Scale Invariance, Inflation and the Present Vacuum Energy of the Universe

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1 The Model in the absence of fermions

The concept of scale invariance appears as an attractive possibility for a fundamental symmetry of nature. In its most naive realizations, such a symmetry is not a viable symmetry, however, since nature seems to have chosen some typical scales.

Here we will find that scale invariance can nevertheless be incorporated into realistic, generally covariant field theories. However, scale invariance has to be discussed in a more general framework than that of standard generally relativistic theories, where we must allow in the action, in addition to the ordinary measure of integration $\sqrt{-g}d^4x$, another one, $\Phi d^4x$, where $\Phi$ is a density built out of degrees of freedom independent of the metric.

For example, given 4-scalars $\varphi_a$ $(a = 1,2,3,4)$, one can construct the density

$$\Phi = \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \partial_\mu \varphi_a \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d$$

(1)

One can allow both geometrical objects to enter the theory and consider

$$S = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x$$

(2)
Here \( L_1 \) and \( L_2 \) are \( \varphi_a \) independent. There is a good reason not to consider mixing of \( \Phi \) and \( \sqrt{-g} \), like for example using \( \sqrt{-g} \). This is because (2) is invariant (up to the integral divergence) under the infinite dimensional symmetry \( \varphi_a \rightarrow \varphi_a + f_a(L_1) \) where \( f_a(L_1) \) is an arbitrary function of \( L_1 \) if \( L_1 \) and \( L_2 \) are \( \varphi_a \) independent. Such symmetry (up to the integral of a total divergence) is absent if mixed terms are present.

We will study now the dynamics of a scalar field \( \phi \) interacting with gravity as given by the action (2) with

\[
L_1 = -\frac{1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \quad L_2 = U(\phi) \tag{3}
\]

\[
R(\Gamma, g) = g^{\mu\nu} R_{\mu\nu}(\Gamma), \quad R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\nu\lambda}, \quad \Gamma^\lambda_{\mu\nu\sigma} = \Gamma^\lambda_{\mu\sigma\nu} + \Gamma^\lambda_{\nu\alpha} \Gamma^\alpha_{\mu\sigma} - \Gamma^\lambda_{\mu\alpha} \Gamma^\alpha_{\nu\sigma} \tag{4}
\]

In the variational principle \( \Gamma^\lambda_{\mu\nu\sigma}g_{\mu\nu} \), the measure fields scalars \( \varphi_a \) and the scalar field \( \phi \) are all to be treated as independent variables. If we perform the global scale transformation (\( \theta = \text{constant} \))

\[
g_{\mu\nu} \rightarrow e^{\theta} g_{\mu\nu} \tag{5}
\]

then (2), with the definitions (3), (4), is invariant provided \( V(\phi) \) and \( U(\phi) \) are of the form

\[
V(\phi) = f_1 e^{\alpha \phi}, \quad U(\phi) = f_2 e^{2\alpha \phi} \tag{6}
\]

and \( \varphi_a \) is transformed according to \( \varphi_a \rightarrow \lambda_a \varphi_a \) (no sum on \( a \)) which means \( \Phi \rightarrow \left( \prod_\alpha \lambda_\alpha \right) \Phi = \lambda \Phi \) such that \( \lambda = e^\theta \) and \( \phi \rightarrow \phi - \frac{a}{\alpha} \). In this case we call the scalar field \( \phi \) needed to implement scale invariance ”dilaton”.

1.1 Equations of Motion

Let us consider the equations which are obtained from the variation of the \( \varphi_a \) fields. We obtain then

\[
A_\mu^a \partial_\mu L_1 = 0 \quad \text{where} \quad A_\mu^a = \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d. \quad \text{Since} \quad \det (A_\mu^a) = \frac{\kappa}{4!} \Phi^5 \neq 0 \quad \text{if} \quad \Phi \neq 0. \quad \text{Therefore if} \quad \Phi \neq 0 \quad \text{we obtain that} \quad \partial_\mu L_1 = 0, \quad \text{or that} \quad L_1 = M, \quad \text{where} \quad M \quad \text{is constant.} \quad \text{This constant} \quad M \quad \text{appears in a self-consistency condition of the equations of motion that allows us to solve for} \quad \chi \equiv \frac{\Phi}{\sqrt{-g}}
\]

\[
\chi = \frac{2U(\phi)}{M + V(\phi)}. \tag{7}
\]

To get the physical content of the theory, it is convenient to go to the Einstein conformal frame where

\[
\overline{g}_{\mu\nu} = \chi g_{\mu\nu} \tag{8}
\]

and \( \chi \) given by (7). In terms of \( \overline{g}_{\mu\nu} \), the non Riemannian contribution (defined as \( \Sigma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \{\lambda_{\mu\nu}\} \) where \( \{\lambda_{\mu\nu}\} \) is the Christoffel symbol), disappears from the equations, which can be written then in the Einstein form \( (R_{\mu\nu}(\overline{g}_{\alpha\beta}) = \text{usual Ricci tensor}) \)

\[
R_{\mu\nu}(\overline{g}_{\alpha\beta}) - \frac{1}{2} \overline{g}_{\mu\nu} R(\overline{g}_{\alpha\beta}) = \frac{\kappa}{2} T^\text{eff}_{\mu\nu}(\phi) \tag{9}
\]

where

\[
T^\text{eff}_{\mu\nu}(\phi) = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \overline{g}_{\mu\nu} \phi_{,\alpha} \phi_{,\beta} \overline{g}^{\alpha\beta} + \overline{g}_{\mu\nu} V_{\text{eff}}(\phi), \quad V_{\text{eff}}(\phi) = \frac{1}{4U(\phi)}(V + M)^2. \tag{10}
\]
If \( V(\phi) = f_1 e^{\alpha \phi} \) and \( U(\phi) = f_2 e^{2\alpha \phi} \) as required by scale invariance, we obtain from (10)

\[
V_{\text{eff}} = \frac{1}{4 f_2^2} (f_1 + M e^{-\alpha \phi})^2
\]  

