The Euclidean three-point function in loop and perturbative gravity

Carlo Rovelli and Mingyi Zhang

Centre de Physique Théorique de Luminy, Case 907, F-13288 Marseille, France

E-mail: rovelli@cpt.univ-mrs.fr and Mingyi.Zhang@cpt.univ-mrs.fr

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Abstract

We compute the leading order of the three-point function in loop quantum gravity, using the vertex expansion of the Euclidean version of the new spin foam dynamics, in the region of $\gamma < 1$. We find results consistent with Regge calculus in the limit $\gamma \rightarrow 0$, $j \rightarrow \infty$. We also compute the tree-level three-point function of perturbative quantum general relativity in position space and discuss the possibility of directly comparing the two results.

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1. Introduction

The difficulty of extracting physical predictions from a background-independent theory is a well-known difficulty of quantum gravity. A strategy to address the problem has been developing in recent years, based on two ideas. The first is to define $n$-point functions over a background by storing the information about the background in the boundary state [1]. In covariant loop gravity [2, 3], this technique yields a definite expression for the theory’s $n$-point functions. The second is to explore the expansion of this expression order by order in the number of interaction vertices [4]. Although perhaps counter-intuitive, this expansion has proven effective in certain regimes; for details see [5, 6]. In particular, the low-energy limit of the two-point function (the ‘graviton propagator’) obtained in this way from the improved–Barrett–Crane spin foam dynamics [7–12] (sometimes denoted as the EPRL/FK model) correctly matches the graviton propagator of pure gravity in a transverse radial gauge (harmonic gauge) [13, 14]. This result has been possible thanks to the introduction of the coherent intertwiner basis [15] and the asymptotic analysis of vertex amplitude [16, 17].

The obvious next step is to compute the three-point function. In this paper, we begin the three-point function analysis. We compute the three-point function from the non-perturbative

1 Unité mixte de recherche du CNRS et des Universités deProvence, de la Méditerranée et du Sud; affilié à la FRUMAN.
theory. As in [14], we work in the Euclidean regime and with the Barbero–Immirzi parameter $0 < \gamma < 1$, where the amplitude defined in [11] and that defined in [12] coincide.

Our main result is as follows. We consider the limit, introduced in [14, 18], where the Barbero–Immirzi parameter is taken to zero $\gamma \to 0$, and the spin of the boundary state is taken to infinity $j \to \infty$, keeping the size of the quantum geometry $A \sim \gamma j$ finite and fixed. This limit corresponds to neglecting Planck scale discreteness effects, at large finite distances. In this limit, the three-point function we obtain exactly matches the one obtained from Regge calculus [19].

This implies that the spin foam dynamics is consistent with a discretization of general relativity, not just in the quadratic approximation, but also to the first order in the interaction terms. This result agrees with the recent one in [18], where Magliaro and Perini show that in the $\gamma \to 0$ regime, the partition function for a two-complex takes the form of a path integral over continuous Regge metrics.

The relation between the Regge and loop three-point function and the three-point function of the weak field perturbation expansion of general relativity around flat space, on the other hand, remains elusive. We compute explicitly the perturbative three-point function in position space in the transverse gauge (harmonic gauge), and we discuss the technical difficulty of comparing this with the Regge/loop one.

The paper is organized as follow. In section 2, we review the ingredients and the assumptions needed to define the $n$-point functions in loop gravity and compute the three-point function. In section 3, we derive the three-point function from perturbative field theory and discuss the relation between this and the loop/Regge one.

2. Three-point function in loop gravity

In this section, we compute the three-point function of the spin foam amplitude in loop quantum gravity at first order in the vertex expansion. We follow closely the techniques developed for the two-point function in [5, 14] and the calculation of the three-point function for the old Barrett–Crane model in [20]. For previous work in this direction, see also [13, 21, 22].

2.1. Boundary formalism

The well-known difficulty of defining $n$-point functions in a general covariant quantum field theory can be illustrated by the following (naive) argument. If the action $S[g]$ and the measure are invariant under coordinate transformations, then

$$W(x_1, \ldots, x_N) \sim \int Dg\, g(x_1) \cdots g(x_N) e^{iS[g]}$$  \hspace{1cm} (1)

is formally independent from $x_n$ (as long as the $x_n$ are distinct), because a change in $x_n$ can be absorbed into a change of coordinates that leaves the integral invariant. This difficulty is circumvented in the weak field approximation as follows. If we want to study the theory around flat space, we have to impose boundary conditions on equation (1) demanding that $g$ goes to flat space at infinity. With this choice, the classical solution that dominates the path integral in the weak field limit is flat spacetime. In flat spacetime, we can choose the preferred Cartesian coordinates $x$ and write the field insertions in terms of these preferred coordinates. Then equation (1) is well defined: the coordinates $x_{\mu}$ are not generally covariant coordinates, but rather Minkowski coordinates giving physical distances and physical time intervals in the background metric picked out by the boundary conditions of the field at infinity. This is the way $n$-point functions are defined for perturbative general relativity. In the
full non-perturbative theory, on the other hand, this strategy is not viable because the integral equation (1) has formally to be taken over arbitrary geometries, where the notion of preferred Cartesian coordinate loses meaning.

The idea for solving this difficulty was introduced in [1] and is explained in detail in [5]. We give here a short account of this formalism, but we urge the reader to look at the original references for a detailed explanation of the approach. Let us begin by picking a surface $\Sigma$ in flat spacetime, bounding a compact region $R$, and approximate equation (1) by replacing $S[g]$ outside $R$ with the linearized action. Then split equation (1) into three integrals: the integral on the field variables in $R$, outside $R$ and on $\Sigma$. Let $\gamma$ be the value of the field on $\Sigma$. Let $W_{\Sigma}[\gamma]$ be the result of the internal integration, at a fixed value $\gamma$ of the field on $\Sigma$:

$$W_{\Sigma}[\gamma] = \int_{g|z=\gamma} Dg \ e^{iS[g]}.$$ \hspace{1cm} (2)

Let $\Psi_{\Sigma}[\gamma]$ be the result of the outside integral. Then we can write

$$W(x_1, \ldots, x_N) \sim \int D\gamma \ W_{\Sigma}[\gamma] \ y(x_1) \cdots y(x_N) \Psi_{\Sigma}[\gamma]$$

$$= \langle W_{\Sigma}[\gamma(x_1) \cdots y(x_N)] \Psi_{\Sigma} \rangle.$$ \hspace{1cm} (3)

Now observe first that because of the (assumed) diff-invariance of the measure and action, $W_{\Sigma}[\gamma]$ is in fact independent from $\Sigma$. That is, $W_{\Sigma} = W$. Second, since the external integral is that of a free theory, $\Psi_{\Sigma}[\gamma]$ will be the vacuum state of the free theory on the surface $\Sigma$. This can be shown to be a Gaussian semiclassical state peaked on the intrinsic and extrinsic geometries of $\Sigma$. The quantities appearing in the formal expression equation (4) are well defined in loop quantum gravity, and this expression can be taken as the starting point for computing $n$-point functions from the background-independent theory.

2.2. The theory

The definition of the non-perturbative quantum gravity theory we use is given for instance in [3]. The Hilbert space of the theory is spanned by spin network states $|\Gamma, \psi\rangle$, where $\Gamma$ is a graph with $L$ links $l$ and $N$ nodes $n$, and $\psi$ is in $\mathcal{H}_\Gamma = L_2(SU(2)^L/SU(2)^N)$. A convenient basis in $\mathcal{H}_\Gamma$ is given by the coherent states $|j, \vec{n}\rangle$ which are the gauge-invariant projections of $SU(2)$ Bloch coherent states [8]. These are labeled by a spin $j_l$ per each link of the graph, and a unit-norm 3-vector $\vec{n}_{nl}$ for each couple node-link of the graph. The dynamics of the theory is determined by the amplitude $W$ defined as a sum over two-complexes or, equivalently [25], as the limit for $\sigma \to \infty$ over the two-complexes $\sigma$ bounded by $\Gamma$, of the amplitude (we follow here [26] for the notation)

$$\langle W_{\sigma}|\Gamma, j, n \rangle = \sum_{j_f} \int dg_{X f} \int d\vec{n}_{ef} \prod f \langle e| df \ tr \left[ \prod_{ee\partial f} P_{ef} \right].$$ \hspace{1cm} (5)

where $e \in \partial f$ is the ordered sequence of the oriented edges around the face $f$ and

$$P_{ef} = g_{s,e} Y_{j_f, \vec{n}_{ef}} Y_{j_f, -\vec{n}_{ef}} Y_{j_f, \vec{n}_{ef}}^{-1}$$ \hspace{1cm} (6)

for an internal edge $e$. For an external edge $e$, namely an edge hitting the boundary $\Gamma$ of $\sigma$,

$$P_{ef} = \langle j_i, -\vec{n}_{al}|Y| j_i, \vec{n}_{al} \rangle \quad \text{or} \quad P_{ef} = g_{s,e} Y_{j_i, \vec{n}_{al}} Y_{j_i, \vec{n}_{al}}^{-1} \hspace{1cm} (7)$$
according to whether the orientation of the edge is incoming or outgoing. Here \( l \) is the link bounding the face \( f \) and \( n \) is the node bounding the edge \( e \). In all these formulas, the notation \( g \) stands for the matrix elements of the group element \( g \) in the appropriate representation.

Here we deal with the Euclidean theory. Then \( g_{ve} = (g^n_e, g^o_e) \in \text{Spin}(4) \sim SU(2) \times SU(2) \) and \( Y \) maps the \( SU(2) \) representations \( j \) into the highest weight \( SU(2) \) irreducible of the \( SO(4) \) representation \( (j^+, j^-) \), where \( j^\pm = \frac{1}{2}(1 \pm \gamma) j \). The matrix elements of \( Y \) are the standard Clebsch–Gordan coefficients.

The amplitude can be written in the form of a path integral by defining the action

\[
S = \sum_f S_f = \sum_f \ln \left( \prod_{e \in f} P_{ef} \right). \tag{8}
\]

Then

\[
\langle W_\sigma | \Gamma, j, n \rangle = \sum_{j_f} \mu \int dg_{ve} \int d\bar{n}_{ef} e^S, \tag{9}
\]

where \( \mu = \sum_f d_j \). This is the form which is suitable for the asymptotic expansion that we use below.

