Transonic flow bifurcations over a double wedge

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Abstract. Inviscid and turbulent airflows over a symmetric flat-sided double wedge are studied numerically. Solutions of the Euler and Navier-Stokes equations are obtained with a finite-volume solver on fine meshes. The solutions demonstrate both symmetric and asymmetric flow regimes in significant bands of the free-stream Mach number at zero angle of attack. The realization of a certain regime depends on the time history of boundary conditions. A physical interpretation of transitions between the symmetric and asymmetric flows is suggested.

1. Introduction
In 1991, a numerical study of inviscid flow over a few asymmetric airfoils revealed nonunique solutions of the system of Euler equations [1]. The non-uniqueness was associated with a hysteresis of the solutions under variation of the angle of attack. Then multiple solutions were obtained for symmetric airfoils at zero angle of attack under variation of the free-stream Mach number [2].

It was noticed in [3, 4] that multiple solutions arise in the cases when an airfoil comprises a flat or nearly flat part. The small curvature of this part causes a formation of local supersonic regions, whose coalescence/rupture proceeds abruptly and results in the flow non-uniqueness. This phenomenon was documented for inviscid and turbulent flows over a number of symmetric and asymmetric airfoils [4-7]. Also, instability of the coalescence/rupture of supersonic regions was demonstrated for airfoils with aileron or spoiler deflections that produce a small local curvature of the profile [8]. Transonic flows over 3D wings whose surfaces comprise nearly flat parts were explored in [9-11].

Nowadays, aerodynamic surfaces comprising nearly flat parts attract a special interest because they often arise as solutions of aerodynamic optimization problems [12, 13]. Meanwhile, physical causes of the flow instability and non-uniqueness remain not quite clear [7].

In the present paper, we study 2D transonic flow over a 8%-thick flat-sided double wedge, which admits a wider bifurcation band than a wedge of smaller thickness examined in [5]. In Sections 2 and 3 we formulate the problem and outline a numerical method. Then in Sections 4 and 5 we study inviscid flow over the wedge at various free-stream Mach numbers and zero angle of attack. Section 6 addresses the same issues for the fully turbulent flow.

2. Formulation of the problem for inviscid flow
We consider a symmetric double wedge determined by the expressions:

\[ y(x) = \begin{cases} 
\pm hx/0.6 & \text{at } 0 \leq x < 0.3, \\
\pm h/2 & \text{at } 0.3 \leq x \leq 0.7, \\
\pm h(1-x)/0.6 & \text{at } 0.7 < x \leq 1.0, 
\end{cases} \]  

(1)
where \( h = 0.08 \) is wedge thickness and \((x, y)\) are Cartesian coordinates non-dimensionalized by wedge length \( l = 0.5 \) m. The inviscid 2D airflow is governed by the system of Euler equations with respect to the density \( \rho(x, y, t) \), static temperature \( T(x, y, t) \) and flow velocity components \( U(x, y, t), V(x, y, t) \), where \( t \) is time.

The outer boundary of a lens-type computational domain is composed by circular arcs \( \Gamma_1 \) and \( \Gamma_2 \) whose minimum distance to the wedge is 40 and maximum distance is 100, see figure 1. On the inflow part \( \Gamma_1 \) of boundary, we prescribe the temperature \( T_\infty = 250 \) K, angle of attack \( \alpha \), and free-stream Mach number \( M_\infty < 1 \), which determine velocity components \( U_\infty = M_\infty a_\infty \cos \alpha, V_\infty = M_\infty a_\infty \sin \alpha \), where \( a_\infty = (\gamma RT)^{1/2} \) is the sound speed and \( R = c_p - c_v/\gamma \) is the specific gas constant, \( \gamma = 1.4 \). The air is treated as a perfect gas whose specific heat at constant pressure \( c_p \) is 1004.4 J/(kg K).

On the outflow part \( \Gamma_2 \) of boundary, we impose the static pressure \( p_\infty = 1.5 \times 10^5 \) N/m\(^2\) related to \( \rho \) and \( T \) by the equation of state \( p = \rho RT \). On wedge (1), the vanishing heat flux and free-slip condition are given. Initial data are either uniform free stream or a nonuniform flow obtained for other values of \( M_\infty \) and \( \alpha \).

3. Numerical method

Solutions of the Euler equations were obtained with ANSYS-18.2 CFX finite-volume solver of second-order accuracy in space and time [14]. An implicit backward Euler scheme is employed for the global time-stepping.

Computational meshes were constituted by triangles whose sizes were crucially decreasing near the wedge for an accurate resolution of shock waves. Mesh cells located at \( y > 0 \) and \( y < 0 \) were perfectly symmetric about the \( x \)-axis. Test computations on uniformly refined meshes of approximately \( 4.2 \times 10^5 \), \( 7.5 \times 10^5 \), and \( 13.5 \times 10^5 \) cells showed a negligible discrepancy between flow fields obtained on the second and third meshes. That is why, to study transonic flow at various \( M_\infty \), we used the mesh of \( 7.5 \times 10^5 \) cells. The time step was set to \( 10^{-5} \) s in order to provide the root-mean-square CFL number smaller than 2. The solver was validated by computation of several benchmark transonic flow problems [8, 11].

