Monotone path-connectedness and solarity of Menger-connected sets in Banach spaces

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Abstract. We prove that every boundedly compact m-connected (Menger-connected) set is monotone path-connected and is a sun in a broad class of Banach spaces (in particular, in separable spaces). We show that the intersection of a boundedly compact monotone path-connected (m-connected) set with a closed ball is cell-like (of trivial shape) and, in particular, acyclic (contractible in the finite-dimensional case) and is a sun. We also prove that every boundedly weakly compact m-connected set is monotone path-connected. In passing, we extend the Rainwater–Simons weak convergence theorem to the case of convergence with respect to the associated norm (in the sense of Brown).

Keywords: sun, acyclic set, cell-like set, monotone path-connected set, Menger connectedness, d-convexity, Menger convexity, Rainwater–Simons theorem.

§ 1. Introduction and main definitions

For a bounded subset \( M \neq \emptyset \) of a real normed linear space \( X \), we let \( m(M) \) denote the intersection of all closed balls containing \( M \) (following Brown [1], \( m(M) \) is referred to as the Banach–Mazur hull, or ball hull, of \( M \)). A set \( M \subset X \) is said to be m-connected (Menger-connected) [1] if \( m(\{x, y\}) \cap M \neq \{x, y\} \) for any \( x, y \in M \). For brevity, we write \( m(\{x, y\}) = m(x, y) \). Despite the name, a closed m-connected subset of an infinite-dimensional space need not be connected.

Let \( k(\tau), 0 \leq \tau \leq 1, \) be a continuous curve in a normed linear space \( X \). Following [2], we say that \( k(\cdot) \) is monotone if \( f(k(\tau)) \) is a monotone function of \( \tau \) for every \( f \in \text{ext} S^* \) (here and in what follows, \( \text{ext} S^* \) is the set of extreme points of the unit sphere \( S^* \) in the dual space).

A closed set \( M \subset X \) is said to be monotone path-connected [3] if any two points of \( M \) can be connected by a continuous monotone curve (an arc) \( k(\cdot) \subset M \).

That m-connected and monotone path-connected sets arise naturally in the study of the connectedness of suns has been demonstrated by Berens and Hetzelt and further by Brown and the author (see [1], [3]–[5]): for a monotone path-connected sun one can answer the long-standing question on the connectedness

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and even the acyclicity) of the intersection of a sun with a ball, thereby substantially supplementing Vlasov’s well-known theorem on the solarity of acyclic sets.

Given \( x \in X \) and \( \emptyset \neq M \subset X \), we let \( P_M x \) denote the metric projection of \( X \) onto \( M \) (the set of best approximations to \( x \) in \( M \)).

A subset \( M \neq \emptyset \) of a normed linear space \( X \) is called a sun [6] if, for every \( x \in X \setminus M \), there is a point \( y \in P_M x \) (a point of luminosity) such that
\[
y \in P_M [(1 - \lambda)y + \lambda x] \quad \text{for all} \quad \lambda \geq 0.
\] (1.1)

The concept of a sun was introduced by Efimov and Stechkin in 1958 for the study of Chebyshev sets, although what they understood by a ‘sun’ is now called a strict protosun (such sets \( M \) are defined by the property that (1.1) holds for all \( x \notin M \) and \( y \in P_M x \)).

A Chebyshev monotone path-connected set is always a sun [7]. Moreover, if \( M \) is monotone path-connected and \( P_M x = \{y\}; x \notin M \), then \( y \) is a point of luminosity. It follows from the definition that monotone path-connectedness is preserved under intersections of a monotone path-connected set with extreme hyperplanes and, moreover, with arbitrary bars (that is, sets formed by intersections of hyperstrips generated by extreme points of the dual sphere [8]). Since every closed ball is a bar, every monotone path-connected set is \( B \)-monotone path-connected, that is, its intersection with any closed (and hence, with any open) ball is monotone path-connected. Therefore [5], a monotone path-connected set is necessarily \( m \)-connected

The converse does not hold. An example was given by Franchetti and Roversi [9]: \( M = M_1 \cup M_2 \), where \( M_i = \{x \in C[0,1] \mid x(0) = i, i = 1, 2\} \). However, for closed sets in \( c_0 \) (see [4]) and boundedly compact sets in an arbitrary separable space \( X \), these properties are equivalent (see Theorem 4.1 below).

We shall be concerned with topological and approximative properties of \( m \)-connected (Menger-connected) sets and monotone path-connected sets and with their solarity. Theorem 4.1 asserts that in a broad class of Banach spaces (in particular, in separable spaces) every boundedly compact \( m \)-connected set is monotone path-connected. Furthermore, the intersection of a boundedly compact monotone path-connected set with a closed ball is cell-like (has the shape of a point) and, in particular, it is acyclic and is a sun. This result partially complements Vlasov’s well-known theorem [6] asserting that every boundedly compact \( P \)-acyclic subset of a Banach space is a sun. The case of weakly compact sets is examined in Theorem 4.2. In the course of the proof of Theorem 4.1 we extend the classical Rainwater–Simons theorem on weak convergence of sequences to the case of convergence in the associated norm (Proposition 3.1).

§ 2. Acyclic and cell-like sets

A homology (cohomology) theory associates with every topological space \( X \) a sequence of abelian groups \( H_k(X), k = 0, 1, 2, \ldots \) (homology groups) and \( H^k(X), k = 0, 1, 2, \ldots \) (cohomology groups), which are homotopy invariants of the space: if two spaces are homotopy equivalent, then their corresponding (co)homology groups are isomorphic. There are several ways to construct (co)homology groups.
We mention the construction based on nerves of covers (proposed by Alexandroff and further extended by Čech), the Vietoris construction based on the concept of true cycles and applicable to metric spaces, and the construction based on the concept of singular chains (continuous images of simplicial chains).

