Classification of unit-vector fields in convex polyhedra with tangent boundary conditions

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Abstract

A unit-vector field $\mathbf{n}$ on a convex three-dimensional polyhedron $\bar{P}$ is tangent if, on the faces of $\bar{P}$, $\mathbf{n}$ is tangent to the faces. A homotopy classification of tangent unit-vector fields continuous away from the vertices of $\bar{P}$ is given. The classification is determined by certain invariants, namely edge orientations (values of $\mathbf{n}$ on the edges of $\bar{P}$), kink numbers (relative winding numbers of $\mathbf{n}$ between edges on the faces of $\bar{P}$), and wrapping numbers (relative degrees of $\mathbf{n}$ on surfaces separating the vertices of $\bar{P}$), which are subject to certain sum rules. Another invariant, the trapped area, is expressed in terms of these. One motivation for this study comes from liquid crystal physics; tangent unit-vector fields describe the orientation of liquid crystals in certain polyhedral cells.

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1 Introduction

A unit-vector field \( \mathbf{n} \) on a convex polyhedron \( \bar{P} \subset \mathbb{R}^3 \) is a map from \( \bar{P} \) to the unit sphere \( S^2 \subset \mathbb{R}^3 \). \( \mathbf{n} \) is said to satisfy tangent boundary conditions, or, more simply, to be tangent, if, on the faces of \( \bar{P} \), \( \mathbf{n} \) is tangent to the faces. Tangent boundary conditions imply that, on the edges of \( \bar{P} \), \( \mathbf{n} \) is parallel to the edges, and therefore that \( \mathbf{n} \) is necessarily discontinuous at the vertices. Let \( P \subset \mathbb{R}^3 \) denote \( \bar{P} \) without its vertices (thus \( \bar{P} \) is the closure of \( P \)). Let \( C^0(P) \) denote the space of continuous tangent unit-vector fields on \( P \). We have the usual notion of homotopic equivalence in \( C^0(P) \); two maps \( \mathbf{n}, \mathbf{n}' \in C^0(P) \) are homotopic, denoted \( \mathbf{n} \sim \mathbf{n}' \), if there exists a continuous map \( H : P \times [0, 1] \rightarrow S^2; (x, t) \mapsto H_t(x) \), such that \( H_0 = \mathbf{n} \), \( H_1 = \mathbf{n}' \).

Here we classify unit-vector fields in \( C^0(P) \) up to homotopy. The paper is organised as follows. To a unit-vector field \( \mathbf{n} \in C^0(P) \) we associate certain homotopy invariants, which we call edge orientations, kink numbers, and wrapping numbers (Section 3). Edge orientations are just the values of \( \mathbf{n} \) on the edges of \( P \) (as noted above, there are two possible values, differing by a sign). Kink numbers are the integer-valued relative winding numbers of \( \mathbf{n} \) between adjacent edges along a face of \( P \). Wrapping numbers are the integer-valued relative degrees of \( \mathbf{n} \) on planar surfaces which separate one vertex of \( P \) from the others. The continuity of \( \mathbf{n} \) imposes sum rules on the kink numbers and wrapping numbers. In Section 4 we construct representative maps for each of the allowed sets of values of the invariants. In Section 5 we show that an arbitrary map \( \mathbf{n} \in C^0(P) \) is homotopic to the reference map with the same values of the invariants. One part of the proof, concerning homotopies on the boundary of \( P \), is deferred to Section 6.

We remark that it is the tangent boundary conditions which substantially determine the classification. In contrast, continuous unit-vector fields satisfying fixed boundary conditions – for simplicity, imagine \( \mathbf{n} \) to be constant on the boundary of \( P \) – are equivalent to continuous maps of \( S^3 \) (the unit ball in \( \mathbb{R}^3 \) with boundary points identified) to \( S^2 \). As is well known, such maps are classified by the Hopf invariant. The absence of a Hopf invariant for tangent unit-vector fields is due to the vertex discontinuities.

The problem considered here is part of a study of extremals of the energy functional

\[
E = \int_P \sum_{j,k=1}^3 \partial_j n_k \partial_j n_k \, d^3r
\]

defined on tangent unit-vector fields in \( C^0(P) \) with square-integrable derivative. Lower bounds for the energy in terms of the invariants, along with
upper bounds for the case where $P$ is a cube, will be reported elsewhere [6].

The study of these extremal maps is motivated in part by the study of liquid crystals in polyhedral cells. In the continuum limit, the average local molecular orientation of a uniaxial nematic liquid crystal may be described by a unit-vector field $n$ (but see below). The energy of a configuration $n$ – the so-called Frank energy – reduces, in a certain approximation (the so-called one-constant approximation), to the expression (1) [2]. Polyhedral liquid crystal cells can be manufactured so that $n$ is approximately tangent to the cell surfaces. The homotopy type of $n$ determines, at least in part, the optical properties of the liquid crystal, and is relevant to the design of liquid crystal displays [8].

In fact, the local orientation of a liquid crystal is only determined up to a sign, as antipodal orientations are physically equivalent. Therefore, it is properly described by a director field, a map from $P$ to the real-projective plane $RP^2$, rather than a unit-vector field. However, because $P$ is simply connected, a continuous director field on $P$ can be lifted to a continuous unit-vector field. The lifted unit-vector field is determined up to an overall sign. As is shown in Section 1, $+n$ and $-n$ belong to distinct homotopy classes; their kink numbers are the same, but their edge orientations and wrapping numbers differ by a sign. By identifying these pairs of homotopy classes, we obtain a classification of continuous tangent director fields on $P$.

Twice-differentiable extremals of (1) are examples of harmonic maps. Harmonic maps between Riemannian polyhedra have been studied by Gromov & Schoen [5] and Eells & Fuglede [3]. In the case where the target manifold has nonpositive Riemannian curvature, results concerning the existence, uniqueness and regularity of solutions of the Euler-Lagrange equations have been established. Harmonic unit-vector fields in $\mathbb{R}^3$ have been studied by Brezis et al [1], also in connection with liquid crystals. The topological classification of liquid crystal configurations in $\mathbb{R}^3$ as well as in domains with smooth boundary has been extensively discussed – see, eg, Mermin [7], de Gennes and Prost [2], and Kléman [1].

We remark that the homotopy classification of tangent unit-vector fields on $P$ may be regarded as the decomposition of $C^0(P)$ into its path-connected components with respect to the compact-open topology. The compact-open topology on $C^0(P)$ is generated by sets $[K, U]$, defined for compact $K \subset P$ and open $U \subset S^2$ by

$$[K, U] = \{n \in C^0(P) \mid n(K) \subset U\}. \tag{2}$$

We note that because $P$ is not compact, the compact-open topology on $C^0(P)$ is distinct from the metric topology on $C^0(P)$, which is induced by the metric.
\[ d(\mathbf{n}, \mathbf{n}') = \sup_{x \in P} |\mathbf{n}(x) - \mathbf{n}'(x)|. \]  

A path \( \mathbf{H}_t \in C^0(P) \) is continuous with respect to the compact-open topology if and only if \( \mathbf{H}_t(x) \) is continuous on \( P \times [0, 1] \). The continuity for \( \mathbf{H}_t \) with respect to the metric topology is a stronger condition; in addition to \( \mathbf{H}_t(x) \) being continuous on \( P \times [0, 1] \), sup \( \sup_{x \in P} |\mathbf{H}_t(x) - \mathbf{H}_t'(x)| \) must vanish as \( t' \) approaches \( t \).

