ON FRACTIONAL DIFFUSION-ADVECTION-REACTION EQUATION IN $\mathbb{R}^*$

V. GINTING† AND Y. LI†

Abstract. We present an analysis of existence, uniqueness, and smoothness of the solution to a class of fractional ordinary differential equations posed on the whole real line that models a steady state behavior of a certain anomalous diffusion, advection, and reaction. The anomalous diffusion is modeled by the fractional Riemann-Liouville differential operators. The strong solution of the equation is sought in a Sobolev space defined by means of Fourier Transform. The key component of the analysis hinges on a characterization of this Sobolev space with the Riemann-Liouville derivatives that are understood in a weak sense. The existence, uniqueness, and smoothness of the solution is demonstrated with the assistance of several tools from functional and harmonic analyses.

Key words. Riemann-Liouville fractional operators, fractional diffusion, advection, reaction, weak fractional derivative, strong solution, regularity.

AMS subject classifications. 26A33, 34A08, 46N20

1. Introduction. Fractional integral and differential operators and fractional differential equations have gained increasingly crucial role as useful tools for modeling various anomalous and nonlocal phenomena. By no means exhaustive, some of the applications include conservation of fluid in a porous medium [26], anomalous diffusion [17], atmospheric advection-dispersion of pollutants [10], continuum mechanics [16], and dynamics in financial markets [21].

Recent years have seen very active investigations on theoretical and numerical analysis of fractional differential equations. The existence of solutions to many types of fractional differential equations have been widely studied by using functional analytic approaches with some aiming at finding analytical/closed form solutions of the problems (see, e.g. [14], [20], [27]). Using functional analytic framework and variational formulations, several numerical schemes for approximating boundary value problems involving fractional differential equations were derived and analyzed (see e.g. [7], [24], and [12]). Moreover, there has been a renewed interest on investigation of fractional Sturm-Liouville boundary value problems on unbounded domains [13].

Among the recurring themes in the aforementioned works is on the wellposedness of the problems under investigation. When posed on a bounded domain, typically a fractional differential equation must be provided with a set of boundary conditions. However, fractional integral and differential operators are inherently nonlocal, and in this regard, the choice of suitable and correct boundary settings to accompany the equation is not immediately clear. Other related topic is on the stability and regularity of the solution, namely, questions about the smoothness of the solution and how it depends on the data. A variety of issues on the wellposedness of the problems and solutions regularity was for example addressed in [23, 2, 5].

The subject of this paper is on the existence, uniqueness, and regularity of stationary fractional ordinary differential equation modeling a certain anomalous diffusion, advection, and reaction on the whole real line, in which the anomalous diffusion is modeled by the fractional Riemann-Liouville derivatives. One can associate this equation as a study of steady state behavior of a time dependent problem containing spatial fractional derivatives (see e.g. [11] and [3]). In giving a proper response, there are several inquiries to address, among which are: 1) What is a suitable functional space inside of which the solution of the equation is to be sought? 2) What should be a good setting to analyze the existence, smoothness, stability of solution?

The central thesis of the current investigation is that a class of fractional Sobolev spaces is a suitable "sandbox" to search for the solutions of the said fractional ordinary differential equations. In particular, we heavily utilize the Sobolev space that is defined by means of Fourier Transform. One of the main results is an ability to relate functions

*Submitted to the editors May, 2018.

Funding: Y. Li was partially supported by the UW Science Initiative Scholarship.

†Department of Mathematics & Statistics, University of Wyoming, Laramie, WY (vginting@uwyo.edu, yli25@uwyo.edu).

This manuscript is for review purposes only.
in this Sobolev space to functions whose Riemann-Liouville derivatives are understood in a weak sense. In fact, we show that the Sobolev space is equal to space of functions whose Riemann-Liouville derivatives are square integrable. Once this is in place, several tools from functional and harmonic analyses are employed to certify the existence and uniqueness of the strong solution of the equation. Furthermore, under an assumption of increasing smoothness of the data, the smoothness of the solution may be revamped as well.

The rest of the paper is organized as follows. An introduction to fractional Riemann-Liouville integral and differential operators and some of their relevant properties are presented in Section 2. After listing several well-established results on Sobolev spaces of real-valued functions in \( \mathbb{R} \), discussion in Section 3 is concentrated on a characterization of \( \tilde{H}^s(\mathbb{R}) \), a Sobolev space that is defined using Fourier Transform. It is achieved through the notion of weak fractional Riemann-Liouville derivatives, whose corresponding functional spaces are shown to be identical to \( \tilde{H}^s(\mathbb{R}) \). An application of the preceding framework to demonstrate existence and uniqueness of a strong solution to a fractional diffusion-advection-reaction in \( \mathbb{R} \) (see (4.1)) is presented in Section 4. The analysis in this section includes the stability and regularity estimates of the solution. Conclusion and future works is presented in Section 5. A list of frequently invoked theorems is given in Appendix A.

Several notations, conventions, definitions, and related facts to be used throughout the paper are collected in this paragraph. We assume all the functions are real valued unless otherwise specified. For a given set \( \Omega \subset \mathbb{R} \), we use characteristic function \( 1_\Omega (x) \) to denote \( 1_\Omega (w(x)) = \begin{cases} w(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}\setminus\Omega, \end{cases} \) for any function \( w \) defined in \( \Omega \) (even though \( w \) may not be defined on \( \mathbb{R}\setminus\Omega \)). Let

\[
\|w\|_{L^p(\Omega)} := \left( \int_\Omega |w(x)|^p \, dx \right)^{1/p}, \quad \text{for } 1 \leq p < \infty, \\
\text{ess sup}\{ |w(x)| : x \in \Omega \}, \quad \text{for } p = \infty.
\]

The Lebesgue spaces \( L^p(\Omega) \) is defined as \{ \( w : \Omega \to \mathbb{R} : \|w\|_{L^p(\Omega)} < \infty \} \). We note that \( L^2(\Omega) \) is a Hilbert space and \( (\cdot, \cdot)_{L^2(\Omega)} \) denotes its usual inner product that generates its norm \( \| \cdot \|_{L^2(\Omega)} \). To simplify presentation, we use \( (\cdot, \cdot) \) when \( \Omega = \mathbb{R} \). \( C_0^\infty(\mathbb{R}) \) denotes the space of all infinitely differentiable functions with compact support in \( \mathbb{R} \). \( \mathbb{N}_0 \) denotes the set of all non-negative integers. Convolution of two functions \( v \) and \( w \) is defined as \( [v * w](t) = \int_\mathbb{R} v(t - x) w(x) \, dx \). Given \( w : \mathbb{R} \to \mathbb{R} \), \( \mathcal{F}(w)(\xi) = \int_\mathbb{R} e^{-2\pi i x \xi} w(x) \, dx \), for \( \xi \in \mathbb{R} \), denotes the Fourier Transform of \( w \). The notation \( \hat{w} \) denotes the Plancherel Transform of \( w \) defined in Theorem A.1, which coincides with \( \mathcal{F}(w) \) if \( w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). The notation \( w^\vee \) denotes the inverse of Plancherel Transform. Given \( h \in \mathbb{R} \), define the translation operator \( \tau_h \) as \( \tau_h w(x) = w(x - h) \). Also, given \( \kappa > 0 \), define the dilation operator \( \Pi_\kappa \) as \( \Pi_\kappa w(x) = w(\kappa x) \). By appropriate change of variable, \( \mathcal{F}(\tau_h w)(\xi) = e^{-2\pi i h \xi} \mathcal{F}(w)(\xi), \mathcal{F}(\Pi_\kappa w)(\xi) = \mathcal{F}(w)(\xi), \) and \( \mathcal{F}(\Pi_\kappa w)(\xi) = \kappa^{-1} \mathcal{F}(w)(\kappa^{-1} \xi) \) for \( 0 \neq \kappa \in \mathbb{R} \). Here \( \overline{z} \) is the usual complex conjugate of \( z \in \mathbb{C} \).

2. Fractional Riemann-Liouville Operators. Definitions and several well-established facts about Riemann-Liouville (in short R-L) integrals and derivatives are laid out in this section, most of them without providing rigorous proofs. They have been recorded in various literatures, for which interested readers may refer to the specific references cited in the statements of the results.

2.1. Fractional Riemann-Liouville Integrals and Their Properties.

Definition 2.1. Let \( w : (a, b) \to \mathbb{R}, (a, b) \subset \mathbb{R} \) and \( \sigma > 0 \). The left and right Riemann-Liouville fractional
integrals of order $\sigma$ are, formally respectively, defined as

\begin{align}
(2.1) \quad & aD_x^{-\sigma}w(x) := \frac{1}{\Gamma(\sigma)} \int_a^x (x-s)^{\sigma-1}w(s) \, ds, \\
(2.2) \quad & xD_b^{-\sigma}w(x) := \frac{1}{\Gamma(\sigma)} \int_x^b (s-x)^{\sigma-1}w(s) \, ds,
\end{align}

where $\Gamma(\sigma)$ is the usual Gamma function. For convenience, we set

\begin{align}
(2.3) \quad D^{-\sigma}w(x) := -\infty D_x^{-\sigma}w(x) \quad \text{and} \quad D^{-\sigma}\ast w(x) := xD_{\infty}^{-\sigma}w(x).
\end{align}

Various aspects of these operators have been investigated in [20].

Remark 2.2. Each $D^{-\sigma}w$ and $D^{-\sigma}\ast w$ can be expressed as a convolution ([20], p. 94), namely,

\begin{align}
D^{-\sigma}w &= \frac{1}{\Gamma(\sigma)} f_1 \ast w, & D^{-\sigma}\ast w &= \frac{1}{\Gamma(\sigma)} f_2 \ast w,
\end{align}

where

\begin{align}
f_1 &= \mathbb{1}_{(0,\infty)}t^{\sigma-1} \quad \text{and} \quad f_2 = \mathbb{1}_{(-\infty,0]}|t|^{\sigma-1},
\end{align}

In particular, if $0 < \sigma < 1$, $f_1, f_2$ can be identified as distributions since they are locally integrable (see, e.g. [19], p. 157).

**Property 2.1.** Let $\mu, \sigma > 0$, $w \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. For any fixed $a, b \in \mathbb{R}$, the following is true

\begin{align}
(2.4) \quad & aD_x^{-\mu}aD_x^{-\sigma}w(x) = aD_x^{-\mu-\sigma}w(x), \quad x > a \\
& xD_b^{-\mu}xD_b^{-\sigma}w(x) = xD_b^{-\mu-\sigma}w(x), \quad b < x.
\end{align}

**Proof.** For a bounded interval $(a, b)$ and $w \in L^p(a, b)$ it has been shown in [14] Lemma 2.3, p. 73 that

\begin{align}
(2.5) \quad & aD_x^{-\mu}aD_x^{-\sigma}w(x) = aD_x^{-\mu-\sigma}w(x) \quad \text{and} \quad xD_b^{-\mu}xD_b^{-\sigma}w(x) = xD_b^{-\mu-\sigma}w(x).
\end{align}

The proof below is shown only for the first equality, the second one follows similarly. For any $x > a$, we could always pick an integer $n$, such that $x \in (a, a+n)$. Notice $w \in L^p(a, a+n)$, applying (2.5), we have

\begin{align}
& aD_x^{-\mu}aD_x^{-\sigma}w(x) = aD_x^{-\mu-\sigma}w(x), \quad x \in (a, a+n).
\end{align}

Since $n$ is arbitrary, this means $aD_x^{-\mu}aD_x^{-\sigma}w(x) = aD_x^{-\mu-\sigma}w(x)$, for $x > a$. The following is an immediate consequence of Property 2.1.

