System of intuitionistic fuzzy differential equations with intuitionistic fuzzy initial values

Ömer Akin and Selami Bayeğ
Department of Mathematics, TOBB Economics and Technology University
Söğütüz Mahallesi, Söğütüz Cd. No:43, 06510 Çankaya/Ankara, Turkey
e-mails: omerakin@etu.edu.tr, sbayeg@etu.edu.tr

Received: 13 December 2017
Accepted: 7 November 2018

Abstract: In this paper, we have studied the system of differential equations with intuitionistic fuzzy initial values under the interpretation of (i,ii)-GH differentiability concepts and Zadeh’s extension principle interpretation. And we have given some numerical examples.

Keywords: Intuitionistic fuzzy sets, Strongly generalized Hukuhara differentiability, Intuitionistic fuzzy initial value problems, Intuitionistic Zadeh’s extension principle.

2010 Mathematics Subject Classification: 03E72.

1 Introduction

Fuzzy set theory was firstly introduced by L. A. Zadeh in 1965 [46]. He defined fuzzy set concept by introducing every element with a function $\mu : X \rightarrow [0, 1]$, called membership function. Later, some extensions of fuzzy set theory were proposed [8, 30, 37]. One of these extensions is Atanassov’s intuitionistic fuzzy set (IFS) theory [8].

In 1983, Atanassov [7] introduced the concept of intuitionistic fuzzy sets and carried out rigorous researches to develop the theory [8–16]. In this set concept, Apart from the membership function, he introduced a new degree $\nu : X \rightarrow [0, 1]$, called non-membership function, such that the sum $\mu + \nu$ is less than or equal to 1. Hence the difference $1 - (\mu + \nu)$ is regarded as degree of hesitation. Since intuitionistic fuzzy set theory contains membership function, non-membership function and the degree of hesitation, it can be regarded as a tool which is more flexible and closer to human reasoning in handling uncertainty due to imprecise knowledge or data.

Intuitionistic fuzzy set and fuzzy set theory have very compelling applications in various fields of science and engineering [1, 3–6, 17, 23, 24, 27–29, 31, 33–36, 38–40, 42, 45].
In literature, there are different approaches for solving fuzzy differential equations. Each method has advantages and disadvantages in the applications. One of the commonly used methods is based on Zadeh’s extension principle. In this method, the fuzzy solution is obtained from the crisp solution by using the well-known Zadeh’s extension principle [22]. However, there is no definition of fuzzy derivative in this approach. Hence some other methods based on fuzzy derivative concept were also proposed and used. One of the earliest methods is Hukuhara differentiability concept [41]. However, this approach has also a weak point which is that the solution becomes fuzzier as time passes by [25]. Hence the length of the support of the fuzzy solution increases. To overcome this disadvantage some methods such as differential inclusions [32] and strongly generalized differentiability concept [19] were coined. The method based on strongly generalized differentiability concept allows us to obtain the solutions with decreasing length of support [18–21]. Hence the drawback of Hukuhara differentiability can be overcome with strongly generalized Hukuhara differentiability concept. Besides, this approach shows to be more favorable in applications [21].

The main goal of this paper is to give solutions to system of differential equations under the special cases of strongly generalized differentiability (GH) concept, (i.e. (i)-, (ii)-GH differentiability) [18–21] and under intuitionistic Zadeh’s extension principle [16].

This paper is prepared as follows. In Section 2, some fundamental definitions and theorems in fuzzy sets and intuitionistic fuzzy sets are given. In Section 3, some definitions and theorems related to (i)-GH and (ii)-GH are extended from fuzzy case to intuitionistic fuzzy case by using the definitions and theorems in Section 2. In Section 4, we study system of differential equation on intuitionistic fuzzy environment under (i)-GH and (ii)-GH differentiability and the intuitionistic Zadeh’s extension principle interpretation. Besides, we give some numerical examples in this section. Finally, we conclude the paper by giving summary and results in Section 6.

2 Preliminaries

Definition 2.1. [8] Let $\mu_A, \nu_A : \mathbb{R}^n \to [0, 1]$ be two functions such that for each $x \in \mathbb{R}^n$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ holds. The set
\[
\tilde{A}^i = \{(x, \mu_A(x), \nu_A(x)) \mid x \in \mathbb{R}^n; \mu_A, \nu_A : \mathbb{R}^n \to [0, 1]\}
\]
is called an intuitionistic fuzzy set in $\mathbb{R}^n$. Here $\mu_A$ and $\nu_A$ are called membership and non-membership functions, respectively.

We will denote set of all intuitionistic fuzzy sets in $\mathbb{R}^n$ by $IF(\mathbb{R}^n)$.

Definition 2.2. [8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. The $\alpha$-cut of $\tilde{A}^i$ is defined as follows:
For $\alpha \in (0, 1]$ $A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\},$
and for $\alpha = 0$
\[
A(0) = cl \left( \bigcup_{\alpha \in [0,1]} A(\alpha) \right).
\]
Definition 2.3. [8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. The $\beta$-cut of $\tilde{A}^i$ is defined as follows:

For $\beta \in (0, 1)$

$$A^\ast(\beta) = \{ x \in \mathbb{R}^n : \nu_A(x) \leq \beta \},$$

and for $\beta = 1$

$$A^\ast(1) = cl \left( \bigcup_{\beta \in [0, 1)} A^\ast(\beta) \right).$$

Definition 2.4. [8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. For $\alpha$ and $\beta \in [0, 1]$ with $0 \leq \alpha + \beta \leq 1$, the set

$$A(\alpha, \beta) = \{ x \in \mathbb{R}^n : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \}$$

is called $(\alpha, \beta)$-cut of $\tilde{A}^i$

Theorem 2.1. [8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. Then

$$A(\alpha, \beta) = A(\alpha) \cap A^\ast(\beta)$$

holds.

Definition 2.5. [43] Let $X$ be a topological space. Let $f$ be a function from $X$ to $\mathbb{R} \cup \{-\infty, \infty\}$.

i) $f$ is called an upper semi-continuous function if for all $k \in \mathbb{R}$, the set $\{ x \in X \mid f(x) < k \}$ is an open set.

ii) $f$ is called a lower semi-continuous function if for all $k \in \mathbb{R}$, the set $\{ x \in X \mid f(x) > k \}$ is an open set.

Definition 2.6. [44] Let $f$ be a function defined on a convex subset $K$ of $\mathbb{R}^n$.

i) $f$ is called a quasi-concave function on $K$ if for all $x, y \in K$ and $t \in [0, 1]$, it holds that

$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\}.$$  

ii) $f$ is called a quasi-convex function on $K$ if for all $x, y \in K$ and $t \in [0, 1]$, it holds that

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$  

Definition 2.7. An intuitionistic fuzzy set $\tilde{A}^i \in IF(\mathbb{R}^n)$ satisfying the following properties is called an intuitionistic fuzzy number in $\mathbb{R}^n$.

1. $\tilde{A}^i$ is a normal set, i.e., $A(1) \neq \emptyset$ and $A^\ast(0) \neq \emptyset$.

2. $A(0)$ and $A^\ast(1)$ are bounded sets in $\mathbb{R}^n$.

3. $\mu_A : \mathbb{R}^n \to [0, 1]$ is an upper semi-continuous function; i.e., $\forall k \in [0, 1], \{ x \in A \mid \mu_A(x) < k \}$ is an open set.

4. $\nu_A : \mathbb{R}^n \to [0, 1]$ is a lower semi-continuous function; i.e., $\forall k \in [0, 1], \{ x \in A \mid \nu_A(x) > k \}$ is an open set.
5. The membership function $\mu_A$ is quasi-concave; i.e., $\forall \lambda \in [0, 1], \forall x, y \in \mathbb{R}^n$

$$\mu_A(\lambda x + (1 - \lambda)y) \geq \min(\mu_A(x), \mu_A(y))$$

6. The non-membership function $\nu_A$ is quasi-convex; i.e., $\forall \lambda \in [0, 1], \forall x, y \in \mathbb{R}^n$

$$\nu_A(\lambda x + (1 - \lambda)y) \leq \max(\nu_A(x), \nu_A(y)); \forall \lambda \in [0, 1],$$

We will denote the set of all intuitionistic fuzzy numbers in $\mathbb{R}^n$ by $IF_N(\mathbb{R}^n)$.