Since the coherent states factorize under the Clebsch–Gordan decomposition, and since the scalar product of coherent states in the representation \( j \) is the \( j \)'s power of that in the fundamental representation, we obtain \( S = S^+ + S^- \) with

\[
S^\pm = \sum_{j_f} 2j^\pm_f \ln \left( -\bar{n}_{ef} |(g^\pm_{ve})^{-1} g_{ve}^\pm | \bar{n}_{ef} \right), \tag{10}
\]

where \( e \) and \( e' \) are the two edges bounding \( f \) and \( v \).

The last ingredient we need are the gravitational field operators \( \gamma(x) \) that enter in equation (4). The gravitational field operator that corresponds to the metric is expressed in loop quantum gravity by the Penrose operator [3]

\[
G_{il}^{ab} = E_i^a \cdot E_l^b, \tag{11}
\]

where \( E_i^a \) is the left invariant vector field acting on the \( h_{la} \) variable of the state vector, namely the \( SU(2) \) group element associated with the link \( a \) bounded by the node \( l \). The key technical observation of [14] is that

\[
\langle W | G_{il}^{ab} | \Gamma, j, n \rangle = \sum_{j_f} \mu \int dg_{ve} \int d\bar{n}_{ef} q_{il}^{ab} e^S, \tag{12}
\]

where \( q_{il}^{ab} = A_{il}^a \cdot A_{il}^b \) and \( A_{il}^a = A_{il}^{a+} + A_{il}^{a-} \):

\[
A_{il}^{a\pm} = \gamma_{ja}^{a\pm}\frac{-\bar{n}_{al} (g^a_{al})^{-1} g_{il}^{a\pm} | \bar{n}_{il} \rangle}{\langle -\bar{n}_{al} | (g^a_{al})^{-1} g_{il}^{a\pm} | \bar{n}_{il} \rangle}. \tag{13}
\]

This is the insertion that we consider below.

2.3. Vertex expansion

The second idea for computing \( n \)-point functions is the vertex expansion [4]. This is the idea of studying the approximation to equation (4) given by the lowest order in the \( \sigma \to \infty \) limit, namely using small graphs and small two-complexes. Here we look at only the first nontrivial term. That is, we take a minimal two-complex, formed by a single vertex. We consider for simplicity the theory restricted to five-valent vertices and four-valent edges. Then the lowest order is given by a two-complex formed by a single five-valent vertex bounded by the
complete graph with five nodes $\Gamma_5$. Labeling the nodes with the indices $a, b, \ldots, = 1, \ldots, 5$, the amplitude of this two-complex for the boundary state $|\Gamma_5, j_{ab}, \vec{n}_{ab}\rangle$ (here $j_{ab} = j_{ba}$, but $\vec{n}_{ab} \neq \vec{n}_{ba}$) reads simply

$$\langle W|\Gamma_5, j_{ab}, \vec{n}_{ab}\rangle = \mu(j) \int_{SU(2)^{10}} \frac{dg_+}{g_+} e^{\sum_{ab} S_{ab}}$$

with

$$S_{ab} = \sum_{\pm} 2 j_{ab}^+ \ln \left( -\vec{n}_{ab} \left( g_{ab}^+ \right)^{-1} g_{ba}^+ |\vec{n}_{ba}\rangle \right).$$

The $\mu(j)$ term comes from the face amplitude and the measure (and cancels at the tree level [20]).

The vertex expansion has appeared counterintuitive to some, on the base of the intuition that the large distance limit of quantum gravity could be reached only by states defined on very fine graphs, and with very fine two-complexes. We are not persuaded by this intuition (in spite of the fact that one of the authors is quite responsible for propagandizing it [27–29]) for a number of reasons. The main one is as follows. It has been shown that under appropriate conditions, equation (9) can approximate a Regge path integral for large spins [26, 30, 31]. Regge calculus is an approximation to general relativity that is good up to the order $O(l^2/\rho^2)$, where $l$ is the typical Regge discretization length and $\rho$ is the typical curvature radius. This implies that Regge theory on a coarse lattice is good as long as we look at small curvatures’ scale. In particular, it is obviously perfectly good on flat space, where in fact it is exact, because the Regge simplices are themselves flat, and is good as long as we look at weak field perturbations of long wavelength. This is precisely the limit in which we want to study the theory here. In this limit, it is therefore reasonable to explore whether the vertex expansion give any sensible result.

Reducing the theory to a single vertex is a drastic simplification of the field theory, which reduces the calculation to one for a system with a finite number of degrees of freedom. Is this reasonable? The answer is in noticing that the same drastic simplification occurs in the analog calculation in QED: at the lowest order, an $n$-point function involves only the Hilbert space of a finite number of particles, which are described by a finite number of degrees of freedom in the classical theory. The genuine field theoretical aspects of the problem, such as renormalization, do not show up at the lowest order, of course.

If we regard the calculation from the perspective of the triangulation dual to the two-complex, what is being considered is a region of spacetime with the geometry of a four-simplex. In the approximation considered the region is flat, but this does not mean that there are no degrees of freedom. In fact, the Hamilton function of general relativity is a nontrivial function of the intrinsic geometry of the boundary, whose variation gives equations that determine the extrinsic geometry as a function of the intrinsic geometry. This relation captures a small finite-dimensional sector of the Einstein-equation dynamics (for a simple example of this, see [32]). This is precisely the component of the dynamics of general relativity captured in this limit. The three-point function in this large wavelength limit describes the correlations between the fluctuations of the boundary geometry of the four-simplex, governed by the quantum version of this restricted Einstein dynamics.

Let us illustrate this dynamics a bit more in detail, both in second order (metric) and first order (tetrad/connection) variables. In metric variables, the intrinsic geometry of a boundary of a four-simplex (formed by glued flat tetrahedra) is uniquely determined by the ten areas $A_{ab}$ of their faces. The extrinsic geometry of the boundary four-simplex is determined by the ten angles $\Phi_{ab}$ between the 4D normals to the tetrahedra. The Einstein equations reduce
in the case of a single simplex to the requirement that this is flat. If the four-simplex is flat, then the ten angles $\Phi_{1ab}$ are well-defined functions

$$\Phi_{ab} = \Phi_{ab}(A_{ab})$$

(16)
of the ten areas $A_{ab}$ (for comparison, if the four-simplex has constant curvature because of a cosmological constant, then the same $A_{ab}$’s determine different $\Phi_{ab}$’s). This dependence captures the restriction of the Einstein equations to a single simplex. In first-order variables, the situation is more complicated. The variables $g$, $j$ and $\vec{n}$ in equation (8) can be viewed as the discretized version of the connection and the tetrad. The vanishing-torsion equation of the first-order formalism, which relates the connection to the tetrad, becomes in the discrete formalism a gluing condition between normals to the faces parallel transported by the group elements.

### 2.4. Boundary vacuum state

Following the general strategy described above, we need a boundary state peaked on the intrinsic as well as on the extrinsic geometry. This state cannot be the state $|\Gamma_5, j_{ab}, \vec{n}_{ab}\rangle$ which is an eigenvalue of boundary areas and therefore is maximally spread in the extrinsic curvature, namely in the 4D dihedral angle between two boundary tetrahedra $\Phi_{ab}$ [33]. Rather, we need a state which is also smeared over spins [34–36].

Following [36], we choose here a boundary state peaked on the intrinsic and extrinsic geometries of a regular four-simplex and defined as follows. The geometry of a flat four-simplex is uniquely determined by the ten areas $A_{ab}$ of its ten faces. Let then $\vec{n}_{ab}(A_{ab})$ be the 20 normals determined up to the arbitrary $SO(3)$ rotations of each quadruplet $\vec{n}_{ab}, \ldots, \vec{n}_{ab}$ by these areas. By this we mean the following. The flat four-simplex determined by the given areas is bounded by five tetrahedra. For each such tetrahedron, the four normals to its four faces in the three-space determined by the tetrahedron determine, up to rotations, the four unit vectors $\vec{n}_{ab}, \ldots, \vec{n}_{ab}$. Using this, we define the boundary state as

$$|\Psi_j\rangle = |\Psi_j (j_{ab})\rangle = \sum_{j_{ab}} c_j (j_{ab}) |\Gamma_5, j_{ab}, n_{ab}(j_{ab})\rangle,$$

(17)

where the coefficients $c_j (j)$ in the large $j$ limit are given by [36]

$$c_j (j_{ab}) = \frac{1}{N} e^{-\frac{1}{\sqrt{N}} \sum_{ab, (cd)} \gamma_{ab}(j_{ab}) \frac{\sqrt{N}}{j_{0}} \frac{\sqrt{N}}{j_{cd}} - j_{0} \sum_{ab} \Phi_0 j_{ab}}.$$

(18)

The coefficients are also given in [4, 5]. $\alpha_{(cd)}$ is a 10 × 10 matrix that has the symmetries of the four-simplex, that is, it can be written in the form $\alpha_{(cd)} = \sum_k \alpha_k P_{(cd)} (k)$, where

$$P_{0}^{(ab)(cd)} = 1 \quad \text{if} \quad (ab) = (cd) \quad \text{and} \quad 0 \quad \text{otherwise},$$

$$P_{0}^{(ab)(cd)} = 1 \quad \text{if} \quad |a = c, b \neq d| \quad \text{or} \quad \text{a permutation},$$

$$P_{1}^{(ab)(cd)} = 1 \quad \text{if} \quad (ab) \neq (cd) \quad \text{and} \quad 0 \quad \text{otherwise},$$

$\Phi_0$ is the background value of the 4D dihedral angles which give the extrinsic curvature of the boundary. $j_0$ is the background value of all the areas. The state is peaked on the areas $j_{ab} = j_0$, which determine a regular four-simplex. The dihedral angles of a flat tetrahedron are $\Phi_0 = \arccos (-\frac{1}{4})$, and we fix $\Phi_0$ to this value. As a consequence, $|\Psi_j\rangle$ is a semiclassical physical state, namely it is peaked on the values of intrinsic and extrinsic geometries that satisfy the (Hamilton) equations of motion (16) of the theory. See [4, 5, 14, 32] for more details.
2.5. Three-point function

Let us now choose the operator insertion. We are interested in the connected component of the quantity

$$\tilde{G}_{lmn}^{abcdef} = \langle G_{lmn}^{abcdef} \rangle,$$

(19)

where $G_{lmn}^{abcdef}$ is the Penrose operator associated with the node $l$ of $\Gamma_5$ and the two links of this node going from $l$ to $a$ and from $l$ to $b$ respectively. The connected component is

$$G_{lmn}^{abcdef} = \langle G_{lmn}^{abcdef} \rangle + 2\langle G_{lmn}^{(i)} G_{mn}^{(j)} \rangle - \langle G_{lmn}^{(i)} G_{mn}^{(j)} \rangle - \langle G_{lmn}^{(i)} G_{mn}^{(j)} \rangle.$$

(20)

We begin by studying the full three-point function equation (19), before subtracting the disconnected components. From equations (4) and (18), and simplifying a bit the notation in a self-explicatory way, this is

$$\tilde{G}_{lmn}^{abcdef} = \sum_j c(j) \langle W \mid G_{lmn}^{abcdef} \mid \Gamma_5, j, n \rangle \sum_j c(j) \langle W \mid \Gamma_5, j, n \rangle.$$

(21)

Using equation (12), this gives

$$\tilde{G}_{lmn}^{abcdef} = \sum_j c(j) \int d^g_{ab} q_{l/m}^{ab} q_{m/n}^{cd} q_{n}^{ef} e^{S} \sum_j c(j) \int d^g_{ab} e^{S},$$

(22)

where the sum over spins is only given by the boundary state, since there are no internal faces.

