Interaction induced delocalization of two particles: large system size calculations and dependence on interaction strength

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Abstract. The localization length $L_2$ of two interacting particles in a one-dimensional disordered system is studied for very large system sizes by two efficient and accurate variants of the Green function method. The numerical results (at the band center) can be well described by the functional form $L_2 = L_1 [0.5 + c(U)] L_1$ where $L_1$ is the one-particle localization length and the coefficient $c(U) \approx 0.074 |U|^2/(1 + |U|)$ depends on the strength $U$ of the on-site Hubbard interaction. The Breit-Wigner width or equivalently the (inverse) life time of non-interacting pair states is analytically calculated for small disorder and taking into account the energy dependence of the one-particle localization length. This provides a consistent theoretical explanation of the numerically found $U$-dependence of $c(U)$.

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1 Introduction

The quantum eigenstates of non-interacting particles in a random potential are localized if the fluctuations of the potential (the disorder strength) are sufficiently strong (for a review see [1]). This phenomenon of Anderson localization is particularly well understood for one-dimensional or quasi one-dimensional geometries where localization even persists for arbitrarily small disorder strength. For this case efficient numerical methods [1] and also powerful analytical theories in terms of the supersymmetric non-linear $\sigma$-model [2,3] and the Fokker-Planck approach for the transfer matrix [4] are available.

Dorokhov [4] and, recently, Shepelyansky [5] considered the case of two interacting particles (TIP) moving in a one-dimensional random potential for which they predicted a strong enhancement of the localization length for the pair-states due to the interaction. While the results of Dorokhov are only valid for the case of a strongly attractive interaction confining both particles together, Shepelyansky also considered a local, attractive or repulsive Hubbard interaction. He claimed that among many states with both particles being localized far away, there are a few pairs-states where the typical distance between the particles is of the order of the one-electron localization length $L_1$ and the center of mass coordinate is characterized by a pair-localization length $L_2 \gg L_1$. Mapping the original problem on a random band-matrix model superimposed with large diagonal elements, he found for the case $L_1 \gg 1$ (length measured in units of the lattice spacing)

$$L_2 \sim \frac{U^2}{t^2} L_1^2$$

(1)

where $U$ is the strength of the Hubbard interaction and $t$ is the value of the hopping matrix element. The disorder strength enters through the value of $L_1$ (see below). This estimate has been confirmed by Imry [7] using a different argument based on the Thouless scaling block picture [8]. A crucial role is played here by the spreading width $\Gamma$ (also called Breit-Wigner width) which is the energy scale over which unperturbed states are mixed due to the interaction. Imry identified this energy scale with a generalized Thouless energy defined as the sensitivity of the energy levels with respect to a change of the boundary conditions for a finite block of size $L_1$. The pair-localization length is then obtained by scaling theory as $L_2/L_1 = \Gamma/\Delta_2$ where $\Delta_2 \sim t/L_1^2$ is the two-particle level spacing in the finite block. Using an ergodic hypothesis for the one-particle eigenfunctions, one can estimate the spreading width by Fermi’s golden rule $\Gamma \sim U^2/(t L_1)$ [9] reproducing Eq. (1).

First numerical studies in terms of finite size transfer matrix calculations [10] and exact diagonalization [11] confirmed the strong enhancement due to the interaction. Borgonovi et al. [12] showed that the enhancement effect also appears in a related model of two interacting kicked rotors for which it is possible to determine directly the quantum time-evolution. Von Oppen et al. [13] introduced an efficient method to calculate the two-particle Green function and based on their numerical results they pro-
posed the scaling relation \( L_2/L_1 \approx 0.5 + 0.054 |U| L_1 \) with a linear dependence on \( U \) contradicting the estimate (3). This behavior was explained by Jacquod et al. [14] who calculated analytically the spreading width for the limit of vanishing disorder. Resuming an infinite series of diagrams they obtained for energies close to the band center \( \Gamma/\Delta_2 \sim L_1 |U|/\sqrt{L^2 + (U/4)^2} \). Therefore the physical arguments of Refs. [6,7,9] did not contradict the results of [3] but the application of Fermi’s golden rule corresponding to lower order perturbation theory and the ergodic hypothesis appeared to be insufficient to determine \( \Gamma \).

