On the composition of integral operators acting in tempered Colombeau algebras

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Abstract
We show that the generalised composition of generalised integral operators is well defined on the space $G^\tau$ Colombeau algebras of tempered generalised functions.

Keywords: generalised integral operators, Colombeau tempered generalised functions, Colombeau tempered generalised ultradistributions

1 Introduction

The extension of the classes of differential and integral equations that can be rigorously set and solved is seminal to current mathematics [9] and vital to its application in diverse domains [13]. In this paper we continue the study of the field of generalised integral operators that was commenced in the context of Sobolev and Schwartz spaces of generalised functions [19, 21], ultradistributions [10, 12] and continued recently within the spaces of Colombeau algebras of generalised functions [2, 3, 5, 6, 15].

In part the motivation for the following analysis arises from a requirement in physics to be able to compose generalised integral operators [8, 17] and the fact that it is not possible to compose generalised integral operators that act within the space of Schwartz distributions [18]. It has been shown that such compositions exist in the Colombeau algebras of compactly supported generalized functions [6]. In this paper we demonstrate via an extension of the Schwartz kernel theorem to the space of bounded linear operators $L(G^\tau, G^\tau)$ on tempered Colombeau algebras that compositions of generalised integral operators are well defined on the space $G^\tau$. More than this we use Hermite function expansions of ultradistributions to demonstrate that a countably infinite number of such compositions is well defined, hence we are able to show that compositions of exponentiated forms of these operators exist in the space $G^\tau_R$ of Colombeau tempered ultradistributions.

1 For a quick understanding of the need to generalise operator composition for quantum theory see section A.1 of Appendix A of Alexander Stottmeister’s thesis On the Embedding of Quantum Field Theory on Curved Spacetimes into Loop Quantum Gravity [16].
2 Mapping from rapidly decreasing to tempered Colombeau algebras

It has been shown that operators $A$ $A: \mathcal{G}_S \rightarrow \mathcal{G}_\tau$
that are defined as
$$A\phi = (A_\varepsilon)(\phi_\varepsilon)_\varepsilon = \left( \int_{\mathbb{R}^n} K_\varepsilon(x,y)\phi_\varepsilon(y)dy \right)_\varepsilon \in \mathcal{G}_\tau$$
(1)
where $\phi \in \mathcal{G}_S$ a Colombeau algebra of rapidly decreasing generalised functions, $\phi_\varepsilon \in S(\Omega)$ the Schwartz space of rapidly decreasing smooth functions, and $\mathcal{G}_\tau$ is a Colombeau algebra of tempered generalised functions, are bounded linear operators: $A \in \mathcal{L}(\mathcal{G}_S,\mathcal{G}_\tau)$ \([6]\). Rigorous definitions of these spaces are given below.

2.1 Simplified Colombeau algebras

Colombeau algebra of rapidly decreasing generalised functions

Let $\Omega$ be an open subset of $\mathbb{R}^d$ where $d \in \mathbb{N}_+$. Consider a smooth function $f \in C^\infty(\Omega)$ and denote
$$\mu_{q,l}(f) := \sup_{x \in \Omega, |\alpha| \leq l} (1 + |x|)^q |\partial^\alpha f(x)|,$$ where $q \in \mathbb{Z}$ and $l \in \mathbb{N}$.

The set
$$\mathcal{E}_S(\Omega) := \left\{ (f_\varepsilon)_\varepsilon \in S(\Omega)^{0,1} \text{ such that } \forall q,l \in \mathbb{N}, \exists n \in \mathbb{N} : \mu_{q,l}(f_\varepsilon) = O(\varepsilon^{-n}) \text{ as } \varepsilon \rightarrow 0 \right\}$$
is a sub-algebra of $S(\Omega)^{0,1}$, where $S(\Omega)$ is the Schwartz space of rapidly decreasing functions and
$$S(\Omega)^{(0,1)} = \left\{ (f_\varepsilon)_\varepsilon \text{ such that } \forall \varepsilon \in (0,1] f_\varepsilon \in S(\Omega) \right\}$$
is the set of nets. The set
$$\mathcal{N}_S(\Omega) := \left\{ (f_\varepsilon)_\varepsilon \in S(\Omega)^{0,1} \text{ such that } \forall q,l \in \mathbb{N}, \forall p \in \mathbb{N} : \mu_{q,l}(f_\varepsilon) = O(\varepsilon^p) \text{ as } \varepsilon \rightarrow 0 \right\}$$
is an ideal in $\mathcal{E}_S(\Omega)$. The factor-algebra
$$\mathcal{G}_S(\Omega) := \mathcal{E}_S(\Omega)/\mathcal{N}_S(\Omega)$$
is referred to as the Colombeau algebra of rapidly decreasing generalised functions (see \([6]\) and references therein).
Colombeau algebras of tempered generalised functions

