GLOBAL EXISTENCE AND BOUNDEDNESS IN A CHEMOTAXIS-STOKES SYSTEM WITH SLOW $p$-LAPLACIAN DIFFUSION

TAO WEIRUN AND LI YUXIANG

Abstract. This paper deals with a boundary-value problem in three-dimensional smooth bounded convex domains for the coupled chemotaxis-Stokes system with slow $p$-Laplacian diffusion

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \nabla \cdot (|\nabla n|^{p-2} \nabla n) - \nabla \cdot (n \nabla c), & x \in \Omega, & t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega, & t > 0, \\
    u_t &= \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, & t > 0, \\
    \nabla \cdot u &= 0, & x \in \Omega, & t > 0
\end{align*}
\]

where $\phi \in W^{2,\infty}(\Omega)$ is the gravitational potential. It is proved that global bounded weak solutions exist whenever $p > \frac{11}{12}$ and the initial data $(n_0, c_0, u_0)$ are sufficiently regular satisfying $n_0 \geq 0$ and $c_0 \geq 0$.

1. Introduction

In this paper, our goal is to establish the existence and the boundedness of the global weak solutions to the following chemotaxis-Stokes system with $p$-Laplacian diffusion

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \nabla \cdot (|\nabla n|^{p-2} \nabla n) - \nabla \cdot (n \nabla c), & x \in \Omega, & t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega, & t > 0, \\
    u_t &= \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, & t > 0, \\
    \nabla \cdot u &= 0, & x \in \Omega, & t > 0
\end{align*}
\]

(1.1)

in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, where the scalar function $n$ represents the density of aerobic bacteria, and $c$ represents the concentration of oxygen. The vector $u = (u_1, u_2, u_3)$ is the fluid velocity field and $P = P(x, t)$ represents the associated pressure of the fluid. The given function $\phi$ stands for the gravitational potential produced by the action of physical forces on the cell. The nonlinear diffusion $\nabla \cdot (|\nabla n|^{p-2} \nabla n)$ is called the slow $p$-Laplacian diffusion if $p > 2$, and called the fast $p$-Laplacian diffusion if $1 < p < 2$.

Chemotaxis describes biased movement of cells in response to the concentration gradient of a diffusible chemical signal. The most famous model used to describe this biochemotactic phenomenon is the Keller-Segel model which was first presented in [20]. The prototype of classical chemotaxis model reads as

\[
\begin{align*}
    n_t &= \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega, & t > 0, \\
    \tau c_t &= \Delta c - c + n, & x \in \Omega, & t > 0
\end{align*}
\]

(1.2)

where $n$ denotes the cell density and $c$ describes the concentration of the chemical signal which is directly produced by cells themselves. In the past 4 decades, this model has been attracted many mathematicians to study its qualitative properties such as critical mass phenomenon, blowup, boundedness, pattern formations and critical sensitivity exponents (e.g. see [2, 5, 14, 17, 18, 30–

2010 Mathematics Subject Classification. 35Q92, 35K55, 35Q35, 92C17.

Key words and phrases. chemotaxis, Navier-Stokes equation, nonlinear diffusion, $p$-Laplacian diffusion, global existence, boundedness.
32, 36, 41, 43] and the references therein). In addition to the original model, a large number of variants of the classical form have also been studied, including the system with the logistic terms (e.g. see [22, 38, 51]), multi-species chemotaxis system (e.g. see [21, 23, 29, 50]), attraction-repulsion chemotaxis system (e.g. see [24, 37]) and so on (e.g. see review articles [1, 13, 15, 16] for the further reading).

The chemotaxis-Navier-Stokes system was first proposed in [40]. Aerobic bacteria such as Bacillus subtilis often live in thin fluid layers near solid-air-water contact line, in which the biology of chemotaxis, metabolism, and cell-cell signaling is intimately connected to the physics of buoyancy, diffusion, and mixing (cf. [40]). Both bacteria and oxygen diffuse through the fluid, and they are also transported by the fluid (cf. [8] and [28]). Taking all these roles into account, the model in [40] reads as

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(c) \nabla c), & x \in \Omega, \ t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - nf(c), & x \in \Omega, \ t > 0, \\
    u_t + \kappa(u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \Phi, & x \in \Omega, \ t > 0, \\
    \nabla \cdot u &= 0, & x \in \Omega, \ t > 0,
\end{align*}
\]

(1.3)

where the domain \( \Omega \subset \mathbb{R}^d \), the vector \( u = (u_1(x,t), u_2(x,t), \cdots, u_d(x,t)) \) is the fluid velocity field and the associated pressure is represented by \( P = P(x,t) \).

In the last decade, the chemotaxis fluid system (1.3) has attracted much attention. In 2010, the author showed in [28] that in certain parameter regimes, the system (1.3) possesses local weak solutions in a bounded domain in \( \mathbb{R}^d, d = 2,3 \) with no-flux boundary condition and in \( \mathbb{R}^2 \) in the case of inhomogeneous Dirichlet conditions for the oxygen. In the same year, the authors proved in [9] that in the two-dimensional case, the Cauchy problem for (1.3) with \( \kappa = 0 \) admits global existence of weak solutions, provided that some further technical conditions are satisfied and the structural conditions on \( \chi \) and \( f \) are satisfied. It was also proved in [9] that under the two-dimensional setting, the chemotaxis-Navier-Stokes system (1.3) with \( \kappa = 1 \) and \( \Omega = \mathbb{R}^3 \) admits global classical solutions near constant steady states. In 2011, global existence of solutions to the Cauchy problem was investigated under certain conditions in [27]. The authors showed there that the chemotaxis-Navier-Stokes system (the system (1.3) with \( \kappa = 1 \)) possesses global weak solutions for large data. In 2012, it was proved in [42] that if \( \chi(s) \equiv 1 \) for \( s \in \mathbb{R} \) and \( f(s) \equiv s \) for \( s \in \mathbb{R} \), then the simplified chemotaxis-Stokes system (the system (1.3) with \( \kappa = 0 \)) possesses at least one global weak solution and the full chemotaxis-Navier-Stokes system (the system (1.3) with any \( \kappa \in \mathbb{R} \)) admits a unique global classical solution under the boundary condition \( \frac{\partial n}{\partial \nu} = \frac{\partial f}{\partial \nu} = u = 0 \) on \( \partial \Omega \) and suitable regularity assumptions on the initial data. In 2013, it was proved in [3] that there exist a global classical solution to the Cauchy problem with \( \Omega = \mathbb{R}^2 \) under the appropriate structural assumptions for \( \chi \) and \( f \). In 2014, the same author of [42] showed in [44] that the general classical solutions obtained in [42] stabilize to the spatially uniform equilibrium \((\bar{n}_0,0,0)\) with \( \bar{n}_0 = \frac{1}{|\Omega|} \int_{\Omega} n_0(x)dx \) as \( t \to \infty \) if \( \Omega \) is a bounded convex domain in \( \mathbb{R}^2 \). In 2015, the authors proved in [52] that such solution converges to the equilibrium \((\bar{n}_0,0,0)\) exponentially in time. In the same year, by deriving a new type of entropy-energy estimate, it was shown in [19] that the restricted condition in [44] that \( \Omega \) is essentially assumed to be convex can be removed. In 2016, the author proved in [25] that global classical bounded solutions to the Cauchy problem exist for regular initial data. In the same year, the author in [46] established global weak solutions of (1.3) in bounded convex domains \( \Omega \subset \mathbb{R}^3 \) with suitable regularity assumptions on the initial data and appropriate assumptions for \( \chi, f \) and \( \phi \). In 2017, the long-term behaviour of eventual energy solution was investigated in [47], which, namely, become smooth on some interval \([T, \infty)\) and uniformly converge in the large-time-limit. For more results of the well-posedness of the Cauchy problem to (1.3) in the whole space we refer the reader to [4, 27, 49, 54].
Recently, a number of papers studied the Keller-Segel system with the linear diffusion replaced by the nonlinear diffusion. In [7], the authors introduced the model

\[
\begin{aligned}
& n_t + u \cdot \nabla n = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (n \chi(c) \nabla c), \quad x \in \Omega, \ t > 0, \\
& c_t + u \cdot \nabla c = \Delta c - nf(c), \quad x \in \Omega, \ t > 0, \\
& u_t + \kappa (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi, \quad x \in \Omega, \ t > 0, \\
& \nabla \cdot u = 0,
\end{aligned}
\]

Under the assumption of porous media, taking \( D(n) = n^{m-1} \), global-in-time solution to the chemotaxis-Stokes system was constructed in [7] for general initial data if \( m \in \left( \frac{3}{2}, 2 \right] \), while the same result holds in three-dimensional setting under the constraint \( m \in \left( \frac{7}{5} + \frac{2m}{12}, 2 \right] \). Intuitively, the nonlinear diffusion \( \Delta u^m \) for \( m > 1 \) can prevent the occurrence of blow up. Under the three-dimensional setting, in [27], global-in-time weak solutions to the Cauchy problem was obtained for \( m = \frac{7}{5} \) under the appropriate structural assumptions for \( \chi \). It was shown in [45] that (1.4) admits a global bounded weak solutions to the chemotaxis-stokes system in bounded convex domains if \( m > \frac{7}{5} \). This partially extended a precedent result which asserted global solvability within the larger range \( m > \frac{\frac{7}{5}}{2} \), but only in a class of weak solutions locally bounded in \( \Omega \times [0, \infty) \) (cf. [39]). It was proved in [53] that the model possesses at least one global weak solution under the condition that \( m \geq \frac{2}{3} \). Recently, it was shown in [48] that the chemotaxis-Stokes system admits a global bounded weak solutions under the assumption that \( m > \frac{9}{8} \). Moreover, the obtained solutions are shown to approach the spatially homogeneous steady state \( \left( \frac{1}{\Omega} \int \Omega n_0, 0, 0 \right) \) in the large time limit. For smaller values of \( m > 1 \), up to now existence results are limited to classes of possibly unbounded solutions (cf. [10]).

In addition to porous media diffusion, \( p \)-Laplacian diffusion has also been considered in the Keller-Segel system. In [6], the authors studied the following \( p \)-Laplacian Keller-Segel model in dimension \( d \geq 3 \):

\[
\begin{aligned}
& n_t = \nabla \cdot (|\nabla n|^{p-2} \nabla n) - \nabla \cdot (n \nabla c), \quad x \in \mathbb{R}^d, \ t > 0, \\
& -\Delta c = n, \quad x \in \mathbb{R}^d, \ t > 0, \\
& n(x, 0) = n_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

and they proved the existence of a uniform in time \( L^\infty \) bounded weak solution for system (1.5) with the supercritical diffusion exponent \( 1 < p < \frac{3d}{d+1} \) in the multi-dimensional space \( \mathbb{R}^d \) under the condition that the \( L^{\frac{d(3-p)}{p}} \) norm of initial data is smaller than a universal constant. They also proved the local existence of weak solutions and a blow-up criterion for general \( L^1 \cap L^\infty \) initial data.

In [35], the authors studied the following chemotaxis-Navier-Stokes system with slow \( p \)-Laplacian diffusion

\[
\begin{aligned}
& n_t + u \cdot \nabla n = \nabla \cdot (|\nabla n|^{p-2} \nabla n) - \nabla \cdot (n \chi(c) \nabla c), \quad x \in \Omega, \ t > 0, \\
& c_t + u \cdot \nabla c = \Delta c - nf(c), \quad x \in \Omega, \ t > 0, \\
& u_t + (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi, \quad x \in \Omega, \ t > 0, \\
& \nabla \cdot u = 0,
\end{aligned}
\]

It is proved that if \( p > \frac{32}{15} \) and under appropriate structural assumptions on \( f \) and \( \chi \), for all sufficiently smooth initial data \((n_0, c_0, u_0)\) the model possesses at least one global weak solution.
In this paper, we shall consider the existence and the boundedness of the global weak solutions for the chemotaxis-Stokes system (1.1) with initial conditions

\[ n(x,0) = n_0(x), \ c(x,0) = c_0(x) \text{ and } u(x,0) = u_0(x), \quad x \in \Omega \]  

(1.7)

and the boundary conditions

\[ |\nabla n|^{p-2} \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \text{ and } u = 0 \quad \text{on } \partial \Omega \]  

(1.8)

in a bounded convex domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary, where \( \nu \) is the exterior unit normal vector on \( \partial \Omega \). As for the initial data, we shall suppose that

\[
\begin{aligned}
  n_0 &\in C^\infty(\overline{\Omega}) \text{ for some } \omega > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega \text{ and } n_0 \not\equiv 0, \\
  c_0 &\in W^{1,\infty}(\Omega) \text{ is nonnegative}, \\
  u_0 &\in D(A^\alpha) \text{ for some } \alpha \in (\frac{3}{4}, 1),
\end{aligned}
\]

(1.9)

where \( A = -P\Delta \) denotes the Stokes operator in \( L^2_0(\Omega) = \{ \varphi \in (L^2(\Omega))^3 | \nabla \cdot \varphi = 0 \} \) with its domain given by \( D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_0(\Omega) \), and with \( P \) representing the Helmholtz projection on \( L^2(\Omega) \) ([33]).