Define the total action as $S_{\text{tot}} = \ln c(j) + S$. Because we want to obtain the large $j$ limit of the spin foam model, we rescale the spins $j_{ab}$ and $j_0$. Then the action goes to $S_{\text{tot}} \rightarrow \lambda S_{\text{tot}}$ and also $q_{l/m}^{ab} \rightarrow \lambda^2 q_{n}^{ab}$. In large $\lambda$ limit, the sum over $j$ can be approximated to the integrals over $j$:

$$\sum_j \mu \int d^g_{ab} q_{l/m}^{ab} e^{S_{\text{tot}}} \approx \int d^2 j d^2 g_{ab} \mu j_{ab} e^{S_{\text{tot}}},$$

(23)

where $\mu$ is the product of the face amplitudes. Thus, (dropping the suffix $\text{tot}$ from now on)

$$\tilde{G}_{lmn}^{abcdef} = \lambda^6 \int d^2 j d^2 g_{ab} \mu j_{ab} q_{l/m}^{cd} q_{n}^{ef} e^{S} \int d^2 j d^2 g_{ab} \mu e^{S}.$$

(24)

The action, measure and insertions are invariant under a $SO(4)$ symmetry; therefore, only four of the five $d^2 g_{ab}$ are independent. We can fix the gauge that one of the group elements $g_{\pm} = 1$, and the integral reduced to $dg = \prod_{a=1}^{4} dg_{a}^{+}dg_{a}^{-}$. This gives the expression

$$\tilde{G}_{lmn}^{abcdef} = \lambda^6 \int d^2 j d^2 g \mu j_{ab} q_{l/m}^{cd} q_{n}^{ef} e^{S} \int d^2 j d^2 g \mu e^{S}.$$

(25)

We simplify the notation by writing this in the simple form

$$\tilde{G} = \lambda^6 \int d^2 j d^2 g \mu l m n e^{S} \equiv \langle lmn \rangle,$$

(26)

where $l = q_{l/m}^{ab}$, $m = q_{cd}$ and $n = q_{n}^{ef}$ are the functions of $j$ and $g$. The connected component then reads

$$G = \langle lmn \rangle + 2\langle l \rangle \langle m \rangle \langle n \rangle - \langle lm \rangle \langle n \rangle - \langle ml \rangle \langle n \rangle - \langle mn \rangle \langle l \rangle,$$

(27)

which is the point of departure for the saddle point expansion.
2.6. Saddle point expansion

To study the asymptotic behavior of equation (26), we use the saddle point expansion [20, 37, 38]. For this, we need the stationary point of the total action $S_{\text{tot}} = \ln c(j) + S$. Here we briefly review the works in [16] and [14]. They discuss the behavior of the critical point and stationary point of $S = S^+ + S^-$ and $S_{\text{tot}}$. We invite readers to read their articles for a full detailed discussion.

The critical point and stationary point of $\text{Re}(S)$ coincide with each other when $\gamma < 1$. For the real part of the action $S$, the critical points are the group element $\hat{g}^\pm$ satisfying the gluing condition

$$R^\pm_a n_{ab} = -R^\pm_b n_{ba},$$

where $R^\pm_a = R(\hat{g}^\pm_a)$ is the spin-$1$ irrep. of SU(2). This means that at the critical point, the geometry of spacetime goes to a classical one in which all tetrahedra glue perfectly. There are four classes of critical points that satisfy condition (28). At the critical points of $\text{Re}(S)$, the action $S$ can be written as $S = iA$, where $A$ is a real function and reduces to Regge-like actions. See [16] and [14]. A unique class of critical points is then selected by the stationary point behavior of $S_{\text{tot}}$.

The stationary points of $\text{Re}(S)$ are the critical points of $\text{Re}(S)$, because of the closure constraint, which is satisfied by the boundary state for large $j_0$. We are interested in the stationary points of $S_{\text{tot}}$ which are not just with respect to the group variables but also with respect to the spin $j$ variables. The stationary point $j_{ab} = j_0$ also selects the class of group stationary point. This is because at the stationary point, $S$ must satisfy

$$-i\gamma \Phi_{ab} + \frac{\partial S(g_0)}{\partial j_{ab}} = 0.$$  \hfill (29)

Therefore, it means that only when $S(g_0) = iS_{\text{Regge}}$ (with a definite sign), this condition can be satisfied. This condition picks the unique class of critical points $g^\pm_0$ of $\text{Re}(S)$, which makes $S(g_0) = iS_{\text{Regge}}$.

We are thus interested in the saddle point expansion of the integrals in equation (26) around the stationary points $(j_0, g^\pm_0)$ described above. According to the general theory, the integral

$$F(\lambda) = \int dx f(x) e^{\lambda S(x)}$$

(30)

can be expanded for large $\lambda$ around the stationary points as follows:

$$F(\lambda) = C(x_0) \left( f(x_0) + \frac{1}{\lambda} \left( \frac{1}{2} f_{ij}(x_0) J^{ij} + D \right) \right) + O\left( \frac{1}{\lambda^2} \right),$$  \hfill (31)

where $x_0$ is the stationary point, $f_{ij}$ is the Jacobian matrix of $f$ and $J = H^{-1} = (S''(x_0))^{-1}$ is the inverse of the Jacobian matrix of the action $S$. A straightforward application of this formula to equation (27) shows that

$$G = 0 + O\left( \frac{1}{\lambda^2} \right).$$  \hfill (32)

This in fact is not surprising because we are computing a three-point function, and this cannot be captured only by the second order of the saddle point expansion. The second order of the saddle point expansion sees only the second derivatives of the action, while the connected component of the three-point function depends on the third derivatives of the action. In fact, the third derivative of the action term can be identified with a Feynman vertex, the inverse of
the second derivative as the propagator and the insertions as the external legs of a Feynman diagram. Then it is clear that to second order, there is no connected component.

Therefore, we need the next order of the saddle point expansion. From equation (30), this is given by

\[ F(\lambda) = C(x_0) \left( f(x_0) + \frac{F_1}{\lambda} + \frac{F_2}{\lambda^2} \right) + O(1/\lambda^3), \]  

(33)

where

\[ F_1 = -\frac{1}{2} f_{ij} J^{ij} + \frac{1}{2} f_i J^i \bar{J} R_{ijkl} - \frac{5}{24} f J^{ji} F_{jk} R_{jkl} + \frac{1}{2} f J^{ik} J_{ijkl}, \]  

(34)

and

\[ F_2 = \frac{1}{8} f_{ijkl} J^{ij} J^{kl} - \frac{5}{12} f_{ijk} J^{jl} J^{km} \bar{R}_{lmn} - \frac{5}{16} f_{ij} J^{km} J^{ln} R_{klmn} \]  

(35)

Here \( R(x) = S(x) - \frac{1}{2} H_{ij}(x - x_0)^i(x - \bar{x})^j \), all functions are computed in \( x_0 \), and the stationary point of \( S(x) \) and the indices indicate derivatives. In the last two equations, we have left understood some symmetrization. For instance, the third term on the right-hand side of equation (34) should read

\[ \frac{5}{48} f (J^{ij} J^{km} J^{ln} + J^{jl} J^{km} J^{ln}) R_{ijkl} R_{lmn}, \]  

(36)

and so on.

Using this, and recalling that here \( f(x) = \mu(x) l(x) m(x) n(x) \), we obtain, up to the order \( O(1/\lambda^2) \),

\[ G_{lmn}^{abcde} = \lambda^2 \left( -R_{ijkl} m_{nm} n_{lp} J^{ij} J^{km} J^{ln} + (l_{ij} m_{ik} n_{j} + l_{ij} m_{ik} n_{j} + l_{ik} m_{ij} n_{j}) J^{ij} \right). \]  

(37)

The first term on the right-hand side resembles the one-vertex diagram with three legs. The second term resembles a four-point function in which two points are identified.

### 2.7. Analytical expression

Equation (37) indicates that we need to get the second and third derivatives of the total action, and the first and second derivative of the insertions. Here we compute these terms.

We use Euler angles to parameterize the \( SU(2) \) group elements \( g^\pm_0 \) around the stationary point:

\[ R^\pm_a = e^{\theta_i^{\pm} J_i} R^\pm_a, \]  

(38)

where \( i = 1, 2, 3, \theta_i \) are the Euler angles, \( J_i \) are the generators of \( SU(2) \) and \( R^\pm_a \) stands for arbitrary irrep. of \( SU(2) \). There are 34 independent variables: 10 areas \( f_{ab} \) of triangles in the four-simplex, and 24 group element parameters in which 12 are for \( g^+ \) and 12 are for \( g^- \). Here we give only some steps to get to the result. The whole results can be found in the appendix.

