MULTIDEGREES, PRIME IDEALS, AND NON-STANDARD GRADINGS

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ABSTRACT. We study several properties of multihomogeneous prime ideals. We show that the multigraded generic initial ideal of a prime has very special properties, for instance, its radical is Cohen-Macaulay. We develop a comprehensive study of multidegrees in arbitrary positive multigraded settings. In these environments, we extend the notion of Cartwright-Sturmfels ideals by means of a standardization technique. Furthermore, we recover or extend important results in the literature, for instance: we provide a multidegree version of Hartshorne’s result stating the upper semicontinuity of arithmetic degree under flat degenerations, and we give an alternative proof of Brion’s result regarding multiplicity-free varieties.

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1. INTRODUCTION

This paper is concerned with several aspects of the theory of multidegrees. The concept of multidegree provides the right generalization of the degree of a variety to a multiprojective setting, and its study goes back to seminal work by van der Waerden [60] in 1929. Multidegrees have found applications in several areas (e.g., algebraic geometry, commutative algebra, combinatorics, convex geometry, and more recently, algebraic statistics), and there is a long list of more recent papers where the notion of multidegree (or mixed multiplicity) plays a fundamental role (see, e.g., [1, 6, 7, 9, 11, 15, 18, 31, 34, 35, 39–42, 46, 47, 56, 58, 59]).

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The goal of this paper is twofold. Firstly, we concentrate on studying several interesting properties that prime ideals enjoy in a multigraded setting. Secondly, we make a comprehensive study of multidegrees in non-standard multigradings.

For organizational purposes, we divide the Introduction into two shorter subsections.

1.1. Prime ideals.

Let $k$ be a field and $S = k[x_{ij} | 1 \leq i \leq p, 0 \leq j \leq m_i]$ be a standard $\mathbb{N}^p$-graded polynomial ring with grading induced by setting $\deg(x_{ij}) = e_i \in \mathbb{N}^p$, where $e_i$ is the $i$-th standard basis vector. Then $S$ is naturally seen as the multihomogeneous coordinate ring of the product of projective spaces $P = P_{k}^{m_1} \times \cdots \times P_{k}^{m_p}$.

Let $P \subset S$ be an $S$-homogeneous prime ideal. A fundamental goal of this paper is to study several properties of $P$. Although our results regarding prime ideals do not involve in principle multidegrees, our proofs heavily depend on the use of this concept. The fact that prime ideals are quite special from a multigraded point of view has already been noticed. To highlight some of these we point out: Castillo–Cid-Ruiz–Li–Montaño–Zhang’s result that the support of the positive multidegrees of $S/P$ forms a discrete polymatroid [7, Theorem A], Brändén–Huh’s result that the volume polynomial (a different encoding of multidegrees) of $S/P$ is Lorentzian [2, Theorem 4.6], and Brion’s result on multiplicity-free varieties [3].

Our first main result is regarding multigraded generic initial ideals. Given a monomial order $>$ on $S$, one may define the multigraded generic initial ideal in analogy with the singly-graded setting (see Section 5 for details).

**Theorem A (Theorem 5.6).** ($k$ infinite) We have that $\sqrt{\text{gin}_{\succ}(P)}$ is a Cohen-Macaulay ideal.

This result can be seen as yet another manifestation of the fact that the initial ideals of primes are “special”. For example:

- In [38], Kalkbrener and Sturmfels showed that the radical of the initial ideal of a prime is equidimensional and connected in codimension one.  
- In [54, Chapter 8], Sturmfels proved that the radical of the initial ideal of a toric ideal is Cohen-Macaulay.  
- In [33], Hoşten and Thomas proved that the associated primes of the initial ideal of a toric ideal satisfy a saturated chain property.

On the other hand, it is not difficult to find a prime ideal $P$ such that $\text{in}_{\succ}(P)$ and its radical are not Cohen-Macaulay (see Remark 5.8). In this sense, Theorem A is a sharp result.

We also discovered an interesting (and perhaps unexpected) behavior of the minimal primary components of $\text{gin}_{\succ}(P)$. Let $J = \{j_1, \ldots, j_k\} \subset [p] = \{1, \ldots, p\}$ be a subset. We denote by $S_{(3)}$ the polynomial subring generated by the variables with degrees $e_{j_1}, \ldots, e_{j_k}$, and we write $I_{(3)} = I \cap S_{(3)}$ for the corresponding contraction of an $S$-homogeneous ideal $I \subset S$. For an ideal $I \subset S$, let $\text{MLength}(I)$ denote the maximal length of the minimal primary components of $I$. The following theorem shows that $\text{MLength}$ cannot increase under the natural projections of $\text{gin}_{\succ}(P)$.

**Theorem B (Theorem 5.9).** ($k$ infinite) We have the inequality

$$\text{MLength} \left( \text{gin}_{\succ} \left( P_{(3)} \right) \right) \leq \text{MLength} \left( \text{gin}_{\succ} \left( P \right) \right).$$
Moreover, for every \( p \in \text{Min}_{S, (3)} \left( \text{S}/\text{gin}_> (P_{(3)}) \right) \), we can find \( q \in \text{Min}_S \left( \text{S}/\text{gin}_> (P) \right) \) such that the length of the \( p \)-primary component of \( \text{gin}_> (P_{(3)}) \) divides the length of the \( q \)-primary component of \( \text{gin}_> (P) \).

To better appreciate the complexity of the process described in Theorem B, the reader is referred to Example 5.11. Also, Theorem B may not hold for non prime ideals (see Remark 5.12).

Let \( X = \text{MultiProj}(S/P) \subseteq \mathbb{P}^m_1 \times_k \cdots \times_k \mathbb{P}^m_p \) be the integral closed subscheme corresponding to \( P \). We also consider the behavior of the multidegrees of \( X \) under the natural projections. It turns out that our proof of Theorem B is a consequence of this study of multidegrees under projections. The projection corresponding to \( J = \{j_1, \ldots, j_k\} \) is given by \( \Pi_J : \mathbb{P} = \mathbb{P}^m_1 \times_k \cdots \times_k \mathbb{P}^m_p \to \mathbb{P}' = \mathbb{P}^m_1 \times_k \cdots \times_k \mathbb{P}^m_{p'} \). For each \( n = (n_1, \ldots, n_p) \in \mathbb{N}^p \) with \( |n| = n_1 + \cdots + n_p = \dim(X) \), one says that \( \text{deg}_P^n (X) \) is the multidegree of \( X \) of type \( n \) with respect to \( P \) (see Section 2 for precise definitions). Geometrically speaking, when \( k \) is algebraically closed, \( \text{deg}_P^n (X) \) equals the number of points (counting multiplicity) in the intersection of \( X \) with the product \( L_1 \times_k \cdots \times_k L_p \subseteq \mathbb{P} \), where \( L_i \subseteq \mathbb{P}^m_i \) is a general linear subspace of dimension \( m_i - n_i \). Let \( \text{MDeg}_P^n (X) \) be the maximum of the multidegrees of \( X \subseteq \mathbb{P} \). The next theorem shows that \( \text{MDeg} \) cannot increase under the natural projections.

**Theorem C (Theorem 4.2).** We have the inequality

\[ \text{MDeg}_P^n (\Pi_J (X)) \leq \text{MDeg}_P (X). \]

Moreover, for any \( d \in \mathbb{N}^k \) with \( |d| = \dim(\Pi_J (X)) \) and such that \( \text{deg}_P^d (\Pi_J (X)) > 0 \), there exists some \( n \in \mathbb{N}^P \) with \( |n| = \dim(X) \) and such that \( \text{deg}_P^n (X) > 0 \) and \( \text{deg}_P^d (\Pi_J (X)) \) divides \( \text{deg}_P^n (X) \).

If we drop the condition that \( P \) is a prime ideal, then one can find fairly simple instances where multidegrees do increase under projections (see Example 4.9).

Finally, we also provide an alternative proof of the following remarkable result of Brion [3] regarding multiplicity-free varieties. We say that \( X \subseteq \mathbb{P} \) is multiplicity-free if \( \text{deg}_P^n (X) \in \{0, 1\} \) for all \( n \in \mathbb{N}^P \) with \( |n| = \dim(X) \). Let \( \text{MSupp}_P (X) = \{n \in \mathbb{N}^P \mid |n| = \dim(X) \text{ and } \text{deg}_P^n (X) > 0\} \) be the support of positive multidegrees of \( X \).

**Theorem D (Theorem 6.6; Brion [3]).** If \( X \subseteq \mathbb{P} = \mathbb{P}^m_1 \times_k \cdots \times_k \mathbb{P}^m_p \) is multiplicity-free, then:

(i) \( X \) is arithmetically Cohen-Macaulay (i.e., \( S/P \) is a Cohen-Macaulay ring).

(ii) \( X \) is arithmetically normal (i.e., \( S/P \) is a normal domain).

(iii) (\( k \) infinite) There is flat degeneration of \( X \) to the following reduced union of multiprojective spaces

\[ H = \bigcup_{n = (n_1, \ldots, n_p) \in \text{MSupp}_P (X)} \mathbb{P}^{n_1}_k \times_k \cdots \times_k \mathbb{P}^{n_p}_k \subseteq \mathbb{P} = \mathbb{P}^m_1 \times_k \cdots \times_k \mathbb{P}^m_p \]

where \( \mathbb{P}^{n_i}_k = \text{Proj} (k[\ldots, x_{i,m_i - n_i}, \ldots, x_{i,m_i}]) \subseteq \mathbb{P}^m_k = \text{Proj} (k[x_1, \ldots, x_{i,m_i}]) \) only uses the last \( n_i + 1 \) coordinates.

Our proof of Theorem D uses techniques quite different from the ones used in [3], and ideas related to the fiber-full scheme ([10, 13, 14]) play an important role. We also point out that our proof is valid for arbitrary fields.
1.2. Non-standard gradings.

Multidegrees in non-standard multigradings are considerably more complicated as they lack a clear multiprojective geometrical content, but they are still important because of the flexibility they provide in terms of applications. One prime example is Knutson–Miller’s result [40, Theorem A] expressing double Schubert polynomials and double Grothendieck polynomials as the multidegree polynomials and $X$-polynomials, respectively, of matrix Schubert varieties with a fine grading. This fine grading takes into account both row and column position; we provide some examples that explore the technicalities of this fine grading (up to certain sign changes) in Section 8. These fine gradings play a key role in the recent proof by Castillo–Cid-Ruiz–Mohammadi–Montaño [8] of a conjecture of Monical–Tokcan–Yong [49] stating the saturated Newton polytope property of double Schubert polynomials.

Let $R = \mathbb{k}[x_1, \ldots, x_n]$ be a positively $\mathbb{N}^p$-graded polynomial ring (that is, $\deg(x_i) \in \mathbb{N}^p \setminus \{0\}$ for all $1 \leq i \leq n$ and $\deg(\alpha) = 0 \in \mathbb{N}^p$ for all $\alpha \in \mathbb{k}$). Let $I \subset R$ be an $R$-homogeneous ideal. The multidegree polynomial $C(R/I; t) = \mathcal{C}(R/I; t_1, \ldots, t_p)$ of $R/I$ is defined in terms of the multigraded Hilbert series of $R/I$ (see Section 2 for details). In the standard multigraded setting it may be seen as an algebraic encoding of the multidegrees of the corresponding multiprojective scheme (see Theorem 2.12).

Our approach is to associate to $I$ an ideal $J$ in a standard $\mathbb{N}^p$-graded polynomial ring $S$ such that $C(R/I; t) = C(S/J; t)$. We call $J$ the standardization of $I$ (see Setup 7.1). This construction is inspired by the techniques step-by-step homogenization introduced by McCullough–Peeva in [45] for singly-graded settings and standardization introduced in [8] for certain multigraded settings. It turns out that, there is a tight relation between $I \subset R$ and its standardization $J \subset S$, as it may be witnessed from the following statements:

(i) $\text{codim}(I) = \text{codim}(J)$.
(ii) $I \subset R$ and $J \subset S$ have the same $\mathbb{N}^p$-graded Betti numbers.
(iii) $\mathcal{X}(R/I; t) = \mathcal{X}(S/J; t)$ and $C(R/I; t) = C(S/J; t)$.
(iv) $R/I$ is a Cohen-Macaulay ring if and only if $S/J$ is a Cohen-Macaulay ring.
(v) Initial ideals are compatible with standardization.
(vi) If $I \subset R$ is a prime ideal and it does not contain any variable, then $J \subset S$ is also a prime ideal.

For more details, see Theorem 7.2.

These results allow us to deduce in Theorem 7.5 that the support of $C(R/P; t)$ is a discrete polymatroid for any prime ideal $P \subset R$, which constitutes an extension of [7, Theorem A].

We introduce the notion of arithmetic multidegree polynomial, which can be seen as a generalization of the notion of arithmetic degree considered by Sturmfels–Trung–Vogel in [55]. It is defined as follows

$$A(R/I; t) = \sum_{P \in \text{Ass}(R/I)} \text{length}_{R_p}(H^0_p((R/I)_P)) \mathcal{C}(R/P; t).$$

Motivated by the work of Hartshorne on the connectivity of Hilbert schemes [29], we show in Theorem 3.8 that arithmetic multidegree is upper semicontinuous under flat degenerations. As a consequence, in the case of Gröbner degenerations, we obtain the coefficient-wise inequality

$$A(R/I; t) \leq_c A(R/\text{in}_w(I); t).$$

This upper semicontinuity result was obtained by Hartshorne [29, Chapter 2] in the singly-graded setting.
In §7.1, we extend the notion of Cartwright–Sturmfels ideals to arbitrary positive multigradings. The original definition was given over a standard multigraded setting by Conca–De Negri–Gorla in a series of papers [15–19]. This was obtained as an abstraction of previous work of Cartwright and Sturmfels [6]; thus the chosen name.

In the current arbitrary positive multigraded setting, we say that \( I \subset R \) is Cartwright–Sturmfels (CS for short) if there is a radical Borel-fixed ideal in the standard \( \mathbb{Z}^p \)-graded polynomial ring \( S \) that has the same \( \mathcal{K} \)-polynomial as \( I \) (see Definition 7.7). It should be mentioned that \( I \subset R \) is CS if and only if its standardization \( J \subset S \) is CS (in the sense of [18]). By utilizing the close relation between \( I \subset R \) and its standardization, in Theorem 7.10, we prove the following statements when \( I \) is CS:

(i) \( \text{in}_{>}(I) \) is radical and CS for any monomial order \( > \) on \( R \); in particular, \( I \) is radical.

(ii) The \( \mathbb{N} \)-graded Castelnuovo-Mumford regularity \( \text{reg}(I) \) is bounded from above by \( p \).

(iii) \( I \) is a multiplicity-free ideal.

(iv) If \( P \subset R \) is a minimal prime of \( I \), then \( P \) is CS.

(v) All reduced Gröbner bases of \( I \) consist of elements of multidegree \( \leq (1,\ldots,1) \in \mathbb{N}^p \). In particular, \( I \) has a universal Gröbner basis of elements of multidegree \( \leq (1,\ldots,1) \in \mathbb{N}^p \).

This shows that CS ideals in arbitrary positive multigradings enjoy similar desirable properties as in the standard multigraded setting.

Finally, as a consequence of our work, we portray a family of multigraded Hilbert functions whose ideals have very rigid properties. It is most convenient to enunciate this result in terms of the multigraded Hilbert scheme of Haiman and Sturmfels [28]. Let \( h : \mathbb{N}^p \to \mathbb{N} \) be a function and consider the corresponding multigraded Hilbert scheme \( HS^h_{R/k} \) parametrizing \( R \)-homogeneous ideals with Hilbert function equal to \( h \).

**Theorem E** (Corollary 7.11). If \( HS^h_{R/k} \) contains a \( k \)-point that corresponds to a CS ideal and a \( k \)-point that corresponds to a prime ideal, then any \( k \)-point in \( HS^h_{R/k} \) corresponds to an ideal that is Cohen-Macaulay and CS.

This theorem can be seen as a generalization of a previous result by Cartwright and Sturmfels [6, Theorem 2.1 and Corollary 2.6].

**Outline.** The basic outline of this paper is as follows. In Section 2, we set the notation and recall basic definitions. In Section 3, we prove the upper semicontinuity of arithmetic multidegree under flat degenerations. Section 4 is dedicated to prove Theorem C. We establish Theorem A and Theorem B in Section 5. We study multiplicity-free varieties and prove Theorem D in Section 6. In Section 7, we develop the new notion of CS ideals and prove Theorem E. Finally, in Section 8, we present several examples of determinantal ideals with a fine grading.

2. Preliminaries

In this preparatory section, we briefly recall some basic concepts and we fix the notation. In particular, we establish the initial stage regarding the main theme of this paper, that of multidegrees. For more details on multidegrees the reader is referred to [11, 48].

We use a multi-index notation. Let \( p \geq 1 \) be a positive integer and, for each \( 1 \leq i \leq p \), let \( e_i \in \mathbb{N}^p \) be the \( i \)-th elementary vector \( e_i = (0,\ldots,1,\ldots,0) \). If \( n = (n_1,\ldots,n_p) \), \( m = (m_1,\ldots,m_p) \in \mathbb{Z}^p \) are two vectors,
we write $\mathbf{n} \succeq \mathbf{m}$ whenever $n_i \geq m_i$ for all $1 \leq i \leq p$, and $\mathbf{n} > \mathbf{m}$ whenever $n_j > m_j$ for all $1 \leq j \leq p$. For any $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$, we set $\mathbf{n}! = n_1! \cdots n_p!$. We write $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$, respectively.

A discrete polymatroid $\mathcal{P}$ on $[p] := \{1, \ldots, p\}$ is a collection of points in $\mathbb{N}^p$ of the following form

$$\mathcal{P} = \left\{ (n_1, \ldots, n_p) \in \mathbb{N}^p \mid \sum_{j \in J} n_j \leq r(J), \forall J \subseteq [p], \sum_{i \in [p]} n_i = r([p]) \right\}$$

with $r : 2^{[p]} \to \mathbb{N}$ being a rank function on $[p]$. A rank function on $[p]$ is a function $r : 2^{[p]} \to \mathbb{N}$ satisfying the following three properties:

(i) $r(\emptyset) = 0$.

(ii) $r(J_1) \leq r(J_2)$ if $J_1 \subseteq J_2 \subseteq [p]$.

