Orthogonal systems of spline wavelets as unconditional bases in Sobolev spaces

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Abstract
We exhibit the necessary range for which functions in the Sobolev spaces $L^p_s$ can be represented as an unconditional sum of orthonormal spline wavelet systems, such as the Battle–Lemarié wavelets. We also consider the natural extensions to Triebel–Lizorkin spaces. This builds upon, and is a generalization of, previous work of Seeger and Ullrich, where analogous results were established for the Haar wavelet system.

KEYWORDS
Sobolev spaces, spline wavelets, Triebel–Lizorkin spaces, unconditional basis

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1 | INTRODUCTION

It is well known that unlike the trigonometric system, the Haar system forms an unconditional basis in $L^p[0, 1]$ for all $1 < p < \infty$ (see [16]). In this paper, we aim to explore the analogous problem in the case of Sobolev (and Triebel-Lizorkin) spaces. More precisely, we seek to answer the following question: For what Sobolev spaces does a given orthonormal spline wavelet system form an unconditional basis?

Let $n \in \mathbb{N} \cup \{0\}$. We consider an orthogonal spline system on the real line, characterized by a scaling function $\Psi_n$ and an associated wavelet $\psi_n$ (both real valued) with the following properties:

(A) $\Psi_n, \psi_n \in C^{n-1}(\mathbb{R})$ (no condition for $n = 0$).
(B) The restriction of $\Psi_n, \psi_n$ to each interval $\left(j, j + \frac{1}{2}\right)$ (for $j \in \mathbb{Z}/2\mathbb{Z}$) is a polynomial of degree at most $n$.
(C) When $n > 0$, there exist positive constants $C$ and $\gamma$ (depending on $n$) such that

$$|\Psi_n^{(\alpha)}(x)| + |\psi_n^{(\alpha)}(x)| \leq Ce^{-\gamma|x|} \quad \text{for all } 0 \leq \alpha \leq n - 1.$$  

(D) \[
\int x^M \psi_n(x) \, dx = 0, \quad \text{for } M = 0, 1, \ldots, n.
\]

We say that $\psi_n$ is of order $n$. When $n = 0$, the Haar wavelet is perhaps the simplest and the most famous example of this type, with
\[ \Psi_0(x) = \mathbb{1}_{[0,1]}, \; \psi_0(x) = \mathbb{1}_{(-\frac{1}{2}, \frac{1}{2})}(x) - \mathbb{1}_{(\frac{1}{2}, 1]}(x), \]

where \( \mathbb{1}_{[a,b]} \) denotes the characteristic function of the interval \([a, b]\). More generally, for \( n \geq 0 \), the Battle–Lemarié wavelets (constructed independently by Battle [1] and Lemarié [14], also investigated by Mallat [15]) are well-known examples of such a system.

For \( k \in \mathbb{N} \cup \{0\} \) and \( \mu \in \mathbb{Z} \), we define

\[ \psi_{n,k,\mu} := \psi_n(2^k \cdot -\mu) \quad \text{and} \quad \psi_{n,-1,\mu} := \sqrt{2} \psi_n(2^k \cdot -\mu). \]

The Battle–Lemarié wavelets form an example of what is known as a multiresolution analysis in wavelet theory. We refer the interested reader to standard texts like [7], Section 5.4, and [30], and Section 3.3 for a more thorough discussion and actual construction of these wavelet systems. For our purposes, it is sufficient to know that they satisfy the properties (A)–(D) above. One must think of the Battle–Lemarié system of order \( n \) as an “orthonormalized” wavelet version of the \( n \)-th order cardinal spline \( N_n \), recursively defined by the relation \( N_0 = \mathbb{1}_{[0,1]} \), and

\[ N_n(x) = (N_{n-1} \ast N_1)(x), \]

for \( n \geq 1 \). In particular, the system

\[ \mathcal{W}_n := \{2^{k/2}\psi_{n,k,\mu} : k \in \mathbb{N} \cup \{-1, 0\}, \mu \in \mathbb{Z}\} \quad (1.1) \]

forms an orthonormal basis in \( L^2(\mathbb{R}) \).

We remark here that there also exist other (nonorthogonal) wavelet systems, which generalize the idea of B-splines, such as the following:

- Chui–Wang wavelets: These wavelets, constructed independently by Chui–Wang [5] and Unser–Aldroubi–Eden [27], retain interscale orthogonality and are compactly supported.
- Biorthogonal wavelets: Introduced by Cohen–Daubechies–Feauveau [6], these wavelets are compactly supported, symmetric, and regular, but nonorthogonal, with a dual basis generated by another compactly supported wavelet.

We refer the reader to [29] for a concise introduction and comparison. In this paper, we will focus on orthogonal wavelet systems, although it might be possible to adapt some of our ideas to the aforementioned systems as well.

Triebel ([24], [26]) showed that the Haar system forms an unconditional basis in Besov spaces \( B^s_{p,q} \) if \( 1 < p, q < \infty \) and \(-1/p' < s < 1/p \) (also see [17]). The endpoint case \( s = 1/p \) (and the dual case \( s = -1/p' \)) can be excluded by noting that all the Haar functions belong to \( B^{1/p}_{p,q} \) if and only if \( q = \infty \).

In the case of Sobolev and Triebel–Lizorkin spaces, we have a dependence on the secondary integrability parameter \( q \) as well. More precisely, it was shown by Triebel [26] that the Haar system forms an unconditional basis in the Sobolev spaces \( L^s_p(1 < p < \infty) \) when \( \max\{-1/p', -1/2\} < s < \min\{1/p, 1/2\} \) (recall that the norm in \( L^s_p \) is given by \( \|f\|_{L^s_p} = \|D^s f\|_{L^p} \), where \( D^s f = F^{-1}[(1 + |\xi|^2)^{s/2} \hat{f}(\xi)] \)). It had been an open question if the Haar system formed an unconditional basis in \( L^s_p \) in the range \( 1/2 \leq s \leq 1/p \) (for \( 1 < p < 2 \)) and \(-1/p' \leq s \leq -1/2 \) (when \( 2 < p < \infty \)). This was answered in the negative in [19], where Seeger and Ullrich established that the aforementioned sufficient condition is also a necessary one. In fact, in [19], the question was settled for the general class of Triebel–Lizorkin spaces \( F^s_{p,q} \) (we recall that by Littlewood–Paley theory, \( L^s_p = F^s_{p,2} \) for \( s \in \mathbb{R} \) and \( 1 < p < \infty \)). In a series of follow-up papers, Garrigós, Seeger, and Ullrich also established slightly better necessary and sufficient ranges for suitable enumerations of the Haar system to form a Schauder basis in Besov and Triebel–Lizorkin spaces (see [11]), including the limiting case for the former (in [12]) and the endpoint case for the latter (see [13]).

It is clear from the above discussion that the Haar system is not a good candidate for an unconditional basis in function spaces of higher order smoothness. This is because the Haar wavelet has poor regularity (it fails to be even continuous). Hence, we turn our attention to orthonormal spline wavelet systems satisfying properties (A)–(D). For such systems, Bourdaud [3] and Triebel [26] proved results analogous to the Haar case for Besov spaces, with a shift in the range of the smoothness parameter domain corresponding to the shift in regularity of the basis functions. More precisely, they proved
FIGURE 1  Domain for an unconditional basis in $L^s_p$ spaces for spline wavelets of order 1

that the system $\mathcal{W}_n$ forms an unconditional basis in $B^s_{p,q}$ if $1 < p, q < \infty$ and $-n - 1/p' < s < n + 1/p$. This result is also optimal, for the case $s \geq n + 1/p$ (and hence the dual case $s \leq -n - 1/p'$) can be excluded by noting that $\mathcal{W}_n \not\subset B^s_{p,q}$ in this range.

Coming to the case of the Sobolev and Triebel–Lizorkin spaces, Triebel ([26], Theorem 2.49, (ii)) showed that the system $\mathcal{W}_n$ (generated by a spline wavelet $\psi_n$ of order $n$, satisfying properties (A)–(D)) forms an unconditional basis in $F^s_{p,q}(1 < p, q < \infty)$ when

$$\max \{-1/p', -1/q', -n\} < s < \min \{1/p, 1/q\} + n$$

(see [21] for related results for splines). It is an open question if in this case too, the aforementioned sufficient condition is also necessary. In this work, we provide a positive answer to this question. In particular, we show that the system $\mathcal{W}_n$ does not form an unconditional basis on the Sobolev space $L^p_s$ for the ranges $1 < p < 2, n + 1/2 \leq s \leq n + 1/p$, and $2 < p < \infty, -n - 1/p' \leq s \leq -n - 1/2$ (see Figure 1).

Our main result is the following:

**Theorem 1.1.** Let $n \in \mathbb{N} \cup \{0\}$ and $1 < p, q < \infty$. The system $\mathcal{W}_n$ (as defined in (1.1)) is an unconditional basis in $F^s_{p,q}$ only if

$$\max \{-1/p', -1/q', -n\} < s < \min \{1/p, 1/q\} + n.$$  

**Remark 1.2.** Since the Haar system corresponds to the case $n = 0$, our result is a generalization of the one in [19] to orthogonal spline wavelet systems of arbitrary order.

