Mathematical model of avian influenza epidemics with burning infected poultry and noticing the success ratio of poultry vaccination

M Kharis, Amidi and A Agoestanto

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Semarang, Semarang, Indonesia

*Corresponding author: kharis.mat@mail.unnes.ac.id

Abstract. Epidemics have a major impact on human life both in the health and economic fields. One such epidemic is avian influenza. This epidemic causes the infected human to get hospitalization. Some action is needed to prevent the extension of this outbreak. Measures undertaken include vaccination in poultry, burning of infected poultry, and quarantine and treatment of infected humans. In this paper, we developed the mathematical model of AI with burning infected poultry and noticing the success ratio of poultry vaccination in a constant population.

1. Introduction

Epidemics have a significant impact on human life both in the health and economic fields. One such epidemic is avian influenza. This epidemic causes the infected human to get hospitalization. Avian Influenza (AI) virus can infect humans [1]. The influenza viruses were known as the flu that attacks poultry and mammals [2]. The AI virus was transmitted through the air by coughing or sneezing, which will lead to an aerosol containing the virus [3]. This epidemic has a great influence on the economic aspects of poultry-related issues [4].

There was mentioned that the influenza A subtype, i.e., the H5N1 virus has have resistance to the drugs given [5]. There was influenza A virus mutations that are resistant to oseltamivir [6]. The mutated virus lethal for high-risk patients. The H5N1 virus has high mutation ability, so require more attention so as not to become an outbreak in poultry and human. One of avian influenza A virus mutation is H7N9 virus that can infect humans [7-12]. The other mutation is the H10N8 virus [13].

Sya'baningtyas et al. developed that a mathematical model of AI with poultry vaccination only in the poultry population with recruitment-death population dynamic. This model assumed that the success ratio of vaccination is 100% [14]. Kharis and Amidi developed a mathematical model of avian influenza epidemics poultry vaccination in constant population. This model used the assumption that the success ratio of vaccination is 100% [15]. In this paper, we analyzed the model with the success ratio of vaccination had nonconstant value. It means the success ratio had the value \( p \) where \( 0 \leq p \leq 1 \).

2. Methods

The method that was used in this research is analysis method of a deterministic mathematical model. Analysis method in this research used some step. The first step is an analysis of the existence of the
equilibrium points. The next step is the stability analysis of the equilibrium point. The next step is making a simulation to clarify the result of the analysis. We do a literacy study before developing the model. In this activity, we determine facts and assumption which will be used to develop the model.

3. Result and Discussion
In this paper, we bounded the model by assuming the human population is constant. In the human population, the birth rate has the same value as the natural death rate. Transfer diagram of AI epidemic is given in Fig. 1.

![Transfer diagram of AI epidemic with vaccination in poultry](image-url)

Figure 1. Transfer diagram of AI epidemic with vaccination in poultry

Where \( N \) is the total number of human, \( S \) is the total number of susceptible human, \( I \) is the total number of infected human, \( R \) is the total number of recovered human, \( N_b \) is the total number of poultry, \( S_b \) is the total number of susceptible poultry, \( I_b \) is the total number of infected poultry, and \( V_b \) is the total number of vaccinated poultry. The meaning of parameters in human population: \( \mu \) means the birth rate in human was assumed same with natural death rate, \( \beta_1 \) means the probability of infectious contact among human, \( \beta_2 \) means the probability of infectious contacts between susceptible human and infected poultry, \( \gamma \) means the recovery rate of infected human, and \( \theta \) means the immunity loss rate. The meaning of parameters in poultry population: \( A \) means the recruitment rate, \( \beta_b \) means the probability of infectious contact among poultry, \( \mu_b \) the means natural death rate in poultry, \( m_b \) the means rate of death by infection in poultry, \( M \) the means death rate of infected poultry by burning, \( \delta \) the means the proportion of susceptible bird to be vaccinated every unit time, and success ratio of vaccination. From Fig. 1, we construct the System (1).

