Stable, finite energy density solutions in the effective theory of non-abelian gauge fields

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(Dated: April 10, 2018)

We consider the gauge fixed partition function of pure $SU(N_c)$ gauge theory in axial gauge following the Halpern’s field strength formalism. We integrate over $3(N_c^2 - 1)$ field strengths using the Bianchi identities and obtain an effective action of the remaining $3(N_c^2 - 1)$ field strengths in momentum space. We obtain the static solutions of the equations of motion (EOM) of the effective theory. The solutions exhibit Gaussian nature in the $z$ component of momentum and are proportional to the delta functions of the remaining components of momentum. The solutions render a finite energy density of the system and the parameters are found to be proportional to fourth root of the gluon condensate. It indicates that the solutions offer a natural mass scale in the low energy phase of the theory.

I. INTRODUCTION

It is well known that non abelian gauge theory (NAGT) has asymptotic freedom \cite{1} which indicates that the theory is almost free at very high energy and at very small distance scale. It also indicates that at very low energy it exhibits the confinement of gauge degrees of freedom and results in the non-zero values of gluon condensates at this low energy phase of the theory \cite{2}. Owing to non linear nature of NAGT its exact quantization is still lacking and as a consequence of which the dynamics of this theory at low energy remains least understood till today. It is believed that stable classical solutions of EOM of the theory may be of use to understand the behaviour of the system at low energy and it may also shed some light on the issue of quantization of the theory at this energy \cite{2}.

The endeavour for obtaining the classical solutions of EOM of NAGT has a long history since its inception. Attempts have been made \cite{3} to cast those equations in the form of Maxwell equations of electrodynamics. The exact periodic solutions of the $SU(2)$ gauge theory have been constructed \cite{3} in Minkowski space-time. There also exists solutions which are the non-abelian analogues of electromagnetic plane waves \cite{6}. A relationship has been established \cite{7} between the solutions of a $\phi^4$ scalar field theory and a class of solutions of EOM of SU(2) gauge theory. The most general self-dual, non-abelian, plane wave solutions have been obtained \cite{8} in NAGT. A comprehensive discussion on the solutions of EOM in Minkowski space-time are given in Ref. \cite{9, 10}.

It has been proposed long before \cite{11} that a non-abelian gauge theory can be formulated in terms of field strengths in axial gauge. The unique inversion of gauge potentials to field strength requires Bianchi identities to be satisfied in the quantized theory. We use Bianchi identities to integrate over the $3(N_c^2 - 1)$ degrees of freedom and obtain an effective action in terms of the remaining $3(N_c^2 - 1)$ field strengths in the momentum space. Then we obtain a solution of the equations of motion of the effective theory. Solutions indicate that the non abelian magnetic fields have Gaussian nature in their $z$ component of the momentum and non abelian electric fields are zero. We expand the effective action around this solution to quadratic order in fluctuation and obtain the energy density of the system by integrating out the fluctuations in the partition function. The energy density has been found to have a minimum with respect to the parameter of the solution and it thus indicates the stability of the obtained solution.

II. GAUGE FIXED EFFECTIVE ACTION

We start with the pure $SU(N_c)$ gauge theory Lagrangian

$$\mathcal{L} = -\frac{1}{4}G^{\mu\nu}G_{\mu\nu} \quad (1)$$

where $G^{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc}A^b_\mu A^c_\nu \,(a, b, c = 1, \ldots, N_c^2 - 1)$ is the field strength of the non-abelian gauge field $A^a_\mu$. Here the axial gauge $A^a_\mu(x) = 0$ is chosen. The unique inversion $A(G)$ makes it possible to change variables to field strengths and the gauge fixed partition function takes the form \cite{11}

$$Z = \int DG\delta[I[G]] \exp \left[ -\frac{i}{4} \int d^4x G^{\mu\nu}\tilde{G}^{\mu\nu} \right] \quad (2)$$