The second-order derivative of the total action gives

\[ \frac{\partial^2 S}{\partial \theta^{\pm} \partial \theta^{\pm}} \bigg|_{\theta = 0} = -\frac{1}{2} \sum_{(b,p,o)} j_{ab} (\delta_{ij} - (n^\pm_{ab})_j (n^\pm_{ab})_j), \]  

(39)

\[ \frac{\partial^2 S}{\partial \theta^{\pm} \partial \theta^{\pm}} \bigg|_{\theta = 0} = \frac{1}{2} \sum_{(b,p,o)} j_{ab} \delta_{ij} (n^\pm_{ab})_j (n^\pm_{ab})_j, \]  

(40)

\[ \frac{\partial^2 S}{\partial \theta^{\pm} \partial \theta^{\pm}} \bigg|_{\theta = 0} = \frac{1}{2} \sum_{(b,p,o)} j_{ab} \delta_{ij} (n^\pm_{ab})_j (n^\pm_{ab})_j, \]  

(41)
The third-order derivative of the total action gives
\[
\frac{\partial^3 S_{\text{tot}}}{\partial j_{ab} \partial j_{cd} \partial j_{ef}} \bigg|_{\theta=0} = \frac{i \partial^3 S_{\text{Regge}}}{\partial j_{ab} \partial j_{cd} \partial j_{ef}}
\]
(42)

\[
\frac{\partial^3 S}{\partial q_{ab}^{\pm} \partial q_{cd}^{\pm} \partial q_{ef}^{\pm}} \bigg|_{\theta=0} = \sum_{b \neq a} \frac{1}{3} \mathrm{i} y^\pm j_{ab} (\delta_{jk} (n_{ab})^+_i + \delta_{ki} (n_{ab})^+_j) - \delta_{ij} (n_{ab})^+_k - 3(n_{ab})^+_l (n_{ab})^+_j (n_{ab})^+_k)
\]
(43)

\[
\frac{\partial^3 S}{\partial q_{ab}^{\pm} \partial q_{ac}^{\pm} \partial q_{bc}^{\pm}} \bigg|_{\theta=0} = -\frac{1}{4} \mathrm{i} y^\pm j_{ac} (n_{ac}^x) - (n_{ac})^+_i (n_{ac})^+_j + \mathrm{i} \varepsilon_{ijk} (n_{ac})^+_k (n_{ac})^+_l)
\]
(44)

\[
\frac{\partial^3 S}{\partial q_{ab}^{\pm} \partial q_{cd}^{\pm} \partial q_{ef}^{\pm}} \bigg|_{\theta=0} = \frac{1}{2} \mathrm{i} y^\pm j_{ac} (n_{ac}^x) + (n_{ac})^+_i (n_{ac})^+_j) (1 - (n_{ac})^+_i (n_{ac})^+_j).
\]
(45)

The first derivatives of the insertions give
\[
\frac{\partial q_{ab}^{\pm}}{\partial j_{ef}} = \gamma^2 \frac{\partial (j_{ac}, j_{nb} n_{ca} \cdot n_{cb})}{\partial j_{ef}} = \gamma^2 \frac{\partial (j_{ac}, j_{nb} \cos \Theta_{ab})}{\partial j_{ef}}
\]
(46)

\[
\frac{\partial q_{ab}^{\pm}}{\partial j_{ef}} \bigg|_{\theta=0} = -\frac{1}{4} \mathrm{i} y^\pm j_{ac} (n_{ac}^x) - (n_{ac})^+_i (n_{ac})^+_j + \mathrm{i} \varepsilon_{ijk} (n_{ac})^+_k (n_{ac})^+_l)
\]
(47)

\[
\frac{\partial q_{ab}^{\pm}}{\partial j_{ef}} \bigg|_{\theta=0} = \frac{1}{2} \mathrm{i} y^\pm j_{ac} (n_{ac}^x) + (n_{ac})^+_i (n_{ac})^+_j) (1 - (n_{ac})^+_i (n_{ac})^+_j).
\]
(48)

The second derivatives of the insertions give
\[
\frac{\partial^2 q_{ab}^{\pm}}{\partial q_{ef}^{\pm} \partial q_{ef}^{\pm}} \bigg|_{\theta=0} = \frac{1}{4} \mathrm{i} y^\pm j_{ac} (n_{ac}^x) + (n_{ac})^+_i (n_{ac})^+_j) (1 - (n_{ac})^+_i (n_{ac})^+_j).
\]
(49)

\[
\frac{\partial^2 q_{ab}^{\pm}}{\partial q_{ef}^{\pm} \partial q_{ef}^{\pm}} \bigg|_{\theta=0} = -\frac{1}{4} \mathrm{i} y^\pm j_{ac} (n_{ac}^x) + (n_{ac})^+_i (n_{ac})^+_j) (1 - (n_{ac})^+_i (n_{ac})^+_j).
\]
(50)

2.8. Numerical results

The derivatives over the spin $j$s can be obtained numerically. For simplicity, we consider only the situation where the boundary is a regular four-simplex. For the total action $S$, the second derivatives give
\[
\frac{\partial^2 S}{\partial j_{ab} \partial j_{ab}} = -\gamma \alpha_0 - \gamma \frac{9}{j_0} \sqrt{\frac{3}{5}}
\]
(51)

\[
\frac{\partial^2 S}{\partial j_{ac} \partial j_{ab}} = \gamma \frac{8}{j_0} \sqrt{\frac{3}{5}}
\]
(52)

\[
\frac{\partial^2 S}{\partial j_{cd} \partial j_{ab}} = -\gamma \frac{\alpha_2}{j_0} - \gamma \frac{3}{j_0} \sqrt{\frac{3}{5}}
\]
(53)
For the third derivatives, only seven of them are independent. They are

\[ \frac{\partial^3 S}{\partial j_{ab} \partial j_{ac} \partial j_{ab}} = -\frac{i}{j_0^3} \sqrt{\frac{3}{5}}, \]

\[ \frac{\partial^3 S}{\partial j_{ac} \partial j_{ac} \partial j_{ab}} = -\frac{1}{j_0^3} \sqrt{\frac{3}{5}}, \]

\[ \frac{\partial^3 S}{\partial j_{cd} \partial j_{ac} \partial j_{ab}} = -\frac{i}{j_0^3} \sqrt{\frac{3}{5}}, \]

\[ \frac{\partial^3 S}{\partial j_{bd} \partial j_{ac} \partial j_{ab}} = -\frac{i}{j_0^3} \sqrt{\frac{3}{5}}, \]

\[ \frac{\partial^3 S}{\partial j_{ad} \partial j_{ac} \partial j_{ab}} = -\frac{i}{j_0^3} \sqrt{\frac{3}{5}}, \]

\[ \frac{\partial^3 S}{\partial j_{ce} \partial j_{ac} \partial j_{ab}} = -\frac{i}{j_0^3} \sqrt{\frac{3}{5}}, \]

\[ \frac{\partial^3 S}{\partial j_{de} \partial j_{ac} \partial j_{ab}} = -\frac{i}{j_0^3} \sqrt{\frac{3}{5}}, \]

For the metric quantities \( q_{a^b} \), when \( a \neq b \), we can find that only five of them are independent. They are

\[ \frac{\partial q_{a^b}^c}{\partial j_{ab}} = \frac{4}{3} \gamma^2 j_0, \quad \frac{\partial q_{a^b}^c}{\partial j_{ac}} = -\frac{2}{3} \gamma^2 j_0, \]

\[ \frac{\partial q_{a^b}^c}{\partial j_{ad}} = -\frac{2}{3} \gamma^2 j_0, \quad \frac{\partial q_{a^b}^c}{\partial j_{cd}} = \frac{1}{3} \gamma^2 j_0, \]

\[ \frac{\partial q_{a^b}^c}{\partial j_{ae}} = \frac{4}{3} \gamma^2 j_0. \]

The second derivatives give

\[ \frac{\partial^2 q_{a^b}^c}{\partial j_{gh} \partial j_{ef}} = \gamma^2 \frac{\partial^2 (j_{ca} j_{cb} n_{ca} \cdot n_{cb})}{\partial j_{gh} \partial j_{ef}} \]

\[ = \gamma^2 \begin{pmatrix} \frac{4}{3} & 1 & -2 & -2 & 1 & -2 & -2 & 1 & 1 & 4 \\ 1 & \frac{4}{3} & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -2 & \frac{4}{3} & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -2 & \frac{4}{3} & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ -2 & \frac{4}{3} & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -2 & \frac{4}{3} & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ -2 & \frac{4}{3} & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ 4 & 1 & -2 & -2 & 1 & -2 & -2 & 1 & 1 & \frac{4}{3} \end{pmatrix}. \]

Rows and columns are labeled by \( j_{gh} \) and \( j_{ef} \), respectively. The order is \( \{ j_{ab}, j_{ac}, j_{ad}, j_{ae}, j_{bc}, j_{bd}, j_{be}, j_{cd}, j_{ce}, j_{de} \} \).
When $a = b$, there is only one non-zero first and second derivative. They are
\[ \frac{\partial q_{ab}}{\partial j_{ab}} = 2\gamma^2 j_0, \quad \frac{\partial^2 q_{ab}}{\partial j_{ab}\partial j_{ab}} = 2\gamma^2. \]

Now, let us look at the dependence of these quantities on $\gamma$ and $j = j_0$. We obtain
\[ \frac{\partial^3 S}{\partial j \partial j \partial j} \sim \frac{\partial \gamma}{\partial j^2}, \quad \frac{\partial^3 S}{\partial \theta \partial \theta \partial \theta} \sim \frac{\partial \gamma}{\partial j}, \quad \frac{\partial^3 S}{\partial \theta \partial \theta \partial \theta} \sim \frac{\partial \gamma}{\partial j}, \quad \frac{\partial^2 q}{\partial j \partial j} \sim \frac{\partial \gamma}{\partial j}, \quad \frac{\partial^2 q}{\partial \theta \partial \theta} \sim \frac{\partial \gamma}{\partial j}, \quad \frac{\partial^2 q}{\partial \theta \partial \theta} \sim \frac{\partial \gamma}{\partial j}. \]

For the three-valent term, the scaling is
\[ \gamma \rightarrow j \rightarrow \gamma \rightarrow j \rightarrow \gamma \rightarrow j. \]

And for the ‘four’-point function terms,
\[ \frac{\partial^3 S}{\partial j \partial j \partial j} \sim \frac{\partial^3 S}{\partial \theta \partial \theta \partial \theta} \sim \frac{\partial^3 S}{\partial \theta \partial \theta \partial \theta} \sim \frac{\partial^2 q}{\partial j \partial j} \sim \frac{\partial^2 q}{\partial \theta \partial \theta} \sim \frac{\partial^2 q}{\partial \theta \partial \theta} \sim \gamma^4 f^4. \]

Consider now the limit which is introduced by Bianchi, Magliaro and Perini [14], i.e. $\gamma \rightarrow 0$, $j \rightarrow \infty$, with fixed physical area $\gamma j = A$. Then the only terms that survive are equations (51) and (52). These terms are precisely those appearing in the Regge calculus three-point function, given in [20].

