ADMISSIBILITY UNDER EXTENSION OF NUMBER FIELDS

DANNY NEFTIN AND UZI VISHNE

Abstract. A finite group $G$ is $K$-admissible if there is a $G$-crossed product $K$-division algebra. In this manuscript we study the behavior of admissibility under extension of number fields $M/K$. While tame admissibility goes down, wild admissibility goes up: if $G$ is a wildly $K$-admissible $p$-group, for an odd prime $p$, having the Grunwald-Neukirch (GN) property over $M$, then $G$ is also wildly $M$-admissible. We generalize this statement to odd order groups $G$ with the GN-property over $M$, provided that the extension locally avoids a certain list of 29 sensitive extensions. On the other hand, we give an example of a quadratic extension $M/K$ with a 2-group which is wildly $K$-admissible, and not even $M$-preadmissible.

We also consider eight possible notions of being $M$-admissible with respect to a subfield $K$, where the field of definition of the division algebra, the maximal subfield or the Galois group is asserted to be $K$. We provide counterexamples for each implication between these notions which is not proved in the text.

1. Introduction

Let $M$ be a number field. A group $G$ is $M$-admissible if there is a $G$-crossed product $M$-division algebra, namely a division algebra $D$ over $M$ with a maximal subfield $L$, which is Galois over $M$ with Galois group $G$.

The basic facts about admissibility of groups, including tame and wild admissibility and preadmissibility, are reviewed in Section 2.

Assuming that a group $G$ is $M$-admissible, a natural question is what would be the field of definition of this property. Fixing a subfield $K$ of $M$, one may ask whether or not $G$ is already $K$-admissible, with a $G$-crossed product over $K$ which remains a division algebra after scalar extension to $M$. Failing this strong assumption it is still possible that $G$ is both $K$ and $M$-admissible; that the $G$-crossed product $D$ is defined over $K$ (namely, $D = D_0 \otimes_K M$ for a suitable division algebra over $K$); that $L$ is defined and Galois over $K$ (namely $L = L_0 \otimes_K M$ where $L_0/K$ is $G$-Galois); or that $L$ is merely defined over $K$.

In Section 3 we study eight variations of $M$-admissibility of a group $G$, with respect to a fixed subfield $K$ of $M$. We provide a complete list of implications between those conditions, and in Section 7 provide counterexamples to every false implication. It turns out that for a cyclic group, the eight conditions are equivalent.

For a prime $v$ of a global field $K$, we denote by $K_v$ the completion of $K$ with respect to $v$. If $L/K$ is a finite Galois extension, $L_v$ denotes the completion of $L$ with respect to some prime divisor of $v$ in $L$.

The basic criterion for admissibility over global fields is due to Schacher:

Theorem 1.1 ([11]). Let $K$ be a global field and $G$ a finite group. Then $G$ is $K$-admissible if and only if there exists a Galois $G$-extension $L/K$ such that for
every rational prime $p | |G|$, there is a pair of primes $v_1, v_2$ of $K$ such that each of $\text{Gal}(L_{v_i}/K_{v_i})$ contains a $p$-Sylow subgroup of $G$.

A $K$-admissible group $G$ is tamely $K$-admissible if there is a division algebra $D$ over $K$ with a maximal subfield $L$ which is $G$-Galois over $K$, and such that $D$ is split by the tamely ramified part of every completion of $L$. It is conjectured that any $K$-admissible group $G$ which is not tamely admissible, is necessarily wildly admissible (see Subsection 2.2 for the definition of wild admissibility and more details). Let $p$ be an odd prime. The conjecture is known to hold if $G$ is a $p$-group satisfying the Grunwald-Wang property (see Subsection 2.3).

Tame admissibility goes down: If a solvable group $G$ is tamely admissible over $M$, then it is also tamely admissible over any subfield $K$. In Section 4 we show that wild admissibility of odd $p$-groups exhibits the opposite behavior: If $G$ is a wildly $K$-admissible $p$-group that has the Grunwald-Wang property over an extension field $M$, then $G$ is also wildly $M$-admissible. On the other hand we give an example of a quadratic extension $M/K$ with a 2-group which is wildly $K$-admissible, and not even $M$-preadmissible.

A necessary condition for a group $G$ to be $k$-admissible is that it is realizable over $k$. When $k$ is a $p$-adic field, the conditions are equivalent. In Section 5 we study realizability of $p$-groups, for odd $p$, under extensions of local fields, preparing later applications to global fields. We show that, except possibly for 28 ‘sensitive’ field extensions $l/k$ for $p = 3$ and one sensitive extension for $p = 5$, every group $G$ that is realizable over $k$ has a subgroup $G_1$ which contains a $p$-Sylow subgroup of $G$ and is realizable over $l$.

Finally, in Section 6 we study admissibility of arbitrary odd order groups under global field extensions. We say that a finite extension $M/K$ of number fields is sensitive if one of its completions $M_v/K_v$ is sensitive. We prove that in a non-sensitive extension $M/K$, if an odd order group $G$ is $K$-admissible and has the Grunwald-Wang property over $M$, then it is $M$-admissible if and only if for every prime divisor $p$ of $|G|$ which has full degree in $M/Q$, the $p$-Sylow subgroup of $G$ is metacyclic and satisfies Liedahl’s condition (Definition 2.6) over $M$.

2. Preliminaries

2.1. Admissible groups.

**Definition 2.1.** Let $L/K$ be a finite extension of fields. The field $L$ is $K$-adquate if it is a maximal subfield in some division algebra whose center is $K$.

Given a group $G$, we shall say $L/K$ is a $G$-extension if $L/K$ is a Galois extension with Galois group $\text{Gal}(L/K) \cong G$.

**Definition 2.2.** Let $K$ be a field and let $G$ be a finite group. The group $G$ is $K$-admissible if there exists a $K$-adequate Galois $G$-extension $L/K$.

Extracting the necessary local conditions for $K$-admissibility from Theorem 1.1, we arrive at the following definition. For a group $G$, $G(p)$ denotes a $p$-Sylow subgroup.
Definition 2.3. Let $K$ be a number field. The group $G$ is $K$-preadmissible if $G$ is realizable over $K$, and there exists a finite set $S = \{v_i(p) : p \mid |G|, i = 1, 2\}$ of primes of $K$, and, for each $v \in S$, a subgroup $G^v \leq G$, such that

1. $v_1(p) \neq v_2(p)$,
2. $G^{v_i(p)} \supseteq G(p)$ for every $p$ and $i = 1, 2$, and
3. $G^v$ is realizable over $K_v$ for every $v \in S$.

(Notice that a $p$-group $G$ is $K$-preadmissible if and only if there is a pair of primes $v_1$ and $v_2$ of $K$, such that $G$ is realizable over $K_{v_1}$ and over $K_{v_2}$.)

Clearly, every $K$-admissible group is also $K$-preadmissible. However the opposite does not hold.

For an extension of fields $L/K$, $\text{Br}(L/K)$ denotes the kernel of the restriction map $\text{res} : \text{Br}(K) \to \text{Br}(L)$. For number fields we have the following isomorphism of groups, where $\Pi_K$ is the set of primes of $K$:

$$\text{Br}(L/K) \cong \left( \bigoplus_{\pi \in \Pi_K} \frac{1}{\gcd_{\pi'}|\pi} \left[ \frac{L_{\pi'}:K_{\pi}}{Z/Z} \right] \right)_0,$$

where $(\cdot)_0$ denotes that the sum of invariants is zero.

2.2. Tame and wild admissibility. We denote by $k_{\text{un}}$ the maximal unramified extension of a local field $k$, and by $k_{\text{tr}}$ the maximal tamely ramified extension.

Definition 2.4. The tamely ramified subgroup $\text{Br}(L/K)_{\text{tr}}$ of $\text{Br}(L/K)$ is the subgroup of algebras which are split by the tamely ramified part of every completion of $L$; namely the subgroup corresponding under the above isomorphism to

$$\left( \bigoplus_{\pi \in \Pi_K} \frac{1}{\gcd_{\pi'}|\pi} \left[ \frac{L_{\pi'} \cap (K_{\pi})_{\text{tr}}:K_{\pi}}{Z/Z} \right] \right)_0.$$

Over a number field $K$, the exponent of a division algebra is equal to its degree, and so $L$ is $K$-adequate if and only if there is an element of order $[L : K]$ in $\text{Br}(L/K)$ (\cite[Proposition 2.1]{*}). Let us refine this observation:

Definition 2.5. We say that a finite extension $L$ of $K$ is tamely $K$-adequate if there is an element of order $[L : K]$ in $\text{Br}(L/K)_{\text{tr}}$.

Likewise, a finite group $G$ is tamely $K$-admissible if there is a tamely $K$-adequate Galois $G$-extension $L/K$.

Let $\mu_n$ denote the set of $n$-th roots of unity in $\mathbb{C}$. For $t$ prime to $n$, let $\sigma_{t,n}$ be the automorphism of $\mathbb{Q}(\mu_n)/\mathbb{Q}$ defined by $\sigma_{t,n}(\zeta) = \zeta^t$ for $\zeta \in \mu_n$.

Definition 2.6. We say that a metacyclic $p$-group satisfies Liedahl’s condition (first defined in \cite{*}) with respect to $K$, if it has a presentation

$$\mathcal{M}(m, n, i, t) = \langle x, y \mid x^m = y^i, y^n = 1, x^{-1}yx = y^t \rangle$$

such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$.

This condition implies $K$-tame-preadmissibility of $G = \mathcal{M}(m, n, i, t)$ (namely, Definition 2.3 holds with realizability within $(K_v)_{tr}$ in 2.3(3)).
Remark 2.7. (Liedahl, [4]) If \( G \) is tamely \( K \)-admissible, then, for every \( p \mid |G| \), the \( p \)-Sylow subgroup \( G(p) \) is metacyclic and satisfies Liedahl’s condition (this follows from the structure of \( \text{Gal}(k_{tr}/k) \), see below). There are no known counterexamples to the opposite implication.

Remark 2.8. Note that if a metacyclic \( p \)-group \( G \) satisfies Liedahl’s Condition over \( M \) which is an extension of \( K \) then \( G \) satisfies Liedahl’s Condition over \( K \).

The following result on realizability of metacyclic \( p \)-groups will be used in Section 6.

Lemma 2.9. Let \( k \) be a \( p \)-adic field. Then the metacyclic \( p \)-group \( G = M(m, n, i, t) \) (see \( (2.2) \)) is realizable over \( k \).

Proof. The proof for \( k \neq \mathbb{Q} \) is in [6]. For \( k = \mathbb{Q}_2 \) we cover 2-groups, so \( m \) and \( n \) are 2-powers and \( t \) is odd. In this case \( G_k(2) \) has the topological presentation \( \langle a, b, c \mid a^2 b^4 [b, c] = 1 \rangle \) (2), i.e. \( G_k(2) \) is isomorphic to the free pro-2 group on three generator modulo the normal closure of the single relation. So the map \( \phi : G_k(2) \to G \) defined by:

\[ a \mapsto x^{-2} y^s, \ b \mapsto x, \ c \mapsto y, \]

is well defined (and surjective) whenever \( (x^{-2} y^s)^2 x^4 [x, y] = 1 \).

Note that as \( t \) is odd, \( t^2 + 1 \equiv 2 \pmod{4} \), \( \frac{t^2 + 1}{2} \) is odd and we can choose an \( s \equiv \frac{1}{\sqrt{2}} \frac{t^2 + 1}{2} \pmod{n} \) so that \( s(t^2 + 1) \equiv \frac{1}{\sqrt{2}} \pmod{n} \). For such \( s \) one can check that:

\[ (x^{-2} y^s)^2 x^4 [x, y] = x^{-4} y^{st^2 + s} x^4 y^{t-1} = y^{s(t^2+1)t^2+t-1} = 1. \]

Thus \( \phi \) is well defined. \( \square \)

The following result is proved for \( p \)-groups in [4, Theorem 30], and in general in [5, Theorem 3.24].

