Extremals for $\alpha$-Strichartz inequalities

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Abstract

A necessary and sufficient condition on the precompactness of extremal sequences for one dimensional $\alpha$-Strichartz inequalities, equivalently $\alpha$-Fourier extension estimates, is established based on the profile decomposition arguments. One of our main tools is an operator-convergence dislocation property consequence which comes from the van der Corput Lemma. Our result is valid in asymmetric cases as well. In addition, we obtain the existence of extremals for non-endpoint $\alpha$-Strichartz inequalities.

1 Introduction

For $\alpha > 1$, we investigate the following symmetric $\alpha$-Strichartz inequality

$$\left\| \left[ D^{\frac{\alpha-2}{2}} \right] \left[ e^{it|\nabla|^\alpha} \right] u \right\|_{L^6_t L^2_x(\mathbb{R}^2)} \leq M_\alpha \| u \|_{L^2_x(\mathbb{R})},$$

where

$$M_\alpha := \sup \left\{ \left\| \left[ D^{\frac{\alpha-2}{2}} \right] \left[ e^{it|\nabla|^\alpha} \right] u \right\|_{L^6_t L^2_x(\mathbb{R}^2)} : \| u \|_{L^2_x(\mathbb{R})} = 1 \right\}$$

is the sharp constant and

$$\left[ e^{it|\nabla|^\alpha} \right] u(x) := \mathcal{F}^{-1} e^{-it|\xi|^\alpha} \mathcal{F}[u](x), \quad \left[ D^s \right] u(x) := \mathcal{F}^{-1} |\xi|^s \mathcal{F}[u](x), \quad \mathcal{F}[u](\xi) := \int_{\mathbb{R}} e^{-ix\xi} u(x) dx,$$

with $\mathcal{F}$ denoting the spatial Fourier transform. This estimate (1) comes from Kenig et al. [23, Theorem 2.3] which is also named Fourier extension estimate. Moreover it says that, for every $\alpha > 1$, there holds the (mixed norm) asymmetric $\alpha$-Strichartz inequality

$$\left\| \left[ D^{\frac{\alpha-2}{2}} \right] \left[ e^{it|\nabla|^\alpha} \right] u \right\|_{L^q_t L^r_x(\mathbb{R}^2)} \leq \tilde{M}_{\alpha,q,r} \| u \|_{L^2_x(\mathbb{R})},$$

where $2/q + 1/r = 1/2$ with the sharp constant $\tilde{M}_{\alpha,q,r}$ defined by

$$\tilde{M}_{\alpha,q,r} := \sup \left\{ \left\| \left[ D^{\frac{\alpha-2}{2}} \right] \left[ e^{it|\nabla|^\alpha} \right] u \right\|_{L^q_t L^r_x(\mathbb{R}^2)} : \| u \|_{L^2_x(\mathbb{R})} = 1 \right\}.$$

We call the pairs $(q,r) = (\infty,2)$ and $(q,r) = (4,\infty)$ endpoint pairs. Otherwise the pairs $(q,r)$ are called non-endpoint pairs.

The symmetries for these $\alpha$-Strichartz inequalities, on the $L^2_x$ side, are time-space translations and scaling as follows

$$\left[ g^{\text{sym}}_n \right] u := \left[ e^{it_n |\nabla|^\alpha} \right] \left( h_n \right)^{-1/2} u \left( \frac{x-x_n}{h_n} \right), \quad (h_n, x_n, t_n) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R};$$

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and the associated group $G^{\text{sym}}$ is defined by
\[ G^{\text{sym}} := \left\{ [g_n^{\text{sym}}] : (h_n, x_n, t_n) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \right\}. \]

To state the results more precisely, we say a sequence of functions $(f_n)$ in $L^2(\mathbb{R})$ is precompact up to symmetries if there exists a sequence of symmetries $([g_n^{\text{sym}}])$ in $G^{\text{sym}}$ such that $([g_n^{\text{sym}}]f_n)$ is precompact in $L^2(\mathbb{R})$. On the other hand, a sequence of functions $(f_n)$ in $L^2(\mathbb{R})$ concentrates at a point $x_0 \in \mathbb{R}$ if for arbitrary $\varepsilon, \rho > 0$, there exists $N \in \mathbb{N}_+$ such that for every $n > N$, there holds
\[ \int_{|x-x_0| \geq \rho} |f_n(x)|^2dx \leq \varepsilon \|f_n\|^2_{L^2(\mathbb{R})}. \]

Meanwhile a sequence of functions $(f_n)$ in $L^2(\mathbb{R})$ is an extremal sequence for $\tilde{M}_{\alpha,q,r}$ if it satisfies
\[ \|f_n\|_{L^2(\mathbb{R})} = 1, \quad \lim_{n \to \infty} \left\| \left| D^{\frac{\alpha-2}{\alpha}} \right| [e^{it|\nabla|^\alpha}] f_n \right\|_{L^q_{t,x}(\mathbb{R}^2)} = \tilde{M}_{\alpha,q,r}. \]

The sharp Fourier restriction theory, more generally the sharp constant theory, has been an important part in harmonic analysis. Readers are referred to the survey [15] and the references therein for some recent progress on sharp Fourier restriction theory. One of the more recent results is

**Theorem A ([7]).** All the extremal sequences for $M_{\alpha}$ are precompact up to symmetries if and only if
\[ M_{\alpha} > \left[ \sqrt{3} \alpha (\alpha - 1) \right]^{-\frac{1}{\alpha}}. \] (3)

In particular, if the strict inequality (3) holds, then there exists an extremal for $M_{\alpha}$. If on the contrary the equality holds in (3), then given any $x_0 \in \mathbb{R}$, there exists an extremal sequence for $M_{\alpha}$ which concentrates at $x_0$.

This result is previously obtained by Brocchi et al. [7, Theorem 1.3]. The proof there uses a variant of Lions’ concentration-compactness lemma from [29, 30] together with a variant of Brézis-Lieb lemma from [6, 28]. As pointed out in [7], various of results with a similar condition to (3) have been studied in recent literature. In our paper, this condition comes from the asymptotic Schrödinger behavior Lemma 6.11, see also Remark 6.2 and Remark 7.3. Roughly speaking, to get the existence of extremals, there may be some strict inequality conditions like (3) to rule out some concentrate-type situations which deduce the loss of compactness. We refer to [11, 16, 17] for more discussions on these type of conditions in the low dimensional sphere and cubic curve cases.

The main purpose of this article is investigating the extremal problems for $\alpha$-Strichartz inequalities by means of profile decomposition arguments. One of our results, Theorem 1.1 below, generalizes the aforementioned Theorem A to asymmetric cases. Furthermore as an application of our profile decomposition consequences, for $\alpha \geq 2$, we also give the existence of extremals for non-endpoint $\alpha$-Strichartz inequalities (14) which will be presented later.2 The key ingredient here to establish this generalized profile decomposition Proposition 1.5 is a conditional dislocation property consequence Proposition 1.3 on the weak operator topology convergence for some $L^2$-unitary operators. Now we state our first main result as follows.

**Theorem 1.1.** For the non-endpoint pairs $(q, r)$, all the extremal sequences for $\tilde{M}_{\alpha,q,r}$ are precompact up to symmetries if and only if
\[ \tilde{M}_{\alpha,q,r} > \left( \frac{\alpha^2 - \alpha}{2} \right)^{-\frac{1}{\alpha}} \tilde{M}_{2,q,r}. \] (4)

---

1In view of Frank-Sabin [17, Remark 2.6], this behavior may also be called approximate symmetries.

2We follow this non-endpoint terminology from Hundertmark-Shao [19].
In particular, if the strict inequality (4) holds, then there exists an extremal for $\hat{M}_{\alpha,q,r}$. If on the contrary the equality holds in (4), then given any $x_0 \in \mathbb{R}$, there exists an extremal sequence for $\hat{M}_{\alpha,q,r}$ which concentrates at $x_0$.

As we have mentioned above, Theorem 1.1 extends the previous result [7, Theorem 1.3]. Meanwhile by taking some symmetries\(^3\), on the Fourier side, Theorem 1.1 claims that if the equality holds in (4) then there exists an extremal sequence which concentrates at one fixed frequency. Thereby this Theorem 1.1, in some sense, also coincides with the result in [17] where the extremal sequence concentrates at two opposite frequencies due to some symmetries of the odd curves.

Here we make some historical remarks first. For the case $\alpha = 2$ in (1), the classical Strichartz inequality (Stein-Tomas inequality for the paraboloid), abundant conclusions have been made: the existence of extremals is proved by Kunze [27] for one dimensional case and by Shao [35] for general dimensions; in low dimensions, up to symmetries, the only extremals are shown to be Gaussians by Foschi [13] and Hundertmark-Zharnitsky [20] independently. Extremals are conjectured to be Gaussians in all dimensions [20]. Meanwhile, on the Stein-Tomas inequality for the sphere, we briefly mention that Christ-Shao [11, 37] give the existence of extremals in low dimensions and Foschi [14] shows that the extremals are constants for two-dimension sphere $S^2$. We refer to [15, 33] and the references therein for more recent results on the sharp Fourier restriction theory in the sphere situation.

As for the case $\alpha = 4$ in (1), Jiang et al. [21, 22] give some dichotomy results on the existence of extremals by using the profile decomposition from [1, 2, 8, 25, 31]. For more general case $\alpha > 1$ in (1), Brocchi et al. [7] resolves the dichotomy in [21] by using a geometric comparison principle developed in [32] which resolves the dichotomy in [22]. As far as we know, there is no extremal result on the asymmetric $\alpha$-Strichartz inequality (2) with general $\alpha > 1$, except for the classical $\alpha = 2$ case in (2) which has been studied in some papers such as [3, 9, 18, 35]. Meanwhile, it should be mentioned that Frank-Sabin [17] has studied the existence of extremals for Airy-Strichartz inequality (odd cubic curve), whose result is also valid for non-endpoint asymmetric cases, by using the missing mass method.

Note that $\alpha > 1$ may not be a natural number in our setting and this fact leads to some barriers. In order to establish the desired linear profile decomposition, one of the main results we should establish is the conditional dislocation property Proposition 1.3 for some unitary operators on $L^2(\mathbb{R})$. We begin with the definitions for the dislocation group and the $L^2$-unitary operators that, maybe non-compact, we are concerned about. For parameters $(h_0, x_0, \xi_0, \theta_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, the unitary operators $g_0$ on $L^2_x(\mathbb{R}^d)$ is defined by

$$[g_0] \phi(x) := g_{\theta_0, \xi_0, x_0, h_0}[\phi](x) := e^{i h_0} \frac{d}{d e^{i x \cdot \xi_0} \phi \left( \frac{x - x_0}{h_0} \right)}.$$

We should point out that the parameter $\theta_0$ is inessential and we use it just because it may be deduced from other parameters on the Strichartz space.

Definition 1.2 (Dislocation group [34]). Let $H$ be a separable Hilbert space and let $G$ be a group of unitary operators on $H$. We said $G$ is a group of dislocations if it satisfies the following condition: for every sequence $([g_n]) \subset G$ does not converge weakly (in weak operator topology) to zero, there exists a renamed strongly convergent subsequence of $([g_n])$ such that the strong limit (in strong operator topology) is not zero.

In the classical case $\alpha = 2$, due to the Galilean invariance of classical Schrödinger equations, the dislocation property for the group generated by non-compact $L^2$-unitary operators is obvious. This

\(^3\)Notice the construction of $\bar{a}_n$ in the proof of Theorem A in Section 6.
potentially crucial fact, when establishing the classical profile decomposition, deduces the orthog-

onality of these decomposed profiles in Strichartz spaces. Hence it is a natural idea to generalize

this dislocation property to the $\alpha$-Strichartz setting. On the other hand, we may do some adaption

along the way we generalize it. The following conditional dislocation property proposition comes

from the method of stationary phase, which is contained in [38, Chapter 8] and [45, Chapter 6], or

more precisely the classical van der Corput Lemma [38, p. 332, Proposition 2].

**Proposition 1.3** (Conditional dislocation property). When $d = 1$, if we assume that for fixed $j \neq k$ either

$$\lim_{n \to \infty} \left( \frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} + (h_n^j + h_n^k) |\xi_n^j - \xi_n^k| \right) = \infty$$

or $(h_n^j, \xi_n^j) \equiv (h_n^k, \xi_n^k)$. Then the group $G$, generated by the $L^2$-symmetries

$$(g_n^j)^{-1} [e^{it_n^j |\nabla|^{\alpha}}] [e^{-it_n^k |\nabla|^{\alpha}}] |g_n^k|,$$  

is a group of dislocations provided that $\alpha \in \mathbb{Z}_+$ and $\alpha \geq 2$. Moreover, the group $G$ is a group of
dislocations for all real numbers $\alpha > 1$.

**Remark 1.4.** As we shall see later in Lemma 3.3, the assumptions in Proposition 1.3 arise naturally
during the construction of linear profile decomposition. Analogous assumptions can also be seen
in [21, Theorem 1.3] and [22, p. 10] as well as some earlier papers such as [1, 8, 25]. We will use
this conditional dislocation property to describe the orthogonality for profiles instead of using the
parameters therein, since we need to deal with the situation that $\alpha$ is not a natural number.

The profile decomposition results are intensively studied and widely used in many topics. Be-
sides some of the aforementioned references such as [1, 2, 8, 25, 31] which establish these profile
decompositions in different analysis situations, the profile decomposition may also be called *bubble decomposition* in the literature due to some geometric background. We refer to [26, p. 359] for a
historical discussion, see also [26, p. 373]. Here with the conditional dislocation property Proposition
1.3 in place, we are able to show the following $\alpha$-Strichartz version linear profile decomposition.

**Proposition 1.5** (Linear profile decomposition for the $\alpha$-Strichartz version). Let $(u_n)$ be a bounded
sequence in $L^2(\mathbb{R})$. Then, up to subsequences, there exist a sequence of operators $([T_n^j])$ defined by

$$[T_n^j] \phi(x) := [e^{-it_n^j |\nabla|^{\alpha}}] \left[ (h_n^j)^{-\frac{1}{2}} e^{i(x - x_n^j) \xi_n^j} \phi \left( \frac{x - x_n^j}{h_n^j} \right) \right]$$

with $(h_n^j, x_n^j, \xi_n^j, t_n^j) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and a sequence of functions $(\phi^j) \subset L^2(\mathbb{R})$ such that for every

$J \geq 1$, we have the profile decomposition

$$u_n = \sum_{j=1}^J [T_n^j] \phi^j + \omega_n^J,$$  

where the decomposition possesses the following properties: firstly the remainder term $\omega_n^J$ has van-
ishing Strichartz norm

$$\lim_{J \to \infty} \lim_{n \to \infty} \sup \left\| D_t^{\alpha-2} \left[ e^{it |\nabla|^{\alpha}} \right] \omega_n^J \right\|_{L^p_t L^q_x(\mathbb{R}^2)} = 0;$$

secondly the sequence of operators $[T_n^j]$ satisfies that if $j \neq k$, there holds the limit-orthogonality property

$$[T_n^k]^{-1} [T_n^j] \to 0$$

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as $n$ goes to infinity in the weak operator topology of $\mathcal{B}(L^2)$; moreover for each $J \geq 1$, we have

$$
\lim_{n \to \infty} \left[ \|u_n\|_{L^2(\mathbb{R})}^2 - \left( \sum_{j=1}^{J} \|\phi_j\|_{L^2(\mathbb{R})}^2 \right) - \|\omega_n\|_{L^2(\mathbb{R})}^2 \right] = 0. 
$$

(9)

Remark 1.6. We should point out that the limit-orthogonality (8) of the operators $[T_n^j]$ is crucial and powerful, especially when combined with the conditional dislocation property Proposition 1.3. By the $L^2$-almost orthogonal identity (9), it can be easily inferred that for every $j \neq k$ in Proposition 1.5, there holds

$$
\lim_{n \to \infty} \left\langle [T_n^j], [T_n^k] \phi^k \right\rangle_{L^2(\mathbb{R})} = 0;
$$

(10)

and for each $j \leq J$, there holds

$$
\lim_{n \to \infty} \left\langle [T_n^j], \omega_n^j \right\rangle_{L^2(\mathbb{R})} = 0.
$$

(11)

Meanwhile, the $\alpha$-Strichartz version profile decomposition Proposition 1.5 is equipped with the following Strichartz-orthogonality for the decomposed linear profiles.

