On Asymptotic Distributions of Estimators Concerned with Fuzzy Random Data as Vague Perceptions of Crisp Random

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Abstract

In this paper, the author investigates the asymptotic distributions of estimators concerned with the class of fuzzy random sets, which is considered as a model of the capricious vague perceptions of a crisp random phenomenon.

First, the class of fuzzy random sets, which has been proposed by Kratzschmer[1, 2, 3, 4, 5], Kwakernaak[6], and the revised one previously proposed by the author as a model of the capricious vague perceptions of a crisp random phenomenon.

The theoretical aspect of this research for investigating asymptotic properties of proposed estimators is mainly inspired by the papers of Kratzschmer[1, 2, 3, 4, 5].

Keywords: fuzzy random sets, capricious vague perception, expectation, strong law of large numbers

1 Metric Spaces of Fuzzy Sets

The class of fuzzy sets adopted in this paper is inspired by Kwakernaak[6], Kruse and Meyer[7], Kruse, Gebhardt and Klawonn[8], and the revised one previously proposed by the author[9, 10, 11, 12].

Let $\mathbb{I}$ be the open interval between 0 and 1, i.e., $\mathbb{I} = (0, 1)$ and $\bar{\mathbb{I}} = [0, 1]$. A fuzzy set $\tilde{U}(u_o)$ as the vague perception of $u_o \in \mathbb{R}^n$ is defined by the triple $\tilde{U}(u_o) = (\mathbb{R}^n, [\tilde{U}(u_o)], s_{\mathbb{I}})$ with $[\tilde{U}(u_o)] = \{[\tilde{U}(u_o)]_\alpha | \alpha \in \mathbb{I}\}$, where $\mathbb{R}^n$ is the $n$-dimensional Euclidean space called the basic space; $s_{\mathbb{I}}$ is the predicate, i.e., $s_{\mathbb{I}} : \mathbb{R}^n \rightarrow \mathcal{S}$ with $\mathcal{S}$ the "universe of discourse" defined by a set of statements, which assigns a proposition $s_{\mathbb{I}}(u) = \{u \in \tilde{U} \text{ coincides with } u_o\}$ to each element $u \in \mathbb{R}^n$(see e.g., [13]); and $[\tilde{U}(u_o)]$ is the family of subsets of $\mathbb{R}^n$ called the set representation of $\tilde{U}(u_o)$, which satisfies

\[ L_\alpha \tilde{U}(u_o) \subseteq [\tilde{U}(u_o)]_\alpha \subseteq L_\pi \tilde{U}(u_o) \quad \text{for any } \alpha \in \mathbb{I}, \]  

where $L_\alpha \tilde{U}(u_o)$ and $L_\pi \tilde{U}(u_o)$ are the strong cut(strong level set) and the level set of $\tilde{U}(u_o)$ at the level $\alpha$(see e.g., [9, 14]). The crisp point $u_o$ in $\mathbb{R}^n$, the vague perception of which gives the fuzzy set $\tilde{U}(u_o)$, is called the original point of $\tilde{U}(u_o)$.

Hereafter, a fuzzy set $\tilde{U}(u_o)$ will be abbreviated as $\tilde{U}$, when no confusion occurs or no special interest in the original point $u_o$ has been concerned.

Definition 1.1. The family of fuzzy sets is denoted by $\mathcal{F}_{cc}(\mathbb{R}^n)$, whose element $\tilde{U} = (\mathbb{R}^n, [\tilde{U}], s_{\mathbb{I}})$ satisfies the following conditions:

(i) Any element $[\tilde{U}]_\alpha$ of the set representation $[\tilde{U}]$ of a fuzzy set $\tilde{U}$ satisfies $[\tilde{U}]_\alpha \in \mathcal{K}_{cc}(\mathbb{R}^n)$, where $\mathcal{K}_{cc}(\mathbb{R}^n)$ is the family of nonempty compact and convex subsets of $\mathbb{R}^n$.

(ii) $[\tilde{U}]_{\alpha = 1} \cap [\tilde{U}]_{\alpha = 1}$ is the element of $\mathcal{K}_{cc}(\mathbb{R}^n)$.

(iii) $[\tilde{U}]_\alpha = L_\pi \tilde{U}$ except for $\alpha \in \bar{\mathbb{I}} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

The support of $\tilde{U}$ is defined by $[\tilde{U}]_0 = cl([\cup_{\alpha \in \bar{\mathbb{I}}} [\tilde{U}]_\alpha]$.

