Non-classical heat conduction problem with non local source

Mahdi Boukrouche∗ Domingo A. Tarzia†

Abstract

We consider the non-classical heat conduction equation, in the domain

\[ D = \mathbb{R}^{n-1} \times \mathbb{R}^+ \]

for which the internal energy supply depends on an integral function in the time variable of the heat flux on the boundary \( S = \partial D \), with homogeneous Dirichlet boundary condition and an initial condition. The problem is motivated by the modeling of temperature regulation in the medium. The solution to the problem is found using a Volterra integral equation of second kind in the time variable \( t \) with a parameter in \( \mathbb{R}^{n-1} \). The solution to this Volterra equation is the heat flux \((y,s) \mapsto V(y,t) = u_x(0,y,t)\) on \( S \), which is an additional unknown of the considered problem. We show that a unique local solution exists, which can be extended globally in time. Finally a one-dimensional case is studied with some simplifications, we obtain the solution explicitly by using the Adomian method and we derive its properties.

Keywords: Nonclassical n-dimensional heat equation, non local sources, Volterra integral equation, existence and uniqueness of solution, integral representation of solution, explicit solution in 1-dimensional case and its properties.

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1 Introduction

Let’s consider the domain \( D \) and its boundary \( S \) defined by

\[ D = \mathbb{R}^{n-1} \times \mathbb{R}^+ = \{(x,y) \in \mathbb{R}^n : x = x_1 > 0, \ y = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\}, \]
\[ S = \partial D = \mathbb{R}^{n-1} \times \{0\} = \{(x,y) \in \mathbb{R}^n : x = 0, \ y \in \mathbb{R}^{n-1}\}. \]

The aim of this paper is to study the following the problem 1.1 with the non-classical heat equation, in the domain \( D \) with non local source, for which the internal energy supply depends on the integral \( \int_0^t u_x(0,y,s)ds \) on the boundary \( S \).

Problem 1.1. Find the temperature \( u \), at \((x,y,t)\) such that it satisfies the following conditions

\[ u_t - \Delta u = -F \left( \int_0^t u_x(0,y,s)ds \right), \quad x = x_1 > 0, \ y \in \mathbb{R}^{n-1}, \ t > 0, \]
\[ u(0,y,t) = 0, \ y \in \mathbb{R}^{n-1}, \ t > 0, \]
\[ u(x,y,0) = h(x,y), \quad x > 0, \ y \in \mathbb{R}^{n-1}, \]

∗Address: Lyon University, F-42023 Saint-Etienne, Institut Camille Jordan CNRS UMR 5208, 23 rue Paul Michelon 42023 Saint-Étienne Cedex 2, France. Mahdi.Boukrouche@univ-st-etienne.fr
†Address: Departamento de Matemática-CONICET, FCE, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina. DTarzia@austral.edu.ar
where $\Delta$ denotes the Laplacian in $\mathbb{R}^n$. This problem is motivated by the modeling of temperature regulation in an isotropic medium, with non-uniform and non-local sources that provide cooling or heating system. According to the properties of the function $F$ with respect to the heat flow $V(y, s) = u_x(0, y, s)$ at the boundary $S$. For example, assuming that

$$V F(V) > 0, \quad \forall V \neq 0, \quad F(0) = 0,$$

with

$$F(V(y, t)) = F \left( \int_0^t V(y, s) ds \right)$$

then, see \[12, 14\], the cooling source occurs when $V(y, t) > 0$ and heating source occurs when $V(y, t) < 0$.

Some references on the subject are \[8\] where $F(V) = F(V)$, \[5, 15, 27, 28\] where the following semi-one-dimension of this nonlinear problem, have been considered. The non-classical one-dimensional heat equation in a slab with fixed or moving boundaries was studied in \[9, 10, 11, 25\]. More references on the subject can be found in \[13, 18, 19, 21, 22\]. To our knowledge, it is the first time that the solution to a non-classical heat conduction of the type of Problem 1.1 is given. Other non-classical problems can be found in \[6\].

The goal of this paper is to obtain in Section 2 the existence and uniqueness of the global solution of the non-classical heat conduction Problem 1.1, which is given through a Volterra integral equation. In Section 3 we obtain the explicit solution of the one-dimensional case of Problem 1.1 with some simplifications, which is obtained by using the Adomian method through a double induction principle.

We recall here the Green’s function for the n-dimensional heat equation with homogeneous Dirichlet’s boundary conditions, given the following expression \[17, 23\]

$$G_1(x, y, t; \xi, \eta, \tau) = \frac{\exp \left[ -\frac{\| y-\eta \|^2}{4(t-\tau)} \right]}{(2\sqrt{\pi(t-\tau)})^{n-1}} G(x, t, \xi, \tau),$$

where $G$ is the Green’s function for the one-dimensional case given by

$$G(x, t, \xi, \tau) = \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}} \quad t > \tau.$$

## 2 Existence results

In this Section, we give first in Theorem 2.1, the integral representation (2.1) of the solution of the considered Problem 1.1, but it depends on the heat flow $V$ on the boundary $S$, which satisfies the Volterra integral equation (2.3) with initial condition (2.4). Then we prove, in Theorem 2.3, under some assumptions on the data, that there exists a unique solution of the Problem 1.1 locally in times which can be extended globally in times.

