Wilson’s Grassmannian and a Noncommutative Quadric

Vladimir Baranovsky, Victor Ginzburg and Alexander Kuznetsov

To Yuri Ivanovich Manin on his 65-th birthday

Abstract

Let the group $\mu_m$ of $m$-th roots of unity act on the complex line by multiplication. This gives a $\mu_m$-action on $\text{Diff}$, the algebra of polynomial differential operators on the line. Following Crawley-Boevey and Holland, we introduce a multiparameter deformation, $\text{Diff}_\tau$, of the smash-product $\text{Diff} \# \mu_m$. Our main result provides natural bijections between (roughly speaking) the following spaces:

1. $\mu_m$-equivariant version of Wilson’s adelic Grassmannian of rank $r$;
2. Rank $r$ projective $\text{Diff}$-modules (with generic trivialization data);
3. Rank $r$ torsion-free sheaves on a ‘noncommutative quadric’ $\mathbb{P}^1 \times \tau\mathbb{P}^1$;
4. Disjoint union of Nakajima quiver varieties for the cyclic quiver with $m$ vertices.

The bijection between (1) and (2) is provided by a version of Riemann-Hilbert correspondence between $\text{Diff}$-modules and sheaves. The bijections between (2), (3) and (4) were motivated by our previous work [BGK]. The resulting bijection between (1) and (4) reduces, in the very special case: $r = 1$ and $\mu_m = \{1\}$, to the partition of (rank 1) adelic Grassmannian into a union of Calogero-Moser spaces, discovered by Wilson. This gives, in particular, a natural and purely algebraic approach to Wilson’s result [W].

Table of Contents

1. Introduction
2. Statement of Results
3. Kashiwara Theorem and De Rham Functor
4. $\text{Diff}_\tau$-module Grassmanian and Sheaves on $\mathbb{P}^1 \times \tau\mathbb{P}^1$
5. Monads and Quiver Varieties
6. Projective $\text{Diff}_\tau$-modules
7. Appendix A: Formalism of Polygraded Algebras
8. Appendix B: The Geometry of $\mathbb{P}^1 \times \tau\mathbb{P}^1$

1 Introduction

Nakajima quiver varieties can be viewed, according to our previous paper [BGK], as spaces parametrizing torsion-free sheaves on a “noncommutative plane”. In the simplest case this yields a relation, first observed by Berest-Wilson [BW], between Calogero-Moser spaces and projective modules over the first Weyl algebra, $\mathcal{D}(\mathbb{C})$, of polynomial differential operators on the line $\mathbb{C}$. The approach to this result (and to its ‘quiver generalizations’) used in [BGK] was purely algebraic and totally different from the approach in [BW]. The latter involved a non-algebraic Baker function and was based heavily on the earlier remarkable discovery by Wilson [W] of a connection between an adelic Grassmannian and Calogero-Moser spaces.
In this paper we reverse the logic used by Berest-Wilson and explain (rather than exploit) the connection between adelic Grassmannians and Quiver varieties by means of ‘noncommutative algebraic geometry’, using the strategy of [BGK].

Our first key observation is that each point of adelic Grassmannian can be viewed as a ‘constructible sheaf’ on the line built up from ‘infinite-rank’ local systems. This way, the correspondence between projective (not holonomic!) \(\mathcal{D}(\mathbb{C})\)-modules and points of the adelic Grassmannian becomes nothing but (a non-holonomic version of) the standard De Rham functor between \(\mathcal{D}\)-modules and constructible sheaves on the line.

The De Rham correspondence works equally well in a more general context of equivariant \(\mathcal{D}\)-modules with respect to a natural action on the line \(\mathbb{C}\) of the group \(\mu_m\) of \(m\)-th roots of unity, by multiplication. Giving a \(\mu_m\)-equivariant \(\mathcal{D}\)-module is clearly the same thing as giving a module over \(\mathcal{D}(\mathbb{C})\#\mu_m\), the smash-product of \(\mathcal{D}(\mathbb{C})\) with the group \(\mu_m\), acting on \(\mathcal{D}(\mathbb{C})\) by algebra automorphisms. Note that in [BGK] any, not only cyclic, finite group \(\Gamma \subset SL_2(\mathbb{C})\) of automorphisms of the Weyl algebra was considered. In order to have a De Rham functor, however, one needs to specify a standard holonomic \(\mathcal{D}\)-module of ‘regular functions’. The choice of such a \(\mathcal{D}\)-module breaks the \(SL_2(\mathbb{C})\)-symmetry of the 2-plane formed by the generators of the first Weyl algebra. Thus, the group \(\Gamma \subset SL_2(\mathbb{C})\) has to have an invariant line in \(\mathbb{C}^2\). This leaves us with the only choice \(\Gamma = \mu_m\).

Below, we will be working not only with the algebra \(\mathcal{D}(\mathbb{C})\#\mu_m\), but with a multi-parameter deformation
\[
\mathcal{D}_\tau = \mathbb{C}(x,y)\#\mu_m/\langle [y,x] = \tau \rangle.
\]
of that algebra introduced by Crawley-Boevey and Holland. Here \(\tau\) (= deformation parameter) is an arbitrary element in the group algebra \(\mathbb{C}[\mu_m]\), and \(\mathbb{C}(x,y)\) stands for the free \(\mathbb{C}\)-algebra of noncommutative polynomials in two variables \(x,y\). Once a De Rham functor between projective \(\mathcal{D}_\tau\)-modules and points of an adelic Grassmannian is established one can construct a ‘Wilson type’ connection between the adelic Grassmannian and Quiver varieties as follows. First, view a projective \(\mathcal{D}_\tau\)-module as a vector bundle on an appropriate noncommutative plane \(\mathbb{A}_{\tau}^2\). Next, extend (see [BGK]) this vector bundle to a (framed) torsion-free sheaf on a completion \(X_\tau \subset \mathbb{A}_{\tau}^2\), a ‘noncommutative projective surface’. Finally, we use a description of framed torsion-free sheaves on \(X_\tau\) in terms of monads.

---

1. During the preparation of the present paper (which was first supposed to be part of [BGK]) another paper by Berest-Wilson appeared, see [BW2]. Our approach is similar to that of [BW2] (we treat more general case of ‘higher rank’ and \(\mu_m\)-equivariant projective modules). However, even in the rank 1 case, in [BW2] the authors do not provide an independent proof of the bijection between Calogero-Moser spaces and projective modules; instead they construct a map inverse to the map constructed in [BW] assuming the latter is already known to be a bijection. An independent direct proof of the bijectivity in the rank 1 case was obtained in the Appendix to [BW2] by M. Van den Bergh, who used some results of [BGK].

We emphasize that, for the reasons explained at the end of the Introduction below, it seems to be impossible to extend the approach of [BW2] (connecting the adelic Grassmannian with rank 1 sheaves on a non-commutative \(\mathbb{P}^1\)) to the higher rank case without replacing \(\mathbb{P}^1\) by a non-commutative surface which fibers over \(\mathbb{P}^1\), like the noncommutative quadric \(\mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{P}^1\) that we are using in the present paper.

2. More generally, our construction of De Rham functor yields a similar correspondence between projective \(\mathcal{D}\)-modules on any smooth algebraic curve \(X\) and points of an appropriately defined adelic Grassmannian attached to the curve (in that case a non-commutative version of projective completion of \(\mathcal{P}^X\) should play the role of \(\mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{P}^1\)). The case of an elliptic curve seems to be especially interesting; we hope to discuss it elsewhere.
(i.e. in terms of linear algebra data) developed in [BGK] to obtain a parametrisation of projective \( \mathbb{D}_\tau \)-modules by points of certain Quiver varieties.

There are several possible choices for a 'compactification' \( X_\tau \) of the noncommutative plane \( \mathbb{A}_2^2 \). In [BGK] we used \( X_\tau = \mathbb{P}_2^2 \), a noncommutative version of projective plane. In the present paper we choose another 'compactification' of \( \mathbb{A}_2^2 \), a noncommutative version, \( \mathbb{P}^1 \times_\tau \mathbb{P}^1 \), of two-dimensional quadric. This choice is essential for our present purposes. Our construction of the extension of a \( \mathbb{D}_\tau \)-module to a torsion free sheaf on \( \mathbb{P}^1 \times_\tau \mathbb{P}^1 \) doesn’t behave well enough in the case of \( \mathbb{P}_2^2 \). On the other hand, the relation of sheaves on \( \mathbb{P}^1 \times_\tau \mathbb{P}^1 \) to quiver varieties is a posteriori equivalent to the one used in [BGK], since the two noncommutative spaces \( \mathbb{P}^1 \times_\tau \mathbb{P}^1 \) and \( \mathbb{P}_2^2 \) can be obtained from each other by "blowing up" and "blowing down" constructions. We will indicate the idea of such a construction in a Remark (above Theorem 5.10) and it will be hinted there how a bijection between torsion-free sheaves on \( \mathbb{P}^1 \times_\tau \mathbb{P}^1 \) and on \( \mathbb{P}_2^2 \) can be established via a noncommutative version of Fourier-Mukai transform, see (5.8).

Our results generalize (and, hopefully, put in context) the results of Wilson [W] in two ways. First, we incorporate a \( \mu_m \)-action. Second, Wilson only considered the rank 1 case, that is the case of rank 1 sheaves on \( \mathbb{P}^1 \times_\tau \mathbb{P}^1 \) in the terminology of the present paper. In Wilson’s situation the whole adelic Grassmanian gets partitioned into a disjoint union of Calogero Moser spaces. In the more general setup of arbitrary rank \( r \geq 1 \) this is no longer true for two reasons. First, in our definition of the adelic Grassmanian we drop the "index zero" condition of ([W], 2.1(ii)) (it has to do with replacing the group \( SL_r \) by \( GL_r \)). This makes our version of adelic Grassmanian somewhat larger than that of [W]. The geometric consequence of "index zero" condition is (in our language) that the restriction of a coherent sheaf to the line \( \mathbb{P}^1 \times \{\infty\} \subset \mathbb{P}^1 \times_\tau \mathbb{P}^1 \) is a vector bundle with the vanishing first Chern class: \( c_1 = 0 \). In the rank 1 case considered by Wilson, any such bundle is necessarily trivial, while this is clearly not true for higher ranks. Thus, our main result says that, for any \( m \geq 1 \) and \( r \geq 1 \), the part of \( (\mu_m \text{-equivariant}) \) rank \( r \) adelic Grassmanian formed by sheaves trivial on the line \( \mathbb{P}^1 \times \{\infty\} \subset \mathbb{P}^1 \times_\tau \mathbb{P}^1 \) can be partitioned into a disjoint union of quiver varieties of type \( \mathbb{A}_{m-1} \).

Remark. We would like to end this Introduction by bringing attention of the reader to a surprising correspondence resulting from comparing [BGK, Theorem 1.3.12] with [EG, Theorem 1.13]. Specifically, let \( \Gamma \subset SL_2(\mathbb{C}) \) be a finite subgroup, and \( \mathbb{D}_\tau(\Gamma) \) the corresponding coordinate ring of the corresponding \( \Gamma \)-equivariant 'non-commutative plane'. Recall that according to [BGK], there is a partition of the moduli space of projective \( \mathbb{D}_\tau(\Gamma) \)-modules \( N \) such that \( [N] = R \) into a disjoint union, according to the ‘second Chern class’ \( c_2(N) := n \in \mathbb{Z} \). On the other hand, given \( n \geq 1 \), let \( \Gamma_n := S_n \times (\Gamma \times \cdots \times \Gamma) \subset Sp(\mathbb{C}^{2n}) \) denote the corresponding wreath product and let \( H_{0,\tau}(\Gamma_n) \) be the Symplectic Reflection algebra attached to \( \Gamma_n \), see [EG, p.249]. Further, Theorem 1.13 of [EG] shows that representation theories of the algebras \( \mathbb{D}_\tau(\Gamma) \) and \( H_{0,\tau}(\Gamma_n) \) are related by the following mysterious bijection:

\(^3\)In [BGK] we use the notation \( B_\tau \) instead of \( \mathbb{D}_\tau(\Gamma) \), and in [EG] we write ‘c’ for what we denote by \( \tau \) in [BGK] (and in the present paper).

3
Isomorphism classes of finitely generated \textbf{projective} $D_r(\Gamma)$-modules $N$ such that $[N] = \text{triv}$ and $c_2(N) = n$ \quad \approx \quad \{ \text{Isomorphism classes of simple } H_{0, r} (\Gamma_n)\text{-modules} \}

Finding a direct conceptual explanation of the bijection above presents a challenging problem. We remark that even in the case of the trivial group $\Gamma$, where the moduli space on each side reduces, as a variety, to the Calogero-Moser space, the bijection is still completely unexplained.

Acknowledgements. We are indebted to Sasha Beilinson for some very useful remarks. The third author was partially supported by RFFI grants 99-01-01144 and 99-01-01204. Also, he would like to express his gratitude to the University of Chicago, where the major part of this paper was written.

2 Statement of Results

From now on, let $\Gamma = \mu_m$ denote the group of $m$-th roots of unity, and $\mathbb{C}_\Gamma$ its group algebra. We fix an embedding $\Gamma = \mu_m \hookrightarrow SL_2(\mathbb{C})$, and let $L$ denote the tautological two-dimensional representation of $\Gamma$ arising from the embedding. We have: $L \cong \epsilon \oplus \epsilon^{-1}$, where $\epsilon$ is a primitive character of $\Gamma$. Let $\{x, y\}$ be a basis of $L^*$ compatible with the above direct sum decomposition, such that $\Gamma$ acts on $x$ by $\epsilon$ and on $y$ by $\epsilon^{-1}$. Write $\mathbb{C}[x]$ for the polynomial algebra on the line with coordinate $x$, and $\mathbb{C}(x)$ for the corresponding field of rational functions. We form the smash-product algebras

$$\mathbb{C}_\Gamma[x] := \mathbb{C}[x]#\Gamma \quad \text{and} \quad \mathbb{C}_\Gamma(x) := \mathbb{C}(x)#\Gamma.$$ 

The standard embedding $\mathbb{C}_\Gamma \hookrightarrow \mathbb{C}[x]#\Gamma$ makes $\mathbb{C}_\Gamma[x]$ into a $\Gamma$-bimodule via left and right multiplication by $\Gamma$. There is a similar $\Gamma$-bimodule structure on $\mathbb{C}_\Gamma(x)$.

Choose and fix a finite-dimensional $\Gamma$-module $W$. There is a natural $\mathbb{C}(x)^\Gamma$-action on $W \otimes_\mathbb{C}_\Gamma(x)$ given by $p : w \otimes f \mapsto p \cdot (w \otimes f) := w \otimes (p \cdot f)$.

**Definition 2.1** A $\Gamma$-invariant vector subspace $U \subset W \otimes_\mathbb{C}_\Gamma(x)$ is called \textbf{primary decomposable} if the following two conditions hold

(a) There exists a $\Gamma$-invariant polynomial $p = p(x)$ such that

$$p \cdot (W \otimes_\mathbb{C}_\Gamma[x]) \subset U \subset \frac{1}{p} \cdot (W \otimes_\mathbb{C}_\Gamma[x]).$$

(b) If $p(x) = \prod_{\mu} (x - \mu)^{k_\mu}$ then the subspace on the left below is compatible with the direct sum decomposition on the right (i.e. $\text{LHS} = \text{sum of its intersections with the direct summands on the RHS}$):

$$\frac{U}{p \cdot (W \otimes_\mathbb{C}_\Gamma[x])} \subset \frac{1}{p} \cdot (W \otimes_\mathbb{C}_\Gamma[x]) = \bigoplus_{\mu} \frac{(x - \mu)^{-k_\mu} W \otimes_\mathbb{C}_\Gamma[x]}{(x - \mu)^{k_\mu} W \otimes_\mathbb{C}_\Gamma[x]}.$$

Define an \textbf{adelic Grassmanian} $\text{Gr}^{\text{ad}}(W)$ to be the set of all primary decomposable subspaces $U \subset W \otimes_\mathbb{C}_\Gamma(x)$.

\textsuperscript{4}this notion is due to Cannings-Holland [CaH].
Our first goal is to relate the adelic Grassmanian to modules over some noncommutative algebra. To that end, we fix an element \( \tau \in \mathbb{C} \Gamma \) and consider the algebra \( D_{\tau} \), see (1.1), to be denoted \( D \) in the future. Let \( D_{\text{frac}} \) be the localization of the algebra \( D \) with respect to the multiplicative system \( \mathbb{C}[x] \setminus \{0\} \) of all nonzero \( \Gamma \)-invariant polynomials in \( x \). This localization has a natural algebra structure extending that on \( D \). Note further that we have a natural embedding \( \mathbb{C} \Gamma(x) \hookrightarrow D_{\text{frac}} \) and, moreover, this embedding yields a vector space isomorphism: \( \mathbb{C} \Gamma(x) \cong D_{\text{frac}}/D_{\text{frac}} \cdot y \). We make \( \mathbb{C} \Gamma(x) \) into a left \( D_{\text{frac}} \)-module by transporting the obvious \( D_{\text{frac}} \)-module structure on \( D_{\text{frac}}/D_{\text{frac}} \cdot y \) via the bijection above. For \( \tau = 1 \), this reduces essentially to the standard action on \( \mathbb{C} \Gamma(x) \) by differential operators.

For any \( \Gamma \)-module \( W \), the space \( W \otimes_{\tau} D_{\text{frac}} \) has an obvious structure of a projective right \( D_{\text{frac}} \)-module.

**Definition 2.2** Let \( G_{W} = GL_{D_{\text{frac}}}(W \otimes_{\tau} D_{\text{frac}}) \) be the group of all (invertible) right \( D_{\text{frac}} \)-linear automorphisms of the \( D_{\text{frac}} \)-module \( W \otimes_{\tau} D_{\text{frac}} \).

