The three different regimes in coulombic friction

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Abstract. de Gennes identified three regimes in the phenomenon of the Langevin equation which includes Coulombic friction. Here we extend and precise this phenomenon to a constant external force.

Keywords: Brownian motion, Coulombic Friction, Langevin equation.

1. Introduction

P-G de Gennes$^3$ studied the Langevin equation under the influence of a dry friction force modelled by the equation

\[ dv = -\frac{1}{2}\Delta \text{sgn}(v)dt + \sqrt{D}dB, \]

the dry friction force with threshold force \( \Delta > 0 \), and \( D > 0 \) is the diffusion coefficient. Here \( B \) is the standard Brownian motion, and \( \text{sgn}(v) = 1 \) if \( v > 0 \), and \( \text{sgn}(v) = -1 \) if \( v < 0 \). Comparing the magnitude of \( \alpha, \Delta \) and \( D \) de Gennes$^3$ identified three different regimes: viscous, partly stuck and stuck.

Later Touchette et al.$^7$ extended de Gennes work by calculating the time-dependent propagator of the Langevin equation

\[ dv = -\frac{1}{2}[\alpha v - a + \Delta \text{sgn}(v)]dt + \sqrt{D}dB, \quad (1) \]

which includes a constant external force \( a \in \mathbb{R} \).

In this paper, we precise and extend de Gennes’s work to the Langevin equation (1) and find again the result of Touchette et al.$^7$ using the trivariate density of Brownian motion, its local and occupation times.

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2. The three different regimes in coulombic friction

If \( v(t) \) is solution of (1), then \( v(T_D) \) satisfies the equation

\[
dv = -\frac{1}{2D} [\alpha v - a + \Delta \text{sgn}(v)] dt + dB.
\]

(2)

It follows that for large time \( T \) the PDF of the velocity \( v(T_D) \) is approximated by the stationary PDF

\[
\frac{1}{Z} \exp \left[ -\frac{1}{\nu} \left( \frac{(v - y)^2}{2\tau} + |v| \right) \right],
\]

where

\[
Z = \frac{1}{2\nu} \left[ \exp\left( \frac{\tau - 2y}{2\nu} \right) G\left( \frac{\tau - y}{\sqrt{\nu \tau}} \right) + \exp\left( \frac{\tau + 2y}{2\nu} \right) G\left( \frac{\tau + y}{\sqrt{\nu \tau}} \right) \right]
\]

is the partition function i.e. the normalization constant. Here and the sequel

\[
G(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left( -\frac{v^2}{2} \right) dv,
\]

\[
\nu = \frac{D}{\Delta}, \quad \tau = \frac{\Delta}{\alpha}, \quad y = \frac{a}{\alpha}.
\]

We say that the stochastic process \( (V_D : D > 0) \) defined in some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) converges in probability distribution as \( D \to 0 \) to the PDF \( f \) if for each couple \( l < r \) of real numbers

\[
\mathbb{P}(l \leq V_D \leq r) \to \int_{l}^{r} f(v) dv, \quad \text{as} \quad D \to 0.
\]

Now we can announce our result.

1) Stuck regime. If \(|a| < \Delta\), then the velocity \( v(T_D) \to 0 \) as \( D \to 0 \). More precisely \( \frac{1}{\nu} v(T_D) \) converges in distribution as \( D \to 0 \) to the PDF

\[
\frac{1 - y^2}{2} \exp \left[ -|v|(1 - \text{sgn}(v)|y|) \right].
\]

Observe that if the constant force \( a = 0 \), then \( y = 0 \) and the limit is

\[
\frac{1}{2} \exp(-|v|).
\]

2) Partly stuck regime. If \(|a| = \Delta\), then the velocity \( v(T_D) \to 0 \) as \( D \to 0 \). More precisely we distinguish two cases.

a) If we consider only the event \( av(T_D) < 0 \), then

\[
\frac{1}{\nu} v(T_D) \to 2 \exp(-2|v|) \mathbf{1}_{[av<0]} \quad \text{as} \quad D \to 0.
\]

b) If we consider only the event \( av(T_D) > 0 \), then

\[
\frac{1}{\nu} v(T_D) \to 2 \frac{\sqrt{2\pi \tau}}{\sqrt{v^2}} \exp(-\frac{v^2}{2\tau}) \mathbf{1}_{[av>0]} \quad \text{as} \quad D \to 0.
\]

Moreover the probability of the event \( av(T_D) > 0 \) tends to 1 as \( D \to 0 \). Hence \( \frac{1}{\nu} v(T_D) \) converges to \( \frac{2}{\sqrt{2\pi \tau}} \exp(-\frac{v^2}{2\tau}) \mathbf{1}_{[av>0]} \).

