An improved approach to find the height of the accretion disk

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Abstract.

In the present work, we propose an improved approach to find the height of the accretion disks around Kerr black holes. The vertical structure of the disk is obtained using a relativistic continuity equation and finally, the result is expressed in terms of the accretion rate and flow parameters. For different models of thin and slim disks, the expressions of the vertical height of the accretion disk are also been reviewed chronologically. We further discuss the basic assumptions, merits, and demerits of these models. Compared to existing approaches, our analysis is more self-consistent and less axiomatic. Further, we find that disk height is directly proportional to accretion rate which could be important to find out the vertical structure of the accretion disk in different black hole accretion states in a hysteresis cycle.

1. Introduction

The vertical structure of an accretion disk is very important to understand the physics of the accretion processes. Observations indicate that the energy spectrum depends on the geometry of the accretion disk which in turn changes with the accretion rate. In general, the vertical structure of the accretion disk does not only depend on the accretion rate but also changes with the geometry of the black holes and the accretion flow parameters. In most cases the vertical structure of an accretion disk is determined from the hydrostatic equilibrium in the vertical direction by balancing the pressure gradient force to the gravitational force. Based on these, several models of vertical structure of the accretion disk have been proposed. Depending on different states of the compact objects, these models are important to understand the vertical structure of the accretion disk. Among them (1) Thick disk: Polish doughnuts [17,26,27], (2) Thin disk [2,6,24,29,30,32,33] (3) Slim disk [2,6,31] (4) Advection dominated accretion flows (ADAF) [3,16,21,22,28,34] (5) Two component advective flow (TCAF) [8,14,23] etc are quite well known. Depending on the accretion rate these models are useful to predict the vertical structure of the accretion disk for the different black hole systems. The predicted solutions of these models are different from each other because they are obtained from different governing equations with different boundary conditions. The standard model for Keplerian accretion disk first proposed by Shakura-Sunyaev in 1973 (SS73) in order to determine the radial and vertical structure of the flow. Almost at the same time, Novikov and Thorne (1973) (NT73) propose an alternative model in the relativistic framework. The height of the accretion disk is presented by taking the gravitational tidal force calculated from the components of the Riemannian tensor.
Expanding the metric equations in several orders of \( z \) around Kerr black hole, Riffert, and Herold (1995) (RH95) [30] propose an algebraic correction to the NT73. Both NT73 and RH95 are well-describe to the vertical structure of the accretion disk, however, it diverges at ISCO, although it is well inside of the disk. To circumvent this problem, Abramowicz, Lanza, and Percival (1997) (ALP97) [2] extended the work by RH95. The singularity problem at the circular photon orbit is removed by using a similar metric expansion in suitably chosen coordinate. The work of ALP97 is well appreciated in the study of the relativistic accretion flows however obtained using the notions of the particle dynamics. Gammie and Popham (1998) further extended the ALP97’s work for thin and advection-dominated flows in a strong rotation limit of a black hole [13]. For the transonic accretion flows, the vertical structure of the disk is also being predicted by Chakrabarti [7] and many others [1, 10, 25]. Adding heat generations (such as viscous heating, turbulence, etc) and radiation pressure, different models of disk height have been proposed in Newtonian, as well as relativistic framework [4, 11, 15, 18]. All of these work are obtained by assuming certain assumptions and approximations and lead us to an approximate vertical structure of the disk. On the contrary, our work is obtained relativistically using the four divergence of the energy-momentum tensor. Interestingly we find that the accretion rate changes the vertical geometry of the accretion disk. This might be helpful in determining the different spectral states of the black hole candidate.

The organization of this paper is as follows. In Section 2, we present the Kerr geometry and relativistic flow variables. In Section 3 we discuss previous vertical structure for thin and slim accretion disks. In Section 4, we present the expression for the height of the accretion disks. Finally, we end up our conclusions in section 5.

2. Kerr geometry and relativistic flow variables

2.1. Units
We consider the units of velocity as \( c \), distance \( \frac{GM}{c^2} \) and time \( \frac{GM}{c^3} \) respectively, where, \( c \), \( G \) and \( M \) are the velocity of light, gravitational constant and the mass respectively.

