SPHERICAL HALL ALGEBRAS OF WEIGHTED PROJECTIVE CURVES

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Abstract. In this article, we deal with the properties of the spherical Hall algebra \( U_{\text{Coh}^D(X)} \) of coherent sheaves with \( D \)-parabolic structures on a smooth projective curve \( X \) of arbitrary genus \( g \) and a given effective divisor \( D \). We provide a shuffle-like presentation of \( U_{\text{Coh}^D(X)} \) and show the existence of some kind of “universal” spherical Hall algebra of genus \( g \). We also prove the algebra \( U_{\text{Coh}^D(X)} \) contains the characteristic functions on all the Harder-Narasimhan strata.

0. Introduction

0.1. The Hall algebra approach to quantum groups developed since around 1990 is based on a deep relationship between simple or affine Lie algebras and certain finite-dimensional hereditary algebras. More precisely, let \( g \) be a Kac-Moody algebra, and let \( \Gamma \) be its Dynkin diagram. Ringel proved in [R] that the Hall algebra \( H_{\text{Rep}^{\overrightarrow{Q}}} \) of the category of representations, over a finite field \( \mathbb{F}_l \), of a quiver \( \overrightarrow{Q} \) whose underlying graph is \( \Gamma \) provides a realization of the “positive part” \( U^+_v(g) \) of the Drinfeld-Jimbo quantum group associated to \( g \), where \( v = l^{-1/2} \). This result was the starting point of Lusztig’s geometric construction of the canonical basis of \( U^+_v(g) \), see [Lu].

0.2. Another natural example of a hereditary category is provided by the category \( \text{Coh}(X) \) of coherent sheaves on a smooth projective curve \( X \). Let \( \text{Bun}_r(X) \) be the set of isomorphism classes of rank \( r \) vector bundles on \( X \) and \( \text{AF}_r \) be the space of all complex valued functions on \( \text{Bun}_r(X) \). A classical observation of A. Weil provides us an expression of \( \text{Bun}_r(X) \) as the double coset space

\[
\text{Bun}_r(X) \cong \text{GL}_r(\mathbb{F}_l(X)) \setminus \text{GL}_r(\mathbb{A}_X)/\text{GL}_r(\mathbb{O}_X)
\]

where \( \mathbb{F}_l(X) \) is the function field of \( X \), \( \mathbb{A}_X \) its ring of adèles and \( \mathbb{O}_X \subset \mathbb{A}_X \) the ring of integer adèles and functions in \( \text{AF}_r \) can be regarded as unramified automorphic forms on \( \text{GL}_r(\mathbb{A}_X) \). In the remarkable paper [Kap], Kapranov initiated the systematic study of the spaces \( \text{AF}_r \) using the language of Hall algebras: \( H_{\text{Bun}(X)} \) is the direct sum over \( r \) of \( \text{AF}_r \) equipped with an associative product and coassociative coproduct via the convolution diagram

\[ \xymatrix{ & & \tilde{\text{Bun}}_r(X) \ar[dl]_q \ar[dr]^p \ar[dr] & \text{Bun}_s(X) \times \text{Bun}_{r-s}(X) \ar[dl]_q \ar[dr]^p & \text{Bun}_r(X) \ar[dl]_q \ar[dr]^p \ar[dl]_q \ar[dr]^p \ar[dl]_q \ar[dr]^p \ar[dl]_q \ar[dr]^p \ar[dl]_q \ar[dr]^p \ar[dl]_q \ar[dr]^p & } \]

where \( \tilde{\text{Bun}}_r(X) \) stands for stack classifying the inclusions \( F \subset G \) of a vector bundle \( F \) of rank \( s \) into a vector bundle \( G \) of rank \( r \) over \( X \). The algebra \( H_{\text{Bun}(X)} \) is generated by all cuspidal functions; Kapranov translated the functional equations satisfied by Eisenstein series associated to such pairs of cuspidal functions into commutation relations between the corresponding generators. Such commutation relations bear a resemblance with those appearing in Drinfeld’s loop-liked realization of quantum affine algebras [Dr1].
The classical Hecke operator associated to a point \( x \in X \) on automorphic forms is translated by the multiplication with the characteristic function of the corresponding torsion sheaf. These Hecke operators naturally form a commutative subalgebra of the Hall algebra \( H_{\text{Coh}(X)} \) which can be identified with the Hall algebra \( H_{\text{Tor}(X)} \) of the category of torsion sheaves of \( X \), and we might call it the global Hecke algebra.

When \( X = \mathbb{P}^1 \), by a theorem of Grothendieck [Gro], any indecomposable vector bundle is a line bundle. Using this we can easily show that there are no cusp functions of rank \( > 1 \). Kapranov's result (see also [BK] for more details) provides an embedding from \( \mathbb{P}^1 \) to the Hall algebra which can be generalized to coherent sheaves: one only need to replace \( "\) with \( \cdot \) and to add the condition that the induced maps \( E(\eta) \rightarrow H_{\text{Bun}(\mathbb{P}^1)} \). The Hall algebras \( H_{\text{Bun}(X)} \) as well as \( H_{\text{Coh}(X)} \) are also explicitly described when \( X \) is an elliptic curve, see [BS2], [Fra]. In this case they are related to double affine Hecke algebras, see [SV2].

0.3. Rather than consider higher genus smooth projective curves, Schiffmann and Burban also study the (spherical) Hall algebra of the category of coherent sheaves on certain “noncommutative smooth projective curves”, called weighted projective lines, introduced by Geigle and Lenzing [GL]. In [S2], Schiffmann constructs, analogously to the works of Ringel and Kapranov, a natural embedding \( H_{\text{Coh}(X)} \) of certain “positive part” of \( U_\mathfrak{g} \) into the Hall algebra of a suitable weighted projective line \( X_\mathfrak{g} \). In particular, this yields a geometric realization of the quantum toroidal algebras of types \( D^{(1,1)}_4, E^{(1,1)}_6, E^{(7,1)}_6, E^{(1,1)}_8 \).

0.4. Let us come back to the case when the genus \( g > 1 \). Now the functional equations are no longer enough to give a presentation of \( H_{\text{Coh}(X)} \). However, if we restrict our attention to a natural subalgebra \( U_{\mathfrak{g}} \) of \( H_{\text{Bun}(X)} \) generated by cuspidal functions of rank one, the algebra \( U_{\mathfrak{g}} \) can be also described combinatorially in terms of shuffle algebras, see [SV2]. One nice feature of such presentation is that \( U_{\mathfrak{g}} \) possesses an integral (or a generic) form \( U_{\mathfrak{g}}^{\mathbb{Z}} \) over the representation ring \( \mathfrak{g} \) of the torus

\[
T_g = \{(\eta_1, \ldots, \eta_{2g}) \in (\mathbb{C}^*)^{2g} | \eta_{2i-1}\eta_{2i} = \eta_{2j-1}\eta_{2j}, \forall i, j\}.
\]

Schiffmann and Vasserot had also shown that the generic spherical Hall algebra \( U_{\mathfrak{g}}^{\mathbb{Z}} \) of a fixed genus \( g \) is in correspondence with the convolution algebra in the equivariant K-theory of the Hilbert Scheme the stacks \( \mathcal{L}_{\mathfrak{g}} \) parametrizing the local systems on \( X_\mathfrak{g} \) of rank \( r \), supported in the formal neighborhood of the trivial local systems on a complex curve \( X_\mathfrak{g} \) of genus \( g \). Such a correspondence is indeed expected of Beilinson and Drinfeld’s version (c.f. [Dr2]) of the geometric Langlands correspondence.

0.5. In this paper, we study the spherical Hall algebras of coherent sheaves on higher genus curves with (quasi)-parabolic structures hence completing the picture. Let \( X \) as before be a smooth projective curve defined over a finite field \( \mathbb{F}_l \). Fix a finite subset \( S \subset \mathbb{X} \) of \( \mathbb{F}_l \)-rational points and an effective divisor \( D = \sum_{p \in S} w_p p \). By definition a (quasi)-parabolic vector bundle \( (E, V^\bullet) \) on \( (X, D) \) is a vector bundle \( E \) on \( X \) equipped with a collection of complete flag of subspaces of the stalks at all points in \( S \):

\[
0 \subset V_{1,p} \subset \cdots \subset V_{w_p-1,p} = \mathcal{E}_p.
\]

This can be also described as a collection of vector bundles \( \mathcal{E}^\bullet = (\mathcal{E}, \mathcal{E}^{(s,p)})_{p \in S, 1 \leq s \leq w_p - 1} \) such that:

\[
\mathcal{E} \subset \mathcal{E}^{(1,p)} \subset \cdots \subset \mathcal{E}^{(w_p,p)} = \mathcal{E}(p)
\]

which can be generalized to coherent sheaves: one only need to replace “\( \subset \)” by arbitrary maps “\( \to \)” and to add the condition that the induced maps \( \mathcal{E}^{(1,p)} \to \mathcal{E}^{(w_p,p)}(p) \) are the natural ones.
Let \( w = \{ w_p \}_{p \in S} \) and \( \text{Coh}^{w,S}(\mathcal{X}), \text{Bun}^{w,S}(\mathcal{X}), \text{Tor}^{w,S}(\mathcal{X}) \) respectively be the category of coherent sheaves, vector bundles and torsion sheaves respectively with parabolic structures associated to \((S, D)\). The category \( \text{Coh}^{w,S}(\mathcal{X}) \) is known to be hereditary. In the case that \( \mathcal{X} = \mathbb{P}^1, \text{Coh}^{w,S}(\mathbb{P}^1) \) is equivalent to the category \( \text{Coh}(\mathcal{X}_{w,S}) \) of coherent sheaves over the corresponding weighted projective line \( \mathcal{X}_{w,S} \).

For every parabolic coherent sheaf \( F^* \in \text{Coh}^{w,S}(\mathcal{X}) \) we can associate its class in the numerical Grothendieck group \( K_S := K'(\text{Coh}^{w,S}) \) given by \((\text{rank} F^*, \text{deg}(F^*))\), here the degree \( \text{deg}(F^*) \) of a parabolic coherent sheaf is defined as the collection \((\text{deg} F^{(i,p)})_{p \in S, 0 \leq i \leq w_p - 1} \). For a class \( \alpha = (r, d) \in K_S \), let \( \text{Bun}_{w,S}^{r,s}(\mathcal{X}) \) be the set consisting of parabolic vector bundles of class \( \alpha \) and \( 1_{1,d}^{ss} \) be the characteristic function of \( \text{Bun}_{1,d}^{w,S} \). The spherical Hall algebra \( U^> := U_{\text{Bun}_{w,S}^{r,s}(\mathcal{X})} \) is defined to be the subalgebra of \( H := H_{\text{Coh}^{w,S}(\mathcal{X})} \) generated by \( 1_{1,d}^{ss} \) for all \((1, d) \in K_S \) for all \( p \in S \).

**0.6.** It is easy to see that, by definition of a parabolic vector bundle, the set \( \text{Bun}_{1,d}^{w,S}(\mathcal{X}) \) is non-empty if and only if \( d \) satisfies \( d(0,p) \leq d(1,p) \leq \cdots \leq d(w_p-1,p) \leq 1 \). We might rewrite \((1, d)\) as \((1, d, a)\) where \( d = d(0,p) \) and \( a(i,p) = d(i,p) - d(i-1,p) \) for all \( 1 \leq i \leq w_p - 1 \) and all \( p \in S \). We call \( a \) the *dimension type of \((1, d) \in K_S \). We denote by \( D_1 \) the set of dimension types \( a \) such that \( \text{Bun}_{1,d}^{w,S}(\mathcal{X}) \neq \emptyset \) for all \( d \in \mathbb{Z} \). The set \( D_1 \) is in one-to-one correspondence to \( \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z} \) via the assignement \( a \mapsto (\sum_{i=1}^{w_p-1} a(i,p))_{p \in S} =: \bar{x} \) and we denote by \( 1_{1,d}^{ss}(\bar{x}) = 1_{1,d,a}^{ss} \). For any \( \bar{x} \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z} \), we define the subalgebra \( U^> (\bar{x}) \) of \( U^> \) generated by \( 1_{1,d}^{ss} \), \((1, d) \in D_1 \). Then we have an isomorphism between the algebra \( U^> (\bar{x}) \) and the spherical Hall algebra \( U^> \) of coherent sheaves without extra structures on \( \mathcal{X} \) which already has a nice presentation in terms of shuffle algebras mentioned above.

**0.7.** In order to describe the whole spherical Hall algebra \( U^> \), we consider the vector space
\[
V = \mathbb{C} \oplus \bigoplus_{r \geq 1} \mathbb{C}[\text{L}(w)]^S_r
\]
where \( \mathbb{C}[\text{L}(w)] = \mathbb{C}[\{ t_{p}^{\pm 1} \}_{p \in S}]/J \) and \( J \) is generated by \( t_p^{w_p} - t_q^{w_q} \) for all \( p, q \in S \). We define the operators \( \Gamma_{r,S}(t; x) \) associated to a sequence \( x \in (\prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z})^r \) on \( V \) as follows: for all \( p \in S \) and \( \bar{x} \in \prod_p \mathbb{Z}/w_p \mathbb{Z} \), we set
\[
\Gamma_{r}(t_p) = \begin{cases} v + (1 - v^2)t_p^{w_p} & \text{if } x_p = 1, \ldots, w_p - 1 \\ 1 & \text{if } x_p = 0 \end{cases}
\]
and
\[
\Gamma_{\bar{x}}(t) := \prod_{p \in S} \Gamma_{x_p}(t_p) \in \mathbb{C}[\text{L}(w)]
\]
For \( r \geq 1 \) and \( x \in (\prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z})^r \), we put
\[
\Gamma^*_{r,S}(t_1, \ldots, t_r; x) := \prod_{(i,j) \in I_r} \prod_{p \in S} \Gamma_{\pi(x_{i+1} - x_i)}(t_p^{t_{i+1}^{x_i+1}})
\]
where \( \pi(x_i - x_j) \) denotes its canonical form in \( \prod_p \mathbb{Z}/w_p \mathbb{Z} \). We define an algebra structure on \( V \) with the multiplication defined on the homogenous components
\[
\begin{align*}
t_1^{a_1+x_1} \cdots t_r^{a_r+x_r} \ast t_1^{\gamma_1+y_1} \cdots t_s^{\gamma_s+y_s} &= \sum_{\sigma \in S_h_{r,s}} \sigma (t_1^{a_1+x_1} \cdots t_r^{a_r+x_r}; \bigcup \mathcal{Y}) \sigma (t_1^{\gamma_1+y_1} \cdots t_s^{\gamma_s+y_s}).
\end{align*}
\]
where \( h_{\mathcal{X}}(t) = v^{2x}y^{x^2} - \sum_{\mathcal{Y}}(t_{\mathcal{X}}(c)\mathcal{X}) \) and \( S_{r,s} \subset S_{r,s} \) stands for the set of all \((r, s)\)-shuffles and extended linearly to the whole \( V \). Its coalgebra structure is given as \((2.29)\). Let us denote
the subalgebra $S_{h(t)}^{w,S}$ of $F_{h(t)}^{w,S}$ generated by the degree one component $F_{h(t)}^{w,S}[1] = \mathbb{C}[L(w)]$.

Our main result shows that the spherical Hall algebra $U^+$ is isomorphic to the subalgebra $S_{h}^{w,S}$ of $V$ generated by $\mathbb{C}[L(w)] \subset V$.

0.8. Observe from such slurry-like presentation of $U^+$ that the structure of spherical Hall algebra only depends on the genus $g$ of the curve $X$ and the marked points $(w, S)$. We provide the existence of the generic form $U^+_{w,S,R_g}$ over $R_g$ as $[SV2]$ in the sense that there exists a specialization map $U^+_{w,S,R_g} \rightarrow U^+_{w,S,X}$ to the spherical Hall algebra $U^+_{w,S,X}$ of a fixed curve $X$ of genus $g_X = g$. Recall that Drinfeld also proved an analogue geometric Langlands correspondence for local systems of rank 2 with unipotent ramification at a finite set of points $S \subset X(\mathbb{F}_l)$. In this case the corresponding automorphic forms should be defined on the space

$$GL_r(\mathbb{F}_l(X)) \backslash GL_r(A_X)/K_S$$

where $K_S = \prod_{x \in X(S)} GL_r(\mathbb{Q}_x) \times \prod_{x \in S} Iw_x$ and $Iw_x \subset GL_r(\mathbb{Q}_x)$ is the subgroup of matrices which are upper triangular mod $x$. Similar to (0.1), we have

$$Bun^w_{x}(X) \cong GL_r(\mathbb{F}_l(X)) \backslash GL_r(A_X)/K_S.$$

We conjecture that the generic spherical Hall algebra $U^+_{S,R_g}$ is isomorphic to the convolution algebra of equivariant $K$-theory on the trivial local systems with $D$-parabolic structure on a complex curve $X_C$.

0.9. One of the common exposition to study the vector bundles on a smooth projective curve relies on the work of Narasimhan and Seshadri (see [Se]) by the concept of (semi)-stability. Mehta and Seshadri also introduced similar notion for the parabolic vector bundles (see [MS], [Se]) by attaching “weights” to the terms in flags. As a first step towards understanding the higher genus spherical Hall algebras of parabolic coherent sheaves, we prove that, following the strategy in [S4], the characteristic functions of all the Harder-Narasimhan strata belong to $U$. Recently Schiffmann provides an explicit polynomial in the Weil numbers for counting the number of indecomposable vector bundles of a fixed rank $r$ and degree $d$, see [S5]. His approach also suits the case of vector bundles with parabolic structures. The subject will be of a companion paper.

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1. Parabolic coherent sheaves on smooth projective curves

In this paper we fix a smooth projective curve $X$ defined over some finite field $k = \mathbb{F}_l$ except the Section 2.7.

1.1. Some notations and generalities on coherent sheaves. We fix a smooth projective curve $X$ defined over some finite field $k = \mathbb{F}_l$ and its category of coherent sheaves $\text{Coh}(X)$. It is of global dimension one, and by a classical theorem of Serre we have $\dim \text{Hom}(F, G) < \infty$ and $\dim \text{Ext}^1(F, G) < \infty$ for any two sheaves $F$ and $G$. Thus $\text{Coh}(X)$ is finitary. Note that sometimes we omit the $X$ since the curve is always fixed and we write $\text{Ext}(F, G) = \text{Ext}^1(F, G)$. Let $\omega_X = T^*X$ be the canonical bundle of $X$, and let $g_X = \dim H^0(\omega_X)$ be the genus of $X$. The Serre duality is a functorial isomorphism

$$\text{Ext}(F, G)^* \cong \text{Hom}(G \otimes \omega_X^{-1}, F)$$

for any coherent sheaves $F$ and $G$.