We define a left \( G_{W} \)-action on the vector space \( W \otimes_{\tau} \mathbb{C} \Gamma(x) \) and on the adelic Grassmanian \( G^{\text{ad}}_{W}(W) \) as follows. First, observe that the natural left \( GL_{D_{\text{frac}}}(W \otimes_{\tau} D_{\text{frac}}) \)-action on \( W \otimes_{\tau} D_{\text{frac}} \) commutes with right multiplication by \( y \), therefore keeps the subspace \( W \otimes_{\tau} D_{\text{frac}} \cdot y \subset W \otimes_{\tau} D_{\text{frac}} \) stable. Hence, there is a well-defined left \( G_{W} \)-action on the quotient \( (W \otimes_{\tau} D_{\text{frac}})/(W \otimes_{\tau} D_{\text{frac}} \cdot y) \). Further, since left \( \Gamma \)-action commutes with right \( D_{\text{frac}} \)-action, and since \( W \) is a projective \( \Gamma \)-module, it follows that we have an isomorphism

\[
\psi : W \otimes_{\tau} \mathbb{C} \Gamma(x) \cong (W \otimes_{\tau} D_{\text{frac}})/(W \otimes_{\tau} D_{\text{frac}} \cdot y),
\]

induced by the isomorphism \( \mathbb{C} \Gamma(x) \cong D_{\text{frac}}/D_{\text{frac}} \cdot y \) considered two paragraphs above. We define the left \( G_{W} \)-action on \( W \otimes_{\tau} \mathbb{C} \Gamma(x) \) by transporting the left \( G_{W} \)-action on the quotient \( (W \otimes_{\tau} D_{\text{frac}})/(W \otimes_{\tau} D_{\text{frac}} \cdot y) \) via the isomorphism \( \psi \). It is straightforward to verify that elements of \( G_{W} \) take primary decomposable subspaces of \( W \otimes_{\tau} \mathbb{C} \Gamma(x) \) into primary decomposable subspaces. This gives a canonical left \( G_{W} \)-action on the adelic Grassmanian \( G^{\text{ad}}_{W}(W) \).

To formulate our first result, recall that there is a canonical isomorphism (due to Quillen) of Grothendieck \( K \)-groups: \( K(\mathbb{C} \Gamma) \cong K(D) \) induced by the functor: \( W \mapsto W \otimes_{\tau} D \). We write \([N] \in K(\mathbb{C} \Gamma)\) for the image of the class of a \( D \)-module \( N \) under the inverse isomorphism \( K(D) \rightarrow K(\mathbb{C} \Gamma) \), and let \( \dim : K(\mathbb{C} \Gamma) \rightarrow \mathbb{Z} \) denote the dimension homomorphism.

Further, there is a distinguished finite collection of codimension one hyperplanes in the vector space \( \mathbb{C} \Gamma \), called the root hyperplanes. One way to define these hyperplanes is to use McKay correspondence. The latter associates to the cyclic group \( \Gamma = \mu_{m} \subset SL_{2}(\mathbb{C}) \) an affine Dynkin graph of type \( \widetilde{A}_{m-1} \) such that the underlying vector space of the group algebra \( \mathbb{C} \Gamma \) gets identified with the dual of the \( C \)-vector space generated by simple roots of the corresponding affine root system. In particular, every root gives a hyperplane in the vector space \( \mathbb{C} \Gamma \).

**Definition 2.3** An element \( \tau \in \mathbb{C} \Gamma \) is called generic if it does not belong to any root hyperplane in \( \mathbb{C} \Gamma \).
Our first theorem below is a noncommutative analogue of a well-known result due to A. Weil, providing a description of algebraic vector bundles on an algebraic curve in terms of an adelic double-coset construction.

**Theorem 2.4** Assume that $\tau$ is generic. Let $W$ be a $\Gamma$-module with $\dim W = r$. The set of isomorphism classes of projective (right) $D$-modules $N$ such that $\dim[N] = r$ is in a canonical bijection with the coset space $G_W \backslash \Gr^{\ad}(W)$.

To explain the main ideas involved in the proof of Theorem 2.4 we need the following

**Definition 2.5** A right $D$-submodule $N \subset W \otimes_{\Gamma} D_{\text{frac}}$ is called fat if there exists a $\Gamma$-invariant polynomial $p(x)$ such that $p \cdot (W \otimes_{\Gamma} D) \subset N \subset \frac{1}{p} (W \otimes_{\Gamma} D)$.

Let $\Gr^{D}(W)$ be the set of all fat right $D$-submodules $N \subset W \otimes_{\Gamma} D_{\text{frac}}$.

Now, the proof goes as follows. First, we check that for generic $\tau$ any projective right $D$-module can be embedded into $W \otimes_{\Gamma} D_{\text{frac}}$ as a fat $D$-submodule. The embedding is unique up to a $G_W$-action. Then, it remains to relate the Grassmanians $\Gr^{D}(W)$ and $\Gr^{\ad}(W)$. To this end, recall that we have equipped the space $\mathbb{C}[x]$ with a canonical structure of left $D$-module, that clearly commutes with right $\Gamma$-action by multiplication.

We introduce a De Rham functor $\mathcal{D}R$ from the category of right $D$-modules to the category of right $\Gamma$-modules as follows

$$N \mapsto \mathcal{D}R(N) := N \otimes D \Gamma[x].$$

Given a non-zero polynomial $p \in \mathbb{C}[x]^{\Gamma}$, we write $p \cdot (W \otimes_{\Gamma} D) := (W \otimes_{\Gamma} (p \cdot D))$. The space $p \cdot (W \otimes_{\Gamma} D)$ has an obvious right $D$-module structure, and we have:

$$\mathcal{D}R(p \cdot (W \otimes_{\Gamma} D)) = p \cdot (W \otimes_{\Gamma} D) \otimes D \Gamma[x] = p \cdot (W \otimes_{\Gamma} \Gamma[x]).$$

Hence the De Rham functor takes any fat $D$-submodule of $W \otimes_{\Gamma} D_{\text{frac}}$ to a $\Gamma$-invariant vector subspace $\mathcal{U} \subset W \otimes_{\Gamma} \Gamma[x]$, such that $p \cdot (W \otimes_{\Gamma} \Gamma[x]) \subset \mathcal{U} \subset \frac{1}{p} (W \otimes_{\Gamma} \Gamma[x])$. Moreover, one can check that this space is primary decomposable.

**Example.** Assume that $\Gamma = \{1\}$ is trivial, i.e. that $m = 1$. Then $\Gamma[x] = \mathbb{C}$ and $\tau \in \Gamma[x] = \mathbb{C}$ is generic if and only if $\tau \neq 0$. In this case the algebra $D$ is isomorphic to $\mathcal{D}(\mathbb{C})$, the algebra of differential operators on the line $\mathbb{C}$, and the algebra $D_{\text{frac}}$ is isomorphic to the algebra of differential operators with rational coefficients. The functor $\mathcal{D}R$ becomes the standard De Rham functor.

We have seen that the De Rham functor yields a map $\mathcal{D}R : \Gr^{D}(W) \rightarrow \Gr^{\ad}(W)$. We also define a map: $\Gr^{\ad}(W) \rightarrow \Gr^{D}(W)$ (in the opposite direction) as follows. Let $\mathcal{U} \subset W \otimes_{\Gamma} \Gamma[x]$ be a primary decomposable subspace. Then the left $D_{\text{frac}}$-action map $D_{\text{frac}} \otimes \Gamma[x] \rightarrow \Gamma[x]$ induces, after tensoring with $W$ and restricting to $\Gamma[x]$, a linear map $a : W \otimes_{\Gamma} D_{\text{frac}} \otimes \mathbb{C} \Gamma[x] \rightarrow W \otimes_{\Gamma} \Gamma[x]$. Let $\text{Diff}_{\tau}(\mathcal{U})$ denote the set of all elements $u \in W \otimes_{\Gamma} D_{\text{frac}}$, such that $a(u \otimes \Gamma[x]) \subset \mathcal{U}$. It is easy to show that $\text{Diff}_{\tau}(\mathcal{U}) \subset W \otimes_{\Gamma} D_{\text{frac}}$ is a fat $D$-submodule. Thus, we obtain a map $\text{Diff}_{\tau} : \Gr^{\ad}(W) \rightarrow \Gr^{D}(W)$. 

6
Theorem 2.6 For generic \( \tau \), the maps \( \mathcal{DR} \) and \( \text{Diff}_\tau \) give mutually inverse bijections \( \text{Gr}^D(W) \cong \text{Gr}^{\text{ad}}(W) \).

Our next step is to interpret the space \( \text{Gr}^D(W) \) using the formalism of noncommutative geometry. To this end, we introduce the algebra

\[
Q = \mathbb{C}(x, z, y, w) \# \Gamma / \left\langle [x, z] = [y, z] = [z, w] = [y, w] = [x, w] = 0, [y, x] = \tau \cdot z w \right\rangle, \tag{2.7}
\]

where for any \( \gamma \in \Gamma \) we put:

\[
\gamma \cdot x \cdot \gamma^{-1} = \epsilon(\gamma) \cdot x, \quad \gamma \cdot y \cdot \gamma^{-1} = \epsilon^{-1}(\gamma) \cdot y, \quad \gamma \cdot z \cdot \gamma^{-1} = z, \quad \gamma \cdot w \cdot \gamma^{-1} = w. \tag{2.8}
\]

Define a grading: \( Q = \oplus_{i,j \geq 0} Q_{i,j} \) on the algebra \( Q \) by letting \( \deg x = \deg z = (1, 0), \deg y = \deg w = (0, 1) \), and \( \deg \gamma = (0, 0) \) for any \( \gamma \in \Gamma \). Thus, \( Q_{0,0} = \mathbb{C} \Gamma \).

When \( \Gamma \) is trivial and \( \tau = 0 \) the algebra \( Q \) reduces to the standard bigraded algebra associated to the quadric \( \mathbb{P}^1 \times \mathbb{P}^1 \) and a pair of line bundles \( L_1 = O(1, 0); L_2 = O(0, 1) \), that is for any \( i, j \geq 0 \), we have \( Q_{i,j} = H^0(\mathbb{P}^1 \times \mathbb{P}^1, L_1^{\otimes i} \otimes L_2^{\otimes j}) \). In this case, the category of coherent sheaves of \( \mathbb{P}^1 \times \mathbb{P}^1 \) can be described as a quotient category of the category of bigraded \( Q \)-modules (see §4).

In the general case of a nontrivial group \( \Gamma \) and arbitrary \( \tau \) a similar quotient category construction may still be applied formally to the bigraded ring \( Q \). Following [AZ], see also [BGK] and §§7-8 below, we will view objects of the resulting quotient category as coherent sheaves on a “noncommutative quadric” \( \mathbb{P}^1 \times \tau \mathbb{P}^1 \).

Note that, in the commutative case \( \tau = 0 \), the equations \( z = 0 \) and \( w = 0 \) give rise to two embeddings \( i_z : \mathbb{P}_z^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) and \( i_w : \mathbb{P}_w^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) of the corresponding factors \( \mathbb{P}^1 \). Thus, one may consider the restriction functor \( i_z^* \) taking coherent sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) to coherent sheaves on \( \mathbb{P}_z^1 \) (and also consider coherent sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) which are trivialized in some formal neighbourhood of \( \mathbb{P}_z^1 \)). In §4 and Appendix A, we show how to extend all relevant concepts to the noncommutative case. The homogeneous coordinate rings of \( \mathbb{P}^1_z \) and \( \mathbb{P}^1_w \) will be replaced by \( \mathbb{C}[y, w] \# \Gamma \) and \( \mathbb{C}[x, z] \# \Gamma \), respectively. The latter algebras are only slightly noncommutative in the sense that the corresponding quotient categories of ‘noncommutative coherent sheaves’ are nothing but the categories of \( \Gamma \)-equivariant coherent sheaves on \( \mathbb{P}^1 \), the ordinary (commutative) projective line. This leads to the following provisional (see Definitions 4.5 and 4.6 for details)

**Definition 2.9** Let \( \text{Gr}^{\mathbb{P}^1 \times \tau \mathbb{P}^1}(W) \) be the set of all (equivalence classes of) coherent sheaves \( E \) on \( \mathbb{P}^1 \times \tau \mathbb{P}^1 \) trivialized in a formal neighbourhood of \( \mathbb{P}_z^1 \) and such that \( i_z^* E \cong W \otimes_{\mathbb{C} \Gamma} O_{\mathbb{P}_z^1} \).

Note that in the commutative situation we have: \( \mathbb{P}^1 \times \mathbb{P}^1 \setminus (\mathbb{P}_z^1 \cup \mathbb{P}_w^1) = \mathbb{A}^2 \) is an affine plane with coordinates \( x, y \). In the noncommutative case we have an algebra isomorphism

\[
D \cong Q / ((z - 1)Q + (w - 1)Q).
\]

Therefore, the algebra \( D \) may be viewed as coordinate ring of a noncommutative affine plane \( j : \mathbb{A}^2 \rightarrow \mathbb{P}^1 \times \tau \mathbb{P}^1 \). This gives rise to a restriction functor \( j^* : \text{coh}(\mathbb{P}^1 \times \tau \mathbb{P}^1) \rightarrow \text{mod}(D) \) taking the category of coherent sheaves on \( \mathbb{P}^1 \times \tau \mathbb{P}^1 \) to \( D \)-modules. It is easy to see that \( j^* \) takes any sheaf trivialized in a neighborhood of \( \mathbb{P}_z^1 \) to a fat \( D \)-submodule of \( W \otimes_{\mathbb{C}} D_{\text{frac}} \).
Theorem 2.10 The ‘restriction’ functor $j^*$ induces a bijection $\text{Gr}^{P_1 \times P_1}(W) \rightarrow \text{Gr}^D(W)$.

We write $j_*$ for an inverse of the bijection $j^*$.

The fourth (and the last) infinite Grassmanian considered in this paper is an affine Grassmanian $\text{Gr}^{\text{aff}}(W)$ introduced below.

Definition 2.11 A right $\Gamma[x]$-submodule $W \subset W \otimes_r \Gamma(x)$ is called fat if there exists a $\Gamma$-invariant polynomial $p(x)$ such that $p(W \otimes_r \Gamma[x]) < W \subset \frac{1}{p}(W \otimes_r \Gamma[x])$.

Define $\text{Gr}^{\text{aff}}(W)$ to be the set of all fat $\Gamma[x]$-submodules in $W \otimes_r \Gamma(x)$.

It is clear that a (right) $\Gamma$-stable subspace $W \subset W \otimes_r \Gamma(x)$ is a fat $\Gamma[x]$-submodule if and only if $W$ is a finitely-generated $\mathbb{C}[x]$-submodule such that $W \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = W \otimes_r \Gamma(x)$. Thus, a fat $\Gamma[x]$-submodule may be viewed as a $\Gamma$-stable lattice in the $\mathbb{C}(x)$-vector space $W \otimes_r \Gamma(x)$. For this reason we refer to $\text{Gr}^{\text{aff}}(W)$ as the affine Grassmanian. The standard relation between loop-Grassmannians and vector bundles on the Riemann sphere, see e.g. [PS], shows that the space $\text{Gr}^{\text{aff}}(W)$ can also be interpreted as the set of all $\Gamma$-equivariant vector bundles on $\mathbb{P}^1$ trivialized in a Zariski neighbourhood of the point $\infty \subset \mathbb{P}^1$ (cf. Definition 4.5), with $W$ being a fiber at $\infty$. Thus the restriction functor $i^*_w$ takes $\text{Gr}^{P_1 \times P_1}(W) \rightarrow \text{Gr}^{\text{aff}}(W)$.

We observe that the group $G_W$, see Definition 2.2, acts naturally on each of the Grassmannians: $\text{Gr}^{\text{ad}}(W)$, $\text{Gr}^D(W)$, and $\text{Gr}^{P_1 \times P_1}(W)$. Specifically, the action on $\text{Gr}^{\text{ad}}(W)$ has been defined earlier, and the action on $\text{Gr}^D(W)$ is induced by the corresponding $G_W$-action on $W \otimes_1 D_{\text{frac}}$. The action on $\text{Gr}^{P_1 \times P_1}(W)$ arises from $G_W$-action on the direct system formed by the sheaves $W \otimes_1 \mathcal{O}(n)\big|_0$, $n = 1, 2, \ldots$. Observe further that the affine Grassmanian $\text{Gr}^{\text{aff}}(W)$ has an action of the subgroup $GL_{\Gamma[x]}(W \otimes_r \Gamma(x)) \subset G_W$ (the group $G_W$ itself does not act on $\text{Gr}^{\text{aff}}$ since it does not preserve the condition to be a $\Gamma[x]$-submodule).

All the objects and the maps we have introduced so far are incorporated in the following diagram

$$\text{Gr}^{\text{ad}}(W) \xrightarrow{\text{Diff}_r} \text{Gr}^D(W) \xrightarrow{j_*} \text{Gr}^{P_1 \times P_1}(W) \xrightarrow{i^*_w} \text{Gr}^{\text{aff}}(W), \quad (2.12)$$

Principal symbol map ‘Symb’: The algebra $D$ has a natural increasing filtration: $\Gamma[x] = F_0 D \subset F_1 D \subset F_2 D \subset \ldots$, where $F_k D$ is the $\Gamma[x]$-submodule generated by $\{1, y, \ldots, y^k\}$. This filtration by the ‘order of differential operator’ extends canonically to a similar filtration $\Gamma(x) = F_0 D_{\text{frac}} \subset F_1 D_{\text{frac}} \subset F_2 D_{\text{frac}} \subset \ldots$, on the algebra $D_{\text{frac}}$, and for the corresponding associated graded algebras we have: $\text{gr}^F D \simeq \Gamma[x, y]$ and $\text{gr}^F D_{\text{frac}} \simeq \Gamma(x)[y]$. The filtration on $D_{\text{frac}}$ also induces an increasing filtration, $F_k(W \otimes_1 D_{\text{frac}}) := W \otimes F_k D_{\text{frac}}$, on the $D_{\text{frac}}$-module $W \otimes_1 D_{\text{frac}}$ such that $\text{gr}^F(W \otimes_1 D_{\text{frac}}) \simeq (W \otimes_1 \Gamma(x))[y]$. 
Now, given a $D$-submodule $N \subset W \otimes \Gamma \frac{D}{\Gamma}$, we put
\[
\text{Symb}(N) := \{ f \in W \otimes \Gamma \frac{D}{\Gamma} | f \cdot y^k + u_{k-1} \in N, \text{ for some } k \in \mathbb{Z} \text{ and some } u_{k-1} \in F_{k-1}(W \otimes \Gamma \frac{D}{\Gamma}) \}.
\]
This is a $\Gamma(x)$-submodule in $W \otimes \Gamma \frac{D}{\Gamma}(x)$ that can be equivalently defined as follows.

Right multiplication by '$y'$ gives rise to a direct system of bijective maps: $\Gamma(x) \xrightarrow{\sim} \Gamma(x) \cdot y \xrightarrow{\sim} \Gamma(x) \cdot y^2 \xrightarrow{\sim} \ldots$, and it is clear that this yields isomorphisms
\[
\Gamma(x) = \Gamma(x) \cdot y^0 \xrightarrow{\sim} \lim_{k} \Gamma(x) \cdot y^k \quad \text{and} \quad W \otimes \Gamma \frac{D}{\Gamma} \Gamma(x) = W \otimes \Gamma \frac{D}{\Gamma} \Gamma(x) \cdot y^0 \xrightarrow{\sim} \lim_{k} (W \otimes \Gamma \frac{D}{\Gamma} \Gamma(x)) \cdot y^k.
\]
Let $\text{gr}^F N \subset \text{gr}^F(W \otimes \Gamma \frac{D}{\Gamma}) \simeq W \otimes \Gamma \frac{D}{\Gamma}(x)[y]$ denote the associated graded of $N$ with respect to the induced filtration $F_\bullet N := N \cap F_\bullet(W \otimes \Gamma \frac{D}{\Gamma})$, and form the direct system: $\text{gr}_0^F N \to \text{gr}_2^F N \to \text{gr}_2^F N \to \ldots$, induced by the $y$-action on $\text{gr}^F N$ (which is not necessarily bijective). Using the identification provided by (2.13), one has: $\text{Symb}(N) = \lim_{k} \text{gr}_k^F N$.
It is clear that the RHS is a $\Gamma(x)$-submodule in $\Gamma(x)$. The assignment: $N \mapsto \text{Symb}(N)$ gives a (discontinuous) map $\text{Symb} : \text{Gr}^D(W) \to \text{Gr}^\text{aff}(W)$.