2 The truncated polyhedron

Let \( \mathbf{v}^a, a = 1, \ldots, v \), denote the vertices of \( P \). Let \( E^b, b = 1, \ldots, e \), denote the edges, and let \( F^c, c = 1, \ldots, f \), denote the faces. We regard \( E^b \) and \( F^c \) as subsets of \( P \).

The truncated polyhedron, denoted \( \hat{P} \), is obtained by cleaving \( P \) along planes which separate the vertices from each other. Explicitly, let \( C^a \subset \mathbb{R}^3 \) be a plane which separates the vertex \( \mathbf{v}^a \) from the vertices \( \mathbf{v}^{b \neq a} \). That is, if \( C^a \) denotes a unit normal to \( C^a \) and \( \mathbf{c}^a \) is a point in \( C^a \), then \((\mathbf{v}^a - \mathbf{c}^a) \cdot \mathbf{C}^a \) and \((\mathbf{v}^{b \neq a} - \mathbf{c}^a) \cdot \mathbf{C}^a \) have opposite signs. For definiteness, we take \( C^a \) to be outwardly oriented, so that \((\mathbf{v}^a - \mathbf{c}^a) \cdot \mathbf{C}^a > 0 \). Let \( R^a \) denote the closed half-space given by

\[ R^a = \{ \mathbf{x} \in \mathbb{R}^3 | (\mathbf{x} - \mathbf{c}^a) \cdot \mathbf{C}^a \leq 0 \}. \]

Then the truncated polyhedron \( \hat{P} \) is given by

\[ \hat{P} = P \cap (\cap_{a=1}^v R^a). \]

\( \hat{P} \) is closed and convex.

\( \hat{P} \) has two kinds of faces, which we call cleaved faces and truncated faces (see Fig 1). The cleaved faces, denoted \( \hat{C}^a \), are given by the intersections of the planes \( C^a \) with \( P \). The truncated faces, denoted \( \hat{F}^c \), are given by the intersections of the faces \( F^c \) of the original polyhedron \( P \) with \( \cap_{a=1}^v R^a \).

\( \hat{P} \) has two kinds of edges, which we call cleaved edges and truncated edges (see Fig 1). The cleaved edges, denoted by \( \hat{B}^{ac} \), are given by the intersections of the cleaved faces \( \hat{C}^a \) and the truncated faces \( \hat{F}^c \). The truncated edges, denoted by \( \hat{E}^b \), are given by the intersections of the original edges \( E^b \) with \( \cap_{a=1}^v R^a \). The boundaries of the truncated faces consist of cleaved edges and truncated edges in alternation.

We will say that a continuous unit-vector field on \( \hat{P} \) satisfies tangent boundary conditions if, on the truncated face \( \hat{F}^c \), the vector field is tangent to
Figure 1: (a) The polyhedron $P$ (b) The cleaved polyhedron $\hat{P}$

$\hat{F}^c$ (note that it need not be tangent on the cleaved faces). Let $C^0(\hat{P})$ denote the space of continuous tangent unit-vector fields on $\hat{P}$. Given $n \in C^0(P)$, let $\hat{n}$ denote its restriction to $\hat{P}$. Then $\hat{n} \in C^0(\hat{P})$.

It turns out that the map $n \mapsto \hat{n}$ induces a one-to-one correspondence between homotopy classes of $C^0(P)$ and $C^0(\hat{P})$.

**Proposition 2.1.** Given $n, n' \in C^0(P)$, let $\hat{n}, \hat{n}' \in C^0(\hat{P})$ denote their restrictions to $\hat{P}$. Then $n \sim n'$ if and only if $\hat{n} \sim \hat{n}'$.

**Proof.** Clearly $n \sim n'$ implies $\hat{n} \sim \hat{n}'$. For the converse, we introduce maps $N, N' \in C^0(P)$ which coincide with $n, n'$ on $\hat{P}$ and are constant along rays in $P - \hat{P}$ through the vertices. These rays are of the form

$$x^a(r, y^a) = ry^a + (1 - r)v^a,$$

where $y^a \in \hat{C}^a$ and $0 < r < 1$. Every $x \in P - \hat{P}$ lies on such a ray and uniquely determines the cleaved face $\hat{C}^a$ through which the ray passes as well as $y^a$ and $r$. Let $N$ by given by

$$N(x) = \begin{cases} n(x), & x \in \hat{P}, \\ n(y^a), & x = x^a(r, y^a). \end{cases}$$

$N'$ is similarly defined, with $n$ replaced by $n'$. 

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Assuming that \( \hat{n} \) and \( \hat{n}' \) are homotopic, it follows that \( N \) and \( N' \) homotopic. Indeed, a homotopy is given by

\[
H_t(x) = \begin{cases} 
\hat{H}_t(x), & x \in \hat{P}, \\
\hat{H}_t(y^a), & x = x^a(r, y^a),
\end{cases}
\]

where \( \hat{H}_t \) is a homotopy between \( \hat{n} \) and \( \hat{n}' \).

Next we show that \( n \) is homotopic to \( \hat{n} \). A homotopy \( H_t \) is given by

\[
H_t(x) = \begin{cases} 
n(x), & x \in \hat{P}, \\
n(y^a), & x = x^a(r, y^a), 0 < r < t, \\
n(x^a((r - t)/(1 - t), y^a)), & x = x^a(r, y^a), t \leq r < 1.
\end{cases}
\]

where \( 0 \leq t \leq 1 \). Clearly \( H_0 = n \) and \( H_1 = N \). It is straightforward to verify that \( H_t(x) \) is continuous for \((x, t) \in P \times [0,1]\) and that it satisfies tangent boundary conditions. A similar argument shows that \( n' \) is homotopic to \( N' \). Thus we have a chain of equivalences, \( n \sim N \sim N' \sim n' \), which establishes the required result.

Thus, the homotopy type of tangent unit-vector fields on \( P \) is determined by the homotopy types of their restrictions to the truncated polyhedron \( \hat{P} \). Because \( \hat{P} \) is closed, the classification of the restricted maps is easier to carry out. For this reason, we determine homotopy classes of \( C^0(\hat{P}) \) in what follows.

### 3 Invariants

Given \( \hat{n} \in C^0(\hat{P}) \), tangent boundary conditions imply that its values on the truncated edges \( \hat{E}^b \) are constant, are tangent to the edges, and therefore are determined up to a sign.

**Definition 3.1.** The edge orientation \( e^b(\hat{n}) \) is the value of \( \hat{n} \) on \( \hat{E}^b \).

The edge orientations are obviously homotopy invariants. Under the antipodal map \( \hat{n} \mapsto -\hat{n} \), the edge orientations obviously change sign.

Kink numbers are relative winding numbers along cleaved edges. Let \( z^{ac}(t), 0 \leq t \leq 1 \), denote a continuous parameterisation of the cleaved edge \( \hat{B}^{ac} \), positively oriented with respect to the outward normal, denoted \( \mathbf{F}^c \), on \( \hat{P}^c \). Let \( \hat{n}^{ac}(t) = \hat{n}(z^{ac}(t)) \). As \( \hat{n}^{ac}(t) \) is tangent to \( \hat{F}^c \), its values are related to \( \hat{n}^{ac}(0) \) by a rotation about \( \mathbf{F}^c \), which we write as

\[
\hat{n}^{ac}(t) = \mathcal{R}(\mathbf{F}^c, \xi^{ac}(t)) \cdot \hat{n}^{ac}(0),
\]

where \( \mathcal{R}(\cdot , \cdot) \) is the rotation about \( \mathbf{F}^c \).
where $\xi^{ac}(t)$ is the angle of rotation. We take $\xi^{ac}(t)$ to be continuous, and fix it uniquely by taking $\xi^{ac}(0) = 0$.

