Impurity Green’s function of the one-dimensional Fermi-gas

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Abstract. We consider the one-dimensional gas of fermions interacting with \(\delta\)-function interaction, at finite positive coupling constant. We compute the time-dependent two-point correlation function of a spin down fermion in a gas of fully polarized fermions, all having spin up. For this correlation function a representation in terms of a Fredholm determinant is obtained.

1. Introduction

Consider the one-dimensional system of quantum particles driven by the secondary quantized Hamiltonian

\[
H = \int_0^L \left( \sum_{s=\uparrow, \downarrow} \partial_x \psi_s^\dagger(x) \partial_x \psi_s(x) + 2c \psi_\uparrow^\dagger(x) \psi_\uparrow(x) \psi_\downarrow^\dagger(x) \psi_\downarrow(x) \right),
\]

(1.1)

where \(c\) is a coupling constant, \(c > 0\). The one-dimensional fields \(\psi_s(x)\) and \(\psi_s^\dagger(x)\) \((s = \uparrow, \downarrow)\) are quantum operators in the Fock space with canonical anti-commutation relations

\[
\psi_s(x) \psi_s^\dagger(x') + \psi_s^\dagger(x') \psi_s(x) = \delta_{ss'} \delta(x - x'),
\]

(1.2)

with all other anti-commutators vanishing. The impurity Green’s function is the two-point correlation function

\[
G_\downarrow(x,t) = \langle \Omega_\uparrow | e^{itH} \psi_\downarrow(x) e^{-itH} \psi_\downarrow^\dagger(0) | \Omega_\uparrow \rangle,
\]

(1.3)

where \(|\Omega_\uparrow\rangle\) is the normalized ground state of \(N\) spin up fermions, \(\langle \Omega_\uparrow | \Omega_\uparrow \rangle = 1\). The ground state is parametrized by the Fermi momentum \(k_F = \pi N/L\). The intermediate states between the two field operators in (1.3) belongs to the sector of the Fock space with \(N + 1\) fermions: \(N\) of spin up and one of spin down. The first quantization Hamiltonian corresponding to (1.1) in this sector describes a gas of \(N\) polarized fermions interacting via the \(\delta\)-function potential with an impurity, which is the spin down fermion. Thus the correlation function (1.3) describes propagation of an impurity in a gas of free Fermi particles.

The original solution of the impurity problem (i.e., the problem of eigenfunctions of the first quantized Hamiltonian in the sector with \(N\) spin up and one spin down fermions) was first given by McGiure [1]. A simpler formulation of the eigenfunctions, in the form of a determinant, was proposed by Edwards [2]; see also the work of Castella and Zotos [3]. A general solution of the eigenstates problem of the
Hamiltonian (1.1) was given in a seminal paper by C. N. Yang \cite{Yang1967} and in a series of papers by Gaudin, see \cite{Gaudin1969} and references therein.

In this paper, using a modified version of the Edwards’ solution for finite $N$ and $L$, we represent the correlation function (1.3) in terms of determinants of $N \times N$ matrices. In the thermodynamic limit, $N,L \to \infty$, with $N/L$ fixed, we obtain a representation for the impurity Green’s function in terms of a Fredholm determinants of linear integral operators of the so-called integrable type. In computing the impurity Green’s function (1.3) here, we follow in many respects the same technique as in the case of infinite coupling, adopting it to the case of finite coupling.

Representations in terms of Fredholm determinants for correlation functions were introduced, on the example of density matrix of the Tonks-Gerardeau gas (impenetrable Bose-gas), by Schultz \cite{Schultz1969} and Lenard \cite{Lenard1970, Lenard1971}. Subsequently, it was realized that the representations of this kind plays an important role in description of correlation functions of one-dimensional quantum solvable models in terms classical integrable systems \cite{Yang1970, Komar1971}. The role of integrable integral operators has been realized in \cite{Baxendale1972}. The whole approach makes it possible to construct asymptotic expansions of correlation functions, e.g., for long times and large distances using the method of matrix Riemann-Hilbert problem \cite{Zamolodchikov1980, Zakharov1981}; for a recent progress, see \cite{Zabrocki2013, Zabrocki2015}.

At the infinite coupling ($c = \infty$) the general, depending on temperature, time, external magnetic field and chemical potential, two-point correlation functions of the two-component Fermi and Bose gases were computed in terms of Fredholm determinants in \cite{Gaudin1969, Yang1970}. Similar results in the context of the Hubbard model and ladder spin (a spin-1/2 Bose-Hubbard) model at the infinite coupling were given in \cite{Yang1970, Yang1971}. Various asymptotic results about these correlation functions, using the Reimann-Hilbert problem, have been obtained in \cite{Zabrocki2013, Zabrocki2015}. Applications of these results to description of physical phenomena are discussed in \cite{Zabrocki2015, Zabrocki2016}.

Recently, a number of new phenomena were predicted for the dynamics and relaxation of the mobile impurity injected into a free Fermi gas \cite{Yang1970, Yang1971}. We expect that our findings will pave a way towards quantitative description of those phenomena.

2. Bethe Ansatz and the impurity problem

The basis in the Fock space of the model is constructed by acting with operators $\psi_s^\dagger(x)$ onto the pseudovacuum $|0\rangle$, defined as

$$\psi_s(x) |0\rangle = 0, \quad \langle 0 | \psi_s^\dagger(x) = 0, \quad \langle 0 | 0 \rangle = 1.$$  \hspace{0.5cm} (2.1)

We say that a state belongs to the sector $(N,M)$ of the Fock space if it contains $N - M$ particles of spin up and $M$ particles of spin down, so that the total number of particles is $N$. The number of particles of each type being conserved separately, an eigenstate of the Hamiltonian (1.1) can be obtained as a linear superposition of the basis states from the same sector. In the sector $(N,M)$ the eigenstates are enumerated by two sets, $\{k\} = k_1, \ldots, k_N$ and $\{\lambda\} = \lambda_1, \ldots, \lambda_M$, of unequal, in each set separately, numbers, called quasi-momenta.

In the sector $(N,M)$ the eigenstates can be written in the form

$$|\Psi_{N,M}(\{k\}; \{\lambda\})\rangle = \int_{[0,L]^N} d^N x \sum_{\{s\}} \Psi_{N,M}^{(s)}(\{k\}, \{\lambda\}; \{x\}) \psi^\dagger_{s_1}(x_1) \cdots \psi^\dagger_{s_N}(x_N) |0\rangle,$$  \hspace{0.5cm} (2.2)
where \( \{ x \} := x_1, \ldots, x_N \) and the wave function \( \Psi_{N,M}^{\{ s \}}(\{ k \}, \{ \lambda \} | \{ x \}) \) is not equal to zero only if \( M \) elements are equal to \( \downarrow \) and \( N - M \) elements are equal to \( \uparrow \) in the set \( \{ s \} := s_1, \ldots, s_N \). The quasi-momenta are solutions of the nested Bethe Ansatz equations:

\[
e^{ik_j L} = \prod_{a=1}^{M} \frac{k_j - \lambda_a + ic/2}{k_j - \lambda_a - ic/2}, \quad j = 1, \ldots, N, \tag{2.3}
\]

\[
\prod_{j=1}^{N} \frac{k_j - \lambda_a + ic/2}{k_j - \lambda_a - ic/2} = \prod_{b=1}^{M} \frac{\lambda_b - \lambda_a + ic}{\lambda_b - \lambda_a - ic}, \quad a = 1, \ldots, M.
\]

The eigenvalue (energy) of the Hamiltonian (1.1) in the sector \((N,M)\) is

\[
E_{N,M}(\{ k \}, \{ \lambda \}) = \sum_{j=1}^{N} k_j^2, \tag{2.4}
\]

where \( k \)th are solutions of (2.3). For what follows we need only the eigenstates belonging to the sectors \((N,0)\) and \((N+1,1)\). The former are given simply by the Slater determinant wave-function while the latter can be constructed using a connection to the problem of mobile impurity moving in a Fermi-gas of \( N \) fermions.

