Testing Seiberg-Witten Theory to All Orders in the Instanton Expansion

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Abstract: In the context of softly-broken $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetric SU($N$) gauge theory, we calculate using semi-classical instanton methods, the lowest order non-trivial terms in the mass expansion of the prepotential for all instanton number. We find exact agreement with Seiberg-Witten theory and thereby achieve the most powerful test yet of this theory. We also calculate the one- and two-instanton contributions completely and also find consistency with Seiberg-Witten theory. Our approach relies on the fact that the instanton calculus admits a nilpotent fermionic symmetry, or BRST operator, whose existence implies that the integrals over the instanton moduli space, which give the coefficients of the prepotential, localize on the space of resolved point-like instantons or what we call “topicons”.

Keywords: Instantons, supersymmetry.
1. Introduction

The remarkable theory of Seiberg and Witten [1] determines the low-energy behaviour of $\mathcal{N} = 2$ supersymmetric gauge theories exactly. In principle the low-energy effective action can also be calculated from first principles via conventional semi-classical methods using instantons. This leads to the idea of testing Seiberg-Witten theory by calculating the instanton effects and comparing these expressions with those extracted from the Seiberg-Witten curve. This idea was pursued most successfully to date in Refs. [2–4] at the one- and two-instanton level for the theory with gauge group SU(2) and Ref. [5] at the one-instanton level in SU($N$). The ultimate goal of this program is to provide an instanton-based “proof” of Seiberg-Witten theory by calculating instanton effects to all orders in the instanton charge. This paper provides the first test of Seiberg-Witten theory to all instanton number and the ultimate goal just articulated suddenly looks feasible. We should mention that there are other semi-classical tests of Seiberg-Witten theory based on matching the “monodromies” of the central charges to the semi-classical spectrum of dyons [6].

In [7] we described a new technique for calculating the instanton contributions to the prepotential of $\mathcal{N} = 2$ supersymmetric gauge theory. In these theories there is an adjoint-valued VEV. Derrick’s Theorem [8] implies that the action of an instanton can always be lowered by shrinking its size. As a consequence in the presence of a VEV instantons are no longer solutions of the equations-of-motion. The way to implement the semi-classical approximation in these circumstances was elucidated by Affleck [9, 10] (see also the in-depth discussion in Refs. [2, 11]) leading to the concept of a constrained instanton. To leading order in the semi-classical expansion the constrained nature of the instanton manifests itself as a non-trivial potential, or instanton effective action, on the moduli space of instantons. One of the main conceptual results of Ref. [7] was the realization that it is actually unnecessary to integrate over the whole moduli space of a constrained instanton, rather one can expand around the exact solutions corresponding to point-like instantons. In order to avoid the singular nature of point-like instantons, Ref. [7] introduced a regularization based on the smooth resolution of the instanton moduli space first described in purely geometrical terms without reference to the gauge theory by Nakajima in Ref. [12]. In this case the exact instanton solutions are smooth and of small, but fixed, size. Since these exact solutions really do have a local action we call them “topicons”.

In order to uncover the “calculus of topicons”, the key idea is to formulate the integral over the original instanton moduli space as a zero-dimensional topological, or cohomological, field theory.\footnote{Subsequent to Nakajima, it was realized by Nekrasov and Schwarz [13] that this smooth resolution of the instanton moduli space arises naturally when the theory is defined on a non-commutative spacetime.} In this formalism, there exists a nilpotent fermionic symmetry, or—depending on the context perhaps “topological, or cohomological, matrix theory” would be more appropriate.\footnote{Considering the context perhaps “topological, or cohomological, matrix theory” would be more appropriate.}
on taste—a BRST operator, and the integrals can be shown to localize on the critical points of the potential; namely, the subspace of point-like instantons. For \( k \)-instantons, this space is simply the space of \( k \) indistinguishable points in \( \mathbb{R}^4 \), or \( \text{Sym}^k \mathbb{R}^4 \)—a much simpler space than the full instanton moduli space \( \mathcal{M}_{k,N} \).

As alluded to above, the problem is that the subspace of point-like instantons arises as a singular subspace of \( \mathcal{M}_{k,N} \) and has singularities of its own when two, or more, points come together in \( \mathbb{R}^4 \). These singularities complicate the issue of localization. In order to resolve the difficulties, in Ref. [7], we suggested that one could consider the spacetime non-commutative version of the \( \text{U}(N) \) theory. After all, the instanton moduli space in the non-commutative theory, \( \mathcal{M}_{k,N}^{(c)} \), is a resolved version of \( \mathcal{M}_{k,N} \) which no longer has singularities [13]. Physically instantons can no longer shrink to zero size and the consequences of Derrick’s Theorem are avoided. In fact in the non-commutative theory there are now exact non-singular instanton solutions even in the presence of VEVs: the topicons. Actually there are \( N \) flavours of topicon obtained by embedding the spacetime non-commutative \( \text{U}(1) \) instanton solutions in each of \( N \) unbroken abelian factors of the gauge group. (For a discussion of instantons in the noncommutative \( \text{U}(1) \) theory see Refs. [13–16] and references therein.)

So the following picture emerges. For instanton charge \( k \), the exact instanton solutions come as a disjoint union of spaces associated to the inequivalent partitions \( k \rightarrow k_1 + \cdots + k_N \), \( k_u \geq 0 \), where each \( k_u \) corresponds to each of \( N \) \( \text{U}(1) \) subgroups picked out by the VEV. Hence, the space of exact solutions, or moduli space of topicons, lying within the larger moduli space is of the form

\[
\mathcal{M}_{k,N} \rightarrow \mathcal{M}_{k,N}^{(c)} \bigg|_{\text{topicon}} = \bigcup_{\text{partitions } k_1 + \cdots + k_N} \mathcal{M}_{k_1,1}^{(c)} \times \cdots \times \mathcal{M}_{k_N,1}^{(c)}. \tag{1.1}
\]

Each factor \( \mathcal{M}_{k_i,1}^{(c)} \) is a smooth resolution of the singular space \( \mathcal{M}_{k,1} = \text{Sym}^k \mathbb{R}^4 \). The picture that we have arrived at qualitatively is one of the main predictions of the localization technique as described in the companion paper [7].

In this paper, following on from [7], we consider the problem of calculating the prepotential in the theory obtained by softly breaking the \( \mathcal{N} = 4 \) supersymmetric gauge theory to \( \mathcal{N} = 2 \) by adding mass terms: the so-called “\( \mathcal{N} = 2^* \) theory”. Using localization we will be able to completely evaluate both the one- and two-instanton contributions to the prepotential of the \( \text{SU}(N) \) theory (or more precisely the \( \text{U}(N) \) spacetime non-commutative version) and the results that we obtain will be consistent with the same quantities extracted from the Seiberg-Witten curve. More ambitiously we will calculate the leading order contribution to the prepotential in an expansion in the supersymmetry breaking mass to all orders in the instanton charge. Once again we find perfect agreement with the predictions of Seiberg-Witten theory. This consistency relies on the hypothesis, first made in Ref. [17], that introducing spacetime non-commutativity
as a device for regulating the singularities of the instanton moduli space does not affect the instanton contributions to the prepotential.\(^3\)

We now describe the predictions of Seiberg-Witten theory. As usual, the prepotential of the \(\mathcal{N} = 2^*\) theory has the form

\[
\mathcal{F} = \mathcal{F}_{\text{pert}} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \mathcal{F}_k e^{2\pi i k \tau},
\]

(1.2)

where the sum is over the contributions from instantons of charge \(k\). These contributions are simply given, up to a multiplicative factor, by the centred instanton partition function [2,3,7,11]:

\[
\mathcal{F}_k = -m^2 \widehat{Z}_{k,N}^{(\mathcal{N} = 2^*)},
\]

(1.3)

where \(m\) is the supersymmetry breaking mass. The partition function \(\widehat{Z}_{k,N}^{(\mathcal{N} = 2^*)}\) is defined as an integral over the suitably supersymmetrized version of the centred \(k\)-instanton moduli space \(\widehat{\mathcal{M}}_{k,N}\).\(^4\) If \(\omega^{(\mathcal{N} = 4)}\) is the \(\mathcal{N} = 4\) supersymmetric volume form, then

\[
\widehat{Z}_{k,N}^{(\mathcal{N} = 2^*)} = \int_{\widehat{\mathcal{M}}_{k,N}} \omega^{(\mathcal{N} = 4)} e^{-S - mS_{\text{mass}}}.
\]

(1.4)

The quantity \(S\) is the instanton effective action which depends on the VEVs parameterizing the Coulomb branch and we find it convenient to explicitly separate out the terms \(S_{\text{mass}}\) which describe the effect of adding the supersymmetry breaking mass \(m\). The fact that there is a non-trivial action on the instanton moduli space is a direct manifestation of the fact that instantons are not exact solutions of the equation-of-motion when the scalar fields have VEVs: rather they should be treated as constrained instantons à la Affleck [9, 10].

The first two instanton contributions were extracted from the Seiberg-Witten curve for this theory via a procedure making extensive use of the underlying Calogero-Moser integrable

\(^3\)Roughly speaking point-like instantons do not couple to the VEV and so resolving them does not lead to any VEV-dependent corrections. As in Ref. [7], we shall find some physically irrelevant, \(i.e.\) VEV independent, differences between the instanton contributions in the commutative and non-commutative theories.

\(^4\)The centred moduli space has the overall position of the instanton configuration factored off: \(\mathcal{M}_{k,N} = \mathbb{R}^4 \times \widehat{\mathcal{M}}_{k,N}\).
The predictions in our notation read:

\[ F_1 = -m^2 \sum_{u=1}^{N} \prod_{v=1 \atop (v \neq u)}^{N} \left( 1 - \frac{m^2}{\phi_{uv}^2} \right), \quad (1.5a) \]

\[ F_2 = -m^2 \sum_{u=1}^{N} \left( \frac{3}{2} T_u(\phi_u)^2 + \frac{1}{4} T_u(\phi_u) \frac{\partial^2 T_u(\phi_u)}{\partial \phi_u^2} \right) \]

\[ - m^4 \sum_{u,v=1 \atop (u \neq v)}^{N} T_u(\phi_u) T_v(\phi_v) \left[ \frac{1}{\phi_{uv}^2} - \frac{1}{2 (\phi_{uv} + m)^2} - \frac{1}{2 (\phi_{uv} - m)^2} \right], \quad (1.5b) \]

where we have defined \( \phi_{uv} = \phi_u - \phi_v \) and make use of the functions

\[ T_u(x) = \prod_{v=1 \atop (v \neq u)}^{N} \left( 1 - \frac{m^2}{(x - \phi_v)^2} \right). \quad (1.6) \]

The instanton contributions for \( k > 2 \) can—at least in principle—be generated order-by-order in instanton number from the recursion relation established in [19]. However another approach, described in Ref. [20], established a very different kind of—and for us more useful—recursion relation for the prepotential. In this work, the prepotential is expanded in powers of \( m^2 \) rather than the instanton factor \( e^{2\pi i \tau} \). Up to irrelevant constant factors

\[ F = \frac{1}{2\pi i} \sum_{n=1}^{\infty} f_n(\tau, \phi_u) m^{2n}. \quad (1.7) \]

The recursion relation can then used to find \( f_n \) in terms of \( f_p, p < n \). The first two terms of the expansion are

\[ f_1 = \frac{1}{2} \sum_{u,v=1 \atop (u \neq v)}^{N} \log \phi_{uv}^2, \quad (1.8a) \]

\[ f_2 = -\frac{E_2(\tau)}{24} \sum_{u,v=1 \atop (u \neq v)}^{N} \frac{1}{\phi_{uv}^2}, \quad (1.8b) \]

where \( E_2(\tau) \) is the regulated Eisenstein series of weight two. This function has an instanton expansion

\[ E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \left( \sum_{d \mid k} d \right) e^{2\pi i k \tau}, \quad (1.9) \]

\(^5\)The following expressions are identical to those in Eq. (5.22) of Ref. [18]. I am grateful to the authors of Ref. [18] for pointing out a typo in their subsequent Eq. (5.23).
where the sum is over the integer divisors \( d \) of \( k \). Putting the mass expansion together with the expansion over instanton number we find

\[
\mathcal{F}_k = m^4 \left( \sum_{d|k} d \right) \sum_{u,v=1}^{N_{u,v}} \frac{1}{(\phi_u - \phi_v)^2} + \mathcal{O}(m^6). \tag{1.10}
\]

Again we emphasize that this is modulo VEV-independent pieces.