**Definition 2.8.** [39] A triangular intuitionistic fuzzy number (TIFN) $\bar{A}^i \in IF_N(\mathbb{R})$ is defined with the following membership and non-membership functions:

$$\mu_A(x) = \begin{cases} 
\frac{x - a_1}{a_2 - a_1}; & a_1 \leq x \leq a_2 \\
\frac{a_3 - x}{a_3 - a_2}; & a_2 \leq x \leq a_3 \\
0; & \text{otherwise}
\end{cases}$$

and

$$\nu_A(x) = \begin{cases} 
\frac{a_2 - x}{a_2 - a_1}; & a_1^* \leq x \leq a_2 \\
\frac{a_3 - x}{a_3 - a_2}; & a_2 \leq x \leq a_3^* \\
1; & \text{otherwise}
\end{cases}$$

Here $a_1^\leq a_1 \leq a_2 \leq a_3 \leq a_3^*$ and it is denoted by $\bar{A}^i = (a_1, a_2, a_3; a_1^*, a_2^*, a_3^*)$. Note that its $\alpha$ and $\beta$ cuts can be obtained as

$$A(\alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_3 - a_2)]$$

and

$$A^*(\beta) = [a_2 + \beta(a_2 - a_1^*), a_2 + \beta(a_3^* - a_2)].$$

**Definition 2.9.** [26] Let $A$ and $B$ be two nonempty subsets of $\mathbb{R}^n$ and $c \in \mathbb{R}$. The Minkowski addition and scalar multiplication of sets are defined as follows:

$$A + B = \{a + b : a \in A \text{ and } b \in B\}$$

and

$$cA = \{ca : a \in A\}$$

**Theorem 2.2.** [26] The family of all compact and convex subsets of $\mathbb{R}^n$ is closed under Minkowski addition and scalar multiplication.

**Definition 2.10.** [2] Let $\bar{A}^i, \bar{B}^i \in IF_N(\mathbb{R}^n)$ and $c \in \mathbb{R} - \{0\}$. Addition and scalar multiplication of fuzzy numbers in $IF_N(\mathbb{R}^n)$ is defined as follows:

$$\bar{A}^i + \bar{B}^i = \bar{C}^i \Leftrightarrow C(\alpha) = A(\alpha) + B(\alpha) \text{ and } C^*(\beta) = A^*(\beta) + B^*(\beta)$$

and

$$c(\bar{A}^i) = \bar{D}^i \Leftrightarrow D(\alpha) = cA(\alpha) \text{ and } D^*(\beta) = cA^*(\beta)$$

144
Definition 2.11. [26] Let \( a_1, a_2, b_1 \) and \( b_2 \in \mathbb{R} \) such that \( A = [a_1, a_2] \) and \( B = [b_1, b_2] \). Basic end-point arithmetic operations of these closed and bounded intervals are as follows:

1. **Addition:** \( A + B = [a_1 + b_1, a_2 + b_2] \)
2. **Subtraction:** \( A - B = [a_1 - b_2, a_2 - b_1] \)
3. **Multiplication:** \( A \cdot B = [\min\{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}, \max\{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}] \)
4. **Division:** Assume the interval \( B \) does not contain zero. Then \( A/B = [\min\{a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2\}, \max\{a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2\}] \)

**Theorem 2.3.** [2] Let \( \bar{A}^i, \bar{B}^i \in IF_N(\mathbb{R}^n) \). Let us define the following distance functions

\[
D_1(\bar{A}^i, \bar{B}^i) = \sup\{d_H(A(\alpha), B(\alpha)) : \alpha \in [0, 1]\}, \\
D_2(\bar{A}^i, \bar{B}^i) = \sup\{d_H(A(\beta), B(\beta)) : \beta \in [0, 1]\}.
\]

Here \( d_H \) is Hausdorff metric [49]. The function \( D(\bar{A}^i, \bar{B}^i) = \max\{D_1(\bar{A}^i, \bar{B}^i), D_2(\bar{A}^i, \bar{B}^i)\} \) defines a metric on \( IF_N(\mathbb{R}^n) \). Hence \( (IF_N(\mathbb{R}^n), D) \) is a metric space.

**Definition 2.12.** Let \( \bar{A}^i, \bar{B}^i \in IF_N(\mathbb{R}^n) \). The Hukuhara difference of \( \bar{A}^i \) and \( \bar{B}^i \) is \( \bar{C}^i \), if it exists, such that

\[ \bar{A}^i \ominus_H \bar{B}^i = \bar{C}^i \iff \bar{A}^i = \bar{B}^i + \bar{C}^i \]

**Definition 2.13.** Let \( \bar{A}^i, \bar{B}^i \in IF_N(\mathbb{R}^n) \). The generalized Hukuhara difference of \( \bar{A}^i \) and \( \bar{B}^i \) is \( \bar{C}^i \), if it exists, such that

\[ \bar{A}^i \ominus_{gH} \bar{B}^i = \bar{C}^i \iff \bar{A}^i = \bar{B}^i + \bar{C}^i \text{ or } \bar{B}^i = \bar{A}^i + (-1)\bar{C}^i \]

**Definition 2.14.** Let \( f : (a, b) \to IF_N(\mathbb{R}) \) be an intuitionistic fuzzy number valued function and \( x_0, x_0 + h \in (a, b) \). \( f \) is called Hukuhara differentiable at \( x_0 \) if there exists an element \( f'_H(x_0) \in IF_N(\mathbb{R}) \) such that for all \( h > 0 \) the following is satisfied

\[ \lim_{h \to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_H(x_0). \]

**Definition 2.15.** Let \( f : (a, b) \to IF_N(\mathbb{R}) \) be an intuitionistic fuzzy number valued function and \( x_0, x_0 + h \in (a, b) \). \( f \) is called strongly generalized Hukuhara differentiable at \( x_0 \) if there exists an element \( f'_{GH}(x_0) \in IF_N(\mathbb{R}) \) such that for all \( h > 0 \) at least one of the followings is satisfied:

1. \( f(x_0 + h) \ominus_H f(x_0) \) and \( f(x_0) \ominus_H f(x_0 - h) \) exist and the following limits exist such that

\[ \lim_{h \to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_{GH}(x_0) \]

2. \( f(x_0) \ominus_H f(x_0 + h) \) and \( f(x_0 - h) \ominus_H f(x_0) \) exist and the following limits exist such that

\[ \lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus_H f(x_0)}{-h} = f'_{GH}(x_0) \]

145
3. \( f(x_0 + h) \ominus_H f(x_0) \) and \( f(x_0 - h) \ominus_H f(x_0) \) exist and the following limits exist such that
\[
\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus_H f(x_0)}{-h} = f'_GH(x_0)
\]

4. \( f(x_0) \ominus_H f(x_0 + h) \) and \( f(x_0) \ominus_H f(x_0 - h) \) exist and the following limits exist such that
\[
\lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_GH(x_0)
\]

**Definition 2.16.** [16] Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a real valued function. The following function
\[
\theta(f(x)) = \begin{cases} 
1, & f(x) \geq 0 \\
0, & f(x) < 0 
\end{cases}
\]
is called the Heaviside step function.

**Definition 2.17.** [16] Let \( X \) and \( Y \) be two sets and \( f : X \to Y \) be a function. Let \( \tilde{A}^i \) be an intuitionistic fuzzy set over \( X \). Then \( f(\tilde{A}^i) \) is an intuitionistic fuzzy set over \( Y \) such that for every \( y \in Y \)
\[
\mu_{f(\tilde{A}^i)}(y) = \begin{cases} 
\sup\{\mu_A(x) : f(x) = y\}; & y \in f(X) \\
0; & y \notin f(X) 
\end{cases}
\]

\[
\nu_{f(\tilde{A}^i)}(y) = \begin{cases} 
\inf\{\nu_A(x) : f(x) = y\}; & y \in f(X) \\
1; & y \notin f(X) 
\end{cases}
\]

### 3. (i)-GH and (ii)-GH differentiability in an intuitionistic fuzzy environment

**Definition 3.1.** Let \( f : [a, b] \to IF_N(\mathbb{R}) \) and its \( \alpha \) and \( \beta \) cuts be given by \( f(t, \alpha) = [f_1(t, \alpha), f_2(t, \alpha)] \) and \( f^*(t, \beta) = [f_1^*(t, \beta), f_2^*(t, \beta)] \) such that the end-points of its \( \alpha \) and \( \beta \) cuts are differentiable at \( t_0 \in (a, b) \).

1. If
\[
f'_GH(t_0, \alpha) = [f'_1(t_0, \alpha), f'_2(t_0, \alpha)]
\]
\[
(f^*)'_GH(t_0, \beta) = [(f_1^*)'(t_0, \beta), (f_2^*)'(t_0, \beta)],
\]
then \( f \) is called (i)-GH differentiable at \( x_0 \) for each \( \alpha, \beta \in [0, 1] \) and is denoted by \( f'_{(i)-GH} \).

2. If
\[
f'_GH(t_0, \alpha) = [f'_2(t_0, \alpha), f'_1(t_0, \alpha)]
\]
\[
(f^*)'_GH(t_0, \beta) = [(f_2^*)'(t_0, \beta), (f_1^*)'(t_0, \beta)],
\]
then \( f \) is called (ii)-GH differentiable at \( t_0 \) for each \( \alpha, \beta \in [0, 1] \) and is denoted by \( f'_{(ii)-GH} \).
Theorem 3.1. [2] Let $\tilde{A}^i \in IF_N(\mathbb{R}^n)$ and $\alpha, \beta \in [0, 1]$ such that the $\alpha$ and $\beta$ cuts of $\tilde{A}^i$ given by $A(\alpha) = \{ x \in \mathbb{R}^n : \mu_A(x) \geq \alpha \}$ and $A^*(\beta) = \{ x \in \mathbb{R}^n : \nu_A(x) \leq \beta \}$. Then the followings hold.