where

$$\delta[I[G]] = \prod_{\mu,a,x} \delta(I^a_\mu(x)), \quad (3)$$

$$I^a_\mu(x) = \partial_\lambda \tilde{G}^{a\lambda\mu} + gC^{abc}A^b_\lambda (G)\tilde{G}^{c\nu\lambda}(x) \quad (4)$$

and

$$\tilde{G}^{a\mu\nu} = \frac{1}{g^2}e_{\mu\nu\lambda}\epsilon^{ab}_c G^{c\lambda\mu} \quad (5)$$
We use the Fourier transform of

\[ G_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-i k.x} A^a_{\mu\nu}(k) \]

and \( A^a_{\mu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-i k.x} A^a_{\mu}(k) \)

and write the solutions of Halpern\(^{11}\) for the gauge potentials in momentum space as

\[ A^a_0(k) = -i \frac{k}{k_z} E^a_z(k), \quad A^a_1(k) = -i \frac{k}{k_z} B^a_x(k), \]
\[ A^a_2(k) = i \frac{k}{k_z} B^a_y(k) \text{ and } A^a_3(k) = 0. \]

where \( E^a(k) \) and \( B^a(k) \) are the non-abelian electric and magnetic field vectors in momentum space respectively. We use eq. (10) and (7) to write the partition function in terms of the fields in momentum space as

\[ Z = \int DG \delta[I[G]] e^{iS[G]}, \]

where

\[ S[G] = -\frac{1}{4} \int \frac{d^4k}{(2\pi)^4} G^a_{\mu\nu}(k) G^{a\mu\nu}(-k) \]
and \( \delta[I[G]] = \prod_{\mu, a, k} \delta(I^a_{\mu}(k)). \)

\[ I^{a\mu}(k) \text{ in momentum space in terms of non abelian electric and magnetic fields read} \]
\[ I^{a0}(k) = -ik_z B^a_x(k), \]
\[ +igC^{abc} \int \frac{d^4k'}{(2\pi)^4} \frac{1}{k_z'} (B^b(k - k') \times B^c(k')) \]
\[ \text{and} \]
\[ I^{a\mu}(k) = -ik_z E^a_{\mu}(k), \]

\[ f_j^a(k) \text{ in terms of the fields } X^a_j(k) \text{ (} j = 1, 2, 3 \text{) are given as} \]
\[ f_j^a(k) = \frac{k_x X_j^a(k) + k_0 X^a_0(k)}{k_z} \]
\[ -gC^{abc} \int \frac{d^4k'}{(2\pi)^4} \frac{k_x(k_z' - k_z)}{k_z' x^b_j(k') X^a_j(k')}, \]
\[ f_2^a(k) = \frac{k_y X^a_2(k) - k_0 X^a_0(k)}{k_z} \]
\[ -gC^{abc} \int \frac{d^4k'}{(2\pi)^4} \frac{k_y(k_z' - k_z)}{k_z' x^b_2(k') X^a_2(k')}, \]
\[ f_3^a(k) = \frac{k_z X^a_3(k) - k_0 X^a_0(k)}{k_z} \]
\[ -gC^{abc} \int \frac{d^4k'}{(2\pi)^4} \frac{k_z(k_z' - k_z)}{k_z' x^b_3(k') X^a_3(k')}, \]

where \( X^a_1 = B^a_x, \ X^a_2 = B^a_y \) and \( X^a_3 = E^a_z. \)

