Revisit to electrical and thermal conductivities, Lorenz number and Knudsen number in thermal QCD in a strong magnetic field

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Abstract

We have explored how the electrical ($\sigma_{el}$) and thermal ($\kappa$) conductivities in a thermal QCD medium get affected in weak-momentum anisotropy arising due to a strong magnetic field ($|q_iB| \gg T^2, |q_iB| \gg m_i^2, q_i$ ($m_i$) is the electric charge (mass) of $i$-th flavor and $T, B$ are the temperature and magnetic field, respectively). This study, in turn, facilitates to understand the duration of strong magnetic field, the ratio $\kappa/\sigma_{el}$, i.e. Wiedemann-Franz law, and the assumption for the local equilibrium by the Knudsen number. We calculate the conductivities by solving the relativistic Boltzmann transport equation in relaxation-time approximation. The interactions among partons are incorporated within the quasiparticle approach at finite $T$ and strong $B$. Somehow we also take the QCD masses to compare the deviations with respect to the noninteracting scenario, which gives sometimes unusually large values, thus validating the use of quasiparticle model. We have found that the electrical conductivity monotonically decreases with the increase of temperature in a magnetic field-driven anisotropy, which is opposite to its behavior in an expansion-driven anisotropy. Whereas the thermal conductivity increases very slowly with the temperature, contrary to its rapid increase in the expansion-driven anisotropic medium, therefore both $\sigma_{el}$ and $\kappa$ may distinguish the origin of anisotropies. The above findings in conductivities are broadly attributed to three factors: Firstly, the weak-momentum anisotropies are generated either by the strong magnetic field or by the asymptotic initial expansion. As a result, the distribution function either gets stretched or squeezed, respectively. Secondly, the phase-space factor, which is severely affected by the strong magnetic field only. Thirdly the relaxation-time of quark with and without the presence of strong magnetic field, which takes into account the dispersion relation. Next we have extracted the time-dependence of initially produced strong magnetic field by $\sigma_{iso}^{\text{el}}$, where the magnetic field expectedly decays slower than in vacuum. However, due to the presence of weak-anisotropy, magnetic field decays relatively faster than in isotropic one. Furthermore for a given $\kappa^{\text{iso}}$, the Knudsen number ($\Omega$) is found to decrease with the temperature, but the presence of expansion-driven anisotropy reduces its magnitude. However, the presence of strong magnetic field raises its value but, remains less than one unlike the much larger value in ideal case. Finally, the ratio, $\kappa/\sigma_{el}$ in magnetic field comes out as a promising tool to probe the anisotropy because it increases linearly with temperature, but with a magnitude larger than in isotropic medium and smaller than in expansion-driven anisotropic medium.

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1 Introduction

Relativistic heavy-ion experiments at RHIC and LHC create a new state of strongly interacting medium, known as quark gluon plasma (QGP) and are continuing to successfully collect the evidences in the form of elliptic flow \[1–3\], jet quenching \[4–6\], dilepton and photon spectra \[7–9\], anomalous quarkonium suppression \[10–12\] etc. for the existence of QGP. The abovementioned predictions were made for the simplest possible phenomenological setting, i.e. fully central collisions, where the baryon number density is negligible and it is expected that due to the symmetric configuration of the collision, no strong magnetic fields are produced. But only a small portion of heavy-ion collisions are truly head-on, most collisions indeed occur with a finite impact parameter or centrality. As a result, the two highly charged ions impacting with a small offset may produce extremely large magnetic fields reaching between \(m_\pi^2 \approx 10^{18}\) Gauss at RHIC and \(15 m_\pi^2\) at LHC \[13\].