In the sector \((N,0)\) the second set of quasi-momenta is empty, \( \{ \lambda \} = \emptyset \), and the Bethe equations for the elements of the first set are given by the first equation in (2.3) with the right-hand side equal to one. For the sake of further convenience in calculations, we denote the quasi-momenta of the first set as \( \{ q \} = q_1, \ldots, q_N \).

The eigenstate (2.2) in this case reads

\[
|\Psi_{N,0}(\{ q \})\rangle = \int_{[0,L]^N} \Psi_{N,0}^{-\uparrow}^{\uparrow} (\{ q \} | \{ x \}) \psi_{\uparrow}^{\dagger}(x_1) \cdots \psi_{\uparrow}^{\dagger}(x_N) |0\rangle, \tag{2.5}
\]

where and the wave-function is

\[
\Psi_{N,0}^{-\uparrow}^{\uparrow} (\{ q \} | \{ x \}) = \frac{1}{N!L^{N/2}} \det_{1 \leq j,l \leq N} (e^{iq_j x_l}). \tag{2.6}
\]

The quasi-momenta are quantized as

\[
q_j = \frac{2\pi L}{m_j}, \quad m_j \in \mathbb{Z}, \tag{2.7}
\]

and in order that the wave function does not vanish, they must all be distinct; the eigenstates in the considered sector are labeled in a unique way by the sets \( \{ q \} \) where \( q_1 < q_2 < \cdots < q_N \).

The state \( |\Omega_{\uparrow}\rangle \) corresponds to the ground-state in the sector \((N,0)\). The ground-state corresponds to the choice \(-k_F \leq q_1 < \cdots < q_N \leq k_F \) for \( N \) odd, and \(-k_F < q_1 < \cdots < q_N \leq k_F \) or \(-k_F \leq q_1 < \cdots < q_N < k_F \) for \( N \) even,

\[
k_F = \pi N/L. \tag{2.8}
\]

In the thermodynamic limit, \( N, L \to \infty \), with the ratio \( D = N/L \) fixed, the quasi-momenta in the set \( \{ q \} \) are uniformly distributed along the interval \([-k_F, k_F]\), with \( k_F = \pi D \); we discuss this in more detail in the section concerning the evaluation of the thermodynamic limit.
Now we turn to construction of the eigenstates in the sector \((N + 1, 1)\). According to (2.2) they have the form
\[
|\Psi_{N+1,1}(\{k\}, \lambda)\rangle = \int_{[0,L]^{N+1}} d^{N+1}x \sum_{j=1}^{N+1} \Psi_{N+1,1}^j(\{k\}, \lambda|\{x\}) \\
\times \psi^\dagger_1(x_1) \cdots \psi^\dagger_j(x_{j-1}) \psi^\dagger_j(x_j) \psi^\dagger_{j+1}(x_{j+1}) \cdots \psi^\dagger_N(x_{N+1}) |0\rangle,
\]
where \(\uparrow_j\) denotes the set of \(N + 1\) spins with all but one spins up, \(\uparrow_j = \uparrow \cdots \uparrow \downarrow \uparrow \cdots \uparrow\), with the down spin standing on the \(j\)th position. We first note that one can rewrite the eigenstates by relabeling coordinates such that field operator with spin down depends on the coordinate \(x_{N+1}\),
\[
|\Psi_{N+1,1}(\{k\}, \lambda)\rangle = \int_{[0,L]^{N+1}} d^{N+1}x \tilde{\Psi}_{N+1,1}(\{k\}, \lambda|\{x\}) \\
\times \psi^\dagger_1(x_1) \cdots \psi^\dagger_{j-1}(x_{j-1}) \psi^\dagger_j(x_j) \psi^\dagger_{j+1}(x_{j+1}) |0\rangle.
\]
The wave-function \(\tilde{\Psi}_{N+1,1}(\{x\})\) (to simplify writing, we lift the dependence of the wave-functions on quasi-momenta where possible) is expressed in terms of the original ones as follows
\[
\tilde{\Psi}_{N+1,1}(\{x\}) = \sum_{j=1}^{N+1} (-1)^{N-j} \Psi_{N+1,1}^j(x_1, \ldots, x_{j-1}, x_{N+1}, x_{j+1}, \ldots, x_N),
\]
and must be an eigenfunction of the first quantized Hamiltonian
\[
\mathcal{H}_{N+1,1} = - \sum_{j=1}^{N+1} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j=1}^{N} \delta(x_i - x_{N+1}),
\]
where \(x_1, \ldots, x_{N+1} \in [0, L]\), with the periodic boundary conditions.

The standard way (see, e.g., [2, 3]) to diagonalize the Hamiltonian (2.12) is to represent it first in the reference frame of the spin down particle, introducing the coordinates
\[
y_j = x_j - x_{N+1}, \quad j = 1, \ldots, N.
\]
Hence, \(x_j = y_j + x_{N+1}, j = 1, \ldots, N\), and therefore
\[
\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, N, \quad \frac{\partial}{\partial x_{N+1}} = i\mathcal{P}_{N+1} - \sum_{j=1}^{N} \frac{\partial}{\partial y_j},
\]
where \(\mathcal{P}_{N+1}\) is the first quantized total momentum operator of \(N + 1\) particles,
\[
\mathcal{P}_{N+1} = \sum_{j=1}^{N+1} \frac{1}{\partial x_j}.
\]
Since \(\mathcal{P}_{N+1}\) is an integral of motion, \([\mathcal{P}_{N+1}, \mathcal{H}_{N+1,1}] = 0\), it can be replaced by its eigenvalues \(P\) when acting on the eigenfunctions,
\[
P = \sum_{j=1}^{N+1} k_j = \frac{2\pi}{L} n, \quad n \in \mathbb{Z}.
\]
The Hamiltonian in the new coordinates reads
\[ H_{N+1,1} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial y_j^2} + \left( P + i \sum_{j=1}^{N} \frac{\partial}{\partial y_j} \right)^2 + 2c \sum_{j=1}^{N} \delta(y_j). \] (2.17)

Noticing that this in fact an \( N \)-particle Hamiltonian and denoting its eigenfunctions as \( \Phi_N(\{y\}) \), we have the following relation between the wave-functions in the two reference frames [3]:
\[ \tilde{\Psi}_{N+1,1}(\{x\}) = e^{iP_{x_{N+1}}\lambda} \Phi_N(x_1 - x_{N+1}, \ldots, x_N - x_{N+1}). \] (2.18)

The functions \( \Phi_N(\{y\}) = \Phi_N(\{k\}, \lambda(\{y\})) \) must be periodic function in each coordinate, and totally antisymmetric with respect to their permutations. The periodicity implies that
\[ \Phi_N(\{y\})|_{y_j=0} = \Phi_N(\{y\})|_{y_j=L}, \quad j = 1, \ldots, N, \] (2.19)
and that the first derivatives, due to the \( \delta \)-function potential in (2.17), must satisfy the conditions
\[ \left. \frac{\partial}{\partial y_j} \Phi_N(\{y\}) \right|_{y_j=0} - \left. \frac{\partial}{\partial y_j} \Phi_N(\{y\}) \right|_{y_j=L} = c \Phi_N(\{y\})|_{y_j=0}, \quad j = 1, \ldots, N. \] (2.20)

The antisymmetry implies that \( \Phi_N(\{y\}) \) is a determinant
\[ \Phi_N(\{y\}) = C_{\Phi_N} \det_{1 \leq j, l \leq N} (\phi_j(y)). \] (2.21)

