On total dominating sets in graphs

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Abstract

A set $S$ of vertices in a graph $G(V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A set $S$ of vertices in a graph $G(V, E)$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element of $S$. The domination number of a graph $G$ denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. Respectively the total domination number of a graph $G$ denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in $G$. An upper bound for $\gamma_t(G)$ which has been achieved by Cockayne and et al. in [1] is: for any graph $G$ with no isolated vertex and maximum degree $\Delta(G)$ and $n$ vertices, $\gamma_t(G) \leq n - \Delta(G) + 1$.

Here we characterize bipartite graphs and trees which achieve this upper bound. Further we present some another upper and lower bounds for $\gamma_t(G)$. Also, for circular complete graphs, we determine the value of $\gamma_t(G)$.

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1 Introduction

Let $G(V, E)$ be a graph. For any vertex $x \in V$, we define the neighborhood of $x$, denoted by $N(x)$, as the set of all vertices adjacent to $x$. The closed neighborhood of $x$, denoted by $N[x]$, is the set $N(x) \cup \{x\}$. For a set of vertices $S$, we define $N(S)$ as the union of $N(x)$ for all $x \in S$, and $N[S] = N(S) \cup S$. The degree of a vertex is the size of its neighborhoods. The maximum degree

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of a graph $G$ is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. Here $n$ will denote the number of vertices of a graph $G$. A set $S$ of vertices in a graph $G(V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A set $S$ of vertices in a graph $G(V, E)$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element of $S$. The domination number of a graph $G$ denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. Respectively the total domination number of a graph $G$ denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in $G$. Clearly $\gamma(G) \leq \gamma_t(G)$, also it has been proved that $\gamma_t(G) \leq 2\gamma(G)$.

An upper bound for $\gamma_t(G)$ has been achieved by Cockayne and et al. in [1] in the following theorems:

**THEOREM A** If a graph $G$ has no isolated vertices, then $\gamma_t(G) \leq n - \Delta(G) + 1$.

**THEOREM B** If $G$ is a connected graph and $\Delta(G) < n - 1$, then $\gamma_t(G) \leq n - \Delta(G)$.

As a result of the above theorems, if $G$ is a graph with $\gamma_t(G) = n - \Delta(G) + 1$, then $\Delta(G) \geq n - 1$. Hence, if $G$ is a $k$-regular graph and $\gamma_t(G) = n - k + 1$, then $G$ is $K_n$. As a result of the above theorems, if $G$ is a graph with $\gamma_t(G) = n - \Delta(G) + 1$, then $\Delta(G) \geq n - 1$. Hence, if $G$ is a $k$-regular graph and $\gamma_t(G) = n - k + 1$, then $G$ is $K_n$. Total domination and upper bounds on the total domination number in graphs were intensively investigated, see e.g. (3), (4).

Here we characterize bipartite graphs and trees which achieve the upper bound in Theorem A. Further we present some another upper and lower bounds for $\gamma_t(G)$. Also, for circular complete graphs, we determine the value of $\gamma_t(G)$.

It is easy to prove that for $n \geq 3$, $\gamma_t(C_n) = \gamma_t(P_n) = \frac{n}{2}$ if $n \equiv 0 \pmod{4}$ and $\gamma_t(C_n) = \gamma_t(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ otherwise.

for the definitions and notations not defined here we refer the reader to texts, such as [2].

## 2 Other bounds for $\gamma_t(G)$

In this section we introduce some other upper bounds for $\gamma_t(G)$.

**Theorem 2.1** Let $G$ be a connected graph, then $\gamma_t(G) \geq \left\lceil \frac{n}{\Delta(G)} \right\rceil$.

**Proof:** Let $S \subseteq V(G)$ be a total dominating set in $G$. Every vertex in $S$ dominates at most $\Delta(G) - 1$ vertices of $V(G) - S$ and dominate at least one of the vertices in $S$. Hence, $|S|((\Delta(G) - 1) + |S| \geq n$. Since, $S$ is an arbitrary total dominating set, then $\gamma_t(G) \geq \left\lceil \frac{n}{\Delta(G)} \right\rceil$. ■
If $G = K_n$, $G = C_{4n}$, or $G = P_{4n}$ then $\gamma_t(G) = \lceil \frac{n}{\Delta(G)} \rceil$, so the above bound is sharp.

**Theorem 2.2** Let $G$ be a graph with $\text{diam}(G) = 2$ then, $\gamma_t(G) \leq \delta(G) + 1$.

**Proof:** Let $x \in V(G)$ and $\text{deg}(x) = \delta(G)$. Since, $\text{diam}(G) = 2$, then $N(x)$ is a dominating set for $G$.

