DIAGONALIZATION OF INDEFINITE SADDLE POINT FORMS

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To the memory of Boris Sergeevich Pavlov

ABSTRACT. We obtain sufficient conditions that ensure block diagonalization (by a direct rotation) of sign-indefinite symmetric sesquilinear forms as well as the associated operators that are semi-bounded neither from below nor from above. In the semi-bounded case, we refine the obtained results and, as an example, revisit the block Stokes Operator from fluid dynamics.

1. INTRODUCTION

Diagonalizing a quadratic form, which is a classic problem of linear algebra and operator theory, is closely related to the search for invariant subspaces for the (bounded) operator associated with the form. In the Hilbert space setting, a particular case where an invariant subspace can be represented as the graph of a bounded operator acting from a given subspace of the Hilbert space to its orthogonal complement, is of special interest. This situation is quite common while studying block operator matrices, where an orthogonal decomposition of the Hilbert space is available by default. In particular, solving the corresponding invariant graph-subspace problem for bounded self-adjoint block operator matrices automatically yields a block diagonalization of the matrix by a unitary transformation. It is important to note that solving the problem is completely nontrivial even in the bounded case: a self-adjoint operator matrix may have no invariant graph subspace (with respect to a given orthogonal decomposition) and, therefore, may not be block diagonalized in this sense, see, e.g., [23, Lemma 4.2].

To describe the block diagonalization procedure in the self-adjoint bounded case in more detail, assume that the Hilbert space \( \mathcal{H} \) splits into a direct sum of its subspaces, \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \), and suppose that \( B \) is a \( 2 \times 2 \) self-adjoint block operator matrix with respect to this decomposition. In the framework of off-diagonal perturbation theory, we also assume that \( B = A + V \), with \( A \) and \( V \) the diagonal and off-diagonal parts of \( B \), respectively.

We briefly recall that the search for an invariant subspace of \( B \) that can be represented as the graph of a bounded (angular) operator \( X \) acting from \( \mathcal{H}_+ \) to \( \mathcal{H}_- \) is known to be equivalent to finding the skew-self-adjoint “roots”

\[
Y := \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}
\]

of the (algebraic) Riccati equation (see, e.g., [3, 30])

\[
AY - YA - YVV^* + V = 0.
\]

Given such a solution \( Y \), one observes that the Riccati equation can be rewritten as the following operator equalities

\[
(A + V)(I_{\mathcal{H}_+} + Y) = (I_{\mathcal{H}_-} + Y)(A + VY) \quad \text{and} \quad (I_{\mathcal{H}_-} - Y)(A + V) = (A - YV)(I_{\mathcal{H}_-} - Y),
\]

with \( A + VY = (A - YV)^* \) block diagonal operators.

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In turn, those operator equalities ensure a block diagonalization of $B$ by the similarity transformation $I \pm Y$, and, as the next step, by the direct rotation $U$ from the subspace $\mathcal{H}_+$ to the invariant graph subspace $\mathcal{G}_+ = \text{Graph}(\mathcal{H}_+, X) := \{x + Xx \mid x \in \mathcal{H}_+\}$ (see [7, 8] for the concept of a direct rotation). Apparently, the direct rotation is given by the unitary operator from the polar decomposition $(I + Y) = U|I + Y|.

Solving the Riccati equation, the main step of the diagonalization procedure described above, attracted a lot of attention from several groups of researchers.

Different ideas and methods have been used to solve the Riccati equation under various assumptions on the (unbounded) operator $B$. For an extensive list of references we refer to [3] and [40] (for matrix polynomial and Riccati equations in finite dimension see [9, 13, 14, 15, 16, 28, 34]). For more recent results, in particular on Dirac operators with Coulomb potential, dichotomous Hamiltonians, and bisectorial operators, we refer to [6, 41, 43, 44].

The most comprehensive results regarding the solvability of the Riccati equation can be obtained under the hypothesis that the spectra of the diagonal part of the operator $B$ restricted to its reducing subspaces $\mathcal{H}_\pm$ are subordinated. For instance, in the presence of a gap separating the spectrum, the Davis-Kahan $\tan 2\Theta$-Theorem [8] can be used to ensure the existence of contractive solutions to the corresponding Riccati equation. In this case, efficient norm bounds for the angular operator can be obtained. The case where there is no spectral gap but the spectra of the diagonal entries have only one-point intersection $\lambda$ has also been treated, see, e.g., [2, 24], [37, Theorem 2.13], [40, Proposition 2.7.13]. Also, see the recent work [30], where, in particular, the decisive role of establishing the kernel splitting property

$$\text{Ker}(B - \lambda) = (\text{Ker}(B - \lambda) \cap \mathcal{H}_+) \oplus (\text{Ker}(B - \lambda) \cap \mathcal{H}_-)$$

in the diagonalization process is discussed, cf. [40, Section 2.7].

In the present paper, we extend the diagonalization scheme to the case of indefinite saddle point forms. Recall that a symmetric saddle point form $b$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a form sum $b = a + v$, where the diagonal part of the form $a$ splits into the difference of two non-negative closed forms in the spaces $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively, and the off-diagonal part $v$ is a symmetric form-bounded perturbation of $a$.

First, we treat the case where the domain of the form $\text{Dom}[b]$ and the form domain $\text{Dom}(|B|^{1/2})$ of the associated operator $B$ defined via the First Representation Theorem for saddle point forms coincide. Putting it differently, we assume that the corresponding Kato square root problem has an affirmative answer. In this case, we show that on the one hand the semi-definite subspaces

$$L_\pm = \text{Ran} \left( E_B(\mathbb{R}_\pm \setminus \{0\}) \right) \oplus (\text{Ker}(B) \cap \mathcal{H}_\pm)$$

reduce both the operator $B$ and the form $b$. On the other hand, the semi-positive subspace $L_+$ is a graph subspace with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, see Theorem 3.3. Under some additional regularity assumptions, we block diagonalize both the form and the associated operator by the direct rotation from the subspace $\mathcal{H}_+$ to the subspace $L_+$.

More generally, we introduce the concept of a block form Riccati equation associated with a given saddle point form and relate its solvability to the existence of graph subspaces that reduce the form. Based on these considerations, we block diagonalize the form by a unitary transformation, provided that some regularity requirements are met as well, see Theorem 6.5.

As an application, we revisit the spectral theory for the Dirichlet Stokes block operator (that describes stationary motion of a viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$) (see [20], cf. [10]),

$$\begin{pmatrix} -\nu \Delta & v^* \text{grad} \\ -v^* \text{div} & 0 \end{pmatrix}$$
in the direct sum of Hilbert spaces \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = L^2(\Omega)^d \oplus L^2(\Omega) \).

The paper is organized as follows.

In Section 2, we introduce the class of saddle point forms and recall the corresponding Representation Theorems for the associated operators.

In Section 3, we discuss reducing subspaces for saddle point forms that are the graph of a bounded operator.

In Section 4, we recall the concept of a direct rotation and define the class of regular graph decompositions.

In Section 5, we block diagonalize the associated operator by a unitary transformation provided that the domain stability condition holds and that the graph decomposition \( \mathcal{H} = \mathcal{L}_+ \oplus \mathcal{L}_- \) into the sum of semi-definite subspaces \( \mathcal{L}_\pm \) given by (1.1) is regular, see Theorem 5.1.

In Section 6, we introduce the concept of a block form Riccati equation and provide sufficient conditions for the block diagonalizability of a saddle point form by a unitary transformation, see Theorem 6.5.

In Section 7, we discuss semi-bounded saddle point forms and illustrate our approach on an example from fluid dynamics.

We adopt the following notation. In the Hilbert space \( \mathcal{H} \) we use the scalar product \( \langle \cdot, \cdot \rangle \) semi-linear the first and linear in the second component. Various auxiliary quadratic forms will be denoted by \( t \). We write \( t[x] \) instead of \( t[x, x] \). \( I_\mathcal{R} \) denotes the identity operator on a Hilbert space \( \mathcal{R} \), where we frequently omit the subscript. If \( t \) is a quadratic form and \( S \) is a bounded operator we define the sum \( t + S \) as the form sum \( \langle \cdot, S \cdot \rangle \) on the natural domain \( \text{Dom}[t] \).

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2. SADDLE POINT FORMS

To introduce the concept of a saddle point form in a Hilbert space \( \mathcal{H} \), we pick up a self-adjoint involution \( J \) given by the operator matrix \([13]\).

\[
J = \begin{pmatrix} I_{\mathcal{H}_+} & 0 \\ 0 & -I_{\mathcal{H}_-} \end{pmatrix}_{\mathcal{H}_+ \oplus \mathcal{H}_-}
\]

with respect to a given decomposition of the Hilbert space \( \mathcal{H} \) into the orthogonal sum of its closed subspaces

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-.
\]

A sesquilinear form \( a \) is called diagonal (with respect to the decomposition \([2.2]\)) if the domain \( \text{Dom}[a] \) is \( J \)-invariant and the form \( a \) “commutes” with the involution \( J \),

\[
a[x, Jy] = a[Jx, y] \quad \text{for} \quad x, y \in \text{Dom}[a],
\]
and the form
\[ \mathbf{a}_J[x, y] = \mathbf{a}[x, Jy] \] on \( \text{Dom}[\mathbf{a}_J] = \text{Dom}[\mathbf{a}] \)
is a closed non-negative form. In particular, the form \( \mathbf{a} \) splits into the difference of closed non-negative forms \( \mathbf{a} = \mathbf{a}_+ \oplus (-\mathbf{a}_-) \) with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \).

