A sharp upper bound for the first eigenvalue of the Laplacian of compact hypersurfaces in rank-1 symmetric spaces

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MS received 10 July 2006; revised 27 September 2006

Abstract. Let $M$ be a closed hypersurface in a simply connected rank-1 symmetric space $\bar{M}$. In this paper, we give an upper bound for the first eigenvalue of the Laplacian of $M$ in terms of the Ricci curvature of $\bar{M}$ and the square of the length of the second fundamental form of the geodesic spheres with center at the center-of-mass of $M$.

Keywords. Hypersurface; center-of-mass; rank-1 symmetric space; Laplacian; eigenvalue.

1. Introduction

Let $(M(\kappa), ds^2)$ denote the simply connected space form of constant curvature $\kappa$ where $\kappa = 0, 1$ or $-1$ and dimension $n \geq 2$. Let $M$ be a closed hypersurface of $M(\kappa)$. When $\kappa = 0$ and $M$ a closed hypersurface of $\mathbb{R}^n$, Bleecker–Weiner [2] proved that the first eigenvalue $\lambda_1(M)$ of the Laplace operator of $M$ satisfies the inequality: 

$$\lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M |A|^2,$$

where $|A|^2$ denotes the square of the length of the second fundamental form of the hypersurface $M$. In [10], Reilly improved this inequality to show that 

$$\lambda_1(M) \leq \frac{n-1}{n \text{vol}(M)} \int_M |H|^2,$$

where $H$ is the mean curvature of the hypersurface $M$. These inequalities of Bleecker–Weiner and Reilly are also sharp for geodesic spheres in $\mathbb{R}^n$. Since then, Reilly’s inequality has been extended to hypersurfaces in other simply connected space forms (see [7] and [8] for details and related results).

While trying to understand these results, we noticed that one can obtain a similar sharp upper bound for the first eigenvalue $\lambda_1(M)$, of closed hypersurfaces $M$ in rank-1 symmetric spaces. Namely $\lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r))$ where $\lambda_1(S(r))$ is the first eigenvalue of the geodesic sphere $S(r)$ with center at a point $p_0$ called the center-of-mass of the hypersurface $M$ and radius $r(q) = d(p_0, q)$.

We refer to [4] and [9] for the basic Riemannian geometry used in this paper.

1.1 Statement of results

To state our results we need the notion of center-of-mass and result on the existence of center-of-mass for measurable subsets of $M$.

Let $(M, g)$ be a complete Riemannian manifold. For a point $p \in M$, we denote by $c(p)$, the convexity radius of $(M, g)$ at $p$. For a subset $A \subseteq B(q, c(q))$ for some $q \in M$, we let $CA$ denote the convex hull of $A$. Let $\exp_q: T_qM \rightarrow M$ be the exponential map.
and \( X = (x_1, x_2, \ldots, x_n) \) be the normal coordinates centered at \( q \). We identify \( CA \) with \( \exp^{-1}(CA) \) and we also denote \( g_q(X, X) = \|X\|^2_q \) for \( X \in T_qM \).

We now state and prove the center-of-mass theorem (see also [6] and [1]).

**Theorem 1.** Let \( A \) be a measurable subset of \((M, g)\) contained in \( B(q_0, c(q_0)) \) for some point \( q_0 \in M \). Let \( G : [0, 2c(q_0)] \rightarrow \mathbb{R} \) be a continuous function such that \( G \) is positive on \((0, 2c(q_0))\). Then there exists a point \( p \in CA \) such that

\[
\int_A G(\|X\|_p)XdV = 0,
\]

where \( X = (x_1, x_2, \ldots, x_n) \) denotes the geodesic normal coordinate system centred at \( p \).

**Proof.** For \( q \in CA \), we define \( v(q) := \int_A G(\|X\|_q)XdV \), where \( X = (x_1, x_2, \ldots, x_n) \) is a geodesic normal coordinate system centred at \( q \).

We shall now show that the continuous vector field \( v \) points inward along the boundary \( \partial CA \) of \( CA \). Then the theorem follows from the Brouwer’s fixed point theorem.

Since \( CA \) is convex, it is contained in the half-space \( H_q := \{X \in T_qM : g(X, v(q)) \leqslant 0\} \) for every \( q \in \partial CA \), where \( v(q) \) denotes the outward normal to \( \partial CA \) at \( q \). This implies that \( g(v(q), v(q)) < 0 \) for all \( q \in \partial CA \). Thus \( v \) points inward along the boundary of \( CA \).

