An intermediate regime for exit phenomena driven by non-Gaussian Lévy noises

Zhihui Yang¹ and Jinqiao Duan²

1. Department of Mathematics
   Western Illinois University
   Macomb, IL 61455, USA.
   E-mail: z-yang2@wiu.edu
2. Department of Applied Mathematics
   Illinois Institute of Technology
   Chicago, IL 60616, USA.
   E-mail: duan@iit.edu

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Abstract

A dynamical system driven by non-Gaussian Lévy noises of small intensity is considered. The first exit time of solution orbits from a bounded neighborhood of an attracting equilibrium state is estimated. For a class of non-Gaussian Lévy noises, it is shown that the mean exit time is asymptotically faster than exponential (the well-known Gaussian Brownian noise case) but slower than polynomial (the stable Lévy noise case), in terms of the reciprocal of the small noise intensity.

Key Words: Stochastic dynamical systems; non-Gaussian Lévy processes; Lévy jump measure; First exit time; Small noise limit

Mathematics Subject Classifications (2000): 60H15, 60F10, 60G17

Dedicated to Professor Ludwig Arnold on the occasion of his 70th birthday

1 Introduction

Although Gaussian processes like Brownian motion have been widely used in modeling fluctuations in engineering and science, it turns out that some complex phenomena involve with non-Gaussian Lévy motions. For instance, it has been argued that diffusion by geophysical turbulence [13] corresponds,
loosely speaking, to a series of “pauses”, when the particle is trapped by a 
coherent structure, and “flights” or “jumps” or other extreme events, when 
the particle moves in the jet flow. Paleoclimatic data [4] also indicates such 
irregular processes.

Lévy motions are thought to be appropriate models for non-Gaussian 
processes with jumps [11]. Let us recall that a Lévy motion \( L(t) \), or \( L_t \), has 
independent and stationary increments, i.e., increments \( \Delta L(t, \Delta t) = L(t + \Delta t) - L(t) \) are stationary (therefore \( \Delta L \) has no statistical dependence on \( t \)) and independent for any non overlapping time lags \( \Delta t \). Moreover, its sample 
paths are only continuous in probability , namely , 
\[
P(|L(t) - L(t_0)| \geq \delta) \to 0 \quad \text{as} \quad t \to t_0 \quad \text{for any positive} \ \delta.
\]
This continuity is weaker than the usual 
continuity in time.

This generalizes the Brownian motion \( B(t) \), as \( B_t \) satisfies all these 
three conditions. But Additionally, (i) Almost every sample path of the 
Brownian motion is continuous in time in the usual sense and (ii) Brownian 
motion’s increments are Gaussian distributed.

SDEs driven by non-Gaussian Lévy noises have attracted much attention 
recently [1, 12]. Although the SDEs driven by Lévy motion may generate 
stochastic flows [9, 1], or generate random dynamical systems in the sense 
of Arnold [2], under certain conditions, this research issue is still under 
development. Recently , mean exit time estimates have been investigated by 
Imkeller and Pavlyukevich [7, 8].

Consider a scalar deterministic ordinary differential equation 
\[
\dot{Y}_t = -U’(Y_t), \quad Y_0 = x, \quad x \in [-b, a], \quad a, b > 0,
\]
where 0 is a asymptotically stable equilibrium. Namely , for any starting point \( x \) in \([-b, a]\), the trajectory \( Y_t \) tends to 0 as 
time \( t \to \infty \). In this case, \( U(\cdot) \) has a minimum at 0.

Now perturb the deterministic dynamical system \( Y_t \) with some small 
random noise. Let us consider a scalar stochastic differential equation (SDE) 
\[
dX^\varepsilon_t = -U’(X^\varepsilon_t)dt + \varepsilon dL_t, \quad X_0 = x, \tag{1}
\]
where \( 0 < \varepsilon \ll 1 \) is the noise intensity, and \( L_t \) is a Lévy process. A scalar 
Lévy process is characterized by a drift parameter \( \theta \), a variance parameter \( d > 0 \) and a non-negative Borel measure \( \nu \), defined on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) and 
concentrated on \( \mathbb{R} \setminus \{0\} \), which satisfies 
\[
\int_{\mathbb{R} \setminus \{0\}} (y^2 \land 1) \nu(dy) < \infty, \tag{2}
\]
or equivalently 
\[
\int_{\mathbb{R} \setminus \{0\}} \frac{y^2}{1 + y^2} \nu(dy) < \infty. \tag{3}
\]
This measure \( \nu \) is the so called Lévy measure or the Lévy jump measure of 
the Lévy process \( L(t) \). We also call \((\theta, d, \nu)\) the generating triplet.
We study the first exit problem for the solution process \( X_t^\varepsilon \) from bounded intervals containing the attracting equilibrium 0, as \( \varepsilon \downarrow 0 \).