2.2. Geometry of Kerr black holes
We consider a steady and axisymmetric accretion disk in the equatorial plane (\( \theta = \frac{\pi}{2} \)) of a Kerr black hole. The geodesic equation in Boyer-Lindquist coordinate (\( r, \theta, \phi, t \)) is given by,

\[
\begin{multline*}
\frac{dS^2}{\Delta} = -\left(1 - \frac{2r}{\Sigma}\right)dt^2 - \left(\frac{4ar^2 \sin^2 \theta}{\Sigma}\right)dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2 + a^2 + \frac{2ra^2 \sin^2 \theta}{\Sigma}) \sin^2 \theta d\phi^2
\end{multline*}
\]

Here \( \Delta = r^2 - 2ar^2 + a^2 \) and \( \Sigma = r^2 + a^2 \cos^2 \theta \), \( a \) is the Kerr parameter

2.3. Energy-momentum tensor and thermodynamic variables
For a perfect fluid, the stress-energy tensor \( T^{\mu\nu} \) is given by,

\[
T^{\mu\nu} = (P + \rho)u^\mu u^\nu + Pg^{\mu\nu}
\]

where, \( u^\mu \) is the four velocity of a fluid element, \( P \) is the pressure and total mass density, \( \rho = \rho_0(1 + U) \), where \( U = \frac{P}{(\gamma - 1)\rho_0} \) being the specific internal energy, \( \gamma = \frac{c_P}{c_v} \) connected to the polytropic index (\( n \)) by \( n = \frac{1}{(\gamma - 1)} \). We further consider an adiabatic equation of state

\[ P = K \rho_0^n \]

Using relativistic definition of the adiabatic sound speed \( a_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_s \) the expression
of the enthalpy \((h)\) is given by,

\[ h = \frac{P + \rho}{\rho_0} = \frac{1}{1 - na_s^2}. \]  

(3)

The other thermodynamical variables can be found in [20].

3. Vertical structure for thin and slim accretion disks

In SS73, the vertical structure for a geometrically thin accretion disk describes by considering the vertical hydrostatic equilibrium : \(\frac{dP}{\rho} = -\frac{v_\phi^2}{R^2}z\). The half-thickness of the disk is therefore obtained as \(H = \frac{a_\phi}{v_\phi}R\), where \(a_\phi\) is the sound speed, \(v_\phi\) is the circular Keplerian velocity at a radius \(R\). This simple relation determines the geometrical structure of the thin disks. The vertical structure roughly varies as \(R^{3/8}\) when the constant viscosity parameter is considered. However, it changes with heat transport. Generally, in the inner region, the ratio \(\frac{H}{R}\) increases towards the center remains increasing outward in the middle region and decreases slightly in the outer region to becomes constant far away. This nature of the vertical structure is being computed by Meyer and Meyer-Hofmeister [19]. However, this thin disk approach becomes inappropriate for a high mass accretion rate. The disk is no longer thin when the mass accretion rate approaches the Eddington rate. Furthermore above Newtonian approach needs to be modified particularly at the inner part of the disk where relativistic corrections are required.

After SS73, NT73 [24] provide a relativistic treatment in approximating the disk structure. The tidal acceleration of gravity is determined using,

\[ \nabla \phi = \frac{\phi}{R^2} = \frac{a_\phi}{v_\phi} \]  

(24)

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By circumventing this problem, ALP97 [2] uses a similar expansion to the hydrodynamic equations.

Further in thin disk, \(\tilde{z} \ll 1\) approximation holds. Expanding the metric coefficient upto second order of \((\tilde{z})^2\), RH95 [30] made some relativistic corrections to the gravitational tidal force. Employing the appropriate correction in the standard disk model NT73, the resulting reduced disk height is obtained from the equation of hydrostatic equilibrium as,

\[ H = \sqrt{\frac{P}{\rho_0}} \sqrt{\frac{r^3}{M}} \sqrt{\frac{(r^2 + a^2)^2 - \Delta a^2}{\gamma^2((r^2 + a^2)^2 + 2\Delta a^2)}}, \]  

(4)

where \(\gamma = \frac{r^2 - 2Mr \pm a(Mr)^{1/2}}{r^2 - 3Mr \pm 2a(Mr)^{1/2}}\) which diverges when \([r^2 - 3Mr \pm 2a(Mr)^{1/2}] = 0\) i.e. at ISCO. The circular photon orbit is well inside the accretion disk and the above equation nicely describes the vertical height particularly in the outer region. The expression for the vertical structure changes with heat generation. Both the models (SS73 and NT73) use the heat generation through the viscous mechanism. The \(r-\phi\) component of the viscous stress tensor is proportional to \(-\alpha P\). The generated heat is radiated away and the disk becomes cool. The efficiency of the cooling and heating mechanism determines the vertical structure of the disk.