(11)

Since we can always perform the transformation \( \phi \to -\phi \) we can choose by convention \( \alpha > O \). We then see that as \( \phi \to \infty, V_{\text{eff}} \to \frac{f_1^2}{4 f_2^2} = \text{const.} \) providing an infinite flat region. Also a minimum is achieved at zero cosmological constant, without fine tuning for the case \( \frac{f_1^2}{M^2} < O \) at the point \( \phi_{\text{min}} = \frac{-1}{\alpha} \ln \frac{f_1}{M} \). Finally, the second derivative of the potential \( V_{\text{eff}} \) at the minimum is \( V_{\text{eff}}'' = \frac{\alpha^2}{2 f_2^2} | f_1 |^2 > O \).

### 2 Interpretations, Generalizations and Physical Applications of the Model

There are many interesting issues that one can raise here. The first one is of course the fact that a realistic scalar field potential, with massive excitations when considering the true vacuum state, is achieved in a way consistent with the idea of scale invariance. An interesting point to be made concerning this is that even though spontaneous symmetry breaking has taken place, no Goldstone boson appears nevertheless when analyzing the theory in its ground state. This interesting and unusual effect is due to the fact that although a locally conserved current can be defined (from Noether’s theorem), this still does not lead to a globally conserved charge, because the currents have an infrared singular behavior that causes scale charge to leak to infinity.

The second point to be raised is that since there is an infinite region of flat potential for \( \phi \to \infty \), we expect a slow rolling inflationary scenario to be viable, provided the universe is started at a sufficiently large value of the scalar field \( \phi \). Notice that a small value of \( \phi \) can provide a long lived almost constant vacuum energy for a long period of time, which can be made concerning this is that even though spontaneous symmetry breaking has taken place, no Goldstone boson appears nevertheless when analyzing the theory in its ground state. This interesting and unusual effect is due to the fact that although a locally conserved current can be defined (from Noether’s theorem), this still does not lead to a globally conserved charge, because the currents have an infrared singular behavior that causes scale charge to leak to infinity.

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Furthermore, one can consider this model as suitable for the present day universe rather than for the early universe, after we suitably reinterpret the meaning of the scalar field \( \phi \). This can provide a long lived almost constant vacuum energy for a long period of time, which can be small if \( f_2^2/4 f_2 \) is small. Such small energy density will eventually disappear when the universe achieves its true vacuum state.

Notice that a small value of \( \frac{f_2}{f_1} \) can be achieved if we let \( f_2 >> f_1 \). In this case \( \frac{f_2}{f_1} << f_1 \), i.e. a very small scale for the energy density of the universe is obtained by the existence of a very high scale (that of \( f_2 \)) the same way as a small fermion mass is obtained in the see-saw
mechanism from the existence also of a large mass scale. It can be shown also that if we take \( f_2 \gg f_1 \), so that the vacuum energy is small, this also produces a drastic suppression in the fermion particle masses (everything can be done consistently with scale invariance, see Ref.3) and a possible correlation between the lightest particle mass and the vacuum energy of the universe is argued for (see the transparencies of this talk which can be found in the web site of this conference). The scale invariant way of introducing fermion masses has also implications concerning the ”cosmic coincidences” problem (see Ref.3).

Finally, this kind of theories can naturally provide a dynamics that interpolates between a high energy density (associated with inflation) and a very low energy density (associated with the present universe). For this consider two scalar fields \( \phi_1 \) and \( \phi_2 \), with normal kinetic terms coupled to the measure \( \Phi \) as it has been done with the simpler model of just one scalar field. Introducing for \( \phi_1 \) a potential \( V_1(\phi_1) = a_1 e^{\alpha_1 \phi_1} \) that couples to \( \Phi \) and another \( U_1(\phi_1) = b_1 e^{2\alpha_1 \phi_1} \) that couples to \( \sqrt{-g} \) as required by scale invariance and the potential for \( \phi_2 \), \( V_2(\phi_2) = a_2 e^{\alpha_2 \phi_2} \) that couples to \( \Phi \) and another \( U_2(\phi_2) = b_2 e^{2\alpha_2 \phi_2} \) that couples to \( \sqrt{-g} \), we arrive (after going through the same steps as those explained in the model with just one scalar, i.e. solving the constraint and going to the Einstein frame) at the effective potential

\[
V_{\text{eff}} = \frac{(V_1(\phi_1) + V_2(\phi_2) + M)^2}{4(U_1(\phi_1) + U_2(\phi_2))}
\]  

(13)

which introduces interactions between \( \phi_1 \) and \( \phi_2 \), although no interactions appeared in the original action (i.e. no direct couplings appeared). If we take then \( \alpha_1 \phi_1 \) very big while \( \phi_2 \) is fixed, then \( V_{\text{eff}} \) approaches the constant value \( \frac{a_1^2}{b_1} \) while if we take \( \alpha_2 \phi_2 \) to be very big while \( \phi_1 \) is kept fixed, then \( V_{\text{eff}} \) approaches the constant value \( \frac{a_2^2}{b_2} \). One of these flat regions of the potential can be associated with a very high energy density, associated with inflation and the other can be very small and associated with the energy density of the present universe. The effective potential (13) provides therefore a dynamics that interpolates naturally between the inflationary phase and the present slowly accelerated universe.

References

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