FORMALITY THEOREM FOR $g$-MANIFOLDS

HSUAN-YI LIAO, MATHIEU STIENON, AND PING Xu

À la mémoire de notre ami Jacky Michéa

ABSTRACT. To any $g$-manifold $M$ are associated two dglas $\bigotimes_k (\Lambda^g T\check{\mathcal{T}}^\ast_M)$ and $\bigotimes_k (\Lambda^g T\check{\mathcal{T}}^\ast_M)$, whose cohomologies $\check{\mathcal{T}}^\ast_M$ and $\check{\mathcal{T}}^\ast_M$ are Gerstenhaber algebras. We establish a formality theorem for $g$-manifolds: there exists an $L_\infty$ quasi-isomorphism $\Phi : \bigotimes_k (\Lambda^g T\check{\mathcal{T}}^\ast_M)$ to $\bigotimes_k (\Lambda^g T\check{\mathcal{T}}^\ast_M)$ whose first Taylor coefficient $(1)$ is equal to the Hochschild–Kostant–Rosenberg map twisted by the square root of the Todd cocycle of the Todd class of the $g$-manifold and $(2)$ induces an isomorphism of Gerstenhaber algebras on the level of cohomology. Consequently, the Hochschild–Kostant–Rosenberg map twisted by the square root of the Todd cocycle of the Todd class of the $g$-manifold $M$ is an isomorphism of Gerstenhaber algebras from $(\check{\mathcal{T}}^\ast_M) \to (\check{\mathcal{T}}^\ast_M)$ to $\check{\mathcal{T}}^\ast_M$.

1. INTRODUCTION

Two differential graded Lie algebras (dglas) are canonically associated with a given smooth manifold $M$: the dglas of polyvector fields $T^\ast_M$ and $T^\ast_M$, which is endowed with the zero differential and the Schouten bracket $[\ ,\ ]$, and the dglas of polydifferential operators $D^\ast_M$ and $D^\ast_M$, which is endowed with the Hochschild differential $d_H$ and the Gerstenhaber bracket $[\ ,\ ]$. Here $D^\ast_M$ denotes the algebra of smooth functions $\mathcal{R} = C^\infty(M)$, $D^\ast_M$ the algebra of differential operators on $M$, and $D^\ast_M$ (with $k \geq 0$) the space of $(k+1)$-differential operators on $M$, i.e. the tensor product $D^\ast_M \otimes \mathcal{R} \otimes \mathcal{R} \otimes D^\ast_M$ of $(k+1)$ copies of the algebra of left $\mathcal{R}$-module $D^\ast_M$. The classical Hochschild–Kostant–Rosenberg (HKR) theorem $[6,7]$ states that the Hochschild–Kostant–Rosenberg map, the natural embedding $\mathrm{hkr} : T^\ast_M \hookrightarrow D^\ast_M$ defined by Equation $(4)$, determines an isomorphism of Gerstenhaber algebras $\mathrm{hkr} : T^\ast_M \cong H^\ast(D^\ast_M, d_H)$ on the cohomology level — the products on $T^\ast_M$ and $D^\ast_M$ are the wedge product and the cup product respectively. However, the HKR map $\mathrm{hkr} : T^\ast_M \hookrightarrow D^\ast_M$ is not a morphism of dglas. Kontsevich’s celebrated formality theorem states that the HKR map $\mathrm{hkr}$ extends to an $L_\infty$ quasi-isomorphism from $T^\ast_M$ to $D^\ast_M$ $[7,12]$. The formality theorem is highly non trivial and has many applications, one of which is the deformation quantization of Poisson manifolds.

In this Note, we study the Gerstenhaber algebra structures associated with a $g$-manifold and we establish a formality theorem for $g$-manifolds. By a $g$-manifold, we mean a smooth manifold equipped with an infinitesimal action of a Lie algebra $g$. In this situation, the analogues of $T^\ast_M$ and $D^\ast_M$ are the Chevalley–Eilenberg complexes $\bigotimes_k (\Lambda^\ast g T\check{\mathcal{T}}^\ast_M)$ and $\bigotimes_k (\Lambda^\ast g T\check{\mathcal{T}}^\ast_M)$, respectively — they are briefly mentioned in Dolgushev’s work $[4, concluding remarks]$. Both of them are naturally dglas (see Lemma $3.1$ and Lemma $3.2$ and their cohomologies are Gerstenhaber algebras.