Before giving the main result, let us first give the definition of weak solution.

**Definition 1.1.** Let

\[
\begin{aligned}
  n &\in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \\
  c &\in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \quad \text{and} \\
  u &\in L^1_{loc}([0,\infty); W^{1,1}(\Omega; \mathbb{R}^3)),
\end{aligned}
\]

(1.10)  (1.11)  (1.12)

be such that \( n \geq 0 \) and \( c \geq 0 \) in \( \Omega \times (0,\infty) \), and

\[ |\nabla n|^{p-1}, \ n|\nabla c|, \text{ and } n|u| \text{ belong to } L^1_{loc}(\overline{\Omega} \times [0,\infty)). \]

(1.13)

We call \((n, c, u)\) a **global weak solution** of the chemotaxis-Stokes system (1.1) with initial condition (1.7) and boundary condition (1.8) if \( \nabla \cdot u = 0 \) in the distribution sense, if

\[
- \int_0^\infty \int_\Omega n \varphi_t - \int_\Omega n_0 \varphi(\cdot, 0) = - \int_0^\infty \int_\Omega |\nabla n|^{p-2} \nabla n \cdot \nabla \varphi + \int_0^\infty \int_\Omega n \nabla c \cdot \nabla \varphi + \int_0^\infty \int_\Omega n u \cdot \nabla \varphi, \]

(1.14)

for all \( \varphi \in C_0^\infty(\Omega \times [0,\infty)) \) fulfilling \( \frac{\partial \varphi}{\partial \nu} = 0 \) on \( \partial \Omega \times (0,\infty) \),

\[
- \int_0^\infty \int_\Omega c \varphi_t - \int_\Omega c_0 \varphi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^\infty \int_\Omega n c \varphi + \int_0^\infty \int_\Omega c u \cdot \nabla \varphi,
\]

(1.15)

for all \( \varphi \in C_0^\infty(\Omega \times [0,\infty)) \), and

\[
- \int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \varphi + \int_0^\infty \int_\Omega n \nabla \phi \cdot \varphi,
\]

(1.16)

for all \( \varphi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^3) \) such that \( \nabla \cdot \varphi \equiv 0 \) in \( \Omega \times (0,\infty) \).

Our main result reads as:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded convex domain with smooth boundary and \( \phi \in W^{2,\infty}(\Omega) \), and

\[ p > \frac{23}{11}. \]

(1.17)
Then for each \( n_0, c_0 \) and \( u_0 \) satisfying (1.9) there exist functions
\[
\begin{aligned}
&\{n \in L^\infty(\Omega \times (0, \infty)), \\
&c \in \bigcap_{s>1} L^\infty((0, \infty); W^{1,s}(\Omega)) \cap C^0(\overline{\Omega} \times [0, \infty)) \cap C^{1,0}(\overline{\Omega} \times (0, \infty)), \\
&u \in L^\infty(\Omega \times (0, \infty)) \cap L^2_{loc}(0, \infty); W^{1,2}_0(\Omega) \cap C^0(\overline{\Omega} \times [0, \infty))
\end{aligned}
\] (1.18)
such that the triple \((n, c, u)\) forms a global weak solution of (1.1), (1.7) and (1.8) in the sense of Definition 1.1.

The rest of this paper is organized as follows. In Section 2, we introduce a family of regularized problems and give some preliminary properties. Based on an energy-type inequality, a priori estimates are given in Section 3. Section 4 to section 7 are several useful uniform estimates. Finally, we give the proof of the main result in Section 8.

2. Regularized problems

The global weak solution of (1.1) is constructed as the limit of appropriate regularized problems. We regularize the cross-diffusive term in (1.1) by introducing a family \((\chi_\varepsilon)_{\varepsilon \in (0,1)} \subset C_0^\infty([0, \infty))\) (cf. [48]) fulfilling
\[
0 \leq \chi_\varepsilon \leq 1 \text{ in } [0, \infty), \quad \chi_\varepsilon \equiv 1 \text{ in } [0, \frac{1}{\varepsilon}] \quad \text{and} \quad \chi_\varepsilon \equiv 0 \text{ in } \left[\frac{2}{\varepsilon}, \infty\right),
\] (2.1)
and letting
\[
F_\varepsilon(s) = \int_0^s \chi_\varepsilon(\sigma) d\sigma, \quad s \geq 0,
\] (2.2)
for \( \varepsilon \in (0,1) \). Then \( F_\varepsilon \in C^\infty([0, \infty)) \) satisfies
\[
0 \leq F_\varepsilon(s) \leq s \quad \text{and} \quad 0 \leq F_\varepsilon'(s) \leq 1, \quad \text{for all } s \geq 0
\] (2.3)
as well as
\[
F_\varepsilon(s) \not\nearrow s \quad \text{and} \quad F_\varepsilon'(s) \not\nearrow 1, \quad \text{for all } s > 0, \quad \text{as } \varepsilon \searrow 0.
\] (2.4)
According to the idea from [48] (see also [39, 53]), let us consider the approximate variants of (1.1), (1.7) and (1.8).

\[
\begin{aligned}
&\partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot \left( (|\nabla n_\varepsilon|^2 + \varepsilon) \nabla n_\varepsilon \right) - \nabla \cdot (n_\varepsilon F_\varepsilon'(n_\varepsilon) \nabla c_\varepsilon), \quad x \in \Omega, \ t > 0, \\
&\partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) c_\varepsilon, \quad x \in \Omega, \ t > 0, \\
&\partial_t u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi, \quad x \in \Omega, \ t > 0, \\
&\nabla \cdot u_\varepsilon = 0, \quad x \in \Omega, \ t > 0, \\
&\frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0, \quad x \in \partial \Omega, \ t > 0, \\
&n_\varepsilon(x, 0) = n_0, \ c_\varepsilon(x, 0) = c_0, \ u_\varepsilon(x, 0) = u_0, \quad x \in \Omega
\end{aligned}
\] (2.5)
for \( \varepsilon \in (0,1) \), where the families of approximate initial data \( n_{0\varepsilon} \geq 0, \ c_{0\varepsilon} \geq 0 \) and \( u_{0\varepsilon} \) have the properties
\[
\begin{aligned}
&\{n_{0\varepsilon}\}_{\varepsilon \in (0,1)} \text{ uniformly bounded in } L^\infty(\Omega), \\
&\sqrt{n_{0\varepsilon}} \in C_0^\infty(\Omega), \quad \|c_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}, \\
&\sqrt{n_{0\varepsilon}} \to \sqrt{n_0} \text{ a.e. in } \Omega \text{ and } W^{1,2}(\Omega) \text{ as } \varepsilon \searrow 0.
\end{aligned}
\] (2.6)

Lemma 2.2. Proof of our result.

Integrating the first equation in (2.5) we obtain (2.10). And an application of the maximum principle to the second equation in (2.5) gives (2.11).

3. An energy-type inequality

In this section, we derive the following energy-type inequality of the approximate problems (2.5) and give some consequence.
Lemma 3.1. Assume $p \geq 2$, then there exist $\kappa > 0$ and $C > 0$ (independent of $\varepsilon$) such that

$$
\frac{d}{dt} \left\{ \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_{\Omega} \frac{\nabla c_\varepsilon}{c_\varepsilon} + \kappa \int_{\Omega} |u_\varepsilon|^2 \right\} + \frac{1}{C} \left\{ \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_{\Omega} \frac{\nabla c_\varepsilon}{c_\varepsilon} + \kappa \int_{\Omega} |u_\varepsilon|^2 \right\} 
+ \frac{1}{C} \left\{ \int |\nabla n_\varepsilon^{1-p}| + \int \frac{\nabla c_\varepsilon^4}{c_\varepsilon^3} + \int |\nabla u_\varepsilon|^2 \right\} \leq C, \quad \text{for all } t > 0. \quad (3.1)
$$

Proof. By means of straightforward computation using the first two equations in (2.5) (cf. [42] for details), we obtain the identity

$$
\frac{d}{dt} \left\{ \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_{\Omega} \frac{\nabla c_\varepsilon}{c_\varepsilon} \right\} + \int_{\Omega} (|\nabla n_\varepsilon|^2 + \varepsilon) \frac{p-2}{p} |\nabla n_\varepsilon|^2 \frac{n_\varepsilon}{c_\varepsilon} + \int c_\varepsilon |D^2 \ln c_\varepsilon|^2
$$

$$
= -\frac{1}{2} \int \frac{\nabla c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon) + \int \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon)
$$

$$
- \frac{1}{2} \int F_\varepsilon (n_\varepsilon) \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{1}{2} \int (\frac{p}{p-1}) \int |\nabla n_\varepsilon^{1-p}|, \quad \text{for all } t > 0. \quad (3.2)
$$

Using the inequalities

$$
\int (|\nabla n_\varepsilon|^2 + \varepsilon) \frac{p-2}{p} |\nabla n_\varepsilon|^2 \frac{n_\varepsilon}{c_\varepsilon} \geq \int \frac{\nabla n_\varepsilon^p}{n_\varepsilon} = \left( \frac{p}{p-1} \right) \int |\nabla n_\varepsilon^{1-p}|, \quad \text{for all } t > 0
$$

and $\int \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2} \leq C_1 \int c_\varepsilon |D^2 \ln c_\varepsilon|^2$ for all $t > 0$ ([42, Lemma 3.3]) with $C_1 = (2 + \sqrt{3})^2$, we can derive from (3.2) that

$$
\frac{d}{dt} \left\{ \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_{\Omega} \frac{\nabla c_\varepsilon}{c_\varepsilon} \right\} + \left( \frac{p}{p-1} \right) \int |\nabla n_\varepsilon^{1-p}|^p + \frac{1}{C_1} \int \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2}
$$

$$
\leq -\frac{1}{2} \int \frac{\nabla c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon) + \int \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon)
$$

$$
- \frac{1}{2} \int F_\varepsilon (n_\varepsilon) \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{1}{2} \int \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu}, \quad \text{for all } t > 0. \quad (3.3)
$$

From the facts that the nonnegativity of $F_\varepsilon$ and $\frac{\partial |\nabla c_\varepsilon|^2}{c_\varepsilon} \leq 0$ on $\partial \Omega \times (0, \infty)$ since $\Omega$ due to convex ([26, Lemma 2.1.1]), we know that the last two summands on the right of (3.2) are nonpositive. This shows that (3.3) leads to the inequality

$$
\frac{d}{dt} \left\{ \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_{\Omega} \frac{\nabla c_\varepsilon}{c_\varepsilon} \right\} + \left( \frac{p}{p-1} \right) \int |\nabla n_\varepsilon^{1-p}|^p + \frac{1}{C_1} \int \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^2}
$$

$$
\leq -\frac{1}{2} \int \frac{\nabla c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon) + \int \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon), \quad \text{for all } t > 0. \quad (3.4)
$$

To estimate the right hand side of (3.4), we make the following computation by using integration by parts, the identity $\int \frac{1}{c_\varepsilon} \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon) = \frac{1}{2} \int \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon)$ and Young’s inequality.
\[-\frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) + \int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon) = \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) - \frac{1}{2} \int_\Omega c_\varepsilon \nabla c_\varepsilon \cdot (\nabla^2 c_\varepsilon \cdot u_\varepsilon) - \frac{1}{2} \int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) = - \frac{1}{2} \int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \leq \frac{1}{2C_1} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} + \frac{C_1}{2} \int_\Omega |c_\varepsilon| \cdot |\nabla u_\varepsilon|^2 \leq \frac{1}{2C_1} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} + C_2 \int_\Omega |\nabla u_\varepsilon|^2, \quad \text{for all } t > 0 \tag{3.5} \]

with \( C_2 = \frac{C_2}{2} s_0 \), where we have used (21.11) in the last inequality. So we obtain

\[
\frac{d}{dt} \left\{ \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \right\} + \left( \frac{p}{p-1} \right) \int_\Omega |\nabla n_\varepsilon^{1-p} n_\varepsilon|^p + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} \leq C_2 \int_\Omega |\nabla u_\varepsilon|^2, \quad \text{for all } t > 0. \tag{3.6} \]