It is worth mentioning that the topic also inspired considerable progress in the understanding of the random band matrix model with superimposed diagonal originally introduced and used by Shepelyansky [15]. Furthermore, in Ref. [18], a sophisticated random matrix model was proposed which works for arbitrary space dimension and takes properly both particle coordinates (relative and center of mass coordinate) into account. This model can be mapped onto an effective supermatrix nonlinear \( \sigma \)-model and it is thus possible to explain features like a logarithmically suppressed diffusion or a logarithmically increasing pair size [18] found previously by Borzoni et al. [12] for the pair-diffusive regime in \( d > 2 \) dimensions where all states are delocalized. Subsequent work was concerned with the role of the level statistics [19] and, very recently, with the fractal structure of the coupling matrix elements due to the interaction [20].

Despite the available evidence in favor for the enhancement effect the general situation is still not really clear, due to different proposals for the dependence of \( L_2 \) on \( W \) and \( U \) [20,21,22] and the claim of Römer et al. that the effect completely vanishes in the limit of infinite system size [23]. This claim, which was contested in [24], is based on the finite size extrapolation of the localization length calculated by a transfer matrix method for finite square samples being put together to an infinite strip.

In this work, we present numerical results (section 2) based on an exact and efficient variant of the Green function method introduced in Ref. [13] that allows to treat rather large system sizes, i.e., \( N \geq 1000 \). This is indeed important for small disorder values in order to perform an accurate finite size extrapolation. We furthermore present a second variant which consists of the recursive Green function technique applied to an effective band matrix Hamiltonian as considered in [13]. In this approach one can indeed take \( N \to \infty \) and the results we find are consistent with those of the finite size extrapolation of the first variant. The issue of an accurate variant of the Green function method is actually of considerable interest since Römer et al. [23] had questioned the original results of von Oppen et al. due to a certain approximation applied in the original approach of Ref. [13]. We find in our calculations qualitative agreement with those results concerning the strong enhancement of the localization length \( L_2 \) and the dependence on \( L_1 \). However, we find nevertheless a quantitative difference concerning the dependence on the interaction strength \( U \) which is only linear for sufficiently small \( U \). To understand this, we reconsider the issue of the determination of the Breit-Wigner width \( \Gamma \) for small disorder (section 3). Improving the \( \Gamma \) estimate of Ref. [14], we can indeed explain the modified \( U \) dependence.

Very recently, we learned of related relevant work [24,26] in which the TIP Green function was evaluated by a decimation method for system sizes up to \( N = 251 \) or \( N = 300 \) [25].

2 Numerical Green function approach

We consider two particles in a disordered system interacting via a local Hubbard-interaction and characterized by the following tightbinding Hamiltonian,

\[
H = -t \sum_{x,y} \left( |x+1,y><x,y| + |x,y+1><x,y| + \text{h.c.} \right) + \sum_{x,y} \left( \varepsilon(x) + \varepsilon(y) + U \delta_{x,y} \right) |x, y><x, y|.
\]

(2)

\( x \) and \( y \) denote the positions of the first and the second particle, respectively. \( t \) is the strength of nearest neighbor coupling matrix element which we put to unity in the following and \( U \) is the value of the on-site interaction. The disorder energies are random, i.e., \( \varepsilon(x) \in [-W/2, W/2] \) with \( W \) being the disorder strength. At vanishing interaction \( U = 0 \) the one-particle eigenstates (at a one-particle energy \( E = 0 \)) are localized with the localization length: \( L_1 \approx 105 W^2 / |U| \). In this work we do not discuss the particular effects of symmetric or anti-symmetric two-particle states (Bosons or Fermions). The on-site Hubbard interaction only acts on the subspace of symmetric states and our results apply therefore to the case of Bosons. However, for the actual calculations and the representation, we find it more convenient to keep all states.

