Van Alphen-de Haas effect for dense cold quark matter in a homogeneous magnetic field

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Abstract

We study the relativistic van Alphen–de Haas effect in cold dense quark matter in a homogeneous magnetic field which arises from oscillations in the effective potential of an underlying extended NJL-model. Finally, we discuss the phase structure of the model in the parameter plane (µ, H) and the solution of the mass gap equation.

1 Introduction

The study of dense matter phenomena is an important branch of physics. Most properties of matter are only weakly affected by changes in external parameters, such as temperature, chemical potential, electromagnetic field etc., when they vary within ranges far from extreme values, i.e. under laboratory conditions. Therefore, each time when a small change in external parameters leads to a considerable response is remarkable and means a discovery of a significant effect. We can mention as examples the quantum Hall effect and oscillations of the magnetic moment predicted in 1930 by Landau [1] and experimentally observed by van Alphen – de Haas [2]. Sixty years have passed since the discovery of this effect, but until this day only quantum oscillations considered in the nonrelativistic case have been discussed extensively. Most attention of researchers dealing with oscillations of the magnetic moment is now focused on the relativistic problem since the results of these studies may be applied to cosmology, astrophysics and high-energy physics [3, 4].

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The consideration of dense quark matter as described by a chemical potential leads to screening of quark charges and possibly to a restoration of chiral symmetry. In order to describe the corresponding chiral phase transition we must have a realistic model. It is well known that models of the Nambu-Jona-Lasinio (NJL) type with inclusion of background fields, temperature and chemical potential are a good laboratory for investigating the nonperturbative phenomenon of spontaneous breaking of chiral symmetry, as well as for describing the low-energy sector of quantum chromodynamics \[5, 6, 7\].

In this paper we shall in particular demonstrate the possibility of the existence of the van Alphen–de Haas effect in quark matter with finite density in a homogeneous constant magnetic field. As model Lagrangian for light quarks we shall use the NJL model where quarks acquire a dynamical mass due to spontaneous breaking of chiral symmetry in the presence of a homogeneous magnetic background field.

2 Effective potential of the NJL model

The Lagrangian of the NJL model with light quarks \(u, d\) of the flavor group \(SU(2)_f\) and the color group \(SU(N_c)\) has the form (flavor and color indices of quarks will be suppressed)

\[
L = \bar{q}i\gamma^\mu D_\mu q + \frac{G}{2N_c}[(\bar{q}q)^2 + (\bar{q}\gamma^5 \vec{\tau}q)^2],
\]

where the covariant derivative is given by \(D_\mu = \partial_\mu - iQA_\mu\), \(Q = diag(e_1, e_2)\), \(G\) is a four-fermion coupling constant, and \(\vec{\tau}\) are isospin Pauli matrices. As is well known, this model is symmetric under continuous chiral transformations.

To investigate the properties of the vacuum of the NJL model it is convenient to employ instead of the quark Lagrangian (1) the quark-meson Lagrangian

\[
L_\sigma = \bar{q}i\gamma^\mu D_\mu q - \bar{q}(\sigma + i\gamma^5 \vec{\tau}\vec{\pi})q - \frac{N_c}{2G}\Sigma^2,
\]

where \(\Sigma^2 = \sigma^2 + \vec{\pi}^2\), which, due to the equations of motion for the fields \(\sigma, \vec{\pi}\), is equivalent to (1). It is convenient to consider the effective action of the model which is given by

\[
\exp\{iS_{\text{eff}}(\mu, \sigma, \vec{\pi})\} = \int D\bar{q}Dq \exp\left(i \int (L_\sigma + \mu\gamma_0 q)\, d^4x\right),
\]

where

\[
\frac{1}{N_c}S_{\text{eff}} = -\int d^4x \frac{(\sigma^2 + \vec{\pi}^2)}{2G} - iT\ln(i\gamma^\mu D_\mu + \mu\gamma_0 - \sigma - i\gamma^5 \vec{\tau}\vec{\pi}),
\]

and \(\mu\) denotes the chemical potential.

