THE CLOSEDNESS OF COMPLETE SUBSEMILATTICES IN FUNCTIONALLY HAUSDORFF SEMITOPOLOGICAL SEMILATTICES

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Abstract. A topologized semilattice $X$ is complete if each non-empty chain $C \subset X$ has $\inf C \in \bar{C}$ and $\sup C \in \bar{C}$. It is proved that for any complete subsemilattice $X$ of a functionally Hausdorff semitopological semilattice $Y$ the partial order $P = \{(x, y) \in X \times X : xy = x\}$ of $X$ is closed in $Y \times Y$ and hence $X$ is closed in $Y$. This implies that for any continuous homomorphism $h : X \to Y$ from a complete topologized semilattice $X$ to a functionally Hausdorff semitopological semilattice $Y$ the image $h(X)$ is closed in $Y$. The functional Hausdorffness of $Y$ in these two results can be replaced by the weaker separation axiom $\vec{T}_2$, defined in this paper.

In this paper we continue to study the closedness properties of complete semitopological semilattices, which were introduced and studied by the authors in [1], [2], [3], [4], [5]. It turns out that complete semitopological semilattices share many common properties with compact topological semilattices, in particular their continuous homomorphic images in Hausdorff topological semilattices are closed.

A semilattice is any commutative semigroup of idempotents (an element $x$ of a semigroup is called an idempotent if $xx = x$).

Each semilattice carries a natural partial order $\leq$ defined by $x \leq y$ iff $xy = x = yx$. Many properties of semilattices are defined in the language of this partial order. In particular, for a point $x \in X$ we can consider its upper and lower sets

$\uparrow x := \{y \in X : xy = x\}$ and $\downarrow x := \{y \in X : xy = y\}$

in the partially ordered set $(X, \leq)$.

A subset $C$ of a semilattice $X$ is called a chain if $xy \in \{x, y\}$ for any $x, y \in C$. A semilattice $X$ is called chain-finite if each chain in $X$ is finite.

A semilattice endowed with a topology is called a topologized semilattice. A topologized semilattice $X$ is called a (semi)topological semilattice if the semigroup operation $X \times X \to X$, $(x, y) \mapsto xy$, is (separately) continuous.

In [13] Stepp proved that for any homomorphism $h : X \to Y$ from a chain-finite semilattice to a Hausdorff topological semilattice $Y$, the image $h(X)$ is closed in $Y$. In [1], the first two authors improved this result of Stepp proving the following theorem.

Theorem 1 (Banakh, Bardyla). For any homomorphism $h : X \to Y$ from a chain-finite semilattice to a Hausdorff semitopological semilattice $Y$, the image $h(X)$ is closed in $Y$.

Topological generalizations of the notion of chain-finiteness are the notions of chain-compactness and completeness, discussed in [5].

A topologized semilattice $X$ is called

- chain-compact if each closed chain in $X$ is compact;
- complete if each non-empty chain $C \subset X$ has $\inf C \in \bar{C}$ and $\sup C \in \bar{C}$.

Here $\bar{C}$ stands for the closure of $C$ in $X$. Chain-compact and complete topologized semilattices appeared to be very helpful in studying the closedness properties of topologized semilattices, see [1], [2], [3], [4], [5]. By Theorem 3.1 [1], a Hausdorff semitopological semilattice is chain-compact if and only if complete (see also Theorem 4.3 [5] for generalization of this characterization to topologized posets). In [1] the first two authors proved the following closedness property of complete topologized semilattices.

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Theorem 2 (Banakh, Bardyla). For any continuous homomorphism \( h : X \to Y \) from a complete topologized semilattice \( X \) to a Hausdorff topological semilattice \( Y \), the image \( h(X) \) is closed in \( Y \).

Theorems 1 and 2 motivate the following (still) open problem.

Problem 1. Assume that \( h : X \to Y \) is a continuous homomorphism from a complete topologized semilattice \( X \) to a Hausdorff semitopological semilattice \( Y \). Is \( h(X) \) closed in \( Y \)?