Correspondingly, a sesquilinear form \( v \) is called off-diagonal if the “anti-commutation relation”
\[ v[x, Jy] = -v[Jx, y] \] for \( x, y \in \text{Dom}[v] \) holds.

We say that a form \( b \) is a saddle point form with respect to the decomposition \( (2.2) \) if it admits the representation
\[ b[x, y] = \mathbf{a}[x, y] + v[x, y], \quad x, y \in \text{Dom}[b] = \text{Dom}[\mathbf{a}], \]
where \( \mathbf{a} \) is a diagonal form, \( v \) is a symmetric off-diagonal form and relatively bounded with respect to \( \mathbf{a}_J \),
\[ |v[x]| \leq \beta \left( \mathbf{a}_J[x] + \|x\|^2 \right), \quad x \in \text{Dom}[v], \]
for some \( \beta \geq 0 \).

We start by recalling the First and Second Representation Theorem adapted here to the case of saddle point forms (see [37, Theorem 2.7], [18], [19], see also [32]).

**Theorem 2.1** (The First Representation Theorem). Let \( b \) be a saddle point form with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \).

Then there exists a unique self-adjoint operator \( B \) such that
\[ \text{Dom}(B) \subseteq \text{Dom}[b] \]
and
\[ b[x, y] = \langle x, By \rangle \quad \text{for all} \quad x \in \text{Dom}[b] \quad \text{and} \quad y \in \text{Dom}(B). \]

We say that the operator \( B \) associated with the saddle point form \( b \) via Theorem 2.1 satisfies the domain stability condition if the Kato square root problem has an affirmative answer. That is,
\[ \text{Dom}[b] = \text{Dom}(|B|^{1/2}). \]

We note that the domain stability condition may fail to hold for form-bounded but not necessarily off-diagonal perturbations of a diagonal form, see [18, Example 2.11] and [12] for counterexamples.

The corresponding Second Representation Theorem can be stated as follows.

**Theorem 2.2** (The Second Representation Theorem). Let \( b \) be a saddle point form with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) and \( B \) the associated operator.

If the domain stability condition \( \text{(2.4)} \) holds, then the operator \( B \) represents this form in the sense that
\[ b[x, y] = \langle |B|^{1/2}x, \text{sign}(B)|B|^{1/2}y \rangle \quad \text{for all} \quad x, y \in \text{Dom}[b] = \text{Dom}(|B|^{1/2}). \]

**Remark 2.3.** Let \( \mathbf{a} \) be a diagonal form and \( A = JA_J \), where \( A_J \) is a self-adjoint operator associated with the closed non-negative form \( \mathbf{a}_J \) in \( (2.3) \). Clearly, the operator \( A \) is associated with the form \( \mathbf{a} \) and the form \( \mathbf{a} \) is represented by \( A \) as well. Notice that \( \mathbf{a}_J \) is associated in the standard sense with the self-adjoint operator \( |A| \).

Next, we present an example of a saddle point form “generated” by an operator.
Example 2.4. Given the decomposition (2.2), suppose that $A_\pm \geq 0$ are self-adjoint operators in $\mathcal{H}_\pm$. Also suppose that
\[ W : \text{Dom}(W) \subseteq \mathcal{H}_+ \to \mathcal{H}_- \]
is a densely defined closable linear operator such that
\[ \text{Dom}(A_+^{1/2}) \subseteq \text{Dom}(W). \]

Let $a$ be the diagonal saddle point form associated with the diagonal operator
\[ A = \begin{pmatrix} A_+ & 0 \\ 0 & -A_- \end{pmatrix}. \]

On $\text{Dom}[b] = \text{Dom}[a] = \text{Dom}(|A|^{1/2})$ consider the form sum
\[ b = a + v, \]
where the off-diagonal symmetric perturbation is given by
\[ v[x, y] = \langle Wx_+, y_- \rangle + \langle x_-, Wy_+ \rangle, \]
\[ x = x_+ \oplus x_- \quad \text{and} \quad y = y_+ \oplus y_-, \]
x, y $\in \text{Dom}(|A_\pm|^{1/2}) \subseteq \mathcal{H}_\pm$.

Lemma 2.5. The form $b$ defined by (2.6) in Example 2.4 is a saddle point form. Moreover, the off-diagonal part $v$ of the form $b$ is infinitesimally form-bounded with respect to the non-negative closed form $a_J$ given by
\[ a_J[x, y] = \langle |A|^{1/2}x, |A|^{1/2}y \rangle \]
on $\text{Dom}[a_J] = \text{Dom}(|A|^{1/2})$.

Proof. From (2.5) it follows that the operator $W$ is $A_+^{1/2}$-bounded (see [21, Remark IV.1.5]) and therefore
\[ \|Wx_+\| \leq a\|x_+\| + b\|A_+^{1/2}x_+\|, \quad x_+ \in \text{Dom}(A_+^{1/2}), \]
for some constants $a$ and $b$. This shows the off-diagonal part $v$ of the form $b$ is relatively bounded with respect to the diagonal form $a_J$ and hence $b$ is a saddle point form.

The last assertion follows from the series of inequalities
\[ |v[x]| \leq 2\|Wx_+, x_-\| \leq 2\|Wx_+\| \cdot \|x_-\| \leq 2(a\|x_+\| + b\|A_+^{1/2}x_+\|) \cdot \|x_-\| \]
\[ = 2a\|x_+\| \cdot \|x_-\| + 2b\sqrt{a_+}[x_+]|x_-| \]
\[ \leq (a^2 + 1)\|x\|^2 + cb^2a_+ [x_+] + \frac{\|x\|^2}{\varepsilon}, \]
x, y $\in \text{Dom}[a] \cap \mathcal{H}_\pm$
valid for all $\varepsilon > 0$. \hfill \square

Remark 2.6. The operator $B$ associated with the saddle point form $b$ from Example 2.4 can be considered a self-adjoint realization of the “ill-defined” Hermitian operator matrix
\[ \hat{B} = \begin{pmatrix} A_+ & W^* \\ W & -A_- \end{pmatrix}. \]

Note that in this case we do not impose any condition on $\text{Dom}(A_-) \cap \text{Dom}(W^*)$, so that the “initial” operator $\hat{B}$ is not necessarily densely defined on its natural domain $\text{Dom}(\hat{B}) = \text{Dom}(A_+) \oplus (\text{Dom}(A_-) \cap \text{Dom}(W^*))$. In particular, we neither require that $\text{Dom}(A_-) \supseteq \text{Dom}(W^*)$, cf. [4], nor that $\hat{B}$ is essentially self-adjoint, cf. [40, Theorem 2.8.1].
We close this section by the observation that semi-bounded saddle point forms are automatically closed is the standard sense.

Recall that a linear set $D \subseteq H$ is called a core for the semi-bounded from below form $b \geq cI_H$ if $D \subseteq \text{Dom}[b]$ is dense in $\text{Dom}[b]$ with respect to the norm $||f||_b = \sqrt{b[f] + (1-c)||f||^2}$, see, e.g., [35, Section VIII.6].

**Lemma 2.7.** Suppose that $b$ is a semi-bounded saddle point form with respect to the decomposition $H = H_+ \oplus H_-$. Then $b$ is closed in the standard sense. In particular, the domain stability condition (2.4) automatically holds.

Moreover, if $D$ is a core for the diagonal part $a$ of the form $b$, then $D$ is also a core for $b$.

**Proof.** Assume for definiteness that $b$ is semibounded from below. Let $a$ and $\nu$ be the diagonal and off-diagonal parts of the form $b$, respectively.

Since the off-diagonal part $\nu$ is relatively bounded with respect to $a_J$, that is,

$$|\nu[x]| \leq \beta(a_+ + a_- + I)[x] = \beta(\langle |A| + I \rangle^{1/2}x, (|A| + I)^{1/2}x), \quad x \in \text{Dom}[a],$$

for some $\beta < \infty$, applying [21, Lemma VI.3.1] shows that $\nu$ admits the representation

$$\nu[x, y] = \langle (|A| + I)^{1/2}x, R(|A| + I)^{1/2}y \rangle, \quad x, y \in \text{Dom}[a],$$

with a bounded operator $R$.

Since $\nu$ is off-diagonal, the operator $R$ is off-diagonal as well, so that

$$JR = -JR.$$

Introducing the form

$$\tilde{b}[x, y] = b[x, y] + \langle x, Jy \rangle, \quad x, y \in \text{Dom}[a],$$

one observes that

$$\tilde{b}[x, y] = \langle (|A| + I)^{1/2}x, (J + R)(|A| + I)^{1/2}y \rangle, \quad x, y \in \text{Dom}[a].$$

Here we used that

$$a[x, y] + \langle x, Jy \rangle = \langle |A|^{1/2}J|A|^{1/2}y \rangle + \langle x, Jy \rangle = \langle (|A| + I)^{1/2}x, J(|A| + I)^{1/2}y \rangle$$

for $x, y \in \text{Dom}[a]$.

