A Formulation of Lattice Gauge Theories for Quantum Simulations

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We examine the Kogut-Susskind formulation of lattice gauge theories under the light of fermionic and bosonic degrees of freedom that provide a description useful to the development of quantum simulators of gauge invariant models. We consider both discrete and continuous gauge groups and adopt a realistic multi-component Fock space for the definition of matter degrees of freedom. In particular, we express the Hamiltonian of the gauge theory and the Gauss law in terms of Fock operators. The gauge fields are described in two different bases, based on either group elements or group representations. This formulation allows for a natural scheme to achieve a consistent truncation of the Hilbert space for continuous groups, and provides helpful tools to study the connections of gauge theories with topological quantum double and string-net models for discrete groups. Several examples, including the case of the discrete $D_3$ gauge group, are presented.

I. INTRODUCTION

The Kogut-Susskind Hamiltonian formulation [1] of lattice gauge theories [2] is experiencing a renewed interest driven both by major developments in numerical techniques, in particular based on the study of tensor networks [3, 4], and by groundbreaking experimental and theoretical achievements in quantum simulation, especially in the field of cold atomic gases trapped in optical lattices (see, for example, [5]).

On the quantum simulation side, many experimental successes have been achieved recently, including the realization of artificial static gauge potentials for both atoms trapped in harmonic potentials [6] and in optical lattices [7–9], as well as the observation of Higgs modes in two-dimensional systems [10].

Furthermore, the fast developments in the control of the interactions among atoms in an optical lattice (as well as other systems, such as trapped ions and superconducting circuits) envision the possibility of obtaining, in the near future, quantum simulations of both Abelian and non-Abelian lattice gauge theories [11–22]. This is of particular interest, for example, for solving problems involving fermions with finite chemical potential (for example, as expected in exotic phases of QCD, such as quark-gluon plasma and color superconductivity [23]). The Euclidean lattice Monte-Carlo simulations of these encounter the sign-problem [25] which is avoided in the framework of quantum simulations, by the replacement of Grassman variables by real fermions. Moreover, quantum simulations allow for real-time dynamics observation, as the Hamiltonian theory takes place in Minkowski spacetime, unlike the statistical correlations obtained in the Euclidean approach.

Concerning the numerical and analytical study of lattice gauge theories, tensor networks allowed the investigation of the spectral properties of the $1 + 1$ dimensional Schwinger model with precisions comparable with the best results available from other techniques [26–30] and, more in general, provide new tools to examine gauge invariant states and their dynamics in higher dimensions as well [31–33].

To the purpose of obtaining a realistic lattice gauge model that can be experimentally implemented with cold atoms, it is useful to embed the Kogut-Susskind Hamiltonian in a lattice model for multi-component fermions and bosons that realizes, as simply as possible, the required local gauge symmetry under a gauge group $G$ whose phenomenology is at quest. To accomplish this task, one of the main requirements is the possibility of building the model starting from a limited number of local degrees of freedom. In particular, we focus on characterizing the matter degrees of freedom in terms of fermionic operators which define a Fock space on each lattice vertex and describe multi-component fermions like the ones customarily used in cold atoms experiments, for example in the context of lanthanide atoms like Ytterbium [34] or Erbium [35] presenting several nuclear hyperfine states which can be addressed separately. This method has already been used in several proposals for quantum simulations of lattice gauge theories [15, 18, 21], applying approaches such as the prepotential formalism [36, 37] and the link model [38–41], in which the gauge degrees of freedom are composed out of bosons or fermions, respectively. In both these approaches, the link is divided into “left” and “right” parts, with two families of such fundamental ingredients. Our approach, on the other hand, suggests considering the link as a whole piece, allowing for some simplification. It also suggest a consistent way of truncating the gauge invariant Hilbert space in a gauge invariant manner.

The purpose of this paper is therefore to review and reformulate the formalism of the Kogut-Susskind theory in terms of realistic fermionic and bosonic degrees of freedom. Our aim is to define the key ingredients for lattice gauge theories, with either continuous or discrete groups, in the atomic and many-body physics perspective: what must a model of interacting atoms on a lattice fulfill to properly manifest local gauge invariance?

The Fock space we adopt to describe the matter fields provides a simple platform to describe matter coupled to either finite (discrete) or continuous gauge groups, including truncated ones. In particular, we require that the internal degrees of freedom of the fermions provide a sufficient number of states to realize the smallest faithful representation of the considered group already at the single-particle level: the single-particle Hilbert space associated to a matter site must have dimension equal to that of the desired representation of the gauge group, usually the fundamental one.

With respect to the gauge field, instead, we will consider a Hilbert space whose dimension equals the group order in the
case of finite groups; whereas for continuous groups we will define an efficient truncation based on its irreducible representations (Theoretically, one could include all the possible representations by using an infinite number of bosonic modes. This should not impose a theoretical difficulty, as quantum field theories are usually described in terms of an infinite number of modes).

The use of a Fock space for the matter suggests a different implementation of the Kogut-Susskind Hamiltonian with respect to previous proposals aimed at simulating pure lattice gauge theories with discrete groups in Josephson junction arrays [42, 44] or in the context of Majorana zero energy modes [45, 46] and parafermions [47] in topological superconductors. In these superconducting setups the charge excitations (meant as violations of the gauge constraint for the pure gauge theory) present indeed a cyclic structure in their local Hilbert space, due to the presence of a Cooper pair condensate. And the same feature appears in the study of matter-coupled gauge theories based on Majorana modes [48]. This structure is well suited to deal with cyclic groups but it is unfeasible for cold atom implementations.

Concerning the gauge bosons, instead, the weak coupling regime of the Kogut-Susskind Hamiltonian for finite groups in the weak interaction limit can be interpreted as a topological order regime of the Kogut-Susskind Hamiltonian for finite groups at the weak coupling limit [49]. This change of basis may be seen as a generalized, non-Abelian Fourier transform, mapping between the group element states) is the group's order, and an identifier within the representation, and an identifier within the representation, presentation based on Majorana modes [43]. This structure is well suited to deal with cyclic groups but it is unfeasible for cold atom implementations.

Throughout this work, the Einstein summation convention (summation on double indices) is used, for all indices but the momentum index. That means that the body-frame angular momenta. Let us go back to the rigid rotator problem. Recall that a rigid rotator’s Hilbert space is three dimensional, and may be described either by three coordinates - the Euler angles $\alpha, \beta, \gamma$, parameterizing the configuration space or simply the $SU(2)$ group elements, or by three integers - $j, m, n$, where $m$ is the eigenvalue of the $z$ component of the angular momentum in the space frame - $J_z$, $n$ is the eigenvalue of the $z$ component of the angular momentum in the body frame - $J_b^z$, and $j$ is the total angular momentum quantum number, shared by both frames - $J^2 = (J^z)^2 = (J_b^z)^2$. The angular momentum operators satisfy the algebra $[1, 53]$: $J_i^a J_j^b = i \hbar \delta^{ab} \delta_{ij} J_k^a$ with $\alpha, \beta = s, b$ and $i, j, k = 1, 2, 3$ or $x, y, z$.