We denote by $\overline{X} = \mathbb{X} \times \text{Spec } k \text{Spec } \overline{k}$ the extension of scalars of $X$ to the algebraic closure. The Galois group $\text{Gal}(\overline{k}/k)$ acts on the set of all points $\overline{X}(k)$. By a closed point of $X$ we will mean a $\text{Gal}(\overline{k}/k)$-orbit of points in $\overline{X}(k)$. The degree $\deg(x)$ of a closed point $x \in X$ is
the number of elements in the associated orbit. Equivalently, if $\mathcal{O}_{X,x}$ stands for the
local ring at $x$ and if $k_x = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field, then $\deg(x) = [k_x : k]$.

For any $d \geq 1$, let $X(\mathbb{F}_d)$ be the set of $\mathbb{F}_d$-rational points of $X$. The Galois group
$\text{Gal}(\mathbb{F}_d/\mathbb{F}_l)$ acts on $X(\mathbb{F}_d)$ and orbits correspond to the closed points whose degree is a
divisor of $d$. We have

$$\#X(\mathbb{F}_d) = \sum_{n\mid d} \sum_{p \in X, \deg(p) = n} n$$

where the sum is over the closed points of $X$.

Let us fix a prime number $p \neq l$ and consider the $p$-adic cohomology group $H^i(\overline{X}, \overline{\mathbb{Q}}_p)$.
Then

$$\dim_{\mathbb{Q}_p}(H^0(\overline{X}, \overline{\mathbb{Q}}_p)) = 1,$$
$$\dim_{\mathbb{Q}_p}(H^1(\overline{X}, \overline{\mathbb{Q}}_p)) = 2g_X,$$
$$\dim_{\mathbb{Q}_p}(H^2(\overline{X}, \overline{\mathbb{Q}}_p)) = 1.$$

Let $F_r$ denote the geometric Frobenius. It acts on $H^i(\overline{X}, \overline{\mathbb{Q}}_p)$, and we denote by
$\alpha_1, \ldots, \alpha_{2g_X}$ its eigenvalues in $H^1(\overline{X}, \overline{\mathbb{Q}}_p)$. We will fix once and for all an embedding
$\mathbb{Q}_p \subset \mathbb{C}$ and consider $\alpha_1, \ldots, \alpha_{2g_X}$ as complex numbers. It is known that all the $\alpha_i$
are algebraic numbers satisfying $|\alpha_i| = l^{1/2}$ and these may be reordered in such a way that
$\alpha_i \alpha_{2g_X-i} = l$ for all $i$. The number of points in $X(\mathbb{F}_l)$ may be expressed in terms of these
Frobenius eigenvalues as

$$\#X(\mathbb{F}_l) = \sum_{i=0}^{2g_X} (-1)^i \text{Tr}(F_r, H^i(\overline{X}, \overline{\mathbb{Q}}_p)) = 1 - \sum_{i} \alpha_i^k + l^k.$$

A convenient way to formulate the above is to introduce the zeta function

$$\zeta_X(t) = \exp(\sum_{k \geq 1} \#X(\mathbb{F}_l) l^k t^k).$$

Then

$$\zeta_X(t) = \prod_{i} \frac{1 - \alpha_i t}{(1-t)(1-lt)}.$$

Any coherent sheaf $\mathcal{F}$ has a canonical torsion subsheaf $\mathcal{T} \subset \mathcal{F}$ and canonical quotient
vector bundle $\mathcal{E} = \mathcal{F}/\mathcal{T}$. Moreover the exact sequence

$$0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{E} \to 0$$

splits, i.e., any coherent sheaf can be decomposed as a direct sum $\mathcal{F} = \mathcal{E} \oplus \mathcal{T}$.

Let $\text{Tor}(X)$ stand for the full subcategory of $\text{Coh}(X)$ consisting of torsion sheaves. It
decomposes into a direct product of blocks

$$\text{Tor}(X) = \prod_{q \in X} \text{Tor}_q$$

where $q$ ranges over the set of closed points of $X$ and $\text{Tor}_q$ is the category of torsion sheaves
supported at $q$. This category is Morita equivalent to the category of finite-dimensional
modules over the local ring $\mathcal{O}_{X,q}$ at $q$ which by smoothness of $X$ is a discrete valuation
ring. In particular, the skyscraper sheaf at $q$, denoted also by $\mathcal{O}_{X,q}$, is the unique simple
sheaf supported at $q$ and more generally for each positive integer $n$, there is a unique
indecomposable torsion sheaf $O_{X,q}^{(n)}$. By Krull-Schmidt theorem, any isomorphism class of torsion sheaf supported at $q$ is in bijection with the set of all partitions via

$$(\lambda_1, \ldots, \lambda_i) = \lambda \mapsto O_{X,q}^{(\lambda)} = O_{X,q}^{(\lambda_1)} \oplus \cdots \oplus O_{X,q}^{(\lambda_i)}.$$

The rank of a coherent sheaf $F$ is the rank of its canonical quotient vector bundle $E$. The degree of a sheaf is the only invariant satisfying $\deg(O_X) = 0$, $\deg(O_{X,q}) = \deg(q) = [k_q; k]$ and which is additive on short exact sequences. Let $\text{Pic}(X)$ be the Picard group of line bundles on $X$. The kernel of the degree map $\text{Pic}(X) \to \mathbb{Z}$ is denoted by $\text{Pic}^0(X)$ which is a finite abelian group.

One consequence of the Serre duality is that for any $T \in \text{Tor}(X)$ and $E \in \text{Bun}(X)$, we have

$$\text{Hom}(T, E) = \text{Ext}(E, T) = \{0\}.$$  
Moreover, $\text{Ext}(O_{X,q}, O_X) \simeq k^{\deg(q)}$ for any closed point $q \in X$ and there is a unique line bundle extension of $O_{X,q}$ by $O_X$ which we denote by $O_{X,q}$.

Let $K(X) = K(\text{Coh}(X))$ be the Grothendieck group of $\text{Coh}(X)$. The degree map descends to a group homomorphism $\deg : K(X) \to \mathbb{Z}$ which satisfies $\deg(O_X(q)) = \deg(q)$ for all $q \in X$. Together with the rank, these define a natural morphism

$$\phi : K(X) \to \mathbb{Z}^2$$

$$F \mapsto \overline{F} = (\text{rank}(F), \deg(F)).$$

The Euler form factors through the map $\phi$ and Riemann-Roch theorem tells us

$$(F, G) = (1 - g_X) \text{rank}(F) \text{rank}(G) + \text{rank}(F) \deg(G) - \deg(F) \text{rank}(G).$$

In fact the kernel of $\phi$ is precisely equal to the kernel of the Euler form. Thus $\mathbb{Z}^2$ can be considered as the numerical Grothendieck group of $\text{Coh}(X)$.

### 1.2. Parabolic Coherent sheaves.

#### 1.2.1. The category of $w$-cycles.

Fix a smooth projective curve $X$ defined over some finite field $k = \mathbb{F}_q$ and its category of coherent sheaves $\text{Coh}(X)$. Following [Len], given a closed point $p \in X$ and a positive integer $w_p$, a $w_p$-cycle $F^\bullet$ concentrated in $p$ is a diagram of objects $F^{(i,p)} \in \text{Coh}(X)$

$$(1.2) \quad \cdots \xrightarrow{\phi^{(i,p)}} F^{(0,p)} \xrightarrow{\phi^{(1,p)}} F^{(1,p)} \xrightarrow{\phi^{(2,p)}} \cdots \xrightarrow{\phi^{(w_p,p)}} F^{(w_p-1,p)} \xrightarrow{\phi^{(w_p,p)}} F^{(w_p,w_p+1,p)} \cdots$$

which is $w_p$-periodic in the sense that $F^{(w_p+i,p)} = F^{(i,p)}(p)$, $\phi^{(w_p+i,p)} = \phi^{(i,p)}(p)$ and moreover the composition of $w_p$ maps $\phi^{(i,p)}(p) \circ \cdots \circ \phi^{(i+1,p)} : F^{(i,p)} \to F^{(i,p)}(p)$ is the natural morphism for all $i \in \mathbb{Z}$. A morphism $u : E^\bullet \to F^\bullet$ between two $w_p$-cycles concentrated at the same point $p$ is a sequence of morphisms $u_i : E^{(i,p)} \to F^{(i,p)}(p)$ which is $w_p$-periodic, i.e., $u_{i+w_p} = u_i$ for each $i$ and such that each diagram

$$\begin{align*}
\xymatrix{ E^{(i,p)} \ar[r]^-{\phi^{(i+1,p)}} & E^{(i+1,p)} \\
F^{(i,p)} \ar[u]^{u_i} & F^{(i+1,p)} \ar[u]_{u_{i+1}} 
}
\end{align*}$$

commutes. We denote by $\text{Coh}^{w_p,(p)}(X)$ the category of all $w_p$-cycles concentrated at $p$. Note that if the sheaf $F^{(i,p)}$ is not torsion free at $p$ for some $i$, then the natural map $F^{(i,p)} \to F^{(i,p)}(p)$ is not injective, hence at least one of the $\phi^\bullet$ is not injective. By the same argument we can see that all the $\phi^{(i,p)}$ are isomorphisms on $X \setminus \{p\}$ and all the $F^{(i,p)}$ have the same generic rank. Thus we define the rank of $F^\bullet$ as the rank of $F^{(0,p)}$. 

We call a $w_p$-cycle $E^\bullet$ concentrated at $p$ locally free if $E^{(i,p)}$ is locally free for all $i \in \mathbb{Z}$. In this case we may hence interpret the $w_p$-cycle $E^\bullet$ as a filtration
\begin{equation}
E^{(0,p)} \subseteq E^{(1,p)} \subseteq \cdots \subseteq E^{(w_p-1,p)} \subseteq E^{(w_p,p)} = E^{(0,p)}(p)
\end{equation}
equivalently as a filtration
\begin{equation}
0 = E^{(0,p)}/E^{(0,p)} \subseteq E^{(1,p)}/E^{(0,p)} \subseteq \cdots \subseteq E^{(w_p-1,p)}/E^{(0,p)} \subseteq E^{(0,p)}/E^{(0,p)}(p)
\end{equation}
of the fibre $E^i_p = E^{(0,p)}(p)/E^{(0,p)}$ of $E^{(0,p)}$ at $p$. In other word, a locally free $w_p$-cycle $E^\bullet$ concentrated at $p$ is nothing but a vector bundle $E = E^{(0,p)}$ with a (quasi-)parabolic structure of length $w_p$ at $p$ in the sense of Mehta and Seshadri [see MS, Se]. Hence the elements in the category $\text{Coh}_{w_p^{-1}}(\mathcal{X})$ can be thought as a notion of coherent sheaves with parabolic structure of length $w_p$ at $p$.

It is obvious that $\text{Coh}_{w_p^{-1}}(\mathcal{X})$ is an abelian category, where the kernel and cokernel of a morphism can be defined componentwise. The compatibilities thus follows from the corresponding ones of $\text{Coh}(\mathcal{X})$. Moreover we have a full exact embedding
\begin{equation}
(-)^\bullet : \text{Coh}(\mathcal{X}) \hookrightarrow \text{Coh}_{w_p^{-1}}(\mathcal{X}), \quad F \mapsto (F)^\bullet = (F = F = \cdots = F \rightarrow F(p)).
\end{equation}
We can therefore identify $\text{Coh}(\mathcal{X})$ with the resulting exact subcategory of $\text{Coh}_{w_p^{-1}}(\mathcal{X})$. Conversely, there are natural exact functors $(-)^{(j,p)} : \text{Coh}_{w_p^{-1}}(\mathcal{X}) \rightarrow \text{Coh}(\mathcal{X})$ defined by $F^j_p = (F^{(j,p)})_{i \in \mathbb{Z}} \mapsto F^{(j,p)}$ for any $j \in \mathbb{Z}$. These two functors $(-)^\bullet$ and $(-)^{(j,p)}$ satisfy the following adjunction properties:
\begin{equation}
\begin{align*}
\text{Hom}_{\text{Coh}_{w_p^{-1}}(\mathcal{X})}((F)^\bullet, G^\bullet) &= \text{Hom}_{\text{Coh}(\mathcal{X})}(F, G^{(0,p)}) \\
\text{Hom}_{\text{Coh}_{w_p^{-1}}(\mathcal{X})}(G^\bullet, (F)^\bullet) &= \text{Hom}_{\text{Coh}(\mathcal{X})}(G^{(w_p-1,p)}, F).
\end{align*}
\end{equation}

Note that $\text{Coh}_{w_p^{-1}}(\mathcal{X})$ is equipped with a natural shift automorphism
\begin{equation}
\tau_p : \text{Coh}_{w^{-1}}(\mathcal{X}) \rightarrow \text{Coh}_{w^{-1}}(\mathcal{X})
\end{equation}
bysending aw_p-cycle $F^\bullet = (F^{(0,p)} \rightarrow \cdots \rightarrow F^{(w_p-1,p)} \rightarrow F^{(0,p)}(p))$ to
\begin{equation}
\tau_p(F^\bullet) = (F^{(1,p)} \rightarrow F^{(2,p)} \rightarrow \cdots \rightarrow F^{(w_p-1,p)} \rightarrow F^{(0,p)}(p) \rightarrow F^{(1,p)}(p)).
\end{equation}
On the other hand, there is a natural action of the Picard group $\text{Pic}(\mathcal{X})$ of $\text{Coh}(\mathcal{X})$ on $\text{Coh}_{w_p^{-1}}(\mathcal{X})$ by
\begin{equation}
\begin{align*}
\text{Pic}(\mathcal{X}) \times \text{Coh}_{w_p^{-1}}(\mathcal{X}) &\rightarrow \text{Coh}_{w_p^{-1}}(\mathcal{X}) \\
(L, F^\bullet) &\mapsto (L \cdot F^\bullet) := (F^{(i,p)} \otimes L)_{i \in \mathbb{Z}}.
\end{align*}
\end{equation}
Clearly these two kinds of automorphisms on $\text{Coh}_{w_p^{-1}}(\mathcal{X})$ commute with each other and satisfy the relation $\tau_p(F^\bullet) = \mathcal{O}_X(p) \cdot F^\bullet$ for all $F^\bullet \in \text{Coh}_{w_p^{-1}}(\mathcal{X})$.

\subsection{Parabolic coherent sheaves.}

Now, let us fix a finite (non-empty) set of closed points (of degree one for simplicity) $S = \{p_1, \ldots, p_N\} \subset \mathcal{X}$ and $w = \{w_p\}_{p \in S}$ a collection of positive integers. We define the category $\text{Coh}^{w,S}(\mathcal{X})$ inductively as follows: set $\text{Coh}^{w,0}(\mathcal{X}) = \text{Coh}(\mathcal{X})$ and let $\text{Coh}^{1}(\mathcal{X})$ be the $w_p$-cycle category obtained from $\text{Coh}^{i-1}(\mathcal{X})$ by inserting weight $w_{p_i}$ in $p_i$. We define $\text{Coh}^{w,S}(\mathcal{X}) := \text{Coh}^{N}(\mathcal{X})$. Or equivalently, an
object $\mathcal{F}^\bullet$ in $\text{Coh}^{w,S}$ is a $N$-dimensional diagram of objects $\mathcal{F}^a \in \text{Coh}(X)$:

$$
\begin{array}{cccc}
\mathcal{F}^a & \mathcal{F}^{a+e_p} & \mathcal{F}^{a+2e_p} & \cdots & \mathcal{F}^{a+we_p} \\
\downarrow \phi^{a+e_p} & \downarrow \phi^{a+2e_p} & \cdots & \downarrow \phi^{a+we_p} \\
\mathcal{F}^{a+e_q} & \mathcal{F}^{a+e_p+e_q} & \cdots & \mathcal{F}^{a+we_p+e_q} \\
\downarrow & \downarrow & \cdots & \downarrow \\
\vdots & \vdots & \cdots & \vdots \\
\mathcal{F}^{a+we_q} & \cdots & \cdots & \cdots & \mathcal{F}^{a+we_p+we_q}
\end{array}
$$

satisfying the periodic properties $\mathcal{F}^{a+we_p} = \mathcal{F}^a$, $\phi^{a+we_p} = \phi^a$ and $\phi^a(\cdot) = \phi^{a+we_p}(\cdot)$ for all $a \in \mathbb{Z}^S$ and $p \in S$, where $e_p$ is the canonical basis of $\mathbb{Z}^S$. A morphism between two objects $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ in $\text{Coh}^{w,S}(X)$ is defined in the obvious way. In fact, the category $\text{Coh}^{w,S}(X)$ is equivalent to the category of coherent sheaves $X$ with $w$-step parabolic structure on $S$ defined by Heinloth in [Hei]. We might as well call an object $\mathcal{F}^\bullet$ in $\text{Coh}^{w,S}(X)$ a parabolic coherent sheaf on $X$. We have the following Lemma:

**Lemma 1.1.** [Len, Lemma 4.2, Theorem 4.3] [Hei, Lemma 2.1, 3.4] $\text{Coh}^{w,S}(X)$ is connected, abelian, noetherian and hereditary.

For $\mathcal{F}^\bullet \in \text{Coh}^{w,S}(X)$, by the same argument as the $w_p$-cycles case, all the $\mathcal{F}^a$ have the same generic rank. We define the rank of $\mathcal{F}^\bullet$ to be the rank of $\mathcal{F}^0$. A parabolic coherent sheaf $\mathcal{F}^\bullet$ is called *locally free or a vector bundle* if all the $\mathcal{F}^a$ are locally free and $\mathcal{F}^\bullet$ is called *torsion* if $\text{rank}(\mathcal{F}^\bullet) = 0$. Any parabolic coherent $\mathcal{F}^\bullet$ over $X$ can be decomposed uniquely (but non-canonically) as $\mathcal{F}^\bullet = \mathcal{F}^\bullet_v \oplus \mathcal{F}^\bullet_i$ where $\mathcal{F}^\bullet_v$ is locally free and $\mathcal{F}^\bullet_i$ is a parabolic torsion sheaf. For $\mathcal{F}^\bullet \in \text{Coh}^{w,S}(X)$, the middle terms of the diagram (1.9) are determined by its frame since the requirement of the composition maps $\phi^{a}(\cdot) \circ \cdots \circ \phi^{a+we_p}: \mathcal{F}^a \to \mathcal{F}^a(p)$ being a natural morphism. The collection $(\text{deg}(\mathcal{F}^a))_{a \in \mathbb{Z}^S}$ is therefore determined by the sub-collection $(\text{deg}(\mathcal{F}^{np_{wp}}))_{p \in S, 0 \leq n_p < w_p}$ and rank($\mathcal{F}^\bullet$). We might define the *multi-degree* of $\mathcal{F}^\bullet$ as the collection $\text{deg}(\mathcal{F}^\bullet) = (\text{deg}(\mathcal{F}^{np_{wp}}))_{p \in S, 0 \leq n_p < w_p}$.

