The Mean Curvature of First-Order Submanifolds in Exceptional Geometries with Torsion

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Abstract

We derive formulas for the mean curvature of associative 3-folds, coassociative 4-folds, and Cayley 4-folds in the general case where the ambient space has intrinsic torsion. Consequently, we are able to characterize those $G_2$-structures (resp., Spin(7)-structures) for which every associative 3-fold (resp. coassociative 4-fold, Cayley 4-fold) is a minimal submanifold.

In the process, we obtain new obstructions to the local existence of coassociative 4-folds in $G_2$-structures with torsion.

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1 Introduction

In their fundamental work on calibrations, Harvey and Lawson [9] defined four new classes of calibrated submanifolds in Riemannian manifolds with special holonomy, summarized in the following table:

| Submanifold          | Ambient Manifold                                      |
|----------------------|--------------------------------------------------------|
| Special Lagrangian $n$-fold | Riemannian $2n$-manifold $(M^{2n}, g)$ with $\text{Hol}(g) \leq \text{SU}(n)$ |
| Associative 3-fold    | Riemannian 7-manifold $(M', g)$ with $\text{Hol}(g) \leq G_2$ |
| Coassociative 4-fold  | Riemannian 7-manifold $(M', g)$ with $\text{Hol}(g) \leq G_2$ |
| Cayley 4-fold         | Riemannian 8-manifold $(M^8, g)$ with $\text{Hol}(g) \leq \text{Spin}(7)$ |

By virtue of being calibrated, each of these submanifolds satisfy a strong area-minimizing property. In particular, they are stable minimal submanifolds. Moreover, by an argument using the Cartan-Kähler Theorem, Harvey and Lawson [9] were able to show that submanifolds of each class exist locally in abundance.

Riemannian manifolds with special holonomy groups function as the background spaces for supersymmetric theories of physics. In this setting, calibrated submanifolds are related to supersymmetric cycles [3]. Special Lagrangian submanifolds lie at the foundation of the SYZ formulation of mirror symmetry [13], and calibrated submanifolds in manifolds with holonomy groups $G_2$ and $\text{Spin}(7)$ are expected to play a similar role in theories of mirror symmetry for such manifolds [1, 8].

In fact, each of the classes of submanifolds described above make sense in an even more general class of ambient spaces: namely, that of (Riemannian) manifolds $M$ equipped with a $G$-structure, for $G = \text{SU}(n)$ or $G_2$ or $\text{Spin}(7)$ as appropriate.

| Submanifold          | Ambient Manifold                                      |
|----------------------|--------------------------------------------------------|
| Special Lagrangian $n$-fold | $2n$-manifold $M^{2n}$ with an $\text{SU}(n)$-structure |
| Associative 3-fold    | 7-manifold $M'$ with a $G_2$-structure                 |
| Coassociative 4-fold  | 7-manifold $M'$ with a $G_2$-structure                 |
| Cayley 4-fold         | 8-manifold $M^8$ with a $\text{Spin}(7)$-structure     |

However, in this generalized setting, such submanifolds need not be minimal. This raises the following:

**Minimality Problem:** Let $M$ be a manifold. Characterize those $G$-structures (for $G = \text{SU}(n)$, $G_2$, $\text{Spin}(7)$) on $M$ for which every submanifold in $M$ of a given class (special Lagrangian, associative, etc.) is a minimal submanifold of $M$.

We will completely solve the Minimality Problem in the contexts of associative 3-folds, and coassociative 4-folds, and Cayley 4-folds by deriving simple formulas for their mean curvature. The case of special Lagrangian 3-folds is addressed in our preprint [2].

Perhaps more fundamentally, in our generalized context the relevant submanifolds need not exist at all, even locally. This raises the natural:

**Local Existence Problem:** Let $M$ be a manifold. Characterize those $G$-structures (with $G$ as above) on $M$ for which submanifolds of a given class (special Lagrangian, etc.) exist locally at every point of $M$. 

In this work, we make progress towards the resolution of the Local Existence Problem in the setting of coassociatives. More precisely, we obtain an explicit obstruction to the local existence of coassociative 4-folds. Analogous obstructions to the local existence of special Lagrangian 3-folds are obtained in [2].

1.1 Results on Associative 3-folds and Coassociative 4-folds

Let \((M^7, \varphi)\) be a 7-manifold with a \(G_2\)-structure \(\varphi \in \Omega^3(M)\). The first-order local invariants of \(\varphi\) are completely encoded in four differential forms, called the torsion forms of the \(G_2\)-structure, denoted
\[
(\tau_0, \tau_1, \tau_2, \tau_3) \in \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \Omega^3.
\]
These are defined by the equations
\[
d\varphi = \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast \tau_3
\]
\[
d \ast \varphi = 4\tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi.
\]

In order to study associative 3-folds and coassociative 4-folds in \(M\), we will break the torsion forms into \(SO(4)\)-irreducible pieces with respect to a certain splitting of \(TM\). Indeed, in \(\S 2.3\), we will decompose \(\tau_0, \tau_1, \tau_2, \tau_3\) into \(SO(4)\)-irreducible components, writing
\[
\tau_0 = \tau_0
\]
\[
\tau_1 = (\tau_1)_A + (\tau_1)_C
\]
\[
\tau_2 = (\tau_2)_A + (\tau_2)_{1,3} + (\tau_2)_{2,0}
\]
\[
\tau_3 = (\tau_3)_{0,0} + (\tau_3)_{0,4} + (\tau_3)_{2,2} + (\tau_3)_{1,3} + (\tau_3)_C
\]

We will refer to the individual pieces
\[
\tau_0, \ (\tau_1)_A, (\tau_1)_C, \ (\tau_2)_A, (\tau_2)_{1,3}, (\tau_2)_{2,0}, \ (\tau_3)_{0,0}, (\tau_3)_{0,4}, (\tau_3)_{2,2}, (\tau_3)_{1,3}, (\tau_3)_C
\]
as refined torsion forms (with respect to a certain splitting of \(TM\)). The mean curvature of associatives and coassociatives can then be expressed purely in terms of the refined torsion. In the sequel, we let \(\sharp: T^*M \rightarrow TM\) denote the usual musical (index-raising) isomorphism.

**Theorem 2.18 (Mean Curvature of Associatives):** The mean curvature vector \(H\) of an associative 3-fold in \(M\) is given by
\[
H = -3[(\tau_1)_C] \frac{\sharp}{2} - \frac{\sqrt{3}}{2} [(\tau_3)_C] \frac{\sharp}{2}
\]
where \(\sharp\) is a particular isometric isomorphism defined (2.18).

In particular, a \(G_2\)-structure on \(M\) has the property that every associative 3-fold in \(M\) is minimal if and only if \(\tau_1 = \tau_3 = 0\). Equivalently, if and only if \(d\varphi = \lambda \ast \varphi\) for some constant \(\lambda \in \mathbb{R}\).

**Theorem 2.21 (Mean Curvature of Coassociatives):** The mean curvature vector \(H\) of a coassociative 4-fold in \(M\) is given by
\[
H = -4[(\tau_1)_A] \frac{\sharp}{2} + \frac{\sqrt{5}}{3} [(\tau_2)_A] \frac{\sharp}{2}
\]
where \(\sharp\) is a particular isometric isomorphism defined in (2.11).

In particular, a \(G_2\)-structure on \(M\) has the property that every coassociative 4-fold in \(M\) is minimal if and only if \(\tau_1 = \tau_2 = 0\). Equivalently, if and only if \(d \ast \varphi = 0\).
These formulas can be regarded as a submanifold analogue of the curvature formulas derived by Bryant [5] for 7-manifolds with $G_2$-structures. In the process of proving Theorem 2.21, we will observe an obstruction to the local existence of coassociative 4-folds:

**Theorem 2.19 (Local Obstruction to Coassociatives):** If a coassociative 4-fold $\Sigma$ exists in $M$, then the following relation holds at points of $\Sigma$:

$$\tau_0 = -\frac{\sqrt{42}}{7} [\tau_3, 0, 0]$$

where $\dagger$ is an isometric isomorphism defined in (2.17).

In particular, if $\tau_3 = 0$ and $\tau_0$ is non-vanishing, then $M$ admits no coassociative 4-folds (even locally).

**Corollary 2.20:** Fix $x \in M$. If every coassociative 4-plane in $T_x M$ is tangent to a coassociative 4-fold, then $\tau_0|_x = 0$ and $\tau_3|_x = 0$.

Note that Theorem 2.19 generalizes the well-known fact that nearly-parallel $G_2$-structures (viz., those with $\tau_1 = \tau_2 = \tau_3 = 0$ and $\tau_0$ non-vanishing) admit no coassociative 4-folds.

### 1.2 Results on Cayley 4-folds

Let $(M^8, \Phi)$ be an 8-manifold with a Spin(7)-structure $\Phi \in \Omega^4(M)$. The first-order local invariants of $\Phi$ are completely encoded in two differential forms, $\tau_1 \in \Omega^1(M)$ and $\tau_3 \in \Omega^3(M)$, called the *torsion forms* of the Spin(7)-structure. They are defined by equation

$$d\Phi = \tau_1 \wedge \Phi + *\tau_3.$$

To study Cayley 4-folds in $M$, we will break the torsion forms into Spin$^h(4)$-irreducible pieces with respect to a certain splitting of $TM$, where here Spin$^h(4) = (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ is the stabilizer of a Cayley 4-plane. Indeed, in §3.3, we will decompose $\tau_1$ and $\tau_3$ into irreducible pieces, writing

$$\tau_1 = (\tau_1)_K + (\tau_1)_L$$
$$\tau_3 = (\tau_3)_K + (\tau_3)_L + (\tau_3)_{0,3,1} + (\tau_3)_{2,1,1} + (\tau_3)_{1,3,0} + (\tau_3)_{1,1,2}$$

and refer to the individual pieces

$$(\tau_1)_K, (\tau_1)_L, (\tau_3)_K, (\tau_3)_L, (\tau_3)_{0,3,1}, (\tau_3)_{2,1,1}, (\tau_3)_{1,3,0}$$

as *refined torsion forms* (with respect to a certain splitting of $TM$). As with associative and coassociative submanifolds, the mean curvature of Cayley submanifolds can be expressed purely in terms of the refined torsion. As before, we let $\sharp: T^*M \rightarrow TM$ denote the musical isomorphism.

**Theorem 3.12 (Mean Curvature of Cayleys):** The mean curvature vector $H$ of a Cayley 4-fold in $M$ is given by

$$H = -[\tau_1]_L^\sharp - \frac{\sqrt{42}}{7} [\tau_3]_L$$

where $\dagger$ is a particular isometric isomorphism defined in Definition 3.11.

In particular, a Spin(7)-structure on $M$ has the property that every Cayley 4-fold in $M$ is minimal if and only if $d\Phi = 0$. 
1.3 Organization

In §2, we study associative 3-folds and coassociative 4-folds in 7-manifolds with $G_2$-structures. We use §2.1 to recall basic aspects of $G_2$ geometry and set notation. In §2.2, we will decompose various $G_2$-modules (viz., $\Lambda^k(\mathbb{R}^7)$ and $\text{Sym}^2(\mathbb{R}^7)$) into $SO(4)$-irreducible pieces, where we think of $SO(4)$ as the stabilizer of an associative (or coassociative) plane. These $SO(4)$-submodules are central to the geometry of associative and coassociative submanifolds, and we will take care to provide explicit descriptions of these submodules as much as possible.

In §2.3, we will define the refined torsion forms by way of the $SO(4)$-irreducible decompositions obtained in §2.2. Once these refined torsion forms are defined, we will express them in terms of a local $SO(4)$-frame in order to perform calculations.

Finally, in §2.4 we will apply this machinery to the study of associative 3-folds, proving Theorem 2.18. Similarly, in §2.5, we turn to coassociative 4-folds, proving Theorem 2.19, Corollary 2.20, and Theorem 2.21.

The structure of §3 is completely analogous. In brief, we use §3.1 to recall the basic aspects of $\text{Spin}(7)$ geometry and set notation. In §3.2, we decompose various $\text{Spin}(7)$-modules into $\text{Spin}^h(4)$-representations, where we continue to write $\text{Spin}^h(4) = (\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2$, regarded as the stabilizer of a Cayley 4-plane. In §3.3, we will define the corresponding refined torsion forms by way of $\text{Spin}^h(4)$-representation theory, and then express them in terms of a local $\text{Spin}^h(4)$-frame. Finally, in §3.4, we will study Cayley 4-folds and prove Theorem 3.12.

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After this work was completed, it was pointed out to us that the mean curvature formula for associative 3-folds, Theorem 2.18, was derived earlier by Paul Reynolds in his 2011 PhD thesis [10]. In that work, Reynolds also derived an expression for the mean curvature of Cayley 4-folds, but his expression is not completely split into irreducible pieces.

2 Associative 3-Folds and Coassociative 4-Folds in $G_2$-Structures

Our goal in this section is to derive formulas (Theorems 2.18 and 2.21) for the mean curvature of associative 3-folds and coassociative 4-folds in 7-manifolds equipped with a $G_2$-structure. We will also derive an obstruction (Theorem 2.19) to the local existence of coassociative 4-folds.

These formulas and obstructions will be phrased in terms of refined torsion forms, which we will define in §2.3. These refined forms are essentially the $SO(4)$-irreducible pieces of the usual torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ of a $G_2$-structure. As such, we will devote §2.2 to the relevant $SO(4)$-representation theory needed to decompose $\tau_0, \tau_1, \tau_2, \tau_3$.

2.1 Preliminaries

In this subsection, we define both the ambient spaces (in §2.1.2) and submanifolds (in §2.1.3) of interest. We also use this subsection to fix notation and clarify conventions.


2.1.1 $G_2$-Structures on Vector Spaces

Let $V = \mathbb{R}^7$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, and an orientation. Let $\{e_1, \ldots, e_7\}$ denote the standard (orthonormal) basis of $V$, and let $\{e^1, \ldots, e^7\}$ denote the corresponding dual basis of $V^*$. The **associative 3-form** is the alternating 3-form $\phi_0 \in \Lambda^3(V^*)$ defined by

$$\phi_0 = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56})$$

The **coassociative 4-form** is the alternating 4-form $*\phi_0 \in \Lambda^4(V^*)$ given by the Hodge dual $*$ of $\phi_0$. Explicitly:

$$*\phi_0 = e^{4567} + e^{23} \wedge (e^{45} + e^{67}) + e^{13} \wedge (-e^{46} + e^{57}) + e^{12} \wedge (-e^{47} - e^{56}).$$

For calculations, it will be convenient to express $\phi_0$ and $*\phi_0$ in the form

$$\phi_0 = \frac{1}{6} \epsilon_{ijk} e^{ijk} \quad \quad *\phi_0 = \frac{1}{24} \epsilon_{ijkl} e^{ijkl}$$

where the constants $\epsilon_{ijk}, \epsilon_{ijkl} \in \{-1, 0, 1\}$ are defined by this formula. For example, $\epsilon_{123} = \epsilon_{145} = 1$ and $\epsilon_{347} = \epsilon_{356} = -1$. Identities involving the $\epsilon$-symbols are given in [5].

**Remark.** The associative and coassociative forms admit simple descriptions via the algebra of the octonions $\mathbb{O}$.

Equip $\mathbb{O} \simeq \mathbb{R}^8$ with the standard (euclidean) inner product and split $\mathbb{O} = \text{Re}(\mathbb{O}) \oplus \text{Im}(\mathbb{O}) \simeq \mathbb{R} \oplus \mathbb{R}^7$, where $\text{Re}(\mathbb{O}):= \text{span}_{\mathbb{R}}(1)$ is the real line and $\text{Im}(\mathbb{O}) := \text{Re}(\mathbb{O})^\perp$ is its orthogonal complement. Under the identification $V \simeq \text{Im}(\mathbb{O})$, the associative and coassociative forms are given by

$$\phi_0(x, y, z) = \langle x, y \times z \rangle$$

$$*\phi_0(x, y, z, w) = \frac{1}{2} \langle x, [y, z, w] \rangle,$$

for $x, y, z \in V$, where $y \times z := \text{Im}(\overline{z}y) = \frac{1}{2}(\overline{z}y - \overline{y}z)$ is the octonionic cross product, and $[y, z, w] := (yz)w - y(zw)$ is the **associator**, measuring the failure of associativity of multiplication in $\text{Im}(\mathbb{O})$. See [9] for a proof.

Consider the $\text{GL}(V)$-action on $\Lambda^3(V^*)$ given by pullback: $A \cdot \gamma := A^* \gamma$ for $A \in \text{GL}(V)$ and $\gamma \in \Lambda^3(V^*)$. It is a classical result of Schouten (see [Bryant 87] for a proof) that the stabilizer of $\phi_0 \in \Lambda^3(V^*)$ is the compact Lie group $G_2$, i.e.:

$$G_2 \cong \{ A \in \text{GL}(V) : A^* \phi_0 = \phi_0 \}.$$ We let $\Lambda^3_+(V^*)$ denote the orbit of $\phi_0 \in \Lambda^3(V^*)$ under this $G_2$-action, i.e.:

$$\Lambda^3_+(V^*) := \{ A^* \phi_0 : A \in \text{GL}(V) \} \cong \frac{\text{GL}(V)}{G_2}.$$ In [5], it is noted that $\Lambda^3_+(V^*) \subset \Lambda^3(V^*) \simeq \mathbb{R}^{35}$ is an open subset with two connected components, each diffeomorphic to $\mathbb{RP}^7 \times \mathbb{R}^{28}$.

The isomorphism $G_2 \cong \{ A \in \text{GL}(V) : A^* \phi_0 = \phi_0 \}$ lets us regard $G_2$ as a subgroup of $\text{GL}(V)$, which in turn lets us view $V \simeq \mathbb{R}^7$ as a faithful $G_2$-representation. It can be shown (see [Bryant 87]) that this $G_2$-representation is irreducible.

