On semi $\theta$-` axioms

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Abstract. The aim of this paper is to introduce some new separation axioms based on the idea of defining new kinds of open sets, namely semi-$\theta$-open sets. We study there basic properties and also the implications of these new separation axioms among themselves and with the known axioms $T_2$, $T_1$ and $T_0$ are obtained.

Keywords: semi-$\theta$-space, semi-$\theta$-$T_0$-space, semi-$\theta$-$T_1$-space, semi-$\theta$-$T_2$-space.

1. Introduction
Assume that $\langle X, \tau_\theta \rangle$ is a topological space and let $A \subseteq X$. The closure and interior of $A$ are denoted by the letters $\varphi$ and $A^\theta$, respectively. A set $A$ is called semi-open(s-open) $[1]$ in a topological space $\langle X, \tau_\theta \rangle$ if there exist open set $U$ with $U \subset A \subset U$. Alternatively, if $A \subset A^\theta$, Semi-closed(s-closed)$[2]$ set is the complement of a semi-open set. The letter $A^\theta$ denotes the semi-closure$[2]$ of $A$, which is the intersection of all semi-closed sets containing $A$. $SO(X, \tau_\theta)$ (resp. $SC(X, \tau_\theta)$, $O(X, \tau_\theta)$, $C(X, \tau_\theta)$) denotes the family of all semi-open (resp. semi-closed, open, closed) sets in $X$. A point $x \in X$ is said to be $\theta$-adherent $[3]$ of a subset $A$ of $X$ if $A \cap U = \emptyset$ for every $U \in O(X, x)$. The $\theta$-closure $[3]$ of $A$ is the set of all $\theta$-adherent points of $A$ and is denoted by $A^\theta$. if $A = A^\theta$ then a set $A$ of $X$ is called $\theta$-closed $[3]$. The $\theta$-open $[3]$ set is the complement of a $\theta$-closed set $\theta O(X, x)(\text{resp. } \theta C(X, x))$ will denotes the class of all $\theta$-open (resp. $\theta$-closed) sets in $X$. We set $\theta O(X, \tau_\theta) = \{ U : x \in U \in \theta O(X, \tau_\theta) \}$ and $\theta C(X, \tau_\theta) = \{ U : x \in U \in \theta C(X, \tau_\theta) \}$. If $A$ is semi-open and semi-closed at the same time, it is called semi-regular (s-regular)$[4]$.

The family of all s-regular sets of $\langle X, \tau_\theta \rangle$ is denoted by $SR(X, \tau_\theta)$. The semi-$\theta$-closure(s-$\theta$-closure) $[4]$ of $A$ denoted by $A^\theta$ is defined to be $\it{The set of all x \in X such that U^\theta \cap A \neq \emptyset}$; for every $U \in SO(X, \tau_\theta)$ with $x \in A$. A subset $A$ is called semi-$\theta$-closed(s-$\theta$-closed) $[5]$ if $A = A^\theta$. semi-$\theta$-open(s-$\theta$-open) is the complement of a semi-$\theta$-closed set. $S\theta C(X, \tau_\theta)$ (resp. $S\theta O(X, \tau_\theta)$) will denotes the class of all semi-$\theta$-closed (semi-$\theta$-open) sets in $X$. We set $S\theta C(X, x) = \{ U : x \in U \in S\theta C(X, x) \}$ and $S\theta O(X, x) = \{ U : x \in U \in S\theta O(X, x) \}$. 
2. Preliminaries

Definition 2.1 [6]
A mapping \( f: (X, \tau_X) \rightarrow (Y, \tau_Y) \) is said to be \( s-\theta \)-irresolute (or quasi-irresolute) if for each \( x \in X \) and each \( U \in S\theta O(Y, f(x)) \), there exists \( V \in S\theta O(X, x) \) such that \( f(U) \subseteq V \), equivalent if each \( A \in S\theta O(y, \tau_y) \) implies \( f^{-1}(A) \in S\theta O(X, \tau_x) \).

Definition 2.2 [3]
A subset \( A \) is said to be \( \theta \)-open set if for each \( x \in A \) there exists an open set \( U \) such that \( x \in U \subseteq U \subseteq A \).

Theorem 2.3
If \( B \in \theta O(X, \tau_X) \) and \( A \in S\theta O(X, \tau_X) \) implies \( A \cap B \) is \( s-\theta \)-open in \( B \).

Theorem 2.4
Assume that \( X_1 \) and \( X_2 \) be topological spaces and \( X = X_1 \times X_2 \) be the topological product. Let \( A_1 \in S\theta O(X_1) \) and \( A_2 \in S\theta O(X_2) \). Then \( (A_1 \times A_2) \in S\theta O(X_1 \times X_2) \).

Remark 2.5
(i) for all \( A \subseteq O(X, \tau_X) \) implies \( A \subseteq SO(X, \tau_X) \).
(ii) for all \( A \subseteq S\theta O(X, \tau_X) \) implies \( A \subseteq SO(X, \tau_X) \).