We define the first exit time from the spatial interval \([-b, a]\), \(a\) and \(b\) positive, as follows:

\[
\sigma(\varepsilon) = \inf\{t \geq 0, \ X_t^\varepsilon \notin [-b, a]\}.
\]

It is known [5] that when the noise is Gaussian Brownian, i.e., the Lévy measure part is absent, the mean exit time of the perturbed system \( X_t^\varepsilon \) is asymptotically exponentially fast

\[
E_x\sigma(\varepsilon) \sim \exp\left(\frac{C}{\varepsilon^2}\right)
\]
for some positive constant \(C\). Note that here and hereafter \(E_x\) is the expectation with respect to the probability law of \(X_t\) starting at \(X_0 = x\).

If the Lévy measure part is not absent, the exit problem has recently been studied. For symmetric \(\alpha\)-stable Lévy noise, i.e., the Lévy process whose Lévy jump measure is \(\nu(dy) = \frac{dy}{|y|^{1+\alpha}}\) with \(0 < \alpha < 2\), Imkeller and Pavlyukevich [7] have shown that the mean exit time is polynomially fast, \(O(\frac{1}{\varepsilon^\alpha})\), in terms of \(\frac{1}{\varepsilon}\). Namely, there exist positive constants \(\varepsilon_0, \gamma \) and \(\delta > 0\) such that for \(0 < \varepsilon \leq \varepsilon_0\),

\[
E_x\sigma(\varepsilon) \sim \frac{\alpha}{\varepsilon^\alpha} \left[ \frac{1}{\alpha} + \frac{1}{b^\alpha} \right]^{-1}(1 + O(\varepsilon^\delta)).
\]

for any \(x \in [-b + \gamma, a - \gamma]\).

Furthermore, for a class of Lévy noise of exponentially light jumps, Imkeller, Pavlyukevich and Wetzel [8] have shown that the mean exit time is exponentially fast, in terms of \(\frac{1}{\varepsilon}\), namely, \(O(\exp(\frac{1}{\varepsilon^\alpha}))\) with \(\alpha \in (0, 1)\), or \(O(\exp(\frac{\ln\varepsilon^{1+\frac{1}{\alpha}}}{\varepsilon}))\) with \(\alpha > 1\).

In this paper, for a class of non-Gaussian Lévy noises, we show that the mean exit time is asymptotically \(O(\frac{1}{\varepsilon^\alpha})\). This is faster than exponential (the well-known Gaussian Brownian noise case) but slower than the polynomial (the stable Lévy noise case). So we have an intermediate regime,

\[
O\left(\frac{1}{\varepsilon^\alpha}\right) < O\left(\frac{\ln\varepsilon}{\varepsilon^\alpha}\right) < \exp\left(\frac{C}{\varepsilon^2}\right),
\]
for \(0 < \varepsilon \ll 1\).

In section 2, we recall the generators for Lévy processes, and then prove the main result. In section 3, we consider two examples of SDEs driven by symmetric Lévy noises, including the \(\alpha\)-stable symmetric Lévy noises.
2 Main results

Let \( L_t \) be a Lévy process with the generating triplet \((\theta, d, \nu)\).

It is known that any Lévy process is completely determined by the Lévy-Khintchine formula (See [1, 11, 10]). This says that for any one-dimensional Lévy process \( L_t \), there exists a \( \theta \in \mathbb{R}, d > 0 \) and a measure \( \nu \) such that

\[
E e^{i\lambda L_t} = \exp\{i\theta \lambda t - dt \frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1 - i\lambda y I\{|y| < 1\}) \nu(dy)\}, \tag{7}
\]

where \( I(S) \) is the indicator function of the set \( S \), i.e., it takes value 1 on this set and takes zero value otherwise.

