Y}}$$

(3.16)

The initial condition of Eq. (3.15) has a very simple physics behind it. It describes the independent (non-correlated) production of the Pomerons at low energy but with the only condition that one Pomeron lives shorter time than the second one (see Fig. 7). If all $n$ Pomerons were emitted by the same dipole (see Fig. 7-b) this condition leads to Glauber factor $1/n!$ leading to $N_0 (\gamma_R | Y = 0) = \gamma = 1 - \exp(-\gamma_R)$, instead of Eq. (3.15). However, if Pomerons are produced as the consequent decays (see Fig. 7-a) the factor is equal to 1. $(-1)^n$ comes from the Glauber screening, resulting in Eq. (3.15). In QCD we have strong evidence that the second case is correct [24].

The analysis of enhanced diagrams we start from the first diagram of Fig. 8. It leads to the following contribution

$$A(Fig. 8) = -\Delta^2 \gamma^2 \int_0^Y dy_1 \int_0^{y_1} dy_2 G(Y - y_1) G^2(y_1 - y_2) G(y_2 - 0)$$

$$= -\Delta^2 \gamma^2 \int_0^Y dy_1 \int_0^{y_1} dy_2 e^{\Delta (Y + y_1 - y_2)}$$

$$= -\gamma^2 e^{2\Delta Y} + \gamma^2 e^{\Delta Y} + \Delta \gamma^2 Y e^{\Delta Y}$$

where $\Gamma(2 \to 1) = \Delta \gamma^2$ (see Fig. 7-a).

Adding Eq. (3.17) to the exchange of the one Pomeron we obtain that the exchange of one Pomeron and the enhanced diagram of Fig. 8 can be written in close form

$$\text{One Pomeron exchange } + A(Fig. 8) = \gamma_R e^{\Delta_R Y} - \gamma^2 e^{2\Delta Y}$$

(3.18)

with

$$\gamma_R = \gamma^{(2)} = \gamma + \gamma^2; \quad \Delta_R = \Delta + \gamma \Delta;$$

(3.19)

It is easy to see the Eq. (3.17) can be viewed as the expansion to first order of Eq. (3.18).

Therefore, the Pomeron loops can be either large (of the order of $Y$) and they can be considered as un-enhanced diagrams or small (of the order of $1/\Delta$) and they can be treated as the renormalization of the Pomeron intercept.

In QCD $\Delta \propto \bar{\alpha}_S$ while $\gamma \propto \alpha^3_S$. Therefore, the renormalization of the Pomeron intercept $\Delta$ is proportional to $\alpha^3_S$. We can neglect this contribution since (i) there a lot of $\alpha^2_S$ corrections to the kernel of the BFKL equation that are much larger than this contribution; and (ii) in the region of $Y \ll 1/\alpha^2_S$, where we can trust our Pomeron calculus (see introduction) $(\Delta_R - \Delta) Y \ll 1$.

Concluding this analysis we can claim that the BFKL Pomeron calculus in zero dimensions is a theory of non-interacting Pomerons with renormalized vertices of Pomeron-particle interaction. In the dipole language, it means that we have a system non-interacting Pomerons with a specific hypothesis on the amplitude of the dipole interactions at low energy. For the problem that we are solving here, namely, when we have one bare Pomeron at low energy, this amplitude is determined by Eq. (3.15).
For such a system we can calculate the scattering amplitude using a method suggested by Mueller, Patel, Salam and Iancu and developed in a number of papers (see [33, 12, 30, 2, 39, 5, 9] and references therein). This method suggests that the scattering amplitude can be calculated using the t-channel unitarity constraints which is written in the following way (assuming that amplitudes at high energy are pure imaginary, \( N = \text{Im} A \)):

\[
N([\ldots]|Y) = N([\ldots]|Y - Y'; P \to nP) \otimes N([\ldots]|Y'; P \to nP)
\]

(3.20)

where \( \otimes \) stands for all needed integrations while \([\ldots]\) describes all quantum numbers (dipole sizes and so on).

