Failure of the Hopf-Oleinik lemma for a linear elliptic problem with singular convection of non-negative divergence

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Abstract

In this paper we study existence, uniqueness, and integrability of solutions to the Dirichlet problem

\[-\text{div}(M(x)\nabla u) = -\text{div}(E(x)u) + f \text{ in } \Omega\]

\[u = 0 \text{ on } \partial\Omega.\]  

(1)

Here \(\Omega \subset \mathbb{R}^N, N \geq 3\), is an open, bounded set, and we assume that \(M \in L^\infty(\Omega)^N\times N\) is elliptic

\[M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega.\]

According to the regularity of the right-hand side datum \(f(x)\) it is natural to search the solution in the energy space \(W^{1,2}_0(\Omega)\) (case of \(f \in H^{-1}(\Omega)\): see, e.g. [21, 16, 1]), or in a larger Sobolev space if \(f\) is singular (see [1]); when \(f \in L^1(\Omega)\), see, for instance, [8], or when \(L^1(\Omega, \delta)\) with \(\delta(x) = d(x, \partial\Omega)\), see, e.g., [7, 13].

In the mentioned references it assumed that the convection term is regular (for instance \(E \in W^{1,\infty}(\Omega)\)) and that it satisfies an additional condition which helps to have a maximum principle:

\[\text{div } E \geq 0 \text{ a.e. on } \Omega.\]  

(2)

1 Introduction

It is well known that many relevant applications lead to the presence of a convection term in the correspondent model which, in its simplest formulation, leads to a boundary value problem for linear elliptic second order equation of the following type:

\[
\begin{cases}
-\text{div}(M(x)\nabla u) = -\text{div}(uE(x)) + f(x) & \text{in } \Omega \\
\quad \quad u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

(M)(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega.

According to the regularity of the right-hand side datum \(f(x)\) it is natural to search the solution in the energy space \(W^{1,2}_0(\Omega)\) (case of \(f \in H^{-1}(\Omega)\): see, e.g. [21, 16, 1]), or in a larger Sobolev space if \(f\) is singular (see [1]); when \(f \in L^1(\Omega)\), see, for instance, [8], or when \(L^1(\Omega, \delta)\) with \(\delta(x) = d(x, \partial\Omega)\), see, e.g., [7, 13].

In the mentioned references it assumed that the convection term is regular (for instance \(E \in W^{1,\infty}(\Omega)\)) and that it satisfies an additional condition which helps to have a maximum principle:

\[\text{div } E \geq 0 \text{ a.e. on } \Omega.\]  

(2)

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More recently, some effort has been devoted to get an existence and regularity theory under more general conditions on the convection term $E$ by different authors (see, e.g. [1], [5] and their references). For instance, solutions in the energy space can be considered under the conditions $|E| \in L^N(\Omega)$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$. In [13] and [12] the authors study the case in which $|E| \in L^N(\Omega)$ and $\text{div} \ E = 0$ in $\Omega$ and $E \cdot n = 0$ on $\partial \Omega$, $f \in L^1(\Omega, \delta)$. See also [15, 20].

In this paper, we will show that (2) makes $\text{div} \ E$ behave like a non-negative potential in the Schrödinger case, and we can apply techniques from that setting. See, for example, [12, 13, 14, 17]. We focus on the case where (2) holds in distributional sense.

The paper is structured as follows. First, in Section 2 we review known results for the case $|E| \in L^N$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$ which were published in [1], were shown there is a unique weak solution of (1) that can be constructed by approximation. In Section 3 we show that if $|E| \in L^2(\Omega)$, $\text{div} \ E \geq 0$, and $f \in L^m(\Omega)$ for some $m > 1$ then the same approximation procedure converges to a weak solution of (1), and we give some a priori bounds for this solution. In Section 4 we show that, if we also assume $f \in L^{\frac{2N}{N+2}}(\Omega)$, then this constructed solution is the unique weak solution of (1).

Then we move to discussing interesting examples that fall in this setting. In Section 5 we focus on the case

$$E(x) = A \frac{x}{|x|^2},$$

which is somehow in the limit of theory since it is not in $L^N(\Omega)$ but it is in $L^r(\Omega)$ for $r \in [1, N)$. In [5] the authors examined the more general class

$$|E| \leq \frac{|A|}{|x|}.$$  \hspace{1cm} (4)

The authors show existence of solutions $u$, where the summability is reduced as $|A|$ is increased. Their results indicate that the sign of $A$ should play a role, but the application of Hardy’s inequality (which they use in a crucial way) is not able to detect this fact. In Theorem 16 we show that if $N > 1$, $f \in L^m(\Omega)$ for suitable $m$, and $A > 0$ then we can use the sign of $\text{div} \ E$ to deduce that the solution $u_A$ of (1) with $E = A \frac{x}{|x|^2}$ satisfies

$$u_A \to 0 \text{ in } L^1(\Omega) \text{ as } A \to +\infty.$$  

By the contrary, when $A < 0$ we cannot improve the result in [5]. Notice that this is similar to the equation $L(u_B) + B u_B = f$, whereas $B \to \infty$ we have $u_B \to 0$.

Lastly, in Section 6, we discuss the case where $E$ is suitably singular only on the boundary. We present an example showing that if $\text{div} \ E$ behaves like $d(x, \partial \Omega)^{-2-\gamma}$ for some $\gamma > 0$ and $f$ is bounded, then the solutions are flat on the boundary, i.e.

$$|u(x)| \leq C \text{dist}(x, \partial \Omega)^\alpha \text{ for some } \alpha > 1.$$  

In particular, this shows that the Hopf-Oleinik lemma, i.e. $\frac{\partial u}{\partial n} < 0$ on $\partial \Omega$, fails in the presence of such singular drift terms $E$. Our example can be easily extended to a more general class of $E$, as we comment in Section 7. Again, we use the fact that $\text{div} \ E$ acts as a potential. However, in the Schrödinger equation it is sufficient that $V(x) \geq C \delta^{-2}$ to get flat solutions, whereas for $E$ we need a strictly larger exponent (see Remark 22). Questions of this type are quite relevant in the framework of linear Schrödinger equations associated to singular potential
since they can be understood as complements to the Heisenberg Incertitude Principle (see, e.g. [10, 11, 12, 13, 18, 14]).

We conclude with some further comments and open problems in Section 7.