\[
\begin{align*}
\frac{dS}{dt} &= \mu N + \theta R - \frac{S (\beta_1 I + \beta_2 I_b)}{N} - \mu S \\
\frac{dI}{dt} &= \frac{S}{N} (\beta_1 I + \beta_2 I_b) - (\mu + \gamma) I \\
\frac{dR}{dt} &= \gamma I - (\theta + \mu) R \\
\frac{dS_b}{dt} &= A - \left[ \beta_b (1 - p) \frac{I_b}{N_b} + \delta p + \mu_b \right] S_b \\
\end{align*}
\]
We assumed that $\beta_1 = \beta_2 = \beta$.

Clear that $\frac{dN}{dt} = 0 \Leftrightarrow N = K > 0, K \in R$ and $R = N - I = S = K - I - S$.

Clear that $\frac{dN_b}{dt} = A - \mu_b N_b - (m_b + M)I_b$. Hence, we get System (2).

\[
\begin{align*}
\frac{dS}{dt} &= (\mu + \theta)K - \frac{\beta}{K}S(I + I_b) - (\mu + \theta)S - \theta I \\
\frac{dN}{dt} &= A - \mu_b N_b - (m_b + M)I_b \\
\frac{dI_b}{dt} &= \beta_b (1 - p) \frac{S_b - I_b}{N_b} - (\mu_b + m_b + M)I_b \\
\frac{dV_b}{dt} &= \delta p(N_b - I_b - V_b) - \mu_b V_b
\end{align*}
\]

The domain of System (2) was defined as

$\Gamma = \{P \in R^3_+ | P = (S, I, N_b, I_b, V_b) \text{ where } 0 \leq S + I \leq K \text{ and } 0 \leq I_b + V_b < N_b\}$

The existence of equilibrium points of System (2) was given in Theorem 1.

**Theorem 1.**

Let $r_0 = \frac{\beta_b (1 - p) \mu_b}{(\mu_b + m_b + M)(\mu_b + \delta p)}$ and $R_0 = \frac{\beta}{\mu + \gamma}$

1. If $r_0 < 1$ and $R_0 < 1$ then System (2) has only one equilibrium point, i.e., nonendemic equilibrium point $P_0 = (S, I, N_b, I_b, V_b) = (K, 0, 0, \frac{\mu_b}{\mu_b}, 0, \frac{A \delta p}{\mu_b} \frac{\mu_b}{(\delta p + \mu_b)})$.

2. If $r_0 < 1$ and $R_0 > 1$ then System (2) has two equilibrium i.e $P_0$ and $P_1 = (S, I, N_b, I_b, V_b)$ where

\[
P_1 = (K(\mu + \gamma) I_2, I_, A - (m_b + M)I_b, \frac{A \delta p}{\mu_b} \frac{\mu_b}{(\delta p + \mu_b)})
\]

3. If $r_0 > 1$ then System (2) has Three equilibrium i.e $P_0, P_1$ and $P_2 = (S, I, N_b, I_b, V_b)$ where

\[
P_2 = (K(\mu + \gamma) I_2, I_, A - (m_b + M)I_b, \frac{A \delta p}{\mu_b} \frac{\mu_b}{(\delta p + \mu_b)})
\]

**Proof:**

The equilibrium points were solution of System (3).

\[(\mu + \theta)K - \frac{\beta}{K}S(I + I_b) - (\mu + \theta)S - \theta I = 0
\]

\[
\frac{\beta}{K}S(I + I_b) - (\mu + \gamma) I = 0
\]
\[ A - \mu_b N_b - (m_b + M)I_b = 0 \]  \hspace{1cm} (3)
\[ \beta_b (1 - p) \frac{N_b - I_b - V_b}{N_b} I_b - (\mu_b + m_b + M)I_b = 0 \]
\[ \delta p (N_b - I_b - V_b) - \mu_b V_b = 0 \]

From the fourth equation of System (3), we get \( I_b = 0 \vee \beta_b (1 - p) \frac{N_b - I_b - V_b}{N_b} = (\mu_b + m_b + M) \).