We write using eq. (13)
\[ \delta[I(G)] = \prod_{a, k} \delta(E^a_z(k) - f_1^a(k)) \delta(E^a_0(k) - f_2^a(k)) \]
\[ \times \delta(B^a_x(k) - f_3^a(k)) \delta(I^a_{\mu}(k)) \]
and then integrating over the fields \( E^a_z, E^a_0 \) and \( B^a_x \) obtain the partition function as

\[ Z = \mathcal{N} \int D\xi e^{iS[X]}, \]

where the factor \( \mathcal{N} = \delta(0) \prod_{k_z} \frac{1}{k_z}. \) The factor of \( \delta(0) \) comes from \( \delta(I^a_{\mu}(G)) \) when \( I^a_{\mu}(G) = 0, \) after performing the integration over the remaining delta functions. The effective action

\[ S[X] = S^{(0)}[X] + S^{(1)}[X], \]

where the quadratic part

\[ S^{(0)}[X] = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \delta^4(p)(M^{-1})_{jk}(p) X^a_j(-p) \]

and the interaction part

\[ S^{(1)}[X] = g \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} C^{abc} \delta^4(p - q - z) X^a_j(p) X^b_k(q) \]
\[ \times \delta(0) \delta(x^b_j(p - q)) X^c(k) \delta(0), \]
\[ + \frac{g^2}{2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4q'}{(2\pi)^4} C^{abc} \delta^4(p - q - z) \]
\[ \times \delta^4(0) \delta(x^b_j(p - q)) X^c(k) \delta(0) \delta(x^b_j(q')) X^c(q') \]

Here \( 3 \times 3 \) matrix \( M^{-1}(p) \) reads

\[ M^{-1}(p) = \left( \frac{p^2 + q^2}{p_z^2} - \frac{p_z}{p^2} \right) \left( \frac{p_z}{p^2} \right) \left( \frac{p_z}{p_z^2} \right), \]
Using the property in eq. (28) we evaluate the integral

\[ \int_{-\infty}^{\infty} \xi^a_j(p_z) dp_z = 0. \]  

We then make the following ansatz:

\[ \xi^a_1(p_z) = \sum_{n=0}^{\infty} y^{a}_1(n) p_z^2 e^{-(p_z - \Delta^2)/\Delta^2}, \]

\[ \xi^a_2(p_z) = \sum_{n=0}^{\infty} y^{a}_2(n) p_z^2 e^{-(p_z - \Delta^2)/\Delta^2} \]

and \( \xi^a_3(p_z) = 0 \)

that are consistent with the property in eq. (28). Here \( \Delta \) is a scale of mass dimension one and \( y^{a}_j(n) (j = 1, 2) \)

satisfy the following equations:

\[ y^{a}_1(n) + \frac{g^2}{2} C^{acd} C^{bcd} y^{b}_2(0) y^{a}_1(n + 1) - y^{a}_1(n + 1) y^{b}_1(0) y^{a}_1(n) = 0, \]

\[ y^{a}_2(n) - \frac{g^2}{2} C^{acd} C^{bcd} y^{b}_1(0) y^{a}_2(n + 1) - y^{a}_1(n + 1) y^{b}_1(0) y^{a}_2(n) = 0. \]

We assume that in case of SU(2) gauge theory

\[ y^{a}_1(0) = \phi \delta^{a1} \quad \text{and} \quad y^{a}_2(0) = \phi \delta^{a3}. \]

Then \( y^{a}_j(n) (j = 1, 2) \) for few different \( n (n \geq 1) \) are as follows:

\[ y^{a}_1(n) : \quad y^{a}_1(1) = -\frac{e^2}{2g^2 \pi^2 \phi}, \quad y^{a}_1(2) = -\frac{e^4}{4g^4 \pi^4 \phi^3}, \]

\[ y^{a}_2(3) = \frac{3e^6}{8g^6 \pi^6 \phi^3}, \cdots, \]

\[ y^{a}_2(0) = 0 \quad \text{and} \quad y^{a}_2(n) = 0 \quad \text{for} \quad n \geq 1. \]

\[ y^{a}_1(n) = 0 \quad \text{and} \quad y^{a}_2(n) = 0 \quad \text{for} \quad n \geq 1. \]

