A PRIORI ESTIMATES AND BIFURCATION OF SOLUTIONS FOR A NONCOERCIVE ELLIPTIC EQUATION WITH CRITICAL GROWTH IN THE GRADIENT

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Abstract: We study nonnegative solutions of the boundary value problem

\[(P_\lambda) \quad -\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H^1_0(\Omega) \cap L^\infty(\Omega),\]

where \(\Omega\) is a smooth bounded domain, \(\mu, c \in L^\infty(\Omega), h \in L^r(\Omega)\) for some \(r > n/2\) and \(\mu, c, h \geq 0\). Our main motivation is to study the "noncoercive" case. Namely, unlike in previous work on the subject, we do not assume \(\mu\) to be positive everywhere in \(\Omega\).

In space dimensions up to \(n = 5\), we establish uniform a priori estimates for weak solutions of \((P_\lambda)\) when \(\lambda > 0\) is bounded away from 0. This is proved under the assumption that the supports of \(\mu\) and \(c\) intersect, a condition that we show to be actually necessary, and in some cases we further assume that \(\mu\) is uniformly positive on the support of \(c\) and/or some other conditions.

As a consequence of our a priori estimates, assuming that \((P_0)\) has a solution, we deduce the existence of a continuum \(C\) of solutions, such that the projection of \(C\) onto the \(\lambda\)-axis is an interval of the form \([0, a]\) for some \(a > 0\) and that the continuum \(C\) bifurcates from infinity to the right of the axis \(\lambda = 0\). In particular, for each \(\lambda > 0\) small enough, problem \((P_\lambda)\) has at least two distinct solutions.

KEYWORDS: Elliptic equation, critical growth in the gradient, a priori estimates, existence, multiplicity, bifurcation, \(L^p_\delta\) spaces, weighted Sobolev inequalities, Hardy inequalities

1. Introduction and main results.

In this article, we consider the following Dirichlet problem:

\[(P_\lambda) \quad \begin{cases} -\Delta u = \mu(x)|\nabla u|^2 + \lambda c(x)u + h(x), \\ u \in H^1_0(\Omega) \cap L^\infty(\Omega). \end{cases}\]

Here \(\Omega \subset \mathbb{R}^n\) is a bounded domain of class \(C^2\), \(\mu, c, h\) are given functions, whose regularity will be specified below, and \(\lambda\) is a real parameter. By a solution, we mean a weak solution in the sense of the usual integral formulation, with test-functions in \(H^1_0(\Omega) \cap L^\infty(\Omega)\). It is known (see [4]) that, under assumption (1.3) below, any solution \(u\) of \((P_\lambda)\) is Hölder continuous in \(\Omega\).
Elliptic equations with a gradient dependence up to the critical (quadratic) growth were studied by Boccardo, Murat and Puel in the 80’s and a large literature on the subject has appeared since then. Many results are known in the case $\lambda = 0$ (see e.g. [3, 19, 2, 13, 17, 14, 12]) or $\lambda < 0$ (see e.g. [7, 8]). We shall here consider the case $\lambda > 0$ and will be concerned with questions of a priori estimates, existence, multiplicity and bifurcation of solutions.

Denote the nonnegative solution set
\[ \Sigma = \{ (\lambda, u) \in [0, \infty) \times C(\overline{\Omega}); \ u \geq 0 \text{ and } u \text{ solves } (P_\lambda) \}. \]

Interesting properties of the set $\Sigma$ were recently established in [4] under the assumptions (with $n \geq 3$):
\begin{align}
(1.1) \quad & \mu \in L^\infty(\Omega), \ c, h \in L^r(\Omega) \text{ for some } r > n/2, \ c, h \not= 0, \\
(1.2) \quad & \mu(x) \geq \mu_0 > 0.
\end{align}

For every $\varepsilon > 0$, it is shown that nonnegative solutions of $(P_\lambda)$ with $\lambda \geq \varepsilon$ satisfy a uniform a priori estimate. Next assume in addition that $(P_0)$ has a (necessarily nonnegative) solution (see Remark 1.1 below for known sufficient conditions). Then it is shown in [4] that there exists a continuum $C \subset \Sigma$, such that the projection of $C$ onto the $\lambda$-axis is an interval of the form $[0, a]$ for some $a > 0$ and that the continuum $C$ bifurcates from infinity to the right of the axis $\lambda = 0$. In particular, for each $\lambda > 0$ small enough, $C$ contains at least two distinct solutions of $(P_\lambda)$.

We note that, when $\mu(x) = \mu$ is a positive constant, multiplicity results (actually up to an explicit value $\lambda_0 > 0$ of $\lambda$) have been obtained before in [18], using the transformation $v = e^{\mu u}$ (see also [1]). We stress that the multiplicity results in [4] allow nonconstant functions $\mu(x)$, in which case such a transformation is not available. However, the coercivity of the function $\mu(x)$, i.e. (1.2), is still needed in [4].

Our main goal here is to establish similar results as in [4] for noncoercive functions $\mu(x)$, namely to allow just $\mu \geq 0$, $\mu \not= 0$ (possibly at the expense of additional assumptions on $c$). The main difficulty here is to establish a priori estimates without assuming (1.2). Indeed, this assumption seems necessary in [4], in order to apply the method of Brezis and Turner [10] based on Hardy-Sobolev inequalities. Therefore, we need some new ideas (see after the statement of the results, for a brief description of the main arguments of our proofs).

We assume:
\begin{align}
(1.3) \quad & \mu, c \in L^\infty(\Omega), \ h \in L^r(\Omega) \text{ for some } r > \max(1, n/2), \ c, h \geq 0.
\end{align}

Also, we shall make the following essential assumption of intersecting supports for $\mu$ and $c$:
\begin{align}
(1.4) \quad & \mu, c \geq \eta \text{ on } B(x_0, \rho) \subset \Omega \text{ for some } \rho, \eta > 0.
\end{align}

In low dimensions $n \leq 2$, this turns out to be sufficient to guarantee a priori estimates.
**Theorem 1.** Let $n \leq 2$ and assume (1.3), (1.4). Then for any $\Lambda_1 > 0$ there exists a constant $M > 0$ such that, for each $\lambda \geq \Lambda_1$, any nonnegative solution of $(P_\lambda)$ satisfies $\|u\|_\infty \leq M$.

We shall see right away that the assumption (1.3) of intersecting supports for $\mu$ and $c$ is also essentially necessary.

**Theorem 2.** Consider problem $(P_\lambda)$ with

$$n = 1, \quad \Omega = (0, 3), \quad \mu(x) = \chi_{(1,2)}, \quad c(x) = \chi_{(2,3)}, \quad h = 0.$$ 

There exists a sequence $\lambda = \lambda_j \to \pi^2/4$ and a sequence $u_j \in H^2(\Omega) \cap H^1_0(\Omega)$ of solutions of $(P_\lambda)$ such that $\|u_j\|_\infty \to \infty$ as $j \to \infty$.

We now turn to the higher dimensional range $3 \leq n \leq 5$. In this case, beside (1.4), we need additional assumptions to guarantee a priori estimates. In our next result, we shall assume, roughly speaking, that $\mu$ is positively bounded below on the support of $c$. However, we do not know presently whether this assumption is technical or not (nor the assumptions in Theorem 4 below). In particular, we do not know if the a priori estimates are still true in dimensions $n \geq 6$. The restriction $n \leq 5$ comes from the fact that our method (see at the end of this section) requires to estimate the function $c(x)u$ in $L^p$ for some $p > n/2$, whereas the $L^p$ estimates that we are able to derive are limited to $p \leq 2n/(n-2)$, due to Sobolev imbeddings or to even more stringent functional inequalities (note that $2n/(n-2)$ and $n/2$ precisely coincide for $n = 6$).