**Theorem 2.1.** The integral representation of a solution of the considered Problem 1.1 is given by the following expression

$$u(x, y, t) = u_0(x, y, t) - \int_0^t \frac{\text{erf} \left( \frac{x-\eta}{2\sqrt{t-\tau}} \right)}{(2\sqrt{\pi(t-\tau)})^{n-1}} \left[ \int_{\mathbb{R}^{n-1}} \exp \left[ -\frac{\| y-\eta \|^2}{4(t-\tau)} \right] F(V(\eta, \tau)) d\eta \right] d\tau \quad (2.1)$$
where

\[
\text{erf}(\zeta) = \left(\frac{2}{\sqrt{\pi}} \int_0^\zeta e^{-X^2}dX\right)
\]

is the error function, with

\[
u_0(x, y, t) = \int_D G_1(x, y, t; \xi, \eta, 0) h(\xi, \eta) d\xi d\eta
\]

(2.2)

and the heat flux \(V(y, t) = u_x(0, y, t)\) on the surface \(x = 0\), satisfies the following Volterra integral equation

\[
V(y, t) = V_0(y, t) - 2 \int_0^t \frac{1}{(2\sqrt{\pi(t-\tau)})^n} \left[ \int_{\mathbb{R}^{n-1}} \exp \left[ -\frac{||y-\eta||^2}{4(t-\tau)} \right] F(\eta, \tau) d\eta \right] d\tau
\]

(2.3)

in the variable \(t > 0\), with \(y \in \mathbb{R}^{n-1}\) is a parameter and

\[
V_0(y, t) = \int_D G_{1,x}(0, y, t; \xi, \eta, 0) h(\xi, \eta) d\xi d\eta,
\]

(2.4)

where the function \((y, t) \mapsto F(V(y, t))\) is defined by (1.4) for \(y \in \mathbb{R}^{n-1}\) and \(t > 0\).

**Proof.** As the boundary condition in Problem (1.1) is homogeneous, we have from (1.1)

\[
u(x, y, t) = \int_D G_1(x, y, t; \xi, \eta, 0) h(\xi, \eta) d\xi d\eta
\]

+ \( \int_0^t \int_D G_1(x, y, t; \xi, \eta, \tau)[-F(V(\eta, \tau))] d\xi d\eta d\tau, \)

(2.5)

and therefore

\[
u_x(x, y, t) = \int_D G_{1,x}(x, y, t; \xi, \eta, 0) h(\xi, \eta) d\xi d\eta
\]

+ \( \int_0^t \int_D G_{1,x}(x, y, t; \xi, \eta, \tau)[-F(V(\eta, \tau))] d\xi d\eta d\tau. \)

(2.6)

From (1.3) (the definition of \(G_1\)) by derivation with respect to \(x\), then taking \(x = 0\) we obtain

\[
\int_D G_{1,x}(0, y, t; \xi, \eta, \tau) F(V(\eta, \tau)) d\xi d\eta = \int_{\mathbb{R}^{n-1}} \frac{F(V(\eta, \tau)) e^{-\frac{||y-\eta||^2}{4(t-\tau)}}}{(t-\tau)^{n+2} (2\sqrt{\pi})^n} \left( \int_0^{\infty} \xi e^{-\frac{\xi^2}{4(t-\tau)}} d\xi \right) d\eta
\]

\[
= \frac{2}{(2\sqrt{\pi(t-\tau)})^n} \int_{\mathbb{R}^{n-1}} F(V(\eta, \tau)) e^{-\frac{||y-\eta||^2}{4(t-\tau)}} d\eta,
\]

(2.7)

as

\[
\int_0^{\infty} \xi e^{-\frac{\xi^2}{4(t-\tau)}} d\xi = 2(t-\tau).
\]

