Quantum digital-to-analog conversion algorithm using decoherence

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Abstract
We consider the problem of mapping digital data encoded on a quantum register to analog amplitudes in parallel. It is shown to be unlikely that a fully unitary polynomial-time quantum algorithm exists for this problem; NP becomes a subset of BQP if it exists. In the practical point of view, we propose a nonunitary linear-time algorithm using quantum decoherence. It tacitly uses an exponentially large physical resource, which is typically a huge number of identical molecules. Quantumness of correlation appearing in the process of the algorithm is also discussed.

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1 Introduction

There have been many conventional electrical and algorithmic implementations for digital-to-analog conversion (DAC) [19]. For a very brief explanation, given a digital data \( f(k) \in \{0, 1\}^n \) for label \( k \in \{0, 1\}^m \), DAC produces an analog signal with an amplitude proportional to \( f(k) \). (We do not give a particular physical meaning to the variable \( k \) although it is often a label of a time slot. It can be regarded as a label of a channel especially in the context of parallel processing.) The DAC problem can be formulated as follows.

Problem 1: DAC problem

Instance: Integers \( m, n \geq 1 \), set \( \{k\} \) of integers \( k \in \{0, 1\}^m \), and function \( f : \{0, 1\}^m \to \{0, 1\}^n \).

Task: Generate analog amplitudes \( V(k) \propto f(k) \) for all \( k \) (here, the proportionality constant is a certain constant independent of \( k \)).

A classical parallel solver for this problem can be intuitively implemented; one may prepare the same converter for each \( k \) although this is not economical. In view of recent development of quantum information processing, it is natural to consider the possibility of using quantum parallelism instead of preparing many identical converters. Then the parallelization will be quite economical. We are seeking for this possibility in this contribution, but we will first notice that it is not straightforward to find a useful quantum algorithm for DAC.

Quantum parallelism plays an essential role in quantum computing [6, 10]. In a conceptual explanation, a unitary operation

\[ U_f : |0\rangle|k\rangle \mapsto |f(k)\rangle|k\rangle \]

corresponding to a function \( f : \{0, 1\}^m \to \{0, 1\}^n \) is often considered (here, \( k \in \{0, 1, \ldots, 2^m-1\} \)). It is tacitly assumed that \( U_f \) is constructed with a reasonable number of single-qubit
Quantum digital-to-analog conversion algorithm using decoherence

and two-qubit unitary operations. Once $U_f$ is constructed, we may apply it to a superposition of quantum states by linearity of a unitary transformation. Given the initial state $|0\rangle \sum_{k=0}^{2^n-1} c_k |k\rangle$ with complex amplitudes $c_k$ satisfying $|c_k|^2 = 1$, the following evolution can be performed:

$$|0\rangle \sum_{k} c_k |k\rangle \mapsto \sum_{k} c_k |f(k)\rangle |k\rangle.$$  \hspace{1cm} (1)

The resultant state is a sort of digital states because $f(k)$ is kept as a digital datum in the left register. Thus this process is an encoding of digital data as a quantum state. For simplicity, we may often choose $c_k = \frac{1}{\sqrt{2^n}}$. To perform DAC in a quantum manner, we need to change the amplitude $c_k$ so that $|c_k|^2$ is proportional to $f(k)$ for each $k$ in parallel. This will however turn out to be a hard problem in the context of computational cost, later in this paper.

There is a conventional quantum algorithm for analog-to-digital and digital-to-analog conversions (ADC and DAC) in a different context: Schmüser and Janzing \cite{22} considered the problem of conversions between a continuous wave function $\psi(x)$ ($x \in [0, L]$ stands for a position) and a quantum state $|\Psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \psi(jL/2^n) |j\rangle$ of an $n$-qubit register. They employed the Jaynes-Cummings model as a physical model and showed procedures for the conversions using several physical time evolutions that are possible under the model. Their algorithm, however, requires an exponentially large interaction strength or an exponential interaction time. Besides, it is probably unnatural to call their conversions as ADC and DAC because $|\Psi\rangle$ has amplitude information in its complex amplitudes. They are rather conceptually close to analog-to-analog conversions in this sense.

Here, we are considering the DAC problem along with Definition \[1\] and going to construct a certain quantum algorithm for it. In this context, there have been several related works: Ventura and Martinez \cite{29} considered a problem to generate a superposition with desired phase factors for the complex amplitudes that have unit absolute values. (Thus their problem is a phase modulation problem.) They developed an algorithm whose cost scales quasi-linearly in the number of components of a superposition; this cost is exponential in the number of qubits. Other related works are those about decomposition of an arbitrary unitary transformation \cite{27, 28, 15, 24, 12}. (They are applicable to constructing a unitary DAC.) They all require exponential number of quantum gates for decomposition. These conventional results suggest that it is unlikely to achieve exponential speedup of DAC by using quantumness as a resource. This is very natural in the sense that $\mathbb{NP}$ will be included in $\mathbb{BQP}$ if such a speedup is possible by using a fully unitary process. It is also unlikely for any parallel computer to perform a parallel DAC within polynomial resources. These points will be discussed in Sec. 2. (See, e.g., Ref. \cite{30} for quantum computational complexity classes, such as $\mathbb{BQP}$.)