Let $A$ be an arbitrary non-trivial abelian group. A space (throughout, all spaces are assumed to be metrizable) is said to be acyclic if its Čech cohomology group with coefficients in $A$ is trivial (it has no cycles other than boundaries). Thus, the definition of acyclicity depends on the group of coefficients. It is worth noting that Alexandroff–Čech homology is not a homology theory because the exactness axiom does not hold, whereas Čech cohomology is a cohomology theory of topological spaces. For a comprehensive account of (co)homology theory on compacta, topological and uniform spaces, the reader is referred to Melikhov’s recent survey [10].

If a (co)homology has compact support (satisfies the compact supports axiom) and the coefficients of the (co)homology group lie in a field, then the notions of homological and cohomological acyclicity coincide [11]. However, this is not the case for an arbitrary abelian coefficient group. For example, the 2-adic solenoid (the inverse limit of the sequence $S^1 \xleftarrow{f} S^1 \xleftarrow{f} \cdots$, where $f = z^2$) is acyclic in the Čech homology with coefficients in the field $\mathbb{Z}_2$, but there is no acyclicity in the Čech cohomology (see, for example, [12]).

Unless otherwise stated, we shall understand acyclicity in the sense of Čech cohomology with coefficients in an arbitrary abelian group.

A compact non-empty space is called an $R_\delta$-set (see, for example, [13], (2.11)) if it is homeomorphic to the intersection of a descending sequence of compact contractible spaces (or compact absolute retracts; see [13], Theorem 2.13). $R_\delta$-sets naturally arise as spaces of solutions of Cauchy problems and of autonomous or non-autonomous differential inclusions [14]–[16]. Results of this kind date back to Aronshain.

A compact space $Y$ is said to be cell-like (or to have the shape of a point) if there are an ANR-space (absolute neighbouring retract) $Z$ and an embedding $i : Y \to Z$ such that $i(Y)$ is contractible in any of its neighbourhoods $U \subset Z$ (see [13], (82.4)). A cell-like set need not be contractible. It follows from Hyman’s well-known characterization of $R_\delta$-sets that all $R_\delta$-sets are cell-like ([17], §4.2 and [18], p. 50). But since every map of a compact space having the shape of a point into an ANR is homotopically trivial, we see that every compact space having the shape of a point (every cell-like space) is contractible in each of its neighbourhoods in any ambient ANR. As a corollary, the classes of $R_\delta$-sets and cell-like compact spaces coincide.

Note that cell-likeness implies acyclicity (with respect to any continuous (co)homology theory; see [17], p. 854), but there are acyclic sets which are not cell-like, as well as cell-like sets which are not path-connected (the topologist’s sine curve).

Following Vlasov [6], if $Q$ denotes some property (such as connectedness), then we say that $M$ is

- $P$-$Q$ if $P_M(x)$ is non-empty and possesses property $Q$ for all $x \in X$;
- $B$-$Q$ if $M \cap B(x, r)$ possesses property $Q$ for all $x \in X$, $r > 0$;
- $\bar{B}$-$Q$ if $M \cap \bar{B}(x, r)$ possesses property $Q$ for all $x \in X$, $r > 0$. 
Thus every closed subset of a finite-dimensional space is $P$-non-empty. In other words, it is an existence set or a proximinal set. Correspondingly, a set $M$ is $P$-acyclic if $P_Mx$ is non-empty and acyclic for every $x$. A set $M$ is $B$-acyclic if the intersection of $M$ with any closed ball is acyclic.

Remark 2.1. $B$-connected sets are also said to be $V$-connected (here $V$ comes from the work of Vlasov, who denoted balls by $V(x, r)$). Our term ‘$B$-connectedness’ agrees with the more conventional notation, where balls are denoted by $B(x, r)$, as well as with the term ‘bounded connectedness’ introduced by Wulbert in the 1960s.

Brown ([19], Corollary 1.6.2) proved that if a boundedly compact subset $M$ of a Banach space is $P$-acyclic (relative to Čech cohomology with coefficients in an arbitrary abelian group), then $M$ is $B$-acyclic.\footnote{Brown [19] actually proved more: if $M$ is a $P$-acyclic approximatively compact subset of a Banach space and the intersection of $M$ with some ball $B$ is compact, then $M \cap B$ is acyclic.} Hence the acyclicity of an arbitrary compact $m$-connected set implies the $P$- and $B$-acyclicity of an arbitrary boundedly compact $m$-connected set $M$. This follows from the fact that the intersection of such a set with an arbitrary ball $B(x, r)$ is compact and $m$-connected.

§ 3. The Banach–Mazur hull. Monotone path-connectedness. Spaces of classes (MeI) and (Ex-$w^*$s).

The associated norm. The Rainwater–Simons theorem

In the present section, we examine the relationship between the Banach–Mazur hull $m(\cdot, \cdot)$ and intervals of functions $[\cdot, \cdot]$ (to be defined in (3.2) below). We introduce a class (MeI) of normed linear spaces (including all separable spaces) in which $m(\cdot, \cdot) = [\cdot, \cdot]$. An important property of the Banach–Mazur hull $m(\cdot, \cdot)$ is that $z \in m(x, y)$ if and only if $z$ lies metrically between $x$ and $y$ with respect to the so-called (Brown-) associated norm $|\cdot|$. Such a norm exists on spaces of class (Ex-$w^*$s), which also includes all separable Banach spaces. This observation enables one to use the machinery of metric convexity (Lemma 5.A). In passing, we extend the well-known Rainwater–Simons theorem on weak convergence of sequences to the case of convergence with respect to the associated norm (Proposition 3.1).