In [4] the first two authors gave the following partial answer to Problem 1.

Theorem 3 (Banakh, Bardyla). For any continuous homomorphism \( h : X \to Y \) from a complete topologized semilattice \( X \) to a sequential Hausdorff semitopological semilattice \( Y \), the image \( h(X) \) is closed in \( Y \).

In this paper we shall show that the answer to Problem 1 is affirmative under the additional condition that the semitopological semilattice \( Y \) satisfies the separation axiom \( \bar{T}^{2\delta} \), which is stronger that the Hausdorff axiom \( T_2 \) and weaker than the axioms \( T_3 \) of regularity and \( \bar{T}_3^{\frac{3}{2}} \) of functional Hausdorffness.

We recall that a topological space \( X \) is defined to satisfy the separation axiom

- \( T_1 \) if for any distinct points \( x, y \in X \) there exists an open set \( U \subset X \) such that \( x \in U \) and \( y \notin U \);
- \( T_2 \) if for any distinct points \( x, y \in X \) there exists an open set \( U \subset X \) such that \( x \in U \subset \bar{U} \subset X \setminus \{y\} \);
- \( T_3 \) if \( X \) is a \( T_1 \)-space and for any open set \( V \subset X \) and point \( x \in V \) there exists an open set \( U \subset X \) such that \( x \in U \subset \bar{U} \subset V \);
- \( T_3^{\frac{3}{2}} \) if \( X \) is a \( T_1 \)-space and for any open set \( V \subset X \) and point \( x \in V \) there exists a continuous function \( f : X \to [0,1] \) such that \( x \in f^{-1}([0,1]) \subset V \).

In these definitions by \( \bar{A} \) we denote the closure of a subset \( A \) in a topological space.

Topological spaces satisfying a separation axiom \( T_i \) are called \( T_i \)-spaces; \( T_2 \)-spaces are called Hausdorff. Now we define a new separation axiom.

Definition 1. A topological space \( X \) is defined to satisfy the separation axiom

- \( T_{2\delta} \) if \( X \) is a \( T_1 \)-space and for any open set \( U \subset X \) and point \( x \in U \) there exists a countable family \( \mathcal{U} \) of closed neighborhoods of \( x \) in \( X \) such that \( \bigcap \mathcal{U} \subset U \).

We shall say that a topological space \( X \) satisfies the separation axiom

- \( \bar{T}_i \) for \( i \in \{1, 2, 2\delta, 3, 3^{\frac{3}{2}}\} \) if \( X \) admits a bijective continuous map \( X \to Y \) to a \( T_i \)-space \( Y \).

The following diagram describes the implications between the separation axioms \( T_i \) and \( \bar{T}_i \) for \( i \in \{1, 2, 3, 3^{\frac{3}{2}}\} \).

\[
\begin{array}{cccccc}
T_{\frac{3}{2}}^{\frac{3}{2}} & \Rightarrow & T_3 & \Rightarrow & T_{2\delta} & \Rightarrow & T_2 & \Rightarrow & T_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bar{T}_{\frac{3}{2}}^{\frac{3}{2}} & \Rightarrow & \bar{T}_3 & \Rightarrow & \bar{T}_{2\delta} & \Rightarrow & \bar{T}_2 & \Rightarrow & \bar{T}_1 
\end{array}
\]

Observe that a topological space \( X \) satisfies the separation axiom \( \bar{T}_{2\delta}^{\frac{3}{2}} \) if and only if it is functionally Hausdorff in the sense that for any distinct points \( x, y \in X \) there exists a continuous function \( f : X \to \mathbb{R} \) with \( f(x) \neq f(y) \). So, each functionally Hausdorff space is a \( \bar{T}_{2\delta} \)-space.

The following theorem gives a partial answer to Problem 1 and is the main result of this paper.