3) Viscous regime. If \(|a| > \Delta\) then as \( D \to 0 \) the velocity \( v(T_D) \) becomes Gaussian with the mean \( (y - \text{sgn}(y)\tau) \) and the variance \( \nu \tau \). More precisely, we have

\[
\frac{v(T_D) - (y - \text{sgn}(y)\tau)}{\sqrt{\nu}} \to \frac{1}{\sqrt{2\pi \tau}} \exp(-\frac{v^2}{2\tau}).
\]
Observe that the asymptotic mean $y - sgn(y)\tau$ is the minimizer of the potential $v \to \frac{(v-y)^2}{2\tau} + |v| := U(v)$.

![Figure 1](image-url)

**Figure 1.** Three scenarios of the stuck regime with $y = 0, 0.4, 0.9$, partly stuck regime with $y = 1$ and $\tau = 1$ and viscous regime with $y = 3$ and $\tau = 1$.

The proof was done in a general case in [4]. For the sake of completeness we recall it. It is sufficient to show the case $a \geq 0$ i.e. $y \geq 0$.

3. **Proof**

3.1. **Stuck regime**

We observe that the potential $U$ attains its minimum $\frac{y^2}{2\tau}$ at $v = 0$. We have

$$
\mathbb{P}(l \leq \frac{v(T_D)}{\nu} \leq r) = \frac{\int_{\nu}^{\nu} \exp(-\frac{U(v)}{\nu})dv}{\int_{-\infty}^{+\infty} \exp(-\frac{U(v)}{\nu})dv}.
$$

Multiplying the denominator and the nominator by $\exp(\frac{y^2}{2\nu})$, and using the change of variable $\frac{v}{\nu}$ we have

$$
\mathbb{P}(l \leq \frac{v(T_D)}{\nu} \leq r) = \frac{\int_{\nu}^{\nu} \exp \left[-|v|(1 - sgn(v))\frac{y}{\tau} - \sqrt{\nu}\frac{y^2}{2\tau}\right] dv}{\int_{-\infty}^{+\infty} \exp \left[-|v|(1 - sgn(v))\frac{y}{\tau} - \sqrt{\nu}\frac{y^2}{2\tau}\right] dv}.
$$
The latter converges to
\[ \frac{\int^r_l \exp \left(-|v|(1 - \text{sgn}(v)\frac{\nu}{2}) \right) dv}{\int^{+\infty}_{-\infty} \exp \left[-|v|(1 - \text{sgn}(v)\frac{\nu}{2}) \right] dv} \]
as \( \nu \to 0 \), which achieves the proof of the stuck regime.

3.2. Partly stuck regime

a) We are going to prove for each \( l < r \leq 0 \) that
\[ \mathbb{P}(l \leq v(T_D) \leq r | v(T_D < 0) \to \int_l^r 2 \exp(2v)dv. \]
We have
\[ \mathbb{P}(l \leq v(T_D) \leq r | v(T_D < 0) = \frac{\int_{\nu}^r \exp \left[-\frac{1}{\nu}(v + \frac{(v-\tau)^2}{2\tau}) \right] dv}{\int_{-\infty}^0 \exp \left[-\frac{1}{\nu}(\nu + \frac{(\nu-\tau)^2}{2\tau}) \right] dv}. \]
Multiplying the denominator and the nominator by \( \exp(\tau^2\nu) \), and using the change of variable \( \frac{v}{\nu} \) we obtain
\[ \frac{\int_l^r \exp \left(2v - \sqrt{v^2/\nu} \right) dv}{\int_{-\infty}^0 \exp \left(2v - \sqrt{v^2/\nu} \right) dv}. \]
The latter converges to
\[ \frac{\int_l^r \exp(2v)dv}{\int_{-\infty}^0 \exp(2v)dv} \]
as \( \nu \to 0 \), which achieves the proof of the part 1.

b) We have, for \( 0 < l < r \),
\[ \mathbb{P}(l \leq v(T_D) \leq r | v(T_D > 0) = \frac{\int_{\sqrt{\nu}}^r \exp \left[\frac{-1}{\nu}(v + \frac{(v-\tau)^2}{2\tau}) \right] dv}{\int_{\sqrt{\nu}}^{+\infty} \exp \left[\frac{-1}{\nu}(v + \frac{(v-\tau)^2}{2\tau}) \right] dv}. \]
Multiplying the denominator and the nominator by \( \exp(-\sqrt{\nu}) \) and using the change of variable \( \frac{v}{\sqrt{\nu}} \) we get the proof of the first part of b).