Theorem 2.10. Let \( K \) be a number field and let \( G \) be a solvable group with metacyclic Sylow subgroups. Then \( G \) is tamely \( K \)-admissible if and only if its Sylow subgroups satisfy Liedehal’s condition.

Moreover, if \( S \) is a finite set of primes of \( K \) and the Sylow subgroups of \( G \) satisfy Liedehal’s Condition, then there is a \( K \)-adequate Galois \( G \)-extension \( N/K \) for which every \( v \in S \) splits completely in \( N \).

Remark 2.11. In the proof of Theorem 2.10 there is a construction of a \( \mathbb{Q} \)-division algebra \( D_0 \) that has a maximal subfield \( L_0 \) so that

1. \( L_0/\mathbb{Q} \) is a \( G \)-extension disjoint to \( K \), i.e. \( L := L_0 K \) is also a \( G \)-extension,
2. \( D := D_0 \otimes \mathbb{Q} K \) remains a division algebra.

Thus, not only \( G \) is \( K \)-admissible but there is also a \( G \)-crossed product division algebra \( D \) and a maximal subfield \( L \) so that both are defined compatibly over \( \mathbb{Q} \).

Corollary 2.12. Let \( K \) be a number field. Let \( G \) be a solvable group such that the rational prime divisors of \( |G| \) do not decompose in \( K \) (have a unique prime divisor in \( K \)). Then \( G \) is \( K \)-admissible if and only if its Sylow subgroups are metacyclic and satisfy Liedehal’s condition.
So far we have discussed tame admissibility. As mentioned earlier \( \mathbb{Q} \)-admissibility is often tame. However over larger number fields this is no longer the case. Let us define wild \( K \)-admissibility:

**Definition 2.13.** A \( G \)-extension \( L/K \) is non-wildly \( K \)-adequate if \( L/K \) is \( K \)-adequate and there is a set \( S = \{ v_i(p) : i = 1, 2, p \mid |G| \} \) of primes of \( K \) such that for every prime \( p \mid |G| \),

1. \( v_1(p) \neq v_2(p) \),
2. both \( v_1(p) \) and \( v_2(p) \) do not divide \( p \), and
3. \( \text{Gal}(L_{v_i(p)}/K_{v_i(p)}) \supseteq G(p) \).

A \( K \)-admissible group \( G \) will be called non-wildly \( K \)-admissible if it has a \( K \)-adequate \( G \)-extension which is non-wildly \( K \)-adequate.

As a counterpart, we have:

**Definition 2.14.** A \( G \)-extension \( L/K \) is wildly \( K \)-adequate if \( L/K \) is \( K \)-adequate and for every set \( T = \{ v_i(p) \mid i = 1, 2, p \mid |G| \} \) of primes of \( K \) for which \( \text{Gal}(L_{v_i(p)}/K_{v_i(p)}) \supseteq G(p) \), there is a prime \( q \mid |G| \) for which \( v_1(q), v_2(q) \mid q \).

A \( K \)-admissible group \( G \) will be called wildly \( K \)-admissible if every \( K \)-adequate \( G \)-extension is wildly \( K \)-adequate.

Note that an extension \( L/K \) (resp. a group \( G \)) is wildly \( K \)-adequate (resp. \( K \)-admissible) if \( L/K \) (resp. \( G \)) is \( K \)-adequate (resp. \( K \)-admissible) but not non-wildly \( K \)-adequate (resp. \( K \)-admissible).

**Remark 2.15.** 1. Theorem 2.10 guarantees that a solvable group which is non-wildly \( K \)-admissible is tamely \( K \)-admissible.
2. In particular, every \( K \)-admissible \( p \)-group which is not metacyclic, is wildly \( K \)-admissible.

2.3. **The GN-property.** Notice that if \( M/K \) is an extension of number fields and \( v \) is a prime of \( M \) (viewed also as a prime of \( K \) by restriction), then \( \text{Gal}(M_v/K_v) \) can naturally be viewed as a subgroup of \( \text{Gal}(M/K) \). A group \( G \) has the GN-property (named after Grunwald-Neukirch) over a number field \( K \) if for every finite set \( S \) of primes of \( K \) and corresponding subgroups \( G^v \leq G \) for \( v \in S \), there is a Galois \( G \)-extension \( M/K \) for which \( \text{Gal}(M_v/K_v) \cong G^v \) for every \( v \in S \). Let us give some examples.

**Definition 2.16.** A finite group is of class \( C \) if it is trivial, or, inductively, a split extension of a group in \( C \) by a cyclic one.

All groups in \( C \) are solvable, but not necessarily nilpotent. Likewise there are nilpotent metacyclic groups that are not in \( C \).

For any field \( K \), let \( G_K \) denote the absolute Galois group of \( K \). The following is a corollary (to Theorems 6.4(b) and 2.5 of [3]) that appears in [5]:

**Corollary 2.17.** Let \( K \) be a number field. Then every group \( G \) in \( C \) of odd order has the GN-property over \( K \). Moreover, for any finite set \( S \) of primes of \( K \) the map

\[
\theta_G : \text{Hom}(G_K, G)_{\text{sur}} \to \prod_{v \in S} \text{Hom}(G_{K_v}, G),
\]
is surjective. Here \(\text{Hom}(G_K,G)_{\text{sur}}\) denotes the set of surjective homomorphisms.

A large set of examples comes from the following Theorem of Neukirch \([7]\) Corollary 2. Let \(m(K)\) denote the number of roots of unity in a number field \(K\).

**Theorem 2.18.** (Neukirch, \([7]\)) Let \(K\) be a number field and \(G\) a group for which \(|G|\) is prime to \(m(K)\). Then \(G\) has the GN-property over \(M\). Moreover the map \(\theta_G\) from Corollary 2.17 is again an epimorphism for every finite set \(S\) of primes of \(K\).

### 2.4. Galois groups of local fields.

Let \(k\) be a \(p\)-adic field of degree \(n\) over \(\mathbb{Q}_p\). Let \(q\) be the size of the residue field \(\overline{k}\), and let \(p^s\) be the size of the group of \(p\)-power roots of unity inside \(k_\text{nr}\). Then

1. \(\text{Gal}(k_{\text{nr}}/k)\) is (topologically) generated by an automorphism \(\sigma\), and isomorphic to \(\mathbb{Z}/p\mathbb{Z}\);
2. \(\text{Gal}(k_{\text{nr}}/k_{\text{un}})\) is (topologically) generated by an automorphism \(\tau\), isomorphic to \(\mathbb{Z}/p^s\mathbb{Z}\) (which is the complement of \(\mathbb{Z}_p\) in \(\hat{\mathbb{Z}}\));
3. The group \(\text{Gal}(k_{\text{nr}}/k)\) is a pro-finite group generated by \(\sigma\) (lifting the above mentioned automorphism) and \(\tau\), subject to the single relation \(\sigma^{-1}\tau\sigma = \tau^q\).

Moreover, \(\sigma\) and \(\tau\) act on \(\mu_{p^n}\) by exponentiation by some \(g \in \mathbb{Z}_p\) and \(h \in \mathbb{Z}_p\), respectively (Note that \(g\) and \(h\) are well defined modulo \(p^s\)).

Let \(\hat{G}_k(p)\) denote the Galois group of the maximal \(p\)-extension of \(k\) inside \(\overline{k}\), over \(k\). Let \(s_0\) be the maximal number such that \(k\) contains roots of unity of order \(p^{s_0}\). Note that if \(s_0 > 0\) then \(n\) must be even.

**Theorem 2.19** (\([12]\) Section II.5.6). When \(p^{s_0} \neq 2\), \(\hat{G}_k(p)\) have the following presentation of pro-\(p\) groups:

\[
\hat{G}_k(p) \cong \left\{ \langle x_1, \ldots, x_{n+2} \mid x_1^{p^{s_0}}[x_1, x_2] \cdots [x_{n+1}, x_{n+2}] = 1 \rangle, \quad \text{if } s_0 > 0 \right\} \cup \left\{ \langle x_1, \ldots, x_{n+1} \rangle, \quad \text{if } s_0 = 0 \right\}.
\]

When \(p^{s_0} = 2\) and \(n\) is odd,

\[
\hat{G}_k(p) \cong \langle x_1, \ldots, x_{n+2} \mid x_1^2 x_2^3 [x_2, x_3] \cdots [x_{n+1}, x_{n+2}] = 1 \rangle.
\]

(see \([2]\) for the remaining cases, which are given in \([12]\) Section II.5.6] as well).

**Theorem 2.20** (\([1]\), see also \([3]\) Theorem 7.5.10). The group \(G_k\) has the following presentation (as a profinite group):

\[
G_k = \langle \sigma, \tau, x_0, \ldots, x_n \mid \tau^q = \tau^q, x_0^q = (x_0, \tau)^g x_1^{p^r} [x_1, x_2] \cdots [x_{n-1}, x_n] \rangle
\]

if \(n\) is even, and

\[
G_k = \langle \sigma, \tau, x_0, \ldots, x_n \mid \tau^q = \tau^q, x_0^q = (x_0, \tau)^g x_1^{p^r} [x_1, y_1] [x_2, x_3] \cdots [x_{n-1}, x_n] \rangle
\]

if \(n\) is odd. Here, the group topologically generated by \(\sigma\) and \(\tau\) is isomorphic to \(\text{Gal}(k_{\text{nr}}/k)\) and the group topologically generated by \(x_0, \ldots, x_n\) is a pro-\(p\) group. The notation \(\langle x_0, \tau \rangle\) stands for \((x_0^{h_0^{-1}} \tau x_0^{h_0^{-1}} \tau \cdots x_0^{h_0^{-1}} \tau)^{\pi_{p^s}}\), where \(\pi_p \in \hat{\mathbb{Z}}\) is an element such that \(\pi \hat{\mathbb{Z}} = \mathbb{Z}_p\). Also, \(y_1\) is a multiple of \(x_1^{\pi_{p^s}(p+1)}\) by an element in the maximal pro-\(p\) quotient of the pro-finite group generated by \(x_1, \sigma^{\pi_2}\) and \(\tau^{\pi_2}\). In particular, in every pro-odd quotient of \(G_k\), \([x_1, y_1]\) is trivial.
3. CONDITIONS ON THE FIELD OF DEFINITION

Let $K$ be a global field, $M/K$ be a finite extension and $G$ a finite group. One way to study the condition

$$(1) \ G \text{ is } M\text{-admissible}$$

is by refining it to require that the field extension or the division algebra be defined over $K$ (we say that a field or an algebra over $M$ is defined over $K$ if it is obtained by extending scalars from $K$ to $M$). We consider the following variants.

$$(2) \text{ there is an } M\text{-adequate Galois } G\text{-extension } L/M \text{ for which } L \text{ is defined over } K,$$

$$(3) \text{ there is an } M\text{-adequate Galois } G\text{-extension } L/M \text{ so that } L = L_0 \otimes M \text{ and } \text{Gal}(L_0/K) = G,$$

$$(4) \text{ there is a } K\text{-division algebra } D_0 \text{ and a Galois } G\text{-extension } L/M \text{ for which } L \text{ is a maximal subfield of } D_0 \otimes M,$$

$$(5) \text{ there is a } K\text{-division algebra } D_0 \text{ and a maximal subfield } L_0 \text{ which is a Galois } G\text{-extension of } K \text{ so that } L_0 \cap M = K \text{ and } L = L_0 M \text{ is a maximal subfield of the division algebra } D = D_0 \otimes M,$$

$$(6) \text{ there is a } K\text{-adequate Galois } G\text{-extension } L_0/K \text{ for which } L_0 M \text{ is an } M\text{-adequate Galois } G\text{-extension},$$

$$(7) \ G \text{ is both } K\text{-admissible and } M\text{-admissible},$$

and finally

$$(8) \text{ there is a } K\text{-adequate } G\text{-extension } L_0/K \text{ for which } L_0 M \text{ is an } M\text{-adequate Galois } G\text{-extension.}$$

We provide a diagrammatic description of each condition, for easy reference. Inclusion is denoted by a vertical line, and diagonal lines show the extension of scalars from $K$ to $M$. A vertical line is decorated by $G$ if the field extension is $G$-Galois. Note that in some cases (3), (5) and (6) the fact that the extension over $K$ is Galois implies the same condition on the extension over $M$.

\[
\begin{array}{cccc}
D & D & D & D \\
L & L & L & D_0 \\
| & | & | & | \\
G & G & G & | \\
M & M & M & M \\
| & | & | & | \\
L_0 & L_0 & L & L \\
M & M & M & M \\
| & | & | & | \\
K & K & K & K \\
\end{array}
\]

(1) (2) (3) (4)
Theorem 3.1. Let $M/K$ be a finite extension of fields and $G$ a finite group. Then the implications in Diagram (3.1) hold, but no others.

\[(3.1)\]

Some implications hold only for wild $K$-admissibility; this will be discussed in Sections 4–6. In Section 7 we give counterexamples for the false implications, with $G$ being a $p$-group in each case.

