Self-dual Vortices in the Generalized Abelian Higgs Model with Independent Chern-Simons Interaction

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Abstract

Self-dual vortex solutions are studied in detail in the generalized abelian Higgs model with independent Chern-Simons interaction. For special choices of couplings, it reduces to a Maxwell-Higgs model with two scalar fields, a Chern-Simons-Higgs model with two scalar fields, or other new models. We investigate the properties of the static solutions and perform detailed numerical analyses. For the Chern-Simons-Higgs model with two scalar fields in an asymmetric phase, we prove the existence of multisoliton solutions which can be viewed as hybrids of Chern-Simons vortices and $CP^1$ lumps. We also discuss solutions in a symmetric phase with the help of the corresponding exact solutions in its nonrelativistic limit. The model interpolating all three models—Maxwell-Higgs, Chern-Simons-Higgs, and $CP^1$ models—is discussed briefly. Finally we study the possibility of vortex solutions with half-integer vorticity in the special case of the model. Numerical results are negative.
I. INTRODUCTION

In the last few years there has been much interest in the (2+1)-dimensional abelian Higgs models and the $CP^1$ model with a Chern-Simons term in the gauge field action, partly because of their possible relevance in condensed matter physics [1]. These models allow classical vortexlike solutions and the analysis of them has become quite active recently [2, 3]. Here particularly interesting are the so-called self-dual systems. Since the discovery of the self-dual charged vortex solutions in relativistic pure Chern-Simons-Higgs system [4], several other self-dual systems including the Chern-Simons term are now known [5, 6, 7]. Of some interest among these is the generalized self-dual abelian Higgs model with independent Chern-Simons interaction, discussed in Refs. [6, 7]. It has two abelian gauge fields, one of which has only the Maxwell term as its kinetic term and the other only the Chern-Simons term. It may be considered as a realistic model if one identifies the Maxwell field as the electromagnetic field and the Chern-Simons field as an effective field arising in a condensed matter system. Besides, as we will see below, it contains a linearized version of the $CP^1$ model with or without a Chern-Simons term as special cases and therefore interpolates abelian Higgs systems and the $CP^1$ model. The present paper is devoted to the detailed study of static solutions in this model, which was not given in Ref. [4].

This model consists of two complex scalar fields, a neutral scalar field, and two gauge fields. Each complex scalar field couples to both gauge fields and so there are four gauge couplings in general. But, with some special choices made for these couplings, the model reduces to simpler systems. For example, for some particular choices [3] it becomes a decoupled sum of self-dual Maxwell Higgs and self-dual
Chern-Simons Higgs systems. More interestingly, it is possible to obtain a system in which two scalar fields interact with a Maxwell or Chern-Simons gauge field (but not with both), while possessing an additional global $SU(2)$ symmetry. This kind of model with a Maxwell gauge field has attracted much attention recently \cite{8} because it admits topological vortex solutions in spite of having a simply connected vacuum manifold, and a class of solutions obtained here interpolates ordinary Ginzburg-Landau vortices and $CP^1$ lumps \cite{8}. We consider the corresponding model with a Chern-Simons gauge field, and find that it has nontopological vortex solutions in a symmetric phase as well as topological vortices in an asymmetric phase (as in the Maxwell case). Again a class of solutions obtained in this case approximate $CP^1$ lumps, and the existence of multivortex solutions can be shown by adapting the method of Wang \cite{9}. There are also more general self-dual systems based on the same set of fields but with no additional global symmetry.

If both gauge fields couple to scalar fields nontrivially, a new kind of models are obtained \cite{8}. If one of the two complex scalar fields decouples and the other couples to both gauge fields, only a broken phase is possible and topological vortices exist. We find that the nature of the vortex solutions is very similar to those of the Ginzburg-Landau system, the Chern-Simons field playing a minor role only. We will briefly discuss its extension to the two complex-scalar-field case in such a way that there is an additional global $SU(2)$ symmetry. Complicated but interesting is the case (f) of Ref. \cite{6}, where all fields couple to one another nontrivially. In ref \cite{6}, it was noted that there might be solutions with a half-integer vorticity (as regards each of the two gauge fields). We will report the result of our numerical work, which is negative.

This paper is organized as follows. In Sec. II, we briefly review the model to be
considered by us, following Ref. [6]. Section III is the major part of the paper. We study static solutions for various interesting cases of this model, based on analytical and numerical means. We summarize our findings in Sec. IV. In the appendix, a $U(1)$ self-dual Maxwell-Chern-Simons Higgs system with additional global $SU(2)$ symmetry is given. This model might be interesting because the solutions interpolate all three cases—Ginzburg-Landau, Chern-Simons, and $CP^1$ solitons.

II. THE MODEL

We consider the (2+1)-dimensional system described by the following Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \kappa e^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + |D_\mu \chi|^2 + |D_\mu \psi|^2 + \frac{1}{2} (\partial_\mu N)^2 - U(\chi, \psi, N),$$  

(1)

where $D_\mu \chi = [\partial_\mu - i(e_1 a_\mu + e_2 A_\mu)] \chi$, $D_\mu \psi = [\partial_\mu - i(e_3 a_\mu + e_4 A_\mu)] \psi$, and

$$U(\chi, \psi, N) = \frac{1}{2} (e_2 |\chi|^2 + \sigma_1 e_4 |\psi|^2 - u^2)^2 + \frac{e_1^2}{\kappa^2} (e_1 |\chi|^2 + \sigma_1 e_3 |\psi|^2 - \sigma_2 \frac{Ke_2}{e_1} N - v^2)^2 + \frac{e_3^2}{\kappa^2} |\psi|^2 (e_1 |\chi|^2 + \sigma_1 e_3 |\chi|^2 - \sigma_2 \frac{Ke_4}{e_3} N - v^2)^2.$$  

(2)

Here, $\sigma_1$ and $\sigma_2$ may assume the values $\pm 1$ independently, $\chi$ and $\psi$ denote complex scalar fields with couplings to both Maxwell and Chern-Simons gauge fields (i.e., $A_\mu$ and $a_\mu$), while $N$ is a neutral scalar. The specific form of the potential (4) leads to a self-dual system for general $\sigma_1, \sigma_2 = \pm 1$, and only the case of $\sigma_1 = \sigma_2 = +1$ was considered in Ref. [6]. We shall now briefly summarize the results of Ref. [6] and extend them to generally allowed values of $\sigma_1, \sigma_2$. The variation of $a_0$ and $A_0$ leads to two Gauss laws

$$\kappa f_{12} = e_1 J^0_\chi + e_3 J^0_\psi,$$

$$\partial_i F^{i0} = e_2 J^0_\chi + e_4 J^0_\psi.$$  

(3)  

(4)
where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, $J_\chi^\mu = -i(\chi^* D^\mu \chi - D^\mu \chi^* \chi)$, and $J_\psi^\mu = -i(\psi^* D^\mu \psi - D^\mu \psi^* \psi)$.