**1.2.3. Automorphisms.** $\text{Coh}^{w,S}(X)$ is equipped with the natural $\mathbb{Z}^S$-action as the shift automorphisms $a: \text{Coh}^{w,S}(X) \to \text{Coh}^{w,S}(X)$ defined by $a \cdot (\mathcal{F}^\bullet) = a \cdot ((\mathcal{F}^b)_{b \in \mathbb{Z}^S}) = (\mathcal{F}^{b+a})_{b \in \mathbb{Z}^S} = a \cdot \mathcal{F}(X)$ for all $a \in \mathbb{Z}^S$. The Picard group $\text{Pic}(X)$ of $\text{Coh}(X)$ also acts on $\text{Coh}^{w,S}(X)$ as automorphisms:

$$
\text{Pic}(X) \times \text{Coh}^{w,S}(X) \to \text{Coh}^{w,S}(X) \\
(L, \mathcal{F}^\bullet) = (\mathcal{F}^b)_{b \in \mathbb{Z}^S} \mapsto (L \cdot \mathcal{F}^\bullet) = (\mathcal{F}^b \otimes L)_{b \in \mathbb{Z}^S}.
$$

Clearly these two actions commute and satisfy the relations $O_X(p) \cdot \mathcal{F}^\bullet = w_p e_p \cdot (\mathcal{F}^\bullet) = \mathcal{F}^\bullet(w_p e_p)$ for all $p \in S$. 
1.2.4. The constant parabolic sheaf. We say a parabolic coherent sheaf $F^\bullet \in \text{Coh}^{w, S}(X)$ is constant if $F^\bullet$ is of the form

$$
\begin{align*}
F^0 = F & \rightarrow F & \rightarrow \cdots & \rightarrow F & \rightarrow F(p) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
F & \rightarrow F & \rightarrow \cdots & \rightarrow F & \rightarrow F(p) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
F(q) & \rightarrow F(q) & \rightarrow \cdots & \rightarrow F(q) & \rightarrow F(p + q)
\end{align*}
$$

(1.11)

for some coherent sheaf $F \in \text{Coh}(X)$ and we denote by $(F)^\bullet$ such a parabolic sheaf. The category $\text{Coh}(X)$ of coherent sheaves over $X$ can be embedded into $\text{Coh}^{(w, S)}(X)$ by sending a coherent sheaf $F$ to the constant parabolic sheaf $(F)^\bullet$. Conversely there is a natural functor $(-)^a : \text{Coh}^{(w, S)}(X) \rightarrow \text{Coh}(X)$ for any $a \in \mathbb{Z}^S$ by sending $F^\bullet = (F^b)_{b \in \mathbb{Z}^S} \mapsto F^a$. These two kinds of functors $(-)^*$ and $(-)^a$ again satisfy the following adjunction properties:

$$
\text{Hom}_{\text{Coh}^{w, S}(X)}((F)^\bullet, G^\bullet) = \text{Hom}_{\text{Coh}(X)}(F, G^0).
$$

and

$$
\text{Hom}_{\text{Coh}^{w, S}(X)}(G^\bullet, (F)^\bullet) = \text{Hom}_{\text{Coh}(X)}(G^w, F)
$$

where $G^w := \mathcal{G}^{\sum_{s \in \mathbb{Z}^S} (w_s - 1) s}$. Similarly the extension spaces of a parabolic coherent sheaves $F^\bullet$ by a parabolic constant sheaf $(G)^\bullet$ for some $G \in \text{Coh}(X)$ have the same adjunction formulas

$$
\text{Ext}_{\text{Coh}^{w, S}(X)}((F)^\bullet, G^\bullet) = \text{Ext}_{\text{Coh}(X)}(F, G^0).
$$

and

$$
\text{Ext}_{\text{Coh}^{w, S}(X)}(G^\bullet, (F)^\bullet) = \text{Ext}_{\text{Coh}(X)}(G^w, F).
$$

A direct consequence of the usual Serre duality and the adjunction properties is that for any $G \in \text{Coh}(X)$, we have

$$
\text{Ext}_{\text{Coh}^{w, S}(X)}((F)^\bullet, (G)^\bullet)^* \simeq \text{Hom}_{\text{Coh}^{w, S}(X)}((G \otimes \omega_X^{-1})^\bullet(-w), F^\bullet).
$$

(1.12)

Let $\mathcal{O}^\bullet := (\mathcal{O}_X)^\bullet$, $\omega_X^\bullet := (\omega_X)^\bullet(w)$ and $\omega_X^{\bullet,k} := (\omega_X^{\otimes k})^\bullet(kw)$. Then we have the following analogue Serre duality:

$$
\text{Ext}_{\text{Coh}^{w, S}(X)}((F)^\bullet, \omega_X^{\bullet,k})^* \simeq \text{Hom}_{\text{Coh}^{w, S}(X)}(\omega_X^{\bullet,k-1}, F^\bullet).
$$

(1.13)

1.2.5. Parabolic line bundles. We call an $L^\bullet \in \text{Coh}^{w, S}(X)$ a parabolic line bundle if $L^\bullet$ is locally free and of rank 1. Let $\text{Pic}^{w, S}(X)$ be the set of all parabolic line bundles. Contrast with Pic$(X)$, the set Pic$^{w, S}(X)$ is not obviously a group. However, we can describe $\text{Pic}^{w, S}(X)$ as follows: Let $\mathbb{L}(w)$ be the rank one abelian group on generators $\{\bar{x}_p \mid p \in S\}$ subject to the relations

$$
w_{p_1} \bar{x}_{p_1} = w_{p_2} \bar{x}_{p_2} = \cdots = w_{p_N} \bar{x}_{p_N} =: \bar{c}.
$$

(1.14)

Clearly $\text{Pic}^{w, S}(X)$ is stable under the action of Pic$^0(X)$. The set Pic$^{w, S}(X)/\text{Pic}^0(X)$ of Pic$^0(X)$-orbits is canonically isomorphic to $\mathbb{L}(w)$ by

$$
\text{Pic}^{w, S}(X)/\text{Pic}^0(X) \rightarrow \mathbb{L}(w)
$$

(1.15)

$$
\text{Pic}^0(X) \cdot L^\bullet \mapsto \deg(L^0)\bar{c} + \sum_{p \in S} \sum_{1 \leq i \leq w_p - 1} i(\deg(L^{i\bar{c}}) - \deg(L^{(i-1)\bar{c}}))\bar{x}_p.
$$
and hence the orbit space \( \text{Pic}^{w,S}(\mathcal{X})/\text{Pic}(\mathcal{X}) \simeq \mathbb{L}(w)/\mathbb{Z} \mathbb{C} \simeq \prod_{p \in S} \mathbb{Z}/w_p\mathbb{Z} \). Thus there is a bijection
\[
\text{Pic}^{w,S}(\mathcal{X}) \to \text{Pic}(\mathcal{X}) \times \prod_{p \in S} \mathbb{Z}/w_p\mathbb{Z}
\]
\[(1.16)\]
\[\mathcal{L}^\bullet \mapsto (\mathcal{L}^0, (\sum_{p \in S} \sum_{1 \leq i \leq w_p - 1} i(\deg(\mathcal{L}^i) - \deg(\mathcal{L}^{i-1})))x_p),\]
where we denote by \( x_p \) again the image of \( x_p \) under the projection \( \pi : \mathbb{L}(w) \to \prod_{p \in S} \mathbb{Z}/w_p\mathbb{Z} \).

1.2.6. Parabolic Torsion Sheaves. We denote by \( \text{Tor}^{w,S}(\mathcal{X}) \) the subcategory of \( \text{Coh}^{w,S}(\mathcal{X}) \) consisting of parabolic torsion sheaves. As in the usual case, it splits into a direct product of blocks
\[
\text{Tor}^{w,S}(\mathcal{X}) = \prod_{q \in \mathcal{X}} \text{Tor}^w_q(\mathcal{X})
\]
where \( \text{Tor}^w_q(\mathcal{X}) \) is the subcategory of parabolic torsion sheaves supported at a single point \( q \in \mathcal{X} \). If \( q \notin S \) and \( T^\bullet \in \text{Tor}^w_q(\mathcal{X}) \), then all \( T^\bullet \) are isomorphic and thus \( T^\bullet = (T^0)^\bullet \). Thus the category \( \text{Tor}^w_q(\mathcal{X}) \) is equivalent to the category of torsion sheaves \( \text{Tor}_q(\mathcal{X}) \) with extra structure supported at a single point \( q \in \mathcal{X} \), and hence equivalent to the category \( \mathcal{O}_{\mathcal{X},q}\)-\text{Mod} of finite-length modules over the local ring \( \mathcal{O}_{\mathcal{X},q} \) of \( q \), a discrete valuation ring. If \( q \in S \), any indecomposable parabolic torsion sheaf supported at \( q \) will be of the form
\[
\mathcal{O}_{\mathcal{X},q}^{\bullet,(k)}(i\epsilon_q) := \mathcal{O}_{\mathcal{X},q}^{\bullet}(i\epsilon_q)/\mathcal{O}_{\mathcal{X},q}^{\bullet}(i - k)\epsilon_q \quad \text{for some } 0 \leq k < i, \text{ i.e. at the } i\epsilon_q \text{ part}
\]
\[
\cdots \to \mathcal{O}_{\mathcal{X},q}^{(d)} \to \cdots \to \mathcal{O}_{\mathcal{X},q}^{(d-1)} \to \cdots \to \mathcal{O}_{\mathcal{X},q}^{(d-1)} \to \mathcal{O}_{\mathcal{X},q}^{(d)} \to \cdots
\]
where \( d = \lceil \frac{k}{w_q} \rceil \) is the smallest integer bigger than \( \frac{k}{w_q} \).

To be more explicitly described, let \( C_n \) denote the quiver of type \( A_n^{(1)} \) with cyclic orientation. We also let \( C_1 \) be the Jordan quiver, i.e., the quiver with one vertex and one arrow. A representation of \( C_1 \) over a field \( k \) is a collection of \( k \)-vector spaces \( V_i \) with \( i \in \mathbb{Z}/n\mathbb{Z} \) together with a collection of linear maps \( x_i : V_i \to V_{i+1} \). Morphisms between two representations \( (V_i, x_i) \) and \( (W_i, y_i) \) are \( k \)-linear maps \( \phi_i : V_i \to W_i \) such that \( \phi_i x_i = y_i \phi_i \). Finally, a representation \( (V_i, x_i) \) is called nilpotent if there exists \( N \gg 0 \) such that \( x_{i+N} \cdots x_i = 0 \) for all \( i \).

The set of isomorphism classes of indecomposable nilpotent representations of \( C_1 \) over \( k \) is in bijection with \( \mathbb{N}^+ \) and we denote by \( [n] \) the class of the indecomposable representation of dimension \( n \). The set of isomorphism classes of representations is thus in bijection with the set of all partitions via the assignment \( \lambda = (\lambda_1, \ldots, \lambda_r) \mapsto [\lambda] = [\lambda_1] + \cdots + [\lambda_r] \). For \( q \notin S \), the assignment \( (\mathcal{O}_{\mathcal{X},q}^{(n)} \to [n]) \) give rise to a Morita equivalence of \( \text{Tor}^w_q(\mathcal{X}) \) and the category of nilpotent representations of the quiver \( C_n \) over \( k_q \).

Now, consider the quiver \( C_n \) for \( n > 1 \). Denote by \( \{\epsilon_i\}_{i \in \mathbb{Z}/n\mathbb{Z}} \) the canonical basis of \( \mathbb{Z}/n\mathbb{Z} \). For each \( i \in \mathbb{Z}/n\mathbb{Z} \) and \( k \in \mathbb{N} \), define the cyclic segment \( [-i; k] \) to be the image of the projection to \( \mathbb{Z}/n\mathbb{Z} \) of the segment \( [i', i' + k - 1] \) for any \( i' \in \mathbb{Z} \), \( i' \equiv i \pmod{n} \). A cyclic multisegment is a finite linear combination \( \mathbf{m} = \sum_{i,k} a_{i,k}(i, k) \) with \( a_{i,k} \in \mathbb{N} \). The isomorphism classes of representations (resp. indecomposable representations) of \( C_n \) over \( k \) are in bijection with the set of cyclic multisegments (resp. cyclic segments). For \( q \in S \), the assignment \( \mathcal{O}_{\mathcal{X},q}^{\bullet,(k)}(i\epsilon_q) \to [i; k] \) give rise to a Morita equivalence between \( \text{Tor}^w_q(\mathcal{X}) \) and the category of nilpotent representations of \( C_w_q \) over the field \( k_q \).

To summarize, we have the following properties:

**Proposition 1.2.** [Hei, Proposition 3.2]
1) Any parabolic torsion sheaf is a direct sum of sheaves of the form
\[ \mathcal{O}_{X,p}^{(i)}(i \epsilon_p) = \mathcal{O}_{X}^{(*)}(i \epsilon_p)/\mathcal{O}_{X}^{(*)}(i - j) \epsilon_p, \ p \in S, \ i, j \in \mathbb{N} \]
and sheaves supported outside \( S \).

2) Any parabolic torsion sheaf \( \mathcal{T}^* \) has a filtration \( \mathcal{T}^*_j \subset \mathcal{T}^*_j \subset \cdots \subset \mathcal{T}^*_1 \) such that \( \mathcal{T}^*_j/\mathcal{T}^*_j \) are isomorphic to one of the following
(a) \( \mathcal{T}^*_j/\mathcal{T}^*_j \cong (\mathcal{O}_{X,q})^* \) and \( q \notin S \).
(b) There is a \( p_0 \in S \) and \( 0 \leq i_0 < w_{p_0} \) such that \( \mathcal{T}^{(i,p)}(i \epsilon_p) \cong \mathcal{O}_{X,p_0} \) if \( i \equiv i_0 \mod w_{p_0} \) and \( p = p_0 \in S \) and 0 otherwise.

3) For \( d > 0 \), any constant parabolic torsion sheaf \( \mathcal{T}^* \) with \( \deg(\mathcal{T}^*) = d \) for all \( \alpha \in \mathbb{Z}^S \) has a filtration \( \mathcal{T}^*_1 \subset \cdots \subset \mathcal{T}^*_d \subset \cdots \subset \mathcal{T}^* \) such that \( \deg(\mathcal{T}^*_i) = i \).

Remark. Note that the notations \( \mathcal{O}_{X,p}^{(*)}(i \epsilon_p) \) are compatible with the notion of the \( \mathbb{Z}^S \)-action and the \( \mathbb{Z}^S \)-action on \( \text{Tor}^{w,\mathbb{Z}}(\mathbb{X}) \) factors through \( \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z} \).

1.2.7. Some homological consequences. We have the following lemma due to Heinloth:

**Lemma 1.3.** [Hei, Lemma 3.5] Let \( \mathcal{T}^* \) be a parabolic torsion sheaf and \( \mathcal{E}^* \) a parabolic vector bundle. Then:

1. \( \text{Hom}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{T}^*, \mathcal{E}^*) = \text{Ext}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{E}^*, \mathcal{T}^*) = \{0\} \).
2. If \( \mathcal{T}^* = \mathcal{O}_{X,p}^{(*)}(i \epsilon_p) \), then
   \[ \dim \text{Ext}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{T}^*, \mathcal{E}^*) = \deg(\mathcal{E}^{(i+1)\epsilon_p}) - \deg(\mathcal{E}^{i \epsilon_p}) \]
   and
   \[ \dim \text{Hom}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{E}^*, \mathcal{T}^*) = \deg(\mathcal{E}^{i \epsilon_p}) - \deg(\mathcal{E}^{(i-1)\epsilon_p}) \).
3. If \( \mathcal{T}^* \) is constant with \( \deg(\mathcal{T}^*) = d \) for all \( \alpha \in \mathbb{Z}^S \), we get
   \[ \dim \text{Ext}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{T}^*, \mathcal{E}^*) = d \cdot \text{rank}(\mathcal{E}^*) = \dim \text{Hom}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{E}^*, \mathcal{T}^*) \).

In other words, the dimension of the spaces of morphisms and extensions between two parabolic coherent sheaves only depends on their ranks and multi-degrees. We might as well interpret the numerical Grothendieck group \( \mathcal{K} := K'(\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})) \) as \( \mathbb{Z} \times \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z} \) by assigning a parabolic coherent sheaf \( \mathcal{F}^* \) to \( (\text{rank}(\mathcal{F}^*), \deg(\mathcal{F}^*)) \). We denote by \( \mathcal{K}^+ \subset \mathcal{K} \) the subset of \( \mathcal{K} \) consisting of the classes of parabolic coherent sheaves.

**Lemma 1.4.** For \( p, q \in S, \ 0 \leq i < w_p \) and \( 0 \leq k < w_q \), we have
\[ \langle \mathcal{O}^*, \mathcal{O}(p)^* \rangle_S = 1 - g_X + \deg(p), \ \langle \mathcal{O}^*, \mathcal{O}(i \epsilon_p) \rangle_S = 1 - g_X, \]
\[ \langle \mathcal{O}^*, \mathcal{O}^{(1)}_p(i \epsilon_p) \rangle_S = \begin{cases} \deg(p) & \text{if } i = 0, \\ 0 & \text{otherwise}, \end{cases} \]
\[ \langle \mathcal{O}^{(1)}_p(i \epsilon_p), \mathcal{O}^* \rangle_S = \begin{cases} - \deg(p) & \text{if } i = 1, \\ 0 & \text{otherwise}, \end{cases} \]
\[ \langle \mathcal{O}^{(1)}_p(i \epsilon_p), \mathcal{O}^{(1)}_q(k \epsilon_q) \rangle_S = \begin{cases} \deg(p) & \text{if } p = q, \ i = k, \\ - \deg(p) & \text{if } p = q, \ i \equiv k - 1 \mod w_p, \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** By adjointness of the functors \((-)^*\) and \((-)^0\), we have
\[ \text{Hom}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{O}^*, \mathcal{O}(p)^*) = \text{Hom}(\mathcal{O}, \mathcal{O}(p)), \ \text{Ext}_{\text{Coh}^{w,\mathbb{Z}}(\mathbb{X})}(\mathcal{O}^*, \mathcal{O}(p)^*) = \text{Ext}(\mathcal{O}, \mathcal{O}(p)). \]
Hence the Euler form can be computed by Riemann-Roch
\[ \langle \mathcal{O}^*, \mathcal{O}(p)^* \rangle_S = \langle \mathcal{O}, \mathcal{O}(p) \rangle = 1 - g_X + \deg(p). \]
Similarly, since $0 \leq i < w_p$, $(\mathcal{O}^*(i\epsilon_p))^0 = (\mathcal{O}^*(-i\epsilon_p))^w = \mathcal{O}$. So
\[
\text{Hom}_{\text{Coh}^w, \mathcal{S}(\mathcal{X})}(\mathcal{O}^*, \mathcal{O}^*(i\epsilon_p)) = \text{Hom}(\mathcal{O}, \mathcal{O}),
\]
\[
\text{Ext}_{\text{Coh}^w, \mathcal{S}(\mathcal{X})}(\mathcal{O}^*, \mathcal{O}^*(i\epsilon_p)) = \text{Ext}_{\text{Coh}^w, \mathcal{S}(\mathcal{X})}(\mathcal{O}^*(-i\epsilon_p), \mathcal{O}^*) = \text{Ext}(\mathcal{O}, \mathcal{O}).
\]
and we get $(\mathcal{O}^*, \mathcal{O}^*(i\epsilon_p)) = \langle \mathcal{O}, \mathcal{O} \rangle = 1 - g\xi$.