**Corollary 2.3.** Let $\mu, \sigma > 0$, and $w \in C_0^\infty(\mathbb{R})$, then

\begin{align}
(2.6) \quad D^{-\mu}D^{-\sigma}w = D^{-(\mu+\sigma)}w \quad \text{and} \quad D^{-\mu}\ast D^{-\sigma}\ast w = D^{-(\mu+\sigma)\ast}w.
\end{align}

**Corollary 2.4.** Let $v, w \in C^\infty(\mathbb{R})$, $\text{supp}(v) \subset (a, +\infty)$, $\text{supp}(w) \subset (-\infty, b)$, $\mu > 0, b > a$. It is true that

\begin{align}
(2.7) \quad (D^{-\mu}v, w) = (v, D^{-\mu}\ast w).
\end{align}
Proof. For a bounded interval \((a, b)\) and \(v, w \in L^2(a, b)\) and \(\sigma > 0\), it has been shown in the corollary of Theorem 3.5, p. 67 of [20], that

\[
(aD_x^{-\sigma}v, w)_{L^2(a, b)} = (v, D^\sigma_x w)_{L^2(a, b)}.
\]  

(2.8)

Notice \(\text{supp}(D^{-\mu}v) \subset (a, +\infty)\) and \(\text{supp}(D^{-\mu^*}w) \subset (-\infty, b)\). By Definition 2.1 and (2.8),

\[
(D^{-\mu}v, w) = (aD_x^{-\mu}v, w)_{L^2(a, b)} = (v, D_b^{-\mu}w)_{L^2(a, b)} = (v, D^{-\mu^*}w).
\]  

(2.9)

\[
\text{Property } 2.2 \text{ (Fourier Transform of R-L Integrals, [20], Theorem 7.1, p.138). Under the assumption that } w \in L^1(\mathbb{R}) \text{ and } 0 < \sigma < 1,
\]

\[
\mathcal{F}(D^{-\sigma}w) = (2\pi i \xi)^{-\sigma} \mathcal{F}(w) \text{ and } \mathcal{F}(D^{-\sigma^*}w) = (-2\pi i \xi)^{-\sigma} \mathcal{F}(w), \quad \xi \neq 0.
\]

This property is equivalently given in [20] Theorem 7.1, p. 138 with a different version of the definition of Fourier Transform up to a sign \(-2\pi\) in exponentiation.

Remark 2.5. \((\mp i \xi)^\sigma\) is understood as equal to \(|\xi|^\sigma e^{\mp \pi i \text{sign}(\xi)/2}\).

The following property is on the commutativity of R-L integrals with translation and dilation operators.

\[
\text{Property } 2.3 \text{ ([20], pp. 95, 96). Under the assumption that } D^{-\mu}w \text{ and } D^{-\mu^*}w \text{ are well-defined, the following is true:}
\]

\[
\tau_h(D^{-\mu}w) = D^{-\mu}(\tau_h w), \quad \tau_h(D^{-\mu^*}w) = D^{-\mu^*}(\tau_h w)
\]

\[
\Pi_{\kappa}(D^{-\mu}w) = \kappa^\mu D^{-\mu}(\Pi_{\kappa} w), \quad \Pi_{\kappa}(D^{-\mu^*}w) = \kappa^\mu D^{-\mu^*}(\Pi_{\kappa} w).
\]

2.2. Fractional Riemann-Liouville Derivatives and Their Properties.

Definition 2.6. Let \((a, b) \subset \mathbb{R}\) and \(w : (a, b) \to \mathbb{R}\). Assume \(\mu > 0\) and \(n\) is the smallest integer greater than \(\mu\) \(\text{i.e., } n - 1 \leq \mu < n\). The left and right Riemann-Liouville fractional derivatives of order \(\mu\) are, formally respectively, defined as

\[
aD_x^\mu w := \frac{d^n}{dx^n} (aD_x^{\mu-n}w(x)), \quad \text{and} \quad \_\_D_x^\mu w := (-1)^n \frac{d^n}{dx^n} (\_\_D_x^{\mu-n}w(x)).
\]

For convenience of notation, we set

\[
D^\mu w = -\infty D_x^\mu w \text{ and } D^\mu^* w = \_\_D_x^\mu w.
\]

Property 2.4 ([14], Lemma 2.4, p. 74 ). For any \(\mu > 0\) and \(w \in L^p(a, b)\), where \((a, b) \subset \mathbb{R}\) is a bounded interval and \(1 \leq p \leq \infty\), then

\[
aD_x^\mu aD_x^{-\mu}w = w(x) \text{ and } \_\_D_x^\mu _\_D_x^{-\mu}w = w(x).
\]

(2.14)

Two immediate consequences of Property 2.4 are stated below.

Corollary 2.7. Let \(\mu > 0\) and \(w \in L^p(\mathbb{R})\) with \(1 \leq p \leq \infty\). For any fixed \(a, b \in \mathbb{R}\),

\[
aD_x^\mu aD_x^{-\mu}w = w(x), \quad \text{for } x > a, \quad \text{and} \quad \_\_D_x^\mu _\_D_x^{-\mu}w = w(x), \quad \text{for } x < b.
\]

\[
\_\_D_x^\mu aD_x^{-\mu}w = w(x), \quad \text{for } x > a, \quad \text{and} \quad \_\_D_x^\mu _\_D_x^{-\mu}w = w(x), \quad \text{for } x < b.
\]
Property 2.5. Let $\mu > 0$ and $(a, b) \subset \mathbb{R}$ be a bounded interval. If $w = aD_x^{-\mu}\psi$ for some $\psi \in L^p(a, b)$ with $1 \leq p \leq \infty$, then

\begin{equation}
(2.16)
  aD_x^{-\mu}D_x^{\mu}w = w(x), \forall x \in (a, b).
\end{equation}

Furthermore, if $w \in C^\infty((a, +\infty))$ and supp$(w) \subset (a, +\infty)$, then

\begin{equation}
(2.17)
  aD_x^{-\mu}D_x^{\mu}w = w(x), \forall x \in (a, +\infty).
\end{equation}

Similarly, if $w = xD_b^{-\mu}\psi$ for some $\psi \in L^p(a, b)$ with $1 \leq p \leq \infty$, then

\begin{equation}
(2.18)
  xD_b^{-\mu}D_b^{\mu}w = w(x), \forall x \in (a, b).
\end{equation}

And furthermore, if $w \in C^\infty((\infty, b))$ and supp$(w) \subset (\infty, b)$, then

\begin{equation}
(2.19)
  xD_b^{-\mu}D_b^{\mu}w = w(x), \forall x \in (\infty, b).
\end{equation}

This property is equivalently stated by [20] (c.f. Theorem 2.3, p. 43 combined with Theorem 2.4, p. 44). As an immediate corollary, we have:

Corollary 2.8. Let $0 < \mu$, and $w \in C_0^\infty(\mathbb{R})$, then

\begin{equation}
(2.20)
  D^{-\mu}D^\mu w = w \quad \text{and} \quad D^{-\mu}D^\mu w = w.
\end{equation}

Property 2.6. Let $0 < \mu$ and $w \in C_0^\infty(\mathbb{R})$, then $D^\mu w, D^{\mu}w \in L^p(\mathbb{R})$ for any $1 \leq p < \infty$.

Proof. The proof is shown only for $D^\mu w$, the other one can be established in a similar fashion. Since $w \in C_0^\infty(\mathbb{R})$, there exists a bounded interval $(a, b - 1)$, such that supp$(w) \subset (a, b - 1)$. When $\mu$ is a positive integer, then $D^\mu w = w^{(\mu)} \in L^p(\mathbb{R})$ for any $1 \leq p < \infty$. Otherwise, we can always choose a non-negative integer $n$ such that $n - 1 < \mu < n$. Since $w \in C_0^\infty(\mathbb{R})$, by Corollary 2.8, $w(x) = D^{-n}v(x)$, where $v(x) = w^{(n)}(x)$ also belonging to $C_0^\infty(\mathbb{R})$. Thus, $D^\mu w = D^\mu D^{-n}v(x)$. Applying Corollary 2.3, we know $D^{-n}v(x) = D^{-\mu}D^{-(n-\mu)}v(x)$. Plugging in back gives

\begin{equation}
(2.21)
  D^\mu w = D^\mu(D^{-\mu}D^{-(n-\mu)}v(x)) = (D^\mu D^{-\mu})D^{-(n-\mu)}v(x).
\end{equation}

Since $(D^\mu D^{-\mu})D^{-(n-\mu)}v(x) = \mathbb{1}_{\{a < x\}}(aD_x^{-\mu}D_x^{-\mu})D^{-(n-\mu)}v(x)$, applying Corollary 2.7 and plugging back into (2.21) yields

\begin{equation}
(2.22)
  D^\mu w = D^{-(n-\mu)}v(x).
\end{equation}

Now we consider decomposition $D^\mu w = f(x) + g(x)$, where

\begin{equation}
(2.23)
  f(x) = \mathbb{1}_{\{x \leq b\}} D^\mu w \quad \text{and} \quad g(x) = \mathbb{1}_{\{x > b\}} D^\mu w.
\end{equation}

In order to show $D^\mu w \in L^p(\mathbb{R})$, we only need to show $f, g \in L^p(\mathbb{R})$. First we claim $f \in L^p(\mathbb{R})$. By (2.22) and definition in (2.13), we know

\begin{equation}
(2.24)
  f(x) = \mathbb{1}_{\{x \leq b\}} D^{-(n-\mu)}v(x) = \mathbb{1}_{\{a < x \leq b\}} aD_x^{-(n-\mu)}v(x),
\end{equation}

and thus (see for example [20], p. 48)

\[ \|f\|_{L^p(\mathbb{R})} = \|aD_x^{-(n-\mu)}v\|_{L^p(a, b)} \leq \frac{(b-a)^{n-\mu}}{(n-\mu)\Gamma(n-\mu)} \|v\|_{L^p(a, b)} < \infty. \]

This manuscript is for review purposes only.
Thus, \( f \in L^p(\mathbb{R}) \). Next it is demonstrated that \( g \in L^p(\mathbb{R}) \). By setting \( \sigma = n - \mu \) and using Definition 2.6, \( g(x) = \mathbb{1}_{\{x > b\}}(\Gamma(\sigma))^{-1}1 \), with

\[
I = \frac{d^n}{dx^n} \int_{-\infty}^{x} (x-s)^{\sigma-1}w(s) \, ds = \frac{d^n}{dx^n} \int_{-\infty}^{b-1} (x-s)^{\sigma-1}w(s) \, ds,
\]

Notice that when \( x > b \),

\[
\left| \frac{d^n}{dx^n} ((x-s)^{\sigma-1}w(s)) \right| \leq |w(s)| \quad \text{and} \quad |w(s)| \text{ is integrable over } (b, \infty).
\]

Therefore, application Dominated Convergence Theorem gives

\[
I = C \int_{-\infty}^{b-1} (x-s)^{\tilde{\sigma}}w(s) \, ds, \quad \text{where } \tilde{\sigma} = \sigma - 1 - n,
\]

and \( C = (\sigma - 1) \cdot (\sigma - 2) \cdots (\sigma - n) \). Applying Hölder’s Inequality to \( I \), for \( x > b \), results in

\[
I \leq |I| \leq C\|(x-s)^{\bar{\sigma}}\|_{L^{\infty}(-\infty,-b)}\|w\|_{L^{1}(-\infty,-b-1)} \leq C(x-b+1)^{\bar{\sigma}}\|w\|_{L^{1}(-\infty,-b-1)}.
\]

Notice that \( \|w\|_{L^{1}(-\infty,-b-1)} = \|w\|_{L^{1}(\mathbb{R})} < \infty \), and since \( \bar{\sigma} < -1 \), it can be easily verified that \( (x-b+1)^{\bar{\sigma}} \in L^p(b, \infty) \) for \( 1 \leq p < \infty \). Taken all these together into account yields \( \|g\|_{L^p(\mathbb{R})} = \|\mathbb{1}_{\{x > b\}}(\Gamma(\sigma))^{-1}1\|_{L^p(\mathbb{R})} = ||(\Gamma(\sigma))^{-1}1\|_{L^p(\mathbb{R}, \infty)} < \infty \). Therefore, \( D^{\mu}w \in L^p(\mathbb{R}) \), which completes the proof.

**Property 2.7** (Fourier Transform of R-L Derivatives [20], p. 137).\( \) If \( \mu > 0 \) and \( w \in C_0^\infty(\mathbb{R}) \), then

(2.25)\[
\mathcal{F}(D^{\mu}w) = (2\pi i)^{\nu}F(w) \quad \text{and} \quad \mathcal{F}(D^{\nu+\mu}w) = -(2\pi i)^{\nu}F(w), \quad \xi \neq 0,
\]

where as in Property 2.2, \( (\mp i)^{\sigma} \) is understood as \( |\xi|^{\sigma} e^{\mp \sigma i \operatorname{sign}(\xi)/2} \).