In this contribution, we develop a nonunitary quantum algorithm for the above-defined DAC problem. We can encode a given instance as a quantum state in Hilbert space while we move to Liouville space (the operator Hilbert space) in subsequent steps in the algorithm. We may also start with a mixed state from the first step. We utilize decoherence processes in addition to standard quantum gates. Thus our model is a nonunitary ensemble quantum computing model. The run time of our algorithm is linear in the input size while the physical resource it consumes is exponential. (It should be noted that quantum computing in Liouville space is not a very rare topic. See, e.g., Refs. \cite{5} \cite{26}.)

As our model is not a standard quantum computer model, it is also of interest to consider quantumness of computation. Although we do not have a direct measure for it, we will discuss quantumness of correlation among registers, in particular, quantum discord \cite{17}.
This paper is organized as follows. We first prove the hardness of the parallel DAC problem in Sec. 2. Then we introduce our algorithm in Sec. 3. Complexity of the algorithm is also discussed in the section. In addition, quantumness of correlation appearing in the algorithm is argued in Sec. 4. Section 5 gives the summary and remarks on the results.

2 Computational difficulty of parallel DAC

As mentioned in Sec. 4 it is unlikely to achieve an exponential speedup of DAC using any parallel computer. This is because the following proposition holds.

Proposition 2. Consider the DAC problem. Suppose we can achieve the amplitudes $V(k)$ for all $k$ within $\text{poly}(m, n)$ time and $\text{poly}(m, n)$ physical resource on average for any $f$. Here, we assume that noise are i.i.d. and the noise intensity for each output $V(k)$ is bounded above by a constant $\varepsilon_{th}$. Then SAT is solvable within polynomial cost on average.

This is easily proved:

Proof. Consider a conjunctive normal form (CNF) $\varphi(x_0, \ldots, x_{m-1})$ with variables $x_i \in \{0, 1\}$. Let us write $k = x_0 \cdots x_{m-1}$ and choose $f$ such that $f(k) = \varphi(k)_{n-1}0_{n-2}0_{n-3} \cdots 0$ (the subscripts are bit labels). We choose $n$ so that $2^{n-1}V_{\text{LSB}} > 2^m(\varepsilon_{th} + \epsilon)$, where $V_{\text{LSB}}$ stands for the least-significant-bit voltage (namely, the smallest nonzero output amplitude of a DAC device) and $\epsilon > 0$ is a certain small constant. Prepare the set $\{0, \ldots, 2^m - 1\}$ for $\{k\}$. Then, after DAC is complete for all $k$, we look at the mixture of resultant amplitudes $V(k)$. Apart from noise, for each $k$, a nonzero analog signal is generated for nonzero $\varphi(k)_{n-1}$ only. This signal corresponds to the most significant bit [namely, the $(n - 1)$th bit] so that its intensity is $2^{n-1}V_{\text{LSB}}$. In contrast, the total noise intensity of the mixture is obviously bounded above by $2^m\varepsilon_{th}$. Therefore, the signal-per-noise ratio is $> 1 + c/\varepsilon_{th}$, which has a constant gap from 1. Hence, on average, a constant number of accumulation of data of the mixture is enough to decide if $\varphi$ is satisfiable. This is owing to the central limit theorem in regard with the sum of random variables [25, 11] (this holds by assumption of i.i.d. noise).

Here we have considered the hardness of DAC in the context of general parallel computation. It is also possible to consider it in the context of quantum computation, as we will discuss next.

2.1 Case of fully unitary quantum process

Although we do not employ a fully unitary process in our algorithm to be shown in Sec. 3 here it is meaningful to consider a quantum DAC described as a unitary map. This will show the hardness of quantum DAC in the context of standard quantum computation. Let us consider the following map as a unitary map realising a quantum DAC.

$$\hat{V} : \frac{1}{\sqrt{N_k}} \sum_k |f(k)||k\rangle \mapsto \sum_k \hat{f}(k)|f(k)||k\rangle$$

where $N_k$ is the size of set $\{k\}$ and $\hat{f}(k)$’s are complex amplitudes satisfying the conditions $|\hat{f}(k)|^2 \propto f(k) + \epsilon$ and $\sum_k |\hat{f}(k)|^2 = 1$. Here, $\epsilon < 1/2^m$ is a small bias; without it, $\hat{V}$$

footnote[1]{It is a tacit assumption that any DAC is designed to satisfy $V_{\text{LSB}} \geq \varepsilon_{th}$ [9, 15]. Thus, setting $n$ to a value slightly larger than $m$ should be enough to satisfy $2^{n-1}V_{\text{LSB}} > 2^m(\varepsilon_{th} + \epsilon)$. Even if this tacit assumption does not hold, an appropriate value of $n$ scales linearly in $m$ as long as $\varepsilon_{th}$ is a constant.
Proposition 3. Suppose there exists a quantum circuit realizing $\hat{V}$ with circuit depth and circuit width $\text{poly}(m,n)$ for any $f$. Then SAT is included in BQP.

