Double degeneracy associated with hidden symmetries in the asymmetric two-photon Rabi model

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In this paper, we uncover the elusive level crossings in a subspace of the asymmetric two-photon quantum Rabi model (tpQRM) when the bias parameter of qubit is an even multiple of the renormalized cavity frequency. Due to the absence of any explicit symmetry in the subspace, this double degeneracy implies the existence of the hidden symmetry. The non-degenerate exceptional points are also given completely. It is found that the number of the doubly degenerate crossing points in the asymmetric tpQRM is comparable to that in asymmetric one-photon QRM in terms of the same order of the constrained conditions. The bias parameter required for occurrence of level crossings in the asymmetric tpQRM is characteristically different from that at a multiple of the cavity frequency in the asymmetric one-photon QRM, suggesting the different hidden symmetries in the two asymmetric QRM.

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I. INTRODUCTION

The simplest interaction between a two-level system (qubit) and a single mode bosonic cavity (oscillator) was described by the quantum Rabi model (QRM) [1, 2], which is thus a fundamental textbook model in quantum optics [3]. It has been demonstrated in many advanced solid devices, such as circuit quantum electrodynamics (QED) system [4, 5], trapped ions [6], and quantum dots [7] from weak coupling to the ultra-strong coupling, even deep strong coupling between the artificial atom and resonators [8-10].

In contrast to the conventional cavity QED system, the artificial qubit appears in modern solid devices usually contains both the splitting $\Delta$ and the bias $\epsilon$ between the two qubit states, thus the so-called asymmetric QRM is ubiquitous. Driven by the proposals and experimental realizations of the various QRM which, the asymmetric two-photon QRM (tpQRM) are also realized or stimulated to explore new quantum effects [11-13]. The typical two asymmetric QRM can be generally written in a unified way as

$$H_p = \Delta \sigma_z + \frac{\epsilon}{2} \sigma_x + \omega a^\dagger a + g \left[ (a^\dagger)^p + a^p \right] \sigma_x,$$

(1)

where the first two terms fully describe a qubit with the energy splitting $\Delta$ and the bias $\epsilon$, $\sigma_{x,z}$ are the Pauli matrices, $a^\dagger$ and $a$ are the creation and annihilation operators with the cavity frequency $\omega$, and $g$ is the qubit-cavity coupling strength. $p = 1, 2$ denote the one-photon and two-photon QRM, respectively. In the superconducting flux qubit [8, 9], $\Delta$ is the tunnel coupling between the two persistent current states, $\epsilon = 2I_p (\Phi - \Phi_0)/2$ with $I_p$ the persistent current in the qubit loop, $\Phi$ the external applied magnetic flux, and $\Phi_0$ the flux quantum. The flux qubit is usually manipulated by the external magnetic flux and the persistent currents.

For the symmetric case ($\epsilon = 0$), the one-photon QRM possesses $\mathbb{Z}_2$-symmetry (parity), i.e. $[H_1^0, \hat{P}_1] = 0$ with the parity operator $\hat{P}_1 = \sigma_z \exp(\pm \pi a^\dagger a)$, whose eigenvalues are $\pm 1$, while the tpQRM has $\mathbb{Z}_4$-symmetry, i.e. $[H_2^0, \hat{P}_2] = 0$ with the parity operator $\hat{P}_2 = \sigma_z \exp(i\pi a^\dagger a/2)$, whose eigenvalues are the quartic roots of unity $\pm 1, \pm i$ [14, 15]. Hence the whole Hilbert space separates therefore into two and four infinite-dimensional subspaces in one-photon QRM and the tpQRM, respectively.

An analytical exact solution of the one-photon QRM has been found by Braak in the Bargmann space representation [10]. It was quickly reproduced in the more familiar Hilbert space using the Bogoliubov operator approach (BOA) by Chen et al. [17]. Moreover, the BOA can be easily extended to the tpQRM, and solutions in terms of a G-function, which shares the common pole structure with Braak’s G-function for the one-photon QRM, are also found. It was soon realized that the G-function can be constructed in terms of the mathematically well-defined Heun confluent function [18]. These studies have stimulated extensive interests in various QRM [19,27]. For more theoretical details in this field, one may refer to recent review articles [28-30].

The presence of the qubit bias term $\frac{\epsilon}{2} \sigma_x$ breaks $\mathbb{Z}_2$-symmetry of the QRM, so no any obvious symmetry remains in the asymmetric QRM [18,21,51], while in the tpQRM, it reduces the original $\mathbb{Z}_4$-symmetry to the $\mathbb{Z}_2$-symmetry. In the asymmetric tpQRM, the $\mathbb{Z}_2$-symmetry corresponding to the parity operator $\hat{P}_2^p = \exp(\pm \pi a^\dagger a)$ only acts in the bosonic Hilbert spaces, the whole Hilbert space then only divides into two invariant subspaces: even and odd number Fock states, which can be still labeled by the Bargmann index $q = 1/4$ and $3/4$ [14].

Level crossing is very helpful to identify the symmetry in quantum systems. The quasi-exact energies in the symmetric QRM, also called Juddian solutions [22], have been found 20 years ago [32]. The Juddian solutions are corresponding to the doubly degenerate states,
and can be constructed with the terminated polynomials. These quasi-exact energies now can also be easily derived with the help of the pole structure of the $G$-function in both the one-photon \cite{16} and the two-photon \cite{17} QRM.

Surprisingly, the level crossing even exists without both the one-photon \cite{16} and the two-photon \cite{17} QRMs. The level crossings appear again in the spectra even without any explicit known symmetry in the system \cite{18,24}. It should be noted that here $\epsilon$ in accord with the standard qubit Hamiltonian \cite{8,9,34,38} is twice of that used in \cite{16,18,24}.

In this section, we revisit the asymmetric one-photon QRM by BOA. We first briefly review the solutions in the BOA framework \cite{17}, then we can describe the level crossings in the BOA alternatively, which is essentially equivalent to the Bargmann space approach. Furthermore, by BOA, we can obtain all the non-degenerate exceptional points in a more concise and complete way. Most importantly, this scheme can be easily extended to the asymmetric tpQRM in the next sections.

A. Solutions in BOA

By two Bogoliubov transformations

\begin{equation}
A = a + g/\omega, \quad B = a - g/\omega,
\end{equation}

the wavefunction can be expressed as the series expansions in terms of $A$ operator

\begin{equation}
|A\rangle = \left( \sum_{n=0}^{\infty} \sqrt{n} \epsilon_n |n\rangle_A \right),
\end{equation}

where $e_n$ and $f_n$ are the expansion coefficients, and also in terms of $B$ operator

\begin{equation}
|B\rangle = \left( \sum_{n=0}^{\infty} (-1)^n \sqrt{n} d_n |n\rangle_B \right),
\end{equation}

with two coefficients $c_n$ and $d_n$. $|n\rangle_A$ and $|n\rangle_B$ are called extended coherent states \cite{40}.

By the Schrödinger equation, we get the linear relation for two coefficients $e_m$ and $f_m$ with the same index $m$ as \cite{17}

\begin{equation}
e_m = \frac{\Delta}{2(m\omega - g^2/\omega + \frac{\epsilon}{2} - E)} f_m, \end{equation}

and the coefficient $f_m$ can be defined recursively,

\begin{equation}
(m+1) f_{m+1} = \frac{1}{2g} \left( m\omega + 3g^2/\omega - \frac{\epsilon}{2} - E - \frac{\Delta^2}{4 (m\omega - g^2/\omega + \frac{\epsilon}{2} - E)} \right) f_m - f_{m-1},
\end{equation}

with $f_0 = 1$. Similarly, the two coefficients $c_m$ and $d_m$ satisfy

\begin{equation}
d_m = \frac{\Delta}{2 (m\omega - g^2/\omega + \frac{\epsilon}{2} - E)} c_m.
\end{equation}
and the recursive relation is given by

\[(m+1)c_{m+1} = \frac{1}{2g} \left( m\omega + 3g^2/\omega + \frac{\epsilon}{2} - E - \frac{\Delta^2}{4(m\omega - g^2/\omega - \frac{\epsilon}{2} - E)} \right) c_m - c_{m-1}, \quad (8)\]

with \(c_0 = 1\).