**Proof.** Using a different version of the Fourier Transform (namely up to a sign \(-2\pi \) in the exponential position), this property was stated without a proof in [20], p.137. The proof below is provided only for completeness. The proof is shown only for the Fourier Transform of left derivative since the right derivative counterpart can be carried out analogously. First notice that if \( \mu \) is a positive integer, integration by parts and a simple calculation give the equality (see, e.g. [15], p. 274). Otherwise, there is a positive integer \( n \), such that \( n - 1 < \mu < n \). Suppose \( \text{supp}(w) \subset (a, \infty) \) with \(-\infty < a < b < \infty \). Using Remark 2.2 and Theorem A.5 gives

(2.26)\[
D^{\mu}w = \mathbb{1}_{\{x > a\}}D^{\mu}xw = \mathbb{1}_{\{x > a\}}aD^{\mu-n}(x)w^{(n)}(x) = D^{-(n-\mu)}w^{(n)}(x).
\]

Notice now \( 0 < n - \mu < 1 \), so using Property 2.2 yields

(2.27)\[
\mathcal{F}(D^{\mu}w) = \mathcal{F}(D^{-(n-\mu)}w^{(n)}) = (2\pi i)^{-n+\mu}F(w^{(n)}), \xi \neq 0.
\]

The proof is completed by recalling that \( \mathcal{F}(w^{(n)}) = (2\pi i)^{n}F(w) \). \hfill \Box

**Property 2.8.** Given \( w \in C_0^\infty(\mathbb{R}) \), \( \mu > 0 \), and a positive integer \( n \) such that \( n - 1 < \mu < n \), then

(2.28)\[
\tau_h(D^{\mu}w) = D^{\mu}(\tau_h w), \quad \tau_h(D^{\mu+n}w) = D^{\mu}(\tau_h w),
\]

\[
\Pi_k(D^{\mu}w) = \kappa^{-\mu}D^{\mu}(\Pi_k w), \quad \Pi_k(D^{\mu+n}w) = \kappa^{-\mu}D^{\mu}(\Pi_k w).
\]

This manuscript is for review purposes only.
Proof. First, since \( w \in C_0^\infty(\mathbb{R}) \), using Remark 2.2 and Theorem A.5 gives \( D^\mu w = D^{-(n-\mu)w(n)} \). According to Property 2.3,
\[
\tau_h(D^{\mu}w) = \tau_h(D^{-(n-\mu)w(n)}) = D^{-(n-\mu)(\tau_h w(n))} = D^{-(n-\mu)(\tau_h w)^{\mu}}(\tau_h w).
\]

Similar argument is used to establish \( \tau_h(D^{\mu*}w) = D^{\mu*}w\). Next, again using Property 2.3,
\[
\Pi_\kappa(D^{\mu}w) = \Pi_\kappa(D^{-(n-\mu)w(n)}) = \kappa^{(n-\mu)}D^{-(n-\mu)}(\Pi_\kappa w(n)) = \kappa^{(n-\mu)}D^{-(n-\mu)}((\Pi_\kappa w)^{(n)}) = \kappa^{(n-\mu)}D^{-(n-\mu)}(\Pi_\kappa w).
\]
Using similar argument, \( \Pi_\kappa(D^{\mu*}w) = \kappa^{(n-\mu)}D^{\mu*}(\Pi_\kappa w) \) follows. \( \square \)

### 3. Characterization of Fractional Sobolev Spaces

In this section, we shall characterize classical fractional Sobolev spaces \( W^{s,\gamma}(\mathbb{R}) \) (namely \( \tilde{H}^s(\mathbb{R}) \)) by giving another equivalent definition using weak fractional R-L derivatives. There is a vast amount of literatures devoted to Sobolev spaces (see, e.g., \([4],[6],[22]\)), thus those well-established results pertaining to subsequent analyses are stated without proof.

#### 3.1. Some Facts on Sobolev Spaces

**Definition 3.1** (Sobolev Spaces on \( \mathbb{R} \) \([8],[1] \) p. 258, \([1] \) p. 250). Let \( m \in \mathbb{N}_0, 1 \leq p \leq \infty \).
\[
W^{m,p}(\mathbb{R}) = \{ w \in L^p(\mathbb{R}) : D^\alpha w \in L^p(\mathbb{R}), \forall 0 \leq \alpha \leq m \},
\]
where \( \alpha \) is integer and \( D^\alpha u \) are weak derivatives. If \( s > 0 \) is a real number and \( m \) is the smallest integer greater than \( s \), the fractional order Sobolev spaces are defined by complex interpolation as
\[
W^{s,p}(\mathbb{R}) = [L^p(\mathbb{R}), W^{m,p}(\mathbb{R})]_{s/m}.
\]

**Definition 3.2** (Sobolev Spaces Via Fourier Transform, e.g. \([6],[22]\)). Given \( s \geq 0 \), let
\[
\tilde{H}^s(\mathbb{R}) = \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |2\pi\xi|^2)^\frac{s}{2} |\hat{w}(\xi)|^2 d\xi < \infty \right\},
\]
where \( \hat{w} \) is Plancherel Transform defined in Theorem A.1. It is endowed with norm
\[
||w||_{\tilde{H}^s(\mathbb{R})} := \left( ||w||^2_{L^2(\mathbb{R})} + ||w||^2_{\tilde{H}^s(\mathbb{R})} \right)^{1/2}, \text{ with } |w|_{\tilde{H}^s(\mathbb{R})} := ||(2\pi\xi)^s \hat{w}||_{L^2(\mathbb{R})}.
\]
It is well-known that \( \tilde{H}^s(\mathbb{R}) \) is a Hilbert space.

**Theorem 3.3** ([22], p. 78). \( C_c^\infty(\mathbb{R}) \) is dense in \( \tilde{H}^s(\mathbb{R}) \).

**Theorem 3.4** ([1], p. 252). Let \( s \geq 0 \). \( u \in W^{s,2}(\mathbb{R}) \) if and only if \( u \in \tilde{H}^s(\mathbb{R}) \). In addition, \( W^{s_1,2}(\mathbb{R}) \subseteq W^{s_2,2}(\mathbb{R}) \) for \( s_1 \geq s_2 \).

**Remark 3.5.** Notice the particular case, if \( s = 0 \), \( L^2(\mathbb{R}) = \tilde{H}^0(\mathbb{R}) = W^{0,2}(\mathbb{R}) \).

This manuscript is for review purposes only.
3.2. Connections between Sobolev Spaces and R-L Derivatives. First, a generalization of the usual integer-order weak derivatives to include weak fractional R-L derivatives is presented.

**Definition 3.6 (Weak Fractional R-L Derivatives).** Let \( s > 0, \) and \( v, w \in L^1_{\text{loc}}(\mathbb{R}) \). The function \( w \) is called weak \( s \)-order left fractional derivative of \( v \), written as \( D^s v = w \), provided

\[
(v, D^s \psi) = (w, \psi), \quad \forall \psi \in C_0^\infty(\mathbb{R})..
\]

In a similar fashion, \( w \) is weak \( s \)-order right fractional derivative of \( v \), written as \( D^s v = w \), provided

\[
(v, D^s \psi) = (w, \psi), \quad \forall \psi \in C_0^\infty(\mathbb{R})..
\]

**Lemma 3.7 (Uniqueness of Weak Fractional R-L Derivatives).** If \( v \in L^1_{\text{loc}}(\mathbb{R}) \) has a weak \( s \)-order left (or right) fractional derivative, then it is unique up to a set of zero measure.

**Proof.** We only show the uniqueness for the left fractional derivative. Assume \( w_1, w_2 \in L^1_{\text{loc}}(\mathbb{R}) \) are both weak \( s \)-order fractional derivatives of \( v \), namely, \((w_1, \psi) = (v, D^s \psi) = (w_2, \psi)\), for all \( \psi \in C_0^\infty(\mathbb{R}) \). This implies \((w_1 - w_2, \psi) = 0\) for all \( \psi \in C_0^\infty(\mathbb{R}) \), whence \( w_1 = w_2 \) a.e. (e.g. [4], Corollary 4.24, p.110).

**Definition 3.8.** Given \( s \geq 0 \), let

\[
\text{\hat{W}}_L^s(\mathbb{R}) = \{ v \in L^2(\mathbb{R}) : D^s v \in L^2(\mathbb{R}) \}, \quad \text{\hat{W}}_R^s(\mathbb{R}) = \{ v \in L^2(\mathbb{R}) : D^s v \in L^2(\mathbb{R}) \},
\]

where \( D^s v \) and \( D^s v \) are understood as the weak fractional derivative of Definition 3.6. A semi-norm

\[
|v|_L := \|D^s v\|_{L^2(\mathbb{R})} \quad \text{for \( \text{\hat{W}}_L^s(\mathbb{R}) \)} \quad \text{and} \quad |v|_R := \|D^s v\|_{L^2(\mathbb{R})} \quad \text{for \( \text{\hat{W}}_R^s(\mathbb{R}) \)},
\]

is given with the corresponding norm

\[
\|v\|_* := (\|v\|_{L^2(\mathbb{R})}^2 + |v|_*^2)^{1/2}, \quad \text{with} \quad * = L, R.
\]

**Remark 3.9.** By convention, \( \text{\hat{W}}_L^0(\mathbb{R}) = \text{\hat{W}}_R^0(\mathbb{R}) = L^2(\mathbb{R}) \). If \( s \) is a positive integer, by definition, \( D^s = D^s \), and \( D^s = (-1)^s D^s \), so \( D^1 = D = D, \) and \( D^s = D^s = -D \).

It is obvious that \( \text{\hat{W}}_L^s(\mathbb{R}) \) and \( \text{\hat{W}}_R^s(\mathbb{R}) \) are normed linear spaces. The following theorem describes a characterization of Sobolev space \( \text{\hat{H}}^s(\mathbb{R}) \) in terms of these spaces.

**Theorem 3.10.** Given \( s \geq 0 \), \( \text{\hat{W}}_L^s(\mathbb{R}) \), \( \text{\hat{W}}_R^s(\mathbb{R}) \) and \( \text{\hat{H}}^s(\mathbb{R}) \) are identical spaces with equal norms.

**Proof.** The proof only demonstrates \( \text{\hat{W}}_L^s(\mathbb{R}) = \text{\hat{H}}^s(\mathbb{R}) \) and their norms equality, noting that the case for \( \text{\hat{W}}_R^s(\mathbb{R}) = \text{\hat{H}}^s(\mathbb{R}) \) can be analogously established. By the construction of respective norms, equality of norms is achieved by showing equality of seminorms.

First we show that \( \text{\hat{H}}^s(\mathbb{R}) \subseteq \text{\hat{W}}_L^s(\mathbb{R}) \). Pick any \( v \in \text{\hat{H}}^s(\mathbb{R}) \), which implies that \( (2\pi i \xi)^s \hat{v} \in L^2(\mathbb{R}) \). In turn, this gives a justification for setting \( \psi_s := (\xi^{2s})^{-1} \hat{v} \) \in \( L^2(\mathbb{R}) \). Furthermore, Theorem A.1 (Plancherel) guarantees that \( \psi_s \in L^2(\mathbb{R}) \). An application of Theorem A.2 gives

\[
(v, D^s \psi) = (\hat{\tau}, D^s \hat{\psi}) = (\hat{\tau}, D^s \hat{\psi}) = (\hat{\tau}, D^s \hat{\psi}) = (\hat{\tau}, D^s \hat{\psi}) = (v_s, \psi).
\]

Next we use Property 2.7 to \( D^s \hat{\psi} \) and utilize Theorem A.2 to yield

\[
(\hat{\tau}, D^s \hat{\psi}) = (\hat{\tau}, (2\pi i \xi)^s \hat{\psi}) = (\hat{\tau}, (2\pi i \xi)^s \hat{\psi}) = (v_s, \psi).
\]

By combining (3.9) with (3.8), we get \((v, D^s \psi) = (v_s, \psi)\) for any \( \psi \in C_0^\infty(\mathbb{R}) \), which according to Definition 3.6 implies that \( D^s v = v_s \), and thus \( v \in \text{\hat{W}}_L^s(\mathbb{R}) \). It is straightforward to see the equality of semi-norms, namely,

\[
|v|_{\text{\hat{H}}^s}(\mathbb{R}) = \|(2\pi i \xi)^s \hat{v}\|_{L^2(\mathbb{R})} = |v|_{L^2(\mathbb{R})} = |v|_L.
\]
It remains now to show \( \hat{H}^s(\mathbb{R}) \supseteq \hat{W}_L^s(\mathbb{R}) \). Pick any \( v \in \hat{W}_L^s(\mathbb{R}) \). By Definition 3.6,

\[
(3.10) \quad (v, D^{*s} \psi) = (D^s v, \psi), \quad \forall \psi \in C_0^\infty(\mathbb{R})
\]

