FORM BOUNDEDNESS OF THE GENERAL SECOND ORDER DIFFERENTIAL OPERATOR

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Abstract. We give explicit necessary and sufficient conditions for the boundedness of the general second order differential operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j + \sum_{j=1}^{n} b_j \partial_j + c$$

with real- or complex-valued distributional coefficients $a_{ij}$, $b_j$, and $c$, acting from the Sobolev space $W^{1,2}(\mathbb{R}^n)$ to its dual $W^{-1,2}(\mathbb{R}^n)$. This enables us to obtain analytic criteria for the fundamental notions of relative form boundedness, compactness, and infinitesimal form boundedness of $\mathcal{L}$ with respect to the Laplacian on $L^2(\mathbb{R}^n)$.

In particular, we establish a complete characterization of the form boundedness of the Schrödinger operator $(i\nabla + \vec{a})^2 + q$ with magnetic vector potential $\vec{a} \in L^2_{loc}(\mathbb{R}^n)$ and $q \in D'(\mathbb{R}^n)$.

1. Introduction

The property of form boundedness, as well as related notions of relative compactness, infinitesimal form boundedness, and subordination of differential operators in Hilbert spaces, are used extensively in mathematical physics, geometry, and PDE, especially in relation to quantum mechanics problems [ChWW, Fel, LL, RS], elliptic differential operators and spectral theory [D2, EE, GT, RSS, Sch, Shi], semigroup theory [DL, LPS, Sim], harmonic maps [Ev], and Markov processes [CWZ, CrZ].

The goal of the present paper is to give an analytic characterization of form boundedness for the general second order differential operator

$$(1.1) \quad \mathcal{L} = \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j + \sum_{j=1}^{n} b_j \partial_j + c,$$

where $a_{ij}$, $b_j$, and $c$ are real- or complex-valued distributions, on the Sobolev space $W^{1,2}(\mathbb{R}^n)$, and its homogeneous counterpart $L^{1,2}(\mathbb{R}^n)$.

One of our motivations is to give a criterion for the relative form boundedness of the operator $\vec{b} \cdot \nabla + q$ with distributional coefficients $\vec{b}$ and $q$ with respect to the Laplacian $\Delta$ on $L^2(\mathbb{R}^n)$. This ensures, in view of the so-called KLMN Theorem (see [EE], Theorem IV.4.2; [RS], Theorem X.17), that $\mathcal{L} = \Delta + \vec{b} \cdot \nabla + q$...
can be defined, under appropriate smallness assumptions on $\vec{b}$ and $q$, as an m-sectorial operator on $L^2(\mathbb{R}^n)$ so that its quadratic form domain coincides with $W^{1,2}(\mathbb{R}^n)$.

In particular, we will obtain a characterization of the relative form boundedness for the magnetic Schrödinger operator

\begin{equation}
\mathcal{M} = (i \nabla + \vec{a})^2 + q,
\end{equation}

with arbitrary vector potential $\vec{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, and $q \in D'(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ with respect to $\Delta$.

Our approach is based on factorization of functions in Sobolev spaces and integral estimates of potentials of equilibrium measures, combined with compensated compactness arguments, commutator estimates, and the idea of gauge invariance. We are able to treat general second order differential operators, and establish an explicit Hodge decomposition for form bounded vector fields. It is worth mentioning that in this decomposition, the irrotational part of the vector field is subject to a more stringent condition than its divergence-free counterpart.

Methods and techniques proposed in the present paper, along with their natural extensions to higher order differential operators and more general $L^p$-inequalities, might be useful in further applications to mathematical physics, dynamics, analysis of phases, and other nonlinear problems (see, e.g., [BB1], [BB2], [D3], [IM]).

For the sake of convenience, let us assume in the Introduction that the principal part of $\mathcal{L}$ is in the divergence form, i.e.,

\begin{equation}
\mathcal{L} u = \text{div}(A \nabla u) + \vec{b} \cdot \nabla u + q u, \quad u \in C^\infty_0(\mathbb{R}^n),
\end{equation}

where $A = (a_{ij})_{i,j=1}^n \in D'(\mathbb{R}^n)^{n \times n}$, $\vec{b} = (b_j)_{j=1}^n \in D'(\mathbb{R}^n)^n$, and $q \in D'(\mathbb{R}^n)$.

We will present necessary and sufficient conditions on $A$, $\vec{b}$, and $q$ which guarantee the boundedness of the sesquilinear form associated with $\mathcal{L}$:

\begin{equation}
|\langle \mathcal{L} u, v \rangle| \leq C \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)}
\end{equation}

where the constant $C$ does not depend on $u, v \in C^\infty_0(\mathbb{R}^n)$. Here $L^{1,2}(\mathbb{R}^n)$ is the completion of (complex-valued) $C^\infty_0(\mathbb{R}^n)$ functions with respect to the norm $\|u\|_{L^{1,2}(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}$.

Equivalently, we characterize all $A$, $\vec{b}$, and $q$ such that

\begin{equation}
\mathcal{L} : L^{1,2}(\mathbb{R}^n) \to L^{-1,2}(\mathbb{R}^n)
\end{equation}

is a bounded operator, where $L^{-1,2}(\mathbb{R}^n) = (L^{1,2}(\mathbb{R}^n))^*$ is a dual Sobolev space. Analogous results are obtained below for the inhomogeneous Sobolev space $W^{1,2}(\mathbb{R}^n) = L^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ as well.
In the special case where $A$, $\vec{b}$ and $q$ are locally integrable, the form boundedness of $L$ may be expressed in the form of the integral inequality

$$
(1.6) \quad \left| \int_{\mathbb{R}^n} (-(A \nabla u) \cdot \nabla v + \vec{b} \cdot \nabla u \, v + qu \, v) \, dx \right| \leq C \|u\|_{L^{1,2}(\mathbb{R}^n)} \|v\|_{L^{1,2}(\mathbb{R}^n)},
$$

where the constant $C$ does not depend on $u, v \in C^\infty_0(\mathbb{R}^n)$. Sometimes it will be convenient to write (1.4) in this form even for distributional coefficients $a_{ij}$, $b_j$, and $q$.

To state our main results, we introduce the class of \textit{admissible measures} $M_{1,2}^1(\mathbb{R}^n)$, i.e., nonnegative Borel measures $\mu$ on $\mathbb{R}^n$ which obey the trace inequality

$$
(1.7) \quad \int_{\mathbb{R}^n} |u|^2 \, d\mu \leq C \|u\|^2_{L^{1,2}(\mathbb{R}^n)}, \quad u \in C^\infty_0(\mathbb{R}^n),
$$

where the constant $C$ does not depend on $u$. For admissible measures $q(x) \, dx$ with nonnegative density $q \in L^1_{\text{loc}}(\mathbb{R}^n)$, we will write $q \in M_{1,2}^1(\mathbb{R}^n)$.

Inequalities of this type (with $\mu$ possibly singular with respect to Lebesgue measure) have been thoroughly studied. A straightforward consequence of (1.7) is that if $\mu \in M_{1,2}^1(\mathbb{R}^n)$ then

$$
(1.8) \quad \int_{|x-y|<r} \, d\mu(y) \leq \operatorname{const} r^{n-2},
$$

for all $r > 0$, $x \in \mathbb{R}^n$, if $n \geq 3$, and $\mu = 0$ if $n = 1, 2$ (see e.g. [M], Sec. 2.4).

A close sufficient condition on $q \in L^1_{\text{loc}}(\mathbb{R}^n)$, $q \geq 0$, which ensures that $q \in M_{1,2}^1(\mathbb{R}^n)$, is provided by the Fefferman–Phong class

$$
(1.9) \quad \int_{|x-y|<r} q^{1+\epsilon} \, dy \leq \operatorname{const} r^{n-2(1+\epsilon)},
$$

where $\epsilon > 0$, and the constant does not depend on $r > 0$, $x \in \mathbb{R}^n$. More precise sufficiency results are due to Chang, Wilson, and Wolff [ChWW].

A complete characterization of the class of admissible measures $M_{1,2}^1(\mathbb{R}^n)$ can be expressed in several equivalent forms: using capacities [M], local energy estimates [KS], pointwise potential inequalities [MV1], or dyadic Carleson measures [V]. These criteria, discussed in Sec. 2 below, employ various degrees of localization of $\mu$, and each of them has its own advantages depending on the area of application.

We now state our main form boundedness criterion. For $A = (a_{ij})$, let $A^t = (a_{ji})$ denote the transposed matrix, and let $\text{Div}: D'(\mathbb{R}^n)^{n \times n} \to D'(\mathbb{R}^n)$ be the row divergence operator defined by

$$
(1.10) \quad \text{Div} (a_{ij}) = \left( \sum_{j=1}^n \partial_j a_{ij} \right)_{i=1}^n.
$$
Theorem I. Let $\mathcal{L} = \text{div}(A \nabla \cdot) + \tilde{b} \cdot \nabla + q$, where $A \in D'(\mathbb{R}^n)^{n \times n}$, $\tilde{b} \in D'(\mathbb{R}^n)^n$, and $q \in D'(\mathbb{R}^n), n \geq 2$. Then the following statements hold.

(i) The sesquilinear form of $\mathcal{L}$ is bounded, i.e., (1.1) holds if and only if $\frac{1}{2} (A + A^t) \in L^\infty(\mathbb{R}^n)^{n \times n}$, and $\tilde{b}$ and $q$ can be represented respectively in the form

(1.11) \[ \tilde{b} = \tilde{\alpha} + \text{Div} F, \quad q = \text{div} \, \tilde{h}, \]

where $F$ is a skew-symmetric matrix field such that

(1.12) \[ F - \frac{1}{2} (A - A^t) \in \text{BMO}(\mathbb{R}^n)^{n \times n}, \]

whereas $\tilde{\alpha}$ and $\tilde{h}$ belong to $L^2_{\text{loc}}(\mathbb{R}^n)^n$, and obey the condition

(1.13) \[ |\tilde{\alpha}|^2 + |\tilde{h}|^2 \in \mathcal{M}^{1,2}(\mathbb{R}^n). \]

(ii) If the sesquilinear form of $\mathcal{L}$ is bounded, then $\tilde{\alpha}$, $F$, and $\tilde{h}$ in decomposition (1.11) can be determined explicitly by

(1.14) \[ \tilde{\alpha} = \nabla (\Delta^{-1} \text{div} \, \tilde{b}), \quad \tilde{h} = \nabla (\Delta^{-1} q), \]

(1.15) \[ F = \Delta^{-1} \text{curl} [\tilde{b} - \frac{1}{2} \text{Div} (A - A^t)] + \frac{1}{2} (A - A^t), \]

where

(1.16) \[ \Delta^{-1} \text{curl} [\tilde{b} - \frac{1}{2} \text{Div} (A - A^t)] \in \text{BMO}(\mathbb{R}^n)^{n \times n}, \]

and

(1.17) \[ |\nabla (\Delta^{-1} \text{div} \, \tilde{b})|^2 + |\nabla (\Delta^{-1} q)|^2 \in \mathcal{M}^{1,2}(\mathbb{R}^n). \]

Remark 1. In the case $n = 2$, we will show that (1.14) holds if and only if $\frac{1}{2} (A + A^t) \in L^\infty(\mathbb{R}^2)^{2 \times 2}, \Delta^{-1} \text{curl} \, \tilde{b} - \frac{1}{2} (A - A^t) \in \text{BMO}(\mathbb{R}^2)^{2 \times 2}$, and $q = 0$.

Remark 2. Expressions like $\nabla (\Delta^{-1} \text{div} \, \tilde{b})$, $\text{div} (\Delta^{-1} \text{curl} \, \tilde{b})$, and $\nabla (\Delta^{-1} q)$ used above which involve nonlocal operators are defined in the sense of distributions. This is possible, as we demonstrate below, since $\Delta^{-1} \text{div} \, \tilde{b}, \Delta^{-1} \text{curl} \, \tilde{b}$, and $\Delta^{-1} q$ can be understood in terms of the convergence in the weak-* topology of $\text{BMO}(\mathbb{R}^n)$ of, respectively, $\Delta^{-1} \text{div} (\psi_N \tilde{b}), \Delta^{-1} \text{curl} (\psi_N \tilde{b})$, and $\Delta^{-1} (\psi_N q)$ as $N \to +\infty$. Here $\psi_N$ is a smooth cut-off function supported on $\{x : |x| < N\}$, and the limits above do not depend on the choice of $\psi_N$.

It follows from Theorem I that $\mathcal{L}$ is form bounded on $L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n)$ if and only if the symmetric part of $A$ is essentially bounded, i.e., $\frac{1}{2} (A + A^t) \in L^\infty(\mathbb{R}^n)^{n \times n}$, and $\tilde{b}_1 \cdot \nabla + q$ is form bounded, where

(1.18) \[ \tilde{b}_1 = \tilde{b} - \frac{1}{2} \text{Div} (A - A^t). \]
In particular, the principal part $P u = \text{div}(A \nabla u)$ is form bounded if and only if
\begin{align}
\frac{1}{2}(A + A^t) &\in L^\infty(\mathbb{R}^n)^{n \times n}, \\
\Delta^{-1} [\text{curl} \text{Div} \frac{1}{2}(A - A^t)] &\in \text{BMO}(\mathbb{R}^n)^{n \times n}.
\end{align}

A simpler condition with $\frac{1}{2}(A - A^t) \in \text{BMO}(\mathbb{R}^n)^{n \times n}$ in place of (1.20) is sufficient, but generally not necessary, unless $n \leq 2$.

Thus, the form boundedness problem for the general second order differential operator in the divergence form (1.3) is reduced to the special case
\begin{equation}
\mathcal{L} = \vec{b} \cdot \nabla + q, \quad \vec{b} \in D'(\mathbb{R}^n), \quad q \in D'(\mathbb{R}^n).
\end{equation}

As a corollary of Theorem I, we deduce that, if $\vec{b} \cdot \nabla + q$ is form bounded, i.e., for all $u, v \in C_0^\infty(\mathbb{R}^n)$,
\begin{equation}
\left| \int_{\mathbb{R}^n} (\vec{b} \cdot \nabla u \ \vec{v} + qu \ \vec{v}) \ dx \right| \leq C \ ||u||_{L^{1.2}(\mathbb{R}^n)} \ ||v||_{L^{1.2}(\mathbb{R}^n)},
\end{equation}
then the Hodge decomposition
\begin{equation}
\vec{b} = \nabla(\Delta^{-1} \text{div} \vec{b}) + \text{Div}(\Delta^{-1} \text{curl} \vec{b})
\end{equation}
holds, where $\Delta^{-1} \text{curl} \vec{b} \in \text{BMO}(\mathbb{R}^n)^{n \times n}$, and
\begin{equation}
\int_{|x-y|<r} [ |\nabla(\Delta^{-1} \text{div} \vec{b})|^2 + |\nabla(\Delta^{-1} q)|^2 ] \ dy \leq \text{const} \ r^{n-2},
\end{equation}
for all $r > 0, x \in \mathbb{R}^n$, in the case $n \geq 3$; in two dimensions, it follows that $\text{div} \vec{b} = q = 0$.

We observe that condition (1.24) is generally stronger than $\Delta^{-1} \text{div} \vec{b} \in \text{BMO}$ and $\Delta^{-1} q \in \text{BMO}$, while the divergence-free part of $\vec{b}$ is characterized by $\Delta^{-1} \text{curl} \vec{b} \in \text{BMO}$, for all $n \geq 2$.

A close sufficient condition of the Fefferman–Phong type can be stated in the following form:
\begin{equation}
\int_{|x-y|<r} [ |\nabla(\Delta^{-1} \text{div} \vec{b})|^2 + |\nabla(\Delta^{-1} q)|^2 ]^{1+\epsilon} \ dy \leq \text{const} \ r^{n-2(1+\epsilon)},
\end{equation}
for some $\epsilon > 0$ and all $r > 0, x \in \mathbb{R}^n$. This is a consequence of Theorem I coupled with (1.9), where $|\nabla(\Delta^{-1} \text{div} \vec{b})|^2 + |\nabla(\Delta^{-1} q)|^2$ is used in place of $q$. Sharper conditions of the Chang–Wilson–Wolff type are readily deduced from Theorem I by combining it with the results of [ChWW].

It is worth mentioning that the class of potentials obeying (1.25) is substantially broader than its subclass
\begin{equation}
\int_{|x-y|<r} (|\vec{b}|^2 + |q|)^{1+\epsilon} \ dy \leq \text{const} \ r^{n-2(1+\epsilon)}.
\end{equation}
The sufficiency of the preceding condition for (1.22) is deduced by a direct application of the original Fefferman–Phong condition and Schwarz’s inequality. More generally, (1.22) clearly follows from a cruder estimate,

\[
\int_{\mathbb{R}^n} |u|^2 (|\vec{b}|^2 + |q|) \, dx \leq \text{const} \, ||u||^2_{L^{1,2}(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n),
\]

which is equivalent to $|\vec{b}|^2 + |q| \in \mathcal{M}^1_{1,2}(\mathbb{R}^n)$.

However, by replacing (1.22) with (1.27), one strongly reduces the class of admissible vector fields $\vec{b}$ and potentials $q$. Various examples of this phenomenon in the case $\vec{b} = 0$ are given in [MV1]. An instructive example for $\vec{b} \cdot \nabla$ in the case $q = 0$ is provided by the vector field

\[
\vec{b}(x) = (x_2(x_1^2 + x_2^2)^{-1}, -x_1(x_1^2 + x_2^2)^{-1}, 0, \ldots 0), \quad x \in \mathbb{R}^n,
\]

where $n \geq 2$. An elementary argument involving polar coordinates and a Fourier series expansion shows that this vector field obeys (1.22). On the other hand, (1.27) fails since $\vec{b} \notin L^2_{\text{loc}}(\mathbb{R}^n)$.

We note in passing that, for $q = 0$, (1.27) is equivalent to the boundedness of the nonlinear quadratic form $\langle |\vec{b} \cdot \nabla u|, u \rangle$ on $L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n)$ (see Sec. 5).

As it turns out, dealing with the linear version $\langle \vec{b} \cdot \nabla u, u \rangle$ is more difficult. The main obstacle in the proof of Theorem 1 is the interaction between the quadratic forms associated with $q - \frac{1}{2} \text{div} \vec{b}$ and the divergence free part of $\vec{b}$ (see Sections 3 and 4). To overcome this difficulty, one needs to distinguish the class of vector fields $\vec{b}$ such that the commutator inequality

\[
(1.28) \quad \left| \int_{\mathbb{R}^n} \vec{b} \cdot (u \nabla \vec{v} - \vec{v} \nabla u) \, dx \right| \leq \text{const} \, ||u||_{L^{1,2}(\mathbb{R}^n)} \, ||v||_{L^{1,2}(\mathbb{R}^n)}
\]

holds for all $u, v \in C_0^\infty(\mathbb{R}^n)$. In the important special case of irrotational fields where $\vec{b} = \nabla f$, the preceding inequality is equivalent to the boundedness of the commutator $[f, \Delta]$ acting from $L^{1,2}(\mathbb{R}^n)$ to $L^{-1,2}(\mathbb{R}^n)$.