Thus taking \(x = 0\) in (2.6) with (2.7) we get (2.3).
Also by (1.5) we obtain

\[
\int_D G_1(x, y, t; \xi, \eta, \tau)F(V(\eta, \tau))d\xi d\eta = \frac{1}{(2(\sqrt{\pi(t-\tau)})^n} \times \\
\times \int_D e^{-\|y-\eta\|^2/(4(t-\tau))} \left[ e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}} \right] F(V(\eta, \tau))d\xi d\eta \\
= \frac{1}{(2(\sqrt{\pi(t-\tau)})^n} \int_{\mathbb{R}^+} \left[ e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}} \right] d\xi \int_{\mathbb{R}^{n-1}} e^{-\|y-\eta\|^2/(4(t-\tau))} F(V(\eta, \tau))d\eta
\]

and by using

\[
\int_0^{+\infty} e^{-\xi^2/(4(t-\tau))} d\xi = 2\sqrt{\frac{\pi(t-\tau)}{t}} \left( \int_{-\infty}^{0} e^{-X^2} dX + \int_{0}^{+\infty} e^{-X^2} dX \right) \\
= \sqrt{\pi(t-\tau)} \left( 1 + \text{erf} \left( \frac{x}{2\sqrt{t-\tau}} \right) \right)
\]

and

\[
\int_0^{+\infty} e^{-\xi^2/(4(t-\tau))} d\xi = 2\sqrt{\frac{\pi(t-\tau)}{t}} \left( \int_{0}^{+\infty} e^{-X^2} dX - \int_{0}^{+\infty} e^{-X^2} dX \right) \\
= \sqrt{\pi(t-\tau)} \left( 1 - \text{erf} \left( \frac{x}{2\sqrt{t-\tau}} \right) \right)
\]
we get

\[
\int_D G_1(x, y, t; \xi, \eta, \tau)F(V(\eta, \tau))d\xi d\eta = \frac{\text{erf} \left( \frac{x}{2\sqrt{t-\tau}} \right)}{(2\sqrt{\pi(t-\tau)})^{n/2}} \int_{\mathbb{R}^{n-1}} e^{-\|y-\eta\|^2/(4(t-\tau))} F(V(\eta, \tau))d\eta.
\]

Taking this formula in (2.3) we obtain (2.1). \(\square\)

To solve the Volterra integral equation (2.3), we rewrite it in the suitable form

**Lemma 2.2.** The Volterra integral equation (2.3) can be rewrite in the following form

\[
V(y, t) = \frac{1}{t(2\sqrt{\pi t})^n} \int_{\mathbb{R}^+} \frac{\xi e^{-\xi^2/(4(t-\tau))}}{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}^{n-1}} e^{-\|\xi-\eta\|^2/(4(t-\tau))} h(\xi, \eta) d\eta \right) d\xi \\
- \frac{2}{(2\sqrt{\pi t})^n} \int_0^t \frac{1}{(t-\tau)^{n/2}} \int_{\mathbb{R}^{n-1}} F(V(\eta, \tau))e^{-\|y-\eta\|^2/(4(t-\tau))} d\eta d\tau.
\] (2.8)

**Proof.** Using the derivative, with respect to \(x\), of (1.5), then taking \(x = 0\) and \(\tau = 0\), then taking the new expression of \(V_0(y, t)\) in the Volterra integral equation (2.3) we obtain (2.8). \(\square\)

**Theorem 2.3.** Assume that \(h \in C(D)\), \(F \in C(\mathbb{R})\) and locally Lipschitz in \(\mathbb{R}\), then there exists a unique solution of the problem locally in times which can be extended globally in times.
Proof. We know from Theorem (2.1) that, to prove the existence and uniqueness of the solution (2.1) of Problem (1.1), it is enough to solve the Volterra integral equation (2.8). So we rewrite it again as follows

\[ V(y, t) = f(y, t) + \int_0^t g(y, \tau, V(y, \tau))d\tau \]  

(2.9)

with

\[ f(y, t) = \frac{1}{t(2\sqrt{\pi})^n} \int_{\mathbb{R}^+} \xi e^{-\frac{\xi^2}{4t}} \left( \int_{\mathbb{R}^{n-1}} e^{-\frac{|y-\eta|^2}{4t}} h(\xi, \eta) d\eta \right) d\xi \]  

(2.10)

and

\[ g(t, \tau, y, V(y, \tau)) = -\frac{2(t - \tau)^{-n/2}}{(2\sqrt{\pi})^n} \int_{\mathbb{R}^{n-1}} F(V(\eta, \tau))e^{-\frac{|y-\eta|^2}{4(t-\tau)}} d\eta. \]  

(2.11)

We have to check the conditions H1 to H4 in Theorem 1.1 page 87, and H5 and H6 in Theorem 1.2 page 91 in [24].