2 Known results when \(|E| \in L^N\)

We define the Sobolev conjugate exponent

\[ m^* = \frac{mN}{N - m} \quad \text{if} \quad m \leq N, \quad \text{and} \quad m^{**} = (m^*)^* = \frac{mN}{N - 2m} \quad \text{if} \quad m \leq \frac{N}{2}. \]

We have that \(m^{**} \in [1, \infty]\) for \(\frac{N}{N+2} \leq m \leq \frac{N}{2}\). Notice that \(m^* \geq 2\) if and only if \(m \geq \frac{2N}{N+2} = (2^*)'\). Notice that, since \(m \geq 1\) we have \(m^* \geq m\). In order to compute explicit a priori estimates, we use the Sobolev embedding constant \(S_p\) such that, for \(1 < p < +\infty\)

\[ S_p \|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}. \] (5)

We point out the relevance of the constants, for \(N > 2\) of \((2^*)' = \frac{2N}{N+2}\). This constant depends only of \(N\). Since we are going to require the Sobolev embedding for \(p = 2\), we assume that \(N \geq 3\). In [1] the author proves the following existence theorem with a priori estimates.

**Theorem 1** ([1]). Let \(f \in L^{\frac{2N}{N+2}}(\Omega)\) and \(|E| \in L^N(\Omega)\). Then, there exists a unique weak solution \(u\) of (1) in the sense that

\[ u \in W^{1,2}_0(\Omega) \] is such that

\[ \int_\Omega M(x)\nabla u \nabla v = \int_\Omega u E(x) \cdot \nabla v + \int_\Omega f(x) v(x), \quad \forall \ v \in W^{1,2}_0(\Omega). \]

and it satisfies:

1. **Logarithmic estimate:**

\[ \left( \int_\Omega |\log(1 + |u|)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{S^2_2\alpha^2} \int_\Omega |E|^2 + \frac{2}{S^2_2\alpha} \int_\Omega |f|, \]

2. **Gradient estimate:** there exists \(C = C(\alpha, N)\) such that

\[ \int_\Omega |\nabla u|^2 \leq C \left( \|E\|_{L^N}^2 + \|f\|_{L^{\frac{2N}{N+2}}}^2 \right). \] (6)

3. **Stampacchia-type summability:** For \(m \in (\frac{2N}{N+2}, \frac{N}{2})\) there exists a constant \(C = C(m, \alpha, N, \|E\|_{L^N})\) such that

\[ \|u\|_{m^*} \leq C\|f\|_m. \] (7)

4. **Stampacchia-type boundedness:** Let \(r > N\) and \(m > \frac{N}{2}\). There exists \(C\) such that

\[ \|u\|_{L^\infty} \leq C(m, r, \alpha, \|f\|_{L^m}, \|E\|_{L^r}). \] (8)

**Remark 2.** The natural theory for this problem in energy space is precisely \(|E| \in L^N(\Omega)\), since in the weak formulation we need to justify a term of the form \(Eu \nabla v\), where \(u, v \in W^{1,2}_0(\Omega)\). This means that \(u \in L^2\) whereas \(\nabla v \in L^2\). So we always have that \(uE \in L^2(\Omega)\).
In [1] the main tool to study the linear problem (1) are the auxiliary non-linear Dirichlet problems
\[
\begin{aligned}
- \text{div}(M(x)\nabla u_n) &= - \text{div}\left(\frac{u_n}{1 + \frac{1}{n}u_n}E_n(x)\right) + f_n(x) &\text{in } \Omega \\
u &= 0 &\text{on } \partial\Omega,
\end{aligned}
\]
where the author take \( f_n = T_n(f) \) a truncation of \( f \) through the family
\[T_n(s) = \begin{cases} s & |s| \leq k, \\ k \text{sign}(s) & |s| > k, \end{cases}\]
and \( E_n = \frac{E}{1 + \frac{1}{n}|E|} \). We will take advantage of a similar approximation.

**Remark 3.** Since the problem is linear, for \( t \in \mathbb{R} \) we have that \( tu \) is solution of
\[- \text{div}(M(x)\nabla [tu]) = - \text{div}([tu]E(x)) + tf(x),\]
and that \( E \) does not change. Thus, using (6)
\[t^2 \int_{\Omega} |\nabla u|^2 \leq C \left( ||E||_{L^N}^2 + t^2 ||f||_{L^2}^2 \right).\]
Dividing by \( t^{-2} \) and taking the limit as \( t \to \infty \) gives
\[\int_{\Omega} |\nabla u|^2 \leq C ||f||_{L^2}^2.\]
Notice that in Theorem 11 we will prove this fact for the case \( \text{div} E \geq 0 \).

### 3 Existence theory when \(|E| \in L^2 \) and \( \text{div} E \geq 0 \)

The structural assumption in this section is the following:
\[
\begin{aligned}
E &\text{ belongs to the Lebesgue space } (L^2(\Omega))^N, \\
\text{div} E &\geq 0 \text{ in } D'(\Omega), \text{ that is } \int_{\Omega} E \cdot \nabla \phi \leq 0, \forall \ 0 \leq \phi \in W_0^{1,2}(\Omega).
\end{aligned}
\]

**Theorem 4.** Assume (11) and
\[f \in L^m(\Omega), \ 1 < m < \frac{N}{2},\]
and let \( p = \min\{2, m^*\} \). Then, there exists a weak solution \( u \) of (1) in the sense that
\[u \in W_0^{1,p}(\Omega) \text{ is such that } \int_{\Omega} M(x)\nabla u \nabla v = \int_{\Omega} u E(x) \cdot \nabla v + \int_{\Omega} f(x) v(x), \forall \ v \in W_0^{1,\infty}(\Omega).\]
Furthermore, it satisfies
\[
\begin{aligned}
\|u\|_{W_0^{1,p}(\Omega)} &\leq C_m \|f\|_{m^*}, &\text{if } 1 < m < \frac{2N}{N+2}; \\
\|u\|_{W_0^{1,2}(\Omega)} + \|u\|_{m^*} &\leq C_m \|f\|_{m^*}, &\text{if } \frac{2N}{N+2} \leq m < \frac{N}{2}.
\end{aligned}
\]
Lemma 7. The proof of Theorem 4 is based on the following approximation lemma.

Thus, we have

Corollary 6. The solutions constructed in Theorem 4 satisfy (7) and (8).

