DESINGULARIZATION OF
TORIC AND BINOMIAL VARIETIES

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Abstract. We give a combinatorial algorithm for equivariant embedded resolution of singularities of a toric variety defined over a perfect field. The algorithm is realized by a finite succession of blowings-up with smooth invariant centres that satisfy the normal flatness condition of Hironaka. The results extend to more general varieties defined locally by binomial equations.

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1. Introduction

We give a simple combinatorial algorithm for equivariant embedded resolution of singularities of a toric variety $X$ (not necessarily normal) defined over a perfect field $k$. The algorithm is realized by a finite succession of blowings-up with smooth invariant centres, such that each
successive transform of $X$ is normally flat along the corresponding centre (condition of Hironaka [Hi1]). We announced this article in [BM5] as “Desingularization algorithms II. Toric and locally binomial varieties”, but have changed the title because it is largely independent of [BM5].

Throughout this paper, $k$ denotes a field. If $X$ is a toric variety over $k$, we denote by $T_X$ the embedded algebraic torus. We say that a morphism of toric varieties $f : X \to Y$ is equivariant if $f|T_X$ is a homomorphism of tori $h : T_X \to T_Y$ and $f$ is equivariant with respect to the homomorphism $h$. (Some basic facts about toric varieties are recalled in Section 2 below.)

**Theorem 1.1** (Equivariant embedded desingularization of a toric variety). Let $X \hookrightarrow M$ denote an equivariant embedding of toric varieties over a perfect field $k$, where $M$ is smooth. Then there is a finite sequence of blowings-up of $M$,

$$M = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\cdots} \pi_{t+1} M_{t+1},$$

such that:

1. The centre $D_j$ of each blowing-up $\pi_{j+1}$ is a smooth $T_{M_j}$-invariant subvariety of $M_j$.
2. Set $X_0 = X$. For each $j = 0, \ldots, t$, let $X_{j+1}$ denote the strict transform of $X_j$ by $\pi_{j+1}$. Then each $C_j := D_j \cap X_j$ is a smooth $T_{X_j}$-invariant subvariety of $X_j$, and $X_j$ is normally flat along $C_j$.
3. For each $j = 0, \ldots, t$, let $E_{j+1}$ denote the exceptional divisor of $\pi_1 \circ \cdots \circ \pi_{j+1}$. Then, for each $j = 0, \ldots, t$, either $C_j \subset \text{Sing} X_j$, or $X_j$ is smooth and $C_j \subset X_j \cap E_j$.
4. $X_{t+1}$ is smooth, and $X_{t+1}, E_{t+1}$ simultaneously have only normal crossings.

The condition $C_j = D_j \cap X_j$ (intersection as subspaces, or subschemes) implies that $X_{j+1} = \text{blowing-up of } X_j$ with centre $C_j$ [Ha, §II.7]. Normal flatness of $X_j$ along $C_j$ means the normal cone of $C_j$ in $X_j$ is flat over $C_j$; this is equivalent to the property that the Hilbert-Samuel function $H_{X_j,x} : \mathbb{N} \to \mathbb{N}$,

$$H_{X_j,x}(l) := \text{length } \frac{O_{X_j,x}}{m_{X_j,x}^{l+1}},$$

is locally constant as a function of $x \in C_j$ [Be]. ($m_{X_j,x}$ denotes the maximal ideal of $O_{X_j,x}$.)
A sequence of blowings-up as in the theorem has the property that, for each \( j \), \( T_{M_j} = T_M \), \( T_{X_j} = T_X \), and the embedding \( X_j \hookrightarrow M_j \) is equivariant.

For each \( j \), every irreducible component of \( C_j \) is a smooth orbit closure of \( X_j \). We can assume that each \( D_j \) is the union (necessarily disjoint) of the smallest orbit closures of \( M_j \) containing the components of \( C_j \). Although Theorem 1.1 could be restated with each \( C_j, D_j \) taken to be orbit closures, we need to allow disconnected centres to have also the following.

**Addendum 1.2** (Canonical equivariant embedded desingularization). For every equivariant embedding \( X \hookrightarrow M \) of toric varieties over a perfect field \( k \), where \( M \) is smooth, there is sequence of blowings-up (1.1) satisfying the conditions of Theorem 1.1, with the following property: If \( \iota : M' \to M \) is an open equivariant embedding, then \( X \) and \( X' = \iota^{-1}(X) \) have the same resolution towers over \( M' \) (not counting isomorphisms in the sequences of blowings-up).

We prove Theorem 1.1 in Section 8 below. Desingularization can be realized by a simple combinatorial algorithm that is canonical up to an ordering of the codimension 1 orbit closures of \( M \) (§9.1). (Every orbit closure of a smooth toric variety is an intersection of codimension 1 orbit closures.) The canonical desingularization algorithm of Addendum 1.2 comes at an extra cost: Additional blowings-up are needed to replace the codimension 1 orbit closures involved by components of exceptional divisors (which are ordered by the sequence of blowings-up). (See §9.2.)

At a comparable cost, we can give a canonical desingularization algorithm for more general toroidal or binomial varieties defined over \( k \). For these classes, the locally toric or binomial structures are related by global divisors (Section 10).

The general desingularization algorithm of [BM3, BM5] can be adapted to varieties over \( k \) that are merely locally toric or locally binomial, but again has additional complexity due to blowings-up needed to guarantee that combinatorial centres chosen locally as in the toric case extend to global smooth centres. The general algorithm, however, strengthens condition (3) in Theorem 1.1: \( C_j \) and \( E_j \) will simultaneously have only normal crossings, for all \( j \). (The simpler toric or binomial algorithms provide only simultaneous normal crossings of \( C_j \) and \( N \cap E_j \), where \( N \) is a local minimal embedding variety of \( X_j \).)
Example 1.3 (cf. [BM5, Example 1.2]). Consider the toric hypersurface $X$ in $k^3$ defined by
\[ z^d - x^{d-1}y^d = 0. \]
Let $n(T)$, $n(CT)$, $n(CB)$ and $n(BM)$ denote the number of blowings-up needed to reduce the maximum order $d$ of $X$, using our toric desingularization algorithm, canonical toric algorithm, canonical binomial algorithm, and the general desingularization algorithm of [BM3, BM5], respectively. Then
\[ n(T) = 1 \]
\[ n(CT), n(CB) \leq d + j \]
\[ n(BM) \leq 2d + k, \]
where $j$ and $k$ are independent of $d$.

The generalizations of our results on toric varieties are presented briefly in Section 10, but Section 6 and §§7.1-7.3 already cover the more general binomial ideals, to prepare the ground.

An application of our results to desingularization of the induced metric and of the Gauss mapping of an embedded binomial variety will be presented in a forthcoming paper.

Our toric desingularization algorithm can be viewed as the combinatorial part of the general algorithm of [BM3, BM5]. From the point of view of resolution of singularities over fields of arbitrary characteristic, the classes of toric or locally binomial varieties present a great simplification because of the trivial existence of smooth subvarieties of maximal contact; cf. Example 2.5 below. Our techniques are of a “differential calculus” nature. The hypothesis that $k$ is perfect is essentially equivalent to the possibility of using calculus in local coordinates. (See Section 3.) Of course, every algebraically closed field is perfect.

Combinatorial desingularization of normal toric varieties goes back to [KKMS]. A normal toric variety $X$ corresponds to a fan $\Sigma$ in a lattice; $X = X(\Sigma)$ is smooth if and only if $\Sigma$ is a regular fan (cf. §2.2 below). Any fan can be refined to a regular fan by iterated star-subdivisions [F, §2.6], [C, §5]; the corresponding morphism of toric varieties is a resolution of singularities (i.e., a proper morphism from a smooth variety to $X$ that is an isomorphism over the complement of the singular locus). Each star-subdivision corresponds to a normalized blowing-up (with centre not necessarily smooth). In the case of an equivariant embedding $X \hookrightarrow M$, this provides, in general, neither equivariant embedded desingularization, nor is the morphism given by (or evidently dominated by) a sequence of blowings-up with centers satisfying
the conditions of our theorem. (See [T, §6.2].) De Concini and Procesi have shown that if \( M' \) is a smooth toric variety birationally equivalent to \( M \), then there is an equivariant birational morphism \( M'' \to M' \), where \( M'' \) is obtained from \( M \) by a finite sequence of blowings-up with centres that are codimension 2 orbit closures [DP].

Theorem 1.1 (and Addendum 1.2) can be simplified in the case that \( X \hookrightarrow M \) is a toric hypersurface in \( M \) (i.e., \( X \) is defined by a principle ideal \( \mathcal{I}_X \) in \( \mathcal{O}_M \)). Moreover, in this case, we do not need to assume that \( k \) is perfect.

**Theorem 1.4** (Equivariant embedded desingularization of a toric hypersurface). Let \( X \hookrightarrow M \) denote an equivariant embedding of toric varieties over a field \( k \), where \( M \) is smooth. Assume that \( X \) is a hypersurface in \( M \). Then Theorem 1.1 (and Addendum 1.2) can be strengthened so that each \( \pi_{j+1} \) is a blowing-up with \( T_M \)-invariant centre \( C_j = D_j \) in \( X_j \).

In the case of a hypersurface, the condition that \( X_j \) is normally flat along \( C_j \) is equivalent to the condition that the order of vanishing of \( \mathcal{I}_{X_j} \) is locally constant on \( C_j \).

A smooth toric variety \( M \) corresponds to a regular fan \( \Sigma \) in a lattice \( L \cong \mathbb{Z}^n \). A blowing-up of \( M \) with smooth \( T_M \)-invariant centre corresponds to a certain star-subdivision of \( \Sigma \). (See Section 4.) A toric hypersurface \( X \hookrightarrow M = M(\Sigma) \) is determined by (the restriction to \( \Sigma \) of) an integer-valued linear function \( \lambda \) on \( M \). (See Lemma 2.6.) Theorem 1.4 translates into a purely combinatorial statement about linear extension of \( \lambda \) to successive star-subdivisions of \( \Sigma \) (Theorem 5.3.)

Our proof of Theorem 1.4 shows that (if \( X_j \) is not already smooth) the locus of maximum order of \( \mathcal{I}_{X_j} \) has only normal crossings and, moreover, each of its irreducible components is an orbit-closure of \( M \). We can resolve the singularities by choosing any of these components as the centre \( C_j \) of the next blowing-up. (Compare with Theorem 1.6.)

A general toric subvariety \( X \subset M \) cannot be desingularized by blowings-up with \( T_M \)-invariant centres that lie in the successive strict transforms of \( X \), as in the hypersurface case. (We say that a toric variety \( X \subset M \) is a toric subvariety of \( M \) if the inclusion \( X \hookrightarrow M \) is an equivariant embedding of toric varieties.)

**Example 1.5.** Let \( X \subset \mathbb{K}^6 \) denote the affine toric variety whose ideal is generated by the binomials

\[
f = w^2 - uv, \quad g = z - xy
\]

(in six variables \( u, v, w, x, y, z \)). Let \( N \subset \mathbb{K}^6 \) denote the smooth subvariety \( \{ g = 0 \} \). Then \( X \hookrightarrow N \) is a minimal embedding; \( N \) is a
toric subvariety of $M$. Desingularization of $X$ is the result of a single blowing-up with centre $C$ defined by

$$u = v = w = 0, \quad z - xy = 0.$$ 

The centre $C$ is $T_X$- or $T_N$-invariant, but is not $(\mathbb{k}^*)^6$-invariant. (It is not a coordinate subspace of $\mathbb{k}^6$.) Let $D = \{u = v = w = 0\} \subset \mathbb{k}^6$. Then $D$ is the smallest orbit closure of $\mathbb{k}^6$ containing $C$; $D$ and $N$ have only normal crossings, and $D \cap N = C$.

An affine toric subvariety of $\mathbb{k}^n$ corresponds to a prime ideal in $\mathbb{k}[x_1, \ldots, x_n]$ generated by binomials (a toric ideal; see Section 2 below). A toric ideal has a distinguished set of binomial generators (that we call a standard basis), uniquely determined by an ordering of the variables (Theorem 6.2). A standard basis is similar to the notion of Gröbner basis, with the difference that the initial monomial of an element of a standard basis is a monomial of lowest degree, rather than of highest degree as in a Gröbner basis.

Standard bases play a key role in resolution of singularities because, in the general case, we can work with the elements of a standard basis in the same way as with a single defining equation in the hypersurface case (or with an arbitrary system of generators in “principalization of an ideal”). The properties needed are made precise in Theorems 7.1, 7.3: The locus of maximum order of the elements of a standard basis coincides with the maximal locus of the Hilbert-Samuel function (the “maximal Samuel stratum”). Moreover, if the Hilbert-Samuel function does not decrease on blowing-up with centre in the Samuel stratum, then the standard basis transforms to the standard basis of the ideal of the strict transform. The following theorem is a corollary of the first assertion (proof in §7.4).

**Theorem 1.6.** Let $X$ denote a toric variety over a perfect field $\mathbb{k}$. Then the maximal Samuel stratum $S$ of $X$ has only normal crossings. Moreover, each irreducible component of $S$ is a closed smooth $T_X$-invariant subspace of $X$.

Unlike the situation in Theorem 1.4 on the hypersurface case, however, it is not, in general, possible to resolve singularities by choosing as each successive centre of blowing-up an arbitrary component of the maximal Samuel stratum (Example 7.6).

We do not know whether one can get an algorithm for equivariant embedded desingularization of a toric variety by blowing up with centre given at each step by some component of the maximal Samuel stratum. (We doubt that this is true.) Our desingularization algorithm involves blowing up with perhaps smaller centres that serve to order
monomials appearing in a standard basis, in preparation for blowing up components of the Samuel stratum as in the hypersurface case.

2. Toric varieties

Let $k$ denote a field. A *toric variety (over $k$)* is an algebraic variety $X$ over $k$ that contains an algebraic torus $T = T_X$ as an open dense subset, and has an action $T \times X \to X$ of $T$ that extends the natural action of $T$ on itself. *We do not assume that $X$ is necessarily normal.* (In general, an *algebraic variety over $k$* means a reduced scheme of finite type over $k$.)

2.1. Affine toric varieties. An affine toric variety is simply an affine variety that is parametrized by a set of Laurent monomials. Consider a subset $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^d$. Each vector $a_i = (a_{i1}, \ldots, a_{id})$ identifies with a monomial $t^{a_i} = t_1^{a_{i1}} \cdots t_d^{a_{id}}$ in the Laurent polynomial ring $k[t^\pm] = k[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}]$. We define the *toric ideal* $I_A \subset k[x] = k[x_1, \ldots, x_n]$ as the kernel of the algebra homomorphism $k[x] \to k[t^\pm], \ x_i \mapsto t^{a_i}$.

(See [S2] Chs. 4, 13.)

Consider the group homomorphism

$$\pi : \mathbb{Z}^n \to \mathbb{Z}^d,$$

$$\gamma = (\gamma_1, \ldots, \gamma_n) \mapsto \gamma_1 a_1 + \cdots + \gamma_n a_n.$$

Any $\gamma \in \mathbb{Z}^n$ can be written uniquely as $\gamma = \gamma^+ - \gamma^-$, where $\gamma^+$ and $\gamma^- \in \mathbb{Z}^n$ are nonnegative and have disjoint supports. See [S2] for the following three lemmas.

**Lemma 2.1.** The toric ideal $I_A$ is spanned (as a $k$-vector space) by the binomials

$$x^{\gamma^+} - x^{\gamma^-}, \ \gamma \in \ker \pi .$$

(A *binomial* means a difference of two monomials.)

**Lemma 2.2.** An ideal in $k[x]$ is toric if and only if it is prime and generated by binomials.

Let $X_A \subset \mathbb{A}^n$ denote the affine variety $V(I_A)$ defined by the toric ideal $I_A$ (where $\mathbb{A}^n$ denotes $n$-dimensional affine space over $k$). A variety of the form $X_A$ is an affine toric variety. (See below.)

Let $\text{rk} \mathcal{A}$ denote the rank of the $d \times n$ matrix with columns $a_1, \ldots, a_n$ (so that $\text{rk} \mathcal{A}$ is the dimension of the lattice $\mathbb{Z} \mathcal{A} \subset \mathbb{Z}^d$ spanned by $a_1, \ldots, a_n$).

**Lemma 2.3.** The Krull dimension of $k[x]/I_A$ equals $\text{rk} \mathcal{A}$. 

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The ring \( \mathbb{k}[x]/I_A \) is isomorphic to the subring \( \mathbb{k}[t^{a_1}, \ldots, t^{a_n}] \) of \( \mathbb{k}[t^\pm] \). The Krull dimension of the latter is the maximum number of algebraically independent monomials \( t^{a_i} \). But a set of monomials \( t^{a_i} \) is algebraically independent if and only if the exponent vectors \( a_i \) are linearly independent (by Lemma 2.1).

**Example 2.4.** Let \( T \) denote the algebraic group whose \( \mathbb{k} \)-rational points are the points of the multiplicative group \( \mathbb{k}^* \) of non-zero elements of \( \mathbb{k} \). The algebraic torus \( T^n \) has the structure of an affine subvariety \( V(I) \) of \( \mathbb{A}^{2n} \) determined by the ideal \( I \subset \mathbb{k}[y, z] = \mathbb{k}[y_1, \ldots, y_n, z_1, \ldots, z_n] \) generated by \( y_j z_j - 1, j = 1, \ldots, n \). \( \mathbb{k}[y, z]/I \cong \mathbb{k}[y^\pm] \).

An algebraic torus is an algebraic group isomorphic to \( T^n \), for some \( n \geq 1 \).