2. The \( \mathcal{N} = 4 \) Instanton Calculus and Localization

The primary goal of this section is to define the centred instanton partition function and construct the fermionic BRST symmetry. We will not provide a completely self-contained description of the instanton calculus: for this one can refer to the more detailed treatments in Refs. [5, 11, 21] along with the companion paper [7].

The ADHM construction [22] involves a set of over-complete collective coordinates that are subject to a set of non-linear constraints. The variables consist of the \( N \times k \) complex matrices \( w_{\dot{\alpha}} \), \( \dot{\alpha} = 1, 2 \), with elements \( w_{u\dot{\alpha}} \) and \( k \times k \) traceless (in order to describe the centred moduli space) Hermitian matrices \( a'_n \).\(^6\) Our conventions are described in the Appendix. The ADHM constraints are

\[
\tau^{c\dot{\alpha}}\beta \left( \overline{w}^{\dot{\beta}} w_{\dot{\alpha}} + \overline{a'}_{\beta\alpha} a'_{\alpha\dot{\alpha}} \right) = \zeta^c \delta_{[k][k]}.
\]

Here, \( \zeta^c \) are real constants which parameterize the spacetime non-commutativity of the gauge theory. For the commutative theory one has \( \zeta^c = 0 \). In order to regulate the singularities of the instanton moduli space, we will consider for the most part the non-commutative theory where \( \zeta^c \) is non-trivial. Without-loss-of-generality, we will choose

\[
\zeta^1 = \zeta^2 = 0, \quad \zeta^3 \equiv \zeta > 0. \tag{2.2}
\]

To complete the construction of the instanton moduli space one takes a quotient of the solutions of (2.1) by the \( U(k) \) symmetry that acts as

\[
w_{\dot{\alpha}} \rightarrow w_{\dot{\alpha}} \mathcal{U}, \quad a'_n \rightarrow \mathcal{U} a'_n \mathcal{U}^\dagger.
\]

\( \mathcal{U} \in U(k) \). As described more fully in [11, 23], the ADHM construction is an example of the hyper-Kähler quotient construction [24] starting from flat space with metric

\[
ds^2 = 8\pi^2 \text{tr}_k \left[ d\overline{w}^{\dot{\alpha}} dw_{\dot{\alpha}} + da'_n da'_n \right]. \tag{2.4}
\]

\(^6\)SU(\( N \)) gauge indices are denoted \( u, v = 1, \ldots, N \) and “instanton” indices are denoted \( i, j = 1, \ldots, k \), where \( k \) is the instanton charge. In addition \( \overline{w}^{\dot{\alpha}} \equiv (w_{\dot{\alpha}})^\dagger \).
by the group action (2.3).

In an $\mathcal{N} = 4$ supersymmetric theory an instanton has a set of Grassmann collective coordinates which parameterize the $8kN$ zero modes of the Dirac operator in the instanton background.\footnote{For a review of the $\mathcal{N} = 4$ instanton calculus see Refs.\ [11, 21].} In the supersymmetric extension of the ADHM construction we first define a set of over-complete Grassmann variables $\{\mu^A, \bar{\mu}^A, \mathcal{M}^{A\alpha}, \mathcal{M}^{\alpha A}\}$. Here $A = 1, \ldots, 4$ is a fundamental index of the $\text{SU}(4)$ $R$-symmetry. The quantity $\mu^A$ is an $N \times k$ matrix with elements $\mu^A_{\alpha a}$, $\bar{\mu}^A$ is a $k \times N$ matrix with elements $\bar{\mu}^A_{ia}$ and $\mathcal{M}^{A\alpha}$ are traceless $k \times k$ matrices with elements $(\mathcal{M}^{A\alpha})_{ij}$. This over-complete set of variables is subject to fermionic analogues of the ADHM constraints:

$$\bar{\mu}^A w_\alpha + \bar{w}_\alpha \mu^A + [\mathcal{M}^{\alpha A}, a^I_{\alpha a}] = 0 . \tag{2.5}$$

As in the $\mathcal{N} = 2$ theory with fundamental hypermultiplets discussed in [7], it is convenient to introduce some additional auxiliary variables. In the $\mathcal{N} = 4$ context these consist of: $\chi_a$, $a = 1 \ldots 6$, a six-vector\footnote{We shall often denote a six-vector by a bold symbol, e.g. $\mathbf{x}$.} of Hermitian $k \times k$ matrices; $D_c$, $c = 1, 2, 3$, three Hermitian $k \times k$ matrices; and $k \times k$ matrices of Grassmann superpartners $\bar{\psi}^a_A$. Using these variables, the $\mathcal{N} = 4$ centred instanton partition function can be written in a completely “linearized” form:

$$\hat{\mathcal{M}}_{U(k,N)}^{(N=4)} = \frac{2^{-5/2k^2+k/2-2kN \pi -10k^2-6kN+6k^2}}{\text{Vol}(U(k))} \int d^{2kN} w d^{2kN} \bar{w} d^{4k} \bar{w} d^8 \bar{w} \exp(-S) . \tag{2.6}$$

Here, the instanton effective action is\footnote{Our conventions are set out in the Appendix.}

$$S = 4\pi^2 \text{tr} \left\{ |w^\alpha \chi + \phi w^\alpha|^2 - [\chi, a^I_{\alpha a}]^2 + \frac{1}{2} \sum_{AB} \left[ \bar{\mu}^A (\mu^B \chi + \phi \mu^B) + \mathcal{M}^{A\alpha} \mathcal{M}^{\alpha B} \chi \right] \right\} + S_{\text{L.m.}} . \tag{2.7}$$

The variables $D_c$ and $\bar{\psi}^a_A$ act as Lagrange multipliers for the bosonic and fermionic ADHM constraints (2.1) and (2.5) through the final term in the action:

$$S_{\text{L.m.}} = -4i\pi^2 \text{tr} \left\{ \bar{\psi}^a_A (\bar{\mu}^A w_\alpha + \bar{w}_\alpha \mu^A + [\mathcal{M}^{\alpha A}, a^I_{\alpha a}]) + D_c \left( \tau^{ca}_{\beta} (w^\beta w_\alpha + \bar{w}^\beta \bar{w}_\alpha) - \zeta^c \right) \right\} . \tag{2.8}$$

In (2.7), $\phi$ is the VEV of the scalar field. In the $\mathcal{N} = 4$ theory, before the mass perturbation is considered, $\phi$ is a six-vector of diagonal (traceless) $N \times N$-dimensional matrices with elements $\phi_u$, $u = 1, \ldots, N$.

An expression for the $\mathcal{N} = 4$ supersymmetric volume form on the resolved centred instanton moduli space $\hat{\mathcal{M}}_{U(k,N)}^{(N=4)}$ is obtained by integrating out the the auxiliary variables $\{\chi_a, D_c, \bar{\psi}^a_A\}$. Notice that after this has been done the volume integral is weighted by an exponential of a
non-trivial function depending on the VEVs $\phi$. If $\phi = 0$ then this function does not vanish: there remains a non-trivial quadrilinear coupling of the Grassmann collective coordinates which was first obtained for the SU($N$) theory in Ref. [21]. This is the major difference between instantons in the $\mathcal{N} = 4$ versus $\mathcal{N} = 2$ theories: in the former, instantons with Grassmann fields turned on are not generally exact solutions of the equations-of-motion [11,21] in contrast to the latter. This means that in the $\mathcal{N} = 2$ theory with zero VEVs the centred instanton partition function vanishes since there are unsaturated Grassmann integrals. In the $\mathcal{N} = 4$ theory with vanishing VEVs the centred instanton partition function does not vanish. Actually it is precisely the Gauss-Bonnet-Chern integral on $\hat{\mathcal{M}}_{k,N}^{(\mathcal{C})}$. The reason is that the quadratic coupling of the Grassmann collective coordinates involves the Riemann tensor and performing the Grassmann integrals pulls down powers of the Riemann tensor contacted in precisely the right way to give the Euler form. On a compact space, therefore, the partition function (with zero VEVs) would yield the Euler characteristic. However, $\hat{\mathcal{M}}_{k,N}^{(\mathcal{C})}$ is not compact since individual instantons can become arbitrarily separated in $\mathbb{R}^4$. To define the Euler characteristic on such a space, we have to cut it off; for instance, on a large sphere of radius $R$:

$$\text{tr}_k(\bar{w}^\alpha w_{\bar{\alpha}} + a'_n a'_n) = R^2. \quad (2.9)$$

As $R \to \infty$, the Euler characteristic receives a bulk contribution given by the Gauss-Bonnet-Chern integral and also a boundary contribution [25]. The Gauss-Bonnet-Chern integral for $\hat{\mathcal{M}}_{1,N}^{(\mathcal{C})}$ was calculated explicitly in [26] with result

$$\hat{\mathcal{Z}}_{1,N}^{(\mathcal{N} = 4)} = \frac{2\Gamma(N + \frac{1}{2})}{\Gamma(N)\Gamma(\frac{1}{2})}. \quad (2.10)$$

The Euler characteristic of $\hat{\mathcal{M}}_{1,N}^{(\mathcal{C})}$ (in the sense defined above) has been computed by Nakajima using Morse theory [12,27] giving the result $N$. This means that the boundary contribution on the large sphere at infinity must be non-vanishing.

When the VEVs are turned on, the instanton effective action (2.7) is modified: instantons become constrained and there is a non-trivial potential on the moduli space. On the mathematical side the resulting situation is rather familiar in the context of the Mathai-Quillen formalism for dealing with representations of Euler classes of vector bundles [28] (for a physicists’ review see Ref. [29].) The VEVs correspond to a vector fields on $\hat{\mathcal{M}}_{k,N}^{(\mathcal{C})}$ and the resulting integral over the moduli space involves the Mathai-Quillen form. In the compact case this form is cohomologous to the Euler class and so yields the Euler characteristic. By making the VEVs very large the integral localizes on the critical points of the vector fields. Using this formalism it is possible to show that the Euler characteristic can be computed either as a bulk integral (the Gauss-Bonnet Theorem) or as a sum over the critical points (the Poincaré-Hopf Theorem). In the following we will basically set up this formalism using the language of topological, or cohomological field theory, where one uses Grassmann variables and BRST operators rather
than forms and exterior derivatives. However, one should keep in mind that our application is rather more complicated than usual due to non-compactness.

Before we proceed, it is useful to take stock of the various symmetries that play a rôle in the $\mathcal{N} = 4$ instanton calculus. Firstly, we have the SO(4) Euclidean Lorentz group with covering group SU(2)$_L \times$ SU(2)$_R$. The $\alpha = 1,2$ ( $\dot{\alpha} = 1,2$) indices are spinor indices of SU(2)$_L$ (SU(2)$_R$). Then we have the SU(4) $\simeq$ SO(6), $R$-symmetry group with spinor indices $A = 1,2,3,4$ and vector indices $a = 1,\ldots,6$. Global U(N) gauge transformations act on quantities with indices $u = 1,\ldots,N$ and finally the ADHM construction involves the auxiliary U($k$) symmetry which acts on indices $i = 1,\ldots,k$.