To treat the right hand side of (3.6), we test the third equation in (2.5) by \( u_\varepsilon \) and use the Sobolev embedding \( W^{1,2}_0(\Omega) \hookrightarrow L^6(\Omega) \) to obtain

\[
\frac{1}{2} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \leq \|\nabla \phi\|_{L^\infty(\Omega)} \|u_\varepsilon\|_{L^6(\Omega)} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \leq C_3 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)} \leq \frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 + \frac{C_3^2}{2} \|n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)}^\frac{2p}{p-1}, \quad \text{for all } t > 0, \tag{3.7} \]

where Cauchy-Schwarz inequality has been used in the last inequality. Since \( p \geq 2 > \frac{11}{7} \), we have \( \frac{1}{p-1} \leq \frac{5(p-1)}{6p} \) and hence \( W^{1,p}(\Omega) \hookrightarrow L^{\frac{6p}{5(p-1)}}(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega) \). Now the Gagliardo-Nirenberg inequality together with (2.10) shows that there exist positive constants \( C_4 > 0 \) and \( C_5 > 0 \) such that

\[
\frac{C_3^2}{2} \|n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)}^\frac{2p}{p-1} \leq C_4 \left\{ \|\nabla n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)} \|n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)} + \|n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)}^\frac{2p}{p-1} \right\} \leq C_4 \left\{ \|\nabla n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)} + \|n_\varepsilon^\theta\|_{L^1(\Omega)} \right\} \leq C_5 \|\nabla n_\varepsilon^{1-1/p}\|_{L^{\frac{p}{p-1}}(\Omega)} + C_5, \quad \text{for all } t > 0, \tag{3.8} \]

where \( \theta = \frac{p-1}{4(p-3)} = \frac{1}{8} + \frac{1}{8(2p-3)} \in \left( \frac{1}{8}, \frac{1}{4} \right) \) thanks to \( p \geq 2 \). Noticing that \( p \geq 2 > \frac{7}{4} \) implies \( \frac{p}{2(2p-3)} < p \), we then use Young’s inequality along with (3.7) and (3.8) to see that there exist
positive constant $C_6 > 0$ such that for all $t > 0$
\[
\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \leq 2C_5 \|\nabla n_\varepsilon^{1-1/p}\|_{L^p(\Omega)}^{2(p-1)} + 2C_5
\]
\[
\leq \frac{1}{2(C_2 + 1)} \left( \frac{p}{p-1} \right)^p \|\nabla n_\varepsilon^{1-1/p}\|_{L^p(\Omega)}^p + C_6
\]  \hspace{1cm} (3.9)

Combining (3.9) with (3.6), we obtain
\[
\frac{d}{dt} \left\{ \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + (C_2 + 1) \int_\Omega |u_\varepsilon|^2 \right\}
\hspace{1cm} + \frac{1}{2} \left( \frac{p}{p-1} \right)^p \int_\Omega |\nabla n_\varepsilon^{1-1/p}|^p + \frac{1}{2C_1} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2
\leq 2(C_2 + 1)C_6, \quad \text{for all } t > 0.
\]  \hspace{1cm} (3.10)

Using the elementary inequality $z \ln z \leq 2z^{\frac{6}{5}}$ for all $z \geq 0$, (3.8) and Young’s inequality, we have
\[
\int_\Omega n_\varepsilon \ln n_\varepsilon \leq 2 \int_\Omega n_\varepsilon^{\frac{6}{5}}
\hspace{1cm} = 2\|n_\varepsilon^{1-1/p}\|^2_{L^{\frac{5p}{3(p-1)}}(\Omega)}
\hspace{1cm} \leq 2\|n_\varepsilon^{1-1/p}\|^2_{L^{\frac{2p}{p-1}}(\Omega)} + 2
\hspace{1cm} \leq \frac{4}{C_5^2} \left( C_5 \|\nabla n_\varepsilon^{1-1/p}\|_{L^p(\Omega)}^p + C_5 \right) + 2
\hspace{1cm} \leq \frac{4}{C_5^2} \left( C_5 \|\nabla n_\varepsilon^{1-1/p}\|_{L^p(\Omega)}^p + 2C_5 \right) + 2
\hspace{1cm} = C_7 \|\nabla n_\varepsilon^{1-1/p}\|_{L^p(\Omega)}^p + C_8, \quad \text{for all } t > 0,
\]  \hspace{1cm} (3.11)

where $C_7 = \frac{4C_5}{C_3^2}$ and $C_8 = \frac{8C_5}{C_3^2} + 2$. And using Young’s inequality and (2.11), we have
\[
\frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \leq \frac{1}{4} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{1}{4} \int_\Omega c_\varepsilon \leq \frac{1}{4} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{8}{4} \Omega.
\]  \hspace{1cm} (3.12)

From Poincaré inequality, we obtain some constant $C_9 > 0$ such that
\[
\int_\Omega |u_\varepsilon|^2 \leq C_9 \int_\Omega |\nabla u_\varepsilon|^2, \quad \text{for all } t > 0.
\]  \hspace{1cm} (3.13)

Combining (3.11)-(3.13), we immediately arrive at
\[
\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \kappa \int_\Omega |u_\varepsilon|^2
\hspace{1cm} \leq C_7 \|\nabla n_\varepsilon^{1-1/p}\|_{L^p(\Omega)}^p + \frac{1}{4} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \kappa C_9 \int_\Omega |\nabla u_\varepsilon|^2 + C_8 + \frac{8}{4} \Omega, \quad \text{for all } t > 0.
\]

This together with (3.10) readily establishes (3.1) upon evident choices of $\kappa$ and $C$. \hspace{1cm} \(\square\)

The following is the direct consequence of Lemma 3.1 and will be used frequently in the sequel.

**Lemma 3.2.** There exist $C > 0$ such that for all $\varepsilon \in (0, 1)$,
\[
\int_t^{t+1} \int_\Omega |\nabla n_\varepsilon^{1-1/p}|^p \leq C, \quad \text{for all } t \geq 0
\]  \hspace{1cm} (3.14)
and
\[ \int_t^{t+1} \int_\Omega |\nabla c_\varepsilon|^4 \leq C, \quad \text{for all } t \geq 0 \] (3.15)
as well as
\[ \int_t^{t+1} \int_\Omega |\nabla u_\varepsilon|^2 \leq C, \quad \text{for all } t \geq 0. \] (3.16)

Proof. All inequalities immediately result from an integration of (3.1) because of (2.11) and the fact that \( \int_\Omega n_\varepsilon \ln n_\varepsilon \leq -\frac{[n]}{\varepsilon} \) for all \( t \geq 0. \)

\[ \square \]

4. Preparing an inductive argument

With Lemma 2.2 and Lemma 3.1 at hand, we can improve the integrability of \( n_\varepsilon \) step by step.

Lemma 4.1. Let \( m_\ast \geq 1, q \geq 2, p > \frac{2q}{p-1}, m > m_\ast \) be such that
\[ 2q(p-1) - p m_\ast + p - 2 \leq m \leq \frac{2q(p-1) - p m_\ast + (2q - 1)(p - 2)}{3}. \] (4.1)
Then for all \( K > 0 \) there exist \( C = C(m_\ast, m, p, q, K) > 0 \) such that if for some \( \varepsilon \in (0, 1) \) there hold
\[ \int_\Omega n_\varepsilon^m (\cdot, t) \leq K, \quad \text{for all } t \geq 0 \] (4.2)
and
\[ \int_t^{t+1} \int_\Omega |\nabla c_\varepsilon (\cdot, s)|^{2q} ds \leq K, \quad \text{for all } t \geq 0, \] (4.3)
then we have
\[ \int_\Omega n_\varepsilon^m (\cdot, t) \leq C, \quad \text{for all } t \geq 0 \] (4.4)
and
\[ \int_t^{t+1} \int_\Omega |\nabla n_\varepsilon^{\frac{m_\ast - 1}{\varepsilon} + 1}|^p \leq C, \quad \text{for all } t \geq 0. \] (4.5)

Proof. Testing the first equation in (2.5) by \( n_\varepsilon^{p-1} \) and using Young’s inequality along with (2.11) we can see that there are positive constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for all \( t > 0, \)
\[ \frac{1}{m} \frac{d}{dt} \int_\Omega n_\varepsilon^m + (m-1) \int_\Omega |\nabla n_\varepsilon|^p n_\varepsilon^{m-2} \leq C_1 (m-1) \int_\Omega n_\varepsilon^{m-1} |\nabla c_\varepsilon| \cdot |\nabla n_\varepsilon| \]
\[ \leq (m-1) \int_\Omega \left( \frac{1}{2} |\nabla n_\varepsilon|^p n_\varepsilon^{m-2} + C_2 (n_\varepsilon^{\frac{2-m}{p} + m-1} |\nabla c_\varepsilon|)^{p'} \right) \]
where \( p' = \frac{p}{p-1}, \) so that
\[ \frac{1}{m} \frac{d}{dt} \int_\Omega n_\varepsilon^m + \frac{m-1}{2} \int_\Omega |\nabla n_\varepsilon|^p n_\varepsilon^{m-2} \leq C_2 (m-1) \int_\Omega (n_\varepsilon^{\frac{2-m}{p} + m-1} |\nabla c_\varepsilon|)^{p'}, \quad \text{for all } t > 0. \] (4.6)
Let \( \beta = \left( \frac{2-m}{p} + m-1 \right) * p' = m + \frac{1}{p-1} - 1 \) and \( \tilde{m} = \frac{m-2}{p} + 1. \) Applying Young’s inequality to (4.6), we obtain \( C_2 > 0 \) such that
\[ \frac{1}{m} \frac{d}{dt} \int_\Omega n_\varepsilon^m + \frac{(m-1)}{2 \tilde{m} p} \int_\Omega |\nabla \tilde{n}_\varepsilon|^p \leq C_2 (m-1) \int_\Omega n_\varepsilon^{\beta} |\nabla c_\varepsilon|^{p'} \]
\[ \leq C_2 (m-1) \int_\Omega (n_\varepsilon^{\frac{2q}{2q-\beta} p} + |\nabla c_\varepsilon|^2 q), \quad \text{for all } t > 0. \] (4.7)
Denote $A_1 = \frac{2m^3}{2q-p}$. If $A_1 \leq m_\ast$, then we can apply Hölder inequality together with (4.2) to (4.7) to obtain $C_3 > 0$ fulfilling
\[
\frac{1}{m} \frac{d}{dt} \int_\Omega n_\varepsilon^m + (\frac{m - 1}{2m^p}) \int_\Omega |\nabla n_\varepsilon|^{2p} \leq C_3 (m - 1)(1 + \int_\Omega |\nabla c_\varepsilon|^{2q}), \quad \text{for all } t > 0. \tag{4.8}
\]

If $A_1 > m_\ast$. Due to our assumption $p \geq 2$, $q \geq 2$ and $m > \frac{2q}{p-1}$ we have
\[
\frac{\bar{m}}{A_1} - \left(\frac{1}{p} - \frac{1}{3}\right) = \frac{m(2q(p - 1) - 3) + (p - 2)(4q - 3)}{6q(m(p - 1) - p + 2)} > 0.
\]
Thus $W^{1,p}(\Omega) \hookrightarrow L^{\frac{A_1}{m}}(\Omega) \hookrightarrow L^{\frac{A_1}{m}}(\Omega)$, accordingly the Gagliardo-Nirenberg inequality together with (2.10) provides $C_4 > 0$ and $C_5 > 0$ such that
\[
\int_\Omega n_\varepsilon^{\frac{\beta_2}{m}2 - \frac{\beta_1}{m}p} \leq C_4 \|\nabla n_\varepsilon\|_{\frac{\beta_1}{m}p} + C_4 \|n_\varepsilon\|_{\frac{\beta_2}{m}p} \leq C_5 \|\nabla n_\varepsilon\|_{\frac{\beta_1}{m}p}^{\theta_1} + 1, \quad \text{for all } t > 0, \tag{4.9}
\]
where
\[
\theta_1 = \frac{3(m + p - 2)(2q(m(p - 1) - p(m_\ast + 1) + m_\ast + 2) + pm_\ast)}{2q(m(p - 1) - p + 2)(3(m + p - 2) + (p - 3)m_\ast)}
\]
satisfies $\frac{\bar{m}}{A_1} = \theta_1(\frac{1}{p} - \frac{1}{3}) + (1 - \theta_1)\frac{\bar{m}}{m_\ast}$. A routine computation shows that $\theta_1 \in (0, 1)$. Indeed, the left inequality of (4.1) ensures that
\[
2q(m(p - 1) - p(m_\ast + 1) + m_\ast + 2) + pm_\ast > 0. \tag{4.10}
\]
Since $q \geq 2$, the left inequality of (4.1) implies
\[
m > \frac{(2q(p - 1) - p)m_\ast}{2q(p - 1)} + \frac{p - 2}{p - 1} = \frac{(p - 1 - \frac{p}{2q})m_\ast}{(p - 1)} + \frac{p - 2}{p - 1} \geq \frac{(3p - 4)m_\ast + p - 2}{4p - 4},
\]
hence we obtain
\[
3(m + p - 2) + (p - 3)m_\ast = 3 \left( m - (m_\ast(1 - \frac{p}{3}) - p + 2) \right) > 3 \left( m_\ast(3p - 4) + \frac{p - 2}{p - 1} - (m_\ast(1 - \frac{p}{3}) - p + 2) \right)
\]
\[
= 3 \left( m_\ast(3p - 7) + 12(p - 2)p \right) > 0. \tag{4.11}
\]
And our assumption $m > \frac{p-2}{p-1}$ and $p \geq 1$ ensure that $m(p - 1) - p + 2 > 0$, this together with (4.10) and (4.11) shows that $\theta_1 > 0$. On the other hand, we claim that
\[
\theta_1 - 1 = \frac{pm_\ast(2q(-(m + 2)p + m + 4) + 3(m + p - 2))}{2q(m(p - 1) - p + 2)(3(m + p - 2) + (p - 3)m_\ast)} < 0. \tag{4.12}
\]
In fact, from $q \geq 2$ and $p \geq 2 > \frac{17}{9}$ we have
\[
2q(-(m + 2)p + m + 4) + 3(m + p - 2) = -(2q(p - 1) - 3)m - (p - 2)(4q - 3)
\]
\[
\leq -(2q(p - 1) - 3) - (p - 2)(4q - 3)
\]
\[
= -2q(3p - 5) + 3(p - 1)
\]
\[
\leq -4(3p - 5) + 3(p - 1)
\]
\[
= 17 - 9p < 0,
\]
this together with \( m > \frac{p-2}{p-1} \), (4.11) and (4.12) confirm that \( \theta_1 < 1 \).