Let $\eta^{ac}$, where $-\pi < \eta^{ac} < \pi$, denote the angle (of smallest magnitude) between $n^{ac}(0)$ and $n^{ac}(1)$, so that

\[
\hat{n}^{ac}(1) = R(F_c, \eta^{ac}) \cdot \hat{n}^{ac}(0). \tag{11}
\]

(Note that since $n^{ac}(0)$ and $n^{ac}(1)$ are parallel to consecutive truncated edges $\hat{E}^b$ and $\hat{E}'^b$, they cannot be parallel to each other, so that $\eta^{ac}$ cannot be a multiple of $\pi$). From (10) and (11), $\xi^{ac}(1)$ and $\eta^{ac}$ differ by an integer multiple of $2\pi$. This integer is the kink number.

**Definition 3.2.** The kink number $k^{ac}(\hat{n})$ is given by

\[
k^{ac}(\hat{n}) = \frac{1}{2\pi} (\xi^{ac}(1) - \eta^{ac}). \tag{12}\]

The kink number $k^{ac}(\hat{n})$ depends continuously on $\hat{n}$, and therefore is an integer-valued homotopy invariant on $C^0(\hat{P})$. It may be regarded as the degree (winding number) of the map of $S^1$ to itself obtained by concatenating $\hat{n}^{ac}(t)$ with a path along which $\hat{n}^{ac}(1)$ is minimally rotated back to $\hat{n}^{ac}(0)$ through their common plane.

Equations (10) and (11) remain valid if $\hat{n}$ is replaced by $-\hat{n}$. Therefore,

\[
k^{ac}(-\hat{n}) = k^{ac}(\hat{n}). \tag{13}\]

The kink numbers on each truncated face satisfy the following sum rule:

**Proposition 3.1.** Given $\hat{n} \in C^0(\hat{P})$ and $\hat{F}^c$ a truncated face of $\hat{P}$ with outward normal $F^c$. Let $q^c(\hat{n})$ denote the number of pairs of consecutive truncated edges of $\hat{F}^c$ on which $\hat{n}$ is oppositely oriented with respect to $F^c$ (ie, $e^b(\hat{n}) \cdot e'^b(\hat{n}) < 0$ for consecutive $\hat{E}^b$ and $\hat{E}'^b$). Then

\[
\sum_{a'} k^{ac}(\hat{n}) = \frac{1}{2} q^c(\hat{n}) - 1, \tag{14}\]

where the sum $\sum_{a'}$ is taken over the cleaved edges $\hat{B}^{ac}$ of $\hat{F}^c$.

**Proof.** Let $z^c(t)$, $0 \leq t \leq 1$, denote a continuous parameterisation of $\partial \hat{F}^c$ (the boundary of $\hat{F}^c$), positively oriented with respect to $F^c$, with $z^c(1) = z^c(0)$. Let $\hat{n}^c(t) = \hat{n}(z^c(t))$, and let

\[
\hat{n}^c(t) = R(F^c, \xi^c(t)) \cdot \hat{n}^c(0), \tag{15}\]
where $\xi^c(t)$ is continuous with $\xi^c(0) = 0$. Along the truncated edges of $\hat{F}^c$, $\xi^c(t)$ is constant. It follows that $\xi^c(1) = \sum_a '\xi^{ac}(1)$. But $\xi^c(1)$ is just $2\pi$ times the winding number of $\hat{n}$ around $\partial\hat{F}^c$. Since $\hat{n}$ is continuous inside $\hat{F}^c$, this winding number vanishes. Therefore

$$\sum_a '\xi^{ac}(1) = 0. \quad (16)$$

Taking the sum $\sum_a '$ in (12), we conclude that

$$\sum_a 'k^{ac}(\hat{n}) = -\sum_a '\frac{1}{2\pi} \eta^{ac}. \quad (17)$$

Without loss of generality, we may assume that $F^c$, the face of the original polyhedron $P$, is a regular polygon ($P$ can be continuously deformed while remaining convex to make $F^c$ regular). In this case, $\hat{n}^{ac}(0)$ and $\hat{n}^{ac}(1)$ are parallel to consecutive edges of a regular polygon. If $\hat{n}^{ac}(0)$ and $\hat{n}^{ac}(1)$ are similarly oriented with respect to $F^c$, then $\eta^{ac} = 2\pi/m$, where $m$ is the number of sides of $F^c$. If they are oppositely oriented, then $\eta^{ac} = 2\pi/m - \pi$. Substituting into (17), and noting that there are $m$ terms in the sum, we obtain the required result (14). \qed

Wrapping numbers classify the homotopy type of $\hat{n}$ on the cleaved faces $\hat{C}^a$. For the explicit definition it will be useful to introduce coordinates on $\hat{C}^a$. Let $z^a(\phi)$ denote a piecewise-differentiable, $2\pi$-periodic parameterisation of $\partial\hat{C}^a$, positively oriented with respect to the outward normal, denoted $C^a$, on $\hat{C}^a$. Let $c^a$ be a point in the interior of $\hat{C}^a$, and let

$$y^a(\rho, \phi) = \rho z^a(\phi) + (1 - \rho)c^a, \quad (18)$$

where $0 \leq \rho \leq 1$.

To a map $\hat{n} \in C^0(\hat{P})$, we associate a continuous map $\nu^a$ from $D^2$, the unit two-disk, to $S^2$, given by

$$\nu^a(\rho, \phi) = \hat{n}(y^a(\rho, \phi)). \quad (19)$$

We construct another continuous map $\nu_0^a : D^2 \rightarrow S^2$ as follows. On the boundary of the disk, $\nu_0^a$ is taken to coincide with $\nu^a$. Along radial lines from the boundary to the centre of the disk, $\nu_0^a$ is taken to trace out the shortest geodesic from its value on the boundary to a fixed value, which we denote by $-s$. (In what follows, we sometimes regard $s$ as the south pole of $S^2$, and $-s$ as the north pole.) Explicitly, let $g_\rho(-s, a)$, where $0 \leq \rho \leq 1,$
denote the shortest geodesic arc from $-s$ to $a$, where the parameter $\rho$ is proportional to arclength. Then $\nu_0^a : D^2 \to S^2$ is given by

$$\nu_0^a(\rho, \phi) = g_\rho(-s, \hat{n}(z^a(\phi))).$$

(20) is well defined provided that the boundary values of $\hat{n}$ are not antipodal to $-s$, i.e. $\hat{n} \neq s$ on $\partial \hat{C}^a$. Since, on $\partial \hat{C}^a$, $\hat{n}$ is tangent to a truncated face, this condition is satisfied provided that

$$s \cdot F^c \neq 0, \quad c = 1, \ldots, f.$$  

(21)

From now on, we assume $s$ is chosen to satisfy (21). Note that, by construction, $s$ is not in the image of $\nu_0^a$.

Given two maps $\nu^a, \nu_0^a : D^2 \to S^2$ which coincide on $\partial D^2$, we may glue them on the boundary to get a continuous map on $S^2$, which we denote by $\nu^a \circ \nu_0^a$. Explicitly, $\nu^a \circ \nu_0^a : S^2 \to S^2$ is given by

$$(\nu^a \circ \nu_0^a)(x, y, z) = \begin{cases} 
\nu^a(\rho, \phi), & z \geq 0, \\
\nu_0^a(\rho, \phi), & z < 0,
\end{cases}$$

(22)

where $(\rho, \phi)$ are the polar coordinates of $(x, y)$. The wrapping number is the degree of this map.

**Definition 3.3.** The wrapping number $w^a(\hat{n})$ is given by

$$w^a(\hat{n}) = \deg(\nu^a \circ \nu_0^a).$$

(23)

The wrapping number depends continuously on $\hat{n}$ (since $\nu^a$ and $\nu_0^a$ do), and therefore is a homotopy invariant.