We shall say that a triple $(K, M, G)$ satisfies Condition (m) if there are $L_0, L, D_0$ and $D$ as required in this condition. In such case we shall also say $(L_0, L, D_0, D)$ realizes Condition (m), omitting $L_0$ or $D_0$ if they are not needed.

Remark 3.2. Let $M/K$ be a finite extension of global fields and $G$ a finite group. Consider the condition

(9) there is a $G$-crossed product $K$-division algebra $D_0$, for which $D = D_0 \otimes M$ is also a $G$-crossed product division algebra.

In the spirit of previous diagrams, this condition can be described by

\[(9)\]

However, (9) is equivalent to Condition (5).

Indeed, let $(L_0, L, D_0, D)$ realize (9). Then $D$ is of index $|G|$ and $D$ is also split by $L' = ML_0$. Therefore $[L': M] = |G|$, $L_0 \cap M = K$ and hence we can take $L'$ to
be the required maximal $G$-subfield of $D$. Thus, $(L_0, L', D_0, D)$ realizes \(5\). The converse implication is obvious, taking $L = L_0 \otimes_K M \subset D_0 \otimes_K M = D$.

Let us go over the immediate implications in diagram \(3.1\).

**Positive part of Theorem \(3.1\).** Fix $K, M, G$. Clearly if $(L_0, L, D_0, D)$ realizes Condition \(3\), $(L_0, L, D_0, D)$ also realizes \(5\) and $(L, D_0, D)$ realizes \(4\), so that \(2 \Rightarrow 1, 3\).

If $(L_0, L, D_0, D)$ realizes \(1\) then $L_0/K$ is a $G$-extension and hence $L = L_0 M/M$ is also a $G$-extension (since $L_0 \cap M = K$). Thus, $(L_0, L, D_0, D)$ realizes \(8\). It is clear that $L_0$ is a field of definition of $L$ (and $\Gal(L_0/K) = G$) and hence $(L_0, L, D)$ realizes \(3\). As $L_0$ is a $K$-adequate $G$-extension and $L$ is an $M$-adequate $G$-extension, $(L_0, L, D_0, D)$ realizes \(4\). Therefore \(6 \Rightarrow 3, 7, 5\).

If $(L_0, L, D)$ realizes Condition \(3\) then $\Gal(L_0/K) = G$, $\Gal(L/M) = G$ (since $L_0 \cap M = K$) and hence $(L_0, L, D_0, D)$ realizes Condition \(2\). If $(L_0, L, D_0, D)$ realizes condition \(8\), clearly $L_0$ is a field of definition of $L$ and hence $(L_0, L, D)$ realizes Condition \(2\).

Clearly when $(K, M, G)$ satisfies either of the conditions \(2, 1, 7\), $G$ is $M$-admissible and hence \(2 \Rightarrow 1\). \(\square\)

**Proposition 3.3.** Let $G$ be a cyclic group. Then Conditions \(1\)–\(8\) are satisfied for any extension of global fields $M/K$.

**Proof.** It is sufficient to show that \(5\) is satisfied. By Chebotarev density Theorem (applied to the Galois closure of $M/K$) there are infinitely many primes $v$ of $K$ that split completely in $M$. Let $v_1, v_2$ be two such primes that are not divisors of 2. By the weak version (prescribing degrees and not local extensions) of the Grunwald-Wang Theorem (see [15]) there is a Galois $G$-extension $L_0/K$ for which $\Gal((L_0)_{v_i}/K_{v_i}) = G$ and thus $L_0$ is $K$-adequate, so there is a division algebra $D_0$ containing $L_0$ as a maximal subfield, and supported by $\{v_i : i = 1, 2\}$. As $v_i$ split completely in $M$ we have $L = L_0 M$ satisfies $\Gal(L_{v_i}/M_{v_i}) = G$ for $i = 1, 2$ and hence $\Gal(L/M) = G$. Finally $D = D_0 \otimes M$ is a division algebra by the choice of the $v_i$. Thus $L$ is $M$-adquate and $(K, M, G)$ satisfies \(5\). \(\square\)

The conditions above can also be considered with respect to tame $K$-admissibility. Let $G$ be a solvable group and $K, M$ number fields. By Remark 2.11 if $G$ is tamely $M$-admissible then there is a tamely $K$-adequate Galois $G$-extension $L_0/K$ for which $L = L_0 M$ is $M$-adequate (and hence tamely $M$-adequate). For $m = 1, \ldots, 8$, let $(m^*)$ denote the condition $(m)$, where every adequate extension is assumed to be tamely adequate, and an admissible group is assumed tamely admissible. More precisely for $m = 4, 5$ we consider

\(1\) there is a $K$-division algebra $D_0$ and a Galois $G$-extension $L/M$ for which $[D] = [D_0 \otimes M] \in \Br(L/M)_{tr}$ and $L$ is a maximal subfield of $D$,

and

\(5\) there is a $K$-division algebra $D_0$ and a maximal subfield $L_0$ which is a Galois $G$-extension of $K$ so that $L_0 \cap M = K$, $D_0 \in \Br(L_0/K)_{tr}$ and $L = L_0 M$ is a maximal subfield of $D = D_0 \otimes M$ (and hence $[D] \in \Br(L/M)_{tr}$).

**Corollary 3.4.** Let $G$ be a solvable group and let $M/K$ be a finite extension of number fields. Then the conditions $(1^*), (8^*)$ are all equivalent.
Proof. With the added conditions the implications given in (3.1) clearly continue to hold. But by Remark 2.11 the implication \((1^* ) \Rightarrow (5^* )\) also holds.

(What is shown there is that for every solvable group that satisfies Liedahl’s condition for every \(p\), there is a \(G\)-extension \(L_0/\mathbb{Q}\) and a division algebra \(D_0 \in \text{Br}(L_0/\mathbb{Q})_{tr}\) over \(\mathbb{Q}\), such that \(D_0 \otimes K\) contains \(L_0 \otimes \mathbb{Q}K\) as a maximal subfield). □

4. NILPOTENT GROUPS

A nilpotent group is a direct product of its Sylow subgroups. At first we shall note that this decomposition is compatible with admissibility and with any of the conditions \((1)–(8)\). Throughout this section, \(G\) will denote a nilpotent group (or, oftentimes, a \(p\)-group).

Remark 4.1. Let \(|G| = p_1^{r_1} \cdots p_k^{r_k}\) and \(G = P_1 \times \cdots \times P_k\) where \(|P_i| = p_i^{r_i}\) for \(i = 1, \ldots, k\). Then for any \(m = 1, \ldots, 8\), \((K, M, G)\) satisfies Condition \((m)\) if and only if \((K, M, P_i)\) satisfies Condition \((m)\) for all \(i = 1, \ldots, k\).

This follows from the following simple observations:

1) For any \(G\)-extension \(T/F\), \(T\) decomposes as \(T_1 \cdots T_k\) so that \(\text{Gal}(T_i/F) = P_i\). Vise versa, given a \(P_i\)-extension \(T_i/F\) for each \(i = 1, \ldots, k\), so that \(T_j \cap (T_1 \cdots \hat{T}_j \cdots T_k) = F\), one has \(\text{Gal}(T_1 \cdots T_k/F) = G\).

2) Any division algebra \(D\) decomposes (uniquely) as \(D_1 \otimes \cdots \otimes D_k\) with \(\text{ind}(D_i) = p_i^{r_i}\). The field \(T\) splits \(D\) if and only if \(T_i\) splits \(D_i\) for all \(i = 1, \ldots, k\).

Thus, one has that:

Proposition 4.2. For every \(m = 1, \ldots, 8\), a nilpotent group \(G\) satisfies Condition \((m)\) with respect to an extension \(M/K\), if and only if each of its Sylow subgroups satisfies this condition.

Proof. Let \(G = P_1 \cdots P_k\) be a decomposition into \(p\)-groups. By Remark 4.1, we have:

(a) \(G\) is \(F\)-admissible if and only if each \(P_i\) is \(F\)-admissible,
(b) \(T/F\) is defined over \(E \subset F\), if and only if each \(T_i/F\) is defined over \(E\), and
(c) \(T\) is obtained by scalar extension from \(E\) to \(F\) of a Galois extension \(T_0/E\) if and only if each \(T_i/F\) is obtained in a similar manner, and
(d) \(D \in \text{Im}(\text{res}_E^F)\) if and only if each \(D_i \in \text{Im}(\text{res}_E^{F_i})\).

This proves the claim, since conditions \((1)–(8)\) only use the terms treated in (a)–(d).

We note that admissibility of \(p\)-groups behave nicely under extensions. In particular, we shall prove:

Proposition 4.3. Let \(p\) be an odd rational prime. Let \(M/K\) be an extension of number fields and \(G\) a wildly \(K\)-admissible \(p\)-group that has the GN-property over \(M\). Then \(G\) is also wildly \(M\)-admissible.

(Following Proposition 4.2 one may assume \(G\) is any odd order nilpotent group).

The proof is based on the following observation:

Lemma 4.4. Let \(p\) be an odd prime, \(G\) a \(p\)-group and \(l/k\) a finite extension of \(p\)-adic fields. If \(G\) is realizable over \(k\) then it is also realizable over \(l\).
Proof. Let $n$ denote the rank $[k : \mathbb{Q}_p]$ and let $t = [l : k]$. If $t = 1$ there is nothing to prove. For $t = 2$, $(l : k, p) = 1$ and hence from a $G$-extension $m/k$ we can form a $G$-extension $m/l$. Now let $t > 2$. Let $\overline{G_k(p)}$ be the Galois group of the maximal $p$-extension of $k$. It is enough to show that $\overline{G_k(p)}$ is a quotient of $\overline{G_l(p)}$.

By Theorem 2.19, $\overline{G_k(p)}$ and $\overline{G_l(p)}$ have the following presentation of pro-$p$ groups:

\[
\overline{G_k(p)} \cong \left\{ \langle x_1, \ldots, x_{n+2} \mid x_1^{p^0} [x_1, x_2] \cdots [x_{n+1}, x_{n+2}] = 1 \rangle, \text{ if } s_0 > 0 \text{, if } s_0 = 0 \right\},
\]

and

\[
\overline{G_l(p)} \cong \left\{ \langle x_1, \ldots, x_{nt+2} \mid x_1^{p^{s_0}} [x_1, x_2] \cdots [x_{nt+1}, x_{nt+2}] = 1 \rangle, \text{ if } s_0' > 0 \text{, if } s_0' = 0 \right\},
\]

where $p^0$ and $p^{s_0}$ are the numbers of $p$-power roots of unity in $k$ and $l$, respectively. Clearly $s_0 \leq s_0'$. If $s_0' = 0$ then we are done since $\langle x_1, \ldots, x_{n+1} \rangle$ is a quotient of $\langle x_1, \ldots, x_{nt+1} \rangle$.

