Critical buckling predictions for plates and stiffened panels from natural frequency measurements

David Kennedy, Kwok Ip Lo
School of Engineering, Cardiff University, Queen’s Buildings, The Parade, Cardiff CF24 3AA, United Kingdom

KennedyD@cardiff.ac.uk

Abstract. The paper establishes a linear relationship between the square of natural frequency and applied compressive loading for a simply supported plate. This relationship enables the critical buckling load to be predicted by extrapolating from the natural frequencies obtained when the plate is lightly loaded. If the fundamental natural frequency and the critical buckling load have different mode shapes, buckling predictions can be made by extrapolating from higher natural frequencies. Shear loading causes a continual change in the mode shape as the loading is increased, and the critical buckling load can be estimated approximately by curve fitting and extrapolation. Similar results are obtained for simply supported stiffened panels.

1. Introduction
Plates and stiffened panels are used in many engineering applications, including aerospace, road and rail vehicles, ships and bridges. Initial imperfections, cracks, delaminations and other types of damage reduce their strength and stiffness properties, including their critical buckling loads. However buckling tests for built-up structures such as aircraft wings are impractical, while predictions from analytical models tend to be imprecise due to difficulties in detecting the location and extent of any damage.

Doyle [1] discussed the transverse vibration of axially compressed bars and frame structures, establishing a linear relationship between the applied load and the square of natural frequency such that the latter reduces to zero at the critical buckling load. Singer et al. [2] developed the Vibration Correlation Technique (VCT) which predicts the critical buckling load by extrapolating the natural frequencies obtained from non-destructive testing as the structure is progressively loaded at pre-buckling levels. The technique has been successfully applied to plates and shells [3], but it has been shown that the linear relationship breaks down in the presence of geometric imperfections [4, 5].

This paper explores the validity of the linear relationship for isotropic simply supported plates and stiffened panels with varying aspect ratios and subjected to different kinds of in-plane loading, including shear. Section 2 outlines the underlying vibration and buckling theory for a simply supported plate and establishes relationships between the natural frequencies and critical buckling loads, taking account of any possible changes in mode shapes as the applied load is increased. The predictions are validated by numerical results in section 3 and are shown also to apply to stiffened panels. The conclusions in section 4 include a discussion of how the critical buckling load can be estimated in cases where the linear relationship breaks down.
2. Vibration of loaded plates

The governing differential equation for an isotropic rectangular plate of length \( a \), width \( b \), mass per unit area \( \mu \), flexural rigidity \( D \), loaded in longitudinal compression \( N_x \), transverse compression \( N_y \) and in-plane shear \( N_{xy} \) per unit width and vibrating with frequency \( \omega \) is [6]

\[
D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \left( N_x \frac{\partial^2 w}{\partial x^2} + 2N_y \frac{\partial^2 w}{\partial x \partial y} + N_{xy} \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial t^2} \right) = 0
\]  

1(1)

2.1. Simply supported plate loaded in compression

If all four edges of the plate are simply supported and \( N_{xy} = 0 \), the solution to equation (1) takes the form

\[
w(x, y, t) = w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \omega t
\]  

(2)

with the natural frequencies \( \omega_{mn} \) given by

\[
\mu \omega_{mn}^2 = D\pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \pi^2 \left( N_x \frac{m^2}{a^2} + N_y \frac{n^2}{b^2} \right)
\]  

(3)

Setting \( N_x = N_y = 0 \) in equation (3) gives the natural frequencies \( \overline{\omega}_{mn} \) of an unloaded plate, of which the lowest is \( \overline{\omega}_{11} \), i.e. the fundamental vibration mode has one half-wave in both the longitudinal and transverse directions. Setting \( \omega_{mn} = 0 \) and \( N_y = 0 \) \( [N_x = 0] \) gives buckling loads \( \overline{N}_{xmn} \) \( [\overline{N}_{ymn}] \) in longitudinal [transverse] compression, of which the lowest has \( m=1 \) \( [m=1] \) and a value of \( m \) \( [n] \) which depends on the aspect ratio of the plate, so that the critical buckling mode may differ from the fundamental vibration mode.

Equation (3) can be written as

\[
\omega_{mn}^2 = \overline{\omega}_{mn}^2 \left( 1 - \frac{N_x}{\overline{N}_{xmn}} \right)
\]  

(4)

indicating a linear relationship between \( \omega_{mn}^2 \) and \( N_x \) \( [N_y] \) such that the natural frequency is reduced to zero when the loading reaches the buckling load \( \overline{N}_{xmn} \) \( [\overline{N}_{ymn}] \). Thus if the critical buckling load has \( m=n=1 \) it can be extrapolated from the fundamental natural frequencies \( \omega_{11} \) obtained when the plate is loaded at lower levels.