We can find a \( \delta > 0 \) such that \( \exp_q(\delta v(q)) \in CA \) for every \( q \in CA \). Then the continuous map \( f_v : CA \rightarrow CA \) defined by

\[
f_v(q) := \exp_q(\delta v(q))
\]

has a fixed point \( p \in CA \) by the Brouwer’s fixed point theorem. Hence \( v(p) = 0 \). This completes the proof of the theorem. \( \square \)

**DEFINITION 1**

The point \( p \) in the theorem is called a center-of-mass of the measurable subset \( A \) with respect to the mass distribution function \( G \).

Before we state our results, we fix some notations that we will be using throughout this paper.

We let \((\bar{M}, ds^2)\) denote any one of the following rank-1 symmetric spaces: the round sphere \( S^n \) with constant sectional curvature \( \frac{1}{4} \), complex projective space \( CP^n \), quaternionic projective space \( HP^n \) and the Cayley projective plane \( CaP^2 \) with sectional curvature \( \frac{1}{4} \leq K_{\bar{M}} \leq 1 \) or their non-compact duals with sectional curvature \(-1 \leq K_{\bar{M}} \leq -\frac{1}{4}\). We also write \( \dim \bar{M} = kn \), where \( k = \dim \mathbb{K} \); \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( Ca \). We also let \( \text{Ric}_{\bar{M}} \) denote the Ricci curvature of \( \bar{M} \) and also remark that \( \text{Ric}_{\bar{M}} \) is constant. The round sphere \( S^n \) with constant sectional curvature 1 is denoted by \( (S^n, Can) \).

Given a point \( p \in M \), we let \( S(r) \) denote the geodesic sphere of radius \( r \) with center \( p \). Let \( \Delta_{S(r)} \) denote the Laplacian of the geodesic sphere \( S(r) \) with respect to the induced metric and \( \lambda_1(S(r)) \) denote the first eigenvalue of \( \Delta_{S(r)} \).
Theorem 2. Let $M$ be a closed hypersurface in a simply connected rank-1 symmetric space $(\bar{M}, ds^2)$ of compact type. Assume that $M$ is contained in a ball of radius $\pi/2$. Let $p_0$ be the center-of-mass of $M$ with respect to the mass distribution function $G(t) = 1/t$. Then

$$\lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r))$$

$$= \frac{1}{\text{vol}(M)} \int_M (|A(r)|^2 + \text{Ric}_{\bar{M}}),$$

(1)

where $r(x) := d(p_0, x)$ is the distance from the point $p_0$ to the point $x$. Furthermore, equality holds in the above inequality iff $M$ is a geodesic sphere with center $p_0$.

Theorem 3. Let $M$ be a closed hypersurface in a simply connected rank-1 symmetric space $(\bar{M}, ds^2)$ of non-compact type. Let $p_0$ be the center-of-mass of $M$ with respect to the mass distribution function $G(t) = 1/t$. Then

$$\lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r))$$

$$= \frac{1}{\text{vol}(M)} \int_M (|A(r)|^2 + \text{Ric}_{\bar{M}}),$$

(2)

where $r(x) := d(p_0, x)$ is the distance from the point $p_0$ to the point $x$. Furthermore, equality holds in the above inequality iff $M$ is a geodesic sphere with center $p_0$.

2. Preliminaries

Let $0 < r < i(\bar{M})$. Let $S(r)$ be a geodesic sphere of radius $r$ in $(\bar{M}, ds^2)$ centred at a point $m \in M$. We identify $S(r)$ with the inverse image $\exp^{-1}(S(r))$ with the metric $\exp^* m(ds^2|_{S(r)})$. Let $\Delta_{S(r)}$ denote the Laplacian of $S(r)$ and $\lambda_1(S(r))$ the first eigenvalue of $\Delta_{S(r)}$.

2.1 First eigenvalue of $\Delta_{S(r)}$

In this section we study the first eigenvalue $\lambda_1(S(r))$ of $\Delta_{S(r)}$. This is also done in [3] and [1].

