Solitary Wave Solutions of the Generalized (3+1)-Dimensional Shallow Water-Like Equation by Using Modified Kudryashov Method

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Abstract

In this study, the generalized (3+1)-dimensional Shallow Water-Like (SWL) equation, which is one of the evolution equations, is taken into consideration. With the help of this evolution equation discussed, the modified Kudryashov method, traveling wave solutions are successfully obtained. In these solutions, graphs of solitary waves to be obtained by giving special values to arbitrary parameters are presented. At the same time, the effect of change of velocity parameter on the behavior on the solitary wave is examined in the solution obtained. The breaking of the wave is discussed. In this study, complex operations and graphic presentations are presented with the use of a ready-made package program.

Keywords: Generalized (3+1)-dimensional Shallow Water-Like equation; Modified Kudryashov method; Traveling wave solution.

Modifiye Kudryashov Metodu Kullanılarak Genelleştirilmiş (3 + 1)-Boyutlu Sığ Su Benzeri Denkleminin Solitary Dalga Çözümleri

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Öz

Bu çalışmada evrim denklemlerinden biri olan genelleştirilmiş (3+1) boyutlu şiş su benzeri denklemi dikkate alınır. Ele alınan bu evrim denklemi modifiye Kudryashov metodu yardımcıyla yürüyen dalgalar çözümleri başarılı bir şekilde elde edilir. Bu çözümlerde keyfi parametrelerle özel değerler verilerek elde edilecek solitary dalgaların grafikleri sunulur. Aynı zamanda elde edilen çözümdede hız parametresinin değişiminin solitary dalgalar üzerindeki davranışlara olan etkisi incelenir. Dalganın kırılma olayı tartışılır. Bu çalışmada karmaşık işlemler ve grafik sunumları hazır paket programının kullanımyla sunulur.

Anahtar Kelimeler: Genelleştirilmiş (3+1) boyutlu Şiş Su Benzeri denklem; Modifiye Kudryashov metodu, Yürüyen dalga çözümü.

1. Introduction

Nonlinear partial differential equations (NLPDEs) have many application areas such as fluid dynamics, hydromagnetic, optics, physics, chemistry, biology and others [1-5]. With the solutions of these NLPDEs and the values given to the special parameters in these solutions, many physical phenomena we encounter in daily life are modelled [6-10]. Therefore, there has been an increasing interest in the solution methods of NLPDEs by many scientists. Especially the methods existing in the last twenty years are updated and applied to these differential equations. With the help of these methods, many traveling wave solutions that satisfy the equation have been obtained [11-14]. Some of the methods are very efficient in NLPDEs and generate solutions from many different types. These properties of the methods are very important for the application area. Some of these methods are \((G'/G)\)-expansion method [15], \((G'/G^2)\)-expansion method [16], \((1/G')\)-expansion method [17-19], \((m+G'/G)\)-expansion method [20], \((m+1/G')\)-expansion method [21], \((G'/G, 1/G)\)-expansion method [22] and so on [23].

\((3+1)\)-dimensional Shallow Water-Like (SWL) equation is a nonlinear evolution equation that has become quite popular recently [24].

\[
  u_{xxy} + 3u_{x}u_{y} + 3u_{x}u_{xy} - u_{yt} - u_{xz} = 0.
\]  

(1)

There have been many studies on this equation recently. The \((G'/G)\)-expansion method was obtained by Zayed in 2010 [25], and the traveling wave solutions were obtained with the help of the generalized binary operator by Zhang in 2017 [26]. In 2019, the traveling wave solutions of Eqn. (1) were obtained by Dusunceli with the help of the Bernoulli sub equation method [24].
Then, by applying the sine-Gordon method, traveling wave solutions in complex form were reached by Baskonus and Eskitascioğlu in 2020 [27].

In this study, we aimed to reach traveling wave solutions for Eqn. (1) with the help of the modified Kudryashov method [28]. At the same time, in the solutions obtained, special values are given to the parameters and presented with the help of graphics.

2. Materials and Methods

2.1. Methodology of the modified Kudryashov method

Assume you have a NLPDE in the form below

\[ T(u, u_x, u_y, u_z, u_{x}, u_{y}, u_{z}, \ldots) = 0, \]  

(2)

where \( T \) is a function in \( u(x, y, z, t) \) and its partial derivatives in which nonlinear terms and highest-order derivatives. We give the basic steps of this method in the following.