However, the induced $G_2$-representations on $\Lambda^k(V^*)$ for $2 \leq k \leq 5$ are not irreducible. Indeed, $\Lambda^2(V^*)$ decomposes into irreducible $G_2$-modules as

$$\Lambda^2(V^*) = \Lambda_2^2 \oplus \Lambda_1^2,$$
where
\[ \Lambda^2 = \{ \beta \in \Lambda^2(V^*) : \ast (\phi_0 \wedge \beta) = 2\beta \} \]
\[ \Lambda^2_{14} = \{ \beta \in \Lambda^2(V^*) : \ast (\phi_0 \wedge \beta) = -\beta \} \]

Similarly, \( \Lambda^3(V^*) \) decomposes into irreducible \( G_2 \)-modules as
\[ \Lambda^3(V^*) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} \]

where
\[ \Lambda^3_1 = \mathbb{R}\phi_0 \]
\[ \Lambda^3_7 = \{ \ast (\alpha \wedge \phi_0) : \alpha \in \Lambda^1 \} \]
\[ \Lambda^3_{27} = \{ \gamma \in \Lambda^3 : \gamma \wedge \phi_0 = 0 \text{ and } \gamma \wedge \ast \phi_0 = 0 \}. \]

In each case, \( \Lambda^k_\ell \) is an irreducible \( G_2 \)-module of dimension \( \ell \). Via the Hodge star \( \ast : \Lambda^k(V^*) \to \Lambda^{7-k}(V^*) \), one can obtain similar decompositions of \( \Lambda^4(V^*) \) and \( \Lambda^5(V^*) \).

In the sequel, we will always equip \( \Lambda^k(V^*) \) with the usual inner product, also denoted \( \langle \cdot, \cdot \rangle \), given by declaring
\[ \{ e^I : I \text{ increasing multi-index} \} \]

(2.1)
to be an orthonormal basis. We let \( \| \cdot \| \) denote the corresponding norm.

For our calculations in \( \S 2.2 \), we will need the \( G_2 \)-equivariant map \( i \), defined on decomposable elements of \( \text{Sym}_2^2(V^*) \) as follows:
\[ i : \text{Sym}_2^2(V^*) \to \Lambda^3(V^*) \]
\[ i(\alpha \circ \beta) = \alpha \wedge \ast (\beta \wedge \ast \phi_0) + \beta \wedge \ast (\alpha \wedge \ast \phi_0). \]

(2.2)
It is shown in [5] that the image of \( i \) is \( \Lambda^3_{27} \), so that the map with restricted image \( i : \text{Sym}_2^2(V^*) \to \Lambda^3_{27} \) is an isomorphism of \( G_2 \)-modules. It is also remarked that with respect to the orthonormal basis \( \{ e^1, \ldots, e^7 \} \) of \( V^* \), one has
\[ i(h_{ij} e^i \circ e^j) = \epsilon_{ijk} h_{ij} e^{jk}. \]

To invert \( i \), one can use the map
\[ j : \Lambda^3_{27}(V^*) \to \text{Sym}_2^2(V^*) \]
\[ j(\gamma)(v, w) = \ast (\iota_v \phi_0 \wedge \iota_w \phi_0 \wedge \gamma) \]

which satisfies \( j \circ i = 8 \text{ Id}_{\text{Sym}_2^2(V^*)} \).

Finally, we remark that from the associative 3-form \( \phi_0 \), one can recover the inner product \( \langle \cdot, \cdot \rangle \) and volume form \( \text{vol} = e^{1-7} \) via
\[ \langle X, Y \rangle \text{vol} = \frac{1}{6} (\iota_X \phi_0 \wedge (\iota_Y \phi_0) \wedge \phi_0 \]n (2.3a)
\[ \text{vol} = \phi_0 \wedge \ast \phi_0. \]n (2.3b)

From these identities, one can show that, in fact, \( G_2 \) preserves both \( \langle \cdot, \cdot \rangle \) and the orientation on \( V \), so we may regard \( G_2 \leq \text{SO}(V, \langle \cdot, \cdot \rangle) \simeq \text{SO}(7) \).
### 2.1.2 $G_2$-Structures on 7-Manifolds

**Definition 2.1.** Let $M$ be an oriented 7-manifold. A $G_2$-structure on $M$ is a differential 3-form $\varphi \in \Omega^3(M)$ such that $\varphi|_x \in \Lambda^3_x(T^*_x M)$ at each $x \in M$. That is, at each $x \in M$, there exists a coframe $u: T_x M \to \mathbb{R}^7$ for which $\varphi|_x = u^*(\phi_0)$.

Intuitively, a $G_2$-structure is a smooth identification of each tangent space $T_x M$ with $\text{Im}(\mathbb{O})$ in such a way that $\varphi|_x$ and $\phi_0$ are aligned: $(T_x M, \varphi|_x) \simeq (\text{Im}(\mathbb{O}), \phi_0)$. We remark that a 7-manifold $M$ admits a $G_2$-structure if and only if it is orientable and spin: see [5] for a proof.

Every $G_2$-structure $\varphi$ on $M$ induces a Riemannian metric $g_\varphi$ and an orientation form $\text{vol}_\varphi$ on $M$ via the formulas (2.3), reflecting the inclusion $G_2 \leq \text{SO}(7)$. We caution, however, that the association $\varphi \mapsto g_\varphi$ is not injective: different $G_2$-structures may induce the same Riemannian metric. For a discussion of this point, see [5].

The first-order local invariants of a $G_2$-structure are completely encoded in a certain $G_2$-equivariant function

$$T: F_{G_2} \to \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27} \simeq \mathbb{R}^{49}$$

called the *intrinsic torsion function*, defined on the total space of the $G_2$-frame bundle $F_{G_2} \to M$ over $M$. We think of $T$ as describing the 1-jet of the $G_2$-structure.

The intrinsic torsion function is somewhat technical to define — the interested reader can find more information in [7] and [12] — but several equivalent reformulations are available. Most conveniently for our purposes: the intrinsic torsion function of a $G_2$-structure is equivalent to the data of the 4-form $d\varphi$ and the 5-form $d*\varphi$. In [5], the exterior derivatives of $\varphi$ and $*\varphi$ are shown to take the form

$$d\varphi = \tau_0 *\varphi + 3\tau_1 \wedge \varphi + *\tau_3 \quad \text{(2.4a)}$$
$$d*\varphi = 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi. \quad \text{(2.4b)}$$

where

$$(\tau_0, \tau_1, \tau_2, \tau_3) \in \Gamma(\Lambda^0(T^* M) \oplus \Lambda^1(T^* M) \oplus \Lambda^2_{14}(T^* M) \oplus \Lambda^3_{27}(T^* M))$$

We refer to $\tau_0, \tau_1, \tau_2, \tau_3$ as the *torsion forms* of the $G_2$-structure.

Following standard conventions, we let $W_1, W_7, W_{14}, W_{27}$ denote the vector bundles $\Lambda^0(T^* M), \Lambda^1(T^* M), \Lambda^2_{14}(T^* M), \Lambda^3_{27}(T^* M)$, respectively. Consider the set $\mathcal{S}$ consisting of the $2^4 = 16$ vector bundles

$$\mathcal{S} = \{0, W_i, W_i \oplus W_j, W_i \oplus W_j \oplus W_k, W_i \oplus W_j \oplus W_{14} \oplus W_{27}: i, j, k \in \{1, 7, 14, 27\}\}.$$

**Definition 2.2.** Let $E \in \mathcal{S}$ be a vector bundle on the list above. We say that a $G_2$-structure belongs to the *torsion class* $E$ if and only if the torsion forms of the $G_2$-structure $(\tau_0, \tau_1, \tau_2, \tau_3) \in \Gamma(W_1 \oplus W_7 \oplus W_{14} \oplus W_{27})$ is valued in $E \subset W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$.

For example, a $G_2$-structure belongs to the torsion class $W_7 \oplus W_{27}$ if and only if $\tau_0 = \tau_2 = 0$.

### 2.1.3 Associative 3-Folds and Coassociative 4-Folds

Let $(M^7, \varphi)$ be a 7-manifold with a $G_2$-structure, and consider a tangent space $(T_x M, \varphi|_x) \simeq (V, \phi_0)$. The vector space $(V, \phi_0)$ possesses two distinguished classes of subspaces — associative 3-planes and coassociative 4-planes (to be defined shortly) — first studied by Harvey and Lawson [9] in their work on calibrations. Indeed, they observed that $\phi_0$ and $*\phi_0$ enjoy the following remarkable property:
Proposition 2.3 ([9]). The associative 3-form $\phi_0$ and coassociative 4-form $*\phi_0$ have co-mass one, meaning that

$$|\phi_0(x, y, z)| \leq 1$$

for every orthonormal set $\{x, y, z, w\}$ in $V \simeq \mathbb{R}^7$.

In light of this proposition, it is natural to examine more closely those 3-planes $A \in \text{Gr}_3(V)$ (respectively, 4-planes $C \in \text{Gr}_4(V)$) for which $|\phi_0(A)| = 1$ (resp., $|*\phi_0(C)| = 1$).

Proposition 2.4 ([9]). Let $A \in \text{Gr}_3(V)$ be a 3-plane in $V$. The following are equivalent:
1. If $\{u, v, w\}$ orthonormal basis of $A$, then $\phi_0(u, v, w) = \pm 1$.
2. For all $u, v, w \in A$, we have $[u, v, w] = 0$.
3. $A = \text{span}\{u, v, u \times v\}$ for some linearly independent set $\{u, v\}$.
If any of these conditions hold, we say that $A$ is an associative 3-plane.

Proposition 2.5 ([9]). Let $C \in \text{Gr}_4(V)$ be a 4-plane in $V$. The following are equivalent:
1. If $\{x, y, z, w\}$ is an orthonormal basis of $C$, then $*\phi_0(x, y, z, w) = \pm 1$.
2. $C^\perp$ is associative.
3. $\phi_0|_C = 0$.
If any of these conditions hold, we say that $C$ is a coassociative 4-plane.

Proofs of the above propositions can be found in [9] and [11].

The $G_2$-action on $V$ induces $G_2$-actions on the Grassmannians $\text{Gr}_k(V)$ of $k$-planes in $V$. While these actions are transitive for $k = 1, 2, 5, 6$, they are not transitive for $k = 3, 4$: indeed, the (proper) subsets consisting of associative 3-planes and coassociative 4-planes are $G_2$-orbits. The corresponding stabilizer, recorded in the following proposition, will play a crucial role in this work:

Proposition 2.6 ([9]). The Lie group $G_2$ acts transitively on the subset of associative 3-planes and on the subset of coassociative 4-planes:

$$\{E \in \text{Gr}_3(V) : |\phi_0(E)| = 1\} \subset \text{Gr}_3(V),$$

$$\{E \in \text{Gr}_4(V) : |*\phi_0(E)| = 1\} \subset \text{Gr}_4(V).$$

In both cases, the stabilizer of the $G_2$-action is isomorphic to $\text{SO}(4)$.

We may finally define our primary objects of interest:

Definition 2.7. Let $(M^7, \varphi)$ be a 7-manifold equipped with a $G_2$-structure $\varphi$. Identify each tangent space $(T_x M, \varphi|_x) \simeq (V, \phi_0)$.

An associative 3-fold in $M$ is a 3-dimensional immersed submanifold $\Sigma \subset M$ for which each tangent space $T_x \Sigma \subset T_x M$ is an associative 3-plane.

Similarly, a coassociative 4-fold in $M$ is a 4-dimensional immersed submanifold $\Sigma \subset M$ for which each tangent space $T_x \Sigma \subset T_x M$ is a coassociative 4-plane.

Note that if $d\varphi = 0$, then $\varphi$ is a calibration whose calibrated 3-planes are the associative 3-planes in $T_x M$. Thus, in this case, an associative 3-fold is a calibrated submanifold, and hence a minimal submanifold of $M$.

Similarly, if $d*\varphi = 0$, then $*\varphi$ is a calibration whose calibrated 4-planes are the coassociative 4-planes in $T_x M$. Thus, in this case, a coassociative 4-fold is a calibrated submanifold, and hence a minimal submanifold of $M$. The “Minimality Problem” described in the introduction asks about the converses of these claims.
2.2 Some SO(4)-Representation Theory

In this subsection, we describe the aspects of SO(4)-representation theory that are relevant to the study of associative 3-folds and coassociative 4-folds. Particularly important for our purposes are SO(4)-irreducible decompositions of \( \Lambda^1(\mathbb{R}^7) \), \( \Lambda^2(\mathbb{R}^7) \), \( \text{Sym}^2(\mathbb{R}^7) \), and \( \Lambda^3(\mathbb{R}^7) \), each of which we will describe in turn.

To begin, recall that the compact Lie group SO(4) is double-covered by the simply-connected group SU(2) \( \times \) SU(2). The complex irreducible representations of SU(2) \( \times \) SU(2) are exactly the tensor products \( V_p \otimes V_q \) of irreducible SU(2)-representations for each factor. The complex irreducible representations of SU(2) are well known to be the spaces of homogeneous polynomials in two variables of fixed degree, \( V_p = \text{Sym}^p(\mathbb{C}(x, y)) \).

Let \( V_{p,q}^\mathbb{C} \) denote \( V_p \otimes V_q \). We think of \( V_{p,q}^\mathbb{C} \) as the space of homogeneous polynomials in \( (x, y; w, z) \) of bidegree \( (p, q) \). When \( p \) and \( q \) have the same parity the representation \( V_{p,q}^\mathbb{C} \) descends to a representation of SO(4), and each of these representations has a real structure induced by the map \( (x, y, w, z) \mapsto (y, -x, z, -w) \). This yields a complete description of the real representations of SO(4).

We work with real representations, letting \( V_{p,q} \) denote the real representation underlying \( V_{p,q}^\mathbb{C} \). In this language, the standard 4-dimensional representation of SO(4) is \( V_{1,1} \), while the adjoint representation \( \mathfrak{so}(4) = V_{2,0} \oplus V_{0,2} \). The ordering of the subscripts is chosen so that the representation \( \Lambda^2(\mathbb{R}^4) \) of SO(4) on the self-dual 2-forms is \( V_{0,2} \).

The Clebsch-Gordan formula applied to each SU(2) representation gives the irreducible decomposition of a tensor product of SO(4)-modules:

\[
V_{p_1,q_1} \otimes V_{p_2,q_2} \cong \bigoplus_{i=0}^{[p_1-p_2]} \bigoplus_{j=0}^{[q_1-q_2]} V_{p_1+p_2-2i, q_1+q_2-2j}.
\]

2.2.1 SO(4) as a subgroup of G2

In our calculations we shall need a concrete realization of SO(4) as the stabilizer of an associative or coassociative plane. Let SO(4) act on \( V \cong \mathbb{R}^7 \) via the identification \( V \cong V_{0,2} \oplus V_{1,1} \), and let \( (e_1, \ldots, e_7) \) be an orthonormal basis of \( V \) such that:

- \( \langle e_1, e_2, e_3 \rangle \cong V_{0,2} \) and \( \langle e_4, e_5, e_6, e_7 \rangle \cong V_{1,1} \),
- The map

\[
e_1 \mapsto e_{45} + e_{67}, \quad e_2 \mapsto e_{46} - e_{57}, \quad e_3 \mapsto -e_{47} - e_{56}
\]

is SO(4)-equivariant.

Then the 3-form

\[
e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56})
\]

is SO(4)-invariant, and thus the action of SO(4) on \( V \) gives an embedding SO(4) \( \subset \) G2. The 3-plane \( \langle e_1, e_2, e_3 \rangle \) is associative and preserved by the action of SO(4), while the 4-plane \( \langle e_4, e_5, e_6, e_7 \rangle \) is coassociative and preserved by the action of SO(4).

2.2.2 Decomposition of 1-Forms on \( V^* \)

Let \( V \) be as above. The decomposition of \( \Lambda^1(V^*) \) into irreducible SO(4)-modules is simply

\[
\Lambda^1(V^*) = A \oplus C,
\]

10
where
\[ A \cong \langle e^1, e^2, e^3 \rangle, \quad C \cong \langle e^4, e^5, e^6, e^7 \rangle. \]

As abstract SO(4)-modules, we have isomorphisms \( A \cong V_{0,2} \) and \( C \cong V_{1,1} \).

**Notation:** We let \( b : V \rightarrow V^* \) via \( X^\flat := \langle X, \cdot \rangle \) denote the usual musical (index-lowering) isomorphism induced by the inner product \( \langle \cdot, \cdot \rangle \) on \( V \), and let
\[ \sharp : V^* \rightarrow V \quad (2.5) \]
denote its inverse. In the sequel, we let \( A^\sharp, C^\sharp \subset V \) denote the images of \( A, C \subset V^* \) under the \( \sharp \) isomorphism.

### 2.2.3 Decomposition of 2-Forms on \( V^* \)

We now seek to decompose \( \Lambda^2(V^*) \) into SO(4)-irreducible submodules. As noted in §2.1 above, \( \Lambda^2(V^*) \) splits into \( G_2 \)-irreducible submodules as
\[ \Lambda^2(V^*) \cong \Lambda^2_2 \oplus \Lambda^2_{14} \quad (2.6) \]
On the other hand, using \( V^* = A \oplus C \), we may also decompose \( \Lambda^2(V^*) \) as
\[ \Lambda^2(V^*) \cong \Lambda^2(A) \oplus (A \otimes C) \oplus \Lambda^2_4(C) \oplus \Lambda^2_7(C). \quad (2.7) \]
We will refine both decompositions (2.6) and (2.7) into SO(4)-submodules.

To begin, note first that as SO(4)-modules, we have that \( \Lambda^2_2(A) \cong \Lambda^2_{+}(C) \cong V_{0,2} \) and \( \Lambda^2_7(C) \cong V_{2,0} \) are irreducible. Thus, it remains only to decompose \( \Lambda^2_2, \Lambda^2_{14}, \) and \( A \otimes C \).

**Definition 2.8.** We define
\[
\begin{align*}
(\Lambda^2_2)_{A} & := \Lambda^2_2 \cap (\Lambda^2(A) \oplus \Lambda^2(C)) = \{ \iota_X \phi_0 : X \in A^\sharp \} \\
(\Lambda^2_2)_{C} & := \Lambda^2_2 \cap (A \otimes C) = \{ \iota_X \phi_0 : X \in C^\sharp \} \\
(\Lambda^2_{14})_{A} & := \Lambda^2_{14} \cap (\Lambda^2(A) \oplus \Lambda^2_{+}(C)) \\
(\Lambda^2_{14})_{1,3} & := \Lambda^2_{14} \cap (A \otimes C) \\
(\Lambda^2_{14})_{2,0} & := \Lambda^2_{14} \cap \Lambda^2(C).
\end{align*}
\]

The reader can check that, in fact, \( \Lambda^2_2(C) \subset \Lambda^2_{14} \), so that \( (\Lambda^2_{14})_{2,0} = \Lambda^2_{14} \cap \Lambda^2(C) = \Lambda^2_2(C). \)

Consider the SO(4)-module isomorphism
\[
L : A \rightarrow (\Lambda^2_2)_{A} \\
L(\alpha) = \iota_{\alpha^\sharp} \phi_0 = *(\alpha \wedge *\phi_0).
\]
For \( \beta \in \Lambda^2(V^*) \), write \( \beta = \beta|_{\Lambda^2_2(A)} + \beta|_{A \otimes C} + \beta|_{\Lambda^2_7(C)} \), where \( \beta|_E \in E \) for \( E \in \{ \Lambda^2_2(A), A \otimes C, \Lambda^2(C) \} \).
Define SO(4)-equivariant maps
\[
L_A : A \rightarrow \Lambda^2(A) \\
L_A(\alpha) = L(\alpha)|_{\Lambda^2(A)} \\
L_C : A \rightarrow \Lambda^2_7(C) \\
L_C(\alpha) = L(\alpha)|_{\Lambda^2_7(C)}
\]
It is straightforward to check that $L_A$ and $L_C$ are well-defined SO(4)-module isomorphisms, and that $L_A = *_A$ coincides with the usual Hodge star operator on $A$. Finally, we define the map

$$W: A \to (\Lambda^2_{14})_A$$
$$W(\alpha) = 2L_A(\alpha) - L_C(\alpha)$$

Again, the reader can check that $W$ is a well-defined SO(4)-module isomorphism. We caution that the maps $L$, $L_C$, and $W$ are not isometries.