Proposition 2.6 [7]
The following are equivalent for a topological space \( (X, \tau_X) \):
1) any set that is closed is also \( s-\theta \)-closed.
2) \( (X, \tau_X) \) is \( s \)-regular.
3) any set that is open is also \( s-\theta \)-open.
4) each closed set is the intersection of \( s \)-regular sets.

Proposition 2.7 [4, 9]
(i) for all \( A \subseteq SO(X, \tau_X) \), implies \( A^s = A^s\theta \).
(ii) for all \( A \subseteq SR(X, \tau_X) \), implies \( A \) is \( s-\theta \)-closed and \( s-\theta \)-open.

Remark 2.8
Assume that \( (X, \tau_X) \) be a topological space and \( A, B \subseteq X \), then
1) \( A \subseteq A^s\theta \).
2) If \( A \subseteq B \) then \( A^s\theta \subseteq B^s\theta \).
3) If \( A \) \( s-\theta \)-closed then \( A = A^s\theta \) [5].

3. Semi-\( \theta -T_0 \), semi-\( \theta -T_1 \) and semi-\( \theta -T_2 \) spaces

Definition 3.1
Assume that \( (X, \tau_X) \) is a topological space. If for each \( a, b \in X \) such that \( a \neq b \) there is a \( s-\theta \)-open (resp. \( s \)-open) set \( W \) of \( X \) containing a but not \( b \), we say \( X \) is semi-\( \theta -T_0 \) (resp. semi-\( T_0 \))[10]).

Remark 3.2
(i) Every \( T_0 \)-space is semi-\( \theta -T_0 \)-space.
Assume that \( X \) be a \( T_0 \)-space and let \( a, b \in X \) with \( a \neq b \). Then there exist \( U \in SO(X, \tau_X) \) [Remark 2.5] such that \( a \in U \) and \( a \notin U^s \) but \( b \notin U^s \) [Proposition 2.6]. Since \( U^s \) is \( s \)-regular, this \( U \) \( s-\theta \)-open. Hence \( X \) is semi-\( \theta -T_0 \).
(ii) As the following example shows, the converse of (i) is not always true.
Example 3.3
Assume that \((X, \tau_X)\) is a topological space such that \(X = \{a, b, c, d\}, \tau_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\). Then there is the semi-open set family.
\[
S\theta O(X) = \{\emptyset, X, \{b, c, d\}, \{a, c, d\}, \{a, d\}, \{b, c\}, \{a, \{b, c\}, \{a, b\}\}\}
\]
Observe that \(X\) is semi-\(\theta\)-\(T_0\) but not \(T_0\)-space.

Theorem 3.4
Assume that \((X, \tau_X)\) is a topological space. We say \(X\) is semi-\(\theta\)-\(T_0\) iff for every \(x, y \in X\) with \(x \neq y\).

Implies \(\{x\}^{s\theta} \neq \{y\}^{s\theta}\).

Proof.
Let \(x, y \in X\) with \(x \neq y\) and \(X\) is semi-\(\theta\)-\(T_0\) space. We shall show that \(\{x\}^{s\theta} \neq \{y\}^{s\theta}\). Since \(X\) is semi-\(\theta\)-\(T_0\), there exists a \(s\)-\(\theta\)-open set \(G\) such that \(x \in G\) but \(y \notin G\). Also \(x \notin X - G\) and \(y \in X - G\) where \(X - G\) is \(s\)-\(\theta\)-closed set in \(X\). Now by definition \(\{y\}^{s\theta}\) is the intersection of all \(s\)-\(\theta\)-closed sets which contain \(y\). Hence, \(y \notin \{y\}^{s\theta}\) but \(x \notin \{y\}^{s\theta}\) as \(x \notin X - G\). Therefore, \(\{x\}^{s\theta} \neq \{y\}^{s\theta}\).

Conversely, for any \(x, y \in X, x \neq y\). And \(\{x\}^{s\theta} \neq \{y\}^{s\theta}\). Then there exists at least one point \(z \in X\) such that \(z \in \{x\}^{s\theta}\) but \(z \notin \{y\}^{s\theta}\). We claim that \(x \notin \{y\}^{s\theta}\). If \(x \in \{y\}^{s\theta}\) then \(x \subseteq \{y\}^{s\theta}\) implies \(\{x\}^{s\theta} \subseteq \{y\}^{s\theta}\). So, \(z \in \{y\}^{s\theta}\), which is a contradiction. Hence, \(x \notin \{y\}^{s\theta}\). Now, \(x \notin \{y\}^{s\theta}\) implies \(x \in X - \{y\}^{s\theta}\) and \(x \notin \{y\}^{s\theta}\) is \(S\theta O(X, \tau_X)\) but \(y \notin X - \{y\}^{s\theta}\).

Observe that \(X\) is a semi-\(\theta\)-\(T_0\) space.

Theorem 3.5
Each \(\theta\)-open subspace of a semi-\(\theta\)-\(T_0\) space is semi-\(\theta\)-\(T_0\)-space.