Let us first recall the well-known properties of the model with vanishing chemical potential and vanishing background magnetic field. Assuming, as usual, that the fields \(\sigma, \vec{\pi}\) do not depend on the space-time, and using the definition

\[
S_{\text{eff}} = -V_{\text{eff}}(\sigma, \vec{\pi})\int d^4x,
\]
we obtain the effective potential

\[ \frac{1}{N_c} V_{\text{eff}} = \frac{\Sigma^2}{2G} + 4i \int \frac{d^4p}{(2\pi)^4} \ln(\Sigma^2 - p^2) = \frac{V_0(\Sigma)}{N_c}. \] (4)

Switching in \( V_{\text{eff}}(\sigma, \pi) \) to the Euclidean metric and introducing a cut-off for the integration region \( p^2 \leq \Lambda^2 \), we obtain

\[ \frac{1}{N_c} V_0(\Sigma) = \frac{\Sigma^2}{2G} - \frac{(8\pi^2)^{-2}}{\Lambda^2} \cdot \left\{ \Lambda^4 \ln(1 + \Sigma^2/\Lambda^2) + \Lambda^2 \Sigma^2 - \Sigma^4 \ln(1 + \Lambda^2/\Sigma^2) \right\}. \] (5)

The condition that the function (5) is stationary is

\[ \frac{\partial V_0(\Sigma)}{\partial \Sigma} = 0. \] (6)

Obviously, for \( G < G_c = \frac{4\pi^2}{\Lambda^2} \) equation (5) has no solutions except the trivial one \( \Sigma = 0 \). On the other hand, for \( G \geq G_c \) a nontrivial solution \( \Sigma = \sigma_0 \) does exist (we choose \( \pi_0 = 0 \), which means that a pion condensate is absent), and the potential \( V_0(\Sigma = \sigma_0) \) has a global minimum at the point \( \Sigma \neq 0 \), signalling spontaneous breakdown of chiral symmetry and the appearance of a quark mass.

In the following, we shall study a gas of quarks with non-zero chemical potential \( \mu \) in the presence of an external homogeneous magnetic field which is determined by the vector potential \( A_\mu = H\delta_\mu x_1 \). In this case the energy spectrum of quarks is given by

\[ E_{i,n}^2 = \Sigma^2 + p_3^2 + 2|e_i|Hn, \] (7)

where \( n = 0, 1, 2, ... \) is the Landau quantum number. Let us next temporarily assume that, together with the chemical potential \( \mu \), a heatbath with temperature \( T \) acts on the system described by the NJL Lagrangian (1). In this case, to obtain the effective potential the following transformation of the integration variable \( p_4 \) must be made \( p_4 \to 2\pi(l+1/2)/\beta \), \( l = 0, \pm 1, \pm 2, ... \), where \( 1/\beta = T \). After that the effective potential of our model can be written as

\[ \frac{1}{N_c} V_{\mu\beta}^{\text{tot}}(\Sigma, |e_i|H) = \frac{1}{N_c} V_{\mu\beta}(\Sigma) + \frac{1}{N_c} V_{0\beta}(\Sigma, |e_i|H) + \frac{1}{N_c} \Delta V(\mu, \beta, \Sigma, |e_i|H), \] (8)

where \( V_{\mu\beta}(\Sigma)/N_c \) denotes the part of the effective potential with \( \mu \neq 0 \) and \( H = 0 \), and \( V_{0\beta}(\Sigma, |e_i|H)/N_c \) is the part with \( \mu = 0 \), \( H \neq 0 \). Finally, \( \Delta V(\mu, \beta, \Sigma, |e_i|H) \) is responsible for the situation in which both \( \mu \) and \( H \) are non-zero. The third part of the effective potential is of particular interest here and takes the form