For finite energy configuration, Eqs. (3) and (4) imply relations among the magnetic flux $\Phi_a = \int d^2 x f_{12}$ and charges $Q_\chi = \int d^2 x J_\chi^0$ and $Q_\psi = \int d^2 x J_\psi^0$, viz.,

\begin{align}
\kappa \Phi_a &= e_1 Q_\chi + e_3 Q_\psi, \\
0 &= e_2 Q_\chi + e_4 Q_\psi.
\end{align}

There is no restriction on $\Phi_A = \int d^2 x F_{12}$. The energy functional is

\begin{align}
E &= \int d^2 x \left[ \frac{1}{2} F_{i0}^2 + \frac{1}{2} F_{12}^2 + |D_0 \chi|^2 + |D_0 \psi|^2 \\
&\quad + |D_i \chi|^2 + |D_i \psi|^2 + \frac{1}{2} (\partial_0 N)^2 + \frac{1}{2} (\partial_i N)^2 + U \right].
\end{align}

After employing the Gauss laws and integrating by parts, one has the bound

\begin{align}
E \geq |v^2 \Phi_a + u^2 \Phi_A|,
\end{align}

where the equality holds if and only if the following self-duality equations are satisfied:

\begin{align}
(D_1 \pm i D_2) \chi &= (D_1 \pm i \sigma_1 D_2) \psi = 0, \\
F_{12} \pm (e_2 |\chi|^2 + \sigma_1 e_4 |\psi|^2 - u^2) &= 0, \\
F_{i0} \pm \sigma_2 \partial_i N &= 0, \\
D_0 \chi \pm i \frac{e_1}{\kappa} \chi (e_1 |\chi|^2 + \sigma_1 e_3 |\psi|^2 - \sigma_2 \frac{\kappa e_2}{e_1} N - v^2) &= 0, \\
D_0 \psi \pm i \frac{e_3}{\kappa} \psi (e_1 |\chi|^2 + \sigma_1 e_3 |\psi|^2 - \sigma_2 \frac{\kappa e_4}{e_3} N - v^2) &= 0.
\end{align}

Note that if $\sigma_1 = -1$, the first equation shows that one of the two complex scalar fields is self-dual while the other is antiself-dual. Also, depending on the value of $\sigma_2$, the sign in front of $N$ gets changed. However, each choice of $(\sigma_1, \sigma_2)$ corresponds to
a different potential as seen in Eq. (2) and all signs in Eq. (3) are correlated in a
given theory. Each self-dual system for given $\sigma_1$ and $\sigma_2$ is in fact not much different
from the others. From Eqs. (1), (2) and (3), we see that the system with $\sigma_1 = -1$
is transformed into that with $\sigma_1 = +1$ under $e_3 \to -e_3$, $e_4 \to -e_4$ and $\psi \to \psi^*$, and
into the system with $\sigma_2 = -1$ by $N \to -N$. Therefore, we put $\sigma_1 = \sigma_2 = +1$ from
now on, except at the end of Sec. IIIA where we will give some further comments.

We are interested in static solitonlike solutions of Eq. (3). Since there are four
different gauge couplings, the model has a rich variety and the nature of solutions
depends on the choice of these couplings. It should be thus desirable to consider
separately various cases corresponding to special choices of couplings. For certain
choices of couplings some of known self-dual systems, including the model of some
recent interest [8], are recovered. For other choices we encounter models which have
not been studied in detail so far.

III. SPECIAL CASES

A. $e_1 = e_3 = 0$

If the couplings associated with $a_\mu$, i.e., $e_1$ and $e_3$, are identically zero, then $a_\mu$
decouples and we have a self-dual Maxwell-Higgs system with two complex scalar
fields and a neutral scalar. This is a generalization of the well-known self-dual
Ginzburg-Landau model [13]. Furthermore, from Eq. (3), the equation for $N$ be-
comes $\nabla^2 N = 2(e_2^2|\chi|^2 + e_4^2|\psi|^2)N$, which means that $N$ may be taken to be zero
identically. Here the potential is effectively

$$U = \frac{1}{2} (e_2 |\chi|^2 + e_4 |\psi|^2 - u^2)^2. \quad (10)$$
If one further restricts oneself to the case $e_2 = e_4$, this model will possess additional global $SU(2)$ symmetry and become in fact identical to the one recently considered by Vachaspati et al., Hindmarsh, and Gibbons et al. [8]. For general $e_2$ and $e_4$, there is no additional global symmetry and we study this general case below.

From Eqs. (7) and (10), finiteness of energy requires that

$$r \rightarrow \infty : \quad e_2 |\chi|^2 + e_4 |\psi|^2 \rightarrow u^2$$

as well as

$$r \rightarrow \infty : \quad D_i \chi \rightarrow 0, \quad D_i \psi \rightarrow 0.$$  \hspace{1cm} (12)

Eq. (11) tell us that on the circle at infinity, scalar fields must lie on an ellipsoid in the 4-dimensional internal space; then, Eq. (12) further restricts the scalar fields to be at most a pure phase there. Hence even if the vacuum manifold is simply connected, there can be vortex solutions characterized by a winding number [8].

Now we consider the self-duality equations (9). On points where $\chi, \psi$ are nonzero, the first equation may be rewritten as

$$i(A_1 + iA_2) = \frac{1}{e_2} (\partial_1 + i\partial_2) \ln \chi = \frac{1}{e_4} (\partial_1 + i\partial_2) \ln \psi,$$

where we chose the upper sign. Then

$$(\partial_1 + i\partial_2) \ln \frac{\psi^{e_2/e_4}}{\chi} = 0,$$

so that

$$w(z) \equiv \frac{\psi^{e_2/e_4}}{\chi}$$ \hspace{1cm} (13)

1For the moment, we consider the case $e_2 > 0, e_4 > 0$ only.

2In fact this phenomenon has been known since seventies even if such model has not been explicitly constructed [10].
is locally analytic in $z = x + iy$. Denoting $z_r(r = 1, \ldots, n_1)$ and $z'_r(r = 1, \ldots, n_2)$ to be the zeros of $\chi$ and $\psi$ respectively, we may then write

$$w(z) = \frac{\prod_{r=1}^{n_2} (z - z'_r)^{e_2/e_4}}{\prod_{r=1}^{n_1} (z - z_r)} h(z), \quad (14)$$

where $h(z)$ has no zero or pole and is regular everywhere. Since $\chi$ and $\psi$ should approach vacuum values as $r \to \infty$, $h(z)$ is at most a constant, say $q$, by Liouville's theorem in complex variable theory. Note that $w(z)$ is multivalued in general, but it is not problematic as long as $\chi$ and $\psi$ are single-valued functions. Now let

$$r \to \infty : \quad \left( \begin{array}{c} \chi \\ \psi \end{array} \right) \to \left( \begin{array}{c} \chi_0 \\ \psi_0 \end{array} \right), \quad (15)$$

where $\chi_0$ and $\psi_0$ are some fixed constants satisfying $e_2|\chi_0|^2 + e_4|\psi_0|^2 = u^2$. If $e_2 = e_4$, all vacuums are equivalent because of global $SU(2)$ symmetry and hence it suffices to choose any convenient values for $\chi_0$ and $\psi_0$, for example, $\psi_0 = 0$ and $|\chi_0| = u/\sqrt{e_2}$. However, if $e_2 \neq e_4$, various choices for the vacuum values are inequivalent and one must consider in particular the case with $\chi_0\psi_0 \neq 0$. In the latter case, from Eq. (13), we have $w(z) \to \text{finite}$ as $r \to \infty$, which implies in view of Eq. (14) that $e_2/e_4 = n_1/n_2$. In other words, regular vortex solutions may exist only if $e_2/e_4$ is rational and the ratio of the number of zeros for $\chi$ and $\psi$, $n_1/n_2$, must have the same fixed value equal to $e_2/e_4$. The flux is then given by

$$\Phi \equiv \oint_{r=\infty} dx^i A_i = \frac{2\pi n_1}{e_2} = \frac{2\pi n_2}{e_4}. \quad (16)$$