A complete characterization of those $\vec{b}$ which obey (1.28) is obtained below (Sec. 4, Lemma 4.6) using the idea of the gauge transformation ([LL], Sec. 7.19; [RS], Sec. X.4):

\[
\nabla \to e^{-i\lambda} \nabla e^{+i\lambda},
\]

where the gauge $\lambda$ is a real-valued function which lies in $L_{\text{loc}}^{1,2}(\mathbb{R}^n)$.

The problem of choosing an appropriate gauge is known to be highly non-trivial. In the present paper, $\lambda$ is picked in a very specific form:

\[
\lambda = \tau \log (P\mu), \quad 1 < 2\tau < \frac{n}{n-2}, \quad n \geq 3,
\]

where $\tau$ is a constant, and $P\mu = (-\Delta)^{-1} \mu$ is the Newtonian potential of the equilibrium measure $\mu$ associated with an arbitrary compact set $e$ of positive capacity, $\text{cap} (e) > 0$ (see the definitions in Sec. 2).
We will verify that, with this choice of $\lambda$, the energy space $L^{1,2}(\mathbb{R}^n)$ is gauge invariant, and the irrotational part $\vec{c} = \nabla(\Delta^{-1}\text{div}\, \vec{b})$ of $\vec{b}$ obeys

$$\int_{\epsilon} |\vec{c}|^2 \, dx \leq \text{const cap}(\epsilon),$$

where the constant does not depend on $\epsilon$. This is known to be equivalent to $|\vec{c}|^2 \in M_{1,2}^{1,2}(\mathbb{R}^n)$ (see Theorem 2.1). In addition, a careful analysis shows that $F = \Delta^{-1}\text{curl}\, \vec{b}$ belongs to $\text{BMO}$, and $\vec{b} = \vec{c} + \text{Div}\, F$. These conditions combined turn out to be necessary and sufficient for (1.28).

At the end of Sec. 4, we give applications to the magnetic Schrödinger operator $\mathcal{M}$ defined by (1.2). We show that $\mathcal{M}$ is form bounded if and only if both $q + |\vec{a}|^2$ and $\vec{a} \cdot \nabla$ are form bounded. Thus, the form boundedness criterion of $\mathcal{M}$ is deduced from Theorem I (see Theorem 4.12).

In Sec. 6, we extend our results to the Sobolev space $W^{1,2}(\mathbb{R}^n)$. In particular, we give necessary and sufficient conditions (Theorem 6.1) for the boundedness of the general second order operator

$$\mathcal{L} : W^{1,2}(\mathbb{R}^n) \to W^{-1,2}(\mathbb{R}^n).$$

This solves the relative form boundedness problem for $\mathcal{L}$, and consequently for the magnetic Schrödinger operator $\mathcal{M}$, with respect to the Laplacian on $L^2(\mathbb{R}^n)$ (see [RS], Sec. X.2). The proofs are based on a localized version of the approach developed in Sec. 4, and in particular involve an inhomogeneous version of the div-curl lemma (see Lemma 6.2 below).

We remark that other fundamental properties of quadratic forms associated with differential operators can be characterized using our methods. For the Schrödinger operator $\mathcal{L} = \Delta + q$, criteria of relative compactness were obtained in [MV1], while the infinitesimal form boundedness expressed by the inequality

$$|\langle \mathcal{L} u, u \rangle| \leq \epsilon \left| \nabla u \right|^2_{L^2(\mathbb{R}^n)} + C(\epsilon) \left| u \right|^2_{L^2(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n),$$

for every $\epsilon > 0$, along with Trudinger's condition where $C(\epsilon) = C \epsilon^{-\beta}$, $\beta > 0$, was characterized in [MV1]. Necessary and sufficient conditions for such properties in the case of the general second order differential operator are discussed in Sec. 7.

2. Preliminaries

By $L^{1,2}(\mathbb{R}^n)$ we denote the energy space (homogeneous Sobolev space) defined in the Introduction as the completion of the complex-valued $C_0^\infty$ functions in the Dirichlet norm. For $n \geq 3$, an equivalent norm on $L^{1,2}(\mathbb{R}^n)$ is given by

$$\left| |u| \right|_{L^{1,2}(\mathbb{R}^n)} = \left| |x|^{-1}u \right|_{L^2(\mathbb{R}^n)} + \left| \nabla u \right|_{L^2(\mathbb{R}^n)}, \quad u \in L^{1,2}(\mathbb{R}^n).$$
By $W^{1,2}(\mathbb{R}^n)$ we denote the space of weakly differentiable (complex-valued) functions on $\mathbb{R}^n$ ($n \geq 1$) such that
\[ ||u||_{W^{1,2}(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)} + ||\nabla u||_{L^2(\mathbb{R}^n)} < +\infty. \]
The dual spaces are respectively $L^{-1,2}(\mathbb{R}^n) = L^{1,2}(\mathbb{R}^n)^*$ and $W^{-1,2}(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)^*$.

For $0 < r < \infty$, denote by $L^r_{\text{unif}}(\mathbb{R}^n)$ all $f \in L^r_{\text{loc}}(\mathbb{R}^n)$ such that
\[ ||f||_{L^r_{\text{unif}}} = \sup_{x_0 \in \mathbb{R}^n} ||\chi_{B_1(x_0)} f||_{L^r(\mathbb{R}^n)} < \infty. \]
We set
\[ m_B(f) = \frac{1}{|B|} \int_B f(x) \, dx \]
for a ball $B \subset \mathbb{R}^n$, and denote by $\text{BMO}(\mathbb{R}^n)$ the class of $f \in L^r_{\text{loc}}(\mathbb{R}^n)$ for which
\[ \sup_{x_0 \in \mathbb{R}^n, \delta > 0} \frac{1}{|B_\delta(x_0)|} \int_{B_\delta(x_0)} |f(x) - m_{B_\delta(x_0)}(f)|^r \, dx < +\infty, \]
for any (or, equivalently, all) $1 \leq r < +\infty$. An inhomogeneous version of $\text{BMO}(\mathbb{R}^n)$ (the so-called local BMO; see [St], p. 264), which we denote by $\text{bmo}(\mathbb{R}^n)$, is defined as the set of $f \in L^r_{\text{unif}}(\mathbb{R}^n)$ such that the preceding condition holds for all $0 < \delta \leq 1$, and additionally
\[ \sup_{x_0 \in \mathbb{R}^n, \delta \geq 1} \frac{1}{|B_\delta(x_0)|} \int_{B_\delta(x_0)} |f(x)|^r \, dx < +\infty. \]
We will also need the space $\text{BMO}^\#(\mathbb{R}^n)$ defined as the set of $f \in L^r_{\text{loc}}(\mathbb{R}^n)$ such that
\[ \sup_{x_0 \in \mathbb{R}^n, 0 < \delta \leq 1} \frac{1}{|B_\delta(x_0)|} \int_{B_\delta(x_0)} |f(x) - m_{B_\delta(x_0)}(f)|^r \, dx < +\infty, \]
for any (or equivalently all) $1 \leq r < +\infty$. Notice that $\text{bmo}(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n) \subset \text{BMO}^\#(\mathbb{R}^n)$.

The corresponding vector- and matrix-valued function spaces are introduced in a similar way. In particular, $\text{BMO}(\mathbb{R}^n)^n$ stands for the class of vector fields $\vec{f} = \{f_j\}_{j=1}^n : \mathbb{R}^n \to \mathbb{C}^n$, such that $f_j \in \text{BMO}(\mathbb{R}^n)$, $j = 1, 2, \ldots, n$. The matrix-valued analogue is denoted by $\text{BMO}(\mathbb{R}^n)^{n \times n}$, etc.

For a matrix field $F = (f_{ij})_{i,j=1}^n \in D'((\mathbb{R}^n)^{n \times n})$, the matrix divergence operator $\text{Div}$ is defined by $\text{Div} F = \left( \sum_{j=1}^n \partial_j f_{ij} \right)_{i=1}^n \in D'(\mathbb{R}^n)^n$. The Jacobian, $D$, is the formal adjoint of $-\text{Div}$ (see, e.g., [IM]):
\[ \langle \text{Div} F, \vec{v} \rangle = -\text{trace} \left( F' \cdot \begin{bmatrix} \vec{v} \end{bmatrix} \right), \quad \vec{v} \in C^\infty_0 (\mathbb{R}^n)^n, \]
where $F' = (f_{ji})_{i,j=1}^n$ is the transposed matrix field. If $F$ is skew-symmetric, i.e., $f_{ij} = -f_{ji}$, then obviously $\text{div} (\text{Div} F) = 0$. 

By $W^{1,2}(\mathbb{R}^n)$ we denote the space of weakly differentiable (complex-valued) functions on $\mathbb{R}^n$ ($n \geq 1$) such that
\[ ||u||_{W^{1,2}(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)} + ||\nabla u||_{L^2(\mathbb{R}^n)} < +\infty. \]
The capacity of a compact set $e \subset \mathbb{R}^n$ is defined by ([LL], Sec. 11.15; [M], Sec. 2.2):

\begin{equation}
(2.1) \quad \text{cap} (e) = \inf \left\{ ||u||^2_{L^1,2(\mathbb{R}^n)} : \quad u \in C_0^\infty (\mathbb{R}^n), \quad u(x) \geq 1 \text{ on } e \right\}.
\end{equation}

For a cube or ball $Q$ in $\mathbb{R}^n$,

\begin{equation}
(2.2) \quad \text{cap} (Q) \simeq |Q|^\frac{1-\frac{2}{n}}{2} \quad \text{if } n \geq 3; \quad \text{cap} (Q) = 0 \quad \text{if } n = 2.
\end{equation}

We will also need the capacity $\text{Cap} (\cdot)$ associated with the Sobolev space $W^{1,2}(\mathbb{R}^n)$ defined by

\begin{equation}
(2.3) \quad \text{Cap} (e) = \inf \left\{ ||u||^2_{W^{1,2}(\mathbb{R}^n)} : \quad u \in C_0^\infty (\mathbb{R}^n), \quad u(x) \geq 1 \text{ on } e \right\},
\end{equation}

for compact sets $e \subset \mathbb{R}^n$. Note that $\text{Cap} (e) \simeq \text{cap} (e)$ if $\text{diam} (e) \leq 1$, and $n \geq 3$. For a cube or ball $Q$ in $\mathbb{R}^n$,

\begin{equation}
(2.4) \quad \text{Cap} (Q) \simeq \frac{1}{|Q|} \quad \text{if } n \geq 3; \quad \text{Cap} (Q) \simeq \left( \log \frac{2}{|Q|} \right)^{-1} \quad \text{if } n = 2,
\end{equation}

provided $|Q| \leq 1$. For these and other properties of capacities, as well as related notions of potential theory we refer to [AH], [M].

We conclude this section with several equivalent characterizations of the class of admissible measures $\mu \in \mathcal{M}^{1,2}_+(\mathbb{R}^n)$ which obey the trace inequality

\begin{equation}
(2.5) \quad \int_{\mathbb{R}^n} |u|^2 \, d\mu \leq c_2 \||u||^2_{L^1,2(\mathbb{R}^n)}, \quad u \in C_0^\infty (\mathbb{R}^n),
\end{equation}

where $c$ is a positive constant which does not depend on $u$.

By $(-\Delta)^{-\frac{1}{2}} \mu = c(n) \int_{\mathbb{R}^n} |x - t|^{-1-n} \, d\mu (t)$ we denote the Riesz potential of order $1$ of the measure $\mu$; here $c(n)$ is a normalization constant which depends only on $n$.

**Theorem 2.1.** Let $\mu$ be a locally finite nonnegative measure on $\mathbb{R}^n$. Then $\mu \in \mathcal{M}^{1,2}_+(\mathbb{R}^n)$ if and only if any one of the following statements hold.

(i) The Riesz potential $(-\Delta)^{-\frac{1}{2}} \mu \in L^2_{\text{loc}}(\mathbb{R}^n)$, and $[(-\Delta)^{-\frac{1}{2}} \mu]^2 \in \mathcal{M}^{1,2}_+(\mathbb{R}^n)$, i.e.,

\begin{equation}
(2.6) \quad \int_{\mathbb{R}^n} |u|^2 \left[ (-\Delta)^{-\frac{1}{2}} \mu \right]^2 \, dx \leq c_1^2 \||u||^2_{L^1,2(\mathbb{R}^n)}, \quad u \in C_0^\infty (\mathbb{R}^n),
\end{equation}

where $c_1 > 0$ does not depend on $u$.

(ii) For every compact set $e \subset \mathbb{R}^n$,

\begin{equation}
(2.7) \quad \mu (e) \leq c_2 \text{cap} (e),
\end{equation}

where $c_2$ does not depend on $e$.

(iii) For every ball $B$ in $\mathbb{R}^n$,

\begin{equation}
(2.8) \quad \int_B \left[ (-\Delta)^{-\frac{1}{2}} \mu_B \right]^2 \, dx \leq c_3 \mu (B),
\end{equation}

where $c_3$ is a positive constant which does not depend on $B$. 


where $d\mu_B = \chi_B \, d\mu$, and $c_3$ does not depend on $B$.

(iv) The pointwise inequality

$$(-\Delta)^{-\frac{1}{2}} \mu(x) < \infty$$

holds a.e., where $c_4$ does not depend on $x \in \mathbb{R}^n$.

(v) For every dyadic cube $P$ in $\mathbb{R}^n$,

$$\sum_{Q \subseteq P} \left[ \frac{\mu(Q)}{|Q|^{1-1/n}} \right]^2 |Q| \leq c_5 \mu(P),$$

where the sum is taken over all dyadic cubes $Q$ contained in $P$, and $c_5$ does not depend on $P$.

Moreover, the least constants $c_i$, $i = 1, \ldots, 5$, are equivalent to the least constant $c$ in (2.5).

Theorem 2.1 follows from the results of [KS], [M], [MV1], and [V].

Remark 3. An analogous characterization holds for admissible measures on the space $W^{1,2}(\mathbb{R}^n)$ in place of $L^{1,2}(\mathbb{R}^n)$. One only needs to replace $(-\Delta)^{-\frac{1}{2}} \mu$ in statements (i), (iii), and (iv) by $(1 - \Delta)^{-\frac{1}{2}} \mu$, the capacity $\text{cap} (\cdot)$ in (ii) by $\text{Cap} (\cdot)$, and restrict oneself to cubes $P$ such that $|P| \leq 1$ in (v).

### 3. Reduction to inequalities for lower order terms

In this section, the form boundedness problem for the general second order differential operator $L$ defined by (1.1) is reduced to the special case of lower order terms, $\vec{b} \cdot \nabla + q$. The latter, in its turn, is shown to be equivalent to the form boundedness of $q - \frac{1}{2} \text{div} \vec{b}$, and the commutator inequality (1.28).

Since the coefficients $A = (a_{ij})$, $\vec{b} = (b_i)$, and $q$ are arbitrary real- or complex-valued distributions, we may assume without loss of generality that $L$ is in the divergence form,

$$L u = \text{div} (A \nabla u) + \vec{b} \cdot \nabla u + q u, \quad u \in C^\infty_0(\mathbb{R}^n),$$

with the same principal part as (1.1). We denote by $A^s = \frac{1}{2} (A + A^t)$ and $A^c = \frac{1}{2} (A - A^t)$ respectively the symmetric and skew-symmetric parts of $A$, and by $\text{Div}$ the row divergence operator acting from $D'(\mathbb{R}^n)^{n \times n}$ to $D'(\mathbb{R}^n)^n$.

**Proposition 3.1.** Suppose $L$ is defined by (3.1), where $A = (a_{ij})_{i,j=1}^n \in D'(\mathbb{R}^n)^{n \times n}$, $\vec{b} = (b_j)_{j=1}^n \in D'(\mathbb{R}^n)^n$, and $q \in D'(\mathbb{R}^n)$, $n \geq 1$. Let $\vec{b}_1 = \vec{b} - \text{Div} A^c$.

Then the following statements are equivalent.

(i) The sesquilinear form associated with $L$ is bounded, i.e.,

$$|\langle L u, v \rangle| \leq C \|u\|_{L^{1,2}(\mathbb{R}^n)} \|v\|_{L^{1,2}(\mathbb{R}^n)}$$

where the constant $C$ does not depend on $u, v \in C^\infty_0(\mathbb{R}^n)$.
(ii) The sesquilinear form associated with $L_1 = \vec{b}_1 \cdot \nabla + q$ is bounded, and $A^s \in L^\infty(\mathbb{R}^n)^{n \times n}$.

Proof. Since $A^c$ is skew-symmetric, $\text{div} (A^c \nabla u) = -\text{Div} A^c \cdot \nabla u$, and consequently
$$\langle \text{div} (A \nabla u), v \rangle = -\langle A^s \nabla u, \nabla v \rangle - \langle \text{Div} A^c \cdot \nabla u, v \rangle,$$
for $u, v \in C_0^\infty(\mathbb{R}^n)$. Hence,
$$\langle L u, v \rangle = -\langle A^s \nabla u, \nabla v \rangle + \langle \vec{b}_1 \cdot \nabla u, v \rangle + \langle q u, v \rangle,$$ where $\vec{b}_1 = \vec{b} - \text{Div} A^c$.

Suppose that the sesquilinear form of $L$ is bounded. Then, replacing $u$ and $v$ in (3.2) respectively by $\bar{u} = e^{it\xi \cdot x} u$ and $\bar{v} = e^{it\xi \cdot x} v$, where $x, \xi \in \mathbb{R}^n$, and $t > 0$, we obtain
$$| -\langle A^s \nabla \bar{u}, \nabla \bar{v} \rangle + \langle \vec{b}_1 \cdot \nabla \bar{u}, \bar{v} \rangle + \langle q \bar{u}, \bar{v} \rangle | \leq C ||\bar{u}||_{L^{1.2}(\mathbb{R}^n)} ||\bar{v}||_{L^{1.2}(\mathbb{R}^n)} \leq C (||u||_{L^{1.2}(\mathbb{R}^n)} + t |\xi| ||u||_{L^2(\mathbb{R}^n)}) (||v||_{L^{1.2}(\mathbb{R}^n)} + t |\xi| ||v||_{L^2(\mathbb{R}^n)})$$
Dividing both sides of the preceding inequality by $t^2$, and letting $t \to +\infty$, we see that the last two terms on the left-hand side tend to 0, which yields
$$\left| \int_{\mathbb{R}^n} (A^s \xi \cdot \xi) u \bar{v} \, dx \right| \leq C |\xi|^2 ||u||_{L^2(\mathbb{R}^n)} ||v||_{L^2(\mathbb{R}^n)}.$$ From this we deduce
$$|A^s(x) \xi \cdot \xi| \leq C |\xi|^2, \quad x, \xi \in \mathbb{R}^n.$$ Clearly, both the real and imaginary parts of $A^s$ obey the preceding inequality, and since $A^s$ is symmetric, their operator norms are bounded by $C$. Hence, necessarily $A^s \in L^\infty(\mathbb{R}^n)^{n \times n}$. The latter is also sufficient for the form boundedness of $\text{div} (A^s \nabla)$. Thus, $L$ is form bounded if and only if $A^s \in L^\infty(\mathbb{R}^n)^{n \times n}$, and $\vec{b}_1 \cdot \nabla + q$ is form bounded. \hfill $\square$

Proposition 3.2. Let $q \in D'(\mathbb{R}^n)$ and $\vec{b} \in D'(\mathbb{R}^n)^n$, $n \geq 1$, and let $L = \vec{b} \cdot \nabla + q$. Then the following statements are equivalent.