**Lemma 2.9.** The decompositions

$$\Lambda^2_7 = (\Lambda^2_7)_A \oplus (\Lambda^2_7)_C$$
$$\Lambda^2_{14} = (\Lambda^2_{14})_A \oplus (\Lambda^2_{14})_{1,3} \oplus (\Lambda^2_{14})_{2,0}$$
$$A \otimes C = (\Lambda^2^*_C) \oplus (\Lambda^2^*_{14})_{1,3}$$

**Lemma 2.9.** The decompositions consist of SO(4)-irreducible submodules.

Thus, the decomposition

$$\Lambda^2(V) = [(\Lambda^2_7)_A \oplus (\Lambda^2_7)_C] \oplus [(\Lambda^2_{14})_A \oplus (\Lambda^2_{14})_{1,3} \oplus (\Lambda^2_{14})_{2,0}]$$

is SO(4)-irreducible and refines (2.6), while

$$\Lambda^2(V) = \Lambda^2(A) \oplus [(\Lambda^2^*_C) \oplus (\Lambda^2^*_{14})_{1,3}] \oplus \Lambda^2_3(C) \oplus \Lambda^2_7(C)$$

is SO(4)-irreducible and refines (2.7).

**Proof.** The decomposition (2.8) follows from the isomorphism $V \to \Lambda^2_7$, $X \mapsto \iota_X(\varphi_0)$ and the irreducible decomposition $V \cong A \oplus C$.

By a dimension count, the SO(4)-invariant subspace $\Lambda^2_{14}$ of $\Lambda^2(V^*)$ must be isomorphic to the SO(4)-module $V_{0,2} \oplus V_{2,0} \oplus V_{1,3}$, while, by the Clebsch-Gordan formula, the space $A \otimes C$ is isomorphic to $V_{1,3} \oplus V_{1,1}$. It follows from Schur’s lemma that the space $(\Lambda^2_{14})_{1,3}$ is isomorphic to the SO(4)-module $V_{1,3}$. The space $(\Lambda^2_{14})_A$ is the image of $A$ under the isomorphism $W$ defined above, so it is an irreducible SO(4)-module, while the space $(\Lambda^2_{14})_{2,0}$ is isomorphic to $\Lambda^2_7(C) \cong V_{2,0}$ so it is an irreducible SO(4)-module. Thus, the decomposition (2.9) consists of irreducible SO(4)-modules.

To see that (2.10) is an irreducible decomposition, note that we have already shown that both $(\Lambda^2_7)_C$ and $(\Lambda^2_{14})_{1,3}$ are irreducible SO(4)-modules.

**Definition 2.10.** We define the map

$$\iota: (\Lambda^2_{14})_A \to A^*$$
$$\beta \mapsto \beta^* = \sqrt{6}(W^{-1}(\beta)^{\sharp})$$

The map $\iota$ is an SO(4)-module isomorphism, and (because of the factor of $\sqrt{6}$) an isometry with respect to the inner product (2.1) on $\Lambda^2(V^*)$.

### 2.2.4 Decomposition of the Quadratic Forms on $V^*$

Before turning to the decomposition of $\Lambda^3(V^*)$, we take a moment to decompose $\text{Sym}^2(V^*)$ into SO(4)-irreducible pieces. To this end, we first use $V^* = A \oplus C$ to split

$$\text{Sym}^2(V^*) \cong \mathbb{R}\text{Id}_A \oplus \text{Sym}^2_0(A) \oplus (A \otimes C) \oplus \mathbb{R}\text{Id}_C \oplus \text{Sym}^2_0(C).$$
Each of these summands is $SO(4)$-irreducible, with the exception of $A$-irreducible summands as

$$A \otimes C \cong (A \otimes C)_{1,3} \oplus (A \otimes C)_C$$

where $(A \otimes C)_{1,3}$ and $(A \otimes C)_C$ are submodules isomorphic to $V_{1,3}$ and $C$, respectively.

Note that we are employing a slight abuse of notation. That is, in §2.2.1 we used $A \otimes C$ to denote a submodule of $\Lambda^2(V^*)$, whereas here in §2.2.2 we are using the same symbol $A \otimes C$ to denote a submodule of $\text{Sym}^2(V^*)$. Abstractly, these two $SO(4)$-modules are isomorphic, as are their irreducible summands. By Schur’s Lemma, there is a one-dimensional family of $SO(4)$-module isomorphisms $(\Lambda^3_0)_C \cong (A \otimes C)_C$ and $(\Lambda^3_{14})_{1,3} \cong (A \otimes C)_{1,3}$.

For computations, we will make use of the particular $SO(4)$-module isomorphism

$$s: (\Lambda^3_0)_C \to (A \otimes C)_C$$

defined as follows. In terms of a basis $\{e^1, \ldots, e^6\}$ of $V^*$ with $A = \text{span}(e^1, e^2, e^3)$ and $C = \text{span}(e^4, e^5, e^6)$, the map $s$ will formally replace $\wedge$ symbols with $\circ$ symbols in each $e^i \wedge e^j$ term with $i < j$. So, for example,

$$s(-e^{15} - e^{26} + e^{37}) = -e^1 \circ e^5 - e^2 \circ e^6 + e^3 \circ e^7.$$

Finally, we remark that $\text{Sym}^2_0(V^*)$ decomposes into irreducible $SO(4)$-modules as

$$\text{Sym}^2_0(V^*) = \text{Sym}^2_0(A) \oplus (A \otimes C)_{1,3} \oplus (A \otimes C)_C \oplus \text{Sym}^2_0(C) \oplus \mathbb{R}E_0,$$  

(2.13)

where

$$E_0 = \text{diag}(4, 4, 4, -3, -3, -3, -3) \in \text{Sym}^2_0(V^*).$$  

(2.14)

### 2.2.5 Decomposition of 3-Forms on $V^*$

We now turn to $\Lambda^3(V^*)$. As noted in §2.1, $\Lambda^3(V^*)$ splits into $G_2$-irreducible submodules as

$$\Lambda^3(V^*) \cong \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}.$$  

(2.15)

The summand $\Lambda^3_3 \cong \mathbb{R}$ is $SO(4)$-irreducible, but the summands $\Lambda^3_1$ and $\Lambda^3_{27}$ are not.

On the other hand, using $V^* \cong A \oplus C$, we also have the decomposition:

$$\Lambda^3(V^*) \cong \Lambda^3(A) \oplus (\Lambda^2(A) \otimes C) \oplus (A \otimes \Lambda^2(C)) \oplus (A \otimes \Lambda^2(C)) \oplus \Lambda^3(C).$$  

(2.16)

Three of these summands are $SO(4)$-irreducible, namely $\Lambda^3(A) \cong V_{0,0}$ and $A \otimes \Lambda^2(C) \cong V_{2,2}$ and $\Lambda^3(C) \cong V_{1,1}$. Meanwhile, the second and third summands $\Lambda^2(A) \otimes C$ and $A \otimes \Lambda^2(C)$ are not.

As in §2.2.1 above, we will refine both (2.15) and (2.16) into $SO(4)$-irreducible submodules, though only the refinement of (2.15) will be used in this work. We begin with (2.15).

**Definition 2.11.** Recall the isomorphism $i: \text{Sym}^2_0(V^*) \to \Lambda^3_{27}$ of (2.2) and recall the $SO(4)$-irreducible splitting of $\text{Sym}^2_0(V^*)$ given in (2.13). We define

$$(\Lambda^3_0)_A := \{*(\alpha \wedge \phi_0): \alpha \in A\} \quad (\Lambda^3_0)_{0,0} := i(E_0) \quad (\Lambda^3_{27})_{1,3} := i((A \otimes C)_{1,3})$$

$$(\Lambda^3_1) := \{*(\alpha \wedge \phi_0): \alpha \in A\} \quad (\Lambda^3_{0,4}) := i(\text{Sym}^2_0(A)) \quad (\Lambda^3_{27})_C := i((A \otimes C)_C)$$

$$(\Lambda^3_2) := \{*(\alpha \wedge \phi_0): \alpha \in C\} \quad (\Lambda^3_{2,2}) := i(\text{Sym}^2_0(C))$$
Lemma 2.12. The decompositions
\[
\Lambda^3_7 = (\Lambda^3_7)_A \oplus (\Lambda^3_7)_C
\]
\[
\Lambda^3_{27} = (\Lambda^3_{27},0,0) \oplus (\Lambda^3_{27},0,4) \oplus (\Lambda^3_{27},2,2) \oplus (\Lambda^3_{27},1,3) \oplus (\Lambda^3_{27},C)
\]
consist of SO(4)-irreducible submodules.

Definition 2.13. Recall the isomorphisms i: \( \text{Sym}^2(V^*) \to \Lambda^3_{27} \) of (2.2) and \( s: (\Lambda^3_7)_C \to (A \otimes C)_C \) of (2.12). We define \( \dagger: (\Lambda^3_{27},0,0) \to \mathbb{R} \) to be the unique vector space isomorphism for which
\[
[i(E_0)]^\dagger = 4\sqrt{42}
\]
where \( E_0 \) is as in (2.14). The map \( \dagger \) is an isometry (due to the choice of \( 4\sqrt{42} \)) with respect to the inner products (2.1).

We will also need the composition of SO(4)-module isomorphisms
\[
\mathbb{C}^2 \to (\Lambda^3_7)_C \to (\Lambda^3_{27})_C \\
X \mapsto \iota_X \phi_0 \mapsto \frac{1}{2\sqrt{3}} (i \circ s)(\iota_X \phi_0).
\]

This map is an isometry due to the factor of \( \frac{1}{2\sqrt{3}} \). We denote the inverse of this isometric isomorphism by
\[
\dagger: (\Lambda^3_{27})_C \to \mathbb{C}^2
\]

Remark. Extend the isomorphism \( L_C: A \to \Lambda^2_+ (C) \) to an isomorphism \( L_C: A \otimes A \to A \otimes \Lambda^2_+ (C) \) by the identity on the first \( A \)-factor, and split \( A \otimes A = \mathbb{R} \oplus \text{Sym}^2(A) \oplus \Lambda^2(A) \). Extend the Hodge star operator \( *_A: A \to \Lambda^2(A) \) to an isomorphism \( *_A: A \otimes C \to \Lambda^2(A) \otimes C \) by the identity on the \( C \)-factor, and recall the decomposition \( A \otimes C = (\Lambda^2_7)_C \oplus (\Lambda^2_{14})_{1,3} \).

Defining
\[
(\Lambda^2(A) \otimes C)_C := *_A[(\Lambda^2_7)_C] \quad (A \otimes \Lambda^2_+ (C))_{0,0} := L_C(\mathbb{R}) \n\]
\[
(\Lambda^2(A) \otimes C)_{1,3} := *_A[(\Lambda^2_{14})_{1,3}] \quad (A \otimes \Lambda^2_+ (C))_{0,4} := L_C(\text{Sym}^2(A)) \n\]
\[
(A \otimes \Lambda^2_+ (C))_A := L_C(\Lambda^2(A))
\]
we obtain decompositions
\[
\Lambda^2(A) \otimes C = (\Lambda^2(A) \otimes C)_C \oplus (\Lambda^2(A) \otimes C)_{1,3} \\
A \otimes \Lambda^2_+ (C) = (A \otimes \Lambda^2_+ (C))_{0,0} \oplus (A \otimes \Lambda^2_+ (C))_{0,4} \oplus (A \otimes \Lambda^2_+ (C))_A
\]
consisting of SO(4)-irreducible submodules.

Remark. The reader can check that some of the above submodules of \( \Lambda^3(V^*) \) are, in fact, equal to one another. Namely, we have the equalities
\[
A \otimes \Lambda^2_+ (C) = (\Lambda^3_{27})_{2,2} \quad (A \otimes \Lambda^2_+ (C))_{0,4} = (\Lambda^3_{27})_{0,4} \\
(\Lambda^2(A) \otimes C)_{1,3} = (\Lambda^3_{27})_{1,3} \quad (A \otimes \Lambda^2_+ (C))_A = (\Lambda^3_7)_A
\]
2.3 The Refined Torsion Forms

Let \((M^{7}, \varphi)\) be a 7-manifold equipped with a G\(_{2}\)-structure \(\varphi\). Fix a point \(x \in M\), choose an arbitrary associative 3-plane \(A^{3} \subset T_x M\), and let \(C^{2} \subset T_x M\) denote its orthogonal coassociative 4-plane. Our purpose in this section is to understand how the torsion of the G\(_{2}\)-structure decomposes with respect to the splitting

\[T_x M = A^3 \oplus C^2.\]

In §2.3.1, we use the decompositions of Lemmas 2.9 and 2.12 to break the torsion forms \(\tau_0, \tau_1, \tau_2, \tau_3\) into SO(4)-irreducible pieces called \textit{refined torsion forms}. Separately, in §2.3.2, we set up the G\(_{2}\)-coframe bundle \(\pi: F_{G_2} \to M\) following [5], repackaging the original G\(_{2}\) torsion forms \(\tau_0, \tau_1, \tau_2, \tau_3\) as a matrix-valued function

\[T = (T_{ij}): F_{G_2} \to \text{Mat}_{7 \times 7}(\mathbb{R}) \simeq \mathbb{R}^{49}.\]

Finally, in §2.3.3, we express the functions \(T_{ij}\) in terms of the (pullbacks of the) refined torsion forms.

2.3.1 The Refined Torsion Forms in a Local SO(4)-Frame

Fix \(x \in M\) and split \(T^*_x M = A \oplus C\) as above. All of our calculations in this subsection will be done pointwise, and we will suppress reference to \(x \in M\). By Lemmas 2.9 and 2.12, the torsion forms \(\tau_0, \tau_1, \tau_2, \tau_3\) decompose into SO(4)-irreducible pieces as

\[
\begin{align*}
\tau_0 &= \tau_0 \quad (2.19a) \\
\tau_1 &= (\tau_1)_A + (\tau_1)_C \quad (2.19b) \\
\tau_2 &= (\tau_2)_A + (\tau_2)_{1,3} + (\tau_2)_{2,0} \quad (2.19c) \\
\tau_3 &= (\tau_3)_{0,0} + (\tau_3)_{0,4} + (\tau_3)_{2,2} + (\tau_3)_{1,3} + (\tau_3)_C \quad (2.19d)
\end{align*}
\]

where here

\[
\begin{align*}
(\tau_1)_A &\in A \\
(\tau_1)_C &\in C \\
(\tau_2)_A &\in (\Lambda^3_{1,4})_A \\
(\tau_2)_{1,3} &\in (\Lambda^3_{1,4})_{1,3} \\
(\tau_2)_{2,0} &\in (\Lambda^3_{1,4})_{2,0} \\
(\tau_3)_{0,0} &\in (\Lambda^3_{27})_{0,0} \\
(\tau_3)_{0,4} &\in (\Lambda^3_{27})_{0,4} \\
(\tau_3)_{2,2} &\in (\Lambda^3_{27})_{2,2} \\
(\tau_3)_{1,3} &\in (\Lambda^3_{27})_{1,3} \\
(\tau_3)_C &\in (\Lambda^3_{27})_C
\end{align*}
\]

We refer to \(\tau_0, (\tau_1)_A, (\tau_1)_C, \ldots, (\tau_3)_C\) as the \textit{refined torsion forms} of the G\(_{2}\)-structure at \(x\) relative to the splitting \(T^*_x M = A \oplus C\).

We now move to express the refined torsion forms in terms of a local SO(4)-frame, for which we will need explicit bases of \((\Lambda^3_{1,4})_A, \ldots, (\Lambda^3_{27})_C\). To that end, let \(\{e_1, \ldots, e_7\}\) be an orthonormal basis for \(T_x M\) for which \(A^3 = \text{span}(e_1, e_2, e_3)\) and \(C^2 = \text{span}(e_4, e_5, e_6, e_7)\). Let \(\{e^1, \ldots, e^7\}\) denote the dual basis for \(T^*_x M\).

Index Ranges: We will employ the following index ranges: \(1 \leq p, q \leq 3\) and \(4 \leq \alpha, \beta \leq 7\) and \(1 \leq i, j, k, \ell, m \leq 7\) and \(1 \leq \delta \leq 8\) and \(1 \leq a \leq 5\).
Definition 2.14. Define the 2-forms
\[\begin{align*}
\Upsilon_1 &= e^{45} + e^{67} & \Omega_1 &= e^{45} - e^{67} & \Delta_1 &= e^{17} + e^{24} & \Delta_5 &= e^{16} + e^{34} \\
\Upsilon_2 &= e^{46} - e^{57} & \Omega_2 &= e^{46} + e^{57} & \Delta_2 &= e^{16} + e^{25} & \Delta_6 &= -e^{17} + e^{35} \\
\Upsilon_3 &= -e^{47} - e^{56} & \Omega_3 &= e^{47} - e^{56} & \Delta_3 &= -e^{15} + e^{26} & \Delta_7 &= -e^{14} + e^{36} \\
\Delta_4 &= -e^{14} + e^{36} & \Delta_8 &= e^{15} + e^{37}
\end{align*}\]

We also define
\[\Gamma_p = 2 * A e^p - \Upsilon_p\]
(no summation).

Lemma 2.15. We have that:
(a) \(\{\Gamma_1, \Gamma_2, \Gamma_3\}\) is a basis of \((\Lambda^2_{14})_A\).
(b) \(\{\Delta_1, \ldots, \Delta_8\}\) is a basis of \((\Lambda^2_{14})_{1.3}\).
(c) \(\{\Omega_1, \Omega_2, \Omega_3\}\) is a basis of \((\Lambda^2_{14})_{2.0}\).

