EXISTENCE RESULTS FOR THE FRACTIONAL Q-CURVATURE PROBLEM ON THREE DIMENSIONAL CR SPHERE

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(Communicated by Zhi-Qiang Wang)

Abstract. In this paper the fractional Q-curvature problem on three dimensional CR sphere is considered. By using the critical points theory at infinity, an existence result is obtained.

1. Introduction and main result. The sphere \(S^{2n+1}\) is the boundary of the unit ball of \(\mathbb{C}^{n+1}\). It is a contact manifold with a standard contact form \(\theta_1\). We denote by \((S^{2n+1}, \theta)\) the contact sphere with its contact form \(\theta\). Let \(K: S^{2n+1} \to \mathbb{R}\) be a \(C^2\) positive function. The prescribed Webster scalar curvature problem on \(S^{2n+1}\) is to find suitable conditions on \(K\) such that \(K\) is the Webster scalar curvature for some contact form \(\tilde{\theta}\) on \(S^{2n+1}\), CR equivalent to \(\theta_1\). If we set \(\tilde{\theta} = u^{2n} \theta_1\), where \(u\) is a smooth positive function on \(S^{2n+1}\), then the above problem is equivalent to solving the following PDE:

\[
\begin{cases}
    L_{\theta_1} u = K u^{1 + \frac{2}{n} - \gamma}, \\
    u > 0,
\end{cases}
\]

where \(L_{\theta_1}\) is the conformal Laplacian of \(S^{2n+1}\).

In recent years, fractional calculus has attracted a lot of mathematicians’ interests. The CR fractional sub-Laplacian \(P_{\gamma}^{\theta_1}\) is defined by Rupert L. Frank, María del Mar González, Dario D. Monticelli, and Jinggang Tan in [28]. In the paper [28], it was shown that one can treat the CR fractional sub-Laplacian as a boundary operator. In [12], the CR fractional sub-Laplacian is viewed as intertwining operator.

On the CR sphere, the general intertwining operator \(P_{\gamma}^{\theta_1}\) is defined by the following property:

\[
|J_\tau|^{\frac{n+1+\gamma}{2n+2}} (P_{\gamma}^{\theta_1} \circ F) \circ \tau = P_{\gamma}^{\theta_1} \left(|J_\tau|^{\frac{n+1+\gamma}{2n+2}} (F \circ \tau)\right), \quad \forall \tau \in \text{Aut}(S^{2n+1})
\]

for each \(F \in C^\infty(S^{2n+1})\).

AMS 2000 Mathematics Subject Classification. Primary: 35H20, 35S05; Secondary: 53C55.

Key words and phrases. CR manifolds, CR fractional sub-Laplacian, Yamabe problem, critical points at infinity, fractional Q-curvature.

The first author is supported by NSF of China (11471170).

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Given another conformal representative $\tilde{\theta} = u^{\frac{2}{n+1}-\frac{1}{n}}\theta_1$ which identifies the CR structure of $S^{2n+1}$, the corresponding operator is defined by

$$P_{\tilde{\gamma}}(\phi) = u^{\frac{n+1}{n+1-\gamma}}P^{\theta_1}(u\phi).$$

For $(S^{2n+1}, \theta)$, we define its fractional Q-curvature as

$$Q^\theta_{\gamma} = P^\theta_{\gamma}(1).$$

Let $\phi = 1$,

$$P^{\theta_1}(u) = u^{\frac{n+1}{n+1-\gamma}}P^\theta_{\gamma}(1).$$

The fractional Q-curvature problem is that for a prescribed function $K$, whether there is a positive function $u$ such that $Q^\theta_{\gamma} = K$ with $\tilde{\theta} = u^{\frac{2}{n+1}-\frac{1}{n}}\theta_1$. This problem is equivalent to the existence of the following fractional nonlinear PDE:

$$P^{\theta_1}(u) = Ku^{\frac{n+1}{n+1-\gamma}}, \quad u > 0 \text{ on } S^{2n+1}. \quad (1)$$

The scalar curvature problem for the Riemannian manifolds has been extensively studied. For instance on Riemannian sphere $S^n$, we consider classical Nirenberg problem

$$-\Delta g_0 u + 1 = Ke^u \text{ on } S^2 \quad (2)$$

$$-\frac{4(n-1)}{n-2} \Delta g_0 u + R_{g_0} u = Ku^{\frac{n+2}{n-2}}, \quad \text{on } S^n, \quad n \geq 3. \quad (3)$$

The requirements of different sufficient conditions are given, which are divided into three main categories:

1. Group invariance conditions. See Moser [45], Escobar-Schoen [26], Bianchi-Egnell [10], Wenxiong Chen [22], Hebey [35].

2. Mountain Path type condition. See Wenxiong Chen and Weiyue Ding [23].

3. Bahri-Coron type condition. In 1991, Bahri-Coron give sufficient conditions when $n = 3$ [6]: Set $K$ is a $C^2$ nondegeneracy and positive Morse function, and

$$\sum_{\nabla K(x) = 0, \Delta K(x) < 0} (-1)^{(x)} \neq (-1)^n. \quad (4)$$

When $n = 2$, the sufficient conditions similar to (4) is given by Chang-Yang [16] [19]. But when $n \geq 4$, The conditions given are generally not sufficient conditions. In particular, Chang-Yang [17] prove that when $K$ approximate to 1, the conditions (4) are sufficient for any dimension.

The scalar curvature problem for the Riemannian manifolds in dimension 2,3, we can also see [6, 19, 34] as well as in high dimension ([9, 40]). Fractional scalar curvature problem for the Riemannian manifolds has been studied by [21, 20]. There are also few works on scalar curvature problem on CR manifolds, see [24, 31, 32, 43, 46].

In this paper, we give Bahri-Coron type condition for fractional Q-curvature problem on $S^3$.

Let $\mathcal{S}^\gamma(S^{2n+1})$ be the closure of $C^\infty_0(S^{2n+1})$ with respect to the quadratic form

$$\int_{S^{2n+1}} f P_{\gamma}^\theta f \theta_1 \wedge (d\theta_1)^n,$$

$$\Sigma = \{u \in \mathcal{S}^\gamma(S^{2n+1})||u|| = \sqrt{\int_{S^{2n+1}} u P_{\gamma} \theta_1 \wedge (d\theta_1)^n} = 1\} \text{ and } \Sigma^+ = \{u \in \Sigma|u \geq 0\}.$$
For \( u \in \mathcal{S}^1(\mathbb{S}^{2n+1}) \), we define

\[
J(u) = \frac{\|u\|^2}{(\int_{\mathbb{S}^{2n+1}} Ku \frac{n+1}{n+1} \wedge (d\theta_1)^n)^{\frac{n+1}{n+1}}}.
\]

If \( u \) is a critical point of the function \( J \) in \( \Sigma^+ \), then \( v = (J(u))^{\frac{n+1}{n+1}} u \) is a solution of (1). However, the functional \( J \) does not satisfy the Palais-Smale condition, that is to say there exist critical points at infinity, which are the limits of noncompact orbits for the gradient flow of \(-J\). Thinking of these sequences as critical points, a natural idea is to expand the functional \( J \) near the sets of such critical points.

In this paper we care the case \( n = 1 \). We state now the main result. If \( K : \mathbb{S}^3 \to \mathbb{R} \) is a \( C^2 \) positive function, we assume \( K \) satisfying condition:

each critical point \( \eta_i \) is a non degenerate critical point of \( K \) and \( \Delta_{\theta_i} K(\eta_i) \neq 0 \).

Denote

\[
I^+ = \{ \eta_i \in \mathbb{S}^3 : \nabla_{\theta_i} K(\eta_i) = 0, -\Delta_{\theta_i} K(\eta_i) > 0 \}.
\]

Assume that \( 2 |I^+| = m \). \( |I^+| \) is the cardinality of \( I^+ \). For simplicity, we assume \( I^+ = \{ \eta_1, \eta_2, \cdots, \eta_m \} \). For any \( l \)-element subset \( \{ \eta_i_1, \cdots, \eta_i_l \} \) of \( I^+ \), \( 1 \leq l \leq m \), we define \( \mu_i = \sum_{j=1}^{l} \text{ind}(K, \eta_j) \) with \( \tau_i = (i_1, \cdots, i_l) \).

**Theorem 1.1.** Let \( \frac{n}{2} \leq \gamma < 1 \), assume \( K \) satisfies (5). Then the problem (1) has a solution provided

\[
\sum_{i=1}^{m} \sum_{\tau_i} (-1)^{\mu_i} \neq -1.
\]

We will prove the theorem by contradiction in section 5. Therefore we assume that equation (1) has no solutions. Our proof is based on a technical Morse Lemma at infinity; it relies the construction of a suitable pseudogradient for \( J \). The (PS) condition is satisfied along the decreasing flow lines of this pseudogradient, as long as these flow lines stay out of the neighborhood of a finite number of critical points of \( K \). Finally we compute the contribution of some critical points at infinity to the changes of topology for the level set of the functional, from this we can achieve a contradiction.

This paper is organized as follows. In the next section, we introduce preliminary result and the general variational framework. In section 3, we give some expansions of the functional and its gradient near the sets of its critical points at infinity. In section 4, we establish the Morse lemma at infinity, which allows us to refine the expansion of the function. In section 5, we give a proof of Theorem 1.1. In Appendices A-C, we show some useful estimates which will be used in our proof of Theorem 1.1.

2. Preliminary results. The Heisenberg group \( \mathbb{H}^1 \) is a Lie group whose underlying manifold is \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) with elements \( u = (x, y, t) \) and whose group law is

\[
u \circ u' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x' y - x y')).
\]

Alternatively, we can use complex coordinates \( z = x + i y \) to denote elements of \( \mathbb{R} \times \mathbb{R} \simeq \mathbb{C} \), so that the group law can be written as

\[
(z', t') \circ (z, t) = (z' + z, t' + t + 2\text{Im} < z', z >_\mathbb{C}),
\]

for \((z', t'), (z, t) \in \mathbb{H}^1\), and \(< \cdot, \cdot >_\mathbb{C} \) is the standard Hermitian inner product in \( \mathbb{C} \).
The CR sphere $S^{2n+1} = \{ \zeta = (\zeta_1, \cdots, \zeta_{n+1}) \in \mathbb{C}^{n+1}, \sum_{j=1}^{n+1} |\zeta_j|^2 = 1 \}$. The standard Euclidean volume element of $S^{2n+1}$ is denoted by $\mathrm{d}S$.

We introduce Cayley transform $\mathcal{C}$ between the Heisenberg group and the CR sphere.

$$\mathcal{C} : \mathbb{H}^n \to S^{2n+1} \setminus \{(0, 0, \cdots, -1)\},$$

$$(z, t) \mapsto (z, t) \mapsto \left( \frac{2z}{1 + |z|^2}, \frac{1 + i t}{1 + |z|^2 + i t} \right).$$

The inverse is given by

$$\mathcal{C}^{-1} : S^{2n+1} \setminus \{(0, 0, \cdots, -1)\} \to \mathbb{H}^n,$$

$$\zeta = (\zeta_1, \cdots, \zeta_{n+1}) \mapsto \left( \frac{\zeta_1}{1 + \zeta_{n+1}}, \cdots, \frac{\zeta_n}{1 + \zeta_{n+1}}, \text{Im} \frac{1 - \zeta_{n+1}}{1 + \zeta_{n+1}} \right).$$

The Jacobian determinant of $\mathcal{C}$ is

$$|\text{Jac}(z, t)| = \frac{2^{2n+1}}{(1 + |z|^2 + i t)^{n+1}}.$$

For any $\lambda > 0$ the dilation $\lambda : \mathbb{H}^1 \to \mathbb{H}^1$ is defined by $\lambda u = \lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ and we denote the homogeneous norm on $\mathbb{H}^1$ by $|u| = |(x, y, t)| = ((x^2 + y^2)^2 + t^2)^{1/4}$. The CR structure on $\mathbb{H}^1$ is given by the left invariant vector field:

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

The standard contact form $\theta_0 = dt + 2(x dx - y dy)$. Haar measure on $\mathbb{H}^1$ is the Lebesgue measure $du = dx dy dt$. Denote $\nabla_{\theta_0} = (X, Y, T)$. In $\mathbb{H}^1$, Taylor polynomials can be written in a special symmetric form. The expansions are similar to Taylor expansions in $\mathbb{R}^n$ but are adjusted to compensate for the different Heisenberg structure. The following formula from [11] gives the Taylor expansions based at the origin. Let $f : \mathbb{H}^1 \to \mathbb{R}$ be a $C^2$ functions. Let the origin be denoted by $0$, $p = (x, y, t)$ be an arbitrary point around $0$. Then,

$$f(p) = f(0) + x(Xf)(0) + y(Yf)(0) + t(Tf)(0) + \frac{x^2}{2} (X^2 f)(0) + \frac{y^2}{2} (Y^2 f)(0) + \frac{xy}{2} (XYf)(0) + \frac{xy}{2} (YXf)(0) + o(|p|^2).$$

The sub-Laplacian on $\mathbb{H}^1$ is the second order differential operator

$$\Delta_{\theta_0} = \frac{1}{4}(X^2 + Y^2).$$

On the CR sphere, the standard contact form $\theta_1 = i \sum_{j=1}^{n+1} (\zeta_j d\bar{\zeta}_j - \bar{\zeta}_j d\zeta_j)$, the sub-gradient is $\nabla_{\theta_1}$, and the sub-Laplacian is defined as

$$\Delta_{\theta_1} = \frac{1}{2} \sum_{j=1}^{n+1} (T_j T_j + T_j T_j),$$

where

$$T_j = \frac{\partial}{\partial \zeta_j} - \bar{\zeta}_j R, \quad R = \sum_{k=1}^{n+1} \zeta_k \frac{\partial}{\partial \zeta_k}.$$

The conformal sub-Laplacian on the sphere is defined as

$$L_{\theta_1} = \Delta_{\theta_1} + \frac{1}{4}.$$
The peculiarity of $L_{\theta_i}$ is its direct relation with $\Delta_{\theta_0}$ via the Cayley transform:
\[
\Delta_{\theta_i} \left( (2|Jac|)^{\frac{1}{2}} (F \circ C) \right) = (2|Jac|^{\frac{1}{2}} (L_{\theta_i} F) \circ C,
\]
where $F : \mathbb{S}^{2n+1} \to \mathbb{C}$ is a smooth function.

The differences between the standard volume elements for $\mathbb{S}^{2n+1}$ and $\mathbb{H}^n$ and the volume forms associated with the standard contact forms $\theta_1$, and $\theta_0$ of these two spaces state as:
\[
\int_{\mathbb{S}^{2n+1}} F \tau \wedge (d\theta_1)^n = 2^{2n+1} n! \int_{\mathbb{S}^{2n+1}} F \, d\zeta
\]
\[
= \int_{\mathbb{H}^n} 2|Jac| F \circ C \tau \wedge (d\theta_0)^n
\]
\[
= 2^{2n} n! \int_{\mathbb{H}^n} 2|Jac| F \circ C \, du.
\]
We refer [12] for details.

We consider the CR fractional operators of order $2\gamma$. For $\gamma \in (0, 1)$, the symbol of the operator on $\mathbb{H}^n$ is
\[
P^\theta_{\gamma} = (2|T|)^{\gamma} \frac{\Gamma \left( \frac{1 + \gamma}{2} + \frac{\Delta_{\theta_0}}{2n+1} \right)}{\Gamma \left( \frac{1}{2} + \frac{\Delta_{\theta_0}}{2n+1} \right)}.
\]
In particular, for $\gamma = 1$, $P^\theta_1 = \Delta_6$ is the CR Yamabe operator on $\mathbb{H}^n$.

$P^\theta_{\gamma}$ also satisfy (setting $Q = 2n + 2$ and $d = 2\gamma$)
\[
|J_t|^{\frac{Q-d}{2}} (P^\theta_{\gamma} \circ f) \circ h = P^\theta_{\gamma} (|J_t|^{\frac{Q-d}{2}} (f \circ h)), \quad \forall h \in \text{Aut}(\mathbb{H}^n),
\]
for each $f \in C^\infty(\mathbb{H}^n)$.

On the CR sphere, the general intertwining operator $P^\theta_{\gamma}$ is defined by the following property:
\[
|J_t|^{\frac{Q-d}{2}} (P^\theta_{\gamma} \circ f) \circ \tau = P^\theta_{\gamma} \left( |J_t|^{\frac{Q-d}{2}} (F \circ \tau) \right), \quad \forall \tau \in \text{Aut}(\mathbb{S}^{2n+1}),
\]
for each $F \in C^\infty(\mathbb{S}^{2n+1})$.

Thus we have
\[
P^\theta_{\gamma} \left( (2|Jac|)^{\frac{Q-d}{2}} (F \circ C) \right) = (2|Jac|^{\frac{Q-d}{2}} (P^\theta_{\gamma} F) \circ C.
\]
For more details, we refer to the well written paper [28] and [12].