Note that $(\mathcal{O}^*(1))(i\epsilon_p)^{jw_p} = \delta_{w_p-i,\mathcal{O}}$. By lemma 1.3 or again by adjointness of functors, we have
\[
\text{Hom}_{\text{Coh}^w, \mathcal{S}(\mathcal{X})}(\mathcal{O}^*(1)(i\epsilon_p), \mathcal{O}^*) = \text{Ext}_{\text{Coh}^w, \mathcal{S}(\mathcal{X})}(\mathcal{O}^*, \mathcal{O}^*(1)(i\epsilon_p)) = 0,
\]
\[
\text{Hom}_{\text{Coh}^w, \mathcal{S}(\mathcal{X})}(\mathcal{O}^*, \mathcal{O}^*(1)(i\epsilon_p)) = \begin{cases} \text{Hom}(\mathcal{O}, \mathcal{O}) & \text{if } i = 0, \\
0 & \text{if } i \neq 0,
\end{cases}
\]
and
\[
\text{Ext}_{\text{Coh}^w, \mathcal{S}(\mathcal{X})}(\mathcal{O}^*(1)(i\epsilon_p), \mathcal{O}^*) = \begin{cases} \text{Ext}(\mathcal{O}, \mathcal{O}) & \text{if } i = 1, \\
0 & \text{if } i \neq 1.
\end{cases}
\]
The lemma follows. \qed

2. The Hall Algebra of Parabolic Coherent Sheaves

2.1. Reminders on Hall Algebras. We briefly recall here the definition of a Hall algebra of a finitary abelian category for reader’s convenience and refer to [S1] for its standard properties. We will say an abelian category $\mathcal{A}$ finitary if for any two objects $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$ all the groups $\text{Ext}^i(\mathcal{F}, \mathcal{G})$ have finite cardinality and are zero for almost all $i$. For instance any abelian category $\mathcal{A}$ which is linear over some finite field $\mathbb{k}$ and which satisfies $\dim\text{Ext}^i(\mathcal{F}, \mathcal{G}) < \infty$ is finitary.

For a $\mathbb{k}$-linear abelian category $\mathcal{A}$ we denote by $K(\mathcal{A})$ its Grothendieck group, defined over $\mathbb{Z}$. If $\mathcal{A}$ is of finite global dimension then we may consider the Euler form
\[
\langle \mathcal{M}, \mathcal{N} \rangle = \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}^i(\mathcal{M}, \mathcal{N}).
\]
Note that the Euler form only depend on the classes of $\mathcal{M}$ and $\mathcal{N}$ in the Grothendieck group and thus it descends to a form $\langle \ , \ , \rangle : K(\mathcal{A}) \otimes K(\mathcal{A}) \to \mathbb{C}$. We also need the symmetrized version $\langle \mathcal{M}, \mathcal{N} \rangle = \langle \mathcal{M}, \mathcal{N} \rangle + \langle \mathcal{N}, \mathcal{M} \rangle$.

Our only interest here is in the case when $\mathcal{A}$ is finitary, abelian and of global dimension one. Let $\mathcal{I}$ be the set of isomorphism classes of objects of $\mathcal{A}$. Let us choose a square root $v$ of $l^{-1}$. As a vector space we have
\[
\mathbf{H}_\mathcal{A} = \{ f : \mathcal{I} \to \mathbb{C} \mid \text{supp}(f) \text{ finite} \} = \bigoplus_{\mathcal{F} \in \mathcal{I}} \mathbb{C} 1_\mathcal{F}
\]
where $1_\mathcal{F}$ denotes the characteristic function of $\mathcal{F} \in \mathcal{I}$. The multiplication is defined as
\[
(f \cdot g)(\mathcal{R}) = \sum_{\mathcal{N} \subseteq \mathcal{R}} v^{-\langle \mathcal{R}, \mathcal{N} \rangle} f(\mathcal{R}/\mathcal{N}) g(\mathcal{N})
\]
and the comultiplication is
\[
\Delta(f)(\mathcal{M}, \mathcal{N}) = \frac{v^{\langle \mathcal{M}, \mathcal{N} \rangle}}{|\text{Ext}^1(\mathcal{M}, \mathcal{N})|} \sum_{\xi \in \text{Ext}^1(\mathcal{M}, \mathcal{N})} f(\mathcal{X}_\xi)
\]
where $\mathcal{X}_\xi$ is the extension of $\mathcal{N}$ by $\mathcal{M}$ corresponding to $\xi$. Note that the coproduct may take values in a completion $\mathbf{H}_\mathcal{A} \hat{\otimes} \mathbf{H}_\mathcal{A}$ of the tensor space $\mathbf{H}_\mathcal{A} \otimes \mathbf{H}_\mathcal{A}$ only. The Hall algebra is graded by the class in the Grothendieck group. We will sometimes write $\Delta_{\alpha, \beta}$ in order to specify the graded components of the coproduct.
The Hall algebra $\mathbf{H}_A$ will become a (topological) bialgebra if we suitably twist the coproduct. To do this, let $K = \mathbb{C}[K(A)]$ be the group algebra of $K(A)$ and for any class $\alpha \in K(A)$ we denote by $\kappa_\alpha$ the corresponding element in $K$. We define an extended Hall algebra $\tilde{\mathbf{H}}_A = \mathbf{H}_A \otimes K$ with relations 

$$\kappa_\alpha \kappa_\beta = \kappa_{\alpha + \beta}, \quad \kappa_0 = 1, \quad \kappa_\alpha 1_F \kappa_{-\alpha}^{-1} = v^{-\langle \alpha, F \rangle} 1_F$$

where $F$ denote the class of $F$ in $K(A)$. The new coproduct is given by the formulas 

$$\tilde{\Delta}(\kappa_\alpha) = \kappa_\alpha \otimes \kappa_\alpha$$

$$\tilde{\Delta}(f) = \sum_{M,N} \Delta(f)(M,N) 1_M \kappa_N \otimes 1_N.$$ 

Then $\tilde{\mathbf{H}}_A$ is a topological algebra. Finally, let 

$$(\ , \ )_G : \tilde{\mathbf{H}}_A \otimes \tilde{\mathbf{H}}_A \to \mathbb{C}$$

be the Green’s Hermitian scalar product defined by 

$$(1_M \kappa_\alpha, 1_N \kappa_\beta)_G = \frac{\delta_{M,N}}{\# \text{Aut}(M)} v^{-\langle \alpha, \beta \rangle}.$$ 

This scalar product is a Hopf pairing, i.e., 

$$(ab, c)_G = (a \otimes b, \tilde{\Delta}(c))_G, \quad a, b, c \in \tilde{\mathbf{H}}_A.$$ 

The restriction of $(\ , \ )_G$ to $\mathbf{H}_A$ is nondegenerate.

Finally, for any class $\gamma \in K(A)$, we set 

$$1_\gamma = \sum_{M \in \gamma} 1_M$$

where the sum ranges over all objects $M$ of class $\gamma$. This sum may be infinite for some categories, so strictly speaking $1_\gamma$ belongs to the formal completion $\hat{\mathbf{H}}_A$ of $\mathbf{H}_A$. The coproduct extends to a map 

$$\Delta : \hat{\mathbf{H}}_A \to \mathbf{H}_A \hat{\otimes} \mathbf{H}_A.$$ 

We have the following useful lemma

**Lemma 2.1.** [S1, Lemma 1.7] We have

$$\Delta(1_\gamma) = \sum_{\gamma = \alpha + \beta} v^{\langle \alpha, \beta \rangle} 1_\alpha \otimes 1_\beta.$$ 

### 2.2. The Hall algebra of Coh($X$).

We collect in this section a few facts about $\mathbf{H}_X := \mathbf{H}_{\text{Coh}(X)}$ and its extended version which can be found, e.g., in [SV2].

#### 2.2.1. The decomposition of Tor($X$) = $\prod_{q \in X} \text{Tor}_q$ gives rise to a decomposition at the level of Hall algebras 

$$\mathbf{H}_{\text{Tor}(X)} = \bigotimes_{q \in X} \mathbf{H}_{\text{Tor}_q},$$

It is well-known that $\mathbf{H}_{\text{Tor}_q}$ is commutative and cocommutative: it is isomorphic to the classical Hall algebra defined over the residue field $k_q$, and is therefore identified with the algebra of symmetric functions:

$$\Psi_q : \mathbf{H}_{\text{Tor}_q} \simeq \Lambda \otimes \mathbb{C}_{v^{\deg(q)}},$$

$$e_{r,q} \mapsto e_r$$

where $e_{r,q} := v^{-\deg(q)r(r-1)} 1_{(O^{(r)}_{2,q})}$, and $\Lambda = \mathbb{C}[y_1, y_2, \ldots]^{S\infty}$ is Macdonald’s ring of symmetric functions, $\{e_r\}_{r \in \mathbb{N}}$ denotes the elementary symmetric polynomial. Under this identification, for any $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Pi$, the element $1_{(O^{(r)}_{2,q})}$ corresponds to
functions (c.f. [Mac]) tells us

\[ p \]

\[ \sum_{j=1}^{r} (j - 1) \lambda_j \] and \( P_\lambda(t) \) is the Hall-Littlewood polynomial. Let \( p_r \in \Lambda \) be the power-sum symmetric function and set

\[ h_{r,q} = \frac{[r]}{r} \Psi_q^{-1}(p_r). \]

Here as usual \([r] = \frac{v^{-r \deg(q)} - v^{-r \deg(q)}}{v^{-\deg(q)} - v^{-\deg(q)}}\) is the \( q \)-integer. The classical theory of symmetric functions (c.f. [Mac]) tells us

\[ h_{r,q} = \frac{[r]}{r} \sum_{\lambda, |\lambda| = r} n(l(\lambda) - 1) \Omega^2_{X}, \]

where \( n(l) = \prod_{j=1}^{r}(1 - v^{-2j \deg(q)}). \)

For \( d \geq 1 \), we set

\[ 1_{0,d} = \sum_{\tau \in \text{Tor}_d} 1_{\tau} \]

and we define the elements \( T_{0,d}, \theta_{0,d} \) of \( \hat{H}_{\text{Tor}} \) via the relations

\[ 1 + \sum_{d \geq 1} 1_{0,d} s^d = \exp \left( \sum_{d} \frac{T_{0,d}}{[d]} s^d \right), \quad 1 + \sum_{d \geq 1} \theta_{0,d} s^d = \exp \left( (v^{-1} - v) \sum_{d \geq 1} T_{0,d} s^d \right). \]

We set also \( 1_{0,0} = T_{0,0} = \theta_{0,0} = 1. \)

**Lemma 2.2.** [S1, Example 4.12, Lemma 4.50, Lemma 4.51] The following hold for all \( d \in \mathbb{N} \)

\( 1 \)

\( \Delta(T_{0,d}) = T_{0,d} \otimes 1 + \kappa_{0,d} \otimes T_{0,d}, \)

\( 2 \)

\[ \Delta(\theta_{0,d}) = \sum_{s=0}^{d} \theta_{0,s} \kappa_{0,d-s} \otimes \theta_{0,d-s}, \]

\( 3 \)

\[ (T_{0,d}, T_{0,d})_G = (v^{-d-1} \# \text{X}(\mathbb{F}_p)[d])/([d] - d). \]

The sets \( \{1_{0,d} \mid d \in \mathbb{N}\} \), \( \{T_{0,d} \mid d \in \mathbb{N}\} \) and \( \{\theta_{0,d} \mid d \in \mathbb{N}\} \) all generate the same subalgebra \( U_X^0 \) of \( H_{\text{Tor}(\text{X})} \). It is known that

\[ U_X^0 = \mathbb{C}[1_{0,1}, 1_{0,2}, \ldots], \]

i.e., the commuting elements \( 1_{0,d} \) for \( d \geq 1 \) are algebraically independent. The same holds also for the collections of generators \( \{T_{0,d}\} \) and \( \{\theta_{0,d}\} \).

2.2.2. Because the sub-category \( \text{Bun}(\text{X}) \) of \( \text{Coh}(\text{X}) \) is exact and extension-closed it gives rise to a subalgebra \( H_{\text{Bun}(\text{X})} \) of \( H_\text{X} \). However, this subalgebra is not stable under the coproduct \( \Delta \).

For \( d \in \mathbb{Z} \) let Pic\(^d(\text{X}) \) be the set of all line bundles over \( \text{X} \) of degree \( d \). Set

\[ 1_{1,d}^{\text{ss}} = 1_{\text{Pic}^d(\text{X})} = \sum_{\mathcal{E} \in \text{Pic}^d(\text{X})} 1_{\mathcal{E}}. \]

Denote by \( U_X^0 \) the subalgebra of \( H_X \) generated by \( \{1_{1,d}^{\text{ss}} \mid d \in \mathbb{Z}\} \). The *spherical Hall algebra* of \( \text{X} \) is the subalgebra \( U_X \) of \( H_X \) generated by \( U_X^0 \) and \( U_X^0 \). We define \( \tilde{U}_X \) as the subalgebra of \( \tilde{H}_X \) generated by \( K_X \) and \( U_X \). It is known to be a topological sub-bialgebra of \( \tilde{H}_X \) (c.f. [SV1, Section 1]). We also set \( U_X^0 = U_X^0 K_X \). The multiplication map again gives an isomorphism \( U_X^0 \otimes U_X^0 \to U_X \).

Since \( \text{Ext}(\mathcal{E}, \mathcal{T}) = 0 \) and \( \text{Hom}(\mathcal{T}, \mathcal{E}) = 0 \) for any vector bundle \( \mathcal{E} \) and torsion sheaf \( \mathcal{T} \), it follows that

\[ 1_{\mathcal{E}} \cdot 1_{\mathcal{T}} = v^{-1(\mathcal{E}, \mathcal{T})} 1_{\mathcal{E} \otimes \mathcal{T}}. \]
Moreover, any coherent sheaf $\mathcal{F}$ is isomorphic to a direct sum $\mathcal{E} \oplus \mathcal{T}$ in a unique fashion. Hence the multiplication map yields isomorphisms

\[(2.2) \quad H_{\text{Bun}(X)} \otimes H_{\text{Tor}(X)} \to H_X, \quad H_{\text{Bun}(X)} \otimes \tilde{H}_{\text{Tor}(X)} \to \tilde{H}_X.\]

Finally, since $\mathcal{T}$ is the only torsion subsheaf of $\mathcal{E} \oplus \mathcal{T}$ of degree $d = \text{deg}(\mathcal{T})$, we may use (2.2) to define a projection $\omega : \tilde{H}_X \to H_{\text{Bun}(X)}$ which satisfies

\[
\omega(u_v u_t \kappa_\alpha) = \begin{cases} 
  u_v & \text{if } u_t = 1 \\
  0 & \text{otherwise}
\end{cases}
\]

where $u_v \in H_{\text{Bun}(X)}$, $u_t \in H_{\text{Tor}(X)}$ and $\alpha \in K_X$.

**Lemma 2.3.** [SV2, Corollary 1.4] Define complex numbers $\psi_k$, for $k \in \mathbb{N}$, by $\omega(\theta_0, k \mathbf{1}_{d+1}) = \xi_k \mathbf{1}_{d+k}$. Then we have

1. for $d \in \mathbb{Z}$ and $k \in \mathbb{N}$,

\[(2.3) \quad \theta_0, k \mathbf{1}_{d+1} = \sum_{s=0}^{k} \xi_s \xi_{1,d+s} \theta_0, k-s,
\]

2. as a series in $\mathbb{C}[[z]]$, we have

\[(2.4) \quad \sum_{k \geq 0} \xi_k z^k = \frac{\zeta_X(z)}{\zeta_X(n^2 z)}.
\]

### 2.3. The Hall algebra of $\text{Coh}_{w,s}(X)$

Let $H := H_{\text{Coh}_{w,s}(X)}$ (resp. $\tilde{H} := \tilde{H}_{\text{Coh}_{w,s}(X)}$) be the Hall algebra(resp. extended Hall algebra) associated to the category $\text{Coh}_{w,s}(X)$. The functor $(-)^* : \text{Coh}(X) \to \text{Coh}_{w,s}(X)$ induces an embedding of $H_X$ into $H$ as a subalgebra. Clearly it holds also for the extended Hall algebras $\tilde{H}_X \subset \tilde{H}$. The Hall algebras $H$ and $\tilde{H}$ have similar features as $H_X$ and $\tilde{H}_X$ do. The fact that the category $\text{Tor}_{w,s}(X)$ is a Serre subcategory of $\text{Coh}_{w,s}(X)$ implies that

\[
\text{H}_{\text{Tor}} = \bigoplus_{\mathcal{T} \in \text{Tor}_{w,s}(X)} \mathbb{C} \mathcal{T}.
\]

The decomposition of $\text{Tor}_{w,s}(X)$ over points of $X$ gives rise to a decomposition at the level of Hall algebras

\[
H_{\text{Tor}} = \bigotimes_{q \in X} H_{\text{Tor}_{w,s}(X)}.
\]

If $q \notin S$, the equivalence $\text{Tor}_{q} \simeq \text{Tor}_{q}$ yields an isomorphism $H_{\text{Tor}_{q}} \simeq H_{\text{Tor}_{q}}$ of bialgebras. For $q \in S$, it follows from [R] and section 1.2.6 that the assignment $E_i \mapsto 1_{C_{x,i}(x_1, x_2, \ldots)}^{(1)}(i \delta_q)$ defines an embedding $U_v^+ (\text{sl}_{w_q}) \hookrightarrow H_{\text{Tor}_{q}}$. This embedding extends to an isomorphism (see [S2, Theorem 4.2])

\[
H_{\text{Tor}_{q}} \simeq U_v^+ (\text{sl}_{w_q}) \otimes_{C_v} \mathcal{Z}
\]

where $\mathcal{Z} = \mathbb{C}[x_1, x_2, \ldots]$ is a central subalgebra and where the element $x_i$ is homogeneous of degree $i \delta_q$. To any $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Pi$ we associate

\[
f_{\lambda,q} = \sum_j 1_{C_{y_j}^{*}(y_q \lambda_j)} \in H_{\text{Tor}_{q}}
\]

and we set

\[
H_{\text{Tor}_{q}}^0 := \bigoplus_{\lambda \in \Pi} C_v f_{\lambda,q}.
\]
The assignment $O_{\lambda_{0},q} := O_{\lambda_{1}}^{(\lambda_{1})} \oplus \cdots \oplus O_{\lambda_{r}}^{(\lambda_{r})} \mapsto f_{\lambda_{q}}$ extends to an algebra isomorphism $\Phi : H_{\mathsf{Tor}}^{0} \cong H_{\mathsf{Tor}}^{0,\mathsf{s}}$. Moreover, $H_{\mathsf{Tor}}^{0,\mathsf{s}}$ is a commutative subalgebra (but not central) of $H_{\mathsf{Tor}}^{0,\mathsf{s}}$ and is freely generated by any of the three sets $\{e_{r,q} : r \in \mathbb{N}^{{\ast}}, \{1_{r,q} : r \in \mathbb{N}^{{\ast}}\}$ and $\{h_{r,q} : r \in \mathbb{N}^{{\ast}}\}$, where $e_{r,q} = \Phi(e_{r,q}), L_{r,q} = f_{(r,q),q}, h_{r,q} = \Phi(h_{r,q})$.