Definition 2.16. Define the 3-forms
\[\begin{align*}
\phi_A &= e^{123} \\
\phi_C &= \sum e^p \wedge \Upsilon_p
\end{align*}\]
\[\begin{align*}
\lambda_{pq} &= e^p \wedge \Omega_q & \nu_\alpha &= (i \circ s)(t_{e_\alpha} \phi_0)
\end{align*}\]

and
\[\begin{align*}
\kappa_1 &= e^1 \wedge \Upsilon_2 - e^2 \wedge \Upsilon_1 \\
\kappa_2 &= e^1 \wedge \Upsilon_3 + e^3 \wedge \Upsilon_1 \\
\kappa_3 &= e^2 \wedge \Upsilon_3 + e^3 \wedge \Upsilon_2 \\
\kappa_4 &= e^1 \wedge \Upsilon_1 - e^2 \wedge \Upsilon_2 \\
\kappa_5 &= e^2 \wedge \Upsilon_2 - e^3 \wedge \Upsilon_3.
\end{align*}\]

Note that \(\varphi = \phi_A + \phi_C\).

Lemma 2.17. We have that:
(a) \(\{6 \phi_A - \phi_C\}\) is a basis of \((\Lambda^3_{27})_{0,0}\).
(b) \(\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}\) is a basis of \((\Lambda^3_{27})_{0,4}\).
(c) \(\{\lambda_{pq}: 1 \leq p, q \leq 3\}\) is a basis of \((\Lambda^3_{27})_{2,2}\).
(d) \(\{\mu_1, \ldots, \mu_8\}\) is basis of \((\Lambda^3_{27})_{1,3}\).
(e) \(\{\nu_4, \nu_5, \nu_6, \nu_7\}\) is a basis of \((\Lambda^3_{27})_C\).

We now express \((\tau_1)_A, (\tau_1)_C\), etc., in terms of the above bases. That is, we define functions \(A_p, B_\alpha\) and \(C_p, D_\delta, E_p\) and \(F, G_\alpha, J_{pq}, L_\delta, M_\alpha\) by:
\[\begin{align*}
(\tau_1)_A &= 6A_p e^p & (\tau_2)_A &= 12C_p \Gamma_p & (\tau_3)_{0,0} &= 12 F (6 \phi_A - \phi_C) \\
(\tau_1)_C &= 6B_\alpha e^\alpha & (\tau_2)_{1,3} &= 12D_\delta \Delta_\delta & (\tau_3)_{0,4} &= 6G_\alpha \kappa_\alpha \\
(\tau_2)_{2,0} &= 12E_p \Omega_p & (\tau_3)_{2,2} &= 12J_{pq} \lambda_{pq} & (\tau_3)_{1,3} &= 12L_\delta \mu_\delta \\
(\tau_3)_C &= 6M_\alpha \nu_\alpha
\end{align*}\]

The various factors of 6 and 12 are included simply for the sake of clearing future denominators.
Note that the bases of Lemmas 2.15 and 2.17 are orthogonal but not orthonormal with respect to the inner product (2.1) on $\Lambda^k(V^*)$. Indeed, we have:

$$
\|\Gamma_\rho\| = \sqrt{6} \quad \|\Omega_\rho\| = \sqrt{2} \quad \|\mu_\delta\| = \sqrt{2} \quad \|\kappa_\alpha\| = 2
$$

Thus, in terms of the isometric isomorphisms (2.5), (2.11), (2.17), (2.18) of §2.2, we have:

$$
[(\tau_1)_A]^2 = 6A_p e_p \quad [(\tau_2)_A]^2 = 12\sqrt{6} C_p e_p \quad [(\tau_3)_{0,0}]^\dagger = 12\sqrt{42} F \quad (2.21a)
$$

$$
[(\tau_1)_C]^2 = 6B_\alpha e_\alpha \quad [(\tau_3)_C]^\dagger = 12\sqrt{3} M_\alpha e_\alpha \quad (2.21b)
$$

We will need these for our calculations in §2.4 and §2.5.

### 2.3.2 The Torsion Functions $T_{ij}$

Let $(M^7, \varphi)$ be a 7-manifold with a $G_2$-structure $\varphi$, and let $g_\varphi$ denote the underlying Riemannian metric. Let $F_{SO(7)} \to M$ denote the oriented orthonormal coframe bundle of $g_\varphi$, and let $\omega = (\omega^1, \ldots, \omega^7) \in \Omega^1(F_{SO(7)}; \mathbb{R}^7)$ denote the tautological 1-form. By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form $\psi \in \Omega^1(F_{SO(7)}; so(7))$, the Levi-Civita connection form of $g_\varphi$, satisfying the First Structure Equation

$$
d\omega = -\psi \wedge \omega.
$$

Let $\pi: F_{G_2} \to M$ denote the $G_2$-coframe bundle of $M$. Restricted to $F_{G_2} \subset F_{SO(7)}$, the Levi-Civita 1-form $\psi$ is no longer a connection 1-form in general. Indeed, according to the splitting $so(7) = g_2 \oplus \mathbb{R}^7$, we have the decomposition

$$
\psi = \theta + 2\gamma,
$$

where $\theta = (\theta_{ij}) \in \Omega^1(F_{G_2}; g_2)$ is a connection 1-form (the so-called natural connection of the $G_2$-structure $\varphi$) and $\gamma \in \Omega^1(F_{G_2}; \mathbb{R}^7)$ is a $\pi$-semi-basic 1-form. Here, we are viewing $\mathbb{R}^7 = \{(\epsilon_{ijk}v_k) \in so(7): (v_1, \ldots, v_7) \in \mathbb{R}^7\}$, so that $\gamma$ takes the form

$$
\gamma = \begin{bmatrix}
0 & 73 & -72 & 75 & -74 & 77 & -76 \\
-73 & 0 & 71 & 76 & -77 & -74 & 75 \\
72 & -71 & 0 & -77 & -76 & 75 & 74 \\
-75 & -76 & 77 & 0 & 71 & 72 & -73 \\
74 & 77 & 76 & -71 & 0 & -73 & 72 \\
-77 & 74 & -75 & -72 & 73 & 0 & 71 \\
76 & -75 & -74 & 73 & 72 & -71 & 0
\end{bmatrix}.
$$

Since $\gamma$ is $\pi$-semibasic, we may write

$$
\gamma_i = T_{ij}\omega^j
$$

for some matrix-valued function $T = (T_{ij}) : F_{G_2} \to \mathrm{Mat}_{7 \times 7}(\mathbb{R})$. The 1-form $\gamma$, and hence the functions $T_{ij}$, encodes the torsion of the $G_2$-structure. In this notation, the first structure equation reads

$$
d\omega_i = -(\theta_{ij} + 2\epsilon_{ijk}\gamma_k) \wedge \omega_j \quad (2.22)
$$
Remark. The reader may wonder how the functions $T_{ij}$ are related to the forms $\tau_0, \tau_1, \tau_2, \tau_3$. In [5], Bryant expresses the torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ in terms of $T_{ij}$ as:

$$
\pi^*(\tau_0) = \frac{24}{7} T_{ii} \\
\pi^*(\tau_1) = \epsilon_{ijk} T_{ij} \omega_k \\
\pi^*(\tau_2) = 4 T_{ij} \omega_i \wedge \omega_j - \epsilon_{ijk\ell} T_{ij} \omega_k \wedge \omega_{\ell} \\
\pi^*(\tau_3) = -\frac{3}{2} \epsilon_{ijk\ell}(T_{ij} + T_{ji}) \omega_{jk\ell} + \frac{18}{7} T_{ii} \sigma.
$$

In the next section, we will exhibit a sort of inverse to this, expressing the $T_{ij}$ in terms of the refined torsion forms $\pi^*(\tau_0), \pi^*(\tau_1), \ldots, \pi^*(\tau_3)$.

### 2.3.3 Decomposition of the Torsion Functions

For our computations in §2.4 and §2.5, we will need to express the torsion functions $T_{ij}$ in terms of the functions $A_p, B_\alpha, \ldots, L_\delta, M_\alpha$. To this end, we will continue to work on the total space of the $G_2$-coframe bundle $\pi: F_{G_2} \to M$, pulling back all of the quantities defined on $M$ to $F_{G_2}$. Following common convention, we systematically omit $\pi^*$ from the notation, so that (for example) $\pi^*(\tau_0)$ will simply be denoted $\tau_0$, etc. Note, however, that $\pi^*(\epsilon^j) = \omega_j$.

To begin, recall that the torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ satisfy

$$d\varphi = \tau_0 \ast \varphi + 3 \tau_1 \wedge \varphi + \ast \tau_3 \\
d \ast \varphi = 4 \tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi.$$

Into the left-hand sides, we substitute $\varphi = \frac{1}{6} \epsilon_{ijk} \omega^{ijk}$ and $\ast \varphi = \frac{1}{24} \epsilon_{ijk\ell} \omega^{ijk\ell}$ and use the first structure equation (2.22) to obtain

$$\epsilon_{ijk\ell} T_{im} \omega^{mjk\ell} = \tau_0 \ast \varphi + 3 \tau_1 \wedge \varphi + \ast \tau_3 \\
-\epsilon_{ijk} T_{im} \omega^{m\ellijk} = 4 \tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi$$

Into the right-hand sides, we again substitute $\varphi = \frac{1}{6} \epsilon_{ijk} \omega^{ijk}$ and $\ast \varphi = \frac{1}{24} \epsilon_{ijk\ell} \omega^{ijk\ell}$, as well as the expansions (2.19) and (2.20).

Upon equating coefficients, we obtain a system of 56 = $\binom{7}{1} + \binom{7}{6}$ linear equations relating the 49 = $7^2$ functions $T_{ij}$ on the left side to the 49 = dim($H^{0,2}(g_2)$) functions $\tau_0, A_p, B_\alpha, \ldots, L_\delta, M_\alpha$ on the right side. One can then use a computer algebra system (we have used MAPLE) to solve this linear system for the $T_{ij}$.

We now exhibit the result, taking advantage of the SO(4)-irreducible splitting

$$\text{Mat}_{7 \times 7}(\mathbb{R}) \cong V^* \otimes V^* \cong (A \otimes A) \oplus 2(A \otimes C) \oplus (C \otimes C) \oplus (A^2) \oplus 2((A \otimes C)_{1,3} \oplus (A \otimes C)c)$$

$$\oplus (\Lambda^2_\perp(C) \oplus \Lambda^2(C) \oplus \mathbb{R} \oplus \text{Sym}_0^2(C))$$

to highlight the structure of the solution.

We have

$$
\frac{1}{2} \begin{bmatrix} 0 & T_{12} - T_{21} & T_{13} - T_{31} \\ T_{21} - T_{12} & 0 & T_{23} - T_{32} \\ T_{31} - T_{13} & T_{32} - T_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_2 + 2C_3 & -(A_2 + 2C_2) \\ -(A_3 + 2C_3) & 0 & A_1 + 2C_1 \\ A_2 + 2C_2 & -(A_1 + 2C_1) & 0 \end{bmatrix},
$$

$$
\frac{1}{2} \begin{bmatrix} 2T_{11} & T_{12} + T_{21} & T_{13} + T_{31} \\ T_{21} + T_{12} & 2T_{22} & T_{23} + T_{32} \\ T_{31} + T_{13} & T_{32} + T_{23} & 2T_{33} \end{bmatrix} = \begin{bmatrix} G_4 & G_1 & G_2 \\ G_1 & G_5 - G_4 & G_3 \\ G_2 & G_3 & -G_5 \end{bmatrix} + \left(-4F + \frac{1}{24} \tau_0\right) \text{Id}_3,
$$

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corresponding to $A \otimes A \cong \Lambda^2(A) \oplus \text{Sym}_2^2(A) \oplus \mathbb{R}$ and

$\begin{bmatrix}
1 & T_{41} + T_{14} & T_{42} + T_{24} & T_{43} + T_{34} \\
2 & T_{51} + T_{15} & T_{52} + T_{25} & T_{53} + T_{35} \\
3 & T_{61} + T_{16} & T_{62} + T_{26} & T_{63} + T_{36} \\
4 & T_{71} + T_{17} & T_{72} + T_{27} & T_{73} + T_{37}
\end{bmatrix} = \begin{bmatrix}
L_4 + L_7 & -L_1 & -L_5 \\
L_3 - L_8 & -L_2 & -L_6 \\
-L_2 - L_5 & -L_3 & -L_7 \\
-L_1 + L_6 & -L_4 & -L_8
\end{bmatrix} + \begin{bmatrix}
-M_5 & -M_6 & M_7 \\
M_4 & M_7 & M_6 \\
-M_7 & M_4 & -M_5 \\
M_6 & -M_5 & -M_4
\end{bmatrix}$

and

$\begin{bmatrix}
1 & T_{41} - T_{14} & T_{42} - T_{24} & T_{43} - T_{34} \\
2 & T_{51} - T_{15} & T_{52} - T_{25} & T_{53} - T_{35} \\
3 & T_{61} - T_{16} & T_{62} - T_{26} & T_{63} - T_{36} \\
4 & T_{71} - T_{17} & T_{72} - T_{27} & T_{73} - T_{37}
\end{bmatrix} = \begin{bmatrix}
D_4 + D_7 & -D_1 & -D_5 \\
D_3 - D_8 & -D_2 & -D_6 \\
-D_2 - D_5 & -D_3 & -D_7 \\
-D_1 + D_6 & -D_4 & -D_8
\end{bmatrix} + \begin{bmatrix}
-B_5 & -B_6 & B_7 \\
B_4 & B_7 & B_6 \\
-B_7 & B_4 & -B_5 \\
B_6 & -B_5 & -B_4
\end{bmatrix}$

corresponding to $A \otimes C \cong (A \otimes C)_{1,3} \oplus (A \otimes C)_C$, and

$\begin{bmatrix}
0 & T_{45} - T_{54} & T_{46} - T_{64} & T_{47} - T_{74} \\
T_{54} = T_{45} & 0 & T_{56} - T_{65} & T_{57} - T_{75} \\
T_{64} = T_{46} & T_{65} - T_{56} & 0 & T_{67} - T_{76} \\
T_{74} = T_{47} & T_{75} - T_{57} & T_{76} - T_{67} & 0
\end{bmatrix} = \begin{bmatrix}
0 & A_1 - C_1 & A_2 - C_2 & -A_3 + C_3 \\
-A_1 + C_1 & 0 & -A_3 + C_3 & -A_2 + C_2 \\
-A_2 - C_2 & A_3 - C_3 & 0 & A_1 - C_1 \\
-A_3 + C_3 & A_2 - C_2 & -A_1 + C_1 & 0
\end{bmatrix}

+ \begin{bmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -E_3 & E_2 \\
-E_2 & E_3 & 0 & -E_1 \\
-E_3 & -E_2 & E_1 & 0
\end{bmatrix}$

and

$\begin{bmatrix}
2T_{44} & T_{45} + T_{54} & T_{46} + T_{64} & T_{47} + T_{74} \\
T_{54} + T_{45} & 2T_{55} & T_{56} + T_{65} & T_{57} + T_{75} \\
T_{64} + T_{46} & T_{65} & 2T_{66} & T_{67} + T_{76} \\
T_{74} = T_{47} & T_{75} + T_{57} & T_{76} + T_{67} & 2T_{77}
\end{bmatrix} = \begin{bmatrix}
-J_{11} - J_{22} + J_{33} & J_{23} + J_{32} & -J_{13} - J_{31} & J_{12} - J_{21} \\
J_{23} + J_{32} & -J_{11} - J_{22} - J_{33} & -J_{21} - J_{12} & -J_{13} + J_{31} \\
-J_{13} - J_{31} & -J_{12} - J_{21} & J_{11} - J_{22} - J_{33} & -J_{23} + J_{32} \\
J_{12} - J_{21} & -J_{13} + J_{31} & -J_{23} + J_{32} & J_{11} + J_{22} + J_{33}
\end{bmatrix} + \left(3F + \frac{1}{24}T_7\right) \text{Id}_4$

corresponding to $C \otimes C \cong \Lambda^2_2(C) \oplus \Lambda^2_2(C) \oplus \mathbb{R} \oplus \text{Sym}_2^2(C)$.

The above relations are more than we need for this work. In fact, we will only make use of the following relations, which can be read off from the above:

$\epsilon_{\alpha \beta \rho} T_{\beta \rho} = -3(B_\alpha + M_\alpha)$  \hspace{1cm} (2.23)

and

$T_{44} + T_{55} + T_{66} + T_{77} = 3F + \frac{1}{24}T_7$  \hspace{1cm} (2.24)

and

$-(T_{45} - T_{54}) - (T_{67} - T_{76}) = -4(A_1 - C_1)$  \hspace{1cm} (2.25a)

$(T_{57} - T_{75}) - (T_{46} - T_{64}) = -4(A_2 - C_2)$  \hspace{1cm} (2.25b)

$(T_{47} - T_{74}) + (T_{56} - T_{65}) = -4(A_3 - C_3)$.  \hspace{1cm} (2.25c)
2.4 Mean Curvature of Associative 3-Folds

In this subsection, we derive a formula (Theorem 2.18) for the mean curvature of an associative 3-fold in an arbitrary 7-manifold $(M, \varphi)$ with $G_2$-structure $\varphi$.

We continue with the notation of §2.3, letting $\pi: F_{G_2} \to M$ denote the $G_2$-coframe bundle of $M$, and $\omega = (\omega_A, \omega_C) \in \Omega^1(F_{G_2}; \mathbb{R}^7) \oplus \mathbb{C}^3$ denote the tautological 1-form. We remind the reader that $\theta = (\theta_{ij}) \in \Omega^1(F_{G_2}; \mathfrak{g}_2)$ is the natural connection 1-form, and that $\gamma = (\gamma_{ij}) \in \Omega^1(F_{G_2}; \mathbb{R}^7)$ is a $\pi$-semibasic 1-form encoding the torsion of $\varphi$. We will continue to write $\gamma_{ij} = \epsilon_{ijk}\gamma_k$ and $\gamma_i = T_{ij}\omega^j$ for $T = (T_{ij}): F_{G_2} \to \text{Mat}_{7 \times 7}(\mathbb{R})$.

Let $f: \Sigma^3 \to M^7$ denote an immersion of an associative 3-fold into $M$, and let $f^*(F_{G_2}) \to \Sigma$ denote the pullback bundle. Let $B \subset f^*(F_{G_2})$ denote the subbundle of coframes adapted to $\Sigma$, i.e., the subbundle whose fiber over $x \in \Sigma$ is

$$B|_x = \{u \in f^*(F_{G_2})|_x: u(T_x\Sigma) = \mathbb{R}^3 \oplus 0\}$$

We recall (Proposition 2.6) that $G_2$ acts transitively on the set of associative 3-planes with stabilizer $\text{SO}(4)$, so $B \to \Sigma$ is a well-defined $\text{SO}(4)$-bundle. Note that on $B$, we have

$$\omega_C = 0.$$ 

For the rest of §2.4, all of our calculations will be done on the subbundle $B \subset F_{G_2}$.

