A Phase Transition for the Metric Distortion of Percolation on the Hypercube

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June, 2003

Abstract

Let $H_n$ be the hypercube $\{0,1\}^n$, and let $H_{n,p}$ denote the same graph with Bernoulli bond percolation with parameter $p = n^{-\alpha}$. It is shown that at $\alpha = 1/2$ there is a phase transition for the metric distortion between $H_n$ and $H_{n,p}$. For $\alpha < 1/2$, asymptotically there is a map from $H_n$ to $H_{n,p}$ with constant distortion (depending only on $\alpha$). For $\alpha > 1/2$ the distortion tends to infinity as a power of $n$. We indicate the similarity to the existence of a non-uniqueness phase in the context of infinite nonamenable graphs.

1 Introduction

The hypercube $H_n$ is the graph with vertex set $\{0,1\}^n$ and edges between two vectors differing in a single coordinate. Bernoulli bond percolation on a graph $G$ with parameter $p$ is a distribution on subgraphs $H \subset G$ where each edge is in $H$ with probability $p$ independently of all other edges. Site percolation is defined similarly except that each vertex is in $H$ with probability $p$ and all edges with endpoints in $H$ are in $H$. $H_{n,p}$ is the graph resulting from bond Bernoulli percolation on $H_n$. We view a graph $G$ as a metric space on its vertex set $V(G)$ with the induced graph metric, i.e. $d(a,b)$ is the length of the shortest path from $a$ to $b$. Our interest is in the effect of Bernoulli percolation on the geometry of the hypercube.

In [6] Hastad, Leighton and Newman show that if $p$ is taken to be constant, then for large $n$ the distortion between $H_n$ and $H_{n,p}$ is at most a constant with probability tending to 1. If a graph is deemed to represent a computer network, then they were interested in the latency of embedding $H_n$ in $H_{n,p}$, which measures the loss of computational power resulting from eliminating a subset of the processors (vertices) or communication channels.
The latency of an embedding between graphs is slightly stronger than metric distortion.

In the classic paper [2] of Ajtai, Komlós and Szemerédi, bond percolation on the hypercube is analyzed. It is shown that if \( p = an^{-1} \) with \( a < 1 \), then all connected components of \( H_{n,p} \) have size polynomial in \( n \) (hence logarithmic in the size of \( H_n \)), while if \( a > 1 \) there is a single giant component of size linear in the size of \( H_n \), and of diameter of order \( n \). When all connected components are small it is evident that there is no map with a small distortion between the cube and any of the components. However, when a giant component exists it is a priori possible that the distortion between the cube and the giant component is constant. Thus there is a gap between \( p = n^{-1} \) and \( p \) constant where it is possible that the distortion is small.

To make things more precise, we use the following measure of the distortion between metric spaces.

**Definition 1.** The distortion of a map \( f \) between metric spaces \((X, d_X), (Y, d_Y)\) is given by \( D(f) = D_+(f)/D_-(f) \)

\[
D_+(f) = \max \left\{ 1, \sup_{a,b \in X} \frac{d_Y(f(a), f(b))}{d_X(a,b)} \right\},
\]
\[
D_-(f) = \inf_{a,b \in X} \max\left\{ 1, \frac{d_Y(f(a), f(b))}{d_X(a,b)} \right\}.
\]

The metric distortion between \( X \) and \( Y \), \( D(X, Y) \), is

\[
\inf_{f: X \to Y} D(f).
\]

Note that since we will deal with functions that are not necessarily 1 to 1, the definition accommodates the possibility that two points \( a, b \) are mapped to the same point in \( Y \). This does not constitute a significant change from other definitions. It should be noted though that this definition is not symmetric, as the functions do not need to have a full image.

Two graphs (or metric spaces) with a small metric distortion between them have several similar properties. For more information on metric distortion see [7] and reference there. Note that the distance between disjoint connected components of a graph is infinite, so only maps that preserve connected components are of interest.

Note that (for bond percolation) a graph \( G \) and \( G_p \) have the same vertex sets, so there is a natural map between the two. However, there is no guarantee (and generally it is not the case) that this map gives a minimal distortion or even gets close to doing that. However, in the case considered here, it will turn out that when the metric distortion is small, there is a map that approximates the identity map which gives a small distortion.
Theorem 2. Let \( p = n^{-\alpha} \).