However, the naive (classical) estimates for the lifetime of these strong magnetic fields show that they only exist for a small fraction of the lifetime of QGP \[14, 15\]. However, the charge transport properties of QGP have been found to significantly extend their lifetime, thus the study of the transport coefficient, mainly, the electrical conductivity (\(\sigma_{el}\)) becomes essential. Here our motivations are of two fold, which complement to each other: first we wish to revisit \(\sigma_{el}\) for an isotropic hot QCD medium in absence of any external field to check how long the magnetic field produced in relativistic heavy ion collisions stays appreciably large, i.e., some sort of time-dependence of externally produced magnetic field. However, the issue about the longevity of the magnetic field is not yet settled. So keeping the uncertainties about the exact nature of magnetic field in mind, if the external magnetic field still remains large till the medium is formed, the transport properties of the medium can then be significantly affected and the effect depends on the magnitude of \(\sigma_{el}\) of the medium in presence of strong magnetic field \((|q_i B| \gg T^2, |q_i B| \gg m_i^2)\), \(q_i\) (\(m_i\)) is the electric charge (mass) of \(i\)-th flavor and \(T, B\) are the temperature and magnetic field, respectively), whose evaluation is exactly our second motivation. Since \(\sigma_{el}\) is responsible for the production of electric current due to the Lenz’s law, its value becomes vital for the strength of Chiral Magnetic Effect \[16\]. Moreover the electrical field in mass asymmetric collisions has overall a preferred direction, which will eventually generate a charge asymmetric flow and the strength of the flow is given by \(\sigma_{el}\) \[17\]. Furthermore, \(\sigma_{el}\) is used as a vital input for many phenomenological applications in RHIC, LHC etc., such as the emission rate of soft photons \[18\], which accounts the raising of the spectra \[19, 20\].

The effects of the magnetic fields on \(\sigma_{el}\) for quark matter have been investigated previously in different models, such as quenched SU(2) lattice gauge theory \[21\], the dilute instanton-liquid model \[22\], the nonlinear electromagnetic currents \[23, 24\], axial Hall
current [25], real-time formalism using the diagrammatic method [26], effective fugacity approach [27] etc. As we know, the external magnetic field modifies the dispersion relation 

\( E_n = \sqrt{p_L^2 + 2n|q_i B| + m_i^2} \) 

quantum mechanically of the charged particle, where the motion along the longitudinal direction \((p_L)\) (with respect to the magnetic field direction) remains the same as for a free particle and only the motion along the transverse direction \((p_T)\) gets quantized in terms of the Landau levels \((n)\). In strong magnetic field limit \((eB >> T^2\) as well as \(eB \gg m^2\)), only the lowest Landau level will be occupied, i.e. \(p_T \approx 0\), and the particle can only move along the direction of the magnetic field, resulting an anisotropy in the momentum space, i.e. \(p_L \gg p_T\). Thus the anisotropic parameter, \(\xi = \frac{(p_T^2)}{2p_L^2} - 1\) comes out to be negative and for a weak-anisotropy \((\xi < 1)\), the distribution function may be approximated by stretching the isotropic one along a certain direction (say, the direction of magnetic field). Thus, to know the effects of strong magnetic field on conductivities in kinetic theory approach, an introduction of anisotropy is automatically needed.

Much earlier than the former one, it was envisaged that the relativistic heavy ion collisions at the initial stage induce a momentum anisotropy in the local rest frame of fireball, due to the asymptotic free expansion of the fireball in the beam direction compared to its transverse direction [28, 29]. Unlike the previous one, here \(p_T\) is greater than \(p_L\), hence the anisotropy parameter becomes positive. Therefore, for a weak-anisotropy \((\xi < 1)\), the distribution of partons can be approximated by squeezing an isotropic one along a certain direction and its effects on many phenomenological and theoretical observations have already been made. For example, the leading-order dilepton and photon yields get enhanced due to the anisotropic component [30–33]. Recently one of us had observed the effect of this kind of anisotropy on the properties of heavy quarkonium bound states [34], the electrical conductivity [35], where the heavy quarkonia are found to dissociate earlier than its counterpart in isotropic one and the electrical conductivity decreases with the increase of anisotropy. Later its relation with the shear viscosity is also explored in [36].

Now we move on to the thermal conductivity \((\kappa)\), which is related to the efficiency of the heat flow or the energy dissipation in a thermal QCD medium. Our intention is to comment on the validity of the local equilibrium assumed in hydrodynamics in terms of Knudsen number \((\Omega)\), which, in turn, is related to the thermal conductivity through the mean free path \((\lambda)\). Similar to the electrical conductivity, we also wish to explore the effect of strong magnetic field on the thermal conductivity by calculating it in the presence of weak-momentum anisotropy caused by the strong magnetic field. A natural question arises about whether we can distinguish the anisotropies from the abovementioned origins through the transport coefficients and knowing that we can improve the knowledge on the transport properties of the medium. This query may be a worth of investigation.