We now exploit splitting $T_xM = T_x\Sigma \oplus (T_x\Sigma)^\perp \simeq \mathbb{A}^3 \oplus \mathbb{C}^3$ to decompose $\theta$ and $\gamma$ into $\text{SO}(4)$-irreducible pieces. To decompose the connection 1-form $\theta \in \Omega^1(B; \mathfrak{g}_2)$, we split

$$\mathfrak{g}_2 \cong [\mathfrak{g}_2 \cap (\Lambda^2(\mathbb{A}) \oplus \Lambda^3_+(-\mathbb{C}))] \oplus [\mathfrak{g}_2 \cap (\mathbb{A} \times \mathbb{C})] \oplus [\mathfrak{g}_2 \cap \Lambda^2_-(\mathbb{C})],$$

so that $\theta$ takes the block form

$$\theta = \begin{bmatrix} \rho(\zeta) & -\sigma^T \\ \sigma & \zeta + \xi \end{bmatrix} = \begin{bmatrix} 0 & 2\zeta_3 & -2\zeta_2 \\ -2\zeta_3 & 0 & 2\zeta_1 \\ 2\zeta_2 & -2\zeta_1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 \\ -\sigma_1 + \sigma_2 \end{bmatrix}.$$ 

Similarly, the 1-form $\gamma \in \Omega^1(B; \mathbb{R}^7)$ breaks into block form as:

$$\gamma = \begin{bmatrix} \gamma_A & -(\gamma_C)^T \\ \gamma_C & (\gamma_A)^+ \end{bmatrix} = \begin{bmatrix} 0 & \gamma_3 & -\gamma_2 & \gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\ -\gamma_3 & 0 & \gamma_1 & \gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\ \gamma_2 & -\gamma_1 & 0 & -\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4 \\ -\gamma_5 & -\gamma_6 & \gamma_7 & 0 & \gamma_1 & \gamma_2 & -\gamma_3 \\ \gamma_4 & \gamma_7 & \gamma_6 & -\gamma_1 & 0 & -\gamma_3 & -\gamma_2 \\ -\gamma_7 & \gamma_4 & -\gamma_5 & -\gamma_2 & \gamma_3 & 0 & \gamma_1 \\ \gamma_6 & -\gamma_5 & -\gamma_4 & \gamma_3 & \gamma_2 & -\gamma_1 & 0 \end{bmatrix}.$$ 

In this notation, the first structure equation (2.22) on $B$ reads:

$$d(\omega_A) = -\left(\begin{bmatrix} \rho(\zeta) & -\sigma^T \\ \sigma & \zeta + \xi \end{bmatrix} + 2\begin{bmatrix} \gamma_A & -(\gamma_C)^T \\ \gamma_C & (\gamma_A)^+ \end{bmatrix}\right) \wedge (\omega_A).$$
In particular, the second line gives

\[ 0 = -(\sigma + 2\gamma^C) \wedge \omega_A \]

or in detail,

\[
\begin{bmatrix}
\sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\
\sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\
-\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\
-\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8
\end{bmatrix} \wedge \begin{bmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4
\end{bmatrix} = -2 \begin{bmatrix}
-\gamma_5 & -\gamma_6 & \gamma_7 \\
\gamma_4 & \gamma_7 & \gamma_6 \\
-\gamma_7 & \gamma_4 & -\gamma_5 \\
\gamma_6 & -\gamma_5 & -\gamma_4
\end{bmatrix} \wedge \begin{bmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4
\end{bmatrix}
\]  \tag{2.26}

Note that on \( B \), the 1-forms \( \sigma_\delta \) and \( \gamma_\alpha \) are semibasic, and we write

\[ \sigma_\delta = S_{\delta p} \omega^p \quad \quad \gamma_\alpha = T_{\alpha p} \omega^p \]

for some function \( S = (S_{\delta p}) \colon B \to V_{1,3} \otimes A \), recalling our index ranges \( 1 \leq p \leq 3 \) and \( 4 \leq \alpha \leq 7 \) and \( 1 \leq \delta \leq 8 \).

Now, the 24 functions \( S_{\delta p} \) and the 12 functions \( T_{\alpha p} \) are not independent: the equation (2.26) amounts to \( 12 = 4 \binom{3}{2} \) linear relations among them. Explicitly:

\[
\begin{bmatrix}
S_{13} - S_{52} \\ S_{23} - S_{62} \\ S_{33} - S_{72} \\ S_{43} - S_{82}
\end{bmatrix}
\begin{bmatrix}
S_{43} + S_{73} + S_{51} \\ S_{33} - S_{83} + S_{61} \\ -S_{23} - S_{53} + S_{71} \\ -S_{13} + S_{63} + S_{81}
\end{bmatrix}
\begin{bmatrix}
-S_{42} - S_{72} - S_{11} \\ -S_{32} + S_{82} - S_{21} \\ S_{22} + S_{52} - S_{31} \\ S_{12} - S_{62} - S_{41}
\end{bmatrix}
= -2 \begin{bmatrix}
T_{63} + T_{72} \\ T_{62} - T_{73} \\ -T_{43} - T_{52} \\ -T_{12} + T_{53}
\end{bmatrix}
\begin{bmatrix}
-T_{53} - T_{71} \\ T_{43} - T_{61} \\ T_{51} - T_{73} \\ T_{41} + T_{72}
\end{bmatrix}
\begin{bmatrix}
T_{52} - T_{61}
\end{bmatrix}
\]

In particular, these relations imply:

\[ S_{41} + S_{71} - S_{12} - S_{53} = -4\epsilon_{40p}T_{op} \] \tag{2.27a}
\[ S_{31} - S_{81} - S_{22} - S_{63} = -4\epsilon_{50p}T_{op} \] \tag{2.27b}
\[ -S_{21} - S_{51} - S_{32} - S_{75} = -4\epsilon_{60p}T_{op} \] \tag{2.27c}
\[ -S_{11} + S_{61} - S_{42} - S_{83} = -4\epsilon_{70p}T_{op} \] \tag{2.27d}

With these calculations in place, we may finally compute the mean curvature of an associative 3-fold:

**Theorem 2.18.** Let \( \Sigma \subset M \) be an associative 3-fold immersed in a 7-manifold \( M \) equipped with a \( G_2 \)-structure. Then the mean curvature vector \( H \) of \( \Sigma \) is given by

\[ H = -3[(\tau_1)_C]^2 - \frac{\sqrt{3}}{2}[(\tau_3)_C]^3. \]

In particular, the largest torsion class of \( G_2 \)-structures \( \varphi \) for which every associative 3-fold is minimal is \( W_1 \oplus W_{14} = W_1 \cup W_{14} \), i.e., the class for which \( d\varphi = \lambda \ast \varphi \) for some \( \lambda \in \mathbb{R} \).

**Proof.** The mean curvature vector may be computed as follows:

\[
\begin{bmatrix}
H_4 \\ H_5 \\ H_6 \\ H_7
\end{bmatrix}
\omega^{123} = \begin{bmatrix}
\psi_{41} & \psi_{42} & \psi_{43} \\ \psi_{51} & \psi_{52} & \psi_{53} \\ \psi_{61} & \psi_{62} & \psi_{63} \\ \psi_{71} & \psi_{72} & \psi_{73}
\end{bmatrix}
\wedge \begin{bmatrix}
\omega_{23} \\ \omega_{31} \\ \omega_{12}
\end{bmatrix}
\mathbf{\wedge} \begin{bmatrix}
\sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\ \sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\ -\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\ -\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8
\end{bmatrix}
\begin{bmatrix}
-\gamma_5 & -\gamma_6 & \gamma_7 \\ \gamma_4 & \gamma_7 & \gamma_6 \\ -\gamma_7 & \gamma_4 & -\gamma_5 \\ \gamma_6 & -\gamma_5 & -\gamma_4
\end{bmatrix}
\wedge \begin{bmatrix}
\omega_{23} \\ \omega_{31} \\ \omega_{12}
\end{bmatrix}
+ 2 \begin{bmatrix}
-\gamma_5 & -\gamma_6 & \gamma_7 \\ \gamma_4 & \gamma_7 & \gamma_6 \\ -\gamma_7 & \gamma_4 & -\gamma_5 \\ \gamma_6 & -\gamma_5 & -\gamma_4
\end{bmatrix}
\wedge \begin{bmatrix}
\omega_{23} \\ \omega_{31} \\ \omega_{12}
\end{bmatrix}
\]  \tag{2.28}
To evaluate the first term in (2.28), we substitute $\sigma_\delta = S_{\delta p}w^p$, followed by (2.27), and finally (2.23), to obtain:

$$\begin{pmatrix} \sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\ \sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\ -\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\ -\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8 \end{pmatrix} \wedge \begin{pmatrix} \omega^{23} \\ \omega^{31} \\ \omega^{12} \end{pmatrix} = \begin{pmatrix} S_{41} + S_{71} - S_{12} - S_{53} \\ S_{31} - S_{81} - S_{22} - S_{63} \\ -S_{21} - S_{51} - S_{32} - S_{73} \\ -S_{11} + S_{61} - S_{42} - S_{83} \end{pmatrix} \omega^{123}
$$

$$= -4 \begin{pmatrix} \epsilon_{40p}T_{0p} \\ \epsilon_{50p}T_{0p} \\ \epsilon_{60p}T_{0p} \\ \epsilon_{70p}T_{0p} \end{pmatrix} \omega^{123} = -12 \begin{pmatrix} B_4 + M_4 \\ B_5 + M_5 \\ B_6 + M_6 \\ B_7 + M_7 \end{pmatrix} \omega^{123}
$$

Similarly, to evaluate the second term in (2.28), we substitute $\gamma_\alpha = T_{\alpha p}w^p$ followed by (2.23) to obtain:

$$2 \begin{pmatrix} -\gamma_5 & -\gamma_6 & \gamma_7 \\ \gamma_4 & \gamma_7 & \gamma_6 \\ -\gamma_7 & \gamma_4 & -\gamma_5 \\ \gamma_6 & -\gamma_5 & -\gamma_4 \end{pmatrix} \wedge \begin{pmatrix} \omega^{23} \\ \omega^{31} \\ \omega^{12} \end{pmatrix} = -2 \begin{pmatrix} \epsilon_{40p}T_{0p} \\ \epsilon_{50p}T_{0p} \\ \epsilon_{60p}T_{0p} \\ \epsilon_{70p}T_{0p} \end{pmatrix} \omega^{123} = -6 \begin{pmatrix} B_4 + M_4 \\ B_5 + M_5 \\ B_6 + M_6 \\ B_7 + M_7 \end{pmatrix} \omega^{123}
$$

We conclude that

$$H_\alpha = -18 B_\alpha - 18 M_\alpha,$$

and so (2.21) yields

$$H = -3[(\tau_1)c]^2 - \sqrt{3} \frac{[(\tau_3)c]^2}{2}.$$

In particular, the largest torsion class for which $H = 0$ for all associatives is the one for which $\tau_1 = \tau_3 = 0$, which is $W_1 \oplus W_{14} = W_1 \cup W_{14}$.  

\section*{2.5 Mean Curvature of Coassociative 4-Folds}

In this subsection, we derive a formula (Theorem 2.21) for the mean curvature of a coassociative 4-fold in an arbitrary 7-manifold $(M, \varphi)$ with $G_2$-structure $\varphi$. In the process, we observe a necessary condition (Theorem 2.19) for the local existence of coassociative 4-folds. We continue with the notation of §2.3.

Let $f: \Sigma^4 \to M^7$ denote an immersion of a coassociative 4-fold into $M$, and let $f^*(F_{G_2}) \to \Sigma$ denote the pullback bundle. Let $B \subset f^*(F_{G_2})$ denote the subbundle of coframes adapted to $\Sigma$, i.e., the subbundle whose fiber over $x \in \Sigma$ is

$$B|_x = \{ u \in f^*(F_{G_2})|_x : u(T_x \Sigma) = 0 \oplus C^7 \}$$

We recall (Proposition 2.6) that $G_2$ acts transitively on the set of coassociative 4-planes with stabilizer $SO(4)$, so $B \to \Sigma$ is a well-defined $SO(4)$-bundle. Note that on $B$, we have

$$\omega_A = 0.$$

For the rest of §2.5, all of our calculations will be done on the subbundle $B \subset F_{G_2}$.
As in §2.4, we use the splitting $T_\nu M = (T_\nu \Sigma)^+ \oplus T_\nu \Sigma \simeq A^2 \oplus C^2$ to decompose $\theta$ and $\gamma$ into SO(4)-irreducible pieces. The result is the identical: the connection 1-form $\theta \in \Omega^1(B; \mathfrak{g}_2)$ takes the block form

$$\theta = \begin{bmatrix} \rho(\zeta) & -\sigma^T \\ \sigma & \zeta + \xi \end{bmatrix} = \begin{bmatrix} 0 & 2\xi_3 & -2\xi_2 \\ -2\xi_3 & 0 & 2\xi_1 \\ 2\xi_2 & -2\xi_1 & 0 \\ \sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\ \sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\ -\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\ -\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8 \end{bmatrix}.$$

and the 1-form $\gamma \in \Omega^1(B; \mathbb{R}^7)$ takes the block form

$$\gamma = \begin{bmatrix} \gamma_A & -\gamma C \\ \gamma_C & (\gamma A)_+ \end{bmatrix} = \begin{bmatrix} 0 & \gamma_3 & -\gamma_2 & \gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\ \gamma_3 & 0 & \gamma_1 & \gamma_7 & -\gamma_6 & \gamma_5 & -\gamma_4 \\ \gamma_2 & -\gamma_1 & 0 & -\gamma_7 & \gamma_6 & \gamma_5 & -\gamma_4 \\ -\gamma_5 & -\gamma_6 & \gamma_7 & 0 & \gamma_1 & \gamma_2 & -\gamma_3 \\ \gamma_7 & \gamma_6 & \gamma_5 & -\gamma_1 & 0 & -\gamma_3 & -\gamma_2 \\ -\gamma_7 & \gamma_4 & -\gamma_5 & -\gamma_2 & \gamma_3 & 0 & \gamma_1 \\ \gamma_6 & -\gamma_5 & -\gamma_4 & \gamma_3 & \gamma_2 & -\gamma_1 & 0 \end{bmatrix}.$$

In this notation, the first structure equation (2.22) on $B$ reads:

$$d\begin{bmatrix} 0 \\ \omega_C \end{bmatrix} = -\left(\begin{bmatrix} \rho(\zeta) & -\sigma^T \\ \sigma & \zeta + \xi \end{bmatrix} + 2\begin{bmatrix} \gamma_A & -\gamma C \\ \gamma_C & (\gamma A)_+ \end{bmatrix}\right) \wedge \begin{bmatrix} 0 \\ \omega_C \end{bmatrix}.$$ 

In particular, the first line gives

$$0 = (\sigma^T + 2(\gamma C)^T) \wedge \omega_C$$

or in full detail,

$$\begin{bmatrix} -\sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 \end{bmatrix} \wedge \begin{bmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \\ \eta^4 \end{bmatrix} = -2 \begin{bmatrix} \gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\ \gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\ -\gamma_7 & \gamma_6 & \gamma_5 & \gamma_4 \\ -\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4 \end{bmatrix} \wedge \begin{bmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \\ \eta^4 \end{bmatrix} \tag{2.29}$$

Note that on $B$, the 1-forms $\sigma_\delta$ and $\gamma_\alpha$ are semibasic, so we can write

$$\sigma_\delta = S_{\delta \alpha} \omega^\alpha \quad \quad \quad \gamma_\beta = T_{\beta \alpha} \omega^\alpha$$

for some function $S = (S_{\delta \alpha}) : B \to V_{1,3} \otimes \mathbb{C}$, recalling our index ranges $1 \leq p \leq 3$ and $4 \leq \alpha, \beta \leq 7$ and $1 \leq \delta \leq 8$.

Note that the 32 functions $S_{\delta \alpha}$ and the 16 functions $T_{\beta \alpha}$ are not independent: the equation (2.29) shows that they satisfy $3(\binom{8}{2}) = 18$ linear relations. Explicitly:

$$\begin{bmatrix} S_{15} - S_{24} & S_{55} - S_{64} & S_{84} + S_{45} + S_{75} - S_{34} \\ S_{16} - S_{34} & S_{56} - S_{74} & S_{54} + S_{24} + S_{46} + S_{76} \\ S_{26} - S_{35} & S_{66} - S_{75} & S_{14} - S_{64} + S_{47} + S_{77} \\ S_{17} - S_{44} & S_{57} - S_{84} & S_{55} - S_{86} + S_{25} + S_{36} \\ S_{27} - S_{45} & S_{67} - S_{85} & -S_{65} + S_{15} + S_{37} - S_{87} \\ S_{37} - S_{46} & S_{77} - S_{86} & -S_{66} - S_{57} + S_{16} - S_{27} \end{bmatrix} = 2 \begin{bmatrix} -T_{74} - T_{55} & -T_{64} + T_{75} & T_{44} + T_{55} \\ -T_{14} - T_{66} & T_{54} + T_{76} & -T_{74} + T_{56} \\ -T_{15} + T_{76} & T_{55} + T_{66} & T_{64} + T_{57} \\ T_{14} - T_{67} & T_{44} + T_{77} & -T_{75} - T_{46} \\ T_{55} + T_{77} & T_{54} + T_{67} & T_{65} - T_{47} \\ T_{56} + T_{47} & T_{46} - T_{57} & T_{66} + T_{77} \end{bmatrix}.$$
We make two observations on this system of equations. First, we observe the relation

$$(T_{44} + T_{55}) + (T_{55} + T_{66}) + (T_{55} + T_{77}) + (T_{66} + T_{77}) + (T_{44} + T_{77}) + (T_{44} + T_{66}) = 0.$$  

Substituting (2.24), this equation simplifies to

$$3F + \frac{1}{24}\tau_0 = 0.$$  

Substituting (2.21), we have proved:

**Theorem 2.19.** If a coassociative 4-fold $\Sigma$ exists in $M$, then the following relation holds at points of $\Sigma$:

$$\tau_0 = -\frac{\sqrt{42}}{7}[(\tau_3)_{0,0}]^\dagger \quad (2.30)$$

In particular, if $\tau_3 = 0$ and $\tau_0$ is non-vanishing (so the torsion takes values in $(W_1 \oplus W_7 \oplus W_{14}) - (W_7 \oplus W_{14})$), then $M$ admits no coassociative 4-folds (even locally).