It was proved by [30] that on $\mathbb{H}^n$: for $q = \frac{Q}{n+1-\gamma}$ and any function $f \in \hat{\mathcal{S}}^\gamma(\mathbb{H}^n)$, it holds that
\[
\|f\|^2_{L^2(\mathbb{H}^n)} \leq C_\gamma \int_{\mathbb{H}^n} f P^\gamma f \, du
\]
where $C_\gamma = 2^{n-\gamma-1} \cdot 2^{(n+1)\gamma} \cdot n!^{\frac{1}{2}} (2n+1)^\gamma \frac{\Gamma((Q-2\gamma)/2)\Gamma^2((Q-2\gamma)/4)}{\Gamma^2((Q-\gamma)/2)\Gamma((Q-\gamma)/2)}$. And all optimizers are translates, dilates or constant multiples of the function
\[
\delta(u) = \delta(z, t) = \left( \frac{1}{(1+|z|^2)^2+t^2} \right)^{\frac{2-2\gamma}{2}}.
\]

We know that, for $\lambda > 0$, $a \in \mathbb{H}^1$ and some suitable choice of $c_0 = C(\gamma) > 0$, the function
\[
\delta_{a, \lambda}(u) = c_0 \lambda^{2-\gamma} \delta(\lambda(a^{-1}u))
\]
satisfies the Euler-Lagrangian equation for
\[ P_{\gamma}^{\theta_0} u = u^{\frac{2+\gamma}{\gamma}}, \quad u > 0 \text{ in } H^1. \]  
(11)
is the fractional CR Yamabe equation introduced in [28]. (10) indicates that (11) is invariant under the scaling and translations.

We introduce the function for each \((\zeta_0, \lambda) \in S^{2n+1} \times (0, +\infty),\)
\[ w_{\zeta_0, \lambda}(\zeta) = |1 + \zeta n+1|^{-(2-\gamma)} \delta_{C^{-1}(\zeta_0), \lambda} \circ C^{-1}(\zeta). \]  
(12)
Using (7) and (8), we have
\[ \int_{S^3} w_{\zeta_0, \lambda}^{2-\gamma} \theta_1 \wedge d\theta_1 = \int_{H^1} \delta_{a_0, \lambda} \theta_0 \wedge d\theta_0, \]
and
\[ \int_{S^3} w_{\zeta_0, \lambda} P_{\gamma}^{\theta_1} w_{\zeta_0, \lambda} \theta_1 \wedge d\theta_1 = \int_{H^1} \delta_{a_0, \lambda} P_{\gamma}^{\theta_0} \delta_{a_0, \lambda} \theta_0 \wedge d\theta_0, \]
where \(a_0 = C^{-1}(\zeta_0), \xi = C^{-1}(\zeta).\) We also have \(P_{\gamma}^{\theta_1} w_{\zeta_0, \lambda} = w_{\zeta_0, \lambda}^{\frac{2+\gamma}{\gamma}}.\)

For any \(\varepsilon > 0\) and \(p \geq 1,\) we set
\[ (\alpha, g, \lambda) = (\alpha_1, \ldots, \alpha_p, g_1, \ldots, g_p, \lambda_1, \ldots, \lambda_p) \in (0, +\infty)^p \times (S^3)^p \times (0, +\infty)^p, \]
\[ \varepsilon_{ij} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d(g_i, g_j)^2 \right)^{-(2-\gamma)}. \]
Let \(V(p, \varepsilon)\) be the subset of \(\Sigma^+\) of the following functions: \(u \in \Sigma^+, \exists(\alpha, g, \lambda),\) such that
\[ \|u - \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i}\| < \varepsilon \]
and \(|J(u) \frac{\partial u}{\partial \alpha_i} \pi_{\gamma} \alpha_i^{\frac{2+\gamma}{\gamma}} K(g_i) - 1| < \varepsilon, \varepsilon_{ij} < \varepsilon, \lambda_i > \varepsilon^{-1}.\) The set \(V(p, \varepsilon)\) has a simple interpretation: It is a neighborhood of the critical points at infinity of the functional \(J\) on \(\Sigma^+.\)

**Definition 2.1** ([3]). We will say that the Palais-Smale condition holds on flowlines in the \(V(p, \varepsilon)\) if, taking an initial data \(u_0\) in \(V(p, \varepsilon),\) with \(\varepsilon_0\) small enough (but fixed), the solution \(u(s, u_0)\) of the differential equation \(\frac{\partial u}{\partial s} = -\partial J(u)\) with initial data \(u_0\) remains outside a \(V(p, \varepsilon_1), \varepsilon_1 > 0,\) which depends only on \(u_0.\)

The failure of (PS) condition is characterized as follows.

**Lemma 2.2.** Assume that (1) has no solutions. Let \(\{u_k\} \subseteq \Sigma^+\) be a sequence such that \(J(u_k) \to 0\) and \(J(u_k)\) is bounded. Then there exists an integer \(p \geq 1,\) a positive sequence \(\varepsilon_k \to 0\) and an extracted subsequence of \(\{u_k\},\) such that \(u_k \in V(p, \varepsilon_k)\).

In order to prove Lemma 2.2, we introduce
\[ I(u) = \frac{1}{2} \|u\|^2 - \frac{2 - \gamma}{4} \int_{S^3} Ku \frac{\partial u}{\partial \alpha_i} \pi_{\gamma} \theta_1 \wedge d\theta_1. \]
Note that \(J(u_k) \to 0\) if and only if \(I'(J(u_k)) \frac{\partial u_k}{\partial \alpha_i} \to 0.\) Then we can follow the first part of [13] by using the functional \(I.\) The proof is by now classical, we can also see [33].

We introduce the minimization problem for \(\varepsilon\) small enough
\[ \min_{u \in V(p, \varepsilon)} \{ \|u - \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i}\|, \alpha_i > 0, g_i \in S^3, \lambda_i > 0 \}. \]  
(13)
Lemma 2.3. For any \( p \geq 1 \), there exists \( \varepsilon_p > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_p \) and \( u \in V(p, \varepsilon) \), the minimization problem (13) has a unique solution \((\bar{u}, \bar{g}, \bar{\lambda})\).

Denoting \( v = u - \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \), \( v \) satisfies

\[
\begin{align*}
\langle v, w_{g_i, \lambda_i} \rangle &= 0, \\
\frac{\partial v}{\partial \lambda_i} &= 0, \\
\frac{\partial v}{\partial g_i} &= 0 \quad i = 1, \ldots, p.
\end{align*}
\]

(14)

Here \( \langle \cdot, \cdot \rangle \) denote the inner product in \( \dot{S}^\gamma(\mathbb{S}^3) \) defined by

\[
\langle u, v \rangle = \int_{\mathbb{S}^3} u P^\gamma \ast v d\theta_1 \wedge d\theta_1.
\]

The proof of Lemma 2.3 follows from Appendix A in [5] with some modulations.

3. The expansion of the function \( J \).

Lemma 3.1. If \( \varepsilon > 0 \) small enough and \( u = \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + v \in V(p, \varepsilon) \), \( v \) satisfies (14), we have

\[
J(u) = \frac{\sum_{i=1}^{p} \alpha_i^2 S}{\left( \sum_{i=1}^{p} \alpha_i^2 S \right)^{\frac{2}{p}}} \left[ 1 - \frac{2 - \gamma}{2} \frac{c_2}{S^2} \sum_{i=1}^{p} \frac{\alpha_i^4 \Delta_\theta K(g_i)}{\sum_{k=1}^{p} \alpha_k^2 S} - \sum_{k=1}^{p} \frac{\alpha_k^2 \Delta_\theta K(g_k) \lambda_i^2}{\sum_{k=1}^{p} \alpha_k^2 S} \right] + (f, v) + Q(v, v) + O(\varepsilon^2) + o(\frac{1}{\lambda_i^2}) + o(||v||^2)
\]

with

\[
(f, v) = \frac{-2 \sum_{k=1}^{p} \alpha_k^2 S}{\sum_{k=1}^{p} \alpha_k^2 S^{\frac{2}{p}}} \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2}{p+2}} v d\theta_1 \wedge d\theta_1,
\]

\[
Q(v, v) = \frac{1}{\sum_{k=1}^{p} \alpha_k^2 S^{\frac{2}{p}}} ||v||^2 - \frac{2 + \gamma}{(2 - \gamma) S^{\frac{2}{p}}} \sum_{k=1}^{p} \alpha_k^2 S \int_{\mathbb{S}^3} K(\zeta) \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right) S^{\frac{2}{p+2}} v^2.
\]

Furthermore \( ||f|| \) is bounded by

\[
||f|| = O \left( \sum_{i=1}^{p} \left( \frac{\nabla_{\theta_i} K(g_i)}{\lambda_i} + \frac{1}{\lambda_i^2} \right) + \sum_{i \neq j} \varepsilon_{ij} (\ln \varepsilon_{ij}^{-1}) \right)^{\frac{2}{2+2}}
\]

The proof of this lemma is provided in Appendix A.

Now, we state the following two lemmas whose proof follow the arguments used to prove similar statements in [6] (also in [3]); see the Appendix of [32] where some necessary modifications are made.

Lemma 3.2. \( Q(v, v) \) is a quadratic form positively definite in

\[
H_\varepsilon(\lambda, \alpha) = \{ v \in \dot{S}^\gamma(\mathbb{S}^3) | \ v \ satisfies \ (14), \ ||v|| \leq \varepsilon \}.
\]
One can follow the idea of the proof of Lemma A.2 in [6] to get a proof of this lemma. We omit the details.

**Lemma 3.3.** For any \( u_0 = \sum_{i=1}^{P} \alpha_i w_{g_i, \lambda} \in V(p, \varepsilon) \), there exists a unique \( \varpi = \varpi(\alpha, g, \lambda) \) which minimizes \( J(u_0 + \varpi) \) with respect to \( \varpi \in H_c(\lambda, a) \) and we have estimate \( \|\varpi\| = O(\|f\|) \).

For a proof of Lemma 3.3, one may follow the idea and similar estimates in the proof of Proposition 5.4 in [3](P191). We omit the details.

Since \( \varpi \) is a minimizer, we have
\[
(f, \varpi) + 2Q(\varpi, \varpi) + o(\|\varpi\|^2) = 0.
\]
It yields
\[
(f, v) + Q(v, v) + o(\|v\|^2) = Q(v - \varpi, v - \varpi) - Q(\varpi, \varpi) + o(\|\varpi\|^2).
\]
From Lemma 3.1 and Lemma 3.3, we state the following lemma which improve the asymptotic behavior of the function \( J \).

**Lemma 3.4.** For any \( p \geq 1 \), there exists \( \varepsilon_p > 0 \) such that for \( u = \sum_{i=1}^{P} \alpha_i w_{g_i, \lambda} + v, \ v \in H_c(\lambda, a) \), we have
\[
J(u) = \frac{\sum_{i=1}^{P} \alpha_i^2 S}{\left( \sum_{i=1}^{P} \alpha_i^4 K(g_i) \right)^{2/3}} \left[ 1 - \frac{2 - \gamma c_2}{2} \frac{1}{S^2} \sum_{i=1}^{P} \alpha_i^2 K(g_i) \lambda_i^2 \right]^{2/3} + Q(v - \varpi, v - \varpi) - Q(\varpi, \varpi) + o(\|\varpi\|^2) + O(\sum_{i \neq j} \varepsilon_{ij}) + O\left( \sum_{i=1}^{P} \frac{1}{\lambda_i^4} \right).
\]

**Lemma 3.5.** Assume \( K \) be a \( C^2 \) positive function satisfying condition (5). For any \( u = \sum_{i=1}^{P} \alpha_i w_{g_i, \lambda} \in V(p, \varepsilon) \), the following estimate holds
\[
J'(u) \left( \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) = 2\lambda(u) \left[ -\sum_{i \neq j} O'(\lambda_j) \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \frac{2 - \gamma}{4} c_2 \alpha_j \frac{K(g_j)}{K(g_j) \lambda_j^2} \right] + o\left( \frac{1}{\lambda_j^4} \right) + o\left( \sum_{i \neq j} \varepsilon_{ij} \right), \tag{15}
\]
\[
J'(u) \left( \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \right) = -2\lambda(u) \alpha_j c_2 \nabla_{\theta_j} K(g_j) + O\left( \sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_j^2} \right). \tag{16}
\]

We will give the proof in Appendix B and Appendix C.

4. **Morse lemma at infinity.** This section is devoted to characterize the critical points at infinity associated to problem (1). The characterization is obtained through the construction of a suitable pseudogradient at infinity in the set \( V(p, \varepsilon) \), depending on a delicate expansion of the gradient of \( J \) near infinity.
Theorem 4.1. There is a covering \( \{ O_1 \} \) and a subset of \( \{ (\alpha_1, g_i, \lambda_i) \} \) of the base space for the bundle \( V(p, \varepsilon) \) and a diffeomorphism \( \xi_1 : V(p, \varepsilon) \to V(p, \varepsilon') \) for some \( \varepsilon' > 0 \) with

\[
\xi_1 \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \overline{\nu}(\alpha, g, \lambda) \right) = \sum_{i=1}^{p} \alpha_i w_{g_i, \tilde{\lambda}_i}
\]

such that

\[
J \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + v \right) = J \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \tilde{\nu}(\alpha, g, \lambda) \right) + \frac{1}{2} J'' \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \tilde{\nu}(\alpha, g, \tilde{\lambda}) \right) V_i \cdot V_i,
\]

(17)

where \( (\alpha, g, \lambda) \in O_1, (\alpha, \tilde{g}, \tilde{\lambda}) \) not depending on \( O_1 \), \( V_i \) orthogonal to \( w_{g_i, \lambda_i} \), \( \frac{\partial w_{g_i, \lambda_i}}{\partial \lambda_i} \).

The proof of this theorem need some technical result. First we give the Morse lemma at infinity by isolating the contribution of \( v - \overline{\nu} \).

Lemma 4.2. For any \( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \in V(p, \varepsilon) \), let

\[
(\bar{\alpha}, \bar{g}, \bar{\lambda}) = (\alpha_1, \ldots, \alpha_p, \bar{g}_1, \ldots, \bar{g}_p, \lambda_1, \ldots, \lambda_p),
\]

there is a neighborhood \( U \) of \( (\bar{\alpha}, \bar{g}, \bar{\lambda}) \) such that

\[
J \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + v \right) = J \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \nu(\alpha, g, \lambda) \right) + \frac{1}{2} J'' \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \nu(\bar{\alpha}, \bar{g}, \bar{\lambda}) \right) V \cdot V
\]

for any \( u = \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + v \in V(p, \varepsilon) \) with \( (\alpha, g, \lambda) \in U \), where \( V = V(\alpha, g, \lambda, v) \) is a \( C^1 \) diffeomorphism that has range orthogonal to

\[
\bigcup_{i=1}^{p} \left\{ \delta_{g_i, \lambda_i}, \frac{\partial \delta_{g_i, \lambda_i}}{\partial \lambda_i}, \frac{\partial \delta_{g_i, \lambda_i}}{\partial \lambda_i'}, \frac{\partial \delta_{g_i, \lambda_i}}{\partial g_i} \right\}
\]

for any \( (\alpha', g', \lambda') \in U \) and \( \| V \| = O(\| v - \nu(\bar{\alpha}, \bar{g}, \bar{\lambda}) \|) \).

The proof of Lemma 4.2 is similar to the proof of Lemma 3.2 in [9] for the Riemannian manifold.

Lemma 4.3. Let \( K \in C^2(\mathbb{S}^3) \) be a positive function satisfying condition (5). For any \( u = \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \in V(p, \varepsilon), \varepsilon \) small enough, Then there exists a vector field \( W' \) so that the following holds: there is a constant \( C > 0 \) such that

\[
-J'(u)(W') \geq C_1 \left( \sum_{i=1}^{p} \left| \frac{\nabla g_i K(g_i)}{\lambda_i} \right| + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right), \tag{18}
\]

\[
-J'(u + \nu) \left( W' + \frac{\partial \nu}{\partial (\alpha, g, \lambda)}(W') \right) \geq C_1 \left( \sum_{i=1}^{p} \left| \frac{\nabla g_i K(g_i)}{\lambda_i} \right| + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right), \tag{19}
\]

where \( C_1 \) is a positive constant, \( \| W' \| \) is bounded.
We remind that $\frac{\partial \pi}{\partial (\alpha, g, \lambda)}(W) = \frac{\partial \pi}{\partial W}$ means the variation of $\bar{v}$ along the direction $W$, where $W = W_0 \delta \alpha + W_g \cdot \delta g + W_\lambda \delta \lambda$ is an increment in the $(\alpha, g, \lambda)$ space.

