Neostability in countable homogeneous metric spaces

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Abstract

Given a countable, totally and positively ordered commutative monoid \( R \), there is a unique countable, universal and ultrahomogeneous metric space taking distances in \( R \). We refer to this space as the \( R \)-Urysohn space, denoted \( U_R \). In this paper, we consider the complete first-order theory of \( U_R \) in a binary relational language consisting of distance inequalities. In the case that \( \text{Th}(U_R) \) has quantifier elimination, we show that many model theoretic properties of \( \text{Th}(U_R) \) are characterized by straightforward properties of \( R \), which are often first-order in the language of monoids. In particular, we show that \( \text{Th}(U_R) \) never has the strict order property and, therefore, we use Shelah’s hierarchy of strong order properties to determine the complexity of \( \text{Th}(U_R) \). We show that the position of \( \text{Th}(U_R) \) in this hierarchy is characterized by a purely algebraic rank describing the archimedean complexity of \( R \). This rank also determines when \( \text{Th}(U_R) \) is simple and, in this case, controls the behavior of nonforking independence.

1 Introduction

In this paper, we consider model theoretic properties of abstract metric spaces obtained as generalizations of the rational Urysohn space. Our results will show that this class of metric spaces exhibits a rich spectrum of complexity in the classification of first-order theories without the strict order property.

The object of focus is informally described as the countable \( R \)-Urysohn space, denoted \( U_R \), where \( R \) is a countable, totally and positively ordered commutative monoid, or distance monoid (see Definition 2.8). We refer to generalized metric spaces taking distances in \( R \) as \( R \)-metric spaces. The space \( U_R \) is then defined to be the unique countable, ultrahomogeneous \( R \)-metric space, which is universal for finite \( R \)-metric spaces. The fact that \( U_R \) exists follows from more general work in [9], which generalizes previous results of Delhommé, Laflamme, Pouzet, and Sauer [11] and Sauer [21]. Alternatively, \( U_R \) can be directly constructed simply by verifying that the class of finite \( R \)-metric spaces is a Fraïssé class (see Remark 2.9). The following are a few examples of such spaces (which are all still standard metric spaces with respect to \((\mathbb{R}^\geq, +, \leq, 0)\)).

Example 1.1.

1. Let \( Q = (\mathbb{Q}^\geq, +, \leq, 0) \) and \( Q_1 = (\mathbb{Q} \cap [0, 1], +_1, \leq, 0) \), where \(+_1\) is addition truncated at 1. Then \( U_Q \) and \( U_{Q_1} \) are, respectively, the rational Urysohn space and rational Urysohn sphere. The completion of the rational Urysohn space is called the Urysohn space, and is the unique complete, separable metric space, which is homogeneous and universal for separable metric spaces. These spaces were originally constructed by Urysohn in 1925 (see [26], [27]).
2. Let \( R_2 = \{0, 1, 2\} \), where \( +_2 \) is addition truncated at 2. Then \( U_{R_2} \) is isometric to the countable random graph or Rado graph (when equipped with the path metric). A directed version of this graph was first constructed by Ackermann in 1937 [20]. The standard graph construction is usually attributed to Erdős and Rényi (1963) [14] or Rado (1964) [20].

3. Generalize the previous example as follows. Fix \( n > 0 \) and let \( R_n = \{0, 1, \ldots, n\} \), where \( +_n \) is addition truncated at \( n \). Let \( \mathcal{N} = (\mathbb{N}, +, \leq, 0) \). We refer to \( U_{R_n} \) as the integral Urysohn space of diameter \( n \), and to \( U_{\mathcal{N}} \) as the integral Urysohn space. Alternatively, in [7], Casanovas and Wagner construct the free \( n \)th root of the complete graph. As with the case \( n = 2 \), equipping this graph with the path metric yields \( U_{R_n} \).

4. Generalize all of the previous examples as follows. Let \( S \subseteq \mathbb{R}_{\geq 0} \) be countable good value set, which means \( 0 \in S \), \( S \) is either unbounded or contains a maximal element, and, for all \( r, s \in S \), if \( r + s \leq \sup S \) then \( r + s \in S \). Let \( S = (S, +, \leq, 0) \), where \( + \) is addition truncated at \( \sup S \). Urysohn spaces of the form \( U_S \) are often used as interesting examples in the study of automorphism groups of countable structures (e.g. [3], [5], [22], [24], [25]).

5. Generalize all of the previous examples as follows. Fix a countable subset \( S \subseteq \mathbb{R}_{\geq 0} \), with \( 0 \in S \). Assume \( S \) is closed under the binary operation \( r + S s := \sup\{x \in S : x \leq r + s\} \) and, moreover, that \( + \) is associative. Let \( S = (S, +, \leq, 0) \). For sets \( S \) closed under \( + \), associativity of \( + \) characterizes the existence of \( U_S \) (see [21], Theorem 5 or [9], Proposition 7.8). This situation is studied in further generality in [9] and [11].

6. For an example of a different flavor, fix a countable linear order \( (R, \leq, 0) \), with least element 0, and let \( R = (R, \text{max}, \leq, 0) \). We refer to \( U_R \) as the ultrametric Urysohn space over \( (R, \leq, 0) \). Explicit constructions of these spaces are given in [14]. Alternatively, \( U_R \) can be viewed as a countable model of the theory of infinitely refining equivalence relations indexed by \( (R, \leq, 0) \). These are standard model theoretic examples, often used to illustrate various behavior in the stability spectrum (see [4], Section III.4).

We will consider model theoretic properties of \( \mathcal{R} \)-Urysohn spaces. In particular, given a countable distance monoid \( \mathcal{R} \), we let \( \text{Th}(U_{\mathcal{R}}) \) be the complete \( L_{\mathcal{R}} \)-theory of \( U_{\mathcal{R}} \), where \( L_{\mathcal{R}} \) is a first-order language consisting of binary relations \( d(x, y) \leq r \), for \( r \in R \). In [9], we construct a “nonstandard” distance monoid extension \( \mathcal{R}^* \) of \( \mathcal{R} \), with the property that any model of \( \text{Th}(U_{\mathcal{R}}) \) is canonically an \( \mathcal{R}^* \)-metric space (see Theorem 2.11 below). We let \( U_{\mathcal{R}} \) denote a sufficiently saturated monster model of \( \text{Th}(U_{\mathcal{R}}) \). Then \( U_{\mathcal{R}} \) is, of course, a \( \kappa^+ \)-universal and \( \kappa \)-homogeneous \( L_{\mathcal{R}} \)-structure, where \( \kappa \) is the saturation cardinal of \( U_{\mathcal{R}} \). It is also true that \( U_{\mathcal{R}} \) is \( \kappa^+ \)-universal as an \( \mathcal{R}^* \)-metric space (see [9], Proposition 8.1). However, we will focus on the case when \( U_{\mathcal{R}} \) is, moreover, \( \kappa \)-homogeneous as an \( \mathcal{R}^* \)-metric space. In other words, we focus on the case when \( \text{Th}(U_{\mathcal{R}}) \) has quantifier elimination, in which case we say that \( \mathcal{R} \) is Urysohn. In [9], Theorem 8.10, we characterize quantifier elimination as a natural continuity property of \( \mathcal{R} \) (see Theorem 2.12 below). This motivates a general schematic for analyzing the model theoretic behavior of \( \text{Th}(U_{\mathcal{R}}) \).

**Definition 1.2.** Let \( \text{RUS} \) denote the class of \( \mathcal{R} \)-Urysohn spaces \( U_{\mathcal{R}} \), where \( \mathcal{R} \) is a Urysohn monoid. We say a property \( P \) of \( \text{RUS} \) is **axiomatizable** (resp. **finitely axiomatizable**) if there is a first-order \( L_{\omega_1, \omega} \)-sentence (resp. \( L_{\omega, \omega} \)-sentence) \( \varphi_P \), in the language of ordered monoids, such that, if \( \mathcal{R} \) is a Urysohn monoid, then \( U_{\mathcal{R}} \) satisfies \( P \) if and only if \( \mathcal{R} \models \varphi_P \).

Although we have relativized this notion of axiomatizability to the class \( \text{RUS} \), it is worth mentioning that there is a first-order sentence \( \varphi_{\text{QE}} \), in the language of ordered monoids, such
that a countable distance monoid \( R \) is Urysohn if and only if \( R \models \varphi_{\text{QE}} \) (see [9, Corollary 8.12]). Therefore, if some property \( P \) is axiomatizable with respect to the class of all \( R \)-Urysohn spaces, then \( P \) is also axiomatizable relative to \( \textbf{RUS} \). This remark will be especially pertinent when we show that certain properties are not axiomatizable (relative to \( \textbf{RUS} \)).

Concerning axiomatizable properties of \( \textbf{RUS} \), we begin with notions around stability and simplicity. In particular, the ultrametric spaces in Example 1.1(6) are well-known to be stable when considered as theories of refining equivalence relations. We also have the random graph as a canonical example of a simple unstable theory. Toward a general understanding of the role of stability and simplicity in Urysohn spaces, we consider, in Section 3, several ternary relations defined on subsets of the monster model \( U_R \), where \( R \) is Urysohn. First to be considered are the notions of independence given by nonforking and nondividing. We state a combinatorial characterization of forking and dividing for complete types in \( \text{Th}(U_R) \), when \( R \) is Urysohn. This characterization is identical to the same result for the complete Urysohn sphere in continuous logic, which was proved in joint work with Caroline Terry [10]. The proof of this result in our present setting closely follows the strategy of [10], and we outline the argument in Appendix A. Finally, in Section 3 we define three more ternary relations on \( U_R \), including the stationary independence relation of free amalgamation of metric spaces, which was used by Tent and Ziegler [24, 25] to analyze the algebraic structure of the isometry groups of \( U_Q \) and \( U_{Q_1} \).

In Section 4 we use this network of ternary relations to prove the following result.

**Theorem A.**

(a) Stability and simplicity are finitely axiomatizable properties of \( \textbf{RUS} \). In particular, given a Urysohn monoid \( R \),

(i) \( \text{Th}(U_R) \) is stable if and only if \( U_R \) is ultrametric, i.e., for all \( r, s \in R \), \( r \oplus s = \max\{r, s\} \);

(ii) \( \text{Th}(U_R) \) is simple if and only if, for all \( r, s \in R \), if \( r \leq s \) then \( r \oplus r \oplus s = r \oplus s \).

(b) Superstability and supersimplicity are not axiomatizable properties of \( \textbf{RUS} \).

This result is a combination of Theorem 4.4, Theorem 4.9, and Corollary 4.17. Concerning part (b), we show that superstability and supersimplicity are detected via relatively straightforward properties of \( R \), but not in a first-order way.

Having established the presence of generalized Urysohn spaces in the most well-behaved regions of classification theory, we then turn to the question of how complicated \( \text{Th}(U_R) \) can be. For example, Theorem 4.4 immediately implies that the rational Urysohn space is not simple. This is a well-known fact, which was observed for the complete Urysohn sphere in continuous logic by Pillay (see [12]). Casanovas and Wagner give a similar argument in [7] to show that \( \text{Th}(U_{R_n}) \) is not simple when \( n \geq 3 \). In [10], it is shown that the complete Urysohn sphere in fact has SOP\(_n\) for all \( n \geq 3 \), and these methods can be easily adjusted to show that, if \( n \geq 3 \), then \( \text{Th}(U_{R_n}) \) is SOP\(_n\) and NSOP\(_{n+1}\). Regarding an upper bound in complexity, it is shown in [10] that the complete Urysohn sphere does not have the fully finite strong order property. Altogether, this work sets the stage for the main result of Section 5, which gives the following upper bound for the complexity of \( \text{Th}(U_R) \).

**Theorem B.** If \( R \) is Urysohn then \( \text{Th}(U_R) \) does not have the finitary strong order property.

This result, which appears again as Corollary 5.2, is obtained by generalizing work in [10], which analyzes when the 2-type of an indiscernible sequence in \( U_R \) can be “cyclically amalgamated” (see Definition 2.6).

In Section 6 we address the region of complexity between simplicity and the finitary strong order property, which, in general, is stratified by Shelah’s SOP\(_n\)-hierarchy. Concerning \( \text{Th}(U_R) \), we
first use the characterizations of stability and simplicity to formulate a purely algebraic notion of the archimedean rank, arch(\(R\)), of a general distance monoid \(R\) (see Definition 6.1). In particular, Th(\(U_R\)) is stable (resp. simple) if and only if arch(\(R\)) \(\leq 1\) (resp. arch(\(R\)) \(\leq 2\)). We then use this rank to pinpoint the exact complexity of Th(\(U_R\)).

Theorem C. If \(R\) is Urysohn and \(n \geq 3\), then Th(\(U_R\)) is SOP\(_n\) if and only if arch(\(R\)) \(\geq n\).

This result appears again as Theorem 6.6, and provides the first class of examples in which the entirety of the SOP\(_n\)-hierarchy has a meaningful interpretation independent of combinatorial dividing lines. As an immediate consequence of the theorem, we obtain that any non-simple \(U_R\) is SOP\(_3\); and we further show that the failure of simplicity also implies TP\(_2\).

In Section 7, we consider the question of elimination of hyperimaginaries. This builds on work of Casanovas and Wagner [7], which was motivated by the search for a theory without the strict order property that does not eliminate hyperimaginaries. In particular, they showed that Th(\(U_{Q_1}\)) is such a theory (although they did not identify their theory as such, see [9, Proposition 9.5]). We adapt their methods to give necessary conditions for elimination of hyperimaginaries and weak elimination of imaginaries for Th(\(U_R\)), where \(R\) is any Urysohn monoid. Finally, we conjecture that these conditions are sufficient, and discuss consequences of this conjecture.

2 Preliminaries

2.1 Classification Theory

In this section, we record the definitions and facts from model theoretic classification theory that will be used in our results. We let \(T\) denote a complete first-order theory and \(M\) a sufficiently saturated monster model of \(T\).

We first specify some notation and conventions, which will apply throughout the paper. We write \(A \subset M\) to denote that \(A\) is a subset of \(M\) and \(|A| < |M|\). We write \(\bar{b} \in M\) to denote that \(\bar{b}\) is a tuple of elements of \(M\). By convention, tuples can be infinite in length, but always smaller in cardinality than \(M\); we will use vector notation \(\bar{a}, \bar{b}, \bar{c}, \ldots\) for tuples of length larger than 1. We use \(\ell(\bar{a})\) to denote the index set of the tuple \(\bar{a}\).

We assume the reader is familiar with basic definitions concerning forking, dividing, stability, and simplicity. See [23] for details. Given \(A, B, C \subset M\), we use the notation \(A \downarrow^d_C B\) (resp. \(A \downarrow^f_C B\)) to denote that tp(\(A/BC\)) does not fork (resp. divide) over \(C\). We will also use the following facts.

Fact 2.1. The following are equivalent.

(i) For all \(A, B, C \subset M\), \(A \downarrow^d_C B\) if and only if \(A \downarrow^f_C B\).

(ii) For all \(A, B, C \subset M\), if \(A \downarrow^d_C B\) and \(b_* \in M\) is a singleton then there is \(A' \equiv_{BC} A\) such that \(A' \downarrow^d_C Bb_*\).

Proof. It is a standard fact that forking and dividing coincide if and only if nonforking independence satisfies the extension property (see e.g. [2]). The difference here is that (ii) only describes extensions obtained from adding a singleton \(b_*\). By induction and finite character of forking, one obtains the full extension property.

The next fact concerns the behavior of nonforking in simple and stable theories. Part (a) is a result of Kim (see e.g. [28, Theorem 2.6.1]). For part (b), see [28, Remark 2.9.6].
Fact 2.2.

(a) The following are equivalent.

(i) $T$ is simple.

(ii) Nonforking satisfies symmetry, i.e. for all $A, B, C \subseteq \mathcal{M}$, $A \downarrow^f_C B$ if and only if $B \downarrow^f_C A$.

(iii) Nonforking satisfies local character, i.e. for all $A \subseteq \mathcal{M}$, there is a cardinal $\kappa(A)$ such that, for all $B \subseteq \mathcal{M}$ there is $C \subseteq B$, with $|C| \leq \kappa(A)$ and $A \downarrow^f_C B$.

(b) Suppose $T$ is simple. Then $T$ is stable if and only if nonforking satisfies stationarity over models, i.e. for all models $M \subseteq \mathcal{M}$ and sets $A, A', B \subseteq \mathcal{M}$, with $M \subseteq B$, if $A \downarrow_M^f B$, $A' \downarrow_M^f B$, and $A' \equiv M A$, then $A' \equiv B A$.

Finally, we define Shelah’s SOP$_n$-hierarchy. Suppose $p(\bar{x}, \bar{y})$ is a type (possibly over parameters), where $\bar{x}$, $\bar{y}$ are (possibly infinite) tuples of the same length. Then $p$ induces a directed graph structure on $M^{\ell(\bar{x})}$, consisting of pairs $(\bar{a}, \bar{b})$ such that $M \models p(\bar{a}, \bar{b})$. The SOP$_n$-hierarchy is defined from combinatorial complexity arising in this directed graph structure.

Definition 2.3.