Similarly, the factor-algebra
\[ G_\tau(\Omega) := E_\tau(\Omega)/N_\tau(\Omega) \]
is referred to as the Colombeau algebra of tempered generalised functions [4]. Here
\[ E_\tau(\Omega) := \{ (f_\varepsilon)_\varepsilon \in O_M(\Omega) \mid \exists q, n \in N : \mu_{-q,l}(f_\varepsilon) = O(\varepsilon^{-n}) \text{ as } \varepsilon \to 0 \} \]
is a sub-algebra of \( O(\Omega) \), where
\[ O_M(\Omega) := \{ f \in C^\infty(\Omega) \mid \forall l \in N \exists q \in N : \mu_{-q,l}(f_\varepsilon) < \infty \} \]
is the algebra of smooth functions with slow growth (also known as the algebra of multiplicators); and

\[ N_\tau(\Omega) := \{ (f_\varepsilon)_\varepsilon \in O_M(\Omega) \mid \forall l \in N \exists q \in N : \mu_{-q,l}(f_\varepsilon) = O(\varepsilon^{p}) \text{ as } \varepsilon \to 0 \} \]
is an ideal in \( E_\tau(\Omega) \).

Colombeau algebras of tempered generalised ultradistributions

Following [14] we define two types of Colombeau algebras of tempered generalised ultradistributions that correspond with the sets of generalised ultradistributions of Roumieu and Beurling type.

**Definition 1** The factor-algebra
\[ G_{\tau R}(\Omega) := E_{\tau R}(\Omega)/N_{\tau R}(\Omega) \]
is defined as the Colombeau algebra of tempered generalised ultradistributions of Roumieu type. Here
\[ E_{\tau R}(\Omega) = E^\{M_p,N_p\}_\exp(\Omega) := \{ (f_\varepsilon)_\varepsilon \in O^\{M_p\}_\exp(\Omega) \mid \exists h, k > 0 : \nu_{h,M_p}(f_\varepsilon) = O(e^{N_* k/\varepsilon}) \text{ as } \varepsilon \to 0 \} , \]
is the set of generalised ultradistributions of Roumieu type, with
\[ O^\{M_p\}_\exp(\Omega) := \{ f \in C^\infty(\Omega) \mid \exists h > 0 : \nu_{h,M_p}(f_\varepsilon) < \infty \} \]
the Roumieu algebra of smooth functions of exponential growth; and

\[ N_{\tau R}(\Omega) = N^\{M_p,N_p\}_\exp(\Omega) := \{ (f_\varepsilon)_\varepsilon \in O^\{M_p\}_\exp(\Omega) \mid \exists h > 0, \forall k > 0 : \nu_{h,M_p}(f_\varepsilon) = O(e^{-N_* k/\varepsilon}) \text{ as } \varepsilon \to 0 \} , \]
is an ideal in \( E_{\tau R}(\Omega) \).
Where as is customary in the theory of ultradistributions [10], we denote $M_p$ a sequence of positive numbers such that $M_0 = 1$ and

(M.1) $M_p^2 \leq M_{p-1}M_{p-1}$ for any $p \in \mathbb{N}_+$;
(M.2) $M_p \leq c H^p M_q M_{p-q}$ for any $p \in \mathbb{N}_0, q \leq p$ and some $c, H \geq 1$;
(M.3) $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$.