(iii) $r(J_1 \cup J_2) + r(J_1 \cap J_2) \leq r(J_1) + r(J_2)$ if $J_1, J_2 \subseteq [p]$.

A comprehensive discussion regarding polymatroids can be found in [51].

**Remark 2.1.** The following statements hold:

(i) Let $\mathcal{P}$ be a discrete polymatroid on $[p]$ with rank function $r : 2^{[p]} \to \mathbb{N}$. Let $m_1, \ldots, m_p$ be positive integers such that $r(\{i\}) \leq m_i$. Then the function $s : 2^{[p]} \to \mathbb{N}$ given by

$$s(J) := \sum_{j \in J} m_j + r([p] \setminus J) - r([p])$$

is a rank function. The discrete polymatroid $\mathcal{P}^*$ on $[p]$ determined by $s$ is said to be a dual of $\mathcal{P}$.

(ii) If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two discrete polymatroids on $[p]$, then the Minkowski sum $\mathcal{P}_1 + \mathcal{P}_2$ is a discrete polymatroid on $[p]$.

*Proof.* (i) See [51, §44.6f].

(ii) See [51, Theorem 44.6, Corollary 46.2c].

Following [49], we say that a polynomial $f = \sum_n c_n \mathbf{t}^n \in \mathbb{Z}[t_1, \ldots, t_p]$ has the Saturated Newton Polytope property (SNP property for short) if the support $\text{supp}(f) = \{ \mathbf{n} \in \mathbb{N}^p \mid c_n \neq 0 \}$ of $f$ is equal to Newton($f$) $\cap \mathbb{N}^p$, where Newton($f$) = ConvexHull($\mathbf{n} \in \mathbb{N}^p \mid c_n \neq 0$) denotes the Newton polytope of $f$; in other words, if the support of $f$ consists of the integer points of a polytope.

**Remark 2.2.** Given a homogeneous polynomial $f \in \mathbb{Z}[t_1, \ldots, t_p]$, if supp($f$) is a discrete polymatroid on $[p]$, then $f$ has the SNP property.

### 2.1. Multidegrees in positive multigradings

Here we briefly review the notion of multidegrees in positive multigradings that are not necessarily standard.

Let $k$ be a field and $R = k[x_1, \ldots, x_n]$ be a positively $\mathbb{N}^p$-graded polynomial ring (that is, $\deg(x_i) \in \mathbb{N}^p \setminus \{\mathbf{0}\}$ for all $1 \leq i \leq n$ and $\deg(\alpha) = \mathbf{0} \in \mathbb{N}^p$ for all $\alpha \in k$). Let $M$ be a finitely generated $\mathbb{Z}^p$-graded $R$-module and $F^* \subseteq \mathbb{Z}^p$-graded free $R$-resolution $F^* : \cdots \to F_1 \to F_{i-1} \to \cdots \to F_1 \to F_0$ of $M$. Let $t_1, \ldots, t_p$ be variables over $\mathbb{Z}$ and consider the polynomial ring $\mathbb{Z}[t] = \mathbb{Z}[t_1, \ldots, t_p]$, where the variable $t_i$ corresponds to the $i$-th elementary vector $e_i \in \mathbb{N}^p$. If we write $F_i = \bigoplus J R(-b_{i,j})$ with $b_{i,j} = (b_{i,j,1}, \ldots, b_{i,j,p}) \in \mathbb{Z}^p$, then we define the Laurent polynomial $[F_i]_t := \sum_j t^{b_{i,j}} = \sum_j t_1^{b_{i,j,1}} \cdots t_p^{b_{i,j,p}}$.
Then, the $\mathcal{K}$-polynomial of $M$ is defined by

$$\mathcal{K}(M; t) := \sum_{i} (-1)^i [F_i]_t.$$  

**Remark 2.3.** Since $R$ is assumed to be positively graded, there is a well-defined notion of Hilbert series

$$\text{Hilb}_M(t) := \sum_{n \in \mathbb{Z}^P} \dim_k ([M]_n) t^n,$$

and then we can write

$$\text{Hilb}_M(t) = \frac{\mathcal{K}(M; t)}{\prod_{i=1}^n (1 - t^{\deg(x_i)})}.$$  

In particular, this shows that $\mathcal{K}(M; t)$ does not depend on the specific free resolution $F_\bullet$ (also, see [48, Theorem 8.34]).

**Definition 2.4.** The multidegree polynomial of a finitely generated $\mathbb{Z}^P$-graded $R$-module $M$ is the homogeneous polynomial $\mathcal{C}(M; t) \in \mathbb{Z}[t]$ given as the sum of all terms in

$$\mathcal{K}(M; 1 - t) = \mathcal{K}(M; 1 - t_1, \ldots, 1 - t_p)$$

having total degree $\text{codim}(M)$, which is the lowest degree appearing.

An $R$-homogeneous ideal $I \subset R$ is said to be multiplicity-free if the coefficients of $\mathcal{C}(R/I; t)$ belong to the set $\{0, 1\}$. We also have the following variation of the multidegree polynomial that was introduced in [15]. This notion was introduced with the goal of measuring the contribution of all the minimal primes (possibly of not maximal dimension), and in the case of Borel-fixed ideals, it does precisely that (see [15, Proposition 3.12]).

**Definition 2.5.** The $\mathcal{G}$-multidegree polynomial of a finitely generated $\mathbb{Z}^P$-graded $R$-module $M$ is the polynomial $\mathcal{G}(M; t) \in \mathbb{Z}[t]$ given as the sum of all terms in $\mathcal{K}(M; 1 - t)$ whose corresponding monomial is minimal in the support of $\mathcal{K}(M; 1 - t)$.

For a polynomial $p \in \mathbb{Z}[t]$, we denote by $[p]_i$ the sum of the terms of $p$ with total degree equal to $i$. So, for any finitely generated $\mathbb{Z}^P$-graded module $M$ with $\text{codim}(M) \geq i$, we obtain that $[\mathcal{K}(M; 1 - t)]_i$ equals $\mathcal{C}(M; t)$ if $\text{codim}(M) = i$ and $0 \in \mathbb{Z}[t]$ otherwise. We collect the following basic facts.

**Remark 2.6.** Let $M$ be a finitely generated $\mathbb{Z}^P$-graded $R$-module. The following statements hold:

(i) $\mathcal{C}(M; t)$ is the sum of the terms of $\mathcal{G}(M; t)$ having total degree equal to $\text{codim}(M)$.

(ii) If $z \in R$ is a homogeneous non-zero-divisor on $M$ with $\deg(z) = (a_1, \ldots, a_p) \in \mathbb{N}^P$, then

(a) $\mathcal{K}(M/zM; t) = (1 - t^a_1 \cdots t^a_p) \mathcal{K}(M; t)$,

(b) $\mathcal{C}(M/zM; t) = \langle \deg(z), t \rangle \mathcal{C}(M; t)$ where $\langle \deg(z), t \rangle = a_1 t_1 + \cdots + a_p t_p$.

(iii) (additivity) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of finitely generated $\mathbb{Z}^P$-graded $R$-modules. Then we have the equality $\mathcal{C}(M; t) = [\mathcal{K}(L; 1 - t)]_i + [\mathcal{K}(N; 1 - t)]_i$ with $i = \text{codim}(M)$.

(iv) (associativity formula)

$$\mathcal{C}(M; t) = \sum_{p \in \text{Ass}_R(M)} \text{length}_{R_p}((M_p) \mathcal{C}(R/P; t)).$$
Remark 7.4

(i) This part follows from the fact that any term in $\mathcal{X}(M;1-t)$ has total degree at least $\text{codim}(M)$ (see [48, Claim 8.54]).

(ii) It can be obtained by simple algebraic manipulations with Hilbert series (see [48, Exercise 8.12]).

(iii) It is a direct consequence of the additivity of Hilbert series and the inequalities $\text{codim}(L) \geq \text{codim}(M)$ and $\text{codim}(N) \geq \text{codim}(M)$.

(iv) This is proved by a standard use of prime filtrations; see [48, Theorem 8.53].

(v) Write $M \cong F/K$ with $F$ a finite rank $\mathbb{Z}[t]$-graded free $R$-module. By considering a Gröbner degeneration we may assume that $K$ is a monomial submodule (see, e.g., [21, Chapter 15]). Then the associativity formula (part (iv)) reduces to the case $M = R/P$ and $P$ a monomial prime ideal. For a monomial prime ideal $P = (x_{i_1}, \ldots, x_{i_m})$, part (ii)(b) yields $\mathcal{C}(R/P; t) = \langle \deg(x_{i_1})t \cdots \deg(x_{i_m})t \rangle$, and so the result follows from the fact that $R$ is positively $\mathbb{N}[t]$-graded. Alternatively, see Remark 7.4. □

We introduce yet another notion of multidegrees of polynomial. It can be seen as a generalization of the notion of arithmetic degree from [55] (also, see [29]). The goal of this notion is to capture the contribution of all the associated primes.

**Definition 2.7.** The arithmetic multidegree polynomial of a finitely generated $\mathbb{Z}[t]$-graded $R$-module $M$ is given by

$$\mathcal{A}(M; t) := \sum_{P \in \text{Ass}(M)} \text{length}_{R_P} \left( H^0_P(M_P) \right) \mathcal{C}(R/P; t).$$

### 2.2. Multidegrees in standard multigradings.

We now concentrate on standard multigradings. The study of standard multigraded algebras is of utmost importance as they correspond with closed subschemes of a product of projective spaces (see, e.g., [7] and the references therein).

Let $S = \mathbb{k}[x_1, \ldots, x_n]$ be a standard $\mathbb{N}[t]$-graded polynomial ring over a field $\mathbb{k}$. That is, the total degree of each variable $x_i$ is equal to one (i.e., for each $1 \leq i \leq n$, we have $\deg(x_i) = e_k_i \in \mathbb{N}[t]$ with $1 \leq k_i \leq p$) and $\deg(\alpha) = 0 \in \mathbb{N}[t]$ for all $\alpha \in \mathbb{k}$.

Suppose that for each $1 \leq i \leq p$ there are exactly $m_i + 1$ variables $x_j$ with degree $e_i$ (i.e., $m_i + 1 = \# \{1 \leq j \leq n \mid \deg(x_j) = e_i\}$). Thus the corresponding multiprojective scheme is the product of projective spaces $\text{MultiProj}(S) = \mathbb{P}_{\mathbb{k}}^{m_1} \times \mathbb{P}_{\mathbb{k}}^{m_2} \cdots \mathbb{P}_{\mathbb{k}}^{m_p}$. In this setting, the irrelevant ideal is given by $\mathfrak{N} := ([S]_{e_1}) \cap \cdots \cap ([S]_{e_p}) \subset S$. An $S$-homogeneous prime ideal $P \subset S$ is said to be relevant if $P \not\supseteq \mathfrak{N}$. Relevant prime ideals are important as they are the ones that have a well-defined geometrical counterpart. For more details on the MultiProj construction, the reader is referred to [7, 37]. The next remark shows that one can always restrict to relevant prime ideals when considering the multidegree polynomial.

**Remark 2.8.** Let $S' = S[x_{n+1}, \ldots, x_{n+p}]$ be a standard $\mathbb{N}[t]$-graded polynomial ring extending the grading of $S$ and with $\deg(x_{n+i}) = e_i$ for all $1 \leq i \leq p$; accordingly, we have that $\text{MultiProj}(S') = \mathbb{P}_{\mathbb{k}}^{m_1+1} \times \mathbb{P}_{\mathbb{k}}^{m_2} \cdots \mathbb{P}_{\mathbb{k}}^{m_p+1}$. Let $P \subset S$ be an $S$-homogeneous (not necessarily relevant) prime ideal, and set $P' = PS$ to be the extension of $P$ to $S'$. Then the following statements hold:

(i) $P' \subset S'$ is a relevant prime ideal.

(ii) $\mathcal{C}(S/P; t) = \mathcal{C}(S'/P'; t)$.

**Proof.** (i) It is clear that $P'$ is a prime ideal, and it is relevant because $x_{n+i} \not\in P'$ for all $1 \leq i \leq p$. 


(ii) Given a $\mathbb{Z}^p$-graded free $S$-resolution $F_\bullet$ of $S/P$, then $F_\bullet \otimes_S S'$ is a $\mathbb{Z}^p$-graded free $S'$-resolution of $S'/P'$ with the same Betti numbers. Therefore, $\mathcal{X}(S/P; t) = \mathcal{X}(S'/P'; t)$ and consequently $\mathcal{C}(S/P; t) = \mathcal{C}(S'/P'; t)$.

Let $M$ be a finitely generated $\mathbb{Z}^p$-graded $S$-module, and set $\mathbb{P} := \text{MultiProj}(S) = \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_p}_k$. The relevant support of $M$ is given by $\text{Supp}_{++}(M) := \text{Supp}(M) \cap \mathbb{P}$. There is a polynomial $P_M(t) = P_M(t_1, \ldots, t_p) \in \mathbb{Q}[t] = \mathbb{Q}[t_1, \ldots, t_p]$, called the \textit{Hilbert polynomial} of $M$ (see, e.g., [31, Theorem 4.1], [11, Theorem 3.4]), such that the degree of $P_M$ is equal to $\tau = \dim \left( \text{Supp}_{++}(M) \right)$ and

$$P_M(\nu) = \dim_k ([M]_\nu)$$

for all $\nu \in \mathbb{N}^p$ such that $\nu \gg 0$. Furthermore, if we write

$$(1) \quad P_M(t) = \sum_{n_1, \ldots, n_p \geq 0} e(n_1, \ldots, n_p) \binom{t_1 + n_1}{n_1} \cdots \binom{t_p + n_p}{n_p},$$

then $0 \leq e(n_1, \ldots, n_p) \in \mathbb{Z}$ for all $n_1 + \cdots + n_p = \tau$.

**Definition 2.9.** Under the notation of (1), we define the following invariants:

(i) For $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$ with $|\mathbf{n}| = \dim \left( \text{Supp}_{++}(M) \right)$, $e(\mathbf{n}; M) := e(n_1, \ldots, n_p)$ is the \textit{mixed multiplicity} of $M$ of type $\mathbf{n}$.

(ii) Let $R$ be an $\mathbb{N}^p$-graded quotient ring of $S$ and $X = \text{MultiProj}(R) \subset \mathbb{P}$ be the corresponding closed subscheme. For each $\mathbf{n} \in \mathbb{N}^p$ with $|\mathbf{n}| = \dim(X)$, $\deg^{R}_{\mathbb{P}}(X) := e(\mathbf{n}; R)$ is the \textit{multidegree of $X$ of type $\mathbf{n}$ with respect to $\mathbb{P}$}.

**Definition 2.10.** For a multiprojective scheme $X \subset \mathbb{P} = \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_p}_k$, the support of positive multidegrees of $X$ is denoted by $\text{MSupp}_{\mathbb{P}}(X) := \{ \mathbf{n} \in \mathbb{N}^p \mid |\mathbf{n}| = \dim(X) \text{ and } \deg^{R}_{\mathbb{P}}(X) > 0 \}$.

**Remark 2.11.** In a multigraded setting one needs to be careful because the notions of multidegrees introduced in Definition 2.4 and Definition 2.9 may not agree. This pathology stems from the fact that the support $\text{Supp}(M)$ and the relevant support $\text{Supp}_{++}(M)$ may be quite different for a $\mathbb{Z}^p$-graded $S$-module $M$. Nevertheless, we have the following unifying result.

**Theorem 2.12** ([11]). Let $M$ be a finitely generated $\mathbb{Z}^p$-graded $S$-module and set $\tau = \dim \left( \text{Supp}_{++}(M) \right)$. If $H^{0, \nu}_R(M) = 0$, then we have equality

$$\mathcal{C}(M; t) = \sum_{\mathbf{n} \in \mathbb{N}^p \atop |\mathbf{n}| = \tau} e(\mathbf{n}; M) t_1^{m_1 - n_1} \cdots t_p^{m_p - n_p}.$$

**Proof.** This is a consequence of [11, Theorem A]. Also, for more details, see [7, Remark 2.9].

For a closed subscheme $X \subset \mathbb{P} = \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_p}_k$, we say that the \textit{multidegree polynomial of $X$} is defined as

$$\mathcal{C}(X; t) := \sum_{\mathbf{n} \in \mathbb{N}^p \atop |\mathbf{n}| = \dim(X)} \deg^{R}_{\mathbb{P}}(X) t_1^{m_1 - n_1} \cdots t_p^{m_p - n_p}.$$

The above theorem implies that $\mathcal{C}(X; t) = \mathcal{C}(R; t)$ for an $\mathbb{N}^p$-graded quotient ring $R$ of $S$ such that $X = \text{MultiProj}(R)$ and $(0:R \mathfrak{J}^{\mathbf{n}}) = 0$.

The following piece of notation will be useful throughout the paper.
Notation 2.13 (Natural projections). Let \( \mathcal{J} = \{ j_1, \ldots, j_k \} \subseteq [p] = \{1, \ldots, p\} \) be a subset.

(i) Let \( \Pi_\mathcal{J} : \mathbb{P}_k^{m_1} \times_k \cdots \times_k \mathbb{P}_k^{m_p} \to \mathbb{P}_k^{m_{j_1}} \times_k \cdots \times_k \mathbb{P}_k^{m_{j_k}} \) denote the natural projection.

(ii) Denote by \( S_{(\mathcal{J})} \) the \( \mathbb{N}^k \)-graded \( k \)-algebra given by

\[
S_{(\mathcal{J})} := \bigoplus_{i_1 \geq 0, \ldots, i_p \geq 0, i_j = 0 \text{ if } j \notin \mathcal{J}} [S]_{(i_1, \ldots, i_p)},
\]

and denote by \( I_{(\mathcal{J})} \) the contraction \( I_{(\mathcal{J})} := I \cap S_{(\mathcal{J})} \) for any \( S \)-homogeneous ideal \( I \subset S \).

Since the coefficients of the multidegree polynomial are non-negative in positive gradings, it becomes natural to address the positivity of these coefficients. The next result characterizes the positivity of multidegrees, and it follows from [7].

Theorem 2.14 ([7]). Let \( P \subset S \) be an \( S \)-homogeneous (not necessarily relevant) prime ideal. Then the support of \( \mathcal{C}(S/P; t) \) is a discrete polymatroid.