If the critical buckling mode has \( m \neq 1 \) or \( n \neq 1 \), it differs from the fundamental vibration mode. The critical buckling load can now be extrapolated from changes in the higher natural frequency \( \omega_{mn} \). As the load is gradually increased from zero there will be one or more discrete changes in the fundamental vibration mode which are accompanied by shifts between the straight line relationships associated with different values of \( m \) and/or \( n \).

2.2. Other loading and edge conditions

If the plate is loaded in shear \( (N_{xy} \neq 0) \) then equation (2) must be replaced by a series solution in order to approximate the skewed deflection pattern. The pure vibration mode is sinusoidal, but as the load is gradually increased the skewing becomes more pronounced, indicating a continual change in mode shape towards the critical buckling mode. Thus the relationship between the square of natural frequency
and the applied loading is no longer linear. In section 3 an attempt will be made to estimate the critical buckling load approximately from the natural frequencies by curve fitting and extrapolation.

An alternative to equation (2) is also required in order to satisfy alternative edge conditions, including the often uncertain elastic restraints associated with imperfect connections to adjacent parts of a built-up structure. Here there is also a non-linear relationship between the square of natural frequency and the applied loading. Similar considerations are needed in the presence of thickness variations, geometric imperfections or localized damage, which will be explored in detail in future publications.

3. Numerical results

Vibration and buckling analysis was performed on plates and stiffened panels using the exact strip software VICONOPT [7], which employs a transcendental stiffness matrix derived from analytical solutions of the governing differential equations of the component plates. Undamped natural frequencies and critical buckling loads and are found using the Wittrick-Williams algorithm [8, 9]. In the simplest (VIPASA) form of the analysis [10], the vibration or buckling mode is assumed to vary sinusoidally in the longitudinal direction, with a half-wavelength which divides exactly into the panel length, so that exact solutions are obtained for simply supported isotropic and orthotropic plates and panels without shear loading. For panels that are anisotropic or loaded in shear, the VICON analysis [11] employs a series solution to couple such modes, using Lagrangian multipliers to approximate the end conditions.

3.1. Simply supported plate loaded in compression

Figure 1 shows the square of the lowest natural frequency for simply supported plates with different aspect ratios \( a/b \) loaded in longitudinal compression. The plots are normalised with respect to the fundamental natural frequency for an unloaded plate \( \bar{\omega}_{11} \) and the critical buckling load \( \bar{N}_{um} \). For plates with aspect ratio 1 and 1.2, the critical buckling mode has \( m = n = 1 \) and a linear relationship is observed. For plates with higher aspect ratios, the critical buckling mode has \( m > 1 \) and \( n = 1 \) and the relationship is piecewise linear, confirming the analysis of [6]. For example, when \( a/b = 3 \), the critical buckling mode has \( m = 3 \) and \( n = 1 \). There is a mode shift from \( m = 1 \) to \( m = 2 \) when \( N_x = 0.64\bar{N}_{x33} \) and a further shift to \( m = 3 \) when \( N_x = 0.86\bar{N}_{x33} \), see figure 2 which also shows contour plots of the mode shapes.

It is noted that if VCT is applied at low load levels, extrapolating the plot of \( (\omega_{mn}/\bar{\omega}_{11})^2 \) will fail to predict the critical buckling load \( \bar{N}_{x33} \), but it can be predicted accurately by extrapolating \( (\omega_{mn}/\bar{\omega}_{11})^2 \).

![Figure 1](image_url)
The fundamental vibration mode and the critical buckling mode for loading in transverse compression both have \( m = n = 1 \) for plates with aspect ratio \( a/b \geq 1 \), so it is easy to predict the critical buckling load from changes in the lowest natural frequency (see figure 3). Mode shifts would however occur for plates with aspect ratios sufficiently less than 1.

3.2. Simply supported plate loaded in shear

When a simply supported plate is loaded in shear, the vibration and buckling modes are skewed (see figure 4) and the sinusoidal solution of equation (2) must be replaced by a series solution. The \((m,n)\) classification is replaced by a longitudinal response parameter \([7]\). Figure 5 illustrates the development of skewing as the shear load is increased, showing that the fundamental vibration mode \((N_{xy} = 0)\) and the critical buckling mode \((N_{xy} = \bar{N}_{xy})\) differ substantially. The relationship between the square of the
lowest natural frequency and the applied load is highly nonlinear (see figure 6), so the critical buckling load $\overline{N}_{xy}$ cannot be found by linear extrapolation.

The VCT was simulated by considering the first $r$ points ($r = 2, 3, 4, 5$) in figure 6 for the plate with aspect ratio 1. A polynomial curve of order $r - 1$ was fitted through the $r$ points and extrapolated to find the load $\overline{N}_{xy}^{est}$ at which the lowest natural frequency becomes zero. This is an estimate of the true critical buckling load $\overline{N}_{xy}$. Table 1 shows that acceptable accuracy is achieved by considering the natural frequencies at 5 points for which the shear loading lies below 25% of the critical buckling load.