2.1.1 $(\bar{M}, ds^2)$ of compact type. If $k = 1$, the geodesic sphere $S(r)$ is isometric to $(M_{n-1} = S^{n-1}, 4 \sin^2(r/2)\text{Can})$. Therefore the first eigenvalue of $S(r)$ is $\lambda_1(S(r)) = \frac{n-1}{4 \sin^2(r/2)}$.

We know that there is a canonical Riemannian submersion

$$\Pi : S(r) \to (M_{n-1}, 4 \sin^2(r/2)ds^2)$$

with connected totally geodesic fibres, where $M_{n-1}$ is the simply connected compact rank-1 symmetric space of dimension $k(n - 1)$.

If $k \geq 2$, we can decompose the Laplacian $\Delta_{S(r)}$ of $S(r)$ as

$$\Delta_{S(r)} = \frac{1}{4 \cos^2(r/2)} \Delta_{(S^{n-1}, \text{Can})} + \frac{1}{4 \sin^2(r/2)} \Delta_{(S^{n-1}, \text{Can})},$$

(3)
We also know that the Euclidean co-ordinate functions \( X_i \)'s, for \( 1 \leq i \leq kn \), are the first eigenfunctions of \( \Delta_{(S^{kn-1}, \text{Can})} \) corresponding to the first eigenvalue \( kn - 1 \) (see [5] for details). Since the fibres are all totally geodesic, when we restrict these eigenfunctions to each fibre, they become eigenfunctions of
\[
\frac{1}{4 \cos^2(r/2)} \Delta_{(S^{kn-1}, \text{Can})}
\]
with eigenvalue
\[
\frac{k - 1}{4 \cos^2(r/2)}.
\]
Hence we get
\[
\Delta_{S(r)} X_i = \left( \frac{kn - 1}{4 \sin^2(r/2)} + \frac{k - 1}{4 \cos^2(r/2)} \right) X_i
\]
for \( 1 \leq i \leq kn \).

Let \( \Delta_H \) denote the horizontal Laplacian of the Riemannian submersion. Since
\[
\Delta_H \mid_{\Pi^* C^\infty(M_{n-1})} = \Pi^* \Delta_{(M_{n-1}, 4 \sinh^2(r/2)dr^2)}
\]
all the eigenfunctions of \( \Delta_{(M_{n-1}, 4 \sinh^2(r/2)dr^2)} \) are also eigenfunctions of \( \Delta_{S(r)} \) with the same eigenvalues. In particular, the first non-zero eigenvalue
\[
\frac{2kn}{4 \sin^2(r/2)}
\]
occurs as an eigenvalue of \( \Delta_{S(r)} \) also. Now
\[
\frac{kn - 1}{\sin^2(r/2)} + \frac{k - 1}{\cos^2(r/2)} < \frac{2kn}{\sin^2(r/2)}
\]
iff
\[
\tan(r/2) < \sqrt{\frac{kn + 1}{k - 1}}.
\]
Since \( k \geq 1 \) and \( n \geq 2 \), we see that \( \sqrt{\frac{kn + 1}{k - 1}} > 1 \). Therefore \( \tan(r/2) < \sqrt{\frac{kn + 1}{k - 1}} \) for \( r < \pi/2 \) and consequently the functions \( X_i \), for \( 1 \leq i \leq kn \), are all first eigenfunctions of \( \Delta_{S(r)} \) with the first eigenvalue
\[
\lambda_1(S(r)) = \frac{kn - 1}{4 \sin^2(r/2)} + \frac{k - 1}{4 \cos^2(r/2)}.
\]

2.1.2 \((\bar{M}, ds^2)\) of non-compact type. We denote by \((\bar{M})^*\) the compact dual of \( \bar{M} \).

If \( k = 1 \), the geodesic sphere \( S(r) \) is isometric to \((M_{n-1})^* = S^{n-1}, 4 \sin^2(r/2)\text{Can})\) and hence the first eigenvalue \( \lambda_1(S(r)) \) of \( S(r) \) is \( \frac{n-1}{4 \sinh^2(r/2)} \).

