Finite volume partition functions
and Itzykson-Zuber integrals

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Abstract

We find the finite volume QCD partition function for different quark masses. This is a generalization of a result obtained by Leutwyler and Smilga for equal quark masses. Our result is derived in the sector of zero topological charge using a generalization of the Itzykson-Zuber integral appropriate for arbitrary complex matrices. We present a conjecture regarding the result for arbitrary topological charge which reproduces the Leutwyler-Smilga result in the limit of equal quark masses. We derive a formula of the Itzykson-Zuber type for arbitrary rectangular complex matrices, extending the result of Guhr and Wettig obtained for square matrices.

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1 Introduction

In general, numerical simulations offer our only access to the QCD partition function. For sufficiently small volumes, however, the partition function is dominated by the constant modes which makes analytic treatment possible. (See, e.g., ref. [4].) In a similar spirit, Leutwyler and Smilga [2] identified a parameter range within which the mass dependence of the partition function is completely determined by the underlying structure of broken chiral symmetry. This range is given as

\[ \frac{1}{\Lambda} \ll L \ll \frac{1}{m_\pi}, \]  

where \( L \) is the linear size of the 4-dimensional Euclidean box, \( \Lambda \) is a typical hadronic mass scale and \( m_\pi \) is the mass of the Goldstone modes (i.e., \( m_\pi \sim \sqrt{m\Lambda} \) for quark mass \( m \)). The lower limit of this range ensures that the partition function is dominated by the Goldstone modes. The upper limit ensures that these modes can be treated as constant modes. If chiral symmetry is broken according to \( SU(N_f) \times SU(N_f) \rightarrow SU(N_f) \), the QCD partition function in the range (1) and for vacuum angle \( \theta \) is given by [2]

\[ Z(M, \theta) = \int_{U \in SU(N_f)} dU \exp \left( V \Sigma \text{Re}(e^{i\theta/N_f} \text{tr} U M) \right). \]  

Here, \( \Sigma \) is the chiral condensate, and \( M \) is the mass matrix which can be taken as diagonal without loss of generality. In the sector with topological charge \( \nu \), the partition function is given as

\[ Z_\nu(M) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\nu\theta} Z(\theta). \]  

For equal quark masses, Leutwyler and Smilga obtained an exact analytic expression for this partition function

\[ Z_\nu(m) = \det_{ij} I_{\nu+j-i}(m V \Sigma), \]  

where \( i \) and \( j \) run from 1, \( \cdots \), \( N_f \). The importance of this partition function lies in the fact that it enables us to determine the volume dependence of the chiral condensate. This is of interest in lattice QCD simulations, where the volume is necessarily finite.

Recently, lattice QCD calculations have been performed to determine the connected and disconnected contributions to the scalar susceptibility [3]. The determination of
these susceptibilities requires differentiation of the partition function with respect to two different masses. In order to determine volume dependence of the scalar susceptibility in the range (1), we require a generalization of (4) to different quark masses. This is the primary objective of the present paper.

In section 2, we analyze the two flavor case and generalize the result to an arbitrary number of flavors. In section 3, we present a generalization of the Itzykson-Zuber formula valid for arbitrary rectangular complex matrices. For square complex matrices, our result reduces to that obtained by Wettig and Guhr [5]. In section 4, we derive the finite volume partition function for different masses in the sector of zero topological charge. We also make a conjecture of the result for arbitrary topological charge and show that it reduces to (4) for the special case of equal quark masses.

2 The finite volume partition function for $N_f = 2$ and its generalization to arbitrary $N_f$.

For two flavors and in the range (1), the QCD-partition function is known for different quark masses. For vacuum angle $\theta$, it is given as

$$Z(\theta) = \frac{2}{V \Sigma \mu} I_1(V \Sigma \mu) , \quad (5)$$

where $I_1$ is a modified Bessel function and where

$$\mu^2 = m_1^2 + m_2^2 + 2m_1m_2 \cos(\theta) . \quad (6)$$

The partition function in the sector with topological charge $\nu$ can be obtained by integrating over $\theta$ according to (3). Remarkably, this integral can be expressed analytically. After some manipulations the result can be written as