If both wavefunctions \(3\) and \(4\) are the true eigenfunction for a non-degenerate eigenstate with eigenvalue \(E\), they should be in principle only different by a complex constant \(z\), i.e. \(|A| = z|B\). Projecting both sides onto the original vacuum state \(|0\rangle\), using \(\sqrt{n!}|0\rangle_A = (-1)^n\sqrt{n!}|0\rangle_B = e^{-(\frac{\epsilon}{2})/2} (\frac{\omega}{2})^n\) and eliminating the ratio constant \(z\) gives

\[\sum_{n=0}^{\infty} c_n (\frac{g}{\omega})^n \sum_{n=0}^{\infty} d_n (\frac{g}{\omega})^n = \sum_{n=0}^{\infty} f_n (\frac{g}{\omega})^n \sum_{n=0}^{\infty} c_n (\frac{g}{\omega})^n, \quad (9)\]

with the help of Eqs. \(9\) and \(11\), one arrives at one-photon G-function

\[G_{1p} = \left( \frac{\Delta}{2} \right)^2 \left[ \sum_{n=0}^{\infty} \frac{f_n}{n\omega - g^2/\omega + \frac{\epsilon}{2} - E} (\frac{g}{\omega})^n \right]
\times \left[ \sum_{n=0}^{\infty} \frac{c_n}{n\omega - g^2/\omega + \frac{\epsilon}{2} - E} (\frac{g}{\omega})^n \right]
- \sum_{n=0}^{\infty} f_n (\frac{g}{\omega})^n \sum_{n=0}^{\infty} c_n (\frac{g}{\omega})^n. \quad (10)\]

This G-function was first derived by Braak \(16\) using Bargmann space approach, and later reproduced by Chen et al. \(17\). We then discuss the level crossing of this asymmetric QRM in terms of BOA framework described above.

### B. Doubly degenerate states

The two types of pole energies appear in the one-photon G-function \(10\) as

\[E^A_N = N\omega - g^2/\omega + \frac{\epsilon}{2}, \quad N = 0, 1, 2, ... \quad (11)\]
\[E^B_M = M\omega - g^2/\omega - \frac{\epsilon}{2}, \quad M = 0, 1, 2, ... \quad (12)\]

They are labeled with the type-A and type-B pole energy, respectively. If

\[\epsilon = (M - N)\omega, \quad (13)\]
these two pole energies are the same

\[E^A_N(g) = E^B_M(g) = \frac{1}{2} (M + N)\omega - g^2/\omega. \quad (14)\]

Note that \(\epsilon\) should be a multiple of the cavity frequency \(\omega\) under the condition \(13\). In this paper, we only consider \(M > N\), so that \(\epsilon\) is positive. For the case of \(M < N\), the extension is achieved straightforwardly by changing \(\epsilon\) into \(-\epsilon\) and interchanging \(M\) and \(N\).

From Eqs. \(3\) \(\sim\) \(7\), one immediately notes that the coefficient \(d_N(d_M)\) would diverge at the same pole energy \(14\). It does not make sense if some coefficients in the series expansion of a wavefunction really become infinity. A normalizable wavefunction should consist of the global property, i.e. the finite inner product, so the series expansion coefficients in the wavefunction \(3\) and \(4\) should be analytic and vanish as or before \(n \rightarrow \infty\).

To achieve a physics state, at the pole energy \(14\), the numerator of right-hand-side of Eq. \(3\) \(\sim\) \(7\) should also vanish, so that \(e_N(d_M)\) remains finite, which result in

\[f_N(M, g) = 0; \quad c_M(N, g) = 0. \quad (15)\]

Note that \(f_N\) and \(c_M\) can be obtained using the following three-terms recurrence relation from \(9\) and \(8\) with energy \(14\), respectively

\[(n + 1) f_{n+1} = \frac{1}{2g} \left[ 4g^2/\omega + (n - M)\omega - \frac{\Delta^2}{4\omega (n - N)} \right] f_n
- f_{n-1}, \quad (16)\]
\[(n + 1) c_{n+1} = \frac{1}{2g} \left[ 4g^2/\omega + (n - N)\omega - \frac{\Delta^2}{4\omega (n - M)} \right] c_n
- c_{n-1}. \quad (17)\]

If \(\Delta, M, N\) are given, two equations in \(15\) would provide the coupling strength \(g\) in the energy spectra where the energy levels intersect with the same pole line described by Eq. \(14\).

A mathematical proof to the conjecture that \(f_N(M, g) = 0\) and \(c_M(N, g) = 0\) could give the same real and positive solutions for the coupling strength \(g\) was given in \(31\). Li and Batchelor \(24\) have analyzed the relation between the number of the exceptional points and the model parameters \((\Delta, \epsilon)\), and numerically found that the number of positive roots from these two equations are the same for integer \(\epsilon/\omega\). But we confine us here to a closed-form proof for small values of \(N\) and \(M\), and numerically confirmation for large \(N\) and \(M\). In the asymmetric tpQRM, similar constrained condition will be derived and we will also do the similar things.
FIG. 1: (Color online) Energy spectrum $E + g^2 + \epsilon/2$ for $\omega = 1$, $\Delta = 1.5$, $\epsilon = 1$ (a) and $\epsilon = 2$ (b) in left panels. The horizontal blue dotted lines correspond to the pole energy ones for $E_N^c$ and the red dashed lines to $E_M^c$. Only the overlapped pole lines with $N > 0$ allow for the true level crossings. The triangles denotes the doubly degenerate crossing points. Circles indicate the non-degenerate exceptional solutions by Eq. (A2). $E_N$ and the red dashed lines to $E_M$. The dotted horizontal blue dotted lines correspond to the pole energy ones (N), respectively. The dotted triangles in the left spectrum.

Because a similar conjecture will be proposed but cannot be proven at the present stage.

To this end, we present our discussions only in terms of the fixed integers $N$ and $M$. In this case, $\epsilon$ is a multiple of the cavity frequency is known immediately, and the remaining task is to show the same crossing points by two equations in Eq. (15), which is illustrated in Appendix A1.

In the left panels of Figs. 1 and 2, we present the energy spectrum $E + g^2/\omega + \frac{\epsilon}{2}$ for $\epsilon = 1$, 2 at $\Delta = 1.5$ and $\Delta = 3$ with $\omega = 1$, respectively. The dotted and dashed horizontal lines denote different types of pole lines. Obviously, if the two types of pole lines cannot coincide, the level crossings cannot happen. $f_N(M, g)$ and $c_M(N, g)$ curves are plotted in the right panels. By Eq. (14), one finds $g = 0.5995$ for $N = 1, M = 2$ at $\Delta = 1.5$, consistent with the spectra in Fig. 1 (a). For $\Delta = 3$, see Eq. (15), no real positive solution can be found in this case, so level crossings cannot occur in the overlapped line with $N = 1$ in Fig. 2 (a).

At $\Delta = 1.5$, we can obtain two solutions for $g$ as 0.4804, 1.0287 for $N = 2$ and $M = 3$ by Eq. (14), agreeing well with two crossing points in the second type-A pole line with $N = 2$ and $M = 3$ shown in Fig. 1 (a). For $\Delta = 3$, we only find one real positive $g = 0.8356$, consistent with the spectra in Fig. 2 (a).

Associated with the overlapped $N = 1$ type-A and $M = 3$ pole lines, one can obtain $g = 0.7806$ for $\Delta = 1.5$, and $g = 0.4330$ for $\Delta = 3$ by Eq. (15), consistent with the crossing points in the calculated spectrum in Fig. 1 (a) and Fig. 2 (a).

For large value of $M$ and $N$, as shown in the right panels of Figs. 1 and 2, both $f_N(M, g)$ and $c_M(N, g)$ curves provide the same zeros for all cases.