For the second part we use the same proof and show that \( \mathbb{P}(av(T_D) > 0) \to 1 \) as \( D \to 0 \).

3.3. Viscous regime

The main tool of the proof is the following well known result see e.g. [1].

**Lemma:** Let \( H \) be any measurable map such that
\[ \int_{-\infty}^{+\infty} \exp(-H(v))dv < +\infty \]
and
\[ \inf \{H(v) : |v - v_0| \geq \delta \} > H(v_0) \]
for some \( v_0 \) and \( \delta > 0 \). Then for any \( \gamma > 0 \),
\[
\nu^{-\gamma} \int_{|v-v_0|\geq \delta} \exp \left[ -\frac{1}{\nu}(H(v) - H(v_0)) \right] dv \to 0
\]
as \( \nu \to 0 \).

Now, let us apply this lemma with \( H(v) = U(v) \) and \( v_0 = y - \tau \) the minimizer of \( U \). We have, for \( l < r \),
\[
P(l \leq \frac{v(T_D) - (y - \tau)}{\sqrt{\nu}} \leq r) = \frac{\int_{l}^{r} \exp \left( -\frac{1}{2\nu}(v-(y-\tau))^2 \right) dv}{\int_{-\infty}^{+\infty} \exp(-\frac{1}{2\nu}U(v)) dv}.
\]
We have for \( v > 0 \), that
\[
U(v) - U(y - \tau) = \frac{(v - (y - \tau))^2}{2\tau}.
\]
If \( l > -\infty \), then for small \( \nu \), we have
\[
\int_{l}^{r} \exp \left[ -\frac{1}{\nu}(U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} = \int_{l}^{r} \exp \left[ -\frac{1}{2\nu}(v-(y-\tau))^2 \right] \frac{dv}{\sqrt{\nu}}
\]
and then
\[
P(l \leq \frac{v(T_D) - (y - \tau)}{\sqrt{\nu}} \leq r) = \int_{l}^{r} \exp(-\frac{v^2}{2\tau}) dv.
\]
If \( l = -\infty \), then
\[
\int_{-\infty}^{r} \exp \left[ -\frac{1}{\nu}(U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} = (1) + (2),
\]
where
\[
(1) = \int_{|v|<0} \exp \left[ -\frac{1}{\nu}(U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} \to 0,
\]
\[
(2) = \int_{[0\leq v \leq \sqrt{\nu}+(y-\tau)]} \exp \left[ -\frac{1}{\nu}(U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}}.
\]
From Lemma (3.3) the term (1) converges to 0. By the change of variable \( z = \frac{v-(y-\tau)}{\sqrt{\nu}} \), the term (2)
\[
(2) = \int_{[-\frac{(y-\tau)}{\sqrt{\nu}} \leq z \leq r]} \exp(-\frac{z^2}{2\tau}) dz
\]
converges to \( \int_{-\infty}^{\infty} \exp(-\frac{z^2}{2\tau}) dz \). By taking \( r = +\infty \), we get
\[
\int_{-\infty}^{+\infty} \exp \left[ -\frac{1}{\nu}(U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} \to \int_{-\infty}^{+\infty} \exp(-\frac{z^2}{2\tau}) dz,
\]
and then
\[
P(l \leq \frac{v(T_D) - (y - \tau)}{\sqrt{\nu}} \leq r) \to \int_{l}^{r} \exp(-\frac{v^2}{2\tau}) \frac{dv}{\sqrt{2\pi \tau}},
\]
which achieves the proof.
4. Time-dependent propagator

Now we drop the coefficient $\frac{1}{2}$ in (1) and we discuss the calculation of the time-dependent propagator of

$$dv = -[\alpha v + a + \Delta \text{sgn}(v)]dt + \sqrt{D}dB.$$  

Using the equality of the laws or the probability distributions of $(\sqrt{DB}(t))$ and $(B(t))$, we derive that

$$\text{Law}(v^{\alpha,a,\Delta,D}(t)) = \text{Law}(v^{\alpha,D,a,\Delta,D}(t)).$$

Hence the propagators $p^{\alpha,a,\Delta,D}(v, t \mid v_0, 0)$ and $p^{\alpha,D,a,\Delta,D}(v, t \mid v_0, 0)$ respectively of $v^{\alpha,a,\Delta,D}(t)$ and $v^{\alpha,D,a,\Delta,D}(t)$ satisfy the relation

$$p^{\alpha,a,\Delta,D}(v, t \mid v_0, 0) = p^{\alpha,D,a,\Delta,D}(v, D(t) \mid v_0, 0).$$

Hence, it sufficient to study the case $D = 1$.