Here \( C_{\Phi_N} \) is a normalization factor to be found later, and the functions \( \phi_j(y) \) can be represented as a superpositions of at least \( N + 1 \) plane waves:
\[ \phi_j(y) = \sum_{l=1}^{N+1} a_{jl} e^{ik_l y}. \] (2.22)

The condition (2.19) implies that
\[ \sum_{l=1}^{N+1} a_{jl} (1 - e^{ik_l L}) = 0, \quad j = 1, \ldots, N; \] (2.23)
and (2.20) that
\[ \sum_{l=1}^{N+1} a_{jl} (ik_l (1 - e^{ik_l L}) - c) = 0, \quad j = 1, \ldots, N. \] (2.24)

Subtracting (2.23) from (2.24) with the factor \( i\lambda - c/2 \), where \( \lambda \) is the quasi-momentum of the auxiliary BA problem, and requiring that all coefficients of the sum in the resulting equation vanish, we recover the first set of BA equations (2.3) of the sector \((N+1, 1)\):
\[ e^{ik_l L} = \frac{k_l - \lambda + ic/2}{k_l - \lambda - ic/2}, \quad l = 1, \ldots, N + 1. \] (2.25)

Recalling the quantization condition for the total momentum (2.16), we also have the equation for \( \lambda \):
\[ \prod_{l=1}^{N+1} \frac{k_l - \lambda + ic/2}{k_l - \lambda - ic/2} = 1. \] (2.26)
Obviously, the obtained equations are exactly the nested Bethe Ansatz equations of the sector \((N + 1, 1)\), see (2.3).

The Bethe Ansatz equations (2.25) can also be written in the form
\[
\cot \frac{k_j L}{2} = \frac{2(k_j - \lambda)}{c}, \quad j = 1, \ldots, N + 1. \tag{2.27}
\]

Introduce the notation
\[
\alpha_j = -\arccot \frac{2(k_j - \lambda)}{c}, \quad j = 1, \ldots, N + 1. \tag{2.28}
\]

We use the convention that \(\arccot x \in [0, \pi], x \in \mathbb{R}\). The quantization conditions for momenta \(\{k\}\) are
\[
k_j = \frac{2\pi}{L} n_j - \frac{2}{L} \alpha_j, \quad n_j \in \mathbb{Z}. \tag{2.29}
\]

Equation (2.26) has the form
\[
\frac{R_N(\lambda)}{Q_{N+1}(\lambda)} = 0, \tag{2.30}
\]
where \(R_N(\lambda)\) and \(Q_{N+1}(\lambda)\) are polynomials in \(\lambda\) of the degrees shown in the subscripts,
\[
R_N(\lambda) = Q_{N+1}(\lambda - ic) - Q_{N+1}(\lambda), \quad Q_{N+1}(\lambda) = \prod_{l=1}^{N+1} \left( k_l - \lambda - \frac{ic}{2} \right). \tag{2.31}
\]

The quantization condition for the quasi-momentum \(\lambda\), which takes \(N + 1\) values, follows from the relation
\[
\sum_{j=1}^{N+1} \alpha_j = -\pi m, \quad m = 0, 1, \ldots, N. \tag{2.32}
\]

The value \(m = 0\) corresponds to \(\lambda = -\infty\); the values \(m = 1, \ldots, N\) correspond to the roots of the polynomial \(R_N(\lambda)\).

To finalize the construction of the wave-function \(\Phi_N(\{y\})\), we have to satisfy the conditions (2.23). It can be easily seen that this can be done by choosing
\[
\phi_j(y) = \chi_j(y) - \chi_{N+1}(y), \quad j = 1, \ldots, N, \tag{2.33}
\]
where
\[
\chi_l(y) = e^{ik_l y + i\alpha_l} = -\frac{2}{c} \left( k_l - \lambda - \frac{ic}{2} \right) e^{ik_l y}, \quad l = 1, \ldots, N + 1. \tag{2.34}
\]

With our choice it is obvious that the wave-function \(\Phi_N(\{y\})\) can also be represented as the following \((N + 1) \times (N + 1)\) determinant:
\[
\Phi_N(\{y\}) = C_{\Phi_N} \begin{vmatrix} \chi_1(y_1) & \cdots & \chi_1(y_N) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \chi_{N+1}(y_1) & \cdots & \chi_{N+1}(y_N) & 1 \end{vmatrix}. \tag{2.35}
\]

Recalling relation (2.18), we find that the function \(\tilde{\Psi}_{N+1,1}(\{x\})\) can be written in the form
\[
\tilde{\Psi}_{N+1,1}(\{x\}) = C_{\Phi_N} \begin{vmatrix} \chi_1(x_1) & \cdots & \chi_1(x_N) & e^{ik_1 x_{N+1}} \\ \vdots & \ddots & \vdots & \vdots \\ \chi_{N+1}(x_1) & \cdots & \chi_{N+1}(x_N) & e^{ik_{N+1} x_{N+1}} \end{vmatrix}. \tag{2.36}
\]
To satisfy the normalization condition \( \langle \Psi_{N+1,1}(\{k\}, \lambda) | \Psi_{N+1,1}(\{k\}, \lambda) \rangle = 1 \) the constant \( C_{\Phi_N} \) has to be chosen such that

\[
|C_{\Phi_N}|^2 = \frac{1}{(N!)^2 L^{N+1}} \left( \sum_{l=1}^{N+1} \prod_{j=1 \atop j \neq l}^{N+1} \left( \frac{1}{\sin^2 \alpha_j} + \frac{4}{L c} \right) \right)^{-1}. \tag{2.37}
\]

We prove (2.37) in the next section.

3. Finite volume calculations

In this section we perform various calculations with the wave-functions obtained in the previous section. We first prove formula (2.37). We have

\[
\langle \Psi_{N+1,1}(\{k\}, \lambda) | \Psi_{N+1,1}(\{k\}, \lambda) \rangle = N! L \int_{[0, L]^N} d^N y |\Phi_N(\{y\})|^2 = |C_{\Phi_N}|^2 (N!)^2 L^{N+1} \det_{1 \leq j, l \leq N} \left( \frac{1}{L} \int_0^L dy \overline{\phi_j(y)} \phi_l(y) \right). \tag{3.1}
\]

To evaluate the entries of the matrix here, we use the following relation, valid for \( k_j \neq k_l \),

\[
\int_0^L dy e^{i(k_j - k_l)y} = \frac{e^{i(k_j - k_l)L} - 1}{i(k_j - k_l)} = -\frac{4 \sin \alpha_j \sin \alpha_l}{c} e^{ia_j - ia_l}, \tag{3.2}
\]

where the second equality follows by considering a difference of two Bethe Anzatz equations (2.27). For the off-diagonal entries, we obtain

\[
\frac{1}{L} \int_0^L dy \overline{\phi_j(y)} \phi_l(y) = \frac{1}{\sin^2 \alpha_{N+1}} - \frac{4}{L c} \quad (j \neq l), \tag{3.3}
\]

and for the diagonal ones, we obtain

\[
\frac{1}{L} \int_0^L dy |\phi_j(y)|^2 = \frac{1}{\sin^2 \alpha_j} + \frac{1}{\sin^2 \alpha_{N+1}} - \frac{8}{L c}. \tag{3.4}
\]

