The dynamics of vortices on $S^2$

near the Bradlow limit

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Abstract

The explicit solutions of the Bogomolny equations for $N$ vortices on a sphere of radius $R^2 > N$ are not known. In particular, this has prevented the use of the geodesic approximation to describe the low energy vortex dynamics. In this paper we introduce an approximate general solution of the equations, valid for $R^2 \gtrsim N$, which has many properties of the true solutions, including the same moduli space $\mathbb{C}P^N$. Within the framework of the geodesic approximation, the metric on the moduli space is then computed to be proportional to the Fubini-Study metric, which leads to a complete description of the particle dynamics.

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1 Introduction

The abelian Higgs model in the plane is one of the most studied examples of a field theory with topological solitons. The solitons are vortices. At critical coupling there are Bogomolny equations, and it is known that there is a $2N$-dimensional manifold of gauge-inequivalent $N$-vortex solutions. This is known as the $N$-vortex moduli space, and denoted $\mathcal{M}_N$. As a manifold, $\mathcal{M}_N \cong \mathbb{C}^N$. There is a natural metric on $\mathcal{M}_N$, arising from the kinetic terms in the Lagrangian of the model, and it has been proved by Stuart that, at least for finite time intervals, geodesic trajectories on the moduli space give a good approximation to the true dynamics of slowly moving vortices.

It is convenient to introduce the standard complex coordinate $z$ on the plane. The locations of the vortices are the $N$ unordered points where the Higgs field vanishes. These points may be regarded as the roots of a monic polynomial in $z$ (monic means that the coefficient of $z^N$ is 1), and the natural coordinates on moduli space are the $N$ complex coefficients of such a polynomial. If a particular geodesic motion is known, then the time-dependence of the polynomial is known, and hence the time-dependence of the roots can be calculated.

Now, a general formula for the metric on moduli space has been given by Samols, but it is not explicit, so only very special geodesics, with a high degree of symmetry, are understood in detail for $N > 2$. One vortex just moves at constant speed along a straight line. The geodesic motion for two vortices has been calculated by Samols numerically. The most interesting phenomenon is that, in a head-on collision, two vortices scatter at right angles. Recently, Manton and Speight have found an explicit metric for $N$ well separated vortices, from which the geodesic motion could be computed.

In this paper we are interested in the opposite limit. It is possible to consider the abelian Higgs model with fields defined on any compact surface. We shall only consider the case of a 2-sphere with its standard round metric, parametrized by its radius $R$. There are again Bogomolny equations and a $2N$-dimensional moduli space of $N$-vortex solutions. As a manifold this is $\mathbb{C}\mathbb{P}^N$. However, there is an important geometrical constraint, discovered by Bradlow. This is that the area of the sphere must be greater than $4\pi N$ for non-trivial solutions of the Bogomolny equations to exist. Equivalently, $R^2 > N$. The metric on moduli space is known to collapse to zero size as the Bradlow limit $R^2 \searrow N$ is approached. We shall be interested here in the case where $R^2$ is slightly greater than $N$. One should think of this as a situation where the vortices are densely squeezed together. We shall present an approximate general solution of the Bogomolny equations, and using this, calculate the metric on moduli space directly from its definition. Again the solutions involve a polynomial, and the natural coordinates are the complex coefficients of the polynomial. We shall find that the metric is that of Fubini-Study, with an overall scale factor that depends on $R^2 - N$.

The geodesics on Fubini-Study are quite simple, and hence, in principle, the motion of vortices can be calculated straightforwardly. However, this does involve finding the roots of polynomials with time-varying coefficients, which is not algebraically trivial for three or more vortices. We shall present examples, mainly of two vortex motion. We should also remark that Stuart’s proof of the validity of the geodesic approximation for vortex motion does not extend automatically to this regime of being close to the Bradlow limit, so our results on vortex motion remain rather formal at this stage. The particle dynamics we obtain at the end is, nevertheless, quite “physical”.

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The paper is structured as follows. In Section 2 we introduce the Bogomolny equations on $S^2$, which is identified with $\mathbb{C}P^1$. As in [4, 13], the equations are defined on complex line bundles over this surface. In Sections 3 and 4 we explain our approximation for $R^2$ close to $N$, and proceed to compute the metric on the moduli space of the approximate solutions. The geodesics of this Fubini-Study metric are then used in Section 5 to give an explicit description of the $N$-vortex dynamics. Some examples of motions are presented in Section 6, and finally in Section 7 a general result concerning the number of vortex collisions is proved.

2 The Bogomolny equations

According to Bradlow’s generalization of the classical vortices over $\mathbb{R}^2$ [2], when the base manifold is the sphere $S^2 \cong \mathbb{C}P^1$, the setup for the problem is a complex line bundle $\pi : E \rightarrow S^2$ of degree $N$ equipped with a hermitian metric $h$. The Higgs field $\phi$ is now a section of this bundle, and the gauge potentials are the local 1-forms of an $h$-compatible connection $D$ on the bundle. We will take the metric on $S^2$ to be $g_R := R^2 \times$ (standard metric on $S^2$), so that the volume of $(S^2, g_R)$ is $4\pi R^2$.

Letting $\mathcal{A} := \{h$-compatible connections on $E\}$ and $\Gamma(E) := \{\text{global } C^\infty \text{ sections of } E\}$, the Bogomolny equations for $(D, \phi) \in \mathcal{A} \times \Gamma(E)$ are (2, [4]):

\begin{align*}
D^{0,1} \phi &= 0 \quad (1) \\
F + \frac{1}{2}(|\phi|^2 - 1)(\text{vol}_R) &= 0 \quad (2)
\end{align*}

where $D^{0,1}$ is the anti-holomorphic part of $D$, $\text{vol}_R \in \Omega^2(S^2, \mathbb{R})$ is the volume form of the metric $g_R$, and $-iF$ is the globally defined curvature form of $D$, so that $F \in \Omega^2(S^2, \mathbb{R})$.

We remark that the problem does not depend essentially on $(E, h)$, because all complex line bundles over $S^2$ of a given degree $N$ are isomorphic. In fact, for another choice $(E', h')$ there will always be an isomorphism $f : E \rightarrow E'$ such that $f^*(h') = h$. It is then not difficult to check that $(D, \phi)$ is a solution of (1) and (2) on $(E', h')$ if and only if $(f^*D, f^{-1} \circ \phi)$ is a solution on $(E, h)$, where $f^*D$ is the pull-back connection on $E$. Notice in particular that $\phi \in \Gamma(E')$ and $f^{-1} \circ \phi \in \Gamma(E)$ have the same zeros on $S^2$, and hence correspond to the same vortex configuration.