5. Time-dependent propagator for $\alpha = a = 0$ using local occupation time

We denote by $\mathbb{P}$ and $\mathbb{P}_{v_0}$ the probability distribution respectively of the trajectories $s \in [0, t] \to v(s)$ of the solution of (1) and the Brownian motion starting from $v_0$.

Under the probability distribution

$$\exp \left( -\Delta \int_0^t \text{sgn}(B_s)dB_s - \frac{t\Delta^2}{2} \right) d\mathbb{P}_{v_0} := f_{\text{sgn}}(B) d\mathbb{P}_{v_0}$$

the process $(B(s) : s \in [0, t])$ is solution of the equation

$$dv = -\Delta \text{sgn}(v)dt + dB, \quad v(0) = v_0.$$  

We simplify the stochastic integral $\int_0^t \text{sgn}(B_s)dB_s$ using Tanaka formula [6]

$$|B_t| = |v_0| + \int_0^t \text{sgn}(B_s)dB_s + 2L_t.$$

Here the local time

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} \int_0^t 1_{|B_s| \leq \varepsilon} ds$$

$$= \frac{1}{2} \int_0^t \delta(B_s)ds.$$

It follows that

$$-\int_0^t \text{sgn}(B_s)dB_s = |v_0| - |B_t| + 2L_t.$$

Now,

$$f_{\text{sgn}}(B) = \exp \left( \Delta(|v_0| - |B_t| + 2L_t) - \frac{t\Delta^2}{2} \right).$$

The densities of $v(t)$ and the Brownian motion $B(t)$ are related by

$$p(v, t \mid v_0) = \mathbb{E}_{v_0} \left[ \delta(B_t - v) \exp(\Delta(|v_0| - |B_t| + 2L_t) - \frac{t\Delta^2}{2}) \right].$$

The latter formula is also known as path integral representation [2]. Hence the law of the solution $v(t)$ is given by the law of $(B_t, L_t)$. 


5.1. Density of Brownian motion and its local time

Set $\Gamma_t = \int_0^t 1[B_s \geq 0]ds$, and

$$h(s, v) = \frac{|v|}{\sqrt{2\pi s^3}} \exp(-\frac{v^2}{2s}), \quad s > 0, v \in \mathbb{R}.$$ 

Karatzas and Shreve [3] have calculated the probability density $\mathbb{P}_{v_0}(B_t \in db, L_t \in dl, \Gamma_t \in d\tau) := p_t(dv, dl, d\tau | v_0)$ of $(B_t, L_t, \Gamma_t)$ as follows. For $v_0 \geq 0$ we have

$$\mathbb{P}_{v_0}(B_t \in db, L_t \in dl, \Gamma_t \in d\tau) = 2h(\tau, l + v_0)h(t - \tau, l - b)dbdl, \quad b < 0,$$

$$\mathbb{P}_{v_0}(B_t \in db, L_t \in dl, \Gamma_t \in d\tau) = 2h(t - \tau, l)h(\tau, l + b + v_0)dbdl, \quad b > 0,$$

$$\mathbb{P}_{v_0}(B_t \in db, L_t = 0, \Gamma_t = t) = \omega(v_0, b, t), \quad b > 0, v_0 \geq 0,$$

where

$$\omega(v_0, b, t) = \gamma_t(b - v_0) - \gamma_t(b + v_0),$$

$$\gamma_t(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2t}).$$

We derive the joint distribution of $(B_t, L_t)$ under $\mathbb{P}_{v_0}$ with $v_0 \geq 0$:

$$\mathbb{P}_{v_0}(B_t \in db, L_t \in dl) = \frac{2}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2l + v_0 - b)^2}{2t}\right) dbdl, \quad b < 0, l > 0,$$

$$\mathbb{P}_{v_0}(B_t \in db, L_t \in dl) = \frac{2}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2l + v_0 + b)^2}{2t}\right) dbdl + \omega(v_0, b, t)\delta(l), \quad b > 0, l \geq 0.$$