1. For every $\alpha \in [0, 1]$, $A(\alpha)$ is a non-empty compact and convex set in $\mathbb{R}^n$.
2. If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A(\alpha_2) \subseteq A(\alpha_1)$.
3. If $(\alpha_n)$ is a non-decreasing sequence in $[0, 1]$ converging to $\alpha$ then
   $$\bigcap_{n=1}^{\infty} A(\alpha_n) = A(\alpha).$$
4. If $(\alpha_n)$ is a non-increasing sequence in $[0, 1]$ converging to 0 then
   $$cl \left( \bigcup_{n=1}^{\infty} A(\alpha_n) \right) = A(0).$$
5. For every $\beta \in [0, 1]$, $A^*(\beta)$ is a non-empty compact and convex set in $\mathbb{R}^n$.
6. If $0 \leq \beta_1 \leq \beta_2 \leq 1$ then $A^*(\beta_1) \subseteq A^*(\beta_2)$.
7. If $(\beta_n)$ is a non-increasing sequence in $[0, 1]$ converging to $\beta$ then
   $$\bigcap_{n=1}^{\infty} A^*(\beta_n) = A^*(\beta).$$
8. If $(\beta_n)$ is a non-decreasing sequence in $[0, 1]$ converging to 1 then
   $$cl \left( \bigcup_{n=1}^{\infty} A^*(\beta_n) \right) = A^*(1).$$

Lemma 3.2. Let $f : (a, b) \to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function and $t_0 \in (a, b)$.

1. If for every $\alpha \in [0, 1]$ the interval $[A_1(\alpha), A_2(\alpha)]$ satisfies the properties (1)-(4) in Theorem 3.1 and for every $\beta \in [0, 1]$ the interval $[A^*_1(\beta), A^*_2(\beta)]$ satisfies the properties (5)-(8) in Theorem 3.1, and
2. If for every $\alpha, \beta \in [0, 1]$, $\lim_{x \to x_0} f(t, \alpha) = [A_1(\alpha), A_2(\alpha)]$ and $\lim_{x \to x_0} f^*(t, \beta) = [A^*_1(\beta), A^*_2(\beta)]$
then there exists an intuitionistic fuzzy number $\tilde{A}$ such that $\lim_{t \to t_0} f(t) = \tilde{A}$. And the $\alpha$ and $\beta$ cuts of $\tilde{A}$ are $[A_1(\alpha), A_2(\alpha)]$ and $[A^*_1(\beta), A^*_2(\beta)]$, respectively.

Proof. Assume the conditions (1) and (2) are satisfied. Since for every $\alpha \in [0, 1]$ the interval $[A_1(\alpha), A_2(\alpha)]$ satisfies the properties (1)-(4) in Theorem 3.1 and for every $\beta \in [0, 1]$ the interval $[A^*_1(\beta), A^*_2(\beta)]$ satisfies the properties (5)-(8) in Theorem 3.1 then there exists an intuitionistic
fuzzy number $\tilde{A}^i$ such that the $\alpha$ and $\beta$ cuts of $\tilde{A}^i$ are $[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$ [2]. Furthermore, since
\[
\lim_{t \to t_0} f(t; \alpha) = A(\alpha) \Rightarrow \lim_{t \to t_0} D_1(f(t; \alpha), A(\alpha)) = 0
\]
and
\[
\lim_{t \to t_0} f^*(t; \beta) = A^*(\beta) \Rightarrow \lim_{t \to t_0} D_2(f^*(t; \beta), A^*(\beta)) = 0,
\]
then by the definition of the metric $D$ we can write that $\lim_{t \to t_0} D(f(t), \tilde{A}^i) = 0$. Hence we obtain that $\lim_{t \to t_0} f(t) = \tilde{A}^i$.

**Theorem 3.3.** Let $f : (a, b) \to IF_N(\mathbb{R})$ and its $\alpha$ and $\beta$-cuts be given by
\[
f(t; \alpha) = [f_1(t; \alpha), f_2(t; \alpha)] \text{ and } f^*(t; \beta) = [f_1^*(t; \beta), f_2^*(t; \beta)].
\]

1. If $f$ is strongly generalized differentiable at $t_0 \in (a, b)$ as in case (1) of Definition 2.15 then for every $\alpha, \beta \in [0, 1]$
\[
f'(t_0, \alpha) = [f_1'(t_0, \alpha), f_2'(t_0, \alpha)] \quad \text{ and } \quad (f^*)'(t_0, \beta) = [(f_1^*)'(t_0, \beta), (f_2^*)'(t_0, \beta)]
\]

2. If $f$ is strongly generalized differentiable at $t_0 \in (a, b)$ as in case (2) of Definition 2.15 then for every $\alpha, \beta \in [0, 1]$
\[
f'(t_0, \alpha) = [f_2'(t_0, \alpha), f_1'(t_0, \alpha)] \quad \text{ and } \quad (f^*)'(t_0, \beta) = [(f_2^*)'(t_0, \beta), (f_1^*)'(t_0, \beta)]
\]

**Proof.** 1. Let $f : (a, b) \to IF_N(\mathbb{R})$, $t_0 \in (a, b)$ be strongly generalized differentiable as in case (1). Then $f(t + h) \ominus_H f(t)$ exists. Let $C(t, h) \in IF_N(\mathbb{R})$ such that $f(t + h) \ominus_H f(t) = C(t, h)$. Then $f(t + h) = f(t) + C(t, h)$ implies $f(t + h, \alpha) = f(t, \alpha) + C(t, h; \alpha)$ and $f^*(t + h, \beta) = f^*(t, \beta) + C^*(t, h; \beta)$. So by the interval operations we can write that
\[
[f_1(t + h, \alpha), f_2(t + h, \alpha)] = [f_1(t, \alpha), f_2(t, \alpha)] + [C_1(t, h; \alpha), C_2(t, h; \alpha)]
\]
and
\[
[f_1^*(t + h, \beta), f_2^*(t + h, \beta)] = [f_1^*(t, \beta), f_2^*(t, \beta)] + [C_1^*(t, h; \beta), C_2^*(t, h; \beta)]
\]
So by the equality of intervals we obtain that
\[
C_1(t, h; \alpha) = f_1(t + h, \alpha) - f_1(t, \alpha), C_2(t, h; \alpha) = f_2(t + h, \alpha) - f_2(t, \alpha)
\]
and
\[
C_1^*(t, h; \beta) = f_1^*(t + h, \beta) - f_1^*(t, \beta), C_2^*(t, h; \beta) = f_2^*(t + h, \beta) - f_2^*(t, \beta).
\]
Similar results can be obtained for \( f(t) \ominus_H f(t - h) \). Hence we obtain
\[
\begin{align*}
\lim_{h \to 0^+} \frac{C_1(t, h; \alpha)}{h} &= \lim_{h \to 0^+} \frac{f_1(t + h, \alpha) - f_1(t, \alpha)}{h} = f_1'(t, \alpha) \\
\lim_{h \to 0^+} \frac{C_2(t, h; \alpha)}{h} &= \lim_{h \to 0^+} \frac{f_2(t + h, \alpha) - f_2(t, \alpha)}{h} = f_2'(t, \alpha) \\
\lim_{h \to 0^+} \frac{C_1^*(t, h; \beta)}{h} &= \lim_{h \to 0^+} \frac{f_1^*(t + h, \beta) - f_1^*(t, \beta)}{h} = (f_1^*)'(t, \beta) \\
\lim_{h \to 0^+} \frac{C_2^*(t, h; \beta)}{h} &= \lim_{h \to 0^+} \frac{f_2^*(t + h, \beta) - f_2^*(t, \beta)}{h} = (f_2^*)'(t, \beta)
\end{align*}
\]
So by Lemma 3.2 we can write that
\[
\begin{align*}
f'(t_0, \alpha) &= [f_1'(t_0, \alpha), f_2'(t_0, \alpha)] \\
(f^*)'(t_0, \beta) &= [(f_1^*)'(t_0, \beta), (f_2^*)'(t_0, \beta)]
\end{align*}
\]

2. The proof can be done in a similar way. \( \square \)

**Theorem 3.4.** Let \( f : (a, b) \rightarrow IF_N(\mathbb{R}) \), \( t_0 \in (a, b) \). If \( f \) is strongly generalized differentiable at \( t_0 \) as in case (3) or (4) of Definition 2.17. Then \( f'(t) \in \mathbb{R} \) for all \( t \in (a, b) \).