Let $\sigma$ denote the composite map, see (2.12) $\sigma : \text{Gr}^\text{ad}(W) \xrightarrow{\text{Diff}} \text{Gr}^D(W) \xrightarrow{j_\ast} \text{Gr}^\mathbb{P}_1 \times \mathbb{P}_1(W) \xrightarrow{i_{w}^\ast} \text{Gr}^\text{aff}(W)$. The following is an enriched version of diagram (2.12)

**Theorem 2.14** Assume that $\tau$ is generic. Then we have a commutative diagram
\[
\begin{array}{ccc}
\text{Gr}^\text{ad}(W) & \xrightarrow{\text{Diff}} & \text{Gr}^D(W) & \xrightarrow{j_\ast} & \text{Gr}^\mathbb{P}_1 \times \mathbb{P}_1(W) \\
\downarrow{\sigma} & & \downarrow{j^*} & & \downarrow{i_{w}^*} \\
\text{Gr}^\text{aff}(W) & & & & \\
\end{array}
\]
where the maps $\text{Diff}^\mathbb{P}_1$, $\text{Diff}^D$, $j_\ast$, and $j^*$ are $G_W$-equivariant bijections, and the maps $i_{w}^*$, $\text{Symb}$, and $\sigma$ are $GL_{\Gamma}(x)(W \otimes \Gamma \frac{D}{\Gamma}(x))$-equivariant.

Finally, we explain how Quiver varieties enter the picture. Given a pair of finite dimensional $\Gamma$-modules $W$, $V$, and an element $\tau \in \Gamma$ define, following Nakajima, a locally closed subvariety of quiver data:
\[
M_\tau^\Gamma(V, W) := \{ (B, I, J) \in \text{Hom}_\Gamma(V, V \otimes L) \bigoplus \text{Hom}_\Gamma(W, V) \bigoplus \text{Hom}_\Gamma(V, W) \}.
\]
formed by the triples $(B, I, J)$ satisfying the following two conditions:

**Moment Map Equation:** $[B, B] + IJ = \tau |_V$;

**Stability Condition:** if $V' \subset V$ is a $\Gamma$-submodule such that $B(V') \subset V' \otimes L$ and $I(W) \subset V' \otimes L$ and $I(W) \subset V'$ then $V' = V$.
The group $G_{\Gamma}(V) = GL(V)^{\Gamma}$ of $\Gamma$-equivariant automorphisms of $V$ acts on $M^{\tau}_{\Gamma}(V,W)$ by the formula: $g(B,I,J) = (gBg^{-1},gI,Jg^{-1})$. Note that this $G_{\Gamma}(V)$-action is free, due to the stability condition.

**Definition 2.17** Let $M^{\tau}_{\Gamma}(V,W) = M^{\tau}(V,W)/G_{\Gamma}(V)$ be the Geometric Invariant Theory quotient, called a (Nakajima) quiver variety.

The affine Grassmanian $Gr_{\aff}(W)$ has a marked point $W_0 = W \otimes_{\Gamma} \mathcal{O}_{P^1_x}$ corresponding to the $\Gamma$-equivariant sheaf $W \otimes \mathcal{O}$ with its natural trivialization in the Zariski neighbourhood of infinity.

**Theorem 2.18** Let $\tau$ be generic. Then there is a canonical bijection

$$\bigsqcup_{V} M^{\tau}_{\Gamma}(V,W) \cong \sigma^{-1}(W_0) \subset Gr^{ad}(W),$$

where $V$ runs through the set of isomorphism classes of all finite dimensional $\Gamma$-modules.

**Example.** Let the group $\Gamma$ be trivial and $\tau \neq 0$. If $W = \mathbb{C}$ and $V = \mathbb{C}^n$, then the corresponding quiver variety is isomorphic to the Calogero-Moser space $CM_n$. Further, the affine Grassmanian reduces to the coset space $\mathbb{C}(x)/\mathbb{C}$, at every point. Thus, Theorem 2.18 implies in this case Wilson’s Theorem saying that $Gr^{ad}(\mathbb{C}) = \bigsqcup_{n \geq 0} CM_n$, is a union of the Calogero-Moser spaces. Note that our proof is purely algebraic (as opposed to [W]) and totally different from that in [W].

This paper is organized as follows. In §3 we prove Theorem 2.6 by means of a $D$-module version of Kashiwara’s Theorem describing $D$-modules concentrated on a point. In §4 we re-interpret $D$-modules in terms of noncommutative geometry, and prove Theorem 2.10 and Theorem 2.14. Sections 5 and 6 contain proofs of Theorems 2.18 and 2.4, respectively. Appendix A deals with modifications that one has to introduce in the formalism of [AZ] in order to be able to work with polygraded algebras. Finally, in Appendix B we prove some technical results on the “noncommutative surface” $\mathbb{P}^1 \times_{\tau} \mathbb{P}^1$, including Serre Duality and Beilinson Spectral Sequence.

### 3 Kashiwara Theorem and De Rham Functor

In this section we prove Theorem 2.6 by reducing it to a deformed version of Kashiwara’s theorem on $D$-modules supported on a single point.

To begin the proof of Theorem 2.6 observe first that any primary decomposable sub-space $p \cdot (W \otimes_{\Gamma} \mathcal{O}[x]) \subset U \subset \frac{1}{p} \cdot (W \otimes_{\Gamma} \mathcal{O}[x])$ is determined by the subspace

$$\frac{p \cdot U}{p^2 \cdot (W \otimes_{\Gamma} \mathcal{O}[x])} \subset \frac{W \otimes_{\Gamma} \mathcal{O}[x]}{p^2 \cdot (W \otimes_{\Gamma} \mathcal{O}[x])}.$$  Similarly, any fat $D$-submodule $p \cdot (W \otimes_{\Gamma} \mathcal{O}[x]) \subset N \subset \frac{1}{p} \cdot (W \otimes_{\Gamma} \mathcal{O}[x])$ is determined by the $D$-submodule

$$\frac{p \cdot N}{p^2 \cdot (W \otimes_{\Gamma} \mathcal{O}[x])} \subset \frac{W \otimes_{\Gamma} \mathcal{O}[x]}{p^2 \cdot (W \otimes_{\Gamma} \mathcal{O}[x])}.$$  Observe further that the De Rham functor $DR$ is right exact and the homological dimension of the
category of D-modules equals 1 (see [CBH]). Moreover, since $W \otimes \Gamma D$ and $p^2 \cdot (W \otimes \Gamma D)$ are projective D-modules we get

$$
\mathcal{D} \mathcal{R} \left( \frac{p \cdot N}{p^2 \cdot (W \otimes \Gamma D)} \right) = \frac{p \cdot \mathcal{D} \mathcal{R}(N)}{p^2 \cdot (W \otimes \Gamma \mathcal{C} \Gamma[x])} \subset \frac{W \otimes \Gamma \mathcal{C} \Gamma[x]}{p^2 \cdot (W \otimes \Gamma \mathcal{C} \Gamma[x])}.
$$

Therefore, to prove Theorem 2.6 it suffices, according to the definitions of $\text{Gr}^D(W)$ and $\text{Gr}^{\text{ad}}(W)$, to show that the functor $\mathcal{D} \mathcal{R}$ induces a bijection between the following sets

(a) the set of D-submodules of $(W \otimes \Gamma D)/p^2 \cdot (W \otimes \Gamma D)$; and

(b) the set of vector subspaces in $(W \otimes \Gamma \mathcal{C} \Gamma[x])/p^2 \cdot (W \otimes \Gamma \mathcal{C} \Gamma[x])$ which are compatible with the direct sum decomposition

$$
\frac{W \otimes \Gamma \mathcal{C} \Gamma[x]}{p^2 \cdot (W \otimes \Gamma \mathcal{C} \Gamma[x])} = \bigoplus_{\mu} \frac{W \otimes \mu \Gamma \mathcal{C} \Gamma[x]}{(x - \mu)^{2k_{\mu}} W \otimes \mu \Gamma \mathcal{C} \Gamma[x]}
$$

where $p(x) = \prod_{\mu}(x - \mu)^{k_{\mu}}$ is a fixed $\Gamma$-invariant polynomial, and where we have used an identification: $\frac{1}{p} \cdot (W \otimes \mu \Gamma D)/p \cdot (W \otimes \Gamma D) \xrightarrow{\sim} (W \otimes \Gamma D)/p^2 \cdot (W \otimes \Gamma D)$ provided by multiplication by $p$ (and similarly for $\mathcal{C} \Gamma[x]$).

Next, equip the vector space $(W \otimes \Gamma \mathcal{C} \Gamma[x])/p^2 \cdot (W \otimes \Gamma \mathcal{C} \Gamma[x])$ with a new $\mathcal{C} \Gamma[x]$-module structure by requiring that $x \in \mathcal{C} \Gamma[x]$ acts on $(W \otimes \Gamma \mathcal{C} \Gamma[x])/(x - \mu)^{2k_{\mu}} W \otimes \mu \Gamma \mathcal{C} \Gamma[x]$ as multiplication by $\mu$, and $\Gamma$ acts as before. In other words, we replace the natural $x$-action by its semisimple part. Let $S(W, p^2)$ denote the result of such a semisimplification. The key observation is that a vector subspace $U \subset (W \otimes \Gamma \mathcal{C} \Gamma[x])/p^2 \cdot (W \otimes \Gamma \mathcal{C} \Gamma[x])$ is compatible with the direct sum decomposition as in (b) above, if and only if $U$ is a $\mathcal{C} \Gamma[x]$-submodule in $S(W, p^2)$. Thus Theorem 2.6 reduces to the assertion that $\mathcal{D} \mathcal{R}$ induces a bijection between

(a) the set of D-submodules of $(W \otimes \Gamma D)/p^2 \cdot (W \otimes \Gamma D)$; and

(b) the set of $\mathcal{C} \Gamma[x]$-submodules in $S(W, p^2)$.

Our next step is to show that the polynomial $p^2 = p(x)^2$ above can be replaced by a simpler polynomial. For any $\mu \in \mathcal{C}$, let $\Gamma_{\mu}$ be the stabilizer of $\mu$ in $\Gamma$, $m_{\mu}$ the order of $\Gamma/\Gamma_{\mu}$ and $p_{\mu}$ the minimal $\Gamma$-semiinvariant polynomial vanishing on $\Gamma \cdot \mu$. In other words, we put

$$
\begin{align*}
\Gamma_{\mu} & := \{1\}, \quad m_{\mu} := m, \quad p_{\mu}(x) := x^m - \mu^m \quad \text{if } \mu \neq 0; \text{ and} \\
\Gamma_{\mu} & := \Gamma, \quad m_{\mu} := 1, \quad p_{\mu}(x) := x \quad \text{if } \mu = 0.
\end{align*}
$$

Then any $\Gamma$-invariant polynomial $p(x)$ can be factored as $p(x) = \prod_{\mu \in \mathcal{C}/\Gamma} p_{\mu}(x)^{s_{\mu}}$. This factorization induces direct sum decompositions:

$$
\frac{W \otimes \Gamma D}{p^2 \cdot (W \otimes \Gamma D)} = \bigoplus_{\mu \in \mathcal{C}/\Gamma} \frac{W \otimes \mu \Gamma D}{p^{2s_{\mu}} \cdot (W \otimes \mu \Gamma D)}; \quad S(W, p^2) = \bigoplus_{\mu \in \mathcal{C}/\Gamma} S(W, p^{2s_{\mu}}).
$$

The following is clear
**Lemma 3.1** For any $D$-submodule $N \subset \bigoplus_{\mu \in \mathbb{C}/T_0} W \otimes_{\tau} D$ we have

$$N = \bigoplus_{\mu \in \mathbb{C}/T_0} \left( N \cap \frac{W \otimes_{\tau} D}{p^{2s_{\mu}}_{\mu} \cdot (W \otimes_{\tau} D)} \right).$$

Due to the above Lemma we may (and will) assume without any loss of generality that $p(x) = p_{\mu}(x)^{2s_{\mu}}$, for some fixed $\mu \in \mathbb{C}$ and some $s_{\mu} = 1, 2, \ldots$. Further, we have $S(W, p^{2s_{\mu}}_{\mu}) \cong S(W, p_{\mu})^{\otimes 2s_{\mu}}$, and this space is, in effect, a module over the quotient algebra $\mathbb{C}\Gamma[x]/\mu p_{\mu} \mathbb{C}\Gamma[x]$. The set of submodules in $S(W, p^{2s_{\mu}}_{\mu})$ may be therefore described by the following result, which is proved by a straightforward computation.

**Lemma 3.2** (a) The correspondence $U \mapsto U \otimes_{\Gamma_{\mu}} \left( \mathbb{C}\Gamma[x]/(x - \mu)\mathbb{C}\Gamma[x] \right)$ establishes a Morita equivalence between the category $\text{Rep}(\Gamma_{\mu})$ of finite-dimensional representations of $\Gamma_{\mu}$ and the category of finite-dimensional $\mathbb{C}\Gamma[x]/\mu \mathbb{C}\Gamma[x]$-modules.

(b) The $\Gamma_{\mu}$-module $U(W, p^{2s_{\mu}}_{\mu})$ corresponding to $S(W, p^{2s_{\mu}}_{\mu})$ via this equivalence, is equal to $W^{\otimes 2s_{\mu}}$, viewed as a vector space (=module over $\Gamma_{\mu} = \{1\}$) if $\mu \neq 0$ and as a $\Gamma$-module $W \otimes_{\mathbb{C}} (\mathbb{C}[x]/x^{2s_{\mu}}\mathbb{C}[x])$, if $\mu = 0$.

Thus, to prove Theorem 2.6 we have to establish a bijection between the following sets

(a) The set of all $D$-submodules of $W \otimes_{\tau} D / p^{2s_{\mu}}_{\mu} \cdot (W \otimes_{\tau} D)$; and

(b) The set of all $\Gamma_{\mu}$-submodules of $U(W, p^{2s_{\mu}}_{\mu})$.

To that end, we introduce the following

**Definition 3.3** Denote by $\text{mod}_{\Gamma_{\mu}}(D)$ the category of all finitely generated $D$-modules $\mathcal{M}$, such that $p_{\mu}(x)$ acts locally nilpotently on $\mathcal{M}$.

If $\mathcal{M}$ is an object in $\text{mod}_{\Gamma_{\mu}}(D)$ then

$$K_{\mu}(\mathcal{M}) := \text{Ker}(x - \mu) \subset \mathcal{M}$$

is a $\Gamma_{\mu}$-module. Moreover, it is clear that the assignment $\mathcal{M} \mapsto K_{\mu}(\mathcal{M})$ gives a functor $K_{\mu} : \text{mod}_{\Gamma_{\mu}}(D) \to \text{Rep}(\Gamma_{\mu})$. Further, consider the induction functor

$$I_{\mu} : \text{Rep}(\Gamma_{\mu}) \to \text{mod}_{\Gamma_{\mu}}(D), \quad U \mapsto U \otimes_{\mathbb{C}\Gamma_{\mu}[x]} D,$$

where $\mathbb{C}\Gamma_{\mu}[x] = \mathbb{C}\Gamma_{\mu} \otimes \mathbb{C}[x]$ and the $\mathbb{C}\Gamma_{\mu}[x]$-module structure on $U \in \text{Rep}(\Gamma_{\mu})$ is given by the standard action of $\Gamma_{\mu}$ and the action of $x$ by the $\mu$-multiplication.

The following theorem is a deformed (and equivariant) analogue of a well-known result of Kashiwara saying that any $D$-module concentrated at a point is the $D$-module direct image of a vector space (= $D$-module on that point).

**Theorem 3.4** (Kashiwara Theorem) Assume that the element $\tau \in \mathbb{C}\Gamma$ involved in the definition of $D$ is generic in the sense of Definition 2.3. Then the functors $K_{\mu}$ and $I_{\mu}$ give mutually inverse equivalences between the categories $\text{mod}_{\Gamma_{\mu}}(D)$ and $\text{Rep}(\Gamma_{\mu})$. 

12
Before we prove this theorem we record a few consequences of the condition: \( \tau \) is generic. For any \( k = 1, 2, \ldots \), and any integers \( 0 \leq a \leq b \), we define elements \( \tau^{(k)}, \tau_{[a,b]} \in \mathbb{C}\Gamma \) by the equations

\[
y^k \cdot \tau = \tau^{(k)} \cdot y^k, \quad \tau_{[a,b]} = \sum_{k=a}^{b} \tau^{(k)}.
\]

The definition yields

\[
y \cdot \tau_{[a,b]} = \tau_{[a+1,b+1]} \cdot y, \quad \text{and} \quad x \cdot \tau_{[a,b]} = \tau_{[a-1,b-1]} \cdot x.
\]  \hspace{1cm} (3.5)

**Lemma 3.6** (a) The element \( \tau \) is generic if and only if for all \( a \leq b \) the element \( \tau_{[a,b]} \in \mathbb{C}\Gamma \) is invertible. Furthermore, in this case for any \( a \in \mathbb{Z} \) the element \( \tau_{[a,a+m-1]} \) acts by a constant (independent of \( a \)) in any representation of \( \Gamma \).

(b) If \( \tau \) is generic then for any \( \mu \in \mathbb{C} \) and all \( a \leq b \) the element \( \sum_{i=a}^{b} \tau_{[i,i+m\mu-1]} \in \mathbb{C}\Gamma \) is invertible.

(c) We have \([y, p_\mu(x)] = \frac{\tau_{[0,m\mu-1]}}{m\mu} \cdot p'_\mu(x)\).

**Proof:** To prove (a) recall (cf. e.g. [CBH]) that McKay correspondence associates to the cyclic group \( \mathbb{Z}/m\mathbb{Z} = \mu_m \) the affine Dynkin graph \( \tilde{A}_{m-1} \). Using an explicit expression for the roots it is easy to deduce the assertion.