Suppose $s_0' > 0$. Let $F_p(y_1, \ldots, y_k)$ denote the free pro-$p$ group of rank $k$ with generators $y_1, \ldots, y_k$. Consider the epimorphism $\phi : \overline{G_l(p)} \to F_p(y_1, \ldots, y_{nt+2})$ that sends $\phi(x_{2i-1}) = 1$ and $\phi(x_{2i}) = y_i$, $i = 1, \ldots, nt+2$. Now as $t > 2$ we have:

\[
\frac{nt + 2}{2} = \frac{n + t}{2} + 1 \geq n + 2
\]

and hence there is a projection $\pi$:

\[
\pi : F_p(y_1, \ldots, y_{nt+2}) \to \overline{G_k(p)}.
\]

Thus $\pi \circ \phi : \overline{G_l(p)} \to \overline{G_k(p)}$ is an epimorphism. We deduce that every epimorphic image of $\overline{G_k(p)}$ is also an epimorphic image of $\overline{G_l(p)}$. $\square$

We can now prove the proposition.

Proof of Proposition 4.3. Since the $p$-group $G$ is wildly $K$-admissible, $G$ is realizable over $K_{v_i}$ for two primes $v_1, v_2$ of $K$ which divide $p$. Let $w_1, w_2$ be divisors of $v_1, v_2$ in $M$, respectively. Then by Lemma 4.4, $G$ is realizable over $M_{w_1}, M_{w_2}$ and hence $M$-preadmissible. As $G$ has the GN-property, $G$ is also $M$-admissible.

By Remark 2.15(2), the wild $K$-admissibility of $G$ implies that $G$ is not realizable over $K_v$ for any prime $v$ which is not a divisor of $p$. By Remark 2.8, $G$ is also not realizable over $M_w$ for any prime $w$ of $M$ which is not a divisor of $p$. Therefore an $M$-adequate $G$-extension must also be wildly $M$-adequate and hence $G$ is wildly $M$-admissible. $\square$

Remark 4.5. As Theorem 2.10 shows, in case of tame admissibility it is not necessary that a tamely $K$-admissible group will also be tamely $M$-admissible. In fact the converse implication holds. Comparison of Liedahl’s condition over $K$ and over $M$ shows that a $p$-group which is tamely $M$-admissible is also tamely $K$-admissible. $\square$

Remark 4.6. Let $K$ be a global field with characteristic $p$. By [14], every $p$-group is $K$-admissible. Thus, Proposition 4.3 also holds for fields of characteristic $p$ in the following sense: any $p$-group $G$ that is $K$-admissible is also $M$-admissible.
If we apply Theorem 2.18 to get a GN-property over \(M\), we have:

**Corollary 4.7.** Let \(p\) be an odd prime. Let \(M/K\) be an extension of number fields so that \(M\) does not contain the \(p\)-roots of unity. Let \(G\) be a \(p\)-group that is wildly \(K\)-admissible. Then \(G\) is wildly \(M\)-admissible.

### 4.1. The case \(p = 2\)

We show that Lemma 4.4 fails for \(p = 2\). Let \(k = \mathbb{Q}_2\), and let \(l = \mathbb{Q}_2(\sqrt{-1})\). The Galois groups of the maximal pro-2 extensions of \(k\) and \(l\) are

\[
\overline{G}_k(2) = \langle a, b, c \mid a^2b^4|b,c| = 1 \rangle
\]

and

\[
\overline{G}_l(2) = \langle x_0, x_1, x_2, x_3 \mid x_0^4[x_0, x_1][x_2, x_3] = 1 \rangle
\]

(see Theorem 2.19).

We give an example of a group (of order \(2^{10}\)) which is a quotient of \(\overline{G}_k(2)\) but not of \(\overline{G}_l(2)\). Thus, as a Galois group, it can be realized over \(k\) but not over \(l\).

Let

\[
G = \langle a, b, c \mid \gamma = [b, a], \beta = [c, a], \alpha = [c, b], a^2 = \beta^2 = \gamma^2 = 1, \alpha, \beta, \gamma \text{ are central}, a^2 = \alpha, b^8 = c^8 = 1 \rangle.
\]

Replacing \(a\) by \(ab^{-2}\), we get the presentation \(\overline{G}_k(2) = \langle a, b, c \mid ab^{-2}ab^2|b,c| = 1 \rangle\). That \(G\) is an image of \(\overline{G}_k(2)\) is seen by noting that \(b^2\) is central in \(G\), so the only relation in \(\overline{G}_k(2)\) translates to \(a^2 = \alpha\). The center of \(G\) is \(\langle \alpha, \beta, \gamma, b^2, c^2 \rangle\), so this group is a central extension of \(G/Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^3\) by \((\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/4\mathbb{Z})^2\). In particular every element of \(G\) has the form \(a^ib^jc^kz\) for some central element \(z\), and \(z^4 = 1\) for every central \(z\).

**Proposition 4.8.** There is no epimorphism from \(\overline{G}_l(2)\) to \(G\).

**Proof.** Let \(\phi: \overline{G}_l(2) \to G\) be an epimorphism. For \(t = 0, 1, 2, 3\), write \(\phi(x_t) = a^{i_t}b^{j_t}c^{k_t}z_t\) where \(z_t \in Z(G)\) and \(i_t, j_t, k_t\) are integers. Projecting further to \(G/Z(G)\), we see that the vectors \((i_t, j_t, k_t) \in (\mathbb{Z}/2\mathbb{Z})^3\), \(t = 0, 1, 2, 3\), must be a spanning set.

Compute that \(x_0^4 \mapsto a^{2i_t}b^{j_t}c^{k_t}z_t^2\), and iterate to get

\[
x_0^4 \mapsto a^{4i_0}b^{j_0}c^{4k_0}z_0^4 = b^{4j_0}c^{4k_0}.
\]

We also have that \([x_0, x_1] \mapsto [a^{i_0}b^{j_0}c^{k_0}, a^{i_1}b^{j_1}c^{k_1}] = a^{k_0j_1-j_0k_1}b^{k_0i_1-i_0k_1}c^{j_0i_1-i_0j_1}\) and similarly for \([x_2, x_3]\), so the single relation of \(\overline{G}_l(2)\) maps to the central element

\[
a^{k_0j_1-j_0k_1+k_2j_3-j_2k_3}b^{k_0i_1-i_0k_1+k_2i_3-i_2k_3}c^{j_0i_1-i_0j_1+j_2i_3-i_2j_3}.
\]

Working modulo 2 from now on, all exponents must be zero, and so we have that

\[
k_0j_1-j_0k_1+k_2j_3-j_2k_3 = 0
\]

\[
k_0i_1-i_0k_1+k_2i_3-i_2k_3 = 0
\]

\[
j_0i_1-i_0j_1+j_2i_3-i_2j_3 = 0
\]

and \(j_0 = k_0 = 0\). Substituting, we get

\[
k_2j_3 = j_2k_3
\]

\[
i_0k_1 = k_2i_3 + i_2k_3
\]

\[
i_0j_1 = j_2i_3 + i_2j_3
\]
Now there are two options. If \( i_0 = 0 \) then
\[
\begin{vmatrix}
 i_1 & j_1 & k_1 \\
 i_2 & j_2 & k_2 \\
 i_3 & j_3 & k_3 \\
\end{vmatrix} = i_1 \begin{vmatrix} j_2 & k_2 \\ j_3 & k_3 \end{vmatrix} + j_1 \begin{vmatrix} i_2 & k_2 \\ i_3 & k_3 \end{vmatrix} + k_1 \begin{vmatrix} i_2 & j_2 \\ i_3 & j_3 \end{vmatrix} = 0,
\]
and the vectors span at most a 2-dimensional subspace. If \( i_0 = 1 \) then \( \begin{vmatrix} j_1 \\ k_1 \end{vmatrix} = i_3 \begin{vmatrix} j_2 \\ k_2 \end{vmatrix} + i_2 \begin{vmatrix} j_3 \\ k_3 \end{vmatrix} \) is a linear combination of two vectors which are dependent by the first equation. It follows that the projection of the \((i_t, j_t, k_t)\) on the second and third entries spans a one-dimensional subspace, so the four vectors cannot span \((\mathbb{Z}/2\mathbb{Z})^3\).

Let us extend this example into an example of an extension of number fields \( M/K \) for which \( G \) is wildly \( K \)-admissible but not \( M \)-admissible and not even \( M \)-preadmissible.

Let \( p \) and \( q \) be two rational primes for which:
1) \( p \equiv 5 \) (mod 8)
2) \( q \equiv 1 \) (mod 8)
3) \( q \) is not a square mod \( p \).

**Proposition 4.9.** Let \( K = \mathbb{Q}(\sqrt{q}) \) and \( M = K(i) \). Then \( G \) is wildly \( K \)-admissible but not \( M \)-preadmissible.

**Proof.** Since \( G \) is a 2-group that is not metacyclic it is realizable only over completions at primes divisors of 2. In particular if \( G \) is \( K \)-admissible then \( G \) is wildly \( K \)-admissible. As 2 splits in \( K \), any prime divisor \( v \) of 2 in \( M \) has a completion \( M_v \cong \mathbb{Q}_2(i) \). By Proposition 4.8 \( G \) is not realizable over \( \mathbb{Q}_2(i) \) and hence not \( M \)-preadmissible. It therefore remains to show \( G \) is \( K \)-admissible.

The rational prime \( p \) is inert in \( K \). Let \( p \) be the unique prime of \( K \) that divides \( p \). We have \( N(p) := |K_p| = p^2 = 1 \) (mod 8). Thus, \( K_p \) has a totally ramified \( C_8 \)-extension. Let \( q_1, q_2 \) be the two prime divisors of 2 in \( K \).

Consider the field extension \( L_0 = K(\mu_8, \sqrt{p})/K \). It has a Galois group
\[
\text{Gal}(L_0/K) \cong (\mathbb{Z}/2\mathbb{Z})^3 \cong G/Z(G).
\]
This extension is ramified only at \( q_1, q_2 \) and \( p \). As \( K_{q_i} \cong \mathbb{Q}_2 \) and \( \text{Gal}(L_0)_{q_i}/K_{q_i} \cong (\mathbb{Z}/2\mathbb{Z})^3, (L_0)_{q_i}/K_{q_i} \) is the maximal abelian extension of \( K_{q_i} \) of exponent 2, for \( i = 1, 2 \). Since \( N(p) \equiv 1 \) (mod 8), \( K_p \) contains \( \mu_8 \) and hence \( \text{Gal}((L_0)_p/K_p) \cong \mathbb{Z}/2\mathbb{Z} \).

Let us show that the following central embedding problem:
\[
0 \rightarrow \mathbb{Z}(G) \cong (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/4\mathbb{Z})^2 \xrightarrow{\pi} G \xrightarrow{\text{mod } Z(G)} G/Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^3 \rightarrow 0,
\]
has a solution. Let \( \pi \) denote the epimorphism \( G \rightarrow G/Z(G) \). By theorems 2.2 and 4.7 in [8], there is a global solution to Problem 4.1 if and only if there is a local solution at every prime of \( K \). There is always a solution at primes of \( K \) which are unramified in \( L_0 \) so it suffices to find solutions at \( p, q_1, q_2 \). Any \( G \)-extension of \( K_{q_i} \).
contains \((L_0)_{q_i}\) (as it the unique \((\mathbb{Z}/2\mathbb{Z})^3\) extensions of \(\mathbb{Q}_2\), \(i = 1, 2\). Since \(G\) is realizable over \(\mathbb{Q}_2\) we deduce the induced local embedding problem:

\[
\pi^{-1}(\text{Gal}((L_0)_{q_i}/K_{q_i})) = G \longrightarrow \text{Gal}((L_0)_{q_i}/K_{q_i}) \cong G/Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^3 \longrightarrow 0,
\]

has a solution for \(i = 1, 2\). Since \((L_0)_p\) is the ramified \(\mathbb{Z}/2\mathbb{Z}\)-extension of \(K_p\), it can be embedded into the totally ramified \(\mathbb{Z}/8\mathbb{Z}\)-extension and hence the local embedding problem at \(p\) has a solution.