(i) The bilinear form associated with $L$ obeys (3.2).

(ii) The following two conditions hold:

(a) For all $u, v \in C_0^\infty(\mathbb{R}^n)$,

$$\left| \langle (q - \frac{1}{2} \text{div} \vec{b}) u, v \rangle \right| \leq C ||u||_{L^{1.2}(\mathbb{R}^n)} ||v||_{L^{1.2}(\mathbb{R}^n)}.$$

(b) For all $u, v \in C_0^\infty(\mathbb{R}^n)$,

$$\left| \langle \vec{b}, \bar{u} \nabla v - v \nabla \bar{u} \rangle \right| \leq C ||u||_{L^{1.2}(\mathbb{R}^n)} ||v||_{L^{1.2}(\mathbb{R}^n)}.$$
Proof. Integration by parts gives
\[
(b \cdot \nabla u + q u, v) = \langle (q - \frac{1}{2} \text{div } b), \bar{u}v \rangle - \frac{1}{2} \langle b, \bar{u} \nabla v - v \nabla \bar{u} \rangle.
\]
Interchanging the roles of \(\bar{u}\) and \(v\), it is easy to see that the bilinear form associated with \(b \cdot \nabla + q\) is bounded if and only if both forms on the right-hand side of the preceding equation are bounded, i.e., both (3.3) and (3.4) hold. □

Remark 4. Inequality (3.3) holds if and only if the inequality
\[
\int_{\mathbb{R}^n} |\nabla \Delta^{-1} (q - \frac{1}{2} \text{div } b)|^2 |u|^2 \, dx \leq C \left| \nabla u \right|^2_{L^2(\mathbb{R}^n)}
\]
is valid, where \(C\) does not depend on \(u \in C_0^\infty(\mathbb{R}^n)\) (see [MV1], Theorem I).

Corollary 3.3. Let \(P u = \text{div} (A \nabla u)\), where \(A = (a_{ij})_{i,j=1}^n \in D'(\mathbb{R}^n)^{n \times n}\). Then
\[
\langle Pu, v \rangle \leq C \|u\|_{L^{1,2}(\mathbb{R}^n)} \|v\|_{L^{1,2}(\mathbb{R}^n)}
\]
for all \(u, v \in C_0^\infty(\mathbb{R}^n)\), if and only if \(A^s \in L^\infty(\mathbb{R}^n)^{n \times n}\), and \(A^c\) obeys the inequality
\[
|\langle \text{Div } A^c, u \nabla v - v \nabla u \rangle| \leq C \|u\|_{L^{1,2}(\mathbb{R}^n)} \|v\|_{L^{1,2}(\mathbb{R}^n)}
\]
for all \(u, v \in C_0^\infty(\mathbb{R}^n)\). The preceding inequality holds if \(A^c \in BMO(\mathbb{R}^n)^{n \times n}\).

Corollary 3.3 follows from Propositions 3.1 and 3.2. The last statement is a consequence of the div-curl lemma [CLMS] (see [1], Sec. 3.8). A more precise necessary and sufficient condition for (3.7) is obtained below.

4. Operators with lower order terms and magnetic Schrödinger operators

In this section, which contains our main results, we consider the form boundedness problem on \(L^{1,2}(\mathbb{R}^n)\), \(n \geq 2\), for the operator
\[
\mathcal{L} = b \cdot \nabla + q
\]
with distributional lower order terms \(\vec{b} \in D'(\mathbb{R}^n)^n\) and \(q \in D'(\mathbb{R}^n)\). Here \(\mathcal{L}\) is initially defined as acting from \(D(\mathbb{R}^n)\) to \(D'(\mathbb{R}^n)\). We deduce necessary and sufficient conditions for the form boundedness of \(\mathcal{L}\), and as a consequence, of the Schrödinger operator \((i \nabla + \vec{a})^2 + q\) with magnetic vector potential \(\vec{a} \in L^2_{\text{loc}}(\mathbb{R}^n)\).

We will need a series of lemmas and propositions.

Proposition 4.1. Let \(\vec{b} \in D'(\mathbb{R}^n)^n, n \geq 2\). Suppose that (3.4) holds. Then, for every cube \(Q\) in \(\mathbb{R}^n\), the following estimates are valid:
\[
\begin{align*}
\|\text{div } b\|_{L^{-1,2}(Q)} &\leq C |Q|^{\frac{1}{2} - \frac{1}{n}} \text{ if } n \geq 3; \quad \text{div } b = 0 \text{ if } n = 2, \\
\|b\|_{L^{-1,2}(Q)} &\leq C |Q|^{\frac{1}{2}} \text{ if } n \geq 2,
\end{align*}
\]
where $C$ does not depend on $Q$.

Proof. Let $v \in C_0^\infty(Q)$, and suppose $u = 1$ on $Q$, $u \in C_0^\infty(\mathbb{R}^n)$ in (3.4). Then
\[
\left| \langle \vec{b}, \bar{u} \nabla v - v \nabla \bar{u} \rangle \right| = \left| \langle \vec{b}, \nabla v \rangle \right| = \left| \langle \text{div} \vec{b}, v \rangle \right| \leq C \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(Q)}.
\]
Taking the infimum over all such $u$ on the right-hand side, we obtain
\[
\left| \langle \text{div} \vec{b}, v \rangle \right| \leq C \text{cap}(Q)^{\frac{1}{2}} \|\nabla v\|_{L^2(Q)}, \quad v \in C_0^\infty(Q),
\]
where the capacity $\text{cap}(\cdot)$ is defined by (2.1). Taking into account (2.2), we deduce from the preceding inequality that $\text{div} \vec{b} = 0$ if $n = 2$, and
\[
\left| \langle \text{div} \vec{b}, v \rangle \right| \leq C |Q|^{\frac{1}{2} - \frac{2}{n}} \|\nabla v\|_{L^2(Q)}, \quad v \in C_0^\infty(Q),
\]
if $n = 3$, which proves (4.2).

Now suppose $v \in C_0^\infty(Q)$, and let us set $u = (x_i - a_i) \eta$ ($i = 1, \ldots, n$), where $a = (a_i)$ is the center of $Q$, $\eta = 1$ on $Q$ and $\eta \in C_0^\infty(2Q)$. Then it is easy to see that $\|\nabla u\|_{L^2(2Q)} \leq C |Q|^{\frac{3}{2}}$. Notice that for such $u$ and $v$,
\[
\langle \vec{b}, \bar{u} \nabla v - v \nabla \bar{u} \rangle = -\langle \text{div} \vec{b}, \bar{u} v \rangle - 2 \langle \vec{b}, v \nabla \bar{u} \rangle
\]
\[
= -\langle \text{div} \vec{b}, (x_i - a_i) v \rangle - 2 \langle \vec{b}, v \rangle.
\]
Using (4.3) with $(x_i - a_i) v$ in place of $v$, and Poincaré’s inequality, we obtain
\[
\left| \langle \text{div} \vec{b}, (x_i - a_i) v \rangle \right| \leq C |Q|^{\frac{1}{2} - \frac{2}{n}} \|\nabla [(x_i - a_i) v]\|_{L^2(Q)}
\]
\[
\leq C |Q|^{\frac{1}{2} - \frac{2}{n}} (\|v\|_{L^2(Q)} + \|(x_i - a_i) \nabla v\|_{L^2(Q)})
\]
\[
\leq C |Q|^{\frac{1}{2}} \|\nabla v\|_{L^2(Q)},
\]
for every $v \in C_0^\infty(Q)$. Hence, for every $i = 1, \ldots, n$,
\[
2 \left| \langle \vec{b}, v \rangle \right| \leq \left| \langle \vec{b}, \bar{u} \nabla v - v \nabla \bar{u} \rangle \right| + \left| \langle \text{div} \vec{b}, (x_i - a_i) v \rangle \right|
\]
\[
\leq C \|\nabla u\|_{L^2(2Q)} \|\nabla v\|_{L^2(Q)} + C |Q|^{\frac{1}{2}} \|\nabla v\|_{L^2(Q)}
\]
\[
\leq C |Q|^{\frac{1}{2}} \|\nabla v\|_{L^2(Q)}.
\]
This yields $\|\vec{b}\|_{L^{-1,2}(Q)} \leq C |Q|^{\frac{1}{2}}$, which completes the proof of Proposition 4.1.

For a fixed cube $Q$ in $\mathbb{R}^n$, we denote by $\{\eta_j\}_{j=0}^\infty$ a smooth partition of unity associated with $Q$; i.e., $\eta_0 \in C_0^\infty(2Q)$, $\eta_j \in C_0^\infty(2^{j+1}Q \setminus 2^j Q)$, $j = 1, 2, \ldots$, so that
\[
0 \leq \eta_j(x) \leq 1, \quad |\nabla \eta_j(x)| \leq C (2^j \ell(Q))^{-1}, \quad j = 0, 1, \ldots
\]
(4.6) \[ \sum_{j=0}^{\infty} \eta_j(x) = 1, \quad x \in \mathbb{R}^n, \]

where \( \ell(Q) \) denotes the side length of \( Q \), and \( C \) depends only on \( n \).

We will need the following proposition.

**Proposition 4.2.** Let \( Q \) be a cube in \( \mathbb{R}^n, n \geq 2 \). Let \( \{ \eta_j \}_{j=0}^{+\infty} \) be the partition of unity associated with \( Q \) defined by above. Then the following estimates hold.

(i) For any \( v \in C_0^\infty(Q) \) and \( j = 0, 1, \ldots, \)

\[ \| \nabla (\eta_j \partial_i \partial_m \Delta^{-1} v) \|_{L^2(2^{j+1}Q)} \leq C 2^{-j(1+\frac{n}{2})} \| \nabla v \|_{L^2(Q)}, \quad i, m = 1, \ldots, n, \]

where \( C \) depends only on \( n \).

(ii) For any \( v \in C_0^\infty(Q) \) and \( j = 0, 1, \ldots, \)

\[ \| \nabla (\eta_j \partial_i \Delta^{-1} v) \|_{L^2(2^{j+1}Q)} \leq C 2^{-j \frac{n}{2}} \| v \|_{L^2(Q)}, \quad i = 1, \ldots, n, \]

where \( C \) depends only on \( n \).

(iii) For any \( v \in C_0^\infty(Q) \) such that \( \int_Q v(x) \, dx = 0 \), and \( j = 2, 3, \ldots, \)

\[ \| \nabla (\eta_j \partial_i \Delta^{-1} v) \|_{L^2(2^{j+1}Q)} \leq C 2^{-j(1+\frac{n}{2})} |Q|^{-\frac{1}{2}} \| v \|_{L^1(Q)}, \quad i = 1, \ldots, n, \]

where \( C \) depends only on \( n \).

(iv) Let \( n \geq 3 \). For any \( v \in C_0^\infty(Q) \) such that \( \int_Q v(x) \, dx = 0 \), and \( j = 2, 3, \ldots, \)

\[ \| \nabla (\eta_j \Delta^{-1} v) \|_{L^2(2^{j+1}Q)} \leq C 2^{-j \frac{n}{2}} |Q|^{-\frac{1}{2}} \| v \|_{L^1(Q)}, \quad i = 1, \ldots, n, \]

where \( C \) depends only on \( n \).

**Proof.** Let \( v \in C_0^\infty(Q) \). Let \( a = a_Q \) be the center of \( Q \), and \( r = \ell(Q) \) its side length. We denote by \( R_i \) the Riesz transforms, and by \( R_i R_m, i, m = 1, \ldots, n, \) the second order Riesz transforms on \( \mathbb{R}^n \) (see [St]). For \( j = 0, 1, \) (4.7) follows from the boundedness of \( R_i R_m \) on \( L^2(\mathbb{R}^n) \), and Poincaré’s inequality:

\[ \| \nabla (\eta_j \partial_i \partial_m \Delta^{-1} v) \|_{L^2(2^{j+1}Q)} \leq \| \nabla \eta_j \|_{L^2(Q)} \| \partial_i \partial_m \Delta^{-1} v \|_{L^2(2^{j+1}Q)} \]

\[ + \| \eta_j \partial_i \Delta^{-1} \nabla v \|_{L^2(2^{j+1}Q)} \]

\[ \leq C \left( r^{-1} \| R_i R_m v \|_{L^2(\mathbb{R}^n)} + \| R_i R_m \nabla v \|_{L^2(\mathbb{R}^n)} \right) \]

\[ \leq C \left( r^{-1} \| v \|_{L^2(Q)} + \| \nabla v \|_{L^2(Q)} \right) \leq C \| \nabla v \|_{L^2(Q)}. \]

For \( j \geq 2 \), one needs estimates of the kernels of the operators \( \partial_i \Delta^{-1} \) and \( \partial_i \partial_m \Delta^{-1} = -R_i R_m \) which are given respectively, up to a constant multiple, by

\[ K^i(x-t) = \frac{(x_i-t_i)}{|x-t|^n}, \quad K^{i,m}(x-t) = \frac{\delta_{im} |x-t|^2 - n (x_i-t_i) (x_m-t_i)}{|x-t|^{n+2}}. \]
Clearly,
\begin{align}
|K^i(x - t) - K^i(x - a)| & \leq C(n) \frac{|t - a|}{|x - t|^n}, \\
|K^{i,m}(x - t) - K^{i,m}(x - a)| & \leq C(n) \frac{|t - a|}{|x - t|^{n+1}},
\end{align}

if $|t - a| < R$, $|x - t| > 2R$. Using the preceding estimates with $R = c(n) 2^j r$, we see that, for $x \in 2^{j+1} Q \setminus 2^{j-1} Q$:

\[
|\partial_i \partial_m \Delta^{-1} v(x)| = \left| \int_Q \left( K^i(x - t) - K^i(x - a) \right) \partial_m v(t) \, dt \right|
\]

\[
\leq \int_Q |K^i(x - t) - K^i(x - a)| |\nabla v(t)| \, dt \leq C r^{1-n} 2^{-jn} ||\nabla v||_{L^1(Q)},
\]

\[
|\nabla \partial_i \partial_m \Delta^{-1} v(x)| = \left| \int_Q \left( K^{i,m}(x - t) - K^{i,m}(x - a) \right) \partial_m \nabla v(t) \, dt \right|
\]

\[
\leq \int_Q |K^{i,m}(x - t) - K^{i,m}(x - a)| |\nabla v(t)| \, dt \leq C r^{1-n} 2^{-j(n+1)} ||\nabla v||_{L^1(Q)}.
\]

Hence,
\[
||\nabla (\eta_j \partial_i \partial_m \Delta^{-1} v)||_{L^2(2^{j+1} Q)} \leq ||\nabla \eta_j (\partial_i \partial_m \Delta^{-1} v)||_{L^2(2^{j+1} Q)}
\]

\[
+ ||\eta_j \partial_i \partial_m \Delta^{-1} \nabla v||_{L^2(2^{j+1} Q)} \leq C r^{-\frac{1}{2}} 2^{-j(1+\frac{q}{n})} ||\nabla v||_{L^1(Q)}
\]

\[
\leq C 2^{-j(1+\frac{q}{n})} ||\nabla v||_{L^2(Q)},
\]

which gives (3.7).

To prove (3.8), notice that for $j = 0, 1$, it follows

\[
||\nabla (\eta_j \partial_i \Delta^{-1} v)||_{L^2(2^{j+1} Q)} \leq ||\nabla \eta_j (\partial_i \Delta^{-1} v)||_{L^2(2^{j+1} Q)}
\]

\[
+ ||\eta_j \partial_i \Delta^{-1} \nabla v||_{L^2(2^{j+1} Q)} \leq C (r^{-1} ||\nabla \Delta^{-1} v||_{L^2(\mathbb{R}^n)} + \sum_{m=1}^n ||R_i R_m v||_{L^2(\mathbb{R}^n)})
\]

\[
\leq C ||\nabla \Delta^{-1} v||_{L^q(Q)} + C ||v||_{L^q(Q)},
\]

where $q = \frac{2n}{n-2}$. Estimating the first term on the right by means of Sobolev’s inequality, we conclude that it is bounded by $C ||v||_{L^2(Q)}$.

If $j = 2, 3, \ldots$, then for $x \in 2^{j+1} Q \setminus 2^{j-1} Q$ we have:

\[
|\nabla \eta_j (\partial_i \Delta^{-1} v(x)| \leq |\nabla \eta_j(x)||\partial_i \Delta^{-1} v(x)| + |\eta_j(x)||\nabla \partial_i \Delta^{-1} v(x)|
\]

\[
\leq C (2^j r)^{-1} \int_Q |K^i(x - t)| |v(t)| \, dt + \sum_{m=1}^n \int_Q |K^{i,m}(x - t)| |v(t)| \, dt
\]

\[
\leq C (2^j r)^{-1} \int_Q |v(t)| \, dt \leq C 2^{-jn} r^{-\frac{q}{2}} ||v||_{L^2(Q)}.
\]
Thus, for all \( j = 0, 1, 2, \ldots \),
\[
\| \nabla (\eta_j \partial_i \Delta^{-1} v) \|_{L^2(2^{j+1}Q)} \leq C 2^{-j} 2^{n} \| v \|_{L^2(Q)}.
\]
which proves (4.8).

The proof of (4.9) for \( j = 2, 3, \ldots \), provided \( \int_Q v(x) dx = 0 \), is similar to that of (4.7). Using estimates (4.11) and (4.12), we deduce that, for \( x \in 2^{j+1}Q \setminus 2^jQ \),
\[
|\nabla (\eta_j \partial_i \Delta^{-1} v)(x)| \leq |\nabla \eta_j(x)| |\partial_i \Delta^{-1} v(x)| + |\eta_j(x)| |\nabla \partial_i \Delta^{-1} v(x)|
\leq C (2^j r)^{-1} \int_Q |K^i(x-t) - K^i(x)| |v(t)| dt
\]
\[
+ C \sum_{m=1}^n \int_Q |K^{i,m}(x-t) - K^{i,m}(x)| |v(t)| dt \leq C 2^{-j(n+1)} |Q|^{-\frac{1}{2}} \int_Q |v(t)| dt.
\]
This yields
\[
\| \nabla (\eta_j \partial_i \Delta^{-1} v) \|_{L^2(2^{j+1}Q)} \leq C 2^{-j(1+\frac{n}{2})} |Q|^{-\frac{1}{2}} \| v \|_{L^2(Q)}.
\]
The proof of (4.10) for \( j = 2, 3, \ldots \) is very similar to that of (4.9), and is omitted here. \( \square \)

For \( N > 0 \), define a smooth cut-off function \( \psi_N(x) = \psi\left(\frac{x}{N}\right) \), where
\[
\psi \in C^\infty_0(\mathbb{R}^n); \quad \psi(x) = 1 \text{ if } |x| \leq \frac{1}{2}, \quad \psi(x) = 0 \text{ if } |x| > 1.
\]

**Lemma 4.3.** Suppose \( \tilde{b} \in D'(\mathbb{R}^n)^n, \ n \geq 2 \). Suppose that (4.3) holds. Then
\[
\tilde{b} = \nabla f + \text{Div } F \text{ in } D'(\mathbb{R}^n)^n,
\]
where
\[
f = \Delta^{-1} \text{div } \tilde{b} \in \text{BMO}(\mathbb{R}^n), \quad F = \Delta^{-1} \text{curl } \tilde{b} \in \text{BMO}(\mathbb{R}^n)^{n \times n}.
\]
Here \( f \) and \( F \) are defined (up to a constant) by, respectively,
\[
f = \lim_{N \to +\infty} f_N, \quad f_N = \Delta^{-1} \text{div } (\psi_N \tilde{b}),
\]
\[
F = \lim_{N \to +\infty} F_N, \quad F_N = \Delta^{-1} \text{curl } (\psi_N \tilde{b}),
\]
in the sense of the convergence in the weak-* topology of BMO(\( \mathbb{R}^n \)). The limits above do not depend on the choice of \( \psi_N \).