Fix \( h \in \mathbb{R} \) and use \( \tau_h \psi \in C_0^\infty(\mathbb{R}) \) in (3.10) to obtain

\[
(3.11) \quad (v, D^{*s}(\tau_h \psi)) = (v, \tau_h (D^{*s} \psi)) = (D^s v, \tau_h \psi),
\]

where Property 2.8 was used. For convenience, set

\[
(3.12) \quad A(z) = |D^{*s} \psi|(-z), \quad B(z) = |D^s \psi|(z), \quad \Psi(z) = \psi(-z).
\]

Then

\[
(3.13) \quad \int_\mathbb{R} v(t) A(h - t) \, dt = \int_\mathbb{R} B(h - t) \, dt, \quad \forall h \in \mathbb{R},
\]

or in other words,

\[
(3.14) \quad [v * A](h) = [B * \Psi](h).
\]

Notice that \( v, \Psi \in L^2(\mathbb{R}) \), and \( A, B \in L^1(\mathbb{R}) \) by Property 2.6. Theorem A.3 applied to (3.14) gives

\[
(3.15) \quad \hat{v} A = \hat{B} \Psi.
\]

Since \( \Psi(z) = \psi(-z) \) and \( \psi \in C_0^\infty(\mathbb{R}) \), then \( \hat{\Psi} = \hat{\psi} = \hat{F}(\psi) \). In a similar fashion, and using Property 2.7, \( \hat{A} = \bar{D}^{*s} \psi = (-2\pi i \xi)^s \hat{F}(\psi) = (2\pi i \xi)^s \hat{F}(\psi) \). Putting these back to (3.15) gives

\[
(3.16) \quad (2\pi i \xi)^s \hat{v} - \bar{D}^s v = 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}).
\]

We claim that (3.16) implies that \( (2\pi i \xi)^s \hat{v} = \bar{D}^s v \). Since \( \psi \in C_0^\infty(\mathbb{R}) \) is arbitrary, choose a non-zero \( \psi \) such that by Theorem A.1 (Plancherel), \( \hat{F}(\psi) \neq 0 \). Without loss of generality, suppose \( |\hat{F}(\psi)|(a) \neq 0 \) at point \( a \in \mathbb{R} \). Since \( \hat{F}(\psi) \) is continuous, there exists an open interval \( (c, d) \) containing \( a \) such that \( \hat{F}(\psi) \neq 0 \) in \( (c, d) \). Notice that \( \Pi \psi \in C_0^\infty(\mathbb{R}) \) and by the Fourier Transform property of dilation operator, \( |\hat{F}(\Pi \psi)(\xi)| = \epsilon^{-1} |\hat{F}(\psi))(\epsilon^{-1} \xi) \) for arbitrary \( \epsilon > 0 \). This means \( \hat{F}(\Pi \psi) \neq 0 \) in \( (\epsilon c, \epsilon d) \). With this fact in place and using \( \Pi \psi \) as a test function in (3.16) implies

\[
(3.17) \quad (2\pi i \xi)^s \hat{v} - \bar{D}^s v = 0, \quad \text{in } (\epsilon c, \epsilon d).
\]

Since \( \epsilon \) is arbitrary, we conclude that

\[
(3.18) \quad (2\pi i \xi)^s \hat{v} = \bar{D}^s v, \quad \text{in } \mathbb{R}.
\]

Therefore \( (2\pi i \xi)^s \hat{v} \in L^2(\mathbb{R}) \), and thus \( v \in \hat{H}^s(\mathbb{R}) \). Furthermore, (3.18) implies \( |v|_{\hat{H}^s(\mathbb{R})} = |v|_{L^2} \).

The preceding theorem reveals that \( D^s v \) and \( D^{*s} v \) always makes sense for \( v \in \hat{H}^s(\mathbb{R}) \), \( s > 0 \). The following results will be utilized later.

**Corollary 3.11.** \( C_0^\infty(\mathbb{R}) \) is dense in \( \hat{W}_L^s(\mathbb{R}) \) and \( \hat{W}_R^s(\mathbb{R}) \).

**Proof.** This is a consequence of Theorem 3.10 and Theorem 3.3. \( \square \)

**Corollary 3.12.** \( v \in \hat{H}^s(\mathbb{R}) \) if and only if there exists a sequence \( \{v_n\} \subset C_0^\infty(\mathbb{R}) \) such that \( \{v_n\}, \{D^s v_n\} \)
are Cauchy sequences in \( L^2(\mathbb{R}) \) with \( \lim_{n \to \infty} v_n = v \). Likewise, \( v \in \hat{H}^s(\mathbb{R}) \) if and only if there exists a sequence \( \{v_n\} \subset C_0^\infty(\mathbb{R}) \) such that \( \{v_n\}, \{D^{*s} v_n\} \)
are Cauchy sequences in \( L^2(\mathbb{R}) \), with \( \lim_{n \to \infty} v_n = v \).

**Proof.** This is a consequence of Theorem 3.10 and Corollary 3.11. \( \square \)

**Remark 3.13.** As a consequence of Corollary 3.12, \( \lim_{n \to \infty} D^s v_n = D^s v \) and \( \lim_{n \to \infty} D^{*s} v_n = D^{*s} v \).
4. Stationary Fractional Diffusion-Advection-Reaction Equations. In this section, we investigate the following Stationary Fractional Diffusion-Advection-Reaction equation: find \( u \in \widetilde{H}^{2-\mu}(\mathbb{R}) \) such that

\[
\begin{align*}
[Lu](x) &= f(x), \quad x \in \mathbb{R}, \\
\text{where} \quad Lu &= pD^{2-\mu}u + qD^{(2-\mu)*}u + aDu + bu, \quad f \in L^2(\mathbb{R}), \\
\text{with} \quad p, q, a, b, \mu \in \mathbb{R}, \quad \text{such that} \quad \mu > 0, p^2 + q^2 \neq 0, \mu \in (0, 1).
\end{align*}
\]

In this equation, \( D^{2-\mu}u, D^{(2-\mu)*}u, Du \) are all understood as weak derivatives. The condition \( p^2 + q^2 \neq 0 \) implies that at least one of \( pD^{2-\mu}u \) or \( qD^{(2-\mu)*}u \) must be present in (4.1), thereby avoiding the classical first order ODEs. Also, we point out that \( \mu > 0 \) plays an important role in determining the regularity of solution to problem (4.1). The main results are stated in Theorem 4.9, Theorem 4.11.

4.1. Several Important Tools. Several results that are crucial in the subsequent analysis are first established.

**Theorem 4.1.** For \( v, w \in C_0^\infty(\mathbb{R}) \) and \( \mu \geq 0 \), it is true that

\[
\begin{align*}
(D^\mu v, D^\mu w) &= (D^{\mu*} v, D^{\mu*} w) = (2\pi)^{2\mu} \int_\mathbb{R} |\xi|^{2\mu} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi, \\
(D^\mu v, D^{\mu*} w) + (D^{\mu*} v, D^\mu w) &= 2 \cos(\mu \pi) (D^\mu v, D^\mu w).
\end{align*}
\]

**Proof.** The two equalities in (4.2) are true when \( \mu = 0 \), so suppose \( \mu > 0 \). Since \( v, w \in C_0^\infty(\mathbb{R}) \), Property 2.6 guarantees that \( D^\mu v, D^\mu w, D^{\mu*} v, D^{\mu*} w \in L^p(\mathbb{R}) \) with \( p \geq 1 \). Using Theorem A.2 (Parseval Formula) and in combination with Property 2.7 give

\[
\begin{align*}
(D^\mu v, D^\mu w) &= (\mathcal{F}(D^\mu v), \mathcal{F}(D^\mu w)) = (2\pi)^{2\mu} \int_\mathbb{R} |\xi|^{2\mu} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi, \\
(D^{\mu*} v, D^{\mu*} w) &= (\mathcal{F}(D^{\mu*} v), \mathcal{F}(D^{\mu*} w)) = (2\pi)^{2\mu} \int_\mathbb{R} |\xi|^{2\mu} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi,
\end{align*}
\]

confirming the first equality in (4.2). In a similar fashion,

\[
\begin{align*}
(D^\mu v, D^{\mu*} w) &= (\mathcal{F}(D^\mu v), \mathcal{F}(D^{\mu*} w)) = (2\pi)^{2\mu} I, \\
(D^{\mu*} v, D^\mu w) &= (\mathcal{F}(D^{\mu*} v), \mathcal{F}(D^\mu w)) = (2\pi)^{2\mu} II,
\end{align*}
\]

where

\[
I = \int_\mathbb{R} (i\xi)^\mu (-i\xi)^{\mu} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi \quad \text{and} \quad II = \int_\mathbb{R} (i\xi)^{\mu} (-i\xi)^\mu \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi.
\]

Upon utilization of Remark 2.5,

\[
\begin{align*}
I &= \int_\mathbb{R} |\xi|^{2\mu} e^{i\text{sign}(\xi)\pi/2} \overline{e^{-i\text{sign}(\xi)\pi/2}} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi = \int_\mathbb{R} |\xi|^{2\mu} e^{i\text{sign}(\xi)\pi} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi, \\
II &= \int_\mathbb{R} |\xi|^{2\mu} e^{-i\text{sign}(\xi)\pi/2} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi = \int_\mathbb{R} |\xi|^{2\mu} e^{-i\text{sign}(\xi)\pi} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi.
\end{align*}
\]
Summation of I and II and decomposition of $\mathbb{R}$ into $(-\infty, 0)$ and $(0, \infty)$ yield

$$I + II = (e^{-\pi i \mu} + e^{\pi i \mu}) \int_{-\infty}^{0} |\xi|^{2\mu} \tilde{\nu}(\xi) \overline{w(\xi)} \, d\xi + (e^{-\pi i \mu} + e^{\pi i \mu}) \int_{0}^{\infty} |\xi|^{2\mu} \tilde{\nu}(\xi) \overline{w(\xi)} \, d\xi,$$

from which the second equality in (4.2) follows. \qed

**Lemma 4.2.** The set $M = \{w : w = Lv, \forall v \in C_0^\infty(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$. Furthermore, the set $\tilde{M} = \{w : w = \tilde{L}v, \forall v \in C_0^\infty(\mathbb{R})\}$ is also dense in $L^2(\mathbb{R})$, where $\tilde{L}v = pD^{2-\mu}v + qD^{2-\mu}v - aDv + bv$.

**Proof.** By Property 2.6, $M \subset L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is a Hilbert space, the density of $M$ is established by invoking Theorem A.4. Furthermore, because $C_0^\infty(\mathbb{R})$ is closed under addition and scalar multiplication, so is $M$, and thus $M$ is a subspace of $L^2(\mathbb{R})$. Therefore all conditions are met for the utilization of Theorem A.4. Using Property 2.7 (Fourier Transform) for $w = Lv$ gives

$$[\mathcal{F}(w)](\xi) = H(\xi)[\mathcal{F}(v)](\xi), \quad H(\xi) = (p(2\pi i \xi)^{2-\mu} + q(-2\pi i \xi)^{2-\mu} + a(2\pi i \xi) + b).$$

Setting $\vartheta = \frac{(2-\mu)\pi \text{sign}(\xi)}{2}$ and following Remark 2.5, $H(\xi)$ is expressed as

$$H(\xi) = (2\pi |\xi|)^{2-\mu} (pe^{i\vartheta} + qe^{-i\vartheta}) + a(2\pi i \xi) + b$$

$$= ((2\pi |\xi|)^{2-\mu}(p+q)\cos(\vartheta) + b) + i((2\pi |\xi|)^{2-\mu}(p-q)\sin(\vartheta) + 2\pi a \xi).$$

If $H(\xi) = 0$, then $\xi$ must satisfy

$$(4.5) \quad \begin{cases} (2\pi |\xi|)^{2-\mu}(p+q)\cos(\vartheta) + b = 0, \\ (2\pi |\xi|)^{2-\mu}(p-q)\sin(\vartheta) + 2\pi a \xi = 0. \end{cases}$$

Notice that $\cos(\vartheta)$ and $\sin(\vartheta)$ can never be zero when $\mu \in (0, 1)$. In such a case, there is at most one $\xi \in \mathbb{R}$ such that $H(\xi) = 0$, thereby confirming that $H(\xi) \neq 0$ a.e. in $\mathbb{R}$.

