A Survey of Quantum Enhancements

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Abstract

In this short survey article we collect the current state of the art in the nascent field of quantum enhancements, a type of knot invariant defined by collecting values of quantum invariants of knots with colorings by various algebraic objects over the set of such colorings. This class of invariants includes classical skein invariants and quandle and biquandle cocycle invariants as well as new invariants.

Keywords: biquandle brackets, quantum invariants, quantum enhancements of counting invariants

2010 MSC: 57M27, 57M25

1 Introduction

Counting invariants, also called coloring invariants or coloring-counting invariants, are a type of integer-valued invariant of knots or other knotted objects (links, braids, tangles, spatial graphs, surface-links etc.). They are defined by attaching elements of some algebraic structure, envisioned as “colors”, to portions of diagrams according to rules, typically stated in the form of algebraic axioms, which ensure that the number of such colorings is unchanged by the relevant diagrammatic moves. Underlying this simplistic combinatorial picture of diagrams and colorings lurks a more sophisticated algebraic structure, a set of morphisms from a categorical object associated to the knotted object to a (generally finite) coloring object. Perhaps the simplest nontrivial example is Fox tricoloring, where the simple rule of making all three colors match or all three differ at each crossing secretly encodes group homomorphisms from the fundamental group of the knot complement to the group of integers modulo 3. Examples of coloring structures include groups, kei, quandles, biquandles and many more.

An enhancement of a counting invariant is a stronger invariant from which the counting invariant can be recovered [3]. One strategy which has proven successful for defining enhancements is to seek invariants \( \phi \) of colored knots; then for a given \( \phi \), the multiset of \( \phi \) values over the set of colorings of our knot is a new invariant of knots whose cardinality is the original counting invariant but which carries more information about the original knot. One of the first such examples was the quandle cocycle invariant introduced in [2], in which integer-valued invariants of quandle-colored knots known as Boltzmann weights are defined using a cohomology theory for quandles. The multiset of such Boltzmann weights is then an invariant of the original uncolored knot; it is stronger than the quandle counting invariant in question since different multisets of Boltzmann weights can have the same cardinality.

A quantum enhancement, then, is a quantum invariant of \( X \)-colored knots for some knot coloring structure \( X \). In [9] these are conceptualized as \( X \)-colored tangle functors, i.e. assignments of matrices of appropriate sizes to the various \( X \)-colored basic tangles (positive and negative crossings, maximum, minimum and vertical strand) which make up tangles via sideways stacking interpreted as tensor product and vertical stacking as matrix composition. In [6] some examples are found via structures known as biquandle brackets, skein invariants modeled after the Kauffman bracket but with coefficients which depend on biquandle colorings at crossings. In [7] biquandle brackets are generalized to include a virtual crossing as a type of smoothing. A special case of biquandle brackets was described independently in [1]. In [4] a type of biquandle bracket whose
skein relation includes a vertex is considered. In [8] biquandle brackets are defined using trace diagrams in order to allow for recursive expansion as opposed to the state-sum definition in [6].

This paper is organized as follows. In Section 2 we survey some knot coloring structures and look in detail at one such structure, biquandles. In Section 3 we see the definition and examples of biquandle brackets as an example of a quantum enhancement. In Section 4 we summarize a few other examples of quantum enhancements, and we end in Section 5 with some questions for future research.

2 Biquandles and Other Coloring Structures

A knot coloring structure is a set $X$ whose elements we can think of as colors or labels to be attached to portions of a knot or link diagram, together with coloring rules chosen so that the number of valid colorings of a knot diagram is not changed by Reidemeister moves and hence defines an invariant. In this section we will look in detail at one such structure, known as biquandles, and then briefly consider some other examples. For more about these topics, see [3].