To begin with, we note that the structure of $m(x, y)$, $x, y \in X$, in $X = C(Q)$ is quite apparent:

$$m(x, y) = [x, y],$$

where

$$[x, y] := \{z \in C(Q) \mid z(t) \in [x(t), y(t)] \ \forall t \in Q\}. \quad (3.1)$$

A similar representation holds in $C_0(Q)$ (where $Q$ is a locally compact Hausdorff space). This follows from a characterization of extreme elements (‘evaluation at a point’) of the unit sphere in the space dual to $C_0(Q)$ (see [20]).

In the general case, the interval $[x, y]$ is defined, in analogy with (3.1), as follows:

$$[x, y] := \{z \in X \mid \min\{\varphi(x), \varphi(y)\} \leq \varphi(z) \leq \max\{\varphi(x), \varphi(y)\} \ \forall \varphi \in \text{ext } S^*\}. \quad (3.2)$$
Following Brown [1] and Franchetti and Roversi [9], we consider the following class of spaces $X$:

$$\text{(MeI)}: \quad m(x, y) = [x, y] \quad \text{for all } x, y \in X.$$ 

The abbreviation (MeI) comes from the description ‘the hull $m(x, y)$ equals the interval $[x, y]$ for all $x, y$’. The author does not know examples of Banach spaces which are not (MeI)-spaces.

The inclusion $m(x, y) \supseteq [x, y]$ holds in any normed linear space $X$ (see, for example, [5]). It is also known [5], [9] that $m(x, y) = [x, y]$ for a broad class of Banach spaces, including those whose smooth points are dense in the unit sphere (this class contains all weakly Asplund spaces and, therefore, all separable spaces).

It is readily verified that the class (MeI) contains the spaces $C(Q)$ for all Hausdorff compact spaces $Q$. In particular, it contains the space $\ell^\infty$ (qua the space of continuous functions on the Stone–Čech compactification of the positive integers). We also note that if a space $X$ is such that $\text{ext} \ S^*$ lies in the closure of the set of $w^*$-semi-denting points of the dual ball $B^*$ (Moreno’s condition), then $[x, y] = m(x, y)$ for all $x, y \in X$. For example, this condition holds for finite-dimensional spaces and spaces with the Mazur intersection property. Recall that a point $f \in S^*$ is called a $w^*$-semi-denting point of the dual ball $B^*$ (see, for example, [21]) if, for every $\varepsilon > 0$, there is a $w^*$-slice $S\ell$ of $B^*$ such that $\text{diam}(\{f\} \cup S\ell) < \varepsilon$. Here $S\ell(B^*, x, \delta) := \{g \in S^* \mid g(x) > 1 - \delta\}$, $0 < \delta < 1$, $x \in X$.

We shall also use the following class of spaces introduced by Franchetti and Roversi [9]:

$$\text{(Ex-} w^*s\text{)}: \quad \text{ext} \ S^* \text{ is } w^*\text{-separable.}$$

In the definition of (Ex-$w^*$s) we always assume that

$$F = (f_i)_{i \in I} \subset \text{ext} \ S^* \text{ is } w^*\text{-dense in } \text{ext} \ S^*, \quad \text{card } I \leq \aleph_0, \quad F = -F.$$ 

The abbreviation (Ex-$w^*$s) is taken from the German ‘die Extrempunktmenge der konjugierten Einheitskugel ist $w^*$-separabel’.

By the Krein–Milman theorem, every space of class (Ex-$w^*$s) has a $w^*$-separable dual ball $B^*$. This is equivalent [22] to saying that $X$ is isometrically isomorphic to a subspace of $\ell^\infty$. It is worth noting that there are examples of spaces of the form $X = C(K)$ (where $K$ is a non-separable Hausdorff compact set) or $X = \ell_1 \oplus \ell_2(\Gamma)$ ($|\Gamma| = c$) such that $X^*$ is $w^*$-separable but the dual unit ball $B^*$ is not.

Furthermore, if $X$ is a separable normed linear space, then the $w^*$-topology of the dual unit ball $B^*$ is metrizable. As a result, the ball $B^*$ is $w^*$-separable ([25], Corollary 3.104). Hence every separable space lies in the class (Ex-$w^*$s). The class (Ex-$w^*$s) also contains the non-separable space $\ell^\infty$ (qua the space of continuous functions on the Stone–Čech compactification $\beta\mathbb{N}$ of the positive integers; this compact set is separable but non-metrizable). Also note that the space $C(Q)$ on a non-separable $Q$ and $c_0(\Gamma)$ on an uncountable $\Gamma$ fail to lie in (Ex-$w^*$s).

It would be interesting to characterize the class (Ex-$w^*$s).

In summary, with regard to spaces of classes (MeI) and (Ex-$w^*$s), we point out that the class (MeI) $\cap$ (Ex-$w^*$s) contains all separable normed linear spaces
(and in particular, all \( C(Q) \) on a metrizable compact set \( Q \)) and the non-separable space \( \ell^\infty \).

Suppose that \( X \in (\text{Ex-}w^*\text{s}) \), \( F = (f_i)_{i \in I} \) is the family of functionals in the definition of \( (\text{Ex-}w^*\text{s}) \), \( (\alpha_i) \subset \mathbb{R}, \alpha_i > 0, i \in I, \) and \( \sum \alpha_i < \infty \). Given \( x \in X \), we set
\[
|x| = \sum_{i \in I} \alpha_i |f_i(x)|. \tag{3.3}
\]
Then \(|\cdot|\) is a norm on \( X \). Following Brown [1], we call it the associated norm on \( X \). Clearly, \(|x| \leq \|x\| \sum \alpha_i \).

The importance of the associated norm is seen from the following result, which is a direct and straightforward generalization of Corollary 3.2 in [1], and was proved by Brown for \( \dim X < \infty \).