Consider the action of the torus \( T^d \) on \( \mathbb{A}^n \) (or \( (\mathbb{k}^*)^d \) on \( \mathbb{k}^n \)) given by

\[
(t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n) \mapsto (t^{a_1} x_1, \ldots, t^{a_n} x_n).
\]

Clearly, \( X_A \) is the closure of the orbit of the point \( (1, \ldots, 1) \). Of course, \( T^n \) acts trivially on \( \mathbb{A}^n \), by

\[
(s, x) = (s_1, \ldots, s_n, x_1, \ldots, x_n) \mapsto (s_1 x_1, \ldots, s_n x_n),
\]

and the embedding \( \iota : X_A \hookrightarrow \mathbb{A}^n \) is equivariant with respect to the homomorphism of tori

\[
\varphi : T^d \to T^n, \quad t \mapsto (t^{a_1}, \ldots, t^{a_n}).
\]

One can then see that \( X_A \) contains an algebraic torus \( T \) (of dimension \( = \text{rk} A \)) as a dense open subset, and that there is an action \( T \times X_A \to X_A \) extending the natural action of \( T \) on itself. (See [KKMS], [GKK] Ch. 5 for the converse statement: An affine variety satisfying the latter condition is of the form \( X_A \).) In fact, if \( X \subset \mathbb{A}^n \) is an affine variety defined by a toric ideal, then \( X \cap T^n \) is a subgroup of \( T^n \).

Let \( d = \text{dim} X \) and let \( A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d \), where the \( a_i \) are the columns of a matrix whose rows form a minimal set of generators of the sublattice of \( \mathbb{Z}^n \) orthogonal to all \( \gamma \in \mathbb{Z}^n \) such that the binomial \( x^{\gamma^+} - x^{\gamma^-} \) belongs to the ideal of \( X \). Then \( \varphi : T^d \to T^n \) (as above) induces an isomorphism onto \( X \cap T^n \).

We will sometimes write a binomial equation \( x^{\gamma^+} - x^{\gamma^-} = 0 \) as \( x^{\gamma} = 1 \) (in the ring of Laurent polynomials \( \mathbb{k}[x^\pm] \)); of course, if \( I \subset \mathbb{k}[x] \) is a toric ideal, then \( I \cdot \mathbb{k}[x^\pm] \) defines the torus \( T_X \).

We present the following as an exercise using the lemmas above.

**Example 2.5** (Nonexistence of maximal contact in positive characteristic \( \mathbb{F}_p \)). Let \( \mathbb{k} \) denote an algebraically closed field of characteristic 2.
Let $X \subset \mathbb{A}^4$ denote the hypersurface
\[ y^2 + x_1^3x_2 + x_2^3x_3 + x_1x_3^7 = 0. \]
Then: (1) The locus of points of order 2 (see Section 3) is the curve $C$ given by
\[ x_1 = t^{15}, \quad x_2 = t^{19}, \quad x_3 = t^7, \quad y = t^{32}, \]
but (2) $C$ lies in no smooth hypersurface. (Hints: (1) The locus of points of order 2 is defined by binomial equations. (2) $C$ satisfies no binomial equation of order 1.)

2.2. Embedded toric varieties. Let $\mathbb{Z}^n$ denote a lattice, and let $\mathbb{R}^n$ denote the real vector space spanned by $\mathbb{Z}^n$.

A fan $\Sigma$ in $\mathbb{Z}^n$ is defined as a (finite) set of cones (rational convex polyhedral cones), such that each face of a cone in $\Sigma$ is also a cone in $\Sigma$, and the intersection of two cones in $\Sigma$ is a face of each. (See [E], [O2].) Every normal toric variety is associated to a fan [KKMS]. Ch. I. A normal toric variety determines the fan; i.e., an equivariant isomorphism of normal toric varieties corresponds to an isomorphism of lattices taking one fan to the other [O1, Thm. 4.1]. A cone $\sigma$ in $\mathbb{R}^n$ corresponds to a normal affine toric variety $U_{\sigma}$. The toric variety associated to a fan $\Sigma$ in $\mathbb{R}^n$ is obtained by glueing together the affine toric varieties $U_{\sigma}$, $\sigma \in \Sigma$; if $\sigma, \tau \in \Sigma$, then $U_{\sigma} \cap U_{\tau} = U_{\sigma \cap \tau}$. Clearly, $X = \bigcup U_{\sigma}$, where the union is over the maximal cones $\sigma \in \Sigma$. ($\sigma$ is maximal if it is not properly contained in another cone of $\Sigma$.) We will need these general considerations only in the case of smooth toric varieties.

Let $\sigma$ denote a cone in $\mathbb{R}^n$. We define the vertices of $\sigma$ as the (unique) generators $e_\rho$ of the semigroups $\rho \cap \mathbb{Z}$, for every edge (1-dimensional face) $\rho$ of $\sigma$. (Then the vertices of $\sigma$ generate $\sigma$.) We will say that $\sigma$ is regular if its vertices $e_{p+1}, \ldots, e_n$ can be completed to a basis $e_1, \ldots, e_p, e_{p+1}, \ldots, e_n$ of the lattice $\mathbb{Z}^n$.

A regular cone $\sigma$ of dimension $n - p$ in $\mathbb{R}^n$ corresponds to a smooth affine toric variety $U_{\sigma} \cong \mathbb{P}^p \times \mathbb{A}^{n-p}$; $U_{\sigma}$ can be realized as the subvariety of $\mathbb{A}^{2p} \times \mathbb{A}^{n-p}$ (with affine coordinates $(u, x) = (u_1, \ldots, u_p, x_1, \ldots, x_p, x_{p+1}, \ldots, x_n)$) defined by the binomial equations
\[ u_ix_i = 1, \quad i = 1, \ldots, p. \]

A fan $\Sigma$ is regular if every cone $\sigma \in \Sigma$ is regular. Smooth toric varieties correspond to regular fans. Consider a lattice $\mathbb{Z}^n$. Let $\Sigma \subset \mathbb{R}^n$ denote a regular fan, and let $M = M(\Sigma)$ denote the smooth toric variety over $\mathbb{C}$ determined by $\Sigma$. Consider cones $\sigma, \tau \in \Sigma$, of dimensions $n - p$, $n - q$, respectively. Choose bases $e_1, \ldots, e_n$ and
$f_1, \ldots, f_n$ of $N$ such that $e_{p+1}, \ldots, e_n$ and $f_{q+1}, \ldots, f_n$ are the vertices of $\sigma$ and $\tau$ (respectively). We can write
\[
e_i = \sum_{j=1}^n a_{ij} f_j, \quad i = 1, \ldots, n,
\]
where $(a_{ij})$ is an integer matrix with determinant $\pm 1$.

Corresponding to the bases $\{e_i\}$ and $\{f_j\}$, we can realize $U_\sigma \hookrightarrow \mathbb{A}^{2p} \times \mathbb{A}^{n-p}$ with coordinates $(u, x)$ as above, and $U_\tau \hookrightarrow \mathbb{A}^{2q} \times \mathbb{A}^{n-q}$ with analogous coordinates $(v, y) = (v_1, \ldots, v_q, y_1, \ldots, y_n)$, in such a way that the transformation from $x$- to $y$-coordinates in the overlap $U_\sigma \cap U_\tau$ is given by the Laurent monomial mapping
\[
y_j = x^{a_j}, \quad \text{where } a_j = (a_{ij}, \ldots, a_{nj}).
\]

A binomial equation $x^\gamma = 1$, $\gamma \in \mathbb{Z}^n$, in the $x$-coordinates of $U_\sigma$ becomes $y^\delta = 1$, where $\gamma_i = \sum_{j=1}^n a_{ij} \delta_j$, $i = 1, \ldots, n$, in $U_\tau$. In other words, if $\lambda : N \to \mathbb{Z}$ is the $\mathbb{Z}$-linear function determined by $\lambda(e_i) = \gamma_i$, $i = 1, \ldots, n$, then $x^\gamma = 1$ in $U_\sigma$ becomes $y^\delta = 1$ in $U_\tau$, where $\delta_j = \lambda(f_j)$, $j = 1, \ldots, n$. This means:

**Lemma 2.6.** A toric hypersurface (not necessarily normal) in $M = M(\Sigma)$ corresponds to (the restriction to $\Sigma$ of) a linear function $\lambda : N \to \mathbb{Z}$.

In general, if $X$ is a toric subvariety of $M$, then there are finitely many linear functions $\lambda_k : N \to \mathbb{Z}$, such that each $U_\sigma$ admits a closed embedding in $\mathbb{A}^{2p} \times \mathbb{A}^{n-p}$, for some $p$, with coordinates $(u, x)$ as above, in such a way that $X \cap U_\sigma$ is defined by the toric ideal generated by $u_j x_j = 1$, $j = 1, \ldots, p$, and $x^{\gamma_k} = 1$, where $\gamma_{ki} = \lambda_k(e_i)$, $i = 1, \ldots, n$, for each $k$.

### 3. Equimultiple locus of a binomial

#### 3.1. Perfect fields and order of vanishing

Let $k$ denote a field. Let $m$ be a maximal ideal of $k[x] = k[x_1, \ldots, x_n]$. Let $f \in k[x]$. The order $\mu_m(f)$ of $f$ at $m$ is defined as the order of $f$ as an element of the local ring $R := k[x]_m$: i.e., the largest $l \in \mathbb{N}$ such that $f \in m_l R$, where $m_R$ denotes the maximal ideal $m \cdot R$ of $R$. (The maximal ideals $m$ of $k[x]$ are the (closed) points $a$ of $n$-dimensional affine space $\mathbb{A}^n = \mathbb{A}^n_k$ over $k$; we will write $\mu_m(f)$ or $\mu_a(f)$, indifferently.)

There is a second natural notion of order at $a$, corresponding to the definition above, after extension to the residue field $\mathbb{F} := R/m_R$. For each $i = 1, \ldots, n$, let $x_i(a) \in \mathbb{F}$ denote the image of $x_i \in k[x]$ in $\mathbb{F}$; and set $x(a) = (x_1(a), \ldots, x_n(a)) \in \mathbb{F}^n$. Let $\mu_{x(a)}(f)$ denote the order
at $x(a)$ of $f$ as an element of $\mathbb{F}[x]$; i.e., the largest $l \in \mathbb{N}$ such that $f \in (x - x(a))^l$, where $(x - x(a))$ denotes the maximal ideal of $\mathbb{F}[x]$ generated by $x_i - x_i(a)$, $i = 1, \ldots, n$.

We will show that the two notions of order above coincide for all maximal ideals $m$ of $\mathbb{k}[x]$ if and only if $\mathbb{k}$ is perfect.

Let $m$ be a maximal ideal of $\mathbb{k}[x] = \mathbb{k}[x_1, \ldots, x_n]$. We use the notation above. If $f(x) \in \mathbb{k}[x]$, let $\overline{f}(x) \in \mathbb{F}[x]$ denote the image of $f(x)$ by the injective homomorphism $\mathbb{k}[x] \to \mathbb{F}[x]$; then $f(x) \in m$ if and only if $\overline{f}(x) \in (x - x(a))$. Therefore, there is an induced homomorphism $\tau_m : R \to \mathbb{F}[x](x - x(a))$.

The completion of $\mathbb{F}[x](x - x(a))$ can be identified with the ring of formal power series $\mathbb{F}[X] = \mathbb{F}[X_1, \ldots, X_n]$, where the canonical injection $\mathbb{F}[x](x - x(a)) \hookrightarrow \mathbb{F}[X]$ is induced by $g(x) \mapsto g(x(a) + X)$, $g(x) \in \mathbb{F}[x]$. Let $T_m : \hat{R} \to \mathbb{F}[X]$ denote the homomorphism of completions induced by $\tau_m$. (Then $\mu_{x(a)}(f) = \mu_X(T_m f) \geq \mu_a(f).$) We will need the following theorem only beginning with Section 7.

**Theorem 3.1.** The homomorphism $T_m : \hat{R} \to \mathbb{F}[X]$ is an isomorphism, for all maximal ideals $m$ of $\mathbb{k}[x]$, if and only if $\mathbb{k}$ is perfect.

**Proof.** Let $m$ denote a maximal ideal of $\mathbb{k}[x]$. For all $k \in \mathbb{N}$, $\tau_m$ induces a homomorphism of finite-dimensional $\mathbb{F}$-algebras,

$$T_m^k : \frac{m_R^k}{m_R^{k+1}} \to \frac{(X)^k}{(X)^{k+1}}$$

(where $(X) = (X_1, \ldots, X_n)$ is the maximal ideal of $\mathbb{F}[X]$); $T_m$ is an isomorphism if and only if $T_m^k$ is an isomorphism for all $k \in \mathbb{N}$.

Suppose that $\mathbb{k}$ is perfect. For each $i = 1, \ldots, n$, let $P_i(t) \in \mathbb{k}[t]$ denote the minimal polynomial of $x_i(a)$. Then $P_i(t) = (t - x_i(a))Q_i(t)$, where $Q_i(t) \in \mathbb{F}[t]$, and $Q_i(0) = P_i(x_i(a)) \neq 0$ [ZS Ch.II, §5]. It is not difficult to see that $(P_1(x_1), \ldots, P_n(x_n)) \cdot \mathbb{k}[x]$ (the ideal in $\mathbb{k}[x]$ generated by $P_1(x_1), \ldots, P_n(x_n)$) is radical, and, as a consequence, that $m_R = (P_1(x_1), \ldots, P_n(x_n)) \cdot R$.

If $\alpha \in \mathbb{N}^n$, let $Q_\alpha(x) := \prod_{i=1}^n P_i(x_i)^{\alpha_i}$; Then

$$Q_\alpha(x) = \xi_\alpha \cdot (x - x(a))^{\alpha} \mod ((x - x(a))^{\alpha+1},$$

where $\xi_\alpha = \prod P_i(x_i)^{\alpha_i} \neq 0$, and it follows that each $T_m^k$ is bijective.

Conversely, suppose that $\mathbb{k}$ is not perfect. Let $\text{char} \mathbb{k} = p \neq 0$. Take $\lambda \in \mathbb{k} \setminus \mathbb{k}^p$ and set $f(x) = x^p - \lambda$ (one variable). Then $f(x)$ is irreducible. Let $m$ denote the maximal ideal generated by $f(x)$. We use the notation above. Let $b$ denote the image of $x$ in $\mathbb{F}$; then $b^p = \lambda$, and

$$\tau_m(f) = (b + (x - b))^p - \lambda = b^p + (x - b)^p - \lambda = (x - b)^p.$$
Therefore, $\tau_m(m_R) \subset (x - b)^p$; in particular $m_R/m_R^2 \to (x - b)/(x - b)^2$ is not an isomorphism. \qed

Remark 3.2. The converse direction in the proof of the theorem shows that the two notions of order introduced above do not coincide for all maximal ideals $m$ when $k$ is not perfect. (The other direction in) the statement of the theorem shows that the two notions coincide for a perfect field.

3.2. Equimultiple locus of a binomial. Let $f(x) \in k[x], x = (x_1, \ldots, x_n)$, be a binomial. Relabelling the variables, let us write

$$f(x) = f(u, v) = u^\alpha - v^\beta,$$

where $x = (u, v) = (u_1, \ldots, u_k, v_1, \ldots, v_{n-k}), \alpha \in \mathbb{N}^k, \beta \in \mathbb{N}^{n-k}, 0 < |\alpha| \leq |\beta|$ and $\alpha_i > 0, i = 1, \ldots, k$.

Let $S_f(0)$ denote the equimultiple locus of the origin for $f$; i.e.,

$$S_f(0) := \{ b \in \mathbb{A}_k^n : \mu_b(f) = \mu_0(f) \}.$$

(Of course, $\mu_0(f) = |\alpha|$.) Clearly, if $\text{char } k = 0$ and $|\alpha| \geq 2$, then

$$S_f(0) = \{(u, v) : u = 0, \mu_u(v^\beta) \geq |\alpha|\}.$$

This is not necessarily true in positive characteristic.

Example 3.3. Let $\text{char } k = p > 0$. Let $f(u, v) = u^p - v^p$ (where $u$ is a single variable). Then $f(u, v) = (u - v^\beta)^p$, so that $\mu_{(\xi, \eta)}(f) = p$ for any $(\xi, \eta)$ such that $\xi = \eta^\beta$.

We will show that (3.1) holds if $f$ is not a $p$'th power in $k[u, v]$, where $p = \text{char } k > 0$. We will, in fact, need a slightly more general result (Theorem 3.4 following). If $a \in \mathbb{A}_k^n$, we let $\mathbb{F}_a$ denote the residue field of $a$, and write $x(a) = (x_1(a), \ldots, x_n(a))$, where $x_i(a)$ denotes the image of $x_i$ in $\mathbb{F}_a$, for each $i$.

Theorem 3.4. Let $p = \text{char } k > 0$. Let

$$f(u, v, w) = w^\tau u^\alpha - v^\beta \in k[u, v, w],$$

where $u = (u_1, \ldots, u_k), v = (v_1, \ldots, v_l), w = (w_1, \ldots, w_{n-k-l})$. Let $a \in \mathbb{A}_k^n$. Assume that $u(a) = 0, \alpha_i > 0 (i = 1, \ldots, k)$, and $w_j(a) \neq 0 (j = 1, \ldots, n - k - l)$. Let $d := \mu_a(f) = |\alpha|$, and let $S_f(a) \subset \mathbb{A}_k^n \setminus \{w = 0\}$ denote the equimultiple locus of $a$; i.e.,

$$S_f(a) = \{ b \in \mathbb{A}_k^n \setminus \{w = 0\} : \mu_b(f) = d \}.$$

If $f$ is not a $p$'th power in $k[u, v, w]$ and $d \geq 2$, then $S_f(a) \subset \{u = 0\}$. (If fact, $S_f(a) \not\subset \{u = 0\}$ if and only if $f$ is a $d$'th power and $d = p^s$.)
Corollary 3.5. Under the hypotheses of Theorem 3.4,
\[ S_f(a) = \{(u, v, w) \in \mathbb{A}_k^n \setminus \{w = 0\} : u = 0, \; \mu_u(v^\beta) \geq d\} . \]

Lemma 3.6. Let \( d = qp^s \), where \( q \not\equiv 0 \mod p \). If \( b \in \mathbb{A}_k^n \), \( w_j(b) \neq 0 \), for all \( j \), and \( c \in \mathbb{F}_b \), then
\[ \mu_{w=w(b)}(w^\gamma(z+c)^d - w^\gamma c^d) = p^i \]
(\( \gamma \) is a single variable).