Proof.
Assume that \(Y\) is a \(\theta\)-open subspace of a semi-\(\theta\)-\(T_0\) space \(X\) and let \(x, y \in Y\) with \(x \neq y\). Then there exists a \(s\)-\(\theta\)-open set \(A\) in \(X\) containing \(x\) or \(y\), say, \(x\) but not \(y\). Now by \[ \text{Theorem 2.3} \], \(A \cap Y\) is \(s\)-\(\theta\)-open set in \(Y\) containing \(x\) but not \(y\). Observe that \(Y\) is semi-\(\theta\)-\(T_0\).

Definition 3.6
Assume that \((X, \tau_X)\) is a topological space. Then \(X\) is semi-\(\theta\)-\(T_1\) (resp. semi-\(T_1\)[10]) if for each \(a, b \in X\) such that \(a \neq b\) there exists a \(s\)-\(\theta\)-open (resp. \(s\)-open) set \(W\) of \(X\) containing \(a\) but not \(b\) and a \(s\)-\(\theta\)-open set \(U\) of \(X\) containing \(b\) but not \(a\).

Remark 3.7
i) Every \(T_1\)-space is semi-\(\theta\)-\(T_1\)-space.
ii) As the following example shows, the converse of (i) is not always true.

Example 3.8
Assume that \((X, \tau_X)\) is a topological space with \(X = \{a, b, c\}, \tau_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, X\}\). Then there is the family of semi-open sets. \(S\theta O(X) = \{\emptyset, X, \{b\}, \{a\}, \{a, c\}, \{b, c\}\}\).

Hence \(X\) is semi-\(\theta\)-\(T_1\) but not \(T_1\)-space.
Theorem 3.9

X is semi-$\theta$-$T_1$ if and only if for all $x \in X$ implies $\{x\}$ is s-$\theta$-closed.

Proof.
Assume that $x, y \in X$ with $x \neq y$. So that $\{x\}$ and $\{y\}$ are s-$\theta$-closed sets and as such $\{x\}^C$ and $\{y\}^C$ are s-$\theta$-open sets. Thus $y \in \{x\}^C$ but $x \notin \{x\}^C$ and $x \in \{y\}^C$ but $y \notin \{y\}^C$. Hence, $X$ is a semi-$\theta$-$T_1$ space.

Conversely, let $X$ be a semi-$\theta$-$T_1$-space and $x$ be any arbitrary point of $X$. If $y \in \{x\}^C$ then $y \neq x$.

Now the space being semi-$\theta$-$T_1$ space and $y$ is a point different from $x$ so that there must exists a s-$\theta$-open set $G$ such that $y \in G$ but $x \notin G$. Thus corresponding to each $y \in \{x\}^C$ there exists $G \in S\theta O(X, \tau_X)$ with $y \in G \subseteq \{x\}^C$. Therefore $\cup \{y; y \neq x\} \subseteq \cup \{G; y \neq x\} \subseteq \{x\}^C$ implies $\{x\}^C \subseteq \cup \{G; y \neq x\} \subseteq \{x\}^C$. Therefore $\{x\}^C = \cup \{G; y \neq x\}$. Since $G$ is s-$\theta$-open and the union of s-$\theta$-open sets, therefor $\{x\}^C \in S\theta O(X, \tau_X)$ that is $\{x\} \in S\theta C(X, \tau_X)$. Since $x$ is arbitrary, it follows that every singleton subset $\{x\}$ of $X$ is s-$\theta$-closed set.

Theorem 3.10

Each $\theta$-open subspace of a semi-$\theta$-$T_1$ space is semi-$\theta$-$T_1$.

Proof.
Assume that $A$ be an $\theta$-open subspace of a semi-$\theta$-$T_1$ space $X$. Let $x \in A$ since $X$ is semi-$\theta$-$T_1$, then there exists $X - \{x\} \in S\theta O(X, \tau_X)$ A being $\theta$-open $A \cap (X - \{x\}) = A - \{x\}$ is s-$\theta$-open in $A$ by [ Theorem 2.3]. Consequently, $\{x\}$ is s-$\theta$-closed in $A$. Hence by [ Theorem 3.9 ], $A$ is semi-$\theta$-$T_1$.

Theorem 3.11

If $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is bijective and s-$\theta$-irresolute mapping from a topological space $X$ into $Y$. If $Y$ is semi-$\theta$-$T_1$, then $X$ is semi-$\theta$-$T_1$ space.

Proof.
Assume that $x, y \in X$ such that $x \neq y$. Since $f$ is an injective implies $f(x) \neq f(y)$ with $f(x), f(y) \in Y$. Since $Y$ is semi-$\theta$-$T_1$-space, there exists $G, H \in S\theta O(Y, \tau_Y)$ such that $f(x) \in G, f(y) \notin G$ and $f(x) \notin H, f(y) \in H$. A gain since $f$ is s-$\theta$-irresolute, $f^{-1}(G)$ and $f^{-1}(H)$ are s-$\theta$-open sets in $X$ such that $x \in f^{-1}(G), y \notin f^{-1}(G)$ and $x \notin f^{-1}(H), y \in f^{-1}(H)$. Hence $X$ is semi-$\theta$-$T_1$ space.