In order to state the formality theorem and the precise relation between these two Gerstenhaber algebras, one must take into consideration the obstruction to the existence of a $g$-invariant affine

Research partially supported by NSF grants DMS-1406668 and DMS-1101827, and NSA grant H98230-14-1-0153.
connection on $M$, the Atiyah cocycle $R_{t,1}^\vee \in g^\vee \otimes \Gamma(T_M^\vee \otimes \text{End} T_M)$, which is a Chevalley–Eilenberg 1-cocycle of the $g$-module $\Gamma(T_M^\vee \otimes \text{End} T_M)$. More precisely, we must call upon its cohomology class, the Atiyah class $\alpha_{M/g} \in H^1_{\text{CE}}(g, \Gamma(T_M^\vee \otimes \text{End} T_M))$, which we introduce in Proposition 4.1.

The Todd cocycle $t_{dM/g} \in \bigoplus_{k=0}^\infty \Lambda^k g^\vee \otimes \Omega^k(M)$ of a $g$-manifold $M$ is defined in terms of the Atiyah cocycle in Equation (5). The corresponding class in Chevalley–Eilenberg cohomology is the Todd class $t_{M/g} \in \bigoplus_{k=0}^\infty H^k_{\text{CE}}(g, \Omega^k(M))$. See Equation (6).

The main results of this Note are a formality theorem for $g$-manifolds and its consequence: a Kontsevich–Duflo type theorem for $g$-manifolds.

**Formality theorem.** Given a $g$-manifold $M$ and an affine torsionfree connection $\nabla$ on $M$, there exists an $L_\infty$ quasi-isomorphism $\Phi$ from the dgla tot $(\Lambda^* g^\vee \otimes_k D_{\text{poly}}^\bullet(M))$ to the dgla tot $(\Lambda^* g^\vee \otimes_k D_{\text{poly}}^\bullet(M))$ whose first ‘Taylor coefficient’ $\Phi_1$ satisfies the following two properties:

1. $\Phi_1$ is, up to homotopy, an isomorphism of associative algebras (and hence induces an isomorphism of associative algebras of the homologies);
2. $\Phi_1$ is equal to the composition $hkr \circ t_{dM/g}^1$ of the HKR map and the action of the square root of the Todd cocycle $t_{dM/g}^1 \in \bigoplus_{k=0}^\infty \Lambda^k g^\vee \otimes \Omega^k(M)$ on tot $(\Lambda^* g^\vee \otimes_k D_{\text{poly}}^\bullet(M))$ by contraction.

**Kontsevich–Duflo type theorem.** Given a $g$-manifold $M$, the map

$$hkr \circ t_{dM/g}^1 : H^*_\text{CE}(g, T_{\text{poly}}^\bullet(M)) \xrightarrow{\partial} T_{\text{poly}}^{*+1}(M) \rightarrow H^*_\text{CE}(g, D_{\text{poly}}^\bullet(M)) \xrightarrow{dH} D_{\text{poly}}^{*+1}(M)$$

is an isomorphism of Gerstenhaber algebras. Here $H^k_{\text{CE}}(g, E^\bullet \xrightarrow{dE} E^{\bullet+1})$ denotes the Chevalley–Eilenberg cohomology of $g$ with coefficients in the complex of $g$-modules $E^\bullet$. It is understood that the square root $t_{dM/g}^1$ of the Todd class $t_{dM/g} \in \bigoplus_{k=0}^\infty H^k_{\text{CE}}(g, \Omega^k(M))$ acts on $H^*_\text{CE}(g, T_{\text{poly}}^\bullet(M)) \xrightarrow{\partial} T_{\text{poly}}^{*+1}(M)$ by contraction.

The theorem above is parallel in spirit to an analogue of Duflo’s Theorem — a classical result of Lie theory — discovered by Kontsevich in complex geometry [7]. Kontsevich observed that, for a complex manifold $X$, the composition $hkr \circ (\text{Td}_X)^{1/2} : H^* (X, \Lambda^* T_X) \xrightarrow{\cong} HH^* (X)$ is an isomorphism of associative algebras. Here $\text{Td}_X$ denotes the Todd class of the tangent bundle $T_X$ and $HH^*(X)$ denotes the Hochschild cohomology groups of the complex manifold $X$, i.e. the groups $\text{Ext}_{\text{O}_{X \times X}}^*(\text{O}_X, \text{O}_X)$. The multiplications on $H^* (X, \Lambda^* T_X)$ and $HH^*(X)$ are the wedge product and the Yoneda product respectively. A detailed proof of Kontsevich’s result appeared in [2]. It is worth mentioning that the map $hkr \circ (\text{Td}_X)^{1/2}$ actually respects the Gerstenhaber algebra structures on both cohomologies; this was brought to light in [2].