To prove (b) note that if \( \mu \neq 0 \) then \( m\mu = m \) and the sum in question equals \(|\tau| \cdot (b - a + 1)\), where \(|\tau| \) is the constant of part (a). Hence this sum is invertible. If \( \mu = 0 \) then \( m\mu = 1 \) and we have: \( \sum_{i=a}^{b} \tau_{[i,i+m\mu-1]} = \tau_{[a,b]} \). Hence this element is invertible also.

Part (c) is proved by a direct computation. \( \Box \)

**Proof of Theorem 3.4:** It follows from the definitions of \( K_\mu \) and \( I_\mu \) that the functor \( K_\mu \) is right adjoint to \( I_\mu \). This gives canonical adjunction morphisms \( U \rightarrow K_\mu(I_\mu(U)) \) and \( I_\mu(K_\mu(M)) \rightarrow M \) for any \( M \in \text{Ob} (\text{mod}_{\Gamma, \mu} \mathcal{D}) \) and \( U \in \text{Ob}(\text{Rep} \Gamma_\mu) \).

To show that \( I_\mu(K_\mu(M)) \rightarrow M \) is an isomorphism, set \( \mathcal{M}_k := \text{Ker} p_\mu(x)^{k+1} \subset \mathcal{M} \) and write \( \mathcal{M}_k := \mathcal{M}_k/\mathcal{M}_{k-1} \), for short. On \( \oplus_k \mathcal{M}_k \), we have the following structure.

First, it is clear that the increasing filtration \( \{ \mathcal{M}_k \}_{k=0,1,\ldots} \) is stable under the action of the subalgebra \( \Gamma[x] \subset \mathcal{D} \), hence each \( \mathcal{M}_k \) is a \( \Gamma[x] \)-module. Further, multiplication by \( p_\mu(x) \) takes \( \mathcal{M}_k \) to \( \mathcal{M}_{k-1} \) and thus induces a map \( p : \mathcal{M}_k \rightarrow \mathcal{M}_{k-1} \). Moreover, the action of \( y \) moves \( \mathcal{M}_k \) to \( \mathcal{M}_{k+1} \) and thus induces a map \( \mathcal{M}_k \rightarrow \mathcal{M}_{k+1} \). Finally, it is clear that the map \( p \) is an embedding of \( \Gamma[x] \)-modules, while \( y \) is a morphism of \( \mathbb{C}[x] \)-modules, and \( p_\mu(x) \) is an isomorphism (because \( p_\mu(x) \) and \( p'_\mu(x) \) are coprime). Let us prove by induction in \( k \) that

\[
(p \cdot y)|_{\mathcal{M}_k} = - \sum_{i=0}^{k} \frac{\tau_{[i,i+m\mu-1]}}{m\mu} \cdot p'_\mu(x), \quad (y \cdot p)|_{\mathcal{M}_{k+1}} = - \sum_{i=1}^{k+1} \frac{\tau_{[i,i+m\mu-1]}}{m\mu} \cdot p'_\mu(x). \hspace{1cm} (3.7)
\]
Applying Lemma 3.6(b), we calculate

\[ p \cdot y \cdot a = y \cdot p \cdot a - [y, p] \cdot a = \sum_{i=1}^{k} \frac{\tau_{[i,i+m-1]}(y,p)}{m_i} \cdot p'_{i}a - \sum_{i=0}^{k} \frac{\tau_{[0,m_i]}(y,p)}{m_i} \cdot p'_{i}a = \sum_{i=1}^{k} \frac{\tau_{[i,i+m-1]}(y,p)}{m_i} \cdot p'_{i}a. \]

Note that since \( \tau \) is generic it follows from Lemma 3.6(b) that the map \( p \cdot y : M_k \to M_k \) is a bijection. On the other hand, \( p \) is injective by definition. Hence \( y \) gives an isomorphism \( M_k \overset{\sim}{\to} M_{k+1} \). It follows that, for any \( b \in M_{k+1} \) there exists an \( a \in M_k \) such that \( b = y \cdot a \). Applying Lemma 3.6(b), we calculate

\[ y \cdot p \cdot b = y \cdot p \cdot y \cdot a = -y \cdot \sum_{i=0}^{k} \frac{\tau_{[i,i+m-1]}(y,p)}{m_i} \cdot p'_{i}a = -\sum_{i=0}^{k} \frac{\tau_{[i,i+m-1]}(y,p)}{m_i} \cdot y \cdot p'_{i}a = -\sum_{i=1}^{k+1} \frac{\tau_{[i,i+m-1]}(y,p)}{m_i} \cdot p'_{i} \cdot y = -\sum_{i=1}^{k+1} \frac{\tau_{[i,i+m-1]}(y,p)}{m_i} \cdot p'_{i} \cdot b \]

(In the third equality we use the fact that the operators \( y : M_k \to M_{k+1} \) and \( p'_{j}(x) : M_k \to M_k \) commute, because their commutator on \( \mathcal{M} \) takes \( M_k \) to \( M_k \), hence induces the zero map \( M_k \to M_{k+1} \)). Thus we have proved (3.7) for any \( k \) and, moreover, we have shown along the way that for any \( k \) the map \( y : M_k \to M_{k+1} \) is an isomorphism. This means that the action of the subalgebra \( \mathbb{C}[y] \subset D \) gives an isomorphism \( \mathcal{M} \cong M_0 \otimes \mathbb{C}[y] \). Further, we have: \( M_0 = \text{Ker} \, p_\mu(x) \) and \( K_\mu(\mathcal{M}) = \text{Ker}(x - \mu) \). We use the equalities:

\[ \text{Ker} \, p_\mu(x) = \bigoplus_{\gamma \in \Gamma / \Gamma_\mu} \text{Ker}(\gamma(x - \mu) \gamma^{-1}) = \bigoplus_{\gamma \in \Gamma / \Gamma_\mu} (\text{Ker}(x - \mu) \gamma^{-1}) = \text{Ker}(x - \mu) \otimes_{\mathbb{C}[\Gamma_\mu]} \mathbb{C}[\Gamma] \]

to conclude that \( I_\mu(K_\mu(\mathcal{M})) \to \mathcal{M} \) is an isomorphism.

To show that the canonical morphism \( f : \mathcal{U} \to K_\mu(I_\mu(\mathcal{U})) \) is an isomorphism, first note that it is clearly injective. Hence we have an exact sequence

\[ 0 \to \mathcal{U} \to K_\mu(I_\mu(\mathcal{U})) \to \mathcal{U}' \to 0 \quad \text{where} \quad \mathcal{U}' := \text{Coker}(f). \]

On the other hand, the functor \( I_\mu \) is exact since \( D \) is a flat \( \Gamma[x] \)-module. Hence, applying the functor \( I_\mu(-) \), we obtain an exact sequence

\[ 0 \to I_\mu(\mathcal{U}) \xrightarrow{\alpha} I_\mu(K_\mu(I_\mu(\mathcal{U}))) \to I_\mu(\mathcal{U}') \to 0. \]

The argument of the first part of the proof applied to the \( D \)-module \( I_\mu(\mathcal{U}) \), shows that the morphism \( \alpha \) above is an isomorphism. Hence \( I_\mu(\mathcal{U}') = 0 \). But this clearly yields \( \mathcal{U}' = 0 \). Thus, the map \( \mathcal{U} \to K_\mu(I_\mu(\mathcal{U})) \) is an isomorphism, and Theorem 3.4 follows. \(\square\)

\textit{End of proof of Theorem 2.6:} The De Rham functor \( DR \) restricted to the set of submodules in \((W \otimes \Gamma, D)/p_{2\mu} \cdot (W \otimes \Gamma, D)\) can be factored as a composition of the equivalence \( K_\mu : \text{mod}_{\Gamma_\mu} D \to \text{Rep} \Gamma_\mu \), and the Morita equivalence \( \text{Rep} \Gamma_\mu \to \text{mod} (\mathbb{C}[\Gamma[x]/p_\mu \cdot \mathbb{C}[\Gamma[x]]) \) of Lemma 3.2 (b). Hence it is an equivalence as well. By a straightforward (but a bit tedious) computation one deduces that

\[ K_\mu((W \otimes \Gamma, D)/p_{2\mu} \cdot (W \otimes \Gamma, D)) \simeq \mathcal{U}(W, p_{2\mu} \cdot \tau) \]

14
where \( \mathcal{U}(W, p_{\mu}^{2\nu}) \) is given by Lemma 3.2 (b). This implies Theorem 2.6, as we have seen in the first half of this section.

Finally, one can check that the map \( \text{Diff}_r \) defined just below the statement of Theorem 2.6 is in effect the inverse bijection: \( \text{Gr}^{ad}(W) \rightarrow \text{Gr}^{D}(W) \). \( \square \)

4 D-module Grassmanian and Sheaves on \( \mathbb{P}^1 \times_{\tau} \mathbb{P}^1 \)

Recall the bigraded algebra \( \mathbb{Q} \) defined in (2.7). Let \( \text{gr}^2(\mathbb{Q}) \) be the category of finitely generated bigraded right \( \mathbb{Q} \)-modules \( M = \bigoplus M_{i,j} \). Let \( \text{tor}^2(\mathbb{Q}) \) denote its Serre subcategory formed by all modules \( M \) such that there exists a pair \( (i_0, j_0) \) such that for any \( i > i_0 \) and \( j > j_0 \) we have \( M_{i,j} = 0 \). Consider the quotient category \( \text{ogr}^2(\mathbb{Q}) = \text{gr}^2(\mathbb{Q})/\text{tor}^2(\mathbb{Q}) \).

The category \( \text{ogr}^2(\mathbb{Q}) \) will be viewed as the category of coherent sheaves on a noncommutative scheme \( \mathbb{P}^1 \times_{\tau} \mathbb{P}^1 \) (see Appendix A for details). Thus, by definition, we put \( \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) := \text{ogr}^2(\mathbb{Q}) \), and we write \( \pi : \text{gr}^2(\mathbb{Q}) \rightarrow \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) \) for the canonical projection functor.

The isomorphism \( \mathcal{D} \cong \mathbb{Q}/((z - 1)\mathbb{Q} + (w - 1)\mathbb{Q}) \) gives rise to a ‘restriction’ functor \( j^* : \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) \rightarrow \text{mod}(\mathcal{D}) \), where the direct limit is taken with respect to the maps \( M_{k,l} \rightarrow M_{k+1,l} \) and \( M_{k,l} \rightarrow M_{k,l+1} \), induced by multiplication by \( z \) and \( w \), respectively.

There are canonical isomorphisms

\[
\mathbb{Q}/z\mathbb{Q} \cong (\mathbb{C}[x] \otimes \mathbb{C}[y,w])/\Gamma, \quad \mathbb{Q}/w\mathbb{Q} \cong (\mathbb{C}[x,z] \otimes \mathbb{C}[y])/\Gamma.
\]

Thus, we obtain the following equivalences of categories: \( \text{ogr}^2(\mathbb{Q}/z\mathbb{Q}) \cong \text{ogr}(\mathbb{C}[x,y,xw]/\Gamma) \), and \( \text{ogr}^2(\mathbb{Q}/w\mathbb{Q}) \cong \text{ogr}(\mathbb{C}[xy,yz]/\Gamma) \), (see Appendix, Corollary 7.5). The two categories on the right can be viewed as the categories of \( \Gamma \)-equivariant coherent sheaves on the ordinary projective line \( \mathbb{P}^1 \). We denote the corresponding copies of \( \mathbb{P}^1 \) by \( \mathbb{P}^1_z \) and \( \mathbb{P}^1_w \), respectively. We have the corresponding push forward and pull back functors:

\[
(i_z)_* : \text{coh}(\mathbb{P}^1_z) \rightarrow \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1), \quad i_z^* : \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) \rightarrow \text{coh}(\mathbb{P}^1_z),
\]

\[
(i_w)_* : \text{coh}(\mathbb{P}^1_w) \rightarrow \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1), \quad i_w^* : \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) \rightarrow \text{coh}(\mathbb{P}^1_w).
\]

Let \( L^1i_*^* \) stand for the first derive functor.

Given a \( \mathcal{D} \)-module \( \mathcal{M} \), we let its support be the support of \( \mathcal{M} \), viewed as a module over the subalgebra \( \mathbb{C}[x] \subset \mathcal{D} \) (more precisely, the union of supports of all \( \mathbb{C}[x] \)-finitely generated submodules in \( \mathcal{M} \)).

**Definition 4.1** (i) Let \( \text{Quot}^{\mathbb{P}^1 \times_{\tau} \mathbb{P}^1}(W) \) be the category of all surjections \( W \otimes_{\tau} \mathcal{O} \rightarrow F \) in the category \( \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) \) such that

\[
i_z^* F = L^1i_*^* F = 0 \quad \text{and} \quad L^1i_w^* F = 0. \tag{4.2}
\]

(ii) Let \( \text{Quot}^{\mathcal{D}}(W) \) be the category of all \( \mathcal{D} \)-module surjections \( W \otimes_{\tau} \mathcal{D} \rightarrow \mathcal{M} \), such that \( \mathcal{M} \) has zero-dimensional support.
Theorem 4.3 The functor $j^*$ takes any object of the category $\text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W)$ to an object of $\text{Quot}^D(W)$ and, moreover, gives an equivalence $\text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W) \sim \text{Quot}^D(W)$.

Proof: First note that $j^*$ is exact, being a direct limit functor. Thus, to show that $j^*$ takes $\text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W)$ to $\text{Quot}^D(W)$ it suffices to show that for any object of $\text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W)$ of the form $W \otimes \mathcal{O} \rightarrow F$, the $D$-module $j^*F$ has zero-dimensional support. Indeed, let $F = \pi(M)$ where $M = \oplus M_{k,l}$. Then conditions $i^*_x F = L^1 i^*_y F = 0$ imply that $z$-multiplication gives an isomorphism $M_{k,l} \sim \rightarrow M_{k+1,l}$ for $k, l \gg 0$. Since $M$ is finitely generated we can choose $k_0$ and $l_0$ such that $\oplus_{k \geq k_0, l \geq l_0} M_{k,l}$ is generated by $M_{k_0,l_0}$ and such that $z: M_{k,l} \rightarrow M_{k+1,l}$ is an isomorphism for all $k \geq k_0, l \geq l_0$. Let $p$ be the characteristic polynomial of the operator $z^{-1} x: M_{k_0,l_0} \rightarrow M_{k_0,l_0}$. Then it is easy to see that $p(x)$ acts locally nilpotently on $j^*F = \lim \rightarrow M_{k,l}$ and hence $j^*$ indeed defines a functor $j^*: \text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W) \rightarrow \text{Quot}^D(W)$.

The assertion that $j^*$ is an equivalence will be proved by constructing a quasi-inverse functor $j_*$. To that end, let $\psi: W \otimes \mathcal{O} \rightarrow \mathcal{M}$ be an object of $\text{Quot}^D(W)$. Since $\mathcal{M}$ has zero-dimensional support there exists a $\Gamma$-invariant polynomial $p(x)$ that acts by zero on the subspace $\psi(W \otimes \mathcal{O} \Gamma) \subset \mathcal{M}$. It is clear that we have $\psi(W \otimes p(x)) = 0$, hence

$$\psi(p \cdot (W \otimes \mathcal{O})) = 0 \quad \text{(4.4)}$$

Let $D_{k,l}$ be the natural increasing bifiltration of $D$ (induced by the bigrading of $\mathcal{Q}$) and

$$\mathcal{M}_{k,l} = \psi(W \otimes D_{k,l}) \subset \mathcal{M},$$

the induced bifiltration of $\mathcal{M}$. It follows from (4.4) that this bifiltration stabilizes with respect to the first index when $k \geq d = \deg p(x)$, that is we have

$$\mathcal{M}_{k,l} = \psi(W \otimes D_{k,l}) = \psi(W \otimes D_{d,l}) = \mathcal{M}_{d,l} \subset \mathcal{M} \quad \text{for } k \geq d \text{ and all } l.$$ 

It follows from the definition that the bifiltration $\mathcal{M}_{k,l}$ is compatible with the bifiltration on $D$. Moreover, it is clearly increasing, finitely generated and exhaustive (because $\psi$ is surjective). Hence $M = \oplus_{k,l} \mathcal{M}_{k,l}$ is a finitely generated $\mathcal{Q}$-module, where the action on $M$ of $x$-generators and $y$-generators of $\mathcal{Q}$ is given by the $x$ and $y$ multiplication maps $M_{k,l} \rightarrow M_{k+1,l}$ and $M_{k,l} \rightarrow M_{k,l+1}$ respectively, and the action of $z$ and $w$ generators is given by the natural embeddings $M_{k,l} \hookrightarrow M_{k+1,l}$ and $M_{k,l} \hookrightarrow M_{k,l+1}$ respectively.

Consider $F = \pi(M)$, a coherent sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. It follows from the definition of $F$ that the $z$-multiplication map $F \rightarrow F(1,0)$ is an isomorphism (since $\mathcal{M}_{k,l} = \mathcal{M}_{k+1,l}$ for $k \geq d$), hence $i^*_z F = L^1 i^*_y F = 0$. On the other hand, the $w$-multiplication map $F \rightarrow F(0,1)$ is an embedding (because $\mathcal{M}_{k,l} \subset \mathcal{M}_{k,l+1}$), hence $L^1 i^*_w F = 0$. Finally note that the map $\psi$ is compatible with the bifiltrations on $W \otimes \mathcal{O}$ and $\mathcal{M}$, hence it gives rise to a map $\psi: W \otimes \mathcal{O} \rightarrow F$ of coherent sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, this map is surjective by definition, hence it gives an object of $\text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W)$. Finally, it is easy to show that this way we obtain a functor $j_*: \text{Quot}^D(W) \rightarrow \text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W)$.

We show that $j_*$ and $j^*$ are quasi-inverse. Let $W \otimes \mathcal{O} \xrightarrow{\psi} F$ be an object of the category $\text{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}(W)$ and let $M = \oplus M_{k,l}$, where $M_{k,l} = H^0(\mathbb{P}^1 \times \mathbb{P}^1, F(k,l))$. Then
$M$ is a bigraded $\mathbb{Q}$-module and it is clear that $\pi(M) = F$. Note that we have exact sequences

$$
0 \to i_zL^1i_z^*F(k + 1, l) \to F(k, l) \to F(k + 1, l) \to i_zi_z^*F(k + 1, l) \to 0,
$$

$$
0 \to i_wL^1i_w^*F(k, l + 1) \to F(k, l) \to F(k, l + 1) \to i_wi_w^*F(k, l + 1) \to 0.
$$

Moreover, applying (4.2) we get

$$
L^1i_z^*F(k + 1, l) \cong (L^1i_z^*F)(l) = 0, \quad i_z^*F(k + 1, l) \cong (i_z^*F)(l) = 0,
$$

$$
L^1i_w^*F(k, l + 1) \cong (L^1i_w^*F)(k) = 0.
$$

Combining these isomorphisms with the above exact sequences and with the definition of $M_{k,l}$, we see that the maps $M_{k,l} \to M_{k+1,l}$ and $M_{k,l} \to M_{k,l+1}$ are an isomorphism and an embedding respectively. Therefore $j^*F = \bigcup_k M_{k,l}$ for any $k$. On the other hand we have $D_{k,l} = H^0(\mathbb{P}^1 \times \tau, \mathcal{O}(k,l))$ by definition and it is clear that the map $j^*\psi$ sends $W \otimes \tau D_{k,l}$ to $M_{k,l}$ and coincides there with the map $H^0(\psi(k,l))$. Thus to show that $j_{k,l}(j^*F) \cong F$ it suffices to show that this map is surjective for all $k$ and $l$ sufficiently large. But this is nothing but the definition of the map $\psi$ being a surjection in the category $\text{coh}(\mathbb{P}^1 \times \tau \mathbb{P}^1)$.