Case of \( I_b = 0 \):

Substitute \( I_b = 0 \) to the third equation, and then we get \( N_b = \frac{A}{\mu_b} \).

Substitute \( I_b = 0 \) and \( N_b = \frac{A}{\mu_b} \) to the last equation, and then we get \( V_b = \frac{A \delta p}{\mu_b (\delta p + \mu_b)} \).

Substitute \( I_b = 0 \) to the second equation, then we get \( I = 0 \vee S = \frac{K (\mu + \gamma)}{\beta} \).

Case of \( I = 0 \):

Substitute \( I_b = 0 \) and \( I = 0 \) to the first equation, and then we get \( S = K \).

Hence, we get \( P_0 = (S, I, N_b, I_b, V_b) = \left( K, 0, \frac{A}{\mu_b}, 0, \frac{A \delta p}{\mu_b (\delta p + \mu_b)} \right) \).

Case of \( I \neq 0 \):

Clear that \( S = \frac{K (\mu + \gamma)}{\beta} \).

Substitute to the first equation then we get \( I = \frac{k (\mu + \theta) [\beta - (\mu + \gamma)]}{\beta (\mu + \gamma + \theta)} \).

Clear that if \( R_0 = \frac{\beta}{\mu + \gamma} > 1 \) then \( I > 0 \).

Hence, we get if \( R_0 > 1 \) then
\[ P_1 = (S, I, N_b, I_b, V_b) = \left( \frac{K (\mu + \gamma)}{\beta}, \frac{k (\mu + \theta) [\beta - (\mu + \gamma)]}{\beta (\mu + \gamma + \theta)}, \frac{A}{\mu_b}, 0, \frac{A \delta p}{\mu_b (\delta p + \mu_b)} \right) \]

The case of \( I_b \neq 0 \):

Clear that \( \beta_b (1 - p) [N_b - (I_b + V_b)] = (\mu_b + m_b + M) \).

\( \iff [\beta_b (1 - p) - (\mu_b + m_b + M)] N_b - \beta_b (1 - p) (I_b + V_b) = 0 \).

From the third equation of System (3), we get
\[ A - \mu_b N_b - (m_b + M)I_b = 0 \iff N_b = \frac{A - (m_b + M)I_b}{\mu_b} \]

Substitute the value of \( N_b \) to the last equation of System (3), and we got \( V_b = \frac{\delta p [A - (\mu_b + m_b + M)I_b]}{\mu_b (\delta p + \mu_b)} \).

From \( [\beta_b (1 - p) - (\mu_b + m_b + M)] N_b - \beta_b (1 - p) (I_b + V_b) = 0 \), we got
\[ A [\beta_b (1 - p) \mu_b - (\mu_b + m_b + M) (\delta p + \mu_b)] = \beta_b (1 - p) \mu_b^2 \]

Let \( r_0 = \frac{(\mu_b + m_b + M) (\mu_b + \delta p)}{\beta_b (1 - p) \mu_b} \).

Hence, if \( r_0 > 1 \) then \( I_b < 0 \).

Clear that for \( r_0 > 1 \), we got
\[ \frac{\beta_b (1 - p) \mu_b}{(\mu_b + m_b + M) (\mu_b + \delta p)} > 1 \iff \beta_b (1 - p) \mu_b - (\mu_b + m_b + M) (\delta p + \mu_b) > 0 \).

Hence, if \( r_0 > 1 \) then \( I_b > 0 \).

Clear that \( A - (m_b + \mu_b + M) I_b \)
\[ A [\mu_b + m_b + M) (\delta p + \mu_b)] = \beta_b (1 - p) \mu_b^2 \]

Hence, \( A - (m_b + \mu_b + M) I_b > 0 \iff r_0 > 1 \).