The generator \( A \) of the process \( L_t \) is the same as infinitesimal generator since Lévy process has independent and stationary increments. Hence \( A \) is defined as \( A\varphi = \lim_{t\to0} \frac{P_t\varphi - \varphi}{t} \) where \( P_t\varphi(x) = E_x\varphi(L_t) \) and \( \varphi \) is any function belonging to the domain of the operator \( A \). Recall the generator \( A \) for \( L_t \) is (See [1, 10])

\[
A\varphi(x) = a\varphi'(x) + \frac{1}{2}d\varphi''(x) + \int_{\mathbb{R}\setminus\{0\}} [\varphi(x+y) - \varphi(x) - I\{|y| < 1\} y\varphi'(x)] \nu(dy).
\tag{8}
\]

Let us find out the generator of \( \varepsilon L_t \).

**Lemma 1.** Let \( L_t \) be a Lévy process with the generating triplet \((a, d, \nu)\). Then for any \( \varepsilon > 0 \), the generator for \( \varepsilon L_t \) is

\[
A^{\varepsilon}\varphi = \varepsilon a\varphi'(x) + \frac{1}{2}\varepsilon^2 d\varphi''(x) + \int_{\mathbb{R}\setminus\{0\}} [\varphi(x+\varepsilon y) - \varphi(x) - \varepsilon I\{|y| < 1\} y\varphi'(x)] \nu(dy).
\tag{9}
\]

**Proof.** Notice that

\[
E e^{i\lambda \varepsilon L_t} = \exp\{i\varepsilon a \lambda - \varepsilon^2 \frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda \varepsilon y} - 1 - i\varepsilon \lambda y I\{|y| < 1\}) \nu(dy)\}
\]

\[
= \exp\{i\varepsilon a \lambda - i\lambda \varepsilon \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\} \nu(dy) - \varepsilon^2 \frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1) \nu(dy)\}
\]

\[
= \exp\{i\lambda a \varepsilon - i\lambda \varepsilon \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\} \nu(dy) + i\lambda \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\} \nu(dy)\}
\]

\[\quad - \varepsilon^2 \frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1 - i\lambda y I\{|y| < 1\}) \nu(dy)\}}.\]

Hence, \( \varepsilon L_t \) is a Lévy process with the generating triplet \((\varepsilon a - \varepsilon \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\} \nu(dy) + \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\} \nu(dy)\), \( \varepsilon^2, \varepsilon \nu(dy) \)). Using the equation (8), it is seen that the generator of \( \varepsilon L_t \) is given in (9).

This completes the proof of Lemma 1. \( \square \)
Remark 1. The generator for the process $X_t^\varepsilon$ in (1) is then
\[
A^\varepsilon \varphi = -U'(x)\varphi'(x) + \varepsilon a\varphi'(x) + \frac{1}{2}\varepsilon^2 d\varphi''(x)
+ \int_{\mathbb{R}\setminus\{0\}} \left[ \varphi(x + \varepsilon y) - \varphi(x) - \varepsilon I\{|y| < 1\} y\varphi'(x) \right] \nu(dy). \tag{10}
\]

We make the following assumptions for the SDE (1):

(A) There exists a function $g_1(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ such that for any $\gamma > 0$
\[
\int_{\mathbb{R}\setminus[-\gamma, \gamma]} \nu(d\frac{u}{\varepsilon})) \leq K(\gamma)g_1(\varepsilon)
where $K(\gamma)$ is some function of $\gamma$.

(B) For any $\delta > 0$, there exists a positive constant $K < \infty$ such that
\[
\int_{\mathbb{R}\setminus[-K, K]} \nu(d\frac{u}{\varepsilon})) \leq \delta g_1(\varepsilon).

(C) There exists a measure $\nu^*(du)$ on $\mathbb{R}\setminus\{0\}$ such that $\frac{1}{g_1(\varepsilon)} \nu(d\frac{u}{\varepsilon}))$
converges weakly to $\nu^*(du)$. The limit measure $\nu^*$ satisfies the condition
that for any Borel set $A \subset \mathbb{R}\setminus\{0\}$ with measure 0, we have $\nu^*(A) = 0$.

