A Soluble Model for Scattering and Decay
in
Quaternionic Quantum Mechanics I: Decay
L.P. Horwitz§
School of Natural Sciences
Institute for Advanced Study
Princeton, N.J. 08540

Abstract. The Lee-Friedrichs model has been very useful in the study of decay-scattering systems in the framework of complex quantum mechanics. Since it is exactly soluble, the analytic structure of the amplitudes can be explicitly studied. It is shown in this paper that a similar model, which is also exactly soluble, can be constructed in quaternionic quantum mechanics. The problem of the decay of an unstable system is treated here. The use of the Laplace transform, involving quaternion-valued analytic functions of a variable with values in a complex subalgebra of the quaternion algebra, makes the analytic properties of the solution apparent; some analysis is given of the dominating structure in the analytic continuation to the lower half plane. A study of the corresponding scattering system will be given in a succeeding paper.

§ Permanent address: School of Physics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Israel; also at Department of Physics, Bar-Ilan University, Ramat Gan, Israel.
1. Introduction.

Adler\(^1,2\) has developed methods in time dependent perturbation theory for the treatment of scattering and decay problems in quaternionic quantum theory. Although the spectrum of the anti-self-adjoint generator of evolution is quaternion defined\(^\dagger\) as positive, the resulting formulas for the scattering matrix display both left and right hand cuts. It has recently been shown\(^3\) that such an anti-self-adjoint operator on a quaternionic Hilbert space has, quite generally, a symmetric effective spectrum, and in case the quaternion defined spectrum is absolutely continuous in \([0,\infty)\), the effective spectrum is absolutely continuous in \((-\infty, \infty)\). In this case, a symmetric conjugate operator exists. In case the anti-self adjoint operator is the generator of evolution, the conjugate operator can be interpreted as a “time operator”; it was shown\(^3\) that there is then no evident contradiction to the definition of a Lyapunov operator\(^4\).

The Lee-Friedrichs model\(^5\) has been very fruitful in the study of decay-scattering systems in the complex Hilbert space\(^6\). The generalized states (elements of a Gel’fand triple, or rigged Hilbert space) associated with the resonances of the scattering matrix and the reduced resolvent, for the case in the the unperturbed Hamiltonian is bounded from below \(^7\) as well as for the case for which, as in a model for the Stark effect, the “unperturbed” Hamiltonian has spectrum \(-\infty\) to \(\infty\) \(^8\), have also been studied using this spectral model. For the study of the consequences of the symmetrical effective spectrum\(^9\) of a quaternionic quantum theory, as well as for the construction of physical models, it would be useful to construct an exactly soluble model of Lee-Friedrichs type in this framework as well. It is the purpose of this paper to construct such a model.

We first utilize (Section 2) the formulation of the decay problem suitable for application of time-dependent perturbation theory, following the methods given by Adler\(^1,2\) and show that a soluble model can be constructed. In Section 3 we formulate the problem in terms of the Laplace transform of the Schrödinger evolution. The resolvent formalism cannot be used directly, since there is no natural imaginary direction associated with an anti-self-adjoint operator on a quaternion Hilbert space. Choosing a direction arbitrarily, and hence a subalgebra \(\mathbf{C}(1,i)\) of the quaternion algebra \(\mathbf{H}\), the Laplace transform (and its inverse) of the Schrödinger equation is well defined. In the complex Hilbert space, for a time-independent Hamiltonian, this procedure is equivalent to the resolvent formalism; for a time dependent generalization of the Lee-Friedrichs model, it has been shown that it provides a useful basis for a well-controlled perturbation theory\(^9\). For the quaternionic analog of the Lee-Friedrichs model, we find that the Laplace transformed amplitude has a structure analogous to that of the resolvent form, but contains a linear combination of a function analytic in the transform variable \(z \in \mathbf{C}(1,i)\) and a function of the negative of its complex conjugate \(-z^*\), corresponding to the contributions of the right- and left-hand cuts. The analytic continuation of the amplitude into the lower half plane is carried out in Section 4, and some results are given for the structure of the functions there that can

\(^\dagger\) For an anti-self-adjoint operator \(\tilde{S}\), the (generalized) eigenvalue equation is \(\tilde{S}f = \bar{f}e\lambda, e \in \text{Im}\mathbf{H}, e^2 = -1\), where \(\mathbf{H}\) is the quaternion algebra spanned over the reals by \(1, i, j, k, ij = k\) and \(i^2 = j^2 = k^2 = -1\). One can choose the quaternion phase\(^1\) of \(f\) such that \(e = i\) and \(\lambda \geq 0\). We call this the quaternion defined spectrum.
dominate the inverse transform. In a succeeding paper, some aspects of formal scattering theory are developed, and it is shown that this model provides a soluble scattering problem as well; it therefore describes a decay-scattering system. A summary and conclusions are given in Section 5.

2. Spectral Expansion

In this section, we follow the methods of Adler\(^1\)\(^2\) for the construction of an effective time dependent perturbation theory; assuming, as for the Lee-Friedrichs model, that the potential has no continuum-continuum matrix elements, we obtain a soluble quaternionic model for a decay-scattering system.

In the framework of the theory of Wigner and Weisskopf\(^10\), the component of the wave function evolving under the action of the full Hamiltonian which remains in the initial state corresponds to the probability amplitude \(A(t)\) for the initial state to persist (the survival amplitude\(^11\)). Let \(|\psi_0\rangle\) be the initial state and \(\tilde{H}\) be the quaternion linear operator which generates the time evolution of the system. Then,

\[
A(t) = \langle \psi_0 | e^{-\tilde{H}t} | \psi_0 \rangle. \tag{2.1}
\]

Let us now suppose that \(\tilde{H}\) has the decomposition

\[
\tilde{H} = \tilde{H}_0 + \tilde{V}, \tag{2.2}
\]

where \(\tilde{V}\) is a small operator which perturbs the eigenstates of \(\tilde{H}_0\). If we consider the eigenstates of \(\tilde{H}_0\) as the quasistable states of the system, (2.2) generates the evolution of an unstable system. In particular, if \(|\psi_0\rangle\) in (2.1) is an eigenstate of \(\tilde{H}_0\), the evolution of \(A(t)\) reflects its instability.

Let us assume that there is just one eigenstate of \(\tilde{H}_0\), with eigenvalue \(E_0\), so that the spectral expansion of \(|\psi(t)\rangle = e^{-\tilde{H}t}|\psi_0\rangle\) is

\[
|\psi(t)\rangle = |\psi_0\rangle e^{-iE_0t}C_0(t) + \int_0^\infty dE |E\rangle e^{-iEt}C_E(t), \tag{2.3}
\]

where

\[
\tilde{H}_0|\psi_0\rangle = |\psi_0\rangle iE_0 \tag{2.4}
\]

and

\[
\langle E|\tilde{H}_0|f\rangle = iE \langle E|f\rangle. \tag{2.5}
\]

We take initial values to be

\[
C_0(0) = 1, \quad C_E(0) = 0. \tag{2.6}
\]

\(^\dagger\) There are operators on the quaternionic Hilbert space with weaker linearity properties, such as complex linear or real linear\(^12\). We restrict ourselves here to the quaternion linear case, so that \(\tilde{H}(fq) = (\tilde{H}f)q\) for all vectors \(f\) in the domain of \(\tilde{H}\).
The transition from the eigenstate of $\tilde{H}_0$ to the continuum is interpreted as the decay of the unstable system. The corresponding amplitude is

$$\langle E | \psi(t) \rangle = e^{-iE t} C_E(t).$$

(2.7)