To determine the two-particle localization length, we consider as in Ref. [13] the two-particle Green function. Since we are interested in the coherent propagation the particles being close, we determine only the Green function matrix elements of doubly occupied states \(|xx>\),

\[
g_{xx} = <xx| (E - H)^{-1} |yy>.
\]

(3)

A priori, for a finite system of size \( N \), the matrix inverse in (3) has to be evaluated for a \( N^2 \times N^2 \) matrix. Fortunately, von Oppen et al. [23] have shown that this problem can be reduced to an effective Green function on an \( N \)-dimensional space because the interaction operator is proportional to the projector on the space of doubly occupied states. The matrix \( g \) in (3) can be calculated [13] from an \( N \times N \)-matrix equation

\[
g = g_0 \frac{1}{1 - g_0 U}, \quad \text{where} \quad g_0 = g \big|_{U=0}.
\]

(4)

The matrix \( g_0 \) is given in terms of the one-particle eigenstates \( \varphi_\alpha(x) \) and the one-particle energies \( E_\alpha \) via,

\[
(g_0)_{xy} = \sum_{\alpha,\beta} \varphi_\alpha(x) \varphi_\beta(x) \frac{1}{E - E_\alpha - E_\beta} \varphi_\beta(y) \varphi_\alpha(y).
\]

(5)
The two-particle localization length $L_2$ is determined by the exponential decay $g_{x_0,x} \sim \exp(-|x - x_0|/L_2)$ corresponding to

$$
\frac{1}{L_2} = -\lim_{N \to \infty} \frac{1}{N} \left\langle \ln \left| \frac{g_{x,x+N}}{g_{x,x}} \right| \right\rangle \quad . \quad (6)
$$

The ensemble average is performed over different disorder realizations and for practical purposes also over some initial sites $x$ close to one boundary. The extra denominator $g_{x,x}$ in (6) is not relevant in the limit $N \to \infty$ but provides a considerable improvement if (6) is evaluated for finite $N$. For vanishing interaction, we expect according to (6) $L_2(U = 0) \approx L_1/2$ [13].

In Ref. [13], Eq. (4) was evaluated for a finite system by employing two approximations. First, von Oppen et al. omitted the first factor $g_0$ and, second, they did not evaluate the full matrix $g_0$ but only a sufficiently large band on which they applied the recursive Green function technique [24, 25] for the matrix inverse in (4). Since $g_0$ is indeed a band matrix of width $\sim L_1$ both approximations seem to be well justified provided $L_1 < L_2/2$. However, the validity of the corresponding results was seriously questioned by Römer et al. [26] due to these approximations and, furthermore, the limit of very small (vanishing) interaction cannot accurately be studied within this approach.

We have evaluated (6) exactly without any approximations. For this we note two important points concerning the numerical precision and the efficiency. First, the multiplication of the band matrix $g_0$ with the matrix inverse in (6) requires that the relative error of the exponentially small matrix elements of $g_0$ far away from the diagonal is small [26]. Otherwise, Eq. (6) provides incorrect results for $L_2$. This in turn requires that the exponential tails of the $\varphi_\alpha(x)$ are accurate over the whole length scale $x = 1, \ldots, N$. According to this, we have determined the one-electron eigenstates by the method of inverse vector iteration [28] which provides the required accuracy by sufficiently increasing the number of iterations. The second point concerns the efficiency. Here the matrix inverse in (6), which costs of the order of $N^3$ operations, is actually not the limiting factor. This is due to the necessary evaluation of $g_0$. The naive application of (6) already costs the order of $N^4$ operations. Even though this number can be reduced by exploiting the exponential decay of the $\varphi_\alpha(x)$ this does not yield any significant improvement for small disorder values when $L_1 \sim 50 - 100$. Fortunately, it is possible to determine $g_0$ exactly with $N^3$ operations. For this we rewrite (6) in the form

$$
(g_0)_{xy} = \sum_\alpha \varphi_\alpha(x) G_{xy}^{(1)}(E - E_\alpha) \varphi_\beta(y) \quad , \quad (7)
$$

where $G_{xy}^{(1)}(E)$ is the one-particle Green function at energy $E$ that can efficiently be determined by $N^2$ operations due to the tridiagonal form of the one-electron Hamiltonian. Since this has to be done for $N$ different energies $E - E_\alpha$, Eq. (7) provides an algorithm with only $N^3$ operations.