Proposition 1.7 (Strichartz-orthogonality of profiles). Furthermore, in the linear profile decomposition Proposition 1.5, for $j \neq k$ there holds

$$
\lim_{n \to \infty} \left\| [D^{\frac{\alpha-2}{\alpha}} e^{i|\nabla|^\alpha} [T_n^j] \phi^j \cdot [D^{\frac{\alpha-2}{\alpha}} e^{i|\nabla|^\alpha} [T_n^k] \phi^k] \right\|_{L^3_{t,x}(\mathbb{R}^2)} = 0. 
$$

(12)

Thus for each $J \geq 1$, by Hölder’s inequality, there holds

$$
\limsup_{n \to \infty} \left( \left\| \sum_{j=1}^{J} [D^{\frac{\alpha-2}{\alpha}} e^{i|\nabla|^\alpha} [T_n^j] \phi^j \right\|_{L^6_{t,x}(\mathbb{R}^2)}^6 - \sum_{j=1}^{J} \left\| [D^{\frac{\alpha-2}{\alpha}} e^{i|\nabla|^\alpha} [T_n^j] \phi^j \right\|_{L^6_{t,x}(\mathbb{R}^2)}^6 \right) = 0. 
$$

(13)

Finally, as an application of our profile decomposition results, we turn to the following estimates. For $\alpha \geq 2$, the result of Kenig et al. [23, Theorem 2.3] and Sobolev inequalities imply the following non-endpoint $\alpha$-Strichartz estimates

$$
\left\| e^{i|\nabla|^\alpha} u \right\|_{L^{2n+2}_{t,x}(\mathbb{R}^2)} \leq \bar{M}_\alpha \|u\|_{L^2(\mathbb{R})},
$$

(14)

where $\bar{M}_\alpha$ is the sharp constant

$$
\bar{M}_\alpha := \sup \left\{ \left\| e^{i|\nabla|^\alpha} u \right\|_{L^{2n+2}_{t,x}(\mathbb{R}^2)} : \|u\|_{L^2(\mathbb{R})} = 1 \right\}.
$$

See, for instance, [23, Theorem 2.4] for analogous arguments. In [19], Hundertmark and Shao give the existence of extremals for some similar non-endpoint Airy-Strichartz inequalities based on the profile decomposition of Airy-Strichartz version. Moreover they also establish the analyticity of these extremals on the Fourier space by using a bootstrap argument. In the spirit of their work and based on the generalized profile decomposition consequences obtained above, we show the existence of extremals for $\bar{M}_\alpha$ as a short incidental result.

Theorem 1.8. For every $\alpha \geq 2$, there exists an extremal for $\bar{M}_\alpha$. 

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The outline of this paper is as follows. In Section 2 we begin with proving the conditional dislocation property Proposition 1.3 which is one of the key ingredients in our paper. Then we extract the frequency and scaling parameters for the desired $\alpha$-Strichartz version linear profile decomposition in Section 3. After that, by using Proposition 1.3, we are able to obtain the time and space translation parameters in Section 4 and further present the $\alpha$-Strichartz version linear profile decomposition in Section 5. Then Section 6 and Section 7 contain the extremal results for symmetric $\alpha$-Strichartz estimates Theorem A and asymmetric $\alpha$-Strichartz estimates Theorem 1.1 respectively. Finally the proof of Theorem 1.8 is provided in Section 8.

We end this section with some notations. Firstly we use the familiar notation $x \lesssim y$ to denote that there exists a finite constant $C$ such that $|x| \leq C|y|$, similarly for $x \gtrsim y$ and $x \sim y$. Sometimes we may show the dependence such as $x \lesssim_{\alpha} y$ for the constant $C = C(\alpha)$ if necessary. Occasionally we may write $\hat{u} := \mathcal{F}[u]$ or $u^\wedge := \mathcal{F}[u]$, similarly for the inverse Fourier transform $\hat{u} = u^\vee := \mathcal{F}^{-1}[u]$. In addition, since there may be different topologies throughout this paper, we use the notation $\rightarrow$ to denote strong convergence and the notation $\rightharpoonup$ to denote weak convergence. More precisely, for a sequence of functions $(f_n) \subset L^p$, we write $f_n \rightarrow f_0$ for the fact that $f_n$ converge to $f_0$ as $n$ goes to infinity in the norm (strong) topology of $L^p$, and write $f_n \rightharpoonup f_0$ for the fact that $f_n$ converge to $f_0$ as $n$ goes to infinity in the weak topology of $L^p$. As for a sequence of operators $(T_n)$ on the space $H$ which means $(\{T_n\}) \subset B(H)$, similarly $T_n \rightarrow T_0$ and $T_n \rightharpoonup T_0$ denote the convergence in the strong operator topology and weak operator topology of $B(H)$ respectively.

2 Dislocation property from van der Corput Lemma

Before to give the linear profile decomposition, we show the conditional dislocation property Proposition 1.3 first since it will be used in the forthcoming work of extracting time-space translation parameters in Section 4. As what we have said before this property is, in some sense but not directly, generalization of the classical Schrödinger dislocation property which comes from the Galilean invariance. Note that the conditional dislocation property Proposition 1.3 has been adapted to the desired profile decomposition Proposition 1.5 when we establish it.

**Proof of Proposition 1.3.** We begin with proving the first conclusion. By a standard approximation argument together with the symmetry of $j$ and $k$, it suffices to prove that if

$$\lim_{n \rightarrow \infty} \left\langle \left[ g_n^{j} \right]^{-1} [e^{i t_n |\nabla|^\alpha}][e^{-i t_n |\nabla|^\alpha}] [g_n^{k}] \phi, \psi \right\rangle \neq 0 \quad (15)$$

for some Schwartz functions $\phi$ and $\psi$ whose Fourier supports are compact, then there exist a unitary operator $G^{jk} \in B(L^2_x)$ and a subsequence for $n$ (also denoted by $n$) such that

$$[g_n^{j}]^{-1} [e^{i t_n |\nabla|^\alpha}][e^{-i t_n |\nabla|^\alpha}] [g_n^{k}] f \rightarrow G^{jk} f \quad (16)$$

as $n \rightarrow \infty$ in the $L^2_x$ norm topology for all Schwartz functions $f$. Note that a simple computation shows

$$2\pi \left\langle \left[ g_n^{j} \right]^{-1} [e^{i t_n |\nabla|^\alpha}][e^{-i t_n |\nabla|^\alpha}] [g_n^{k}] \phi(x) \right\rangle = \left[ g_n^{j} \right]^{-1} (h_n^{j})^{-\frac{1}{2}} \int_{\mathbb{R}} e^{i x\xi} \left| \int_{\mathbb{R}} e^{i \frac{|\xi - \xi_n^{j}|^2}{e_n^{j}}} \frac{i (\xi - h_n^{j} \xi_n^{j})}{(h_n^{j})^2} \phi(\xi) d\xi \right|$$

$$= \left( \frac{h_n^{j}}{h_n^{k}} \right)^{-\frac{1}{2}} \left| \int_{\mathbb{R}} e^{i x h_n^{j} (\xi - \xi_n^{j})} \left( \int_{\mathbb{R}} e^{i \frac{|\xi - \xi_n^{k}|^2}{e_n^{k}}} \frac{i (\xi - h_n^{k} \xi_n^{k})}{(h_n^{k})^2} \phi(\xi) d\xi \right) \right| \quad (17)$$

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We first eliminate the case \( \lim_{n \to \infty} \left( \frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} \right) = \infty \). Due to the fact that the operators in condition (15) are unitary operators on \( L^2_x \), it is easy to conclude

\[
2\pi \left\langle \left[ g_n^j \right]^{-1} [e^{it\Delta_x^j}] [e^{-it\Delta_x^k}] [g_n^k] \phi, \psi \right\rangle = (h_n^j h_n^k)^{-\frac{1}{2}} \left( \int_{\mathbb{R}} e^{\frac{x-x_j}{h_n^j}} [e^{it\Delta_x^j} \phi] \wedge (\xi) d\xi, \int_{\mathbb{R}} e^{\frac{x-x_k}{h_n^k}} [e^{it\Delta_x^k} \psi] \wedge (\xi) d\xi \right)
\]

\[
= (h_n^j h_n^k)^{-\frac{1}{2}} \left( \Phi_n^j \left( \frac{x-x_j}{h_n^j} \right), \Phi_n^k \left( \frac{x-x_k}{h_n^k} \right) \right).
\]

(18)

Notice that \( \Phi_n^j \in L^2 \) which implies

\[
\lim_{R \to \infty} \int_{|y| > R} |\Phi_n^j(y)|^2 dy = 0.
\]

Hence if we setting

\[
B_n^j(R) := \left\{ x : \left| \frac{x-x_j}{h_n^j} \right| \leq R \right\}, \quad B_n^k(R) := \left\{ x : \left| \frac{x-x_k}{h_n^k} \right| \leq R \right\}
\]

and considering (18) with the integral on \( \mathbb{R} \setminus B_n^j(R), \) Hölder’s inequality will give a bound as follows

\[
2\pi \left\langle \left[ g_n^j \right]^{-1} [e^{it\Delta_x^j}] [e^{-it\Delta_x^k}] [g_n^k] \phi, \psi \right\rangle_{\mathbb{R} \setminus B_n^j(R)} \leq \left( \int_{|y| > R} |\Phi_n^j(y)|^2 dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |\Phi_n^j(y)|^2 dy \right)^{\frac{1}{2}}.
\]

Similar approach also works for the integral on \( \mathbb{R} \setminus B_n^k(R) \) in (18). Thus, by the fact that \( \Phi_n^j \) and \( \Phi_n^k \) are \( L^\infty_x \) functions, we aim to show the following estimate

\[
\lim_{n \to \infty} (h_n^j h_n^k)^{-\frac{1}{2}} \left| B_n^j(R) \cap B_n^k(R) \right| = 0,
\]

(22)

which will lead to a contradiction to the assumption (15). One observation we need is

\[
\left| B_n^j(R) \cap B_n^k(R) \right| \leq C_R \min \left\{ h_n^j, h_n^k \right\}.
\]

Then we obtain the desired estimate (22) immediately since \( h_n^j / h_n^k \) goes to either zero or infinity. Consequently, we can assume \( h_n^j \sim h_n^k \) from now on.

Next we eliminate the case \( \lim_{n \to \infty} (h_n^j + h_n^k)|\xi_j^k - \xi_n^j| = \infty \). By the Plancherel theorem and the fact that these operators are unitary operators in on \( L^2(\mathbb{R}) \), we conclude

\[
\left\langle \left[ g_n^j \right]^{-1} [e^{it\Delta_x^j}] [e^{-it\Delta_x^k}] [g_n^k] \phi, \psi \right\rangle_x = \left\langle [e^{-it\Delta_x^k}] [g_n^k] \phi, [e^{-it\Delta_x^j}] [g_n^j] \psi \right\rangle_x \\
\sim \left\langle e^{-it\Delta_x^k} [g_n^k] \phi, e^{-it\Delta_x^j} [g_n^j] \psi \right\rangle_x \\
= (h_n^j h_n^k)^{1/2} \left\langle \hat{\phi} \left( h_n^j \xi - h_n^k \xi_n^k \right), \hat{\psi} \left( h_n^k \xi - h_n^j \xi_n^j \right) \right\rangle_\xi.
\]

Thus the assumption \( h_n^j \sim h_n^k \) gives

\[
\left\langle \left[ g_n^j \right]^{-1} [e^{it\Delta_x^j}] [e^{-it\Delta_x^k}] [g_n^k] \phi, \psi \right\rangle_x \sim \left\langle \hat{\phi} \left( \frac{h_n^k}{h_n^j} (\xi - h_n^j \xi_n^j) \right), \hat{\psi} \left( \xi - h_n^j \xi_n^j \right) \right\rangle_\xi
\]

\[
\sim \left\langle \hat{\phi} \left( \xi_n^j \right), \hat{\psi} \left( \xi_n^j \right) \right\rangle_\xi.
\]

(19)
\[
\sim \left\langle \hat{\phi} \left( \xi - h_n^k \xi_n^k \right), \hat{\psi} \left( \frac{h_n^j}{h_n^k} \xi - h_n^k \xi_n^k \right) \right\rangle_{\xi}.
\]

Then the condition (15) together with the assumption that \( \hat{\phi} \) and \( \hat{\psi} \) have compact supports imply, up to subsequences, the following

\[
\lim_{n \to \infty} (h_n^j + h_n^k) \left| \xi_n^k - \xi_n^j \right| = c_2, \quad c_2 < \infty.
\]

Hence we can assume that \((h_n^j, \xi_n^j) \equiv (h_n^k, \xi_n^k) \equiv (h_n, \xi_n)\) from now on.