Now we will further demonstrate explicitly that any crossing point found above is corresponding to a doubly degenerate state in the BOA framework. At the crossing point, looking at (13), since both the numerator $f_N(g)$ and denominator vanish, $e_N$ would be arbitrary. If we set $e_N = -\frac{4g}{\Delta}f_{N-1}$, (18) from Eq. (16) we know $f_{N+1} = 0$, further $e_{N+1} = 0$, and all coefficients $f_k$ and $e_k$ for $k > N + 1$ vanish. So the infinite series expansion in the wavefunction (14) terminates with finite $N$ as

$$|A\rangle_N = \left( \sum_{R=0}^{N} \sum_{n=0}^{M} \sqrt{R \cdot N} |c_n| |n\rangle_A \right).$$

Similarly, the infinite series expansion in the wavefunction (16) terminates with finite $M$ as

$$|B\rangle_M = \left( \sum_{n=0}^{M} \sum_{R=0}^{N-1} (-1)^R \sqrt{R \cdot N} |d_n| |n\rangle_B \right),$$

where

$$d_M = -\frac{4g}{\Delta}c_{M-1}.$$
At this stage, we can simply discuss the number of the doubly degenerate crossing points associated with the given \( N \) type-A pole line. \( f_N(M > N, x = 4g^2) \) derived by Eq. (16) is a polynomial with \( N \) terms. Its zero would generally give around \( N \) roots, indicating that there are around \( N \) doubly degenerate crossing points along the \( N \) type-A pole line in the energy spectra. Note that for large \( \Delta \), the number of the roots could be slightly less than \( N \), as shown in Fig. 2. For small \( \Delta \), we can actually have just \( N \) roots.

C. Non-degenerate exceptional points

The non-degenerate exceptional points can be generated if only one energy level intersects with the energy pole line alone. In principle, all non-degenerate states including non-degenerate exceptional ones can be obtained by the G-function [10] because it is built based on the proportionality [19], only excluding the degenerate states. These states have been first analyzed for the symmetric QRM with the Bargmann space technique in [13] and later in [11–43]. We believe that the BOA has advantages with regard to the non-degenerate exceptional solutions, which cannot be found with any ansatz.

Note from G-function [10] that, at the pole energy either (11) or (12), the denominator of the associated term treated specially. For a physics state, to avoid the diverging term, we may cut off (11) or (12), the denominator of the associated term later in [41–43]. We believe that the BOA has advantages with regard to the non-degenerate exceptional points associated with the Type-A pole lines. Particularly, \( f_N = 0 \) or \( c_M = 0 \) is implied Eq. (21) or \( G_{1p}^{non,1B} = 0 \), thus can be also used to give the same non-degenerate exceptional points in a simpler way. Just as pointed out in Ref [24], for noninteger \( \epsilon \), a subset of the non-degenerate exceptional points associated with the pole lines can be given by the vanishing coefficients \( f_m \) or \( c_m \), equivalently, using Eq. (21) or \( G_{1p}^{non,1B} = 0 \) here. However Eq. (21) and \( G_{1p}^{non,1B} = 0 \) fail at integer \( \epsilon \) including \( \epsilon = 0 \), because \( f_N = 0 \) or \( c_M = 0 \) actually results in the doubly degenerate states, which results in nonzero G-function in this case.

Interestingly, for integer \( \epsilon \), two types of pole line may merge together. At the same pole energy [13], the second non-degenerate exceptional G-function Eq. (22) would be further modified as

\[
G_{1p}^{non,2A} = \left[ \left( \frac{g}{\omega} \right)^N + \sum_{n=N+1}^{\infty} \frac{\Delta f_n}{2\omega (n-N)} \left( \frac{g}{\omega} \right)^n \right]
\times \left[ \sum_{n=0}^{\infty} \frac{\Delta c_n}{2(n\omega-g^2/\omega-\epsilon)} \left( \frac{g}{\omega} \right)^n \right]
- \sum_{n=0}^{N-1} f_n \left( \frac{g}{\omega} \right)^n \sum_{n=0}^{\infty} c_n \left( \frac{g}{\omega} \right)^n = 0, \quad (22)
\]

with the initial condition \( e_N = 1 \). The non-degenerate exceptional G-functions \( G_{1p}^{non,1B} \) and \( G_{1p}^{non,2B} \) associated with the type-B pole line can be obtained similarly by modifying the other infinite summation, which are not shown here.

Two non-degenerate exceptional G-functions [21] and [22] provide different exceptional solutions, which comprise the full non-degenerate exceptional points associated with the Type-A pole lines. Particularly, \( f_N = 0 \) or \( c_M = 0 \) is implied Eq. (21) or \( G_{1p}^{non,1B} = 0 \) here. However Eq. (21) and \( G_{1p}^{non,1B} = 0 \) fail at integer \( \epsilon \) including \( \epsilon = 0 \), because \( f_N = 0 \) or \( c_M = 0 \) actually results in the doubly degenerate states, which results in nonzero G-function in this case.
In the end of this section, we would like to point out that the previous main results in the asymmetric QRM based on the Bargmann space approach, see Ref. \texttt{[24]} and reference therein, can be well described in the BOA framework in a self-contained way. The asymmetric tpQRM has not been studied in the literature, much less the level crossings irrelevant to the explicit symmetry, to our knowledge. Note that the G-function by the direct application of the Bargmann space approach to the tpQRM \texttt{[44]} has no pole structure, and thus could not give qualitative insight into the behavior of the spectral collapse \texttt{[15]} and, in particular, has so far not been derived using the Bargmann space method in the literature. Therefore, it is perhaps irreplaceable, at the moment, to employ the BOA to study the asymmetric tpQRM, which is the main topic of this paper.

III. ASYMMETRIC TWO-PHOTON RABI MODEL AND SOLUTIONS USING BOA

For convenience, we rewrite the Hamiltonian $H^2_z$ on the $\sigma_z$ basis by rotating it around the $y$-axis with an angle $\pi/2$. The transformed Hamiltonian is given by the following matrix form

$$H^2_{z,r} = \left( \begin{array}{cc} \omega a^\dagger a + g(a^{12} + a^2) + \frac{3}{2} & -\frac{\Delta}{2} - \frac{\omega}{2} \\ -\frac{\omega}{2} - \frac{\Delta}{2} & \omega a^\dagger a - g(a^{12} + a^2) - \frac{3}{2} \end{array} \right).$$

The Hamiltonian above is connected with $su(1, 1)$ Lie algebra

$$K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{4}), K_+ = \frac{1}{2}a^{12}, K_- = \frac{1}{2}a^2,$$

which obey spin-like commutation relations $[K_0, K_\pm] = \pm K_\pm, [K_+, K_-] = -2K_0$. The quadratic invariant Casimir operator is given by

$$C_2 = K_+K_- + K_0(1 - K_0).$$

Then we apply a squeezing operator $S_1 = e^{\frac{\sqrt{2}}{4}(a^2 - a^{12})}$ to diagonalize the bosonic part of the above Hamiltonian and the parameter $r$ is to be fixed later. In terms of the $K_0, K_\pm$, the transformed Hamiltonian is derived as

$$H^2_{z,r} = \left( \begin{array}{cc} \beta (2K_0) + \frac{\omega}{2} - \frac{\Delta}{2} & \frac{\Delta}{2} \\ \frac{\Delta}{2} & H_{22} \end{array} \right),$$

where $\beta = \omega \sqrt{1 - 4(\frac{g}{3})^2} < \omega$ can be termed as the renormalized cavity frequency owing to the fact that it is just a $g$-dependent pre-factor of the free photon number operators $2K_0$, and $\beta = \omega$ if $g = 0$. It will be shown later that $\beta$ plays a key role in two-photon QRM. The second diagonal element is

$$H_{22} = (2\omega \cosh 2r - 4g \sinh 2r)K_0 + (\omega \sinh 2r - 2g \cosh 2r)(K_+ + K_-) - \frac{\epsilon + \omega}{2},$$

and the squeezing parameter

$$r = \frac{1}{4} \ln \left( \frac{1 - 2g/\omega}{1 + 2g/\omega} \right).$$

It is obvious that the coupling strength $g < \omega / 2$ leads to a real squeezing parameter.

