New inclusion sets for singular values

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Abstract

In this paper, for a given matrix \(A = (a_{ij}) \in \mathbb{C}^{n \times n}\), in terms of \(r_i\) and \(c_i\), where

\[ r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad c_i = \sum_{j=1, j\neq i}^{n} |a_{ji}|, \]

some new inclusion sets for singular values of the matrix are established. It is proved that the new inclusion sets are tighter than the Geršgorin-type sets (Qi in Linear Algebra Appl. 56:105-119, 1984) and the Brauer-type sets (Li in Comput. Math. Appl. 37:9-15, 1999). A numerical experiment shows the efficiency of our new results.

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1 Introduction

Singular values and the singular value decomposition play an important role in numerical analysis and many other applied fields [3–8]. First, we will use the following notations and definitions. Let \(N := \{1, 2, \ldots, n\}\), and assume \(n \geq 2\) throughout. For a given matrix \(A = (a_{ij}) \in \mathbb{C}^{n \times n}\), we define \(a_i = |a_{ii}|, \quad s_i = \max\{r_i, c_i\}\) for any \(i \in N\) and \(u_i = \max\{0, u\}\), where

\[ r_i := \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad c_i := \sum_{j=1, j\neq i}^{n} |a_{ji}|. \]

In terms of \(s_i\), the Geršgorin-type, Brauer-type and Ky Fan-type inclusion sets of the matrix singular values are given in [1, 2, 9, 10], we list the results as follows.

Theorem 1 If a matrix \(A = (a_{ij}) \in \mathbb{C}^{n \times n}\), then

(i) (Geršgorin-type, see [1]) all singular values of \(A\) are contained in

\[ C(A) := \bigcup_{i=1}^{n} C_i \quad \text{with} \quad C_i = \{(a_i - s_i), (a_i + s_i)\} \in \mathbb{R}; \] (1)

(ii) (Brauer-type, see [2]) all singular values of \(A\) are contained in

\[ D(A) := \bigcup_{i=1}^{n} \bigcup_{j=1, j\neq i}^{n} \{z \geq 0 : |z - a_i||z - a_j| \leq s_is_j\}; \] (2)
(iii) (Ky Fan-type, see [2]) let $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be a nonnegative matrix satisfying $b_{ij} \geq \max(|a_{ij}|, |a_{ji}|)$ for any $i \neq j$, then all singular values of $A$ are contained in

$$E(A) : = \bigcup_{i=1}^{n} \{ z \geq 0 : |z - a_{ii}| \leq \rho(B) - b_{ii} \}.$$ 

We observe that all the results in Theorem 1 are based on the values of $s_i = \max (r_i, c_i)$, if $r_i \ll c_i$ or $r_i \gg c_i$, all these singular value localization sets in Theorem 1 become very crude. In this paper, we give some new singular value localization sets which are based on the values of $r_i$ and $c_i$. The remainder of the paper is organized as follows. In Section 2, we give our main results. In Section 3, a numerical experiment is given to show the efficiency of our new results.

2 New inclusion sets for singular values

Based on the idea of Li in [2], we give our main results as follows.

**Theorem 2** If a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then all singular values of $A$ are contained in

$$\Gamma(A) : = \Gamma_1(A) \cup \Gamma_2(A),$$

where

$$\Gamma_1(A) : = \bigcup_{i=1}^{n} \{ \sigma \geq 0 : |\sigma^2 - |a_{ii}|^2| \leq |a_{ii}| r_i(A) + \sigma c_i(A) \}$$

and

$$\Gamma_2(A) : = \bigcup_{i=1}^{n} \{ \sigma \geq 0 : |\sigma^2 - |a_{ii}|^2| \leq |a_{ii}| c_i(A) + \sigma r_i(A) \}.$$ 

**Proof** Let $\sigma$ be an arbitrary singular value of $A$. Then there exist two nonzero vectors $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ such that

$$\sigma x = A^* y \quad \text{and} \quad \sigma y = Ax. \quad (3)$$

Denote

$$|x_p| = \max \{ |x_i|, 1 \leq i \leq n \}, \quad |y_q| = \max \{ |y_i|, 1 \leq i \leq n \}.$$ 

Now, we assume that $|x_p| \leq |y_q|$, the $q$th equations in (3) imply

$$\sigma x_q - a_{qq} y_q = \sum_{j \neq q} a_{jq} y_j, \quad (4)$$

$$\sigma y_q - a_{qq} x_q = \sum_{j \neq q} a_{qj} x_j. \quad (5)$$
Solving for $y_q$ we can get

$$\left(\sigma^2 - a_{qq} \bar{a}_{qq}\right) y_q = a_{qq} \sum_{j \neq q} \bar{a}_{jq} y_j + \sigma \sum_{j \neq q} a_{jq} x_j.$$