Denoting \( u_j = 1/\sin^2 \alpha_j - 4/Lc, j = 1, \ldots, N + 1 \), we see that the \( N \times N \) matrix in the determinant in (3.1) is the sum of a diagonal matrix, with entries \( u_j \delta_{jl} \), and of a matrix of rank one, with all entries equal to \( u_{N+1} \). Hence,

\[
\det_{1 \leq j, l \leq N} \left( \int_0^L dy \overline{\phi_j(y)} \phi_l(y) \right) = \det_{1 \leq j, l \leq N} (u_j \delta_{jl} + u_{N+1}) \]

\[
= u_1 \cdots u_N \left( 1 + \left( u_1^{-1} + \cdots + u_N^{-1} \right) u_{N+1} \right) = \sum_{j=1}^{N+1} u_j^{-1} \prod_{j=1}^{N+1} u_j, \tag{3.5}
\]

and formula (2.37) follows.
Next we consider the form-factors of the operator \( \psi_\downarrow(x) \) relevant to the correlation function (1.3). Specifically, these are its matrix elements between the states belonging to the sectors \((N,0)\) and \((N+1,1)\). We have

\[
\langle \Psi_{N,0}\{q\}\rangle |e^{itH} \psi_\downarrow(x)e^{-itH}\rangle |\Psi_{N+1,1}\{k\},\lambda\rangle = \langle \Psi_{N,0}\{q\}\rangle |\psi_\downarrow(0)|\Psi_{N+1,1}\{k\},\lambda\rangle \exp \left\{ i \sum_{j=1}^{N} \tau(q_j) - i \sum_{j=1}^{N+1} \tau(k_j) \right\},
\]

(3.6)

where \( \tau(q) = tq^2 - xq \). Using expressions (2.5) and (2.35) for the wave-functions involved, for the form-factor we obtain the following representation:

\[
\langle \Psi_{N,0}\{q\}\rangle |\psi_\downarrow(0)|\Psi_{N+1,1}\{k\},\lambda\rangle = N! \int_{[0,L]^N} d^N x \Psi_{N,0}^{\uparrow\cdots\uparrow}(x) \Phi_N(x) = N! C_{\Phi_N} \frac{F_N\{q\}\{k\}}{LN/2},
\]

(3.7)

where

\[
F_N\{q\}\{k\} = \det_{1 \leq j,l \leq N} \left( \int_0^L dx e^{-iq_jx} \phi_l(x) \right).
\]

(3.8)

Taking into account that \( e^{iq_jL} = 1 \) and \( e^{ik_lL} = e^{-2ik_lL} \), the integrals are evaluated as follows:

\[
\int_0^L dx e^{-iq_jx} \chi_i(x) = \frac{e^{i\alpha_i}}{\sin \alpha_i} \int_0^L dx e^{i(k_l-q_j)x} = \frac{\sin \alpha_j}{i(k_j-q_l)} \left( e^{i(k_l-q_j)L} - 1 \right) = \frac{2}{q_j-k_l}.
\]

Hence,

\[
F_N\{q\}\{k\} = \det_{1 \leq j,l \leq N} \left( \frac{2}{q_j-k_l} - \frac{2}{q_j-k_{N+1}} \right).
\]

(3.10)

We also note that the function \( F_N\{q\}\{k\} \) can be written as an \((N+1) \times (N+1)\) determinant,

\[
F_N\{q\}\{k\} = \begin{vmatrix} 2(q_1-k_1)^{-1} & \cdots & 2(q_1-k_{N+1})^{-1} \\ \vdots & \ddots & \vdots \\ 2(q_N-k_1)^{-1} & \cdots & 2(q_N-k_{N+1})^{-1} \\ 1 & \cdots & 1 \end{vmatrix},
\]

(3.11)

which can be obtained when using (2.35) instead of (2.21).

We now ready to consider the correlation function on the finite lattice, defined as a diagonal matrix element of the two-point field operator in the sector \((N,0)\),

\[
G_{4,N}(x,t\{q\}) = \langle \Psi_{N,0}\{q\}\rangle |e^{itH} \psi_\downarrow(x)e^{-itH}\psi_\downarrow^\dagger(0)|\Psi_{N,0}\{q\}\rangle.
\]

(3.12)

This correlation function can be written as the sum over all states (i.e., all distinct solutions of the Bethe Ansatz equations) in the sector sector \((N+1,1)\):

\[
G_{4,N}(x,t\{q\}) = \sum_{\{k\},\lambda} \left| \langle \Psi_{N,0}\{q\}\rangle |\psi_\downarrow(0)|\Psi_{N+1,1}\{k\},\lambda\rangle \right|^2 \times \exp \left\{ i \sum_{j=1}^{N} \tau(q_j) - i \sum_{j=1}^{N+1} \tau(k_j) \right\}.
\]

(3.13)
Using (3.7) and taking into account (2.37), we have

$$G_{\downarrow,N}(x,t\{|q\}) = \frac{1}{L^{2N+1}} \sum_{\{k\},\lambda} \frac{u^{-1}_1 \cdots u^{-1}_{N+1}}{u^{-1}_1 + \cdots + u^{-1}_{N+1}}$$

$$\times F_N^2(\{q\}|\{k\}) \exp \left\{ i \sum_{j=1}^{N} \tau(q_j) - i \sum_{j=1}^{N+1} \tau(k_j) \right\}. \quad (3.14)$$

Here the quantities $u_1, \ldots, u_{N+1}$ are

$$u_j = \frac{1}{\sin^2 \alpha_j} + a, \quad j = 1, \ldots, N + 1, \quad (3.15)$$

where for a later convenience we introduced the notation $a = 4/Lc$.

The sum (3.14) is defined as the sum over all distinct sets of integers $\{n\} = n_1, \ldots, n_{N+1}$ and the integer $m = 0, \ldots, N$, which label the solutions of the Bethe Ansatz equations,

$$\sum_{\{k\},\lambda} = \sum_{n_1 \in \mathbb{Z}}^{n_{N+1} \in \mathbb{Z}} \sum_{m=0}^{N} \cdot \quad (3.16)$$

Since every term of the sum in (3.14), being totally symmetric with respect to permutations of $k_j$’s, vanishes as soon as any two of $k_j$’s coincide, and since $k_i = k_j$ if $n_i = n_j$, the sum over ordered sets of integers $\{n\}$ can be replaced by independent sums over these integers,

$$\sum_{n_1 \in \mathbb{Z}, n_{N+1} \in \mathbb{Z}} \rightarrow \frac{1}{(N + 1)!} \sum_{n_1 \in \mathbb{Z}} \cdots \sum_{n_{N+1} \in \mathbb{Z}}. \quad (3.17)$$

However, since the Bethe Ansatz equations for the momenta $\{k\}$ and the quasi-momentum $\lambda$ are coupled, the $k_j$’s and $\lambda$ depend on all these integers, $k_j = k_j(\{n\}, m)$, $\lambda = \lambda(\{n\}, m)$, so some further transformations in (3.14) are required to handle the summations. Namely, it would be convenient that the sum over each individual $n_j$ implies the summation over the admissible values of the momentum $k_j$ only; it would be useful also to have an interpretation of the sum of $u^{-1}_j$’s standing in the denominator in (3.14), which prevents a factorized summation over admissible values of $k_j$’s in the multiple sum.