The electronic contribution of the thermal conductivity and the electrical conductivity
are not completely independent rather their ratio is equal to the product of Lorenz number $(L)$ and temperature, widely known as Wiedemann-Franz law. In fact, the ratio, $\kappa/\sigma_{el}$ has approximately the same value for different metals at the same temperature. But, it diverges in quasi-one-dimensional metallic phase with decreasing temperature [37], reaching a value much larger than that found in conventional metals, near the insulator-metal transition [38], thermally populated electron-hole plasma in graphene [39] etc. Recently the temperature dependence of the Lorenz number is calculated for the two-flavor quark matter in NJL model [40] and for the strongly interacting QGP medium [41]. In the metallic phase, the electronic contribution to thermal conductivity was much smaller than what would be expected from the Wiedemann-Franz law, which can be explained in terms of independent propagation of charge and heat in a strongly correlated system. However, in this work we intend to observe how the ratio gets affected due to the presence of an ambient strong magnetic field.

In this work, we have evaluated both the conductivities in kinetic theory approach, where the relativistic Boltzmann transport (RBT) equation is employed and is being solved by the relaxation-time approximation (RTA), where, as such there is no scope to incorporate the interaction among the partons. We circumvent the problem by incorporating the interactions among partons through their dispersion relations, known as quasiparticle model (QPM), in their distribution functions. The quasiparticle masses are conveniently obtained from their respective self energies, which, in turn, depends on the temperature and the magnetic field. Thus the presence of magnetic field affects both electrical and thermal conductivities. However, as a base line, we also compute the conductivities with the current quark masses (noninteracting), which give unusually large values, thus motivates us to use the QPM.

In brief, we have observed that the electrical conductivity and the thermal conductivity of the hot QCD medium increase in the presence of strong magnetic field-driven weak-momentum anisotropy, contrary to the decrease of the counterparts in the expansion-driven anisotropic medium. The opposite behavior in two anisotropic mediums may help to distinguish the origin of anisotropy in a thermal medium produced at the initial stage of ultrarelativistic heavy ion collision. From the relative behavior of thermal conductivity and electrical conductivity, we have noticed that the ratio, $\kappa/\sigma_{el}$ in a strong magnetic field shows linear enhancement with temperature, whose magnitude and slope are larger than in isotropic medium and smaller than in expansion-driven anisotropic medium, thus describes different Lorenz numbers ($\kappa/(\sigma_{el}T)$). We have also observed that the presence of strong magnetic field makes the Knudsen number slightly larger (but, remains less than one unlike the much larger value in ideal case) than its value in the isotropic medium.

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1If one can solve RBT equation with the collisional integral ($C[f]$), one can then incorporate the interactions through the matrix element.
The present work is organized as follows. In section 2, we have formulated the electrical conductivity where we first revisit the electrical conductivity for an isotropic thermal medium in subsection 2.1. Then we proceed to calculate it for the anisotropic thermal mediums due to expansion-induced anisotropy and magnetic field-induced anisotropy in subsection 2.2, and then their results are discussed in the QCD model involving current quark masses. Similarly in section 3 we have determined the thermal conductivity, where we first revisit its form for an isotropic thermal medium in subsection 3.1, which is followed by the calculation of thermal conductivity for the abovementioned anisotropic thermal mediums in subsection 3.2., and then we have discussed their results in the QCD model involving current quark masses. We have studied the applications of aforesaid conductivities in section 4. In section 5, we have discussed the quasiparticle model and calculated the quasiparticle mass in the presence of a strong magnetic field, and then explained the results on electrical conductivity, thermal conductivity, Wiedemann-Franz law and Knudsen number using the quasiparticle model. Finally, we have concluded in section 6.