We say that $u_n$ is a weak solution of (9) if $u \in W^{1,2}_0(\Omega)$ is such that

$$
\int_\Omega M(x)\nabla u_n \nabla v = \int_\Omega \frac{u_n}{1 + \frac{1}{n}|u_n|^q} E_n(x) \cdot \nabla v + \int_\Omega f_n(x) v(x), \quad \forall v \in W^{1,2}_0(\Omega).
$$

The existence of a weak solution if $E_n \in L^2(\Omega)^N$ is a consequence of the Schauder theorem. The proof of Theorem 4 is based on the following approximation lemma.

Lemma 7. Let $u_n$ be any weak solution of (15) with $E_n = E$, (11), (12), and $f_n = T_n(f)$. Then, for any weak solution $u_n$ of (15) we have that

$$
\begin{align*}
\|u_n\|_{W^{1,m^*}_0(\Omega)} & \leq C_m \|f\|_{m^*}, & \text{if } 1 < m < \frac{2N}{N+2}; \\
\|u_n\|_{W^{1,2}_0(\Omega)} + \|u_n\|_{m^*} & \leq C_m \|f\|_{m^*}, & \text{if } \frac{2N}{N+2} \leq m \leq \frac{N}{2}.
\end{align*}
$$

(16)

where

$$
C_m \text{ does not depend on } E.
$$

(17)

Hence, up to a subsequence, $\{u_n\}$ converges weakly in $L^{m^*}$.

Proof. Our proof is the same of [4], since we will see that the contribution of new term on $E$ is a negative number. We use $T_k(u_n)T_k(u_n)|^{2\gamma-2}$ as test function in (15), $\gamma = \frac{m^*}{2}$; we repeat it is possible since every $T_k(u_n)$ has exponential summability. Note that $2\gamma - 1 > 0$ since $m > 1$. Thus, we have

$$
\int_\Omega M(x)\nabla u_n \nabla (T_k(u_n)|T_k(u_n)|^{2\gamma-2}) = \int_\Omega \frac{u_n}{1 + \frac{1}{n}|u_n|^\frac{4}{3}} E(x) \cdot \nabla (T_k(u_n)|T_k(u_n)|^{2\gamma-2}) + \int_\Omega f_n(x) T_k(u_n)|T_k(u_n)|^{2\gamma-2}.
$$

To study the second integral, we define the function

$$
H_\gamma(s) = \int_0^s \frac{t|t|^{2\gamma-2}}{1 + \frac{1}{n}|t|^\gamma} dt.
$$

It is easy to check that $H_\gamma(s) \geq 0$ for all $s \in \mathbb{R}$. Thus, using the sign condition on $\text{div } E$ we have that

$$
\begin{align*}
\int_\Omega \frac{u_n}{1 + \frac{1}{n}|u_n|^\frac{4}{3}} E(x) \cdot \nabla (T_k(u_n)|T_k(u_n)|^{2\gamma-2}) & = \int_\Omega (2\gamma - 1) \frac{T_k(u_n)|T_k(u_n)|^{2\gamma-2}}{1 + \frac{1}{n}|T_k(u_n)|^\gamma} E(x) \cdot \nabla T_k(u_n) \\
& = \int_\Omega H_\gamma(T_k(u_n)) E(x) \cdot \nabla T_k(u_n) = \int_\Omega E(x) \cdot \nabla |H_\gamma(T_k(u_n))| \leq 0.
\end{align*}
$$

5
Hence, we have that
\[ \int_{\Omega} M(x) \nabla u_n \nabla (T_k(u_n)|T_k(u_n)|^{2\gamma - 2}) \leq \int_{\Omega} f_n(x) T_k(u_n)|T_k(u_n)|^{2\gamma - 2}, \]
which is the starting point of [4], and we get the estimates
\[
\begin{cases}
\|T_k(u_n)\|_{W^{1,m}_0(\Omega)} \leq C_m \|f\|_m, & \text{if } 1 < m < \frac{2N}{N+2}; \\
\|T_k(u_n)\|_{W^{1,2}_0(\Omega)} + \|T_k(u_n)\|_{m,\infty} \leq \tilde{C}_m \|f\|, & \text{if } \frac{2N}{N+2} \leq m \leq \frac{N}{2}.
\end{cases}
\]
Letting \( k \to \infty \) we recover (16).

With this lemma, we can pass to the limit to prove Theorem 4.

**Proof of Theorem 4.** Up to subsequences, the sequence \( \{T_k(u_n)\} \) constructed above, weakly converges (in \( W^{1,m}_0(\Omega) \) or in \( W^{1,2}_0(\Omega) \cap L^{m^*}(\Omega) \)) and it is possible to pass to some \( u \) (note that \( u \in L^{m^*}(\Omega) \)). Recall that \( E \in (L^2)^N \). In order to pass to the limit in
\[ \int_{\Omega} \frac{u_n}{1 + \frac{1}{n}|u_n|^2} E_n(x) \cdot \nabla v \]
in (15) we need
\[ 1 \geq \frac{1}{m^*} + \frac{1}{N} + \frac{1}{2} \]
That is equivalent to \( m \geq \frac{2N}{N+2} \). Thus we pass also to the limit in (16). \( \square \)

**Remark 8.** Note that, once more it is possible to develop an approximate method in order to prove the existence when \( E \in L^r \). Indeed, let \( E_0 \in L^r \), \( r > 1 \) and \( E_n \in L^2 \) converging to \( E_0 \) in \( L^r \). Define now \( u_n \) in the corresponding way, we can use the statement of (17), so that we can say that estimates (14) still hold for this new sequence \( \{u_n\} \) and once more we can pass to the limit, and we prove the existence if
\[ 1 \geq \frac{1}{r} + \frac{1}{m} - \frac{2}{N} \]
We can provide further a priori estimates when \( \text{div } E \geq 0 \)

**Proposition 9.** The solutions constructed in Theorem 4 satisfy the following additional estimates:

1. **(L¹ estimate)** If \( \text{div } E \in L^1(\Omega) \) then we have that
\[ \int_{\Omega} |u| \text{div } E \leq \int_{\Omega} |f|. \] (18)

2. **(L^m estimate)** If \( \text{div } E \geq c_0 > 0 \) and \( m > 1 \) then
\[ \|u\|_{L^m} \leq \frac{m}{m - 1} \frac{\|f\|_{L^m}}{c_0}. \] (19)

We will later take advantage of (18) and present several extensions. See, e.g., Lemma 19 where we extend the result to \( \text{div } E \in L^1_{loc} \).
Remark 10. Notice that (19) blows up as \( m \to 1 \). In fact, it is known that the case \( m = 1 \) does not satisfy such an estimate.