We will now define the particular pair $(E, h)$ which is to be used in the remainder of this paper. Let $U_1 = \mathbb{C}P^1 \setminus \{[0, 1]\}$, $U_2 = \mathbb{C}P^1 \setminus \{[1, 0]\}$, where we use the standard homogeneous coordinates $[z_0, z_1]$ for points on $\mathbb{C}P^1$, and let $\varphi_i : U_i \rightarrow \mathbb{C}$ be the standard complex charts of $\mathbb{C}P^1$ with transition functions $\varphi_1 \circ \varphi_2^{-1} = \varphi_2 \circ \varphi_1^{-1} : z \mapsto 1/z$. Define $g_{ij} : U_i \cap U_j \rightarrow U(1)$ by

$$g_{21} \circ \varphi_2^{-1}(z) = (z/|z|)^N, \quad g_{12} = 1/g_{21}, \quad g_{11} = g_{22} = 1.$$  

Since the $g_{ij}$ satisfy the usual cocycle conditions, it is possible to construct a complex line bundle $\pi : E \rightarrow \mathbb{C}P^1$ with trivializations $\psi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}$ such that $\psi_i \circ \psi_j^{-1}(p, y) = (p, g_{ij}(p)y)$. The hermitian metric $h$ on $E$ is defined by requiring the unitarity of the trivializations $\psi_i$, that is $|\psi_i^{-1}(p, y)|^2_h = |y|^2$; it is well defined because $g_{ij}$ has values in $U(1)$.

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We should now check that \( \deg E = N \). Define the real valued 1-forms \( A_i \in \Omega^1(U_i, \mathbb{R}) \) by

\[
A_i = \varphi_i^* A \quad \text{with} \quad A := \frac{-iN}{2(1 + |z|^2)}(\bar{z} \, dz - z \, d\bar{z}) \in \Omega^1(\mathbb{C}, \mathbb{R}) .
\]

On \( U_1 \cap U_2 \) one has

\[
(\varphi_2^{-1})^*(A_1 - A_2) = \left( \frac{1}{z} \right)^* A - A = \imath \left( \frac{|z|}{z} \right)^N d \left( \frac{z}{|z|} \right)^N = \imath (\varphi_2^{-1})^*(g_{12} \, dg_{21}) ,
\]

or equivalently

\[
(-iA_1) - (-iA_2) = g_{12} \, dg_{21} ,
\]

which shows that the local forms \(-iA_1\) and \(-iA_2\) define a connection \( D_N \) on \( E \). The curvature \(-iF_N\) of \( D_N \) is a global 2-form on \( \mathbb{CP}^1 \) determined by \( F_N = dA_j \) on \( U_j \). In particular, one can compute that

\[
(\varphi_1^{-1})^*F_N = dA = \frac{iN}{(1 + |z|^2)^{2}} dz \wedge d\bar{z} = (\varphi_1^{-1})^* \left( \frac{1}{2 \, \vol \sqrt{\mathbb{R}}} \right) ,
\]

and hence

\[
\deg E := \frac{i}{2\pi} \int_{\mathbb{CP}^1} (-iF_N) = \frac{1}{2\pi} \int_{\mathbb{C}} (\varphi_1^{-1})^*F_N = N .
\]

Integrating equation (2) over \( \mathbb{CP}^1 \), and using that \( \int_{\mathbb{CP}^1} F = 2\pi \deg E = 2\pi N \), it is clear that for \( R^2 < N \) the Bogomolny equations have no solution, and that for \( R^2 = N \) any solution \((D, \phi)\) must satisfy \( \phi = 0 \) and \( F = \frac{1}{2} \vol \sqrt{\mathbb{R}} = F_N \). Since we have already constructed a connection \( D_N \) on \( E \) with curvature \(-iF_N\), we have an explicit solution of the Bogomolny equations for the case \( R^2 = N \) (which is called the Bradlow limit), and it can be shown that it is unique up to gauge transformations.

For \( R^2 > N \) Bradlow has shown \cite{Bradlow}, in a more general context, that for any solution \((D, \phi)\) of (1) and (2), the section \( \phi \) has exactly \( N \) zeros (which are called vortices), counting multiplicities. Moreover, the moduli space \( \mathcal{M}_N \) of these solutions up to gauge tranformations, is parametrized by the positions in \( \mathbb{CP}^1 \) of these \( N \) vortices. Since the vortices are indistinguishable, this moduli space is identified with \( \left( \mathbb{CP}^1 \right)^N / \mathcal{E}_N \), where \( \mathcal{E}_N \) is the group of permutations of \( N \) elements.

Now consider the map \( \Upsilon : \left( \mathbb{CP}^1 \right)^N / \mathcal{E}_N \to \mathbb{CP}^N \) defined in homogeneous coordinates by

\[
\left[ [u^1, v^1], \ldots, [u^N, v^N] \right] \mapsto \left[ \ldots, \sum_{\sigma \in \mathcal{E}_N} u^{\sigma(1)} \cdots u^{\sigma(k)} u^{\sigma(k+1)} \cdots u^{\sigma(N)} , \ldots \right]_{0 \leq k \leq N} .
\]

With some care, one can verify that \( \Upsilon \) is a bijection. In fact, its inverse may be described in the following way. Given \([w_0, \ldots, w_N] \in \mathbb{CP}^N\), consider the non-zero polynomial

\[
P(z) = \sum_{k=0}^{N} (-1)^k \frac{N!}{k!(N-k)!} w_k z^{N-k} ,
\]

which has degree \( l \leq N \). Calling \( z_1, \ldots, z_l \) the complex roots of \( P(z) \), one has

\[
\Upsilon^{-1} \left( [w_0, \ldots, w_N] \right) = \left[ [1, z_1], \ldots, [1, z_l], [0, 1], \ldots, [0, 1] \right] .
\]

Using this bijection, \( \mathcal{M}_N \) can also be identified with \( \mathbb{CP}^N \).