Following [19], we introduce a suitable
framework to quantify the failure of unconditional convergence. For \( k \geq 0 \), we define the spline wavelet frequency of \( \psi_{n,k,\mu} \) to be \( 2^k \). For any subset \( E \) of the system \( \mathcal{W} \), let \( SF(E) \) denote the spline wavelet frequency set of \( E \). In other words,

\[
SF(E) = \{2^k : k \geq 0, \text{ there exists } \mu \in \mathbb{Z} \text{ with } \psi_{n,k,\mu} \in E\}.
\]

We denote by \( P_E \) the orthogonal projection to the subspace spanned by \( \{g : g \in E\} \) (which is closed in \( L^2(\mathbb{R}) \)). In particular, for a Schwartz function \( f \),

\[
P_E f = \sum_{\psi_{n,k,\mu} \in E} \langle f, \psi_{n,k,\mu} \rangle \psi_{n,k,\mu}.
\]

Now for any \( A \subset \{2^j : j = 0, 1, \ldots\} \), we set

\[
G(F^s_{p,q}, A) = \sup \{ \|P_E\|_{F^s_{p,q}} : SF(E) \subset A \}.
\]

For \( \Lambda \in \mathbb{N} \), we define the lower wavelet projection number

\[
\gamma_*(F^s_{p,q}, \Lambda) = \inf \{ G(F^s_{p,q}, A) : \#A \geq \Lambda \}.
\]

As \( \psi_{n,k,\mu} \notin F^s_{p,q} \) for \( s \geq 1/p + n \), we have that \( \gamma_*(F^s_{p,q}, \Lambda) = \infty \). By duality, \( \gamma_*(F^s_{p,q}, \Lambda) = \infty \) for \( s \leq -n + 1/p' \). In our discussion throughout, we shall assume that \( \Lambda \) is large enough (say \( \Lambda > 50 \)).

The approach used in [19] to establish the necessary range for unconditional convergence in the case of the Haar basis was the quantification of the growth rate \( G(F^s_{p,q}, A) \) in terms of the cardinality of \( A \). In particular, to give precise lower bounds for \( \gamma_*(F^s_{p,q}, \Lambda) \), the authors constructed a suitable test function, by first considering a sum of the translates of a smooth compactly supported function \( \eta \) at a fixed dyadic scale and then taking a randomized sum of the functions hence constructed at different scales, dictated by the frequency of the given set \( A \). The Sobolev (or Triebel–Lizorkin) norm of the test function was controlled by introducing enough separation between the translates at the same scale (Proposition 4.1). This separation and the compact support of the Haar wavelet was also used to ensure that at each scale, a given translate of the wavelet interacted with exactly one translate of the test function. Finally, by choosing \( \eta \) to be an odd function and exploiting the antisymmetry of the Haar wavelet (with respect to the midpoint of the associated interval), the authors were able to avoid cancelation and get the different interactions to add up, yielding the desired lower bounds.

In this paper, we use the same example as above, and verify that this approach also works for the wavelet systems we consider. In [19], the authors had the advantage of working with the Haar wavelet, which can be written down in a very simple closed form and is compactly supported. However, in our paper, we do not use any explicit formulas for the wavelets (which can get very tedious as the order increases). Neither are our wavelets compactly supported. The novelty of this paper lies in identifying the properties hidden behind the deceptively simple form of the Haar wavelet, which make the example in [19] work, and adapting them to our setting. Moreover, exponential decay (property (C)) is only slightly worse than being compactly supported, and can be essentially dealt with by increasing the separation between the different translates. Consequently, we obtain some tail terms (absent in [19]), which need to be carefully considered.

1.1 Notation

We shall use the notation \( A \lesssim B \), or \( B \gtrsim A \), if \( A \leq CB \) for a positive constant \( C \) depending only on \( p, q, s \) and the wavelet \( \psi_n \) under consideration. Also, if both \( A \lesssim B \) and \( B \lesssim A \), we shall use the notation \( A \approx B \).

1.2 Plan of the paper

In Section 2, we briefly discuss the characterization of Triebel–Lizorkin spaces via compactly supported local means, which is quite suitable for our problem. In Section 3, we reformulate the properties of the orthogonal spline wavelets in a quantitative form. In Section 4, we state and prove a technical lemma. This is in preparation for defining a suitable family of test functions in \( F^s_{p,q} \), which we do for \( p > q \) and \( s \leq -1/q' - n \) in Section 5. In Section 6, we establish a few
preliminary estimates and lower bounds for the interactions of the test functions with the members \( \psi_{n,k,\mu} \) of the wavelet family. In Sections 7 and 8, we prove the existence of the desired lower bounds for \( \gamma_s(F_{p,q}^s, \Lambda) \) when \( s < -1/q' - n \) and \( s = -1/q' - n \), respectively.

2 SOME BACKGROUND ON TRIEBEL–LIZORKIN SPACES

We briefly discuss the characterization of Triebel–Lizorkin spaces via “local means” (termed so in [25], Section 2.4.6), which will be useful for our purposes.

The usual way to define Triebel–Lizorkin spaces is via a smooth dyadic decomposition of unity. Let \( \varphi_0 \) be a smooth function supported in \([-3/2, 3/2]\) such that \( \varphi_0 \equiv 1 \) on \([-4/3, 4/3]\). We set \( \varphi = \varphi_0(\cdot) - \varphi_0(2\cdot) \), so that \( \varphi_0 + \sum_{k \in \mathbb{N}} \varphi_0(2^{-k}\cdot) \equiv 1 \). Defining \( \widehat{L_0f} = \varphi_0 \hat{f} \) and \( \widehat{L_kf} = \varphi(2^{-k}\cdot) \hat{f} \) for a Schwartz function \( f \), we obtain an inhomogeneous Littlewood–Paley decomposition

\[
 f = \sum_{k=0}^{\infty} L_k f,
\]

with convergence in \( S'(\mathbb{R}) \) and in all \( L^p \) spaces. For \( 0 < p < \infty \), \( 0 < q \leq \infty \), and \( s \in \mathbb{R} \), the Triebel–Lizorkin space \( F_{p,q}^s(\mathbb{R}) \) is defined as the collection of all tempered distributions \( f \in S'(\mathbb{R}) \) such that

\[
 \|f\|_{F_{p,q}^s} = \left( \left\| \sum_k 2^{ksq} |L_k f|^q \right\|_{L^p} \right)^{1/q} < \infty,
\]

with the usual modification when \( q = \infty \). We now define another pair of functions \( \phi_0 \) and \( \phi \) such that \( |\hat{\phi}_0| > 0 \) on \((-\epsilon, \epsilon)\) and \( |\hat{\phi}| > 0 \) on the set \( \{ \xi : \epsilon/4 < |\xi| < \epsilon \} \). We also assume that

\[
 \int y^m \phi(y) \, dy = 0, \tag{2.1}
\]

for \( m = 0, \ldots, M_1 \) where \( M_1 \in \mathbb{N} \) is such that \( M_1 + 1 > s \). It can be proved using vector valued singular integrals (see [25], Section 2.4.6) that

\[
 \|f\|_{F_{p,q}^s} \approx \left( \left\| \sum_k 2^{ksq} |\phi_k * f|^q \right\|_{L^p} \right)^{1/q}, \tag{2.2}
\]

with \( \phi_k(x) = 2^k \phi(2^k \cdot) \). The above characterization allows for compactly supported \( \phi \) and \( \phi_0 \), termed as “local means.”

3 PROPERTIES OF THE ORTHOGONAL SPLINE WAVELETS

For \( k, \mu \in \mathbb{Z} \), we define

\[
 I_{k,\mu} := [2^{-k} \mu, 2^{-k}(\mu + 1)]
\]

and

\[
 x_{k,\mu} := 2^{-k} \mu + 2^{-k-1}.
\]

The Haar wavelet generates a system that can be easily written down explicitly. Unfortunately, these formulas become extremely complicated when \( n > 0 \). Moreover, \( \psi_n \) is no longer compactly supported in this case. However, on a closer inspection, one can isolate the primary properties of the Haar system on which the arguments in [19] are based. These are the following:
(1) Each $\psi_{0,k,\mu}$ is supported on the dyadic interval $I_{k,\mu}$.
(2) $\psi_{0,k,\mu}$ is antisymmetric around $x_{k,\mu}$.

The test functions are then constructed by taking a sum of compactly supported functions $\eta_{k,\mu}$ centered around $x_{k,\mu}$ for $0 \leq \mu \leq 2^k - 1$. The first property ensures enough separation so that each wavelet translate $\psi_{0,k,0}$ interacts with exactly one translate of $\eta$ at scale $2^{-k}$, namely $\eta_{k,\mu_0}$. The second property is exploited by considering $\eta_{k,\mu}$ to be odd, so that the contributions from both halves of $I_{k,\mu}$ get added up.

In our case, even though $\psi_n$ does not have compact support, it is only slightly worse: $\psi_{n,k,\mu}$ (and its derivatives) decay exponentially off of $I_{k,\mu}$ (property (C)). Thus, by introducing enough separation (as determined by the decay rate), we can still ensure that the interaction of $\eta_{k,\mu}$ with $\psi_{n,k,\mu'}$ is negligibly small when $\mu \neq \mu'$.