Now, the right inequality of (4.1) ensures that
\[
\frac{A_1}{m} \theta_1 - p = \frac{p^2(3m - p(2q - 1)(m_\ast + 3) + 2(q(m_\ast + 6) - 3))}{(2(p - 1)q - p)(3m + p - 2) + (p - 3)m_\ast)} = \frac{3p^2 \left( m - \frac{2q(p - 1) - 2m_\ast + (p - 2)(2q - 1)}{3} \right)}{(2(p - 1)q - p)(3m + p - 2) + (p - 3)m_\ast)} < 0,
\]
this together with (4.9) and Young’s inequality shows that
\[
C_2(m - 1) \int_\Omega n_\varepsilon^{\frac{2q}{2q-p}} \leq C_2 C_5(m - 1)(\| \nabla n_\varepsilon^{\frac{A_1}{m}} \|^p_{p} \theta_1 + 1) \leq \frac{(m - 1)}{4m^p} \| \nabla n_\varepsilon^{\frac{A_1}{m}} \|_{p}^p + C_6(m - 1), \quad \text{for all } t > 0, \quad (4.13)
\]
with some \( C_6 \geq C_2 > 0 \).

Substituting (4.13) into (4.7), we obtain
\[
\frac{1}{m} \frac{d}{dt} \int_\Omega n_\varepsilon^m + \frac{(m - 1)}{4m^p} \int_\Omega |\nabla n_\varepsilon^{\tilde{m}}|^p \leq C_6(m - 1)(1 + \int_\Omega |\nabla \varepsilon^{2q}) \quad \text{for all } t > 0. \quad (4.14)
\]

We can generate a linear absorption term in (4.14). Noticing \( \tilde{m} = \frac{m-2}{p} + 1 \), we have the inequality
\[
\frac{\tilde{m}}{m} - \left( \frac{1}{p} - \frac{1}{3} \right) = \frac{p - 2}{m^p} + \frac{1}{3} > 0, \quad (4.15)
\]
so there holds \( W^{1,p}(\Omega) \hookrightarrow L^{\frac{m}{\tilde{m}}}(\Omega) \hookrightarrow L^{\frac{m_\ast}{m}}(\Omega) \). In addition, define \( \theta_2 = \frac{3(m - p - 2)(m - m_\ast)}{m(3(m - m_\ast) + 3(p-2) + 3p + (p - 2))} \)
satisfying \( \frac{\tilde{m}}{m} = \theta_2 \left( \frac{1}{p} - \frac{1}{3} \right) + \left( 1 - \theta_2 \right) \frac{m_\ast}{m_\ast} \) and \( \theta_2 \in (0, 1) \) thanks to \( \theta_2 - 1 = \frac{3(m - p - 2)}{m(3(m - m_\ast) + 3(p-2) + 3p + (p - 2))} < 0 \).

Using the Gagliardo-Nirenberg inequality, (4.2) and Young’s inequality, we obtain \( C_7 > 0 \) and \( C_8 > 0 \) such that
\[
\int_\Omega n_\varepsilon^m = \| n_\varepsilon^{\tilde{m}} \|_{m}^{\frac{m}{m}} \leq C_7 \| \nabla n_\varepsilon^{\frac{\tilde{m}}{m}} \|_{\theta_2} \| n_\varepsilon \|_{m_\ast}^{\frac{m}{m}} (m - 2 - \frac{1}{\theta_2}) \leq C_8 \| \nabla n_\varepsilon \|_{\theta_2} \| n_\varepsilon \|_{m_\ast} + C_7 \| n_\varepsilon \|_{m_\ast}^{1 - \frac{1}{\theta_2}} \leq C_8 \| \nabla n_\varepsilon \|_{\theta_2} + 1 \leq C_8 \| \nabla n_\varepsilon \|_{p} + 2, \quad \text{for all } t > 0, \quad (4.16)
\]
where we have used \( \frac{\tilde{m}}{m} \theta_2 - p = -\frac{m_\ast(p + m_\ast + 3(p - 2))}{m(m + p - 2 - m_\ast) + m_\ast(p - 2)} \) < 0 in the last inequality.

A combination of (4.14) and (4.16) shows that
\[
\frac{1}{m} \frac{d}{dt} \int_\Omega n_\varepsilon^m + \frac{(m - 1)}{8m^p C_8} \int_\Omega n_\varepsilon^m + \frac{(m - 1)}{8m^p} \int_\Omega |\nabla n_\varepsilon|^p \leq C_6(m - 1)(1 + \int_\Omega |\nabla \varepsilon^{2q}) + \frac{(m - 1)}{4m^p}, \quad (4.17)
\]
for all \( t > 0 \). Let \( y(t) = \int_\Omega n_\varepsilon^m \cdot t, t > 0, \) and \( h(t) = C_6(m - 1)(1 + \int_\Omega |\nabla \varepsilon^{2q}) + \frac{(m - 1)}{4m^p}, t > 0 \).

Then (4.17) can be rewritten as
\[
\frac{1}{m} y'(t) + \frac{(m - 1)}{8m^p C_8} y(t) + \frac{(m - 1)}{8m^p} \int_\Omega |\nabla n_\varepsilon|^p \leq h(t), \quad \text{for all } t > 0. \quad (4.18)
\]
In view of (4.3), we have
\[
\int_t^{t+1} h(s) ds \leq C_9 = C_6(m - 1)(1 + K + \frac{(m - 1)}{4m^p}), \quad \text{for all } t > 0. \quad (4.19)
\]
Obviously, (4.17) implies
\[ \frac{1}{m} \frac{d}{dt} \int_{\Omega} n_\varepsilon^m + \frac{(m-1)}{8m^pC_8} \int_{\Omega} n_\varepsilon^m \leq C_6(m-1)(1 + \int_{\Omega} |\nabla c_\varepsilon(\cdot, t)|^{2q}) + \frac{(m-1)}{4m^p}, \quad \text{for all } t > 0. \quad (4.20) \]

According to an elementary lemma on decay in linear first-order ODEs with suitably decaying inhomogeneities (see e.g. [34, Lemma 3.4]), we can deduce from (4.19) and (4.20) that \( y(t) < C_{10} \) for some \( C_{10} > 0 \). The boundedness of \( y(t) \) together with (4.18) and (4.19) shows that
\[ (m-1) \frac{1}{8m^p} \int_t^{t+1} \int_{\Omega} |\nabla n_\varepsilon^m|^{p} \leq \frac{1}{m} C_{10} + \int_t^{t+1} h(t) \leq \frac{C_{10}}{m} + C_9, \quad \text{for all } t > 0, \]
so that indeed both (4.4) and (4.5) hold with some suitably large \( C = C(m_*, m, p, q, K) > 0 \).

\[ \square \]

**Remark 4.1.** The set of \( m \) that satisfies (4.1) in Lemma 4.1 is not empty. On the one hand, the right-hand side of (4.1) is larger than the left-hand side:
\[ \left( \frac{2q(p-1) - p}{3} m_* + (2q-1)(p-2) \right) - \left( \frac{2q(p-1) - p}{2q(p-1)} \right) m_* + \frac{2q-2}{p-1} \]
\[ = \frac{(2(p-1)q - p)((2q(p-1) - 3)m_* + 6q(p-2))}{6(p-1)q} > 0. \]

One the other hand, the right-hand side of (4.1) is bigger than 1:
\[ \left( \frac{2q(p-1) - p}{3} m_* + (2q-1)(p-2) \right) - 1 \]
\[ = \frac{1}{3} \left( 6q(p-2) + 3(1-p) + m_* (2q(p-1) - p) \right) \]
\[ \geq \frac{1}{3} \left( 6q(p-2) + 3(1-p) + 2q(p-1) - p \right) \]
\[ = \frac{1}{3} (2q(4p - 7) + 3 - 4p) \]
\[ \geq \frac{1}{3} (4(4p - 7) + 3 - 4p) \]
\[ = \frac{1}{3} (12p - 25) > 0, \]

since \( p > \frac{25}{12} \).

Based on Lemma 4.1, we can iteratively get the following conclusion.

**Lemma 4.2.** Let \( p > \frac{25}{12} \) and
\[ \begin{cases} r = \frac{9(p-2)}{7-3p}, & \frac{25}{12} < p < \frac{7}{3}, \\ r \in (1, \infty), & p \geq \frac{7}{3}. \end{cases} \quad (4.21) \]

Then for all \( m \in [1, r) \) there exist \( C = C(m) > 0 \) such that for all \( \varepsilon \in (0, 1) \),
\[ \int_{\Omega} n_\varepsilon^m(\cdot, t) \leq C, \quad \text{for all } t \geq 0 \]
and
\[ \int_t^{t+1} \int_{\Omega} |\nabla n_\varepsilon^{m-2} + 1|^{p} \leq C, \quad \text{for all } t \geq 0. \]
Proof. Based on (2.10) and Lemma 3.2, we can apply Lemma 4.1 to \( q = 2 \) and define \( (m_k)_{k \in \mathbb{N}_0} \subset \mathbb{R} \) by letting \( m_0 = 1 \) and

\[
m_{k+1} = (p - \frac{4}{3})m_k + 3(p - 2), \quad \text{for } k \geq 0.
\]

If \( p \geq \frac{7}{3} \), then \( m_{k+1} - m_k \geq 1 \), and hence the sequence \( (m_k)_{k \in \mathbb{N}_0} \) is strictly increasing with \( m_k \not\to \infty \) as \( k \to \infty \).

If \( p \in \left(\frac{25}{12}, \frac{7}{3}\right) \), we can deduce from (4.24) that

\[
m_k = (p - \frac{4}{3})^k \frac{12p - 25}{3p - 7} + \frac{9(p - 2)}{7 - 3p},
\]

therefore the sequence \( (m_k)_{k \in \mathbb{N}_0} \) is strictly increasing with an upper bound \( \frac{9(p-2)}{7-3p} \) as \( k \to \infty \) since \( p - \frac{4}{3} \in (0, 1) \) and \( \frac{12p-25}{3p-7} < 0 \). So by means of an interpolation argument it is clear that we only need to prove (4.22) and (4.23) for \( m = m_k \) and each \( k \in \mathbb{N}_0 \). For this purpose, we first point out that the case \( k = 0 \) can be proved by the combination of (2.10) and Lemma 3.2, so that using mathematical induction, it suffices to prove that if for any \( k \in \mathbb{N}_0 \) there hold

\[
\int_\Omega n_{\varepsilon}^{m_k}(-, t) \leq C_1(m) \quad \text{and} \quad \int_\Omega \int_t^{t+1} |\nabla n_{\varepsilon}^{m_k-2} + 1|^p \leq C_1(m), \quad \text{for all } t \geq 0 \text{ and each } \varepsilon \in (0, 1)
\]

with some \( C_1(m) > 0 \), then we can find \( C_2(m) > 0 \) satisfying

\[
\int_\Omega n_{\varepsilon}^{m_{k+1}}(-, t) \leq C_2(m) \quad \text{and} \quad \int_\Omega \int_t^{t+1} |\nabla n_{\varepsilon}^{m_k+1-2} + 1|^p \leq C_2(m), \quad \text{for all } t \geq 0 \text{ and each } \varepsilon \in (0, 1).
\]

To this end, we note that the requirements (4.2) and (4.3) from Lemma 4.1 are satisfied with \( m_* = m_k \) and \( q = 2 \) thanks to the first inequality in (4.25) and (3.15). Applying Lemma 4.1 to \( m = m_{k+1} \), we arrive at (4.26).

5. Improving Estimate for \( \nabla c_\varepsilon \)

It follows from Lemma 4.2 that if \( p \geq \frac{7}{3} \), we can using the Moser iteration method to get the \( L^\infty \) estimate of \( n_{\varepsilon} \). But for \( p < \frac{7}{3} \), we couldn’t use the Moser iteration, since (4.22) holds only for \( m < \frac{9(p-2)}{7-3p} \). For \( p < \frac{7}{3} \), in order to obtain (4.22) for all \( m \geq 1 \), we first need to improve the integrability of \( \nabla c_{\varepsilon} \), which enables us to use Lemma 4.1. Noting that in the proof of Lemma 4.2, we use Lemma 4.1 only for \( q = 2 \) in (4.3). We first give the improving estimate for \( u_{\varepsilon} \).