Therefore, Problem 4.1.1 has a solution. Problem 4.1.1 is a central embedding problem with kernel of exponent 4 and it therefore follows from [8] Theorem 6.4.f that there is a solution to Problem 4.1.1 with prescribed local conditions at \(S = \{q_1, q_2\}\). Thus, \(L_0/K\) can be embedded in a Galois \(G\)-extension \(L/K\) for which \(\text{Gal}(L_{q_i}/K_{q_i}) \cong G\), for \(i = 1, 2\). The field \(L\) is clearly \(K\)-admissible and hence \(G\) is \(K\)-admissible.

Remark 4.10. Note that there is no \(G\)-extension of \(\mathbb{Q}\) with Galois group \(G\) at \(\mathbb{Q}_2\). This follows from the observation that \(G\) has \(\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^2\) as a quotient. Since \(A = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^2\) is the Galois group of the maximal abelian exponent-2 extension of \(\mathbb{Q}_2\) over \(\mathbb{Q}_2\), the existence of an \(A\)-extensions of \(\mathbb{Q}\) with Galois group \(A\) at \(\mathbb{Q}_2\) implies there exist a \(\mathbb{Z}/8\mathbb{Z}\)-extension of \(\mathbb{Q}\) in which 2 is inert. But Wang’s counterexample shows there is no such extension (see [16]).

5. Realizability under extension of local fields

Realizability of a group \(G\) as a Galois group over a field \(k\) is clearly a necessary condition for \(k\)-admissibility. When \(k\) is a local field, the conditions are equivalent since a division algebra of index \(n\) is split by every extension of degree \(n\).

In this section we study realizability of groups under field extensions, assuming the fields are local.

5.1. Totally ramified extensions. We first note what happens under prime to \(p\) local extensions:

Lemma 5.1. Let \(G_1\) be a subgroup of \(G\) that contains a \(p\)-Sylow subgroup of \(G\) and is realizable over the \(p\)-adic field \(k\). Let \(l/k\) be a finite extension for which \([l:k], p\) = 1. Then there is a subgroup \(G_2 \leq G_1\) that contains a \(p\)-Sylow subgroup of \(G\) and is realizable over \(l\).

Proof. Indeed if \(m/k\) is a \(G_1\)-extension then \(ml/l\) is a Galois extension with Galois group \(G_2\) which is a subgroup of \(G_1\) and for which \([m : l \cap m] = |G_2|\). Since \([l \cap m : k], p\) = 1, any \(p^s\) \([m:k] = |G_1|\) also divides \(p^s\) \([m:l \cap m] = |G_2|\). Thus \(G_2\) must also contain a \(p\)-Sylow subgroup of \(G\).  

The case where \(p\) divides the degree \([l:k]\) is more difficult. Let us consider next totally ramified extensions:
Lemma 5.2. Let $p \neq 2$. Let $G$ be a group, $k$ a $p$-adic field with $n = [k : \mathbb{Q}_p]$ and $l/k$ a totally ramified finite extension. Assume furthermore that $l/k$ is not the extension $\mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9)/\mathbb{Q}_3$. If $G$ is realizable over $k$ then $G$ is also realizable over $l$.

Remark 5.3. This shows that if $G$ has a subgroup $G_1$ that contains a $p$-Sylow of $G$ and is realizable over $k$ then $G$ also has a subgroup $G_2$ that contains a $p$-Sylow of $G$ and is realizable over $l$ (moreover, $G_1$ is realizable over $l$).

Proof. Let $l/k$ be a totally ramified extension of degree $r = [l:k]$. We shall construct an epimorphism $G_l \rightarrow G_k$. For this we shall consider the presentations given in Theorem 2.20. Denote the parameters of $k$ by $n, q, s, g$ and $h$. Then the degree of $l$ over $\mathbb{Q}_p$ is $nr$ and its residue degree remains $q$. Denote the rest of the parameters of $l$ by $s', g'$ and $h'$ (the parameters that correspond to $s, g$ and $h$ in Theorem 2.20). Then by Theorem 2.20, $G_l$ has the following presentation (as a profinite group):

$$G_l = \langle \sigma, \tau, x_0, \ldots, x_{nr} \mid \tau^q = \tau^q, x_0^q = \langle x_0, \tau \rangle^{g'} x_1^{g'} \cdot x_2 \ldots \cdot x_{nr-1}, x_{nr} \rangle,$$

if $nr$ is even and

$$G_l = \langle \sigma, \tau, x_0, \ldots, x_{nr} \mid \tau^q = \tau^q, x_0^q = \langle x_0, \tau \rangle^{g'} x_1^{g'} \cdot x_2 \ldots \cdot x_{nr-1}, x_{nr} \rangle,$$

if $nr$ is odd, where $y_1$ is defined there (in manner which depends on the base field). Let $P_k$ (resp. $P_l$) be the subgroup of $G_k$ (resp. $G_l$) topologically generated by $x_0, \ldots, x_n$ (resp. $x_0, \ldots, x_{nr}$) and let $D_k$ (resp. $D_l$) be the subgroup topologically generated by $\sigma$ and $\tau$. Note that as $k$ and $l$ have the same residue degree (same $q$), $D_k \cong D_l$.

Let us construct the epimorphism from $G_l$ to $G_k$ in several steps. First send $x_0$ and every $x_k$ with $k$ odd in the presentation of $G_l$ to 1, we get an epimorphism

$$G_l \rightarrow D_l \ast F_p([\frac{nr-1}{2}]) \cong D_k \ast F_p([\frac{nr-1}{2}]),$$

where $[\gamma]$ denotes the smallest integer $\geq \gamma$. Let us continue under the assumption $[\frac{nr-1}{2}] \geq n + 1$. Then there is an epimorphism $F_p([\frac{nr-1}{2}]) \rightarrow F_p(n + 1)$. We therefore obtain epimorphisms:

$$G_l \rightarrow D_k \ast F_p(n + 1) \rightarrow D_k \ast P_k \rightarrow G_k.$$

The numerical condition $[\frac{nr-1}{2}] \geq n + 1$ fails if and only if:

1. $r = 1$, or
2. $r = 2$, or
3. $r = 3$ and $n = 1$.

The case $r = 1$ is trivial. Since $p$ is an odd prime, the cases $r = 2$ and $r = 3$ are done by Lemma 5.1 unless $p = 3$ and $[l:k] = r = 3$, in which case $n = 1$, so $k = \mathbb{Q}_3$. If $l \neq \mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9)$ then $l \cap k_{9r} = k$ and the parameters $g, h, s$ in the presentation of $G_k$ remain the same in the presentation of $G_l$. In such case there is an epimorphism from $G_l$ onto $G_k$ whose kernel is generated by $(x_2, x_3)$. 

So the case $k = \mathbb{Q}_3$ and $l = \mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9)$ remains open. This will be one of several sensitive cases.
5.2. The sensitive cases.

**Definition 5.4.** We call the extension \( l/k \) *sensitive* if it is one of the following:

1. \( k = \mathbb{Q}_3 \) and \( l \) is the totally ramified 3-extension \( \mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9) \),
2. \( k = \mathbb{Q}_5 \) and \( l = \mathbb{Q}_5(\rho_{11}) \) is the unramified 5-extension,
3. \([k:\mathbb{Q}_3] = 1, 2, 3 \) and \( l/k \) is the unramified 3-extension,
4. \( k = \mathbb{Q}_3 \) and \( l = \mathbb{Q}_3(\rho_7) \) is the unramified 6-extension.

**Remark 5.5.** There are 29 sensitive field extensions. Indeed, there is a single extension in each of cases (1), (2) and (4). In case (3) the extension is unramified, so it suffices to count the ground field \( k \), which we do by dimensions over \( \mathbb{Q}_3 \). In dimension 1 there is of course one case, and there are \( \left| \mathbb{Q}_3^x / \mathbb{Q}_3^2 \right| - 1 = 3 \) quadratic extensions.

Since the abelianization of \( \overline{G_{\mathbb{Q}_3}(3)} \) is \( \mathbb{Z}_2^3 \), there are \( \frac{3^2 - 1}{2} = 4 \) Galois cubic extensions of \( \mathbb{Q}_3 \). For every non-Galois cubic extension \( k \) there is a unique \( S_3 \)-Galois extension of \( \mathbb{Q}_3 \) (generated over \( k \) by the square root of the discriminant).

The \( S_3 \)-Galois extensions of \( \mathbb{Q}_3 \) are in one-to-one correspondence to the normal subgroups of \( G_{\mathbb{Q}_3} \) with quotient \( S_3 \). The number of such subgroups is the number of epimorphisms from \( G_{\mathbb{Q}_3} \) to \( S_3 \), divided by \( |\text{Aut}(S_3)| = 6 \). When counting such epimorphisms \( \varphi: G_{\mathbb{Q}_3} \rightarrow S_3 \), we may assume the generators are in \( S_3 \), which simplifies the presentation a great deal. Since \( p^s = q = 3 \) and we may assume \( g = 1 \) and \( h = -1 \), the presentation is

\[
G_{\mathbb{Q}_3} = \langle \sigma, \tau, x_0, x_1 \mid \tau^3, x_0^3 = (x_0 \tau x_0^{-1} \tau)^2 \sigma^3(x_0, y_1) \rangle.
\]

However, since \( x_1 \) is a pro-3 element, we may assume \( x_1^3 = 1 \). Since all elements of order 3 in \( S_3 \) commute with each other, we may assume \( [x_1, y_1] = 1 \) (see the theorem for more details on \( y_1 \)). Since \( \tau \) is a pro-3 element, \( \varphi(\tau) \) has order at most 2, so \( \varphi(x_0 \tau x_0^{-1} \tau) \) is a commutator, whose order must divide 3. Exponentiation by \( \frac{3}{2} \) squares such elements. So we count epimorphisms from

\[
\langle \sigma, \tau, x_0, x_1 \mid \tau^2 = x_0^3 = x_1^3 = 1, \tau^3, x_0^3 = (x_0 \tau x_0^{-1} \tau)^2 \rangle
\]
to \( S_3 \). If \( \tau = 1 \) there are 6 epimorphisms. So assume \( \tau \neq 1 \). The relations give \( [\sigma, \tau] = 1 \), so \( \sigma \) is either 1 or \( \tau \). Moreover, it turns out that \( (x_0 \tau x_0^{-1} \tau)^2 = x_0 \) for every \( x_0 \) of order dividing 3. For each possible value of \( \tau \) we get 8 epimorphisms with \( \sigma = 1 \) and 2 more with \( \sigma = \tau \). There are 3 involutions, providing us with 6 + 3 \cdot 10 = 36 epimorphisms all together. Dividing by the number of automorphisms, we have 6 Galois extensions of \( \mathbb{Q}_3 \) with Galois group \( S_3 \).

Each Galois extension of this type contains 3 non-Galois cubic extensions of \( \mathbb{Q}_3 \), since this is the number of involutions in \( S_3 \). So we have 3 \cdot 6 = 18 non-Galois cubic extensions. Summing up, we have \( 1 + 1 + (1 + 3 + (4 + 18)) + 1 = 29 \) altogether.

Let us formulate the problem in case (1) for odd order groups:

**Remark 5.6.** Given a field \( F \) denote by \( G'_F \) the Galois group that corresponds to the maximal pro-odd Galois extension of \( F \). In the \( p \)-adic case, for odd \( p \), this is obtained from the presentation of \( G'_k \) (see Theorem 2.20) simply by dividing by the 2-part of \( \sigma \) and \( \tau \). In such case we get a presentation of \( G'_F \) by identifying \( \sigma_2 = \tau_2 = 1 \). We get that \( y_1 \) is a power of \( x_1 \) and hence \( [x_1, y_1] = 1 \).
Question 5.7. Let $l/k$ be the sensitive extension $\mathfrak{q}$. Then $q = 3$; also $p^s = 3$ so we can choose $h = -1$. For $l$ we have $p^{s'} = 9$, but $\tau(\zeta_9 + \zeta_9^{-1}) = \zeta_9 + \zeta_9^{-1}$, so $h' = -1$ as well. Theorem 2.20 gives us the presentations:

$$G_k' = \langle \sigma, \tau, x_0, x_1 | \tau^\sigma = \tau^3, x_0^\sigma = (x_0, \tau)x_1^3 \rangle,$$

where $\sigma, \tau$ are of order prime to 2 and $\langle x_0, \tau \rangle = (x_0\tau x_0^{-1}\tau)\mathbb{Z}_p$, which has order a power of 3 in every finite quotient. Does the following hold: Let $G$ be an epimorphic image of $G_k'$, is there necessarily a subgroup $G(p) \leq G' \leq G$ so that $G'$ is an epimorphic image of $G_k'$?