### III. SOLUTIONS OF EOM

Equations of motion

\[ (M^{-1})_{kl} \dot{X}^a_m = \frac{\delta S[X]}{\delta X^a_j(p)} |_{X = \bar{X}} \]

take the form

\[ \left( M^{-1} \right)_{kl} \dot{X}^a_k = -g C^{abc} \frac{\partial}{\partial \xi^c} \frac{1}{p_z} \int \frac{d^4 q}{(2\pi)^4} \frac{\tilde{b}^{(3)}_{jklm}(p, q)}{q_z} \dot{X}^b_m(q) \]

\[ \times \left( \frac{1}{(2\pi)^4} \int d^4 q \frac{d^4 q'}{2q_z} \right) \tilde{X}^c_m(q') \]

\[ \times \left( \frac{1}{2} \int d^4 q \frac{d^4 q''}{2q_z} \right) \tilde{b}^{(4)}_{jklmn} \frac{\xi^a_j(p_z) - \xi^a_k(p_z) - \xi^a_l(p_z) + \xi^a_m(p_z)}{p_z + q_z + q''_z} \]

\[ \times \dot{X}^b_l(q') \dot{X}^c_m(q'') \right) = 0, \]

where

\[ \tilde{b}^{(3)}_{jklm}(p, q) = b^{(3)}_{jklm}(p) + b^{(3)}_{jklm}(p + q) - b^{(3)}_{jmlk}(q), \]

\[ \tilde{b}^{(4)}_{jklmn} = b^{(4)}_{jklmn} - b^{(4)}_{jlmkn} + b^{(4)}_{jmlkn} - b^{(4)}_{jmnkl}. \]

We assume a solution of eq. (24) as

\[ \dot{X}^a_j(p) = (2\pi)^3 \delta^{(3)}(p_\perp) \xi^a_j(p_\parallel) \]

where \( p_\perp = (p_0, p_x, p_y) \) and

\[ \xi^a_j(z) \rightarrow 0 \quad \text{as} \quad |z| \rightarrow \infty. \]

Upon substitution of this solution into eq. (24), it takes the form

\[ \left\{ -\xi^a_1(-p_\parallel) \delta_{n1} - \xi^a_2(-p_\parallel) \delta_{n2} + \xi^a_3(-p_\parallel) \delta_{n3} \right\} \]

\[ -\frac{g^2}{2p_z} C^{abc} C^{bcd} \tilde{b}^{(4)}_{jklmn} \int_{-\infty}^{\infty} dq_z dq' \frac{\xi^a_j(q_z) \xi^a_k(q'_z) \xi^a_l(q''_z) \xi^a_m(q'''_z)}{p_z + q_z + q''_z + q'''_z} \]

\[ \times \frac{1}{(2\pi)^4} \int d^4 q \frac{d^4 q'}{2q_z} \frac{d^4 q''}{2q'_{z}} \frac{d^4 q'''}{2q''_{z}} \tilde{b}^{(4)}_{jklmn} \]

\[ \times \dot{X}^b_l(q') \dot{X}^c_m(q'') \right) = 0. \]

Using the property in eq. (25), we evaluate the integral

\[ \int_{-\infty}^{\infty} \xi^a_j(p_z) dp_z = i\pi \delta^{a1}(-\lambda). \]

With the use of this result, eq. (25) simplifies to

\[ \left\{ -\xi^a_1(-p_\parallel) \delta_{n1} - \xi^a_2(-p_\parallel) \delta_{n2} + \xi^a_3(-p_\parallel) \delta_{n3} \right\} \]

\[ + \frac{g^2}{2p_z} C^{abc} C^{bcd} \tilde{b}^{(4)}_{jklmn} \xi^a_j(0) \xi^a_k(0) \xi^a_l(0) \xi^a_m(0) \]

\[ - \xi^a_1(-p_\parallel) \xi^a_2(0) = 0. \]