The sequence $M^*_p := M_p/p!$ with $M^*_0 = 1$, the associated function

$$M(\rho) := \sup_{p \in \mathbb{N}_0} \frac{\rho^p}{M_p}, \quad \rho > 0$$

and the growth function

$$M^*(\rho) := \sup_{p \in \mathbb{N}_0} \frac{\rho^p}{M^*_p}, \quad \rho > 0.$$  

For a smooth function $f \in C^\infty(\Omega)$ and we denote

$$\nu_{h,M}(f) := \sup_{x \in \Omega, \alpha, \beta \in \mathbb{N}_0^d} \frac{h^{\alpha_1 + |\beta|} |x^\beta \partial^\alpha f(x)|}{M_{|\alpha|} M_{|\beta|}}, \quad h > 0.$$  

Definition 2 The factor-algebra

$$G_{TB}(\Omega) := \mathcal{E}_{TB}(\Omega)/\mathcal{N}_{TB}(\Omega)$$

is defined as the Colombeau algebra of tempered generalised ultradistributions of Beurling type. Here

$$\mathcal{E}_{TB}(\Omega) = \mathcal{E}_{\exp}^{(M_p, N_p)}(\Omega) := \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{O}_{\exp}^{(M_p)}(\Omega)^{[0,1]} \text{ such that } \forall h > 0, \exists k > 0 : \nu_{h,M}(f_\varepsilon) = O(e^{N^*(k/\varepsilon)}) \text{ as } \varepsilon \to 0 \right\},$$

is the set of generalised ultradistributions of Beurling type, with

$$\mathcal{O}_{\exp}^{(M_p)}(\Omega) := \left\{ f \in C^\infty(\Omega) \text{ such that } \forall h > 0 : \nu_{h,M}(f_\varepsilon) < \infty \right\},$$

the Beurling algebra of smooth functions of exponential growth; and

$$\mathcal{N}_{TB}(\Omega) = \mathcal{N}_{\exp}^{(M_p, N_p)}(\Omega) := \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{O}_{\exp}^{(M_p)}(\Omega)^{[0,1]} \text{ such that } \forall h, k > 0 : \nu_{h,M}(f_\varepsilon) = O(e^{-N^*(k/\varepsilon)}) \text{ as } \varepsilon \to 0 \right\},$$

is an ideal in $\mathcal{E}_{TB}(\Omega)$.

Generalised constants

We will also use the factor-ring of generalised constants:

$$\bar{K} := \mathcal{E}_M(K)/\mathcal{N}(K),$$

for $K = \mathbb{C}, \mathbb{R}$ or $\mathbb{R}_+$, where

$$\mathcal{E}_M(K) := \left\{ (C_\varepsilon)_\varepsilon \in K^{(0,1)} \text{ such that } \exists n \in \mathbb{N} : |C_\varepsilon| = O(\varepsilon^{-n}) \text{ as } \varepsilon \to 0 \right\}$$

and

$$\mathcal{N}(K) := \left\{ (C_\varepsilon)_\varepsilon \in K^{(0,1)} \text{ such that } \forall p \in \mathbb{N} : |C_\varepsilon| = O(\varepsilon^{p}) \text{ as } \varepsilon \to 0 \right\}.$$
2.2 Inclusions

Note that we have the following inclusions:
\[ \mathcal{G}_T(\Omega) \subset \mathcal{G}_{TR}(\Omega) \subset \mathcal{G}_{TB}(\Omega). \]

We also note the inclusions
\[ S'_{\tau R}(\Omega) \subset \mathcal{G}_{\tau R}(\Omega) \quad \text{and} \quad S'_{\tau B}(\Omega) \subset \mathcal{G}_{\tau B}(\Omega), \]
where we denote \( S'_{\tau R} \) the space of ultradistributions of Roumieu type and \( S'_{\tau B} \) the space of ultradistributions of Beurling type. Indeed, for \( \varphi \in S_{\tau R}(\Omega) \) and \( f \in S'_{\tau R}(\Omega) \) we have \[ \varphi = \sum_n a_n h_n \quad \text{and} \quad f = \sum_n b_n h_n, \]
where \( h_n \) are Hermite functions, which form an orthonormal basis of \( L^2(\mathbb{R}^d) \) and Hermite coefficients \( a_n \) and \( b_n \) satisfy estimates
\[ |a_n| \leq e^{-M(\sqrt{n}h)} \quad \text{and} \quad |b_n| \leq e^{M(\sqrt{n}h)}. \]
Define
\[ f_\varepsilon = \sum_n e^{-\varepsilon M(\sqrt{n}h)} b_n h_n \equiv \sum_n f^\varepsilon_n h_n, \]
where
\[ |f^\varepsilon_N| = |e^{-\varepsilon M^2(\sqrt{n}h)} b_n| \leq C_\varepsilon e^{-M(\sqrt{n}h)} \]
and therefore \( f_\varepsilon \in S_{\tau R}(\Omega) \) for each \( \varepsilon > 0 \). We observe that
\[ (f_\varepsilon)_{\varepsilon} = \left( \sum_n f^\varepsilon_n h_n \right)_{\varepsilon} \in \mathcal{G}_{\tau R}(\Omega), \]
since
\[ \forall n, \varepsilon \quad |f^\varepsilon_n| \leq e^{M(\sqrt{n}h)}. \]