For $\hat{n} \in C^1(\hat{P})$ (i.e., $\hat{n}$ is continuously differentiable on $\hat{P}$), we derive an integral formula for the wrapping number. We take $z^a(\phi)$ to be piecewise-$C^1$, so that the derivative of $\nu^a \circ \nu_0^a$ is piecewise continuous. Then

$$\deg(\nu^a \circ \nu_0^a) = \frac{1}{4\pi} \int_{S^2} (\nu^a \circ \nu_0^a)^* \omega,$$

(24)

where $\omega$ is the rotationally invariant area-form on $S^2$, normalised so that $\int_{S^2} \omega = 4\pi$, and $(\nu^a \circ \nu_0^a)^*$ denotes the pull-back. From (22),

$$w^a(\hat{n}) = \deg(\nu^a \circ \nu_0^a) = \frac{1}{4\pi} \int_{D^2} (\nu^a)^* \omega - (\nu_0^a)^* \omega = \frac{1}{4\pi} \int_{\hat{C}^a} \hat{n}^* \omega - \frac{1}{4\pi} \int_{D^2} \nu_0^a \omega.$$

(25)
By construction, \( \nu^a_0 \) takes values in \( \{ S^2 - s \} \) (the two-sphere with the point \( s \) removed). Let \( \gamma \) denote a one-form on \( \{ S^2 - s \} \) for which

\[
d\gamma = \omega \text{ on } \{ S^2 - s \}.
\]

For example, we may take \( \gamma = (1 - \cos \alpha)d\beta \), where \( (\alpha, \beta) \) are spherical polar coordinates on \( S^2 \) with south pole at \( s \). Applying Stokes’ theorem to the second integral in (25), we get

\[
w^a(\hat{n}) = \frac{1}{4\pi} \left( \int_{C^a} \hat{n}^* \omega - \int_{\partial C^a} \hat{n}^* \gamma \right).
\]

From (27), it is clear that wrapping numbers change sign under the antipodal map \( \hat{n} \to -\hat{n} \),

\[
w^a(-\hat{n}) = -w^a(\hat{n}).
\]

In fact, (28) holds for all maps in \( C^0(\hat{P}) \), since any map in \( C^0(\hat{P}) \) is homotopic to a \( C^1 \)-map in \( C^0(\hat{P}) \).

If \( s \) is a regular value of \( \hat{n} \) on \( \hat{C}^a \) – that is, if the derivative \( d\hat{n} \) restricted to \( \hat{C}^a \) has full rank on \( \hat{n}^{-1}(s) \) – then \( \hat{n}^{-1}(s) \) is a finite set, and we can express the wrapping number as a signed count of preimages \( y^a_* \) of \( s \) on \( \hat{C}^a \). We have that

\[
\hat{C}^a = \left( \hat{C}^a - \sum_{y^2} U_\epsilon(y^a_*) \right) + \sum_{y^2} U_\epsilon(y^a_*),
\]

where \( U_\epsilon(y^a_*) \) is an \( \epsilon \)-neighbourhood of \( y^a_* \). Substituting (29) into the integral formula (27), the contribution from the first term in (29) vanishes due to Stokes’ theorem, while each neighbourhood \( U_\epsilon(y^a_*) \) in the second term contributes \( \text{sgn det } d\hat{n}(y^a_*) \), where the determinant is computed with respect to positively oriented coordinates on \( \hat{C}^a \) and \( S^2 \). Then

\[
w^a(\hat{n}) = \sum_{y^2} \text{sgn det } d\hat{n}(y^a_*).
\]

Next we use (27) to show that the sum of the wrapping numbers vanishes.

**Proposition 3.2.** Given \( \hat{n} \in C^0(\hat{P}) \),

\[
\sum_{a=1}^v w^a(\hat{n}) = 0.
\]
Proof. We have that
\[ \sum_{a=1}^{v} w^a(\hat{n}) = \sum_{a=1}^{v} \int_{\hat{C}^a} \hat{n}^* \omega - \sum_{a=1}^{v} \int_{\partial \hat{C}^a} \hat{n}^* \gamma. \] (32)

The boundary of the truncated polyhedron \( \hat{P} \) is given by
\[ \partial \hat{P} = \sum_{a=1}^{v} \hat{C}^a + \sum_{c=1}^{f} \hat{F}^c. \] (33)

Since \( \partial(\partial \hat{P}) = 0 \),
\[ \sum_{a=1}^{v} \partial \hat{C}^a + \sum_{c=1}^{f} \partial \hat{F}^c = 0. \] (34)

The second integral in (32) may then be rewritten as
\[ \sum_{a=1}^{v} \int_{\partial \hat{C}^a} \hat{n}^* \gamma = - \sum_{c=1}^{f} \int_{\partial \hat{F}^c} \hat{n}^* \gamma = - \sum_{c=1}^{f} \int_{\hat{F}^c} \hat{n}^* \omega, \] (35)
where in the second equality we have used Stokes’ theorem and (26) (this is justified since \( \hat{n} \neq s \) on \( \hat{F}^c \)). Substituting (35) into (32), we get
\[ \sum_{a=1}^{v} w^a(\hat{n}) = \left( \sum_{a=1}^{v} \int_{\hat{C}^a} + \sum_{c=1}^{f} \int_{\hat{F}^c} \right) \hat{n}^* \omega = \int_{\partial \hat{P}} \hat{n}^* \omega. \] (36)

Since \( \omega \) is closed, the last expression vanishes. Therefore
\[ \sum_{a=1}^{v} w^a(\hat{n}) = 0. \] (37)

This result applies to all maps in \( C^0(\hat{P}) \), as every map in \( C^0(\hat{P}) \) is homotopic to a \( C^1 \)-map in \( C^0(\hat{P}) \).

The wrapping number depends on the choice of \( s \in S^2 \). For \( \hat{n} \in C^1(\hat{P}) \), an alternative, convention-independent invariant is the real-valued quantity
\[ \Omega^a(\hat{n}) = \int_{\hat{C}^a} \hat{n}^* \omega = 4\pi w^a(\hat{n}) + \int_{\partial \hat{C}^a} \hat{n}^* \gamma. \] (38)

We call \( \Omega^a \) the trapped area at the vertex \( a \). It plays a central role in estimates of lower bounds for the energy \( [11, 11, 13] \).
The second term in the expression (38) for the trapped area can be expressed in terms of the kink numbers and edge orientations, as we now show. We have that

$$\hat{n}(\partial \hat{C}^a) = \sum_j k^{ac} S^{1c} + K^a,$$

where, in the first term, $S^{1c}$ denotes the unit circle in $S^2$ normal to $F^c$, positively oriented with respect to $F^c$, and the sum $\sum_j$ is taken over the cleaved edges $\hat{B}^{ac}$ of $\partial \hat{C}^a$. From (26), the integral of $\gamma$ around $S^{1c}$ is given by $-2\pi \text{sgn}(F^c \cdot s)$. The second term in (39), $K^a$, is the geodesic polygon in $S^2$ with vertices $e^{b_1}(\hat{n}), \ldots, e^{b_m}(\hat{n})$, where the indices $b_r$ label the truncated edges $\hat{E}^{b_1}, \ldots, \hat{E}^{b_m}$ of $\partial \hat{C}^a$, consecutively ordered with respect to the outward normal. Suppose first that $K^a$ has just three vertices, which we denote $a$, $b$ and $c$ for convenience. From (26),

$$\int_{K^a} \gamma = A(a, b, c) - 4\pi \sigma(a, b, c),$$

where $A(a, b, c)$ is the oriented area of $K^a$, with the interior of $K^a$ chosen so that $|A(a, b, c)| < 2\pi$, and $\sigma(a, b, c) = \pm 1, 0$ according to whether $s$ is outside $K^a$ (in which case $\sigma = 0$) or inside $K^a$ (in which case $\sigma$ is the orientation of $\partial K^a$ about $s$). Explicitly, $A(a, b, c)$ is given by

$$A(a, b, c) = 2\text{arg}((1 + a \cdot b + b \cdot c + c \cdot a) + i(a \times b) \cdot c),$$

where $\text{arg}$ is taken between $-\pi$ and $\pi$ (41) is equivalent to the standard expression $\alpha + \beta + \gamma - \pi$ for the area of a unit spherical triangle with interior angles $\alpha$, $\beta$ and $\gamma$. $\sigma(a, b, c)$ is given by

$$\sigma(a, b, c) = \begin{cases} 
\text{sgn}((a \times b) \cdot s), & s \in K^a, \\
0, & s \notin K^a.
\end{cases}$$