The total transition probability is

$$\int_0^\infty |\langle E | \psi(t) \rangle|^2 dE = \int_0^\infty |C_E(t)|^2 dE = 1 - |C_0(t)|^2$$

(2.8)

The Schrödinger equation for the evolution,

$$\frac{\partial}{\partial t} |\psi(t)\rangle = -\tilde{H} |\psi(t)\rangle,$$

(2.9)

with (2.3), (2.4) and (2.5), becomes

$$|\psi_0\rangle e^{-iE_0 t} C_0'(t) + \int_0^\infty dE |E\rangle e^{-iE t} C_E'(t)$$

$$= -\tilde{V} |\psi_0\rangle e^{-iE_0 t} C_0(t) - \int_0^\infty dE \tilde{V} |E\rangle e^{-iE t} C_E(t).$$

(2.10)

We take the scalar product in turn with $\langle \psi_0 \rangle$ and $\langle E \rangle$ to obtain the coupled differential equations

$$C_0'(t) = -e^{iE_0 t} \langle \psi_0 | \tilde{V} | \psi_0 \rangle e^{-iE_0 t} C_0(t)$$

$$- \int_0^\infty dE e^{iE_0 t} \langle \psi_0 | \tilde{V} | E \rangle e^{-iE t} C_E(t),$$

(2.11)

and

$$C_E'(t) = -e^{iE t} \langle E | \tilde{V} | \psi_0 \rangle e^{-iE_0 t} C_0(t)$$

$$- \int_0^\infty dE' e^{iE t} \langle E | \tilde{V} | E' \rangle e^{-iE' t} C_{E'}(t).$$

(2.12)

Integrating these equations over a very short interval in $t$, covering $t = 0$, one sees that if we take $C_0(t), C_E(t)$ to be zero for $t \leq 0$, the conditions (2.6) can be realized in the form $C_0(0+) = 1, C_E(0+) = 0$, by adding a term $\delta(t)$ to the right hand side of (2.11).

We now assume a condition on $\tilde{V}$ parallel to that of the Lee-Friedrichs model in the complex quantum theory, i.e., that

$$\langle E | \tilde{V} | E' \rangle = 0.$$  

(2.13)

The imaginary complex part of $\langle \psi_0 | \tilde{V} | \psi_0 \rangle$, as for the Lee-Friedrichs model of the complex quantum theory, contributes a level shift (i.e., it appears only in the combination $E_0 +$
where $iv$ is the complex part of $\tilde{V}[\text{cf.}(2.16)]$, but the imaginary quaternionic part contributes to the quaternionic phase of the (sub-asymptotic) decay amplitude*. We shall retain the term $\langle \psi_0 | \tilde{V} | \psi_0 \rangle$ here to display its effect. The condition (2.13) is essential to the solubility of the model; it corresponds to neglecting final state interactions, often (as in radiative processes) a good approximation to the physical problem in the complex quantum theory. Eqs. (2.11), (2.12) then become

$$C_0'(t) = -e^{iE_0 t} \langle \psi_0 | \tilde{V} | \psi_0 \rangle e^{-iE_0 t} C_0(t)$$

$$- \int_0^{\infty} dE e^{iE_0 t} \langle \psi_0 | \tilde{V} | E \rangle e^{-iEt} C_E(t) + \delta(t)$$

(2.14)

and

$$C_E'(t) = -e^{iEt} \langle E | \tilde{V} | \psi_0 \rangle e^{-iE_0 t} C_0(t).$$

(2.15)

To make use of the methods of complex analysis, we decompose all of the quaternion-valued functions in (2.14), (2.15) into their symplectic components\(^2,12,13\), i.e.,

$$C_0(t) = C_{0\alpha}(t) + jC_{0\beta}(t)$$

$$C_E(t) = C_{E\alpha}(t) + jC_{E\beta}(t)$$

$$\langle E | \tilde{V} | \psi_0 \rangle = -\langle \psi_0 | \tilde{V} | E \rangle^* \equiv V_{E\alpha} + jV_{E\beta},$$

$$\langle \psi_0 | \tilde{V} | \psi_0 \rangle = iv + ju,$$

(2.16)

where, since $\langle \psi_0 | \tilde{V} | \psi_0 \rangle^* = -\langle \psi_0 | \tilde{V} | \psi_0 \rangle$, $v$ is real, and $u \in \mathbb{C}(1, i)$; the asterisk is used here to denote the quaternion conjugation (it then corresponds to complex conjugation on the complex subalgebra $\mathbb{C}(1, i)$). Then, Eqs. (2.14) and (2.15) become

$$C_{0\alpha}'(t) = -ivC_{0\alpha}(t) + u^* e^{2iE_0 t} C_{0\beta}(t)$$

$$+ \int_0^{\infty} dE e^{i(E_0 - E)t} V_{E\alpha}^* C_{E\alpha}(t)$$

$$+ \int_0^{\infty} dE e^{i(E_0 + E)t} V_{E\beta}^* C_{E\beta}(t) + \delta(t),$$

(2.17)

$$C_{0\beta}'(t) = -e^{-2iE_0 t} u C_{0\alpha}(t) + iv C_{0\beta}(t)$$

$$- \int_0^{\infty} dE e^{-i(E_0 + E)t} V_{E\beta} C_{E\alpha}(t)$$

$$+ \int_0^{\infty} dE e^{-i(E_0 - E)t} V_{E\alpha} C_{E\beta}(t)$$

and

* Since we have already chosen the quaternionic phase of $|\psi_0\rangle$ so that the eigenvalue of $\tilde{H}_0$ is $iE$, with $E > 0$, there is no further freedom to rearse $\langle \psi_0 | \tilde{V} | \psi_0 \rangle$ so that it lies in $\mathbb{C}(1, i)$. 

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\[ C'_{E\alpha}(t) = -e^{i(E-E_0)t}V_{E\alpha}C_{0\alpha}(t) + e^{i(E+E_0)t}V_{E\beta}^*C_{0\beta}(t) \]
\[ C'_{E\beta}(t) = -e^{-i(E+E_0)t}V_{E\beta}C_{0\alpha}(t) - e^{-i(E-E_0)t}V_{E\alpha}^*C_{0\beta}(t). \]  

We now define the Fourier transforms

\[ C_{0\alpha}(t) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega e^{i(E_\alpha - \omega)t}C_{0\alpha}(\omega) \]
\[ C_{0\beta}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega e^{-i(E_\alpha + \omega)t}C_{0\beta}(\omega) \]
\[ C_{E\alpha}(t) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega e^{i(E_\alpha - \omega)t}C_{E\alpha}(\omega) \]
\[ C_{E\beta}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega e^{-i(E_\alpha + \omega)t}C_{E\beta}(\omega) \]
\[ i\delta(t) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega e^{i(E_\alpha - \omega)t} \]

Note that it follows from the conditions \( C_0(t), C_E(t) = 0 \) for \( t \leq 0 \), that the complex-valued functions \( C_{0\alpha}(\omega), C_{0\beta}(\omega), C_{E\alpha}(\omega), C_{E\beta}(\omega) \) are analytic in the upper half complex \( \omega \)-plane (upper half plane analyticity is equivalent to the condition that \( C_0(t), C_E(t) \) vanish for \( t \leq 0 \)).