Theorem 4. For any continuous homomorphism \( h : X \to Y \) from a complete topologized semilattice \( X \) to a semitopological semilattice \( Y \) satisfying the separation axiom \( \bar{T}_{2\delta} \), the image \( h(X) \) is closed in \( Y \).

Theorem 4 will be proved in Section 2 after some preliminary work made in Sections 1. As a by-product of the proof we obtain the following useful result.

Theorem 5. For any complete semitopological semilattice \( X \) satisfying the separation axiom \( \bar{T}_{2\delta} \), the partial order \( \leq := \{(x, y) \in X \times X : xy = x\} \) of \( X \) is a closed subset of \( X \times X \).

Remark 1. The completeness of \( X \) is essential in Theorem 5 by [9], there exists a metrizable semitopological semilattice \( X \) whose partial order is not closed in \( X \times X \), and for every \( x \in X \) the upper set \( \uparrow x \) is finite.
1. The closedness of the partial order of a semitopological semilattice

In this section we shall prove Theorem 7 implying Theorem 5. In the proof we shall use the following lemma.

Lemma 1. Each non-empty subsemilattice $S$ of a complete topologized semilattice $X$ has $\inf S \in \bar{S}$.

Proof. By transfinite induction, we shall prove that for any non-zero cardinal $\kappa$ the following condition holds:

$(*)_{\kappa}$ each non-empty subsemilattice $S \subset X$ of cardinality $|S| \leq \kappa$ has $\inf S \in \bar{S}$.

For any finite cardinal $\kappa$ the statement $(*)_{\kappa}$ is true (as each finite semilattice contains the smallest element).

Assume that for some infinite cardinal $\kappa$ and any non-zero cardinal $\lambda < \kappa$ the statement $(*)_{\lambda}$ has been proved. To prove the statement $(*)_{\kappa}$, choose any subsemilattice $S \subset X$ of cardinality $|S| = \kappa$. Write $S = \{x_\alpha\}_{\alpha \in \kappa}$. For every ordinal $\beta \in \kappa$ consider the subsemilattice $S_\beta$ generated by the set $\{x_\alpha : \alpha \leq \beta\}$. Since $\lambda := |S_\beta| \leq |\omega + \beta| \leq \kappa$, by the condition $(*)_{\lambda}$, the semilattice $S_\beta$ has $\inf S_\beta \in \bar{S}_\beta \subset \bar{S}$. For any ordinals $\alpha < \beta$ in $\kappa$ the inclusion $S_\alpha \subset S_\beta$ implies $\inf S_\beta \leq \inf S_\alpha$. By the $k$-completeness of $X$, the chain $C = \{\inf S_\alpha : \alpha \in \kappa\} \subset \bar{S}$ has $\inf C \in C \subset \bar{S}$. We claim that $\inf C = \inf S$. Indeed, for any $\alpha \in \kappa$ we get $\inf C \leq \inf S_\alpha \leq x_\alpha$, so $\inf C$ is a lower bound for the set $S$ and hence $\inf C \leq \inf S$. On the other hand, for any lower bound $b$ of $S$ and every $\alpha \in \kappa$, we get $S_\alpha \subset S \cup b$ and hence $b \leq \inf S_\alpha$. Then $\inf C = \inf \{\inf S_\alpha : \alpha \in \kappa\} \geq b$ and finally $\inf C = \inf S$.\hfill $\square$

The following theorem is technically the most difficult result of this paper.

Theorem 6. Let $X$ be a complete subsemilattice of a semitopological semilattice $Y$ satisfying the separation axiom $T_{2\beta}$. Then the partial order $P := \{(x, y) \in X \times X : xy = x\}$ of $X$ is closed in $Y \times Y$.