Definition 1.2. The support function of the fuzzy set $\tilde{U}$ is defined by

\[ \tilde{sp}(\tilde{U}, \alpha, x) = \begin{cases} sp(x, [\tilde{U}]_\alpha) & \text{for } \alpha \in \mathbb{I} = \mathbb{I} \cup \{1\} \\ 0 & \text{for } \alpha = 0 \end{cases} \]  

for $(\tilde{U}, \alpha, x) \in \mathcal{F}_{cc}(\mathbb{R}^n) \times \bar{\mathbb{I}} \times S^{n-1}$, where $S^{n-1}$ is a unit sphere in $\mathbb{R}^n$, and the support function $sp(x, [\tilde{U}]_\alpha)$ of $[\tilde{U}]_\alpha$ is given by $sp(x, [\tilde{U}]_\alpha) = sup(x, u) | u \in [\tilde{U}]_\alpha$ (see e.g., [15]) with the inner product $(x, u) \in \mathbb{R}^n$.

Let $(\mathbb{R}, \mathcal{B}_B(\mathbb{R}), \mu_B)$ be the complete measure space, where $\mathcal{B}_B(\mathbb{R})$ is the completion of the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ generated by the standard topology of $\mathbb{R}$, and $\mu_B$ is the Lebesgue measure. Then, $(\mathbb{I}, \mathcal{B}_B(\mathbb{I}), \mu_I)$ is the relative measure space, where $\mathcal{B}_B(\mathbb{I})$ is the relative $\sigma$-algebra given by the restriction.
of $\mathcal{B}_0(\mathbb{R})$, i.e., $\mathcal{B}_0(\mathbb{R}) = \{\mathbb{1} \cap B | B \in \mathcal{B}_0(\mathbb{R})\}$ with $\mathbb{1} \in \mathcal{B}_0(\mathbb{R})$, and $\mu_\mathbb{1}$ is the relative Lebesgue measure on $\mathbb{1}$ ($\mu_\mathbb{1}(\mathbb{1}) = 1$). Then, it can be shown that $\mathcal{S}(\mathbb{1}, \alpha, x)$ is $\mathcal{B}_0(\mathbb{R}) \otimes \mathcal{B}_0(\mathbb{R})$ measurable.

The measurability of $\tilde{U} \in \mathcal{F}_c(\mathbb{R}^n)$ concerned with the product $\sigma$-algebra $\mathcal{B}_0(\mathbb{R}) \otimes \mathcal{B}_0(\mathbb{R})$ is the base of the integrability of $\tilde{U}$. Let $p \in [1, +\infty]$. Then, we can define the integrability of $\mathcal{S}(\tilde{U}, \alpha, x)$ of the order $p$ with respect to $\mu_\mathbb{1} \otimes \mu_{\mathbb{R}^n}$ by

$$\int |\mathcal{S}(\tilde{U}, \alpha, x)|^p d\mu_\mathbb{1}(\alpha) \otimes \mu_{\mathbb{R}^n}(\alpha) < +\infty.$$  

The family of integrable fuzzy sets of the order $p$ is denoted by $\mathcal{F}_c^p(\mathbb{R}^n)$. Let $\tilde{U}_1$ and $\tilde{U}_2$ be the elements of $\mathcal{F}_c^p(\mathbb{R}^n)$. If

$$\int |\mathcal{S}(\tilde{U}_1, \alpha, x) - \mathcal{S}(\tilde{U}_2, \alpha, x)|^p d\mu_\mathbb{1}(\alpha) \otimes \mu_{\mathbb{R}^n}(\alpha) = 0$$

holds, we represent this binary relation by $\tilde{U}_1 \sim \tilde{U}_2$. Then, it can be shown that $\sim$ is the equivalence relation. Hence, we can define the quotient set of $\mathcal{F}_c^p(\mathbb{R}^n)$ as follows:

$$\mathcal{F}_c^p(\mathbb{R}^n) = \mathcal{F}_c^p(\mathbb{R}^n)/\sim.$$  

(3)

The mapping $\tilde{U} \mapsto \mathcal{S}(\tilde{U}, \cdot, \cdot)$ is an isomorphism of $\mathcal{F}_c^p(\mathbb{R}^n)$ onto cone of $\mathcal{B}_0(\mathbb{R}) \otimes \mathcal{B}_0(\mathbb{R})$ measurable functions, preserving the semi-linear structure,

$$\mathcal{S}(\lambda \cdot \tilde{U} + \mu \cdot \tilde{V}, \cdot, \cdot) = \lambda \cdot \mathcal{S}(\tilde{U}, \cdot, \cdot) + \mu \cdot \mathcal{S}(\tilde{V}, \cdot, \cdot)$$  

for $\lambda, \mu \geq 0$ and $\tilde{U}, \tilde{V} \in \mathcal{F}_c^p(\mathbb{R}^n)$.