With the assumptions for \( \xi_n \) and \( h_n \) at hand, we can turn the expression (20) into

\[
\left\langle [g_n^j]^{-1} [e^{it_n^j h_n^j \xi}]^\alpha [e^{-it_n^k h_n^k \xi}] [g_n^k] \phi, \psi \right\rangle = (2\pi h_n)^{-1} \left\langle \Phi_n^j \left( \frac{x - x_n^j}{h_n^j} \right), \Phi_n^k \left( \frac{x - x_n^k}{h_n^k} \right) \right\rangle.
\]

Then just as what we have done above, recalling the condition (21), a similar changing of variables argument and the assumption (15) imply, up to subsequences, that

\[
\lim_{n \to \infty} \frac{x_n^j - x_n^k}{h_n} = c_3, \quad |c_3| < \infty.
\]

On the other hand, we can turn the expression (19) into

\[
\left\langle [g_n^j]^{-1} [e^{it_n^j h_n^j \xi}]^\alpha [e^{-it_n^k h_n^k \xi}] [g_n^k] \phi, \psi \right\rangle = (2\pi h_n)^{-1} \left\langle \tilde{\Phi}_n^j \left( \frac{x}{h_n} + \frac{i t_n^j}{(h_n)^\alpha} \right), \tilde{\Phi}_n^k \left( \frac{x}{h_n} + \frac{i t_n^k}{(h_n)^\alpha} \right) \right\rangle, \quad (23)
\]

where the function \( \tilde{\Phi}_n^j \) is defined by the following

\[
\tilde{\Phi}_n^j \left( \frac{x}{h_n} + \frac{i t_n^j}{(h_n)^\alpha} \right) := e^{i \left( \frac{\xi x}{h_n^j} + \frac{t_n^j}{(h_n)^\alpha} \right) \xi} e^{-\left( \frac{\xi x}{h_n^j} + \frac{t_n^j}{(h_n)^\alpha} \right) \xi} e^{-i \left( \frac{\xi x}{h_n^k} + \frac{t_n^k}{(h_n)^\alpha} \right) \xi} e^{i \left( \frac{\xi x}{h_n^k} + \frac{t_n^k}{(h_n)^\alpha} \right) \xi},
\]

and similarly for the definition of \( \tilde{\Phi}_n^k \). Then we still have the fact \( \tilde{\Phi}_n^j \in L^2 \) and further

\[
\lim_{R \to \infty} \int_{|y| > R} |\tilde{\Phi}_n^j|^2 dy = 0.
\]

Analogously, the expression (23) and a changing of variables argument imply, up to subsequences, that

\[
\lim_{n \to \infty} \frac{t_n^j - t_n^k}{(h_n)^\alpha} = c_4, \quad |c_4| < \infty
\]

based on the assumption (15). Moreover, we can turn the expression (17) into

\[
\left| [g_n^j]^{-1} [e^{it_n^j h_n^j \xi}]^\alpha [e^{-it_n^k h_n^k \xi}] [g_n^k] \phi(x) \right| = \frac{1}{2\pi} \left| \int_{\mathbb{R}} e^{i \Phi_n^{jk}(x, \xi)} \phi(\xi) d\xi \right|,
\]

where

\[
\Phi_n^{jk}(x, \xi) := \xi \left( x + \frac{x_n^j - x_n^k}{h_n} \right) - \frac{i t_n^j - i t_n^k}{(h_n)^\alpha} \xi + h_n \xi_n^m. \quad (24)
\]

It is obvious that \( \int_{\mathbb{R}} e^{i \Phi_n^{jk}(x, \xi)} \phi(\xi) d\xi \in L^\infty \). Next we are going to use the method of stationary phase to obtain the decay estimates of this oscillatory integral. To begin with, analysing piece by piece if necessary, we can assume \( \xi + h_n \xi_n > 0 \) without loss of generality and rewrite \( \Phi_n^{jk}(x, \xi) \) as

\[
\Phi_n^{jk}(x, \xi) = \xi \left( x + \frac{x_n^j - x_n^k}{h_n} \right) + \sum_{m=1}^{\alpha \over (m)} \frac{(\alpha) t_n^j - t_n^k}{(h_n)^\alpha} (\xi_n)^{\alpha-m} (\xi)^m. \quad (25)
\]
\[ \sum_{m=1}^{\alpha} a_{m,j,k}^m(\xi)^m, \]

where \( a_{m,j,k}^m \) are the coefficients of the \( m \)-order term \( (\xi)^m \) in the expression of \( \Phi_n^{jk} \), except for the case \( m = 1 \) where \( x + a_{1,j,k} \) is the coefficient of \( \xi \). Note that we have ignored the constant term \( (\xi)^0 \) here and in the computation (17). This term is easy to manage due to the compactness of \( S^1 \), which will be shown after the definition (26). We are going to prove that, after passing to a subsequence, each of the coefficients \( a_{m,j,k}^m \) goes to some constant \( c_{m,j,k} \neq \infty \) as \( n \) goes to infinity. Then this result gives the desired function

\[ \Phi^{jk}(x, \xi) := \xi x + \sum_{m=1}^{\alpha} c_{m,j,k}^m(\xi)^m \]

satisfying \( \lim_{n \to \infty} \Phi_n^{jk}(x, \xi) = \Phi^{jk}(x, \xi) \). Thereby we get the desired operator \( G^{jk} \) defined as

\[ G^{jk} f(x) := \frac{e^{i\theta^{jk}}}{2\pi} \int_{\mathbb{R}} e^{i\Phi^{jk}(x, \xi)} \hat{f}(\xi) d\xi, \tag{26} \]

which satisfies the equation (16). It should be pointed out that the term \( e^{i\theta^{jk}} \), which we do not pay much attention to it before, comes from the parameters involved and the compactness of \( S^1 \) due to the fact \( e^{i\theta^{jk}} = 1 \). We also remark that the lack of the term \( e^{ix} \) in (26), compared with the expression in (17), comes from the assumption \( \Theta^{jk}_n \equiv \zeta_n^{jk} \equiv \xi_n \).

It remains to be proved that for each \((m, j, k)\) there exists \( |c_{m,j,k}| < \infty \) satisfying, after passing to a subsequence,

\[ \lim_{n \to \infty} a_{m,j,k}^n = c_{m,j,k}. \]

If on the contrary for some fixed \( m \) there holds \( \lim_{n \to \infty} a_{m,j,k}^n = \infty \). Take

\[ m_0 := \max\{m : \lim_{n \to \infty} a_{m,j,k}^n = \infty\}. \tag{27} \]

We break the proof into two cases \( m_0 = 1 \) and \( m_0 > 1 \). For the case \( m_0 = 1 \), we have the following limit relation

\[ \Phi_n^{jk}(x, \xi) = (x + a_{1,j,k}^1) \xi + \Phi_n^{1,j,k}(\xi) \to (x + \infty) \xi + \Phi_n^{1,j,k}(\xi) \]

as \( n \) goes to infinity. Here the function \( \Phi_n^{1,j,k}(\xi) \) whose coefficients are bounded is the limit function of \( \Phi_n^{1,j,k}(\xi) \). Since the parameters \( a_{1,j,k}^1 \) just deduce translations for \( \xi \) on the Fourier side, the assumption of compact Fourier supports property and the Plancherel theorem imply that

\[ \lim_{n \to \infty} \left\langle [g_n^{\ell}]^{-1} e^{it_n^\ell} [\nabla]^\alpha [e^{-i t_n^k} |\nabla|^\alpha] [\hat{g}_n^k][\phi, \psi] \right\rangle = 0. \]

This is a contradiction to the condition (15). For the case \( m_0 > 1 \), by the compactness of \( \text{supp}(\hat{\phi}) \), we deduce the following estimates

\[ \left| \frac{d}{d\xi} \Phi_n^{jk}(x, \xi) \right| \sim |a_{m_0,j,k}^n|, \quad \left| \frac{d}{d\xi} \Phi_n^{jk}(x, \xi) \right| < \infty \]

for \( \xi \in \text{supp}(\hat{\phi}) \) and \( n \) large enough. Therefore the classical van der Corput Lemma [38, p. 334, Corollary] implies

\[ \left\| [g_n^{\ell}]^{-1} e^{it_n^\ell} [\nabla]^\alpha [e^{-i t_n^k} |\nabla|^\alpha] [\hat{g}_n^k][\phi] \right\|_{L^\infty} \lesssim \phi |a_{m_0,j,k}^n| - \frac{1}{m_0} \to 0 \]
as \( n \to \infty \) and thus \([g_n^j]^{-1}[e^{it_n^j|\nabla|^\alpha}][e^{-it_n^j|\nabla|^\alpha}[g_n^k]\phi \to 0 \) in \( L^2_t \) as \( n \) goes to infinity. This is a contradiction to (15) and finishes the proof of the first conclusion.

Now, we turn to the second conclusion. Actually the strategy is similar to the proof above for the first conclusion. We still can get the expression (24) even if \( \alpha > 1 \) is a real number. Here we divide the proof into two parts: up to subsequences, either

\[
\lim_{n \to \infty} |h_n \xi_n| \to \infty
\]  

or \( \lim_{n \to \infty} h_n \xi_n = c_5 \) with \( |c_5| < \infty \). For the latter case, indeed we have got the desired operator \( G^{jk} \) satisfying (16) defined as follows

\[
G^{jk} f(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x+c_5)\xi - ic_4 \xi + c_3^m} \tilde{f}(\xi) d\xi.
\]

Hence our last target is to deal with the case \( h_n \xi_n \to \infty \) as \( n \) goes to infinity. This time, noticing the compact Fourier supports assumption, we should change (25) to the following series

\[
\Phi_n^{jk}(x, \xi) = \xi \left( x + \frac{x_n^j - x_n^k}{h_n} \right) + \sum_{m=1}^{\infty} a_n^{m,j,k}(\xi)^m
\]

for \( n \) large enough, since we have the assumption (28) which can guarantee the uniform convergency of this series. Meanwhile, we can investigate further about the coefficients \( a_n^{m,j,k} \) by using this assumption. Define \( m_0 \) as in (27). Then if \( m_0 \geq 2 \), the assumption (28) will imply

\[
\lim_{n \to \infty} a_n^{2,j,k} = \infty.
\]

Again, the classical van der Corput Lemma will give the decay estimate

\[
\left\| [g_n^j]^{-1}[e^{it_n^j|\nabla|^\alpha}][e^{-it_n^j|\nabla|^\alpha}[g_n^k]\phi \right\|_{L^2_t} \lesssim \phi \left| a_n^{2,j,k} \right|^{-\frac{1}{\alpha}} \to 0
\]

as \( n \to \infty \). If \( m_0 = 1 \) we do the same arguments as the proof for the first conclusion aforementioned. Analogously when \( a_n^{1,j,k} \) and \( a_n^{2,j,k} \) are both bounded, we can assume that up to subsequences

\[
\lim_{n \to \infty} a_n^{1,j,k} = c^{1,j,k}, \quad \lim_{n \to \infty} a_n^{2,j,k} = c^{2,j,k}.
\]

Then the desired operator, similar to the expression (26), is given by

\[
\tilde{G}^{jk} f(x) := \frac{e^{i\delta^{jk} k}}{2\pi} \int_{\mathbb{R}} e^{i\tilde{\Phi}^{jk}(x, \xi) \tilde{f}(\xi) d\xi}, \quad \tilde{\Phi}^{jk}(x, \xi) := (x + c^{1,j,k} \xi + c^{2,j,k}(\xi)^2.
\]

Therefore we totally complete the proof of these two conclusions. \( \square \)

3 First-step decomposition: frequency and scaling

Usually the profile decomposition results are obtained by following two steps: first for the frequency-scaling parameters based on some refinement of Strichartz estimates which can be deduced by the bilinear restriction estimates from \([40, 43, 44]\), and second for the time-space translations by using

\[\text{Recall that the binomial coefficient } \binom{\alpha}{m} := \alpha(\alpha-1) \cdots (\alpha-m+1)/m! \text{ is well-defined for } \alpha \notin \mathbb{Z}.\]
some weak convergence arguments which will be further discussed in Section 4 later. There may be some papers providing slightly different procedures by using similar ingredients such as [42, Appendix A] and [26, Theorem 4.26]. We refer to [39] for a brief discussion on the $L^2$-based linear profile decomposition and a generalization in the $L^p$ setting, see also [4] for some recent results on the $L^p$-generalization.

In this section, we present the first-step decomposition by following the proofs in [7, 21], similar method can also be seen in some earlier papers [8, 24]. It is convenient to give the following dyadic intervals in $\mathbb{R}$ to do some dyadic analysis.

**Definition 3.1 (Dyadic intervals).** Given $j \in \mathbb{Z}$, the dyadic intervals of length $2^j$ in $\mathbb{R}$ is defined by

$$D_j := \{2^j[k, k+1) : k \in \mathbb{Z}\};$$

and we use $D := \bigcup_{j \in \mathbb{Z}} D_j$ to denote the set of all the dyadic intervals in $\mathbb{R}$.

**Proposition 3.2 (\(\alpha\)-refined Strichartz).** For any $p > 1$, we have

$$\left\| \mathcal{D}^{\alpha/2} \left[ e^{it|\xi|^\alpha} f \right] \mathcal{D}^{\alpha/2} \right\|_{L^p_x(L^q_t(\mathbb{R}^2))} \lesssim_{\alpha,p} \left( \sup_{\tau} \|\tau|^{1/2-\frac{1}{p}} \|\mathcal{F}(\hat{f})(\tau)\|_{L^p(\mathbb{R})} \right)^{\frac{1}{2}} \|f\|_{L^3(\mathbb{R})}^{\frac{p}{2}},$$

where $\tau$ denotes an interval in $\mathbb{R}$ with the length $|\tau|$. Moreover, we can restrict $\tau$ to be dyadic intervals.

**Proof of Proposition 3.2.** We adapt the proofs in [21, Lemma 1.2] and [7, Section 2] by using the Whitney decomposition and Hausdorff-Young inequality instead of bilinear restriction estimates aforementioned since we are dealing with the one dimensional case now. See also [8, 24] for different methods using Fefferman-Phong’s weighted inequality from [12].

Notice that we can normalize $\sup_{\tau \subset D} |\tau|^{1/2-1/p} \|\mathcal{F}(\hat{f})(\tau)\|_{L^p(\mathbb{R})} = 1$ for given $p > 1$. This implies that the following inequality

$$\int_I |\hat{f}|^p d\xi \leq |I|^{1-p/2}$$

holds for all dyadic intervals $I \in \{2^j[k, k+1) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$. In our proof here we aim to show that

$$\left\| \mathcal{D}^{\alpha/2} \left[ e^{it|\xi|^\alpha} f \right] \mathcal{D}^{\alpha/2} \right\|_{L^p_x(L^q_t(\mathbb{R}^2))} \lesssim \int_{\mathbb{R}^2} \frac{|\hat{f}(\xi)\hat{g}(\eta)|^\frac{p}{2}}{|\xi - \eta|^\frac{p}{2}} d\xi d\eta.$$  \hspace{1cm} (32)

Getting the desired result (30) from the estimates (31) and (32) is a standard application of Whitney decomposition. Since the details of this process can be found in [21, Lemma 1.2] and [7, Proposition 2.7], we omit the detailed proof of this part for avoiding too much repetition.

