AUTOMORPHIC INTEGRALS WITH LOG-POLYNOMIAL PERIOD FUNCTIONS AND ARITHMETICAL IDENTITIES

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Abstract. Building on the works of S. Bochner on equivalence of modular relation with functional equation associated to the Dirichlet series, K. Chandrasekharan and R. Narasimhan obtained new equivalences between the functional equation and some arithmetical identities. Sister Ann M. Heath considered the functional equation in the Hawkins and Knopp context and showed its equivalence to two arithmetical identities associated with entire modular cusp integrals involving rational period functions for the full modular group. In this paper we use techniques of Chandrasekharan and Narasimhan and extend the results of Sister Ann M. Heath to entire automorphic integrals involving rational period functions on discrete Hecke group. Moreover, we establish equivalence of two arithmetical identities with a functional equation associated with automorphic integrals involving log-polynomial-period functions on the Hecke groups.

1. Introduction

In 1951, Bochner introduced in [1] and showed that the modular relation

$$f(x) = x^{-\delta} g \left( \frac{1}{x} \right) + P(x), \quad 0 < x < \infty,$$

where $f$ and $g$ are exponential series, $P$ is a “residual” function and $\delta > 0$, is equivalent to the functional equation

$$\Gamma(s) \phi(s) = \Gamma(\delta - s) \psi(\delta - s), \quad (1.1)$$

where $\Gamma(s)$ is the standard gamma function, $\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$ and $\psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s}$. Here, he assumes that $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences of real numbers, with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$$

and

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty,$$

and $\{a_n\}, \{b_n\}$ are two sequences of complex numbers not identically zero.

Building on the works of Bochner, Chandrasekharan and Narasimhan [2] showed that the functional equation (1.1) is equivalent to two identities given by

$$\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} a_n (x - \lambda_n)^{\rho} = \left( \frac{1}{2\pi} \right)^{\rho} \sum_{n=1}^{\infty} \left( \frac{x}{\lambda_n} \right)^{\frac{\lambda n}{\mu}} b_n J_{\delta + \rho} \left( 4\pi \sqrt{\mu_n} x \right) + Q_\rho(x), \quad (1.2)$$

for $x > 0$ and $\rho \geq 2\beta - \delta - \frac{1}{2}$, where, the prime notation $'$ on the summation sign indicates that if $\rho = 0$ and $x = \lambda_n$ for some positive integer $n$, then we count only $\frac{1}{2} a_n$,

$$Q_\rho(x) = \frac{1}{2\pi i} \int_{c} \frac{\chi(s)(2\pi)^s x^{s+\rho}}{\Gamma(s + \rho + 1)} ds,$$

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\[ \sum_{n=1}^{\infty} \frac{|b_n|}{\mu_{n,\rho}} < \infty, \]  

\[ J_\rho(z) \text{ denotes the Bessel function of the first kind of order } \nu, \text{ and} \]

\[ \left( -\frac{1}{s} \frac{d}{ds} \right)^\rho \left[ \frac{1}{s} \sum_{n=1}^{\infty} a_n e^{-s\sqrt{n}} \right] = 2^{3\rho + \delta} \Gamma(\delta + \rho + \frac{1}{2}) \pi^{\frac{\delta}{2}} \sum_{n=1}^{\infty} \frac{b_n}{(s^2 + 16\pi^2 \mu_n)^{\delta + \rho + \frac{1}{2}}} + R_\rho(s), \]  

for \( \Re s > 0 \), where

\[ R_\rho(s) = \frac{1}{2\pi i} \oint_C \frac{\chi(z)(2\pi)^2 \Gamma(2z + 2\rho + 1)2^{-\rho}}{\Gamma(z + \rho + 1)} s^{-2z - 2\rho - 1} dz, \]

for \( \Re s > 0 \), \( \rho \) integral, \( \rho - \delta - \frac{1}{2} \) and \( \rho \geq 2\beta - \delta - \frac{1}{2} \).

Sister Ann Heath, in \[ [13] \] showed the equivalence of the above forms for the full modular group.

In our work here, we will show that these results hold for a more general Hecke Groups. Our notations will be different from the ones shown above. The structure of our paper is as follows: in section 2 we review some preliminary concepts which we use in the later sections. In section 3 we state and proof our main results which are in Theorem 3.1 and Theorem 3.2. In section four we give some conclusions and discussions.

2. Preliminaries

In this section, we recall some definitions, results proved elsewhere, and make some conventions for our later discussion.

For \( \lambda \in \mathbb{R}^+ \), the Hecke group \( G(\lambda) \) is defined as the subgroup of \( SL_2(\mathbb{R}) \) given by

\[ G(\lambda) = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \]

Note that \( G(\lambda) \) can be viewed as a group of linear fractional transformations generated by \( S(z) = z + \lambda \) and \( T(z) = -\frac{1}{z} \). In this context, we see that \( G(\lambda) \) acts on the Riemann sphere as linear fractional transformation, that is \( Mz = \frac{cz + b}{cz - a} \) for \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda) \), and \( z \in \mathbb{C} \cup \{\infty\} \), thus \( M \) and \( -M \) can be identified as the same matrices. E. Hecke \[ [8] \] showed the groups \( G(\lambda) \) is discrete (operate discontinuously) as linear fractional transformations on the upper half plane \( \mathcal{H} = \{ z = x + iy : y > 0 \} \) if and only if either

\[ \lambda > 2 \text{ or } \lambda = \lambda_p := 2 \cos \left( \frac{\pi}{p} \right), \text{ with } 3 \leq p \in \mathbb{N} \cup \{\infty\}. \]

Clearly \( G(\lambda_3) = \Gamma(1) \) is the full modular group and \( G(\lambda_\infty) = \Gamma_0 \) is the theta group and is denoted by \( \Gamma_\theta \).

A complex valued function \( \nu \) defined on \( G(\lambda) \) is called a multiplier system with weight \( 2k \) for \( G(\lambda) \) if \( |\nu(M)| = 1 \) for all \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda) \), and \( \nu(M) \) satisfies the consistency condition

\[ \nu(M_3)(c_3z + d_3)^{2k} = \nu(M_1)\nu(M_2) \left( c_1M_2z + d_1 \right)^{2k} \left( c_2z + d_2 \right)^{2k} \]

for all \( M_1, M_2 \in G(\lambda), M_1M_2 = M_3, M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, j = 1, 2, 3 \in \mathcal{H} \), where \( k \in \mathbb{R} \).

A holomorphic function \( F \) is called entire automorphic integral of weight \( 2k \) with multiplier system \( \nu \) for \( G(\lambda) \) if it admits an exponential series expansion

\[ F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i mz/\lambda}, \]

(2.1)
where $\Im z = y > 0$ and $a_m = O(m^{\delta})$, for some $\delta \in \mathbb{R}^+$, and satisfies the transformations
\[ \tilde{\nu}(S\lambda)F(z + \lambda) = F(z), \quad \nu(S\lambda) = e^{2\pi i \kappa}, \quad 0 \leq \kappa < 1 \] (2.2)
and
\[ \bar{\nu}(T)z^{-2k}F\left(\frac{-1}{z}\right) = F(z) + q(z) \quad \forall z \in \mathcal{H}, \] (2.3)
where $q(z) = \sum_{j=1}^{N} z^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} (\log z)^t$, $\alpha_j, \beta_{jt} \in \mathbb{C}$. 

The function $q(z)$ in (2.3) is called the log-polynomial period function of the entire automorphic integral function $F$.

A. Hassen in [4] has completely characterized the log-polynomial period functions for entire automorphic integrals of weight $2k$, $k \in \mathbb{R}$ on the discrete Hecke group $G(\lambda)$ for the following cases:

1. $k \geq 1$, $\nu(S\lambda) = 1$
2. $k > 0$, $\nu(S\lambda) \neq 1$
3. $k \geq 0$, $2k \in \mathbb{Z}$, $\nu(S\lambda) = 1$ and $\nu(T) \neq 1$
4. $k \leq 0$, $\nu(S\lambda) \neq 1$.