Proof. Given two points $x, y \in Y$ we should prove that $(x, y) \in P$ if for any neighborhoods $O_x$ and $O_y$ of $x$ and $y$ in $Y$ there are points $x' \in O_x \cap X$ and $y' \in O_y \cap X$ with $x' \leq y'$. By transfinite induction, for every infinite cardinal $\kappa$ we shall prove the following statement:

$(*)_{\kappa}$ for any families $U_x, U_y$ of closed neighborhoods of $x$ and $y$ with $\max\{|U_x|, |U_y|\} \leq \kappa$ there are points $x' \in X \cap \bigcap U_x$ and $y' \in X \cap \bigcap U_y$ such that $x' \leq y'$.

Claim 1. The statement $(*)_{\omega}$ holds.

Proof. Fix any countable families $(U_n)_{n \in \omega}$ and $(V_n)_{n \in \omega}$ of closed neighborhoods of the points $x, y$, respectively. Replacing each set $U_n$ by $\bigcap_{i \leq n} U_i$, we can assume that $U_{n+1} \subset U_n$ for all $n \in \omega$. By the same reason, we can assume that the sequence $(V_n)_{n \in \omega}$ is decreasing. For every $n \in \omega$ denote by $U_n$ and $V_n$ the interiors of the sets $U_n$ and $V_n$ in $Y$.

By induction we shall construct sequences $(x_n)_{n \in \omega}$ and $(y_n)_{n \in \omega}$ of points of $X$ such that for every $n \in \omega$ the following conditions are satisfied:

$(1_n)$ $x_n \leq y_n$;

$(2_n)$ $\{x_1 \cdots x_n, x_1 \cdots x_n x\} \subset U^*_x$ for all $i \leq n$;

$(3_n)$ $\{y_1 \cdots y_n, y_1 \cdots y_n y\} \subset V^*_y$ for all $i \leq n$.

To choose the initial points $x_0, y_0$, find neighborhoods $U'_0 \subset U^*_0$ and $V'_0 \subset V^*_0$ of $x$ and $y$ in $Y$ such that $U'_0 x \subset U^*_0$ and $V'_0 y \subset V^*_0$. By our assumption, there are points $x_0 \in X \cap U'_0$ and $y_0 \in X \cap V'_0$ such that $x_0 \leq y_0$. The choice of the neighborhoods $U'_0$ and $V'_0$ ensures that the conditions $(2)_0$ and $(3)_0$ are satisfied.

Now assume that for some $n \in \mathbb{N}$ points $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{n-1}$ of $X$ are chosen so that the conditions $(1_{n-1}) \sim (3_{n-1})$ are satisfied. The condition $(2_{n-1})$ implies that for every $i \leq n$ we have the inclusion $x_i \cdots x_{n-1} x = x_i \cdots x_{n-1} x \in U^*_i$ (if $i = n$, then we understand that $x_i \cdots x_{n-1} x = x$). Using the continuity of the shift $s_x : Y \to Y, s_x : z \mapsto xz$, we can find a neighborhood $U'_i \subset Y$ of $x$ such that $x_i \cdots x_{n-1} (U'_i \cup U^*_i x) \subset U^*_i$ for every $i \leq n$. By analogy, we can find a neighborhood $V'_i \subset Y$ of $y$ such that $y_i \cdots y_{n-1} (V'_i \cup V^*_i y) \subset V^*_i$ for every $i \leq n$. By our assumption, there are points $x_n \in X \cap U'_n$ and $y_n \in X \cap V'_n$ such that $x_n \leq y_n$. The choice of the neighborhoods $U'_n$ and $V'_n$ ensures that the conditions $(2)_n$ and $(3)_n$ are satisfied. This completes the inductive step.