Define the, in some sense, extension operator $[E_\alpha]$ by

$$[E_\alpha] f(t,x) := 2\pi \mathcal{D}^{\alpha/2} \left[ e^{it|\xi|^\alpha} f \right] \mathcal{D}^{\alpha/2} = \int_\mathbb{R} e^{i\xi - it|\xi|^\alpha} f(\xi) d\xi.$$  \hspace{1cm} (31)

Then we investigate the following bilinear forms

$$[E_\alpha] f [E_\alpha] g(t,x) = \int_{\mathbb{R}^2} e^{i\xi + \eta - it(|\xi|^\alpha + |\eta|^\alpha)} |\xi|^\frac{\alpha}{\alpha - \frac{2}{p}} |\eta|^\frac{\alpha - 2}{\alpha} \hat{f}(\xi)\hat{g}(\eta) d\xi d\eta.$$  \hspace{1cm} (33)

Consider the changing of variables

$$(\xi, \eta) \mapsto (u,v) := (\xi + \eta - |\xi|^\alpha - |\eta|^\alpha).$$
Recall that for fixed \((u_0, v_0)\), the graph of the function \(u = \xi + \eta\) is a line and the graph of \(v = -|\xi| - |\eta|^\alpha\) is a “circle” in some sense. This implies that the map defined in (33) is an at most 2-to-1 map from \(\mathbb{R}^2\) to the region \(Q := \{(u, v) : -v \geq 2^{1-\alpha}|u|^\alpha\}\) which comes from the convexity. Further the Jacobian is given by

\[ J(u, v) = J^{-1}(\xi, \eta) = \frac{\partial(u, v)}{\partial(\xi, \eta)} = \alpha(\xi|\xi|^{\alpha - 2} - \eta|\eta|^{\alpha - 2}). \]

Thus we conclude

\[ \left| [E_\alpha f] [E_\alpha g](t, x) \right| \leq 2 \left| \int_Q e^{ixu + itv} |\xi\eta|^{\alpha-2} \hat{f}(\xi) \hat{g}(\eta) J^{-1}(u, v)du dv \right|, \]

where \((\xi, \eta)\) is a function of \((u, v)\) via the change of variables (33) above. By the symmetry, we can assume \(|\eta| \leq |\xi|\) without loss of generality. Using the Hausdorff-Young inequality and then changing variables back to \((\xi, \eta)\) we deduce the following

\[ \| [E_\alpha f] [E_\alpha g] \|_{L^3_x L^2_t(\mathbb{R}^2)}^{3/2} \leq \left\| |\xi\eta|^{\alpha-2} \hat{f}(\xi) \hat{g}(\eta) J^{-1}(u, v) \right\|_{L^3_x L^2_t(\mathbb{R}^2)}^{3/2} = \left\| |\xi\eta|^{\alpha-2} |J(\xi, \eta)|^{1/2} \hat{f}(\xi) \hat{g}(\eta) \right\|_{L^3_x L^2_t(\mathbb{R}^2)}^{3/2}. \] (34)

To estimate the norm above, our next target is the Jacobian factor

\[ \hat{J}(\xi, \eta) := |\xi\eta|^{\alpha-2} |J(\xi, \eta)|^{1/2} = \frac{|\xi\eta|^{\alpha-2}}{\alpha(|\xi|^{\alpha} - |\eta|^{\alpha-2})^{1/2}}. \]

If \(\xi \eta \leq 0\), it is easy to see that

\[ \hat{J}(\xi, \eta) = \frac{|\xi\eta|^{\alpha-2}}{\alpha(|\xi|^{\alpha} + |\eta|^{\alpha-1})^{1/2}} \lesssim_\alpha (|\xi| + |\eta|)^{-\frac{1}{2}} = |\xi - \eta|^{-\frac{1}{2}}. \]

If \(\xi \eta > 0\) and \(|\xi| \geq |\eta|\), then we have

\[ |\xi|^{\alpha - 1} - |\eta|^{\alpha - 1} \sim_\alpha (|\xi| - |\eta|)|\xi|^{\alpha - 2}. \]

This estimate leads to

\[ \hat{J}(\xi, \eta) = \frac{|\xi\eta|^{\alpha-2}}{\alpha(|\xi|^{\alpha} - |\eta|^{\alpha-1})^{1/2}} \lesssim_\alpha \frac{|\xi\eta|^{\alpha-2}}{|\xi|^{\alpha-1} |\xi - \eta|^{1/2}} \leq |\xi - \eta|^{-\frac{1}{2}}. \]

If \(\xi \eta > 0\) and \(|\xi| < |\eta|\), by the symmetry, analogously as above we can obtain \(\hat{J}(\xi, \eta) \lesssim_\alpha |\xi - \eta|^{-\frac{1}{2}}\). In summary, we know that

\[ \hat{J}(\xi, \eta) \lesssim_\alpha |\xi - \eta|^{-\frac{1}{2}} \]

holds uniformly in \(\xi\) and \(\eta\). Taking this into the expression (34), we get the desired estimate (32). \(\square\)

Based on the refined Strichartz estimate Proposition 3.2, we can extract the frequency and scaling parameters by following a standard approach in [21], similar argument can also be seen in [8]. We omit the detailed proof of the following Lemma 3.3 here, since it is too long but essentially the same as [21, Lemma 5.1] and [8, Lemma 3.3].
Lemma 3.3. Let \( \{u_n\}_{n \geq 1} \) be a sequence of functions with \( \|u_n\|_{L^2_x(\mathbb{R})} \leq 1 \). Then up to subsequences, for any \( \delta > 0 \), there exist

\[
N = N(\delta), \quad \{(\rho_n^\beta, \xi_n^\beta)_{1 \leq \beta \leq N}\} \subset (0, \infty) \times \mathbb{R}, \quad \{(f_n^\beta)_{1 \leq \beta \leq N}\} \subset L^2_x(\mathbb{R})
\]
such that

\[
u_n = \sum_{\beta=1}^N f_n^\beta + q_n^N
\]
and there exists a compact set \( K = K(N) \in \mathbb{R} \) such that for every \( 1 \leq \beta \leq N \) there holds

\[
(\rho_n^\beta) \frac{1}{\rho_n} |\hat{f}_n^\beta (\rho_n^\beta \xi + \xi_n^\beta)| \leq C_\delta \mathbb{I}_K (\xi).
\]
Here the sequence \( (\rho_n^\beta, \xi_n^\beta) \) satisfies that if \( \beta \neq \gamma \) then

\[
\lim_{n \to \infty} \left( \frac{\rho_n^\beta}{\rho_n^\gamma} + \frac{\rho_n^\gamma}{\rho_n^\beta} + \frac{\xi_n^\beta - \xi_n^\gamma}{\rho_n^\beta} + \frac{\xi_n^\gamma - \xi_n^\beta}{\rho_n^\gamma} \right) = \infty.
\]
The remainder term \( q_n^N \) has a negligible Strichartz norm

\[
\left\| |D|^{\frac{n-2}{n}} |e^{i(|\xi|)}| q_n^N \right\|_{L^6_x} \leq \delta;
\]
and furthermore, if for each \( 1 \leq N' \leq N \) we generally define

\[
q_n^{N'} := q_n^N + f_n^{N-1} + \ldots + f_n^{N'+1},
\]
then we have the \( L^2 \)-almost orthogonal identity

\[
\lim_{n \to \infty} \left( \|u_n\|_{L^2}^2 - \sum_{\beta=1}^{N'} \|f_n^{\beta}\|_{L^2}^2 + \|q_n^{N'}\|_{L^2}^2 \right) = 0.
\]

Remark 3.4. We should remark that in the proof of Lemma 3.3, by the construction, we know that the Fourier supports of \( f_n^\beta \) and \( q_n^N \) are mutually disjoint. This crucial fact also implies the conclusion (39). On the other hand, define operators \( [\tilde{G}_n^\beta] \) on the Fourier side by

\[
[\tilde{G}_n^\beta] [\hat{f}] (\xi) := (\rho_n^\beta)^{\frac{1}{2}} [\hat{f} (\rho_n^\beta \xi + \xi_n^\beta)].
\]
Then the conclusion (37) means that, in view of the conditional dislocation property Proposition 1.3, the sequence of operators satisfy

\[
[\tilde{G}_n^\beta][G_n^\beta]^{-1} \to 0 \quad \text{and} \quad [\tilde{G}_n^\beta][G_n^\beta]^{-1} \to 0
\]
as \( n \to \infty \) for every \( \beta \neq \gamma \). Or equivalently on the spatial side, define

\[
[G_n^\beta] f(x) := \mathcal{F}^{-1} [\tilde{G}_n^\beta] [\mathcal{F} f(x)] = (\rho_n^\beta)^{-\frac{1}{2}} e^{-ix(\rho_n^\beta)^{-1} \xi_n^\beta} f \left( \frac{x}{\rho_n^\beta} \right).
\]
Then the conclusion (37) implies that \( [G_n^\beta][G_n^\beta]^{-1} \) and \( [G_n^\beta][G_n^\beta]^{-1} \) goes to zero as \( n \) go to infinity in the weak operator topology of \( \mathcal{B}(L^2) \) for \( \beta \neq \gamma \). This comes from the dual approach on \( L^2(\mathbb{R}) \) and Plancherel theorem as follows

\[
\left< \left[ \tilde{G}_n^\beta][G_n^\beta]^{-1} [\hat{f}], \hat{g} \right] \right>_\xi = \left< \left[ G_n^\beta]^{-1} [\hat{f}], [\tilde{G}_n^\beta][G_n^\beta]^{-1} \hat{g} \right] \right>_\xi = \left< \mathcal{F} [G_n^\beta][G_n^\beta]^{-1} f, \mathcal{F} [G_n^\beta][G_n^\beta]^{-1} g \right> \]
\[
\sim \left< [G_n^\beta][G_n^\beta]^{-1} f, [G_n^\beta][G_n^\beta]^{-1} g \right> = \left< [G_n^\beta][G_n^\beta]^{-1} f, g \right>_x.
\]
4 Second-step decomposition: time and space translations

After the first-step decomposition Lemma 3.3, indeed we have obtained the desired frequency and scaling parameters. Hence in this section, we are devoted to getting the time and space translation parameters. Recall that the dislocation property (or equivalently the Galilean invariance) always play an important role in the classical case [2, 5, 8, 31]. However this Galilean invariance is not valid in our $\alpha$-Strichartz setting and also note that $\alpha$ may not be a natural number. Thus our strategy is using the conditional dislocation property Proposition 1.3 obtained in Section 2.

To begin this section, one ingredient we need is the following local restriction Lemma 4.1. Then we are ready to further decompose the functions $f_n$ obtained in the first-step decomposition and get the time-space translation parameters in Lemma 4.3.

**Lemma 4.1 (Localized restriction).** For $4 < q < 6$ and $\hat{F} \in L^\infty(B(\xi_0, R))$ with some $R > 0$, we have

$$\| [D^{\frac{\alpha-2}{2}} |e^{i[t\nabla|^\alpha]} F] \|_{L^q_{t,x}} \leq C_{q,R} \| \hat{F} \|_{L^\infty(B(\xi_0, R))}.$$ \hspace{1cm} (40)

**Proof of Lemma 4.1.** Similarly as what we have done in the proof of Proposition 3.2, the desired estimate is equivalent to the following bilinear form

$$\left\| \int_{B(\xi_0, R)} \int_{B(\xi_0, R)} e^{i\langle \xi+\eta \rangle - i\langle \xi\rangle \alpha} |\xi|^{-\alpha} |\eta|^{-\alpha} \hat{F}(\xi) \hat{F}(\eta) d\xi d\eta \right\|_{L^q_{t,x}} \lesssim_{q,R} \| \hat{F} \|_{L^\infty(B(\xi_0, R))}^2.$$ \hspace{1cm} (41)

By changing of variables

$$(u, v) := (\xi + \eta, -|\xi|\alpha - |\eta|\alpha),$$

using the Hausdorff-Young inequality and then changing the variables back, we conclude that the left hand side of (40) is bounded by

$$C \left( \int_{B(\xi_0, R) \times B(\xi_0, R)} |\hat{F}(\xi) \hat{F}(\eta)|^r |\xi|^{\frac{(\alpha-2)r'}{2}} |\eta|^{\frac{(\alpha-2)r'}{2}} |J(\xi, \eta)|^{r-1} d\xi d\eta \right)^{\frac{1}{r}}.$$ \hspace{1cm} (42)

where

$$r := \frac{q}{2} \in (2, 3), \quad J(\xi, \eta)^{-1} := \alpha(|\xi|^{\alpha-2} - |\eta|^{\alpha-2}).$$

We then consider the Jacobian factor

$$\tilde{J}(\xi, \eta) := |\xi|^{\frac{(\alpha-2)r'}{2}} |J(\xi, \eta)|^{r-1} = \frac{|\xi|^{\frac{(\alpha-2)(r'-1)}{2}}}{|\xi|^{\alpha-2} - |\eta|^{\alpha-2}}.$$

It is easy to see that $\tilde{J}$ can only has singularity at the following singular line

$$\xi = \eta.$$

By investigating the order of the singularity of $\tilde{J}$ at this singular line, we know that

$$\left\| \tilde{J}(\xi, \eta) \right\|_{L^1_{\xi,\eta}(B(\xi_0, R) \times B(\xi_0, R))} \lesssim_{R, r'} 1.$$ \hspace{1cm} (43)

Therefore we can control (42) by

$$C_{q,R} \| \hat{F} \|_{L^\infty(B(\xi_0, R))}^2,$$

which leads to the desired result (40) and thereby the proof is completed. \hfill $\Box$
**Definition 4.2** (Limit-orthogonality for sequences of operators). For fixed $j \neq k$, we say that two sequences of operators $(g_n^j)$ and $(g_n^k)$ in $B(L^2)$ are limit-orthogonal if

$$[g_n^j]^{-1}g_n^k \to 0, \quad n \to \infty.$$  

**Lemma 4.3** (Time-space translations). Let $F := (f_n)_{n \geq 1}$ be a sequence of $L^2(\mathbb{R})$ functions. Define the unitary operators $[\tilde{G}_n]$ and $[G_n]$ on $L^2(\mathbb{R})$ by

$$[\tilde{G}_n]f(x) := (\rho_n)^{\frac{1}{2}}f(\rho_n x + \xi_n), \quad [G_n]f(x) := \mathcal{F}^{-1}[\tilde{G}_n]\mathcal{F}f(x) = (\rho_n)^{-\frac{1}{2}}e^{-i\frac{\xi_n}{\rho_n}x}f\left(\frac{x}{\rho_n}\right).$$

If we assume that the following condition

$$\left| [\tilde{G}_n][\hat{f}_n](\xi) \right| \leq \hat{F}(\xi), \quad \hat{F} \in L^\infty(K)$$

holds for some compact set $K \subset \mathbb{R}$ independent of $n$. Then up to subsequences, there exist

$$(s_n^j, y_n^j)_{j \geq 1} \subset \mathbb{R} \times \mathbb{R}, \quad \{(\phi^j)_{j \geq 1}\} \subset L^2(\mathbb{R}), \quad [g_n^j]|\phi(x) := [e^{-i\xi_n^j|\nabla|^\alpha}]\phi(x - y_n^j)$$

such that the operators $[g_n^j][G_n]^{-1}$ satisfy the following limit-orthogonality property

$$[G_n][g_n^j]^{-1}g_n^k[G_n]^{-1} \to 0, \quad n \to \infty$$

for every $j \neq k$. Meanwhile, for every $M \geq 1$ there exist $e_n^M \in L^2(\mathbb{R})$ and the decomposition

$$f_n(x) = \sum_{j=1}^M [g_n^j][G_n]^{-1}\phi^j(x) + e_n^M(x)$$

with the vanishing Strichartz norm estimate for the remainder

$$\lim_{M \to \infty} \lim_{n \to \infty} \left\| D_{\alpha-2}^{\alpha-2} [e^{it|\nabla|^\alpha}]e_n^M \right\|_{L^6_{t,x}} = 0. \quad (44)$$

Furthermore, for every $M \geq 1$ we have the $L^2$-almost orthogonal identity

$$\lim_{n \to \infty} \left( \|f_n\|^2_{L^2} - \sum_{j=1}^M \|\phi^j\|^2_{L^2} + \|e_n^M\|^2_{L^2} \right) = 0. \quad (45)$$

**Proof of Lemma 4.3.** We adopt some ideas from [21, Lemma 5.2] and [8, Section 3], while similar approaches also arise in earlier papers [1, 25] and some of the references aforementioned. However as we have stated before, to generalize these classical arguments into our $\alpha$-Strichartz setting, we should use the conditional dislocation property Proposition 1.3. Take $P := (P_n)_{n \geq 1}$ with

$$\tilde{P}_n(\xi) := [\tilde{G}_n][\hat{f}_n](\xi).$$

Let $\mathcal{W}(P)$ be the set of weak limits in $L^2(\mathbb{R})$ for subsequences of $[G_n][g_n]^{-1}[G_n]^{-1}P_n$ defined by

$$\mathcal{W}(P) := \left\{ w-\lim_{n \to \infty} [G_n][g_n]^{-1}[G_n]^{-1}P_n(x) : (s_n, y_n) \in \mathbb{R}^2 \right\}, \quad [g_n]|\phi(x) := [e^{-i\xi_n|\nabla|^\alpha}]\phi(x - y_n),$$

and then define

$$\mu(P) := \sup \{ \|\phi\|_{L^2} : \phi \in \mathcal{W}(P) \}.$$
To get the desired decomposition (43), our strategy is to get the decomposition for $P_n$ as follows

$$P_n(x) = \sum_{j=1}^{M} [G_n][g_n^j][G_n]^{-1}\phi^j(x) + p_n^M(x),$$

and then set $e_n^M(x) := [G_n]^{-1}p_n^M(x) = \sqrt{\rho_n}e^{ix_n}p_n^M(p_n x)$. Similarly define the following

$$\mathbb{E}^M := (p_n^M)_{n \geq 1}, \quad \mathbb{E}^M := (e_n^M)_{n \geq 1}.$$

Firstly, we claim that if the conclusion (44) in Lemma 4.3 is replaced by

$$\lim_{M \to \infty} \mu(\mathbb{E}^M) = \lim_{n \to \infty} \mu(\mathbb{E}^M) = 0,$$

then this lemma is true even if we do not have the assumption that $K$ is a compact set independent of $n$. We show this claim as follows.