Speaking of symmetry, although the Battle–Lemarié wavelet of order $n$ is known to be symmetric (antisymmetric) around $1/2$ when $n$ is odd (even), we do not rely on this property in our argument, in order to make it applicable to general settings. Let us consider the unit interval $[0,1]$ (for the other dyadic intervals can be obtained from this case by appropriate scaling and translation). $\psi_n$ is represented by (possibly different) polynomials of degree $n$ on $[0,1/2]$ and $[1/2,1]$. However, the condition that $\psi_n \in C^{n-1}$ forces the nonleading left and right coefficients to be equal. This takes care of the cancelation of the lower order polynomial terms, provided the lower moments of the test function disappear, which is indeed chosen to be so by construction. Further, by considering a translation of $\psi_n$, if necessary (see Remark 6.1), we can assume that the leading coefficients of the left and right polynomial representation of $\psi_n$ around $1/2$ are not equal. Then by choosing a test function $\eta$ such that $y^n\eta(y)$ is odd, we can still get the interactions to add up, yielding nonzero lower bounds. In the endpoint case, we use a slight generalization of this idea, choosing $\eta$ to be even or odd depending on the signs of the leading coefficients of $\psi_n$ around $0$ with respect to each other.

In the paper henceforth, $n \in \mathbb{N}$ shall remain fixed and be understood from the context. Consequently, we denote $\psi_n(x)$ by $\psi(x)$ and $\psi_{n,k,\mu}$ by $\psi_{k,\mu}$. The following lemma is a quantitative interpretation of Properties (A) and (C).

**Lemma 3.1.** Let $\theta \in \mathbb{Z}$. Suppose $\psi = \psi_n$ is represented by

$$
\psi(x) = A_{n,L}^\theta \left( x - \frac{\theta}{2} \right)^n + A_{n-1,L}^\theta \left( x - \frac{\theta}{2} \right)^{n-1} + \cdots + A_0^\theta,
$$

on $\left[ \frac{\theta-1}{2}, \frac{\theta}{2} \right]$, and

$$
\psi(x) = A_{n,R}^\theta \left( x - \frac{\theta}{2} \right)^n + A_{n-1,R}^\theta \left( x - \frac{\theta}{2} \right)^{n-1} + \cdots + A_0^\theta,
$$

on $\left[ \frac{\theta}{2}, \frac{\theta+1}{2} \right]$. Then we have

(i)

$$
A_j^{\theta,L} = A_j^{\theta,R}
$$

for $j = 0,1,\ldots,n-1$.

(ii)

$$
\max \{|A_n^{\theta,L}|, |A_n^{\theta,R}|\} \leq 4Ce^{-1/2^\gamma},
$$

where $C$ and $\gamma$ are as defined in Property (C).

**Proof.** By virtue of the fact that $\psi \in C^j$ for $j = 0,1,\ldots,n-1$ (Property (A)), we have that

$$
\lim_{x \to \theta/2^-} \psi^{(j)}(x) = \lim_{x \to \theta/2^+} \psi^{(j)}(x),
$$

and
which yields

\[ A_j^{\beta,L} = A_j^{\beta,R} \]

for \( 0 \leq j \leq n - 1 \), thus proving (i).

For proving (ii), we use Property (C) with \( \alpha = n - 1 \) to obtain

\[ \left| n! A_n^{\beta,R} \left( x - \frac{\theta}{2} \right) + (n - 1)! A_{n-1}^{\beta,R} \right| \leq C e^{-\gamma |x|} \]  

(3.3)

for \( x \in \left[ \frac{\theta}{2}, \frac{\theta+1}{2} \right] \). In particular, substituting \( x = \theta/2 \), we get

\[ \left| A_{n-1}^{\beta,R} \right| \leq \frac{C}{(n-1)!} e^{-\gamma \frac{\theta}{2}}. \]  

(3.4)

Now, we substitute \( x = \frac{\theta+1}{2} \) in (3.3) and use the triangle inequality, along with (3.4), which yields

\[ \left| \frac{A_n^{\beta,R}}{2} \right| \leq \frac{C}{n!} \left( e^{-\gamma \frac{\theta+1}{2}} + e^{-\gamma \frac{\theta}{2}} \right) \leq 2C e^{\gamma/2} e^{-\gamma \frac{\theta}{2}}. \]

This establishes the required upper bound for \( |A_n^{\beta,R}| \). The proof for the same for \( |A_n^{\beta,L}| \) proceeds similarly, except \( x \) now lies in the interval \( \left[ \frac{\theta-1}{2}, \frac{\theta}{2} \right] \) and we substitute \( x = \frac{\theta-1}{2} \) in the last step. \( \square \)

4  |  BOUNDEDNESS OF TEST FUNCTIONS

We now prepare the ground for the definition of test functions to be used to establish the desired lower bounds. The arguments used in this section are identical to those in [19], Section 4. Nevertheless, we include them here for completeness. Throughout this section, we fix \( m \in \mathbb{N} \).

We will use the local means characterization of Triebel–Lizorkin spaces, as described in Section 2. To this effect, we consider smooth functions \( \phi_0 \) and \( \phi \), both supported in \((-1/2, 1/2)\) so that \( |\hat{\phi}_0(\xi)| > 0 \) for \( |\xi| \leq 1 \) and \( |\hat{\phi}(\xi)| > 0 \) for \( 1/4 \leq |\xi| \leq 1 \). We also assume that the cancelation condition (2.1) holds for \( \phi \), for \( M_1 \in \mathbb{N} \) with \( M_1 + 1 > s \). We set \( \phi_k = 2^k \phi(2^k \cdot) \) for \( k \in \mathbb{N} \). We shall use the characterization of \( \mathcal{F}_{\beta,R}^s \) using the \( \phi_k \), as defined in (2.2).

Let \( \eta \in C^\infty(\mathbb{R}) \) be supported in \((-1/2, 1/2)\) such that \( \int x^M \eta(x) \, dx = 0 \) for \( M = 0, 1, \ldots, n + 2 \). Let \( \mathcal{Q}^m \) be a finite set of nonnegative integers \( \geq m \), such that \( \# \mathcal{Q}^m \geq 2^m \). For each \( l \in \mathcal{Q}^m \), let \( P_l^m \) denote a set of \( K_0 2^{m-l} \)-separated points in \([0, K_0]\), where \( K_0 \in \mathbb{N} \) is a fixed positive integer to be decided later. More precisely, we have \( P_l^m = \{ x_{l,1}, \ldots, x_{l,N(l)} \} \) with \( N(l) \leq 2^{l-m} \) and \( x_{l,v} < x_{l,v+1} \) with \( x_{l,v+1} - x_{l,v} \geq K_0 2^{m-l} \). For each \( l \in \mathcal{Q}^m \), let

\[ \mathcal{E}_l^m = \{ v : x_{l,v} \in P_l^m \}. \]

Next, we define

\[ \eta_{l,v} = \eta(2^l(x - x_{l,v})). \]

For a sequence \( \{a_{l,v}\} \) with \( \sup_{l,v} |a_{l,v}| \leq 1 \), we define

\[ g_m(x) = \sum_{l \in \mathcal{Q}^m} 2^{-lM} \sum_{v \in \mathcal{E}_l^m} a_{l,v} \eta_{l,v}(x). \]
If the families \( \mathcal{Q}^m \), \((m \in \mathbb{N})\) are disjoint, we define

\[
g = \sum_{m \in \mathbb{N}} \hat{\beta}_m g_m
\]

for \( \hat{\beta}_m \in \mathbb{R} \).

The following proposition is identical to Proposition 4.1 in [19] (which was in turn a modification of the corresponding result in [4]).

**Proposition 4.1.** Let \( s > -n - 2 \).

(i) For \( 1 \leq p, q < \infty \), we have

\[
\|g_m\|_{F^s_{p,q}} \lesssim_{p,q,s} \left( \left\| \sum_{l \in \mathcal{Q}^m} \sum_{\nu \in \mathcal{H}^m} a_{l,\nu} 1_{l,\nu} \right\|^q \right)^{1/q} \]

and

\[
\|g\|_{F^s_{p,q}} \lesssim_{p,q,s} \left( \left\| \sum_{m \geq 1} \|\hat{\beta}_m\|^q \sum_{l \in \mathcal{Q}^m} \sum_{\nu \in \mathcal{H}^m} a_{l,\nu} 1_{l,\nu} \right\|^q \right)^{1/q} .
\]

Here, \( 1_{l,\nu} \) denotes the characteristic function of the interval centered at \( x_{l,\nu} \) of radius \( 2^{-l} \).

(ii) For \( 1 \leq q \leq p < \infty \), we have

\[
\|g_m\|_{F^s_{p,q}} \lesssim_{p,q,s} (2^{-m \#(\mathcal{Q}^m)})^{1/q}
\]

and

\[
\|g\|_{F^s_{p,q}} \lesssim_{p,q,s} \left( \sum_{m \geq 1} \|\hat{\beta}_m\|^{2^{-m \#(\mathcal{Q}^m)}} \right)^{1/q} .
\]

**Proof.** (i) is a consequence of the fact that \( \{\eta_{l,\nu}\}_{l,\nu} \) form a family of smooth atoms in the sense of Frazier and Jawerth ([10], Theorem 4.1 and Section 12). We use the pairwise disjointness of the sets \( \mathcal{Q}^m \) here.