Proof.
\[ w^\gamma(z+c)^d - w^\gamma c^d = w^\gamma(z^p^s + c^p^s)^q - w^\gamma c^d \]
\[ = w^\gamma(z^d + qz^{(q-1)p^s}c^p^s + \cdots + qz^{p^s(p-1)p^s}) . \]
The result follows. \( \square \)

Proof of Theorem 3.4. Consider \( b \in S_f(a) \); say \( (u(b), v(b), w(b)) = (\xi, \eta, \zeta) \). Then \( \mu_{(\xi, \eta, \zeta)}(f) \geq \mu_u(f) = d \). Assume that \( \xi \neq 0 \). We will get a contradiction.

First suppose that \( k > 1 \). We can assume that \( \xi_1 \neq 0 \), where \( \xi = (\xi_1, \ldots, \xi_k) \). Write
\[ f(u, v, w) = w^\gamma((u - \xi + \xi^\alpha - \xi^\alpha) - ((v - \eta + \eta^\beta - \eta^\beta) + (w^\gamma \xi^\alpha - \eta^\beta)) , \]
and
\[ ((u - \xi + \xi^\alpha - \xi^\alpha) = (u - \xi)^\alpha + \sum_{1 \leq |\beta| < |\alpha|} \binom{\alpha}{\beta} (u - \xi)^{\alpha - \beta} \xi^\beta . \]
Consider \( \delta = (\alpha_1, 0, \ldots, 0) \). Then
\[ \binom{\alpha}{\delta} (u - \xi)^{\alpha - \delta} \xi^\delta = \xi_1^{\alpha_1}(u_2 - \xi_2)^{\alpha_2} \cdots (u_k - \xi_k)^{\alpha_k} , \]
so that
\[ \mu_{(\xi, \eta, \zeta)}(f) \leq \alpha_2 + \cdots + \alpha_k = d - \alpha_1 < d ; \]
a contradiction.

It remain to consider the case that \( k = 1 \); i.e., \( f = w^\gamma u^d - v^\beta \), where \( u \) is a single variable. Write \( d = d_1 p^s \), where \( d_1 \not\equiv 0 \mod p \). As above, write
\[ f(u, v, w) = w^\gamma((u - \xi + \xi^d - \xi^d) - ((v - \eta + \eta^\beta - \eta^\beta) + (w^\gamma \xi^d - \eta^\beta)) . \]
By Lemma 3.6, the first summand of the right-hand side has order \( p^s \) in \( u - \xi \); therefore, \( \mu_u(f) \leq p^s \). So, if \( d_1 \not\equiv 1 \), then \( \mu_{(\xi, \eta, \zeta)}(f) < d ; \) a contradiction.
On the other hand, suppose that \( d_1 = 1 \); i.e., \( d = p^s \). Then
\[
f(u, v, w) = w^\gamma (u - \xi)^{p^s} - (v - \eta + \gamma)^{\beta} - \eta^\beta + (w^\gamma \xi^{p^s} - \eta^\beta).
\]

We can assume that \( b \in \{ f = 0 \} \); i.e., \( \zeta^\gamma \xi^{p^s} - \eta^\beta = 0 \). We consider two cases:

1. \( \beta_j \not\equiv 0 \mod p^s \), for some \( j \). Say that \( \beta_1 \not\equiv 0 \mod p^s \); i.e.,
\[
\beta_1 = qp^t, \text{ where } q \not\equiv 0 \mod p \text{ and } 0 \leq t < s. \text{ Then}
\]
\[
(v - \eta + \gamma)^{\beta} - \eta^\beta = (v_1 - \eta_1 + \eta_1)^{\beta_1} \prod_{i \neq 1} (v_i - \eta_i + \eta_i)^{\beta_i} - \eta^\beta.
\]

By Lemma 3.6, the coefficient of \((v_1 - \eta_1)^{p^t}\) in this expansion is nonzero. Therefore, \( \mu_{(\xi, \eta, \zeta)}(f) \leq p^t < d \); a contradiction.

2. \( \beta_j \equiv 0 \mod p^s \), for all \( j \). Then
\[
f(u, v, w) = (w - \zeta + \zeta^\gamma^s (u - \xi)^{p^s} + \xi^{p^s}) - (v - \eta + \gamma)^{\beta}
\]
\[
= ((w - \zeta + \zeta^\gamma - \zeta^\gamma)^{\xi^{p^s}} + (w - \zeta + \zeta^\gamma (u - \xi)^{p^s} - ((v - \eta + \gamma)^{\beta} - \eta^\beta).
\]

The second summand here has order at least \( p^s \) (with respect to \((u - \xi, v - \eta, w - \zeta)\)), and the third summand also has order at least \( p^s \), since all \( \beta_j \equiv 0 \mod p^s \). Thus \((w - \zeta + \zeta^\gamma - \zeta^\gamma)^{\xi^{p^s}}\) has order at least \( p^s \) in \( w - \zeta \). If all \( \gamma_j \equiv 0 \mod p^s \), then \( f \) is a \( p^s \)th power; a contradiction to the hypothesis of the theorem. On the other hand, if some \( \gamma_j \not\equiv 0 \mod p^s \), then we can show that \((w - \zeta + \zeta^\gamma - \zeta^\gamma)^{\xi^{p^s}}\) has order \( < p^s \), using Lemma 3.6 as in the case (1); a contradiction. \( \square \)

**Remark 3.7.** Let \( k \) be a field. Let \( f(x) \in k[x] = k[x_1, \ldots, x_n] \). If \( X = (X_1, \ldots, X_n) \), then \( f(x + X) \in k[x][X] \); say, \( f(x + X) = \sum f_\alpha (x)X^\alpha \). For each \( \alpha \in \mathbb{N}^n \), \( f_\alpha (x) \) is called the Hasse derivative of \( f \) of order \( \alpha \). Let \( f(u, v, w) \) be a binomial as in Theorem 3.3. By Corollary 3.5, the equimultiple locus \( S_f(a) \) has the structure of a subspace (or subscheme) of \( \mathbb{A}_k^N \setminus \{w = 0\} \) given by the vanishing of all Hasse derivatives of \( f \) (or all Hasse derivatives with respect to \((u, v)\)) of orders \( < d \). (In particular, the order \( \mu_a(f) \) is a Zariski upper-semicontinuous function of \( a \).) According to §3.1 above, the analogous result for general polynomials requires the hypothesis that \( k \) be perfect.

### 4. Blowing up and strict transform

Let \( \Sigma \) denote a regular fan in a lattice \( N \cong \mathbb{Z}^n \). (See §2.2.) Let \( \Delta \) be a cone in \( \Sigma \), and let \( e_\Delta \in N \) denote the sum of the vertices of \( \Delta \). We call \( e_\Delta \) the barycentre of \( \Delta \). We define the star-subdivision \( \Sigma' \) of \( \Sigma \) as the smallest refinement of \( \Sigma \) that includes \( e_\Delta \) as a vertex.
Let $M = M(\Sigma)$ denote the smooth toric variety over $k$ corresponding to $\Sigma$. Let $\lambda : N \to \mathbb{Z}$ be a $\mathbb{Z}$-linear function, and let $X \hookrightarrow M$ denote the corresponding toric hypersurface in $M$ (Lemma 2.6).

4.1. **Affine case.** $M = \mathbb{A}^n = \mathbb{A}^n_\mathbb{Z}$. We can assume that $N = \mathbb{Z}^n$ and that $\Sigma$ is given by the cone generated by the standard basis vectors $e_1, \ldots, e_n$. In the affine coordinates $x = (x_1, \ldots, x_n)$ of $\mathbb{A}^n$, the ideal of $X$ is generated by the binomial $x^\gamma - 1$ (in $k[x^\pm]$), where $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\gamma_i = \lambda(e_i)$, $i = 1, \ldots, n$ (Lemma 2.1 and §2.2).

The $\mathbb{T}^n$-invariant subspaces of $\mathbb{A}^n$ are simply the coordinate subspaces $Z_\Delta := \{x_i = 0, \ i \in \Delta\}$, where $\Delta \subset \{1, \ldots, n\}$. Fix such $\Delta$, and consider the blowing-up $\pi$ of $M' \to M = \mathbb{A}^n$ with centre $Z_\Delta$. This blowing-up can be described combinatorially as follows. We identify $\Delta$ with the face of $\Sigma$ spanned by $e_i$, $i \in \Delta$. Then $M' = M(\Sigma')$, where $\Sigma'$ is the star-subdivision of $\Sigma$ determined by $\Delta$. For each $i \in \Delta$, let $\sigma_i \in \Sigma'$ denote the cone with vertices $e_j$, $j \in \Delta \setminus \{i\}$, and $e_\Delta$. (The $\sigma_i$, $i \in \Delta$, are the maximal cones of $\Sigma'$.) For each $i \in \Delta$, $U_{\sigma_i}$ has affine coordinates $x = (x_1, \ldots, x_n)$ with respect to which $\pi : U_{\sigma_i} \to \mathbb{A}^n$ is given by the following substitution rules: For each $j \in \Delta \setminus \{i\}$, substitute $x_i x_j$ for $x_j$, and leave the remaining variables $x_j$ unchanged. (For economy of notation, we use the same symbols for the variables before and after blowing up.)

Let $\gamma_\Delta = \lambda(e_\Delta) = \sum_{j \in \Delta} \gamma_j$. It is easy to see that the strict transform $X'$ of $X$ by $\pi$ is defined in each affine chart $U_{\sigma_i}$, $i \in \Delta$, by the toric ideal generated by $x'^\gamma - 1$, where $\gamma' = (\gamma'_1, \ldots, \gamma'_n)$ and $\gamma'_j = \gamma_j$, $j \neq i$, and $\gamma'_i = \gamma_\Delta$. In other words, $X'$ is defined by (the restriction to $\Sigma'$ of) the same linear function $\lambda : N \to \mathbb{Z}$ as before blowing-up.

4.2. **General case.** Let $\Delta$ denote a cone in $\Sigma$. Then $\Delta$ determines a smooth invariant subspace $Z_\Delta$ of $M = M(\Sigma)$: Consider any cone $\sigma \in \Sigma$ such that $\Delta$ is a face of $\sigma$; choose a basis $e_1, \ldots, e_n$ of $N$ so that $e_{p+1}, \ldots, e_n$ are the vertices of $\sigma$. We can realize $U_\sigma$ as a closed subvariety of $\mathbb{A}^{2p} \times \mathbb{A}^{n-p}$ with affine coordinates $(x, u)$ as in §2.2. Then $Z_\Delta \cap U_\sigma$ is defined by the equations $u_i x_i = 1$, $i = 1, \ldots, p$, and $x_j = 0$, $j \in \Delta$.

Let $\pi : M' \to M = M(\Sigma)$ denote the blowing-up with centre $Z_\Delta$. The $M' = M(\Sigma')$, where $\Sigma'$ is the star-subdivision of $\Sigma$ determined by $\Delta$. The strict transform $X'$ of $X$ by $\pi$ is the toric hypersurface of $M'$ determined by (the restriction to $\Sigma'$ of) the linear function $\lambda$. 

DESINGULARIZATION OF TORIC AND BINOMIAL VARIETIES 15
5. Equivariant resolution of singularities of a toric hypersurface

In this section, we prove Theorem 1.4; we translate Theorem 1.4 into a purely combinatorial statement (Theorem 5.3 below) which we prove using a simple algorithm. Our desingularization algorithm for a toric hypersurface corresponds to the “combinatorial part” of the desingularization algorithm of [BM3]. (See [BM3, p. 260], [BM5, §3.3(1)] and also [BM1, §4], as well as Step 2(b) in the proof of Theorem 8.5 below.) Throughout this section, we will use the language of Section 4 without further notice.

5.1. Affine case. We first consider the special case of an affine toric hypersurface $X \subset \mathbb{A}^n = \mathbb{A}^n_\mathbb{Z}$. The toric ideal corresponding to $X$ is generated by a binomial $x^\gamma - 1$; i.e., by a binomial $f(x) = x^{\gamma^+} - x^{\gamma^-}$ in the affine coordinates $x = (x_1, \ldots, x_n)$ of $\mathbb{A}^n$, where $\gamma \in \mathbb{Z}^n$ (Lemma 2.1). We can assume that $|\gamma^-| \leq |\gamma^+|$; then $d := |\gamma^-|$ is the maximum order $\mu_0(f)$ of $f$ (or $X$). (The maximum order of a binomial is, of course, taken at the origin.)

The locus of maximum order $d$ of $X$ is the equimultiple locus $S_f(0)$ of $0$. By Corollary 3.5,

$$S_f(0) = \{ x \in \mathbb{A}^n : x_i = 0 \text{ if } \gamma_i < 0, \text{ and } \mu_x(x^{\gamma^+}) \geq d \}. $$

Let $Z_\Delta := \{ x_i = 0, \ i \in \Delta \}$, where $\Delta \subset \{1, \ldots, n\}$. Then $Z_\Delta \subset S_f(0)$ if and only if

$$i \in \Delta \text{ if } \gamma_i < 0 \quad (i.e., \ \gamma^-_\Delta = d), \quad \gamma^+_\Delta \geq d = \gamma^-_\Delta \quad (i.e., \ \gamma_\Delta \geq 0)$$

(recall that $\gamma_\Delta := \sum_{i \in \Delta} \gamma_i$; likewise for $\gamma^+, \gamma^-$), and

$$S_f(0) = \bigcup Z_\Delta ,$$

where the union is over the minimal subsets $\Delta$ of $\{1, \ldots, n\}$ satisfying (5.1). In other words, the union in (5.2) is over the subsets $\Delta$ of $\{1, \ldots, n\}$ such that

$$\gamma^-_\Delta = d , \quad 0 \leq \gamma_\Delta < \gamma_i , \text{ for all } i \in \Delta \text{ such that } \gamma_i \geq 0 . $$

(In particular, $\gamma_i \neq 0$, for all $i \in \Delta$.)

Consider the blowing-up of $\mathbb{A}^n$ with centre $Z_\Delta \subset S_f(0)$. Let $X'$ denote the strict transform of $X$. Then (in the notation of §4.1) $X'$ has order $\leq d$ at the origin of each chart chart $U_{\sigma_i}, i \in \Delta$, and therefore at every point.
Now suppose that $Z_\Delta$ is a component of $S_f(0)$. It follows from (5.3) that $X'$ has order $< d$ throughout each $U_\sigma$, where $\gamma_i < 0$, and, in a chart $U_\sigma$, $\gamma_i > 0$, if $\mu_f'(0) = d$, where $f'(x) = x^{(\gamma)^+} - x^{(\gamma)^-}$, then

$$d \leq |(\gamma)^+| < |\gamma^+|.$$ 

It follows that we can reduce the order over every chart by a finite number of analogous blowings-up. At each step, the centre of blowing-up extends to a global closed smooth invariant subspace, as we will observe in §5.2 below. The main point is the following: For any $\Delta \subset \{1, \ldots, n\}$ (i.e., for any face $\Delta$ of $\Sigma$, in the language of §4.1), set

$$d_\Delta := \min\{\gamma^-_\Delta, \gamma^+_\Delta\},$$

$$\Omega_\Delta := \max\{\gamma^-_\Delta, \gamma^+_\Delta\}.$$

(In particular, $d_\Sigma = d$. Then $Z_\Delta$ is a component of $S_f(0)$ above if and only if

$$d_\Delta = d,$$

$$d_{\Delta_1} < d,$$ for every proper subface $\Delta_1$ of $\Delta$.

Moreover, if $\Sigma'$ is the star-subdivision of $\Sigma$ determined by such a face $\Delta$, then

$$(d_\sigma, \Omega_\sigma) < (d_\Sigma, \Omega_\Sigma)$$

(with respect to the lexicographic ordering of pairs), for every cone $\sigma \in \Sigma'$.

5.2. **General case.** We now consider a general toric hypersurface $X \subset M = M(\Sigma)$ over $k$, as in Section 4. For any cone $\sigma \in \Sigma$, write

$$\gamma^+_{\sigma} := \sum_{e \in \sigma, \lambda(e) > 0} \lambda(e), \quad \gamma^-_{\sigma} := \sum_{e \in \sigma, \lambda(e) < 0} (-\lambda(e))$$

(where "$e \in \sigma$" means that $e$ is a vertex of $\sigma$). Set

$$d_\sigma := \min\{\gamma^-_{\sigma}, \gamma^+_{\sigma}\}, \quad \sigma \in \Sigma,$$

$$d(\lambda, \Sigma) := \max_{\sigma \in \Sigma} d_\sigma.$$ 

**Definitions 5.1.** We will say that a cone $\Delta \in \Sigma$ (or the star-subdivision of $\Sigma$ determined by $\Delta$) is

1. **admissible** if $d_\Delta = d(\lambda, \Sigma)$.
2. **minimal** if it is admissible and $d_{\Delta_1} < d(\lambda, \Sigma)$, for every proper subface $\Delta_1$ of $\Delta$. 
Clearly, $d(\lambda, \Sigma)$ is the maximum order of $X$ (if $d(\lambda, \Sigma) > 0$). We recall that the centre of blowing up $Z_\Delta$ corresponding to the star-subdivision of $\Sigma$ determined by a cone $\Delta$ is a smooth closed invariant subspace of $M$. (Therefore, $Z_\Delta$ simultaneously has only normal crossings with respect to all smooth invariant subspaces of $M$.) If $\Delta$ is admissible, then $X$ assumes its maximum order $d(\lambda, \Sigma)$ at each point of $Z_\Delta$ (so $X$ is normally flat along $Z_\Delta$). If $\Delta$ is minimal, then $Z_\Delta$ is also a component of the locus of points of $X$ of maximum order. In particular, we obtain the following version of Theorem 1.5.

**Proposition 5.2.** The locus of maximum order (the equimultiple locus $S$) of a toric hypersurface $X \subset M$ has only normal crossings. Moreover, $S$ is a union of global smooth closed $T_M$-invariant components.

The following theorem is a combinatorial restatement of Theorem 1.4.