Proof. The proof is conceptually similar to the previous one. Consider a CNF $\varphi : \{0,1\}^m \to \{0,1\}$ and choose $f$ such that $f(k) = \varphi(k)_{n-1}0_{n-2}0_{n-3} \cdots 0_0$ where $k$ is an $m$-bit string. Let us choose $n \geq 2$. First, we prepare the initial state $|0\rangle|0\rangle$ where the left register consists of $n$ qubits and the right register consists of $m$ qubits. Second, we apply $H^\otimes m$ to the right register where $H = (|0\rangle\langle 0| + |1\rangle\langle 1|)/\sqrt{2}$ is an Hadamard transform. The state becomes $(1/\sqrt{2^m})|0\rangle\sum_{k=0}^{2^m-1}|k\rangle$. Third, because $f$ is implementable as a classical boolean circuit, it is straight-forward to transform it to a quantum circuit using the right register as its input and the left register as its output. This circuit changes the state into $(1/\sqrt{2^m})\sum_{k=0}^{2^m-1}|f(k)\rangle|k\rangle$. As a convention, the internal cost to apply $\hat{f}$ is not considered since it is a sort of an oracle. Fourth, we apply $\hat{V}$ to the state and obtain $\sum_k \hat{f}(k)|f(k)\rangle|k\rangle$. Then we measure the right register. There are two cases: (i) In case $\varphi$ is satisfiable, among $m$-qubit states $|k\rangle$, those dis-satisfying $\varphi$ individually have relative population $< 1/2^{m+n-1}$ by the assumption on $\hat{f}(k)$. Thus the probability $p_s$ to obtain a bit string satisfying $\varphi$ by measuring the right register is larger than $1/(1+\mu)$ with $\mu = (2^m - 1)/2^{m+n-1}$. Because we chose $n \geq 2$, we have $p_s > 2/3$. (ii) In case $\varphi$ is not satisfiable, measuring the right register produces a uniformly random number from $\{k\}$. Fifth, we use a classical circuit to verify the resultant $m$-bit string. Case (i) results in “yes” with probability $> 2/3$ and case (ii) results in “no” with probability 1. By definition of BQP, the proof is now completed.

Here is an alternative statement of the proposition:

Corollary 4. There exist functions $f$ for which there is no quantum circuit to implement $\hat{V}$ with circuit depth and circuit width $\text{poly}(m,n)$ unless $\text{NP} \subseteq \text{BQP}$.

3 Nonunitary algorithm

We have seen the hardness of parallel DAC in the previous section. As it has turned out to be unlikely that a polynomial-time unitary quantum algorithm exists, we seek for the possibility of polynomial-time nonunitary quantum algorithm. As it has also turned out to be unlikely to have a polynomial-cost parallel algorithm in a more general sense, we may compromise with increase in physical resource. In particular, a massive parallelism of an ensemble of identical systems like a molecular ensemble is considered a realistic resource in our strategy. Our algorithm is entirely constructed as a completely-positive trace-preserving linear map (a CPTP map) and hence physically feasible (see, e.g., Refs. [2, 4] for the CPTP condition).

We first revisit two decoherence maps that are very common in quantum physics [16, 7]. These maps are used in our algorithm.

Quantum digital-to-analog conversion algorithm using decoherence

becomes nonunitary when $f(k)$ is always zero. (Of course, such an $\epsilon$ is negligible in case there is nonzero $f(k)$.) Under these settings, the existence of a quantum circuit realising $\hat{V}$ is manifest: one can use a known method [21] to implement an arbitrary unitary operation although this requires an exponential number of basic quantum gates in general.

Let us now show that the following proposition holds, which indicates the hardness of implementing $\hat{V}$. (Here, it should be recalled that $k$ is an $m$-bit string and $f(k)$ is an $n$-bit string. As for the definition of BQP, see, e.g., Ref. [30].)

◮ Proposition 3. Suppose there exists a quantum circuit realizing $\hat{V}$ with circuit depth and circuit width $\text{poly}(m,n)$ for any $f$. Then SAT is included in BQP.
Definition 5. For probability $p$ and a $d \times d$ density matrix $\rho$, the depolarization map $\Lambda_{\text{dpl}}$ is defined as

$$\Lambda_{\text{dpl}} : \rho \mapsto (1 - p)\rho + p(\text{Tr}\rho)I/d$$

with $I$ the $d \times d$ identity matrix. We also write

$$\Lambda_{\text{dpl}}(p, \rho) = (1 - p)\rho + p(\text{Tr}\rho)I/d.$$ 

Definition 6. For probability $p$ and a $d \times d$ density matrix $\rho$, the dephasing map $\Lambda_{\text{ph}}$ is defined as

$$\Lambda_{\text{ph}} : \rho \mapsto (1 - p)\rho + p\left(\sum_{i=0}^{d-1} \langle i | \rho | i \rangle | i \rangle \langle i | \right).$$

We also write

$$\Lambda_{\text{ph}}(p, \rho) = (1 - p)\rho + p\left(\sum_{i=0}^{d-1} \langle i | \rho | i \rangle | i \rangle \langle i | \right).$$

In the following, we are going to present our algorithm. First, we introduce a subroutine beforehand.