Proof. By Remark 2.8, we may adjoin new variables to \( S \) and assume that \( P \subset S \) is a relevant prime ideal. Consider the closed subscheme \( X = \text{MultiProj}(S/P) \subset P = \mathbb{P}_k^{m_1} \times_k \cdots \times_k \mathbb{P}_k^{m_p} \). Then Theorem 2.12 implies that \( \mathcal{C}(S/P; t) = \sum_{\mathbf{n} \in \mathbb{N}^p} \deg_{\mathbb{F}}^{m - \mathbf{n}}(X) t^\mathbf{n} \in \mathbb{N}[t_1, \ldots, t_p] \).

Fix \( \mathbf{n} \in \mathbb{N}^p \) with \( |\mathbf{n}| = \text{codim}(P) \). From [7, Theorem A] we obtain that \( \deg_{\mathbb{F}}^{m - \mathbf{n}}(X) > 0 \) if and only if \( \sum_{j \in \mathcal{J}} m_j - \sum_{j \notin \mathcal{J}} n_j \leq \dim (\Pi_\mathcal{J}(X)) \) for all \( \mathcal{J} \subset [p] \). We have that \( r : 2^{[p]} \to \mathbb{N}, r(\mathcal{J}) := \dim (\Pi_\mathcal{J}(X)) \) is a rank function (see [7, Proposition 5.1]). Equivalently, we get that \( \deg_{\mathbb{F}}^{m - \mathbf{n}}(X) > 0 \) if and only if

\[
\sum_{j \in \mathcal{J}} n_j \leq s(\mathcal{J}) := \sum_{j \in \mathcal{J}} m_j + r([p] \setminus \mathcal{J}) - r([p]) \quad \text{for all } \mathcal{J} \subset [p],
\]

where \( s : 2^{[p]} \to \mathbb{N} \) is a rank function by Remark 2.1(i). Therefore the support of \( \mathcal{C}(S/P; t) \) is a discrete polymatroid, as claimed. \( \square \)

3. A MULTIDEGREE VERSION OF A RESULT OF HARTSHORNE

In this section, we extend some results of Hartshorne into the world of multidegrees. More precisely, by extending the results of [29, Chapter 2], we study the behavior of the arithmetic multidegree under flat degenerations (Definition 2.7). We shall see that the arithmetic multidegree is the multigraded extension of the numbers \( n_i \)'s introduced by Hartshorne in [29, Chapter 2].

3.1. The functor \( R^i \). Here we recall the definition of the functors \( R^i \) and some of their basic properties. Essentially the same construction was considered by Schenzel [50], under the name of dimension filtration, in his study of Cohen-Macaulay filtered modules.

Let \( R \) be a Noetherian ring and \( M \) be a finitely generated \( R \)-module.

Definition 3.1. For any \( i \geq 0 \), we set

\[
R^i(M) := \{ m \in M \mid \text{codim}(\text{Supp}_R(m)) \geq i \}.
\]

We first pause to point out some of the basic properties of the functor \( R^i \).

Remark 3.2. The following statements hold:
Remark 3.2

(i) $R^i(M)$ is an $R$-module.

(ii) $R^i$ is a left exact functor.

(iii) We have the equality $R^i(M) = H^0_{J^i(M)}(M)$ where

$$J^i(M) := \bigcap_{P \in \text{Ass}_R(M), \ \text{codim}(P) \geq i} P$$

is the intersection of the associated primes of $M$ of codimension $\geq i$. By convention, we set $J^i(M) = R$ if there is no $P \in \text{Ass}_R(M)$ with $\text{codim}(P) \geq i$.

(iv) Let $0 = \bigcap_j M_j$ be an irredundant primary decomposition of the submodule $0 \subset M$, and suppose that $M_j$ is $P_j$-primary. Then $R^i(M)$ is the intersection of those $M_j$ such that $\text{codim}(P_j) < i$.

(v) $\text{Ass}_R(R^i(M)) = \{ P \in \text{Ass}_R(M) \mid \text{codim}(P) \geq i \}$.

(vi) $\text{Ass}_R(M/R^i(M)) = \{ P \in \text{Ass}_R(M) \mid \text{codim}(P) < i \}$.

(vii) If $N$ is an $R$-submodule of $M$ supported on codimension $\geq i$, then $N \subset R^i(M)$.

Proof. (i) and (ii) are immediate from the definition of $R^i$.

(iii) It is clear that $R^i(M) \supseteq H^0_{J^i(M)}(M)$. Let $m \in R^i(M)$. By definition, $\text{codim}(\text{Ann}_R(m)) = \text{codim}(\text{Supp}_R(m)) \geq i$, and so it follows that $\text{Ass}_R(R/\text{Ann}_R(m)) \subseteq \{ P \in \text{Ass}_R(M) \mid \text{codim}(P) \geq i \}$. This implies that $J^i(M) \subseteq \sqrt{\text{Ann}_R(m)}$ and consequently $m \in H^0_{J^i(M)}(M)$. Thus we have the claimed equality.

(iv), (v) and (vi) follow from the already proved part (iii) and known behavior of the zeroth local cohomology with respect to associated primes and primary decompositions (see [21, Proposition 3.13]).

(vii) follows directly from the definition of $R^i$.

The proposition below extends [29, Proposition 2.2].

Lemma 3.3. Let $\phi : A \to B$ be homomorphism of Noetherian rings which is either faithfully flat or a localization map with $B = W^{-1}A$ and $W \subset A$ a multiplicatively closed set. Let $R$ be a finitely generated $A$-algebra and $M$ be a finitely generated $R$-module. Set $R_B = R \otimes_A B$ and $M_B = M \otimes_A B$. Then

$$R^i(M_B) \cong R^i(M) \otimes_A B$$

for all $i \geq 0$.

Proof. From Remark 3.2(iii) and the base change property of local cohomology (in either case $\phi$ is flat), we obtain

$$R^i(M) \otimes_A B = H^0_{J^i(M)}(M) \otimes_A B \cong H^0_{J^i(M)R_B}(M_B).$$

First, we assume that $\phi : A \to B = W^{-1}A$ is a localization map. Then, by basic properties of localizations, $\text{Ass}_{R_B}(M_B) = \{ PR_B \mid P \in \text{Ass}_R(M) \text{ and } P \not\supseteq W \}$ and $\text{codim}(PR_B) = \text{codim}(P)$ for any $P \in \text{Spec}(R)$. As a consequence, the equality $J^i(M)R_B = J^i(M_B)$ follows, and so we get $R^i(M) \otimes_A B \cong H^0_{J^i(M)R_B}(M_B) = H^0_{J^i(M_B)}(M_B) = R^i(M_B)$.

Next, we assume that $\phi : A \to B$ is faithfully flat. By [44, Theorem 23.2] the associated primes of $M_B$ are given by

$$\text{Ass}_{R_B}(M_B) = \bigcup_{P \in \text{Ass}_R(M)} \text{Ass}_{R_B}(R_B/PR_B).$$
On the other hand, since $R \rightarrow R_B$ is faithfully flat, it follows that $\text{codim}(J^i(M)) = \text{codim}(J^i(M)R_B)$ (see [43, (4.D) Theorem 3, page 28] and [43, (13.B) Theorem 19(3), page 79]). This yields the equality $\sqrt{J^i(M)R_B} = \sqrt{J^i(M_B)}$ (see Remark 3.2(iii)). So, we obtain $R^i(M) \otimes_A B \cong H^{0}_{J^i(M)R_B}(M_B) = H^{0}_{J^i(M_B)}(M_B) = R^i(M_B)$, and the result follows. \hfill $\square$

We now make the connection between the functors $R^i$ and the arithmetic multidegree introduced in Definition 2.7.

**Definition-Proposition 3.4.** Let $k$ be a field and $R = k[x_1, \ldots, x_n]$ be a positively $\mathbb{N}^p$-graded polynomial ring. For any $i \geq 0$, we define the $i$-th truncation multidegree polynomial by

$$\ell^i(M; t) := [\ell(R^i(M); t)]_i = \begin{cases} \ell(R^i(M); t) & \text{if codim}(R^i(M)) = i \\ 0 & \text{otherwise.} \end{cases}$$

Then one has the equality

$$\ell(M; t) = \sum_{i \geq 0} \ell^i(M; t).$$

**Proof.** For any $P \in \text{Ass}_R(M)$ with $\text{codim}(P) = i$, Remark 3.2(iii) yields the following equality

$$R^i(M)_P = H^0_{P}(M_P).$$

Thus by Remark 3.2(v) and the associativity formula of multidegrees (see Remark 2.6(iv)), we obtain

$$[\ell(R^i(M); t)]_i = \sum_{P \in \text{Ass}_R(M)} \text{length}_{R_P}(H^0_{P}(M_P)) \ell(R/P; t).$$

Hence summing over $i \geq 0$ gives the claimed equality. \hfill $\square$

### 3.2. Behavior under flat degenerations

In this subsection, we study the functor $R^i$ under a flat degeneration. The following setup is assumed throughout the present subsection.

**Setup 3.5.** Let $A$ be a universally catenary Noetherian domain. Let $R = A[x_1, \ldots, x_n]$ be a positively $\mathbb{N}^p$-graded polynomial ring over $A$. Let $Q = \text{Quot}(A)$ be the field of fractions of $A$. For any $p \in \text{Spec}(A)$, let $\kappa(p) := A_p/pA_p$ be the residue field at $p$.

Notice that, if $M$ is a finitely generated $Z^p$-graded $R$-module and $p \in \text{Spec}(A)$, then $M \otimes_A \kappa(p)$ is a finitely generated $Z^p$-graded module over the positively $\mathbb{N}^p$-graded polynomial ring $R \otimes_A \kappa(p) \cong \kappa(p)[x_1, \ldots, x_n]$. The goal of this subsection is to study the behavior of $\ell(M \otimes_A \kappa(p); t)$ when varying $p \in \text{Spec}(A)$. We have the following generic base change property of $R^i$.

**Proposition 3.6 ([29, Proposition 2.3]).** Let $M$ be a finitely generated $Z^p$-graded $R$-module. Then there is a dense open subset $\mathcal{U} \subset \text{Spec}(A)$ such that $R^i(M) \otimes_A \kappa(p) \cong R^i(M \otimes_A \kappa(p))$ for all $i \geq 0$ and $p \in \text{Spec}(A)$.

In the polynomial ring $Z[t] = Z[t_1, \ldots, t_p]$ we consider a coefficient-wise order. Let $\geq_c$ be the partial order on $Z[t]$ such that for any two polynomials $f = \sum_n f_n t^n$ and $g = \sum_n g_n t^n$ in $Z[t]$ we have $f \geq_c g$ if and only if $f_n \geq g_n$ for all $n \in \mathbb{N}^p$. The next proposition shows that the arithmetic multidegree polynomial cannot decrease under flat degenerations.
Proposition 3.7. Let $M$ be a finitely generated $\mathbb{Z}^p$-graded $R$-module, and suppose that $M$ is flat over $A$. Then we have the coefficient-wise inequality

$$A(M \otimes_A Q; t) \leq_c A(M \otimes_A \kappa(p); t)$$

for all $p \in \text{Spec}(A)$.

Proof. Fix $p \in \text{Spec}(A)$. By [30, Exercise II.4.11] or [27, Proposition 7.1.7] there is a discrete valuation ring $V$ of $Q$ that dominates $A_p$; that is, $A_p \subset V$ and $pA_p = n \cap A_p$ where $n$ is the closed point of $\text{Spec}(V)$. We have a field extension $\kappa(p) \hookrightarrow \kappa(n)$ and by utilizing Lemma 3.3, we obtain that

$$R^i(M_V \otimes_V \kappa(n)) = R^i(M \otimes_A \kappa(p)) \otimes_{\kappa(p)} \kappa(n),$$

where $M_V = M \otimes_A V$. Since multigraded Hilbert functions are preserved under base field extensions, we get the equality of polynomials $A(M \otimes_A \kappa(p); t) = A(M_V \otimes_V \kappa(n); t)$. Therefore, we can substitute $A$ by $V$, and for the rest of the proof we assume that $A$ is a discrete valuation ring with maximal ideal $p$.

Over the discrete valuation ring $A$, a module is $A$-flat if and only if it is $A$-torsion-free. Since $M$ is $A$-flat by assumption, we have $P \cap A = 0$ for all $P \in \text{Ass}_R(M)$. Then Remark 3.2(vi) imply that both $R^i(M)$ and $M/R^i(M)$ are $A$-flat for all $i \geq 0$. From Lemma 3.3, we get $R^i(M) \otimes_A Q = R^i(M \otimes_A Q)$. The $A$-flatness of $R^i(M)$ yields the equalities

$$c^i(M \otimes_A Q; t) = c^i(M \otimes_A Q; t) = c^i(M \otimes_A \kappa(p); t),$$

and so we obtain $\text{codim}(R^i(M) \otimes_A \kappa(p)) \geq i$. The $A$-flatness of $M/R^i(M)$ gives the injection

$$R^i(M) \otimes_A \kappa(p) \hookrightarrow M \otimes_A \kappa(p).$$

Then we obtain the injection $R^i(M) \otimes_A \kappa(p) \hookrightarrow R^i(M \otimes_A \kappa(p))$ (see Remark 3.2(vii)), and so the additivity and positivity of multidegree polynomials give the inequality

$$c^i(M \otimes_A \kappa(p); t) \leq_c c^i(M \otimes_A \kappa(p); t)$$

(see Remark 2.6(iii)(v)). By combining everything together we get the inequality

$$c^i(M \otimes_A Q; t) = c^i(M \otimes_A Q; t) \leq_c c^i(M \otimes_A \kappa(p); t) = c^i(M \otimes_A \kappa(p); t).$$

This concludes the proof of the proposition.

We are now ready to present the main result of this section.

Theorem 3.8. Assume Setup 3.5. Let $M$ be a finitely generated $\mathbb{Z}^p$-graded $R$-module, and suppose that $M$ is flat over $A$. Then the function

$$\text{Spec}(A) \to \mathbb{Z}[t], \quad p \mapsto A(M \otimes_A \kappa(p); t)$$

is upper semicontinuous with respect to the component-wise order $\geq_c$ on $\mathbb{Z}[t]$.

Proof. It suffices to show that

$$E_h := \{ p \in \text{Spec}(A) \mid A(M \otimes_A \kappa(p); t) \leq_c h \}$$

is an open subset for an arbitrary $h \in \mathbb{Z}[t]$. Let $h \in \mathbb{Z}[t]$. The topological Nagata criterion for openness (see, e.g., [44, Theorem 24.2]) says that $E_h$ is open if and only if the following two conditions are satisfied:
(i) If $p,q \in \text{Spec}(A)$, $p \in E_h$ and $p \supseteq q$, then $q \in E_h$.

(ii) If $p \in E_h$, then $E_h$ contains a non-empty open subset of $V(p) \subset \text{Spec}(A)$.

Condition (i) follows from Proposition 3.7. Condition (ii) is obtained by utilizing Proposition 3.6 and Grothendieck’s generic freeness lemma (see, e.g., [44, Theorem 24.1]). This establishes the result of the theorem.

3.3. Behavior under Gröbner degenerations. For the sake of completeness, we restate our results in terms of Gröbner degenerations.

**Theorem 3.9.** Let $k$ be a field, $R = k[x_1, \ldots, x_n]$ be a positively $\mathbb{N}^p$-graded polynomial ring and $>$ be a monomial order (or a weight order) on $R$. For any $R$-homogeneous ideal $I \subset R$, we have that

$$A(R/I; t) \leq_c A(R/\text{in}_>(I); t).$$

**Proof.** It is a direct consequence of [21, Theorem 15.17] and Proposition 3.7.

This result portrays a different behavior of the arithmetic multidegree, since we always have the equalities $\mathcal{C}(R/I; t) = \mathcal{C}(R/\text{in}_>(I); t)$ and $\mathfrak{g}(R/I; t) = \mathfrak{g}(R/\text{in}_>(I); t)$. These two equalities follow from the fact that these two notions of multidegree are determined by the Hilbert series (see Definition 2.4 and Definition 2.5).

4. Multidegrees of projections

In this section, we study the multidegrees of all the possible natural projections of an integral multiprojective scheme $X \subset \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_p}_k$. We show that the maximal multidegree of a projection is always smaller or equal than the maximal multidegree of $X$. Even more, we prove that any multidegree of a projection must divide some multidegree of $X$. The setup below is used throughout this section.

**Setup 4.1.** Let $k$ be a field and $S = k[x_1, \ldots, x_n]$ be a standard $\mathbb{N}^p$-graded polynomial ring. Suppose that $\text{MultiProj}(S) = \mathbb{P} = \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_p}_k$. Let $X = \text{MultiProj}(R) \subset \mathbb{P}$ be an integral closed subscheme and $R$ be an $\mathbb{N}^p$-graded quotient ring of $S$ that is a domain.

For a closed subscheme $Z \subset \mathbb{P}' = \mathbb{P}^{r_1}_k \times_k \cdots \times_k \mathbb{P}^{r_k}_k$, we consider the following number

$$\text{MDeg}_{\mathbb{P}'}(Z) := \max \{ \deg_{\mathbb{P}'}^n(Z) \mid n \in \mathbb{N}^k \text{ with } |n| = \dim(Z) \}.$$

The main result of this section is the theorem below.

**Theorem 4.2.** Assume Setup 4.1. Let $\mathcal{J} = \{j_1, \ldots, j_k\} \subset [p]$ and set $\mathbb{P}' = \mathbb{P}^{m_{j_1}}_k \times_k \cdots \times_k \mathbb{P}^{m_{j_k}}_k$. Then we have the inequality

$$\text{MDeg}_{\mathbb{P}'}(\mathcal{I}(\mathcal{P})) \leq \text{MDeg}_{\mathbb{P}}(\mathcal{X}).$$

Moreover, for any $d \in \mathbb{N}^k$ with $|d| = \dim(\mathcal{I}(\mathcal{P}))$ and such that $\deg_{\mathbb{P}'}^d(\mathcal{I}(\mathcal{P})) > 0$, there exists some $n \in \mathbb{N}^p$ with $|n| = \dim(\mathcal{X})$ and such that $\deg_{\mathbb{P}}^n(\mathcal{X}) > 0$ and $\deg_{\mathbb{P}}^d(\mathcal{I}(\mathcal{P}))$ divides $\deg_{\mathbb{P}}^n(\mathcal{X})$.

Before providing the proof of the theorem we need some preparatory results.