On the other hand, if one of the vacuum values, say $\psi_0$, is zero we can have a totally different behaviors for the $e_2 \neq e_4$ case, which is absent for the case $e_2 = e_4$. From Eq. (13), $w \to 0$ as $r \to \infty$ and this implies only the condition $e_2 n_2 < e_4 n_1$. There is no need for $e_2/e_4$ to be rational here and, except the above inequality, no
condition on $n_1/n_2$, either. This can be understood once if one notices that the vacuum manifold for $\psi$ is trivial in this case. Now suppose that

$$\psi \rightarrow \frac{1}{r^n} \quad \text{as} \quad r \rightarrow \infty.$$  

Then the flux will be, from Eq. (9),

$$\Phi = \frac{2\pi n_1}{e_2} = \frac{2\pi(n_2 + \alpha)}{e_4},$$

or $\alpha$ is given by

$$\alpha = \frac{e_4}{e_2} n_1 - n_2 > 0.$$  

There is a further non-trivial equation to be satisfied by $|\chi|$ (or by $|\psi|$). Eliminating gauge fields from the self-duality equations (9), we have in fact

$$\nabla^2 \ln |\chi|^2 - 2e_2(e_2|\chi|^2 + e_4|\psi|^2 - u^2) = 4\pi \sum_{r=1}^{n_1} \delta(\mathbf{x} - \mathbf{x}_r),$$

where $|\psi| = (|w||\chi|)^{e_4/e_2}$. If $e_2 = e_4 \equiv e$, we can simplify this equation as

$$\nabla^2 \eta + 2eu^2(1 - e^\eta) = \nabla^2 \ln \left( \prod_{r=1}^{n_1} |z - z_r|^2 + \prod_{r=1}^{n_2} |z - z'_r|^2 \right),$$

where $\eta = \ln[e(|\chi|^2 + |\psi|^2)/u^2]$. Analytic solutions to Eq. (19) or (20) are not available and we should resort to numerical analysis to go further. The simplest case of $e_2 = e_4$, with additional global $SU(2)$ symmetry, gives rise to some interesting consequences, but they were extensively discussed already in Ref. [8].

For generic case, some numerical solutions have been studied by us for specific couplings and vorticities. In Fig. 1(a) we have plotted rotationally symmetric solutions with unit vorticity when the ratio $p \equiv \frac{e_4}{e_2}$ is equal to 1, or 2, while choosing the point $|\chi_0|^2 \equiv u^2/2e_2$ on the ellipse of possible vacuum values. In addition, we

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3 However, see the next section.
studied solutions for other $n$ and $p$ values and we found that the shapes are more or less the same for different $p$-values except that the vortex sizes become a little smaller for larger $p$. In Fig 1(b) we have plotted $e_{n/2}$ for the case $n_1 = 1$ and $n_2 = 0$ with $e_2 = e_4 \equiv e$. Note that the approximation $e_{n/2} = 1$ is excellent for $|q| = 10$ and good even for $|q| = 5$. Therefore the solutions well approximate $CP^1$ lumps for large $|q|$, as noted in Ref. [8]. For the number of free parameters in the general solution (which need not be rotationally symmetric), one can use the index theorem and this was already done in Ref. [7] for the case with $\chi_0 \psi_0 \neq 0$. The number is equal to $2(n_1 + n_2)$.

Closing the discussions, we comment on the case with $e_2 e_4 < 0$. From Eqs. (13) and (14), we now find that

$$\frac{1}{w} = \chi_{v}|e_2/e_4| = \prod_{r=1}^{n_1} (z - z_r) \prod_{r=1}^{n_2} (z - z'_r)|e_2/e_4|h_0,$$

where $h_0$ is a constant. Then $n_1$ and $n_2$ should be zero because $1/w$ approaches a finite value as $r \to \infty$. This implies that the energy $E = u^2|\Phi_A|$ is zero and there is no nontrivial solution satisfying the self-duality equations. However, as we mentioned in Sec. II, there is another self-dual system which corresponds to $\sigma_1 = -1$ in Eq. (3), and this system is essentially identical to that with $e_2 e_4 > 0$ and $\sigma_1 = +1$.

B. $e_2 = e_4 = 0$

With $e_2$ and $e_4$ set to zero, the Maxwell $U(1)$ field decouples from matters fields, and so does the neutral field $N$. It is evidently the Chern-Simons counterpart of the model considered in the case A, and similar analysis can be used to find soliton

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4Readers may consult discussions given in the case B for the relevance of the $CP^1$ model here.
solutions allowed for this case. Here we have the potential

\[ U = \frac{1}{2} (e_1^2 |\chi|^2 + e_2^2 |\psi|^2)(e_1 |\chi|^2 + e_3 |\psi|^2 - v^2)^2, \]  

(21)

and, if \( e_1 = e_3 \), there is an additional global \( SU(2) \) symmetry in the system. From the self-duality equations (9), we may conclude, just as in the case A (see Eq. (14)), that

\[ w(z) \equiv \frac{\psi}{\chi} = q \prod_{r=1}^{n_2} (z - z'_r)^{e_1/e_3} \prod_{r=1}^{n_1} (z - z_r), \]  

(22)

where \( q \) is an arbitrary complex constant and \( z_r (r = 1, \ldots, n_1) \) and \( z'_r (r = 1, \ldots, n_2) \) are zeros of \( \chi \) and \( \psi \), respectively. In the present case, however, the situation is more complicated because there is also a symmetric phase as well as an asymmetric or broken phase; in the symmetric phase, we have an additional possibility of nontopological soliton solutions.

First, we concentrate on the topological soliton solutions allowed in the broken phase, for which the analysis is almost an exact parallel to that of the case A. For example, the statements below Eq. (14) apply also to this case with \( e_2 \) and \( e_4 \) replaced by \( e_1 \) and \( e_3 \), respectively. In particular, with \( e_1 = e_3 \equiv e \), additional global \( SU(2) \) symmetry allows us to assume, without loss of generality, that

\[ r \to \infty : \quad \left( \begin{array}{c} \chi \\ \psi \end{array} \right) \to \left( \begin{array}{c} \chi_0 \\ 0 \end{array} \right), \quad |\chi_0|^2 = \frac{v^2}{e_1}, \]  

(23)

and we can follow Gibbons et al. [8] closely. An immediate observation is that the function \( w(z) \equiv \psi/\chi \) should vanish at infinity and Eq. (22) reduces to

\[ w(z) = \frac{Q_n(z)}{P_n(z)}, \]  

(24)

where \( P_n(z) = \prod_{r=1}^{n} (z - z_r) \), and \( Q_n(z) \) is a polynomial of \( z \) of order not larger than \( n - 1 \). Further, eliminating gauge field in Eq. (9) yields the following equation for
\[ |\chi| \text{ (note that } |\psi|^2 = |\omega|^2 |\chi|^2) : \]
\[ \nabla^2 \ln |\chi|^2 - \frac{4e^4}{\kappa^2} (|\chi|^2 + |\psi|^2) \left( |\chi|^2 + |\psi|^2 - \frac{v^2}{e} \right) = 4\pi \sum_{r=1}^{n} \delta(x - x_r) \]  
(25)

Choosing to work with dimensionless quantities, we make the replacements \( \chi \rightarrow v\chi/\sqrt{e}, \psi \rightarrow v\psi/\sqrt{e}, a_{\mu} \rightarrow 2v^2 a_{\mu}/\kappa, \) and then Eq. (25) reads
\[ \nabla^2 \ln |\chi|^2 - 4(e\eta - 1) = 4\pi \sum_{r=1}^{n} \delta(x - x_r). \]  
(26)

To facilitate the analysis of Eq. (26), we introduce the quantity
\[ \eta = \ln(|\chi|^2 + |\psi|^2) \]  
(27)
so that we can recast Eq. (26) as
\[ \nabla^2 \eta - e\eta(e\eta - 1) = \rho, \]  
(28)

where we defined
\[ \rho = \nabla^2 \ln(|P_n|^2 + |Q_n|^2). \]  
(29)