1. If \( \alpha < 1/2 \) then there is some constant \( c = c(\alpha) \) such that
\[
P(D(H_n, H_{n,p}) < c) \to 1
\]
as \( n \to \infty \).

2. If \( \alpha > 1/2 \) then there is some constant \( \beta = \beta(\alpha) > 0 \) such that
\[
P(D(H_n, H_{n,p}) < n^\beta) \to 0
\]
as \( n \to \infty \).

A similar theorem holds for site percolation, as well as for mixed percolation (with threshold at \( p_b p_s = n^{-1} \)). The essentially identical proof will be omitted.

It follows from the definition of the latency of an embedding (see [6]) that for \( p = n^{-\alpha} \) with \( \alpha < 1/2 \) the latency is polynomial in \( n \). This follows from Theorem 2 as the latency is at most \( n^{D(f)} \). It is plausible (though not proved here) that when \( \alpha > 1/2 \) the latency is super-polynomial in \( n \).

The phase transition in this context is linked below to the \( p_c < p_u \) conjecture in context of infinite nonamenable graphs. In this light it is interesting to compare the behavior of the metric distortion of percolation in the hypercube to the behavior in other natural graphs. For the \( n \)-box in \( \mathbb{Z}^d \) the distortion between the box and the giant component is \( O(\log n) \). While not constant, this is small in relation to the diameter and there is no additional phase transition beyond the percolation threshold. For percolation on the complete graph, the metric distortion is given by the diameter of the giant component. On the hypercube the giant component has diameter \( \theta(n) \) a.s. for any \( p > p = an^{-1} \) with \( a > 1 \).

Regarding the proof, the main idea for \( p = n^{-\alpha} \) with \( \alpha < 1/2 \), is that two vertices that are nearby in \( H \) are likely to stay nearby in \( H_{n,p} \). Thus the identity map is almost good enough, as it gives a constant distortion for most pairs of vertices. To find a map that gives a constant distortion requires a somewhat finer choice of the map, though the distance from \( x \) to \( f(x) \) is still bounded.

When \( \alpha > 1/2 \) the proof is based on showing that while the cube contains “a lot” of geodesic cycles, the giant component for \( \alpha > 1/2 \) is locally very treelike and contains much fewer geodesic cycles and thus distortion is generated.

In this regime there is a constant probability that two adjacent vertices that are both in the giant component are not joined by a short path in \( H_{n,p} \),
but will be at distance proportional to the diameter. Of course, since for
the cube (and much more generally, [2] [3]) the giant component is known
to be unique, the distance between the two vertices is finite. However, there
is no longer a direct correlation between the metric of the cube and that
of the giant component. As giant components are the finite graph analogue
of infinite percolation clusters, this behavior can be seen as the finite graph
analogue of non-uniqueness of infinite clusters. For graphs where \( p_u \) — the
threshold for uniqueness of the infinite cluster — is strictly greater than \( p_c \),
there is a regime where two nearby vertices have a positive probability of
being in disjoint infinite components. That strict inequality was established
in many cases of nonamenable infinite graphs, see e.g. [8] [5]. We believe
that the distortion in this case is indeed linear in the diameter. However,
since we have not eliminated possible maps that are far from preserving the
geometry we only give a lower bound of \( n^\beta \) for some \( \beta \).

Thus we propose that an analogue in finite graphs of having non-unique
infinite components is that the metric in the giant component is very different
from the original graph metric. In particular, with probability bounded away
from 0, neighbors in the graphs are at distance of the order of the diameter
in the giant component. For more on percolation on finite graphs see [3].

An additional aspect of the phase transition at \( \alpha = 1/2 \) is the routing
complexity. The routing problem is to find a path in \( H_{n,p} \) between two given
vertices, preferably a short one. The routing problem may be defined under
one of several models, the strictest of which involves starting at \( x \) and only
being allowed to query edges incident on vertices that have already been
reached from \( x \). We refer to this as the local model.