2 Electrical conductivity

Transport coefficients such as electrical conductivity and thermal conductivity of a hot QCD system can be determined using different models and approaches namely relativistic Boltzmann transport equation [36, 42–44], the Chapman-Enskog approximation [41, 45], the correlator technique using Green-Kubo formula [22, 46, 47] and lattice simulation [48–51]. However, we will employ the relativistic Boltzmann transport equation with the relaxation-time approximation to calculate the electrical conductivity for both isotropic and anisotropic hot QCD mediums in subsections 2.1 and 2.2, respectively.

2.1 Electrical conductivity for an isotropic thermal medium

When an isotropic and hot medium of quarks, antiquarks and gluons in thermal equilibrium is disturbed infinitesimally by an electric field, an electric current $J_\mu$ is induced as

$$J_\mu = \sum_i q_i g_i \int \frac{d^3p}{(2\pi)^3 \omega_i} p_\mu [\delta f_i^q(x, p) + \delta f_i^{\bar{q}}(x, p)],$$

(1)

where the summation is over three flavors ($u$, $d$ and $s$) and $q_i$, $g_i$ and $\delta f_i^q$ ($\delta f_i^{\bar{q}}$) are the electric charge, degeneracy factor and infinitesimal change in the distribution function for the quark (antiquark) of $i$th flavor, respectively. In our calculations we will be using $\delta f_i^q = \delta f_i^{\bar{q}} = \delta f_i$ for zero chemical potential. According to Ohm’s law, the longitudinal
component of the spatial part of four-current is directly proportional to the external electric field and the proportionality factor is known as the electrical conductivity,

$$J = \sigma_{cl}E.$$  \hfill (2)

The infinitesimal change in quark distribution function is defined as $$\delta f_i = f_i - f_i^{iso},$$ where $$f_i^{iso}$$ is the equilibrium distribution function in the isotropic medium for $$i$$th flavor,

$$f_i^{iso} = \frac{1}{e^{\beta \omega_i} + 1},$$ \hfill (3)

with $$\omega_i = \sqrt{p^2 + m_i^2}$$. It is possible to obtain $$\delta f_i$$ from the relativistic Boltzmann transport equation (RBTE) \[52\],

$$p^\mu \frac{\partial f_i(x,p)}{\partial x^\mu} + q_i F^{\rho\sigma} p_\sigma \frac{\partial f_i(x,p)}{\partial p^\rho} = C[f_i(x,p)],$$ \hfill (4)

where $$F^{\mu\nu}$$ denotes the electromagnetic field strength tensor and the collision term, $$C[f_i(x,p)]$$ is given in the relaxation-time approximation as

$$C[f_i(x,p)] \simeq -\frac{p_\mu u^\nu}{\tau_i} \delta f_i(x,p),$$ \hfill (5)

where $$u^\nu$$ is the four-velocity of fluid in the local rest frame and the relaxation-time ($$\tau_i$$) for $$i$$th flavor in a thermal medium is given \[53\] by

$$\tau_i = \frac{1}{5.1T\alpha_i^2 \log (1/\alpha_s)[1 + 0.12(2N_i + 1)]}. \hfill (6)$$

To take into account the effect of the electric field, we use only $$\mu = i$$ and $$\nu = 0$$ and vice versa components of the electromagnetic field strength tensor, i.e. $$F^{0i} = -E$$ and $$F^{i0} = E$$ in our calculation, thus the RBTE eq. (4) takes the following form,

$$q_i E \cdot \frac{\partial f_i(x,p)}{\partial p_0} + q_i p_0 E \cdot \frac{\partial f_i^{iso}(x,p)}{\partial p} = -\frac{p_0}{\tau_i} \delta f_i,$$ \hfill (7)

which gives the solution, $$\delta f_i$$,

$$\delta f_i = 2q_i \tau_i \beta \frac{E \cdot p}{\omega_i} f_i^{iso}(1 - f_i^{iso}).$$ \hfill (8)

Now substituting the value of $$\delta f_i$$ in eq. (1) for zero chemical potential, we obtain the electrical conductivity for the thermal isotropic medium,

$$\sigma_{cl}^{iso} = \frac{2\beta}{3\pi^2} \sum_i^g q_i^2 \int dp \frac{p^4}{\omega_i^2} \tau_i f_i^{iso}(1 - f_i^{iso}),$$ \hfill (9)

which can now be used to show how the magnetic field varies with time in the isotropic thermal conducting medium. According to electrodynamics, the magnetic field created due to the spatial variation of the electric field, rapidly changes over time. However for an medium with substantial value of electrical conductivity, the momentary magnetic field would induce an electric current which ultimately would help to enhance the lifetime of the strong magnetic field.
2.2 Electrical conductivity for an anisotropic thermal medium