For any $d \geq 0$, we denote by the same symbol the image of the $1_{0,d}$ under the inclusion map $H_{\mathsf{Tor}}(\mathbb{X}) \subset H_{\mathsf{Tor}}(\mathbb{X})$ induced by the embedding $(-)^{\bullet} : \mathsf{Coh} (\mathbb{X}) \hookrightarrow \mathsf{Coh}^{\mathsf{w},S} (\mathbb{X})$ and similarly for $\theta_{0,d}, T_{0,d}$. Denote by $U_{\mathbb{X}}^{0}$ the subalgebra of $H_{\mathsf{Tor}}(\mathbb{X})$ generated by $\{1_{0,d} : d \in \mathbb{N}\}$. Finally, for any $q \in \mathbb{Z}$ and $0 \leq i < w_{q}$, we set $T_{q}(i) := 1_{(w_{q})}(\frac{i}{w_{q}})$ to simplify the notations. Let $U_{\mathbb{X}}^{0}$ be the subalgebra of $H_{\mathsf{Tor}}(\mathbb{X})$ generated by $U_{\mathbb{X}}^{0}$ and $T_{q}(i) : q \in \mathbb{Z}, 0 \leq i < w_{q}$.

Let $H_{\mathsf{Bun}}^{\mathsf{w},S} := H^{\mathsf{w},S} (\mathbb{X})$. Again, $H_{\mathsf{Bun}}^{\mathsf{w}}$ is a sub-algebra of $H$ which is not stable under the coproduct $\Delta^{S}$. The decomposition $\Phi^{\bullet} = \mathcal{E}^{\bullet} \oplus T^{\bullet}$ gives rise to isomorphisms $H_{\mathsf{Bun}}^{\mathsf{w}} \otimes H_{\mathsf{Tor}}^{0,\mathsf{s}} \to H_{\mathsf{Bun}}^{\mathsf{w}} \otimes H_{\mathsf{Tor}}^{0,\mathsf{s}} \otimes K \to H$ defined by the multiplication map. We denote by $\omega$ be the projection $\tilde{H} \to H_{\mathsf{Bun}}^{\mathsf{w}}$, which is nothing but the restriction of functions on vector bundles.

For any $\vec{x} = \vec{x}^{d} + \pi(\vec{x}) \in \mathbb{L}(w)$, let $\mathsf{Pic}^{\mathsf{w},S}(\mathbb{X})$ be the subset of $\mathsf{Pic}^{\mathsf{w},S}(\mathbb{X})$ consisting of all parabolic line bundles of type $\vec{x}$ with respect to the decomposition (1.15) and we set

$$1^{\mathsf{ss}}_{\mathsf{Bun}}(\vec{x}) = 1^{\mathsf{ss}}_{\mathsf{Bun}}(\pi(\vec{x})) = \sum_{\mathcal{L}^{\bullet} \in \mathsf{Pic}^{\mathsf{w},S}(\mathbb{X})} 1_{\mathcal{L}^{\bullet}}.$$ 

We may write $1^{\mathsf{ss}}_{\mathsf{Bun}}(\vec{x}) = 1^{\mathsf{ss}}_{\mathsf{Bun}}(0)$ which is the image of $1^{\mathsf{ss}}_{\mathsf{Bun}}$ via the inclusion $H_{\mathsf{Bun}}(\mathbb{X}) \subset H_{\mathsf{Bun}}^{\mathsf{w}}$. Denote by $U_{\mathcal{S}}$ the subalgebra of $H$ generated by $\{1_{1,d}^{\mathsf{ss}}(\vec{x}) : d \in \mathbb{Z}, \vec{x} \in \prod_{i \in S} \mathbb{Z} / w_{i} \mathbb{Z}\}$. We define analogously the spherical Hall algebra of $H_{\mathsf{Bun}}^{\mathsf{w},S}(\mathbb{X})$ to be the subalgebra $U$ of $H$ generated by $U^{0}$ and $U^{\mathsf{w}}$. Similarly for $U_{\mathcal{S}}$.

2.4. Some commutation relations.

Lemma 2.4. (BS, Lemma 5.16) For $p \in S$, we set $T^{(na_{p} + j)}_{p}(i) = 1_{(w_{p} + j)}(i)$ for $0 \leq j \leq p$. Then,

$$T^{(na_{p} + j)}_{p}(i) = \begin{cases} T_{p}^{(j)}(i), & \text{if } n = 0 \\ 1_{(O_{p}^{(a_{p})})^{\bullet} \cdot i_{p}} T_{p}^{(j)}(i) - v^{2 \deg(p)} T_{p}^{(j)}(i) 1_{(O_{p}^{(a_{p})})^{\bullet} \cdot i_{p}}, & \text{if } n > 0 \end{cases}$$

Proposition 2.5.

(2.6) $H(1^{\mathsf{ss}}_{1,0}) = 1^{\mathsf{ss}}_{1,0} \otimes 1 + \sum_{d \geq 0, \vec{x} \in \prod_{i \in S} \mathbb{Z} / w_{i} \mathbb{Z}} \theta_{0,d,\vec{x}} \kappa_{1,-d,-\vec{x}} \otimes 1^{\mathsf{ss}}_{1,-d,\vec{x}}.$

where

$$\theta_{0,d,\vec{x}} = \sum_{S' = S \cup S'_{2}} \left( \prod_{p \in S'_{1}} v^{- \deg(p)} T_{p}^{(a_{p})}(0) \times \theta_{0,d} \prod_{p \in S'_{2}} v^{\deg(p)} T_{p}^{(a_{p})}(0) \right)$$

where, write $\vec{x} = \sum_{p \in S} a_{p} \vec{x}_{p} \in \prod_{i \in S} \mathbb{Z} / w_{i} \mathbb{Z}$, $S' = \{p \in S : \text{a}_{p} \neq 0\}.$
Moreover, and similarly by the adjointness of \((L^\bullet)^\ast\). Note that a morphism \(\theta\)
the Hall number \(P\) (2.9)
\[ S \rightarrow (2.8) 0 \]
Now, using the formula of Lemma 2.4 and the fact (c.f. [S1, Example 4.1 2]) that
where the second sum rangers over all distinct points \(x\).
\[ \sum\limits_{i=1}^n (1 - v^{2 \deg(x_i)}) 1_{\mathcal{O}_{x_i}^{(n_i)}} \]
where the second sum ranges over all distinct points \(x_1, \ldots, x_m \in X\) and positive integers \(n_1, \ldots, n_m\) such that \(\sum_{i=1}^m n_i \deg(x_i) = d\), we deduce the formula (2.7).

We set the quantum commutator \([a, b] = ab - v^{-(a,b)_S} ba\), where \((a, b)_S\) is the symmetrized Euler form for homogeneous elements \(a\) and \(b\). Note that if \([b, c] = [b, c] = 0\), then we have
\[ [[a, b], c] = [[[a, c], b] \]
and similarly if \([a, c] = [a, c] = 0\), then
\[ [[a, b], c] = [a, [b, c]]. \]
And in general, we have
\[ [[a, b], c] = [a, [b, c]] + v^{- (b, c)_S} [a, c] b - v^{-(a, b)_S} b [a, c]. \]
\[ [ab, c] = a [b, c] + v^{- (b, c)_S} [a, c] b. \]

**Lemma 2.6.** For any \(p \in S\), \(0 \leq i < w_p\) and \(0 < j < w_p\), we have
\[ v_{p}^{j-1}T_{p}^{(j)}(i) = \cdots \cdots \cdots \cdots \cdots \cdot T_{p}^{(1)}(i - j) \cdots T_{p}^{(1)}(i). \]
Proof. Using the equivalence $\text{Tor}_p^{w_*} S \simeq C_{w_*}$, we are reduced to a computation in the context of quivers. Let us assume that $i < j$. For any $1 \leq k < j$ we have, from lemma 1.4,

$$\langle O_p^{(i)}(0), O_p^{(j)}(-k) \rangle_{S} = 0,$$

$$(O_p^{(i)}(0), O_p^{(j)}(-k) \rangle_{S} = -\text{deg}(p),$$

$$\text{Hom}_{\text{Coh}^{w_*}(X)}(O_p^{(i)}(0), O_p^{(j)}(-k) \rangle = \text{Hom}_{\text{Coh}^{w_*}(X)}(O_p^{(i)}(-k), O_p^{(j)}(0)) = 0,$$

$$\text{Ext}_{\text{Coh}^{w_*}(X)}(O_p^{(i)}(-k), O_p^{(j)}(0)) = 0,$$

$$\text{Ext}_{\text{Coh}^{w_*}(X)}(O_p^{(i)}(0), O_p^{(j)}(-k) \rangle = k_p.$$ 

Thus

$$1 O_p^{(i)}(0) 1 O_p^{(j)}(-k) = v_p(1 O_p^{(i)}(0) \oplus O_p^{(j)}(-k), 1 O_p^{(i)}(0), 1 O_p^{(j)}(-k)),$$

$$1 O_p^{(i)}(-k) 1 O_p^{(j)}(0) = 1 O_p^{(i)}(-k) 1 O_p^{(j)}(0).$$

Hence $T_p^{(i)}(0) T_p^{(j)}(-k) = v_p T_p^{(i)}(-k) T_p^{(j)}(0) = v_p T_p^{(i)}(0)$ and the lemma follows by induction and shifting. 

$\square$

Lemma 2.7. For any $p \in S$, $0 \leq i < k < w_p$, $d \in \mathbb{Z}$, $0 < j < w_p$, we have the following relations:

$$\omega(T_p^{(i)}(v_p^{1} 1_{1,0}^{ss}(k_jp))) = \begin{cases} v_p^{1} 1_{1,0}^{ss}(k+j)p) & \text{if } i-k \equiv j \mod w_p \\ 0 & \text{otherwise} \end{cases}$$

where $v_p = v^\text{deg}(p)$. 

Proof.\quad Up to a shift, we can always assume the case $d = k = 0$. As explained in the proof of lemma 1.4, if $i \neq 0, 1$, we have

$$\langle O^*, O_p^{(i)}(i\epsilon_p) \rangle_{S} = 0,$$

$$\text{Hom}_{\text{Coh}^{w_*}(X)}(O^*, O_p^{(i)}(i\epsilon_p)) = \text{Hom}_{\text{Coh}^{w_*}(X)}(O^*, O_p^{(i)}(i\epsilon_p)) = 0,$$

$$\text{Ext}_{\text{Coh}^{w_*}(X)}(O_p^{(i)}(i\epsilon_p), O^*) = 0,$$

And it is also true for any $(L)^*$ with $L \in \text{Pic}^0(X)$ instead of $O^*$. Hence we have

$$[T_p^{(i)}(i), 1_{1,0}^{ss}] = [T_p^{(i)}(i), 1_{1,0}^{ss}] = 0.$$

For the case $i = 0$, we have

$$\langle O^*, O_p^{(i)}(0) \rangle_{S} = 0,$$

$$\text{Hom}_{\text{Coh}^{w_*}(X)}(O^*, O_p^{(i)}(0)) = \text{Ext}_{\text{Coh}^{w_*}(X)}(O^*, O_p^{(i)}(0)) = 0,$$

$$\text{Ext}_{\text{Coh}^{w_*}(X)}(O^*, O_p^{(i)}(0)) = 0,$$

Thus

$$T_p^{(i)}(0) 1 O^* = v_p^{-2} 1 O^* \oplus O_p^{(i)}(0),$$

$$1 O^* T_p^{(i)}(0) = v_p^{-1} 1 O^* \oplus O_p^{(i)}(0).$$

The same is true for any parabolic line bundle $(L)^*$ with $L \in \text{Pic}^0(X)$ instead of $O^*$ and hence $T_p^{(i)}(0) 1_{1,0}^{ss} = v_p^{-1} 1_{1,0}^{ss} T_p^{(i)}(0)$. Similarly, for $i = 1$, we have

$$T_p^{(i)}(1) 1 O^* = v_p(1 O^* \oplus O_p^{(i)}(\epsilon_p) + 1 O^*(\epsilon_p),$$

$$1 O^* T_p^{(i)}(1) = 1 O^* \oplus O_p^{(i)}(\epsilon_p).$$

Therefore we have

$$T_p^{(i)}(1) 1_{1,0}^{ss} - v_p 1_{1,0}^{ss} T_p^{(i)}(1) = v_p 1_{1,0}^{ss}(\vec{x}_p).$$

To summarize,
\[ [T_p^{(1)}(i), 1_{SS}^{i,s}(k\vec{x}_p)] = \begin{cases} 
0 & \text{if } i \not\equiv k + 1 \mod w_p, \\
vp_{p, i} 1_{SS}^{i,s}((k + 1)\vec{x}_p) & \text{if } i \equiv k + 1 \mod w_p. 
\end{cases} \]

Now let us compute \([T_p^{(1)}(j), 1_{SS}^{i,s}]\) using lemma 2.6. If \(i = 0\), \(1_{SS}^{i,s}\) commutes with all \(T_p^{(1)}(0 - j + 1)\) for all \(0 < j < w_p\) and \((O^\bullet, T_p^{(1)}(0 - j + 1)) = 0\) for all \(j\). Thus
\[
\llbracket \cdots [T_p^{(1)}(0), T_p^{(1)}(0 - j + 1)] \cdots [T_p^{(1)}(0), 1_{SS}^{i,s}] \rrbracket
= \llbracket \cdots [T_p^{(1)}(0), 1_{SS}^{i,s}] \rrbracket [T_p^{(1)}(0), [T_p^{(1)}(0), 1_{SS}^{i,s}]]
= \llbracket \cdots [T_p^{(1)}(0), 1_{SS}^{i,s}] \rrbracket [T_p^{(1)}(0), [T_p^{(1)}(0), 1_{SS}^{i,s}]]
= \llbracket \cdots [T_p^{(1)}(0), 1_{SS}^{i,s}] \rrbracket [T_p^{(1)}(0), [T_p^{(1)}(0), 1_{SS}^{i,s}]]
= \cdots = v_p^{i-1} 1_{SS}^{1,0}(i\vec{x}_p) T_p^{(1)}(0) + v_p^{i+1} T_p^{(1)}(0) 1_{SS}^{i,s}(i\vec{x}_p)
= (v_p^{i-1} - v_p^{i+1}) v_p^{i} 1_{SS}^{1,0}(i\vec{x}_p) T_p^{(1)}(0),
\]
where the last equality holds since \(i \not\equiv 0, w_p - 1\) in the cases. Finally, we only need to compute
\[
[\cdots [1_{SS}^{i,s}(i\vec{x}_p) T_p^{(1)}(0), T_p^{(1)}(i - j + 1)] \cdots [T_p^{(1)}(i - j + 1)]
= [\cdots [1_{SS}^{i,s}(i\vec{x}_p) T_p^{(1)}(0), T_p^{(1)}(i - j + 1)] + v_p [1_{SS}^{i,s}(i\vec{x}_p), T_p^{(1)}(i - j + 1)] T_p^{(1)}(0)] T_p^{(1)}(0)
= \cdots T_p^{(i - j + 1)}
= [\cdots [1_{SS}^{i,s}(i\vec{x}_p) T_p^{(1)}(0), T_p^{(1)}(i - j + 1)] T_p^{(1)}(0)] T_p^{(1)}(0)
= \cdots = 1_{SS}^{1,0} T_p^{(1)}(0), \cdots, T_p^{(1)}(i - j + 1),
\]
where the third equality and the later ones hold since \(i < w_p + i - j\).

To summarize, we have
\[
[v_p^{j-1} T_p^{(j)}(i), 1_{SS}^{i,s}] = \begin{cases} 
0 & \text{if } j = i, \\
v_p^{j-1} 1_{SS}^{i,s}(i\vec{x}_p) & \text{if } 0 < i < j, \\
v_p^{i+1} v_p^{2i-1} 1_{SS}^{i,s}(i\vec{x}_p) T_p^{(j-i)}(0) & \text{otherwise}.
\end{cases}
\]

The lemma follows by shifting. \(\square\)
Lemma 2.8. Let $d \in \mathbb{N}$, $e \in \mathbb{Z}$ and $\tilde{x}, \tilde{y}, \tilde{z} \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z}$. Then $\omega(\theta_{0,d,\tilde{x}}(\tilde{z})1_{1,e}^s(\tilde{y}))$ does not vanish only if $z_p - y_p - x_p \mod w_p$ for all $p \in S$. In this case, we have

$$
\omega(\theta_{0,d,\tilde{x}}(\tilde{z})1_{1,e}^s(\tilde{y})) = \prod_{p \in S} v_p(v_p^{-1} - v_p)\xi_d 1_{1,s+d}(\tilde{y} + \tilde{x}).
$$

Proof. Applying the lemma 2.7 to the expression of $\theta_{0,d,\tilde{x}}$ in the lemma 2.7, the necessary condition for $\omega(\theta_{0,d,\tilde{x}}(\tilde{z})1_{1,e}^s(\tilde{y})) \neq 0$ is clear. So we assume that $\tilde{z} - \tilde{y} \neq 0$ and $z_p - y_p - x_p \mod w_p$ for all $p \in S$. Then

$$
\omega(\theta_{0,d,\tilde{x}}(\tilde{z})1_{1,e}^s(\tilde{y})) = \sum_{S' = S_1 \cup S_2} \left( \prod_{p \in S_1} v_p^0 \prod_{p \in S_2} (-v_p^2) \right)\xi_d 1_{1,s+d}(\tilde{y} + \tilde{x})
= \left( \prod_{p \in S} v_p^0(v_p^{-1} - v_p) \right)\xi_d 1_{1,s+d}(\tilde{y} + \tilde{x}),
$$

where $S' = \{ p \in S | z_p - y_p \neq 0 \}$. 