4. Inviscid flow bifurcations

First, we solved the problem at \( M_\infty = 0.8340 \) and zero angle of attack \( \alpha = 0 \) using the free-stream parameters for initialization of solution. Computations showed a convergence of the time-dependent flow to a steady state which is symmetric about the \( x \)-axis and exhibits two local supersonic regions (LSRs) on each side of the wedge. The obtained solution was then used as initial data for flow computations step-by-step at increasing \( M_\infty \) from 0.8340 to 0.8393. In this symmetric flow regime with four LSRs, the lift coefficient vanishes, see figure 2 where sketches next to curves/segments illustrate locations and number of supersonic regions in the flow.

Then we performed computations at \( M_\infty = 0.8420, \alpha = 0 \) using again the free-stream parameters for initialization of solution. The obtained steady symmetric flow exhibited a large local supersonic region
(LSR) on each side of wedge. This flow was used as initial data for flow computations step-by-step at decreasing \( M_\infty \) from 0.8420 to 0.8394, see segment 3 in figure 2.

\[ \text{Figure 2. Lift coefficient } C_L \text{ of wedge (1) at } \alpha=0 \text{ versus the Mach number } M_\infty: \]
\[ 1 \text{ and } 3 \text{ – symmetric inviscid flow, } 2 \text{ and } 4 \text{ – asymmetric inviscid flow.} \]

At free-stream Mach numbers that are close to a “singular” value \( M_\infty \approx 0.8393 \), the symmetric flows are unstable, so that a very small perturbation of the angle of attack triggers a transition to an asymmetric flow with three LSRs, see curves 2 and 4 in figure 2. For instance, a positive impulse of \( \alpha \) at \( M_\infty=0.8393 \) results in a transition to the steady asymmetric flow depicted in figure 3.

\[ \text{Figure 3. Mach number contours } M(x,y)=\text{const}, \]
\[ \text{where } M=(U^2+V^2)^{1/2}/a, \text{ in the asymmetric inviscid flow with positive lift at } M_\infty=0.8393, \alpha=0. \]

The obtained asymmetric flow was then used for computations step-by-step at smaller and larger \( M_\infty \). This scenario made it possible to determine the bifurcation band

\[ 0.8351 \leq M_\infty \leq 0.8413, \]  

in which the asymmetric flows (with positive or negative lift) are stable with respect to small perturbations. At Mach numbers \( M_\infty \) smaller than 0.8351, computations show a transition to the symmetric flow with four LSRs. This means a transition from curve 2 or 4 in figure 2 to segment 1. On the other hand, if \( M_\infty \) exceeds 0.8413, then computations demonstrate a transition from curve 2 or 4 to segment 3, i.e., from an asymmetric flow to symmetric one with two large LSRs.

5. Interpretation of transitions between different flow regimes

5.1. Transition from an asymmetric steady flow to symmetric one with four LSRs

A transition from curve 2 or 4 to segment 1 in figure 2 at decreasing \( M_\infty \) can be interpreted using oblique shock relations as follows. Left endpoints of the curves correspond to \( M_\infty=0.8351 \) and flow fields in which the oblique shock wave (SW) hits the wedge immediately downstream of the rear
corner, see figure 4. The angle β made by SW with the rear ramp of wedge is $\beta_1 \approx 53.5^\circ$, whereas Mach number in front of SW on the wedge is $M_1 =1.39$. For such $\beta_1$ and $M_1$, the oblique shock relations [15] admit a regular reflection of SW in agreement with the flow pattern displayed in figure 4.

![Figure 4](image)

Figure 4. A fragment of the asymmetric inviscid flow with positive lift at $M_\infty =0.8351$, $\alpha =0$: Mach number on the upper side of wedge versus $x$ (top) and isoMachlines in the flow (bottom).

Meanwhile, if $M_\infty$ is decreased from 0.8351 to, e.g., 0.8348, then computations show that SW shifts upstream and hits the horizontal part of wedge. As a consequence, the angle β made by SW with the wedge jumps from $\beta_1$ to $\beta_2 \approx 79^\circ$, whereas Mach number in front of SW drops from $M_1$ to $M_2 =1.11$. For these $\beta_2$ and $M_2$, a steady regular reflection of SW cannot exist due to the oblique shock relations [15]. Also, a steady curved SW that would terminate on the horizontal part of wedge near corner $x=0.7$, $x=0.04$ cannot exist; indeed, such a SW would approach the wedge in the normal direction, producing subsonic flow behind it, and this would contradict the closely spaced sonic line emanated from the corner. That is why SW must move upstream at a significant distance from the corner. This means a rupture of the LSR on the upper side of wedge, followed by a formation of two LSRs symmetric to those on the lower side.