Now assume that $W \otimes \tau \to \mathcal{M}$ is an object of $\text{Quot}^D(W)$. Then by definition of $j_{k,l}$ we have $\mathcal{M}_{k+1,l} = \mathcal{M}_{k,l}$ for $k \geq d$ and all $l$, and when $k \geq d$ is fixed the filtration $\mathcal{M}_{k,l}$ of $\mathcal{M}$ is exhaustive. Hence $\mathcal{M} = \lim_k \mathcal{M}_{k,l}$, that is $j^*j_{k,l}(\mathcal{M}) = \mathcal{M}$.

Now we give a more rigorous version of Definition 2.9. Let $E$ be a coherent sheaf on $\mathbb{P}^1 \times \tau \mathbb{P}^1$ such that $i_*^*E \cong W \otimes \tau \mathcal{O}_{\mathbb{P}^1}$. Recall that $\epsilon$ denotes a fixed primitive character: $\Gamma = \mu_m \to \mathbb{C}^*$.

**Definition 4.5**

(i) *We say that the sheaf $E$ is trivialized in a neighborhood of $\mathbb{P}^1_\tau$ if we are given embeddings*

$$
(W \otimes \epsilon^{-n})\otimes_\tau \mathcal{O}(-n, 0) \overset{\phi}{\to} E \overset{\psi}{\to} (W \otimes \epsilon^n)\otimes_\tau \mathcal{O}(n, 0)
$$

*such that the composite*

$$
\psi \phi \in \text{Hom}((W \otimes \epsilon^{-n})\otimes_\tau \mathcal{O}(-n, 0), (W \otimes \epsilon^n)\otimes_\tau \mathcal{O}(n, 0)) \cong \text{Hom}_\Gamma(W, (W \otimes \epsilon^{2n})\otimes_\tau \mathcal{Q}_{2n,0})
$$

*equals multiplication by $P(x, z)^2$, where $P(x, z) \in \mathbb{C}[x, z]$ is a $\Gamma$-semi-invariant homogeneous polynomial of degree $n$, such that $P(1,0) = 1$.*

(ii) *We call two trivializations $(\phi, \psi)$ and $(\phi', \psi')$ of the sheaf $E$ equivalent if there exists a pair of $\Gamma$-semi-invariant homogeneous polynomials $\tilde{q}(x, z)$ and $\tilde{q}'(x, z)$ such that $\tilde{q}(1,0) = \tilde{q}'(1,0) = 1$ and the following diagram commutes*

\[\begin{array}{ccc}
(W \otimes \epsilon^{-n})\otimes_\tau \mathcal{O}(-n, 0) & \overset{\phi}{\longrightarrow} & E \\
\downarrow \tilde{q} & & \downarrow \tilde{q} \\
(W \otimes \epsilon^{-n''})\otimes_\tau \mathcal{O}(-n'', 0) & \overset{\phi'}{\longrightarrow} & (W \otimes \epsilon^{n''})\otimes_\tau \mathcal{O}(n'', 0) \\
\downarrow \tilde{q}' & & \downarrow \tilde{q}' \\
(W \otimes \epsilon^{-n''})\otimes_\tau \mathcal{O}(-n'', 0) & \overset{\phi'}{\longrightarrow} & E \\
\downarrow \tilde{q}' & & \downarrow \tilde{q}' \\
(W \otimes \epsilon^{n''})\otimes_\tau \mathcal{O}(n'', 0) & \overset{\psi'}{\longrightarrow} & (W \otimes \epsilon^{n''})\otimes_\tau \mathcal{O}(n', 0)
\end{array}\]
Remark. We can always replace a trivialization by an equivalent one with \( n \equiv 0 \mod m \), thus getting rid of \( e^n \) and \( e^{-n} \) factors in the definition and making the polynomial \( P(x, z) \) \( \Gamma \)-invariant.

Definition 4.6 Let \( \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \) be the set of all equivalence classes of trivializations in a neighborhood of \( \mathbb{P}^1_z \) of coherent sheaves \( E \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) such that \( i^*_z E \cong W \otimes_{\tau} \mathcal{O}_{\mathbb{P}^1} \).

Proof of Theorem 2.10 (bijection between \( \text{Gr}^{\mathbb{P}^1}(W) \) and \( \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \)): For any \( \Gamma \)-invariant polynomial \( p(x) = \sum_{k=0}^{d} a_k x^k \), let \( \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \subset \text{Gr}^{\mathbb{P}^1}(W) \) denote the set of all sheaves admitting a trivialization \( (\phi, \psi) \) with \( \psi \circ \phi = P(x, z)^2 \), where \( P(x, z) \) is the homogenization of \( p(x) \), that is \( P(x, z) = \sum_{k=0}^{d} a_k x^k z^{d-k} \). Then we have

\[
\text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) = \bigcup_{p(x)} \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W).
\]

We will show that the functor \( j^* \) induces a bijection between \( \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \subset \text{Gr}^{\mathbb{P}^1}(W) \) and the subset \( \text{Gr}^{\mathbb{P}^1}(W) \subset \text{Gr}^{\mathbb{P}^1}(W) \) formed by all \( \mathbb{D} \)-submodules (or, equivalently, quotient modules) of \( \frac{1}{p} \cdot (W \otimes_{\tau} \mathcal{D})/p \cdot (W \otimes_{\tau} \mathcal{D}) \).

Let \( \text{Quot}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \subset \text{Quot}^{\mathbb{P}^1}(W) \) be the subset formed by surjections: \( W \otimes_{\tau} \mathcal{O} \twoheadrightarrow F \), which send the image of the map: \( W \otimes_{\tau} \mathcal{O}(-2n, 0) \xrightarrow{P(x, z)^2} W \otimes_{\tau} \mathcal{O} \) to zero in \( F \). We may identify the set \( \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \) with \( \text{Quot}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \) via the assignment

\[
\left\{ W \otimes_{\tau} \mathcal{O}(-n, 0) \xrightarrow{\phi} E \xrightarrow{\psi} W \otimes_{\tau} \mathcal{O}(n, 0) \right\} \mapsto \left\{ W \otimes_{\tau} \mathcal{O} \twoheadrightarrow \text{Coker} \left( E(-n, 0) \xrightarrow{\phi(-n, 0)} W \otimes_{\tau} \mathcal{O} \right) \right\}.
\]

Hence, Theorem 4.3 implies that the functor \( j^* \) provides an identification of the set \( \text{Quot}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \) with the subset \( \text{Quot}^{\mathbb{P}^1}(W) \subset \text{Quot}^{\mathbb{P}^1}(W) \) formed by all surjections \( W \otimes_{\tau} \mathcal{D} \twoheadrightarrow \mathcal{M} \) which send \( p^2 \cdot (W \otimes_{\tau} \mathcal{D}) \) to zero (in \( \mathcal{M} \)). On the other hand, to any object

\[
W \otimes_{\tau} \mathcal{D} \twoheadrightarrow \text{Coker} \left( p \cdot N \xrightarrow{p\psi} W \otimes_{\tau} \mathcal{D} \right).
\]

in \( \text{Gr}^{\mathbb{P}^1}(W) \) we associate the quotient

\[
W \otimes_{\tau} \mathcal{D} \twoheadrightarrow \text{Coker} \left( p \cdot N \xrightarrow{p\psi} W \otimes_{\tau} \mathcal{D} \right).
\]

This yields an identification of \( \text{Quot}^{\mathbb{P}^1}(W) \) with \( \text{Gr}^{\mathbb{P}^1}(W) \). Therefore, we get a bijection \( \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \cong \text{Gr}^{\mathbb{P}^1}(W) \).

Note that, for any polynomial \( p(x) \) dividing \( q(x) \), the map \( j^* \) commutes with the natural embeddings \( \text{Gr}^{\mathbb{P}^1 \times \mathbb{P}^1}(W) \hookrightarrow \text{Gr}^{\mathbb{P}^1}(W) \) and \( \text{Gr}^{\mathbb{P}^1}(W) \hookrightarrow \text{Gr}^{\mathbb{P}^1}(W) \). The assertion of Theorem 2.10 follows. \( \square \)

Finally we prove Theorem 2.14. Recall that the pull back functor \( i^*_w \) takes any sheaf trivialized in a neighborhood of \( \mathbb{P}^1_z \) to a sheaf on \( \mathbb{P}^1_w \) trivialized in a neighborhood of the point \( P \), which is the same as a \( \Gamma \)-equivariant sheaf on the ordinary projective line \( \mathbb{P}^1 \).
Moreover, the category $qgr$ evidently $G_P$-embeddings. There is a canonical isomorphism of functors $P\to A$.

A coherent sheaf $E$ is given on the line $\mathbb{P}^1_w$ by the equation $w = 0$ and on the line $\mathbb{P}^1_z$ by the equation $z = 0$. Let $i_P^*: \{P\} \to \mathbb{P}^1_w$ and $i_z^*: \{P\} \to \mathbb{P}^1_z$ denote the embeddings. There is a canonical isomorphism of functors

$$(i_P^*)^*\circ i_w^* \cong (i_P^*)^*\circ i_z^*: \text{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1) \to \text{coh}(P) = \text{Rep}(\Gamma).$$

Let $W, V$ be a pair of $\Gamma$-modules as in Definition 2.17.

**Definition 5.1** A coherent sheaf $E$ on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ is called $W$-framed provided it is equipped with two isomorphisms $i_z^*E \cong W \otimes_\tau O_{\mathbb{P}^1_w}$ and $i_w^*E \cong W \otimes_\tau O_{\mathbb{P}^1_z}$, which agree at the point $P$.

Let $\mathcal{M}_{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(V, W)$ denote the set of isomorphism classes of $W$-framed torsion free sheaves $E$ (for the definition of ‘torsion free’ see [BGK, Def. 1.1.4]) on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ such that $H^1(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1, -1)) \cong V$.

**Theorem 5.2** The set $\mathcal{M}_{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(V, W)$ is in a natural bijection with the quiver variety $\mathcal{M}_{\mathbb{P}^1}(V, W)$.

**Sketch of proof:** The proof is essentially the same as that of Theorem 1.3.10 of [BGK], §4. So, we will skip most of the details and only mention the points that are different from [BGK], §4.

5 Monads and Quiver Varieties

The two lines $\mathbb{P}^1_w$ and $\mathbb{P}^1_z$ in $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ intersect at the point $P$, corresponding to the quotient algebra

$$\mathbb{Q}/(z\mathbb{Q} + w\mathbb{Q}) \cong (\mathbb{C}[x] \otimes \mathbb{C}[y])/\Gamma.$$
The first difference is that, in the present situation, the monad representing a framed sheaf has a form slightly different from the one used in [BGK]. Specifically, for any point \((B_1, B_2, I, J)\) of the quiver variety, our monad is now given by the following complex

\[
0 \to V \otimes \mathcal{O}(0, -1) \oplus W \otimes \mathcal{O} \to (V \otimes e^{-1}) \otimes \mathcal{O}(0, -1) \oplus (V \otimes \epsilon) \otimes \mathcal{O}(0, -1) \to V \otimes \mathcal{O} \to 0,
\]

and furthermore there is a canonical exact sequence

\[
a = (B_1 z - x, B_2 w - y, Jzw), \quad b = -(B_2 w - y, B_1 z - x, I).
\]

Second, whenever the functor \(i^*\) (the restriction to the line at infinity in \(\mathbb{P}^2\)) was used in [BGK], it should be replaced by a pair of functors \(i_x^*, i_w^*\).

Third, ([BGK], Lemma 4.2.12) should be replaced by the following isomorphisms

\[
\begin{align*}
H^0(\mathbb{P}^1 \times \mathbb{P}^1, E(0, -1)) &= H^0(\mathbb{P}^1 \times \mathbb{P}^1, E(0, -1)) = H^0(\mathbb{P}^1 \times \mathbb{P}^1, E(0, -1)) = 0, \\
H^2(\mathbb{P}^1 \times \mathbb{P}^1, E) &= H^2(\mathbb{P}^1 \times \mathbb{P}^1, E(0, -1)) = H^2(\mathbb{P}^1 \times \mathbb{P}^1, E(0, -1)) = 0,
\end{align*}
\]

and furthermore there is a canonical exact sequence

\[
0 \to H^0(\mathbb{P}^1 \times \mathbb{P}^1, E) \to W \overset{f_E}{\to} V \to H^1(\mathbb{P}^1 \times \mathbb{P}^1, E) \to 0.
\]

Fourth, the Beilinson spectral sequence takes the following form (see Appendix B)

\[
E_1^{p,q} = \left\{ \right. \Ext^q(\mathcal{O}(0,1), E) \otimes \mathcal{O}(-1,-1) \left. \right\} \otimes \mathcal{O}(0,-1) \\
\Ext^q(\mathcal{O}(0,1), E) \otimes \mathcal{O}(-1,-1)
\]

We apply this spectral sequence to obtain a monadic description of an arbitrary framed coherent sheaf \(E\). Using (5.5) we see that, for any \(W\)-framed sheaf \(E\) such that \(H^1(\mathbb{P}^1 \times \mathbb{P}^1, E(-1,-1)) \cong V\), the spectral sequence takes the form

\[
V \otimes \mathcal{O}(0, -1) \to (V \otimes e^{-1}) \otimes \mathcal{O}(0, -1) \to H^1(E) \otimes \mathcal{O}
\]

Further, one shows that one can replace \(H^1(E) = \operatorname{Coker} f_E\) and \(H^0(E) = \operatorname{Ker} f_E\) by \(V\) and \(W\) respectively and lift the differential \(d_2^{-2,1} : E_2^{-2,1} \to E_2^{0,0}\) to a morphism \(V \otimes \mathcal{O}(0, -1) \to W \otimes \mathcal{O}\). Finally, replacing the spectral sequence with the total complex one obtains the desired monadic description (5.3) of the sheaf \(E\). We leave for the
reader to check that the maps in (5.3) take form (5.4) for an appropriately chosen quiver data \((B_1, B_2, I, J)\).

**Remark.** There is an alternative way to prove Theorem 5.2. using the following trigraded algebra

\[ S := \mathbb{C}(\xi, \eta, \zeta, x, z, y, w)\# \Gamma \]

where \(\hat{\phi}\) takes form (5.4) for an appropriately chosen quiver \((1, 3, 10)\).

Let \(X\) be the corresponding noncommutative variety (i.e., such that \(\text{coh}(X) = \text{qgr}^3(S)\)). Then we have a diagram

\[
P^2_\tau \xrightarrow{p} X \xrightarrow{q} \mathbb{P}^1 \times_\tau \mathbb{P}^1, \tag{5.8}
\]

where the morphism \(q\) is a noncommutative analog of the blowup of the point \(P\) on \(\mathbb{P}^1 \times_\tau \mathbb{P}^1\), and the morphism \(p\) is a noncommutative analog of the blowup of a pair of points on the line at infinity. One can show that Fourier–Mukai type functors:

\[
q_*p^* : \text{coh}(\mathbb{P}^2_\tau) \rightarrow \text{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1), \quad \text{and} \quad p_*q^* : \text{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1) \rightarrow \text{coh}(\mathbb{P}^2_\tau)
\]

induce mutually inverse bijections between the corresponding sets of (isomorphism classes of) \(W\)-framed torsion free sheaves. Theorem 5.2 is now immediate from ([BGK], Theorem 1.3.10).

We now turn to the proof of Theorem 2.18. Let \(E\) be a \(W\)-framed torsion free coherent sheaf on \(\mathbb{P}^1 \times_\tau \mathbb{P}^1\) such that \(H^1(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1, -1)) \cong V\). Then \(E\) can be represented as the cohomology sheaf of monad (5.3). We consider the following maps

\[
(V \otimes \epsilon) \otimes_1 \mathcal{O}(0, -1) \bigoplus (W \otimes \epsilon^{-n}) \otimes_1 \mathcal{O}(-n, 0) \xrightarrow{\Phi} (V \otimes \epsilon^{-1}) \otimes_1 \mathcal{O}(-1, 0) \xrightarrow{\Psi} (W \otimes \epsilon^n) \otimes_1 \mathcal{O}(n, 0),
\]

\[
\Phi = (0, -(B_1z - x)I, P(x, z)), \quad \Psi = (-zwJ(B_1z - x), 0, P(x, z)), \tag{5.9}
\]

where \((B_1z - x)\) stands for the cofactor matrix (i.e., the matrix formed by the \((n-1) \times (n-1)\) minors in the matrix \(B_1z - x\), taken with appropriate sign) and \(P(x, z) = \det(B_1z - x)\).

It is easy to see that \(\Psi \cdot a = b \cdot \Phi = 0\). Thus \(\Phi\) and \(\Psi\) induce morphisms

\[
(W \otimes \epsilon^{-n}) \otimes_1 \mathcal{O}(-n, 0) \xrightarrow{\phi} E \xrightarrow{\psi} (W \otimes \epsilon^n) \otimes_1 \mathcal{O}(n, 0).
\]

Furthermore, it is easy to show that the composite \(\psi \circ \phi : (W \otimes \epsilon^{-n}) \otimes_1 \mathcal{O}(-n, 0) \rightarrow (W \otimes \epsilon^n) \otimes_1 \mathcal{O}(n, 0)\) equals multiplication by \(P(x, z)^2\). Thus \((\phi, \psi)\) is a trivialization of \(E\) in a neighborhood of \(\mathbb{P}^1_z\). Finally, it is easy to see that the trivialization of \(i_{uw}^* E \cong W \otimes_1 \mathcal{O}_{\mathbb{P}^1_u}\) takes form

\[
(W \otimes \epsilon^{-n}) \otimes_1 \mathcal{O}(-n) \xrightarrow{P(x, z)} W \otimes_1 \mathcal{O} \xrightarrow{P(x, z)} (W \otimes \epsilon^n) \otimes_1 \mathcal{O}(n).
\]
This trivialization is equivalent to the trivial one, thus the map $i^*_w : \text{Gr}^{P^1 \times P^1}(W) \to \text{Gr}^{\text{aff}}(W)$ takes $(E, \phi, \psi)$ to the base point $W_0$. Thus we obtain an embedding
\[
\beta : \bigsqcup_V \mathcal{M}_{P^1 \times P^1}(V, W) \to (i^*_w)^{-1}(W_0) \subset \text{Gr}^{P^1 \times P^1}(W).
\]

Theorem 2.18 is an immediate consequence of Theorem 5.2 and the following result.