Note that for a $p$-group $G$ the claim was proved in Lemma 4.4.

Remark 5.8. Quotients of $G_k'$ with $\tau = 1$ can in fact be covered: the group $\langle \sigma, x_0, x_1 | x_0^3 = x_1 \rangle = \langle \sigma, x_1 \rangle$ is covered by $\sigma \mapsto \sigma$, $\tau \mapsto 1$, $x_0 \mapsto 1$, $x_1 \mapsto 1$. This corresponds to fields whose ramification index over $k$ is a 3-power. In particular if $l/k$ is a $p$-extensions we have an epimorphism $G_l \to G_k$.

5.3. Extensions of local fields. We can now approach the general case:

**Proposition 5.9.** Let $p$ be an odd prime. Let $l/k$ be a non-sensitive extension of $p$-adic fields with degree $r$. Let $G$ be a group that is realizable over $k$. Then there is a subgroup $G_1 \leq G$ that contains a $p$-Sylow subgroup of $G$ and is realizable over $l$.

**Proof.** Let $n, q$ be as defined above for $k$. Let $f = [l : k] = f_p f_{p'}$ where $f_p$ is a $p$-power and $f_{p'}$ is prime to $p$. There is an unramified $C_f$-extension $m'/k$ which lies in $l$, and then $l/m'$ is totally ramified. Denote by $m$ the subfield of $m'$ which is fixed by $C_{f_p}$. Let $r' = \frac{l}{f_{p'}}, m = [l : m]$. By Lemma 5.1 there is a subgroup $G_0 \leq G$ that contains a $p$-Sylow subgroup of $G$ and an epimorphism $\phi : G_m \to G_0$. The list of sensitive cases satisfies that if $l/k$ is non-sensitive and $m/k$ is unramified and prime to $p$, then $l/m$ is also non-sensitive and therefore we can assume without loss of generality that $m = k$, $G = G_0$, i.e $f_{p'} = 1$, $f = f_p$, and $r' = r$.

We shall construct an epimorphism from $G_l$ to $G$. Let $s', g', h'$ be the invariants $s, g, h$ in Theorem 2.20 that correspond to $l$, and let $n = [k : \mathbb{Q}_p]$. Theorem 2.20 gives the following presentation of $G_l$:

$$G_l = \langle \sigma, \tau, x_0, \ldots, x_{nr} | \tau^\sigma = \tau^{q^s}, x_0^\sigma = (x_0, \tau)^{q^s}x_1^{q^s} x_2^{q^s} \cdots [x_{nr-1}, x_{nr}] \rangle,$$

if $nr$ is even and

$$G_l = \langle \sigma, \tau, x_0, \ldots, x_{nr} | \tau^\sigma = \tau^{q^s}, x_0^\sigma = (x_0, \tau)^{q^s}x_1^{q^s} x_2^{q^s} \cdots [x_{nr-1}, x_{nr}] \rangle,$$

if $nr$ is odd.

Let $P_k$ (resp. $P_l$) be the subgroup topologically generated by $x_0, \ldots, x_n$ (resp. $x_0, \ldots, x_{nr}$) in $G_k$ (resp. $G_l$) and let $D_k \leq G_k$ (resp. $D_l \leq G_l$) be the subgroup topologically generated by $\sigma, \tau$ in $G_k$ (resp. in $G_l$).

Once again, we can form an epimorphism $G_l \to D_l \ast F_p(\mathbb{Z}_{nr-1})$. Though, as oppose to Lemma 5.2 $D_l \not\cong D_k$. However, $D_l$ can be viewed as a subgroup of $D_k$ of index $f(= f_p)$. 


We claim there is an epimorphism
\[ D_l \ast F_p(1) = \langle \sigma, \tau, x \mid \tau^\sigma = \tau^{q^l} \rangle \rightarrow D_k = \langle \sigma_k, \tau_k \mid \tau_k^{\sigma_k} = \tau^q \rangle, \]
where \( x \) is the generator that corresponds to the free pro-\( p \) group on one generator \( F_p(1) = \mathbb{Z}_p \). Decompose \( \langle \sigma \rangle \) into its \( p \)-primary part generated by \( \sigma_p \) and its complement generated by \( \sigma_p' \) so that \( \sigma = \sigma_p \sigma_p' \). Then
\[
\langle \sigma, \tau, x \mid \tau^\sigma = \tau^{q^l} \rangle = \langle \sigma_p, \sigma_p', \tau, x \mid \tau^{\sigma_p} = \tau^{q^l \sigma_p'^{-1}}, [\sigma_p, \sigma_p'] = 1 \rangle,
\]
so that in the latter presentation \( \sigma_p \) generates a \( p \)-group and \( \sigma_p' \) generates a pro-cyclic group of prime to \( p \) order. Let us divide by the relations \( x^f = \sigma_p \) and \( \tau^x = \tau^{q^l \sigma_p'^{-1}/f} \), where \( \sigma_p'^{-1}/f \) is well defined since \( f \) is a \( p \)-power. We then obtain an epimorphism:
\[ D_l \ast F_p(1) \rightarrow \langle x, \sigma_p', \tau \mid \tau^x = \tau^{q^l \sigma_p'^{-1}/f}, [x, \sigma_p'] = 1 \rangle. \]
The latter group is isomorphic to \( D_k \) by mapping \( \sigma_k \rightarrow x \sigma_p', \tau_k \rightarrow \tau \) (decomposing \( \sigma_k \) into a \( p \)-primary part and a complement). Let us now use this claim to construct an epimorphism \( G_l \rightarrow G_k \). Similarly to Lemma 5.2 we have an epimorphism \( G_l \rightarrow D_l \ast F_p([\frac{nr-1}{2}]) \).

Assuming \( [\frac{nr-1}{2}] \geq n + 2 \) we have an epimorphism:
\[
(5.1) \quad G_l \rightarrow D_l \ast F_p([\frac{nr-1}{2}]) \cong (D_l \ast F_p(1)) \ast F_p([\frac{nr-1}{2}] - 1) \rightarrow \]
\[ \rightarrow (D_l \ast F_p(1)) \ast F_p(n + 1) \rightarrow D_k \ast P_k \rightarrow G_k. \]

The numerical condition \( [\frac{nr-1}{2}] \geq n + 2 \) fails if and only if
(1) \( r = 4, 5 \) and \( n = 1 \);
(2) \( r = 3 \) and \( n = 1, 2, 3 \);
(3) \( r = 1, 2 \).

The cases \( r = 1, 2, 4 \) are covered by Lemmas 5.1. We are left with cases \( r = 3, 5 \). For \( r = 5, n = 1 \) so \( k = \mathbb{Q}_3 \) and by Lemma 5.2 we may assume \( l/k \) is not totally ramified, so \( l/k \) is the unramified 5-extension which is sensitive.

Let \( r = 3 \). Lemma 5.1 covers the case \( p \neq 3 \), so we may assume \( p = 3 \). Note that \( f \mid r \). If \( f = 1 \), Lemma 5.2 applies, except for \( l = \mathbb{Q}_3(\zeta_9 + \bar{\zeta}_9) \) and \( k = \mathbb{Q}_3 \), which is sensitive. If \( f = 3 \), then \( l/k \) is the unramified 3-extension and \( n = [k : \mathbb{Q}_3] = 1, 2, 3 \) which are all sensitive. \( \square \)

6. Admissibility under field extension

Let \( M/K \) be a finite extension of number fields. In this section we shall examine cases in which \( K \)-admissibility induces \( M \)-admissibility. We have seen (Remark 4.5) that tame \( K \)-admissibility does not imply \( M \)-admissibility even in the case of a \( p \)-group. We conjecture those are all the cases. Some examples are given in Section 7.

Our first question along these lines is whether or not failure to remain admissible can always be explained by a rational prime that remains prime in the extension. Namely:
**Question 6.1.** Let $G$ be a $K$-admissible group which is not $M$-admissible. Is there necessarily a rational prime $p$ with a unique prime divisor in $M$, such that $G(p)$ satisfies Liedahl’s condition over $K$ but does not satisfy it over $M$?

In other words, let $G$ be a $K$-admissible group and assume that for every prime $p | |G|$ one of the following conditions hold:

1. $p$ decomposes in $M$,
2. $G(p)$ satisfies Liedahl’s condition over $M$.

Does this imply that $G$ is $M$-admissible?

The constructions of Section 5 will eventually lead to a positive answer to Question 6.1 for odd order groups $G$ with the GN-property except for some ‘sensitive’ extensions $M/K$.

6.1. **Extensions of global fields.** We say that a finite extension $M/K$ of number fields is sensitive if one of its completions $M_{v}/K_{v}$ (at a prime divisor of 3 or 5) is sensitive.

We conclude with the following positive answer to question 6.1.

**Theorem 6.2.** Let $M/K$ be a non-sensitive extension and $G$ a $K$-admissible group of odd order, that has the GN-property over $M$. Then $G$ is $M$-admissible if and only if for every $p | |G|$, either $p$ has more than one divisor in $M$ or $G(p)$ is metacyclic that satisfies Liedahl’s condition over $M$.

**Proof.** If $G$ is $M$-admissible we have already seen that for every $p | |G|$, either there is more than one extension or $G(p)$ is a tamely $M$-admissible group (see Remark 2.7). Let us show that under the latter condition, $G$ is $M$-admissible. We claim that for every prime $p | |G|$ one can choose two distinct primes $w_{1}(p), w_{2}(p)$ of $M$ and choose corresponding groups $H_{i}(p), H_{2}(p)$ so that $H_{i}(p)$ contains a $p$-Sylow subgroup of $G$ and is realizable over $M_{w_{i}(p)}, i = 1,2$. Furthermore, one can choose all the primes $w_{i}(p), i = 1,2, p | |G|$ to be distinct. This will show that $G$ is $M$-admissible.

As $G$ is $K$-admissible, for every $p | |G|$ there are two options:

1. there are two prime divisors $v_{1}, v_{2}$ of $p$ in $K$ and two subgroups $G(p) \leq G_{i} \leq G$ so that $G_{i}$ is realizable over $K_{v_{i}}, i = 1,2$.
2. $G(p)$ is realizable over $K_{v}$ for $v$ which is not a divisor of $p$.

In case (1), by Proposition 5.2, for any prime divisor $w$ of $v_{1}$ or $v_{2}$ there is a subgroup $G(p) \leq H_{w} \leq G$ that is realizable over $M_{w}$. Choose two such primes $w_{1}(p), w_{2}(p)$ and set $H_{i}(p) := H_{w_{i}(p)}$ (the subgroups Proposition 5.2 constructs).

In case (2), $G(p)$ is metacyclic, which is the case treated in Lemma 2.9. If $p$ has more than one prime divisor in $M$ then there are two primes $w_{1}(p), w_{2}(p)$ so that $w_{i}(p)$ is a divisor of $p$ and $H_{i}(p) := G(p)$ is realizable over $M_{w_{i}(p)}, i = 1,2$.

If $p$ has a unique prime divisor in $M$ then $G(p)$ is assumed to satisfy Liedahl’s condition. Liedahl’s condition implies that there are infinitely many primes $w(p)$ for which $G(p)$ is realizable over $M_{w(p)}$ (see Theorem 28 and Theorem 30 in [1]). Thus, we can choose two primes $w_{1}(p), w_{2}(p)$ for every prime $p | |G|$ that has only one prime divisor in $M$, so that the primes $w_{i}(p), i = 1,2$, are not divisors of any prime $q | |G|$ and are all distinct. For such $p$, we also choose $H_{i}(p) := G(p)$.