Now for every $i \in \omega$ consider the chain $C_i = \{x_1 \cdots x_n : n \geq i\} \subset U^*_i$ in $X$. By the completeness of $X$, this chain has $\inf C_i \in X \cap C_i \subset X \cap \overline{\bigcup C_i} \subset X \cap U_i$. Observing that $\inf C_i \leq x_i x_{i+1} \cdots x_n \leq x_{i+1} \cdots x_n$ for all $i > n$, we see that $\inf C_i$ is a lower bound of the chain $C_{i+1}$ and hence $\inf C_i \leq \inf C_{i+1}$. By the completeness
of $X$, for every $i \in \omega$ the chain $D_i := \{\inf C_i : j \geq i\} \subseteq U_i$ has $\sup D_i \in X \cap D_i \subseteq X \cap U_i$. Since the sequence $(\inf C_i)_{i \in \omega}$ is increasing, we get $\sup D_0 = \sup D_i \in X \cap U_i$ for all $i \in \omega$. Consequently, $\sup D_0 \in X \cap \bigcap_{i \in \omega} U_i$.

By analogy, for every $k \in \omega$ consider the chain $E_k = \{y_k, \ldots, y_n : n \geq i\} \subseteq V_i$. By the completeness of $X$, this chain has $\inf E_i \in X \cap E_i \subseteq X \cap V_i$. By the completeness of $X$, for every $i \in \omega$ the chain $F_i := \{\inf E_i : j \geq i\} \subseteq V_i$ has $\sup F_i \in X \cap F_i \subseteq X \cap V_i$. Since the sequence $(\inf E_i)_{i \in \omega}$ is increasing, we get $\sup F_0 = \sup F_i \in X \cap V_i$ for all $i \in \omega$. Consequently, $\sup F_0 \in X \cap \bigcap_{i \in \omega} V_i$.

To finish the proof of Claim 1 it suffices to show that $\sup D_0 \leq \sup F_0$. The inductive conditions $(1_n), n \in \omega$, imply that $\inf C_i \leq \inf E_i$ for all $i \in \omega$ and $\sup D_0 = \sup \{\inf C_i : i \in \omega\} \leq \sup \{\inf E_i : \sup F_0 \in X \cap V_i\}$.

Now assume that some uncountable cardinal $\kappa$ we have proved that the property $(\star \lambda)$ holds for any infinite cardinal $\lambda < \kappa$. To prove the property $(\star \kappa)$, fix families $(U_\alpha)_{\alpha \in \kappa}$ and $(V_\alpha)_{\alpha \in \kappa}$ of closed neighborhoods of $x$ and $y$ in $Y$, respectively.

By transfinite induction, we shall construct sequences $(x_\alpha)_{\alpha \in \kappa}$ and $(y_\alpha)_{\alpha \in \kappa}$ of points of $X$ such that for every $\alpha \in \kappa$ the following conditions are satisfied:

$(1_\alpha)$ $x_\alpha \leq y_\alpha$;

$(2_\alpha)$ $\{x_\alpha, \ldots, x_{\alpha-1}, x_\alpha \cdot \cdot \cdot x_{\alpha-n}, x\} \subseteq U_\alpha$ for any sequence of ordinals $\alpha_0 < \alpha_1 < \cdot \cdot \cdot < \alpha_n = \alpha$ with $n \in \mathbb{N}$;

$(3_\alpha)$ $\{y_\alpha, \ldots, y_{\alpha-1}, y_\alpha \cdot \cdot \cdot y_{\alpha-n}, y\} \subseteq V_\alpha$ for any sequence of ordinals $\alpha_0 < \alpha_1 < \cdot \cdot \cdot < \alpha_n = \alpha$ with $n \in \mathbb{N}$.

To start the inductive construction, choose neighborhoods $U_0, V_0 \subseteq Y$ of the points $x$ and $y$, such that $U_0 \cup V_0 \subseteq U$ and $V_0 \cup U_0 \subseteq V_0$. By our assumption there are points $x_0 \in X \cap U_0$ and $y_0 \in X \cap V_0$ such that $x_0 \leq y_0$.