Consequently, in order to prove (4.1), it suffices to show that

\[
\left\| \left( \sum_{l \in \mathcal{Q}^m} \sum_{\nu \in \mathcal{H}^m} 1_{l,\nu} \right)^q \right\|_{p,q,s} \lesssim_{p,q,s} (2^{-m \#(\mathcal{Q}^m)})^{1/q}
\]

(recall that \( \sup_{l,\nu} |a_{l,\nu}| \leq 1 \)). Let \( G_l(x) = \sum_{\nu \in \mathcal{H}^m} 1_{l,\nu}(x) \) and \( G(x) = (\sum_{l \in \mathcal{Q}^m} |G_l(x)|^q)^{1/q} \). To prove the desired inequality for the \( L^p \) norm of \( G \), we use the dyadic version of the Fefferman–Stein interpolation theorem for \( L^q \) and \( BMO \) (see [23], Chapter 4). Here, we use the fact that \( p \geq q \). Thus, it is enough to show that both the \( L^q \) and the \( BMO_{dyad} \) norms of \( G \) are bounded by \( (2^{-m \#(\mathcal{Q}^m)})^{1/q} \). This follows almost immediately for the former. For the \( BMO_{dyad} \) norm, we need to show that

\[
\sup_{J} \inf_{c \in \mathbb{R}} \frac{1}{|J|} \int_J |G(y) - c| \, dy \lesssim (2^{-m \#(\mathcal{Q}^m)})^{1/q},
\]

(4.4)
where the supremum is taken over all dyadic intervals \( J \). We fix \( J \) and denote its midpoint by \( x_J \). We define

\[
c_{J,l} = \begin{cases} 
\sum_{\nu \in \mathfrak{S}_m} \mathbb{1}_{l,\nu}(x_J) & \text{if } |J| \leq 2^{-l} \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
c_J = \left( \sum_{l \in \mathcal{L}_m} c_{J,l}^q \right)^{1/q}.
\]

Then

\[
\frac{1}{|J|} \int_J |G(y) - c_J| \, dy = \frac{1}{|J|} \int_J \left| \left( \sum_{l \in \mathfrak{S}_m} G_l(y)^q \right)^{1/q} - \left( \sum_{l \in \mathfrak{S}_m} c_{J,l}^q \right)^{1/q} \right| \, dy
\]

\[
\leq \frac{1}{|J|} \int_J \left( \sum_{l \in \mathfrak{S}_m} |G_l(y) - c_{J,l}|^q \right)^{1/q} \, dy
\]

\[
\leq \left( \sum_{l \in \mathfrak{S}_m} \frac{1}{|J|} \int_J |G_l(y) - c_{J,l}|^q \, dy \right)^{1/q},
\]

where we have used the triangle inequality in \( \ell^q \) and Hölder’s inequality on the interval \( J \). Now for \( |J| \leq 2^{-l} \) and \( y \in J \), we have that \( G_l(y) = c_{J,l} \). Also, as \( c_{J,l} = 0 \) for \( |J| > 2^{-l} \), we get

\[
\frac{1}{|J|} \int_J |G(y) - c_J| \, dy \leq \left( \sum_{l \in \mathfrak{S}_m, 2^{-l} < |J|} \frac{1}{|J|} \int_J |G_l(y)|^q \, dy \right)^{1/q}.
\]

But as the points in \( \mathfrak{S}_m \) are \( K_0 2^{m-l} \)-separated, by the definition of \( G_l(y) \), we have

\[
\int_J |G_l(y)|^q \, dy \leq \begin{cases} 
2^{-l} & \text{if } 2^{-l} < |J| \leq 2^{m-l} \\
2^{-m} |J| & \text{if } 2^{m-l} < |J|.
\end{cases}
\]

Hence,

\[
\sum_{l \in \mathfrak{S}_m, 2^{-l} < |J|} \frac{1}{|J|} \int_J |G_l(y)|^q \, dy \leq \sum_{l: 2^{-l} < |J| \leq 2^{m-l}} (2^l |J|)^{-1} + \sum_{l \in \mathcal{L}_m} 2^{-m}
\]

\[
\lesssim (1 + 2^{-m} \#(\mathfrak{S}_m)).
\]

This proves (4.4), as \( (\mathfrak{S}_m) \geq 2^m \).

Finally, (4.2) can be proven by using the second assertion in (i), (4.3), and the triangle inequality in \( L^{p/q} \), noting that \( p \geq q \).

\[
\square
\]

5  |  DEFINITION OF THE TEST FUNCTIONS FOR THE NON-ENDPOINT CASE

In this section, we define the test functions to be used to establish the lower bounds in the non-endpoint case. Our example is essentially the same as the one used in [19], except we take care to increase the separation between the translates at each dyadic scale (by a factor of \( K_0 \)), to allow the exponential decay of the spline wavelet to kick in. Consequently, our function lives on \([0, K_0]\), rather than the unit interval. We now present the details.
Let \( \eta \) be a \( C^\infty \) function supported in \((-2^{-5}, 2^{-5})\). We require \( \eta \) to be odd for even \( n \), and even for odd \( n \), so that \( x^n \eta(x) \) is always odd. Furthermore, let

\[
\int x^M \eta(x) \, dx = 0, \tag{5.1}
\]

for \( M = 0, 1, \ldots, n + 2 \) and let

\[
\int_0^{\frac{1}{2}} x^\alpha \eta(x) \, dx \geq 1. \tag{5.2}
\]

Let \( A \) be an arbitrary nonempty set of wavelet frequencies and \( \Lambda, N \) be such that

\[
\Lambda < \#A + 1 \text{ and } 2^N \leq \#A < 2^{N+1}.
\]

\( N \) and \( \eta \) will remain fixed henceforth. For \( k = 0, 1, 2, \ldots \) and \( \mu \in \mathbb{Z} \), we define

\[
\eta_{k, K_0 \mu}(y) = \eta(2^k + N(y - 2^{-k} K_0 \mu - 2^{-k-1})).
\]

Let \( r_k \) denote the \( k \)-th Rademacher function on \([0,1] \). For \( t \in [0,1] \) and \( 2^k \in A \), let

\[
Y_k(y) = 2^{N(-s+1/q)} 2^{k-1} \sum_{\mu=0}^{2^k-1} \eta_{k, K_0 \mu}(y) \tag{5.4}
\]

and

\[
f_t(y) = 2^{-N/q} \sum_{2^k \in A} r_k(t) 2^{-k} Y_k(y). \tag{5.5}
\]

**Lemma 5.1.** Let \( s > -n - 2 \) and \( 1 \leq q \leq p < \infty \). Then, we have

\[
\|f_t\|_{F_{p,q} s} \lesssim_{p,q,s} 1
\]

uniformly in \( t \in [0,1] \).

**Proof.** We write \( f_t \) in the expanded form

\[
f_t = \sum_{2^k \in A} 2^{-(k+N)s} \sum_{\mu=0}^{2^k-1} r_k(t) \eta(2^{k+N}(y - 2^{-k} K_0 \mu - 2^{-k-1})).
\]

We now set \( m = N, \mathcal{Q}^N = \{ k + N : 2^k \in A \} \), and apply Proposition 4.1, (ii). The lemma now follows as \( 2^{-N} \#(\mathcal{Q}^N) \lesssim 1 \) and the points \( \{2^{-k} K_0 \mu + 2^{-k-1} : 0 \leq \mu \leq 2^k - 1\} \) are \( K_0 2^{m-l} \)-separated, for \( l = k + N \).

\( \square \)

## 6  |  A FEW PRELIMINARY ESTIMATES

In this section, we require \( \phi \) (as defined in Section 4) to be supported on \((-2^{-4}, 2^{-4})\) such that

\[
\int x^M \phi(x) \, dx = 0 \tag{6.1}
\]
for $M = 0, 1, \ldots, n + 1$, and $\|\phi\|_{L^1} \leq 1$. Let $\phi_k = 2^k \phi(2^k \cdot)$. We define $\Phi_1(x) = \int_{-\infty}^{x} \phi(t) \, dt$ and for $j = 2, \ldots, n + 1$, let

$$
\Phi_j(x) = \int_{-\infty}^{x} \Phi_{j-1}(t) \, dt
$$

be the $j$th order primitive of $\phi$, also supported in $(-2^{-4}, 2^{-4})$. Further, let

$$
\psi(x) = A^L_n \left( x - \frac{1}{2} \right)^n + A_{n-1} \left( x - \frac{1}{2} \right)^{n-1} + \cdots + A_0
$$

(6.2)
on $[0, \frac{1}{2}]$, and

$$
\psi(x) = A^R_n \left( x - \frac{1}{2} \right)^n + A_{n-1} \left( x - \frac{1}{2} \right)^{n-1} + \cdots + A_0
$$

(6.3)on $\left[ \frac{1}{2}, 1 \right]$.

**Remark 6.1.** The equality of the nonleading coefficients above follows from Lemma 3.1, (i). In principle, it is possible that the left and right leading coefficients ($A^L_n$ and $A^R_n$, respectively) are also equal. All the same, we claim that there exists some $\theta \in \mathbb{Z}$ so that $A^L_\theta \neq A^R_\theta$ (where $A^L_\theta$ and $A^R_\theta$ are as defined in Lemma 3.1). For if not, then it follows that $A^L_\theta = A^R_\theta := A_n$ for all $\theta \in \mathbb{Z}$. Here, we use the fact that the leading coefficient of a polynomial is invariant under translation. Then Part (ii) of Lemma 3.1 implies that $|A_n| \leq 4Ce^{\gamma/2}e^{-\gamma |\frac{1}{2}|}$ for all $\theta \in \mathbb{Z}$ and thus $A_n = 0$, which would contradict the fact that $\psi_n$ is a spline wavelet of order $n$.