**Lemma 3.1.** Let \( X \) be a Banach space of class (MeI) \( \cap (\text{Ex-}w^*\text{s}) \) (in particular, \( X \) can be any separable Banach space) and let \( x, y \in X \). Then the following conditions are equivalent:

a) \( z \in m(x, y) \);

b) \(|f_i(x) - f_i(y)| = |f_i(x) - f_i(z)| + |f_i(z) - f_i(y)| \) for all \( i \in I \), where \( F = (f_i)_{i \in I} \) is the family in the definition of \( (\text{Ex-}w^*\text{s}) \);

c) \(|x - y| = |x - z| + |z - y|\).

**Proof.** Let \( z \in m(x, y) \). We have \( X \in \text{(MeI)} \) and hence, by definition, \( f(z) \in [f(x), f(y)] \) for all \( f \in \text{ext } S^* \). In particular, condition b) is satisfied, and hence, so is condition c) by (3.3). Conversely, assume that \( z \) is \(|\cdot|\)-between \( x \) and \( y \). Clearly, \(|f_i(x - y)| \leq |f_i(x - z)| + |f_i(z - y)|\). Hence, by (3.3),
\[
|f_i(x) - f_i(y)| = |f_i(x) - f_i(z)| + |f_i(z) - f_i(y)|
\]
for all \( i \). As a corollary,
\[
|f(x) - f(y)| = |f(x) - f(z)| + |f(z) - f(y)|
\]
for all \( f \in \text{ext } S^* \) since \( (f_i) \) is \( w^* \)-dense in \( \text{ext } S^* \). Finally, \( z \in m(x, y) \) because \( X \in \text{(MeI)} \). \( \square \)

The following result may be regarded as an extension of the well-known Rainwater–Simons theorem (see, for example, §3.11.8.5 in [25]) to the case of convergence in the associated norm \(|\cdot|\) on spaces of class \( (\text{Ex-}w^*\text{s}) \) (in particular, on separable spaces). The Rainwater–Simons theorem states that a bounded sequence \( (x_n) \) in a Banach space \( X \) converges weakly to \( x \in X \) if and only if the sequence \( (f(x_n)) \) converges to \( f(x) \) for every functional \( f \) in an arbitrary fixed James boundary for \( X \) (for example, for all \( f \in \text{ext } S^* \)). Thus, even though weak convergence is non-metrizable in general, there is a norm on \( X \in (\text{Ex-}w^*\text{s}) \) with respect to which the convergence of sequences is equivalent to weak convergence.

We recall that a subset \( A \) of the dual unit sphere \( S^* \) in \( X^* \) is called a (James) boundary for \( X \) if, for every \( x \in X \), there is an \( f \in A \) such that \( f(x) = \|x\| \) (see, for example, §3.11.8 in [25]). It is an easy consequence of the Krein–Milman theorem that the set \( \text{ext } S^* \) of extreme points of the dual unit ball is a boundary for \( X \).
Proposition 3.1. Let $X \in (\text{Ex-}w^*\text{s})$ be a Banach space, $F := (f_i)_{i \in I} \subset \text{ext } S^*$ the family of functionals in the definition of $(\text{Ex-}w^*\text{s})$, and $(x_n)$ a bounded sequence in $X$. Consider the following conditions:

a) $x_n \xrightarrow{| \cdot |} x$;

b) $f_i(x_n) \to f_i(x)$ for any $i \in I$;

c) $x_n \overset{w}{\to} x$.

Then conditions a) and b) are equivalent and each of them follows from c). If $X^*$ is separable, then all three conditions are equivalent.

Remark 3.1. The separability of $X^*$ is essential in b) $\Rightarrow$ c). Indeed, let $X = \ell^1$. Consider all finite sequences in $\ell^\infty$ consisting of zeros and $\pm 1$. This set is countable and $w^*$-dense in $\text{ext } S^*$. However, the convergence of elements of $\ell^1$ on these sequences does not imply their weak convergence. This fact was noted by Borodin during a discussion of the results of this paper.

Remark 3.2. Proposition 3.1 implies that $(X, | \cdot |)$ is always a Schur space with respect to the associated norm $| \cdot |$ (recall that $X$ is a Schur space if all weakly convergent sequences in $X$ are norm convergent; $\ell^1$ is a classical example of a Schur space). Indeed, by the definition of the associated norm, $|x| > \alpha_i|f_i(x)|$ for all $i$. Hence every $f_i \in F$ lies in $X^*_|$. It now follows from assertion b) of Proposition 3.1 that if $(x_n)$ $w| \cdot |$-converges, then $(x_n)$ $| \cdot |$-converges. Thus, for a bounded sequence $(x_n)$,

$$x_n \overset{w| \cdot |}{\to} x \iff x_n \overset{| \cdot |}{\to} x.$$  

As a result, $w| \cdot |$-compactness in $(X, | \cdot |)$ coincides with strong $| \cdot |$-compactness. In particular, every reflexive $(X, | \cdot |)$ is finite-dimensional.

This observation comes from correspondence with O. Nygaard, to whom the author is sincerely grateful.

Proof of Proposition 3.1. The implication a) $\Rightarrow$ b) is quite clear: if $x_n \overset{| \cdot |}{\to} x$ (in the norm $| \cdot |$), then the sum $\sum \alpha_i|f_i(x_n) - f_i(x)|$ is small for all sufficiently large $n$; as a corollary, for every fixed $j$ the difference $|f_j(x_n) - f_j(x)|$ is also small for such $n$.

Let us prove b) $\Rightarrow$ a). For any $n$, we split the sum $\sum_{i \in I} \alpha_i|f_i(x_n) - f_i(x)|$ into two sums: over $i \leq N$ and $i > N$ ($N$ will be chosen later). By hypothesis, the sequence $(x_n)$ is uniformly bounded. Hence we have $|f_i(x_n) - f_i(x)| \leq C$ in the second sum (where $C$ is independent of $i, n$). Therefore the second sum is bounded above by $\sum_{i > N} C\alpha_i < \infty$. Given any $\varepsilon > 0$, we choose an $N$ such that the second sum is smaller than $\varepsilon$. The first sum is finite, and there we choose large $n$.