We prove a priori estimates under the assumption of \( \text{div} \ E \geq 0 \) for bounded (or even smooth) \( E \), which we now know will hold for approximations.

Proof of Proposition 9. Assume first that \( E \in (L^N)^N \), and \( f \in L^m \) for \( m \geq \frac{2N}{N+2} \). Then, we can deal with the unique solution \( u \in W_0^{1,2}(\Omega) \) that exists by Theorem 1. Due to the construction by approximation in Theorem 4 the estimates pass to the limit in the construction. Take \( h \in W^{1,\infty}(\mathbb{R}) \) such that \( h(0) = 0 \). We take \( v = h(u) \) as a test function we can write

\[
\alpha \int_{\Omega} h'(u) |\nabla u|^2 \leq \int_{\Omega} M(x) \nabla u \cdot \nabla h(u) = \int_{\Omega} uE \cdot \nabla h(u) + \int_{\Omega} fh(u).
\]

We can write

\[
u \nabla h(u) = uh'(u) \nabla u = \nabla F(u)
\]

where \( F(s) = \int_0^s \tau h'(\tau)d\tau \). Hence,

\[
\alpha \int_{\Omega} h'(u) |\nabla u|^2 \leq \int_{\Omega} E \cdot \nabla F(u) + \int_{\Omega} fh(u).
\]

Now we prove both items

- **Item 1.** Since \( \text{div} \ E \in L^1(\Omega) \) we can integrate by parts again to deduce

\[
\alpha \int_{\Omega} h'(u) |\nabla u|^2 + \int_{\Omega} F(u) \text{div} E \leq \int_{\Omega} fh(u). \tag{20}
\]

Let us consider \( h_\varepsilon(s) = T_\varepsilon(s) / \varepsilon \). Then \( h_\varepsilon' \geq 0 \) and \( |h_\varepsilon| \leq 1 \) and, hence, in (20)

\[
\int_{\Omega} F_\varepsilon(u) \text{div} E \leq \int_{\Omega} |f|.
\]

It is clear that \( F_\varepsilon(s) \to |s| \) a.e. as \( \varepsilon \to 0 \). Then,

\[
\int_{\Omega} |u| \text{div} E \leq \int_{\Omega} |f|.
\]

- **Item 2.** Let us take, for \( m > 1 \), \( h(s) = |s|^{m-1} \) then

\[
F(s) = (m-1) \int_0^s |\tau|^{m-2} \text{sign}(\tau) \tau d\tau = \frac{m-1}{m} s^m.
\]

Hence, going back to (20)

\[
c_0 \frac{m-1}{m} \|u\|_{L^m}^m \leq \frac{m-1}{m} \int_{\Omega} |u|^m \text{div} E \leq \int_{\Omega} f |u|^{m-1} \leq \|f\|_{L^m} \|u\|_{L^m}^{m-1}.
\]

Hence, we simplify

\[
\|u\|_{L^m} \leq \frac{m}{m-1} \frac{\|f\|_{L^m}}{c_0}.
\]
4 Comparison principle and uniqueness

To show uniqueness of solutions we prove a weak maximum principle.

**Theorem 11.** Let \( f \in L^{\frac{N}{N-2}}(\Omega) \) and (11). Then, if \( u \in W^{1,2}_0(\Omega) \) is a solution of (13) then

\[
\|\nabla u^+\|_2 \leq \frac{1}{\alpha \delta_2} \|f^+\|_{\frac{2N}{N+2}}.
\]

Hence, there is, at most, one solution of (13) in \( W^{1,2}_0(\Omega) \). Furthermore, if \( f \geq 0 \) then \( u \geq 0 \).

We first prove the following lemma

**Lemma 12.** Let \( m, r > 1 \), \( E \in L^r(\Omega) \) with \( 0 \leq \div E \in D'(\Omega) \). Then, we have that

\[
-\int_{\Omega} E \nabla v \geq 0 \quad \forall \ 0 \leq v \in W^{1,r}_0(\Omega).
\]

**Proof.** By definition of having a sign in distributional sense, for \( 0 \leq \varphi \in C^\infty_c(\Omega) \), we have that

\[
-\int_{\Omega} E \nabla \varphi = \langle \div E, \varphi \rangle \geq 0.
\]

For \( 0 \leq v \in W^{1,r}_0(\Omega) \), we can find a sequence \( 0 \leq \varphi_n \in C^\infty_c(\Omega) \), such that \( \varphi_n \to v \) in \( W^{1,r}_0(\Omega) \). In particular, \( \nabla \varphi \to \nabla v \) in \( L^r(\Omega) \). We can pass to the limit in the estimate. \( \square \)

**Proof of Theorem 11.** Let \( u \) be a solution. Take \( \rho_n \) a family of non-negative mollifiers, and use \( v_n = \rho_n \ast u^+ \) as a test function. Passing to the limit in \( n \) and applying the previous lemma

\[
a \int_{\Omega} |\nabla u^+|^2 \leq \int_{\Omega} E \nabla u^+ \frac{u^2}{2} + \int_{\Omega} f u^+ \leq \|f\|_{(2, r)} \|u^+\|_{2^*} \leq \frac{1}{S} \|f\|_{(2, r)} \|\nabla u^+\|_2.
\]

We recover the estimate. \( \square \)

**Lemma 13.** Let \( E \in L^r(\Omega)^N \) for \( r > 1 \) with \( \div E \geq 0 \) in \( D'(\Omega) \). Then, there exists a sequence \( E_n \in W^{1,\infty}(\Omega) \) with \( \div E_n \geq 0 \) such that \( E \to E_n \) in \( L^r(\Omega)^N \).