Based on the squeezing transformation, we propose the corresponding wavefunction as

$$|\Psi_A\rangle^g = \left( \sum_{m=0}^{\infty} \sqrt{[2(m + q - \frac{1}{4})]}e^{(q)}_{m} |q, m\rangle_A \right),$$

where the new basis $|q, m\rangle_A = S_A |q, m\rangle$ with $|q, m\rangle$ is the Fock state. The coefficients $e^{(q)}_{m}$ and $f^{(q)}_{m}$ are to be determined in the following.

In the case of the Lie algebra considered here, $K_0|q, 0\rangle_A = q|q, 0\rangle_A$ where $q = \frac{1}{4}$ and $\frac{3}{4}$ divide the whole Hilbert space $H$ into even and odd sectors and label them, respectively. For the even subspace, $H_{1/4} = \{a^{1n} |0\rangle, n = 0, 2, 4, \ldots \}$, and for the odd subspace, $H_{3/4} = \{a^{1n} |0\rangle, n = 1, 3, 5, \ldots \}$, corresponding to even or odd Fock number basis. The Casimir element $C_2 = \frac{1}{16}$ in both cases. The Bargmann index $q$ allows us to deal with both cases independently.

The $su(1, 1)$ Lie algebra operators satisfy

$$K_0 |q, n\rangle_A = (n + q) |q, n\rangle_A,$$

$$K_+ |q, n\rangle_A = \sqrt{(n + q + \frac{3}{4})(n + q - \frac{1}{4})} |q, n + 1\rangle_A,$$

$$K_- |q, n\rangle_A = \sqrt{(n + q - \frac{3}{4})(n + q - \frac{1}{4})} |q, n - 1\rangle_A.$$

Projecting both sides of the Schrödinger equation onto $|q, n\rangle_A$ gives a linear relation between coefficients $e^{(q)}_{m}$ and $f^{(q)}_{m}$,

$$e^{(q)}_{n} = \frac{\Delta/2}{2\beta(n + q) - E + \frac{\epsilon + \omega}{2}} f^{(q)}_{n},$$

and a three-term linear recurrence relation is given by
\[ f_{n+1}^{(q)} = \frac{2(2\omega^2 - \beta^2)(n + q) - \beta(E + \frac{\omega}{2}) - \frac{\Delta^2 n^{2/4}}{2(\beta(n+q) - E - \frac{\omega}{2})}}{8g\omega(n + q + \frac{1}{4})(n + q + \frac{3}{4})} f_n^{(q)} - \frac{1}{4(n + q + \frac{1}{4})(n + q + \frac{3}{4})} f_{n-1}^{(q)}. \]  

(30)

All coefficients \( f_n^{(q)} \) and \( c_n^{(q)} \) can be calculated with initial conditions \( f_0^{(q)} = 0 \) and \( f_1^{(q)} = 1 \).

We then apply the second squeezing operator \( S_B = e^{-\frac{\Delta}{2}(a^2 - a^*2)} \) to the Hamiltonian \[24\] and suggest the wavefunction as

\[ |\Psi_B\rangle = \left( \frac{\Delta}{2} \right)^{\frac{1}{2}} \sum_{m=0}^{\infty} (-1)^m \sqrt{[2(m + q - \frac{1}{4})]!} c_m^{(q)} |q, m\rangle_B, \]

(31)

Left-multiplying the vacuum state \( |q, 0\rangle \) to the extended squeezed state \( |q, m\rangle_A \) and \( |q, m\rangle_B \), we can obtain the inner product

\[ \langle q, 0 | q, m \rangle_A = \frac{(-\tanh r)^m}{\sqrt{\cosh r}} \frac{\sqrt{[2(m + q - \frac{1}{4})]!}}{2^m m!}, \]

\[ \langle q, 0 | q, m \rangle_B = \frac{(\tanh r)^m}{\sqrt{\cosh r}} \frac{\sqrt{[2(m + q - \frac{1}{4})]!}}{2^m m!}. \]  

(34)

If both wavefunction \( |\Psi_A\rangle \) and \( |\Psi_B\rangle \) for the same \( q \) are the true eigenfunction for a non-degenerate eigenstate with eigenvalue \( E \), they should be proportional with each other, i.e. \( |\Psi_A\rangle = z |\Psi_B\rangle \), where \( z \) is a complex constant. Projecting both sides of this identity onto the original vacuum state \( |q, 0\rangle \), we obtain a transcendentental function below defined as G-function

\[ G^{(q)} = \left( \frac{\Delta}{2} \right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{f_m^{(q)} \Omega_m^{(q)}}{2\beta(m + q) + \frac{\omega}{2} - E} \]

\[ \times \left[ \sum_{m=0}^{\infty} \frac{c_m^{(q)} \Omega_m^{(q)}}{2\beta(m + q) + \frac{\omega}{2} - E} \right] \]

\[ - \sum_{m=0}^{\infty} f_m^{(q)} c_m^{(q)} \sum_{m=0}^{\infty} c_m^{(q)} c_m^{(q)}, \]  

(35)

with

\[ \Omega_m^{(q)} = \frac{(-\tanh r)^m}{\sqrt{\cosh r}} \frac{\sqrt{[2(m + q - \frac{1}{4})]!}}{2^m m!}. \]

\[ G^{(q)} \] gives the regular spectrum in the \( q \) subspace of the asymmetric tpQRM.

If \( \epsilon = 0 \), the G-function for the symmetric tpQRM \[17\] is recovered. The zeros of the \( G^{(q)} \)-function give the regular spectrum in the \( q \) subspace of the asymmetric tpQRM.

From Eqs. \[29\] and \[32\], we find the G-function diverges when its denominators vanishes, the condition of the denominators being zero can be obtained as

\[ E_m^A = 2\beta(m + q) + \frac{\epsilon - \omega}{2}, \]

(36)

and

\[ E_m^B = 2\beta(m + q) - \frac{\epsilon + \omega}{2}, \]

(37)

with \( m = 0, 1, 2, ... \). They are also labeled as two types (A and B) pole energies, similar to the asymmetric QRM.

G-curves at \( q = 1/4 \) and \( 3/4 \) for \( \epsilon = 0.4, g = 0.35 \), and \( \Delta = 3 \) with \( \omega = 1 \) are plotted in Fig. 3. The zeros are easily detected. As usual, one can check it easily with numerics, an excellent agreement can be achieved. The poles given in Eqs. \[30\] and \[37\] are marked with vertical lines. The G-curves indeed show diverging behavior when approaching the poles.

In the limit of \( g \to \omega/2 \), the type-A pole energies are squeezed into a single finite value \( E_m^A(g = \omega/2) = \frac{\omega + \epsilon}{2} \), and type-B pole energies into \( E_m^B(g = \omega/2) = - \frac{\omega + \epsilon}{2} \). It seems that there are two kinds of collapse energies \( \frac{\omega + \epsilon}{2} \). But actually, at \( g = \omega/2 \), our obtained energy levels tend to the smaller one \( - \frac{\omega - |\epsilon|}{2} \), except some low lying states which split off from the continuum. The ground-state...
is always separated from the continuum by a finite excitation gap. In the spectrum shown in the next sections, we will indeed observe that the plotted energy levels always collapse to the smaller one at \( g = \omega / 2 \), which generate more non-degenerate exceptional points when \( g \to \omega / 2 \). However, we cannot rule out the possibility that some energy levels would stay between two limit energies, \( \frac{\omega - \Delta}{2} \) and \( \frac{\omega + \Delta}{2} \), and thus these levels could not collapse. Since the analytical solution at \( g = \omega / 2 \) in the asymmetric tpQRM is lacking, the collapse issue in this model is somehow challenging.

### IV. NON-DEGENERATE EXCEPTIONAL SOLUTIONS IN THE ASYMMETRIC TPQRM

As outlined in the Sec. II (c) for the asymmetric one-photon QRM, we can easily find the non-degenerate exceptional solutions in the spectra for the asymmetric tpQRM by the pole structures of the G-function. When the energy levels cross the pole lines, the coefficients in the G-function would diverge, and therefore should be treated specially. For any real physical systems, the wavefunction should be analytic, so the numerators \( f_m^{(q)} \) in Eq. (29) or \( c_m^{(q)} \) Eq. (30) should also vanish, which further gives the condition for the model parameters \( g, \Delta, \epsilon \), for fixed value of \( m \) associated with one pole line.