The $D$ matrices are merely the wavefunctions $[53, 54]$:

$$\langle \alpha, \beta, \gamma | jmn \rangle = \frac{1}{\sqrt{2j + 1}} \frac{1}{2\pi} D_{jm}^{\alpha \beta \gamma}$$  

This change of basis may be seen as a generalized, non-Abelian Fourier transform, mapping between the Representation States $| jmn \rangle$ and Group Element States $| \alpha, \beta, \gamma \rangle$.

One may generalize this relation to any group $G$ by replacing $\alpha, \beta, \gamma$ by the general $g \in G$, with

$$\langle g | jmn \rangle = \frac{\sqrt{\dim (j)}}{|G|} D_{jm}^{g} (g)$$

where the normalization follows from the great orthogonality theorem, $\dim (j)$ is the dimension of the $j$th representation and $|G|$ is the order of $G$.

We can check consistency for finite groups by counting the dimensions of both spaces. The number of representation states $| jmn \rangle$ is $\sum_j \dim (j)^2$. The number of group elements (or, group element states) is the group’s order, $|G|$. According to a theorem in the theory of finite groups, these two are equal $[55]$.

Finally, let us consider the analogs of the space and body angular momenta. Let us go back to the rigid rotator problem. Denote by $U^{j}$ a rotation matrix (group element) expressed in the representation $j$. This satisfies the following commutation relations with the angular momentum operators: $U^{j}_{mn} J^i_j = \delta_{jk} U^{j}_{mn}$ and $U^{j}_{nm} J^i_j = U^{j}_{mk} (T^i_k)(j)$, with $T^i_k$ being the $j$th representation of the $i$ component of angular momentum. That means that the body-frame angular momenta generate left transformations on group elements, and the ones

$$D^{\alpha \beta \gamma}_{jm} (g) = \langle jm | \Theta_g | jn \rangle$$  

In other words -

$$D^{j}_{mn} (g) = \langle jm | \Theta_g | jn \rangle$$

The paper is organized as follows: in section II, we shall introduce a lattice theory containing only matter (fermions) with a global gauge invariance. This gauge element states) is the group's order, and an identifier within the representation, and an identifier within the representation, presentation based on Majorana modes [43]. This structure is well suited to deal with cyclic groups but it is unfeasible for cold atom implementations.

Let $G$ be any finite or compact Lie group. We denote its elements by $g \in G$. These may be represented by matrix representations labeled by $j$, denoted as $D^j (g)$. We restrict ourselves to unitary representations, i.e. such that satisfy:

$$D^j_{nm} (g^{-1}) = D^j_{nm} (g)$$  

Next, we define the ”group states”. These are labeled by the representation $j$ and an identifier within the representation, $m$ - both may be sets of indices. These states transform (”undergo rotations”) according to their representation. If we denote the unitary operator corresponding to a group element by $\Theta_g$, then

$$\Theta_g | jm \rangle = D^j_{nm} (g) | jn \rangle$$
of the space-frame - right ones. Recall that the body and space
algebras had opposite signs for the structure constants, and if
we wish to have the same sign in both of them, we can simply
define $R_i \equiv J_i^r$ and $L_i \equiv -J_i^b$. Then, generalizing for a
compact $G$ with structure constants $f_{abc}$ we get that the left and
right generators $L_i, R_i$, satisfy the group’s algebra
\[
\begin{align*}
[L_i, L_j] &= i f_{abc} L_c \\
[R_i, R_j] &= i f_{abc} R_c \\
[L_i, R_j] &= 0
\end{align*}
\]  
(6)
as well as
\[
\begin{align*}
[U_{im}, L_i] &= - (T_i^j)_{mk} U_{kn}^j \\
[U_{im}, R_i] &= U_{mk}^j (T_i^j)_{kn}
\end{align*}
\]  
(7)
where the change of sign has been absorbed. The algebra and
the commutation relations with the group elements may be
proven from one another using the Jacobi identity.
Next, we may define the finite transformation operators,
$\Theta^L = \phi^{\alpha L}$ and $\Theta^R = \phi^{\alpha R}$, such that
\[
\begin{align*}
\Theta^L_{im} U^j_{mn} \Theta^L_{kn} &= D_{mk} \left( g^{-1} \right) U^j_{kn} \\
\Theta^R_{im} U^j_{mn} \Theta^R_{kn} &= U^j_{mk} D_{kn} \left( g \right)
\end{align*}
\]  
(8)
This relation holds for a finite $G$ as well, where there are no
infinitiesmial transformations. Thus, this can be taken as the
most general rule for left and right transformations, for any $G$.

III. FERMIONS AND GLOBAL GAUGE THEORIES

A. Symmetries and the Hamiltonian

We begin, as usual, with a theory of fermions on a square
lattice, having a global gauge symmetry with respect to the
group $G$. On each vertex $n$ of the lattice (see Figure [I]), we
define a spinor $\psi_n$. The spinor belongs to a given faithful rep-}
resentation (say, the $j$th) of $G$, and thus has dim $(j)$ components
$\psi_n^j$, which we shall call group, gauge or color components.
Terms like $\psi_n^a \psi_m^b$, in which the group indices are not explicit-
ly written, should be understood as scalar products in group
space: $\psi_n^a \psi_m^b = \sum_i \overline{\psi}_n^i \psi_m^i$. On the other hand, the physical
index $n$ may be neglected when on-site properties are addressed,
as we shall do first.

A Gauge transformation on a fermionic operator, with re-
spect to the group element $g$, is the result of acting with the
operator $\Theta^G_{i,j}$, and is defined as
\[
\begin{align*}
\Theta^G_{i,j} \psi_n^j & = \psi_n^j D^{i,j}_{n a} (g) \\
\Theta^G_{i,j} \psi_n^a & = D^{i,j}_{a b} (g^{-1}) \psi_n^b.
\end{align*}
\]  
(9)

Hence we can construct a general globally gauge invariant
Hamiltonian with nearest-neighbor interactions,
\[
H = \sum_n M_n \psi_n^a \psi_n^a + \sum_{n, \hat{k}} \left( \epsilon_{n, \hat{k}} \psi_n^a \psi_{n+\hat{k}}^a + H.c. \right)
\]  
(10)
where $k$ runs over the dimensions ($\hat{k}$ are the lattice vectors).
One may, of course, generalize, and include many represen-
tations (types of spinors), as long as all the group indices are
contracted properly.

B. The Transformation Operators

**Proposition 1.** The transformation operators can be written
explicitly in terms of the fermionic operators. Using
\[
q^j (g) = -i \log \left( D^j (g) \right)
\]  
(11)
one may define
\[
\Theta^G_{i,j} = e^{q^j \delta_{i,\alpha} \delta_{g,\beta} \psi_n^\alpha} \det (g^{-1})^N
\]  
(12)
where $N = 0$ for a vertex in the even sublattice and $N = 1$ for
the odd one, and $\det (g^{-1}) \equiv \det \left( D^j (g^{-1}) \right)$ (with $j$ being the
fermionic matter’s representation).