We have the canonical Riemannian submersion
\[
\Pi : (S(r), ds^2|_{S(r)}) \to ((M_{n-1})^*, 4 \sinh^2(r/2)ds^2)
\]
with connected totally geodesic fibres.
If \( k \geq 2 \), we can decompose the Laplacian \( \Delta_{S(r)} \) as

\[
\Delta_{S(r)} = -\frac{1}{4\cosh^2(r/2)} \Delta_{(S^{k-1}, \text{Can})} + \frac{1}{4\sinh^2(r/2)} \Delta_{(S^{k-1}, \text{Can})}.
\]

We know that the euclidean coordinate functions \( X_i \)'s, for \( 1 \leq i \leq kn \), are eigenfunctions of \( \Delta_{S(r)} \) with eigenvalue

\[
\lambda_1(S(r)) = \frac{kn - 1}{4\sinh^2(r/2)} - \frac{k - 1}{4\cosh^2(r/2)}.
\]

Since

\[
\Delta_{H^1\Pi^*C^\infty((\tilde{M}_{n-1})^*)} \Pi^*\Delta_{((\tilde{M}_{n-1})^*, 4\sinh^2(r/2)ds^2)}
\]

all the eigenfunctions of \( \Delta_{((\tilde{M}_{n-1})^*, 4\sinh^2(r/2)ds^2)} \) are also eigenfunctions of \( \Delta_{S(r)} \) with the same eigenvalues. In particular, the first non-zero eigenvalue

\[
\frac{2kn}{4\sinh^2(r/2)}
\]

occurs as an eigenvalue of \( \Delta_{S(r)} \) also. Now

\[
\frac{kn - 1}{4\sinh^2(r/2)} - \frac{k - 1}{4\cosh^2(r/2)}
\]

will be the first non-zero eigenvalue of \( \Delta_{S(r)} \) so long as

\[
\frac{kn - 1}{4\sinh^2(r/2)} - \frac{k - 1}{4\cosh^2(r/2)} < \frac{2kn}{4\sinh^2(r/2)}.
\]

Since the inequality above is valid for all \( r > 0 \), we get that

\[
\lambda_1(S(r)) = \frac{kn - 1}{4\sinh^2(r/2)} - \frac{k - 1}{4\cosh^2(r/2)}
\]

for all \( r > 0 \).

2.1.3 Geometry of \( S(r) \). We will now relate \( \lambda_1(S(r)) \) of \( S(r) \) with the square of the length of the second fundamental form of \( S(r) \) and the Ricci curvature of \( \tilde{M} \).

Let \( \gamma \) be a geodesic starting at the point \( p \). Let \( R_{\gamma'(t)}: T_{\gamma'(t)}M \to T_{\gamma'(t)}M \) be the symmetric endomorphism defined by \( R_{\gamma'(t)}(v) := R(v, \gamma'(t))\gamma'(t) \), where \( R \) is the curvature tensor of \( \tilde{M} \).

For \( 0 < t < i(\tilde{M}) \), we let \( A(t) \) denote the Weingarten map \( A(\gamma(t)) \) of the smooth hypersurface \( S(t) \) at the point \( \gamma(t) \). It is known that these family of symmetric endomorphisms \( A(t) \) satisfy the Riccati equation

\[
A' + A^2 + R_{\gamma'} = 0
\]

along the geodesic \( \gamma \). Therefore, by taking the trace of these endomorphisms we get

\[
-\text{Tr}(A'(r)) = \text{Tr}(A^2(r)) + \text{Tr}(R_{\gamma'(r)})
\]

\[
= |A(r)|^2 + \text{Ric}_{\mathcal{M}}(\gamma', \gamma')
\]
where \(|A(r)|^2\) is the square of the length of the second fundamental form of the geodesic sphere \(S(r)\) and \(\text{Ric}_{\bar{M}}\) is the Ricci curvature of \(\bar{M}\).

Let \(E_2, E_3, \ldots, E_{kn}\) be an orthonormal basis of \(T_{y(r)}S(r)\) such that the vectors \(E_i\), for \(2 \leq i \leq k\), are tangent to the fibre of the canonical Riemannian submersion \[\Pi: S(r) \to \begin{cases} (\bar{M}_{n-1}, ds^2) & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ (\bar{M}^*_{n-1}, ds^2) & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type.} \end{cases}\]

Then an easy Jacobi field computation shows that
\[A(r)E_i = \begin{cases} \cot rE_i & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ \coth rE_i & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type} \end{cases}\]
for \(2 \leq i \leq k\) and
\[A(r)E_i = \begin{cases} \frac{1}{2} \cot(r/2)E_i & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ \frac{1}{2} \coth(r/2)E_i & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type} \end{cases}\]
for \(k + 1 \leq i \leq kn\). Therefore
\[\text{Tr}(A(r)) = \begin{cases} k(n - 1) \cot r & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ k(n - 1) \coth r & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type} \end{cases}\]
and \(-\text{Tr}A'(r) = \lambda_1(S(r))\).