Therefore, we can conclude that in the Bianchi–Magliaro–Perini limit, the three-point function of loop quantum gravity matches the Regge calculus one.

With an analogous ‘dimensional’ analysis, we can check that for the four-point and five-point functions, the spin foam model gives perturbative Regge calculus result in the same limit. For the four-point function, the Regge part has the scale of $O(\gamma^5 f^5)$, and others have the scale of $O(\gamma^k j^k)$, $k > 5$. For the four-point function, it is the same. The scale of the Regge part is $O(\gamma^6 f^6)$.

Therefore, it appears that $\gamma$ scales the amplitude of the ‘un-gluing’ fluctuation. It also measures the difference between area bivectors $A^H$ and group generators $J^H$. The $\gamma \rightarrow 0$ limit corresponds to $J^{IJ} = A^{IJ}$ [11, 39].
3. Three-point function in perturbative quantum gravity

In this section, we give for completeness the analytic expression of the three-point function in position space, at tree level, in the harmonic gauge. We briefly review the main definitions and notations on perturbative quantum general relativity, based on [37, 40, 41]. We show only the result in this paper. More details are given in the appendix.

3.1. Definitions

Perturbative quantum gravity describes the quantum gravitational field as a tensor field in a flat background spacetime. This is a weak field expansion that does not address the problem of the full consistency of the theory, but it gives nevertheless a credible approximation in the very low energy regime. Therefore, a consistent full theory of quantum gravity should match the perturbative results in the low energy limit.

Here we focus on the Euclidean spacetime and take background spacetime to be flat; i.e. the metric of the background is $\delta_{\mu\nu}$. The definition of the gravitation field $h_{\mu\nu}(x)$ is

$$ h_{\mu\nu}(x) = g_{\mu\nu}(x) - \delta_{\mu\nu}, \quad (53) $$

where $g_{\mu\nu}(x)$ is the total metric and $x$ is a Cartesian coordinate which covers the background spacetime manifold.

Since we use a path integral formalism to write the quantum theory of perturbative gravitation field, we need to rewrite the Einstein–Hilbert (EH) action (without cosmological constant)

$$ S = \frac{1}{16\pi G} \int dx \sqrt{g} R \quad (54) $$

in terms of the field $h_{\mu\nu}(x)$. Under general coordinate transformation, the gravitation field $h_{\mu\nu}$ has a gauge freedom, with a structure similar to that in the electromagnetic field case. To compute the symmetric three-point function, we choose the harmonic gauge

$$ \partial_{\mu} h_{\mu\nu} = \frac{1}{2} \delta_{\nu} h. \quad (55) $$

where $h \equiv h_{\nu}^\nu$. We consider only the pure gravity situation, without matter. In this case, the linearization of the Einstein equations reads

$$ \partial_{\rho} \partial^\rho h_{\mu\nu} = \frac{1}{2} \delta_{\mu\nu} \partial_{\rho} \partial^\rho h. \quad (56) $$

Taking the trace for both sides, we have

$$ \partial_{\rho} \partial^\rho h = 0 \quad \text{and} \quad \partial_{\rho} \partial^\rho h_{\mu\nu} = 0. \quad (57) $$

Using this and the gauge fixing, the EH action becomes (only keeping the three-valent terms)

$$ S_3 = \frac{1}{64\pi G} \int dx (h^{\sigma\rho} \partial_{\sigma} h^{\mu\nu} \partial_{\rho} h_{\mu\nu} - 2h_{\mu\rho} \partial^\sigma h^{\mu\nu} \partial^\rho h_{\nu\sigma}). \quad (58) $$

3.2. Three-point function

The three-point function at the tree-level leading order is defined as follows:

$$ G_{\mu_1, \nu_1, \mu_2, \nu_2, \sigma_1, \sigma_2}(x_1, x_2, x_3) = \frac{1}{Z} \int Dh \ e^{iS_3} \ iS_3 h_{\mu_1\nu_1}(x_1) h_{\nu_2\sigma_2}(x_2) h_{\sigma_1\mu_2}(x_3), \quad (59) $$

where $Z = \int Dh \ \exp(iS_2)$ and

$$ S_2 = \frac{1}{64\pi G} \int d^4 z \left( \partial^\sigma h^{\mu\rho} \partial_{\sigma} h_{\mu\rho} - \frac{1}{2} \partial_{\rho} h \partial^\rho h \right). \quad (60) $$
The terms in $S_3$ are quite analogous; let us focus on the first one, namely $h^\rho_\sigma \partial_\sigma h^{\mu\nu} \partial_\mu h_{\mu\nu}$. Using the Wick contraction method, we obtain

$$G_{\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2}(x_1, x_2, x_3) = \frac{i}{64\pi G} \int Dh \exp(iS_2) \int d^4z h^\rho_\sigma(z) \partial_\sigma h^{\mu\nu}(z) h_{\mu_2\nu_2}(x_1) h_{\nu_1\sigma_2}(x_2) h_{\sigma_1\sigma_2}(x_3)$$

$$= \frac{\kappa}{2} \int d^4z \left( D^\rho_{\mu_1\mu_2}(z-x_1) \partial_\sigma D^{\mu_2\nu_1}(z-x_2) \partial_\mu D_{\mu_\nu,\sigma_1\sigma_2}(z-x_3) \right)$$

$$+ \frac{\kappa}{2} \int d^4z \left( D^\rho_{\mu_1\mu_2}(z-x_1) \partial_\sigma D^{\mu_2\nu_1}(z-x_2) \partial_\mu D_{\mu_\nu,\sigma_1\sigma_2}(z-x_3) \right)$$

$$+ \frac{\kappa}{2} \int d^4z \left( D^\rho_{\mu_1\nu_2}(z-x_1) \partial_\sigma D^{\nu_2\mu_2}(z-x_2) \partial_\mu D_{\mu_\nu,\sigma_1\sigma_2}(z-x_3) \right)$$

$$+ \frac{\kappa}{2} \int d^4z \left( D^\rho_{\mu_1\nu_2}(z-x_1) \partial_\sigma D^{\nu_2\mu_2}(z-x_2) \partial_\mu D_{\mu_\nu,\sigma_1\sigma_2}(z-x_3) \right)$$

$$+ \frac{\kappa}{2} \int d^4z \left( D^\rho_{\nu_1\nu_2}(z-x_1) \partial_\sigma D^{\nu_2\mu_2}(z-x_2) \partial_\mu D_{\mu_\nu,\sigma_1\sigma_2}(z-x_3) \right)$$

$$+ \frac{\kappa}{2} \int d^4z \left( D^\rho_{\nu_1\nu_2}(z-x_1) \partial_\sigma D^{\nu_2\mu_2}(z-x_2) \partial_\mu D_{\mu_\nu,\sigma_1\sigma_2}(z-x_3) \right), \quad (61)$$

where $\kappa = \sqrt{32\pi G}$ and $D_{\mu_\nu,\rho\sigma}(x-y)$ is the graviton propagator in position space, which is

$$D_{\mu_\nu,\rho\sigma}(x-y) = -\frac{1}{8\pi^2} \frac{1}{|x-y|^2} \left( \delta_{\rho\sigma} \delta_{\mu\nu} + \delta_{\rho\nu} \delta_{\mu\sigma} - \delta_{\rho\mu} \delta_{\nu\sigma} \right)$$

$$= -\frac{1}{8\pi^2} \frac{1}{|x-y|^2} \Delta_{\mu_\nu,\rho\sigma}. \quad (62)$$

We do not write the non-connected terms because they are equal to zero by gauge symmetry.

Since all the terms in equation (61) have a similar form, we focus on the first one. This reads

$$\int d^4z D^\rho_{\mu_1\mu_2}(z-x_1) \partial_\sigma D^{\mu_2\nu_1}(z-x_2) \partial_\mu D_{\mu_\nu,\sigma_1\sigma_2}(z-x_3)$$

$$= \frac{1}{2(2\pi)^6} \int d^4z \left( \frac{1}{|z-x_1|^2} \frac{1}{|z-x_2|^2} \frac{1}{|z-x_3|^2} \right)$$

$$\Delta^\rho_{\mu_1\mu_2} \Delta^{\mu_2\nu_1} \Delta_{\mu_\nu,\sigma_1\sigma_2} \quad (63)$$

The difficulty is to solve the integral in equation (63). The asymmetric form of the integral comes from the derivatives in the perturbative EH action (58). Fortunately, we can change the derivative variables and take the derivatives out of the integral, turning it into a three-point function in $\lambda\phi^3$ theory. For equation (63), it turns into

$$\frac{\Delta^\rho_{\mu_1\mu_2} \Delta^{\mu_2\nu_1} \Delta_{\mu_\nu,\sigma_1\sigma_2}}{2(2\pi)^6} \frac{\partial}{\partial x_3^\mu} \frac{\partial}{\partial x_3^\nu} G_{\lambda\phi^3}(x_1, x_2, x_3), \quad (64)$$

where

$$G_{\lambda\phi^3}(x_1, x_2, x_3) = \int \frac{d^4z}{|z-x_1|^2 |z-x_2|^2 |z-x_3|^2} \quad (65)$$

According to a theorem in [42], for a scalar three-point function, which is rotation, translation and dilation covariant, it must have the form $G(x_1, x_2, x_3) = C x_{12}^{\alpha\beta} x_{23}^{\gamma\delta} x_{31}^{\epsilon\eta}$ in general, where $x_{ij} = |x_i - x_j|$, $C$ is a constant. Then

$$G_{\lambda\phi^3}(x_1, x_2, x_3) = \frac{C}{|x_1 - x_2|^2 |x_2 - x_3|^2 |x_3 - x_1|^2}. \quad (66)$$