**Proof.** We use a similar decomposition to [22, Theorem 1.5] (done there for \( r = 2 \)). First, define

\[
\begin{cases}
-\Delta p^{(1)} = \div E \quad \text{in } \Omega, \\
p^{(1)} = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

By well-known results we get a unique solution \( p^{(1)} \in W^{1,r}_0(\Omega) \). Take \( E^{(1)} = \nabla p^{(1)} \). Lastly, take \( E^{(2)} = E - E^{(1)} \in L^r(\Omega) \). Notice that \( \div E^{(2)} = 0 \). Due to [19], \( E^{(2)} \) admits a divergence-free extension to \( L^r(\mathbb{R}^d) \), which we denote \( \tilde{E}^{(2)} \). We can take a family of mollifiers \( \rho_n \), and \( E^{(2)} = \tilde{E}^{(2)} * \rho_n \in W^{1,\infty}(\Omega) \). Now let \( 0 \leq g^{(1)} = \div E^{(1)} \in W^{-1, r'}(\Omega) \). Let \( g_n^{(1)} \geq 0 \) be a sequence of \( C^\infty_c(\Omega) \) functions with \( g_n^{(1)} \to g^{(1)} \) in \( L^r(\Omega) \). Take \( p^{(1)}_n \) the unique solution to

\[
\begin{cases}
-\Delta p^{(1)}_n = g^{(1)}_n \quad \text{in } \Omega, \\
p^{(1)}_n = 0 \quad \text{in } \partial \Omega.
\end{cases}
\]

Finally, define \( E^{(1)}_n = \nabla p^{(1)}_n \in W^{1,\infty}(\Omega) \). It is now easy to see that \( E^{(i)}_n \to E^{(i)} \) in \( L^r(\Omega)^N \) for \( i = 1, 2 \), and the proof is complete. \( \square \)
**Theorem 14.** Let \( f \in L^m(\Omega) \) and \( E \in L'(\Omega) \) such that \( 0 \leq \text{div } E \in \mathcal{D}'(\Omega) \) and

\[
\begin{cases}
\frac{1}{\min\{2^*, m^{**}\}} + \frac{1}{r} \leq 1 & \text{if } \frac{2N}{N+2} \leq m \leq \frac{N}{2} \\
\frac{1}{2^*} + \frac{1}{r} \leq 1 & \text{otherwise}
\end{cases}
\]  
(22)

Then, taking \( q = \min\{2, m^*\} \) (using formally \( m^* = \infty \) for \( m \geq N \)) there exists a solution of \( u \in W^{1,q}_0(\Omega) \) such that

\[
\int_{\Omega} M(x) \nabla u \nabla v = \int_{\Omega} E \nabla v + \int_{\Omega} f v, \quad \forall v \in W^{1,q}_0(\Omega).
\]  
(23)

Furthermore, if \( m \geq \frac{2N}{N+2} \) and \( r \geq N \) it is the unique solution of (13).

**Proof.** Let \( f_k = T_k(f) \) where \( T_k \) is the cut-off function. We consider \( E_k \) constructed in Lemma 13. By Proposition 9 there exists a unique weak \( u_k \) solution of (13). Since the \( \cdot^* \) operation is monotone, then \( q^* = \min\{2^*, m^{**}\} \). The sequence \( u_k \) is uniformly bounded in \( W^{1,q}_0(\Omega) \). Therefore, by the Sobolev embedding theorem, it is uniformly bounded on \( L^{q^*_*}(\Omega) \). Up to a subsequence, there exists \( u \in W^{1,q}_0(\Omega) \) such that

\[
\nabla u_k \rightharpoonup \nabla u \quad \text{in } L^{q}(\Omega)
\]

\[
u_k \rightharpoonup u \quad \text{in } L^{q^*_*}(\Omega).
\]

Since \( M \in L^\infty(\Omega)^{N \times N} \), \( E_k \to E \in L'(\Omega)^N \) strongly and (22) we have that

\[
M(x) \nabla u_k \rightharpoonup M(x) \nabla u \quad \text{in } L^q(\Omega)
\]

\[
u_k E_k \rightharpoonup u E \quad \text{in } L^1(\Omega).
\]

Therefore, we can pass to the limit in the weak formulation for \( v \in W^{1,\infty}_0(\Omega) \). If \( m \geq \frac{2N}{N+2} \) and \( r \geq N \), then \( u E \in L^2(\Omega) \), and it is a solution of (13) by approximation. \( \square \)

5 Convection with singularity at one point

With the approach developed in this paper we are able to study the special situation

\[
E = A \frac{x}{|x|^2}
\]  
(24)

which is somehow in the limit of theory since it is not in \( L^N(\Omega) \), but it is in \( L^r(\Omega) \) for \( r \in [1,N] \). In [5] the authors examined the framework of drifts such that

\[
|E| \leq \frac{|A|}{|x|}.
\]  
(25)

The authors show existence of solutions \( u \) under (25), where the summability is reduced as \( |A| \) is increased. They prove

**Theorem 15** ([5]). Let \( f \in L^m(\Omega) \) and \( |E| \leq |A|/|x| \). Then, there exists a solution \( u \) the solution of (1) and

1. If \( |A| < \frac{a(N-2m)}{m} \) and \( m \in [\frac{2N}{N+2}, \frac{N}{2}] \) then \( u \in W^{1,2}_0(\Omega) \cap L^{m^{**}}(\Omega) \).
2. If $|A| < \frac{\alpha(N-2m)}{m}$ and $m \in (1, \frac{2N}{N+2})$ then $u \in W^{1,m}_0(\Omega)$.

3. If $|A| < \alpha(N-2)$ and $m = 1$ then $\nabla u \in (M^{N^*}(\Omega))^N$ and $u \in W^{1,q}_0(\Omega)$, for every $q < \frac{N}{N-1}$.

Above, $M^{N^*}$ denotes the Marcinkiewicz space (see [5] for the definition and some properties). The argument in [5] is based on Hardy’s inequality

$$\int_{\mathbb{R}^N} |u|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2.$$  \hfill (26)

We are able to extend this result to distinguish depending on the sign of $A$. Our result is the following

**Theorem 16.** Let $f \in L^m(\Omega)$ for some $m > 1$ and (24). Then, there exists a solution $u_A$ of (23), and it satisfies the estimates in Proposition 9. Furthermore, $u_A \to 0$ as $A \to \infty$ in the sense that

$$\int_{\Omega} \frac{|u_A(x)|}{|x|^2} \leq \frac{1}{A(N-2)} \int_{\Omega} |f|.$$  \hfill (27)

We point out that, if $m > \frac{2N}{N+2}$, we have furthermore $u_A E \in L^2(\Omega)$.