$$Z_\nu(N_f = 2) = \frac{2}{x_2^2 - x_1^2} \frac{1}{\det \begin{vmatrix} I_\nu(x_1) & x_1 I'_\nu(x_1) \\ I_\nu(x_2) & x_2 I'_\nu(x_2) \end{vmatrix}} \quad (7)$$

where

$$x_k = m_k V \Sigma . \quad (8)$$
This suggests the generalization to three flavors

\[ Z_\nu(N_f = 3) = \frac{16}{(x_2^2 - x_1^2)(x_3^2 - x_2^2)(x_3^2 - x_1^2)} \det \begin{vmatrix} I_\nu(x_1) & x_1 I'_\nu(x_1) & x_1^2 I''_\nu(x_1) \\ I_\nu(x_2) & x_2 I'_\nu(x_2) & x_2^2 I''_\nu(x_2) \\ I_\nu(x_3) & x_3 I'_\nu(x_3) & x_3^2 I''_\nu(x_3) \end{vmatrix}, \]

where the numerical prefactor is chosen such that for equal masses the result of Leutwyler is reproduced. The generalization to an arbitrary number of flavors is now obvious. We define a Vandermonde determinant as

\[ \Delta(x^2) = \prod_{k<l} (x_k^2 - x_l^2), \]

and an \( N_f \times N_f \) matrix as

\[ A_{kl} = x_k^{l-1} I_{\nu}^{(l-1)}(x_k), \quad k, l = 1, \ldots, N_f. \]

The partition function is then given as

\[ Z_\nu(m_1, \cdots, m_{N_f}) = C_{N_f} \frac{\det A}{\Delta(x^2)}, \]

where the normalization constant

\[ C_{N_f} = 2^{N_f(N_f-1)/2} \prod_{k=1}^{N_f} (k - 1)! \]

is determined by the limit of equal quark masses.

In section 4, we shall prove this formula for the special case \( \nu = 0 \). It will be shown that this formula reduces to the Leutwyler-Smilga finite volume partition function for arbitrary \( \nu \).

3 The Itzykson-Zuber integral for complex rectangular matrices

In this section we offer a derivation of the extension of the Itzykson-Zuber integral to the case of arbitrary complex rectangular matrices using the diffusion equation method. The result for square matrices has also been obtained in \[5\]. Our derivation is patterned on the argument for Hermitean matrices, which has been discussed widely in the literature. (See, e.g., \[6\].) The expression which will be required in section 4 for the calculation of the finite volume partition function for different masses is given in \([53]\).
Let \( \sigma \) and \( \rho \) be arbitrary complex (i.e., non-Hermitian) \( N_1 \times N_2 \) matrices. Without loss of generality, we assume that \( \nu \equiv N_1 - N_2 \geq 0 \). Any such matrix can be diagonalized in the form

\[
\sigma = U^\dagger S V \tag{14}
\]

where

\[
S = \left( \begin{array}{c} \hat{S} \\ 0 \end{array} \right), \tag{15}
\]

and where the square diagonal matrix \( \hat{S} = \text{diag}(s_1, \ldots, s_{N_2}) \) has nonnegative real entries. The matrices \((U, V)\) parameterize the coset space \( U(N_1) \times U(N_2)/[U(1)]^{N_2} \). Similarly, \( \rho \) can be written as \( \rho = U'^\dagger RV' \).

In the following, we evaluate the integral

\[
\frac{1}{(\pi t)(N_1 N_2)} \int d\mu(U, V) \exp \left( -\frac{1}{t} \text{tr}[(\sigma - \rho)^\dagger(\sigma - \rho)] \right) \tag{16}
\]

using the diffusion equation method and exploiting the invariance of the measure, \( d\mu(U, V) \), which is taken to be the Haar measure of \( U(N_1) \times U(N_2)/[U(1)]^{N_2} \). The function

\[
F(\rho, t) = \frac{1}{(\pi t)(N_1 N_2)} \int d[\sigma] \exp \left( -\frac{1}{t} \text{tr}[(\sigma - \rho)^\dagger(\sigma - \rho)] \right) \varphi(\sigma) \tag{17}
\]

with integration measure \( d[\sigma] = \prod_{m=1}^{N_1} \prod_{n=1}^{N_2} d\text{Re}\sigma_{mn} d\text{Im}\sigma_{mn} \) satisfies the diffusion equation

\[
\sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \frac{\partial^2}{\partial \rho_{mn} \partial \rho_{mn}^\dagger} F(\rho, t) = \frac{\partial}{\partial t} F(\rho, t). \tag{18}
\]