3 Vortices near the Bradlow limit

Although we have an accurate description of the moduli space $\mathcal{M}_N$, the explicit form of the solutions $(D, \phi)$ of (1) and (2) is not known. In particular, this has prevented any successful attempt to describe the dynamics of the vortices by means of the well-known geodesic approximation. The purpose of this paper is to show that by replacing the exact Bogomolny equations by two other conditions, which should be a good approximation near the Bradlow limit $R^2 \searrow N$, one can obtain the solutions explicitly; they also have $\mathbb{CP}^N$ as their moduli space, and furthermore the dynamics of these “pseudo-vortices” is completely computable in the framework of the geodesic approximation.

Since for $R^2 = N$ the pair $(D_N, 0) \in \mathcal{A} \times \Gamma(E)$ is an exact solution of (1) and (2), we may expect that for $R^2$ close to $N$ the solutions $(D, \phi)$ will have $D \approx D_N$ (after a gauge transformation if necessary). We therefore impose $D = D_N$ and look for $\phi \in \Gamma(E)$ such that:

- $(D_N, \phi)$ is a solution of (1), i.e. $D^0_{N,1} \phi = 0$; (4)
- $(D_N, \phi)$ satisfies (2) “on average”, i.e. $\int_{\mathbb{CP}^1} \left( F_N + \frac{1}{2}(|\phi|^2 - 1) \text{vol}_R \right) = 0$; (5)

(We note in passing that eq.(4) is analogous to the equation for electron wavefunctions of the first Landau level in the uniform background magnetic field $F_N$; eq.(5) is then a wavefunction normalization condition.)

We will now find the explicit solutions of (4) and (5), and then describe their moduli space. Using the local trivialization $\psi_1$ of $E$ and the chart $\varphi_1$ of $\mathbb{CP}^1$ defined before, the equation $D^0_{N,1} \phi = 0$ over the domain $U_1$ is the same as $(\bar{\partial} - iA^{0,1}) \phi_1 = 0$, or explicitly, using (3):

$$\frac{\partial \phi_1}{\partial \bar{z}} = \frac{-Nz}{2(1 + |z|^2)} \phi_1,$$

where $\phi_1 \in C^\infty(\mathbb{C})$ is the representative of $\phi$ with respect to $\psi_1$.

Equation (6) has the general solution $\phi_1 = f(z)(1 + |z|^2)^{-N/2}$, with $f$ holomorphic on $\mathbb{C}$. The section $\phi$ of $E$ determined by $\phi_1$, which is only defined over $U_1$, has a representative $\phi_2$ with respect to $\psi_2$ given by $\phi_2(z) = g_{12}(z) \phi_1(\frac{1}{z})$, which is smooth on $\mathbb{C} \setminus \{0\}$. But since we are looking for global solutions of (4), $\phi$ must be extensible to all of $\mathbb{CP}^1$, and this will happen iff $\phi_2(z)$ is smoothly extensible to $\mathbb{C}$. Writing $f$ as a Taylor series, it is then not difficult to check that this requires that the coefficient of $z^n$ vanishes for all $n > N$. Thus any solution $\phi$ of (4) must have a representative $\phi_1$ over $U_1$ of the form

$$\phi_1(z) = \frac{a_0 z^N + \cdots + a_N}{(1 + |z|^2)^{N/2}},$$

and conversely any $\phi_1$ of this form determines a global section $\phi$ of $E$ which is a solution of (4) over $U_1$, and by continuity over all of $\mathbb{CP}^1$.

If $\phi$ is represented by $\phi_1$ as in (7), then the representative $\phi_2$ will be

$$\phi_2(z) = \frac{a_0 + \cdots + a_N z^N}{(1 + |z|^2)^{N/2}}.$$
and hence, as for (1) and (2), any solution $\phi$ of (4) has exactly $N$ zeros over $\mathbb{C}P^1$, counting multiplicities.

We now turn to condition (5). Using that $-iF_N$ is the curvature form of a degree $N$ bundle, (5) is equivalent to

$$4\pi(R^2 - N) = \int_{\mathbb{C}P^1} |\phi|^2 \text{vol}_R = \int_{\mathbb{C}} |\phi_1|^2 \frac{2iR^2}{(1 + |z|^2)^2} \, dz \wedge d\bar{z} = 4\pi R^2 \sum_{k=0}^{N} \frac{k!(N-k)!}{(N+1)!} |a_k|^2,$$

where the last integral is calculated in the appendix for $\phi_1$ of the form (7). We can therefore conclude that $\phi_1$ represents a solution $\phi$ of (4) and (5) iff $\phi_1$ is of the form (7) and satisfies the normalization condition

$$\sum_{k=0}^{N} \frac{k!(N-k)!}{(N+1)!} |a_k|^2 = 1 - \frac{N}{R^2}.$$

Calling $D \subset \mathcal{A} \times \Gamma(E)$ the subspace of solutions of (4) and (5), we thus get a bijection $\alpha : D \rightarrow S^{2N+1} \subset \mathbb{C}^{N+1}$ that maps each $\phi \in D$, represented by a $\phi_1$ like in (7), to the point

$$\left(1 - \frac{N}{R^2}\right)^{-1/2} \left( \ldots, \left( \frac{k!(N-k)!}{(N+1)!} \right)^{1/2} a_k, \ldots \right)_{0 \leq k \leq N}. \quad (8)$$

The following step is to determine when two solutions in $D$ are gauge equivalent. Let therefore $(D_N, \phi)$ and $(D_N, \tilde{\phi})$ be a pair of solutions, an suppose $g : \mathbb{C}P^1 \rightarrow U(1)$ is a gauge transformation on $E$ that takes one into the other. Using the usual transformation rule for connection forms under $g$, and the key fact that the connection is fixed, it is readily shown that $g$ must be constant. So $\phi = e^{i\beta} \tilde{\phi}$ for some $\beta \in \mathbb{R}$. Since the converse is clear, we conclude that $(D_N, \phi)$ and $(D_N, \tilde{\phi})$ are gauge equivalent iff

$$\tilde{\phi} = e^{i\beta} \phi \iff \tilde{\phi}_1 = e^{i\beta} \phi_1 \iff \tilde{c} = e^{i\beta} c$$

for some $\beta \in \mathbb{R}$, where $c, \tilde{c} \in S^{2N+1}$ are the images of $(D_N, \phi)$ and $(D_N, \tilde{\phi})$ under the bijection $\alpha$. Furthermore, notice that this last condition is also equivalent to $\pi(\tilde{c}) = \pi(c)$ in $\mathbb{C}P^N$, where $\pi : S^{2N+1} \rightarrow \mathbb{C}P^N$ is the usual principal $U(1)$-bundle.