2. **Preliminary: Chevalley–Eilenberg cohomology**

Let $g$ be a Lie algebra over $\mathbb{k}$ ($\mathbb{k}$ is $\mathbb{R}$ or $\mathbb{C}$). Given a $g$-module $E$, one may consider the Chevalley–Eilenberg cochain complex

$$\cdots \rightarrow \Lambda^{p-1} g^\vee \otimes_k E \xrightarrow{d_{CE}} \Lambda^p g^\vee \otimes_k E \xrightarrow{d_{CE}} \Lambda^{p+1} g^\vee \otimes_k E \rightarrow \cdots,$$

where $d_{CE}$ is the Chevalley–Eilenberg differential. More generally, given a bounded below complex of left $g$-modules

$$\cdots \rightarrow E^{p-1} \xrightarrow{d_E} E^p \xrightarrow{d_E} E^{p+1} \rightarrow \cdots,$$
we may consider the double complex:

\[
\begin{array}{c}
\cdots \rightarrow \Lambda^{p-1} E^q \otimes_k E^{q+1} \\
\vdots \rightarrow \Lambda^p E^q \otimes_k E^{q+1} \rightarrow \Lambda^{p+1} E^q \otimes_k E^{q+1} \rightarrow \cdots
\end{array}
\]

where \( d_{\text{CE}} \) is the Chevalley–Eilenberg differential corresponding to the \( \mathfrak{g} \)-module structure on \( E^\bullet \). By definition, the Chevalley–Eilenberg cohomology of \( \mathfrak{g} \) with coefficients in the complex of \( \mathfrak{g} \)-modules \((E^\bullet, d_E)\) is the total cohomology of the double complex above:

\[
H_{\text{CE}}^k(\mathfrak{g}, \mathfrak{g}^\bullet \otimes_k E^\bullet) = H^k(\text{tot}(\Lambda^k \mathfrak{g}^\vee \otimes_k E^\bullet))
\]

### 3. Hochschild–Kostant–Rosenberg theorem for \( \mathfrak{g} \)-manifolds

#### 3.1. Polyvector fields

Let \( M \) be a \( \mathfrak{g} \)-manifold with infinitesimal action given by a Lie algebra morphism \( \varphi : \mathfrak{g} \rightarrow \mathfrak{X}(M) \). It is well known that the space of polyvector fields \( T^\bullet_{\text{poly}}(M) = \bigoplus_{n=-1}^{\infty} T_{\text{poly}}^n(M) \) on \( M \), together with the wedge product and the Schouten bracket \([\cdot,\cdot]\), forms a Gerstenhaber algebra. Moreover, the \( \mathfrak{g} \)-action on \( M \) and the Schouten bracket together determine a \( \mathfrak{g} \)-module structure on \( T^k_{\text{poly}}(M) \) for each \( k \geq -1 \):

\[
\forall a \in \mathfrak{g}, \quad \gamma \in T^k_{\text{poly}}(M).
\]

Therefore \( \cdots \rightarrow T^k_{\text{poly}}(M) \overset{0}{\rightarrow} T^{k+1}_{\text{poly}}(M) \rightarrow \cdots \) is a complex of \( \mathfrak{g} \)-modules. Its Chevalley–Eilenberg cohomology

\[
H_{\text{CE}}^k(\mathfrak{g}, T^\bullet_{\text{poly}}(M)) = H^k(\text{tot}(\Lambda^k \mathfrak{g}^\vee \otimes_k T^\bullet_{\text{poly}}(M)))
\]

is the total cohomology of the double complex:

\[
\begin{array}{c}
\cdots \rightarrow \Lambda^{p-1} T^q_{\text{poly}}(M) \\
\vdots \rightarrow \Lambda^p T^q_{\text{poly}}(M) \rightarrow \Lambda^{p+1} T^q_{\text{poly}}(M) \rightarrow \cdots
\end{array}
\]