![Critical buckling modes for plates of different aspect ratios](image1)

**Figure 4.** Critical buckling modes for plates of different aspect ratios $a/b$ loaded in shear.

![Vibration (0 ≤ $N_{xy}$ < $\overline{N}_{xy}$) and buckling ($N_{xy} = \overline{N}_{xy}$) modes](image2)

**Figure 5.** Vibration ($0 \leq N_{xy} < \overline{N}_{xy}$) and buckling ($N_{xy} = \overline{N}_{xy}$) modes of a simply supported square plate loaded in shear $N_{xy}$. 
Figure 6. Non-dimensional plot of lowest natural frequency squared versus shear load for plates of different aspect ratios.

Table 1. Non-dimensional estimates $\hat{N}_{xy,est}/\hat{N}_{xy}$ of critical shear buckling load of a simply supported square plate, obtained by extrapolating from the first $r$ points in figure 6.

| Point ($r$) | $N_{xy}/\hat{N}_{xy}$ at point $r$ | $\hat{N}_{xy,est}/\hat{N}_{xy}$ |
|------------|-----------------------------------|---------------------------------|
| 2          | 0.060                             | 19.773                          |
| 3          | 0.121                             | 1.092                           |
| 4          | 0.181                             | 1.063                           |
| 5          | 0.242                             | 1.006                           |

Figure 7. Non-dimensional plot of lowest natural frequency squared versus longitudinal compression for stiffened panels of different aspect ratios.
Table 2. Non-dimensional estimates $\frac{\bar{N}_{xy}^{est}}{\bar{N}_{xy}}$ of critical shear buckling load of a simply supported square stiffened panel, obtained by extrapolating fitted polynomials of order $r-1$.

| Point ($r$) | $\frac{N_{xy}}{\bar{N}_{xy}}$ at point $r$ | $\frac{N_{xy}^{est}}{\bar{N}_{xy}}$ |
|------------|---------------------------------|---------------------------------|
| 2          | 0.066                           | 35.714                          |
| 3          | 0.131                           | 1.524                           |
| 4          | 0.197                           | 1.357                           |
| 5          | 0.263                           | 1.109                           |
| 6          | 0.328                           | 1.037                           |

3.3. Stiffened panel loaded in compression or shear
Figure 7 shows the square of the lowest natural frequency for the stiffened panel whose cross section is shown in figure 7. The stiffeners have half the thickness of the skin, web height $0.06b$, flange width $0.02b$ and spacing $0.2b$. Panels with different aspect ratios $a/b$ are loaded in longitudinal compression. The plots are normalised with respect to the square of the fundamental natural frequency for an unloaded panel $\bar{\omega}^2$ and the critical buckling load $\bar{N}_{xy}$. The figure shows that, when $a/b = 1$ or 1.2, the lowest natural frequency has a global mode while critical buckling occurs in a local mode. There is a single mode shift as the critical buckling load is approached. For higher aspect ratios the fundamental natural frequency and critical buckling load have identical global modes and a linear relationship is observed.

When the panel is loaded in shear there is a gradual change of mode shape as the loading is increased, and the natural frequencies vary in a similar way to those of a single plate. Table 2 illustrates that the critical buckling load can be estimated with reasonable accuracy by extrapolating from the natural frequencies obtained as the load is increased to 33% of its critical value.

4. Conclusions
This study has confirmed the Vibration Correlation Technique for predicting the critical buckling load of a simply supported plate or stiffened panel loaded in compression or shear. Three kinds of behaviour have been identified.

Firstly, when the fundamental vibration mode and the critical buckling mode are identical, there is a linear relationship between the square of natural frequency and the applied load. Hence the critical buckling load can be accurately predicted from measurements of the lowest natural frequency of the unloaded and lightly loaded structure. Such behaviour is observed for a simply supported square plate in longitudinal or transverse compression.

Secondly, when the fundamental vibration mode and the critical buckling mode differ, but the lowest natural frequency exhibits discrete mode shifts as the load is increased, the critical buckling load can be accurately predicted from measurements of the lowest and selected higher natural frequencies of the unloaded and lightly loaded structure. Such behaviour is observed for a long simply supported plate and for stiffened panels which exhibit local buckling modes in longitudinal compression.

Thirdly, when there is a smooth transition from the fundamental vibration mode to the critical buckling mode as the loading is increased, for example on account of skewing due to shear loading, there is a non-linear relationship between the square of natural frequency and applied load. The critical buckling load can be predicted with reasonable accuracy by extrapolating a polynomial curve fitted to the squares of natural frequencies obtained when the structure is lightly loaded.

Although all the illustrative results presented here have been for perfect structures, the methods are also expected to prove useful for non-destructive monitoring and assessment of damaged structures whose natural frequencies and critical buckling loads have been degraded.
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