For the purpose of this work we consider only the multiplier system which $\nu(S\lambda) = 1$ (or $\nu(T) = 1$).

In [10] Paul C. Pasles showed Riemann-Hecke-Bochner correspondence for entire automorphic integrals on the Hecke groups. We need a modified result of Pasles [10] (see Theorem 3.1), which we state and prove in Theorem 2.1.

**Theorem 2.1.** Let $k \in \mathbb{R}^+ \cup \{0\}$, $\lambda \geq 2$ or $\lambda = 2 \cos \left(\frac{\pi}{p}\right)$, $p \in \mathbb{Z}$, $p \geq 3$. Suppose that $F$ is an entire automorphic integral of weight $2k$, multiplier system $\nu$ and $\nu(S\lambda) = 1$. Let $q(z)$ be log-polynomial period function of $F$ on $G(\lambda)$ and let $F$ has Fourier expansion of the form
\[ F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi imz/\lambda}, \] for $z \in \mathcal{H}$, where

(i) $a_m = O(m^{\beta})$ for some $\beta > 0$, $m \to \infty$.

(ii) Let $\varphi(s) = \sum_{m=1}^{\infty} a_m m^{-s}$, $s = \sigma + it$. And put
\[ \Phi_F(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi(s), \quad \text{for } \sigma > \beta + 1. \]

Then

(A) $\Phi_F(s)$ has a meromorphic continuation to the whole complex plane with at most a finite number of poles given by:
\[ \Phi_F(s) = D_k(s) + M_k(s) + L_k(s), \]
where
\[ D_k(s) = \int_{1}^{\infty} \left\{ F(iy) - a_0 \right\} y^{s-1} ds + i^2 \nu(T) \int_{1}^{\infty} \left\{ F(iy) - a_0 \right\} y^{2k-s-1} ds, \] (2.4)
\[ M_k(s) = a_0 \left\{ \frac{i^{2k} \nu(T)}{s - 2k} - \frac{1}{s} \right\}, \]  

and
\[ L_k(s) = - \sum_{j=1}^{N} i^{2k+\alpha_j} \sum_{t=0}^{M_j} b_j t^j \left( \frac{i\pi}{2} \right)^{t-l} \frac{(-1)^{t+l}!}{(2k + \alpha_j - s)^{t+l+1}}, \]

for \( \text{Res} > 2k + \max_j |Re\alpha_j| \).

(B) \( \Phi_F(s) \) is bounded uniformly and absolutely in each \( \sigma \) in a lacunary vertical strips of the form
\[ S(\sigma_1, \sigma_2, t_0) = \{ s : \sigma_1 \leq \sigma \leq \sigma_2, |\Im s| = t \geq t_0 \}, \]

where \( \sigma_1, \sigma_2 \in \mathbb{R} \) and \( t_0 > \max_j |3\alpha_j| \).

Moreover,
(C) \( \Phi_F(s) \) satisfies the functional equation
\[ \Phi_F(2k - s) = e^{\pi i k} \nu(T) \Phi_F(s). \]

**Proof.** Suppose \( F \) is an entire automorphic integral of weight \( 2k \), \( k \in \mathbb{R} \), multiplier system \( \nu \) and associated LPPF \( q(z) \). For \( \text{Res} > 2k + \max_j |Re\alpha_j| \), applying Mellin transform of \( F \), we have
\[ \Phi_F(s) = \int_0^\infty (F(iy) - a_0) y^s \frac{dy}{y} = \int_1^\infty (F(iy) - a_0) y^s \frac{dy}{y} + \int_0^1 (F(iy) - a_0) y^s \frac{dy}{y}. \]

Taking \( y \mapsto \frac{1}{y} \) in the later integral and applying the transformation law in (2.3), we get
\[ \int_0^1 (F(iy) - a_0) y^s \frac{dy}{y} = \nu(T)i^{2k} \int_1^\infty (F(iy) - a_0) y^{2k-s-1} dy + \nu(T)i^{2k} a_0 \int_1^\infty y^{2k-s-1} dy \]
\[ + \nu(T)i^{2k} \int_1^\infty q(iy)y^{2k-s-1} dy - \frac{a_0}{s} \]
\[ = \nu(T)i^{2k} \int_1^\infty (F(iy) - a_0) y^{2k-s-1} dy + a_0 \left[ \frac{\nu(T)i^{2k}}{s - 2k} - \frac{1}{s} \right] \]
\[ + \nu(T)i^{2k} \int_1^\infty q(iy)y^{2k-s-1} dy. \]

Thus
\[ \Phi_F(s) = D_k(s) + M_k(s) + L_k(s), \]

where
\[ D_k(s) = \int_1^\infty (F(iy) - a_0) y^{s-1} dy + \nu(T)i^{2k} \int_1^\infty (F(iy) - a_0) y^{2k-s-1} dy, \]
\[ M_k(s) = a_0 \left[ \frac{\nu(T)i^{2k}}{s - 2k} - \frac{1}{s} \right], \]

and
\[ L_k(s) = \nu(T)i^{2k} \int_1^\infty q(iy)y^{2k-s-1} dy. \]
$D_κ(s)$ is entire and converges uniformly (absolutely) on compact subset of $C_r$, for large $Re(s)$ and $D_κ(s)$ satisfies the functional equation $D_κ(2k - s) = e^{πik}\nu(T)D_κ(s)$. Similarly $M_κ(s)$ is meromorphic with simple poles at $(s = 2k, 0)$, and satisfies the functional equation $M_κ(2k - s) = e^{πik}\nu(T)M_κ(s)$. In order to show the meremorphic continuation of $Φ_F(s)$ we need the meromorphic continuation of $L_κ(s)$ to $C$. To see this, note that

$$L_κ(s) = \nu(T)i^{2k}\int_1^∞ q(iy)y^{2k-s-1}dy$$

$$= \nu(T)i^{2k}\int_1^∞ \sum_{j=1}^N (iy)^{α_j} \sum_{l=0}^{M_l} β_{jl}(log iy)^l y^{2k-s-1}dy.$$ 

Using the fact that $log iy = log y + i arg(iy) = log y + i\frac{π}{2}$ for $y > 0$, and applying the binomial theorem on $(\frac{iπ}{2} + log y)^l$, we can rewrite the above equation as

$$L_κ(s) = \nu(T)\sum_{j=1}^N t^{2k+α_j} \sum_{l=0}^{M_l} β_{jl}\left(\frac{iπ}{2}\right)^l \int_1^∞ y^{2k+α_j-s-1}(log y)^l dy.$$ 

Integration by parts now yields

$$L_κ(s) = \nu(T)\sum_{j=1}^N t^{2k+α_j} \sum_{l=0}^{M_l} β_{jl}\left(\frac{iπ}{2}\right)^l \int_1^∞ y^{2k+α_j-s-1}(log y)^l dy.$$ 

for $Re s > 2k + max_j Reα_j$. Thus $L_κ(s)$ is meremorphic in $C$ with poles of order $M_j + 1$ at $s = 2k + α_j$. Thus $Φ_F(s) = D_κ(s) + M_κ(s) + L_κ(s)$ is meremorphic on $C$ with finite simple poles at $s = 2k, s = 0$ and poles of order $M_j + 1$ at $s = 2k + α_j$. This completes the prove of (A).