For \( p = n^{-\alpha} \) with \( \alpha < 1/2 \), it turns out that there is a polynomial time (in
\( n \)) algorithm for routing in the local model, outputting a path of length \( O(n) \)
whenever \( x, y \) are in the same component. This is a consequence of the fact
that distance between neighboring vertices is typically bounded, and so to get
from \( x \) to a nearby \( y \) only a ball of constant radius (and polynomial volume)
needs to be explored. On the other hand when \( \alpha > 1/2 \) no polynomial
algorithm exists. For more details on complexity of routing in the hypercube
and other scenarios see [4].

We now proceed to present the proof of Theorem [2]. The proof is separated
into the super-critical and the sub-critical cases, each requiring a different
approach.
2 The Super-Critical Case: $\alpha < 1/2$

To illustrate the proof we first show a weaker result, namely that the typical distortion of the identity map is constant.

**Proposition 3.** Let $p = n^{-\alpha}$ with $\alpha < 1/2$. For some $l = l(\alpha)$, with probability tending to 1 the percolation distance between two neighbors in $H_n$ is $T$ most $2l + 1$.

**Sketch of Proof.** We consider paths of length $2l + 1$ between them making $l$ steps in distinct coordinates, making a step in the coordinate that differs $x$ from $y$, and then retracing the first $l$ steps to reach $y$. The number of such paths is roughly $n^l$ and each is open with probability $p^{2l+1}$. The expectation of $X$: the number of open paths is $n^{(1-2\alpha)l-\alpha}(1+o(1))$, and for some $l$ the exponent is positive.

A pair of paths is unlikely to have a large intersection. The second moment of $X$ can be approximated by the square of the expectation. The largest error term comes from pairs of paths that intersect only in their first step, so that $\mathbb{E}X^2 = (\mathbb{E}X)^2(1+n^{2\alpha-1}+o(n^{2\alpha-1}))$. The second moment method (see [1]) yields that

$$\mathbb{P}(X = 0) \leq 1 - \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} = (1 + o(1))n^{2\alpha-1} \to 0.$$ 

\[\square\]

Similarly, the idea in the following proof is that there are many paths of some length between nearby vertices, and that one of them is very likely to be open. However, in order to get an exponential bound on the probability of failure we choose the set of paths we consider carefully.

Fix $l$ so that $(1-2\alpha)l > 9\alpha$, which is possible as $\alpha < 1/2$. Set $m = \left\lfloor \frac{n-1}{l+2} \right\rfloor$ and fix throughout this section some partition of the coordinates into disjoint sets of size $m$ denoted $A, B, C_1, \ldots, C_l$ (with at least one coordinate left unused). A vertex in $H_{n,p}$ will be called *good* if it has at least $2m$ vertices at distance 2 (in $H_{n,p}$) differing only in $A$-coordinates. We first show that good vertices are dense.

**Lemma 4.** Let $p = n^{-\alpha}$ for some $\alpha < 1/2$. With probability tending to 1, every vertex $v$ has is some good vertex $u$ in $H_{n,p}$ differing from $v$ only in a single $B$-coordinate.

**Proof.** First we show that the probability that a vertex is good tends to 1 as $n \to \infty$, with $\alpha$ fixed. To this end we count open paths of length 2 using
A-coordinates starting at \( v \). If there are more that \( 4m \) such paths, then they connect \( v \) to at least \( 2m \) vertices and \( v \) is good. The probability that \( v \) is incident on at least \( m \cdot p/2 = c_1 n^{1-\alpha} \) open \( A \)-edges tends to 1 as \( n \to \infty \). On this event there are at least \((m-1)c_1 n^{1-\alpha}\) edges that extend these to paths of length 2. With probability tending to 1 at least fraction \( p/2 \) of these are open giving a total of at least \( c_2 n^{2-2\alpha} \) open paths. Since for \( n \) large enough, \( c_2 n^{2-2\alpha} > 4m \), the probability that \( v \) is good tends to 1.