At this stage, we repeat some of the arguments in the proof of Theorem 3.10. Specifically, choose $0 \neq \varphi \in C_0^\infty(\mathbb{R})$, so that by Theorem A.1 (Plancherel), $\mathcal{F}(\varphi) \neq 0$. On the account of continuity of $\mathcal{F}(\varphi)$, there exists $(a, b) \subset \mathbb{R}$ such that $\mathcal{F}(\varphi) \neq 0$ in $(a, b)$. Choose $\epsilon > 0$, and let $v \in C_0^\infty(\mathbb{R})$ such that $v = \Pi_{\epsilon} \varphi$. It is true that $[\mathcal{F}(v)](\xi) = \epsilon^{-1}[\mathcal{F}(\varphi)](\epsilon^{-1} \xi)$ and thus $\mathcal{F}(v) \neq 0$ in $(\epsilon a, \epsilon b)$. This and in combination with the fact that $H(\xi) \neq 0$ a.e. in $\mathbb{R}$ implies $\mathcal{F}(w) \neq 0$ a.e. in $(\epsilon a, \epsilon b)$, or equivalently, $\mathcal{F}(w) \neq 0$ a.e. in $(a, b)$.

Let $g \in L^2(\mathbb{R})$ such that $(g, w) = 0$ for any $w \in M$. By Theorem A.4, the density of $M$ is confirmed if this equation implies that $g = 0$. Given $w \in M$ and any fixed $y \in \mathbb{R}$, and using the translation operator, set

$$G(y) = (g, \tau_y w) = \int_{\mathbb{R}} g(x)w(x-y) \, dx = \int_{\mathbb{R}} g(y-z)w(-z) \, dz,$$

where a change of variable was used to get the last term in the above equality. Notice that Property 2.8 implies that $\tau_y w = \tau_y Lv = L(\tau_y v)$, where it is true that $\tau_y v \in C_0^\infty(\mathbb{R})$ for $v \in C_0^\infty(\mathbb{R})$. This means $\tau_y w \in M$ and thus $G(y) = 0$ for every $y \in \mathbb{R}$. This fact along with an application of Theorem A.3 yields $0 = \hat{g}w = \hat{g}\mathcal{F}(w)$. However, as noted earlier, $\mathcal{F}(w) \neq 0$ a.e. in $(\epsilon a, \epsilon b)$, so it must be that $\hat{g} = 0$ a.e. in $(\epsilon a, \epsilon b)$. Because $\epsilon > 0$ is arbitrary, $\hat{g} = 0$ in any open interval, and thus $\hat{g} = 0$ in $\mathbb{R}$. Another use of Theorem A.1 (Plancherel) concludes that $g = 0$, implying the density of $M$ in $L^2(\mathbb{R})$.

Density of $\tilde{M}$ is shown by repeating the foregoing arguments using $\tilde{L}$. \qed
Lemma 4.2 is the basis for computing \( \|w\|_{L^2(\mathbb{R})} \) for \( w \in M \) whose representation can be either \( w = Lv \) or \( w = L(\Pi_{1/\delta}v), \, v \in C_0^\infty(\mathbb{R}) \). The results are stated in Lemma 4.3 and Lemma 4.5.

**Lemma 4.3.** For \( w = Lv \), with \( v \in C_0^\infty(\mathbb{R}) \), the following norm equality holds,

\[
\|w\|_{L^2(\mathbb{R})}^2 = \sum_{j=1}^5 C_j \| D^{\sigma_j} v \|_{L^2(\mathbb{R})}^2,
\]

where

\[
\begin{align*}
C_1 &= p^2 + q^2 + 2pq \cos(\sigma_1 \pi), \quad \sigma_1 = 2 - \mu, \\
C_2 &= 2a(q - p) \cos(\sigma_2 \pi), \quad \sigma_2 = \frac{1}{2}(3 - \mu), \\
C_3 &= a^2, \quad \sigma_3 = 1, \\
C_4 &= 2b(p + q) \cos(\sigma_4 \pi), \quad \sigma_4 = \frac{1}{2}(2 - \mu), \\
C_5 &= b^2, \quad \sigma_5 = 0.
\end{align*}
\]

**Proof.** By definition,

\[
\|w\|_{L^2(\mathbb{R})}^2 = (Lv, Lv) = I + II + III,
\]

where

\[
\begin{align*}
I &= (pD^{2-\mu}v + qD^{(2-\mu)*}v, pD^{2-\mu}v + qD^{(2-\mu)*}v), \\
II &= (aDv + bv, aDv + bv), \\
III &= 2(pD^{2-\mu}v + qD^{(2-\mu)*}v, aDv + bv).
\end{align*}
\]

In the following, we compute I, II, III separately. The idea is that we would like to shift the exponents in the fractional derivatives by using basic properties of R-L operators, so that Theorem 4.1 can be utilized.

Application of Theorem 4.1 shows that

\[
\begin{align*}
I &= (pD^{2-\mu}v + qD^{(2-\mu)*}v, pD^{2-\mu}v + qD^{(2-\mu)*}v) \\
&= (p^2 + q^2)(D^{\sigma_1}v, D^{\sigma_1}v) + 2pq \cos((2 - \mu)\pi)(D^{\sigma_1}v, D^{\sigma_1}v) \\
&= C_1 \| D^{\sigma_1}v \|_{L^2(\mathbb{R})}^2.
\end{align*}
\]

An integration by parts shows that \((aDv, bv) = -(av, Dv)\) and thus \((aDv, v) = 0\). This means

\[
\begin{align*}
II &= (aDv, aDv) + (bv, bv) + 2(aDv, bv) = C_3 \| D^{\sigma_3}v \|_{L^2(\mathbb{R})}^2 + C_5 \| D^{\sigma_5}v \|_{L^2(\mathbb{R})}^2.
\end{align*}
\]

Moreover, we make a decomposition \( III = 2bIII_1 + 2aIII_2 \), with

\[
\begin{align*}
III_1 &= p(D^{2-\mu}v, v) + q(D^{(2-\mu)*}v, v) \quad \text{and} \quad III_2 = p(D^{2-\mu}v, Dv) + q(D^{(2-\mu)*}v, Dv).
\end{align*}
\]
The following calculation for III_1 is performed:

\[
\text{III}_1 = p(D^{-\mu}v^{(2)}, v) + q(D^{-\mu}v^{(2)}, v) \quad \text{(by Remark 2.2 and Theorem A.5)}
\]
\[
= p(D^{-\mu/2}D^{-\mu/2}v^{(2)}, v) + q(D^{-\mu/2}D^{-\mu/2}v^{(2)}, v) \quad \text{(by Corollary 2.3)}
\]
\[
= p(D^{-\mu/2}v^{(2)}, D^{-\mu/2}v^{(2)}) + q(D^{-\mu/2}v^{(2)}, D^{-\mu/2}v^{(2)}) \quad \text{(by Corollary 2.4)}
\]
\[
= p(D^2D^{-\mu/2}v, D^{-\mu/2}v) + q(D^2D^{-\mu/2}v, D^{-\mu/2}v) \quad \text{(by Remark 2.2 and A.5)}
\]
\[
= p(D^4v, D^{2\sigma}v) + q(D^{2\sigma}v, D^{2\sigma}v) \quad \text{(int. by parts and Definition 2.6)}
\]
\[
= (p + q)\cos(\sigma\pi)\|D^{2\sigma}v\|^2_{L^2(\mathbb{R})}. \quad \text{(by Theorem 4.1)}
\]

Similar calculation is performed for III_2, after first integrating it by parts and using Definition 2.6:

\[
\text{III}_2 = -p(D^{3-\mu}v, v) + q(D^{3-\mu}v, v) \quad \text{(int. by parts and Definition 2.6)}
\]
\[
= -p(D^{3-\mu}v^{(3)}, v) + q(-D^{-\mu}v^{(3)}, v) \quad \text{(by Remark 2.2 and A.5)}
\]
\[
= -p(v^{(3)}, D^{-\mu}v) + q(-v^{(3)}, D^{-\mu}v) \quad \text{(by Corollary 2.4)}
\]
\[
= -p(D^{1-\mu}/2D^{1+\mu}/2v^{(4)}, D^{-\mu}v) + q(D^{1-\mu}/2D^{1+\mu}/2v^{(4)}, D^{-\mu}v) \quad \text{(by Corollary 2.3)}
\]
\[
= -p(D^{1-\mu}/2v^{(4)}, D^{1+\mu}/2v^{(4)}) \quad \text{and (by Remark 2.2 and A.5)}
\]
\[
= -p(D^{1+\mu}/2v^{(4)}, D^{1-\mu}/2v^{(4)}) \quad \text{and (by Corollary 2.4)}
\]
\[
= -p(D^4D^{1-\mu}/2v, D^{1+\mu}/2v) \quad \text{and (by Remark 2.2 and A.5)}
\]
\[
= (q - p)\cos(\sigma\pi)\|D^{2\sigma}v\|^2_{L^2(\mathbb{R})}. \quad \text{(by Theorem 4.1)}
\]

This completes the proof. \[\square\]

Remark 4.4. It is worth noting that from the construction \(C_1 > 0\) (because \(|\cos(\sigma\pi)| < 1, p^2 + q^2 \neq 0\), \(C_3 \geq 0\), \(C_5 > 0\) (because \(b \neq 0\), \(\cos(\sigma\pi) < 0\), and \(\cos(\sigma\pi) < 0\). However \(C_2, C_4\) may be negative, and \(C_2 \geq 0\) only when \(a(q - p) \leq 0\), \(C_4 \geq 0\) only when \(a(p + q) \leq 0\). Discussion on different cases of \(C_2, C_4\) is relegated to Lemma 4.6, Lemma 4.7 and Lemma 4.8.

Lemma 4.5. Let \(\delta > 0\), \(\varphi \in C^\infty_0(\mathbb{R})\) and \(w = L(\Pi_{1/\delta}\varphi)\). Then

\[
(4.11) \quad \|w\|^2_{L^2(\mathbb{R})} = \sum_{j=1}^5 \frac{C_j}{\delta^{2\sigma_j-1}} \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_j} |\hat{\varphi}|^2 \, d\xi.
\]

Proof. By Property 2.8, \(|D^{\sigma_j}(\Pi_{1/\delta}\varphi))(x) = \delta^{-\sigma_j} [\Pi_{1/\delta}(D^{\sigma_j}\varphi)](x) = [D^{\sigma_j}\varphi](x/\delta)|\), so using Lemma 4.3 along with appropriate change of variable in the integration yields

\[
\|w\|^2_{L^2(\mathbb{R})} = \sum_{j=1}^5 \frac{C_j}{\delta^{2\sigma_j-1}} \|D^{\sigma_j}(\Pi_{1/\delta}\varphi)\|^2_{L^2(\mathbb{R})} = \sum_{j=1}^5 \frac{C_j}{\delta^{2\sigma_j-1}} \|D^{\sigma_j}\varphi\|^2_{L^2(\mathbb{R})},
\]

from which (4.11) is obtained through application of Property 2.7 (Fourier Transform) and Theorem A.1 (Plancherel). \[\square\]
Recall from Remark 4.4, \( C_1, C_3, C_5 \) are non-negative, however \( C_2, C_4 \) may be positive or non-positive, and this presents a constraint in guaranteeing the existence of solutions to (4.1). Therefore, different cases for \( C_2, C_4 \) are treated separately to help materialize the conclusion in Theorem 4.9. Lemma 4.6, Lemma 4.7, and Lemma 4.8 below show different representations of norm of \( w = L((\Pi_1/\alpha \varphi)) \) according to different cases of \( C_2, C_4 \). More precisely, we discuss three different cases:

\[
(1) \ C_2 \geq 0, C_4 < 0, \quad (2) \ C_2 < 0, C_4 \geq 0, \quad (3) \ C_2 < 0, C_4 < 0.
\]

The case \( C_2 \geq 0, C_4 \geq 0 \) is treated in a straightforward manner later on.