Lemma 5.1. Let \( p \in \left(\frac{545}{264}, \frac{7}{3}\right) \). Then there exists \( \delta_1(p) > 0 \) such that for all \( m > 1 \) fulfilling \( m > \frac{9(p-2)}{7-3p} - \delta_1(p) \) and any \( K > 0 \), if

\[
\int_\Omega n_{\varepsilon}^{m}(-, t) \leq K, \quad \text{for all } t \geq 0,
\]

then there exist \( C(m, p, K) > 0 \) such that

\[
\int_\Omega |u_{\varepsilon}(-, t)|^{2\left(\frac{mp}{3}+m+p-2\right)} \leq C(m, p, K), \quad \text{for all } t \geq 0.
\]

Proof. Let \( q = 2\left(\frac{mp}{3} + m + p - 2\right) \) and

\[
\rho(\tau) = 4(3 + p)\tau^2 - (33 - 6p)\tau - 18(p - 2), \quad \tau \in \mathbb{R},
\]

then we have

\[
\frac{3}{2} \left(\frac{1}{m} - \frac{1}{q}\right) - 1 = -\frac{\rho(m)}{4m(m(p + 3) + 3(p - 2))}.
\]
Lemma 5.2. Let $p \geq 2$. Then for all $m \geq 1$ and any $K > 0$, if
\[
\int_{\Omega} n_{\varepsilon}^{m}(\cdot, t) \leq K, \quad \text{for all } t \geq 0
\]
and
\[ \int_t^{t+1} \int_\Omega \left| \nabla n_{\epsilon}^{\frac{m-2}{p}+1} \right|^p \leq K, \quad \text{for all } t \geq 0, \tag{5.12} \]
then there exist \( C(m, p, K) > 0 \) such that
\[ \int_t^{t+1} \int_\Omega n_{\epsilon}^{\frac{m}{p}+m+p-2} \leq C(m, p, K), \quad \text{for all } t \geq 0. \tag{5.13} \]

**Proof.** Let \( \alpha = \frac{mp}{3} + m + p - 2 \) and \( \bar{m} = \frac{m-2}{p} + 1 \), then
\[ \frac{\bar{m}}{\alpha} - \left( \frac{1}{p} - \frac{1}{3} \right) = \frac{(m+3)p-6}{3(m(p+3)+3(p-2))} > 0, \]
and hence \( W^{1,p}(\Omega) \hookrightarrow L_{\bar{m}}^p(\Omega) \hookrightarrow L_{\alpha}^p(\Omega) \). Define
\[ \theta = \frac{3(m+p-2)}{m(p+3)+3(p-2)} \in (0, 1), \]
then \( \theta \) satisfies
\[ \frac{\bar{m}}{\alpha} = \theta \left( \frac{1}{p} - \frac{1}{3} \right) + (1 - \theta) \frac{\bar{m}}{m} \quad \text{and} \quad \frac{\alpha}{\bar{m}} \theta = p. \]

Using (5.11), (5.12) and the Gagliardo-Nirenberg inequality, we obtain \( C_1 > 0 \) such that
\[ \int_\Omega n_\epsilon^\alpha = \left\| n_\epsilon^{\frac{\bar{m}}{m}} \right\|_{m}^\alpha \leq C_1 \left\| \nabla n_\epsilon^{\frac{\bar{m}}{m}} \right\|_p \left\| n_\epsilon \right\|_m^{(1-\theta)} + C \left\| n_\epsilon \right\|_m \]
\[ \leq C_1 K^{\frac{\alpha(1-\theta)}{m}} \left\| \nabla n_\epsilon^{\frac{\bar{m}}{m}} \right\|_p + C_1 K^{\frac{\alpha}{m}} \]
\[ \leq C_1 K^{\frac{\bar{m}}{m}} \left\| \nabla n_\epsilon^{\frac{\bar{m}}{m}} \right\|_p + C_1 K^{1+\frac{p}{3}+\frac{p-2}{m}}, \quad \text{for all } t > 0, \]
which further implies
\[ \int_t^{t+1} \int_\Omega n_\epsilon^\alpha \leq C_1 K^{\frac{\bar{m}}{m}} \int_t^{t+1} \int_\Omega \left| \nabla n_\epsilon^{\frac{m-2}{p}+1} \right|^p + C_1 K^{1+\frac{p}{3}+\frac{p-2}{m}} \]
\[ \leq C_1 K^{1+\frac{p}{3}+\frac{p-2}{m}}, \quad \text{for all } t \geq 0. \]

With Lemma 5.1 and Lemma 5.2 at hand, we can improve the estimate for \( \nabla c_\epsilon \).

**Lemma 5.3.** Let \( p \in \left( \frac{545}{261}, \frac{7}{3} \right) \) and let \( \delta_1(p) > 0 \) be as in Lemma 5.1. Then for all \( m > \frac{9(p-2)}{7-3p} - \delta_1(p) \) and any \( K > 0 \), if for any \( \epsilon \in (0, 1) \) there hold
\[ \int_\Omega n_\epsilon^m(\cdot, t) \leq K, \quad \text{for all } t \geq 0, \tag{5.14} \]
and
\[ \int_t^{t+1} \int_\Omega \left| \nabla n_\epsilon^{\frac{m-2}{p}+1} \right|^p \leq K, \quad \text{for all } t \geq 0, \tag{5.15} \]
then one can find \( C(m, p, K) > 0 \) with the property that
\[ \int_t^{t+1} \int_\Omega \left| \nabla c_\epsilon \right|^{2\left( \frac{m}{3} + m + p - 2 \right)} \leq C(m, p, K), \quad \text{for all } t \geq 0. \tag{5.16} \]

**Proof.** Similar as the proof of Lemma 6.3 in [48](we only need to set \( q = \frac{mp}{3} + m + p - 2 \) here), then we can get the desired result.
6. $L^\infty$ Bounds for $n_\varepsilon$ When $p > \frac{23}{11}$

According to Lemma 5.3, we can improve the estimate of Lemma 4.1.

**Lemma 6.1.** Let $p \in \left(\frac{545}{201}, \frac{7}{2}\right)$, $m_* > \frac{9(p-2)}{7-3p} - \delta_1(p)$ with $\delta_1(p) > 0$ taken from Lemma 5.1. Then for all $m > 1$ fulfilling

\[ m \leq \frac{1}{9} \left\{ 2(p+3)(p-1)m_*^2 + 3(4p^2 - 5p - 8)m_* + 9(2p-5)(p-2) \right\}, \tag{6.1} \]

and any $K > 0$, if for any $\varepsilon \in (0,1)$ there hold

\[ \int_{\Omega} n_\varepsilon^{m_*}(\cdot,t) \leq K, \quad \text{for all } t \geq 0 \tag{6.2} \]

and

\[ \int_t^{t+1} \int_{\Omega} |\nabla n_\varepsilon^{m_*+1}|^p \leq K, \quad \text{for all } t \geq 0, \tag{6.3} \]

then one can pick $C(m_*,m,p,K) > 0$ such that

\[ \int_{\Omega} n_\varepsilon^{m}(\cdot,t) \leq C(m_*,m,p,K), \quad \text{for all } t \geq 0 \tag{6.4} \]

and

\[ \int_t^{t+1} \int_{\Omega} |\nabla n_\varepsilon^{m_*+1}|^p \leq C(m_*,m,p,K), \quad \text{for all } t \geq 0. \tag{6.5} \]

**Proof.** Since $m_* > \frac{9(p-2)}{7-3p} - \delta_1(p)$, we can apply Lemma 5.3 to $q = \frac{m_*p}{3} + m_* + p - 2$ to see that for some $C_1(m,p,K) > 0$ there holds

\[ \int_t^{t+1} \int_{\Omega} |\nabla c_\varepsilon|^{2q} \leq C_1(m,p,K), \quad \text{for all } t \geq 0. \tag{6.6} \]

Substituting $q = \frac{m_*p}{3} + m_* + p - 2$ into the right-hand side of (4.1), we obtain

\[
\frac{2q(p-1) - p}{3} m_* + (2q - 1)(p-2) = \frac{1}{9} \left\{ 2(p+3)(p-1)m_*^2 + 3(4p^2 - 5p - 8)m_* + 9(2p-5)(p-2) \right\}.
\]

Then for any $m > 1$ satisfies (6.1), an application of Lemma 4.1 shows that both inequalities in (6.4) and (6.5) hold for some suitably large $C(m_*,m,p,K) > 0$ due to (6.1) and (6.2). \[ \square \]

In order to find out through its condition (6.1), how far Lemma 6.1 can improve the regularity of $n_\varepsilon$ and $\nabla n_\varepsilon$, let us first demonstrate the following observation, which highlight the role of restriction $p > \frac{23}{11}$ made in Theorem 1.1.

**Lemma 6.2.** For $p \in (2, \frac{7}{2})$, let

\[ \psi(m) = \frac{1}{9} \left\{ 2(p+3)(p-1)m_*^2 + 3(4p^2 - 5p - 8)m + 9(2p-5)(p-2) \right\}, \quad p \in \mathbb{R}. \tag{6.7} \]

Then

\[ \psi\left(\frac{9(p-2)}{7-3p}\right) > \frac{9(p-2)}{7-3p} \quad \text{if and only if } \quad p > \frac{23}{11}, \tag{6.8} \]

and there exist $\delta_2(p) > 0$ and $\Gamma > 1$ such that

\[ \psi(m) > m\Gamma, \quad \text{for all } m > \frac{9(p-2)}{7-3p} - \delta_2(p). \tag{6.9} \]
Proof. Since \( p > 2 \) and
\[
\psi\left(\frac{9(p - 2)}{7 - 3p}\right) - \frac{9(p - 2)}{7 - 3p} = \frac{16(p - 2)(11p - 23)}{(7 - 3p)^2} > 0,
\]
we directly obtain (6.8). To prove (6.9), we let
\[
\tilde{\psi}(m) = \frac{\psi(m)}{m}, \text{ for all } m > 0,
\]
this together with (6.8) yields that \( C_1 = \tilde{\psi}\left(\frac{9(p - 2)}{7 - 3p}\right) - 1 \) is positive. The continuity of \( \tilde{\psi}(m) \) allows us to pick \( \delta_2 = \delta_2(p) > 0 \) such that
\[
\tilde{\psi}(m) \geq \Gamma = 1 + \frac{C_1}{2}, \text{ for all } m \in \left(\frac{9(p - 2)}{7 - 3p}, \frac{9(p - 2)}{7 - 3p} - \delta_2(p)\right].
\]
Using (6.7) and (6.10), we obtain
\[
\tilde{\psi}'(m) = \frac{2}{9}(p - 1)(p + 3) - \frac{(p - 2)(2p - 5)}{m^2}, \text{ for all } m > 0.
\]
If \( p \leq \frac{5}{2} \), then \( \tilde{\psi}'(m) \geq \frac{2}{9}(p - 1)(p + 3) > 0 \) for \( m \in (0, \infty) \). If \( p > \frac{5}{2} \), then for \( m \geq \frac{9(p - 2)}{7 - 3p} \), there holds
\[
\tilde{\psi}'(m) \geq \frac{2}{9}(p - 1)(p + 3) - (p - 2)(2p - 5)\left(\frac{7 - 3p}{9(p - 2)}\right)^2 = \frac{1}{81}\left(129p + \frac{1}{p - 2} - 176\right) > 0.
\]
In both cases, we obtain that \( \tilde{\psi}' > 0 \) on \( \left[\frac{9(p - 2)}{7 - 3p}, \infty\right) \) and hence \( \tilde{\psi} \geq \Gamma \) on \( \left[\frac{9(p - 2)}{7 - 3p} - \delta_2(p), \infty\right) \) by (6.11).

**Lemma 6.3.** Let \( p \in \left(\frac{23}{11}, \frac{7}{3}\right) \). Then for all \( m > 1 \), there exist \( C = C(m) > 0 \) such that
\[
\int_{\Omega} n_{\varepsilon}^{m}(\cdot, t) \leq C(m), \quad \text{for all } t \geq 0
\]
and
\[
\int_{t}^{t+1} \int_{\Omega} |\nabla n_{\varepsilon}^{m-2+1}|^p \leq C(m), \quad \text{for all } t \geq 0,
\]
\[
\int_{t}^{t+1} \int_{\Omega} |\nabla n_{\varepsilon}^{m-2+1}|^p \leq C(m), \quad \text{for all } t \geq 0.
\]
Proof. Since \( p > \frac{23}{11} > \frac{115}{26} \), we can pick \( m_0 \in (1, \frac{9(p - 2)}{7 - 3p}) \) such that
\[
m_0 > \frac{9(p - 2)}{7 - 3p} - \min\{\delta_1(p), \delta_2(p)\},
\]
where \( \delta_1(p) \) and \( \delta_2(p) \) are given by Lemma 6.1 and Lemma 6.2 respectively. Define
\[
m_k = \psi(m_{k-1}), \quad k \in \mathbb{N} = \{1, 2, 3, \ldots\},
\]
with \( \psi : \mathbb{R} \to \mathbb{R} \) given by (6.7) in Lemma 6.2. Combining (6.14) with Lemma 6.2, an inductive argument shows that
\[
m_k \geq \Gamma^k m_0, \quad \text{for all } k \in \mathbb{N}
\]
with \( \Gamma > 1 \) given by Lemma 6.2, whence in particular \( m_k \to \infty \) as \( k \to \infty \).