Clear that \( A - m_b I_b = \frac{\beta_b (1 - p) \mu_b^2 A}{(m_b + M) (\beta_b (1 - p) \mu_b - (\mu_b + m_b + M) (\delta p + \mu_b)) + \beta_b (1 - p) \mu_b^2} \)

Hence, \( V_b > 0 \) and \( N_b > 0 \) if \( r_0 > 1 \). Clear that
V_b = \frac{A\delta p(\mu_b + m_b + M)}{[(m_b + M)[\beta_b(1 - p)\mu_b - (\mu_b + m_b + M)(\delta p + \mu_b)] + \beta_b(1 - p)\mu_bA]}

and \( N_b = \frac{[(m_b + M)[\beta_b(1 - p)\mu_b - (\mu_b + m_b + M)(\delta p + \mu_b)] + \beta_b(1 - p)\mu_b^2]}{\beta_b(1 - p)\mu_bA} \).

From the second equation of System (3), we got \( S = \frac{K(\mu + \theta)I_b}{\beta(I_b + I_s)} \).

The value of S and \( I_b \) were substituted to the first equation, we got

\( \beta(\mu + \gamma + \theta)I^2 + [K(\mu + \theta)(\mu + \gamma) + \beta(\mu + \gamma + \theta)I_b - \beta K(\mu + \theta)]I - \beta K(\mu + \theta)I_b = 0 \).

Let \( C_1 = \beta(\mu + \gamma + \theta), C_2 = K(\mu + \theta)[(\mu + \gamma) - \beta] + \beta(\mu + \gamma + \theta)I_b, C_3 = -\beta K(\mu + \theta)I_b \).

Clear that \( C_1 > 0 \) and \( C_3 < 0 \).

Clear that \( C_2^2 - 4C_1C_3 > 0 \) for every sign of \( C_2 \), so we got

\( I_1 = \frac{-C_2 - \sqrt{C_2^2 - 4C_1C_3}}{2C_1} < 0 \) and \( I_2 = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1} > 0 \).

Hence, we got
\( P_2 = (S, I, N_b, I_b, V_b) = \left(\frac{K(\mu + \gamma)I_b^2}{(\mu_b + m_b + M)I_b}, \frac{A - (\mu_b + m_b + M)I_b}{\mu_bI_b}, \frac{\delta p[A - (\mu_b + m_b + M)I_b]}{\mu_b(\delta p + \mu_b)}\right) \)

where \( I_b = \frac{[(m_b + M)[\beta_b(1 - p)\mu_b - (\mu_b + m_b + M)(\delta p + \mu_b)] + \beta_b(1 - p)\mu_b^2]}{\beta_b(1 - p)\mu_bA} \)

and \( I_2 = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1} \) with \( C_1 = \beta(\mu + \gamma + \theta), C_2 = K(\mu + \theta)[(\mu + \gamma) - \beta] + \beta(\mu + \gamma + \theta)I_b, \) and \( C_3 = -\beta K(\mu + \theta)I_b \).

The Stability of equilibrium points of System (2) was given at Theorem 2.

**Theorem 2.**

Let \( r_0 = \frac{\beta_b(1 - p)\mu_b}{(\mu_b + m_b + M)(\mu_b + \delta p)} \) and \( R_0 = \frac{\beta}{\mu + \gamma} \).

1. If \( r_0 < 1 \) and \( R_0 < 1 \) then \( P_0 \) is locally asymptotically stable.
2. If \( r_0 < 1 \) and \( R_0 > 1 \) then \( P_0 \) is unstable and \( P_1 \) is locally asymptotically stable.
3. If \( r_0 > 1 \) then both \( P_0 \) and \( P_1 \) are unstable.