Since the spectrum of $J$ consists of no more than two points $\pm 1$ and the operator $R$ is off-diagonal, the interval $(-1, 1)$ belongs to the resolvent set of the bounded operator $J + R$. In particular, $J + R$ has a bounded inverse, see [24, Remark 2.8]. Since $|A| + I$ is strictly positive, applying the First Representation Theorem [18, Theorem 2.3] shows that the self-adjoint operator $\tilde{B} = (|A| + I)^{1/2}(J + R)(|A| + I)^{1/2}$ is associated with the semi-bounded form $b$ and is semi-bounded as well. Taking into account the one-to-one correspondence between closed semi-bounded forms and semi-bounded self-adjoint operators proves that the form $b$ is closed, so is $b$ as a bounded perturbation of a closed form.

To show that any core for the diagonal part $a$ is also a core for $b$, we remark first that since $b$ is semi-bounded from below, the diagonal part $a$ of the form $b$ is semi-bounded from below as well. Indeed, otherwise, the form $a_-$ is not bounded and therefore there is a sequence $x_n \in \text{Dom}[a_-], ||x_n|| = 1$, such that $a_-[x_n] \to \infty$. In this case,

$$b[0 \oplus x_n] = -a_-[x_n] \to -\infty,$$

which contradicts the assumption that $b$ is a semi-bounded from below form.

Now, since $b$ is closed, by [21, Theorem VI.2.23], the domain stability condition (2.4) holds. This means that the Sobolev (Hilbert) spaces $H^1_A$ and $H^1_B$ associated with the operators $A$ and $B$ coincide. Hence $D$ is dense in $H^1_A$ if and only if it is dense in $H^1_B$ (w.r.t. the natural topology on the form domain). In other words, $D$ is a core for the form $b$ whenever it is a core for the form $a$. 

□
3. Reducing Subspaces

Recall that a closed subspace $K$ reduces a self-adjoint operator $T$ if

$$Q T \subseteq T Q,$$

where $Q$ is the orthogonal projection in $\mathcal{H}$ onto $K$ (see [21, Section V.3.9]).

This notion of a reducing subspace $K$ means that $K$ and its orthogonal complement $K^\perp$ are invariant for $T$ and the domain of the operator $T$ splits as

$$\text{Dom}(T) = (\text{Dom}(T) \cap K) \oplus (\text{Dom}(T) \cap K^\perp).$$

Next, we introduce the corresponding notion for sesquilinear forms.

**Definition 3.1.** We say that a closed subspace $K$ of a Hilbert space $\mathcal{H}$ reduces a symmetric densely defined quadratic form $t$ with domain $\text{Dom}[t] \subseteq \mathcal{H}$ if

1. $Q(\text{Dom}[t]) \subseteq \text{Dom}[t]$ and
2. $t[Q u, v] = t[u, Q v]$ for all $u, v \in \text{Dom}[t],$

where $Q$ is the orthogonal projection onto $K$.

A short computation shows that a closed subspace $K$ reduces a symmetric densely defined quadratic form $t$ if and only if

$$(3.1) \quad Q(\text{Dom}[t]) \subseteq \text{Dom}[t] \quad \text{and} \quad t[Q^\perp u, Q v] = 0 \quad \text{for all} \quad u, v \in \text{Dom}[t].$$

In particular, $K$ reduces the form $b$ if and only if the orthogonal complement $K^\perp$ does.

Taking this into account, along with saying that a closed subspace $K$ reduces a form, we also occasionally say that the orthogonal decomposition $\mathcal{H} = K \oplus K^\perp$ reduces the form.

The following lemma shows that under the domain stability condition, the concepts of reducibility for the form and the associated (representing) self-adjoint operator coincide.

**Lemma 3.2.** Assume that $b$ is a saddle point form with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $B$ the associated operator. Suppose that the domain stability condition (2.4) holds.

Then a closed subspace $K$ reduces the form $b$ if and only if $K$ reduces the operator $B$.

**Proof.** Assume that $K$ reduces the form $b$. Denote by $Q$ the orthogonal projector onto $K$. In this case,

$$\text{Dom}[b] = \text{Dom}[a] = \text{Dom}(|B|^{1/2})$$

and

$$Q \left( \text{Dom}(|B|^{1/2}) \right) \subseteq \text{Dom}(|B|^{1/2}).$$

Moreover,

$$b[Q x, y] = b[x, Q y] \quad \text{for all} \quad x, y \in \text{Dom}[b].$$

Since the form $b$ is represented by $B$, we have

$$\langle |B|^{1/2} Q x, \text{sign}(B)|B|^{1/2} y \rangle = \langle |B|^{1/2} x, \text{sign}(B)|B|^{1/2} Q y \rangle \quad \text{for all} \quad x, y \in \text{Dom}(|B|^{1/2}).$$

In particular,

$$\langle Q x, B y \rangle = \langle B x, Q y \rangle \quad \text{for all} \quad x, y \in \text{Dom}(B).$$

Since $B$ is self-adjoint, this means that $Q y \in \text{Dom}(B)$ and that

$$Q By = B Q y \quad \text{for all} \quad x \in \text{Dom}(B),$$

which shows that $K$ reduces the self-adjoint operator $B$.

To prove the converse, suppose that $K$ reduces the operator $B$. By [42, Satz 8.23], the decomposition also reduces both operators $|B|^{1/2}$ and $\text{sign}(B)$. Together with

$$\text{Dom}[b] = \text{Dom}[a] = \text{Dom}(|B|^{1/2})$$
this means that
\[ Q(\text{Dom}[b]) \subseteq \text{Dom}[b] \]
and that \( Q \) commutes with \( \text{sign}(B) \) and \( |B|^{1/2} \). Thus,
\[ b[Qu, v] = \langle |B|^{1/2}Qu, \text{sign}(B)|B|^{1/2}v \rangle = \langle |B|^{1/2}u, \text{sign}(B)|B|^{1/2}v \rangle = b[u, Qv], \]
which shows that \( \mathcal{R} \) reduces the form \( b \). \( \square \)

The theorem below generalizes a series of results of \([1, 2, 24, 37]\), cf. \([40\text{, Section 2.7}]\), and provides a canonical example of a semi-definite reducing subspace for a saddle-point form.

**Theorem 3.3.** Let \( b \) be a saddle point form with respect to the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) and \( B \) the associated operator. Assume that the form \( b \) satisfies the domain stability condition (2.4).

Then the subspace \( \text{Ker}(B) \cap \mathcal{H}_+ \) reduces both the form \( b \) and the operator \( B \). In particular, the kernel of \( B \) splits as
\[ \text{Ker}(B) = (\text{Ker}(B) \cap \mathcal{H}_+) \oplus (\text{Ker}(B) \cap \mathcal{H}_-), \]
the semi-definite subspaces
\[ \mathcal{L}_\pm = (\text{Ran} \ E_B((\mathbb{R}_\pm) \setminus \{0\})) \oplus (\text{Ker}(B) \cap \mathcal{H}_\pm) \]
are complimentary, and the orthogonal decomposition
\[ \mathcal{H} = \mathcal{L}_+ \oplus \mathcal{L}_- \]
reduces both the form \( b \) and the associated operator \( B \).

Moreover, the subspace \( \mathcal{L}_+ \) is a graph of a linear contraction \( X : \mathcal{H}_+ \to \mathcal{H}_- \).

**Proof.** Assume temporarily that \( \text{Ker}(B) = \{0\} \). Then the orthogonal decomposition \( \mathcal{H} = \mathcal{L}_+ \oplus \mathcal{L}_- \) is spectral. Therefore \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) reduce the operator \( B \) and, by the domain stability condition and Lemma 3.2, the form \( b \) as well.

To complete the proof under the assumption \( \text{Ker}(B) = \{0\} \), we check that \( \mathcal{L}_\pm \) are graph subspaces. Denote by \( P \) the orthogonal projection onto \( \mathcal{H}_+ \) and let \( Q = E_B(\mathbb{R}_+) \) be the spectral projection of \( B \) onto its positive subspace. Introducing the sequence of self-adjoint operators
\[ B_n = B + \frac{1}{n} J, \quad J = \begin{pmatrix} I_{\mathcal{H}_+} & 0 \\ 0 & -I_{\mathcal{H}_-} \end{pmatrix}, \quad n \in \mathbb{N}, \]
one observes that
\[ \lim_{n \to \infty} B_n \varphi = B \varphi, \quad \varphi \in \text{Dom}(B). \]

By \([35\text{, Theorem VIII.25}]\), the sequence of operators \( B_n \) converges to \( B \) in the strong resolvent sense, and therefore, by \([35\text{, Theorem VIII.24}]\),
\[ \text{s-lim}_{n \to \infty} E_{B_n}(\mathbb{R}_+) = E_B(\mathbb{R}_+), \]
since 0 is not an eigenvalue of \( B \).