We have used two variants of the Greens function method. The first is based on a finite size extrapolation (FSE) to
For small disorder values it is quite difficult to arrive at this regime. In Fig. 2, we compare for different disorder values, 1/2 ≤ U ≤ 1, the interval 40 ≤ b ≤ 200. The dashed lines correspond to the values obtained by the second method. To our knowledge, this is the first method (finite size extrapolation).

We find overall agreement between both methods and L2(b) indeed coincides with L2 for b ≥ b_c ≈ 5 L1. For small disorder values it is quite difficult to arrive at this regime. For b < b_c, the values of L2(b) are typically larger than the values obtained by the first method. To our knowledge, the approach described above is indeed the first method to determine directly the TIP localization length for quasi infinite system size without the side effects of a bag-interaction. This is possible, because the cutoff is applied on an effective Hamiltonian and the neglected matrix elements are indeed exponentially small if the block size b is sufficiently large. The results shown in Fig. 2 provide therefore an additional confirmation of the validity of the above discussed extrapolation scheme.

For a systematic study of the dependence on W and U, we used the first variant based on the FSE scheme which appears to be more efficient, especially for large values of L1. For the scope of this paper, we studied the band center E = 0 where the localization properties are symmetric with respect to the sign of U. We considered for the disorder values 1.0 ≤ W ≤ 7.0 and interaction strengths 0.0 ≤ U ≤ 2.0 at least system sizes up to N = 500 and for W ≤ 1.75 even sizes up to N = 1000. (For W = 1.0 and U = 1.5, 2.0, we have also calculated two data points with N = 1400.) Most of the data points (for the finite size values L2(N)) were determined with a relative error smaller than 2%. For the largest system sizes and smallest disorder values the relative error is 3–5%. To verify the scaling relation L2/L1 = (a + b U) L1 suggested by Song et al. [22], we show in Fig. 3 the ratio L2/L1 as a function of L1 where L2 has been obtained by FSE from L2(N).

The line behavior in L1 is qualitatively indeed confirmed but the errors for the smaller disorder values do not allow to exclude a behavior of the type (L2/L1 − 0.5) ∝ L1^α with α < 1. A corresponding fit indeed gives α ≈ 0.9 but this depends also on the chosen offset 0.5 (the fit with the offset 0.55 gives α ≈ 1.0). We mention that the slight deviations from the linear behavior can also be well described by an ansatz of the type (L2/L1 − 0.5) ∝ L1/ln(C L1) suggested by Borgonovi et al. [11]. However, since the precision of the data does not permit to distinguish significantly between this and the linear behavior, we do not enter into more details here. For U = 0, we confirm the previous observation [24, 25] of a slight enhancement E2(U = 0)/L1 ≈ 0.5 − 0.7 which is presumably due to the energy average in Eq. 5.6.

In Fig. 4, we also show the dependence of L2 on the disorder strength W (for U = 1 and E = 0). Previously, Song et al. [22] found a behavior L2 ∝ W^{−2.9} and as we can see the overall slope in Fig. 4 is indeed comparable to this behavior. However, we find for small and large W values significant deviations due to the curvature in the curve of ln(L2) versus ln(W). This is due to the constant term in the above scaling relation. Actually, the data can be extremely well fitted by L2 ∝ L1 (0.55 + 0.038 L1) for the whole interval 1.0 ≤ W ≤ 7.0 and one indeed finds the asymptotic behavior L2 ∝ W^{−4} for small W (L1 ≫ 10) and L2 ∝ W^{−2} for larger values of W (1 < L1 < 10).

To extract the dependence on U, we determined the slope c(U) in the linear fit L2/L1 = a + c(U) L1 which is compared in Fig. 5 with the numerical data. The slope
c(U) itself is shown in the insert of Fig. 4 as a function of U. Apparently, the U dependence is not linear for the whole interval 0 ≤ U ≤ 2.0. This linear behavior was observed by v. Oppen et al. for U ≤ 1.0 where the discrepancy is still quite moderate. At U = 1.0 their values are about 40% larger than ours. We believe that this is due to finite size effects and the applied approximations.

Also the estimate c(U) ∝ |U|/√(1+(U/4)^2) based on the analytical calculation of the Breit-Wigner width for W = 0 [14] only agrees with the numerical data for sufficiently small U. In the next section, we will try to explain this disagreement and reconsider the determination of the Breit-Wigner width.