Subroutine $S_1$

Input: An $(m + n)$-qubit pure state

$$|\psi_{f k}\rangle = |f_{n-1}f_{n-2} \cdots f_0\rangle^L |k\rangle^R,$$

where $f_i \in \{0, 1\}$ ($i = 0, \ldots, n - 1$) and $k \in \{0, 1\}^m$. Superscripts L and R stand for left and right registers.

Output: $(m + 2n)$-qubit mixed state

$$\tilde{\rho}_{f k} = |f_{n-1}\rangle\langle f_{n-1}| \otimes \cdots \otimes |f_0\rangle\langle f_0| \otimes |k\rangle\langle k|$$

$$\otimes \left[\left(\frac{I}{2}\right) \delta(f_{n-1}) + (|0\rangle\langle 0|) \delta(f_{n-1} - 1)\right] \otimes \cdots \otimes \left[\left(\frac{I}{2}\right) \delta(f_0) + (|0\rangle\langle 0|) \delta(f_0 - 1)\right].$$

Construction:

(i) Attach $n$ ancillary qubits in the state $|0_{n-1} \cdots 0_0\rangle$ to the input.

(ii) For each $i$, select $f_i$ and the corresponding ancilla qubit $|0_i\rangle$ to make a pair and apply the 0-controlled-Hadamard gate ($C_0H$) with the control bit $f_i$ and the target bit $0_i$. Here, $C_0H = |0\rangle\langle 0| \otimes H + |1\rangle\langle 1| \otimes I$. The resultant state of each pair is $|f_i\rangle[(|+\rangle\delta(f_i) + (|0\rangle\langle 0|) \delta(f_i - 1)]$ with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

(iii) For each qubit of register $R$, apply the fully dephasing map $\Lambda_{\text{ph}}(p = 1)$.

(iv) For each $i$, apply the fully dephasing map $\Lambda_{\text{ph}}(p = 1)$ to the qubits in the pair made in (ii) individually. The resultant state of each pair is $|f_i\rangle\langle f_i| \otimes \left[\left(\frac{I}{2}\right) \delta(f_i) + (|0\rangle\langle 0|) \delta(f_i - 1)\right]$.

One can regard Subroutine $S_1$ as a map

$$\Lambda_1 : \mathcal{H}_2^m \otimes \mathcal{S}(\mathcal{H}_2^m) \rightarrow \mathcal{S}(\mathcal{H}_2^{m+2n}),$$

where $\mathcal{H}_2$ is a two-dimensional Hilbert space and generally $\mathcal{S}(\mathcal{H})$ is a space of density matrices (namely, a state space) acting on Hilbert space $\mathcal{H}$. $\Lambda_1$ maps $|\psi_{f k}\rangle$ to $\tilde{\rho}_{f k}$. Then, it is straightforward to find the following fact.
Remark 7. Map $\Lambda_1$ has the property:

$$\Lambda_1(|\psi_{f,k}\rangle + |\psi_{f',k}\rangle) = \Lambda_1(|\psi_{f,k}\rangle) + \Lambda_1(|\psi_{f',k}\rangle),$$

where $|\psi_{f,k}\rangle = |f_{n-1} \cdots f'_0\rangle^L |k\rangle^R$.

This is because, for $k \neq k'$, step (iii) erases the terms with factors $|k\rangle^R |k'\rangle$ and its Hermitian conjugates in a density matrix represented in the computational basis.

One can also regard Subroutine $S_1$ as a map

$$\tilde{\Lambda}_1 : \mathcal{S}(\mathcal{H}_2^{\otimes m+n}) \to \mathcal{S}(\mathcal{H}_2^{\otimes m+2n}).$$

It maps $|\psi_{f,k}\rangle |\psi_{f,k}\rangle$ to $\rho_{f,k}$. We can easily reach the following fact simply by linearity of the operations used in the subroutine.

Remark 8. Map $\tilde{\Lambda}_1$ has the property:

$$\tilde{\Lambda}_1(|\psi_{f,k}\rangle |\psi_{f,k}\rangle + |\psi_{f',k}\rangle |\psi_{f',k}\rangle) = \tilde{\Lambda}_1(|\psi_{f,k}\rangle |\psi_{f,k}\rangle) + \tilde{\Lambda}_1(|\psi_{f',k}\rangle |\psi_{f',k}\rangle).$$

Let us now introduce the main routine of our algorithm.

**Algorithm NONUNITARY_QUANTUM_DAC:**

**Input:** Integers $m, n \geq 1$ and function $f : \{0, 1\}^m \to \{0, 1\}^n$.

**Output:** A mixture of component states, labeled by $k$, each of which possesses information of analog amplitude $V(k) \propto f(k)$. $V(k)$ can be derived by using an appropriate observable for each $k$.