Fix some \( v \) and let \( u_1, \ldots, u_m \) be \( u \)'s neighbors along \( B \) coordinates. Since the \( u_i \)'s all differ in their \( B \)-coordinates, the events of the \( u_i \)'s being good are independent. For large enough \( n \) each \( u_i \) is good with probability at least \( 1 - 3^{-l} \) and then the probability that none of the \( u_i \)'s is good is at most \( 3^{-ml} \ll 2^{-n} \). Thus, a union bound over all vertices \( v \) shows that with probability tending to 1 every \( v \) has a good neighboring \( u \). \( \square \)

**Proof of Theorem 2(1).** We construct a map \( f \) that takes each vertex \( x \) to some good neighbor of \( x \) by changing a single \( B \) coordinate. By the above Lemma, with probability tending to 1 such a map exists. Since \( d(x, f(x)) = 1 \), it follows that \( d(f(x), f(y)) \geq d(x, y) - 2 \) and so \( D_-(f) > 1/3 \). To show that distances do not increase by more then some constant factor, it suffices to show that with probability tending to 1 the percolation distance between \( f(x) \) and \( f(y) \) for all neighboring \( x, y \) is bounded by \( 2l + 9 \).

The distance between \( f(x) \) and \( f(y) \) is at most 3. First, fix some coordinate \( e \) where \( f(x) \) and \( f(y) \) differ. In the remainder there are some slight variations according as to which set \( e \) is part of. Since \( f(x) \) is good it has \( 2m \) vertices at percolation distance 2, differing only in the \( A \)-coordinates. Let \( x_1, \ldots, x_m \) be \( m \) of those that do not differ from \( f(x) \) in the \( e \) coordinate (If \( e \in A \) there are at least \( m \) such neighbors; Otherwise all \( 2m \) are suitable). Similarly define \( y_1, \ldots, y_n \). Note that \( x_i \) differs from \( y_i \) in at most 7 coordinates.

We look for an open path from \( x_i \) to \( y_i \) of length \( 2l + 9 \) for some \( i \), and bound the probability that there is no such path. The second moment method is used to bound the probability of having no path for some \( i \). The sets of paths we will consider are chosen so that distinct \( i \)'s will be independent.

To each \( i \) we associate a unique coordinate \( b_i \in B \). If \( e \in B \) we use instead of \( e \) the remaining coordinate not in any of the sets. For a sequence \( c = (c_1, \ldots, c_l) \) of coordinates with \( c_k \in C_k \) we define the path \( P_i(j) \) from \( x_i \) to \( y_i \) as the path making \( l \) steps in the coordinates given by \( c \), making a step in the coordinate \( b_i \), making steps in the coordinates where \( x_i \) and \( y_i \) differ, and then retracing the \( l + 1 \) steps given by \( i \) and \( c \) in reversed order, thereby arriving at \( y_i \). If \( e \in C_k \) for some \( k \) then we again replace it by the unused
coordinate.

We now claim that paths $P_i(c)$ and $P_i'(c')$ for $i \neq i'$ are disjoint. Indeed, suppose that some $v$ is in some $P_i(c)$. According to the $e$ coordinate of $v$ we can say whether $v$ is in the first or second half of the path. Assume w.log. it is the first. By the distance from $f(x)$ we can determine the exact position of $v$ in the path. Finally, either $v$ differs from $f(x)$ in the $b_i$ coordinate, and then we can say what $i$ is, or it differs from $f(x)$ in exactly 2 $a$ coordinates and then we can say which is $x_i$. In any case, we can determine $i$ from $v$.

Let $X$ be the number of open paths from $x_i$ to $y_i$.

$$\mathbb{E}X = p^{2l+9}m^l = cn^{(1-2\alpha)l-9\alpha},$$

where $c$ is a constant depending only on $l$. For $l = l(\alpha)$ as specified above the the exponent is positive, and so the expected number of paths tends to infinity. The second moment of $X$ is given by

$$\mathbb{E}X^2 = \sum_{P,P'} p^{4l+18-|P \cap P'|} \leq (\mathbb{E}X)^2 \left( \sum_k p^{-2k}m^{-k} + m^{-l}p^{-2l-9} \right) \leq (\mathbb{E}X)^2(1 + cn^{2\alpha-1}).$$

And so the probability of having $X > 0$ tends to 1 as $n \to \infty$. In particular, for large enough $n$, the probability of having no open path for any of the $i$ is less then $(3-l)^m$. In this case there is an open path of length $2l + 13$ between $f(x)$ and $f(y)$. Since the number of pairs $x, y$ is $n2^n \ll 3^{3m}$, a union bound shows that asymptotically almost surely distances increase by a factor of no more than $2l + 13$.

$$\Box$$

3 The Sub-Critical Case: $\alpha > 1/2$

For two vertices $x, y$ at some fixed distance from each other, the number of paths of length $l$ between them grows like $n^{l/2}$, and so the probability of having an open path tends to 0 as $n \to \infty$. This shows that there is no map with bounded $d(x, f(x)$ with small distortion. However, the image of a map need not cover all pairs $x, y$, and there will be some pairs where paths will exist.