\[ \frac{1}{N_c} \Delta V(\mu, \beta, \Sigma, |e_i|H) = - \sum_{i=1}^2 \frac{|e_i|HT}{\pi^2} \int_0^\Lambda dp_3 \sum_{n=0}^\infty \alpha_n \{ \ln[(1 + e^{-\beta(E_{i,n} + \mu})] \times [1 + e^{-\beta(E_{i,n} - \mu})] \}, \] (9)

\[ \alpha_n = 2 - \delta_{n0}. \]
Since we shall consider in this paper only the case of cold quark matter in a magnetic
background field \( H \), we let \( T \) in (9) tend to zero which finally yields

\[
\frac{1}{N_c} \Delta V(\mu, \Sigma, |e_i|H) = - \sum_{i=1}^{2} \frac{|e_i|H}{\pi^2} \int_0^\Lambda dp \sum_{n=0}^{\infty} \alpha_n \theta(\mu - E_{i,n})(\mu - E_{i,n})
\]

This representation will now be used for examining the phase structure of the NJL model.

3 van Alphen - de Haas oscillations

Next we will show that in the range of small magnetic fields the expression \( \Delta V(\mu, \Sigma, |e_i|H) \)
is an oscillating function of \(|e_i|H|\) which cannot be easily seen from the expression (10).
Indeed, equation (10) is not quite convenient for extracting relevant physical informations.
In order to resolve this problem, let us apply Poisson’s summation formula

\[
\sum_{n=0}^{\infty} \alpha_n \Phi(n) = 2 \sum_{n=0}^{\infty} \alpha_n \int_0^\infty \Phi(x) \cos(2\pi kx) \, dx
\]

(11)
to this equation. By integrating first over the momentum variables in (10), we find

\[
\frac{1}{N_c} \Delta V(\mu, \Sigma, |e_i|H) = - \frac{1}{\sqrt{2\pi}} \theta(\mu - \Sigma) \sum_{i=1}^{2} \sum_{k=1}^{\infty} \left( \frac{|e_i|H}{\pi k} \right)^{3/2} \{ S_-(a_i, b_i) \cos \pi k b_i - C_-(a_i, b_i) \sin \pi k b_i \},
\]

(12)

where

\[ S_-(a_i, b_i) = S(\pi k a_i) - S(\pi k b_i), \quad C_-(a_i, b_i) = C(\pi k a_i) - C(\pi k b_i). \]

The functions \( C(x) \) and \( S(x) \) are given by Fresnel’s integrals

\[
\left( \begin{array}{c} S(x) \\ C(x) \end{array} \right) = \frac{1}{\sqrt{2\pi}} \int_0^x dy \left( \frac{\sin y}{\cos y} \right),
\]

(13)

and \( a_i = \frac{\mu^2}{|e_i|H}, \quad b_i = \frac{(\Sigma^2 + \xi \Delta^2)}{|e_i|H}, \quad \xi \in [0, 1] \). Finally, after doing some calculations we obtain for
the oscillating component of \( \Delta V(\mu, \Sigma, |e_i|H) \) the expression

\[
\frac{1}{N_c} \Delta V_{osc}(\mu, \Sigma, |e_i|H) = \frac{\mu \theta(\mu - \Sigma)}{4 \pi^{3/2}} \sum_{i=1}^{2} \sum_{k=1}^{\infty} \left( \frac{|e_i|H}{\pi k} \right)^{3/2} \{ Q(\pi k a_i) \cos(2\pi k \omega_i + \frac{\pi}{4}) + P(\pi k a_i) \cos(2\pi k \omega_i - \frac{\pi}{4}) \},
\]

(14)

where \( \omega_i = \frac{\mu^2 - \Sigma^2}{2|e_i|H} \) are the oscillation frequencies.