Now assume that for some ordinal $\beta < \kappa$ we have constructed points $x_\alpha, y_\alpha$ for all ordinals $\alpha < \beta$ so that the conditions $(1_\alpha)-(3_\alpha)$ are satisfied. For any $n \in \mathbb{N}$ and any sequence of ordinals $\alpha_0 < \cdot \cdot \cdot < \alpha_n = \beta$ use the inductive condition $(2_{\alpha_n-1})$ and the continuity of the shift $s_\beta : Y \to Y$ for finding a neighborhood $U_{\alpha_n-1} \subseteq Y$ of $x$ such that

$$x_\alpha \cdot \cdot \cdot x_{\alpha-n}, U_{\alpha_n-1} \subseteq U_{\alpha_n-1} \cdot \cdot \cdot U_{\alpha_0} \subseteq U_{\alpha_0}.$$ 

If $n = 1$, then we assume that $x_\alpha \cdot \cdot \cdot x_{\alpha-n}, U_{\alpha_n-1} \subseteq U_{\alpha_n-1} \cdot \cdot \cdot U_{\alpha_0} \subseteq U_{\alpha_0}$.

In similar way choose a neighborhood $V_{\alpha_n-1} \subseteq Y$ of $y$ such that

$$y_\alpha \cdot \cdot \cdot y_{\alpha-n}, V_{\alpha_n-1} \subseteq V_{\alpha_n-1} \cdot \cdot \cdot V_{\alpha_0} \subseteq V_{\alpha_0}.$$ 

Since $Y$ is a $T_\infty$-space, there are countable families $U_{\alpha_n-1} \subseteq U_{\alpha_n-1}$ and $V_{\alpha_n-1} \subseteq V_{\alpha_n-1}$ of closed neighborhoods of $x$ and $y$ respectively such that $\bigcap U_{\alpha_n-1} \subseteq U_{\alpha_n-1}$ and $\bigcap V_{\alpha_n-1} \subseteq V_{\alpha_n-1}$. Also choose countable families $U_\beta$ and $V_\beta$ of closed neighborhoods of $x$ and $y$ in $Y$ such that $\bigcap U_\beta \subseteq U_\beta$ and $\bigcap V_\beta \subseteq V_\beta$.

Now consider the families

$$U_\beta = \bigcup_{n=0}^{\infty} \{U_{\alpha_0}, \ldots, \alpha_n \cdot \cdots \cdot \alpha_0 : \alpha_0 < \cdot \cdot \cdot < \alpha_n = \beta\} \text{ and } V_\beta = \bigcup_{n=0}^{\infty} \{V_{\alpha_0}, \ldots, \alpha_n \cdot \cdots \cdot \alpha_0 : \alpha_0 < \cdot \cdot \cdot < \alpha_n = \beta\}$$

and observe that $\lambda := \max\{|\beta|, |\beta|\} \leq |\omega + \beta| < \kappa$. By the inductive assumption, the condition $(\star \kappa)$ holds. So we can find points $x_\alpha \in X \cap \bigcap U_\beta$ and $y_\alpha \in X \cap \bigcap V_\beta$ such that $x_\alpha \leq y_\beta$. Observe that for any $n \in \mathbb{N}$ and any sequence of ordinals $\alpha_0 < \cdot \cdot \cdot < \alpha_n = \beta$ we have

$$x_\alpha \cdot \cdot \cdot x_{\alpha-n} \subseteq x_\alpha \cdot \cdot \cdot x_{\alpha-n-1} \cdot \bigcap U_{\alpha-n} \subseteq x_\alpha \cdot \cdot \cdot x_{\alpha-n-1} \cdot \bigcap U_{\alpha-n} \subseteq U_{\alpha-n} \subseteq U_{\alpha_0}.$$ 

By analogy we can prove that $x_\alpha \cdot \cdot \cdot x_{\alpha-n} \subseteq U_{\alpha_0}$, which means that the inductive condition $(2_\beta)$ is satisfied. By analogy we can check that the inductive condition $(3_\beta)$ is satisfied. This completes the inductive step.