The part with the determinant is related to the staggering of
fermions, and will be shortly explained. But first, let us prove
that this, indeed, generates the required transformation.

**Proof.** First, note that as we consider unitary representations,
$\det (g^{-1})^* = \det (g^{-1})^{-1}$, and thus the determinant part
will not contribute to the calculation of $\Theta^G_{i,j} \psi_n^a \Theta^G_{i,j}$. Thus,
we define $\Theta^G_{i,j} \equiv \Theta^G_{i,j} \det (g^{-1})^N$ and proceed without the
determinant.

Second, we wish to prove that $\Theta^G_{i,j}$ is, indeed, a unitary op-
erator. For that, it is sufficient to check that the exponent in
Eq. (12), is anti-Hermitian. Note that both $\Theta^G_{i,j}$ and its expon-
et are number conserving. Thus, it is sufficient to check matrix elements of the local fermionic Fock space. Since
the representation matrices are unitary, their logarithm is anti-Hermitian and \( q_{ab} = q_{ba} \). Denote \( A_{ab} = i q_{ab} \). This is anti-Hermitian: \( A_{ab} = -A_{ba}^* \). Now we turn to the calculation of matrix elements.

Let us denote by \( \langle I, n \rangle = \psi_{i_1}^\dagger \ldots \psi_{j_n}^\dagger (\Omega) \) an element of the basis of the Fock space for the fermions in a given lattice site, where \( |\Omega\rangle \) is the local vacuum state, \( n \leq \dim (j) \) and we assume that all the participating indices are different. Then,

\[
\langle I, n | \psi_a^\dagger A_{ab} \psi_b | I, n \rangle = 
- A_{ab} \langle \Omega | \psi_{i_1} \ldots \psi_{j_n} \psi_{i_1}^\dagger \ldots \psi_{j_n}^\dagger | \Omega \rangle + 
A_{ab} \langle \Omega | \psi_{i_1} \ldots \psi_{j_n} \psi_{i_1}^\dagger \ldots \psi_{j_n}^\dagger | \Omega \rangle = 
- A_{ab} \delta_{b_1 \ldots b_n} + A_{ab} \delta_{a_1 \ldots a_n} 
\]

where \( \delta_{b_1 \ldots b_n} \) is the generalized Kronecker delta, equal to 0 if there are repeating indices in the lower/upper sets or if the two sets are different, and to \( \pm 1 \) if the upper sequence is respectively an even or odd permutation of the lower one. On the other hand, and similarly,

\[
\langle J, n | \psi_a^\dagger A_{ab} \psi_b | I, n \rangle = 
- A_{ab} \langle \Omega | \psi_{j_1} \ldots \psi_{j_n} \psi_{i_1}^\dagger \ldots \psi_{j_n}^\dagger | \Omega \rangle + 
A_{ab} \langle \Omega | \psi_{j_1} \ldots \psi_{j_n} \psi_{i_1}^\dagger \ldots \psi_{j_n}^\dagger | \Omega \rangle = 
- A_{ab} \delta_{a_1 \ldots a_n} + A_{ab} \delta_{b_1 \ldots b_n} 
\]

But now, using the fact that \( A \) is anti-Hermitian, we get

\[
\langle J, n | \psi_a^\dagger A_{ab} \psi_b | I, n \rangle = - A_{ab} \delta_{a_1 \ldots a_n} + A_{ab} \delta_{b_1 \ldots b_n} = 
- A_{ab} \delta_{b_1 \ldots b_n} + A_{ab} \delta_{a_1 \ldots a_n} = A_{ab} \delta_{a_1 \ldots a_n} - A_{ab} \delta_{b_1 \ldots b_n} = 
- \langle I, n | \psi_a^\dagger A_{ab} \psi_b | I, n \rangle^* 
\]

and thus the exponent is anti-Hermitian and \( \Theta_\theta^{O,j} \) is unitary. Besides the unitarity requirement, it is also helpful from a technical aspect, since now the inverse operator is easy to calculate using the Campbell-Baker-Hausdorff formula, resulting in

\[
\Theta_\theta^{O,j} \psi_a^\dagger \Theta_\theta^{O,j} = \psi_b^\dagger (e^{\theta \theta})_{ab} = \psi_b^\dagger D_{ba} (g) 
\]

as required, and this completes the proof. □

C. Staggered Fermions

Now let us consider the staggering. Staggered fermions have been suggested by Kogut and Susskind [11][56] as a method to solve the problem of fermionic doubling in the continuum limit of lattice theories. It involves the decomposition of continuum spinors into several lattice sites, such that, for example, in the case of two spin-component spinors (disregarding the gauge components, which are not affected by staggering) the even sites correspond to particles and the odd ones to anti-particles. This allows to define a lattice analogy of the Dirac-Sea, which is a state in which all the even (particle) vertices are empty, while the odd (anti-particle) vertices are full.

A meaningful staggered fermions prescription must fulfill three requirements:

1. The masses have to change alternately - i.e., \( M_n = (-1)^n M \). This contributes to the "Dirac Sea" picture in terms of the masses.

2. The charges may change alternately as well, depending on the gauge group.

3. The tunneling coefficients \( \epsilon \) are dictated by the continuum limit requirements, in order to obtain Dirac equation. This, of course, applies only for theories for which a continuum limit is expected - i.e., when \( G \) is continuous.

Let us see how the second requirement is fulfilled by the definition of the \( \Theta_\theta^{O,j} \) operators. Consider an empty vertex, \( |\Omega\rangle \). This is not affected by \( \Theta_\theta^{O,j} \), and thus its transformation law is

\[
|\Omega\rangle \rightarrow \det (g^{-1})^N |\Omega\rangle 
\]

That means that an empty even vertex is invariant under the transformation (no particles - no charges), but an empty odd vertex is multiplied by \( \det (g^{-1}) \) - i.e., a conjugate-trivial transformation law.

On the other hand, what happens for a completely full site? We expect the complete opposite. Suppose \( \dim (j) = n \), and define the state

\[
|n\rangle = \frac{1}{n!} \epsilon_{i_1 \ldots i_n} \psi_{i_1}^\dagger \ldots \psi_{i_n}^\dagger |\Omega\rangle 
\]

**Proposition 2**. Under a gauge transformation,

\[
|n\rangle \rightarrow \det (g) \det (g^{-1})^N |n\rangle 
\]

**Proof.** Let us neglect the staggering for a while, and consider only even vertices. We have:

\[
\det (g) |n\rangle = \frac{1}{n!} \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} D_{i_1 j_1} \cdots D_{i_n j_n} \psi_{i_1}^\dagger \ldots \psi_{i_n}^\dagger |\Omega\rangle 
\]

where \( |i\rangle, |j\rangle \) and \( |k\rangle \) are three permutations of the gauge indices and the matrix \( D \) is the representation of the group element \( g \) in the faithful representation of the fermions.