[Note that \( \nabla^2 \ln |z - z_r|^2 = 4\pi \delta(x - x_r). \)] Since \( |\chi|^2 + |\psi|^2 \) approaches 1 as \( r \rightarrow \infty \), we here expect that \( \eta \rightarrow 0 \) as \( r \rightarrow \infty \). Following ref. [8], we define
\[ u_1 = \nabla^2 \ln(|P_n|^2 + |Q_n|^2) - \sum_{r=1}^{n} \ln(|z - z_r|^2 + \mu), \]  
(30)
where \( \mu(>0) \) is for the moment an arbitrary constant. Setting \( \eta = u_1 + \bar{\eta} \) then gives
\[ \nabla^2 \bar{\eta} - h + e^{u_1} e^{\bar{\eta}} (1 - e^{u_1} e^{\bar{\eta}}) = 0 \]  
(31)
with \( h = 4 \sum_{r=1}^{n} \mu(|z - z_r|^2 + \mu)^{-2}. \) Eq. (31) corresponds to the variational problem associated with the functional
\[ \mathcal{A}(\bar{\eta}) = \int d^2 x \left[ \frac{1}{2} |\nabla \bar{\eta}|^2 + (e^{u_1+\bar{\eta}} - 1)^2 + 2h\bar{\eta} \right], \]  
(32)
where $\tilde{\eta}$ should vanish sufficiently fast as $r \to \infty$. With $Q_n \equiv 0$, this is in fact the action used by Wang \[9\] to prove the existence of multivortex solutions in the self-dual pure Chern-Simons model with one complex scalar field. His method can be adapted to the present case. Wang’s proof for $Q_n = 0$ rests essentially on showing that $A(\tilde{\eta})$ is a coercive functional and hence $A(\tilde{\eta})$ has a unique minimum. Other parts differing from the $Q_n = 0$ case having been checked already by Gibbons et al. \[8\] for sufficiently large $\mu$, the only crucial step needed in establishing coercivity is to estimate the quantity $\int d^2x e^{2u_1}(e^{\tilde{\eta}} - 1)^2$. Wang proved for $Q_n = 0$ that

$$\int d^2x e^{2u_1}(e^{\tilde{\eta}} - 1)^2 \geq \frac{1}{2} \int \left( \frac{|\tilde{\eta}|}{1 + |\tilde{\eta}|} \right)^2 - C,$$  \hspace{1cm} (33)$$

where $C$ is a positive constant. However, it is not difficult to see that the same argument can be used also for the case $Q_n \neq 0$ and hence Eq. (33) holds even if $Q_n \neq 0$. We conclude that for every choice of polynomial $P_n$ and $Q_n$, a unique solution exists and the scalar fields may be reconstructed from $\eta$ via

$$\begin{pmatrix} \chi \\ \psi \end{pmatrix} = \frac{1}{\sqrt{1 + |w|^2}} \begin{pmatrix} 1 \\ w \end{pmatrix} e^{\eta/2} \prod_{r=1}^{n} \frac{z - z_r}{|z - z_r|} = \frac{1}{\sqrt{|P_n|^2 + |Q_n|^2}} \begin{pmatrix} P_n \\ Q_n \end{pmatrix} e^{\eta/2}$$  \hspace{1cm} (34)$$

up to gauge transformations. The gauge field $a_i$ is then readily obtained using Eq. (3). The moduli space of solutions is just $C^{2n}$, the $4n$-dimensional space parametrized by the complex coefficients specifying the polynomials $P_n$ and $Q_n$.

We note that the equation of the form (34) has been given for the case $A$ in Ref. \[8\]. The index theorem analysis of Lee, Min and Rim \[7\] on the number of zero modes is also consistent with what we have shown rigorously here.

\[5\]Explicitly, the free parameters can be identified with $p_k, q_k \ (k = 0, 1, \ldots, n - 1)$ when one writes $P_n(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0$ and $Q_n(z) = q_{n-1}z^{n-1} + \cdots + q_1 + q_0$. 
A characteristic feature of Chern-Simons vortices is that, unlike the Ginzburg-Landau vortices, they carry nonzero angular momentum. The angular momentum is

\[ J = -\frac{\kappa}{2e^2} \int d^2x \varepsilon^{ij} x^i (D_0 \chi^* D_j \chi + D_0 \psi^* D_0 \psi + \text{c.c.}) \]

\[ = \int d^2x \varepsilon^{ij} x^i \left( \frac{f_{12}}{|\chi|^2 + |\psi|^2} \left[ |\chi|^2 (a_j - \partial_j \arg \chi) + |\psi|^2 (a_j - \partial_j \arg \psi) \right] \right), \quad (35) \]

where we used the self-duality equations on the second expression. Now suppose we restrict ourselves to solutions with \( P_n(z) = z^{n_1}, Q_n(z) = qz^{n_2}, (n_2 < n_1) \), where \( q \) is a constant, in Eq. (34). This corresponds to working with the ansatz

\[ \chi = |\chi| e^{i n_1 \theta}, \]
\[ \psi = |\psi| e^{i n_2 \theta}, \]
\[ a_i = \varepsilon^{ij} \frac{x_j r^2}{r^2} \left( g - \frac{n_1 + n_2}{2} \right), \quad (36) \]

where \( |\chi|, |\psi| \) and \( g \) are functions of \( r \) and \( |\psi|/|\chi| = qr^{n_2-n_1} \). Then Eq. (33) becomes

\[ J = \frac{\pi \kappa}{e^2} \left[ g^2(\infty) - g^2(0) \right] + \frac{\pi \kappa}{e^2} (n_1 - n_2) \int_0^\infty dr g' \frac{r^{2n_1} - |q|^{2n_2}}{r^{2n_1} + |q|^{2n_2}}. \quad (37) \]

Obviously, \( g(0) = (n_1 + n_2)/2 \) from Eq. (36), while the finite energy condition requires that \( g(\infty) = -\frac{1}{2}(n_1 - n_2) \). Therefore, the first term on the right hand side of Eq. (37) is equal to \(-\pi \kappa n_1 n_2/e^2 \). But the second term cannot be explicitly evaluated unless \( \psi \) vanishes identically. For the special cases corresponding to \( \psi \equiv 0 \), we have

\[ J = -\frac{\pi \kappa n_1^2}{e^2}, \quad (38) \]