Extend the Schouten bracket \([\cdot,\cdot]\) on \( T^\bullet_{\text{poly}}(M) \) to \( \Lambda^k \mathfrak{g}^\vee \otimes_k T^\bullet_{\text{poly}}(M) \) as follows:

\[
[\alpha \otimes \chi, \beta \otimes \psi] = (-1)^{qp} \alpha \wedge \beta \otimes [\chi, \psi]
\]

for any \( \alpha \otimes \chi \in \Lambda^n \mathfrak{g}^\vee \otimes_k T^n_{\text{poly}}(M) \) and \( \beta \otimes \psi \in \Lambda^2 \mathfrak{g}^\vee \otimes_k T^q_{\text{poly}}(M) \). The following lemma can be easily verified.
Lemma 3.1. The graded \( k \)-vector space \( \text{tot} \left( \Lambda \cdot g^\vee \otimes_k T_{\text{poly}}^\bullet (M) \right) \), together with the Chevalley-Eilenberg differential \( d_{CE} \), the wedge product \( \wedge \) and the bracket defined by Equation (1) is a differential Gerstenhaber algebra. As a consequence, \( H^*_\text{CE} (\mathfrak{g}, \mathcal{T}_{\text{poly}}^\bullet (M)) \xrightarrow{d_H} \mathcal{T}_{\text{poly}}^{*+1} (M) \) is a Gerstenhaber algebra.

3.2. Polydifferential operators. On a smooth manifold \( M \), one also has the dgl of polydifferential operators, \( D^\bullet_{\text{poly}} (M) \).

Let \( M \) be a manifold, let \( \mathcal{R} \) denote its algebra of smooth functions \( C^\infty (M) \), and let \( D^0_{\text{poly}} (M) \) denote the algebra of differential operators on \( M \). Denote by \( D^k_{\text{poly}} (M) , k \geq 0 \), the space of \( (k+1) \)-differential operators on \( M \), i.e., the tensor product \( D^0_{\text{poly}} (M) \otimes \cdots \otimes D^0_{\text{poly}} (M) \) of \( (k+1) \) copies of the left \( \mathcal{R} \)-module \( D^0_{\text{poly}} (M) \). Denote also by \( D^{-1}_{\text{poly}} (M) \) the space of smooth functions \( \mathcal{R} = C^\infty (M) \).

It is well known that endowing \( D^\bullet_{\text{poly}} (M) \) the space of smooth functions \( \mathcal{R} = C^\infty (M) \) with the Hochschild differential \( d_H \), the cup product \( D^k_{\text{poly}} (M) \otimes D^l_{\text{poly}} (M) \xrightarrow{\cup} D^{k+l+1}_{\text{poly}} (M) \), and the Gerstenhaber bracket \( [ , ] \) makes it a Gerstenhaber algebra up to homotopy \( [3] \).

Following our earlier notations, now assume that \( M \) is a \( \mathfrak{g} \)-manifold with infinitesimal action \( \varphi : g \rightarrow \mathfrak{X} (M) \). Analogously to the polyvector field case, the Lie algebra \( \mathfrak{g} \) acts on \( D^\bullet_{\text{poly}} (M) \) by:

\[
a \cdot \mu = [\varphi (a), \mu] \quad \forall \ a \in \mathfrak{g}, \ \mu \in D^\bullet_{\text{poly}} (M).
\]

Since the Gerstenhaber bracket satisfies the graded Jacobi identity, this infinitesimal \( \mathfrak{g} \)-action on \( D^\bullet_{\text{poly}} (M) \) is compatible with the Hochschild differential. Consequently \( \cdots \to D^k_{\text{poly}} (M) \xrightarrow{d_H} D^{k+1}_{\text{poly}} (M) \to \cdots \) is a complex of \( \mathfrak{g} \)-modules, and therefore we have the Chevalley–Eilenberg cohomology

\[
H^*_\text{CE} (\mathfrak{g}, D^\bullet_{\text{poly}} (M)) \xrightarrow{d_H} H^{*+1} (\text{tot} (\Lambda \cdot g^\vee \otimes_k D^\bullet_{\text{poly}} (M))),
\]

which is, by definition, the total cohomology of the double complex

\[
\begin{array}{ccc}
& \cdots & \\
\uparrow & \cdots & \uparrow \\
\Lambda^{p-1} g^\vee \otimes_k D^{q+1}_{\text{poly}} (M) & \xrightarrow{d_{CE}} & \Lambda^p g^\vee \otimes_k D^{q+1}_{\text{poly}} (M) \\
\uparrow & \cdots & \uparrow \\
\Lambda^{p-1} \otimes d_H & \cdots & \Lambda^p \otimes d_H \\
\uparrow & \cdots & \uparrow \\
\cdots & \cdots & \cdots
\end{array}
\]