**Theorem 5.3.** There is a finite succession of admissible star-subdivisions,

$$\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_t,$$

such that $d(\lambda, \Sigma_t) = 0$.

**Remark 5.4.** For each $l \geq 0$, let $X_l$ denote the toric hypersurface in $M_l := M(\Sigma_l)$ determined by the function $\lambda$ on $\Sigma_l$. Then $X_{t+1}$ is the strict transform of $X_t$ by the blowing-up $M_{t+1} \to M_t$ determined by the star-subdivision $\Sigma_{t+1}$ of $\Sigma_t$. The condition $d(\lambda, \Sigma_t) = 0$ means (in the notation of §§2.2, 4.2) that, in each chart $U_\sigma$ of $M_t$, $X_t$ is defined by a binomial $x^\gamma - 1$, where $\gamma_i \geq 0$ for all $i$. Therefore, $X_t$ is smooth and simultaneously has only normal crossings with respect to the collection of exceptional hypersurfaces (all given by coordinate subspaces in the local charts).

Our remarks in §5.1 show that we can simply use the following combinatorial algorithm to prove Theorem 5.3.

**Algorithm 5.5.** For each $l \geq 0$, let $\Sigma_{t+1}$ be any minimal star-subdivision of $\Sigma_t$.

**Questions 5.6.** Let us say that a sequence of star-subdivisions, $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_t$, is **resolving** if $d(\lambda, \Sigma_t) = 0$.

(1) Does any resolving sequence of star-subdivisions have length greater than or equal to the length of some resolving sequence of minimal star-subdivisions?

(2) It is not difficult to give an example of an affine toric hypersurface for which there are resolving sequences of minimal star-subdivisions of different lengths. Do we get a resolving sequence of shortest length by
taking, at each step, the star-subdivision based on any minimal cone \( \Delta \) with the smallest value of \( \Omega_\Delta = \max\{\gamma^-_\Delta, \gamma^+_\Delta\} \).

6. Standard basis of a toric or binomial ideal

Our goal in this section is to show that a standard basis of a toric or more general binomial ideal is given by binomials. (Compare [S1].) Let \( k \) denote a field and let \( k[x] = k[x_1, \ldots, x_n] \).

6.1. Binomial ideal. Let \( I \) denote an ideal in \( k[x] \).

Definition 6.1. We say that \( I \) is a binomial ideal if:

1. \( I \) is generated by binomials \( x^\gamma + x^{-\gamma}, \gamma \in \mathbb{N} \).
2. \( k[x]/I \) contains no nilpotents.
3. If \( f(x) \in k[x] \) and \( x_if(x) \in I \) (for some \( i = 1, \ldots, n \)), then \( f(x) \in I \).

A toric ideal is a special case of a binomial ideal. (If \( X \) is the affine toric variety \( V(I) \) corresponding to a toric ideal \( I \), then property (3) above expresses the density of \( \mathcal{T}X = X \cap \mathbb{T}^n \) in \( X \).) Note that we use “binomial ideal” in a more restrictive sense than [ES].

If \( I \subset k[x] \) is a binomial ideal, we call \( X = V(I) \subset \mathbb{A}^n \) an affine binomial variety. Affine binomial varieties share many of the properties of affine toric varieties.

6.2. Distinguished point. An affine binomial variety \( X \subset \mathbb{A}^n = \mathbb{A}^n_k \) has a distinguished point \( a \): Let \( I \subset k[x] \) denote the binomial ideal corresponding to \( X \). (After permuting the variables if necessary) we can assume that \( x_{n-m+1}, \ldots, x_n \) are precisely the variables \( x_i \) that vanish nowhere on \( X \). Let us relabel the variables \( (x_1, \ldots, x_n) \) as \( (x, y) = (x_1, \ldots, x_{n-m}, y_1, \ldots, y_m) \). Then it is easy to see that \( I \) (as an ideal in \( k[x, y] \)) is generated by binomials of the form

\[
1 - y^\gamma, \quad \gamma \in \mathbb{Z}^m, \\
x^{\alpha} - x^\beta y^\gamma, \quad \alpha, \beta \in \mathbb{N}^{n-m}, \quad \gamma \in \mathbb{Z}^m, \quad 1 \leq |\alpha| \leq |\beta|.
\]

Let \( a \) denote the point \((\mathbf{0}, \mathbf{1})\), where \( \mathbf{0} = (0, \ldots, 0) \in \mathbb{A}^{n-m}, \mathbf{1} = (1, \ldots, 1) \in \mathbb{A}^m \).

In the toric case, the distinguished point \( a \) belongs to the unique closed orbit of \( X \) (in particular, \( a \) belongs to the closure of every orbit). The closed orbit of \( X \) is an algebraic torus isomorphic to \( T_Y \), where \( Y \subset \mathbb{A}^m \) is the toric variety defined by the ideal generated by all \( 1 - y^\gamma \in I \), and \( a \) is, in fact, its identity element [F, Ch. 3]

Let \( X \subset \mathbb{A}^n \) be an affine binomial variety, as above. Consider any affine subspace (i.e., coordinate subspace) \( \mathbb{A}^q \) of \( \mathbb{A}^n (q \leq n) \), and let
The torus $\mathbb{T}^q \subset \mathbb{A}^q$ denote the standard torus. Then (as in the toric case), $X \cap \mathbb{T}^q$ is a subgroup of $\mathbb{T}^q$. In particular, $X \cap \mathbb{T}^q$ is smooth.

It follows that (in the notation above), the affine variety $Y \subset \mathbb{A}^n$ defined by the ideal generated by all $1 - y^i \in I$ is a smooth affine binomial variety (toric, if $I$ is a toric ideal).

**Lemma 6.2.** If $X \subset \mathbb{A}^n$ is an affine binomial variety, then $X \cap \mathbb{T}^n$ is dense in $X$.

**Proof.** Let $I \subset k[x_1, \ldots, x_n]$ be the binomial ideal of $X$. If $f(x) \in k[x]$ vanishes on $X \cap \mathbb{T}^n$, then $x_1 \cdots x_nf(x)$ vanishes on $X$, so that $x_1^r \cdots x_n^rf(x)^r \in I$, for some positive integer $r$. Therefore, $f(x)^r \in I$, by Definition 6.1(3), so that $f(x) \in I$, by (2). □

**Lemma 6.3.** Let $X \subset \mathbb{A}^n$ be an affine binomial variety. Then the distinguished point $a \in X$ belongs to the closure of $X \cap \mathbb{T}^n$, for every affine subspace $\mathbb{A}^q \subset \mathbb{A}^n$ such that $X \cap \mathbb{T}^q \neq \emptyset$.

**Proof.** Let $\mathbb{A}^q$ be an affine subspace of $\mathbb{A}^n$, and let $I \subset k[x, y^+]$ denote the ideal of $X$ (using the notation above). (After permuting the variables $x_1, \ldots, x_{n-m}$ if necessary) we can assume that $x = (u_1, \ldots, u_{n-q}, v_1, \ldots, v_{q-m})$ ($q \geq m$) and that $\mathbb{A}^q = \{u = 0\}$. (Otherwise, $X \cap \mathbb{T}^n = \emptyset$.) Let $W \subset \mathbb{A}^n$ denote the open subset defined by $v_i \neq 0$ and $y_j \neq 0$, for all $i = 1, \ldots, q - m$ and $j = 1, \ldots, m$. Then $X \cap \mathbb{T}^q = X \cap \{u = 0\} \cap W$ is a smooth subvariety of $\{u = 0\} \cap W$ defined by those binomials in $I$ which involve only the variables $v_i, y_j$.

Let $Z \subset X$ denote the binomial variety defined by the latter binomials. (In particular, $X \cap \{u = 0\} \cap W = Z \cap \{u = 0\} \cap W$.) $Z \cap W$ is a smooth subvariety of $W$. Since $X \cap \mathbb{T}^n$ is dense in $X$, by Lemma 6.2, $a \in X \cap \mathbb{T}^n \subset Z \cap W$; therefore, $a$ is in the closure of $Z \cap \{u = 0\} \cap W$, as required. □

### 6.3. Diagram of initial exponents.

Let $A$ be a commutative ring with identity. Let $A[x] = A[x_1, \ldots, x_q]$. If $\alpha = (\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q$, put $|\alpha| = \alpha_1 + \cdots + \alpha_q$. We will use the total order on $\mathbb{N}^q$ that is given by the lexicographic ordering of $(q + 1)$-tuples $(|\alpha|, \alpha_1, \ldots, \alpha_q)$. Let $F = \sum_{\alpha \in \mathbb{N}^q} F_\alpha X^\alpha \in A[x]$, where $x^\alpha := x^{\alpha_1} \cdots x^{\alpha_q}$. Let supp $F := \{\alpha : F_\alpha \neq 0\}$. The initial exponent $\exp F$ is the smallest element of supp $F$. (Exp $F := \infty$ if $F = 0$.) If $F \neq 0$ and $\alpha = \exp F$, then $F_\alpha X^\alpha$ is called the initial monomial $\exp F$ of $F$.

Let $I$ be an ideal in $A[x]$. The *diagram of initial exponents* $\mathfrak{N}(I) \in \mathbb{N}^q$ is defined as

$$\mathfrak{N}(I) := \{\exp F : F \in I \setminus \{0\}\}.$$

Clearly, $\mathfrak{N}(I) + \mathbb{N}^q = \mathfrak{N}(I)$. It follows that there is a smallest finite subset $\mathfrak{V}$ of $\mathfrak{N}(I)$ (the vertices of $I$) such that $\mathfrak{N}(I) = \mathfrak{V} + \mathbb{N}^q$. 

6.4. The vertices of the diagram of initial exponents of a toric or binomial ideal are represented by binomials. In the remainder of Section 6, we assume that $I$ is any binomial ideal in $k[x, y] = k[x_1, \ldots, x_{n-m}, y_1, \ldots, y_m]$ and that $X = V(I)$ has nonempty intersection with $\{0\} \times \mathbb{T}^m \subset \mathbb{A}^{n-m} \times \mathbb{A}^m$. (For desingularization of toric varieties, we will be interested only the in special case that $I$ is a toric ideal and $a := (0, 1)$ is the distinguished point, as above.) We write binomials in $I$ in the form $x^\alpha - x^{\beta} y^\gamma$, where $\gamma \in \mathbb{Z}^m$; i.e., we consider $I$ as an ideal in $k[x, y^\pm]$. Then any binomial in $I$ either involves $y$ alone, or is of the form $x^\alpha - x^{\beta} y^\gamma$, where $\alpha, \beta$ are nonzero elements of $\mathbb{N}^{n-m}$.

Let $J \subset k[y^\pm]$ denote the ideal generated by all binomials $1 - y^\gamma \in I$, and let $B$ denote the quotient ring $k[y^\pm]/J$. If $f \in k[x, y^\pm]$, let $f_J \in B[x]$ denote the element induced by $f$. Let $I_J$ denote the ideal in $B[x]$ induced by $I$.

We consider $B[x]$ as a subring of the ring of formal power series $B[x]$, and write $\mathfrak{N}(I_J) \subset \mathbb{N}^{n-m}$ for the diagram of initial exponents of $I_J \cdot B[x]$.

**Lemma 6.4.** (1) If $f \in I$ and $\text{supp} f_J \subset \mathbb{N}^{n-m} \setminus \mathfrak{N}(I_J)$, then $f \in J \cdot k[x, y^\pm]$.

(2) If $\alpha$ is a vertex of $\mathfrak{N}(I_J)$, then $\alpha = \exp G_J$, where $G \in I$ is a binomial of the form $x^\alpha - x^{\beta} y^\gamma$, with $\beta \in \mathbb{N}^{n-m}$, $\alpha < \beta$, and $\gamma \in \mathbb{Z}^m$.

**Proof.** The assertion (1) is obvious. For (2), let

$$\mathfrak{N} := \{ \exp f_J \in \mathbb{N}^{n-m} : f = x^\alpha - x^{\beta} y^\gamma \in I, \alpha, \beta \in \mathbb{N}^{n-m}, 0 < \alpha < \beta, \gamma \in \mathbb{Z}^m \} ,$$

The assertion (2) means that $\mathfrak{N}(I_J) = \mathfrak{N}$. Clearly, $\mathfrak{N} \subset \mathfrak{N}(I_J)$. Consider $f \in I$. Then we can write

$$f = \sum_{i=1}^{t} g_i f_i \pmod{J \cdot k[x, y^\pm]} ,$$

where each $g_i \in k[x, y^\pm]$ and each $f_i$ is a binomial

$$f_i = x^{\alpha^i} - x^{\beta^i} y^{\gamma^i} \in I ,$$

in which $\alpha^i, \beta^i \in \mathbb{N}^{n-m}$, $\alpha^i < \beta^i$, and $\gamma^i \in \mathbb{Z}^m$. Set

$$\delta := \min_i \{ \exp g_i f_J + \alpha^i \} .$$

If $\delta = \infty$, then $f_J = 0$; i.e., $f \in J \cdot k[x, y^\pm]$. Suppose that $\delta < \infty$. Then $\delta \leq \exp f_J$. If $\exp f_J = \delta$, then $\exp f_J \in \mathfrak{N}$. Suppose that $\delta < \exp f_J$. Let

$$\Lambda := \{ i : \exp g_i f_J + \alpha^i = \delta \} .$$
If \( i \in \Lambda \), then \( \text{mon } g_{i,l} = \mu_i x^{\delta - \alpha_i} \), where \( \mu_i \) is a nonzero element of \( B \); take \( \lambda_i \in k[y^\pm] \) such that \( \mu_i \) is induced by \( \lambda_i \). Then \( \sum_{i \in \Lambda} \lambda_i \in J \). We can assume that \( \Lambda = \{1, \ldots, s\} \), where \( 2 \leq s \leq t \). Write
\[
\begin{align*}
g_i' &:= g_i - \lambda_i(y) x^{\delta - \alpha_i}, \quad i = 1, \ldots, s, \\
g_i' &:= g_i, \quad i = s + 1, \ldots, t.
\end{align*}
\]
Then, modulo \( J \cdot k[x, y^\pm] \),
\[
\begin{align*}
f &= \sum_{i=1}^{t} g_i' f_i + \sum_{j=1}^{s} \lambda_j(y) x^{\delta - \alpha_j} f_j \\
(6.2) &= \sum_{i=1}^{t} g_i' f_i + \sum_{j=2}^{s} \lambda_j(y) (x^{\delta - \alpha_j} f_j - x^{\delta - \alpha_1} f_1).
\end{align*}
\]
For each \( j = 2, \ldots, s \),
\[
x^{\delta - \alpha_j} f_j - x^{\delta - \alpha_1} f_1 = \pm y^{\sigma^j} (x^{\tilde{\alpha}} - x^{\tilde{\beta}} y^{\tilde{\gamma}}),
\]
where \( \sigma^j, \tilde{\gamma}^j \in \mathbb{Z}^m \) and
\[
\tilde{\alpha}^j = \min \{\delta - \alpha^j + \beta^j, \delta - \alpha^1 + \beta^1\}, \\
\tilde{\beta}^j = \max \{\delta - \alpha^j + \beta^j, \delta - \alpha^1 + \beta^1\}.
\]
Note that \( \exp g_i' > \exp g_{i,j}, \ i = 1, \ldots, s \) and \( \tilde{\alpha}^j > \delta, \ j = 2, \ldots, s \). If \( \delta - \alpha^j + \beta^j = \delta - \alpha^1 + \beta^1 \), for some \( j \), then \( x^{\tilde{\alpha}} - x^{\tilde{\beta}} y^{\tilde{\gamma}} = x^{\tilde{\alpha}} (1 - y^{\tilde{\gamma}}) \in I \), so that \( 1 - y^{\tilde{\gamma}} \in J \) (by Definition 6.1(3)). Therefore, (6.2) is a new representation of \( f \) of the same form as (6.1) and, if \( \delta' \) denotes the analogue of \( \delta \) for this new representation, then
\[
\delta < \delta' < \exp f_j.
\]
By induction, \( f \) has a representation of the form (6.1) with \( \delta = \exp f_j \), and the result follows. \( \square \)

6.5. **The standard basis is given by binomials.** Let \( \alpha^i, i = 1, \ldots, s \), denote the vertices of \( \mathfrak{N}(I_J) \). We associate to \( \alpha^i, \ldots, \alpha^s \) a partition of \( \mathbb{N}^{n-m} \): Set \( \Delta_i := (\alpha^i + \mathbb{N}^{n-m}) \setminus \cup_{j=1}^{i} \Delta_j, \ i = 1, \ldots, s \), and put \( \Delta_0 := \mathbb{N}^{n-m} \setminus \cup_{i=1}^{s} \Delta_i \). Clearly, \( \mathfrak{N}(I_J) = \bigcup_{i=1}^{s} \Delta_i \) and \( \Delta_0 = \mathbb{N}^{n-m} \setminus \mathfrak{N}(I_J) \). Let \( A \) denote the ring of Laurent polynomials \( k[y^\pm] \). By Lemma 6.4(2), for each \( i = 1, \ldots, s \), there is a binomial \( G^i = x^{\alpha_i} - x^{\beta_i} y^{\gamma_i} \) in \( I \subset k[x, y^\pm] = A[x] \) that represents \( \alpha^i \).

**Theorem 6.5** (Hironaka division). (1) For each \( F \in A[x] \subset A[x] \), there are unique \( Q_i, R \in A[x] \) such that \( \alpha^i + \supp Q_i \subset \Delta_i \) (\( i =
will show that, under this assumption, we obtain (1) with the stronger
\[ F = \sum_{i=1}^{s} G^i Q_i + R. \]

(2) Moreover, if \( F = x^\delta H(y) \), where \( \delta \in \mathbb{N}^{n-m} \) and \( H(y) \) is a Laurent monomial, then the remainder \( R \) has the same form.

(3) For each \( i = 1, \ldots, s \), there is a unique binomial
\[ F^i = x^{\alpha^i} - x^{\beta^i} y^{\gamma^i} \in I, \]
where \( \beta^i \in \mathbb{N}^{n-m} \setminus \mathfrak{M}(I_j) \), \( \alpha^i < \beta^i \), and \( \gamma^i \in \mathbb{Z}^m \).