### IV. STABILITY OF THE SOLUTION

To address the issue of the stability of the obtained solution we compute the vacuum energy density \( \epsilon(\phi) \) of the system. The solutions will be stable if \( \epsilon(\phi) \) has a minimum with respect to \( \phi \) and it remains bounded even for limited fluctuation of \( \phi \) around the minimum. From trace anomaly \ref{22} we know that the trace of the energy momentum tensor

\[ \Theta^a_\mu = \frac{\beta(g)}{2g^2} G^{a\alpha\beta} G^{\alpha\beta}, \]

where \( \beta(g) = \frac{\mu^2}{8g^2} \) is negative for pure NAGT. Lorentz invariance requires that \( \Theta^{\mu\nu} \) is also consistent with the fact that \( \Theta^{00} = \epsilon \). We therefore obtain \( \Theta^{00} = 4\epsilon \) and then the use of eq. (38) gives

\[ \epsilon = \frac{\beta(g)(\mu)}{8g^2} - \frac{1}{2} G^{a\alpha\beta} G^{a\alpha\beta} \]

(39)
In the following we first expand the action around the solutions to quadratic order in fluctuations and then proceed to compute \( \langle 0 \mid G_{\alpha\beta}^{\mu
u}G_{\alpha'\beta'}^{\mu'\nu'} \mid 0 \rangle \) using this expanded action around the solutions. Solutions in eq. (42) are dependent on a mass scale \( \Delta \) which is dynamic in nature. It will be shown later using a rough estimate that it falls inversely as \( g \gg 1 \) and goes to zero when \( g \rightarrow 0 \). Since \( \Delta \) turns out to be small in the non-perturbative \((g \gg 1)\) region, we take \( \Delta \ll 1 \) and expand eq. (42) in the following in powers of \( \Delta \):

\[
\xi_j^{(p)}(p_z) \approx \sqrt{\pi} \Delta \delta(p_z - \Delta) \tilde{\xi}_j^{(p)} \quad \text{for} \quad j = 1, 2
\]

where \( \tilde{\xi}_j^{(p)} = \frac{y_j^{(p)}(n) \Delta^{2n}}{n!} \). (40)

We introduce 3\((N_c^2 - 1)\) sources and write the partition function of eq. (16) as

\[
Z[J] = \int \mathcal{D}G \delta[I[G]] e^{iS[G,J]},
\]

where

\[
S[G,J] = \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{4} G_{\mu\nu}^{a}(k) G_{a\mu\nu}^{\ast}(-k) + X_j^{a}(k) J_j^{\ast}(-k) \right\}.
\]

Consider an operator \( O(E^a, B^a) \) which is a functional of non abelian electric and magnetic fields. The vacuum expectation value

\[
\langle 0 \mid O(E^a, B^a) \mid 0 \rangle = \frac{1}{Z[0]} \int \mathcal{D}G \delta[I[G]] O(E^a, B^a) e^{iS[G,J]}
\]

Then after performing the integration over \( E^a, E^a \) and \( B^a \) as discussed in section II we obtain

\[
\langle 0 \mid O(E^a, B^a) \mid 0 \rangle = \frac{N}{Z[0]} \int \mathcal{D}X \tilde{O}(X_j^{a}) e^{iS[X,J]} = \frac{1}{Z[0]} \tilde{O} \left( \frac{1}{\delta \delta J} \right) Z[J] \mid _{J=0},
\]

where the action of eq. (15) is modified as

\[
S[X,J] = S[X] + \int \frac{d^4p}{(2\pi)^4} X_j^{a}(p) J_j^{\ast}(-p).
\]

Moreover, according to eq. (14) \( J_j^{a}(p) \) are functions of \( X_j^{a} \) \((j = 1, 2, 3)\), so we use

\[
\tilde{O}(X_j^{a}) = O(J_j^{a}, X_j^{a}).
\]

To compute the vacuum average of the operator we adopt the stationary phase approximation in \( Z[J] \):

\[
Z[J] = N \int \mathcal{D}X e^{iS[X,J]}.
\]