Extend the cup product \( \cup \) and the Gerstenhaber bracket \( [ , ] \) to \( \Lambda^\bullet g^\vee \otimes_k D^\bullet_{\text{poly}} (M) \) as follows:

\[
(\alpha \otimes \xi) \cup (\beta \otimes \eta) = (\alpha \otimes \beta) \otimes (\xi \cup \eta)
\]

\[
[\alpha \otimes \xi, \beta \otimes \eta] = (\alpha \otimes [\xi, \eta] + (\beta \otimes [\alpha, \eta])
\]

for any \( \alpha \otimes \xi \in \Lambda^{p_1} g^\vee \otimes_k D^{q_1}_{\text{poly}} (M) \) and \( \beta \otimes \eta \in \Lambda^{p_2} g^\vee \otimes_k D^{q_2}_{\text{poly}} (M) \).

Again the following lemma is immediate.

Lemma 3.2.  
(1) The graded \( k \)-vector space \( \text{tot}(\Lambda^\bullet g^\vee \otimes_k D^\bullet_{\text{poly}} (M)) \), together with the differential \( d_{CE} + \text{id} \otimes d_H \) and the Gerstenhaber bracket \( [ , ] \) defined by Equation (1), is a dgl.

(2) The graded \( k \)-vector space \( H^*_\text{CE} (\mathfrak{g}, D^\bullet_{\text{poly}} (M)) \xrightarrow{d_H} H^{*+1} (\text{tot} (\Lambda^\bullet g^\vee \otimes_k D^\bullet_{\text{poly}} (M))) \), together with the cup product and the Gerstenhaber bracket defined by Equations (2) and (3), is a Gerstenhaber algebra.
3.3. Hochschild–Kostant–Rosenberg theorem. Given a smooth manifold \( M \), there is a natural embedding \( \text{hkr} : T^{\bullet}_{\text{poly}}(M) \hookrightarrow D^{\bullet}_{\text{poly}}(M) \), called Hochschild–Kostant–Rosenberg map, and defined by

\[
\text{hkr}(X_1 \wedge \cdots \wedge X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}, \quad \forall X_i \in \mathfrak{X}(M),
\]
where \( S_k \) is the symmetric group on \( k \) objects. The Hochschild–Kostant–Rosenberg theorem for smooth manifolds states that \( \text{hkr} \) is a quasi-isomorphism, i.e. the induced morphism in cohomology \( \text{hkr} \) is an isomorphism of vector spaces \([6, 7]\).

Suppose we are given a \( \mathfrak{g} \)-manifold \( M \). Then the map \( \text{id} \otimes \text{hkr} : \Lambda^\bullet \mathfrak{g}^\vee \otimes_k T^{\bullet}_{\text{poly}}(M) \to \Lambda^\bullet \mathfrak{g}^\vee \otimes_k D^{\bullet}_{\text{poly}}(M) \) is a morphism of double complexes. Abusing notations, the induced morphism on Chevalley–Eilenberg cohomologies will also be denoted by \( \text{hkr} \).

**Proposition 3.3** ([8]). Let \( M \) be a \( \mathfrak{g} \)-manifold. The Hochschild–Kostant–Rosenberg map

\[
\text{hkr} : H^\bullet_{\text{CE}}(\mathfrak{g}, T^{\bullet}_{\text{poly}}(M)) \to H^\bullet_{\text{CE}}(\mathfrak{g}, D^{\bullet}_{\text{poly}}(M))
\]

is a quasi-isomorphism of vector spaces.

The proof is a straightforward spectral sequence computation relying on the classical Hochschild–Kostant–Rosenberg theorem for smooth manifolds.

4. Atiyah class of a \( \mathfrak{g} \)-manifold

The Atiyah class was originally introduced by Atiyah for holomorphic vector bundles \([1]\). Atiyah classes can also be defined for Lie algebroid pairs \([3]\) and dg vector bundles \([10]\). In this section, we introduce the notions of Atiyah class and Todd class of a \( \mathfrak{g} \)-manifold.