Calling $\mathcal{M}_N$ the moduli space of solutions of (4) and (5) up to gauge transformations, and $p : D \rightarrow \mathcal{M}_N$ the natural projection, we therefore have:

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\alpha} & S^{2N+1} \\
\downarrow p & & \downarrow \pi \\
\mathcal{M}_N & \xrightarrow{\tilde{\alpha}} & \mathbb{C}P^N
\end{array} \quad (9)$$

where $\tilde{\alpha}$, defined by the commutativity of the diagram, is, like $\alpha$, a bijection. The right-hand side of this diagram is a concrete model for the space of solutions $\mathcal{D}$ and its moduli space.
4 The metric on the moduli space

Using the usual prescriptions of the geodesic approximation (first described in \cite{10}), we will now obtain the metric $m$ on $\mathcal{M}_N$ which, within the framework of this approximation, determines the dynamics of the “pseudo-vortices” (which from now on will be just called vortices).

Suppose one has a curve $\gamma$ in $\mathcal{D}$:

$$t \mapsto \mathcal{D}_N, \phi(t) \in \mathcal{D} \mapsto \alpha \mapsto (w_0(t), \ldots, w_N(t)) \in S^{2N+1}.$$

A natural hermitian metric on $\mathcal{D}$ is defined by

$$\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle_{\gamma(t)} := \frac{1}{2} \int_{\mathbb{C}P^1} h(\dot{\phi}(t), \dot{\phi}(t)) \text{vol}_R,$$

where the dot stands for the time derivative. Notice that in this case, as opposed to what happens in \cite{12}, the gauge potentials do not contribute to the metric, since the connection in our space $\mathcal{D}$ is fixed, and thus time independent.

Writing $\phi_1(t) = a_0(t) z^N + \cdots + a_N(t)$ for the usual representative of $\phi(t)$, using the unitarity of $\psi_1$, and noting that

$$\dot{\phi}_1 = (1 + |z|^2)^{-N/2} (\dot{a}_0 z^N + \cdots + \dot{a}_N),$$

one has that

$$\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle_{\gamma(t)} = \frac{1}{2} \int_{\mathbb{C}P^1} |\dot{\phi}_1|^2 \frac{2iR^2}{(1 + |z|^2)^2} d\bar{z} \wedge dz = 2\pi R^2 \sum_{k=0}^{N} k!(N-k)! \hat{a}_k \hat{\bar{a}}_k =$$

$$= 2\pi (R^2 - N) \sum_{k=0}^{N} \hat{w}_k \hat{\bar{w}}_k,$$

again using the integral calculated in the appendix. We conclude that the hermitian $L^2$ metric on $\mathcal{D}$ corresponds via the map $\alpha$ to the restriction to $S^{2N+1}$ of the canonical hermitian metric on $\mathbb{C}^{N+1}$, up to the constant factor $2\pi(R^2 - N)$. This metric will also be called $\langle \cdot , \cdot \rangle$.

According to the usual procedure, the metric $m$ on $\mathcal{M}_N$ is induced from $\langle \cdot , \cdot \rangle$ on $\mathcal{D}$ in the following way. Given $q \in \mathcal{D}$ and a tangent vector $\frac{d\gamma}{dt} \in T_q \mathcal{D}$, let $(\frac{d\gamma}{dt})_\perp$ be its component perpendicular to the subspace of $T_q \mathcal{D}$ formed by the vectors tangent to curves on $\mathcal{D}$ which are pure gauge transformations, that is perpendicular to $\text{ker}(p_*)q$. Then

$$(p^*m)_q(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}) := \langle (\frac{d\gamma}{dt})_\perp, (\frac{d\gamma}{dt})_\perp \rangle_q.$$

We will now compute the metric on $\mathbb{C}P^N$ corresponding to $m$ by the identification $\tilde{\alpha}$. It will also be called $m$.

Using the diagram (9), the subspace $\text{ker}(p_*)q \subset T_q \mathcal{D}$ corresponds to the subspace $\text{ker}(\pi_*)_{\alpha(q)} \subset T_{\alpha(q)}S^{2N+1}$. Given $w \in S^{2N+1} \subset \mathbb{C}^{N+1}$, we have that $\text{ker}(\pi_*)w$ is the one-dimensional real subspace generated by the vector $\frac{d}{dt}(e^{it}w)(0) = iw$. Therefore, given a tangent
where $\mu$.

Therefore $160$. Since $\pi$

vector $\frac{d\gamma}{dt} = (\dot{\gamma}_0, \ldots, \dot{\gamma}_N) \in T_w S^{2N+1} \subset T_w \mathbb{C}P^{2N+1}$, we have

$$\left(\frac{d\gamma}{dt}\right)_\perp = \frac{d\gamma}{dt} - \frac{\langle \frac{d\gamma}{dt}, w \rangle \langle w, w \rangle}{\langle w, w \rangle} w,$$

so

$$\langle \left(\frac{d\gamma}{dt}\right)_\perp, \left(\frac{d\gamma}{dt}\right)_\perp \rangle_w = \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle_w - \frac{\langle \frac{d\gamma}{dt}, w \rangle \langle w, \frac{d\gamma}{dt} \rangle}{\langle w, w \rangle} = 2\pi(R^2 - N) \sum_{j,k=0}^N (\delta_{jk} - \bar{w}_j w_k) \dot{\gamma}_j \dot{\gamma}_k,$$

where the computation in the last step uses that $\langle w, w \rangle = 2\pi(R^2 - N)$, since $w \in S^{2N+1}$. Thus $\pi^*m$ is the restriction to $S^{2N+1}$ of the 2-tensor in $\mathbb{C}P^{2N+1}$

$$2\pi(R^2 - N) \sum_{j,k=0}^N (\delta_{jk} - \bar{w}_j w_k) dw_j \otimes d\bar{w}_k.$$