1. A type $p(\bar{x}, \bar{y})$ admits infinite chains if there is a sequence $(\bar{a}^l)_{l<\omega}$ in $\mathcal{M}$ such that $M \models p(\bar{a}^l, \bar{a}^m)$ for all $l \leq m$.

2. Given $n > 0$, a type $p(\bar{x}, \bar{y})$ omits $n$-cycles if

$$p(\bar{x}^1, \bar{x}^2) \cup p(\bar{x}^2, \bar{x}^3) \cup \ldots \cup p(\bar{x}^{n-1}, \bar{x}^n) \cup p(\bar{x}^n, \bar{x}^1)$$

is inconsistent.

3. Given $n \geq 3$, $T$ has the $n$-strong order property (SOP$_n$) if there is a formula $\varphi(\bar{x}, \bar{y})$, which admits infinite chains and omits $n$-cycles.

Recall that $T$ has the strict order property if there is a formula $\varphi(\bar{x}, \bar{y})$, with an infinite chain, that defines a partial order on $M^{\ell(\bar{x})}$. On the other hand if a formula $\varphi(\bar{x}, \bar{y})$ omits $n$-cycles for all $n > 0$ then, by taking the transitive closure of $\varphi$, one obtains an $\bigvee$-definable partial order. Therefore, the SOP$_n$-hierarchy can be viewed as a yardstick measuring how close $T$ comes to having the strict order property. Between the strict order property and the SOP$_n$-hierarchy, there is room for several distinct variations of strong order properties. The following three examples will be of interest to us.

Definition 2.4. $T$ has the strong order property (SOP) if there is a type $p(\bar{x}, \bar{y})$, which admits infinite chains and omits $n$-cycles for all $n > 0$. If $\ell(\bar{x})$ is finite then we say finitary strong order property (FSOP). If $p(\bar{x}, \bar{y})$ is a formula then we say fully finite strong order property (FFSOP).

As a remark on notation, we note that some sources use the acronym SOP for the strong order property, while other use this for the strict order property (which is stronger than each order property defined above). We have chosen SOP$_\omega$ for the strong order property because a straightforward exercise shows that $T$ has the strong order property if and only if it has SOP$_n$ for all $n \geq 3$. The reader may also wonder about SOP$_1$ and SOP$_2$. These notions do exist, and are defined as tree properties, rather than in analogy to SOP$_n$ for $n \geq 3$. However, if one were to apply Definition 2.3 with $n = 1$ and $n = 2$, the resulting notions would be equivalent to, respectively, “$T$ has an infinite model” and “$T$ has the order property”. Altogether, to avoid this confusion arising from acronyms, we use the following rank.
**Definition 2.5.** Define the **strong order rank of** $T$, denoted $\text{SO}(T)$, as follows.

(i) $\text{SO}(T) = 0$ if $T$ has finite models.

(ii) $\text{SO}(T) = 1$ if $T$ has infinite models, but does not have the order property (i.e. is stable).

(iii) $\text{SO}(T) = 2$ if $T$ has the order property, but does not have SOP$_3$.

(iv) Given $n \geq 3$, $\text{SO}(T) = n$ if $T$ has SOP$_n$, but does not have SOP$_{n+1}$.

(v) $\text{SO}(T) = \omega$ if $T$ has SOP$_n$ for all $n \geq 3$ (i.e SOP$_\omega$), but does not have FSOP.

(vi) $\text{SO}(T)$ is undefined if $T$ has FSOP.

The following observation illustrates that this definition of strong order rank is not as ad hoc as it might seem.

**Definition 2.6.** Suppose $\mathcal{I} = (\bar{a}^i)_{i < \omega}$ is an indiscernible sequence in $M$.

1. Let $\text{NP}(\mathcal{I}) = \{i \in \ell(\bar{a}^0) : a^0_i \neq a^1_i\}$ be the set of **non-parameter indices** of $\mathcal{I}$.

2. Given $n > 0$, $\mathcal{I}$ is **$n$-cyclic** if $\text{tp}(\bar{a}^0, \bar{a}^1)$ does not omit $n$-cycles.

By a standard application of Ramsey’s theorem and the Ehrenfeucht-Mostowski type (see [23, Lemma 7.1.1]), we obtain the following uniformization of strong order rank.

**Fact 2.7.**

(a) Given $n > 0$, $\text{SO}(T) < n$ if and only if every infinite indiscernible sequence in $M$ is $n$-cyclic.

(b) $\text{SO}(T)$ is defined (i.e. $T$ does not have FSOP) if and only if, for any indiscernible sequence $\mathcal{I}$, if $\text{NP}(\mathcal{I})$ is finite then $\mathcal{I}$ is $n$-cyclic for some $n > 0$.

Part (a) of the previous fact was observed by Scow, when $n = 3$, and the proof is the same for all $n > 0$. Note that, when $n = 2$, we recover the familiar fact that $T$ does not have the order property if and only if every indiscernible sequence is an indiscernible set. The use of $\text{NP}(\mathcal{I})$ in part (b) is simply a technical way to avoid indiscernibility with respect to a parameter set, as the definition of FSOP allows the use of types over parameters.

### 2.2 Generalized Metric Spaces

In this section, we define distance monoids and generalized Urysohn spaces. We then briefly summarize the first-order setting for the theories of these structures, as well as the characterization of quantifier elimination from [9].

**Definition 2.8.**

1. Let $\mathcal{L}_{om} = \{\oplus, \leq, 0\}$ be the language of ordered monoids. An $\mathcal{L}_{om}$-structure $\mathcal{R} = (R, \oplus, \leq, 0)$ is a **distance monoid** if

   (i) $(R, \leq, 0)$ is a linear order with least element 0;

   (ii) $\oplus$ is a commutative and associative binary operation on $R$, which preserves $\leq$ and has identity 0.
2. Suppose $\mathcal{R}$ is a distance monoid. Given a set $A$ and a function $d : A \times A \rightarrow R$, we call $d$ an $\mathcal{R}$-metric on $A$ if

(i) for all $x, y \in A$, $d(x, y) = d(y, x)$, and $d(x, y) = 0$ if and only if $x = y$;

(ii) for all $x, y, z \in A$, $d(x, z) \leq d(x, y) \oplus d(y, z)$.

In this case, $(A, d)$ is an $\mathcal{R}$-metric space.

3. Given a distance monoid $\mathcal{R}$, let $\mathcal{K}_R$ denote the class of finite $\mathcal{R}$-metric spaces.

4. Given a countable distance monoid $\mathcal{R}$, let $\mathcal{U}_\mathcal{R}$ denote the unique (up to isomorphism) countable Fraïssé limit of $\mathcal{K}_\mathcal{R}$. We call $\mathcal{U}_\mathcal{R}$ the $\mathcal{R}$-Urysohn space. $\mathcal{U}_\mathcal{R}$ is the unique (up to isometry) $\mathcal{R}$-metric space, which is ultrahomogeneous and universal for finite $\mathcal{R}$-metric spaces.

Remark 2.9. The reader may verify that, given a countable distance monoid $\mathcal{R}$, $\mathcal{K}_\mathcal{R}$ is a Fraïssé class. Indeed, the only nontrivial verification is in the amalgamation property, and one may simply use the natural generalization of free amalgamation of metric spaces (see [9, Definition 7.11]). As a tangential remark, we note that the associativity axiom could be omitted when defining $\mathcal{R}$-metric spaces and the class $\mathcal{K}_\mathcal{R}$. However, in this more general setting, associativity is crucial in proving the existence of free amalgamations of $\mathcal{R}$-metric spaces. In fact, $\mathcal{K}_\mathcal{R}$ is a Fraïssé class if and only if $\oplus$ is associative. See [9, Section 7], [11], and [21] for more details.

Definition 2.10. Suppose $\mathcal{R}$ is a countable distance monoid.

1. Let $\mathcal{L}_\mathcal{R} = \{d(x, y) \leq r : r \in R\}$, where $d(x, y) \leq r$ is a binary relation. We interpret $\mathcal{R}$-metric spaces as $\mathcal{L}_\mathcal{R}$-structures in the obvious way.

2. Let $\text{Th}(\mathcal{U}_\mathcal{R})$ denote the complete $\mathcal{L}_\mathcal{R}$-theory of $\mathcal{U}_\mathcal{R}$. Let $\mathcal{U}_\mathcal{R}$ be a sufficiently saturated monster model of $\mathcal{U}_\mathcal{R}$.

If $\mathcal{R}$ is a countably infinite distance monoid, then it is easy to see that saturated models of $\text{Th}(\mathcal{U}_\mathcal{R})$ cannot be considered coherently as $\mathcal{R}$-metric spaces. For example, $\mathcal{U}_\mathbb{Q}$ contains points of infinite, infinitesimal, or irrational distance. However, in [9], we prove the following result.

Theorem 2.11. [9] Suppose $\mathcal{R}$ is a countable distance monoid. Then there is a distance monoid extension $\mathcal{R}^* = (R^*, \oplus, \leq, 0)$ of $\mathcal{R}$ such that:

(a) Given $M \models \text{Th}(\mathcal{U}_\mathcal{R})$ and $a, b \in M$, there is a unique $\alpha = \alpha(a, b) \in R^*$ such that, for all $r \in R$, $M \models d(a, b) \leq r$ if and only if $\alpha \leq r$. Moreover, if $d_M : M \times M \rightarrow R^*$ is such that $d_M(a, b) = \alpha(a, b)$, then $(M, d_M)$ is an $\mathcal{R}^*$-metric space.

(b) $\mathcal{R}^*$ satisfies the following analytic properties.

(i) $(R^*, \leq, 0)$ is a Dedekind complete linear order with a maximal element.

(ii) For all $\alpha, \beta \in R^*$, if $\alpha < \beta$ then there is some $r \in R$ such that $\alpha \leq r < \beta$.

(iii) For all $\alpha, \beta, \gamma \in R^*$, if $\gamma \leq r \oplus s$ for all $r, s \in R$, with $\alpha \leq r$ and $\beta \leq s$, then $\gamma \leq \alpha \oplus \beta$.

The properties of $\mathcal{R}^*$ listed in the previous result are essentially all we will need for the subsequent work. In light of property (b)(i), we adopt the convention that, when considering $\emptyset$ as a subset of $R^*$, we let $\inf \emptyset = \sup R^*$ and $\sup \emptyset = 0$.

We refer the reader to [9] for explicit descriptions of $\mathcal{R}^*$ and its construction. The essential idea is that $R^*$ is the set of quantifier-free 2-types consistent with $\text{Th}(\mathcal{U}_\mathcal{R})$. In particular, as a set,
$R^*$ depends only on $(R, \leq, 0)$. Given a model $M \models \text{Th}(U_R)$, $d_M(a, b)$ is simply the quantifier-free 2-type realized by $(a, b)$. The operation $\oplus$ is extended to $R^*$ by defining $\alpha \oplus \beta$ to be the largest $\gamma \in R^*$ such that the 3-type defining a triangle with side lengths $\alpha$, $\beta$, and $\gamma$ is consistent.

For example, if $R$ is finite then $R^* = R$. If we consider $Q$, then $(Q^{\geq 0})^*$ can be identified with $\mathbb{R}^{\geq 0} \cup \{q^+ : q \in \mathbb{Q}^{\geq 0}\} \cup \{\infty\}$, where $q^+$ denotes an infinitesimal cut to the right of $q$, and $\infty$ is an infinite element. In particular, it is important to emphasize that $R^*$ is not an elementary extension of $R$. For example, there is there is a single positive infinitesimal $0^+$ in $Q^*$. Moreover, for any $R$, any non-maximal element of $R$ will have an immediate successor in $R^*$.

A main result in [9] is the following characterization of quantifier elimination for $\text{Th}(U_R)$.

**Theorem 2.12.** [9] Suppose $R$ is a countable distance monoid. Then $\text{Th}(U_R)$ has quantifier elimination if and only if, for all nonzero $\alpha \in R^*$, if $\alpha$ has no immediate predecessor then, for all $s \in R$, $\alpha \oplus s = \sup\{x \oplus s : x < \alpha\}$.

Recall that we define a Urysohn monoid to be a countable distance monoid $R$ such that $\text{Th}(U_R)$ has quantifier elimination. In [9, Section 9], we show that this situation includes most natural examples arising in the literature (e.g. each space in Example 1.1, except the full generality of (5)). Using quantifier elimination and Theorem 2.11 we have the following conclusion.

**Proposition 2.13.** Suppose $R$ is a Urysohn monoid and $M \models \text{Th}(U_R)$ is $\kappa$-saturated. Then $(M, d_M)$ is a $\kappa$-homogeneous and $\kappa^+$-universal $R^*$-metric space, i.e., any isometry between subspaces of $M$, of cardinality less than $\kappa$, extends to an isometry of $M$; and any $R$-metric space of cardinality at most $\kappa$ is isometric to a subspace of $M$.

We will primarily apply this fact to the monster model $U_R$. To ease notation, we will use $d$ for the $R^*$-metric $d_{U_R}$ on $U_R$ given by Theorem 2.11(a). Note that if we restrict $d$ to $U_R$ (considered as an $R^*$-subspace of $U_R$), then $d$ agrees with the original $R$-metric on $U_R$, which, in turn, agrees with the $R^*$-metric $d_{U_R}$ on $U_R$ given by Theorem 2.11(a).

Next, we define for $R^*$ an analog of the absolute value of the difference between between two distances.

**Definition 2.14.** Given a distance monoid $R$ and $\alpha, \beta \in R^*$, define

$$|\alpha \oplus \beta| := \inf\{x \in R^* : \alpha \leq \beta \oplus x \text{ and } \beta \leq \alpha \oplus x\}.$$  

The continuity property in Theorem 2.11(b)(iii) ensures that this operation is well behaved (e.g. $|\alpha \oplus \beta| \leq \gamma$ if and only if $\alpha \leq \beta \oplus \gamma$ and $\beta \leq \alpha \oplus \gamma$). See [9, Section 6] for more details.

Finally, we define natural multiplicative operations on elements of $R^*$.

**Definition 2.15.** Given $\alpha \in R^*$ and $n > 0$, we define

$$n\alpha := \underbrace{\alpha \oplus \ldots \oplus \alpha}_{n \text{ times}}$$

and

$$\frac{1}{n}\alpha := \inf\{\beta \in R^* : \alpha \leq n\beta\}.$$  

These notions allow us to treat $R^*$ as a module over the semiring $(\mathbb{N}, +, \cdot)$, but not necessarily over $(\mathbb{Q}^{\geq 0}, +, \cdot)$. For example, if $S = \{0, 1, 3, 4\}, +_S, \leq, 0\}$, then $\frac{1}{4}(1 \oplus 3) = 3$ and $\frac{1}{2}1 \oplus \frac{1}{2}3 = 4$. However, the following properties will be sufficient for our results. The proof is left to the reader.

**Proposition 2.16.** Suppose $\mathcal{U}$ is a countable distance monoid.

(a) Fix $X,Y \subseteq R^*$ and suppose $\alpha = \inf X$ and $\beta = \inf Y$. Then $\alpha \oplus \beta = \inf\{x \oplus y : x \in X, y \in Y\}$.

(b) For any $\alpha, \beta \in R^*$ and $n > 0$, $\frac{1}{n}(\alpha \oplus \beta) \leq \frac{1}{n}\alpha \oplus \frac{1}{n}\beta$.  

8
3 Notions of Independence

In this section, we consider various ternary relations on subsets of $\mathbb{U}_R$, where $R$ is a Urysohn monoid. The first such relations are nonforking and nondividing independence. Toward a characterization of these notions, we define the following distance calculations.

**Definition 3.1.** Fix a countable distance monoid $R$. Given $C \subset \mathbb{U}_R$ and $b_1, b_2 \in \mathbb{U}_R$, we define

\[
d_{\text{max}}(b_1, b_2/C) = \inf_{c \in C} (d(b_1, c) \oplus d(c, b_2))
\]

\[
d_{\text{min}}(b_1, b_2/C) = \max \left\{ \sup_{c \in C} |d(b_1, c) \oplus d(c, b_2)|, \frac{1}{3} d(b_1, b_2) \right\}.
\]

Note that $d_{\text{max}}(b_1, b_2/C)$ is the distance between $b_1$ and $b_2$ in the free amalgamation of $b_1C$ and $b_2C$ over $C$ (equivalently, the largest possible distance between realizations of $\text{tp}(b_1/C)$ and $\text{tp}(b_2/C)$). On the other hand, $d_{\text{min}}$ does not have as straightforward of an interpretation, and has to do with the behavior of indiscernible sequences in $\mathbb{U}_R$. We use these values to give a completely combinatorial description of $\downarrow^d$ and $\downarrow^f$, which, in particular, shows that forking and dividing are the same for complete types in $\text{Th}(\mathbb{U}_R)$.

**Theorem 3.2.** Suppose $R$ is a Urysohn monoid. Given $A, B, C \subset \mathbb{U}_R$, $A \downarrow^d_C B$ if and only if $A \downarrow^f_C B$ if and only if for all $b_1, b_2 \in B$,

\[
d_{\text{max}}(b_1, b_2/AC) = d_{\text{max}}(b_1, b_2/C) \quad \text{and} \quad d_{\text{min}}(b_1, b_2/AC) = d_{\text{min}}(b_1, b_2/C).
\]

This characterization is identical to the characterization of forking and dividing, given in [10], for the complete Urysohn sphere as a metric structure in continuous logic. Moreover, the proof of Theorem 3.2 is very similar to [10]. The work lies in showing that, with only minor modifications, the methods in [10] can be reformulated and applied in classical logic to $\text{Th}(\mathbb{U}_R)$. We give an outline of the proof, and the necessary modifications, in Appendix A.

