Effect of ansatz on soliton propagation pattern in photorefractive crystals

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Abstract. Soliton propagation and some related parameter in a photorefractive crystal are considered. This study analyzes the soliton propagation on a photorefractive crystal having both linear and quadratic electro-optical effects through a time-dependent model of nonlinear dynamics equation using the numerical split-step Fourier method. In conducting analysis, we apply various models of ansatz as initial conditions of the optical beam envelope. Based on the studies conducted, ansatz influences the soliton propagation pattern in the photorefractive crystals. Besides, we found ansatz in secant-hyperbolic and Gaussian functions as the most appropriate model for realizing solitons in crystals. Finally, the photorefractive effect supports the evolution of soliton before it achieved a level of stability.

1. Introduction

Soliton is one of the three branches of nonlinear science. It has attracted attention during the last decade and developed rapidly in various physics branches such as particle physics, plasma, fluids, solids state, condensed matter, biophysics (dynamics of protein DNA), optics, and others. [1-3]. In optics, although still at a theoretical level, solitons (after this referred to as optical soliton) are an exciting topic for research because they have potential applications in developing high-quality optical telecommunication systems in the future: example, in optical switching, routing, waveguiding, optical communications, and signal processing [4-7]. Optical solitons have three specifications: spatial, temporal, and Spatio-temporal solitons [3,5,6]. In literature, spatial solitons are solitary waves (or soliton) that are bounded in space. Temporal solitons are determined in the time domain, whereas Spatio-temporal solitons are limited in space and time [5]. Each regime has its uniqueness but still shares the same characteristic. Soliton arises when the nonlinearity and dispersive properties balance each other in a medium [6]. A slightly different view is shown by the optical spatial soliton [8-9].
Optical spatial solitons form a balance between diffraction and the nonlinearity properties of a medium [1-7]. The nonlinearity properties of an optical medium are known to vary, including known as Kerr nonlinearity (generated by optical Kerr effect), nonlocal nonlinearity, cubic-quintic nonlinearity, and photorefractive nonlinearity (caused by the photorefractive effect, through photoconductivity events and their combination with the electro-optical effect in materials) [3,6,10-12]. Optical spatial solitons formed in a photorefractive medium have recently attracted much attention due to their realization at low optical power and the saturable nature of the photorefractive nonlinearity [5,10]. The photorefractive media’s nonlinear nature arises through the factor of refractive index change caused by linear electro-optical effects on the photoconductivity of a material [2,5,11,12]. Segev et al., who became pioneers in the study of photorefractive solitons on one occasion, proved that the nonlinear nature of a photorefractive medium could be supported by refractive index changes caused by the quadratic electro-optical effect [2,11]. Several types of photorefractive media that have linear and quadratic electro-optical effects, or a combination of the two are known, among other, ferroelectric crystal KTaxNb1-xO3 (KTN), single-crystal Pb(Zn1/3Nb2/3)O3-PbTiO3(PZN-PT), and single-crystal Pb[(Mg1/3Nb2/3)O3](1-x)-(PbTiO3)x (PMN-PT) which are known to have both linear and quadratic electro-optical effects [2,5,10,11].

This paper discusses the characteristics of soliton propagation in a photorefractive medium. The type of media used as the discussion object is single-crystal PMN-0.33PT (Lead Magnesium Niobate with 33 mol % Lead Titanate). Principally, the soliton propagation problem in this medium was observed by Katti in 2019 through an analysis of a time-dependent nonlinear dynamics equation [11]. In this paper, we present numerical analysis results of the soliton propagation in a similar medium, PMN-0.33PT crystals, which are known to have both linear and quadratic electro-optical effects by utilizing the split-step Fourier method. The analysis is performed by applying various ansatz (or assumptions) to the initial condition of an envelope to support a soliton solution. In carried out, we observed the effect of ansatz on the soliton propagation pattern in photorefractive crystals.

2. Theoretical Model

This study’s theoretical model is a time-dependent nonlinear dynamics equation based on Katti’s 2019 investigation [11]. Following Katti, the dynamics equation is obtained by considering the direction of optical beam propagation in the crystal along the z-axis and is permitted to diffract at one of the other axes (x-direction). For simplicity of calculations, only the one-dimensional nonlinear diffraction theory is accounted for, where the variation into the y-axis is negligible. Additionally, assuming that the centre axis of the crystal, the incident beam’s polarization direction, and the external bias electric field are all oriented along the x-coordinate. Here, the incident beam is expressed as a slowly varying envelope $E = \varphi(x,z) \exp(ikz)$, where $k = k_n = (2\pi/\lambda_o)n_z$, $\lambda_o$ is the free space wavelength (or vacuum), and $n_z$ is the unperturbed index of refractive. Mathematically, the dynamical evolution of the envelope in photorefractive crystals is governed by the following equations [2,11]:

$$
\left\{ \frac{1}{2k} \frac{\partial^2}{\partial x^2} + i \frac{\partial}{\partial z} - \frac{k_n^4 r_{\text{eff}} E_{\varphi}}{2} - \frac{k_n^4 g_{\text{eff}} e_0^2 (e_r - 1)^2 E_{\varphi}}{2} \right\} \varphi(x,z) = 0
$$