**Definition 1.** Let $X$ be a set. A biquandle structure on $X$ is a pair of binary operations $◃$, $▹ : X \times X \rightarrow X$ satisfying the following axioms:

(i) For all $x \in X$, we have $x ◃ x = x ▹ x$,

(ii) The maps $S : X \times X \rightarrow X \times X$ and $\alpha_x, \beta_x : X \rightarrow X$ for each $x \in X$ defined by

$$\alpha_x(y) = y ▹ x, \quad \beta_x(y) = x ◃ y \quad \text{and} \quad S(x, y) = (y ▹ x, x ◃ y)$$

are invertible, and

(iii) For all $x, y, z \in X$, we have the exchange laws:

$$
\begin{align*}
(x ◃ y) ◃ (z ▹ y) &= (x ▹ y) ◃ (z ▹ y) \quad (iii.i) \\
(x ▹ y) ▹ (z ◃ y) &= (x ◃ y) ▹ (z ◃ y) \quad (iii.ii) \\
(x ▹ y) ▹ (z ◃ y) &= (x ◃ y) ▹ (z ▹ y) \quad (iii.iii)
\end{align*}
$$

The biquandle axioms encode the Reidemeister moves using a coloring scheme with elements of $X$ coloring semi-arcs in an oriented link diagram with the pictured coloring rules at crossings:

Then using the following generating set of oriented Reidemeister moves,
the following theorem is then easily verified:

**Theorem 1.** Given an oriented link diagram $D$ with a coloring by a biquandle $X$, for any diagram $D'$ obtained from $D$ by a single Reidemeister move, there is a unique coloring of $D'$ by $X$ which agrees with the coloring on $D$ outside the neighborhood of the move.

Hence we obtain:

**Corollary 2.** The number of colorings of a knot or link diagram $D$ by a biquandle $X$ is an integer-valued invariant of the knot or link $K$ represented by $D$, called the biquandle counting invariant and denoted by $\Phi^Z_X(K)$.

**Example 1.** (Alexander biquandles) A straightforward example of a biquandle structure is to let $X$ be any commutative ring with identity $R$ with choice of units $s, t$ and define binary operations

$$
\begin{align*}
x \triangleright y &= tx + (s-t)y \\
x \triangleleft y &= sx.
\end{align*}
$$

For instance, setting $X = \mathbb{Z}_5$ with $t = 3$ and $s = 2$, we obtain biquandle operations

$$
\begin{align*}
x \triangleright y &= 3x + (2 - 3)y = 3x + 4y \\
x \triangleleft y &= 2x.
\end{align*}
$$

To compute the biquandle counting invariant $\Phi^Z_X(K)$ for an oriented knot or link $K$ represented by a diagram $D$, we can then solve the system of the linear equations obtained from the crossings of $D$ using the coloring rule above. For example, the $(4, 2)$-torus link has system of coloring equations below.

$$
\begin{align*}
3x_1 + 4x_8 &= x_2 \\
2x_8 &= x_7 \\
3x_8 + 4x_1 &= x_5 \\
2x_1 &= x_4 \\
3x_3 + 4x_6 &= x_4 \\
2x_6 &= x_5 \\
3x_6 + 4x_3 &= x_7 \\
2x_3 &= x_2
\end{align*}
$$
Then row-reducing over \( \mathbb{Z}_5 \) we have

\[
\begin{bmatrix}
3 & 4 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 4 & 2 \\
4 & 0 & 0 & 0 & 4 & 0 & 0 & 3 \\
2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 4 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 3 & 4 & 0 \\
0 & 4 & 2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and the space of colorings is one-dimensional, so \( \Phi_X^Z(K) = |\mathbb{Z}_5| = 5 \). This distinguishes \( K \) from the unlink of two components, which has \( \Phi_X^Z(U_2) = |\mathbb{Z}_5|^2 = 25 \) colorings by \( X \).

A coloring of a diagram \( D \) representing an oriented knot or link \( K \) by biquandle \( X \) determines a unique homomorphism \( f : B(K) \to X \) from the fundamental biquandle of \( K \), \( B(K) \), to \( X \). Hence the set of colorings may be identified with the homset \( \text{Hom}(B(K), X) \). In particular, an \( X \)-labeled diagram \( D_f \) can be identified with an element of \( \text{Hom}(B(K), X) \), and we have

\[
\Phi_X^Z(K) = |\text{Hom}(B(K), X)|.
\]

See [3] for more about the fundamental biquandle of an oriented knot or link.