The rest of this section focuses on three more notions of independence, which will be useful in understanding stability and simplicity.

**Definition 3.3.** Suppose $R$ is a countable distance monoid.

1. Given $a \in \mathbb{U}_R$ and $C \subset \mathbb{U}_R$, define $d(a, C) = \inf \{ d(a, c) : c \in C \}$.

2. Given $A, B, C \subset \mathbb{U}_R$, define

\[
A \downarrow^\text{dist}_C B \iff d(a, BC) = d(a, C) \quad \text{for all } a \in A;
\]

\[
A \downarrow^\otimes_C B \iff d(a, b) = d_{\text{max}}(a, b/C) \quad \text{for all } a \in A, b \in B;
\]

\[
A \downarrow^\text{max}_C B \iff d_{\text{max}}(b_1, b_2/AC) = d_{\text{max}}(b_1, b_2/C) \quad \text{for all } b_1, b_2 \in B.
\]

The relation $\downarrow^\text{dist}$ has obvious significance as a notion of independence in metric spaces. The relation $A \downarrow^\otimes_C B$ should be viewed as asserting that, as $R^s$-metric spaces, $ABC$ is isometric to the free amalgamation of $AC$ and $BC$ over $C$. The final relation $\downarrow^\text{max}$ is a reasonable simplification of the characterization of $\downarrow^f$ given by Theorem 3.2. This relation will play a major role in the case when $\text{Th}(\mathbb{U}_R)$ is simple.

Finally, we make some observations on the general relationship between these various notions of independence. In [25], Tent and Ziegler define a stationary independence relation as a ternary
relation satisfying invariance, monotonicity, symmetry, (full) transitivity, (full) existence, and stationarity (over all sets). Their definitions are for finite subsets of some countable structure, and can easily be adapted to small subsets of a monster model \(M\) of some theory \(T\).

**Proposition 3.4.**

(a) If \(M\) is the monster model of a complete theory \(T\), and \(\downarrow\) is a stationary independence relation on \(M\), then \(\downarrow\) implies \(\downarrow^f\).

(b) Suppose \(\mathcal{R}\) is a Urysohn monoid.

(i) \(\downarrow^f\) implies \(\downarrow^{\text{max}}\).

(ii) \(\downarrow^\otimes\) is a stationary independence relation on \(\mathcal{U}_\mathcal{R}\), and so \(\downarrow^\otimes\) implies \(\downarrow^f\).

(iii) \(\downarrow^\otimes\) implies \(\downarrow^{\text{dist}}\).

(iv) \(\downarrow^{\text{dist}}\) satisfies local character.

**Proof.** Part (a). This is a nice exercise in the style of [2]. A proof can be found in [10, Theorem 4.1].

Part (b). Claim (i) is immediate from Theorem 3.2.

For claim (ii), the verification of invariance, monotonicity, symmetry, and transitivity are all straightforward. Some of these are verified in [25]. The existence axiom is simply asserting that free amalgamations exist, which is a standard exercise. We remark, however, that transitivity and full existence require the continuity provided by Proposition 2.16(a). Finally, stationarity follows from quantifier elimination.

For claim (iii), fix \(A, B, C \subset \mathcal{U}_\mathcal{R}\), with \(A \downarrow^\otimes B\). Given \(a \in A\) and \(b \in B\), we have

\[d(a, C) = \inf_{c \in C} d(a, c) \leq d_{\text{max}}(a, b/C) = d(a, b).\]

Therefore \(d(a, BC) = d(a, C)\), and so \(A \downarrow^{\text{dist}} B\).

For claim (iv), fix \(A, B \subset \mathcal{U}_\mathcal{R}\). We may assume \(A, B \neq \emptyset\). We show that there is \(C \subseteq B\) such that \(|C| \leq |A| + n_0\) and \(A \downarrow^{\text{dist}} C\). It suffices to show that for all \(a \in A\), there is \(C_a \subseteq B\) such that \(|C_a| \leq n_0\) and \(a \downarrow^{\text{dist}} C_a\). We will then set \(C = \bigcup_{a \in A} C_a\).

Fix \(a \in A\). If there is some \(b \in B\) such that \(d(a, b) = d(a, B)\) then set \(C_a = \{b\}\). Otherwise, define \(X = \{r \in R : d(a, B) = r\}\). Given \(r \in X\), we claim that is some \(b_r \in B\) such that \(d(a, b_r) = r\). Indeed, this follows simply from the observation that any non-maximal \(r \in R\) has an immediate successor in \(R^*\). Set \(C_a = \{b_r : r \in X\}\). By assumption and Theorem 2.11(b)(ii), for any \(b \in B\) there is some \(r \in X\), with \(r < d(a, b)\). We have \(b_r \in C\) and \(d(a, b_r) \leq r < d(a, b)\), as desired. \(\square\)

4 Urysohn Spaces of Low Complexity

4.1 Stability

In this section, we characterize the Urysohn monoids \(\mathcal{R}\) for which \(\text{Th}(\mathcal{U}_\mathcal{R})\) is stable. We recall the following definition from [9].

**Definition 4.1.** A countable distance monoid \(\mathcal{R} = (R, \oplus, \leq, 0)\) is **ultrametric** if, for all \(r, s \in R\), \(r \oplus s = \max\{r, s\}\).
It is easy to verify that ultrametric monoids are Urysohn (see [9, Proposition 9.3]). The goal of this section is to show that, given a Urysohn monoid \( R \), \( \text{Th}(U_R) \) is stable if and only if \( R \) is ultrametric. The heart of this fact lies in the observation that ultrametric spaces correspond to refining equivalence relations. In particular, if \((A, d)\) is an ultrametric space, then for any distance \( r, d(x, y) \leq r \) is an equivalence relation on \( A \). Altogether, the result that ultrametric monoids yield stable Urysohn spaces recovers classical results on theories of equivalence relations (see [4, Section III.4]). Therefore, our work focuses on the converse, which says that stable Urysohn spaces must be ultrametric. We will also emphasize the relationship to nonforking, and so it will be useful to have the following simplification of \( d_{\text{max}} \) for ultrametric Urysohn spaces.

**Proposition 4.2.** Suppose \( R \) is an ultrametric monoid. Fix \( C \subset U_R \) and \( b_1, b_2 \in U_R \).

(a) If \( d(b_1, c) \neq d(b_2, c) \) for some \( c \in C \) then \( d_{\text{max}}(b_1, b_2/C) = d(b_1, b_2) \).

(b) If \( d(b_1, c) = d(b_2, c) \) for all \( c \in C \) then \( d_{\text{max}}(b_1, b_2/C) = d(b_1, C) \).

The characterization of stability will combine Proposition 4.2 with the following observations.

**Lemma 4.3.** Suppose \( R \) is a Urysohn monoid.

(a) If \( R \) is ultrametric then \( \downarrow \circ \) coincides with \( \downarrow \text{dist} \).

(b) If \( \downarrow \text{dist} \) is symmetric then \( R \) is ultrametric.

**Proof.** Part (a). Suppose \( R \) is ultrametric. By Proposition 3.4(b)(iii), it suffices to show \( \downarrow \text{dist} \) implies \( \downarrow \circ \). Suppose \( A, B, C \subset U_R \) are such that \( A \downarrow_C^{\text{dist}} B \). We want to show that for all \( a \in A \) and \( b \in B \), \( d(a, b) = d_{\text{max}}(a, b/C) \). By Proposition 4.2 it suffices to assume \( d(a, c) = d(b, c) \) for all \( c \in C \), and prove that \( d(a, b) = d(a, C) \). Note that \( A \downarrow_C^{\text{dist}} B \) implies \( d(a, BC) = d(a, C) \), and so we have \( d(a, b) \geq d(a, C) \). On the other hand, if \( d(a, b) > d(a, C) \) then there is \( c \in C \) such that \( d(a, b) > d(a, c) \). But then \( d(b, c) = \max\{d(a, b), d(a, c)\} = d(a, b) > d(a, c) \), which contradicts our assumptions.

Part (b). Suppose \( \downarrow \text{dist} \) is symmetric. Fix \( r, s \in R \). There are \( a, b, c \in U_R \) such that \( d(a, b) = \max\{r, s\} \), \( d(a, c) = \min\{r, s\} \), and \( d(b, c) = r \oplus s \). Then \( d(a, b) \geq d(a, c) \), and so \( a \downarrow_c^{\text{dist}} b \). By symmetry, we have \( b \downarrow_c^{\text{dist}} a \), which means \( \max\{r, s\} = d(a, b) \geq d(b, c) = r \oplus s \). Therefore \( r \oplus s = \max\{r, s\} \), and we have shown that \( R \) is ultrametric. \( \square \\

**Theorem 4.4.** Given a Urysohn monoid \( R \), the following are equivalent.

(i) \( \text{Th}(U_R) \) is stable.

(ii) \( \downarrow f \) coincides with \( \downarrow \text{dist} \).

(iii) \( \downarrow f \) coincides with \( \downarrow \circ \).

(iv) \( R \) is ultrametric, i.e. for all \( r, s \in S \), if \( r \leq s \) then \( r \oplus s = s \).

**Proof.** (iv) \( \Rightarrow \) (iii): Suppose \( R \) is ultrametric. By Proposition 3.4(b)(ii), it suffices to show that \( \downarrow f \) implies \( \downarrow \circ \). So suppose \( A \downarrow f B \) and fix \( a \in A, b \in B \). We want to show \( d(a, b) = d_{\text{max}}(a, b/C) \). By Theorem 3.2 we have \( d(b, C) = d_{\text{max}}(b, b/C) \leq \max\{d(b, a), d(a, b)\} = d(a, b) \). Suppose, toward a contradiction, that \( d(a, b) < d_{\text{max}}(a, b/C) \). By Proposition 4.2, it follows that \( d(a, c) = d(b, c) \) for all \( c \in C \), and so \( d_{\text{max}}(a, b/C) = d(b, C) \leq d(a, b) \), which is a contradiction. \( \square \)

11
Lemma 4.6. Given a countable distance monoid following useful inequality.

(iii) ⇒ (i): If \( \downarrow^f \) coincides with \( \downarrow^\oplus \) then \( \downarrow^f \) satisfies symmetry and stationarity by Proposition 3.4(b)(ii). Therefore \( \text{Th}(\mathcal{U}_\mathcal{R}) \) is stable by Fact 2.2.

(i) ⇒ (iv): Suppose \( \mathcal{R} \) is not ultrametric. Then we may fix \( r \in R \) such that \( r < r \oplus r \). We show that the formula \( \varphi(x_1, x_2, y_1, y_2) := d(x_1, y_2) \leq r \) has the order property. Define a sequence \( (a^1_l, a^2_l)_{l<\omega} \) such that, given \( l < m \), \( d(a^1_l, a^2_l) = r \oplus r \), and all other distances are \( r \). This clearly satisfies the triangle inequality. We have \( \varphi(a^1_l, a^1_m, a^2_m) \) if and only if \( l \leq m \).

(iv) ⇒ (ii): Combine (iv) ⇒ (iii) with Lemma 4.3(a).

(ii) ⇒ (iv): If \( \downarrow^f \) coincides with \( \downarrow^\text{dist} \) then \( \downarrow^f \) satisfies local character by Proposition 3.4(b)(iv). Then \( \downarrow^\text{dist} \) is symmetric by Fact 2.2(a), and so \( \mathcal{R} \) is ultrametric by Lemma 4.3(b). □

Looking back at this characterization, it is worth pointing out that (i), (ii), and (iii) could all be obtained from (iv) by showing that, when \( \mathcal{R} \) is ultrametric, both \( \downarrow^\text{dist} \) and \( \downarrow^\oplus \) satisfy the axioms characterizing nonforking in stable theories (c.f. [28, Theorem 2.6.1, Remark 2.9.6]). In this way, the above theorem could be entirely obtained without using the general characterization of nonforking given by Theorem 3.2.

4.2 Simplicity

Our next goal is an analogous characterization of simplicity for \( \text{Th}(\mathcal{U}_\mathcal{R}) \), when \( \mathcal{R} \) is an Urysohn monoid. We will obtain similar behavior in the sense that simplicity of \( \text{Th}(\mathcal{U}_\mathcal{R}) \) is detected by both a simplification in the characterization of forking, and also low complexity in the arithmetic behavior of \( \mathcal{R} \). However, unlike the stable case, the class of simple Urysohn spaces contains much more than just the classic examples of refining equivalence relations or random graphs. Moreover, the characterization of forking and dividing, given by Theorem 3.2, will be crucial for our results.

We begin by defining the preorder on \( R^* \) given by archimedean equivalence.

Definition 4.5. Suppose \( \mathcal{R} \) is a distance monoid.

1. Define the relation \( \preceq \) on \( R^* \) such that \( \alpha \preceq \beta \) if and only if \( \alpha \leq n\beta \) for some \( n > 0 \).
2. Define the relation \( \sim \) on \( R^* \) such that \( \alpha \sim \beta \) if and only if \( \alpha \leq \beta \) and \( \beta \leq \alpha \).
3. Given \( \alpha, \beta \in R^* \), write \( \alpha < \beta \) if \( \beta \not\preceq \alpha \), i.e. if \( n\alpha < \beta \) for all \( n > 0 \).

Throughout this section, we will use the fact that, given a countable distance monoid \( \mathcal{R} \), if \( b \in \mathbb{U}_\mathcal{R} \) and \( C \subset \mathbb{U}_\mathcal{R} \) then \( d_{\max}(b, b/C) = 2d(b, C) \) (see Proposition 2.16(a)). We also note the following useful inequality.

Lemma 4.6. Given a countable distance monoid \( \mathcal{R} \), if \( C \subset \mathbb{U}_\mathcal{R} \) and \( b_1, b_2, b_3 \in \mathbb{U}_\mathcal{R} \) then

\[
d_{\max}(b_1, b_3/C) \leq d_{\max}(b_1, b_2/C) \oplus d_{\min}(b_2, b_3/C).
\]

Proof. For any \( c \in C \), we have

\[
d_{\max}(b_1, b_3/C) \leq d(b_1, c) \oplus d(b_3, c)
\]

\[
\leq d(b_1, c) \oplus d(b_2, c) \oplus |d(b_2, c) \oplus d(b_3, c)|
\]

\[
\leq d(b_1, c) \oplus d(b_2, c) \oplus d_{\min}(b_2, b_3/C),
\]

which proves the result. □

We now focus on the ternary relation \( \downarrow^f \). When \( \mathcal{R} \) is Urysohn, we have that \( \downarrow^f \) implies \( \downarrow^\text{max} \). Our next result characterizes when these two relations coincide.
**Proposition 4.7.** Suppose $\mathcal{R}$ is a Urysohn monoid. The following are equivalent.

(i) $\downarrow^f$ coincides with $\downarrow^{d_{\text{max}}}$.

(ii) For all $r, s \in R$, if $r \leq s$ then $r \oplus r \oplus s = r \oplus s$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose (ii) fails, and fix $r, s \in R$, with $r \leq s$ and $r \oplus s < 2r \oplus s$. Define the space $(X, d)$ such that $X = \{a, b_1, b_2, c, c'\}$ and

\[
\begin{align*}
    d(a, b_1) &= d(b_1, c) = d(b_1, c') = r, & d(b_1, b_2) &= r \oplus s, \\
    d(b_2, c) &= d(b_2, c') = s, & d(a, b_2) &= 2r \oplus s, \\
    d(a, c) &= d(a, c') = d(c, c') = 2r.
\end{align*}
\]

It is straightforward to verify that $(X, d)$ is an $\mathcal{R}^*$-metric space, and so we may assume $(X, d)$ is a subspace of $\mathbb{U}_\mathcal{R}$. Let $C = \{c, c'\}$. First, note that

- $d_{\text{max}}(b_1, b_2/C) = r \oplus s \leq d(a, b_1) \oplus d(a, b_2)$,
- $d_{\text{max}}(b_1, b_2/C) = 2r \leq d(a, b_1) \oplus d(a, b_1)$, and
- $d_{\text{max}}(b_2, b_2/C) = 2s \leq d(a, b_2) \oplus d(a, b_2)$.

Therefore $a \downarrow_{C}^{d_{\text{max}}} b_1 b_2$. So to show the failure of (i), we show $a \downarrow_C^f b_1 b_2$. Indeed, we have

- $|d(b_1, c) \ominus d(b_2, c)| = |r \ominus s| \leq s$,
- $|d(b_1, c') \ominus d(b_2, c')| = |r \ominus s| \leq s$, and
- $d(b_1, b_2) = r \oplus s \leq 3s$.

Altogether, this implies $d_{\text{min}}(b_1, b_2/C) \leq s$. Therefore, since $r \oplus s < 2r \oplus s$, we have

\[d_{\text{min}}(b_1, b_2/C) \leq s < |(2r \oplus s) \ominus r| = |d(a, b_1) \ominus d(a, b_2)|.\]

So $a \downarrow_C^f b_1 b_2$ by Theorem 3.2.