To prove $Φ_F(2k - s) = e^{πik}\nu(T)Φ(s)$ it is enough to show that $L_κ(2k - s) = e^{πik}\nu(T)L_κ(s)$. For this we need

**Lemma 2.1.** If $F$ and $q$ are as in Theorem (2.1) then

$$q\left(\frac{-1}{z}\right) = -(-z)^{2k}\nu(T)q(z), \forall z \in \mathcal{H}. \quad (2.8)$$

**Proof.** By taking $z \to \frac{-1}{z}$ in (2.3), we have

$$F(z) = \nu(T)(-z)^{-2k}\left[\nu(T)z^{2k}F(z) + \nu(T)z^{2k}q(z) + q\left(\frac{-1}{z}\right)\right].$$

Thus,

$$F(z) - \nu^2(T)(-z)^{-2k}z^{2k}F(z) = \nu^2(-z)^{-2k}z^{2k}q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right) = i^{4k}\nu^2(T)q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right) = (i^2\nu(T))^2q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right) = q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right).$$
Therefore,
\[ F(z) - \nu^2(T)(i^{2k})^2 F(z) = q(z) + \nu(T)(-z)^{-2k} q \left( \frac{-1}{z} \right). \]
From the consistency condition on \( \nu \), we see that \( \nu^2(T)(i^{2k})^2 = 1 \) and hence
\[ q(z) + \nu(T)(-z)^{-2k} q \left( \frac{-1}{z} \right) = 0, \ \forall z \in \mathcal{H}, \]
as desired.

Now from \( L_k(s) = i^{2k} \nu(T) \int_1^\infty y^{2k-s-1} q(iy) dy \), we have,
\[ L_k(2k - s) = i^{2k-s} \nu(T) \int_1^\infty y^{s-1} q(iy) dy. \]
Applying Lemma 2.1 and replace \( y \) by \( \frac{1}{y} \) in the integral we obtain
\[ i^{-2k} \nu(T) L_k(2k - s) = -i^{2k} \nu(T) \int_0^1 y^{2k-s-1} q(iy) dy. \]
Then by definition of \( q \) we have
\[ i^{-2k} \nu(T) L_k(2k - s) = -i^{2k} \nu(T) \int_0^1 \sum_{j=1}^N (iy)^{\alpha_j} \sum_{l=0}^{M_j} \beta_{jl} (\log iy)^l y^{2k-s-1} dy. \]

Using \( \log iy = \frac{i\pi}{2} + \log y \) and applying the binomial theorem, the substitution \( y \mapsto \frac{1}{y} \), and integration by parts we have,
\[ i^{-2k} \nu(T) L_k(2k - s) = -i^{2k} \nu(T) \int_0^1 \sum_{j=1}^N (i^j)^{\alpha_j+2k} \sum_{l=0}^{M_j} \beta_{jl} \sum_{l=0}^t \frac{i\pi}{2} \left( \frac{i\pi}{2} \right)^{t-l} \frac{l!}{|s-(2k+\alpha_j)|^{l+1}} dy \]
Hence
\[ L_k(s) = \nu(T) \sum_{j=1}^N (i^j)^{\alpha_j+2k} \sum_{l=0}^{M_j} \beta_{jl} \sum_{l=0}^t \frac{i\pi}{2} \left( \frac{i\pi}{2} \right)^{t-l} \frac{l!}{|s-(2k+\alpha_j)|^{l+1}}. \]
Therefore, \( L_k(2k - s) = e^{\pi i k} \nu(T) L_k(s) \). This complete the prove of (C).

Finally to show the boundedness condition (B), note that the rational functions \( M_k \) and \( L_k \) are bounded in lacunary vertical strips which do not contain poles of \( M_k \) and \( L_k \). The boundedness of \( D_k \) follows from the fact that \( a_m = O(m^\beta) \) for \( \sigma > 1 + \beta \) implies
\[ \left| \int_1^\infty [F(iy) - a_0] y^{s-1} dy \right| = O(\zeta(\sigma - \beta) \Gamma(\sigma)). \]
This complete the proof of (B) and that of the theorem.
Finally before we state and proof Theorem 3.1 we state Perron’s Formula as the following Lemma. (See [2] for details.) We shall use the convention of writing $\int_{(b)}$ for $\int_{b-i\infty}^{b+i\infty}$.

**Lemma 2.2.** Let $\sigma_0$ be the abscissa of absolute convergence for $\varphi(s) = \sum_{m=1}^{\infty} a_m \lambda_m^{-s}$ and $\{\lambda_m\}$ be a sequence of positive real numbers tending to $\infty$ as $m \to \infty$. Then for $k \geq 0, \sigma > 0$ and $\sigma > \sigma_0$,

$$\frac{1}{\Gamma(k+1)} \sum_{\lambda_m \leq x} \nu_{\lambda_m} (x - \lambda_m)^k = \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(s) \varphi(s) x^{s+k}}{\Gamma(s+k+1)} ds,$$

(2.9)

where the prime $'$ on the summation sign indicates that if $k = 0$ and $x = \lambda_m$ for some positive integer $m$, then we count only $\frac{1}{2}a_m$.

### 3. Main results

In this section we state and prove the main results of the paper. Here we establish two arithmetical identities associated to entire automorphic integrals involving log-polynomial period functions on the discrete Hecke group $G(\lambda)$, analogous to Chandrasekharan and Narasimhan and show that they are equivalent to the functional equation $2.7$,

$$\Phi_F(2k-s) = e^{\pi i k} \nu(T) \Phi_F(s).$$

**Theorem 3.1.** Let $\Phi_F(s), q(z)$ and $F(z)$ be as in Theorem 2.7 Then the functional equation

$$\Phi_F(2k-s) = e^{\pi i k} \nu(T) \Phi_F(s)$$

(3.1)

is equivalent to the arithmetical identity

$$\frac{1}{\Gamma(\rho+1)} \sum_{0 \leq \lambda_m \leq x} a_m (x - \lambda_m)^\rho = \frac{e^{-\pi i k} T}{\lambda} \left( \frac{2\pi}{\lambda} \right)^{\rho} \sum_{m=1}^{\infty} a_m \left( \frac{x}{\lambda_m} \right) \frac{x^{\frac{i\lambda_m}{\lambda}}}{\nu(T)} J_{2k+\rho} \left( \frac{4\pi \sqrt{mx}}{\lambda} \right)$$

$$+ \left( \frac{2\pi}{\lambda} \right)^{2k} \frac{\nu(T) e^{\pi i k} a_0 x^{2k+\rho}}{\Gamma(2k+\rho+1)} - \frac{a_0 x^\rho}{\Gamma(\rho+1)}$$

$$+ \left( \frac{2\pi}{\lambda} \right)^{2k+\alpha_j} \frac{x^{2k+\rho+\alpha_j}}{\Gamma(2k+\alpha_j+\rho+1)} \nu(T) e^{\pi i k} \sum_{j=1}^{N} \sum_{i=0}^{M_j} \beta_{j,i} \left( \frac{i\pi}{2} \right)^t,$$

(3.2)

for $x > 0, \rho \in \mathbb{Z}_{\geq 0}, \alpha_j, \beta_{j,i} \in \mathbb{C}, \beta_{j,i}M_j \neq 0$ and $N, M_j \in \mathbb{Z}$, provided $\rho \geq 2\beta - 2k$, where $a_m = O(m^\beta)$.

**Proof of (3.1) implies (3.2).**

**Proof.** Applying Lemma 2.2 for $\rho > 0, b \geq \beta$ and $a_m = O(m^\beta)$ we have

$$\frac{1}{\Gamma(\rho+1)} \sum_{0 \leq \lambda_m \leq x} \nu_{\lambda_m} (x - \lambda_m)^\rho = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(s) \varphi(s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds.$$  

(3.3)

Here the prime notation $'$ indicates that if $m = x, a_m$ is to be multiplied by $\frac{1}{2}$ and $\varphi(s) = \sum_{m=1}^{\infty} a_m m^s$ with $\sum_{m=1}^{\infty} |a_m| m^\beta < \infty$.