Definition 3.12

Assume that $(X, \tau_X)$ is a topological space. Then $X$ is semi-$\theta$-$T_2$ (resp. semi-$\theta$-$T_2$ [10]) if for every $a, b \in X$ with $a \neq b$. There exist $U, W \in S\theta O(X, \tau_X)$ (resp. $U, W \in S\theta O(X, \tau_X)$) such that $a \in W$ and $b \in U$ with $W \cap U = \emptyset$.

Remark 3.13
(i) Every $T_2$-space is semi-$\theta$-$T_2$.

Assume that $X$ be a $T_2$-space and let $a, b \in X$ with $a \neq b$. Then there is $U$ and $V$, which are s-open sets. [ Remark 2.5] with $a \in U, b \in V$ and $U^s \cap V^s = \emptyset$ [ Proposition 2.6]. Since $U^s, V^s \in S\theta O(X, \tau_X)$, then $U^s, V^s \in S\theta O(X, \tau_X)$. Observe that $X$ is semi-$\theta$-$T_2$

(ii) As the following example shows, the converse of (i) is not always true.

Example 3.14

Assume that $(X, \tau_X)$ be a topological space with $X = \{1, 2, 3\}$, $\tau_X = \{\emptyset, X, \{2\}, \{1, 3\}, \{2\}\}$. Then there is the semi-open set family $S\theta O(X) = \{\emptyset, X, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{3\}, \{2\}\}$. 

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Observe that $X$ is semi-$\theta$-$T_2$ but not $T_2$-space.

**Remark 3.15**

1) per semi-$\theta$-$T_0$-space is also a semi-$T_0$-space.
2) per semi-$\theta$-$T_1$-space is also a semi-$T_1$-space.
3) per semi-$\theta$-$T_2$-space is also a semi-$T_2$-space.

**Theorem 3.16**

If $X$ is semi-$\theta$-$T_2$ then the diagonal $\Delta$ in $X \times X$ is s-$\theta$-closed.

**Proof.**

Assume that $X$ is semi-$\theta$-$T_2$, to proof $\Delta$ is s-$\theta$-closed i.e to proof $(X \times X) - \Delta$ is s-$\theta$-open. $(x, y) \in (X \times X) - \Delta$. As $(x, y) \notin \Delta$, $x \neq y$. Since $X$ is semi-$\theta$-$T_2$ there exist $G, H \in S\theta O (X, \tau_x)$ with $x \in G$, $y \in H$ and $G \cap H = \emptyset$, $\Delta \cap G = \emptyset$. implies that $(G \times H) \cap \Delta = \emptyset$. And so $G \times H \subset (X \times X) - \Delta$. Further $(x, y) \in G \times H$ and $G \times H$ is a s-$\theta$-open set in $X \times X$ [by Theorem 2.4]. Consequently, $(X \times X) - \Delta$ is s-$\theta$-open Set in $X \times X$. Observe that $\Delta$ is s-$\theta$-closed.

**Theorem 3.17**

If $f: (X, \tau_x) \rightarrow (Y, \tau_y)$ is bijective, s-$\theta$-irresolute mapping. If $Y$ is semi-$\theta$-$T_2$, implies $X$ is semi-$\theta$-$T_2$ space.

**Proof.**

Let $x, y \in X$ with $x \neq y$. Since $f$ is an injective, implies $f(x), f(y) \in Y$ with $f(x) \neq f(y)$. Since $Y$ is semi-$\theta$-$T_2$-space, there exists $G, H \in S\theta O(Y, \tau_y)$ with $f(x) \in G, f(y) \in H$ and $f(x) \notin H, f(y) \in H$, $H \cap G = \emptyset$. A gain since $f$ is s-$\theta$-irresolute, $f^{-1}(G), f^{-1}(H) \in S\theta O(X, \tau_x)$ such that $x \in f^{-1}(G), y \notin f^{-1}(G)$ and $x \notin f^{-1}(H), y \in f^{-1}(H), f^{-1}(H) \cap f^{-1}(G) = \emptyset$. Hence $X$ is semi-$\theta$-$T_2$ space.

**Theorem 3.18**

Let $(X, \tau_x)$ and $(Y, \tau_y)$ be two topological spaces. Then the product space $X \times Y$ is a semi-$\theta$-$T_2$-space if and only if for every $X$ and $Y$ are semi-$\theta$-$T_2$-space.