(6)

Taking the absolute value on both sides of the equation and using the triangle inequality yield

$$|\sigma^2 - |a_{qq}|^2| \leq |a_{qq}| \sum_{j \neq q} |\bar{a}_{jq}| |y_j| + \sigma \sum_{j \neq q} |a_{jq}| |x_j|. \quad (7)$$

Then we can get

$$|\sigma^2 - |a_{qq}|^2| \leq |a_{qq}| c_q(A) + \sigma r_q(A).$$

Similarly, if $|y_q| \leq |x_p|$, we can get

$$|\sigma^2 - |a_{pp}|^2| \leq |a_{pp}| r_p(A) + \sigma c_p(A).$$

Thus, we complete the proof. □

**Remark 1** Since

$$|a_{ii}| r_i(A) + \sigma c_i(A) \leq (|a_{ii}| + \sigma) s_i$$

and

$$|a_{ii}| c_i(A) + \sigma r_i(A) \leq (|a_{ii}| + \sigma) s_i,$$

the results in Theorem 2 are always better than the results in Theorem 1(i).

**Theorem 3** If a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then all singular values of $A$ are contained in

$$\Omega(A) := \Omega_1(A) \cup \Omega_2(A) \cup \Omega_3(A),$$

where

$$\Omega_1(A) := \bigcup_{i \neq j} \{\sigma \geq 0 : |\sigma^2 - |a_{ii}|^2||\sigma^2 - |a_{jj}|^2| \leq (|a_{ii}| r_i(A) + \sigma c_i(A)) (|a_{jj}| r_j(A) + \sigma c_j(A))\},$$

$$\Omega_2(A) := \bigcup_{i \neq j} \{\sigma \geq 0 : |\sigma^2 - |a_{ii}|^2||\sigma^2 - |a_{jj}|^2| \leq (|a_{ii}| c_i(A) + \sigma r_i(A)) (|a_{jj}| c_j(A) + \sigma r_j(A))\},$$

$$\Omega_3(A) := \bigcup_{i \neq j} \{\sigma \geq 0 : |\sigma^2 - |a_{ii}|^2||\sigma^2 - |a_{jj}|^2| \leq (|a_{ii}| c_i(A) + \sigma r_i(A)) (|a_{jj}| c_j(A) + \sigma r_j(A))\}.$$
and

\[ \Omega_4(A) := \bigcup_{i \neq j} \{ \sigma \geq 0 : |\sigma^2 - |a_{qi}|^2 || \sigma^2 - |a_{pj}|^2 | \leq \left( |a_{qi}|r_q(A) + \sigma c_q(A) \right) \left( |a_{pj}|c_q(A) + \sigma r_q(A) \right) \}. \]

**Proof** Let \( \sigma \) be an arbitrary singular value of \( A \). Then there exist two nonzero vectors \( x = (x_1, x_2, \ldots, x_n)^T \) and \( y = (y_1, y_2, \ldots, y_n)^T \) such that

\[ \sigma x = A^* y \quad \text{and} \quad \sigma y = Ax. \] (8)

Denote \( \omega_i = \max(|x_i|, |y_i|) \). Let \( q \) be an index such that \( \omega_q = \max(|\omega_i|, i \in N) \). Obviously, \( \omega_q \neq 0 \). Let \( p \) be an index such that \( \omega_p = \max(|\omega_i|, i \in N, i \neq q) \).

Case I: We suppose \( \omega_q = |x_q|, \omega_p = |x_p| \), similar to the proof of Theorem 2, the \( q \)th equations in (8) imply

\[ |\sigma^2 - |a_{qq}|^2 | |\sigma^2 - |a_{pp}|^2 | \leq \left( |a_{qq}|^{\sum_{j=1,j \neq q}^n} |a_{jq}| |y_j| + \sigma \sum_{j=1,j \neq q}^n |a_{jq}| |x_j| \right) \omega_p. \] (9)

Similarly, the \( p \)th equations in (8) imply

\[ |\sigma^2 - |a_{pp}|^2 | |\sigma^2 - |a_{qq}|^2 | \leq \left( |a_{pp}|^{\sum_{j=1,j \neq p}^n} |a_{jq}| |y_j| + \sigma \sum_{j=1,j \neq p}^n |a_{jq}| |x_j| \right) \omega_q. \] (10)