As initial condition, we choose an invariant function, i.e., \( \varphi(\sigma) \) is a function of the eigenvalues of \( \sigma \) only. Then, using the invariance of the measure, we find immediately that \( F(\rho, t) \) is a symmetric function of the eigenvalues of \( \rho \).

In order to proceed, we express both the Laplacian and the integration measure in the ‘polar coordinates’ introduced by the diagonalization (14),

\[
d[\sigma] = \Omega d[S] j^2(S) d\mu(U, V) \tag{19}
\]

where \( d[S] = \prod_{n=1}^{N_2} ds_n \). Here, the constant \( \Omega \), which depends on the convention adopted for the measure of the group, will be fixed later. The Jacobian is given by \( J(S) \equiv j^2(S) \)
with
\[ j(S) = \prod_{n=1}^{N_2} s_n^{N_1 - N_2 + 1/2} \Delta(\hat{S}^2) \]  
(20)
and \( \Delta(\hat{S}^2) \) is the Vandermonde determinant defined in (10). Because \( F \) is an invariant function, it satisfies a diffusion equation which involves only the radial part of the Laplacian:
\[ \sum_{n=1}^{N_2} \frac{1}{J(R) \partial r_n} J(R) \frac{\partial}{\partial r_n} F = 4 \frac{\partial F}{\partial t} . \]  
(21)
This equation can be simplified materially with the introduction of a reduced wave function,
\[ f(R, t) = j(R) F(R, t) . \]  
(22)
Now, (21) reduces to
\[ \sum_{n=1}^{N_2} \left( \frac{\partial^2}{\partial r_n^2} - \frac{1}{j(R)} \left[ \frac{\partial^2}{\partial r_n^2} j(R) \right] \right) f(R, t) = 4 \frac{\partial f}{\partial t} . \]  
(23)
Remarkably, this differential equation is separable.\(^1\) Performing the differentiations of \( j(R) \), it can be rewritten as
\[ \sum_{n=1}^{N_2} \left( \frac{\partial^2}{\partial r_n^2} - \frac{4\nu^2 - 1}{4} \frac{1}{r_n^2} \right) f(R, t) = 4 \frac{\partial f}{\partial t} . \]  
(24)
Because of the presence of the factor \( j(R) \) in (22), \( f(R, t) \) is an antisymmetric function of the eigenvalues. Therefore, the solution of (24) is given by an integral over a Slater determinant
\[ f(R, t) = \int d[S] \frac{1}{N_2!} \det_{k,l} |g(r_k, s_l; t)| j(S) \varphi(S) , \]  
(25)
where \( g(r, s; t) \) is the kernel of
\[ \frac{\partial^2}{\partial r^2} g - \frac{4\nu^2 - 1}{4r^2} g = 4 \frac{\partial g}{\partial t} . \]  
(26)
The kernel can be expressed in terms of the regular eigenfunctions of the Bessel equation of order \( \nu \):
\[ u'' + \left[ k^2 - (4\nu^2 - 1)/4r^2 \right] u = 0 . \]  
(27)
\(^1\)In \[\text{[5]}\], a separable equation was obtained for \( N_1 = N_2 \) by the substitution of \( \Delta(R^2) F(R, t) \) instead of \( j(R) F(R, t) \).
This is obtained following a separation of variables which leads to a time dependence of the form \( \exp(-k^2t/4) \). The regular eigenfunctions are given by

\[
u_k(r) \sim \sqrt{kr} J_\nu(kr),
\]

where \( J_\nu \) is a Bessel function. Recalling the orthogonality relation for Bessel functions, the kernel of (26) can be written as

\[
g(r, s; t) = \theta(t) \int_0^\infty dk k \sqrt{rs} e^{-k^2t/4} J_\nu(kr) J_\nu(ks).
\]

This can be evaluated to give

\[
g(r, s; t) = \theta(t) \frac{2}{t} \sqrt{rs} \exp\left(-\frac{r^2 + s^2}{t}\right) I_\nu\left(\frac{2rs}{t}\right).
\]

Finally, we equate the definition of \( F(\rho, t) \) in (17) with its expression in terms of the kernel (30) as given by (22) and (25). Since this equality is valid for an arbitrary choice of the initial condition \( \varphi(\sigma) \), the integrands of \( d[S] \) must be the same.