Here we will mainly discuss two types of momentum anisotropies, which may arise in the very early stages of ultrarelativistic heavy ion collisions. The first one is due to the preferential flow in the longitudinal direction compared to the transverse direction and the second one is due to the creation of strong magnetic field. We will first revisit the former one.

2.2.1 Expansion-induced anisotropy

At early times, the QGP created in the heavy ion collision experiences larger longitudinal expansion than the radial expansion and this develops a local momentum anisotropy. For the weak-momentum anisotropy ($\xi < 1$) in a particular direction (say $n$), the distribution function is written

$$f_{\text{aniso}, i}(p; T) = \frac{1}{e^{\beta \sqrt{p^2 + \xi (p \cdot n)^2 + m_i^2}} + 1},$$

(10)

which can be expanded in a Taylor series, and up to $O(\xi)$ it takes the following form,

$$f_{\text{aniso}, i} = f_{i}^{\text{iso}} - \frac{\xi \beta (p \cdot n)^2}{2 \omega_i} f_{i}^{\text{iso}} (1 - f_{i}^{\text{iso}}).$$

(11)

The anisotropic parameter ($\xi$) is generically defined in terms of the transverse and longitudinal components of momentum as

$$\xi = \frac{(p_L^2)}{2 (p_T^2)} - 1,$$

(12)

where $p_L = p \cdot n$, $p_T = p - n \cdot (p \cdot n)$, $p \equiv (p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta)$, $n = (\sin \alpha, 0, \cos \alpha)$, $\alpha$ is the angle between z-axis and direction of anisotropy, $(p \cdot n)^2 = p^2 c(\alpha, \theta, \phi) = p^2 (\sin^2 \alpha \sin^2 \theta \cos^2 \phi + \cos^2 \alpha \cos^2 \theta + 2 \alpha \sin \theta \cos \theta \cos \phi)$. For $p_T \gg p_L$, $\xi$ takes positive value, which explains the removal of particles with a large momentum component along $n$ direction due to the faster longitudinal expansion than the transverse expansion [28].

Now we are going to observe how the weak-momentum anisotropy affects the electrical conductivity of the thermal medium. Thus, after solving the RBTE (4) for the anisotropic distribution function, we get $\delta f_i$ as

$$\delta f_i = \frac{2 \tau_i \beta q_i E \cdot p}{\omega_i} \left[ f_{i}^{\text{iso}} (1 - f_{i}^{\text{iso}}) + \frac{\xi c(\alpha, \theta, \phi)}{2} \left\{ \frac{\beta p^2}{\omega_i} f_{i}^{\text{iso}} (1 - f_{i}^{\text{iso}}) ight. ight.$$

$$\left. + \frac{2 \beta p^2}{\omega_i} f_{i}^{\text{iso}} (1 - f_{i}^{\text{iso}}) - \frac{p^2}{\omega_i} f_{i}^{\text{iso}} (1 - f_{i}^{\text{iso}}) \right\},$$

(13)
which is then substituted in eq. (1) to yield the expression of electrical conductivity,

$$\sigma_{\text{cl,ex}}^{\text{aniso}} = \frac{2\beta}{3\pi^2} \sum_i g_i q_i^2 \int \frac{dp}{\omega_i^2} \tau_i f_i^{\text{iso}}(1 - f_i^{\text{iso}}) - \frac{\xi\beta^2}{9\pi^2} \sum_i g_i q_i^2 \int \frac{dp}{\omega_i^2} \tau_i f_i^{\text{iso}}(1 - f_i^{\text{iso}})$$

\[
+ \frac{2\xi\beta^2}{9\pi^2} \sum_i g_i q_i^2 \int \frac{dp}{\omega_i^2} \tau_i f_i^{\text{iso}}(1 - f_i^{\text{iso}}) - \xi\beta \sum_i g_i q_i^2 \int \frac{dp}{\omega_i^2} \tau_i f_i^{\text{iso}}(1 - f_i^{\text{iso}}) 
\]