Indeed, if $\mu(\mathbb{P}) = 0$, then we can take $\phi^j = 0$ for all $j$ and the claim is proved. Otherwise if $\mu(\mathbb{P}) > 0$, we take $\phi^1 \in \mathcal{W}(\mathbb{P})$ such that

$$\|\phi^1\|_{L^2} \geq \frac{\mu(\mathbb{P})}{2} > 0.$$  

By the definition of $\mathcal{W}(\mathbb{P})$, there exists a sequence $(s_n^1, y_n^1) \in \mathbb{R}^2$ such that, up to extracting a subsequence, we have

$$[G_n][g_n^1][G_n]^{-1}P_n \to \phi^1$$

in $L^2(\mathbb{R})$ as $n$ goes to infinity. Setting $p_n^1 := P_n - [G_n][g_n^1][G_n]^{-1}\phi^1$, then we obtain

$$\lim_{n \to \infty} (\|P_n\|_{L^2} - ||\phi^1||_{L^2} - \|p_n^1\|_{L^2}) = 0,$$

due to the weak convergency (48) and the fact that $L^2(\mathbb{R})$ is a Hilbert space. Notice that all these operators involved are unitary operators on $L^2$. Therefore the almost orthogonal identity (45) holds for $M = 1$. Next, we replace $P_n$ by $p_n^1$ and then do this process again. If $\mu(\mathbb{P}^1) > 0$, we get the function $\phi^2$, the sequence of parameters $(s_n^2, y_n^2)$ and the sequence of functions $\mathbb{P}^2$. Moreover, we have one more conclusion as follows: the sequence of operators

$$[G_n][g_n^2][G_n]^{-1} \to 0$$

in $\mathcal{B}(L^2)$ as $n$ goes to infinity. Indeed if this conclusion is not true, then the dislocation property Proposition 1.3 asserts that, up to subsequences, there exists an isometric $[g^{1,2}]$ on $L^2(\mathbb{R})$ satisfying

$$[G_n][g_n^2][G_n]^{-1} \to [g^{1,2}]$$

in $\mathcal{B}(L^2)$ as $n$ goes to infinity. Therefore the following relation

$$[G_n][g_n^2][G_n]^{-1}p_n^1 = ([G_n][g_n^2][G_n]^{-1})[G_n][g_n^1][G_n]^{-1}p_n^1$$

and the weak convergency fact (48) imply that $\phi^2 = 0$, which means $\mu(\mathbb{P}^2) = 0$. This is a contradiction. Iterating this process leads to

$$p_n^1 := P_n - [G_n][g_n^1][G_n]^{-1}\phi^1, \quad [G_n][g_n^1][G_n]^{-1}P_n \to \phi^1, \quad \|\phi^1\|_{L^2} \geq \frac{\mu(\mathbb{P})}{2} > 0;$$

$$p_n^2 := p_n^1 - [G_n][g_n^2][G_n]^{-1}\phi^2, \quad [G_n][g_n^2][G_n]^{-1}p_n^1 \to \phi^2, \quad \|\phi^2\|_{L^2} \geq \frac{\mu(\mathbb{P}^1)}{2} > 0;$$

$$\vdots$$

$$p_n^j := p_n^{j-1} - [G_n][g_n^j][G_n]^{-1}\phi^j, \quad [G_n][g_n^j][G_n]^{-1}p_n^{j-1} \to \phi^j, \quad \|\phi^j\|_{L^2} \geq \frac{\mu(\mathbb{P}^{j-1})}{2} > 0;$$

$$\vdots$$
A diagonal process yields a sequence of functions \((\phi^j)_{j \geq 1}\) and a family of operators \([g^k_n]\) satisfying the orthogonal conclusion (42) for the case \(j = k + 1\). By the construction, we get the decomposition identity (46) and the almost orthogonal identities (45). To prove the desired claim, it remains for us to show the conclusion (42) for all \(j \neq k\) and the estimate (47). We show the estimate (47) first. Recall that \(\|f_n\|_{L^2}\) is uniformly bounded. Then (45) implies

\[
\sum_{j=1}^{M} \|\phi^j\|_{L^2}^2 \leq \limsup_{n \to \infty} \|f_n\|_{L^2}^2 \leq C.
\]

Hence we know that the positive series \(\sum_j \|\phi^j\|_{L^2}^2\) is convergent and further \(\lim_{n \to \infty} \|\phi^j\|_{L^2} = 0\). On the other hand, by the construction we have

\[
\mu(\mathbb{P}^M) \leq 2\|\phi^{M+1}\|_{L^2},
\]

which gives the desired estimate (47). Now we turn to the conclusion (42). Indeed, the more general case \(j = k + m(m \in \mathbb{Z}_+)\) comes from the basic case \(j = k + 1\), the following identity

\[
p^k_{n} = p^k_{n} - [G_n][g^k_{n}]^{-1}[G_n]^{-1}\phi^{k+1} - \cdots - [G_n][g^{k+m-1}_{n}]^{-1}\phi^{k+m-1},
\]

and an inductive argument. For the case \(j = k + m(m \in \mathbb{Z}_-)\), if there does not hold the following

\[
[G_n][g^j_{n}]^{-1}[g^k_{n}]^{-1} \rightarrow 0
\]

in \(\mathcal{B}(L^2)\) as \(n\) goes to infinity, then by the dislocation property Proposition 1.3 we can assume

\[
[G_n][g^j_{n}]^{-1}[g^k_{n}]^{-1} \rightarrow [g^{j,k}], \quad [g^{j,k}] \in \mathcal{B}(L^2), \quad [g^{j,k}] \neq 0
\]

in \(\mathcal{B}(L^2)\) as \(n\) goes to infinity. In this case we obviously have \([g^{j,k}]^{-1} = [g^{k,j}]\). Hence we can investigate the sequence \([G_n][g^k_{n}]^{-1}[g^j_{n}]^{-1}[G_n]^{-1}\) and turn the case \(j = k + m(m \in \mathbb{Z}_-)\) into the case \(j = k + m(m \in \mathbb{Z}_+)\) which we have already proved. Therefore we complete the proof of the claim.

To totally finish the proof of this Lemma 4.3, our next target is to get the desired conclusion (44) from the estimate (47) by using the localized restriction estimate Lemma 4.1.

Notice that we have the compact set \(K\) and the operators \([g^k_n]\) do not change the support on the Fourier side. It means that when we get the above decomposition with conclusion (47), on the Fourier side, all the processes are taken place on this compact set \(K\). Therefore, we conclude \(\tilde{\phi}^l \in L^\infty(K)\) and further \(p^M_n \in L^\infty(K)\). Since the Fourier support for \(e^M_n\) is not ideal, we first use some scaling skills as follows

\[
\left\| D^{\frac{\alpha-2}{6}} [e^{i\ell|\nabla|}n] e^M_n \right\|_{L^6_{t,x}} = \left\| D^{\frac{\alpha-2}{6}} [e^{i\ell|\nabla|}n] [\sqrt{p_n} e^{i\ell|\nabla|}n] e^M_n (p_n)^{-1} \right\|_{L^6_{t,x}}
\]

\[
\leq \left\| D^{\frac{\alpha-2}{6}} [e^{i\ell|\nabla|}n] (e^{i\ell|\nabla|}n) e^M_n \right\|_{L^6_{t,x}}.
\]

Then we investigate the function

\[
\omega^M_n(x) := e^{i\ell|\nabla|}n p^M_n (x),
\]

with the Fourier support information \(\text{supp}(\omega^M_n) \subset K + (p_n)^{-1} \xi_n\). The Hölder’s inequality and the Bernstein inequality imply that

\[
\left\| D^{\frac{\alpha-2}{6}} [e^{i\ell|\nabla|}n] \omega^M_n \right\|_{L^6_{t,x}} \lesssim K \left\| D^{\frac{\alpha-2}{6}} [e^{i\ell|\nabla|}n] \omega^M_n \right\|_{L^q_{t,x}}^{q/6} \left\| e^{i\ell|\nabla|}n \omega^M_n \right\|_{L^\infty_{t,x}}^{1-q/6}
\]

for \(4 < q < 6\). Meanwhile, Lemma 4.1 gives the following estimate

\[
\left\| D^{\frac{\alpha-2}{6}} [e^{i\ell|\nabla|}n] \omega^M_n \right\|_{L^1_{t,x}} \lesssim 1,
\]

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which is independent of $n$ and $M$. Hence, to get the desired result (44), it suffices to prove

$$\lim_{M \to \infty} \limsup_{n \to \infty} \left\| e^{it|\nabla|^\alpha} \omega_n^M \right\|_{L^\infty_{t,x}} = 0.$$  

Moreover by (47), it suffices to prove the following claim

$$\limsup_{n \to \infty} \left\| e^{it|\nabla|^\alpha} \omega_n^M \right\|_{L^\infty_{t,x}} \lesssim_K \mu(\mathbb{R}^M). \tag{49}$$

Indeed, choose an even function $\mathbb{I}_K \in C_c^\infty(\mathbb{R})$ satisfying $\mathbb{I}_K = 1$ on $K$ and choose $(a_n, b_n)$ such that

$$\left\| e^{it|\nabla|^\alpha} \omega_n^M \right\|_{L^\infty_{t,x}} = \left\| e^{i(a_n|\nabla|^\alpha)\omega_n^M (b_n)} \right\|.

Define

$$\mathbb{I}_{K_n}(x) := \mathbb{I}_K \left( x - (\rho_n)^{-1} \xi_n \right), \quad \Omega_n^M(t, x) := [e^{it|\nabla|^\alpha}] \omega_n^M (x).$$

It follows that

$$\Omega_n^M(t, x) = \mathcal{F}^{-1} e^{-it|\xi|} \mathbb{I}_{K_n}(\xi) \mathcal{F} \omega_n^M(x), \quad \Omega_n^M(a_n, x + b_n) \in \mathcal{H}(\mathbb{P}^M).$$

Then using some basic properties for the spatial Fourier transform $\mathcal{F}$ and $\mathbb{I}_K$, by Hölder’s inequality we can control the $\left\| \Omega_n^M \right\|_{L^\infty_{t,x}}$ term as follows

$$\left\| \Omega_n^M \right\|_{L^\infty_{t,x}} = \left\| \Omega_n^M(a_n, b_n) \right\| = \left\| \mathcal{F}^{-1} \mathbb{I}_{K_n} \mathcal{F} \Omega_n^M \right\| \left( a_n, b_n \right)$$

$$\sim \lim_{n \to \infty} \left\| \mathbb{I}_{K_n} \mathcal{F} \Omega_n^M \right\| \left( a_n, b_n \right)$$

$$\leq \lim_{n \to \infty} \left\| \mathbb{I}_{K_n} \mathcal{F} \Omega_n^M \right\| \left( \mu(\mathbb{P}^M) \right) \lesssim K \mu(\mathbb{P}^M).$$

Therefore we can obtain the desired result (49) and finish the proof. \hfill \Box

**Remark 4.4.** As has been pointed out in [21, Remark 5.3], we can make a reduction in Lemma 4.3 when

$$\lim_{n \to \infty} (\rho_n)^{-1} \xi_n = a, \quad |a| < \infty.$$  

In this case we can assume $\xi_n \equiv 0$ since we can replace $e^{i(x(\rho_n)^{-1} \xi_n) \phi_j}$ by $e^{i(xa) \phi_j}$, put the difference into the error term and then regard $e^{i(xa) \phi_j}$ as the new $\phi'$.  

### 5 Profile decomposition of alpha-Strichartz version

In this section, with the two steps of decomposition Lemma 3.3 and Lemma 4.3 at hand, we are able to show the desired $\alpha$-Strichartz version profile decomposition results Proposition 1.5 and the Strichartz-orthogonality of profiles Proposition 1.7. It should be pointed out that, in the proof of Proposition 1.7, we use the conditional dislocation property Proposition 1.3 once more to coordinate the limit-orthogonal property conclusion (8) in Proposition 1.5.