It turns out that all this can be achieved by considering the momenta $k_j$ as functions of $\lambda$, namely, $k_j = k_j(\lambda)$, in which $\lambda$ is allowed to take arbitrary (real) values; the solutions of Bethe Ansatz equations (2.25) and (2.26) correspond to the values $\lambda = \Lambda_m, m = 0, \ldots, N$. These values will follow as roots of certain (transcendent) equation imposed on the set of functions $\{k(\lambda)\}$. To be more precise, let us introduce a function $z_n(\lambda)$, where $n \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$, as the solution of the equation

$$z_n(\lambda) = \pi n + \arccot \left( a z_n(\lambda) - \frac{2 \lambda}{c} \right). \quad (3.18)$$

As it can be easily seen (e.g., using graphical interpretation by taking cotangent of both sides) that this equation, for every value of $n$, defines a single-valued, continuous, monotonously increasing function of $\lambda$. The last property can be seen
by taking the derivative in $\lambda$ of $z(n; \lambda)$, expressing it in terms of $z(n; \lambda)$,

$$\partial_\lambda z_n(\lambda) = \frac{2/c}{1 + a + (a z_n(\lambda) - 2\lambda/c)^2}. \quad (3.19)$$

Now, given a set of integers $\{n\} = n_1, \ldots, n_{N+1}$, we define the set of functions $\{k(\lambda)\} = k_1(\lambda), \ldots, k_{N+1}(\lambda)$ by setting

$$k_j(\lambda) = \frac{2}{L} z_{n_j}(\lambda), \quad j = 1, \ldots, N + 1. \quad (3.20)$$

Clearly, here each $k_j(\lambda)$ depends solely on $n_j$, as desired. Furthermore, let us introduce functions $\alpha_j(\lambda)$ and $u_j(\lambda)$ by defining them by (2.28) and (3.15), respectively, where $k_j$ is replaced by $k_j(\lambda)$. Namely, we set $\alpha_j(\lambda) := \alpha(k_j, \lambda)$ and $u_j(\lambda) := u(k_j, \lambda)$, where

$$\alpha(k, \lambda) = -\arccot \left( \frac{2(k - \lambda)}{c} \right),$$

$$u(k, \lambda) = \frac{1}{\sin^2 \alpha(k, \lambda)} + a = 1 + a + \left( \frac{2(k - \lambda)}{c} \right)^2. \quad (3.21)$$

Relation (3.19) then implies

$$\partial_\lambda k_j(\lambda) = \frac{a}{u_j(\lambda)}. \quad (3.22)$$

Hence, introducing a $\lambda$-dependent “total momentum” $P(\lambda) = \sum_j k_j(\lambda)$, we have

$$\sum_{j=1}^{N+1} u_j^{-1}(\lambda) = a^{-1} \partial_\lambda P(\lambda). \quad (3.23)$$

This expression clarify the meaning the sum of $u_j^{-1}$’s standing in the denominator of the expression under the sum sign in (3.14) if we recall relation (2.28), which implies that the solutions of Bethe Ansatz equation in our treatment are restored at the values $\lambda = \Lambda_0, \ldots, \Lambda_N$, that is

$$\sum_{j=1}^{N+1} \alpha_j(\Lambda_m) = -\pi m, \quad m = 0, 1, \ldots, N. \quad (3.24)$$

An equivalent way to impose this condition is to set

$$P(\Lambda_m) = \frac{2\pi}{L} \sum_{j=1}^{N+1} n_j + \frac{2\pi}{L} m. \quad (3.25)$$

Hence, for given values of the set of integers $\{n\}$ and integer $m$, the sum of $u_j^{-1}$’s is exactly the derivative of the total momentum at $\lambda = \Lambda_m$,

$$\sum_{j=1}^{N+1} u_j^{-1} = a^{-1} P'(\Lambda_m), \quad (3.26)$$

where we used the notation $P'(\Lambda_m) := \partial_\lambda P(\lambda)|_{\lambda=\Lambda_m}$.

To show how these considerations make it possible to factorize of the summation in (3.14), let us consider the sum over $m$ at some fixed set of values of the integers $n_1, \ldots, n_{N+1}$. Using (3.26) for the denominator and regarding the remaining part
(the numerator) of the summands as some trial function \( f(\lambda) \), we can transform this sum using the following chain of identities involving the Dirac \( \delta \)-function:

\[
\sum_{m=0}^{N} \frac{f(\Lambda_m)}{P'(\Lambda_m)} = \int_{-\infty}^{+\infty} d\lambda \sum_{m=0}^{N} \frac{\delta(\lambda - \Lambda_m)}{P'(\lambda)} f(\lambda) = \int_{-\infty}^{+\infty} d\lambda \sum_{m=0}^{N} \frac{\delta(P(\lambda) - P(\Lambda_m)) f(\lambda)}{P'(\lambda)} = \frac{L}{2} \int_{-\infty}^{+\infty} d\lambda \sum_{m=0}^{N} \delta \left( \sum_{j=1}^{N+1} \alpha_j(\lambda) + \pi m \right) f(\lambda) = \frac{L}{4\pi} \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} ds \frac{1-e^{(N+1)\pi s}}{1-e^{\pi s}} \exp \left\{ is \sum_{j=1}^{N+1} \alpha_j(\lambda) \right\} f(\lambda). \tag{3.27}
\]

Here at the last step we replaced the Dirac \( \delta \)-function by its Fourier transform, to bring the dependence on the \( k_j(\lambda)'s \) (defining the \( \alpha_j(\lambda)'s \)) in a factorized form.

As a result, taking into account (3.17), we have the following representation for the correlation function of the finite system

\[
G_{4, N}(x, t|\{q\}) = \frac{aL}{4\pi} \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} ds \frac{1-e^{(N+1)\pi s}}{1-e^{\pi s}} \Xi_N(x, t|\{q\}, \lambda; s), \tag{3.28}
\]

where

\[
\Xi_N(x, t|\{q\}, \lambda; s) = \frac{(N+1)!}{L^{2N+1}} \sum_{k_1(\lambda)} \cdots \sum_{k_{N+1}(\lambda)} \prod_{j=1}^{N+1} \frac{1}{u_j(\lambda)}
\]

\[
\times F_N^2(\{q\}|\{k\}) \exp \left\{ i \sum_{j=1}^{N} \tau(q_j) - i \sum_{j=1}^{N+1} [\tau(k_j(\lambda)) - s\alpha_j(\lambda)] \right\}. \tag{3.29}
\]

Here each sum is performed over all the values of the momentum \( k_j(\lambda) \) defined in (3.20) as the corresponding \( n_j \) runs over all integer values.

The expression (3.29) can be written in terms of \( N \times N \) determinants, using the procedure of “insertion of the summation in the determinant” similarly to the infinite coupling case [18, 34]. Using the total symmetry in permutation of momenta \( k_1, \ldots, k_{N+1} \) of the general term of the sum in (3.29), one of the functions \( F_N(\{q\}|\{k\}) \), when represented as the \( (N+1) \times (N+1) \) determinant (3.11), can be replaced by the product

\[
F_N(\{q\}|\{k\}) \longrightarrow (N+1)! \prod_{j=1}^{N} \frac{2}{q_j - k_j}, \tag{3.30}
\]

while the second one, when using (3.10), can be written as a sum of two terms

\[
F_N(\{q\}|\{k\}) = \left[ \det_{1 \leq j, l \leq N} \left( \frac{2}{q_j - k_l} - \frac{2}{q_j - k_{N+1}} \right) - \det_{1 \leq l \leq N} \left( \frac{2}{q_j - k_l} \right) \right] + \det_{1 \leq j, l \leq N} \left( \frac{2}{q_j - k_l} \right). \tag{3.31}
\]

Since the matrix with the entries \( 2/(q_j - k_{N+1}) \) is of rank one, the first term in this sum is a homogeneous linear function of its entries while the second term is
independent of this momentum. This allows one easily perform the summation with respect to \( k_{N+1} \). The summation with respect to the remaining momenta \( k_1, \ldots, k_N \) are performed in the usual way.