Let \( M \) be a \( \mathfrak{g} \)-manifold with infinitesimal action \( \mathfrak{g} \ni a \mapsto \hat{a} \in \mathfrak{X}(M) \). Given an affine connection \( \nabla \) on \( M \), the Atiyah 1-cocycle associated with \( \nabla \) is defined as the map \( R^\nabla_{1,1} : \mathfrak{g} \times \mathfrak{X}(M) \to \text{End}_\mathbb{R} \mathfrak{X}(M) \) given by

\[
R^\nabla_{1,1}(a, X) = \mathcal{L}_\hat{a} \circ \nabla_X - \nabla_X \circ \mathcal{L}_\hat{a} - \nabla_{\mathcal{L}_\hat{a} X},
\]
where \( a \in \mathfrak{g}, X \in \mathfrak{X}(M) \), and \( \mathcal{R} = C^\infty(M) \).

Following \([3]\), we prove the following

**Proposition 4.1.**  
1. The Atiyah cocycle \( R^\nabla_{1,1} \in \mathfrak{g}^\vee \otimes \Gamma(T^\vee_M \otimes \text{End}_M T_M) \) is a Chevalley–Eilenberg 1-cocycle of the \( \mathfrak{g} \)-module \( \Gamma(T^\vee_M \otimes \text{End}_M T_M) \).
2. The cohomology class \( \alpha_M/\mathfrak{g} \in H^1_{\text{CE}}(\mathfrak{g}, \Gamma(T^\vee_M \otimes \text{End}_M T_M)) \) of the 1-cocycle \( R^\nabla_{1,1} \) does not depend on the choice of connection \( \nabla \).

The cohomology class \( \alpha_M/\mathfrak{g} \) is called the Atiyah class of the \( \mathfrak{g} \)-manifold \( M \). It is the obstruction class to the existence of a \( \mathfrak{g} \)-invariant connection on \( M \), i.e. an affine connection \( \nabla \) on \( M \) satisfying

\[
[\hat{a}, \nabla_X Y] = \nabla_{[\hat{a}, X]} Y + \nabla_X [\hat{a}, Y]
\]
for all \( a \in \mathfrak{g} \) and \( X, Y \in \mathfrak{X}(M) \).

**Proposition 4.2.** Let \( M \) be a \( \mathfrak{g} \)-manifold. The Atiyah class \( \alpha_M/\mathfrak{g} \) of \( M \) vanishes if and only if there exists a \( \mathfrak{g} \)-invariant connection on \( M \).

Note that if \( \mathfrak{g} \) is a compact Lie algebra, \( \alpha_M/\mathfrak{g} \) vanishes since \( \mathfrak{g} \)-invariant connections always exist.

The Todd class of complex vector bundles plays an important role in the Riemann–Roch theorem. In our context, the Todd cocycle of a \( \mathfrak{g} \)-manifold \( M \) is the Chevalley–Eilenberg cocycle

\[
\text{td}_{M/\mathfrak{g}} = \det \left( \frac{R^\nabla_{1,1}}{1 - e^{-R^\nabla_{1,1}}} \right) \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M),
\]

(5)
with \( \Omega^k(M), k \geq 0 \), being the natural \( \mathfrak{g} \)-module. Its corresponding Chevalley–Eilenberg cohomology class is the **Todd class** \( \text{Td}_{M/\mathfrak{g}} \). Alternatively
\[
\text{Td}_{M/\mathfrak{g}} = \det \left( \frac{e^{\alpha(M/\mathfrak{g})} - 1}{e^{\alpha(M/\mathfrak{g})}} \right) \in \bigoplus_{k=0} H^k_{\text{CE}}(\mathfrak{g}, \Omega^k(M)).
\]
Since the Lie algebra \( \mathfrak{g} \) is finite dimensional, the above expression for the Todd class \( \text{Td}_{M/\mathfrak{g}} \) reduces to a finite sum.