3 Mapping between tempered Colombeau algebras

Since generalised integral operators of form \[ A\phi = (A_\varepsilon)_{\varepsilon}(\phi_{\varepsilon})_{\varepsilon} = \left( \int_{\mathbb{R}^n} K_\varepsilon(x,y)\phi_{\varepsilon}(y)dy \right)_{\varepsilon} \in \mathcal{G}_T \]
are defined as bounded linear operators from \( \mathcal{G}_S \) to \( \mathcal{G}_T \), in order to compose such operators we demonstrate that their extensions
\[ A : \mathcal{G}_T \rightarrow \mathcal{G}_T \]
can be well-defined. Such maps can be represented by nets \( (A_\varepsilon)_{\varepsilon} \) of linear continuous maps
\[ A = (A_\varepsilon)_{\varepsilon} \in \mathcal{L}(\mathcal{O}_M, \mathcal{O}_M)^{0,1}, \]
where these nets are defined to be of moderate growth if
\[ \forall \ell \in \mathbb{N} \exists (C_\varepsilon)_{\varepsilon} \in \mathcal{E}_M(\mathbb{R}^+), \exists p, q, \ell' \in \mathbb{N} \]
such that
\[ \forall f \in \mathcal{O}_M \mu_{-p,\ell}(A_\varepsilon f) \leq C_{\varepsilon} \mu_{-q,\ell'}(f). \]

We then note that the net \((\phi_\varepsilon)_\varepsilon\) where \(\phi_\varepsilon \in \mathcal{O}_M\) has an associated net
\[ \phi_\varepsilon^{\gamma} := \phi_\varepsilon e^{-\gamma|x|^2} \in S \]
if \(\gamma \in \mathbb{R}_+\) with
\[ \lim_{\gamma \to 0} \phi_\varepsilon^{\gamma} = \phi_\varepsilon \in S'. \]

In considering the nature of this limit it is helpful to note that \(G^\infty_r \cap S' = \mathcal{O}_M\) where \(G^\infty_r\) is the subspace of regular elements of \(G_r\) and that the closure \(\bar{S} = \mathcal{O}_M\) with convergence in \(S'\). We then define
\[ A\phi = (A_\varepsilon)_\varepsilon(\phi_\varepsilon^{\gamma})_\varepsilon|_{\gamma=\varepsilon} = \left( \left( \int_{\mathbb{R}^n} K_\varepsilon(x,y)\phi_\varepsilon(y)dy \right)_\varepsilon \right)_{\varepsilon=\varepsilon} \in \mathcal{E}_r \]
where we have used a double regularisation but for simplicity defined \(\gamma = \varepsilon\). For any \(\alpha\) there exists \(q_1, q_2 \) and \(q_3 \in \mathbb{N}\) such that
\[ |\partial_\alpha A_\varepsilon \phi_\varepsilon^{\gamma}| \leq C(1 + |x|)^{q_1} e^{-q_2} \gamma^{-q_3}|_{\gamma=\varepsilon} = C(1 + |x|)^{q_1} e^{-q} \]
where \(q = q_2 + q_3\), therefore \(A_\varepsilon \in \mathcal{L}(\mathcal{O}_M, \mathcal{O}_M)\).