We define a set $I_i$ for all $i \in \{1, \cdots, p\}$. We divide it in three cases:

1. For all $i \in \{1, \cdots, p\}$, there exists a suitable constant $C > 0$ such that
\[
\sum_{j \neq i} \varepsilon_{ij} \leq \frac{C}{\lambda_i^2},
\]
In this case, we define $I_i = \emptyset$ the empty set.

2. $\lambda_i$ is the largest concentration with
\[
\sum_{j \neq i} \varepsilon_{ij} > \frac{C}{\lambda_i^2},
\]
then in this case we define $I_i = \{i\}$.

3. $\lambda_i$ satisfies (21), but it is not the largest concentration, i.e., we have another $j$ such that $\lambda_j > \lambda_i$ and $\lambda_j$ satisfies (21). In this case we define
\[
I_i = \left\{ k \mid \lambda_k \geq \lambda_i, \sum_{j \neq k} \varepsilon_{kj} > \frac{C}{\lambda_k^2} \right\}.
\]

Suppose $I_i = \{k_1, k_2, \cdots, k_m\}$ such that $k_1 = i$ and $\lambda_{k_1} \leq \lambda_{k_2} \leq \cdots \leq \lambda_{k_m}$. We define $\mu_{k_s} = 2^{s-1}, s = 1, \cdots, m$.

Claim. There holds
\[
\left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i, \lambda_i}}{\partial g_i} \right) \right| + \frac{c'}{c} \sum_{k \in I_i} \mu_k \left( \frac{\partial w_{g_k, \lambda_k}}{\partial \lambda_k} \right) \geq c \frac{|\nabla \theta_i K(g_i)|}{\lambda_i} \sum_{j \neq i} \varepsilon_{ij} + 1 \frac{1}{\lambda_i^2},
\]
where $c$ is the constant in (16) and $c' > 0$ is a suitable constant.

Proof of the claim. In the case (1), by (16), we have
\[
\left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i, \lambda_i}}{\partial g_i} \right) \right| \geq c \frac{|\nabla \theta_i K(g_i)|}{\lambda_i} - \frac{1}{c} \left( \sum_{j \neq i} \varepsilon_{ij} + 1 \frac{1}{\lambda_i^2} \right) \geq c \frac{|\nabla \theta_i K(g_i)|}{\lambda_i} - 1 \frac{1}{c \lambda_i^2}.
\]

In the case (2), then for $\lambda_j > \lambda_i$, there holds
\[
\sum_{k \neq j} \varepsilon_{kj} \leq \frac{C}{\lambda_j^2}.
\]
Notice that
\[
\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -(2 - \gamma) \varepsilon_{ij} \left( 1 - \frac{2 \lambda_j}{\lambda_i} \varepsilon_{ij} \right).
\]
If $\lambda_i$ and $\lambda_j$ are comparable or $\lambda_j \leq \lambda_i$ (in this case, $|\lambda_j/\lambda_i| \leq C$), then
\[
\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -(2 - \gamma) \varepsilon_{ij} (1 + o(1)).
\]
If they are not comparable, say $\lambda_i = o(\lambda_j)$, then
\[
\left| \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \right| = O(\varepsilon_{ij}) \leq \frac{C}{\lambda_j^2} = o\left( \frac{1}{\lambda_j^2} \right).
\]
Thus by (21), there holds
\[ -\sum_{j \neq i} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \geq \frac{2 - \gamma}{2} \sum_{j \neq i} \varepsilon_{ij} \geq \frac{2 - \gamma}{2} \frac{C}{\lambda_i^2}. \] (29)

Hence, by choosing a large $C$, it holds that
\[
J'(u) \left( \lambda_i \frac{\partial w_{g_i,\lambda_i}}{\partial \lambda_i} \right) = 2\lambda(u) \left[ O'(-\sum_{j \neq i} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}) + \frac{2 - \gamma}{4} c_2 \alpha_i \frac{\Delta \theta_i K(g_i)}{K(g_i)\lambda_i^2} (1 + o(1)) + o(\sum_{j \neq i} \varepsilon_{ij}) \right] \geq C_0 \sum_{j \neq i} \varepsilon_{ij},
\] (30)

where $C_0 > 0$ is a constant depending on $\gamma$. Then from the first inequality of (24) and (30), we have
\[
\left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i,\lambda_i}}{\partial \lambda_i} \right) \right| + \frac{c'}{c} J'(u) \left( \lambda_i \frac{\partial w_{g_i,\lambda_i}}{\lambda_i} \right) \geq c' \left| \frac{\nabla \theta_i K(g_i)}{\lambda_i} - \frac{1}{c} \frac{1}{\lambda_i} \right| \geq \frac{1}{C_0}.
\] (31)

In the case (3), by a simple computation, we observe that for $s > t$,
\[ -\mu_k \lambda_k \frac{\partial \varepsilon_{k_s,k_t}}{\partial \lambda_k} - \mu_k \lambda_k \frac{\partial \varepsilon_{k_s,k_t}}{\partial \lambda_k} \geq \frac{2 - \gamma}{2} \frac{\varepsilon_{k_s,k_t}}{1 + o(1)}. \] (32)

By using (15) and choosing a constant $c'$ which depends on $\gamma$, there holds
\[
\left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i,\lambda_i}}{\partial \lambda_i} \right) \right| + \frac{c'}{c} \sum_{k=1}^{m} \mu_k J'(u) \left( \lambda_k \frac{\partial w_{g_k,\lambda_k}}{\lambda_k} \right) \geq c' \left| \frac{\nabla \theta_i K(g_i)}{\lambda_i} - \frac{1}{c} \frac{1}{\lambda_i} \right|
\] (33)

For a proof of (33), from (15), (26) and that the functional $\lambda(u)$ has positive lower bound, we have
\[
\sum_{k \in I_i} \mu_k J'(u) \left( \lambda_k \frac{\partial w_{g_k,\lambda_k}}{\lambda_k} \right)
= \sum_{k \in I_i} \mu_k 2\lambda(u) \left[ -O'(\lambda_k \frac{\partial \varepsilon_{k_i,k_t}}{\lambda_k}) + \frac{2 - \gamma}{4} c_2 \alpha_k \frac{\Delta \theta_i K(g_k)}{K(g_k)\lambda_k^2} \right] + o(\sum_{j \neq k} \varepsilon_{kj})
\geq c\gamma \sum_{k \in I_i} \mu_k \lambda_k \left( \frac{2 - \gamma}{4} c_2 \alpha_k \frac{\Delta \theta_i K(g_k)}{K(g_k)\lambda_k^2} \right)
= \sum_{k \in I_i, j \in I_i} c\gamma \left( -\mu_k \lambda_k \frac{\partial \varepsilon_{k_j,k_t}}{\partial \lambda_k} - \mu_j \lambda_j \frac{\partial \varepsilon_{k_j,k_t}}{\partial \lambda_j} \right) + \sum_{k \in I_i} c\gamma \left( -\mu_k \lambda_k \frac{\partial \varepsilon_{k_j,k_t}}{\partial \lambda_k} \right)
\geq c\gamma \sum_{k \in I_i, j \in I_i} \sum_{j \neq k} \varepsilon_{kj} + \sum_{k \in I_i} \mu_k \lambda_k \left( -\lambda_k \frac{\partial \varepsilon_{k_j,k_t}}{\partial \lambda_k} \right)
\[
+ \sum_{k=\lambda_k=0(\lambda_j), k \not\in I} \left(-\lambda_k \frac{\partial \varepsilon_{kj}}{\partial \lambda_k}\right) + \sum_{k \in I_i} \mu_k \lambda(u) \left(\frac{2 - \gamma}{4} c_2 \alpha_k \frac{\Delta \theta_j K(g_k)}{K(g_k) \lambda_k^2}\right)
\]
\[\geq \frac{c_2}{2} \sum_{k \in I_i} \sum_{j \neq k} \varepsilon_{kj} \geq \frac{c_2}{2} \sum_{k \in I_i} \lambda_k^{-\frac{4}{k}},\]

where \(0 < c_\gamma < 1\). In the last two estimates, we have used the inequalities (27) and (28) and choose the constant \(C\) large enough. \(\lambda_i \sim \lambda_j\) means that \(\lambda_i\) and \(\lambda_j\) are comparable. It completes the proof of claim. \(\square\)

**Proof of Lemma 4.3.** For the sake of simplicity, we assume
\[
\lambda_1 \leq \cdots \leq \lambda_p.
\]
We note that when we construct the vector field \(W'\) satisfying the estimate (18), then by the same method as in [8] and [9], we can prove (19). So in the following we only need to construct the vector field \(W'\) satisfies the inequality (18). We divide it into four cases.

**Case 1.** Suppose there holds
\[
\left| \frac{\nabla \theta_i K(g_i)}{\lambda_1} \right| > \frac{2}{c^2 \lambda_1^2}.
\]
In this case, by (23) we have
\[
\left| J'(u) \left(\frac{1}{\lambda_1} \frac{\partial w_{g_i, \lambda_i}}{\partial g_i}\right) \right| + \left| J'(u) \left(\frac{1}{\lambda_1} \frac{\partial w_{g_i, \lambda_i}}{\partial g_i}\right) \right| + \frac{c}{c} \sum_{k \in I_i} \mu_k J'(u) \left(\lambda_k \frac{\partial w_{g_k, \lambda_k}}{\partial \lambda_k}\right)
\]
\[\geq - \sum_{j \neq i} O'\left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}\right) + \frac{B}{\lambda_i^2} (1 + o(1)) + c \sum_{j \neq i} \varepsilon_{ij} + c \left| \frac{\nabla \theta_i K(g_i)}{\lambda_i} \right| + c \left| \frac{\nabla \theta_i K(g_i)}{\lambda_i} \right| \left(1 - \frac{1}{c} \frac{1}{\lambda_i^2}\right)
\]
\[\geq - \sum_{j \neq i} O'\left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}\right) + \frac{B}{\lambda_i^2} + c \left| \frac{\nabla \theta_i K(g_i)}{\lambda_i} \right| + o(\sum_{j \neq i} \varepsilon_{ij}). \tag{36}\]

\[
\Gamma_i = J'(u) \left(\lambda_i \frac{\partial w_{g_i, \lambda_i}}{\partial g_i}\right) + \left| J'(u) \left(\frac{1}{\lambda_1} \frac{\partial w_{g_i, \lambda_i}}{\partial g_i}\right) \right| + \frac{c}{c} \sum_{k \in I_i} \mu_k J'(u) \left(\lambda_k \frac{\partial w_{g_k, \lambda_k}}{\partial \lambda_k}\right)
\]
\[\geq - \sum_{j \neq i} O'\left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}\right) + \frac{B}{\lambda_i^2} + c \left| \frac{\nabla \theta_i K(g_i)}{\lambda_i} \right| + o(\sum_{j \neq i} \varepsilon_{ij}). \tag{36}\]

Here and in sequel we denote \(B > 0\) a constant which may vary in different places, We define \(\nu_k = 2^{k-1}\). By simple computation, we have
\[
- \left(\nu_j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \nu_i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}\right) \geq D_\gamma \varepsilon_{ij}, \text{ if } j > i, \ D_\gamma = \frac{2 - \gamma}{2}.
\]
So we get
\[
\sum_{i=1}^{p} \nu_i \Gamma_i \geq B \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{B}{\lambda_i^2} + B \sum_{i=1}^{p} \left| \frac{\nabla \theta_i K(g_i)}{\lambda_i} \right|. \tag{37}\]
This can be rewritten in the following form

\[
J'(u) \left( \sum_{i=1}^{p} \gamma_i \lambda_i \frac{\partial w_{g_i \lambda_i}}{\partial \lambda_i} \right) + \sum_{i=1}^{p} \beta_i \left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i \lambda_i}}{\partial g_i} \right) \right| \\
\geq B \left( \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i=1}^{p} \left| \nabla \theta_1 K(g_i) \right| \right),
\]

(38)

where \(\gamma_i, \beta_i\) are bounded nonnegative constants depending on \(\mu_i, \nu_i\) and \(\gamma\).

We now define the vector field by

\[
W' = - \left( \sum_{i=1}^{p} \gamma_i \lambda_i \frac{\partial w_{g_i \lambda_i}}{\partial \lambda_i} \right) - \sum_{i=1}^{p} \beta_i \left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i \lambda_i}}{\partial g_i} \right) \right| \cdot \nabla \theta_1 K(g_i) \left| \nabla \theta_1 K(g_i) \right|.
\]

(39)

From the estimate (38), we have (18).

**Case 2.** Suppose there holds

\[
- \sum_{j \neq i} O' \left( \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \right) > \frac{4(\Delta g_1 K(g_1) + 1)}{\lambda_1^2}.
\]

(40)

In this case, as in (36) we define

\[
\Gamma_i = J'(u) \left( \lambda_i \frac{\partial w_{g_i \lambda_i}}{\partial \lambda_i} \right) + \left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i \lambda_i}}{\partial g_i} \right) \right| + \frac{c}{\tilde{c}} \sum_{k \in I_i} \mu_k J'(u) \left( \lambda_k \frac{\partial w_{g_k \lambda_k}}{\partial \lambda_k} \right) + B J'(u) \left( \lambda_1 \frac{\partial w_{g_1 \lambda_1}}{\partial \lambda_1} \right)
\]

(41)

A similar construct of \(W'\) can be done and the proof of (18) is repeated as the case 1 word by word by some mirror modifications.

**Case 3.** Suppose there holds

\[
\sum_{j \neq i} \varepsilon_{ij} \geq C \lambda_1^2, \quad C \text{ a suitable constant.}
\]

(42)

In this case, we define

\[
\Gamma_i = J'(u) \left( \lambda_i \frac{\partial w_{g_i \lambda_i}}{\partial \lambda_i} \right) + \left| J'(u) \left( \frac{1}{\lambda_i} \frac{\partial w_{g_i \lambda_i}}{\partial g_i} \right) \right| + \frac{c}{\tilde{c}} \sum_{k \in I_i} \mu_k J'(u) \left( \lambda_k \frac{\partial w_{g_k \lambda_k}}{\partial \lambda_k} \right)
\]

(43)

Since there holds

\[
- \sum_{i=1}^{p} \nu_i \sum_{j \neq i} O' \left( \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \right) \geq B \sum_{j \neq i} \varepsilon_{ij},
\]

so we can define vector field \(W'\) and give a similar proof of (18) as in case 1.

**Case 4.** Suppose there holds

\[
\sum_{j \neq i} \varepsilon_{ij} < C \lambda_1^2, \quad \left| \nabla \theta_1 K(g_1) \right| \lambda_1 < \frac{2}{\tilde{c}^2 \lambda_1^2}.
\]

(44)

the above proof extends as follows.

**Subcase 1.** Suppose in the sequence \(\lambda_1 \leq \cdots \leq \lambda_p\), there exists \(i_1\) such that for some \(0 \leq r \leq p - i_1\), there holds
\[
\sum_{s=0}^{r} \sum_{j \leq i_1 + r, j \neq i_1 + s} \varepsilon_{i_1 + r, j} \geq \frac{C}{\lambda_{i_1}^2}, \quad \text{or} \quad \frac{|\nabla_{\theta_i} K(g_{i_1})|}{\lambda_{i_1}} \geq \frac{2}{c^2 \lambda_{i_1}^2}. \tag{45}
\]

We note that for a choice of \((i_1, r_0)\) satisfying (45), then all of \((i_1, r)\) with \(r_0 \leq r \leq p - i_1\) satisfies (45). Similarly to the case 1 and case 3, we can define a vector field \(W(i_1, r)\) in \(\text{span}_{i_1} \{ \frac{\partial u_{g_1}, \lambda_1}{\partial \lambda_1}, \frac{\partial u_{g_1}}{\partial g_1} \} \) such that

\[
\|W(i_1, r)\| \leq C
\]

and

\[
-J'(u)W(i_1, r) \geq B \left( \sum_{s=0}^{r} \sum_{j \leq i_1 + r, j \neq i_1 + s} \varepsilon_{i_1 + s, j} + \sum_{s=0}^{r} \frac{1}{\lambda_{i_1 + s}} \right) + \sum_{s=0}^{r} \frac{|\nabla_{\theta_i} K(g_{i_1 + s})|}{\lambda_{i_1 + s}} - \frac{1}{c} \sum_{s=0}^{r} \sum_{j \geq i_1 + r + 1} \varepsilon_{i_1 + s, j}. \tag{46}
\]

Assume \(i_1\) is the smallest subscript satisfying (45). Then by choosing \(r = p - i_1\), we have

\[
-J'(u)W(i_1, p - i_1) \geq B \left( \sum_{k=i_1, j \neq k}^{P} \varepsilon_{j} + \sum_{j \geq i_1} \frac{1}{\lambda_{j}} + \sum_{j \geq i_1} \frac{|\nabla_{\theta_i} K(g_{j})|}{\lambda_{j}} \right). \tag{47}
\]

If \(i_1 = 1\), we obtain the result of (18). Otherwise, for integer \(l \in [1, i_1)\), there holds

\[
\sum_{k \geq l, j \neq k} \sum_{j \geq i_1} \varepsilon_{j} \leq \frac{C}{\lambda_{l}^2} \quad \text{and} \quad \frac{|\nabla_{\theta_i} K(g_{j})|}{\lambda_{j}} \leq \frac{2}{c^2 \lambda_{j}^2}. \tag{48}
\]

From (47) and (48), then (18) is true if

\[
\sum_{k=i_1, j \neq k}^{P} \varepsilon_{j} + \sum_{j \geq i_1} \frac{1}{\lambda_{j}} + \sum_{j \geq i_1} \frac{|\nabla_{\theta_i} K(g_{j})|}{\lambda_{j}} \geq B \quad \text{for some} \quad B > 0. \tag{49}
\]

If (49) is not hold, then there holds

\[
\sum_{k=i_1, j \neq k}^{P} \varepsilon_{j} + \sum_{j \geq i_1} \frac{1}{\lambda_{j}} + \sum_{j \geq i_1} \frac{|\nabla_{\theta_i} K(g_{j})|}{\lambda_{j}} = o \left( \frac{1}{\lambda_{l}^2} \right). \tag{50}
\]

Combining with (48), we have for \(j \leq i_1 - 1, \lambda_j |\nabla_{\theta_i} K(g_{j})| \leq \frac{2}{c^2}, |\nabla_{\theta_i} K(g_{j})| = o(1)\).