That c) $\Rightarrow$ b) is clear. We now assume that $X^*$ is separable and prove that b) $\Rightarrow$ c). Note that the following assertions about a Banach space $X$ are equivalent:

1) $X$ has a separable boundary;

2) the boundary $\text{ext } B^*$ is separable;

3) $X^*$ is separable.

Here, the implications 3) $\Rightarrow$ 2), 2) $\Rightarrow$ 1) are straightforward (the separability of a set in a normed linear space is inherited by all its subsets), and the first assertion implies the last in view of the well-known Godefroy–Rodé theorem.
(see, for example, [25], Theorem 3.122), which states that \( B^* = \text{conv} \| \cdot \| A \) (here \( A \) is a separable boundary for \( X \)). As a corollary, the ball \( B^* \) is separable and so is the space \( X^* \).

Further, we shall need the concept of an \((I)\)-generating set introduced by Fonf and Lindenstrauss. By definition, a set \( C \subset B^* \) \((I)\)-generates the dual ball \( B^* \) if

\[
B^* = \text{conv} \left( \bigcup_i \text{conv} w^* C_i \right)
\]  

(3.4)

for any representation \( C = \bigcup C_i \) as a countable union of sets \( C_i \). In this definition, ‘\( I \)’ comes from the Latin \textit{intermedius} and is explained by the fact that

\[
B^* = \text{conv} C \implies C \ (I)\text{-generates} \ B^* \implies B^* = \text{conv} w^* C.
\]

We put \( C_i := \{ f_1, \ldots, f_i \} \), \( i \in I \) (where \( F := (f_i)_{i \in I} \) is the family of functionals in the definition of \((\text{Ex}-w^*\text{s})\)). Clearly, \( F = \bigcup C_i \). By the Kadets–Fonf–Godefroy–Rodé theorem,

\[
B^* = \text{conv} \| \cdot \| \text{ext} B^*.
\]  

(3.5)

The space \( X^* \) is separable, and hence so is \( \text{ext} B^* \). By the definition of \((\text{Ex}-w^*\text{s})\) we have \( \text{ext} S^* \subset F^{w^*} \) and, therefore, \( \text{ext} S^* \subset \left( \bigcup_i \text{conv} w^* C_i \right) \). Finally, \( F \) \((I)\)-generates the ball \( B^* \) by (3.5) and (3.4).

It remains to use a result obtained independently by Nygaard [23] and Kalenda [24]. It asserts that if \( C \subset B^* \) \((I)\)-generates the ball \( B^* \), then \( C \) is a Rainwater set, that is, a set with the following property: if a bounded sequence \( (x_n) \subset X \) converges pointwise on \( C \), then \( (x_n) \) converges weakly. \( \Box \)

\section*{§ 4. Statement of the main results}

Here are the main results of the paper.

**Theorem 4.1.** Let \( X \in (\text{MeI}) \cap (\text{Ex}-w^*\text{s}) \) be a Banach space (in particular, \( X \) can be any separable Banach space) and let \( M \subset X \) be closed and \( m \)-connected. Assume that at least one of the following conditions holds.

\begin{enumerate}
  \item \( M \) is boundedly compact (in the norm \( \| \cdot \| \) of \( X \)).
  \item \( M \) is \( | \cdot | \)-closed and \( m(x, y) \ | \cdot | \)-compact for all \( x, y \in X \).
  \item \( m(x, y) \) is \( \| \cdot \| \)-compact for all \( x, y \in X \).
\end{enumerate}

Then \( M \) is monotone path-connected.

If in addition \( M \) is boundedly compact, then \( M \) is \( P \)- and \( B \)-cell-like, \( P \)- and \( B \)-acyclic (relative to any continuous (co)homology theory) and is a sun.

If \( X \) is finite-dimensional, then moreover \( M \) is \( P \)- and \( B \)-contractible.

Recall [26] that a set \( M \subset X \) has a continuous multiplicative (additive) \( \varepsilon \)-selection of the metric projection for all \( \varepsilon > 0 \) if and only if \( M \) is \( \hat{B} \)-contractible (\( B \)-contractible if \( M \) is approximatively compact). It now follows from Theorem 4.1 that the metric projection onto an \( m \)-connected (monotone path-connected) closed subset of a finite-dimensional space has a continuous multiplicative (additive) \( \varepsilon \)-selection for all \( \varepsilon > 0 \). There is some hope that a similar result holds in the infinite-dimensional setting.
For spaces with linear ball embedding [8] (in particular, for $\ell^1(n), C(Q)$, where $Q$ is a metrizable compact set, and $C_0(Q)$) Theorem 4.1 partially extends the results of Balashov and Ivanov ([27], Theorem 2.9 and Lemma 4.18) on path-connectedness of $R$-weakly convex sets (in the sense of Vial): the intersection of an $R$-weakly convex set with a closed or open ball is always m-connected in a space with linear ball embedding ([8], Theorem 4.1).

Theorem 4.1 has the following corollary.

Corollary 4.1. Let $X \in (\text{MeI}) \cap (\text{Ex-w}^*\text{s})$ be a Banach space and let $M \subset X$ be closed. Assume that for some $x \notin M$ the set $P_M x$ is compact and m-connected. Then $M$ is monotone path-connected, $P$- and $B$-cell-like, $P$- and $B$-acyclic and is a sun.

We note that if $M \subset X \in (\text{MeI})$ is m-connected, then so is $P_M x$.