The key idea behind enhancements of counting invariants is the observation that it’s not just the number of colorings of a diagram which is invariant, but the set of colored diagrams itself. More precisely, given a biquandle \( X \) and an oriented knot or link diagram \( D \), the set of \( X \)-colorings of \( D \) is an invariant of \( K \) in the sense that changing \( D \) by a Reidemeister move yields a set of colorings of \( D' \) in one-to-one correspondence with the set of colorings of \( D \). Then any invariant \( \phi \) of \( X \)-colored oriented knot or link diagrams can give us a new invariant of the original knot or link, namely the multiset of \( \phi \)-values over the set of colorings of \( D \). We call such an invariant an enhancement of the counting invariant.

**Example 2.** Perhaps the simplest enhancement is the image enhancement, which sets \( \phi \) for a coloring of a diagram to be the size of the image sub-biquandle of the coloring. For example, the trefoil knot has nine colorings by the Alexander biquandle \( X = \mathbb{Z}_3 \) with \( t = 2 \) and \( s = 1 \). Three of these colorings are monochromatic, while six are surjective colorings. Then the counting invariant value \( \Phi_X^Z(3_1) = 9 \) is enhanced to the multiset \( \Phi_{X,\text{Im}}^{\text{im},M}(3_1) = \{1,1,1,3,3,3,3,3,3\} \). For convenience, we can convert the multiset to a polynomial by converting the multiplicities to coefficients and the elements to powers of a formal variable \( u \), so the image enhanced invariant becomes \( \Phi_{X,\text{Im}}^{\text{im},M}(3_1) = 3u + 6u^3 \). This notation, adapted from [2], has the advantage that evaluation of the polynomial at \( u = 1 \) yields the original counting invariant. See [3] for more about enhancements.

**Example 3.** The earliest example of an enhancement of the counting invariant is the family of quandle 2-cocycle invariants, introduced in [2]. In this type of enhancement, we consider biquandles \( X \) with operation \( x \triangledown y = x \), known as quandles, and consider maps \( \phi : X \times X \to A \) where \( A \) is an abelian group. For each \( X \)-coloring of an oriented knot or link diagram \( D \), we obtain a contribution \( +\phi(x,y) \) at positive crossings and \( -\phi(x,y) \) at negative crossings as depicted:
The sum of such contributions over all crossings in an $X$-colored diagram is called the Boltzmann weight of the colored diagram. The conditions on $\phi$ which make the Boltzmann weight unchanged by $X$-colored Reidemeister moves can be expressed in terms of a cohomology theory for quandles: the Boltzmann weight is preserved by Reidemeister III moves if $\phi$ is a rack 2-cocycle and preserved by Reidemeister I moves if $\phi$ evaluates to zero on degenerate cochains, which form a subcomplex; invariance under Reidemeister II moves is automatic from the way the Boltzmann weights are defined. The quotient of rack cohomology by the degenerate subcomplex yields quandle cohomology. In particular, 2-coboundaries always yield a Boltzmann weight of zero, so cohomologous cocycles define the same enhancement. See [2, 3] for more.

Other examples of knot coloring structures include but are not limited to the following:

- **Groups.** Any finite group $G$ defines a counting invariant consisting of the set of group homomorphisms from the knot group, i.e., the fundamental group of the knot complement, to $G$.

- **Quandles.** As mentioned in Example 3, quandles are biquandles $X$ whose over-action operation is trivial, i.e. for all $x, y \in X$ we have $x \triangledown y = x$. Introduced in [5], the fundamental quandle of a knot determines both the knot group and the peripheral structure, and hence determines the knot up to ambient homeomorphism.

- **Biracks.** Biracks are biquandles for framed knots and links, with the Reidemeister I move replaced with the framed version. To get invariants of unframed knots and links using biracks, we observe that the lattice of framings of a link is an invariant of the original link; then the lattice of, say, birack colorings of the framings of an unframed link $L$ forms an invariant of $L$.

For each of these and other coloring structures, enhancements can be defined, resulting in new invariants.

### 3 Biquandle Brackets

A biquandle bracket is a skein invariant for biquandle-colored knots and links. The definition was introduced in [6] (and independently, a special case was introduced in [11]) and has only started to be explored in other recent work such as [8, 7, 4].