In parallel to the asymmetric one-photon QRM, the first non-degenerate exceptional G-function associated with the N-th type-A pole lines (36) for the asymmetric tpQRM is easily given by

\[
G_{non,1A}^{(q)} = \left[ \sum_{n=0}^{N-1} \frac{\Delta f_n^{(q)} \Omega_n^{(q)}/2}{2 \beta (n - N)} + \epsilon N \Omega_N^{(q)} \right] \\
\times \left[ \sum_{n=0}^{\infty} \frac{\Delta c_n^{(q)} \Omega_n^{(q)}/2}{2 \beta (n - N) - \epsilon} \right] \\
- \sum_{n=0}^{N-1} f_n^{(q)} \Omega_n^{(q)} \sum_{n=0}^{\infty} c_n^{(q)} \Omega_n^{(q)} = 0, \quad (38)
\]

where

\[
e^{(q)}_N = - \frac{4g \omega}{\Delta \beta} f_N^{(q)} (N-1), \quad (39)
\]

and that associated with the M-th type-B pole lines (37) reads

\[
G_{non,1B}^{(q)} = \left[ \sum_{n=0}^{N-1} \frac{\Delta f_n^{(q)} \Omega_n^{(q)}/2}{2 \beta (n - M) + \epsilon} \right] \\
\times \left[ \sum_{n=0}^{\infty} \frac{\Delta c_n^{(q)} \Omega_n^{(q)}/2}{2 \beta (n - M) + \epsilon} + d_M \Omega_M^{(q)} \right] \\
- \sum_{n=0}^{M-1} f_n^{(q)} \Omega_n^{(q)} \sum_{n=0}^{\infty} c_n^{(q)} \Omega_n^{(q)} = 0, \quad (40)
\]

where

\[
d_M^{(q)} = - \frac{4g \omega}{\Delta \beta} c_M^{(q)} (M-1). \quad (41)
\]

Note that by Eq. 39 [Eq. 41], all the remaining coefficients for \( n > N \) \( n > M \) vanish. Zeros of the first non-degenerate exceptional G-functions are equivalent to \( f_N^{(q)} = 0 \) or \( c_M^{(q)} = 0 \). Obviously, the later ones are obviously simpler in practical calculations, while the former ones are more conceptually interesting, both can give the same solutions.

Similarly, the second non-degenerate exceptional G-function associated with the type-A pole lines (38) is

\[
G_{non,2A}^{(q)} = \left[ \Omega_N^{(q)} + \sum_{n=N+1}^{\infty} \frac{\Delta f_n^{(q)} \Omega_n^{(q)}/2}{2 \beta (n - N)} \right] \\
\times \left[ \sum_{n=0}^{\infty} \frac{\Delta c_n^{(q)} \Omega_n^{(q)}/2}{2 \beta (n - N) - \epsilon} \right] \\
- \sum_{n=N+1}^{\infty} f_n^{(q)} \Omega_n^{(q)} \sum_{n=0}^{\infty} c_n^{(q)} \Omega_n^{(q)} = 0, \quad (42)
\]

where we have set \( e_N^{(q)} = 1 \), and the coefficients \( e_n^{(q)} N = 0 \) and \( f_n^{(q)} N = 0 \). By the recurrence relations and the pole energy, all other coefficients can be obtained. The second non-degenerate exceptional G-function associated with
FIG. 4: (Color online) (a) The spectra for the first 5 levels and the non-degenerate exceptional solutions for the asymmetric tpQRM with $\omega = 1$, $\Delta = 2$, $\epsilon = 1$, $q = 1/4$. The blue dashed lines are $E_{m=0,1}$ by Eq. (35) and red dashed lines are $E_{m=1,2}$ by Eq. (37). The inset on an enlarged scale shows the avoided crossing instead of the true level crossing. $f_{N=1}^{(1/4)}$, $c_M^{(1/4)}$ curves are exhibited in (b), whose zeros are indicated by open triangles, agreeing with the same symbols in the spectra (a). The non-degenerate exceptional G-function $G_{n=2}^{(1/4)}$ in Eq. (45) for $M = 2$, and $G_{n=2}^{(1/4)}$ in Eq. (46) for $N = 1, 0$ are given in (c), (d), and (e), respectively. Their zeros are denoted by open circles, which are excellently consistent with the non-degenerate exceptional points indicated by the same symbols in the spectra (a).

The type-B pole lines (37) can be obtained in a straightforward way as

$$G_{n=2}^{(1/4)} = \left[ \sum_{n=0}^{\infty} \frac{\Delta f_n^{(q)}(1)_{M}^{(q)}/2}{2\beta(n-M) + \epsilon} \right] \times \left[ \Omega_M^{(q)} + \sum_{n=M+1}^{\infty} \frac{\Delta c_n^{(q)}(1)_{M}^{(q)}/2}{2\beta(n-M)} \right] - \sum_{n=0}^{M} f_n^{(q)}(1)_{n}^{(q)} \sum_{n=M+1}^{\infty} c_n^{(q)}(1)_{n}^{(q)} = 0, \quad (43)$$

where $d_{M}^{(q)} = 1$, and the coefficients $d_{n=M}^{(q)} = 0$ and $c_{n=M}^{(q)} = 0$.

We plot the spectra in Fig. 4 (a) for the parameters $q = 1/4, \Delta = 2, \epsilon = 1$ with $\omega = 1$. The crossing points of the energy levels and the pole lines (35) and (37), known as non-degenerate exceptional points, are marked with open symbols. All these non-degenerate exceptional points can be confirmed analytically. The solutions by the coefficient polynomial equations $f_{N=1}^{(1/4)} = 0$ and $c_{M=1,2}^{(1/4)} = 0$ are indicated by open triangles in Fig. 4 (b), and denoted with the same symbols in Fig. 4 (a). 7 zeros of the non-degenerate exceptional G-functions (42) and (43) corresponding to 7 open circles in Fig. 4 (c-e) are indicated by the 7 same symbols in Fig. 4 (a).

As revealed on an enlarged scale in the inset of Fig. 4 (a) and (b) that two open triangles do not coincide, indicating an avoided crossing at this bias parameter $\epsilon = 1$. We will show in the left panels of Fig. 5 at $\epsilon = 1.0954$ in the next section that, the two open triangles also obtained from $f_{N=1}^{(1/4)} = 0$ and $c_{M=2}^{(1/4)} = 0$ eventually can meet. Thus it should be very interesting to see how an avoided crossing essentially turns to a true level crossing when $\epsilon = 1 \rightarrow 1.0954$. 

V. DOUBLY DEGENERATE STATES IN ASYMMETRIC TWO-PHOTON QRM

In the asymmetric tpQRM, can we also find level crossings in the same $q$ subspace? According to the pole energies (36) and (37), if $E_{M}^{A} = E_{N}^{B}$, then

$$\epsilon = 2\beta(M-N), \quad (44)$$

the same pole energy takes

$$E = (M + N + 2q)\beta - \frac{1}{2} \omega. \quad (45)$$

Interestingly, Eq. (44) entails $\epsilon$ to be an even multiple of the renormalized cavity frequency $\beta$, in contrast to the asymmetric one-photon QRM where $\epsilon$ should be an multiple of the cavity frequency $\omega$ under the condition (18) for level crossings. It makes sense that only the two-photon process is involved in the two-photon model, while the single photon process in the one-photon model.

Without loss of generality, we also only consider $M > N$ here. From Eq. (38), one immediately note that the coefficient $c_{M}^{(q)}$ (in (29), $d_{M}^{(q)}$ in (22)) would diverge at the same pole energy (44). Similar to the asymmetric QRM case, the series expansion coefficients in the wavefunction (28) and (51) should be analytic and vanish as or before $n \rightarrow \infty$.