**Proof.**

Assume that $X$ and $Y$ is semi-$\theta$-$T_2$-spaces and let $x, y \in X \times Y, x \neq y$. Let $x = (a, b)$ and $y = (c, d)$. Without any loss of generality, suppose that $a \neq c$ and $b \neq d$. Since $a, c \in X$ with $a \neq c$, and $X$ is semi-$\theta$-$T_2$ there exist $U, V \in S\theta O(X, \tau_x)$ such that $U \cap V = \emptyset$ with $a \in U, c \in V$. Likewise let $G, H \in S\theta O(Y, \tau_y)$ such that $G \cap H = \emptyset$ with $b \in G, \quad d \in H$. Then $U \times G$ and $V \times H$ are s-$\theta$-open sets [Theorem 2.4] in $X \times Y$ containing $x$ and $y$ respectively. Also, $(U \times G) \cap (V \times H) = (U \cap V) \times (G \cap H) = \emptyset$. Hence $X \times Y$ is semi-$\theta$-$T_2$.

**Remark 3.19**

semi-$\theta$-$T_2$ $\Rightarrow$ semi-$\theta$-$T_1$ $\Rightarrow$ semi-$\theta$-$T_0$.

**Theorem 3.20** [8]

The following are equivalent for a topological space $(X, \tau_x)$:

1) $(X, \tau_x)$ is semi-$\theta$-$T_2$;
2) $(X, \tau_x)$ is semi-$\theta$-$T_1$;
3) $(X, \tau_x)$ is semi-$\theta$-$T_0$;

**Proof.**

It is enough to prove (3) $\Rightarrow$ (1): For any points $a \neq b$, let $V \in S\theta O(X, \tau_x)$ with $a \in V, \quad b \notin V$. After that, there is $U \in S\theta O(X, \tau_x)$ such that $a \in U \subset U^s \subset V$. By [Proposition 2.7] $U^s \in S\theta O(X, \tau_x)$ then $U^s \in S\theta O(X, \tau_x), a \in U^s$ and also $X - U^s \in S\theta O(X, \tau_x), b \in X - U^s$. Therefore, $X$ is semi-$\theta$-$T_2$. 
4. Semi-$\theta$-R spaces and semi-$\theta$-N spaces

Definition 4.1
Assume that $(X, \tau_X)$ be a topological space, we call $X$ is s-$\theta$-regular iff for each $F \in S\theta C(X, \tau_X)$ with $x \notin F$, then there is $W_1, W_2 \subseteq S\theta O(X, \tau_X)$ such that $x \in W_1$, $F \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$.

Example 4.2
let $X = \{1,2,3\}$, $\tau_X = \{ \emptyset, X, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3\} \}$. And $C(X, \tau_X) = \tau_X$
$S\theta O(X, \tau_X) = \{ \emptyset, X, \{1,2\}, \{2,3\}, \{1,3\} \}$, $S\theta C(X, \tau_X) = \{ \emptyset, X, \{1,2\}, \{1,3\} \}$.
Observe that $X$ is semi-$\theta$-R-space.

Remark 4.3
In previous example notes that $X$ is not semi-$\theta$-$T_0$, not semi-$\theta$-$T_1$, and not semi-$\theta$-$T_2$, so the semi-$\theta$-$R$ is not necessarily semi-$\theta$-$T_0$ or semi-$\theta$-$T_1$ or semi-$\theta$-$T_2$.

i.e. (semi-$\theta$-$R$ $\neq$ semi-$\theta$-$T_0 \land$ semi-$\theta$-$R$ $\neq$ semi-$\theta$-$T_1 \land$ semi-$\theta$-$R$ $\neq$ semi-$\theta$-$T_2$).

Theorem 4.4
Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces. Then the product space $X \times Y$ is a semi-$\theta$-R-space if and only if each $X$ and $Y$ are semi-$\theta$-R-space.
Proof
assume that $X$ and $Y$ are semi-$\theta$-R-space, we will two show that $X \times Y$ is semi-$\theta$-R-space
Let $(x, y) \in X \times Y$ and $A$ semi-$\theta$-closed set in $X \times Y$ ; $(x, y) \notin A$
After that, there is $F_1$ semi-$\theta$-closed set in $X$ and $F_2$ semi-$\theta$-closed in $Y$; $F_1 \times F_2 \subseteq A$ and $(x, y) \notin F_1 \times F_2$.
$\Rightarrow x \notin F_1$ or $y \notin F_2$.

$\therefore X$ is a semi-$\theta$-R-space, then there exists $U_1, V_1$ are semi-$\theta$-open set in $X$, $U_1 \cap V_1 = \emptyset$, ($x \in U_1$ and $F_1 \subseteq V_1$).
$\therefore Y$ is a semi-$\theta$-R-space, then there exists $U_2$, $V_2$ are semi-$\theta$-open set in $Y$, $U_2 \cap V_2 = \emptyset$.

$(y \in U_2$ and $F_2 \subseteq V_2)$. Then $U_1 \times Y, V_1 \times Y$ are semi-$\theta$-open set in $X \times Y$;
$(U_1 \times Y) \cap (V_1 \times Y) = (U_1 \cap V_1) \times Y = \emptyset \times Y = \emptyset$. ($x \in U_1 \times Y$ and $F_1 \times F_2 \subseteq V_1 \times Y$).