Hence, we arrive at

\[
\int d\mu(U,V) \exp\left(-\frac{1}{t} \text{tr}[(\sigma - \rho)^\dagger(\sigma - \rho)]\right)
= \frac{t^{N_1N_2-N_2} 2^{N_2} \pi^{N_1N_2} \Omega}{\prod_{k=1}^{N_2} (r_k s_k)^\nu} \frac{1}{N_2!} \det_{k,l} \exp\left(-\frac{r_k^2 + s_l^2}{t}\right) I_\nu\left(\frac{2r_k s_l}{t}\right) \left/\Delta(S^2)\Delta(R^2)\right.
\]

Here, the value of \( \Omega \) follows from the normalization integral calculated in [10]:

\[
\Omega = \frac{\pi^{N_1N_2} 2^{N_2}}{\prod_{j=1}^{N_2} j!(j + \nu - 1)!},
\]

where we have used the convention that \( \int d\mu(U, V) = 1 \).

This result enables us to calculate the Itzykson-Zuber integral for arbitrary complex matrices

\[
\int d\mu(U, V) \exp\left(\text{Re tr} U^\dagger S V R\right) = C_{N_1} C_{N_2} \prod_{k=0}^{\nu-1} \frac{1}{k!} \prod_{k=1}^{N_2} (r_k s_k)^\nu \det_{k,l} |I_\nu(r_k s_l)| \left/\Delta(S^2)\Delta(R^2)\right.
\]

When \( \nu = 0 \), the product of factorials in this expression is understood to be 1. The constants \( C_{N_1} \) and \( C_{N_2} \) can be evaluated to be

\[
C_n = 2^{n(n-1)/2} \prod_{k=1}^n (k - 1)!
\]

In the special case \( N_1 = N_2 \) for which \( \nu = 0 \), our expression reduces to the result of Guhr and Wettig [3] apart from a normalization constant.
4 The finite volume partition function for different masses

The finite volume effective partition function of QCD in the sector with topological charge \( \nu \), defined in (3) and (2), can be written as an integral over \( U(N_f) \) instead of \( SU(N_f) \) [2]

\[
Z_\nu = \int_{U(N_f)} d\mu(U) (\det U)^\nu \exp \left( V \Sigma \Re \text{tr} MU^\dagger \right). \tag{35}
\]

For \( \nu = 0 \), this result can be obtained from (33) by taking \( R = V \Sigma M \) and \( S = 1_{N_f} \). Because of the singularity as \( S \to 1 \), the final result requires a careful analysis of this limit.

In (33), we take \( S = 1 + \delta S \) and expand the modified Bessel functions to order \( (\delta S)^{N_f} \).

The result is

\[
I_\nu(r_k s_l) = \sum_{j=1}^{N_f} \frac{r_k^{j-1}}{(j-1)!} I_\nu^{(j-1)}(r_k) \delta s_l^{j-1} + \mathcal{O}(\delta S^{N_f}), \tag{36}
\]

which can be written as the product of the matrix \( A \) defined in (11) and the matrix

\[
B_{jl} = \frac{1}{(j-1)!} \delta s_l^{j-1}. \tag{37}
\]

The determinant of \( B \) can be written as

\[
\det(B) = \frac{1}{\prod_{k=1}^{N_f} (k-1)!} \Delta(\delta S) = \frac{1}{C_{N_f}} \Delta \left( (1 + \delta S)^2 \right), \tag{38}
\]

with the normalization constant defined in (34). Hence

\[
Z_\nu(M) = C_{N_f} \frac{\det A}{\Delta(R^2)}, \tag{39}
\]

which is simply the result conjectured in (12) above. This result is now proved for \( \nu = 0 \).