(D) There exists a function $g_2(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and a positive constant $K < \infty$ such that
\[
d\varepsilon^2 + \int_{\mathbb{R}} \frac{u^2}{1 + u^2} \nu(d\frac{u}{\varepsilon})) < Kg_2(\varepsilon).
\]
And, there exists some $n > 0$ such that $(g_2(\varepsilon))^n \leq g_1(\varepsilon)$.

(E) There exists a $g_3(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and a positive constant $K < \infty$ such that
\[
\int_{\mathbb{R}} \frac{u}{1 + u^2} \nu(d\frac{u}{\varepsilon})) < Kg_3(\varepsilon).
\]

We consider a special class of symmetric Lévy measures on $\mathbb{R}$ for $0 < \alpha < 2$:
\[
\nu(du) = f(\ln|u|) \frac{du}{|u|^{1+\alpha}}, \tag{11}
\]
where $f$ is a nonnegative measurable function on $\mathbb{R}$ such that this $\nu$ is a
Lévy measure, i.e., it satisfies the above condition (2):
\[
\int_{\mathbb{R}\setminus\{0\}} \frac{u^2}{1 + u^2} f(\ln|u|) \frac{du}{|u|^{1+\alpha}} < \infty. \tag{12}
\]
Being symmetric, this Levy measure $\nu(du)$ automatically satisfies the condition (E).

Let $\delta > 0$. Define $G^\delta = \{x \in [-b, a] : \inf_{t\geq0} \min(|Y(0, x, t) - a|, |Y(0, x, t) - (-b)|) \geq \delta\}$.
Theorem 1. Consider a class of symmetric Lévy measures on $\mathbb{R}$ in (11). Assume that the conditions (A) – (E) be satisfied. Especially the condition (C) is satisfied, namely, there exists a positive function $\tilde{f}(\varepsilon)$ such that

$$\frac{f(\ln \frac{|u|}{\varepsilon})}{\tilde{f}(\varepsilon)} \frac{du}{|u|^{1+\alpha}} \rightarrow \nu^*(du).$$

weakly (as Borel measures on $\mathbb{R}$) to a certain measure $\nu^*$. Then for any $x \in G^\delta$, we have

$$\lim_{\varepsilon \downarrow 0} \tilde{f}(\varepsilon) E_x \sigma(\varepsilon) = \frac{1}{\nu^*(\mathbb{R} \setminus [-b, a])},$$

or for $\varepsilon \downarrow 0$,

$$E_x \sigma(\varepsilon) \sim \frac{1}{\nu^*(\mathbb{R} \setminus [-b, a])} \frac{1}{\tilde{f}(\varepsilon)}.$$

Proof. Let us rewrite the generator for the one dimensional Markov process $X_t^\varepsilon$. The generator $A^\varepsilon$ depending on parameter $\varepsilon$ for the one dimensional Markov process $X_t^\varepsilon$ in equation (11) can be rewritten as in the following, using Lemma 1 and (10),

$$A^\varepsilon f(x) = -U'(x)f'(x) + \varepsilon a f'(x) - \varepsilon \int_{\mathbb{R}} u I\{|u| < 1\} \nu(du) f'(x) + \frac{d\varepsilon^2}{2} f''(x)$$

$$+ \int_{\mathbb{R}} [f(x + u) - f(x) - u I\{|u| < 1\} \nu(du)] f'(x) + \int_{\mathbb{R}} \frac{u}{1 + |u|^2} \nu(\frac{u}{\varepsilon}) f'(x)$$

$$+ \frac{d\varepsilon^2}{2} f''(x) + \int_{\mathbb{R}} [f(x + u) - f(x) - \frac{u}{1 + |u|^2} f'(x)] \nu(\frac{u}{\varepsilon})$$

$$= U^\varepsilon(x) f'(x) + \frac{d\varepsilon^2}{2} f''(x) + \int_{\mathbb{R}} [f(x + u) - f(x) - \frac{u}{1 + |u|^2} f'(x)] \nu(\frac{u}{\varepsilon}).$$

Here

$$U^\varepsilon(x) = -U'(x) + \varepsilon a - \varepsilon \int_{\mathbb{R}} u I\{|u| < 1\} \nu(du) + \int_{\mathbb{R}} \frac{u}{1 + |u|^2} \nu(\frac{u}{\varepsilon}).$$

The result follows a similar idea in the proof of the Assertion in [6].