Now consider the Kähler form $\mu$ associated to $m$. It is defined, as usual, using the imaginary part of $m$: $\mu = -\text{Im} \ m \in \Omega^2(\mathbb{C}P^N, \mathbb{R})$. We have

$$\pi^*\mu = \pi^*(-\text{Im} \ m) = -\text{Im} \ (\pi^*m) = 2\pi(R^2 - N) \left(\frac{i}{2} \sum_{j,k=0}^N (\delta_{jk} - \bar{w}_j w_k) dw_j \wedge d\bar{w}_k \right)_{S^{2N+1}}$$

$$= 2\pi(R^2 - N) \left(\frac{i}{2} \sum_{j,k=0}^N \left(\frac{\delta_{jk}}{|w_0|^2 + \cdots + |w_N|^2} - \frac{\bar{w}_j w_k}{(|w_0|^2 + \cdots + |w_N|^2)^2}\right) dw_j \wedge d\bar{w}_k \right)_{S^{2N+1}}$$

$$= 2\pi(R^2 - N) \left(\frac{i}{2} \partial \bar{\partial} \log(|w_0|^2 + \cdots + |w_N|^2) \right)_{S^{2N+1}} = 2\pi(R^2 - N) \pi^* \mu_{\text{FS}}$$

where $\mu_{\text{FS}}$ is the Fubini-Study form on $\mathbb{C}P^N$, and the last equality is a well-known result [8, p. 160]. Since $\pi$ and $(\pi_*)_w$ are both surjective,

$$\pi^* \mu = \pi^*(2\pi(R^2 - N) \mu_{\text{FS}}) \quad \text{implies} \quad \mu = 2\pi(R^2 - N) \mu_{\text{FS}}.$$

Therefore $m = 2\pi(R^2 - N) m_{\text{FS}}$, where $m_{\text{FS}}$ is the Fubini-Study metric on $\mathbb{C}P^N$, because a hermitian metric is uniquely determined by its Kähler form.

5 Vortex dynamics

Having determined the metric $m$ on the moduli space $\mathcal{M}_N = \mathbb{C}P^N$, we will now proceed to explicitly describe its geodesics, which provide an approximate description of the low-energy particle dynamics. Note that $m \propto m_{\text{FS}}$ implies that the geodesics of $m$ are exactly the Fubini-Study geodesics. These are well-known [8, p. 277] but nevertheless we will rederive them here again.

Let $\pi : \mathbb{C}P^{N+1} \setminus \{0\} \longrightarrow \mathbb{C}P^N$ be the natural projection and $\chi_0 : U_0 \longrightarrow \mathbb{C}P^N$ one of the standard charts of $\mathbb{C}P^N$, where $U_0 = \{[w^0, \ldots, w^N] \in \mathbb{C}P^N : w^0 \neq 0\}$. Calling $(c^1, \ldots, c^N)$ the coordinate functions of this chart, then by definition of the Fubini-Study metric we have on $U_0$:

$$\mu_{\text{FS}} = \frac{i}{2} \partial \bar{\partial} \log(1 + |c|^2 + \cdots + |c|^2) = \frac{i}{2} \sum_{j,k} h_{j,k} dc^j \wedge dc^k$$

and

$$m_{\text{FS}} = h_{j,k} dc^j \otimes dc^k,$$
with
\[ h_{jk} = \frac{\delta_{jk}}{1 + |c|^2} - \frac{c^k c^j}{(1 + |c|^2)^2} \, . \]  
(11)

For a general Kähler metric the geodesic equations have the simplified form \([14, p. 4]\):
\[ \ddot{c}^k = -\frac{\partial h_{js}}{\partial c^j} h^{ks} \dot{c}^l \dot{c}^s \, , \quad \text{where} \quad h^{ks} := (sk \text{ entry of } [h_{ij}]^{-1}) \, . \]

In our case \( h^{ks} = (1 + |c|^2)(\delta^{sk} + c^k c^s) \), and so the geodesic equations for \((\mathbb{C}P^N, m_{FS})\) in the chart \(\chi_0\) are
\[ \ddot{c} = \frac{2 < \dot{c}, c >}{1 + < c, c >} \dot{c} \, , \]
where \(c(t)\) is a curve in \(\mathbb{C}^N\) and \(< \cdot, \cdot >\) is the canonical hermitian product.

Now consider curves in \(S^{2N+1} \subset \mathbb{C}^{N+1}\) of the form:
\[ \gamma(t) = \sin(\omega t) y + \cos(\omega t) x \, , \quad t \in \mathbb{R} \]
(13)
where \(x, y \in \mathbb{C}^{N+1}\) are orthonormal with respect to the canonical hermitian product of \(\mathbb{C}^{N+1}\). If \(x^0 \neq 0\), then \(\pi \circ \gamma(t) \notin U_0\) only for a discrete set \(D\) of non-zero values of \(t\), and a short computation shows that, in \(\mathbb{R} \setminus D\),
\[ c(t) := \chi_0 \circ \pi \circ \gamma(t) = c(0) + \dot{c}(0) \frac{x^0 \sin(\omega t)}{\omega (y^0 \sin(\omega t) + x^0 \cos(\omega t))} \]
(14)
where
\[ c^k(0) = \frac{x^k}{x^0} \quad \text{and} \quad \dot{c}^k(0) = \frac{\omega (y^k x^0 - x^k y^0)}{(x^0)^2} \, , \quad k = 1, \ldots, N \, . \]
(15)

One can verify directly that this \(c(t)\) satisfies (12), and therefore \(\pi \circ \gamma(t)\) is a geodesic for \(t\) in \(\mathbb{R} \setminus D\), and by continuity for all \(t\). If \(x^0 = 0\), a similar computation in one of the other standard charts of \(\mathbb{C}P^N\) would establish that, also in this case, \(\pi \circ \gamma(t)\) is a geodesic.