$E_{\varphi}$ is the accumulation of an internal electric field or often known as a space-charge field mediated by an external bias field in a material through the photoconductivity event (schematic can be seen in reference 12) [2,11]. $r_{\text{eff}}$ and $g_{\text{eff}}$ are the linear and quadratic electro-optical effect coefficients, respectively. $e_0$ and $e_r$ are representations of the vacuum permittivity and the relative dielectric constant of materials [11].
Mathematically, the dynamical evolution equation of envelopes (Eq. 1) is obtained from a model of optical beam dynamics expressed by the Helmholtz equation [4,13]. Another side, the value of the space-charge field is determined through the following the charge transport model of Kukhtarev et al. [11,14], i.e.

\[
\frac{\partial N'_D}{\partial t} = (s_i I + \beta)(N'_D - N'_A) - \gamma n N'_D
\]  

(2)

\[
\frac{\partial (e_i \varphi E_w)}{\partial x} = \rho_i
\]

(3)

\[
\frac{\partial J}{\partial x} + \frac{\partial \rho_i}{\partial t} = 0
\]

(4)

\[
\rho_i = e\left(N'_D - n - N'_A\right)
\]

(5)

\[
J = e \mu n E_w + K_B T \mu \frac{dn}{dx}
\]

(6)

\(I = |\varphi|^2\) is the emitted optical beam intensity, \(\beta\) is the dark-carrier generation rate, \(s_i\) is the cross-section photoionization, \(\gamma\) is the recombination rate, \(e\) is the charge of the electron, \(\mu\) is the mobility of the electron, \(K_B\) is the Boltzmann’s constant, \(\rho_i\) is the charge density, \(N_D\) is the donor concentration, \(N_A\) is the acceptor concentration, \(N'_D\) is the ionized donor density, \(n\) is the electron density, and \(T\) is the absolute temperature. Overall, charge transport models (Eq. 2 – Eq. 6) can be simplified by utilizing several assumptions. Firstly, in moderate intensities, the electron’s density is negligible concerning ionized donor or acceptor [11], from this assumption, \(N'_D \approx N'_A\). Secondly, carrier recombination time is assumed to be very small (or negligible) compared to the dielectric response time. Hence \(\partial N'_D / \partial t = 0\), it has consequences \((s_i I + \beta)(N'_D - N'_A) = \gamma n N'_A\) [11,14]. Physically, the model of Kukhtarev et al. describes the charge transport process of the electron in crystals when it interacts with light (known as a photoconductivity event) [12,14]. The result of this interaction is a space-charge field where its combination with electro-optical effects produce the refractive index modulation as an essential indicator for forming nonlinear properties of a material (crystals) and realizing solitons [2,4,5,10-12]. From this charge transport model can be obtained a mathematical relationship between the space-charge field and the optical beam intensity emitted into crystals [11], i.e.

\[
e \mu \frac{\partial}{\partial x} \left[ (I + I_d) E_w \right] + K_B T \mu \frac{\partial^2 I}{\partial x^2} + e_i \varphi E_w \frac{\gamma N'_A}{s_i (N'_D - N'_A)} \frac{\partial^2 E_w}{\partial t^2} = 0
\]

(7)

it is obtained from the substitution process of \(\rho_i\) from Eq. (3), \(J\) from Eq. (6), and or by utilizing previous assumptions into the model of continuity equation (Eq. 4). Furthermore, this relationship is integrated once by applying steady-state conditions \((E(t \to \infty) = \frac{E_0}{(1+I_d) / (1+I_d)} - K_B T e (I+I_d) \partial E_w / \partial x)\), and we get:

\[
E_w + \frac{K_B T}{e (I + I_d)} \frac{\partial I}{\partial x} + \frac{C I_d}{I + I_d} \frac{\partial E_w}{\partial t} = \frac{E_0}{I + I_d} \frac{I + I_d}{(1 + I_d)}
\]