As a result, the quantity \( \Xi_N(x, t|\{q\}, \lambda; s) \) is given in terms of determinants of \( N \times N \) matrices:

\[
\Xi_N(x, t|\{q\}, \lambda; s) = \det(S - R) + (G(x, t; \lambda; s) - 1) \det S. \tag{3.32}
\]

The matrix \( S = S(x, t|\{q\}, \lambda; s) \) has entries

\[
S_{jl} = \frac{4e^{i(\tau(q_j) + \tau(q_l))/2}}{L^2} \sum_{k(\lambda)} u(k(\lambda), \lambda)(k(\lambda) - q_j)(k(\lambda) - q_l) e^{-i\tau(k(\lambda)) + i\sigma_0(k(\lambda), \lambda)} \tag{3.33}
\]

and the matrix \( R = R(x, t|\{q\}, \lambda; s) \), of rank one, has entries

\[
R_{jl} = \frac{4e^{i(\tau(q_j) + \tau(q_l))/2}}{L^3} \sum_{k(\lambda)} u(k(\lambda), \lambda)(k(\lambda) - q_j) \sum_{k(\lambda)} u(k(\lambda), \lambda)(k(\lambda) - q_l) e^{-i\tau(k(\lambda)) + i\sigma_0(k(\lambda), \lambda)} \tag{3.34}
\]

The function \( G(x, t; \lambda; s) \) is

\[
G(x, t; \lambda; s) = \frac{1}{L} \sum_{k(\lambda)} e^{-i\tau(k(\lambda)) + i\sigma_0(k(\lambda), \lambda)} u(k(\lambda), \lambda). \tag{3.35}
\]

The functions \( \alpha(k, \lambda) \) and \( u(k, \lambda) \) are defined in (3.21); the dependence on \( x \) and \( t \) is contained in the function \( \tau(k) = tk^2 - xk \).

4. Results in the thermodynamic limit

The thermodynamic limit is the limit in which \( L, N \to \infty \), with the ratio \( N/L \) kept fixed. A convenient property of the representation (3.25) is that in the limit it contains a Dirac \( \delta \)-function:

\[
\frac{1 - e^{i(N+1)\pi s}}{1 - e^{i\pi s}} \to 2\delta(s), \quad N \to \infty. \tag{4.1}
\]

Hence, in the deriving a thermodynamic limit of the correlation function we can restrict ourselves in obtaining that of the quantity \( \Xi_N(x, t|\{q\}, \lambda; s)|_{s=0} \).

To derive an thermodynamic limit expression for this quantity, let us study the matrices \( S = S(x, t|\{q\}, \lambda; s) \) and \( R = R(x, t|\{q\}, \lambda; s) \) entering representation (3.32), specifying everywhere \( s = 0 \). We first consider the matrix \( S \). It is useful to consider separately its off-diagonal and diagonal entries. For the off-diagonal entries, since \( q_j \neq q_l \) for \( j \neq l \), we can use the relation

\[
\frac{1}{k(\lambda) - q_j} \frac{1}{k(\lambda) - q_l} = \frac{1}{q_j - q_l} \left( \frac{1}{k(\lambda) - q_j} - \frac{1}{k(\lambda) - q_l} \right), \tag{4.2}
\]

and represent them in the form

\[
S_{jl} = \frac{2}{L} e^{i(\tau(q_j) + \tau(q_l))/2} \frac{E(q_j|\lambda) - E(q_l|\lambda)}{q_j - q_l}, \quad j \neq l. \tag{4.3}
\]

Here the function \( E(q|\lambda), q \in \mathbb{R} \), is given by the expression

\[
E(q|\lambda) = \frac{2}{L} \sum_{k(\lambda)} \frac{1}{k(\lambda) - q} \left( \frac{e^{-i\tau(k(\lambda))} u(k(\lambda), \lambda)}{u(k(\lambda), \lambda)} - \frac{e^{-i\tau(q)} u(q, \lambda)}{u(q, \lambda)} \right) + \frac{2(q - \lambda) e^{-i\tau(q)}}{c u(q, \lambda)}. \tag{4.4}
\]
For \( q = q_j \), where \( q_j \) is one of the momenta in the set \( \{ q \} \) of the sector \( (N,0) \), it evaluates as

\[
E(q_j|\lambda) = \frac{2}{L} \sum_{k(\lambda)} \frac{\text{e}^{-i\tau(k(\lambda))}}{(k(\lambda) - q_j)u(k(\lambda), \lambda)},
\]

so (4.3) reproduces (3.33) for \( j \neq l \). Note, that the function \( E(q|\lambda) \) is well-defined for arbitrary real values of \( q \); the expression in (4.4) follows upon extracting a formal singularity at \( k(\lambda) = q_j \) of the sum over \( k(\lambda) \) in (4.3), using a summation formula for \( \sum_{k(\lambda)} 1/(k(\lambda) - q_j) \), see formula (A.10) of the appendix. Similarly, for the diagonal entries of the matrix \( S \) we obtain

\[
S_{jj} = \frac{4\epsilon^{-i\tau(q_j)}}{L^2} \sum_{k(\lambda)} \frac{1}{(k(\lambda) - q_j)^2}
\times \left\{ \frac{\text{e}^{-i\tau(k(\lambda))}}{u(k(\lambda), \lambda)} - \frac{\text{e}^{-i\tau(q_j)}}{u(q_j, \lambda)} \right\}
\times \left[ 1 + (k(\lambda) - q_j) \left( -i\tau'(q_j) - \frac{u'(q_j, \lambda)}{u(q_j, \lambda)} \right) \right]
+ \left[ 1 + 2a + \frac{4(q_j - \lambda)^2}{c^2} + a(q_j - \lambda) \left( -i\tau'(q_j) - \frac{u'(q_j, \lambda)}{u(q_j, \lambda)} \right) \right]
\times \frac{1}{u(q_j, \lambda)}
= 1 + \frac{2}{L} e^{i\tau(q_j)} \partial_q E(q|\lambda)|_{q=q_j} = 1 + \frac{2}{L} \epsilon^{i\tau(q)} \partial_q E(q|\lambda)|_{q=q_j}.
\]

Here the first equality follows from (3.33) at \( j = l \) by extracting a double-pole singularity using a summation formula for \( \sum_{k(\lambda)} 1/(k(\lambda) - q_j)^2 \), see (A.13), and the second equality follows just by comparing the result with (4.4). Introducing the function

\[
V(q, q') = \frac{2}{L} e^{i(\tau(q) + \tau(q'))/2} \frac{E(q|\lambda) - E(q'|\lambda)}{q - q'}, \quad q, q' \in \mathbb{R},
\]

we conclude that

\[
S_{jl} = \delta_{jl} + V(q_j, q_l).
\]

In other words, \( S = I + V \), where the matrix \( V \) has entries \( V_{jl} = V(q_j, q_l) \).

In similar manner, introducing the function

\[
R(q, q') = \frac{1}{L} e^{i(\tau(q) + \tau(q'))/2} E(q|\lambda)E(q'|\lambda), \quad q, q' \in \mathbb{R},
\]

for the entries of the matrix \( R \) we have

\[
R_{jl} = R(q_j, q_l).
\]

We note that, as a result, the matrices \( V \) and \( R \) are expressed in terms of the functions \( V(q, q') \) and \( R(q, q') \), which are well-defined for their arguments taking arbitrary real values. We also note that the entries of the matrices \( V \) and \( R \) are all of order \( 1/L \), as \( L \) is large.