On the other hand, it is not difficult to verify that every geodesic of \((\mathbb{C}P^N, m_{FS})\) can be written as \(\pi \circ \gamma\), where \(\gamma\) has the form (13). Although one could give a general, chart-independent argument for this, for later convenience we will proceed unnaturally. Namely, using (15), one may simply check that given any \(c(0) \in \mathbb{C}^N\) and \(\dot{c}(0) \in T_{c(0)} \mathbb{C}^N \cong \mathbb{C}^N\), the geodesic \(\pi \circ \gamma(t)\) with
\[ x = (1 + |c(0)|^2)^{-1/2} \left( 1, c(0) \right) \]
\[ y = \omega^{-1} (1 + |c(0)|^2)^{-1/2} \left( -\frac{< \dot{c}(0), c(0) >}{1 + |c(0)|^2}, \dot{c}(0) - \frac{< \dot{c}(0), c(0) >}{1 + |c(0)|^2} c(0) \right) \]
\[ \omega = (1 + |c(0)|^2)^{-1/2} \left( |\dot{c}(0)|^2 - \frac{|< \dot{c}(0), c(0) >|^2}{1 + |c(0)|^2} \right)^{1/2} \]
(16)
has initial position and velocity \(\chi_0^{-1}(c(0))\) and \((\chi_0^{-1})_*(\dot{c}(0))\), respectively. This shows that every geodesic starting in \(U_0\) is of the form \(\pi \circ \gamma(t)\). Similar calculations in the other standard charts would extend the result to all of \(\mathbb{C}P^N\).
We now note two general properties of the geodesics $\pi \circ \gamma(t)$. Firstly, using (11) and (15), one can compute that the velocity of the geodesic, which is a constant of motion, is $|\omega|$. Secondly, notice that all the geodesics of $(\mathbb{CP}^N, m_{FS})$ are periodic. It is not difficult to show that for $\omega \neq 0$ the period is $\pi/|\omega|$.

We will now use our knowledge of the geodesics on the moduli space $(\mathbb{CP}^N, m)$ to give an explicit description of the vortex dynamics.

Recall from Section 3 that the solutions $(D_N, \phi) \in \mathcal{D}$ with the vortices (zeros of $\phi$) in positions $\varphi_1^{-1}(z_1), \ldots, \varphi_1^{-1}(z_N)$ in $\mathbb{CP}^1$, are represented by a function $\phi_1$ of the form (7), where we now have

$$a_0 z^N + \cdots + a_N \propto (z - z_1) \cdots (z - z_N),$$

and therefore

$$a_k = A (-1)^k S_k(z_1, \ldots, z_N), \quad k = 0, \ldots, N$$

where the $S_j$ are the usual elementary symmetric polynomials, and $A$ is a normalization factor which is non-zero for $R^2 > N$. Thus such solutions $(D_N, \phi) \in \mathcal{D}$ correspond by $\pi \circ \alpha$ to (see (8))

$$\left[ \ldots, (-1)^k \frac{k!(N-k)!}{(N+1)!} \right]^{1/2} S_k(z_1, \ldots, z_N), \ldots \right]_{0 \leq k \leq N} \in \mathbb{CP}^N \cong \mathcal{M}_N$$

and by $\chi_0 \circ \pi \circ \alpha$ to $c = (c_1, \ldots, c_N) \in \mathbb{C}^N$, where

$$c^k = (-1)^k \binom{N}{k}^{-1/2} S_k(z_1, \ldots, z_N).$$

(17)

Inverting these relations, we can obtain the positions of the $N$ vortices as a function of the coordinates $c^k$ of a given point in the moduli space $\mathbb{CP}^N$. In particular, to the geodesics $c(t)$ of the form (14) corresponds a motion of the vortices determined by:

$$w^N + \sum_{k=1}^N \binom{N}{k}^{1/2} c^k(t) w^{N-k} = (w - z_1(t)) \cdots (w - z_N(t))$$

(18)

where the $z_i(t)$ are the coordinates of the vortices in the chart $(\varphi_1, U_1)$ of $\mathbb{CP}^1$. Thus, since we know all the geodesics of $(\mathbb{CP}^N, m)$, we can determine all the possible $N$-vortex motions by finding the roots of polynomials of degree $N$ — either analytically for $N \leq 4$ or numerically for $N > 4$.

Now suppose we are given initial positions $z_i(0)$ and initial velocities $\dot{z}_i(0)$ for the vortices, where we assume that the $z_i(0)$ are all different. Through (17) and its derivative we can get the corresponding values $c(0), \dot{c}(0) \in \mathbb{C}^N$, then use (16) to determine which geodesic $c(t)$ corresponds to this initial data, and finally solve (18) to get the motions $z_i(t)$. This general procedure has been used to obtain the various special vortex motions shown below.

We remark that, because (17) is a local diffeomorphism only in the region where the vortices do not coincide, only in this region can we guarantee that the final result $z_i(t)$ has indeed the prescribed initial velocities. This is why we take the $z_i(0)$ all different. If the $z_i(0)$ are not all different there are some values of $\dot{z}_i(0)$ that do not correspond to any $z_i(t)$ coming from a geodesic motion.
6 Examples of motions

Using the method described in the previous Section, we now give a few examples of 2-and 3-vortex motions on the sphere. The trajectories are shown in the complex plane through the use of the stereographic projection \( \varphi_1 : S^2 \setminus \{N\} \rightarrow \mathbb{C} \). The particular initial positions and velocities used in each case are listed in Table 1.

|     | 1(a) | 1(b) | 1(c) | 1(d) | 2(a) | 2(b) | 2(c) | 2(d) | 3   |
|-----|------|------|------|------|------|------|------|------|-----|
| \( z_1(0) \) | 1 + i | 1 + i | 1 | −2i/\sqrt{3} | 1/2 | 1/2 | 1/2 | 1/2 | \( a > 0 \) |
| \( \dot{z}_1(0) \) | −1 − i | −1 − i | −3 | 2i/\sqrt{3} | \( i \) | \( i \) | \( i \) | \( i \) | \( i \) |
| \( z_2(0) \) | 0 | −1 − i | −1 | 1 + i/\sqrt{3} | 2 | 2 | 2 | 2 | \( −a \) |
| \( \dot{z}_2(0) \) | 0 | 1 + i | 1 | −1 − i/\sqrt{3} | 0.6i | 3.7i | 4i | 4.5i | \( −i \) |
| \( z_3(0) \) | — | — | — | −1 + i/\sqrt{3} | — | — | — | — | — |
| \( \dot{z}_3(0) \) | — | — | — | 1 − i/\sqrt{3} | — | — | — | — | — |

Table 1: Initial positions and velocities

To help with the interpretation of the figures, we recall that the stereographic projection has the property of mapping circles of \( S^2 \) (not necessarily great circles) to circles and straight lines in the plane. The inverse \( \varphi_1^{-1} \) has the converse property. Also the circle of unit radius is always shown in a dashed line; the northern (southern) hemisphere of \( S^2 \) projects to the exterior (interior) of that circle.