We note that for any \(\alpha\) there exist \(q_1, q_2 \in \mathbb{N}\) such that
\[
|\partial_\alpha A_\varepsilon f_\varepsilon| = |\partial_\alpha A_\varepsilon(\phi_\varepsilon e^{-\varepsilon|y|^2})| \\
\leq C \int_{\mathbb{R}^n} (1 + |x|)^{q_1} (1 + |y|)^{q_2} |f_\varepsilon(y)| \, dy \\
\leq C (1 + |x|)^{q_1} \sup_y (1 + |y|)^{q_2} \phi_\varepsilon(y) e^{-\varepsilon|y|^2} \phi_\varepsilon(y) \, dy \\
\leq C (1 + |x|)^{q_1} \mu_{-q_2,0}(\phi_\varepsilon) e^{-(q_2+q_3)/2},
\]
therefore for any \(\alpha\)
\[ (1 + |x|)^{-q_1} |\partial_\alpha A_\varepsilon(\phi_\varepsilon e^{-\varepsilon|y|^2})| \leq C \varepsilon^{-q} \mu_{-q_2,0}(\phi_\varepsilon) \]
and thus for any \(l\) there exist \(p\) and \(q'\) such that
\[ \mu_{-p,l}(A_\varepsilon(\phi_\varepsilon e^{-\varepsilon|y|^2})) \leq C \varepsilon^{-q} \mu_{-q',0}(\phi_\varepsilon) \]
Now for any \(\phi = (\phi_\varepsilon)_\varepsilon \in G_r\) we define
\[
A\phi := \left( A_\varepsilon(\phi_\varepsilon e^{-\varepsilon|y|^2}) \right)_\varepsilon + N_r
= \left( \int_{\mathbb{R}^n} K_\varepsilon(x,y)\phi_\varepsilon(y)e^{-\varepsilon|y|^2} \, dy \right)_\varepsilon + N_r
\]
and we have that
\[ A \in \mathcal{L}(G_r, G_r). \]
4 Composition of generalised integral operators on tempered Colombeau algebras

**Theorem 1** Let generalised integral operators \( A_1, A_2 \) be defined by formula (2), so that \( A_1, A_2 \in \mathcal{L} (\mathcal{G}_\tau, \mathcal{G}_\tau) \). Their composition \( A_2 \circ A_1 \in \mathcal{L} (\mathcal{G}_\tau, \mathcal{G}_\tau) \) is a generalised integral operator with the kernel

\[
K_\varepsilon(x, y) = \int_{\mathbb{R}^n} K_\varepsilon^2(x, z) K_\varepsilon^1(z, y) e^{-\varepsilon |z|^2} dz \in \mathcal{G}_\tau (\mathbb{R}^{2n}).
\]

**Proof** For any \( \alpha, \beta \) there exist \( q_1, q_2 \in \mathbb{N} \) such that

\[
|\partial_\alpha x \partial_\beta y K_\varepsilon(x, y)| = |\partial_\alpha x \partial_\beta y \int_{\mathbb{R}^n} K_\varepsilon^2(x, z) K_\varepsilon^1(z, y) e^{-\varepsilon |z|^2} dz|
\leq C (1 + |x|)^{q_1} (1 + |y|)^{q_2} \int_{\mathbb{R}^n} (1 + |z|)^{q_1+q_2} e^{-\varepsilon |z|^2} dz
\leq C (1 + |x|)^{q_1} (1 + |y|)^{q_2} \varepsilon^{-(q_1+q_2)/2}, \tag{3}
\]

therefore \( K_\varepsilon \in \mathcal{G}_\tau (\mathbb{R}^{2n}) \).

Furthermore

\[
(A_2 \circ A_1) \phi = \int_{\mathbb{R}^n} K_\varepsilon(x, y) \phi_\varepsilon(y) e^{-\varepsilon |y|^2} dy
= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} K_\varepsilon^2(x, z) K_\varepsilon^1(z, y) e^{-\varepsilon |z|^2} dz \right] \phi_\varepsilon(y) e^{-\varepsilon |y|^2} dy
= \int_{\mathbb{R}^n} K_\varepsilon^2(x, z) \left[ \int_{\mathbb{R}^n} K_\varepsilon^1(z, y) \phi_\varepsilon(y) e^{-\varepsilon |y|^2} dy \right] e^{-\varepsilon |z|^2} dz
= A_2 (A_1 \phi).
\]

Estimate (3) implies the following extension.

**Corollary 1** Let generalised integral operator \( A \in \mathcal{L} (\mathcal{G}_\tau, \mathcal{G}_\tau) \) be defined by formula (2). Then \( A^k \) is well-defined in \( \mathcal{L} (\mathcal{G}_\tau, \mathcal{G}_\tau) \) for any \( k \) and the operator

\[
e^A := I + \sum_{k=1}^{\infty} A^k / k!
\]

is well-defined in \( \mathcal{L} (\mathcal{G}_{\tau R}, \mathcal{G}_{\tau R}) \).

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