Regarding states with the energy (45), the numerator of right-hand-side of (30) (32) should also vanish, so that $c_{N}^{(q)} (d_{M}^{(q)})$ remains finite, which requires

$$f_{N}^{(q)}(M, g) = 0; c_{M}^{(q)}(N, g) = 0. \quad (46)$$

Note that $f_{N}^{(q)}$ and $c_{M}^{(q)}$ can be respectively obtained from the recurrence relations (30) and (33) by using the same pole energy (45).

$$f_{n+1}^{(q)} = \frac{2\omega^2(n+q) - \beta^2(n+M+2q) + \frac{\lambda^2}{4(n(n+q)}} f_{n}^{(q)} - \frac{1}{4(n(q+1/4)(n+q+1/4)} f_{n-1}^{(q)}, \quad (47)$$
\[ e_{n+1}^{(q)} = \frac{2\omega^2 (n+q) - \beta^2 (n+N+2q)}{4g\omega(n+q + \frac{1}{4})(n + q + \frac{3}{4})} c_{n}^{(q)} - \frac{1}{4(n+q + \frac{1}{4})(n + q + \frac{3}{4})} c_{n-1}^{(q)} . \] (48)

FIG. 5: (Color online) Energy spectrum are list in low panels. The black lines are energy levels, the blue dashed lines are \( E_{N}^A \) and the red dashed lines are \( E_{N}^B \). Open triangles indicate the doubly degenerate level crossings. \( f(q)_{N-1}(M, g) \) (blue) and \( c(q)_{N}(N = 1, g) \) (red) curves are displayed in the upper panels. \( (g, \epsilon) = (1/4, 1.0954) \) (left), \( (1/4, 1.8516) \) (middle), and \( (3/4, 2.4944) \) (right). Zeros are the same for both curves. \( \Delta = 2 \) and \( \omega = 1 \).

Similar to the asymmetric one-photon QRM, we conjecture that both \( f(q)_{N=1}(M, g) = 0 \) and \( c(q)_{M}(N = 1, g) = 0 \) could give the same positive real \( \epsilon \) and \( g \) under the constrained condition \( [14] \), leading to levels crossing at the same pole energy. While it would be interesting to rigorously prove the conjecture in the two-photon case mathematically, we also confine us here to an analytical closed-form proof only for small values of \( N \) and \( M \), and numerically confirmation for large \( N \) and \( M \), in searching for physically reasonable coupling strength \( g \). Similar to the asymmetric QRM, we also present our discussions only in terms of fixed values of \( N \) and \( M \), but here \( \epsilon \) cannot be determined independently, and would be determined together with \( g \) by Eqs. \( [14] \) and \( [16] \).

In Appendix A3, we analytically prove that, for some small values of \( N \) and \( M \), both \( f(q)_{N}(M, g) = 0 \) and \( c(q)_{M}(N, g) = 0 \) in \( [16] \) give the same values for \( \epsilon \) and \( g \). Two energy levels cross the corresponding pole lines at the same values of \( \epsilon \) and \( g \), where the two pole lines also cross. Thus true level crossings also happen in the asymmetric tpQRM. Compare to the one-photon QRM where \( \epsilon \) can be determined independently, in the asymmetric tpQRM, we need to solve two equations simultaneously to determine \( \epsilon \) and \( g \).

We show the energy spectrum of the asymmetric tpQRM at \( N = 1 \), \( \Delta = 2 \) with \( \omega = 1 \), for \( q = 1/4, M = 2 \) (left), \( q = 1/4, M = 3 \) (middle), and \( q = 3/4, M = 3 \) (right) in the low panels of Fig. 5. The corresponding values of \( \epsilon \) are just those determined by Eq. \( [16] \), which in turn are \( \epsilon = 1.0954, 1.8516 \), and \( 2.4944 \) from left to right. Interestingly, one level crossing point indicated by the open triangle really appears in each spectra, confirming the analytical prediction.

The upper panels in Fig. 6 present the curves for \( f(q)_{N=2}(M, g) \) and \( c(q)_{M}(1, g) \). It is clear that the zeros of both functions are the same, and are consistent with the coupling strength at the level crossing points. For example, for \( q = 1/4, \Delta = 2 \), two energy levels cross exactly at \( g = 0.4183 \) by Eq. \( [16] \). This analytical findings is in excellent consisten with numerical results presented in the left panels of Fig. 5. This agreements also applies to the middle and right panels. As expected, the type-A pole lines \( E_{N=1}^A \) and the type-B pole lines \( E_{N=1}^B \) also cross at the degenerate points in the low panels of Fig. 6.

For \( N = 1 \), no matter what is the value of \( M > N \), from Eq. \( [17] \), we can at most find one solution for \( g \), which is \( M \)-dependent. For \( N > 1 \), \( f(q)_{N}(M, g) = 0 \) is a polynomial equation with \( N \) terms, which would give more than one solutions for \( g \), and further corresponding solutions for \( \epsilon \) in terms of Eq. \( [16] \).

As shown in left panel of Fig. 6 for \( \Delta = 2, \omega = 1, q = 1/4, N = 2 \) and \( M = 3 \) , both \( f_{N=2}^{(1/4)}(3, g) = 0 \) and \( c_{M=3}^{(1/4)}(2, g) = 0 \) yield the same solutions for \( g \), in turn are \( 0.3015, 0.4686 \) by Eq. \( [17] \), and two values of \( \epsilon \) are then determined accordingly. We then plot the energy spectrum for these two values of \( \epsilon \) in the middle and right panels of Fig. 6 for \( \Delta = 2 \). The level crossings are clearly shown at the analytical predicted coupling strength. Note that the \( N = 2 \) type-A pole line and the \( M = 3 \) type-B pole line indeed cross at the same doubly degenerate points.

Finally, the doubly degenerate states at the true level crossing points can be expressed explicitly in terms of the
BOA as

\[ |\Psi_A|^q = \left( \sum_{m=0}^{N} \sqrt{2(m + q - \frac{1}{2})} |c_m(q)|^2 |q, m\rangle_A \right), \]

and

\[ |\Psi_B|^q = \left( \sum_{m=0}^{N} \sqrt{2(m + q - \frac{1}{2})} |d_m(q)|^2 |q, m\rangle_B \right), \]

respectively, where \( c_m(q) \) and \( d_m(q) \) are given by Eqs. (39) and (41). Because these two wavefunctions are not obtained from the G-function based on the proportionality, so they are different, leading to doubly degenerate states. Both wavefunction terminates at finite terms, so they are the quasi-exact solutions of the asymmetric tpQRM.

VI. DISCUSSIONS

From the spectrum in Figs. 5 and 6, one might speculate that level crossings seldom happen in the asymmetric tpQRM. Actually it is not that case. If we incorporate Eq. (44) required by the level crossings, we may plot the similar spectra graph as Figs. 1 and 2 in one-photon case. In doing so, we calculate the energy as a function of \( g \), and at the same time \( \epsilon \) also changes as Eq. (41). To display the level crossings in asymmetric tpQRM more clearly, we can make the pole lines horizontal, thus we plot the normalized energy \( E' = \frac{E + \omega/2 - q + \epsilon}{\sqrt{3}} \) as a function of \( g \) and simultaneously varying \( \epsilon = k\beta \) in Fig. 7 for \( k = 0, 1, 2, 4 \) at \( \Delta = 2 \) with \( \omega = 1, q = 1/4 \).

When \( \epsilon \) is an even multiple of the normalized cavity frequency \( \beta \) entailed in Eq. (41), i.e. \( k \) is an even integer including the symmetric case \( k = 0 \), we find that the two equations in Eq. (46) result in the same positive solutions for the coupling strength, as indicated with open triangles (a), (c), and (d). One can note that the level crossings happen regularly. The crossing points at the \( N = 1 \) type-A pole line in (c) and (d) are just corresponding to those in Figs. 5 (a) and (b), while the two crossing points at the \( N = 2 \) type-A pole line in (c) to those in Fig. 6.

However, if \( k \) is not an even integer, no level crossings happen, a lot of non-degenerate exceptional points emerges instead. As exhibited in Fig. 7 (b) for \( k = 1 \), the open triangles correspond to the non-degenerate exceptional points by Eqs. (63) or (64), while the open circles to those by Eqs. (62) and (63).