**Proof.** Since $N \geq 3$ we know that $|E| \in L^2(\Omega)$ and that

$$\text{div } E = r^{1-N} \frac{\partial}{\partial r}(r^{N-1} A r^{-1}) = \frac{A(N-2)}{|x|^2}$$  \hfill (27)

is non-negative, and it is in $L^1(\Omega)$. Then, we have satisfied the existence theory of Theorem 4. Due to Proposition 9 and (27) the estimate follows.  \hfill \Box

## 6 Convection with singularity on the boundary

The aim of this section is to understand the case where $E$ is regular inside $\Omega$ but blows up towards $\partial \Omega$. For the sake of simplicity we present an example, which as mentioned in Section 7 can be generalized, but the computations become quite technical. Let us consider $\varphi_1$ the first eigenfunction of $-\Delta$ with Dirichlet boundary conditions, i.e.,

$$\begin{cases}
-\Delta \varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega,

\varphi_1 = 0 & \text{on } \partial \Omega.
\end{cases}$$

We normalize it so that $\|\nabla \varphi_1\|_{L^\infty} = 1$. It is known that there exists $C > 0$ such that

$$0 < C \text{dist}(x, \partial \Omega) \leq \varphi_1(x) \leq C^{-1} \text{dist}(x, \partial \Omega), \quad \forall x \in \Omega,$$

and near $\partial \Omega$ we have that

$$|\nabla \varphi_1(x)| \geq C > 0.$$  \hfill (28)

We focus our efforts on the particular case

$E = -\varphi_1^{-1-\gamma} \nabla \varphi_1,$

for some $\gamma > 0$,

and $f \in L^\infty(\Omega)$, the space of bounded functions with compact support in $\Omega$. The aim of this section is to prove
Theorem 17. Let $E$ be given by (28), $M = I$ and $f \in L^\infty(\Omega)$. Then, there exists a unique $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that $uE \in L^\infty(\Omega)$ and $u$ is a weak solution in the sense that (13) holds. Furthermore, $u$ is flat on the boundary in the sense that
\[ |u(x)| \leq C_\alpha \text{dist}(x, \partial\Omega)^\alpha, \quad \text{for a.e. } x \in \Omega. \] (29)

We will give the proof below. First, we prove positivity in the interior.

Proposition 18. In the assumptions of Theorem 17 if $f \geq 0$ and $\int_\Omega f > 0$, then $u > 0$ in $\Omega$.

Proof. Let $\Omega_\eta = \{ x \in \Omega : d(x, \Omega) > \eta \}$. Consider $u_\eta$ the solution of (1) with $E$ given by (28) and $u_\eta = 0$ in $\partial\Omega_\eta$. Notice that $E$ is smooth on $\Omega_\eta$ for $\eta > 0$. Since we already know from Theorem 11 that $u \geq 0$ in $\Omega$, the classical comparison principle in $\Omega_\eta$ ensures that $u_\eta \leq u$ for any $\eta > 0$. Take $\eta > 0$ small enough so that $\int_{\Omega_\eta} f > 0$. Then, by the “classical” strong maximum principle we get $u_\eta > 0$ in $\Omega_\eta$, and the proof is complete.

It is immediate to compute that
\[ \text{div } E = (1 + \gamma)\varphi_1^{-2-\gamma}|\nabla \varphi_1|^2 - \varphi_1^{-1-\gamma}\Delta \varphi_1 = (1 + \gamma)\varphi_1^{-2-\gamma}|\nabla \varphi_1|^2 + \lambda_1 \varphi_1^{-\gamma}. \]
Hence div $E(x) \geq c \text{dist}(x, \partial\Omega)^{-2-\gamma}$ near the boundary. Notice that $E$ and div $E$ are not in $L^1(\Omega)$. We start the proof with a lemma.

Lemma 19. In the assumptions of Theorem 14, assume furthermore that div $E \in L^1_{\text{loc}}(\Omega)$. Then $u \text{div } E \in L^1(\Omega)$, still satisfying estimate (18).

Proof. We consider the approximating sequence for Theorem 14. For the approximation we know that $\int_{\Omega} |u_n| \text{div } E_n \leq \int_{\Omega} |f|$.

Let us fix $K \subset \Omega$. We have that $\int_{K} |u_n| \text{div } E_n \leq \int_{\Omega} |f|$.

Since we know that $\text{div } E_n \to \text{div } E$ in $L^1(K)$, we have that, up to a further subsequence, $\text{div } E_n$ converges a.e. in $K$. Hence, applying Fatou’s lemma
\[ \int_{K} |u| \text{div } E \leq \int_{\Omega} |f|. \]
Since this estimate is uniform in $K$, we can take $K_h = \{ x \in \Omega : \text{dist}(x, \partial\Omega) \geq h \}$ and deduce, as $h \to 0$, that (18) holds.

The solution found in Theorem 17 is unique in a certain class. We provide a uniqueness result extending Theorem 11, which can itself be generalised to a larger framework.

Lemma 20. Assume that $u \in H^1_0(\Omega)$, $E \in L^\infty_{\text{loc}}(\Omega)$, $u|E| \in L^2(\Omega)$, div $E \geq 0$ distributionally, and $f \in L^{\frac{2}{\alpha+2}}(\Omega)$. Then
\[ \| \nabla u^+ \|_2 \leq \frac{1}{\alpha S_2} \| f^+ \|^{\frac{\alpha}{\alpha+2}}. \]
In particular, there is at most one weak solution in $H^1_0(\Omega)$ of the (1).
Proof. We want to repeat the argument in Theorem 11, i.e., taking \( v = u \) in the weak formulation and using that
\[
- \int_{\Omega} u E \cdot \nabla u_+ \geq 0.
\]

We prove this formula by approximation. Take \( \eta \in C_{c}^{\infty}(\Omega) \). There exists \( K \Subset \Omega \) and \( \phi_m \in C_{0}^{\infty}(K) \) such that
\[
\phi_m \rightarrow u_+ \eta \text{ in } H_{0}^{1}(\Omega).
\]
We have that
\[
- \int_{\Omega} \phi_m E \cdot \nabla \phi_m = \left\langle \text{div } E, \frac{\phi_m^2}{2} \right\rangle \geq 0.
\]

Since \( E \in L^{\infty}(K) \), we pass to the limit to deduce
\[
- \int_{\Omega} (u_+ \eta) E \cdot \nabla (u_+ \eta) \geq 0.
\]