Then the derivatives outside of the integral give the final results. Let us introduce some notations. Focus on an equilateral four-simplex. $|x_1 - x_2| = |x_2 - x_3| = |x_3 - x_1| = L$, $x_1^2 = x_2^2 = x_3^2 = \frac{2}{3} L^2$ and $x_i \cdot x_j |_{i \neq j} = -\frac{1}{3} L^2$. Writing

$$I_{ij} = \frac{\partial}{\partial x_i^\alpha} \frac{\partial}{\partial x_j^\beta} G_{\lambda\phi^3}(x_1, x_2, x_3), \quad (67)$$

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we have, for instance,
\[ I_{12}^{\mu\nu} = \frac{4}{9L^6} \left( x_1^\mu x_3^\nu + x_3^\mu x_1^\nu - 2x_2^\mu x_1^\nu - 2x_1^\mu x_2^\nu - 5x_1^\mu x_1^\nu + 7x_1^\mu x_2^\nu + x_2^\mu x_3^\nu + 4x_2^\mu x_1^\nu - 5x_2^\mu x_2^\nu \right) \]
and similarly for the other components. This allows us to write the three-point function explicitly:

\[
G_{\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2}(x_1, x_2, x_3) = \frac{1}{2(2\pi)^2} \left( I_{\mu_2\nu_2}^{\rho}\Delta_{\rho\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\nu}\Delta_{\nu\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_1}\Delta_{\sigma_1\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_2}\Delta_{\sigma_2\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} \right)
- \frac{1}{2} \left( I_{\mu_2\nu_2}^{\rho}\Delta_{\rho\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\nu}\Delta_{\nu\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_1}\Delta_{\sigma_1\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_2}\Delta_{\sigma_2\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} \right)
= \frac{\kappa}{2} \frac{1}{2(2\pi)^2} \left( I_{\mu_2\nu_2}^{\rho}\Delta_{\rho\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\nu}\Delta_{\nu\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_1}\Delta_{\sigma_1\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_2}\Delta_{\sigma_2\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} \right)
- \frac{1}{2} \left( I_{\mu_2\nu_2}^{\rho}\Delta_{\rho\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\nu}\Delta_{\nu\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_1}\Delta_{\sigma_1\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} + I_{\mu_2\nu_2}^{\sigma_2}\Delta_{\sigma_2\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2} \right),
\]

(69)

3.3. Comparison between the perturbative and loop three-point functions

The comparison of the three-point function computed here with the one computed in the previous section is not easy. In order to compare the expectation values, we need to identify the Penrose operators $G_{\mu\nu}^{ab}$ with the metric field. The Penrose operator has a clear geometrical interpretation [33]: it is the scalar product of the flux operator across the boundary triangles $a$ and $b$ of the boundary tetrahedron $l$ of a four-simplex-like spacetime region. It can therefore be immediately compared with quantities well defined in Regge geometry: areas of triangles and angles between triangles.

The direct comparison with the metric operator, on the other hand, is tricky, since the areas and angles of simplices are the nonlocal functions of the metric. In addition, the $n$-point functions are computed in the linearized theory in a certain gauge. The loop theory defines implicitly a gauge in two steps. First, the boundary operators are naturally defined in a ‘time’ gauge, with respect to the foliation defined by the boundary. Second, the remaining gauge freedom is fixed by the boundary state [6, 43].

Tentatively, we may write

\[
G_{\mu_1\mu_2}^{ab} = E_a^\mu \cdot E_b^\mu = \det(q)q^{ij}(x)N^{ma}_i(x)N^{mb}_j(x),
\]

(70)

where $N^{ma}_i$ is the normal one-form to the triangle $(n, a)$ in the plane of the tetrahedron $a$, normalized to the coordinate area of the triangle, in the background geometry, and $q_{ij}$ is the three-metric induced on the boundary. More precisely, we can use the two-form $B^{ab}_{\mu\nu}$ associated with the $(n, a)$ triangle and write

\[
G_{\mu_1\mu_2}^{ab} = 2s_{\rho\sigma}^a s^{\mu\nu}_b B_{\mu\nu}^{ab} B_{\rho\sigma}^{ab}.
\]

(71)

This is the way the loop operator was identified with the perturbative gravitational field in [14]. The same simple-minded identification does not appear to work for the three-point function, as shown by an explicit numerical calculation given in appendix C, if we use the numerical values for the boundary state found in [14]. Since the loop calculation matches the Regge one, the inconsistency is not related to the specific of the loop formalism, and is therefore of secondary interest here.

The problem of the consistency between Regge calculus [19] and continuum perturbative quantum gravity field theory has been discussed in [44–46]. The consistency between Regge
calculus and continuum theory is based on the relation between the Regge action $S_{\text{Regge}}$ and the EH action $S_{\text{EH}}$. $S_{\text{Regge}}$ can be derived from $S_{\text{EH}}$ [44], and $S_{\text{Regge}}$ yields back $S_{\text{EH}}$ with a correction in the order of $O(l^2/\rho^2)$ [45], where $l$ is the typical length of a four-simplex and $\rho$ is the Gauss radius which stands for the intrinsic curvature. In the limit $l \to 0$ or $\rho \to \infty$, $S_{\text{Regge}} \to S_{\text{EH}}$. In our calculation, we use the limit $\rho \to \infty$, as we have mentioned in section 2.3. Then we can use the regular way to calculate the graviton $n$-point function, i.e. adding $n$ $h_{\mu\nu}$s into the path integral as insertions and change the action $S_{\text{EH}} \to S_{\text{EH}} + O(l^2/\rho^2)$ [45]. Perturbative Regge calculus is given by the strong coupling expansion [46]. The expansion around the saddle point in loop gravity corresponds to the strong coupling expansion in Regge calculus.

We also point out here that in [4, 5], the traceless gauge $h_{\mu}^\mu = 0$ was assumed, but this may not be consistent with the gauge choice implicit in the use of the Penrose field operator. If we take this into account in the definition of the two-point function given in [14] since $E_n^\mu$ is a densitized operator, we obtain

\begin{equation}
G_{abcd}^{mn} = \langle E_m^a \cdot E_n^b E_m^c \cdot E_n^d \rangle - \langle E_m^a \cdot E_m^b \rangle \langle E_n^c \cdot E_n^d \rangle + O(h^3),
\end{equation}

which is certainly not the standard two-point function. For the three-point function case, the relation is even more complicated.

An additional source of uncertainty in the relation between the flux variables $E_n^\mu$ and $g_{\mu\nu}$ is given by the correct identification of the normals. Above we have assumed $E_n^\mu$ is a densitized operator, we obtain

\begin{align}
G_{abcd}^{mn} &= \langle \text{det}(g(x_m))g_{\mu\nu}(x_m)\text{det}(g(x_n))g_{\rho\sigma}(x_n)(N^a_m)^\mu(N^b_m)^\nu(N^c_n)^\rho(N^d_n)^\sigma \\
&\quad - \langle \text{det}(g(x_m))g_{\mu\nu}(x_m)\rangle \langle \text{det}(g(x_n))g_{\rho\sigma}(x_n)\rangle (N^a_m)^\mu(N^b_m)^\nu(N^c_n)^\rho(N^d_n)^\sigma \rangle \langle \text{det}(g(x_m))g_{\mu\nu}(x_m)\rangle \langle \text{det}(g(x_n))g_{\rho\sigma}(x_n)\rangle (N^a_m)^\mu(N^b_m)^\nu(N^c_n)^\rho(N^d_n)^\sigma \rangle + O(h^3),
\end{align}

where the normals $N^a_n$ are those of the background geometry. But in the boundary state we used $N^a_n = j_m n^a_n(j(h))$, where the normals are determined by the areas of the entire four-simplex. This gives an extra dependence on the metric: $\text{det}(g)N^a_n(j(h(x))) = N^a_n(j(h(x))) + O(h^3)$.

Because of these various technical complications, a direct comparison with the weak field expansion in $g_{\mu\nu}$ requires more work. On the other hand, it is not clear that this work is of real interest, since the key result of the consistency of the loop dynamics with the Regge one is already established.

4. Conclusion

We have computed the three-point function of loop quantum gravity, starting from the background-independent spin foam dynamics, at the lowest order in the vertex expansion. We have shown that this is equivalent to the one of perturbative Regge calculus in the limit $\gamma \to 0, j \to \infty$ and $\gamma j = A$.

Given the good indications on the large distance limit of the $n$-point functions for the Euclidean quantum gravity, we think that the most urgent open problem is to extend these results to the Lorentzian case, and to the theory with matter [47, 48] and cosmological constant [49–51].

Among the problems that we leave open are the following. (i) We have computed the three-point function in position space from perturbative quantum gravity, treated as a flat-space
quantum field theory. We have found that we cannot use here the techniques of [4, 5, 14, 21] to compare this with the loop calculation, because of technical complications in comparing the expansion. These can be traced to the different gauges in which the calculations are performed, to the traceless condition $h_{\mu}^{\mu} = 0$ which in general is not satisfied and to the fact that the normals have a non-trivial relation with the field $N_0^{\alpha} = N_\alpha^a(h)$. (ii) The boundary vacuum state and the parameters $\alpha^{(ab)(cd)}$ introduced in equation (18) should be better understood and checked. A possibility is to compute them from the first principle, using the unitary condition $\langle W | \Psi_\beta \rangle = 1$. (iii) The gauge implicit in the use of the loop formalism is not completely clear. In weak field expansion, the De Donder-like (harmonic) gauge turns out to be consistent for the lattice graviton propagator [46, 53] and with the radial structure of the loop calculation [54]. But the extension of this to higher $n$-point functions is not clear.