**Proof:**

The Jacobian matrix of System (2) was given below

\[
J(P) = \begin{bmatrix}
\frac{-\beta(1 + \mu - (\mu + \theta)}{K} & -\theta - \frac{\beta S}{K} & 0 & -\frac{\beta S}{K} & 0 \\
\frac{\beta(1 + \mu)}{K} & \frac{\beta S}{K} & (\mu + \gamma) & 0 & 0 \\
0 & 0 & \frac{-\mu_b}{K} & 0 & 0 \\
0 & 0 & \frac{\beta_b(1 - p)l_4I_b + V_b}{N_b} & \frac{\beta_b(1 - p)(N_b - 2l_4 - V_b)}{N_b} & (\mu_b + m_b + M) \\
0 & 0 & 0 & \frac{-\delta p}{N_b} & -\delta p \\
\end{bmatrix}
\]

For \( P_0 \), we get eigenvalues of \( J(P_0) \):

\( \lambda_1 = -\theta + \mu, \lambda_2 = \beta - (\mu + \gamma) = (\mu + \gamma)(R_0 - 1), \lambda_3 = -\mu_b, \lambda_4 = -(-\delta p + \mu_b) \), and

\( \lambda_5 = \frac{1}{\delta p + \mu_b} \left[ \beta_b(1 - p)\mu_b - (\mu_b + m_b + M)(\delta p + \mu_b) \right] = (\mu_b + m_b + M)(r_0 - 1) \).

Hence, \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are negative, \( \lambda_2 < 0 \) if \( R_0 < 1 \) and \( \lambda_2 > 0 \) if \( R_0 > 1 \).

For \( P_1 \), we get characteristics equation of Matrix \( J(P_1) \):

\[
\frac{1}{(\mu + \gamma + \theta)} \left[ (\lambda + \mu_b)(\lambda + \delta p + \mu_b) \left( \frac{\lambda - \beta_b(1 - p)\mu_b}{\delta p + \mu_b} \right) \right] (\lambda^2 + BA + C) = 0
\]
where $A = \mu + \gamma + \theta, B = (\mu + \theta)(\mu + \theta + \gamma) + (\mu + \gamma)[\beta - (\mu + \gamma)],$ and $C = (\mu + \theta)(\mu + \theta + \gamma)[\beta - (\mu + \gamma)]$

Clear that $A > 0$ for every $R_0, B > 0$ and $C > 0$ if $R_0 > 1$.

From $(\lambda + \mu_B)(\lambda + \delta p + \mu_B) \left( \lambda - \frac{\beta_B(1 - p)\mu_B - (\mu_B + m_B + M)(\delta p + \mu_B)}{(\delta p + \mu_B)} \right) = 0$

We got $\lambda_1 = -\mu_B, \lambda_2 = -\delta p + \mu_B$, and $\lambda_3 = \frac{\beta_B(1 - p)\mu_B - (\mu_B + m_B + M)(\delta p + \mu_B)}{(\delta p + \mu_B)}$.

Hence, $\lambda_1 < 0$ and $\lambda_2 < 0, \lambda_3 < 0$ if $R_0 < 1$ and $\lambda_3 > 0$ if $R_0 > 1$.

From $A\lambda^2 + B\lambda + C = 0$ where $A > 0$ for every $R_0, B > 0$ and $C > 0$ if $R_0 > 1$, we got

$\lambda_4 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$ and $\lambda_5 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$.

Because of $A > 0$ and $C > 0$ then $B^2 - 4AC < B^2$ and because of $B > 0$ then $\text{Re}(\lambda_4) < 0$ and $\text{Re}(\lambda_5) < 0$ for every sign of $B^2 - 4AC$.

For $P_2$, it was complicated to determine the eigenvalues of Jacobian matrix $J(P_2)$, so we suspended it.

4. Conclusion

From the analysis above, we get the dynamic of the mathematics model of AI epidemic with vaccination where this activity has a success ratio. We also got the reproduction number, which can be used to determine whether the epidemic spread widely or vanish. For the next research, we propose to make the mathematics model for a nonconstant population in both populations.

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