A geodesic cycle in a graph is an isometric embedding of a cycle in the graph. The cube has a great number of geodesic cycles passing through any given vertex. These can be found by making any number of steps in distinct
coordinates, and then repeating the same sequence of coordinates to return to the starting point.

A geodesic cycle (as well as a geodesic path) is a structure that is roughly preserved by a map with a small distortion. The key idea of the proof is to show that the percolated cube does not have many cycles, restricting the possible maps.

**Lemma 5.** If a map from $G$ to $H$ has distortion $D$, and $G$ has a geodesic cycle $C$ of length $2l$ passing through a vertex $v$, then there is a simple (self avoiding) cycle $C' \subset H$ of length $l' \in [l/2D, DL]$ so that $f(v)$ is at distance at most $2D^2$ from $C'$.

The key idea is to look at a cycle in $H$ that is the “image” of $C$, and remove any small loops it has to extract a simple cycle.

**Proof.** Recall that the distortion is $D = D_+/D_-$. Suppose the vertices of $C$ are $v_0, v_1, \ldots, v_l$ with $v_0 = v_1 = v$, and their images in $H$ are $x_i = f(v_i)$.

Since $d(x_i, x_{i+1}) \leq D_+$, there are simple paths of length at most $D_+$ between them, and these paths may be joined to form a (not necessarily simple) cycle $C_0$ in $H$ of length at most $D_+ l$, passing through all the $x_i$'s in order. The paths from $x_i$ to $x_{i+1}$ for various $i$'s will be denoted as arcs.

Next we extract from $C_0$ a simple cycle by a erasing loops it includes. Note that the procedure we use is not the standard procedure known as loop erasure. Suppose some vertex $u$ appears twice in $C_0$. Since $u$ cannot be used twice in the same arc, it appears in distinct arcs. If $u$ appears in the arc $(x_i, x_{i+1})$ and in $(x_j, x_{j+1})$ then there is a path of length at most $2D_+$ between $x_i$ and $x_j$. However, $d(x_i, x_j) \geq D_- d(v_i, v_j)$, and so $d(v_i, v_j) \leq 2D$.

Since the $v_i$'s form a geodesic cycle, breaking $C_0$ at the two occurrences of $u$ results in one segment containing all but at most $2D$ of the $x_i$'s.

Since the above holds for any 2 appearances of $u$ in $C_0$, for each vertex $u \in C_0$ there is a single segment of $C_0$ starting and ending at $u$, including all appearances of $u$, and including at most $2d$ of the $x_i$'s. We now repeatedly remove from $C_0$ the longest such segment corresponding to any repeated vertex. This choice guarantees that if a loop starting and ending at $u$ is removed, then $u$ itself will not be removed later. Since at each step we remove at most $2D$ of the $x_i$'s and create a vertex that cannot be removed at a later stage, eventually a simple cycle is left of length at least $l/2D$.

Finally, if $x_0 = f(v)$ is not in the remaining cycle, it is in a loop that was removed from the cycle at some stage. Since the length of each such loop is at most $2DD_+$, it follows that $f(v)$ is close to the cycle. $\square$

**Lemma 6.** Let $p = n^{-\alpha}$ with $\alpha > 1/2$, and $\gamma < 2\alpha - 1$. For large enough $n$, the probability that an vertex $v$ is included in an open simple cycle in $H_{n,p}$ of
length $2l \in [2n^\beta, 2n^\gamma]$ is at most $n^{1+n^\beta(\beta + 1-2\alpha)}$. Furthermore, the probability that $v$ is within distance $\delta$ of such a cycle is at most $n^{\delta+1+n^\beta(\beta + 1-2\alpha)}$.