To describe the soft-breaking to $\mathcal{N} = 2$ by mass terms, we first have to restrict the non-zero elements of the SO(6) vector of VEVs, $\phi_a$, to $a = 1,2$. Since the mass terms break $\mathcal{N} = 4$ to $\mathcal{N} = 2$, the $R$-symmetry is broken from SU(4) to SU(2) $\times$ U(1). In order to describe the symmetry breaking, it is expedient to decompose SO(6) vector indices as

$$a \rightarrow (a', \dot{a}), \quad a' = 1,2, \quad \dot{a} = 3,4,5,6,$$

and SU(4) spinor indices as$^{10}$

$$A \rightarrow (A', \dot{A}), \quad A' = 1,2, \quad \dot{A} = 3,4.$$

The unbroken SU(2) factor of the $R$-symmetry is then indexed by $A'$. The mass deformation is obtained by adding the following term to the instanton effective action [4, 11]:

$$mS_{\text{mass}} = -\pi^2 M_{AB} \text{tr}_k (2\bar{\mu}^A \mu^B + \mathcal{M}^{\alpha A} \mathcal{M}^{\beta B}),$$

where

$$M_{AB} = m \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order that the mass deformation preserves $\mathcal{N} = 2$ supersymmetry, the components $\phi_a$ of the VEV must vanish. The remaining variables $\phi_{a'}, a' = 1,2$, parameterize the Coulomb branch of the $\mathcal{N} = 2^*$ theory. We will find that the prepotential depends holomorphically on the variables defined by

$$\phi = i\phi_1 - \phi_2 \equiv -\frac{i}{2} \hat{A}^B \hat{\Sigma}_{AB} \cdot \phi.$$  

$^{10}$Where necessary we will raise and lower $A'$ and $\hat{A}$ indices with the $\epsilon$ tensor with $\epsilon_{21} = \epsilon^{12} = \epsilon_{43} = \epsilon^{34} = 1$, $\epsilon_{12} = \epsilon^{21} = \epsilon_{34} = \epsilon^{43} = -1$. 

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Here, $\phi$ is then a diagonal $N \times N$ traceless matrix with elements $\phi_{a\dot{a}}$. With the mass term $S_{\text{mass}}$ added to the instanton effective action we have

$$\Gamma^{(N=4)}_{k,N} \rightarrow \Gamma^{(N=2')}_{k,N} = \int_{\Omega_{k,N}} \omega^{(N=4)} \left( - S - m S_{\text{mass}} \right)$$

which gives the $k$-instanton coefficient of the prepotential via (1.3).

Before mass deforming, the instanton effective action (2.7) is invariant under eight supersymmetries corresponding to precisely half the number of the $\mathcal{N} = 4$ theory reflecting the fact that an instanton is a BPS configuration that breaks half the supersymmetries of the underlying field theory. On the full set of variables, the transformations are

$$\begin{align*}
\delta a'_{a\dot{a}} &= i \bar{\xi}_{aA} \mathcal{M}^A_{a\dot{a}} , \\
\delta w_\dot{a} &= i \bar{\xi}_{A\dot{a}} \mu^A , \\
\delta \chi &= - \Sigma^{AB} \bar{\zeta}_{aA} \bar{\psi}^a_{\dot{B}} , \\
\delta \bar{\psi}^a_A &= 2 \bar{\Sigma}_{aBA} [\chi_a, \chi_B] \bar{\delta}^a_{\dot{B}} - i D^c R^{ca}_{\beta} \bar{\delta}^a_{\dot{B}} , \\
\delta D^c &= - i R^{ca}_{\beta} \bar{\xi}_{aA} \Sigma^{AB} \left[ \bar{\psi}^a_A, \chi_B \right].
\end{align*}$$

Once we have added the mass term, and set $\phi_{a\dot{a}} = 0$, only four supersymmetries corresponding to $\bar{\xi}_{aA'}$ remain as symmetries.

One can interpret the partition function $\Gamma^{(N=4)}_{k,N}$ as the dimensional reduction of an $\mathcal{N} = (0,1)$ supersymmetric gauged linear $\sigma$-model in six dimensions [11,21]. In this interpretation $\chi$ is the U($k$) six-dimensional gauge field forming a vector multiplet of supersymmetry along with $\bar{\psi}^a_A$ and $D^c$ and the non-commutativity parameters $\zeta^c$ arise as Fayet-Illiopolos terms for the abelian subgroup of U($k$). These variables have no kinetic term (in six dimensions) and on integrating them out one recovers a non-linear $\sigma$-model with the hyper-Kähler space $\mathfrak{M}_{k,N}^{(6)}$ as target.

The fermionic symmetry, or “BRST operator”, for the $\mathcal{N} = 4$ instanton calculus was constructed in [26].\textsuperscript{11} It is simply given by a certain combination of the supersymmetries. Firstly, from the supersymmetry transformations we can define corresponding supercharges via $\delta = \xi_{aA} Q^{aA}$. The fermionic symmetry we are after is then generated by the BRST operator

$$Q = \epsilon_{aA'} Q^{aA'}$$

which gives

$$\begin{align*}
Q a'_{a\dot{a}} &= i \epsilon_{A\dot{a}} \mathcal{M}^A_{a\dot{a}} , \\
Q w_\dot{a} &= i \epsilon_{A\dot{a}} \mu^A , \\
Q \chi &= - \Sigma^{AB} \epsilon_{A\dot{a}} \bar{\psi}^a_{\dot{B}} , \\
Q D^c &= - i R^{ca}_{\beta} \epsilon_{aA} \Sigma^{AB} \left[ \bar{\psi}^a_A, \chi_B \right].
\end{align*}$$

\textsuperscript{11}For related work in the context of the $\mathcal{N} = 2$ with fundamental hypermultiplets see Refs. [7,30].
More precisely, to obtain (2.7), one has to integrate out the variables cohomological field theory.

In the above, we have defined the following quantities (the “equations” in the language of

\[ Q^2 \ast = 2i(\delta_\chi \ast + \delta_\phi \ast) , \]

where \( \delta_\chi \) and \( \delta_\phi \) are infinitesimal \( U(k) \) and \( U(N) \) transformations generated by

\[ \chi = i\chi_1 - \chi_2 \equiv -\frac{i}{2} \epsilon \hat{A} \hat{B} \Sigma_{\hat{A}\hat{B}} \cdot \chi , \]

and \( \phi \) in (2.15), respectively, e.g. \( \delta_\chi w_\alpha = w_\dot{\alpha} \chi \) and \( \delta_\phi w_\dot{\alpha} = \phi w_\dot{\alpha} \).

It is now possible to write the instanton effective action (2.7) in a manifestly \( Q \)-exact way. This was done in Ref. [26] in a slightly more general form where the variables \( \{ \chi_\alpha, D^c, \bar{\psi}_A \} \) have “kinetic terms” with a coupling constant \( g_0^{-2} \) rather than being auxiliary. However, the kinetic terms can be removed by a careful re-scaling by \( g_0 \), and then taking the limit \( g_0 \to \infty \). We now describe the result of this procedure. The construction is greatly facilitated by introducing some additional auxiliary variables \( \{ H_\alpha^A, F_\dot{A}, \bar{F}_\dot{A} \} \) which linearize the fermionic symmetry:

\[
\begin{align*}
Q M^A_\alpha &= H^A_\alpha , & Q H^A_\alpha &= 2i[ M^A_\alpha, \chi ] , \\
Q \mu^A &= F^A , & Q F^A &= 2i(\mu^A \chi + \phi \mu^A) , \\
Q \bar{\mu}^\dot{A} &= \bar{F}^\dot{A} , & Q \bar{F}^\dot{A} &= -2i(\chi \bar{\mu}^\dot{A} + \bar{\mu}^\dot{A} \phi) .
\end{align*}
\]

One can then show that the instanton effective action is \( Q \)-exact:

\[ S = Q \Xi , \]

where

\[
\Xi = 4\pi^2 \text{tr} \left\{ \frac{1}{2}\delta^A_A \bar{w}^\dot{\alpha} \Sigma_{AB} \cdot (\mu^B \chi + \phi \mu^B) + \frac{1}{4}\delta^{A'}_A \bar{a}^{\hat{\alpha}\alpha} \Sigma_{AB}^{'A'} \cdot [M^{'B'}_A, \chi] \\
+ \frac{1}{4}\mathcal{M}^{'A'}_A \left( \frac{1}{2} H^A_\alpha - h^A_\alpha \right) + \frac{1}{2}\bar{F}_A \left( \frac{1}{2} F^A - \bar{f}^A \right) + \frac{1}{4} \left( \frac{1}{2} F^A - \bar{f}^A \right) \mu^A_\alpha \\
+ \delta^A_A \bar{\psi}^\beta \bar{w}_\beta \bar{\bar{w}}^A + \bar{a}^{\hat{\alpha}\alpha} \bar{a}^{\hat{\beta}\beta} \bar{a}^{\hat{\gamma}\gamma} \right\} .
\]

In the above, we have defined the following quantities (the “equations” in the language of cohomological field theory)

\[
\begin{align*}
h^A_\alpha &= -2i\delta^A_A \Sigma^{AA'} \cdot [a^A_\alpha, \chi] , \\
f^A &= -2i\delta^A_A \Sigma^{AA'} \cdot (w_\alpha \chi + \phi w_\alpha) , \\
\bar{f}^\dot{A} &= -2i\epsilon_\alpha^A \Sigma^{AA'} \cdot (\chi \bar{w}^\dot{\alpha} + \bar{w}^\dot{\alpha} \phi) .
\end{align*}
\]

More precisely, to obtain (2.7), one has to integrate out the variables \( \{ H_\alpha^A, F_\dot{A}, \bar{F}_\dot{A} \} \) which appear quadratic in the action. This is equivalent to setting

\[ H_\alpha^A \to h^A_\alpha , \quad F^A \to f^A , \quad \bar{F}_\dot{A} \to \bar{f}^\dot{A} . \]
We have, up till now, not discussed the $\mathcal{N} = 4 \rightarrow 2$ breaking mass term $S_{\text{mass}}$. The question is how this term modifies the picture we have established of the BRST operator $\mathcal{Q}$ and a $\mathcal{Q}$-exact action? The answer involves a mass-dependent deformation of the BRST operator itself which we denote $\mathcal{Q}_m$, while $\Xi$ remains unaffected:

$$S + mS_{\text{mass}} = \mathcal{Q}_m \Xi.$$  \hspace{1cm} (2.27)

For this to work we must set $\phi_a = 0$ which ensures that the resulting set-up has $\mathcal{N} = 2$ supersymmetry. The action of the deformed symmetry is equal to (2.19) and (2.22) up to the following changes:

$$\begin{align*}
\mathcal{Q}_m H^\alpha_A &= 2i[\mathcal{M}^\alpha_A, \chi] + \varpi^A_B \mathcal{M}^{\tilde{B}}_\alpha, \\
\mathcal{Q}_m F^\hat{A} &= 2i(\mu^\hat{A}\chi + \phi \mu^{\hat{A}}) + \varpi^A_B \mu^{\tilde{B}}, \\
\mathcal{Q}_m F^{\hat{A}} &= -2i(\chi \tilde{\mu}^\hat{A} + \tilde{\phi} \mu^{\hat{A}}) + \varpi^{\hat{A}}_B \mu^{\tilde{B}}, \\
\mathcal{Q}_m \psi^{\hat{A}}_\alpha &= 2\Sigma_{ab}^{\hat{A}} [\chi_a, \chi_b] \delta^{\hat{A}}_B + \bar{\Sigma}_{a\hat{A}B}^B \epsilon^{a\hat{A}} \varpi^b_a \chi_b, \\
\mathcal{Q}_m \bar{\psi}^\hat{A}_\dot{\alpha} &= 2\Sigma_{ab}^{\hat{A}} [\chi_a, \chi_b] \delta^{\hat{A}}_B + \bar{\Sigma}_{a\hat{A}B}^B \epsilon^{a\hat{A}} \varpi^b_a \chi_b, \hspace{1cm} (2.28)
\end{align*}$$

where $\varpi$ is a specific, mass-dependent, infinitesimal generator of a transformation in the unbroken SU(2) $\times$ U(1) $R$-symmetry group. For the spinor and vector representations of SU(4), respectively,

$$\begin{align*}
\varpi^A_B &= \frac{m}{8} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}, & \varpi_a^b &= \frac{m}{8} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}. \hspace{1cm} (2.29)
\end{align*}$$

Note that $\varpi_a^b \phi_b = 0$, as it should so that the mass deformation is consistent with the VEV.