To evaluate the integral on the right side of (3.3), for $2k - b < \sigma < b$, where $s = \sigma + it$, consider rectangular
region with vertices $2k - b \pm iT$ and $b \pm iT$, having positive orientation. Let $T$ be large positive number so that Stirling’s formula can be used in approximating $\Gamma(s)$ as $|s| \to \infty$. Using Stirling’s formula and the Phragmen-Lindelof theorem we can show the integrals along the horizontal paths tend to 0 as $T \to \infty$. Hence with the substitution, $(\frac{2\pi}{x})^s \Phi_F(s) = \Gamma(s)\varphi(s)$, the right hand side of (3.3) may be written as

$$
\frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(s)\varphi(s)}{\Gamma(s + \rho + 1)} ds = \frac{1}{2\pi i} \int_{(b)} \left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s)x^{s+\rho}\frac{ds}{\Gamma(s + \rho + 1)}
$$

$$
= \frac{1}{2\pi i} \int_{(2k-b)} \left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s)x^{s+\rho}\frac{ds}{\Gamma(s + \rho + 1)}
$$

$$
+ \sum_{s \in \text{Pole set of } \Phi_F(s)} \text{Res} \left\{ \left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s)x^{s+\rho} \right\}.
$$

Put

$$
\omega_1(x, \lambda) = \frac{1}{2\pi i} \int_{(2k-b)} \left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s)x^{s+\rho}\frac{ds}{\Gamma(s + \rho + 1)}
$$

and

$$
\omega_2(x, \lambda) = \sum_{s \in \text{Pole set of } \Phi_F(s)} \text{Res} \left\{ \left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s)x^{s+\rho} \right\}.
$$

To evaluate $\omega_1(x, \lambda)$ we begin by using the functional equation

$$
\Phi_F(2k - s) = e^{2\pi i \nu(T)} \Phi(s).
$$

$\omega_1(x, \lambda)$ can be written as

$$
\omega_1(x, \lambda) = \frac{1}{2\pi i} \int_{(2k-b)} \left(\frac{2\pi}{\lambda}\right)^s \frac{e^{-2\pi i \nu(T)}x^{s+\rho}}{\Gamma(s + \rho + 1)} ds.
$$

By substituting

$$
\Phi_F(2k - s) = \left(\frac{2\pi}{\lambda}\right)^s \Gamma(2k - s) \sum_{m=1}^{\infty} \frac{a_m}{m^{2k-s}}
$$

and replacing $s$ by $2k - s$, to evaluate $\omega_1(x, \lambda)$ we can use the integral in (2, page 5) given by the formula

$$
\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{2s-\nu-1}{\Gamma(\nu-\frac{1}{2}s+1)} x^{-s} ds = \frac{J_\nu(x)}{x^\nu},
$$

provided that $0 < C \leq \nu + 1$, $\nu > 0$.

With simple algebraic manipulation we obtain

$$
\omega_1(x, \lambda) = \left(\frac{2\pi}{\lambda}\right)^{-\rho} e^{-2\pi i \nu(T)} \sum_{m=1}^{\infty} \frac{a_m}{m^{2k+\rho}} J_{2k+\rho} \left(\frac{4\pi \sqrt{m} x}{\lambda}\right),
$$

provided that $\rho \geq 2\beta - 2k$.

To complete evaluation of the integral in (3.3) we calculate the residues in $\omega_2(x, \lambda)$. Note that $\Phi_F$ has poles at $s = 0, 2k$ resulting from $M_k$ and at $s = 2k + \alpha_j$ with order $M_j + 1$ from $L_k$. Thus calculating the residues of the functions we obtain,

$$
\omega_2(x, \lambda) = \left(\frac{2\pi}{\lambda}\right)^{2k} e^{2\pi i \nu(T)} \frac{x^{2k+\rho}}{\Gamma(2k + \rho + 1)} - \frac{a_0 x^\rho}{\Gamma(\rho + 1)}
$$

$$
+ \left(\frac{2\pi}{\lambda}\right)^{2k+\alpha_j} \frac{x^{2k+\rho+\alpha_j}}{\Gamma(2k + \rho + 1 + \alpha_j)} \nu(T) e^{2\pi i \nu(T)} \sum_{j=1}^{N} \left(\frac{i\pi}{2}\right)^t,
$$

(3.5)
for $\rho \in \mathbb{Z}_{\geq 0}$, $\alpha_j, \beta_{j\ell} \in \mathbb{C}$ and $\beta_{j\ell}M_j \neq 0$.

The integral in right of (3.3) is equal to the combination of the expressions in (3.4) and (3.5) respectively, thus it follows the identity (3.2).

**Proof of converse of the theorem:** Suppose $F$ be entire automorphic integral function. $F(z)$ has Fourier series expansion of the form

$$F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi im/\lambda}.$$ \hspace{1cm} (3.6)

For $z \in \mathcal{H}$, where $a_m = O(m^\gamma)$ as $m \to \infty$, for some $\gamma > 0$. Put

$$\Phi_F(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \sum_{m=1}^{\infty} a_m m^{-s}.$$ \hspace{1cm} (3.7)

For $z \in \mathcal{H}$, $F$ satisfies the relation

$$\bar{\nu}(T) z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z) \forall z \in \mathcal{H},$$ \hspace{1cm} (3.8)

where

$$q(z) = \sum_{j=1}^{N} \sum_{t=0}^{M_j} z^{\alpha_j} \beta_{j\ell}(\log z)^t, \alpha_j, \beta_{j\ell} \in \mathbb{C}.$$

Now using (3.6) and (3.7) we have

$$z^{-2k} \bar{\nu}(T) \sum_{m=0}^{\infty} a_m e^{-2\pi im/\lambda z} = \sum_{m=0}^{\infty} a_m e^{2\pi imz/\lambda} + q(z).$$

Letting $z = \frac{iy\lambda}{2\pi}, y > c, c \in \mathbb{R}^+$, we have

$$\left(\frac{2\pi}{iy\lambda}\right)^{2k} \bar{\nu}(T) \sum_{m=0}^{\infty} a_m e^{-\frac{4\pi^2 m}{y^2}} = \sum_{m=0}^{\infty} a_m e^{-my} + q\left(\frac{iy\lambda}{2\pi}\right).$$ \hspace{1cm} (3.9)

In proving the converse, it is enough to show that the identity in (3.2) implies (3.8). Since $q(z)$ is entire in $\mathcal{H}$ and the series expansion for $F$ in (3.6) is uniformly convergent in compact subsets of $\mathcal{H}$, the automorphic relation in (3.7) will follow by analytic for $\forall z \in \mathcal{H}$. Following the method used in Chandraseharan and Narasimhan [2], we multiply the identity in (3.2) through out by $y^{\rho+1} e^{-xy}$ with $y > 0$, integrate relative to $x$ from 0 to $\infty$. To this end consider four integrals (separate) corresponding to the terms occurring in (3.2) as

$$\varpi_1(y, \lambda) = \int_0^{\infty} \left\{ \frac{1}{\Gamma(\rho + 1)} \sum_{0 \leq m \leq x} a_m (x - m)^\rho \right\} y^{\rho+1} e^{-xy} dx.$$ \hspace{1cm} (3.10)

$$\varpi_2(y, \lambda) = \int_0^{\infty} \left\{ e^{-\pi k \bar{\nu}(T) \left(\frac{2\pi}{\lambda}\right)^{-\rho}} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{2k-\rho} \frac{J_{2k+\rho}}{\Gamma(2k + \rho + 1)} \right\} y^{\rho+1} e^{-xy} dx.$$ \hspace{1cm} (3.11)

$$\varpi_3(y, \lambda) = \int_0^{\infty} \left\{ e^{\pi k} \left(\frac{2\pi}{\lambda}\right)^{2k} \frac{a_0 \nu(T) x^{2k+\rho}}{\Gamma(2k + \rho + 1)} - \frac{a_0 x^{\rho}}{\Gamma(\rho + 1)} \right\} y^{\rho+1} e^{-xy} dx.$$ \hspace{1cm} (3.12)