These imply that: for \(j \leq i_1 - 1, g_{j}\) is close to a critical point of \(K\) which we denoted by \(\eta_j, \lambda_j d(g_{j}, \eta_j) = O(1)\).

If \(i \leq i_1 - 1, j \leq i_1 - 1, \eta_i = \eta_j, \) we have

\[
|\inf(\lambda_i, \lambda_j) d(g_{i}, g_{j})| = O(1).
\]

So if \(j < i, \)

\[
o(1) = \varepsilon_{ij} \geq C(\frac{\lambda_{j}}{\lambda_{i}})^{2-\gamma} \quad \text{and} \quad \varepsilon_{ij} \leq C(\frac{\lambda_{j}}{\lambda_{i}})^{2} = o(\frac{1}{\lambda_{j}^2}).
\]
and for $1 < j \leq i_1 - 1$ the holds \( \frac{1}{\lambda_j^2} = o\left( \frac{1}{\lambda_i^2} \right) \). If $i \leq i_1 - 1$, $j \leq i_1 - 1$, $\eta_i \neq \eta_j$, we have
\[
\varepsilon_{ij} = O\left( \frac{1}{\lambda_i \lambda_j} \right)^{2-\gamma} = o\left( \frac{1}{\lambda_i^2} \right).
\]
Thus together with (50), we have
\[
\sum_{i \neq j} \varepsilon_{ij} = o\left( \frac{1}{\lambda_i^2} \right).
\]
Combining (48) and (50), we have
\[
\sum_{i = 1}^{p} \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} = \sum_{i \geq i_1} \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} + \sum_{i < i_1} \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} \leq \frac{c}{\lambda_i^2}.
\]
So
\[
\frac{1}{\lambda_i^2} \geq B \left( \sum_{i = 1}^{p} \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} + \sum_{i = 1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]
Since $g_1$ is close to a critical point of $K$ which we denoted by $\eta_1$ and $\Delta_{\theta_i} K(\eta_1) \neq 0$, we get
\[
-J'(u) \left( \lambda_1 \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1} \right) = O\left( \sum_{j \neq 1} \lambda_1 \frac{\partial \varepsilon_{ij}}{\partial \lambda_1} \right) - c \frac{\Delta_{\theta_i} K(\eta_1)}{\lambda_i^2} (1 + o(1))
\]
\[
= -c \frac{\Delta_{\theta_i} K(\eta_1)}{\lambda_i^2} + o\left( \frac{1}{\lambda_i^2} \right).
\]
If $-\Delta_{\theta_i} K(\eta_1) \geq c > 0$, it holds that
\[
-J'(u) \left( \lambda_1 \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1} \right) = -c \frac{\Delta_{\theta_i} K(\eta_1)}{\lambda_i^2} + o\left( \frac{1}{\lambda_i^2} \right)
\]
\[
\geq c \left( \sum_{i = 1}^{p} \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} + \sum_{i = 1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]
Now we define the vector field
\[
W' = \lambda_1 \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1},
\] (51)
which satisfies (18).
If $-\Delta_{\theta_i} K(\eta_1) \leq -c < 0$, we define
\[
W' = -\lambda_1 \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1},
\] (52)
which also satisfies (18).
Subcase 2. Assume that indices $i_1$ satisfying (45) do not exist, i.e., for any $l \in \{1, \ldots, p\}$
\[
\sum_{k = l}^{p} \sum_{i \neq k}^{\varepsilon_{ik}} \leq C \frac{1}{\lambda_i^2} \quad \text{and} \quad \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} \leq \frac{2}{c^2 \lambda_i^2}.
\] (53)
By a direct argument, when $i < j$, $\eta_i = \eta_j$, we get
\[
\inf(\lambda_i, \lambda_j) d(g_i, g_j) = O(1).
\]
Thus under the condition (53), for some \( i < j \)
\[
o(1) = \varepsilon_{ij} \leq C(\frac{\lambda_i}{\lambda_j})^{2-\gamma} \text{ and } \varepsilon_{ij} \leq \frac{C}{\lambda_i^2} = o\left(\frac{1}{\lambda_i^2}\right).
\]
The construct of vector field is same as (51) or (54) in the previous subcase. In fact, we have reach that if two \( \lambda \)'s for example \( \lambda_i \) and \( \lambda_j \) are not comparable, the vector field \( W' \) can be defined to satisfy (18). So in this subcase, we can assume that \( \inf_{i \neq j} d(g_i, g_j) \geq d_0 > 0 \) and all the \( \lambda \)'s are comparable. Thus we have
\[
\sum_{j \neq i} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \sum_{j \neq i} -(2-\gamma)\varepsilon_{ij}(1+o(1)) = o\left(\frac{1}{\lambda_i}\right).
\]
Therefore, we have
\[
-J'(u) \left( \lambda_i \frac{\partial w_{g_i, \lambda_i}}{\partial \lambda_i} \right) = -c \frac{\Delta g_i K(\eta_i)}{\lambda_i^2} + o\left(\frac{1}{\lambda_i}\right).
\]
If for some \( \eta_i \) satisfies \( -\Delta g_i K(\eta_i) \leq -c < 0 \), we define
\[
W' = \lambda_i \frac{\partial w_{g_i, \lambda_i}}{\partial \lambda_i}.
\]  
If for all \( \eta_i \), \( -\Delta g_i K(\eta_i) \geq c > 0 \), Now the construct of vector field is same as
\[
W' = \sum_{i=1}^{p} \lambda_i \frac{\partial w_{g_i, \lambda_i}}{\partial \lambda_i}.
\]
Since in the cases 1-4, we can adjust some constants to insure that the union of these four case is the whole discussed space, by using a partition of unity, we can define the final vector field \( W' \) satisfying (18) at all. The proof of Lemma 4.3 is complete.

Lemma 4.4. For any \( u = \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \in V(p, \varepsilon') (\varepsilon' < \frac{\varepsilon}{2}) \), there exist \( \tilde{g} = (\tilde{g}_1, \cdots, \tilde{g}_p) \) and \( \tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_p) \) such that
\[
J \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \bar{v}(\alpha, g, \lambda) \right) = J \left( \sum_{i=1}^{p} \alpha_i w_{\tilde{g}_i, \tilde{\lambda}_i} \right)
\]
and the following two statements
\[
\sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \rightarrow 0 \iff \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \rightarrow 0,
\]  
\[
d(\tilde{g}_i, g_i) \rightarrow 0 \text{ as } \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \rightarrow 0.
\]

Proof. The proof is similar to the one given in [9] and [32].

By lemma 4.3, the vector field \( W' \) is Lipschitz. Hence, there is a 1-parameter group \( h_s \) generated by \( W' \) satisfying
\[
\begin{align*}
\frac{\partial}{\partial s} h_s (\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}) &= W' (h_s (\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i})), \\
h_0 (\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}) &= \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}.
\end{align*}
\]
For different critical points \( \eta_i \)'s of \( K \) with \( -\Delta g_i K(\eta_i) \geq c > 0 \), \( i \in \{1, \cdots, p\} \) and \( \delta < \frac{1}{4} \min\{\text{dist}(\eta_i, \eta_j)\} \), we define \( V_\delta(\eta_1, \cdots, \eta_p) \) to be the set of \((g, \lambda)\) satisfying \( g_i \in B_\delta(\eta_i), i = 1, \cdots, p \).
Similarly, we consider the vector field

$$J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}))$$

and

$$J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \bar{v}(s)))$$

are decreasing functions of $s$. Since $J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \bar{v}) \leq J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}))$, there is at most one solution of the equation

$$J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i})) = J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \bar{v})).$$

(58)

By using Lemma 4.3 and a similar proof as in [6], we can see that the flow line $h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i})$ satisfies the (PS) condition if $u_0 = \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \notin V_\delta(\eta_1, \cdots, \eta_p)$. i.e., for a small fixed $\varepsilon_0 > 0$, there is $\varepsilon_1 > 0$ such that $h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i})$ remains outside $V(p, \varepsilon_1)$ when $s \geq \varepsilon_0$.

The cases in which there could be no solution of (58) are $h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i})$ exits from $V(p, \varepsilon_1)$ or the decreasing flow goes to critical points at infinity.

If $h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i})$ exits from $V(p, \varepsilon_1)$, the flow line have to travel from $V(p, \frac{\varepsilon_1}{2})$ to $V(p, \varepsilon_1)$. By Lemma 4.3, $\frac{\partial}{\partial s} J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}))$ is lower bounded by a constant $\delta_1 > 0$ and $d(\partial V(p, \varepsilon_1), \partial V(p, \frac{\varepsilon_1}{2})) = \delta_2 > 0$. Since $\|W'\| \leq C$, then $J(h_s(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}))$ decreases at least $\frac{\delta_1 \delta_2}{2}$. However,

$$J(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}) - J(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} + \bar{v}) \to 0, \quad \varepsilon \to 0.$$

We can choose $\varepsilon > 0$ sufficiently small, there is a solution of (58).

If $\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \in V_{\delta}(\eta_1, \cdots, \eta_p)$, by Lemma 4.3, it will take an infinity time for the flow to go to infinity. Therefore, at least for a subsequence $s_k, s_k \to +\infty$,

$$\varepsilon_{ij}(s_k) + \sum_{i=1}^{p} \frac{1}{\lambda_i}(s_k) \to 0.$$

This implies $\bar{v}(s_k) \to 0$ and $J(u(s_k)) - J(\bar{u}(s_k)) \to 0$. Thus

$$\lim_{s \to +\infty} J(u(s)) = \liminf_{s \to +\infty} J(\bar{u}(s)) < J(\bar{u}).$$

By continuity, (58) must have a solution.

Similarly, we consider the vector field $-W'$ and the flow line $h_{-s}(\sum_{i=1}^{p} \alpha_i w_{\tilde{g}_i, \tilde{\lambda}_i})$.

It is easy to know that there is a unique solution for

$$J(h_{-s}(\sum_{i=1}^{p} \alpha_i w_{\tilde{g}_i, \tilde{\lambda}_i}) + \bar{v}(h_{-s}(\sum_{i=1}^{p} \alpha_i w_{\tilde{g}_i, \tilde{\lambda}_i}))) = J(h_{-s}(\sum_{i=1}^{p} \alpha_i w_{\tilde{g}_i, \tilde{\lambda}_i})).$$

Set

$$h_s\left(\sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i}\right) = \sum_{i=1}^{p} \alpha_i w_{g_i(s), \lambda_i(s)},$$

and take $(\tilde{g}_i, \tilde{\lambda}_i) = (g_i(s), \lambda_i(s))$, we have (55).

As for (56) and (57), we note that

$$W' = \sum_{i=1}^{p} \alpha_i \left(\frac{1}{\lambda_i(s)} \frac{\partial w_{g_i(s), \lambda_i(s)}}{\partial g_i(s)} \right) (\lambda_i(s) \delta_i(s)) + \sum_{i=1}^{p} \alpha_i \left(\lambda_i(s) \frac{\partial w_{g_i(s), \lambda_i(s)}}{\partial \lambda_i(s)} \right) \tilde{\lambda}_i(s) \delta_i(s),$$

where $\delta_i(s)$ and $\tilde{\lambda}_i(s)$ denote the action of $W'$ on the variables $g_i$ and $\lambda_i$. We have $|\lambda_i \delta_i| \leq C, \tilde{\lambda}_i \leq C, i = 1, \cdots, p$. Then

$$\left|\frac{\partial \varepsilon_{ij}(s)}{\partial s}\right| = \frac{\partial \varepsilon_{ij}(s)}{\partial \lambda_i} \frac{\partial \lambda_i(s)}{\partial s} + \frac{\partial \varepsilon_{ij}(s)}{\partial \lambda_j} \frac{\partial \lambda_j(s)}{\partial s} + \frac{\partial \varepsilon_{ij}(s)}{\partial g_i} \frac{\partial g_i(s)}{\partial s} + \frac{\partial \varepsilon_{ij}(s)}{\partial g_j} \frac{\partial g_j(s)}{\partial s} \leq C \varepsilon_{ij}(s).$$
Thus,
\[ e^{-Cs \epsilon_{ij}} \leq \epsilon_{ij}(s) \leq e^{Cs \epsilon_{ij}} \leq \frac{\lambda_i(s)}{\lambda_i(0)} \leq e^{Cs}, |g_i(s) - g_i| \leq \frac{e^{Cs}}{\lambda_i(0)}. \]

Since the \( s \) satisfying (58) is bounded, we get (56) and (57).

Following from Lemma 4.4, for any \( \epsilon_1 > 0 \) small, there are \( \epsilon > 0 \) and \( \epsilon_2 > 0 \) such that
\[ V(p, \epsilon) \xrightarrow{h \to} V(p, \epsilon_1) \xrightarrow{h \to} V(p, \epsilon_2) \supseteq V(p, \epsilon_1). \]

From Lemma 4.2, Lemma 4.4 and this fact, we can prove Theorem 4.1.

5. The proof of main theorem. For technical reasons, we introduce for \( \epsilon_0 > 0 \)
small enough, the following subset of \( \Sigma \) as \( V_{\epsilon_0}(\Sigma^+) = \{ u \in \Sigma, ||u - u||_{L^\infty} < \epsilon_0 \}. \)

By Theorem 4.1, there is a covering \( \{ O_i \} \) of the base space of the bundle \( V(p, \epsilon) \) such that Theorem 4.1 holds on each \( \{ O_i \} \). For \( u = \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} + v \in V(p, \epsilon) \), we consider transformation of coordinates \( \varphi : (\alpha, g, \lambda, v) \to (\tilde{\alpha}, \tilde{g}, \tilde{\lambda}, \tilde{v}) \), so that (17) holds.

We first define a vector field on \( V(p, \epsilon) \) by using a partition of unity \( \eta_i \) on the base space of \( (\alpha, g, \lambda, V) \) as \( X = W - \sum_{i=1}^n \eta_i \tilde{v}_i \), where \( W(\alpha, g, \lambda, V) = W'(\alpha, \tilde{g}, \tilde{\lambda}) \).

Then the vector field in the variables \( (\alpha, g, \lambda, v) \) is defined as \( X = Z \circ \varphi \). By direct computation, in every open set \( O_i \) we have
\[ -J'(\sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} + v)(Z) = -J'(\sum_{i=1}^p \alpha_i w_{g_i, \lambda_i})(W') + J''(u_0) V_i \cdot V_i + O(\|v\|^2), \]
where \( u_0 = \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} + \tilde{v}(\alpha, g, \lambda) \).