In fact, $s \in K^a$ if and only if $(a \times b) \cdot s$, $(b \times c) \cdot s$ and $(c \times a) \cdot s$ all have the same sign. If $K^a$ has $m > 3$ vertices, we may represent it as a sum of geodesic triangles $K_j^a$ with vertices $e^{b_1}(\hat{n}), e^{b_2}(\hat{n}), e^{b_{j+1}}(\hat{n})$, with $2 \leq j \leq m - 1$.

These considerations are summarised in the following:

**Proposition 3.3.** Given a cleaved face $\hat{C}^a$ with truncated edges $\hat{E}^{b_1}, \ldots, \hat{E}^{b_m}$ consecutively ordered with respect to the outward orientation. The trapped area (38) is given by

$$\Omega^a = 4\pi w^a - 2\pi \sum_c \text{sgn}(F^c \cdot s) k^{ac} + \sum_{j=2}^{m-1} \left(A(e^{b_1}, e^{b_j}, e^{b_{j+1}}) - 4\pi \sigma(e^{b_1}, e^{b_j}, e^{b_{j+1}}) \right),$$

where the sum $\sum_c' \text{sgn}(F^c \cdot s) k^{ac}$ is taken over the cleaved edges $\hat{B}^{ac}$ of $\hat{C}^a$, and $A$ and $\sigma$ are given by (41) and (42) respectively.
4 Representatives

Let
\[ \text{Inv} = \{ \varepsilon^b, \kappa^{ac}, w^a \} \]  
(43)
denote the set of homotopy invariants on \( C^0(\hat{P}) \) defined in Section \ref{section}. Let \( I = (\varepsilon^b, \kappa^{ac}, \omega^a) \) denote a set of values of Inv which satisfies the sum rules (14) and (37). In what follows, we construct a representative map \( \hat{n}_I \in C^0(P) \) for which
\[ \text{Inv}(\hat{n}_I) = I. \]  
(44)

We first define \( \hat{n}_I \) on the edges of \( \hat{P} \). On the truncated edges, \( \hat{n}_I \) is determined by the edge orientations, \( \varepsilon^b \).
\[ \hat{n}_I(x) = \varepsilon^b, \quad x \in \hat{E}^b. \]  
(45)

On the cleaved edges, \( \hat{n}_I \) is determined up to homotopy by the edge orientations and the kink numbers, \( \kappa^{ac} \). Let \( z^{ac}(t), \quad 0 \leq t \leq 1, \) denote a parameterisation of \( \hat{B}^{ac} \), positively oriented with respect to \( F^c \). Let the endpoints \( z^{ac}(0) \) and \( z^{ac}(1) \) lie on consecutive truncated edges \( \hat{E}^b \) and \( \hat{E}^b' \) respectively. Let \( \eta^{ac} \in (-\pi, \pi) \) denote the angle from \( \varepsilon^b \) to \( \varepsilon^b' \), as in (11). Then on \( \hat{B}^{ac} \), we take \( \hat{n}_I \) to be given by
\[ \hat{n}_I(z^{ac}(t)) = R(F^c, (\eta^{ac} + 2\pi\kappa^{ac})t) \cdot \varepsilon^b. \]  
(46)

To extend \( \hat{n}_I \) to the faces of \( \hat{P} \), it is convenient to introduce polygonal-polar coordinates. Let \( f^c \) be a point in the interior of the truncated face \( \hat{F}^c \). We parameterise \( \hat{F}^c \) by
\[ y^c(\rho, z^c) = \rho z^c + (1 - \rho)f^c, \]  
(47)
where \( 0 \leq \rho \leq 1 \) and \( z^c \in \partial\hat{F}^c \). By a radial chord, we mean the segment obtained by taking \( z^c \) fixed in (47), and letting \( \rho \) vary between 0 and 1. Similarly, let \( c^a \) be a point in the interior of the cleaved face \( \hat{C}^a \). We parameterise \( \hat{C}^a \) by
\[ y^a(\rho, z^a) = \rho z^a + (1 - \rho)c^a, \]  
(48)
Radial chords on \( \hat{C}^a \) are defined as for \( \hat{F}^c \).

On \( \hat{F}^c \), we define \( \hat{n}_I \) along radial chords by contracting its boundary values to a constant. Explicitly, we note that (43) and (46) determine \( \hat{n}_I \) on \( \partial\hat{F}^c \). We regard \( \hat{n}_I \) on \( \partial\hat{F}^c \) as a continuous map of \( S^1 \) to itself (the image lies in \( S^1 \)), the great circle orthogonal to \( F^c \). Since the kink numbers \( \kappa^{ac} \) satisfy the sum rule (14), this map has zero winding number, and therefore
is contractible. That is, there exists a continuous unit-vector field $\mathbf{h}(\mathbf{z}^c)$ tangent to $\hat{F}^c$ such that $\hat{h}^3_0(\mathbf{z}^c) = \mathbf{n}_z(\mathbf{z}^c)$ and $\hat{h}^3_1$ is constant. Let

$$
\hat{h}_z(y^c(\rho, \mathbf{z}^c)) = \hat{h}^3_0(\mathbf{z}^c).
$$

(49)

On $\hat{C}^a$, we note that (44) determines the values of $\mathbf{n}_z$ on $\partial \hat{C}^a$, where $\rho = 1$. We define $\mathbf{n}_z$ for $\frac{1}{2} \leq \rho < 1$ by contracting its boundary values along shortest geodesics on $S^2$ to $-s$.

$$
\mathbf{n}_z(y^a(\rho, \mathbf{z}^a)) = g_{\rho - 1}(-s, \mathbf{n}_z(y^a(1, \mathbf{z}^a))), \quad \frac{1}{2} \leq \rho < 1,
$$

(50)

where $g_{\rho}(-s, a), 0 \leq \tau \leq 1$, denotes the shortest geodesic from $-s$ to $a$ (as in (20)). For $\rho \leq \frac{1}{2}$, we insert a covering of $S^2$ with multiplicity given by the wrapping number $\omega^a$. Explicitly, let $\mathbf{z}(\phi)$ be a $2\pi$-periodic parameterisation of $\partial \hat{C}^a$, and let

$$
\mathbf{n}_z(y^a(\rho, \mathbf{z}(\phi))) = \sin 2\pi \rho \cos \omega^a \phi \mathbf{\xi} + \sin 2\pi \rho \sin \omega^a \phi \mathbf{\eta} + \cos 2\pi \rho s, \quad 0 \leq \rho < \frac{1}{2},
$$

(51)

where $\mathbf{\xi}$ and $\mathbf{\eta}$ are orthonormal vectors in the plane perpendicular to $s$ with $\mathbf{\xi} \times \mathbf{\eta} = -s$. Let $(\alpha, \beta)$ denote polar coordinates on $S^2$ with south pole at $s$. Identifying $S^2$ with the region $\rho \leq \frac{1}{2}$ on $\hat{C}^a$ via $\alpha = (\pi - \alpha)/2\pi, \mathbf{z}^a = \mathbf{z}^a(\beta)$, then (51) corresponds to the $S^2$-map $(\alpha, \beta) \mapsto (\alpha, \omega^a(\beta))$, which has degree $\omega^a$. It is readily verified from (23) that $w^a(\mathbf{n}_z) = \omega^a$.