**Theorem 3.5.** Let \( f \) and \( g \) be intuitionistic fuzzy number valued functions. If \( f \) and \( g \) are both (i)-GH differentiable or both (ii)-GH differentiable, then

1. \( (f + g)'(t, \alpha) = f'_1(t, \alpha) + g'_1(t, \alpha) \)
2. \( (f + g)'(t, \beta) = f'_2(t, \beta) + g'_2(t, \beta) \)

**Proof.** 1. Let \( f, g \in IF_N(\mathbb{R}) \). Let \( f(t, \alpha) = [f_1(t, \alpha), f_2(t, \alpha)] \) and \( f^*(t, \beta) = [f_1^*(t, \beta), f_2^*(t, \beta)] \) be \( \alpha \) and \( \beta \) cuts of \( f \) ; and \( g(t, \alpha) = [g_1(t, \alpha), g_2(t, \alpha)] \) and \( g^*(t, \beta) = [g_1^*(t, \beta), g_2^*(t, \beta)] \) be \( \alpha \) and \( \beta \) cuts of \( g \). Assume \( f \) and \( g \) be (i)-GH differentiable then we can write that
\[
(f + g)'(t, \alpha) = [(f + g)_1'(t, \alpha), (f + g)_2'(t, \alpha)] = [f_1'(t, \alpha) + g_1'(t, \alpha), f_2'(t, \alpha) + g_2'(t, \alpha)]
\]
and
\[
((f + g)^*)'(t, \beta) = [((f + g)_1^*)'(t, \beta), ((f + g)_2^*)'(t, \beta)] = [f_1^*(t, \beta) + (g_1^*)'(t, \beta), f_2^*(t, \beta) + (g_2^*)'(t, \beta)]
\]
2. The proof can be done in a similar way. \( \square \)
4 Application to a system of intuitionistic fuzzy differential equations

In this section we will study the following system of first order differential equations in intuitionistic fuzzy environment under (i,ii)-GH differentiability and the intuitionistic Zadeh’s extension principle interpretation.

\[
\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)
\]

4.1 Solving a system of intuitionistic fuzzy differential equations under (i,ii)-GH differentiability interpretation

Let \( A \) and \( B \) be non-zero real numbers. Let us consider the following system of intuitionistic fuzzy differential equation:

\[
\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)
\]

with the following triangular intuitionistic fuzzy initial values \( x(t_0) = (a_1, a_2, a_3; a_1^*, a_2^*, a_3^*) \) and \( y(t_0) = (b_1, b_2, b_3; b_1^*, b_2^*, b_3^*) \). So in terms of \( \alpha \) and \( \beta \) cuts we can write this system as follows:

\[
\frac{dx(t, \alpha)}{dt} = Ay(t, \alpha)
\]

\[
\frac{dy(t, \alpha)}{dt} = Bx(t, \alpha)
\]

\[
x(t_0, \alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]
\]

\[
y(t_0, \alpha) = [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]
\]

and

\[
\frac{dx^*(t, \beta)}{dt} = Ay^*(t, \beta)
\]

\[
\frac{dy^*(t, \beta)}{dt} = Bx^*(t, \beta)
\]

\[
x^*(t_0, \beta) = [a_2 + \beta(a_1^* - a_2), a_2 + \beta(a_3^* - a_2)]
\]

\[
y^*(t_0, \beta) = [b_2 + \beta(b_1^* - b_2), b_2 + \beta(b_3^* - b_2)]
\]

**Case 1:** If \( x \) and \( y \) are (i)-differentiable, we can write that

\[
[x_1(t, \alpha), x_2(t, \alpha)] = A[y_1(t, \alpha), y_2(t, \alpha)]
\]

\[
[y_1(t, \alpha), y_2(t, \alpha)] = B[x_1(t, \alpha), x_2(t, \alpha)]
\]

\[
x(t_0, \alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]
\]

\[
y(t_0, \alpha) = [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]
\]

So by using Heaviside step function we can obtain that
Case 2: So by using Heaviside step function we can obtain that

\[
x'_1(t) = \theta(A)(y_1(t, \alpha) - y_2(t, \alpha)) + y_2(t, \alpha)
\]
\[
x'_2(t) = \theta(A)(y_2(t, \alpha) - y_1(t, \alpha)) + y_1(t, \alpha)
\]
\[
y'_1(t) = \theta(B)(x_1(t, \alpha) - x_2(t, \alpha)) + x_2(t, \alpha)
\]
\[
y'_2(t) = \theta(B)(x_2(t, \alpha) - x_1(t, \alpha)) + x_1(t, \alpha)
\]
\[
x_{1}(t_0, \alpha) = a_1 + \alpha(a_2 - a_1)
\]
\[
x_{2}(t_0, \alpha) = a_3 + \alpha(a_2 - a_3)
\]
\[
y_{1}(t_0, \alpha) = b_1 + \alpha(b_2 - b_1)
\]
\[
y_{2}(t_0, \alpha) = b_3 + \alpha(b_2 - b_3)
\]

and

\[
(x'_1^*) = \theta(A)(y'_1^*(t, \beta) - y'_2^*(t, \beta)) + y'_2^*(t, \beta)
\]
\[
(x'_2^*) = \theta(A)(y'_2^*(t, \beta) - y'_1^*(t, \beta)) + y'_1^*(t, \beta)
\]
\[
(y'_1^*) = \theta(B)(x'_1^*(t, \beta) - x'_2^*(t, \beta)) + x'_2^*(t, \beta)
\]
\[
(y'_2^*) = \theta(B)(x'_2^*(t, \beta) - x'_1^*(t, \beta)) + x'_1^*(t, \beta)
\]

\[
x^*_1(t_0, \beta) = a_2 + \beta(a_1^* - a_2)
\]
\[
x^*_2(t_0, \beta) = a_2 + \beta(a_3^* - a_2)
\]
\[
y^*_1(t_0, \beta) = b_2 + \beta(b_1^* - b_2)
\]
\[
y^*_2(t_0, \beta) = b_2 + \beta(b_3^* - b_2)
\]

Case 2: If \( x \) and \( y \) are (ii)-differentiable we can write that

\[
[x'_2^*, x'_1^*] = A[y_1(t, \alpha), y_2(t, \alpha)]
\]
\[
[y'_2^*, y'_1^*] = B[x_1(t, \alpha), x_2(t, \alpha)]
\]
\[
x(t_0, \alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]
\]
\[
x(t_0, \alpha) = [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]
\]

So by using Heaviside step function we can obtain that

\[
x'_1(t) = \theta(A)(y_1(t, \alpha) - y_2(t, \alpha)) + y_2(t, \alpha)
\]
\[
x'_2(t) = \theta(A)(y_2(t, \alpha) - y_1(t, \alpha)) + y_1(t, \alpha)
\]
\[
y'_1(t) = \theta(B)(x_1(t, \alpha) - x_2(t, \alpha)) + x_2(t, \alpha)
\]
\[
y'_2(t) = \theta(B)(x_2(t, \alpha) - x_1(t, \alpha)) + x_1(t, \alpha)
\]
\[
x_{1}(t_0, \alpha) = a_1 + \alpha(a_2 - a_1)
\]
\[
x_{2}(t_0, \alpha) = a_3 + \alpha(a_2 - a_3)
\]
\[
y_{1}(t_0, \alpha) = b_1 + \alpha(b_2 - b_1)
\]
\[
y_{2}(t_0, \alpha) = b_3 + \alpha(b_2 - b_3)
\]

and

\[
(x'_1^*) = \theta(A)(y'_1^*(t, \beta) - y'_2^*(t, \beta)) + y'_2^*(t, \beta)
\]
\[
(x'_2^*) = \theta(A)(y'_2^*(t, \beta) - y'_1^*(t, \beta)) + y'_1^*(t, \beta)
\]
\[
(y'_1^*) = \theta(B)(x'_1^*(t, \beta) - x'_2^*(t, \beta)) + x'_2^*(t, \beta)
\]
\[
(y'_2^*) = \theta(B)(x'_2^*(t, \beta) - x'_1^*(t, \beta)) + x'_1^*(t, \beta)
\]
\[x_1^*(t_0, \beta) = a_2 + \beta(a_1^* - a_2)\]
\[x_2^*(t_0, \beta) = a_2 + \beta(a_2^* - a_2)\]
\[y_1^*(t_0, \beta) = b_2 + \beta(b_1^* - b_2)\]
\[y_2^*(t_0, \beta) = b_2 + \beta(b_3^* - b_2)\]

**Case 3:** If \(x\) is (i)- and \(y\) is (ii)-differentiable we can write that
\[
[x_1', x_2'] = A[y_1(t, \alpha), y_2(t, \alpha)]
\]
\[
[y_1', y_2'] = B[x_1(t, \alpha), x_2(t, \alpha)]
\]
\[x(t_0, \alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]\]
\[y(t_0, \alpha) = [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]\]