**Theorem 3.** Let $3 \leq n \leq 5$ and let (1.3), (1.4) be satisfied. Assume that there exists a $C^2$ domain $\omega \subset \Omega$ and a constant $\mu_0 > 0$, such that

$$\mu \geq \mu_0 \text{ on } \omega \text{ and } \text{Supp}(c) \subset \overline{\omega}.$$ 

If $n = 5$, assume in addition that

$$c(x) \leq C_1 [\text{dist}(x, \partial \omega)]^\sigma, \quad x \in \omega, \quad \text{for some } C_1, \sigma > 0.$$ 

Then for any $\Lambda_1 > 0$ there exists a constant $M > 0$ such that, for each $\lambda \geq \Lambda_1$, any nonnegative solution of $(P_\lambda)$ satisfies $\|u\|_\infty \leq M$.

As usual, Supp is here understood in the sense of essential support. As a special case of Theorem 3, we see that the a priori estimate holds for instance if $3 \leq n \leq 5$, $c$ is compactly supported and $\mu \geq \mu_0 > 0$ on a neighborhood of $c$.

We now give additional results in the case of dimension $n = 3$, which is rather special. Indeed, without assuming (1.5), we can then obtain a priori estimates under various, relatively mild, assumptions either on $\mu$ or $c$. 

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Theorem 1. Let $n \leq 2$ and assume (1.3), (1.4). Then for any $\Lambda_1 > 0$ there exists a constant $M > 0$ such that, for each $\lambda \geq \Lambda_1$, any nonnegative solution of $(P_\lambda)$ satisfies $\|u\|_\infty \leq M$.

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As usual, Supp is here understood in the sense of essential support. As a special case of Theorem 3, we see that the a priori estimate holds for instance if $3 \leq n \leq 5$, $c$ is compactly supported and $\mu \geq \mu_0 > 0$ on a neighborhood of $c$.

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**Theorem 4.** Let \( n = 3 \) and (1.3), (1.4) be satisfied. Assume in addition that either
\[
\text{(1.7)} \quad c(x) \leq C_1 [\text{dist}(x, \partial \Omega)]^\sigma, \quad x \in \Omega, \quad \text{for some } C_1, \sigma > 0,
\]
or
\[
\text{(1.8)} \quad \text{Supp}(\mu) \subset \subset \Omega,
\]
or
\[
\text{(1.9)} \quad \mu(x) \geq c_1 [\text{dist}(x, \partial \Omega)]^\sigma \quad \text{on a neighborhood of } \partial \Omega \text{ for some } c_1 > 0 \text{ and } \sigma < 2.
\]
Then for any \( \Lambda_1 > 0 \) there exists a constant \( M > 0 \) such that, for each \( \lambda \geq \Lambda_1 \), any nonnegative solution of \((P_\lambda)\) satisfies \( \|u\|_\infty \leq M \).

As a consequence of Theorems 1, 3, 4 and of [4, Theorem 1.2] (see also the proof of Theorem 1.3 in [4]), one deduces the following result on existence, multiplicity and bifurcation.

**Theorem 5.** Assume that \((P_0)\) has a solution and let the assumptions of Theorem 1 (resp. Theorem 3, 4) be in force. Then there exists a continuum \( \mathcal{C} \subset \Sigma \), such that the projection of \( \mathcal{C} \) onto the \( \lambda \)-axis is an interval of the form [0, \( a \)] for some \( a > 0 \) and that the continuum \( \mathcal{C} \) bifurcates from infinity to the right of the axis \( \lambda = 0 \). In particular, for each \( \lambda > 0 \) small enough, problem \((P_\lambda)\) has at least two distinct nonnegative solutions.

**Remarks 1.1.** (a) The existence of a solution for \((P_0)\) is known under various smallness assumptions. For instance, in [12], it is assumed that
\[
\|\mu\|_\infty \|h\|_{n/2} < S_n^2 = \inf \left\{ \frac{\|\nabla \phi\|_2^2}{\|\phi\|_2^2} ; 0 \neq \phi \in H^1_0(\Omega) \right\}
\]
(where \( S_n \) is the best constant in the Sobolev inequality and \( n \geq 3 \)). In the particular case \( \mu(x) = \mu_0 > 0 \) and \( h \geq 0 \), a more precise sufficient condition for the existence of a solution for \((P_0)\) is given [1] by
\[
\mu < \inf \left\{ \frac{\int_\Omega |\nabla \phi|^2 \, dx}{\int_\Omega h(x) \phi^2 \, dx} ; 0 \neq \phi \in H^1_0(\Omega) \right\}.
\]
(b) We stress that some of our results could be extended to the case when the function \( c \) belongs to \( L^p \) for suitable \( p \) (instead of \( L^\infty \)). However, since our main motivation here is to treat the case of noncoercive \( \mu \), we have left this aside from simplicity.

To prove a priori estimates without the coercivity assumption (1.2), we shall combine various ingredients. We first reduce the desired uniform bounds to estimating the term \( c(x)u \) in a suitable \( L^q \) space (namely, \( q > n/2 \); see Proposition 1). This is done by
using exponential test-functions and auxiliary unknowns, in the spirit of, e.g., Boccardo-Murat-Puel [7]. To derive the necessary $L^q$ estimates, we first get (Lemma 5) some basic weighted $L^1$ estimates for $u$ and for the right-hand side of $(P_\lambda)$. This is obtained by using the quantitative Hopf Lemma in Brezis-Cabrè [9], and a test-function which solves a singular auxiliary problem (originally from Crandall-Rabinowitz-Tartar [11]). We then apply smoothing effects in $L^p_\delta$, the Lebesgue spaces weighted by the distance to the boundary, along with suitable interpolation. Note that such smoothing effects were used before for problems without gradient terms (see [6, 15, 22]) but not, as far as we know, for problems with (critical) gradient terms. In higher dimensional cases, we also need to take advantage of the weighted $H^1$ bound guaranteed by the estimate of the right-hand side. To this end, we rely on suitable weighted Sobolev and Hardy inequalities (different from those in [4] or [10]); see Proposition 2 in section 2.4 below.

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2. Proofs of a priori estimates.

In the rest of the article, we denote by

$$\delta(x) = \delta_\Omega(x) = \text{dist}(x, \partial \Omega)$$

the function distance to the boundary (the subscript $\Omega$ will be dropped when no confusion arises). Conjugate exponents will be denoted by $'$ (i.e., $p' = p/(p - 1)$, $1 < p < \infty$). For $1 \leq p \leq \infty$, the $L^p(\Omega)$ and $L^p_\delta(\Omega)$ norms will be denoted by $\| \cdot \|_p$ and $\| \cdot \|_{p,\delta}$. Recall that

$$\|u\|_{p,\delta} = \left( \int_\Omega |u(x)|^p \delta(x) \, dx \right)^{1/p}, \quad 1 \leq p < \infty.$$ 

The notation will be also used for $0 < p < 1$ although it is not a norm in this case.