Taking into account that the operator \( B_n \) is associated with the form \( b_n \) given by
\[ b_n[x, y] := b[x, y] + \frac{1}{n} \langle x, Jy \rangle \]
and that the interval \((-1/n, 1/n)\) belongs to its resolvent set, one applies the Tan 2Θ-Theorem \([19\text{, Theorem 3.1}]\) to conclude that
\[ \|Q - E_{B_n}(\mathbb{R}_+)\| < \frac{\sqrt{2}}{2} \]
(3.3)

Since (3.2) holds, one also gets the weak limit
\[ \text{w-lim}_{n \to \infty} (Q - E_{B_n}(\mathbb{R}_+)) = Q - E_B(\mathbb{R}_+). \]
(3.4)
Using the principle of uniform boundedness, see [21, Equation (3.2), Chapter III], one obtains from (3.3) and (3.4) the bound

\[ \|Q - E_B(\mathbb{R}^+)\| \leq \liminf_{n \to \infty} \|Q - E_{B_n}(\mathbb{R}^+)\| \leq \frac{\sqrt{2}}{2}. \]

Hence, \( L_+ \) is the graph subspace \( \text{Graph}(\mathcal{H}_+, X) \) with \( X \) a contraction, see [22, Corollary 3.4]. The orthogonal complement \( L_- \) is then the graph subspace \( \text{Graph}(\mathcal{H}_-, -X^*) \).

We now treat the general case (of a non-trivial kernel).

First, we check that the semi-positive subspaces \( L_{\pm} \) reduce the operator \( B \), and thus also the form \( b \).

It is clear that both \( L_\pm \) are invariant for \( B \). It is also clear that the subspaces \( L_{\pm} \) are complementary if and only if the kernel splits as

\[ \text{Ker}(B) = (\text{Ker}(B) \cap \mathcal{H}_+) \oplus (\text{Ker}(B) \cap \mathcal{H}_-). \]  

To prove (3.5), recall (see [37, Theorem 2.13]) that the kernel of \( B \) can be represented as

\[ \text{Ker}(B) = (\text{Ker}(A_+) \cap K_+) \oplus (\text{Ker}(A_-) \cap K_-), \]

where \( A_\pm \) are self-adjoint non-negative operators associated with the forms \( a_\pm \) and the subspaces \( K_+ \) and \( K_- \) are given by

\[ K_\pm := \{ x_\pm \in \text{Dom}[a_\pm] \mid b[x_+, x_-] = 0 \text{ for all } x_\pm \in \text{Dom}[a_\pm] \} \subseteq \mathcal{H}_\pm. \]

Hence \( \text{Ker}(B) \cap \mathcal{H}_\pm = \text{Ker}(A_\pm) \cap K_\pm \) and (3.5) follows.

Next, in view of (3.5), since \( \mathcal{H} \) naturally splits as

\[ \mathcal{H} = \text{Ran} E_B(\mathbb{R}^+) \oplus \text{Ker}(B) \oplus \text{Ran} E_B(\mathbb{R}^-), \]

one gets

\[
\text{Dom}(B) = (\text{Dom}(B) \cap \text{Ran} E_B(\mathbb{R}^+)) \oplus (\text{Ker}(B) \cap \mathcal{H}_+)
\oplus (\text{Dom}(B) \cap \text{Ker}(B) \cap \mathcal{H}_-)
\oplus (\text{Dom}(B) \cap \text{Ran} E_B(\mathbb{R}^-)).
\]

This representation shows that the domain \( \text{Dom}(B) \) splits as

\[ \text{Dom}(B) = (\text{Dom}(B) \cap L_+) \oplus (\text{Dom}(B) \cap L_-). \]  

Summing up, we have shown that \( L_{\pm} \) are \( B \)-invariant mutually orthogonal subspaces such that (3.7) holds. That is, the subspaces \( L_{\pm} \) reduce the operator \( B \) and therefore the form \( b \).

To complete the proof, we now need to check that \( L_+ \) (and thus also \( L_- \)) is a graph subspace with a contractive angular operator.

By [22, Corollary 3.4], it is sufficient to show that

\[ \|Q - P\| \leq \frac{\sqrt{2}}{2}, \]

where \( Q \) and \( P \) are the orthogonal projection onto \( \mathcal{H}_+ \) and \( L_+, \) respectively.

We will prove (3.8) by reducing the problem to the one where the kernel is trivial.

First we show that \( \text{Ker}(B) \) reduces the operator \( A \). Indeed, by (3.6) we have \( \text{Ker}(B) \subseteq \text{Ker}(A) \), so that \( \text{Ker}(B) \) is invariant for \( A \). Hence, \( \text{Ker}(B)^\perp \) is invariant for \( A \) as well. It remains to check that \( \text{Dom}(A) \) splits as

\[ \text{Dom}(A) = (\text{Dom}(A) \cap \text{Ker}(B)) \oplus (\text{Dom}(A) \cap \text{Ker}(B)^\perp). \]

Indeed, since \( \text{Ker}(B) \) reduces \( B \), by [42, Satz 8.23], the subspace \( \text{Ker}(B) \) also reduces \( |B|^{1/2} \).

By the required domain stability condition, this implies that \( \text{Ker}(B) \) reduces \( |A|^{1/2} \) and, by [42, Satz 8.23] again, also \( |A| \). Thus (3.9) holds by observing that \( \text{Dom}(A) = \text{Dom}(|A|) \).

We now complete the proof that \( L_+ = \text{Graph}(\mathcal{H}_+, X) \) is a graph subspace for a contraction \( X \).
Taking into account that the subspace \( \tilde{H} := \text{Ker}(B) \) reduces both \( A \) and \( B \), denote by \( \tilde{A} := A|_{\tilde{H}} \) and \( \tilde{B} := (B)|_{\tilde{H}} \) the corresponding parts of \( A \) and \( B \), respectively. In particular, \( \tilde{A} \) and \( \tilde{B} \) are self-adjoint operators and \( \text{Ker}(\tilde{B}) = \{0\} \).

In view of the kernel splitting (3.6), a simple reasoning shows that \( \tilde{H} \) splits as \( \tilde{H} = \tilde{H}_+ \oplus \tilde{H}_- \) with \( \tilde{H}_+ := H_+ \cap \tilde{H} \) and \( \tilde{H}_- := H_- \cap \tilde{H} \), and that the operator \( \tilde{A} \) is represented as the diagonal block matrix

\[
\tilde{A} = \begin{pmatrix}
\tilde{A}_+ & 0 \\
0 & -\tilde{A}_-
\end{pmatrix}
\]

with

\[
\sup \text{spec}(-\tilde{A}_-) \leq 0 \leq \inf \text{spec}(\tilde{A}_+).
\]

In this case the corresponding sesquilinear symmetric form \( \tilde{a} \) also splits into the difference of two non-negative forms. The restriction \( \tilde{b} = b|_{\tilde{H}} \) is clearly seen to be a saddle point form associated with the self-adjoint operator \( \tilde{B} \). Since \( \text{Ker}(\tilde{B}) = \{0\} \), by the above reasoning, we get the inequality

\[
\|\tilde{Q} - E_{\tilde{B}}(\mathbb{R}_+)\| \leq \frac{\sqrt{2}}{2},
\]

where \( \tilde{Q} \) is the orthogonal projection onto \( \tilde{H}_+ \) and \( E_{\tilde{B}}(\mathbb{R}_+) \) is the spectral projection of \( \tilde{B} \) for the positive part. In particular, as in the previous case, \( \text{Ran} E_{\tilde{B}}(\mathbb{R}_+) = \text{Graph}(\tilde{H}_+, \tilde{X}) \) is the graph of a linear contraction \( \tilde{X} : \tilde{H}_+ \to \tilde{H}_- \).

Denoting by \( X \) the extension of the operator \( \tilde{X} \) by zero on \( \text{Ker}(B) \cap H_+ \) and taking into account that by (3.6)

\[
A|_{\text{Ker}(B) \cap H_+} = B|_{\text{Ker}(B) \cap H_+} = 0,
\]

we obviously get that \( L_+ = \text{Graph}(H_+, X) \). Observing that the extended operator \( X \) is also a contraction completes the proof. \( \square \)

4. Regular embeddings and Direct Rotations

Given the orthogonal decomposition

\[
H = H_+ \oplus H_-,
\]

suppose that Hilbert spaces \( \tilde{H}_\pm \) are continuously embedded in \( H_\pm \),

(4.1)

\[
\tilde{H}_\pm \hookrightarrow H_\pm,
\]

so that their direct sum \( \tilde{H} = \tilde{H}_+ \oplus \tilde{H}_- \) is also continuously embedded in \( H = H_+ \oplus H_- \).

Suppose that a subspace \( G_+ \) can be represented as a graph of a bounded operator \( X \) from \( H_+ \) to \( H_- \) and let

(4.2)

\[
H = G_+ \oplus G_-
\]

be the corresponding decomposition with \( G_- = (G_+)^\perp \), the graph of the bounded operator \( -X^* : H_- \to H_+ \).

**Definition 4.1.** We say that the graph decomposition (4.2) is \( \tilde{H} \)-regular (with respect to the embedding) if the linear sets

\[
G_\pm = G_\pm \cap \tilde{H}
\]

naturally embedded in \( \tilde{H} \) are closed complimentary graph subspaces in the Hilbert space \( \tilde{H} \) with respect to the decomposition \( \tilde{H} = \tilde{H}_+ \oplus \tilde{H}_- \).
Denote by $P$ and $Q$ the orthogonal projections onto the subspaces $H_+$ and $G_+$, respectively.