- The function \( f \) is defined and continuous for all \((y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^+\), so H1 holds.
- The function \( g \) is measurable in \((t, \tau, y, x)\) for \(0 \leq \tau \leq t < +\infty, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}\), and continuous in \(x\) for all \((y, t, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}^+, g(y, t, \tau, x) = 0\) if \(\tau > t\), so here we need the continuity of

\[ V(\eta, \tau) \mapsto F(V(\eta, \tau)) = F \left( \int_0^\tau V(\eta, s)ds \right), \]

which follows from the hypothesis that \(F \in \mathcal{C}(\mathbb{R})\). So H2 holds.
- For all \(k > 0\) and all bounded set \(B \subset \mathbb{R}\), we have

\[
|g(y, t, \tau, X)| \leq \frac{2}{(2\sqrt{\pi})^n} \sup_{X \in B} |F(X)|(t - \tau)^{-n/2} \int_{\mathbb{R}^{n-1}} e^{-\|y-\eta\|^2/(4(t-\tau))} d\eta \\
\leq \frac{2}{(2\sqrt{\pi})^n} \sup_{X \in B} |F(X)|(t - \tau)^{-n/2}(2\sqrt{\pi}(t-\tau))^{n-1} \\
= \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \frac{1}{\sqrt{(t-\tau)}}
\]

thus there exists a measurable function \(m\) given by

\[ m(t, \tau) = \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \frac{1}{\sqrt{(t-\tau)}} \]  

(2.12)

such that

\[ |g(y, t, \tau, X)| \leq m(t, \tau) \quad \forall 0 \leq \tau \leq t \leq k, \quad X \in B \]  

(2.13)

and satisfies

\[
\sup_{t \in [0, K]} \int_0^t m(t, \tau)d\tau = \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \sup_{t \in [0, k]} \int_0^t \frac{1}{\sqrt{t-\tau}}d\tau \\
= \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \sup_{t \in [0, k]} \left(-2\sqrt{(t-\tau)}\right)_{0}^{t} \\
= \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \sup_{t \in [0, k]} \leq 2\sqrt{\frac{k}{\pi}} \sup_{X \in B} |F(X)| < \infty,
\]

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so $H3$ holds.

- Moreover we have also

$$
\lim_{t \to 0^+} \int_0^t m(t, \tau) d\tau = \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \lim_{t \to 0^+} \int_0^t \frac{d\tau}{\sqrt{t - \tau}} = \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \lim_{t \to 0^+} (2\sqrt{t}) = 0,
$$

and

$$
\lim_{t \to 0^+} \int_0^{T+t} m(t, \tau) d\tau = \frac{1}{\sqrt{\pi}} \sup_{X \in B} |F(X)| \lim_{t \to 0^+} (2\sqrt{t}) = 0.
$$

- For each compact subinterval $J$ of $\mathbb{R}^+$, each bounded set $B$ in $\mathbb{R}^{n-1}$, and each $t_0 \in \mathbb{R}^+$, we set

$$
\mathcal{A}(t, y, V(\eta)) = |g(t, \tau; y, V(\eta)) - g(t_0, \tau; y, V(\eta))|.
$$

$$
\mathcal{A}(t, y, V(\eta)) = \frac{2}{(2\sqrt{\pi})^n} \int_J \left| \int_{\mathbb{R}^{n-1}} e^{-\frac{|y-\eta|^2}{4(t-\tau)}} F(V(\eta, \tau)) (t-\tau)^{-n/2} - e^{-\frac{|y-\eta|^2}{4(t_0-\tau)}} F(V(\eta, \tau)) (t_0-\tau)^{-n/2} d\eta \right| d\tau
$$

as the function $\tau \mapsto V(\eta, \tau)$ is continuous then

$$
\tau \mapsto \int_0^\tau V(\eta, s) ds
$$

is $C^1(\mathbb{R})$ and is in the compact $B \subset \mathbb{R}$ for all $\eta \in \mathbb{R}^{n-1}$, so by the continuity of $F$ we get $F(V(\eta, \tau)) \subset F(B)$, that is there exists $M > 0$ such that $|F(V(\eta, \tau))| \leq M$ for all $(\eta, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$. So

$$
\sup_{V(\eta) \in \mathcal{C}(J,B)} \mathcal{A}(t, y, V(\eta)) \leq \frac{2M}{(2\sqrt{\pi})^n} \sup_{V(\eta) \in \mathcal{C}(J,B)} \left| \int_{\mathbb{R}^{n-1}} \frac{e^{-\frac{|y-\eta|^2}{4(t-\tau)}}}{\sqrt{(t-\tau)^n}} - \frac{e^{-\frac{|y-\eta|^2}{4(t_0-\tau)}}}{\sqrt{(t_0-\tau)^n}} d\eta \right|
$$

using that

$$
\int_{\mathbb{R}^{n-1}} \exp\left[\frac{-||y-\eta||^2}{4(t-\tau)}\right] d\eta = \left(2\sqrt{\pi(t-\tau)}\right)^{n-1}
$$

we obtain

$$
\sup_{V(\eta) \in \mathcal{C}(J,B)} \mathcal{A}(t, y, V(\eta)) \leq \frac{2M}{(2\sqrt{\pi})^n} \sup_{V(\eta) \in \mathcal{C}(J,B)} \left| \frac{(2\sqrt{\pi(t-\tau)})^{n-1}}{(\sqrt{t-\tau})^n} - \frac{(2\sqrt{\pi(t_0-\tau)})^{n-1}}{(\sqrt{t_0-\tau})^n} \right|
$$

thus

$$
\sup_{V(\eta) \in \mathcal{C}(J,B)} \mathcal{A}(t, y, V(\eta)) \leq \frac{M}{\sqrt{\pi}} \sup_{V(\eta) \in \mathcal{C}(J,B)} \left| \sqrt{t_0-\tau} - \sqrt{t-\tau} \right| \sqrt{t-\tau(t_0-\tau)}.
$$
Thus we deduce that

$$\lim_{t \to t_0} \int_J \sup_{V(\eta) \in C(J,B)} A(t, y, V(\eta)) d\tau = 0.$$ 

So H4 holds.