**Construction:**

1. Generate either of the pure state

$$|\psi_o\rangle = (1/\sqrt{2m}) \sum_{k=0}^{2^m-1} |f(k)\rangle^L |k\rangle^R$$

or the mixed state

$$\rho_o = (1/2^m) \sum_{k=0}^{2^m-1} |f(k)\rangle^L |f(k)\rangle \otimes |k\rangle^R |k\rangle.$$  \hfill (3)

Here, $|\psi_o\rangle$ can be generated from $(I^{\otimes n} \otimes H^{\otimes m}) |0_{n-1} \cdots 0_0\rangle |0_{m-1} \cdots 0_0\rangle^R = (1/\sqrt{2m}) \sum_k |0\rangle^L |k\rangle^R$.

As $f$ is a boolean function, it is easy to construct a quantum circuit $C_f$ mapping $|0\rangle^L |k\rangle^R$ to $|f(k)\rangle^L |f(k)\rangle^R$ for each $k$.

$\rho_o$ can be generated from $|0\rangle^L |0\rangle \otimes (I/\sqrt{2})^{\otimes m} = (1/2^m) \sum_k |0\rangle^L |0\rangle \otimes |k\rangle^R |k\rangle$. By linearity of a quantum circuit, $C_f$ maps each $|0\rangle^L |0\rangle \otimes |k\rangle^R |k\rangle$ to $|f(k)\rangle^L |f(k)\rangle \otimes |k\rangle^R |k\rangle$.

2. Apply Subroutine $S_1$ to the state (either $|\psi_o\rangle$ or $\rho_o$). In either of the cases, the resultant state is

$$\rho_1 = \frac{1}{2^m} \sum_{k=0}^{2^m-1} \left\{ |f_{n-1}(k)\rangle |f_{n-1}(k)\rangle \otimes \cdots \otimes |f_0(k)\rangle |f_0(k)\rangle \otimes |k\rangle |k\rangle \right. \otimes \left[ \left( \frac{1}{2} \right)^{d(f_{n-1}(k)) + (0)(0)d(f_0(k) - 1)} \right]^{a_n-1} \otimes \cdots \left[ \left( \frac{1}{2} \right)^{d(f_0(k)) + (0)(0)d(f_0(k) - 1)} \right]^{a_0} \right\},$$

where $f_i(k)$ is the $i$th bit of $f(k)$ and $a_i$ stands for the $i$th ancilla qubit.
3. For each $i \in \{0, \ldots, n-1\}$, apply $\Lambda_{dpl}(p = p_i)$ to $a_i$ with $p_i = 1 - 2^{i-n+1}$. Let us write $q_i = 2^{i-n+1}$. The resultant state is

$$
\rho_2 = \frac{1}{2^m} \sum_{k=0}^{2^m-1} \left\{ \left| f_{n-1}(k) \right\rangle \langle f_{n-1}(k) | \otimes \cdots \otimes \left| f_0(k) \right\rangle \langle f_0(k) | \right|^L \otimes \left| k \right\rangle^R \left\langle k \right| \right\}^{2^m-1} \otimes \\
\cdots \otimes \left[ \left( \frac{i}{2} \right) \delta(f_{n-1}(k)) + p_{n-1} \left( \frac{i}{2} \right) \delta(f_{n-1}(k) - 1) + q_{n-1}(0) \langle 0 | \delta(f_{n-1}(k) - 1) \right]^{2^m-1} \otimes \\
\cdots \otimes \left[ \left( \frac{i}{2} \right) \delta(f_0(k)) + p_0 \left( \frac{i}{2} \right) \delta(f_0(k) - 1) + q_0(0) \langle 0 | \delta(f_0(k) - 1) \right]^{2^m-1} \right\}.
$$

(5)

This is the output of the algorithm. As for the observable for deriving analog amplitudes from the output, see description \(\heartsuit\).

\(\heartsuit\). As an observable, we employ \(\sum_{i=0}^{n-1} Z_i\) where \(Z = |0 \rangle \langle 0| - |1 \rangle \langle 1|\) is the Pauli Z matrix and each operator \(Z_i\) acts on the corresponding ancilla qubit \(a_i\). The analog amplitude of the ancilla register for each \(k\) is obtained by

$$
V(k) = \text{Tr} \left( (P_R^k P_R^k) (I^L \otimes \sum_{i=0}^{n-1} Z_i) \right),
$$

where \(P_k = |k \rangle \langle k|\) is a projector. (Here, we do not normalize the trace of \(P_R^k \rho_2 P_R^k\). This is because the signal intensity of a state is proportional to its physical population in practice.) This equation indicates that we project \(\rho_2\) onto the subspace of \(|\cdot \rangle |k \rangle \langle \cdot|\) and perform the ensemble- or time-average measurements of polarizations \(Z_i\) in order to obtain \(V(k)\). It is straightforward to find that

$$
V(k) = \frac{1}{2^m} \sum_{i=0}^{n-1} q_i \delta(f_i(k) - 1)
\propto f(k).
$$

(6)

With the process we have seen, seemingly only one signal can be derived from \(\rho_2\). Nevertheless this is not the case in practice. In case we initially start from \(\rho_0\), many identical copies of the same system can be simultaneously input into the algorithm. Therefore, it is valid to assume that we may have a mixture of many identical copies of \(\rho_2\) in the output.