We extend $\mathbf{n}_z$ to the interior of $\hat{P}$ along radial lines by contracting its boundary values to a constant. Explicitly, we note that (44), (50) and (51) determine $\mathbf{n}_z$ on $\partial \hat{P}$. From (50), the integral of $\mathbf{n}_z$ over $\partial \hat{P}$ is given by the sum of the wrapping numbers $\omega^a$. By assumption, this sum vanishes, so that

$$
\int_{\partial \hat{P}} \mathbf{n}_z \omega = 0
$$

(52)

(we can take $\mathbf{n}_z$ to be piecewise-differentiable on $\partial \hat{P}$, so that $\mathbf{n}_z \omega$ is piecewise-continuous). Regarding $\partial \hat{P}$ as a topological two-sphere, we may regard $\mathbf{n}_z$ on $\partial \hat{P}$ as a degree-zero map on $S^2$. There exists a contraction to a constant map. Let $\mathbf{h}_t : \partial \hat{P} \to S^2$, where $0 \leq t \leq 1$, be such a contraction, ie $\mathbf{h}_0$ is continuous, $\mathbf{h}_0 = \mathbf{n}_z$ and $\mathbf{h}_1 = s$, constant. Let $p$ be a point in the interior of $\hat{P}$, and let

$$
x(r, y) = ry + (1 - r)p,
$$

(53)

where $0 \leq r \leq 1$. Then we define $\mathbf{n}_z$ in $\hat{P}$ by

$$
\mathbf{n}_z(x(r, y))) = \mathbf{h}_r(y).
$$

(54)

Let $c^{a*}$ denote the interior point of the cleaved face $\hat{C}^{a*}$. Setting $\rho = 0$ in (51), we see that $\mathbf{n}_z(c^{a*}) = s$. Without loss of generality, and for future
convenience, we choose the homotopy \( \hat{h}_t \) so that \( \hat{h}_t(c^{a_1}) = s \) for all \( 0 \leq t \leq 1 \). Therefore, from (54),

\[
\hat{n}_I(x, c^{a_1}) = s.
\] (55)

We note that the construction of \( \hat{n}_I \) is not completely explicit, in that we make use of the contractibility of degree-zero maps on \( S^1 \) and \( S^2 \) without specifying these contractions explicitly. An explicit prescription for these contractions (which is valid for all \( S^n \)) is described by, eg, Whitehead [9] (of course, for \( S^1 \), the contraction is easily constructed).

5 Classification

Our main result is that the invariants, \( \text{Inv} \), classify maps in \( C^0(\hat{P}) \) up to homotopy.

**Theorem 1.** Let \( \hat{n}, \hat{n}' \in C^0(\hat{P}) \). Then \( \hat{n} \sim \hat{n}' \) if and only if \( \text{Inv}(\hat{n}) = \text{Inv}(\hat{n}') \).

**Proof.** Since \( \text{Inv}(\hat{n}) \) is homotopy invariant, it is clear that \( \hat{n} \sim \hat{n}' \) only if \( \text{Inv}(\hat{n}) = \text{Inv}(\hat{n}') \). For the converse, it suffices to show that \( \hat{n} \) is homotopic to the representative map \( \hat{n}_I \), where \( I = \text{Inv}(\hat{n}) \).

It will be convenient to use the polyhedral-polar coordinates \( x(r, y) \) on \( \hat{P} \) given by (53), where \( 0 \leq r \leq 1 \) and \( y \in \partial \hat{P} \). The sets \( r = \) constant interpolate between the boundary \( \partial P (r = 1) \) and the interior point \( p (r = 0) \). Let \( \hat{P}(a, b) \) denote the polyhedral shell \( a \leq r \leq b \). With an abuse of notation, we shall sometimes write, for the sake of brevity, \( \hat{n}(r, y) \), rather than \( \hat{n}(x(r, y)) \), and similarly for other maps in \( C^0(\hat{P}) \).

To show that \( \hat{n} \sim \hat{n}_I \), we argue as follows. First, we deform \( \hat{n} \) to a map \( \hat{n}_1 \) which coincides with a radially scaled copy of \( \hat{n}_I \) on the outer shell \( \hat{P}(1, 1) \) and which is constant, equal to \( s \), on the inner shell \( \hat{P}(0, \epsilon) \), where \( \epsilon > 0 \) is specified below. The dependence of \( \hat{n}_1 \) on the original map \( \hat{n} \) is confined to the polyhedral bubble \( \hat{P}(0, \epsilon) \). Then, we create a radial channel through the outer shell, inside of which the map is made to be constant, equal to \( s \). The polyhedral bubble is made to evaporate through this channel. The channel is then removed, leaving a map which is a radially scaled copy of \( \hat{n}_I \) on \( \hat{P}(0, \frac{1}{2}) \) and which is constant, equal to \( s \), on \( \hat{P}(0, \frac{1}{2}) \). A final rescaling produces \( \hat{n}_I \).

A schematic description of these deformations is shown in Fig 2. Details of the argument follow below.

Without loss of generality, we may assume that \( \hat{n} \) coincides with \( \hat{n}_I \) on \( \partial \hat{P} \); this is demonstrated in the following section (see Prop 6.1). Then for
any \(0 < \epsilon < \frac{1}{2}\), \(\hat{n}\) is homotopic to a map \(\hat{n}_1 \in C^0(\hat{P})\) given by

\[
\hat{n}_1(r, y) = \begin{cases} 
\hat{n}_I(2r - 1, y), & \frac{1}{2} \leq r \leq 1, \\
\hat{s}, & \epsilon \leq r < \frac{1}{2}
\end{cases}
\] (56)

for \(\epsilon \leq r \leq 1\). Note that, from (55), \(\hat{n}_I(0, y) = \hat{s}\), so that \(\hat{n}_1\) is continuous at \(r = \frac{1}{2}\). For \(r < \epsilon\), \(\hat{n}_1\) is given by

\[
\hat{n}_1(r, y) = \begin{cases} 
\hat{n}_I(2(\epsilon - r)/\epsilon, y), & \frac{1}{2}\epsilon \leq r < \epsilon, \\
\hat{n}(2r/\epsilon, y), & 0 \leq r < \frac{1}{2}\epsilon.
\end{cases}
\] (57)

See Fig 2(b). In fact, the particular form for \(r \leq \epsilon\) will not concern us in what follows. A homotopy between \(\hat{n}_I\) and \(\hat{n}_1\) is given by

\[
H_t(r, y) = \begin{cases} 
\hat{n}_I(\sigma(r), y), & 1 - \frac{1}{2}t \leq r \leq 1, \\
\hat{n}_I(1 - t, y), & 1 - (1 - \epsilon)t \leq r < 1 - \frac{1}{2}t, \\
\hat{n}_I(\tau_t(r), y), & 1 - (1 - \frac{1}{2}\epsilon)t \leq r < 1 - (1 - \epsilon)t, \\
\hat{n}(\upsilon_t(r), y), & r < 1 - (1 - \epsilon/2)t,
\end{cases}
\] (58)

where

\[
\sigma(r) = 2r - 1, \\
\tau_t(r) = 1 + 2((1 - r) - (1 - \frac{1}{2}\epsilon)t)/\epsilon, \\
\upsilon_t(r) = r/(1 - (1 - \frac{1}{2}\epsilon)t).
\] (59)

Consider the set \(T\) given by

\[
T = \{x(r, y^{a*}(\rho, z^{a*}))| r \geq \frac{1}{2}, \rho \leq \frac{1}{2}\},
\] (60)

where \(y^{a*}(\rho, z^{a*})\) denotes the polygonal-polar coordinates \([48]\) on \(\hat{C}^{a*}\). \(T\) represents a channel in the outer shell \(\hat{P}^{1/2}, 1\) through the cleaved face \(\hat{C}^{a*}\). The central axis of \(T\), where \(\rho = 0\), is given by \((1 - r)c^{a*} + r\mathbf{p}, r \geq \frac{1}{2}\).