5.2. Transition from an asymmetric steady flow to the symmetric one with two LSRs

Let us consider, e.g., the asymmetric flow that corresponds to the right endpoint of curve 2 in figure 2 ($M_\infty =0.8413$). In this flow, the first of two LSRs formed beneath wedge (1) is terminated by a nearly normal SW at $x=0.647$. Behind SW, the flow is subsonic and Mach number is $M_1 =0.90$, see figure 5. The subsonic flow then accelerates from $M_1$ up to $M=1$ at the corner $x=0.7$ due to a convergence of streamlines. The convergence $(h_1 - h_2)/h_1$ is 0.9% according to the numerical solution; this value agrees with an isentropic formula which links the Mach number to the ratio $h_2/h_1$ of cross sections of a streamtube [15].

Meanwhile, if $M_\infty$ is increased from 0.8413 to, e.g., 0.8416, then the normal SW shifts downstream and becomes stronger. As a consequence, Mach number $M_2$ behind SW becomes smaller than $M_1$, while the convergence of streamlines remains about 0.9%. Then the isentropic formula shows that subsonic flow cannot speed up from $M_2$ to $M=1$ at $x=0.7$. Therefore, SW must become unsteady and migrate downstream to a position in which it hits the wedge behind the corner. This means a coalescence of LSRs on the lower side of wedge and formation of a single LSR symmetric to that on the upper side.
5.3. Transitions from symmetric to asymmetric flows

The numerical solutions show that transitions from symmetric to asymmetric flows are accompanied by zero streamline displacements from the x-axis upward or downward in the wake. For instance, a transition to flow with positive lift (curve 2 in figure 2) is accompanied by a deflection of zero streamline from the x-axis downward. The deflection correlates with shrinking LSRs beneath wedge (1) like a deployment of an aileron does [8]. The shrunk LSRs and increased distance between them mean higher flow stability. In addition, the downward-deflected zero streamline promotes a shift of the oblique SW downstream from the corner on upper side of wedge, see figure 4; the latter contributes to flow stability as well.

The displacements of zero streamline imply an interaction between LSRs located above and beneath the wedge. To confirm an important role of the interaction, we solved the problem for a truncated wedge, which coincides with (1) at 0≤x<0.8 and terminates with a vertical segment −h/3≤y≤h/3 at x=0.8. Computations showed that in this case the bifurcation band is longer than band (2) due to a stronger interaction between the upper and lower LSRs via an expanded wake region.

6. Turbulent flow bifurcations

Solutions of the unsteady Reynolds-averaged Navier-Stokes equations were obtained with ANSYS-18.2 CFX solver using the Shear Stress Transport (SST) k-ω and Spalart-Allmaras turbulence models [16, 17]. The free-stream turbulence level was set to 1%. The Reynolds number Re based on the length l of wedge (1) was 1.1×10^6. An unstructured mesh of 783744 cells was constituted by quadrangles in 38 layers on the wedge and by triangles in the remaining region. The non-dimensional thickness y' of first mesh layer on the wedge was less than 1.

The obtained solutions demonstrate turbulent flow bifurcations similar to ones discussed above for inviscid flow. Apart from the bifurcations, the SST k-ω turbulence model reveals self-exciting oscillations (buffet onset) which arise in asymmetric flows at M∞>0.8392 due to instability of a shock-induced boundary layer separation from the rear ramp of wedge. Figure 6 depicts margins of the oscillating lift coefficient at various M∞. In symmetric flows, oscillations arise at M∞>0.8428. The oscillations are periodic in time and may exhibit an intricate character. The frequency of oscillations depends on M∞ and does not exceed 40 s⁻¹. As seen from figure 6, the bifurcation band

\[ 0.8376 \leq M_\infty \leq 0.8420, \]

is shifted to larger M∞ as compared to (2). At Mach numbers that are close to the “singular” value M_∞=0.8404, the symmetric flows are unstable, so that a very small perturbation of α triggers a transition to an asymmetric flow. Computations using the Spalart-Allmaras turbulence model showed bifurcations in a shorter band 0.8380≤M_∞≤0.8416 and did not expose flow oscillations unless M_∞>0.8435.
Figure 6. Margins of the lift coefficient oscillations in asymmetric and symmetric turbulent flows over wedge (1) at $\alpha=0$, $Re=1.1\times10^7$, SST $k$-\$\omega$ turbulence model.

7. Conclusion
The numerical study demonstrated bifurcations of inviscid and turbulent flows over the flat-sided double wedge at zero angle of attack in the bands (2) and (3), respectively. Transitions from asymmetric to symmetric flow regimes occur at the ends of the bands. Transitions from symmetric to asymmetric flows occur at Mach numbers $M_\infty$ that lie in the bands and are close to a “singular” value (0.8393 for the inviscid flow and 0.8404 for turbulent one at $Re=1.1\times10^7$). The transitions are interpreted using isentropic and oblique shock relations. In addition to the bifurcations, turbulent flow exhibits a buffet onset at sufficiently large $M_\infty$.

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