$$\varpi_4(y, \lambda) = \int_0^{\infty} \left\{ \left(\frac{2\pi}{\lambda}\right)^{2k+\alpha_j} \frac{x^{2k+\rho + \alpha_j}}{\Gamma(2k + \rho + \alpha_j + 1)} \nu(T) e^{\pi k} \sum_{j=1}^{M_j} \beta_{j\ell} \left(\frac{i\pi}{2}\right)^t \right\} y^{\rho+1} e^{-xy} dx.$$ \hspace{1cm} (3.13)
The evaluation of \( \varpi_1 \) and \( \varpi_2 \) follows as Chandrasekharan and Narasimhan found in [2]. Interchange of summation and integration being permitted for \( \rho \geq 2\beta - 2k - \frac{1}{2} \). Then

\[
\varpi_1(y, \lambda) = \frac{y^{\rho} + 1}{\Gamma(\rho + 1)} \int_0^\infty x^\rho e^{-xy} dx + \frac{y^{\rho} + 1}{\Gamma(\rho + 1)} \sum_{1 \leq m \leq x} a_m \int_m^\infty (x - \rho)^\rho e^{-xy} dx.
\]

Integration by substitution and using the standard integral representation for \( \Gamma(s) \) we get

\[
\varpi_1(y, \lambda) = \sum_{m=0}^\infty a_m e^{-xy}.
\]

To compute \( \varpi_2 \) we apply the integral formula

\[
\int_0^\infty e^{-xy} J_\nu(a \sqrt{x}) x^{\frac{\nu}{2}} dx = \frac{2^{\nu} a^\nu}{(2y)^{\frac{\nu}{2} + 1}} e^{-\frac{a^2}{2y}},
\]

provided that \( y > 0, Re(\nu) > -1, a > 0 \). And with the substitution \( \frac{4xy}{\lambda} = a, \nu = \rho + 2k \) after simplifying, we obtain

\[
\varpi_2(y, \lambda) = \left( \frac{2\pi}{\lambda y} \right)^{2k} e^{-\pi i k} \nu(T) \sum_{m=1}^\infty a_m e^{-\frac{4xy}{\lambda^2} m}.
\]

The integral in \( \varpi_3(y, \lambda) \) is evaluated using the integral representation for \( \Gamma(s) \). Therefore, after simplifying, we see that

\[
\varpi_3(y, \lambda) = \left( \frac{2\pi}{\lambda y} \right)^{2k} a_0 e^{\pi i k} \nu(T) - a_0.
\]

To evaluate \( \varpi_4(y, \lambda) \) since each term is integrable the inter change of the integration and the finite double sum is valid. Then integration by substitution and using the integral representation of \( \Gamma(z) \) we obtain

\[
\varpi_4(y, \lambda) = \left( \frac{2\pi i}{\lambda y} \right)^{2k} \nu(T) \sum_{j=1}^N \left( \frac{2\pi i}{\lambda y} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_j (\frac{i\pi}{2})^t
\]

\[
= \left( \frac{-1}{i\lambda y/2\pi} \right)^{2k} \nu(T) \sum_{j=1}^N \left( \frac{-1}{\lambda y i/2\pi} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_j (\frac{i\pi}{2})^t.
\]

For \( y > 0 \), we see that \( \log \left( \frac{1}{iy} \right) + \log y = \frac{1}{iy} \). Then for \( l \in \mathbb{N} \cup \{0\} \) and applying the binomial theorem we have

\[
\left( \log \left( \frac{-1}{iy} \right) + \log y \right)^t = \sum_{l=0}^t \binom{t}{l} \left( \log \left( \frac{-1}{iy} \right) \right)^{t-l} (\log y)^l.
\]

If \( l = 0 \) we have \( \left( \log \left( \frac{1}{iy} \right) \right)^t = \left( \log \left( \frac{1}{iy} \right) + \log y \right)^t \). Now replacing \( \frac{1}{iy} \) by \( \frac{1}{iy \lambda / 2\pi} \) and substituting for \( \left( \frac{1}{iy} \right)^t \) in (3.12) yields

\[
\varpi_4(y, \lambda) = \left( \frac{-1}{i\lambda y/2\pi} \right)^{2k} \nu(T) \sum_{j=1}^N \left( \frac{-1}{\lambda y i/2\pi} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_j \left( \log \left( \frac{-1}{iy \lambda / 2\pi} \right) \right)^t.
\]

Observe that \( \varpi_4(y, \lambda) = \left( \frac{-1}{i\lambda y / 2\pi} \right)^{2k} \nu(T) q \left( \frac{-1}{i\lambda y / 2\pi} \right) \), where \( q(z) \) is the log-polynomial period function in (3.7). Then applying Lemma 2.1 we have \( \varpi_4(x, \lambda) = \left( \frac{-1}{i\lambda y / 2\pi} \right)^{2k} \nu(T) q \left( \frac{-1}{i\lambda y / 2\pi} \right) = -q \left( \frac{iy \lambda}{2\pi} \right) \). Combining the
results in (3.9), (3.10), (3.11) and (3.12) we showed that the identity in (3.2) implies
\[
\sum_{m=0}^{\infty} a_m e^{-my} = \left(\frac{2\pi}{\lambda y}\right)^2 \sum_{m=1}^{\infty} a_m e^{-\frac{4\pi m^2}{\lambda y}} + \left(\frac{2\pi}{\lambda y}\right)^2 a_0 e^{\pi ik} \nu(T) + \nu(T) \sum_{j=1}^{N} \left(\frac{-1}{\lambda y/2\pi}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} \left(\log\left(\frac{-1}{iy/\lambda y/2\pi}\right)\right)^{t}. 
\] (3.14)

Observe that
\[
\left(\frac{-1}{i\lambda y/2\pi}\right)^{2k} \nu(T) \sum_{j=1}^{N} \left(\frac{-1}{\lambda y/2\pi}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} \left(\log\left(\frac{-1}{iy/\lambda y/2\pi}\right)\right)^{t} = -q\left(\frac{i\lambda y}{2\pi}\right).
\]

Since \(F(z)\) is holomorphic and has a Fourier series expansion \(F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi imz/\lambda}\) in \(\mathcal{H}\). Then with \(z = \frac{i\lambda y}{2\pi}\) written as (3.14), by the identity theorem, then the automorphic relation (2.3) follows \(\forall z \in \mathcal{H}\). This completes the proof of the theorem.

**Theorem 3.2.** Let \(\Phi_F(s), F(z)\) and \(q(z)\) be as in Theorem 2.1. Then the functional equation
\[
\Phi_F(2k-s) = e^{\pi ik} \nu(T) \Phi_F(s)
\] (3.1)
is equivalent to the arithmetical identity
\[
\left(\frac{-1}{y} \frac{d}{dy}\right)^{\rho} \left(\frac{1}{y} \sum_{m=1}^{\infty} a_m e^{-y\sqrt{m}}\right) = \frac{2^\rho}{e^{\sqrt{\pi}}} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{8\pi}{\lambda}\right)^{2k} \nu(T) e^{-\pi ik} \times
\sum_{m=0}^{\infty} \frac{a_m}{y^{2 + \left(\frac{2\pi}{\lambda}\right)^2 m}}^{2k+\rho+\frac{1}{2}} + \frac{2^\rho}{e^{\sqrt{\pi}}} \Gamma(2k + \rho + \alpha_j + \frac{1}{2}) \left(\frac{8\pi}{\lambda y^2}\right)^{2k} \nu(T) e^{\pi ik} \times
\sum_{j=1}^{N} \left(\frac{8\pi i}{\lambda y}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} \left(\frac{i\pi}{2}\right)^{t} - \frac{2^\rho+1}{e^{\sqrt{\pi}}} \frac{a_0}{y^{2\rho+1}} \Gamma\left(\rho + \frac{1}{2}\right),
\] (3.2)
provided that \(\Re y \in \mathbb{R}^+, \alpha_j, \beta_{j,t} \in \mathbb{C}, \rho \in \mathbb{Z}_{\geq 0}\), and \(\rho > \beta - 2k + \frac{1}{2}\),
where \(\sum_{m=1}^{\infty} |a_m| < \infty\).