**Example 1.** Consider the case of the 1-dimensional abelian Lie algebra \( \mathfrak{g} = \mathbb{R} \) acting on the real line \( M = \mathbb{R} \). The infinitesimal action is uniquely determined by a vector field \( Q = q(x) \frac{d}{dx} \in \mathfrak{X}(\mathbb{R}) \). The Chevalley–Eilenberg complex \( (\Lambda^* \mathfrak{g} \otimes \Gamma(T_M^* \otimes \text{End} T_M), d_{\text{CE}}) \) is then isomorphic to the 2-term complex
\[
0 \longrightarrow C^\infty(\mathbb{R}) \xrightarrow{d_Q} C^\infty(\mathbb{R}) \longrightarrow 0,
\]
where the map \( d_Q \) is given by
\[
d_Q(f) = \frac{d(fq)}{dx} = f'q +fq',
\]
for \( f \in C^\infty(\mathbb{R}) \). Let \( \nabla \) be the trivial affine connection on the manifold \( M = \mathbb{R} \), i.e. \( \nabla \frac{d}{dx} = 0 \). Under the above isomorphism, the Atiyah 1-cocycle \( R_{1,1}^{\nabla} \) is simply the second order derivative of \( q \):
\[
R_{1,1}^{\nabla} = q'' \in C^\infty(\mathbb{R}) \cong \mathfrak{g}^\vee \otimes \Gamma(T_M^* \otimes \text{End} T_M).
\]
As a consequence, the Atiyah class vanishes if and only if there exists a smooth function \( y \) defined on the whole real line and satisfying the differential equation \( qy' + q'y = q'' \). For instance, if \( Q = x^2 \frac{d}{dx} \), the Atiyah class is non-trivial since no function \( y \in C^\infty(\mathbb{R}) \) satisfies \( x^2 \frac{dy}{dx} + 2xy = 2 \) and therefore there exists no \( Q \)-invariant connection on \( \mathbb{R} \).

5. **Formality theorem and Kontsevich–Duflo theorem for \( \mathfrak{g} \)-manifolds**

The main results of this Note are a formality theorem for \( \mathfrak{g} \)-manifolds and its consequence: a Kontsevich–Duflo type theorem for \( \mathfrak{g} \)-manifolds.

**Theorem 5.1** (Formality theorem for \( \mathfrak{g} \)-manifolds). Given a \( \mathfrak{g} \)-manifold \( M \) and an affine torsionfree connection \( \nabla \) on \( M \), there exists an \( L_\infty \) quasi-isomorphism \( \Phi \) from the dgla \( \text{tot}(\Lambda^* \mathfrak{g}^\vee \otimes_k T_{\text{poly}}^* (M)) \) to the dgla \( \text{tot}(\Lambda^* \mathfrak{g}^\vee \otimes_k D_{\text{poly}}^* (M)) \) whose first ‘Taylor coefficient’ \( \Phi_1 \) satisfies the following two properties:

1. \( \Phi_1 \) is, up to homotopy, an isomorphism of associative algebras (and hence induces an isomorphism of associative algebras of the cohomologies);
2. \( \Phi_1 \) is equal to the composition \( \text{hkr} \circ \text{td}_{M/\mathfrak{g}}^{\frac{1}{2}} \) of the HKR map and the action of the square root of the Todd cocycle \( \text{td}_{M/\mathfrak{g}}^{\frac{1}{2}} \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M) \) on \( \text{tot}(\Lambda^* \mathfrak{g}^\vee \otimes_k T_{\text{poly}}^* (M)) \) by contraction.