(ii) $\Rightarrow$ (i). Assume $\mathcal{R}$ satisfies (ii). By Theorem 2.11(b)(iii), it follows that we have

(ii) for all $\alpha, \beta \in R^*$, if $\alpha \leq \beta$ then $2\alpha \oplus \beta = \alpha \oplus \beta$.

In particular, $2\alpha = 3\alpha$ for all $\alpha \in R^*$, which then implies $2\alpha = n\alpha$ for all $\alpha \in R^*$ and $n > 1$.

In order to prove (i), it suffices by Theorem 3.2 to show $\downarrow_{C}^{d_{\text{max}}}$ implies $\downarrow_f$. So suppose $A \downarrow_C^{d_{\text{max}}} B$. Suppose, toward a contradiction, that $A \downarrow_C^f B$. By Theorem 3.2, there are $a \in A$ and $b_1, b_2 \in B$ such that $d_{\text{min}}(b_1, b_2/C) < |d(a, b_1) \ominus d(a, b_2)|$. Without loss of generality, we assume $d(a, b_1) \leq d(a, b_2)$, and so we have

\[d(a, b_1) \ominus d_{\text{min}}(b_1, b_2/C) < d(a, b_2). \tag{†}\]

**Case 1:** $\alpha := \frac{1}{2}d(b_1, b_2) \leq d(a, b_1)$.

By (†),

\[d(a, b_1) \ominus \alpha < d(a, b_2) \leq d(b_1, b_2) \ominus d(a, b_1) \leq 3\alpha \ominus d(a, b_1) = 2\alpha \ominus d(a, b_1),\]

which contradicts (ii)\(^*\).

**Case 2:** $d(a, b_1) < \frac{1}{2}d(b_1, b_2)$.
Suppose, toward a contradiction, that \( d_{\text{max}}(a,b_1/C) \sim d(a,b_1) \). Since \( 2d(a,b_1) = nd(a,b_1) \) for all \( n > 1 \), it follows that \( d_{\text{max}}(a,b_1/C) \leq 2d(a,b_1) \). Combining this observation with (†) and Lemma 4.6, we have

\[
d(a,b_1) \oplus d_{\text{min}}(b_1,b_2/C) < d(a,b_2) \\
\leq d_{\text{max}}(a,b_1/C) \oplus d_{\text{min}}(b_1,b_2/C) \\
\leq 2d(a,b_1) \oplus d_{\text{min}}(b_1,b_2/C),
\]

which, since \( d(a,b_1) < \frac{1}{2}d(b_1,b_2) \leq d_{\text{min}}(b_1,b_2/C) \), contradicts (ii)*.

So we have \( d(a,b_1) < d_{\text{max}}(a,b_1/C) \). Moreover, \( d_{\text{max}}(a,b_1/C) \leq d_{\text{max}}(b_1,b_1/C) \oplus d(a,b_1) \) by Lemma 4.6. It follows that \( d_{\text{max}}(a,b_1/C) \leq d_{\text{max}}(b_1,b_1/C) \), and so \( d(a,b_1) < d_{\text{max}}(b_1,b_1/C) \). But then \( d(a,b_1) \oplus d(a,b_1) < d_{\text{max}}(b_1,b_1/C) \), which contradicts \( A \downarrow d_{\text{max}}^* B \).

The previous result uses an algebraic condition on \( R \) to isolate when \( \downarrow f \) "reduces" to \( \downarrow d_{\text{max}} \), in the sense that \( d_{\text{min}} \) can be omitted from the characterization of \( \downarrow f \). It is worth observing that this already indicates good model theoretic behavior, since \( d_{\text{max}} \) is a much more natural operation than \( d_{\text{min}} \). Our next result shows that this same algebraic condition on \( R \) yields a relationship between \( \downarrow f \) and \( \downarrow \text{dist} \).

Lemma 4.8. Suppose \( R \) is a Urysohn monoid and, for all \( r,s \in R \), if \( r \leq s \) then \( r \oplus r \oplus s = r \oplus s \). Then \( \downarrow \text{dist} \) implies \( \downarrow f \).

Proof. Suppose \( A \downarrow \text{dist}^* B \). By Proposition 4.7, it suffices to show \( \downarrow \text{dist} \) implies \( \downarrow d_{\text{max}} \). So we fix \( a \in A \) and \( b_1,b_2 \in B \) and show \( d_{\text{max}}(b_1,b_2/C) \leq d(a,b_1) \oplus d(a,b_2) \).

Without loss of generality, assume \( d(a,b_1) \leq d(a,b_2) \). Since \( A \downarrow \text{dist}^* B \), we have \( d(a,C) \leq d(a,b_1) \), which means \( d_{\text{max}}(a,a/C) \leq 2d(a,b_1) \). As in the proof of Proposition 4.7, if \( \alpha, \beta \in R^* \) then \( \alpha \leq \beta \) implies \( \alpha \oplus \alpha \oplus \beta = \alpha \oplus \beta \). Combining these observations with Lemma 4.6, we have

\[
d_{\text{max}}(b_1,b_2/C) \leq d_{\text{max}}(a,b_1/C) \oplus d(a,b_2) \\
\leq d_{\text{max}}(a,a/C) \oplus d(a,b_1) \oplus d(a,b_2) \\
\leq 3d(a,b_1) \oplus d(a,b_2) \\
= d(a,b_1) \oplus d(a,b_2). \]

We can now give the characterization of simplicity for \( U_R \). The reader should compare the statement of this result to the statement of Theorem 4.4.

Theorem 4.9. Given a Urysohn monoid \( R \), the following are equivalent.

(i) \( \text{Th}(U_R) \) is simple.

(ii) \( \downarrow \text{dist} \) implies \( \downarrow f \).

(iii) \( \downarrow f \) coincides with \( \downarrow d_{\text{max}} \).

(iv) For all \( r,s \in R \) if \( r \leq s \) then \( r \oplus r \oplus s = r \oplus s \).

Proof. (i) \( \Rightarrow \) (iv): Suppose (iv) fails, and fix \( r,s \in R \) such that \( r \leq s \) and \( r \oplus s < 2r \oplus s \). Define the space \((X,d)\) such that \( X = \{a,b_1,b_2,c\} \) and

\[
d(a,b_1) = d(a,c) = r, \quad d(b_1,c) = 2r, \\
d(a,b_2) = d(b_2,c) = s, \quad d(b_1,b_2) = r \oplus s.
\]
Let $B = \{ b_1, b_2 \}$. Then we have $d(a, b_1) \oplus d(a, b_2) = r \oplus s < 2r \oplus s = d(b_1, c) \oplus d(b_2, c)$, and so $a \nleq \ell c B$ by Theorem 3.2. On the other hand $d_{\text{max}}(a, a/Bc) = 2r = d(a, c) \oplus d(a, c)$, and so $B \nleq \ell c a$. Therefore $\text{Th}(\mathcal{U}_R)$ is not simple by Fact 2.2(a).

$(iv) \Rightarrow (ii)$: By Lemma 1.8.

$(ii) \Rightarrow (i)$: If $(ii)$ holds then, by Proposition 3.4(b)(iv), $\nleq f$ satisfies local character. Therefore $\text{Th}(\mathcal{U}_R)$ is simple by Fact 2.2(a).

$(iii) \Leftrightarrow (iv)$: By Proposition 1.7.

Combining previous results, we have the following picture of how the four ternary relations $\nleq f$, $\leq d_{\text{max}}$, $\leq \odot$, and $\leq \text{dist}$ interact in $\text{Th}(\mathcal{U}_R)$. (Arrows of the form “$\Rightarrow$” indicate the implication cannot be reversed; no arrow indicates no implication in either direction.)

\[
\begin{align*}
\text{Th}(\mathcal{U}_R) & \text{ is stable:} & \leq \odot & \Leftrightarrow \leq \text{dist} & \Leftrightarrow \leq f & \Leftrightarrow \leq d_{\text{max}} \\
\text{Th}(\mathcal{U}_R) & \text{ is simple and unstable:} & \leq \odot & \Rightarrow \leq \text{dist} & \Rightarrow \leq f & \Rightarrow \leq d_{\text{max}} \\
\text{Th}(\mathcal{U}_R) & \text{ is not simple:} & \leq \odot & \Rightarrow \leq f & \Rightarrow \leq d_{\text{max}} & \Rightarrow \leq \text{dist}
\end{align*}
\]

The final result of this section is motivated by the distance monoid $\mathcal{R}_n = (\{0, 1, \ldots, n\}, +, n, \leq, 0)$ in the case when $n = 1, 2$ (see Example 1.1(3)). Recall that $\mathcal{U}_R$ can be viewed as the countable random graph. Moreover, $\mathcal{U}_{R_2}$ is simply a countably infinite complete graph, and therefore its theory is interdefinable with the theory of infinite sets in the empty language. $\text{Th}(\mathcal{U}_{R_1})$ and $\text{Th}(\mathcal{U}_{R_2})$ are both classical examples in which nonforking is as uncomplicated as possible. In particular, $A \nleq f C B$ if and only if $A \cap B \subseteq C$ (see e.g. [23] Exercise 7.3.14). We generalize this behavior as follows.

**Definition 4.10.** A distance monoid $\mathcal{R}$ is **metrically trivial** if $r \oplus s = \sup R$ for all $r, s \in R^{>0}$.

The following properties of metrically trivial monoids are easy to verify.

**Proposition 4.11.** Let $\mathcal{R}$ be a countable distance monoid.

(a) $\mathcal{R}$ is metrically trivial if and only if $r \leq s \oplus t$ for all $r, s, t \in R^{>0}$.

(b) If $\mathcal{R}$ is metrically trivial then $\mathcal{R}^*$ is metrically trivial.

(c) If $\mathcal{R}$ is metrically trivial then $\mathcal{R}$ is a Urysohn monoid.

In particular, property $(a)$ says that $\mathcal{R}$ is metrically trivial if and only if $\mathcal{R}$-metric spaces coincide with graphs whose edges are arbitrarily colored by nontrivial elements of $R$. Therefore $\text{Th}(\mathcal{U}_R)$ is, roughly speaking, the theory of a randomly colored graph, with color set $R^{>0}$.

**Theorem 4.12.** Given a Urysohn monoid $\mathcal{R}$, the following are equivalent.

(i) $\mathcal{R}$ is metrically trivial.

(ii) For all $A, B, C \subseteq \mathcal{U}_R$, $A \nleq f C B$ if and only if $A \cap B \subseteq C$.

Proof. $(i) \Rightarrow (ii)$. Suppose $\mathcal{R}$ is metrically trivial, and fix $A, B, C \subseteq \mathcal{U}_R$. If $A \nleq f C B$ then $A \cap B \subseteq C$ (this is true in any theory). So suppose $A \nleq f C B$. Note that metrically trivial monoids clearly satisfy condition (iv) of Theorem 4.9, and so we must have $A \nleq d_{\text{max}} C B$. Fix $a \in A$, $b_1, b_2 \in B$ such that $d(a, b_1) \oplus d(a, b_2) < d_{\text{max}}(b_1, b_2/C)$.

15
Since $\mathcal{R}^*$ is metrically trivial, we may assume, without loss of generality, that $d(a, b_1) = 0$, i.e. $a = b_1$. We then have $d(b_1, b_2) < d_{\text{max}}(b_1, b_2/C)$ which, in particular, means $b_1 \notin C$. Altogether, $b_1 \in (A \cap B) \setminus C$.

$(ii) \Rightarrow (i)$: Suppose, $\mathcal{R}$ is not metrically trivial. Then there is $r \in R^{>0}$ such that $r \nmid r < \text{sup} R$. Fix $a, b \in U_R$ such that $d(a, b) = r$. Then $\{a\} \cap \{b\} = \emptyset$. On the other hand, $d(a, b) \nmid d(a, b) < d_{\text{max}}(b, b/\emptyset)$, and so $a \nmid f b$.

Note that, up to isomorphism, there is a unique nontrivial, ultrametric, and metrically trivial distance monoid, namely, $R_1$. Therefore, all other metrically trivial monoids yield simple unstable Urysohn spaces. However, there is evidence to suggest that, in a quantifiable sense, these monoids form a negligible portion of the simple unstable case. See Remark 6.16

4.3 Non-axiomatizable Properties

Summarizing previous results, we have shown that the following properties (and thus all of their equivalent formulations) are each finitely axiomatizable as properties of $\text{RUS}$.

1. $\text{Th}(U_R)$ is stable.

2. $\text{Th}(U_R)$ is simple.

3. Nonforking in $\text{Th}(U_R)$ coincides with equality.

In this section, we show that supersimplicity and superstability are characterized as properties of $\mathcal{R}$, but not in an axiomatizable way. We will need the following technical observations about $\mathcal{R}^*$, which easily follow from its construction in [9, Section 3].

Proposition 4.13. Suppose $\mathcal{R}$ is a distance monoid.

(a) If $X \subseteq R^*$ is nonempty and $\inf X \in R \cup \{\text{sup} R^*\}$, then $\inf X \in X$.

(b) $\mathcal{R}$ is well-ordered if and only if $R^* = R \cup \{\text{sup} R^*\}$.

Theorem 4.14. If $\mathcal{R}$ is a Urysohn monoid, and $\text{Th}(U_R)$ is simple, then the following are equivalent.

(i) $\text{Th}(U_R)$ is supersimple.

(ii) For all $\alpha \in R^* \setminus R$, if $\alpha < \text{sup} R^*$ then $\alpha < \alpha \nmid \alpha$.

Proof. $(i) \Rightarrow (ii)$. Suppose $(ii)$ fails. Fix $\alpha \in R^* \setminus R$ such that $\alpha < \text{sup} R^*$ and $\alpha \nmid \alpha = \alpha$. Let $X_\alpha = \{u \in R : \alpha < u\}$. Then $\alpha = \inf X_\alpha$ by Theorem 2.11$(b)(iii)$. We define a space $\{a\} \cup \{b_u : u \in X_\alpha\}$ such that $d(a, b_u) = u$ and, for $u \neq v$, $d(b_u, b_v) = \max\{u, v\}$. It is easy to check that this space is an $\mathcal{R}^*$-metric space, and therefore we may assume it is a subspace of $U_R$.

Let $B = \{b_u : u \in X_\alpha\}$. To show that $\text{Th}(U_R)$ is not supersimple, it suffices to show that $a \nmid f C$ for all finite $C \subseteq B$. Fix $C \subseteq B$ finite. Let $t = \min\{u \in X_\alpha : b_u \in C\}$. If $t \nmid t \leq s_1 \nmid s_2$ for all $s_1, s_2 \in X_\alpha$, then $t \nmid t \leq \alpha \nmid \alpha = \alpha < t$, which is a contradiction. Therefore, there are $s_1, s_2 \in X_\alpha$ such that $s_1 \nmid s_2 < t \nmid t$. Next, for any $u \geq t$, we have

\[ t \nmid t \leq u \nmid u = d(b_{s_1}, b_u) \nmid d(b_{s_2}, b_u), \]

and so $d_{\text{max}}(b_{s_1}, b_{s_2}/C) \geq t \nmid t$. Altogether, we have $b_{s_1}, b_{s_2} \in B$ and

\[ d(a, b_{s_1}) \nmid d(a, b_{s_2}) < d_{\text{max}}(b_{s_1}, b_{s_2}/C), \]

16
and so a $\downarrow^f B$, as desired.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Since $T_R$ is simple, it suffices by Theorem [4.9(iii)] to fix $a \in U_R$ and $B \subseteq U_R$ and find some finite $C \subseteq B$ such that $\bar{a} \downarrow^d C$. Let $a = (a_1, \ldots, a_n)$ and fix $a_i$. We claim that there is some $b_i \in B$ such that $d(a_i, b_i) \leq 2d(a_i, B)$. If $d(a_i, B) \not\in R$ and $d(a_i, B) < \sup R^*$ then this follows from (ii). Otherwise, by Proposition [4.13(a)], we can in fact find $b_i \in B$ such that $d(a_i, b_i) = d(a_i, B)$.

Let $C = \{b_1, \ldots, b_n\}$. To show $\bar{a} \downarrow^d C$, we fix $a_i$ and $b, b' \in B$, and show

$$d_{\max}(b, b'/C) \leq d(a_i, b) \oplus d(a_i, b').$$

Without loss of generality, assume $d(a_i, b) \leq d(a_i, b')$, and note that $d(a_i, b_i) \leq 2d(a_i, b)$ by choice of $b_i$. Using Theorem [4.9(iv)], we have

$$d_{\max}(b, b'/C) \leq d(b, b_i) \oplus d(b_i, b') \leq d(a_i, b_i) \oplus d(a_i, b_i) \oplus d(a_i, b) \oplus d(a_i, b') \leq 5d(a_i, b) \oplus d(a_i, b') = d(a_i, b) \oplus d(a_i, b').$$

Remark 4.15. From the previous theorem and Proposition [4.13(c)], we see that if $\text{Th}(U_R)$ is simple and $R$ is well-ordered then $\text{Th}(U_R)$ is supersimple. For finite $R$, this conclusion also follows from a more general result of Koponen [17].