We fix a $\theta \in \mathbb{Z}$ for which $A^L_\theta \neq A^R_\theta$ and define the translate $\tilde{\psi}_n := \psi_n(\cdot + \theta/2)$. Then, observe that the left and right leading coefficients of $\tilde{\psi}_n$ around $1/2$ are given by $A^L_n$ and $A^R_n$, respectively, which are not equal by our assumption. We can now prove our estimates for $\tilde{\psi}_n$ instead of $\psi_n$. This would not affect our main theorems as both $\psi_n$ and $\tilde{\psi}_n$ generate the same wavelet system $\mathcal{W}_n$.

Hence, without loss of generality, we can assume that

$$
A^L_n \neq A^R_n
$$

(6.4)
and in particular, that

$$
A^L_n \neq 0.
$$

(6.5)

**Lemma 6.2.** There exists $c_0 \in (0, 1)$ and a subinterval $J \subset [1/4, 3/4]$ with nonempty interior so that

$$
|\phi \ast \psi(x)| \geq c_0
$$

for $x \in J$.

**Proof.** We observe that the support of $\phi$ is contained in $[x - 1, x]$, whenever $x \in [1/4, 3/4]$. For such $x$, we have

$$
\phi \ast \psi(x) = \int_{x-1/2}^{x} \phi(y)\psi(x - y) \, dy + \int_{x-1}^{x-1/2} \phi(y)\psi(x - y) \, dy.
$$

We observe that $x - y$ lies in $[0, \frac{1}{2}]$ in the first integral and in $\left[ \frac{1}{2}, 1 \right]$ in the second one. Hence, we can use (6.2) and (6.3) in the left and right integrals, respectively. Now, for $j = 0, \ldots, n - 1$, we have
The last expression is easily seen to be 0 by (6.1). Thus, all the lower degree terms cancel, and we have

\[
\phi \ast \psi(x) = A^L_n \int_{x-1/2}^x \left( x - y - \frac{1}{2} \right)^n \phi(y) \, dy + A^R_n \int_{x-1}^{x-1/2} \left( x - y - \frac{1}{2} \right)^n \phi(y) \, dy.
\]

Now performing an integration by parts \( n \) times, along with the observation that the boundary terms are all zero, gives

\[
\phi \ast \psi(x) = (-1)^n n! \left[ A^L_n \int_{x-1/2}^x \Phi_n(y) \, dy + A^R_n \int_{x-1}^{x-1/2} \Phi_n(y) \, dy \right].
\]

We thus conclude that

\[
\phi \ast \psi(x) = (-1)^n n! \left[ A^L_n \Phi_{n+1}(x) - A^R_n \Phi_{n+1}(x - 1) + (A^R_n - A^L_n) \Phi_{n+1} \left( x - \frac{1}{2} \right) \right].
\]

In particular, for \( x \in [1/4, 3/4] \), we have that \( \phi \ast \psi(x) = (-1)^n n! (A^R_n - A^L_n) \Phi_{n+1} \left( x - \frac{1}{2} \right) \). Using (6.4), we conclude that there exists \( c_0 \in (0, 1) \) (depending on \( \psi_n \) and \( \phi \)) and a subinterval \( J \subset [1/4, 3/4] \) so that

\[
|\phi \ast \psi(x)| \geq c_0
\]

for \( x \in J \).

We again use \( K_0 \) to denote a fixed positive integer (to be decided later), which shall depend only on the wavelet \( \psi_n \). For \( k \in \mathbb{N} \cup \{0\} \) and \( \mu \in \mathbb{Z} \), let \( J_{k,K_0} = 2^{-k} K_0 \mu + 2^{-k} J \) (where \( J \) is as in Lemma 6.2). We then have

\[
|\phi_k \ast \psi_{k,K_0}(x)| \geq c_0
\]

for \( x \in J_{k,K_0} \). We note that \( J_{k,K_0} \) is an interval of length \( \geq 2^{-k} \).

**Proposition 6.3.** Let \( Y_k \) be as defined in (5.4). Then, for \( K_0 \) large enough, we have

\[
\left| \frac{\tilde{A}}{2} 2^{N(-s+1/q-n-1)} \int_{-1/2}^0 y^n \eta(y) \, dy \right| \leq \left| 2^k \left\langle Y_k, \psi_{k,K_0} \right\rangle \right| \leq 2 \left| \tilde{A} \right| 2^{N(-s+1/q-n-1)} \int_{-1/2}^0 y^n \eta(y) \, dy.
\]

Here, \( \tilde{A} = A^R_n - A^L_n \) and depends only on the wavelet \( \psi_n \).

**Proof.** Using the definition of \( Y_k \), we get

\[
2^k \left\langle Y_k, \psi_{k,K_0} \right\rangle = 2^{N(-s+1/q)} \sum_{\mu' = 0}^{2^k-1} 2^k \left\langle \eta_{k,K_0 \mu'}, \psi_{k,K_0} \right\rangle.
\]

Now, we have

\[
2^k \left\langle \eta_{k,K_0 \mu'}, \psi_{k,K_0} \right\rangle = 2^k \int \eta(2^{k+N}(x - 2^{-k} K_0 \mu' - 2^{-k-1}) \psi(2^k x - K_0 \mu) \, dx
\]

\[
= \int_0^1 \eta \left( 2^N \left( y - \frac{1}{2} \right) \right) \psi \left( y + \frac{A}{2} \right) \, dy
\]
\[ \int_0^{1/2} \eta \left( 2^N \left( y - \frac{1}{2} \right) \right) \psi \left( y + \frac{\lambda}{2} \right) dy + \int_{1/2}^1 \eta \left( 2^N \left( y - \frac{1}{2} \right) \right) \psi \left( y + \frac{\lambda}{2} \right) dy, \]

where \( \lambda = 2K_0(\mu' - \mu) \). We observe that \( y + \frac{\lambda}{2} \) lies in \( \left[ \frac{\lambda}{2}, \frac{\lambda+1}{2} \right] \) in the first integral and in \( \left[ \frac{\lambda+1}{2}, \frac{\lambda+2}{2} \right] \) in the second one. Hence, we can use formulations (3.1) and (3.2) of \( \psi \) (with \( \theta = \lambda + 1 \)), for the left and right integrals, respectively. By arguing as in the proof of Lemma 6.2, using (5.1) instead of (6.1), it is easy to see that the lower degree terms cancel out, and we obtain

\[ 2^k \langle \eta_{k,\mu''}, \psi_{k,\mu} \rangle = A_{n}^{\lambda+1, L} \int_0^{1/2} \eta \left( 2^N \left( y - \frac{1}{2} \right) \right) \left( y - \frac{1}{2} \right)^n dy + A_{n}^{\lambda+1, R} \int_{1/2}^1 \eta \left( 2^N \left( y - \frac{1}{2} \right) \right) \left( y - \frac{1}{2} \right)^n dy. \]

Applying a change of variables, we get

\[ 2^k \langle \eta_{k,\mu''}, \psi_{k,\mu} \rangle = A_{n}^{\lambda+1, L} \int_{-1/2}^0 y^n \eta(2^N y) dy + A_{n}^{\lambda+1, R} \int_{-1/2}^{1/2} y^n \eta(2^N y) dy \]

\[ = 2^{-(n+1)N} \left( A_{n}^{\lambda+1, L} \int_{-1/2}^0 y^n \eta(y) dy + A_{n}^{\lambda+1, R} \int_{-1/2}^{1/2} y^n \eta(y) dy \right) \]

\[ = 2^{-(n+1)N} (A_{n}^{\lambda+1, L} - A_{n}^{\lambda+1, R}) \int_0^{1/2} y^n \eta(y) dy, \]

where in the last step we have used the fact that \( y^n \eta(y) \) is odd.

When \( \mu = \mu' \) and \( \lambda = 0 \), we conclude that

\[ 2^k \langle \eta_{k,\mu''}, \psi_{k,\mu} \rangle = 2^{-(n+1)N} A_{n} \int_{0}^{1/2} y^n \eta(y) dy. \]

When \( \mu \neq \mu' \), we take absolute values and use the exponential decay of the leading coefficients (part (ii) of Lemma 3.1) to obtain

\[ |2^k \langle \eta_{k,\mu''}, \psi_{k,\mu} \rangle| \leq 8C 2^{-(n+1)N} e^{y} e^{-y|\lambda|/2} \int_{0}^{1/2} y^n \eta(y) dy \]

\[ = 8C 2^{-(n+1)N} e^{y} e^{-yK_0|\mu' - \mu|} \int_{0}^{1/2} y^n \eta(y) dy. \]

Combining the two estimates above, we have

\[ |2^k \langle Y_k, \psi_{k,\mu} \rangle| = 2^{N(-s+1/q)} \sum_{\mu' = 0}^{2^k-1} 2^k \langle \eta_{k,\mu''}, \psi_{k,\mu} \rangle \geq 2^{N(-s+1/q)} \left( 2^k \left| \langle \eta_{k,\mu''}, \psi_{k,\mu} \rangle \right| - \sum_{\mu' \neq \mu} 2^k \langle \eta_{k,\mu''}, \psi_{k,\mu} \rangle \right) \]

\[ \geq 2^{N(-s+1/q)} 2^{-(n+1)N} \left( \int_{0}^{1/2} y^n \eta(y) dy \right) \left( |A| - 8C \sum_{\mu' \neq \mu} e^{y} e^{-yK_0|\mu' - \mu|} \right) \]
Similarly, by using triangle inequality, we have

\[ |2^k \langle Y_k, \Psi_{k,K_0}\mu \rangle| \leq 2^{N(-s+1/q)} \left( 2^k |\langle \eta_{k,K_0\mu}, \Psi_{k,K_0\mu} \rangle| + \left| \sum_{\mu' \neq \mu} 2^k \langle \eta_{k,K_0\mu'}, \Psi_{k,K_0\mu} \rangle \right| \right) \leq 2^{N(-s+1/q)} 2^{-(n+1)N} \left( \int_0^{1/2} y^n \eta(y) \, dy \right) \left( |A| + 8C \sum_{\mu' \neq \mu} e^{\gamma K_0 |\mu'-\mu|} \right). \]