With the definitions (2.19), Eqs. (2.17) and (2.18) become

\[-i = -i(\omega - E_0 - v)C_{0\alpha}(\omega) + u^*C_{0\beta}(\omega) + \int_0^\infty dE (-V_{E\alpha}^*C_{E\alpha}(\omega) + V_{E\beta}^*C_{E\beta}(\omega)) \]
\[ 0 = i(\omega + E_0 + v)C_{0\beta}(\omega) + uC_{0\alpha}(\omega) + \int_0^\infty dE (V_{E\beta}C_{E\alpha}(\omega) + V_{E\alpha}C_{E\beta}(\omega)), \]  

and

\[ C_{E\alpha}(\omega) = \frac{i}{E - \omega}(V_{E\alpha}C_{0\alpha}(\omega) + V_{E\beta}^*C_{0\beta}(\omega)) \]
\[ C_{E\beta}(\omega) = \frac{i}{E + \omega}(V_{E\beta}C_{0\alpha}(\omega) - V_{E\alpha}^*C_{0\beta}(\omega)), \]  

where the division is well-defined for \( \omega \) in the upper half plane. Substituting (2.21) into (2.20), one obtains the closed form

\[ h(\omega)C_{0\alpha}(\omega) + g(-\omega^*)^*C_{0\beta}(\omega) = 1 \]
\[ g(\omega)C_{0\alpha}(\omega) - h(-\omega^*)^*C_{0\beta}(\omega) = 0, \]

where
\[ h(\omega) = \omega - E_0 - v + \int_{0}^{\infty} dE \left( \frac{|V_{E\alpha}|^2}{E - \omega} - \frac{|V_{E\beta}|^2}{E + \omega} \right), \]
\[ g(\omega) = -iu + \int_{0}^{\infty} dE V_{E\alpha} V_{E\beta} \left( \frac{1}{E - \omega} + \frac{1}{E + \omega} \right). \] (2.23)

Note that \( h(\omega) \) and \( g(\omega) \) are analytic in the upper half plane; it therefore follows that \( h(-\omega^*) \) and \( g(-\omega^*) \) are also analytic functions of \( \omega \) in the upper half plane.

The Eqs. (2.22) provide a closed solution for the coefficients \( C_{0\alpha}(\omega), C_{0\beta}(\omega) \) in the domain of analyticity \( \text{Im} \ \omega > 0 \). The time dependent coefficients \( C_{0\alpha}(t), C_{0\beta}(t) \) can then be obtained by inverting the Fourier transform, integrating on a line infinitesimally above the real axis in the \( \omega \) plane (implicit in the relations (2.19)). We wish to emphasize that unlike the Lee-Friedrichs model with semi-bounded unperturbed Hamiltonian, the cut is not just a half line. In accordance with what we have found for the effective spectrum of an anti-self-adjoint operator in ref. 3, these results reflect the presence of both a left and right hand cut\(^14\). The integrals can be estimated by lowering the contour into the lower half plane (the second Riemann sheet)\(^8\), where the analytic continuation of the first sheet functions are well-defined, to obtain the contribution of leading singularities. We study this analytic continuation in Section 4.

3. Laplace Transforms

In this section we study the Laplace transform of the amplitude \( A(t) \) defined in (2.1). As we have pointed out above, the direct application of resolvent methods is complicated by the quaternion structure of the Hilbert space. However, the Laplace transform nevertheless brings the equations to a form which is similar to that of the resolvent method in the complex Hilbert space.

The amplitude \( A(t) \) satisfies the equation

\[ -\frac{\partial}{\partial t} A(t) = \langle \psi_0 | (\tilde{H} + \tilde{V}) | \psi(t) \rangle \]
\[ = iE_0 A(t) + \langle \psi_0 | \tilde{V} | \psi(t) \rangle, \] (3.1)

where we have used (2.4).

Inserting the completeness relation

\[ 1 = |\psi_0 \rangle \langle \psi_0 | + \int_{0}^{\infty} dE \langle E \rangle \langle E | \]

between \( \tilde{V} \) and \( |\psi(t) \rangle \), this is

\[ -\frac{\partial}{\partial t} A(t) = iE_0 A(t) + \int_{0}^{\infty} dE \langle \psi_0 | \tilde{V} | E \rangle A_E(t), \]
\[ + \langle \psi_0 | \tilde{V} | \psi_0 \rangle A(t), \] (3.2)

where
The amplitude $A_E(t)$ satisfies

$$A_E(t) = \langle E|\psi(t)\rangle. \quad (3.3)$$

$$-\frac{\partial}{\partial t} A_E(t) = \langle E|\tilde{H}_0 + \tilde{V}|\psi(t)\rangle$$

$$= iE A_E(t) + \langle E|\tilde{V}|\psi_0\rangle A(t), \quad (3.4)$$

where we have used the completeness relation again, taking into account the restriction (2.13) on the matrix elements that constitute the soluble model, i.e., that $\tilde{V}$ connects the continuum only to the discrete eigenstate. We now define the Laplace transform:

$$\hat{A}(z) = \int_0^\infty dt \ e^{izt} A(t), \quad (3.5)$$

and a similar expression for $\hat{A}_E(z)$. Clearly, $\hat{A}(z)$ exists and is analytic in the upper half $z$-plane, since by the Schwarz inequality

$$|A(t)| \leq \|\psi_0\|\|e^{-\tilde{H}t}\psi_0\| = \|\psi_0\|^2,$$

i.e., $A(t)$ is bounded. The upper half plane analyticity of $\hat{A}_E(z)$ follows from that of $\hat{A}(z)$, as we shall see below.

The inverse Laplace transform follows as for the usual complex function space, since the quaternion structure of $A(t)$ does not affect the left transform of $\hat{A}(z)$. For $C$ a line from $-\infty + i\epsilon$ to $+\infty + i\epsilon$, above the real axis in the $z$-plane, we assert that

$$A(t) = \frac{1}{2\pi} \int_C dz \ e^{-izt} \hat{A}(z). \quad (3.6)$$

For $t < 0 (\hat{A}(z) \to 0$ for $\text{Im}z \to i\infty$), the contour can be closed in the upper half-plane, so that, by upper half-plane analyticity, $A(t) = 0$. For $t \geq 0$,

$$A(t) = \frac{1}{2\pi} \int_C dz e^{-izt} \int_0^\infty dt' e^{izt'} A(t')$$

$$= \frac{1}{2\pi} \int_0^\infty dt' e^{i(t-t')} \int_{-\infty}^\infty dx \ e^{-ix(t-t')} A(t'), \quad (3.7)$$

and, since $\int_{-\infty}^\infty dx \ e^{-ix(t-t')} = 2\pi \delta(t-t')$, the assertion is verified.

The Laplace transform of Eq. (3.2) is the

$$1 + iz\hat{A}(z) = iE_0\hat{A}(z) - \int_0^\infty dE \ V_{E\alpha}^* \hat{A}_E(z)$$

$$+ j \int_0^\infty dE \ V_{E\beta} \hat{A}_E(-z^*) + iv\hat{A}(z) + ju\hat{A}(-z^*), \quad (3.8)$$

\[\hat{z} \text{ Note that the first term on the left hand side is due to integration by parts, i.e.,}\]

$$\int_0^\infty dt \ e^{izt} \left(-\frac{\partial}{\partial t} A(t)\right) = -e^{izt} A(t)|_0^\infty + iz \int_0^\infty dt \ e^{izt} A(t) = 1 + iz\hat{A}(z).$$
and of Eq. (3.4),

$$iz \hat{A}_E(z) = iE \hat{A}_E(z) + V_{E\alpha} \hat{A}(z) + jV_{E\beta} \hat{A}(-z^*).$$  \hspace{1cm} (3.9)

These formulas reduce to the results for the complex case if $V_{E\beta}$ and $u$ are zero. The contribution of $\langle \psi_0 | \hat{V} | \psi_0 \rangle$ is then, as is clear from (3.8), just an energy shift.