Proof. First we estimate the number of simple cycles of length $2l$ in $H_n$ that include $v$. Since a cycle makes an even number of moves in each coordinate, the steps of a cycle of length $2l$ can be partitioned into $l$ pairs of (non-consecutive) moves in the same coordinate. The number of partitions of $2l$ elements into pairs is $(2l-1)!!$. The number of ways to choose a coordinate for each of the pairs is $n^l$. The probability that a cycle of length $2l$ is open is $p^{2l} = n^{-2\alpha l}$, so the probability of $v$ being in an open cycle of length $2l$ is at most $(2l-1)!!(n^{1-2\alpha})^l$.

This bound is decreasing in $l$ as long as $2l < n^{2\alpha-1}$, so when summing over $l$, for large enough $n$ the first term is the largest:

$$\sum_{l=n^\beta}^{n^\gamma} (2l-1)!!(n^{1-2\alpha})^l < n(2n^\beta-1)!!(n^{1-2\alpha})^{n^\beta} < n^{1+n^\beta(\beta + 1-2\alpha)}.$$

Finally, the number of vertices within distance $\delta$ of $v$ is at most $n^\delta$, and the second claim follows from the first. \qed

Proof of Theorem 2(2). Fix positive constants $\beta, \gamma$ so that $\beta + \gamma < 2\alpha - 1$ and $\gamma > 3\beta$. Assume that there is a map $f : H_n \to H_{n,p}$ with distortion at most $n^{\beta}$. Since vertices at distance greater than $n^\beta$ are mapped to distinct vertices, the range of $f$ has size at least $2^{2n-n^\beta}$. Since every vertex in $H_n$ is in a geodesic cycle of length $n^\gamma$, every vertex in the range is at distance at most $2^{n^2\beta}$ from a simple cycle of length $l \in [n^{\gamma-\beta}, 2n^{\gamma+\beta}]$.

However, Lemma 6 bounds the probability that a vertex has this property, and thus the expected number of such vertices. The probability of having at least $2^{n-n^\beta}$ vertices close to such cycles is at most

$$\frac{2^n n^{2n^{2\beta} + 1 + (\gamma - \beta + 1 - 2\alpha)n^{\gamma - \beta}}}{2^n n^{-n^\beta}} = n^{2n^{2\beta} + n^\beta + 1 + (\gamma - \beta + 1 - 2\alpha)n^{\gamma - \beta}}.$$

This tends to 0 as $n \to \infty$ since the dominant term in the exponent is $n^{\gamma - \beta}$ with a negative coefficient.

Thus the probability of having a map with distortion $n^\beta$ tends to 0. The above constraints allow taking $\beta = (2\alpha - 1)/4 - \epsilon$. \qed

4 Problems

• What is the critical window? What happens if $p = an^{-1/2}$? Is there some $a_c$ so that above $a_c$ the distortion is constant or perhaps logarithmic?
mic in $n$ while below it the distortion is logarithmic in $n$ or even larger? What do the above proofs yield in this case?

- When $\alpha > 1/2$, what is the true behavior of the distortion? Since the diameter of $H_n$ is $n$ the distortion can not be larger, but the proof only gives a lower bound of $n^\beta$ for any $\beta < (2\alpha - 1)/4$.

- When $\alpha < 1/2$, what is the true distortion? The proof of Theorem 2 gives a bound $D_+ = O((1/2 - \alpha)^{-1})$. The proof of Proposition 3 suggests a bound of $D_+ = 1 + \lfloor \frac{\alpha}{1/2 - \alpha} \rfloor$. We believe this is the correct uniform bound on the distortion.

- What is the minimal latency of an embedding of $H_n$ in $H_{n,p}$ when $\alpha < 1/2$? Show that it is super-polynomial in $n$. Does it grow exponentially in $n$?

- Let $G$ be a vertex transitive graph. Perform $p$-bond percolation on $G$. Assume that the median distance between neighbors in $G$, after percolation is $k$, show that the metric distortion between $G$ and the percolation giant component is at least $k$.

- In particular, assume a vertex transitive graph $G$ has girth $g$, is it true that the distortion between $G$ and a percolation subgraph of $G$, with $p$ bounded away from 1, is a.s. at least $g$?

Acknowledgments. Thanks to Eran Ofek and Udi Wieder for useful discussions.

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