Moreover, we recall (see [4]) that problem $(P_\lambda)$ admits no nontrivial nonnegative solutions for $\lambda > \gamma_1$, where $\gamma_1 > 0$ is the first eigenfunction of the problem

$$-\Delta \varphi = \gamma_1 c(x) \varphi, \quad \varphi \in H^1_0(\Omega).$$

2.1. Reduction to a suitable $L^q$ estimate.

The goal of this subsection is to reduce the proof of $L^\infty$ estimate to a suitable $L^q$ estimate. We shall prove:

**Proposition 1.** Assume (1.3). Set $q = n/2$ if $n \geq 3$ or fix any $q \in (1, \infty)$ if $n \leq 2$. Let $\varepsilon, \alpha, M > 0$. There exists $C_1 > 0$ such that, for any $\lambda \in [0, \gamma_1]$ and any nonnegative solution of $(P_\lambda)$,

$$\|cu^{1+\varepsilon}\|_q + \|u\|_\alpha \leq M \implies \|u\|_\infty \leq C_1.$$ 

The proposition is a direct consequence of the following two lemmas.
Lemma 1. Assume (1.3). There exists $K = K(r, n, \|\mu\|_\infty) > 0$ with the following property. For any $M_1 > 0$, there exists $C_1 > 0$ such that, for any $\lambda \in [0, \gamma_1]$ and any nonnegative solution of $(P_\lambda)$,

$$\|e^{Ku}\|_1 \leq M_1 \implies \|u\|_\infty \leq C_1.$$ 

Lemma 2. Assume (1.3). Let $q$ be as in Proposition 1 and let $\varepsilon, \alpha, k, M > 0$. There exists $M_1 > 0$ such that, for any $\lambda \in [0, \gamma_1]$ and any nonnegative solution of $(P_\lambda)$,

$$\|cu^{1+\varepsilon}\|_q + \|u\|_\alpha \leq M \implies \|e^{ku}\|_1 \leq M_1.$$ 

Proof of Lemma 1. For any $k > 0$, we have $e^{ku} - 1 \in H^1_0(\Omega)$, due to $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$. We claim that

$$(2.1) \quad -\Delta(e^{ku} - 1) = ke^{ku}((\mu(x) - k)|\nabla u|^2 + \lambda c(x)u + h(x))$$

in the weak sense in $\Omega$ (i.e., with test functions in $H^1_0(\Omega) \cap L^\infty(\Omega)$). Indeed, setting $w = e^{ku} - 1$ and $F = \mu|\nabla u|^2 + \lambda cu + h \in L^1(\Omega)$, we compute, for any $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \nabla w \cdot \nabla \phi = \int_\Omega ke^{ku} \nabla u \cdot \nabla \phi = \int_\Omega \left(\nabla(ke^{ku}\phi) - k^2 e^{ku} \phi \nabla u\right) \cdot \nabla u.$$ 

On the other hand, using $ke^{ku}\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ as a test-function in $(P_\lambda)$, we have

$$\int_\Omega \nabla(ke^{ku}\phi) \cdot \nabla u = \int_\Omega ke^{ku}\phi F.$$ 

It follows that

$$\int_\Omega \nabla w \cdot \nabla \phi = \int_\Omega ke^{ku}(F - k|\nabla u|^2)\phi,$$

hence the claim $(2.1)$.

Now choosing $k = \|\mu\|_\infty$, as a consequence of $(2.1)$, we have

$$-\Delta(e^{ku} - 1) \leq g := ke^{ku}(\lambda c(x)u + h(x))$$

in the weak sense in $\Omega$. Since $r > \max(1, n/2)$, we may fix $r_1 \in (\max(1, n/2), r)$ and $r_2 \in (1, \infty)$ such that $\frac{1}{r_1} = \frac{1}{r} + \frac{n}{r_2}$. Using the maximum principle, the standard $L^{r_1}$-$L^\infty$ estimate for the Dirichlet Laplacian with $r_1 > n/2$ (see, e.g., [23, Proposition 47.5]), Hölder’s inequality and $e^u \geq u$, we then obtain

$$\|e^{ku} - 1\|_\infty \leq C(r_1, \Omega)\|g\|_{r_1} \leq kC(r_1, \Omega)[\lambda\|c\|_\infty \|e^{(k+1)u}\|_{r_1} + \|e^{ku}\|_{r_2}\|h\|_{r_2}].$$

The Lemma follows with $K = \max((k + 1)r_1, kr_2)$. 

□
Proof of Lemma 2. If \( n \leq 2 \), we may assume \( q < r \) without loss of generality. Fix \( M > 0 \) and assume

\[
\|cu^{1+\varepsilon}\|_q + \|u\|_\alpha \leq M. \tag{2.2}
\]

For \( k > 0 \), testing problem \((P_\lambda)\) with \( e^{ku} - 1 \in H^1_0(\Omega) \cap L^\infty(\Omega) \), we get

\[
\int_\Omega ke^{ku}|\nabla u|^2 = \int_\Omega (e^{ku} - 1)(\mu(x)|\nabla u|^2 + \lambda c(x)u + h(x)).
\]

Assume \( k \geq 1 + \|\mu\|_\infty \) without loss of generality. Then

\[
\int_\Omega e^{ku}|\nabla u|^2 \leq \int_\Omega (e^{ku} - 1)(\lambda c(x)u + h(x)).
\]

Set \( s = q' \) and note that \( s = n/(n-2) \) if \( n \geq 3 \). By Sobolev’s inequality, we get

\[
\|e^{ku/2} - 1\|_{2s}^2 \leq C \int_\Omega |\nabla(e^{ku/2} - 1)|^2 \leq C\frac{k^2}{4} \int_\Omega (e^{ku} - 1)(\lambda c(x)u + h(x)). \tag{2.3}
\]

Here and in the rest of the proof, \( C \) denotes a generic constant independent of \( u, \lambda \) and \( k \).

Also, using \( X - 1 \leq 2(\sqrt{X} - 1)^2 + C \) for all \( X \geq 1 \), we have

\[
\|e^{ku} - 1\|_s \leq C\|e^{ku/2} - 1\|_s^2 + C \leq C\|e^{ku/2} - 1\|_{2s}^2 + C. \tag{2.4}
\]

Next, for any \( \tau \in (0, 1) \), by (2.3), (2.4), \( \lambda \leq \gamma_1 \) and \( s' = q \), we have

\[
\|e^{ku} - 1\|_s - C \leq Ck^2\left\| \frac{e^{ku} - 1}{u^\tau} \right\|_s \left( \|cu^{1+\tau}\|_q + \|hu^\tau\|_q \right). \tag{2.5}
\]

Recalling \( q < r \) and choosing \( \tau = \min(1, \varepsilon, \alpha(r - q)/(rq)) \), we deduce from (2.2) that

\[
\|hu^\tau\|_q^q \leq \int_\Omega h^\tau + \int_\Omega u^{\tau q/(r-q)} \leq \|h\|_r^r + |\Omega| + \int_\Omega u^\alpha \leq C + M^\alpha
\]

and

\[
\|cu^{1+\tau}\|_q \leq \|c\|_q + \|cu^{1+\varepsilon}\|_q \leq C + M.
\]

It thus follows from (2.5) that

\[
\|e^{ku} - 1\|_s \leq k^2C_0(M)\left\| \frac{e^{ku} - 1}{u^\tau} \right\|_s + C.
\]

where \( C_0(M) \) is a positive constant depending on \( M \) but otherwise independent of \( u, \lambda \).

Now observe that, for any \( A > 0 \),

\[
\left\| \frac{e^{ku} - 1}{u^\tau} \right\|_s \leq \left\| \frac{e^{ku} - 1}{u^\tau} \chi\{u^\tau \geq A\} \right\|_s + \left\| \frac{e^{ku} - 1}{u^\tau} \chi\{u^\tau < A\} \right\|_s \leq A^{-1}\|e^{ku} - 1\|_s + C(A, k, \tau).
\]

Choosing \( A = 2k^2C_0(M) \), we get

\[
\|e^{ku} - 1\|_s \leq C_1(M, k, \alpha, \varepsilon),
\]

which implies the desired estimate.