Now one can show that $\mathcal{Q}_m$ is nilpotent up to infinitesimal transformations in the U($k$) and U($N$) symmetry groups, as before, but now, in addition, an infinitesimal transformation in the unbroken SU(2) $\times$ U(1) $R$-symmetry group with generator $\varpi$. It is a standard argument to show that the partition function—at least formally—localizes on the critical points of the action $\mathcal{Q}_m \Xi$. Consider the more general integral

$$\tilde{Z}_{k,N}^{(\mathcal{N}=2^*)}(s) = \int_{\mathcal{M}_{k,N}} \omega^{(\mathcal{N}=4)}(s) \exp \left( -s^{-1} \mathcal{Q}_m \Xi \right).$$  \hspace{1cm} (2.30)

We then have

$$\frac{\partial \tilde{Z}_{k,N}^{(\mathcal{N}=2^*)}(s)}{\partial s} = s^{-2} \int_{\mathcal{M}_{k,N}} \omega^{(\mathcal{N}=4)}(s) \mathcal{Q}_m \left\{ \Xi \exp \left( -s^{-1} \mathcal{Q}_m \Xi \right) \right\},$$  \hspace{1cm} (2.31)

using the fact that $\mathcal{Q}_m^2 \Xi = 0$. Since the volume form is invariant under supersymmetry (as proved in Ref. [11, 31]), U($N$), U($k$) and SU(4) $R$-symmetry, it is $\mathcal{Q}_m$-invariant and so the
right-hand side of (2.31) vanishes. Consequently, $\hat{Z}^{(N-2^*)}_{k,N}(s)$ is independent of $s$ and, therefore, it can be evaluated in the limit $s \to 0$ where the integral is dominated by the critical points of $Q_m \Xi$. Since the result is independent of $s$, under favourable circumstances—which will be shown to hold in the present application—the Gaussian approximation is exact (for references to this kind of localization in the physics literature see Refs. [29,32–34] and references therein).

Notice in this formalism the dependence on the anti-holomorphic component of the VEV $\phi^*$ and the non-commutativity couplings $\zeta_c$, resides solely in $\Xi$. Hence the derivative of the integral with respect to either of these parameters is $Q_m$-exact and so the integral cannot depend on either $\phi^*$ and $\zeta_c$. On the other hand, the dependence of the integrals on the holomorphic component of the VEVs $\phi$ and the mass $m$ is through the operator $Q_m$ and so there is every reason to expect the integrals to depend on these parameters, as we will find.

Having established the idea of localization, we now investigate exactly on what submanifolds the integrals localize. The critical points are the zeros of (2.7),

$$\left| w_\alpha \chi + \phi w_\alpha \right|^2 - \left[ \chi, a'_n \right]^2 ,$$

which requires

$$w_\alpha \chi + \phi w_\alpha = \left[ \chi, a'_n \right] = 0 .$$

Up to the $U(k)$ auxiliary symmetry, there are a set of discrete critical-point sets associated to the inequivalent partitions

$$k \to k_1 + k_2 + \cdots + k_N .$$

For a given partition, each $i \in \{1, 2, \ldots, k\}$ is associated to a given $u$ by a map $u_i$ as follows:

$$\left\{ \begin{array}{c} 1, 2, \ldots, k_1, \underbrace{k_1+1, \ldots, k_1+k_2}_{u=2}, \ldots, \\
\underbrace{k_1+\cdots+k_{u-1}+1, \ldots, k_1+\cdots+k_u}_{u}, \ldots, \\
\underbrace{k_1+\cdots+k_{u-1}+1, \ldots, k_1+\cdots+k_N}_{u=N} \end{array} \right\}$$

and the variables have a block diagonal-form

$$\chi_{ij} = -\phi_{ui} \delta_{ij} , \quad w_{ui\alpha} \propto \delta_{uu_i} , \quad (a'_n)_{ij} \propto \delta_{uu_iu_j} .$$

The critical-point sets have a very suggestive form. Imposing the ADHM constraints implies that in the $u^{th}$ block the constraints are those of $k_u$ instantons in a non-commutative $U(1)$ gauge theory. The critical submanifold associated to $\{k_1, \ldots, k_N\}$ is then simply

$$\mathcal{M}_{k_1,1}^{(k)} \times \cdots \times \mathcal{M}_{k_N,1}^{(k)} \bigg/ \mathbb{R}^4 ,$$

where the quotient is by the overall centre of the instanton. The factors $\mathcal{M}_{k,1}^{(k)}$ are the $k$-instanton moduli space in the spacetime non-commutative theory with gauge group $U(1)$. As
we mentioned in the introduction these spaces are smooth resolutions of the symmetric product Sym$^k \mathbb{R}^4$. We interpret (2.37) as the moduli space of a composite configuration of topicons involving $k_u$ of flavour $u$.

3. One Instanton

We now use the localization technique to evaluate the centred one-instanton partition function. The details are very similar to the $\mathcal{N} = 2$ case with fundamental hypermultiplets described in Ref. [7].

The instanton effective action has $N$ critical points, corresponding to a single topicon of arbitrary flavour labelled by $v \in \{1, 2, \ldots, N\}$, at which (2.36)

$$\chi = -\phi_v , \quad w_{u\dot{\alpha}} \propto \delta_{uv} .$$

(3.1)

Note that $\alpha'_n = 0$ in the one-instanton sector. With the choice of non-commutativity parameters in (2.2), the ADHM constraints (2.1) are solved with

$$w_{u\dot{\alpha}} = \sqrt{\zeta} e^{i\theta} \delta_{uv} \delta_{\dot{\alpha} \dot{\beta}},$$

(3.2)

for an arbitrary phase angle $\theta$. The integrals over $w_{u\dot{\alpha}}$ are then partially saturated by the $\delta$-function ADHM constraints that arise once $D^c$ are integrated out. A trivial integral over the phase angle $\theta$ remains:

$$\int d^2w_v d^2\bar{w}_v \prod_{c=1}^{3} \delta \left( 1/2 \tau_{\dot{\alpha} \dot{\beta}} (\bar{w}_{\dot{\alpha}} w_{v\dot{\beta}} - \zeta \delta_{\dot{\alpha} \dot{\beta}}) \right) = 8\pi \zeta^{-1} .$$

(3.3)

Once the Lagrange multipliers $\bar{\psi}_{\dot{\alpha}} A^A$ are integrated out the resulting Grassmann $\delta$-functions saturate the integrals over $\{\mu^A_u, \bar{\mu}^A_u\}$:

$$\int d\mu^A_v d\bar{\mu}^A_v \prod_{\dot{\alpha}=1}^{2} \delta \left( \bar{w}_{v\dot{\alpha}} \mu^A_v + w_{v\dot{\alpha}} \bar{\mu}^A_v \right) = \zeta ,$$

(3.4)

for each $A = 1, \ldots, 4$. The remaining variables, $\{w_{u\dot{\alpha}}, \mu^A_u, \bar{\mu}^A_u\}, u \neq v$, are all treated as Gaussian fluctuations around the critical point. To this order, the instanton effective action (2.7) is

$$S = 4\pi^2 \left\{ \zeta \chi^2 + \sum_{u=1}^{N} \left( |\phi_{uv}|^2 |w_{u\dot{\alpha}}|^2 + 1/2 \bar{\mu}^A_u (\Sigma \cdot \phi_{uv} - M)_{AB} \mu^B_u \right) \right\} + \cdots .$$

(3.5)

where $\phi_{uv} \equiv \phi_u - \phi_v$. The integrals are easily done. Note that the integral over $\chi$ yields a factor of $\zeta^{-3}$ which cancels against the factors of $\zeta$ arising from (3.3) and (3.4) so the final result
is, as expected, independent of $\zeta$. Summing over the $N$ critical-point sets gives the centred
one-instanton partition function

$$\hat{Z}_{1,N}^{(N=2*)} = \sum_{v=1}^{N} \prod_{u=1}^{N} \frac{\det_{4}(\Sigma \cdot \phi_{uv} - M)}{\phi_{uv}^4} = \sum_{v=1}^{N} \prod_{u=1}^{N} \left(1 - \frac{m^2}{(\phi_v - \phi_u)^2}\right).$$  \hspace{1cm} (3.6)

Notice that the resulting expression is holomorphic in $\phi$ and independent of $\zeta^c$ as expected.

When $m = 0$, our result (3.6), is simply the integer $N$: precisely the Euler characteristic of $\hat{M}_{1,N}^{(c)}$. This is because the VEV is equivalent to introducing a Morse potential on the moduli
space in the language of the Mathai-Quillen formalism. In the one-instanton example the critical-point set is a set of discrete points, precisely $N$ of them, and therefore the centred
instanton partition function computes the Euler characteristic. For $k > 1$ the critical-point set will include non-compact components and so the partition function will no longer compute a
topological index.

4. Two Instantons

We now evaluate the centred two-instanton partition function using localization. Once again
some of the details are similar to the $N = 2$ with fundamental hypermultiplets discussed in [7].

There are two kinds of critical submanifolds. The first in which $u_1 < u_2$ and the second
when $u_1 = u_2$; i.e. two topicons of different flavours and two of the same flavour, respectively.
We consider the contributions in the next two subsections.

4.1 Topicons of different flavour

For two topicons of flavour $u_1$ and $u_2$, with $u_1 < u_2$, the critical submanifold is

$$\mathcal{M}_{u_1,u_2}^{(c)} \times \mathcal{M}_{u_1,u_2}^{(c)}/\mathbb{R}^4.$$  \hspace{1cm} (4.1)

On this submanifold, the ADHM constraints are solved with

$$w_{u_1u_2} = \sqrt{\zeta} e^{i\theta_i} \delta_{u_1,u_2}, \quad a_n' = \frac{1}{2} \begin{pmatrix} Y_n & 0 \\ 0 & -Y_n \end{pmatrix}. $$  \hspace{1cm} (4.2)

The two phase angles $\theta_i$, $i = 1, 2$, are not genuine moduli since they can be separately rotated
by transformations in the subgroup $U(1)^2 \subset U(2)$ of the auxiliary group. The variables $Y_n$ are
the genuine moduli representing the relative positions of the two topicons. The corresponding solution of the fermionic ADHM constraints \((2.5)\) on the critical submanifold is

\[
\begin{align*}
\mu^A &= \bar{\mu}^A = 0, \\
\mathcal{M}_{\alpha}^A &= \frac{1}{2} \begin{pmatrix} \xi_{\alpha}^A & 0 \\ 0 & -\xi_{\alpha}^A \end{pmatrix},
\end{align*}
\] (4.3)

where \(\xi_{\alpha}^A\) are the eight relative supersymmetric modes of the two topicons.