Or
$X \times U_2, X \times V_2$ are semi-$\theta$-open set in $X \times Y$;
$(X \times U_2) \cap (X \times V_2) = X \times (U_2 \cap V_2) = X \times \emptyset = \emptyset$. ($x \in X \times U_2$ and $F_1 \times F_2 \subseteq X \times V_2$).

In both cases we have $X \times Y$ is semi-$\theta$-$R$-space.

$\Rightarrow$ Suppose that $X \times Y$ is semi-$\theta$-$R$-space, to prove $X$ and $Y$ are semi-$\theta$-$R$-space.
Let $x \in X$ and $F_1$ semi-$\theta$-closed; $x \notin F_1$ and $y \in Y$ and $F_2$ semi-$\theta$-closed; $y \notin F_2$.

$\Rightarrow (x, y) \in X \times Y$ and $F_1 \times F_2$ are semi-$\theta$-closed; $(x, y) \notin F_1 \times F_2$.
$\therefore X \times Y$ is a semi-$\theta$-R-space, then there exists $U_1 \times V_1, U_2 \times V_2$ semi-$\theta$-open, $(U_1 \times V_1) \cap (U_2 \times V_2) = \emptyset$. ($(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2) \subseteq F_1 \times F_2$).
Then there exists $U_1, U_2$ semi-$\theta$-open, $U_1 \cap U_2 = \emptyset$. ($x \in U_1$ and $F_1 \subseteq U_2$). Then $X$ is semi-$\theta$-$R$-space.

And there exists $V_1, V_2$ semi-$\theta$-open, $V_1 \cap V_2 = \emptyset$. ($y \in V_1$ and $F_2 \subseteq V_2$). Then $Y$ is semi-$\theta$-$R$-space.

Theorem 4.5
Assume that $(X, \tau_X)$ be a topological space, The space $X$ is semi-$\theta$-$R$ if and only if for each $x \in X$ and each $W \subseteq S\theta O(X, \tau_X)$, there exists $U \subseteq S\theta O(X, \tau_X)$ such that $x \in U \subseteq U_{\subseteq}^{\subseteq} \subseteq W'$.

Proof
Suppose that $X$ is semi-$\theta$-regular. Let $x \in X$, $W$ is semi-$\theta$-open; $x \in W$ Then $x \notin X - W$ and $X - W$ is semi-$\theta$-closed.

$\therefore X$ is semi-$\theta$-$R$. Then there is the semi-$\theta$-open $U, V$ set ; $U \cap V = \emptyset$, $(x \in U$ and $X - W \subseteq V)$

$\therefore U \cap V = \emptyset \Rightarrow U \subseteq X - V$ We have , $U \subseteq X - V$ and $X - V \subseteq W$

$\Rightarrow U^{s\theta} \subseteq X - V^{s\theta}$ \hspace{1cm} \text{[ Remark 2.8 (ii) ]}

$\Rightarrow U^{s\theta} \subseteq X - V$ \hspace{1cm} \text{[ Remark 2.8 (iii) ]}

$\Rightarrow U^{s\theta} \subseteq X - V$ and $X - V \subseteq W$. \hspace{1cm} \text{[ Remark 2.8 (i) ]}

$\iff \supseteq$ suppose the condition of theorem satisfy, to proof $X$ is semi-$\theta$-$R$-space.

Let $x \in X$ and $F$ semi-$\theta$-closed set in $X$, $x \notin F$

$\Rightarrow x \in X - F$, $X - F$ semi-$\theta$-open since $(F$ is semi-$\theta$-closed) $\therefore$ there exists $U$ is semi-$\theta$-open ; $x \in U \subseteq U^{s\theta} \subseteq X - F$ \hspace{1cm} ( by hypothesis) $\Rightarrow U^{s\theta} \subseteq X - F \Rightarrow F \subseteq X - U^{s\theta}$ \hspace{1cm} ( since $A \subseteq B \iff B^c \subseteq A^c$)

But, $X - U^{s\theta}$ semi-$\theta$-open since $U^{s\theta}$ semi-$\theta$-closed , say $X - U^{s\theta} = V$

$\Rightarrow x \in U \text{ and } U \subseteq V \text{ and } U \cap V = \emptyset$, since $(U \subseteq U^{s\theta}$ and $U^{s\theta} \cap X - U^{s\theta} = \emptyset \Rightarrow U \cap V = \emptyset)$

Then $X$ is semi-$\theta$-$R$-space.

**Definition 4.6**

Assume that $(X, \tau_X)$ be a topological space, we call $X$ is semi-$\theta$-normal (semi-$\theta$-$N$) iff every $F_1, F_2 \in S\theta C(X, \tau_X)$ with $F_1 \cap F_2 = \emptyset$, there exist $W_1, W_2 \in S\theta O(X, \tau_X)$ with $W_1 \cap W_2 = \emptyset$ such that $F_1 \subseteq W_1$, $F_2 \subseteq W_2$.

**Example 4.7**

Assume that $X = \{1, 2, 3\}$, $\tau_x = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$. Such that $S\theta O(X) = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1\}, \{2\}, \{1\}, \{2\}, \{1\}\}$.