The previous characterization of supersimplicity is not as natural as some of our other results. However, it can be used to give a very natural characterization of superstability.

Theorem 4.16. Suppose $R$ is a Urysohn monoid and $\text{Th}(U_R)$ is stable. The following are equivalent.

(i) $\text{Th}(U_R)$ is $\omega$-stable.

(ii) $\text{Th}(U_R)$ is superstable.

(iii) $R$ is well-ordered.

Proof. (ii) $\Leftrightarrow$ (iii): Since $\text{Th}(U_R)$ is stable, it follows from Theorem [4.4(iv)], Theorem [4.14] and Proposition [4.13(b)] that $\text{Th}(U_R)$ is supersimple if and only if $R$ is well-ordered. Therefore the result follows since supersimplicity and superstability coincide for stable theories.

(i) $\Rightarrow$ (ii): This is true for any theory.

(iii) $\Rightarrow$ (i): Suppose $R$ is well-ordered. Consider $\text{Th}(U_R)$ as the theory of infinitely refining equivalence relations $d(x, y) \leq r$, indexed by $(R, \leq, 0)$. It is also common to refer to this situation as “expanding equivalence relations”. This example is well-known in the folklore to be $\omega$-stable. The case that $R = (N, \leq, 0)$ is credited to Shelah (see e.g. [15]).

Corollary 4.17. Supersimplicity and superstability are not axiomatizable properties of $\text{RUS}$. 

Proof. Since “superstable” is equivalent to “stable and supersimple”, and stability is finitely axiomatizable, it is enough to show that superstability is not axiomatizable. Suppose, toward a contradiction, that there is an $L_{\omega_1, \omega}$-sentence $\varphi$ in $L_{om}$ such that, for any Urysohn monoid $R$, $\text{Th}(U_R)$ is superstable if and only if $R \models \varphi$. After adding constants $(c_i)_{i<\omega}$ to $L_{om}$, and conjuncting with $\varphi_{QE}$ along with a sentence axiomatizing distance monoids with universe $(c_i)_{i<\omega}$, we obtain an $L_{\omega_1, \omega}$-sentence $\varphi^*$ in $L_{om}$ such that, for any $L_{om}$-structure $R$, $R \models \varphi^*$ if and only if $R$ is a
countable, ultrametric, well-ordered, distance monoid. By classical results in infinitary logic (see e.g. [18 Corollary 4.28]), it follows that there is some $\mu < \omega_1$ such that any model of $\varphi^*$ has order type at most $\mu$. This is clearly a contradiction, since any ordinal can be given the structure of an ultrametric distance monoid (c.f. Example [1.1(6)]).

5 Cyclic Indiscernible Sequences

So far our results have been motivated by choosing a particular kind of good behavior for $\text{Th}(\mathcal{U}_R)$ and then characterizing when this behavior happens. In this section, we give a uniform upper bound for the complexity of $\text{Th}(\mathcal{U}_R)$ for any Urysohn monoid $\mathcal{R}$. In particular, we will show that if $\mathcal{R}$ is a Urysohn monoid then $\text{Th}(\mathcal{U}_R)$ does not have the finitary strong order property. We will accomplish this by proving the following theorem.

**Theorem 5.1.** Suppose $\mathcal{R}$ is a Urysohn monoid and $I = (\bar{a}^i)_{i<\omega}$ is an indiscernible sequence in $\mathcal{U}_R$ of tuples of possibly infinite length. If $|NP(I)| = n < \omega$ then $I$ is $(n+1)$-cyclic.

From this and Fact 2.7, we obtain the following corollary.

**Corollary 5.2.** If $\mathcal{R}$ is a Urysohn monoid then $\text{Th}(\mathcal{U}_R)$ does not have FSOP.

The proof of Theorem 5.1 closely follows the strategy of [10 Section 3.1], in which it is shown that the complete Urysohn sphere in continuous logic does not have FFSOP.

For the rest of the section, we fix a Urysohn monoid $\mathcal{R}$. The key tool we use to prove Theorem 5.1 is the following test for when an indiscernible sequence in $\mathcal{U}_R$ is $n$-cyclic.

**Lemma 5.3.** Suppose $\mathcal{R}$ is a Urysohn monoid and $(\bar{a}^i)_{i<\omega}$ is an indiscernible sequence in $\mathcal{U}_R$. Given $i, j \in \ell(\bar{a}^0)$, set $e_{i,j} = d(a^0_i, a^0_j)$. Given $n \geq 2$, $(\bar{a}^i)_{i<\omega}$ is $n$-cyclic if and only if, for all $i_1, \ldots, i_n \in \ell(\bar{a}^0)$, $e_{i_n, i_1} \leq e_{i_1,i_2} + e_{i_2,i_3} + \ldots + e_{i_{n-1},i_n}$.

**Proof.** After some minor attention to detail, this can be proved via a direct generalization of [10 Lemma 3.7]. We sketch the setup.

Fix an indiscernible sequence $I = (\bar{a}^i)_{i<\omega}$ and some $n \geq 2$. We let $p(\bar{x}, \bar{y}) = tp(\bar{a}^0, \bar{a}^1)$ and set

$q(\bar{x}^1, \ldots, \bar{x}^n) = p(\bar{x}^1, \bar{x}^2) \cup p(\bar{x}^2, \bar{x}^3) \cup \ldots \cup p(\bar{x}^{n-1}, \bar{x}^n) \cup p(\bar{x}^n, \bar{x}^1)$.

Then $I$ is $n$-cyclic if and only if $q$ is consistent. Let $X = \bar{x}^1 \cup \ldots \cup \bar{x}^n$. Note that, by quantifier elimination, $q$ is determined by a partial symmetric function $f : X \times X \rightarrow R^*$, which we define as follows:

(i) $\text{dom}(f)$ is the symmetric closure of

$$\{(x^l_i, x^m_j) : i, j \in \ell(\bar{a}^0), l, m < \omega, \text{ and } m \in \{l, l+1\} \text{ or } (l, m) = (1, n)\};$$

(ii) given $(x^l_i, x^m_j) \in \text{dom}(f)$, we set $f(x^l_i, x^m_j) = d(a^0_l, a^0_j)$ if $(l, m) \notin \{(1, n), (n, 1)\}$ and $f(x^l_i, x^m_j) = d(a^0_l, a^0_j)$ if $(l, m) = (1, n)$.

Altogether, by Proposition 2.13 $q$ is consistent if and only if $f$ can be extended to an $\mathcal{R}^*$-pseudometric on $X$.

Claim: $f$ can be extended to an $\mathcal{R}^*$-pseudometric on $X$ if and only if, for all $m > 0$, $f$ is $m$-transitive, i.e., for all $z_0, z_1, \ldots, z_m \in X$, if $(z_0, z_m) \in \text{dom}(f)$ and $(z_i, z_{i+1}) \in \text{dom}(f)$ for all $0 \leq i < m$, then

$$f(z_0, z_m) \leq f(\bar{z}) := f(z_0, z_1) \oplus f(z_1, z_2) \oplus \ldots \oplus f(z_{m-1}, z_m).$$
Proof: The forward direction is trivial from the triangle inequality. For the reverse direction, we define an $R^s$-pseudometric $d_f$ on $X$ by setting
\[
d_f(x, y) = \inf \{ f(\bar{z}) : m > 0, \bar{z} = (z_0, \ldots, z_m), \ z_0 = x, \ z_m = y, \ (z_i, z_{i+1}) \in \text{dom}(f) \}.
\]
Using Proposition 2.16(a), it is straightforward to show that $d_f$ satisfies the triangle inequality. Moreover, $d_f$ extends $f$ since $f$ is $m$-transitive for all $m > 0$.

Altogether, we have that $I$ is $n$-cyclic if and only if $f$ is $m$-transitive for all $m > 0$. The rest of the argument now follows exactly as in [10, Lemma 3.7].

The final tools needed for Theorem 5.1 are the following observations concerning transitivity properties of indiscernible sequences.

**Lemma 5.4.** Suppose $I = (\bar{a}^i)_{i < \omega}$ is an indiscernible sequence in $U_R$. Given $i, j \in \ell(\bar{a}^0)$, set $\epsilon_{i,j} = d(a^0_i, a^1_j)$. Fix $n \geq 2$ and $i_1, \ldots, i_n \in \ell(\bar{a}^0)$.

(a) $\epsilon_{i_1,i_n} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

(b) If $i_s = i_t$ for some $1 \leq s < t \leq n$, then $\epsilon_{i_1,i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

(c) If $i_s \not\in \text{NP}(I)$ for some $1 \leq s \leq n$, then $\epsilon_{i_1,i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

**Proof.** For parts (a) and (b), see [10, Lemma 3.5].

Part (c). First, if $i_s \not\in \text{NP}(I)$ then $a^0_{i_s} = a^2_{i_s}$. Therefore, for any $j \in \ell(\bar{a}^0)$, we have
\[
\epsilon_{i_s,j} = d(a^0_{i_s}, a^1_j) = d(a^2_{i_s}, a^1_j) = d(a^0_{i_s}, a^1_{i_s}) = \epsilon_{j,i_s}.
\]
So if $s = 1$ or $s = n$ then the result follows immediately from part (a). Suppose $1 < s < n$. Then, using part (a), we have $\epsilon_{i_1,i_1} \leq \epsilon_{i_1,i_s} \oplus \epsilon_{i_s,i_1} = \epsilon_{i_1,i_s} \oplus \epsilon_{i_s,i_n} = \epsilon_{i_1,i_s} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

We can now prove the main result of this section.

**Proof of Theorem 5.1.** Let $R$ be a Urysohn monoid and fix an indiscernible sequence $I$ in $U_R$, with $|\text{NP}(I)| = n < \omega$. We want to show $I$ is $(n+1)$-cyclic. We may assume $n \geq 1$ and so, by Lemma 5.3, it suffices to fix $i_1, \ldots, i_{n+1} \in \ell(\bar{a}^0)$ and show $\epsilon_{i_1,i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n+1},i_{n+1}}$. By Lemma 5.4(c), we may assume $i_s \in \text{NP}(I)$ for all $1 \leq s \leq n+1$. Therefore, there are $1 \leq s < t \leq n+1$ such that $i_s = i_t$, and so the result follows from Lemma 5.4(b).\[\square\]

6 Strong Order Rank

Suppose $R$ is a Urysohn monoid. Summarizing our previous results, we have shown that $\text{Th}(U_R)$ is NFSOP and, moreover, stability and simplicity are both possible for $\text{Th}(U_R)$. In this section, we address the region of NSOP theories, which are not simple. In general, this region is stratified by Shelah’s SOP$_n$-hierarchy, which we have formulated as strong order rank (see Definition 2.5).

6.1 Calculating the rank

First, we observe that the results of Sections 4.1 and 4.2 can be restated as follows:

(i) $\text{Th}(U_R)$ is stable if and only if for all $r, s \in R$, if $r \leq s$ then $r \oplus s = s$.

(ii) $\text{Th}(U_R)$ is simple if and only if for all $r, s, t \in R$, if $r \leq s \leq t$ then $r \oplus s \oplus t = s \oplus t$.\[19\]
This motivates the following definition.

**Definition 6.1.** Let \( \mathcal{R} \) be a distance monoid. The **archimedean rank of** \( \mathcal{R} \), denoted \( \text{arch}(\mathcal{R}) \), is the minimum \( n < \omega \) such that, for all \( r_0, r_1, \ldots, r_n \in R \), if \( r_0 \leq r_1 \leq \ldots \leq r_n \) then

\[
r_0 \oplus r_1 \oplus \ldots \oplus r_n = r_1 \oplus \ldots \oplus r_n.
\]

If no such \( n \) exists, set \( \text{arch}(\mathcal{R}) = \omega \).

We have shown that, for Urysohn monoids \( \mathcal{R} \), \( \text{Th}(\mathcal{U}_\mathcal{R}) \) is stable if and only if \( \text{arch}(\mathcal{R}) \leq 1 \) and \( \text{Th}(\mathcal{U}_\mathcal{R}) \) is simple if and only if \( \text{arch}(\mathcal{R}) \leq 2 \). In particular, since stability (for general theories) is equivalent to strong order rank at most 1, we have \( \text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) \leq 1 \) if and only if \( \text{arch}(\mathcal{R}) \leq 1 \).

The goal of this section is to extend this result, and show \( \text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) = \text{arch}(\mathcal{R}) \) for any Urysohn monoid \( \mathcal{R} \). We begin by refining previous results on cyclic indiscernible sequences. Throughout the section, we fix a Urysohn monoid \( \mathcal{R} \).

**Definition 6.2.** Fix \( n \geq 2 \) and \( \alpha_1, \ldots, \alpha_n \in R^* \). Let \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \).

1. \( \bar{\alpha} \) is **diagonally indiscernible** if there is an indiscernible sequence \( (\bar{a}^t)_t < \omega \) in \( \mathcal{U}_\mathcal{R} \) such that
   
   \( \ell(\bar{a}^0) = n; \)
   
   \( \text{for all } 1 \leq t < n, \ d(\bar{a}^t_0, \bar{a}^t_1) = \alpha_t; \)
   
   \( d(\bar{a}^n_1, \bar{a}^n_0) = \alpha_n. \)

2. \( \bar{\alpha} \) is **transitive** if \( \alpha_n \leq \alpha_1 \oplus \ldots \oplus \alpha_{n-1} \).

**Proposition 6.3.** Given \( n > 1 \), the following are equivalent.

\( (i) \) \( \ell(\bar{a}^0) = n; \)

\( (ii) \) **Every infinite indiscernible sequence in** \( \mathcal{U}_\mathcal{R} \) **is** \( n \)-cyclic.

\( (iii) \) **Every diagonally indiscernible sequence of length** \( n \) **in** \( \mathcal{R}^* \) **is transitive.**

**Proof.** Recall that \( (i) \) and \( (ii) \) are equivalent in any theory by Fact 2.7 \( a \). Therefore, we only need to show \( (ii) \) and \( (iii) \) are equivalent.

\( (ii) \Rightarrow (iii) \): Fix a diagonally indiscernible sequence \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \) in \( \mathcal{R}^* \), witnessed by an indiscernible sequence \( (\bar{a}^t)_t < \omega \) in \( \mathcal{U}_\mathcal{R} \). By \( (ii) \), \( (\bar{a}^t)_t < \omega \) is \( n \)-cyclic and so there is some \( (c^1, \ldots, c^n) \) such that \( (c^t, c^{t+1}) \equiv (\bar{a}^0, \bar{a}^1) \equiv (\bar{c}^0, \bar{c}^1) \) for all \( 1 \leq t < n \). In particular,

\[
\alpha_n = d(\bar{a}^n_1, \bar{a}^n_0) = d(c^1_1, c^n_1) \leq d(c^1_1, c^2_1) \oplus \ldots \oplus d(c^n_{n-1}, c^n_n) = \alpha_1 \oplus \ldots \oplus \alpha_{n-1}.
\]

Therefore \( \bar{\alpha} \) is transitive.
(iii) ⇒ (ii): Suppose there is an indiscernible sequence $\mathcal{I} = (\bar{a}^l)_{l<\omega}$ in $\mathbb{U}_R$, which is not $n$-cyclic. Given $i, j \in \ell(\bar{a}^0)$, let $\epsilon_{i,j} = d(a_i^0, a_j^0)$. By Lemma 5.3, there are $i_1, \ldots, i_n \in \ell(\bar{a}^0)$ such that

$$\epsilon_{i_n,i_1} > \epsilon_{i_1,i_2} \oplus \cdots \oplus \epsilon_{i_{n-1},i_n}. $$

By Lemma 5.4(b), it follows that the map $t \mapsto i_t$ is injective. Given $l < \omega$, define $\bar{b}^l = (a_{i_1}, \ldots, a_{i_n})$. Then $\ell(\bar{b}^0) = n$ and $\mathcal{J} = (\bar{b}^l)_{l<\omega}$ is an indiscernible sequence. Let $\alpha_n = \epsilon_{i_n,i_1}$ and, given $1 \leq t < n$, let $\alpha_t = \epsilon_{i_t,i_{t+1}}$. Then, for any $t < n$, we have $d(\bar{b}^0,\bar{b}^1_{i_{t+1}}) = d(a_{i_t}^0, a_{i_{t+1}}^0) = \alpha_t$. Moreover, $d(\bar{b}^0,\bar{b}^1_{i_1}) = d(a_{i_1}^0, a_{i_1}^0) = \alpha_n$. Therefore $\mathcal{J}$ witnesses that $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a non-transitive diagonally indiscernible sequence.

Next, we prove two technical lemmas.

**Lemma 6.4.** Suppose $n > 1$ and $(\alpha_1, \ldots, \alpha_n)$ is a diagonally indiscernible sequence in $\mathbb{R}^*$. Then, for any $1 \leq i < n$, we have $\alpha_n \leq \alpha_1 \oplus \cdots \oplus \alpha_{n-1} \oplus 2\alpha_i$.