The metric $\rho_p(\tilde{U}, \tilde{V})$ for any $\tilde{U}, \tilde{V} \in \mathcal{F}_c^p(\mathbb{R}^n)$ is defined by

$$\rho_p(\tilde{U}, \tilde{V}) = \left(\int |\mathcal{S}(\tilde{U}, \alpha, x) - \mathcal{S}(\tilde{V}, \alpha, x)|^p d\mu_\mathbb{1}(\alpha) \otimes \mu_{\mathbb{R}^n}(x)\right)^{1/p}.$$  

For every $p \in [1, +\infty)$, $L^p(\mathbb{I} \times \mathbb{R}^n, \mathcal{B}_0(\mathbb{R}) \otimes \mathcal{B}_0(\mathbb{R}))$ is a separable Banach space with respect to the measure space $(\mathbb{I} \times \mathbb{R}^n, \mathcal{B}_0(\mathbb{R}) \otimes \mathcal{B}_0(\mathbb{R}), \mu_\mathbb{1} \otimes \mu_{\mathbb{R}^n})$ (see e.g., [16]), where $\|\cdot\|$ is the usual $L^p$-norm, i.e.,

$$\|f\|_p = \int f(\alpha, x)^p d\mu_\mathbb{1}(\alpha) \otimes \mu_{\mathbb{R}^n}(x)$$  

for any $f \in \mathcal{F}_c^p(\mathbb{R}^n)$. For every $p \in [1, +\infty)$, we can embed $\mathcal{F}_c^p(\mathbb{R}^n)$ isomorphically into the separable Banach $L^p(\mathbb{I} \times \mathbb{R}^n, \|\cdot\|_p)$-space by the mapping defined by

$$j_{\mathcal{F}_c^p(\mathbb{R}^n)}(\tilde{U}) = \mathcal{S}(\tilde{U}, \alpha, x)$$  

and

$$\left\|j_{\mathcal{F}_c^p(\mathbb{R}^n)}(\tilde{U}) - j_{\mathcal{F}_c^p(\mathbb{R}^n)}(\tilde{V})\right\|_p = \rho_p(\tilde{U}, \tilde{V})$$  

for any $\tilde{U}, \tilde{V} \in \mathcal{F}_c^p(\mathbb{R}^n)$. From the definition of $\mathcal{F}_c^p(\mathbb{R}^n)$ given by (3), $j_{\mathcal{F}_c^p(\mathbb{R}^n)}$ is injective. Furthermore, using the property (4) of the support function for $\mathcal{F}_c^p(\mathbb{R}^n)$, we have

$$j_{\mathcal{F}_c^p(\mathbb{R}^n)}(\lambda \cdot \tilde{U} + \mu \cdot \tilde{V}) = \lambda \cdot j_{\mathcal{F}_c^p(\mathbb{R}^n)}(\tilde{U}) + \mu \cdot j_{\mathcal{F}_c^p(\mathbb{R}^n)}(\tilde{V})$$  

for $\lambda, \mu \geq 0$ and $\tilde{U}, \tilde{V} \in \mathcal{F}_c^p(\mathbb{R}^n)$. Using the embedding defined by (6), it can be shown that $(\mathcal{F}_c^p(\mathbb{R}^n), \rho_p)$ is a complete separable metric space for $p \in [1, +\infty)[17]$. 

### 2 Vague Perception of Random Phenomena

In order to consider the vague perceptions of random phenomena, two types of randomness should be considered, one of which is the randomness due to the capricious person’s feelings and another of which is that of the phenomena themselves. Let $(\Omega_1, \mathcal{A}_1, P_1)$ be a probability space describing the randomness of capricious persons’ minds, and let $(\Omega_2, \mathcal{A}_2, P_2)$ be a probability space, on which an original random point $u_0 \in \mathbb{R}^n$ as the model of a random phenomenon is defined. Then, the fuzzy random set as a capricious vague perception of the original random point $u_0$ should be defined on $(\Omega, \mathcal{A}, P) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, P_1 \times P_2)$.

Let here $A_{ij}$ $(i = 1, 2, \ldots, M; j = 1, 2, \ldots, M)$ be the disjoint subsets in $\mathcal{A}_0$ describing the $i$-th vague perception of the $j$-th cluster of random phenomena, and hence it follows

$$\Omega = \bigcup_{j=1}^M \bigcup_{i=1}^{M_j} A_{ij}.$$  

Then, a fuzzy random set $\tilde{U}(\omega) \in \mathcal{F}_c^p(\mathbb{R}^n)$ is given by

$$\tilde{U}(\omega) = \sum_{j=1}^M \sum_{i=1}^{M_j} 1_{A_{ij}}(\omega) \cdot \tilde{U}_{ij},$$  

where

$$1_{A_{ij}}(\omega) = \begin{cases} 1 & \text{if } \omega \in A_{ij} \\ 0 & \text{otherwise.} \end{cases}$$  