Furthermore,
\[
\nabla f = \lim_{N \to +\infty} \nabla f_N, \quad \text{Div } F = \lim_{N \to +\infty} \text{Div } F_N \text{ in } D'(\mathbb{R}^n)^n,
\]
\[
\text{curl } (\nabla f) = 0, \quad \text{div } (\text{Div } F) = 0, \quad \Delta f = \text{div } \tilde{b}, \quad \Delta F = \text{curl } \tilde{b}.
\]
Proof. By Proposition 4.1, (3.4) implies (4.3). It follows that the latter inequality holds with \( \psi_N \vec{b} \) in place of \( \vec{b} \), i.e., for every cube \( Q \),

\[
(4.21) \quad ||\psi_N \vec{b}||_{L^{1,2}(Q)} \leq C|Q|^{\frac{1}{2}},
\]

where \( C \) does not depend on \( Q \) and \( N \). This is a consequence of the inequality \( ||(\nabla \psi_N ) \nu||_{L^2(\mathbb{R}^n)} \leq C(n) ||\nabla \nu||_{L^2(\mathbb{R}^n)} \), for \( \nu \in C_0^\infty(\mathbb{R}^n) \), which follows from Poincaré’s inequality.

We observe that \( f_N \) and \( F_N \) given respectively by (4.17) and (4.18) are well-defined in terms of distributions. Moreover, by (4.21), \( \psi_N \vec{b} \in L^{-1,2}(\mathbb{R}^n) \), and hence \( f_N \in L^2(\mathbb{R}^n) \), \( F_N \in L^2(\mathbb{R}^n)^{n \times n} \).

Our next step is to show that, for \( i, m = 1, 2, \ldots, n \),

\[
(4.22) \quad ||\partial_i \partial_m \Delta^{-1} (\psi_N \vec{b})||_{L^{-1,2}(Q)} \leq C|Q|^{\frac{1}{2}},
\]

where \( C \) does not depend on \( Q \) and \( N \).

Notice that \( \partial_i \partial_m \Delta^{-1} (\psi_N \vec{b}) \) is well-defined in \( D'(\mathbb{R}^n)^n \). Let \( \{\eta_j\}_{j=0}^\infty \) be the partition of unity \((4.3) - (4.6)\) associated with a cube \( Q \) in \( \mathbb{R}^n \). Then

\[
\langle \partial_i \partial_m \Delta^{-1} (\psi_N \vec{b}), \vec{v} \rangle = \langle \vec{b}, \psi_N \Delta^{-1} \partial_i \partial_m \vec{v} \rangle = \sum_{j=0}^\infty \langle \psi_N \vec{b}, \eta_j \partial_i \partial_m \Delta^{-1} \vec{v} \rangle,
\]

for every \( \vec{v} \in C_0^\infty(Q)^n \), where the sum on the right contains only a finite number of nonzero terms.

Then by (4.21) and statement (i) of Proposition 4.2

\[
\left| \langle \psi_N \vec{b}, \partial_i \partial_m \Delta^{-1} \vec{v} \rangle \right| \leq \sum_{j=0}^\infty \left| \langle \psi_N \vec{b}, \eta_j \partial_i \partial_m \Delta^{-1} \vec{v} \rangle \right|
\]

\[
\leq c \sum_{j=0}^\infty 2^{j\frac{n}{2}} |Q|^{\frac{1}{2}} ||\nabla (\eta_j \partial_i \partial_m \Delta^{-1} \vec{v})||_{L^2(2j+1,Q)}
\]

\[
\leq C |Q|^{\frac{1}{2}} ||\nabla \nu||_{L^2(Q)},
\]

i.e., (4.22) holds. In particular,

\[
||\nabla f_N||_{L^{-1,2}(Q)} \leq C|Q|^{\frac{1}{2}}, \quad ||D(F_N)||_{L^{-1,2}(Q)} \leq C|Q|^{\frac{1}{2}}.
\]

This gives:

\[
||f_N - m_Q(f_N)||_{L^2(Q)}^2 \leq c ||\nabla f_N||_{L^{-1,2}(Q)}^2 \leq C|Q|,
\]

\[
||F_N - m_Q(F_N)||_{L^2(Q)}^2 \leq c ||D(F_N)||_{L^{-1,2}(Q)}^2 \leq C|Q|,
\]

where \( C \) does not depend on \( Q \) and \( N \). Hence,

\[
\sup_N ||f_N||_{BMO(\mathbb{R}^n)} < \infty, \quad \sup_N ||F_N||_{BMO(\mathbb{R}^n)^{n \times n}} < \infty.
\]
We now show that both \( \{ f_N \} \) and \( \{ F_N \} \) converge in the weak-* topology of BMO (considered as the dual of \( \mathcal{H}^1 \); see [St]) respectively to \( f \in \text{BMO}(\mathbb{R}^n) \), and \( F \in \text{BMO}(\mathbb{R}^n)^{n \times n} \) (defined up to an additive constant). We will then deduce that \( \Delta f = \text{div} \, \vec{b} \) and \( \Delta F = \text{curl} \, \vec{b} \) in the distributional sense, and set

\[
f = \Delta^{-1} \text{div} \, \vec{b}, \quad F = \Delta^{-1} \text{curl} \, \vec{b}.
\]

Let us prove the weak-* convergence for the sequence \( \{ f_N \} \) in \( \text{BMO}(\mathbb{R}^n) \). For \( \{ F_N \} \), the argument is quite similar. Since \( \{ f_N \} \) is uniformly bounded in the BMO-norm, it is enough to verify that it forms a Cauchy sequence in the weak-* topology of BMO on a dense family of \( C^\infty \)-functions in \( \mathcal{H}^1(\mathbb{R}^n) \).

Suppose that \( v \in C^\infty(\mathbb{R}^n) \) is supported on a cube \( Q \), and \( \int_Q v(x) \, dx = 0 \). Then using the same partition of unity \( \{ \eta_j \} \) one can easily check that

\[
\left| \int_{\mathbb{R}^n} (f_N - f) \, \tilde{v} \, dx \right| \leq \sum_{j \geq N_0} \left| \langle (\psi_N - \psi_M) \, \vec{b}, \, \eta_j \, \nabla \Delta^{-1} v \rangle \right|,
\]

where \( N_0 \to +\infty \) as \( M, N \to +\infty \). We deduce from (4.21):

\[
\left| \langle (\psi_N - \psi_M) \, \vec{b}, \, \eta_j \, \nabla \Delta^{-1} v \rangle \right| \leq c \, 2^j \, Q^{\frac{1}{2}} \, ||\nabla (\eta_j \, \nabla \Delta^{-1} v)||_{L^2(2|Q|)}.
\]

By statement (iii) of Proposition 4.2, we have

\[
\|\nabla (\eta_j \, \nabla \Delta^{-1} v)\|_{L^2(2|Q|)} \leq c \, 2^{-j(1 + \frac{1}{p})} |Q|^{-\frac{1}{2}} \|v\|_{L^p(Q)}, \quad j \geq N_0,
\]

where \( c \) does not depend on \( j, Q, \) and \( v \). Thus,

\[
\left| \langle (\psi_N - \psi_M) \, \vec{b}, \, \eta_j \, \nabla \Delta^{-1} v \rangle \right| \leq c \, 2^{-j} \, ||v||_{L^p(Q)}, \quad j \geq N_0,
\]

and consequently,

\[
\sum_{j \geq N_0} \left| \langle (\psi_N - \psi_M) \, \vec{b}, \, \eta_j \, \nabla \Delta^{-1} v \rangle \right| \leq c \, ||v||_{L^p(Q)} \sum_{j \geq N_0} 2^{-j}, \quad j \geq N_0.
\]

Using the preceding estimates and letting \( M, N \to +\infty \) so that \( N_0 \to +\infty \), we see that \( \{ f_N \} \) is a Cauchy sequence in the weak-* topology of BMO. In particular,

\[
\lim_{N \to +\infty} \int_{\mathbb{R}^n} f_N \, \tilde{v} \, dx = \int_{\mathbb{R}^n} f \, \tilde{v} \, dx, \quad v \in C^\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} v \, dx = 0,
\]

where \( f \in \text{BMO}(\mathbb{R}^n) \).

To show that the limit in (4.24) does not depend on the choice of the cut-off functions \( \psi_N \), and for future reference, we now demonstrate that, for every \( v \in C^\infty(\mathbb{R}^n) \) supported on a cube \( Q \) such that \( \int_Q v \, dx = 0 \), it follows

\[
\int_{\mathbb{R}^n} f \, \tilde{v} \, dx = -\sum_{j=0}^\infty \langle \vec{b}, \, \eta_j \, \nabla (\Delta^{-1} v) \rangle.
\]
Notice that, by (4.3) and statement (iii) of Proposition 4.2,
\[ \sum_{j \geq M} \left| \langle \vec{b}, \eta_j \nabla \Delta^{-1} v \rangle \right| \leq C \sum_{j \geq M} |2^j Q|^{\frac{1}{2}} \| \nabla (\eta_j \nabla \Delta^{-1} v) \|_{L^2(2^j Q)} \]
\[ \leq C \| v \|_{L^1(Q)} \sum_{j \geq M} 2^{-j}, \]
for every \( M \geq 1 \). Moreover, by (4.21), a similar estimate holds with \( \psi_N \vec{b} \) in place of \( \vec{b} \), and \( C \) which does not depend on \( M \) and \( N \).

Clearly, (4.25) holds with \( \psi_N \vec{b} \) in place of \( \vec{b} \), and, for \( N \) large,
\[ \sum_{0 \leq j \leq M} \langle \vec{b}, \eta_j \nabla \Delta^{-1} v \rangle = \sum_{0 \leq j \leq M} \langle \psi_N \vec{b}, \eta_j \nabla \Delta^{-1} v \rangle. \]

By picking \( M \) and \( N \) large enough, and taking into account the above estimates together with (4.24), we arrive at (4.25).

We observe that (4.25) with \( \text{div} \vec{v} \) in place of \( v \) yields
\[ \langle \nabla f, \vec{v} \rangle = - \int_{\mathbb{R}^n} f \, \text{div} \vec{v} \, dx = \sum_{j=0}^{\infty} \langle \vec{b}, \eta_j \nabla (\Delta^{-1} \text{div} \vec{v}) \rangle, \]
for every \( \vec{v} \in C_0^\infty(\mathbb{R}^n)^n \) supported on \( Q \). Hence, \( \nabla f \in D'(\mathbb{R}^n)^n \), and
\[ \nabla f = \lim_{N \to +\infty} \nabla f_N \quad \text{in} \quad D'(\mathbb{R}^n)^n, \quad \text{curl} (\nabla f) = 0 \quad \text{in} \quad D'(\mathbb{R}^n)^n \times n. \]

Moreover, for every \( v \in C_0^\infty(\mathbb{R}^n) \),
\[ \langle \Delta f, v \rangle = \lim_{N \to +\infty} \langle f_N, \Delta v \rangle = - \lim_{N \to +\infty} \langle \psi_N \vec{b}, \nabla v \rangle = - \langle \vec{b}, \nabla v \rangle, \]
which gives \( \Delta f = \text{div} \vec{b} \) in \( D'(\mathbb{R}^n)^n \).

In a completely analogous fashion, one verifies that \( F_N \to F \) in the weak-* topology of BMO,
\[ \lim_{N \to +\infty} \text{Div} F_N = \text{Div} F \quad \text{in} \quad D'(\mathbb{R}^n)^n, \]
and \( \Delta F = \text{curl} \vec{b} \) in \( D'(\mathbb{R}^n)^n \times n \), \( \text{div} (\text{Div} F) = 0 \). Moreover, \( F \) is a skew-symmetric matrix field since \( F_N \) is skew-symmetric for every \( N \).

We are now in a position to establish decomposition (4.15) for vector fields which obey (3.4). Let us set \( \vec{c} = \nabla f \) and \( \vec{d} = \text{Div} F \). Using a standard decomposition for \( \vec{v} \in C_0^\infty(\mathbb{R}^n)^n \),
\[ \vec{v} = \nabla (\Delta^{-1} \text{div} \vec{v}) + \text{Div} (\Delta^{-1} \text{curl} \vec{v}), \]
we deduce:
\[ \langle \nabla f_N, \vec{v} \rangle = - \langle f_N, \text{div} \vec{v} \rangle = \langle \psi_N \vec{b}, \nabla (\Delta^{-1} \text{div} \vec{v}) \rangle \]
\[ = \langle \psi_N \vec{b}, \vec{v} \rangle - \langle \psi_N \vec{b}, \text{Div} (\Delta^{-1} \text{curl} \vec{v}) \rangle. \]
Hence,
\[
\langle \vec{c}, \vec{v} \rangle = \lim_{N \to +\infty} \langle \nabla f_N, \vec{v} \rangle = \lim_{N \to +\infty} \langle \psi_N \vec{b}, \vec{v} \rangle - \lim_{N \to +\infty} \langle \psi_N \vec{b}, \text{Div} \,(\Delta^{-1} \text{curl} \, \vec{v}) \rangle
\]
\[
= \langle \vec{b}, \vec{v} \rangle - \lim_{N \to +\infty} \langle \text{Div} \, F_N, \vec{v} \rangle = \langle \vec{b}, \vec{v} \rangle - \langle \vec{d}, \vec{v} \rangle.
\]
This completes the proof of Lemma 4.3. \qed

**Corollary 4.4.** Denote by \( P \) and \( Q \) respectively the operators
\[
P = \nabla (\Delta^{-1} \text{div}), \quad Q = \text{Div} \,(\Delta^{-1} \text{curl})
\]
defined on the class of vector fields \( \vec{b} \) which obey (3.4). Then \( P \) and \( Q \) are bounded complementary projections, that is, both \( Pb \) and \( Q\vec{b} \) satisfy (3.4), \( P(P\vec{b}) = P\vec{b}, \) \( Q(Q\vec{b}) = Q\vec{b}, \) and \( P\vec{b} + Q\vec{b} = \vec{b}. \)

**Proof.** Suppose \( \vec{b} \in D'(\mathbb{R}^n), \) and (3.4) holds. Let \( \vec{c} = Pb \) and \( \vec{d} = Q\vec{b}. \) By Lemma 4.3 \( \vec{c} + \vec{d} = \vec{b}. \) Moreover, \( \text{curl} \vec{b} = \text{curl} \vec{c}, \) and \( \vec{d} = \text{Div} \, F, \) where \( F = \Delta^{-1} \text{curl} \vec{d} \in BMO(\mathbb{R}^n)^{n \times n}. \)

Then, for every \( u, v \in C_0^\infty(\mathbb{R}^n), \)
\[
\left| \langle \vec{d}, \vec{u} \nabla v - v \nabla \vec{u} \rangle \right| = \left| \text{trace} \, \left( F, D[\vec{u} \nabla v - v \nabla \vec{u}] \right) \right| \leq C \left| \nabla u \right|_{L^2(\mathbb{R}^n)} \left| \nabla v \right|_{L^2(\mathbb{R}^n)},
\]
by the div-curl lemma \cite{CLMS}. (If \( n = 2, \) this is equivalent to the Jacobian estimate in \( \mathcal{H}^1(\mathbb{R}^2); \) for \( n \geq 3, \) it follows by the commutator estimates involving Riesz transforms. See \cite{T}, Sec. 3.8, and the proof of Theorem 4.8 below.) Thus, (3.4) holds with \( \vec{d}, \) and hence \( \vec{c}, \) in place of \( \vec{b}. \)

It remains to verify that \( P(P\vec{b}) = P\vec{b}. \) By the preceding estimate and Proposition 4.4 applied to \( \vec{d}, \) it follows
\[
\left( \vec{d} \right)_{L^{-1,2}(Q)} \leq C \left| Q \right|^{\frac{1}{2}},
\]
for every cube \( Q. \) Then, obviously,
\[
\left( \nabla \psi_N \cdot \vec{d} \right)_{L^{-1,2}(Q)} \leq C N^{-1} \left| Q \right|^{\frac{1}{2}}.
\]
where \( C \) does not depend on \( Q \) and \( N > 0. \)

From this we will deduce
\[
\lim_{N \to +\infty} \langle \nabla \psi_N \cdot \vec{d}, \Delta^{-1} \text{div} \, \vec{v} \rangle = 0, \quad \vec{v} \in C_0^\infty(\mathbb{R}^n)^n.
\]
Observe that \( \nabla \psi_N(x) = 0 \) unless \( \frac{N}{2} \leq |x| \leq N, \) and thus, for \( \vec{v} \in C_0^\infty(Q)^n, \)
\[
\langle \nabla \psi_N \cdot \vec{d}, \Delta^{-1} \text{div} \, \vec{v} \rangle = \sum_{M_1 \leq j \leq M_2} \langle \nabla \psi_N \cdot \vec{d}, \eta_j \Delta^{-1} \text{div} \, \vec{v} \rangle,
\]
where \( M_1, M_2 \to +\infty \) as \( N \to +\infty. \) Recall that \( \eta_j \) is supported on \( 2^{j+1}Q \setminus 2^{j-1}Q \) for \( j \geq 1. \) Hence, \( \nabla \psi_N \cdot \eta_j \) is supported on \( \{2^{j+1}Q \setminus 2^{j-1}Q\} \cap \{\frac{N}{2} \leq |x| \leq N\}. \) We may assume without loss of generality that \( |a_Q| < 2^{j+\ell}(Q) \) for \( N \) large,
where \( a_Q \) denotes the center of \( Q \). Then clearly, \( c_1(n)\frac{N}{|Q|} \leq 2^j \leq c_2(n)\frac{N}{|Q|} \).

In other words, for a fixed \( Q \), the sum above contains a bounded number of terms which does not depend on \( N \).