$$+ \frac{\beta^2}{9\pi^2} \sum_i g_i q_i^2 \int \frac{dp}{\omega_i^2} \tau_i f_i^{\text{iso}}(1 - f_i^{\text{iso}}),$$

where the first term in R.H.S. is the electrical conductivity for an isotropic medium. So in terms of $\sigma_{\text{cl}}^{\text{iso}}, \sigma_{\text{cl,ex}}^{\text{aniso}}$ is written as

$$\sigma_{\text{cl,ex}}^{\text{aniso}} = \sigma_{\text{cl}}^{\text{iso}} - \xi \left[ \frac{\beta^2}{9\pi^2} \sum_i g_i q_i^2 \int \frac{dp}{\omega_i^2} \tau_i f_i^{\text{iso}}(1 - f_i^{\text{iso}}) \left\{ 1 - 2f_i^{\text{iso}} + \frac{1}{\beta\omega_i} \right\} \right]$$

\[
- \frac{\beta}{9\pi^2} \sum_i g_i q_i^2 \int \frac{dp}{\omega_i^2} \tau_i f_i^{\text{iso}}(1 - f_i^{\text{iso}}). \tag{15}
\]

2.2.2 Life-span of magnetic field

Earlier people had thought that the magnetic field generated in the heavy ion collision decays instantly. However in the presence of transport coefficient such as electrical conductivity, the lifetime of magnetic field may be elongated. To reaffirm this, we are going to see the variation of magnetic field using the value of electrical conductivity that we have calculated above for both isotropic and anisotropic mediums.

Thus for a charged particle moving in $x$-direction, a magnetic field will be produced in the perpendicular direction of the particle trajectory, say $z$-direction. According to the Maxwell’s equations, the magnetic field created along $z$-direction is expressed, as a function of time and electrical conductivity [55] for an isotropic medium as

$$eB_{\text{medium}}^{\text{iso}} = \frac{e^2 b \sigma_{\text{cl}}^{\text{iso}}}{8\pi(t-x)^2} e^{-\frac{\sigma_{\text{cl}}^{\text{iso}}}{\gamma(t-x)}} \hat{z}. \tag{16}$$

However for an anisotropic medium, the expression for $eB$ is not available, so we assumed the same expression by replacing $\sigma_{\text{cl}}^{\text{iso}} \rightarrow \sigma_{\text{cl,ex}}^{\text{aniso}},$

$$eB_{\text{medium}}^{\text{aniso}} = \frac{e^2 b \sigma_{\text{cl,ex}}^{\text{aniso}}}{8\pi(t-x)^2} e^{-\frac{\sigma_{\text{cl,ex}}^{\text{aniso}}}{\gamma(t-x)}} \hat{z}. \tag{17}$$

For the sake of comparison, the magnetic field produced in vacuum [55] is given by,

$$eB_{\text{vacuum}} = \frac{e^2 b\gamma}{4\pi \left\{ b^2 + \gamma^2(t-x)^2 \right\}^{3/2}} \hat{z}, \tag{18}$$

9
where $b$ and $\gamma$ denote the impact parameter and the Lorentz factor of heavy ion collision, respectively. In equations (16) and (17), the electrical conductivity is taken as a function of time through the cooling law, $T^3 \propto t^{-1}$, where initial time and temperature are set at 0.2 fm and 390 MeV, respectively. From figures 1 and 2, which are plotted at $x = 0$ for the centre of mass energies 200 GeV and 2.76 TeV, respectively, we see that the magnetic field in the isotropic conducting medium decays very slowly as compared to the vacuum. At initial time, the fluctuation of magnetic field in a thermal medium is quite high, however after certain time, it gradually stabilizes.

However for a conducting medium in the presence of weak-momentum anisotropy ($\xi = 0.6$), we have investigated (from figure 3) that the lifetime of existence of a nearly stable magnetic field in the anisotropic thermal medium is slightly less than in the isotropic thermal medium, whereas at initial time, this difference in the variations of magnetic field in two mediums is less illustrious.