Due to the boundedness of \( \Omega \), it suffices to show that for all \( k \in \mathbb{N} \) there exists \( C_1(k) > 0 \) such that for all \( \varepsilon \in (0, 1) \),
\[
\int_{\Omega} n_{\varepsilon}^{m_k}(\cdot, t) \leq C_1(k) \quad \text{and} \quad \int_{t}^{t+1} \int_{\Omega} |\nabla n_{\varepsilon}^{m_k-2+1}|^p \leq C_1(k), \quad \text{for all } t \geq 0.
\]
This can be proven again by an iterative approach. Indeed, for \( k = 0 \), (6.17) can be derived from Lemma 4.2, since \( p > \frac{23}{11} > \frac{115}{12} \) and \( m_0 \in (1, \frac{9(p - 2)}{7 - 3p}) \). If (6.17) holds for some \( k_0 \geq 0 \) and some
We note that (6.15) and (6.16) ensure that \( m_k \geq m_0 > \frac{9(p-2)}{4-3p} - \delta_1(p) \), and again since \( p \in \left( \frac{545}{261}, \frac{7}{3} \right) \), Lemma 6.1 provides \( C_2 > 0 \) such that

\[
\int_{\Omega} n_{\varepsilon}^m(\cdot, t) \leq C_1(k) \quad \text{and} \quad \int_{t}^{t+1} \int_{\Omega} |\nabla n_{\varepsilon}^{m-1}|^p \leq C_1(k), \quad \text{for all } t \geq 0
\]

with

\[
m = \frac{1}{9} \left\{ 2(2p+3)(p-1)m_k^2 + 3(4p^2 - 5p - 8)m_k + 9(2p - 5)(p-2) \right\}.
\]

From (6.15) we know that \( m = \psi(m_k) = m_{k_0+1} \), which means (6.17) also holds for \( k = k_0 + 1 \) and thereby completes the proof.

Using Lemma 6.3 and Lemma 5.1, we can obtain further regularity of \( u_{\varepsilon} \) and \( \nabla c_{\varepsilon} \) through a standard regularization feature of the heat semigroup.

**Lemma 6.4.** Let \( p \in \left( \frac{23}{11}, \frac{7}{3} \right) \) and \( m > 1 \). Then there exist \( C(m) > 0 \) such that for all \( \varepsilon \in (0, 1) \), there hold

\[
\int_{\Omega} |u_{\varepsilon}(\cdot, t)|^m \leq C(m), \quad \text{for all } t \geq 0,
\]

and

\[
\int_{\Omega} |\nabla c_{\varepsilon}(\cdot, t)|^m \leq C(m), \quad \text{for all } t \geq 0,
\]

**Proof.** (6.18) is a direct consequence of a combination of Lemma 6.3 with Lemma 5.1. Based on (6.18) and Lemma 6.3, using the well-known results on gradient regularity in semilinear heat equation ([18]), we can arrive at (6.19).

A Moser-type iterative procedure which was similar to [35, Lemma 4.1], shows the boundedness of \( n_{\varepsilon}(\cdot, t) \) in \( L^\infty(\Omega) \) for all \( t \geq 0 \). For readers’ convenience, we shall give the detailed proof here.

**Lemma 6.5.** If \( p \geq \frac{23}{11} \), then there exists \( C > 0 \) such that for all \( \varepsilon \in (0, 1) \) we have

\[
||n_{\varepsilon}(\cdot, t)||_{L^\infty(\Omega)} \leq C, \quad \text{for all } t \geq 0.
\]

**Proof.** We now recursively define

\[
m_k = 2m_{k-1} + 2 - p, \quad k \in \mathbb{N} = \{1, 2, 3, \cdots \},
\]

with

\[
m_0 > p - 2.
\]

We note that (6.21) and (6.22) ensure that \( \{m_k\}_{k \in \mathbb{N}} \) is a nonnegative strictly increasing sequence and

\[
m_k \nearrow \infty \quad \text{as} \quad k \to \infty,
\]

and moreover there holds

\[
c_1 2^k \leq m_k \leq c_2 2^k, \quad \text{for all} \quad k \in \mathbb{N},
\]

where \( c_1 > 0 \) and \( c_2 > 0 \) which, as all constants \( C_1, C_2, \cdots \) appearing below, are independent of \( k \). Define

\[
\theta_k = 2 \cdot \frac{m_k + p - 2}{m_k + p'} > 2,
\]

then

\[
\theta_k' = \frac{\theta_k}{\theta_k - 1} \in (1, 2) \quad \text{and} \quad \frac{1}{\theta_k'} > \frac{1}{2} > \frac{p'}{4}.
\]

Our goal is to derive a recursive inequality for

\[
M_k = \sup_{t \in (0, \infty)} \int_{\Omega} n_{\varepsilon}^{-m_k}(x, t)dx, \quad k \in \mathbb{N},
\]
where \( \bar{n}_\varepsilon(x,t) = \max\{n_\varepsilon(x,t),1\} \) for \( x \in \Omega \) and \( t \in [0, \infty) \). To this end, we may test the second equation of (2.5) by \( m_k n_k^{m_k-1} \) and employ Young’s inequality to obtain for \( k \geq 1 \)

\[
\frac{d}{dt} \int_\Omega n_\varepsilon^{m_k} + m_k(m_k - 1) \int_\Omega n_\varepsilon^{m_k-1} (|\nabla n_\varepsilon^2| + \varepsilon) \frac{p-2}{2} |\nabla n_\varepsilon|^2 \\
= m_k(m_k - 1) \int_\Omega n_\varepsilon^{m_k-1} F_\varepsilon(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon \cdot \nabla n_\varepsilon \\
\leq C_1 m_k(m_k - 1) \int_\Omega n_\varepsilon^{m_k-1} |\nabla c_\varepsilon| \cdot |\nabla n_\varepsilon| \\
= C_1 m_k(m_k - 1) \int_\Omega \frac{m_k - 2}{p} |\nabla n_\varepsilon| \cdot n_\varepsilon^{m_k-2-1} |\nabla c_\varepsilon| \\
\leq C_1 m_k(m_k - 1) \int_\Omega \left[ \frac{1}{2C_1} n_\varepsilon^{m_k-2} |\nabla n_\varepsilon|^p + p - 1 \left( \frac{p}{2C_1} \right)^{\frac{1}{p-1}} \cdot \left( \frac{2 - m_k + m_k - 1}{n_\varepsilon^{p} |\nabla c_\varepsilon|} \right)^{\frac{p}{p-1}} \right]
\]

for all \( t > 0 \). This together with the Hölder inequality, (6.26) and Lemma 6.4 shows that there exists some \( C_2 > 0 \) and \( C_3 > 0 \) such that

\[
\frac{d}{dt} \int_\Omega n_\varepsilon^{m_k} + m_k(m_k - 1) \int_\Omega n_\varepsilon^{m_k-1} |\nabla n_\varepsilon|^{p-1+1} p \\
= \frac{d}{dt} \int_\Omega n_\varepsilon^{m_k} + m_k(m_k - 1) \int_\Omega n_\varepsilon^{m_k-2} |\nabla n_\varepsilon|^p \\
\leq C_2 m_k(m_k - 1) \int_\Omega \left( \frac{2 - m_k + m_k - 1}{n_\varepsilon^p |\nabla c_\varepsilon|} \right)^{p'} \\
= C_2 m_k(m_k - 1) \int_\Omega n_\varepsilon^{m_k-2+p'} |\nabla c_\varepsilon|^{p'} \\
\leq C_2 m_k(m_k - 1) |\Omega|^{\frac{1}{p'}} \theta_k^{(p'-p)} \left( \int_\Omega n_\varepsilon^{(m_k-2+p')\theta_k} \right)^{\frac{1}{\theta_k}} \cdot \left( \int_\Omega |\nabla c_\varepsilon|^{q'} \right)^{\frac{q'}{p'}} \\
\leq C_3 m_k(m_k - 1) \left( \int_\Omega n_\varepsilon^{(m_k-2+p')\theta_k} \right)^{\frac{1}{\theta_k}}, \text{ for all } t > 0. \tag{6.28}
\]

We first estimate the right-hand side of (6.28). Noting that \( \frac{m_k+p-2}{2} = m_k-1 \) and \( \frac{m_k-2+p'}{m_k+p-2} = 2 \) by (6.21) and (6.25) respectively. This together with the Gagliardo-Nirenberg inequality enables us to find \( C_4 > 0 \) such that

\[
C_3 \left( \int_\Omega n_\varepsilon^{(m_k-2+p')\theta_k} \right)^{\frac{1}{\theta_k}} = C_3 \|n_\varepsilon^{m_k+p-2}\|^{\frac{2p}{2p}} \|\frac{2p}{2p} \|^{\frac{2p}{2p}} \\
\leq C_4 \|\nabla n_\varepsilon\|^{\frac{m_k+p-2}{p}} \cdot \|n_\varepsilon\|^{\frac{m_k+p-2}{p}} + C_4 \|n_\varepsilon\|^{\frac{m_k+p-2}{p}} \|\frac{2p}{2p} \|^{\frac{2p}{2p}} \\
= C_4 \|\nabla n_\varepsilon\|^{\frac{m_k+p-2}{p}} \cdot \|n_\varepsilon^{m_k-1}\|^{\frac{2p}{p}} + C_4 \|n_\varepsilon^{m_k-1}\|^{\frac{2p}{p}}, \tag{6.29}
\]

for all \( t > 0 \), with

\[
a = \frac{6}{p} - \frac{3}{2p} = \frac{9}{2p} + \in (0,1). \]
In the view of (6.27), we can apply Young’s inequality to (6.29) to obtain

\[ C_3 \left( \int_\Omega n_\varepsilon^{(m_k-2+p')\theta_k} \right)^{\frac{1}{\theta_k}} \]
\[ \leq C_4 M_{k-1}^{4(1-a)\eta_k} \left( \int_\Omega |\nabla n_\varepsilon|_p^{m_k+p-2} \right)^{\frac{2a}{\theta_k}} + C_4 M_{k-1}^{\frac{4}{\eta_k}} \]
\[ \leq C_4 \left( \eta \int_\Omega |\nabla n_\varepsilon|_p^{m_k+p-2} \right)^{\frac{2a}{\theta_k}} + C_4 \left( M_{k-1}^{\frac{4}{\eta_k}} \right)^{\frac{4(1-a)}{\eta_k-2a}} + C_4 M_{k-1}^{\frac{4}{\eta_k}}, \quad \text{for all } t > 0 \quad (6.30) \]

where \( \eta = \frac{1}{4C_4} (\frac{p}{m_k+p-2})^p > 0 \) and

\[ C_\eta = \left( \frac{\theta_k}{2a} \right)^{\frac{1}{\theta_k}-1} \cdot \frac{\theta_k}{\theta_k-1} = \frac{\theta_k-2a}{\theta_k} \left( \frac{\theta_k}{2a} \eta \right)^{\frac{2a}{\theta_k-2a}}. \]

Letting

\[ \tilde{b} = 2^{\frac{a}{1-a}} > 1, \quad (6.31) \]

then by (6.24) we have

\[ C_\eta \leq \eta^{\frac{2a}{\theta_k-2a}} = \left( 4C_4 \right)^{\frac{2a}{\theta_k-2a}} \left( \frac{m_k+p-2}{p} \right)^{\frac{2a}{\theta_k-2a}} \]
\[ \leq \left( 4C_4 \right)^{\frac{2a}{\theta_k-2a}} \left( m_k^{\frac{2a}{\theta_k-2a}}p + 1 \right) \]
\[ \leq \left( 4C_4 \right)^{\frac{a}{1-a}} 2^{\frac{a}{1-a}} m_k^{\frac{2a}{\theta_k-2a}} \]
\[ \leq C_5 \tilde{b} \quad (6.32) \]

since \( \frac{a}{1-a} > 0 \) and \( \theta_k > 2 \). A combination of (6.30)-(6.32) shows that

\[ C_3 \left( \int_\Omega n_\varepsilon^{(m_k-2+p')\theta_k} \right)^{\frac{1}{\theta_k}} \]
\[ \leq \frac{1}{4} \left( \frac{p}{m_k+p-2} \right)^p \int_\Omega |\nabla n_\varepsilon|_p^{m_k+p-2} \right)^{p} \]
\[ + C_4 C_5 \tilde{b}^k M_{k-1}^{\frac{4(1-a)}{\eta_k-2a}} + C_4 M_{k-1}^{\frac{4}{\eta_k}}, \quad \text{for all } t > 0 \quad (6.33) \]