*Proof.* Let $(\bar{a}^l)_{l<\omega}$ be an indiscernible sequence in $\mathbb{U}_R$, which witnesses that $(\alpha_1, \ldots, \alpha_n)$ is diagonally indiscernible. Given $1 \leq i, j \leq n$, let $\epsilon_{i,j} = d(a_i^0, a_j^0)$. Note that, if $1 \leq i < n$ then $\epsilon_{i,i+1} = \alpha_i$ and, moreover,

$$\epsilon_{i+1,i+1} = d(a_{i+1}^0, a_{i+1}^0) \leq d(a_{i+1}^1, a_i^0) \oplus d(a_i^0, a_{i+1}^2) = 2\alpha_i. $$

If $i < n - 1$ then, using Lemma 5.4(a), we have

$$\alpha_n = d(a_1^2, a_n^1) \leq d(a_1^2, a_{i+1}^3) \oplus d(a_{i+1}^3, a_i^0) \oplus d(a_i^0, a_{i+1}^1) \leq \alpha_1 \oplus \cdots \oplus \alpha_{n-1} \oplus 2\alpha_i. $$

On the other hand, if $i = n - 1$ then, using Lemma 5.4(a), we have

$$\alpha_n = d(a_1^1, a_n^0) \leq d(a_1^1, a_n^2) \oplus d(a_n^2, a_1^0) = \epsilon_{1,n} \oplus \epsilon_{n,n} \leq \alpha_1 \oplus \cdots \oplus \alpha_{n-1} \oplus 2\alpha_{n-1}. $$

**Lemma 6.5.** Suppose $n \geq 2$ and fix $r_1, \ldots, r_n \in R$ such that $r_1 \leq r_2 \leq \cdots \leq r_n$. Then

$$(r_2, \ldots, r_n, \alpha_1 \oplus r_2 \oplus \cdots \oplus r_n)$$

is a diagonally indiscernible sequence.

*Proof.* Fix $r_1, \ldots, r_n \in R$, with $r_1 \leq r_2 \leq \cdots \leq r_n$. Define the sequence $(\bar{a}^l)_{l<\omega}$, such that $\ell(\bar{a}^0) = n$ and, given $k \leq l < \omega$ and $1 \leq i, j \leq n$,

$$d(a_i^k, a_j^l) = \begin{cases} r_j \oplus r_{j+1} \oplus \cdots \oplus r_i & \text{if } k < l \text{ and } i \geq j, \text{ or } k = l \text{ and } i > j \\ r_{i+1} \oplus r_{i+2} \oplus \cdots \oplus r_j & \text{if } k < l \text{ and } i < j. \end{cases} $$

Given $1 \leq i < n$, we have $d(a_i^0, a_{i+1}^1) = r_i$ and $d(a_n^0, a_1^1) = r_1 \oplus r_2 \oplus \cdots \oplus r_n$. Therefore, it suffices to verify that this sequence satisfies the triangle inequality. This verification follow from a tedious, but routine, case analysis, which crucially depends on the assumption that $r_1 \leq r_2 \leq \cdots \leq r_n$. □

We now have all of the pieces necessary to prove the main result of this section.

**Theorem 6.6.** If $\mathcal{R}$ is a Urysohn monoid then $\text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) = \text{arch}(\mathcal{R})$. 21
Proof. First, note that if SO(Th(USR)) \(> n\) and arch(\(R\)) \(> n\) for all \(n < \omega\) then, by Corollary 5.2 and our conventions, we have SO(Th(USR)) = \(\omega = \text{arch}(R)\).

Therefore, it suffices to fix \(n \geq 1\) and show SO(Th(USR)) \(\geq n\) if and only if arch(\(R\)) \(\geq n\). Note that arch(\(R\)) = 0 if and only if \(R\) is the trivial monoid, in which case \(\cup R\) is a single point. Conversely, if \(R\) is nontrivial then \(\cup R\) is clearly infinite. So arch(\(R\)) < 1 if and only if Th(USR) has finite models, which is equivalent to SO(Th(USR)) < 1. So we may assume \(n \geq 2\).

Suppose arch(\(R\)) \(\geq n\). Then there are \(r_1, \ldots, r_n \in R\) such that \(r_1 \leq r_2 \leq \ldots \leq r_n\) and \(r_2 \oplus \ldots \oplus r_n < r_1 \oplus r_2 \oplus \ldots \oplus r_n\). By Lemma 6.5 \((r_2, \ldots, r_n, r_1 \oplus \ldots \oplus r_n)\) is a non-transitive diagonally indiscernible sequence of length \(n\). Therefore SO(Th(USR)) \(\geq n\) by Proposition 6.3.

Finally, suppose SO(\(R\)) \(\geq n\). By Proposition 6.3 there is a non-transitive diagonally indiscernible sequence \((\alpha_1, \ldots, \alpha_n)\) in \(R^*\). Let \(\beta_1, \ldots, \beta_{n-1}\) be an enumeration of \(\alpha_1, \ldots, \alpha_{n-1}\), with \(\beta_1 \leq \ldots \leq \beta_{n-1}\). Then, using Lemma 6.4 we have

\[
\beta_1 \oplus \ldots \oplus \beta_{n-1} < \alpha_n \leq 2\beta_1 \oplus \beta_1 \oplus \ldots \oplus \beta_{n-1},
\]

which implies \(\beta_1 \oplus \ldots \oplus \beta_{n-1} < \beta_1 \oplus \beta_1 \oplus \ldots \oplus \beta_{n-1}\). By Theorem 2.11(b)(iii), we may fix \(r_1, \ldots, r_{n-1} \in R\) such that \(\alpha_1 \leq r_i\) and

\[
r_1 \oplus \ldots \oplus r_{n-1} < \beta_1 \oplus \beta_1 \oplus \ldots \oplus \beta_n \leq r_1 \oplus r_1 \oplus \ldots \oplus r_n.
\]

By Theorem 2.11(b)(ii), we may assume \(r_1 \leq \ldots \leq r_{n-1}\). Therefore, arch(\(R\)) \(\geq n\).

Note that, as archimedean rank is clearly a first-order property of distance monoids, we have that, for all \(n < \omega\), “SO(Th(USR)) = \(n\)” is a finitely axiomatizable property of RUS.

6.2 Further Remarks on Simplicity

Recall that Section 4.2 resulted in the equivalence: Th(USR) is simple if and only if arch(\(R\)) \(< 2\). Therefore, combined with Theorem 6.6 we have the following corollary.

Corollary 6.7. If \(R\) is a Urysohn monoid, and Th(USR) is not simple, then Th(USR) has SOP_3.

In general, non-simple theories without SOP_3 are scarce. Indeed, there are essentially only three known examples, which are all described in [16]. A similar phenomenon in model theoretic dividing lines is related to the question of non-simple theories, which have neither TP_2 nor the strict order property. In particular, there are no known examples of such theories. Since we have shown that Th(USR) never has the strict order property, it is worth proving that any non-simple Th(USR) has TP_2. We refer the reader to [8] for the definition of TP_2, and recall the fact that simple theories do not have TP_2.

Theorem 6.8. If \(R\) is a Urysohn monoid, and Th(USR) is not simple, then Th(USR) has TP_2.

Proof. Suppose Th(USR) is not simple. By Theorem 4.9 we may fix \(r, s \in R\) such that \(r \leq s\) and \(r \oplus s < 2r \oplus s\). Let \(A = (a^{i,j}_{m,n}, a^{k,l}_n)_{i,j<\omega}\). We define \(d\) on \(A \times A\) such that

\[
d(a^{i,j}_{m,n}, a^{k,l}_n) = \begin{cases} r & \text{if } m = n = 1 \text{ and } (i, j) \neq (k,l) \\ s & \text{if } m = n = 2 \text{ and } (i, j) \neq (k,l) \\ r \oplus s & \text{if } m \neq n, \text{ and } i \neq k \text{ or } j = l \\ 2r \oplus s & \text{if } m \neq n, \text{ and } i = k, \text{ and } j \neq l. \end{cases}
\]
To verify the triangle inequality for $d$, fix a non-degenerate triangle $\{a_{m}^{i,j}, a_{m}^{k,l}, a_{p}^{g,h}\}$ in $A$. Let $\alpha = d(a_{m}^{i,j}, a_{n}^{k,l})$, $\beta = d(a_{m}^{i,j}, a_{p}^{g,h})$, and $\gamma = d(a_{n}^{k,l}, a_{p}^{g,h})$. Without loss of generality, we may assume $m = n$. If $m = p$ then $\alpha = \beta = \gamma$ and so the triangle inequality holds. If $m \neq p$ then $\alpha \in \{r, s\}$ and $\beta, \gamma \in \{r \oplus s, 2r \oplus s\}$, so the triangle inequality holds.

We may assume $A \subseteq \mathbb{U}_{\mathcal{R}}$. Define the formula

$$\varphi(x, y_{1}, y_{2}) := d(x, y_{1}) \leq r \land d(x, y_{2}) \leq s.$$  

We show that $A$ and $\varphi(x, y_{1}, y_{2})$ witness TP$_{2}$ for $\text{Th}(\mathcal{U}_{\mathcal{R}})$.

Fix a function $\sigma : \omega \to \omega$ and, given $n < \omega$ and $i \in \{1, 2\}$, set $b_{i}^{n} = a_{i}^{n, \sigma(n)}$. Let $B = (b_{1}^{n}, b_{2}^{n})_{n < \omega}$. To show that $\{\varphi(x, b_{1}^{n}, b_{2}^{n}) : n < \omega\}$ is consistent, it suffices to show that the function $f : B \to \{r, s\}$, such that $f(b_{1}^{n}) = r$ and $f(b_{2}^{n}) = s$, describes a one-point metric space extension of $B$ (in general, such functions are called Katětov maps, see [19]). In other words, we must verify the inequalities $|f(u) \oplus f(v)| \leq d(u, v) \leq f(u) \oplus f(v)$ for all $u, v \in B$. For this, we have:

- for all $n < \omega$, $|f(b_{1}^{n}) \oplus f(b_{2}^{n})| \leq s$, $f(b_{1}^{n}) \oplus f(b_{2}^{n}) = r \oplus s$ and $d(b_{1}^{n}, b_{2}^{n}) = r \oplus s$;
- for all distinct $m, n < \omega$, $|f(b_{1}^{m}) \oplus f(b_{1}^{n})| = 0$, $f(b_{1}^{m}) \oplus f(b_{1}^{n}) = 2r$, and $d(b_{1}^{m}, b_{1}^{n}) = r$;
- for all distinct $m, n < \omega$, $|f(b_{2}^{m}) \oplus f(b_{2}^{n})| = 0$, $f(b_{2}^{m}) \oplus f(b_{2}^{n}) = 2s$, and $d(b_{2}^{m}, b_{2}^{n}) = s$;
- for all distinct $m, n < \omega$, $|f(b_{1}^{m}) \oplus f(b_{2}^{n})| \leq s$, $f(b_{1}^{m}) \oplus f(b_{2}^{n}) = r \oplus s$, and $d(b_{1}^{m}, b_{2}^{n}) = s$.

Next, we fix $n < \omega$ and $i < j < \omega$ and show that $\varphi(x, a_{1}^{n,i}, a_{2}^{n,j}) \land \varphi(x, a_{1}^{n,i}, a_{2}^{n,j})$ is inconsistent. Indeed, if $c$ realizes this formula then we have

$$d(c, a_{2}^{n,i}) \oplus d(c, a_{1}^{n,j}) \leq r \oplus s < 2r \oplus s = d(a_{2}^{n,i}, a_{1}^{n,j}).$$  

\[\square\]

### 6.3 Examples

In this section, we give tests for calculating the strong order rank of $\text{Th}(\mathcal{U}_{\mathcal{R}})$, when $\mathcal{R}$ is a Urysohn monoid. We also simplify the calculation in the case when $\mathcal{R}$ is archimedean.

**Definition 6.9.** Let $\mathcal{R}$ be a distance monoid.

- 1. Given $\alpha, \beta \in \mathcal{R}^{*} \text{ define } \lfloor \frac{\alpha}{\beta} \rfloor = \inf\{n < \omega : \alpha \leq^{*} n\beta\}, \text{ where, by convention, we let } \inf\emptyset = \omega.$
- 2. Given $t \in \mathcal{R}$, define $[t] = \{x \in \mathcal{R} : x \sim t\}$. Define

$$\text{arch}_{\mathcal{R}}(t) = \sup \left\{ \left\lfloor \frac{r}{s} \right\rfloor : r, s \in [t] \right\},$$

where, by convention, we let $\sup \emptyset = \omega$.
- 3. $\mathcal{R}$ is **archimedean** if $R^{\geq 0} = [t]$ for some $t \in \mathcal{R}$.

We record the following properties.

**Proposition 6.10.** Suppose $\mathcal{R}$ is a distance monoid.

- (a) For any $t \in \mathcal{R}$, $[t]$ is a convex subset of $\mathcal{R}$, which is closed under $\oplus$.
- (b) $\mathcal{R}$ is archimedean if and only if for all $r, s \in R^{\geq 0}$ there is some $n < \omega$ such that $r \leq ns$.  

23
(c) Given $t \in R$, 
\[
\operatorname{arch}_R(t) = \left \lceil \frac{\sup [t]}{\inf [t]} \right \rceil ,
\]
where $\sup [t]$ and $\inf [t]$ are calculated in $R^*$.  

Proof. These are routine to verify, although we note that part (c) uses Proposition 2.16(a). \qed

Proposition 6.11. Suppose $\mathcal{R}$ is a distance monoid.

(a) $\operatorname{arch}(\mathcal{R}) \geq \max \{\operatorname{arch}_R(t) : t \in R\}$.  

(b) If $\mathcal{R}$ is archimedean then, for any $t \in R^{>0}$, 
\[
\operatorname{arch}(\mathcal{R}) = \operatorname{arch}_R(t) = \left \lceil \frac{\sup R^{>0}}{\inf R^{>0}} \right \rceil .
\]

Proof. Part (a). It suffices to fix $t \in R$ and $r, s \in [t]$, with $s < r$, and show that if $n < \omega$ is such that $ns < r$, then $\operatorname{arch}(\mathcal{R}) > n$. Since $r, s \in [t]$, there is some $m < \omega$ such that $r \leq ms$, and so we have $ns < ms$. It follows that $ns < (n+1)s$, which gives $\operatorname{arch}(\mathcal{R}) > n$.

Part (b). Fix $t \in R^{>0}$. Since $\mathcal{R}$ is archimedean, we have $[t] = R^{>0}$, and so the second equality follows from Proposition 6.10(c). To show the first inequality, it suffices by part (a) to show $\operatorname{arch}(\mathcal{R}) \leq \operatorname{arch}_R(t)$. We may assume $\operatorname{arch}_R(t) = n < \omega$. In particular, for any $r, s \in R^{>0}$, we have $s \leq nr$. Therefore, for any $r_0, r_1, \ldots, r_n \in R$, with $0 < r_0 \leq r_1 \leq \ldots \leq r_n$, we have 
\[
\begin{align*}
\begin{array}{c}
\end{array}
\end{align*}
\]

Example 6.12.

1. A large, natural family of Urysohn monoids is the class of convex monoids, defined in [9] Section 9. In particular, fix a countable ordered abelian group $G = (G, +, \leq, 0)$ and let $G^{>0}$ be the distance monoid of nonnegative elements of $G$. Next, fix a convex subset $I \subseteq G^{>0}$ such that if $r + s = \sup I$ for some $r, s \in I$, then $\sup I \in I$. Let $R = I \cup \{0\}$, and define a distance monoid structure $\mathcal{R} = (R, \ominus, \leq, 0)$, where, given $r, s \in R$, $r \ominus s = \min \{r + s, \sup I\}$. Then $\mathcal{R}$ is Urysohn (see [9] Proposition 9.3). If we further assume that $G$ is an archimedean, then $\mathcal{R}$ will be archimedean as well. Therefore, we have 
\[
\operatorname{SO}(\operatorname{Th}(U_\mathcal{R})) = \operatorname{arch}(\mathcal{R}) = \left \lceil \frac{\sup I}{\inf I} \right \rceil .
\]

2. Using the previous example, we can calculate the model-theoretic complexity of many classical examples of Urysohn spaces. In particular, using the notation of Example 1.1 we have 
\[
\begin{align*}
(i) \quad & \operatorname{SO}(\operatorname{Th}(U_{\mathcal{Q}})) = \operatorname{SO}(\operatorname{Th}(U_{\mathcal{Q}_1})) = \operatorname{SO}(\operatorname{Th}(U_{\mathcal{N}})) = \omega; \\
(ii) \quad & \text{given } n > 0, \operatorname{SO}(\operatorname{Th}(U_{\mathcal{R}_n})) = n.
\end{align*}
\]
Recall that, using acronyms, rank $\omega$ is the same as SOP and NFSOP; and rank $n \geq 3$ is the same as SOP and NSOP.

3. We give an example which shows that, in Proposition 6.11(a), the inequality can be strict. Consider $\mathcal{S} = (\{0, 1, 2, 5, 6, 7\}, +, \leq, 0)$. The reader may verify that $+_{\mathcal{S}}$ is associative on $\mathcal{S}$. Note that $1$ and $5$ are representatives for the two nontrivial archimedean classes in $\mathcal{S}$, and $\operatorname{arch}_\mathcal{S}(1) = 2 = \operatorname{arch}_\mathcal{S}(5)$. However, $1 +_{\mathcal{S}} 5 < 1 +_{\mathcal{S}} 1 +_{\mathcal{S}} 5$, and so $\operatorname{arch}(\mathcal{S}) \geq 3$. In fact, a direction calculation shows $\operatorname{arch}(\mathcal{S}) = 3$.  