We have not proved the result (12) for arbitrary \( \nu \). However, we offer one non-trivial check of this conjecture by demonstrating that (12) reduces to the finite volume partition function of Leutwyler and Smilga in the limit of equal masses. We start with the expression

\[
A_{kj} = C_{N_f}^{1/N_f} r_k^{j-1} I_\nu^{(j-1)}(r_k). \tag{40}
\]

Using the recursion relation

\[
\frac{d I_\nu}{dr} = I_{\nu+1} + \frac{\nu}{r} I_\nu \tag{41}
\]
and adding a suitable multiple of the column to the left of the column in question, we arrive at the matrix

\[ A_{kj} \rightarrow C_{N_f}^{1/N_f} r_j^{j-1} I_{\nu+j-1}(r_k) . \]  

This is evidently correct in going from the first to the second column and, hence, true in general. The fact that the coefficient of \( I_{\nu+1} \) in (41) is 1 guarantees that the determinant will not be affected by this rearrangement.

To realize the limit \( r_k \to r \equiv mV\Sigma \), we write \( r_k = r + \delta r_k \) and expand each element in \( A \) in a Taylor series through order \( \delta r_k^{N_f-1} \). The result is that

\[ A_{kj} = C_{N_f}^{1/N_f} \sum_{p=1}^{N_f} \frac{1}{(p-1)!} \frac{d^{p-1}}{dr^{p-1}} \left[ r_j^{j-1} I_{\nu+j-1}(r) \right] \delta r_k^{p-1} . \]  

This can be written as the product of the matrix with elements

\[ M_{jp} = C_{N_f}^{1/N_f} \frac{d^{p-1}}{dr^{p-1}} \left[ r_j^{j-1} I_{\nu+j-1}(r) \right] \]  

and the matrix \( B_{jp} \equiv (\delta r_k)^{p-1}/(p-1)! \). As in (38), we have

\[ \det(B) = \prod_{p=1}^{N_f} \frac{1}{(p-1)!} \Delta(\delta R) = \frac{1}{C_{N_f}} \left( \frac{1}{r} \right)^{N_f(N_f-1)/2} \Delta((R + \delta R)^2) . \]  

The determinant of the matrix \( M \) can be simplified by using the recursion relation

\[ \frac{dI_{\nu}}{dr} = I_{\nu-1} - \frac{\nu}{r} I_{\nu} \]  

and adding a suitable multiple of the row immediately above the row in question. As before, this rearrangement does not alter the determinant. This leaves us with

\[ M_{kj} \rightarrow C_{N_f}^{1/N_f} r_j^{j-1} I_{\nu+j-k} . \]  

Now, the factor \( r^j \) can be extracted from each column and used to eliminate the \( r \) dependence in the prefactor. Thus, we arrive at the final result

\[ \det(A) = \det(M) \det(B) = \det_{k,j} I_{\nu+j-k} , \]  

which is precisely the result in [2] as given in (4).
5 Conclusions

We have obtained the finite volume QCD partition function for different quark masses in the range $1/\Lambda \ll L \ll 1/m_{\pi}$. This result generalizes the finite volume partition function obtained previously by Leutwyler and Smilga for the case of equal quark masses. In order to derive this result, we were led to generalize the Itzykson-Zuber integral to arbitrary rectangular complex matrices. The integral for square matrices, first obtained by Guhr and Wettig, leads immediately to the proof in the sector of zero topological charge. Based on the result for two flavors and the general result for $\nu = 0$, we have conjectured the result for arbitrary $\nu$ and $N_f$. As a decidedly nontrivial check of this conjecture, we have shown that the result of Leutwyler and Smilga emerges in the limit of equal quark masses.

We wish to note a remarkable coincidence. Consider the Itzykson-Zuber formula for complex rectangular matrices (with $\nu$ equal to the difference between the number of rows and columns) in the same limit considered for $\nu = 0$. Up to a factor, this leads to the finite-volume partition function in the sector of topological charge $\nu$. While we can offer no explanation of this coincidence, it may be useful to note that, in the chiral limit, the joint eigenvalue density of the random matrix model associated with the finite volume partition function depends only on the combination $\nu + N_f$ \[1\]. Clearly, more work lies ahead in this area.

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