Including the fluctuations around the critical-point solution, we write

\[
\begin{align*}
a_n' &= \left( \frac{1}{2} Y_n \begin{bmatrix} a_n'_{12} \\ -\frac{1}{2} Y_n \end{bmatrix} \right), \\
\mathcal{M}_{\alpha}^A &= \begin{pmatrix} \frac{1}{2} \xi_{\alpha}^A & \mathcal{M}_{\alpha}^A \end{pmatrix}_{12} \begin{pmatrix} \frac{1}{2} \xi_{\alpha}^A & \mathcal{M}_{\alpha}^A \end{pmatrix}_{21} - \frac{1}{2} \xi_{\alpha}^A. 
\end{align*}
\] (4.4)

In addition, we have the following fluctuations \([w_{\alpha}]_{pq} \equiv w_{u_p q \alpha}, [\mu^A]_{pq} \equiv \mu^A_{u_p q} \) and \([\bar{\mu}^A]_{pq} \equiv \bar{\mu}^A_{u_p q}\), for \(p, q = 1, 2\) and \(2, 1\), as well as the auxiliary variables \(\chi\). It is convenient to make the shift

\[
\chi \rightarrow \chi - \begin{pmatrix} \phi_{u_1} & 0 \\ 0 & \phi_{u_2} \end{pmatrix}, \quad \chi = \begin{pmatrix} [\chi]_{12} \\ [\chi]_{21} \end{pmatrix}
\]

so that \(\chi = 0\) on the critical submanifold. We then integrate over the Lagrange multipliers \(D^c\) and \(\bar{\psi}_{\dot{A}}\) which impose the ADHM constraints \((2.1)\) and \((2.5)\). The two diagonal components of the constraints (in \(i, j\) “instanton” indices) are the ADHM constraints of the two single topicons. The off-diagonal components vanish on the critical-point set and must therefore be expanded to linear order in the fluctuations. Before we write down the constraints, it is necessary to weed-out the fluctuations which correspond to U(2) “gauge transformations” of the critical-point solution. This can be done by imposing the condition that the fluctuations are orthogonal to infinitesimal U(2) transformations acting on the critical-point solution which lie in the coset U(2)/U(1)^2. In turn this is done by inserting the following \(\delta\)-functions and Jacobian into the partition function:

\[
\text{Vol} \left[ \frac{U(2)}{U(1)^2} \right] (\zeta + Y^2)^2 \cdot \prod_{p, q=1}^{2} (\psi e^{i\theta_q} [\bar{\psi}^A]_{pq} + \psi e^{-i\theta_p} [w^2]_{pq} + (-1)^p [\bar{\psi}^A]_{pq} Y_{\alpha \dot{a}}) .
\]

When these gauge-fixing conditions are put together with the ADHM constraints they can be written in a unified way:

\[
\begin{align*}
\sqrt{\zeta} e^{i\theta_q} [\bar{\psi}^A]_{pq} + (-1)^p [\bar{\psi}^A]_{pq} Y_{\alpha_1} &= 0, \\
\sqrt{\zeta} e^{-i\theta_p} [w^2]_{pq} + (-1)^p [\bar{\psi}^A]_{pq} Y_{\alpha_2} &= 0,
\end{align*}
\] (4.7a, 4.7b)

for \(p, q = 1, 2\) and \(2, 1\). In the Grassmann sector, the off-diagonal fermionic ADHM constraints are

\[
\begin{align*}
\sqrt{\zeta} e^{i\theta_q} [\bar{\mu}^A]_{pq} + (-1)^p [\mu^A]_{pq} Y_{\alpha_1} &= (-1)^p [a_{\alpha_1}]_{pq} \xi_{\alpha}^A, \\
\sqrt{\zeta} e^{-i\theta_p} [\bar{\mu}^A]_{pq} + (-1)^p [\mu^A]_{pq} Y_{\alpha_2} &= (-1)^p [a_{\alpha_2}]_{pq} \xi_{\alpha}^A,
\end{align*}
\] (4.8a, 4.8b)
where $Y_{\alpha \dot{\alpha}} = Y_n \sigma_{n \alpha \dot{\alpha}}$, etc. Notice the similarity between the left-hand sides of (4.7a)-(4.7b) and (4.8a)-(4.8b). This arises as a consequence of the fact that once the bosonic fluctuations are gauge fixed, both they, and the Grassmann fluctuations, are geometrically related to tangent vectors to the instanton moduli space at the critical point.\(^\text{12}\) We will use the ADHM constraints (4.7a)-(4.7b) and (4.8a)-(4.8b) to eliminate the fluctuations $[\alpha'_n]_{p q}$ and $[\mathcal{M}^A_{\alpha}]_{p q}$, $p, q = 1, 2$ and $2, 1$. When this done a non-trivial Jacobian factor of $Y^8$ results. Putting this together with the Jacobian factor in (4.6) gives the function

$$Y^8 (\zeta + Y^2)^2$$

that will be required later.

The next problem is to expand the the instanton effective action (2.7) to Gaussian order in the fluctuations. First the bosonic pieces. To Gaussian order around the critical point we decompose

$$S_b = S^{(1)}_b + S^{(2)}_b + S^{(12)}_b + \cdots,$$  \hspace{1cm} (4.10)

where $S^{(i)}_b$ include all the terms that pertain separately to each of the topicons:

$$\frac{1}{4 \pi^2} S^{(i)}_b = \zeta [\chi]_i^2 + \sum_{u=1}^{N} \phi^2_{u_1 u_2} \left| w_{u_1 u_2} \right|^2,$$  \hspace{1cm} (4.11)

while the third term describes the interactions between the topicons:

$$\frac{1}{4 \pi^2} S^{(12)}_b = \phi^2 \left( 1 + \zeta / Y^2 \right) \left( [\bar{w}^\alpha]_{21} [w_{\dot{\alpha}}]_{12} + [\bar{w}^\dot{\alpha}]_{12} [w_{\alpha}]_{21} \right) + 2 (\zeta + Y^2) [\chi]_{21} \cdot [\chi]_{12},$$  \hspace{1cm} (4.12)

where we have defined

$$\phi \equiv \phi_{u_1 u_2}.$$  \hspace{1cm} (4.13)

The fermionic part of the action has a similar decomposition at Gaussian order

$$S_f = S^{(1)}_f + S^{(2)}_f + S^{(12)}_f + \cdots,$$  \hspace{1cm} (4.14)

where

$$\frac{1}{2 \pi^2} S_f^{(i)} = -\frac{1}{8} M_{AB} \xi_i^A \xi_i^B + \sum_{u=1}^{N} \tilde{\mu}^A_{i u} \left( \Sigma \cdot \phi_{u_1 u_2} - M \right)_{AB} \mu^B_{i u},$$  \hspace{1cm} (4.15)

and

$$\frac{1}{2 \pi^2} S_f^{(12)} = [\tilde{\mu}^A_{21} (\phi - M)_{AB} [\mu^B]_{12} - [\mu^A]_{21} (\phi - M)_{AB} [\tilde{\mu}^B]_{12}$$

$$+ [\mathcal{M}^{\alpha A}_{21} (\phi - M)_{AB} [\mathcal{M}^B_{\alpha}]_{12} + [\mathcal{M}^{\dot{\alpha} A}]_{21} [\chi_{AB}]_{12} \xi_\alpha^B + \xi^\alpha \chi_{AB} [\mathcal{M}^B_{\alpha}]_{12}.\]$$

\(^\text{12}\)More precisely, for each $A$ the Grassmann collective coordinates are to symplectic tangent vectors to the hyper-Kähler instanton moduli space: see Ref. [11].

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In the above, and in much of the following, we use the notation
\[ \phi_{AB} \equiv \Sigma_{AB} \cdot \phi, \quad \chi_{AB} \equiv \Sigma_{AB} \cdot \chi. \] (4.17)

The two sets of relative supersymmetric modes \( \xi^A_{\alpha} \) with \( A = \hat{A} = 3, 4 \) and \( A = A' = 1, 2 \), respectively, are treated differently. The integrals over \( \xi^A_{\alpha} \) are saturated by the mass terms in (4.15), whereas those over \( \xi^{A'}_{\alpha} \) are saturated by interactions with the bosonic fluctuations in (4.16). The integrals over the fluctuations \([\mu^A]_{pq}\) and \([\bar{\mu}^A]_{pq}, p \neq q\), are simplified by shifting them by appropriate amounts of \( \xi^{A'}_{\alpha} \) in order to “complete the square”. This gives the equivalent form
\[
\frac{1}{2\pi^2} S_t^{(12)} = (1 + \zeta/Y^2) \left\{ [\bar{\mu}^A]_{21} (\phi - M)_{AB} [\mu^B]_{12} - [\mu^A]_{21} (\phi - M)_{AB} [\bar{\mu}^B]_{12} \right\} \\
+ (\xi + Y^2)^{-1} \xi^{A'} \left\{ \phi^{A'B'} [a'_{\alpha\hat{\alpha}}]_{21} [\bar{a}'^{\hat{\alpha}\beta}]_{12} - [a'_{\alpha\hat{\alpha}}]_{21} Y^{\hat{\alpha}\beta} [\chi_{A'B'}]_{12} \\
- [\chi_{A'B'}]_{21} Y_{\alpha\hat{\alpha}} [\bar{a}'^{\hat{\alpha}\beta}]_{12} - \zeta [\chi_{A'C}]_{21} (\phi - M)^{-1CD} [\chi_{DB'}]_{12} \right\} \xi_{B'}. \] (4.18)

In (4.18) we should substitute for \([a'_{\alpha\hat{\alpha}}]_{pq}\) by using the ADHM constraints (4.7a)-(4.7b).

Now we begin to integrate. First of all, the integrals over \( w_{u\hat{\alpha}}, \mu^A_{u1}, \bar{\mu}^{A'}_{1u} \), for \( u \neq u_2 \), and \([\chi],[\chi]_1\) completely decouple from the remaining integrals (and similarly for \( 1 \leftrightarrow 2 \)). These integrals are identical to the single instanton integrals done in Section 3. What results is the non-trivial factor
\[
\prod_{i=1}^{2} \prod_{u=1}^{N} \left( 1 - \frac{m^2}{\phi^{2u_i}} \right). \] (4.19)

Now we describe the remaining integrals over the fluctuations the couple the two topicons. The integrals over the Grassmann fluctuations \([\mu^A]_{pq}\) and \([\bar{\mu}^A]_{pq}\) produces the factors\(^{13}\)
\[
(1 + \zeta/Y^2)^8 \phi^{*4}(m^2 - \phi^2)^2. \] (4.20)

Next we integrate over the relative supersymmetric modes \( \xi^A_{\alpha} \). The four \( \xi^{A'}_{\alpha} \) are saturated by the mass terms in (4.15) leaving the four \( \xi^{A'}_{\alpha} \) to be saturated by the interactions in (4.18). Integrating out these latter four variables produces a number of terms, however, many of them involve odd functions of some of the components of the six-vector \([\chi]_{pq}\) and will subsequently

\(^{13}\)In the following we shall not indicate the appropriate multiplicative numerical factors but simply collect them in the final expression.
integrate to zero. The only terms which subsequently lead to non-zero contributions are

\[
\zeta^2 (\zeta + Y^2)^{-2} \left\{ Y^{-4} \phi^{*2} \left( ([\bar{\omega}]^{(1)}_{21}[w_{\alpha}]_{12} + [\bar{\omega}]^{(2)}_{12}[w_{\alpha}]_{21})^2 - 4[\bar{\omega}]^{(1)}_{12}[w_{\alpha}]_{12}[\bar{\omega}]^{(2)}_{21}[w_{\alpha}]_{21} \right) - \frac{4m^2}{(m^2 - \phi^2)^2} \sum_{\alpha = 3}^{6} [\chi_{\alpha}]_{21}[\chi_{\alpha}]_{12}^2 + 2[\chi_{3}]_{21}[\chi_{5}]_{21}[\chi_{3}]_{12}[\chi_{5}]_{12} + 2[\chi_{4}]_{21}[\chi_{6}]_{21}[\chi_{4}]_{12}[\chi_{6}]_{12} \right\} \quad (4.21)
\]

Now we turn to the bosonic fluctuations \([w_{\alpha}]_{pq}\) and \([\chi]_{pq}\), \(p, q = 1, 2\) and \(2, 1\). The integrals involve (4.21) as an insertion into the Gaussian integrals. The result, when amalgamated with the non-trivial factors in Eqs. (4.9) and (4.20), is

\[
m^2 \left(1 - \frac{m^2}{\phi^2} \right)^2 \left[ \frac{1}{\phi^2 - \frac{1}{2(\phi + m)^2}} - \frac{1}{2(\phi - m)^2} \right] \frac{\zeta^2}{(\zeta + Y^2)^4} \quad (4.22)
\]

We can now integrate over the relative position of the topicons:

\[
\int d^4Y \frac{\zeta^2}{(\zeta + Y^2)^4} = \frac{\pi^2}{6} \quad (4.23)
\]

Finally putting all the non-trivial factors together with the correct numerical factors and restoring the notation \(\phi \equiv \phi_{u_1 u_2}\), gives the final contribution of the critical-point set to the centred instanton partition function

\[
2m^2 \left(1 - \frac{m^2}{\phi_{u_1 u_2}^2} \right)^2 \left[ \frac{1}{\phi_{u_1 u_2}^2 - \frac{1}{2(\phi_{u_1 u_2} + m)^2}} - \frac{1}{2(\phi_{u_1 u_2} - m)^2} \right] \prod_{i=1}^{2} \prod_{i=1}^{N} \left(1 - \frac{m^2}{\phi_{u_i u_i}^2} \right) \quad (4.24)
\]

Notice that the result is holomorphic in the VEVs and independent of \(\zeta^c\) as expected by our general cohomological argument. Summing over the \(\frac{1}{2} N(N - 1)\) critical-point sets of this type gives the following contribution to the partition function:

\[
m^2 \sum_{\substack{u, v = 1 \atop (u \neq v)}}^{N} T_u(\phi_u)T_v(\phi_v) \left[ \frac{1}{\phi_{u v}^2 - \frac{1}{2(\phi_{u v} - m)^2}} - \frac{1}{2(\phi_{u v} + m)^2} \right] \quad (4.25)
\]

where we have written the answer in terms of the functions \(T_u(x)\) defined in (1.6).