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Appendix A. From EH action to equation (58)

We follow the work of Modesto [41] and Bianchi and Modesto [55]. We split the total metric $g_{\mu\nu}$ into the background metric $\delta_{\mu\nu}$ and the fluctuation $h_{\mu\nu}$, as in (53). From this we can obtain its inverse $g^{\mu\nu}$ and square root of its determinant $\sqrt{|\det (g)|}$:

$$g^{\mu\nu} = \sum_{n=0}^{\infty} (-)^n (h^n)^{\mu\nu}$$

$$\sqrt{|\det (g)|} = \prod_{k=1}^{\infty} \left( \sum_{m=0}^{\infty} \frac{(-)^{m(k+1)}}{m!(2k)^m} (h^k)^{\kappa\kappa} \right)^m.$$  

Because the EH action contains a Ricci scalar and a Ricci scalar contains a Christopher symbol, we write the symbol based on $h_{\mu\nu}$:

$$\Gamma_{\nu\rho}^{\sigma} = \frac{1}{2} \sum_{n_1=0}^{\infty} (-)^{n_1} (h^{n_1})^{\sigma\rho} (\partial_\nu h_\rho^{\sigma} + \partial_\rho h_\sigma^{\nu} - \partial_\sigma h_{\nu\rho}).$$

Then we can finally obtain the Lagrangian based on $h_{\mu\nu}$:

$$L = \frac{1}{16\pi G} \sqrt{|\det (g)|} R$$

$$= \frac{1}{16\pi G} \prod_{k=1}^{\infty} \left( \sum_{m=0}^{\infty} \frac{(-)^{m(k+1)}}{m!(2k)^m} (h^k)^{\kappa\kappa} \right)^m$$

$$\times \left( \frac{1}{2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-)^{n_1+n_2} (h^n)^{\mu\nu} (\partial_\nu (h^{n_1})^{\rho\sigma} \partial_\rho h_\nu^{\sigma} + \partial_\sigma (h^{n_1})^{\rho\sigma} \partial_\rho h_\sigma^{\nu} - \partial_\sigma (h^{n_1})^{\rho\sigma} \partial_\rho h_{\nu\sigma} + (h^{n_1})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\nu\mu} - (h^{n_1})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\sigma\mu} - (h^{n_1})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\nu\sigma} - (h^{n_1})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\sigma\mu} \right)$$

$$- \frac{1}{2} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} (-)^{n_2+n_3} (h^n)^{\mu\nu} (\partial_\nu (h^{n_2})^{\rho\sigma} \partial_\rho h_\nu^{\sigma} + \partial_\sigma (h^{n_2})^{\rho\sigma} \partial_\rho h_\sigma^{\nu} - \partial_\sigma (h^{n_2})^{\rho\sigma} \partial_\rho h_{\nu\sigma} + (h^{n_2})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\nu\mu} - (h^{n_2})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\sigma\mu} - (h^{n_2})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\nu\sigma} - (h^{n_2})^{\rho\sigma} \partial_\sigma \partial_\rho h_{\sigma\mu} \right).$$
\[ + (h^{\mu \nu})^{\alpha \beta} \partial_{\alpha} \partial_{\beta} h_{\sigma \rho} + (h^{\rho \sigma})^{\mu \nu} \partial_{\rho} \partial_{\sigma} h_{\mu \nu} - (h^{\mu \nu})^{\alpha \beta} \partial_{\alpha} \partial_{\beta} h_{\sigma \rho} \]
\[ + \frac{1}{4} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-)^{m(k+1)} \frac{m!}{(2k)^m} \left( h^{k} \right)^{m} \left( (h^{k})^{k} \right) \left( (h^{k})^{k} \right) \left( (h^{k})^{k} \right) \]
\[ \times \partial_{\rho} \partial_{\sigma} h_{\mu \nu} + \partial_{\rho} \partial_{\sigma} h_{\mu \nu} - \partial_{\rho} \partial_{\sigma} h_{\mu \nu} \]
\[ \times \left( \partial_{\rho} h_{\mu \nu} + \partial_{\sigma} h_{\mu \nu} - \partial_{\rho} h_{\mu \nu} \right) \]
\[ \times \left( \partial_{\rho} h_{\mu \nu} + \partial_{\sigma} h_{\mu \nu} - \partial_{\rho} h_{\mu \nu} \right) \]
\[ \left( \partial_{\mu} h_{\nu \beta} + \partial_{\nu} h_{\beta \mu} - \partial_{\beta} h_{\mu \nu} \right) \]
\[ \left( \partial_{\mu} h_{\nu \beta} + \partial_{\nu} h_{\beta \mu} - \partial_{\beta} h_{\mu \nu} \right) \]  

(A.4)

Before simplifying the Lagrangian to equation (58), we emphasize that we use the harmonic gauge (55) and the Einstein equation without matter field (57). We also assume that the integral over a total derivative vanishes. Under these assumptions, we find that the terms like
\[ h_{\mu \nu} \partial^{\rho} h_{\mu \nu} \partial^{\rho} h, h \partial^{\rho} h \partial^{\rho} h, h_{\mu \nu} \partial^{\rho} h_{\mu \sigma} \partial^{\rho} h_{\nu \sigma} \]
vanish after integration in the action. We use
\[ h_{\mu \nu} \partial^{\rho} h_{\mu \nu} \partial^{\rho} h \]
as an example to prove this:
\[ \int d^{4}x h_{\mu \nu} \partial^{\rho} h_{\mu \nu} \partial^{\rho} h = \int d^{4}x \frac{1}{2} \left( h_{\mu \nu} \partial^{\rho} h_{\mu \nu} \partial^{\rho} h + h_{\mu \nu} \partial^{\rho} h_{\mu \nu} \partial^{\rho} h \right) \]
\[ = \int d^{4}x \frac{1}{2} \partial^{\rho} (h_{\mu \nu} h_{\mu \nu}) \partial^{\rho} h \]
\[ = - \int d^{4}x \frac{1}{2} h_{\mu \nu} h_{\mu \nu} \partial^{\rho} \partial^{\rho} h \]
\[ = 0. \]

Now we can simplify equation (A.4). For the three-point function, we just need the terms with the form \[ h \partial^{\rho} h \partial^{\rho} h \]. We denote it \[ L_{3} \] and simplify it case by case. For \( km = 0 \),
\[ \prod_{k=1}^{\infty} \sum_{m=0}^{\infty} (-)^{m(k+1)} \frac{m!}{(2k)^m} \left( h^{k} \right)^{m} = \prod_{k=1}^{\infty} \left( (-)^{0(k+1)} \frac{m!}{0!(2k)^m} \left( (h^{k})^{k} \right) \right) = \prod_{k=1}^{\infty} 1 = 1. \]
Then
\[ L_{3}^{(0)} = \frac{1}{16\pi G} \frac{1}{64\pi G} \left( h^{\sigma \rho} \partial_{\sigma} h^{\mu \nu} \partial_{\mu} h_{\nu \rho} - 2h^{\mu \nu} \partial_{\mu} h_{\nu \rho} \partial^{\beta} h + 2h^{\mu \nu} \partial^{\beta} h_{\nu \rho} \partial^{\sigma} h_{\mu \beta} \right). \]

For \( km = 1 \),
\[ \prod_{k=1}^{\infty} \sum_{m=0}^{\infty} (-)^{m(k+1)} \frac{m!}{(2k)^m} \left( h^{k} \right)^{m} = \frac{(-)^{0(1+1)}}{1!(2)^1} \left( h^{1} \right) = \frac{1}{2} h^{1}. \]
Then
\[ L_{3}^{(1)} = \frac{1}{16\pi G} \frac{1}{2} h = - \frac{1}{64\pi G} h \partial_{\sigma} h_{\nu \rho} \partial^{\sigma} h^{\rho \sigma}. \]

For \( km = 2 \),
\[ \prod_{k=1}^{\infty} \sum_{m=0}^{\infty} (-)^{m(k+1)} \frac{m!}{(2k)^m} \left( h^{k} \right)^{m} = \frac{(-)^{0(2+1)}}{2!(2)^2} \left( h^{2} \right)^{2} + \frac{(-)^{0(1+1)}}{1!(4)^1} \left( h^{1} \right)^{4} \]
\[ = \frac{1}{8} h^{2} - \frac{1}{4} h^{4}. \]
Then
\[ L_{3}^{(2)} = 0. \]
Thus, we can obtain $L_3$:

$$
L_3 = \frac{1}{64\pi G} \left( h^{\alpha \rho} \partial_\alpha h^{\nu \sigma} \partial_\nu h_{\rho \sigma} - 2 h^{\mu \nu} \partial_\mu h_{\beta \rho} \partial_\beta h + 2 h^{\mu \nu} \partial_\rho h_{\nu \sigma} \partial_\sigma h_{\rho \mu} - h_{\rho \mu} \partial_\rho h_{\nu \sigma} \partial_\sigma h^{\nu \mu} \right)
$$

$$
= \frac{1}{64\pi G} \left( h^{\alpha \rho} \partial_\alpha h^{\nu \sigma} \partial_\nu h_{\rho \sigma} + h \partial_\rho h^{\nu \sigma} \partial_\nu h_{\rho \sigma} - h^{\mu \nu} \partial_\mu h_{\beta \rho} \partial_\beta h + 2 h^{\mu \nu} \partial_\rho h_{\nu \sigma} \partial_\sigma h_{\rho \mu} + 2 h^{\mu \nu} \partial_\rho h_{\nu \sigma} \partial_\sigma h^{\nu \mu} \right)
$$

$$
= \frac{1}{64\pi G} \left( h^{\alpha \rho} \partial_\alpha h^{\nu \sigma} \partial_\nu h_{\rho \sigma} - 2 h_{\rho \mu} \partial_\rho h^{\nu \sigma} \partial_\sigma h_{\nu \mu} \right).
$$

(A.5)

This is (58).