**Theorem 5.10** The map $\beta : \bigsqcup_V \mathcal{M}_{P^1 \times P^1}(V, W) \to (i^*_w)^{-1}(W_0)$ is a bijection.

**Proof:** Let $E$ be a coherent sheaf on $P^1 \times P^1$ with a trivialization $(\phi, \psi)$ in a neighborhood of $P^1_w$. Then $E$ has a canonical $W$-framing on $P^1_w$ (given by restricting the trivialization). If, in addition, $i^*_w(E) = W_0$ then the sheaf $E$ acquires a canonical $W$-framing on $P^1_w$.

Moreover, the framings agree at the point $P$, hence we obtain a map
\[
\alpha : (i^*_w)^{-1}(W_0) \to \bigsqcup_V \mathcal{M}_{P^1 \times P^1}(V, W).
\]

We now show that both $\alpha \circ \beta = \text{Id}$ note that the $W$-framings of the sheaf $E$ on $P^1_w$ and $P^1_w$ induced by the trivialization (5.9) coincide with the canonical $W$-framings.

In order to prove $\beta \circ \alpha = \text{Id}$ one needs to show that any trivialization of $E$ which gives the canonical $W$-framing of $E$ on $P^1_w$ is equivalent to the trivialization (5.9). Indeed, let $E$ be the cohomology of monad (5.3) and consider an arbitrary trivialization $(W \otimes e^{-n'})_{\otimes r} \otimes \mathcal{O}(-n', 0) \xrightarrow{\phi'} E \xrightarrow{\psi'} (W \otimes e^{n'})_{\otimes r} \otimes \mathcal{O}(n', 0)$ of $E$ in a neighborhood of $P^1_w$. Applying diagram (5.3) to compute $\text{Hom}((W \otimes e^{-n'})_{\otimes r} \otimes \mathcal{O}(-n', 0), E)$ and $\text{Hom}(E, (W \otimes e^{n'})_{\otimes r} \otimes \mathcal{O}(n', 0))$ we see that the morphisms $\phi'$ and $\psi'$ can be lifted to morphisms
\[
(V \otimes e)_{\otimes r} \mathcal{O}(0, -1) \xrightarrow{\oplus} (W \otimes e^{-n'})_{\otimes r} \mathcal{O}(-n', 0) \xrightarrow{\Phi'} (V \otimes e^{-1})_{\otimes r} \mathcal{O}(-1, 0) \xrightarrow{\Psi'} (W \otimes e^n)_{\otimes r} \mathcal{O}(n', 0) \xrightarrow{\oplus} W \otimes_r \mathcal{O}
\]
such that
\[
b \cdot \Phi' = \Psi' \cdot a = 0. \tag{5.11}
\]
Moreover, the lift $\Phi'$ of $\phi$ is unique, while the lift $\Psi'$ of $\psi'$ is unique up to a summand of the form $\lambda \cdot b$, where $\lambda \in \text{Hom}(V \otimes_r \mathcal{O}, (W \otimes e^{n'})_{\otimes r} \mathcal{O}(n', 0))$. Let
\[
\Phi' = (\Phi'_1, \Phi'_2, \Phi'_3), \quad \Psi' = (\Psi'_1, \Psi'_2, \Psi'_3)
\]
be the components of $\Phi'$ and $\Psi'$ with respect to the direct sum decomposition
\[
(V \otimes e)_{\otimes r} \mathcal{O}(0, -1) \oplus (V \otimes e^{-1})_{\otimes r} \mathcal{O}(-1, 0) \oplus W \otimes_r \mathcal{O}.
\]
Since the trivialization $(\phi', \psi')$ of $E$ restricts to the canonical $W$-framing of $E_{P^1_w}$ it follows that
\[
\Phi'_3 = \Psi'_3 = P'(x, z),
\]

22
where \( P'(x, z) \) is a homogeneous polynomial of certain degree \( n' \), such that \( P'(1, 0) = 1 \). Furthermore, the vanishing:

\[
\hom((W \otimes \epsilon^{n'}) \otimes \mathcal{O}(-n', 0), (V \otimes \epsilon^{-1}) \otimes \mathcal{O}(0, -1)) = 0
\]

implies that \( \Phi'_1 = 0 \). On the other hand, using the freedom in the choice of \( \lambda \) one can make \( \Psi'_2 = 0 \). Then, equations (5.11) yield

\[
(B_1 z - x)\Phi'_2 + IP'(x, z) = 0, \quad \Psi'_1(B_1 z - x) + P'(x, z)zwJ = 0.
\]

Multiplying the first equation by \((B_1 z - x)\) on the left and the second by \((B_1 z - x)\) on the right we obtain

\[
P(x, z)\Phi'_2 = -P'(x, z)(B_1 z - x)I, \quad \Psi'_1 P(x, z) = -zwJ(B_1 z - x)P'(x, z),
\]

where \( P(x, z) = \det(B_1 z - x) \). Thus, we see that the trivialization \((\phi', \psi')\) is equivalent to trivialization (5.9) via the equivalence given by the polynomials \( P(x, z) \) and \( P'(x, z) \). \( \square \)

6 Projective D-modules.

In this section we prove Theorem 2.4. Throughout, we will assume the parameter \( \tau \) to be generic. We begin with a description of projective \( D_{\text{frac}} \)-modules.

**Proposition 6.1** Assume that \( \tau \) is generic. Then

(i) Any projective finitely generated \( D_{\text{frac}} \)-module has the form: \( \mathcal{M} \cong W \otimes \Gamma D_{\text{frac}} \), for a finite dimensional \( \Gamma \)-module \( W \).

(ii) Two \( D_{\text{frac}} \)-modules \( W \otimes \Gamma D_{\text{frac}} \) and \( W' \otimes \Gamma D_{\text{frac}} \) are isomorphic if and only if \( \dim W = \dim W' \).

**Proof:** Let \( e = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C} \Gamma \) denote the averaging idempotent. Consider the subalgebra \( eD_{\text{frac}}e \subset D_{\text{frac}} \). It is clear that this algebra is isomorphic to the algebra of differential operators on \( \mathbb{C}/\Gamma \) with rational coefficients. Set: \( \xi = ex^{m} \), and \( \eta = ex^{1-m}y \). We have an isomorphism

\[
eD_{\text{frac}}e \cong \mathbb{C}(\xi)/\langle [\eta], [\xi] = [\tau] \rangle
\]

(\( \xi \) can be considered as a coordinate on \( \mathbb{C}/\Gamma \) and \( \eta \) as a vector field on \( \mathbb{C}/\Gamma \) generating the algebra of differential operators). We see that \( eD_{\text{frac}}e \) is a skew polynomial ring over the field \( \mathbb{C}(\xi) \), hence it is Euclidean. Therefore, \( eD_{\text{frac}}e \) is a principal ideal domain, hence any projective \( eD_{\text{frac}}e \)-module is free.

We claim next that the algebras \( D_{\text{frac}} \) and \( eD_{\text{frac}}e \) are Morita equivalent. To prove this, observe first that since \( \Gamma \)-action on \( \mathbb{C} \setminus \{0\} \) is free, the field \( \mathbb{C}(x) \) is a Galois extension of \( \mathbb{C}(x)^\Gamma \), with \( \Gamma \) being the Galois group. It follows that the algebra \( \mathbb{C}(x)^\Gamma = \mathbb{C}(x)#\Gamma \) is a simple \( \mathbb{C}(x)^\Gamma \)-algebra. Hence, \( \mathbb{C}(x)^\cdot e \cdot \mathbb{C}(x) \), a two-sided ideal in \( \mathbb{C}(x) \), must be equal to \( \mathbb{C}(x)^\Gamma \). We see that there exist elements \( a_j, b_j \in \mathbb{C}(x) \), \( j = 1, \ldots , l \), such that \( \sum a_j \cdot e \cdot b_j = 1 \). Therefore, since \( a_j, b_j \in \mathbb{C}(x) \subset D_{\text{frac}} \), we deduce \( D_{\text{frac}} \cdot e \cdot D_{\text{frac}} = D_{\text{frac}} \).
This implies, by a standard argument, that the functor $N \mapsto N \otimes_{\mathcal{D}_{\text{frac}}} e \mathcal{D}_{\text{frac}}$ provides a Morita equivalence between the algebras $e \mathcal{D}_{\text{frac}}$ and $\mathcal{D}_{\text{frac}}$. Our claim follows.

Using the Morita equivalence we deduce that any projective $\mathcal{D}_{\text{frac}}$-module is isomorphic to

$$(e \cdot \mathcal{D}_{\text{frac}} \cdot e)^{\oplus r} \otimes_{\mathcal{D}_{\text{frac}}} e \mathcal{D}_{\text{frac}} \cong (e \cdot \mathcal{D}_{\text{frac}})^{\oplus r} \cong (\text{triv}^{\oplus r}) \otimes_{\Gamma} \mathcal{D}_{\text{frac}},$$

where $\text{triv} = e^0$ is the trivial 1-dimensional $\Gamma$-module. This proves the first part of the Proposition.

To prove the second part, let $W \cong \oplus W_i \otimes e^i$ be a decomposition of $W$ with respect to the irreducible $\Gamma$-modules $e^i$. Then, $W \otimes_{\gamma} \mathcal{D}_{\text{frac}}$ goes, under Morita equivalence to

$$(W \otimes_{\gamma} \mathcal{D}_{\text{frac}}) \otimes_{\mathcal{D}_{\text{frac}}} \mathcal{D}_{\text{frac}} \cong W \otimes_{\gamma} \mathcal{D}_{\text{frac}} e \cong \bigoplus W_i \otimes e_i \mathcal{D}_{\text{frac}} e,$$

where $e_i \in \mathcal{C} \Gamma$ is the projector onto $e^i$. Now it is easy to see that $e_i \mathcal{D}_{\text{frac}} e$ is a free rank 1 $e \mathcal{D}_{\text{frac}} e$-module (with $e \cdot x^i$ being a generator). Hence

$$\bigoplus W_i \otimes e_i \mathcal{D}_{\text{frac}} e \cong (e \mathcal{D}_{\text{frac}} e)^{\oplus \dim W}.$$ 

In particular, it follows that $W \otimes_{\gamma} \mathcal{D}_{\text{frac}}$ and $W' \otimes_{\gamma} \mathcal{D}_{\text{frac}}$ are isomorphic $\mathcal{D}_{\text{frac}}$-modules if and only if $\dim W = \dim W'$. □

Recall that we have a natural isomorphism $K(\mathcal{C} \Gamma) \sim \rightarrow K(\mathcal{D})$, $W \mapsto W \otimes_{\gamma} \mathcal{D}$. Let $[N] \in K(\mathcal{C} \Gamma)$ denote the class of a $\mathcal{D}$-module $N$ under the inverse isomorphism, and write $\dim : K(\mathcal{C} \Gamma) \rightarrow \mathbb{Z}$ for the dimension homomorphism. Proposition 6.1 can be reformulated as follows.

**Corollary 6.2** We have $K(\mathcal{D}_{\text{frac}}) = \mathbb{Z}$. Moreover, the morphism $K(\mathcal{D}) \rightarrow K(\mathcal{D}_{\text{frac}})$ induced by the localization functor $N \mapsto N \otimes_{\mathcal{D}} \mathcal{D}_{\text{frac}}$ gets identified with the dimension homomorphism $\dim : K(\mathcal{C} \Gamma) \rightarrow \mathbb{Z}$.

**Lemma 6.3** If $N$ is a projective finitely generated $\mathcal{D}$-module such that $[N] = W$, then $N \otimes_{\mathcal{D}} \mathcal{D}_{\text{frac}} \cong W \otimes_{\gamma} \mathcal{D}_{\text{frac}}$.

**Proof:** It is clear that $N \otimes_{\mathcal{D}} \mathcal{D}_{\text{frac}}$ is a projective finitely generated $\mathcal{D}_{\text{frac}}$-module, hence Proposition 6.1 (i) yields: $N \otimes_{\mathcal{D}} \mathcal{D}_{\text{frac}} \cong W' \otimes_{\gamma} \mathcal{D}_{\text{frac}}$ for some $W'$. Moreover, Corollary 6.2 implies that $\dim W' = \dim W$. Finally, Proposition 6.1 (ii) shows that $N \otimes_{\mathcal{D}} \mathcal{D}_{\text{frac}} \cong W \otimes_{\gamma} \mathcal{D}_{\text{frac}}$. □

**Lemma 6.4** Let $W$ be a $\Gamma$-module. Any projective finitely generated $\mathcal{D}$-module $N$ with $\dim [N] = \dim W$ can be embedded into $W \otimes_{\gamma} \mathcal{D}_{\text{frac}}$, as a fat $\mathcal{D}$-submodule. Furthermore, the embedding is unique up to the action of the group $G_W$. 24
Proof: To prove the existence of embedding we consider the natural map \( N \to N \otimes_D D_{\text{frac}} = N \otimes_{\mathbb{C}T[x]} \mathcal{C}^\prime(x) \). Since \( N \) is projective it follows that \( N \) is torsion free (as \( \mathbb{C}T[x] \)-module), hence the above map is an embedding. Further, by Lemma 6.3 it follows that \( N \otimes_D D_{\text{frac}} \cong W \otimes_r D_{\text{frac}} \). Finally, taking an arbitrary set of generators (over \( D \)) of \( N \subset W \otimes_r D_{\text{frac}} \) and denoting by \( p_1(x) \) some \( \Gamma \)-invariant multiple of all their denominators we see that

\[
N \subset \frac{1}{p_1} (W \otimes_r D) \subset W \otimes_r D_{\text{frac}}.
\]

Similarly, considering the dual \( D \)-module one can check that there exists a \( \Gamma \)-invariant polynomial \( p_2(x) \) such that

\[
p_2 \cdot (W \otimes_r D) \subset N \subset W \otimes_r D_{\text{frac}}.
\]

Finally, taking \( p(x) = p_1(x)p_2(x) \) we see that \( N \) is a fat \( D \)-submodule in \( W \otimes_r D_{\text{frac}} \).

Now assume that we have two embeddings \( \psi_1, \psi_2 : N \hookrightarrow W \otimes_r D_{\text{frac}} \). Tensoring with \( D_{\text{frac}} \) we obtain two isomorphisms \( \psi_1, \psi_2 : N \otimes_D D_{\text{frac}} \longrightarrow W \otimes_r D_{\text{frac}} \). Then \( g = \psi_2 \circ \psi_1^{-1} \in GL_{D_{\text{frac}}}(W \otimes_r D_{\text{frac}}) = G_W \) and it is clear that \( \psi_2 = g \cdot \psi_1 \). \( \square \)

Proof of Theorem 2.4: It follows from Lemma 6.4 that any projective \( D \)-module \( N \) such that \( \dim[N] = r \) can be embedded into \( W \otimes_r D_{\text{frac}} \) as a fat \( D \)-submodule. On the other hand, for generic \( \tau \) the homological dimension of the algebra \( D \) equals 1 (see [CBH]), hence any fat \( D \)-submodule \( N \subset W \otimes_r D_{\text{frac}} \) is projective. Moreover, it is clear that we have \( N \otimes_D D_{\text{frac}} = W \otimes_r D_{\text{frac}} \), hence by Proposition 6.1 and Corollary 6.2 we deduce: \( \dim[N] = \dim W = r \). Finally, by Lemma 6.4 two fat \( D \)-submodules in \( W \otimes_r D_{\text{frac}} \) are isomorphic as \( D \)-modules if and only if they are conjugate by the action of the group \( G_W \). It follows that the set of isomorphism classes of projective \( D \)-modules \( N \) with \( \dim[N] = r \) is in a natural bijection with the coset space \( G_W \backslash \text{Gr}^D(W) \). It remains to apply the isomorphisms of diagram (2.12). \( \square \)

7 Appendix A: Formalism of Polygraded Algebras

Let \( A = \oplus_{p \geq 0} A_p \) be a graded algebra over a field \( \mathbb{k} \). Let \( \text{gr}(A) \) denote the category of graded finitely generated \( A \)-modules. For any \( n \in \mathbb{Z} \) and any \( M \in \text{gr}(A) \) let \( M_{\geq n} = \oplus_{p \geq n} M_p \) be the tail of \( M \). An element \( x \in M \) is called torsion if \( x \cdot A_{\geq n} = 0 \), for some \( n \). A module \( M \) is called torsion if every element of \( M \) is torsion. Let \( \text{tor}(A) \) denote the full subcategory of \( \text{gr}(A) \) formed by all torsion \( A \)-modules. Then \( \text{tor}(A) \) is a dense subcategory, hence one can consider the quotient category \( \text{qgr}(A) = \text{gr}(A) / \text{tor}(A) \). If \( A \) is commutative and generated over \( A_0 \) by \( A_1 \) then by the Serre Theorem the category \( \text{qgr}(A) \) is equivalent to the category \( \text{coh}(X) \) of coherent sheaves on \( X = \text{Proj} (A) \), the projective spectrum of the algebra \( A \).

In the series of papers [AZ], [YZ], etc., a formalism has been developed that allows to consider the category \( \text{qgr}(A) \) as a category of coherent sheaves in the case when \( A \) is a noncommutative graded algebra. This means that the category \( \text{qgr}(A) \) shares many of the general properties of categories of coherent sheaves, provided the algebra \( A \) satisfies some ‘reasonable’ properties. In this case one says that \( \text{qgr}(A) \) is the category of coherent sheaves on a noncommutative algebraic variety \( X \) and denotes it by \( \text{coh}(X) \).
We extend the formalism of [AZ] to the poly-graded case as follows. Let \( A = \bigoplus_{p \in \mathbb{N}^r} A_p \) be an \( \mathbb{N}^r \)-graded algebra (we will denote vector indices by bold letters). Let \( \text{gr}^r(A) \) denote the category of finitely generated \( \mathbb{Z}^r \)-graded \( A \)-modules. For any \( n \in \mathbb{Z}^r \) and any \( M \in \text{gr}^r(A) \) let \( M_{\geq n} = \bigoplus_{p \geq n} M_p \) be the tail of \( M \), where \( p = (p_1, \ldots, p_r) \geq n = (n_1, \ldots, n_r) \) if and only if \( p_i \geq n_i \) for all \( 1 \leq i \leq r \). An element \( x \in M \) is called torsion if \( x \cdot A_{\geq n} = 0 \), for some \( n \). A module \( M \) is called torsion if every its element is torsion. Let \( \text{tor}^r(A) \) denote the full subcategory of \( \text{gr}^r(A) \) formed by all torsion \( A \)-modules. Thus, \( \text{tor}^r(A) \) is a fat subcategory. We let \( \text{qgr}^r(A) = \text{gr}^r(A)/\text{tor}^r(A) \) be a polygraded counterpart of the category.