We choose \( K_0 \) large enough so that

\[ \left| 8C \sum_{\mu' \neq \mu} e^{\gamma K_0 |\mu'-\mu|} \right| \leq \frac{|A|}{2}, \]

which gives us the desired result. \( \square \)

**Proposition 6.4.** For \( x \in J_{k, K_0\mu} \) and \( \mu \neq \mu' \), we have that

\[ |\phi_k * \Psi_{k,K_0\mu'}(x)| \leq C e^{\gamma/8} e^{-\gamma K_0 |\mu'-\mu|}. \]

**Proof.**

\[ |\phi_k * \Psi_{k,K_0\mu'}(x)| = \int |\phi_k(y) \Psi(2^k(x-y) - K_0 \mu') \, dy| = \int |\phi(y) \Psi(x_1 - y - K_0 \mu') \, dy|, \]

where \( x_1 = 2^k x \in J_{0, K_0\mu} \subset K_0\mu + [1/4, 3/4] \). We observe that

\[ |x_1 - y - K_0 \mu'| \geq |K_0(\mu - \mu')| - |x_1 - y - K_0 \mu| \geq |K_0(\mu - \mu')| - 7/8. \]

Combining this with the fact that

\[ |\Psi(x)| \leq C e^{-\gamma |x|}, \]

we obtain

\[ |\phi_k * \Psi_{k,K_0\mu'}(x)| \leq C \int |\phi(y)| e^{-\gamma K_0 |\mu'-\mu|} e^{\gamma/8} \, dy \leq C e^{\gamma/8} e^{-\gamma K_0 |\mu'-\mu|}. \] \( \square \)

### 7 LOWER BOUNDS FOR THE NON-ENDPOINT CASE

In this section, we prove the following, which can be interpreted as a quantitative version of Theorem 1.1 for the non-endpoint case.

**Theorem 7.1.** Let \( \Lambda > 10 \) and let \( \gamma_*(F_{p,q}^s, \Lambda) \) be as defined in (1.2).

1. For \( 1 < p < q < \infty, 1/q + n < s < 1/p + n \), we have

\[ \gamma_*(F_{p,q}^s, \Lambda) \geq p,q,s \Lambda^{s-1/q-n}. \]
2. For \( 1 < q < p < \infty, -1/p' - n < s < -1/q' - n \), we have

\[
\gamma_s(F_{p,q}^s, \Lambda) \gtrsim p, q, s \Lambda^{-1/q' - n - s}.
\]

In other words, the magnitude of \( \mathcal{G}(F_{p,q}^s, A) \) depends on the cardinality of \( A \) alone.

**Remark 7.2.** The statements for (1) and (2) above are equivalent, by a standard argument using the duality of the Triebel–Lizorkin spaces

\[
(F_{p,q}^s)^* = F_{p', q'}^{-s}.
\]

We refer the reader to [19], section 2.3, for the details. Consequently, it suffices to prove only the second assertion above.

For \( t \in [0, 1] \), we define the operator

\[
T_t f(x) = \sum_{2^j \in A} r_j(t) \sum_{\mu = 0}^{2^j - 1} 2^j \langle f, \psi_{j,K_0,\mu} \rangle \psi_{j,K_0,\mu}(x).
\]

The following proposition is the main ingredient in the proof of Theorem 7.1.

**Proposition 7.3.** Let \( 1 < q < p < \infty \) and \(-n - 1 < s \leq -1/q' - n\). Let \( f_t \) as in (5.5). Then there exists a \( c > 0 \) such that

\[
\left( \int_0^1 \int_0^1 \left\| T_{t_1} f_{t_2} \right\|_{F_{p,q}^s}^q \ dt_1 dt_2 \right)^{1/q} \geq c 2^{N(-s - 1/q' - n)}.
\]

**Proof.** We can rewrite the left-hand side of the above inequality as

\[
\left( \int_0^1 \int_0^1 \left\| \sum_{k=0}^{\infty} 2^{ksq} |\phi_k * T_{t_1} f_{t_2}|^q \right\|_{L^p([-1,K_0])}^q \ dt_1 dt_2 \right)^{1/q}.
\]

Restricting the innermost function to the interval \([-1, K_0]\) and using Hölder’s inequality (with \( p \geq q \)), we can bound the expression from below by a positive constant times

\[
\left( \int_0^1 \int_0^1 \left\| \sum_{2^k \in A} 2^{ksq} |\phi_k * T_{t_1} f_{t_2}|^q \right\|_{L^q([-1,K_0])}^q \ dt_1 dt_2 \right)^{1/q}.
\]

For a fixed \( x \), we have

\[
\phi_k * T_{t_1} f_{t_2}(x) = 2^{-N/q} \sum_{2^j \in A} \sum_{2^l \in A} r_j(t_1) r_l(t_2) 2^{-ls} \sum_{\mu = 0}^{2^j - 1} \langle \chi_l, \psi_{j,K_0,\mu} \rangle \phi_k * \psi_{j,K_0,\mu}(x).
\]
By Khinchine’s inequality,

\[
\left( \int_0^1 \int_0^1 |\phi_k \ast (T_1 f_{i_2})(x)|^q \, dt_1 \, dt_2 \right)^{1/q} \geq c(q) 2^{-N q / 2} \left( \sum_{2^k \in A} \left| \sum_{2^{j-k} \in A} 2^{j-1} \sum_{\mu=0}^{2^{j-k}-1} \langle Y_k, \psi_{j, K_{01}} \rangle \phi_k \ast \psi_{j, K_{01}}(x) \rangle^2 \right) \right)^{1/2}.
\]

For a given \(2^k \in A\), we consider only the terms with \(j = k\) and \(l = k\), and get

\[
\left( \int_0^1 \int_0^1 |\phi_k \ast (T_1 f_{i_2})(x)|^q \, dt_1 \, dt_2 \right)^{1/q} \geq 2^{-N q / 2} \left| \sum_{\mu=0}^{2^k-1} 2^k \langle Y_k, \psi_{K_{01}} \rangle \phi_k \ast \psi_{K_{01}}(x) \rangle \right|.
\]

Now for \(x \in J_{k, K_{01}}\), we have

\[
\left| \sum_{\mu'=0}^{2^k-1} 2^k \langle Y_k, \psi_{K_{01}} \rangle \phi_k \ast \psi_{K_{01}}(x) \rangle \right| \geq \left| \sum_{\mu'=0}^{2^k-1} 2^k \langle Y_k, \psi_{K_{01}} \rangle \phi_k \ast \psi_{K_{01}}(x) \rangle \right| - 4 \sum_{\mu'=0}^{2^k-1} 2^k \langle Y_k, \psi_{K_{01}} \rangle \phi_k \ast \psi_{K_{01}}(x) \rangle \right| \]

which, using Proposition 6.3, can be bounded from below by a positive constant times

\[
\frac{1}{2} \left| \sum_{\mu'=0}^{2^k-1} 2^k \langle Y_k, \psi_{K_{01}} \rangle \phi_k \ast \psi_{K_{01}}(x) \rangle \right| \geq \frac{1}{2} \left| \sum_{\mu'=0}^{2^k-1} 2^k \langle Y_k, \psi_{K_{01}} \rangle \phi_k \ast \psi_{K_{01}}(x) \rangle \right| - 4 C \left| \sum_{\mu'=0}^{2^k-1} 2^k \langle Y_k, \psi_{K_{01}} \rangle \phi_k \ast \psi_{K_{01}}(x) \rangle \right| \]

Continuing with the proof, we can bound (7.1) from below by

\[
\left( \sum_{2^k \in A} 2^{k q} \left| \left( \int_0^1 \int_0^1 |\phi_k \ast (T_1 f_{i_2})(x)|^q \, dt_1 \, dt_2 \right) \right|^{1/q} \right)^{1/q} \]

where we have used (5.2) and (5.3), and the fact that \(|J_{k, K_{01}}| \geq 2^{-k}\) in the last step. \(\square\)
7.1 Growth of $\gamma_s(F_{p,q}^s, \Lambda)$ for $s < -1/q' - n$

We take $A$ as in (5.3). Let $f_i$ be as in (5.5), so that $\|f_i\| \leq 1$. By Proposition 7.3, there exist $t_1, t_2$ in [0,1] so that

$$\|T_{t_1}f_{t_2}\|_{F_{p,q}^s} \geq 2^{N(-s-1/q'-n)}.$$

Hence,

$$\|T_{t_1}\|_{F_{p,q}^s} \geq c_{p,q,s}2^{N(-s-1/q'-n)}.$$

Now we let

$$E^\pm := \{\psi_{k,K,\mu} : 2^k \in A, r_k(t_1) = \pm 1, \mu = 0, \ldots, 2^k - 1\}.$$

Then we have

$$T_{t_1} = P_{E^+} - P_{E^-}$$

and we conclude that at least one of $P_{E^+}$ or $P_{E^-}$ has operator norm bounded from below by $c_{p,q,s}2^{N(-s-1/q'-n)}$. Since $SF(E^\pm) \subset A$, we get

$$Q(F_{p,q}^s, A) \gtrsim 2^{N(-s-1/q'-n)}$$

for $s < -1/q' - n$ and the asserted lower bound for $Q(F_{p,q}^s, A)$ follows in this range.