**Lemma 4.3.** Consider the natural projection $\Pi_{[p-1]}: \mathbb{P} \to \mathbb{P}' = \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_{p-1}}_k$ and let $Y = \Pi_{[p-1]}(X)$ be the corresponding image. Let $\Phi: X \to Y$ be the restriction of $\Pi_{[p-1]}$ to $X$ and $Y$. If $\dim(X) = \dim(Y)$,
then
\[ \deg_{\mathcal{F}}^{n,0}(X) = \deg(\Phi) \deg_{\mathcal{F}}^p(Y) \]
for all \( n \in \mathbb{N}^{p-1} \) with \( |n| = \dim(X) \); here \( \deg(\Phi) = [K(X) : K(Y)] \) denotes the degree of \( \Phi \).

**First proof.** Let \( r = \dim(X) \). We write the Hilbert polynomial of \( X \) as follows
\[
P_X(t_1,\ldots,t_p) = \sum_{n \in \mathbb{N}^p} e(n) \left( \frac{t_1 + n_1}{n_1} \right) \cdots \left( \frac{t_p + n_p}{n_p} \right).
\]
Choose a positive integer \( \omega \) such that \( P_X(\nu) = \dim_k ([R]_{\nu}) \) for all \( \nu \in \mathbb{N}^p \) satisfying \( \nu \geq \omega \cdot 1 \in \mathbb{N}^p \). Let \( A \subset R \) be the standard \( \mathbb{N}^{p-1} \)-graded \( \mathbb{k} \)-algebra given by
\[ A = R_{\{[p-1]\}} = \bigoplus_{\nu_1,\ldots,\nu_{p-1} \geq 0} [R]_{\nu_1,\ldots,\nu_{p-1},0}. \]
Consider the finitely generated \( \mathbb{N}^{p-1} \)-graded \( A \)-module \( M = \bigoplus_{\nu_1,\ldots,\nu_{p-1} \geq 0} [R]_{\nu_1,\ldots,\nu_{p-1},\omega} \). By construction the Hilbert polynomial of \( M \) is given by
\[ P_M(t) = P_X(t,\omega) = \sum_{n \in \mathbb{N}^{p-1}, m \in \mathbb{N}} e(n,m) \left( \frac{t_1 + n_1}{n_1} \right) \cdots \left( \frac{t_{p-1} + n_{p-1}}{n_{p-1}} \right) \left( \omega + m \right). \]
Regrouping the terms yields the following expression
\[ P_M(t) = \sum_{n \in \mathbb{N}^{p-1}} \left( \sum_{|n| + m \leq r} \left( \sum_{m=0}^{r-|n|} e(n,m) \left( \frac{\omega + m}{m} \right) \right) \left( \frac{t_1 + n_1}{n_1} \right) \cdots \left( \frac{t_{p-1} + n_{p-1}}{n_{p-1}} \right). \]
This forces the equality \( e(n;M) = e(n,0) = \deg_{\mathcal{F}}^{n,0}(X) \) for all \( n \in \mathbb{N}^{p-1} \) with \( |n| = r \). Since \( M \) is a torsion-free module over the domain \( A \), the associativity formula for mixed multiplicities gives the equalities \( \deg_{\mathcal{F}}^{n,0}(X) = e(n;M) = \text{rank}_A(M)e(n;A) = \text{rank}_A(M) \deg_{\mathcal{F}}^p(Y) \) for all \( n \in \mathbb{N}^{p-1} \) with \( |n| = r \). As \( \dim(X) = \dim(Y) \), the morphism \( \Phi \) is generically finite. Let \( \phi : \text{Proj}(R) \rightarrow \text{Spec}(A) \) be the natural morphism obtained by considering \( R \) as an \( \mathbb{N} \)-graded ring after setting \( [R]_\mu = \bigoplus_{\nu_1,\ldots,\nu_{p-1} \geq 0}[R]_{\nu_1,\ldots,\nu_{p-1},\mu} \) for all \( \mu \in \mathbb{N} \). Notice that \( \phi \) is a generically finite since \( \Phi \) is, and that \( \deg(\Phi) = \deg(\phi) \). Let \( \eta \) be the the generic point of \( \text{Spec}(A) \) and \( Q = \text{Quot}(A) \) be the field of fractions of \( A \). Then the generic fiber \( Z = \phi^{-1}(\eta) = \text{Proj}(R \otimes_A Q) \) is zero-dimensional (and thus affine, see, e.g., [25, Proposition 5.20]) and we have the equality
\[ \deg(\phi) = \dim_Q(\phi^{-1}(\eta)) = \dim_Q(\mathcal{H}^0(Z,O_Z(\mu))) \]
for all \( \mu \in \mathbb{Z} \). In particular, this implies that \( \deg(\Phi) = \dim_Q(\mathcal{H}^0(Z,O_Z(\omega))) = \dim_Q(\mathcal{H}^0(Z,O_Z(\omega))) \).
Hence, \( \text{rank}_A(M) \), after possibly increasing \( \omega \) and making it large enough. By combining everything, we obtain the claimed equality \( \deg_{\mathcal{F}}^{n,0}(X) = \deg(\Phi) \deg_{\mathcal{F}}^p(Y) \) for all \( n \in \mathbb{N}^{p-1} \) with \( |n| = r \).

**Second proof.** For each \( 1 \leq i \leq p-1 \), let \( L_i = O_{\mathcal{F}'}(e'_i) \) be the line bundle on \( \mathcal{F}' = P_k^{m_1} \times \cdots \times P_k^{m_{p-1}} \) corresponding with the \( i \)-th elementary vector \( e'_i \in \mathbb{N}^{p-1} \). To simplify notation, set \( f = \Pi_{[p-1]} \) and let \( r = \dim(X) \). From the projection formula (see [24, Proposition 2.5]) and the fact that \( f_*[X] = \deg(\Phi)[Y] \)
(see [24, §1.4]), we obtain the equalities
\[
\deg_{\mathbb{P}}^{\mathbb{A}}(X) = \int_{\mathbb{P}} c_1(f^*(L_1))^n_1 \cdots c_1(f^*(L_{p-1}))^{n_{p-1}} \cap [X] \\
= \deg(\Phi) \int_{\mathbb{P}} c_1(L_1)^n_1 \cdots c_1(L_{p-1})^{n_{p-1}} \cap [Y] \\
= \deg(\Phi) \deg_{\mathbb{P}}^{\mathbb{A}}(Y)
\]
for all \( n = (n_1, \ldots, n_{p-1}) \in \mathbb{N}^{p-1} \) with \( |n| = r \). So, the result follows. \( \square \)

We shall use a standard type of generic Bertini theorem, much in the spirit of [23, §1.5], that is quite flexible and useful for our purposes. To that end, we introduce the following notation.

**Notation 4.4.** Let \( \{y_0, \ldots, y_{\ell}\} \) be a basis of the \( k \)-vector space \([S]_{e_p}\). Consider a purely transcendental field extension \( L := k(z_0, \ldots, z_{\ell}) \) of \( k \), and set \( S_L := S \otimes_k L \), \( R_L := R \otimes_k L \) and \( X_L := X \otimes_k L = \text{MultiProj}(R_L) \subset \mathbb{P}_L := \mathbb{P} \otimes_k L = \mathbb{P}^{m_1}_L \times \mathbb{P} \cdots \times \mathbb{P}^{m_p}_L \). We say that \( y := z_0y_0 + \cdots + z_{\ell}y_{\ell} \in [S_L]_{e_p} \) is the *generic element* of \([S_L]_{e_p}\).

**Remark 4.5.** If \( k(\xi) \) is a purely transcendental field extension over \( k \), then \( R \otimes_k k(\xi) \) is also a domain (see, e.g., [7, Remark 3.9]).

**Remark 4.6** ([23, Corollary 1.5.9, Corollary 1.5.10]). Assume \( \dim(X) \geq 1 \). Then \( X_L \cap V(y) \) is also an integral scheme.

**Proof.** We may restrict to an affine open subscheme \( U = \text{Spec}(A) \) with \( A \) a domain, and assume that the restriction to \( A \) of one of the \( y_i \)'s, say \( y_0 \), becomes a unit. Notice that \( U_L \cap V(y) = \text{Spec}(A \otimes_k L)/y \) and that \( (A \otimes_k L)/y \) is a localization of \( A[z_0, z_1, \ldots, z_{\ell}]/(z_0 + \frac{y_1}{y_0}z_1 + \cdots + \frac{y_{\ell}}{y_0}z_{\ell}) \). From the fact that \( (z_0 + \frac{y_1}{y_0}z_1 + \cdots + \frac{y_{\ell}}{y_0}z_{\ell}) \) is a prime ideal, it follows that \( (A \otimes_k L)/y \) is a domain. This concludes the proof. \( \square \)

We recall the following result from [7] that will be essential.

**Theorem 4.7.** Assume Setup 4.1 and Notation 4.4, and suppose that \( \dim(\Pi_p(X)) \geq 1 \). Then the following statements hold:

(i) \( \dim(\Pi_3(X_L \cap V(y))) = \min \{ \dim(\Pi_3(X)), \dim(\Pi_{3 \cup (p)}(X)) - 1 \} \) for all \( \mathcal{J} \subset [p] \).

(ii) \( \deg_{\mathbb{P}_L}^{n-e_p}(X_L \cap V(y)) = \deg_{\mathbb{P}}^{n-e_p}(X_L) \) for all \( n = (n_1, \ldots, n_p) \in \mathbb{N}^p \) with \( |n| = \dim(X) \) and \( n_p \geq 1 \).

**Proof.** We may assume that the field \( k \) is infinite; indeed, by changing \( k \) by a purely transcendental field extension we can keep all the assumptions (see Remark 4.5).

Let \( x \in [R]_{e_p} \) be a general element. From [7, Theorem 3.7], we obtain that
\[
\dim(\Pi_3(X \cap V(x))) = \min \{ \dim(\Pi_3(X)), \dim(\Pi_{3 \cup (p)}(X)) - 1 \}
\]
for all \( \mathcal{J} \subset [p] \). As a consequence of [11, Lemma 3.7, Lemma 3.9], we get that \( \deg_{\mathbb{P}}^{n-e_p}(X \cap V(x)) = \deg_{\mathbb{P}}^{n}(X) \) for all \( n = (n_1, \ldots, n_p) \in \mathbb{N}^p \) with \( |n| = \dim(X) \) and \( n_p \geq 1 \). Therefore, the results of both parts (i) and (ii) follow by applying [7, Lemma 3.10] and by utilizing the two latter formulas for cutting with a general element \( x \in [R]_{e_p} \). \( \square \)

We are now ready for the proof of the main result of this section.
**Proof of Theorem 4.2.** Notice that it suffices to consider the case $\mathfrak{I} = [p-1] \subset [p]$. Let $Y$ be the image of $X$ under the natural projection $\Pi_{(p-1)} : P \to \mathbb{P}^{m_1}_k \times \cdots \times \mathbb{P}^{m_{p-1}}_k$. We proceed by induction on $s = \dim(X) - \dim(Y)$. If $s = 0$, then the result follows directly from Lemma 4.3.

Suppose that $s > 0$. This implies that $\dim(\Pi_{(p)}(X)) \geq 1$ since the function $r(\mathfrak{I}) = \dim(\Pi_{(p)}(X))$ is a rank function (see [7, Proposition 5.1]). We substitute $X$ and $Y$ by $X_{\mathfrak{L}}$ and $Y_{\mathfrak{L}}$, respectively, and we consider the generic element $y$ in $[S_\mathfrak{L}]_{e_{\mathfrak{L}}}$. Let $X' = X_{\mathfrak{L}} \cap V(Y) \subset F_{\mathfrak{L}}$ and $Y' = \Pi_{(p-1)}(X') \subset \mathbb{P}_{\mathfrak{L}}^3$. From Remark 4.6 and Theorem 4.7, we obtain that $X'$ is an integral scheme and the equalities

$$\dim(X') = \dim(X) - 1 \quad \text{and} \quad \dim(Y') = \min(\dim(X) - 1, \dim(Y)) = \dim(Y).$$

This implies that $Y'$ and $Y_{\mathfrak{L}}$ have the same dimension, and thus we necessarily get $Y' = Y_{\mathfrak{L}}$ because $Y' \subset Y_{\mathfrak{L}}$ by construction. Also, for all $n = (n_1, \ldots, n_p) \in \mathbb{N}^p$ such that $[n] = \dim(X)$ and $n_p \geq 1$, we have that $\deg_{\mathbb{P}_{\mathfrak{L}}^3}(X') = \deg_{\mathfrak{L}}(X)$. Therefore, since $\dim(X') = \dim(Y') = s - 1$, we can apply the inductive hypothesis to the scheme $X'$ and the corresponding image $Y' = Y_{\mathfrak{L}}$ under the projection $\Pi_{(p-1)}$. This completes the proof of the theorem. □

We illustrate the result of Theorem 4.2 with the following example that constitutes a variation of [7, Example 1.1].

**Example 4.8.** Let $A = k[v_1, v_2, v_3] [w_1, w_2, w_3]$ be $\mathbb{N}^3$-graded with $\deg(v_1) = 0 \in \mathbb{N}^3$ and $\deg(w_1) = e_1 \in \mathbb{N}^3$. Let $S = k[x_0, \ldots, x_3] [y_0, \ldots, y_3] [z_0, \ldots, z_3]$ be $\mathbb{N}^3$-graded with $\deg(x_i) = e_i$, $\deg(y_i) = e_2$ and $\deg(z_i) = e_3$. Consider the $\mathbb{N}^3$-graded $k$-algebra homomorphism

$$\varphi : S \to A, \quad \begin{align*}
x_0 &\mapsto w_1, \quad x_1 \mapsto v_1^3 w_1, \quad x_2 \mapsto v_1^2 w_1, \quad x_3 \mapsto v_1 w_1, \\
y_0 &\mapsto w_2, \quad y_1 \mapsto v_1 w_2, \quad y_2 \mapsto v_2 w_2, \quad y_3 \mapsto (v_1^2 + v_2^2) w_2, \\
z_0 &\mapsto w_3, \quad z_1 \mapsto v_1^3 w_3, \quad z_2 \mapsto v_1^2 w_3, \quad z_3 \mapsto v_1 w_3.
\end{align*}$$

Let $P \subset S$ be the $\mathbb{N}^3$-graded prime ideal $P = \text{Ker}(\varphi)$, which is given by

$$P = \left( x_1 - x_2, y_1 z_0 - y_0 z_1 - y_2 z_2, y_2 z_0 - y_0 z_2, x_2 z_0 - x_0 z_1, y_1^2 + y_2^2 - y_0 y_3, x_3 y_0 - x_0 y_1, x_2 y_0 - x_1 y_1, x_0 x_2 - x_3^2, y_0 y_2 z_1 + y_2^2 z_2 - y_0 y_1 z_2, x_1 y_2 z_1 - x_2 y_2 z_2, x_0 y_2 z_1 - x_3 y_1 z_2, x_0 y_2 z_2 - x_3 y_1 z_1 \right).$$

Let $X = \text{MultiProj}(S/P) \subset \mathbb{P}^3_k \times_k \mathbb{P}^3_k \times_k \mathbb{P}^3_k$ be the corresponding integral closed subscheme. The multidegree polynomial of $X$ is equal to

$$c(X; t_1, t_2, t_3) = 2 t_1^3 t_2^3 + 4 t_1^2 t_2^2 t_3 + 2 t_1^2 t_2 t_3^2 + 2 t_1^3 t_3^2 + 4 t_1^2 t_2 t_3^2.$$
Here one may see that the multidegrees of the projections behave as predicted by Theorem 4.2. The computations of this example can be carried out by hand or by utilizing Macaulay2 [26].

The next simple example shows that, if we drop the condition of Theorem 4.2 that \( \mathbb{R} \) is a domain, then the multidegrees can grow arbitrarily under projections.

**Example 4.9.** Suppose \( S = k[x_0, x_1, x_2][y_0, y_1, y_2] \) is \( \mathbb{N}^2 \)-graded with \( \deg(x_i) = e_i \) and \( \deg(y_1) = e_2 \). Let \( a > 0 \) be a positive integer. Consider the monomial ideal \( J = (x_0^a, x_0x_1, x_1y_0, y_0^a) \subset S \) and its corresponding closed subscheme \( Z = \text{MultiProj}(S/J) \subset \mathbb{P}_{k}^2 \times_k \mathbb{P}_{k}^2 \). One can check that

\[
\mathcal{C}(Z; t_1, t_2) = t_1t_2 \quad \text{and} \quad \deg_{\mathbb{P}_{k}^2}^{a}(\Pi(Z)) = a.
\]

This follows at once from the fact \( J = (x_0, y_0) \cap (x_0^2, x_1, y_0^a) \) is a primary decomposition of \( J \).

As a direct consequence of Theorem 4.2, we obtain that the projection of a multiplicity-free variety is also multiplicity-free. A closed subscheme \( Z \subset \mathbb{P} = \mathbb{P}_{k}^1 \times_k \cdots \times_k \mathbb{P}_{k}^r \) is said to be *multiplicity-free* if \( \deg_{\mathbb{P}}^n(Z) \) is either 0 or 1 for all \( n \in \mathbb{N}^r \) with \( |n| = \dim(Z) \) (that is, \( \text{MDeg}_{\mathbb{P}}^n(Z) = 1 \)).

**Corollary 4.10.** If \( X \subset \mathbb{P} \) is multiplicity-free, then \( \Pi_j(X) \) is multiplicity-free for all \( j \subset [p] \).

## 5. The multidegrees of prime ideals

In this section, we investigate several special properties that are enjoyed by the multidegrees of prime ideals. We use the following setup throughout this section.

**Setup 5.1.** Let \( k \) be a field and \( S = k[x_1, \ldots, x_n] \) be a standard \( \mathbb{N}^r \)-graded polynomial ring.

The theorem below proves that the multidegree polynomial and the \( S \)-multidegree polynomial coincide for a prime ideal.