We remind that \( V_i \) orthogonal to \( w_{g_i, \lambda_i}, \sum_{i=1}^p \frac{\partial w_{g_i, \lambda_i}}{\partial g_i}, \sum_{i=1}^p \frac{\partial w_{g_i, \lambda_i}}{\partial g_i} \), so by computation,
\[ J''(u_0) V_i \cdot V_i = 2 \lambda (u_0) \|V_i\|^2 - 4 \lambda (u_0) \sum_{i=1}^p P_i u_0 V_i \int_{\mathbb{R}^3} K u_0^{2+\frac{2\gamma}{\gamma - 2}} V_i \]
\[ + \frac{4(4 - \gamma)}{2 - \gamma} \lambda (u_0) \int_{\mathbb{R}^3} K u_0^{2+\frac{2\gamma}{\gamma - 2}} V_i^2 - \frac{2(2 + \gamma)}{2 - \gamma} \lambda (u_0) \sum_{i=1}^p P_i u_0 V_i \int_{\mathbb{R}^3} K u_0^{2+\frac{2\gamma}{\gamma - 2}} V_i^2. \]

Using the estimates in Appendix A and \( Q(v, v) \) is positive definite, we obtain \( J''(u_0) V_i \cdot V_i \) is positive definite. So in \( V(p, \epsilon) \), if \( \epsilon \) small enough, there holds
\[ -J'\left( \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} + v \right)(Z) \geq C \left( \sum_{i=1}^p \frac{\|\nabla g_i K(\tilde{g}_i)\|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \bar{\epsilon}_{ij} \right). \]

We now suppose that the functional \( J \) has no critical point and there holds \( \|J'(u)\| \geq C \) for \( u \notin V(p, \tilde{\epsilon}) \). On \( V(p, \tilde{\epsilon}) \), define the vector field \( -J' \), and then also via a partition of unity of the two sets \( V(p, \epsilon) \) and \( V(p, \tilde{\epsilon}) \) to build a global vector field \( \tilde{Z}(u) \) on \( V_{\epsilon_0}(\Sigma^+) \) from \( Z \) and \( -J' \). It is easy to see
\[ -J'(u)(Z) \geq C \left( \sum_{i=1}^p \frac{\|\nabla g_i K(\tilde{g}_i)\|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \bar{\epsilon}_{ij} \right). \]

It is important to insure that any flow line generated by the vector field \( \tilde{Z} \) with initial condition \( u \in V_{\epsilon_0}(\Sigma^+) \) remains in \( V_{\epsilon_0}(\Sigma^+) \). We have the following lemma.

**Lemma 5.1.** \( V_{\epsilon_0}(\Sigma^+) \) is invariant under the flow generated by \( \tilde{Z}(u) \).
Proof. It is sufficient to prove that \( V_{\omega}(\Sigma^+) \) is invariant under the negative gradient flow of \( J \).

Suppose \( u_0 \in V_{\omega}(\Sigma^+) \),

\[
\begin{aligned}
\frac{du(s)}{ds} &= -2\lambda(u(s))u(s) + 2\lambda(u(s))^{\frac{4}{3}}P_\gamma^{-1}Ku(s)^{\frac{2+\gamma}{\gamma}}, \\
u(0) &= u_0.
\end{aligned}
\]

Then

\[
e^{-\int_0^t 2\lambda(u(t))dt}u(s) = u_0 + \int_0^s e^{-\int_0^t 2\lambda(u(t))dt}2\lambda(u(t))^{\frac{4}{3}}P_\gamma^{-1}Ku(t)^{\frac{2+\gamma}{\gamma}}dt.
\]

Therefore,

\[
u(s) = e^{-\int_0^s 2\lambda(u(t))dt}u_0^+ - e^{-\int_0^s 2\lambda(u(t))dt}u_0^- + e^{-\int_0^s 2\lambda(u(t))dt}\int_0^s e^{\int_0^t 2\lambda(u(t))dt}2\lambda(u(t))^{\frac{4}{3}}P_\gamma^{-1}Ku(t)^{\frac{2+\gamma}{\gamma}}dt.
\]

Hence,

\[
u^-(s) \leq e^{-\int_0^s 2\lambda(u(t))dt}u_0^-.
\]

Set

\[
f(s) = e^{-\frac{4}{3}\int_0^s 2\lambda(u(t))dt}\left|\frac{u_0}{\lambda}\right|^{\frac{1}{\gamma}}.
\]

Then \( |\nu^-(s)|^{\frac{4}{3\gamma}} \leq f(s) \),

\[
f'(s) = -\frac{8\lambda}{2-\gamma}e^{-\frac{4}{3}\int_0^s 2\lambda(u(t))dt}\left|\frac{u_0}{\lambda}\right|^{\frac{1}{\gamma}} \leq 0.
\]

Therefore,

\[
|\nu(s)|^{\frac{4}{3\gamma}} < \varepsilon_0 \text{ for all } s > 0. 
\]

Next, we study the concentration phenomenon of the functional \( J \).

Lemma 5.2. Assume that (1) has no solution. Then, the set of critical point at infinity of \( J \) in \( \Sigma^+ \) lie in \( \bigcup_{\eta \in V_{\omega}(\Sigma^+)} V_{\omega}(\eta_1, \cdots, \eta_p) \).

Proof. From the fact that outside \( \cup_{\eta \in V_{\omega}(\Sigma^+)} V_{\omega}(\eta_1, \cdots, \eta_p) \) and lemma 2.2, we know that if \( u_0 \in V_{\omega}(\Sigma^+) \), there is \( p \in N^+ \) and \( s_0 > 0 \) such that if \( \eta(s, u_0) \) denotes the flow line of the vector field \( Z \) with initial condition \( u_0 \), that is \( \eta(s, u_0) \) satisfies

\[
\begin{aligned}
\frac{\partial}{\partial s} \eta(s, u_0) &= \tilde{Z}(\eta(s, u_0)), \\
\eta(0, u_0) &= u_0,
\end{aligned}
\]

\( \eta(s, u_0) \) is in \( V_{\omega}(\Sigma^+) \) for \( s \geq s_0 \), since in \( V_{\omega}(\Sigma^+) \), \( J(u) > 0 \).

Assume that for any \( s \geq s_0 \), \( \eta(s, u_0) \) is in \( V_{\omega}(\Sigma^+) \) but outside \( V_{\omega}(\eta_1, \cdots, \eta_p) \). It means that \( \eta(s, u_0) \) satisfies (PS) condition for \( s \geq s_0 \). By the construction of the vector field \( W' \), in all the cases of the vector field defined with negative coefficients in direction \( \lambda \)'s, then we have \( \lambda_{\text{max}} = \max_{i=1, \cdots, p} \lambda_i(s) \leq c \), where \( c \) depends only on \( (s_0, u_0) \). But in the cases of \( W' \) being defined by \( W' = \lambda_1 \frac{\partial u_{\omega}}{\partial \lambda_1} \), then \( \lambda_1 \) is not comparable with \( \lambda \) for \( i \geq 2 \), then \( \lambda_1(s) < \lambda_2 \leq \cdots \leq \lambda_p \) since the vector field does not increase the variables \( \lambda_i \) for \( i \geq 2 \) (in fact \( \lambda_i(s) = \lambda_i(0) \)). In these cases, all \( \lambda \)'s are bounded as well. The only case is that the \( \lambda \)'s are comparable, \( g_i \) and \( g_j \).
converges to the different critical points $\eta_i \neq \eta_j$ for all $i \neq j$. This is the case that the flow line entries the set $V_\delta(\eta_1, \cdots, \eta_p)$. If $\gamma(s, u_0)$ remains out of $V_\delta(\eta_1, \cdots, \eta_p)$, then, $-J'(\gamma(s, u_0))((\dot{Z}(s, u_0)) \geq C$ since $\{c, \gamma_i \leq c, g_i \in S^1\}$ is a compact set. Hence, $J(\eta(s, u_0)) = J(\gamma(0, u_0)) + \int_0^s J'(u)\dot{Z}(u) \leq J(\eta(0, u_0)) - C(s - s_0)$ tends to $-\infty$, when $s \to +\infty$, a contradiction to the fact that $J$ is lower bounded. 

**Lemma 5.3.** For any $u = \sum_{i=1}^p \alpha_i w_{\eta_i, \lambda_i} + v$ in $V_\delta(\eta_1, \cdots, \eta_p)$, we have the following expansion of $J(u)$ after changing the variables:

$$J(u) = S \left( \sum_{i=1}^p \frac{1}{K(\eta_i)} \right)^\frac{2}{2-\gamma} \left( 1 - \frac{2 - \gamma c_2}{2S^\frac{2}{\gamma}} \sum_{i=1}^p \frac{\alpha_i^{2-\gamma} \Delta_{\eta_i} K(g_i)}{\sum_{k=1}^p \alpha_k^{2-\gamma} K(g_k)\lambda_i^2} \right) + O(\sum_{i<j} \varepsilon_{ij}) + O(\sum_{i=1}^p \frac{1}{\lambda_i^2}) + \frac{1}{2} J'' \left( \sum_{i=1}^p \alpha_i w_{\eta_i, \lambda_i} + \bar{v}(\alpha, g, \lambda) \right) V_i \cdot V_i,$$

where $g_i^+ = (g_i^+)_{\eta_i}$, $g_i^- = (g_i^-)_{\eta_i}$ are the coordinates of $g_i$ near $\eta_i$, along the stable and unstable manifold for $K$ and $h = (h_1, \cdots, h_{p-1}) \in \mathbb{R}^{p-1}$ with $h_i = h_i(\alpha_1, \cdots, \alpha_p)$, $i = 1, \cdots, p-1$ are independent functions.

**Proof.** From Lemma 3.1 and Theorem 4.1, we have

$$J(u) = \sum_{i=1}^p \frac{1}{K(\eta_i)} \left( 1 - \frac{2 - \gamma c_2}{2S^\frac{2}{\gamma}} \sum_{i=1}^p \frac{\alpha_i^{2-\gamma} \Delta_{\eta_i} K(g_i)}{\sum_{k=1}^p \alpha_k^{2-\gamma} K(g_k)\lambda_i^2} \right) + O(\sum_{i<j} \varepsilon_{ij}) + O(\sum_{i=1}^p \frac{1}{\lambda_i^2}) + \frac{1}{2} J'' \left( \sum_{i=1}^p \alpha_i w_{\eta_i, \lambda_i} + \bar{v}(\alpha, g, \lambda) \right) V_i \cdot V_i,$$

From the proof of Lemma 4.3, we have $|\nabla_{\eta_i} K(g_i)| = o(1), \varepsilon_{ij} = o(\frac{1}{\lambda_i^2})$,

$$|J(u)^{\frac{2-\gamma}{2}} \alpha_i^{\frac{2\gamma}{2-\gamma}} K(g_i) - 1| < \varepsilon,$$

the expansion of the functional $J$ can be rewritten as follows:

$$J(u) = \sum_{i=1}^p \frac{1}{K(\eta_i)} \left( 1 - \frac{2 - \gamma c_2}{2S^\frac{2}{\gamma}} \sum_{k=1}^p \frac{1}{K(g_k)\lambda_i^2} \right) + \frac{1}{2} J'' \left( \sum_{i=1}^p \alpha_i w_{\eta_i, \lambda_i} + \bar{v}(\alpha, g, \lambda) \right) V_i \cdot V_i.$$

Except the term

$$g(\alpha, g) = \frac{\sum_{i=1}^p \alpha_i^{2-\gamma} K(g_i)}{\sum_{i=1}^p \alpha_i^{2-\gamma} K(g_i)},$$

all others are positive on the right hand side of the above equality. Since $g(\alpha, g)$ is homogeneous in the variable $\alpha$, we have a degenerated critical point $(\bar{\alpha}_1, \cdots, \bar{\alpha}_p)$ which satisfies

$$\frac{\bar{\alpha}_i^2 K(g_i)}{\bar{\alpha}_j^2 K(g_j)} = 1.$$

This critical point has an index equal to $p - 1$ (since the critical point corresponds to a maximum).
On the other hand, \( g(a, g) \) has a single critical point \( \eta = (\eta_{\beta_1}, \eta_{\beta_2}, \cdots, \eta_{\beta_p}) \) in the \( g \) variable. Thus, using the Morse lemma, after a change of variables, we have have the following normal form,

\[
J(u) = \left( \sum_{i=1}^{p} \frac{1}{K(\eta_{\beta_i})^{\frac{1}{2\gamma}}} \right) \left( S - |h|^2 + \sum_{i=1}^{p} (|g_i|^2 - |g_i^{-1}|^2) + c \sum_{i=1}^{p} \frac{1}{\chi_i^2} \right) + ||V||^2.
\]

For any \( l \)-tuple \( \tau_l = (i_1, \cdots, i_l) \), \( 1 \leq i_j \leq m_1, j = 1, \cdots, l \), let

\[
c(\tau_l) = \left( \sum_{j=1}^{l} \frac{1}{K(\eta_{\beta_{i_j}})^{\frac{1}{2\gamma}}} \right)^2
\]
denote the associated critical value. We only consider a simple situation, where for any \( \tau \neq \tau', c(\tau) \neq c(\tau') \), and thus order as \( c(\tau_1) < \cdots < c(\tau_k) \).

From Lemma 5.3 and a deformation lemma (see [3] and [7]) or directly the critical group theory (see [15]), we have

**Lemma 5.4.** If \( c(\tau_{l-1}) < a < c(\tau_l) < b < c(\tau_{l+1}) \), for any coefficient group \( G \), then

\[
H_q(J_b, J_a) = \begin{cases} 
0, & q \neq k(\tau_l), \\
G, & q = k(\tau_l), 
\end{cases}
\]

where \( k(\tau_l) = 4l - 1 - \sum_{j=1}^{l} \text{ind}(K, \xi_{i_j}) \).

If \( X \) is a topological set, then \( \chi(X) \) is its Euler-Poincare characteristic with rational coefficients.

**Proof of the theorem.** Since we assumed that \( (1) \) has no solution, \( V_{\varepsilon_0}(\Sigma^+) \) is retract by deformation of \( \Sigma^+ \). \( \Sigma^+ \) is contractible, so \( \chi(V_{\varepsilon_0}(\Sigma^+)) = 1 \). By Lemma 5.4 and the Morse lemma, we have

\[
\chi(V_{\varepsilon_0}(\Sigma^+)) = \sum_{l=1}^{m_1} \sum_{\tau_l=(i_1,\cdots,i_l), \eta_{i_j} \in \mathcal{I}^+} (-1)^{4l-1-\sum_{j=1}^{l} \text{ind}(K, \eta_{i_j})}
\]
is a contradiction. Therefore, \( (1) \) has a solution \( u_0 \in V_{\varepsilon_0}(\Sigma^+) \).

We claim that \( u_0 > 0 \), when \( \varepsilon_0 \) is small enough. Otherwise, we can write \( u_0 = u_0^+ - u_0^- \). Multiplying equation \( (1) \) by \( u_0^- \) and integrating, using the fact that \( u_0 \in V_{\varepsilon_0}(\Sigma^+) \), we derive

\[
||u_0^-||^2 \leq C_1 ||u_0^-||^{\frac{4}{1-\gamma}} \leq C_2 ||u_0^-||^{\frac{4}{1-\gamma}}.
\]

Hence, either \( u_0^- = 0 \), or \( ||u_0^-|| \geq C \), where \( C > 0 \). Thus we have a contradiction with \( \varepsilon_0 \) small enough. Therefore \( u_0^- = 0 \) and \( u_0 > 0 \).

**6. Appendix.** We first introduce some well-known inequalities which are from Taylor expansion and some computations.

**Lemma 6.1.** For \( \alpha \geq 3 \), there exists a constant \( M > 0 \), such that for any \( (a, b) \in \mathbb{R}^2 \), there holds

\[
||(a+b)^\alpha - a^\alpha - \alpha a^{\alpha-1} b - \frac{\alpha(\alpha-1)}{2} a^{\alpha-2} b^2|| \leq M(||b||^\alpha + ||a||^{\alpha-3} \inf(||a||^3, ||b||^3)).
\]

In the following four lemmas, we assume \( \alpha > 0 \).
Lemma 6.2 ([3]). There exists a constant $M$, such that for any $(a, b) \in \mathbb{R}^2$, $a > 0, a + b > 0$,
\[
| (a + b)^\alpha - a^\alpha - aa^{\alpha - 1}b | \leq M (|b|^\alpha + |a|^\alpha - 2 \inf |a|^2 b^2 ).
\] (60)

Lemma 6.3 ([3]). There exists a constant $M$, such that for any $(a_1, \cdots , a_p) \in \mathbb{R}^p$,
\[
\left| \left( \sum_{i=1}^p a_i \right)^\alpha - \sum_{i=1}^p a_i^\alpha - \alpha \sum_{i \neq j} a_i^{\alpha - 1}a_j \right| \leq M \left( \sum_{i \neq j} |a_i|^{\alpha - 1} \inf |a_i|, |a_j| \right).
\] (61)

Lemma 6.4 ([3]). There exists a constant $M$, such that for any $(a_1, \cdots , a_p) \in \mathbb{R}^p$,
\[
\left| \left( \sum_{i=1}^p a_i \right)^\alpha - \sum_{i=1}^p a_i^\alpha - \alpha \sum_{i \neq j} a_i^{\alpha - 1}a_j \right| \leq M \left( \sum_{i \neq j} \inf |a_i|^{\alpha - 2}, |a_j|^{\alpha - 2} \inf (a_i^2, a_j^2) + \sum_{i \neq j} \inf |a_i|^{\alpha - 1}, |a_j|^{\alpha - 1} \sup |a_i|, |a_j| \right).
\] (62)

Lemma 6.5 ([3]). There exists a constant $M$, such that for any $(a_1, \cdots , a_p) \in \mathbb{R}^p$,
\[
\left| \left( \sum_{i=1}^p a_i \right)^\alpha - \sum_{i=1}^p a_i^\alpha - \alpha \sum_{i \neq j} a_i^{\alpha - 1}a_j \right| \leq M \left( \sum_{i \neq j, i \neq k} |a_i|^{\alpha - 1} \inf |a_i|, |a_k| \right) + \sum_{i \neq j} |a_i|^{\alpha - 2} \inf (a_i^2, a_j^2) + \sum_{i \neq j} \inf |a_i|^{\alpha - 1}, |a_j|^{\alpha - 1} \sum_{i \neq j} |a_i| \right).
\] (63)

6.1. Appendix A. We set
\[
J(u) = \frac{\|u\|^2}{\left( \int_{\mathbb{S}^3} K(\zeta) u^{2+\gamma} \, d\theta_1 \wedge d\theta_1 \right)^{2+\gamma}} = \frac{N}{D}.
\]

We first expand the numerator $N$ as follows,
\[
N = \|u\|^2 = \int_{\mathbb{S}^3} P^\theta_1 w_1 \, d\theta_1 \wedge d\theta_1
\]
\[
= \int_{\mathbb{S}^3} P^\theta_1 (\sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} + v) (\sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} + v) \, d\theta_1 \wedge d\theta_1
\]
\[
= \sum_{i=1}^p \alpha_i^2 \int_{\mathbb{S}^3} P^\theta_1 w_{g_i, \lambda_i} w_{g_i, \lambda_i} \, d\theta_1 \wedge d\theta_1 + \sum_{i \neq j} \alpha_i \alpha_j \int_{\mathbb{S}^3} P^\theta_1 w_{g_i, \lambda_i} w_{g_j, \lambda_j} \, d\theta_1 \wedge d\theta_1
\]
\[
+ \int_{\mathbb{S}^3} P^\theta_1 v v \, d\theta_1 \wedge d\theta_1,
\]
all the other terms are zero since $v$ satisfies conditions. From now on, we denote $a_i = C^{-1}(g_i), \xi = C^{-1}(\zeta).