(8)

where \(C = e_i \varphi e N'_A / e \mu \beta (N'_D - N'_A)\) [11].
Eq. (8) is a simple form of the mathematical relationship between the space-charge field with an optical beam emitted into the crystals. In another way, its read as,

$$\frac{\partial E_{sc}}{\partial t} + \frac{I + I_d}{Cl_d} E_{sc} = \left( E_0 \left( \frac{I + I_d}{I + I_d} \right) - \frac{K_n T e}{e(1 + I_d)} \right) \frac{I + I_d}{Cl_d}$$

(9)

its obtained by applying the assumption that the beam envelope is slowly varying and $E (t \rightarrow 0) = E_0$ [2,11]. Furthermore, this model (Eq. 9) is solved by utilizing a solving method of 1-order differential equations, that is when a differential equation is defined,

$$\psi' + p \psi = Q$$

(10)

have a solution,

$$\psi = e^{-1} \left[ Q e^{i} dx + \Omega e^{i} \right] \Omega \text{ is a constant}$$

(11)

with $1 = \int p dx$ [15]. Through this method, the mathematical form of the space-charge field can be expressed as follows [11]:

$$E_{sc} = E_0 \exp \left( \frac{(I + I_d)t}{Cl_d} \right) + E_0 \left( \frac{I - I_d}{I + I_d} \right) \exp \left( \frac{(I + I_d)t}{Cl_d} \right)$$

(12)

The first and second terms of this equation represent the drift process, while the third term is associated with the diffusion process [2]. In this case, drift and diffusion have an essential role in the dynamical process of electrons when there is irradiation or illumination to produce a space-charge field (photoconductivity event) [12]. Therefore, if assumed that there is a strong bias, the term diffusion ($K_n T / e$) can be ignored [11]. So, the space-charge field read as [11]:

$$E_{sc} = E_0 \left( \frac{I + I_d}{I + I_d} \right) + E_0 \left( \frac{I - I_d}{I + I_d} \right) \left( \exp \left( \frac{(I + I_d)t}{Cl_d} \right) \right)$$

(13)

In Eq. (13), $E_0$ and $I_n = I (x \rightarrow \infty, z)$ are the space-charge field and soliton intensity when $x \rightarrow \infty$ [2,11]. Mathematically, $E_0$ can be described as $\pm V/W$, where $V$ is the applied external voltage [2]. $I_d$ is the dark irradiance (intensity in the dark regime), and $I = I(x, z) = (n_z / 2 \eta_0) |\phi|^2$ is the soliton intensity with $\eta_0 = (\mu / \omega_0)^{1/2}$ [2]. Next, substitute Eq. (13) to Eq. (1), it obtained the time-dependent nonlinear (or soliton) dynamics equation (Katti’s model):

$$iu_x + \frac{1}{2} u_{ss} - \beta_1 \left( 1 + \rho \left( \frac{|u|^2 - \rho}{{|u|^2}} \right) \exp \left( - \left(1 + |u|^2 \right) \tau \right) \right) u$$

$$- \beta_2 \left( 1 + \rho \left( \frac{|u|^2 - \rho}{{|u|^2}} \right) \exp \left( - \left(1 + |u|^2 \right) \tau \right) \right) \frac{2}{|u|^2} u = 0$$

(14)

Here $\beta_1 = (k_n x_0)^2 n_s r_{eff} E_0 / 2$ and $\beta_2 = (k_n x_0)^2 n_s' r_{eff} e_0^2 (e, -1)^2 E_0^2 / 2$ are the linear and quadratic electro-optical effect coefficients in the dimensionless form [11]. Mathematically, this dynamics equation (Eq. 14) is obtained by applying several coordinate transformations into a dimensionless form, such as $\tau = t / C_s$, $s = x / x_0$, $\xi = z / k_n^2 x_0$ and $\phi = 2 \eta_0 I_d / n_z$ \( \xi^2 \) $u$, where $x_0$ is an arbitrary spatial scaling width [11].
In this paper, bright solitons were selected as the observation category, namely solitons with a maximum peak intensity [6]. For the reviewed model of bright soliton dynamics is

\[ iu_t + \frac{1}{2} u_{xx} - \beta_1 \left( 1 + \left| u \right|^2 \right) e^{-\left| u \right|^2} u - \beta_2 \left( 1 + \left| u \right|^2 \right) e^{-\left| u \right|^2} u = 0 \quad \text{(15)} \]

its obtained by taking the definition of \( \rho = I_s/I_d = 0 \) in Eq. (14) [11]. In the next stage, a numerical analysis of the bright soliton dynamics equation (Eq. 15) was solved using the split-step Fourier method into Eq. (15) (for examples and benchmarks of this method, see Ref. [16]). Various ansatz models, as the initial condition of the envelope \( (u_0) \), will be applied. The effect of ansatz on the soliton propagation pattern was analyzed by involving the related physical parameters.