Having all this in mind, we are now ready to address the problem of deriving the thermodynamic limit of the quantity \( \sum_{N,\{q\}, \{s\}} |\Xi_N(x, t(\{q\}, \{s\}))|_{s=0}^\Lambda \). Consider the determinant of the matrix \( S = I + V \); the case of the matrix \( S = I + V - R \) is similar. We have

\[
\det(I + V) = \sum_{p=0}^N \frac{1}{p!} \prod_{q_1 \in \{q\}} \cdots \prod_{q_p \in \{q\}} \det_{1 \leq a, b \leq p} V(q_a, q_b).
\]
Here each summation is performed over elements in the set \{q\}; in the thermodynamic limit the summations turn into integrations. To define the integrals, we need now specify the values of the momenta in the set \{q\}. Recall that we are interested in the correlation function of the ground-state \(|\Omega_\uparrow\rangle = |\Psi_{N,0}(\{q\})\rangle\), in which the momenta in the set \{q\} satisfy the relations \(q_{j+1} - q_j = 2\pi/L\), \(q_1 \geq -k_F\), \(q_N \leq k_F\), where \(k_F = \pi N/L\). Hence,

\[
\frac{1}{L} \sum_{q_j \in \{q\}} \rightarrow \frac{1}{2\pi} \int_{-k_F}^{k_F} dq
\]  

(4.12)

Correspondingly, let us introduce the function

\[
\mathcal{V}(q, q') = \lim_{L,N \to \infty} \frac{L}{2\pi} \mathcal{V}(q, q').
\]  

(4.13)

Then, evaluating the limit, we obtain

\[
\lim_{L,N \to \infty} \det(I + V) = \sum_{p=0}^{\infty} \frac{1}{p!} \int_{[-k_F, k_F]^p} dq^p \det_{1 \leq a, b \leq p} \mathcal{V}(q_{ja}, q_{jb}) = \det(1 + \mathcal{V}).
\]  

(4.14)

Essentially similarly, we have

\[
\lim_{L,N \to \infty} \det(I + V - R) = \det(1 + \mathcal{V} - \mathcal{R}),
\]  

(4.16)

where the kernel of the operator \(\mathcal{R}\) is defined by

\[
\mathcal{R}(q, q') = \lim_{L,N \to \infty} \frac{L}{2\pi} \mathcal{R}(q, q').
\]  

(4.17)

The kernels are

\[
\mathcal{V}(q, q') = \frac{1}{\pi} e^{i(\tau(q) + \tau(q'))/2} \frac{e(q|\lambda) - e(q'|\lambda)}{q - q'},
\]  

(4.18)

\[
\mathcal{R}(q, q') = \frac{1}{2\pi} e^{i(\tau(q) + \tau(q'))/2} e(q|\lambda) e(q'|\lambda),
\]  

where the function \(e(q|\lambda) = e(q|x, t; \lambda)\) is the thermodynamic limit of the function \(E(q|\lambda)\),

\[
e(q|\lambda) = \lim_{L \to \infty} E(q|\lambda).
\]  

(4.19)

If we also define the function \(g(x, t|\lambda)\) as the thermodynamic limit of the function \(G(x, t; \lambda; s)|_{s=0}\),

\[
g(x, t; \lambda) = \lim_{L \to \infty} G(x, t; \lambda; 0),
\]  

(4.20)

then we have the following expression for the quantity \(\Xi_N(x, t|\{q\}, \lambda; s)|_{s=0}\) in the thermodynamic limit:

\[
\lim_{L,N \to \infty} \Xi_N(x, t|\{q\}, \lambda; 0)|_{s=0} = \det(1 + \mathcal{V} - \mathcal{R}) + (g(x, t; \lambda) - 1) \det(1 + \mathcal{V}).
\]  

(4.21)
We recall that the linear integral operators $\mathcal{V}$ and $\mathcal{R}$ depend on $x$, $t$, and $\lambda$ via the functions defining their kernels; they also depend implicitly on the Fermi momentum $k_F$, defining the interval $[-k_F,k_F]$ where these operators act.

To finalize our calculation we have to derive the functions $e(q|\lambda)$ and $g(x,t;\lambda)$. These functions contains summation over all values of the momentum $k(\lambda) = 2z_n(\lambda)/L$, where $z_n(\lambda)$ is the solution of (3.18), as $n$ runs over all integer values,

$$\sum_{k(\lambda)} f(k) = \sum_{n \in \mathbb{Z}} f(2z_n(\lambda)/L). \quad (4.22)$$

In the thermodynamic limit the sum turns into an integral. To obtain the limit, we need to find a distribution of the values of the momentum $k(\lambda)$. In the thermodynamic limit the sum turns into an integral. To obtain the limit, we have to derive the functions $e(q|\lambda)$ and $g(x,t;\lambda)$.

$$\rho([k(\lambda)]_n) = \frac{1}{L([k(\lambda)]_{n+1} - [k(\lambda)]_n)}. \quad (4.23)$$

The sum turns into an integral by the rule

$$\frac{1}{L} \sum_{k(\lambda)} \rightarrow \int_{-\infty}^{+\infty} dk \rho(k). \quad (4.24)$$

Using that $z_{n+1}(\lambda) = z_n(\lambda) + (2\rho([k(\lambda)]_n))^{-1}$ and recalling that $a = 4/Lc$, from (3.18) we get

$$z_{n+1}(\lambda) - z_n(\lambda) = \frac{1}{1 + (az_n(\lambda) - 2\lambda/c)^2} a 2\rho([k(\lambda)]_n) + O(a^2). \quad (4.25)$$

Replacing the left-hand side by $(2\rho([k(\lambda)]_n))^{-1}$, we obtain

$$2\pi\rho([k(\lambda)]_n) = 1 + \frac{a}{1 + 4([k(\lambda)]_n - \lambda)^2/c^2} + O(a^2). \quad (4.26)$$

In the thermodynamic limit we have $a \to 0$, so we obtain $\rho(k) = (2\pi)^{-1}$. Taking also into account the explicit expression for the function $u(k,\lambda)$, see (3.21), where we have to put $a = 0$ in the limit, for the function $g(x,t;\lambda)$ we obtain

$$g(x,t;\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{e^{-ir(k)}}{1 + 4(k - \lambda)^2/c^2}. \quad (4.27)$$

In turn, for the function $e(q|\lambda) = e(q|x,t;\lambda)$ we have

$$e(q|\lambda) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \frac{e^{-ir(k)}}{(k - q)(1 + 4(k - \lambda)^2/c^2)} + \frac{2(q - \lambda)e^{-ir(q)}}{\pi c (1 + 4(q - \lambda)^2/c^2)}. \quad (4.28)$$

where the integral must be understood in the sense of its principal value.

Thus, we may now summarize our results about the correlation function (1.3).

In the thermodynamic limit the correlation function is given in terms of an integral of Fredholm determinants:

$$G_\downarrow(x,t) = \frac{2}{\pi c} \int_{-\infty}^{+\infty} d\lambda \left\{ \det (1 + \mathcal{V} - \mathcal{R}) + (g(x,t;\lambda) - 1) \det (1 + \mathcal{V}) \right\}. \quad (4.29)$$
The integral operators $\mathcal{V} = \mathcal{V}(x, t; \lambda)$ and $\mathcal{R} = \mathcal{R}(x, t; \lambda)$ act on functions in the interval $[-k_F, k_F]$, see (4.13), and possess kernels

$$\mathcal{V}(q, q') = \frac{e_+(q|\lambda)e_-(q') - e_-(q)e_+(q'|\lambda)}{\pi(q-q')} , \quad \mathcal{R}(q, q') = \frac{e_+(q|\lambda)e_+(q'|\lambda)}{2\pi}.$$  

The functions defining the kernels are

$$e_+(q|\lambda) = e_-(q|\lambda), \quad e_-(q) = e^{i\tau(q)/2}.$$  

The functions $g(x, t; \lambda)$ and $e(q|\lambda)$ are given by (4.27) and (4.28), respectively, and $\tau(q) = tq^2 - xq$.