Lemma A.1. We have
\[
\int_{\mathbb{S}^3} P^\theta_1 w_{g_i, \lambda_i} w_{g_j, \lambda_j} \, d\theta_1 = \int_{\mathbb{H}^1} P^\theta_0 \delta_{a_i, \lambda_i} \delta_{a_j, \lambda_j} \delta_0 \wedge d\theta_0 = S^2,
\]
where $S$ is the sharp Sobolev constant given by
\[ S = \inf_{u \in S^1_0(\mathbb{H}^1)} \frac{\int_{\mathbb{H}^1} |\nabla^2 u|^2 \, d\theta_0}{\left( \int_{\mathbb{H}^1} u^2 \, d\theta_0 \right)^{2/3}} , \quad 2^* = \frac{4}{2 - \gamma}. \]

Lemma A.2. It holds that for $i \neq j$
\[ \int_{\mathbb{H}^1} w_{g_{ij},\lambda_i} \, d\theta_0 = O(\varepsilon_{ij}). \]
Here $O'(h)$ means that when $|h| << 1$, there exist two constants $C_1, C_2 > 0$ such that $C_1 |h| \leq O'(h) \leq C_2 |h|.$

Proof. Let $\xi = (x, y, t), a_i = (x_i, y_i, t_i) \in \mathbb{H}^1.$
\[ I = \int_{\mathbb{H}^1} w_{g_{ij},\lambda_i,\lambda_j} \, d\theta_0 \]
\[ = \int_{\mathbb{H}^1} \left( \frac{\lambda_i^{2+\gamma} \lambda_j^{2-\gamma}}{\theta_0} \right) \, d\theta_0 \]
\[ = \frac{1}{\lambda_i^{2+\gamma} \lambda_j^{2-\gamma}} \int_{\mathbb{H}^1} \left[ (1 + \lambda_i^2 |x - x_i|^2 + \lambda_j^2 |y - y_j|^2)^2 + \lambda_i^4 (t - t_i + 2x_iy - 2xy_j)^2 \right] \frac{1}{\lambda_i^{2+\gamma} \lambda_j^{2-\gamma}} \]
\[ \times f_1(x, y, t)^{2-\gamma}, \]
where
\[ f_1(x, y, t) = \left( 1 + \left( \frac{\lambda_j}{\lambda_i} \right)^2 \left( |x - \lambda_i (x_i - x_j)|^2 + |y + \lambda_i (y_i - y_j)|^2 \right) \right)^2 \]
\[ + \left( \frac{\lambda_j}{\lambda_i} \right)^4 \left( t + \lambda_i^2 (t_i - t_j + 2x_iy - 2xy_j) + 2\lambda_i (y_i - y_j)x - 2\lambda_i (x_i - x_j)y \right)^2. \]
Let $\mu = \max(\frac{\lambda_i^2}{\lambda_j^2}, \lambda_i\lambda_j|d_{ij}|^2)$, where
\[ d_{ij} = d(a_i, a_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (t_i - t_j + 2x_iy_i - 2xy_j)^2}. \]
First we assume $\mu = \frac{\lambda_i^2}{\lambda_j^2}$. If $|\xi| \leq \frac{1}{4} \frac{\lambda_j}{\lambda_i}$, we use Taylor expansions in $\mathbb{H}^1$,
\[ f_1(x, y, t) = \left( 1 + \lambda_i^{-1} Xf(0)x + \lambda_i^{-1} Yf(0)y + \lambda_i^{-1} T_0 f(0)t + O(\left( \frac{\lambda_i}{\lambda_j} \right)^2 |\xi|^2) \right). \]
Denote
\[ g_1 = \left( 1 + \lambda_i^2 (t_i - t_j + 2x_iy_i - 2xy_j)^2 \right)^{1/2} \]
\[ f_1(x, y, t)^{-(2-\gamma)} = g_1^{-(2-\gamma)} \left[ 1 - (2 - \gamma)g_1^{-1} Xf(0)x - (2 - \gamma)g_1^{-1} Yf(0)y \right. \]
\[ - (2 - \gamma)g_1^{-1} T_0 f(0)t + O(\left( \frac{\lambda_i}{\lambda_j} \right)^2 |\xi|^2) \right]. \]
Thus it yields
\[ \int_{B(0, \frac{1}{4} \frac{\lambda_j}{\lambda_i})} \frac{\theta_0 \wedge d\theta_0}{\left( (1 + |x|^2 + |y|^2)^2 + t^2 \right)^{2+\gamma}} = \int_{\mathbb{H}^1} \frac{\theta_0 \wedge d\theta_0}{\left( (1 + |x|^2 + |y|^2)^2 + t^2 \right)^{2+\gamma}}. \]
The case where \( \varepsilon = \lambda \) is similar to the case \( \mu = \lambda \). Then we consider the third case \( \mu = \lambda_i \lambda_j |d_{ij}|^2 \). In this case,

\[
I = c_0 \int_{\mathbb{H}^1} \frac{\theta_0 \wedge d\theta_0}{(1 + |x|^2 + |y|^2)^{\frac{2+\gamma}{2}}} f_2(x, y, t)^{2-\gamma}.
\]

where

\[
f_2(x, y, t)^2 = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i}(|x + \lambda_i(x - x_j)|^2 + |y + \lambda_i(y - y_j)|^2) \right)^2
\]

\[+ \left( \frac{\lambda_j}{\lambda_i} \right)^2 \left( t + \lambda_i^2(t_i - t_j + 2x_i y_i - 2x_j y_j) + 2\lambda_i(y_i - y_j)x - 2\lambda_i(x_i - x_j)y \right)^2.
\]

Without loss of generality, we assume \( \lambda_i \geq \lambda_j \), therefore

\[
f_2(x, y, t) = g_2 \left[ 1 + g_2^{-1}X f(0)x + g_2^{-1}Y f(0)y + g_2^{-1}T_0 f(0)t + O(\frac{\lambda_j}{\lambda_i})^2 \right].
\]

\[
g_2 = \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |x_i - x_j|^2 + \lambda_i \lambda_j |y_i - y_j|^2 \right)^2 + \lambda_i^2 \lambda_j^2 \left( t_i - t_j + 2x_i y_i - 2x_j y_j \right)^2 + \frac{\lambda_j}{\lambda_i}.
\]

By the same arguments used in the first case, we obtain

\[
\int_{|\xi| \leq \frac{\pi}{2\gamma}} \frac{\theta_0 \wedge d\theta_0}{(1 + |x|^2 + |y|^2)^{\frac{2+\gamma}{2}}} f_2(x, y, t)^{2-\gamma} = O'(\varepsilon).\]

Let

\[B_1 := \{ \xi \in \mathbb{H}^1 \mid |\xi + \lambda_i d_{ij}| \leq \frac{1}{10} \lambda_i d_{ij} \}.\]
This completes the proof.

Lemma A.3. \((1)\) It holds that
\[
\int_{\mathbb{S}^3} w_{g_i}^2 \, d\theta_1 \wedge d\theta_0 = \int_{\mathbb{S}^3} \delta_{a_i,\lambda_i} \delta_{b_i,\lambda_i} \theta_0 \wedge d\theta_0 = O(\varepsilon_{ij}^{-2}) \ln \varepsilon_{ij}^{-1}).
\]

\[(2)\] Let \(\alpha, \beta > 1\), such that \(\alpha + \beta = \frac{4}{2-\gamma}, \theta = \inf (\alpha, \beta)\), it holds that
\[
\int_{\mathbb{S}^3} w_{g_i}^\alpha \, d\theta_1 \wedge d\theta_0 = \int_{\mathbb{S}^3} \delta_{a_i,\lambda_i} \delta_{b_i,\lambda_i} \theta_0 \wedge d\theta_0 = O(\varepsilon_{ij}^\alpha (\ln \varepsilon_{ij}^{-1})^{2-2\theta}).
\]

Let us consider the denominator \(D\) of \(J\),
\[
D^{\frac{2-\gamma}{2}} = \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} + v \right)^{\frac{2-\gamma}{2}} \theta_1 \wedge d\theta_0.
\]

Lemma A.4. If \(|v| < \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i}\),
\[
\int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} + v \right)^{\frac{4}{2-\gamma}}
= \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} \right)^{\frac{1}{1-\gamma}} + \frac{4}{2-\gamma} \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} \right)^{\frac{2+\gamma}{2-\gamma}} v^2 + O(||v||^3).
\]

Proof. Using (59), we have
\[
\left| \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} + v \right)^{\frac{4}{2-\gamma}} - \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} \right)^{\frac{1}{1-\gamma}} \right|
\]
\[
- \frac{4}{2-\gamma} \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} \right)^{\frac{2+\gamma}{2-\gamma}} v^2 + \frac{2(2+\gamma)}{(2-\gamma)^2} \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} \right)^{\frac{2\gamma}{2-\gamma}} v^2
\]
\[
\leq M_2 \left( \int_{\mathbb{S}^3} |v|^{\frac{4}{2-\gamma}} + \int_{\mathbb{S}^3} \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} \right)^{\frac{2\gamma}{2-\gamma}} \inf \left( \left( \sum_{i=1}^p \alpha_i w_{g_i,\lambda_i} \right)^3, |v|^3 \right) \right)
\]
\[
= O(||v||^3).
\]
In order to get more information from Lemma A.4, we now estimate the first three terms on the right hand side of (64).

**Lemma A.4.1.**

\[
\int_{S^3} K(\zeta) \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right) \frac{4}{\gamma} \theta_1 \wedge d\theta_1 = \sum_{i=1}^{p} \alpha_i^4 \left( -K(g_i) S^2 \frac{\epsilon_2}{\lambda_i^4} \right) + O' \left( \sum_{i \neq j} \epsilon_{ij} \right) + o \left( \frac{1}{\lambda_i^4} \right),
\]

where \( c_2 = 4c_0 \frac{4}{\gamma} \int_{H^1} \frac{(x^2+y^2)^{\theta_0} \wedge d\theta_0}{(1+|x|^2+|y|^2)^{2+\epsilon}}. \)

**Proof.** From definition and (7), there holds

\[
\int_{S^3} K(\zeta) \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right) \frac{4}{\gamma} \theta_1 \wedge d\theta_1 = \int_{H^1} \tilde{K}(\xi) \left( \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i} \right) \frac{4}{\gamma} \theta_0 \wedge d\theta_0,
\]

where \( \tilde{K} = K \circ \mathcal{C}. \) From (62) and Lemma A.3, we get

\[
\left| \int_{H^1} \tilde{K}(\xi) \left( \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i} \right) \frac{4}{\gamma} \theta_0 \wedge d\theta_0 \right|
\]

\[
- \int_{H^1} \tilde{K}(\xi) \left( \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i} \right) \frac{4}{\gamma} \theta_0 \wedge d\theta_0 \right| \leq M_1 \sum_{i \neq j} \int_{H^1} \sup(\alpha_i \delta_{a_i, \lambda_i}, \alpha_j \delta_{a_j, \lambda_j}) \frac{4}{\gamma} \theta_0 \wedge d\theta_0
\]

\[
+ M_4 \sum_{i \neq j} \int_{H^1} \inf(\alpha_i \delta_{a_i, \lambda_i}, \alpha_j \delta_{a_j, \lambda_j}) \frac{4}{\gamma} \theta_0 \wedge d\theta_0
\]

\[
\leq 2M_4 \sum_{i \neq j} \alpha_i \alpha_j \frac{4}{\gamma} \int_{H^1} \delta_{a_i, \lambda_i} \delta_{a_j, \lambda_j} \theta_0 \wedge d\theta_0
\]

\[
= O \left( \sum_{i \neq j} \epsilon_{ij} \frac{4}{\gamma} \ln \epsilon_{ij}^{-1} \right).
\]

Then we have

\[
\int_{H^1} \tilde{K}(\xi) \delta_{a_i, \lambda_i} \frac{4}{\gamma} \theta_0 \wedge d\theta_0 = \int_{H^1} \left( \tilde{K}(\xi) - \tilde{K}(a_i) \right) \delta_{a_i, \lambda_i} \frac{4}{\gamma} \theta_0 \wedge d\theta_0 + \tilde{K}(a_i) S^2 \frac{\epsilon_2}{\lambda_i^4}
\]

\[
= \tilde{K}(a_i) S^2 \frac{\epsilon_2}{\lambda_i^4} + \int_{B(a_i, \varepsilon)} \left( \tilde{K}(\xi) - \tilde{K}(a_i) \right) \delta_{a_i, \lambda_i} \frac{4}{\gamma}
\]

\[
+ \int_{B^c(a_i, \varepsilon)} \left( \tilde{K}(\xi) - \tilde{K}(a_i) \right) \delta_{a_i, \lambda_i} \frac{4}{\gamma}
\]

and

\[
\int_{B^c(a_i, \varepsilon)} \delta_{a_i, \lambda_i} \theta_0 \wedge d\theta_0 = O \left( \frac{1}{\lambda_i^4} \right).
\]
By Taylor expansion, there holds
\[ \int_{B(a_i, \varepsilon)} (\tilde{K}(\xi) - \tilde{K}(a_i)) \delta^{\frac{1}{2+\gamma}} \delta a_i \wedge d\theta_i = \frac{c_2 \Delta_{\theta_i} \tilde{K}(a_i)}{4 \lambda_i^2} + o\left( \frac{1}{\lambda_i} \right). \]

Finally we estimate for \( i \neq j \), by Lemma A.2, Lemma A.3 and Taylor expansion and Young inequality,
\[
\int_{\mathbb{H}^1} \tilde{K}(\xi)(\delta a_i, \lambda_i) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \\
= \int_{\mathbb{H}^1} \tilde{K}(a_i)(\delta a_i, \lambda_i) \frac{2+\gamma}{2+\gamma} \delta a_i + \int_{B(a_i, \varepsilon)} (\tilde{K}(\xi) - \tilde{K}(a_i)) (\delta a_i, \lambda_i) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \\
+ \int_{B(a_i, \varepsilon)} (\tilde{K}(\xi) - \tilde{K}(a_i)) (\delta a_i, \lambda_i) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \\
= O(\varepsilon_{ij}) + O(\varepsilon_{ij}^{2+\gamma} \ln \varepsilon_{ij}^{-1}) + o\left( \frac{1}{\lambda_i^2} \right) + O\left( \frac{1}{\lambda_i^{2+\gamma}} \right). \]