3 Breit-Wigner Width and U dependence

The delocalization effect of two interacting particles is related to the finite life time τ = Γ^{-1} of the product states |αβ> with <x_1 x_2|αβ> = φ_α(x_1) φ_β(x_2) [2]. The interaction gives rise to transitions |αβ> → |γδ> which can be viewed as random hops of typical size L_1. For short time scales when quantum interference effects can be neglected one therefore obtains a diffusive dynamics [2] with the diffusion constant D ∼ L_1^2 τ^{-1} = L_1^2 Γ. Following a general argument developed in Ref. [14] and applied to the TIP case in Refs. [14], one can estimate the localization length due to quantum interference effects. According to this the classical diffusive behavior is only valid for time scales smaller than the Heisenberg time corresponding to a wave packet of width √Dt, i. e.

\[ t < t_H(t) = ν_{eff} √Dt. \]  

(9)

Here ν_{eff} is the density of states per length such that (L ν_{eff})^{-1} is the level spacing in a block of size L. For the TIP, we have ν_{eff} ≈ ν_2 L_1 with ν_2(E) being the energy dependent two-particle density of states. ν_2(E) is proportional to the inverse bandwidth 1/(8 + 2W) ∼ 1 with a logarithmic singularity at the band center (for W = 0). The density ν_{eff} corresponds to the number of well coupled states with the same center of mass coordinate but with different relative coordinates (up to a maximal value ∼ L_1).

At t ≈ t_c = ν_{eff}^2 D, when the condition [14] ceases to be valid, the discrete energy spectrum can be resolved and localization with a localization length L_2 ∼ √Dt_c ∼ ν_{eff} D sets in. This general relation for quasi 1d systems has also been obtained in a more rigorous way using the supersymmetric non-linear σ-model [8]. In view of this, the Breit-Wigner width Γ and L_2 are related by

\[ L_2 ∼ ν_{eff} D ∼ L_1^2 Γ ν_2. \]  

(10)

Jacquod et al. [14] determined Γ using diagrammatic perturbation theory (in U) for the case of vanishing disorder, W = 0, and finite system size N. They argued that one obtains a good estimate of Γ for finite W by replacing N with L_1 due to the ballistic dynamics in 1d for length scales up to L_1. For energies close to the band center and moderate interaction strengths U ≲ 1, their result reads

\[ L_2 ∼ L_1^2 \frac{|U|}{1 + (U/4)^2}. \]  

(11)

As we can see in the insert of Fig. 5 our numerical data agrees with this behavior only for very small values of U. We therefore feel that it is justified to reconsider the issue of the Breit-Wigner width which can be calculated from
the energy dependent local density of states
\[ \rho_{\alpha\beta}(E) = -\frac{1}{\pi} \text{Im} < \alpha\beta | (E + i0 - H)^{-1} | \alpha\beta > , \] (12)
with $|\alpha\beta>$ as above. Using Schur's formula to perform the matrix inverse, we rewrite (12) as
\[ \rho_{\alpha\beta}(E) = -\frac{1}{\pi} \text{Im} \left[ E + i0 - E_{\alpha} - E_{\beta} - i\alpha\beta | \alpha\beta > + i\Gamma_{\alpha\beta}/2 \right]^{-1} , \] (13)
\[ \Gamma_{\alpha\beta} = \Gamma_{\alpha\beta}^{(0)} + i \Gamma_{\alpha\beta}^{(1)} \]
\[ = -2 < \alpha\beta | \hat{U} (E + i0 - \hat{H}) | \alpha\beta > . \] (14)