We expand the action about the classical solution and keep terms to order \((X - \bar{X})^2\):

\[
S[X,J] = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \Phi_j^{(p)}(p) \Phi^{\ast}_j(-p) + \int \frac{d^4p}{(2\pi)^4} J_j^{\ast}(p) \Phi_j^{(p)}(-p) + O(\Phi^3)
\]

where \( \Phi_j^{(p)}(p) = X_j^{p}(p) - \bar{X}_j^{p}(p) \). \( \Phi_j^{(p)}(p) \) is assumed as a very slowly varying function of \( p \) over the momentum scale \( (\Delta \ll 1) \). Then

\[
(M^{-1})_{j,k}^{(ab)}(p) = \delta^{ab} (M^{-1})_{jk}(p) - \frac{\sqrt{\pi} g}{p_z(p_z + \Delta)} C^{abc} \left[ \xi_j \left( 2p_y (\delta_j \delta_k - \delta_j \delta_k) - p_0 (\delta_j \delta_k - \delta_j \delta_k) \right) + \xi_j \left( 2p_y (\delta_j \delta_k - \delta_j \delta_k) + p_0 (\delta_j \delta_k - \delta_j \delta_k) \right) - p_0 (\delta_j \delta_k - \delta_j \delta_k) \right]
\]

\[
+ \frac{g^2 \pi}{p_z(p_z + 2\Delta)} C^{cde} \left[ \xi_j \xi_j \xi_j \xi_j - 2 \delta_j \delta_k \xi_j \xi_j + (\delta_j \delta_k - \delta_j \delta_k) \xi_j \xi_j \xi_j \xi_j + (\delta_j \delta_k - \delta_j \delta_k) \xi_j \xi_j \xi_j \xi_j \right].
\]

Then we change the integration variable in eq. (17) from \( X \) to \( \Phi \) and integrate over \( \Phi \) to obtain the final result as

\[
Z[J] = N |\text{Det}M^{-1}|^{-\frac{1}{2}} \exp \left\{ iS[\bar{X}] + i \sqrt{\pi} \Delta \tilde{\xi} \tilde{\xi} J_j^{\ast}(0, -\Delta) \right\}
\]

\[
- \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} Q^{(ab)}_{jk}(p) J_j^{\ast}(-p) J_k^{(p)},
\]

where

\[
Q^{(ab)}_{jk}(p) = \frac{1}{2} \left[ (M^{-1})^{(ab)}_{jk}(p) + (M^{-1})^{(ba)}_{jk}(p) \right]
\]

and the convention \( J_j^{p}(p) = J_j^{p}(p_z, p_z) \) has been adopted. We take the operator as

\[
O = G_{\mu\nu}^{a}(0) G_{\mu\nu}^{a}(q) = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} G_{\mu\nu}^{a}(p) G_{\mu\nu}^{a}(q)
\]

and use eq. (14) to obtain its average as

\[
\langle 0 \mid G_{\mu\nu}^{a}(0) G_{\mu\nu}^{a}(0) \mid 0 \rangle = O_2 + O_3 + O_4,
\]

where the expressions for \( O_2, O_3 \) and \( O_4 \) are as follows:

\[
O_2 = 2\pi \Delta^2 \left( \xi_j \xi_j + \xi_j \xi_j \right) - \frac{2i}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} (M^{-1})_{jk}(p) Q^{(a)}_{j,k}(p),
\]

\[
O_3 = 4i g C^{abc} \frac{\sqrt{\pi}}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} b^{(3)}_{jk}(p)
\]

\[
\times \left[ \xi_j^{(a)}(p) + \xi_j^{(b)}(p) \right]
\]

\[(55)\]
\[ O_4 = 2g^2 C_{ab1c} C_{ab2c} \frac{h_{kjm}}{(2\pi)^{16}} \left[ -\pi^2 \xi_{bj}^c \xi_{bs} \xi_{km}^c \right] \]

\[ + i\pi \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \left\{ Q^{(cb_1)}_{m} (p) \xi_{bs} \right\} \]

\[ + Q^{(cb_1)}_{m} (p) \xi_{bs} + Q^{(cb_2)}_{b} (p) \xi_{bj}^c \]