- For all compact $I \subset \mathbb{R}^+$, for all function $\psi \in C(I, \mathbb{R}^n)$, and all $t_0 > 0$,

$$|g(t, \tau; \psi(\tau)) - g(t_0, \tau, \psi(\tau))| = \frac{2}{(2\sqrt{\pi})^n} \left| \int_{\mathbb{R}^{n-1}} \mathcal{F}(\psi(\tau)) \left( e^{-\frac{|y-n|^2}{4(t-\tau)}} - e^{-\frac{|(y-n)^2}{4(t_0-\tau)}} \right) d\eta \right|$$

as $\mathcal{F} \in C(\mathbb{R})$ and $\psi \in C(I, \mathbb{R}^n)$ then there exists a constant $M > 0$ such that $|\mathcal{F}(\psi(\tau))| \leq M$ for all $\tau \in I$. Then we obtain as for H4, that

$$\lim_{t \to t_0} \int_I |g(t, \tau; \psi(\tau)) - g(t_0, \tau, \psi(\tau))| d\tau = 0.$$ 

So H5 holds.

- Now for each constant $K > 0$ and each bounded set $B \subset \mathbb{R}^{n-1}$ there exists a measurable function $\varphi$ such that

$$|g(y, t, \tau, x) - g(y, t, \tau, X)| \leq \varphi(t, \tau)|x - X|$$

whenever $0 \leq \tau \leq t \leq K$ and both $x$ and $X$ are in $B$. Indeed as $F$ is assumed locally Lipschitz function in $\mathbb{R}$ there exists constant $L > 0$ such that

$$|\mathcal{F}(x) - \mathcal{F}(X)| \leq L(\tau)|x - X| \quad \forall (x, X) \in B^2$$

with $L(\tau) = L\tau$. Then we have

$$|g(y, t, \tau, x) - g(y, t, \tau, X)| = \frac{2}{(2\sqrt{\pi})^n} \left| \int_{\mathbb{R}^{n-1}} (t - \tau)^{-n/2} e^{-\frac{|y-n|^2}{4(t-\tau)}} (\mathcal{F}(x) - \mathcal{F}(X)) d\eta \right|$$

\[\leq \frac{2}{(2\sqrt{\pi})^n} \left( \int_{\mathbb{R}^{n-1}} e^{-\frac{|y-n|^2}{4(t-\tau)}} d\eta \right) (t - \tau)^{-n/2} L\tau|x - X| \leq \frac{L\tau}{\sqrt{\pi(t-\tau)}}|x - X|,\]

then $\varphi(t, \tau) = \frac{L\tau}{\sqrt{\pi(t-\tau)}}$. We have also for each $t \in [0, k]$ the function $\varphi \in L^1(0, t)$ as a function of $\tau$ and

$$\int_t^{t+l} \varphi(t + \tau) d\tau = \frac{L}{\sqrt{\pi}} \int_t^{t+l} \frac{\tau d\tau}{\sqrt{t + l - \tau}} = \frac{L}{\sqrt{\pi}} \int_1^0 (u^2 - t - l) du = \frac{Ll}{\sqrt{\pi}} (l + t - \frac{1}{3}) \to 0 \quad \text{with } l \to 0$$

where $u = \sqrt{t + l - \tau}$.

So H6 holds. All the conditions H1 to H6 are satisfied with (2.14) and (2.15).

Thus from [24] (Theorem 1.1 page 87, Theorem 1.2 page 91 and Theorem 2.3 page 97) there exists a unique local times solution of the Volterra integral equation (2.3) which can be extended globally in times. Then the proof of this theorem is complete. ☐
The one-dimensional case of Problem 1.1