Appendix B. The total analytical expression of the derivatives in section 2.7

The second order of the total action gives

$$
\frac{\partial^2 S_{\text{tot}}}{\partial j_{ab} \partial j_{cd}} \bigg|_{\theta = 0} = - \frac{\gamma \alpha^{(ab)(cd)}}{\sqrt{\delta_{ab} \delta_{cd}}} + \frac{\partial^2 S_{\text{Regge}}}{\partial j_{ab} \partial j_{cd}}
$$

(B.1)

$$
\frac{\partial^2 S}{\partial \theta^\pm \partial \theta^\pm} \bigg|_{\theta = 0} = - \frac{1}{2} \gamma^\pm \sum_{(b \neq a)} j_{ab} (\delta_{ij} - (n_{ab}^+) (n_{ab}^-))
$$

(B.2)

$$
\frac{\partial^2 S}{\partial \theta^\pm \partial \theta^\pm} \bigg|_{\theta = 0} = \frac{1}{2} \gamma^\pm j_{ab} (\delta_{ij} - (n_{ab}^+) (n_{ab}^-))_j - i \epsilon_{ijk} (n_{ab}^+) k).
$$

(B.3)

The third order of the total action gives

$$
\frac{\partial^3 S_{\text{tot}}}{\partial j_{ab} \partial j_{cd} \partial j_{ef}} \bigg|_{\theta = 0} = \frac{1}{4} \frac{\partial^3 S_{\text{Regge}}}{\partial j_{ab} \partial j_{cd} \partial j_{ef}}
$$

(B.4)

$$
\frac{\partial^3 S}{\partial \theta^\pm \partial \theta^\pm \partial \theta^\pm} \bigg|_{\theta = 0} = \sum_{b \neq a} \frac{1}{6} \gamma^\pm j_{ab} (\delta_{jk} (n_{ab}^+) + \delta_{ik} (n_{ab}^-))_j
$$

$$
+ \delta_{ij} (n_{ab}^+) k - 3 (n_{ab}^+) (n_{ab}^-)_j (n_{ab}^-)_k)
$$

(B.5)

$$
\frac{\partial^3 S}{\partial \theta^\pm \partial \theta^\pm \partial \theta^\pm} \bigg|_{\theta = 0} = \frac{1}{4} \gamma^\pm j_{ab} (\delta_{jk} (n_{ab}^+) + \delta_{ik} (n_{ab}^-))_j - 2 (n_{ab}^+) (n_{ab}^-)_j (n_{ab}^-)_k
$$

$$
+ i \epsilon_{ijk} (n_{ab}^+) j + \epsilon_{ikj} (n_{ab}^+) j (n_{ab}^-) k)
$$

(B.6)

$$
\frac{\partial^3 S}{\partial \theta^\pm \partial \theta^\pm \partial \theta^\pm} \bigg|_{\theta = 0} = \frac{1}{4} \gamma^\pm j_{ab} (\delta_{jk} (n_{ab}^+) + \delta_{ik} (n_{ab}^-))_j - 2 (n_{ab}^+) (n_{ab}^-)_j (n_{ab}^-)_k
$$

$$
+ i \epsilon_{ijk} (n_{ab}^+) j (n_{ab}^+) k + i \epsilon_{ikj} (n_{ab}^+) j (n_{ab}^-) k)
$$

(B.7)

$$
\frac{\partial^3 S}{\partial \theta^\pm \partial \theta^\pm \partial \theta^\pm} \bigg|_{\theta = 0} = \frac{1}{4} \gamma^\pm j_{ab} (\delta_{jk} (n_{ab}^+) + \delta_{ik} (n_{ab}^-))_j - 2 (n_{ab}^+) (n_{ab}^-)_j (n_{ab}^-)_k
$$

$$
+ i \epsilon_{ijk} (n_{ab}^+) j (n_{ab}^+) k + i \epsilon_{ikj} (n_{ab}^+) j (n_{ab}^-) k)
$$

(B.8)
\[
\frac{\partial^3 S}{\partial j_{cd} \partial \theta_{i} \partial \theta_{l}^\pm} \bigg|_{\theta=0} = \frac{1}{2} \gamma^2 \frac{\partial}{\partial j_{cd}} \left( \sum_{(b=pq)} f_{ab} (\delta_{ij} - (n_{ab}^\pm)_{j} (n_{ab}^\pm)_{j}) \right)
\]
\[\text{(B.9)}\]

\[
\frac{\partial^3 S}{\partial j_{cd} \partial \theta_{i} \partial \theta_{l}^\pm} \bigg|_{\theta=0} = \frac{1}{2} \gamma^2 \frac{\partial}{\partial j_{cd}} \left( f_{ab} (\delta_{ij} - (n_{ab}^\pm)_{j} (n_{ab}^\pm)_{j} - i\varepsilon_{ijk} (n_{ab}^\pm)_{k}) \right)
\]
\[\text{(B.10)}\]

The first derivatives of the insertions give
\[
\frac{\partial q_{ab}^{\pm}}{\partial j_{ef}} = \gamma^2 \frac{\partial (j_{ca} j_{cb} n_{ca} \cdot n_{cb})}{\partial j_{ef}} = \gamma^2 \frac{\partial (j_{ca} j_{cb} \cos \theta_{cab})}{\partial j_{ef}}
\]
\[\text{(B.11)}\]

\[
\frac{\partial q_{ab}^{\pm}}{\partial \theta_{i}^\pm} \bigg|_{\theta^\pm=0} = -\frac{1}{2} i\gamma^2 \gamma^\mp j_{na} j_{ab} ((n_{ab}^\pm)_{j} - (n_{na}^\pm)_{j} (n_{na}^\pm)_{j} + i\varepsilon_{ijk} (n_{ab}^\pm)_{j} (n_{na}^\pm)_{k})
\]
\[\text{(B.12)}\]

\[
\frac{\partial q_{ab}^{\pm}}{\partial \theta_{i}^\pm} \bigg|_{\theta^\pm=0} = -\frac{1}{2} i\gamma^2 \gamma^\mp (n_{na}^\pm)_{j} + (n_{nb}^\pm)_{j} - (n_{na}^\pm)_{j} (n_{nb}^\pm)_{j}
\]
\[\text{(B.13)}\]

The second derivatives of the insertions give
\[
\frac{\partial^2 q_{ab}^{\pm}}{\partial j_{ef} \partial \theta_{i}^\pm} \bigg|_{\theta=0} = \frac{1}{4} \gamma^2 \gamma^\mp j_{na} j_{ab} \left( (n_{na}^\pm)_{j} (n_{na}^\pm)_{j} + (n_{nb}^\pm)_{j} (n_{na}^\pm)_{j} - 2(n_{na}^\pm)_{j} (n_{na}^\pm)_{j} - i(n_{na}^\pm)_{j} (n_{na}^\pm)_{j} \right)
\]
\[\text{(B.14)}\]

\[
\frac{\partial^2 q_{ab}^{\pm}}{\partial j_{ef} \partial \theta_{i}^\pm} \bigg|_{\theta=0} = \frac{1}{4} \gamma^2 \gamma^\mp j_{na} j_{ab} \left( 2(n_{na}^\pm)_{j} (n_{nb}^\pm)_{j} + 2(n_{nb}^\pm)_{j} (n_{na}^\pm)_{j} - 2(n_{na}^\pm)_{j} (n_{na}^\pm)_{j} - i(n_{na}^\pm)_{j} (n_{na}^\pm)_{j} \right)
\]
\[\text{(B.15)}\]
Here the parameters \( G \) values of these parameters given in [14]:

\[
\alpha_k = \begin{cases} 
0, & \text{if } k = 0, \\
\frac{1}{2304\sqrt{3}}, & \text{if } k = 1, \\
-\frac{7}{9216\sqrt{3}}, & \text{if } k = 2.
\end{cases}
\]

Appendix C. Numerical comparison

Let us write some explicit terms of the loop three-point function:

\[
\begin{align*}
\frac{\partial^2 G_{ab}}{\partial \theta^i \partial \theta^j} & = \frac{1}{2} \gamma^2 \gamma^+ j_{ab} \left( (n_{ab})_j (n_{na})_i - (n_{ab})_j (n_{na})_i \right) \\
& + i \epsilon_{ijkl} (n_{ab})_k (n_{na})_l - (n_{ab})_j (n_{na})_i \epsilon_{ijkl} (n_{na})_m
\end{align*}
\]

(B.17)

\[
\begin{align*}
\frac{\partial^2 G_{ab}}{\partial \theta^i \partial \theta^j} & = \frac{1}{4} \gamma^2 \gamma^+ j_{ab} \left( \delta_{ij} - (n_{ab})_j (n_{na})_i - i \epsilon_{ijkl} (n_{na})_k \right) \\
& \times (\delta_{ij} - (n_{ab})_j (n_{na})_i - i \epsilon_{ijkl} (n_{na})_k)
\end{align*}
\]

(B.18)

\[
\begin{align*}
\frac{\partial^2 G_{ab}}{\partial \theta^i \partial \theta^j} & = \frac{1}{4} \gamma^2 \gamma^+ j_{ab} \left( \delta_{ij} - (n_{ab})_j (n_{na})_i - i \epsilon_{ijkl} (n_{na})_k \right) \\
& \times (\delta_{ij} - (n_{ab})_j (n_{na})_i - i \epsilon_{ijkl} (n_{na})_k)
\end{align*}
\]

(B.19)

\[
\begin{align*}
\frac{\partial^2 G_{ab}}{\partial \theta^i \partial \theta^j} & = \frac{1}{2} \gamma^2 \gamma^+ j_{ab} \left( \delta_{ij} - (n_{ab})_j (n_{na})_i - i \epsilon_{ijkl} (n_{na})_k \right) \\
& \times (\delta_{ij} - (n_{ab})_j (n_{na})_i - i \epsilon_{ijkl} (n_{na})_k)
\end{align*}
\]

(B.20)

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then we obtain
\[
G_{123}^{44444} = -\frac{35\sqrt{5}A^4}{391378894848},
\]
\[
G_{123}^{44445} = \frac{5\sqrt{5}A^4}{195689447424},
\]
\[
G_{123}^{44455} = \frac{5\sqrt{5}A^4}{97844723712}.
\]

On the other hand, from perturbation theory, we obtain
\[
(G_{123}^{44444})_{QFT} = -\frac{3}{16384000}C,
\]
\[
(G_{123}^{44445})_{QFT} = \frac{13}{65536000}C,
\]
\[
(G_{123}^{44455})_{QFT} = \frac{11}{16384000}C,
\]
where \(C\) is a constant. The ratios of the two give
\[
\frac{G_{123}^{44444}}{(G_{123}^{44444})_{QFT}} / \frac{G_{123}^{44445}}{(G_{123}^{44445})_{QFT}} = \frac{91}{24},
\]
\[
\frac{G_{123}^{44444}}{(G_{123}^{44444})_{QFT}} / \frac{G_{123}^{44455}}{(G_{123}^{44455})_{QFT}} = \frac{77}{12},
\]
which do not match. The use of the values for the \(\beta\) coefficients given in equation (C.4) is of course rather questionable and should not be taken too seriously. The main purpose of this computation is to show that the expectation values can indeed be computed completely explicitly.

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