3. **Proof of Theorems 2 and 3**

We will now prove Theorems 2 and 3.

Let \(M\) be a closed hypersurface in \(\bar{M}\) contained in a ball of radius \(i(\bar{M})/2\) where \(i(\bar{M}) = \infty\) if \(\bar{M}\) is non-compact.

By the center-of-mass theorem (Theorem 1), there exists a point \(p_0\) such that \(\int_M \frac{1}{r}X = 0\).

Since \(X = (x_1, x_2, \ldots, x_{kn})\), we see that \(\int_M \frac{1}{r}x_i = 0\) for \(1 \leq i \leq kn\).

Let \(f_i := \frac{x_i}{r}\) for \(1 \leq i \leq kn\). Since \(\int_M f_i = 0\), we can use these functions \(f_i\)'s as test functions in the Rayleigh quotient. Therefore

\[\lambda_1(M) \int_M f_i^2 \leq \int_M \|\nabla^M f_i\|^2\]

for \(1 \leq i \leq kn\), where \(\nabla^M\) denotes the gradient in \(M\). Since \(\sum_{i=1}^{kn} x_i^2 = r^2\), we see that \(\sum_{i=1}^{kn} f_i^2 = 1\). Hence

\[\lambda_1(M) \text{vol}(M) \leq \sum_{i=1}^{kn} \int_M \|\nabla^M f_i\|^2.\]

If we now show that
\[\sum_{i=1}^{kn} \int_M \|\nabla^M f_i\|^2 \leq \int_M \lambda_1(S(r))\]
then we are through.
By Green's identity, \( \| \nabla M f_i \|^2 = f_i \Delta_M f_i - \Delta_M(f_i^2) \) for \( 1 \leq i \leq kn \). Therefore,

\[
\int_M \| \nabla M f_i \|^2 = \int_M f_i \Delta_M f_i - \int_M \Delta_M(f_i^2).
\]

Since the boundary of \( M \) is empty, it follows from the divergence theorem that \( \int_M \Delta_M(f_i^2) = 0 \) for \( 1 \leq i \leq kn \). Thus \( \int_M \| \nabla M f_i \|^2 = \int_M f_i \Delta_M f_i \).

Let us now recall that \( \Delta = -\frac{\partial^2}{\partial r^2} - \text{Tr}(A) \frac{\partial}{\partial r} + \Delta_M \), where \( \Delta \) is the Laplacian of \((\bar{M}, ds^2)\), \( A \) is the Weingarten map of the hypersurface \( M \) and \( \nu \) is the unit outward normal to \( M \). Using this identity, we write that\( \Delta_M f_i = \Delta f_i + \frac{\partial^2}{\partial r^2} f_i + \text{Tr}(A)(\nabla f_i, \nu) \).

We will now compute \( \Delta f_i \), \( (\nabla f_i, \nu) \) and \( \frac{\partial^2}{\partial r^2} f_i / \partial \nu^2 \).

We decompose the Laplacian \( \Delta \) of \( \bar{M} \) as

\[
\Delta = -\frac{\partial^2}{\partial r^2} - \text{Tr}(A(r)) \frac{\partial}{\partial r} + \Delta_S(r)
\]

along the radial geodesics starting at \( p_0 \). Now \( \frac{\partial}{\partial r}(\bar{M}) = 0 \) for \( 1 \leq i \leq kn \). Therefore \( \Delta f_i = \Delta_S(r) f_i \). In \$2\$, we have shown that \( \Delta_S(r) f_i = \lambda_1(S(r)) f_i \) for \( 0 < r < i(\bar{M})/2 \).