2.5. The constant term map. Let us introduce the so-called constant term map as follows. For $r \geq 1$, we set

$$
J_r : \mathbb{U}^G_\Sigma^{[r]} \rightarrow \mathbb{U}^{[1]} \otimes \cdots \otimes \mathbb{U}^{[1]}
$$

and denote by $J : \mathbb{U}^G \rightarrow \bigoplus_r (\mathbb{U}^{[1]} \otimes \cdots \otimes \mathbb{U}^{[1]})^{\hat{G}}$ the sum of the maps $J_r$. Writing

$$
J(u) = \sum_{\mathcal{L}_1^*, \ldots, \mathcal{L}_r^*} u(\mathcal{L}_1^*, \ldots, \mathcal{L}_r^*)1_{\mathcal{L}_1^*} \otimes \cdots \otimes 1_{\mathcal{L}_r^*},
$$

we have

$$
u(\mathcal{L}_1^*, \ldots, \mathcal{L}_r^*) = \frac{1}{(l - 1)^r} (J(u), 1_{\mathcal{L}_1^*} \otimes \cdots \otimes 1_{\mathcal{L}_r^*})_G = \frac{1}{(l - 1)^r} (u, 1_{\mathcal{L}_1^*} \otimes \cdots \otimes 1_{\mathcal{L}_r^*})_G
$$

which coincides with the standard notion of constant term in the theory of automorphic forms (up to the factor $(l - 1)^{-r}$). Since the image of $J_r$ lies in $(\mathbb{U}^{[1]} \otimes \cdots \otimes \mathbb{U}^{[1]})^{\hat{G}}$ and $\mathbb{U}^{[1]} = \bigoplus (\mathbb{U}^{[1]} \otimes \mathbb{U}^{[1]})$, the function $u(\mathcal{L}_1^*, \ldots, \mathcal{L}_r^*)$ only depends on the respective degree $(d_1, \tilde{x}_1), \ldots, (d_r, \tilde{x}_r)$ of the parabolic line bundles $\mathcal{L}_1^*, \ldots, \mathcal{L}_r^*$.

Lemma 2.9. [SV2, Lemma 1.5] The constant term map $J : \mathbb{U}^G \rightarrow \bigoplus_r (\mathbb{U}^{[1]} \otimes \cdots \otimes \mathbb{U}^{[1]})^{\hat{G}}$ is injective.

Let $\mathcal{G}_r$ be the symmetric group on $r$ letters. If $w \in \mathcal{G}_r$ and $P(z_1, \ldots, z_r)$ a function in $r$ variables, then we set $wP(z_1, \ldots, z_r) = P(z_{w(1)}, \ldots, z_{w(r)})$. Let

$$
Sh_{r,s} = \{ w \in \mathcal{G}_{r+s} | w(i) < (j) \text{ if } 1 \leq i < j \leq r \text{ or } r < i < j \leq r + s \}
$$

be the set of $(r,s)$-shuffles, i.e., the set of minimal length representatives of the left cosets in $\mathcal{G}_{r+s}/\mathcal{G}_r \times \mathcal{G}_s$. For any $w \in Sh_{r,s}$, write

$$
I_w = \{ (i,j) \mid 1 \leq i < j \leq r + s, \ w^{-1}(i) > r \geq w^{-1}(j) \}.
$$

Let us first compute explicitly the $J_2$. For any $\tilde{z}_i = (z_{i,p})_{p \in S} \in \prod_p \mathbb{Z}/w_p \mathbb{Z}$ for $i = 1, 2$, we set

$$
S_{\tilde{z}_1, \tilde{z}_2} = \{ p \in S | z_{1,p} - z_{2,p} \neq 0 \}.
$$

We have

$$
J_2(1_{1,d_1}(\tilde{z}_1)1_{1,d_2}(\tilde{z}_2)) = \omega^{\otimes 2} \sum_{\sigma \in \mathcal{G}_2} \tilde{\Delta}_{\sigma^{-1}(1)}(1_{1,d_1}(\tilde{z}_1))1_{1,d_2}(\tilde{z}_2)) = \omega^{\otimes 2} \sum_{\sigma \in \mathcal{G}_2} \tilde{\Delta}_{\sigma^{-1}(1)}(1_{1,d_1}(\tilde{z}_1))\tilde{\Delta}_{\sigma^{-1}(2)}(1_{1,d_2}(\tilde{z}_2)),
$$

(2.19)
where by definition \((\delta_1, \delta_2)\) is the standard bases of \(\mathbb{Z}^2\) and in above we have made use of the fact that \(\tilde{\Delta}\) is a morphism of algebras. Using (2.7) we get

\[
\tilde{\Delta}_{d_{k+1}}(1_{1,d_1}^{ss} (\tilde{z})) = \sum_{d_1, \ldots, d_k} \sum_{x_1, \ldots, x_k} \left( \theta_{0, d_1, x_1} (\tilde{z}) \kappa_{1, d - d_1 + \tilde{z} - \bar{x}_1} \otimes \cdots \right.
\]

\[
\otimes \theta_{0, d_k, x_k} (\tilde{z} - \sum_{j < k} \bar{x}_j') \kappa_{1, d + \tilde{z} - \sum_{j=1}^{s} d_j + x_j} \otimes \cdots
\]

\[
\otimes 1_{1, d - \sum_{j=1}^{s} d_j} (\tilde{z} - \sum_{j=1}^{d} 1) \otimes 1 \otimes \cdots \otimes 1
\]

(2.20)

where the sum is taken over all \(d_1, \ldots, d_k \geq 0\) and \(\bar{x}_1, \ldots, \bar{x}_k \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z}\). Thus the lemma 2.8 yields

\[
\omega^{\otimes 2} \left( \tilde{\Delta}_{d_1} (1_{1, d_1}^{ss} (\tilde{z}_1)) \tilde{\Delta}_{d_2} (1_{1, d_2}^{ss} (\tilde{z}_2)) + \tilde{\Delta}_{d_3} (1_{1, d_3}^{ss} (\tilde{z}_3)) \tilde{\Delta}_{d_4} (1_{1, d_4}^{ss} (\tilde{z}_4)) \right)
\]

\[
= 1_{1, d_1}^{ss} (\tilde{z}_1) \otimes 1_{2, d_2}^{ss} (\tilde{z}_2) + \omega^{\otimes 2} \left( \sum_{s_1, \bar{x}_1} \theta_{0, s_1, x_1} (\tilde{z}_1') \kappa_{1, d_1 - s_1 + \tilde{z}_1 - \bar{x}_1} 1_{1, d_2}^{ss} (\tilde{z}_2) \otimes 1_{1, d_3 - s_1} (\tilde{z}_3 - \bar{x}_1) \right)
\]

where the sum is taken over all \(s_1 \geq 0\) and \(\bar{x}_1 \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z}\). By Lemma 2.8, the summand will not vanish under \(\omega\) if and only if \(x_{1,p} \equiv s_1 - z_2, p \mod w_p\) for \(p\) lies in some subset \(S' \subset S_{\bar{z}_1 - \bar{z}_2}\) and \(x_{1,p} = 0\) for \(p \in S\setminus S'\). For any subset \(S' \subset S_{\bar{z}_1 - \bar{z}_2}\), we have

\[
\langle O^* (\bar{z}_1 - \bar{x}_1), O^* (\bar{z}_2) \rangle = \langle O^* (\bar{z}_2 - \bar{z}_1 + \bar{x}_1) \rangle = 2(1 - g_{\bar{z}}) - \sum_{p \in S_{\bar{z}_1 - \bar{z}_2} \setminus S'} 1
\]

Therefore (2.21) becomes

\[
1_{1, d_1}^{ss} (\tilde{z}_1) \otimes 1_{2, d_2}^{ss} (\tilde{z}_2) + v^{2g_{\bar{z}} - 2} \sum_{s_1, \bar{x}_1} \xi_{s_1} \cdot \sum_{S' \subset S_{\bar{z}_1 - \bar{z}_2}} \left( \prod_{p \in S'} (1 - v_p^2) \prod_{p \in S_{\bar{z}_1 - \bar{z}_2} \setminus S'} v_p \right)
\]

\[
\cdot 1_{1, d_2 - s_1}^{ss} (\tilde{z}_2) \otimes \sum_{p \in S'} x_{1,p} \bar{x}_p \otimes 1_{1, d_3 - s_1} (\tilde{z}_3 - \bar{x}_1)
\]

(2.22)

where \(x_{1,p} \equiv z_{1,p} - z_{2,p} \mod w_p\) for all \(p \in S_{\bar{z}_1 - \bar{z}_2}\). We introduce the automorphism \(\gamma\) of \(U^\Lambda [1]\) by \(\gamma(1_{1,d}^{ss}) = 1_{1,d+1}^{ss}\) and let us denote by \(\gamma_i\) the operator \(\gamma\) acting on the \(i\)th component of the tensor product. We also define \(\gamma_p (1_{1,d}^{ss} (\tilde{z})) = 1_{1,d}^{ss} (\tilde{z} + \bar{x}_p)\) for all \(p \in S\) and similarly for \(\gamma_i p\)'s. Finally, we set

\[
\Gamma_p^a (\gamma_i) = \begin{cases} v_p + (1 - v_p^2) (\frac{\gamma_{i,p}}{v_p})^a & \text{if } a \neq 0, \\ 1 & \text{if } a = 0 \end{cases}
\]

Using these notions and lemma 2.3, we can rewrite above as

\[
v^{2g_{\bar{z}} - 2} \left( \sum_{s_1, \bar{x}_1} \xi_{s_1} \left( \frac{\gamma_{1}}{\gamma_{2}} \right)^{s_1} \prod_{p \in S} \Gamma_p^{\gamma_{1,p} / \gamma_{2}} \right) \prod_{p \in S} \Gamma_p^{\gamma_{1,p} / \gamma_{2}} 1_{1, d_1}^{ss} (\tilde{z}_1) \otimes 1_{1, d_1}^{ss} (\bar{z}_1)
\]

\[
= v^{2g_{\bar{z}} - 2} \frac{\xi_{S} (\gamma_1 \gamma_2 - 1)}{\xi_{S} (v^2 \gamma_1 \gamma_2 - 1)} \prod_{p \in S} \Gamma_p^{\gamma_{1,p} / \gamma_{2}} 1_{1, d_1}^{ss} (\tilde{z}_1) \otimes 1_{1, d_1}^{ss} (\bar{z}_1)
\]

(2.24)

Put \(h_S (z) = v^{2g_{\bar{z}} - 2} \frac{\xi_{S} (z)}{\xi_{S} (v^2 z)}\) and \(\Gamma_p (z) = \prod_{p \in S} \Gamma_p^{\gamma_p / \gamma_{2}} (z_p)\). By (2.20) and the above computation for \(J_2\), we have

\[
J_r (1_{1,d_1}^{ss} (\tilde{z}_1) \cdots 1_{1,d_r}^{ss} (\tilde{z}_r)) = \sum_{\sigma \in \Sigma_r} \prod_{(i,j) \in I_r} h_{S_{\sigma, i,j}} (\gamma_{1,j} / \gamma_{2,j}) \Gamma_{x_{i,j}} (\gamma_{1,j} / \gamma_{2,j}) \sigma (1_{1,d_1}^{ss} (\bar{z}_1) \cdots 1_{1,d_r}^{ss} (\bar{z}_r))
\]

where \(\bar{x}_{i,j} = \pi (\bar{z}_{\sigma, i,j} - \bar{z}_{\sigma, i,j})\).
2.6. Shuffle presentation. Let \( h(t) \in \mathbb{C}(t) \) be a fixed rational function. We consider the so-called shuffle algebra \( F_h(t) \) as follows. As a vector space

\[
F_h(t) = \bigoplus_r \mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}][(t_1/t_2, \ldots, t_{r-1}/t_r)],
\]

and the multiplication is given by

\[
P(t_1, \ldots, t_r) \ast Q(t_1, \ldots, t_s) = \sum_{\sigma \in \text{Sh}_r,s} h_{\sigma}(t_1, \ldots, t_{r+s}) \sigma(P(t_1, \ldots, t_r)Q(t_{r+1}, \ldots, t_{r+s}))
\]

where

\[
h_{\sigma}(t_1, \ldots, t_{r+s}) = \prod_{(i,j) \in I_{\sigma}} h(t_i/t_j)
\]

and the rational function \( h(t_i/t_j) \) is developed as a Laurent series in \( t_1/t_2, \ldots, t_{r-1}/t_r \).

We equip also \( F_h(t) \) with a coproduct \( \Delta : F_h(t) \to F_h(t) \otimes F_h(t) \) defined by

\[
\Delta_{m,n}(t_1^{i_1} \cdots t_r^{j_r}) = \prod_i t_i^{i_1} \cdots t_i^{i_{m+n}} \otimes \sum_{\sigma \in \text{Sh}(m_n)} h_{\sigma}(t_1, \ldots, t_{r+m+n}) \sigma(P(t_1, \ldots, t_r)Q(t_{r+1}, \ldots, t_{r+m+n})),
\]

\( \Delta = \bigoplus_{m,n} \Delta_{m,n} \).

Let \( S_{h(t)} \) be the subalgebra of \( F_h(t) \) generated by the degree one component \( F_h(t)[1] = \mathbb{C}[t_r^{\pm 1}] \).

Now let us consider another shuffle algebra as follows. Let \( g(t) \in \mathbb{C}(t) \) be a rational function. Again, for \( r \geq 1 \), we put \( g(t_1, \ldots, t_r) = \prod_{1 \leq j \leq r} g(t_i/t_j) \). Denote by

\[
\text{Sym}_r : \mathbb{C}(t_1, \ldots, t_r) \to \mathbb{C}(t_1, \ldots, t_r)^{\mathfrak{S}_r}, \quad P(t_1, \ldots, t_r) \mapsto \sum_{w \in \mathfrak{S}_r} wP(t_1, \ldots, t_r)
\]

the standard symmetrization operator and consider the weighted symmetrization

\[
\Xi_r : \mathbb{C}[t_1^\pm, \ldots, t_r^\pm] \to \mathbb{C}(t_1, \ldots, t_r)^{\mathfrak{S}_r}, \quad P(t_1, \ldots, t_r) \mapsto \text{Sym}_r(g(t_1, \ldots, t_r)P(t_1, \ldots, t_r)).
\]

Let \( A_r \) be the image of \( \Xi_r \) and \( A_{g(t)} = \mathbb{C}1 \oplus \bigoplus_{r \geq 1} A_r \). We endow the space \( A_{g(t)} \) with the structure of an associative algebra with the product defined as

\[
P(t_1, \ldots, t_r) \ast Q(t_1, \ldots, t_s) = \sum_{w \in \text{Sh}_r,s} w(P(t_1, \ldots, t_r)Q(t_{r+1}, \ldots, t_{r+s})),
\]

where the product ranges of all the \((i,j)\) with \( 1 \leq i \leq r \) and \( r+1 \leq j \leq r+s \). Note the by construction the algebra \( A_{g(t)} \) is generated by the subspace \( A_1 = \mathbb{C}[t_r^{\pm 1}] \).

For any \( \tilde{x} \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z} \) fixed, we denoted by \( U_X^S(\tilde{x}) \) the subalgebra of \( U^S \) generated by \( 1_{1,d}^e(\tilde{x}) \) for all \( d \in \mathbb{Z} \). Let \( g_X(t) = \) be a rational function satisfying the equation \( h_X(t) = g_X(t^{-1})/g_X(t) \). If \( \tilde{x} = 0 \), we get \( U_X(0) = U_X^S \). One can deduce from the above computations or [SV2, Proposition 1.6] with shifting that for any \( \tilde{x} \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z} \), the assignment \( 1_{1,d}(\tilde{x}) \mapsto t_1^d \mapsto t_1^d \) in degree one extends to algebra isomorphism

\[
U_X^S(\tilde{x}) \simeq S_{h_X}(t) \simeq A_{g_X}(t).
\]

Using the functional equation for zeta functions

\[
\zeta_X(v^2t) = (vt)^2 g_X(vt^{-1}) \zeta_X(t^{-1}),
\]

one can check that \( t^{g_X-1} \zeta_X(t^{-1}) \) is a solution of the equation \( h_X(t) = g_X(t^{-1})/g_X(t) \). The same is also true of \( t^{g_X-1}k(t) \) for any function \( k(z) \) satisfying \( k(t) = k(t^{-1}) \). It will be more convenient to set

\[
\zeta_X(t) = \zeta_X(t)(1 - v^{-2}t)(1 - v^{-2}t^{-1}) = \frac{1 - v^{-2}t^{-1}}{1 - z} \prod_{i=1}^{g_X} (1 - \alpha_i t)(1 - \overline{\alpha_i} t),
\]

\[
g_X(t) = t^{g_X-1} \zeta_X(t^{-1}).
\]
We now fix the above choice for $g_x(t)$ and write $A = A_{g_x(t)}$. Then we have an algebra isomorphism $U^> (\vec{x}) \simeq A$ for all $\vec{x} \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z}$.

Now, consider the group algebra $\mathbb{C}[L(w)]$ of $L(w)$ which can be described as

$$C[L(w)] = C[ t_p^{\pm 1}, t_p^{\pm 1} ] / J$$

where $J$ is generated by $t_p^{w_1} - t_p^{w_j}$ for all $i \neq j$ and we put $t = t_p^{w_1}$ for all $p \in S$. For any $C$ and $\vec{x}$, we denote the subalgebra $\mathbb{C}[t^{\pm 1}]$ of $\mathbb{C}[L(w)]$ by $\mathbb{C}[t^{\pm 1}] = \mathbb{C}[t^{\pm 1}] \otimes \mathbb{C}[L(w)]$. There is a natural embedding $\mathbb{C}[t^{\pm 1}] \hookrightarrow \mathbb{C}[L(w)]$ by $t \mapsto t$. We can identify $U^> [1]$ with $C[L(w)]$ via the assignment

$$1_{1,d}(\vec{x}) \mapsto t_{d+1}^{x_1} \cdots t_{d+1}^{x_n}.$$

Thus we have

$$\left( U^> [1]^{\otimes r} \cong C[L(w)]^{\otimes r} \right)$$

(2.33)

$$\Gamma_{x^p} (t_p) = \begin{cases} v + (1 - v^2) t_p^{x_p} & \text{if } x_p = 1, \ldots, w_p - 1 \\ 1 & \text{if } x_p = 0 \end{cases}$$

(2.34)

and

$$\Gamma_\vec{x} (t) = \prod_{p \in S} \Gamma_{x_p} (t_p) \in \mathbb{C}[L(w)]$$

(2.35)

For $r \geq 1$ and $\vec{x} \in \left( \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z} \right)^r$, we put

$$\Gamma^\sigma_{x^p, S} (t_1, \ldots, t_r; \vec{v}) = \prod_{(i,j) \in I_{\sigma}} \prod_{p \in S} \Gamma_{(x_{p(i-1)} - x_{p(i-1)})} (t_i, t_j, \vec{v})$$

(2.36)

Now we define a shuffle-like (called weighted shuffle) algebra on the vector space

$$\mathbf{F}^w_{h(t)} = \bigoplus_{r \geq 1} C[L(w)]^{\otimes r}$$

(2.37)

with the multiplication defined as

$$I^{d+1 + \vec{v}_1, \ldots, d+1 + \vec{v}_s} \ast I^{1 + \vec{y}_1, \ldots, 1 + \vec{y}_s}$$

(2.38)

and extended linearly to the whole $\mathbf{F}^w_{h(t)}$. Its coalgebra structure is given as (2.29). Let us denote the subalgebra $S_{x^p, S}^{w, S}$ of $\mathbf{F}^w_{h(t)}$ generated by the degree one component $\mathbf{F}^w_{h(t)} [1] = C[L(w)]$. Then the following theorem holds by the computations in the section 2.5.