Representation (4.29) is our main result. Let us discuss it in the limit $c \to \infty$. Setting $\lambda = -(c/2)\cot \vartheta$, and sending $c$ to infinity, we have

$$\lim_{c \to \infty} g(x, t; \lambda) = \sin^2 \vartheta \cdot g_{\infty}(x, t), \quad g_{\infty}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{-i\tau(k)},$$  

and

$$\lim_{c \to \infty} e(q|\lambda) = \frac{\sin^2 \vartheta}{\pi} \int_{-\infty}^{+\infty} dk \frac{e^{-i\tau(k)}}{(k-q)} + \sin \vartheta \cos \vartheta e^{-i\tau(q)} =: e_\infty(q|\vartheta).$$  

Evaluating the limit, for the Green’s function we obtain

$$\lim_{c \to \infty} G_{\vartheta}^{+}(x, t) = \frac{1}{\pi} \int_0^{\pi} d\vartheta \left\{ \det (1 + \mathcal{V}_{\infty} - \mathcal{R}_{\infty}) + (g_{\infty}(x, t) - 1) \det (1 + \mathcal{V}_{\infty}) \right\},$$  

where the operators $\mathcal{V}_{\infty} = \mathcal{V}_{\infty}(x, t; \vartheta)$ and $\mathcal{R}_{\infty} = \mathcal{R}_{\infty}(x, t; \vartheta)$ possess kernels

$$\mathcal{V}_{\infty}(q, q') = \frac{\ell_+(q|\vartheta)\ell_-(q') - \ell_-(q)\ell_+(q'|\vartheta)}{\pi(q-q')} , \quad \mathcal{R}_{\infty}(q, q') = \frac{\ell_+(q|\vartheta)\ell_+(q'|\vartheta)}{2\pi \sin^2 \vartheta},$$  

and the functions $\ell_+(q|\vartheta)$ and $\ell_-(q)$ are

$$\ell_+(q|\vartheta) = \ell_-(-q) = e_\infty(q|\vartheta), \quad \ell_-(q) = e_-(q).$$  

The representation (4.31) is the correlation function $G_{\vartheta}^{(\pm)}(x, t, \hbar, B)$ of paper [18] at zero temperature in the case of a negative magnetic field $B$ (see [18], Eqs. (6.9)–(6.11)), evaluated at vanishing chemical potential, $\hbar = 0$, and $B \to 0^-$. Hence, the known result for the correlation function [13] at $c = \infty$ is reproduced.

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Appendix A. Summation formulas

To transform the matrices $S$ and $R$ in a form regular in the thermodynamic limit we have used two summation formulas. Consider the function

$$g(z, \lambda) = \sum_n \frac{1}{z - z_n(\lambda)}, \quad z \in \mathbb{C},$$  

(A.1)
where the sum is taken over all solutions of (3.18). It is assumed that the summation is organized in a such way that the sum is convergent, thus defining \( g(z, \lambda) \) as a meromorphic function in \( z \) with simple poles at points \( z_n(\lambda), \ n \in \mathbb{Z} \), with the principal parts \( 1/(z - z_n(\lambda)) \) at these points. We are interested in finding an explicit form of \( g(z, \lambda) \). The summation formulas will follow from expressions for \( g(z, \lambda) \) and \( \partial_z g(z, \lambda) \) at \( z = Lq/2 \), where \( q \) is a solution of the Bethe Ansatz equations in the sector \((N, 0)\), given by (2.7).

To find \( g(z, \lambda) \) it is useful to look at \( z_n(\lambda) \)'s as zeros of some entire function (see, e.g., [35], Vol. II, Chap. 10). In our case this entire function can be easily inferred from (3.18) to be

\[
\begin{align*}
f(z, \lambda) &= \cos z - \left( az - \frac{2\lambda}{c} \right) \sin z. \\
\end{align*}
\] (A.2)

Recalling that \( a = 4/Lc \), we have the functional relation

\[
\begin{align*}
f(z + \pi, \lambda) &= -f(z, \lambda - 2\pi/L). \\
\end{align*}
\] (A.3)

Correspondingly, zeros satisfy

\[
\begin{align*}
z_{n+1}(\lambda) &= \pi + z_n(\lambda - 2\pi/L), \\
\end{align*}
\] (A.4)

that can also be seen directly from (3.18). Writing \( z_n(\lambda) = \pi n + z_0(\lambda - 2\pi n/L) \), and taking into account that \( z_0(\lambda) \in [0, \pi] \), one can construct from (3.18) an estimate for \( z_n(\lambda) \) as \( n \to \pm \infty \). Namely, we have

\[
\begin{align*}
z_{n+1}(\lambda) - z_n(\lambda) &= \pi + O(1/n), \quad n \to \pm \infty. \\
\end{align*}
\] (A.5)

This estimate allows one to represent the entire function \( f(z, \lambda) \) as an infinite product. Indeed, the estimate shows that the series \( \sum_n 1/z_n^2(\lambda) \) converges, and we are in a situation, similar to that of the well-known case of the sine function (with the only difference that \( z = 0 \) is not a zero of \( f(z, \lambda) \)). By Borel’s theorem have the following infinite product formula

\[
\begin{align*}
f(z, \lambda) &= e^{2\lambda z/c} \prod_n \left( 1 - \frac{z}{z_n(\lambda)} \right) e^{z/z_n(\lambda)}. \\
\end{align*}
\] (A.6)

Comparing (A.1) and (A.6) we conclude (see also [35], Vol. II, Sec. 51, ex. 2) that

\[
\begin{align*}
g(z, \lambda) &= \frac{\partial_z f(z, \lambda)}{f(z, \lambda)}. \\
\end{align*}
\] (A.7)

Substituting (A.2) into (A.7), we obtain that the function \( g(z, \lambda) \) has the following explicit form

\[
\begin{align*}
g(z, \lambda) &= -\left( 1 + a \right) \sin z + \left( az - \frac{2\lambda}{c} \right) \cos z. \\
\end{align*}
\] (A.8)

Note that this expression is in agreement with the special case \( \lambda \to \pm \infty \), in which the zeros \( z_n(\lambda) \) tend to those of \( \sin z \), and hence \( g(z, \lambda) \to \cot z \).

Setting \( z = Lq_j/2 \), where \( q_j \) is a momentum belonging to the set momenta \( \{q\} \) of the sector \((N, 0)\), we have

\[
\begin{align*}
\frac{2}{L} \sum_{k(\lambda)} \frac{1}{k(\lambda) - q_j} &= \sum_n \frac{1}{z_n(\lambda) - Lq_j/2} = -g(z, \lambda) \big|_{z = Lq_j/2}. \\
\end{align*}
\] (A.9)
Since \( q_j = 2\pi m_j/L \), \( m_j \in \mathbb{Z} \), we have \( \sin Lq_j/2 = \sin \pi m_j = 0 \) and \( \cos Lq_j/2 = \cos \pi m_j = (-1)^{m_j} \), so we obtain the first summation formula

\[
\frac{2}{L} \sum_{k(\lambda)} \frac{1}{k(\lambda) - q_j} = \frac{2(q_j - \lambda)}{c}, \tag{A.10}
\]

where we have used that \( a = 4/Lc \).

The second formula, relevant for the diagonal entries of the matrix \( S \), follows from the derivative of the function \( g(z, \lambda) \),

\[
\frac{4}{L^2} \sum_{k(\lambda)} \frac{1}{(k(\lambda) - q_j)^2} = -\frac{\partial z g(z, \lambda)}{\partial z} \bigg|_{z=Lq_j/2}. \tag{A.11}
\]

We have

\[
\partial z g(z, \lambda) = \frac{\partial^2 f(z, \lambda) f(z, \lambda) - (\partial_z f(z, \lambda))^2}{f^2(z, \lambda)} = -\frac{1 + 2a + (az - 2\lambda/c)^2 + a^2 \sin^2 z}{(\cos z - (az - 2\lambda/c) \sin z)^2}.
\]

Hence,

\[
\frac{4}{L^2} \sum_{k(\lambda)} \frac{1}{(k(\lambda) - q_j)^2} = 1 + 2a + \frac{4(q_j - \lambda)^2}{c^2}. \tag{A.13}
\]

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