Step 1. Using the wave transmutation

\[ u(x, y, z, t) = U(\xi), \quad \xi = x + ky + mz - wt, \]  

(3)

we can transform it the following nODE for \( U(\xi) \):

\[ S(U, U', U'', \ldots) = 0, \]  

(4)

where Eqn. (4) is the ODE, where \( k, w, m \) are constants. Here \( w \) is a physical quantity and is the speed parameter of the wave.

Step 2. We assume that Eqn. (4) has the formal solution

\[ U(\xi) = a_0 + \sum_{i=1}^{n}(a_iQ(\xi)^i + a_{-i}Q(\xi)^{-i}), \]  

(5)

where \( a_i, a_{-i}, \quad i=1, \ldots, n \) are constants to be determined, such that \( a_n \neq 0 \) or \( a_{-n} \neq 0 \), and \( Q(\xi) \) is the solution of the equation

\[ Q'(\xi) = [Q^2(\xi) - Q(\xi)] \ln a, \]  

(6)

Eqn. (6) has solutions
\[ Q(\xi) = \frac{1}{1 \pm a^\xi}, \quad (7) \]

where \( a > 0, a \neq 1 \) is a real number.

**Step 3.** In Eqn. (4), a positive integer \( n \) is calculated according to the balance principle.

**Step 4.** Substitute Eqn. (5) with Eqn. (6) into Eqn. (4), we compute all the required derivatives \( U', U'', \ldots \) of the function \( U(\xi) \). Thus, we get a polynomial of \( Q^j(\xi), (j = 0, 1, 2, \ldots) \).

Computed polynomial, we add all the terms of the same powers of \( Q^j(\xi) \) and equal them to zero, we get a system of algebraic equations that can be solved by a computer package program to attain the unknown parameters \( a_i, a_i, \quad i = \{1, \ldots, n\}, k \) and \( w \). As a result, we get exact solutions of the Eqn. (2).

### 2.2. Application of modified Kudryashov method

We consider Eqn. (1). By using

\[ u(x, y, z, t) = U(\xi), \quad \xi = x + ky + mz - wt, \quad (13) \]

Inserting Eqn. (13) into Eqn. (1), we obtain

\[ kU^{(4)} + 6kU'U'' + (kw - m)U'' = 0, \quad (14) \]

Once the Eqn. (14) is integrated

\[ kU''' + 3k(U')^2 + (kw - m)U' = 0. \tag{15} \]

In the Eqn. (15), we get the balancing term \( n = 2 \) and by considering in the Eqn. (5),

\[ U(\xi) = a_0 + a_1Q(\xi) + a_2Q(\xi)^{-1} + b_1Q(\xi)^2 + b_2Q(\xi)^{-2}, \quad (16) \]