In terms of the components $\hat{A}_\alpha(z)$, $\hat{A}_\beta(z)$, $\hat{A}_{E\alpha}(z)$ and $\hat{A}_{E\alpha}(z)$, where (we shall see below that the $\alpha$ components are analytic functions of $z$, and the $\beta$ components, in spite of the notation, are analytic functions of $z^*$)

$$\hat{A}(z) = \hat{A}_\alpha(z) + j\hat{A}_\beta(z),$$

$$\hat{A}_E(z) = \hat{A}_{E\alpha}(z) + j\hat{A}_{E\beta}(z),$$  \hspace{1cm} (3.10)

one obtains

$$1 + iz \hat{A}_\alpha(z) = iE_0 \hat{A}_\alpha(z) - \int_0^\infty dE \ V_{E\alpha}^* \hat{A}_{E\alpha}(z)$$
$$- \int_0^\infty dE \ V_{E\alpha}^* \hat{A}_{E\beta}(-z^*) + iv \hat{A}_\alpha(z) - u^* \hat{A}_\beta(-z^*),$$

$$-iz^* \hat{A}_\beta(z) = -iE_0 \hat{A}_\beta(z) - \int_0^\infty dE \ V_{E\alpha} \hat{A}_{E\alpha}(z)$$
$$+ \int_0^\infty dE \ V_{E\beta} \hat{A}_{E\alpha}(z) - iz^* \hat{A}_\alpha(z) + u^* \hat{A}_\beta(-z^*),$$  \hspace{1cm} (3.11)

and

$$iz \hat{A}_{E\alpha}(z) = iE \hat{A}_{E\alpha}(z) + V_{E\alpha} \hat{A}_\alpha(z) - V_{E\beta} \hat{A}_\beta(-z^*)$$
$$-iz^* \hat{A}_{E\beta}(z) = -iE \hat{A}_{E\beta}(z) + V_{E\alpha} \hat{A}_\beta(z) + V_{E\beta} \hat{A}_\alpha(-z^*)$$  \hspace{1cm} (3.12)

We now study the relation between the amplitudes $\hat{A}(z)$, $\hat{A}_E(z)$ and $C_0(z)$, $C_E(z)$, and show that the equations (3.8) and (3.9) are equivalent to (2.20) and (2.21).

The transforms defined in (2.19) have the property

$$C_{E\alpha}(t) + jC_{E\beta}(t) = -\frac{1}{2\pi i} e^{iEt} \int_{-\infty}^{\infty} d\omega \{ e^{-i\omega t} C_{E\alpha}(\omega) + e^{i\omega t} jC_{E\beta}(\omega) \},$$  \hspace{1cm} (3.13)

a relation valid for the $C_{E_0}(t)$ coefficients as well. The transforms act with opposite sign of the frequency on the $\alpha$ and $\beta$ parts of the $C$-amplitudes; they are, however, analytic in the upper half $\omega$-plane by construction. On the other hand, the transform (3.5), also analytic in the upper half $z$-plane, for

$$A(t) = A_\alpha(t) + jA_\beta(t),$$  \hspace{1cm} (3.14)

has the form

$$\hat{A}(z) = \int_0^\infty dt \ e^{izt} A_\alpha(t) + j \int_0^\infty dt \ e^{-iz^*t} A_\beta(t),$$  \hspace{1cm} (3.15)
and hence, with (3.10), we see that

\[ \hat{A}_\alpha(z) = \int_0^\infty dt \ e^{izt} A_\alpha(t) \]

\[ \hat{A}_\beta(z) = \int_0^\infty dt \ e^{-iz^*t} A_\beta(t). \]

The second of (3.16) is an analytic function of \( z^* \) in the upper half \( z \)-plane (for which \( -z^* \) is also in the upper half plane).

This form is, in fact a general one\textsuperscript{15} for quaternion-valued left analytic functions of a variable in the complex subalgebra \( \mathbb{C}(1,i) \). To see this, consider the Cauchy-Riemann relations for the existence of a left-acting derivative with respect to a variable in the complex subalgebra for

\[ f(z) = f_\alpha(z) + jf_\beta(z) \]

\[ = u_\alpha + iv_\alpha + j(u_\beta + iv_\beta). \]

Then,

\[ \frac{\partial f}{\partial x} = \frac{\partial u_\alpha}{\partial x} + i \frac{\partial v_\alpha}{\partial x} + j \left( \frac{\partial u_\beta}{\partial x} + i \frac{\partial v_\beta}{\partial x} \right) \]

\[ \frac{1}{i} \frac{\partial f}{\partial y} = \frac{1}{i} \frac{\partial u_\alpha}{\partial y} + \frac{\partial v_\alpha}{\partial y} - j \left( \frac{1}{i} \frac{\partial u_\beta}{\partial y} + \frac{\partial v_\beta}{\partial y} \right) \]

so that if the left derivative of \( f(z) \) exists,

\[ \frac{\partial u_\alpha}{\partial x} = \frac{\partial v_\alpha}{\partial y} \quad \frac{\partial v_\alpha}{\partial x} = -\frac{\partial u_\alpha}{\partial y} \]

and

\[ \frac{\partial u_\beta}{\partial x} = -\frac{\partial v_\beta}{\partial y} \quad \frac{\partial v_\beta}{\partial x} = \frac{\partial u_\beta}{\partial y}. \]

The relations (3.18) correspond to anti-Cauchy-Riemann conditions, implying that \( f_\beta \) is an analytic function of \( z^* \) [note that with the convention we have used, i.e., of writing \( j \) to the left of the \( \beta \) part of the function, right analyticity results in normal Cauchy-Riemann conditions for both parts; later we shall study a function with \( j \) appearing to the right of the \( \beta \)-part, for which the normal Cauchy-Riemann conditions for left analyticity apply to both parts].

From (2.3), we have the relations

\[ A(t) = e^{-iE_0t} C_0(t), \]

\[ A_E(t) = e^{-iEt} C_E(t). \]

It then follows from (2.19) that the Laplace transform of \( A(t) \) is
\[ \hat{A}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left( \frac{C_{0\alpha}(\omega)}{\omega - z} - j \frac{C_{0\beta}(\omega)}{\omega - (-z^*)} \right). \] (3.20)

The functions \( C_{0\alpha}(\omega), C_{0\beta}(\omega) \) are analytic in the upper half \( \omega \)-plane, and for smooth functions \( C_{0\alpha}(t), C_{0\beta}(t) \), vanish for \( \Im \omega \to \infty \). Hence, one can complete the contour in (3.20) to obtain

\[ \hat{A}_{\alpha}(z) = iC_{0\alpha}(z) \]
\[ \hat{A}_{\beta}(z) = -iC_{0\beta}(-z^*), \] (3.21)

consistent with the upper half plane left analyticity of \( \hat{A}(z) \). The functions \( \hat{A}_{E\alpha}(z), \hat{A}_{E\beta}(z) \) are related in the same way to \( C_{E\alpha}(z), C_{E\beta}(-z^*) \).