**4.2 Topicons of the same flavour**

There are \(N\) critical-points describing two topicons of the same flavour \(u_1 = u_2 \equiv v \in \{1, \ldots, N\}\). On the critical submanifold \(\{w_{u_1}, a'_{\alpha}\}\) and \(\{\mu_{u_1}, \bar{\mu}^A_{u_1}, \mathcal{M}^A_{\alpha}\}\) satisfy the ADHM
constraints, (2.1) and (2.5), respectively, of two instantons in a non-commutative U(1) theory. In other words the critical submanifold is simply

\[ \hat{\mathcal{M}}^{(2)}_{2,1}. \]  

(4.26)

The remaining variables all vanish and are treated as fluctuations.

As previously, it is convenient to shift the auxiliary variable \( \chi \) by its critical-point value:

\[ \chi \rightarrow \chi - \phi_v^{1[2] \times [2]} . \]  

(4.27)

We now expand in the fluctuations \( \{ w_{ui\dot{a}}, \mu^A_{ui}, \bar{\mu}^A_{ui} \} \), for \( u \neq v \). Since all the components of the ADHM constraints are non-trivial at leading order the fluctuations decouple from the \( \delta \)-functions which impose the constraints (2.1) and (2.5). The fluctuation integrals only involve the integrand \( \exp(-S - mS_{\text{mass}}) \), where the action is expanded to Gaussian order around the critical submanifold. However, it is important, as we shall see below, to leave \( \chi \) arbitrary rather than set it to its critical-point value; namely, \( \chi = 0 \) (after the shift (4.27)). The fluctuation integrals produce the non-trivial factor

\[ G(\chi) = \prod_{u=1}^{N} \det_2 \left( 1_{[2] \times [2]} - m^2 (\chi + \phi_v^{1[2] \times [2]})^{-2} \right) = T_v(\phi_v - \lambda_1)T_v(\phi_v - \lambda_2) . \]  

(4.28)

Here, \( \lambda_i, i = 1, 2 \), are the eigenvalues of the \( 2 \times 2 \) matrix \( \chi \) and \( T_u(x) \) was defined in (1.6).

The remaining integral is of the form

\[ \int_{\hat{\mathcal{M}}^{(2)}_{2,1}} \omega^{(N=4)} e^{-S - mS_{\text{mass}}} G(\chi) . \]  

(4.29)

where \( G(\chi) \) is the non-trivial function (4.28). There are two types of contribution depending on whether the mass term \( S_{\text{mass}} \) is employed to saturate any Grassmann integrals. For the first \( S_{\text{mass}} \) is not used and then it is easy to see that the SU(4) symmetry of the resulting integral means that any insertion of powers of \( \lambda_i \) integrate to zero. Hence, only the value of \( G(\chi) \) at \( \chi = 0 \) contributes. The second occurs when the mass terms are used to saturate the integrals over the four Grassmann collective coordinates left over from the set of eight \( \{ \mu^A, \bar{\mu}^A, M_\alpha^A \} \) once the ADHM constraints (2.5) have been imposed. What is left is the volume form of the \( \mathcal{N} = 2 \) theory since, when the coordinates \( \{ \mu^A, \bar{\mu}^A, M_\alpha^A \} \) are set to zero, the instanton effective action reduces to that of the \( \mathcal{N} = 2 \) theory denoted \( S^{(N=2)} \). This integral is precisely the same as the one that appeared in Ref. [7] where we argued that only terms quadratic in the expansion.
of $\mathcal{G}(\chi)$ contribute. Hence, (4.29) is

$$
\int_{\mathcal{M}_{2,1}} \omega^{(N=4)} e^{-S} \rho^2 \mathcal{G}(\chi)
= T_v(\phi_v)^2 \int_{\mathcal{M}_{2,1}} \omega^{(N=4)} e^{-S} m^2 \left( \frac{\partial T_v(\phi_v)}{\partial \phi_v} \right)^2 \int_{\mathcal{M}_{2,1}} \omega^{(N=2)} e^{-S} \lambda_1 \lambda_2
$$

(4.30)

$$
+ \frac{1}{2} m^2 T_v(\phi_v) \frac{\partial^2 T_v(\phi_v)}{\partial \phi_v^2} \int_{\mathcal{M}_{2,1}} \omega^{(N=2)} e^{-S} (\lambda_1^2 + \lambda_2^2)
.$$  

In the above, the first term here involves the centred instanton partition on the non-commutative $U(1)$ two-instanton moduli space. This integral can be evaluated explicitly using the formulae of the Appendix in Ref. [7]. However, from our earlier discussion we also know that this partition function is precisely the Gauss-Bonnet-Chern integral on $\mathcal{M}_{2,1}$ [11] and this four-dimensional space is the Eguchi-Hanson manifold [35] whose Gauss-Bonnet-Chern integral is well known to be $\frac{3}{2}$ [25]. The other terms were evaluated in [7]:

$$
\int_{\mathcal{M}_{2,1}} \omega^{(N=2)} e^{-S} \lambda_1 \lambda_2 = 0, \quad \int_{\mathcal{M}_{2,1}} \omega^{(N=2)} e^{-S} (\lambda_1^2 + \lambda_2^2) = \frac{1}{2}.
$$

(4.31)

Hence, the final result for the contributions from two topicons of the same flavour to the centred instanton partition function is

$$
\sum_{u=1}^{N} \left( \frac{3}{2} T_u(\phi_u)^2 + \frac{1}{4} m^2 T_u(\phi_u) \frac{\partial^2 T_u(\phi_u)}{\partial \phi_u^2} \right).
$$

(4.32)

Finally, summing (4.32) and (4.25) we have the centred two-instanton partition function

$$
\hat{Z}_{2,N}^{(N=2^*)} = \sum_{u=1}^{N} \left( \frac{3}{2} T_u(\phi_u)^2 + \frac{1}{4} m^2 T_u(\phi_u) \frac{\partial^2 T_u(\phi_u)}{\partial \phi_u^2} \right)
$$

$$
+ m^2 \sum_{u,v=1}^{N} T_u(\phi_u) T_v(\phi_v) \left[ \frac{1}{\phi_{uv}^2} - \frac{1}{2(\phi_{uv} - m)^2} - \frac{1}{2(\phi_{uv} + m)^2} \right].
$$

(4.33)

Using the relation (1.3), we find precisely the Seiberg-Witten prediction for the $k = 2$ instanton coefficient of the prepotential (1.5b).

## 5. Arbitrary Instanton Number

Now we proceed to the wholly more ambitious proposition of calculating the prepotential for instanton charge $k > 2$. In fact we shall show that it is possible to calculate the terms of $O(m^4)$.
in the prepotential for all instanton charge. These are the first non-trivial, i.e. VEV-dependent, terms in the expansion of the instanton portion of the prepotential in $m^2$. The prediction from Seiberg-Witten theory follows from Ref. [20] and is quoted in (1.10). The verification of this prediction will provide by far the most stringent test of Seiberg-Witten theory to date.

The key step in evaluating the $\mathcal{O}(m^4)$ contribution to $\mathcal{F}_k$ is the following:

**Proposition:** The $\mathcal{O}(m^2)$ contribution to $\hat{\mathcal{Z}}_{k,N}^{(N=2^*)}$ comes exclusively from $k$ topicons of the same flavour.

**Proof:** First of all, it is easy to see that if the component of the critical-point set corresponds to a partition with at least 3 non-trivial blocks, i.e. some $k_{u_1}, k_{u_2}, k_{u_3}, \ldots, k_{u_p} > 0$, with $p > 2$, then the contribution must be at least $\mathcal{O}(m^6)$. The reason is that the supersymmetric modes associated to each flavour of topicon, $\text{tr}_{k_{u_\ell}} M_A^{\alpha A}, \ell = 1, \ldots, p$, number in total $4(p-1)$. The integrals over these variables must be saturated by mass terms, giving a factor of $m^2(p-1)$ in the contribution to $\hat{\mathcal{Z}}_{k,N}^{(N=2^*)}$. So for $p > 2$ these give at least $\mathcal{O}(m^6)$ contributions to the prepotential.

The argument above fails when there are precisely two blocks, i.e. just two flavours of topicons, and this case must be considered more explicitly. We have already seen by explicit computation in the case $k = 2$ in Section 4.1, and in particular (4.25), that this contribution is actually $\mathcal{O}(m^4)$ in partition function and so $\mathcal{O}(m^6)$ in the prepotential. However, when one examines the reason for this it is because the term in square brackets in (4.25) vanishes like $\mathcal{O}(m^2)$. In other words, in order to see the result relies on a delicate cancellation between terms. So we must prove something similar occurs in a partition $k \rightarrow k_1 + k_2$ with $k > 2$ and this, unfortunately, requires us to perform the integrals over the fluctuations explicitly.

First of all, let us separate out the variables $\{w_{ui\alpha}, \mu^A_{ui}, \bar{\mu}^A_{iu}\}, u \neq u_1, u_2$. As we have seen in Section 4.1, the integrals over these fluctuations decouples from the the rest and they will play no rôle in the following argument. The remaining variables then have an obvious $2 \times 2$ block form, where the blocks are of size $k_1$ and $k_2$. We will indicate the diagonal block by $[a'_{n}]_{p}$, $p = 1, 2$, and the off-diagonal blocks by $[a'_n]_{pq}$, $p, q = 1, 2$ and $2, 1$, etc. The two block-diagonal elements satisfy the ADHM constraints of a charge $k_1$, respectively $k_2$, $U(1)$ instanton which parameterize the critical submanifold. The off-diagonal variables are the treated as fluctuations.

Each of the diagonal blocks has 8 supersymmetric modes. However, since we are considering

\begin{itemize}
\item The traces here are taken in each block and the $-4$ arises because of the overall traceless condition which implies $\sum_{\ell} \text{tr}_{k_{u_\ell}} M^{\alpha A}_{\ell} = 0$.
\item Note that $[a'_{n}]_{pq}$ is a $k_p \times k_q$ matrix, while $[w_{ui}]_{pq}$ is a $1 \times k_q$ matrix, etc.
\end{itemize}

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the centred instanton moduli space only the 8 relative supersymmetric modes defined by
\[ M_0^A = \frac{1}{k} \begin{pmatrix} k_21_{[k_1]×[k_1]} & 0 \\ 0 & -k_11_{[k_2]×[k_2]} \end{pmatrix} \xi^A \]
are relevant. At leading order around the critical submanifold, the integrals over the four modes \( \xi^k_\alpha \) must be lifted by the mass terms \( S_{mass} \) yielding a factor of \( m^2 \). Hence we must show that the remaining integral vanishes at \( O(m^0) \) in order to complete the proof. We now focus on the other four relative supersymmetric modes \( \xi^A_\alpha \). The integrals over these variables must be lifted via interactions with the fluctuations as we found in the case \( k_1 = k_2 = 1 \) in Section 4.1. In fact the following analysis is very similar to that in Section 4.1 but with the added complication that we have to keep track of the additional non-trivial matrix structure for \( k_1, k_2 > 1 \).