Recall that as long as it is known that $G_+$ is a graph subspace, there exists a unique unitary operator $U$ on $H$ that maps $H_+$ to $G_+$, such that

$$ U P = Q U, $$

the diagonal entries of which (in its block matrix representation with respect to the decomposition $H = H_+ \oplus H_-$) are non-negative operators, see [8]. In this case the operator $U$ is called the direct rotation from the subspace $H_+ = \text{Ran}(P)$ to the subspace $G_+ = \text{Ran}(Q)$.

**Lemma 4.2.** Suppose that the graph decomposition (4.2) is $H$-regular with respect to the embedding (4.1). Let $U$ and $U$ be the direct rotation from $H_+$ to $G_+$ in the space $H$ and from $H_+$ to $G_+$ in $H$, respectively. Then

$$ \dot{U} = U|_{H_+}. $$

**Proof.** Since the graph decomposition (4.2) is $H$-regular, it follows that $G_+ \cap H_+$ is the graph of a bounded operator $\tilde{X}: H_+ \to H_-$, Therefore, $G_+ \cap H_+$ is the graph of $-\tilde{X}^*$. Clearly,

$$ \tilde{X} = X|_{H_+} \quad \text{and} \quad (-\tilde{X}^*) = (-X^*)|_{H_-}. $$

In particular,

$$ X^* X|_{H_+} = \tilde{X}^* \tilde{X} \quad \text{and} \quad XX^*|_{H_-} = \tilde{X} \tilde{X}^*. $$

A classic Neumann series argument shows that

$$ (t I + X^* X)^{-1}|_{H_+} = (t I + X^* X)^{-1} $$

for $|t|$ is large enough. Taking into account the continuity of the embedding, one extends (4.3) for all $t > 0$ by analytic continuation. Next, using the formula for the fractional power (see, e.g., [21] Ch. V, eq. (3.53))

$$ T^{-1/2} = \frac{1}{\pi} \int_0^\infty t^{-1/2}(T + t I)^{-1} dt $$

valid for any positive self-adjoint operator $T$ and taking $T = (I + X^* X)|_{H_+}$ first and then $T = I + \tilde{X}^* \tilde{X}$ in the Hilbert spaces $H_+$ and $H$, respectively, from (4.3) one deduces that

$$ (I + X^* X)^{-1/2}|_{H_+} = (I + X^* X)^{-1/2} $$

(4.4)

Analogously,

$$ (I + XX^*)^{-1/2}|_{H_-} = (I + \tilde{X} \tilde{X}^*)^{-1/2}. $$

(4.5)

Since the direct rotation $U$ admits the representation

$$ U = \begin{pmatrix} (I + X^* X)^{-1/2} & -X^*(I + XX^*)^{-1/2} \\ X(I + X^* X)^{-1/2} & (I + XX^*)^{-1/2} \end{pmatrix}, $$

cf. [3] 8 [22], see also [39] Proof of Proposition 3.3], and analogously

$$ \dot{U} = \begin{pmatrix} (I + \tilde{X}^* \tilde{X})^{-1/2} & -\tilde{X}^*(I + \tilde{X} \tilde{X}^*)^{-1/2} \\ X(I + \tilde{X} \tilde{X}^*)^{-1/2} & (I + \tilde{X} \tilde{X}^*)^{-1/2} \end{pmatrix}, $$

the assertion follows from (4.4) and (4.5). □

**Remark 4.3.** If $G_+$ is the graph of a bounded operator $X$ from $H_+$ to $H_-$, introduce

$$ Y = \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}_{H_+ \oplus H_-}. $$

Then the direct rotation $U$ is just the unitary operator from the polar decomposition of the operator $I + Y$,

$$ (I + Y) = U|I + Y|. $$
Observe that the $\hat{\mathcal{H}}$-regularity of the decomposition $\mathcal{H} = \text{Graph}(\mathcal{H}_+, X) \oplus \text{Graph}(\mathcal{H}_-, -X^*)$ can equivalently be reformulated in purely algebraic terms that invoke mapping properties of the operators $I \pm Y$ only. That is, the graph space decomposition \( (5.2) \) is $\hat{\mathcal{H}}$-regular if and only if the operators $I \pm Y$ are algebraic/ topologic automorphisms of $\hat{\mathcal{H}}$, see \cite{[30]} Lemma 3.1, Remark 3.2.

5. Block-Diagonalization of Associated Operators by a Direct Rotation

One of the main results of the current paper is as follows.

**Theorem 5.1.** Let $b$ be a saddle point form with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $B$ the associated operator. Assume that the form $b$ satisfies the domain stability condition \( (2.4) \).

Suppose that the decomposition

\[
\mathcal{H} = \mathcal{L}_+ \oplus \mathcal{L}_-
\]

referred to in Theorem 3.3 is $\mathcal{H}_A^1$-regular.

Then the form $b$ and the associated operator $B$ can be block diagonalized by the direct rotation $U$ from the subspace $\mathcal{H}_+$ to the reducing graph subspace $\mathcal{L}_+$. That is,

(i) the form

$$\hat{b}[f, g] = b[U f, U g], \quad f, g \in \text{Dom}[\hat{b}] = \text{Dom}[b]$$

is a diagonal saddle point form with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$,

$$\hat{b} = \hat{b}_\pm \oplus (-\hat{b}_-),$$

with $\hat{b}_\pm = \pm \hat{b}|_{\mathcal{H}_\pm}$. In particular, the non-negative the forms $\hat{b}_\pm$ are closed;

(ii) the associated operator $\hat{B}$ can be represented as the diagonal operator matrix,

$$\hat{B} = U^* B U = \begin{pmatrix} \hat{B}_+ & 0 \\ 0 & -\hat{B}_- \end{pmatrix}_{\mathcal{H}_+ \oplus \mathcal{H}_-};$$

(iii) the non-negative closed the forms $\hat{b}_\pm$ are in one-to-one correspondence to the non-negative self-adjoint operators $\hat{B}_\pm$.

If, in addition, the form $b$ is semi-bounded, then the hypotheses that $b$ satisfies the domain stability condition and that the decomposition \( (5.1) \) is regular are redundant.

**Proof.** Note that in the Hilbert space $\mathcal{H}_A^1$ the form $b$ can be represented by a bounded operator $B$, such that

\[
b[x, y] = \langle x, B y \rangle_{\mathcal{H}_A^1}, \quad x, y \in \text{Dom}[b].
\]

Let $\hat{U}$ denote the direct rotation from $\mathcal{H}_+ \cap \mathcal{H}_A^1$ to $\mathcal{L}_+ \cap \mathcal{H}_A^1$ in the Sobolev space $\mathcal{H}_A^1$. By Lemma 4.2 one has $\hat{U} = U|_{\mathcal{H}_A^1}$ and therefore

$$b[U x, U y] = \langle \hat{U} x, B \hat{U} y \rangle_{\mathcal{H}_A^1} = \langle x, (\hat{U})^* B \hat{U} y \rangle_{\mathcal{H}_A^1}.$$  

Since the decomposition \( (5.1) \) reduces $b$, it follows that $(\hat{U})^* B \hat{U}$ is a diagonal operator matrix in the Sobolev space $\mathcal{H}_A^1$ with respect to the decomposition $\mathcal{H}_A^1 = (\mathcal{H}_+ \cap \mathcal{H}_A^1) \oplus (\mathcal{H}_- \cap \mathcal{H}_A^1)$. The corresponding subspaces $\mathcal{L}_\pm$ are non-negative subspaces for the operator $B$, so that

$$\hat{U}^* B \hat{U} = \begin{pmatrix} \hat{B}_+ & 0 \\ 0 & -\hat{B}_- \end{pmatrix}_{(\mathcal{H}_+ \cap \mathcal{H}_A^1) \oplus (\mathcal{H}_- \cap \mathcal{H}_A^1)},$$

where $\hat{B}_\pm$ are non-negative bounded operators in $\mathcal{H}_\pm \cap \mathcal{H}_A^1$. Since

$$b[U x_{\pm}, U y_{\pm}] = \pm \langle x_{\pm}, \hat{B}_\pm y_{\pm} \rangle_{\mathcal{H}_A^1 \cap \mathcal{H}_\pm},$$

the form $b$ is $\hat{H}_\pm$-regular. Therefore, the hypothesis that $b$ is semi-bounded is redundant.
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one observes that $b[Ux \pm, Uy \pm],\ x \pm, y \pm \in \mathcal{H}_+ \cap \mathcal{H}_A^1$ defines a sign-definite closed form on $\mathcal{H}_+ \cap \mathcal{H}_A^1$. This proves (i).

On the other hand,

$$b[Ux, Uy] = \langle x, U^* Bu \rangle_{\mathcal{H}}, \ x \in \text{Dom}[b], \ y \in U^{-1}(\text{Dom}(B)).$$

In particular, one has

$$b[Ux \pm, Uy \pm] = \langle x \pm, \pm \tilde{B} \pm y \pm \rangle_{\mathcal{H}}, \ x \pm \in \text{Dom}[a \pm], y \pm \in U^{-1}(\text{Dom}(B)) \cap \mathcal{H}_ \pm,$$

which shows (ii).