(9)

$\tilde{U}_{ij}$ in (8) is a collection of fuzzy sets representing the realized values of $\tilde{U}(\omega)$, and given by

$$\tilde{U}_{ij} = \left(\mathbb{R}^n, [\tilde{U}_{ij}], s_{\tilde{U}_{ij}}\right) \in \mathcal{F}_c^p(\mathbb{R}^n)$$  

with

$$[\tilde{U}_{ij}] = \left\{1_{\tilde{U}_{ij}}(\alpha) | \alpha \in \mathbb{R}^n\right\},$$  

where $s_{\tilde{U}_{ij}}$ is the predicate associated with the statement such as

$$s_{\tilde{U}_{ij}}(u) = \left\{u \in \tilde{U}_{ij} \text{ coincides with the original random point } u_0\right\}.$$  

(11)

Assuming that there exists a fuzzy random set $\tilde{U}(\omega) \in \mathcal{F}_c^p(\mathbb{R}^n)$ such that

$$\lim_{M \to \infty} \rho_p\left(\tilde{U}(\omega), \tilde{U}(\omega)\right) = 0 \text{ a.s.}$$

or equivalently

$$\tilde{U}(\omega) = \lim_{M \to \infty} \tilde{U}(\omega) \text{ a.s.}$$

the following definition is obtained:
Definition 2.1. A fuzzy random set $\tilde{U}(\omega)$ on $(\Omega, \mathcal{A}, P)$ obtained as the vague capricious perception of an original random point $u_0(\omega^{(2)})$ on $(\Omega, \mathcal{A}; P_2)$ is defined by

\[ \tilde{U}(\omega) = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \mathbf{1}_{\lambda_j}(\omega) \cdot \tilde{U}_{ij} \quad \text{a.s.,} \]  

(12)

where $\{\tilde{U}_{ij}; i = 1, 2, \cdots, M_j; j = 1, 2, \cdots\}$ is a collection of realized fuzzy sets given by (10), and $\mathbf{1}_{\lambda_j}(\omega)$ is a characteristic function defined by (9).

Hence, we have the relation between the predicates of $\tilde{U}$ and $\tilde{U}_{ij}$ such that

\[ s_{\tilde{U}(\omega)} = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \mathbf{1}_{\lambda_j}(\omega) \cdot s_{\tilde{U}_{ij}} \quad \text{a.s.} \]  

(13)

or equivalently

\[ s_{\tilde{U}(\omega)}(u) = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \mathbf{1}_{\lambda_j}(\omega) \cdot s_{\tilde{U}_{ij}}(u) \quad \text{a.s.} \]  

(14)

and

\[ [\tilde{U}]_0 = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \mathbf{1}_{\lambda_j}(\omega) \cdot [\tilde{U}_{ij}]_0 \quad \text{a.s.} \]  

(15)

Let $\mathcal{A}$ be the sub $\sigma$-algebra of $\mathcal{A}_0$ generated from $\{\lambda j; 1, 2, \cdots, M_j; j = 1, 2, \cdots\}$. Then, the measurability of $\tilde{U}$ is given through

\[ \tilde{U}^{-1}(G) = \left\{ \omega \in \Omega \mid \tilde{U}(\omega) \in G \in \mathcal{B}_{\tau_{\rho_r}} \right\} \in \mathcal{A} \subset \mathcal{A}_0 \]  

(16)

for $\forall G \in \mathcal{B}_{\tau_{\rho_r}}$, where $\mathcal{B}_{\tau_{\rho_r}}$ is the $\sigma$-algebra generated by the topology $\tau_{\rho_r}$ induced from $\rho_r$.

3 Expectation of $\tilde{U}(\omega)$

As described in Section 2, a vague capricious perception of the original random point $u_0(\omega^{(2)})$ is modeled by the fuzzy random set $\tilde{U}(\omega)$ given by (12). In order to explore the reasonable defining method for the expectation $\mathcal{E}[\tilde{U}]$ of $\tilde{U}(\omega)$, consider first the statement given by

\[ s_{\mathcal{E}[\tilde{U}]}(x) = \left\{ x \in \mathcal{E}[\tilde{U}] \right\} \text{ coincides with the expectation of } u_0 \right\}. \]  

(17)

Then, the above statement may be given by a composite statement such as

\[ s_{\mathcal{E}[\tilde{U}]}(x) = \bigvee_{\omega \in \mathcal{E}[\tilde{U}]} \left\{ x = E(u) \right\} \wedge \left\{ u(\omega) = u_0(\omega^{(2)}) \text{ a.s. on } (\Omega, \mathcal{A}, P) \right\}, \]  