Thus, by (4.30),

\[
\left| \langle \nabla \psi_N \cdot \vec{d}, \Delta^{-1} \text{div} \, \vec{v} \rangle \right| \leq \sum_{M_1 \leq j \leq M_2} \left| \langle \nabla \psi_N \cdot \vec{d}, \eta_j \Delta^{-1} \text{div} \, \vec{v} \rangle \right|
\]

\[
\leq C N^{-1} |2^{j+1}Q|^{\frac{1}{2}} \sum_{M_1 \leq j \leq M_2} ||\nabla (\eta_j \Delta^{-1} \text{div} \, \vec{v})||_{L^2(2^{j+1}Q)}.
\]

By statement (ii) of Proposition 4.2,

\[
||\nabla (\eta_j \Delta^{-1} \text{div} \, \vec{v})||_{L^2(2^{j+1}Q)} \leq C 2^{-j^{\frac{n}{2}}} ||\vec{v}||_{L^2(Q)}.
\]

Combining the preceding estimates we obtain

\[
\left| \langle \nabla \psi_N \cdot \vec{d}, \Delta^{-1} \text{div} \, \vec{v} \rangle \right| \leq C N^{-1} |Q|^{\frac{1}{2}} ||\vec{v}||_{L^2(Q)},
\]

and hence (4.31) holds.

By Lemma 4.3, \( \text{div} \, \vec{d} = 0 \), and so integration by parts yields

\[
\langle \psi_N \vec{d}, \nabla (\Delta^{-1} \text{div} \, \vec{v}) \rangle = -\langle \nabla \psi_N \cdot \vec{d}, \Delta^{-1} \text{div} \, \vec{v} \rangle - \langle \psi_N \text{div} \, \vec{d}, \Delta^{-1} \text{div} \, \vec{v} \rangle,
\]

\[
= -\langle \nabla \psi_N \cdot \vec{d}, \Delta^{-1} \text{div} \, \vec{v} \rangle.
\]

Thus,

\[
\langle P\vec{d}, \vec{v} \rangle = \lim_{N \to +\infty} \langle \psi_N \vec{d}, \nabla (\Delta^{-1} \text{div} \, \vec{v}) \rangle = 0,
\]

i.e., \( P\vec{d} = 0 \). Consequently, \( P(P\vec{b}) = P(\vec{b} - \vec{d}) = P\vec{b} \), i.e., \( P \), and hence \( Q \), is a projection. \( \square \)

**Lemma 4.5.** Suppose \( \vec{b} \in D'(\mathbb{R}^n)^n \) and \( q \in D'(\mathbb{R}^n) \), \( n \geq 2 \). Suppose \( \vec{b} \cdot \nabla + q \) is form bounded on \( L^{1,2}(\mathbb{R}^n) \), i.e., both (3.3) and (3.4) hold. Then \( q = \text{div} \, \vec{b} = 0 \) if \( n = 2 \). In the case \( n \geq 3 \),

(4.32) \( \Delta^{-1} \text{div} \, \vec{b} \in \text{BMO}(\mathbb{R}^n) \), \( \Delta^{-1} q \in \text{BMO}(\mathbb{R}^n) \),

where \( \Delta^{-1} \text{div} \, \vec{b} \) and \( \Delta^{-1} q \) are defined (up to a constant) by, respectively,

(4.33) \( \Delta^{-1} \text{div} \, \vec{b} = \lim_{N \to \infty} \Delta^{-1}(\psi_N \text{div} \, \vec{b}) \), \( \Delta^{-1} q = \lim_{N \to \infty} \Delta^{-1}(\psi_N q) \),

in terms of the convergence in the weak-* topology of \( \text{BMO}(\mathbb{R}^n) \).

Furthermore, for every cube \( Q \) in \( \mathbb{R}^n \),

(4.34) \( \int_Q \left( |\nabla (\Delta^{-1} \text{div} \, \vec{b})|^2 + |\nabla (\Delta^{-1} q)|^2 \right) \, dx \leq C |Q|^{1-\frac{n}{2}} \),

where \( C \) does not depend on \( Q \).
Remark 5. We have already defined $\Delta^{-1} \text{div} \vec{b}$ in Lemma 4.3 by $\Delta^{-1} \text{div} \vec{b} = \lim_{N \to \infty} \Delta^{-1} \text{div} (\psi_N \vec{b})$. However, as we will show below, this definition is consistent with (4.33) under the assumptions of Lemma 4.5.

Proof. By (3.3),
\[ |\langle q - \frac{1}{2} \text{div} \vec{b}, \bar{u} v \rangle| \leq C \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad u, v \in C_0^\infty(\mathbb{R}^n). \]

Letting $v \in C_0^\infty(Q)$ and $u \in C_0^\infty(2Q)$, $u = 1$ on $Q$, in the preceding inequality, and taking the infimum over all such $u$, as in the proof of Proposition 4.1, we arrive at the estimate
\[ (4.35) \quad |\langle q - \frac{1}{2} \text{div} \vec{b}, v \rangle| \leq C \text{cap}(Q) \|\nabla v\|_{L^2(Q)}, \quad v \in C_0^\infty(Q). \]

In two dimensions, $\text{div} \vec{b} = 0$ by Proposition 4.1, and $\text{cap}(Q) = 0$ by (2.2). Hence, we see from (4.35) that $\langle q, v \rangle = 0$ for every $v \in C_0^\infty(\mathbb{R}^n)$, i.e., $q = 0$.

Let us now consider the case $n \geq 3$. By (4.35),
\[ |\langle q - \frac{1}{2} \text{div} \vec{b}, v \rangle| \leq C |Q|^{\frac{1}{2} - \frac{n}{4}} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad v \in C_0^\infty(Q). \]

Notice that by Proposition 4.1,
\[ (4.36) \quad |\langle \text{div} \vec{b}, v \rangle| \leq C |Q|^{\frac{1}{2} - \frac{n}{4}} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad v \in C_0^\infty(Q). \]

Combining the preceding estimates, we obtain
\[ (4.37) \quad |\langle q, v \rangle| \leq C |Q|^{\frac{1}{2} - \frac{n}{4}} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad v \in C_0^\infty(Q). \]

Thus,
\[ (4.38) \quad \|\text{div} \vec{b}\|_{L^{1, 2}(Q)} + \|q\|_{L^{1, 2}(Q)} \leq C |Q|^{\frac{1}{2} - \frac{n}{4}}. \]

This obviously implies
\[ (4.39) \quad \|\psi_N \text{div} \vec{b}\|_{L^{1, 2}(Q)} + \|\psi_N q\|_{L^{1, 2}(Q)} \leq C |Q|^{\frac{1}{2} - \frac{n}{4}}, \]

where $C$ does not depend on $Q$ and $N$.

We now set
\[ \tilde{f}_N = \Delta^{-1} (\psi_N \text{div} \vec{b}), \quad g_N = \Delta^{-1} (\psi_N q), \]

which are well-defined in $D'(\mathbb{R}^n)$. Note that $\tilde{f}_N$ differs slightly from $f_N = \Delta^{-1} \text{div} (\psi_N \vec{b})$ used in Lemma 4.3. We will deduce from (4.39)
\[ (4.40) \quad \|\nabla \tilde{f}_N\|_{L^{1, 2}(Q)} + \|\nabla g_N\|_{L^{1, 2}(Q)} \leq C |Q|^{\frac{1}{2}}, \]

where $C$ does not depend on $Q$ and $N$. It is enough to estimate only the second term on the left-hand side; the first one is treated analogously simply by putting $\text{div} \vec{b}$ in place of $q$. 

Notice that, for every \( \vec{v} \in C_0^\infty(Q)^n \),
\[
\langle \nabla \Delta^{-1}(\psi_N q), \vec{v} \rangle = -\langle q, \psi_N \Delta^{-1} \text{div} \vec{v} \rangle = -\sum_{j=0}^\infty \langle \psi_N q, \eta_j \Delta^{-1} \text{div} \vec{v} \rangle,
\]
where the sum on the right contains only a finite number of nonzero terms.

Now using (4.39) and statement (ii) of Proposition 4.2, we estimate
\[
|\langle \nabla \Delta^{-1}(\psi_N q), \vec{v} \rangle| \leq C \sum_{j=0}^\infty |\langle \psi_N q, \eta_j \Delta^{-1} \text{div} \vec{v} \rangle| \leq C |Q| \frac{1}{2^{j+1}} \sum_{j=0}^\infty 2^{-j} \leq C |Q| \frac{1}{2^j} \sum_{j=0}^\infty \langle \eta_j \Delta^{-1} \text{div} \vec{v} \rangle.
\]

This proves (4.40), from which it is immediate that
\[
\left\| \tilde{f}_N - m_Q(f_N) \right\|_{L^2(Q)}^2 + \left\| g_N - m_Q(g_N) \right\|_{L^2(Q)}^2 \leq C \left( \left\| \nabla \tilde{f}_N \right\|_{L^2(Q)}^2 + \left\| \nabla g_N \right\|_{L^2(Q)}^2 \right) \leq C |Q|.
\]

Thus,
\[
\sup_N (\left\| \tilde{f}_N \right\|_{\text{BMO}(\mathbb{R}^n)} + \left\| g_N \right\|_{\text{BMO}(\mathbb{R}^n)}) < +\infty.
\]

We can now follow the argument of Lemma 4.3 to demonstrate that
\[
\tilde{f}_N \to \tilde{f} = \Delta^{-1} \text{div} \vec{b}, \quad g_N \to g = \Delta^{-1} q,
\]
in the sense of the weak-* topology of BMO(\( \mathbb{R}^n \)). Note that we also have to verify \( \tilde{f} = f \) where \( f = \lim_{N \to \infty} f_N \).

Let us indicate some changes that are needed here. We are now utilizing estimates (4.39) in place of (4.21). By using statement (iv), rather than statement (iii), of Proposition 4.2, one deduces that \( \{g_N\} \) is a Cauchy sequence. Hence, \( g \) is defined (up to a constant) by
\[
g = \Delta^{-1} q = \lim_{N \to \infty} \Delta^{-1}(\psi_N q) \in \text{BMO}(\mathbb{R}^n),
\]
in the sense of the weak-* BMO convergence. Moreover, using (4.39) together with statement (iv) of Proposition 4.2, we obtain, exactly as in the proof of (4.25), that for any \( v \in C_0^\infty(Q) \) such that \( \int_Q v \, dx = 0 \),
\[
\langle \Delta^{-1} q, v \rangle = \int_{\mathbb{R}^n} g \, \vec{v} \, dx = \sum_{j=0}^\infty \langle q, \eta_j \Delta^{-1} \text{div} v \rangle.
\]
From the above equations we see that
\[ \nabla g = \lim_{N \to \infty} \nabla \Delta^{-1}(\psi_N \cdot q) \] in \( D'(\mathbb{R}^n) \),
and, for every \( \vec{v} \in C_0^\infty(Q)^n \),
\[ \langle \nabla g, v \rangle = -\int_{\mathbb{R}^n} g \, \text{div} \, \vec{v} \, dx = -\sum_{j=0}^\infty \langle q, \eta_j \Delta^{-1} \text{div} \, \vec{v} \rangle. \]
Hence, \( \text{div} \, \nabla g = q \) in \( D'(\mathbb{R}^n) \).

Obviously, analogous statements hold with \( \text{div} \, \vec{b} \) and \( \tilde{f} \) in place of \( q \) and \( g \), respectively. It remains only to justify Remark 5 above. We show that \( \tilde{f} = f \) in the BMO sense, i.e.,
\[ \int_{\mathbb{R}^n} \tilde{f} \, \vec{v} \, dx = \int_{\mathbb{R}^n} f \, \vec{v} \, dx, \]
for every \( v \in C_0^\infty(\mathbb{R}^n) \) such that \( \int_{\mathbb{R}^n} v \, dx = 0 \). Notice that
\[ \langle \tilde{f}_N, v \rangle = \langle f_N, v \rangle - \langle \nabla \psi_N \cdot \vec{b}, \Delta^{-1} v \rangle. \]
It suffices to check
\[ (4.41) \quad \lim_{N \to \infty} \langle \nabla \psi_N \cdot \vec{b}, \Delta^{-1} v \rangle = 0, \]
where \( v \) is supported on a cube \( Q \), and \( \int_Q v \, dx = 0 \). Notice that
\[ \langle \nabla \psi_N \cdot \vec{b}, \Delta^{-1} v \rangle = \sum_{M_1 \leq j \leq M_2} \langle \nabla \psi_N \cdot \vec{b}, \eta_j \Delta^{-1} v \rangle, \]
where \( M_1, M_2 \to \infty \) as \( N \to \infty \). As was shown in the proof of Corollary 4.4 for a fixed \( Q \), the sum above contains a uniformly bounded number of nonzero terms.

We recall that by Proposition 4.1 \( ||\vec{b}||_{L^{1,2}(Q)} \leq C |Q|^\frac{1}{2} \), and hence
\[ ||\nabla \psi_N \cdot \vec{b}||_{L^{1,2}(Q)} \leq C N^{-1} |Q|^\frac{1}{2}, \]
where \( C \) does not depend on \( Q \) and \( N \). It follows,
\[ |\langle \nabla \psi_N \cdot \vec{b}, \Delta^{-1} v \rangle| \leq C \sum_{M_1 \leq j \leq M_2} |\langle \nabla \psi_N \cdot \vec{b}, \eta_j \Delta^{-1} v \rangle| \]
\[ \leq C N^{-1} \sum_{M_1 \leq j \leq M_2} |2^{j+1}Q|^\frac{1}{2} \| \nabla (\eta_j \Delta^{-1} v) \|_{L^2(2^{j+1}Q)}. \]
Applying statement (iv) of Proposition 4.2 we conclude:
\[ |\langle \nabla \psi_N \cdot \vec{b}, \Delta^{-1} v \rangle| \leq C N^{-1} ||v||_{L^1(Q)}, \]
which yields (4.41). Thus, \( \tilde{f} = f \) in BMO(\( \mathbb{R}^n \)).
Then, for every $\vec{v} \in C^\infty_0(Q^n)$,

\begin{align}
\langle \nabla f, \vec{v} \rangle &= -\int_{\mathbb{R}^n} f \nabla \vec{v} \, dx = -\sum_{j \geq 0} \langle \text{div} \ b, \eta_j \Delta^{-1} \text{div} \ \vec{v} \rangle, \\
\langle \nabla g, \vec{v} \rangle &= -\int_{\mathbb{R}^n} g \nabla \vec{v} \, dx = -\sum_{j \geq 0} \langle q, \eta_j \Delta^{-1} \text{div} \ \vec{v} \rangle.
\end{align}

This is verified exactly as in the proof of (4.25), using (4.39) together with statement (iv), rather than statement (iii), of Proposition 4.2.

We are now in a position to obtain the estimate

\begin{equation}
\langle \nabla \Delta^{-1} (\text{div} \ b), \vec{v} \rangle \leq C |Q|^{\frac{1}{2} - \frac{1}{n}} \|\vec{v}\|_{L^2(Q)}, \quad \vec{v} \in C^\infty_0(Q^n).
\end{equation}

Indeed, by (4.42) and statement (ii) of Proposition 4.2,

\begin{equation}
\langle \nabla f, \vec{v} \rangle \leq C \sum_{j \geq 0} |2^{j+1}Q| \|\eta_j \Delta^{-1} \text{div} \ \vec{v} \|_{L^2(2^{j+1}Q)} \sum_{j \geq 0} 2^{-j}.
\end{equation}

Taking the supremum over all $\vec{v} \in C^\infty_0(Q^n)$ in (4.44), we obtain

\begin{equation}
\|\nabla (\Delta^{-1} \text{div} \ b)\|_{L^2(Q)} \leq C |Q|^{\frac{1}{2} - \frac{1}{n}}.
\end{equation}

Analogously, we deduce from (4.43),

\begin{equation}
\|\nabla (\Delta^{-1} q)\|_{L^2(Q)} \leq C |Q|^{\frac{1}{2} - \frac{1}{n}}.
\end{equation}

Combining the preceding estimates, we arrive at (4.34).

We now establish the main lemma, whose proof makes use of the idea of the magnetic gauge invariance.

**Lemma 4.6.** Let $\vec{b} \in D'(\mathbb{R}^n)$, $n \geq 2$. Suppose that, for all $u, v \in C^\infty_0(\mathbb{R}^n)$,

\begin{equation}
\left| \langle \vec{b}, \bar{u} \nabla v - v \nabla \bar{u} \rangle \right| \leq C \|u\|_{L^{1.2}(\mathbb{R}^n)} \|v\|_{L^{1.2}(\mathbb{R}^n)}.
\end{equation}

Then $\vec{c} = \nabla (\Delta^{-1} \text{div} \ \vec{b}) \in L^2_{\text{loc}}(\mathbb{R}^n)$, and $|\vec{c}|^2 \in \mathcal{M}^{1,2}_{+}(\mathbb{R}^n)$, i.e.,

\begin{equation}
\int_{\mathbb{R}^n} |u|^2 |\vec{c}|^2 \, dx \leq C_1 \|u\|_{L^{1.2}(\mathbb{R}^n)}^2,
\end{equation}

where $C_1$ does not depend on $u \in C^\infty_0(\mathbb{R}^n)$. If $n = 2$, then $\vec{c} = 0$.

**Proof.** Suppose that (4.47) holds. Then by continuity the bilinear form on the left-hand side can be extended to all $u, v \in L^{1.2}(\mathbb{R}^n)$. Let $v$ be a nonnegative function such that $\lambda = \log v$ has the property:

\begin{equation}
\nabla \lambda = \frac{\nabla v}{v} \in L^2_{\text{loc}}(\mathbb{R}^n).
\end{equation}
Moreover, we need $\lambda$ to be chosen so that the energy space $L^{1,2}(\mathbb{R}^n)$ be invariant under the gauge transformation:

\begin{equation}
(4.49) \quad u \to \tilde{u} = e^{i\lambda} u, \quad v \to \tilde{v} = e^{i\lambda} v, \quad ||\tilde{u}||_{L^{1,2}(\mathbb{R}^n)} \approx ||u||_{L^{1,2}(\mathbb{R}^n)}, \quad ||\tilde{v}||_{L^{1,2}(\mathbb{R}^n)} \approx ||v||_{L^{1,2}(\mathbb{R}^n)}.
\end{equation}

We set

\begin{equation}
(4.50) \quad ||\tilde{u}||_{L^{1,2}(\mathbb{R}^n)} \approx ||u||_{L^{1,2}(\mathbb{R}^n)}, \quad ||\tilde{v}||_{L^{1,2}(\mathbb{R}^n)} \approx ||v||_{L^{1,2}(\mathbb{R}^n)}.
\end{equation}

To deduce (4.49), we notice that by Theorem 2.4, it suffices to show that, for every compact set $e \subset \mathbb{R}^n$,

\begin{equation}
(4.51) \quad \int_e |\tilde{e}|^2 \, dx \leq \text{const} \, \text{cap} (e).
\end{equation}

Without loss of generality we may assume that $\text{cap} (e) > 0$, since otherwise $|e| = 0$, and hence the preceding inequality is obvious. Denote by $\mu = \mu_e$ the equilibrium measure associated with $e$. Let $P\mu(x) = (-\Delta)^{-1}\mu$ denote the Newtonian potential of $\mu$.