Figure 1:

(a) Two colliding vortices, one of which is at rest.
   One of the particles describes a great circle on the sphere that passes through the static position of the other.
(b) Head-on collision of two vortices with the same speed.
There are two collisions at antipodal points. The total trajectory is the union of two great circles.

(c) Head-on collision of two vortices with different initial speeds.
Again two collisions occur. The total trajectory is the union of a great circle and another circle.

(d) Symmetrical collision of three vortices with equal speeds.
The three vortices collide twice at antipodal points. The total trajectory is the union of three great circles.

![Fig. 2(a)](image_a)
![Fig. 2(b)](image_b)
![Fig. 2(c)](image_c)
![Fig. 2(d)](image_d)

**Figure 2:**
Except for 2(c), no collisions occur, and each vortex returns to its initial position after one period. The degenerate case 2(c) is the same as 1(c) — one great circle and another circle — in a different orientation.

![Fig. 3](image_3)

**Figure 3 (with a=2):**
No collision takes place and the vortices exchange positions after one period. The coordinate $c^1(t)$ of expressions (14) and (18) is always zero. The coordinate $c^2(t)$ is of the form $-a^2 + 2a(1 + a^4)/(2a^3 + i\omega \cot(\omega t))$. Taking one of the roots $z(t) = x(t) + iy(t)$ of (18) and eliminating $t$ from the system $x(t), y(t)$, one obtains that the trajectory on the plane has the simple equation $(x^2 + y^2)^2 + (1/a^2 - a^2)(x^2 - y^2) - 1 = 0$. These are special cases of Cassini’s ovals and, when projected back to the sphere, look like the joint of a tennis ball.
7 Coincident particles and collisions

In this Section we start by finding an algebraic condition which determines when a point in the moduli space \( \mathcal{M}_N \cong \mathbb{CP}^N \) corresponds to a vortex configuration where at least two of the vortices coincide. We then use this condition to show that, for a system of \( N \) vortices starting in different positions with arbitrary initial velocities, there are at most \( 2N - 2 \) collisions during one period of the motion.

Using diagram (9) and the definition (8) of the bijection \( \alpha \), it is not difficult to recognize that a point \([w_0, \ldots, w_N] \in \mathbb{CP}^N\) corresponds by \( \tilde{\alpha}^{-1} \) to the equivalence class in \( \mathcal{M}_N \) of a solution \( \phi \) represented by

\[
\phi_1(z) = A (1 + |z|^2)^{-N/2} \sum_{k=0}^{N} \frac{w_k}{(k!(N-k)!)^{1/2}} z^{N-k},
\]

where \( A \) is a non-zero normalization factor. Thus, asserting that \([w_0, \ldots, w_N]\) corresponds to a solution with at least two coincident vortices is equivalent to saying that one of the following conditions holds:

- \( P(z) := \sum_{k=0}^{N} \frac{w_k}{(k!(N-k)!)^{1/2}} z^{N-k} \) has a root of multiplicity at least two;
- \( w_0 = w_1 = 0 \), which corresponds to a double zero of \( \phi \) at \([0, 1] \in \mathbb{CP}^1\).

We now use the following result, whose proof is at the end of this section:

**Lemma:** For any \( n \in \mathbb{N} \), there is a homogeneous polynomial \( S \) in \( n + 1 \) variables of degree \( 2n - 2 \), such that \( S(a_0, \ldots, a_n) = 0 \) if and only if \( \sum_{k=0}^{n} a_k z^{n-k} \) has a multiple root or \( a_0 = a_1 = 0 \).

An explicit formula for \( S \) is given in the proof and we note that, up to a sign, \( S \) coincides with the discriminant of \( \sum_{k=0}^{n} a_k z^{n-k} \) whenever \( a_0 \neq 0 \).

Using this lemma, it is clear that the points \([w_0, \ldots, w_N] \in \mathbb{CP}^N\) which correspond to at least two coincident vortices, are exactly those of the algebraic hypersurface \( H \) of degree \( 2N - 2 \) in \( \mathbb{CP}^N \) determined by the equation

\[
\tilde{S}(\ldots, w_k, \ldots) := S(\ldots, \frac{w_k}{(k!(N-k)!)^{1/2}} \ldots) = 0.
\]

As we have seen in Section 5, any motion of \( N \) vortices in \( S^2 \) corresponds to a Fubini-Study geodesic in \( \mathbb{CP}^N \), and these are all of the form \( t \mapsto \pi(\sin(\omega t)y + \cos(\omega t)x) \), with \( x, y \in \mathbb{C}^{N+1} \) orthonormal and \( \pi \) being the projection from \( \mathbb{C}^{N+1} \) to \( \mathbb{CP}^N \). By the discussion above, it is also clear that this motion will have a collision of two or more vortices iff the corresponding geodesic intersects \( H \). But since this geodesic lies on the projective line \( L = \pi(\text{span}_\mathbb{C}\{x, y\}) \subset \mathbb{CP}^N \), and does not intersect itself during one period, we conclude that the number of collisions is not bigger than the cardinality of \( L \cap H \).

It is, however, a standard result in algebraic geometry that either \( L \subset H \) or \( \#(L \cap H) \leq \deg H = 2N - 2 \). In fact, denoting \( x = (x_0, \ldots, x_N) \) and \( y = (y_0, \ldots, y_N) \) in \( \mathbb{C}^{N+1} \), it follows
that a point $\pi(ux + vy)$ in $L$, with $(u, v) \in \mathbb{C}^2 \setminus \{0\}$, belongs to $H$ iff

$$Q(u, v) := \tilde{S}(ux_0 + vy_0, \ldots, ux_N + vy_N) = 0.$$  

But since $\tilde{S}$ is homogeneous of degree $2N - 2$, so is $Q$, and therefore there is a factorization (see [5], p. 31)

$$Q(u, v) = \prod_{i=1}^{2N-2} (\alpha_i u + \beta_i v), \quad \text{for some } (\alpha_i, \beta_i) \in \mathbb{C}^2.$$  

If $Q$ is identically zero we have $L \subset H$. If $Q$ is not identically zero then the roots of $Q$ are $(\beta_i, -\alpha_i) \neq 0 \forall i$, and it is apparent that $L \cap H$ consists of the points $\pi(\beta_i x - \alpha_i y)$, which are at most $2N - 2$.