which is in agreement with the result of Ref. [4].
Let us consider the case with $n_1 = 1$ and $n_2 = 0$ in more detail. For this case we may write
\[
\left( \begin{array}{c} \chi \\ \psi \end{array} \right) = \frac{1}{\sqrt{r^2 + |q|^2}} \left( \begin{array}{c} re^{i\theta} \\ q \end{array} \right) e^{\eta/2},
\]
where we have set $z - z_1 = re^{i\theta}$. It describes a vortex-like structure centered at $z = z_1$, with size and orientation determined by the complex parameter $q$. Using Eq. (28), we can determine the asymptotic behavior of $\eta$:
\[
e^n = 1 - \frac{4|q|^2}{r^4} - \frac{8|q|^2(8 + |q|^2)}{r^6} + O\left(\frac{1}{r^8}\right).
\]
This shows that if $q \neq 0$, scalar fields approach the vacuum values like $O(1/r^4)$, contrary to the $q = 0$ case for which the limit are approached exponentially. For $q \neq 0$ the magnetic field shows also a power fall-off at large $r$, being given by
\[
f_{12} = \frac{2|q|^2}{r^4} + \frac{4|q|^2(8 + |q|^2)}{r^6} + O\left(\frac{1}{r^8}\right).
\]
On the other hand, the behaviors for smaller $r$ are found as
\[
e^n = c_0 + \frac{c_0}{4} \left( \frac{4}{|q|^2} - c_0(c_0 - 1) \right) r^2 + O(r^4),
\]
\[
f_{12} = -\frac{1}{2} c_0(c_0 - 1) - \frac{1}{8} (1 - 2c_0) \left( \frac{4}{|q|^2} - c_0(c_0 - 1) \right) r^2 + O(r^4),
\]
where $c_0$ is a constant (which is equal to 1 if $q = 0$). Note that the magnetic field is nonzero at the origin for $q \neq 0$. Hence its behaviors are different from those of Ref. [4] at both short and large distances. Another interesting feature is that the solution approximates a $CP^1$ lump when $|q| \gg 1$. This was noted in Ref. [8] for the case A, and we can apply the same argument here. Namely, for $|q| \gg 1$, we have $\rho = 4|q|^2/(r^2 + |q|^2)^2 \approx 0$ and $\eta \approx 0$ so that the scalar fields lie effectively on the vacuum manifold $S^3$,
\[
\left( \begin{array}{c} \chi \\ \psi \end{array} \right) \approx \frac{1}{\sqrt{r^2 + |q|^2}} \left( \begin{array}{c} re^{i\theta} \\ q \end{array} \right).
\]
This represents a \( CP^1 \) lump and we may say that the topological solitons of this model interpolate between Chern-Simons vortices and \( CP^1 \) lumps. We have verified that the angular momentum \( J \), given by the expression \((37)\), approaches zero as we let \( q \to \infty \).

Some numerical results are shown in Fig. 2. In Fig 2(a) we plot rotationally symmetric solutions for the coupling ratios \( p \equiv \frac{\alpha}{\epsilon_1} = 1 \) and 2, while choosing \( |\chi(\infty)|^2 = |\chi_0|^2 \equiv v^2/2\epsilon_1 \). As in the case \( A \), sizes become smaller for larger \( p \) and the magnetic field gets confined in a narrow ring when \( p \) is large. The values of the magnetic field are zero at the origin, which is a characteristic feature of Chern-Simons vortices \([4]\). In Fig. 2(b) we plot the solutions of the form \((39)\) for \( |q| = 0.2, 1, 5, \) and 10. It is clear that the solutions approximate \( CP^1 \) lumps when \( |q| \gtrsim 5 \) just as in the case \( A \). The magnetic field here shows interesting behaviors; for small \( |q| \) it is ring-shaped (as in Fig. 2(a)) even if it does not vanish at the origin, while it looks similar to that in Fig. 1(a) for large \( |q| \). In this way, we see clearly that our solution indeed interpolates Chern-Simons vortices and \( CP^1 \) lumps.

Now we turn to nontopological soliton solutions allowed in symmetric phase. While Eq. \((22)\) is still valid, there is no restriction on \( n_1 \) and \( n_2 \) because the vacuum manifold is trivial. Let the asymptotic behaviors of scalar fields be such that

\[
|\chi| \sim \frac{1}{r^{\alpha}}, \quad |\phi| \sim \frac{1}{r^{\beta}},
\]

(44)

where \( \alpha, \beta \) are positive constants. Then, by Eq. \((22)\), \( \alpha \) and \( \beta \) are not independent but related by

\[
e_3(n_1 + \alpha) = e_1(n_2 + \beta).
\]

(45)

This relation is due to the fact that there is only one gauge field in this case. The
flux is given by

$$\Phi = \frac{2\pi(n_1 + \alpha)}{e_1} = \frac{2\pi(n_2 + \beta)}{e_3}. \quad (46)$$

At present there is no rigorous existence proof of nontopological solitons in Chern-Simons theory even with one scalar field, although their existence is strongly supported by numerical analysis. The situation is similar for the present case. Here we expect that the number of zero modes be larger than 4n, as we have seen in the model with one scalar field [4].

We have done some analysis on nontopological solitons in the case of $e_1 = e_3 = e$, assuming the form (39). Even if we do not know the exact solution, we can find approximate solutions for $|q| \gg 1$ by making use of the solutions for the corresponding non-relativistic self-dual model. Formally, in the non-relativistic limit, Eq. (25) reduces to

$$\nabla^2 \ln |\chi|^2 + |\chi|^2 + |\psi|^2 = 4\pi \sum_{r=1}^{n} \delta(x - x_r), \quad (47)$$

Other than the Liouville-type solutions obtained by setting $|\chi| = |\psi|$, the present author found that Eq. (47) admits a family of exact solutions [12] of the form

$$\left(\begin{array}{c}
\chi \\
\psi
\end{array}\right) = \frac{\sqrt{12}(P\partial_z Q - Q\partial_z P)}{(|P|^2 + |Q|^2)^{3/2}} \left(\begin{array}{c}
P \\
Q
\end{array}\right), \quad (48)$$

where $P, Q$ are polynomials of $z$ sharing no common zeros and $x_r$'s (see the right hand side of Eq. (47)) can be identified with zeros of $P(P\partial_z Q - Q\partial_z P)$. For our purpose, we take $P(z) = e^{\pi i/3}q_0^{-1/3}z$ and $Q(z) = e^{\pi i/3}q_0^{2/3}$ so that

$$\left(\begin{array}{c}
\chi \\
\psi
\end{array}\right) = \frac{\sqrt{12}|q|}{(r^2 + |q|^2)^{3/2}} \left(\begin{array}{c}
r e^{i\theta} \\
q
\end{array}\right), \quad (49)$$

Eq. (49) will be a good approximate solution of the relativistic system when $|q| \gg 1$. We have found the asymptotic behaviors $|\chi| \sim r^{-2}$ and $|\psi| \sim r^{-3}$, and so we identify
\( \alpha = 2 \) and \( \beta = 3 \) for this solution. The gauge field \( a_i \) is given by Eq. (36) and \( g(r) \) has limiting values \( g(\infty) = -5/2 \) and \( g(0) = -3/2 \). With these and using Eq. (37), we can calculate the angular momentum for this solution,

\[
J = \frac{4\pi\kappa}{e^2} - \frac{3\pi\kappa}{e^2|q|^2} \\
\simeq \frac{4\pi\kappa}{e^2}, \quad |q| \gg 1.
\]

Therefore, in symmetric phase, we find a non-zero angular momentum even if \( |q| \gg 1 \), contrary to the case in broken phase. Fig. 3(a) shows the plot of rotationally symmetric nontopological soliton solutions for \( \alpha = 2.1 \) and 3 when \( p = 2 \). In Fig. 3(b) we plot solutions of the form (39), for \( |q| = 5 \) and 10. Note that for \( |q| = 10 \), it is well approximated by the exact solutions of nonrelativistic self-duality equations. It gives \( \alpha = 2.056 \), which is not far from the nonrelativistic value \( \alpha = 2 \).