The correct implementation of this leads in our case to the following formula (see also [39, 42, 9])

\[
N^\text{MPSI}_0(Y) = 1 - \exp \left\{ -\gamma^BA \frac{\partial}{\partial \gamma_R^1} \frac{\partial}{\partial \gamma_R^2} \right\} \frac{1}{\gamma^BA e^{\Gamma(1-2)Y} \gamma_R^1} \frac{1}{\gamma^BA e^{\Gamma(1-2)Y} \gamma_R^2} \Gamma \left( 0, \frac{1}{\gamma^BA e^{\Gamma(1-2)Y}} \right)
\]

(3.21)

where \( N^\text{MFA} (Y, \gamma_R) \) is given by Eq. (3.14) (see also Eq. (3.12)) in the mean field approximation and \( \gamma^BA \propto \alpha_S^2 \) is the scattering amplitude at low energies which is described by the Born approximation in perturbative QCD. The difference of Eq. (3.21) from the original MPSI approach is the fact that this equation does not depend on the value of \( Y' \) and, because of this, we do not need to choose \( Y' = Y/2 \) for the best accuracy.

Substituting Eq. (3.16) in Eq. (3.21) we obtain

\[
N^\text{MPSI}_0 (\gamma^BA|Y) = 1 - \exp \left( \frac{1}{\gamma^BA e^{\Gamma(1-2)Y}} \right) \Gamma \left( 0, \frac{1}{\gamma^BA e^{\Gamma(1-2)Y}} \right)
\]

(3.22)

\( \Gamma(0, x) \) is the incomplete gamma function (see formulae 8.350 - 8.359 in [33]).

We claim that Eq. (3.22) is the solution to our problem. One can easily see that \( N_0 (\gamma|Y) \to 1 \) at high energies in contrast to the exact solution with Hamiltonian of Eq. (1.3). The exact solution leads to the amplitude that vanishes at high energy (see [1, 8]). As have been mentioned the solution depends crucially on the initial condition for the scattering amplitude at low energies. For Eq. (3.22) this amplitude is equal to

\[
N^\text{MPSI}_0 (\gamma|Y = 0) = \sum_{n=1}^{\infty} (-1)^{n+1} n! \left( \gamma^BA \right)^n
\]

(3.23)

with \( \gamma^BA \propto \alpha_S^2 \). This equation reminds us the ultraviolet renormalons contribution and calls for better understanding of the non-perturbative contribution.

We can rewrite Eq. (3.21) in more convenient form using the Cauchy formula for the derivatives, namely,

\[
\frac{\partial^n Z^\text{MFA}(\gamma_R|Y)}{\partial \gamma_R^n} = n! \frac{1}{2\pi i} \oint_C \frac{Z^\text{MFA}(\gamma'_R|Y)}{\gamma_R^{n+1}} d\gamma'_R;
\]

(3.24)

Contour \( C \) in Eq. (3.24) is a circle with a small radius around \( \gamma_R = 0 \). However, since function \( Z \) does not grow at large \( \gamma_R \) for \( n \leq 1 \) we can close our contour \( C \) on the singularities of function \( Z \). We will call this new contour \( C_R \).

\[
N^\text{MPSI}_0 (Y) = 1 - \exp \left\{ -\gamma^BA \frac{\partial}{\partial \gamma_R^1} \frac{\partial}{\partial \gamma_R^2} \right\} \frac{1}{\gamma^BA e^{\Gamma(1-2)Y} \gamma_R^1} \frac{1}{\gamma^BA e^{\Gamma(1-2)Y} \gamma_R^2} \Gamma \left( 0, \frac{1}{\gamma^BA e^{\Gamma(1-2)Y}} \right)
\]

(3.25)
Here we introduce new variables $\tilde{\gamma}_1^R = \gamma_1^R \exp(\Gamma(1 \to 2)(Y - Y'))$ and $\tilde{\gamma}_2^R = \gamma_2^R \exp(\Gamma(1 \to 2)Y')$. In these new variables

$$Z_{MFA}(\tilde{\gamma}_1^R) = \frac{1}{1 + \tilde{\gamma}_1^R}; \quad Z_{MFA}(\tilde{\gamma}_2^R) = \frac{1}{1 + \tilde{\gamma}_2^R} \quad (3.26)$$

Closing the integration on the poles $\tilde{\gamma}_1^R = -1$ and $\tilde{\gamma}_2^R = -1$ we obtain the formula of Eq. (3.22).