**Proof of Proposition 1.5.** Using the Lemma 3.3 with $\frac{\delta}{2}$ and then using Lemma 4.3 properly, we can obtain the decomposition

$$u_n(x) = \sum_{\beta=1}^{N} \left( \sum_{j=1}^{M} \left[ g_{\beta}^j \right] \left[ G_{\beta}^j \right]^{-1} \phi_{\beta,j} (x) \right) + e_{n,M_1,\ldots,M_N}(x), \tag{50}$$
where the remainder term is
\[ e_{n,M_1,\ldots,M_N}^N(x) := \sum_{\beta=1}^N e_{n}^{M_\beta} + q_n^N \]
and the operators in (50) are defined by
\[ [G_n^\beta] f(x) := (\rho_n^\beta)^{-\frac{3}{2}} e^{-i\alpha(\rho_n^\beta)^{-1}} \xi_n^\beta f \left( \frac{x}{\rho_n^\beta} \right), \quad [g_n^\beta,j] f(x) := [e^{-i\beta,j} \xi_n^\beta] f(x - y_n^\beta). \]
Here, for each \(1 \leq \beta \leq N\), we choose \(M_\beta\) to guarantee that for all \(M \geq M_\beta\) there holds
\[
\lim_{n \to \infty} \left\| [D_x^{n-\frac{3}{2}}] e_{n}^{M_\beta} \right\|_{L_2} \leq \frac{\delta}{2N}.
\]
This is realizable since we have the vanishing Strichartz norm estimate (44) for the remainder in Lemma 4.3. Therefore, by combining the negligible Strichartz norm estimate (38) for the remainder in Lemma 3.3 with \(\frac{\delta}{2}\), we have the following norm estimate for the remainder term
\[
\lim_{n \to \infty} \left\| [D_x^{n-\frac{3}{2}}] e_{n}^{N,M_1,\ldots,M_N} \right\|_{L_2} \leq \delta.
\] (51)

As for the sequence of operators \([G_n^\gamma][g_n^{\gamma,k}]^{-1}[g_n^{\beta,j}][G_n^\beta]^{-1}\), we will investigate the limit of this sequence in the weak operator topology of \(B(L^2)\) as \(n\) goes to infinity. If \(\beta = \gamma\), then the limit-orthogonal conclusion (42) in Lemma 4.3 implies
\[
[G_n^\gamma][g_n^{\gamma,k}]^{-1}[g_n^{\beta,j}][G_n^\beta]^{-1} \to 0
\] (52)
in \(B(L^2)\) as \(n\) goes to infinity. If \(\beta \neq \gamma\), then the dual approach and Plancherel theorem give
\[
\lim_{n \to \infty} \left( [G_n^\gamma][g_n^{\gamma,k}]^{-1}[g_n^{\beta,j}][G_n^\beta]^{-1} f, g \right)_x = \lim_{n \to \infty} \left( [g_n^{\beta,j}][G_n^\beta]^{-1} f, [g_n^{\gamma,k}][G_n^\gamma]^{-1} g \right)_x \\
\sim \lim_{n \to \infty} \left( \mathcal{F}[g_n^{\beta,j}][G_n^\beta]^{-1} f, \mathcal{F}[g_n^{\gamma,k}][G_n^\gamma]^{-1} g \right)_x,
\] (53)
where \(f\) and \(g\) can be assumed to be Schwartz functions with compact Fourier supports. Note that the operators \([g_n^{\beta,j}]\) and \([g_n^{\gamma,k}]\) do not change the Fourier supports. Hence the conclusion (37) for the frequency and scaling parameters in Lemma 3.3, recalling the Remark 3.4, implies the limit value in (53) is zero and further
\[
[G_n^\gamma][g_n^{\gamma,k}]^{-1}[g_n^{\beta,j}][G_n^\beta]^{-1} \to 0
\] (54)
in \(B(L^2)\) as \(n\) goes to infinity.

For the \(L^2\)-orthogonality, combing the \(L^2\)-almost orthogonal identities (39) and (45), we conclude
\[
\lim_{n \to \infty} \left( \|u_n\|_{L^2}^2 - \sum_{\beta=1}^N \left( \sum_{j=1}^{M_\beta} \|\phi^{\beta,j}\|_{L^2}^2 + \|e^{M_\beta}_n\|_{L^2}^2 \right) - \|q_n^N\|_{L^2}^2 \right) = 0.
\]
Recall that the Fourier supports of \(e_n^N\) and \(e_n^{M_\beta}\) are mutually disjoint which comes from the fact that the operators \([g_n^{\beta,j}]\) do not change the Fourier support and the Remark 3.4. Therefore we conclude
\[
\lim_{n \to \infty} \left( \|u_n\|_{L^2}^2 - \left( \sum_{\beta=1}^N \sum_{j=1}^{M_\beta} \|\phi^{\beta,j}\|_{L^2}^2 \right) - \|e_n^{N,M_1,\ldots,M_N}\|_{L^2}^2 \right) = 0.
\] (55)
Notice that the parameters $N$ and $M_\beta$ depend only on $\delta$. Hence by enumerating the pairs $(\beta, j)$

$$\{(\beta, j) < (\gamma, k)\} := \{\beta + j < \gamma + k \text{ or } \beta + j = \gamma + k \text{ and } \beta < \gamma\},$$

and relabeling the pairs $(\beta, j)$, we can define

$$(h_n^\beta, x_n^\beta, s_n^\beta, t_n^\beta) := \left(1/\beta_n, -y_n^\beta j, \xi_n^\beta, \phi_n^\beta j\right), \quad [T(h_n^\beta, x_n^\beta, s_n^\beta, t_n^\beta)] := [g_n^\beta j][G_n^\beta]^{-1}, \quad \omega_n^J := e_n^{N,M_1,\ldots,M_N}.$$ 

Then, after a classical diagonal process, we obtain the desired decomposition (6) by (50); the limit-orthogonality conclusion (8) comes from the weak operator topology convergence (54) and (52); meanwhile the $L^2$-almost orthogonal identity (9) comes from (55). Therefore it remains for us to prove the Strichartz norm estimate (7) for the remainder term in view of the enumeration (56). Actually this is a standard consequence. By using the Strichartz-orthogonality of profiles Proposition 1.7, a $3\varepsilon$ trick will give this desired result (7). We omit the details here for simplicity and the reader can find similar proofs in [36, p. 107] and [25, p. 371].

In order to investigate the extremal problem for the $\alpha$-Strichartz estimates and complete the proof for Proposition 1.5, we need to show the Strichartz-orthogonality of profiles. Indeed, the limit-orthogonality property (8) for $[T_n^\beta]$ in Proposition 1.5 and the conditional dislocation property Proposition 1.3 imply this desired conclusion.

**Proof of Proposition 1.7.** Without loss of generality, we focus on the Schwartz functions $\phi^j$ and $\phi^k$ whose Fourier supports are compact. Based on the conclusion (37) for the frequency and scaling parameters in Lemma 3.3, we first deal with the case

$$\lim_{n \to \infty} \frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} = \infty.$$ 

Direct computation gives the following

$$\Phi_n^{j,k}(t, x) := [D_{\alpha-s}]^2[e^{i|x|\alpha}] [T_n^j]\phi^j(x) \cdot [D_{\alpha-s}]^2[e^{i|x|\alpha}] [T_n^k]\phi^k(x)$$

$$= (h_n^{\beta j})^{-\frac{1}{2}} \frac{\alpha-s}{\alpha} e^{-ix\xi_n^j \xi_n^k} \int_{\mathbb{R}} |\xi| \frac{\alpha-s}{\alpha} e^{-i\xi^x \xi_n^j \xi_n^k} |e^{i(\xi^t \xi_n^j \xi_n^k \phi^j)(\xi)}| d\xi$$

$$\times (h_n^{\beta k})^{-\frac{1}{2}} \frac{\alpha-s}{\alpha} e^{-ix\xi_n^j \xi_n^k} \int_{\mathbb{R}} |\xi| \frac{\alpha-s}{\alpha} e^{-i\xi^x \xi_n^j \xi_n^k} |e^{i(\xi^t \xi_n^j \xi_n^k \phi^k)(\xi)}| d\xi$$

$$= (h_n^{\beta j})^{-\frac{1}{2}} \frac{\alpha-s}{\alpha} \Phi_n^j \frac{t-t_n^j}{(h_n^{\alpha j})^\alpha} + \frac{x-x_n^j}{h_n^{\alpha j}} \cdot (h_n^{\beta k})^{-\frac{1}{2}} \frac{\alpha-s}{\alpha} \Phi_n^k \frac{t-t_n^k}{(h_n^{\alpha k})^\alpha} + \frac{x-x_n^k}{h_n^{\alpha k}}$$

By the $\alpha$-Strichartz estimate (1) we have

$$\lim_{R \to \infty} \int_{|s|+|y|>R} |\Phi_n^j(s, y)|^6 d\gamma ds = 0.$$ 

Therefore setting

$$B_n^j(R) := \left\{(t, x) : \frac{t-t_n^j}{(h_n^{\alpha j})^\alpha} + \frac{x-x_n^j}{h_n^{\alpha j}} \leq R \right\},$$

Hölder’s inequality gives

$$\int_{\mathbb{R} \setminus B_n^j(R)} |\Phi_n^{j,k}|^2 dx dt \leq \left( \int_{\{|s|+|t|>R\}} |\Phi_n^j|^6 dx dt \right)^{\frac{1}{6}} \left( \int_{\mathbb{R}} |\Phi_n^k|^6 dx dt \right)^{\frac{1}{6}}.$$
and analogously for $\mathbb{R} \setminus B_n^k(R)$. Thus we are reduced to proving

$$\lim_{n \to \infty} \left( h_n^j h_n^k \right)^{-\frac{1}{2}} \left| B_n^j(R) \cap B_n^k(R) \right| = 0$$

(58)
due to the fact that $\Phi_n^j$ and $\Phi_n^k$ are $L^\infty_{t,x}$ functions. By the observation

$$\left| B_n^j(R) \cap B_n^k(R) \right| \leq C_R \min \left\{ (h_n^j)^{1+\alpha}, (h_n^k)^{1+\alpha} \right\},$$

the desired estimate (58) follows immediately since $h_n^j / h_n^k$ goes to either zero or infinity. Hence we can assume $h_n^j \sim h_n^k$ from now on. Then we turn to investigate the case

$$\lim_{n \to \infty} \left( h_n^j + h_n^k \right) \left| \xi_n^j - \xi_n^k \right| = \infty.$$ 

By symmetry we may assume $\lim_{n \to \infty} h_n^j \left| \xi_n^j - \xi_n^k \right| = \infty$, thus from the expression (57) we conclude

$$\Phi_n^{j,k}(t, x) = \left( h_n^j \right)^{-\frac{\alpha+1}{2}} e^{-i x \cdot \xi_n^j} \frac{1}{2\pi} \int_{\mathbb{R}} \left| \xi \right|^{\alpha-2} e^{i \frac{(h_n^j)^2 \xi_n^j \cdot \xi}{h_n^j}} \left| \xi \right|^\alpha e^{i \frac{(h_n^j)^2 \xi_n^j - \xi_n^k}{h_n^j}} \left[ e^{i \left( h_n^j \xi_n^j \phi^j \right)^\wedge}(\xi) \right] d\xi$$

$$\times \left( h_n^k \right)^{-\frac{\alpha+1}{2}} e^{-i x \cdot \xi_n^k} \frac{1}{2\pi} \int_{\mathbb{R}} \left| \xi \right|^{\alpha-2} e^{i \frac{(h_n^k)^2 \xi_n^k \cdot \xi}{h_n^k}} \left| \xi \right|^\alpha e^{i \frac{(h_n^k)^2 \xi_n^k - \xi_n^j}{h_n^k}} \left[ e^{i \left( h_n^k \xi_n^k \phi^k \right)^\wedge}(\xi) \right] d\xi$$

$$= \left( h_n^j \right)^{-\frac{\alpha+1}{2}} \tilde{\Phi}_n^j \left( t - \frac{t_j}{(h_n^j)^\alpha}, x - \frac{h_n^j \xi_n^j}{h_n^j} \right) \tilde{\Phi}_n^k \left( \frac{t - t_k}{(h_n^k)^\alpha}, x - \frac{h_n^k \xi_n^k}{h_n^k} \right).$$

(59)

Based on the expression (59), the assumption $h_n^j \sim h_n^k$ gives

$$\| \Phi_n^{j,k} \|_{L^1_{t,x}} \sim \left\| \tilde{\Phi}_n^j \left( t - \frac{t_j}{(h_n^j)^\alpha}, x - \frac{h_n^j \xi_n^j}{h_n^j} \right) \tilde{\Phi}_n^k \left( \frac{t - t_k}{(h_n^k)^\alpha}, x - \frac{h_n^k \xi_n^k}{h_n^k} \right) \right\|_{L^1_{t,x}}.$$ 

Similarly we still have the following estimate

$$\lim_{R \to \infty} \int_{\|s\| + \|y\| > R} |\tilde{\Phi}_n^j(s, y)|^6 dy ds = 0.$$ 

By imitating the argument above, we can get the desired result (12) too. Hence we turn to the case

$$\lim_{n \to \infty} \left( h_n^j + h_n^k \right) \left| \xi_n^j - \xi_n^k \right| = a, \quad a < \infty.$$ 

Recall the construction of the linear profile decomposition and the label in (56). Therefore, due to the conclusion (37) in Lemma 3.3, it remains for us to deal with the case $\beta = \gamma$ in view of the label (56). Consequently, we can assume $(h_n^j, \xi_n^j) \equiv (h_n^k, \xi_n^k) \equiv (h_n, \xi_n)$ from now on.

Since the case $\xi_n \equiv 0$ is much easier, recalling the Remark 4.4, we may further assume

$$\lim_{n \to \infty} h_n^j \xi_n^j = \infty$$

(60)

without loss of generality. By changing the variables in $\| \Phi_n^{j,k} \|_{L^1_{t,x}}$, we turn to investigate

$$\tilde{\Phi}_n^{j,k}(t, x) := \int_{\mathbb{R}} \left| \xi + h_n \xi_n \right|^{-\frac{\alpha+2}{2}} e^{i \left( \frac{h_n}{(h_n^j)^\alpha} \xi + h_n \xi_n \phi^j \right)} \left[ e^{i \left( h_n \xi_n \phi^j \right)^\wedge}(\xi) \right] d\xi$$

$$\times \int_{\mathbb{R}} \left| \xi + h_n \xi_n \right|^{-\frac{\alpha+2}{2}} e^{i x \xi \cdot \xi + h_n \xi_n \phi^k} \left[ e^{i \left( h_n \xi_n \phi^k \right)^\wedge}(\xi) \right] d\xi.$$ 

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and the conditional dislocation property of van der Corput Lemma, we always have the following estimates

\[ \int_{\mathbb{R}} e^{i(\frac{x_k - x_{-1}}{h_n} - \frac{t_n}{h_n})}\zeta \Psi_{n,k,t}(\xi) |\xi + h_n\xi_n|^{\frac{\alpha - 2}{3}} \phi^k(\xi) d\xi \cdot \int_{\mathbb{R}} e^{i(t - \frac{t_n}{h_n})\zeta \Psi_{n,t}(\xi) |\xi + h_n\xi_n|^{\frac{\alpha - 2}{3}} \phi^l(\xi) d\xi =: A_n^{j,k}(t, x) \cdot B_n^l(t, x). \]

To get the desired conclusion (12), it suffices to show that the following estimate

\[ \lim_{n \to \infty} \left\| A_n^{j,k}(t, x) B_n^l(t, x) \right\|_{L^3_{t,x}} = 0 \quad (61) \]

holds for all \( j \neq k \). Just as what we have done in Proposition 1.3, based on the assumption (60), we can rewrite

\[ \Psi_{j,k,l}(\xi) = \sum_{m=1}^{\infty} a_{n,m,j,k,l}(\xi)^m, \quad \Psi_{n,t}(\xi) = \sum_{m=1}^{\infty} a_{n,m,t}(\xi)^m. \]