Closing this subsection we remark two things. First, this model may be viewed as a linearized version of the \( CP^1 \) model with the Chern-Simons term. The potential has been chosen by demanding self-duality, and then the symmetric phase becomes degenerate with the asymmetric phase. In this sense, this model interpolates two popular field-theoretic models useful for high-\( T_c \) superconductivity [1], namely, the \( CP^1 \) model and the Chern-Simons-Higgs model. Second, while the case \( A \) interpolates Ginzburg-Landau vortices and \( CP^1 \) lumps, the case we just discussed interpolates Chern-Simons vortices and \( CP^1 \) lumps (in broken phase). Then it is natural to expect that \( U(1) \) Maxwell-Chern-Simons theory [3] with two complex scalar fields interpolates all three kinds of solitons. This model is briefly discussed in the appendix.
C. $e_3 = e_4 = 0$

In this case, the field $\psi$ decouples from the rest and the potential reduces to

$$U = \frac{1}{2}(e_2|\chi|^2 - u^2)^2 + \frac{e_1^2}{\kappa^2}|\chi|^2 \left( e_1|\chi|^2 - \frac{e_1}{e_2}u^2 - \frac{\kappa e_2}{e_1}N \right)^2,$$

(51)

where we have adjusted the field $N$ by a suitable constant shift. Assuming $e_2 > 0$, the vacuum configuration then clearly corresponds to

$$|\chi|^2 = \frac{u^2}{e_2}, \quad N = 0.$$

(52)

So only the broken phase is possible. This case is different from the previous two cases in that both Maxwell and Chern-Simons gauge fields couple to a single complex scalar $\chi$ simultaneously. From Eqs. (5) and (6), we note that

$$\kappa \Phi_a = e_1 Q_\chi = 0.$$

(53)

Therefore, there is no charged vortex as noted in Ref. [3]. If $e_2 = 0$, this model reduces to the pure Chern-Simons model which is known to have charged vortices. Hence, $e_2 = 0$ is a singular point, i.e., the theory with $e_2 = 0$ has distinct behaviors from that with $e_2 \neq 0$.

We find it convenient to work with dimensionless quantities by making the replacements

$$x_\mu \rightarrow \frac{x_\mu}{\sqrt{2e_2 u}}, \quad \chi \rightarrow \frac{u}{\sqrt{e_2}} \chi, \quad N \rightarrow \frac{e_1^2 u^2}{\kappa e_2} N,$$

$$a_\mu \rightarrow \frac{\sqrt{2e_2 u}}{e_1} a_\mu, \quad A_\mu \rightarrow \sqrt{\frac{2}{e_2}} u A_\mu.$$

Then, after eliminating gauge fields, self-duality equations yield the following equation for $|\chi|$ and $N$ (here $\xi = e_1^4 u^2 / \kappa^2 e_2^3$):

$$\nabla^2 \ln |\chi|^2 - 2\xi |\chi|^2 (|\chi|^2 - N - 1) - |\chi|^2 + 1 = 4\pi \sum_{r=1}^{n} \delta(x - x_r),$$

$$\nabla^2 N - |\chi|^2 (|\chi|^2 - N - 1) = 0,$$

(54)
For these coupled equations, no analytic solution is available at present. For rotationally symmetric solutions, we write

\[ |\chi| = f(r)e^{in\theta}, \quad N = N(r), \]
\[ a_i = \frac{\varepsilon^{ij}x^j}{r^2}g_1(r), \quad A_i = \frac{\varepsilon^{ij}x^j}{r^2}[g_2(r) - n], \quad (55) \]

with various radial functions here satisfying the boundary conditions

\[ nf(0) = g_1(0) = N'(0) = 0, \]
\[ g_2(0) = n, \]
\[ g_1(\infty) = g_2(\infty) = h(\infty) = 0, \]
\[ f(\infty) = 1. \quad (56) \]

For these solutions, the angular momentum is given as

\[ J = -\frac{1}{\sqrt{2}e_2^2} \int d^2x\varepsilon^{ij}x^i(D_0\chi^*D_j\chi + D_0\chi D_j\chi^* + \varepsilon^{jk}F_{0k}F_{12}) \]
\[ = -\frac{1}{\sqrt{2}e_2^2} \int d^2x\varepsilon^{ij}x^i \left[ \frac{\varepsilon^{jk}x^k}{r^3}g_1'(g_1 + g_2) + \frac{\varepsilon^{jk}x^k}{r^3}g_1g_2' \right] \]
\[ = \frac{1}{\sqrt{2}e_2^2} \int_0^\infty dr2\pi \left( \frac{1}{2}g_1^2 + g_1g_2 \right)' \]
\[ = 0. \quad (57) \]

This null result is expected since there is no charged vortices in this model.

In Fig. 4, some numerical results are given for rotationally symmetric solutions with \( n = 1 \). Note that the profiles of \( |\chi| \) and \( F_{12} \) are similar to those found in the ordinary abelian Higgs model. The Chern-Simons magnetic field \( f_{12} \) shows an interesting behavior. It is positive for small \( r \), then changes sign and goes to 0 as
\( r \to \infty \). This is expected because there is no charged vortex, i.e., \( \Phi_a = 0 \), as we noted above.

The following comment on this model may be useful. One may view this model as the ordinary self-dual abelian Higgs model modified by the addition of the Chern-Simons term, and the effect is to make the neutral scalar \( N \) nontrivial. But the main features of this model are essentially the same as those of the abelian-Higgs model: there is only broken phase, and no charged or spinning vortex exists.

**D. \( e_1 = e_3, e_2 = e_4 \)**

Here, two complex scalar fields \( \chi \) and \( \psi \) enter the theory in such a way that an additional global \( SU(2) \) symmetry may be present. The potential is

\[
U = \frac{1}{2}(e_2|\Psi|^2 - u^2) + \frac{e_1}{\kappa^2}|\Psi|^2 \left( e_1|\Psi|^2 - \kappa \frac{e_2}{e_1}N \right),
\]

where \( \Psi = \begin{pmatrix} \chi \\ \psi \end{pmatrix} \). The solution of this case will possess properties corresponding to a juxtaposition of the cases A and C; in particular, there is no charged or spinning vortex solution. As we have seen in the cases A and B, \( SU(2) \) global symmetry enables us to write \( w(z) = \frac{\psi}{\chi} = \frac{Q_n(z)}{P_n(z)} \), where \( P_n, Q_n \) are the same as before. Then the resulting self-duality equations can be read from Eqs. (28) and (54),

\[
\nabla^2 \eta - 2\xi e^n(e^n - N - 1) - e^n + 1 = \rho,
\]

\[
\nabla^2 N - e^n(e^n - N - 1) = 0,
\]

where \( \eta \) and \( \rho \) are defined by Eqs. (27) and (29). In this case, however, a rigorous existence proof for \( \rho = 0 \) case is not available and, strictly speaking, it is just a conjecture that the solution exists for each \( P_n(z) \) and \( Q_n(z) \) in Eq. (54).
Without going further on this model, we just make one comment. As before we may consider this model as a linearized version of $CP^1$ model to which a Chern-Simons gauge field is coupled. A neutral field $N$ appears to make the system self-dual. From the self-duality equations (59), it is easy to see that for solutions of the form Eq. (39),

$$u \simeq 0,$$

$$N \simeq 0 \quad \text{when } |q| \gg 1$$

and they again approximate $CP^1$ lumps as expected.

E. $e_1 = e_3$, $e_2 = -e_4$ and $u^2 = 0$

Finally, we will consider the case with $e_1 = e_3$ and $e_2 = -e_4$, while setting $u^2 = 0$. This case is interesting because both gauge fields couple to scalar fields in a nontrivial manner, and there are both symmetric and asymmetric phases which are degenerate. The potential reads

$$U = \frac{e_2^2}{2}(|\chi|^2 - |\psi|^2)^2 + \frac{e_1^2}{\kappa^2}|\chi|^2 \left[ e_1(|\chi|^2 + |\psi|^2) - \frac{\kappa e_2}{e_1} N - v^2 \right]^2$$

$$+ \frac{e_1^2}{\kappa^2}|\psi|^2 \left[ e_1(|\chi|^2 + |\psi|^2) + \frac{\kappa e_2}{e_1} N - v^2 \right]^2$$

and we have a symmetric phase if the vacuum corresponds to $|\chi| = |\psi| = 0$ and $N = (\text{an arbitrary constant})$, and an asymmetric phase with the vacuum described by $|\chi|^2 = |\psi|^2 = v^2/2e_1$ and $N \equiv 0$.