So by using Heaviside step function we can obtain that
\[
x_1'(t) = \theta(A)(y_1(t, \alpha) - y_2(t, \alpha)) + y_2(t, \alpha)
\]
\[
x_2'(t) = \theta(A)(y_2(t, \alpha) - y_1(t, \alpha)) + y_1(t, \alpha)
\]
\[
y_1'(t) = \theta(B)(x_1(t, \alpha) - x_2(t, \alpha)) + x_2(t, \alpha)
\]
\[
y_2'(t) = \theta(B)(x_2(t, \alpha) - x_1(t, \alpha)) + x_1(t, \alpha)
\]
\[x_1(t_0, \alpha) = a_1 + \alpha(a_2 - a_1)\]
\[x_2(t_0, \alpha) = a_3 + \alpha(a_2 - a_3)\]
\[y_1(t_0, \alpha) = b_1 + \alpha(b_2 - b_1)\]
\[y_2(t_0, \alpha) = b_3 + \alpha(b_2 - b_3)\]

and
\[
(x_1^*)'(t) = \theta(A)(y_1^*(t, \beta) - y_2^*(t, \beta)) + y_2^*(t, \beta)
\]
\[
(x_2^*)'(t) = \theta(A)(y_2^*(t, \beta) - y_1^*(t, \beta)) + y_1^*(t, \beta)
\]
\[
(y_1^*)'(t) = \theta(B)(x_1^*(t, \beta) - x_2^*(t, \beta)) + x_2^*(t, \beta)
\]
\[
(y_2^*)'(t) = \theta(B)(x_2^*(t, \beta) - x_1^*(t, \beta)) + x_1^*(t, \beta)
\]
\[x_1^*(t_0, \beta) = a_2 + \beta(a_1^* - a_2)\]
\[x_2^*(t_0, \beta) = a_2 + \beta(a_2^* - a_2)\]
\[y_1^*(t_0, \beta) = b_2 + \beta(b_1^* - b_2)\]
\[y_2^*(t_0, \beta) = b_2 + \beta(b_3^* - b_2)\]

**Case 4:** If \(x\) is (ii)- and \(y\) is (i)-differentiable we can write that
\[
[x_2', x_1'] = A[y_1(t, \alpha), y_2(t, \alpha)]
\]
\[
[y_1', y_2'] = B[x_1(t, \alpha), x_2(t, \alpha)]
\]
\[x(t_0, \alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]\]
\[x(t_0, \alpha) = [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]\]
In terms of $\alpha$ cuts we obtain

\begin{align*}
x'_2(t) &= \theta(A)(y_1(t, \alpha) - y_2(t, \alpha)) + y_2(t, \alpha) \\
x'_1(t) &= \theta(A)(y_2(t, \alpha) - y_1(t, \alpha)) + y_1(t, \alpha) \\
y'_1(t) &= \theta(B)(x_1(t, \alpha) - x_2(t, \alpha)) + x_2(t, \alpha) \\
y'_2(t) &= \theta(B)(x_2(t, \alpha) - x_1(t, \alpha)) + x_1(t, \alpha) \\
x_1(t_0, \alpha) &= a_1 + \alpha(a_2 - a_1) \\
x_2(t_0, \alpha) &= a_3 + \alpha(a_2 - a_3) \\
y_1(t_0, \alpha) &= b_1 + \alpha(b_2 - b_1) \\
y_2(t_0, \alpha) &= b_3 + \alpha(b_2 - b_3)
\end{align*}

and

\begin{align*}
(x^*_2)'(t) &= \theta(A)(y^*_1(t, \beta) - y^*_2(t, \beta)) + y^*_2(t, \beta) \\
(x^*_1)'(t) &= \theta(A)(y^*_2(t, \beta) - y^*_1(t, \beta)) + y^*_1(t, \beta) \\
y^*_1(t, \beta) &= \alpha_1 + \beta(a^*_1 - a_1) \\
x^*_2(t, \beta) &= \alpha_2 + \beta(a^*_3 - a_2) \\
y^*_1(t_0, \beta) &= b_1 + \beta(b^*_1 - b_1) \\
y^*_2(t_0, \beta) &= b_2 + \beta(b^*_2 - b_2)
\end{align*}

**Example 4.1.1:** Let us find the intuitionistic fuzzy solution of the following system of differential equations

\[
\frac{dx(t)}{dt} = 5y(t), \quad \frac{dy(t)}{dt} = 3x(t)
\]

with the following triangular intuitionistic fuzzy initial values $x(0) = (1, 2, 3; -2, 2, 6)$ and $y(0) = (-2, -1, 0; -4, -1, 2)$ under (i,ii)-GH differentiability. For not being too repetitive we will give the solutions only for Case 1 and Case 3.

**Case 1:** If $x$ and $y$ are both (i)-GH differentiable we can write that $\alpha$ and $\beta$ cuts of the system as follows:

\begin{align*}
x'_1(t) &= 5y_1(t, \alpha) \\
x'_2(t) &= 5y_2(t, \alpha) \\
y'_1(t) &= 3x_1(t, \alpha) \\
y'_2(t) &= 3x_2(t, \alpha) \\
x_1(t_0, \alpha) &= 1 + \alpha \\
x_2(t_0, \alpha) &= 3 - \alpha \\
y_1(t_0, \alpha) &= -2 + \alpha \\
y_2(t_0, \alpha) &= -\alpha
\end{align*}
By solving these two systems as in classical ordinary differential systems we obtain the solutions as follows:

\[
\begin{align*}
(x_1')'(t) &= 5y_1'(t, \beta) \\
(x_2')'(t) &= 5y_2'(t, \beta) \\
(y_1')'(t) &= 3x_1'(t, \beta) \\
(y_2')'(t) &= 3x_2'(t, \beta) \\
x_1^0(t, 0) &= 2 - 4\beta \\
x_2^0(t, 0) &= 2 + 4\beta \\
y_1^0(t, 0) &= -1 - 3\beta \\
y_2^0(t, 0) &= -1 + 3\beta \\
\end{align*}
\]

and

\[
\begin{align*}
x_1(t, \alpha) &= \frac{1}{10} e^{-3\sqrt[5]{t-t}} (-3\sqrt[5]{\alpha e^t} + 5\alpha e^t + 3\sqrt[5]{5\alpha e^{6\sqrt[5]{t-t}}} + 5\alpha e^{6\sqrt[5]{t-t}} - 5e^t + 15e^{3\sqrt[5]{t-t}} + 5e^{3\sqrt[5]{5t+2t}} - 3\sqrt[5]{e^{6\sqrt[5]{t-t}} - 5e^{6\sqrt[5]{t-t}}}) \\
x_2(t, \alpha) &= -\frac{1}{10} e^{-3\sqrt[5]{t-t}} (-3\sqrt[5]{\alpha e^t} + 5\alpha e^t + 3\sqrt[5]{5\alpha e^{6\sqrt[5]{t-t}}} + 5\alpha e^{6\sqrt[5]{t-t}} + 3\sqrt[5]{e^t - 5e^t} - 15e^{3\sqrt[5]{t-t}} - 5e^{3\sqrt[5]{5t+2t}} - 3\sqrt[5]{e^{6\sqrt[5]{t-t}} - 5e^{6\sqrt[5]{t-t}}}) \\
y_1(t, \alpha) &= \frac{1}{6} e^{-3\sqrt[5]{t-t}} (-\sqrt[5]{\alpha e^t} + 3\alpha e^t + \sqrt[5]{5\alpha e^{6\sqrt[5]{t-t}}} + 3\alpha e^{6\sqrt[5]{t-t}} + \sqrt[5]{e^t - 3e^t} - 9e^{3\sqrt[5]{t-t}} + 3e^{3\sqrt[5]{5t+2t}} - \sqrt[5]{e^{6\sqrt[5]{t-t}} - 3e^{6\sqrt[5]{t-t}}}) \\
y_2(t, \alpha) &= -\frac{1}{6} e^{-3\sqrt[5]{t-t}} (-\sqrt[5]{\alpha e^t} + 3\alpha e^t + \sqrt[5]{5\alpha e^{6\sqrt[5]{t-t}}} + 3\alpha e^{6\sqrt[5]{t-t}} + \sqrt[5]{e^t - 3e^t} + 9e^{3\sqrt[5]{t-t}} - 3e^{3\sqrt[5]{5t+2t}} - \sqrt[5]{e^{6\sqrt[5]{t-t}} - 3e^{6\sqrt[5]{t-t}}}) \\
\end{align*}
\]
Figure 1. End points of $x(t, 0)$ and $x^*(t, 1)$ for Case 1.

Figure 2. Graph of $\alpha$ cut of the solution $x$ for Case 1.

Figure 3. Graph of $\beta$ cut of the solution $x$ for Case 1.
Figure 4. The intersected region is $(\alpha, \beta)$ cut of the solution $x$ for Case 1.

Figure 5. End points of $y(t, 0)$ and $y^*(t, 1)$ for Case 1.

Figure 6. Graph of $\beta$ cut of the solution $y$ for Case 1.
Figure 7. Graph of $\beta$ cut of the solution $y$ for Case 1.