\[ + Q^{(cb_1)}_{b} (p) \xi_{bs} \]

\[ + Q^{(cb_1)}_{b} (p) \xi_{bs} \]

\[ + \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{p^2q^2} \left\{ Q^{(cb_1)}_{m} (p) Q^{(cb_1)}_{m} (q) \right\} \]

\[ + Q^{(cb_1)}_{m} (p) Q^{(cb_1)}_{m} (q) + Q^{(cb_1)}_{m} (p) Q^{(cb_2)}_{b} (q) \]

where the values of the parameters are given in the following TABLE I and TABLE II.

\[ \begin{array}{c|c|c|c}
   d_0 & 1.38161 \times 10^{-10} & m_0 & 4.00046 \\
   d_1 & 8.45785 \times 10^{-8} & m_1 & 0.99998 \\
   d_2 & 5.15311 \times 10^{-10} & m_2 & 2.999873 \\
   d_3 & 2.56904 \times 10^{-10} & m_3 & 1.00214 \\
   d_4 & 5.65533 \times 10^{-10} & m_4 & 4.00008 \\
   d_5 & 3.54694 \times 10^{-8} & m_5 & 1.0009 \\
   d_6 & 2.13056 \times 10^{-17} & m_6 & 3.00199 \\
\end{array} \]

TABLE I. divergent parameters

\[ \begin{array}{c|c|c|c}
   d_0 & 3.3454 \times 10^{-12} & f_1 & 6.73501 \times 10^{-11} \\
   d_1 & 39.4784 & f_2 & 1.23925 \times 10^{-14} \\
   d_2 & 9.2334 \times 10^{-12} & f_3 & 9.9540 \times 10^{-16} \\
   d_3 & 1.84434 \times 10^{-14} & f_4 & 4.99795 \\
   d_4 & 1.81293 \times 10^{-14} & f_5 & 3.0001 \\
   d_5 & 8.5485 \times 10^{-14} & f_6 & 3.99762 \\
\end{array} \]

TABLE II. finite parameters

In general we consider a scalar operator \( O(x) \) and the renormalized partition function with sources reads as

\[ Z[\chi] = \int D\Phi \delta(I[\Phi]) \exp \left\{ iS + i \int d^4x (O(x) \right. \]

\[ \left. + \{ 0 | O(x) | 0 \} \chi(x) \right\} \] (58)

where, \( \{ 0 | O(x) | 0 \} \) contains the required counter terms to remove the divergences from the vacuum average of \( O(x) \). The renormalized average is given as

\[ \{ 0 | O(x) | 0 \}_r = \frac{1}{i} \frac{\delta \ln Z}{\delta \chi} |_{\chi=0} = \{ 0 | O(x) | 0 \} + \{ 0 | O(x) | 0 \}_ct \] (59)

Using the minimal subtraction scheme we choose \( \{ 0 | G^{\mu\nu} G_{\mu\nu} | 0 \} \) appropriately and obtain the renormalized vacuum average as

\[ \langle G^{\mu\nu} G_{\mu\nu} \rangle = -A g^2 \phi^4 + B \Delta^2 \phi^2 + C \Delta^4 \] (60)

where

\[ A = h_0 - g^2 \frac{f_1}{s^{n_1}} + g^4 \frac{f_2}{s^{n_2}}, \]

\[ B = h_1 - g^2 \frac{f_3}{s^{n_1}} + g^4 \frac{f_4}{s^{n_4}}, \]

\[ C = h_2 g^2 + f_5 \frac{f_6}{s^{n_6}}. \] (61)

Here the scale parameter must be of the form \( s = \mu / \mu_0 \), where \( \mu_0 \) is the scale of minimal subtraction and \( \mu \) (< \( \mu_0 \)) is the scale of interest at low energy. We then use eq.(39) to obtain the energy density for \( SU(2) \) theory as

\[ \epsilon(\phi) = \left[ \frac{\beta(\phi)}{8 g^3} \right] \left( A g^2 \phi^4 - B \Delta^2 \phi^2 - C \Delta^4 \right) \] (62)
Since, \( f \) malized gluon condensate at the minima