**Definition 2.** Let $X$ be a biquandle and $R$ a commutative ring with identity. A biquandle bracket $\beta$ over $X$ and $R$ is a pair of maps $A, B : X \times X \to R^\times$ assigning units $A_{x,y}$ and $B_{x,y}$ to each pair of elements of $X$ such that the following conditions hold:

(i) For all $x \in X$, the elements $-A_{x,x}^2 B_{x,x}^{-1}$ are all equal, with their common value denoted by $w$,

(ii) For all $x, y \in X$, the elements $-A_{x,y} B_{x,y}^{-1} A_{x,y}^{-1} B_{x,y}$ are all equal, with their common value denoted by $\delta$, and

(iii) For all $x, y, z \in X$ we have

\[
\begin{align*}
A_{x,y} A_{y,z} & A_{x,z} y, z, \sigma y & = & & A_{x,y} A_{y,z} x, \sigma x A_{x,z} y, z, y, z, \sigma y, z \\
A_{x,y} B_{y,z} B_{x,z} y, z, \sigma y & = & & & B_{x,y} B_{y,z} x, \sigma x A_{x,z} y, z, y, z, \sigma y, z \\
B_{x,y} A_{y,z} B_{x,z} y, z, \sigma y & = & & & B_{x,y} A_{y,z} x, \sigma x B_{x,z} y, z, y, z, \sigma y, z \\
A_{x,y} A_{y,z} B_{x,z} y, z, \sigma y & = & & & A_{x,y} A_{y,z} x, \sigma x B_{x,z} y, z, y, z, \sigma y, z \\
B_{x,y} A_{y,z} A_{x,z} y, z, \sigma y & & & & + A_{x,y} B_{y,z} A_{x,z} y, z, \sigma y, z \\
+ A_{x,y} B_{y,z} A_{x,z} y, z, \sigma y & & & & + \delta A_{x,y} B_{y,z} A_{x,z} y, z, \sigma y, z, y, z, \sigma y, z \\
+ \delta B_{x,y} B_{y,z} A_{x,z} y, z, \sigma y & & & & + B_{x,y} B_{y,z} A_{x,z} y, z, \sigma y, z, y, z, \sigma y, z \\
B_{x,y} A_{y,z} A_{x,z} y, z, \sigma y & & & & = B_{x,y} A_{y,z} x, \sigma x A_{x,z} y, z, y, z, \sigma y, z.
\end{align*}
\]
We can specify a biquandle bracket $\beta$ over a ring $R$ and finite biquandle $X = \{x_1, \ldots, x_n\}$ by giving an $n \times 2n$ block matrix with entries in $R$ whose left block lists the $A_{x,y}$ values and whose right block lists the $B_{x,y}$ values:

$$
\beta = \begin{bmatrix}
A_{x_1,x_1} & A_{x_1,x_2} & \cdots & A_{x_1,x_n} & B_{x_1,x_1} & B_{x_1,x_2} & \cdots & B_{x_1,x_n} \\
A_{x_2,x_1} & A_{x_2,x_2} & \cdots & A_{x_2,x_n} & B_{x_2,x_1} & B_{x_2,x_2} & \cdots & B_{x_2,x_n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{x_n,x_1} & A_{x_n,x_2} & \cdots & A_{x_n,x_n} & B_{x_n,x_1} & B_{x_n,x_2} & \cdots & B_{x_n,x_n}
\end{bmatrix}.
$$

The biquandle bracket axioms are the conditions required for invariance of the state-sum value obtained by summing the products of smoothing coefficients and powers of $\delta$ and $w$ for each Kauffman state of an $X$-colored diagram under $X$-colored Reidemeister moves. More precisely, we write skein relations

and assign $\delta$ to be the value of a component in a smoothed state, $w$ the value of a positive kink and $w^{-1}$ the value of a negative kink.

More formally, we have:

**Definition 3.** Let $\beta$ be a biquandle bracket over a finite biquandle $X$ and commutative ring with identity $R$ and let $D$ be an oriented knot or link diagram. Then for each $X$-coloring $D_f$ of $D$, let $\beta(D_f)$ be the state-sum value obtained by summing over the set of Kauffman states the products of smoothing coefficients $\phi_{x,y} \in \{A_{x,y}^{\pm 1}, B_{x,y}^{\pm 1}\}$ at each crossing as determined by the colors and smoothing type times $\delta^k w^n \delta^{-p}$ where $k$ is the number of components in the state, $n$ is the number of negative crossings and $p$ is the number of positive crossings. That is, for each $X$-coloring $D_f$ of $D$, we have

$$
\beta(D_f) = \sum_{\text{Kauffman States}} \left( \prod_{\text{crossings}} \phi_{x,y} \right) \delta^k w^n \delta^{-p}.
$$