(18)

where $\mathcal{E}[\tilde{U}]$ is the admissible class of possible random original points given by

\[ \mathcal{E}[\tilde{U}] = \left\{ u \mid u(\omega) = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \mathbf{1}_{\lambda_j}(\omega) \cdot u_{ij} \text{ with } u_{ij} \in \mathbb{R}^\infty \right\}, \]  

and $E(u)$ is given by

\[ E(u) = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} u_{ij} \cdot P(A_{ij}) \]  

when it exists. Then, the statement “$u(\omega) = u_0(\omega^{(2)})$ a.s. on $(\Omega, \mathcal{A}, P)$” in (18) is given by

\[ s_{\mathcal{E}[\tilde{U}]}(u_{ij}) = \left\{ u_{ij} \right\} \text{ coincides with } u_0 \right\}. \]  

(19)

Therefore, applying the concept of the fuzzy logic[6], the truth value of $s_{\mathcal{E}[\tilde{U}]}(x)$ is given by

\[ t\left(s_{\mathcal{E}[\tilde{U}]}(x)\right) = \sup_{\omega \in \mathcal{E}[\tilde{U}]} \left\{ \inf_{j=1,2,\cdots} \min_{i=1,2,\cdots} \text{ ess inf } \left(\tilde{U}_{ij}\right)(u_{ij}) \right\} x = E(u) \right\}, \]  

where $\text{ ess inf }_{\omega \in \mathcal{E}[\tilde{U}]} \left(\tilde{U}_{ij}\right)(u_{ij})$ is given by the supremum of $a$ satisfying $\left(\tilde{U}_{ij}\right)(u_{ij}) \geq a$ for $\omega \in A_{ij} \setminus N_0$. Then, we can confirm

\[ \left\{ x \mid t\left(s_{\mathcal{E}[\tilde{U}]}(x)\right) > a \right\} \subseteq \sum_{j=1,2,\cdots} \sum_{i=1,2,\cdots} \left(\tilde{U}_{ij}\right)(u_{ij}) \subseteq \sum_{j=1,2,\cdots} \sum_{i=1,2,\cdots} \left(\tilde{U}_{ij}\right)(u_{ij}) \]  

for any $x \in \mathcal{I}$, where $E[\tilde{U}]$ is given by

\[ E[\tilde{U}] = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \left[ \tilde{U}_{ij} \right]_0 \cdot P(A_{ij}). \]  

Then, the following definition of the expectation of a fuzzy random set $\tilde{U}$ may be reasonable:
Definition 3.1. Let \( \tilde{U} = (\mathbb{R}^n, [\tilde{U}], s_{\tilde{U}}) \) be a fuzzy random set given by (12). Then, the expectation of \( \tilde{U} \) is given by

\[
\mathbb{E}[\tilde{U}] = \left( \mathbb{R}^n, [\mathbb{E}[\tilde{U}]], s_{\mathbb{E}[\tilde{U}]} \right) \in \mathcal{F}^\mathbb{P}_c(\mathbb{R}^n)
\]

with

\[
[\mathbb{E}[\tilde{U}]] = \left\{ E[[\tilde{U}]_\alpha], \alpha \in I \right\},
\]

where \( s_{\mathbb{E}[\tilde{U}]} \) is the predicate associated with the statement given through (17); and \( [\mathbb{E}[\tilde{U}]] \) is the set representation of \( \mathbb{E}[\tilde{U}] \) given through (19).

Let here \( A_j (j = 1, 2, \cdots) \) be the sequence of subsets given by

\[
A_j = \bigcup_{i=1}^{M_j} A_{ij},
\]

then we have

\[
\Omega = \bigcup_{j=1}^{\infty} A_j \quad (A_i \cap A_j = \emptyset \text{ for } i \neq j).
\]

Let \( S \) be the sub \( \sigma \)-algebra generated from \( A_1, A_2, \cdots \). Then, because of the \( S \)-measurability of \( E(u|S) \),

\[
E(u|S)(\omega) = u_j \quad \text{(some const. for } \forall \omega \in A_j),
\]

and from the property of the conditional expectation, it follows

\[
\sum_{j=1}^{\infty} E(u_j|S) \cdot P(S \cap A_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} u_{ij} \cdot P(S \cap A_{ij}) \quad \text{for } \forall S \in S.
\]

Then, using (22) we have

\[
E(u|S) = \sum_{j=1}^{\infty} I_{A_j}(\omega) \cdot u_j \quad \text{a.s.},
\]

and

\[
u_j = \frac{1}{P(A_j)} \sum_{i=1}^{M_j} P(A_{ij}) \cdot \tilde{U}_{ij},
\]

which suggests

\[
E(\tilde{U}|S) = \sum_{j=1}^{\infty} I_{A_j}(\omega) \cdot \tilde{U}_j \quad \text{a.s.}
\]

with

\[
\tilde{U}_j = \frac{1}{P(A_j)} \sum_{i=1}^{M_j} P(A_{ij}) \cdot \tilde{U}_{ij}.
\]