For further reference, we give the following elementary interpolation Lemma.
Lemma 3. Let $\omega$ be any open subset of $\mathbb{R}^n$ and $\phi \in L^\infty(\omega)$ be such that $\phi > 0$ a.e. Let $1 \leq q \leq m \leq r < \infty$ and let $b, d \geq 0$ and $\gamma \in \mathbb{R}$ be such that

\begin{equation}
\gamma \geq d - \frac{(r - m)(b + d)}{r - q},
\end{equation}

We have, for any measurable function $v$ on $\omega$,

\[ \int_\omega \phi^\gamma |v|^m \, dx \leq C \left( \int_\omega \phi^{-b} |v|^q \, dx \right)^\theta \left( \int_\omega \phi^d |v|^r \, dx \right)^{1-\theta}, \quad \theta = \frac{r - m}{r - q} \]

(assuming both integrals on the RHS to be finite).

Proof. It suffices to notice that $m = \theta q + (1 - \theta)r$, $\gamma \geq -\theta b + (1 - \theta)d$ and to use Hölder’s inequality and $\phi \in L^\infty(\omega)$.

2.2. Weighted $L^q$ estimates.

In the rest of section 2, when $u$ is a solution of $(P_\lambda)$ we set

\begin{equation}
f = f_u := \mu(x)|\nabla u|^2 + \lambda c(x)u + h(x).
\end{equation}

Also, $C$ will denote a generic constant independent of $u$ and of $\lambda \in [\Lambda_1, \gamma_1]$, but possibly depending on all the other data and parameters (the dependence on some particular parameters will be emphasized if necessary).

We start with an integral a priori estimate of $u$ on a ball where both $\mu$ and $c$ are positively bounded below.

Lemma 4. Assume (1.3), (1.4) and $0 < \Lambda_1 \leq \lambda \leq \gamma_1$. Then any nonnegative solution $u$ of $(P_\lambda)$ satisfies

\begin{equation}
\int_{B_{\rho/2}(x_0)} e^{nu} \leq C(\eta, \rho).
\end{equation}

Proof. Arguing similarly as in the proof of Lemma 1, we have, in the distribution sense,

\[ -\Delta(e^{nu}) = \eta e^{nu}((\mu(x) - \eta)|\nabla u|^2 + \lambda c(x)u + h(x)) \geq \Lambda_1 \eta^2 e^{nu}u \quad \text{in } B := B_\rho(x_0), \]

where we used (1.3) for the last inequality. Therefore, for all $A > 0$, there exists $C(A) > 0$ such that

\[ -\Delta(e^{nu}) \geq Ae^{nu} - C(A) \quad \text{in } B. \]

Denote respectively by $\lambda_{1,B}$ and $\varphi_{1,B}$ the first Dirichlet eigenvalue and eigenfunction of $B$ and take $A = 1 + \lambda_{1,B}$. Testing this inequality with $\varphi_{1,B} \in H^1_0(B) \cap C(\overline{B})$, we easily obtain

\[ \int_{B_{\rho/2}(x_0)} e^{nu} \leq C \int_B e^{nu} \varphi_{1,B} \leq C. \]

The next lemma gives our primary a priori estimates for $u$. 

\[ \square \]
**Lemma 5.** Assume (1.3), (1.4) and \(0 < \Lambda_1 \leq \lambda \leq \gamma_1\). Then any nonnegative solution \(u\) of \((P_\lambda)\) satisfies the following a priori estimates:

\[
(2.9) \quad \int_\Omega |\nabla u|^2 \mu(x) \delta(x) \, dx \leq C,
\]

\[
(2.10) \quad \|u\|_{p, \delta} \leq C(p), \quad p < (n + 1)/(n - 1),
\]

\[
(2.11) \quad \|\delta^{-\gamma} u\|_1 \leq C(\gamma), \quad 0 < \gamma < 1.
\]

**Proof.** It is well known (see [9]) that

\[
(2.12) \quad u \geq c_1 \cdot \left( \int_\Omega f \delta \right) \delta \quad \text{in } \Omega.
\]

for some constant \(c_1 = c_1(\Omega) > 0\). From (2.8), (2.12), we obtain

\[
(2.13) \quad \|f\|_{1, \delta} = \int_\Omega f \delta \leq C \left( \int_{B_{\rho/2}(x_0)} u \right) \left( \int_{B_{\rho/2}(x_0)} \delta \right)^{-1} \leq C,
\]

hence in particular (2.9). By the \(L^1_\delta-L^p_\delta\) estimate for the Dirichlet Laplacian (see [15]), we deduce (2.10) from (2.13).

Next, for each \(\gamma \in (0, 1)\), it is known (see [23, Lemma 10.4] and cf. the references in [23]), that there exists a function \(\xi \in H^1_0(\Omega)\) such that \(-\Delta \xi = \delta - \gamma\) in \(\mathcal{D}'(\Omega)\) and such that \(\xi \leq C\delta\) in \(\Omega\). (Note that the RHS \(\delta^{-\gamma}\) belongs to \(L^1(\Omega)\); see after (A.5).) Actually, for any \(v \in H^1_0(\Omega) \cap L^\infty(\Omega)\), we have

\[
\int_\Omega \nabla v \cdot \nabla \xi = \int_\Omega v \delta^{-\gamma}.
\]

Testing \((P_\lambda)\) with \(\xi\) and using (2.13), we obtain

\[
\int_\Omega u \delta^{-\gamma} = \int_\Omega \nabla u \cdot \nabla \xi = \int_\Omega f \xi \leq C \int_\Omega f \delta \leq C,
\]

hence (2.11). 

\[
\square
\]

**2.3. Proof of Theorem 1 and of Theorem 4 under assumption (1.8) or (1.7).**

**Proof of Theorem 3 under assumption (1.8).** Since \(\text{Supp}(\mu) \subset \subset \Omega\) by assumption, estimate (2.9) implies \(\int_\Omega |\nabla u|^2 \mu(x) \, dx \leq C\). This combined with (2.11) guarantees that
\[ \|f\|_1 \leq C. \] We then deduce from the standard \( L^1 - L^p \) estimate for the Dirichlet Laplacian that

\[ \|u\|_p \leq C(p), \quad p < n/(n-2). \]

Since \( n/(n-2) > n/2 \) for \( n = 3 \) and since \( c \in L^\infty(\Omega) \), the desired conclusion follows from Proposition 1. \( \square \)

**Proof of Theorem 1 and of Theorem 3 under assumption (1.7).** We assume \( n = 2 \) or \( 3 \) (the case \( n = 1 \) can be handled with obvious modifications). Take \( \gamma = 0 \) if \( n = 2 \) and any \( \gamma > 0 \) if \( n = 3 \). We claim that, for \( \varepsilon > 0 \) small,

\[ (2.14) \quad \int_\Omega \delta^\gamma u^{a+\varepsilon} \leq C(\varepsilon). \]

To this end, we interpolate between (2.10) and (2.11), applying Lemma 3 with \( \phi = \delta_\Omega \),

\[ m = \frac{n}{2} + \varepsilon, \quad q = 1, \quad r = \frac{n+1}{n-1} - \varepsilon, \quad b = 1 - \varepsilon, \quad d = 1, \]

with \( \varepsilon > 0 \) small (note that \( q \leq m \leq r \)). Taking \( \varepsilon > 0 \) small enough, condition (2.6) is true provided

\[ \gamma > 1 - 2 \left( \frac{n+1}{n-1} - \frac{n}{2} \right) = 1 - \frac{2(n+1) - n(n-1)/2}{2} = \frac{n(n-3)}{2} \]

and the claim follows.