if Eqn. (16) is written in Eqn. (15) and if necessary adjustments are made, the following systems of equations can be written:
\[ \text{Const:} \quad m \log[a] a_2 - kw \log[a] a_2 - k \log[a] a_2 - 6k \log[a] a_2 - 3k \log[a] a_2^2 \\
+ 6k \log[a] b_2 + 24k \log[a] a_2 b_2 - 24k \log[a] b_2 = 0, \]
\[
\frac{1}{Q[\xi]}: \quad -m \log[a] a_2 + kw \log[a] a_2 + k \log[a] a_2 - 6k \log[a] a_2^2 + 2m \log[a] b_2 \\
- 2k \log[a] b_2 - 14k \log[a] b_2 - 12k \log[a] a_2 b_2 + 12k \log[a] b_2 a_2 b_2 = 0, \]
\[
\frac{1}{Q[\xi]}^2: \quad 3k \log[a] b_2 - 2m \log[a] b_2 + 2k \log[a] a_2 b_2 + 8k \log[a] b_2 = 0, \]
\[
\frac{1}{Q[\xi]}^3: \quad 12k \log[a] b_2 = 0, \]
\[
Q[\xi]: \quad m \log[a] a_1 - kw \log[a] a_1 - k \log[a] a_1 + 12k \log[a] a_1^2 a_1 a_2 \\
- 12k \log[a] a_1 a_2 b_1 - 12k \log[a] a_1^2 a_1 b_2 + 48k \log[a] b_2 = 0, \]
\[
Q[\xi]^2: \quad -m \log[a] a_1 + kw \log[a] a_1 + 7k \log[a] a_1 + 3k \log[a] a_1^2 a_1 \\
- 6k \log[a] a_2 a_2^2 + 2m \log[a] a_2 b_1 - 24k \log[a] a_2 b_1 - 8k \log[a] b_1 \\
+ 24k \log[a] b_1 a_2 b_2 - 24k \log[a] b_1 b_2 = 0, \]
\[
Q[\xi]^3: \quad -12k \log[a] a_1 a_1 - 6k \log[a] a_1 a_1 - 2m \log[a] a_1 b_1 + 2k \log[a] a_1 b_1 \\
+ 38k \log[a] a_1 b_2 + 12k \log[a] a_1 b_2 a_2 b_1 - 12k \log[a] b_1 b_2 = 0, \]
\[
Q[\xi]^4: \quad 6k \log[a] a_1 a_1 + 3k \log[a] a_1 a_1 + 54k \log[a] a_1 a_1 b_1 - 24k \log[a] a_1 a_1 b_1 + 12k \log[a] a_1 b_1 b_2 = 0, \]
\[
Q[\xi]^5: \quad 24k \log[a] a_1 b_1 + 12k \log[a] a_1 b_1 a_2 b_2 - 24k \log[a] a_1 b_1 b_2 = 0, \]
\[
Q[\xi]^6: \quad 12k \log[a] b_2 = 0. \]

\[ a_1, a_2, b_1, b_2 \text{ and } m, k, w \text{ constants are obtained from Eqn. (17) the system utilizing a software program.} \]

**Case 1:** If

\[ a_1 = -2 \log[a], \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad m = k \left( w + \log[a] \right), \]

replacing values Eqn. (18) into Eqn. (16), we get traveling wave soliton for Eqn. (1)
$u_1(x, y, z, t) = -\frac{2\log[a]}{1 + a} + a_0.$ \hspace{1cm} (19)

![Graph showing 3D, 2D, and contour graphs for Eqn. (19)]

**Figure 1**: 3D, 2D and contour graphs for the Eqn. (19) for $a_0 = 5, w = 1, k = 0.1, y = 1, z = 1, a = 2$

### 3. Results and Discussions

In this study, we have obtained the traveling wave solution of the SWL equation with the modified Kudryashov method. It can be said that the solution obtained by this method is general from the solutions obtained in $(1/G')$-expansion method. This is usually because the base of the exponential function contains an arbitrary parameter. It is the “$e$” expression defined as the base exponential function in the $(1/G')$-expansion method. However, the term without exponential expression is constant in modified Kudryashov method, while it contains arbitrary parameters in $(1/G')$-expansion method [29]. When the solutions obtained are examined physically, let’s examine the effect of the change of velocity parameter on the traveling wave solution obtained. The “$w$” expression in the classical wave transformation is a parameter representing the frequency of the wave and therefore its speed. We can present the effect of velocity on the wave with the following 3D simulation provided that other parameters except “$w$” are taken as constant Fig. 2 as seen in the simulation, as the speed increases, the changes in the behavior of the wave and the refraction phenomenon occur. Here is $w = 2.05$. The distortions at the end point of the value of the wave and the breaking event at $w = 2.055$ are clearly seen. In the future, the effect of other parameters on the wave can be observed.
Figure 2: Simulation graphs of the Eqn. (19) for $a_0 = 5, k = 0.1, y = 1, z = 1, \alpha = 2$

4. Conclusions

In this study, the traveling wave solution of the generalized SWL equation has been successfully obtained. Solitary wave solutions were created for specific values of arbitrary parameters in the traveling wave solution. 2D, 3D and contour graphics of these solitary waves are presented. At the same time, the effect of the change of velocity parameter on the behavior on the solitary wave in the solution obtained. Fig. 2 also presented and discussed. In addition, the value in the speed parameter at which the breakage of the wave occurred was determined. It was concluded that the modified Kudryashov method is valid, reliable and applicable. In future studies, many studies can be done on non-linear evolution equations with the help of this method.

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