**Corollary 2.20.** Fix $x \in M$. If every coassociative 4-plane in $T_xM$ is tangent to a coassociative 4-fold, then $\tau_0|_x = 0$ and $\tau_3|_x = 0$.

**Proof.** The hypotheses imply that equation (2.30) holds for all coassociative 4-planes at $x \in M$. Thus, we have a $G_2$-invariant linear relation between $\tau_0|_x$ and $\tau_3|_x$. This implies that $\tau_0|_x = 0$ and $\tau_3|_x = 0$ by Schur’s Lemma.

Second, we observe the three relations

$$(S_{17} - S_{67}) + (S_{26} + S_{56}) + (-S_{35} + S_{85}) - (S_{44} + S_{74}) = -4(T_{45} - T_{54}) - 4(T_{67} - T_{76}) \quad (2.31a)$$

$$(S_{14} + S_{25} + S_{36} + S_{47} = 4(T_{57} - T_{75}) - 4(T_{46} - T_{64}) \quad (2.31b)$$

$$(S_{54} + S_{65} + S_{76} + S_{87} = 4(T_{47} - T_{74}) + 4(T_{56} - T_{65}) \quad (2.31c)$$

We may now compute the mean curvature of a coassociative 4-fold:

**Theorem 2.21.** Let $\Sigma \subset M$ be a coassociative 4-fold immersed in a 7-manifold $M$ equipped with a $G_2$-structure. Then the mean curvature vector $H$ of $\Sigma$ is given by

$$H = -4[(\tau_1)_{A}]^2 + \frac{\sqrt{6}}{3}[(\tau_2)_{A}]^2.$$  

In particular, the largest torsion class of $G_2$-structures $\varphi$ for which every coassociative 4-fold is minimal is $W_1 \oplus W_{27}$, i.e., the class for which $d*\varphi = 0$.

**Proof.** Let $\beta_{\alpha} := (*_{\mathcal{C}}(\omega^{\alpha}) \in \Omega^3(B)$ and $\text{vol}_{\mathcal{C}} = \omega^{4567}$. The mean curvature vector may be computed
as follows:

\[
\begin{bmatrix}
H_1^1 \\
H_2^2 \\
H_3^3
\end{bmatrix}
\text{vol}_C = \begin{bmatrix}
\psi_{14} & \psi_{15} & \psi_{16} & \psi_{17} \\
\psi_{24} & \psi_{25} & \psi_{26} & \psi_{27} \\
\psi_{34} & \psi_{35} & \psi_{36} & \psi_{37}
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\sigma_5 & \sigma_6 & \sigma_7 & \sigma_8
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix} + 2 \begin{bmatrix}
\gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
\gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\
-\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix}
\]

(2.32)

To evaluate the first term in (2.32), we substitute \( \sigma_\delta = S_{\delta \rho \omega}^p \), followed by (2.31), and finally (2.25), to obtain:

\[
\begin{bmatrix}
-\sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\sigma_5 & \sigma_6 & \sigma_7 & \sigma_8
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix} = 4 \begin{bmatrix}
-(T_{45} - T_{54}) - (T_{67} - T_{76}) \\
(T_{57} - T_{75}) - (T_{46} - T_{64}) \\
(T_{47} - T_{74}) + (T_{56} - T_{65})
\end{bmatrix} \text{vol}_C
\]

\[
= 16 \begin{bmatrix}
-A_1 + C_1 \\
-A_2 + C_2 \\
-A_3 + C_3
\end{bmatrix} \text{vol}_C
\]

Similarly, to evaluate the second term in (2.32), we substitute \( \gamma_\alpha = T_{\alpha \rho \omega}^p \) followed by (2.25) to obtain:

\[
2 \begin{bmatrix}
\gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
\gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\
-\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix} = 2 \begin{bmatrix}
-(T_{45} - T_{54}) - (T_{67} - T_{76}) \\
(T_{57} - T_{75}) - (T_{46} - T_{64}) \\
(T_{47} - T_{74}) + (T_{56} - T_{65})
\end{bmatrix} \text{vol}_C
\]

\[
= 8 \begin{bmatrix}
-A_1 + C_1 \\
-A_2 + C_2 \\
-A_3 + C_3
\end{bmatrix} \text{vol}_C
\]

We conclude that

\[
H_p = -24A_p + 24C_p
\]

and so (2.21) yields

\[
H = -4[(\tau_1) A]^2 + \frac{\sqrt{6}}{3} [(\tau_2) A]^2.
\]

In particular, the largest torsion class for which \( H = 0 \) for all coassociatives is the one for which \( \tau_1 = \tau_2 = 0 \), which is \( W_1 \oplus W_{27} \).

\[\diamond\]

3 Cayley 4-Folds in Spin(7)-Structures

3.1 Preliminaries

In this subsection, we define both the ambient spaces and the submanifolds of interest. This subsection will also be used to fix notation and conventions.
3.1.1 Spin(7)-Structures on Vector Spaces

Let \( V = \mathbb{R}^8 \) equipped with the standard inner product \( \langle \cdot , \cdot \rangle \), norm \( \| \cdot \| \), and an orientation. Let \( \{ e_1, \ldots, e_8 \} \) denote the standard (orthonormal) basis of \( V \), and let \( \{ e^1, \ldots, e^8 \} \) denote the corresponding dual basis of \( V^* \). The Cayley 4-form is the alternating 4-form \( \Phi_0 \in \Lambda^4(\Lambda^8) \) defined by

\[
\Phi_0 = e^{1234} + (e^{12} + e^{34}) \wedge (e^{56} + e^{78}) + (e^{13} - e^{24}) \wedge (e^{57} - e^{68}) + (-e^{14} - e^{23}) \wedge (e^{58} + e^{67}) + e^{5678}.
\]

For calculations, it will be convenient to express \( \Phi_0 \) in the form

\[
\Phi_0 = \frac{1}{24} \Phi_{ijkl} e^{ijkl},
\]

where the constants \( \Phi_{ijkl} \in \{-1, 0, 1\} \) are defined by this formula and skew-symmetry.

The stabilizer of \( \Phi_0 \) under the pullback action of \( GL(V) \) is isomorphic to Spin(7),

\[
\text{Spin}(7) \cong \{ A \in GL(V) : A^* \Phi_0 = \Phi_0 \}.
\]

Using the isomorphism (3.1), we regard Spin(7) as a subgroup of \( GL(V) \). As a Spin(7)-module, \( V \) is isomorphic to the spinor representation of Spin(7), and is thus irreducible. The induced representation on \( \Lambda^k(\Lambda^8) \) for \( 2 \leq k \leq 6 \) is reducible, and there are irreducible decompositions [4]

\[
\begin{align*}
\Lambda^2(\Lambda^8) &\cong \Lambda_7^2 \oplus \Lambda_3^2, \\
\Lambda^3(\Lambda^8) &\cong \Lambda_8^3 \oplus \Lambda_{48}^3, \\
\Lambda^4(\Lambda^8) &\cong \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4 \oplus \Lambda_{35}^4.
\end{align*}
\]

We will need the definitions of the irreducible pieces of \( \Lambda^2(\Lambda^8) \) and \( \Lambda^3(\Lambda^8) \), they are

\[
\begin{align*}
\Lambda_7^2 &= \{ \beta \in \Lambda^2(\Lambda^8) \mid *(\Phi_0 \wedge \beta) = 3\beta \}, \\
\Lambda_3^2 &= \{ \beta \in \Lambda^2(\Lambda^8) \mid *(\Phi_0 \wedge \beta) = -\beta \} \cong \text{spin}(7), \\
\Lambda_8^3 &= \{ *(\Phi_0 \wedge \alpha) \mid \alpha \in \Lambda^3(\Lambda^8) \}, \\
\Lambda_{48}^3 &= \{ \gamma \in \Lambda^3(\Lambda^8) \mid \Phi_0 \wedge \gamma = 0 \}.
\end{align*}
\]

The irreducible decompositions of the spaces \( \Lambda^5(V^*) \) and \( \Lambda^6(V^*) \) can be obtained by applying the Hodge star operator to the decompositions of \( \Lambda^3(V^*) \) and \( \Lambda^2(V^*) \) respectively.

3.1.2 Spin(7)-Structures on 8-Manifolds

Let \( M \) be an oriented 8-manifold. A Spin(7)-structure on \( M \) is a differential 4-form \( \Phi \in \Omega^4(M) \) such that at each \( x \in M \), and after identifying \( T_x M \) with \( V = \mathbb{R}^8 \), the 4-form \( \Phi|_x \in \Lambda^4(T^*_x M) \) lies in the \( GL(V) \)-orbit of \( \Phi_0 \). That is, at each \( x \in M \) there exists a coframe \( u : T_x M \to \mathbb{R}^8 \) for which \( \Phi|_x = u^* (\Phi_0) \).

The first order local invariants of a Spin(7)-structure are completely encoded in a Spin(7)-equivariant function

\[
T : F_{\text{Spin}(7)} \to \Lambda^1 \oplus \Lambda_{48}^3
\]

called the intrinsic torsion function, defined on the total space of the Spin(7)-frame bundle \( F_{\text{Spin}(7)} \to M \) over \( M \).
By a result of Fernández [6], the intrinsic torsion function of a Spin(7)-structure is equivalent to the data of the 5-form $d\Phi$. It follows from the decomposition of $\Lambda^5(T^*M) \oplus \Lambda^3_{48}(T^*M)$ described in the previous section that the exterior derivative of $\Phi$ takes the form

$$d\Phi = \tau_1 \wedge \Phi + *\tau_3,$$

where $(\tau_1, \tau_3) \in \Gamma(\Lambda^1(T^*M) \oplus \Lambda^3_{48}(T^*M))$. We refer to the forms $\tau_1$ and $\tau_3$ as the torsion forms of the Spin(7)-structure.

### 3.1.3 Cayley 4-Folds

Let $(M, \Phi)$ be an 8-manifold with a Spin(7)-structure, and consider a tangent space $(T_xM, \Phi|_x) \cong (V, \Phi_0)$. The vector space $(V, \Phi_0)$ possesses a distinguished class of subspaces — the Cayley 4-planes — first studied by Harvey and Lawson [9] in their work on calibrations.

**Proposition 3.1** ([9]). The Cayley 4-form $\Phi_0$ has co-mass one, meaning that $$|\Phi_0(x, y, z, w)| \leq 1$$

for every orthonormal set $\{x, y, z, w\}$ in $V \cong \mathbb{R}^7$.

**Proposition 3.2** ([9]). Let $K \in Gr_4(V)$ be a 4-plane in $V$. The following are equivalent

1. If $\{x, y, z, w\}$ is an orthonormal basis of $K$, then $\Phi_0(x, y, z, w) = \pm 1$.
2. $\text{Im}(x \times y \times z \times w) = 0$ for any basis $x, y, z, w$ of $K$.
3. $y \times z \times w \in K$ for all $y, z, w \in K$.

The Spin(7)-action on $V$ induces Spin(7)-actions on the Grassmannians $Gr_k(V)$ of $k$-planes in $V$. These actions are transitive for $k = 1, 2, 3, 5, 6, 7$, but the action is not transitive when $k = 4$. Indeed, the (proper) subset consisting of Cayley 4-planes is a Spin(7)-orbit.

**Proposition 3.3.** The Lie group Spin(7) acts transitively on the subset of Cayley 4-planes

$$\{E \in Gr_4(V) : |\Phi_0(E)| = 1\} \subset Gr_4(V)$$

with stabiliser isomorphic to $SU(2) \times SU(2) \times SU(2)/\{\pm \text{Id}\}$.

**Notation:** We will denote the group $SU(2)^3/\{\pm \text{Id}\}$ by Spin$^h(4)$.

### 3.2 Some Spin$^h(4)$-Representation Theory

The Lie group Spin$^h(4)$ is double-covered by the simply-connected group $SU(2)^3$. The complex irreducible representations of $SU(2)^3$ are exactly the tensor products $V_p \otimes V_q \otimes V_r$ of irreducible $SU(2)$-representations for each factor. The complex irreducible representations of $SU(2)$ are well known to be the spaces of homogeneous polynomials in two variables of fixed degree, $V_p = \text{Sym}^p(\mathbb{C}[x, y])$.

Let $V^C_{p,q,r}$ denote $V_p \otimes V_q \otimes V_r$. We think of $V^C_{p,q,r}$ as the space of homogeneous polynomials in $(u, v; x, y; w, z)$ of tridegree $(p, q, r)$. When $p + q + r$ is even the representation $V^C_{p,q,r}$ descends to a representation of Spin$^h(4)$, and each of these representations has a real structure induced by the map $(u, v, x, y, w, z) \mapsto (v, -u, y, -x, z, -w)$. This yields a complete description of the real representations of Spin$^h(4)$. We work with real representations, letting $V_{p,q,r}$ denote the real representation underlying $V^C_{p,q,r}$. 27
The Clebsch-Gordan formula applied to each SU(2) representation gives the irreducible decomposition of a tensor product of Spin^h(4)-modules:

\[ V_{p_1,q_1,r_1} \otimes V_{p_2,q_2,r_2} \cong \bigoplus_{i=0}^{\lfloor|p_1-p_2|\rfloor} \bigoplus_{j=0}^{\lfloor|q_1-q_2|\rfloor} \bigoplus_{k=0}^{\lfloor|r_1-r_2|\rfloor} V_{p_1+p_2-2i,q_1+q_2-2j,r_1+r_2-2k}. \]

### 3.2.1 Spin^h(4) as a subgroup of Spin(7)

In our calculations we shall need a concrete realization of Spin^h(4) as the stabilizer of a Cayley plane. Let Spin^h(4) act on \( V \cong \mathbb{R}^8 \) via the identification \( V \cong V_{1,1,0} \oplus V_{0,1,1} \). Define a basis \((e_1,\ldots,e_8)\) of \( V \) by

\[
\begin{align*}
    e_1 &= i(-ux + vy), \\
    e_2 &= ux + vy, \\
    e_3 &= -i(uy + vx), \\
    e_4 &= -uy + vx, \\
    e_5 &= i(xw - yz), \\
    e_6 &= xw + yz, \\
    e_7 &= -i(xz +yw), \\
    e_8 &= -xz + yw.
\end{align*}
\]

Then the 4-form

\[
e^{1234} + (e^{12} + e^{34}) \wedge (e^{56} + e^{78}) + (e^{13} - e^{24}) \wedge (e^{57} - e^{68}) + (-e^{14} - e^{23}) \wedge (e^{58} + e^{67}) + e^{5678}
\]

is Spin^h(4)-invariant, and thus the action of Spin^h(4) on \( V \) gives an embedding Spin^h(4) \( \subset \) Spin(7). The 4-plane \((e_1,e_2,e_3,e_4)\) is Cayley and is preserved by the action of Spin^h(4).

### 3.2.2 Decomposition of 1-forms on \( V \)

Let \( V \) be as above. We have

\[
\Lambda^1(V^*) = K \oplus L,
\]

where

\[
K = \langle e_1, e_2, e_3, e_4 \rangle,
\]

\[
L = \langle e_5, e_6, e_7, e_8 \rangle.
\]

As abstract Spin^h(4)-modules, \( K \cong V_{1,1,0} \) and \( L \cong V_{0,1,1} \).

### 3.2.3 Decomposition of 2-forms on \( V \)

We now seek to decompose \( \Lambda^2(V^*) \) into Spin^h(4)-irreducible submodules. As noted in §3.1 above, \( \Lambda^2(V^*) \) splits into Spin(7)-irreducible submodules as

\[
\Lambda^2(V^*) \cong \Lambda^2_+ \oplus \Lambda^2_{21}.
\]

(3.3)

On the other hand, using \( V^* \cong K \oplus L \), we may also decompose \( \Lambda^2(V^*) \) as

\[
\Lambda^2(V^*) \cong \Lambda^2_+(K) \oplus \Lambda^2_-(K) \oplus (K \otimes L) \oplus \Lambda^2_+(L) \oplus \Lambda^2_-(L).
\]

(3.4)

We will refine both decompositions (3.3) and (3.4) into irreducible Spin^h(4)-modules.

To begin, note first that as Spin^h(4)-modules, we have that \( \Lambda^2_+(K) \cong V_{2,0,0}, \Lambda^2_-(L) \cong V_{0,0,2}, \)

and \( \Lambda^2_+(K) \cong \Lambda^2_2(L) \cong V_{0,2,0} \) are irreducible. Thus, it remains only to decompose \( \Lambda^2_2, \Lambda^2_{21}, \) and \( K \otimes L \).
Definition 3.4. We define

\[(\Lambda_{2}^{3})_{0,2,0} := \Lambda_{2}^{3} \cap (\Lambda_{+}^{2} (K) \oplus \Lambda_{+}^{2} (L)) ,\]
\[(\Lambda_{2}^{3})_{1,0,1} := \Lambda_{2}^{3} \cap (K \otimes L) ,\]
\[(\Lambda_{21}^{3})_{2,0,0} := \Lambda_{21}^{2} \cap \Lambda_{2}^{3} (K) ,\]
\[(\Lambda_{21}^{3})_{0,0,2} := \Lambda_{21}^{2} \cap \Lambda_{2}^{3} (L) ,\]
\[(\Lambda_{21}^{3})_{0,2,0} := \Lambda_{21}^{2} \cap (\Lambda_{+}^{2} (K) \oplus \Lambda_{+}^{2} (L)) ,\]
\[(\Lambda_{21}^{3})_{1,2,1} := \Lambda_{21}^{2} \cap (K \otimes L) .\]

The reader can check that, in fact, \(\Lambda_{2}^{3} (K) \subset \Lambda_{21}^{3} (K)\) and \(\Lambda_{2}^{3} (L) \subset \Lambda_{21}^{3} (L)\), so that \((\Lambda_{21}^{3})_{2,0,0} = \Lambda_{-}^{3} (K)\) and \((\Lambda_{21}^{3})_{0,0,2} = \Lambda_{-}^{3} (L)\).

Lemma 3.5. The decompositions

\[\Lambda_{2}^{3} = (\Lambda_{2}^{3})_{0,2,0} \oplus (\Lambda_{2}^{3})_{1,0,1}\]
\[\Lambda_{21}^{3} = \Lambda_{2}^{3} (K) \oplus (\Lambda_{21}^{3})_{0,2,0} \oplus \Lambda_{2}^{3} (L) \oplus (\Lambda_{21}^{3})_{1,2,1}\]

consist of Spin\(^{h}\) (4)-irreducible submodules.