24
The last counterexample shows that, given a distance monoid $\mathcal{R}$, if $\text{arch}(\mathcal{R}) \geq n$ then we cannot always expect to have some $t \in \mathcal{R}$ with $\text{arch}_\mathcal{R}(t) \geq n$. On the other hand, we do have the following property.

**Proposition 6.13.** Suppose $\mathcal{R}$ is a distance monoid. If $n < \omega$ and $\text{arch}(\mathcal{R}) \geq n$ then there is some $t \in \mathcal{R}^{>0}$ such that $||t|| \geq n$.

**Proof.** Suppose $\text{arch}(\mathcal{R}) \geq n$. We may clearly assume $n \geq 2$. Fix $r_1, \ldots, r_n \in \mathcal{R}$, such that $r_1 \leq \ldots \leq r_n$ and $r_2 \oplus \ldots \oplus r_n < r_1 \oplus \ldots \oplus r_n$. Given $1 \leq i \leq n$, let $s_i = r_i \oplus \ldots \oplus r_n$. Since $r_1 \leq \ldots \leq r_n$, we have $s_i \in [r_n]$ for all $i$. We prove, by induction on $i$, that $s_{i+1} \leq s_i$. The base case $s_2 < s_1$ is given, so assume $s_{i+1} < s_i$. Suppose, for a contradiction, that $s_{i+1} \leq s_i$. Then

$$s_i = r_i \oplus s_{i+1} \leq r_i \oplus s_{i+2} \leq r_{i+1} \oplus s_{i+2} = s_{i+1},$$

which contradicts the induction hypothesis. Altogether, we have $||r_n|| \geq n$. \hfill $\Box$

Combining this result with Corollary 5.2, we obtain the following numeric upper bound for the strong order rank of $\text{Th}(\mathcal{U}_\mathcal{R})$.

**Corollary 6.14.** If $\mathcal{R}$ is a Urysohn monoid then $\text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) \leq |\mathcal{R}^{>0}|$.

For the final result of this section, we consider a fixed integer $n > 0$. Note that any finite distance monoid is Urysohn (by general Fraïssé theory or Proposition 9.3). We have shown that if $\mathcal{R}$ is a distance monoid, with $|\mathcal{R}^{>0}| = n$, then $1 \leq \text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) \leq n$. The next result shows that, moreover, there are unique (up to isomorphism) distance monoids, with $n$ nontrivial elements, of maximal and minimal rank.

**Theorem 6.15.** Fix $n > 0$ and suppose $\mathcal{R}$ is a distance monoid, with $|\mathcal{R}^{>0}| = n$.

(a) $\text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) = 1$ if and only if $\mathcal{R} \cong (\{0,1,\ldots,n\}, \max, \leq, 0)$.

(b) $\text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) = n$ if and only if $\mathcal{R} \cong \mathcal{R}_n = (\{0,1,\ldots,n\}, +_n, \leq, 0)$.

**Proof.** Part (a). We have already shown that $\text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) = 1$ if and only if $\mathcal{R}$ is ultrametric, in which case the result follows. Indeed, if $\mathcal{R}$ is ultrametric, then $\mathcal{R} \cong (S, \max, \leq, 0)$ for any linear order $(S, \leq, 0)$ with least element 0 and $n$ nonzero elements.

Part (b). We have already observed that $\text{arch}(\mathcal{R}_n) = n$, so it suffices to assume $\text{arch}(\mathcal{R}) = n$ and show $\mathcal{R} \cong \mathcal{R}_n$. Since $\text{arch}(\mathcal{R}) = n$, it follows from Proposition 6.13 the there is $t \in \mathcal{R}^{>0}$, with $||t|| \geq n$, and so $\mathcal{R}^{>0} = [t]$. Therefore $\mathcal{R}$ is archimedean and $\text{arch}_\mathcal{R}(t) = n$. If $r = \min \mathcal{R}^{>0}$ and $s = \max \mathcal{R}^{>0}$ then we must have $(n-1)r < s = nr$, and so $\mathcal{R}^{>0} = \{r, 2r, \ldots, nr\}$. From this, we clearly have $\mathcal{R} \cong \mathcal{R}_n$. \hfill $\Box$

**Remark 6.16.** Pursuing the natural line of questioning opened by Theorem 6.15 we fix $1 \leq k \leq n$ and define $\text{DM}(n,k)$ to be the number (modulo isomorphism) of distance monoids $\mathcal{R}$ such that $|\mathcal{R}^{>0}| = n$ and $\text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) = k$ (equivalently, $\text{arch}(\mathcal{R}) = k$). In particular, Theorem 6.15 asserts that, for all $n > 0$, $\text{DM}(n,1) = \text{DM}(n,n) = 1$. On the other hand, using direct calculations and induction, one may show that $\text{DM}(n,k) > 1$ for all $1 < k < n$. We make the following conjectures.

(a) Given a fixed $k > 1$, the sequence $(\text{DM}(n,k))_{n=k}^\infty$ is strictly increasing.

(b) Given a fixed $n > 2$, the sequence $(\text{DM}(n,k))_{k=1}^n$ is (strictly) unimodal.

Using exhaustive calculation, part (b) has been confirmed for $n \leq 6$ and, moreover, the maximal value of the sequence is attained at $k = 2$. Model theoretically, this is interesting since it demonstrates the existence of many more simple unstable Urysohn spaces beyond the metrically trivial ones. Indeed, for a fixed $n \geq 2$, exactly one of the $\text{DM}(n,2)$ rank 2 monoids, with $n$ nontrivial elements, is metrically trivial.
7 Imaginaries and Hyperimaginaries

In this section, we give some partial results concerning the question that originally motivated Casanovas and Wagner [7] to consider the Urysohn spaces $\mathcal{U}_\mathcal{R}_n$. In particular, if we replace $\mathcal{R}_n$ with $\mathcal{S}_n = (S_n, +, 1, \le)$, where $S_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$, then, as $\mathcal{R}_n \cong \mathcal{S}_n$, we can essentially think of $\mathcal{U}_\mathcal{R}_n$ and $\mathcal{U}_\mathcal{S}_n$ as “isomorphic” Urysohn spaces. The advantage of working with $\mathcal{S}_n$ is that we can coherently define an $L_S$-theory $T_\infty = \bigcup_{n<\omega} \text{Th}(\mathcal{U}_{\mathcal{S}_n})$, where $S = \mathbb{Q} \cap [0, 1]$. In [7], Casanovas and Wagner show that $T_\infty$ does not eliminate hyperimaginaries. In [9, Section 9], we verified that $T_\infty = \text{Th}(\mathcal{U}_{\mathcal{Q}_1})$. This is not observed in [7], although the authors do describe the non-eliminable hyperimaginaries as resulting from infinitesimal distance.

We will refine and generalize the results of [7] for arbitrary Urysohn monoids, in order to obtain necessary conditions for elimination of hyperimaginaries and weak elimination of imaginaries. We assume the reader is familiar with the basic definitions and facts concerning these notions (see [6, 23]). Given a complete theory $T$, a monster model $\mathbb{M}$, a tuple $\bar{a} \in \mathbb{M}$, and a type-definable equivalence relation $E(\bar{x}, \bar{y})$, with $\ell(\bar{a}) = \ell(\bar{x})$, we let $[\bar{a}]_E$ denote the $E$-equivalence class of $\bar{a}$, and use $\bar{a}_E$ to denote the hyperimaginary determined by $[\bar{a}]_E$.

Definition 7.1.

1. Given an imaginary $e$, a canonical parameter (resp. weak canonical parameter) for $e$ is a finite real tuple $\bar{c} \in \text{dcl}^eq(e)$ (resp. $\bar{c} \in \text{acl}^eq(e)$) such that $e \in \text{dcl}^eq(\bar{c})$.

2. $T$ has (weak) elimination of imaginaries if every imaginary has a (weak) canonical parameter.

In lieu of a formal definition, we recall the following characterization for elimination of hyperimaginaries (see [6]).

Proposition 7.2. The following are equivalent.

(i) $T$ has elimination of hyperimaginaries.

(ii) Let $E(\bar{x}, \bar{y})$ be a 0-type-definable equivalence relation, with $\bar{x} = (x_i)_{i<\mu}$, and fix a real tuple $\bar{a} = (a_i)_{i<\mu}$. Then there is a sequence $(E_i(\bar{x}^i, \bar{x'}^i))_{i<\lambda}$ of 0-definable $n_i$-ary equivalence relations, with $\bar{x}^i = (x_{j_1}^i, \ldots, x_{j_n}^i)$ and $\bar{y}^i = (y_{j_1}^i, \ldots, y_{j_n}^i)$ for some $j_1 < \ldots < j_{n_i} < \mu$, such that, for all $\bar{b}, \bar{b'} \models \text{tp}(\bar{a})$, $E_i(\bar{b}, \bar{b'})$ holds if and only if, for all $i < \lambda$, $E_i(\bar{b}, \bar{b'})$ holds.

We will first verify that, if $\mathcal{R}$ is a nontrivial Urysohn monoid, then $\text{Th}(\mathcal{U}_\mathcal{R})$ does not have elimination of imaginaries. Specifically, we will show that, as is often the case with Fraïssé limits in symmetric relational languages, finite imaginaries are not eliminated.

Lemma 7.3. Let $\mathbb{M}$ be a monster model of a complete first-order theory $T$. Assume that $\text{acl}(C) = C$ for all $C \subset \mathbb{M}$. Fix $n > 1$. Given $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{M}^n$ and $f \in \text{Sym}(1, \ldots, n)$, let $\bar{a}^f = (a_{f(1)}, \ldots, a_{f(n)})$. Let $E_n$ be the 0-definable equivalence relation on $\mathbb{M}^n$ such that, given $\bar{a}, \bar{b} \in \mathbb{M}^n$,

$$E_n(\bar{a}, \bar{b}) \iff \bar{b} = \bar{a}^f \text{ for some } f \in \text{Sym}(1, \ldots, n).$$

Suppose $\bar{a} \in \mathbb{M}^n$ is a tuple of pairwise distinct elements such that $\bar{a}^f \equiv \bar{a}$ for all $f \in \text{Sym}(1, \ldots, n)$. Then $\bar{a}_{E_n}$ does not have a canonical parameter.
Proof. Fix $n > 1$ and $E_n$ as in the statement. Let $\bar{a} \in M^n$ be a tuple of distinct elements such that $\bar{a}^f \equiv \bar{a}$ for all $f \in \text{Sym}(1, \ldots, n)$. Let $e = \bar{a}_{E_n}$. For any $f \in \text{Sym}(1, \ldots, n)$, we may fix $\sigma_f \in \text{Aut}(M)$ such that $\sigma_f(\bar{a}) = \bar{a}^f$. For any other $g \in \text{Sym}(1, \ldots, n)$, we have $\sigma_f(\bar{a}^g) = \bar{a}^{fg} \sim_{E_n} \bar{a}$. Therefore $\sigma_f(e) = e$.

Suppose, toward a contradiction, that $\bar{c} \in \text{dcl}_a(e)$ and so, since $\sigma_f(e) = e$ for all $f$, we have $\sigma_f(\bar{c}) = \bar{c}$ for all $f$.

Case 1: $\bar{c}$ contains $a_i$ for some $1 \leq i \leq n$.

Since $n > 1$ we may fix $j \neq i$ and let $f$ be a permutation of $\{1, \ldots, n\}$ such that $f(i) = j$. Then $\sigma_f(\bar{c}) = \bar{c}$ implies $a_i = a_j$, which is a contradiction.

Case 2: $\bar{c}$ is disjoint from $\bar{a}$.

Since $\text{acl}(\bar{c}) = \bar{c}$, it follows that $tp(\bar{a}/\bar{c})$ has infinitely many realizations, and so we may fix $\bar{a}' \equiv_{\bar{c}} \bar{a}$ such that $\bar{a}' \setminus \bar{a} \neq \emptyset$. If $\sigma \in \text{Aut}(M/\bar{c})$ is such that $\sigma(\bar{a}) = \bar{a}'$ then, by assumption, $\sigma(e) = e$ and so $E_n(\bar{a}, \bar{a}')$ holds, which is a contradiction.

We now fix a nontrivial Urysohn monoid $R$.

Corollary 7.4. $\text{Th}(U_R)$ does not have elimination of imaginaries.

Proof. We verify that $\text{Th}(U_R)$ satisfies the conditions of Lemma 7.3. First, the fact that $\text{acl}(C) = C$ for all $C \subseteq U_R$ follows from quantifier elimination and that the Fraïssé class $K_R$ has disjoint amalgamation. Next, given $n > 0$, if we fix some $r \in R^{\geq 0}$ then there is $\bar{a} = (a_1, \ldots, a_n) \in U_R$ such that $d(a_i, a_j) = r$ for all $i \neq j$. In particular, $\bar{a}^l \equiv \bar{a}$ for all $f \in \text{Sym}(1, \ldots, n)$.

Definition 7.5. Suppose $E(x, y)$ is a 0-invariant unary equivalence relation on $U_R$. Define $\Gamma(E) \subseteq R^*$ such that $\alpha \in \Gamma(E)$ if and only if there are $a, b \in U_R$ such that $E(a, b)$ and $d(a, b) = \alpha$. Let $\alpha(E) = \sup \Gamma(E)$.

Proposition 7.6. Suppose $E(x, y)$ is a 0-invariant unary equivalence relation on $U_R$.

(a) $\Gamma(E)$ is closed downwards.

(b) If $\alpha \in \Gamma(E)$ then $2\alpha \in \Gamma(E)$.

Proof. Fix $\alpha \in \Gamma(E)$ and let $a, b \in U_R$ be such that $E(a, b)$ holds and $d(a, b) = \alpha$. To prove the claims, it suffices to fix $\beta \in R^*$, with $\beta \leq 2\alpha$, and show $\beta \in \Gamma(E)$. Given such a $\beta$, there is some $b' \equiv_a b$, with $d(b, b') = \beta$. We have $E(a, b)$ and $E(a, b')$, which gives $E(b, b')$. Therefore $\beta \in \Gamma(E)$, as desired.

Lemma 7.7. Suppose $E(x, y)$ is a 0-type-definable unary equivalence relation. Then, for all $a, b \in U_R$, $E(a, b)$ holds if and only if $d(a, b) \leq \alpha(E)$.

Proof. By definition of $\alpha(E)$, we have that $E(a, b)$ implies $d(a, b) \leq \alpha(E)$. Conversely, suppose first that $a, b \in U_R$ are such that $d(a, b) = \beta < \alpha(E)$. Then $\beta \in \Gamma(E)$ by Proposition 7.6(a), and so there are $a', b' \in U_R$ such that $E(a', b')$ and $d(a', b') = \beta$. Then $(a, b) \equiv (a', b')$ by quantifier elimination, so $E(a, b)$ holds. Therefore, we have left to show that $d(a, b) = \alpha(E)$ implies $E(a, b)$.

By quantifier elimination, it suffices to show $\alpha(E) \in \Gamma(E)$. If $\alpha(E)$ has an immediate predecessor in $R^*$ then this is immediate. So we may assume $\alpha(E)$ has no immediate predecessor. Then, by definition of $\alpha(E)$, the type

$$E(x, y) \cup \{d(x, y) \leq r : r \in R, \alpha(E) \leq r\} \cup \{d(x, y) > r : r \in R, r < \alpha(E)\}$$

is finitely satisfiable, and so $\alpha(E) \in \Gamma(E)$.
Definition 7.8.

1. An element $\alpha \in R^*$ is idempotent if $\alpha \oplus \alpha = \alpha$.

2. Define $\text{eq}(\mathcal{R}) \subseteq R$ such that $r \in \text{eq}(\mathcal{R})$ if and only if $r$ is idempotent and $0 < r < \sup R^*$.

3. Define $\text{heq}(\mathcal{R}) \subseteq R^*$ such that $\alpha \in \text{heq}(\mathcal{R})$ if and only if $\alpha$ is idempotent, $\alpha < \sup R^*$, $\alpha \notin R$, and $\alpha$ is not approximated from above by idempotent elements of $R$.

Theorem 7.9. Suppose $\mathcal{R}$ is a nontrivial Urysohn monoid.

(a) The 0-definable unary equivalence relations on $\mathbb{U}_\mathcal{R}$ consist precisely of equality, the trivial relation, and $d(x, y) \leq r$ for $r \in \text{eq}(\mathcal{R})$.

(b) If $\text{eq}(\mathcal{R}) \neq \emptyset$ then $\text{Th}(\mathcal{U}_\mathcal{R})$ does not have weak elimination of imaginaries.

(c) If $\text{heq}(\mathcal{R}) \neq \emptyset$ then $\text{Th}(\mathcal{U}_\mathcal{R})$ does not have elimination of hyperimaginaries.