We now easily deduce from (2.14) that

\[ (2.15) \quad \int_\Omega c^{n/2} u^{a+\varepsilon} \leq C. \]

Indeed, if \( n = 2 \), this is true due to \( \gamma = 0 \) and \( c \in L^\infty(\Omega) \). If \( n = 3 \), since \( c \leq C_1 \delta^\sigma \) by assumption (1.7), this follows from (2.14) with \( \gamma = \sigma n/2 > 0 \).

The desired conclusion is now a direct consequence of Proposition 1. \( \square \)

**2.4. Weighted Sobolev and Hardy inequalities and proof of Theorem 3 and of Theorem 4 under assumption (1.9).**

To go further, we want to exploit the weighted \( H^1 \) nature of estimate (2.9). To this end, a key role will be played by the following weighted Sobolev and Hardy inequalities.

**Proposition 2.** Let \( n \geq 2 \) and let \( \omega \subset \mathbb{R}^n \) be a bounded open set of class \( C^2 \) and denote \( \delta(x) = \text{dist}(x, \partial \omega) \). Let \( 1 \leq p < n \), \( a > p - 1 \) and \( k \geq 0 \).

(i) Set \( p^* = np/(n-p) \). We have

\[ (2.16) \quad \left( \int_\omega \delta^{na/(n-p)} |v|^{p^*} \right)^{p/p^*} \leq C \left( \int_\omega \delta^k |v| \right)^p + C \int_\omega \delta^a |

\[ \nabla v|^{p} \]
for all $v \in W^{1,p}(\omega)$.

(ii) We have

\begin{equation}
\int_\omega \delta^{a-p} |v|^p \leq C \left( \int_\omega \delta^k |v| \right)^p + C \int_\omega \delta^a |\nabla v|^p
\end{equation}

for all $v \in W^{1,p}(\omega)$.

Related results of Hardy and/or weighted Sobolev type have been known for a long time, see for instance [20, 5, 21, 16] and the references therein. However, we haven’t found a suitable reference for this specific statement. Actually, the available results seem to involve $|v|^p$ instead of $|v|$ on the RHS, which is not sufficient for our needs, or they may impose other restrictions such as lower bound on $k$ (although they can be more general in terms of weights or of domain regularity). We thus give a proof in appendix.

Note that, since $u$ is not assumed to vanish on the boundary, the restriction $a > p-1$ is necessary in assertion (ii) (notice that if one would take $0 \leq a \leq p-1$ with, for instance, $v \equiv 1$, then the LHS integral would become divergent, contradicting the finiteness of the RHS).

**Proof of Theorem 3.** Let $\omega$ be as in the statement of the theorem. By (1.5) and (2.9), we have

\begin{equation}
\int_\omega \delta_{\omega} |\nabla u|^2 \leq \int_\omega \delta_{\Omega} |\nabla u|^2 \leq C.
\end{equation}

Applying Propostion 2(ii) with $p = 2$, $a = 1 + \varepsilon$, $k = 0$, if follows from (2.18) and (2.11) that

\begin{equation}
\int_\omega \delta^{-1+\varepsilon}_{\omega} u^2 \leq C(\varepsilon).
\end{equation}

If $n = 3$, we directly deduce that, for $\varepsilon > 0$ small,

\[ \int_{\Omega} c^{n/2} u^{2+\varepsilon} \leq C \int_{\omega} u^{2+\varepsilon} \leq C. \]

Next assume $n = 4$ or 5. Applying Proposition 2(i) with $p = 2$, $a = 1 + \varepsilon$, $k = 0$, it follows from (2.18) and (2.11) that

\begin{equation}
\int_\omega \delta^{n(1+\varepsilon)/(n-2)} u^{2n/(n-2)} \leq C.
\end{equation}

We now interpolate between (2.19) and (2.20). To this end, we use Lemma 3 with $\phi = \delta_\omega$,

\[ m = \frac{n}{2} + \varepsilon, \quad q = 2, \quad r = \frac{2n}{n-2}, \quad b = 1 - \varepsilon, \quad d = \frac{n(1 + \varepsilon)}{n-2}, \]
for \( \varepsilon > 0 \) small (note that \( q \leq m \leq r \)). It follows that

\[
(2.21) \quad \int_\omega \delta_\omega^{\gamma} u^{\frac{n}{2} + \varepsilon} \leq C
\]

whenever \( \gamma \) satisfies condition (2.6). Taking \( \varepsilon > 0 \) small enough, this is true provided

\[
\gamma > \frac{n}{n - 2} - \frac{2(n - 1)}{n - 2} \left( \frac{2n}{n - 2} - \frac{n - 2}{2} \right) \frac{n - 2}{4} = \frac{n}{n - 2} - \frac{n(6 - n)}{n - 2} \frac{n - 1}{4}
\]

i.e., \( \gamma > n(n - 5)/4 \).

If \( n = 4 \), we may take \( \gamma = 0 \) in (2.21). Choosing \( \varepsilon > 0 \) sufficiently small and using \( \text{supp}(c) \subset \overline{\Omega} \) and \( c \in L^\infty(\Omega) \), we get

\[
\int_\Omega c^{n/2} u^{\frac{n}{2} + \varepsilon} \leq C \int_\omega u^{\frac{n}{2} + \varepsilon} \leq C.
\]

Finally, if \( n = 5 \), by assumption (1.6), we have \( c \leq C_1 \delta_\omega \chi_{\overline{\Omega}} \). We may take \( \gamma = \sigma n/2 > 0 \) in (2.21). Choosing \( \varepsilon > 0 \) sufficiently small, we get

\[
\int_\Omega c^{n/2} u^{\frac{n}{2} + \varepsilon} \leq C \int_\omega \delta_\omega^{\sigma n/2} u^{\frac{n}{2} + \varepsilon} \leq C.
\]

The conclusion follows from Proposition 1.

---

**Proof of Theorem 4 under assumption (1.9).** By assumption, there exists \( \eta > 0 \) such that the tubular neighborhood \( \Omega_\eta = \{ x \in \Omega; \; \delta_\Omega(x) < \eta \} \) is \( C^2 \)-smooth and

\[
\mu(x) \geq c_1 \delta_\Omega^\sigma(x) \text{ on } \overline{\Omega}_\eta.
\]

for some \( \sigma < 2 \) and \( c_1 > 0 \). Without loss of generality, we can assume \( \sigma > 1 \). By (2.9) we have

\[
(2.22) \quad \int_{\Omega_\eta} \delta_\Omega^{\sigma + 1} |\nabla u|^2 \leq \int_{\Omega_\eta} \delta_\Omega^{\sigma + 1} |\nabla u|^2 \leq C.
\]

Applying Proposition 2(ii) with \( p = 2 \), \( a = 1 + \sigma \), \( k = 0 \), this along with (2.11) guarantees

\[
\int_{\Omega_\eta} \delta_\Omega^{\sigma - 1} u^2 \leq C.
\]