We can also estimate the number of the doubly degenerate crossing points associated with the given \( N \) type-A pole line. For any \( M > N \), generally there are around \( N \) crossing points due to the polynomial equation with \( N \) terms, \( f_N^{(q)}(M, g) = 0 \) in Eq. (40), the detailed polynomial equations is derived from Eq. (47). This is to say, associated with \( N \) type-A pole line, we generally have

around \( N \) degenerate crossing points for both asymmetric one-photon and two-photon QRMs. In Appendix A3, we have employed the constrained conditions in the asymmetric both one- and two-photon QRMs, and numerically found that they have nearly the same numbers of level crossing points in the range of integers of \( N \) and \( M \) in each case. Therefore, we could reach a conclusion that the number of the doubly degenerate crossing points in asymmetric tpQRM would be twice of that in the asymmetric QRM due to two Bargmann indices in the former model.

Braak proposed a new criterion of integrability that if the eigenstates of a quantum system can be uniquely labeled by \( i = i_1 + i_2 \) quantum numbers, where \( i_1 \) and \( i_2 \) are the numbers of the discrete and continuous degree of freedom, then it is integrable [14]. Both symmetric QRM and tpQRM may be considered integrable in terms of this criterion. As the bias term of qubit sets in, the integrability will be violated in the asymmetric QRM. However, if \( \epsilon \) matches the multiple of the cavity frequency, the in-

FIG. 7: Energy spectrum \( E' = \frac{E + \omega/2 - q + \epsilon/3}{\sqrt{3}} \) for \( \omega = 1, \Delta = 2, q = 1/4, \epsilon = k\beta \). (a) \( \epsilon = 0 \), (b) \( \epsilon = \beta \), (c) \( \epsilon = 2\beta \), and (d) \( \epsilon = 4\beta \). Note particularly that \( \epsilon \) changes with \( g \) along \( g \)-axis. The horizontal blue dotted lines correspond to the pole energy ones for \( E_A \) and the red dashed lines to \( E_B \). Only the overlapped pole lines with \( N > 0 \) allow for the true level crossings. The triangles denotes the doubly degenerate crossing points in (a), (c), and (d). In (b), the triangles are obtained from \( f_N^{(q)} = 0 \) and \( e_M^{(q)} = 0 \) or equivalently from Eqs. (63) and (64), while open circles from Eqs. (62) and (63), all of them correspond to non-degenerate exceptional points.
integrability can be recovered in the asymmetric QRM, by using the hidden symmetry instead of the parity number. As shown in Figs. 1 and 2, the regular level crossings reappear when $\epsilon/\omega$ is an integer, similar to that in the symmetric QRM which is considered to be integrable \[16\]. However, the asymmetric tpQRM with fixed $\epsilon$ is always non-integrable because the energy levels cannot be uniquely labeled by the only continuous degree of freedom. As displayed in the spectrum in Figs. 5 and 6 with special $\epsilon$'s, there is no regular level crossings in sharp contrast to the integrable symmetric tpQRM \[14\]. Of course, if $\epsilon$ changes as $k/\beta$ with $k$ an even integer, the regular level crossings reappear in the asymmetric tpQRM as shown in Fig. 7 and it can be reconsidered to be integrable.

In the asymmetric QRM, the effort to look for the hidden symmetry responsible for the level crossings in the same $\epsilon$, continues to be a great interest \[16\], \[47\]. Since the doubly degenerate states within the same $q$ subspace also exist in the asymmetric tpQRM, which is definitely not owing to an explicit symmetry. It should be also interesting to rigorously find hidden symmetry in the asymmetric tpQRM in the near future.

**VII. CONCLUSION**

In this paper, we have studied both the asymmetric QRM and the asymmetric tpQRM by the BOA in a unified way. The previously observed level crossing when the bias parameter $\epsilon$ is a multiple of cavity frequency in the asymmetric QRM is illustrated by a closed-form proof for low orders of the constrained polynomial equations in a transparent manner. For the asymmetric tpQRM, the biased term breaks original $Z_4$ symmetry to $Z_2$ symmetry, so the Hilbert space only divides into even and odd bosonic number state subspaces. In each subspace, we derived the transcendental equation, called G-function, and obtain the regular spectrum exactly. The coefficients at the pole energy vanish in two different ways, giving two kinds of non-degenerate exceptional G-functions, by which all non-degenerate exceptional points can be detected.

Very interestingly, the true level crossings can also happen in the same $q$ subspace in the asymmetric tpQRM if the qubit bias parameter $\epsilon$ is an even multiple of the $g$-dependent renormalized cavity frequency, in contrast to the asymmetric one-photon QRM where $\epsilon$ can be simply a multiple of the cavity frequency. We argue that the even multiple is originated from the two-photon process involved in the two-photon model. The doubly degenerate points can be also located analytically, similar to the asymmetric QRM. The number of the doubly degenerate points within the same subspace in the asymmetric tpQRM should be comparable with that in asymmetric QRM. The subspace in the asymmetric tpQRM has no any explicit symmetry, the newly found double degeneracy thus also implies the hidden symmetry. The hidden symmetry in the asymmetric QRM could be identified at the same integer $\epsilon/\omega$, while in the asymmetric tpQRM at the same integer $\epsilon/(2\beta)$. The latter constraint on the parameter space for the occurrence of the double degeneracy is illuminating in searching for a conserved operator in two-photon case. The present results may shed some lights on the different nature of the hidden symmetries in the two asymmetric QRMs.

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**Appendix A: Demonstration for the same physical solutions of the two equations in the constrained conditions in two asymmetric QRMs**

In this Appendix, we first present a closed-form proof for the conjecture that $f_N(M, g) = 0$ and $c_M(N, g) = 0$ in Eq. (15) could give the same real and positive solutions for the coupling strength $g$ with small numbers of $N$ and $M$ in the asymmetric one-photon QRM. In parallel, we then provide a closed-form proof for the conjecture that $f_N^{(2)}(M, g) = 0$ and $c_M^{(2)}(N, g) = 0$ in Eq. (16) could give the same real and positive solutions for the coupling strength $g$ with small numbers of $N$ and $M$ in the asymmetric tpQRM. Finally, we provide numerical confirmations on the conjecture with large range of integers $N$ and $M$ in two asymmetric QRMs. We set $\omega = 1$ in both models for simplicity in the whole Appendix.

1. **Analytical proof for the small order of the constrained conditions in asymmetric one-photon QRM**

Since $f_0 = 1$, we begin with the $N = 1$ type-A pole energy, Eq. (10) becomes

$$f_1(M, g) = \frac{1}{2g} \left( 4g^2 + \frac{1}{4}\Delta^2 - M \right),$$

its zero is simply

$$g = \frac{1}{2} \sqrt{M - \left( \frac{\Delta}{2} \right)^2}, \quad (A1)$$

which is dependent on $M$. If $\Delta > 2\sqrt{M}$, no real solution exists, so the level crossing does not occur along the $N = 1$ pole line.

If we set $M = 2$ i.e. $\epsilon = 1$, we have

$$g = \frac{1}{2} \sqrt{2 - \left( \frac{\Delta}{2} \right)^2}, \quad (A2)$$

The second equation in (15) $c_2 (1, g) = 0$ yields
resulting in
\[ g = \frac{1}{2} \sqrt{2 - \left(\frac{\Delta}{2}\right)^2}, \]
which is exactly the same as Eq. (A2), the solution for \( f_1(2, g) = 0 \). It follows that two energy levels intersect with the same pole line at the same coupling strength \( g \) in the spectra, indicating a true energy level crossing.

For \( N = 2 \) type-A pole energy, the first equation in Eq. (A10) \( f_2(M, g) = 0 \) becomes (we set \( x = 4g^2 \) for simplicity)
\[
\left( x + \frac{1}{4}\Delta^2 - M + 1 \right) \left( x + \frac{1}{8}\Delta^2 - M \right) - x = 0, \tag{A3}
\]
yielding
\[
x = \left( M - \frac{3}{16}\Delta^2 \right) \pm \sqrt{\left( \frac{\Delta^2}{16} - 1 \right)^2 + (M - 1)}. \tag{A4}
\]
If \( M = 3 \), i.e. \( \epsilon \) is still 1, the solutions then read
\[
g = \frac{1}{8} \sqrt{-3(\Delta^2 - 16) \pm \left( (\Delta^2 - 16)^2 + 512 \right)}, \tag{A5}
\]
On the other hand, the second equation in Eq. (A10) \( c_3(2, g) = 0 \) is
\[
\left( x + \frac{\Delta^2}{4} \right) \left( x + \frac{\Delta^2}{8} - 1 \right) \left( x + \frac{\Delta^2}{12} - 2 \right) - x = 0,
\]
\[
-2x \left( x + \frac{\Delta^2}{12} - 2 \right) = 0. \tag{A6}
\]
which interestingly gives the same solutions as in Eq. (A3), consistent with the conjecture. Here an unphysical solution \( x = -\frac{1}{12}\Delta^2 \) is omitted.