First of all, consider the issue of gauge fixing. As in Section 4.1 we will fix the gauge by demanding that the fluctuations are orthogonal to \( U(k) \) transformations acting on the critical-point solution. This is convenient because then, as we saw in Section 4.1, the ADHM constraints along with the extra gauge-fixing constraint imply that the bosonic fluctuations satisfy the fermionic ADHM constraints at the critical point. The gauge-fixing conditions are
\[ \left[ \bar{\omega}^\alpha_{\mu_\alpha} \right]_{p, q} + \left[ \bar{\omega}^\alpha_{\mu_\beta} \right]_{p, q} + \left[ \bar{\alpha}^\alpha_{\beta} \right]_{p, q} \alpha^\alpha_{p, q} = 0 \]  
\[ (5.2) \]
Hence, the off-diagonal ADHM constraints plus gauge-fixed conditions are
\[ \left[ \bar{\omega}^\alpha_{\mu_\alpha} \right]_{p, q} + \left[ \bar{\omega}^\alpha_{\mu_\beta} \right]_{p, q} + \left[ \bar{\alpha}^\alpha_{\beta} \right]_{p, q} \alpha^\alpha_{p, q} = 0 \]  
\[ (5.3) \]
with no sum on \( p, q \). Similarly the off-diagonal fermionic ADHM constraints are
\[ \left[ \bar{\omega}^\alpha_{\mu_\alpha} \right]_{p, q} + \left[ \bar{\omega}^\alpha_{\mu_\beta} \right]_{p, q} + \left[ \bar{\alpha}^\alpha_{\beta} \right]_{p, q} \alpha^\alpha_{p, q} = 0 \]  
\[ (5.4) \]
Note the similarity between the left-hand sides of (5.3) and (5.4).

The total number of constraints (5.3) and (5.4) is \( 8k_1k_2 \) and \( 16k_1k_2 \) respectively, matching precisely the number for the fluctuations \( \alpha^\alpha_{n, p} \) or \( M^{A, A}_p, p, q = 1, 2 \) and \( 2, 1 \). Hence, we can use the constraints to eliminate these variables. The action for the fluctuations follows by expanding (2.7) to Gaussian order. In the bosonic sector, the relevant terms are
\[ \frac{1}{4\pi^2} S^{(12)}_b = \text{tr} \left\{ \phi^2 \left( \bar{\omega}^\alpha , -w^\alpha \right)_{21} \Delta \left( w^\alpha \right)_{12} + 2\chi_{21} \cdot \Omega \chi_{12} \right\} \]  
\[ (5.5) \]
where we have defined the following linear operators which depend on the critical-point topicon collective coordinates
\[ A^\alpha_{\beta, 12} \left( w^\alpha \right)_{12} = \left[ \bar{\omega}^\beta_{1, 12} w^\alpha_{1} \right]_{12} + \left[ \bar{\omega}^\alpha_{1, 12} w^\beta_{1} \right]_{12} \]  
\[ B_{\beta, 21} \left( w^\alpha \right)_{21} = \left[ \bar{\omega}^\alpha_{2, 21} w^\alpha_{1} \right]_{21} + \left[ \bar{\omega}^\alpha_{2, 21} w^\beta_{1} \right]_{21} \]  
\[ \Theta \left( \chi \right)_{12} = \Theta \left( \alpha^\alpha_{\alpha, \beta} \right)_{12} - \left[ \alpha^\alpha_{\alpha, \beta} \right]_{12} \Theta \]  
\[ (5.6) \]
in terms of which\textsuperscript{16}

\[
\Omega = \frac{1}{2}(A^\beta B_\beta + \bar{Y}^{\dot{\alpha}\alpha} Y_{\alpha\dot{\alpha}}), \quad \Delta = 1 + B_\alpha \bar{Y}^{-1\dot{\alpha}\alpha} Y^{-1}_{\alpha\beta} A^\beta. 
\]  

(5.7)

In the above, \(\Theta\) is an arbitrary \(k_1 \times k_2\) matrix. The resulting machinations are greatly simplified by noting that for an arbitrary \(k_1 \times k_2\) matrix \(\Theta\)

\[
(A^\alpha B_\beta + \bar{Y}^{\dot{\alpha}\alpha} Y_{\alpha\dot{\alpha}}) \Theta = [\bar{w}^\dot{\alpha} w_\beta + \bar{a}^{\dot{\alpha}\alpha} a_{\alpha\dot{\beta}}]_1 \Theta - [\bar{w}^\dot{\alpha} w_\beta - a_{\alpha\dot{\beta}} \bar{a}^{\dot{\alpha}\alpha}]_2 \\
- [\bar{a}^{\dot{\alpha}\alpha}]_1 \Theta [a'_{\alpha\dot{\beta}}]_2 - [a'_{\alpha\dot{\beta}}]_1 \Theta [\bar{a}^{\dot{\alpha}\alpha}]_2
\]

(5.8)

where we employed the ADHM constraints of the critical-point solutions:

\[
[\bar{w}^\dot{\alpha} w_\beta + \bar{a}^{\dot{\alpha}\alpha} a'_{\alpha\dot{\beta}}]_p = \frac{1}{2} \xi^{\dot{\alpha}} \tau^{\dot{\alpha}} [\Theta]_p [\bar{\Theta}]_{\dot{\alpha}}. 
\]

(5.9)

which defines the \(k_p \times k_p\) matrices \([\sigma]_p\). In addition, (5.8) implies

\[
\Omega \Theta = [\sigma]_1 \Theta + \Theta [\sigma]_2 - 2[a'_{\alpha\dot{\beta}}]_1 \Theta [a'_{\alpha\dot{\beta}}]_2. 
\]

(5.10)

Now we turn to the Grassmann fluctuations. The fermionic ADHM constraints (5.4) are used to eliminate the fluctuations \([M^A_{\alpha \dot{A}}]_{pq}\). Notice, the right-hand sides of these constraints involves the Grassmann collective coordinates of the critical-point solution. To leading order in the mass we can ignore the coupling of the fluctuations to all but the relative supersymmetric modes \(\xi^A_{\alpha} \) in (5.1). The reason is that these modes are not lifted by any other effects, while using the constraints to lift other modes of the critical-point solution inevitably ends up costing powers of \(m\). At leading order in our mass expansion we can also set \(m = 0\) in the action of the fluctuations:

\[
\frac{1}{2\pi^2} S^{12}_t = \text{tr} \left\{ \left( \bar{\mu}^A, -\mu^A \right)_{21} \phi_{AB} \left( \mu^B, \bar{\mu}^B \right)_{12} \right\} \\
+ \left[ \left( \bar{\mu}^A, -\mu^A \right)_{21} B_\alpha \bar{Y}^{-1\dot{\alpha}\alpha} - \xi_{\alpha}^C \left[ X_{CDB}, 21 \phi^{-1DA} \right] \phi_{AB} \right] \\
\times \left[ Y^{-1}_{\alpha\beta} A^\beta \left( \mu^B, \bar{\mu}^B \right)_{12} - Y^{-1}_{\alpha\beta} \xi^B_{\dot{\alpha} \dot{\beta}} \right]_{12} \xi^F_{\dot{\alpha}} + \phi^{-1BE}_{\alpha} \left[ X_{EF}, 12 \xi^F_{\dot{\alpha}} \right] \right\} .
\]

(5.11)

Now we shift the fluctuations \([\mu^A]_{pq}\) and \([\bar{\mu}^A]_{pq}\) by appropriate amounts of \(\xi^A_{\alpha}\) in order to “complete the square”. This leaves the terms which are responsible for the lifting the relative

\textsuperscript{16}To connect with Section 4.1, in the case \(k_1 = k_2 = 1\) we have \(\Omega = \zeta + Y^2\) and \(\Delta = 1 + \zeta/Y^2\).
supersymmetric modes $\xi_{\alpha'}^{A'}$. The expression involves a quadratic interaction of $\xi_{\alpha'}^{A'}$:

$$
\frac{1}{4\pi^2}S_{\text{int}} = \xi^{A'}_{\alpha} \text{tr}\left\{ \phi_{A'B'}[a'_{\alpha}],12 \Omega^{-1}[\delta^{\alpha\beta}_{1}],12 - [a'_{\alpha}]_{21} \Omega^{-1}Y^{\delta\beta} [\chi_{A'B'}]_{12} - [\chi_{A'B'}]_{21} Y_{\alpha\alpha} \Omega^{-1}[\delta^{\alpha\beta}]_{12} - \delta^{\alpha}_{\beta} [\chi_{ABC}]_{21} \phi^{-1CD}(1 - Y_{\nu} \Omega^{-1}Y_{\nu})[\chi_{DB'}]_{12} \right\} \xi_{\beta}^{B'}.
$$

(5.12)

In our conventions

$$
Y_{\alpha\alpha} = Y_{n} \sigma_{n\alpha} , \quad Y^{\alpha\alpha} = Y_{n} \sigma_{n\alpha} , \quad Y^{-1}_{\alpha\alpha} Y^{\alpha\beta} = \delta_{\alpha}^{\beta} , \quad Y^{-1}_{\alpha\alpha} Y_{\alpha\beta} = \delta_{\alpha}^{\beta} .
$$

(5.13)

The integrals over the four relative supersymmetric modes are saturated by pulling down two powers of the action (5.12). This yields an expression which is quartic in the bosonic fluctuations. The constraints (5.3) are used to eliminate $[a'_{\alpha\nu}]_{pq}$ in favour of $[w_{\alpha}]_{pq}$. The remaining expression is rather complicated, however, many of the terms will subsequently integrate to zero since they are odd functions in some of the components of the bosonic fluctuations. Ignoring these terms, the relevant part of the integrand is

$$
\phi^{*2} \left( F_{\beta}^{\alpha} F_{\beta}^{\alpha} + F_{\beta}^{\alpha} F_{\alpha}^{\beta} \right)
$$

$$
- \frac{2}{\phi^{*2}} \sum_{\alpha, \beta = 1}^{6} \text{tr}\left\{ [\chi_{\alpha}]_{21} (1 - Y_{\nu} \Omega^{-1}Y_{\nu})[\chi_{\beta}]_{12} \right\} \text{tr}\left\{ [\chi_{\nu}]_{21} (1 - Y_{\nu} \Omega^{-1}Y_{\nu})[\chi_{\alpha}]_{12} \right\} + \ldots ,
$$

(5.14)

where

$$
F_{\beta}^{\alpha} = \text{tr}\left\{ (\bar{w}^{\dot{\alpha}} , - w^{\dot{\alpha}})_{21} B_{\beta} Y^{-1\dot{\alpha}} \Omega^{-1} Y^{-1 \beta} A^{\dot{\alpha}} (\bar{w}_{\dot{\alpha}}) \right\} .
$$

(5.15)

What remains is to integrate over the Gaussian fluctuations $[w]_{pq}$ and $[\chi]_{pq}$, which are governed by the action (5.5), with with (5.14), a quartic function of the fluctuations, as an integrand. Let us concentrate on the first term in (5.14), quartic in the fluctuations $[w]_{pq}$. These variables have a “propagator”

$$
\frac{1}{\phi^{*2}} \Delta^{-1} \delta^{\alpha}_{\dot{\alpha}} \Delta^{-1} .
$$

(5.16)

Performing the Gaussian integrals yields two types of term: the first consisting of a product of two “trace” factors and a second with a single trace. Let us dispense with the first. The contributions are of the form of the first set of terms in (5.14) with

$$
F_{\beta}^{\alpha} \longrightarrow \frac{1}{\phi^{*2}} \text{tr}\left( B_{\alpha} Y^{-1\dot{\alpha}} \Omega^{-1} Y^{-1 \beta} A^{\dot{\alpha}} \Delta^{-1} \right) .
$$

(5.17)

Now we use the identity

$$
Y^{-1 \beta} A^{\dot{\alpha}} \Delta^{-1} B_{\beta} Y^{-1\dot{\alpha}} = \delta_{\alpha}^{\beta} - \frac{1}{2} Y_{\beta\dot{\alpha}} \Omega^{-1} Y^{\dot{\alpha}} = \delta_{\beta}^{\alpha} (1 - Y_{\nu} \Omega^{-1}Y_{\nu})
$$

(5.18)
to prove that the right-hand of (5.17) is proportional to $\delta^\alpha_\beta$. Given this and the combination of $F_{\alpha}^\beta$ in (5.14), we see that the double trace terms do not contribute. The single trace contribution is

$$
\frac{1}{\phi^2} \text{tr} \left( B_\alpha \tilde{Y}^{-1\dot{\alpha}} \Omega^{-1} Y^{-1} A^\dot{\alpha} \Delta^{-1} B_\gamma \tilde{Y}^{-1\dot{\gamma}} \Omega^{-1} Y^{-1} A^\dot{\gamma} \Delta^{-1} \\
+ B_\alpha \tilde{Y}^{-1\dot{\alpha}} \Omega^{-1} Y^{-1} A^\dot{\alpha} \Delta^{-1} B_\gamma \tilde{Y}^{-1\dot{\gamma}} \Omega^{-1} Y^{-1} A^\dot{\gamma} \Delta^{-1} \right)
$$

$$
= \frac{6}{\phi^2} \text{tr} \left\{ (1 - Y_n \Omega^{-1} Y_n) \Omega^{-1} (1 - Y_n \Omega^{-1} Y_n) \Omega^{-1} \right\}.
$$

where in the second line we used the identity (5.18).