**Theorem 5.2.** Assume Setup 5.1. Let \( \mathcal{P} \subset S \) be an \( S \)-homogeneous prime ideal. Then we have the equality

\[
\mathcal{C}(S/\mathcal{P}; \mathbf{t}) = \mathcal{S}(S/\mathcal{P}; \mathbf{t}).
\]

**Proof.** Choose a monomial order \( \succ \) on \( S \). By using [38], we obtain that \( \sqrt{\text{in}(\mathcal{P})} \) is equidimensional and connected in codimension one (see also [36, Appendix 1]). For any monomial prime ideal \( \mathcal{L} = (x_{i_1}, \ldots, x_{i_k}) \subset S \), we have

\[
\mathcal{C}(S/\mathcal{L}; \mathbf{t}) = \mathbf{t}^{\mathbf{a}} \quad \text{and} \quad \mathcal{S}(S/\mathcal{L}; \mathbf{t}) = \mathbf{t}^{\mathbf{a}},
\]

where \( \mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r \) with \( a_i = |\{ l \mid \deg(x_{i_l}) = e_i \}| \) (see Remark 2.6).

Let \( \mathcal{P}_1, \ldots, \mathcal{P}_h, \mathcal{P}_{h+1}, \ldots, \mathcal{P}_{h+s} \) be the associated primes of \( \text{in}(\mathcal{P}) \), where \( \mathcal{P}_1, \ldots, \mathcal{P}_h \) are the minimal ones. We choose a prime filtration

\[
0 = M_0 \subset M_1 \subset \cdots \subset M_r = S/\text{in}(\mathcal{P})
\]

where \( M_j/M_{j-1} \cong S/L_j \) and \( L_j \subset S \) is a monomial prime ideal (see, e.g., [21, Proposition 3.7]). All the associated primes of \( \text{in}(\mathcal{P}) \) must appear among the \( L_j \)'s and all the \( L_j \)'s must contain a minimal prime of \( \text{in}(\mathcal{P}) \). Therefore, by utilizing the additivity of Hilbert functions over this prime filtration and the above remarks, we obtain

\[
\mathcal{C}(S/\text{in}(\mathcal{P}); \mathbf{t}) = \mathcal{S}(S/\text{in}(\mathcal{P}); \mathbf{t}) = \sum_{j=1}^{h} \text{length}_{S_{L_j}}((S/\text{in}(\mathcal{P}))_{P_j}) \mathcal{C}(S/P_j; \mathbf{t});
\]
indeed: if $L_{k_1} \subseteq L_{k_2}$ then $\mathcal{C}(S/L_{k_1}; t)$ divides $\mathcal{C}(S/L_{k_2}; t)$; all the minimal $P_j$’s have the same codimension; and by localizing we get that the number of times that a minimal $P_j$ appears among the $L_k$’s is equal to the length of the Artinian local ring $(S/in(P))_{P_j}$. This concludes the proof because $S/P$ and $S/in(P)$ have the same multigraded Hilbert function. □

The remainder of this section is dedicated to multigraded generic initial ideals. Thus we further specify the notation by fixing the following setup.

**Setup 5.3.** Let $k$ be an infinite field, $\mathbf{m} = (m_1, \ldots, m_p) \in \mathbb{Z}_+^p$ be a vector of positive integers, and $S = k[x_{1,j} \mid 1 \leq i \leq p, 0 \leq j \leq m_i]$ be a standard $\mathbb{N}_+^p$-graded polynomial with $\deg(x_{1,j}) = e_i \in \mathbb{N}_+^p$. Hence we have MultiProj$(S) = P = \mathbb{P}_k^ {m_1} \times_k \cdots \times_k \mathbb{P}_k^ {m_p}$. The group naturally acting on $S$ as the group of multigraded $k$-algebra isomorphisms is $G = GL_{m_1+1}(k) \times \cdots \times GL_{m_p+1}(k)$, where $GL_{m_i+1}(k)$ is the group of invertible $(m_i+1) \times (m_i+1)$ matrices over $k$. The *Borel subgroup* of $G$ is given by $B = B_{m_1+1}(k) \times \cdots \times B_{m_p+1}(k)$, where $B_{m_i+1}(k)$ is the subgroup of $GL_{m_i+1}(k)$ consisting of upper triangular invertible matrices. Let $> \in S$ be a monomial order on $S$ that satisfies $x_{i,0} > x_{i,1} > \cdots > x_{i,m_i}$ for all $1 \leq i \leq p$.

Much as in the singly-graded case (see [21, §15.9]), we have the following definition.

**Definition 5.4.** Let $I \subseteq S$ be an $S$-homogeneous ideal. The *multigraded generic initial ideal* $\text{gin}_{>}(I)$ of $I$ with respect to $>$ is the ideal $\text{in}_{>}(g(I))$, where $g$ belongs to a Zariski dense open subset $\mathfrak{U} \subseteq G$.

Whenever the used monomial order $>$ is clear from the context, we shall write $\text{in}(I)$ and $\text{gin}(I)$ instead of $\text{in}_{>}(I)$ and $\text{gin}_{>}(I)$, respectively. An $S$-homogeneous ideal $I \subseteq S$ is said to be *Borel-fixed* if $g(I) = I$ for all $g \in B$. Similarly to the singly-graded setting, it can be shown that $\text{gin}(I)$ is Borel-fixed (see [21, Theorem 15.20]).

**Remark 5.5.** One can extend the known properties of Borel-fixed ideals in the singly-graded setting (see [21, Chapter 15]) to obtain the following statements:

(i) Any Borel-fixed prime ideal in $S$ is of the form $P_a = (x_{i,j} \mid 1 \leq i \leq p$ and $0 \leq j < a_i)$ for some $a = (a_1, \ldots, a_p) \in \mathbb{N}_+^p$ (see [15, Lemma 3.1]).

(ii) An $S$-homogeneous Borel-fixed ideal $I \subseteq S$ is a monomial ideal and all its associated primes are also Borel-fixed (see [15, Lemma 3.2]).

The following theorem shows that the radical of the multigraded generic initial ideal of any prime is Cohen-Macaulay. This demonstrates a very special behavior of prime ideals in a multigraded setting.

**Theorem 5.6.** Assume **Setup 5.3.** Let $P \subseteq S$ be an $S$-homogeneous prime ideal. Then $\sqrt{\text{gin}(P)}$ is a Cohen-Macaulay ideal.

**Proof.** Let $J = \sqrt{\text{gin}(P)} \subseteq S$. Let $P_1, \ldots, P_h$ be the minimal primes of $J$. As a consequence of [38], all the $P_k$’s have the same codimension. Each $P_k$ is Borel-fixed, and so we can write $P_k = (x_{i,j} \mid 1 \leq i \leq p, 0 \leq j < a_{k,i})$. 


for some \( a_k = (a_{k,1}, \ldots, a_{k,p}) \in \mathbb{N}^P \) with \( |a_k| = \text{codim}(\mathcal{P}) \). With this notation we now have

\[
\mathcal{C}(S/P; t) = \mathcal{C}(S/\text{gin}(\mathcal{P}); t) = \sum_{k=1}^{h} \text{length}_{S_{p_k}}((S/\text{gin}(\mathcal{P}))_{p_k}) t^{a_k} \quad \text{and} \quad \mathcal{C}(S/J; t) = \sum_{k=1}^{h} t^{a_k}.
\]

The Alexander dual of \( \mathcal{J} \) is a monomial ideal \( K \subset S \) given by \( K = (x_{a_1}, \ldots, x_{a_h}) \) where

\[
x_{a_k} = \prod_{1 \leq i \leq p, 0 \leq j < a_{k,i}} x_{i,j};
\]

see [32, Corollary 1.5.5]. The Eagon-Reiner theorem (see, e.g., [32, Theorem 8.1.9]) yields that \( \mathcal{J} \) is Cohen-Macaulay if and only if \( K \) has a linear resolution. Thus we concentrate on proving that \( K \) has a linear resolution.

From Theorem 2.14, we obtain that the support of \( \mathcal{C}(S/P; t) \), which coincides with the set of lattice points \( \{a_1, \ldots, a_h\} \subset \mathbb{N}^P \), is a discrete polymatroid. We construct the following monomial ideal

\[
\mathcal{M} = (x_{a_1}, \ldots, x_{a_h}) \quad \text{with} \quad x_{a_k} = x_{1,0}^{a_{k,1}} x_{2,0}^{a_{k,2}} \cdots x_{p,0}^{a_{k,p}}
\]

in the polynomial subring \( \mathbb{K}[x_{1,0}, x_{2,0}, \ldots, x_{p,0}] \subset S \). By construction and following the notation of [32, §12.6], we say that \( M \) is a polymatroidal ideal, and so the ideal \( M \) has a linear resolution by [32, Theorem 12.6.2, Proposition 8.2.1]. Notice that \( K \) can be seen naturally as the polarization of \( M \) by sending \( x_{a_k} \) to \( x_{a_k} \). Finally, by standard properties of polarization (see [32, §1.6]), it follows that \( K \) also has a linear resolution. This concludes the proof of the theorem.

\[\square\]

**Remark 5.7.** For the more combinatorially inclined reader it should be the mentioned that the same proof of Theorem 5.6 shows that the simplicial complex associated to the radical monomial ideal \( \sqrt{\text{gin}(\mathcal{P})} \) is shellable (see [32, Proposition 8.2.5]).

**Remark 5.8.** One may find many examples of a prime ideal \( \mathcal{P} \) and a monomial order \( > \) such that \( \sqrt{\text{in}_{>}(\mathcal{P})} \) is not Cohen-Macaulay. For a simple instance, see [20, Remark 2.8(1)]. Also, for more examples of this type and related results, see the book [4, Chapter 5].

For an ideal \( \mathcal{I} \subset S \), we consider the following invariant

\[
\text{MLength}(\mathcal{I}) := \max \{ \text{length}_{S_p}((\mathcal{I}/p)_{p}) \mid p \in \text{Min}_S(\mathcal{I}) \}
\]

that measures the maximal length of the minimal primary components of \( \mathcal{I} \). The next theorem shows that, for a given \( S \)-homogeneous prime ideal, the multiplicities of the minimal primary components of the multigraded generic initial ideal have a very restrictive behavior with respect to the natural projections.

**Theorem 5.9.** Assume Setup 5.3. Let \( \mathcal{P} \subset S \) be an \( S \)-homogeneous prime ideal. Let \( \mathcal{J} = \{j_1, \ldots, j_k\} \subset [p] \) be a subset. Then we have the inequality

\[
\text{MLength} \left( \text{gin} \left( \mathcal{P}(\mathcal{J}) \right) \right) \leq \text{MLength} \left( \text{gin} \left( \mathcal{P} \right) \right).
\]

Moreover, for a given minimal prime \( p \in \text{Min}_S(\mathcal{J}/\text{gin}(\mathcal{P}(\mathcal{J}))) \) of \( \text{gin}(\mathcal{P}(\mathcal{J})) \), we can find a minimal prime \( q \in \text{Min}_S(\mathcal{S}/\text{gin}(\mathcal{P})) \) of \( \text{gin}(\mathcal{P}) \) such that the length of the \( p \)-primary component of \( \text{gin}(\mathcal{P}(\mathcal{J})) \) divides the length of the \( q \)-primary component of \( \text{gin}(\mathcal{P}) \).
Proof. As in Remark 2.8, we can add new variables \( x_{i,m_i+1} \) to \( S \) with \( \deg(x_{i,m_i+1}) = e_i \), and then assume that \( P \) is relevant. By further specifying \( x_{i,m_i} > x_{i,m_i+1} \), we may see that \( \text{gin}(P) \) will not involve the new variables \( x_{i,m_i+1} \). So, without any loss of generality, we assume that \( P \) is a relevant prime ideal.

Let \( I = \text{gin}(P) \subset S \), and \( P_1, \ldots, P_h \) be the minimal primes of \( I \). For each \( 1 \leq k \leq h \), we know that \( P_k = (x_{i,j} \mid 1 \leq i \leq p, 0 \leq j < a_{k,i}) \) for some \( a_k = (a_{k,1}, \ldots, a_{k,p}) \in \mathbb{N}^p \) with \( |a_k| = \text{codim}(P) \). Hence by utilizing the associativity formula for multidegrees (Remark 2.6(iv)) and Remark 2.6(ii), we obtain

\[
(2) \quad \mathcal{C}(S/P; \mathbf{t}) = \mathcal{C}(S/I; \mathbf{t}) = \sum_{k=1}^{h} \text{length}_{S_{P_k}}((S/I)_{P_k}) t^{a_k}.
\]

Let \( X = \text{MultProj}(S/P) \subset \mathbb{P} \) be the integral closed subscheme corresponding with \( P \). Set \( r = \dim(X) = m_1 + \cdots + m_p - \text{codim}(P) \). From Theorem 2.12 it follows that

\[
\mathcal{C}(S/P; \mathbf{t}) = \sum_{n \in \mathbb{N}^p} \deg_{P}(X) | \mathbf{t}^{m-n}.
\]

Then comparing coefficients yields the equality

\[
\text{length}_{S_{P_k}}((S/I)_{P_k}) = \deg_{P}^{m-a_k}(X)
\]

for all \( 1 \leq k \leq h \).

Set \( P' = \mathbb{P}_{k_1}^{m_1} \times_k \cdots \times_k \mathbb{P}_{k_p}^{m_p} \). By applying the same argument to the projection corresponding to \( \mathcal{J} \), we obtain that for any minimal prime \( p \subset S(\mathcal{J}) \) of \( \text{gin}(P(\mathcal{J})) \), there exists some \( d \in \mathbb{N}^k \) with \( |d| = \dim(P(\mathcal{J})) \) such that

\[
\text{length}_{S(\mathcal{J})}((S(\mathcal{J})/\text{gin}(P(\mathcal{J})))_p) = \deg_{P'}^{d}(\mathbb{P}(\mathcal{J})).
\]

Therefore, the result follows as a consequence of Theorem 4.2. \( \square \)

We single out the following result that was obtained in (2).

Remark 5.10. Let \( P \subset S \) be an \( S \)-homogeneous prime ideal. Let \( I = \text{gin}(P) \subset S \). Let \( P_1, \ldots, P_h \) be the minimal primes of \( I \) and \( Q_1, \ldots, Q_h \) be the corresponding minimal primary components. The non-zero coefficients of \( \mathcal{C}(S/P; \mathbf{t}) \) are in one-to-one correspondence with the multiplicities \( \text{length}_{S_{P_k}}((S/Q_k)_{P_k}) \).

The next example shows that the minimal primary components of \( \text{gin}(P(\mathcal{J})) \) may be determined by embedded primary components of \( \text{gin}(P) \). However, Theorem 5.9 still gives a tight relation between the minimal primary components of \( \text{gin}(P(\mathcal{J})) \) and \( \text{gin}(P) \).

Example 5.11. Here we continue using the prime ideal \( P \subset S \) from Example 4.8, where \( S \) is the \( \mathbb{N}^2 \)-graded polynomial ring \( S = k[x_0, \ldots, x_3] [y_0, \ldots, y_3] [z_0, \ldots, z_3] \) with \( \deg(x_i) = e_1, \deg(y_i) = e_2 \) and \( \deg(z_i) = e_3 \). The corresponding multidegree polynomial is given by

\[
\mathcal{C}(R/P; t_1, t_2, t_3) = 2 t_1 t_2 t_3^2 + 4 t_1^2 t_2 t_3 + 2 t_1^3 t_2^2 t_3 + 2 t_1^2 t_2^3 t_3 + 4 t_1 t_2^2 t_3^2.
\]

As we have pointed out before in this section, since \( \text{gin}(P) \) is Borel-fixed and \( \sqrt{\text{gin}(P)} \) is equidimensional (we proved that it is even Cohen-Macaulay in Theorem 5.6), it follows that the multidegree polynomial determines the minimal primes of \( \text{gin}(P) \). More precisely, we have the following correspondence between
terms of $\mathcal{C}(R/P; t)$ and minimal primary components of $\text{gin}(P)$:

$$
\begin{align*}
2t_1^3t_2^3 & \iff (x_0, x_1, x_2^2, y_0, y_1, y_2) =: M_1 \\
4t_1^3t_2^3 & \iff (x_0, x_1^2, x_1x_2, x_2^3, y_0, y_1, y_2) =: M_2 \\
2t_1^3t_2^3 & \iff (x_0, x_1, x_2^2, y_0, y_1, z_0) =: M_3 \\
2t_1^2t_2^3 & \iff (x_0, x_1^2, y_0, y_1, y_2, z_0) =: M_4 \\
4t_1^3t_2^3 & \iff (x_0, x_1^2, y_0, y_1^2, z_0, z_1) =: M_5.
\end{align*}
$$

One can compute the whole expression of $\text{gin}(P)$ by utilizing Macaulay2 [26], and one may check that $\text{gin}(P)$ has a total of 9 primary components and that all the embedded ones have codimension 7. We consider the projection $\pi = [2,3]$. The corresponding projected ideal is given by

$$
Q = P_{[2,3]} = (y_3z_0 - y_0z_4 - y_2z_2, y_2z_0 - y_0z_2, y_1^2 - y_0y_3) \subset T = S_{[2,3]}.
$$

We already saw that the multidegree polynomial of this projection is equal to

$$
\mathcal{C}(T/Q; t_2, t_3) = 2t_2^3 + 4t_2^2t_3 + 2t_2t_3^2.
$$

The correspondence between terms of $\mathcal{C}(T/Q; t_2, t_3)$ and minimal primary components of $\text{gin}(Q)$ is depicted below:

$$
\begin{align*}
2t_2^3 & \iff (y_0, y_1, y_2) \\
4t_2^2t_3 & \iff (y_0^2, y_0y_1, y_1^3, z_0) \\
2t_2t_3^2 & \iff (y_0^3, z_0, z_1).
\end{align*}
$$

Again, $\text{gin}(Q)$ can be computed by using Macaulay2 [26], or just by realizing that $\text{gin}(Q) = \text{gin}(P)_{[2,3]}$. It turns out that $\text{gin}(Q)$ has only three primary components and that all of them are minimal. Notice that $M_1 \cap T = (y_0, y_1, y_2)$, $M_2 \cap T = (y_0, y_1, z_0)$, $M_3 \cap T = (y_0, z_0, z_1)$, $M_4 \cap T = (y_0, y_1, y_2, z_0)$ and $M_5 \cap T = (y_0, y_1^2, z_0, z_1)$. So, it follows that all the primary components of $\text{gin}(Q)$ come from embedded components of $\text{gin}(P)$, and yet the length of any (minimal) component of $\text{gin}(Q)$ divides the length of some minimal component of $\text{gin}(P)$; as predicted by Theorem 5.9.