We finally conclude that, either the vortices have a motion with at least two of them coincident for all time, which corresponds to $L \subset H$, or there are at most $2N - 2$ collisions in one period.

**Proof of the lemma:**

This lemma is a slight generalization of well-known algebraic results, as stated for example in [5], p. 168] or [5], p. 178]. Consider $P(z) = \sum_{k=0}^{n} a_k z^{n-k}$ and its derivative $P'(z) = \sum_{k=1}^{n} (n - k + 1)a_{k-1} z^{n-k}$, and form the usual resultant

$$\mathcal{R}_{P,P'}(a_0, \ldots, a_n) := \begin{vmatrix} a_0 & \cdots & a_n & \cdots & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ na_0 & \cdots & a_n & \cdots & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ na_0 & \cdots & a_n & \cdots & \cdots \end{vmatrix}$$

where there are $n - 1$ rows with the coefficients of $P$ and $n$ rows with the coefficients of $P'$, so that the matrix is $(2n-1) \times (2n-1)$. Applying the usual expansion to the first column of this determinant we get

$$\mathcal{R}_{P,P'}(a_0, \ldots, a_n) = a_0 S(a_0, \ldots, a_n)$$

with

$$S(a_0, \ldots, a_n) = \begin{vmatrix} a_0 & \cdots & a_n & \cdots & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ (n-1)a_1 & \cdots & a_{n-1} & \cdots & \cdots \\ na_0 & \cdots & a_{n-1} & \cdots & \cdots \\ na_0 & \cdots & a_{n-1} & \cdots & \cdots \end{vmatrix} + (-1)^n n \begin{vmatrix} a_1 & \cdots & a_n & \cdots & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_0 & \cdots & a_{n-1} & \cdots & \cdots \\ na_0 & \cdots & a_{n-1} & \cdots & \cdots \\ na_0 & \cdots & a_{n-1} & \cdots & \cdots \end{vmatrix}.$$
Expanding again the first columns of the determinants in \( S(0, a_1, \ldots, a_n) \) we get

\[
S(0, a_1, \ldots, a_n) = (-1)^n (n-1) a_1 \mathcal{R}_{Q,Q'} + (-1)^n n a_1 \mathcal{R}_{Q,Q'}' = (-1)^n (2n - 1) a_1 \mathcal{R}_{Q,Q'}'
\]

where \( Q(z) = \sum_{k=1}^n a_k z^{n-k} \).

Now, if \( a_0 \neq 0 \), by standard results in basic algebra (see [1, 2]),

\[
\mathcal{R}_{P,P'} = (-1)^{n(n-1)/2} a_0 D(P),
\]

where \( D(P) \) is the discriminant of \( P \), and therefore

\[
S(a_0, \ldots, a_n) = 0 \iff D(P) = 0 \iff P \text{ has a multiple root}.
\]

If \( a_0 = 0 \) then \( P(z) = Q(z) \) and

\[
S(a_0, \ldots, a_n) = S(0, a_1, \ldots, a_n) = 0 \iff a_1 = 0 \text{ or } \mathcal{R}_{Q,Q'} = 0.
\]

But when \( a_1 \neq 0 \), by the same algebraic results, \( \mathcal{R}_{Q,Q'} = 0 \iff Q = P \text{ has a multiple root}.

We finally conclude that

\[
S(a_0, \ldots, a_n) = 0 \iff \text{P has a multiple root or } a_0 = a_1 = 0.
\]

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In this appendix we compute integrals of the form
\[
\int_C \phi \bar{\psi} \frac{2iR^2}{(1 + |z|^2)^2} \, dz \wedge d\bar{z} = \int_{\mathbb{R}^2} \phi \bar{\psi} \frac{4R^2}{(1 + x^2 + y^2)^2} \, dx \, dy
\]
where \( \phi = (1 + |z|^2)^{-N/2}(a_0 z^N + \cdots + a_N) \) and \( \psi = (1 + |z|^2)^{-N/2}(b_0 z^N + \cdots + b_N) \).

Write
\[
\phi \bar{\psi} = \sum_{k,j=0}^{N} a_{N-j} b_{N-k} f_{kj}(z), \quad \text{with} \quad f_{kj}(z) = \bar{z}^k z^j (1 + |z|^2)^{-N}.
\]

Using polar coordinates and integration by parts,
\[
I(k,j,N) := \int_{\mathbb{R}^2} \frac{f_{kj}(z)}{(1 + |z|^2)^2} \, dx \, dy = \int_0^{2\pi} e^{i(j-k)\theta} d\theta \int_0^\infty \frac{r^{k+j}}{(1 + r^2)^{N+1}} \, r \, dr = \\
= 2\pi \delta_{jk} \int_0^\infty \frac{r^{2k+1}}{(1 + r^2)^{N+2}} \, dr = \delta_{jk} \frac{-\pi}{N+1} \int_0^\infty \frac{r^{2k} \, d\left(\frac{1}{(1 + r^2)^{N+1}}\right)}{(1 + r^2)^{N+1}} \, dr = \\
= \delta_{jk} \frac{2k\pi}{N+1} \int_0^\infty \frac{r^{2k-1}}{(1 + r^2)^{N+1}} \, dr = \delta_{jk} \frac{k}{N+1} I(k-1, k-1, N-1)
\]
where the vanishing of the boundary terms in the integration by parts is valid for \( N \geq k \geq 1 \).

Since
\[
I(0,0,N-k) = 2\pi \int_0^\infty \frac{r}{(1 + r^2)^{N-k+2}} \, dr = \frac{-\pi}{N-k+1} \left[ \frac{1}{(1 + r^2)^{N-k+1}} \right]^\infty_0 = \frac{\pi}{N-k+1}
\]
we have
\[
I(k,j,N) = \delta_{jk} \frac{k(k-1)\cdots 1}{(N+1)\cdots (N+2-k)} I(0,0,N-k) = \frac{k!(N-k)!}{(N+1)!} \pi \delta_{jk}.
\]

The final result is therefore
\[
\int_C \phi \bar{\psi} \frac{2iR^2}{(1 + |z|^2)^2} \, dz \wedge d\bar{z} = 4R^2 \sum_{k=0}^N I(k,k,N) a_{N-k} b_{N-k} = 4\pi R^2 \sum_{k=0}^N \frac{k!(N-k)!}{(N+1)!} a_k b_k.
\]
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