Now $S = N(x) \cup \{x\}$ is a total dominating set for $G$ and $|S| = \delta(G) + 1$. Hence, $\gamma_t(G) \leq \delta(G) + 1$. $\blacksquare$

As we know, $\gamma_t(C_5) = 3$ and also $\delta(C_5) = 2$, $\text{diam}(C_5) = 2$ then $\gamma_t(C_5) = \delta(C_5) + 1$. Hence, the above bound is sharp.

**Theorem 2.3** If $G$ is a connected graph with the girth of length $g(G) \geq 5$ and $\delta(G) \geq 2$, then $\gamma_t(G) \leq n - \lfloor \frac{g(G)}{2} \rfloor + 1$.

**Proof:** Let $G$ be a connected graph with $g(G) \geq 5$ and let $C$ be a cycle of length $g(G)$. Remove $C$ from $G$ to form a graph $G'$. Suppose an arbitrary vertex $v \in V(G')$, since $\delta(G) \geq 2$, then $v$ has at least two neighbors say $x$ and $y$. Let $x, y \in C$. If $d(x, y) \geq 3$, then replacing the path from $x$ to $y$ on $C$ with the path $x, v, y$ reduces the girth of $G$, a contradiction. If $d(x, y) \leq 2$, then $x, y, v$ are on either $C_3$ or $C_4$ in $G$, contradicting the hypothesis that $g(G) \geq 5$.

Hence, no vertex in $G'$ has two or more neighbors on $C$. Since $\delta(G) \geq 2$, the graph $G'$ has minimum degree at least $\delta(G) - 1 \geq 1$. Then $G'$ has no isolated vertex. Now let $S'$ be a $\gamma_t$-set for $C$. Then $S = S' \cup V(G')$ is a total dominating set for $G$. Hence, $\gamma_t(G) \leq n - \lfloor \frac{g(G)}{2} \rfloor + 1$ (note that $\gamma_t(C) \leq \lfloor \frac{g(C)}{2} \rfloor + 1$). $\blacksquare$

## 3 Bipartite graphs with $\gamma_t(G) = n - \Delta(G) + 1$

In this section we characterize the bipartite graphs achieving the upper bound in the theorem A.

**Theorem 3.4** Let $G$ be a bipartite graph with no isolated vertices. Then $\gamma_t(G) = n - \Delta(G) + 1$ if and only if $G$ is a graph in form of $K_{1,t} \cup rK_2$ for $r \geq 0$.

**Proof:** If $G$ is $K_{1,t} \cup rK_2 (r \geq 0)$, clearly $\gamma_t(G) = n - \Delta(G) + 1$. Now let $G$ be a bipartite graph with partitions $A \cup B$ and $x \in A$ where $\text{deg}(x) = \Delta(G) = t$.

We continue our proof in four stages:

**Stage 1:** We claim that for every vertex $y \in A - \{x\}$, $N(y) - N(x) \neq \emptyset$. If it is not true, there exists a vertex in $A - \{x\}$, say $y$, such that $N(y) \subseteq N(x)$. So let $u \in N(y)$, the set $S = V - (N(x) \cup \{y\}) \cup \{u\}$ is a total dominating set.
and \( |S| = n - \Delta(G) \), a contradiction. So we have \( n \geq 2|A| + \Delta(G) - 1 \).

**Stage 2:** For every vertex \( y \in A \), let \( u_y \in N(y) \). Clearly the set \( S = A \cup (\cup_{y \in A} \{u_y\}) \) is a total dominating set for \( G \) and \( |S| \leq 2|A| \), so \( \gamma_t(G) \leq 2|A| \).

Now let \( y \in A - \{x\} \) such that \( |N(y) - N(x)| \geq 2 \). Hence, we have:

\[
n \geq 2|A| + \Delta(G) = (\gamma_t(G) + \Delta(G) - 1) \geq 2|A| + \Delta(G)
\]

\[
\Rightarrow \gamma_t(G) \geq 2|A| + 1,
\]

a contradiction. Hence, for every vertex \( y \in A - \{x\} \), \( |N(y) - N(x)| = 1 \).

**Stage 3:** Let \( y \in A - \{x\} \) and \( N(y) \cap N(x) \neq \emptyset \). Let \( u \in N(y) \cap N(x) \). Now, \( S = (V - N(x) \cup \{y\}) \cup \{u\} \) is a total dominating set and \( |S| = n - \Delta(G) \). So, \( \gamma_t(G) \leq n - \Delta(G) \), a contradiction.

**Stage 4:** Let \( y, z \in A - \{x\} \) and \( N(y) \cap N(z) \neq \emptyset \). Now \( S = (V - (\{z\} \cup N(x))) \cup \{u\} \), where \( u \in N(x) \), is a total dominating set and \( |S| = n - \Delta(G) \).