Proof. Part (a). First, if $r \in \text{eq}(\mathcal{R})$ then $d(x, y) \leq r$ is an 0-definable equivalence relation. Conversely, suppose $E(x, y)$ is a 0-definable equivalence relation. By Lemma 7.7, $E(x, y)$ is equivalent to $d(x, y) \leq \alpha(E)$. If $\alpha(E) = 0$ then $E$ is equality, and if $\alpha(E) = \sup R^*$ then $E$ is trivial. Therefore, we may assume $0 < \alpha(E) < \sup R^*$. We want to show $\alpha(E) \in \text{eq}(\mathcal{R})$. Since $\alpha(E) \in \Gamma(E)$, we have $2\alpha(E) \in \Gamma(E)$ by Proposition 7.6 (b). Therefore $\alpha(E)$ is idempotent. It remains to show that $\alpha(E) \in R$. But this follows since $d(x, y) \leq \alpha(E)$ is definable.

Part (b). Suppose we have $r \in \text{eq}(\mathcal{R})$. Let $E_r(x, y)$ denote the equivalence relation $d(x, y) \leq r$. Fix $a \in \mathbb{U}_\mathcal{R}$ and let $e = a_{E_r}$ and $X = [a]_{E_r}$. We fix a finite real tuple $\bar{c}$ and show that $\bar{c}$ is not a weak canonical parameter for $e$.

Case 1: There is some $c \in \bar{c} \cap X$.

For any $b \in X$, we may fix $\sigma_b \in \text{Aut}(\mathbb{U}_\mathcal{R})$ such that $\sigma_b(c) = b$. Then $\sigma_b \in \text{Aut}(\mathbb{U}_\mathcal{R}/e)$, and we have shown that any element of $X$ is in the orbit of $c$ under $\text{Aut}(\mathbb{U}_\mathcal{R}/e)$. Since $X$ is infinite, it follows that $\bar{c} \notin \text{acl}^{\text{eq}}(e)$.

Case 2: $\bar{c} \cap X = \emptyset$.

Let $\alpha = \min\{d(a, c) : c \in \bar{c}\}$. Then $r < \alpha$, by assumption of this case. Moreover, we may find $a' \in \mathbb{U}_\mathcal{R}$ such that $a' \equiv a$ and $d(a, a') = \alpha$. If $\sigma \in \text{Aut}(\mathbb{U}_\mathcal{R}/\bar{c})$ is such that $\sigma(a) = a'$ then, as $\alpha > r$, we have $\sigma(e) \neq e$. Therefore $e \notin \text{dcl}^{\text{eq}}(\bar{c})$.

Part (c). Suppose we have $\alpha \in \text{heq}(\mathcal{R})$. Suppose, toward a contradiction, that $\text{Th}(\mathcal{U}_\mathcal{R})$ eliminates hyperimaginaries. Fix a singleton $a \in \mathbb{U}_\mathcal{R}$. Since $d(x, y) \leq \alpha$ is a 0-type-definable equivalence relation, it follows from Proposition 7.2 that there is a sequence $(E_i(x, y))_{i < \lambda}$ of 0-definable unary equivalence relations such that for any $b, b' \models \text{tp}(a)$, $d(b, b') \leq \alpha$ if and only if, for all $i < \lambda$, $E_i(b, b')$ holds. By part (a), there are $r_i \in R$, for $i < \lambda$, such that $r_i$ is idempotent and $E_i(x, y)$ is equivalent to $d(x, y) \leq r_i$. Since $\text{Th}(\mathcal{U}_\mathcal{R})$ has a unique 1-type over $\emptyset$, we have shown that $d(x, y) \leq \alpha$ is equivalent to $\text{dcl}^{\text{eq}}(\bar{c})$. Therefore $\bar{c} \cap X = \emptyset$.

Returning to the results of [7], we observe that $0^+ \in \text{heq}(\mathcal{Q}_1)$, and so failure of elimination of hyperimaginaries for $\text{Th}(\mathcal{U}_{\mathcal{Q}_1})$ is a special case of the previous result. Note also that $0^+ \in \text{heq}(\mathcal{Q})$ and so $\text{Th}(\mathcal{U}_\mathcal{Q})$ also fails elimination of hyperimaginaries. It is worth mentioning that Casanovas and Wagner carry out an analysis of 0-definable equivalence relations of any arity in $\text{Th}(\mathcal{U}_{\mathcal{Q}_1})$. From this analysis it is easy to conclude that, for all $n > 0$, $\text{Th}(\mathcal{U}_{\mathcal{Q}_n})$ has weak elimination of imaginaries, which implies the same result for $\text{Th}(\mathcal{U}_{\mathcal{Q}_1})$.  

28
For future work, we conjecture that these necessary conditions for elimination of hyperimaginaries and weak elimination of imaginaries are also sufficient.

**Conjecture 7.10.** Suppose \( \mathcal{R} \) is a Urysohn monoid.

(a) \( \text{Th}(U_{\mathcal{R}}) \) has weak elimination of imaginaries if and only if \( \text{eq}(\mathcal{R}) = \emptyset \).

(b) \( \text{Th}(U_{\mathcal{R}}) \) has elimination of hyperimaginaries if and only if \( \text{heq}(\mathcal{R}) = \emptyset \).

Regarding consequences of this conjecture, we first make the following observations.

**Proposition 7.11.** Suppose \( \mathcal{R} \) is a Urysohn monoid. If \( \text{heq}(\mathcal{R}) \neq \emptyset \) then \( \text{SO}(\text{Th}(U_{\mathcal{R}})) = \omega \).

Proof. Suppose \( \alpha \in \text{heq}(\mathcal{R}) \). Fix \( \beta \in R^* \) such that \( \alpha < \beta \) and, for all \( r \in R \), if \( r < r \oplus r \). Fix \( n > 0 \). Then \( n\alpha = \alpha < \beta \) so, by Theorem 2.11(b), there is some \( t \in R \) such that \( \alpha < t \) and \( nt < \beta \). Then \( nt < 2nt \), which implies \( \text{arch}(\mathcal{R}) > n \). \( \square \)

The purpose of Casanovas and Wagner’s work in \([7]\) was to demonstrate the existence of a theory without the strict order property that does not eliminate hyperimaginaries. Our previous work slightly sharpens this upper bound of complexity to without the finitary strong order property. On the other hand, if Conjecture 7.10(b) is true then, combined with Proposition 7.11, we would conclude that generalized Urysohn spaces provide no further assistance in decreasing the complexity of this upper bound. In other words, a consequence of our conjecture is that if \( \text{SO}(\text{Th}(U_{\mathcal{R}})) < \omega \) then \( \text{Th}(U_{\mathcal{R}}) \) eliminates hyperimaginaries. An outlandish, but nonetheless open, conjecture could be obtained from this statement by replacing \( \text{Th}(U_{\mathcal{R}}) \) with an arbitrary theory \( T \). Concerning the converse of this statement, note that, if Conjecture 7.10(b) holds, then \( \text{Th}(U_{\mathcal{N}}) \) would eliminate hyperimaginaries, while still having strong order rank \( \omega \). As a side note, we have observed that \( \text{Th}(U_{\mathcal{N}}) \) is small, and so at least eliminates finitary hyperimaginaries (see \([6]\) Theorem 18.14)).

### A Forking and Dividing in Generalized Urysohn Spaces

In this appendix, we outline the proof of Theorem 3.2. Our work is essentially a direct translation of \([10]\), and so we will give a brief outline of the arguments. First, we summarize a “guide” for translating from \([10]\).

1. In \([10]\), we consider \( U_1 \), which is a monster model of the theory of the complete Urysohn sphere as a metric structure in continuous logic. In this case, \( U_1 \) is a complete, \( \kappa^+ \)-universal and \( \kappa \)-homogeneous metric space of diameter 1, where \( \kappa \) is the density character of \( U_1 \). Given a tuple \( \bar{a} = (a_1, \ldots, a_n) \in U_1 \) and a subset \( C \subseteq U_1 \), by quantifier elimination, the type \( tp_x(\bar{a}/C) \) is completely determined by

   \[
   \{d(x, x_j) = d(a_i, a_j) : 1 \leq i, j \leq n\} \cup \{d(x_i, c) = d(a_i, c) : 1 \leq i \leq n, c \in C\},
   \]

   where, given variables \( x, y \) and \( r \in [0, 1] \), \( d(x, y) = r \) denotes the condition \( |d(x, y) - r| = 0 \).

2. In this section, we consider \( U_{\mathcal{R}} \), which is a monster model of the theory of the \( \mathcal{R} \)-Urysohn space as a relational structure in classical logic. In this case, \( U_{\mathcal{R}} \) is a \( \kappa^+ \)-universal and \( \kappa \)-homogeneous \( \mathcal{R}^* \)-metric space, where \( \kappa \) is the cardinality of \( U_{\mathcal{R}} \). Given a tuple \( \bar{a} = (a_1, \ldots, a_n) \in U_{\mathcal{R}} \) and a subset \( C \subseteq U_{\mathcal{R}} \), by quantifier elimination, the type \( tp_x(\bar{a}/C) \) is completely determined by

   \[
   \bigcup_{1 \leq i, j \leq n} d(x_i, x_j) = d(a_i, a_j) \cup \bigcup_{1 \leq i \leq n, c \in C} d(x_i, c) = d(a_i, c),
   \]
where, given variables $x, y$ and $\alpha \in R^*$, $d(x, y) = \alpha$ denotes the type
\[
\{d(x, y) \leq r : r \in R, \alpha \leq r\} \cup \{d(x, y) > r : r \in R, r < \alpha\}.
\]

To characterize dividing, we first apply our recurring theme that consistency of complete types is completely determined by the triangle inequality. As a result, we can strengthen the usual “finite character of dividing”, and show that dividing is always detected by three points.

**Lemma A.1.** Given $A, B, C \subseteq \mathbb{U}_R$, $A \mathrel{\downarrow^d_C} B$ if and only if $a \mathrel{\downarrow^d_C} b_1b_2$ for all $a \in A$ and $b_1, b_2 \in B$.

**Proof.** This is a direct translation of [10, Lemma 3.11]. □

From this result, we see that, in order to understand dividing, it is enough to consider indiscernible sequences of 2-tuples.

**Definition A.2.** Fix $C \subseteq \mathbb{U}_R$ and $b_1, b_2 \in \mathbb{U}_R$. Define $\Gamma(b_1, b_2/C) \subseteq R^*$ such that $\gamma \in \Gamma(b_1, b_2/C)$ if and only if there is a $C$-indiscernible sequence $(b_1', b_2')_{l<\omega}$, with $(b_1', b_2') = (b_1, b_2)$, such that $d(b_1', b_2') = \gamma$.

**Lemma A.3.** Given $C \subseteq \mathbb{U}_R$ and $a, b_1, b_2 \in \mathbb{U}_R$, $a \mathrel{\downarrow^d_C} b_1b_2$ if and only if for all $i, j \in \{1, 2\}$,
\[
d(b_1, a) \oplus d(a, b_j) \geq \sup \Gamma(b_i, b_j/C) \quad \text{and} \quad |d(b_1, a) \oplus d(a, b_j)| \leq \inf \Gamma(b_i, b_j/C).
\]

**Proof.** This is a direct translation of [10, Lemma 3.14]. □

From this result, we see that an explicit characterization of dividing rests on an explicit calculation of $\Gamma(b_1, b_2/C)$, which can be given via the values $d_{\max}$ and $d_{\min}$ (see Definition 3.1).

**Lemma A.4.** Given $C \subseteq \mathbb{U}_R$ and $b_1, b_2 \in \mathbb{U}_R$,
\[
\Gamma(b_1, b_2/C) = \{\gamma \in R^* : d_{\min}(b_1, b_2/C) \leq \gamma \leq d_{\max}(b_1, b_2/C)\}.
\]

**Proof.** This is the most technical part of the characterization of dividing, and is a direct translation of [10, Lemma 3.17]. For the right-to-left inclusion, we fix $\gamma$ between $d_{\min}(b_1, b_2/C)$ and $d_{\max}(b_1, b_2/C)$, and explicitly construct a $C$-indiscernible sequence $(b_1', b_2')_{l<\omega}$, with $(b_1', b_2') = (b_1, b_2)$ and $d(b_1', b_2') = \gamma$. This construction can be copied directly from [10, Lemma 3.17].

For the left-to-right inclusion, suppose $(b_1', b_2')_{l<\omega}$ is a $C$-indiscernible sequence, with $d(b_1', b_2') = \gamma$. To verify $d_{\min}(b_1, b_2/C) \leq \gamma \leq d_{\max}(b_1, b_2/C)$, we make two observations. First, for any $c \in C$,
\[
|d(b_1, c) \oplus d(b_2, c)| = |d(b_1', c) \oplus d(b_2', c)| \leq \gamma \leq d(b_1', c) \oplus d(b_2', c) = d(b_1, c) \oplus d(b_2, c).
\]

Second, $d(b_1, b_2) = d(b_1', b_2') \leq d(b_1', b_2') \oplus d(b_2', b_2') \oplus d(b_1', b_2') = 3\gamma$, and so $\frac{1}{3}d(b_1, b_2) \leq \gamma$. □

Combining the previous results, we obtain the full characterization of dividing.

**Theorem A.5.** Given $A, B, C \subseteq \mathbb{U}_R$, $A \mathrel{\downarrow^d_C} B$ if and only if for all $b_1, b_2 \in B$,
\[
d_{\max}(b_1, b_2/AC) = d_{\max}(b_1, b_2/C) \quad \text{and} \quad d_{\min}(b_1, b_2/AC) = d_{\min}(b_1, b_2/C).
\]

Finally, to prove Theorem 3.2, we show that forking and dividing are the same for complete types. To prove this it suffices, by Fact 2.1, to prove the following theorem.

**Theorem A.6.** Fix subsets $B, C \subseteq \mathbb{U}_R$ and a singleton $b_\ast \in \mathbb{U}_R$. For any $A \subseteq \mathbb{U}_R$, if $A \mathrel{\downarrow^d_C} B$ then there is $A' \equiv_{BC} A$ such that $A' \mathrel{\downarrow^d_C} Bb_\ast$. 

30
Once again, we give an outline of the proof of this result, which closely follows \cite{10}. Fix $B, C \subseteq U_R$ and $b_* \in U_R$.

**Definition A.7.** Given $a \in U_R$, define $U(a) := \inf_{b \in BC}(d(a, b) \oplus d_{\min}(b_*, b/BC))$.

The motivation for this definition is the observation, which follows easily from Theorem A.5, that, given $a \in U_R$ with $a \downarrow^d_C B$, if $a'$ realizes a nondividing extension of $tp(a/BC)$ to $BCb_*$, then $d(a', b_*) \leq U(a)$. Toward the proof of Theorem A.6, the key technical tool is the following result.

**Proposition A.8.**

(a) If $a, a' \in U_R$ are such that $a \downarrow^d_C B$, $a' \equiv_{BC} a$, and $d(a', b_*) = U(a)$, then $a' \downarrow^d_C Bb_*$. 

(b) If $a \in U_R$ is such that $a \downarrow^d_C B$ then $\sup_{b \in BC} |d(a, b) \ominus d(b_*, b)| \leq U(a) \leq d_{\max}(a, b_*/BC)$. 

(c) If $a_1, a_2 \in U_R$ are such that $a_1a_2 \downarrow^d_C B$ then $|U(a_1) \ominus U(a_2)| \leq d(a_2, a_2) \leq U(a_1) \oplus U(a_2)$.

**Proof.** This argument is quite technical, and can be translated directly from \cite{10} Lemma 3.22] and \cite{10} Lemma 3.23. In particular, the claims involve checking a large number of triangle inequalities, which heavily rely on the characterization of $\downarrow^d_C$ given by Theorem A.5 as well as the following useful inequalities:

$$d_{\max}(b_1, b_3/C) \leq d_{\max}(b_1, b_2/C) \ominus d_{\min}(b_2, b_3/C)$$

$$d_{\min}(b_1, b_3/C) \leq d_{\min}(b_1, b_2/C) \oplus d_{\min}(b_2, b_3/C),$$

where $b_1, b_2, b_3 \in U_R$ and $C \subseteq U_R$ are arbitrary. Recall that the first inequality was shown in Lemma A.6. The proof the second inequality is similar, and relies on Proposition 2.16(b), as well as the fact that $|\alpha \ominus \beta|$ is an $R^*$-metric on $R^*$ (see \cite{9} Proposition 6.3)).

We can now prove Theorem A.6 which completes the proof of Theorem 3.2.

**Proof of Theorem A.6.** Fix variables $\bar{x} = (x_a)_{a \in A}$ and define the type $p(\bar{x}) := tp_{\bar{x}}(A/BC) \cup \{d(a, b) \equiv U(a) : a \in A\}$. If $A' \equiv_{BC} A$ and, by Proposition A.8(a), we have $A' \downarrow_C^d Bb_*$ for all $a' \in A$, which gives $A' \downarrow_C^d Bb_*$ by Lemma A.1. Therefore, it suffices to show $p(\bar{x})$ is consistent, which means verifying the triangle inequalities in the definition. The nontrivial triangles to check either have distances $\{d(a, b), d(b, b_*), U(a)\}$ for some $a \in A$ and $b \in BC$, or $\{d(a_1, a_2), U(a_1), U(a_2)\}$ for some $a_1, a_2 \in A$. Therefore, the triangle inequality follows, respectively, from parts (b) and (c) of Proposition A.8.

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