As an immediate consequence, we have the following

**Theorem 5.2** (Kontsevich–Duflo type theorem for \( \mathfrak{g} \)-manifolds). Given a \( \mathfrak{g} \)-manifold \( M \), the map
\[
\text{hkr} \circ \text{Td}_{M/\mathfrak{g}}^{\frac{1}{2}} : H^\bullet_{\text{CE}}(\mathfrak{g}, T_{\text{poly}}^* (M)) \xrightarrow{\partial} T_{\text{poly}}^{\bullet+1} (M) \xrightarrow{\partial} H^\bullet_{\text{CE}}(\mathfrak{g}, D_{\text{poly}}^* (M)) \xrightarrow{\partial} D_{\text{poly}}^{\bullet+1} (M)
\]
is an isomorphism of Gerstenhaber algebras. It is understood that the square root \( \text{Td}_{M/\mathfrak{g}}^{\frac{1}{2}} \) of the Todd class \( \text{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0} H^k_{\text{CE}}(\mathfrak{g}, \Omega^k(M)) \) acts on \( H^\bullet_{\text{CE}}(\mathfrak{g}, T_{\text{poly}}^* (M)) \xrightarrow{\partial} T_{\text{poly}}^{\bullet+1} (M) \) by contraction.
Theorem 5.1 follows from a more general result of ours, a formality theorem for Lie pairs, whose detailed proof will appear in a forthcoming revision of [8]. A pair of Lie algebroids (or Lie pair in short) consists of a Lie algebroid $L$ and a Lie subalgebroid $A$ of $L$. Given any Lie pair, our formality theorem for Lie pairs establishes an $L_\infty$ quasi-isomorphism $\Phi$ from the polyvector fields ‘on the pair’ to the polydifferential operators ‘on the pair.’ The first ‘Taylor coefficient’ $\Phi_1$ of the $L_\infty$ quasi-isomorphism $\Phi$ preserves the associative algebra structures up to homotopy and admits an explicit description in terms of the Hochschild–Kostant–Rosenberg map and the Todd cocycle of the Lie pair. Now every $\mathfrak{g}$-manifold $M$ determines in a canonical way a matched pair: $(\mathfrak{g} \ltimes M, T_M)$ [11, Example 5.5] [9]. The notation $\mathfrak{g} \ltimes M$ refers to the transformation Lie algebroid arising from the infinitesimal $\mathfrak{g}$-action on $M$. Therefore, we can form a Lie pair $(L, A)$, where $L = (\mathfrak{g} \ltimes M) \ltimes T_M$ and $A = \mathfrak{g} \ltimes M$. For this particular pair, the polyvector fields and polydifferential operators reduce to $\text{tot} \left( \Lambda^\bullet \mathfrak{g}^\vee \otimes_k T_{\text{poly}}(M) \right)$ and $\text{tot} \left( \Lambda^\bullet \mathfrak{g}^\vee \otimes_k D_{\text{poly}}(M) \right)$ respectively. Theorem 5.1 then follows from our formality theorem for Lie pairs [8].

To the best of our knowledge, the first construction of an $L_\infty$ quasi-isomorphism from the dgla $\text{tot} \left( \Lambda^\bullet \mathfrak{g}^\vee \otimes_k T_{\text{poly}}(M) \right)$ to the dgla $\text{tot} \left( \Lambda^\bullet \mathfrak{g}^\vee \otimes_k D_{\text{poly}}(M) \right)$ can be credited to Dolgushev [4, concluding remarks].

Applications of Theorem 5.1 to the deformation quantization of $\mathfrak{g}$-manifolds will be considered elsewhere.

Acknowledgements

The authors thank Ruggero Bandiera, Martin Bordemann, Vasily Dolgushev, Olivier Elchinger, Marco Manetti, Boris Shoikhet, Jim Stasheff, and Dima Tamarkin for inspiring discussions and useful comments. Mathieu Stiénon would like to express his gratitude to the Université Paris–Diderot for its hospitality while work on this project was underway.

References

1. Michael F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957), 181–207. MR 0086359
2. Damien Calaque and Michel Van den Bergh, Hochschild cohomology and Atiyah classes, Adv. Math. 224 (2010), no. 5, 1839–1889. MR 2646112
3. Zhuo Chen, Mathieu Stiénon, and Ping Xu, From Atiyah classes to homotopy Leibniz algebras, Comm. Math. Phys. 341 (2016), no. 1, 309–349. MR 3439299
4. Vasily Dolgushev, Covariant and equivariant formality theorems, Adv. Math. 191 (2005), no. 1, 147–177. MR 2102846 (2006c:53101)
5. Murray Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. (2) 78 (1963), 267–288. MR 0161898
6. Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. 102 (1962), 383–408. MR 0142598
7. Maxim Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003), no. 3, 157–216. MR 2062626 (2005i:53122)
8. Hsuan-Yi Liao, Mathieu Stiénon, and Ping Xu, Formality theorem and Kontsevich–Duflo type theorem for Lie pairs, ArXiv e-prints (2016).
9. K. C. H. Mackenzie, Drinfeld doubles and Ehresmann doubles for Lie algebroids and Lie bialgebroids, Electron. Res. Announc. Amer. Math. Soc. 4 (1998), 74–87. MR 1650045
10. Rajan A. Mehta, Mathieu Stiénon, and Ping Xu, The Atiyah class of a dg-vector bundle, C. R. Math. Acad. Sci. Paris 353 (2015), no. 4, 357–362. MR 3319134
11. Tahar Mokri, Matched pairs of Lie algebroids, Glasgow Math. J. 39 (1997), no. 2, 167–181. MR 1460632
12. Dmitry E. Tamarkin, Operadic proof of M. Kontsevich’s formality theorem, ProQuest LLC, Ann Arbor, MI, 1999, Thesis (Ph.D.)–The Pennsylvania State University. MR 2699544