Next observe that \( \delta_{\Omega_\eta}(x) = \delta_\Omega(x) \) on \( \Omega_{\eta/2} \), hence, recalling \( \sigma > 1 \),

\[
(2.23) \quad \int_{\Omega_{\eta/2}} \delta_\Omega^{\sigma - 1} u^2 \leq C.
\]
On the other hand, by (2.11), we have, for all \( \varepsilon > 0 \),
\[
\int_{\Omega_{n/2}} \delta_{\varepsilon}^{-1} u \leq C(\varepsilon).
\]

We now interpolate between (2.23) and (2.24). (It can be checked that working, instead of the Hardy-type estimate (2.23), with a weighted Sobolev estimate deduced from (2.22) and Proposition 2(i), would not improve the conditions.) To this end, we apply Lemma 3 with \( \omega = \Omega_{n/2}, \phi = \delta_{\Omega} \),
\[
m = \frac{3}{2} + \varepsilon, \quad q = 1, \quad r = 2, \quad \gamma = 0, \quad b = 1 - \varepsilon, \quad d = \sigma - 1,
\]
with \( \varepsilon > 0 \) small (note that \( q \leq m \leq r \)). Taking \( \varepsilon \in (0, 1/4) \) small enough, condition (2.6) is true due to
\[
\sigma - 1 - \left(2 - \frac{3}{2}\right)\sigma = \frac{\sigma}{2} - 1 < 0,
\]
hence
\[
\int_{\Omega_{n/2}} u^{(3/2)+\varepsilon} \leq C.
\]
Since, owing to (2.10), \( n = 3 \) and \( \varepsilon < 1/4 \), we also have
\[
\int_{\Omega \setminus \Omega_{n/2}} u^{(3/2)+\varepsilon} \leq C \left( \int_{\Omega \setminus \Omega_{n/2}} \delta_{\Omega} u^{2-\varepsilon} \right)^{(3/2)+\varepsilon}/(2-\varepsilon) \leq C,
\]
we finally obtain
\[
\int_{\Omega} c^{3/2} u^{\frac{3}{2}+\varepsilon} \leq C
\]
and conclude by Proposition 1.

\[3. \text{ Proof of Theorem 5.}\]

For each integer \( j \geq 1 \), we seek \( u = u_j \) under the form
\[
u(x) = \begin{cases} 
 jx, & 0 \leq x < 1, \\
 j + \log(1 + j(x - 1)), & 1 \leq x < 2, \\
 A_j \sin((\frac{\pi}{2} + \varepsilon_j)(3 - x)), & 2 \leq x \leq 3,
\end{cases}
\]
where \( A_j, \varepsilon_j > 0 \) are to be determined. The function \( u \) satisfies
\[
-u''(x) = \begin{cases} 
 0, & 0 < x < 1, \\
 u^2, & 1 < x < 2, \\
 (\frac{\pi}{2} + \varepsilon_j)^2 u, & 2 < x < 3.
\end{cases}
\]
Moreover \( u(0) = u(3) = 0 \) and \( u, u' \) are continuous at \( x = 1 \). It thus suffices to choose \( A_j \) and \( \varepsilon_j \) in such a way as to ensure the continuity of \( u \) and \( u' \) at \( x = 2 \). We compute

\[
u(2^-) = j + \log(j + 1), \quad u(2^+) = A_j \sin\left(\frac{\pi j}{2} + \varepsilon_j \right) = A_j \cos \varepsilon_j
\]

and

\[
u'(2^-) = \frac{j}{j + 1}, \quad u'(2^+) = -\left(\frac{\pi j}{2} + \varepsilon_j \right) A_j \cos \left(\frac{\pi j}{2} + \varepsilon_j \right) = \left(\frac{\pi j}{2} + \varepsilon_j \right) A_j \sin \varepsilon_j.
\]

The conditions \( u(2^-) = u(2^+) \) and \( u'(2^-) = u'(2^+) \) are then equivalent to

\[(3.1) \quad A_j \cos \varepsilon_j = j + \log(j + 1).\]

and

\[(3.2) \quad \left(\frac{\pi j}{2} + \varepsilon_j \right) \tan \varepsilon_j = \frac{j}{j + 1} \left( j + \log(j + 1) \right)^{-1}.
\]

Since the function \( s \mapsto \left(\frac{\pi j}{2} + s \right) \tan s \) is strictly increasing from \([0, \pi/2)\) onto \([0, +\infty)\), there exists (a unique) \( \varepsilon_j \in (0, \pi/2) \) satisfying (3.2) and \( A_j \) is then directly given by (3.1). Finally, we observe that \( \|u_j\|_\infty \geq u_j(1) = j \) and that \( \varepsilon_j \to 0 \) as \( j \to \infty \). The result is proved. \( \square \)

**Appendix: Proof of Proposition 2**

By density (and Fatou’s Lemma) it suffices to establish (2.16) and (2.17) for \( v \in C^1(\omega) \) with \( v \geq 0 \), which we assume in the sequel. In this proof, \( C \) will denote a generic positive constant independent of \( v \). We proceed in several steps.

**Step 1.** Fix \( 1 \leq p < n \), \( a > p - 1 \) and \( k \geq 0 \). Denote by \( \lambda_1 \) and \( \varphi \) the first eigenvalue and eigenfunction of \( -\Delta \) in \( H^1_0(\omega) \). (If \( \omega \) is not connected, we take \( \varphi \) to be positive and \( L^1 \)-normalized on each connected component of \( \omega \).) By elliptic regularity, we have \( \varphi \in W^{2,q}(\omega) \cap C^2(\omega) \) for all \( q \in (1, \infty) \). Then, as a consequence of the Hopf Lemma, there exist constants \( c_1, c_2, \eta > 0 \), such that

\[(A.1) \quad c_1 \delta(x) \leq \varphi(x) \leq c_2 \delta(x), \quad x \in \omega
\]

and

\[(A.2) \quad c_1 \leq |\nabla \varphi| \leq c_2 \quad \text{on} \ \omega_\eta = \{x \in \omega; \ \delta(x) \leq \eta \}.
\]

In particular, by (A.1), it is equivalent to prove (2.16) and (2.17) with \( \varphi \) instead of \( \delta \).