Indeed, let $u, v \in P_M x$. Since $M$ is m-connected, there is a $z \in m(u, v) \cap M$, $z \neq u, v$. Assume that $z \notin S(x, \|x-u\|)$. Since $\text{ext} S^*$ is a boundary for $X$, any point outside a closed ball (the ball $B(x, \|x-u\|)$ in our setting) can be strictly separated from it by an extreme functional. Hence $z \notin [u, v]$. This is a contradiction because $z \in m(u, v)$ and one always has $m(\cdot, \cdot) = [\cdot, \cdot]$ in $X \in (\text{MeI})$.

For weakly compact sets we have the following result.

Theorem 4.2. Let $X$ be a separable Banach space and let $\emptyset \neq M \subset X$ be boundedly weakly compact. Assume that $M$ is m-connected. Then $M$ is monotone path-connected.

The following question remains open. By a well-known theorem of Vlasov [6], a $P$-acyclic boundedly compact subset of a Banach space is a sun.\footnote{The proof of Vlasov’s theorem depends on the classical Eilenberg–Montgomery fixed point theorem (see, for example, [13], Corollary (32.12)) in which acyclicity is understood in the sense of Alexandroff–Čech homology groups with coefficients in a field (in the paper by Eilenberg–Montgomery acyclicity is understood in the sense of Vietoris cycles and homology groups over a field). However, Čech homology groups are known to be isomorphic to Vietoris homology groups on the category of compact metrizable spaces, and the notions of homological and cohomological acyclicity coincide if the coefficients of the homology (cohomology) groups lie in a field and the support is compact.} It would be interesting, especially in view of Theorem 4.2, to extend Vlasov’s theorem to the case of boundedly weakly compact sets, at least in the separable setting. The difficulty here is as follows. Although any weakly compact subset $M$ of a separable Banach space is metrizable and the topology on $M$ generated by the metric agrees with the weak topology on $M$, the Eilenberg–Montgomery theorem is inapplicable because the metric projection to $M$ is only norm-to-weak upper semicontinuous (the support is the same, but the topologies on it are different). We also note in passing that the problem of the solarity of weakly compact Chebyshev sets has not yet been solved in the general setting.

§ 5. Proofs

Proof of Theorem 4.1. To prove the first assertion in Theorem 4.1, we use some of the ideas in [9]. The monotone path-connectedness is established in [3], and the
remaining assertions follow from Lemma 5.2, which will be proved below. The finite-dimensional case is a consequence of assertion a) and a theorem of Brown [28], which says that the intersection of an m-connected closed subset of a finite-dimensional $X$ with a closed ball is n-connected for all $n \in \mathbb{Z}_+$ (that is, every map of a $k$-sphere, $k \leq n$, to this set extends continuously to the $(k + 1)$-dimensional ball). Hence, according to a well-known characterization of absolute retracts (see, for example, [29], Theorem 11.1), $M$ is contractible and locally contractible. Note [4] that condition b) of Theorem 4.1 holds a fortiori in the space $X = c_0$.

We shall use the following auxiliary result.

Recall that a set $M$ is said to be metrically convex or d-convex (Menger-convex) [30] with respect to a metric $d$ if, for any distinct points $x, y \in M$, the set $M \setminus \{x, y\}$ contains a point $z$ lying d-between $x$ and $y$, that is, $d(x, y) = d(x, z) + d(z, y)$. The following result was established by Menger [30] (see also [31], p. 24).

**Lemma 5.A.** Let $(Y, d)$ be a complete Menger-convex metric space. Then, for all $x$ and $y$ in $Y$, there is an isometry $f : [0, d(x, y)] \to Y$ such that $f(0) = x$ and $f(d(x, y)) = y$. In particular, $Y$ is path-connected.

*Proof of Theorem 4.2.* We can assume that $M$ is bounded. It is well known (see, for example, [25], Proposition 3.107]) that if $M$ is a weakly compact subset of a Banach space with $w^*$-separable dual $X^*$, then $M$ with the relative weak topology is metrizable. It is also known that every weakly compact subset of a Banach space is weakly complete (sequentially weakly complete). By Proposition 3.1, in spaces of class (Ex-$w^*$s), and hence, in any separable space [5], $w$-completeness implies $\cdot$-$\cdot$-completeness with respect to the (Brown-) associated norm $\cdot$. Now in order to apply Lemma 5.A it remains to observe that the m-connectedness of a set in a separable space is equivalent to its $\cdot$-$\cdot$-convexity (see Lemma 3.1).

Thus, by Lemma 5.A, any two points of $M$ can be connected by an arc which is the range of an isometry. Applying Lemma 3.1, we see that $f_i(k(t))$ is a monotone function of $t$ for any $i \in I$. Now let $f \in \text{ext } S^*$. Since $X$ is separable, there is a sequence $(f_i) \subset F$ which is $w^*$-convergent to $f$. Hence the function $f(k(t))$ is also monotone. □

Suppose that $X \in (\text{Mel}) \cap (\text{Ex-}w^*s)$ and $F = (f_i) \subset \text{ext } S^*$ is a family ($w^*$-dense in $\text{ext } S^*$) in the definition of (Ex-$w^*$s). Given any $n \in \mathbb{N}$, we define a bounded linear operator $s_n : X \to \ell^\infty(n)$ by the formula

$$s_n(x) = (f_1(x), \ldots, f_n(x)).$$

Note that $\|s_n(x)\| \leq \|x\|$ and $\|s_n(x)\| \to \|x\|$ as $n \to \infty$ since $\text{exp } S^*$ is a James boundary for $X$, that is, $\|x\| = \max\{f(x) \mid f \in \text{ext } S^*\}$.

The first assertion in Lemma 5.1 is proved by a slight modification of the argument given by Franchetti and Roversi [9].

**Lemma 5.1.** Let $X \in (\text{Mel}) \cap (\text{Ex-}w^*s)$ be a Banach space (in particular, $X$ can be any separable Banach space) and let $\emptyset \neq M \subset X$ be boundedly compact. Then the following assertions hold.
1) If $M$ is m-connected in $X$, then $s_n(M)$ is monotone path-connected in $\ell^\infty(n)$ for all $n \in \mathbb{N}$.