**Lemma 4.6.** With \( \{C_j, \sigma_j\}_{j=1}^5 \) defined in Lemma 4.3, assume \( C_2 \geq 0, C_4 < 0 \) and \( b^2 > -C_4 \alpha^{2(\sigma_5-\sigma_4)} \), where \( \alpha > 0 \) satisfies

\[
\sum_{j=1}^{4} \frac{C_j}{\alpha^{2\sigma_j-1}} > 0.
\]

Then for \( w = L((\Pi_1/\alpha \varphi)) \), with \( \varphi \in C_0^\infty(\mathbb{R}) \), we have

\[
\|w\|_{L^2(\mathbb{R})}^2 = \sum_{\ell \neq 4} I_\ell + \sum_{\ell=1}^3 II_\ell + III_1,
\]

where

\[
I_\ell = Q_{1,\ell} \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_\ell} |\hat{\varphi}|^2 \, d\xi, \quad \ell = 1, 2, 3, 5,
\]

\[
II_\ell = Q_{2,\ell} \left( \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_\ell} |\hat{\varphi}|^2 \, d\xi - \int_{|\xi| > 1} |2\pi \xi|^{2\sigma_\ell} |\hat{\varphi}|^2 \, d\xi \right), \quad \ell = 1, 2, 3,
\]

\[
III_1 = Q_{3,1} \left( \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_1} |\hat{\varphi}|^2 \, d\xi - \int_{|\xi| \leq 1} |2\pi \xi|^{2\sigma_1} |\hat{\varphi}|^2 \, d\xi \right),
\]

with constants \( Q_{1,1} > 0, Q_{1,5} > 0 \) and the rest of \( Q_{j,\ell} \geq 0 \).

**Proof.** Since \( 0 < C_1, 0 < 2\sigma_1 - 1 = \max_{j=1,2,3,4} \{2\sigma_j - 1\} \), there always exists a sufficiently small positive number \( \alpha \) such that inequality (4.12) holds true. With this \( \alpha \), condition \( b^2 > -C_4 \alpha^{2(\sigma_5-\sigma_4)} \) equivalently implies \( \frac{C_2}{\alpha^{2\sigma_2-1}} + \frac{C_1}{\alpha^{2\sigma_1-1}} > 0 \). Since \( C_4 < 0 \), from (4.12), there exist non-positive numbers \( A_1, A_2, A_3 \) such that

\[
\sum_{j=1}^3 A_j = \frac{C_4}{\alpha^{2\sigma_4-1}}, \quad \frac{C_1}{\alpha^{2\sigma_1-1}} + A_1 > 0, \quad \frac{C_2}{\alpha^{2\sigma_2-1}} + A_2 > 0, \quad \frac{C_3}{\alpha^{2\sigma_3-1}} + A_3 > 0.
\]

With \( \alpha \) in place of \( \delta \) and \( \frac{C_1}{\alpha^{2\sigma_1-1}} \) substituted by \( \sum_{j=1}^3 A_j \) in Lemma 4.5, and by adding and subtracting appropriate terms, one has

\[
\|w\|_{L^2(\mathbb{R})}^2 = \sum_{\ell \neq 4} I_\ell + \sum_{\ell=1}^3 II_\ell + III_1,
\]

This manuscript is for review purposes only.
where all the terms are as in (4.14), with

\[ Q_{1,\ell} = \frac{C_\ell}{\alpha^{2\sigma_{\ell}-1}} + A_\ell, \quad \ell = 1, 2, 3, \quad Q_{1,5} = \sum_{j=4}^{5} \frac{C_j}{\alpha^{2\sigma_j-1}}, \]

\[ Q_{2,\ell} = -A_\ell, \quad \ell = 1, 2, 3, \]

\[ Q_{3,1} = -\frac{C_4}{\alpha^{2\sigma_4-1}}. \]

**Lemma 4.7.** With \( \{C_j, \sigma_j\}_{j=1}^{5} \) defined in Lemma 4.3, assume \( C_2 < 0, \ C_4 \geq 0, \) and \( b^2 > -\sum_{j=2}^{4} C_j \alpha^{2(\sigma_j-\sigma_{j'})} \), where \( \alpha > 0 \) satisfies

\[ \sum_{j=1}^{2} \frac{C_j}{\alpha^{2\sigma_j-1}} > 0. \]

Then for \( w = L(\Pi_{1/\alpha}\varphi) \), with \( \varphi \in C_0^\infty(\mathbb{R}) \), we have

\[ \|w\|_{L^2(\mathbb{R})}^2 = \sum_{\ell=1}^{5} I_\ell + \sum_{\ell=3,4,5} \Pi_\ell + \Pi_1, \]

where

\[ I_\ell = Q_{1,\ell} \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_\ell} |\tilde{\varphi}|^2 \, d\xi, \quad \ell = 1, 3, 4, 5, \]

\[ \Pi_\ell = Q_{2,\ell} \left( \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_\ell} |\tilde{\varphi}|^2 \, d\xi - \int_{|\xi| \leq 1} |2\pi \xi|^{2\sigma_\ell} |\tilde{\varphi}|^2 \, d\xi \right), \quad \ell = 3, 4, 5, \]

\[ \Pi_1 = Q_{3,1} \left( \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_1} |\tilde{\varphi}|^2 \, d\xi - \int_{|\xi| > 1} |2\pi \xi|^{2\sigma_1} |\tilde{\varphi}|^2 \, d\xi \right), \]

with constants \( Q_{1,1} > 0, \ Q_{1,5} > 0 \) and the rest of \( Q_{4,\ell} \geq 0 \).

**Proof.** Since \( 0 < C_1, 0 < 2\sigma_1 - 1 = \max_{j=1,2} \{2\sigma_j - 1\} \), there always exists a sufficiently small positive number \( \alpha \) such that (4.15) holds true. With this \( \alpha \), condition \( b^2 > -\sum_{j=2}^{4} C_j \alpha^{2(\sigma_j-\sigma_{j'})} \) equivalently implies

\[ \sum_{j=2}^{5} \frac{C_j}{\alpha^{2\sigma_j-1}} > 0, \]

Since \( C_2 < 0 \), from inequality (4.18), there exist non-positive numbers \( B_3, B_4, B_5 \) such that

\[ \sum_{j=3}^{5} B_j = \frac{C_2}{\alpha^{2\sigma_2-1}}, \quad \frac{C_3}{\alpha^{2\sigma_3-1}} + B_3 \geq 0, \quad \frac{C_4}{\alpha^{2\sigma_4-1}} + B_4 \geq 0, \quad \frac{C_5}{\alpha^{2\sigma_5-1}} + B_5 > 0. \]

With \( \alpha \) in place of \( \delta \) and \( \frac{C_j}{\alpha^{2\sigma_j-1}} \) substituted by \( \sum_{j=3} C_j B_j \) in Lemma 4.5, and by adding and subtracting appropriate
terms, one has
\[
\|w\|^2_{L^2(\mathbb{R})} = \sum_{j=1}^{5} \frac{C_j}{\alpha^{2\sigma_j-1}} \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_j}|\hat{\varphi}|^2 \, d\xi + \left( \sum_{j=3}^{5} B_j \right) \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_5}|\hat{\varphi}|^2 \, d\xi \\
= \sum_{\ell=1}^{5} I_\ell + \sum_{\ell=3,4,5} II_\ell + III_1,
\]
where all the terms are as in (4.17), with
\[
Q_{1,1} = \sum_{\ell=1}^{2} \frac{C_\ell}{\alpha^{2\sigma_\ell-1}}, \quad Q_{1,\ell} = \frac{C_\ell}{\alpha^{2\sigma_\ell-1}} + B_\ell, \quad \ell = 3, 4, 5, \\
Q_{2,\ell} = -B_\ell, \quad \ell = 3, 4, 5, \\
Q_{3,1} = -\frac{C_2}{\alpha^{2\sigma_2-1}}.
\]

**Lemma 4.8.** With \( \{C_j, \sigma_j\}_{j=1}^{5} \) defined in Lemma 4.3, assume \( C_2 < 0, C_4 < 0, \) and \( b^2 > -\sum_{j=2,4} C_j \alpha^{2(\sigma_j-\sigma_3)} \), where \( \alpha > 0 \) satisfies
\[
(4.19) \quad \sum_{j=1,2,4} \frac{C_j}{\alpha^{2\sigma_j-1}} > 0.
\]

Then for \( w = L(\Pi_{1/\alpha} \varphi) \), with \( \varphi \in C_0^\infty(\mathbb{R}) \), we have
\[
(4.20) \quad \|w\|^2_{L^2(\mathbb{R})} = \sum_{\ell=1,3,5} I_\ell + \sum_{\ell=2,4} II_\ell + \sum_{\ell=2,4} III_\ell,
\]
where
\[
I_\ell = Q_{1,\ell} \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_\ell}|\hat{\varphi}|^2 \, d\xi, \quad \ell = 1, 3, 5, \\
II_\ell = Q_{2,\ell} \left( \int_{\mathbb{R}} |2\pi \xi|^{2\sigma_\ell}|\hat{\varphi}|^2 \, d\xi - \int_{|\xi| \geq 1} |2\pi \xi|^{2\sigma_\ell}|\hat{\varphi}|^2 \, d\xi \right), \quad \ell = 2, 4, \\
III_\ell = Q_{3,\ell} \left( \int_{|\xi| \leq 1} |2\pi \xi|^{2\sigma_\ell}|\hat{\varphi}|^2 \, d\xi - \int_{|\xi| < 1} |2\pi \xi|^{2\sigma_\ell}|\hat{\varphi}|^2 \, d\xi \right), \quad \ell = 2, 4,
\]
with constants \( Q_{1,1} > 0, Q_{1,5} > 0 \) and the rest of \( Q_{1,\ell} \geq 0. \)

**Proof.** Since \( 0 < C_1, 0 < 2\sigma_1 - 1 = \max_{j=1,2,4} \{2\sigma_j - 1\} \), there always exists a sufficiently small positive number \( \alpha \) such that (4.19) holds true. With this \( \alpha \), condition \( b^2 > -\sum_{j=2,4} C_j \alpha^{2(\sigma_j-\sigma_3)} \) equivalently implies \( \sum_{j=2,4} \frac{C_j}{\alpha^{\sigma_j-\sigma_3}} > 0 \). With \( \alpha \) in place of \( \delta \) in Lemma 4.5, and by adding and subtracting appropriate terms, one has
\[
(4.22) \quad \|w\|^2_{L^2(\mathbb{R})} = \sum_{\ell=1,3,5} I_\ell + \sum_{\ell=2,4} II_\ell + \sum_{\ell=2,4} III_\ell,
\]
This manuscript is for review purposes only.
where all the terms are as in (4.21), with
\[
Q_{1,1} = \sum_{j=1,2,4} \frac{C_j}{\alpha^{2\sigma_j-1}}, \quad Q_{1,3} = \frac{C_3}{\alpha^{2\sigma_3-1}}, \quad Q_{1,5} = \sum_{j=2,4,5} \frac{C_j}{\alpha^{2\sigma_j-1}}.
\]
\[
Q_{2,\ell} = -\frac{C_\ell}{\alpha^{2\sigma_\ell-1}}, \quad \ell = 2, 4,
\]
\[
Q_{5,\ell} = -\frac{C_\ell}{\alpha^{2\sigma_\ell-1}}, \quad \ell = 2, 4.
\]

\[\square\]

Notice that in view of \(\sigma_1 > \cdots > \sigma_5\), Lemma 4.6 implies that each \(I_\ell\), \(II_\ell\), \(III_1\) is at least non-negative. The same situation applies to Lemma 4.7 and Lemma 4.8.

4.2. Existence, Uniqueness, and Regularity of the Solution. At this stage, we are ready to prove the existence and uniqueness of strong solutions to problem (4.1). The following theorem implies that, roughly speaking, if \(|b|\) is big enough compared to other coefficients, there always exists a unique solution \(u \in \tilde{H}^{2-\mu}(\mathbb{R})\) to problem (4.1).