Figure 8. The intersected region is $(\alpha, \beta)$ cuts of the solution $y$ for Case 1.
Case 3: Let $x$ be (i)-GH differentiable and $y$ be (ii)-GH differentiable. Then we can write that $\alpha$ and $\beta$ cuts of the system as follows:
\[ x_1'(t) = 5y_1(t, \alpha) \]
\[ x_2'(t) = 5y_2(t, \alpha) \]
\[ y_1'(t) = 3x_2(t, \alpha) \]
\[ y_2'(t) = 3x_1(t, \alpha) \]
\[ x_1(t_0, \alpha) = 1 + \alpha \]
\[ x_2(t_0, \alpha) = 3 - \alpha \]
\[ y_1(t_0, \alpha) = -2 + \alpha \]
\[ y_2(t_0, \alpha) = -\alpha \]

and
\[ (x_1^*)'(t) = 5y_1^*(t, \beta) \]
\[ (x_2^*)'(t) = 5y_2^*(t, \beta) \]
\[ (y_1^*)'(t) = 3x_2^*(t, \beta) \]
\[ (y_2^*)'(t) = 3x_1^*(t, \beta) \]
\[ x_1^*(t_0, \beta) = 2 - 4\beta \]
\[ x_2^*(t_0, \beta) = 2 + 4\beta \]
\[ y_1^*(t_0, \beta) = -1 - 3\beta \]
\[ y_2^*(t_0, \beta) = -1 + 3\beta \]

By solving these two systems as in classical ordinary differential systems we obtain the solutions as follows:
\[ x_1(t, \alpha) = \frac{1}{10} e^{-t}(6\sqrt{5}\alpha e^t \sin(3\sqrt{5}t) + 10\alpha e^t \cos(3\sqrt{5}t) + 5e^{2t} - 6\sqrt{5}e^t \sin(3\sqrt{5}t)) - 10e^t \cos(3\sqrt{5}t) + 15) \]
\[ x_2(t, \alpha) = \frac{1}{10} e^{-t}(-6\sqrt{5}\alpha e^t \sin(3\sqrt{5}t) - 10\alpha e^t \cos(3\sqrt{5}t) + 5e^{2t} + 6\sqrt{5}e^t \sin(3\sqrt{5}t)) + 10e^t \cos(3\sqrt{5}t) + 15) \]
\[ y_1(t, \alpha) = \frac{1}{6} e^{-t}(-2\sqrt{5}\alpha e^t \sin(3\sqrt{5}t) + 6\alpha e^t \cos(3\sqrt{5}t) + 3e^{2t} + 2\sqrt{5}e^t \sin(3\sqrt{5}t)) - 6e^t \cos(3\sqrt{5}t) - 9) \]
\[ y_2(t, \alpha) = \frac{1}{6} e^{-t}(2\sqrt{5}\alpha e^t \sin(3\sqrt{5}t) - 6\alpha e^t \cos(3\sqrt{5}t) + 3e^{2t} - 2\sqrt{5}e^t \sin(3\sqrt{5}t)) + 6e^t \cos(3\sqrt{5}t) - 9) \]

and
\[ x_1^*(t, \beta) = \frac{1}{10} e^{-t}(-18\sqrt{5}\beta e^t \sin(3\sqrt{5}t) - 40\beta e^t \cos(3\sqrt{5}t) + 5e^{2t} + 15) \]
\[ x_2^*(t, \beta) = \frac{1}{10} e^{-t}(18\sqrt{5}\beta e^t \sin(3\sqrt{5}t) + 40\beta e^t \cos(3\sqrt{5}t) + 5e^{2t} + 15) \]
\[ y_1^*(t, \beta) = \frac{1}{6} e^{-t}(8\sqrt{5}\beta e^t \sin(3\sqrt{5}t) - 18\beta e^t \cos(3\sqrt{5}t) + 3e^{2t} - 9) \]
\[ y_2^*(t, \beta) = \frac{1}{6} e^{-t}(-8\sqrt{5}\beta e^t \sin(3\sqrt{5}t) + 18\beta e^t \cos(3\sqrt{5}t) + 3e^{2t} - 9) \]
Figure 11. End points of $x(t, 0)$ and $x^*(t, 1)$ for Case 3.

Figure 12. Graph of $\alpha$ cut of the solution $x$ for Case 3.

Figure 13. Graph of $\beta$ cut of the solution $x$ for Case 3.
Figure 14. End points of $y(t, 0)$ and $y^*(t, 1)$ for Case 3.

Figure 15. Graph of $\beta$ cut of the solution $y$ for Case 3.

Figure 16. Graph of $\beta$ cut of the solution $y$ for Case 3.
Figure 17. Membership $\mu$ and non-membership $\nu$ functions of the solution $x$ for Case 3.

Figure 18. Membership $\mu$ and non-membership $\nu$ functions of the solution $y$ for Case 3.
4.2 Solving a system of intuitionistic fuzzy differential equations under Zadeh’s extension principle interpretation

Let $A$ and $B$ be positive real numbers. Now let us consider the following system of intuitionistic fuzzy differential equation

$$\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)$$

with the following triangular intuitionistic fuzzy numbers $x(t_0) = (a_1, a_2, a_3; a_1^*, a_2^*, a_3^*)$ and $y(t_0) = (b_1, b_2, b_3; b_1^*, b_2^*, b_3^*)$ under intuitionistic Zadeh’s Extension Principle. Firstly we need to find the crisp solution of

$$\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)$$

with $x(t_0) = a_2$ and $y(t_0) = b_2$. The crisp solution of this system is

$$x(t) = \frac{e^{-\sqrt{AB}t} \left( a_2 \sqrt{B} \left( e^{2\sqrt{AB}t} + 1 \right) + \sqrt{A} b_2 \left( e^{2\sqrt{AB}t} - 1 \right) \right)}{2\sqrt{B}}$$

$$y(t) = \frac{e^{-\sqrt{AB}t} \left( a_2 \sqrt{B} \left( e^{2\sqrt{AB}t} - 1 \right) + \sqrt{A} b_2 \left( e^{2\sqrt{AB}t} + 1 \right) \right)}{2\sqrt{A}}$$

We know that when we replace crisp initial values with intuitionistic fuzzy ones we obtain the intuitionistic fuzzy solution by the intuitionistic Zadeh’s extension principle. Hence we can write the following $\alpha$ and $\beta$ cuts:

$$x(t, \alpha) = \frac{e^{-\sqrt{AB}t} \left( \sqrt{B} (1 + e^{2\sqrt{AB}t}) \left[ a_1 + (a_2 - a_1)\alpha, a_3 + (a_2 - a_3)\alpha \right] + \sqrt{A} (1 + e^{2\sqrt{AB}t}) \right)}{2\sqrt{B}}$$

$$\left[ b_1 + (b_2 - b_1)\alpha, b_3 + (b_2 - b_3)\alpha \right]$$

$$y(t, \alpha) = \frac{e^{-\sqrt{AB}t} \left( \sqrt{B} (1 + e^{2\sqrt{AB}t}) \left[ a_1 + (a_2 - a_1)\alpha, a_3 + (a_2 - a_3)\alpha \right] + \sqrt{A} (1 + e^{2\sqrt{AB}t}) \right)}{2\sqrt{A}}$$

$$\left[ b_1 + (b_2 - b_1)\alpha, b_3 + (b_2 - b_3)\alpha \right]$$

and

$$x^*(t, \beta) = \frac{e^{-\sqrt{AB}t} \left( \sqrt{B} (1 + e^{2\sqrt{AB}t}) \left[ a_2 + (a_1^* - a_2)\beta, a_2 + (a_3^* - a_2)\beta \right] + \sqrt{A} (1 + e^{2\sqrt{AB}t}) \right)}{2\sqrt{B}}$$

$$\left[ b_2 + (b_1^* - b_2)\beta, b_2 + (b_3^* - b_2)\beta \right]$$

$$y^*(t, \beta) = \frac{e^{-\sqrt{AB}t} \left( \sqrt{B} (1 + e^{2\sqrt{AB}t}) \left[ a_2 + (a_1^* - a_2)\beta, a_2 + (a_3^* - a_2)\beta \right] + \sqrt{A} (1 + e^{2\sqrt{AB}t}) \right)}{2\sqrt{A}}$$

$$\left[ b_2 + (b_1^* - b_2)\beta, b_2 + (b_3^* - b_2)\beta \right]$$
So by end-point interval arithmetics and Heaviside function we the following obtain the points of \(x(t, \alpha)\) and \(x^*(t, \beta)\) as follows:

\[
x_1(t, \alpha) = \frac{e^{-\sqrt{ABt}}}{2\sqrt{B}}(\sqrt{B}(1 + e^{2\sqrt{ABt}})(\theta(\sqrt{B}(1 + e^{2\sqrt{ABt}})(a_1 + (a_2 - a_1)\alpha - (a_3 + (a_2 - a_3)\alpha)
+a_3 + (a_2 - a_3)\alpha)) + (\sqrt{A}(1 - e^{2\sqrt{ABt}})\theta(\sqrt{A}(1 - e^{2\sqrt{ABt}})(b_1 + (b_2 - b_1)\alpha
-(b_3 + (b_2 - b_3)\alpha) + b_3 + (b_2 - b_3)\alpha))
\]