Now we turn to the contribution coming from the term quartic in $[\chi_a]_{pq}$ in (5.14). From (5.5), the “propagator” for the $[\chi_a]_{pq}$ fluctuations is

$$
\frac{1}{2} \delta_{ab} \Omega^{-1}.
$$

Hence the contribution from these terms has the form

$$
- \frac{6}{\phi^2} \text{tr} \left\{ (1 - Y_n \Omega^{-1} Y_n) \Omega^{-1} (1 - Y_n \Omega^{-1} Y_n) \Omega^{-1} \right\}.
$$

Remarkably (5.19) precisely cancels the contribution from (5.21), generalizing the vanishing of the square bracket in (4.25) for $m = 0$. This completes the proof that the contribution from the critical submanifolds $\hat{\mathcal{M}}(\zeta)_{k_1} \times \hat{\mathcal{M}}(\zeta)_{k_2} / \mathbb{R}^4$ to the partition function vanishes to $\mathcal{O}(m^2)$. QED

Finally we consider the $N$ cases when the critical submanifold is $\hat{\mathcal{M}}(\zeta)_{k,1}$ corresponding to $k$ topicons all of the same flavour. This is the situation considered in Section 4.2 for $k = 2$ and the generalization to $k > 2$ is reasonably straightforward. We focus on topicons of flavour $v$. Integrating over the fluctuations $\{w_{\mu A}, \mu^A_{\mu}, \bar{\mu}^A_{\mu}\}, u \neq v$ produces the factor

$$
\mathcal{G}(\chi) = \prod_{i=1}^k T_v(\phi_v - \lambda_i),
$$

where $\lambda_i$ are the eigenvalues of the $k \times k$ matrix $\chi$. The remaining integral is then

$$
\int \hat{\mathcal{M}}_{k,1} (N=4) e^{-S-mS_{\text{mass}}} \mathcal{G}(\chi).
$$

We can now expand the factor $\exp -mS_{\text{mass}}$ in powers of the mass. Due to the counting of fermion zero modes, non-trivial terms arise when only even powers of this term are pulled down. A term of order $m^{2p}$ carries with it $4p$ factors of the Grassmann collective coordinates indexed by $A = 3, 4$, i.e. collective coordinates in the set $\{\mu^A, \bar{\mu}^A, M^A\}$. Consequently there is a mismatch between the remaining Grassmann collective coordinates $\{\mu^A, \bar{\mu}^A, M^A\}$ and
\{\mu^A, \bar{\mu}^A, \mathcal{M}^A\}. The excess of the latter set by 4p means that after integrating them out the integrand will contain a factor of \((\chi^\dagger)^{2p}\) times an SU(4)-invariant function of \(\chi\). Consequently in order to have a non-trivial integral we must expand \(G(\chi)\) to \((2p)\)th order in \(\chi\), or, equivalently, its eigenvalues \(\{\lambda_i\}\).

To order \(m^2\), we are interested in the first two terms in the expansion. The first is simply
\[
T_v(\phi_v)^k \int_{\bar{\mathfrak{M}}_{k,1}} \omega^{(N=4)} e^{-S},
\]
while the second term is of the form
\[
\frac{1}{2}m^2 \sum_{i,j=1}^k \frac{\partial^2 G(\chi)}{\partial \lambda_i \partial \lambda_j} \bigg|_{\chi=0} \int_{\bar{\mathfrak{M}}_{k,1}} \omega^{(N=4)} e^{-S(S_{\text{mass}})^2 \lambda_i \lambda_j}.
\]
However, from the form of \(G(\chi)\) in (5.22) one readily shows that
\[
\frac{\partial^2 G(\chi)}{\partial \lambda_i \partial \lambda_j} \bigg|_{\chi=0},
\]
is actually \(O(m^2)\) so that the only contribution at \(O(m^2)\) comes from (5.24) alone. The only mass dependence is in the pre-factor and expanding to \(O(m^2)\) one finds
\[
-km^2 \sum_{d | k} \sum_{u,v=1}^{N} \frac{1}{(\phi_u - \phi_v)^2} \int_{\bar{\mathfrak{M}}_{k,1}} \omega^{(N=4)} e^{-S}.
\]
We recognize the integral over \(\bar{\mathfrak{M}}_{k,1}\) as the Gauss-Bonnet-Chern integral of the resolved instanton moduli space. While this integral has not been explicitly evaluated for all \(k\), there are strong indirect arguments which give the general formula for \(k\)-instantons [26,36]:
\[
\hat{Z}_{k,1}^{(N=4)} = \int_{\bar{\mathfrak{M}}_{k,1}} \omega^{(N=4)} e^{-S} = \sum_{d | k} \frac{1}{d},
\]
where the sum is over the integer divisors of \(k\).

Summing over the \(N\) types of topicon gives our final result for the partition function to \(O(m^2)\):
\[
\hat{Z}_{k,N}^{(N=2^7)} = -m^2 k \left( \sum_{d | k} \frac{1}{d} \right) \sum_{u,v=1}^{N} \frac{1}{(\phi_u - \phi_v)^2} + O(m^4),
\]
modulo an irrelevant constant. Using the fact that \(k \sum_{d | k} d^{-1} = \sum_{d | k} d\), and employing (1.3), we have for the \(O(m^4)\) contribution to the prepotential
\[
\mathcal{F}_k = m^4 \left( \sum_{d | k} \frac{1}{d} \right) \sum_{u,v=1}^{N} \frac{1}{(\phi_u - \phi_v)^2} + O(m^6).
\]
This matches the prediction from Seiberg-Witten theory (1.10) exactly.
6. Discussion

We have already noted that our results are in precise agreement with the predictions coming from Seiberg-Witten Theory. We can also compare the results of our calculation at the one-instanton level with gauge group SU(2) with the brute-force integral over the instanton moduli space in the commutative theory performed in Ref. [4]. Written in our notation the result in that reference is\(^{17}\)

\[
\mathcal{F}_1 = -\frac{m^2}{2} + \frac{2m^4}{\phi^2} \tag{6.1}
\]

to compare with our result in the non-commutative theory (from (1.3) and (3.6))

\[
\mathcal{F}_1^{nc} = -2m^2 \left(1 - \frac{m^2}{\phi^2}\right) . \tag{6.2}
\]

The mismatch between (6.1) and (6.2) can be understood in precisely the way similar mismatches between the commutative and non-commutative expressions for the prepotential were explained in [7]. As described in [7] the integral over the resolved instanton moduli space of the non-commutative theory misses contributions from the singularities of the instanton moduli space in the commutative theory. At the one-instanton level the contribution to the centred instanton partition function from the singularity is independent of the VEV and was calculated in Ref. [26] for arbitrary \(N\):

\[
\mathcal{S}_{1,N} = -\frac{2\Gamma(N + \frac{1}{2})}{\Gamma(N)\Gamma(\frac{1}{2})} , \tag{6.3}
\]

so equal to \(-\frac{3}{2}\) for \(N = 2\): precisely accounting for the mismatch between (6.1) and (6.2). However, this mismatch does not lead to any physical difference between the commutative and non-commutative theories since it is independent of the VEV.

The localization that we have described can be given a nice visual interpretation involving D-branes in Type II string theory. It is now well established that instantons in U(\(N\)) gauge theory correspond to a certain decoupling limit of a configuration of D-instantons in the vicinity of \(N\) D3-branes. More generally it is useful to consider the Dp-D(p + 4)-brane system. The instanton calculus is then obtained as the dimensional reduction of the theory on the \(k\) Dp-branes which is some U(\(k\)) gauge theory that can be formulated as a theory with eight supercharges in a maximum dimension of six. This explains why the \(\mathcal{N} = 4\) instanton partition function is the dimensional reduction of a gauged linear \(\sigma\)-model in six dimensions with \(\mathcal{N'} = (1, 0)\) supersymmetry [11, 21]. In general this theory has a Higgs branch which corresponds to a situation where all the Dp-branes have been absorbed onto the D(p + 4)-branes, a Coulomb branch where all the Dp-branes move off into the bulk, as well as various mixed branches describing intermediate situations. To describe the Coulomb branch of the D(p + 4)-brane theory one separates

\(^{17}\)In order to compare our formulae to those in Ref. [4] replace \(\phi \rightarrow \sqrt{2}v\) where \(\phi = \phi_1 - \phi_2\).
these branes in their transverse space. On top of this, the gauge theory on the D\((p + 4)\)-branes can be made non-commutative by turning on certain background fields. The background field act as Fayet-Illiopolos parameters in the D\(p\)-brane theory which lifts the Coulomb and mixed branches leaving only the Higgs branch: the D\(p\)-brane are forced onto the D\((p + 4)\)-branes. But since the D\((p + 4)\)-branes are separated the D\(p\)-branes must choose which out of the \(N\) to be absorbed on. This is precisely the picture that lies behind the combinatorics of the partitions in (1.1). The effective theory on each D\((p + 4)\)-brane is then a non-commutative U(1) theory as suggested by the exact component of the instanton moduli space in (1.1). The resulting U(1) integrals are still non-trivial because we have to take account of interactions between D\(p\)-branes living on different D\((p + 4)\)-branes as we have seen in our calculations.

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Appendix A: Conventions

We frequently write a 4-vector \(x_n\) as the \(2 \times 2\) matrices

\[
x_{\alpha \dot{\alpha}} = x_n \sigma_{n \alpha \dot{\alpha}}, \quad \bar{x}^{\dot{\alpha} \alpha} = x_n \bar{\sigma}_{n \alpha}^{\dot{\alpha}},
\]

where \(\sigma_{n \alpha \dot{\alpha}}\) are the components of four \(2 \times 2\) matrices \(\sigma_n = (i \tau^c, 1_{[2] \times [2]}), \) with \(\tau^c, c = 1, 2, 3\) being the three Pauli matrices, and \(\bar{\sigma}_n \equiv \sigma_n^\dagger = (-i \tau^c, 1_{[2] \times [2]})\) with components \(\bar{\sigma}_{n \alpha}^{\dot{\alpha}}\). Spinor indices are raised and lowered using the \(\epsilon\)-tensor as in [37].

The \(\mathcal{N} = 4\) instanton calculus makes frequent use of the \(\Sigma\)-matrices of SO(6) (\(\simeq SU(4)\)). These can be thought as Clebsch-Gordon coefficients which relate the spinor, anti-spinor and vector representations. We think of \(\Sigma^{AB}\) and \(\bar{\Sigma}_{AB}\) as six-vectors of \(4 \times 4\) matrices. In our conventions we have

\[
\Sigma^{AB} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad i \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad i \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

(A.2)
and
\[
\begin{align*}
\Sigma_{AB} &= -\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}, \\
i \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, & \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, & \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]
\(\text{(A.3)}\)

We also define
\[
\Sigma_{ab}^{\ A\ B} = \frac{1}{4}(\Sigma_a^{ AC}\Sigma_{bCB} - \Sigma_b^{ AC}\Sigma_{aCB}) , \\
\bar{\Sigma}_{ab}^{\ A\ B} = \frac{1}{4}(\bar{\Sigma}_a^{AC}\Sigma^{CB}_{b} - \bar{\Sigma}_b^{AC}\Sigma^{CB}_{a}) .
\]
\(\text{(A.4)}\)

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