**Remark 5.12.** If we drop the prime condition, then the result of Theorem 5.9 may not hold. Indeed, let $J = (x_0^2, x_1x_2, y_0, y_0^3) \subset S$ be the $\mathbb{N}^2$-graded ideal of Example 4.9, then we have that $\text{MLeng}(\text{gin}(J)) = 1$ and $\text{MLeng}(\text{gin}(J_{[2]})) = \alpha$.

### 6. Multiplicity-Free Varieties

The main goal of this section is to obtain an alternative proof of the result of Brion [3] regarding multiplicity-free varieties. For organizational purposes we divide the section into two subsections.

#### 6.1. Cohomology and associated primes under flat degenerations

This subsection contains some technical results that will be needed in our treatment of multiplicity-free varieties. Here we study the behavior of cohomology and associated primes under flat degenerations. Our approach is inspired by ideas related to the fiber-full scheme ([10, 13, 14]).

**Proposition 6.1 (cf. [13, Theorem A]).** Let $(A, n, [k])$ be a Noetherian local ring with residue field $k$. Let $Y \subset P^r_A$ be a closed subscheme and $\mathcal{F}$ be a coherent sheaf on $Y$. Suppose that $\mathcal{F}$ is flat over $A$. Let $Y = Y \times_{\text{Spec}(A)} \text{Spec}(k) \subset P^n_k$ be the closed subscheme given as the special fiber and $F = \mathcal{F} \otimes_A k$ be the corresponding coherent sheaf on $Y$. 
If \( \text{Hom}_A(H^i(Y,F), H^{i+1}(Y,F)) = 0 \) for all \( i \geq 0 \), then the following statements hold:

(i) \( H^i(Y, \mathfrak{F}) \) is \( A \)-flat for all \( i \geq 0 \).

(ii) The natural map \( H^i(Y, \mathfrak{F}) \otimes_A B \to H^i(Y \times \text{Spec}(A), \mathfrak{F} \otimes_A B) \) is an isomorphism for all \( i \geq 0 \) and any \( A \)-algebra \( B \).

**Proof.** For any \( A \)-algebra \( B \), set \( \mathcal{Y}_B := Y \times \text{Spec}(A) / \text{Spec}(B) \) and \( \mathfrak{F}_B := \mathfrak{F} \otimes_A B \).

First, we prove that for any Artinian quotient ring \( C \) of \( A \) both statements of the proposition hold. We proceed by induction on length(\( C \)). The base case length(\( C \)) = 1 is clear because we necessarily have \( C = k \). Thus, we assume that length(\( C \)) > 1. We may choose an ideal \( a \subset C \) generated by a socle element of \( C \) to obtain a short exact sequence

\[
0 \to a \to C \to C' \to 0
\]

with \( a \cong k \) and length(\( C' \)) = length(\( C \)) − 1. By tensoring with \( - \otimes_A \mathfrak{F} \), since \( \mathfrak{F} \) is flat over \( A \), we obtain the short exact sequence

\[
0 \to F \to \mathfrak{F}_C \to \mathfrak{F}_{C'} \to 0.
\]

Consequently, we get an induced long exact sequence in cohomology

\[
\cdots \to H^{i-1}(\mathcal{Y}_{C'}, \mathfrak{F}_{C'}) \xrightarrow{\delta_{i-1}} H^i(Y, F) \to H^i(\mathcal{Y}, \mathfrak{F}_C) \to H^i(\mathcal{Y}_{C'}, \mathfrak{F}_{C'}) \xrightarrow{\delta_i} H^{i+1}(Y, F) \to \cdots
\]

The inductive hypothesis yields the isomorphism \( H^i(\mathcal{Y}, \mathfrak{F}_C) \otimes_{C'} k \cong H^i(Y, F) \). Then the map \( \delta_i \) gets the following factorization

\[
H^i(\mathcal{Y}_{C'}, \mathfrak{F}_{C'}) \to H^i(\mathcal{Y}_{C'}, \mathfrak{F}_{C'}) \otimes_{C'} k \cong H^i(Y, F) \xrightarrow{\delta_i} H^{i+1}(Y, F)
\]

where \( \beta_i \) is a \( k \)-linear map. The condition \( \text{Hom}_A(H^i(Y, F), H^{i+1}(Y, F)) = 0 \) implies that \( \beta_i = 0 \), hence \( \delta_i = 0 \). For all \( i \geq 0 \), we have proved that the natural map \( H^i(\mathcal{Y}, \mathfrak{F}_C) \to H^i(\mathcal{Y}_{C'}, \mathfrak{F}_{C'}) \) is surjective, and the natural map \( H^i(\mathcal{Y}_{C'}, \mathfrak{F}_{C'}) \to H^i(Y, F) \) is surjective by induction. Therefore, it follows that the natural map

\[
H^i(\mathcal{Y}, \mathfrak{F}_C) \to H^i(Y, F)
\]

is also surjective for all \( i \geq 0 \). Finally, by applying [30, Theorem 12.11], we obtain both statements for \( C \). This establishes the proposition for any Artinian quotient ring of \( A \).

For each \( q \geq 1 \), let \( \mathcal{Y}_q = \mathcal{Y} \times \text{Spec}(A) / \text{Spec}(A/q) \) and \( \mathfrak{F}_q = \mathfrak{F} \otimes_A A/q \). By the above step we have that \( H^i(\mathcal{Y}_q, \mathfrak{F}_q) \) is flat over \( A/q \). The Theorem on Formal Functions (see [30, Theorem III.11.1], [53, Tag 02OC]) yields the following isomorphism

\[
H^i(\mathcal{Y}, \mathfrak{F}) \otimes_A \hat{A} \cong \lim_{\leftarrow} H^i(\mathcal{Y}_q, \mathfrak{F}_q).
\]

So by applying [53, Tag 0912] to the inverse system \( (H^i(\mathcal{Y}_q, \mathfrak{F}_q))_{q \geq 1} \), we obtain that \( H^i(\mathcal{Y}, \mathfrak{F}) \otimes_A \hat{A} \) is a flat \( A \)-module. Hence \( H^i(\mathcal{Y}, \mathfrak{F}) \) is \( A \)-flat for all \( i \geq 0 \). This settles part (i) of the proposition. Then part (ii) follows from (i) (see, e.g., [14, Lemma 2.6]). \( \square \)

**Notation 6.2.** Given projective scheme \( Y \subset \mathbb{P}_k^n = \text{Proj}(B) \) over a field \( k \), we denote by \( I_Y \subset B \) the corresponding saturated homogeneous ideal.

**Lemma 6.3.** Let \( Y \subset \mathbb{P}_k^n = \text{Proj}(B) \) be a projective scheme over a field \( k \). The following statements hold:
(i) For $1 \leq i \leq r$, the ideal $I_Y$ has an associated prime of codimension $i$ if and only if the following limit
\[
\lim_{\nu \to \infty} \frac{\dim_k \left( H^{r-1}_\nu(Y, O_Y(-\nu)) \right)}{\nu^{r-i}/(r-i)!} \in \mathbb{Z}_+
\]
is a positive integer.

(ii) $Y$ is Cohen-Macaulay and equidimensional if and only if $H^1(Y, O_Y(-\nu)) = 0$ for all $i < \dim(Y)$ and $\nu \gg 0$.

**Proof.** Let $m = [B]_+$ be the graded irrelevant ideal of $B$. Set $C = B/I_Y$.

(i) We recall the following known properties of Ext modules:

- codim$(\text{Ext}^1_B(C, B)) \geq i$;
- a prime $p \in \text{Spec}(B)$ of codimension $i$ is an associated prime of $C$ if and only if it is a minimal prime of $\text{Ext}^1_B(C, B)$;

see, e.g., [22, Theorem 1.1]. Since $\delta_i = \dim \left( \text{Ext}^1_B(C, B) \right) \leq r + 1 - i$, we consider the usual truncated notion of multiplicity
\[
e_{r+1-i} \left( \text{Ext}^1_B(C, B) \right) = \lim_{\nu \to \infty} \frac{\dim_k \left( \left[ \text{Ext}^1_B(C, B) \right]_\nu \right)}{\nu^{r-i}/(r-i)!} = \begin{cases} e \left( \text{Ext}^1_B(C, B) \right) & \text{if } \delta_i = r + 1 - i \\ 0 & \text{otherwise} \end{cases}
\]

(see, e.g., [52]). Then the graded local duality theorem (see [5, Theorem 3.6.19]) implies that
\[
e_{r+1-i} \left( \text{Ext}^1_B(C, B) \right) = \lim_{\nu \to \infty} \frac{\dim_k \left( H^{r+1-i}_\nu(C) \right)}{\nu^{r-i}/(r-i)!}.
\]

There is a short exact sequence $0 \to C \to \bigoplus_{\nu < Z} H^0(Y, O_Y(\nu)) \to H^1_\nu(C) \to 0$ and an isomorphism
\[
H^{r+1-i}_\nu(C) \cong \bigoplus_{\nu < Z} H^1(Y, O_Y(\nu)) \quad \text{for all } j \geq 1;
\]

see [21, Theorem A4.1]. As a consequence, we have the equality
\[
e_{r+1-i} \left( \text{Ext}^1_B(C, B) \right) = \lim_{\nu \to \infty} \frac{\dim_k \left( H^{r-1}_\nu(Y, O_Y(-\nu)) \right)}{\nu^{r-i}/(r-i)!}.
\]

Finally, the result follows because $e_{r+1-i} \left( \text{Ext}^1_B(C, B) \right)$ is a positive integer if and only if $\text{Ext}^1_B(C, B)$ has dimension $r + 1 - i$ (i.e., when $C$ has an associated prime of codimension $i$).

(ii) This is the content of the proof of [30, Theorem III.7.6(b)]. Alternatively, we can argue as follows. Let $y \in Y$ be a point and $p \in \text{Spec}(B)$ the corresponding prime in $B$. By definition we have that $O_{Y,y} = \{ f, g \in C \text{ homogeneous elements, } g \notin pC, \deg(f) = \deg(g) \}$. Let $C_p$ be the localization of $C$ at all the homogeneous elements not in $p$ (we follow the notation of [5, §1.5]); this is a local ring ring with *maximal ideal* $pC_p$. By [5, Exercise 2.1.27], $C_p$ is Cohen-Macaulay if and only if $C_p$ is. Let $w \in [C]_1$ be a linear form not in $pC$. Notice that we have the equality $C_p = O_{Y,y}[w, w^{-1}]$ and that $w$ can be seen as an indeterminate over $O_{Y,y}$. Therefore, $O_{Y,y}$ is Cohen-Macaulay if and only if $C_p$ is Cohen-Macaulay.

The same arguments in the proof of part (i) imply that $H^i(Y, O_Y(-\nu)) = 0$ for all $i < \dim(Y), \nu \gg 0$ if and only if $\text{Supp}_B \left( \text{Ext}^i_B(C, B) \right) \subseteq \{ m \}$ for all $i < \dim(Y)$. The latter condition is equivalent to the following two conditions:

(a) all minimal primes of $I_Y$ have codimension equal to $r - \dim(Y)$;
(b) $C_p$ is Cohen-Macaulay for all $p \in V(I_Y) \setminus \{m\} \subset \text{Spec}(B)$; indeed, this follows from the local duality theorem and the depth sensitivity of local cohomology (see [5, Theorem 3.5.7]). So, the proof of part (ii) is complete.

By combining the above results we obtain the following theorem.

**Theorem 6.2.** Let $(A,n,k)$ be a Noetherian local ring with residue field $k$. Let $Y \subset \mathbb{P}_A^n$ be a closed subscheme and suppose that $Y$ is flat over $A$. Let $q \in \text{Spec}(A)$ and $Z = Y \times_{\text{Spec}(A)} \text{Spec}(k(q)) \subset \mathbb{P}_k^n$ be the special fiber. Let $q \in \text{Spec}(A)$ and $Z = Y \times_{\text{Spec}(A)} \text{Spec}(k(q)) \subset \mathbb{P}_k^n$.

If $\text{Hom}_k(H^i(Y, O_Y(−\nu)), H^{i+1}(Y, O_Y(−\nu))) = 0$ for all $i \geq 0$ and $\nu \gg 0$, then the following statements hold:

(i) $Y$ is Cohen-Macaulay and equidimensional if and only if $Z$ is both.

(ii) For $1 \leq i \leq r$, $I_Y$ has an associated prime of codimension $i$ if and only if $I_Z$ has one.

**Proof.** By Proposition 6.1, we obtain the following equality

$$\dim_{k}(H^i(Y, O_Y(−\nu))) = \dim_{k(q)}(H^i(Z, O_Z(−\nu)))$$

for all $i \geq 0$ and $\nu \gg 0$. Then the statements of the theorem follow from Lemma 6.3.

6.2. **Properties of multiplicity-free varieties.** Throughout this subsection we shall use the following setup.

**Setup 6.5.** Let $k$ be a field and $S = k[x_{i,j} \mid 1 \leq i \leq p, 0 \leq j \leq m_i]$ be a standard $\mathbb{N}P$-graded polynomial ring over $k$, such that MultiProj$(S) = \mathbb{P} = \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_p}_k$. Let $X = \text{MultiProj}(R) \subset \mathbb{P}$ be an integral closed subscheme and $R$ be an $\mathbb{N}P$-graded quotient ring of $S$ that is a domain.

Our next goal is to obtain an alternative proof of the following beautiful result.

**Theorem 6.6 (Brion [3]).** Assume Setup 6.5. If $X \subset \mathbb{P}$ is multiplicity-free, then:

(i) $X$ is arithmetically Cohen-Macaulay.

(ii) $X$ is arithmetically normal.

(iii) ($k$ infinite) There is a flat degeneration of $X$ to the following reduced union of multiprojective spaces

$$H = \bigcup_{n=(n_1,\ldots,n_p) \in M_{\text{Supp}}(X)} \mathbb{P}^{n_1}_k \times_k \cdots \times_k \mathbb{P}^{n_p}_k \subset \mathbb{P} = \mathbb{P}^{m_1}_k \times_k \cdots \times_k \mathbb{P}^{m_p}_k$$

where $\mathbb{P}^{n_i}_k = \text{Proj}(k[x_{i,m_i}−n_i, \ldots, x_{i,m_i}]) \subset \mathbb{P}^{m_i}_k = \text{Proj}(k[x_{i,0}, \ldots, x_{i,m_i}])$ only uses the last $n_i + 1$ coordinates.

The proof of this theorem is divided in several steps. We first consider the case of one-dimensional multiplicity-free varieties.

**Lemma 6.7.** Suppose that $\dim(X) = 1$ and that $X \subset \mathbb{P}$ is multiplicity-free. Then $X$ is isomorphic to the image of a diagonal morphism $\mathbb{P}_k^1 \rightarrow (\mathbb{P}_k^1)^e \subset \mathbb{P}$ for some $0 < e \leq p$.

**Proof.** The basic idea is to express $X$ as a blow-up (see [30, Theorem II.7.17]). We have that $\Pi_i(X) = \text{Proj}(R_{(i)})$ with $R_{(i)} = \bigoplus_{k \geq 0} \mathbb{R}(k, e_i)$ (see Notation 2.13) and the natural surjection $R_{(i)} \otimes_k \cdots \otimes_k R_{(p)} \rightarrow R$ of standard $\mathbb{N}P$-graded $k$-algebras, and so as a consequence we obtain that the natural morphism $X \rightarrow$
\[ \Pi_1(X) \times_k \cdots \times_k \Pi_p(X) \] is a closed immersion. By Corollary 4.10, for each \( 1 \leq i \leq p \), we have that \( \Pi_i(X) \) equals either \( \mathbb{P}^1_k \) or \( \text{Spec}(k) \). Since \( X \hookrightarrow \Pi_1(X) \times_k \cdots \times_k \Pi_p(X) \) is a closed immersion, we may assume that \( X \subset (\mathbb{P}^1_k)^e \subset \mathbb{P} \) for some \( 0 < e \leq p \). Let \( \alpha : X \subset (\mathbb{P}^1_k)^e \to \mathbb{P}^1_k \) be the projection into the last component and \( \beta : X \subset (\mathbb{P}^1_k)^e \to (\mathbb{P}^1_k)^{e-1} \) into the first \( e-1 \) components. Due to Lemma 4.3, \( \alpha \) is a birational morphism, and so there exists a dense open subset \( \mathcal{U} \subset \mathbb{P}^1_k \) such that \( \alpha_{|\mathcal{U}} : \mathcal{U} = \alpha^{-1}(\mathcal{U}) \to \mathcal{U} \) is an isomorphism. Then the composition \( \beta \circ \alpha_{|\mathcal{U}} \) gives a morphism \( \mathcal{U} \to (\mathbb{P}^1_k)^{e-1} \) and \( X \) can be realized as its graph. That is, we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & (\mathbb{P}^1_k)^e \\
\downarrow \beta & & \downarrow \phi \\
\mathbb{P}^1_k & \to & (\mathbb{P}^1_k)^{e-1}.
\end{array}
\]

Set \( \mathbb{P}^1_k = \text{Proj}(B) \) with \( B = k[z_0, z_1] \). The morphism \( \phi \) is given by \( e-1 \) pairs of homogeneous polynomials \( \{f_{l,0}, f_{l,1}\} \subset B \) with \( \delta_i = \text{deg}(f_{l,0}) = \text{deg}(f_{l,1}) \) and \( \gcd(f_{l,0}, f_{l,1}) = 1 \). Let \( Y \subset \mathbb{P}' = (\mathbb{P}^1_k)^{e-1} \) be the image of \( \phi \). As a consequence of Corollary 4.10, \( Y \) is multiplicity-free. The degree formula of [12, Corollary 3.9] yields the equalities \( \delta_i = \deg_{\mathbb{P}'}(Y) \deg(\phi) = 1 \) for all \( 1 \leq i \leq e-1 \), where \( e_i = n \in \mathbb{N}^{e-1} \) denotes the standard basis vector. Therefore, up to a \( k \)-linear change of coordinates we may assume that \( f_{l,0} = z_0 \) and \( f_{l,1} = z_1 \). This shows that \( X \) is isomorphic to the image of the diagonal morphism \( \mathbb{P}^1_k \to (\mathbb{P}^1_k)^e \subset \mathbb{P} \). \( \square \)

The next proposition deals with the Hilbert polynomial of \( X \). This is the main step in the proof of Theorem 6.6, and it is the place where we utilize the tools developed in §6.1.

**Proposition 6.8.** Under the notation and assumptions of Theorem 6.6, we have \( P_X(t) = P_H(t) \).

**Proof.** We proceed by induction on \( \dim(X) \).

If \( \dim(X) = 0 \), then the result is clear since we have \( P_X(t) = 1 = P_H(t) \).