Furthermore, it is possible to derive multiple \(V(k)\)’s at once when the system we employ is a bulk-ensemble molecular spin system with a magnetic resonance facility. There is a strategy called the fetching algorithm [31, 13]. Consider the case where the qubits in the ancilla register \(A\) are decoupled to each other while they are individually coupled to the remaining part \(LR\). Then, we measure each ancilla qubit \(a_i\) by a free-induction-decay (FID) measurement. The Fourier spectrum (or the FID spectrum) of each measurement exhibits splitting of peaks (each subpeak corresponds to each \(k\)). The magnitude of each subpeak in the FID spectrum of \(a_i\) corresponds to \(v_i(k) = \text{Tr}(P_k Z_i) \rho_2\). Each \(V(k)\) is obtained by \(V(k) = \sum_{i=0}^{n-1} v_i(k)\). A drawback is that there are possibly so many nonzero \(v_i(k)\)’s that corresponding subpeaks are not clearly separated. The spectrum resolution for this strategy needs to be exponentially fine in general.

There are a few things that should be taken into account. Our algorithm has been constructed as a nonunitary algorithm using decoherence processes. Thus the output state is a mixed
state even if the initial state is a pure state. In addition, the output state is a quantum state and it is not directly usable as an alternative of a classical DAC output. To derive classical analog amplitudes, we need to perform average measurements as described in\textsection 3.

### 3.1 A little specific explanation

It will clarify how our algorithm works if we focus on the evolution of a single component state. Consider the case of $n=3$. We begin with the end of step 1 of the algorithm. Let us pick up $|101\rangle^L|\cdot\rangle^R$ among the component states $|f_2(k)f_1(k)f_0(k)\rangle^L|k\rangle^R$. Here, we have assumed that one of $f_2f_1f_0$'s is 101.

In step 2, Subroutine $S_1$ is applied. In this subroutine, an ancilla register $|0^a(0^a0^a)\rangle^A$ is attached and the $C\theta H$ operation is applied to each pair of $f_i$ and $a_i$. This changes the component state we are tracking to $|101\rangle^L|\cdot\rangle^R|0+0\rangle^A$. Then the dephasing operations are applied in this subroutine. The component state of our interest evolves into $(|101\rangle^L|101\rangle^R\otimes|\cdot\rangle^R\otimes|0^a\rangle^A\otimes(\frac{1}{2}I^\sigma_a^x)^{a_0}0\rangle^A)$.

Then, in step 3, depolarizing operations (with individually different parameter values) are applied to the ancillary qubits. The component state we are tracking becomes $\left\{0^a\langle 0|\otimes(\frac{1}{2}I^\sigma_a^x)^{a_0}\otimes[(1-1/2^2)(I/2)+1/2^2]|0\rangle^A\right\}^A$.

Finally, at the measurement stage, the average measurement is performed with the observable $Z_2+Z_1+Z_0$ (together with projections onto component states). For the component state we are tracking, this results in the output signal $\text{Tr}Z_2|0^a\langle 0|+\text{Tr}Z_1(I/2)^{a_1}+\text{Tr}Z_0[(1-1/2^2)(I/2)+1/2^2]|0\rangle^A=1\times(1/2^3)+0\times(1/2^3)+1\times(1/2^2)$. This is proportional to the value 101.

It is easy to consider the evolution of any other component state among $|f_2(k)f_1(k)f_0(k)\rangle^L|k\rangle^R$'s. The above explanation has been for the case of $n=3$, but larger $n$ does not make much difference in the explanation.

So far we have introduced our algorithm and also given a specific explanation. We will next discuss the computational cost of our algorithm.

### 3.2 Computational cost

The run time of the algorithm is $O(m+n)$ for all the steps except for the process of applying $C_f$ in the first step. Thus the dominant factor is the circuit depth of $C_f$ implementing function $f$. This is however hidden behind a query cost in the convention of computational complexity theory \[1\]. Hence we can state that the time cost of our algorithm is $O(m+n)$.

As for space, we use only $m+2n$ qubits apart from those used inside $C_f$. Although $C_f$ may use a certain number of ancilla qubits internally, this is also hidden behind a query cost. Thus we can state that our algorithm uses $m+2n$ qubits. It should be noted that counting qubits appearing in the algorithm is quite superficial as this does not reflect the resource needed for average measurements.

As for the query cost, $f$ is called only once as a function acting on a superposition or a mixed state. Thus we use only one query in our algorithm.