Substituting the relations (3.21) into (3.11) and (3.12), one obtains

\[ -i = -i(z - E_0 - v)C_{0\alpha}(z) + u^*C_{0\beta}(z) \]
\[ + \int_{0}^{\infty} dE (-V_{E\alpha}^* C_{E\alpha}(z) + V_{E\beta}^* C_{E\beta}(z)) \]
\[ 0 = i(E_0 + v - z^*)C_{0\beta}(-z^*) + uC_{0\alpha}(-z^*) \]
\[ + \int_{0}^{\infty} dE (V_{E\beta} C_{E\alpha}(-z^*) + V_{E\alpha} C_{E\beta}(-z^*)), \] (3.22)

and

\[ C_{E\alpha}(z) = \frac{i}{E - z}(V_{E\alpha} C_{0\alpha}(z) + V_{E\beta}^* C_{0\beta}(z)) \]
\[ C_{E\beta}(-z^*) = \frac{i}{E - z^*}(V_{E\beta} C_{0\alpha}(-z^*) - V_{E\alpha}^* C_{0\beta}(-z^*)). \] (3.23)

The replacement of \(-z^*\) by \(z\) in the second of each of (3.22) and (3.23) corresponds to an analytic continuation of these equations inside the upper half plane, where the functions are analytic. This replacement brings these expressions into correspondence with (2.20) and (2.21).

Let us now return to the quaternionic equations (3.8) and (3.9). Substituting the second into the first to eliminate the amplitude \( \hat{A}_E(z) \), we obtain

\[ 1 + ih(z)\hat{A}(z) = -ijg(-z^*)\hat{A}(-z^*), \] (3.24)

where \( h \) and \( g \) are the functions defined in (2.23). Note that although it appears that \( g(z) \), as defined by (2.23), is an even function of its argument, it is, in fact, defined in the upper half plane, and has a cut along the real axis, so that the formal reflection of \( z \) takes the function out of its original domain of definition. The domain of definition of the function can be extended to the lower half plane by analytic continuation, which we shall discuss below. The same remark is applicable to the function \( h(z) \), which appears, in (2.23), to have a trivial reality property, i.e., that \( h(z)^* = h(z^*) \); this operation, however, also shifts the variable from the upper to the lower half plane. The value of the extension
of this function to the lower half plane also involves analytic continuation, which we shall discuss below.

If \( V_{E\beta} \) vanishes, then \( g(z) \) vanishes, and the result (3.24) coincides precisely with the well-known result for complex quantum theory, usually obtained by means of the method of resolvents\(^6,7\). The quaternionic part of the transition matrix element couples the amplitude to its reflection in the imaginary axis of the complex energy plane, and corresponds to the effect of the negative energy part of the effective spectrum of the anti-self-adjoint Hamiltonian\(^5\).

Since \( h(z), g(z) \) and \( \hat{A}(z) \) are analytic in the upper half plane, Eq. (3.24) can be shifted to the point \(-z^*\), providing the second coupled equation

\[
1 + ih(-z^*) \hat{A}(-z^*) = -ijg(z) \hat{A}(z).
\]

(3.25)

Solving for \( \hat{A}(-z^*) \) from the (3.25), and substituting into (3.24), one obtains the relation

\[
H(z) \hat{A}(z) = iG(z),
\]

(3.26)

where

\[
H(z) = \overline{h}(z)h(z) + \overline{g}(z)g(z)
\]

(3.27)

and

\[
G(z) = \overline{h}(z) + \overline{g}(z)j,
\]

(3.28)

and we have used the notation

\[
\overline{h}(z) \equiv h(-z^*), \quad \overline{g}(z) \equiv g(-z^*).
\]

(3.29)

The functions \( \overline{h}(z) \) and \( \overline{g}(z) \) are analytic functions of \( z \) if \( h(z) \) and \( g(z) \) are; hence \( H(z) \) and \( G(z) \) are analytic functions of \( z \) as well. The function \( g(z) \) contains the complex quantity \( u \), the quaternionic part of \( \langle \psi_0 | \tilde{V} | \psi_0 \rangle \), which therefore contributes to the quaternionic part of the decay amplitude \( A(t) \).

4. Analytic Continuation

The inverse of the Laplace transform (3.6) can be well-approximated when there are singularities in the analytic continuation of the amplitude \( \hat{A}(z) \) into the lower half plane, close to the real axis, by deforming the contour of the integration (3.6) below the real axis. Since, in Eq. (3.26), \( G(z) \) depends on functions which can be taken to have regular analytic continuation to the lower half plane, such singularities arise, in general, from zeros of \( H(z) \).

We shall study in this section the analytic properties of \( h(z) \) and \( g(z) \) to see how zeros of \( H(z) \) could arise. To do this, it is convenient to put these functions into a somewhat simpler form by defining numerator functions in the integrals that can be continued from the interval \([0, \infty)\) to \((-\infty, \infty)\). As pointed out in ref. 5, the effective negative spectrum of \( \tilde{H}_0 \) can be represented by the identification (we use a double bar on the Dirac kets to indicate the spectral representation for which the argument may be positive or negative), for \( E > 0 \),

\[
\langle -E \| = -j \langle E \|.
\]

(4.1)
The matrix element
\[ V_E = \langle E | \tilde{V} | \psi_0 \rangle \] (4.2)
then has the extension
\[ V_- = \langle -E | \tilde{V} | \psi_0 \rangle = -j \langle E | \tilde{V} | \psi_0 \rangle, \] (4.3)
so that, for \( E > 0 \),
\[ V_{-E\alpha} = V_{E\beta} \]
and
\[ V_{-E\beta} = -V_{E\alpha}. \] (4.4)
Since the limit of \( V_{-E\alpha} \) as \( E \to 0_+ \) is not necessarily equal to the limit of \( V_{E\alpha} \) as \( E \to 0_+ \) (we do not require continuity across zero), (4.4) does not place any restriction on the relation between \( V_{E\alpha} \) and \( V_{E\beta} \).

With these identifications, it follows that
\[ \int_{-\infty}^{0} dE \left[ \frac{|\langle E | \tilde{V} | \psi_0 \rangle_\alpha|^2}{E - z} \right] = \int_{\infty}^{0} dE \left[ \frac{|\langle -E | \tilde{V} | \psi_0 \rangle_\alpha|^2}{E + z} \right] = - \int_{0}^{\infty} dE \left[ \frac{|V_{E\beta}|^2}{E + z} \right], \] (4.5)
so that we can write
\[ h(z) = z - E_0 - v + \int_{-\infty}^{\infty} dE \left( \frac{X(E)}{E - z} \right), \] (4.6)
where
\[ X(E) = |V_{E\alpha}|^2, \] (4.7)
for \( E \in (-\infty, \infty) \).

In a similar way, since (by (4.4))
\[ V_{-E\alpha} V_{-E\beta} = -V_{E\beta} V_{E\alpha}, \]
we may write
\[ g(z) = \int_{-\infty}^{\infty} dE \left( \frac{Y(E)}{E - z} \right) - iu, \] (4.8)
where
\[ Y(E) = -Y(-E) = V_{E\alpha} V_{E\beta}. \] (4.9)