Then the multiset of $\beta(D_f)$-values over the set of $X$-colorings of $D$ is denoted

$$
\Phi_X^{\beta,M}(D) = \{\beta(D_f) \mid D_f \in \text{Hom}(\mathcal{B}(K), X)\}.
$$
The same data may be expressed in “polynomial” form (scare quotes since the exponents are not necessarily integers but elements of \(R\)) as
\[
\Phi^\beta_X(D) = \sum_{D_f \in \text{Hom}(B(K), X)} u^{\beta(D_f)}.
\]

Hence we have the following theorem (see [6]):

**Theorem 3.** Let \(X\) be a finite biquandle, \(R\) a commutative ring with identity and \(\beta\) a biquandle bracket over \(X\) and \(R\). Then for any oriented knot or link \(K\) represented by a diagram \(D\), the multiset \(\Phi^\beta_M(D)\) and the polynomial \(\Phi^\beta_X(D)\) are unchanged by Reidemeister moves and hence are invariants of \(K\).

**Example 4.** A biquandle bracket in which \(A_{x,y} = B_{x,y}\) for all \(x, y \in X\) defines a biquandle 2-cocycle \(\phi \in H^2_B(X)\). In this case the polynomial invariant \(\Phi^\phi_X(D)\) is the product of the biquandle 2-cocycle enhancement \(\Phi^\phi_X(K)\) with the Kauffman bracket polynomial of \(K\) evaluated at \(A = -1\). See [6] for more details.

**Example 5.** A biquandle bracket \(\beta\) over the biquandle of one element \(X = \{x_1\}\) is a classical skein invariant. For example, the biquandle bracket \(\beta\) over \(R = \mathbb{Z}[A^{\pm 1}]\) with \(A_{x_1,x_1} = A\) and \(B_{x_1,x_1} = A^{-1}\) (and hence \(\delta = -A^2 - A^{-2}\) and \(w = -A^3\)) is the Kauffman bracket polynomial.

Thus, biquandle brackets provide an explicit unification of classical skein invariants and biquandle cocycle invariants. Even better though, there are biquandle brackets which are neither classical skein invariants nor cocycle invariants, but something new.

**Example 6.** Let \(R = \mathbb{Z}_7\) and \(X = \mathbb{Z}_2 = \{1, 2\}\) with the biquandle operations \(x \triangleright y = x \triangleright y = x + 1\) (note that we are using the symbol 2 for the class of zero in \(\mathbb{Z}_2\) so that our row and column numberings can start with 1 instead of 0). Then via a computer search, one can check that
\[
\beta = \begin{bmatrix}
1 & 5 & 3 & 1 \\
4 & 1 & 5 & 3 
\end{bmatrix}
\]
defines a biquandle bracket, with \(\delta = -(3)^{-1} - 1^{-1} 3 = -5 - 3 = -1 = 6\) and \(w = -(1)^2 (3)^{-1} = -5 = 2\). The skein relations at positive crossings are as shown:

Let us illustrate in detail the computation of \(\Phi^\beta_M(L)\) where \(L\) is the oriented Hopf link with two positive crossings. There are four \(X\)-colorings of the Hopf link and indeed of every classical link – the unenhanced counting invariant with this choice of coloring biquandle \(X\) can detect component number of classical links.
and nothing else. However, the biquandle bracket enhancement gives us more information.

For each coloring, we compute the state-sum value:

\[ \text{(1)(1)} + \text{(1)(3)} + \text{(3)(1)} + \text{(3)(3)} = 2^2 = 1; \]

\[ \text{(1)(1)} + \text{(1)(3)} + \text{(3)(1)} + \text{(3)(3)} = 2^2 = 1; \]
yields \([(5)(4)6^2 + (1)(4)6 + (5)(5)6 + (1)(5)6^2]2^{-2} = (6 + 3 + 3 + 5)2 = 6;
\]

yields \([(4)(5)6^2 + (5)(5)6 + (4)(1)6 + (5)(1)6^2]2^{-2} = (6 + 3 + 3 + 5)2 = 6, and

yields \([(1)(1)6^2 + (3)(1)6 + (1)(3)6 + (3)(3)6^2]2^{-2} = (1 + 4 + 4 + 2)2 = 1.\]

Then the multiset form of the invariant is \(\Phi^{\beta,M}_{X}(L) = \{1,1,6,6\}\), or in polynomial form we have \(\Phi^{\beta}_{X}(L) = 2u + 2u^6\). Since the unlink of two components has invariant value \(\Phi^{\beta,M}_{X}(U_2) = \{6,6,6,6\}\), the enhanced invariant is stronger than the unenhanced counting invariant.