Let us consider now the one dimensional case of Problem 1.1 for the temperature defined by Problem 3.1. Find the temperature $u$ at $(x,t)$ such that it satisfies the following conditions

$$u_t - u_{xx} = -F \left( \int_0^t u_x(0,s)ds \right), \quad x > 0, \quad t > 0,$$

$$u(0,t) = 0, \quad t > 0,$$

$$u(x,0) = h(x), \quad x > 0.$$ 

Taking into account that

$$\int_0^t G(x,t,\xi,\tau)d\xi = \text{erf} \left( \frac{x}{2\sqrt{t-\tau}} \right)$$

thus the solution of the problem 3.1 is given by

$$u(x,t) = u_0(x,t) - \int_0^t \text{erf} \left( \frac{x}{2\sqrt{t-\tau}} \right) F \left( \int_0^\tau W(\sigma)d\sigma \right) d\tau$$

with

$$u_0(x,t) = \int_0^t G(x,t,\xi,0)h(\xi)d\xi$$

and $W(t) = u_x(0,t)$ is the the solution of the following Volterra integral equation

$$W(t) = V_0(t) - \int_0^t \sqrt{\pi(t-\tau)} F \left( \int_0^\tau W(\sigma)d\sigma \right) d\tau$$

where

$$V_0(t) = \frac{1}{2\sqrt{\pi t^{3/2}}} \int_0^{+\infty} \xi e^{-\xi^2/4t}h(\xi)d\xi = \frac{2}{\sqrt{\pi t}} \int_0^{+\infty} \eta e^{-\eta^2} h(2\sqrt{t}\eta)d\eta.$$ 

For the particular case

$$h(x) = h_0 > 0 \text{ for } x > 0, \text{ and } F(W) = \lambda W \text{ for } \lambda \in \mathbb{R},$$

then we have

$$u_0(t,x) = h_0 \text{erf} \left( \frac{x}{2\sqrt{t}} \right)$$

and the integral equation 3.4 becomes

$$W(t) = \frac{h_0}{\sqrt{\pi t}} - \lambda \int_0^t \frac{\int_0^\tau W(\sigma)d\sigma}{\sqrt{\pi(t-\tau)}} d\tau.$$
Lemma 3.1. Assume (3.6) holds. The solution of problem 3.1 is given by
\[ u(x,t) = h_0 \text{erf} \left( \frac{x}{2\sqrt{t}} \right) - \lambda \int_0^t \text{erf} \left( \frac{x}{2\sqrt{t-\tau}} \right) U(\tau)d\tau \] (3.9)
where \( U \) is given by
\[ U(t) = \frac{h_0}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} d\tau \] (3.10)
and \( g \) is the solution of the Volterra integral equation
\[ g(t) = 1 - \frac{2\lambda}{\sqrt{\pi}} \int_0^t g(\tau) \sqrt{t-\tau} d\tau. \] (3.11)
Moreover, the heat flux on \( x = 0 \) is given by
\[ u_x(0,t) = \frac{h_0}{\sqrt{\pi t}} - h_0 \lambda \int_0^t g(\tau)d\tau, \quad t > 0. \] (3.12)

Proof. We set
\[ U(t) = \int_0^t W(\tau)d\tau \] (3.13)
thus the function \( U \) satisfies the following new Volterra integral equation
\[ U(t) = 2h_0 \sqrt{\frac{t}{\pi}} - \frac{\lambda}{\sqrt{\pi}} \int_0^t \int_0^\tau \frac{U(\sigma)}{\sqrt{\tau-\sigma}} d\sigma d\tau \]
\[ = 2h_0 \sqrt{\frac{t}{\pi}} - \frac{2\lambda}{\sqrt{\pi}} \int_0^t U(\tau) \sqrt{t-\tau} d\tau, \quad t > 0 \] (3.14)
by using the following equality
\[ \int_\sigma^t \frac{d\tau}{\sqrt{\tau-\sigma}} = 2\sqrt{t-\sigma}, \quad 0 < \sigma < t. \] (3.15)

From [3], p.229, the solution \( t \mapsto U(t) \) of the integral equation (3.14) is given by (3.10) where \( g \) is the solution of the Volterra equation (3.11).

From (3.11) we obtain that
\[ \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} d\tau = 2\sqrt{t} - \lambda \int_0^t g(\tau) \sqrt{t-\tau} d\tau \] (3.16)
using the following equality
\[ \int_\sigma^t \frac{\sqrt{\tau-\sigma}}{\sqrt{t-\tau}} d\tau = (t-\sigma) \int_0^1 \frac{\sqrt{\xi}}{\sqrt{1-\xi}} d\xi = (t-\sigma)B\left(\frac{3}{2}, \frac{1}{2}\right) \]
\[ = (t-\sigma) \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\pi}{2} (t-\sigma) \] (3.17)
where \( B \) and \( \Gamma \) are the classical Beta and Gamma functions defined below.

Therefore, we have that
\[ U(t) = 2h_0 \sqrt{\frac{t}{\pi}} - \lambda h_0 \int_0^t g(\tau)(t-\tau)d\tau \] (3.18)
and then the heat flux on \( x = 0 \) is given by \( u_x(0,t) = W(t) = U'(t) \), that is (3.12) holds. 