Since, by Theorem 3.1 the functional equation (3.1) is equivalent to (3.2), it would be enough to show that the functional equation (3.1) implies (3.2) and that (3.2) in turn implies the identity (3.2).

**Proof of implication:** First we begin by showing that (3.1) implies (3.2). Let \(\rho\) be an integer, \(\rho \geq 0\) and \(\varphi(s) = \sum_{m=1}^{\infty} a_m m^{-\varphi}\) with \(a_m = \mathcal{O}(m^{\beta})\). Then for \(\gamma > 0, \gamma \geq \beta\) and applying the version of Perron’s in Lemma 2.2 we have
\[
\frac{1}{\Gamma(\rho+1)} \sum_{0 \leq m \leq x} a_m (x-m)^\rho = \frac{1}{2\pi i} \int_{(\gamma)} \frac{\varphi(s) \Gamma(s) x^{\rho+\rho}}{\Gamma(s+\rho+1)} ds.
\] (3.3)
As Chandrasekharan and Narasimhan in [2] we multiply (3.3) by $e^{-y\sqrt{x} \frac{1}{2}}$ and integrating with respect to the variable $x$ on $[0, \infty)$. For $\gamma > 2k$, $\text{Res} = \gamma$ is the vertical path of integration. Thus, we need to prove the identity (3.2) follows from the equation

$$
\int_0^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} \left\{ \frac{1}{\Gamma(\rho + 1)} \sum_{0 \leq m \leq x} \left( a_m(x - m)^\rho \right) \right\} dx
$$

$$
= \int_0^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} \left\{ \frac{1}{2\pi i} \int_\gamma \frac{\varphi(s) \Gamma(s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds \right\} dx,
$$

(3.4)

where $\rho \in \mathbb{Z}_{\geq 0}, \rho \geq \gamma - 2k + \frac{1}{2}$, and $y \in \mathbb{R}^+$. Chandrasekharan and Narasimhan [2] for $\lambda_n \in \mathbb{R}^+$, $\lambda_n \to \infty$ as $n \to \infty$ showed the identity

$$
\sum_{n=1}^\infty a_n \int_{\lambda_n}^\infty \frac{(x-n)^\rho}{\Gamma(\rho + 1)} e^{-y\sqrt{x} x^{-\frac{1}{2}}} dx = 2(-2)^\rho \left( \frac{1}{y} \int_0^\infty \right) \left[ \frac{1}{y} \sum_{n=1}^\infty a_n e^{-y\sqrt{x} n} \right].
$$

(3.5)

Then by the identity in (3.5) with $\lambda_n = m$, we have

$$
\sum_{m=1}^\infty a_m \int_m^\infty \frac{(x-m)^\rho}{\Gamma(\rho + 1)} e^{-y\sqrt{x} x^{-\frac{1}{2}}} dx = 2(-2)^\rho \left( \frac{1}{y} \int_0^\infty \right) \left[ \frac{1}{y} \sum_{m=1}^\infty a_m e^{-y\sqrt{m} m} \right].
$$

(3.6)

Now on the left side of (3.6), the interchange of integration and summation is justified for $\rho \geq 0$. Then

$$
\int_0^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} \left\{ \frac{1}{\Gamma(\rho + 1)} \sum_{0 \leq m \leq x} \left( a_m(x - m)^\rho \right) \right\} dx = \frac{a_0}{\Gamma(\rho + 1)} \int_0^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} dx
$$

$$
+ \frac{1}{\Gamma(\rho + 1)} \times
$$

$$
\sum_{1 \leq m \leq x} a_m \int_m^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} (x - m)^\rho dx.
$$

Thus applying (3.6) we have

$$
\int_0^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} \left\{ \frac{1}{\Gamma(\rho + 1)} \sum_{0 \leq m \leq x} \left( a_m(x - m)^\rho \right) \right\} dx = \frac{2^{\rho+1}}{\sqrt{\pi} y^{2\rho+1} a_0} \frac{\Gamma(\rho + \frac{1}{2})}{\Gamma(\rho + 1)}
$$

$$
+ 2^{\rho+1} \left( \frac{-1}{\rho} \int_0^\infty \right) \left[ \frac{1}{y} \sum_{m=1}^\infty a_m e^{-y\sqrt{m} m} \right].
$$

(3.7)

Now assuming $\Phi_F(s) = \left( \frac{2\pi}{\lambda} \right)^{-s} \Gamma(s) \varphi(s)$, where $\text{Res} > \beta$ the right-hand side of (3.4) may be written as

$$
W(y, \lambda) = \int_0^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} \left\{ \frac{1}{2\pi i} \int_\gamma \left( \frac{2\pi}{\lambda} \right)^s \frac{\Phi_F(s) x^{s+\rho}}{\Gamma(s+\rho+1)} \right\} ds \right\} dx,
$$

for $\gamma \geq \beta$. We may interchange the order of integration for $\rho \geq 0$. Then we have

$$
W(y, \lambda) = \frac{1}{2\pi i} \int_{\gamma} \left\{ \left( \frac{2\pi}{\lambda} \right)^s \frac{\Phi_F(s)}{\Gamma(s+\rho+1)} \int_0^\infty e^{-y\sqrt{x} x^{-\frac{1}{2}}} ds \right\} ds.
$$

Using integration by substitution and applying the Legendre duplication formula

$$
\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma \left( z + \frac{1}{2} \right),
$$

(3.7)
The series on (3.9) converges absolutely for \( \Re y \) and substitute in to the integral of \( w \) and using the integral formula

\[
\Phi_F(s) = \frac{\Gamma(s + \rho + \frac{1}{2})}{\sqrt{\pi}} e^{2\pi i \frac{\rho - 1}{2}} \sum_{m=1}^{\infty} \frac{a_m}{m^{s+1}}.
\]

To evaluate this integral we shall consider a rectangular region with vertices \( 2k - \gamma \pm iT \) and \( \gamma \pm iT, s = \sigma + it \) on its interior. Using Cauchy’s residue theorem, the line of integration oriented positively to \( \text{Res} = 2k - \gamma \). If \( \rho \geq \gamma - 2k + \frac{1}{2} \) the poles of \( \Gamma(s + \rho + \frac{1}{2}) \) lie to the left of the vertical path. Then the poles of the integrand in the interval \([2k - \gamma, \gamma]\) arise from the function \( \Phi_F(s) \). Applying Stirling’s formula and the Phragmen-Lindelof theorem the integrals along both horizontal paths tend to 0 as \( T \to \infty \), and thus \( W(y, \lambda) \) may be written as

\[
W(y, \lambda) = w_1(y, \lambda) + w_2(y, \lambda),
\]

where

\[
w_1(y, \lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{2\pi}{\lambda} \Phi_F(s) \frac{\Gamma(s + \rho + \frac{1}{2})}{y^{2\rho+2\rho+1}} ds,
\]

and

\[
w_2(y, \lambda) = \sum_{s \in \rho(s) \setminus \mathbb{R}} \text{Res} \left\{ \left( \frac{2\pi}{\lambda} \right)^s \Phi_F(s) \frac{\Gamma(s + \rho + \frac{1}{2})}{y^{2\rho+2\rho+1}} \right\}.
\]