The assertion (iii) now follows from (i) and (5.3).

Next, we prove the last assertion of the theorem. Denote by $Q$ the orthogonal projection onto the subspace $L_+$. Then

$$Q = \begin{pmatrix} (I_{\mathcal{H}_+} + X^X)^{-1} & (I_{\mathcal{H}_+} + X^X)^{-1}X^* X(I_{\mathcal{H}_+} + X^X)^{-1} \\ X(I_{\mathcal{H}_+} + X^X)^{-1} & X(I_{\mathcal{H}_+} + X^X)^{-1}X^* \end{pmatrix}$$

and

$$Q_\perp = \begin{pmatrix} X^*(I_{\mathcal{H}_-} + XX^*)^{-1}X & -X^*(I_{\mathcal{H}_-} + XX^*)^{-1}X \\ -(I_{\mathcal{H}_-} + XX^*)^{-1}X & (I_{\mathcal{H}_-} + XX^*)^{-1}X \end{pmatrix}.$$

Note that since the subspace $L_+ \cap \mathcal{H}_- \cap \text{Dom}[b]$ is bijective on $L_+ \cap \text{Dom}[b]$. Therefore, the operator

$$(I - Y^2)^{-1} = \begin{pmatrix} (I_{\mathcal{H}_+} + X^X)^{-1} & 0 \\ 0 & (I_{\mathcal{H}_-} + XX^*)^{-1} \end{pmatrix}$$

maps $\text{Dom}[b]$ into itself.

Again, since $I + XX^*$ is bijective on $\mathcal{H}_- = \text{Dom}[b] \cap \mathcal{H}_-$ and $Q_\perp$ maps $\text{Dom}[b]$ into itself, it follows from (5.3) that $X$ maps into $\text{Dom}[b] \cap \mathcal{H}_+$ into $\text{Dom}[b] \cap \mathcal{H}_-$ and $X^*$ maps $\text{Dom}[b] \cap \mathcal{H}_-$ into $\text{Dom}[b] \cap \mathcal{H}_+$. Thus, $Y$ leaves the form domain $\text{Dom}[b]$ invariant and so do the operators $I + Y, I - Y$ and $I - Y^2$.

Summing up, both $(I - Y^2)$ and $(I - Y^2)^{-1}$ map $\text{Dom}[b]$ into itself. That is, the restriction of the map

$$I - Y^2 = (I - Y)(I + Y)$$

on $\text{Dom}[b]$ is bijective on $\text{Dom}[b]$. In particular $I + Y$ is bijective on $\text{Dom}[b]$ and by Remark 4.3 the decomposition $\mathcal{H} = \mathcal{L}_+ \oplus \mathcal{L}_-$ is $\mathcal{H}_A^1$-regular, which completes the proof.

\[ \Box \]

6. The Riccati Equation

The existence of a reducing graph subspace for a saddle point form, as, for instance, in Theorem 5.1, is closely related to the solvability of the associated block form Riccati equation.

**Hypothesis 6.1.** Suppose that $b$ is a saddle point form with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Assume that a subspace $\mathcal{G}_+$ is the graph of a bounded operator $X : \mathcal{H}_+ \to \mathcal{H}_-$ and that $Y$ is the skew-symmetric off-diagonal operator $Y$

$$Y = \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}_{\mathcal{H}_+ \oplus \mathcal{H}_-}$$

**Theorem 6.2.** Assume Hypothesis 6.1. Assume, in addition, that the orthogonal decomposition $\mathcal{H} = \mathcal{G}_+ \oplus \mathcal{G}_-$ is $\mathcal{H}_A^1$-regular.
Then the decomposition $\mathcal{H} = \mathcal{G}_+ \oplus \mathcal{G}_-$ reduces the form $b$ if and only if the skew-symmetric off-diagonal operator $Y$ is a solution to the block form Riccati equation

\begin{equation}
(6.1) \quad a[f, Yg] + a[Yf, g] + b[Yf, Yg] + v[f, g] = 0, \quad f, g \in \text{Dom}[a],
\end{equation}

\[\text{Ran}(Y|_{\text{Dom}[a]}) \subseteq \text{Dom}[a].\]

**Proof.** The proof of this theorem is a direct combination of the following two lemmas. \hfill \Box

**Lemma 6.3.** Assume Hypothesis 6.1 and suppose that $\mathcal{H} = \mathcal{G}_+ \oplus \mathcal{G}_-$ reduces $b$.

If the Sobolev space $\mathcal{H}^1_\mathcal{A}$ is $Y$-invariant, then $Y$ is a solution to the block form Riccati equation (6.1).

**Proof.** Assume that the decomposition reduces $b$. Since $\text{Dom}[b] = \text{Dom}[a] = \mathcal{H}^1_\mathcal{A}$ (as a set), the $Y$-invariance of the Sobolev space $\mathcal{H}^1_\mathcal{A}$ implies that $X$ and $X^*$ map $\text{Dom}[a_+] = \text{Dom}(A^{1/2}_\mathcal{A})$ into $\text{Dom}[a_-] = \text{Dom}(A^{1/2}_\mathcal{A})$, and vice versa, respectively. Denote by $Q$ the orthogonal projection onto $\mathcal{G}(\mathcal{H}_+, X)$. By (3.1), we have

\begin{equation}
(6.2) \quad 0 = b[Q^+(-X^*y \oplus y), Q(x \oplus Xx)] = b[-X^*y \oplus y, x \oplus Xx],
\end{equation}

\[x \in \text{Dom}[a_+], \quad y \in \text{Dom}[a_-].\]

Taking into account that $b = a + v$, where $a$ and $v$ are the diagonal and off-diagonal parts, respectively, and that $a = a_+ \oplus (-a_-)$, the equality (6.2) shows that $X$ is a solution to the Riccati equation

\begin{equation}
(6.3) \quad a_+[-X^*y, x] - a_-[y, Xx] + v[-X^*y, Xx] + v[y, x] = 0,
\end{equation}

\[x \in \text{Dom}[a_+], \quad y \in \text{Dom}[a_-].\]

Set

\[f = x_+ \oplus x_-, \quad g = y_+ \oplus y_-, \quad x_\pm, y_\pm \in \text{Dom}[a_\pm],\]

combine the Riccati equation (6.3) with $x = y_+$, $y = x_-$ plugged in, and the complex conjugate of (6.3) with $x = y_+$, $y = x_-$ plugged in, to get

\[a[f, Yg] + a[Yf, g] + b[Yf, Yg] + v[f, g]\]

\[= a \left[ \begin{array}{cc} x_+ & 0 \\ 0 & -X^* \\ \end{array} \right] \left[ \begin{array}{c} y_+ \\ 0 \\ y_- \\ \end{array} \right] + a \left[ \begin{array}{cc} 0 & -X^* \\ X & 0 \\ \end{array} \right] \left[ \begin{array}{c} x_+ \\ 0 \\ y_- \\ \end{array} \right] + v \left[ \begin{array}{cc} 0 & -X^* \\ X & 0 \\ \end{array} \right] \left[ \begin{array}{c} x_+ \\ 0 \\ y_- \\ \end{array} \right]
\]

\[= a_+[-X^*y_-, x_+] - a_-[-X^*y_+, y_-] + a_+[-X^*y_+, y_-] - a_-[-X^*y_+, y_-]
\]

\[+ v[-X^*y_-, Xy_+] + v[Xy_+, -X^*y_+] + v[y_-, y_+] + v[y_-, y_+]
\]

\[= a_+[-X^*y_-, x_+] - a_-[-X^*y_+, y_-] + v[-X^*y_-, Xy_+] + v[y_-, y_+]
\]

\[+ a_+[-X^*y_+, y_-] - a_-[-X^*y_-, y_+] + v[-X^*y_-, Xy_+] + v[y_+, y_-]
\]

\[= 0,
\]

which shows that $Y$ is a solution of the block Riccati equation (6.1). \hfill \Box

**Lemma 6.4.** Assume Hypothesis 6.1. Suppose that $b$ is a saddle point form with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and that $\mathcal{H} = \mathcal{G}_+ \oplus \mathcal{G}_-$.

If $Y$ solves to the block form Riccati equation (6.1) and $\mathcal{H}^1_\mathcal{A} \subseteq \text{Ran}(I - Y)|_{\mathcal{H}^1_\mathcal{A}}$, then the decomposition $\mathcal{H} = \mathcal{G}_+ \oplus \mathcal{G}_-$ reduces the form $b$. \hfill \Box
Proof. Let $Q$ denote the orthogonal projection onto $\mathcal{G}(\mathcal{H}_+, X)$. Recall that $Q$ is given by the block matrix (6.4)

\[
Q = \begin{pmatrix}
(I_{\mathcal{H}_+} + X^*X)^{-1} & (I_{\mathcal{H}_+} + X^*X)^{-1}X^* \\
X(I_{\mathcal{H}_+} + X^*X)^{-1} & X(I_{\mathcal{H}_+} + X^*X)^{-1}X^*
\end{pmatrix}_{\mathcal{H}_+ \oplus \mathcal{H}_-}.
\]

By hypothesis, one has that $(I - Y)\text{Dom}[a] \supseteq \text{Dom}[a]$. Since $Y$ is a solution of the Riccati equation (6.1), then necessarily $(I - Y)\text{Dom}[a] \subseteq \text{Dom}[a]$. Thus, $I - Y$ is bijective on $\text{Dom}[a]$. So is the operator

\[
I - Y^2 = (I - Y)J(I - Y)J = \begin{pmatrix}
(I_{\mathcal{H}_+} + X^*X) & 0 \\
0 & (I_{\mathcal{H}_-} + XX^*)
\end{pmatrix},
\]

where the involution $J$ is given by (2.1).