**Remark 7.4.** Like the corresponding argument in [19], the above proof is probabilistic in nature. In [20], Seeger and Ullrich explicitly constructed subsets of the Haar system for which the corresponding projections have large operator norms. It might be of interest to try to adapt this deterministic approach to the case of orthogonal spline wavelets as well.

8 LOWER BOUNDS FOR THE ENDPOINT CASE

In this section we prove the lower bounds for the endpoint cases $s = 1/q + n$ and $s = -1/q' - n$. We still have failure of unconditional convergence here, but with a new phenomenon: The growth rate $Q(F_{p,q}^{n+1/q}, A)$ also depends upon the density of the set $\log_2(A) = \{k : 2^k \in A\}$ on intervals of length $\log_2(#A)$. We define for any $A$ with $#A \geq 2$,

$$\underline{Z}(A) = \min_{2^m \in A} \#\{k \in \log_2(A) : |k - m| \leq \log_2(#A)\}.$$

Then, the following is the analog of Theorem 7.1 for the endpoint cases:

**Theorem 8.1.** Let $A \subset \{2^n : n \geq 0\}$ be a set of large enough cardinality.

1. For $1 < p < q < \infty$,

$$Q(F_{p,q}^{n+1/q}, A) \gtrsim_{p,q} \log_2(#A)^{1/q} \underline{Z}(A)^{1-1/q}.$$

2. For $1 < q < p < \infty$,

$$Q(F_{p,q}^{-n-1/q'}, A) \gtrsim_{p,q} \log_2(#A)^{1-1/q'} \underline{Z}(A)^{1/q}.$$

As a corollary, we obtain the following estimates for the lower wavelet projection numbers.
Corollary 8.2. For $\Lambda$ large enough, we have the following:

1. For $1 < p < q < \infty$,
   \[
   \gamma_\ast(F_n^{p+1/q}, \Lambda) \gtrsim_{p,q} \log_2(\Lambda)^{1/q}.
   \]

2. For $1 < q < p < \infty$,
   \[
   \gamma_\ast(F_n^{q-1/q'}, \Lambda) \gtrsim_{p,q} \log_2(\Lambda)^{1−1/q}.
   \]

By Remark 7.2, it suffices in this case as well to prove only the second assertion of Theorem 8.1. Let $\# A \geq 64$ and $N \geq 3$ be such that $4^N \leq 8^{N−1} \leq \# A \leq 8^N$. Using the definition of $\mathcal{Z}(A)$, we can find $M_N$ disjoint intervals $I_i = (n_i − 3N, n_i + 3N)$ with midpoints $n_i \in \log_2(A)$ ($1 \leq i \leq M_N$) and $M_N \geq 8^{N−1}/6N \geq 4^N$, such that each $I_i$ contains at least $\mathcal{Z}(A)$ points in $\log_2(A)$. By a pigeonholing argument, each $I_i$ contains a subinterval $\tilde{I}_i$ of length $N$ with at least $\mathcal{Z}(A)/6$ points in $\log_2(A)$. The upshot is that we have essentially reduced our problem to proving the following:

Theorem 8.3. Let $\# A \geq 4^N$. Suppose there exist $4^N$ disjoint intervals $I_\kappa (1 \leq \kappa \leq 4^N)$, each of length $N$, with $I_\kappa \cap \log_2(A) \neq \emptyset$. Let

\[
Z = \frac{1}{4N} \sum_{\kappa=1}^{4^N} \#(I_\kappa \cap \log_2(A)).
\]

Then, for $1 < q \leq p < \infty$, we have that

\[
G(F_n^{−n−1/q'} ; A) \geq c(p,q)N^{1−1/q}Z^{1/q}. \tag{8.1}
\]

In order to show (8.1) for the endpoint case, we need to construct a suitable family of test functions. To this effect, let $\eta$ denote a $C^\infty$ function supported in $(-2^{-5}, 2^{-5})$ satisfying the conditions (5.1) and (5.2). However, the parity of $\eta$ would be decided later in the argument.

Let $b_\kappa$ denote the largest integer in $I_\kappa$ and let

\[
\mathcal{Q} = \{b_\kappa + N : \kappa = 1, 2, ..., 4^N\}
\]

and for $\tau = 0, 1, ..., N−1$, let

\[
\mathcal{Q}^{N+\tau} = \{b_\kappa + \tau : \kappa = 1, 2, ..., 4^N\}.
\]

Then, $\mathcal{Q}^{N+\tau}$ are disjoint sets, each of cardinality $4^N$. Further, for $l \in \mathcal{Q}$, we define

\[
H_\kappa(x) = \sum_{\tau=0}^{N−1} 2^{(\tau−N)(n+1)} \sum_{\rho \in \mathbb{N} : 0 < 2^{N+b_\kappa+2}\rho < 1} \eta(2^{b_\kappa+\tau}(x − 2^{N+2−b_\kappa}K_0\rho)).
\]

Finally, for $t \in [0, 1]$, let

\[
f_t(x) = \sum_{\kappa=1}^{4^N} r_{b_\kappa+N}(t)2^{(b_\kappa+N)/q'}H_\kappa(x),
\]

where $r_j$ denotes the $j$th Rademacher function with $j \in \mathbb{N}$ and $q > 1$. 
Lemma 8.4. Let $1 < q \leq p < \infty$. Then we have
\[
\|f_t\|_{F^{-1/q',-n}} \leq C(p,q)N^{1/q}.
\]

Proof. Define $g_{\tau,t}$ to be
\[
g_{\tau,t} = \sum_{l \in \mathbb{Z}^N + \tau} 2^{l(n+1)/q} \sum_{\tau = 0}^{N-1} 2^{(\tau-N)(n+1)} \sum_{\rho \in \mathbb{Z}: 0 < 2^{N+\tau+2-l}\rho < 1} r_{l+\tau}(t) \eta(2^l(x - 2^{N+\tau+2-l}K_0 \rho)).
\]

Then, we can write
\[
f_t = \sum_{\tau = 0}^{N-1} 2^{(\tau-N)/q} g_{\tau,t}.
\]

This sets the stage to apply Proposition 4.1 with $m = N + \tau$. It is clear that the points $2^{N+\tau+2-l}K_0 \rho$ are $K_0 2^{m-l}$-separated. Using (4.2) with $\beta_{N+\tau} = 2^{(\tau-N)/q}$, we get
\[
\|f_t\|_{F^{-1/q',-n}} \lesssim_p \left( \sum_{\tau = 0}^{N-1} (2^{(\tau-N)/q})^q 2^{-\tau-N} \#(\mathbb{Z}^{N+\tau}) \right)^{1/q} \lesssim N^{1/q}.
\]

For $\kappa = 1, 2, ..., 4^N$, we define
\[
\mathcal{A}(\kappa) = I_\kappa \cap \log_2(A),
\]
\[
\mathcal{B}(\kappa) = \{(j, \mu) : j \in \mathcal{A}(\kappa), \mu \in 2^{-b_\kappa + N+2}\mathbb{Z}, 1 \leq \mu < 2^j\} \tag{8.2}
\]
and
\[
\mathcal{B} = \bigcup_{\kappa=1}^{4^N} \mathcal{B}(\kappa).
\]

For $t \in [0, 1]$, we also define
\[
T_t f(x) = \sum_{(j,\mu) \in \mathcal{B}} r_j(t) 2^j \langle f, \psi_j, K_0 \mu \rangle \psi_j, K_0 \mu (x).
\]

Proposition 8.5. For $1 < q < p < \infty$, there exists $c(p,q) > 0$ such that for $K_0$ and $N$ large enough, we have
\[
\left( \int_0^1 \int_0^1 \|T_{t_1} f_{t_2}\|_{F^{-1/q',-n}}^q \, dt_1 \, dt_2 \right)^{1/q} \geq c(p,q) N^{1/q}.
\]

Proof. As in the non-endpoint case, it suffices to show that
\[
\left( \int_0^1 \int_0^1 \left( \sum_{k \in \mathcal{A}(k)} \left( \sum_{l \in \mathbb{Z}} 2^{l(q-1)/q} \|\hat{\psi}_k \ast T_{t_1} f_{t_2}\|_{L^q([-1,K_0])} \right)^q \right)^{1/q} \, dt_1 \, dt_2 \right) \geq c(p,q) N^{1/q}.
\]
Interchanging integrals and applying Khinchine’s inequality, we see that the above follows if we show

\[
\left( \sum_{\kappa} \sum_{k \in \Psi(k)} 2^{kq(-n-1/q')} \left\| \sum_{j} \sum_{\kappa'} \sum_{\mu:(j,\kappa') \in \Psi} 2^j \langle 2^{(b\kappa+N)(n+1/q')} H_{\kappa'}, \psi, \kappa, \mu \rangle \phi_k * \psi, \kappa, \mu \rangle \right\|_{L^q([-1, K_0])} \right) \geq c(p,q)NZ^{1/q}.
\]