**Theorem 2.10.** For any $\vec{x} = (\vec{x}_p)_{p \in S} \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z}$, the assignment $1^w_{1,d}(\vec{x}) \mapsto t_{d+1}^{x_i}$ in degree one extends to an algebra isomorphism

$$U^>_S \cong S^{w, S}_{h_x(t)}.$$

(2.39)
Note that the algebra $\mathcal{S}_h^S$ is just a shuffle-like algebra since $\Gamma$ is not a rational function anymore but an operator which depends on the "weight" sequences $\mathbf{a}$. However, if once we fix a weight sequence, the structure functions will become rational which is the best the author can get. The other remark is that, even we fix a weight sequence, the structure functions are not a ratio of some rational functions. The author doesn’t know yet how to construct a similar algebra structure for $A_g(t)$.

2.7. The generic form. Observe that the algebra $U^r = U^r_{w,S,X}$ only depend on the genus $g_X$, the Weil numbers $\alpha_1, \ldots, \alpha_{2g_X}$ of $X$, the number of marked points and their weights $(w,S)$. It possesses a generic form in the following sense. Let us fix $g \geq 0$ and consider the torus
\[ T_g = \{(\eta_1, \ldots, \eta_{2g}) \in (\mathbb{C}^\times)^{2g} | \eta_{2i-1}\eta_{2i} = \eta_{2j-1}\eta_{2j}, \; \forall i,j\}. \]
The group
\[ W_g = \mathfrak{S}_g \ltimes (\mathfrak{S}_2)^g \]
naturally acts on $T_g$ and the collections $\{\alpha_1, \ldots, \alpha_{2g}\}$ defines a canonical element $\alpha_X$ in the quotient $T_g(\mathbb{C})/W_g$. Let $R_g = \mathbb{Q}[T_g]^{W_g}$ and $K_g$ be its localization at the multiplicative set generated by $\{l^s l|s \geq 1\}$ where by definition $l(\alpha_1, \ldots, \alpha_{2g}) = \alpha_{2i-1}\alpha_{2i}$ for any $1 \leq i \leq g$. For any choice of smooth projective curve $X$ of genus $g_X = g$ there is a natural map $K_g \to \mathbb{C}$, $f \mapsto f(\alpha_X)$. We define, using the construction of $F_h^S$, the $K_g$-algebra $U^r_{w,S}$. Note that $U^0 = U^0_{w,S,X}$ has an obvious generic form $U^0_{w,S,K_g}$. We can define also $U_{w,S,K_g} = U^0_{w,S,K_g} \otimes U^r_{w,S,K_g}$. The bialgebra structure and Green's bilinear form both depend polynomially on the $\{\alpha_1, \ldots, \alpha_{2g}\}$ and hence may be defined over $K_g$. Let $U_{w,s,K_g}$ be the $R_g$-subalgebra of $U_{w,S,K_g}$ generated by $R_g[\mathfrak{L}(w)] \subset U^r_{w,S,K_g}$. By construction $U_{w,s,K_g}$ is a torsion-free integral form of $U^r_{w,S,K_g}$ in the sense that $U_{w,s,K_g} \otimes R_g = U_{w,s,K_g}$. Moreover, there exists a natural specialization map
\[ (\cdot)_{\mathbb{C}} : U_{w,s,R_g} \to U^r_{w,s,K_g}, \quad t_i^{d+\bar{x}} \mapsto 1_{s,i_d}(\bar{x}) \]
to a fixed curve $X$ of genus $g_X = g$. To summarize,

**Theorem 2.11.** There exists an $R_g$-Hopf algebra $U_{w,s,R_g}$ equipped with a Hopf pairing
\[ (\cdot, \cdot) : U_{w,s,R_g} \otimes U_{w,s,R_g} \to K_g \]
generated by elements $1_{R_g,0,p}, 1_{R_g,1,\bar{s}}(\bar{x}), d \geq 1, p \in S, 0 \leq i \leq w_p - 1, s \in \mathbb{Z}, \bar{x} \in \prod_{p \in S} \mathbb{Z}/w_p \mathbb{Z}$, having the following property: for any smooth connected projective curve $X$ of genus $g_X = g$ defined over a finite field $k$ there is a specialisation morphism of Hopf algebras
\[ \Psi_X : U_{w,s,R_g} \otimes R_g \mathbb{C} \to U_{w,S,X}. \]

3. HARDER-Narasimhan strata

3.1. Parabolic degree and the stability condition. Following [MS], we can define the stability condition for the parabolic coherent sheaves by introducing the notion of parabolic degree. Let $\chi = (\chi_{i,p})_{p \in S, 0 \leq i \leq w_p - 1}$ be a collection of real numbers such that $0 \leq \chi_{w_p-1,p} < \chi_{w_p-2,p} < \cdots < \chi_{1,p} < 1$ for all $p \in S$. For any parabolic coherent sheaf $F^*$ of degree $d = (d^s_p)_{p \in S, 0 \leq i \leq w_p - 1}$, we define the parabolic degree of $F^*$ as
\[ \text{par deg}_\chi F^* = \text{deg} F^0 + \sum_{p \in S} \sum_{1 \leq i \leq w_p - 1} \chi_i \eta_{p}(d^{s_p} - d^{(i-1)s_p}). \]
The parabolic slope of $F^*$ is defined as
\[ \mu_\chi(F^*) = \frac{\text{par deg}_\chi F^*}{\text{rank} F^*} \in \mathbb{Q} \cup \{\infty\}. \]
A parabolic sheaf $F^\bullet$ is said to be semistable of slope $\nu$ if $\mu_\chi(F^\bullet) = \nu$ and if $\mu_\chi(G^\bullet) \leq \nu$ for any subsheaf $G^\bullet$ of $F^\bullet$. If the above condition holds with $<$ instead of $\leq$ then we say that $F^\bullet$ is stable. By definition the parabolic slope coincides with the usual notion of slope of a coherent sheaf without extra structure, i.e., any coherent sheaf $F \in \operatorname{Coh}(X)$ we have $\mu(F) = \mu_\chi((F)^\bullet)$, where $\mu(F) = \frac{\deg F}{\operatorname{rank} F}$.

We will fix once and for all the rest of the section one particular $\chi$ with $\chi_{l,p} = \frac{w_r-i}{w_p}$ and write $\mu = \mu_\chi$ but all the results of this subsection remain the same for other stability conditions. Let $w = \operatorname{l.c.m.}(\{w_p\}_{p \in S})$ as before and we set $\chi_S(F^\bullet) := \frac{1}{w} \sum_{k=0}^{w-1} \chi_S(F^\bullet \otimes \omega_{X}^{*,-k})$, where $\chi_S(F^\bullet) = \dim \operatorname{Hom}_{\operatorname{Coh}^{w,S}(X)}(O_X^\bullet, F^\bullet) - \dim \operatorname{Ext}_{\operatorname{Coh}^{w,S}(X)}(O_X^\bullet, F^\bullet)$ is the Euler characteristic of $\operatorname{Coh}^{w,S}(X)$. Then we have following analogue Riemann-Roch:

**Proposition 3.1.** For any $F^\bullet \in \operatorname{Coh}^{w,S}(X)$, we have

\[
\chi_S(F^\bullet) = \operatorname{rank}(F^\bullet)\chi_S(O_X^\bullet) + \operatorname{par deg}_\chi(F^\bullet)
\]

and $\chi_S(O_X^\bullet) = w(1-g_X) + \frac{w}{w_p} \sum_{p \in S}(\frac{1}{w_p} - 1) = -\frac{w}{w} \operatorname{par deg}(\omega_{X}^{*,-1})$. Moreover, we have an analogue formula for the average Euler form

\[
\frac{1}{w} \sum_{k=0}^{w-1} (F^\bullet \otimes \omega_{X}^{*,-k}, G^\bullet)_S
\]

\[
= \frac{\chi_S(O_X^\bullet)}{w} \operatorname{rank} F^\bullet \operatorname{rank} G^\bullet + \operatorname{rank} F^\bullet \operatorname{par deg} G^\bullet - \operatorname{rank} G^\bullet \operatorname{par deg} F^\bullet.
\]

We denote by $C_{\nu}$ the full subcategory of $\operatorname{Coh}^{w,S}(X)$ whose objects are semistable parabolic sheaves of slope $\nu$. As an example, we have $C_{\infty} = \operatorname{Tor}^{w,S}(X)$. The fundamental properties of the categories $C_{\nu}$ are listed below:

**Proposition 3.2.** [Se, Proposition 11, Théorème 8 and 12, Chap. 3] The following hold:

i) the categories $C_{\nu}$ are abelian, artinian and noetherian,

ii) $\operatorname{Hom}_{\operatorname{Coh}^{w,S}(X)}(C_{\nu}, C_{\mu}) = 0$ if $\nu > \mu$,

iii) any parabolic coherent sheaf $F^\bullet$ admits a unique filtration

\[
0 \subsetneq F_1^\bullet \subsetneq \cdots \subsetneq F_I^\bullet = F^\bullet
\]

satisfying the following conditions: $F_i^\bullet/F_{i+1}^\bullet$ is semistable for all $i$ and

$\mu(F_i^\bullet/F_{i+1}^\bullet) < \cdots < \mu(F_{I-1}^\bullet/F_{I}^\bullet) < \mu(F_{I}^\bullet)$.

The filtration (3.3) is called the Harder-Narasimhan (or HN) filtration of $F^\bullet$ and the factors $F_1^\bullet, \ldots, F_I^\bullet$ are called the semistable factors of $F^\bullet$. We also define the HN-type of $F^\bullet$ to be $HN(F^\bullet) = (\alpha_1, \ldots, \alpha_I)$ with $\alpha_i = \frac{F_i^\bullet}{F_{i+1}^\bullet}$. Here $\overline{\alpha}$ is the class of a parabolic sheaf $G^\bullet$ in $K_S^\bullet := \{(r, \mathbf{d}) \mid r \geq 1 \text{ or } r = 0, \mathbf{d} > 0\}$, here we write $\mathbf{d} > 0$ if $d^{(l,p)} > 0$ for all $p \in S, 0 \leq i \leq w_p - 1$ and $\mathbf{d} \neq 0$. Note that the weight $\alpha := \alpha_1 + \cdots + \alpha_I$ of the HN-type of $F^\bullet$ is equal to $\overline{\alpha}$. We will sometimes make use of the Harder-Narasimhan polygon of $\overline{\alpha}$ of $F^\bullet$. 

where \( \mathcal{I}_S \) is the set of isomorphism classes of semistable parabolic sheaves. So \( \mathbf{1}_\alpha \in \mathcal{I}_S = \{ (F, \mu) \mid \mu(\alpha_1) < \cdots < \mu(\alpha_l) \} \), where \( \alpha \) runs through the set of all possible HN-types, i.e. tuples \( \alpha = (\alpha_1, \ldots, \alpha_l) \) with \( \alpha_i \in K_1^+ \). If \( \mathbf{1}_\alpha = (\alpha) \), then \( S_\alpha \) is the set of isomorphism classes of semistable parabolic sheaves of class \( \alpha \). Let us denote by \( \mathbf{1}_\alpha \in H \) be the characteristic function of the set of parabolic sheaves of a fixed HN-type \( \alpha \). For \( \alpha \in K_1^+ \), we will simply denote by \( \mathbf{1}_n^\alpha \) the characteristic function of \( S_\alpha \). Thus, by the uniqueness of HN filtration of a given parabolic sheaf we can easily deduce

**Proposition 3.4.** For any HN-type \( \mathbf{1}_\alpha = (\alpha_1, \ldots, \alpha_l) \), we have

\[
\mathbf{1}_{\mathbf{1}_\alpha} = \nu \sum_i \mathbf{1}_{\alpha_i} \mathbf{1}_{\mathbf{1}_{\alpha_i}}^{\mathbf{1}_{\alpha_i}}.
\]

Following [S4], we will use the stratification of HN-types to define a completion of the Hall algebra \( H \) and \( H \otimes H \). For any integer \( n \in \mathbb{Z} \) we set \( \mathbf{1}_\alpha \geq n \) if \( \mathbf{1}_\alpha = (\alpha_1, \ldots, \alpha_l) \) with \( \mu(\alpha_1) \geq n \). Let \( C_{\geq n} \) be the full subcategory of \( \text{Coh}^{w.S}(X) \) generated by \( C_n \) for all \( n \geq n \). It is clear that \( F^* \in C_{\geq n} \) if and only if the HN-type \( \mathbf{1}_\alpha \) of \( F^* \) satisfies \( \alpha \geq n \). Let \( H^{<n} \) be the subspace of \( H \) consisting of functions supported on the complement of \( \bigcup_{n \geq n} S_\alpha \), so that we have \( H = H^{\geq n} \oplus H^{<n} \).

Now let us fix a class \( \alpha \in K_1^+ \). There is a surjective linear map of vector spaces \( j_{\nu} : H[\alpha] \to H^{\geq n}[\alpha] \) inducing an isomorphism \( \pi_n : H[\alpha]/H^{<n}[\alpha] \to H^{\geq n}[\alpha] \). The canonical embedding \( H^{<m}[\alpha] \to H^{<n}[\alpha] \) for any \( m \leq n \) induces a commutative diagram

\[
\begin{array}{ccc}
H[\alpha]/H^{<m}[\alpha] & \xrightarrow{\pi_n} & H^{\geq n}[\alpha] \\
\downarrow & & \downarrow \phi_{m,n} \\
H[\alpha]/H^{<n}[\alpha] & \xrightarrow{\pi_n} & H^{\geq n}[\alpha]
\end{array}
\]

Obviously \( (H^{\geq n}[\alpha], \phi_{m,n}) \) forms a projective system and we can define

\[
\tilde{H}_S[\alpha] := \lim_{\nu} H^{\geq \nu}[\alpha].
\]

Since each \( H^{\geq n}[\alpha] \) is finite dimensional and since \( H[\alpha] = \bigcup_n H^{\geq n}[\alpha] \), we may view \( \tilde{H}_S[\alpha] \) as the set of infinite sums \( \sum_{\nu} u_{F^*} \mathcal{F}^\nu \) with \( u_{F^*} \in \mathbb{C}, \mathcal{F}^\nu = \alpha, \) that is, \( \tilde{H}_S[\alpha] = \{ f : \mathcal{I}_S \to \mathbb{C} \} = \prod_{\nu, F^* \in \mathcal{I}_S} \mathbb{C} \chi_{F^*} \), as a vector space, where we have denoted by \( \mathcal{I}_S \subset \mathcal{I}_S \) the set of all parabolic coherent sheaves of class \( \alpha \).
For the sake of convenience we also denote by \( \text{jet}_n \) the canonical morphism \( \hat{H}_S[\alpha] \to H^{\geq n}[\alpha] \). By the universal property of the projective limit there is an injective linear map \( H[\alpha] \to \hat{H}_S[\alpha] \) and since the map \( H[\alpha] \to H^{<n}[\alpha] \) splits, we may consider \( H^{<n} \) as a subspace of \( \hat{H}_S[\alpha] \) via the inclusion \( H^{<n}[\alpha] \to H[\alpha] \to \hat{H}_S[\alpha] \). So the projection \( \text{jet}_n : \hat{H}_S[\alpha] \to H^{\geq n}[\alpha] \) is an idempotent morphism. Let us denote \( r_n = 1 - \text{jet}_n \). Then any element \( h \in \hat{H}_S[\alpha] \) can uniquely be written as \( \text{jet}_n(h) + r_n(h) \) where \( \text{jet}_n(h) \in H^{\geq n} \) and \( r_n(h) = 0 \). Thus, the space \( H[\alpha] \) as a subset of \( \hat{H}_S[\alpha] \) can be identified with the set of those sequences \( h = (h_n) \) for which \( r_n(h_n) = 0 \) for \( n > 0 \). We define
\[
(3.5) \quad \hat{H}_S := \bigoplus_{\alpha \in K^+_S} \hat{H}_S[\alpha].
\]
In a similar way, for \( \alpha, \beta \in K^+_S \) the sequence of vector spaces
\[
(3.6) \quad (H^{\geq n}[\alpha] \otimes H^{\geq m}[\beta]) = \ker(H[\alpha]/H^{<n}[\alpha] \otimes H[\beta]/H^{<m}[\beta])
\]
forms a projective system and we set
\[
(3.7) \quad H[\alpha]\hat{\otimes}H[\beta] := \lim_{n,m} H^{\geq n}[\alpha] \otimes H^{\geq m}[\beta].
\]
As before as well, \( H[\alpha]\hat{\otimes}H[\beta] \) can be identified with the space of all infinite sums \( \sum F^*, \rho^*, u_{\alpha^*, \rho^*}, [F^*] \otimes \rho^* \) with \( F^* = \alpha, \rho^* = \beta \) and \( u_{\alpha^*, \rho^*}, \rho^* \in \mathbb{C} \). Finally, for \( \gamma \in K^+_S \), we set
\[
\gamma \in H[\alpha]\hat{\otimes}H[\beta] = \bigoplus_{\alpha, \beta \in K^+_S, \alpha + \beta = \gamma} (H[\alpha] \hat{\otimes} H[\beta])
\]
and
\[
\gamma \in H[\alpha] \hat{\otimes} H[\beta] = \bigoplus_{\gamma \in K^+_S} (H[\alpha] \hat{\otimes} H[\beta]).
\]
Proposition 3.5. In the notation as above the following properties hold

i) \( \hat{H}_S \) and \( H \hat{\otimes} H \) are associative algebras,

ii) For any \( \alpha, \beta \in K^+_S \) we have \( \Delta_{\alpha, \beta} (H[\alpha] \otimes H[\beta]) \subset H[\alpha] \otimes H[\beta] \).