To evaluate \( w_1(y, \lambda) \) recall the functional equation

\[
\Phi_F(2k - s) = e^{\pi i k \nu(T)} \Phi_F(s)
\]

and substitute in to the integral of \( w_1(y, \lambda) \). Then we have

\[
w_1(y, \lambda) = \tilde{\nu}(T) e^{-\pi i k} \frac{2^{2\rho+1}}{y^{2\rho+1}} \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{2\pi}{\lambda} \Phi_F(2k - s) \frac{\Gamma(s + \rho + \frac{1}{2})}{y^{2\rho+1}} ds \right\}.
\]

Now replace \( \theta \) by \( 2k - s \) and then setting \( \Phi_F(\theta) = \left( \frac{2\pi}{\lambda} \right)^{-\theta} \Gamma(\theta) \sum_{m=1}^{\infty} a_m m^{-\theta} \). By organizing the expressions \( w_1(y, \lambda) \) may be written as

\[
w_1(y, \lambda) = \tilde{\nu}(T) e^{-\pi i k} \frac{2^{2\rho+1}}{y^{2\rho+1}} \sum_{m=1}^{\infty} a_m \left\{ \frac{1}{2\pi i} \int_{\gamma} \Gamma(\theta) \Gamma(2k - \theta + \rho + \frac{1}{2}) \left( \frac{16m\pi^2}{\lambda^2 y^2} \right)^{-\theta} d\theta \right\}.
\]

The interchange of the integration with the summation is justified for \( \rho > \beta - 2k + \frac{1}{2} \). Replace \( \theta \) by \(-\theta\) and using the integral formula

\[
\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(-s) \Gamma(\beta + s) \frac{s^t}{ds} ds = \Gamma(\beta)(1 + t)^{-\beta},
\]

where \( 0 > \gamma > Re(1 - \beta) \) and \( |arg t| < \pi \). After simplifying, we obtain

\[
w_1(y, \lambda) = \tilde{\nu}(T) e^{-\pi i k} \frac{2^{2k+2\rho+1}}{\sqrt{\pi}} \left( 2k + \rho + \frac{1}{2} \right) \left( \frac{2\pi}{\lambda} \right)^{2k} \sum_{m=1}^{\infty} a_m \left( \frac{y^2 + \left( \frac{4\pi}{\lambda} \right)^2 m^2}{y^{2k+2\rho+1}} \right)^{2k+\rho+\frac{1}{2}}.
\]

The series on (3.9) converges absolutely for \( Re \in \mathbb{R}^+ \), provided \( \rho > \beta - 2k + \frac{1}{2} \). To proceed evaluation of \( w_2(y, \lambda) \) recall that in Theorem 2.1 we have that \( \Phi_F(s) = D_k(s) + M_k(s) + L_k(s) \), where \( D_k(s), M_k(s) \) and \( L_k(s) \) are functions in (2.3), (2.5), and (2.6) respectively. For \( \rho \geq \gamma - 2k + \frac{1}{2} \) the function \( \left( \frac{2\pi}{\lambda} \right)^s \frac{\Gamma(s + \rho + \frac{1}{2})}{y^{2\rho+2\rho+1}} \) is analytic in the region bounded by \( \text{Res} = 2k - \gamma \) and \( \text{Res} = \gamma \), thus the poles of the integrand in the interval \([2k - \gamma, \gamma]\) arise from \( \Phi_F(s) \). We also note that \( D_k(s) \) is entire, \( M_k(s) \) has
simple poles at (s = 2k, 0) and \( L_k(s) \) has a pole at \( s = 2k + \alpha_j \) of order \( M_j + 1 \), where \( \alpha_j \in \mathbb{C}, M_j \in \mathbb{Z}_{\geq 0} \). Therefore, by computing the residues of \( M_k \) and \( L_k \) after simplifying, we obtain

\[
w_2(y, \lambda) = \frac{2^{2\rho+1}}{\sqrt{\pi}y^{\rho+1}} a_0 \nu(T) e^{\pi i k} \left( \frac{8\pi}{\lambda y^2} \right)^{2k} \Gamma \left( 2k + \rho + \frac{1}{2} \right) \frac{1}{\sqrt{\pi}y^{\rho+1}} a_0 \Gamma \left( \rho + \frac{1}{2} \right) + \frac{2^{2\rho+1}}{\sqrt{\pi}y^{\rho+1}} \nu(T) e^{\pi i k} \left( \frac{8\pi}{\lambda y^2} \right)^{2k} \Gamma \left( 2k + \rho + \alpha_j + \frac{1}{2} \right) \sum_{j=1}^{N} \epsilon_{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} \left( \frac{i\pi}{2} \right)^t.
\]

(3.10)

There for by rewriting \( W(y, \lambda) \) as combination of \([3.9]\) and \([3.10]\) and substituting the respective expressions in to the right-hand side of \([3.4]\), with simple rearrangement, we obtain the identity \([3.2]\), provided that \( \rho \in \mathbb{Z}_{\geq 0}, \rho \geq \beta - 2k + \frac{1}{2} \) and \( y \in \mathbb{R}^+ \), where \( \sum_{m=1}^{\infty} \frac{|a_m|}{m^s} < \infty \).

**Proof of the converse:** To prove the converse it suffices to show that \([3.2]\) implies \([3.2]\). As technique of Chandrasekharan and Narasimhan in [2] multiply \([3.2]\) by \( e^{\nu \sqrt{T}} \), with \( \nu > 0 \), and \( x > 0 \) and integrate the expression along a vertical path \( \text{Res} = \theta \), where \( \theta > 0 \). The left hand side of the transformed equation can be evaluated using the formula

\[
\sum_{m=1}^{\infty} a_m \frac{1}{2\pi i} \int_{(s)} e^{y \sqrt{T}} \left( -\frac{1}{2} \right) dy \left( \frac{1}{y^2} \right) dy = \frac{1}{\Gamma(\rho + 1)} \sum_{m \leq x} \left( x - m \right)^{\rho + 2 - \rho},
\]

(3.11)

while the right hand side of \([3.2]\) we calculate the integral of each terms one by one. So put

\[
h_1(x, \lambda) = \frac{1}{2\pi i} \int_{(s)} e^{y \sqrt{T}} \left( \frac{2^\rho}{\sqrt{\pi}} \nu(T) i^{-2k+2k \sqrt{T}} \Gamma \left( 2k + \rho + \frac{1}{2} \right) \left( \frac{2\pi}{\lambda} \right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \right) \Gamma \left( \rho + \frac{1}{2} \right) \nu(T) e^{\pi i k} \left( \frac{8\pi}{\lambda y^2} \right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \left( \frac{1}{y^2} \right) dy,
\]

\[
h_2(x, \lambda) = \frac{1}{2\pi i} \int_{(s)} e^{y \sqrt{T}} \left( \frac{2^\rho}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \right) \Gamma \left( \rho + \frac{1}{2} \right) \nu(T) e^{\pi i k} \left( \frac{8\pi}{\lambda y^2} \right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \left( \frac{1}{y^2} \right) dy,
\]

\[
h_3(x, \lambda) = \frac{1}{2\pi i} \int_{(s)} e^{y \sqrt{T}} \left( \frac{2^\rho}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \right) \Gamma \left( \rho + \frac{1}{2} \right) \nu(T) e^{\pi i k} \left( \frac{8\pi}{\lambda y^2} \right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \left( \frac{1}{y^2} \right) dy,
\]

and

\[
h_4(x, \lambda) = \frac{1}{2\pi i} \int_{(s)} e^{y \sqrt{T}} \left( \frac{2^\rho}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \right) \Gamma \left( \rho + \frac{1}{2} \right) \nu(T) e^{\pi i k} \left( \frac{8\pi}{\lambda y^2} \right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{y^{2(\rho+1)}} \frac{2^{2k}}{\sqrt{\pi}} \frac{y^{2k}}{\lambda^{2k}} \left( \frac{1}{y^2} \right) dy.
\]