Here $\hat{U} = \hat{P} \sum_x |xx><xx| \hat{P}$ is the interaction operator and $\hat{H} = \hat{P} \hat{H} \hat{P}$ with the projector $\hat{P} = 1 - |\alpha\beta><\alpha\beta|$. Eq. (13) corresponds to the Lorentzian or Breit-Wigner form of the local density of states provided that the energy dependence of $\Gamma_{\alpha\beta}$ is not too weak. The imaginary part $\Gamma_{\alpha\beta}^{(1)}$ is the width of the Lorentzian. In the following, we replace $\hat{H}$ by $H$, and we first evaluate (14) for the case of vanishing disorder. For this we only need the projected Green function $\tilde{g}(k)$ due to the appearance of $\hat{U}$ in (13). In view of Eq. (13), we first determine
\[ (g_{0})_{xy} = <x|x|E + i0 - H_{0})^{-1}|y>y > \]
\[ \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \ e^{i(k-x-y)} \tilde{g}_{0}(k) \] (15)
with
\[ \tilde{g}_{0}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \ \frac{1}{E + i0 + 2|\cos(q) + \cos(k - q)|} \]
\[ = -\frac{1}{\sqrt{4\cos^{2}(k/2) - E^{2}}} . \] (16)
The Green function at $W = 0$ and $U \neq 0$ is then given by
\[ g_{xy} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \ e^{i(k-x-y)} \frac{\tilde{g}_{0}(k)}{1 - U \tilde{g}_{0}(k)} . \] (17)
From this and Eq. (14), we obtain
\[ \Gamma_{\alpha\beta} = -2U^{2} \sum_{x,y} \varphi_{\alpha}^{*}(x) \varphi_{\beta}^{*}(x) g_{xy} \varphi_{\alpha}(y) \varphi_{\beta}(y) . \] (18)

Inserting the plane wave eigenstates for $\varphi_{\alpha}$ or $\varphi_{\beta}$, we exactly recover the result of Ref. [14] for the Breit-Wigner width. This shows that the diagrammatic approach of [14] is equivalent to our above approximations (replacing $\hat{H}$ by $H$ and continuum limit for $k$).

The generalization to the disordered case essentially gives rise to two modifications. Using diagrammatic perturbation theory in the disorder, one can first evaluate the average of the Green functions (15) and (16) which amounts to the replacement $E + i0 \rightarrow \epsilon + i\gamma$ where $\gamma$ is determined by a Dyson equation. For weak disorder one finds $\gamma \sim W^{2}$ (with eventual logarithmic corrections at the band center). In the following discussion, we neglect the effect of this small $\gamma$ which essentially regularizes (14) at the singularity. The second, more important, modification due to finite disorder concerns the eigenfunctions $\varphi_{\alpha}(x)$ in Eq. (15). Here the energy dependence of the one particle localization length plays an important role. To see this, we use the toy ansatz
\[ \varphi_{\alpha}(x) \approx \frac{1}{\sqrt{L_{1,\alpha}}} e^{-|x-x_{\alpha}|/L_{1,\alpha} + ik_{\alpha} x} \] (19)
with $E_{\alpha} = -2\cos k_{\alpha}$ and an energy dependent localization length $L_{1,\alpha} \approx L_{1} \sin^{2}(k_{\alpha})$. This ansatz essentially corresponds to a particle moving ballistically with a well defined momentum inside the localization domain around $x_{\alpha}$. This is indeed reasonable because in one dimension the mean free path is of the same order as the localization length. However, the momentum has to be larger than its typical uncertainty, i.e. $|k_{\alpha}| \gtrsim \Delta k \sim L_{1,\alpha}^{-1/3}$. Therefore the ansatz (19) is valid for momenta with $|\sin k_{\alpha}| \gtrsim L_{1,\alpha}^{-1/3}$ corresponding to energies not being close to the band center. Inserting (19) in (15) and choosing $x_{\alpha} \approx x_{\beta}$, we obtain
\[ \Gamma_{\alpha\beta} \approx -2U^{2} \frac{1}{L_{1,\alpha} + L_{2,\alpha}} \frac{\tilde{g}_{0}(k_{\alpha} + k_{\beta})}{1 - U \tilde{g}_{0}(k_{\alpha} + k_{\beta})} \] (20)
giving rise to the Breit-Wigner width (for $E = 0$)
\[ \Gamma_{\alpha\beta} \approx \frac{U^{2}/(4L_{1})}{\sin^{2}k_{\alpha} + \sin^{2}k_{\beta} \cos^{2}(|k_{\alpha} + k_{\beta}|/2) + (U/4)^{2}} . \] (21)