Taking the imaginary part of (4.6), one obtains
\[ \text{Im} h(z) = \text{Im} \left( 1 + \int_{-\infty}^{\infty} dE \left( \frac{X(E)}{|E - z|^2} \right) \right), \] (4.10)
so that there are no zeros of \( h(z) \) for \( z \) in the upper half plane (this is therefore true of \( \overline{h}(z) \) as well).
Taking \( z \) onto the real axis from above, one finds, from (2.23) with \( z \) in place of \( \omega \),
\[
h(\lambda) = \begin{cases} 
\lambda - E_0 - v + \pi i |V_{\lambda\alpha}|^2 + P \int_0^\infty dE \left( \frac{|V_{E\alpha}|^2}{E-\lambda} - \frac{|V_{E\beta}|^2}{E+\lambda} \right), & \text{for } z \to \lambda > 0; \\
-\lambda - E_0 - v + \pi i |V_{\lambda\beta}|^2 + P \int_0^\infty dE \left( \frac{|V_{E\alpha}|^2}{E+\alpha} - \frac{|V_{E\beta}|^2}{E-\lambda} \right), & \text{for } z \to -\lambda < 0,
\end{cases}
\]
(4.11)
where the principal part restriction is not necessary in one term of each of the integrals. Since \( |V_{-\lambda\alpha}|^2 = |V_{\lambda\beta}|^2 \), this result follows, equivalently, from (4.6) as well. If we now assume that we have chosen a model (specified by \( v, u \) and the functions \( V_{E\alpha} \) and \( V_{E\beta} \)) for which \( X(E) \) is the boundary value of a function \( X(z) \) analytic in the lower half plane in a large enough domain below the real axis to cover the analytic structure we must study, \( h(z) \) has the analytic continuation in the lower half plane
\[
h^{\text{II}}(z) = z - E_0 - v + \int_{-\infty}^{\infty} dE \frac{X(E)}{E - z} + 2\pi i X(z).
\]
(4.12)
We see that taking the limit of \( h^{\text{II}}(z) \) onto the real axis from below, one obtains exactly (4.11); a term \(-\pi i |V_{\lambda\alpha}|^2 \) for \( z \to \lambda > 0 \), or \(-\pi i |V_{\lambda\beta}|^2 \) for \( z \to -\lambda < 0 \), emerges from the integral, cancelling half of the additional term \( 2\pi i X(z) \). Hence \( h(z) \) and \( h^{\text{II}}(z) \) are functions analytic, respectively, in some domain in the upper half plane and in some domain of the lower half plane, with a common segment of the real axis on which they approach the same values; a simple application of the Cauchy theorem demonstrates that they form one analytic function, and hence are unique analytic continuations of each other. It then follows that
\[
\text{Im} h^{\text{II}}(z) = 2\pi \text{Re} X(z) + \text{Im} z \left( 1 + \int_{-\infty}^{\infty} dE \frac{X(E)}{|E - z|^2} \right),
\]
(4.13)
which, if \( \text{Re} X(z) > 0 \), to be expected if \( z \) is close to the real axis, can vanish for \( \text{Im} z \) negative. We shall assume for the remainder of the analysis, for simplicity, that \( X(E) \ll E_0 \) for every \( E \), so that the assumption that the root of \( h^{\text{II}}(z) \) has small imaginary part is justified.

In a similar way, one finds the analytic continuation of \( g(z) \) to be
\[
g^{\text{II}}(z) = \int_{-\infty}^{\infty} dE \frac{Y(E)}{E - z} + 2\pi i Y(z) - iu,
\]
(4.14)
where we have assumed \( Y(E) \) to be the boundary value of an analytic function \( Y(z) \), analytic in the same domain as \( X(z) \). There are no model independent statements we can make about \( g^{\text{II}}(z) \) since \( Y(E) \) has no definite sign or reality properties; we shall therefore further assume, for the purposes of our present discussion, that \( V_{E\beta} \ll V_{E\alpha} \), and that \( u \) is also small, so that the principal contribution to the contour integral (3.6) comes from values of \( z \) close to roots of the analytic continuation of the first term, \( \overline{h}(z)h(z) \), in the expression (3.27) for \( H(z) \). With this assumption, only a small adjustment in the value of a root of this term is required to cancel the contribution of the second term as well. We therefore study the zeros of \( \overline{h}(z)h(z) \).
It follows from (4.12) that the root $\zeta$ must satisfy

$$
\text{Re} h^{II}(\zeta) = 0 = \text{Re}\zeta\left(1 - \int_{-\infty}^{\infty} dE \frac{X(E)}{|E - \zeta|^2}\right) - E_0 - v + \int_{-\infty}^{\infty} dE E \frac{X(E)}{|E - \zeta|^2} - 2\pi \text{Im} X(\zeta),
$$

(4.15)

so that a zero of the real part can only occur at

$$
\omega_0 = \text{Re}\zeta \approx E_0 + v.
$$

(4.16)

If $|v| < E_0$, it follows from (4.12) and (4.13) (and the assumption that $X(E)$ is small) that a zero can only form in $h^{II}(z)$ below the positive real axis. The vanishing of the imaginary part then requires, according to (4.13), that

$$
\text{Im}\zeta \approx -\pi \text{Re} X(\zeta) \approx -\pi |V\omega_0\alpha|^2,
$$

(4.17)

where we understand that $\omega_0 \approx E_0 + v$ is positive, and we have approximated $X(\zeta)$ by its value on the real axis. This result is obtained from (4.13) by noting that the integral, for small values of $\text{Im} z$, contains an approximation to a representation of the $\delta$-function, i.e.(for $\epsilon > 0$),

$$
\lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon^2 + \epsilon^2} = \pi \delta(x).
$$

Hence,

$$
\frac{|Imz|}{|E - z|^2} \approx \pi \delta(E - \text{Re}z);
$$

substituting this back into (4.13), with $\text{Im} h^{II}(z) = 0$, one obtains (4.17). Using the same approximation, this result may be obtained directly from (4.12); the integral, for small negative $\text{Im} z$ contributes an imaginary term which is approximately $-\pi i X(\omega_0)$.

We now show that there is a second leading contribution to the inverse transform (3.6) when the contour is deformed into the lower half plane. The function

$$
\overline{h}(z) \equiv h(-z^*)^* = -z - E_0 - v + \int_{-\infty}^{\infty} dE \frac{X(E)}{E + z}
$$

(4.18)

is analytic in the upper half plane and (following the argument leading to (4.10) for $h(z)$) has no zeros for $\text{Im}z \neq 0$; its analytic continuation to the lower half $z$-plane, obtained following the procedure resulting in (4.12), is

$$
\overline{h}^{II}(z) = -z - E_0 - v + \int_{-\infty}^{\infty} dE \frac{X(E)}{E + z} - 2\pi i \overline{X}(z),
$$

(4.19)

where $\overline{X}(z)$ is a function analytic in the lower half plane, in a sufficiently large domain, with boundary value $X(-\lambda)$ on the real axis, for $z \to \lambda$. Since, from (4.19), we have the relation

$$
\text{Im}\overline{h}^{II}(z) = -\text{Im}z \left(1 + \int_{-\infty}^{\infty} dE \frac{X(E)}{|E + z|^2}\right) - 2\pi \text{Re}\overline{X}(z),
$$

(4.20)
\( \overline{h}^{II}(z) \) can vanish for \( \text{Im}z < 0 \).

The real part of \( \overline{h}^{II}(z) \) is

\[
\text{Re}\overline{h}^{II}(z) = -\text{Re}z \left( 1 - \int_{-\infty}^{\infty} dE \frac{X(E)}{|E + z|^2} \right) - E_0 - v \\
+ \int_{-\infty}^{\infty} dE \frac{X(E)}{|E + z|^2} + 2\pi \text{Im}X(z),
\]

and can therefore vanish at the root \( \zeta \) for

\[
\text{Re}\zeta \approx -\omega_0 \approx -(E_0 + v).
\]

From (4.20), using the procedure described above to obtain (4.17), we find that

\[
\text{Im}\zeta \approx -\pi X(\omega_0),
\]

where we have approximated \( X(\zeta) \) by its value on the real axis, \( X(-\omega_0) = X(\omega_0) \). As for (4.17), this result follows from (4.20) by using the approximation

\[
-\text{Im}z \int_{-\infty}^{\infty} dE \frac{X(E)}{|E + z|^2} \approx \pi X(-\text{Re}z)
\]

for \( \text{Im}z \) small. Hence the decay rate associated with the residue of the second pole is the same as that of the first.