3 Examples

We look at some applications of the above Theorem 1.

Example 1: $\alpha$-stable symmetric Lévy Noise
This is a special case of Theorem 1 above with \( f(\cdot) \equiv 1 \). Consider \( X_\varepsilon \) defined in equation (1), where the Lévy process \( L_t \) is characterized by

\[
Ee^{i\lambda L_t} = \exp\{-dt\frac{\chi^2}{2} + t \int_{\mathbb{R}\setminus\{0\}} \left( e^{i\lambda u} - 1 - i\lambda uI\{|u| < 1\} \right) \frac{1}{|u|^{1+\alpha}}(du)\}
\]

Here \( d \geq 0 \) and \( 0 < \alpha < 2 \) are some constants. The Lévy process \( \nu(du) = \frac{e^\alpha}{|u|^{1+\alpha}}(du) \). This is a so called \( \alpha \)-stable symmetric Lévy process, and it is heavy tailed and has infinite mass due to the strong intensity of small jumps.

Now, we try to verify the conditions in Theorem 1. Notice that \( \nu(du) = \frac{e^\alpha}{|u|^{1+\alpha}}(du) \). It can be verified that there exist \( g_1(\varepsilon) = g_2(\varepsilon) = g_3(\varepsilon) = \varepsilon^\alpha \), and \( \nu^*(du) = \frac{1}{|u|^{1-\alpha}}du \) such that the conditions (A)-(D) are satisfied. Hence, for any \( x \in \mathbb{R}^d \), we have

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha E_x \sigma(\varepsilon) = \frac{1}{\int_a^\infty \frac{1}{|u|^{1+\alpha}}du + \int_{-\infty}^b \frac{1}{|u|^{1+\alpha}}du} = \alpha \frac{1}{a^\alpha} \frac{1}{b^\alpha}^{-1}.
\]

Thus

\[
E_x \sigma(\varepsilon) \sim \alpha \frac{1}{a^\alpha} + \frac{1}{b^\alpha}^{-1} \frac{1}{\varepsilon^\alpha}.
\]

This is the result that Imkeller and Pavlyukevich [7] obtained earlier.

**Example 2: A symmetric Lévy Noise**

Consider \( X_\varepsilon \) defined in equation (1), with a special symmetric Lévy process \( L_t \) that is characterized by

\[
Ee^{i\lambda L_t} = \exp\{-dt\frac{\chi^2}{2} + t \int_{\mathbb{R}\setminus\{0\}} \left( e^{i\lambda u} - 1 - i\lambda uI\{|u| < 1\} \right) \nu(du)\}
\]

Here \( d \geq 0 \) and \( \nu(du) = f(\ln |u|)\frac{du}{|u|^{1+\alpha}}, 0 < \alpha < 2 \) with \( f(\ln |u|) = \frac{1}{|\ln |u||+1} \). Such a \( \nu(du) \) is a Levy measure satisfying the the condition (2).

We claim that there exist \( g_1(\varepsilon) = \frac{1}{\varepsilon^{\alpha}} \) and \( \nu^*(du) = \frac{1}{u^{1-\alpha}}du \) such that the conditions (A)-(D) are satisfied.

To verify condition (A), it is sufficient to show that for any \( r > 0 \) there exists some function of \( r, K(r) \) that \( \int_r^\infty \nu(d(\frac{u}{\varepsilon})) \leq K(r)(\frac{\varepsilon^\alpha}{\ln \varepsilon}) \) for \( \varepsilon \) small enough. We take a constant \( C = |\min\{\ln r, 0\}| \). Notice that