In particular, the operators $I_{\mathcal{H}_+} + X^*X$ and $I_{\mathcal{H}_-} + XX^*$ are bijective on $\text{Dom}[a_+]$ and $\text{Dom}[a_-]$, respectively. Since $I - Y$ is bijective on $\text{Dom}[a]$, one also observes that $X$ maps $\text{Dom}[a_+]$ into $\text{Dom}[a_-]$ and that $X^*$ maps $\text{Dom}[a_-]$ into $\text{Dom}[a_+]$. Taking into account the explicit representation (6.4), one concludes that the operator $Q$ maps $\text{Dom}[b] = \text{Dom}[a]$ into itself.

Therefore, for any $\tilde{y} \in \text{Dom}[b]$, there exists an $x \in \text{Dom}[a_+]$ such that

\[
Q\tilde{y} = x \oplus Xx.
\]

Similarly, for any $\tilde{x} \in \text{Dom}[b]$ there exists a $y \in \text{Dom}[a_-]$ such that

\[
Q^\perp\tilde{x} = -X^*y \oplus y.
\]

Assuming that $x \in \text{Dom}[a_+]$ and $y \in \text{Dom}[a_-]$, we have

\[
b[Q^\perp\tilde{x}, Q\tilde{y}] = b[-X^*y \oplus y, x \oplus Xx]
\]

\[
= a[-X^*y \oplus y, x \oplus Xx] + v[-X^*y \oplus y, x \oplus Xx]
\]

\[
= a_+[-X^*y, x] - a_-[y, Xx] + v[-X^*y, Xx] + v[y, x]
\]

\[
= a\begin{pmatrix}
x \\
X
\end{pmatrix}, \begin{pmatrix}
0 & X^*
\end{pmatrix} \begin{pmatrix}
y \\
0
\end{pmatrix} + v\begin{pmatrix}
n & X^*
\end{pmatrix} \begin{pmatrix}
x \\
0
\end{pmatrix}
\]

\[
= a[f, Yg] + a[Yf, g] + v[Yf, Yg] + v[f, g]
\]

\[
= 0,
\]

where we have used the block Riccati equation (6.1) for $f = x \oplus 0$ and $g = 0 \oplus y$ on the last step. This implies that

\[
b[Q^\perp\tilde{x}, Q\tilde{y}] = 0 \quad \text{for all} \quad \tilde{x}, \tilde{y} \in \text{Dom}[b],
\]

and therefore, the graph subspace $\mathcal{G}_+ = \mathcal{G}(\mathcal{H}_+, X)$ reduces the form $b$ (see (5.1)).

Now we are ready to present a generalization of Theorem 5.1 (i) that yields the block diagonalization of a saddle point form, provided that the latter has a reducing subspace.

**Theorem 6.5.** Assume Hypothesis 6.7. Suppose that the graph decomposition $\mathcal{H} = \mathcal{G}_+ \oplus \mathcal{G}_-$ reduces the form $b$ and let $U$ be the direct rotation from the subspace $\mathcal{H}_+$ to the reducing subspace $\mathcal{G}_+$. Also assume that the decomposition is $\mathcal{H}_0^\perp$-regular.

Then

\[
\hat{b}[f, g] = b[Uf, Ug], \quad f, g \in \text{Dom}[\hat{b}] = \text{Dom}[b],
\]

is a diagonal form with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. 

\[\square\]
Proof. Due to Theorem 6.2, the Riccati equation (6.1) holds if and only if the decomposition \( H = G_+ \oplus G_- \) reduces the form. Then a straightforward computation shows that

\[
b[(I + Y)f, h] = a[f, (I - Y)h] + y[Yf, (I - Y)h], \quad f, h \in \text{Dom}[a].
\]

Then, taking \( h = (I - Y)^{-1}g \) with \( g \in \text{Dom}[a] \), one obtains that

\[
(6.5) \quad b[(I + Y)f, (I - Y)^{-1}g] = a[f, g] + y[Yf, g], \quad f, g \in \text{Dom}[a].
\]

Since the form \( a \) is diagonal, and both the form \( v \) and the operator \( Y \) are off-diagonal, it follows that the form \( b[f, g] = b[(I + Y)f, (I - Y)^{-1}g] \) on \( \text{Dom}[b] = \text{Dom}[a] \), is a diagonal form. Since \( U = (I + Y)|I + Y|^{-1} = (I - Y)^{-1}|I - Y| \) and \( |I - Y| = |I + Y| \) is a diagonal operator, the equation (6.5) yields

\[
b[Uf, Ug] = a[I + Y]^{-1}f, |I - Y|g] + y[YI + Y]^{-1}f, |I - Y|g], \quad f, g \in \text{Dom}[a],
\]

provided that \( |I + Y| = (I - Y^2)^{1/2} \) is bijective on \( \text{Dom}[a] \). This required bijectivity of \( |I + Y| \) follows along similar lines as in the proof of Lemma 4.2.

It should be noted that the proof of Theorem 6.5 compared to the one of Theorem 5.1(i), neither requires the domain stability condition to hold nor the semi-definiteness of the corresponding reducing graph subspaces \( G_\pm \). If, however, the domain stability condition holds, the proof of Theorem 5.1(i) shows that the diagonalization procedure for the unbounded form \( b \) in \( H \) can be reduced to the one of the corresponding bounded self-adjoint operator \( B \) in the Sobolev space \( H_A \) (see 5.2). The form \( b \) then splits into the sum of two diagonal forms \( \pm \hat{b}_\pm \),

\[
\hat{b} = \hat{b}_+ \oplus (-\hat{b}_-),
\]

that are not necessarily semi-bounded. However, if the saddle point form \( b \) is a priori semi-bounded, the domain stability condition holds automatically and the corresponding diagonal forms \( \pm \hat{b}_\pm \) are semi-bounded and closed. In other words, in this case the statement of Theorem 6.5 can naturally be extended to the format of Theorem 5.1.

7. Some Applications

In this section, having in mind applications of the developed formalism to the study of the block Stokes operator from fluid dynamics, cf. [10, 17, 20], we focus on the class of saddle-point forms provided by Example 2.4 in the semi-bounded situation.

We start by the following compactness result that may be of independent interest.

Lemma 7.1. Let \( b \) be the saddle-point form from Example 2.4 and \( B \) the associated operator. Assume that \( A_+ > 0 \) and that the operator \( A_- \) is bounded and has compact resolvent. Then the positive spectral subspace of the operator \( B \) is a graph subspace,

\[
\text{Ran}(E_B((0, \infty))) = \text{Graph}(H_+, X)
\]

with \( X : H_+ \rightarrow H_- \) a compact contraction.

If, in addition, \( A_+^{-1} \) is in the Schatten-von Neumann ideal \( \mathcal{S}_p \), then \( X \) belongs to \( \mathcal{S}_{2p} \).

Proof. By Theorem 3.3

\[
\text{Ran}(E_B((0, \infty))) \oplus (\text{Ker}(B) \cap H_+) = G(H_+, X),
\]

with \( X \) a contraction.

By [37, Theorem 1.3] we have that

\[
\text{Ker}(B) = (\text{Ker}(A_+) \cap K_+) \oplus (\text{Ker}(A_-) \cap K_-),
\]

where

\[
K_\pm = \{ x_\pm \in H_\pm \mid v[x_\pm] = 0 \text{ for all } x_\pm \in \text{Dom}[a_\pm] \}.
\]
Therefore, if $A_+ > 0$, then $\text{Ker}(B) \cap \mathcal{H}_+ = \{0\}$, which proves that 
\[
\text{Ran}(E_B((0, \infty)) = \text{Graph}(\mathcal{H}_+, X).
\]

Since the reducing subspace $\text{Ran}(E_B((0, \infty))$ is a graph subspace, the form Riccati equation (6.1) holds. Notice that the Riccati equation (6.1) can also be rewritten as the following quadratic equation
\[
(A_+^{-1/2}X^*WA_+^{-1/2})A_+^{1/2}X^* = ((W + A_-X)A_+^{-1/2})^*.
\]