For every prime $p | |G|$ we have chosen two primes $w_{1}(p), w_{2}(p)$ so that all primes are distinct. This proves the claim and hence the Theorem. □
Another application of the "local" discussion in Section 5.3 is the following implication for wild amissiblility. We shall show that for non-sensitive extensions with a group that has the GN-property, Condition (7) realized wildly implies Condition (4).

**Proposition 6.3.** Let $M/K$ be a non-sensitive extension and $G$ an odd order group that has a GN-property over $M$

Assume there is a wildly $K$-adequate $G$-extension. Then there is an $M$-adequate $G$-extension $L/M$ and a division algebra $D$ with $L$ as a maximal subfield so that $D$ is defined over $K$, i.e. $D \in \text{Im}(\text{res}_K^L)$.

**Proof.** Fix $p \mid |G|$. Suppose $v_1(p), v_2(p)$ are primes of $K$ that divide $p$, so that there are two subgroups $G(p) \leq G_1, G_2 \leq G$ for which $G_i$ is realizable over $K_{v_i}$. Let $t_p = |G(p)|$ and let $s_p = \min(|M_w:K_v|: v \in \{v_1(p), v_2(p)\}, w \mid v)$. One can form the $K$-division algebra $D_p$ that has the following invariants:

$$\text{ind}(v_1(p))(D_p) = \frac{1}{s_p t_p}, \text{ind}(v_2(p))(D_p) = -\frac{1}{s_p t_p}$$

and $\text{ind}(v_i)(D_p) = 0$ otherwise. Now $D_{p,M} := D_p \otimes_K M$ has exponent $t_p$ and it is supported on prime divisors of $v_1(p), v_2(p)$ in $M$. But by Proposition 5.9 every prime divisor $w$ of either of $v_1(p), v_2(p)$ has a corresponding subgroup $G_w \supseteq G(p)$ of $G$ that is realizable over $M_w$.

Now, as $G$ has the GN-property over $M$, there is a $G$-extension $L/M$ so that for every $p \mid |G|$ and every $w$ that divides $v_i(p)$, $i = 1, 2$, $\text{Gal}(L_w/M_w) \cong G_w$. Then $L$ splits $D_{p,M}$ for every $p$ and hence also $D = \otimes_{p \mid |G|} D_{p,M}$ for every $p$. But $D_{p,M}$ has exponent (and hence index) that is divisible by $t_p$, for every $p \mid |G|$ and hence $D$ has exponent divisible by $\text{lcm}_{p \mid |G|} t_p = |G|$. Hence, $\text{ind}(D) = \text{exp}(D) = |G|$ and $L$ is a $G$-maximal subfield of $D$. It is left to notice that $D$ is also the restriction of $D_0 = \otimes_{p \mid |G|} D_p$. We conclude $(K, M, G)$ satisfies (4). \hfill \square

**Remark 6.4.** This shows that for non-sensitive $G$-extensions with an odd order group $G$ that has a GN-property over $M$, Case (7) realized with a wild extension (even without requiring the existence of $L$) implies Case (4).

Recall, $m(M)$ denotes the number of roots of unity in $M$. Applying Theorem 2.18 we get the following consequence:

**Corollary 6.5.** Let $M/K$ be a non-sensitive extension of number fields and let $G$ be a wildly $K$-admissible group with $(|G|, m(M)) = 1$. Then $G$ is wildly $M$-admissible and moreover there is an $M$-adequate $G$-extension $L/M$ and a division algebra $D$ with $L$ as a maximal subfield so that $D$ is defined over $K$.

**Remark 6.6.** Note that as Proposition 5.9 does not hold for tame extensions (in fact a converse statement holds). Thus, Theorem 6.2, Proposition 6.3, Remark 6.4 and Corollary 6.5 do not continue to hold for non-wild extensions. We shall see examples in the following section.

### 7. Examples

In this section we give counterexamples for all the implications not claimed in Theorem 6.1. Let us first show that neither (4) nor (7) imply any other condition.
except (1). For this, by the implication Diagram 3.1 it is sufficient to show that
(1) \not\Rightarrow (7), (7) \not\Rightarrow (4), (4) \not\Rightarrow (2) and that (7) \not\Rightarrow (2). We will show that (7) \not\Rightarrow (1)
by demonstrating that (3) \not\Rightarrow (4). In fact an example for (3) \not\Rightarrow (1) will show
no other condition, except (5), implies Condition (4). To complete showing that
any other implication does not hold we should also prove (3) \not\Rightarrow (7), (3) \not\Rightarrow (3),
(3) \not\Rightarrow (7) and (3) \not\Rightarrow (8).

Remark 7.1. If \( F_1 \) and \( F_2 \) are field extensions of \( F \) such that \( L = F_1 \otimes_F F_2 \) is a
field, and \( F_1/F \) and \( L/F_1 \) are Galois, then \( L \) is Galois over \( F \).

Example 7.2 ((1), (7) \not\Rightarrow (2)). Let \( p \equiv 1 \mod 4 \), \( G = (\mathbb{Z}/p\mathbb{Z})^3 \) and \( K = \mathbb{Q}(i, \sqrt{p}) \).
Note that \( p \) splits in \( K \). Denote the prime divisors of \( p \) in \( K \) by \( v_1, v_2 \).

Let \( \overline{K_v}^{ab}(p) \) be the maximal abelian pro-p extension of \( K_v \). By local class field
theory the Galois group Gal(\( \overline{K_v}^{ab}(p)/K_v \)) is isomorphic to the pro-p completion
of the group \( K_v^* \), and thus has rank \( [K_v : \mathbb{Q}_p] + 1 = 3 \) (see [13], Chapter 14,
Section 6). This can also be seen by taking the abelinization of the Galois group
in Theorem 2.19.

Since \( K_{v_1} = K_{v_2} = \mathbb{Q}_p(\sqrt{p}) \) this shows \( G \) is realizable over \( K_{v_1}, K_{v_2} \) and hence by
the Grunwald-Wang Theorem, \( G \) is \( K \)-admissible. Let \( M \) be a \( K \)-adequate Galois
\( G \)-extension of \( K \) so that \( M_{v_1} \) is the maximal abelian extension of exponent \( p \) of
\( K_{v_1} \), namely the unique \( G \)-extension of \( K_{v_1} \). One also notices that \( G \) is realizable
over \( M_{v_1}, M_{v_2} \) and hence \( G \) is \( M \)-admissible. We deduce that \( (K, M, G) \) satisfies
condition (4). To show that \( (K, M, G) \) satisfies (4) it is sufficient to notice that
\( v_1, v_2 \) have unique prime divisors \( w_1, w_2 \) in \( M \). Every division algebra \( D \) whose
invariants are supported in \( \{ w_1, w_2 \} \) is \( K \)-uniformly distributed and hence \( D \in \text{Im}(\text{res}_K^M) \). Take \( D \) with

\[
\text{inv}_{w_1}(D) = \frac{1}{p^3}, \quad \text{inv}_{w_2}(D) = -\frac{1}{p^3}
\]

and \( \text{inv}_w(D) = 0 \) for any other prime \( w \) of \( M \). We then have \( D \in \text{Im}(\text{res}_K^M) \), \( D \) is a
\( G \)-crossed product division algebra and hence \( (K, M, G) \) satisfies (4).

Let us show (2) is not satisfied. Suppose on the contrary that there is a triple
\( (L_0, L, D) \) realizing (2). By Remark 7.1, \( L/K \) is Galois and

\[
\text{Gal}(L/K) \cong \text{Gal}(L/L_0) \times \text{Gal}(L/M) \cong G \times \phi G
\]

via some homomorphism \( \phi : G \to \text{Aut}(G) = \text{GL}_3(\mathbb{F}_p) \). As \( G \) is a \( p \)-group, \( \phi \) is a
homomorphism into some \( p \)-Sylow subgroup \( P \) of \( \text{GL}_3(\mathbb{F}_p) \). These are all conjugate,
so we can choose a basis \( \{v_1, v_2, v_3\} \) of \( \mathbb{F}_p^3 \) for which \( P \) is the Heisenberg group (in
other words the unipotent radical of the standard Borel subgroup), generated by
the transformations:

\[
\phi_x(a, b, c) = (a + b, b, c), \quad \phi_y(a, b, c) = (a, b + c, c), \quad \phi_u(a, b, c) = (a + c, b, c)
\]

which correspond to the matrices

\[
x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Note that $P$ has the presentation

$$P = \langle x, y, u \mid x^p = y^p = u^p = [x, u] = [y, u] = 1, [y, x] = u \rangle.$$  

Every subgroup of the form $\mathbb{F}_p^2 \rtimes G$ is a maximal subgroup of $G \rtimes G$ and thus the Frattini subgroup $\Phi$ of $G \rtimes G$ is contained in $1 \rtimes G$. Now the subgroup $H = \langle v_1, v_2 \rangle \leq G$ is invariant under the action of $P$ and hence under the action of $G$ via $\phi$. So, $G \rtimes_\phi H \leq G \rtimes_\phi G$ is a maximal subgroup and $\Phi \leq 1 \rtimes H$. This shows that $\dim_\mathbb{F}_p G/\Phi \geq 4$ and thus $G \rtimes_\phi G$ is not generated by less than 4 elements. Therefore $G \rtimes G$ is not realizable over $\mathbb{Q}_p(\sqrt{p})$.

On the other hand both $L/M$ and $M/K$ have full rank at $w_i$ and $v_i$ and hence $\text{Gal}(L_{w_i}/K_{v_i}) = G \rtimes G$ which is a contradiction as $G \rtimes G$ is not realizable over $K_{v_i}$. Thus, $(K, M, G)$ can not satisfy Condition (2).

**Example 7.3** ($8 \nleq 7$, $8 \nleq 5$). Let $p \equiv 1 \pmod{4}$ and $K = \mathbb{Q}(i)$, and let $v_1, v_2$ be the two prime divisors of $p$ in $K$. Let $G = \mathbb{F}_p^2$ and $P = \mathbb{F}_p \wr (\mathbb{Z}/p\mathbb{Z})$ so that $P = G \rtimes \langle x \rangle$ where $x^p = 1$. Since $P$ is a wreath product of abelian groups it has a generic extension over $K$ and hence $P$ has the GN-property over $K$ (see [10]). The maximal $p$-extension $\overline{\mathbb{Q}_p}(p)$ has Galois group $\overline{\text{Gal}}(\mathbb{Q}_p(p)) := \text{Gal}(\overline{\mathbb{Q}_p(p)}/\mathbb{Q}_p)$ which is a free pro-$p$ group on two generators. As $P$ is generated by two elements it is realizable over $\mathbb{Q}_p$. Combining this with the GN-property we have a $P$-extension $L/K$ for which $\text{Gal}(L_{v_i}/K_{v_i}) = P$ for $i = 1, 2$. Let us choose $M = L^G$ the $G$-fixed subfield of $L$.

Then clearly $L/M$ is an $M$-adequate extension which is defined over $K$ since

$$\text{Gal}(L/K) \cong \text{Gal}(M/K) \rtimes \text{Gal}(L/M).$$

The subfield $L_0 = L^x$ is $K$-adequate since $[(L_0)_{v_i} : K_{v_i}] = p^2$ for $i = 1, 2$ and hence $(K, M, G)$ satisfies Condition (5).

(We write $(L_0)_{v_i}$ even though $L_0/K$ is not Galois, since $v_i$ has a unique prime divisor in $L_0$ for $i = 1, 2$.)