We use the identity \( \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} \delta_{k_1 \ldots k_n} = \epsilon_{k_1 \ldots k_n} \), to eliminate two Levi-Civita symbols. Summing over \( |k\rangle \) there are \( n! \) summands, and the generalized delta is either 1 or \(-1\) for each of them according to \( |k\rangle \) and \( |j\rangle \) being even or odd permutations one of the other. Accounting also for the anticommutation relations of the \( \psi^\dagger \) we obtain:

\[
\frac{1}{n!} \delta_{k_1 \ldots k_n} \psi_{j_1}^\dagger \ldots \psi_{j_n}^\dagger |\Omega\rangle = \psi_{j_1}^\dagger \ldots \psi_{j_n}^\dagger .
\]
Substituting this relation in the Eq. (20) we have:

$$\det (g) |n\rangle = \frac{1}{n!} \epsilon_{i_1 \ldots i_n} D_{i_1 j_1} \cdots D_{i_n j_n} |\Omega\rangle$$ (21)

which is exactly the result of a transformation on each fermionic operator separately. For odd vertices just multiply by the missing determinant. □

Thus, we get that a fully occupied even vertex undergoes the transformation as an empty odd vertex and vice versa - as expected from the Dirac sea picture.

As another example of the charge conjugation due to staggering, consider a site occupied by a single fermion,

$$|a\rangle \equiv \psi_a^\dag |\Omega\rangle \quad (22)$$

There we get

$$|a\rangle \rightarrow D_{ba}^\dagger (g) \det (g^{-1})^N |b\rangle \quad (23)$$

This has to be compared to the state $|a'\rangle \equiv \psi_a |n\rangle$, for which:

$$|a'\rangle \rightarrow D_{ab}^\dagger (g) \det (g^{-1})^N |b'\rangle = D_{ba}^\dagger (g) \det (g)^{-N} |b'\rangle \quad (24)$$

where the charge conjugation may be explicitly seen.

D. Some Examples

As final remarks for this section, we shall consider examples for several gauge groups.

Let us consider first the smallest discrete non-Abelian group, the Dihedralsal group $D_3 = S_3$, corresponding to the symmetry group of the equilateral triangle or, equivalently, to the permutations of three elements. The group has six elements and the only possible faithful irreducible representation has dimension 2. In this case, $\det (g) = \pm 1$ and we obtain that a doubly occupied odd site, as well as an empty even site, undergo the transformation in the trivial representation (without a determinant), whereas the opposite cases undergo it in the parity representation (with a determinant).

Another important case is the one of compact Lie groups. Consider $SU(N)$ or $SU(N)$ for example, for which infinitesimal transformations can be defined, and the group representations can be written in terms of Hermitian generators $T^j$ in the form $D_{ab} (g) = e^{i \alpha_q T^j}$. In this way we can define:

$$q_{ab}^j = \alpha_q \cdot T_{ab}^j \quad (25)$$

and one can easily extract the Hermitian gauge charge $Q^j_i$, defined (as usual) by:

$$\Theta_q^{Q,j} = e^{i \alpha_q Q^j} \quad (26)$$

For $SU(N)$ all the determinants are 1, thus the staggering plays no role in the charge definition, and we get that the charges are

$$Q^j_{SU(N)} = \psi_a^\dagger T_{ab}^j \psi_b \quad (27)$$

as in [11]. This defines a fermionic generalization of the Schwingera√\text{57}. For example, for $SU(2)$, where in the fundamental representation $j = 1/2$ we have $T^{j=1/2} = \sigma^2/2$, it is easy to check that for both a full and an empty vertex, the charge is zero (a singlet state), in accordance with the fact that there is only one trivial representation.

For $U(1)$, on the other hand (or the Abelian subgroup of $U(N)$), we have:

$$\Theta_q^{Q,j} = e^{i (q^j \psi - \bar{\psi})} \quad (28)$$

This results in the well-known Abelian staggered charge:

$$Q_{U(1)} = \psi^\dagger \psi - \frac{1}{2} (1 - (-1)^N) \quad (29)$$

Thus, for $U(N)$ there are both Abelian and non-Abelian charges corresponding to the $Q_{U(1)}$ and $Q_{SU(N)}$ presented above.

IV. LOCAL GAUGE SYMMETRY: THE GAUGE FIELDS

A. Inclusion of a gauge Field

We now have all the ingredients for lifting the gauge symmetry to be local. For that, one simply has to introduce ”connections” - group elements of the form $U^j$ presented earlier on every link (see Figure [1]), where the representation $j$ corresponds to the one chosen for the fermions. The modified Hamiltonian is thus

$$H = \sum_n M_n \psi^\dagger_n \psi_n + \sum_{n,k} \left( \epsilon_n, \epsilon_k \psi^\dagger_n U_{n,k}^j \psi_{n+k} + H.c. \right) \quad (30)$$

where summation on internal (group) indices is, again, assumed:

$$\psi^\dagger_n U_{n,k}^j \psi_{n+k} = \left( \psi^\dagger_n \right)_{a} U_{n,k}^j \left( \psi_{n+k} \right)_b \quad (31)$$

The staggering procedure is unaffected by this modification.

The operators $U^j$ act on the links of the square lattice which are characterized by a Hilbert space identical to the one described in section [11] on each link there is a ”generalized rigid rotator”, playing the role of the gauge field in the theory. Thus it is easy to verify that this model is, indeed, local gauge invariant - i.e., the gauge transformation involves different group elements in different vertices of the lattice. Define on each vertex the transformation

$$\Theta_{\sigma} = \prod_{i} \Theta_q^{\sigma}_{\sigma_i} \prod_{i} \Theta_q^{\sigma}_{i} \Theta_q^{\sigma}_{\bar{\sigma}} \quad (32)$$

with $i, \sigma$ being the links ingoing and outgoing to/from the vertex. This is the generator of gauge transformations. Acting on a $U^j$ operator with $g$ on the left and $h$ to the right results in

$$U_{mn}^j \rightarrow D_{nm}^j (g^{-1}) U_{mn}^j D_{nm}^j (h) \quad (33)$$
Combining it with the fermionic part, which we already know, we see that this Hamiltonian is, indeed, gauge invariant.