The dominant cost is obviously the cost for average measurements to obtain the values of polarizations in the measurement stage. Since each intensity has the factor $1/2^m$ as shown in Eq. \[4\], exponentially many identical data is needed to obtain the signal intensity in the presence of noise. In case noise is a random one, accumulation of $L$ data results in the signal per noise ratio $\propto L/2^m : \sqrt{L}$. Hence $O(2^{2m})$ identical data are needed to overcome noise. Thus we need a bulk ensemble of $O(2^{2m})$ identical systems for ensemble averaging or $O(2^{2m})$ data acquisitions for time averaging.
4 Quantumness of correlation in the algorithm

It is often discussed if a quantum correlation like entanglement [18, 8] and quantum discord [17] is a source of computational power of quantum computers although there is no definite answer presently [8, 4]. In our algorithm, there is clearly a large amount of entanglement for non-constant f (i.e., f such that f(k) is not same for all k) when we choose a pure-state process by employing |ψo⟩ in step 1. The controversial case is when we choose a mixed-state process by employing ρo [Eq. (3)], ρ1 [Eq. (4)], and ρ2 [Eq. (5)] because they are written in the form of a sum of products of diagonal states. There is however, a certain quantumness: In Subroutine S.1, firstly the Cℋ operation is applied to each pair of f's and its corresponding ancilla qubit. This operation changes the ancilla register state to |a(k)⟩A = \( \otimes \sum_j \delta(f(k)) \otimes |k⟩L \otimes |a(k)⟩A \).

The entire state at this point is
\[
\tilde{ρ} = (1/2^n) \sum_{k=0}^{2^n-1} |f(k)⟩A \otimes |k⟩L \otimes |a(k)⟩A,
\]

where Π's are mutually orthogonal projectors acting on A.

Quantum discord \( D_\Pi^A(ρ_{AB}) \) of state \( ρ_{AB} \) is defined in the following way [17] in general for a system consisting of two subsystems A and B. Here, \( Π^A = \{ Π_j^A \} \) where \( Π_j^A \)'s are mutually orthogonal projectors acting on A.

\[
D_\Pi^A(ρ_{AB}) = S(ρ^A) - S(ρ_{AB}) + \min_{Π^A} S(ρ_{AB}|Π^A), \tag{7}
\]

where \( S(ρ_{AB}) \) is the von Neumann entropy of the density matrix of AB, \( S(ρ^A) \) is that of the reduced density matrix of A, and \( S(ρ_{AB}|Π^A) \) is a conditional entropy given by \( S(ρ_{AB}|Π^A) = \sum_j p_j S(σ_j) \) with \( p_j = \text{Tr}(Π_j^A ⊗ I^B ρ_{AB}) \) and \( σ_j = (Π_j^A ⊗ I^B ρ_{AB} Π_j^A ⊗ I^B)/p_j \). It should be noted that we employ the base-two logarithm when calculating entropies.

Our interest is the quantities \( D_\Pi^B(\tilde{ρ}) \) and \( D_\Pi^A(\tilde{ρ}) \). For an explicit calculation, we need to specify f and perform a numerical computation to find the minimum in Eq. (7). Here, a rather simple case is considered for the sake of readability of equations. Let us consider a clock function \( f_c \) such that \( f_c(k) = 0 \) for a half of k’s and \( f_c(k) = 2^{n-1} \) for the remaining k’s.

\[ 2 \text{ In general, the von Neumann entropy of a density matrix } ρ \text{ is calculated as } S(ρ) = -\text{Tr} ρ \log_2 ρ = -\sum λ \log_2 λ \text{ where } λ \text{ are the eigenvalues of } ρ. \]
Quantum digital-to-analog conversion algorithm using decoherence

First we calculate the von Neumann entropy for \( \tilde{\rho} \) and reduced density matrices \( \tilde{\rho}^{LR} \) and \( \tilde{\rho}^{A} \). It is straightforward to obtain

\[
S(\tilde{\rho}) = S(\tilde{\rho}^{LR}) = m.
\]

It is also easy to calculate \( S(\tilde{\rho}^{A}) \):

\[
S(\tilde{\rho}^{A}) = S\left( \frac{|+\rangle\langle+| + |0\rangle\langle0|}{2} \otimes |\cdots+\rangle\langle+\cdots+| \right) = S\left( \frac{|+\rangle\langle+| + |0\rangle\langle0|}{2} \right)
\]

where

\[
\lambda_{\pm} = \frac{1}{2} \pm \frac{\sqrt{m^2 - 2m}}{2}.
\]

We are now going to find the values of \( D^{\text{LR}}(\tilde{\rho}) \) and \( D^{\text{A}}(\tilde{\rho}) \). The easier one is \( D^{\text{LR}}(\tilde{\rho}) \). We have

\[
D^{\text{LR}}(\tilde{\rho}) = S(\tilde{\rho}^{LR}) - S(\tilde{\rho}) + \min_{\Pi^{LR}} S(\tilde{\rho}^{LR} \Pi^{LR}) = \min_{\Pi^{LR}} S(\tilde{\rho}^{LR} \Pi^{LR}).
\]

This is obviously \( \geq 0 \) by the property of von Neumann entropy and it is possible to achieve zero: Consider computational basis vectors \( |lr\rangle \) of LR and projectors \( \Pi_{lr}^{LR} = |lr\rangle\langle lr| \). Then \( S(\tilde{\rho}^{LR} \Pi^{LR}) \) vanishes because, for each \( lr \), the projected state \( (\Pi_{lr}^{LR} \otimes I^{A} \tilde{\rho}^{LR} \Pi_{lr}^{LR} \otimes I^{A})/\rho_{lr} \) is a pure state. Consequently,

\[
D^{\text{LR}}(\tilde{\rho}) = 0.
\]

Although this discord has vanished, the other one is shown to be nonzero as below.