9
We recall here the well known Beta an Gamma functions defined respectively by

\[ B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \quad x > 0, \quad y > 0, \]
\[ \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \quad x > 0, \]

We will use in the next theorem the well known relations

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(x+1) = x\Gamma(x) \quad \forall x > 0, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(n+1) = n! \quad \forall n \in \mathbb{N}, \]

and in particular the following one

**Lemma 3.2.** For all integer \( n \geq 1 \) we have

\[ \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{n}, \]

and we use the definition

\[ (2n-1)!! = (2n-1)(2n-3)(2n-5) \cdots 5 \cdot 3 \cdot 1 \]

for compactness expression.

**Proof.** For \( n = 1 \) we get \( \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \) which is true. By induction we obtain that

\[ \Gamma\left(n+1+\frac{1}{2}\right) = \Gamma\left(n+\frac{1}{2}\right) + 1 = \left(n+\frac{1}{2}\right) \frac{(2n-1)!!}{2^n} \sqrt{\pi} = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\pi}, \]

thus the lemma is true. \( \square \)

**Corollary 3.3.** For all integer \( n \geq 0 \) we have also

\[ \Gamma\left(3n+5+\frac{1}{2}\right) = \frac{(6n+9)!!}{2^{3n+5}} \sqrt{\pi} \]

\[ B\left(\frac{3}{2}, 3n+4\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3(n+1)+1)}{\Gamma(3n+5+\frac{1}{2})} = \frac{(3(n+1))! 2^{3(n+1)+1}}{(6n+9)!!} \]

\[ B\left(\frac{3}{2}, 3n+\frac{5}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3n+\frac{5}{2})}{\Gamma(3(n+1)+1)} = \frac{\pi(6n+3)!!}{(3(n+1))! 2^{3(n+1)}} \]

which will be useful in the next Lemma.

We need previously some preliminary simple results in order to obtain the solution of the integral equation (2.1)

\[ \int_0^t \sqrt{t-\tau} d\tau = \frac{2}{3} t^{3/2}, \quad \int_0^t t^{3/2} \sqrt{t-\tau} d\tau = \frac{\pi}{24} t^3, \quad (3.19) \]
\[
\int_0^t \tau^3 \sqrt{t-\tau} d\tau = \frac{2^4 3!}{9!!} t^{9/2}, \quad \int_0^t \tau^{9/2} \sqrt{t-\tau} d\tau = \frac{\pi 9!!}{2^{9}6!} t^6, \quad (3.20)
\]

\[
\int_0^t \tau^6 \sqrt{t-\tau} d\tau = \frac{2^7 6!}{15!!} t^{15/2}, \quad \int_0^t \tau^{15/2} \sqrt{t-\tau} d\tau = \frac{\pi 15!!}{2^{9}9!} t^{9}, \quad (3.21)
\]

which can be generalized by the following ones:

**Lemma 3.4.** For all integer \( n \geq 0 \) we have

\[
\int_0^t \tau^{2n+3} \sqrt{t-\tau} d\tau = \frac{2^{3n+4} (3(n + 1))_!}{(6n + 9)!!} t^{3(2n+3)/2}, \quad (3.22)
\]

\[
\int_0^t \tau^{3(2n+1)+1} \sqrt{t-\tau} d\tau = \frac{\pi (6n + 3)!!}{2^{3(n+1)}(3(n + 1))!} t^{3(n+1)} \quad (3.23)
\]

**Proof.** Taking the change of variable \( \tau = t \xi \) in (3.22) using Corollary 3.3 we get

\[
\int_0^t \tau^{2n+3} \sqrt{t-\tau} d\tau = t^{3(2n+3)/2} \int_0^1 \xi^{3n+3} (1 - \xi)^{3/2} d\xi = t^{3(2n+3)/2} \int_0^1 \xi^{3(n+4)-1} (1 - \xi)^{3/2-1} d\xi
\]

\[
= t^{3(2n+3)/2} B \left( \frac{3}{2}, 3n + 4 \right) = \frac{(3n+1)!! 2^{3(n+1)+1}}{(6n + 9)!!} t^{3(n+1)}
\]

and

\[
\int_0^t \tau^{3(2n+1)+1} \sqrt{t-\tau} d\tau = t^{3(2n+1)/2} \int_0^1 \xi^{3(n+4)+1} (1 - \xi)^{3/2-1} d\xi
\]

\[
= t^{3(2n+1)/2} B \left( \frac{3}{2}, 3n + 5 \right)
\]

\[
= \frac{\pi (6n + 3)!!}{(3(n + 1))! 2^{3(n+1)}(3(2n + 1))!!} t^{3(n+1)}
\]

thus the (3.22)-(3.23) hold. \( \square \)

Now, we will obtain the explicit solution of the integral equation:

\[
y(t) = 1 - \frac{2\lambda}{\sqrt{\pi}} \int_0^t y(\tau) \sqrt{t-\tau} d\tau, \quad t > 0, \quad (3.24)
\]

by using Adomian decomposition method [1, 2, 30] through a serie expansion.