The evaluation of \( h_1(y, \lambda) \) follows as in [2]. The inter change of integration and summation being justified for \( \rho > \beta - 2k + \frac{1}{2} \) in the right-hand side of \( h_1(x, \lambda) \). Applying the integral formula (see [2], page 22)

\[
\frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} e^{|s|} |s|^{-\nu} \frac{ds}{(a^2 + s^2)^{\nu + \frac{1}{2}}} = J_\nu(ab) \left( \frac{b}{a} \right)^{\nu} \Gamma(\nu + \frac{1}{2}),
\]

(3.12)

where \( w > 0, \nu > -\frac{1}{2}, a > 0, b > 0 \). After simplifying each expressions, we obtain

\[
h_1(x, \lambda) = \nu(T) e^{-\pi i k} \left( \frac{\lambda}{2\pi} \right)^{\rho} 2^{-\rho} \sum_{m=1}^{\infty} \frac{a_m}{2^{k+1}} J_{2\rho+1} \left( \frac{4\pi \sqrt{m\lambda} x}{\lambda} \right).
\]

(3.13)

To evaluate \( h_2(x, \lambda) \) we rewrite as

\[
h_2(x, \lambda) = \frac{2^\rho}{\sqrt{\pi}} a_0 \nu(T) e^{-\pi i k} \left( \frac{8\pi}{\lambda} \right)^{2k} \Gamma \left( 2k + \rho + \frac{1}{2} \right) \frac{1}{2\pi i} \int_{(s)} e^{y \sqrt{T}} \left( \frac{1}{y^2} \right) dy.
\]

Put

\[
g_1(y) = \frac{1}{2\pi i} \int_{(s)} e^{y \sqrt{T}} \left( \frac{1}{y^2} \right) dy.
\]
To compute \( g_1(y) \) we use the integral representation of reciprocal gamma function Hankel’s formula in \([15, page 246]\) given by

\[
\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{t-z} dt = \frac{1}{\Gamma(z)}, \quad \text{where } \Re z > 0, \sigma > 0. \tag{3.14}
\]

and applying the formula in \([3.7]\) for \( \frac{1}{\Gamma(4k+2\rho+1)} \). After simplification we obtain

\[
h_2(x, \lambda) = 2^{\rho-\alpha_0} \nu(T) e^{\pi i k} \left( \frac{2\pi}{\lambda} \right)^2 \frac{x^{2k+\rho}}{\Gamma(2k+\rho+1)}. \tag{3.15}
\]

We evaluate \( h_3(x, \lambda) \) applying the techniques analogous used in evaluating \( h_2(y, \lambda) \). Thus, we conclude that

\[
h_3(x, \lambda) = \frac{2a_0}{2^\rho} \frac{x^\rho}{\Gamma(\rho+1)}. \tag{3.16}
\]

Next we consider \( h_4(x, \lambda) \) the interchange of the integration and the finite double sums is justified and we write as;

\[
h_4(x, \lambda) = \frac{2^\rho}{\sqrt{\pi}} \nu(T) e^{\pi i k} \left( \frac{8\pi}{\lambda} \right)^{2k} \Gamma(2k+\rho+1) \sum_{j=1}^{N} \left( \frac{8\pi}{\lambda} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left( \frac{i\pi}{2} \right)^t \times \frac{1}{2\pi i} \int_{\theta} e^{y\sqrt{\pi}} \frac{x^{2k+\rho}}{y^{4k+2\rho+\alpha_j+1}} dy. \tag{3.17}
\]

Now put

\[
g_2(y) = \frac{1}{2\pi i} \int_{\theta} \frac{e^{y\sqrt{\pi}}}{y^{4k+2\rho+\alpha_j+1}} dy.
\]

Applying \([3.14]\) in \( g_2(y) \) and using formula \([3.7]\) for \( \frac{1}{\Gamma(4k+2\rho+1\alpha_j+1)} \). By substituting the expression derived for \( g_2(y) \) and simplifying, we obtained

\[
h_4(x, \lambda) = 2^{\rho} \nu(T) e^{\pi i k} \left( \frac{2\pi}{\lambda} \right)^2 \frac{x^{2k+\rho}}{\Gamma(2k+\rho+1)} \sum_{j=1}^{N} \left( \frac{2\pi i x}{\lambda} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left( \frac{i\pi}{2} \right)^t. \tag{3.18}
\]

Therefore, we have

\[
\frac{1}{\Gamma(\rho+1)} \sum_{m \leq x} a_m(x-m)^\rho 2^{-\rho} = h_1 + h_2 + h_3 + h_4. \tag{3.18}
\]

By substituting the expressions \([3.13], [3.15], [3.16], \) and \([3.17]\) in to \( h_1, h_2, h_3, \) and \( h_4 \) respectively and multiplying both sides of the equation \([3.18]\) by \( 2^\rho \) the identity in \([3.2]\) holds for \( \rho \geq 2\beta - 2k + \frac{1}{2} \). This completes the proof of the converse, since \([3.2]\) is equivalent to \([3.1]\) as we shown in Theorem \(3.1\).

4. Conclusion and Discussion

In this paper we have used techniques of Chandrasekharan and Narasimhan and extended the results of Sister Ann M. Heath to entire automorphic integrals involving rational period functions on discrete Hecke group. Moreover, we have established equivalence of two arithmetical identities with a functional equation associated with automorphic integrals involving log-polynomial-period functions on the Hecke groups. As noted in the introduction part of this paper Sister Ann M. Heath considered the functional equation in the Hawkins and Knopp context and showed its equivalence to two arithmetical identities associated with entire modular cusp integrals involving rational period functions for the full modular group.
References

[1] Bochner, S., Connection between functional equations and modular relations, and functions of exponential type, J. Indian Math. Soc., 16 (1952), 99-102.

[2] Chandrasekharan, K. and Narasimhan, R., Hecke’s functional equation and arithmetical identities, The Annals of Mathematics, second series, Vol. 74, No. 1, 1961, 1-23.

[3] Hardy, G. H., The General Theory of Dirichlet’s Series, Cambridge Tracts in Math. and Math. Physics, no. 18, Cambridge, London, 1915.

[4] Hassen, Abdulkadir, Log-polynomial period function for Hecke groups, Ramanujan Journal 3, 119-151, (1999).

[5] Hawkins, J and Knopp, M., A Hecke correspondence theorem for automorphic integrals with rational period functions, Illinois Journal of Mathematics, 36, no. 2 (1992), 178-207.

[6] Hekce, E., Dirichlet series, modular functions, and quadratic forms, Edwards Brothers, Inc., Ann Arbor, MI 1938.

[7] Knopp, M. I., Rational period functions of the modular group, Duke Mathematical Journal, 45, No. 1, March 1978, 47-62.

[8] Knopp M. I., Rational period functions of the modular group II, Glasgow Math. Jan. 22 (1981), 185-197.

[9] Knopp, M. I., Some new results on the Eichler cohomology of automorphic forms, Bull. Ann. Math, Soc. 80(4), 607-631(174).

[10] Pasles, P. C., A Hecke Correspondence Theorem for Non analytic Automorphic Integrals, Journal of Number theory 83, 256-281 (2000).

[11] Pasles, P. C., Nonanalytic automorphic integrals on the Hecke groups, Acta Arith. (1999), 155-171.

[12] Roseler, W. A Hecke Correspondence Theorem for Automorphic Integrals with symmetric Rational period Functions on Hecke Groups, Franklin and Marshall college, Lancaster, PA 17604.

[13] Sister Ann M. Heath, Dirichlet series with functional equations and arithmetical identities, Springer Science +Business Media New York 2015.

[14] Titchmarsh, E. C., ” Theory of Functions” 2nd ed., Clarendon Oxford, 1939.

[15] Whittaker, E. T., and Watson, G.N., A Course of Modern Analysis, 4th Ed., Cambridge, London, 1950.

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