Now since $G$ is an abelian group of rank $p > 2$, $G$ is not realizable over $K_{v_1}, K_{v_2} \cong \mathbb{Q}_p$ and hence not $K$-admissible. It follows that $(K, M, G)$ does not satisfy Condition (7). In order for $(K, M, G)$ to satisfy Condition (3) there should be a $G$-extension $L_0/K$ for which $L_0M$ is $M$-adequate. In particular, $\text{Gal}(L_0M_{v_i}/M_{v_i}) \cong G$ and hence $\text{Gal}(L_{v_i}/K_{v_i}) \cong G$ which contradicts the fact that $G$ is not realizable over $K_{v_i} \cong \mathbb{Q}_p$. Thus $(K, M, G)$ does not satisfy Condition (3) either.

**Example 7.4** ($3 \nleq 7$, $3 \nleq 5$). Let $p \equiv 1 \pmod{4}$ and $v$ be its unique prime divisor in $K = \mathbb{Q}(\sqrt{p})$. Let $M = \mathbb{Q}(\sqrt{p}, i)$ and $G = (\mathbb{Z}/p\mathbb{Z})^2$.

By the Grunwald-Wang Theorem ([15]) there is a Galois $G$-extension $L_0/K$ for which $\text{Gal}(L_0{v}/K_{v}) = G$. Then $L/M$, where $L = L_0M$, is a Galois $G$-extension for which $\text{Gal}(L_{v_i}/M_{v_i}) = G$ for each of the two prime divisors $v_1, v_2$ of $v$ in $M$. Thus $L$ is $M$-adequate and $(K, M, G)$ satisfies Condition (3). But as $p$ has a unique prime divisor in $K$ and $G$ is not metacyclic, $G$ is not $K$-admissible and hence $(K, M, G)$ does not satisfy Condition (7).

Let us also show that $(K, M, G)$ does not satisfy Condition (5). Assume, on the contrary, that $(L_0, L, D_0, D)$ realizes (5). Then, as $L_0$ is $K$-adequate there are two primes $w_1, w_2$ of $K$ for which $[(L_0)_{w_i}:K_{w_i}] = |G|$. Without loss of generality we
assume $w_1 \neq v$ (otherwise take $w_2$). Then $\text{Gal}(L_{w_1}/M_{w_1}) \cong G$ since $(L_0)_{w_1} \cap M_{w_1} = K_{w_1}$. This is a contradiction since tamely ramified extensions (such as $L_{w_1}/M_{w_1}$) have metacyclic Galois groups. Thus $(K, M, G)$ does not satisfy Condition (8).

Remark 7.5. Let us also show that $(K, M, G)$ does not satisfy (3) (so that this example will also show that (3) $\not\rightarrow$ (1)). Assume on the contrary that there are $D_0, D$ and $L$ as required in (3). Since $D$ contains $L$ as a maximal subfield, $\text{Gal}(L_{w_i}/M_{w_i}) = G$ and $\text{inv}_{v_i}(D) = \frac{m_i}{p}$ where $(m_i, p) = 1$, for $i = 1, 2$. Note that $G$ is realizable over $M_v$ only for divisors $v$ of $p$, so that $\text{inv}_v(D) = \frac{m_v}{p^2}$ for suitable $m_v \in \mathbb{Z}$ for any $u \neq v_1, v_2$. Now, since $D$ is in the image of the restriction, we have $m_1 = m_2$. The sum of $M$-invariants of $D$ is an integer and hence $p | m_1 + m_2 = 2m_1$ which contradicts $(m_1, p) = 1$.

Example 7.6 (4) $\not\rightarrow$ (7). Let $p$ be any odd prime, and $q$ a prime $\equiv 1 \pmod{p}$. Let $K = \mathbb{Q}(\sqrt{p})$, so that $q$ splits (completely) in $K$. Let $v$ be the prime divisor of $p$ in $K$ and $w$ a prime divisor of $q$ in $K$. Let $M$ be a $\mathbb{Z}/p\mathbb{Z}$-extension of $K$ in which $v$ splits completely and $w$ is inert. Let $G = (\mathbb{Z}/p\mathbb{Z})^3$.

Consider the $K$-division algebra $D_0$ whose invariants are:

$$\text{inv}_v(D_0) = \frac{1}{p^2}, \quad \text{inv}_w(D_0) = -\frac{1}{p^2}$$

and $\text{inv}_w(D_0) = 0$ for any other prime $u$ of $K$. Now $D = D_0 \otimes_K M$ has $M$-invariants $\text{inv}_v(D) = \frac{1}{p}$ for the prime divisors $v_1, v_2, \ldots, v_p$ of $v$ in $M$, $\text{inv}_w(D) = -\frac{1}{p^2}$ for the prime divisor $w'$ of $w$ and $\text{inv}_u(D) = 0$ for any other prime $u$ of $M$. Note that $G$ is realizable over $M_{v_i} \cong K_v$ and since $q \equiv 1 \pmod{p}$, $(\mathbb{Z}/p\mathbb{Z})^3$ is realizable over $M_{w'}$. By the Grunwald-Wang-Stein Theorem, there is a Galois $G$-extension $L/M$ for which:

$$\text{Gal}(L_{v_i}/M_{v_i}) = G \quad \text{for} \quad i = 1, \ldots, p, \quad \text{and} \quad \text{Gal}(L_{w'}/M_{w'}) = (\mathbb{Z}/p\mathbb{Z})^2.$$ 

Thus $L$ is a maximal subfield of $D$ and $(K, M, G)$ satisfies condition (4). Since $p$ has a unique prime divisor in $K$ and $G$ is not metacyclic we deduce $G$ is not $K$-admissible and hence $(K, M, G)$ does not satisfy Condition (7).

Example 7.7 (5) $\not\rightarrow$ (1). Let $p \geq 13$ be a prime for which $p \equiv 1 \pmod{4}$. Let $K = \mathbb{Q}(\mu_p)$ and $M = \mathbb{Q}(\mu_{4p^2}) = \mathbb{Q}(i, \mu_{p^2})$. Let $G$ be the following metacyclic group of order $p^8$:

$$(7.1) \quad G = \left\langle x, y \mid x^p = y^{p^2} = 1, \quad x^{-1}yx = y^{p+1} \right\rangle.$$ 

Note that $p$ splits in $\mathbb{Q}(i)$ and has exactly two prime divisors $v_1, v_2$ in $M$. Let $u$ be the unique prime divisor of $p$ in $K$.

Let us first show that $(K, M, G)$ does not satisfy Condition (1). As $M$ does not satisfy Liedahl’s condition, $G$ is not realizable over $M_v$ for any $v \neq v_1, v_2$. Assume on the contrary there is an $M$-adequate $G$-extension $L/M$ and an $M$-division algebra $D$ which is defined over $K$ and has a maximal subfield $L$. Then necessarily: $\text{inv}_{v_1}(D) = \text{inv}_{v_2}(D) = \frac{a}{p^2}$ for some $(a, p) = 1$. But as the sum of invariants of $D$ is 0 and $G$ is not realizable over any other $v$ we have $p | 2a$ or $p | a$. Contradiction.

We claim that $(K, M, G)$ satisfies Condition (6). Let $\sigma_{p+1} \in \text{Gal}(\mathbb{Q}(\mu_{p^2})/\mathbb{Q})$ be the automorphism that sends $\sigma_{p+1}(\zeta) = \zeta^{p+1}$ where $\zeta$ is a primitive root of unity.
of order $p^2$. Thus $\sigma_{p+1}$ fixes $\mu_p$ and hence $\sigma_{p+1} \in \text{Gal}(\mathbb{Q}(\mu_p^2)/K)$. As $G$ satisfies Liedahl’s condition over $K$, $G$ is realizable over infinitely many primes of $K$ (see the proof Theorem 29 in [4] or [5] Theorem 3.24), so choose one such prime $w$ which is not a divisor of $p$. Since $[K_u:Q_p] = p - 1$, $G$ is also realizable over $K_u$. Moreover, we claim there is a $G$-extension $L_w^p/K_u$ for which $M_u \cap L_w^0 = K_u$. The following Lemma proves even more.

Lemma 7.8. Let $k = \mathbb{Q}_p(\mu_p)$, and let $G$ be the group defined in (7.1). Then, given a $G$-extension $m/k$, there is a $G$-extension $l/k$ for which $m \cap l = k$.

Proof. For any $G$-extension $l/k$ we note that $\text{Gal}(l \cap m/k)$ is an epimorphic image of $G$ and as such it is either $G$ or an abelian group. Thus if $l$ intersects with $m$ non-trivially then it also intersects with $m' = m^{(p^0)}$ (the fixed field of $y^p$ which also corresponds to the abelianization of $G$). We note that $\text{Gal}(m'/k) = (\mathbb{Z}_p) \times (\mathbb{Z}/p\mathbb{Z})$. The maximal abelian group realizable over $k$ is of rank $p - 1$, and since $p - 1 \geq 4$ there is a $(\mathbb{Z}/p\mathbb{Z})^2$-extension $l'/k$ which is disjoint from $m'$ and for which the epimorphism $\pi : G_k \to \text{Gal}(l'/k)$ splits through a free pro-$p$ group of rank $2$. Thus $l'$ is disjoint from $m'$ and hence to $m$. Embedding $l'$ into a $G$-extension produces a $G$-extension which is disjoint to $m$. This is possible since the following embedding problem for $G_k$:

$$
\begin{array}{c}
G_k \\
| \quad | \quad | \\
F_p(\mathbb{Z}/2) \\
| \quad | \quad | \\
G \\
& (\mathbb{Z}/p\mathbb{Z})^2 \\
& 0,
\end{array}
$$

splits through a free pro-$p$ group of large enough rank and hence has a surjective solution. □

By Corollary 2.17, there is a $G$-extension $L_0/K$ for which $\text{Gal}((L_0)_w/K_w) = G$ and $(L_0)_u = L_0^0$. So $L_0$ is $K$-adequate. Let $L = L_0M$. As $M_u \cap L_0^0 = K_u$ we have $\text{Gal}(L_u/M_u) = G$ for $i = 1, 2$. Thus $L/M$ is an $M$-adequate $G$-extension and $(K, M, G)$ satisfies Condition (6).

References

[1] JANNSEN, U., WINGBERG, K., Die Struktur der absoluten Galoisgruppe $p$-adischer Zahlkörper. Invent. Math. 70 (1982/83), no. 1, 71-98.
[2] LABUTE, J. P., Classification of Demushkin groups. Canad. J. Math. 19 (1967), 106-132.
[3] LIEDAHL, S., Maximal subfields of $Q(i)$-division rings. Pacific J. Math. 175 (1996), no. 1, 147-160.
[4] LIEDAHL, S., Presentations of metacyclic $p$-groups with applications to $K$-admissibility questions. J. Algebra 169 (1994), no. 3, 965-983.
[5] NEFTIN, D., Admissibility and Realizability over Number fields. Submitted. See also: arXiv:0904.3772v1.
[6] NEFTIN, D., Admissibility as an equivalence relation. arXiv:0910.4156.
[7] NEUKIRCH, J., On solvable number fields. Invent. Math. 53 (1979), no. 2, 135-164.
[8] J. NEUKIRCH, Uber das Einbettungsproblem der algebraischen Zahlentheorie. Invent. Math. 21 (1973), 59-116.
[9] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of number fields. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 323. Springer-Verlag, Berlin (2000).
[10] Saltman, D., Generic Galois extensions. Proc. Nat. Acad. Sci. U.S.A. 77 (1980), 3, part 1, 1250-1251.
[11] Schacher, M., Subfields of division rings. I. J. Algebra 9 (1968) 451-477.
[12] Serre, J.-P., Galois Cohomology, Springer, 1964 (English trans. 1996).
[13] J. P. Serre, Local fields. Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979.
[14] Stern, L., On the admissibility of finite groups over global fields of finite characteristic. J. Algebra 100 (1986), no. 2, 344-362.
[15] S. Wang, On Grunwald’s theorem. Ann. of Math. (2) 51 (1950), 471-484.
[16] S. Wang, A counter-example to Grunwald’s theorem. Ann. of Math. (2) 49 (1948), 1008-1009.

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel
E-mail address: neftind@tx.technion.ac.il

Department of Mathematics, Bar Ilan University, Ramat Gan 52900, Israel
E-mail address: vishne@math.biu.ac.il