Example 6 is merely a small toy example meant to illustrate the computation of the invariant, of course. Biquandle brackets over larger biquandles and larger (finite or infinite) rings have already proved their utility as powerful knot and link invariants, with cocycle invariants at one extreme (information concentrated in the coloring) and skein invariants at the other (information concentrated in the skein relations). So far, the known examples of biquandle brackets which are neither classical skein invariants nor cocycle invariants have been largely found by computer search; it is our hope that other examples can be found by more subtle methods.

4 Other Quantum Enhancements

Biquandle brackets are one example of a more general phenomenon known as quantum enhancements, broadly defined as quantum invariants of \(X\)-colored knots or other knotted structures for an appropriate coloring structure \(X\). In this section we collect a few other recent examples of quantum enhancements.
In [7], the author together with coauthors K. Oshiro, A. Shimizu and Y. Yaguchi defined \textit{biquandle virtual brackets}, a generalization of biquandle brackets which includes a virtual crossing as a type of smoothing, i.e.,

\[
\begin{align*}
\begin{array}{c}
\begin{xy}
(0,0)*{\ar^{x}}; (0,3)*{\ar^{y}} **@{-},
(0,5)*{\ar^{x}}; (0,8)*{\ar^{y}} **@{-},
(0,8)*{\ar^{y}}; (0,5)*{\ar^{x}} **@{-},
\end{xy}
\end{array}
\end{align*} = A_{x,y}
\begin{align*}
\begin{array}{c}
\begin{xy}
(0,0)*{\ar^{x}}; (0,3)*{\ar^{y}} **@{-},
(0,5)*{\ar^{x}}; (0,8)*{\ar^{y}} **@{-},
(0,8)*{\ar^{y}}; (0,5)*{\ar^{x}} **@{-},
\end{xy}
\end{array}
\end{align*} + B_{x,y}
+ V_{x,y}
\begin{align*}
\begin{array}{c}
\begin{xy}
(0,0)*{\ar^{x}}; (0,3)*{\ar^{y}} **@{-},
(0,5)*{\ar^{x}}; (0,8)*{\ar^{y}} **@{-},
(0,8)*{\ar^{y}}; (0,5)*{\ar^{x}} **@{-},
\end{xy}
\end{array}
\end{align*} + U_{x,y}
\begin{align*}
\begin{array}{c}
\begin{xy}
(0,0)*{\ar^{x}}; (0,3)*{\ar^{y}} **@{-},
(0,5)*{\ar^{x}}; (0,8)*{\ar^{y}} **@{-},
(0,8)*{\ar^{y}}; (0,5)*{\ar^{x}} **@{-},
\end{xy}
\end{array}
\end{align*}
\end{align*}
\]

A biquandle bracket is then a biquandle virtual bracket in which the virtual coefficients are all zero. This framework gives another way of recovering the biquandle cocycle invariants, this time without the factor of the Kauffman bracket evaluated at \(-1\), by having classical smoothing coefficients all zero. Examples of these invariants are shown to be able to detect mirror image and orientation reversal. In particular, these are examples of invariants of classical knots and links which are defined in a way that fundamentally requires virtual knot theory; it is our hope that these invariants can provide a reason for classical knot theorists to care about virtual knot theory.

In [8], the author together with coauthor N. Oyamaguchi addressed the issue of how to compute biquandle brackets in a recursive term-by-term expansion as opposed to the state-sum approach described in Section 3. Our method uses \textit{trace diagrams}, trivalent spatial graphs with decorations carrying information about smoothings which enable maintaining a biquandle coloring throughout the skein expansion.