So, \( \gamma_t(G) \leq n - \Delta(G) \), a contradiction. Hence, \( G \) is a graph in form of \( K_{1,t} \cup rK_2 \).

**COROLLARY 3.1** Let \( T \) is a Tree. Then \( \gamma_t(T) = n - \Delta(T) + 1 \) if and only if \( T \) is a star.

## 4 Total domination numbers of circular complete graphs

If \( n \) and \( d \) are positive integers with \( n \geq 2d \), then circular complete graph \( K_{n,d} \) is the graph with vertex set \( \{v_0, v_1, \ldots, v_{n-1}\} \) in which \( v_i \) is adjacent to \( v_j \) if and only if \( d \leq |i - j| \leq n - d \). In this section we determine the total domination of circular complete graphs. It is easy to see that \( K_{n,1} \) is the complete graph \( K_n \) and \( K_{n,2} \) is a circle on \( n \) vertices, therefore we assume that \( d \geq 3 \).

**Theorem 4.5** For \( n \geq 4d - 2 \) and \( d \geq 3 \), \( \gamma_t(K_{n,d}) = 2 \).

**Proof:** Clearly, \( \gamma_t(K_{n,d}) \geq 2 \). Let \( S = \{v_0, v_{2d-1}\} \). We will show that \( S \) is a total dominating set for \( K_{n,d} \). Since \( n \geq 4d - 2 \) and \( 2d - 1 \leq 2d \), then \( 2d - 1 \leq n - d \). Also \( 2d - 1 \geq d \) since \( d \geq 3 \). Thus \( d \leq 2d - 1 \leq n - d \) and \( v_0v_{2d-1} \in E(K_{n,d}) \). By definition of \( K_{n,d} \), \( v_0 \) is adjacent to each of the vertices \( v_d, v_{d+1}, \ldots, v_{n-d} \).

Now for each \( 1 \leq i \leq d - 1 \) we have

\[
n - d + i - (2d - 1) = n - 3d + i + 1 \geq 4d - 2 - 3d + i + 1 \geq d
\]
and 
\[ n - d + i - (2d - 1) = n - 3d + i + 1 \leq n - 3d + d = n - 2d < n - d. \]

Thus \( v_{2d - 1} \) is adjacent to each of the vertices \( v_{n - d + 1}, \ldots, v_{n - 1} \). On the other hand, for each \( 1 \leq i \leq d - 1 \) we have
\[ 2d - 1 - i \leq 2d - 2 \leq 3d - 2 \leq n - d \]
and
\[ 2d - 1 - i \geq 2d - 1 - d + 1 = d. \]

Hence \( v_{2d - 1} \) is adjacent to each of the vertices \( v_0, v_1, \ldots, v_{d - 1} \) and so \( S \) is a total dominating set for \( K_{n,d} \) and \( \gamma_t(K_{n,d}) = 2. \)

**Theorem 4.6** For \( 3d \leq n \leq 4d - 3 \) and \( d \geq 3 \), \( \gamma_t(K_{n,d}) = 3. \)

**Proof:** Let \( S = \{v_0, v_d, v_{2d - 1}\} \). We prove that \( S \) is a \( \gamma_t(K_{n,d}) \)-set. Since \( d \leq 2d - 2 \leq n - d \), \( G[S] \) contains no isolated vertices. Clearly \( v_0 \) and \( v_d \) are adjacent to each of the vertices \( v_d, v_{d+1}, \ldots, v_{n - d} \) and \( v_{2d}, v_{2d+1}, \ldots, v_{n-d} \) respectively. For \( 1 \leq i \leq d - 1 \) we have
\[ 2d - 1 - i \leq 2d - 1 - d + 1 = d \]
and
\[ 2d - 1 - i \leq 2d - 2 \leq 2d \leq n - d \]

Thus \( v_{2d - 1} \) is adjacent to each of the vertices \( v_1, v_2, \ldots, v_{d-1} \). Hence \( S \) is a total dominating set for \( K_{n,d} \) and so \( \gamma_t(K_{n,d}) \leq 3. \) Now we prove that there is no total dominating set for \( K_{n,d} \) of size 2. Let \( S' = \{u, v\} \) be a \( \gamma_t(K_{n,d}) \)-set. Without loss of generality, let \( u = v_0 \) and \( v = v_j \). Clearly \( d \leq j \leq n - d \).

Since \( v_0 v_{n - d + 1} \notin E(K_{n,d}) \), \( d \leq n - d + 1 - j \leq n - d \) and so \( 1 \leq j \leq d + 1 \). Thus \( j = d \) or \( j = d + 1 \). In both cases, \( S' \) is not a total dominating set since \( v_2, v_3, \ldots, v_{d-1} \) are not dominated by \( S' \) a contradiction. This completes the proof.

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