\[
x_2(t, \alpha) = \frac{e^{-\sqrt{ABt}}}{2\sqrt{B}}(\sqrt{B}(1 + e^{2\sqrt{ABt}})(\theta(\sqrt{B}(1 + e^{2\sqrt{ABt}})(a_3 + (a_2 - a_3)\alpha - (a_1 + (a_2 - a_1)\alpha)
+a_1 + (a_2 - a_1)\alpha)) + (\sqrt{A}(1 - e^{2\sqrt{ABt}})\theta(\sqrt{A}(1 - e^{2\sqrt{ABt}})(b_3 + (b_2 - b_3)\alpha
-(b_1 + (b_2 - b_1)\alpha) + b_1 + (b_2 - b_1)\alpha))
\]

\[
y_1(t, \alpha) = \frac{e^{-\sqrt{ABt}}}{2\sqrt{A}}(\sqrt{B}(1 + e^{2\sqrt{ABt}})(\theta(\sqrt{B}(1 + e^{2\sqrt{ABt}})(a_1 + (a_2 - a_1)\alpha - (a_3 + (a_2 - a_3)\alpha)
+a_3 + (a_2 - a_3)\alpha)) + (\sqrt{A}(1 + e^{2\sqrt{ABt}})\theta(\sqrt{A}(1 + e^{2\sqrt{ABt}})(b_1 + (b_2 - b_1)\alpha
-(b_3 + (b_2 - b_3)\alpha) + b_3 + (b_2 - b_3)\alpha))
\]

\[
y_2(t, \alpha) = \frac{e^{-\sqrt{ABt}}}{2\sqrt{A}}(\sqrt{B}(1 + e^{2\sqrt{ABt}})(\theta(\sqrt{B}(1 + e^{2\sqrt{ABt}})(a_3 + (a_2 - a_3)\alpha - (a_1 + (a_2 - a_1)\alpha)
+a_1 + (a_2 - a_1)\alpha)) + (\sqrt{A}(1 + e^{2\sqrt{ABt}})\theta(\sqrt{A}(1 + e^{2\sqrt{ABt}})(b_3 + (b_2 - b_3)\alpha
-(b_1 + (b_2 - b_1)\alpha) + b_1 + (b_2 - b_1)\alpha))
\]

and

\[
x_1^*(t, \beta) = \frac{e^{-\sqrt{ABt}}}{2\sqrt{B}}(\sqrt{B}(1 + e^{2\sqrt{ABt}})(\theta(\sqrt{B}(1 + e^{2\sqrt{ABt}})(a_2 + (a_1^* - a_2)\beta - (a_2 + (a_3^* - a_2)\beta)
+a_2 + (a_3^* - a_2)\beta)) + (\sqrt{A}(1 - e^{2\sqrt{ABt}})\theta(\sqrt{A}(1 - e^{2\sqrt{ABt}})(b_2 + (b_1^* - b_2)\beta
-(b_2 + (b_3^* - b_2)\beta) + b_2 + (b_3^* - b_2)\beta))
\]

\[
x_2^*(t, \beta) = \frac{e^{-\sqrt{ABt}}}{2\sqrt{B}}(\sqrt{B}(1 + e^{2\sqrt{ABt}})(\theta(\sqrt{B}(1 + e^{2\sqrt{ABt}})(a_2 + (a_3^* - a_2)\beta - (a_2 + (a_1^* - a_2)\beta)
+a_2 + (a_1^* - a_2)\beta)) + (\sqrt{A}(1 - e^{2\sqrt{ABt}})\theta(\sqrt{A}(1 - e^{2\sqrt{ABt}})(b_2 + (b_3^* - b_2)\beta
-(b_2 + (b_1^* - b_2)\beta) + b_2 + (b_1^* - b_2)\beta))
\]

\[
y_1^*(t, \beta) = \frac{e^{-\sqrt{ABt}}}{2\sqrt{A}}(\sqrt{B}(1 + e^{2\sqrt{ABt}})(\theta(\sqrt{B}(1 + e^{2\sqrt{ABt}})(a_2 + (a_1^* - a_2)\beta - (a_2 + (a_3^* - a_2)\beta)
+a_2 + (a_3^* - a_2)\beta)) + (\sqrt{A}(1 + e^{2\sqrt{ABt}})\theta(\sqrt{A}(1 + e^{2\sqrt{ABt}})(b_2 + (b_1^* - b_2)\beta
-(b_2 + (b_3^* - b_2)\beta) + b_2 + (b_3^* - b_2)\beta))
\]
with the following triangular intuitionistic fuzzy initial values

\[ y_2(t, \beta) = \frac{e^{-\sqrt{A}Bt}}{2\sqrt{A}} (\sqrt{B}(-1 + e^{2\sqrt{A}Bt})(\theta(\sqrt{B}(-1 + e^{2\sqrt{A}Bt})(a_2 + (a_3 - a_2)\beta) - (a_2 + (a_1^* - a_2)\beta) + a_2 + (a_1^* - a_2)\beta)) + (\sqrt{A}(1 + e^{2\sqrt{A}Bt})\theta(\sqrt{A}(1 + e^{2\sqrt{A}Bt})(b_2 + (b_3^* - b_2)\beta) \right. \\
\left. -(b_2 + (b_1^* - b_2)\beta) + b_2 + (b_1^* - b_2)\beta))) \]

**Example 4.2.1.** Let us find the intuitionistic fuzzy solution of the following system of differential equations

\[
\frac{dx(t)}{dt} = y(t), \quad \frac{dy(t)}{dt} = 20x(t)
\]

with the following triangular intuitionistic fuzzy initial values \( x(0) = (1, 1, 3; -2, 1, 6) \) and \( y(0) = (-2, 3, 0; -4, 3, 2) \). We can obtain the end points of \( \alpha \) and \( \beta \) cuts of the solution as follows:

\[
x_1(t, \alpha) = -\frac{1}{6} e^{-3\sqrt{5}t} - t(-\sqrt{5}ae^t + 3ae^t + \sqrt{5}ae^{6\sqrt{5}t} + 3ae^{6\sqrt{5}t} + t + \sqrt{5}e^t \\
-3ae^t + 9e^{3\sqrt{5}t} - 3e^{3\sqrt{5}t} - \sqrt{5}e^{6\sqrt{5}t} - 3e^{6\sqrt{5}t})x_1(t, \alpha) \\
= e^{-2\sqrt{5}t} \left( (e^{4\sqrt{5}t} - 1) \left( 2\theta(1 + e^{4\sqrt{5}t}) - 2 \right) \\
+ 2\sqrt{5} \left( e^{4\sqrt{5}t} + 1 \right) \left( 2\theta(1 + e^{4\sqrt{5}t}) + 1 \right) \right) y_1(t, \alpha) \\
= \frac{1}{2} e^{-2\sqrt{5}t} \left( (e^{4\sqrt{5}t} + 1) \left( 3 - 5\theta(1 + e^{4\sqrt{5}t}) \right) + 2\sqrt{5} \left( e^{4\sqrt{5}t} - 1 \right) \right) y_2(t, \alpha) \\
= \frac{1}{2} e^{-2\sqrt{5}t} \left( (e^{4\sqrt{5}t} + 1) \left( 5\theta(1 + e^{4\sqrt{5}t}) - 2 \right) + 2\sqrt{5} \left( e^{4\sqrt{5}t} - 1 \right) \right)
\]

and

\[
x_1^*(t, \beta) = e^{-2\sqrt{5}t} \\
\left( (e^{4\sqrt{5}t} - 1) \left( 2 - 6\theta(1 + e^{4\sqrt{5}t}) \right) + 2\sqrt{5} \left( e^{4\sqrt{5}t} + 1 \right) (6 - 8\theta(2\sqrt{5}(1 + e^{4\sqrt{5}t})) \right)
\]

\[
x_2^*(t, \beta) = e^{-2\sqrt{5}t} \left( 2\sqrt{5} \left( e^{4\sqrt{5}t} + 1 \right) \left( 8\theta(2\sqrt{5}(1 + e^{4\sqrt{5}t})) - 2 \right) + 2 \left( e^{4\sqrt{5}t} - 1 \right) \right)
\]

\[
y_1^*(t, \beta) = \frac{1}{2} e^{-2\sqrt{5}t} \\
\left( 2\sqrt{5} \left( e^{4\sqrt{5}t} - 1 \right) (6 - 8\theta(2\sqrt{5}(1 + e^{4\sqrt{5}t})) \right) + \left( e^{4\sqrt{5}t} + 1 \right) \left( 2 - 6\theta(1 + e^{4\sqrt{5}t}) \right)
\]

\[
y_2^*(t, \beta) = \frac{1}{2} e^{-2\sqrt{5}t} \\
\left( 2\sqrt{5} \left( e^{4\sqrt{5}t} - 1 \right) \left( 8\theta(2\sqrt{5}(1 + e^{4\sqrt{5}t})) - 2 \right) \right) + \left( e^{4\sqrt{5}t} + 1 \right) (6\theta(1 + e^{4\sqrt{5}t}) - 4))
\]
Figure 19. Graph of $\alpha$ cut of the solution $x$ for Example 4.2.1.

Figure 20. Graph of $\beta$ cut of the solution $x$ for Example 4.2.1.