From (55) and (56), it follows that \(\hat{n}_1 = \hat{s}\) along this axis. We show that \(\hat{n}_1\) is homotopic to a map \(\hat{n}_2\) which is equal to \(\hat{s}\) throughout \(T\), and which coincides with \(\hat{n}_1\) for \(r < \frac{1}{2}\) and for \(y \notin \hat{C}^{a*}\). A homotopy \(\hat{H}_i(r, y)\) is given by \(\hat{n}_1(r, y)\) for \(r < \frac{1}{2}\) or \(y \notin \hat{C}^{a*}\), and for \(r \geq \frac{1}{2}\) and \(y \in \hat{C}^{a*}\) by

\[
\hat{H}_i(r, y^{a*}(\rho, z^{a*})) = \begin{cases} 
\hat{n}_1(r, y^{a*}((2\rho - t)/(2t - 2), z^{a*})), & t/2 < \rho \leq 1, \\
\hat{s}, & 0 \leq \rho \leq t/2.
\end{cases}
\] (61)
Figure 2: Homotopy from $\hat{n}$ to $\hat{n}_I$. Polyhedral shells $\hat{P}(a,b)$ are represented schematically as spherical shells. (a) $\hat{n}$ coincides with $\hat{n}_I$ on $\partial \hat{P}$. The marked point is $c^{a*}$, where $\hat{n}_I = s$. (b) $\hat{n}_1$. Note that $\hat{n}_1 = s$ along the outer half of the ray from $c^{a*}$ to the centre. (c) $\hat{n}_2$ is equal to $s$ in the channel. (d) The polyhedral bubble, $P(0,\epsilon)$, is floated through the channel. (e) $\hat{n}_3$ (f) The channel is removed to obtain $\hat{n}_4$. (g) $\hat{n}_I$
where \( z^{a_\ast} \in \partial \hat{C}^{a_\ast} \). Let \( \hat{n}_2 = \hat{H}_1 \). Then, for \( r \geq \frac{1}{2} \),

\[
\hat{n}_2(r, y^{a_\ast}(\rho, z^{a_\ast})) = \begin{cases} 
\hat{n}_1(r, y^{a_\ast}(2\rho - 1, z^{a_\ast})), & \frac{1}{2} < \rho \leq 1, \\
\hat{s}, & 0 \leq \rho \leq \frac{1}{2}.
\end{cases}
\]

(62)

\( \hat{n}_2 \) is constant, equal to \( \hat{s} \), in the inner shell \( \hat{P}(\epsilon, \frac{1}{2}) \) as well as in \( T \). See Fig. 2(c).

Next we deform \( \hat{n}_2 \) so that it is constant, equal to \( \hat{s} \), throughout the whole inner polyhedron \( \hat{P}(0, \frac{1}{2}) \). This is accomplished by displacing the polyhedral bubble in which \( \hat{n}_1 \) is varying from \( \hat{P}(0, \epsilon) \) through the shell \( \hat{P}(\epsilon, \frac{1}{2}) \) and then through the channel \( T \). Let \( u \) be parallel to the axis of \( T \), i.e., proportional to \( c^{a_\ast} - p \), with \( |u| \) sufficiently large so that

\[
\{ \hat{P}(0, \epsilon) + tu \} \cap \hat{P} = \emptyset.
\]

(63)

Choose \( \epsilon \) sufficiently small so that

\[
\{ \hat{P}(0, \epsilon) + tu \} \cap \hat{P} \subset \hat{P}(0, \frac{1}{2}) \cup T, \quad 0 \leq t \leq 1.
\]

(64)

Let

\[
\hat{H}_1(x) = \begin{cases} 
\hat{n}_2(x - tu), & x \in \{ \hat{P}(0, \epsilon) + tu \} \cap \hat{P}, \\
\hat{s}, & x \in \hat{P}(0, \epsilon) \text{ and } x \notin \{ \hat{P}(0, \epsilon) + tu \}, \\
\hat{n}_2(x), & \text{otherwise}.
\end{cases}
\]

(65)

See Fig. 2(d). (64) guarantees that \( \hat{H}_1(x) \) is continuous, as \( \hat{n}_2 \) is continuous and is constant, equal to \( \hat{s} \), throughout \( \hat{P}(0, \frac{1}{2}) \cup T \). Let \( \hat{n}_3 = \hat{H}_1 \). From (63) and (62), it follows that \( \hat{n}_3 \) is constant, equal to \( \hat{s} \), on \( \hat{P}(0, \epsilon) \) and that it coincides with \( \hat{n}_2 \) in \( \hat{P}(\epsilon, 1) \). See Fig. 2(e). By applying the inverse of the homotopy (61), with \( \hat{n}_1 \) replaced by \( \hat{n}_3 \), we can collapse the channel \( T \) to obtain a map \( \hat{n}_4 \) (see Fig. 2(f)) given by

\[
\hat{n}_4(r, y) = \begin{cases} 
\hat{n}_2(2r - 1, y), & \frac{1}{2} \leq r \leq 1, \\
\hat{s}, & r < \frac{1}{2}.
\end{cases}
\]

(66)

Then

\[
\hat{H}_t(r, y) = \begin{cases} 
\hat{n}_2((2r - (1 - t))/(1 + t), y), & \frac{1}{2}(1 - t) \leq r \leq 1, \\
\hat{s}, & \rho < \frac{1}{2}(1 - t)
\end{cases}
\]

(67)

describes a homotopy of \( \hat{n}_4 \) to \( \hat{n}_T \). \( \square \)
6 Surface homotopies

An intermediate step in the proof of Theorem 1 is the fact that maps in $C^0(\hat{P})$ can be deformed to coincide with their associated representative maps on $\partial \hat{P}$. This is summarised by the following:

**Proposition 6.1.** Let $\hat{n} \in C^0(\hat{P})$, with $I = \text{Inv}(\hat{n})$. Then $\hat{n}$ is homotopic to a map $\hat{n}'$ for which $\hat{n}' = \hat{n}_I$ on $\partial \hat{P}$.

To prove Proposition 6.1, we make use of the fact that deformations of $\hat{n}$ on the edges of $\hat{P}$ can be extended to deformations of $\hat{n}$ on the faces, and, similarly, deformations of $\hat{n}$ on the faces of $\hat{P}$ can be extended to deformations of $\hat{n}$ on $\hat{P}$ itself. For completeness, we give an argument below which covers both cases (of course, a similar result holds generally on manifolds with boundary).