To make the fields dimensionless, we make the replacements

$$x_\mu \rightarrow \frac{\sqrt{e_1}}{e_2 v} x_\mu, \quad \chi \rightarrow \frac{v}{\sqrt{2e_1}} \chi, \quad \psi \rightarrow \frac{v}{\sqrt{2e_1}} \psi,$$

$$N \rightarrow \frac{e_1^2 v^2}{2\kappa e_2} N, \quad a_\mu \rightarrow \frac{e_2 v}{e_1^{3/2}} a_\mu, \quad A_\mu \rightarrow \frac{v}{\sqrt{e_1}} u A_\mu.$$
Then after eliminating gauge fields from the self-duality equations we obtain the following set of equations:

\[
\nabla^2 \ln |\chi|^2 = \xi (|\chi|^2 + |\psi|^2)(|\chi|^2 + |\psi|^2 - 2) + |\chi|^2 - |\psi|^2 + 4\pi \sum_{r=1}^{n_1} \delta(x - x_r),
\]

\[
\nabla^2 \ln |\psi|^2 = \xi (|\chi|^2 + |\psi|^2)(|\chi|^2 + |\psi|^2 - 2) + |\chi|^2 - |\psi|^2 + 4\pi \sum_{r=1}^{n_2} \delta(x - x'_r),
\]

\[
\nabla^2 N = - (|\chi|^2 - |\psi|^2)(|\chi|^2 + |\psi|^2 - 2) + N(|\chi|^2 + |\psi|^2),
\]

where \( \xi \equiv \frac{e_1^3 v^2}{\kappa^2 e_2^2} \). Being nontrivially coupled and highly nonlinear, analytic solutions to these equations appear to be out of question. But with the ansatz

\[ |\chi| = |\psi|, \quad N = 0, \]

we can reduce the above three equations to the following single equation

\[
\nabla^2 \ln |\chi|^2 - 4\xi |\chi|^2(|\chi|^2 - 1) = 4\pi \sum_{r=1}^{n_1} \delta(x - x_r),
\]

which is identical to that appearing in the self-dual Chern-Simons model with one scalar field [4]. (See also Eq. (25).) It is unclear whether there exist solutions other than these. Especially, it will be interesting to see whether there is a solution with half integer vorticity for both \( \Phi_a \) and \( \Phi_A \) [6].

Before investigating some of these issues, we first calculate quantum numbers of rotationally symmetric configurations, which are specified as

\[
\chi = f_1(r)e^{in_1\theta}, \quad \psi = f_2(r)e^{in_2\theta},
\]

\[
a_i = \varepsilon^{ij} \frac{x^j}{r^2} \left( g_1(r) - \frac{n_1 + n_2}{2} \right), \quad A_i = \varepsilon^{ij} \frac{x^j}{r^2} \left( g_2(r) - \frac{n_1 - n_2}{2} \right),
\]

\[
N = N(r).
\]
To ensure finite energy, various functions here are subject to following boundary conditions: at spatial infinity,
\[
\begin{align*}
f_1(\infty) &= f_2(\infty) = 1 \text{ or } 0, \\
g_1(\infty) &= -\frac{\alpha + \beta}{2}, \quad g_2(\infty) = -\frac{\alpha - \beta}{2}, \\
N(\infty) &= N_0, \\
\end{align*}
\]  
(65)

while, at \( r = 0 \),
\[
\begin{align*}
n_1 f_1(0) &= n_2 f_2(0) = N'(0) = 0, \\
g_1(0) &= \frac{n_1 + n_2}{2}, \quad g_2(0) = \frac{n_1 - n_2}{2}. \\
\end{align*}
\]  
(66)

[In Eq. (65), we must set \( \alpha = \beta = N_0 = 0 \) for solutions in the broken phase.] Then, as usual, we find the flux values
\[
\begin{align*}
\Phi_a &= 2\pi \left( \frac{n_1 + n_2}{2} + \frac{\alpha + \beta}{2} \right), \\
\Phi_A &= 2\pi \left( \frac{n_1 - n_2}{2} + \frac{\alpha - \beta}{2} \right), \\
\end{align*}
\]  
(68)

while the angular momentum \( J \) is given by
\[
\begin{align*}
J &= -\frac{v}{2\sqrt{e_1 e_2}} \int d^2 x \varepsilon^{ij} x^i (D_0 \chi \ast D_j \chi + D_0 \psi \ast D_j \psi + \text{c.c.} + 2\varepsilon^{jk} F_{0k}F_{12}) \\
&= \frac{v}{2\sqrt{e_1 e_2}} \int d^2 x \left( \frac{2}{\sqrt{\xi}} g_1 \frac{g_1'}{r} + 2g_2 \partial_0 F^{i0} - 2F_{0i} \frac{x^i}{r} g_2' \right) \\
&= \frac{2\pi v}{\sqrt{e_1 e_2}} \left( \frac{g_1^2}{2\sqrt{\xi}} - g_2 r A_0' \right) \bigg|_{0}^{\infty} \\
&= -\frac{Q^2}{\pi \kappa} + \frac{\alpha + \beta}{e_1} Q \\
&= \frac{Q^2}{\pi \kappa} - \frac{n_1 + n_2}{e_1} Q, \\
\end{align*}
\]  
(69)
where \( Q = Q_\chi + Q_\psi (=2Q_\chi) \), from Eq. (5) denote the total electric charge. Note that \( J \) does not depend on couplings associated with the the Maxwell field; this is natural since Chern-Simons interaction is responsible for non-zero angular momentum.

Finally we turn to self-duality equations in Eq. (62) and discuss the possibility of vortex solutions with half-integer vorticity (i.e. \( \Phi_a = 2\pi(\frac{1}{2} + \frac{\alpha - \beta}{2}) \) and \( Phi_A = 2\pi(\frac{1}{2} + \frac{\alpha - \beta}{2}) \)). Since Eq. (62) is very complicated we made some numerical studies assuming the rotationally symmetric ansatz (64) with \( n_1 = 1, n_2 = 0 \). Specifically we tried relaxation methods using the package COLSYS [15] as well as shooting methods, both unidirectional and bidirectional [16]. But we failed to find such solutions for both topological and nontopological cases. A plausible reason for our negative finding might be that the boundary condition \( N(\infty) \) may well be inconsistent with one of the Gauss laws which requires that \( Q_\chi = Q_\psi \). As we shoot the fields near the origin, we found that \( N \) is either logarithmically divergent or becomes singular at a finite value of \( r \). [Note that with the ansatz (63), \( Q_\chi = Q_\psi \) is automatically satisfied.] Of course, as we have already seen in one instance (at the end of the subsection A), that we have appropriate self-duality equations does not imply the actual existence of a non-trivial solution for which the given energy bound is saturated. In spite of what we have observed, however, there remains some possibility that such solutions do exist and our numerical analysis is incomplete; for example, solutions may be found with initial parameters which we did not sweep. Therefore, it is a bit hasty to conclude that there is no vortex solution with half-integer vorticities in this model.

IV. SUMMARY

In this paper we have studied various interesting special cases in the \( U(1) \times U(1) \) self-dual Maxwell-Chern-Simons model. They include abelian Higgs system with two
complex scalar fields (the case A) and its Chern-Simons counterpart (the case B).

The latter can be regarded as a linearized version of the $CP^1$ model with the Chern-Simons term (while maintaining self-duality) and as such it interpolates two familiar field theory systems appearing in the discussion of high-$T_c$ superconductivity [1].

We proved the existence of multisoliton solutions for the topological soliton case, along the lines taken in Refs. [8] and [9]. In the symmetric phase we discussed the properties of nontopological soliton solutions by utilizing some known exact solutions of the corresponding nonrelativistic model. A more general field theory model interpolating the self-dual systems of the cases A and B is given in the appendix.