Next, we set \( N = 1, M = 3 \), thus \( \epsilon = 2 \). \( f_1(3, g) = 0 \) gives
\[
g = \frac{1}{4} \sqrt{12 - \Delta^2}. \tag{A7}
\]
By \( c_3(1, g) = 0 \), we have
\[
\left( x + \frac{1}{4}\Delta^2 + 1 \right) \left( x + \frac{1}{8}\Delta^2 \right) \left( x + \frac{1}{12}\Delta^2 - 1 \right) - x = 0,
\]
\[
-2x \left( x + \frac{1}{12}\Delta^2 - 1 \right) = 0.
\]
Its solutions are
\[
x = 3 - \frac{1}{4}\Delta^2; x = -\frac{\Delta^2}{48} \left( 5 \pm \sqrt{1 - \frac{96}{\Delta^2}} \right).
\]
Note that the second root is not a positive real value, and so omitted. The first root gives exactly the same \( g \) in Eq. (A6).

2. Analytical proof for the small order of the constrained conditions in asymmetric tpQRM

In this Appendix, we present a closed-form proof for the conjecture that \( f^{(q)}_1(M, g) = 0 \) and \( c^{(q)}_M(N, g) = 0 \) in Eq. (10) could give the same real and positive solutions for the coupling strength \( g \) with small numbers of \( N \) and \( M \) in the asymmetric tpQRM.

For the most simply case, we set \( N = 1 \), then \( f^{(q)}_1(M, g) = 0 \) gives
\[
4q - (2M + 4q) \beta^2 + \frac{\Delta^2}{8} = 0, \tag{A8}
\]
then the location of the degenerate point is obtained
\[
\beta^2 = \frac{2q + \Delta^2/16}{M + 2q}. \tag{A9}
\]
If set \( M = 2 \), \( c^{(q)}_2(1, g) = 0 \) gives
\[
\left[ 4(q + 1)(1 - \beta^2) + \frac{1}{8}\Delta^2 \right] \times \left[ 2(2q + 1)(1 - \beta^2) + \frac{1}{16}\Delta^2 - 2 \right] - 4(q + \frac{3}{4})(q + \frac{3}{4})(1 - \beta^2) = 0,
\]
we then have
\[
\beta^2 = \frac{2q + \Delta^2/16}{2 + 2q}, \tag{A10}
\]
which is the same as that in Eq. (A8) for \( M = 2 \), consistent with our conjecture.

Next, we set \( N = 2, M = 3 \). \( f^{(q)}_2(3, g) = 0 \) gives
\[
\left[ 4(1 - \beta^2)(2 + q) + \frac{\Delta^2}{8} - 4 \right] \times \left[ 2(1 - \beta^2)(3 + 2q) + \frac{\Delta^2}{16} - 6 \right] - 4(1 - \beta^2)(q + \frac{1}{4})(q + \frac{3}{4}) = 0.
\]
The solutions at \( q = \frac{1}{4} \) are
\[
\beta^2 = \frac{1}{3} + \frac{23\Delta^2}{2016} \pm \frac{1}{126} \sqrt{\frac{25\Delta^4}{256} + \frac{21}{2}\Delta^2 + 1008},
\]
and at \( q = \frac{3}{4} \) are
\[
\beta^2 = \frac{5}{11} + \frac{29\Delta^2}{3168} \pm \frac{1}{363} \sqrt{\frac{20\Delta^4}{8712} + \frac{49\Delta^4}{3168^2}}.
\]
while $c_3^{(q)}(2, g) = 0$ results in
\[
\left\{ 2 + q + \frac{\Delta^2}{32} (1 - \beta^2) \right\} \left[ 2 (1 - \beta^4) (3 + 2q) + \frac{\Delta^2}{16} - 2 \right] \\
-(q + \frac{5}{4})(q + \frac{7}{4}) \times \left[ 4 (1 - \beta^2) (1 + q) + \frac{\Delta^2}{24} - 4 \right] \\
- 4 (1 - \beta^2) (2 + q) + \frac{\Delta^2}{8} \right\} (q + \frac{1}{4})(q + \frac{3}{4}) = 0.
\]

If $q = \frac{1}{4}$, the solutions are
\[
\beta^2 = \frac{1}{3} + \frac{23\Delta^2}{2016} \pm \frac{1}{126} \sqrt{\frac{25\Delta^4}{256} + \frac{21}{2} \Delta^2 + 1008},
\]
\[
\beta_3^2 = 1 + \frac{\Delta^2}{120}.
\]

If $q = \frac{3}{4}$, the solutions are
\[
\beta^2 = \frac{5}{11} + \frac{29\Delta^2}{3168} \pm \sqrt{\frac{20}{363} + \frac{\Delta^2}{8712} + \frac{49\Delta^4}{3168^2}},
\]
\[
\beta_3^2 = 1 + \frac{\Delta^2}{168}.
\]

Omitting the unreasonable solutions $\beta_3$, we can find that both $f_2^{(q)}(3, g) = 0$ and $c_3^{(q)}(2, g)$ give the same crossing coupling strengths for $q = 1/4$ and $3/4$ respectively.

\[
y_{1,2}^{(1/4)} = \frac{1}{2} \sqrt{\frac{2}{3} - \frac{23}{2016} \Delta^2 \pm \frac{\sqrt{25\Delta^4 + 2688\Delta^2 + 258048}}{2016}},
\]
(A11)
\[
y_{1,2}^{(3/4)} = \frac{1}{2} \sqrt{\frac{6}{11} - \frac{29}{3168} \Delta^2 \pm \frac{\sqrt{49\Delta^4 + 1152\Delta^2 + 552960}}{3168}},
\]
(A12)

which also agree well with our conjecture.

3. Numerical confirmation for the two conjectures in both asymmetric QRMs

We extensively demonstrate that, for large $N$ and $M (> N)$, the two equations in either Eq. (15) or Eq. (40) give the same physics solutions in both asymmetric QRMs. We set $N$ from 1 to 10 and $M$ from 2 to 20 for both one-photon QRM and tpQRM in the $q = 1/4$ subspace at $\Delta = 2$. First, we find that physics solutions from $f_N = 0$ and $c_M = 0$ are exactly the same in either case, confirming the conjectures numerically. Second, there are 715 level crossings points for both cases, indicating $N$ roots in the $N$ order polynomial equations in both models at $\Delta = 2$. Generally, the root number is equal to or slightly less than $N$ for any $\Delta$. This is to say, for any values of $\Delta$, the numbers of the level crossings are generally nearly the same for the same ranges of $N$ and $M$ in asymmetric QRMs.

In Fig. 8 the doubly degenerate level crossing points are visualized in a three-dimensional (3D) view in $(\epsilon, g, E)$-space for the asymmetric one-photon QRM and in $(\epsilon/\beta, g, E)$-space for the asymmetric tpQRM at $\Delta = 2$. It is interesting to draw planes for level crossings in both cases, as $\epsilon$ is simply scaled by a $g$-dependent factor, $2\beta$, in the two-photon case. In the original 3D $(\epsilon, g, E)$-space, all the degenerate crossing points in asymmetric QRM are confined in equally spaced integer $\epsilon/\omega$ planes, while those in asymmetric tpQRM are actually locked in different cylindrical surfaces with integer $\epsilon/(2\beta)$. Those different constrained surfaces in the model parameter spaces for the occurrence of the double degeneracy in two models should be considered in the definition of conserved operators and the detection of hidden symmetries.