Thus, since for each vertex \( n \) we have \( [H, \Theta_{g,n}] = 0 \), \( \Theta_{g,n} \) is a local symmetry, decomposing the Hilbert space of the theory into separate sectors, corresponding to eigenvalues of these operators. These sectors correspond to static charge configurations. Let us set them all to 1, i.e., for every \( n \), any physical state \( |\text{phys}\rangle \) is invariant by the local transformations:

\[
\Theta_{g,n} |\text{phys}\rangle = |\text{phys}\rangle \tag{34}
\]

This is the Gauss’s law which defines the physical, gauge-invariant subspace of the Hilbert space. In case of continuous groups, one may rephrase the Gauss’s law in terms of generators and charges,

\[
G_n = \sum_i R_{g,i} + \sum_o L_{g,o} - Q_n \tag{35}
\]

where the generators \( L \) and \( R \) fulfill the algebra relations \( [L, R] = 0 \) and \( Q \) in defined in \( (25) \). Eq. \( (35) \) constitutes the Gauss’s law for all the charges in the theory and, in particular, in the case of Abelian charges, one can easily verify that \( R = -L \), and thus a discrete version of the divergence of the generators is obtained. Therefore, we may identify the group generators with an electric field, whereas their conjugate variables constitute the vector potentials. Having both right and left generators in non-Abelian groups thus corresponds to having right and left electric fields, due to the fact that in such theories the link - or, in other words, the gauge field - is charged under the gauge group, unlike in Abelian theories (compare the charged \( SU(3) \) gluons with the chargeless \( U(1) \) photon, for example).

Generalizing this to any valid group \( G \), we deduce that the representation basis is appropriate for describing charges and electric fields, while the group element basis is most suited, for example, for the discussion of magnetic vortices in topological models.

B. The gauge field dynamics

To complete the picture, we wish to make the gauge field dynamic. This is done in the usual way, by adding the simplest local and gauge-invariant terms, involving the gauge field only, to the Hamiltonian. The first term it is possible to introduce is constituted by string operators which change the state of the gauge bosons on a closed path by adding a loop of electric flux; in particular these terms may be written in terms of Wilson loops - traces of products of group elements \( U^j \). Locality leads us to choose the smallest closed paths to obtain operators acting on each lattice plaquette \( p \). With these operators we define the magnetic part of the Hamiltonian,

\[
H_B = \frac{1}{2g^2} \sum_p \left( \text{Tr} \left( U_1^j U_2^j U_3^j U_4^j \right) + H.c. \right) \tag{36}
\]

where the numbers 1 - 4 are taken according to the plaquette orientation convention presented in Figure 1. The representation \( j \) is the same along all links (otherwise the matrix product in group space is not defined). In \( SU(N) \) theories, for example, one usually chooses the fundamental representation, but other choices are possible too. For the group \( D_3 \), instead, there is only one faithful irreducible representation and thus the matrices (and the matter fields) are in it.

\( g \) is the gauge field coupling constant (such as the electron charge for quantum electrodynamics). The name “magnetic Hamiltonian” is due to the fact that this term generates the magnetic energy in the continuum limit for compact Lie groups. This is a well-known fact, and we shall review it explicitly in the group element representation later on.

Furthermore, the plaquette operator \( W = U_1^j U_2^j U_3^j U_4^j \) plays the role of an Aharonov-Bohm transformation that a fermion, moving in a closed loop around the plaquette, undergoes. This allows an interpretation of the eigenvalues of \( W \) in terms of the presence of different magnetic vortices on the plaquette. In particular the previous plaquette operator commutes with the Gauss’s law, therefore, it is possible to describe the physical subspace of the theory in terms of the eigenstates of all the plaquette operators \( W \); namely, the eigenvalues of \( \text{Tr}W \) represent all the possible magnetic charges in the theory which correspond to the conjugacy classes of the group \( G \).

The magnetic terms in the Hamiltonian constitute a self interacting term for the gauge field, but it is not sufficient to define its dynamics, since the operators \( W \) commute with both the Gauss’s law and the tunneling terms \( (31) \).

The dynamics of the field must be introduced by adding in the Hamiltonian interactions that depends on the conjugate variables of group elements - which are the representations. By considering single-link terms only, the most general interaction of this kind may be written in the form

\[
H_E = \frac{g^2}{2} \sum_j \sum_{\text{links}} E(j) \Pi_j \tag{37}
\]

where \( \Pi_j \) is the projector onto the \( j \) representation, and \( E(j) \) is a function of the representation only. \( H_E \) commutes with the Gauss’s law but does not commute with the magnetic part, thus providing a non-trivial dynamics to the gauge field also in the absence of matter.

\( H_E \) can be regarded as the electric part of the Hamiltonian. In order to see that, consider \( G = SU(N) \), for example. Then a proper choice would be \( E(j) = C_2(j) \) - i.e., \( H_E \) is a sum of local quadratic Casimir operators (for \( SU(2) \), e.g., \( E(j) = j(j + 1) \)). Then we get

\[
H_E = \frac{g^2}{2} \sum_{\text{links}} E^2 \tag{38}
\]

with the quadratic Casimir operator \( E^2 = L^2 = R^2 \). And we recognize the sum over the squares of the electric field all over the lattice as the well known form of electric energy.

Thus, the Hamiltonian we have for the gauge field,

\[
H_{KS} = H_E + H_B \tag{39}
\]

is a generalization of the well-known Kogut-Susskind Hamiltonian \([11,58,59]\) for lattice gauge theories with compact Lie
gauge groups. The Kogut-Susskind Hamiltonian, its corresponding Wilson action and their continuum limit are the sources of the $g^2$ factors in $H_E$ and $H_G$.

As a final remark, note that in case that $G$ is Abelian, or has an Abelian subgroup ($U(1)$ or $\mathbb{Z}_2$), one can add a term corresponding to the Abelian charge in the electric energy. For example, the square of the electric field in $U(1)$, or the Abelian electric field in $U(N)$ for $N > 1$.

C. Realization in Representation space

Analogously to the construction of the fermionic transformation operators, we would like to obtain the group elements $U^j$ and the transformation operators $\Theta^g, \Theta^R$ in terms operators acting within the gauge field local Hilbert space.

Recall from section II, that this Hilbert space consists of the states $|jmn\rangle$. Suppose these are created out of the local "vacuum" $|\Omega\rangle$ using the creation operators $a^{\dagger}_{mn}$:

$$a^{\dagger}_{mn} |\Omega\rangle = |jmn\rangle.$$  

These operators are not only introduced for the convenience of writing, but they also constitute a suitable language when considering a physical realization of lattice gauge theories. In particular, several proposals to simulate lattice gauge theories in cold atomic setups rely on the simultaneous presence of different atomic species to mimic the matter and gauge fields in the theory [14–16, 18, 21]. Thus it becomes natural to associate Fock operators also to the links of the lattice gauge model. In principle the operators $a, a^\dagger$ may be either bosonic or fermionic; the statistics however is not relevant, as long as there is only one boson or fermion per link. Therefore in the following we choose to work in this particular regime, with a single gauge particle for each link representing the gauge field. As will be shortly seen, all the required operators in the theory consist of a single creation and a single annihilation operator, thus conserving the number of particles on the link and canceling the effect of their statistics. Finally, for the sake of simplicity and without any loss of generality, we will call these gauge particles "bosons"; irrespectively of their nature, in agreement with the usual bosonic representation of gauge fields.