Note that the differences of the coefficients

\[ b_{n,m,j,k} := a_{n,m,j,k,t} - a_{n,m,t} = \left( \frac{\alpha}{m} \right) \frac{t_n^j - t_n^k}{(h_n)^{\alpha}} |h_n\xi_n|^m - m \]

are independent of \( t \) since the difference of the functions

\[ \Psi_{j,k,l}(\xi) - \Psi_{n,t}(\xi) = \frac{t_n^j - t_n^k}{(h_n)^{\alpha}} |\xi + h_n\xi_n|^\alpha \]

is independent of \( t \). Meanwhile the assumption (60) implies \( b_{n,m+1,j,k} \ll b_{n,m,j,k} \) for \( n \) large enough. Hence we have, after passing to a subsequence, the following condition

\[ \lim_{n \to \infty} \left( \left| \tilde{b}_{n,k}^j \right| + \left| \tilde{b}_{n,j}^k \right| \right) = \infty, \quad \tilde{b}_{n,j}^k := \frac{x_n^j - x_n^k}{h_n}, \quad \tilde{b}_{n,j}^k := \frac{x_n^k - x_n^j}{h_n} \]

(62) due to the limit-orthogonality property (8) in Proposition 1.5 and the conditional dislocation property Proposition 1.3. Again, the method of stationary phase will provide the decay estimates of \( A_n^{j,k}(t, x) \) and \( B_n^l(t, x) \). Combining the trivial size estimates and the oscillation estimates deduced by the classical van der Corput Lemma, we always have the following estimates

\[ |A_n^{j,k}(t, x)| \lesssim_{\phi^k} \min \left\{ |h_n\xi_n|^{\frac{\alpha - 2}{3}}, t - \frac{t_n^k - t_n^j}{(h_n)^{\alpha}} \right\} \]

(63)

and

\[ |B_n^l(t, x)| \lesssim_{\phi^l} \min \left\{ |h_n\xi_n|^{\frac{\alpha - 2}{3}}, |t||h_n\xi_n|^{\frac{\alpha - 2}{3}} \right\}. \]

(64)

To get the non-stationary bounds, we decompose the spatial space into

\[
A_t := \left\{ x : \left| x + \frac{x_n^j - x_n^k}{h_n} - a_{n,1,j,k,t} \right| \lesssim_{\phi^k} \left| t - \frac{t_n^k - t_n^j}{(h_n)^{\alpha}} \right| |h_n\xi_n|^{\alpha - 2} \right\}, \\
B_t := \left\{ x : \left| x - a_{n,1,j}^l \right| \lesssim_{\phi^l} |t||h_n\xi_n|^{\alpha - 2} \right\}, \\
C_t := \mathbb{R} \setminus (A_t \cup B_t),
\]

where the implicit constants in the definitions of \( A_t \) and \( B_t \) may depend on the Fourier supports of \( \phi^k \) and \( \phi^l \). Note that

\[
A_t = \left\{ \left| x - a_{n,1,j}^l - \tilde{b}_{n,j}^k \right| \lesssim_{\phi^k} \left| t - \frac{t_n^k - t_n^j}{(h_n)^{\alpha}} \right| |h_n\xi_n|^{\alpha - 2} \right\}
\]

(22)
by the definition of $\overline{b}^{1,j,k}_n$. On the other hand if $x \in C_t$, we always have

$$|h_n \xi_n|^{-2} \left| t - \frac{t^n_j - t^n_k}{(h_n)^a} \right| \left| x + \frac{x^n_j - x^n_k}{h_n} - a^n_{1,j,k,t} \right| \lesssim \phi_k 1, \quad \left| \frac{|t|}{h_n \xi_n} |n^{-2} | x - a^n_{1,t} \right| \lesssim \phi_j 1.$$

Hence on $\mathbb{R} \times C_t$, we can use the classical van der Corput Lemma to obtain the non-stationary bounds

$$|A^{1,j}_n(t, x)| \lesssim \phi_k \left| h_n \xi_n \right|^{a-2} \left| x - a^n_{1,j,k} \right|, \quad |B^{1}_n(t, x)| \lesssim \phi_j \left| h_n \xi_n \right|^{a-2} \left| x - a^n_{1,t} \right| \quad (65)$$

Combining the estimates (63), (64) and (65), together with the condition (62), we can get the desired result (61) by following an analogous argument in [21, Lemma 6.1, Case 2].

The details for the remaining proof are very long but essentially the same as [21, Lemma 6.1, Case 2]. For the convenience of the reader and avoiding too much redundancy, we provide part of the details and the rest of the proof will be sketchy. Split the time space into $\mathbb{R} = \tau_0^- \cup \tau_0^0 \cup \tau_0 \cup \tau_0^+ \cup \tau_0^+$ where

$$\tau_0 := \left[ -|h_n \xi_n|^{2-\alpha}, |h_n \xi_n|^{2-\alpha} \right], \quad \tau_n := \left[ \frac{t^n_k - t^n_j}{(h_n)^a} - |h_n \xi_n|^{2-\alpha}, \frac{t^n_k - t^n_j}{(h_n)^a} + |h_n \xi_n|^{2-\alpha} \right],$$

and

$$\tau_0^- := (-\infty, -|h_n \xi_n|^{2-\alpha}], \quad \tau_n := \left[ |h_n \xi_n|^{2-\alpha}, \frac{t^n_k - t^n_j}{(h_n)^a} + |h_n \xi_n|^{2-\alpha} \right], \quad \tau_n^+ := \left[ \frac{t^n_k - t^n_j}{(h_n)^a} + |h_n \xi_n|^{2-\alpha}, \infty \right].$$

For simplicity we use the notation $I(\tau_0, A_t)$ to denote the integral of $|A^{1,j}_n B_n^1|^3$ on the domain $\tau_0 \times A_t$. In other words, we define

$$I(\tau_0, A_t) := \int_{t \in \tau_0} \int_{x \in A_t} |A^{1,j}_n B_n^1|^3 \, dx \, dt.$$  

Similarly for the notations $I(\tau_n, B_t)$, $I(\tau_n^+, C_t)$ and so on. Taking (62) into consideration, the rest of this proof is divided into two parts.

**Case A:** If there holds

$$\lim_{n \to \infty} |b^{2,j,k}_n| = \lim_{n \to \infty} \left| \frac{t^n_k - t^n_j}{(h_n)^a} (h_n \xi_n)^{a-2} \right| = \infty, \quad (66)$$

which means that $\left| \frac{t^n_j - t^n_k}{(h_n)^a} \right| \gg |h_n \xi_n|^{-a+2}$ for $n$ large enough. Then the desired result comes from a similar process in the proof of [21, Lemma 6.1, Case 2a1]. By taking a subsequence, together with the symmetry of positive and negative cases, we may assume that $\left| \frac{t^n_j - t^n_k}{(h_n)^a} \right| > 0$ without loss of generality. From the estimates (63) and (64), it is not hard to show that

$$I(\tau_0, B_t) \lesssim \left| \frac{t^n_k - t^n_j}{(h_n)^a} (h_n \xi_n)^{a-2} \right|^{-\frac{1}{2}}, \quad I(\tau_n, B_t) \lesssim \left| \frac{t^n_k - t^n_j}{(h_n)^a} (h_n \xi_n)^{a-2} \right|^{-\frac{1}{2}}.$$  

And meanwhile some computation gives

$$I(\tau_0^-, B_t) \lesssim \left| \frac{t^n_k - t^n_j}{(h_n)^a} (h_n \xi_n)^{a-2} \right|^{-\frac{1}{2}}, \quad I(\tau_n^+, B_t) \lesssim \left| \frac{t^n_k - t^n_j}{(h_n)^a} (h_n \xi_n)^{a-2} \right|^{-\frac{1}{2}}.$$
For the term $I(\tilde{\tau}_n, B_t)$, recall the indefinite integral

$$\int t^{-1/2}(a - t)^{-3/2}dt = \frac{2\sqrt{t}}{a\sqrt{a - t}} + C.$$ 

Therefore by the condition $\frac{|t^j - t^k|}{(n h_{\alpha})^\alpha} \gg |h_{n\xi_n}|^{-(\alpha - 2)}$ we can obtain

$$I(\tilde{\tau}_n, B_t) \lesssim |h_{n\xi_n}|^{-(\alpha - 2)} \frac{2\sqrt{(t^k_n - t^j_n)/(h_{\alpha}) - (h_{n\xi_n})^{-\alpha - 2}}}{(t^j_n - t^k_n)/(h_{\alpha})^\alpha \sqrt{(h_{n\xi_n})^{-(\alpha - 2)}}} \lesssim \left| \frac{t^j_n - t^k_n}{(h_{\alpha})^\alpha (n h_{\alpha}^{\alpha - 2})} \right|^{-\frac{1}{2}}.$$ 

Hence we conclude that $I(\mathbb{R}, B_t) \to 0$ as $n \to \infty$ due to the condition (66). Analogous arguments give the estimate $I(\mathbb{R}, A_t) \to 0$ as $n \to \infty$. Finally, by combining the non-stationary bounds (65) and further splitting $\tilde{\tau}_n$ into

$$\tilde{\tau}_n := \left| n_{\alpha} \right|^{2\alpha - \frac{t^j_n - t^k_n}{2(h_{\alpha})^\alpha}} \left[ t^j_n - t^k_n \right] \cup \left[ \frac{t^j_n - t^k_n}{2(h_{\alpha})^\alpha}, \frac{t^j_n - t^k_n}{(h_{\alpha})^\alpha} - |h_{n\xi_n}|^{2\alpha - \frac{t^j_n - t^k_n}{2}} \right],$$

we can obtain the estimate $I(\mathbb{R}, C_t) \to 0$ as $n \to \infty$ and finish the proof of this case.

**Case B:** If $|b_{n,j,k}^1| \leq C_0$ for some fixed $C_0 > 0$ and

$$\lim_{n \to \infty} \left| \tilde{\tau}_n^{1,j,k} \right| = \lim_{n \to \infty} \left| \frac{x^j_n - x^k_n - \alpha(t^j_n - t^k_n)(\xi_n)^{\alpha - 1}}{h_{n\xi_n}} \right| = \infty.$$

Analogously the desired result comes from a similar process in the proof of [21, Lemma 6.1, **Case 2aII**]. We may assume $b_{n,j,k}^1 > 0$ at first. In this case, the corresponding decomposition for the time space is $\mathbb{R} := \tilde{\tau}_K^+ \cup \tilde{\tau}_K^-$ for a large constant $K \gg C_0$, where

$$\tilde{\tau}_K^+ := \{ t : \left| (h_{n\xi_n})^{-\alpha - 2} t \right| \geq K \}.$$ 

Note that on $\tilde{\tau}_K^+$ it holds $|t^j_n - t^k_n|/(h_{\alpha})^\alpha \ll |t|$ and on $\tilde{\tau}_K^- \times B_t$ it holds

$$|x - \alpha(h_{n\xi_n})^{\alpha - 1}| \ll \left| \tilde{b}_{n,j,k}^1 \right|$$

for $n$ large enough. These two facts together with the stationary bounds (63) and (64) imply

$$I(\tilde{\tau}_K^+, B_t) \lesssim K^{-1}, \quad I(\tilde{\tau}_K^-, B_t) \lesssim K^{1/2} \left| \tilde{b}_{n,j,k}^1 \right|^{-3}.$$ 

The first estimate for $I(\tilde{\tau}_K^+, B_t)$ is uniform in all large $n$ and is going to zero as $K$ goes to infinity; while the second estimate for $I(\tilde{\tau}_K^-, B_t)$ is going to zero as $n$ goes to infinity. Thereby, after a similar argument for $I(\mathbb{R}, A_t)$, we can obtain the following two estimates

$$\lim_{n \to \infty} I(\mathbb{R}, A_t) = 0, \quad \lim_{n \to \infty} I(\mathbb{R}, B_t) = 0.$$ 

For the term $I(\mathbb{R}, C_t)$, we should use the non-stationary bounds (65) too. The result $I(\tilde{\tau}_K^+, C_t) \lesssim K^{-1}$ is not hard to obtain. Hence it remains for us to estimate $I(\tilde{\tau}_K^-, C_t)$. Just as what we have done in **Case A**, for $t \in \tilde{\tau}_K^-$ if we split $C_t$ further into $C_t = B_t^- \cup B_t^+ \cup A_t^- \cup A_t^+$ for $n$ large enough where

$$B_t^- := (-\infty, a_{n,t}^{-1} - |t| |h_{n\xi_n}|^{\alpha - 2}], \quad B_t^+ := \left[ a_{n,t}^{1} + |t| |h_{n\xi_n}|^{\alpha - 2}, a_{n,t}^{1} + \frac{\tilde{b}_{n,j,k}}{2} \right],$$
and

\[ A^+_t := \left[ a_{nt}^{1,t} + \hat{a}_{nt}^{1,j,k} + \left| t - \frac{t_n^k - t_n^j}{(h_n)^\alpha} \right| |h_n|^{-\alpha/2} \right], \]

with

\[ A^-_t := \left[ a_{nt}^{1,t} + \hat{a}_{nt}^{1,j,k} + \left| t - \frac{t_n^k - t_n^j}{(h_n)^\alpha} \right| |h_n|^{-\alpha/2}, +\infty \right), \]

then we can get the estimate \( I(\mathbb{R}, C_t) \to 0 \) as \( n \to \infty \) by estimating the integral piece by piece. Therefore, we conclude the desired result (61) and finish the proof.

\( \square \)

6 Extremals for symmetric alpha-Strichartz estimate

With the linear profile decomposition Proposition 1.5, the Strichartz-orthogonality of profiles Proposition 1.7 and the following asymptotic Schrödinger behavior Lemma 6.1 in place, we are ready to give the proof of the desired extremal result Theorem A for symmetric \( \alpha \)-Strichartz estimate. This arguments can be directly used in asymmetric cases, which will be shown in Section 7 later.

**Lemma 6.1** (Asymptotic Schrödinger behavior). If \( \|\phi\|_{L^2_t(\mathbb{R})} = 1 \) and \( \lim_{n \to \infty} |\xi_n| = \infty \), then we have

\[
\lim_{n \to \infty} \left\| D_{x_0}^{\alpha/2} \right[ e^{it|\nabla|^\alpha} e^{i(\xi_n \cdot x)} \phi \left|_{L^2_{t,x}(\mathbb{R}^2)} \right\| = \left( \frac{\alpha^2 - \alpha}{2} \right)^{-1/2} \|e^{-it\Delta} \phi\|_{L^2_{t,x}(\mathbb{R}^2)}. \tag{67} \]

**Proof of Lemma 6.1.** This is a standard consequence by changing of variables and the dominated convergence theorem. While the dominating function, which can be chosen as

\[
F(t, x) := \left\{ \begin{array}{ll}
C_\phi[(1 + |t|)(1 + |x|)]^{-1/2}, & |x| \lesssim \phi |t|; \\
C_\phi[(1 + |t|)(1 + |x|)]^{-1/2}, & |x| \gtrsim \phi |t|,
\end{array} \right.
\]

also comes from the classical van der Corput Lemma. The readers can see [21, Proposition 7.1] or [36, Remark 1.7] for more details about the very similar arguments in other contexts. This idea can also be seen in some earlier papers \([10] \) and \([41]\). Note that the assumption \( \lim_{n \to \infty} |\xi_n| = \infty \) can guarantee the uniform convergency when we use a series which is similar to (29). We omit the proof here.