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\[ H \equiv 2 \sqrt{\frac{P(0)}{\rho_0(0)}} \sqrt{\frac{r^3}{M}} \sqrt{\frac{B}{C}}, \]  

(5)

where \(B = 1 - \frac{3M}{r} + \frac{2a\sqrt{M}}{r^{3/2}}, \ C = 1 - \frac{4a\sqrt{M}}{r^{3/2}} + \frac{3a^2}{r^2}\). However, the divergence nature of the height expression remains unaltered as it follows from NT73.

To circumvent this problem, ALP97 [2] uses a similar expansion to the hydrodynamic equations. Following the expansion scheme in terms of the order of \(\tilde{z}\) they provide an approximate expression for disk height both in the Newtonian and relativistic regimes. In the relativistic approach, the result based on the following assumptions (i) 4-velocity component : \(U_R = \frac{(\tilde{z})}{(\tilde{z})}\)
V_0(r) + O^2(cos \theta), \quad U_\theta = \cos \theta U_1(R) + O^3(cos \theta), \quad U_t = \mathcal{E}, \quad U_\phi = -\mathcal{L}. 

(ii) pressure density : 
\rho(R, \theta) = P_0(R)\left[1 \right. \left(-\frac{(\cos^2 \theta)}{\cos^2 \theta}\right), \quad \rho(r, z) = \rho(r)

(iii) magnitude 
\rho \frac{dP}{dr}
is small and (iv) the notion of the conserve quantities in particle dynamics around the Kerr black holes is used. Employing the above assumptions in the hydrodynamic equations the expression for the vertical structure can be simplified as,

\[ -2P_0 + \left( \frac{H}{R} \right)^2 \left( \frac{1}{R^2} \right)(L_\ast^2 + U_1^2 - \Delta V_0 \frac{dU_1}{dR}) = 0 \] 

where \( L_\ast^2 = L^2 - a^2(E^2 - 1) \). Note that \( L = -u_\phi \) is the conserved angular momentum and \( \mathcal{E} = u_t \) is the conserved energy of particle. However for the accretion flows, the corresponding conserve quantities are \( h u_t = -\mathcal{E} \) and \( h u_\phi = \mathcal{L} \) where \( h \) is the enthalpy. Therefore we modify the above relation accordingly and find the expression for vertical height as,

\[ H = \frac{a_s R^2}{\sqrt{\gamma - a_s^2}} \frac{2}{H_D} \] 

where \( H_D = (1 - na_s^2)(l^2 - a^2)\mathcal{E}^2 + a^2 + U_1^2 - \Delta V_0 \frac{dU_1}{dR} \). Further modification can be obtained by incorporating the term \( \frac{1}{\rho} \frac{dP}{dr} \) (see reference [9]).

4. Our approach

The energy momentum tensor \( T^{ab} \) satisfies generally covariant form of the conservation equation in curved space-time: \( \nabla_a T^{ab} = 0 \), where \( \nabla_a \) represent the covariant derivative. Using Gauss’s theorem we get,

\[ \int_{\text{Space-time}} d^4 x \nabla_a J^a = \int_{\text{Boundary}} dS n_a J^a \] 