Suppose first that $n \geq 3$. Then

\[ P\mu(x) = c(n) \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{n-2}}, \quad x \in \mathbb{R}^n. \]

We set

\begin{equation}
(4.52) \quad \lambda = \tau \log(P\mu), \quad v = (P\mu)^\tau, \quad 1 < 2\tau < \frac{n}{n-2}.
\end{equation}

We observe that $v \in L^{1,2}(\mathbb{R}^n)$ by [MV1], Proposition 2.5. Clearly, $\nabla(e^{i\lambda} u) = (i u \nabla \lambda + \nabla u) e^{i\lambda}$. Consequently, for every $u \in L^{1,2}(\mathbb{R}^n)$,

\[ ||e^{i\lambda} u||_{L^{1,2}(\mathbb{R}^n)} \leq (||u \nabla \lambda||_{L^2(\mathbb{R}^n)} + ||u||_{L^{1,2}(\mathbb{R}^n)}). \]

Note that $\nabla \lambda = \tau \frac{\nabla(P\mu)}{P\mu}$. Hence, by [MV1], Proposition 2.7,

\[ ||u \nabla \lambda||_{L^2(\mathbb{R}^n)} \leq 2\tau ||u||_{L^{1,2}(\mathbb{R}^n)}. \]

From this it follows

\begin{equation}
(4.53) \quad (1 + 2\tau)^{-1} ||u||_{L^{1,2}(\mathbb{R}^n)} \leq ||e^{i\lambda} u||_{L^{1,2}(\mathbb{R}^n)} \leq (1 + 2\tau) ||u||_{L^{1,2}(\mathbb{R}^n)}.
\end{equation}

Using similar estimates for $e^{-i\lambda} v$, we deduce (4.50). Moreover,

\begin{equation}
(4.54) \quad ||\tilde{u}||_{L^{1,2}(\mathbb{R}^n)} \leq (1 + 2\tau) ||u||_{L^{1,2}(\mathbb{R}^n)}, \quad ||\tilde{v}||_{L^{1,2}(\mathbb{R}^n)} \leq (1 + 2\tau) ||v||_{L^{1,2}(\mathbb{R}^n)}.
\end{equation}

Applying (4.54) and (4.51) with $\tilde{u}$ and $\tilde{v}$ in place of $u$ and $v$, we get

\[ \left| \langle \tilde{b}, \tilde{u} \nabla \tilde{v} - \tilde{v} \nabla \tilde{u} \rangle \right| \leq C ||\tilde{u}||_{L^{1,2}(\mathbb{R}^n)} ||\tilde{v}||_{L^{1,2}(\mathbb{R}^n)} \leq C (1 + 2\tau)^2 ||u||_{L^{1,2}(\mathbb{R}^n)} ||v||_{L^{1,2}(\mathbb{R}^n)}. \]

Notice that

\[ \overrightarrow{u} \nabla \tilde{v} - \tilde{v} \nabla \overrightarrow{u} = \tilde{u} \nabla v - v \nabla \tilde{u} - 2i \tilde{u} v \nabla \lambda. \]

Combining the preceding estimates, we obtain

\[ 2 \left| \langle \tilde{b}, \tilde{u} v \nabla \lambda \rangle \right| \leq \left| \langle \tilde{b}, \tilde{u} \nabla v - v \nabla \tilde{u} \rangle \right| + \left| \langle \tilde{b}, \tilde{u} \nabla \tilde{v} - \tilde{v} \nabla \tilde{u} \rangle \right|. \]
Observe that $v \nabla \lambda = \nabla v$. Thus, we arrive at the inequality
\begin{equation}
\left| \langle \bar{\lambda}, \bar{u} \nabla v \rangle \right| \leq C \frac{1+(1+2\tau)^2}{2} \|u\|_{L^{1,2}(\mathbb{R}^n)} \|v\|_{L^{1,2}(\mathbb{R}^n)},
\end{equation}
where $v = (P\mu)^\tau$. From the preceding estimate and (4.47), we deduce:
\begin{equation}
\left| \langle \bar{b}, (\bar{u} \nabla v + v \nabla \bar{u}) \rangle \right| \leq C (2 + (1 + 2\tau)^2) \|u\|_{L^{1,2}(\mathbb{R}^n)} \|v\|_{L^{1,2}(\mathbb{R}^n)}.
\end{equation}
This yields
\begin{equation}
\left| \langle (\text{div} \bar{b}) u, v \rangle \right| \leq C (2 + (1 + 2\tau)^2) \|u\|_{L^{1,2}(\mathbb{R}^n)} \|v\|_{L^{1,2}(\mathbb{R}^n)},
\end{equation}
where $u \in L^{1,2}(\mathbb{R}^n)$ and $v = (P\mu)^\tau$. By Proposition 2.5 in [MV1],
\begin{equation}
\|v\|_{L^{1,2}(\mathbb{R}^n)} = \tau (2\tau - 1)^{-\frac{1}{2}} \text{cap} (e)^{\frac{1}{2}}.
\end{equation}
Hence by (4.57),
\begin{equation}
\left| \langle (\text{div} \bar{b}) u, v \rangle \right| \leq c(\tau) C \|u\|_{L^{1,2}(\mathbb{R}^n)} \text{cap} (e)^{\frac{1}{2}},
\end{equation}
where $c(\tau)$ depends only on $\tau$, and $C$ is the constant in (4.47).

By letting $e = 2Q$ in (4.58), where $Q$ is a cube in $\mathbb{R}^n$, and taking into account that in this case $P\mu = 1$ on $Q$, and $\text{cap} (2Q) \simeq |Q|^\frac{1}{2} - \frac{1}{2}$, we see that
\begin{equation}
\left| \langle \text{div} \bar{b}, \bar{u} \rangle \right| \leq C \|u\|_{L^{1,2}(Q)} |Q|^\frac{1}{2} - \frac{1}{2},
\end{equation}
for every $u \in C^\infty_0(Q)$, $n \geq 3$. (The preceding estimate was already proved in Proposition 4.1.) As in the proof of Lemma 4.3, estimate (4.59) yields
\begin{equation}
\Delta^{-1} (\text{div} \bar{b}) = \lim_{N \to \infty} \tilde{f}_N \in \text{BMO}(\mathbb{R}^n), \quad \tilde{f}_N = \Delta^{-1} (\psi_N \text{div} \bar{b}),
\end{equation}
where the limit is understood in the sense of the weak-* convergence in BMO. Moreover, by (1.45), $\nabla \Delta^{-1} (\text{div} \bar{b}) \in L^2_{\text{loc}}(\mathbb{R}^n)$, and
\begin{equation}
\|\nabla (\Delta^{-1} \text{div} \bar{b})\|_{L^2(Q)} \leq C |Q|^\frac{1}{2} - \frac{1}{2},
\end{equation}
for every cube $Q$ if $n \geq 3$.

It remains to show that (4.58), with arbitrary $u \in L^{1,2}(\mathbb{R}^n)$ and $v = (P\mu)^\tau$, yields
\begin{equation}
\int_\epsilon \left| \nabla \Delta^{-1} (\text{div} \bar{b}) \right|^2 dx \leq C \text{cap} (e),
\end{equation}
where $C$ does not depend on the compact set $e$.

This is verified analogously to the proof of the necessity part of Theorem 2.2 in [MV1], where $u$ and $v$ in (4.57) were picked exactly as above. For the sake of convenience, we outline the rest of the proof as follows.
For $\vec{\phi} \in C_0^\infty(\mathbb{R}^n)^n$, we set

$$u = v^{-1} (\Delta^{-1} \text{div} \vec{\phi}), \quad v = (P\mu)^\tau.$$ 

Then by Lemma 2.6 in [MV1], $u \in L^{1,2}(\mathbb{R}^n)$, and

$$\|u\|_{L^{1,2}(\mathbb{R}^n)} \leq \|\nabla(\Delta^{-1} \text{div} \vec{\phi})\|_{L^2(\mathbb{R}^n, v^{-2}dx)} \leq (1 + \tau)^{\frac{1}{2}} (4\tau + 1)^{\frac{1}{2}} \|u\|_{L^{1,2}(\mathbb{R}^n)}$$

where

$$\|f\|_{L^2(\mathbb{R}^n, v^{-2}dx)} = \left( \int_{\mathbb{R}^n} |f|^2 (P\mu)^{-2\tau} dx \right)^{\frac{1}{2}}$$

is the weighted $L^2$-norm of $f = \nabla(\Delta^{-1} \text{div} \vec{\phi})$. Hence, by (4.57),

$$\langle \text{div} \vec{b}, \Delta^{-1} \text{div} \vec{\phi} \rangle \leq c(\tau) C \text{cap} (e) \|\nabla(\Delta^{-1} \text{div} \vec{\phi})\|_{L^2(\mathbb{R}^n, v^{-2}dx)}^2.$$

Integrating by parts, we get

$$\langle \text{div} \vec{b}, \Delta^{-1} \text{div} \vec{\phi} \rangle = -\langle \nabla(\Delta^{-1} \text{div} \vec{b}), \vec{\phi} \rangle.$$ 

Thus,

$$\|\nabla(\Delta^{-1} \text{div} \vec{b})\|_{L^2(\mathbb{R}^n, v^{-2}dx)} \leq c(\tau) C \text{cap} (e) \|\vec{\phi}\|_{L^2(\mathbb{R}^n, v^{-2}dx)}.$$ 

Notice that, for $\tau$ picked according to (4.52), the weight $v^{-2}$ belongs to the Muckenhoupt class $A_2(\mathbb{R}^n)$, and its Muckenhoupt bound depends only on $\tau$ and $n$ by Proposition 2.9 in [MV1]. Hence, the Calderon–Zygmund operator $\nabla(\Delta^{-1} \text{div})$ is bounded on $L^2(\mathbb{R}^n, v^{-2}dx)$, and

$$\|\nabla(\Delta^{-1} \text{div} \vec{\phi})\|_{L^2(\mathbb{R}^n, v^{-2}dx)} \leq c(\tau, n) \|\vec{\phi}\|_{L^2(\mathbb{R}^n, v^{-2}dx)}.$$ 

Combining the preceding inequalities, we get

$$\|\nabla(\Delta^{-1} \text{div} \vec{b})\|_{L^2(\mathbb{R}^n, v^{-2}dx)} \leq c(\tau, n) C \text{cap} (e) \|\vec{\phi}\|_{L^2(\mathbb{R}^n, v^{-2}dx)},$$

for all $\vec{\phi} \in C_0^\infty(\mathbb{R}^n)^n$.

Since $v^{-2} \in A_2(\mathbb{R}^n)$, and $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, v^{-2}dx)$, we deduce from the preceding inequality:

$$\int_{\mathbb{R}^n} |\nabla(\Delta^{-1} \text{div} \vec{b})|^2 v^2 dx \leq c(\tau, n) C^2 \text{cap} (e),$$

where $C$ is the constant in (4.37). Using the fact that $v = (P\mu)^\tau \geq 1$ $dx$-a.e. on $e$, we obtain (4.61).

If $n = 2$, then by Proposition 4.1, (4.47) yields $\text{div} \vec{b} = 0$, and hence $\vec{c} = \nabla(\Delta^{-1} \text{div} \vec{b}) = 0$. \hfill $\square$
Lemma 4.7. Let $\vec{b} \in D(\mathbb{R}^n)$, $n \geq 2$. Then the inequality
\begin{equation}
(4.62) \quad \left| \langle \vec{b}, \vec{u} \nabla v - v \nabla \vec{u} \rangle \right| \leq c \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad u, v \in C^\infty_0(\mathbb{R}^n),
\end{equation}
holds if and only if
\begin{equation}
(4.63) \quad \vec{b} = \vec{c} + \text{Div } F,
\end{equation}
where $\vec{c}$ obeys (4.48), and $F \in \text{BMO}(\mathbb{R}^n)^{n \times n}$ is a skew-symmetric matrix field.

Moreover, if (4.47) holds then (4.63) is valid with $\vec{c} = \nabla (\Delta^{-1} \text{div } \vec{b})$ obeying (4.48), and $F = \Delta^{-1} \text{curl } \vec{b} \in \text{BMO}(\mathbb{R}^n)^{n \times n}$.

In the case $n = 2$, it follows that $\vec{c} = 0$ in the statements above.

Proof. Suppose first that $n \geq 3$. To prove the “if” part, suppose that (4.63) holds, i.e., $\vec{b} = \vec{c} + \vec{d}$, where $\vec{d} = \text{Div } F$ is divergence free, $F \in \text{BMO}(\mathbb{R}^n)^{n \times n}$, and $\vec{c}$ satisfies (4.48). Then $\vec{c} \cdot \nabla$ is form bounded, since by Schwarz’s inequality and (4.48), we have:
\begin{equation}
(4.64) \quad \langle \vec{c} \cdot \nabla u, \ n \rangle \leq ||u \vec{c}||_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}.
\end{equation}
The preceding inequality obviously yields (4.47).

It remains to show that
\begin{equation}
(4.65) \quad \langle \vec{d}, \vec{u} \nabla v - v \nabla \vec{u} \rangle \leq c \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad u, v \in C^\infty_0(\mathbb{R}^n).
\end{equation}

Let us start with the case $n = 3$. Then equivalently we have $\vec{d} = \text{curl } \vec{F}$, where $\vec{F} \in \text{BMO}(\mathbb{R}^3)^3$. It follows
\begin{equation}
(4.66) \quad \langle \vec{d}, \vec{u} \nabla v - v \nabla \vec{u} \rangle = 2 \langle \vec{F}, \nabla \vec{u} \times \nabla v \rangle \leq c \|\vec{F}\|_{\text{BMO}(\mathbb{R}^3)^3} \|\nabla \vec{u} \times \nabla v\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \leq c_1 \|\vec{F}\|_{\text{BMO}(\mathbb{R}^3)^3} \|\nabla u\|_{L^2(\mathbb{R}^3)} \|\nabla v\|_{L^2(\mathbb{R}^3)}.
\end{equation}
The last inequality is based on a standard compensated compactness argument using commutators with Riesz transforms [CLMS].

Similarly, for $n \geq 3$, we have $\vec{d} = \text{Div } F$, where $F = (f_{ij})_{i,j=1}^n$ is skew-symmetric, and hence (see Sec. 2)
\begin{equation}
(4.67) \quad \langle \vec{d}, \vec{u} \nabla v - v \nabla \vec{u} \rangle = -\text{trace } \langle F^t, D(\vec{u} \nabla v - v \nabla \vec{u}) \rangle = -\sum_{i,j=1}^n \langle f_{ij}, \partial_i \vec{u} \partial_j v - \partial_j \vec{u} \partial_i v \rangle.
\end{equation}
The inequality
\begin{equation}
|\langle f_{ij}, \partial_i \vec{u} \partial_j v - \partial_j \vec{u} \partial_i v \rangle| \leq c \|F\|_{\text{BMO}(\mathbb{R}^n)^{n \times n}} \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}
\end{equation}
now follows again from the $\mathcal{H}^1$ estimates for commutators with Riesz transforms (see [CLMS]).
If \( n = 2 \), then as was indicated above (4.48) yields \( \bar{c} = 0 \). Hence, \( \bar{b} = (\partial_2 g, -\partial_1 g) \), where \( g \in \text{BMO}(\mathbb{R}^2) \). Now (4.47) follows from the well-known \( H^1 \) inequality for the Jacobian determinant [CLMS]:

\[
\left| (\bar{b} \cdot \nabla u, v) \right| = \left| \langle g, \partial_2 \bar{u} \partial_1 v - \partial_1 \bar{u} \partial_2 v \rangle \right| \leq C \left\| u \right\|_{L^1,2(\mathbb{R}^n)} \left\| v \right\|_{L^1,2(\mathbb{R}^n)}.
\]

This proves the “if” part of Lemma 4.7.

To prove the converse, notice that by Proposition 4.1 and Lemma 4.3 it follows that one can set \( \bar{c} = \nabla (\Delta^{-1} \text{div} \bar{b}) \) and \( \bar{d} = \text{Div} (\Delta^{-1} \text{curl} \bar{b}) \in \text{BMO}(\mathbb{R}^n)^n \). Finally, by Lemma 4.6 we deduce that (4.48) holds.

We are now in a position to obtain the main result of this section.

**Theorem 4.8.** Let \( n \geq 2 \). Let \( \mathcal{L} = \bar{b} \cdot \nabla + q \), where \( \bar{b} \in D'(\mathbb{R}^n) \), and \( q \in D'(\mathbb{R}^n) \). Then the following statements hold.

(i) The bilinear form associated with \( \mathcal{L} \) is bounded on \( L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n) \) if and only if

\[
\bar{b} = \bar{c} + \text{Div} F, \quad q = \text{div} \bar{h},
\]

where \( F \in \text{BMO}(\mathbb{R}^n)^{n\times n} \) is a skew-symmetric matrix field, and \( |\bar{c}|^2 + |\bar{h}|^2 \in \mathcal{M}^{1,2}_+ (\mathbb{R}^n) \), i.e.,

\[
\int_{\mathbb{R}^n} (|\bar{c}|^2 + |\bar{h}|^2) |u|^2 \, dx \leq C \left\| \nabla u \right\|_{L^2(\mathbb{R}^n)}^2,
\]

for all \( u \in C^\infty_0(\mathbb{R}^n) \).

(ii) If \( \mathcal{L} \) is form bounded, then in statement (i) one can set

\[
\bar{c} = \nabla (\Delta^{-1} \text{div} \bar{b}), \quad F = \Delta^{-1} \text{curl} \bar{b}, \quad \bar{h} = \nabla (\Delta^{-1} q),
\]

so that (4.65) and (4.66) are valid, and \( F \) is skew-symmetric with entries in \( \text{BMO}(\mathbb{R}^n) \).

(iii) If \( n = 2 \) then the bilinear form of \( \mathcal{L} \) is bounded if and only if \( \text{div} \bar{b} = q = 0 \), and \( \bar{b} = (\partial_2 g, -\partial_1 g) \), where \( g = \Delta^{-1} \text{rot} \bar{b} \in \text{BMO}(\mathbb{R}^2) \).

**Proof.** Let us first prove the sufficiency part of statement (i). To see that \( q = \text{div} \bar{h} \) is form bounded on \( L^{1,2}(\mathbb{R}^n) \), we use integration by parts, Schwarz’s inequality, and (4.66):

\[
|\langle gu, v \rangle| = |\langle \bar{h}, \nabla (\bar{u} v) \rangle| = |\langle \bar{h}, v \nabla \bar{u} + \bar{u} \nabla v \rangle| \leq C \left\| u \bar{h} \right\|_{L^2(\mathbb{R}^n)} \left\| \nabla \bar{v} \right\|_{L^2(\mathbb{R}^n)} + \left\| \bar{u} \bar{h} \right\|_{L^2(\mathbb{R}^n)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \nabla u \right\|_{L^2(\mathbb{R}^n)} \left\| \nabla v \right\|_{L^2(\mathbb{R}^n)},
\]

for every \( u, v \in C^\infty_0(\mathbb{R}^n) \).
Suppose now that $\vec{b}$ is represented in the form (4.65), i.e., $\vec{b} = \vec{c} + \vec{d}$, where $\vec{c}$ satisfies inequality (4.66), and $\vec{d} = \text{Div } F$. Then by Schwarz’s inequality and (4.66), it follows:

$$|\langle \vec{c} \cdot \nabla u, v \rangle| \leq \|v \vec{c}\|_{L^2(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)},$$

which proves that $\vec{c} \cdot \nabla$ is form bounded.