In the polygraded situation we have to make the following modifications in the definitions used in [AZ]. First, a \( \mathbb{Z}^r \)-graded \( k \)-module \( V = \bigoplus_{p \in \mathbb{Z}^r} V_p \) should be called left bounded if \( V = V_{\geq n} \) for some \( n \in \mathbb{Z}^r \) (such \( n \) is called a left bound for \( V \)). Similarly, \( V \) should be called right bounded if \( V_{\geq n} = 0 \) for some \( n \in \mathbb{Z}^r \) (such \( n \) is called a right bound for \( V \)). Note, that a finitely generated module \( M \) over a finitely generated algebra \( A \) is torsion if and only if it is both left and right bounded. Thus \( \text{tor}^r(A) \) is the category of bounded \( \mathbb{Z}^r \)-graded \( A \)-modules.

Most essential changes involve the definition of property \( \chi_i(M) \), cf. ([AZ], Definition 3.2). First, introduce the following notation. For each \( i = 1, \ldots, r \), write \( e_i \in \mathbb{Z}^r \) for the \( i \)-th basis vector, and let \( I \subset \{1, \ldots, r\} \) denote a nonempty subset of indices. For any \( M \in \text{gr}^r(A) \) put

\[
M^I_n = \left( \bigoplus_{p \geq n, \ p_i = n_i \text{ for } i \in I} M_p \right) = M_{\geq n} / \sum_{i \in I} M_{\geq n + e_i}.
\]

**Definition 7.1** We say that property \( \chi_i(M) \) holds for a \( \mathbb{Z}^r \)-graded \( A \)-module \( M \) provided \( \text{Ext}^j(A^{\{k\}}_0, M) \) is bounded for all \( j \leq i \) and all \( 1 \leq k \leq r \).

We say that property \( \chi_i \) holds for the graded algebra \( A \) provided property \( \chi_i(M) \) holds for every finitely generated \( \mathbb{Z}^r \)-graded \( A \)-module \( M \).

We say that property \( \chi \) holds for \( A \) provided property \( \chi_i \) holds for every \( i \).

In [AZ], a graded algebra \( A \) was said to be regular of dimension \( d \) if the following holds

(0) \( A \) is connected (i.e. \( A_0 = k \));
(1) \( A \) has finite global dimension \( d \);
(2) \( A \) has polynomial growth;
(3) \( A \) is Gorenstein, that is \( \text{Ext}^i_{\text{mod}(A)}(k, A) = \begin{cases} \mathbb{k}[l], & \text{if } i = d \\ 0, & \text{otherwise} \end{cases} \)

It was demonstrated in [AZ] that, for regular algebras \( A \), the category \( \text{qgr}(A) \) has good properties, in particular, one can compute cohomology of the sheaves \( \mathcal{O}(i) = \pi(A(i)) \), where \( \pi : \text{gr}(A) \to \text{qgr}(A) \) is the projection functor and \( (i) \) stands for the degree-shift by \( i \in \mathbb{Z} \). Further, in [BGK] we explained that one can replace conditions (0) and (3) above by the following conditions:
(0’) $A_0$ is a finite dimensional semisimple $k$-algebra;

(3’) $A$ is generalized Gorenstein, that is $\text{Ext}^i_{\text{mod}(A)}(\mathbb{k}, A) = \begin{cases} R[l], & \text{if } i = d \\ 0, & \text{otherwise} \end{cases}$

where $R$ is a finite dimensional $A_0$-bimodule isomorphic to $A_0$ as right $A_0$-module.

In this paper we will need a further generalization of the notion of regular algebra to the setup of polygraded algebras. To this end, one has to replace condition (3’) above by the following condition:

(3'') $A$ is strongly Gorenstein with parameters $d = (d_1, \ldots, d_r)$, $l = (l_1, \ldots, l_r)$, such that $d = \sum_{i=1}^r d_i$. This means that for any subset $I \subset \{1, \ldots, r\}$ we have

$$\text{Ext}^i_{\text{mod}(A)}(A^I_0, A) = \begin{cases} (R_I \otimes_{A_0} A^I_0)(1_I), & \text{if } i = d_I \\ 0, & \text{otherwise} \end{cases}$$

where $d_I = \sum_{k \in I} d_k$, $1_I = \sum_{k \in I} l_k e_k$, $R_I = \bigotimes_{k \in I} \epsilon^k$ (tensor product over $A_0$),

where $\epsilon^k$ are $A_0$-bimodules isomorphic to $A_0$ as right $A_0$-modules, and such that $\epsilon^k \otimes_{A_0} \epsilon^l \cong \epsilon^l \otimes_{A_0} \epsilon^k$ as $A_0$-bimodules.

Now, with all these modifications made, one can verify that most of the results of [AZ] can be extended to $\mathbb{N}^r$-graded algebras by the same arguments as in [AZ]. In particular, we have an analog of ([AZ], Theorem 8.1).

**Theorem 7.2** Let $A$ be a $\mathbb{N}^r$-graded noetherian regular algebra of dimension $d$ over a semisimple algebra $A_0$. Let $\mathcal{O}(p) = \pi(A(p)) \in \text{qgr}^r(A) = \text{coh}(X)$. Then

1. Property $\chi$ holds for $A$.
2. $H^0(X, \mathcal{O}(p)) = A_p$ and $H^0(X, \mathcal{O}(p)) = 0$, for all $p \geq 0$.
3. The cohomological dimension of the category $\text{coh}(X) = \text{qgr}^r(A)$ equals $d - r$. $\square$

**Remark.** As opposed to the single-graded case studied in [AZ], in the polygraded case it is impossible to determine the cohomology of the sheaves $\mathcal{O}(p)$ for nonpositive $p$ without some extra information about the structure of the algebra $A$ (it is necessary to know the $A^I_0$-module structure on $A^I_n$ for all $n \geq 0$ and $I \subset \{1, \ldots, r\}$).

**Definition 7.3** We say that an $\mathbb{N}^r$-graded algebra $A$ is strongly generated by its first component if for any $1 \leq i \leq r$ both maps below are surjective for any $p \geq 0$

$$A_{e_i} \otimes A_p \to A_{p+e_i} \quad \text{and} \quad A_p \otimes A_{e_i} \to A_{p+e_i}.$$ 

**Remark.** It is easy to see that any $\mathbb{N}$-graded algebra which is generated by its first component is strongly generated. Thus in the case $r = 1$ we obtain nothing new.

An element $p = (p_1, \ldots, p_r) \in \mathbb{N}^r$ is called strictly positive if $p_i > 0$ for all $1 \leq r$. 

27
Proposition 7.4 If $A$ is an $\mathbb{N}^r$-graded noetherian algebra strongly generated by its first component and satisfying the $\chi$-condition, then for any strictly positive $p$ the shift functor $s(M) = M(p)$ in the category $\text{gr}^r(A)$ is ample in the sense of [AZ], (4.2.1).

Proof: It follows from an $\mathbb{N}^r$-graded analog of the Theorem 4.5 of loc.cit. that the collection of shift functors $s_i(M) = M(e_i)$, $i = 1, \ldots, r$ is ample. Now let $E$ be an object of $\text{gr}^r(A)$. Then it follows from the ampleness of the collection $(s_i)$ that there exists a surjection $\bigoplus_{i=1}^P \mathcal{O}(-l_i) \rightarrow E$ for some $l_i \geq 0$. Now for each $l_i$ we can choose $k_i \in \mathbb{N}$ such that $k_i \cdot p \geq l_i$. Then the strong regularity of the algebra $A$ implies that the canonical map

$$A_{k_i \cdot p - l_i} \otimes_{A_0} \mathcal{O}(-k_i \cdot p) \rightarrow \mathcal{O}(-l_i)$$

is surjective. Further, since $A_{k_i \cdot p - l_i}$ is a finitely generated $A_0$-module it follows that we have a surjection $\bigoplus_{i=1}^P \mathcal{O}(-k_i \cdot p)^{\oplus m_i} \rightarrow E$ and part (a) of the ampleness property for the functor $s$ follows.

Part (b) of the ampleness for the functor $s$ follows trivially from the ampleness of the collection $s_i$.

Remark. For any strictly positive $p$ we put $\Delta_p(A) := \bigoplus_{k=0}^{\infty} A_{k \cdot p}$. Thus, $\Delta_p(A)$ is a single-graded subalgebra of $A$. The following is immediate from Proposition 7.4 and ([AZ], Theorem 4.5).

Corollary 7.5 If $A$ is an $\mathbb{N}^r$-graded noetherian algebra strongly generated by its first component and satisfying the condition $\chi$, then for any strictly positive $p$ the algebra $\Delta_p(A)$ is noetherian, satisfies the condition $\chi$ and we have an equivalence of categories $\text{gr}^r(A) \cong \text{gr}(\Delta_p(A))$.

Remark. We would like to emphasize that, inspite of Corollary 7.5, the above developed formalism of quotient categories for polygraded algebras does not reduce to that for single-graded algebras. The point is that though algebras $A$ and $\Delta_p(A)$ give rise to equivalent quotient categories, the algebra $\Delta_p(A)$ may not be regular or Koszul, for instance, even when $A$ is.

8 Appendix B: The Geometry of $\mathbb{P}^1 \times_{\tau} \mathbb{P}^1$

The goal of this Appendix is to study the homological properties of the algebra $Q$, see (2.7), and to establish Serre Duality and Beilinson Spectral Sequence for $\mathbb{P}^1 \times_{\tau} \mathbb{P}^1$.

Proposition 8.1 The bigraded algebra $Q$ is noetherian and is strongly generated by its first component. Furthermore, $Q$ is regular of dimension 4.

To prove this Proposition we introduce some notation. Given a semisimple algebra $A_0$ and an $A_0$-bimodule $M$, we write $T_{A_0}(M)$ for the tensor algebra of $M$ over $A_0$.  

28
Definition 8.2 Let $A$ be a $\mathbb{N}^r$-graded algebra generated by $\oplus_{i=1}^r A_{ei}$ over a semisimple algebra $A_0$. We say that $A$ is quadratic if $A = T_{A_0}(\oplus_{i=1}^r A_{ei})/\langle R \rangle$, where $\langle R \rangle$ denotes the two-sided ideal generated by a graded vector subspace $R = \oplus_{1 \leq i,j \leq r} R_{e_i+e_j}$ (called ‘quadratic relations’).

Assume that $A$ is a quadratic $\mathbb{N}^r$-graded algebra. Let $A^1$ denote its quadratic dual algebra (with respect to the total grading). Then $A^1$ is also a quadratic $\mathbb{N}^r$-graded algebra. Recall that the algebra $A$ is called Koszul if the following Koszul complex $K^\bullet(A)$ is exact

$$
\cdots \to \oplus_{1 \leq i,j \leq r} (A^1_{ei+ej})^* \otimes A_0 A(-e_i-e_j) \to \oplus_{1 \leq i \leq r} (A^1_{ei})^* \otimes A_0 A(-e_i) \to A \to A_0 \to 0.
$$

Definition 8.3 We call the algebra $A$ strongly Koszul if for any subset $I \subset \{1, \ldots, r\}$ the following partial Koszul complex $K^\bullet_I(A)$ is exact

$$
\cdots \to \oplus_{i,j \in I} (A^1_{ei+ej})^* \otimes A_0 A(-e_i-e_j) \to \oplus_{i \in I} (A^1_{ei})^* \otimes A_0 A(-e_i) \to A \to A^I_0 \to 0.
$$

It is clear from the definition of the quadratic dual algebra that $(A^1)^*_I$ is dual to $A^I_0$, where $I = \{1, \ldots, r\} \setminus I$. Thus if $A$ is strongly Koszul then for any $I \subset \{1, \ldots, r\}$ the algebra $A^I_0$ is Koszul as well. Fix $d = (d_1, \ldots, d_r)$, and for any subset $I$, write $d_I = \sum_{i \in I} d_i e_i$.

Definition 8.4 We say that $A^1$ is strongly Frobenius of index $d$ if the following holds

(i) $A^1_p = 0$ unless $0 \leq p \leq d$;

(ii) The component $A^1_{d_I}$ of $A^1$ is isomorphic to $A_0$ as right $A_0$-module, for any subset $I \subset \{1, \ldots, r\}$;

(iii) The multiplication map: $A^1_p \otimes A_0 A^1_{d_I-p} \to A^1_{d_I}$ gives a nondegenerate pairing, for any $0 \leq p \leq d_I$.

Proposition 8.5 The algebra $Q^1$ is strongly Koszul and $Q^1$ is strongly Frobenius of index $(2, 2)$. Moreover, we have

$$
Q^1 \big|_{i,j} = \begin{cases}
\mathbb{C} \Gamma, & (i,j) = (0,0) \\
\mathbb{C} \langle \xi, \zeta \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (1,0) \\
\mathbb{C} \langle \xi \wedge \zeta \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (2,0) \\
\mathbb{C} \langle \eta, \omega \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (0,1) \\
\mathbb{C} \langle \xi \wedge \eta, \xi \wedge \omega, \zeta \wedge \eta, \zeta \wedge \omega \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (1,1) \\
\mathbb{C} \langle \xi \wedge \zeta \wedge \eta, \xi \wedge \zeta \wedge \omega \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (2,1) \\
\mathbb{C} \langle \eta \wedge \omega \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (0,2) \\
\mathbb{C} \langle \xi \wedge \eta \wedge \omega, \zeta \wedge \eta \wedge \omega \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (1,2) \\
\mathbb{C} \langle \xi \wedge \zeta \wedge \eta \wedge \omega \rangle \otimes \mathbb{C} \Gamma, & (i,j) = (2,2) \\
0, & \text{otherwise}
\end{cases}
$$

where $\xi$ is a generator of $\Gamma$-bimodule $\epsilon^{-1}$, $\eta$ is a generator of $\Gamma$-bimodule $\epsilon$, and $\zeta, \omega$ are each generators of the trivial $\Gamma$-bimodule.
Proof: In the proof we consider the algebra $Q$ as an algebra, depending on a parameter $\tau$. We will indicate the value of $\tau$ by a superscript. For example, $Q^0$ stands for the algebra $Q$ with $\tau = 0$.

First it is easy to show that for any $\tau$ the components of the dual algebra are given by the above formulas. Further note that for $\tau = 0$ we have an isomorphism $Q^0 \cong \mathbb{C}[x, z, y, w] \# \Gamma$. In this case it is quite easy to show that $Q^0$ is strongly Koszul. Finally we note that we may view the family of partial Koszul complexes $K^*_i(Q^\tau)$ of the algebras $Q^\tau$ as a family of varying (with $\tau$) differentials on the partial Koszul complex $K^*_i(Q^0)$. Since the complex is exact for $\tau = 0$ the same is true for all values of $\tau$ close enough to zero. However, the algebras $Q^\tau$ and $Q^{\alpha, \tau}$ are isomorphic for any $\alpha \in \mathbb{C}^*$. Thus, $Q^\tau$ is strongly Koszul for any $\tau$.

Similarly, to show that $(Q^\tau)^!$ is strongly Frobenius for any $\tau$ we note that it is true for $\tau = 0$. Further, we consider the family of pairings $(Q^\tau)^!_p \otimes_{\Gamma} (Q^\tau)^!_{d_{I} - p} \to (Q^\tau)^!_{d_{I}}$ as a family of varying (with $\tau$) pairings $(Q^0)^!_p \otimes_{\Gamma} (Q^0)^!_{d_{I} - p} \to (Q^0)^!_{d_{I}}$. Since the pairings are nondegenerate for $\tau = 0$ the same is true for all values of $\tau$ close enough to zero. However, the algebras $(Q^\tau)^!$ and $(Q^{\alpha, \tau})^{!}$ are isomorphic for any $\alpha \in \mathbb{C}^*$. Thus, $(Q^\tau)^!$ is strongly Frobenius for any $\tau$. \hfill $\square$

Proposition 8.6 If an $\mathbb{N}^r$-graded algebra $A$ is strongly Koszul and the dual algebra $A^!$ is strongly Frobenius of index $(d_1, \ldots, d_r)$ then $A$ is strongly Gorenstein with parameters $d = (d_1, \ldots, d_r)$ and $l = (d_1, \ldots, d_r)$.

Proof: If $A$ is strongly Koszul then the partial Koszul complex $K^*_i(A)$ can be considered as a projective resolution of $A_0^!$. It follows that $\operatorname{Ext}^*_{\operatorname{mod}(A)}(A_0^!, A)$ coincides with the cohomology of complex

$$0 \to A \to \oplus_{i \in I} A^!_{d_1} \otimes_{A_0} A(-e_i) \to \cdots \to \oplus_{i \in I} A^!_{d_I - e_i} \otimes_{A_0} A(d_I - e_i) \to A^!_{d_I} \otimes_{A_0} A(d_I) \to 0.$$

On the other hand, the strong Frobenius property of the algebra $A^!$ shows that $A^!_p \cong A^!_{d_I} \otimes_{A_0} (A^!_{d_I - p})^*$ as $A_0$-bimodule. Hence the above complex is isomorphic to the complex $A^!_{d_I} \otimes_{A_0} K^*_i(A)(d_I)$ truncated at the rightmost term. Therefore, it has a single nonzero cohomology group in degree $d_I$, which is isomorphic to $A^!_{d_I} \otimes_{A_0} A_0^! (d_I)$. It follows that $A$ satisfies the strong Gorenstein property with parameters $(d, d)$ and with $R_I = A^!_{d_I}$. \hfill $\square$

Proof of Proposition 8.1: It is clear that $Q$ is strongly generated by its first component. So, it remains to prove regularity and the noetherian property.

First, note that $Q^0_{(0,0)} = \mathbb{C} \Gamma$ is a semisimple algebra. Thus (0') holds. Second, we have to show that $Q^\tau$ is noetherian. This follows from the fact that $Q^\tau$ can be represented as a consecutive Ore extension of the base field $\mathbb{C}$. Further, it is easy to show that $\dim_{\mathbb{C}} Q^\tau_{i,j} = (i + 1)(j + 1)|\Gamma|$. In particular, $Q^\tau$ has polynomial growth. Thus (2) holds.

The strong Gorenstein property (3') for the algebra $Q^\tau$ follows immediately from Proposition 8.5 and Proposition 8.6. The Gorenstein parameters are given by: $d = (2, 2)$ and $l = (2, 2)$. 