Hence it ensures that the solutions are stable. We can estimate the value of \( \Delta \) which is as follows: The renormalized gluon condensate at the minima \( \phi = \pm \phi_0 \) is

\[
\langle G^2 \rangle_0 = \Delta^4 \left( C + \frac{B^2}{4g^2A} \right) s^{4\Delta} \left( f_6 + \frac{f_4}{4f_2} \right) .
\]

Since, \( f_6 \gg \frac{f_4}{f_2} \) we obtain

\[
\Delta \approx \frac{s^4}{g} \left[ \frac{4\pi (\alpha_s G^2)_0}{f_6} \right]^{1/4} .
\]

When \( g \gg 1 \), we obtain using \( \langle \alpha_s G^2 \rangle_0 = 0.0314 \text{ GeV}^4 \) \[13\]

\( \Delta = 6.79s^2/g \text{ GeV} \). It indicates that even in the non-perturbative region \( (g \gg 1) \) \( \Delta \) remains less than one. When \( g \to 0 \), surely \( \Delta \to 0 \) because the gluon condensate vanishes in the perturbative limit of NAGT. Therefore, in the non-perturbative region \( \Delta \) decreases inversely as \( g \), however it goes to zero in the perturbative region. So our idea of retaining terms to order \( \Delta^4 \) indicates that we have worked to obtain energy density of the system at an energy scale which lies in the non-perturbative phase of the theory. The use of \( \Delta \) in eq.(63) gives

\[
\phi_0 \approx \frac{s^4}{g} \left[ \frac{\pi f_4^2 (\alpha_s G^2)_0}{f_2 f_6} \right]^{1/4} .
\]

Therefore, \( \phi_0 \to 5.85s^2/g^2 \) when \( g \gg 1 \) and \( \phi_0 \to 0 \) when \( g \to 0 \).

V. CONCLUSION

We have taken the gauge fixed partition function of NAGT in axial gauge using the field strength formalism as given by Halpern. We have integrated out the variables \( E^\alpha_0, E^\nu_0 \) and \( B^2 \) using Bianchi Identities in momentum space and have obtained an effective action of the remaining \( 3(N^2 - 1) \) variables \( E^\alpha, B^2 \) and \( B^\nu \). We have obtained a static solution of EOM of the effective theory in terms of parameters \( \phi \) and \( \Delta \). The solutions in the momentum space exhibit a Gaussian nature in the \( z \) component of momentum and are proportional to Dirac delta functions of the remaining components of momentum. To check the stability of the solutions first we expand the effective action around the solutions to quadratic order in fluctuations and then compute the vacuum average \[0 \ G^{\mu\nu}(0)G^\mu_{\nu}(0) | 0 \] using this expanded action around the solutions. Then we use trace anomaly to obtain the energy density as a function of \( \phi \). The energy density has minima at \( \phi = \pm \phi_0 \), where \( \phi_0 \) is proportional to the fourth root of the gluon condensate \( \langle \alpha_s G^2 \rangle_0 \), which is evaluated at the minima of the energy density. Since the energy density does not change appreciably for small fluctuation around these minima, the obtained solutions are stable and make the energy density finite. It is known that the gluon condensate vanishes when computed using the method of small coupling constant perturbation. So, \( \phi_0 \) tends to zero in the perturbative limit. Since \( \phi_0 \) is non-vanishing in the non-perturbative limit, it acts as a mass scale in the low energy limit of NAGT. The parameter \( \Delta \) is also found to be proportional to \( \sqrt{\langle \alpha_s G^2 \rangle_0} \). Hence, \( \Delta \) remains non-zero in the non-perturbative phase and vanishes in the perturbative phase of the theory.

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