**Lemma 6.1.** Let $Q \subset \mathbb{R}^k$ be compact and convex with boundary $\partial Q$, and let $S$ be a topological space with subspace $T$. Let $C^0(Q)$ denote the space of continuous maps from $Q$ to $S$ which map $\partial Q$ to $T$, and let $C^0(\partial Q)$ denote the space of continuous maps of $\partial Q$ to $T$. Given $n \in C^0(Q)$, let $\partial n \in C^0(\partial Q)$ denote its restriction to $\partial Q$. Suppose that $\partial n$ is homotopic to $\nu' \in C^0(\partial Q)$. Then $n$ is homotopic to some $n' \in C^0(Q)$ with $\partial n' = \nu'$.

**Proof.** Introduce polygonal-polar coordinates on $Q$. I.e, let $q$ be a point in the interior of $Q$, and let $u(\lambda, v) = \lambda v + (1 - \lambda)q$, where $0 \leq \lambda \leq 1$ and $v \in \partial Q$. Given $n \in C^0(Q)$, we write, by an abuse of notation but for the sake of brevity, $n(\lambda, v) = n(u(\lambda, v))$, and similarly for other maps in $C^0(Q)$. Let $h_t$ be a homotopy from $\partial n$ to $\nu'$. Let $H_t$ be given by

$$ H_t(\rho, v) = \begin{cases} h_{2\rho + t - 2}(v), & 1 - \frac{1}{2}t < \rho \leq 1, \\ n(\rho/(1 - \frac{1}{2}t), v), & \rho \leq 1 - \frac{1}{2}t. \end{cases} \quad (68) $$

Let $n' = H_1$. Then $n$ is homotopic to $n'$, and $\partial n' = \nu'$.

**Proof of Proposition 6.1.** Let $C^0(\partial \hat{P})$ denote the space of continuous tangent unit-vector fields on the boundary of $\hat{P}$ (so that $\hat{n}(y)$ is tangent to $\partial \hat{P}$ at $y$). Given $\hat{n} \in C^0(\hat{P})$, let $\partial \hat{n} \in C^0(\partial \hat{P})$ denote its restriction to $\partial \hat{P}$.

From Lemma 6.1 it suffices to show that

$$ \partial \hat{n} \sim \partial \hat{n}_I, \quad (69) $$

where we have the usual notion of homotopic equivalence in $C^0(\partial \hat{C})$. We establish (69) in two steps, first deforming $\partial \hat{n}$ to coincide with $\partial \hat{n}_I$ on the
edges of $\partial \hat{P}$, and then deforming it further to coincide with $\partial \hat{n}_I$ on the faces of $\partial \hat{P}$.

Since $\hat{n}$ and $\hat{n}_I$ have the same edge orientations (ie, $e^b(\hat{n}) = e^b$), they coincide on truncated edges, and therefore coincide on the endpoints of the cleaved edges $\hat{B}^{ac}$. Since $\hat{n}$ and $\hat{n}_I$ have the same kink numbers, there is a homotopy between the restrictions of $\hat{n}$ and $\hat{n}_I$ to the cleaved edges. (Explicitly, if, on $\hat{B}^{ac}$, $\hat{n}$ is represented by an angle $\theta^{ac}(s)$ in the plane tangent to $\hat{F}^c$, with $0 \leq s \leq 1$, and $\hat{n}_I$ is similarly represented by $\theta^{ac}(s)$ with $\theta^{ac}(0) = \theta^{ac}(0)$, then $k^{ac} = k^{ac}$ implies that $\theta^{ac}(1) = \theta^{ac}(1)$, and a homotopy is given by $(1-t)\theta^{ac}(s) + t\theta^{ac}(s)$.) By Lemma 6.1 these homotopies on $\hat{B}^{ac}$ can be extended to homotopies which will be convenient in what follows.

es on the faces of $\hat{P}$, and therefore to a homotopy $\hat{h}$ on $\partial \hat{P}$. Let $\hat{\nu}' = \hat{h}_1$. By construction, $\hat{\nu}'$ coincides with $\partial \hat{n}_I$ on the edges of $\partial \hat{P}$.

Next, we construct homotopies from $\hat{\nu}'$ to $\partial \hat{n}_I$ on the faces of $\hat{P}$. On the truncated face $\hat{F}^c$, $\hat{\nu}'$ may be represented by an angle $\theta^c(\nu')$ in the plane tangent to $\hat{F}^c$. The sum rule (13) ensures that $\theta^c(\nu')$ is continuous. $\partial \hat{n}_I$ may be similarly represented by $\theta^c(\nu')$. By construction, $\theta^c(\nu')$ and $\theta^c(\nu')$ agree on $\hat{F}^c$ up to addition of a multiple of $2\pi$, which we can take to vanish. A homotopy between them on $\hat{F}^c$ is given by $(1-t)\theta^c(\nu') + t\theta^c(\nu')$.

Homotopies from $\hat{\nu}'$ to $\partial \hat{n}_I$ on the cleaved faces may be constructed as follows. Let $y^a(\rho, z^a)$ be the polygonal-polar coordinates on $\hat{C}^a$ given by (48), with $0 \leq \rho \leq 1$ and $z^a \in \partial \hat{C}^a$. We first deform $\hat{\nu}'$ so that it agrees with $\partial \hat{n}_I$ for $\rho \geq \frac{1}{2}$. A homotopy is given by

$$\hat{n}_I^a(\rho, z^a) = \begin{cases} 
\partial \hat{n}_I(2\rho - 1, z^a), & 1 - \frac{1}{2}t < \rho \leq 1, \\
\partial \hat{n}_I(5 - 4\rho - 3t, z^a), & 1 - \frac{3}{4}t < \rho \leq 1 - \frac{1}{2}t, \\
\hat{\nu}'(\rho/(1 - \frac{3}{4}t), z^a), & 0 \leq \rho \leq 1 - \frac{3}{4}t.
\end{cases}$$

(70)

Let $\hat{\nu}'' = \hat{n}_I^a$. Then $\hat{\nu}''$ coincides with $\hat{n}_I$ for $\rho \geq \frac{1}{2}$.

The region $\rho \leq \frac{1}{2}$ on $\hat{C}^a$ is a topological two-disk. On the boundary, where $\rho = \frac{1}{2}$, $\hat{\nu}''$ and $\partial \hat{n}_I$ are both constant, equal to $-s$ (cf (50) and (20).)

By identifying points on the boundary, we may regard $\hat{\nu}''$ and $\hat{n}_I$ as maps on $S^2$ which preserve a marked point $-s$. The fact that $w^a(\hat{n}_I) = \omega^a$ implies that these maps have the same degree, and therefore are homotopic. Thus there exists a homotopy on $\rho \leq \frac{1}{2}$ which takes $\hat{\nu}''$ to $\partial \hat{n}_I$ and which is equal to $-s$ for $\rho = \frac{1}{2}$. This establishes a homotopy between $\hat{\nu}''$ and $\partial \hat{n}_I$ on $\hat{C}^a$.

Together, the homotopies on truncated faces and cleaved faces give a homotopy from $\hat{\nu}''$ to $\partial \hat{n}_I$. The chain of equivalences $\hat{n} \sim \hat{\nu}' \sim \hat{\nu}'' \sim \partial \hat{n}_I$ in $C^0(\partial \hat{P})$ gives the required result.
7 Concluding remarks

The problem considered here may be generalised to \( n > 3 \) dimensions. Generalisations suggested by liquid crystal applications include normal boundary conditions (ie, on the faces of \( P \), \( n \) is required to be orthogonal to the faces), and periodic boundary conditions on a cubic domain from which a polyhedral domain has been excised (this corresponds to an array of liquid crystal cells with polyhedral geometries). It may be our results apply to nonconvex polyhedra as well.

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