We next treat $\vec{d} \cdot \nabla$. By Proposition 3.2 in the case $q = 0$, this is equivalent to both (3.3) and (3.4) with $\vec{d}$ in place of $\vec{b}$. Since $\text{div } \vec{d} = 0$, the first condition becomes vacuous, and so it suffices to verify (4.64). The latter inequality holds by Lemma 4.7 since $\vec{d} = \text{Div } F$ where $F \in \text{BMO}(\mathbb{R}^n)^{n \times n}$, and $F$ is skew-symmetric. Combining the preceding estimates we conclude that $L = \vec{c} \cdot \nabla + \vec{d} \cdot \nabla + q$ is form bounded.

Conversely, if the bilinear form of $L$ is bounded, then by Lemma 4.3 and Lemma 4.5, decomposition (4.65) holds, where $\vec{c}$, $F$, and $\vec{h}$ are given by (4.67). Furthermore, it follows that (4.66) holds for $n \geq 3$, and (iii) is valid if $n = 2$. The proof of Theorem 4.8 is complete. □

**Corollary 4.9.** Let $n \geq 3$. Let $L = \vec{b} \cdot \nabla + q$, where $\vec{b} \in D'(\mathbb{R}^n)^n$, and $q \in D'(\mathbb{R}^n)$. Then the bilinear form of $L$ is bounded on $L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n)$ if and only if decomposition (4.65) is valid where $\vec{c}$, $F$, and $\vec{h}$ are given by (4.67), $F = \Delta^{-1} \text{curl } \vec{b} \in \text{BMO}(\mathbb{R}^n)^{n \times n}$, and the locally finite measure

$$d\mu(x) = \left( |\nabla(\Delta^{-1} q)|^2 + |\nabla(\Delta^{-1} \text{div } \vec{b})|^2 \right) dx$$

is subject to any one of the following equivalent conditions:

(i) For every compact set $e \subset \mathbb{R}^n$,

$$\mu(e) \leq C \text{ cap } (e),$$

where the capacity $\text{cap } (\cdot)$ is defined by (2.1).

(ii) For any cube $P$ in $\mathbb{R}^n$,

$$\int_P \left[ (-\Delta)^{-\frac{1}{2}}(\chi_P d\mu) \right]^2 dx \leq C \mu(P),$$

where $C$ does not depend on $P$.

(iii) For a.e. $x \in \mathbb{R}^n$,

$$(-\Delta)^{-\frac{1}{2}} \left[ (-\Delta)^{-\frac{1}{2}} \mu \right]^2 (x) \leq C (-\Delta)^{-\frac{1}{2}} \mu(x) < +\infty.$$  

(iv) For any dyadic cube $P$ in $\mathbb{R}^n$,

$$\sum_{Q \subseteq P} \frac{\mu(Q)^2}{|Q|^{1-\frac{2}{n}}} \leq C \mu(P),$$
where the sum is taken over all dyadic cubes \( Q \) contained in \( P \), and \( C \) does not depend on \( P \).

Corollary 4.9 follows by coupling Theorem 4.8 with Theorem 2.1.

**Corollary 4.10.** Let \( n \geq 3 \). Let \( \mathcal{L} = \vec{b} \cdot \nabla + q \), where \( \vec{b} \in D'(\mathbb{R}^n) \) and \( q \in D'(\mathbb{R}^n) \).

(i) If the sesquilinear form of \( \mathcal{L} \) is bounded on \( L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n) \), then (4.65) and (4.67) hold and, for every ball \( B \) in \( \mathbb{R}^n \),

\[
\int_B \left[ |\nabla (\Delta^{-1} \text{div} \vec{b})|^2 + |\nabla \Delta^{-1} q|^2 \right] dx \leq C |B|^{1-\frac{2}{n}}.
\]

(ii) Conversely, if (4.65) and (4.67) are valid where

\[
\int_B \left[ |\nabla (\Delta^{-1} \text{div} \vec{b})|^2 + |\nabla \Delta^{-1} q|^2 \right]^{1+\epsilon} dx \leq C |B|^{1-\frac{2(1+\epsilon)}{n}}
\]

for some \( \epsilon > 0 \), and \( \Delta^{-1} \text{curl} \vec{b} \in \text{BMO}(\mathbb{R}^n)^{n \times n} \), then the sesquilinear form of \( \mathcal{L} \) is bounded on \( L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n) \).

**Remark 6.** Inequality (4.68), together with Poincaré’s inequality, yields that \( \Delta^{-1} \text{div} \vec{b} \in \text{BMO}(\mathbb{R}^n) \) and \( \Delta^{-1} q \in \text{BMO}(\mathbb{R}^n) \).

Statement (i) of Corollary 4.10 is immediate from Theorem 4.8 and (2.2), whereas (ii) follows by combining Corollary 4.9 with the Fefferman–Phong condition (1.9) applied to \( d\mu(x) = \left( |\nabla (\Delta^{-1} \text{div} \vec{b})|^2 + |\nabla (\Delta^{-1} q)|^2 \right) dx \). Sharper sufficient conditions are deduced from Corollary 4.9 in the same way by making use of the conditions due to Chang, Wilson, and Wolff [ChWW] applied to \( d\mu \).

The following statement is a consequence of Theorem 4.8 and Lemma 4.3.

**Corollary 4.11.** Let \( \vec{b} \in D'(\mathbb{R}^n) \) and \( q \in D'(\mathbb{R}^n) \), \( n \geq 2 \). Then the operator \( \vec{b} \cdot \nabla + q \) is form bounded on \( L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n) \) if and only if, for every cube \( Q \) in \( \mathbb{R}^n \),

\[
||\vec{b}||_{L^{1,2}(Q)} \leq \text{const} |Q|^{\frac{1}{2}}
\]

and both

\[
\text{div} \vec{b} : L^{1,2}(\mathbb{R}^n) \rightarrow L^{-1,2}(\mathbb{R}^n), \quad q : L^{1,2}(\mathbb{R}^n) \rightarrow L^{-1,2}(\mathbb{R}^n),
\]

are bounded multiplication operators.

We conclude this section with a form boundedness criterion for the magnetic Schrödinger operator

\[
\mathcal{M} = (i \nabla + \vec{a})^2 + q
\]

with real-valued magnetic vector potential \( \vec{a} \). As a direct consequence of Theorem 4.8 we establish the following form boundedness criterion for \( \mathcal{M} \).
Theorem 4.12. Let $\vec{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $q \in D'(\mathbb{R}^n)$, $n \geq 2$. Then the operator

$$ M = (i \nabla + \vec{a})^2 + q \text{ is form bounded on } L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n) \text{ if and only if both }$$

$$q + |\vec{a}|^2 \text{ and } \vec{a} \cdot \nabla \text{ are form bounded. More precisely, in order that}$$

$$|\langle M u, v \rangle| \leq C ||u||_{L^{1,2}(\mathbb{R}^n)} ||v||_{L^{1,2}(\mathbb{R}^n)}, \quad u, v \in C_0^\infty(\mathbb{R}^n),$$

it is necessary and sufficient that

$$\vec{a} = \vec{c} + \text{Div } F, \quad q + |\vec{a}|^2 = \text{div } \vec{h},$$

where $F$ is a skew-symmetric matrix field whose entries belong to $\text{BMO}(\mathbb{R}^n)$, and $|\vec{c}|^2 + |\vec{h}|^2 \in \mathcal{M}^{1,2}_+(\mathbb{R}^n)$.

Moreover, one can define $\vec{c}$, $F$, and $\vec{h}$ in representation (4.13) constructively as $\vec{c} = \nabla (\Delta^{-1} \text{div } \vec{a}), \ F = \Delta^{-1} \text{curl } \vec{a}$, and $\vec{h} = \nabla \Delta^{-1} (q + |\vec{a}|^2)$.

In the case $n = 2$, $M$ is form bounded on $L^{1,2}(\mathbb{R}^2) \times L^{1,2}(\mathbb{R}^2)$ if and only if $\text{div } \vec{a} = 0$, and $q + |\vec{a}|^2 = 0$, where $\vec{a} = (\partial_2 g, -\partial_1 g)$, and $g \in \text{BMO}(\mathbb{R}^2)$.

Remark 7. This characterization simplifies under the Coulomb gauge hypothesis $\text{div } \vec{a} = 0$ (see [RS], Sec. X. 4). Then, for the form boundedness of $(i \nabla + \vec{a})^2 + q$ on $L^{1,2}(\mathbb{R}^n) \times L^{1,2}(\mathbb{R}^n)$, $n \geq 3$, it is necessary and sufficient that $q + |\vec{a}|^2$ be form bounded, and $\vec{a} = \text{Div } F$, where $F = \Delta^{-1} \text{curl } \vec{a} \in \text{BMO}(\mathbb{R}^n)^{n \times n}$.

Remark 8. The above characterization of the form boundedness of $M$ holds if one replaces the assumption $\vec{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$ by $q + |\vec{a}|^2 \in L^1_{\text{loc}}(\mathbb{R}^n)$.

5. An estimate for $\langle |\vec{b} \cdot \nabla u|, u \rangle$

In this section we prove the following statement for the nonlinear quadratic form $\langle |\vec{b} \cdot \nabla u|, u \rangle$ which holds for every open set $\Omega \subset \mathbb{R}^n$, in particular $\Omega = \mathbb{R}^n$.

Proposition 5.1. Let $\vec{b} \in L^1_{\text{loc}}(\Omega)^n$. Then the best constant in the inequality

$$(5.1) \quad \int_\Omega |\vec{b} \cdot \nabla u| \, d\bar{x} \leq C \||\nabla u||^2_{L^2(\Omega)}, \quad u \in C_0^\infty(\Omega),$$

satisfies the estimates

$$(5.2) \quad C \leq c \leq 2\sqrt{n} \, C,$$

where $c^2$ is the best constant in the inequality

$$(5.3) \quad \int_\Omega |\vec{b}|^2 |u|^2 \, d\bar{x} \leq c^2 \||\nabla u||^2_{L^2(\Omega)}, \quad u \in C_0^\infty(\Omega).$$

Remark 9. The constant $c$ in the previous inequality coincides with the norm of the multiplier operator $\vec{b}: L^{1,2}(\Omega) \to L^2(\Omega)^n$ where $L^{1,2}(\Omega)$ is a (homogeneous) Sobolev space defined as the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm $||\nabla u||_{L^2(\Omega)}$. 
Proof. The lower bound in (5.2) is obvious. Let us prove the upper bound. For real-valued $u$, inequality (5.1), combined with the well-known estimate $||\nabla|u||_{L^2(\Omega)} \leq ||\nabla u||_{L^2(\Omega)}$ (see [LL], Sec. 7.8), yields

$$\int_{\Omega} |\vec{b} \cdot \nabla u|^2 \, dx \leq C ||\nabla u||_{L^2(\Omega)}^2, \quad u \in C_0^\infty(\Omega).$$

Consequently, for complex-valued $u$,

$$\int_{\Omega} |\vec{b} \cdot \nabla u|^2 \, dx \leq 2C ||\nabla u||_{L^2(\Omega)}^2, \quad u \in C_0^\infty(\Omega).$$

For every $\epsilon > 0$, by (5.5),

$$\int_{\Omega} |(\vec{b} \cdot \nabla)(\epsilon u \pm \epsilon^{-1} v)^2| \, dx \leq 2C ||\nabla u \pm \epsilon^{-1} v||_{L^2(\Omega)}^2.$$

Hence,

$$4 \int_{\Omega} |(\vec{b} \cdot \nabla)(u v)| \, dx = \int_{\Omega} |(\vec{b} \cdot \nabla) \left((\epsilon u + \epsilon^{-1} v)^2 - (\epsilon u - \epsilon^{-1} v)^2\right)| \, dx$$

$$\leq 4C \left(\epsilon^2 ||\nabla u||_{L^2(\Omega)}^2 + \epsilon^{-2} ||\nabla v||_{L^2(\Omega)}^2\right).$$

Minimizing over $\epsilon$, we get:

$$\int_{\Omega} |(\vec{b} \cdot \nabla)(u v)| \, dx \leq 2C ||\nabla u||_{L^2(\Omega)} ||\nabla v||_{L^2(\Omega)}.$$

We now set

$$u(x) = e^{i \langle \omega, \xi \rangle} h(x), \quad h \in C_0^\infty(\Omega),$$

where $\omega \in S^{n-1}$ and $\xi \in \mathbb{R}^n$. We estimate:

$$\int_{\Omega} |\langle \omega, \vec{b} \rangle| |h v| \, dx \leq 2C ||h||_{L^2(\Omega)} ||\nabla v||_{L^2(\Omega)} + O(|\xi|^{-1}).$$

Letting $|\xi| \to +\infty$ gives

$$\int_{\Omega} |\langle \omega, \vec{b} \rangle|^2 |v|^2 \, dx \leq 4C^2 ||\nabla v||_{L^2(\Omega)}.$$

Integrating the preceding inequality over $S^{n-1}$ and using the identity

$$\int_{S^{n-1}} |\langle \omega, \vec{b} \rangle|^2 \, ds_\omega = \frac{1}{n} |S^{n-1}| |\vec{b}|^2,$$

we arrive at:

$$\int_{\Omega} |\vec{b}|^2 |v|^2 \, dx \leq 4nC^2 ||\nabla v||_{L^2(\Omega)}.$$

The proof of Proposition 5.1 is complete. \qed
Corollary 5.2. The best constant in the inequality (5.1) satisfies the estimates:

\[ \frac{1}{2} C \leq \sup_{e \subset \Omega} \frac{||\vec{b}||_{L^2(e)}}{\text{cap} (e, \Omega)^{\frac{1}{2}}} \leq 2\sqrt{n} C, \]

where the supremum is taken over all compact sets \( e \subset \Omega \) of positive capacity defined by

\[ \text{cap} (e, \Omega) = \inf \left\{ \left\| \nabla u \right\|_{L^2(\Omega)}^2 : u \in C_0^\infty (\Omega), \ u \geq 1 \ on \ e \right\}. \]

Corollary 5.2 follows from Proposition 5.1 and \[M\], Sec. 2.5.

6. Form boundedness on the Sobolev space \( W^{1,2}(\mathbb{R}^n) \)

In this section, we obtain the form boundedness criterion for the general second order differential operator \( \mathcal{L} \) on the Sobolev space \( W^{1,2}(\mathbb{R}^n) \), \( n \geq 2 \).

As was noticed above, without loss of generality we may assume that \( \mathcal{L} \) is in the divergence form: \( \mathcal{L} = \text{div} (A \nabla) + \vec{b} \cdot \nabla + q \), where \( A \in D'(\mathbb{R}^{n \times n}) \), \( \vec{b} \in D'(\mathbb{R}^n)^n \) and \( q \in D'(\mathbb{R}^n) \), \( n \geq 2 \). Then clearly the sesquilinear inequality

\[ |\langle \mathcal{L} u, v \rangle| \leq C \left\| u \right\|_{W^{1,2}(\mathbb{R}^n)} \left\| v \right\|_{W^{1,2}(\mathbb{R}^n)} \]

holds for all \( u, v \in C_0^\infty (\mathbb{R}^n) \) if and only if the operator \( \mathcal{L} \) (or, more precisely, its unique extension from \( C_0^\infty \) to \( W^{1,2} \)),

\[ \mathcal{L} : W^{1,2}(\mathbb{R}^n) \rightarrow W^{-1,2}(\mathbb{R}^n) \]

is bounded.

We notice that Proposition 3.1 holds for \( W^{1,2} \) in place of \( L^{1,2} \), with obvious modifications in the proof. In particular, the condition \( A^c = \frac{1}{2}(A + A^t) \in L^\infty (\mathbb{R}^{n \times n}) \) is necessary for the form boundedness of \( \mathcal{L} \) on \( W^{1,2} \), whereas \( A^c = \frac{1}{2}(A - A^t) \) can be included in \( \vec{b} \) by letting \( \vec{b}_1 = \vec{b} - \text{Div} A^c \), exactly as in the case of the homogeneous Sobolev space. In other words, it suffices to consider the form boundedness problem on \( W^{1,2}(\mathbb{R}^n) \) for \( \mathcal{L} = \vec{b} \cdot \nabla + q \).

Recall that \( \text{BMO}^\#(\mathbb{R}^n) \) stands for the space of \( f \in L^1_{\text{loc}} (\mathbb{R}^n) \) such that

\[ \sup_{x_0 \in \mathbb{R}^n, 0 < \delta \leq 1} \frac{1}{|B_\delta(x_0)|} \int_{B_\delta(x_0)} |f(x) - m_{B_\delta(x_0)}(f)| \, dx < +\infty. \]

Theorem 6.1. Let \( \vec{b} \in D'(\mathbb{R}^n)^n \), \( q \in D'(\mathbb{R}^n) \), and let \( \mathcal{L} = \vec{b} \cdot \nabla + q \), \( n \geq 2 \). Then (6.1) holds if and only if \( \vec{b} \) and \( q \) can be represented respectively in the form:

\[ \vec{b} = \vec{c} + \text{Div} F, \quad q = \text{div} \vec{h} + \gamma, \]
where $F$ is a skew-symmetric matrix field such that $F \in \text{BMO}^\#(\mathbb{R}^n)^{n \times n}$, and $(|\vec{c}|^2 + |\vec{h}|^2 + |\gamma|) \, dx$ is an admissible measure for $W^{1,2}(\mathbb{R}^n)$, i.e.,

$$
(6.4) \quad \int_{\mathbb{R}^n} (|\vec{c}|^2 + |\vec{h}|^2 + |\gamma|) \, |u|^2 \, dx \leq c \||u||^2_{W^{1,2}(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n).
$$

Moreover, in the decomposition (6.3) and condition (6.4) one can set

$$
(6.5) \quad \vec{c} = -\nabla((1 - \Delta)^{-1}\text{div} \vec{b}) + (1 - \Delta)^{-1} \vec{b}, \quad F = -(1 - \Delta)^{-1}\text{curl} \vec{b},
$$

$$
(6.6) \quad \vec{h} = -\nabla((1 - \Delta)^{-1}q), \quad \gamma = (1 - \Delta)^{-1}q.
$$

Furthermore, (6.4) holds with $(1 - \Delta)^{-1}\text{div} \vec{b}|^2 + (1 - \Delta)^{-1}\vec{b}|^2$ in place of $|\vec{c}|^2$.

**Proof.** Suppose that $\vec{b}$ is given by (6.3) where $F$ is a skew-symmetric matrix field such that $F \in \text{BMO}^\#(\mathbb{R}^n)^{n \times n}$, and $\vec{c}$, $\vec{h}$, and $\gamma$ satisfy (6.4). The boundedness of the bilinear form associated with $q$ and $\vec{c} \cdot \nabla$ follows easily using integration by parts and Schwarz’s inequality:

$$
||\vec{c} \cdot \nabla u + qu, v|| \leq ||\vec{c} \cdot \nabla u, v|| + ||\vec{h}, \vec{u} \nabla v + v \nabla \vec{u}|| + ||\gamma, \vec{u} v||
$$

$$
\leq ||\vec{c}||_{L^2(\mathbb{R}^n)} ||\nabla u||_{L^2(\mathbb{R}^n)} + ||\vec{h}||_{L^2(\mathbb{R}^n)} ||\nabla v||_{L^2(\mathbb{R}^n)} + ||\gamma||_{L^2(\mathbb{R}^n)} ^{1/2} ||u||_{L^2(\mathbb{R}^n)} ||\gamma||_{L^2(\mathbb{R}^n)} ^{1/2} ||v||_{L^2(\mathbb{R}^n)}
$$

$$
\leq C ||u||_{W^{1,2}(\mathbb{R}^n)} ||v||_{W^{1,2}(\mathbb{R}^n)}, \quad u, v \in C_0^\infty(\mathbb{R}^n).
$$

We next prove the boundedness of the bilinear form associated with the divergence free part of $\vec{b}$ given by $\vec{d} = \text{Div} F$. This may be viewed as an inhomogeneous version of the div-curl lemma [CLMS]. The proof is based on a localization principle, combined with an appropriate extension of BMO($B$) functions originally defined on a ball $B \subset \mathbb{R}^n$.