Figure 21. Graph of $\alpha$ cut of the solution $y$ for Example 4.2.1.
Figure 22. Graph of $\beta$ cut of the solution $y$ for Example 4.2.1.

Figure 23. Membership $\mu$ and non-membership $\nu$ functions of the solution $x$ for Example 4.2.1.
5 Summary and conclusions

The main goal of this paper is to give solutions to system of differential equations with intuitionistic fuzzy initial values under the interpretation of (i,ii)-GH differentiability and intuitionistic Zadeh’s extension principle concept. To do this we have firstly extended some theorems and definitions about (i,ii)-GH differentiability in fuzzy set theory in Section 3. Later, we have given a procedure to find the solutions to system of ordinary differential equations with triangular intuitionistic fuzzy initial values in Section 4. And we have given some numerical results in Section 4.

Under (i,ii)-GH differentiability concept or Zadeh’s extension principle interpretation, we have observed that the endpoints of $\alpha$ or $\beta$ of solutions may switch on subintervals where crisp solution exists. That is why, this fact makes the solution to be exists locally on subintervals. To cope with this Heaviside function can be used to write the endpoints of $\alpha$ or $\beta$ of the solutions.

References

[1] Akın, Ö., & Bayeğ, S. (2017). Intuitionistic fuzzy initial value problems - an application, *Hacettepe Journal of Mathematics and Statistics*, Doi:10.15672/HJMS.2018.598.
[2] Akın, Ö., & Bayeğ, S. (2017). Initial Value Problems in Intuitionistic Fuzzy Environment. Proceedings of the 5th International Fuzzy Systems Symposium, 14–15, Ankara, Turkey.

[3] Akın, Ö., Khaniyev, T., Öruç, Ö. & Türkşen, I. B. (2013). An algorithm for the solution of second order fuzzy initial value problems. Expert Systems with Applications, 40 (3), 953–957.

[4] Akın, Ö., Khaniyev, T., Bayeğ, S. & Türkşen (2016). Solving a Second Order Fuzzy Initial Value Problem Using The Heaviside mapping. Turk. J. Math. Comput. Sci., 4, 16–25.

[5] Akın, Ö., & Oruç, Ö. (2012). A prey predator model with fuzzy initial values. Hacettepe Journal of Mathematics and Statistics, 41, 387–395.

[6] Amrahov, Ş. E., Khastan, A., Gasilov, N. & Fatullayev, A. G. (2016). Relationship between Bede–Gal differentiable set-valued mappings and their associated support mappings, Fuzzy Sets and Systems, 295, 57–71.

[7] Atanassov K. T. (1983). Intuitionistic Fuzzy Sets, VII ITKR Session, Sofia, 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: Int. J. Bioautomation, 2016, 20(S1), S1–S6.

[8] Atanassov, K. T. (1986). Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1), 87–96.

[9] Atanassov, K. T. (1994). Operators over interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 64 (2), 159–174.

[10] Atanassov, K. T. (1995). Remarks on the intuitionistic fuzzy sets - III, Fuzzy Sets and Systems, 75 (3), 401–402.

[11] Atanassov, K. T. (1989). More on intuitionistic fuzzy sets. Fuzzy Sets and Systems, 33 (1), 37–45.

[12] Atanassov, K. T. (1988). Remark on the intuitionistic fuzzy logics, Fuzzy Sets and Systems, 95 (1), 127–129.

[13] Atanassov, K. T. (2000). Two theorems for intuitionistic fuzzy sets, Fuzzy Sets and Systems, 110(2), 267–269.

[14] Atanassov, K. T., Kacprzyk, J. & Szmidi, E. (2003). Separability of intuitionistic fuzzy sets, Fuzzy Sets and Systems, 64, 285–292.

[15] Atanassov, K. T., & Georgiev, C. (1993). Intuitionistic fuzzy Prolog, Fuzzy Sets and Systems, 53(2), 121–128.

[16] Atanassova, L. (2007). On intuitionistic fuzzy versions of L. Zadeh’s extension principle. Notes on Intuitionistic Fuzzy Sets, 13(3), 33–36.
[17] Barros, L. C., Bassanezi, R. C. & Tonelli, P. A. (2000). Fuzzy modelling in population dynamics, *Ecological Modelling*, 128(1), 27–33.

[18] Bede, B. (2010). Solutions of fuzzy differential equations based on generalized differentiability, *Communication in Mathematical Analysis*, 9, 22–41.

[19] Bede, B., & Gal, S. G. (2005). Generalizations of the differentiability of fuzzy-number-valued mappings with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, 151(3), 581–599.

[20] Bede, B., Rudas, I. J., & Bencsik, B. (2007). First order linear differential equations under generalized differentiability, *Information Sciences*, 177, 1648–1662.

[21] Bede, B., Stefanini, L. (2010). Generalized differentiability of fuzzy-valued functions, *Fuzzy Sets and Systems*, 9, 22–41.

[22] Buckley, J. J. & Feuring, T. (2000). Fuzzy differential equations, *Fuzzy Sets and Systems*, 110, 43–54.

[23] Casasnovas, J. & Rossell, F. (2005). Averaging fuzzy bio polymers, *Fuzzy Sets and Systems*, 152 (1), 139–158.

[24] De, S. K., Biswas, R., & Roy, A. R. (2001). An application of intuitionistic fuzzy sets in medical diagnosis, *Fuzzy Sets and Systems*, 117 (2), 209–213.

[25] Diamond, P. (2000). Stability and periodicity in fuzzy differential equations, *IEEE Transactions on Fuzzy Systems*, 8, 583–590.

[26] Diamond, P. & Kloeden, P. (1994). Metric Spaces of Fuzzy Sets, World Scientific Publishing, MA.

[27] Dost, S., & Brown, L. M. (2005). Intuitionistic textures revisited, *Hacettepe Journal of Mathematics and Statistics*, 34, 115–130.

[28] Duman, O. (2010). Statistical fuzzy approximation to fuzzy differentiable mappings by fuzzy linear operators, *Hacettepe Journal of Mathematics and Statistics*, 39, 497–514.

[29] Gasilov, N., Amrahov, Ş. E. & Fatullayev, A. G. (2014). Solution of linear differential equations with fuzzy boundary values, *Fuzzy Sets and Systems*, 257, 169–183.

[30] Goguen, J. A. (1967). L-fuzzy sets, *Journal of Mathematical Analysis and Applications*, 18(1), 145–174.

[31] Gouyandeha, Z., Allahviranloo, T., Abbasbandy, S. & Atefeh, A. (2017). A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform, *Fuzzy Sets and Systems*, 309, 81–97.
[32] Hullermeier, E. (1997). An approach to modelling and simulation of uncertain dynamical systems, *Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems*, 5, 117–137.

[33] Kharal, A. (2009). Homeopathic drug selection using intuitionistic fuzzy sets, *Homeopathy*, 98 (1), 35–39.

[34] Lei, Q., & Xu, Z. (2015). Fundamental properties of intuitionistic fuzzy calculus, *Knowledge-Based Systems*, 76, 1–16.

[35] Li, D. F. (2005). Multi-attribute decision making models and methods using intuitionistic fuzzy sets. *Journal of Computer and System Sciences*, 70(1), 73–85.

[36] Li, D. F., & Cheng, C. T. (2002). New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. *Pattern Recognition Letters*, 23(1–3), 221–225.

[37] Mendel, J. M. (2007). Advances in type-2 fuzzy sets and systems, *Information Sciences*, 177, 84–110.

[38] Mondal, S. P., Banerjee, S. & Roy, T. K. (2013). First Order Linear Homogeneous Ordinary Differential Equation in Fuzzy Environment, *Int. J. Pure Appl. Sci. Technol.*, 14(1), 16–26.

[39] Mondal, S. P., & Roy, T. K. (2014). First order homogeneous ordinary differential equation with initial value as triangular intuitionistic fuzzy number, *Journal of Uncertainty in Mathematics Science*, 1–17.

[40] Oberguggenberger, M. & Pittschmann, S. (1999). Differential equations with fuzzy parameters, *Mathematical and Computer Modelling of Dynamical Systems*, 5, 181–202.

[41] Puri, L. M., & Ralescu, D. (1983). Differentials of fuzzy functions, *Journal of Mathematical Analysis and Applications*, 91(2), 552–558.

[42] Shu, M.H., Cheng, C.H., & Chang, J. R. (2006). Using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly, *Microelectronics Reliability*, 46(12), 2139–2148.

[43] Simon, C. P. & Blume, L. E. (2004). Mathematics for Economics, W. W. Norton Company, New York.

[44] Tiel, J. V. (2004). *Convex Analysis: An Introductory Text*, John Wiley & Sons Ltd., New York.

[45] Ye, J. (2009). Multicriteria fuzzy decision-making method based on a novel accuracy mapping under interval valued intuitionistic fuzzy environment. *Expert Systems with Applications*, 36(3), 6899–6902.

[46] Zadeh, L. A. (1965). Fuzzy sets, *Information and Control*, 8(3), 338–353.