Now we expand
\[
\int_{\Omega} (u_+ \eta) E \cdot \nabla (u_+ \eta) = \int_{\Omega} u_+^2 \eta E \cdot \nabla \eta + \int_{\Omega} u_+ \eta^2 E \cdot \nabla u_+
\]

Now we take \( \eta_m \not\rightarrow 1 \). In particular \( \eta_m(x) = \eta_0(m \varphi_1(x)) \) where \( \eta_0 \) is non-decreasing, \( \eta_0(s) = 0 \) if \( s \leq 1 \) and \( \eta_0(s) = 1 \) if \( s > 2 \). Clearly \( \| \nabla \eta_m \|_{L^{\infty}} \leq Cm \). Since \( u_+ \in H_{0}^{1}(\Omega) \), then \( u_+(x)/\varphi_1(x) \in L^{2}(\Omega) \) by Hardy's inequality. And we compute
\[
\left| \int_{\Omega} u_+^2 \eta_m \cdot E \nabla \eta_m \right| \leq \int_{\varphi_1(x) \leq \frac{1}{m}} \frac{u_+}{\varphi_1} \| u E \| Cm \leq C \int_{\varphi_1(x) \leq \frac{1}{m}} \frac{u_+}{\varphi_1} \| u E \| \rightarrow 0
\]
since \( \frac{u_+}{\varphi_1} |u E| \in L^{1}(\Omega) \) and the size of the domain tends to zero. We conclude, by Dominated Convergence that
\[
0 \geq \int_{\Omega} (u_+ \eta_m) \cdot E \nabla (u_+ \eta_m) \rightarrow \int_{\Omega} u_+ E \cdot \nabla u_+ = \int_{\Omega} u E \cdot u_+.
\]

We are finally ready to prove the result.

**Proof of Theorem 17.** The uniqueness claim is proven in Lemma 20. We now prove the existence and bounds by approximation. We can assume, without loss of generality, that \( f \geq 0 \), and construct approximations of \( E \) given by
\[
E_{\ell} = -(\varphi_1 + \frac{1}{\ell})^{-1-\gamma} \nabla \varphi_1.
\]

Clearly \( E_{\ell} \in L^{\infty}(\Omega) \). These satisfy the assumptions of Theorem 4. Hence, there exists a weak solution \( u_{\ell} \in H_{0}^{1}(\Omega) \) of (1) where \( E = E_{\ell} \). We compute
\[
\text{div } E_{\ell} = (1 + \gamma)(\varphi_1 + \frac{1}{\ell})^{-2-\gamma} |\nabla \varphi_1|^2 + \lambda_1 (\varphi_1 + \frac{1}{\ell})^{-1-\gamma} \varphi_1.
\]

This is non-negative. Hence, due to Theorem 11 we have that
\[
\| \nabla u_{\ell} \|_{L^{2}} \leq C \| f \|_{L^{\infty}}.
\]

Splitting the behaviour near the boundary and away from the boundary, it is easy to see that \( \text{div } E_{\ell} \geq c_0 > 0 \) uniformly. Therefore, due to Proposition 9 we have that
\[
\| u_{\ell} \|_{L^{\infty}} \leq \frac{\| f \|_{L^{\infty}}}{c_0}.
\]

(30)
Now we must construct barrier functions. Select a single $\alpha > 1$ and the barrier

$$U = \frac{1}{\alpha}(\varphi_1 + \frac{1}{\ell})^\alpha.$$  

We drop the dependence on $\ell$ and $\alpha$ to make the presentation below more readable. Plugging it into the equation we get

$$-\Delta U + \text{div}(UE_\ell) = -\Delta U + \nabla U \cdot E_\ell + U \text{div} E_\ell$$

$$= -(\alpha - 1)(\varphi_1 + \frac{1}{\ell})^{\alpha - 2}\nabla \varphi_1^2 + \lambda_1(\varphi_1 + \frac{1}{\ell})^{\alpha - 1}\varphi_1$$

$$+ (1 + \gamma)(\varphi_1 + \frac{1}{\ell})^{\alpha - 2}\nabla \varphi_1^2 + \lambda_1(\varphi_1 + \frac{1}{\ell})^{\alpha - 1}\varphi_1$$

$$\geq \left(\gamma(\varphi_1 + \frac{1}{\ell})^{-\gamma} - (\alpha - 1)\right)(\varphi_1 + \frac{1}{\ell})^{\alpha - 2}\nabla \varphi_1^2.$$  

This is non-negative if $\varphi_1(x) + \frac{1}{\ell} \leq (\frac{\alpha - 1}{\gamma})^{-\frac{1}{\alpha}}$. There exists $\eta_\alpha > 0$ small enough such that

$$f(x) = 0 \quad \text{and} \quad \varphi_1(x) \leq \frac{1}{\ell}(\frac{\alpha - 1}{\gamma})^{-\frac{1}{\alpha}}, \quad \forall x \text{ such that } \text{dist}(x, \partial\Omega) \leq \eta_\alpha.$$  

We will use the neighbourhood of the boundary $A_\alpha = \{ x \in \Omega : \text{dist}(x, \partial\Omega) < \eta_\alpha \}$. Also, we consider the candidate super-solution

$$\overline{\pi}(x) = U(x) \left(\frac{\alpha}{\min_{\text{dist}(x, \partial\Omega) = \eta_\alpha} \varphi_1(x)} + \frac{\alpha}{c_0 \min_{\text{dist}(x, \partial\Omega) \geq \eta_\alpha} \varphi_1(x)} \|f\|_{L^\infty}\right).$$

We denote the constant on the right-hand side as $C_\alpha$. Using the first term of $C_\alpha$, $\overline{\pi} \geq u$ when $\text{dist}(x, \partial\Omega) = \eta_\alpha$. Also, $\overline{\pi} = \frac{1}{\ell} \geq 0 = u$ on $\partial\Omega$. Let us call

$$\overline{f} = -\Delta \overline{\pi} + \text{div}(\overline{\pi}E_\ell).$$

By the previous computations, if $\ell \geq 2(\frac{\alpha - 1}{\gamma})^{\frac{1}{\alpha}}$, we have $\overline{f} \geq 0 = f$ in $A_\alpha$, and clearly $\overline{f} \in L^\infty(A_\alpha)$. Hence, due to Theorem 11 we have that

$$0 \leq u_\ell(x) \leq \overline{\pi}(x), \quad x \in A_\alpha.$$  

Also, due to (30) and the second part of $C_\alpha$, we have that

$$0 \leq u_\ell(x) \leq \overline{\pi}(x), \quad x \in \Omega \setminus A_\alpha.$$  