### 3.3 The generating functional for the multiparticle production with Pomeron loops

Using the result of the previous section we will derive the formula in the MPSI approach for the general functional, defined by Eq. (3.3). This formula is based on the solution of Eq. (3.2) but without the secondary derivatives. Such a solution gives the MFA approximation to our problem and we denote it as $Z_{MFA}(w, \bar{w}, v|Y)$. The equation for $Z_{MFA}(w, \bar{w}, v|Y)$ looks as follows

$$Z(w, \bar{w}, v|Y) = \frac{w e^{-\Gamma(1\to2)Y}}{1 + w(e^{-\Gamma(1\to2)Y} - 1)} + \frac{\bar{w} e^{-\Gamma(1\to2)Y}}{1 + \bar{w}(e^{-\Gamma(1\to2)Y} - 1)} - \frac{(w + \bar{w} - v)e^{-\Gamma(1\to2)Y}}{1 + (w + \bar{w} - v)(e^{-\Gamma(1\to2)Y} - 1)} \quad (3.27)$$

Using the renormalized $\gamma$ of Eq. (3.13) we can rewrite Eq. (3.27) in a different form, namely,

$$Z_{MFA}(\gamma_R, \bar{\gamma}_R, \gamma_{in,R}|Y) = \frac{1}{1 + \gamma_R e^{\Gamma(1\to2)Y}} + \frac{1}{1 + \bar{\gamma}_R e^{\Gamma(1\to2)Y}} - \frac{1}{1 + \xi_R e^{\Gamma(1\to2)Y}} \quad (3.28)$$

where we use a notation $\xi = 1 - w - \bar{w} + v = \gamma + \bar{\gamma} - \gamma_{in}$ and

$$\xi_R = \gamma_R + \bar{\gamma}_R - \gamma_{in,R} = \frac{\xi}{1 - \xi}; \quad \xi = \frac{\xi_R}{1 + \xi_R}; \quad (3.29)$$

The first of Eq. (3.29) is the definition of $\gamma_{in,R}$.
The general formula for the amplitude in the MPSI approach has the form

\[
N^{MPSI} (\gamma_{in}^BA, \gamma_{in}^{BA}|Y) = \left( \exp \left\{ -\gamma_{in}^{BA} \frac{\partial}{\partial \gamma_{in}} \frac{\partial}{\partial \gamma_{in}^R} - \gamma_{in}^{BA} \frac{\partial}{\partial \gamma_{in}^R} \frac{\partial}{\partial \gamma_{in}^R} + \gamma_{in}^{BA} \frac{\partial}{\partial \gamma_{in}^R} \frac{\partial}{\partial \gamma_{in}^R} \right\} - 1 \right) (3.30)
\]

\[
Z^{MFA} (\gamma_{in}^R, \gamma_{in}^N|Y - Y') Z^{MFA} (\gamma_{in}^R, \gamma_{in}^N|Y') |_{\gamma_{in}^R = \gamma_{in}^N = 0} = \left( \exp \left\{ -\gamma_{in}^{BA} \frac{\partial}{\partial \gamma_{in}} \frac{\partial}{\partial \gamma_{in}^R} - \gamma_{in}^{BA} \frac{\partial}{\partial \gamma_{in}^R} \frac{\partial}{\partial \gamma_{in}^R} + \gamma_{in}^{BA} \frac{\partial}{\partial \gamma_{in}^R} \frac{\partial}{\partial \gamma_{in}^R} \right\} - 1 \right) (3.30)
\]

where \(\gamma_{in}^{BA}\) and \(\gamma_{in}^{BA}\) are the elastic and inelastic amplitudes of interaction of two dipoles at low energy which are calculated in QCD in the Born approximation. The plus sign in Eq. (3.30) in front of \(\gamma_{in}^{BA} \frac{\partial}{\partial \gamma_{in}} \frac{\partial}{\partial \gamma_{in}^R} \) reflects the fact that the Pomeron loop with two cut Pomerons does not have a negative contribution unlike in the case of uncut Pomerons. The sign between the exponent and unity in Eq. (3.30) could be easily checked noticing the first term of the expansion of the exponent correctly reproduces the positive contribution of \(\gamma_{in}\) (cut Pomeron).

The nice feature of this equation that one can see that the result does not depend on the value of an arbitrary chosen rapidity \(Y'\). Using the explicit form for \(Z^{MFA}\) given by Eq. (3.28) we can calculate \(Z^{MPSI}\) in closed form denoting by

\[
G (x) \equiv \exp \left( \frac{1}{x} \right) \frac{\partial}{\partial x} \Gamma \left( 0, \frac{1}{x} \right)
\]

\(N^{MPSI}\) is equal to

\[
N^{MPSI} (\gamma_{in}^{BA}, \gamma_{in}^{BA}|Y) = 2 \left( 1 - G (\gamma_{in}^{BA} e^{\Gamma(1-2)} Y) \right) - \left( 1 - G \left( (2 \gamma_{in}^{BA} - \gamma_{in}^{BA}) e^{\Gamma(1-2)} Y \right) \right)
\]

\[(3.32)\]

The useful formulae for getting Eq. (3.32) are the following

\[
\frac{\partial^k \partial^{l_1} \partial^{l_2}}{\partial \gamma_{in} \partial \gamma_{in}^R \partial \gamma_{in}^R} \frac{1}{1 + \gamma + \gamma - \gamma_{in}} = (-1)^{l_1 + l_2} (l_1 + l_2 + k)!
\]

\[(3.33)\]

and equations 8.350 - 8.359 for the incomplete gamma function \(\Gamma (0, x)\) in [13].