Remark 6.6. It follows from Theorem 6.5(1) and Lemma 6.4(1), that \( I \) is generated by the binomials \( F^i = x^{\alpha^i} - x^{\beta^i} y^{\gamma^i} \), modulo \( J \cdot A[x] \). We call \( \{ F^i \} \) the standard basis of \( I \) modulo \( J \cdot A[x] \).

Proof of Theorem 6.5. It is enough to prove (1) in the case that \( F = x^\delta H(y) \), where \( \delta \in \mathbb{N}^{n-m} \) and \( H(y) \in k[y^\pm] \) is a Laurent monomial. We will show that, under this assumption, we obtain (1) with the stronger conclusion of (2).

Consider any \( F \in A[x] \). Clearly, there are unique \( Q_i(F), R(F) \in A[X] \) such that \( \alpha^i + \text{supp} Q_i(F) \in \Delta_i \), \( \text{supp} R(F) \in \Delta_0 \), and \( F = \sum_{i=1}^{s} X^{\alpha^i} Q_i(F) + R(F) \). Set
\[ (6.3) \quad E(F) := F - \sum G^i Q_i(F) - R(F) = \sum (X^{\alpha^i} - G^i) Q_i(F). \]

Put \( E^0(F) = F \) and \( E^j(F) = E(E^{j-1}(F)) \), \( j \geq 1 \).

Now assume that \( F = x^\delta H(y) \), as in (2), where \( \delta \neq 0 \) and \( H \neq 0 \). There are two cases:

(i) If \( \delta \not\in \mathfrak{M}(I_j) \), then \( Q_i(F) = 0 \), \( (i = 1, \ldots, s) \), \( R(F) = F \) and \( E(F) = 0 \). In this case, \( F = \sum G^i Q_i + R \), where \( Q_i = Q_i(F) = 0 \), \( i = 1, \ldots, s \), and \( R = R(F) = F \).

(ii) Suppose that \( \delta \in \mathfrak{M}(I_j) \). Then there is a unique \( i \) such that \( \delta \in \Delta_i \); i.e., \( \delta = \alpha^i + \epsilon \), where \( \epsilon \in \mathbb{N}^{n-m} \). Therefore,
\[ x^\delta H(y) = x^\epsilon (x^{\alpha^i} - x^{\beta^i} y^{\gamma^i}) H(y) + x^{\eta^i + \epsilon} y^{\zeta^i} H(y). \]

So \( R(F) = 0 \) and \( E(F) = x^{\eta^i + \epsilon} y^{\zeta^i} H(y) \). Moreover,
\[ \exp F = \alpha^i + \epsilon < \eta^i + \epsilon = \exp E(F). \]

In case (ii), we can continue to divide. We claim there exists \( t \in \mathbb{N} \setminus \{0\} \) such that:

(a) For each \( j = 0, \ldots, t \), \( E^j(F) \) is a nonzero element of \( A[x] \) of the form \( x^{\delta_j} H_j(y) \), where \( \delta_j \in \mathbb{N}^{n-m} \) and \( H_j(y) \) is a Laurent monomial;
(b) $E^{j+1}(F) = 0$.

Suppose that this is not so. Then, for all $j = 0, 1, \ldots$, $R(E_j(F)) = 0$ and $E_j(F) = x^\delta H_j(y) \in A[x]$, where $\delta_{j+1} > \delta_j$. Therefore, $F = x^\delta H(y) \in I \cdot A[x]$, and it follows that $x^\delta H(y)$ belongs to the ideal generated by $I$ in the localization of $A[x] = k[x, y^\pm]$ at the point $(0, 1)$ (by faithfull flatness of completion of a Noetherian local ring). Then $1$ belongs to this ideal (by Definition 6.1(3)); a contradiction.

Assertions (1) and (2) follow.

It follows from (2) that, if $F = x^\alpha$ (where $1 \leq i \leq s$), then the remainder $R$ is a nonzero element of $A[x]$ of the form $x^{\beta_i} y^\gamma_i$, where $\beta_i \in \mathbb{N}^{n-m}$, $\beta_i > \alpha_i$, and $\gamma_i \in \mathbb{Z}^m$. (3) follows. □

Remark 6.7. Let $V(J) \subset \mathbb{A}^m$ denote the affine binomial variety determined by $J$. Then $V(J) \cap \mathbb{T}^m$ is smooth. (If $a = (0, 1)$ is the distinguished point of $X$, then $V(J) \subset \mathbb{T}^m$.) Let $\tilde{I}_b$ denote the ideal generated by $I$ in the completion $\mathcal{O}_{\mathbb{A}^n, b} \cong \mathbb{F}_b[X, Y]$ at any (closed) point $b$ (notation of §3.1). If follows from Theorem 6.5(3) that, at any $b \in V(J) \cap \mathbb{T}^m$, the vertices of $\mathcal{N}(\tilde{I}_b)$ are given by $(\alpha^i, 0) \in \mathbb{N}^{n-m} \times \mathbb{N}^m$, $i = 1, \ldots, s$, together with $k$ elements $(0, \gamma^j) \in \{0\} \times \mathbb{N}^m$ of order $|\gamma^j| = 1$, where $k$ is the codimension of $V(J)$ in $\mathbb{A}^m$ at $b$.

7. SAMUEL STRATIFICATION OF A TORIC VARIETY

Subsections 7.2 and 7.3 below isolate the properties of the Hilbert-Samuel function that play an important part in embedded resolution of singularities (here in the context of a toric or binomial variety). Theorem 7.1 asserts that the maximal Samuel stratum of an affine binomial variety is the simultaneous equimultiple locus of a standard basis of the binomial ideal, and Theorem 7.2 describes the behaviour of the Hilbert-Samuel function on blowing up with permissible invariant centre. Theorems 7.1, 7.2 are analogues for binomial varieties over perfect fields of theorems in [Hi2], [BM2], [BM3], [BM4].

Subsections 7.4 and 7.5 describe the structure of the Samuel stratificiation of a general embedded toric variety $X$ over a perfect field. In particular, we prove Theorem 1.6 and we show that the components of the maximal Samuel stratum satisfy the conditions for the centres of blowing up given in our main theorem 1.1.

7.1. Hilbert-Samuel function. Let $R$ denote a Noetherian local ring with maximal ideal $m$. The Hilbert-Samuel function $H_R : \mathbb{N} \to \mathbb{N}$ is defined by

$$H_R(l) = \text{length}rac{R}{m^{l+1}}, \quad l \in \mathbb{N}.$$
If $R$ is a $k$-algebra, then $H_R(l) = \dim_k R/m^l$, for all $l$. We partially order the set of functions $\mathbb{N}^\mathbb{N} := \{ H : \mathbb{N} \to \mathbb{N} \}$ as follows: If $H, H' \in \mathbb{N}^\mathbb{N}$, then $H \leq H'$ means that $H(l) \leq H'(l)$, for all $l \in \mathbb{N}$.

Let $I$ denote an ideal in a formal power series ring $k[[x]] = k[[x_1, \ldots, x_n]]$. Then, for all $l \in \mathbb{N}$,

\begin{equation}
H_{[x^l]/I}(l) = \#\{ \alpha \in \mathbb{N}^n : \alpha \notin \mathfrak{N}(I), |\alpha| \leq l \}
\end{equation}

(by Hironaka’s formal division theorem; cf. §6.3 and Theorem 6.5). It follows from (7.1) that (for fixed $n$), any non-increasing sequence of Hilbert-Samuel functions $H_{[x^l]/I}$ stabilizes [BM2 Thm. 5.2.1].

If $I \subset k[x]$ is a principal ideal of order $\mu$, then, by (7.1), $H_{[x^l]/I}(l) = \binom{n+\mu}{\mu}$, if $l < \mu$, and $H_{[x^l]/I}(l) = \binom{n+\mu}{\mu} - \binom{n+l-\mu}{\mu}$, if $l \geq \mu$.

We define the Hilbert-Samuel function $H_{X,b}$ of a Noetherian local-ringed space $X = (|X|, \mathcal{O}_X)$ at a point $b$ as the Hilbert-Samuel function of the local ring $\mathcal{O}_{X,b}$.

7.2. Hilbert-Samuel function and equimultiple locus. Let $I \subset k[x, y] = k[x_1, \ldots, x_{n-m}, y_1, \ldots, y_m]$ denote a binomial ideal, where $\alpha = (\underline{0}, 1)$ is the distinguished point of the corresponding affine binomial variety $X \subset \mathbb{A}^n$.

Let $J \subset k[y^\pm]$ denote the ideal generated by the binomials in $I$ involving $y$ alone, and let $Q$ denote the smooth binomial subvariety of $\mathbb{A}^n$ defined by the ideal $J \cdot k[x, y^\pm]$. Let

$$F^i = x^{\alpha^i} - x^{\beta^i} y^{\gamma^i} \in I, \quad i = 1, \ldots, s,$$

denote the standard basis of $I$ mod $J \cdot k[x, y^\pm]$, as in Theorem 6.5. For each $i$, if $|\alpha^i| = 1$, then $x^{\alpha^i} = x_{j(i)}$, for some $j(i) \in \{1, \ldots, n-m\}$, and $x_{j(i)}$ occurs (to nonzero power) in no monomial $x^{\alpha^j}$, $j \neq i$, and in no monomial $x^{\beta^j}$.

Let $N = N(I)$ denote the smooth binomial subvariety of $Q$ defined by the binomials $F^i(x, y)$, for all $i$ such that $|\alpha^i| = 1$; i.e., $N$ is the smooth binomial subvariety of $\mathbb{A}^n$ defined by the binomials $1 - y^\gamma \in J$ together with $x^{\alpha^i} - x^{\beta^i} y^{\gamma^i}$, for all $i = 1, \ldots, s$ such that $|\alpha^i| = 1$

Then $N$ is a minimal embedding submanifold of $X$ (i.e, a smooth variety of smallest dimension in which $X$ can be embedded; $\dim N$ is determined by $H_{X,a}$).

After reordering the indices $i$ if necessary, we can assume that

1. $|\alpha^i| \geq 2$, $i = 1, \ldots, t$, and $|\alpha^i| = 1$, $i = t+1, \ldots, s$, where $t \leq s$;
2. $x_1, \ldots, x_r$, where $r \leq (n-m) - (s-t)$, are those variables occurring (to nonzero power) in some $x^{\alpha^i}$, $i = 1, \ldots, t$.
The mapping $b \mapsto H_{X,b}$ from $X$ to $\mathbb{N}^n$ is Zariski upper-semicontinuous (cf. [Be], [BM2], [BM3]). We define the Samuel stratum $S_X(b)$ of $b \in X$ as $S_X(b) := \{ c \in X : H_{X,c} = H_{X,b} \}$.

**Theorem 7.1.**

1. $H_{X,b} \leq H_{X,a}$, for all $b \in X$.
2. The Samuel stratum $S_X(a)$ is the closed subset $S_{\{F_i\}}(a)$ of $Q$ given by the simultaneous equimultiple locus of the binomials $F^i(x,y) = x^{\alpha^i} - x^{\beta^i} y^{\gamma^i}$, $i = 1, \ldots, s$; i.e., the closed subset of $N$ defined by

   $$x_j = 0, \quad j = 1, \ldots, r,$$
   $$\mu(x^{\beta^i}) \geq |\alpha^i|, \quad i = 1, \ldots, t.$$

**Proof.** First suppose that $X$ is toric. Since every orbit of $X$ is adherent to the distinguished point $a$, it is enough to prove the assertions in some neighbourhood of $a$. (1) is then a restatement of semicontinuity. (2) can be proved exactly in the same way as [BM2, Theorem 5.3.1], using Theorem 3.1 above (cf. Remark 3.7).

Now consider a general affine binomial variety $X$. As above, (1) and (2) hold in a neighbourhood of the distinguished point $a$. For every affine subspace $\mathbb{A}^q$ of $\mathbb{A}^n$, $X \cap \mathbb{T}^q$ is adherent to $a$, by Lemma 6.3. It therefore suffices to show that the Hilbert-Samuel function is constant on $X \cap \mathbb{T}^q$.

To prove the latter, we use the notation of the proof of Lemma 6.3. Let $K \subset k[v^\pm, y^\pm]$ denote the ideal generated by the binomials in $I$ which involve $(v, y)$ alone. As in §6.5, let $B = k[v^\pm, y^\pm]/K$ and let $I_K$ denote the ideal generated by $I$ in $B[x] \subset B[x]$. If follows from (7.1) and Remark 6.7 that, at any point of $X \cap \mathbb{T}^q$, the Hilbert-Samuel function of $X$ is completely determined by the codimension of $X \cap \mathbb{T}^q$ in $\mathbb{A}^q$ and the vertices of $\mathfrak{M}(I_K) \subset N^{n-q}$. □

**Corollary 7.2.** The Samuel stratum $S_X(a)$ has only simple normal crossings. Each component of $S_X(a)$ is the intersection with $N$ of an affine subspace of $\mathbb{A}^n$ that is transverse to $N$. In particular, in the toric case, each component of $S_X(a)$ is the closure of an orbit of $T_N$ (acting on $N$).

**7.3. Hilbert-Samuel function and strict transform.** We continue to use the notation of the preceding subsection. Let $C$ denote a component of $S_X(a)$. The $C = D \cap N$, where $D = \{ x_j = 0 : j \in \Delta \}$, \{1, \ldots, r\} $\subset \Delta \subset \{1, \ldots, n - m\}$, and $\Delta$ includes no $x_j(i) = x^{\alpha_i}$ with $|\alpha^i| = 1$ (i.e., with $i = t + 1, \ldots, s$).

Let $\pi : U' \to \mathbb{A}^n$ denote the blowing-up of $\mathbb{A}^n$ with centre $D$. Let $N' = N(I)'$ and $X'$ denote the strict transforms of $N = N(I)$ and $X$.
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(respectively) by $\pi$. $U'$ is covered by affine charts $U_{x_i} \cong \mathbb{A}^n$, $i \in \Delta$, where, for each $i$, $U_{x_i}$ has affine coordinates $(x_1, \ldots, x_{n-m}, y_1, \ldots, y_m)$ with respect to which $\pi$ is given by the following substitution rules: For each $j \in \Delta \setminus \{i\}$, substitute $x_i x_j$ for $x_j$, and leave the remaining coordinates $x_j$ (and all $y$-coordinates) unchanged. Clearly, $X' \cap U_{x_i} \subset N' \cap U_{x_i}$ are affine binomial subvarieties of $U_{x_i}$, for each $i$. In the toric case, $X' \subset N'$ are toric subvarieties of $U'$.

Fix $i \in \Delta$. Let $I'$ denote the ideal of $X' \cap U_{x_i}$. Let $a'$ denote the distinguished point of $X' \cap U_{x_i}$, and let $J'$ denote the analogue of $J$ above. By general properties of the standard basis (see [BM3, Lemma 3.22]), $I'$ is generated by $J'$ together with the "strict transforms" $F_j' = x_i^{-|\alpha'_i|} F_j$ of all binomials $F_j$. Each $F_j'$ is a binomial

$$F_j' = x^{\alpha''_j} - x^{\beta''_j} y^{\gamma''_j}.$$  

(Certain $F_j'$ may belong to $J'$.)

**Theorem 7.3.** In the preceding notation,

1. $h_{X', a'} \leq h_{X, a}$.
2. $h_{X', a'} = h_{X, a}$ if and only if $a' \in N(I') = N(I)$, $|\alpha''_i| = \mu_{a'}(F_j') = |\alpha'_i|$, $i = 1, \ldots, s$, and the $F''_i$, $i = 1, \ldots, t$, are the elements of the standard basis of $I'$ mod $J'$ of order $\geq 2$.

The proof follows that of [BM2, Theorem 7.3].

**7.4. Samuel stratification.** Let $M$ be a smooth variety. Let $E$ denote a finite collection of smooth (Zariski-) closed subsets of $M$ having only normal crossings. Assume that $E$ includes $M$, and that $E$ is closed under intersection. (For example, let $M$ be a smooth toric variety, and let $E$ denote the collection of orbit closures of $M$.)

**Lemma 7.4** (cf. [BM5 Cor. 1.17]). Let $Y$ be a closed subset of $M$. Assume that:

1. $Y$ has only normal crossings.
2. Every component $Z$ of the germ $Y_a$ of $Y$ at any point $a \in Y$ is of the form $Z = (Y \cap E)_a$, where $E \in E$.

Then each irreducible component of $Y$ is smooth.

**Proof.** Let $a \in Y$. Write the germ $Y_a$ as a union of irreducible components $Y_a = \cup Z_E$, where $Z_E = (Y \cap E)_a$ and $E$ is the intersection of all elements of $E$ containing $Z_E$. We will show that each $Z_E$ extends to an irreducible component of $Y \cap E$.

Consider any total ordering of $E$. Define $e(a) := \max\{E : Z_E \text{ is a component of } Y_a\}$. Clearly, $a \mapsto e(a)$ is (Zariski-) upper semicontinuous on $Y$, and its maximum locus is smooth.
On the other hand, given \( a \) and a component \( Z_E \) of \( Y_a \), we can choose the ordering of \( \mathcal{E} \) so that \( E = \max \mathcal{E} \). It follows that \( Z_E \) extends to a closed smooth subset of \( Y \). \( \square \)

**Example 7.5.** Let \( M \) be a smooth toric variety over a perfect field \( k \), and let \( \mathcal{E} \) denote the collection of orbit closures of \( M \). Let \( X \) be a toric subvariety of \( M \), and let \( S \) denote the maximal Samuel stratum of \( X \) (i.e., the locus of maximum values of the mapping \( b \mapsto H_{X,b} \)). Then, for every \( E \in \mathcal{E} \), \( Y := S \cap E \) satisfies the hypotheses of Lemma 7.4 (by Theorem 7.1).

Theorem 1.6 is an immediate consequence of Theorem 7.1 and Lemma 7.4.