Then, the conditional expectation of \( \tilde{U} \) concerned with \( S \) should be given as follows:

\[
\mathbb{E}[\tilde{U}|S] = \left( \mathbb{R}^n, [\mathbb{E}[\tilde{U}|S]], s_{\mathbb{E}[\tilde{U}|S]} \right)
\]

with

\[
[\mathbb{E}[\tilde{U}|S]] = \left\{ E[[\tilde{U}]_\alpha|S], \alpha \in I \right\}.
\]

Proposition 3.1. Let \( \tilde{U} = (\mathbb{R}^n, [\tilde{U}], s_{\tilde{U}}) \) be a fuzzy random set given by (12). Then, it follows

\[
\mathbb{E}[\tilde{U}] = \mathbb{E}[\mathbb{E}[\tilde{U}|S]]
\]

where \( \mathbb{E}[\tilde{U}|S] \) is given by (23).

4 Asymptotic Distributions of Estimators

Given a \( \mathcal{A}-\mathcal{B}_{\tau_p} \)-measurable fuzzy random set \( \tilde{U} \), its distribution(or image measure) is a probability measure on \( \mathcal{B}_{\tau_p} \) defined by

\[
P_{\tilde{U}}(G) = P(\tilde{U}^{-1}(G))
\]

for any \( G \in \mathcal{B}_{\tau_p} \).

Let be \( \tilde{U}_i \in \mathcal{B}_{\tau_p} \) \( (i = 1, 2, \cdots, n) \), and let the fuzzy random sets \( [\tilde{U}^{(i)}; i = 1, 2, \cdots, n] \) satisfy the following condition:

\[
P_{[\tilde{U}^{(i)}; i = 1, 2, \cdots, n]}(G_1, G_2, \cdots, G_n)
\]

\[
= P(\tilde{U}^{(1)} \in G_1, \tilde{U}^{(2)} \in G_2, \cdots, \tilde{U}^{(n)} \in G_n)
\]

\[
= P(\tilde{U}^{(1)} \in G_1) \cdot P(\tilde{U}^{(2)} \in G_2) \cdots P(\tilde{U}^{(n)} \in G_n)
\]

\[
= P_{\tilde{U}^{(1)}}(G_1) \cdot P_{\tilde{U}^{(2)}}(G_2) \cdots P_{\tilde{U}^{(n)}}(G_n),
\]

and consequently

\[
P_{[\tilde{U}^{(i)}; i = 1, 2, \cdots, n]}(G_1, G_2, \cdots, G_n) = \prod_{i=1}^{n} P_{\tilde{U}^{(i)}}(G_i).
\]

Then, \( \{\tilde{U}^{(i)}; i = 1, 2, \cdots, n\} \) are said to be independent for arbitrary \( n(=1, 2, \cdots) \). Furthermore, if \( \{\tilde{U}^{(i)}; i = 1, 2, \cdots, n\} \) are distributed as \( \tilde{U} \), i.e.,

\[
P_{\tilde{U}^{(i)}}(G) = P_{\tilde{U}}(G),
\]

\( \{\tilde{U}^{(i)}; i = 1, 2, \cdots, n\} \) are called as identically distributed. Therefore, \( \{\tilde{U}^{(i)}; i = 1, 2, \cdots, n\} \) are independent, and identically distributed(i.i.d.) if

\[
P_{[\tilde{U}^{(i)}; i = 1, 2, \cdots, n]}(G_1, G_2, \cdots, G_n) = \prod_{i=1}^{n} P_{\tilde{U}^{(i)}}(G_i)
\]

\[
= \prod_{i=1}^{n} P_{\tilde{U}}(G_i).
\]

Since every fuzzy set \( \tilde{U} \in \mathcal{F}^\mathbb{P}_c(\mathbb{R}^n) \) is embedded into the separable Banach space \( L^p(\Omega \times \mathbb{R}^n, \|\|p) \) by the mapping (6), we can make use of the standard strong law of large numbers in a separable Banach space.

Proposition 4.1 (Corollary 7.10 in [18]). Let \( (B, \|\|) \) be a separable Banach space and let \( \{X^{(n)}; n = 1, 2, \cdots\} \) be a sequence of i.i.d. Borel random variables distributed as \( X \) with values in \( B \). Then,

\[
\frac{1}{N} \sum_{n=1}^{N} X^{(n)} - E(X) \to 0 \quad \text{a.s. as } N \to \infty
\]

if and only if \( E(\|X\|) < +\infty \).