If \( \dim(X) = 1 \), then \( X \) is isomorphic to the image of a diagonal morphism \( \mathbb{P}^1_k \to (\mathbb{P}^1_k)^e \subset \mathbb{P} \) by Lemma 6.7. Thus, we get \( P_X(t) = 1 + \sum_{i \in \text{MSupp}(X)} t_i = P_H(t) \).

Suppose that \( d = \dim(X) \geq 2 \). Here we use the conventions of Notation 4.4. Write \( R = S/P \) and notice that \( P_L := P \otimes_k L \subset S_L \) is again a prime ideal (see Remark 4.5). One way of relating \( X \) and \( H \) is by using the multigraded generic initial ideal (see Section 5). Let \( \mathcal{T} > 0 \) be a monomial order on \( S \) that satisfies \( y_0 > \cdots > y_t \), where \( y_j = x_{p,j} \). Let \( I = \text{gin}(P_L) \subset S_L \). Let \( P_1, \ldots, P_n \) be the minimal primes of \( I \) and \( Q_1, \ldots, Q_h \) be the corresponding minimal primary components. Since \( X \) is assumed to be multiplicity-free, Remark 5.10 yields that \( \mathbb{P}_k = P_k \) for all \( 1 \leq k \leq h \). Set \( J = Q_1 \cap \cdots \cap Q_h = \sqrt{I} \subset S_L \). By construction we have \( H_L := H \otimes_k L \cong \text{MultiProj}(S_L/I) \). Let \( G := \text{MultiProj}(S_L/I) \). There exists a one-parameter flat family \( X \subset \mathbb{P}_L \times_A L^d \) such that \( X_L \) is the general fiber and \( G \) is the special fiber (see [21, §15.8]). Then we write

\[
P_X(t) = P_G(t) = \sum_{n \in \mathbb{N}^p, \sum n_i \leq t} e(n) \binom{t_1 + n_1}{n_1} \cdots \binom{t_p + n_p}{n_p}
\]

and

\[
P_H(t) = \sum_{n \in \mathbb{N}^p, \sum n_i \leq t} h(n) \binom{t_1 + n_1}{n_1} \cdots \binom{t_p + n_p}{n_p}.
\]
Since $G_{\text{red}} = H_L$, it follows that $\dim (\Pi_2(G)) = \dim (\Pi_2(H))$ for all $\mathfrak{J} \subset [p]$. From [7, Proposition 3.1], we have that $\dim (\Pi_1(X))$ equals the same holds for $G$ and $H$. Suppose that $\dim (\Pi_p(X)) \geq 1$. We have the equalities

$$P_{X_L \cap V(y)}(t) = P_{G \cap V(y)}(t) = \sum_{n \in \mathbb{N}_0} \epsilon(n) \left( \frac{t_1 + n_1}{n_1} \ldots \frac{t_{p-1} + n_{p-1}}{n_{p-1}} \frac{t_p + n_p - 1}{n_p - 1} \right)$$

and

$$P_{H_L \cap V(y)}(t) = \sum_{n \in \mathbb{N}_0} h(n) \left( \frac{t_1 + n_1}{n_1} \ldots \frac{t_{p-1} + n_{p-1}}{n_{p-1}} \frac{t_p + n_p - 1}{n_p - 1} \right).$$

As no monomial of $J$ involves the variable $y_\ell$, we obtain $J + (y) = \bigcap_{k=1}^h (Q_k + (y))$. This implies that

$$H_L \cap V(y) = \bigcup_{n=\{n_1, \ldots, n_p\} \in \text{MSupp}(X)} P_{L_1}^{n_1} \times \ldots \times L_p^{n_p-1} \subset P_L.$$ 

On the other hand, $X_L \cap V(y)$ is an integral scheme, and Theorem 4.7 and [7, Theorem A] yield that $\text{MSupp}_{P_L}(X_L \cap V(y)) = \{ n - e_p \mid n \in \text{MSupp}_P(X) \text{ and } n_p \geq 1 \}$. Therefore, the inductive hypothesis gives the equality $P_{X_L \cap V(y)}(t) = P_{H_L \cap V(y)}(t)$ and so we obtain that $\epsilon(n) = h(n)$ for all $n_p \geq 1$.

For $1 \leq i < p$, we can repeat the above arguments and consider the $i$-th component of $P = P_{k_1}^{m_1} \times \ldots \times_k P_{k_p}^{m_p}$. So, we conclude that $\epsilon(n) = h(n)$ for all $n \neq 0$. Thus $c = P_X(t) - P_H(t)$ is a constant polynomial. We have a short exact sequence $0 \to J/I \to S_L/I \to S_L/J \to 0$. Hence $P_{J/I}(t) = c$ and the corresponding coherent sheaf $\mathcal{R} = (J/I)^{-}\mathcal{O}$ on $G$ is supported on dimension zero. By Theorem 5.6, $S_L/J$ is Cohen-Macaulay, and thus $H_L$ is Cohen-Macaulay.

Let $s : P_L = P_{L_1}^{m_1} \times \ldots \times L_p^{m_p} \to P_{L_1}^r$ be the Segre embedding, where $r = (m_1 + 1) \cdots (m_p + 1) - 1$.

Let $Z = s(X_L)$, $Y = s(G)$ and $Y' = s(H_L)$. Let $\mathfrak{S} = s \times L A_1^p : P_{L_1} \times L A_1^p \to P_{L_1}^r \times L A_i^p$, and notice that $Y = \mathfrak{S}(X) \subset P_{L_1}^r \times L A_i^p$ is a one-parameter flat family with general fiber $Z \subset P_{L_1}^r$ and special fiber $Y \subset P_{L_1}^r$. We get a short exact sequence $0 \to s_* \mathcal{R} \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0$. As $s_* \mathcal{R}$ is supported on dimension zero, we have $H^i(Y, (s_* \mathcal{R})(\nu)) = 0$ for all $i \geq 1, \nu \in Z$. On the other hand, Lemma 6.3(ii) yields $H^i(Y', \mathcal{O}_{Y'}(\nu)) = 0$ for all $i < d, \nu \gg 0$. From the induced long exact sequence in cohomology

$$\cdots \to H^i(Y, (s_* \mathcal{R})(\nu)) \to H^i(Y, \mathcal{O}_{Y'}(\nu)) \to H^i(Y', \mathcal{O}_{Y'}(\nu)) \to \cdots$$

we obtain that $H^i(Y, \mathcal{O}_{Y'}(\nu)) = 0$ for all $1 \leq i < d, \nu \gg 0$. Since $\dim(Y) = d \geq 2$, it follows that the condition of Theorem 6.4 holds for our current $Y$. The ideal $I_Z$ does not have an associated prime of codimension $\tau$ because $Z$ is an integral scheme, and so Theorem 6.4(ii) implies that $I_Y$ does not have an associated prime of codimension $\tau$. Therefore $s_* \mathcal{R} = 0$ and $Y = Y'$.

Finally, we have shown that $c = P_X(t) - P_H(t) = 0$, and this concludes the proof of the proposition. \qed

We can now complete our proof of Brion’s result.

**Proof of Theorem 6.6.** Let $P \subset S$ be a prime ideal with $X = \text{MultiProj}(S/P)$. Let $p$ be a monomial order on $S$ and assume that $k$ is infinite (see Remark 4.5). As in Remark 2.8 and the beginning of the proof of Theorem 5.9, we may adjoin new variables to $S$ in such a way that all the associated primes of $\text{gin}(P)$ are relevant. Let $I = \text{gin}(P)$ and $J = \sqrt{\text{gin}(P)}$. Since $X$ is multiplicity-free, Remark 5.10 implies that $J$ is the intersection of the minimal primary components of $I$. By Proposition 6.8, we have the equalities...
\[ \text{Proposition 7.9} \]

as we can always make \(57\); notice that the proof of \(\text{Theorem 6.6}\) only implies that \(\text{Theorem 7.10}\) (i) and \(\text{Proposition 7.9}\). We have that \(\text{Theorem 6.6}\) is a normal domain.

\[\text{Setup 7.1.}\]

Let \(T = S[z_1, \ldots, z_b]\) be an \(\mathbb{N}P\)-graded polynomial ring over \(S\) with a surjective \(S\)-algebra homomorphism \(\psi : T \rightarrow R, z_1 \rightarrow w_1\). Since \(\operatorname{rank}_R(\bar{R}) = 1\), the associativity formula for mixed multiplicities yields the equality \(e(n; \bar{R}) = e(n; R)\) for all \(n \in \mathbb{N}P\) with \(|n| = \dim(X)\). Hence we consider the multiplicity-free prime ideal \(Q = \ker(\psi) \subset T\). Choose a lexicographical monomial order \(>^t\) on \(T\) such that \(z_1 >^t x_{i,j}\) for all \(1 \leq i \leq b, 1 \leq j \leq p, 0 \leq j \leq m_i\). The assumption \(w_1 \not\in R\) would imply that a monomial of the form \(z_1^{p_1}\) with \(g_1 \geq 2\) is among the minimal generators of \(in_{>^t}(Q)\). However, this would contradict the fact that \(in_{>^t}(Q)\) is radical by \(\text{Theorem 7.10}(i)\) and \(\text{Proposition 7.9}\); notice that the proof of \(\text{Proposition 7.9}\) only depends on the fact that \(\text{gin}(P)\) is radical, which we already proved. So, \(R\) is a normal domain and the proof of the remaining part (ii) is complete.

Finally, we restate Brion’s theorem in a more algebraic language.

\[\text{Theorem 6.9.}\]

Assume \(\text{Setup 6.5}\). If \(P \subset S\) is an \(S\)-homogeneous multiplicity-free prime ideal, then:

(i) \(S/P\) is Cohen-Macaulay.

(ii) \(S/P\) is a normal domain.

(iii) \((k\text{ infinite})\) \(\text{gin}_{>^t}(P)\) is a radical monomial ideal for any monomial order \(>^t\) on \(S\).

**Proof.** This follows from the proof of \(\text{Theorem 6.6}\) as we can always make \(P\) a relevant ideal. \(\square\)

\section{Standardization of ideals}

In this section, we develop a process of standardization of ideals in an arbitrary positive (possibly non-standard) multigrading. Here we use a mixture of the techniques step-by-step homogenization introduced in [45] for singly-graded settings and standardization introduced in [8] for certain multigraded settings. The following setup and construction is used throughout this section.

\[\text{Setup 7.1.}\]

Let \(k\) be a field and \(R = k[x_1, \ldots, x_n]\) be a positively \(\mathbb{N}P\)-graded polynomial ring. For \(1 \leq i \leq n\), let \(\ell_i = |\deg(x_i)|\) be the total degree of the variable \(x_i\). Let

\[ S = k[y_{1,i} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq \ell_i] \]

be a standard \(\mathbb{N}P\)-graded polynomial ring such that

\[ \deg(x_i) = \sum_{j=1}^{\ell_i} \deg(y_{1,j}) \text{ for all } 1 \leq i \leq n. \]

We define the \(\mathbb{N}P\)-graded \(k\)-algebra homomorphism

\[ \phi : R = k[x] \rightarrow S = k[y], \quad \phi(x_i) = y_{1,1}y_{1,2} \cdots y_{1,\ell_i}. \]
For an \( R \)-homogeneous ideal \( I \subset R \), we say that the extension \( J = \phi(I)S \) is the standardization of \( I \), as \( J \subset S \) is an \( S \)-homogeneous ideal in the standard \( \mathbb{N}^p \)-graded polynomial ring \( S \). Let \( t = \{t_1, \ldots, t_p\} \) be variables indexing the \( \mathbb{Z}^p \)-grading, where \( t_i \) corresponds with \( e_i \in \mathbb{Z}^p \). Given a finitely generated \( \mathbb{Z}^p \)-graded \( R \)-module \( M \) and a finitely generated \( \mathbb{Z}^p \)-graded \( S \)-module \( N \), by a slight abuse of notation, we consider both multidegree polynomials \( \mathcal{C}(M; t) \) and \( \mathcal{C}(N; t) \) as elements of the same polynomial ring \( \mathbb{Z}[t] = \mathbb{Z}[t_1, \ldots, t_p] \).

The following theorem contains some of the basic and desirable properties that the standardization process satisfies.

**Theorem 7.2.** Assume Setup 7.1. Let \( I \subset R \) be an \( R \)-homogeneous ideal and \( J = \phi(I)S \) be its standardization. Then the following statements hold:

(i) \( \text{codim}(I) = \text{codim}(J) \).

(ii) \( I \subset R \) and \( J \subset S \) have the same \( \mathbb{N}^p \)-graded Betti numbers.

(iii) \( \mathcal{K}(R/I; t) = \mathcal{K}(S/J; t) \) and \( \mathcal{C}(R/I; t) = \mathcal{C}(S/J; t) \).

(iv) \( R/I \) is a Cohen-Macaulay ring if and only if \( S/J \) is a Cohen-Macaulay ring.

(v) Let \( > \) be a monomial order on \( R \) and \( >' \) be a monomial order on \( S \) which is compatible with \( \phi \) (i.e., if \( f, g \in R \) with \( f > g \), then \( \phi(f) \) > \( \phi(g) \)). Then \( \text{in}_{>'}(J) = \phi(\text{in}_>(I))S \).

(vi) If \( I \subset R \) is a prime ideal and it does not contain any variable, then \( J \subset S \) is also a prime ideal.

**Proof.** Let \( T = \mathbb{k}[x, y] \cong R \otimes_{k} S \) be its natural \( \mathbb{N}^p \)-grading induced from the ones of \( R \) and \( S \). We see \( R \) and \( S \) as subrings of \( T \). We consider the quotient ring \( T/IT \) and notice that \( \{x_i - \prod_{j=1}^{\ell_i} y_{i,j} \}_{1 \leq i \leq n} \) is a regular sequence of homogeneous elements on \( T/IT \). We also have the following natural isomorphism

\[
\frac{T}{IT + \left( \{x_i - \prod_{j=1}^{\ell_i} y_{i,j} \}_{1 \leq i \leq n} \right)} \cong S/J.
\]

As the natural inclusion \( R \hookrightarrow T \) is a polynomial extension, we have that \( \dim(T/IT) = \dim(R/I) + \dim(S) = \dim(R) + \dim(S) - \text{codim}(I) \) and that \( T/IT \) is Cohen-Macaulay if and only if \( R/I \) is Cohen-Macaulay. So, by cutting out with the regular sequence described above, we obtain that \( \dim(S/J) = \dim(T/IT) - \dim(R) = \dim(S) - \text{codim}(I) \) and that \( S/J \) is Cohen-Macaulay if and only if \( T/IT \) is Cohen-Macaulay. This completes the proofs of parts (i) and (iv).

Let \( F_\bullet: \cdots \rightarrow F_1 \xrightarrow{f_1} F_0 \) be a \( \mathbb{Z}^p \)-graded free \( R \)-resolution of \( R/I \). Since \( \{x_i - \prod_{j=1}^{\ell_i} y_{i,j} \}_{1 \leq i \leq n} \) is a regular sequence on both \( T \) and \( T/IT \), it follows that

\[
\text{Tor}_k^R \left( T/IT, T/\left( \{x_i - \prod_{j=1}^{\ell_i} y_{i,j} \}_{1 \leq i \leq n} \right) \right) = 0 \quad \text{for all } k > 0,
\]

and so \( G_\bullet = F_\bullet \otimes_R T/(\{x_i - \prod_{j=1}^{\ell_i} y_{i,j} \}_{1 \leq i \leq n}) \) provides (up to isomorphism) a \( \mathbb{Z}^p \)-graded free \( S \)-resolution of \( S/J \). The identification of \( G_\bullet \) as a resolution of \( S \)-modules is the same as \( \phi(F_\bullet) \) (more precisely, \( G_\bullet \) has the same Betti numbers as \( F_\bullet \) and the \( i \)-th differential matrix of \( G_\bullet \) is given by the substitution \( \phi(f_i) \) of the differential matrix \( f_i \)). We obtained the result of part (ii). Therefore, by definition, we have the equalities \( \mathcal{K}(R/I; t) = \mathcal{K}(S/J; t) \) and \( \mathcal{C}(R/I; t) = \mathcal{C}(S/J; t) \) that settle part (iii).
To show part (v) we can use Buchberger’s algorithm (see, e.g., [21, Chapter 15]). Indeed, we can perform essentially the same steps of the algorithm in a set of generators of I and the corresponding set of generators for J; for instance, for any two polynomials \( f, g \in R \) we have the following relation of S-polynomials \( S(\phi(f), \phi(g)) = \phi(S(f, g)) \).

Lastly, we concentrate on the proof of part (vi). Suppose that \( I \subset R \) is a prime ideal not containing any variable. It then follows that \( T/IT \) is a domain and that \( \{ y_{1,2} \cdots y_{1,t}, x_i \} \) is a regular sequence on \( T/IT \), and so [21, Exercise 10.4] implies that \( T/(IT, x_1 - \prod_{j=1}^{t_j} y_{1,j}) \) is also a domain. Finally, by repeating iteratively this argument we obtain that \( T/(IT + (\{x_i - \prod_{j=1}^{t_j} y_{1,j} \}_{1 \leq i \leq n})) \cong S/J \) is a domain. This concludes the proof of part (vi).

\begin{proof}
\end{proof}

\begin{remark}
Without too many changes the above arguments could be used to define the standardization of a module. In that case, one could proceed by standardizing a presentation matrix of a module.
\end{remark}

A direct known consequence of the above theorem is the following remark.

\begin{remark}
For any \( R \)-homogeneous ideal \( I \subset R \) the multidegree polynomial \( C(R/I; t) \) has non-negative coefficients. Indeed, it follows from Theorem 7.2(iii) and the (well-known) fact that multidegrees are non-negative in a standard multigraded setting.
\end{remark}

The following theorem shows that the support of the multidegree polynomial is a discrete polymatroid for prime ideals in a polynomial ring with arbitrary positive multigrading. It is a consequence of [7, Theorem A].

\begin{theorem}
Assume Setup 7.1. Let \( P \subset R \) be an \( R \)-homogeneous prime ideal. Then the support of the multidegree polynomial \( C(R/P; t) \) is a discrete polymatroid.
\end{theorem}

\begin{proof}
Let \( \mathcal{L} = \{ i \mid x_i \in P \} \) be the set of indices such that the corresponding variables belong to \( P \). We consider the polynomial ring \( R' = \mathbb{k}[x_i \mid i \notin \mathcal{L}] \subset R \). Let \( P' \subset R' \) be the (unique) ideal that satisfies the condition \( P = P' R + (x_i \mid i \in \mathcal{L}) \). By construction \( P' \subset R' \) does contain any variable. Since \( R/P \cong R'/P' \), it follows that \( P' \) is also a prime ideal. As a consequence of Theorem 7.2(vi) the standardization \( Q = \phi(P'R) \subset S \) is a prime ideal.