These expressions differ by the $k$-dependent localization length from the result of Ref. [14]. Since $\Gamma_{\alpha\beta}^{(1)}$ depends strongly on the momenta $k_{\alpha}$ and $k_{\beta}$, we determine the average with respect to these momenta [14]:
\[ \Gamma = \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_{\alpha} \int_{-\pi}^{\pi} dk_{\beta} \delta(E - E_{\alpha} - E_{\beta}) / \nu_{2}(E) . \] (22)
Using Eq. (10), relating $\Gamma$ with $L_{2}$, we can estimate for $E \rightarrow 0$ the two particle localization length as
\[ L_{2} \approx c(U) L_{2}^{2} \],
\[ c(U) = A \frac{2}{\pi^{2}} \int_{0}^{\pi} dk \ \frac{1}{(\sin^{2}k + k_{c}^{2})^{2}} \frac{(U/4)^{2}}{\sin^{2}k + (U/4)^{2}} \]
\[ = 2A \frac{U}{\pi k_{c}} \left[ \frac{1}{|U|} \frac{s(U)}{s(\frac{U}{4})} \frac{s(\frac{U}{2})}{s(\frac{U}{4})} + 4k_{c} \frac{s(k_{c})}{s(\frac{U}{4})} \right] \] (23)
where $s(x) = 1 + x^{2}$, $A$ is a numerical prefactor and $k_{c}$ is a cutoff value to regularize the integral for small $k$ where the ballistic ansatz (19) is invalid. For the values $k_{c} = 0.25$ and $A = 0.028$ the $U$-dependence of (23) fits very well the numerical data for the slope $c(U)$ which can be seen in Fig. 3. We can considerably simplify the somewhat lengthy expression (23) by neglecting the quadratic corrections $U^{2}$ and $k_{c}^{2}$ and slightly modifying $A$.
\[ c(U) \approx 0.074 \frac{|U|}{|U| + 1} . \] (24)
This approximation is numerically very accurate with an
total relative error smaller than 1% for \( |U| \leq 2.0 \). The
linear behavior of \( c(U) \) for small \( |U| \) is due to a com-
bination of the logarithmic singularity in the density of
states at \( E \rightarrow 0 \) and of large values of \( \Gamma^{(1)}_{\alpha,\beta} \sim U \) if
\[
\cos((k_\alpha + k_\beta)/2) \approx \pm U.
\]

In view of the agreement between the numerical data and
the theoretical estimate for \( c(U) \), we conclude that the ideas of
diffusively moving particle pairs for short time scales finally becoming localized due to quantum interference [14] [15] [16] [17] can indeed quantitatively explain the deloca-
larization effect. For this it is important to evaluate care-
fully the Breit-Wigner width by taking into account the
energy dependence of the one particle localization length.

Despite this agreement we want to emphasize that the result [18] is nevertheless based on several qualitative ar-
guments. Actually, the application of the relation [19] is somewhat problematic because both the Breit-Wigner
width and the one-particle localization length do not have unique values due to their energy dependence. It is a priori
not clear if the simple average [20] is really sufficient and
accurate. Furthermore, Eqs. (23), (24) depend on the ar-
tificial cutoff parameter \( k_c \). Theoretically, we expect that
\( k_c \sim L^{-1/3} \) because of the invalidity of the ballistic ansatz
[19] for small momenta \(|k| < k_c \). Numerically, the case
\( k_c = 0.25 \) indeed corresponds to \( L_1 \approx 50 - 100 \) the largest
considered \( L_1 \) values. However, the resulting dependence on \( L_1 \), i. e. \( L_2 \sim L_1^{1/3} \) clearly disagrees with the numerical
data. We attribute this to the fact that for small momenta according to \( L_1, \alpha \approx L_1 \sin^2 k_\alpha \) the effective size of the ran-
don hops is strongly reduced. This feature is not properly
taken into account in the relation [19]. Therefore it would be
interesting to carefully generalize this relation to the
case where \( \Gamma \) and \( L_1 \) have complicated distributions in-
stead of unique values.