Thus, the decomposition

\[\Lambda^{2} (V) = \left[ (\Lambda_{2}^{3})_{0,2,0} \oplus (\Lambda_{2}^{3})_{1,0,1} \right] \oplus [\Lambda_{2}^{3} (K) \oplus (\Lambda_{21}^{3})_{0,2,0} \oplus \Lambda_{2}^{3} (L) \oplus (\Lambda_{21}^{3})_{1,2,1}]\]
refines (3.3), while the decomposition

\[\Lambda^{2} (V) = \Lambda_{+}^{2} (K) \oplus \Lambda_{-}^{2} (K) \oplus \left[ (\Lambda_{21}^{3})_{1,2,1} \oplus (\Lambda_{2}^{3})_{1,0,1} \right] \oplus \Lambda_{+}^{2} (L) \oplus \Lambda_{-}^{2} (L)\]
refines (3.4).

3.2.4 Decomposition of 3-forms on \(V\)

We now decompose \(\Lambda^{3} (V^*)\) into Spin\(^{h}\) (4)-irreducible submodules. As noted in §3.1.1, the Spin(7)-irreducible decomposition of \(\Lambda^{3} (V^*)\) is

\[\Lambda^{3} (V^*) \cong \Lambda_{0}^{3} \oplus \Lambda_{48}^{3}. \tag{3.5}\]

On the other hand, using \(V^* \cong K \oplus L\), we also have the decomposition

\[\Lambda^{3} (V^*) \cong \Lambda_{2}^{3} (K) \oplus \left( (\Lambda_{+}^{2} (K) \oplus \Lambda_{-}^{2} (K)) \otimes L \right) \oplus \left( K \otimes (\Lambda_{+}^{2} (L) \oplus \Lambda_{-}^{2} (L)) \right) \oplus \Lambda^{3} (L) . \tag{3.6}\]

The summands \(\Lambda_{2}^{3} (K) \cong K \cong V_{1,1,0}\) and \(\Lambda_{2}^{3} (L) \cong L \cong V_{0,1,1}\) are irreducible, while the others are not. We have the following decompositions as Spin\(^{h}\) (4)-modules:

\[\Lambda_{-}^{2} (K) \otimes L \cong V_{0,3,1} \oplus V_{0,1,1}, \quad K \otimes \Lambda_{+}^{2} (L) \cong V_{1,3,0} \oplus V_{1,1,0},\]
\[\Lambda_{+}^{2} (K) \otimes L \cong V_{2,1,1}, \quad K \otimes \Lambda_{-}^{2} (L) \cong V_{1,1,2}.\]

We may use the Spin\(^{h}\) (4)-invariant decomposition \(V \cong K \oplus L\) to decompose the space \(\Lambda_{3}^{3}\) into Spin\(^{h}\) (4)-modules as \(\Lambda_{3}^{3} = (\Lambda_{3}^{3})_{K} \oplus (\Lambda_{3}^{3})_{L}\), where

\[(\Lambda_{3}^{3})_{K} := \{ * (\alpha \wedge \Phi_{0}) : \alpha \in K \} \cong K, \quad (\Lambda_{3}^{3})_{L} := \{ * (\alpha \wedge \Phi_{0}) : \alpha \in L \} \cong L.\]

Comparing the decompositions (3.5) and (3.6) with the decompositions of their summands, it follows that the space \(\Lambda_{48}^{3}\) decomposes into Spin\(^{h}\) (4)-modules as

\[\Lambda_{48}^{3} \cong V_{0,3,1} \oplus V_{2,1,1} \oplus V_{1,3,0} \oplus V_{1,1,2} \oplus V_{0,1,1} \oplus V_{1,1,0}. \tag{3.7}\]
Definition 3.6. Let \((A^3_{48})_{i,j,k}\) denote the \(V_{i,j,k}\) summand in the decomposition (3.7). We will also denote \((A^3_{48})_{1,1,0}\) and \((A^3_{48})_{0,1,1}\) by \((A^3_{48})_K\) and \((A^3_{48})_L\) respectively.

3.3 The Refined Torsion Forms

Let \((M^8, \Phi)\) be an 8-manifold with a \(\text{Spin}(7)\)-structure. Fix a point \(x \in M\), choose an arbitrary Cayley 4-plane \(K^2 \subset T_xM\), and let \(L^2 \subset T_xM\) denote its orthogonal 4-plane. Our purpose in this section is to understand how the torsion of the \(\text{Spin}(7)\)-structure decomposes with respect to the splitting

\[T_xM = K^2 \oplus L^2.\]

3.3.1 The Refined Torsion Forms in a Local \(\text{Spin}^h(4)\) Frame

Fix \(x \in M^8\) and split \(T^*_xM = K \oplus L\) as above. All calculations in this subsection perform pointwise, and we suppress reference to \(x \in M\). By the decompositions in §3.2, the torsion forms \(\tau_1\) and \(\tau_3\) decompose into \(\text{Spin}^h(4)\)-irreducible pieces as follows:

\[
\begin{align*}
\tau_1 &= (\tau_1)_K + (\tau_1)_L, \quad (3.8a) \\
\tau_3 &= (\tau_3)_K + (\tau_3)_L + (\tau_3)_{0,3,1} + (\tau_3)_{2,1,1} + (\tau_3)_{1,3,0} + (\tau_3)_{1,1,2}; \quad (3.8b)
\end{align*}
\]

where \((\tau_1)_* \in (\Lambda^1)_*, \) and \((\tau_3)_* \in (\Lambda^3_{48})_*\).

We refer to \((\tau_1)_K, (\tau_1)_L, \ldots, (\tau_3)_{1,1,2}\) as the refined torsion forms of the \(\text{Spin}(7)\)-structure at \(x\) relative to the splitting \(T^*_xM = K \oplus L\).

We next express the refined torsion forms in terms of a local \(\text{Spin}^h(4)\)-frame. Let \(\{e_1, \ldots, e_8\}\) be an orthonormal basis for \(T_xM\) for which \(K^2 = \text{span} (e_1, e_2, e_3, e_4)\) and \(L^2 = \text{span} (e_5, e_6, e_7, e_8)\). Let \(\{e^1, \ldots, e^8\}\) denote the dual basis for \(T^*_xM\).

Index Ranges: We will employ the following index ranges: \(1 \leq p, q \leq 4\) and \(4 \leq r, s \leq 8\) and \(1 \leq a, b \leq 7\) and \(1 \leq i, j, k, \ell, m \leq 8\) and \(1 \leq \alpha, \beta \leq 12\).

Definition 3.7. Define the 2-forms

\[
\begin{align*}
\Theta_1 &= e^{12} - e^{34}, \\
\Theta_2 &= e^{13} + e^{24}, \\
\Theta_3 &= e^{14} - e^{23},
\end{align*}
\]

\[
\begin{align*}
\Gamma_1 &= e^{12} + e^{34}, \\
\Gamma_2 &= e^{13} - e^{24}, \\
\Gamma_3 &= -e^{14} - e^{23},
\end{align*}
\]

\[
\begin{align*}
\Omega_1 &= e^{56} - e^{78}, \\
\Omega_2 &= e^{57} + e^{68}, \\
\Omega_3 &= -e^{58} + e^{67},
\end{align*}
\]

\[
\begin{align*}
\Upsilon_1 &= e^{56} + e^{78}, \\
\Upsilon_2 &= e^{57} - e^{68}, \\
\Upsilon_3 &= e^{58} + e^{67}.
\end{align*}
\]

Lemma 3.8. We have that:

(a) \(\{\Theta_1, \Theta_2, \Theta_3\}\) is a basis of \((\Lambda^2_{27})_{2,0,0}\).

(b) \(\{\Omega_1, \Omega_2, \Omega_3\}\) is a basis of \((\Lambda^3_{27})_{0,0,2}\).

(c) \(\{\Gamma_1 - \Upsilon_1, \Gamma_2 - \Upsilon_2, \Gamma_3 - \Upsilon_3\}\) is a basis of \((\Lambda^3_{27})_{0,2,0}\).

(d) \(\{\Gamma_1 - \Upsilon_1, \Gamma_2 - \Upsilon_2, \Gamma_3 - \Upsilon_3\}\) is a basis of \((\Lambda^3_{27})_{0,2,0}\).

(e) \(\{\Gamma_1 + \Upsilon_1, \Gamma_2 + \Upsilon_2, \Gamma_3 + \Upsilon_3\}\) is a basis of \((\Lambda^3_{27})_{0,2,0}\).

Definition 3.9. Define the 3-forms

\[
\begin{align*}
v_i &= t e_i \Phi_0, \\
\rho_p &= v_p - 7 *_K e_p, \\
\rho_r &= v_r - 7 *_L e_r,
\end{align*}
\]
functions

Lemma 3.10. We have that:

(a) \( \{v_p: 1 \leq p \leq 4\} \) is a basis of \((\Lambda_3^2)_{1,0}\) and \(\{v_{p+4}: 1 \leq p \leq 4\} \) is a basis of \((\Lambda_3^3)_{0,1,1}\).
(b) \( \{\rho_p: 1 \leq p \leq 4\} \) is a basis of \((\Lambda_3^3)_{1,1,0}\) and \(\{\rho_{p+4}: 1 \leq p \leq 4\} \) is a basis of \((\Lambda_3^3)_{0,1,1}\).
(c) \( \{\mu_a: 1 \leq a \leq 12\} \) is a basis of \((\Lambda_{48}^2)_{2,1,1}\) and \(\{\nu_a: 1 \leq a \leq 12\} \) is a basis of \((\Lambda_{48}^3)_{1,1,2}\).
(d) \( \{\lambda_i: 1 \leq i \leq 8\} \) is basis of \((\Lambda_{48}^3)_{0,3,1}\) and \(\{\kappa_i: 1 \leq i \leq 8\} \) is basis of \((\Lambda_{48}^3)_{1,3,0}\).

Definition 3.11. Let \( \dagger \) denote the inverse of the isometric isomorphism

\[
L^2 \rightarrow (\Lambda_{48}^3)_{L}, \quad e_r \mapsto \frac{1}{\sqrt{42}} \rho_r.
\]

We now express the refined torsion forms in terms of the above bases. That is, we define functions \(A_p, B_r\) and \(C_q, D_s, E_\beta, F_\gamma, X_i, Y_j\) by

\[
(\tau_1)_K = 32A_pe^p, \\
(\tau_1)_L = 32B_re^r, \\
(\tau_3)_K = 16C_q\rho_q, \\
(\tau_3)_L = 16D_s\rho_s, \\
(\tau_3)_{2,1,1} = 8E_\beta\mu_\beta, \\
(\tau_3)_{1,1,2} = 8F_\gamma\nu_\gamma, \\
(\tau_3)_{0,3,1} = 8X_i\lambda_i, \\
(\tau_3)_{1,3,0} = 8Y_j\kappa_j.
\]

For future use, we record the formulae

\[
[(\tau_1)_L]^\dagger = 32B_re^r, \\
[(\tau_3)_L]^\dagger = 16\sqrt{42}D_se_s.
\]
3.3.2 The Torsion Functions $T_{ai}$

Let $(M^8, \Phi)$ be an 8-manifold with a Spin(7)-structure, and let $g_\Phi$ denote the underlying Riemannian metric. Let $F_{SO(8)} \to M$ denote the oriented orthonormal coframe bundle of $g_\Phi$, and let $\omega = (\omega_1, \ldots, \omega_8) \in \Omega^1(F_{SO(8)}; V)$ denote the tautological 1-form. By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form $\psi \in \Omega^1(F_{SO(8)}; \mathfrak{so}(8))$, the Levi-Civita connection form of $g_\Phi$, satisfying the First Structure Equation

$$d\omega = -\psi \wedge \omega. \quad (3.11)$$

Let $\pi : F_{Spin(7)} \to M$ denote the Spin(7)-coframe bundle of $M$. Restricted to $F_{Spin(7)} \subset F_{SO(8)}$, the Levi-Civita 1-form $\psi$ is no longer a connection 1-form. Indeed, according to the Spin(7)-invariant splitting $\mathfrak{so}(8) = \mathfrak{spin}(7) \oplus \mathbb{R}^7$, we have the decomposition

$$\psi = \theta + 2\gamma, \quad (3.12)$$

where $\theta = (\theta_{ij}) \in \Omega^1(F_{Spin(7)}; \mathfrak{spin}(7))$ is a connection 1-form (the so-called natural connection of the Spin(7)-structure $\Phi$) and $\gamma \in \Omega(F_{Spin(7)}; \mathbb{R}^7)$ is a $\pi$-semibasic 1-form. The inclusion $\mathbb{R}^7 \hookrightarrow \mathfrak{so}(8)$ is given by

$$(\gamma_1, \ldots, \gamma_7) \mapsto \begin{bmatrix}
0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\
-\gamma_1 & 0 & \gamma_3 & -\gamma_2 & \gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
-\gamma_2 & -\gamma_3 & 0 & \gamma_1 & -\gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\
-\gamma_3 & \gamma_2 & -\gamma_1 & 0 & -\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4 \\
-\gamma_4 & -\gamma_5 & -\gamma_6 & -\gamma_7 & 0 & \gamma_1 & \gamma_2 & -\gamma_3 \\
-\gamma_5 & \gamma_4 & \gamma_7 & -\gamma_6 & -\gamma_1 & 0 & -\gamma_3 & -\gamma_2 \\
-\gamma_6 & -\gamma_7 & \gamma_4 & -\gamma_5 & -\gamma_2 & \gamma_3 & 0 & \gamma_1 \\
-\gamma_7 & \gamma_6 & -\gamma_5 & -\gamma_4 & \gamma_3 & \gamma_2 & -\gamma_1 & 0
\end{bmatrix}.$$  

Since $\gamma$ is $\pi$-semibasic, we may write

$$\gamma_a = T_{ai} \omega_i, \text{ for } 1 \leq a \leq 7$$

for some matrix-valued function $T = (T_{ai}) : F_{Spin(7)} \to \text{Hom}(V, \mathbb{R}^7)$. The functions $T_{ai}$ encode the torsion of the Spin(7)-structure: they are equivalent to the torsion forms $\tau_1$ and $\tau_3$ via the isomorphism

$$\text{Hom}(V, \mathbb{R}^7) \cong \mathbb{A}^1_8 \oplus \mathbb{A}^3_{48}. \quad (3.13)$$

3.3.3 Decomposition of the Torsion Functions

For our computations in §3.4, we will need to express the torsion function $T_{ai}$ in terms of the functions $A_p, B_r, \ldots, Y_j$. To this end, we continue to work on the total space of the Spin(7)-coframe bundle $\pi : F_{Spin(7)} \to M$, pulling back all quantities defined on $M$ to $F_{Spin(7)}$. Following common convention, and similarly to §2.3.3, we omit $\pi^*$ from the notation.

The torsion forms $\tau_1$ and $\tau_3$ satisfy

$$d\Phi = \tau_1 \wedge \Phi + *\tau_3.$$

Into the left-hand side we substitute $\Phi = \frac{1}{24} \Phi_{ijkl} \omega_{ijkl}$ and use the first structure equation (3.11), together with the decomposition (3.12), to express the $d\omega$ terms in terms of the torsion functions.
$T_{ai}$. Into the right-hand side we again substitute $\Phi = \frac{1}{24} \Phi_{ijkl} \omega_{ijkl}$, as well as the expansions (3.8) and (3.9).

Upon equating coefficients, we obtain a system of 56 linear equations relating the 56 functions $T_{ai}$ on the left-hand side to the 56 functions $A_p, B_r, \ldots, Y_j$ on the right-hand side. One can then use a computer algebra system (we have used MAPLE) to solve this linear system for the $T_{ai}$. We now exhibit the result, taking advantage of the Spin$^h$ (4)-invariant isomorphism

$$\text{Hom} \left( V, \mathbb{R}^7 \right) \cong (V_{0,2,0} \oplus V_{1,0,1}) \otimes (K \oplus L)$$

to highlight the structure of the solution.

We find

$$\begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{31} & T_{32} & T_{33} & T_{34}
\end{bmatrix} = \begin{bmatrix}
2Y_5 & -2Y_6 & -2Y_7 & 2Y_8 \\
Y_6 - 3Y_2 & -3Y_1 - Y_7 & 3Y_4 + Y_6 & 3Y_3 - Y_5 \\
-3Y_1 + Y_7 & Y_6 + 3Y_2 & 3Y_3 + Y_5 & -3Y_4 + Y_6
\end{bmatrix}
+ \begin{bmatrix}
-4C_2 + A_2 & 4C_1 - A_1 & -4C_4 + A_4 & 4C_3 - A_3 \\
-4C_3 + A_3 & 4C_4 - A_4 & 4C_1 - A_1 & -4C_2 + A_2 \\
-4C_4 + A_4 & -4C_3 + A_3 & 4C_2 - A_2 & 4C_1 - A_1
\end{bmatrix},$$

corresponding to $V_{0,2,0} \otimes K \cong V_{1,3,0} \oplus V_{1,1,0}$, and

$$\begin{bmatrix}
T_{15} & T_{16} & T_{17} & T_{18} \\
T_{25} & T_{26} & T_{27} & T_{28} \\
T_{35} & T_{36} & T_{37} & T_{38}
\end{bmatrix} = \begin{bmatrix}
-2X_3 & -2X_4 & -2X_5 & 2X_6 \\
X_5 + 3X_2 & -X_5 - 3X_1 & X_4 + 3X_8 & X_3 + 3X_7 \\
-X_5 + 3X_1 & -X_5 + 3X_2 & X_3 - 3X_7 & -X_4 + 3X_8
\end{bmatrix}
+ \begin{bmatrix}
-4D_2 + B_2 & 4D_1 - B_1 & -4D_4 + B_4 & 4D_3 - B_3 \\
-4D_3 + B_3 & 4D_4 - B_4 & 4D_1 - B_1 & -4D_2 + B_2 \\
4D_4 - B_1 & 4D_3 - B_3 & -4D_2 + B_2 & -4D_1 + B_1
\end{bmatrix},$$

corresponding to $V_{0,2,0} \otimes L \cong V_{0,3,1} \oplus V_{0,1,1}$, and

$$\begin{bmatrix}
T_{31} & T_{42} & T_{43} & T_{44} \\
T_{51} & T_{52} & T_{53} & T_{54} \\
T_{61} & T_{62} & T_{63} & T_{64} \\
T_{71} & T_{72} & T_{73} & T_{74}
\end{bmatrix} = \begin{bmatrix}
-2E_{12} + E_4 & 2E_{11} + E_3 & -2E_6 - E_{10} & -2E_5 + E_9 \\
2E_{11} - E_3 & 2E_{12} + E_4 & 2E_5 + E_9 & -2E_6 + E_{10} \\
-2E_2 - E_{10} & 2E_2 + E_9 & -2E_8 - E_4 & 2E_7 - E_5 \\
-2E_9 + 2E_1 & -2E_2 - E_{10} & -2E_7 - E_3 & -2E_8 + E_4
\end{bmatrix}
+ \begin{bmatrix}
3D_1 + B_1 & 3D_2 - B_1 & 3D_3 - B_3 & 3D_4 + B_4 \\
3D_2 + B_2 & 3D_1 + B_1 & 3D_4 + B_4 & 3D_3 + B_3 \\
3D_3 + B_3 & -3D_4 + B_4 & 3D_1 + B_1 & -3D_2 - B_2 \\
3D_4 + B_4 & 3D_3 + B_3 & -3D_2 - B_2 & -3D_1 - B_1
\end{bmatrix},$$

corresponding to $V_{1,0,1} \otimes K \cong V_{2,1,1} \oplus V_{0,1,1}$, and

$$\begin{bmatrix}
T_{45} & T_{46} & T_{47} & T_{48} \\
T_{55} & T_{56} & T_{57} & T_{58} \\
T_{65} & T_{66} & T_{67} & T_{68} \\
T_{75} & T_{76} & T_{77} & T_{78}
\end{bmatrix} = \begin{bmatrix}
2F_4 - F_{10} & -2F_3 + F_9 & 2F_4 + F_{12} & 2F_7 - F_{11} \\
-2F_3 - F_9 & -2F_4 + F_{10} & -2F_7 - F_{11} & 2F_8 - F_{12} \\
2F_2 + F_{12} & -2F_1 - F_{11} & 2F_6 + F_{10} & 2F_5 - F_9 \\
2F_1 - F_{11} & 2F_2 - F_{12} & 2F_5 + F_9 & F_{10} - 2F_6
\end{bmatrix}
+ \begin{bmatrix}
-3C_1 - A_1 & 3C_2 + A_2 & 3C_3 + A_3 & -3C_4 - A_4 \\
-3C_2 + A_2 & -3C_1 - A_1 & -3C_4 - A_4 & -3C_3 - A_3 \\
-3C_3 - A_3 & 3C_4 + A_4 & -3C_1 - A_1 & 3C_2 + A_2 \\
3C_4 + A_4 & 3C_3 + A_3 & -3C_2 - A_2 & -3C_1 - A_1
\end{bmatrix},$$

corresponding to $V_{1,0,1} \otimes L \cong V_{1,1,2} \oplus V_{1,1,0}$.
3.4 Mean Curvature of Cayley 4-Folds

In this subsection we derive a formula (Theorem 3.12) for the mean curvature of a Cayley 4-fold in an arbitrary 8-manifold \((M, \Phi)\) with \(\text{Spin}(7)\)-structure.