Theorem 1.6 generalizes Proposition 5.2. Unlike the case of a toric hypersurface, however, in general it is not possible to desingularize \( X \) by blowing up with each successive centre an arbitrary component of the maximal Samuel stratum.

**Example 7.6** (J. Adamus). Let \( X \) denote the toric subvariety of \( \mathbb{A}^6 \) defined by the ideal \( I \) generated by the binomials

\[
F = u^d - x^{d-1} y^d z^{d+1}, \\
G = v^d - x^d y^{d+1} z^{d-1}, \\
H = w^d - x^{d+1} y^{d-1} z^d,
\]

where \( d \geq 2 \). (In the notation above, \( J = 0 \) and \( F, G, H \) form the standard basis of \( I \).) The maximal Samuel stratum of \( X \) has 3 components, each given by setting \( u, v, w \) to 0 along with any 2 of the remaining variables \( x, y, z \).

Consider the strict transform \( X_1 = X' \) of \( X_0 = X \) by blowing up \( \mathbb{A}^6 \) with centre any of these components; say, \( u = v = w = y = z = 0 \). Then \( X_1 \subset U_y \cup U_z \), where \( U_y, U_z \) are affine charts defined as in §8.3. (For example, \( U_y \to U := \mathbb{A}^6 \) is given by substituting \( x, y, yz, yu, yv, yw \) for \( x, y, z, u, v, w \) (respectively). Now consider two further blowings-up whose centres are (components of the maximal Samuel strata) given in \( U_y \) by \( u = v = w = x = z = 0 \), and then in \( U_{yz} := (U_y)_z \) by \( u = v = w = y = z = 0 \). Then the toric ideal of the strict transform \( X_3 \) in the affine chart \( U_{yz} \) is generated again by the original binomials \( F, G, H \).

7.5. **Combinatorial nature of the Samuel strata.** Let \( M = M(\Sigma) \) denote a smooth toric variety over a perfect field \( k \), corresponding to a regular fan \( \Sigma \) in a lattice \( \cong \mathbb{Z}^n \) (cf. §2.2). We recall that \( M \) is covered by affine toric varieties \( U_\sigma \) determined by the maximal cones \( \sigma \) of \( \Sigma \).
If $\sigma, \tau$ are maximal cones, then $Z_{\sigma\tau} := U_\sigma \setminus U_\tau$ has only normal crossings. $Z_{\sigma\tau}$ is the union of the orbit closures of $U_\sigma$ that are not in $U_\tau$. Moreover, $Z_{\sigma\tau} = \bigcup E_j$, where the $E_j$ are the intersections with $Z_{\sigma\tau}$ of the orbit closures of $M$.

**Remark 7.7.** If $N$ is a smooth toric subvariety of $M$, then:

1. Every orbit closure of $N$ is the intersection with $N$ of an orbit closure of $M$ that is transverse to $N$. (This follows from the affine case.)
2. The intersection with $N$ of an orbit closure of $M$ has only normal crossings, and each of its irreducible components is an orbit closure of $N$ (in particular, smooth).

The following lemma generalizes Remark 7.7.

**Lemma 7.8.** Let $X$ denote a closed toric subvariety of $M = M(\Sigma)$, and let $S$ denote the maximal Samuel stratum of $X$. Let $E$ be an orbit closure of $M$ and let $C$ be an irreducible component of $S \cap E$. Then:

1. $C$ is smooth.
2. If $D$ denotes the smallest orbit closure of $M$ containing $C$, then $X \cap D = C$.

**Proof.** (1) is true, by Lemma 7.4 and Example 7.5.

Let $D$ be the smallest orbit closure of $M$ containing $C$. We first show that $C$ is open and closed in $X \cap D$: By Theorem 7.1, in any chart $U_\sigma$ where the distinguished point $a_\sigma$ of $X$ belongs to $C$, $C \cap U_\sigma = (X \cap D) \cap U_\sigma$. Moreover, $N_\sigma = S \cap U_\sigma$. Therefore, $C \cap U_\sigma = D \cap X \cap U_\sigma$. In other words, if $C \cap U_\sigma \neq \emptyset$, then $C \cap U_\sigma = (X \cap D) \cap U_\sigma$.

Now consider two charts $U_\sigma, U_\tau$. If $C \cap (U_\sigma \cap U_\tau) \neq \emptyset$, then $C \cap U_\tau \neq \emptyset$, so that $C \cap U_\tau = (X \cap D) \cap U_\tau$. On the other hand, suppose that $C \cap U_\tau \subset Z_{\sigma\tau}$. Then $C \subset E_j$ for some $j$, so that $D \subset E_j$, by the definition of $D$. Therefore, $D \cap U_\tau = \emptyset$; i.e., $(X \cap D) \cap U_\tau = \emptyset$. This proves (2). \qed

### 8. Proof of the main theorem

In this section, we prove Theorem 1.1. Let $X \hookrightarrow M$ denote an equivariant embedding of toric varieties, where $M$ is smooth. If $H \in \mathbb{N}^N$, we let $S_H(X)$ denote the Samuel stratum $S_H(X) := \{b \in X : H_{X,b} = H\}$. (In particular, if $a \in X$, then $S_X(a) = S_{H_{X,a}}(X)$.)

Let $H$ denote the maximum Hilbert-Samuel function of $X$. We will say that a sequence of blowings-up (1.1) is $H$-permissible if conditions
(1) and (2) of Theorem 1.1 are satisfied and, in addition, \(C_j \subset S_H(X_j)\), \(j = 0, \ldots, t\) (i.e., each \(X_j\) has maximum Hilbert-Samuel function \(H\), and each \(C_j\) lies in the maximal Samuel stratum of \(X_j\)).

**Theorem 8.1.** Let \(H\) denote the maximum Hilbert-Samuel function of \(X\). Then there is an \(H\)-permissible sequence of blowings-up (1.1) such that \(S_H(X_{t+1}) = \emptyset\).

By the stabilization theorem for the Hilbert-Samuel function ([BM2 Thm. 5.2.1]; cf. §7.1 above), it follows from Theorem 8.1 that there is a finite sequence of blowings-up (1.1) satisfying conditions (1), (2) of Theorem 1.1, and also:

\(3')\) Each \(C_j \subset \text{Sing} X_j\).

\(4')\) \(X_{t+1}\) is smooth.

We prove Theorem 8.1 below. Since each (reduced) component of the exceptional divisor \(E_{t+1}\) is a codimension one orbit closure of \(M_{t+1}\), it will then follow from Theorem 8.9 (in §8.4 below) that, beginning with a sequence of blowings-up (1.1) satisfying (1), (2), (3') and (4'), we can make a further sequence of blowings-up where each \(C_j \subset X_j \cap E_j\), after which (4) is satisfied.

### 8.1. Marked monomial ideal

It is convenient to use some of the structure of Wlodarczyk [W]; in particular, his notion of “marked ideal” in a simple monomial setting. Note, however, that our *marked monomial ideals* below are marked by more structure than the marked ideals of [W].

**Definition 8.2.** A *marked monomial ideal* is a quintuple

\[ \mathcal{H} = (M, N, P, \mathcal{H}, e) \]

where

- \(M\) is a smooth toric variety,
- \(N\) is a smooth closed toric subvariety of \(M\),
- \(P\) is a smooth (closed) invariant subvariety of \(N\),
- \(\mathcal{H} = \mathcal{H}_1 + \cdots + \mathcal{H}_r\), where each \(\mathcal{H}_i \subset \mathcal{O}_M\) is a product of principal ideals defining codimension one orbit closures of \(M\) that simultaneously have only normal crossings with \(N\) and do not contain \(P\),
- \(e\) is a positive integer.

A *monomial ideal* \(\mathcal{H} \subset \mathcal{O}_M\) means an ideal of the form \(\mathcal{H}_1 + \cdots + \mathcal{H}_r\), where each \(\mathcal{H}_i \subset \mathcal{O}_M\) is a product of principal ideals defining codimension one orbit closures of \(M\).
Note that, in Definition 8.2, if \( \dim N > 0 \), then the codimension one orbit closures of \( M \) that are involved in \( \mathcal{H} \) are transverse to \( N \). It follows that the restriction of \( \mathcal{H} \) to \( N \), \( \mathcal{H} \cdot \mathcal{O}_N \) is a monomial ideal in \( \mathcal{O}_N \) and, for all \( a \in P \), \( \mu_a(\mathcal{H} \cdot \mathcal{O}_P) = \mu_a(\mathcal{H} \cdot \mathcal{O}_N) = \mu_a(\mathcal{H}) \) (where \( \mu_a \) denotes the order at \( a \)).

Let \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \) be a marked monomial ideal. We define the support of \( \mathcal{H} \):

\[
\text{supp} \mathcal{H} := \{ a \in P : \mu_a(\mathcal{H}) \geq e \}.
\]

Then \( \text{supp} \mathcal{H} \) is a closed subset of \( N \) that has only normal crossings. (It is a union of orbit closures.) By Lemma 7.4, every irreducible component of \( \text{supp} \mathcal{H} \) is smooth; in fact, every irreducible component of \( \text{supp} \mathcal{H} \) is an orbit closure of \( N \) (cf. §7.5).

We say that \( \mathcal{H} \) has maximal order if \( \mu_a(\mathcal{H}) \leq e \), for all \( a \in P \).

Let \( \pi = \pi_D : M' \to M \) be a blowing-up with centre \( D \), where \( D \) is a smooth invariant subvariety of \( M \). We say that \( \pi \) is permissible for \( \mathcal{H} \) if \( C := D \cap N \) is smooth and \( C \subset \text{supp} \mathcal{H} \).

**Definition 8.3.** Let \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \) be a marked monomial ideal, and let \( \pi = \pi_D : M' \to M \) be a blowing-up that is permissible for \( \mathcal{H} \). The transform \( \mathcal{H}' \) of \( \mathcal{H} \) by \( \pi \) is a marked monomial ideal \( \mathcal{H}' = (M', N', P', \mathcal{H}', e') \), where

- \( N' \) is the strict transform of \( N \) by \( \pi \) (so that \( \pi|N' : N' \to N \) is the blowing-up \( \pi_C \) of \( N \) with centre \( C = D \cap N \)),
- \( P' \) is the strict transform of \( P \),
- \( \mathcal{H}' = \mathcal{I}_{\pi^{-1}(D)} \cdot \pi^*(\mathcal{H}) \), where \( \mathcal{I}_{\pi^{-1}(D)} \subset \mathcal{O}_{M'} \) denotes the principle ideal defining the exceptional divisor \( \pi^{-1}(D) \subset M' \) (a smooth invariant hypersurface in \( M' \)),
- \( e' = e \).

Note that \( \mathcal{H}'|N' = \mathcal{H}' \cdot \mathcal{O}_{N'} \) coincides with \( \mathcal{I}_{\pi^{-1}(N')}^{-1}(C) \cdot (\mathcal{H}|N)' \).

A permissible sequence of blowings up for \( \mathcal{H} \) means a sequence of blowings-up

\[
M = M_0 \leftarrow_{\pi_1} M_1 \leftarrow_{\pi_2} \cdots \leftarrow_{\pi_{t+1}} M_{t+1} ,
\]

where, for each \( j = 0, \ldots, t \), \( \pi_{i+1} \) is permissible for \( \mathcal{H}_i = (M_i, N_i, P_i, \mathcal{H}_i, e_i) \) and \( \mathcal{H}_{i+1} = (M_{i+1}, N_{i+1}, P_{i+1}, \mathcal{H}_{i+1}, e_{i+1}) \) denotes the transform of \( \mathcal{H}_i \) by \( \pi_{i+1} \). (We set \( \mathcal{H}_0 = \mathcal{H} \)).

A resolution of singularities of \( \mathcal{H} \) means a sequence of permissible blowings-up (8.1) such that \( \text{supp} \mathcal{H}_{t+1} = \emptyset \).

**Example 8.4.** Suppose that \( M \) is affine; say \( X \subset M \subset \mathbb{A}^n \). We use the notation of §7.2 above. (In particular, \( N \subset \mathbb{A}^n \) denotes a minimal embedding submanifold of \( X \).) We can assume that \( N \subset M \).
Consider the standard basis elements \( x^{\alpha} - x^{\beta} y^{\gamma} \) of \( I \mod J \cdot k[x, y] \) of orders \( |\alpha| \geq 2 \). We can assume that the variables \( x \) are listed in two blocks \( x = (z, u) \), where \( z \) consists of the essential variables of the initial monomials \( x^{\alpha} \); i.e., those \( x \)-variables which occur (to nonzero power) in some \( x^{\alpha} \), \( |\alpha| \geq 2 \). So, for each \( i \), we write \( x^{\alpha_i} = z^{\alpha_i} \) and \( x^{\beta_i} = z^{\xi_i} u^{\eta_i} \). Set \( e_i := |\alpha_i| - |\xi_i| \).

Each \( x_j = 0 \) is a codimension one orbit closure of \( \mathbb{A}^n \) that intersects \( M \) (respectively, \( N \)) transversely in a codimension one \( T_M \) (respectively, \( T_N \)) orbit closure. Define \( P \subset N \) by \( z = 0 \), set \( e := \prod e_i \), and let \( \mathcal{H} \subset \mathcal{O}_M \) denote the ideal generated by the monomials \( (u^{\eta_i})^{\prod_{j \neq i} e_j} \). Then \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \) is a marked monomial ideal.

Let \( a \in X \) be the distinguished point. Then \( H := H_{X,a} \) is the maximum Hilbert-Samuel function of \( X \). It follows from Theorems 7.1 and 7.3 that a sequence of blowings-up (8.1) of \( M \) is \( H \)-permissible if and only if it is permissible for \( \mathcal{H} \). Moreover, if (8.1) is a resolution of singularities of \( \mathcal{H} \), then \( S_H(X_{i+1}) = 0 \).

**Theorem 8.5.** Let \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \) be a marked monomial ideal. Then \( \mathcal{H} \) admits a resolution of singularities.

We will prove this theorem in §8.4. In §8.3, we reduce Theorem 8.1 to Theorem 8.5. In these theorems, \( M = M(\Sigma) \) denotes a smooth toric variety over a perfect field \( k \), corresponding to a fan \( \Sigma \), and \( X \) denotes a closed toric subvariety of \( M \).

**Remark 8.6.** Theorem 8.5 in the special case \( P = N \) and \( e = 1 \) is the principalization theorem for a marked monomial ideal (cf. [BM5, V, W]). The principalization theorem for a toric (or binomial) ideal follows easily from Theorem 8.5 (without using the Hilbert-Samuel function and Theorems 7.1, 7.3), and implies a weaker version of the main theorem 1.1 (without condition (3) or normal flatness), as in [EnV, W].

**8.2. Reduction to resolution of singularities of a marked monomial ideal.**

**Lemma 8.7.** Theorem 8.5 implies Theorem 8.1.

**Proof.** Assume Theorem 8.5. Let \( H \) denote the maximal Hilbert-Samuel function of \( X \). As in Section 2, we cover \( M \) by the affine toric varieties \( U_{\sigma} \) corresponding to the maximal cones \( \sigma \in \Sigma \). Set \( X_{\sigma} := X|U_{\sigma} \). For each maximal cone \( \sigma \), let \( \mathcal{H}_{\sigma} = (U_{\sigma}, N_{\sigma}, P_{\sigma}, \mathcal{H}_{\sigma}, e_{\sigma}) \) denote the marked monomial ideal as defined in Example 8.4 for \( X_{\sigma} \subset U_{\sigma} \). Number the (finitely many) maximal cones: \( \sigma^{(1)}, \sigma^{(2)}, \ldots \).

By Theorem 8.5, there exists a resolution of singularities of \( \mathcal{H}_{\sigma^{(1)}} := \mathcal{H}_{\sigma^{(1)}} \). By Lemmas 7.4, 7.9 (applied recursively), there is an \( H \)-permissible
sequence (8.1) of blowings-up of $M$ such that, over $U_{\sigma(1)}$, this sequence restricts to the preceding resolution of singularities of $H^{(1)}$.

This sequence of blowings-up of $M$, restricted to $U_{\sigma(2)}$, is permissible for $H_{\sigma(2)}$. Let $H^{(2)}$ denote the transform of $H_{\sigma(2)}$ by this sequence.

We can now resolve the singularities of $H^{(2)}$ using Theorem 8.5. As before, the sequence of blowings-up involved is obtained by restriction of an $H$-permissible sequence of blowings-up of $M^{(1)} := M_{t+1}$. Each centre of blowing-up has empty intersection with the inverse image of $U_{\sigma(1)}$.

Now let $H^{(3)}$ denote the transform of $H_{\sigma(3)}$ and continue in the same way . . .

\[ \square \]

8.3. Resolution of singularities of a marked monomial ideal.

Definition 8.8. Sum of marked monomial ideals. Consider marked monomial ideals $H_1 = (M, N, P, H_1, e_1)$ and $H_2 = (M, N, P, H_2, e_2)$. We define

\[ H_1 + H_2 = (M, N, P, H_1^{e_1} + H_2^{e_2}, e_1e_2) \]

It is easy to check the following lemma.

Lemma 8.9.

1. $\text{supp}(H_1 + H_2) = \text{supp}H_1 \cap \text{supp}H_2$.

2. A sequence of blowings-up of $M$ is permissible for $H_1 + H_2$ if and only if it is permissible for both $H_1$ and $H_2$.

3. The transforms by such a sequence satisfy $(H_1 + H_2)' = H_1' + H_2'$.

Proof of Theorem 8.5. The proof is by induction on $\text{dim } P$. The assertion is trivial in the case $\text{dim } P = 0$.

Step 1. Resolution of singularities of a marked monomial ideal $H = (M, N, P, H, e)$ of maximal order. Let $\sigma$ denote a maximal cone of $\Sigma$, and set $H_\sigma = H|U_\sigma := (U_\sigma, N_\sigma, P_\sigma, H_\sigma, e)$, where $N_\sigma := N \cap U_\sigma$, $P_\sigma := P \cap U_\sigma$, $H_\sigma := H|U_\sigma$. Then $H_\sigma$ is generated by monomials in $u$, where $u$ denotes a certain block of the affine coordinates which simultaneously have only normal crossings with $N_\sigma$, and none of which vanish on $P_\sigma$ (cf. Example 8.4). By Remark 7.8 or Lemma 7.9 (with $X = N$), any permissible sequence of blowings-up for $H_\sigma$ is the restriction of a permissible sequence of blowings-up for $H$.