**Remark 6.2.** Based on the existence of extremals for \( M_2 \), Lemma 6.1 implies

\[
M_\alpha \geq \left( \frac{\alpha^2 - \alpha}{2} \right)^{-1/2} M_2.
\]

Meanwhile we pointed out that the non-precompactness, in view of Theorem A, is different from the non-existence of extremals. Indeed when \( \alpha = 2 \), Foschi [13] and Hundertmark-Zharntisky [20] independently show that the sharp constant \( M_2 = 12^{-12} \) and the only extremals are Gaussians up to symmetries. Based on this result, Lemma 6.1 implies

\[
M_\alpha \geq [\sqrt{3} \alpha (\alpha - 1)]^{-1/2}.
\]

**Proof of Theorem A.** Let \( \{u_n\}_{n \geq 1} \) be an extremal sequence for \( M_\alpha \). Then, up to subsequences, by the profile decomposition Proposition 1.5 we can decompose \( u_n \) into linear profiles as

\[
u_n = \sum_{j=1}^J T_{n,j}^{\nu} \phi^j + \omega^j_n.
\]
Due to the vanishing Strichartz norm estimate (7) for the remainder term in Proposition 1.5, we obtain that for arbitrary $\epsilon > 0$, there exists $N_\epsilon$ such that for all $N \geq N_\epsilon$ and all $n \geq N_\epsilon$,

$$M_\alpha - \epsilon \leq \left\| \frac{D^{\alpha - 2}}{n} \sum_{j=1}^{N} |e^{it\xi_n^\alpha} [T_n^j] \phi_j | \right\|_{L_t^6 L_x^\infty}.$$ 

Hence the Strichartz-orthogonality of profiles Proposition 1.7 gives the following

$$M_\alpha^6 - C_\alpha \epsilon \leq \sum_{j=1}^{N} \left\| [D^{\alpha - 2}] |e^{it\xi_n^\alpha} [T_n^j] \phi_j | \right\|_{L_t^6 L_x^\infty}^6.$$

Take $j_0$ by

$$\max \left\{ \left\| [D^{\alpha - 2}] |e^{it\xi_n^\alpha} [T_n^j] \phi_j | \right\|_{L_t^6 L_x^\infty}^6 : 1 \leq j \leq N_0 \right\}.$$

The $\alpha$-Strichartz estimate (1) implies

$$M_\alpha^6 - C_\alpha \epsilon \leq \sum_{j=1}^{N} \left\| [D^{\alpha - 2}] |e^{it\xi_n^\alpha} [T_n^j] \phi_j | \right\|_{L_t^6 L_x^\infty}^6 \leq M_\alpha^6 \sum_{j=1}^{N} \left\| \phi_j^j \|_{L_t^6 L_x^\infty}^2 \leq M_\alpha^6 \left( \sum_{j=1}^{N} \left\| \phi_j^j \right\|_{L_t^6 L_x^\infty}^2 \right)^3 \leq M_\alpha^6,$$

(68)

where the last inequality comes from the fact that the $L^2$-orthogonal identity (9) which leads to

$$\sum_{j=1}^{\infty} \left\| \phi_j^j \right\|_{L_t^6 L_x^\infty}^2 \leq \lim_{n \to \infty} \left\| u_n \right\|_{L_t^6 L_x^\infty}^2 = 1.$$ 

(69)

This fact also deduces $\lim_{j \to \infty} \left\| \phi_j^j \right\|_{L_t^6 L_x^\infty} = 0$. Consequently, we can choose $j_0$ independent of $\epsilon$. Hence we may let $\epsilon \to 0$ in (68) to obtain

$$\left\| \phi_j^{j_0} \right\|_{L_t^6 L_x^\infty} = 1,$$

which means $\phi_j = 0$ for all $j \neq j_0$ due to the inequality (69). Therefore by the linear profile decomposition we conclude

$$\lim_{n \to \infty} \left\| [D^{\alpha - 2}] [e^{it\xi_n^\alpha}] [T_n^{j_0}] \phi_j^{j_0} \right\|_{L_t^6 L_x^\infty} = \lim_{n \to \infty} \left\| [D^{\alpha - 2}] [e^{it\xi_n^\alpha}] [e^{i(t\xi_n^\alpha) \xi_n^{j_0}}] \phi_j^{j_0} \right\|_{L_t^6 L_x^\infty} = M_\alpha.$$

This suggests that $e^{ixh_n^{j_0} \xi_n^{j_0}} \phi_j^{j_0}(x) \in L_t^6 L_x^\infty$ is an extremal sequence where either $\lim_{n \to \infty} |h_n^{j_0} \xi_n^{j_0}| \to \infty$ or $h_n^{j_0} \xi_n^{j_0} \equiv 0$. For the case $h_n^{j_0} \xi_n^{j_0} \equiv 0$, we get the desired extremal function $\phi_j^{j_0}$. However for the case $|h_n^{j_0} \xi_n^{j_0}| \to \infty$, we should do more investigation as follows. Indeed as we will see, Lemma 6.1 gives the desired conclusion and finishes the proof.

If we have the strict inequality (3), then the case $|h_n^{j_0} \xi_n^{j_0}| \to \infty$ is ruled out by Lemma 6.1 and hence all the extremal sequences for $M_\alpha$ are precompact up to symmetries. On the other hand if

$$M_\alpha = [\sqrt{3\alpha} (\alpha - 1)]^{-\frac{1}{4}},$$

then, after normalizing, $\tilde{u}_n(x) := \sqrt{n} e^{in^2 \xi_n} e^{-|n(x-x_0)|^2}$ will give an extremal sequence that is not precompact up to symmetries, since $\tilde{u}_n$ goes to zero up to symmetries in the weak topology of $L^2(\mathbb{R})$. Meanwhile, it is easy to check that $(\tilde{u}_n)$ concentrates at $x_0$. \qed
7 Extremals for asymmetric alpha-Strichartz estimate

As what we have stated before, our method can produce some extremal results for the asymmetric $\alpha$-Strichartz estimates (2) as well. This mainly because that the profile decomposition Proposition 1.5 is simultaneously equipped with the Strichartz-orthogonality Proposition 1.7 for all the profiles. By imitating the arguments in the proof of Theorem A, it not hard to see that the desired asymmetric $\alpha$-Strichartz result Theorem 1.1 is a standard consequence of the following two lemmas which are generalizations of the estimates (13) and (67) respectively.

Lemma 7.1. Let $(q, r)$ be non-endpoint pairs and $\tilde{N} \geq 1$. For the profiles in Proposition 1.5, if $q \geq r$ then

$$\lim_{n\to\infty} \left| \tilde{N} \sum_{j=1}^{\tilde{N}} \left[ D^{-\frac{2}{q}} \left| e^{it|\nabla|^{\alpha}} \right| T_j \phi \right] \right|^r \lesssim \left| \sum_{j=1}^{\tilde{N}} \lim_{n\to\infty} \left[ D^{-\frac{2}{q}} \left| e^{it|\nabla|^{\alpha}} \right| T_j \phi \right] \right|^r ;$$

while if $q \leq r$ then

$$\lim_{n\to\infty} \left| \tilde{N} \sum_{j=1}^{\tilde{N}} \left[ D^{-\frac{2}{q}} \left| e^{it|\nabla|^{\alpha}} \right| T_j \phi \right] \right|^q \lesssim \left| \sum_{j=1}^{\tilde{N}} \lim_{n\to\infty} \left[ D^{-\frac{2}{q}} \left| e^{it|\nabla|^{\alpha}} \right| T_j \phi \right] \right|^q .$$

Lemma 7.2. If $\| \phi \|_{L^q_x(\mathbb{R})} = 1$ and $\lim_{n\to\infty} |\xi_n| = \infty$, then we have

$$\lim_{n\to\infty} \left| \left[ D^{-\frac{2}{q}} \left| e^{it|\nabla|^{\alpha}} \right| e^{i\xi_n \phi} \right] \right|_{L^q_t L^r_x} = \left( \frac{\alpha^2 - \alpha}{2} \right)^{-\frac{q}{4}} \tilde{M}_{2,q,r} \| e^{-it\Delta} \phi \|_{L^q_t L^r_x} .$$

Remark 7.3. As in Remark 6.2 the non-precompactness up to symmetries, equivalently the equality

$$\tilde{M}_{n,q,r} = \left( \frac{\alpha^2 - \alpha}{2} \right)^{-\frac{q}{4}} \tilde{M}_{2,q,r} ,$$

does not mean the non-existence of extremals. For the sharp constant $\tilde{M}_{2,q,r}$ with asymmetric pairs $(q, r)$, recalling that the existence of extremals have been proved in [35], the known results are pretty few even if there are many excellent works such as [3, 9, 18]. Indeed, up to now, we are only aware of the case $\tilde{M}_{2,8,4} = 2^{-1/4}$ as shown in the aforementioned three papers. As far as we know, there is no higher dimensional result for the asymmetric sharp constant $\tilde{M}_{2,q,r}$.

We are not planing to show all the detailed proofs of these two lemmas since the arguments are standard. Instead, there will be some useful references for the readers who are interested in the further details. For the Lemma 7.1, indeed it is a corollary of the Strichartz-orthogonality estimate (12). This fact may be not as obvious as getting the estimate (13) from (12), since $q \neq r$ and they may be not natural numbers at all. However this difficult has been overcome by using the interpolation arguments and some floor function techniques, see [35, Lemma 1.6] for more details. Also notice that the conclusions in Lemma 7.1 are inequalities instead of equalities compared with the estimate (13).

While Lemma 7.2 may seem easy to accept since it is obvious an asymmetric generalization of Lemma 6.1. Indeed here we only need to give the $L^q_t L^r_x$-dominating function

$$F(t, x) := \begin{cases} C_\phi (1 + |t|)^{\frac{2}{q}} (1 + |x|)^{-\frac{2\alpha}{q}}, & |x| \lesssim_\phi |t| ; \\ C_\phi (1 + |t|)^{-\frac{3}{q}} (1 + |x|)^{-\frac{2\alpha}{q}}, & |x| \gtrsim_\phi |t| . \end{cases}$$

As what we have shown in the proof of Lemma 6.1, the detailed arguments can be founded in [21, Proposition 7.1] and [36, Remark 1.7], or some earlier papers such as [10] and [41].
8 Extremals for non-endpoint alpha-Strichartz estimate

In this section, we provide the proof of Theorem 1.8 by following the arguments in [19]. It is obvious that we only need to prove the following profile decomposition of non-endpoint $\alpha$-Strichartz version Proposition 8.1, which indeed is a standard consequence of the aforementioned linear profile decomposition results Proposition 1.5 and Proposition 1.7.

**Proposition 8.1.** Let $(u_n)$ be a bounded sequence in $L^2(\mathbb{R})$. Then, up to subsequences, there exist a sequence of operators $[T_n^J]$ defined by

$$[T_n^J]\phi(x) := [e^{-it_n^J|\nabla|^\alpha}] \left( (h_n^J)^{-\frac{\alpha}{2}} \phi \left( \frac{x - x_n^J}{h_n^J} \right) \right)$$

with $(h_n^J, x_n^J, t_n^J) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and a sequence of functions $\phi^J \in L^2(\mathbb{R})$ such that for every $J \geq 1$, we have the profile decomposition

$$u_n = \sum_{j=1}^{J} [T_n^J]\phi^J + \omega_n^J,$$

where the decomposition possesses the following properties: firstly the remainder term $\omega_n^J$ has vanishing Strichartz norm

$$\lim_{J \to \infty} \limsup_{n \to \infty} \left\| [e^{it|\nabla|^\alpha}]\omega_n^J \right\|_{L^{2a+2}(\mathbb{R}^2)} = 0; \quad (70)$$

secondly the sequence of operators $[T_n^J]$ satisfies that if $j \neq k$, there holds the limit-orthogonality property

$$[T_n^J]^* [T_n^k] \to 0 \quad (71)$$

as $n$ goes to infinity in the weak operator topology of $\mathcal{B}(L^2)$; for each $J \geq 1$, we have

$$\lim_{n \to \infty} \left[ \|u_n\|_{L^2(\mathbb{R})}^2 - \left( \sum_{j=1}^{J} \|\phi^J\|_{L^2(\mathbb{R})}^2 \right) - \|\omega_n^J\|_{L^2(\mathbb{R})}^2 \right] = 0; \quad (72)$$

moreover for every $j \neq k$ there holds the Strichartz-orthogonality of profiles

$$\lim_{n \to \infty} \left\| [e^{it|\nabla|^\alpha}] [T_n^J]\phi^J \cdot [e^{it|\nabla|^\alpha}] [T_n^k]\phi^k \right\|_{L^{2a+2}_{t,x}(\mathbb{R}^2)} = 0. \quad (73)$$

**Remark 8.2.** In this case, without the frequency parameters $\xi_n$, the limit-orthogonality property (71) holds up to subsequences if and only if

$$\limsup_{n \to \infty} \left( \frac{h_n^j}{h_n^k} + \frac{|t_n^j - t_n^k|}{(h_n^k)^\alpha} + \frac{|t_n^j - t_n^k|}{(h_n^k)^\alpha} + \frac{|x_n^j - x_n^k|}{h_n^j} + \frac{|x_n^j - x_n^k|}{h_n^k} \right) = \infty.$$

This conclusion can be seen in the proof of the conditional dislocation property Proposition 1.3. Note that the condition above is symmetric in the indices $j$ and $k$.

**Proof of Proposition 8.1.** It is not hard to see that the vanishing norm estimate (70) follows from the remainder term estimate (7) in Proposition 1.5 and Sobolev inequalities. To eliminate the frequency parameters, as shown in the proof of [19, Theorem 2.4], the key point is to deduce the following estimate

$$\lim_{|\xi_n| \to \infty} \left\| [e^{it|\nabla|^\alpha}] [e^{i(\cdot)\xi_n}] \phi \right\|_{L^{2a+2}_{t,x}} = 0. \quad (73)$$
Then the highly oscillatory terms, which mean the terms $[T_j^\varepsilon]|\phi(x)$ with
\[
\lim_{n\to\infty} |h_n^j\xi_n| = \infty,
\]
in Proposition 1.5 can be reorganized into the remainder term. After that, the desired Strichartz-orthogonality (72) of these profiles is much easier to established due to the lack of frequency parameters, see also [19, Lemma 2.7] for further details. Other conclusions come from Proposition 1.5 and Proposition 1.7 accordingly.

To obtain the estimate (73), we can follow similar arguments in the proof of Lemma 6.1, see also [19, Theorem 2.4]. Indeed we could rewrite
\[
\left| e^{it|\nabla|^\alpha}|e^{i(\cdot)\xi_n}\phi(x)\right| = \frac{1}{2\pi} \left| \int_{\mathbb{R}} e^{ix\xi + it\Phi_n(\xi)}\hat{\phi}(\xi)d\xi \right|
\]
with
\[
\Phi_n(\xi) := \sum_{m=1}^{\infty} \left( \frac{\alpha}{m} \right) (\xi_n)^{\alpha - m}\xi^m.
\]
Notice that by density we can assume $\phi$ to be a Schwartz function with compact Fourier support and the assumption $|\xi_n| \to \infty$ can guarantee the uniform convergency of $\Phi_n$ for $n$ large enough. Hence by changing of variables and van der Corput Lemma, the dominated convergence theorem implies the desired conclusion (73).

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