Eq. (3.32) allows us to calculate the cross section with fixed multiplicity of produced particles. Namely, the cross section for the processes with \(k < n >\) particles in the final state, where \(< n >\) is the mean multiplicity in our reaction, can be calculated as

\[
\sigma_k (Y) = \frac{1}{k!} \left( \frac{\partial^k}{\partial (\gamma_{in}^{BA})^k} N^{MPSI} (\gamma_{in}^{BA}, \gamma_{in}^{BA}|Y) \right) |_{\gamma_{in}^{BA} = 0} \cdot (\gamma_{in}^{BA} = 2 \gamma_{BA})^k
\]

\[(3.34)\]

Here we use that \(\gamma_{in}^{BA} = 2 \gamma_{BA}\) in the Born approximation of QCD.

It is interesting to check the general equation (see Eq. (7.32)) calculating two known cases: the diffractive dissociation process and the total inelastic cross section. The first one can be calculated using Eq. (3.32) with \(\gamma_{in}^{BA} = 0\). The answer is

\[
N^{MPSI}_{diss} (\gamma_{BA}|Y) = 2 N_0^{MPSI} (\gamma_{BA}|Y) - N_0^{MPSI} \left( 2 \gamma_{BA}|Y \right)
\]

\[(3.35)\]

where \(N_0\) is given by Eq. (3.22). Eq. (3.33) is a direct consequence of the unitarity constraints (see Eq. (2.16)). As you can see from Eq. (3.32) the same formula determines the diffractive production in the mean field approximation.
The value of the total inelastic cross section, that is equal the sum of diffractive production and inelastic cross section, stems from Eq. (3.32) for $\gamma_{in}^{BA} = 2 \gamma_{BA}$ and it is equal to

$$N_{total \ inelastic}^{MPSI} (\gamma_{BA} | \ Y) = 2 N_0^{MPSI} (\gamma_{BA} | \ Y) = N_{diff}^{MPSI} (\gamma_{BA} | \ Y) + N_{inel}^{MPSI} (\gamma_{BA} | \ Y)$$

(3.36)

which is actually the unitarity constraint itself (see Eq. (2.16)). One can see that the inelastic cross section is determined by

$$N_{inel}^{MPSI} (\gamma_{BA} | \ Y) = 1 - G \left( 2 \gamma_{BA} e^{\Gamma(1-\gamma)} Y \right)$$

(3.37)

4. Conclusions

In this paper we introduce a new generating function for the processes of multiparticle production both for the mean field approximation (see Eq. (2.14) and for the general case (see Eq. (3.3)). For general case where the Pomeron loops has been taken into account, we obtain the linear evolution equation for the generating function (see Eq. (3.3)) while in the mean field approximation we proved both the linear evolution equation (see Eq. (2.26)) and the non-linear equation (see Eq. (2.34)). The last one is the generalization of Kovchegov-Levin equation for diffractive production to a general case of the processes with arbitrary multiplicities. Since this equation is proven for the general QCD case we hope that the equations for the general generating function can be proven for the real QCD evolution.

The second result of the paper is the new method of summing the Pomeron loops. We argued that the sum of all Pomeron diagrams, including loops, in the kinematic region of Eq. (1.2) can be reduced to the diagrams of the Pomeron exchanges without interactions between Pomeron if we renormalize the amplitude of low energy interaction. Based on this result we suggest a generalization of the Mueller-Patel-Salam-Iancu method of summation of the Pomeron loops. In particular, we calculated the new generating function for the inelastic processes in the improved MPSI approximation (see Eq. (3.32)).

We would like to stress that we firmly believe that the scattering amplitude, calculated using this method, leads to a correct answer to the old problem: the high energy asymptotic behaviour of the scattering amplitude at ultra high energies beyond of the BFKL Pomeron calculus in kinematic region of Eq. (1.2).

We hope that both results will lead to new simplifications in the case the BFKL Pomeron calculus in QCD (in two transverse dimensions). The general case of the BFKL Pomeron calculus in QCD will be addressed in a separate paper. We would like also to mention that this case has been started to discuss in [14].

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