2) If $s_n(M)$ is m-connected in $\ell^\infty(n)$ for all $n \in \mathbb{N}$, then $M$ is monotone path-connected in $X$.

**Proof.** 1) We can assume that $M$ is compact. The assertion about $s_n(M)$ follows from Lemma 3.1 because every m-connected subset of a finite-dimensional space is monotone path-connected.

2) Assume that $s_n(M)$ is m-connected in $\ell^\infty(n)$ for all $n$. Take $x, y \in M$, $x \neq y$. We can assume that there is a $\nu > 0$ such that $\|s_n(x) - s_n(y)\| \geq \varepsilon > 0$ for $n \geq \nu$. Since $s_n(M)$ is m-connected, we can choose a point $z_n \in M$ such that $s_n(z_n) \in \text{m}(s_n(x), s_n(y))$ and

$$\|s_n(z_n) - s_n(x)\| = \|s_n(z_n) - s_n(y)\| = \frac{1}{2}\|s_n(x) - s_n(y)\| \geq \frac{\varepsilon}{2} > 0.$$ 

The last equality follows from the continuity by Lemma 5.1 applied to the closed m-connected set $s_n(M)$. Letting $n$ approach infinity, we have $z_n \to z \in M$ since $M$ is compact. Clearly, $\|s_n(z_n - x)\| \to \|z - x\|$ and $\|s_n(z_n - y)\| \to \|z - y\|$. Hence $z \neq x$, $z \neq y$. It remains to show that $z \in \text{m}(x, y)$. For every fixed $n \in \mathbb{N}$ it follows from Lemma 3.1 that

$$\min[f_i(x), f_i(y)] \leq f_i(z_n) \leq \max[f_i(x), f_i(y)], \quad i = 1, \ldots, n,$$

because $s_n(z_n) \in \text{m}(s_n(x), s_n(y))$. Letting $n \to \infty$, we have

$$\min[f_i(x), f_i(y)] \leq f_i(z) \leq \max[f_i(x), f_i(y)], \quad i \in I,$$

whence $z \in \text{m}(x, y)$ by Lemma 3.1. The monotone path-connectedness of the m-connected set $M$ now follows from Theorem 4.1. □

Note that assertion 2) of Lemma 5.1 is not used in the proofs of Theorem 4.1 and Lemma 5.2.

**Lemma 5.2.** Let $X \in (\text{Mel}) \cap (\text{Ex-w}^\ast s)$ be a Banach space (in particular, $X$ may be separable or $X = \ell^\infty$) and let $\varnothing \neq M \subset X$ be boundedly compact and m-connected. Then $M$ is P- and B-cell-like, that is, each of the sets

$$P_M x, \quad M \cap B(x, r), \quad x \in X, \quad r > 0,$$

is cell-like. In particular, $M$ is P- and B-acyclic (with respect to any continuous (co)homology theory).

Before proving Lemma 5.2, we recall that an inverse system (see, for example, [32], p. 56) is a family $\mathcal{S} = \{X_\alpha, \pi^\beta_\alpha, \Sigma\}$, where the set $\Sigma$ is partially ordered by a relation $\prec$, $X_\alpha$ is a topological Hausdorff space, and $\pi^\beta_\alpha : X_\beta \to X_\alpha$, $\alpha \prec \beta$, are continuous maps such that $\pi^\alpha_\alpha = \text{id}_{X_\alpha}$ and $\pi^\beta_\alpha \pi^\gamma_\beta = \pi^\gamma_\alpha$ for all $\alpha \prec \beta \prec \gamma$. By definition, the inverse limit of $\mathcal{S}$ is given by

$$\lim_{\alpha} \mathcal{S} = \left\{(x_\alpha) \in \prod_{\alpha \in \Sigma} X_\alpha | \pi^\beta_\alpha(x_\beta) = x_\alpha \quad \forall \alpha \prec \beta\right\}.$$
If $\pi_\alpha : \varprojlim S \to X_\alpha$ is the restriction of the projection $p_\alpha : \prod_{\alpha \in \Sigma} X_\alpha \to X_\alpha$ onto the $\alpha$-axis (the canonical projection), then $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$ for all $\alpha < \beta$.

Assertion 1) of the following lemma is contained, for example, in [15], and assertion 2) is obtained in [33] (see also [16]). Note that assertion a) follows from the classical Tikhonov compactness theorem, and assertion b) is well known and due to Kurosh and Steenrod.

**Lemma 5.B.** 1) Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system. Then $\varprojlim S$ is a closed subset of $\prod_{\alpha \in \Sigma} X_\alpha$. Moreover, the following assertions hold.

a) If $X_\alpha$ is compact for every $\alpha \in \Sigma$, then $\varprojlim S$ is compact.

b) If $X_\alpha$ is compact and non-empty for every $\alpha \in \Sigma$, then $\varprojlim S$ is compact and non-empty.

c) If $X_\alpha$ is a continuum for every $\alpha \in \Sigma$, then $\varprojlim S$ is a continuum.

d) If $X_\alpha$ is compact and acyclic \(^3\) for every $\alpha \in \Sigma$, then $\varprojlim S$ is compact and acyclic.

e) If $X_\alpha$ is metrizable for every $\alpha \in \Sigma$ and $\Sigma$ is countable, then $\varprojlim S$ is metrizable.

2) If $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ is a countable inverse system and each $X_\alpha$ is compact and cell-like, then the inverse limit $\varprojlim S$ is compact and cell-like.