3. Numerical Results and Discussion

Soliton categories discussed in this paper are the optical spatial solitons in a photorefractive crystal, which have both linear and quadratic electro-optical effects (single crystal PMN-0.33PT). The type of optical spatial soliton in the discussion is a bright soliton, an optical spatial soliton with a maximum peak intensity [6]. The time-dependent of the bright soliton dynamics equation (Eq. 15) was solved using the numerical split-step Fourier method. In this section, the results of the numerical calculations are presented for various ansatz models in the initial conditions of the optical beam envelope. The effect of ansatz on the soliton propagation pattern was used as a topic of discussion.

For investigating the bright soliton propagation, first, the physical parameter values of the PMN-0.33 PT photorefractive crystal are defined as input the calculation. The coefficients of the linear and quadratic electro-optical effects are \( \beta_1 = 15.4612 \) and \( \beta_2 = 1.1724 \). Empirically, it is obtained by definition \( \beta_1 = (k_0 x_0)^2 n^2 r_e g_0/2 \) and \( \beta_2 = (k_0 x_0)^2 n^2 r_e g_0^2 (e - 1)^2 E_0^2/2 \) when the initial intensity value is \( E_0 = 1 \times 10^3 \text{ V/m} \), the free space wavelength or vacuum \( \lambda_0 = 632.8 \text{ nm} \), the arbitrary spatial scaling width \( x_0 = 20 \mu \text{m} \), the unperturbed index of refractive \( n_e = 2.562 \), the coefficient \( r_e = 182 \times 10^{-12} \text{ m/V} \), \( e_{eff} = e - 1 \) \( = 1.38 \times 10^{-16} \text{ m}^2/\text{V}^2 \), \( l \) (crystal width) = 1 cm, and \( V \) (emf bias) = 1000 V. For the simulation process, soliton propagation set for a distance of \( z = 10 \text{ m} \), with a total number of iterations \( \text{step num} = 350 \), and stepsize (change in iterations) \( h = 0.003 \). The ansatz models which chosen as the initial condition for the envelope, among other, \( u_0 = u = 0, \tau = \text{sech}(s), u_0 = \exp(-s^2), u_0 = \sinh(s), u_0 = \cosh(s), u_0 = \tanh(s) \), and other functions. Besides, the investigate carried out on several combinations of the ansatz models. The following is an illustration of the simulation results:

**Figure 1.** Soliton propagation in the PMN-0.33PT photorefractive crystals for the ansatz models: (a1)(a2) sech(s), (b1)(b2) exp(-s^2), and (c1)(c2) tanh(s) with contour.
Figure 2. Soliton propagation in the PMN-0.33PT photorefractive crystals for a combination of the ansatz models: (d1)(d2) sech^2(s), (e1)(e2) \( \exp(-s^2) \)^2, and (f1)(f2) tanh^2(s) with contour.

Figure 3. Soliton propagation in the PMN-0.33PT photorefractive crystals for a combination of the ansatz models: (g1)(g2) sech(s)\exp(-s^2), (h1)(h2) sech(s)tanh(s), and (i1)(i2) tanh(s) \exp(-s^2) with contour.

Figure 4. Soliton propagation in the PMN-0.33PT photorefractive crystals for the ansatz models: (j1) sinh(s), (k1) cosh(s), and (l1) another function, i.e., \exp(s)

From the simulation, we are known that the secant-hyperbolic \( \text{sech}(s) \) and Gaussian \( \exp(-s^2) \) come to be a model of ansatz, which most appropriate as an initial condition of the envelope to provide a bright soliton solution in photorefractive crystals (Figure 1: a1 and b1). Meanwhile, the different model doesn’t give a bright soliton solution (Figure 4). The ansatz model of \( \text{tanh}(s) \) is a bit unique, where this ansatz applicable offers like dark soliton solution, i.e., a soliton with a minimum intensity [6]
Physically, soliton occurs in a medium when the nonlinearity and dispersive properties balance each other [6,8,9]. In this case, the inaccuracy of the ansatz model causes the media’s nonlinearity to be weak and have reduced resistance to dispersion. As a result, the soliton’s peak tends to be exhausted and dispersed [6], as shown in Figure 4. Furthermore, the combination of several ansatz models offers a unique soliton solution (Figures 2 and 3). For squares, ansatz $\text{sech}(s)$, $\exp(-s^2)$, and $\tanh(s)$ give higher stability of a soliton solution in crystals, seen in the comparison between the intensity quality in Figure 1 and Figure 2. On the other hand, the ansatz combinations such as $\text{sech}(s)\tanh(s)$, and $\tanh(s)\exp(-s^2)$ also provide a unique solution, were showed the 2-soliton solution for a photorefractive crystal (Figure 3, h1 and i1). Overall, the combination of the ansatz provided a soliton in the crystal. Furthermore, we investigated the differences between the ansatz in secant-hyperbolic $\text{sech}(s)$ and Gaussian $\exp(-s^2)$ models on a bright soliton propagation process (Figure 5).