this together with (6.33) yields

\[ \frac{d}{dt} \int_\Omega n_\varepsilon^{m_k} + \frac{m_k(m_k-1)}{4} \left( \frac{p}{m_k+p-2} \right)^p \int_\Omega |\nabla n_\varepsilon|_p^{m_k+p-2+1} \right)^{p} \]
\[ \leq C_4 C_\eta m_k(m_k-1) \left( M_{k-1}^{\frac{4}{\eta_k}} \right)^{\frac{4(1-a)}{\eta_k-2a}} + C_4 m_k(m_k-1) M_{k-1}^{\frac{4}{\eta_k}} \]
\[ \leq C_4 C_5 m_k(m_k-1) \tilde{b}^k M_{k-1}^{\frac{4(1-a)}{\eta_k-2a}} + C_4 m_k(m_k-1) M_{k-1}^{\frac{4}{\eta_k}}, \quad \text{for all } t > 0. \quad (6.34) \]
We can generate a linear absorption term in (6.34) again by the routine method. Indeed, we can use the Gagliardo-Nirenberg inequality to estimate
\[
\int_{\Omega} n_{\varepsilon}^{m_k} = \|n_{\varepsilon}^{m_k + p - 2} \|_{p}^{\frac{p}{m_k + p - 2}} \|n_{\varepsilon}^{m_k + p - 2} \|^p_{m_k + p - 2} \leq C_6 \|\nabla n_{\varepsilon}^{m_k + p - 2} \|_{p}^{\frac{p}{m_k + p - 2}} \|n_{\varepsilon}^{m_k + p - 2} \|^p_{m_k + p - 2} (1-b) + C_6 \|n_{\varepsilon}^{m_k + p - 2} \|^p_{m_k + p - 2}
\]
\[
= C_6 \|\nabla n_{\varepsilon}^{m_k + p - 2} \|_{p}^{\frac{p}{m_k + p - 2}} \|n_{\varepsilon}^{m_k + p - 2} \|^p_{m_k + p - 2} (1-b) + C_6 \|n_{\varepsilon}^{m_k + p - 2} \|^p_{m_k + p - 2} \leq C_6 \|\nabla n_{\varepsilon}^{m_k + p - 2} \|_{p}^{\frac{p}{m_k + p - 2}} \|n_{\varepsilon}^{m_k + p - 2} \|^p_{m_k + p - 2} + C_6 M_{k-1}^{m_k + p - 2}, \quad \text{for all } t > 0 \tag{6.35}
\]
with
\[
b = \frac{2}{m_k + p - 2} + \frac{2}{p} \left( 1 - \frac{m_k - 1}{m_k} \right) \in (0, 1),
\]
since \(1 - \frac{m_k - 1}{m_k} > 0\) and \(\frac{m_k + p - 2}{p} > 0\). Using Young’s inequality, we can derive from (6.35) that
\[
\int_{\Omega} n_{\varepsilon}^{m_k} \leq C_6 \|\nabla n_{\varepsilon}^{m_k + p - 2} \|_{p}^{\frac{p}{m_k + p - 2}} + C_6 M_{k-1}^{m_k + p - 2} + C_6 M_{k-1}^{2m_k + 2}, \quad \text{for all } t > 0 \tag{6.36}
\]
this together with (6.34) yields
\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon}^{m_k} + \frac{m_k(m_k - 1)}{4C_6} \left( \frac{p}{m_k + p - 2} \right)^p \int_{\Omega} n_{\varepsilon}^{m_k} \leq C_4 C_5 m_k(m_k - 1) \tilde{b}^k M_{k-1}^{\frac{4(1-a)}{p}} + C_4 m_k(m_k - 1) M_{k-1}^{\frac{p}{m_k + p - 2}} + \frac{m_k(m_k - 1)}{2} \left( \frac{p}{m_k + p - 2} \right)^p M_{k-1}^{2m_k + 2}, \quad \text{for all } t > 0. \tag{6.37}
\]
Since \(\theta_k > 2\) and \(a \in (0, 1)\), we conclude that
\[
\frac{4(1-a)}{\theta_k - 2a} = \frac{4}{\theta_k} \cdot \frac{1 - a}{1 - a + \frac{2}{\theta_k}} < \frac{4}{\theta_k} < 2.
\]
This enables us to infer from (6.37) that
\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon}^{m_k} + \frac{m_k(m_k - 1)}{4C_6} \left( \frac{p}{m_k + p - 2} \right)^p \int_{\Omega} n_{\varepsilon}^{m_k} \leq 2C_4 C_5 m_k(m_k - 1) \tilde{b}^k M_{k-1}^2 + \frac{m_k(m_k - 1)}{2} \left( \frac{p}{m_k + p - 2} \right)^p M_{k-1}^{2m_k + 2}, \quad \text{for all } t > 0
\]
which can be rewritten in the form of
\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon}^{m_k} \leq 2C_4 C_5 m_k(m_k - 1) \tilde{b}^k M_{k-1}^2 + \frac{m_k(m_k - 1)}{2} \left( \frac{p}{m_k + p - 2} \right)^p \left( M_{k-1}^2 - \frac{1}{2C_6} \int_{\Omega} n_{\varepsilon}^{m_k} \right), \quad \text{for all } t > 0.
\]
An integration of this ODI shows that
\[
M_k \leq \max \{ \int_{\Omega} n_{0\varepsilon}^{m_k}, 2C_6 \left( 4C_4 C_5 \tilde{b}^k \left( \frac{m_k + p - 2}{p} \right)^p + 1 \right) M_{k-1}^2 \}. \tag{6.38}
\]
Therefore, in the case when \(2C_6 \left(4C_4C_7b^k \left(\frac{m_k + p-2}{p} \right)^p + 1 \right) M_k^2 \rightarrow \frac{1}{\epsilon} \int_\Omega n_{\epsilon k}^m \) holds for infinitely many \(k \geq 1\), we obtain

\[
\sup_{t \in (0, \infty)} \left( \int_\Omega n_{\epsilon k}^{m_{k-1}} \right)^{1/m_{k-1}} \leq \left( \int_\Omega n_{\epsilon 0}^{m_k} \right)^{1/m_{k-1}}
\]

for all such \(k\), and hence conclude that

\[
\|n_{\epsilon}(t)\|_\infty \leq \|n_{0\epsilon}\|_\infty, \quad \text{for all } t > 0,
\]

because \(\frac{m_k}{2m_{k-1}} \rightarrow 1\) as \(k \rightarrow \infty\) according to (6.21) and (6.23).

Conversely, upon enlarging \(C_6\) if necessary we may assume that

\[
M_k \leq 2C_6 \left(4C_4C_7b^k \left(\frac{m_k + p-2}{p} \right)^p + 1 \right) M_{k-1}^2, \quad \text{for all } k \geq 1.
\]

Then from (6.24) and the definition of \(\tilde{b}\) (6.31), we see that there exist \(C_8 = 32C_4C_5C_6 > 0\), and \(C_9 = C_2^pC_8 > 0\) such that

\[
M_k \leq 2C_6 \left(4C_4C_7b^k \left(\frac{m_k + p-2}{p} \right)^p + 1 \right) M_{k-1}^2 \\
\leq 16C_4C_5C_6\tilde{b}^k \left(\frac{m_k + 1}{p} \right)^p M_{k-1}^2 \\
\leq C_8\tilde{b}^k \left(c_2^pM_k^p \right)^p M_{k-1}^2 \\
\leq C_9 \left(b^pM_k^p \right)^k M_{k-1}^2 \\
= C_9\tilde{d}^k M_{k-1}^2, \quad \text{for all } k \geq 1,
\]

where \(\tilde{d} = \tilde{b} \cdot 2^p = 2^p \left(\frac{n}{m-1}+1 \right) = 2^\frac{n}{m-1} > 1\). By a straight forward induction, this yields

\[
M_k \leq C_9^{\sum_{j=0}^{k-1} 2j} \cdot \tilde{d}^{\sum_{j=0}^{k-1} (k-j)2j} \cdot M_0^{2k}
\]

for all \(k \geq 1\). Using (6.24), the identities \(\sum_{j=0}^{k-1} 2j \leq 2^k\) and \(\sum_{j=0}^{k-1} (k-j)2^j = 2^{k+1}\), we have

\[
\lim_{k \rightarrow \infty} \frac{1}{m_k} \cdot 2^k \leq \frac{1}{C_1}, \quad \lim_{k \rightarrow \infty} \frac{1}{m_k} \sum_{j=0}^{k-1} 2^j \leq \frac{1}{C_1} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{1}{m_k} \sum_{j=0}^{k-1} (k-j)2^j = 2^{k+1} \leq \frac{2}{C_1}.
\]

Finally, a combination of (6.27), (6.39) and (6.40) concludes that

\[
\|n_{\epsilon}(t)\|_\infty \leq C_9^{\frac{1}{m_k}} \cdot \tilde{d}^\frac{n}{m-1} \|n_{0\epsilon}\|_\infty, \quad \text{for all } t > 0.
\]

Based on the regularities we have obtained, we can obtain the following estimates.

**Lemma 6.6.** ([48, Lemma 8.4]) There exist \(\theta \in (0, 1)\) with the property that one can find \(C > 0\) such that for all \(\epsilon \in (0, 1)\),

\[
\|c_{\epsilon}\|_{C^\theta(U \times [t, t+1])} \leq C, \quad \text{for all } t \geq 0
\]

and

\[
\|u_{\epsilon}\|_{C^\theta(U \times [t, t+1])} \leq C, \quad \text{for all } t \geq 0,
\]

and that for all \(\tau > 0\) it is possible to choose \(C(\tau) > 0\) fulfilling

\[
\|\nabla c_{\epsilon}\|_{C^\theta(U \times [t, t+1])} \leq C(\tau), \quad \text{for all } t \geq \tau.
\]

Let us now come up with one statement on time regularity of \(n_{\epsilon}\) in a straightforward way.
Lemma 6.7. There exists $C > 0$ such that for all $\varepsilon \in (0,1)$ we have
\[
\int_0^T \| \partial_t n_\varepsilon(\cdot,t) \|_{W^{1,p}(\Omega)}^{p'} \leq C(T + 1), \quad \text{for all } T > 0, \tag{6.44}
\]
where $p' = \frac{p}{p-1}$.

Proof. For $t > 0$ and $\varphi \in C^\infty(\overline{\Omega})$, multiplying the first equation in (2.5) by $\varphi$, integrating by parts and using the Hölder’s inequality, we obtain
\[
\begin{align*}
\int_\Omega \partial_t n_\varepsilon(\cdot,t) \varphi &= \int_\Omega \left( |\nabla n_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla n_\varepsilon \cdot \nabla \varphi + \int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon) \nabla c_\varepsilon \cdot \nabla \varphi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \varphi \\
&\leq \int_\Omega \left( |\nabla n_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} |\nabla n_\varepsilon| \cdot |\nabla \varphi| + C \| n_\varepsilon \nabla c_\varepsilon \|_{p'} \| \nabla \varphi \|_p + \| n_\varepsilon u_\varepsilon \|_{p'} \| \nabla \varphi \|_p \\
&\leq C_1 \left( |\nabla n_\varepsilon|^2 + \varepsilon \right)^{\frac{p-1}{2}} \| \nabla c_\varepsilon \|_{p'} + \| n_\varepsilon \nabla c_\varepsilon \|_{p'} + \| n_\varepsilon u_\varepsilon \|_{p'} \| \nabla \varphi \|_p
\end{align*}
\]
with some $C_1 > 0$. Lemma 3.2 implies
\[
\int_0^T \int_\Omega |\nabla c_\varepsilon|^4 \leq C_2(T + 1), \quad \text{for all } T > 0 \tag{6.45}
\]
with some $C_2 > 0$. Using $\frac{1}{6} + \frac{1}{2} < \frac{1}{5} + \frac{3}{10} < \frac{1}{p'}$, (6.45), Lemma 6.3, Lemma (6.5), Lemma 6.4, Hölder’s inequality and Young’s inequality, we obtain $C_3 > 0$, $C_4 > 0$, and $C_5 > 0$ such that
\[
\begin{align*}
\int_0^T \| \partial_t n_\varepsilon(\cdot,t) \|_{W^{1,p}(\Omega)}^{p'} &\leq C_3 \left( \int_0^T \int_\Omega |\nabla n_\varepsilon|^2 + 1 \right)^{\frac{p-1}{2}} + \int_0^T \int_\Omega |n_\varepsilon \nabla c_\varepsilon|^{p'} + \int_0^T \int_\Omega |n_\varepsilon u_\varepsilon|^{p'} \\
&\leq C_4 \left( \int_0^T \int_\Omega |\nabla n_\varepsilon|^p + 1 \right) + \int_0^T \int_\Omega |n_\varepsilon|^6 + \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 \\
&+ \int_0^T \int_\Omega |n_\varepsilon|^6 + \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + |\Omega|T \\
&\leq C_5(T + 1), \quad \text{for all } T > 0. \tag{6.46}
\end{align*}
\]
□

7. EXISTENCE OF A GLOBAL BOUNDED WEAK SOLUTION

In this section we construct global bounded weak solutions for (1.1), (1.7) and (1.8). Based on the estimates we collected in the previous sections, we can get the following.