For the two inner summations, we only consider terms with \( j = k \) and \( \kappa' = \kappa \). Then, the left-hand side of the above expression is bounded from below by

\[
\left( \sum_{\kappa} \sum_{k \in \Psi(k)} 2^{kq(-n-1/q')} \left\| \sum_{\mu:(k,\mu) \in \Psi(\kappa)} 2 \langle H_{\kappa}, \psi, k, \mu \rangle \phi_k * \psi, k, \mu \rangle \right\|_{L^q([-1, K_0])} \right) \geq c(p,q)NZ^{1/q}.
\]

Setting

\[
\zeta_{k,\kappa,\mu}(x) = \eta(2^{b_k+\tau}(x - 2^{N+2-b_k}K_0\rho)),
\]

we can write

\[
\langle H_{\kappa}, \psi, k, \mu \rangle = \sum_{0 \leq \tau \leq N-1} \sum_{\rho \in \mathbb{N}: 0 < 2^{N+2-b_k}\rho < 1} \langle \zeta_{k,\kappa,\mu}, \psi, k, \mu \rangle.
\]

We recall that by (8.2), \( \mu \) is of the form \( \mu = \mu_m = 2^{k-b_k+N+2}m \) for some \( m \in \mathbb{N} \). Hence,

\[
2^k \langle \zeta_{k,\kappa,\mu}, \psi, k, \mu_m \rangle = 2^k \int \eta(2^{b_k+\tau}(x - 2^{N+2-b_k}K_0\rho)) \psi(2^k x - K_0\mu_m) \, dx
\]

\[
= \int \eta(2^{b_k+\tau-k}(y - 2^{N+2-b_k}K_0(\rho - m))) \psi(y) \, dy.
\]

Setting \( \lambda = 2^{N+2-b_k+k}K_0(\rho - m) \), we observe that the range of the above integral is contained in \( \left[ \frac{\lambda-1}{2}, \frac{\lambda+1}{2} \right] \). This is due to the observation that \( b_k + \tau - k \geq 0 \) for \( k \in \Psi(\kappa) \), combined with the fact that \( \eta \) is supported in \( \left[ -2^{-5}, 2^{-5} \right] \). Hence, we can use the spline formulations (3.1) and (3.2) for \( \psi \) on \( \left[ \frac{\lambda-1}{2}, \frac{\lambda}{2} \right] \) and \( \left[ \frac{\lambda}{2}, \frac{\lambda+1}{2} \right] \), respectively, and argue in the same way as in the proof of Proposition 6.3.

In the light of the moment cancelation condition (5.1) (for \( \eta \)) and symmetry of the lower order coefficients of \( \psi \) (Lemma 3.1, (i)), the integrals involving the lower degree terms cancel. We then have

\[
\int \eta \left( 2^{b_k+\tau-k} \left( y - \frac{\lambda}{2} \right) \right) \psi(y) \, dy
\]

\[
= \int_{\lambda-1}^{\lambda} \eta \left( 2^{b_k+\tau-k} \left( y - \frac{\lambda}{2} \right) \right) A_{nL}^\lambda \left( y - \frac{\lambda}{2} \right)^n \, dy + \int_{\frac{\lambda}{2}}^{\frac{\lambda+1}{2}} \eta \left( 2^{b_k+\tau-k} \left( y - \frac{\lambda}{2} \right) \right) A_{nR}^\lambda \left( y - \frac{\lambda}{2} \right)^n \, dy
\]

\[
= 2^{-(b_k+\tau-k)(n+1)} \left[ \int_{\frac{1}{2}}^{0} \eta(y)A_{nL}^\lambda y^n \, dy + \int_{0}^{\frac{1}{2}} \eta(y)A_{nR}^\lambda y^n \, dy \right].
\]

Because of the rapid decay of \( \psi \), the major contribution comes in the case when \( \rho = m \) (in which case \( \lambda = 0 \)), provided we choose \( \eta \) in a suitable way, so as to prevent unwanted cancelation. It is here that we choose the parity of \( \eta \) to our advantage. More precisely, we choose \( \eta \) such that

\[
y^n \eta(y) = \begin{cases} 
\text{odd, if } A_{nL}^0 \text{ and } A_{nR}^0 \text{ are of opposite signs} \\
\text{even, if } A_{nL}^0 \text{ and } A_{nR}^0 \text{ are of the same sign.}
\end{cases}
\]
Such a choice ensures that for $\lambda = 0$, we have
\[
\left| \int \eta(2^{b\kappa+\tau-k}y)\psi(y) \, dy \right| \geq |A_n^{0,R}| 2^{-(b\kappa+\tau-k)(n+1)} \left( \int_0^{\frac{1}{2}} y^n \eta(y) \, dy \right) > 0.
\]

Here, we also note that $A_n^{0,R} = A_n^{1.L} = A_n^L$, by virtue of the fact that the leading coefficient of a polynomial (the restriction of $\psi$ to $[0, \frac{1}{2}]$ in this case) is invariant under translation and consequently, $A_n^{0,R} \neq 0$ due to assumption (6.5).

When $\lambda \neq 0$, it is a nonzero integer multiple of $K_0$. Hence the exponential decay of the leading coefficients $A_{\lambda, L}^n$ and $A_{\lambda, R}^n$ (Lemma 3.1, (ii)) kicks in, and we get the bound
\[
\left| \int \eta \left( 2^{b\kappa+\tau-k} \left( y - \frac{\lambda}{2} \right) \right) \psi(y) \, dy \right| \leq 8C 2^{-(b\kappa+\tau-k)(n+1)} e^{-\gamma \frac{\lambda}{2}} \left( \int_0^{\frac{1}{2}} y^n \eta(y) \, dy \right).
\]

We now combine the two estimates above and use the triangle inequality as in the proof of Lemma 6.3. For $K_0$ large enough, this yields
\[
\frac{|A_n^{0,R}|}{2} 2^{-(b\kappa+\tau-k)(n+1)} \left( \int_0^{\frac{1}{2}} y^n \eta(y) \, dy \right) \leq 2^k \sum_{\rho \in \mathbb{N}} \langle \zeta_{\kappa, \rho}, \psi_{k, K_0 \mu_m} \rangle \leq 2 |A_n^{0,R}| 2^{-(b\kappa+\tau-k)(n+1)} \left( \int_0^{\frac{1}{2}} y^n \eta(y) \, dy \right).
\]

Thus,
\[
\frac{|A_n^{0,R}|}{2} N 2^{(k-b\kappa-N)(n+1)} \left( \int_0^{\frac{1}{2}} y^n \eta(y) \, dy \right) \leq 2^k \langle H_{\kappa, \psi_{k, K_0 \mu}} \rangle \leq 2 |A_n^{0,R}| N 2^{(k-b\kappa-N)(n+1)} \left( \int_0^{\frac{1}{2}} y^n \eta(y) \, dy \right).
\]

The intervals $J_{k, K_0 \mu}$ (where $J_{k, K_0 \mu}$ is as in (6.6)) are disjoint. Using Proposition 6.4 and arguing as in the non-endpoint case (proof of Proposition 7.3), we conclude that for $K_0$ large enough, (8.3) is bounded from below by
\[
c \left( \sum_k \sum_{m \in \mathbb{N}} 2^k q(-n-1/q') \sum_{0 < 2^{k} + b\kappa - N < 2^k} \int_{J_{k, K_0 \mu}} \left| 2^{(b\kappa+N)(n+1)/q'} N 2^{(k-b\kappa-N)(n+1)} \int_0^{\frac{1}{2}} y^n \eta(y) \, dy \right| ^q \, dx \right)^{1/q}.
\]

The measure of $\cup_{m \in \mathbb{N}, 0 < 2^{k} + b\kappa - N < 2^k} J_{k, K_0 \mu}$ is about $2^{b\kappa-N-k}$. Hence, the above expression is bounded from below by
\[
c' \left( \sum_k \sum_{m \in \mathbb{N}} 2^k q(-n-1/q') 2^{b\kappa-N-k} \left( 2^{(b\kappa+N)(n+1)/q'} N 2^{(k-b\kappa-N)(n+1)} \right)^q \right)^{1/q} \gtrsim \left( \sum_k \sum_{m \in \mathbb{N}} 2^{-2N q} \right)^{1/q} \gtrsim NZ^{1/q}.
\]

This finishes our proof. \[\square\]

Finally, we have all the ingredients ready to prove Theorem 8.3.

**Proof.** By Proposition 8.5, there exist $t_1, t_2 \in [0, 1]$ such that $\|T_{t_1} f_{t_2}\|_{\mathcal{F}^{1/q'-n}} \gtrsim NZ^{1/q}$ but $\|f_{t_2}\|_{\mathcal{F}^{1/q'-n}} \lesssim N^{1/q}$. Consequently,
\[
\|T_{t_1}\|_{\mathcal{F}^{1/q'-n}} \gtrsim c_{p,q} N^{1-1/q} Z^{1/q}.
\]
Defining $E^\pm$ as in (7.2) and recalling that $T_{t_1} = P_{E^+} - P_{E^-}$, we get

$$\max_{\pm} \| P_{E^\pm} \|_{F^{-1/q'}-n} \geq \frac{c_{p,q}}{2} N^{1-1/q} Z^{1/q}.$$ 

Thus, $Q(F^{-1/q'-n},A) \gtrsim N^{1/q} Z^{1/q}$. 

\[\square\]

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