**Theorem 3.5.** The solution of the integral equation (3.24) is given by the following expression

\[
y(t) = I(t) - \sqrt{\frac{2}{\pi}} J(t), \quad t > 0, \quad (3.25)
\]

with

\[
I(t) = \sum_{n=0}^{+\infty} \frac{(\lambda^{2/3} t)^{3n}}{(3n)!} \quad (3.26)
\]

and

\[
J(t) = \sum_{n=0}^{+\infty} \frac{(\lambda^{2/3} t)^{3(2n+1)/2}}{(3(2n+1)))!!} \quad (3.27)
\]

are series with infinite radii of convergence.
Proof. Following the idea of \cite{3,7,16,20,26,29,31,32,33} we propose, for the solution of the integral equation (3.24), the following series of expansion functions given by

\[ y(t) = \sum_{n=0}^{+\infty} y_n(t), \]  

and we obtain the following recurrence expressions

\[ y_0(t) = 1, \quad y_n(t) = -\frac{2\lambda}{\sqrt{\pi}} \int_{0}^{t} y_{n-1}(\tau) \sqrt{t - \tau} \, d\tau, \quad \forall n \geq 1. \]  

Then we get

\[ y_1(t) = -\frac{2\lambda}{\sqrt{\pi}} \int_{0}^{t} \sqrt{t - \tau} \, d\tau = -\frac{4\lambda}{3\sqrt{\pi}} \tau^{3/2} = -\frac{\sqrt{2} \lambda^{2/3} t^{3/2}}{3!!}, \]  

\[ y_2(t) = -\frac{2\lambda}{\sqrt{\pi}} \int_{0}^{t} \left( -\frac{4\lambda}{3\sqrt{\pi}} \tau^{3/2} \right) \sqrt{t - \tau} \, d\tau = \frac{8\lambda^2}{3\pi} \int_{0}^{t} \tau^{3/2} \sqrt{t - \tau} \, d\tau = \frac{\lambda^2 \sqrt{3}}{3!!}. \]  

The first step of the double induction principle is just verified by (3.25) taking into account (3.30) (3.31). The second step, we suppose by induction hypothesis that we have

\[ J_{2n}(t) = \frac{\lambda^{2n}}{(3n)!} t^{3n}, \quad J_{2n+1}(t) = -\frac{2^{3n+2}}{(3(2n+1))!!} \frac{\lambda^{2n+1}}{\sqrt{\pi}} t^{3(2n+1)} \]  

Therefore, we obtain

\[ J_{2n+2}(t) = -\frac{2\lambda}{\sqrt{\pi}} \int_{0}^{t} y_{2n+1}(\tau) \sqrt{t - \tau} \, d\tau = \frac{\lambda^{2n+2}}{\pi} \frac{2^{3n+3}}{(6n+3)!!} \int_{0}^{t} \tau^{3(2n+3)} \sqrt{t - \tau} \, d\tau \]

\[ = \frac{\lambda^{2n+2}}{\pi} \frac{2^{3(n+1)}}{(6n+3)!!} \frac{\lambda^{3(n+1)}}{(3(n+1))!} \]  

\[ = \frac{\lambda^{2n+2}}{(3(n+1))!} t^{3(n+1)} \]  

and

\[ Y_{2n+3}(t) = -\frac{2\lambda}{\sqrt{\pi}} \int_{0}^{t} y_{2n+2}(\tau) \sqrt{t - \tau} \, d\tau = -\frac{2\lambda^{2n+3}}{(3(n+1))!} \frac{\sqrt{\pi}}{\pi} \int_{0}^{t} \tau^{3n+3} \sqrt{t - \tau} \, d\tau \]

\[ = -\frac{(3(n+1))!! \sqrt{\pi}}{(6n+9)!!} \frac{\lambda^{2n+3}}{(3(n+1))!} t^{3n+3} \]

\[ = -\frac{2^{3(n+1)+2}}{(3(n+1))!} \sqrt{\pi} \frac{\lambda^{2n+3} t^{3n+3}}{2} \]

This ends the proof. \(\square\)
Remark 3.6. Taking $t \to 0^+$ in (3.14), (3.12), and (3.11), we obtain

\begin{align*}
W'(0^+) &= +\infty, & W'(0^+) &= -\infty, \\
U(0^+) &= 0, & U'(0^+) &= +\infty, \\
g(0^+) &= 1, & g'(0^+) &= 0.
\end{align*}

So we deduce that the heat flux $W$ and the total heat flux $U$, and also $g$ are positive functions in a neighbourhood of $t = 0$.

Conclusion: We have obtained the global solution of a non-classical heat conduction problem in a semi-$n$-dimensional space. Moreover, for the one-dimensional case we have obtained the explicit solution by using the Adomian method with a double induction principle.

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