**Step 2.** We claim that

\[(A.3) \quad \int_\omega \delta^{a-p} v^p \leq C \int_\omega \delta^{a+2-p} v^p + C \int_\omega \delta^a |\nabla v|^p.
\]
Set $\beta = a + 1 - p > 0$. For $\varepsilon > 0$, we set $\psi = \varphi + \varepsilon$ (the dependence on $\varepsilon$ being omitted for brevity). Using
\[
\Delta \left( \frac{\psi^{1+\beta}}{1+\beta} \right) = \nabla \cdot \left( \psi^\beta \nabla \varphi \right) = \beta \psi^{\beta-1} |\nabla \varphi|^2 + \psi^\beta \Delta \varphi = \beta \psi^{\beta-1} |\nabla \varphi|^2 - \lambda_1 \psi^\beta \varphi
\]
and noting that $v^p \psi^\beta \nabla \varphi \in (W^{1,q}(\omega))^n$ for any $q \in [1, \infty)$, we may apply the divergence theorem to deduce
\[
I_\varepsilon := \varepsilon^\beta \int_{\partial \omega} v^p \frac{\partial \varphi}{\partial \nu} \, d\sigma = \int_{\partial \omega} v^p \psi^\beta \frac{\partial \varphi}{\partial \nu} \, d\sigma = \int_\omega \nabla \cdot \left( v^p \nabla \left( \frac{\psi^{1+\beta}}{1+\beta} \right) \right) \, dx
\]
\[
= p \int_\omega v^{p-1} \psi^\beta \nabla v \cdot \nabla \varphi + \int_\omega v^p [\beta \psi^{\beta-1} |\nabla \varphi|^2 - \lambda_1 \psi^\beta \varphi].
\]
Therefore, by (A.2) and Young’s inequality (if $p > 1$, or directly if $p = 1$), we deduce
\[
c_1^2 \beta \int_{\omega \setminus \partial \omega} \psi^{\beta-1} v^p \leq \beta \int_\omega \psi^{\beta-1} |\nabla \varphi|^2 v^p
\]
\[
\leq I_\varepsilon + \lambda_1 \int_\omega \psi^\beta \varphi v^p + p \int_\omega (\psi^{(\beta-1)(p-1)/p} v^{p-1}) (\psi^{(\beta+p-1)/p} |\nabla v|)
\]
\[
\leq I_\varepsilon + \lambda_1 \int_\omega \psi^\beta \varphi v^p + \frac{c_1^2 \beta}{2} \int_\omega \psi^{\beta-1} v^p + C \int_\omega \psi^{\beta+p-1} |\nabla v|^p.
\]
(here and below, $C$ is independent of $\varepsilon$, as well as of $v$). Consequently,
\[
c_1^2 \beta \int_{\omega \setminus \partial \omega} \psi^{\beta-1} v^p \leq I_\varepsilon + \lambda_1 \int_\omega \psi^\beta \varphi v^p + \frac{c_1^2 \beta}{2} \int_\omega \psi^{\beta-1} v^p + C \int_\omega \psi^{\beta+p-1} |\nabla v|^p.
\]
Since $\int_{\omega \setminus \partial \omega} \psi^{\beta-1} v^p \leq C \int_{\omega \setminus \partial \omega} \psi^\beta \varphi v^p$ due to (A.1), we deduce that
\[
\int_\omega (\varphi + \varepsilon)^{\beta-1} v^p \leq I_\varepsilon + C \int_\omega (\varphi + \varepsilon)^\beta \varphi v^p + C \int_\omega (\varphi + \varepsilon)^{\beta+p-1} |\nabla v|^p.
\]
Since $I_\varepsilon \to 0$ as $\varepsilon \to 0^+$, (A.3) follows by letting $\varepsilon \to 0^+$ and using the monotone convergence theorem on the LHS (which in particular yields the convergence of the integral on the LHS of (A.3)).

**Step 3.** We next claim that, for any $r \geq 0$,
\[
(A.4) \quad \int_\omega \varphi^{a-p} v^p \leq C \int_\omega \varphi^r v^p + C \int_\omega \varphi^a |\nabla v|^p.
\]
It suffices to prove (A.4) for $r = a - p + 2j$ with $j$ integer $\geq 1$. This is true for $j = 1$ by Step 2. Assume it is true for some integer $j \geq 1$. By Step 2 (with $a$ replaced by $a + 2j$), we have
\[
\int_\omega \varphi^{a-p+2j} v^p \leq C \int_\omega \varphi^{a-p+2(j+1)} v^p + C \int_\omega \varphi^{a+2j} |\nabla v|^p
\]
\[
\leq C \int_\omega \varphi^{a-p+2(j+1)} v^p + C \int_\omega \varphi^a |\nabla v|^p.
\]
Therefore,

$$\int_\omega \varphi^{a-p} v^p \leq C \int_\omega \varphi^{a-p+2j} v^p + C \int_\omega \varphi^a |\nabla v|^p \leq C \int_\omega \varphi^{a-p+2(j+1)} v^p + C \int_\omega \varphi^a |\nabla v|^p$$

and (A.4) follows by induction.

**Step 4.** We claim that

(A.5) $$\left( \int_\omega \varphi^{na/(n-p)} v^p \right)^{p/p'} \leq C \int_\omega \varphi^{a-p} v^p + C \int_\omega \varphi^a |\nabla v|^p.$$  

Let $$s \geq 1, d > 0$$ to be chosen and let $$\phi = \varphi^d$$. Note that $$|\nabla \phi| = d |\nabla \varphi| \varphi^{d-1} \leq C \varphi^{d-1}.$$ It is well known that $$\varphi^{d-1} \in L^1(\omega)$$ (but this follows for instance from (A.3) with $$p = 1, a = d$$ and $$v \equiv 1$$). Therefore, $$\varphi v^s \in W^{1,1}(\omega).$$ Set $$m = n' = n/(n-1)$$. Observe that $$p' > m$$ due to $$1 < p < n$$. By the standard Sobolev inequality in $$W^{1,1}$$, we have

$$\int_\omega \varphi^{m} v^{sm} = \|\varphi v^s\|^m_1 \leq C \|\nabla (\varphi v^s)\|^m_1 + C \|\varphi v^s\|^m_1,$$

hence

(A.6) $$\int_\omega \varphi^{dm} v^{sm} \leq C \|\varphi v^{s-1} \nabla v\|^m_1 + C \|v^s \nabla \phi\|^m_1 + C \|\varphi v^s\|^m_1.$$  

If $$p = 1$$, then (A.5) follows from (A.6) with the choice $$s = 1, d = a$$. Thus assume $$p > 1$$.

By Hölder’s inequality, for any $$\theta \in [0, 1]$$, we have

$$\|\varphi v^{s-1} \nabla v\|^m_1 \leq C \left( \int_\omega \varphi^{(1-\theta)p} v^{(s-1)p'} \right)^{m/p'} \left( \int_\omega \varphi^{\theta p} |\nabla v|^p \right)^{m/p}.$$  

Choosing

$$s = (n-1)p/(n-p) > 1, \quad \theta = (n-p)/((n-1)p) \in (0, 1), \quad d = (n-1)a/(n-p),$$

we get $$(1-\theta)p' = m, (s-1)p' = sm = p^*, \theta p = a/d$$ and $$(p'/m)' = p^*/m$$. By Young’s inequality, we then have

(A.7) $$\|\varphi v^{s-1} \nabla v\|^m_1 \leq \frac{1}{4} \int_\omega \varphi^{dm} v^{p'} + C \left( \int_\omega \varphi^a |\nabla v|^p \right)^{p'/p}.$$  

Next, by Hölder’s inequality, for any $$q > 1, \tau \in [0, 1]$$ and $$\lambda \in \mathbb{R},$$ we have

$$\|v^s \nabla \phi\|^m_1 + \|\varphi v^s\|^m_1 \leq C \|\varphi^{d-1} v^s\|^m_1$$  

(A.8) $$\leq C \left( \int_\omega \varphi^{(d-1)\lambda q} v^{\tau sq} \right)^{m/q} \left( \int_\omega \varphi^{(d-1)(1-\lambda)q' v^{(1-\tau)s q'}} \right)^{m/q'}.$$
Choosing
\[ q = (ms - p)/(s - p) = p' > 1, \quad \tau = m/q = 1 - \frac{n - p}{(n - 1)p} \in (0, 1), \]
we get \( \tau sq = sm = p^* \) and \( (1 - \tau) sq' = p \). If \( d \neq 1 \), we also choose \( \lambda = md/(d - 1)q \) and we obtain \( (d - 1)\lambda q = dm \) and
\[
(d - 1)(1 - \lambda)q' = (d - 1)p \left(1 - \frac{md}{(d - 1)q}\right) = (d - 1)p - md(p - 1) = \frac{d(n - p)}{n - 1} - p = a - p.
\]
(If \( d = 1 \), \( \lambda \) is irrelevant in (A.8).) By Young’s inequality, we then have
\[
(A.9) \quad C\|v^s \nabla \phi\|_1^n + C\|\phi^s\|_1^n \leq \frac{1}{4} \int_\omega \varphi^{dm} v^{p^*} + C \left( \int_\omega \varphi^{a-p} v^p \right)^{p^*/p}.
\]
Claim (A.5) now follows by combining (A.6), (A.7) and (A.9) (recalling \( sm = p^* \) and \( dm = na/(n - p) \)).