Now

\[
f_i \frac{\partial^2}{\partial \nu^2} f_i = \frac{1}{2} \frac{\partial}{\partial \nu}(\nabla f_i^2, \nu) - (\nabla f_i, \nu)^2
\]

\[
= \frac{1}{2} \frac{\partial^2}{\partial \nu^2} (f_i^2) - \left( \frac{\partial f_i}{\partial \nu} \right)^2.
\]

Therefore

\[
\sum_{i=1}^{kn} f_i \frac{\partial^2}{\partial \nu^2} f_i = -\left( \sum_{i=1}^{kn} \left( \frac{\partial f_i}{\partial \nu} \right)^2 \right)
\]

and

\[
\sum_{i=1}^{kn} \int_M f_i \frac{\partial^2}{\partial \nu^2} f_i = -\sum_{i=1}^{kn} \int_M \left( \frac{\partial f_i}{\partial \nu} \right)^2.
\]

Similarly,

\[
\sum_{i=1}^{kn} \int_M \text{Tr}(A) f_i (\nabla f_i, \nu) = \frac{1}{2} \int_M \text{Tr}(A) \frac{\partial}{\partial \nu} \left( \sum_{i=1}^{kn} f_i^2 \right)
\]

\[
= 0.
\]

We substitute these quantities in \( \sum_{i=1}^{kn} \int_M f_i \Delta_M f_i \) to get

\[
\sum_{i=1}^{kn} \int_M f_i \Delta_M f_i = \sum_{i=1}^{kn} \int_M f_i \Delta f_i + \int_M f_i \frac{\partial^2}{\partial \nu^2} f_i + \sum_{i=1}^{kn} \int_M \text{Tr}(A) f_i (\nabla f_i, \nu)
\]

\[
= \int_M \lambda_1(S(r)) \sum_{i=1}^{kn} f_i^2 - \sum_{i=1}^{kn} \int_M \left( \frac{\partial f_i}{\partial \nu} \right)^2.
\]
\[ \int_M \lambda_1(S(r)) - \sum_{i=1}^{kn} \int_M \left( \frac{\partial f_i}{\partial \nu} \right)^2 \leq \int_M \lambda_1(S(r)) \]

\[ \leq \int_M \lambda_1(S(r)) \]

\[ = \int_M (|A(r)|^2 + \text{Ric}_M). \]

This completes the proof of the inequality.

For the equality part of the proof, we notice that equality holds in the above inequality iff \( \frac{\partial f_i}{\partial \nu} = 0 \), for \( 1 \leq i \leq kn \), at all points in \( M \). This is true iff the unit outward normal field is same as the radial vector field \( \nabla r \). Hence the equality holds iff \( M \) is a geodesic sphere of radius \( d(p_0, M) \) with center \( p_0 \). This completes the proof of theorems 2 and 3.

**Remark 1.** The analogue of Bleecker–Weiner [2] result is not known in rank-1 symmetric spaces. However we have analogous results of Theorems 2 and 3 in \( \mathbb{R}^n \).

**Theorem 4.** Let \( M \) be a closed hypersurface in \( \mathbb{R}^n \). Let \( p_0 \) be the center-of-mass of \( M \) with respect to the mass distribution function \( G(t) = 1/t \). Then

\[ \lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r)) \]

\[ = \frac{1}{\text{vol}(M)} \int_M |A(r)|^2 \]

\[ = \frac{n-1}{\text{vol}(M)} \int_M \frac{1}{r^2(x)}. \]

where \( r(x) := d(p_0, x) \) is the distance from the point \( p_0 \) to the point \( x \). Furthermore, equality holds in the above inequality iff \( M \) is a geodesic sphere with center \( p_0 \).

**Proof.** Let us first observe that the inequality 4 in the proof of theorems 2 and 3 is valid in all Riemannian manifolds in which the functions \( \frac{x_i}{r} \) are first eigenfunctions of the geodesic sphere \( S(r) \). This is true in \( \mathbb{R}^n \). Further, the first eigenvalue \( \lambda_1(S(r)) \) of the geodesic sphere \( S(r) \) in \( \mathbb{R}^n \) is \( \frac{n-1}{r^2} \). Therefore

\[ \sum_{i=1}^{kn} \int_M f_i \Delta_M f_i \leq \int_M \lambda_1(S(r)) \]

\[ = (n-1) \int_M \frac{1}{r^2(x)}. \]

Hence

\[ \lambda_1(M) \leq \frac{n-1}{\text{vol}(M)} \int_M \frac{1}{r^2(x)}. \]

**Remark 2.** The analogue of Bleecker–Weiner [2] result in rank-1 symmetric spaces and their relation with the results of Theorems 2, 3 and 4 will be discussed in a subsequent paper.
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