Observe that $X$ is semi-$\theta$-$N$ but not semi-$\theta$-$R$.

**Remark 4.8**

In preview example notes that $(X, \tau_X)$ is not semi-$\theta$-$R$ and its semi-$\theta$-$N$ So that: (semi-$\theta$-$N \Rightarrow$ semi-$\theta$-$R$)

also (semi-$\theta$-$N \Rightarrow$ semi-$\theta$-$T_1$) and (semi-$\theta$-$N \Rightarrow$ semi-$\theta$-$T_2$)

Furthermore,

(semi-$\theta$-$R \Rightarrow$ semi-$\theta$-$N$) and (semi-$\theta$-$T_1 \Rightarrow$ semi-$\theta$-$N$) and (semi-$\theta$-$T_2 \Rightarrow$ semi-$\theta$-$N$) and (semi-$\theta$-$T_0 \Rightarrow$ semi-$\theta$-$N$).

**Theorem 4.9**

Assume that $(X, \tau_x)$ be a topological space, the space $X$ is semi-$\theta$-$N$ if and only if for each $s$-$\theta$ closed sub set $F \subseteq X$ and $s$-$\theta$-open set $W$ containing $F$, there exists an $s$-$\theta$-open set $U$ such that $F \subseteq U \subseteq A^{s\theta} \subseteq W$.

Proof.

Suppose $X$ is semi-$\theta$-$N$ and $F \subseteq X, F$ is semi-$\theta$-closed.

Let $W$ semi-$\theta$-open; $F \subseteq W \Rightarrow F \cap X - W = \emptyset$ and $X - W$ is semi-$\theta$-closed.

$\therefore X$ is semi-$\theta$-$N \Rightarrow \exists U, V$ are semi-$\theta$-open; $U \cap V = \emptyset$, $(F \subseteq U, X - W \subseteq V)$

$\Rightarrow X - V \subseteq W$

$\Rightarrow U \subseteq X - V$ since $U \cap V = \emptyset$.

$\Rightarrow U^{s\theta} \subseteq X - V^{s\theta} \Rightarrow U^{s\theta} \subseteq X - V$ \hspace{1cm} \text{[ by Remark 2.8 (iii) ]}

$\Rightarrow U \subseteq U^{s\theta} \subseteq X - V \Rightarrow F \subseteq U$ and $U \subseteq U^{s\theta} \subseteq X - V$ and $X - V \subseteq W$

$\Rightarrow F \subseteq U \subseteq U^{s\theta} \subseteq W$. 


Suppose the condition of theorem satisfy, to proof $X$ is semi-$\theta$-N.

Let $F, E$ are semi-$\theta$-closed; $F \cap E = \emptyset \Rightarrow F \subseteq X - E, X - E$ is semi-$\theta$-open.

$\Rightarrow$ there exists $U$ semi-$\theta$-open; $F \subseteq U \subseteq \overline{U^\theta} \subseteq X - E$

$\Rightarrow \overline{U^\theta} \subseteq X - E \Rightarrow U \subseteq X - \overline{U^\theta}$

But, $X - \overline{U^\theta}$ semi-$\theta$-open since $U^\theta$ semi-$\theta$-closed, say $X - \overline{U^\theta} = V$

$\Rightarrow E \subseteq V = X - \overline{U^\theta}$ and $F \subseteq U$ and $U \cap V = \emptyset$ (since $U \subseteq \overline{U^\theta}$ and $\overline{U^\theta} \cap X - \overline{U^\theta} = \emptyset$

$\Rightarrow U \cap V = \emptyset$). Hence $X$ is semi-$\theta$-N.

5. Some New Separation Axioms

**Definition 5.1**

Assume that $(X, \tau_X)$ is a topological space. Let $A \subset X$ we say that $A$ is semi-$\theta$-Difference (resp. $\theta$-Difference) set if there are $U, V \in S\theta O(X, \tau_X)$ (resp. $U, V \in \theta O(X, \tau_X)$) such that $U \neq X$ and $A = U - V$.

**Remark 5.2**

(i) Every semi-$\theta$-open set $G \neq X$ is a semi-$\theta$-D-set since $G = G - \emptyset$.

(ii) Every $\theta$-open set $G \neq X$ is a $\theta$-D-set since $G = G - \emptyset$.

(iii) The class of all semi-$\theta$-D (\$\theta$-D) sets in $X$ will be denoted by $S\theta D(X, \tau_X)$ (resp. $\theta D(X, \tau_X)$).

**Definition 5.3**

Assume that $(X, \tau_X)$ is a topological space, we say $X$ is semi-$\theta$-$D_0$ (resp. $\theta$-$D_0$) if there exists $W \in S\theta D(X, a)$ (resp. $W \in \theta D(X, a)$) with $b \notin W$, for all $a, b \in X$ such that $a \neq b$.

**Theorem 5.4**

Assume that $(X, \tau_X)$ be a topological space. That a space $X$ is semi-$\theta$-$D_0$ iff $X$ is semi-$\theta$-$T_0$.