We continue with the notation of §3.3, letting \(\pi : F_{\text{Spin}(7)} \to M\) denote the \(\text{Spin}(7)\)-coframe bundle of \(M\) and \(\omega = (\omega_K, \omega_L) \in \Omega^1 \left( F_{\text{Spin}(7)}; \mathbb{K}^2 \oplus \mathbb{L}^2 \right)\) denote the tautological 1-form. We remind the reader that \(\theta = (\theta_{ij}) \in \Omega^1 \left( F_{\text{Spin}(7)}; \text{spin}(7) \right)\) is the natural connection 1-form, and that \(\gamma = (\gamma_{ij}) \in \Omega^1 \left( F_{\text{Spin}(7)}; \mathbb{R}^7 \right)\) is a \(\pi\)-semibasic 1-form encoding the torsion of \(\Phi\). Here \(\gamma_{ij}\) refers to the image of \((\gamma_1, \ldots, \gamma_7)\) under the map \(\mathbb{R}^7 \to \mathfrak{so}(8)\) defined by (3.13). We have \(\gamma_{ai} = T_{ai} \omega_i\) for \(T = (T_{ai}) : F_{\text{Spin}(7)} \to \text{Hom} \left( V, \mathbb{R}^7 \right)\).

Let \(f : \Sigma^4 \to M\) be an immersion of a Cayley 4-fold into \(M\), and let \(f^* \left( F_{\text{Spin}(7)} \right) \to \Sigma\) denote the pullback bundle. Let \(B \subset f^* \left( F_{\text{Spin}(7)} \right)\) denote the subbundle of coframes adapted to \(\Sigma\), i.e., the subbundle whose fibre over \(x \in \Sigma\) is

\[
B|_x = \{ u \in f^* \left( F_{\text{Spin}(7)} \right)|_x : u(T_x \Sigma) = \mathbb{K}^2 \oplus 0 \}.
\]

We recall (Proposition 3.3) that \(\text{Spin}(7)\) acts transitively on the set of Cayley 4-planes, with stabilizer isomorphic to \(\text{Spin}^h(4)\). It follows that \(B\) is a well-defined \(\text{Spin}^h(4)\)-bundle over \(\Sigma\). Note that on \(B\) we have

\[
\omega_L = 0.
\]

We may exploit the splitting \(T_x M = T_x \Sigma \oplus (T_x \Sigma)^\perp \cong \mathbb{K}^2 \oplus \mathbb{L}^2\) to decompose \(\theta\) and \(\gamma\) into \(\text{Spin}^h(4)\)-irreducible pieces. To decompose the connection 1-form \(\theta \in \Omega^1 \left( B; \text{spin}(7) \right)\), we split

\[
\text{spin}(7) \cong \Lambda^2_{21} \cong \Lambda^2_\perp (\mathbb{K}) \oplus (\Lambda^2_{21})_{0,2,0} \oplus \Lambda^2 (\mathbb{L}) \oplus (\Lambda^3_{21})_{1,2,1},
\]

so that \(\theta\) takes the block form

\[
\theta = \begin{bmatrix}
\begin{array}{cccccc}
\chi + \rho_K(\zeta) & -\sigma_T \\
\sigma & \xi + \rho_L(\zeta) \\
\end{array}
\end{bmatrix} = \\
\begin{bmatrix}
0 & \chi_1 + \zeta_1 & \chi_2 + \zeta_2 & \chi_3 - \zeta_3 & 2\sigma_1 - \sigma_7 & 2\sigma_2 - \sigma_8 & 2\sigma_3 - \sigma_{11} & -\sigma_6 + 2\sigma_{12} \\
-\chi_1 - \zeta_1 & 0 & -\chi_3 - \zeta_3 & \chi_2 - \zeta_2 & -2\sigma_2 - \sigma_8 & 2\sigma_1 + \sigma_7 & -\sigma_6 - 2\sigma_{12} & -\sigma_5 - 2\sigma_{11} \\
-\chi_2 - \zeta_2 & \chi_3 + \zeta_3 & 0 & -\chi_1 + \zeta_1 & -2\sigma_3 - \sigma_7 & -2\sigma_4 - \sigma_8 & -2\sigma_5 - \sigma_{11} & \sigma_8 + 2\sigma_{10} \\
-\chi_3 - \zeta_3 & -\chi_2 - \zeta_2 & \chi_1 - \zeta_1 & 0 & 2\sigma_4 - \sigma_6 & -2\sigma_3 + \sigma_5 & 2\sigma_6 - \sigma_{10} & -\sigma_7 - 2\sigma_9 \\
-2\sigma_1 + \sigma_7 & 2\sigma_2 + \sigma_8 & 2\sigma_3 + \sigma_5 & -2\sigma_4 + \sigma_6 & 0 & -\xi_1 - \xi_7 & -\xi_2 - \xi_7 & -\xi_3 - \xi_7 \\
-2\sigma_2 + \sigma_8 & -2\sigma_1 - \sigma_7 & 2\sigma_4 + \sigma_6 & 2\sigma_5 - \sigma_7 & \xi_1 + \xi_7 & 0 & -\xi_3 - \xi_7 & -\xi_2 - \xi_7 \\
-2\sigma_3 + \sigma_7 & \sigma_6 + 2\sigma_{12} & \sigma_7 + 2\sigma_9 & -\sigma_8 + 2\sigma_{10} & -\xi_2 + \xi_7 & \xi_3 + \xi_7 & 0 & -\xi_1 - \xi_7 \\
-\sigma_6 - 2\sigma_{12} & \sigma_5 + 2\sigma_7 & -\sigma_8 - 2\sigma_{10} & -\sigma_7 - 2\sigma_9 & -\xi_3 + \xi_7 & \xi_2 - \xi_7 & -\xi_1 + \xi_7 & 0 \\
\end{bmatrix}
\]

Similarly, the 1-form \(\gamma \in \Omega^1 \left( B; \mathbb{R}^7 \right)\) breaks into block form as

\[
\gamma = \begin{bmatrix}
\phi_K(\gamma_{0,2,0}) & - (\gamma_{1,0,1})^T \\
\gamma_{1,0,1} & \phi_L(\gamma_{0,2,0}) \\
\end{bmatrix} = \\
\begin{bmatrix}
0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\
-\gamma_1 & 0 & \gamma_3 & -\gamma_2 & \gamma_6 & -\gamma_7 & -\gamma_4 & -\gamma_5 \\
-\gamma_2 & -\gamma_3 & 0 & \gamma_1 & \gamma_6 & -\gamma_7 & -\gamma_4 & -\gamma_5 \\
-\gamma_3 & -\gamma_2 & \gamma_1 & 0 & \gamma_6 & -\gamma_7 & -\gamma_4 & -\gamma_5 \\
\gamma_4 & -\gamma_7 & -\gamma_6 & \gamma_7 & 0 & \gamma_1 & \gamma_2 & -\gamma_3 \\
\gamma_5 & \gamma_4 & \gamma_7 & \gamma_6 & -\gamma_1 & 0 & -\gamma_3 & -\gamma_2 \\
\gamma_6 & -\gamma_6 & \gamma_4 & -\gamma_5 & -\gamma_2 & \gamma_3 & 0 & \gamma_1 \\
\gamma_7 & -\gamma_6 & -\gamma_5 & -\gamma_4 & -\gamma_1 & \gamma_2 & -\gamma_3 & 0 \\
\end{bmatrix}
\]
In this notation, the first structure equation (3.11) on \(B\) reads
\[
d\left(\begin{array}{c}
\omega_K \\
0
\end{array}\right) = -\left(\begin{array}{cc}
\chi + \rho_K(\zeta) & -\sigma^T \\
\sigma & \xi + \rho_L(\zeta)
\end{array}\right) + 2\left(\begin{array}{cc}
\phi_K(\gamma_{0,2,0}) & -\gamma_{1,0,1}^T \\
\gamma_{1,0,1} & \phi_L(\gamma_{0,2,0})
\end{array}\right) \wedge \left(\begin{array}{c}
\omega_K \\
0
\end{array}\right).
\tag{3.15}
\]
In particular, the second line gives
\[
(\sigma + 2\gamma_{1,0,1}) \wedge \omega_K = 0,
\tag{3.16}
\]
or in detail
\[
\begin{bmatrix}
-2\sigma_1 + \sigma_7 & 2\sigma_2 + \sigma_8 & 2\sigma_3 + \sigma_5 & -2\sigma_4 + \sigma_6 \\
-2\sigma_2 + \sigma_8 & -2\sigma_1 - \sigma_7 & 2\sigma_4 + \sigma_6 & 2\sigma_3 - \sigma_5 \\
-\sigma_5 + 2\sigma_{11} & \sigma_6 + 2\sigma_{12} & \sigma_7 + 2\sigma_9 & -\sigma_8 + 2\sigma_{10} \\
\sigma_6 - 2\sigma_{12} & \sigma_5 + 2\sigma_{11} & -\sigma_8 - 2\sigma_{10} & -\sigma_7 + 2\sigma_9
\end{bmatrix}
\wedge
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4
\end{bmatrix}
= -2
\begin{bmatrix}
-\gamma_4 & -\gamma_5 & -\gamma_6 & \gamma_7 \\
-\gamma_5 & \gamma_4 & \gamma_7 & \gamma_6 \\
-\gamma_6 & -\gamma_7 & \gamma_4 & -\gamma_5 \\
-\gamma_7 & \gamma_6 & -\gamma_5 & -\gamma_4
\end{bmatrix}
\wedge
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4
\end{bmatrix}.
\tag{3.17}
\]

Note that on \(B\) the 1-forms \(\sigma\) and \(\gamma\) are semibasic. We have
\[
\sigma_\alpha = S_{\alpha p}\omega_p, \quad \gamma_r = T_{rq}\omega_q,
\]
for some function \(S = (S_{\alpha p}) : B \to \mathcal{V}_{1,2,1} \otimes \mathcal{K}\), recalling our index ranges \(1 \leq \alpha \leq 12\) and \(1 \leq p, q \leq 4\) and \(4 \leq r \leq 8\).

Now, the 48 functions \(S_{\alpha p}\) and the 16 functions \(T_{rq}\) are not independent: equation (3.17) amounts to 24 linear relations among them, 8 of which involve only the functions \(S_{\alpha p}\). In our calculation of the mean curvature vector of a Cayley 4-fold we shall require only the following 4 of the relations:
\[
\begin{bmatrix}
S_{8,2} - 2S_{3,3} + 2S_{4,4} - S_{5,3} + 2S_{1,1} - 2S_{2,2} - S_{6,4} - S_{7,1} \\
-S_{8,1} - 2S_{3,4} - 2S_{4,3} + S_{5,4} + 2S_{1,2} + 2S_{2,1} - S_{6,3} + S_{7,2} \\
S_{8,4} - 2S_{1,11} - 2S_{1,22} + S_{5,1} - 2S_{3,3} - 2S_{10,4} - S_{6,2} - S_{7,3} \\
S_{8,3} - 2S_{1,12} + 2S_{1,21} - S_{5,2} - 2S_{9,4} + 2S_{10,3} - S_{6,1} + S_{7,4}
\end{bmatrix}
= 6
\begin{bmatrix}
T_{7,4} + T_{4,1} + T_{5,2} + T_{6,3} \\
-T_{7,3} - T_{3,2} + T_{5,1} - T_{6,4} \\
T_{7,2} - T_{4,3} + T_{5,4} + T_{6,1} \\
T_{7,1} + T_{4,4} + T_{5,3} - T_{6,2}
\end{bmatrix}.
\tag{3.18}
\]

**Theorem 3.12.** Let \(\Sigma \subset M\) be a Cayley 4-fold immersed in an 8-manifold \(M\) with Spin(7)-structure. Then the mean curvature vector \(H\) of \(\Sigma\) is given by
\[
H = -[\langle \tau_1 \rangle_L^2 - \sqrt{37} \langle \tau_3 \rangle_L^2]^{\dagger}.
\]
In particular, the largest torsion class of Spin(7)-structures for which every Cayley 4-fold is minimal is the class of torsion-free Spin(7)-structures.

**Proof.** Let \(\beta_p = *K\omega_p \in \Omega^1(B)\), and let \(\text{vol}_K = \omega_{1234}\). The components of the mean curvature
vector $H$ of $\Sigma$ may be computed as follows:

\[
\begin{bmatrix}
H_5 \\
H_6 \\
H_7 \\
H_8
\end{bmatrix}_{\text{vol}_K} = 
\begin{bmatrix}
\psi_{51} & \psi_{52} & \psi_{53} & \psi_{54} \\
\psi_{61} & \psi_{62} & \psi_{63} & \psi_{64} \\
\psi_{71} & \psi_{72} & \psi_{73} & \psi_{74} \\
\psi_{81} & \psi_{82} & \psi_{83} & \psi_{84}
\end{bmatrix}
\wedge
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix}
\wedge
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix}
\]

(3.19)

To evaluate the first term in (3.19), we substitute $\sigma_\alpha = S_{\alpha p} \omega_p$, followed by (3.18), and finally (3.14), to obtain:

\[
\begin{bmatrix}
-2S_{11} + S_{7,1} + 2S_{22} + 2S_{8,2} + 2S_{13} + 5S_{5,3} - 2S_{44} + S_{64} \\
-2S_{21} + S_{8,1} - 2S_{12} + 2S_{43} + 5S_{6,3} + 2S_{34} - S_{45} \\
- S_{51} + 2S_{11} + S_{6,2} + 2S_{12} + 2S_{7,2} + 5S_{9,3} + S_{84} + 2S_{104} \\
S_{61} - 2S_{121} + S_{5,2} + 2S_{112} - S_{8,3} - 2S_{103} - S_{7,4} + 2S_{94}
\end{bmatrix}_{\text{vol}_K} = 
\begin{bmatrix}
-24B_5 - 72D_5 \\
-24B_6 - 72D_6 \\
-24B_7 - 72D_7 \\
-24B_8 - 72D_8
\end{bmatrix}_{\text{vol}_K}.
\]

Similarly, to evaluate the second term in (3.19), we substitute $\gamma_r = T_{rq} \omega_q$, followed by (3.14), to obtain:

\[
2
\begin{bmatrix}
-T_{41} - T_{52} - T_{63} + T_{74} \\
-T_{51} + T_{42} + T_{73} + T_{64} \\
-T_{61} - T_{72} + T_{43} - T_{54} \\
-T_{71} + T_{62} - T_{53} - T_{44}
\end{bmatrix}_{\text{vol}_K} = 
\begin{bmatrix}
-8B_5 - 24D_5 \\
-8B_6 - 24D_6 \\
-8B_7 - 24D_7 \\
-8B_8 - 24D_8
\end{bmatrix}_{\text{vol}_K}.
\]

We conclude that $H_r = -32B_r - 96D_r$, and so (3.10) yields that

\[
H = -[(\tau_1)_L]^2 - \frac{\sqrt{2}}{7}[(\tau_3)_L]^2
\]

\[\diamondsuit\]

### 3.4.1 The Second Fundamental Form of a Cayley 4-fold

The mean curvature vector is not the only part of the second fundamental form of a Cayley 4-fold that can be written in terms of the torsion forms of the ambient Spin(7)-structure. For a Cayley 4-fold, the second fundamental form is naturally a section of a vector bundle modeled on the Spin$^h$ (4)-representation

\[
\text{Sym}^2 (K) \otimes L \cong (V_{2,2,0} \oplus \mathbb{R}) \otimes L \cong V_{2,3,1} \oplus V_{0,3,1} \oplus V_{2,1,1} \oplus L,
\]

(3.20)

and so the second fundamental form naturally decomposes into four pieces, with the piece corresponding to the $L$ summand equal to the mean curvature vector.

By calculations similar to those performed in the proof of Theorem 3.12, it is possible to show that the piece of the second fundamental form corresponding to the $V_{0,3,1}$ summand in (3.20) is identically zero, while the piece corresponding to the $V_{2,1,1}$ summand in (3.20) is proportional to the refined torsion form $(\tau_3)_{2,1,1}$. 

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