Now, we calculate the density of the solution $[3]$ as follows. If $v < 0$, then

$$p(v, t | v_0) = \mathbb{E}_{v_0} \left[ \delta(B(t) - v) \exp\left(\Delta(v_0 - |B_t| + 2L_t) - \frac{t\Delta^2}{2}\right) \right]$$

$$:= \exp\left(\Delta(v_0 + v) - \frac{t\Delta^2}{2}\right) \mathbb{E}_{v_0} [\delta(B(t) - v) \exp(2\Delta L_t)]$$

$$= 2 \exp\left(\Delta(v_0 + v) - \frac{t\Delta^2}{2}\right) \int_0^{+\infty} \frac{2l - v_0}{\sqrt{2\pi t^3}} \exp(2\Delta l - \frac{(2l - v_0)^2}{2t}) dl$$

$$= \exp(\Delta(v_0 + v) - \frac{t\Delta^2}{2}) \frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{(v_0 - v)^2}{2t}\right) + \Delta \int_0^{+\infty} \exp(\Delta l - \frac{(l + v_0 - v)^2}{2t}) \frac{dl}{\sqrt{2\pi t^3}}$$

After some calculation we obtain

$$\int_0^{+\infty} \exp(\Delta l - \frac{(l + v_0 - v)^2}{2t}) \frac{dl}{\sqrt{2\pi t^3}} = \exp(\frac{\Delta^2 t}{2}) \exp(\Delta(v - v_0)) F\left(\frac{v - v_0 + \Delta t}{\sqrt{t}}\right),$$

where $F(v) = \int_{-\infty}^v \frac{\exp\left(-\frac{u^2}{2}\right)}{\sqrt{2\pi}} du$. Finally for $v_0 \geq 0, v < 0$, we have

$$p(v, t | v_0) = \left(\exp(-\frac{t\Delta^2}{2}) \gamma_t(v_0 - v) \exp(\Delta(v_0 - v)) + F\left(\frac{v - v_0 + \Delta t}{\sqrt{t}}\right)\right) \Delta \exp(2\Delta v).$$
If \( v > 0 \), then
\[
p(v, t | v_0) = \mathbb{E}_{v_0} \left[ \delta(B(t) - v) \exp \left( \Delta(v_0 - |B_0| + 2L_t) - \frac{t\Delta^2}{2} \right) \right]
\]
\[
= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \mathbb{E}_{v_0} \left[ \delta(B(t) - v) \exp(2\Delta L_t) \right]
\]
\[
= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \left[ 2 \int_0^{+\infty} \frac{(2l + v + v_0)}{\sqrt{2t^3\pi}} \exp(2\Delta l - \frac{(2l + v + v_0)^2}{2t}) dl + \omega(v_0, v, t) \right]
\]
\[
= \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \omega(v_0, v, t)
\]
\[
+ \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \int_0^{+\infty} \frac{(l + v + v_0)}{\sqrt{2t^3\pi}} \exp(\Delta l - \frac{(l + v + v_0)^2}{2t}) dl
\]
\[
= \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \omega(v_0, v, t) + \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \left[ \frac{1}{\sqrt{2t\pi}} \exp(-\frac{(v_0 + v)^2}{2t}) \right]
\]
\[
+ \Delta \int_0^{+\infty} \exp(\Delta l - \frac{(l + v + v_0)^2}{2t}) \frac{dl}{\sqrt{2t\pi}}
\]
\[
= \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \omega(v_0, v, t) + \frac{1}{\sqrt{2t\pi}} \exp(-\frac{t\Delta^2}{2}) \exp(\Delta(v_0 - v)) \exp(-\frac{(v_0 + v)^2}{2t})
\]
\[
+ \Delta \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \int_0^{+\infty} \exp(\Delta l - \frac{(l + v + v_0)^2}{2t}) \frac{dl}{\sqrt{2t\pi}}
\]

From some calculation we obtain
\[
\int_0^{+\infty} \exp(\Delta l - \frac{(l + v + v_0)^2}{2t}) \frac{dl}{\sqrt{2t\pi}} = \exp(\frac{\Delta^2 t}{2}) \exp(-\Delta(v + v_0)) F(\frac{\Delta t - (v + v_0)}{\sqrt{t}}).
\]