Eventually, we deduce that for any $\alpha > 1$ we that

$$0 \leq u_\ell(x) \leq \frac{C_\alpha}{\alpha}(\varphi_1 + \frac{1}{\ell})^\alpha, \quad \forall x \in \Omega \text{ and } \ell \geq 2(\frac{\alpha - 1}{\gamma})^{\frac{1}{\alpha}}.$$  

In particular, picking $\alpha = \gamma + 1$ we deduce that

$$|u_\ell E_\ell| \leq \frac{C_{\gamma + 1}}{\gamma + 1}\|\nabla \varphi_1\|_{L^\infty} = \frac{C_{\gamma + 1}}{\gamma + 1}.$$  

We deduce that, up to a subsequence,

$$u_\ell \to u \text{ a.e. and strongly in } L^2 \quad \text{ and } \quad u_\ell \rightharpoonup u \text{ weakly in } H^1_0(\Omega).$$

This implies that $u_\ell E_\ell \to u E$ a.e. And hence $u E$ is bounded. Passing to the limit in the weak formulation by the Dominated Convergence Theorem, the result is proven.  

$$\square$$
Remark 21. Notice that the construction of the super-solution above can be done in any dimension $N \geq 1$. However, most of the results in the rest of the paper are only available for $N \geq 3$.

Remark 22. For Schrödinger-type equations $-\Delta u + Vu = f$, it is known that if the potential $V$ is greater than $\text{dist}(x, \partial \Omega)^{-2}$ and $f$ is compactly supported, then $u$ is flat on the boundary, in the sense that $|u| \leq C \text{dist}(x, \partial \Omega)^{1+\varepsilon}$. This means that $\partial_n u = 0$ on $\partial \Omega$. This means that it satisfies Dirichlet and Neumann homogeneous boundary conditions. And it can be extended by 0 outside $\Omega$ with higher regularity than $H^1$. In contrast, the exponent $\gamma$ in the above result cannot be taken as $\gamma = 0$ in order to get flat solutions. Indeed, the convection term $E \cdot \nabla \varphi_1$, in the above computations, is more singular than the term $\varphi_1 \text{div} E$. A very explicit example can be done in one dimension: if we consider $E = -Cx^{-1}$ then this drift does not generate flat solutions since if we take $U(x) = x^\alpha$ then

$$-U'' + (EU)' = (-\alpha x^{\alpha-1} - Cx^{\alpha-1})' = -(\alpha + C)(\alpha - 1)x^{\alpha-2},$$

and this is a supersolution only if $\alpha \leq 1$.

Corollary 23. In the hypothesis of Theorem 17 replace $f \in L^\infty_c(\Omega)$ by $|f(x)| \leq C \text{dist}(x, \partial \Omega)^{\omega}$ for $0 \leq \omega \leq \gamma + 1$.

Then

$$|u(x)| \leq \text{dist}(x, \partial \Omega)^{\alpha} \quad \text{for all } \alpha \in (1, \gamma + 2 - \omega).$$

Proof. We maintain the notation of the proof of Theorem 17. We have already shown that, on a neighbourhood of the boundary,

$$-\Delta U + \text{div}(UE_m) \geq \frac{\gamma}{2}(\varphi_1 + \frac{1}{m})^{\alpha - 2 - \gamma}|\nabla \varphi_1|^2 \geq c_1 (\varphi_1 + \frac{1}{m})^{\alpha - 2 - \gamma} \geq c_2 |f|.$$

For $\alpha$ in the range $(1, \gamma + 2 - \omega)$, we can take as a supersolutions for the approximating sequence

$$\varpi(x) = U(x) \left( \frac{1}{c_2} + \frac{\alpha}{c_0 \min_{\text{dist}(x, \partial \Omega) = \eta_0} \varphi_1(x)^{\alpha}} + \frac{\alpha}{c_0 \min_{\text{dist}(x, \partial \Omega) \geq \eta_0} \varphi_1(x)^{\alpha}} \|f\|_{L^\infty} \right).$$

And the rest of the proof remains as in Theorem 17.

7 Further remarks, extensions, and open problems

1. We point that the proofs of our estimates can be extended to many non-linear settings.

2. Theorem 17 admits many generalisations. For instance, one can consider the case $|E| \leq c_0 \text{dist}(x, \partial \Omega)^{-\gamma - 1}$ with $\text{div} E \geq c_1 \text{dist}(x, \partial \Omega)^{-\gamma - 2}$ up to suitable conditions on the constants. Also, the techniques in this paper could be extended to the situation where $\text{dist}(x, \partial \Omega)$ is replaced by $\text{dist}(x, \Gamma)$ with a suitable part $\Gamma \subset \partial \Omega$. The case $\Gamma$ an interior manifold can also be studied.

3. Including a non-negative potential. The same analysis can be performed on the equation

$$-\text{div}(M(x) \nabla u) + a(x)u = -\text{div}(uE(x)) + f(x)$$
when $a \geq 0$. As above, our approach allows for less regularity in $a$ than most previous literature, e.g. $a \in L^1_{\text{loc}}(\Omega)$. Furthermore, one will then obtain

$$\int_{\Omega} |u|(a + \text{div } E) \leq \int_{\Omega} |f|.$$ 

Hence, one can reduce the hypothesis to $a + \text{div } E \geq 0$ in the whole analysis.

4. The study of $a \equiv 1$ is useful in the study of the evolution problem

$$u_t - \text{div}(M(x)\nabla u) + \text{div}(uE(x)) = 0.$$ 

For the study of this problem one can write $u_t + Au = 0$ where

$$Au = - \text{div}(M(x)\nabla u) + \text{div}(uE(x)).$$ 

In order to obtain solutions in semigroup form in $L^p$ (where $1 \leq p \leq +\infty$), following the theory of accretive operators, it is sufficient that,

$$\|u\|_{L^p} \leq \|u + \lambda Au\|_{L^p}.$$ 

Letting $f = u + \lambda Au$, this is precisely what we have proven above, where $M = \lambda I$ and $a \equiv 1$. See also [6].

5. We point out that when $|E| \leq |A|/|x|$, we have that, if $m > \frac{2N}{N+2}$ then $u|E| \in L^2(\Omega)$. It seems possible to extend the uniqueness result (20) to this setting.

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