**Lemma A.4.2.** We have
\[
\int_{\mathbb{H}^1} K(\xi) \left( \sum_{i=1}^p \alpha_i w g_{\lambda_i} \right) \frac{2+\gamma}{2+\gamma} v \wedge d\theta_i \leq C_0 \int_{\mathbb{H}^1} \tilde{K}(\xi) \left( \sum_{i=1}^p \alpha_i \delta a_i, \lambda_i \right) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i, \]
where \( C_0 \) is a constant depending on \( \gamma \). From (61), Hölder inequality and Lemma A.3, we get
\[
\left| \int_{\mathbb{H}^1} \tilde{K}(\xi) \left( \sum_{i=1}^p \alpha_i \delta a_i, \lambda_i \right) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i - \sum_{i=1}^p \alpha_i \delta a_i, \lambda_i \right| \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \\
\leq M_3 \sum_{i \neq j} \int_{\mathbb{H}^1} (\alpha_i \delta a_i, \lambda_i) \frac{2+\gamma}{2+\gamma} \delta a_i, \lambda_i \wedge d\theta_i \\
\leq M_3 \sum_{i \neq j} \int_{\mathbb{H}^1} \left[ (\alpha_i \delta a_i, \lambda_i) \frac{2+\gamma}{2+\gamma} \delta a_i, \lambda_i \right] \frac{2+\gamma}{2+\gamma} \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \\
= M_3 \sum_{i \neq j} \left( \int_{\mathbb{H}^1} (\alpha_i \delta a_i, \lambda_i) \frac{4(2\gamma-2)}{2+\gamma} (\alpha_i \delta a_i, \lambda_i, \alpha_j \delta a_j, \lambda_j) \frac{2+\gamma}{2+\gamma} \frac{4}{2+\gamma} \delta a_i \wedge d\theta_i \right) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \\
\leq M_3 \sum_{i \neq j} \left( \int_{\mathbb{H}^1} (\alpha_i \delta a_i, \lambda_i) \frac{4}{2+\gamma} \left( \int_{\mathbb{H}^1} (\alpha_i \delta a_i, \lambda_i, \alpha_j \delta a_j, \lambda_j) \frac{4}{2+\gamma} \right) \delta a_i \wedge d\theta_i \right) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \\
= O(\varepsilon_{ij} \ln \varepsilon_{ij}^{-1} \frac{2+\gamma}{2+\gamma} \left( \int_{\mathbb{H}^1} (\alpha_i \delta a_i, \lambda_i, \alpha_j \delta a_j, \lambda_j) \frac{4}{2+\gamma} \right) \delta a_i \wedge d\theta_i \right) \frac{2+\gamma}{2+\gamma} \delta a_i \wedge d\theta_i \right).
Since \( v \) satisfies (14), we have
\[
\begin{align*}
\int_{\mathbb{R}^3} \tilde{K}(\xi)(\delta_{a_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v &= \int_{\mathbb{R}^3} \tilde{K}(\xi) - \tilde{K}(a_i)) \cdot (\delta_{a_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v \\
&= \int_{B(a_i, \varepsilon)} \left( \nabla_{\theta_0} \tilde{K}(a_i)(\xi - a_i) + O(|\xi - a_i|^2) \right) (\delta_{a_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v \\
&\quad + \int_{B^c(a_i, \varepsilon)} (\tilde{K}(\xi) - \tilde{K}(a_i)) (\delta_{a_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v \\
&\leq C \left( \frac{\nabla_{\theta_0} \tilde{K}(a_i)}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{2+\gamma}} \right) \|v\|.
\end{align*}
\]

\[ \square \]

**Lemma A.4.3.** We have
\[
\int_{\mathbb{R}^3} \tilde{K}(\xi) \left( \sum_{i=1}^P \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2+\gamma}{\gamma}} v^2 = \sum_{i=1}^P \alpha_i^{\frac{2+\gamma}{\gamma}} K(g_i) \int_{\mathbb{R}^3} w_{g_i, \lambda_i}^{\frac{2+\gamma}{\gamma}} v^2 \\
+ O(||v||^2) \left( \sum_{i=1}^P \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} + \sum_{i=1}^P \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{2+\gamma}{\gamma}} (\ln \varepsilon_{ij}^{-1})^2 \right).
\]

**Proof.** Using (61), we get
\[
\begin{align*}
\int_{\mathbb{R}^3} \tilde{K}(\xi) \left( \sum_{i=1}^P \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2+\gamma}{\gamma}} v^2 \theta_1 \wedge d\theta_1 - \int_{\mathbb{R}^3} \tilde{K}(\xi) \sum_{i=1}^P (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \\
\leq M_3 \sum_{i \neq j} \int_{\mathbb{R}^3} (\alpha_i \delta_{a_i, \lambda_i})^{\frac{3-\gamma}{\gamma}} \inf(\alpha_i \delta_{a_i, \lambda_i}, \alpha_j \delta_{a_j, \lambda_j}) v^2.
\end{align*}
\]

By Hölder inequality and Lemma A.3, we can easily get,
\[
\sum_{i \neq j} \int_{\mathbb{R}^3} (\alpha_i \delta_{a_i, \lambda_i})^{\frac{3-\gamma}{\gamma}} \inf(\alpha_i \delta_{a_i, \lambda_i}, \alpha_j \delta_{a_j, \lambda_j}) v^2 = O \left( \sum_{i \neq j} \varepsilon_{ij}^{\frac{2+\gamma}{\gamma}} (\ln \varepsilon_{ij}^{-1})^2 \|v\|^2 \right).
\]

Now, we compute
\[
\begin{align*}
\int_{\mathbb{R}^3} \tilde{K}(\xi) (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \theta_1 \wedge d\theta_1 \\
= \int_{\mathbb{R}^3} \tilde{K}(g_i) (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \theta_1 \wedge d\theta_1 + \int_{\mathbb{R}^3} (\tilde{K}(\xi) - \tilde{K}(a_i)) (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \theta_1 \wedge d\theta_1 \\
\leq \int_{\mathbb{R}^3} \tilde{K}(g_i) (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \theta_1 \wedge d\theta_1 + \int_{B(a_i, \varepsilon)} (\tilde{K}(\xi) - \tilde{K}(a_i)) (\delta_{a_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \\
= \int_{\mathbb{R}^3} \tilde{K}(g_i) (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \theta_1 \wedge d\theta_1 + \int_{B(a_i, \varepsilon)} (\tilde{K}(\xi) - \tilde{K}(a_i)) (\delta_{a_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \\
\quad + \int_{B^c(a_i, \varepsilon)} (\tilde{K}(\xi) - \tilde{K}(a_i)) (\delta_{a_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \\
= \int_{\mathbb{R}^3} K(g_i) (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\gamma}} v^2 \theta_1 \wedge d\theta_1 + O \left( \frac{1}{\lambda_i^{2+\gamma}} \right) \|v\|^2
\end{align*}
\]
Therefore combining Lemma A.1, Lemma A.2 and Lemma A.3, we have
\[ J = \left( \nabla_{\theta_i} \tilde{K}(a_i)(\xi - a_i) + O(\|\xi - a_i\|^2) \right) \left( \delta_{a_i, \lambda_i} \right)^{2\gamma / \gamma - 2} v^2 \]
\[ = \int_{\mathbb{S}^3} K(g_i)(\alpha_i w_{g_i, \lambda_i})^{2\gamma / \gamma - 2} v^2 \theta_1 \wedge d\theta_1 + O\left( \left( \frac{1}{\lambda_i^{2\gamma / \gamma - 2}} \right) \|v\|^2 + O\left( \left( \frac{|\nabla_{\theta_i} K(g_i)|}{\lambda_i} \right) \|v\|^2 \right) \right). \]

Now we can complete the Proof of Lemma 3.1.

\[ D \frac{\partial}{\partial r} \]
\[ = \sum_{i=1}^{p} \alpha_i^{4\gamma} \left( K(g_i) S^{\frac{\gamma}{4}} \right) + c_2 \Delta_{\theta_i} K(g_i) + \frac{4}{2 - \gamma} \int_{\mathbb{S}^3} K(\zeta) \left[ \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right]^{2\gamma / \gamma - 2} v \]
\[ + \frac{2(2 + \gamma)}{(2 - \gamma)^2} \int_{\mathbb{S}^3} K(\zeta) \left[ \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right]^{2\gamma / \gamma - 2} v^2 + O'\left( \sum_{i \neq j} \varepsilon_{ij} \right) + o\left( \frac{1}{\lambda_i^2} \right) + O(\|v\|^3) \]
\[ = \left( \sum_{i=1}^{p} \alpha_i^{4\gamma} K(g_i) S^{\frac{\gamma}{4}} \right) \left[ 1 + \left( \sum_{i=1}^{p} \alpha_i^{4\gamma} K(g_i) S^{\frac{\gamma}{4}} \right)^{-1} \frac{4}{2 - \gamma} \int_{\mathbb{S}^3} K(\zeta) \left[ \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right]^{2\gamma / \gamma - 2} v \right. \]
\[ + \left( \sum_{i=1}^{p} \alpha_i^{4\gamma} K(g_i) S^{\frac{\gamma}{4}} \right)^{-1} \int_{\mathbb{S}^3} K(\zeta) \left[ \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right]^{2\gamma / \gamma - 2} v^2 \]
\[ + O'\left( \sum_{i \neq j} \varepsilon_{ij} \right) + o\left( \frac{1}{\lambda_i^2} \right) + O(\|v\|^3) \right]. \]

Therefore combining Lemma A.1, Lemma A.2 and Lemma A.3, we have
\[ J(u) = \frac{\sum_{i=1}^{p} \alpha_i^2 S^{\frac{\gamma}{4}}}{\left( \sum_{i=1}^{p} \alpha_i^{4\gamma} K(g_i) \right) \left( \sum_{i=1}^{p} \alpha_i^{4\gamma} \Delta_{\theta_i} K(g_i) \right)} \left[ 1 - \frac{2 - \gamma}{2} \sum_{i=1}^{p} \frac{\alpha_i^{4\gamma}}{S^{\frac{\gamma}{4}}} K(g_i) \lambda_i^{2\gamma / \gamma - 2} \right. \]
\[ - \frac{2}{\sum_{k=1}^{p} \alpha_k^{4\gamma} K(g_k) S^{\frac{\gamma}{4}}} \int_{\mathbb{S}^3} K(\zeta) \left[ \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right]^{2\gamma / \gamma - 2} v + \frac{1}{\sum_{k=1}^{p} \alpha_k^2 S^{\frac{\gamma}{4}}} \|v\|^2 \]
\[ - \frac{2 + \gamma}{(2 - \gamma) S^{\frac{\gamma}{4}}} \sum_{k=1}^{p} \alpha_k^{4\gamma} K(g_k) \int_{\mathbb{S}^3} K(\zeta) \sum_{i=1}^{p} \left( \alpha_i w_{g_i, \lambda_i} \right)^{2\gamma / \gamma - 2} v^2 \]
\[ + O'\left( \sum_{i \neq j} \varepsilon_{ij} \right) + o\left( \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \right) + o(\|v\|^2) \right]. \]

This is the estimate of \( J(u) \) in Lemma 3.1.

6.2. Appendix B. We define
\[ \lambda(u) = \left( \int_{\mathbb{S}^3} K(\zeta) u^{2\gamma / \gamma - 2} \theta_1 \wedge d\theta_1 \right)^{-\frac{(2 - \gamma)}{\gamma}}, \]
Thus we have

**Lemma B.1.**

Proof of (15).

Let us consider the proof.

By direct computation,

\[ \lambda'(u)W = -2\lambda(u)^{\frac{2+\gamma}{2+2\gamma}} \int_{\mathbb{R}^3} K(\zeta) u^{\frac{2+\gamma}{2+2\gamma}} W \, d\theta_1 \, d\theta_1. \]

Thus we have

\[ J'(u)W = \lambda'(u)W \int_{\mathbb{R}^3} P_\gamma uu \theta_1 \, d\theta_1 + 2\lambda(u) \int_{\mathbb{R}^3} P_\gamma uu \theta_1 \, d\theta_1 \]

\[ = 2\lambda(u) \left( \int_{\mathbb{R}^3} P_\gamma uu W - \lambda(u)^{\frac{2+\gamma}{2+2\gamma}} \int_{\mathbb{R}^3} K(\zeta) u^{\frac{2+\gamma}{2+2\gamma}} W \int_{\mathbb{R}^3} P_\gamma uu \right). \]

Since for \( u_0 = \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} \in V(p, \varepsilon) \subset \Sigma^+ \), we have \( \int_{\mathbb{R}^3} P_\gamma u_0 u_0 \theta_1 \, d\theta_1 = 1 \). Thus

\[ J'(u_0)W = 2\lambda(u_0) \left( \int_{\mathbb{R}^3} P_\gamma u_0 W - \lambda(u_0)^{\frac{2+\gamma}{2+2\gamma}} \int_{\mathbb{R}^3} K(\zeta) u_0^{\frac{2+\gamma}{2+2\gamma}} W \right). \quad (65) \]

We first take \( W = \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \) in (65), we obtain

\[ J'(u) \left( \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) = 2\lambda(u) \left( \int_{\mathbb{R}^3} P_\gamma \left( \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} \right) \left( \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) \right. \]

\[ - \lambda(u)^{\frac{2+\gamma}{2+2\gamma}} \int_{\mathbb{R}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2+\gamma}{2+2\gamma}} \left( \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) \right). \]

In the remainder of the Appendix B, we will give some lemmas to complete the proof of (15).

**Lemma B.1.** We have

\[ \int_{\mathbb{R}^3} P_\gamma (w_{g_i, \lambda_i}) \lambda_i \frac{\partial w_{g_i, \lambda_i}}{\partial \lambda_i} \theta_1 \, d\theta_1 = 0. \]

**Proof.** Since \( \int_{\mathbb{R}^3} P_\gamma (w_{g_i, \lambda_i}) w_{g_i, \lambda_i} \theta_1 \, d\theta_1 = S^2 \) is independent of \( \lambda_i \), we get the result.

**Lemma B.2.** For \( i \neq j \), we have

\[ \int_{\mathbb{R}^3} P_\gamma (w_{g_i, \lambda_i}) \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \theta_1 \, d\theta_1 = O(\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j}). \]

In the proof of this result, the idea is same as that in the proof of Lemma A.2. The details are omitted.

**Lemma B.3.** There holds

\[ \int_{\mathbb{R}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2+\gamma}{2+2\gamma}} \left( \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) \]

\[ = -\alpha_j^{\frac{2+\gamma}{2+2\gamma}} \frac{2 - \gamma}{4} c_2 \frac{\Delta_{g_j} K(g_j)}{\lambda_j^2} + \sum_{i \neq j} O(\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j}) + o(\sum_{i \neq j} \varepsilon_{ij}) + o \left( \frac{1}{\lambda_j^2} \right). \]
Proof. Using (63), we have

\[
\left| \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2+\gamma}{\lambda_i}} \left( \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) \right| \\
- \int_{\mathbb{S}^3} K(\zeta) \left[ \sum_{i=1}^{p} (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\lambda_i}} + \frac{2 + \gamma}{2} (\alpha_j w_{g_j, \lambda_j})^{\frac{2+\gamma}{\lambda_i}} \left( \sum_{i \neq j} \alpha_i w_{g_i, \lambda_i} \right) \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right] \leq M_5 \left( \int_{\mathbb{S}^3} K(\zeta) \sum_{i \neq j, i \neq k} (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{\lambda_i}} \inf((\alpha_i w_{g_i, \lambda_i}, \alpha_k w_{g_k, \lambda_k}) \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) \\
+ M_5 \left( \int_{\mathbb{S}^3} K(\zeta) \sum_{i \neq j} (\alpha_j w_{g_j, \lambda_j})^{\frac{2+\gamma}{\lambda_j}} \inf((\alpha_j w_{g_j, \lambda_j})^2, (\alpha_j w_{g_j, \lambda_j})^2 \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) \right). 
\]

Together with Lemma B.3.1, Lemma B.3.2, Lemma B.3.3, Lemma B.3.4 and Lemma B.3.5 below, we get a complete proof. \( \square \)

**Lemma B.3.1.** We have

\[
\int_{\mathbb{S}^3} K(\zeta) w_{g_j, \lambda_j}^{\frac{2+\gamma}{\lambda_j}} \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \theta_1 \wedge d \theta_1 = - \frac{2 - \gamma}{2} \Delta_{\partial_1} K(g_1) \left( 1 - \frac{1}{\lambda_i^2} \right) + o \left( \frac{1}{\lambda_i^2} \right). 
\]