THEOREM 4.9. Consider problem (4.1) with \(\{C_j, \sigma_j\}_{j=1}^5\) defined in Lemma 4.3. For either of the following cases, (i) \(C_2 \geq 0, C_4 \geq 0\) or

(ii) \(C_2 \geq 0, C_4 < 0, b^2 > -C_4 \alpha^{2(\sigma_5-\sigma_4)}, \) and \(\alpha > 0\) with \(\sum_{j=1}^4 \frac{C_j}{\alpha^{2\sigma_j-1}} > 0\), or

(iii) \(C_2 < 0, C_4 \geq 0, b^2 > -\sum_{j=2}^4 C_j \alpha^{2(\sigma_5-\sigma_j)}, \) and \(\alpha > 0\) with \(\sum_{j=1}^2 \frac{C_j}{\alpha^{2\sigma_j-1}} > 0\), or

(iv) \(C_2 < 0, C_4 < 0, b^2 > -\sum_{j=2,4} C_j \alpha^{2(\sigma_5-\sigma_j)}, \) and \(\alpha > 0\) with \(\sum_{j=1,2,4} \frac{C_j}{\alpha^{2\sigma_j-1}} > 0\),

there exists a unique solution \(u \in \tilde{H}^{2-\mu}(\mathbb{R})\) that satisfies (4.1). Furthermore,

\[
\|u\|_{\tilde{H}^{2-\mu}(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})},
\]

for some positive constants \(C > 0\) depending only on \(L\).

Proof. These four different cases are discussed together in a unified way as follows.

Fix \(f \in L^2(\mathbb{R})\) in (4.1). Since \(b \neq 0\) in (4.1), for each case in the theorem, Lemma 4.2 guarantees that there is a Cauchy sequence \(\{w_n\} \subset M \subset L^2(\mathbb{R})\) such that

\[
\lim_{n \to \infty} \|w_n - f\|_{L^2(\mathbb{R})} = 0,
\]

where \(w_n = Lu_n\) for certain sequence \(\{u_n\} \subset C_0^\infty(\mathbb{R})\). Now we intend to show that equation (4.24) implies both \(\{u_n\}, \{D^{2-\mu}u_n\}\) are actually Cauchy sequences in \(L^2(\mathbb{R})\) under each case in Theorem 4.9.

To do so, we compute \(\|w_n - w_m\|_{L^2(\mathbb{R})}\) in the following in terms of \(\{\varphi_n\}\), where \(\{\varphi_n\} = \{\Pi_\beta u_n\}\) (namely rewrite sequence \(\{u_n\}\) as \(\{\Pi_{1/\beta}\varphi_n\}\)), with \(\beta > 0\) chosen in this way: \(\beta = 1\) for case (i), \(\beta = \alpha\) for cases (ii), (iii), (iv). Let us note carefully that case (i), (ii), (iii), (iv) allow us to apply Lemma 4.3, Lemma 4.6, Lemma 4.7 and Lemma 4.8 accordingly. By doing so, we have

\[
\|w_n - w_m\|_{L^2(\mathbb{R})}^2 = \sum_{\ell=1,5} P_{1,\ell} \int_\mathbb{R} |2\pi \xi|^{2\alpha_\ell} |\hat{\varphi}_n - \hat{\varphi}_m|^2 \, d\xi + \text{Remainder},
\]

This manuscript is for review purposes only.
where Remainder ≥ 0, while \( P_{1,1} \) and \( P_{1,5} \) are both strictly positive, with

\[
P_{1,1} = \begin{cases} 
C_1 & \text{for case (i)}, \\
Q_{1,1} & \text{for case (ii)}, \\
Q_{1,1} & \text{for case (iii)};
\end{cases} \quad P_{1,5} = \begin{cases} 
C_5 & \text{for case (i)}, \\
Q_{1,5} & \text{for case (ii)}, \\
Q_{1,5} & \text{for case (iii)}.
\end{cases}
\]

Given any \( \epsilon > 0 \), there exists a positive integer \( N \) such that, for \( n, m > N \), it is true that \( \| w_n - w_m \|_{L^2(\mathbb{R})}^2 < \epsilon \). Since every term in equation (4.25) is nonnegative with the first two are positive, it means

\[
\sum_{\ell=1,5} P_{1,\ell} \int_{\mathbb{R}} |2\pi\xi|^{2\sigma_{\ell}} |\tilde{\varphi}_n - \tilde{\varphi}_m|^2 \, d\xi < \epsilon, \quad \ell = 1, 5.
\]

Based on the fact that

\[
|D^{\sigma_{\ell}} u_n - D^{\sigma_{\ell}} u_m|_{L^2(\mathbb{R})}^2 = \frac{1}{\beta^{2\sigma_{\ell}-1}} \int_{\mathbb{R}} |2\pi\xi|^{2\sigma_{\ell}} |\tilde{\varphi}_n - \tilde{\varphi}_m|^2 \, d\xi, \quad \ell = 1, 5,
\]

it is concluded that

\[
|D^{\sigma_{\ell}} u_n - D^{\sigma_{\ell}} u_m|_{L^2(\mathbb{R})}^2 \leq \frac{\epsilon}{\beta^{2\sigma_{\ell}-1} P_{1,\ell}}, \quad \ell = 1, 5.
\]

Recall that \( \sigma_1 = 2 - \mu \) and \( \sigma_5 = 0 \), so this last inequality implies that \( \{D^{2-\mu} u_n\} \) and \( \{u_n\} \) are Cauchy sequences for each case in Theorem 4.9. Denoting the limit by

\[
u = \lim_{n \to \infty} u_n,
\]

then Corollary 3.12 gives \( u \in \hat{H}^{2-\mu}(\mathbb{R}) \). Furthermore, Theorem 3.4 and Theorem 3.10 guarantee the existence of \( D^{2-\mu} u, D^{(2-\mu)^{\ast}} u, \) and \( Du \), therefore \( u \) is the solution of (4.1). Actually, by revisiting (4.24), it is seen that

\[
f = \lim_{n \to \infty} w_n = \lim_{n \to \infty} \left( pD^{2-\mu} u_n + qD^{(2-\mu)^{\ast}} u_n + aD u_n + b u_n \right) = pD^{2-\mu} u + qD^{(2-\mu)^{\ast}} u + aD u + b u.
\]

To estimate the norm of \( u \), we revisit (4.25) to get

\[
\|w_n\|_{L^2(\mathbb{R})}^2 = \sum_{j=1}^5 C_j \|D^{\sigma_j} u_n\|_{L^2(\mathbb{R})}^2 = \sum_{\ell=1,5} P_{1,\ell} \int_{\mathbb{R}} |2\pi\xi|^{2\sigma_{\ell}} |\tilde{\varphi}_n|^2 \, d\xi + \text{Remainder}.
\]

Simply by noticing that Remainder ≥ 0, while \( P_{1,1} > 0 \), \( P_{1,5} > 0 \) and using equation (4.26), we obtain

\[
\|w_n\|_{L^2(\mathbb{R})}^2 \geq \sum_{\ell=1,5} P_{1,\ell} \int_{\mathbb{R}} |2\pi\xi|^{2\sigma_{\ell}} |\tilde{\varphi}_n|^2 \, d\xi = \sum_{\ell=1,5} \beta^{2\sigma_{\ell}-1} P_{1,\ell} \|D^{\sigma_{\ell}} u_n\|_{L^2(\mathbb{R})}^2 \geq \frac{1}{C^2} \sum_{\ell=1,5} \|D^{\sigma_{\ell}} u_n\|_{L^2(\mathbb{R})}^2,
\]

This manuscript is for review purposes only.
where \( \frac{1}{C} = \left( \min_{\ell=1,5} \{ \beta^{2\ell - 1} P_{1, \ell} \} \right)^{1/2} \). By taking the limit as \( n \to \infty \), the last inequality produces

\[
\sum_{\ell=1,5} \| D^{\ell} u \|_{L^2(\mathbb{R})}^2 \leq C^2 \| f \|_{L^2(\mathbb{R})}^2.
\]

Taking the root at both sides, by Definition 3.8 and Theorem 3.10, we get

\[
\| u \|_{\tilde{H}^2-\mu(\mathbb{R})} \leq C \| f \|_{L^2(\mathbb{R})}. \tag{4.29}
\]

For the uniqueness of solution, to the contrary, let \( u_1, u_2 \in \tilde{H}^{2-\mu}(\mathbb{R}) \) be solutions of (4.1) under each same case. This means \( L(u_1 - u_2) = 0 \), which by the stability estimate (4.29) yields \( \| u_1 - u_2 \|_{\tilde{H}^{2-\mu}(\mathbb{R})} = 0 \), implying \( u_1 = u_2 \) a.e., hence the uniqueness of the solution of (4.1). This completes the whole proof. \( \square \)

Remark 4.10. A closer look of the above proof indicates that Theorem 4.9 can be established for ordinary differential equations that use \( \tilde{L} \) (see Lemma 4.2). This is because each \( \{ C_i, \sigma_i \} \) (see Lemma 4.3) corresponding to \( \tilde{L} \) is equal to the one obtained for \( L \).

Once we have established the existence of solutions, now we are ready to discuss the regularity of solutions, it turns out that the smoothness of solutions are exactly determined by the source function \( f \).

Theorem 4.11. Under the same condition in Theorem 4.9 and if \( f \in \tilde{H}^m(\mathbb{R}) \), then there is a unique \( u \in \tilde{H}^{2-\mu+m}(\mathbb{R}) \), where \( m \in \mathbb{N}_0 \) and

\[
\| u \|_{\tilde{H}^{2-\mu+m}(\mathbb{R})} \leq C \| f \|_{\tilde{H}^m(\mathbb{R})}, \tag{4.30}
\]

for some positive constant \( C \) depending only on \( L \).

Proof. This theorem is established by induction on \( m \), noting that the case \( m = 0 \) has been proven in Theorem 4.9 (\( \tilde{H}^0(\mathbb{R}) = L^2(\mathbb{R}) \) by convention). Assume the statement of theorem is true for a positive integer \( m \).

Let \( f \in \tilde{H}^{m+1}(\mathbb{R}) \), which means \( f, Df \in \tilde{H}^m(\mathbb{R}) \). By the induction assumption, there are \( u, v \in \tilde{H}^{2-\mu+m}(\mathbb{R}) \) such that

\[
Lu = f, \quad Lv = Df. \tag{4.31}
\]

Furthermore, using Definition 3.6,

\[
(f, D^* \psi) = (Df, \psi), \forall \psi \in C_0^\infty(\mathbb{R}) \quad \text{(Recall } D^* \text{ from Remark 3.9).} \tag{4.32}
\]

In the following, the intention is to demonstrate that actually \( u \in \tilde{H}^{3-\mu+m}(\mathbb{R}) \).

Since \( u, v \in \tilde{H}^{2-\mu+m}(\mathbb{R}) \), by Corollary 3.12 and Theorem 3.4, there exist sequences \( \{ u_n \}, \{ v_n \} \subset C_0^\infty(\mathbb{R}) \) such that

\[
\lim_{n \to \infty} \| u_n - u \|_{L^2(\mathbb{R})} = 0, \quad \lim_{n \to \infty} \| D^s u_n - D^s u \|_{L^2(\mathbb{R})} = 0, \quad \forall s \in [0, 2 - \mu], \tag{4.33}
\]

\[
\lim_{n \to \infty} \| v_n - v \|_{L^2(\mathbb{R})} = 0, \quad \lim_{n \to \infty} \| D^s v_n - D^s v \|_{L^2(\mathbb{R})} = 0, \quad \forall s \in [0, 2 - \mu].
\]

Convergence of these sequences justifies the following equalities:

\[
(f, D^* \psi) = (Lu, D^* \psi) = \lim_{n \to \infty} (Lu_n, D^* \psi) \quad \forall \psi \in C_0^\infty(\mathbb{R}), \tag{4.34}
\]

\[
(Df, \psi) = (Lv, D^* \psi) = \lim_{n \to \infty} (Lv_n, D^* \psi) \quad \forall \psi \in C_0^\infty(\mathbb{R}).
\]
Using the definition of $L$,

\[
(L u_n, D^s \psi) = p(D^{-\mu} u_n^{(2)}, D^s \psi) - q(D^{-\mu} u_n^{(2)}, D \psi) - a(Du_n, D \psi) - b(u_n, D \psi) \quad \text{(by Remark 2.2 & Theorem A.5)}
\]

\[
= p(Du_n, D^{(2-\mu)} \psi) + q(Du_n, D^{2-\mu} \psi) - a(Du_n, D \psi) + b(Du_n, \psi). \quad \text{(int. by parts & Corollary 2.4)}
\]

Taking limit as $n \to \infty$ of this last equation and using the first equality in (4.34) give

\[
(f, D^s \psi) = (Du, D^{(2-\mu)} \psi) + q(Du, D^{2-\mu} \psi) - a(Du, D \psi) + b(Du, \psi)
\]

(4.35)

\[
= (Du, p D^{(2-\mu)} \psi + q D^{2-\mu} \psi - a D \psi + b \psi)
\]

\[
= (Du, \tilde{L} \psi), \quad \forall \psi \in C^\infty_0(\mathbb{R}),
\]

where $\tilde{L}$ is as defined in Lemma 4.2. Similarly, second equality in (4.34) yields

\[
(D f, \psi) = (L v, \psi) = (v, \tilde{L} \psi), \quad \forall \psi \in C^\infty_0(\mathbb{R}).
\]

We substitute (4.35) and (4.36) back into (4.32) to obtain

\[
(D u - v, \tilde{L} \psi) = 0, \quad \forall \psi \in C^\infty_0(\mathbb{R}).
\]

(4.37)

Since Lemma 4.2 confirms that $\tilde{L} \psi$ is dense in $L^2(\mathbb{R})$, it is concluded that $Du - v = 0$ or $Du = v$, and thus $Du \in \tilde{H}^{2-\mu+m}(\mathbb{R})$. This means there is $w \in L^2(\mathbb{R})$ such that $(Du, D^{(\sigma+m)} \psi) = (w, \psi)$ for any $\psi \in C^\infty_0(\mathbb{R})$. Furthermore,

\[
(Du, D^{(2-\mu+m)} \psi) = \lim_{n \to \infty} (Du_n, D^{(2-\mu+m)} \psi)
\]

(4.38)

\[
= \lim_{n \to \infty} (u_n, D^{(3-\mu+m)} \psi)
\]

\[
= (u, D^{(3-\mu+m)} \psi).
\]

Therefore $(u, D^{(3-\mu+m)} \psi) = (w, \psi), \forall \psi \in C^\infty_0(\mathbb{R})$, which by Theorem 3.10, implies $u \in \tilde{H}^{3-\mu+m}(\mathbb{R})$.