Finally if \( v > 0, v_0 \geq 0 \), then
\[
p(v, t | v_0) = \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \omega(v_0, v, t) + \frac{1}{\sqrt{2t\pi}} \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \exp(-\frac{(v_0 + v)^2}{2t})
\]
\[
+ \Delta \exp(-2\Delta v) F(\frac{\Delta t - (v + v_0)}{\sqrt{t}})
\]
\[
= \frac{1}{\sqrt{2t\pi}} \exp(\Delta(v_0 - v) - \frac{t\Delta^2}{2}) \exp(-\frac{(v_0 - v)^2}{2t}) + \Delta \exp(-2\Delta v) F(\frac{\Delta t - (v + v_0)}{\sqrt{t}})
\]
\[
= \left( \exp(\Delta(v_0 + v) - \frac{t\Delta^2}{2}) \gamma_t(v - v_0) + F(\frac{\Delta t - (v + v_0)}{\sqrt{t}}) \right) \Delta \exp(-2\Delta v).
\]

Finally we have for \( v, v_0 \in \mathbb{R} \), that
\[
p(v, t | v_0) = q(v, t | v_0) \exp(-2\Delta|v|)
\]

where
\[
q(v, t | v_0) = \Delta \left( \exp(\Delta(|v_0| + |v|) - \frac{t\Delta^2}{2}) \gamma_t(v - v_0) + F(\frac{\Delta t - (|v| + |v_0|)}{\sqrt{t}}) \right).
\]

Observe that \( q(v, t | v_0) \) is symmetric, i.e. \( q(v, t | v_0) = q^\Delta(v_0, t | v) \). In the language of linear diffusion \( m(v) = \exp(-2\Delta|v|) \) is the speed measure of the linear diffusion \[3\].
6. The case $a \neq 0$

In this case the probability distribution $\mathbb{P}$ of the solution

$$dv = -[\Delta sgn(v) + a]dt + dB, \quad v(0) = v_0,$$

is also absolutely continuous with respect to $\mathbb{P}_{v_0}$ (the probability distribution of the Brownian motion starting from $v_0$). We have

$$\frac{d\mathbb{P}}{d\mathbb{P}_{v_0}}(B) = \exp\left(- \int_0^t (\Delta sgn(B_s) + a)dB_s - \frac{1}{2} \int_0^t (\Delta sgn(B_s) + a)^2ds\right).$$

After some calculation we have

$$- \int_0^t (\Delta sgn(B_s) + a)dB_s = \Delta(|v_0| - |B_t| + a(v_0 - B_t)) + 2\Delta L_t,$$

$$\int_0^t (\Delta sgn(B_s) + a)^2ds = (\Delta^2 + a^2)t + 2a\Delta(2\Gamma_t - t).$$

It follows that

$$p(v, t | v_0) = \exp\left[\Delta(|v_0| - |v| + a(v_0 - v)) - \frac{(\Delta - a)^2}{2} \right] \mathbb{E}_{v_0}[\delta(B_t - v) \exp(2\Delta L_t - 2a\Delta \Gamma_t)].$$

Then $p(v, t | v_0)$ is calculated using the trivariate probability distribution $p_t(db, dl, d\tau)$ of $(B_t, L_t, \Gamma_t)$ as follows:

$$p(v, t | v_0) = \exp\left[\Delta(|v_0| - |v| + a(v_0 - v)) - \frac{(\Delta - a)^2}{2} \right] \int_0^\infty \int_0^t \exp(2\Delta(l - a\tau))p_t(v, dl, d\tau).$$

7. The general case

Similarly as above the density of the solution of

$$dv = -[\alpha v + \Delta sgn(v) + a]dt + dB, \quad v(0) = v_0,$$

is

$$p(v, t | v_0) = \exp\left[\Delta(|v_0| - |v| + a(v_0 - v)) - \frac{(\Delta - a)^2}{2} + \frac{\alpha t}{2} \right] \int_0^\infty \int_0^t \int_0^\infty \int_0^\infty \exp(2\Delta l - 2a\Delta \tau - \frac{\alpha^2}{2}b_2 - \alpha\Delta|b_1| - a\alpha b_1)p_t(v, dl, d\tau, db_1, d|b_1|, db_2),$$

where $p_t(db, dl, d\tau, db_1, d|b_1|, db_2)$ is the probability density of

$$(B_t, L_t, \Gamma_t, \int_0^t B_s ds, \int_0^t |B_s| ds, \int_0^t B_s^2 ds).$$

8. Conclusion

We have precised and extended the three different regimes of the Langevin equation which includes a viscous friction force, a Coulombic friction and a constant external force. Moreover we find again its time-dependent propagator using the density of Brownian motion, its local and occupation times.
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