**Remark 5.5**

Any $\theta$-$D_0$ space is also a semi-$\theta$-$D_0$-space, but this is not always the case.

**Definition 5.6**

Assume that $(X, \tau_X)$ be a topological space, then $X$ is called semi-$\theta$-$D_1$ (resp. $\theta$-$D_1$) if there exists $W, U \in S\theta D(X, \tau_X)$ (resp. $W, U \in \theta D(X, \tau_X)$) with $a \in W$ but $b \notin W, b \in U$ but $a \notin U$ for all $a, b \in X$ such that $a \neq b$.

**Theorem 5.7**

Let $(X, \tau_X)$ be a topological space, then the space $X$ is semi-$\theta$-$D_1$ if and only if $X$ is semi-$\theta$-$T_1$.

**Definition 5.8**

Assume that $(X, \tau_X)$ be a topological space, then $X$ is called semi-$\theta$-$D_2$ (resp. $\theta$-$D_2$) if there exists $W, U \in S\theta D(X, \tau_X)$ (resp. $W, U \in \theta D(X, \tau_X)$) with $a \in W$ but $b \notin W, b \in U$ but $a \notin U$ such that $W \cap U = \emptyset$, for all $a, b \in X, a \neq b$.

**Theorem 5.9**

Let $(X, \tau_X)$ be a topological space, then $X$ is called semi-$\theta$-$D_2$ iff $X$ is semi-$\theta$-$T_2$.

**Remark 5.10**

1) Every $\theta$-$D_2$ is $\theta$-$D_1$.
2) Every semi-$\theta$-$D_2$ is semi-$\theta$-$D_1$.
3) Every semi-$\theta$-$T_2$ is semi-$\theta$-$D_2$.
4) Every \( \theta-T_2 \) is \( \theta-D_2 \).

**Theorem 5.11**
If \( f: (X, \tau_x) \to (Y, \tau_y) \) is a surjective function, \( s-\theta \)-irresolute and \( S \) is a semi-\( \theta \)-D-set in \( Y \), then \( f^{-1}(S) \) is a semi-\( \theta \)-D-set in \( X \).

**Proof.**
Assume that \( S \in SBD(Y, \tau_y) \). Then exists \( U, V \in SBD(Y, \tau_y) \) such that \( S = U \cap V \) and \( U \neq Y \).
Since \( f \) is \( s-\theta \)-irresolute, implies \( f^{-1}(U), f^{-1}(V) \in SBD(X, \tau_x) \). Since \( U \neq Y \), we get \( f^{-1}(U) \neq X \). Clearly \( f^{-1}(S) = f^{-1}(U) \cap f^{-1}(V) \).
Observe that \( f^{-1}(S) \) is a semi-\( \theta \)-D-set in \( X \).

**Theorem 5.12**
Let \( f: (X, \tau_x) \to (Y, \tau_y) \) bijective, \( s-\theta \)-irresolute. If \( Y \) is semi-\( \theta \)-\( D_1 \), then \( X \) is semi-\( \theta \)-\( D_1 \).

**Proof.**
Assume that \( Y \) is semi-\( \theta \)-\( D_1 \) space. We will two show that \( X \) is semi-\( \theta \)-\( D_1 \).
Let \( x, y \in X \) such that \( x \neq y \). Since \( Y \) is semi-\( \theta \)-\( D_1 \) and \( f \) is injective, implies \( S_x \) and \( S_y \) are semi-\( \theta \)-D-sets of \( S \) that contain \( f(x) \) and \( f(y) \), respectively, such that \( f(x) \notin S_y \), \( f(y) \notin S_x \), by the [Theorem 5.11], \( f^{-1}(S_x), f^{-1}(S_y) \in SBD(X, \tau_x) \). As a result, \( X \) is a semi-\( \theta \)-\( D_1 \) space.

**Remark 5.13**
The following diagram shows the relation between \( \theta-T_0 \), \( \theta-T_1 \), \( \theta-T_1 \), semi-\( \theta \)-\( T_0 \), semi-\( \theta \)-\( T_1 \), semi-\( \theta \)-\( T_2 \), semi-\( \theta \)-\( D_0 \), semi-\( \theta \)-\( D_1 \), semi-\( \theta \)-\( D_2 \) spaces:

```
  --  --  --
semi--  semi--  semi--
semi--  semi--  semi--
```

**Conclusion**
In this paper it was concluded that there is a relationship between the known axioms of separation \( T_0 \)-space, \( T_1 \)-space, \( T_2 \)-space and the axioms of separation of type semi-\( \theta \)-\( T_0 \)-space, semi-\( \theta \)-\( T_1 \)-space, semi-\( \theta \)-\( T_2 \)-space. There is also a relationship between the separation axioms of type semi-\( T_0 \)-space, semi-\( T_1 \)-space, semi-\( T_2 \)-space and the separation axioms of type semi-\( \theta \)-\( T_0 \)-space, semi-\( \theta \)-\( T_1 \)-space, semi-\( \theta \)-\( T_2 \)-space.

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