Note also that for \( 1 \ll x \) we have the representations

\[ P(x) = x^{-1} - 3x^{-3}/4 + ..., \]
\[ Q(x) = -x^{-2}/2 + 15x^{-4}/8 + \ldots \]

Equation (14) is an exact expression for the oscillations of the effective potential of an ideal relativistic quark gas at \( T = 0 \). The main difference of the relativistic case from the nonrelativistic one (discussed in QED by Landau in 1930 ([1, 2, 3, 9])) is the oscillation frequency \( \omega_i = \frac{\mu^2 - \Sigma^2}{2|e_i|H} \) to be compared with the nonrelativistic expression \( \omega_0 = \frac{\Sigma|\mu - \Sigma|}{|e_i|H} \). Further differences are the dependence of the dynamical quark mass \( \Sigma(\mu, H) \) on the chemical potential \( \mu \) and the external background field \( H \) ([5, 7, 10, 11]). From (14) we can easily derive the contribution to the magnetization

\[ \Delta M = -\frac{1}{\beta} \frac{\partial}{\partial H} \frac{1}{N_c} \Delta V_{osc}(\mu, \Sigma, |e_i|H) \]

which is obviously an oscillating function of \( \omega_i \) (van Alphen - de Haas effect).

### 4 Phase structure of the NJL model

Next we shall examine the phase structure of the NJL model. For this purpose it is convenient to rewrite the third component of the effective potential given in (10) in a more useful form which is completely equivalent to (13). To this end, let us integrate over the momentum variable which yields

\[ \frac{1}{N_c} \Delta V(\mu, \Sigma, |e_i|H) = -\frac{1}{\pi^2} \sum_{i=1}^{2} |e_i|H \sum_{n=0}^{\infty} \alpha_n \theta(\mu - \sqrt{\Sigma^2 + 2|e_i|Hn}) \]

\[ \times \left\{ \mu \sqrt{\mu^2 - \Sigma^2 - 2|e_i|Hn} + (\Sigma^2 + 2|e_i|Hn) \ln \frac{\mu + \sqrt{\mu^2 - \Sigma^2 - 2|e_i|Hn}}{\sqrt{\Sigma^2 + 2|e_i|Hn}} \right\}. \quad (15) \]

It is convenient to consider in some detail the various terms in the gap equation

\[ \frac{1}{N_c} \partial V_{\mu \Sigma} = \frac{1}{N_c} \partial V_{\mu} + \frac{1}{N_c} \partial V_{\Sigma} + \frac{1}{N_c} \partial \Delta V_{\mu \Sigma} \]

which leads us to find some new properties concerning the phase structure of the model. After some calculations we find for the first contribution

\[ \frac{1}{N_c} \partial V_{\mu}(\Sigma) = \frac{\Sigma}{\pi^2} \left\{ F(\Sigma) + \theta(\mu - \Sigma) \left[ \mu \sqrt{\mu^2 - \Sigma^2 - \Sigma^2 \ln \frac{\mu + \sqrt{\mu^2 - \Sigma^2}}{\Sigma}} \right] \right\} \]

where \( F(\Sigma) = \frac{\pi^2}{6} - \frac{\lambda^2}{2} + \frac{\Sigma^2}{2} \ln (1 + \frac{\Sigma^2}{2}) \).

For the second term we have

\[ \frac{1}{N_c} \partial V_{0}(\Sigma, |e_i|H) \]

\[ = -\frac{\Sigma}{\pi^2} \left\{ \frac{1}{8\pi} \sum_{i=1}^{2} (2|e_i|H) \left[ \ln \frac{\Gamma(k_i/2)}{\sqrt{2\pi}} + k_i - \frac{k_i - 1}{2} \ln \frac{k_i}{2} \right] \right\}. \quad (18) \]
where \( k_i = \frac{\Sigma^2}{|e_i|H} \), and \( \Gamma(x) \) is the gamma function.