30
Finally, it follows from [Hu] that the global dimension of \(Q\) equals the length of the minimal free resolution of \(Q_{(0,0)}^r\). But the Koszul complex \(K^*(Q)\) provides such resolution of length 4; hence the global dimension of \(Q\) is bounded by 4 from above. On the other hand, since \(Q\) is Gorenstein with parameters \(d = (2,2), l = (2,2)\) it follows that \(\text{Ext}^4(Q_{(0,0)}, Q) \neq 0\), hence the global dimension equals 4.

Thus, the cohomological dimension of the category \(\text{coh}(\mathbb{P}^1 \times \mathbb{P}^1) = \text{agr}^2(Q)\) equals 2, and it is clear that we have

\[
H^p(\mathbb{P}^1 \times \mathbb{P}^1, O(i, j)) = \begin{cases} 
Q_{i,j}, & \text{if } p = 0 \text{ and } i, j \geq 0 \\
\epsilon^{-1} \otimes Q_{-2-i,0}^\ast \otimes_i O_{0,j}, & \text{if } p = 1 \text{ and } i \leq -2, j \geq 0 \\
\epsilon \otimes Q_{-2-j, -1}^\ast \otimes_i O_{i,0}, & \text{if } p = 1 \text{ and } i \geq 0, j \leq -2 \\
Q_{-2-i,-2-j}^\ast, & \text{if } p = 2 \text{ and } i, j \leq -2 \\
0, & \text{otherwise}
\end{cases}
\]

for all \(i, j \geq 0\).

More generally, we prove Lemma 8.7

\[
H^p(\mathbb{P}^1 \times \mathbb{P}^1, O(i, j)) = \begin{cases} 
Q_{i,j}, & \text{if } p = 0 \text{ and } i, j \geq 0 \\
\epsilon^{-1} \otimes Q_{-2-i,0}^\ast \otimes_i O_{0,j}, & \text{if } p = 1 \text{ and } i \leq -2, j \geq 0 \\
\epsilon \otimes Q_{-2-j, -1}^\ast \otimes_i O_{i,0}, & \text{if } p = 1 \text{ and } i \geq 0, j \leq -2 \\
Q_{-2-i,-2-j}^\ast, & \text{if } p = 2 \text{ and } i, j \leq -2 \\
0, & \text{otherwise}
\end{cases}
\]

Sketch of Proof. In order to compute the global cohomology of \(O(i, j)\) for not necessarily positive values of \((i, j)\) we use partial Koszul complexes. In more detail, the projections to the category \(\text{coh}(\mathbb{P}^1 \times \mathbb{P}^1)\) of the partial Koszul complexes yield exact sequences

\[
0 \rightarrow \epsilon \otimes O(-2, 0) \rightarrow Q_{1,0} \otimes_i O(-1, 0) \rightarrow O \rightarrow 0,
\]

\[
0 \rightarrow \epsilon^{-1} \otimes O(0, -2) \rightarrow Q_{0,1} \otimes_i O(0, -1) \rightarrow O \rightarrow 0
\]

(we used here the fact that \(Q_{I}^\dagger \in \text{tor}(Q)\) for any nonempty \(I \subset \{1, 2\}\), and that \(Q_{0,1}^\dagger \approx Q_{0,1}^\ast, Q_{1,0}^\ast \approx Q_{1,0}^\dagger, Q_{2,0}^\ast \approx Q_{2,0}^\dagger \approx \epsilon\)).

To complete the proof of the Lemma we apply descending induction in \((i, j)\) using the above sequences twisted by \((i+2, j)\) and \((i, j+2)\) respectively, and the fact that the multiplication map \(Q_{i,0} \otimes_i Q_{0,j} \rightarrow Q_{i,j}\) is an isomorphism of \(\Gamma\)-bimodules.

Serre Duality for \(\mathbb{P}^1 \times \mathbb{P}^1\). A natural approach to Serre Duality theorems for noncommutative schemes corresponding to regular noncommutative algebras would be via the concept of balanced dualizing complex (see [Y, YZ]). Generalizing this concept to the case of \(\mathbb{N}^r\)-graded algebras does not seem to be straightforward however. The reason is that, while the notion of dualizing complex easily extends to the polygraded case, it is rather difficult to find the relevant meaning of “balanced” in this case. The problem is similar to that of computing the cohomology of sheaves \(O(p)\) for nonpositive values of \(p\), see Remark following Theorem 7.2.

In the special case of \(\mathbb{P}^1 \times \mathbb{P}^1\) these problems can be circumvented as follows. We consider the algebra \(A = \Delta_{(1,1)}(Q)\). It follows from Proposition 8.1 and Corollary 7.5 that this algebra is noetherian and satisfies condition \(\chi\). Moreover, by Corollary 7.5,
Theorem 7.2 and Proposition 8.1 the cohomological degree of the category \( \text{qgr}^2(A) = \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) \) equals 2. Hence we can use ([YZ], Theorem 2.3) which implies that the category \( \text{coh}(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1) \) enjoys the Serre duality with dualizing sheaf defined by
\[
\omega^0 = \pi\left( \bigoplus_{k=0}^{\infty} H^2(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1, \mathcal{O}(-k, -k))^* \right).
\]
But Lemma 8.7 yields
\[
\pi\left( \bigoplus_{k=0}^{\infty} H^2(\mathbb{P}^1 \times_{\tau} \mathbb{P}^1, \mathcal{O}(-k, -k))^* \right) \cong \pi\left( \bigoplus_{k=0}^{\infty} Q_{k-2, k-2} \right) \cong \mathcal{O}(-2, -2).
\]
Thus the dualizing sheaf on \( \mathbb{P}^1 \times_{\tau} \mathbb{P}^1 \) is isomorphic to \( \mathcal{O}(-2, -2) \).

**Beilinson Spectral Sequence.** In the noncommutative setting, an analogue of Beilinson Spectral Sequence has been introduced in [KKO] for a certain class of graded Koszul algebras, using a double Koszul bicomplex. Below, we explain how to adapt the approach of [KKO] to the case of \( \mathbb{N}^r \)-graded Koszul algebras. We will freely use the notation and definitions of [KKO], in particular, the notion of Yang-Baxter operator.

An exact functor from a tensor category \( T \) to the tensor category of vector spaces will be called a 'noncommutative' fiber functor if this functor is compatible with the tensor product structures and associativity constraint, but is not necessarily compatible with the commutativity constraint. Given a Yang-Baxter operator on a finite-dimensional vector space, one can construct as has been explained in [Lyu] (see also [KKO], section 8), a tensor category \( T \) equipped with a 'noncommutative' fiber functor. Then the category of (either graded, or \( \mathbb{N}^r \)-graded, or \ldots) commutative algebras in the category \( T \) gives a class of (graded, \( \mathbb{N}^r \)-graded, \ldots) noncommutative algebras in the category of vector spaces. The class of noncommutative algebras thus obtained shares a lot of properties of the category of commutative algebras. For example, for any two algebras in the class their tensor product admits a canonical algebra structure.

**Remark.** Instead of Yang-Baxter operator in a vector space one may start with a \( A_0 \)-invariant Yang-Baxter operator in a finitely generated \( A_0 \)-bimodule, for any semisimple finite dimensional algebra \( A_0 \). Then we obtain an \( A_0 \)-linear tensor category \( T \) with a functor to the category of \( A_0 \)-bimodules. The category of commutative \( A_0 \)-algebras in \( T \) gives a class of noncommutative \( A_0 \)-algebras.

**Example 8.8** Consider a free right \( \Gamma \)-module \( V \) of rank 4 with generators \( x, y, z, w \) and endow it with a \( \Gamma \)-bimodule structure as in (2.8). Then the \( \Gamma \)-linear operator \( V \otimes_{\Gamma} V \to V \otimes_{\Gamma} V \) defined on the generators as
\[
\begin{align*}
x \otimes y &\mapsto y \otimes x - \frac{\tau}{2} z \otimes w - \frac{\tau}{2} w \otimes z, \\
y \otimes x &\mapsto x \otimes y + \frac{\tau}{2} z \otimes w + \frac{\tau}{2} w \otimes z, \\
u \otimes v &\mapsto v \otimes u \quad \text{otherwise}
\end{align*}
\]
is a Yang-Baxter operator. It is easy to see that the algebra \( Q \) comes from a bigraded commutative algebra in the tensor category corresponding to this Yang-Baxter operator.
Let $A$ be an $\mathbb{N}^r$-graded algebra obtained in such a way. Then $A \otimes_{A_0} A$ is an $\mathbb{N}^r \oplus \mathbb{N}^r$-graded algebra and the maps $p_1^*(a) = a \otimes 1$, $p_2^*(a) = 1 \otimes a$ are homomorphisms of algebras $A \to A \otimes_{A_0} A$. Let $X$ denote the noncommutative variety, corresponding to the algebra $A$ and let $X \times X$ denote the noncommutative variety, corresponding to the algebra $A \otimes_{A_0} A$. Thus $qgr^r(A) = \text{coh}(X)$, $qgr^{2r}(A \otimes_{A_0} A) = \text{coh}(X \times X)$.

Now, if $M$ is a right $\mathbb{N}^r$-graded $A$-module we define $p_1^* M = M \otimes_A (A \otimes_{A_0} A)$. Then $p_1^* M$ is a right $(\mathbb{N}^r \oplus \mathbb{N}^r)$-graded $A \otimes_{A_0} A$-module. It is clear that $p_1^* M(p, q) = M_{p} \otimes_{A_0} A_q$, hence for any $M \in \text{tor}^r(A)$ we have $p_1^* M \in \text{tor}^{2r}(A \otimes_{A_0} A)$. Thus $p_1^*$ can be considered as a functor $qgr^r(A) \to qgr^{2r}(A \otimes_{A_0} A)$, that is, a functor: $\text{coh}(X) \to \text{coh}(X \times X)$.

Similarly, if $M = \oplus_{p, q} M_{p, q}$ is a right $(\mathbb{N}^r \oplus \mathbb{N}^r)$-graded $(A \otimes_{A_0} A)$-module then we define $((p_2)_* M)_q = \Gamma(X, \pi(\oplus_p M_{p, q}))$, where

$$\Gamma(X, \pi(\bullet)) = \text{Hom}_{qgr^r(A)}(\pi(A), \pi(\bullet)),$$

and the $A$-module structure of $\oplus_{p, q} M_{p, q}$ is obtained from the homomorphism $p_1^*$. It is clear that $(p_2)_* M = \oplus_q ((p_2)_* M)_q$ is an $\mathbb{N}^r$-graded $A$-module. Furthermore, if $M \in \text{tor}^{2r}(A \otimes_{A_0} A)$ then it is clear that $(p_2)_* M \in \text{tor}^r(A)$. Thus $(p_2)_*$ can be considered as a functor $qgr^{2r}(A \otimes_{A_0} A) \to qgr^r(A)$, that is, a functor: $\text{coh}(X \times X) \to \text{coh}(X)$.

Now, if $N$ is an $(\mathbb{N}^r \oplus \mathbb{N}^r)$-graded $(A \otimes_{A_0} A)$-bimodule then $M \mapsto (p_2)_*(p_1^* M \otimes_{A \otimes_{A_0} A} N)$ gives a functor $\Phi_N : qgr^r(A) \to qgr^r(A)$, that is, a functor: $\text{coh}(X) \to \text{coh}(X)$.

**Lemma 8.9**

(i) Let $(\Delta_A)_{p, q} = A_{p+q}$, and $\Delta_A = \oplus_{p, q \geq 0} (\Delta_A)_{p, q}$. Then there is a natural isomorphism of functors $\Phi_{\Delta_A} \cong \text{Id}$.

(ii) If $N_1, N_2$ are $A$-bimodules and $N_1 \otimes_{A_0} N_2$ has a canonical structure of an $(A \otimes_{A_0} A)$-bimodule then

$$\Phi_{N_1 \otimes_{A_0} N_2} (M) = \Gamma(X, \pi(M \otimes_A N_1)) \otimes_{A_0} N_2.$$

**Proof:**

(i) Let $M' = \Phi_{\Delta_A}(M)$. Note that $\Delta_A$, considered as an $A$-module, is isomorphic to $\oplus_{q \in \mathbb{N}^r} A(q)_{\geq 0}$. Hence $M'_q = \Gamma(X, \pi(M \otimes_A A(q)_{\geq 0}))$. On the other hand, it is clear that

$$\pi(M \otimes_A A(q)_{\geq 0}) \cong \pi(M \otimes_A A(q)) \cong \pi(M(q)) \cong \pi(M(q)).$$

This means that $M' = \oplus_q \Gamma(X, \pi(M)(q))$, hence $\pi(M') \cong \pi(M)$. Furthermore, it is clear that the isomorphism that we have constructed gives an isomorphism of functors $\Phi_{\Delta_A} \to \text{Id}$.

(ii) Let $M' = \Phi_{N_1 \otimes_{A_0} N_2}(M)$. Then

$$M'_q = \Gamma(X, \pi(M \otimes_A N_1 \otimes_{A_0} (N_2)_q)) = \Gamma(X, \pi(M \otimes_A N_1)) \otimes_{A_0} (N_2)_q,$$

hence $M' = \Gamma(X, \pi(M \otimes_A N_1)) \otimes_{A_0} N_2$. \hfill \Box

**Remark.** It is clear that $\Delta_A$ can be endowed with an algebra structure. Furthermore, it is easy to show that $qgr^{2r}(\Delta_A) \cong qgr^r(A)$. Finally, the multiplication in $A$ gives an epimorphism $A \otimes_{A_0} A \to \Delta_A$. This way, one may view $\Delta_A$ as a diagonal embedding $\Delta_X : X \hookrightarrow X \times X$.  

33
Once we have a diagonal $X \hookrightarrow X \times X$, we could apply standard techniques, provided one finds a resolution of diagonal. If $A$ is Koszul one may obtain a resolution of diagonal as follows. Consider the double Koszul bicomplex of $A$:

\[
\cdots \xrightarrow{d_R} \bigoplus_{i,j} A(e_i + e_j) \otimes (e_i + e_j) \otimes A(-e_i - e_j) \xrightarrow{d_L} \bigoplus_{i} A(e_i) \otimes A(-e_i) \xrightarrow{d_R} A \otimes A
\]

where both $d_R$ and $d_L$ are induced by the differential in the Koszul complex of $A$. Write

\[
K^p(A) = \text{Ker} \left( A(-p) \otimes (A^p)^* \rightarrow \bigoplus_{\{i \mid e_i \leq p\}} A(e_i - p) \otimes (A^p_{e_i})^* \right),
\]

for the cohomology of the truncated Koszul complex. Using the Koszul property of the algebra $A$ and mimicking the proof ([KKO], Proposition 4.7) we deduce

**Proposition 8.10** The following complex is exact

\[
\cdots \xrightarrow{d_R} \bigoplus_{i,j} K^{e_i + e_j} \otimes A(-e_i - e_j) \xrightarrow{d_L} \bigoplus_{i} K^{e_i} \otimes A(-e_i) \xrightarrow{d_R} A \otimes A \rightarrow \Delta_A \rightarrow 0.
\]

where the map $A \otimes A \rightarrow \Delta_A$ is given by the multiplication in $A$. \hfill \Box

Let $Q^p = \pi(K^p(A))^*$. Combining 8.10 with 8.9 we obtain the Beilinson spectral sequence.

**Corollary 8.11** Assume that $A$ is Koszul and $A^1$ is Frobenius. Then for any $F \in X$ there exists a spectral sequence with the first term

\[
E^{p,q}_1 = \bigoplus_{\{p \mid |p|=p\}} \text{Ext}^q(Q^p,F) \otimes A_0 \otimes (\mathcal{O}(-p)) \Rightarrow E^{p,q}_\infty = \begin{cases} F, & i = 0 \\ 0, & \text{otherwise} . \end{cases}
\]

In the special case of the algebra $Q$, the only non-vanishing components of $Q^p$ are

\[
Q^0 = \mathcal{O}, \quad Q^{e_1} = e^{-1} \otimes \mathcal{O}(1,0), \quad Q^{e_2} = e^{-1} \otimes \mathcal{O}(0,1), \quad Q^{e_1+e_2} = \mathcal{O}(1,1)
\]

Thus, Beilinson spectral sequence takes the form of (5.7).
References

[AZ] M. Artin, J.J. Zhang: Noncommutative projective schemes Adv. Math. 109 (1994), 228-287.

[BGK] V. Baranovsky, V. Ginzburg, A. Kuznetsov: Quiver varieties and a noncommutative \( \mathbb{P}^2 \), ArXiv:math.AG/0103068, to appear in Compositio Mathem.

[BW] Yu. Berest, G. Wilson: Automorphisms and ideals of the Weyl algebra. Math. Ann. 318 (2000), 127–147.

[BW2] Yu. Berest, G. Wilson: Ideal classes of the Weyl algebra and noncommutative projective geometry. Int. Math. Res. Not. (2002), no. 26, 1347–1396.

[CaH] R. C. Cannings, M. P. Holland: Right ideals of rings of differential operators. J. Algebra 167 (1994), 116–141.

[CaH2] R. C. Cannings, M. P. Holland: Limits of compactified Jacobians and \( D \)-modules on smooth projective curves. Adv. Math. 135 (1998), 287–302.

[CBH] W. Crawley-Boevey, M.P. Holland: Noncommutative deformations of Kleinian singularities. Duke Math. J. 92 (1998), 605–635.

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Invent. Math. 147 (2002), 243-348, [arXiv:math.AG/0011114 ].

[Hu] L. Huishi: Global dimension of graded local rings. Comm. Algebra 24 (1996), 2399–2405.

[KKO] A. Kapustin, A. Kuznetsov, D. Orlov: Noncommutative Instantons and Twistor Transform. Comm. Math. Phys. 221 (2001), 385–432.

[Lyu] V. Lyubashenko: Hopf algebras and vector symmetries. Uspekhi Math. Nauk 41, 185–186 (1986).

[PS] A. Pressley, G. Segal: Loop groups. Oxford Mathematical Monographs. Oxford Univ. Press, 1986.

[W] G. Wilson: Collisions of Calogero-Moser particles and an adelic Grassmannian. Invent. Math. 133 (1998), 1–41.

[Y] A. Yekutieli: Dualizing complexes over noncommutative algebras. J. Algebra 153 (1992), 41–84.

[YZ] A. Yekutieli, J.J. Zhang: Serre duality for noncommutative projective schemes. Proc. Amer. Math. Soc. 125 (1997), 697-707.

V.B.: Department of Mathematics, California Institute of Technology, Pasadena CA 91125, USA; baranovs@caltech.edu

V.G.: Department of Mathematics, University of Chicago, Chicago IL 60637, USA; ginzburg@math.uchicago.edu

A.K.: Institute for Information Transmission Problems, Russian Academy of Sciences, 19 Bolshoi Karetnyi, Moscow 101447, Russia; sasha@kuznetsov.mccme.ru