Let us start with the construction of $U^j$ in terms of the operators $a, a^\dagger$. Motivated by equation (5), we can deduce that in group element space,

$$\langle g | U^j_{mn} | h \rangle = D^j_{mn}(g) \delta_{gh}.$$  

(41)

From this starting point, we may derive the representation space form of $U^j$ (later we will show the other way around, as a consistency check).

Proposition 3. The $U^j$ matrix elements are given by

$$U^j_{mn} = \sum_{JK} \sqrt{\frac{\dim (J)}{\dim (K)} } \langle JM | jmn \rangle \langle KN' | JM' jm' \rangle a^{\dagger}_{NN} \, a^{\dagger}_{MM'}.$$  

(42)

where $J, K$ are representations of $G$, the sum over the related gauge components $M, M', N$ and $N'$ is understood hereafter, and $\langle KN | JM jm \rangle, \langle KN' | JM' jm' \rangle$ are Clebsch-Gordan coefficients.

This operator changes, in general, the representation associated to the gauge field; in other words, it changes the electric flux carried by the link, acting like an exponential of the vector potential as expected from the definition of the tunneling term (31).

Proof. We wish to calculate the matrix elements of $U^j_{mn}$ in the representation basis, using equations (41) and (5). Using the Clebsch-Gordan series, one obtains

$$\langle KNN' | U^j_{mn} | JM'M' \rangle = \frac{\sqrt{\dim (J) \dim (K)}}{|G|} \sum_{g,l} D^{K}_{N'}(g) D^{L}_{JL}(g) \langle JM jm | IL \rangle \langle IL' | JM' jm' \rangle$$  

(43)

Using the great orthogonality theorem for summation over the group elements, one obtains

$$\langle KNN' | U^j_{mn} | JM'M' \rangle = \sqrt{\frac{\dim (J)}{\dim (K)}} \langle JM jm | KN \rangle \langle KN' | JM' jm' \rangle$$  

(44)

which proves the proposition. □

Next, let us consider the transformation operator.

Proposition 4. The left and right transformation operators are respectively:

$$\Theta^L_g = \sum_j a^{\dagger}_{ml} D^j_{nn}(g) a^j_{nl},$$  

(45)

$$\Theta^R_g = \sum_j a^{\dagger}_{ml} D^j_{mn}(g) a^j_{nl}.$$  

(46)

Proof. First, let us check that these operators correctly implement the composition rules of the transformations. Since these operators act block-diagonally in the representations, it is sufficient to prove it for a generic representation $j$. Thus,

$$\Theta^L_g \Theta^L_h = \sum_{j} a^{\dagger}_{ml} D^j_{nn}(g) a^j_{nl} a^{\dagger}_{ac} D^j_{bc}(h) a^j_{bc}$$  

(47)

However, $a^{\dagger}_{ml} a^j_{bc} = \delta_{ml} \delta_{bc} \mp \delta_{ac} a^j_{bc}$, where the $\mp$ sign holds for bosons or fermions respectively. The second term vanishes due to the multiplication by $a^j_{bc}$ (recall that there is only a single particle on the link at a time). Thus we get a product of the representation matrices which leads to the desired result:

$$\Theta^L_g \Theta^L_h = \sum_{j} a^{\dagger}_{ml} D^j_{nn} (gh) a^j_{nl} = \Theta^L_{gh}$$  

(48)

Also, note that thanks to the unitarity of the representations, $\Theta^L_{g^{-1}} = \Theta^L_g$. And thus, by the former argument,

$$\Theta^L_{g} \Theta^L_{g^{-1}} = \Theta^L_{1} = \sum_j a^{\dagger}_{lm} a^j_{mn} = 1$$  

(49)

using the constraint of exactly one particle per link, thus unitarity is proven. Similar arguments lead to $\Theta^R_g \Theta^L_h = \Theta^R_{gh}$ and $\Theta^R_g \Theta^R_h = 1$. 


Finally, one has to verify the transformation law, i.e. whether
\[
\langle KN' | \Theta^L_g U_{mm}^i \Theta^R_g | JM'M' \rangle = \\
\langle KN' | U^i_m D^j_{mn} (g) | JM'M' \rangle
\] (50)
and similarly for the left transformations. Calculating the LHS, and disregarding the normalization factor and the left Clebsch-Gordan, which are not affected by the transformation, one obtains
\[
\langle KN' | J M' j n \rangle D^K_{N'N'} (g) D^M_{M'M'} (g^{-1}) = \\
\langle KN' | \Theta^L_g |KN'\rangle \langle J M' j n | (\Theta^L_g |JM' \rangle \otimes | j n)\rangle = \\
\langle KN' | (JM') \otimes \Theta^L_g | j n \rangle = \\
\langle KN' | J M' j k \rangle \langle J M' j k | (JM') \otimes \Theta^L_g | j n \rangle = \\
\langle KN' | J M' j k \rangle D^j_{mn} (g)
\] (51)
where completeness had been used. This completes the proof for the right transformations, and similar arguments for the left ones complete everything. □

Let us now perform a consistency check and see what we get for the case of a compact Lie $G$, $U(N)$ or $SU(N)$, for example. In this case, for example,
\[
\Theta^L_g = \sum_j a^j_{ml} e^{-i \alpha_j T^j} a^j_{nl} = \sum_j e^{-i \alpha_j T^j} a^j_{nl}
\] (52)
(the creation and annihilation operators may be lifted to the exponent due to the fact we have a single particle in our Hilbert space). On the other hand, we expect the relation
\[
\Theta^L_g = e^{i \alpha L}
\] (53)
and thus we can extract the generators
\[
L = -\sum_j a^j_{ml} T^j_{mn} a^j_{nl}
\] (54)

This equation shows that the plaquette term is constituted by the sum of the projectors over all the possible group elements obtained as the oriented product of the transformations $g_i$ associated to the plaquette edges. Each projector in the sum is weighted with the character $\chi_j (C) = \text{Tr} \left( D^j (g_p) \right)$ of the related group element in the $j$th (usually the fundamental) representation. Thus, the product of the elements along the square edges, $g_1 g_2 g_3^{-1} g_4^{-1} \equiv g_p$, constitutes the transformation that a matter particle undergoes when it moves around the plaquette. For non-Abelian groups, $g_p$ is mapped into $h^{-1} g_p h$ by a gauge transformation $\Theta_h$ applied on the first vertex of the plaquette (note that as the $T$ matrices are hermitian, $T^* = T^T$). Similarly, one obtains
\[
R = \sum_j a^j_{ml} T^j_{mn} a^j_{nl}
\] (55)
These two sets of operators satisfy the required algebra (6) and commutation relations (7). These may be proven either directly, or by the use of an infinitesimal transformation and the Jacobi identity.