As we can see from figures 1, 2 and 3, the decay of magnetic field with time is very slow in conducting medium and it nearly remains strong. So, it is plausible to explore the effect of strong magnetic field-induced anisotropy on the thermal medium.
Figure 2: Comparison between the variations of magnetic field with time in an isotropic thermal conducting medium and in a vacuum for two values of the impact parameter (a) $b = 4$ fm and (b) $b = 7$ fm, with the Lorentz factor ($\gamma$) is 1380 for Pb+Pb collision at LHC energy $\sqrt{s} = 2.76$ TeV.

Figure 3: Comparison between the variations of magnetic field with time in isotropic and anisotropic thermal conducting mediums for impact parameter $b = 7$ fm and Lorentz factor $\gamma = 100$. 
2.2.3 Magnetic field-induced anisotropy

In the presence of an extremely strong magnetic field, the quarks are confined to only lowest Landau levels (LLLs), because they could not be excited to the higher Landau levels (HLLs) due to very high energy gap $\sim \mathcal{O}(\sqrt{eB})$. Thus the motion of quark is restricted to only one spatial dimension (along the direction of magnetic field) unlike the gluons who still move in three spatial dimensions.

The distribution function for quark at finite temperature and strong magnetic field is given by

$$f_{\text{aniso}}(\mathbf{p}'; T) = \frac{1}{e^{\beta\sqrt{p'^2 + \xi (\mathbf{p}' \cdot \mathbf{n})^2 + m_i^2}} + 1}, \quad (19)$$

where $\xi$ is the anisotropic parameter which characterizes the distribution of particles in a strong magnetic field, $\mathbf{p}' = (0, 0, p_3)$ and $\mathbf{n} = (\sin \alpha, 0, \cos \alpha)$. In the strong magnetic field regime, the anisotropy is mainly produced by the magnetic field, so the direction of anisotropy coincides with the direction of magnetic field ($z$-direction). Thus one can set $\alpha = 0$, which yields $(\mathbf{p}' \cdot \mathbf{n})^2 = p_3^2$. From the definition of $\xi$ in eq. (12), it is evident that, $\xi$ will approach negative value for a medium embedded in a magnetic field with a very large strength, because in this case the momentum along the direction of anisotropy dominates over the momentum perpendicular to the direction of anisotropy. i.e. there will be more number of particles with large longitudinal component of momentum along the direction of anisotropy than along the transverse direction of anisotropy ($p_T \ll p_L$).

For very small $\xi$, the Taylor series expansion of eq. (19) upto $\mathcal{O}(\xi)$ yields

$$f_{\text{aniso}} = f_{\text{aniso}}^{\xi=0} = f_i^{\xi=0} - \frac{\xi \beta p_3^2}{2\omega_i} f_i^{\xi=0} (1 - f_i^{\xi=0}), \quad (20)$$

where $\xi$-independent part of the quark distribution function is given by

$$f_i^{\xi=0} = \frac{1}{e^{\beta \omega_i} + 1}, \quad (21)$$

with $\omega_i = \sqrt{p_3^2 + m_i^2}$.

In the strong magnetic field (SMF) limit, the quark momentum is assumed to be purely longitudinal [56]. Therefore when the thermal medium is disturbed infinitesimally by an electric field, an electromagnetic current is induced in the longitudinal direction ($3$- or $z$-direction) in the SMF limit as

$$J_3 = \sum_i q_i g_i \int \frac{d^3p}{(2\pi)^3 \omega_i} p_3 [\delta f_i^u(\mathbf{x}, \mathbf{p}) + \delta f_i^d(\mathbf{x}, \mathbf{p})], \quad (22)$$

unlike $\mathbf{J}$ in the absence of magnetic field. In eq. (22), we have used new notations relevant for the calculations in SMF limit as $\mathbf{x} = (x_0, 0, 0, x_3)$ and $\mathbf{p} = (p_0, 0, 0, p_3)$. In this case
the electrical conductivity can be obtained from the third component of current in Ohm’s law,

\[ J_3 = \sigma_{el} E_3 \, . \]  \hspace{1cm} (23)