Proof. To show that the composition map \( \hat{H}_S[\alpha] \otimes \hat{H}_S[\beta] \to \hat{H}_S[\alpha + \beta] \) by the rule
\[
(\sum u_{\alpha^*} [H^*]) \otimes (\sum u_{\beta^*} [G^*]) \mapsto (\sum u_{\alpha^*} u_{\beta^*} \cdot [H^*][G^*])
\]
is well-defined, we need to show that for a fixed parabolic sheaf \( F^* \) of class \( F^* = \alpha + \beta \), there are finitely many exact sequence
\[
0 \to G^* \to F^* \to H^* \to 0
\]
such that \( \overline{F^*} = \alpha \) and \( \overline{G^*} = \beta \). To see this, let \( (\gamma_1, \ldots, \gamma_r) \) be the HN type of \( F^* \) and set \( \nu = \mu(\gamma_1) \). We claim that any quotient sheaf \( H^* \) of \( F^* \) belongs to \( \text{Coh}^w_{\geq \nu} S \). Let
\[
M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = H^*
\]
be the HN filtration of \( H^* \). Then on one hand we have \( H^* \subset \text{Coh}^{w,S}_{\geq \mu(\gamma_1) + 1}(X) \) while on the other hand there is a surjective map \( F^* \to H^*/M_{r-1} \). Since \( H^*/M_{r-1} \) is semistable, this implies that \( \mu(H^*/M_{r-1}) \geq \nu \) and hence \( H^* \subset \text{Coh}^{w,S}_{\geq \nu} X \). Since there are finitely many such sheaves \( H^* \) of class \( \beta \) and for each such \( H^* \) there are only finitely many maps \( F^* \to H^* \), we conclude that there are finitely many exact sequences (3.2).

In a similar way, the map
\[
\prod_{\alpha_1 + \beta_1 = \gamma_1} H[\alpha_1] \hat{\otimes} H[\beta_1] \otimes \prod_{\alpha_2 + \beta_2 = \gamma_2} H[\alpha_2] \hat{\otimes} H[\beta_2]
\]
(3.9)
\[
\to \prod_{\alpha_1 + \beta_1 = \gamma_1, \alpha_2 + \beta_2 = \gamma_2} H[\alpha_1 + \alpha_2] \hat{\otimes} H[\beta_1 + \beta_2]
\]
is convergent since for a given $[\mathcal{F}] \otimes [G] \in H[\gamma_1] \otimes H[\gamma_2]$ there are finitely many surjective maps $\mathcal{F} \to M^\bullet$ and $G \to N^\bullet$, where $M^\bullet$ and $N^\bullet$ are parabolic coherent sheaves satisfying $\overline{M^\bullet} + \overline{N^\bullet} = \gamma_1$. 

The proof for the property ii) is completely analogous. 

\section{3.3. Characteristic functions of semistables.} Consider the elements 

$$1_{\alpha} = \sum_{\mathcal{F} \in \mathcal{I}_s} 1_{\mathcal{F}}, \quad 1_{\alpha}^\text{vec} = \sum_{\mathcal{V} \in \mathcal{I}_s^\text{vec}} 1_{\mathcal{V}},$$

where the second sum ranges over all isomorphism classes of locally free parabolic sheaves of class $\alpha$. Using the proposition 3.4 we have the following identities:

\begin{equation}
1_{\alpha} = \sum_{\alpha \in X_{\alpha}^s} \nu_{\sum_{i<j}(\alpha_i, \alpha_j)} s 1_{\alpha_1}^\text{ss} \cdots 1_{\alpha_s}^\text{ss}, \quad 1_{\alpha}^\text{vec} = \sum_{\alpha \in Y_{\alpha}^s} \nu_{\sum_{i<j}(\alpha_i, \alpha_j)} s 1_{\alpha_1}^\text{ss} \cdots 1_{\alpha_s}^\text{ss},
\end{equation}

where $X_{\alpha}$ is the set of all HN types of weight $\alpha$ and $Y_{\alpha}$ is the set of all HN types $\bar{\alpha} = (\alpha_1, \ldots, \alpha_i)$ of weight $\alpha$ for which $\mu(\alpha_i) < \infty$. Note that $1_{\alpha}^\text{vec} \neq 0$ implies $d^0 \leq d^p \leq \cdots \leq d^{|w|-1} \leq d^0 + r$ for all $p \in S$. We may write $d = (d^0, m)$ where, for all $p \in S$, we set $m_{i,p} = d^i - d^0$ for all $i$, and $m = (m(p))_{p \in S}$ with $m(p) = (0 = m_{0,p} \leq m_{1,p} \leq \cdots \leq m_{|w|-1,p} \leq m_{w,p})$, called a \textit{dimension type} at $p$. We also write $|m(p)| = \sum_{i=1}^{w} m_{i,p}$ and $|m| = \sum_{p \in S} |m(p)|$. Finally, we denote by $D_r$ the set of collections $m$ such that, for each $p \in S$, $0 = m_{0,p} \leq m_{1,p} \leq \cdots \leq m_{|w|-1,p} \leq m_{w,p} = r$ equipped with the lexicographical order.

Let us denote by $\widehat{U}_S$ the completion of $U$ in $\widehat{H}_S$, that is, $\widehat{U}_S[\alpha] = \varprojlim U[\alpha]/(U[\alpha] \cap H^{<n}[\alpha])$. The aim in this section is to prove

\begin{theorem}
For any $\alpha \in K_+^S$, we have $1_{\alpha}^\text{ss} \in U$.
\end{theorem}

\begin{remark}
The proof is completely parallel to the non-parabolic case as shown in [S4]. The only difference in our argument is that we use the lexicographical order on our multi-degree.
\end{remark}

\begin{proposition}
For any $\alpha \in K_+^S$, we have $1_{\alpha}^\text{ss} \in \widehat{U}_S$.
\end{proposition}

\begin{proof}
We may use Reinke’s inversion formula (see [Rei]) to write

$$1_{\alpha}^\text{ss} = \sum_{\beta} (-1)^{s-1} \nu_{\sum_{i<j}(\beta_i, \beta_j)} s 1_{\beta_1} \cdots 1_{\beta_s}$$

where the sum ranges over all tuples $\beta = (\beta_1, \ldots, \beta_s)$ of elements of $K_+^S$ satisfying $\mu(\sum_{i=0}^{s} \beta_i) > \mu(\alpha)$ for all $k = 1, \ldots, s$. The above sum converges in $\widehat{H}_S$. Since $\widehat{U}_S$ is a subalgebra of $\widehat{H}_S$, the proposition will be proved if we can show that $1_{\alpha} \in \widehat{U}_S$ for all $\alpha$. Furthermore, any parabolic coherent sheaf $\mathcal{F}^\bullet$ of class $\mathcal{F}^\bullet = \alpha$ can be decomposed into the direct sum of a parabolic vector bundle and a parabolic torsion sheaf. If $\alpha = (r, d, m)$ with $m \notin D_r$, then there exist a locally subsheaf $\mathcal{E}^\bullet_r$ of maximal rank $r$ and maximal parabolic degree $d = (d, m')$ such that $m' \in D_r$. Thus

$$1_{r,d,m} = \sum_{d' \geq 0, m' \in D_r} 1_{(r-d', 0, d', m, m')} s 1_{r,d-d', m-m'+m'} 1_{0,d', m-m'+m'}$$

and $1_{0,d', m-m'+m'} \in U$ for all $d' > 0$ and $m'' \leq m'$, it suffices in fact to prove that $1_{r,d,m}^\text{vec} \in \widehat{U}_S$ for all $d \in \mathbb{Z}, m \in D_r$.

We will argue by induction on the rank $r$. The cases of $r = 0, 1$ are obvious by definition. So let $r > 1$ and let us assume the proposition holds for all $r' < r$. We have to show that for any multi-degree $d = (d, m)$ and any $n \in \mathbb{Z}$ we have

\begin{equation}
1_{r,d,m}^\text{vec} \in U + H^{<n}.
\end{equation}
Let us fix $n$ and argue by induction on multi-degree $d = (d, m)$. If $d < nr - \frac{w}{2} \sum_{p \in S} \frac{w_p - 1}{w_p}$, then there is no parabolic vector bundle of rank $r$ and multi-degree $d$ may belong to $C_{\geq n}$, hence $1_{r,d}^{\text{vec}} \in H^{<n}$. Now let us fix some $d$ and assume that (3.11) holds for all $d' < d$. Choose $N < n - K$, where $K = \text{par deg}(\omega^1_X) + \frac{w}{2} \sum_{p \in S} \frac{w_p - 1}{w_p} = \frac{w + 2}{2} \sum_{p \in S} (1 - \frac{1}{w_p}) + 2(g_X - 1)$ and let $d'$ be a multi-degree with $d' = N$. Consider the product

$$1_{r-1,d-d' \cdot 1_{1,d'}^{\text{vec}}} = \sum_{r,d} c_{r,d}[F^*]$$

where

$$c_{r,d} = v^{-((r-1,d-d'),(1,d'))} \sum_{L^* \in \text{Bun}_{r,d}} \frac{\#\{L^* \hookrightarrow F^*\}}{\# \text{Aut}(L^*)} = v^{-((r-1,d-d'),(1,d'))} \sum_{L^* \in \text{Bun}_{r,d}} \frac{\#\{L^* \hookrightarrow F^*\}}{v^2 - 1}.$$ 

Let us decompose $F^* = L^*_r \oplus T^*_r$ into the direct sum of a parabolic vector bundle and a parabolic torsion sheaf. Let us assume $F^* \in C_{\geq n}$. Then $F^* \in C_{\geq N + \text{par deg}(\omega^1_X) + \frac{w}{2} \sum_{p \in S} \frac{w_p - 1}{w_p}}$ and thus $\text{Ext}_{\text{Coh}_{w,S}(X)}(L^*, F^*) = 0$ by the Serre’s duality. Then $\text{dim} \text{Hom}_{\text{Coh}_{w,S}(X)}(L^*, F^*) = ((1, d'), (r, d))s$. Since any nonzero map from a parabolic line bundle to a parabolic vector bundle is an embedding, we deduce that

$$\#\{L^* \hookrightarrow F^*\} = v^{-2 \text{dim} \text{Hom}_{C_{\geq w,S}(X)}(L^*, F^*)} = v^{-2 \text{dim} \text{Hom}_{C_{\geq w,S}(X)}(L^*, T^*_r)}$$

and, once the class of $L^*$ is fixed as $(1, d')$, this number only depends on the multi-degree of $T^*_r$, by the proposition 1.3. Hence there exists nonzero constants $c_{d', \nu}$ for $d'' \geq 0$ such that

$$1_{r-1,d-d' \cdot 1_{1,d'}^{\text{vec}}} = c_0 1_{r,d}^{\text{vec}} + \sum_{d''} c_{d', \nu} 1_{r-,d-d''}^{\text{vec}} \cdot 1_{0,d''} + H^{<n}.$$ 

We can rewrite the above equation as

$$c_0 1_{r,d}^{\text{vec}} = 1_{r-1,d-d' \cdot 1_{1,d'}^{\text{vec}}} - \sum_{d''} c_{d', \nu} 1_{r-,d-d''}^{\text{vec}} \cdot 1_{0,d''} + H^{<n}$$

and now by the induction hypothesis we have $1_{r-1,d-d'} \in \bar{U}_S$ and $1_{r,d}^{\text{vec}} \in \bar{U}_S$ for all $d'' > 0$. Hence 3.11 follows.

Now we have to show that $1_{\alpha}^{ss}$ actually belongs to $U$ and not only to $\bar{U}_S$. By proposition 3.7 there exists for all $n$ an element $v_n \in H^{<n}$ such that $u_n := 1_{\alpha}^{ss} + v_n \in U$. We may further decompose $v_n = \sum_{\underline{a}} v_{n, \underline{a}}$ according to the HN type $\underline{a}$. The set of $\underline{a}$ for which $v_{n, \underline{a}}$ is nonzero is finite since $v_n \in \bar{H}$. To prove the theorem 3.6, we need the following two lemmas.

**Lemma 3.8.** There exists $n < 0$ such that for any HN-type $\underline{a} = (\alpha_1, \ldots, \alpha_l)$ of weight $\alpha$ satisfying $\mu_\chi(\alpha_1) < n$, we have $\mu_\chi(\alpha_{i+1}) - \mu_\chi(\alpha_i) > \text{par deg}(\omega^1_X)$ for some $1 \leq i \leq l$.

**Proof.** Let $\underline{a} = (\alpha_1, \ldots, \alpha_l)$ be as above. We have par deg($\alpha$) = rank($\alpha$) rank($\alpha_1$) + $\cdots$ + rank($\alpha_l$) rank($\alpha_1$). If $\mu_\chi(\alpha_{i+1}) < n$ and $\mu_\chi(\alpha_{i+1}) - \mu_\chi(\alpha_i) \leq \text{par deg}(\omega^1_X)$ for all $i$, then

$$\text{par deg}(\alpha) < \text{rank}(\alpha_1)n + \text{rank}(\alpha_2)(n + \text{par deg}(\omega^1_X)) + \cdots + \text{rank}(\alpha_l)(n + \text{par deg}(\omega^1_X))(l - 1) \leq \sum_{i=0}^{l} \text{rank}(\alpha_i)(n + \text{par deg}(\omega^1_X)) \leq l \text{rank}(\alpha).$$


This is impossible for $n$ sufficiently negative. \hfill \Box

**Lemma 3.9.** Let $\mathcal{F}^\bullet \in \text{Coh}^{w,S}(\mathcal{X})$ be a parabolic coherent sheaf of class $\alpha$ and of HN type $(\alpha_1, \ldots, \alpha_l)$. Assume that $\mu_\chi(\alpha_{i+1}) - \mu_\chi(\alpha_i) > \text{par deg}(\omega_{\mathcal{X}}^{\bullet,1})$ for some $i$. Then $1_{\mathcal{F}^\bullet} = m \circ \Delta_{\beta,\gamma}(1_{\mathcal{F}^\bullet})$ for $\beta = \alpha_1 + \cdots + \alpha_i$ and $\gamma = \alpha_{i+1} + \cdots + \alpha_l$.

**Proof.** Let $\mathcal{F}_1^\bullet \subset \cdots \subset \mathcal{F}_p^\bullet = \mathcal{F}^\bullet$ be the HN filtration of $\mathcal{F}^\bullet$. Since $\mathcal{F}_1^\bullet \in C_{\geq \mu_\chi(\alpha_{i+1})}$ and $\mathcal{F}^\bullet / \mathcal{F}_{i+1}^\bullet \in C_{\leq \mu_\chi(\alpha_i)}$ while $\mu_\chi(\alpha_{i+1}) - \mu_\chi(\alpha_i) > \text{par deg}(\omega_{\mathcal{X}}^{\bullet,1})$, we have

$$\text{Ext}_{\text{Coh}^{w,S}(\mathcal{X})}(\mathcal{F}_{i+1}^\bullet, \mathcal{F}^\bullet / \mathcal{F}_{i+1}^\bullet) = 0.$$ 

It follows that $\mathcal{F}^\bullet \simeq \mathcal{F}^\bullet_{i+1} \oplus \mathcal{F}^\bullet / \mathcal{F}_{i+1}^\bullet$. Moreover, $1_{\mathcal{F}^\bullet} / \mathcal{F}_{i+1}^\bullet \mathcal{F}_i^\bullet = v^{-}(\mathcal{F}^\bullet / \mathcal{F}_{i+1}^\bullet) \otimes 1_{\mathcal{F}_i^\bullet}$ since there is a unique subsheaf of $\mathcal{F}^\bullet$ isomorphic to $\mathcal{F}_i^\bullet$. Hence the lemma will be proved once we show that $\Delta_{\beta,\gamma}(1_{\mathcal{F}^\bullet}) = v^{(\mathcal{F}^\bullet / \mathcal{F}_{i+1}^\bullet)} \otimes 1_{\mathcal{F}_{i+1}^\bullet}$. But this last equation is a consequence of the fact that there exists a unique subsheaf of $\mathcal{F}^\bullet$ of class $\gamma$, namely $\mathcal{F}^\bullet_{i+1}$.

Now we are ready to finish the proof of Theorem 3.6. Let us choose some $n \ll 0$ as in Lemma 3.8. Let $A$ be the finite set of all $\alpha$ for which $v_n \alpha$ is nonzero and let $\alpha_0^\bullet$ be the lower boundary of the convex hull of elements of $A$.

![Figure 1. The convex hull of a set of HN polygons.](image1)

Thus $\alpha_0^\bullet = (\alpha_0^1, \ldots, \alpha_0^m)$ is also a convex path in $K_\mathcal{S}$ of weight $\alpha$. Moreover $\mu_\chi(\alpha_0^1) \leq n$ so that the conclusion of Lemma 3.8 applies. Choose $i$ such that $\mu_\chi(\alpha_{i+1}^0) - \mu_\chi(\alpha_i^0) > \text{par deg}(\omega_{\mathcal{X}}^{\bullet,1})$ and set $\beta = \alpha_1^0 + \cdots + \alpha_i^0$ and $\gamma = \alpha_{i+1}^0 + \cdots + \alpha_m^0$. By Lemma 3.9, $\Delta_{\beta,\gamma}(1_{\mathcal{F}^\bullet}) = 0$ for all parabolic sheaves $\mathcal{F}^\bullet$ whose HN polygon does not lie below the segment $\beta$.

![Figure 2. Choice of the vertex $\beta$.](image2)
This implies $\Delta_{\beta,\gamma}(v_n) = 0$ for all HN type $\alpha$ whose associated polygon does not pass through the point $\beta$. Furthermore, by Lemma 3.9 again, $m \circ \Delta_{\beta,\gamma}(v_n) = v_n$ for any HN type $\alpha$ whose polygon does pass through $\beta$. Hence

$$m \circ \Delta_{\beta,\gamma}(u_n) = m \circ \Delta_{\beta,\gamma}(\sum_{\alpha} v_n, \alpha) = \sum_{\alpha \in Z_\beta} v_n, \alpha,$$

where $Z_\beta$ is the set of all HN types passing through $\beta$. Since $u_n$ belongs to $U$, which is stable under the coproduct, we deduce that $\sum_{\alpha \in Z_\beta} v_n, \alpha$ belongs to $U$ as well. Hence the same holds for $u'_n = \sum_{\alpha \in Z_\beta} v_n, \alpha$. Note that $u'_n$ contains strictly fewer terms than $u_n$. Arguing as above repeatedly we obtain better and better approximations of $1_n$ by elements of $U$ until we arrive at $1_n \in U$ and the Theorem 3.6 is proved. \hfill $\Box$

The combination of Theorem 3.6 and Proposition 3.4 yields the following:

**Corollary 3.10.** For any HN type $\alpha$ we have $1_n, \alpha \in U$.

**Remark.** The above proof actually shows that $\hat{U} \cap H = U$.

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