To establish the estimate, assumption in the induction argument gives

\[
\|u\|^2_{\tilde{H}^{2-\mu+m}(\mathbb{R})} \leq C_1 \|f\|^2_{\tilde{H}^{(\sigma)}(\mathbb{R})}, \quad \|v\|^2_{\tilde{H}^{2-\mu+m}(\mathbb{R})} \leq C_2 \|Df\|^2_{\tilde{H}^{(\sigma)}(\mathbb{R})},
\]

for certain positive constants $C_1, C_2$. By norms equality stated in Theorem 3.10,

\[
\|u\|^2_{L^2(\mathbb{R})} + \|D^{2-\mu+m} u\|^2_{L^2(\mathbb{R})} \leq C_1^2 \left( \|f\|^2_{L^2(\mathbb{R})} + \|D^{\sigma} f\|^2_{L^2(\mathbb{R})} \right),
\]

(4.40)

and

\[
\|v\|^2_{L^2(\mathbb{R})} + \|D^{2-\mu+m} v\|^2_{L^2(\mathbb{R})} \leq C_2^2 \left( \|Df\|^2_{L^2(\mathbb{R})} + \|D^{\sigma} (Df)\|^2_{L^2(\mathbb{R})} \right).
\]

(4.41)

This manuscript is for review purposes only.
Since $v = Du$, (4.40) and (4.41) can be used to give
\[
\|u\|_{L^2(\mathbb{R})}^2 + \|D^{3-\mu+m}u\|_{L^2(\mathbb{R})}^2 = \|u\|_{L^2(\mathbb{R})}^2 + \|D^{3-\mu+m}v\|_{L^2(\mathbb{R})}^2 \\
\leq C_1^2 \left(\|f\|_{L^2(\mathbb{R})}^2 + \|D^m f\|_{L^2(\mathbb{R})}^2\right) + C_2^2 \left(\|Df\|_{L^2(\mathbb{R})}^2 + \|D^m (Df)\|_{L^2(\mathbb{R})}^2\right) \\
= C_1^2 \|f\|_{H^m(\mathbb{R})}^2 + C_2^2 \|Df\|_{H^m(\mathbb{R})}^2 \\
\leq (C_1^2 + C_2^2) \int_\mathbb{R} (1 + |2\pi \xi|^{2m}) (1 + |2\pi \xi|^{2}) |\hat{f}(\xi)|^2 d\xi \\
= (C_1^2 + C_2^2) (J_1 + J_2),
\]
where
\[
J_1 = \int_{|2\pi \xi|<1} (1 + |2\pi \xi|^{2m}) (1 + |2\pi \xi|^{2}) |\hat{f}(\xi)|^2 d\xi \\
J_2 = \int_{|2\pi \xi|\geq1} (1 + |2\pi \xi|^{2m}) (1 + |2\pi \xi|^{2}) |\hat{f}(\xi)|^2 d\xi.
\]
Therefore
\[
(4.42) \quad \|u\|_{H^{3-\mu+m}(\mathbb{R})} < \sqrt{3(C_1^2 + C_2^2)} \|f\|_{H^{m+1}(\mathbb{R})}.
\]
The uniqueness of solutions directly follows from (4.42) as was done in Theorem 4.9.

A closer look at the proof of Theorem 4.11 (see (4.37)) reveals a possibility for a stronger conclusion, namely for $f \in \tilde{H}^m(\mathbb{R})$, $L(D^m u) = D^n f$, where $n \in \mathbb{N}_0$ and $0 \leq n \leq m$. By repeated application of Theorem 4.11 for $m = 0, 1, 2, \cdots$, infinite differentiability of $u$ can be deduced as follows.

**Corollary 4.12.** Under hypothesis of Theorem 4.9 and if $f \in C^\infty(\mathbb{R})$, then $u \in C^\infty(\mathbb{R})$.

**Proof.** Since $u \in \tilde{H}^{2-\mu+m}(\mathbb{R})$ for $m = 0, 1, 2, \cdots$ by Theorem 4.11, Sobolev Embedding Theorem ([25], p. 220) implies $u \in C^k(\mathbb{R})$ for each $k = 1, 2, 3, \cdots$.

5. **Conclusion.** With the utilization of weak fractional R-L derivatives and appropriate fractional Sobolev space, we have established the existence and uniqueness of the strong solution to problem (4.1), together with its stability estimate and regularity. The result suggests the suitability of utilizing fractional Sobolev spaces to analyze fractional R-L differential equations. The whole framework laid out in this paper is applicable in a straightforward manner to ordinary differential equations that use Caputo fractional derivatives. This is mainly due to the strategy of using $C_0^\infty(\mathbb{R})$ for which Riemann-Liouville derivative coincides with Caputo derivative. We intend to adopt the main idea in the present paper to investigate fractional boundary value problems that can include non-constant coefficients.

**Appendix A. Several Pertinent Theorems.**

**Theorem A.1** (Plancherel Theorem (see eg. [18] p. 187)). Given $w \in L^2(\mathbb{R})$, there is a unique $\hat{w} \in L^2(\mathbb{R})$ so that the following properties hold:
- If $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{w} = \mathcal{F}(w)$.
- For every $w \in L^2(\mathbb{R})$, $\|w\|_2 = \|\hat{w}\|_2$.
- The mapping $w \rightarrow \hat{w}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

**Theorem A.2** ([8], p. 189). Given $v, w \in L^2(\mathbb{R})$, then $(v, w) = (\hat{v}, \hat{w})$, and $w = (\hat{w})^\vee$.

**Theorem A.3** ([9], p. 204). If $v \in L^2(\mathbb{R})$, then $\hat{v} = \hat{v} \ast \hat{w} = \hat{v} \hat{w} \in L^2(\mathbb{R})$.

This manuscript is for review purposes only.
Theorem A.4 ([22], Theorem 4.3-2, p. 191). Let \((X, (\cdot, \cdot))\) be a Hilbert space and let \(Y\) be a subspace of \(X\). \(\overline{Y} = X\) if and only if 0 in \(X\) is the only one satisfying \((x, y) = 0\) for all \(y \in Y\).

Theorem A.5 ([4], Proposition 4.20, p. 107). Let \(v \in C^k_c(\mathbb{R})\) for \(k \geq 1\) and \(w \in L^1_{\text{loc}}(\mathbb{R})\). Then \(v * w \in C^k(\mathbb{R})\) and \(D^\alpha (v * w) = (D^\alpha v) * w\). In particular, if \(v \in C^\infty_c(\mathbb{R})\), \(w \in L^1_{\text{loc}}(\mathbb{R})\), then \(v * w \in C^\infty(\mathbb{R})\).

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, Sobolev spaces, vol. 140 of Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, second edition, 2003.
[2] B. Barumer, M. Kovács, M. M. Meerschaert, and H. Sankaranarayanan, Boundary conditions for fractional diffusion, J. Comput. Appl. Math., 336 (2018), pp. 408–424.
[3] D. A. Benson, M. M. Meerschaert, and J. Revielle, Fractional calculus in hydrologic modeling: A numerical perspective, Adv. Water Resour., 51 (2013), pp. 479–497. 35th Year Anniversary Issue.
[4] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York, 2011.
[5] O. Defterli, M. DElia, Q. Du, M. Gunzburger, R. Lehoucq, and M. M. Meerschaert, Fractional diffusion on bounded domains, Fract. Calc. Appl. Anal., 18 (2015), pp. 342–360.
[6] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math., 136 (2012), pp. 521–573.
[7] V. J. Ervin and J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Methods Partial Differ. Equ., 22 (2006), pp. 558–576.
[8] L. C. Evans, Partial differential equations, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2010.
[9] C. Gasquet and P. Witomski, Fourier analysis and applications, vol. 30 of Texts in Applied Mathematics, Springer-Verlag, New York, 1999. Filtering, numerical computation, wavelets, Translated from the French and with a preface by R. Ryan.
[10] A. Goulart, M. Lazo, J. Suarez, and D. Moreira, Fractional derivative models for atmospheric dispersion of pollutants, Physica A: Statistical Mechanics and its Applications, 477 (2017), pp. 9–19.
[11] F. Izsák and B. J. Szekeres, Models of space-fractional diffusion: A critical review, Applied Mathematics Letters, 71 (2017), pp. 38–43.
[12] B. Jin, R. D. Lazarov, J. E. Pasciak, and W. Rundell, Variational formulation of problems involving fractional order differential operators, Math. Comput., 84 (2015), pp. 2665–2700.
[13] H. Khosravian-Arab, M. Dehghan, and M. Eslahchi, Fractional Sturm-Liouville boundary value problems in unbounded domains: Theory and applications, J. Comput. Phys., 299 (2015), pp. 526–560.
[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and applications of fractional differential equations, vol. 204 of North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, 2006.
[15] T. W. Körner, Fourier analysis, Cambridge University Press, Cambridge, 1988.
[16] F. Mainardi, Fractional Calculus: Some basic problems in continuum and statistical mechanics, Springer Vienna, Vienna, 1997, pp. 291–348.
[17] R. Metzler and J. Klafter, The random walk’s guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep., 339 (2000), pp. 1 – 77.
[18] W. Rudin, Real and complex analysis, McGraw-Hill Book Co., New York, third ed., 1987.
[19] W. Rudin, Functional analysis, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, second ed., 1991.
[20] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional integrals and derivatives, Gordon and Breach Science Publishers, Yverdon, 1993.
[21] E. Scalas, R. Gorenflo, and F. Mainardi, Fractional calculus and continuous-time finance, Physica A: Statistical Mechanics and its Applications, 284 (2000), pp. 370 – 384.
[22] L. Tartar, An introduction to Sobolev spaces and interpolation spaces, vol. 3 of Lecture Notes of the Unione Matematica Italiana, Springer, Berlin; UMI, Bologna, 2007.
[23] H. Wang and D. Yang, Wellposedness of variable-coefficient conservative fractional elliptic differential equations, SIAM J. Numer. Anal., 51 (2013), pp. 1088–1107.
[24] H. Wang, D. Yang, and S. Zhu, Inhomogeneous dirichlet-boundary-value problems of space-fractional diffusion equations and their finite element approximations, SIAM J. Numer. Anal., 52 (2014), pp. 1292–1310.
[25] D. Werner, Funktionalanalysis, Springer-Verlag, Berlin, extended ed., 2011.
[26] S. W. Wheatcraft and M. M. Meerschaert, Fractional conservation of mass, Adv. Water Resour., 31 (2008), pp. 1377–1381.
[27] Y. Zhou, J. Wang, and L. Zhang, Basic theory of fractional differential equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. Second edition [of MR3287248].