\[
\int_r^\infty \nu(d(\frac{u}{\varepsilon}))/\left(\frac{\varepsilon^\alpha}{\ln \varepsilon}\right) = \int_r^\infty \frac{1}{u^{1+\alpha}|1 + \frac{\ln u}{\ln \varepsilon} + \frac{1}{\ln \varepsilon}}du \\
\leq \int_r^\infty \frac{1}{u^{1+\alpha}|1 + \frac{C}{\ln \varepsilon} + \frac{1}{\ln \varepsilon}}du \\
= \int_r^\infty \frac{1}{u^{1+\alpha}|1 + \frac{1-C}{\ln \varepsilon}}du = \frac{1}{\alpha|1 + \frac{1-C}{\ln \varepsilon}|r^\alpha} \\
\leq \frac{2}{\alpha} r^\alpha.
\]
for $\varepsilon$ sufficiently small. To verify Condition (B), we notice that for $K > 1$, 
\[ \int_k^\infty \nu(d(u))/\ell \leq \int_k^\infty \frac{1}{u^{1+\alpha}} du \] 
which is smaller than any $\delta > 0$, if $K$ is big enough. To verify condition (C), it is sufficient to show that for any $r > 0$, 
\[ \lim_{\varepsilon \to 0} \int_r^\infty \frac{1}{u^{1+\alpha}} \nu(d(u)) = \int_r^\infty \frac{1}{u^{1+\alpha}} du. \]
This can be done by a Lebesgue convergence theorem. Finally, let us verify condition (D). Since $\varepsilon^2 < \frac{\varepsilon^\alpha}{\ln \varepsilon}$ for $0 < \varepsilon << 1$ and the measure $\nu$ is symmetrical, we only need to show 
\[ \int_0^\infty \frac{u^2}{1+u^2} \nu(d(u))/\ell < K \]
for some constant $K$. Notice that 
\[ \int_0^\infty \frac{u^2}{1+u^2} \nu(d(u))/\ell = 
\begin{equation}
\begin{aligned}
&\int_0^\infty \frac{u^2}{1+u^2} \nu(d(u))/\ell \\
&= \int_0^\infty \frac{u^2}{1+u^2} \nu(d(u))/\ell \\
&< \int_0^\infty \frac{u^{1-\alpha}}{1+\frac{u}{\ln \varepsilon}} du + \int_0^1 \frac{u^{1-\alpha}}{1+\frac{u}{\ln \varepsilon}} du + \int_1^\infty \frac{u^{1-\alpha}}{1+\frac{u}{\ln \varepsilon}} du \\
&< K_1.
\end{aligned}
\end{equation} \]
for some constant $K_1$ if $\varepsilon$ is sufficiently small. And, 
\[ \int_0^1 \frac{u^{1-\alpha}}{1+\frac{u}{\ln \varepsilon}} du < \int_0^\infty \frac{u^{1-\alpha}}{1+\frac{u}{\ln \varepsilon}} du = \frac{1}{2-\alpha} [1 - (\sqrt{\varepsilon})^{2-\alpha}] < K_2. \]
for some constant $K_2$ if $\varepsilon$ is small enough. Finally, 
\[ \int_0^\infty \frac{1}{u^{1+\alpha}} du < \int_1^\infty \frac{1}{u^{1+\alpha}} = \frac{1}{\alpha}. \]
Therefore, condition (D) is satisfied. By Theorem[4] we conclude that for any $x \in G^\delta$, we have 
\[ \lim_{\varepsilon \to 0} \frac{\varepsilon^\alpha}{\ln \varepsilon} E_x \sigma(\varepsilon) = \frac{1}{\int_{\alpha}^{\infty} \frac{1}{\alpha^{1+\alpha}} du + \int_{\infty}^{-\infty} \frac{1}{\alpha^{1+\alpha}} du} = \alpha \left( \frac{1}{\alpha} + \frac{1}{b^\alpha} \right)^{-1}. \]
So 
\[ E_x \sigma(\varepsilon) \sim \alpha \left( \frac{1}{\alpha} + \frac{1}{b^\alpha} \right)^{-1} \left( \frac{\ln(\varepsilon)}{\varepsilon^\alpha} \right). \]
This mean exit time is asymptotically $O\left(\frac{\ln \varepsilon}{\varepsilon^\alpha}\right)$. It is faster than exponential (the well-known Gaussian Brownian noise case [5]) but slower than polynomial (the stable Lévy noise case [7]; see also Example 1 above). Namely, for $0 < \varepsilon \ll 1$,

$$O\left(\frac{1}{\varepsilon^\alpha}\right) < O\left(\frac{\ln \varepsilon}{\varepsilon^\alpha}\right) < \exp\left(\frac{C}{\varepsilon^2}\right).$$ (13)

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