After completing the inductive construction, for every $\alpha \in \kappa$ consider the subsemilattices

$$C_\alpha := \{x_\alpha, \ldots, x_{\alpha-n}, \cdot \alpha < \alpha_0 < \cdots < \alpha_n = \kappa\} \text{ and } E_\alpha := \{y_\alpha, \ldots, y_{\alpha-n}, \cdot \alpha < \alpha_0 < \cdots < \alpha_n = \kappa\}$$

of $X$. The inductive conditions $(2_\beta)$ and $(3_\beta)$ for $\beta > \alpha$ imply that $C_\alpha \subseteq U_\alpha \subseteq U_\alpha \subseteq V_\alpha$. By Lemma 1 the semilattices $C_\alpha$ and $E_\alpha$ have $\inf C_\alpha \in X \cap C_\alpha \subseteq X \cap U_\alpha$ and $\inf E_\alpha \in X \cap E_\alpha \subseteq X \cap V_\alpha$. For any ordinals $\alpha < \beta$ in $\kappa$ the inclusions $C_\beta \subseteq C_\alpha$ and $E_\beta \subseteq E_\alpha$ imply that the transfinite sequences $(\inf C_\alpha)_{\alpha \in \kappa}$ and $(\inf E_\alpha)_{\alpha \in \kappa}$ are non-decreasing. By the completeness of $X$ for every $\alpha \in \kappa$ the chains $D_\alpha = \{\inf C_\beta : \alpha < \beta < \kappa\} \subseteq X \cap U_\alpha$ and $F_\alpha = \{\inf E_\beta : \alpha < \beta < \kappa\} \subseteq X \cap V_\alpha$ have $\sup D_\alpha \subseteq X \cap D_\alpha \subseteq X \cap U_\alpha$ and $\sup F_\alpha \subseteq X \cap F_\alpha \subseteq X \cap V_\alpha$. Taking into account that the transfinite sequences $(\inf C_\alpha)_{\alpha \in \kappa}$ and $(\inf E_\alpha)_{\alpha \in \kappa}$ are non-decreasing, we conclude that $\sup D_0 = \sup D_\alpha \subseteq X \cap U_\alpha$ and $\sup F_0 = \sup E_\alpha \subseteq V_\alpha$. Consequently $\sup D_0 \subseteq X \cap \bigcap_{\alpha \in \kappa} U_\alpha$ and $\sup F_0 \subseteq X \cap \bigcap_{\alpha \in \kappa} V_\alpha$. 


To finish the proof of the property \((\ast, \kappa)\), it suffices to show that \(\sup D_0 \leq \sup F_0\). The inductive conditions \((1, \alpha)\), \(\alpha \in \kappa\), imply that \(\inf C_\alpha \leq \inf E_\alpha\) for all \(\alpha \in \omega\) and then

\[
\sup D_0 = \sup \{\inf C_\alpha : \alpha \in \kappa\} \leq \sup \{\inf E_\alpha : \alpha \in \kappa\} = \sup F_0.
\]

By the Principle of Transfinite Induction, the property \((\ast, \kappa)\) holds for all infinite cardinals \(\kappa\).

Denote by \(U_x\) and \(U_y\) the families of closed neighborhoods of the points \(x\) and \(y\), respectively. Taking into account that the space \(Y\) is Hausdorff, we conclude that \(\{x\} = \bigcap U_x\) and \(\{y\} = \bigcap U_y\). The property \((\ast, \kappa)\) for \(\kappa = \max\{|U_x|, |U_y|\}\) implies that there are points \(x' \in X \cap \bigcap U_x = X \cap \{x\}\) and \(y' \in X \cap \bigcap U_y = X \cap \{y\}\) such that \(x' \leq y'\). It follows that \(x = x' \leq y' = y\) and hence \((x, y) \in P\).

Theorem 7 implies its own self-improvement.

**Theorem 7.** Let \(X\) be a complete subsemilattice of a semitopological semilattice \(Y\) satisfying the separation axiom \(T_{2\delta}\). Then the partial order \(P := \{(x, y) \in X \times X : xy = x\}\) of \(X\) is closed in \(Y X\) and the semilattice \(X\) is closed in \(Y\).