Where \( J^a = T^{ab} \xi_b \), \( \xi^a \) be a killing vector and \( n^a \) is unit normal vector. Solving the above integral equation, the expression for mass accretion rate \( \dot{M} \) for time independent flows can be obtained as,

\[ \dot{M} = 2\pi \int_{-H}^{H} n_r J^r \sqrt{|G|} d\zeta \quad \text{where} \quad G = g_{zz}(g_{\phi\phi}g_{tt} - g_{t\phi}^2) \]

\[ = 2\pi \mathcal{E} \int_{-H}^{H} (\rho_0 u^r) \sqrt{\frac{R^2 \Delta}{R^2 + z^2}} \sqrt{\frac{z^2(R^2 + z^2) + R^2 \Delta}{R^2(R^2 + z^2) + z^2 \Delta}} d\zeta, \]

where both the quantities specific energy \( \mathcal{E} \) and mass flux \( \dot{M} \) are conserved in the flow [20] and \( u^r = \frac{d}{d\tau} \) is the radial component of the flow velocity \( r = \sqrt{R^2 + z^2} \). From the above equation...
mass accretion rate $\dot{M}$ can be obtained if functional variation of $(\rho_0 u^r)$ is known. In particular, in this case, $(\rho_0 u^r)$ is function of both $r$ and $z$ and varies with the specific energy and angular momentum $(\mathcal{E}, l)$ of the flow. For most simplest case one can evaluate the above integral by assuming (i) $\rho_0 u^r$ constant and (ii) expanding the integrand upto the order of $(\frac{z}{R})^2$. For better accuracy, one can, of course, take higher-order terms of $(\frac{z}{R})$ in the integral equation (9). Further, the result can also be improved by taking the Gaussian type distribution of $(\rho_0 u^r)$ along the vertical direction. This case is discussed elsewhere in future communication (for details please see [9]).

Here we only consider the simplest case and obtain $\dot{M}$ as,

$$\dot{M} = \dot{M}_0 \mathcal{E} \int_{-H/R}^{H/R} [1 + \frac{K_R}{\Delta_R} (\frac{z}{R})^2] d(\frac{z}{R}),$$

where $\dot{M}_0 = 2\pi \Delta R (\rho_0 u^r)$, $\Delta_R = R^2 - 2R + a^2$, $K_R = 3R - 2(a^2 + 1)$ and the coefficient of $(\frac{z}{R})^2$ is approximated upto $\frac{1}{16}$. Now integrating the above equation we obtained a cubic equation of $(\frac{H}{R}) (< 1)$,

$$\frac{1}{A} (\frac{H}{R})^3 + (\frac{H}{R}) - X_0 = 0,$$

$$\therefore \quad H = (\frac{\dot{M}}{\dot{M}_0})(\frac{R}{3\mathcal{E}})\left[1 - \frac{X_0^2}{A}\right],$$

where $X_0 = \frac{\dot{M}}{2\dot{M}_0\mathcal{E}^2}$, $\frac{1}{A} = \frac{K_R}{3\Delta_R}$. The above expression gives the vertical height of the accretion disk. Notice that the vertical height is directly proportional to the mass accretion rate ($\dot{M}$). It also depends on the geometry of the black hole($g_{\mu \nu}$), and accretion flow parameters($\mathcal{E}, l$)(see also equation (9)). The dependence on disk angular momentum $l$ comes through $u^r$. The $\dot{M}_0$ varies with the $\rho_0 u^r$ which in general changes with the angular momentum of the disk. One can also see that the spin of the black hole, particularly at high values, has a non-negligible effect in determining the vertical structure of the accretion disk. Further, in the soft state, intermediate state and hard state of the black hole, the accretion rate changes appreciably. Therefore the spectral state and the corresponding vertical structure of the accretion disk can be obtained from our result.

5. Discussions

In this work, we tried to give an attempt to find out the vertical structure of the accretion disk in the general relativistic framework using the continuity equation. So far, the disk height is mostly calculated considering the vertical hydrostatic equilibrium condition. We discuss some of the important vertical structures available in the literature. A comparative study has also been made based on their basic assumptions, consistency, merits, and demerits. We find that with respect to the previous results, our analysis is more consistent and with less axioms and therefore can be used for the general cases.

Further, it might be help full in determining the spectral variation of black hole candidates. The spectral transition and the corresponding spectral indices change from the soft state to the intermediate state and then to the hard state. The transition of these states mainly depends on the accretion rate. High accretion rate changes the geometry of the accretion disk. Interestingly from our result, we find that the vertical height is directly proportional to the mass accretion rate. It also varies with the geometry of the black hole, and accretion flow parameters. Therefore we think that our result would be important to determine the vertical structure of the accretion disk in different black hole candidates in their hysteresis loop.
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