Lemma 7.1. Let $p > \frac{24}{17}$. Then there exist $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)$ satisfying $\varepsilon_j \searrow 0$ as $j \to \infty$, a null set $N \subset (0,\infty)$ and a triple $(n,c,u)$ of functions $n,c : \Omega \times (0,\infty) \to [0,\infty)$ and $u : \Omega \times (0,\infty) \to \mathbb{R}^3$
such that

\[ n_\varepsilon(\cdot, t) \to n(\cdot, t) \quad \text{a.e. in } \Omega \text{ for all } t \in (0, \infty) \setminus N, \]  
(7.1)

\[ n_\varepsilon \xrightarrow{\ast} n \quad \text{in } L^\infty(\Omega \times (0, \infty)), \]  
(7.2)

\[ n_\varepsilon \to n \quad \text{in } L^p_{\text{loc}}(\Omega \times (0, \infty)), \]  
(7.3)

\[ \nabla n_\varepsilon \rightharpoonup \nabla n \quad \text{in } L^p_{\text{loc}}(\Omega \times (0, \infty)), \]  
(7.4)

\[ |\nabla n_\varepsilon|^{p-2} \nabla n_\varepsilon \rightharpoonup |
abla n|^{p-2} \nabla n \quad \text{in } L^p_{\text{loc}}(\Omega \times (0, \infty)) \]  
(7.5)

\[ c_\varepsilon \to c \quad \text{in } C^0_{\text{loc}}(\Omega \times [0, \infty)), \]  
(7.6)

\[ c_\varepsilon \xrightarrow{\ast} c \quad \text{in } L^\infty((0, \infty); W^{1,s}(\Omega)) \quad \text{for all } s \in (1, \infty), \]  
(7.7)

\[ \nabla c_\varepsilon \rightharpoonup \nabla c \quad \text{in } C^0_{\text{loc}}(\Omega \times [0, \infty)), \]  
(7.8)

\[ u_\varepsilon \to u \quad \text{in } C^0_{\text{loc}}(\Omega \times [0, \infty)), \]  
(7.9)

\[ u_\varepsilon \xrightarrow{\ast} u \quad \text{in } L^\infty(\Omega \times (0, \infty)) \quad \text{and} \]  
(7.10)

\[ \nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)). \]  
(7.11)

as \( \varepsilon = \varepsilon_j \searrow 0 \), where \( p' = \frac{p}{p-1} \). Moreover, \((n, c, u)\) forms a global weak solution of (1.1), (1.7), (1.8) in the sense of Definition 1.1, and we have

\[ \int_\Omega n(\cdot, t) = \int_\Omega n_0, \quad \text{for all } t \in (0, \infty) \setminus N. \]  
(7.12)

**Proof.** From Lemma 3.2 and Lemma 6.5 we have that for all \( T > 0 \) there holds

\[
\int_0^T \int_\Omega |\nabla n_\varepsilon|^p \, dx \, dt = \int_0^T \int_\Omega \left| \frac{p}{p-1} n_\varepsilon^{\frac{p}{p-1}} \nabla \nabla_\varepsilon \right|^p \, dx \, dt \\
\leq \left( \frac{p}{p-1} \right)^p \|n_\varepsilon\|_{L^p((0,T) \times \Omega)}^{p-1} \int_0^T \int_\Omega \left| \nabla \nabla_\varepsilon \right|^p \, dx \, dt \\
\leq C(T+1).
\]
(7.13)

By Lemma 6.7, we have

\[ \|(n_\varepsilon)_t\|_{L^{p'}([0,T];(W^{1,p}(\Omega))^*}) \leq C(T+1), \quad \text{for all } T > 0. \]  
(7.14)

By Lemma 6.5, (7.13) and (7.14), an application of Aubin-Lions lemma yields \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) such that \( \varepsilon_j \searrow 0 \) as \( j \to \infty \) and

\[ n_\varepsilon \to n \quad \text{in } L^p_{\text{loc}}(\Omega \times [0, \infty)) \quad \text{and a.e. in } \Omega \times (0, \infty), \]  
(7.15)

as \( \varepsilon = \varepsilon_j \searrow 0 \) with some nonnegative function \( n \) defined on \( \Omega \times (0, \infty) \). So (7.15) ensures (7.3) and (7.1) follows by using the Arzela-Ascoli theorem. Moreover, (7.13) and (7.15) yield (7.4). From Lemma 6.5 and (7.15) we know that (7.2) is valid. Now with Lemma 6.5, (7.15) and (7.2) at hand, we can use [35, Lemma 6.2] to obtain

\[ |\nabla n_\varepsilon|^{p-2} \nabla n_\varepsilon \rightharpoonup |\nabla n|^{p-2} \nabla n \quad \text{in } L^{p'}_{\text{loc}}(\Omega \times [0, \infty)), \]  

hence we arrive at (7.5), and this is enough to warrant that

\[ (|\nabla n_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla n_\varepsilon \rightharpoonup |\nabla n|^{p-2} \nabla n \quad \text{in } L^{p'}_{\text{loc}}(\Omega \times [0, \infty)). \]  
(7.16)

Based on the priori estimates provided Lemma 3.2, Lemma 6.4 and Lemma 6.6, using the Arzela-Ascoli theorem twice we can achieve (7.6)-(7.11). We can derive from (2.5) that \( \int_\Omega n_\varepsilon = \int_\Omega n_0 \), this together with rneq2 and Lebesgue’s Dominated Convergence Theorem yields (7.12). \( \square \)

**Proof of Theorem 1.1.** Theorem 1.1 is a direct consequence of a combination of Lemma 6.4, Lemma 6.6 and Lemma 7.1.
ACKNOWLEDGMENTS

The authors are supported in part by NSF of China (No. 11671079, No. 11701290, No. 11601127 and No. 11171063), and NSF of Jiangsu Province (No. BK20170896).

REFERENCES

[1] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), pp. 1663–1763.
[2] V. Calvez, L. Corrias, and M. A. Ebde, Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension, Comm. Partial Differential Equations, 37 (2012), pp. 561–584.
[3] M. Chae, K. Kang, and J. Lee, Existence of smooth solutions to coupled chemotaxis-fluid equations, Discrete Contin. Dyn. Syst., 33 (2013), pp. 2271–2297.
[4] Global existence and temporal decay in Keller-Segel models coupled to fluid equations, Comm. Partial Differential Equations, 39 (2014), pp. 1205–1235.
[5] Y.-S. Choi and Z.-a. Wang, Prevention of blow-up by fast diffusion in chemotaxis, J. Math. Anal. Appl., 362 (2010), pp. 553–564.
[6] W. Cong and J.-G. Liu, A degenerate p-Laplacian Keller-Segel model, Kinet. Relat. Models, 9 (2016), pp. 687–714.
[7] M. Di Francesco, A. Lorz, and P. Markowich, Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior, Discrete Contin. Dyn. Syst., 28 (2010), pp. 1437–1453.
[8] C. Dombrowski, L. Cisneros, S. Chatkaew, R. Goldstein, and J. Kessler, Self-concentration and large-scale coherence in bacterial dynamics, Physical Review Letters, 93 (2004), p. 098103.
[9] R. Duan, A. Lorz, and P. Markowich, Global solutions to the coupled chemotaxis-fluid system, Comm. Partial Differential Equations, 35 (2010), pp. 1635–1673.
[10] R. Duan and Z. Xiang, A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion, Int. Math. Res. Not. IMRN, (2014), pp. 1833–1852.
[11] D. Fujihara and H. Morimoto, An Lr-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), pp. 685–700.
[12] Y. Giga, Solutions for semilinear parabolic equations in Lp and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations, 62 (1986), pp. 186–212.
[13] T. Hillen and K. Painter, A user’s guide to PDE models for chemotaxis, J. Math. Biol., 58 (2009), pp. 183–217.
[14] S. Hittmeir and A. Jüngel, Cross diffusion preventing blow-up in the two-dimensional Keller-Segel model, SIAM J. Math. Anal., 43 (2011), pp. 997–1022.
[15] D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein., 105 (2003), pp. 103–165.
[16] ———, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II, Jahresber. Deutsch. Math.-Verein., 106 (2004), pp. 51–69.
[17] D. Horstmann and G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, European J. Appl. Math., 12 (2001), pp. 159–177.
[18] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations, 215 (2005), pp. 52–107.
[19] J. Jiang, H. Wu, and S. Zheng, Global existence and asymptotic behavior of solutions to a chemotaxis-fluid system on general bounded domains, Asymptot. Anal., 92 (2015), pp. 249–258.
[20] E. Keller and L. Segel, Initiation of slime mold aggregation viewed as an instability, Journal of Theoretical Biology, 26 (1970), pp. 399–415.
[21] Y. Li, Global bounded solutions and their asymptotic properties under small initial data condition in a two-dimensional chemotaxis system for two species, J. Math. Anal. Appl., 429 (2015), pp. 1291–1304.
[22] Y. Li and J. Lankeit, Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion, Nonlinearity, 29 (2016), pp. 1564–1595.
[23] Y. Li and Y. Li, Finite-time blow-up in higher dimensional fully-parabolic chemotaxis system for two species, Nonlinear Anal., 109 (2014), pp. 72–84.
[24] ———, Blow-up of nonradial solutions to attraction–repulsion chemotaxis system in two dimensions, Nonlinear Anal. Real World Appl., 30 (2016), pp. 170–183.
[25] ———, Global boundedness of solutions for the chemotaxis-Navier-Stokes system in \( \mathbb{R}^2 \), J. Differential Equations, 261 (2016), pp. 6570–6613.
[26] P.-L. Lions, Résolution de problèmes elliptiques quasilinéaires, Arch. Rational Mech. Anal., 74 (1980), pp. 335–353.
[27] J.-G. Liu and A. Lorz, A coupled chemotaxis-fluid model: global existence, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28 (2011), pp. 643–652.
[28] A. Lorz, Coupled chemotaxis fluid model, Math. Models Methods Appl. Sci., 20 (2010), pp. 987–1004.
[29] Y. Lou, Y. Tao, and M. Winkler, Approaching the ideal free distribution in two-species competition models with fitness-dependent dispersal, SIAM J. Math. Anal., 46 (2014), pp. 1228–1262.
[30] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., 5 (1995), pp. 581–601.
[31] Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, J. Inequal. Appl., 6 (2001), pp. 37–55.
[32] T. Nagai, T. Senba, and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac., 40 (1997), pp. 411–433.
[33] H. Sohr, The Navier-Stokes equations. An elementary functional analytic approach, Birkhäuser, Basel, 2001.
[34] C. Stinner, C. Surulescu, and M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, SIAM J. Math. Anal., 46 (2014), pp. 1969–2007.
[35] W. Tao and Y. Li, Global weak solutions for the three-dimensional chemotaxis-navier-stokes system with slow p-laplacian diffusion, Nonlinear Anal. Real World Appl., 45 (2019), pp. 26–52.
[36] Y. Tao, Global dynamics in a higher-dimensional repulsion chemotaxis model with nonlinear sensitivity, Discrete Contin. Dyn. Syst. Ser. B, 18 (2013), pp. 2705–2722.
[37] Y. Tao and Z.-A. Wang, Competing effects of attraction vs. repulsion in chemotaxis, Math. Models Methods Appl. Sci., 23 (2013), pp. 1–36.
[38] Y. Tao and M. Winkler, A chemotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source, SIAM J. Math. Anal., 43 (2011), pp. 685–704.
[39] Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), pp. 157–178.
[40] I. Tuval, L. Cisneros, C. Dombrowski, C. Wolgemuth, J. Kessler, and R. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proceedings of the National Academy of Sciences of the United States of America, 102 (2005), pp. 2277–2282.
[41] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations, 248 (2010), pp. 2889–2905.
[42] Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, Comm. Partial Differential Equations, 37 (2012), pp. 319–351.
[43] Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl. (9), 100 (2013), pp. 748–767.
[44] Stabilization in a two-dimensional chemotaxis-Navier-Stokes system, Arch. Ration. Mech. Anal., 211 (2014), pp. 455–487.
[45] Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity, Calc. Var. Partial Differential Equations, 54 (2015), pp. 3789–3828.
[46] Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), pp. 1329–1352.
[47] M. Winkler, How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?, Trans. Amer. Math. Soc., 369 (2017), pp. 3067–3125.
[48] Global existence and stabilization in a degenerate chemotaxis-Stokes system with mildly strong diffusion enhancement, J. Differential Equations, 264 (2018), pp. 6109–6151.
[49] Q. Zhang, Local well-posedness for the chemotaxis-Navier-Stokes equations in Besov spaces, Nonlinear Anal. Real World Appl., 17 (2014), pp. 89–100.
[50] Q. Zhang and Y. Li, Global existence and asymptotic properties of the solution to a two-species chemotaxis system, J. Math. Anal. Appl., 418 (2014), pp. 47–63.
[51] Boundedness in a quasilinear fully parabolic Keller-Segel system with logistic source, Z. Angew. Math. Phys., 66 (2015), pp. 2473–2484.
[52] Convergence rates of solutions for a two-dimensional chemotaxis-Navier-Stokes system, Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), pp. 2751–2759.
[53] Global weak solutions for the three-dimensional chemotaxis-Navier-Stokes system with nonlinear diffusion, J. Differential Equations, 259 (2015), pp. 3730–3754.
[54] Q. Zhang and X. Zheng, Global well-posedness for the two-dimensional incompressible chemotaxis-Navier-Stokes equations, SIAM J. Math. Anal., 46 (2014), pp. 3078–3105.
Institute for Applied Mathematics, School of Mathematics, Southeast University, Nanjing 211189, P.R. China
E-mail address: taoweiruncn@163.com

Institute for Applied Mathematics, School of Mathematics, Southeast University, Nanjing 211189, P.R. China
E-mail address: lieyx@seu.edu.cn