Since the variables \( x_i \) with indices in \( \mathcal{L} \) form a regular sequence on \( R/P'R \), we obtain the equation

\[
C(R/P; t) = \prod_{i \in \mathcal{L}} (\deg(x_i), t) \cdot C(R/P'R; t)
\]

where \( (\deg(x_i), t) = a_{1,i} t_1 + \cdots + a_{L,P} t_P \in \mathbb{N}[t] \) after writing \( (a_{1,i}, \ldots, a_{L,P}) = \deg(x_i) \in \mathbb{N}^P \) (see Remark 2.6(ii)). From Theorem 2.14 we have that the support of \( C(S/Q; t) \) is a discrete polymatroid, and then Theorem 7.2(iii) implies that the support of \( C(R/P'R; t) \) is also a discrete polymatroid. On the other hand, it is clear that the support of each linear polynomial \( (\deg(x_i), t) \) is a discrete polymatroid. Finally, Remark 2.1(ii) yields that the support of \( C(R/P; t) \) is a discrete polymatroid.
\end{proof}

\begin{corollary}
Assume Setup 7.1 with \( k \) an infinite field. Let \( P \subset R \) be an \( R \)-homogeneous prime ideal, and \( Q = \phi(P)S \) be its standardization. Then \( \sqrt{\gin(Q)} \) is a Cohen-Macaulay ideal.
\end{corollary}

\begin{proof}
By Theorem 7.5 and Theorem 7.2(iii), the support of \( C(S/Q; t) \) is a discrete polymatroid. One may notice that the proof of Theorem 5.6 only depends on the fact that the support of the multidegree
polynomial of the ideal is a discrete polymatroid. Therefore, by using the same arguments applied to $Q$ (which may not be prime), we conclude that $\sqrt{\text{gin}(Q)}$ is a Cohen-Macaulay ideal. 

7.1. Cartwright-Sturmfels ideals in positive multigradings. The purpose of this subsection is to extend the notion of Cartwright-Sturmfels ideals to arbitrary positive multigradings. This family of ideals was defined and studied in a series of papers [15–19] over a standard multigraded setting.

**Definition 7.7.** An $R$-homogeneous ideal $I \subset R$ is said to be Cartwright-Sturmfels (CS for short) if there exists an $S$-homogeneous radical Borel-fixed ideal $K \subset S$ such that $\mathcal{K}(R/I; t) = \mathcal{K}(S/K; t)$.

The first basic (yet important) observation that one can make is the following lemma.

**Lemma 7.8.** Let $I \subset R$ be an $R$-homogeneous ideal and $J = \phi(I)S$ be its standardization. Then, $I$ is CS (in the sense of Definition 7.7) if and only if $J$ is CS (in the sense of [18]).

**Proof.** By Theorem 7.2(iii), we have $\mathcal{K}(R/I; t) = \mathcal{K}(S/J; t)$. Thus any $S$-homogeneous ideal $K \subset S$ has the same $\mathcal{K}$-polynomial as $I \subset R$ if and only if it has the same multigraded Hilbert function as $J \subset S$. So, the equivalence is clear because $J$ is CS when there is a radical Borel-fixed ideal with the same multigraded Hilbert function as $J$. □

**Proposition 7.9.** Let $P \subset R$ be a multiplicity-free prime ideal, then $P$ is CS. 

**Proof.** As in the proof of Theorem 7.5, we may get rid of the variables that belong to $P$. Let $\mathcal{L} = \{i | x_i \in P\}$ be the set that indexes the variables belonging to $P$. Write $P = P' + \langle x_i | i \in \mathcal{L} \rangle$ with $P' \subset R$ only involving the variables not in $\mathcal{L}$. Let $T = R[z_i | i \in \mathcal{L}]$ be a positively $\mathbb{N}^P$-graded polynomial ring extending the grading of $R$ and with $\deg(z_i) = \deg(x_i)$. Let $\mathcal{Q} = P'T + \langle x_i - z_i | i \in \mathcal{L} \rangle \subset T$. Notice that $\mathcal{Q}$ is a prime ideal containing no variable and that $\mathcal{K}(T/\mathcal{Q}; t) = \mathcal{K}(R/P; t)$. Let $W = S[\ell_{i,j} | i \in \mathcal{L}, 1 \leq j \leq \ell_i]$ be a standard $\mathbb{N}^P$-graded polynomial ring where we consider the corresponding standardization $\mathcal{Q} \subset W$ of $\mathcal{Q} \subset T$. From Theorem 7.2(iii)(vi) the standardization $\mathcal{Q} \subset W$ is a multiplicity-free prime ideal. Then Theorem 6.9(iii) implies that $\mathcal{Q}$ is CS (as we may extend the field $k$; see Remark 4.5). Let $Q \subset S$ be the standardization of $P \subset R$. Since ideal extension $QW \subset W$ has the same multigraded Hilbert function as $\mathcal{Q} \subset W$, it follows that $QW$ is CS, and so [18, Proposition 2.7] implies that $Q \subset S$ is also CS. Finally, $P \subset R$ is CS by Lemma 7.8. □

The theorem below lists some important basic properties of CS ideals in $R$.

**Theorem 7.10.** Assume Setup 7.1. Let $I \subset R$ be an $R$-homogeneous CS ideal. Then the following statements hold:

(i) $\text{in}_T(1)$ is radical and CS for any monomial order $>_T$ on $R$; in particular, $I$ is radical.

(ii) The $\mathbb{N}$-graded Castelnuovo-Mumford regularity $\text{reg}(1)$ is bounded from above by $p$.

(iii) $I$ is a multiplicity-free ideal.

(iv) If $P \subset R$ is a minimal prime of $I$, then $P$ is CS.

(v) All reduced Gröbner bases of $I$ consist of elements of multidegree $\leq (1, \ldots, 1) \in \mathbb{N}^P$. In particular, $I$ has a universal Gröbner basis of elements of multidegree $\leq (1, \ldots, 1) \in \mathbb{N}^P$.

**Proof.** (i) It follows from Theorem 7.2(v) and [19, Proposition 3.4(2)].

(ii) It is a consequence of Theorem 7.2(ii) and [19, Proposition 3.4(3)].
(iii) It follows from Theorem 7.2(iii) and [19, Proposition 3.6(1)].

(iv) Since \(I\) is multiplicity-free, we conclude that \(P\) also is. We then obtain the result from Proposition 7.9. (v) It follows from Theorem 7.2(v) and [19, Proposition 3.4(6)]. □

Finally, we point out some interesting consequences of our work, that signal very rigid properties of certain multigraded Hilbert schemes. To that end, we use the multigraded Hilbert scheme of Haiman and Sturmfels [28]. For a given function \(h : \mathbb{N}^P \to \mathbb{N}\), the multigraded Hilbert scheme \(HS_{R/k}^h\) parametrizes all \(R\)-homogeneous ideals \(I \subset R\) such that \(\dim_{k}(|R/I|_v) = h(v)\) for all \(v \in \mathbb{N}^P\). We have the following rigidity result that extends [6, Theorem 2.1 and Corollary 2.6].

**Corollary 7.11.** Assume Setup 7.1. Let \(h : \mathbb{N}^P \to \mathbb{N}\) be a function and consider the corresponding multigraded Hilbert scheme \(HS_{R/k}^h\). If \(HS_{R/k}^h\) contains a \(k\)-point that corresponds to a CS ideal and a \(k\)-point that corresponds to a prime ideal, then any \(k\)-point in \(HS_{R/k}^h\) corresponds to an ideal that is Cohen-Macaulay and CS.

**Proof.** Let \([P],[H] \in HS_{R/k}^h\) be \(k\)-points in \(HS_{R/k}^h\) such that \(P \subset R\) is a prime ideal and \(H \subset R\) is a CS ideal. Let \(I \subset R\) be an \(R\)-homogeneous ideal such that \([I] \in HS_{R/k}^h\). From the definition, it follows that \(I\) and \(P\) are both CS. We may assume that \(k\) is an infinite field and keep the primeness of \(P\) (see Remark 4.5). Let \(J \subset S\) and \(Q \subset S\) be the standardizations of \(I\) and \(P\), respectively. Since \(J\) and \(Q\) are CS, both \(\text{gin}(J)\) and \(\text{gin}(Q)\) are radical, and actually are equal. Then \(\text{gin}(J) = \text{gin}(Q)\) is Cohen-Macaulay by Corollary 7.6. Finally, \(J \subset S\) is Cohen-Macaulay and Theorem 7.2(iv) implies that \(I \subset R\) is also Cohen-Macaulay; which concludes the proof. □

8. Examples of determinantal ideals with a fine grading

The goal of this section is to present examples of multigraded ideals in non-standard positive multigraded polynomial rings that have a multiplicity-free multidegree. The examples we selected are determinantal ideals.

We start with generic determinantal ideals with the finest multidegree structure. Let \(k\) be a field and set \(R = k[x_{i,j}] | (i,j) \in [m] \times [n]\) with \(m \leq n\). We give \(R\) a multigraded structure by setting \(\deg(x_{i,j}) = e_i + f_j \in \mathbb{N}^m \oplus \mathbb{N}^n\) where \(e_1,\ldots,e_m\) and \(f_1,\ldots,f_n\) are the canonical bases of \(\mathbb{N}^m\) and \(\mathbb{N}^n\), respectively. The determinantal ideal \(I_t\) of the \(r\)-minors of the \(m \times n\) generic matrix \(X = (x_{i,j})\) is then \(R\)-homogeneous.

We have:

**Theorem 8.1.** The \(k\)-algebra \(R/I_m\) has a multiplicity-free multidegree with respect to the \(\mathbb{N}^m \oplus \mathbb{N}^n\)-grading.

A combinatorial formula for the multidegree of \(R/I_t\) is described in [48, Chap.15 and 16] (up to certain sign changes). Indeed, by [48, Theorem 15.40 and Corollary 16.30], the multidegree of a larger family of determinantal ideals, the Schubert determinantal ideals, is given in terms of pipe-dreams (i.e., special tiling of the rectangles by crosses and elbow joints). So one could derive Theorem 8.1 from the results in [48]. Nevertheless, we present below a self-contained proof of Theorem 8.1. To simplify the notation, we set

\[ \mathcal{K}_{m,n} = \mathcal{K}(R/I_m; t_1,\ldots,t_m,s_1,\ldots,s_n) \quad \text{and} \quad \mathcal{E}_{m,n} = \mathcal{E}(R/I_m; t_1,\ldots,t_m,s_1,\ldots,s_n). \]

Indeed, we prove the following formula from which Theorem 8.1 follows immediately.
Theorem 8.2.

\[ c_{m,n} = \sum_{(a,b) \in T} t^a b^b \]

with \( T = \{ (a,b) \in \mathbb{N}^m \times \{0,1\}^n \mid |a| + |b| = n - m + 1 \} \).

For the proof Theorem 8.2 we denote by \( H_{m,n} \) the right hand side of the equality, i.e.

\[ H_{m,n} = \sum_{(a,b) \in T} t^a b^b \]

with \( T = \{ (a,b) \in \mathbb{N}^m \times \{0,1\}^n \mid |a| + |b| = n - m + 1 \} \). Firstly we observe that \( H_{m,n} \) satisfies the following recursion:

**Lemma 8.3.** For all \( n > m > 1 \), one has \( H_{m,n} = s_n H_{m,n-1} + t_m H_{m,n-1} + H_{m-1,n-1} \).

Proof. The first addend corresponds to the sum of all the monomials in \( H_{m,n} \) that contain \( s_n \). The second corresponds to the sum of all the monomials in \( H_{m,n} \) that do not contain \( s_n \) and contain \( t_m \). Finally the third addend corresponds to the sum of all the monomials in \( H_{m,n} \) that do not contain \( s_n \) and \( t_m \). Here it is important to observe that \( n - m + 1 = (n - 1) - (m - 1) + 1 \).

Secondly we observe that also \( c_{m,n} \) satisfies the same recursion of Lemma 8.3.

**Lemma 8.4.** For all \( n > m > 1 \), one has \( c_{m,n} = s_n c_{m,n-1} + t_m c_{m,n-1} + c_{m-1,n-1} \).

Proof. The \( m \)-minors of \( X \) form a Gröbner basis of \( I_m \) with respect to the diagonal term order. Let \( J_{m,n} = \text{in}(I_{m,n}) \) be the monomial ideal generated by the main diagonals of the \( m \)-minors of \( X \). Then

\[ K_{m,n} = K(R / J_{m,n}; t_1, \ldots, t_m, s_1, \ldots, s_n) \]

and similarly for \( c_{m,n} \). We have a decomposition \( J_{m,n} = x_{m,n} J_{m-1,n-1} + J_{m,n-1} \) from which it follows that \( J_{m,n} + (x_{m,n}) = J_{m,n-1} + (x_{m,n}) \) and \( J_{m,n} : (x_{m,n}) = J_{m-1,n-1} \). Therefore we have an induced short exact sequence

\[ 0 \to R / J_{m-1,n-1} (-\deg(x_{m,n})) \to R / J_{m,n} \to R / (J_{m,n-1} + (x_{m,n})) \to 0 \]

and hence

\[ K_{m,n} = (1 - t_m s_n) K_{m,n-1} + t_m s_n K_{m-1,n-1}. \]

Replacing \( t \) with \( 1 - t \) and \( s \) with \( 1 - s \) and extracting the homogeneous component of degree \( n - m + 1 \) we obtain

\[ c_{m,n} = (t_m + s_n) c_{m,n-1} + c_{m-1,n-1}, \]

which concludes the proof.

Now we are ready to prove Theorem 8.2.

**Proof of Theorem 8.2.** By Lemma 8.3 and Lemma 8.4 the polynomials \( H_{m,n} \) and \( c_{m,n} \) satisfy the same recursive relation. Hence it suffices to check that they agree when \( m = 1 \) and when \( m = n \). When \( m = 1 \) the ideal \( I_m \) is generated by a regular sequence of elements of degree \( e_j + f_j \) with \( j = 1, \ldots, m \). In this case, \( c_{1,n} = \prod_{j=1}^n (t_j + s_j) \) which clearly coincides with \( H_{1,n} \). When \( m = n \) then \( I_m \) is generated by
single element of degree \( \sum_{i=1}^{m} e_i + \sum_{j=1}^{n} f_j \), and hence \( c_{m,m} = \sum_{i=1}^{m} t_i + \sum_{j=1}^{n} s_j \) which coincides with \( H_{m,m} \). This concludes the proof of the theorem. \( \Box \)

**Remark 8.5.** As a consequence of Theorem 8.1 and Proposition 7.9, it follows that \( I_m \subset R \) is a CS ideal.

**Remark 8.6.** By inspecting the Eagon-Northcott resolution of \( I_m \), we may compute that the \( \mathbb{N} \)-graded Castelnuovo-Mumford regularity \( I_m \) is given by \( \text{reg}(I_m) = m + n \). Indeed, by coarsening the \( \mathbb{N}^{m} \oplus \mathbb{N}^{n} \)-grading of \( R \) into an \( \mathbb{N} \)-grading, we obtain that each variable \( x_{i,j} \) has degree 2 and that the ideal \( I_m \) is generated in degree 2. Since the Eagon-Northcott complex has linear maps in the \( x_{i,j} \)'s and length \( n - m \), it follows that \( \text{reg}(I_m) = n + m \). This shows that the upper bound of Theorem 7.10(ii) is sharp.

**Remark 8.7.** From the recursion of \( \mathcal{K} \)-polynomials stated in (3), we may compute the following formula for the \( \mathcal{K} \)-polynomial of \( I_m \):

\[
\mathcal{K}(R/I_m; t,s) = 1 - t_1 t_2 \cdots t_m \sum_{j=0}^{n-m} (-1)^j h_j(t) e_{m+j}(s)
\]

where \( h_j(t) \) is the complete symmetric polynomial of degree \( j \) in \( t_1, \ldots, t_m \) (sum of all monomials of degree \( j \)) and \( e_j(s) \) is the elementary symmetric polynomial of degree \( j \) in \( s_1, \ldots, s_n \) (sum of all squarefree monomials of degree \( j \)). Also, the formula for \( \mathcal{K}(R/I_m; t,s) \) may be obtained from the Eagon-Northcott resolution of \( I_m \).

Finally, we point out that most determinantal varieties do not have a multiplicity-free multidegree with respect to the \( \mathbb{N}^{m} \oplus \mathbb{N}^{n} \)-grading.

**Remark 8.8.** For \( 1 < r < m \), the \( k \)-algebra \( R/I_r \) does not have a multiplicity-free multidegree with respect to the \( \mathbb{N}^{m} \oplus \mathbb{N}^{m} \)-grading. For \( 2 < r < m \), this happens already with respect to the \( \mathbb{N}^{m} \) (row) grading and the \( \mathbb{N}^{n} \) (column) grading individually, see [19]. We have a significantly different behavior when \( r = 2 \): \( R/I_2 \) has a multiplicity free multidegree with respect to the row or column grading, but one can check by utilizing [48] or by direct computations that for \( m, n > 2 \) the multidegree polynomial has coefficients > 1 with respect to the \( \mathbb{N}^{m} \oplus \mathbb{N}^{n} \)-grading. For example, when \( m = n = 3 \) and \( r = 2 \) the multidegree polynomial is

\[
c(R/I_2, t_1, t_2, t_3, s_1, s_2, s_3) = t_1^2 t_2^2 + t_1^2 t_2 t_3 + t_1^2 t_2 s_1 + 2 t_1 t_2 t_3 s_1 + t_1 t_2 s_1^2 + t_1^2 s_1 s_2 + 2 t_1 t_2 s_1 s_2 + 2 t_1 s_1 s_2 s_3 + t_1 s_1^2 s_2 + t_1^2 s_2 s_3 + s_1^2 s_2^2 + \cdots \text{ symmetric terms}.
\]

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