**Lemma 6.2.** Suppose $\vec{d} = \text{Div} F$ in $D'(\mathbb{R}^n)^n$, where $F$ is a skew-symmetric matrix function such that $F \in \text{BMO}^\#(\mathbb{R}^n)^{n \times n}$. Then the inequality

$$
(6.7) \quad ||\vec{d} \cdot \nabla u, v|| \leq C ||u||_{W^{1,2}(\mathbb{R}^n)} ||v||_{W^{1,2}(\mathbb{R}^n)}, \quad u, v \in C_0^\infty(\mathbb{R}^n),
$$

holds where $C$ does not depend on $u$ and $v$.

**Proof.** We first prove a localized version of (6.7),

$$
(6.8) \quad ||\vec{d} \cdot \nabla u, v|| \leq C ||\nabla u||_{L^2(B_1(x_0))} ||\nabla v||_{L^2(B_1(x_0))},
$$

where the constant $C$ does not depend on $u, v \in C_0^\infty(B_1(x_0))$, and $x_0 \in \mathbb{R}^n$.

For a domain $\Omega \subset \mathbb{R}^n$, denote by BMO($\Omega$) the space of functions $f \in L^1_{\text{loc}}(\Omega)$ such that

$$
\sup_{B \subset \Omega} \frac{1}{|B|} \int_B |f - m_B(f)| \, dx < +\infty,
$$

where $m_B(f)$ denotes the average of $f$ over the ball $B$. The space BMO($\Omega$) is a Banach space with the norm

$$
||f||_{\text{BMO}(\Omega)} = \sup_{B \subset \Omega} \frac{1}{|B|} \int_B |f - m_B(f)| \, dx.
$$
where the supremum is taken over all balls $B$ in $\Omega$.

Since $F \in \text{BMO}^\#(\mathbb{R}^n)^{n \times n}$, it follows that $F \in \text{BMO}(B_1(x_0))$ for every $x_0 \in \mathbb{R}^n$, and

$$\sup_{x_0 \in \mathbb{R}^n} ||F||_{\text{BMO}(B_1(x_0))^{n \times n}} < +\infty.$$ \hfill (6.9)

By replacing $u$ and $v \in C_0^\infty(B_1(x_0))$ in (6.8) with $u(x - x_0)$ and $v(x - x_0)$ respectively, one can assume without loss of generality that $x_0 = 0$, and $F \in \text{BMO}(B_1(0))$. Denote by $\tilde{F}$ an extension of $F$ from $B_1(0)$ to $\mathbb{R}^n$ such that

$$||\tilde{F}||_{\text{BMO}(\mathbb{R}^n)^{n \times n}} \leq c ||F||_{\text{BMO}(B_1(0))^{n \times n}},$$ \hfill (6.10)

where $c$ depends only on $n$. To construct such an extension one can use a reflection in the boundary. (See, e.g., [J] where this is done for very general domains $\Omega \subset \mathbb{R}^n$.)

Note that both $F$ and $\tilde{F}$ are skew-symmetric. Hence by (6.10) and the version of the div-curl lemma used above (see the proof of Lemma 4.7),

$$2 \left| \langle \vec{d} \cdot \nabla u, v \rangle \right| = \left| \int_{B_1(0)} \text{trace} \ F \cdot \{ \partial_i \bar{u} \partial_j v - \partial_j \bar{u} \partial_i v \} \ dx \right|$$

$$= \left| \int_{B_1(0)} \text{trace} \ \tilde{F} \cdot \{ \partial_i \bar{u} \partial_j v - \partial_j \bar{u} \partial_i v \} \ dx \right|$$

$$\leq C ||F||_{\text{BMO}(\mathbb{R}^n)^{n \times n}} ||\nabla u||_{L^2(B_1(0))} ||\nabla v||_{L^2(B_1(0))},$$

where $C$ depends only on $n$. Taking into account (6.9), we conclude that (6.8) holds for every $u, v \in C_0^\infty(B_1(x_0))$ with a constant which does not depend on $u, v$, and $x_0$.

To prove (6.11), suppose $u, v \in C_0^\infty(B_R(x_0))$, $R > 1$. Pick a sequence of functions $\{\zeta_i\}_{i=1}^{\infty}$ so that

$$\sum_i \zeta_i(x)^2 = 1, \quad \sum_i |\nabla \zeta_i(x)|^2 \leq c(n) \quad \text{on } B_R(x_0),$$ \hfill (6.11)

$$\sum_i \zeta_i^2 \in C^\infty(\mathbb{R}^n), \quad \zeta_i \in C_0^\infty(B_1(x_i)), \quad i = 1, 2, \ldots.$$ \hfill (6.12)

Here $x_i$ is a cubic lattice of equidistant points in $\mathbb{R}^n$ with grid distance equal to $\frac{1}{2^{\sqrt{n}}}$. (See, e.g., [MV4], the proof of Lemma 3.1).

Now integration by parts gives

$$\langle \vec{d} \cdot \nabla u, v \rangle = \sum_i \langle \vec{d} \cdot \nabla u, \zeta_i^2 v \rangle$$

$$= \sum_i \langle \vec{d} \cdot \nabla (\zeta_i u), \zeta_i v \rangle - \frac{1}{2} \sum_i \langle \vec{d} \cdot \nabla (\zeta_i^2), \bar{u} v \rangle$$

$$= \sum_i \langle \vec{d} \cdot \nabla (\zeta_i u), \zeta_i v \rangle.$$
In the last line we have used $\sum_i \nabla (\zeta_i^2) = 0$ on $B_R(x_0)$ which follows from (6.11).

Suppose now that (6.8) holds. Then from the preceding equation we deduce:
\[
\left| \langle \bar{a} \cdot \nabla u, v \rangle \right| \leq C \sum_i \left| \langle \bar{a} \cdot \nabla (\zeta_i u), \zeta_i v \rangle \right|
\]
\[
\leq C \sum_i \| \nabla (\zeta_i u) \|_{L^2(\mathbb{R}^n)} \| \nabla (\zeta_i v) \|_{L^2(\mathbb{R}^n)}
\]
\[
\leq C \sum_i \| \nabla (\zeta_i u) \|_{L^2(\mathbb{R}^n)}^2 + C \sum_i \| \nabla (\zeta_i v) \|_{L^2(\mathbb{R}^n)}^2.
\]

We estimate the first term on the right-hand side using (6.11):
\[
\sum_i \| \nabla (\zeta_i u) \|_{L^2(\mathbb{R}^n)}^2 \leq C \sum_i \| \zeta_i \nabla u \|_{L^2(\mathbb{R}^n)}^2 + C \sum_i \| (\nabla \zeta_i) u \|_{L^2(\mathbb{R}^n)}^2
\]
\[
\leq C \| u \|_{W^{1,2}(\mathbb{R}^n)}^2.
\]

where $C$ does not depend on $u$. A similar estimate holds for the second term which involves $v$. Note that without loss of generality we may assume that $\max (\| u \|_{W^{1,2}(\mathbb{R}^n)}, \| v \|_{W^{1,2}(\mathbb{R}^n)}) \leq 1$. Hence, $\left| \langle \bar{a} \cdot \nabla u, v \rangle \right| \leq C,$ which yields (6.7). This concludes the proof of Lemma 6.2.

It follows from the preceding estimates for $q$ and $\bar{c}$, and Lemma 6.2 that $\mathcal{L} = \bar{b} \cdot \nabla + q$ is form bounded provided (6.3) holds with $F \in \text{BMO}^\#(\mathbb{R}^n)^{n \times n}$ and $\bar{c}, \bar{h}, \gamma$ satisfying (6.4).

It remains to prove the converse for $\mathcal{L} = \bar{b} \cdot \nabla + q$ is form bounded provided (6.3) holds with $F \in \text{BMO}^\#(\mathbb{R}^n)^{n \times n}$ and $\bar{c}, \bar{h}, \gamma$ satisfying (6.4).

An analogue of Proposition 4.1 states that, if (6.14) holds, then the following estimates are valid:
\[
\langle (q - \frac{1}{2} \text{div} \bar{b}) u, v \rangle \leq C \| u \|_{W^{1,2}(\mathbb{R}^n)} \| v \|_{W^{1,2}(\mathbb{R}^n)},
\]
\[
\langle \bar{b}, \bar{u} \nabla v - v \nabla \bar{u} \rangle \leq C \| u \|_{W^{1,2}(\mathbb{R}^n)} \| v \|_{W^{1,2}(\mathbb{R}^n)},
\]

for all $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

An analogue of Proposition 4.1 states that, if (6.14) holds, then the following estimates are valid:
\[
\| \text{div} \bar{b} \|_{W^{-1,2}(Q)} \leq C |Q|^{\frac{1}{2} - \frac{n}{2}} \quad \text{if } n \geq 3,
\]
\[
\| \text{div} \bar{b} \|_{W^{-1,2}(Q)} \leq C \left( \log \frac{2}{|Q|} \right)^{-\frac{1}{2}} \quad \text{if } n = 2,
\]
\[
\| \bar{b} \|_{W^{-1,2}(Q)} \leq C |Q|^\frac{1}{2} \quad \text{if } n \geq 2,
\]

for every cube $Q$ in $\mathbb{R}^n$ such that $\ell(Q) \leq 1$. The only change that is needed in the proof is that, for the capacity $\text{Cap}(\cdot)$ associated with $W^{1,2}(\mathbb{R}^n)$, which
is defined by (2.3), we have \( \text{Cap} (Q) \simeq \left( \frac{2}{|Q|} \right)^{-1} \) for \( n \geq 2 \) and \( \ell(Q) \leq 1 \) by (2.2). (Note that in two dimensions, contrary to the case of \( L^{1,2}(\mathbb{R}^2) \), \( \vec{b} \) is no longer required to be divergence free.)

It now follows from (6.17), as in the proofs of Lemma 4.3 and 4.5, that decomposition (6.3) holds where \( \vec{c} \), \( F \), \( \vec{h} \), and \( \gamma \) are given by (6.5) and (6.6) respectively, and \( F \in \text{BMO}^\#(\mathbb{R}^n)^{n \times n} \). Furthermore, using a direct analogue of Lemma 4.6 for \( W^{1,2}(\mathbb{R}^n) \), we deduce from (6.14) that \( \text{div} \vec{b} \) is form bounded on \( W^{1,2}(\mathbb{R}^n) \), i.e.,

\[
|\langle (\text{div} \vec{b}) u , v \rangle| \leq C \| u \|_{W^{1,2}(\mathbb{R}^n)} \| v \|_{W^{1,2}(\mathbb{R}^n)},
\]

for all \( u, v \in C_0^\infty(\mathbb{R}^n) \). Hence, by (6.13),

\[
|\langle g u , v \rangle| \leq C \| u \|_{W^{1,2}(\mathbb{R}^n)} \| v \|_{W^{1,2}(\mathbb{R}^n)},
\]

for all \( u, v \in C_0^\infty(\mathbb{R}^n) \).

The preceding inequality, by Theorem 4.2 in [MV1], yields

(6.18) \[
\int_{\mathbb{R}^n} (|\nabla (1 - \Delta)^{-1} q|^2 + |(1 - \Delta)^{-1} q|) \, |u|^2 \, dx \leq C \| u \|^2_{W^{1,2}(\mathbb{R}^n)},
\]

for all \( u \in C_0^\infty(\mathbb{R}^n) \). Note that, according to [MV1] (Sec. 4, Remark 3), it is possible to put \(|(1 - \Delta)^{-1} q|^2\) in place of \(|(1 - \Delta)^{-1} q|\) in (6.18). The same argument with \( \text{div} \vec{b} \) in place of \( q \) gives

\[
\int_{\mathbb{R}^n} (|\nabla (1 - \Delta)^{-1} \text{div} \vec{b}|^2 + |(1 - \Delta)^{-1} \vec{b}|^2) \, |u|^2 \, dx \leq C \| u \|^2_{W^{1,2}(\mathbb{R}^n)},
\]

for all \( u \in C_0^\infty(\mathbb{R}^n) \). The proof of Theorem 6.1 is complete. \( \square \)

7. Infinitesimal Form Boundedness and Relative Compactness

In this section, we discuss infinitesimal form boundedness and relative compactness properties (see [RS], [Sch]) for the general second order differential operator \( \mathcal{L} \). Since the coefficients of \( \mathcal{L} \) are arbitrary real- or complex-valued distributions, as above, we may assume without loss of generality that \( \mathcal{L} \) is in the divergence form \( \mathcal{L} = -\text{div} (A \nabla u) + \vec{b} \cdot \nabla + q \) where \( A \in D'(\mathbb{R}^n)^{n \times n} \), \( \vec{b} \in D'(\mathbb{R}^n)^n \), and \( q \in D'(\mathbb{R}^n) \).

The operator \( \mathcal{L} \) is said to be relative form bounded with respect to the Laplacian on the (complex-valued) \( L^2(\mathbb{R}^n) \) space if

(7.1) \[
|\langle \mathcal{L} u , u \rangle| \leq \epsilon \| \nabla u \|^2_{L^2(\mathbb{R}^n)} + C(\epsilon) \| u \|^2_{L^2(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n),
\]

for some \( \epsilon > 0 \) and \( C(\epsilon) > 0 \). This is obviously equivalent to the boundedness of the sesquilinear form on \( W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n) \), which was characterized in Theorem 6.1.
However, in many applications it is of interest to distinguish the class of $\mathcal{L}$ for which (7.1) holds with relative bound zero, i.e., for every $\epsilon > 0$ and some $C(\epsilon) > 0$. In this case, $\mathcal{L}$ is said to be *infinitesimally form bounded* with respect to $-\Delta$ on $L^2(\mathbb{R}^n)$. For the potential energy operator $q \in D'(\mathbb{R}^n)$, the infinitesimal form boundedness with respect to $-\Delta$ was characterized recently in [MV4]. Here we state the corresponding result for $\mathcal{L}$.

Notice that, from the proof of Proposition 3.1 applied to (7.1), it is immediate that the symmetric part $A^s = \frac{1}{2}(A + A^t)$ must be equal to zero, while the skew-symmetric part $A^c = \frac{1}{2}(A - A^t)$ can be incorporated into $\vec{b}$ by letting $\vec{b}_1 = \vec{b} - \text{Div} A^c$, and considering (7.1) for $\vec{b}_1 \cdot \nabla + q$. Thus, without loss of generality it suffices to treat the operator $\mathcal{L} = \vec{b} \cdot \nabla + q$.

**Theorem 7.1.** Let $\mathcal{L} = \vec{b} \cdot \nabla + q$, where $\vec{b} \in D'(\mathbb{R}^n)^n$ and $q \in D'(\mathbb{R}^n)$, $n \geq 2$. Then (7.1) holds for every $\epsilon > 0$ if and only if $\vec{b}$ and $q$ can be represented in the form (6.3), where $F$ has vanishing mean oscillation, i.e.,

$$\lim_{\delta \to +0} \sup_{Q: |Q| \leq \delta} \frac{1}{|Q|} \int_Q |F - m_Q(F)| \, dx = 0,$$

and

$$\lim_{\delta \to +0} \sup_{\delta \to +0} \left\{ \frac{\int_Q |u|^2 \, d\mu}{||\nabla u||^2_{L^2(Q)}} : u \in C_0^\infty(Q), \, u \neq 0, \, |Q| \leq \delta \right\} = 0,$$

where $d\mu = (|\vec{c}|^2 + |\vec{h}|^2 + |\gamma|) \, dx$. Moreover, $\vec{c}$, $F$, $\vec{h}$, and $\gamma$ can be defined respectively by (6.5), (6.6).

The proof of Theorem 7.1 follows by combining the approach of [MV4], which is based on a localization argument, with the form boundedness criterion obtained above.

**Remark 10.** Analytic criteria for (7.3) to hold are discussed in [MV4].

**Remark 11.** Trudinger’s condition where $C(\epsilon) = C(\epsilon \beta, \beta > 0$, in (7.1), and inequalities of Nash’s type,

$$|\langle \mathcal{L} u, u \rangle| \leq C ||\nabla u||^2_{L^2(\mathbb{R}^n)} ||u||^{2(1-\gamma)}_{L^1(\mathbb{R}^n)}, \, u \in C_0^\infty(\mathbb{R}^n),$$

where $\gamma \in (0, 1)$, can be characterized using our approach as well; see [MV4] where this is done for $\vec{b} = 0$.

Finally, we state a criterion for the *relative compactness* property which requires additional conditions at infinity.

**Theorem 7.2.** Let $\mathcal{L} = \vec{b} \cdot \nabla + q$, where $\vec{b} \in D'(\mathbb{R}^n)^n$ and $q \in D'(\mathbb{R}^n)$, $n \geq 2$. Then the operator $\vec{b} \cdot \nabla + q$ is relatively compact with respect to $-\Delta$ on $L^2(\mathbb{R}^n)$.
if and only if \( \vec{b} \) can be represented in the form (6.3), where \( F \in \text{VMO}(\mathbb{R}^n) \times \mathbb{R}^n \), i.e.,

\[
\lim_{\delta \to +0} \sup_{Q: |Q| \leq \delta} \frac{1}{|Q|} \int_Q |F - m_Q(F)| \, dx = 0,
\]

(7.5)

\[
\lim_{\delta \to +\infty} \sup_{Q_0: |Q_0| \geq \delta} \frac{1}{|Q_0|} \int_{Q_0} |F - m_{Q_0}(F)| \, dx = 0,
\]

(7.6)

and

\[
\lim_{\delta \to +0} \sup \left\{ \int_Q |u|^2 \, d\mu : u \in C_0^\infty(Q), \ u \neq 0, \ |Q| \leq \delta \right\} = 0,
\]

(7.7)

\[
\lim_{\delta \to +\infty} \sup \left\{ \int_{Q_0} |u|^2 \, d\mu : u \in C_0^\infty(Q_0), \ u \neq 0, \ |Q_0| \geq \delta \right\} = 0,
\]

(7.8)

where \( Q_0 \) denotes a cube centered at the origin, and \( d\mu = (|\vec{c}|^2 + |\vec{h}|^2 + |\gamma|) \, dx \).

Moreover, \( \vec{c} \), \( F \), \( \vec{h} \), and \( q \) can be defined respectively by (6.5), (6.6).

The proof of Theorem 7.2 is based on the form boundedness criterion obtained in the previous section, and is analogous to the case \( \vec{b} = 0 \) treated in \[MV1\].

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