Finally, the third contribution is given by

\[
\frac{1}{N_c} \frac{\partial \Delta V(\mu, \Sigma, |e_i|H)}{\partial \Sigma} = -\frac{2\Sigma}{\pi^2} \sum_{i=1}^{2} |e_i|H \sum_{n=0}^{\infty} \alpha_n \theta \left( \mu - \sqrt{\Sigma^2 + 2|e_i|Hn} \right)
\times \left\{ \frac{-\mu}{(\mu^2 - \Sigma^2 - 2|e_i|Hn)} + \ln \left( \frac{\mu + \sqrt{\mu^2 - \Sigma^2 - 2|e_i|Hn}}{\sqrt{\Sigma^2 + 2|e_i|Hn}} \right) \right\}.
\]

(19)

It follows from Eqs. (17)-(19) that the gap equation (16) takes the form

\[
\Sigma \cdot (\Phi(\mu, \Sigma, |e_i|H)) = \Sigma \cdot (\phi_\mu(\Sigma) + \phi_0(\Sigma, |e_i|H) + \Delta \phi(\mu, \Sigma, |e_i|H)) = 0,
\]

where the function \( \Phi(\mu, \Sigma, |e_i|H) \) is represented as a sum of three terms corresponding to the R.H.S. of Eq. (16).

To investigate the phase structure of the model it is convenient to consider in the parameter plane \((\mu, h)\), 
\[ h = \sqrt{\text{max}(2|e_i|H)}, \]
the critical curves \( \mu(k)(h) \) defined as solutions of the equation \( \Phi(\mu, 0, |e_i|H) = 0 \) for the case where the dynamical quark mass \( \Sigma \) vanishes and chiral symmetry will be restored. It is then useful to divide the part of the parameter plane in which \( \mu \geq 0, h \geq 0 \) into regions \( \Omega_k \)

\[
\Omega_k = \{ (\mu, h) : h\sqrt{k} \leq \mu \leq h\sqrt{k+1} \}
\]

(21)

limited from above by the critical curve \( \bar{\mu}(k)(h) \). It is obvious that in the region \( \Omega_0 \) only the first term under the summation sign in the contribution from (19) is different from zero, in \( \Omega_1 \) the first and second terms are different from zero, and so on. We can show that for fixed \( h \) the dynamical quark mass \( \Sigma(\mu) \) is monotonically decreasing in \( \mu \). In particular, in discrete points \( \mu_k = h\sqrt{k}, \ k = 1,2,..., \) its derivative turns out to have jumps \( C_k(h) - C_{k-1}(h) \neq 0 \) where \( C_k(h) \) is defined by

\[
C_k(h) = -\left( \frac{\partial \Delta \phi}{\partial \mu} \right)_{\mu=\mu_k},
\]

(22)

and for all \( k \) one has \( C_k > C_{k+1} (k = 0, 1, 2, \ldots) \) and \( C_k < 1 \). Thus in the region \( 0 < \mu < \mu(k)(h) \) we obtain a nontrivial solution \( \Sigma(\mu, |e_i|H) \neq 0 \), whereas for \( \mu > \mu(k)(h) \) the gap equation has only the trivial solution. The corresponding behaviour is shown schematically in the Fig.1.

5 Summary

In this work we have investigated dense cold quark matter in a homogeneous magnetic field using an extended NJL-model. Here, we were interested in the case that chiral symmetry is broken so that quarks get a dynamical mass \( \Sigma(\mu, |e_i|H) \). In particular, we have shown that the effective potential of the model is an oscillating function with frequency \( \omega_i = \frac{\mu^2 - \Sigma^2}{2|e_i|H} \). In
the case $\Sigma(\mu, |e_i|H) \sim \mu$ the relativistic frequency reduces to the well known nonrelativistic expression $\omega_0 = \frac{\Sigma(\mu - \Sigma)}{eH}$. Finally, the phase structure of the NJL model in the parameter plane $(\mu, h)$ and the influence of the oscillations on the $\mu$-dependence of the dynamical quark mass $\Sigma(\mu, |e_i|H)$ have been shortly discussed.

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Figure 1: Schematical representation of the solution $\Sigma(\mu, |e_i|H)$ of the gap equation (16) as a function of $\mu$ for fixed $H$ with a given critical value $\mu_c = \bar{\mu}(k)(h)$.