With $e_3 = e_4 = 0$, the resulting self-dual model consists of a complex scalar, a neutral scalar, the Maxwell field, and the Chern-Simons field. It has no charged vortex because of the Gauss law and has more or less the same properties as those of the ordinary abelian Higgs model. The role of the Chern-Simons field is to modify field configurations such that neutral field $N$ may have nontrivial values. Extension of this case to a system with two complex scalar fields was also discussed.

Finally we considered a model where all scalar fields and gauge fields couple to one another nontrivially. We calculated the quantum numbers for rotationally symmetric configurations, and showed that the ansatz $|\chi| = |\psi|$ leads back to the self-duality equations found in the ‘minimal’ self-dual Chern-Simons system [4]. We performed numerical analysis to find vortex solutions with half-integer vorticities and the result was negative. However, there is still a room for further examination.

As we mentioned in the introduction, this generalized abelian Higgs model with independent Chern-Simons gauge term may have some relevance in describing real physical systems. Then a variety of vortex structures we discussed in this paper
are expected to have some useful physical applications. These issues are under investigation.

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APPENDIX

Here, we briefly describe the self-dual topologically massive $U(1)$ gauge theory with two complex scalar fields and a neutral one, extending the model of Ref. [6] such that the model may have an additional global $SU(2)$ symmetry. The Lagrangian is given by

$$
L = -\frac{1}{4} F_{\mu \nu}^2 + \frac{1}{4} \varepsilon^{\mu \nu \rho} F_{\mu \nu} A_\rho + |D_\mu \Psi|^2 + \frac{1}{2} (\partial_\mu N)^2 - U(\Psi, N),
$$

(70)

where

$$
D_\mu \Psi = (\partial_\mu - ieA_\mu) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},
$$

and

$$
U(\Psi, N) = \frac{1}{2} (e|\Psi|^2 + \kappa N - ev^2)^2 + e^2 N^2 |\Psi|^2
$$

(71)

with

$$
|\Psi|^2 = |\phi_1|^2 + |\phi_2|^2.
$$
It has global $SU(2)$ symmetry in addition to local $U(1)$ symmetry. Gauss law reads

$$- \partial_i F_{i0} + \kappa F_{12} - e J^0 = 0,$$

where

$$J_0 = -i(\Psi^\dagger D_0 \Psi - D_0 \Psi^\dagger \Psi).$$

Integrating over space, we obtain the usual relation

$$\kappa \Phi = e Q,$$

where $Q = \int d^2 x J_0$. It is not difficult to show that the energy functional

$$E = \int d^2 x \left[ \frac{1}{2} F^2_{i0} + \frac{1}{2} F^2_{12} + |D_0 \Psi|^2 + |D_i \Psi|^2 + \frac{1}{2} (\partial_0 N)^2 + \frac{1}{2} (\partial_i N)^2 + U \right]$$

satisfies the following inequality

$$E \geq |e v^2 \Phi|,$$

where the equality holds if and only if the following self-duality equations are satisfied (static fields assumed):

$$(D_1 \pm iD_2) \Psi = 0,$$

$$F_{i0} \pm \partial_i N = 0,$$

$$(D_0 \mp i e N) \Psi = 0,$$

$$F_{12} \pm (\kappa N + e |\Psi|^2 - e v^2) = 0.$$

For a finite energy, we must here demand that

$$r \to \infty : \quad |\Psi|^2 \to v^2, \quad N \to 0 \quad \text{asymmetric phase},$$
or

\[ r \to \infty : \quad |\Psi| \to 0, \quad N \to \frac{\epsilon v^2}{\kappa} \quad \text{symmetric phase.} \]

In the asymmetric phase, the vacuum manifold is \( S^3 \). Therefore, as in the case A of Sec. III, the phase of the scalar field \( \Psi \) fibers the vacuum manifold as a \( U(1) \) bundle over \( CP^1 \simeq S^2 \), and so a given sector of the theory is specified by a choice of a point on \( S^2 \) and the winding number of the scalar field around the fiber over that point.

If only one complex scalar field were present, the model would interpolate the Ginzburg-Landau model and the pure Chern-Simons model (as noted in Ref. [6]). Hence, in the case of the above model (with \( SU(2) \) global symmetry), it should be clear that the case A and the case B of Sec. III can be obtained in the appropriate limiting cases. In other words, the solutions in this model will interpolates three kinds of solitons solutions, i.e., Ginzburg-Landau vortices, Chern-Simons vortices and \( CP^1 \) lumps.
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**Figure captions**

Figure 1. Plot of solutions for the case A. (a) Rotationally symmetric solutions with $n = 1$, for $p(\equiv e_4/e_2) = 1$ (dotted lines) and for $p = 2$ (solid line). We choose $|\chi(\infty)|^2 = |\chi_0|^2 \equiv u^2/2e_2$. Lines approaching zero asymptotically are magnetic fields $F_{12}$. The solid line approaching 1 is $|\chi|$, that approaching $1/\sqrt{2}$ is $|\psi|$, and dashed line approaching 1 is $|\chi| (= |\psi|)$ for $p = 1$. (b) The functions $e^{\eta}/2$ (approaching 1) and $F_{12}$ (approaching 0) for $n_1 = 1$ and $n_2 = 0$ in the
\[ e_2 = e_4 = e \] case. The dotted, solid, dashed, and dot-dashed lines correspond to \(|q| = 0.2, 1, 5, \text{ and } 10\), respectively.

Figure 2. Plot of solutions in an asymmetric phase for the case B. (a) Rotationally symmetric solutions with \(n = 1\) for \(p(\equiv e_3/e_1) = 1\) (dotted lines) and for \(p = 2\) (solid line). We choose \(|\chi(\infty)|^2 = |\chi_0|^2 \equiv v^2/2e_1\). Lines approaching zero asymptotically are magnetic fields \(f_{12}\). The solid line approaching 1 is \(|\chi|\), that approaching \(1/\sqrt{2}\) is \(|\psi|\), and the dashed line approaching 1 is \(|\chi| (= |\psi|)\) for \(p = 1\). (b) \(e^{n/2}\) (approaching 1) and \(f_{12}\) (approaching 0) for solutions having the form in Eq. (39). Dotted, solid, dashed, and dot-dashed lines correspond to \(q = 0.2, 1, 5, \text{ and } 10\), respectively.

Figure 3. Plot of solutions in the symmetric phase for the case B. (a) Solid lines are for \(n = 0\) and \(\alpha = 2.1\) with \(p \equiv e_3/e_1 = 2\). Dashed lines are for \(n = 1\) and \(\alpha = 3\) with \(p = 2\). Among the same types of lines having values less than 1, upper lines correspond to \(|\chi|\) and the other to \(|\psi|\). The magnetic field \(f_{12}\) is ring-shaped in both cases. (b) The functions \(e^{n/2}\) (starting at \(\sqrt{12}/10\)) and \(f_{12}\) for solutions of the form in Eq. (39). The dashed line is for \(|q| = 5\), and solid line for \(|q| = 10\). The dotted line is the plot of the function \(\sqrt{12}|q|/(r^2 + |q|^2)\) for \(|q| = 10\), a solution of nonrelativistic self-duality equations.

Figure 4. Plot of rotationally symmetric solutions with unit vorticity for the case C. Dashed and solid lines are for \(\xi = 1\) and 2, respectively. \(|\chi|\) are represented by lines approaching 1, and \(N\) by negative definite lines. Lines starting at 1 represent \(F_{12}\); lines having both positive and negative values represent \(f_{12}\).