**Proof.** Since \(Y\) satisfies the separation axiom \(T_{2\delta}\) and the class of \(T_{2\delta}\)-spaces is closed under taking subspaces and Tychonoff products, the space \(Y\) admits a weaker topology \(\tau\) such that \((Y, \tau)\) is a \(T_{2\delta}\)-space and any continuous map \(Y \to Z\) to a \(T_{2\delta}\)-space \(Z\) remains continuous in the topology \(\tau\). This implies that for every \(a \in Y\) the shift \(s_a : Y \to Y, s_a : y \mapsto ay\), is continuous with respect to the topology \(\tau\) and hence \(Y\) endowed with the topology \(\tau\) is a semitopological semilattice.

Since the identity map \(Y \to (Y, \tau)\) is continuous, the completeness of the subsemilattice \(X \subset Y\) implies the completeness of \(X\) with respect to the subspace topology inherited from \(\tau\). Applying Theorem 6, we conclude that the partial order \(P = \{(x, y) \in X \times X : xy = x\}\) is closed in the square of the space \((Y, \tau)\) and hence is closed in the original topology of \(Y\) (which is stronger than \(\tau\)).

By Lemma 1 the semilattice \(X\) has the smallest element \(s \in X\). Observing that \(X = \{x \in Y : (s, x) \in P\}\), we see that the set \(X\) is closed in \(Y\).

2. Proof of Theorem 4

Let \(h : X \to Y\) be a continuous homomorphism from a complete topologized semilattice \(X\) to a semitopological semilattice \(Y\) satisfying the separation axiom \(T_{2\delta}\). We need to show that the subsemilattice \(h(X)\) is closed in \(Y\). This will follow from Theorem 7 as soon as we show that the semitopological semilattice \(h(X)\) is complete.

Given any non-empty chain \(C \subset h(X)\), we should show that \(C\) has \(\inf C \in \bar{C}\) and \(\sup C \in \bar{C}\).

For every \(c \in C\) consider the closed subsemilattice \(S_c := h^{-1}(<c)\) in \(X\). By Lemma 1 the semilattice \(S_c\) has \(\inf S_c \in \bar{S}_c = S_c\), which means that \(\inf S_c\) is the smallest element of \(S_c\). It follows from \(h(S_c) = \tau c\) that \(h(\inf S_c) \in h(S_c) = \tau c\) and hence \(c \leq h(\inf S_c)\). On the other hand, for any \(x \in h^{-1}(c)\) we get \(\inf S_c \leq x\) and hence \(h(\inf S_c) \leq h(x) = c\) and finally \(h(\inf S_c) = c\).

It is clear that for any \(x \leq y \in C\), we get \(S_x \subset S_y\) and hence \(\inf S_x \leq \inf S_y\). This means that \(D = \{\inf S_x : x \in C\}\) is a chain in \(X\). By the completeness of \(X\), the chain \(D\) has \(\inf D \in \bar{D}\) and \(\sup D \in \bar{D}\). The continuity of the homomorphism \(h\) implies that \(h(\inf D) \in h(\bar{D}) \subset h(\bar{D}) = \bar{C}\) and \(h(\sup D) \in h(\bar{D}) \subset h(\bar{D}) = \bar{C}\). It remains to show that \(h(\inf D) = \inf C\) and \(h(\sup D) = \sup C\).

Taking into account that \(h\) is a semilattice homomorphism, we can show that \(h(\inf D)\) is a lower bound of the set \(C = h(D)\) in \(Y = h(X)\). For any other lower bound \(b \in h(X)\) of \(C\), we see that \(C \subset \uparrow b, D \subset h^{-1}(\uparrow b) = S_b\) and hence \(\inf S_b \leq \inf D\), which implies \(b \leq h(\inf S_b) \leq h(\inf D)\). So, \(\inf C = h(\inf D)\). By analogy we can prove that \(\sup C = h(\sup D)\).
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