An important property of this representation is the ability to truncate the $U^j$: note that each of the summands in (42) is actually gauge invariant on its own, and thus one could, for example, consider only $J \leq J_{\text{max}}$ and truncate the series in a gauge invariant manner. The result will be, indeed, a non-unitary $U^j$ operator, but yet it will be gauge invariant with all the required transformation properties. As a consequence, one will have to truncate accordingly (such that $J \leq J_{\text{max}}$) the transformation operators of proposition 4 as well.

### D. Realization in Group Element space

A better insight about the magnetic part of the Hamiltonian can be gained by exploiting the space of group elements. First, we shall convert, as a consistency check, the $U^j$ from the previous subsection to this basis. Using (5) and completeness relations, one obtains that:
\[
\langle g | U_{mn}^j | h \rangle = \frac{1}{|G|} \sum_J \dim (J) \text{Tr} \left( D^j (g^{-1} h) \right) D^j_{mn} (g)
\] (56)
Note that $\sum_J \dim (J) \text{Tr} \left( D^j (g^{-1} h) \right)$ is the character of the regular representation of $g^{-1} h$, which is equal to $|G|$ if $g^{-1} h = 1$ and vanishes otherwise (55). Thus we recover, unsurprisingly, Eq. (11).

A straightforward calculation allows us to rewrite $H_B$ in the form:
\[
H_B = -\frac{1}{2g^2} \sum_p \sum_{g_1 g_2 g_3 g_4} \left( \text{Tr} \left( D^j (g_1 g_2 g_3^{-1} g_4^{-1}) \right) \right) \langle g_1 g_2 g_3^{-1} g_4^{-1} | g_1 g_2 g_3^{-1} g_4^{-1} \rangle + H.c.
\] (57)

This equation shows that the plaquette term is constituted by the sum of the projectors over all the possible group elements obtained as the oriented product of the transformations $g_i$ associated to the plaquette edges. Each projector in the sum is weighted with the character $\chi_j (C) = \text{Tr} \left( D^j (g_p) \right)$ of the related group element in the $j$th (usually the fundamental) representation. Thus, the product of the elements along the square edges, $g_1 g_2 g_3^{-1} g_4^{-1} \equiv g_p$, constitutes the transformation that a matter particle undergoes when it moves around the plaquette. For non-Abelian groups, $g_p$ is mapped into $h^{-1} g_p h$ by a gauge transformation $\Theta_h$ applied on the first vertex of the plaquette. Therefore, to rewrite the Hamiltonian in a gauge-invariant form, we must consider the effect of these gauge transformations; in particular, by exploiting the gauge invariance of $\text{Tr} \left( D^j (g_p) \right)$, the plaquette operator may be expressed as:
\[
H_B = -\frac{1}{2g^2} \sum_p \sum_{C} \chi_j (C) \Pi_{C,p} + H.c.
\] (58)
where $C \equiv \{ h^{-1} g_C h, h \in G \}$ labels the conjugacy classes of the group $G$, and $\Pi_{C,p} = \sum_{g_p \in C} |g_p \rangle \langle g_p|$ projects the plaquette state into the conjugacy class $C$ and is a gauge-invariant operator.
Therefore the magnetic term of the plaquette \( p \) associates the energy \( -\chi_j(C)/(2g^2) \) to each conjugacy class \( C \). The ground state of the magnetic Hamiltonian is defined by a configuration in which all the plaquettes are in the state associated with the identity operator, because the identity is the only group element which maximizes \( \chi_j(C) \) to the value \( \text{dim}(j) \).

All the other conjugacy classes, instead, define different kinds of localized magnetic vortices that are localized excitations over the ground states associated with a mass proportional to \( \text{dim}(j) - \chi_j(C) \).

The ground state for the gauge field defined by the Hamiltonian \( H_B \) in the gauge-invariant sector and in the absence of matter corresponds to the ground state of a quantum double model \([49]\) associated to the same group. Also the excited states, defined by the presence of magnetic vortices, are the same in both models. The quantum double Hamiltonian, however, associates the same energy gap to all the species of magnetic vortices since it is constituted by the projector \( \Pi_{1,p} \) over the identity only. The main difference in the magnetic sector of the two Hamiltonians is therefore constituted by the spectrum of the excitations. Furthermore the quantum double model implements dynamically the gauge invariance, which may be violated by the presence of gapped charged excitations. These excitations would play the role of static charges in the Kogut-Susskind Hamiltonian, where, instead, the matter particles are associated to charge excitations transforming under the group representation \( j \).

Hence we recover the well known fact that the weak limit, \( g \to 0 \), of the Kogut-Susskind Hamiltonian, corresponding to the deconfined phase, is related to the quantum double model. The pure-gauge theory (without matter) is represented in the Kogut-Susskind Hamiltonian, where, instead, the matter particles are associated to charge excitations transforming under the group representation \( j \).

Quantum double models are a particular example of string-net models \([51]\). In particular the mapping from quantum double to the string-net models is obtained by interpreting the irreducible representations of the group as “string types” on the link of the string-net model \([52]\). By exploiting the representation basis is therefore possible to interpret the plaquette operators in \( H_B \) in terms of the plaquette operators in the string-net model.

The plaquette operator \([36]\) transforms each link by effectively multiplying its state with the irreducible representation \( j \) as described by Eq. \([44]\). This is a particular example of the plaquette operators appearing in the string net models which are, in general, associated to all the irreducible representations. Therefore the Kogut-Sussking Hamiltonian reproduces one of the possible choices in the definition of the plaquette terms in the string-net model: the one in which only the representation \( j \) appears. However, if \( j \) is the fundamental representation of the group, the powers of the plaquette operator \( W \) may be rewritten in terms of plaquette operators for all the other irreducible representations. This intuitively shows the equivalence of the gauge field Hamiltonian and the string-net plaquette operators: they generate the same topological phase, as rigorously proved through the equivalence of the ground states with the quantum double model, evident in the group elements basis. In particular the string-net model provides an effective description of the scale invariant fixed point of the deconfined phase of the associated lattice gauge theory \([51]\).

Beside finite groups it is also interesting to recall the form of the plaquette interactions in the continuous \( U(1) \) theory (compact QED). In this case, there is a single representation and the group elements are simply phases. Thus one obtains the usual Abelian Kogut-Susskind magnetic energy,

\[
H_B = -\frac{1}{g^2} \sum_p \cos (\phi_1 + \phi_2 - \phi_3 - \phi_4) \tag{59}
\]

Recall that the group elements (or more precisely, for compact groups, the group parameters) correspond to the vector potential. One can see that the argument of the cosine is a lattice, discrete curl, and thus expansion of the cosine for a continuum limit will result in a \( B^2 \) term, corresponding to the usual Quantum-Electrodynamic magnetic energy term \([58]\).