Of course, $\text{supp}H \cap U_\sigma = \emptyset$ unless $H_\sigma$ is generated by monomials of degree $\geq e$. In the latter case, $a_\sigma \in \text{supp}H$, where $a_\sigma$ denotes the distinguished point of $N_\sigma$.

Suppose that $\text{supp}H \cap U_\sigma \neq \emptyset$. Then $\mu_{a_\sigma}(H_\sigma) = e$, so that $H_\sigma$ is not generated by monomials all of degree $> e$. Say that $H_\sigma$ is generated by monomials $u^{e_1}$, $|\xi| \geq e$. Write $u = (z, w)$, where $z$ consists of the
"essential variables" – the variables each of which occur (with nonzero power) in some monomial \( u^{\xi} \) with \( |\xi| = e \).

For each \( i \), write \( u^{\xi_i} = z^{n_i} w^{\xi_i} \). Define \( Q_\sigma \subset P_\sigma \) by \( z = 0 \) ("maximal contact subspace") , and let \( \mathcal{C}(\mathcal{H}_\sigma) \) denote the marked monomial ideal

\[
(U_\sigma, N_\sigma, Q_\sigma, \mathcal{C}(\mathcal{H}_\sigma), e_\sigma) = \sum_i (U_\sigma, N_\sigma, Q_\sigma, (w^{\xi_i}), e - |\eta_i|).
\]

Then \( supp \mathcal{H}_\sigma = supp \mathcal{C}(\mathcal{H}_\sigma) \subset Q_\sigma \), a blowing-up of \( U_\sigma \) is permissible for \( \mathcal{H}_\sigma \), if and only if it is permissible for \( \mathcal{C}(\mathcal{H}_\sigma) \), and the transforms \( \mathcal{H}_\sigma' \) and \( \mathcal{C}(\mathcal{H}_\sigma)' \) by a permissible blowing-up satisfy \( \mathcal{C}(\mathcal{H}_\sigma)' = \mathcal{C}(\mathcal{H}_\sigma') \).

Therefore, a sequence of blowings-up of \( U_\sigma \) is permissible for \( \mathcal{H}_\sigma \), if and only if it is permissible for \( \mathcal{H}_\sigma' \).

In particular, after any sequence of permissible blowings-up for \( \mathcal{H}_\sigma \), writing \( \mathcal{H}_\sigma' \) and \( \mathcal{C}(\mathcal{H}_\sigma)' \) for the transforms, we conclude that \( \mathcal{H}_\sigma' = supp \mathcal{C}(\mathcal{H}_\sigma)' \subset Q'_\sigma \), and \( \mathcal{H}_\sigma', \mathcal{C}(\mathcal{H}_\sigma)' \) have the same sequences of permissible blowings-up. Moreover (as above), any sequence of permissible blowings-up extends to a sequence of permissible blowings-up for \( \mathcal{H}_\sigma' \).

By induction on dimension, since \( dim Q_\sigma < dim P_\sigma \), there is a sequence of blowings-up of \( M \), permissible for \( \mathcal{H}_\sigma \), which restricts to a resolution of singularities of \( \mathcal{H}_\sigma \).

We can now argue as in the proof of Lemma 8.7 to complete the proof of Step 1: Number the maximal cones of \( \Sigma : \sigma^{(1)}, \sigma^{(2)}, \ldots \). As above, there is a sequence of blowings-up of \( M \), permissible for \( \mathcal{H}^{(1)} := \mathcal{H} \), which restricts to a resolution of singularities of \( \mathcal{H}_{\sigma^{(1)}} \). Let \( \mathcal{H}^{(2)} \) denote the transform of \( \mathcal{H}^{(1)} \) by this sequence; likewise, \( (\mathcal{H}_{\sigma^{(2)}})^{(2)} \) and \( \mathcal{C}(\mathcal{H}_{\sigma^{(2)}})^{(2)} \), for each \( \sigma \). Write \( \pi^{(1)} : M^{(2)} \to M^{(1)} = M \) for the composite of the sequence of permissible blowings-up.

Suppose that \( supp \mathcal{H}^{(2)} \cap (\pi^{(1)})^{-1}(U_{\sigma^{(2)}}) \neq \emptyset \). Then \( \mathcal{H}^{(2)} \) has maximal order, \( supp \mathcal{H} \cap U_{\sigma^{(2)}} \neq \emptyset \), \( (\mathcal{H}_{\sigma^{(2)}})^{(2)} \) and \( \mathcal{C}(\mathcal{H}_{\sigma^{(2)}})^{(2)} \) have the same permissible sequences, and each permissible sequence for \( (\mathcal{H}_{\sigma^{(2)}})^{(2)} \) extends to a permissible sequence for \( \mathcal{H}^{(2)} \) whose centres are disjoint from the inverse images of \( U_{\sigma^{(2)}} \). Therefore, there is a further permissible sequence of blowings-up, after which \( supp \mathcal{H}^{(3)} \) is disjoint from the inverse image of \( U_{\sigma^{(1)}} \cup U_{\sigma^{(2)}} \). Etc.

Step 2. Resolution of singularities of a marked monomial ideal \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \), in general.

(a) Reduction to the monomial case. We can factor \( \mathcal{H} \) as \( \mathcal{H} = \mathcal{D}(\mathcal{H}) \cdot \mathcal{Q}(\mathcal{H}) \), where \( \mathcal{D}(\mathcal{H}) \) is a product of principal ideals defining codimension one orbit closures transverse to \( N \) (not containing \( P \)), and \( \mathcal{Q}(\mathcal{H}) \) is not divisible by any such principal ideal (nor, then, by any principle ideal
defining a codimension one orbit closure). Define
\[ \nu_H := \max\{\mu_b(Q(H)) : b \in \text{supp } H\}, \]
and set
\[ Q(H) := (M, N, P, Q(H), \nu_H), \]
\[ D(H) := (M, N, P, D(H), e - \nu_H). \]

We define the \textit{companion ideal} of \( H \) as the marked monomial ideal
\[ G(H) = \begin{cases} Q(H) + D(H), & \text{if } \nu_H < e, \\ Q(H), & \text{if } \nu_H \geq e. \end{cases} \]
(cf. [BM3, 4.23], [BM5, §2.6], [W, Def. 3.0.10]). Then \( G(H) \) is a marked monomial ideal of maximal order.

By the \textit{monomial case}, we mean that \( H = D(H) \).

By Lemma 8.8,
\[ \text{supp } G(H) = \text{supp } Q(H) \bigcap \text{supp } D(H) = \text{supp } Q(H) \bigcap \text{supp } H, \]
any permissible sequence of blowings-up for \( G(H) \) is permissible for \( H \),
and the transforms \( H', G(H)', Q(H)', D(H)' \) by such a sequence satisfy

(8.2)
\[ \text{supp } G(H)' = \text{supp } Q(H)' \bigcap \text{supp } D(H)' = \text{supp } Q(H)' \bigcap \text{supp } H'. \]

Note that if \( H \) has maximal order, then \( H \) and \( G(H) \) have the same permissible sequences and the same supports throughout such a sequence.

By Step 1 above, there is a resolution of singularities of \( G(H) \). The sequence of blowings-up involved is permissible for \( H \). Let \( H_1 \) denote the transform of \( H \) by this sequence. It follows from (8.2) that \( \nu_{H_1} < \nu_H \). We can now repeat the procedure: let \( H_2 \) be the transform of \( H_1 \) by a resolution of singularities of \( G(H_1) \), \ldots The process terminates after a finite number of steps, when we arrive at a marked monomial ideal \( H_m \) with either \( \text{supp } H_m = \emptyset \) or \( \nu_{H_m} = 0 \). In the former case, we have a resolution of singularities of \( H_m \), and in the latter it is enough to resolve the singularities of \( D(H_m) \); i.e., we have reduced to the monomial case.

(b) \textit{Monomial case}. \( H = (M, N, P, \mathcal{H}, e) \), where \( \mathcal{H} = D(H) \). Suppose that \( C \) is an irreducible component of \( \text{supp } H \). Then \( C \) is smooth, and \( C = N \cap D \), where \( D \) is the smallest orbit closure of \( M \) containing \( C \). Clearly, if \( H' = (M', N', P', \mathcal{H}', e) \) is the transform of \( H \) by the blowing-up with centre \( D \), then \( \mathcal{H}' = \mathcal{D}(H') \). We will show that, after finitely many such blowings-up, the transform of \( H \) has empty support.
The proof is the same as that of desingularization of a toric hypersurface (with $D(\mathcal{H})$ playing the part of $x^+_{k+}$ in §5.1 above), or that of [BM3, Case (b), p. 260], [BM5, S3.3(4)], [W, Step 2b, p. 20]: Let $\sigma$ denote a maximal cone of $\Sigma$, and define $\mathcal{H}_\sigma$ as in Step 1 above. Then $\mathcal{H}_\sigma$ is generated by a monomial $u^\omega = u_1^{\omega_1} \cdots u_q^{\omega_q}$, where $u = (u_1, \ldots, u_q)$ is a block of the affine coordinates, as in Step 1. If $C \cap U_\sigma \neq \emptyset$, then $C \cap U_\sigma \subset P_\sigma$ can be described as $u_{j_1} = \cdots = u_{j_l} = 0$, where $1 \leq j_k \leq q$ ($k = 1, \ldots, l$),
\[
\omega_{j_1} + \cdots + \omega_{j_l} \geq e,
\omega_{j_1} + \cdots + \hat{\omega}_{j_k} + \cdots + \omega_{j_l} < e,
\]
where $\hat{\omega}_{j_k}$ means that $\omega_{j_k}$ is deleted). After blowing up with centre $D$, the strict transform $(N_\sigma)'$ lies in the union of the affine charts corresponding to the variables $u_{j_k}$. In the chart corresponding to $u_{j_k}$, $\mathcal{H}'_\sigma$ is generated by $u'^\omega$, where
\[
\omega'_j = \omega_j, \quad j \neq j_k,
\omega'_{j_k} = \omega_{j_1} + \cdots + \omega_{j_l} - e < \omega_{j_k};
\]
i.e., $|\omega'| < |\omega|$. Therefore, the support of the transform of $\mathcal{H}$ is disjoint from the inverse image of $U_\sigma$ after at most $|\omega| - e + 1$ such blowings-up over $U_\sigma$. □

8.4. Transformation to normal crossings. The following result is needed to complete our proof of Theorem 1.1.

**Theorem 8.10.** Let $X \hookrightarrow M$ denote an equivariant embedding of smooth toric varieties over a perfect field $k$. Let $\Theta$ denote a set of smooth invariant hypersurfaces in $M$. Then there is a finite sequence of blowings-up (1.1) such that:

1. The centre $D_j$ of each blowing-up $\pi_{j+1}$ is a smooth invariant subvariety of $M_j$.
2. Set $X_0 = X$ and $\Theta_0 = \Theta$. For each $j = 0, \ldots, t$, let $X_{j+1}$ denote the strict transform of $X_j$ by $\pi_{j+1}$, and let $\Theta_{j+1}$ denote the set of strict transforms of elements of $\Theta_j$. Then, for each $j = 0, \ldots, t$, $C_j := D_j \cap X_j$ is a smooth invariant subvariety of $X_j$, $D_j$ is the smallest invariant subvariety of $M_j$ containing $C_j$, and each component of $C_j$ is the intersection of $X_j$ and certain components of elements of $\Theta_j$.
3. $X_{t+1}$ is disjoint from all elements of $\Theta_{t+1}$.

The proof of Theorem 8.10 parallels the preceding part of this section. If $a \in X_j$, let $s(a)$ denote the number of elements of $\Theta_j$ containing
a. Let \( s := \max_{a \in X} s(a) \). For any sequence of blowings-up (1.1) satisfying conditions (1) and (2) of Theorem 8.9, let \( S_\Theta(X_j) := \{ a \in X_j : s(a) = s \} \). Then, for each \( j \), \( S_\Theta(X_j) \) has only normal crossings. As in Lemma 7.9, if \( E \) is an orbit closure of \( M_j \) and \( C \) is an irreducible component of \( S_\Theta(X_j) \cap E \), then \( C \) is smooth and \( X_j \cap D = C \), where \( D \) is the smallest orbit closure of \( M \) containing \( C \).

A sequence of blowings-up (1.1) satisfying conditions (1) and (2) of Theorem 8.10 will be called \( \Theta \)-permissible. Theorem 8.10 is an obvious consequence of the following.

**Theorem 8.11.** There is a \( \Theta \)-permissible sequence of blowings-up (1.1), such that \( S_\Theta(X_{t+1}) = \emptyset \).

**Proof.** First consider \( M \) affine, as in Example 8.4. Since \( X \) is smooth, \( N = X \) and each standard basis element \( x^{\alpha_i} - x^{\beta_i} y^{\gamma_i} \) of \( I \mod J \cdot k[x, y^z] \) is of order 1; i.e., \( x^{\alpha_i} = x_{j(i)} \) for some \( j(i) \). We can assume that the variables are listed in two blocks \( x = (z, u) \), where \( z \) consists of the variables \( x_{j(i)} \), and each \( x^{\beta_i} \) is a monomial in \( u \) (that we denote \( u^{\beta_i} \)).

Set \( P = N \). Each \( H \in \Theta \) is defined by \( z_k = 0 \), for some \( k \), or by \( u_l = 0 \), for some \( l \). Let \( \mathcal{H} \subset \mathcal{O}_M \) denote the ideal generated by \( u^{\beta_k} \) (for all such \( k \)) and \( u_l \) (for all such \( l \)). Then \( \mathcal{H} = (M, N, P, \mathcal{H}, 1) \) is a marked monomial ideal.

A sequence of blowings-up (8.1) of \( M \) is \( \Theta \)-permissible if and only if it is permissible for \( \mathcal{H} \). Moreover, if (8.1) is a resolution of singularities of \( \mathcal{H} \), then \( S_\Theta(X_{t+1}) = \emptyset \).

Theorem 8.11 follows from Theorem 8.5 as in §8.2 above. \( \square \)

9. **Algorithm for canonical equivariant desingularization**

Our proof of Theorem 1.1 in Section 8 provides an algorithm for equivariant embedded desingularization of a toric variety, but allows certain noncanonical choices for the centres of blowing up. The purpose of this section is to give an algorithm for canonical equivariant resolution of singularities, as well as an invariant that tracks the progress of the algorithm towards desingularization. Both the algorithm and the invariant can be described in a very simple way. The proofs in this section, however, depend on techniques of [BM3] or [BM5] that we do not redevelop, even though they can be considerably simplified for toric (or binomial) varieties (over an arbitrary perfect field). We do not see how to obtain the results here using Wlodarczyk’s approach to canonicity [W], although we use his language of marked ideals.
The algorithm presented here corresponds to Theorem 8.1, and thus to Theorem 1.1 with condition (4) replaced by \((4')\) \(X_{t+1}\) is smooth. (See Theorem 8.1 ff.) The same algorithm can be run for Theorem 8.10 to get all the conditions of Theorem 1.1.

In §9.1, we describe an algorithm for choosing a centre of blowing up locally, that is implicit in the proof of Theorem 8.5. The local algorithm is canonical modulo an ordering of the codimension one orbit closures of \(M\).

Our proofs of Lemma 8.7 and of Step 1 in Theorem 8.5 use the orbit structure of the torus action (in particularly, Lemma 7.9) to globalize the local choice of centre of blowing up. In §9.2, we show how to replace this argument by a global algorithm that is canonical modulo an ordering of the codimension one orbit closures.

In §9.3, we give an algorithm for canonical equivariant embedded desingularization (Addendum 1.2), by a simple variation of the preceding construction: The only change is that, in Step 2 of Theorem 8.5, the marked monomial ideal \(D(H)\) and thus the companion ideal \(G(H)\) are defined using only those codimension one orbit closures which arise as (reduced) components of the exceptional divisors. At any stage of the resolution process, the “exceptional hypersurfaces” (the strict transforms of the exceptional divisors of all previous blowings-up) are ordered by their “years of birth”. (See Definitions 9.1.)

Let \(M = M(\Sigma)\) denote a smooth toric variety over a perfect field \(k\), as in Section 8.

**Definitions 9.1.** Ordering of invariant hypersurfaces and exceptional divisors. Consider any totally ordered subset \(\mathcal{E}\) of the set of smooth invariant hypersurfaces in \(M\); say \(\mathcal{E} = \{H_1, \ldots, H_q\}\). Associate to any subset \(I\) of \(\mathcal{E}\), a finite sequence \(\delta(I) = (\delta_1, \ldots, \delta_q)\), where, for each \(i = 1, \ldots, q\), \(\delta_i = 0\) if \(H_i \not\in I\), and \(\delta_i = 1\) if \(H_i \in I\). We totally order the subsets \(I\) of \(\mathcal{E}\) using the lexicographic order \(\text{lex}\) of the sequences \(\delta(I)\); i.e., \(I_1 < I_2\) means that \(\text{lex} \delta(I_1) < \text{lex} \delta(I_2)\). (For example, \(\emptyset \leq I\), for all \(I \subset \mathcal{E}\).)

Suppose that \(\pi : M' \to M\) is a blowing-up whose centre \(D\) is a smooth invariant subset of \(M\) (\(\text{codim} D \geq 2\)). Set \(\mathcal{E}' := \{H'_1, \ldots, H'_q, H'_{q+1}\}\), where \(H'_i\) is the strict transform of \(H_i\), \(i = 1, \ldots, q\), and \(H'_{q+1} = \pi^{-1}(D)\).