**Proof of Lemma 5.2.** We can assume that $M$ is compact. Let $\Omega$ be the family of all finite subsets of $F \subset \text{ext } S^*$. The set of all finite sequences of positive integers is countable, and so is $\Omega$. The set $\Omega$ is directed by inclusion: $A \prec B$ if and only if $A \subset B$. Given $A \in \Omega$, we define a map $s_A : X \to \ell^\infty(A)$ by setting

$$s_A(x) = \{f(x) \mid f \in A\}.$$ 

Then $s_A$ is a bounded linear operator. Given $A, B \in \Omega$, $A \supset B$, we have a natural restriction map $\ell^\infty(A) \to \ell^\infty(B)$. The family $\{s_A \mid A \in \Omega\}$ forms an inverse system with respect to these restriction maps. By assertion 1) of Lemma 5.1, each $s_A(M)$ is a compact m-connected subset of $\ell^\infty(A)$. Then Brown’s theorem ([19], Theorem 1) yields that each $s_A(M)$ is infinitely connected and, therefore, cell-like and acyclic.

The following construction is well known in the theory of inverse spectra. Let $(X, \pi) = (X_\alpha, \pi_\alpha^\beta, \Sigma)$ be an inverse system and let $Z$ be an arbitrary fixed set. Assume that for every $\alpha \in \Sigma$ we are given a map $g_\alpha : Z \to X_\alpha$ such that $\pi_\alpha^\beta g_\beta = g_\alpha$ whenever $\alpha \leq \beta$. Then there is a unique map $g : Z \to X_\infty := \varprojlim (X, \pi)$ such that $\pi_\alpha \circ g = g_\alpha$ for all $\alpha$. Moreover, if $Z$ is a topological space and the $g_\alpha$ are continuous, then so is $g : Z \to X_\infty$.

Since $\{s_A \mid A \in \Omega\}$ is an inverse system, we have $s_A = s_{BA} \circ s_B$ for $A, B \in \Omega$, $A \subseteq B$, where $s_{BA}$ is the restriction map (see [34], p. 428). Therefore the map $g : M \to M_\infty := \varprojlim s_A(M)$ is continuous (by the above) and injective (see [34], (6.9)). We claim that $g$ is surjective.

Assume that

$$(x_A)_{A \in \Omega} \in \varprojlim s_A(M) \subset \prod_{A \in \Omega} s_A(M).$$

\(^3\)With respect to any continuous cohomology theory.
For every $A \in M$ we choose a $y_A \in M$ such that $s_A(y_A) = x_A$. Then $(y_A)_{A \in \Omega}$ is a net in $M$. Since $M$ is compact, this net has a cluster point $y \in M$. It remains to prove that $s_A(y) = x_A$ for all $A \in \Omega$. Suppose that $A, B \in \Omega$, $B \supset A$. Then $s_A(y_B) = s_B(y_B)|_A = x_A$. Passing to the limit over a subnet, we see that $s_A(y) = x_A$.

Thus $g : M \rightarrow M_\infty$ is a continuous bijection. It is well known ([32], p. 169) that every continuous map from a compact space to a Hausdorff space is closed. Therefore every continuous bijective map from a compact space to a Hausdorff space is a homeomorphism. Since the inverse limit $M_\infty$ is Hausdorff ([32], p. 171), this assertion applies in our case. Thus $M$ is homeomorphic to $M_\infty$.

It is well known that cell-likeness (as well as acyclicity) is a topological property ([13], p. 439). Since $M \simeq M_\infty$, Lemma 5.3 yields that $M$ is cell-like (and hence acyclic) because each $s_A(M)$ has this property by the theorem of Brown mentioned above. Assertion 2) of Lemma 5.3 is applicable because the set of all finite sequences of positive integers is countable. □

The answer to the converse question about the monotone path-connectedness of $B$-acyclic ($P$-acyclic) sets is unknown in the general case. It is worth noting that for every $n \geq 3$ there is a finite-dimensional space $X_n$ containing a non-monotone path-connected Chebyshev set (which is a fortiori a sun and a $P$- and $B$-acyclic set). Indeed, consider the class of spaces $X_n$ such that $\overline{\text{ext}} \ S^* = S^*$. Phelps [35] showed that $\overline{\text{ext}} \ S^* = S^*$ for a given space $X_n$ if and only if each convex bounded closed subset of $X_n$ is representable as an intersection of closed balls (in other words, $X_n \in (MIP)$, that is, $X$ satisfies the Mazur intersection property). As a corollary, the monotone path-connectedness of a closed set in such a space is equivalent to its convexity. Furthermore, for every $n \geq 3$ Tsar’kov [36] constructed an example of a space $X'_n$ with $\overline{\text{ext}} \ S^* = S^*$ containing an unbounded non-convex Chebyshev set $M'$ (all bounded Chebyshev sets in this $X'_n$ are convex). Thus $M'$ is an example of a non-monotone path-connected $B$-acyclic ($P$-acyclic) set (a Chebyshev sun).

In connection with this problem we point out that Brown [1] introduced an important class of normed linear spaces: the so-called $(BM)$-spaces (see also [28], [37], [9]), which have proved very natural in the problem of the m-connectedness of suns. $(BM)$-spaces (in particular, $\ell^\infty(n)$ and $c_0$) are ‘good’ in the following sense: every boundedly compact sun in such a space is m-connected (see [1], [4]) and, therefore, monotone path-connected. By Vlasov’s theorem, a $P$-acyclic boundedly compact set is always a sun. Hence, in $(BM)$-spaces, $P$-acyclicity implies monotone path-connectedness. We also note [38] that in a finite-dimensional polyhedral space $X_n$ each sun is monotone path-connected if and only if $X_n \in (BM)$. It is known that $\ell^1(n) \notin (BM)$, $n \geq 3$, whence $\ell^1(n)$ contains a non-monotone path-connected sun. However, it is unknown whether such a sun is $P$-acyclic (or even $P$-connected).

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