V. AN EXAMPLE: THE \( D_3 \) GROUP

In order to summarize the full construction of the Kogut-Susskind Hamiltonian we present here the example provided by the smallest discrete non-Abelian group: the dihedral group \( D_3 \). The six elements of the group are generated by the rotation \( \xi_{2\pi/3} \) of the angle \( 2\pi/3 \) and the inversion symmetry \( \sigma \) of the equilateral triangle. In particular it admits a fundamental representation \( j = 2 \) in terms of \( 2 \times 2 \) orthogonal matrices:

\[
D^{(2)}(\xi_\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}, \tag{60}
\]

\[
D^{(2)}(\xi_\alpha\sigma) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & -\cos \alpha \end{pmatrix}, \tag{61}
\]

with \( \alpha = 0, 2\pi/3, 4\pi/3 \). The other two irreducible representations are the trivial representation \( D^0(g) = 1 \) and the parity representation \( D^p(g) = \det(g) \). Six states are therefore required to define the link Hilbert space and from the representation matrices straightforward to obtain the left and right transformation operators \( \Theta^L \) and \( \Theta^R \) in Eqs. \([45,46]\).

The particle fields transform under the \( j = 2 \) representation and can be described through the spinor \( \psi = (\psi_\uparrow, \psi_\downarrow) \). The corresponding charge operators in Eq. \([42]\) read:

\[
\Theta^{(2)}_{\xi_\alpha} = e^{i(\psi_\downarrow^\dagger \psi_\downarrow - \psi_\uparrow^\dagger \psi_\uparrow)} = 1 - (1 - \cos \alpha) (n_\uparrow + n_\downarrow) + \sin \alpha (\psi_\uparrow^\dagger \psi_\downarrow - \psi_\downarrow^\dagger \psi_\uparrow) + 2(1 - \cos \alpha) n_\uparrow n_\downarrow \tag{62}
\]

for the rotations having determinant equal to 1, and

\[
\Theta^{(2)}_{\xi_\alpha\sigma} = [1 + (\cos \alpha - 1) n_\uparrow - (\cos \alpha + 1) n_\downarrow + \sin \alpha (\psi_\uparrow^\dagger \psi_\downarrow + \psi_\downarrow^\dagger \psi_\uparrow)] (-1)^N \tag{63}
\]
where we adopted the notation $n_i = \psi_i^\dagger \psi_i$. In particular $\Theta^{D}_2 = (1 - 2n_1)(-1)^N$.

From the previous operators it is easy to define the Gauss law on each vertex of the lattice. The operators $U^{(2)}_{pm}$ in (42) can be evaluated, instead, from the Clebsch-Gordan coefficients (calculated using the technique in (62)):

$$\langle I, 2m|2n \rangle = \delta_{mn},$$  
$$\langle p, 2m|2n \rangle = \epsilon_{mn},$$  
$$\langle 2n, 2m|l \rangle = \delta_{ml}/\sqrt{2},$$  
$$\langle 2n, 2m|p \rangle = \epsilon_{nm}/\sqrt{2},$$  
$$\langle 2n, 2m|2l \rangle = (\delta_{11} \sigma^z_{nm} - \delta_{12} \sigma^x_{nm}) / \sqrt{2},$$  

where $\sigma^x$ and $\sigma^z$ are Pauli matrices for the indices $n, m$.

Finally the spectrum of the plaquette term for the group $D_3$ is composed by three conjugacy classes corresponding to the identity, the rotations of $2\pi/3$ and $4\pi/3$ and the inversions. A detailed analysis of the quantum double model for this group may be found, for example, in [63].

VI. SUMMARY

We have described a method to construct lattice gauge theories out of basic atomic-like ingredients. This applies both for continuous, compact, Lie groups (Kogut-Susskind Hamiltonian theories) as well as finite (discrete) groups. This may provide the background for both quantum simulations and new numerical approaches for such theories.

The matter degrees of freedom are represented by fermions, occupying the vertices of the lattice, and described by usual second-quantization Fock space fermionic operators - spinors in Group/Gauge space. Other possible quantum numbers of the matter - such as spin and flavor - are independent of the gauge symmetry and thus were neglected, but they may be introduced in a straightforward way if required by the analysis of a specific model.

The gauge field degrees of freedom, on the other hand, are slightly different than in the "conventional" formulations. These are described by local Hilbert spaces on the links of the lattice, consisting of a set of possible modes with a total occupation number one. Thus a Fock space is not strictly required for these local Hilbert spaces. Nevertheless we described the link dynamics in terms of annihilation and creation operators, both for ease of notation and to provide a useful description for potential quantum simulations of the theories. Traditionally we refer to the gauge fields as bosons, but since their number operators do not exceed one, and only pairs of such operators are involved in the dynamics, their statistics does not play a role and thus they may be replaced by fermions either.

These "bosonic" modes, or particles, correspond to the different representation states of the gauge group. Hence, for finite groups, the local bosonic Hilbert spaces are finite, while for other groups the full Hilbert space is infinite, but it may be consistently truncated to a few representation states, as long as all the modes belonging to the participating representations are included.

We have also reviewed the possibility to represent the group states in a different basis - the group element basis, connected to the representation basis by a unitary transformation which is a generalized Fourier transform, with the introduction of the generalized Wigner matrices. As the representation plays the role of an electric field, we deduce that group elements, as the conjugate quantum numbers, play the role of the vector potential. Indeed, the representation basis is more suitable for describing theories with matter, which is always in a given representation, and is related to the gauge invariance through the Gauss’s law, generated by the electric fields.

On the other hand, the group element basis is useful to analyze the content of the magnetic sector of the gauge theory and, in the case of discrete groups, allows the mapping of the deconfined phase of the theory into a quantum double model where the excitations of the plaquette terms correspond to magnetic vortices described by the conjugacy classes of the gauge group.

The lattice system has been described in terms of a square lattice, for various reasons: first, to relate this study to conventional high energy physics lattice gauge theories; second, to allow for simple generalization to any dimension (note that the dimension of the system was not addressed in the general framework); and third, but not less important, for the sake of simplicity. However, generalizations to other lattices should be straightforward (one should note that bipartite lattices are required for the staggering of fermions, although other approaches for lattice fermions may also be utilized).

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[1] J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).
[2] K. G. Wilson, Phys. Rev. D 10, 2445 (1974).
[3] F. Verstraete and J. I. Cirac, Phys. Rev. A 70, 060302 (2004).
[4] N. Schuch, J. I. Cirac and D. Pérez-García, Ann. Phys. 325, 2153 (2010).
[5] M. Lewenstein, A. Sanpera, and V. Ahufinger, Ultracold atoms in optical lattices: simulating quantum many-body systems (Oxford, Oxford University Press, 2012).
[6] Y. J. Lin, R. L. Compton, K. Jimenez-Garcia, J. V. Porto, and I. B. Spielman, Nature 462, 628 (2009).
[7] N. Goldman, G. Juzeliunas, P. Ohberg and I. B. Spielman, arXiv:1308.6533 (2013).
[8] M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes and I. Bloch, Phys. Rev. Lett. 111, 185301 (2013).