Since \( \mathbb{E}[\tilde{U}] \in \mathcal{F}^\mathbb{P}_c(\mathbb{R}^n) \), and using (19) it can be rewritten as follows

\[
\mathbb{E}[\tilde{U}] = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \tilde{U}_{ij} \cdot P(A_{ij}).
\]
Then, substituting \( \tilde{s}(\bar{U}(n), \alpha, x) \) and \( \tilde{s}(\bar{U}, \alpha, x) \) for \( X(n) \) and \( X \) respectively in Proposition 4.1, and using (2) and (4), we have

\[
\frac{1}{N} \sum_{n=1}^{N} \tilde{s}(\bar{U}(n), \alpha, x) = \frac{1}{N} \tilde{s}\left(\sum_{n=1}^{N} \bar{U}(n), \alpha, x \right)
\]

\[
= \tilde{s}(1, \alpha, x)
\] (25)

and

\[
E(\tilde{s}(\bar{U}, \alpha, x)) = \sum_{j=1}^{\infty} \sum_{i=1}^{M_j} \tilde{s}(\bar{U}_{ij}, \alpha, x) \cdot P(\omega_i)
\]

\[
= \tilde{s}(E(\bar{U}), \alpha, x).
\] (26)

Then, if

\[
E(||\bar{U}||_p) = E(\rho_p(\bar{U}, [0, 0]))
\]

using (25), (26) and (7), we have

\[
\left\| \tilde{s}\left(\frac{1}{N} \sum_{n=1}^{N} \bar{U}(n), \alpha, x \right) - \tilde{s}(E(\bar{U}), \alpha, x) \right\|_p
\]

\[
= \rho_p\left(\frac{1}{N} \sum_{n=1}^{N} \bar{U}(n), E(\bar{U})\right) \rightarrow 0 \text{ a.s. as } N \rightarrow \infty.
\]

Hence, for the sequence of i.i.d. fuzzy random sets \( \{\bar{U}(n); n = 1, 2, \ldots\} \) distributed as \( \bar{U} \) with its expectation \( E(\bar{U}) \), we have

\[
P_p\left(\bar{U}_N, E(\bar{U})\right) \rightarrow 0 \text{ a.s. as } N \rightarrow \infty,
\]

where \( \bar{U}_N \) is the heuristic estimator of \( E(\bar{U}) \) given by

\[
\bar{U}_N(\alpha) = \frac{1}{N} \sum_{n=1}^{N} \bar{U}(n)(\alpha).
\]

Let \( D \) be a metric space and \( M(D) \) the space of all measures defined on \( \mathcal{B}_D \), where \( \mathcal{B}_D \) is the Borel \( \sigma \)-field of \( D \), i.e., the smallest \( \sigma \)-algebra of subsets of \( D \) which contains all the open subsets of \( D \), and \( C(D) \) stands for the space of all bounded continuous functions on \( D \). An element \( \mu \in M(D) \) is non-negative, countably additive set function defined on \( \mathcal{B}_D \) with the property \( \mu(D) = 1 \). It can be verified that \( M(D) \) is topologized by defining a base of open neighborhoods for any point \( \mu \). Consider the family of set functions of the form

\[
V_p(f_1, f_2, \ldots, f_k; \epsilon_1, \epsilon_2, \ldots, \epsilon_k)
\]

\[
= \{ \nu; \nu \in M(D) \left| \int f_i \, dv - \int f_i \, d\mu \right| < \epsilon_i, i = 1, 2, \ldots, k \},
\]

where \( f_1, f_2, \ldots, f_k \) are elements from \( C(D) \) and \( \epsilon_1, \epsilon_2, \ldots, \epsilon_k \) are positive numbers. Then, it can be confirmed that the family of the sets obtained by varying \( k, f_1, f_2, \ldots, f_k, \epsilon_1, \epsilon_2, \ldots, \epsilon_k \) satisfies the axioms of a base for a topology. We shall refer to this as the “weak topology” in \( M(D) \). It is clear that the net \( \{\mu_m\} \) of measures converges in the weak topology to a measure \( \mu \) if and only if

\[
\int f \, d\mu_m \rightarrow \int f \, d\mu \quad \text{as } m \rightarrow \infty
\]

for every \( f \in C(D) \). In such case we shall say that \( \mu_m \) converges “weakly” to \( \mu \) or

\[
\mu_m \rightharpoonup \mu
\]

in symbol.