Proof. There hold

\[
\int_{\mathbb{S}^3} K(\zeta) w_{g_i, \lambda_i}^{\frac{2+\gamma}{\lambda_i}} \lambda_i \frac{\partial w_{g_i, \lambda_i}}{\partial \lambda_i} \theta_1 \wedge d \theta_1 = \int_{\mathbb{S}^3} \hat{K}(\xi) (\delta_{a_1, \lambda_1})^{\frac{2+\gamma}{\lambda_1}} \lambda_i \frac{\partial \delta_{a_1, \lambda_1}}{\partial \lambda_i} \theta_0 \wedge d \theta_0 \\
= \frac{2 - \gamma}{4} \lambda_i \frac{\partial}{\partial \lambda_i} \int_{\mathbb{S}^3} \hat{K}(\xi) (\delta_{a_1, \lambda_1})^{\frac{2+\gamma}{\lambda_1}} \theta_0 \wedge d \theta_0 \\
= \frac{2 - \gamma}{4} \lambda_i \frac{\partial}{\partial \lambda_i} \int_{\mathbb{S}^3} \hat{K}(\xi) \left( \delta_{a_1, \lambda_1} \right)^{\frac{2+\gamma}{\lambda_1}} \frac{1}{(1 + |x|^2 + |y|^2)^2 + t^2} \theta_0 \wedge d \theta_0 \\
= - \frac{2 - \gamma}{4} \Delta_{\theta_0} \hat{K}(a_1) \int_{|\xi| < \varepsilon} (x^2 + y^2) \theta_0 \wedge d \theta_0 \\
- \frac{2 - \gamma}{4} \Delta_{\theta_0} \hat{K}(a_1) \int_{|\xi| \geq \varepsilon} (x^2 + y^2) \theta_0 \wedge d \theta_0 \\
+ o \left( \int_{|\xi| < \varepsilon} \frac{1}{\lambda_i^2} \frac{1}{(1 + |x|^2 + |y|^2)^2 + t^2} \right) \\
= - \frac{2 - \gamma}{8} \Delta_{\theta_0} \hat{K}(a_1) \frac{1}{\lambda_i^2} + o \left( \frac{1}{\lambda_i^2} \right). 
\]

\( \square \)

**Lemma B.3.2.** For \( i \neq j \), we have

\[
\int_{\mathbb{S}^3} K(\zeta) w_{g_j, \lambda_j}^{\frac{2+\gamma}{\lambda_j}} \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \theta_1 \wedge d \theta_1 = O'((\lambda_i \frac{\partial e_{ij}}{\partial \lambda_i}) + O \left( \frac{1}{\lambda_i^{2-\gamma} \lambda_j^{2+\gamma}} \right) + o \left( \frac{1}{\lambda_i^2} \right). 
\]
Lemma B.3.3. For $i \neq j$, there holds
\[
\int_{\mathbb{R}^3} K(\zeta) w_{\gamma, \lambda_1} \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
= O(\sum_{i \neq j, i \neq k} \left( \frac{1}{\lambda_i^{2-\gamma} \lambda_j^{2+\gamma}} \right) + O(\varepsilon_{ij}^{2-\gamma} \ln \varepsilon_{ij}^{-1}) + o \left( \frac{1}{\lambda_i^{2-\gamma}} \right).
\]

Proof. We have the following computations
\[
\int_{\mathbb{R}^3} K(\zeta) w_{\gamma, \lambda_1} \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
= \int_{\mathbb{R}^3} \tilde{K}(\xi)(\delta_{a_j, \lambda_1}) \frac{\partial \delta_{a_i, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
= \int_{\mathbb{R}^3} \tilde{K}(a_j)(\delta_{a_j, \lambda_1}) \frac{\partial \delta_{a_i, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
+ \int_{B(a_j, c)} (\tilde{K}(\xi) - \tilde{K}(a_j))(\delta_{a_j, \lambda_1}) \frac{\partial \delta_{a_i, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
= O(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_j}) + O(\varepsilon_{ij}^{2-\gamma} \ln \varepsilon_{ij}^{-1}) + o \left( \frac{1}{\lambda_i^{2-\gamma}} \right).
\]

By some similar computations, we also get the following result.

Lemma B.3.4.
\[
\int_{\mathbb{R}^3} K(\zeta) \sum_{i \neq j, i \neq k} (\alpha_i w_{g_1, \lambda_1}) \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
= O \left( \sum_{i \neq j} \varepsilon_{ij}^{2-\gamma} 2 \ln \varepsilon_{ij}^{-1} \right) + \sum_{i \neq k} \varepsilon_{ik}^{2-\gamma} 2 \ln \varepsilon_{ik}^{-1} \right).
\]

Proof. From direct computation, we have $|\lambda_j \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_j}| \leq w_{g_1, \lambda_1}$. Then
\[
\int_{\mathbb{R}^3} K(\zeta) \sum_{i \neq j, i \neq k} (\alpha_i w_{g_1, \lambda_1}) \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
\leq \sum_{i \neq j, i \neq k} \int_{\mathbb{R}^3} \sum_{i \neq j, i \neq k} (\alpha_i w_{g_1, \lambda_1}) \frac{\partial w_{g_1, \lambda_1}}{\partial \lambda_1} \partial_1 \wedge d\theta_1
\]
\[
\leq C \left( \sum_{i \neq j, i \neq k} \varepsilon_{ij}^{2-\gamma} + \varepsilon_{ik}^{2-\gamma} \right) \lambda_j \partial_1 \wedge d\theta_1
\]
\[
= O \left( \sum_{i \neq j} \varepsilon_{ij}^{2-\gamma} 2 \ln \varepsilon_{ij}^{-1} \right) + \sum_{i \neq k} \varepsilon_{ik}^{2-\gamma} 2 \ln \varepsilon_{ik}^{-1} \right).
\]
Similarly, we have the following result, the proof is omitted.

**Lemma B.3.5.**

\[
\int_{\mathbb{S}^3} K(\zeta) \sum_{i \neq j} (\alpha_j w_{g_j, \lambda_j})^{\frac{2n-2}{2n-1}} \inf((\alpha_i w_{g_i, \lambda_i})^2, (\alpha_j w_{g_j, \lambda_j})^2) \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \theta_1 \wedge d\theta_1 \\
= O \left( \sum_{i \neq j} \varepsilon_{ij} \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \right).
\]

By using the lemmas above, we have

\[
J'(u) \left( \lambda_j \frac{\partial w_{g_j, \lambda_j}}{\partial \lambda_j} \right) = 2\lambda(u) \left[ -\sum_{i \neq j} O' \left( \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \right) + \frac{2 - \gamma}{4} \alpha_j \frac{\Delta_{g_j} K(g_j)}{K(g_j)} \lambda_j^2 \\
+ o \left( \frac{1}{\lambda_j} \right) + o \left( \sum_{i \neq j} \varepsilon_{ij} \right) \right].
\]

So we obtain the desired estimate of (15).

### 6.3. Appendix C.

In this section, we take \( W = \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \) in (65) and complete the proof of (16).

\[
J'(u) \left( \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \right) = 2\lambda(u) \left( \int_{\mathbb{S}^3} P_{g_j} \left( \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} \right) \left( \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \right) \\
- \lambda(u) \frac{2}{\lambda_j} \int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2n-2}{2n-1}} \left( \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \right) \right).
\]

By the same reason of Lemma B.1, we have the following result.

**Lemma C.1.** We have

\[
\int_{\mathbb{S}^3} P_{g_j} (w_{g_j, \lambda_j}) \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \theta_1 \wedge d\theta_1 = 0.
\]

**Lemma C.2.** We have

\[
\int_{\mathbb{S}^3} P_{g_j} (w_{g_j, \lambda_j}) \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \theta_1 \wedge d\theta_1 = O' \left( \frac{1}{\lambda_j} \frac{\partial \varepsilon_{ij}}{\partial g_j} \right).
\]

**Proof.** By some similar computations as in the proof of Lemma B.2, we can complete the proof of Lemma C.2. The details are omitted. \( \square \)

**Lemma C.3.** We have

\[
\int_{\mathbb{S}^3} K(\zeta) \left( \sum_{i=1}^p \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2n-2}{2n-1}} \left( \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \right) \\
= \frac{2 - \gamma}{2} \nabla_{\theta_1} K(g_j) + \sum_{i \neq j} O' \left( \frac{1}{\lambda_j} \frac{\partial \varepsilon_{ij}}{\partial g_j} \right) + O(\sum_{i \neq j} \varepsilon_{ij}) + O \left( \frac{1}{\lambda_j^2} \right).
\]
Proof. Using (63), we have
\[
\left| \int_{\mathbb{R}^3} K(\zeta) \left( \sum_{i=1}^{p} \alpha_i w_{g_i, \lambda_i} \right)^{\frac{2+\gamma}{2}} \left( \frac{1}{\lambda_j} \frac{\partial w_{g_i, \lambda_i}}{\partial g_j} \right) \right|
\]
\[
- \int_{\mathbb{R}^3} K(\zeta) \left[ \sum_{i=1}^{p} (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{2}} + \frac{2 + \gamma}{2} \sum_{i \neq j} (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{2}} \left( \frac{1}{\lambda_j} \frac{\partial w_{g_i, \lambda_i}}{\partial g_j} \right) \right] \frac{1}{\lambda_j} \frac{\partial w_{g_i, \lambda_i}}{\partial g_j}
\]
\[
\leq M_5 \left( \int_{\mathbb{R}^3} K(\zeta) \sum_{i \neq j, i \neq k} (\alpha_i w_{g_i, \lambda_i})^{\frac{2+\gamma}{2}} \inf((\alpha_i w_{g_i, \lambda_i}, \alpha_k w_{g_k, \lambda_k}) \frac{1}{\lambda_j} \frac{\partial w_{g_i, \lambda_i}}{\partial g_j}) \right)
\]
\[
+ M_5 \left( \int_{\mathbb{R}^3} K(\zeta) \sum_{i \neq j} (\alpha_j w_{g_j, \lambda_j})^{\frac{2+\gamma}{2}} \inf((\alpha_i w_{g_i, \lambda_i})^2, (\alpha_j w_{g_j, \lambda_j})^2) \frac{1}{\lambda_j} \frac{\partial w_{g_i, \lambda_i}}{\partial g_j} \right).
\]
Together with Lemma C.3.1, Lemma C.3.2, Lemma C.3.3, Lemma C.3.4, and Lemma C.3.5, we get the desired results. \qed

Lemma C.3.1. We have
\[
\int_{\mathbb{R}^3} K(\zeta) (w_{g_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \theta_1 \wedge d\theta_1 = \frac{2 - \gamma}{4} c_2 \frac{\nabla_{\theta_1} K(g_j)}{\lambda_j} + O(\frac{1}{\lambda_j}).
\]

Proof. We understand the vectors in the following formula as row vectors, then there holds
\[
\int_{\mathbb{R}^3} K(\zeta) (w_{g_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \theta_1 \wedge d\theta_1
\]
\[
= \left[ \int_{\mathbb{R}^3} \tilde{K}(\xi) (\delta_{a_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial \delta_{a_j, \lambda_j}}{\partial a_j} \theta_0 \wedge d\theta_0 \right] d\mathcal{C}^{-1}_{\theta_1}. \]
On the right hand side, we have
\[
\int_{\mathbb{R}^3} \tilde{K}(\xi) (\delta_{a_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial \delta_{a_j, \lambda_j}}{\partial a_j} \theta_0 \wedge d\theta_0
\]
\[
= \int_{B(a_j, \epsilon)} (\tilde{K}(\xi) - \tilde{K}(a_j)) (\delta_{a_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial \delta_{a_j, \lambda_j}}{\partial a_j},
\]
\[
+ \int_{B^c(a_j, \epsilon)} (\tilde{K}(\xi) - \tilde{K}(a_j)) (\delta_{a_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial \delta_{a_j, \lambda_j}}{\partial a_j}
\]
\[
= \int_{B(a_j, \epsilon)} \nabla_{\theta_0} \tilde{K}(a_j) (\xi - a_j) (\delta_{a_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial \delta_{a_j, \lambda_j}}{\partial a_j} + O(\int_{B(a_j, \epsilon)} (\delta_{a_j, \lambda_j})^{\frac{2+\gamma}{2}} |\xi - a_j|^2)
\]
\[
+ \int_{B^c(a_j, \epsilon)} (\tilde{K}(\xi) - \tilde{K}(a_j)) (\delta_{a_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial \delta_{a_j, \lambda_j}}{\partial a_j} \theta_0 \wedge (d\theta_0)^n
\]
\[
= \frac{2 - \gamma}{4} c_2 \frac{\nabla_{\theta_0} \tilde{K}(a_j)}{\lambda_j} + O(\frac{1}{\lambda_j}). \qed
\]

Lemma C.3.2. For \( i \neq j \), we have
\[
\int_{\mathbb{R}^3} K(\zeta) (w_{g_j, \lambda_j})^{\frac{2+\gamma}{2}} \frac{1}{\lambda_j} \frac{\partial w_{g_j, \lambda_j}}{\partial g_j} \theta_1 \wedge d\theta_1 = O'(\frac{1}{\lambda_j} \frac{\partial \xi_{ij}}{\partial g_j}) + o(\xi_{ij}) + O(\frac{1}{\lambda_j^{2+\gamma} \lambda_j^{-\gamma}}).\]
Proof. Similarly as in the proof of Lemma C.3.1, we have
\[
\int_{\mathbb{H}^3} K(\zeta)(w_{g_i, \lambda_i})^{\frac{2\gamma}{2+\gamma}} \frac{1}{\lambda_j} \partial w_{g_j, \lambda_j} \, \theta_0 \wedge \theta_1 \quad \text{and}
\int_{\mathbb{H}^3} \tilde{K}(\xi)(\delta_{a_i, \lambda_i})^{\frac{2\gamma}{2+\gamma}} \frac{1}{\lambda_j} \partial \delta_{a_j, \lambda_j} \, \theta_0 \wedge \theta_0
\]
\[
= \tilde{K}(a_i) \frac{1}{\lambda_j} \partial \theta_0 \int_{\mathbb{H}^3} (\delta_{a_i, \lambda_i})^{\frac{2\gamma}{2+\gamma}} \frac{1}{\lambda_j} \partial \delta_{a_j, \lambda_j} \, \theta_0 \wedge \theta_0
\]
\[
+ \int_{B(a_i, r)} (\tilde{K}(\xi) - \tilde{K}(a_i))(\delta_{a_i, \lambda_i})^{\frac{2\gamma}{2+\gamma}} \frac{1}{\lambda_j} \partial \delta_{a_j, \lambda_j} \, \theta_0 \wedge \theta_0
\]
\[
+ \int_{B'(a_i, r)} (\tilde{K}(\xi) - \tilde{K}(a_i))(\delta_{a_i, \lambda_i})^{\frac{2\gamma}{2+\gamma}} \frac{1}{\lambda_j} \partial \delta_{a_j, \lambda_j} \, \theta_0 \wedge \theta_0
\]
\[
= O\left( \frac{1}{\lambda_j} \sigma_{ij} \right) + o(\varepsilon_{ij}) + O\left( \frac{1}{\lambda_j^{2+\gamma} \lambda_j^{2-\gamma}} \right).
\]

Similarly we have the following result, but its proof is more simple.

Lemma C.3.3. For \( i \neq j \), we have
\[
\int_{\mathbb{H}^3} K(\zeta)(w_{g_i, \lambda_i})^{\frac{2\gamma}{2+\gamma}} w_{g_j, \lambda_j} \frac{1}{\lambda_j} \partial w_{g_j, \lambda_j} \, \theta_0 \wedge \theta_0 = O(\varepsilon_{ij}).
\]

Lemma C.3.4. We have
\[
\int_{\mathbb{H}^3} K(\zeta) \sum_{i \neq j, j \neq k} (\alpha_i w_{g_i, \lambda_i})^{\frac{2\gamma}{2+\gamma}} \inf(\alpha_i w_{g_i, \lambda_i}, \alpha_k w_{g_k, \lambda_k}) \frac{1}{\lambda_j} \partial w_{g_j, \lambda_j} \, \theta_0 \wedge \theta_0
\]
\[
= O\left( \sum_{i \neq j} \varepsilon_{ij}^{\frac{2\gamma}{2+\gamma}} \ln \varepsilon_{ij}^{-1} + \sum_{i \neq k} \varepsilon_{ik}^{\frac{2\gamma}{2+\gamma}} \ln \varepsilon_{ik}^{-1} \right).
\]

Proof. Through direct computation, we have \( \frac{1}{\lambda_j} \sigma_{ij} \leq w_{g_j, \lambda_j} \). Using Lemma A.2, we get the result.

We also have the following result, its proof is omitted.

Lemma C.3.5. We have
\[
\int_{\mathbb{H}^3} K(\zeta) \sum_{i \neq j} (\alpha_j w_{g_j, \lambda_j})^{\frac{2\gamma}{2+\gamma}} \inf((\alpha_i w_{g_i, \lambda_i})^2, (\alpha_j w_{g_j, \lambda_j})^2) \frac{1}{\lambda_j} \partial w_{g_j, \lambda_j} \, \theta_0 \wedge \theta_0
\]
\[
= O\left( \sum_{i \neq j} \varepsilon_{ij}^{\frac{2\gamma}{2+\gamma}} \ln \varepsilon_{ij}^{-1} \right).
\]

By using the lemmas above, we have
\[
J'(u) \left( \frac{1}{\lambda_j} \partial w_{g_j, \lambda_j} \right) = -2\lambda(u) \alpha_j c_2 \frac{\nabla g_i K(g_j)}{K(g_j) \lambda_j} + O\left( \sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_j} \right).
\]
This is the desired estimate of (16).

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Received July 2017; revised July 2017.

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