Multiplying inequalities (9) with (10), we have

\[ |\sigma^2 - |a_{qq}|^2 | |\sigma^2 - |a_{pp}|^2 | \leq \left( |a_{qq}|^{\sum_{j=1,j \neq q}^n} |a_{jq}| |y_j| + \sigma \sum_{j=1,j \neq q}^n |a_{jq}| |x_j| \right) \left( |a_{pp}|^{\sum_{j=1,j \neq p}^n} |a_{jq}| |y_j| + \sigma \sum_{j=1,j \neq p}^n |a_{jq}| |x_j| \right) \omega_q. \]

Case II: We suppose \( \omega_q = |y_q|, \omega_p = |y_p| \), similar to the proof of Theorem 2, the \( q \)th equations in (8) imply

\[ |\sigma^2 - |a_{qq}|^2 | |\sigma^2 - |a_{pp}|^2 | \leq \left( |a_{qq}|^{\sum_{j=1,j \neq q}^n} |a_{jq}| |y_j| + \sigma \sum_{j=1,j \neq q}^n |a_{jq}| |x_j| \right) \omega_p. \] (11)

Similarly, the \( p \)th equations in (8) imply

\[ |\sigma^2 - |a_{pp}|^2 | |\sigma^2 - |a_{qq}|^2 | \leq \left( |a_{pp}|^{\sum_{j=1,j \neq p}^n} |a_{jq}| |y_j| + \sigma \sum_{j=1,j \neq p}^n |a_{jq}| |x_j| \right) \omega_q. \] (12)

Multiplying inequalities (11) with (12), we have

\[ |\sigma^2 - |a_{qq}|^2 | |\sigma^2 - |a_{pp}|^2 | \leq \left( |a_{qq}|c_p(A) + \sigma r_p(A) \right) \left( |a_{qq}|c_q(A) + \sigma r_q(A) \right). \]
Case III: We suppose $\omega_q = |y_q|$, $\omega_p = |x_p|$, similar to the proof of Theorem 2, the $q$th equations in (8) imply

$$|\sigma^2 - |a_{qq}|^2|\omega_q \leq |a_{qq}| \sum_{j=1,j \neq q}^{n} |a_{jq}| |y_j| + \sigma \sum_{j=1,j \neq q}^{n} |a_{jq}| |x_j|$$

$$\leq \left(|a_{qq}| \sum_{j=1,j \neq q}^{n} |a_{jq}| + \sigma \sum_{j=1,j \neq q}^{n} |a_{jq}|\right)\omega_q. \quad (13)$$

Similarly, the $p$th equations in (8) imply

$$|\sigma^2 - |a_{pp}|^2|\omega_p \leq \left(|a_{pp}| \sum_{j=1,j \neq p}^{n} |a_{jp}| + \sigma \sum_{j=1,j \neq p}^{n} |a_{jp}|\right)\omega_q. \quad (14)$$

Multiplying inequalities (13) with (14), we have

$$|\sigma^2 - |a_{pp}|^2| |\sigma^2 - |a_{qq}|^2| \leq \left(|a_{pp}| r_p(A) + \sigma c_p(A)\right)\left(|a_{qq}| c_q(A) + \sigma r_q(A)\right).$$

Case IV: We suppose $\omega_q = |x_q|$, $\omega_p = |y_p|$, similar to the proof of Cases I, II, III, we can get

$$|\sigma^2 - |a_{pp}|^2| |\sigma^2 - |a_{qq}|^2| \leq \left(|a_{pp}| c_p(A) + \sigma r_p(A)\right)\left(|a_{qq}| c_q(A) + \sigma r_q(A)\right).$$

Thus, we complete the proof. \qed

**Remark 2** Since

$$\left(|a_{ii}| r_i(A) + \sigma c_i(A)\right)\left(|a_{jj}| r_j(A) + \sigma c_j(A)\right) \leq \left(|a_{ii}| + \sigma\right)\left(|a_{jj}| + \sigma\right)s_{ijs_j},$$
$$\left(|a_{ii}| c_i(A) + \sigma r_i(A)\right)\left(|a_{jj}| c_j(A) + \sigma r_j(A)\right) \leq \left(|a_{ii}| + \sigma\right)\left(|a_{jj}| + \sigma\right)s_{ijs_j},$$
$$\left(|a_{ii}| r_i(A) + \sigma c_i(A)\right)\left(|a_{jj}| c_j(A) + \sigma r_j(A)\right) \leq \left(|a_{ii}| + \sigma\right)\left(|a_{jj}| + \sigma\right)s_{ijs_j}$$

and

$$\left(|a_{ii}| r_i(A) + \sigma c_i(A)\right)\left(|a_{jj}| c_j(A) + \sigma r_j(A)\right) \leq \left(|a_{ii}| + \sigma\right)\left(|a_{jj}| + \sigma\right)s_{ijs_j},$$

the results in Theorem 3 are always better than the results in Theorem 1(ii).