**Step 5.** We claim that, for each \( \ell \in [1, p] \), there exists \( b_\ell \geq 0 \) (possibly depending also on \( k \)), such that
\[
(A.10) \quad \int_\omega \varphi^b v^\ell \leq C \left( \int_\omega \varphi^k v \right)^\ell/p + C \left( \int_\omega \varphi^a |\nabla v|^p \right)^{\ell^*/p}.
\]
When \( \ell = 1 \), this is trivially true with \( b_1 = k \).

Next assume that (A.10) is true for some given \( \ell \in [1, p] \). It is then true for \( \ell \) replaced with \( \ell^* = n\ell/(n - \ell) \). Indeed, letting \( \bar{a} = \max(a, \ell + b_\ell) > \ell - 1 \), and using (A.5) with \( \ell, \bar{a} \) instead of \( p, a \) and Hölder’s inequality, we obtain
\[
\int_\omega \varphi^{{n\bar{a}}/(n-\ell)} v^{{\ell^*}} \leq C \left( \int_\omega \varphi^{\bar{a}-\ell} v^\ell \right)^{\ell^*/\ell} + C \left( \int_\omega \varphi^{\bar{a}} |\nabla v|^\ell \right)^{\ell^*/\ell} \leq C \left( \int_\omega \varphi^{b_\ell} v^\ell \right)^{\ell^*/\ell} + C \left( \int_\omega \varphi^{a} |\nabla v|^p \right)^{\ell^*/p} \leq C \left( \int_\omega \varphi^k v \right)^{\ell^*/p} + C \left( \int_\omega \varphi^a |\nabla v|^p \right)^{\ell^*/p}
\]
that is, (A.10) with \( b_{\ell^*} = n\bar{a}/(n - \ell) \). By Hölder’s inequality, Property (A.10) is then true for any \( \ell \in [1, \ell^*] \).

Now, for any integer \( i \in \{0, \ldots, n - 1\} \), set \( p_i = n/(n - i) \), and observe that \( p_{i+1} := p^*_i = np_i/(n - p_i) \) for \( i \in \{0, \ldots, n - 2\} \) (due to \( 1/p_i = 1 - (i/n) \)). Next let \( j \in \{0, \ldots, n - 2\} \) be the unique integer such that \( p_j \leq p < p_{j+1} \). By induction,
starting from $\ell = p_0 = 1$, the previous paragraph guarantees that (A.10) is true for any $\ell \in [1, p_{j+1}] \supseteq [1, p]$ and the claim follows.

**Conclusion.** Combining (A.4) for $r = b_p$ and (A.10) for $\ell = p$, we obtain

$$
\int_{\omega} \varphi^{a-p} v^p \leq C \int_{\omega} \varphi^{b_p} v^p + C \int_{\omega} \varphi^a \vert \nabla v \vert^p \leq C \left( \int_{\omega} \varphi^k u \right)^p + C \int_{\omega} \varphi^a \vert \nabla v \vert^p,
$$

which proves assertion (ii). Assertion (i) then follows from (A.5) and assertion (ii). □

**REFERENCES**

[1] B. Abdellaoui, A. DallAglio, I. Peral, Some remarks on elliptic problems with critical growth in the gradient, *J. Differential Equations* 222 (2006), 21-62 & Corr. *J. Differential Equations* 246 (2009), 2988-2990.

[2] N. Alaa, M. Pierre, Weak solutions of some quasilinear elliptic equations with data measures, *SIAM J. Math. Anal.* 24 (1993), 23-35.

[3] A. Alvino, P.L. Lions, G. Trombetti, Comparison results for elliptic and parabolic equations via Schwarz symmetrization, *Ann. Inst. H. Poincaré, Analyse non linéaire* 7(1990), 37-65.

[4] D. Arcoya, C. de Coster, L. Jeanjean, K. Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient, preprint ArXiV 1304.3066 (2013).

[5] A. Avantaggiati, On compact embedding theorems in weighted Sobolev spaces, *Czech. Math. J.* 29 (1979), 635-648.

[6] M.-F. Bidaut-Véron, L. Vivier, An elliptic semilinear equation with source term involving boundary measures: the subcritical case *Rev. Mat. Iberoamericana* 16 (2000), 477-513.

[7] L. Boccardo, F. Murat, J.-P. Puel, Existence of bounded solutions for nonlinear elliptic unilateral problems, *Ann. Mat. Pura Appl.* 152 (1988), 183-196.

[8] L. Boccardo, F. Murat, J.-P. Puel, $L^\infty$ estimate for some nonlinear elliptic partial differential equations and application to an existence result, *SIAM J. Math. Anal.* 23 (1992), 326-333.

[9] H. Brezis, X. Cabré, Some simple nonlinear PDEs without solutions, *Boll. Unione Mat. Ital.* (8) 1-B (1999), 223-262.

[10] H. Brezis, R.E.L. Turner, On a class of superlinear elliptic problems, *Comm. Partial Differ. Equations* 2 (1977), 601-614.

[11] M.G. Crandall, P.H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. Partial Differential Equations* 2 (1977), 193-222.

[12] V. Ferone, F. Murat, Nonlinear problems having quadratic growth in the gradient: an existence result when the source term is small, *Nonlinear Anal. TMA* 42 (2000), 1309-1326.

[13] V. Ferone, M.R. Posteraro, On a class of quasilinear elliptic equations with quadratic growth in the gradient, *Nonlinear Anal. TMA* 20 (1993), 703-711.

[14] V. Ferone, M.R. Posteraro, J.M. Rakotoson, $L^\infty$-estimates for nonlinear elliptic problems with p-growth in the gradient, *J. Ineq. Appl.* 2 (1999), 109-125.
[15] M. Fila, Ph. Souplet, F. Weissler, Linear and nonlinear heat equations in $L^q_\delta$ spaces and universal bounds for global solutions, *Math. Ann.* 320 (2001), 87-113.

[16] S. Filippas, V. Maz'ya, A. Tertikas, Critical Hardy-Sobolev inequalities. *J. Math. Pures Appl.* (9) 87 (2007), 37-56.

[17] N. Grenon-Isselkou, J. Mossino, Existence de solutions bornées pour certaines équations elliptiques quasilinéaires, *C. R. Math. Acad. Sci. Paris* 321 (1995), 51-56.

[18] L. Jeanjean, B. Sirakov, Existence and multiplicity for elliptic problems with quadratic growth in the gradient, *Comm. Part. Diff. Equ.* 38 (2013), 244-264.

[19] C. Maderna, C. Pagani, S. Salsa, Quasilinear elliptic equations with quadratic growth in the gradient, *J. Differential Equations* 97 (1992), 54-70.

[20] J. Nečas, Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle, *Ann. Scuola Normale Sup. Pisa* 16 (1962), 305-326.

[21] B. Opic, A. Kufner, Hardy-type inequalities. Pitman Research Notes in Mathematics Series, 219. Longman Scientific & Technical, Harlow, 1990.

[22] P. Quittner, Ph. Souplet, A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces, *Arch. Rational Mech. Anal.* 174 (2004), 49-81.

[23] P. Quittner, Ph. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhauser Advanced Texts, 2007, 584 p.+xi. ISBN: 978-3-7643-8441-8