**Theorem 4.1** (Theorem 7.1 in Chapter 2 of [19]). Let \( X_1, X_2, \ldots \) be i.i.d. random elements of \( \Omega \) into a separable metric space \( D \), and let \( \mu \) be the common induced measure. Let \( \mu_m^{\alpha} \) be the sample distribution based on \( X_1, X_2, \ldots \) at \( \omega \). Then,

\[
P(\omega : \mu_m^{\alpha} \Rightarrow \mu) = 1.
\]

Since \( (\mathcal{F}_m^D(\mathbb{R}^n), \rho_p) \) is a complete separable Banach space and so, a separable metric space) for every \( p \in [1, \infty) \), Theorem 4.1 can be applicable to the sequence of i.i.d. fuzzy random sets \( \{\bar{U}(n); n = 1, 2, \ldots\} \).

**Proposition 4.2**. Let \( p \in [1, +\infty) \) and let \( \{\bar{U}(n); n = 1, 2, \ldots\} \) be a sequence of i.i.d. fuzzy random sets distributed as \( \bar{U} \). Let \( P_m^{\alpha} \) be the sample distributions based on \( \bar{U}(1), \bar{U}(2), \ldots, \bar{U}(m) \), defined by

\[
P_m^{\alpha}(\bar{U}_{jk}) = \frac{\# \{ i \in [1, 2, \ldots, m] \mid \bar{U}(i)(\omega) = \bar{U}_{jk} \}}{m},
\]

where \( \bar{U}_{jk} \) is given by (10). Then,

\[
P(\omega : P_m^{\alpha} \Rightarrow P_{\bar{U}}) = 1 \quad \text{as } m \rightarrow \infty.
\]

For the general investigations when the classical limit theorems for random variables in \( \mathbb{R} \) could be preserved for random elements in Banach spaces, the concept of types of Banach spaces has been introduced.

Let \( \{X(n); n = 1, 2, \ldots\} \) be a sequence of independent Bochner-integrable Borel random variable with values in a Banach space \( B \). A real Banach space \( (B, \| \cdot \|) \) is called that of type \( p \in [1, 2] \) if there exists some constant \( C \) such that for every finite sequence \( \{X(n); n = 1, 2, \ldots, N\} \) of independent Bochner-integrable random elements in \( B \) satisfying \( E(X(0)) = 0 \) and \( E(\|X(n)\|^p) < \infty (n = 1, 2, \ldots, N) \), the inequality

\[
E\left(\left\| \sum_{n=1}^{N} X(n) \right\|^p \right) \leq C \sum_{n=1}^{N} E(\|X(n)\|^p)
\]

holds (see e.g. [20]).
Theorem 4.2 (Theorem 10.5 in [18]). Let $(B, \|\cdot\|)$ be a separable Banach space and let $(X^n; n = 1, 2, \cdots)$ be a sequence of i.i.d. Borel random variable distributed as $X$ with values in $B$ of type 2. Then, if $E(X) = 0$ and $E(||X||^2) < \infty$,
\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X^n \rightarrow Z
\]
weakly as $N \to \infty$, where $Z$ is a Gaussian random element in $B$.

It is well known that $L^p(\mathbb{I} \times \mathbb{N}^{a-1}, \|\cdot\|_p)$ is a real separable Banach space of type $p$ in the case of $p \in [1, 2]$ and of type 2 if $p \in [2, \infty)$. Hence, we have the following result.

Proposition 4.3. Let $p \in [2, \infty)$ and $(\tilde{U}^n; n = 1, 2, \cdots)$ be a sequence of i.i.d. fuzzy random sets distributed as $\tilde{U}$ with its expectation $\mathcal{E}[\tilde{U}]$. Then, there exists a Gaussian random variable $Z$ in $L^p(\mathbb{I} \times \mathbb{N}^{a-1}, \|\cdot\|_p)$ such that
\[
\sqrt{N}\mathcal{E}[\tilde{U}^n] - \sqrt{N}\mathbb{E}\mathcal{E}[\tilde{U}] \rightarrow Z \quad \text{as} \quad N \to \infty,
\]
weakly and using the continuous mapping theorem
\[
\rho_p\left(\sqrt{N}\mathcal{E}[\tilde{U}^n], \sqrt{N}\mathcal{E}[\tilde{U}]\right) \rightarrow \|Z\|_p \quad \text{as} \quad N \to \infty,
\]
weakly.

5 Conclusions

In this paper, the author has investigated the asymptotic distributions of estimators concerned with the class of fuzzy random sets, which is considered as a model of the capricious vague perception of a crisp random phenomenon.

First, the class of fuzzy random sets, which has been proposed by author as a model of the capricious vague perception of a crisp random phenomenon, has been refined from the practical point of view. Secondly, using the refined class of fuzzy random sets, the expectations of fuzzy random sets have been rebuild. Applying the standard strong law of large numbers for the random elements in a separable Banach space, the convergence properties of estimators for expectations of fuzzy random variables have been rebuild. Finally, asymptotic distributions concerned with the proposed estimators have been investigated, applying the central limit theorem for the random elements in a separable Banach space.

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