![Figure 5](image)

**Figure 5.** Bright soliton evolution in the PMN-0.33PT photorefractive crystal for ansatz models: (a) $\text{sech}(s)$ and (b) $\exp(-s^2)$.

Based on Figure 5, ansatz in secant-hyperbolic is seen as an ansatz model faster to balance media dispersion disturbances. This figure shows the ansatz offers a soliton width curve that more stable than the Gaussian model. It indicates that both of the ansatz model, $\text{sech}(s)$ and $\exp(-s^2)$, can produce a bright soliton solution for a photorefractive crystal, but the difference in the responding rate to changes in nonlinear properties and media dispersion.

Figure 5 also presents the evolution of the bright soliton propagation process in a photorefractive crystal. Change occurs before the bright soliton reaches a level of stability. Its physically due to the space-charge field at certain time intervals, namely before the stable level has a low magnitude. Therefore, the resulting photorefractive effect capacity is also low [11]. It has a consequence: a minimum soliton width is formed (the width of the soliton narrows) as in the evolutionary process shown by the shape of the descending curve in the figure. From this information, it is known that photorefractive crystals do not directly respond to a soliton’s formation (having a particular response time) [11].

4. Conclusion
Based on discussion results, it can be concluded that ansatz influences the soliton propagation pattern in photorefractive crystals. Ansatz in secant-hyperbolic $\text{sech}(s)$ and Gaussian $\exp(-s^2)$ functions is the most appropriate model for realizing solitons in a photorefractive crystal. Besides, ansatz in secant-hyperbolic is a model that is faster to balance media dispersion disturbances and is familiarly applied in a soliton study. Inaccuracy at the application of an ansatz model causes the nonlinearity properties of the medium to be weak and has tended to decrease in resistance to dispersion so that the resulting soliton peaks tend to be exhausted and dispersed, or in other words, the soliton can’t exist (or appear). Finally, solitons in a photorefractive crystal were found to undergoes evolution before achieving a level of stability.
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References
[1] Stegeman G I and Segev M 1999 Universality and Diversity Science 286 1518-23
[2] Hao L, Wang Q and Hou C 2004 Journal of Modren Optics 61 (15) 1236-1245
[3] Chen Z, Segev M and Christodoulides D N 2012 Rep. Prog. Phys. 75 086401
[4] Katti A and Yadav R A 2017 Phys. Lett. A 381 3 166-70
[5] Katti A, Yadav R A and Prasad A 2018 Wave Motion 77 64-76
[6] Agrawal G P 2013 Nonlinear Fiber Optics, 5th Edition (San Diedro. CA: Academic Press. Inc).
[7] Agrawal G P 2001 Applications of Nonlinear Fiber Optics (San Diedro: Academic Press. Inc).
[8] Kivshar Y S and Agrawal 2003 Optical Solitons: Form Fibers to Photonic Crystals (San Diedro: Academic Press. Inc.)
[9] Kajzar F and Reinisch R 2006 Beam Shaping and Control with Nonlinear Optics (New York: Academic Publisher)
[10] Katti A 2020 Optik 206 164212
[11] Katti A 2019 Chaos, Solitons & Fractals 26 23-31
[12] Marinova V, Lin S H, Liu R C and Hsu K Y 2016 Photorefractive Effect: Principles, Material, and Near-Infrared Holography (John Wiley & Sons)
[13] Christodoulides D N and Carvalho M I 1995 JOSA B 12 9 1628-1633
[14] Kukhtarev N V, Markov V B, Odulov S G, Soskin M S and Vinetskii V L 1978 Ferroelectrics 22 1 949-960
[15] Arfken G B, Weber H J, and Harris F E 2013 Mathematical Methods for Physicists, 7th Edition (New York: Academic Publisher)
[16] Ripai A, Abdullah Z, Syafwan M, Hidayat W 2020 Jurnal Ilmu Fisika 12 105-112