4 Conclusion

In this work, we have presented and applied two new accu-
rate and efficient variants of the Greens function method
originally introduced by von Oppen et al. [13] to study
the TIP localization problem [6]. Our results for the TIP
localization length \( L_2 \) can be well fitted (Fig. 6) by the
functional form \( L_2 = L_1(0.5 + c(U) L_1) \). The behavior of
the slope \( c(U) \approx 0.074 |U|/(1 + |U|) \) is determined by the \( U \)-dependence of the Breit-Wigner width \( \Gamma \) of non-
interacting pair states. For this we presented an accurate
estimate of \( \Gamma \) extending former work of Jacqod et at. [14].

We think that our results provide important additional
evidence for the delocalization effect as such. In particular,
we find for \( U = 2.0 \) and \( W = 1.0 \) an enhancement factor
\( L_2/(2 L_1) \approx 11.5 \). Our data is in qualitative agreement
with former results of Song et al. [12], who directly used the
less efficient recursive Green function technique for smaller system sizes (\( N \leq 200 \)), and with very recent work
[21, 22] based on the decimation method (\( N \leq 251 \) and
\( N \leq 300 \)). In view of this the original claim of Römer et
al. [23] that there is no delocalization effect for infinite
system size can no longer be maintained. To understand
the transfer matrix data on which this claim was based, we
remind that the considered disorder value \( W = 3.0 \)
was relatively large such that \( L_2 \) and \( L_1 \) are nearly equal.
Using, the Green function method one can still measure
the enhancement because \( L_2 > L_1/2 \). However, in the
transfer matrix approach there is a direct competition of
\( L_2 \) with \( L_1 \) [21]. Furthermore, even for smaller disorder,
the finite size behavior of the transmission eigenvalues is
very subtle and one has to be careful about the finite size
extrapolation here [12].

While the delocalization effect is now well established,
the situation is less clear concerning the functional de-
pendence of \( L_2 \) on \( L_1 \). The formerly observed powerlaw
\( L_2 \sim L_1^\alpha \) with \( \alpha \approx 1.45 - 1.65 \) [11, 21, 22] was obtained
by a fit ansatz without constant term. According to our
above discussion (see Fig. 6) it is numerically not obvious
to distinguish this powerlaw from the functional form we
proposed above. Wantal et. al. [20] gave an argument in
favor of the former with \( \alpha = 1.5 \). This argument is based
on the multifractal properties of the interaction induced
coupling matrix elements in combination with an estimate
of \( \Gamma \) using Fermi’s golden rule. We believe that this
analysis is indeed important and very relevant to the prob-
lem. Actually, our result (21) for the Breit-Wigner width
contains a strong dependence on the initial one-particle
states due to partial momentum conservation. This leads
to strong fluctuations of \( \Gamma \) which are presumably directly
related to the multifractal statistics of the coupling matrix
elements. However, the analytical calculations of Ref. [13]
and of section 3 clearly show that the simple application
of Fermi’s golden rule is not sufficient and the numerical
data do not show the corresponding behavior \( L_2 \sim U^2 \).
To understand these issues in more detail further work is
necessary.

We emphasize that our numerical data and the esti-
mate [23] are valid for moderate interaction strengths
\( |U| \leq 2.0 \). For larger values of \( U \) one expects that \( L_2 \) will be
reduced due to the particular projector structure of the
interaction operator \( U \). Wantal et al. [20] presented a
duality transformation mapping the case of \( |U| \geq 1 \) to a
similar problem with \( |U| \ll 1 \) and a different reference
basis. According to this \( L_2(U) \) should obey the duality
relation \( L_2(U) \approx L_2(\sqrt{24} |U|) \) [20].

Finally, we mention that the numerical trick to eval-
uate efficiently the matrix \( g_0 \) via Eq. (6) also works in
higher dimensions, even though the gain is less spectacu-
lar. In \( d \) dimensions and a system of total size (volume)
\( N \) one can calculate \( \mathcal{G}^{(1)}(E) \) by the recursive Green
function method which provides an algorithm to evaluate
Eq. (6) with \( N^{3+d(d-1)/d} \) operations. In particular the case of two
dimensions is important due to recent claims of Ortu-
no et al. and Cuevas [33] for a delocalization transition for
two interacting particles in \( d = 2 \). We think it is necessary
to consider larger system sizes as in Ref. [13] in order to
decide whether there is a real transition or a very strong
delocalization with a finite but very large two-particle lo-
calization length as it was argued by Shepelyansky [1].
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