Indeed, since
\[
\begin{align*}
\langle (A_+^{1/2} + A_-^{-1/2}X^*W)X^*y, A_+^{1/2}x \rangle &= \langle ((W + A_-X)A_+^{-1/2})^*y, A_+^{1/2}x \rangle, \\
x &\in \text{Dom}[a_+] \subseteq \mathcal{H}_+, \quad y \in \text{Dom}[a_-] \subseteq \mathcal{H}_-,
\end{align*}
\]
and
\[
\begin{align*}
\langle -X^*y, x \rangle - \langle y, A_-Xx \rangle &= -\langle WX^*y, x \rangle + \langle y, Wx \rangle, \\
x &\in \text{Dom}[a_+], \quad y \in \text{Dom}[a_-] = \mathcal{H}_-,
\end{align*}
\]
equation (7.1) can be rewritten as
\[
\langle (A_+^{1/2} + A_-^{-1/2}X^*W)X^*y, A_+^{1/2}x \rangle = \langle ((W + A_-X)A_+^{-1/2})^*y, A_+^{1/2}x \rangle.
\]

To complete the proof of the lemma, one observes that $A_+ + X^*W$ is similar to $\hat{B}_+$ and since the kernel of $B$ is trivial, the kernel of the operator $A_+ + X^*W$ is trivial as well. Hence, the kernel of the Fredholm operator
\[
F = I + A_+^{-1/2}X^*WA_+^{-1/2}
\]
is also trivial (here we used that the operator $WA_+^{-1/2}$ is bounded and that $A_+^{-1/2}$ is compact). Hence $F$ has a bounded inverse and then, from (7.2), we get that
\[
(7.3) \quad X^* = A_+^{-1/2}[F^{-1}((W + A_-X)A_+^{-1/2})^*].
\]

Since $A_+^{-1/2}$ is compact, it follows that $X^*$ is compact, so is $X$. From this representation it also follows that $A_+^{-1/2}$ and $X$ share the same Schatten class membership.
\[\square\]

**Remark 7.2.** Note that in the situation of Lemma 7.1 in the particular case where the off-diagonal part $W$ of the operator matrix (2.7) is a bounded operator from (7.3) it also follows (see, e.g., [42, Satz 3.23]) that $X$ belongs to the same Schatten-Von Neumann ideal $\mathcal{S}_p$ as $A_+^{-1}$ does, cf. [40, Corollary 2.9.2].

As an illustration consider the following example.
Example 7.3 (The Stokes operator revisited). Assume that $\Omega$ is a bounded $C^2$-domain in $\mathbb{R}^d$, $d \geq 2$. In the direct sum of Hilbert spaces

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where $\mathcal{H}_+ = L^2(\Omega)^d$ is the “velocity space” and $\mathcal{H}_- = L^2(\Omega)$ the “pressure space”, introduce the block Stokes operator $S$ via the symmetric sesquilinear form

$$s[v \oplus p, u \oplus q] = \nu \langle \nabla v, \nabla u \rangle - v^* \langle \text{div} v, q \rangle - v^* \langle p, \text{div} u \rangle,$$

where $\nabla$ denotes the component-wise application of the standard gradient operator defined on the Sobolev space $H^1_0(\Omega)$, with $\nu > 0$ and $v^* \geq 0$ parameters.

It is easy to see that the Stokes operator $S$ defined as the self-adjoint operator associated with the saddle-point form $s$, is the Friedrichs extension of the operator matrix

$$\hat{S} = \begin{pmatrix} -\nu \Delta & v^* \nabla \\ -v^* \text{div} & 0 \end{pmatrix}_{\mathcal{H}_+ \oplus \mathcal{H}_-},$$

defined on

$$\text{Dom}(\hat{S}) = (H^2(\Omega) \cap H^1_0(\Omega))^d \oplus H^1_0(\Omega).$$

Here $\Delta = \Delta \cdot I_d$ is the vector-valued Dirichlet Laplacian, with $I_d$ the identity operator in $\mathbb{C}^d$, $\text{div}$ is the maximal divergence operator from $\mathcal{H}_+$ to $\mathcal{H}_-$ on $\text{Dom}(\text{div}) = \{ v \in L^2(\Omega)^d \mid \text{div} v \in L^2(\Omega) \}$, and $(-\nabla)$ is its adjoint.

It is also known that the closure of the operator matrix

$$S = \begin{pmatrix} -\nu \Delta & v^* \nabla \\ -v^* \text{div} & 0 \end{pmatrix}_{\mathcal{H}_+ \oplus \mathcal{H}_-},$$

naturally defined on a slightly different domain

$$\text{Dom}(S) = (H^2(\Omega) \cap H^1_0(\Omega))^d \oplus H^2(\Omega) \supset \text{Dom}(\hat{S})$$

is self-adjoint (see [10]), which yields another characterization for the operator $S = S(\nu, v^*)$.

Proposition 7.4. Let $\lambda_1(\Omega)$ be the first eigenvalue of the Dirichlet Laplacian on the bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. Then

(i) the positive spectral subspace of the Stokes operator $S$ can be represented as the graph of a contractive operator $X : L^2(\Omega)^d \to L^2(\Omega)$ with

$$\|X\| \leq \tan \left( \frac{1}{2} \arctan \text{Re}^* \right) < 1,$$

where

$$\text{Re}^* = \frac{2v^*}{\nu \sqrt{\lambda_1(\Omega)}};$$

(ii) $S(\nu, 0)$ is a closed operator.
(ii) the operator $X$ belongs to the Schatten-von Neumann ideal $\mathfrak{S}_p$ for any $p > d$;

(iii) the corresponding direct rotation $U$ from the “velocity subspace” $L^2(\Omega)^d$ to the positive spectral subspace of the Stokes operator $S$ maps the domain of the form onto itself. That is,

$$U \left( H^1_0(\Omega)^d \oplus L^2(\Omega) \right) = H^1_0(\Omega)^d \oplus L^2(\Omega).$$

In particular, the form (7.4) and the Stokes operator $S$ can be block diagonalized by the unitary transformation $U$.

**Proof.** (i). Due to the embedding

$$\text{Dom}\left(-\Delta \right)^{1/2} = H^1_0(\Omega)^d \subset \{ v \in L^2(\Omega)^d \mid \text{div } v \in L^2(\Omega) \} = \text{Dom}(\text{div}),$$

the entries of the operator matrix $\hat{S}$ satisfy the hypothesis of Example 2.4, so that the sesquilinear form $s$ is a saddle-point form by Lemma 2.5. The first part of the assertion (i) then follows from Lemma 7.1.

To complete the proof of (i) it remains to check the estimate (7.6).

Recall that if $P$ and $Q$ are orthogonal projections and $\text{Ran}(Q)$ is the graph of a bounded operator $X$ from $\text{Ran}(P)$ to $\text{Ran}(P^\perp)$, then the operator angle $\Theta$ between the subspaces $\text{Ran}(P)$ and $\text{Ran}(Q)$ is a unique self-adjoint operator in the Hilbert space $\mathcal{H}$ with the spectrum in $[0, \pi/2]$ such that

$$\sin^2 \Theta = P Q^\perp |_{\text{Ran}(P)}.$$

In this case,

$$\|X\| = \tan \|\Theta\|$$

(see, e.g., equation (3.12) in [22]).

Using the estimate [20]

$$\tan 2\|\Theta\| \leq \frac{2 v_*}{\nu \sqrt[2/d]{\lambda_1(\Omega)}}$$

for the operator angle $\Theta$ between the “velocity subspace” $\mathcal{H}_+ = L^2(\Omega)^d$ and the positive spectral subspace $\mathcal{L}_+ = \text{Ran}(E_S((0, \infty)))$ of the Stokes operator, one gets the bound (7.6) as a consequence of (7.9).

(ii). Denote by $\lambda_k(\Omega)$ the $k$th-eigenvalue counting multiplicity of the Dirichlet Laplacian on the domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. By the Weyl’s law, the following asymptotics

$$\lambda_k(\Omega) \sim \frac{4 \pi^2 k^{2/d}}{(|B_d||\Omega|)^{2/d}} \quad (\text{as } k \to \infty)$$

holds, see, e.g., [5] Theorem 5.1] (here $|B_d|$ is the volume of the unit ball in $\mathbb{R}^d$ and $|\Omega|$ is the volume of the domain $\Omega$). Hence, the resolvent of the vector-valued Dirichlet Laplacian $\Delta$ belongs to the ideal $\mathfrak{S}_p$ for any $p > d/2$. Then, by Lemma 7.1 we have that

$$X \in \mathfrak{S}_p, \quad \text{for any } p > d,$$

which completes the proof of (ii).

(iii). Since $s$ is a semi-bounded saddle-point form, one can apply Theorem 5.1 to justify (7.8) as well as the remaining statements of the proposition. \qed

**Remark 7.5.** The first part of the assertion (i) is known. It can be verified, for instance, by combing Theorem 2.7.7, Remark 2.7.12 and Proposition 2.7.13 in [40].

The generalized Reynolds number $\text{Re}^* = \frac{2 v_*}{\nu \sqrt[2/d]{\lambda_1(\Omega)}}$ given by (7.7) has been introduced by Ladyzhenskaya in connection with her analysis of stability of solutions of the 2D-Navier-Stokes equations in bounded domains [27]. To the best of our knowledge, the estimate (7.6), the Schatten class membership $X \in \mathfrak{S}_p, p > d$, as well the mapping property (7.8) of the direct rotation
U are new. We also note that the diagonalization of S by a similarity transform has already been discussed and the one by a unitary operator has been indicated, see [40] Theorem 2.8.1.

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