We now establish comparison results between $\Gamma(A)$ and $\Omega(A)$.

**Theorem 4** If a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then

$$\sigma(A) \in \Omega(A) \subseteq \Gamma(A).$$

**Proof** Let $z$ be any point of $\Omega_3(A)$. Then there are $i, j \in N$, $i \neq j$, such that $z \in \Omega_3(A)$, i.e.,

$$|\sigma^2 - |a_{ii}|^2| |\sigma^2 - |a_{jj}|^2| \leq \left(|a_{ii}| r_i(A) + zc_i(A)\right)\left(|a_{jj}| c_j(A) + zr_j(A)\right). \quad (15)$$
If \(|a_{ii}|r_i(A) + zc_i(A))((|a_{jj}|c_j(A) + zr_j(A)) = 0\), then
\[ |z^2 - |a_{ii}|^2| = 0 \]
or
\[ |z^2 - |a_{jj}|^2| = 0. \]

Therefore, \(z \in \Gamma_1(A) \cup \Gamma_2(A)\). Moreover, if \((|a_{ii}|r_i(A) + zc_i(A))((|a_{jj}|c_j(A) + zr_j(A)) > 0\), then from inequality (15), we have
\[ \frac{|z^2 - |a_{ii}|^2|}{|a_{ii}|r_i(A) + zc_i(A)} \frac{|z^2 - |a_{jj}|^2|}{|a_{jj}|c_j(A) + zr_j(A)} \leq 1. \]  
(16)

Hence, from inequality (16), we have that
\[ \frac{|z^2 - |a_{ii}|^2|}{|a_{ii}|r_i(A) + zc_i(A)} \leq 1 \]
or
\[ \frac{|z^2 - |a_{jj}|^2|}{|a_{jj}|c_j(A) + zr_j(A)} \leq 1. \]

That is, \(z \in \Gamma_1(A)\) or \(z \in \Gamma_2(A)\), i.e., \(z \in \Gamma(A)\).

Similarly, if \(z\) is any point of \(\Omega_1(A)\) or \(\Omega_2(A)\), we can get
\[ \sigma(A) \in \Omega_1(A) \subseteq \Gamma(A) \]
and
\[ \sigma(A) \in \Omega_2(A) \subseteq \Gamma(A). \]

Thus, we complete the proof. \(\square\)

3 Numerical example

Example 1 Let
\[
A = \begin{bmatrix}
1 & 4 \\
0.1 & 0.5
\end{bmatrix}.
\]

The singular values of \(A\) are \(\sigma_1 = 4.1544\) and \(\sigma_2 = 0.0241\). From Figure 1, it is easy to see that Theorem 2 is better than Theorem 1 for certain examples. In Figure 2, we can see that the results in Theorem 3 are tighter than the results in Theorem 2, which is analyzed in Theorem 4.
4 Conclusion

In this paper, some new inclusion sets for singular values are given. Theoretical analysis and numerical example show that these estimates are more efficient than recent corresponding results in some cases.
Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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References
1. Qi, L.: Some simple estimates of singular values of a matrix. Linear Algebra Appl. 56, 105-119 (1984)
2. Li, L.: Estimation for matrix singular values. Comput. Math. Appl. 37, 9-15 (1999)
3. Brualdi, R.A.: Matrices, eigenvalues, and directed graphs. Linear Multilinear Algebra 11, 143-165 (1982)
4. Golub, G., Kahan, W.: Calculating the singular values and pseudo-inverse of a matrix. SIAM J. Numer. Anal. 2, 205-224 (1965)
5. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
6. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)
7. Johnson, C.R.: A Geršgorin-type lower bound for the smallest singular value. Linear Algebra Appl. 112, 1-7 (1989)
8. Johnson, C.R., Szulc, T.: Further lower bounds for the smallest singular value. Linear Algebra Appl. 272, 169-179 (1998)
9. Li, W., Chang, Q.: Inclusion intervals of singular values and applications. Comput. Math. Appl. 45, 1637-1646 (2003)
10. Li, H-B., Huang, T-Z., Li, H.: Inclusion sets for singular values. Linear Algebra Appl. 428, 2220-2235 (2008)