We turn into the calculation of \( D^{\text{A}}(\tilde{\rho}) \). As we know

\[
D^{\text{A}}(\tilde{\rho}) = S(\tilde{\rho}^{A}) - S(\tilde{\rho}) + \min_{\Pi^{A}} S(\tilde{\rho}^{A} \Pi^{A})
\]

now we need to compute \( \min_{\Pi^{A}} S(\tilde{\rho}^{A} \Pi^{A}) \). Because there is no phase factor in the amplitudes of component states in \( \tilde{\rho} \), we can choose \( \Pi^{A} = \{|\phi_{1}\rangle\otimes|\cdots+\rangle|\cdots+\rangle, |\phi_{2}\rangle\otimes|\cdots+\rangle|\cdots+\rangle\} \) with \( |\phi_{1}\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle \) and \( |\phi_{2}\rangle = \sin \theta |0\rangle - \cos \theta |1\rangle \), and perform a minimization over \( \theta \). After a tedious but straightforward calculation, we find

\[
\min_{\Pi^{A}} S(\tilde{\rho}^{A} \Pi^{A}) = m + \min_{\theta} \left[ -H\left( \frac{(\cos \theta + \sin \theta)^2}{4} + \frac{\cos^2 \theta}{2} \right) + \frac{1}{2} H\left( \frac{(\cos \theta + \sin \theta)^2}{2} + \frac{1}{2} H(\cos^2 \theta) \right) \right],
\]

where \( H(x) = -x \log_2 x - (1-x) \log_2 (1-x) \) is the binary entropy function. We used a simple numerical search to find the minimum. The result is \( \min_{\Pi^{A}} S(\tilde{\rho}^{A} \Pi^{A}) \approx m - 0.399124 \). In addition, we have \( -\sum_{\pm} \lambda_{\pm} \log_2 \lambda_{\pm} \approx 0.600876 \). Substituting these values into Eq. (8), we obtain

\[
D^{\text{A}}(\tilde{\rho}) \approx 0.201752.
\]

Thus, we have found a nonzero quantum discord. Although there is no entanglement when we choose the mixed-state process from the first in our algorithm, a certain quantum correlation still exists. In the above calculation, we have chosen a simple clock function for \( f \) for simplicity, but in general nonvanishing quantum discord is involved owing to the nonorthogonality of \( |0\rangle \) and \( |+\rangle \) when \( f \) is a nonconstant function.
5 Concluding remarks

We have considered a parallel DAC problem handling digital data generated by a boolean function $f : \{0, 1\}^m \rightarrow \{0, 1\}^n$ ($m, n \geq 1$ are integers). This problem has been proven to be a difficult problem under a natural assumption on physical noise, as shown in Proposition 2 by reducing SAT to the problem. A unitary quantum DAC has also been proven to be hard in the sense that a polynomial-size quantum DAC circuit cannot be constructed for certain instances unless $\mathsf{NP}$ is a subset of $\mathsf{BQP}$. Thus it is unlikely to find an algorithm to perform parallel DAC using a resource scaling polynomially in $m$ and $n$ in physically feasible computational models.

On the basis of the above observation, we have developed an algorithm to perform DAC using nonunitary processes. In particular depolarization and dephasing channels have been utilized. The resultant state becomes a mixture of quantum states. A desired analog amplitude is derived from each of them by using a projection and polarization measurements. The algorithm runs with linear time, linear space, and a single query apart from the cost to pick up each amplitude from its output.

A drawback of our algorithm is the required physical resource: we need either a massive bulk-ensemble system or a many-time accumulation to derive the classical analog amplitudes. Despite this drawback, we have discussed that our algorithm is feasible for implementation using a molecular spin system when $m \lesssim 38$.

We have also discussed on quantumness of correlation among registers used in our algorithm. As we have seen, we may choose either a pure-state process or a mixed-state process for the first part of our algorithm. Entanglement exists for the former case while it does not for the latter case. The mixed-state process, however, involves quantum discord. Hence our algorithm is quantum even when a mixed-state process is employed in the sense that it uses a quantum state possessing quantum correlation.

Our strategy has been a nonstandard quantum processing using decoherence. There might be other approaches as quantum DAC has not been widely studied so far. Although its hardness as a computational problem has been proved here, it is to be hoped that different physical models and algorithmic strategies will be tried for more economical constructions.

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