We wish now to estimate the contour integral (3.6) when the contour is deformed into the lower half plane \( \sharp \). The integral is given by

\[
\frac{1}{2\pi} \int_{-\infty+ie}^{\infty+ie} dz e^{-izt} \hat{A}(z) \approx \sum \text{residues} \frac{1}{h^{II}(z)h^{II}(z)} G^{II}(z) + \text{background},
\]

where we neglect, for simplicity, the second term of \( H^{II}(z) \) in the denominator, and the background contribution consists of the contour integral in the lower half plane below the pole contributions; with a suitable choice of the weight functions, for example, for \( X(z), Y(z) \) Hardy class in the lower half plane on lines with sufficiently large negative imaginary part, the background contribution can be made as small as we wish (as in the Pietenpol model\(^8 \) in the complex Hilbert space), given \( \epsilon > 0 \), for \( t \geq \epsilon \).

\(^{\sharp} \) For large real part, the denominator function \( H(z) \) goes as \( O(z^2) \), and the numerator \( G(z) \) as \( O(z) \). At the ends of the long rectangle that brings the integral to the lower half plane, the contribution of the vertical contour is therefore bounded by \( x^{-1} \int dy \ e^{yt} \), where \( y \) is the (negative) imaginary part of \( z \), \( x \) is the very large real part of \( z \), and the integral is taken between zero and the distance into the lower half plane that the contour is carried. These contributions therefore vanish for \( |x| \to \infty \), even for \( t \to 0 \).
To do this, we must also study the analytic continuation of

$$\overline{g}(z) \equiv g(-z^*)^* = \int_{-\infty}^{\infty} dE \frac{Y^*(E)}{E + z^*} + iu^*.$$  \hspace{1cm} (4.26)

Using the property $Y(-E) = -Y(E)$, it is convenient to rewrite (4.26) as

$$\overline{g}(z) = \int_{-\infty}^{\infty} dE \frac{Y^*(E)}{E - z} + iu^*.$$  \hspace{1cm} (4.27)

The analytic continuation is then given, following the procedure used to obtain (4.12) and (4.14), by

$$\overline{g}^I(z) \approx P \int_{-\infty}^{\infty} dE \frac{Y^*(E)}{E - z} + iu^* + 2\pi iY(z),$$  \hspace{1cm} (4.28)

where $Y(z)$ is a function analytic in the lower half-plane with the same domain of analyticity as $X(z)$, $Y(z)$, and with boundary value $Y^*(\lambda)$ in the limit $z \to \lambda$ onto the real axis. At the zero $\zeta$ of $h^I(z)$, close to the positive real axis, near $E = \omega_0$,

$$\overline{g}^I(\zeta) \approx P \int_{-\infty}^{\infty} dE \frac{Y^*(E)}{E - \omega_0} + iu^* + 2\pi iY^*(\omega_0).$$  \hspace{1cm} (4.29)

At the root $\overline{\zeta}$, close to the negative real axis, near $E = -\omega_0$,

$$\overline{g}^I(\overline{\zeta}) \approx P \int_{-\infty}^{\infty} dE \frac{Y^*(E)}{E + \omega_0} + iu^* + 2\pi iY^*(-\omega_0)$$

$$= \int_{-\infty}^{\infty} dE \frac{Y^*(E)}{E - \omega_0} + iu^* - 2\pi iY^*(\omega_0).$$  \hspace{1cm} (4.30)

We are now in a position to evaluate the pole contributions to (4.25). The residue at $\zeta$ contributes a term which is approximately

$$\frac{e^{-i\zeta t}}{h^I(\zeta)} G^I(\zeta) = e^{-i\zeta t} \left( 1 + \frac{\overline{g}^I(\zeta)}{h^I(\zeta)} \right),$$  \hspace{1cm} (4.31)

where we have neglected small terms in the residue of $h^I(z)$.

The zero of the denominator in the neighborhood of $z \sim \overline{\zeta}$ is due to the factor $\overline{h}^I(z)$ in $H^I(z)$. In this case, however, the first term of $G^I(\zeta)$ vanishes, so that the contribution of the negative energy pole is

$$-e^{i\zeta t} \frac{\overline{g}^I(\overline{\zeta})}{h^I(\overline{\zeta})} j,$$  \hspace{1cm} (4.32)

the quaternionic reflection of the resonance below the positive part of the real line. The exponential decay rate of both of these contributions is given by (4.17), i.e., for

$$\zeta = \omega_0 - i(\gamma/2),$$  \hspace{1cm} (4.33)
the decay rate $\gamma$ is given by

$$\gamma = 2\pi|V_{\omega_0\alpha}|^2. \quad (4.34)$$

The oscillation frequencies of the two terms have, however, opposite sign.

From (4.12) and (4.19), we see that $h^{II}(\zeta)$ and $\overline{h}^{II}(\zeta)$ are both well-approximated by $-2\omega_0$; we write (4.29) and (4.30) as

$$\overline{h}^{II}(\zeta) \approx b + \pi i Y^*(\omega_0) \quad (4.35)$$

and

$$\overline{h}^{II}(\zeta) \approx b - \pi i Y^*(\omega_0), \quad (4.36)$$

where

$$b = P \int_{-\infty}^{\infty} dE \frac{Y^*(E)}{E - \omega_0} + iu^*. \quad (4.37)$$

In terms of these quantities,

$$A(t) \approx e^{-i\zeta t} - \frac{1}{2\omega_0} \left\{ e^{-i\zeta t} (b + \pi i Y^*(\omega_0)) - e^{i\zeta^* t} (b - \pi i Y^*(\omega_0)) \right\} j, \quad (4.38)$$

and hence

$$|A(t)|^2 \approx e^{-\gamma t} \left\{ 1 + \frac{1}{2\omega_0^2} [b|^2 + \pi^2 |Y(\omega_0)|^2]ight.$$

$$- (|b|^2 - \pi^2 |Y(\omega_0)|^2) \cos 2\omega_0 t - 2\text{Re}(Y(\omega_0)\pi b \sin 2\omega_0 t) \right\}; \quad (4.39)$$

the term in square brackets is non-negative. It is clear from (4.38) that in the limit $t \to 0$, the approximate expression for $A(t)$ does not approach unity (the approximations we have made in arriving at this expression do not affect the quaternionic amplitude in this order).

We see, moreover, that $|A(t)|^2$ approaches a value slightly larger than unity as $t \to 0$, in agreement with the usual result\(^6\), implied by the fact that the exact initial decay curve starts as $1 - O(t^2)$ in the neighborhood of $t = 0$ (this follows from the anti-self-adjoint property of $\tilde{H}$). We have argued, however, that since the background contribution can be made as small as we wish, the pole contributions must be as accurate as we wish, for $t > \epsilon \geq 0$. The non-cancellation of the quaternionic pole residues is an indication that the series expansion of $A(t)$ for very small $t$ is not smoothly connected to the non-perturbative exponential functions which can represent $A(t)$ with, in principle, arbitrary accuracy for non-vanishing $t$. We shall discuss this phenomenon a little further in the next section.