These are equivalent to the original state-sum biquandle brackets but can allow for faster hand computation as well as for allowing moves and diagram reduction during the course of the expansion. Technical conditions are identified for which trace moves (e.g., moving a strand over, under or through a trace) are permitted depending on certain algebraic conditions being satisfied by the bracket coefficients.

In [4], another skein relation is used in the biquandle bracket setting, superficially similar to the biquandle
virtual brackets described above but with the virtual smoothing replaced with a 4-valent vertex.

\[
\begin{align*}
  x & = A_{x,y} + B_{x,y} + C_{x,y} \\
  y & = D_{x,y} + E_{x,y} + F_{x,y}
\end{align*}
\]

This family of quantum enhancements includes Manturov’s parity bracket invariant as special case, as well as the biquandle brackets defined in Section 3.

In [9], the author together with coauthor V. Rivera (a high school student at the time) introduced the notion of quantum enhancements in the form of \(X\)-colored TQFTs or \(X\)-colored tangle functors for the case of involutory biracks \(X\). These are given by matrices \(X_{x,y}^{\pm 1}, I, U\) and \(N\) over a commutative ring with identity \(R\) corresponding to the basic \(X\)-colored tangles

\[
\begin{align*}
  X_{x,y} & \quad X_{x,y}^{-1} & I & \quad N & \quad U
\end{align*}
\]

such that the tensor equations representing the \(X\)-colored Reidemeister moves and planar isotopy moves are satisfied, where we interpret vertical stacking as matrix product and horizontal stacking as tensor (Kronecker) product. Computer searches for solutions to these equations proved inefficient even for small rings, so we considered \(X\)-colored braid group representations as a first step. Indeed, biquandle brackets have so far been the best method for producing examples of this type of quantum enhancement. For example, the biquandle bracket in example [6] defines the following quantum enhancement:

\[
X_{11} = X_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad X_{12} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad X_{21} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},
\]

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 & 4 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}.
\]

To compute a quantum enhancement in this format, our \(X\)-colored oriented diagrams \(D_f\) are decomposed into matrix products of tensor products of the five basic tangles which are then replaced with the appropriate matrices and multiplied to obtain \(1 \times 1\) matrices, i.e., ring elements. These are then multiplied by the appropriate writhe correction factor \(w^{a-p}\) and collected into a multiset. For example, the Hopf link with
The enhanced invariant for the Hopf link in Example 6 is then the multiset
\[
\begin{align*}
(U \otimes U)(I \otimes X_{yx} \otimes I)(I \otimes X_{xy} \otimes I)(N \otimes N)
\end{align*}
\]

as in Example 6.

5 Questions

We end this short survey with some questions and directions for future research.

- As mentioned in [6], biquandle 2-cocycles define biquandle brackets, and the operation of componentwise multiplication of the biquandle bracket matrix of a 2-cocycle with a bracket representing a 2-coboundary yields a biquandle bracket representing a cohomologous 2-cocycle. Weirdly, this also works with biquandle brackets which do not represent 2-cocycles, extending the equivalence relation of cohomology to the larger set of biquandle brackets. What exactly is going on here?

- So far, biquandle brackets over finite biquandles have been found primarily by computer search using finite coefficient rings. We would like to find examples of biquandle brackets over larger finite biquandles and over larger rings, especially infinite polynomial rings.

- The first approach for generalization, examples of which have been considered in [7] and [4], is to apply the biquandle bracket idea to different skein relations. One may find that skein relations which do not yield anything new in the uncolored case can provide new invariants in various biquandle-colored cases.

- In addition to biquandle brackets, we would like to find other examples of quantum enhancements. Possible avenues of approach include representations of biquandle-colored braid groups, biquandle-colored Hecke algebras, biquandle-colored TQFTs and many more.

- Like the Jones, Homflypt and Alexander polynomials, every biquandle bracket should be susceptible to Khovanov-style categorification, providing another infinite source of new knot and link invariants.

- Since biquandle brackets contain both classical skein invariants and cocycle enhancements as special cases, we can ask which other known (families of) knot and link invariants are also describable in this way or recoverable from biquandle bracket invariants. For example, can every Vassiliev invariant be obtained as a coefficient in some biquandle bracket invariant over a polynomial ring, like the coefficients of the Jones polynomial?

- Finally, we can define quantum enhancements for coloring structures other than biquandles and for knotted objects other than classical knots. The possibilities are limitless!
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