For example, take \(\mathcal{E} = \emptyset\) and consider any sequence of blowings-up (1.1) whose centres are smooth invariant subvarieties. Set \(\mathcal{E}_0 := \mathcal{E}\) and \(\mathcal{E}_{j+1} := \mathcal{E}'_j\), \(j = 0, \ldots, t\). Thus each \(\mathcal{E}_j\) is the set of strict transforms (in \(M_j\)) of the exceptional divisors \(\pi_i^{-1}(D_{i-1})\), \(i = 1, \ldots, j\), ordered by “year of birth”.
9.1. **Local calculation of the centre of blowing up.** Consider the affine case \( X \subset M \subset \mathbb{A}^n \), as in Example 8.4. Let \( \mathcal{E} \) denote the set of codimension one orbit closures of \( M \), with a fixed total ordering. Let \( a \in X \) denote the distinguished point. Let \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \) be the marked monomial ideal defined in Example 8.4. (In the language of BM3, we would call \( \mathcal{H} \) a “presentation” of the Hilbert-Samuel function of \( X \) at \( a \).)

We will extract from the proof of Theorem 8.1, an algorithm for choosing a centre of blowing up \( D \), as well as a local “invariants” \( \text{inv}(a) \), \( \mu(a) \) and \( J(a) \). \( \text{inv}(a) \) is a finite sequence \((H_{X,a}, \nu_2, \ldots, \nu_{q+1})\), where \( \nu_2, \ldots, \nu_q \) are positive rational numbers, \( \nu_{q+1} = 0 \) or \( \infty \), and \( q \leq \dim N \) (the minimal embedding dimension of \( X \)); \( \mu(a) \) is a rational number defined in the case that \( \nu_{q+1} = 0 \); \( J(a) \) is a subset of \( E \).

Recall that a sequence of blowings-up (8.1) is \( \mathcal{H} \)-permissible if and only if it is permissible for \( \mathcal{H} \). Moreover, if (8.1) is a resolution of singularities of \( \mathcal{H} \), then \( S_{\mathcal{H}}(X_{t+1}) = \emptyset \).

Therefore, we consider resolution of singularities of an arbitrary marked monomial ideal \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \) in the affine case \( M \subset \mathbb{A}^n \).

We will define a centre of blowing up \( D_{\mathcal{H}} \), as well as “invariants” \( \text{inv}_{\mathcal{H}}(a) \), \( \mu_{\mathcal{H}}(a) \), \( J_{\mathcal{H}}(a) \), by induction on \( \dim P \).

Then, to define \( \text{inv}(a) \) and \( D \) for \( X \subset M \), we take \( \mathcal{H} \) as in Example 8.4 and set \( D := D_{\mathcal{H}} \),

\[
\text{inv}(a) := (H_{X,a}, 1, \ldots, 1, \text{inv}_{\mathcal{H}}(a)),
\]

where there are codim \((P \subset N) - 1\) ones, \( \mu(a) := \mu_{\mathcal{H}}(a) \), and \( J(a) := J_{\mathcal{H}}(a) \). (codim \((P \subset N) \) means the codimension of \( P \) in \( N \).)

Let \( \mathcal{H} = (M, N, P, \mathcal{H}, e) \) denote a marked monomial ideal, where \( M \subset \mathbb{A}^n \).

1. **Monomial case \( \mathcal{H} = D(\mathcal{H}) \).** Define \( \text{inv}_{\mathcal{H}}(a) := 0 \) and \( \mu_{\mathcal{H}}(a) = \mu_a(\mathcal{H})/e \). By Step 2(b) in §8.3, each irreducible component \( Z \) of \( \text{supp} \mathcal{H} \) is of the form \( Z = N \cap D \), where \( D := \{ H_i \in \mathcal{E} : Z \subset H_i \} \) (the smallest orbit closure of \( M \) containing \( Z \)). Write \( Z = Z_I \), where \( I := \{ H_i \in \mathcal{E} : Z \subset H_i \} \).

Consider a blowing up \( \pi : M' \to M \) with centre \( D = D_I \) corresponding to a component \( Z_I \) of \( \text{supp} \mathcal{H} \). Let \( a' \) denote a distinguished point of the strict transform \( N' \) (in an affine chart of \( M' \), as described in Step 2(b)). If \( a' \in \text{supp} \mathcal{H}' \), then \( 1 \leq \mu_{\mathcal{H}}(a') \leq \mu_{\mathcal{H}}(a) - 1/e \).

Take \( J_{\mathcal{H}}(a) := \max \{ I : Z_I \text{ is a component of } \text{supp} \mathcal{H} \} \), and set \( D_{\mathcal{H}} := D_{J_{\mathcal{H}}(a)} \).

2. If \( \mathcal{H} \neq D(\mathcal{H}) \), we consider two cases:
(a) $\mathcal{H}$ is of maximal order. (Then $\mathcal{H} \neq \emptyset$.) Define $\mathcal{C}(\mathcal{H}) = (M, N, Q, \mathcal{C}(\mathcal{H}), e_{\mathcal{C}(\mathcal{H})})$ as in Step 1 of the proof of Theorem 8.5. (Then $\mathcal{C}(\mathcal{H})$ is not of maximal order.) A resolution of singularities of $\mathcal{C}(\mathcal{H})$ is a resolution of singularities of $\mathcal{H}$.

Since $\dim Q < \dim P$, $\inv_{\mathcal{C}(\mathcal{H})}(a)$, $\mu_{\mathcal{C}(\mathcal{H})}(a)$, $J_{\mathcal{C}(\mathcal{H})}(a)$, and a centre of blowing up $D_{\mathcal{C}(\mathcal{H})}$ are determined by induction. Set $D_{\mathcal{H}} := D_{\mathcal{C}(\mathcal{H})}$, $\inv_{\mathcal{H}}(a) := (1, \ldots, 1, \inv_{\mathcal{C}(\mathcal{H})}(a))$ (where there are $\dim(P - 1)$ ones), $\mu_{\mathcal{H}}(a) := \mu_{\mathcal{C}(\mathcal{H})}(a)$ (if $\inv_{\mathcal{C}(\mathcal{H})}(a)$ ends with 0), and $J_{\mathcal{H}}(a) := J_{\mathcal{C}(\mathcal{H})}(a)$.

(b) $\mathcal{H}$ is not of maximal order. If $\mathcal{H} = 0$ (zero ideal), set $\inv_{\mathcal{H}}(a) := \infty$, $J_{\mathcal{H}}(a) := \emptyset$, and take $D_{\mathcal{H}} :=$ the smallest orbit closure $D$ of $M$ such that $P = N \cap D$. In this case, if $\pi : M' \to M$ is the blowing-up with centre $D$, then $\supp H' = \emptyset$.

On the other hand, if $\mathcal{H} \neq 0$, then we construct the companion ideal $\mathcal{G}(\mathcal{H})$ as in Step 2(a) of the proof of Theorem 8.5. Then $\mathcal{G}(\mathcal{H})$ is of maximal order. Set

$$\inv_{\mathcal{H}}(a) := \left(\frac{\nu_{\mathcal{H}}}{e}, \inv_{\mathcal{C}(\mathcal{H})}(a)\right),$$

$\mu_{\mathcal{H}}(a) := \mu_{\mathcal{C}(\mathcal{H})}(a)$ (if $\inv_{\mathcal{C}(\mathcal{H})}(a)$ ends with 0), $J_{\mathcal{H}}(a) := J_{\mathcal{C}(\mathcal{H})}(a)$, and take $D_{\mathcal{H}} := D_{\mathcal{C}(\mathcal{H})}$.

This completes the inductive definition of $\inv(a)$, $\mu(a)$, $J(a)$, and the centre of blowing up $D$. After any sequence of blowings-up (1.1) satisfying conditions (1) and (2) of Theorem 1.1, we can repeat the algorithm above in any affine chart of $M_{t+1}$, using the induced order of $E_{t+1}$, where $E_0 = E$ (Definitions 9.1).

The invariants $\inv$, $\mu$ and $J$ and the centre of blowing up defined above depend only on the local equivariant isomorphism class of $X \subset M$ and the ordering of $E$ [BM3, BM5]. The following theorem is easy to prove using the construction above. (See, for example, [BM3, p.260], [BM5 §3.3].)

**Theorem 9.2.** Let $a \in X \subset M \subset \mathbb{A}^n$, and let $\pi : M' \to M$ denote the blowing-up with centre $D$ (as above). For every distinguished point $a'$ of $X' \subset M'$, either $\inv(a') < \inv(a)$, or $\inv(a)$ ends with 0, $\inv(a') = \inv(a)$ and $\mu(a') < \mu(a)$.

**Remarks 9.3.** The invariant $\inv(a)$ is a simplification of the general desingularization invariant $\inv_X(a)$ of [BM3, BM5]. The latter is a finite sequence $(H_X, s_1(a), \nu_2(a), s_2(a), \ldots, s_q(a), \nu_{q+1}(a))$, where the terms $s_i(a)$ count certain exceptional hypersurfaces that do not necessarily have normal crossings with respect to the maximal contact subspaces. In the case of toric varieties, the normal crossings condition
is automatic, and the terms $s_i(a)$ (reflecting blowings-up needed to obtain normal crossings) are unnecessary. See [BM3, §1] for a discussion of this point. Example 1.3 illustrates the effect on the complexity of the desingularization algorithm.

The construction of $C(H)$ (Step 1 in the proof of Theorem 8.5, and Case 2(a) above) allows passage to a maximal contact subspace $Q$ of arbitrary codimension in $P$, whereas the general construction of [BM3], [BM5], [W] involves inductive steps of codimension 1 – the extra $\nu_i$ terms thus involved in the general definition of $\text{inv}_X$ are the sequences of ones appearing in the definition of $\text{inv}$ above.

9.2. Global algorithm. Let $E$ denote the set of codimension one orbit closures of $M = M(\Sigma)$, with any fixed total ordering. Let $X \subset M$ denote a toric subvariety of $M$. We cover $M$ by the affine toric varieties $U_\sigma$ corresponding to the maximal cones $\sigma$ of $\Sigma$. For each maximal cone $\sigma$, set $X_\sigma := X|_{U_\sigma}$ and let $a_\sigma$ denote the distinguished point of $X_\sigma$.

For each $\sigma$, we consider the invariants $\text{inv}(a_\sigma)$, $J(a_\sigma)$, and the centre of blowing up $D_\sigma$ constructed as in §9.1 for $X_\sigma \subset U_\sigma$.

**Theorem 9.4.** There is a unique smooth closed invariant subspace $D$ of $M$ such that $D|U_\sigma = D_\sigma$ for every maximal cone $\sigma \in \Sigma$ which realizes the maximum (lexicographic) value of $(\text{inv}(a_\sigma), J(a_\sigma))$ (over the maximal cones $\sigma$).

The invariant subspace $D$ of the theorem is the smallest closed invariant subspace of $M$ containing the centre of blowing up that is prescribed by the desingularization algorithm of [BM3], [BM5] in the simplified version sufficient for toric varieties. Locally, the algorithm works as in §9.1 above; semicontinuity properties of the local invariant guarantee that its maximum locus is a global smooth closed subspace.

**Desingularization algorithm.** Set $M_0 = M$, $X_0 = X$, and $E_0 = E$. Define a sequence of blowings-up (1.1) by applying Theorem 9.4 successively to each strict transform $X_j \subset M_j$ (with the set $E_j$ ordered as in Definitions 9.1), in order to define the centre $D_j$ of the next blowing-up $\pi_{j+1}$.

The algorithm terminates when $X_{t+1}$ is smooth. (It terminates because of Theorem 9.2.) In order to obtain the normal crossings condition (4) of Theorem 1.1, we rerun the preceding algorithm in the context of Theorem 8.10 (rather than Theorem 8.1, as above).

9.3. Canonical equivariant embedded desingularization. The desingularization algorithm of the preceding subsection is canonical modulo the ordering of the set $E$ of codimension one orbit closures of $M$. We
can obtain Addendum 1.2 by a simple change in the algorithm: Take $E_0 = \emptyset$, and define $E_j$ over any sequence of blowings-up (1.1) satisfying conditions (1) and (2) of Theorem 1.1, as in Definitions 9.1; i.e., each $E_j$ is the set of strict transforms in $M_j$ of the exceptional divisors $\pi^{-1}_i(D_{i-1})$, $i = 1, \ldots, j$, ordered by year of birth.

We run the algorithm exactly as before, with one change: At any stage $j$ of the process, $D(H)$ is a product of principal ideals defining codimension one orbit closures (transverse to $N$ and not containing $P$, in the notation of Step 2(a) of the proof of Theorem 8.5) which are irreducible components of the elements of the exceptional set $E_j$. For example, in “year zero”, $E_0 = \emptyset$, so that, if $H = (M, N, P, H, e)$, then $G(H) = Q(H) = (M, N, P, H, \nu_H)$. (See Theorem 8.5, Step 2(a).)

All the constructions and proofs in Sections 8 and 9 (above) carry over with this change. The effect of the change is that extra blowings-up are introduced to “relabel” orbit closures as exceptional divisors before they can be factored out to define the companion ideals. There is an important effect on the complexity of the desingularization algorithm – see Example 1.3.

10. Toroidal and binomial varieties

**Definitions 10.1.** Let $X \hookrightarrow M$ be an embedding of algebraic varieties over $k$, where $M$ is smooth. We will say that $X \hookrightarrow M$ is:

(1) *locally toric* if, for every (closed) point $a \in X$, there is an open neighbourhood $U$ of $a$ in $M$, an (equivariant) embedding of affine toric varieties $Y \hookrightarrow V \subset \mathbb{A}^n_k$ (where $V$ is smooth), and an étale morphism $\eta : U \to V$ such that $\eta(X|U) = Y|\eta(U)$.

(2) *toroidal* if there is a collection $E = \{H_i\}$ of smooth irreducible hypersurfaces in $M$ having only normal crossings, and (1) holds with the additional condition that $\eta^{-1}(T_Y) = U \cup H_i$.

(3) *locally binomial* if we weaken (1) by assuming only that $Y \hookrightarrow V \subset \mathbb{A}^n_k$ is an embedding of affine binomial varieties (§6.1).

(4) *binomial* if there is a collection $E = \{H_i\}$ of smooth irreducible hypersurfaces in $M$ having only normal crossings, and (3) holds with the additional condition that $\eta^{-1}(V \cap \mathbb{T}^n) = U \cup H_i$.

In each case, the étale morphism $\eta$ involved will be called a *local model*.

Let $X \hookrightarrow M$ be a toroidal (or binomial) embedding, and let $E = \{H_i\}$ denote the associated collection of divisors.

**Definition 10.2.** A smooth subvariety $D$ of $M$ will be called *combinatorial* if
(1) each irreducible component of $D$ is an irreducible component of an intersection of divisors $H_i$;
(2) for every local model $\eta : U \to V$, there is a smooth invariant subset $D_\eta$ of $V$ such that $D \cap U = \eta^{-1}(D_\eta)$.

(In analogy with the toric case, we use “invariant subset” of an affine binomial variety $V \subset \mathbb{A}^n$ to mean an intersection of $V$ with a $\mathbb{T}^n$-invariant subset of $\mathbb{A}^n$.)

Consider a blowing-up $\pi : M' \to M$ with combinatorial centre $D$. Let $X'$ denote the strict transform of $X$ by $\pi$. Let $\mathcal{E}'$ denote the collection of smooth irreducible hypersurfaces in $M'$ comprising the strict transforms $H'_i$ of all $H_i \in \mathcal{E}$, together with the irreducible components of $\pi^{-1}(D)$. Then $X' \hookrightarrow M'$ is a toroidal (or binomial) embedding (with respect to $\mathcal{E}'$).

Our analysis of the Samuel stratification in Section 7, and proofs of resolution of singularities in Sections 8, 9 go through for a toroidal (or binomial) embedding over a perfect field $k$, exactly as before, with components of intersections of divisors in $\mathcal{E}$ playing the part of orbit closures in $M$ in the toric case. In particular, we have the following analogue of Theorem 1.1 and Addendum 1.2.

**Theorem 10.3.** Let $X \hookrightarrow M$ (where $M$ is smooth) denote a toroidal (or binomial) embedding over a perfect field $k$. Then there is a finite sequence of blowings-up of $M$

\[(10.1) \quad M = M_0 \leftarrow \pi_1 \leftarrow \cdots \leftarrow \pi_{t+1} M_{t+1},\]

such that:

(1) The centre $D_j$ of each blowing-up $\pi_{j+1}$ is combinatorial.
(2) Denote by $X_j$ the successive strict transforms of $X_0 = X$. Then, for each $j = 0, \ldots, t$, $C_j := D_j \cap X_j$ is smooth and $X_j$ is normally flat along $C_j$.
(3) For each $j = 0, \ldots, t$, either $C_j \subset \text{Sing} X_j$ or $X_j$ is smooth and $C_j \subset X_j \cap E_j$, where $E_j$ denotes the exceptional divisor of $\pi_1 \circ \cdots \circ \pi_j$.
(4) $X_{t+1}$ is smooth and $X_{t+1}, E_{t+1}$ simultaneously have only normal crossings.

Moreover, for every embedding $X \hookrightarrow M$ as above (with collection of divisors $\mathcal{E}$), there is a sequence of blowings-up (10.1) satisfying conditions (1)-(4), with the following property: If $\iota : M' \hookrightarrow M$ is an open toroidal (or binomial) embedding (with respect to $\mathcal{E}$), then $X$ and $X' = \iota^{-1}(X)$ have the same resolution towers over $M'$ (not counting isomorphisms in the sequences of blowings-up).
In the case of a locally toric (or locally binomial) embedding over a perfect field, the general desingularization algorithm of [BM3, BM5] applies to give canonical embedded resolution of singularities, with the additional property that, for every local model \( \eta : U \rightarrow V \), the sequence of strict transforms of \( X \mid U \) is induced by an embedded toric (or binomial) desingularization of \( Y \hookrightarrow V \) (notation of Definitions 10.1). The general algorithm involves additional complexity. (See Section 1 and Remarks 9.3.)

It would be interesting to find good generalizations of the results in this paper to the larger class of “binomial varieties” of [ES] (varieties defined locally by ideals that are generated by polynomials with at most two terms).

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