The probability of survival of the unstable state has small oscillatory interference terms of twice the frequency ($E_0 + v$), with coefficients proportional to the square of $V_{E\alpha}V_{E\beta}$; the oscillation in the survival probability could conceivably provide an observable residual quaternionic effect. Note that these oscillations do not correspond to those of the time dependent perturbation expansion for small $t$ which lead to convergence to the Golden Rule\(^2\).
If the complex part of \( \langle \psi_0 | \tilde{V} | \psi_0 \rangle \) is large and negative, i.e., \( v < -E_0 \), the corresponding complex Hilbert space problem develops a bound state due to the interaction (the complex pole moves to the left and onto the negative real axis). For the quaternionic problem, however, the pole condition now becomes

\[
\omega_1 = -\text{Re} \zeta_- \approx E_0 + v. \tag{4.40}
\]

The pole is, in this case, under the negative real axis, at \( z = \zeta_- \), where (with the same argument leading to (4.17))

\[
\text{Im} \zeta_- \approx -\pi |V_{\omega_1 \beta}|^2. \tag{4.41}
\]

This very small width corresponds to a resonance that is an almost-bound state. The occurrence of a resonance in quaternionic quantum theory replacing the bound state, in a corresponding situation, in the complex quantum theory, has been discussed by Adler\(^1,2\)

both for the one-dimensional and spherically symmetric three dimensional potential problems.

The residue of this pole at \(-\omega_1\) contributes to the survival amplitude a term given approximately by

\[
e^{-i\zeta_- t} \left( 1 + \frac{g^{II}(\zeta_-)}{h^{II}(\zeta_-)} j \right), \tag{4.42}
\]

where \( h^{II}(\zeta_-) \approx 2\omega_1 \). As in the previous case, there is a second pole, now under the positive real axis, for which \( -z^* = \zeta_- \), i.e., \( z = -\zeta^* = \zeta_- \). The contribution of this pole to the survival amplitude is approximately given by

\[
-e^{i\zeta^* t} \frac{g^{II}(\zeta_-)}{h^{II}(\zeta_-)} j, \tag{4.43}
\]

corresponding to a resonance with positive frequency and very small amplitude, which is the quaternionic reflection of the almost-bound state on the negative part of the real axis.

5. Conclusions

We have shown that a soluble model for the description of a quaternionic quantum mechanical system undergoing decay according the description of Wigner and Weisskopf\(^10\) can be constructed in the same way as the Lee-Friedrichs model\(^5\) of the complex quantum theory.

The existence of the effective spectrum\(^3\) in \((\infty, \infty)\) of the anti-self-adjoint Hamiltonian implies that the problem of the analysis of the complex structure of the amplitude is quite similar to that of the Pietenpol model\(^8\), an idealization of the Stark Hamiltonian in complex quantum theory. In this model, the spectrum of the “unperturbed” Hamiltonian has an absolutely continuous part in \((\infty, \infty)\), and the Wigner-Weisskopf theory achieves an almost irreversible law of evolution (there is no branch point to contribute to long time deviations from exponential behavior).

Carrying out an approximate analysis (assuming the discrete-continuous matrix elements of the the perturbing potential are small, and that those of the quaternionic part
are even smaller) of the analytic properties of the (analytically continued) amplitudes in the lower half plane, we find that for perturbations resulting in an energy shift of the real part that does not become negative, the unperturbed bound state becomes a resonance of the usual type (although the amplitude has a quaternionic phase); there is, however, a reflection resonance of small amplitude, with negative frequency, below the negative part of the real axis. This reflected resonance results in an oscillatory interference term which could provide an observable quaternionic residual effect. If, on the other hand, the energy shift induced by the diagonal part of the perturbation moves the pole to a position below the negative part of the real axis, in place of the bound state that would be found in the corresponding complex theory, one finds a rather narrow resonance which becomes a bound state in the limit in which the quaternionic part of the potential vanishes. Its quaternionic reflection on the positive part of the real axis is a positive frequency resonance of small amplitude which vanishes, along with the width of the negative energy resonance, with the quaternionic part of the potential. The occurrence, in quaternionic quantum theory, of a resonance replacing a corresponding bound state in the complex quantum theory, has been discussed by Adler\textsuperscript{1,2} both for the one-dimensional and spherically symmetric three dimensional potential problems.

As we have seen in the previous section, the quaternionic part of the pole residues at $\zeta$ and $\bar{\zeta} = -\zeta^*$ do not cancel to leave an approximate unity residue as $t \rightarrow 0$ as one might expect. The non-cancelling part comes from the behavior of the quaternionic part $\mathcal{g}(z)$ of the amplitude near the real axis, in particular, from a numerator function (absorptive part) which emerges from the integral due to the $\delta$-function that forms as $z$ approaches the real axis.

To understand this behavior, consider the Pietenpol model of the complex theory\textsuperscript{8}. For a Hamiltonian of the form $H = H_0 + V$, where $H_0$ has continuous spectrum from $-\infty$ to $\infty$, and a bound state $\phi$ embedded at $E_0 > 0$, the singularities of the function $\langle \phi | R(z) | \phi \rangle$, where $R(z)$ is the resolvent $(z - H)^{-1}$, are determined by

$$h(z) \langle \phi | R(z) | \phi \rangle = 1, \quad (5.1)$$

where

$$h(z) = z - E_0 - v + \int_{-\infty}^{\infty} \frac{|\langle E | V | \phi \rangle|^2}{E - z}, \quad (5.2)$$

and $v$ is the expectation value of $V$ in the state $\phi$. The analytic continuation of this equation to the lower half plane, as for eq.(4.12), has the form

$$h^{II}(z) = z - E_0 - v + \int_{-\infty}^{\infty} dE \frac{X(E)}{E - z} + 2\pi i X(z), \quad (5.3)$$

where $X(E) = |\langle E | V | \phi \rangle|^2$, the boundary value of the analytic function $X(z)$. Following the arguments we have given above, one sees that $h^{II}(z) = 0$ for

$$z = \zeta \cong -\pi i X(\omega_0), \quad (5.4)$$
for $\omega_0 \cong E_0 + v$. This result corresponds to a nonperturbative width for the decaying system. On the other hand, the time dependent perturbation expansion of the amplitude $A(t)$ results in

$$e^{i(E_0+v)t}A(t) = 1 - I(t) + \ldots,$$

where

$$I(t) = \int_{-\infty}^{\infty} dE \frac{X(E)}{(E - E_0 - v)^2} \{i(E - E_0 - v)t + 1 - e^{i(E-E_0-v)t}\}.$$  \hfill (5.5)

It then follows that

$$|A(t)|^2 \cong 1 - 4 \int_{-\infty}^{\infty} dE \frac{X(E)}{(E - E_0 - v)^2} \sin^2\left[\frac{(E - E_0 - v)t}{2}\right],$$

$$\cong 1 - 2\pi X(E_0 + v)t$$

providing a perturbative width in agreement with the nonperturbative width obtained in (5.4); the condition for this result, corresponding to the Golden Rule, is that $t$ be greater than $1/E_0$. However, as we have pointed out above, the non-perturbative calculation of the time dependence of the amplitude, as represented by the pole residues, can be made as accurate as we wish. The perturbation expansion, in low order, is therefore not an accurate representation of the decay law, in this case, for any $t > \epsilon \geq 0$.

A similar phenomenon occurs for the spectrum of the Stark effect. The bound states of an atomic system, such as hydrogen, are destroyed by the Stark potential (see ref. 8 for a discussion) for any electric field, no matter how small. The perturbative treatment of the energy levels, however, results in a very good approximation of the level shift from the initially bound states to the peaks of the spectral enhancements in the continuous spectrum\textsuperscript{16}.

Some development of a formal scattering theory and the properties of this model for the description of scattering will be given in a succeeding publication.

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