Due to dimensional reduction in the presence of a strong magnetic field, the density of states in two spatial directions perpendicular to the direction of magnetic field can be written in terms of \(|q_i B|\) as

\[ \int \frac{d^3p}{(2\pi)^3} \rightarrow \frac{|q_i B|}{2\pi} \int \frac{dp_3}{2\pi} . \]  \hspace{1cm} (24)

The infinitesimal perturbation around the equilibrium distribution function \( i.e. \delta f_i = f_i - f_{\text{aniso}}^i \) due to the action of external magnetic field is obtained from the relativistic Boltzmann transport equation in RTA, in conjunction with the strong magnetic field limit,

\[ p_0 \frac{\partial f_i}{\partial x^0} + p_3 \frac{\partial f_i}{\partial x^3} + q_i F^0_3 p_3 \frac{\partial f_i}{\partial p_0} + q_i F^3_0 p_0 \frac{\partial f_i}{\partial p_3} = - \frac{p_0}{\tau_{Bi}^0} \delta f_i , \] \hspace{1cm} (25)

where \( \tau_{Bi}^0 \) denotes the relaxation time for quark in the presence of strong magnetic field.

In the LLL approximation, \( 1 \equiv 2 \) (\( g \equiv q\bar{q} \)) scattering process is dominant over \( 2 \equiv 2 \) (\( gq \equiv gq \)) scattering process \([59, 60]\). In the dominant process \( (1 \equiv 2) \), for the quark momentum \( \sim O(T) \), the momentum-dependent relaxation time is calculated \([60]\) as

\[ \tau_{Bi}^B = \frac{\omega_i}{\alpha_s C_2 m_i^2} \left( \frac{e^{\beta\omega_i} - 1}{e^{\beta\omega_i} + 1} \right) \left[ 1 \left\{ \int \frac{dp_3'}{\omega_i'} \left( \frac{1}{e^{\beta\omega_i'} + 1} \right) \right\} \right] , \] \hspace{1cm} (26)

where \( C_2 \) is the Casimir factor and the primed notations are used for antiquark. Now solving the RBTE \((25)\), we obtain the value of \( \delta f_i \) as

\[ \delta f_i = \frac{2\pi B q_i E_3 p_3}{\omega_i} \left[ f_i^{\xi=0}(1 - f_i^{\xi=0}) + \frac{\xi}{2} \left\{ \frac{\beta p_i^2}{\omega_i} f_i^{\xi=0} (1 - f_i^{\xi=0}) + \frac{2\beta p_i^2}{\omega_i} f_i^{\xi=0} (1 - f_i^{\xi=0}) \right\} \right] , \] \hspace{1cm} (27)

which, after substituting in eq. \((22)\) for zero chemical potential, we get the electrical conductivity of the anisotropic thermal QCD medium in the presence of a strong magnetic field,

\[ \sigma_{\text{el,B}}^{\text{aniso}} = \frac{\beta}{\pi^2} \sum_i g_i q_i^2 |q_i B| \int \frac{dp_3}{\omega_i^2} \frac{p_i^2}{\omega_i^2} \tau_i^B f_i^{\xi=0} (1 - f_i^{\xi=0}) \]

\[ - \frac{\xi \beta^2}{2\pi^2} \sum_i g_i q_i^2 |q_i B| \int \frac{dp_3}{\omega_i^2} \frac{p_i^2}{\omega_i^2} \tau_i^B f_i^{\xi=0} (1 - f_i^{\xi=0}) \left\{ 1 - 2f_i^{\xi=0} + \frac{1}{\beta \omega_i} \right\} \]

\[ + \frac{\xi \beta}{2\pi^2} \sum_i g_i q_i^2 |q_i B| \int \frac{dp_3}{\omega_i^2} \frac{p_i^2}{\omega_i^2} \tau_i^B f_i^{\xi=0} (1 - f_i^{\xi=0}) , \] \hspace{1cm} (28)
where the first term in R.H.S. is the $\xi$-independent term. Decomposing into $\xi = 0$ and $\xi \neq 0$ terms, eq. (28) takes the following form,

$$
\sigma_{\text{el,B}}^{\text{aniso}} = \sigma_{\text{el}}^{\xi=0} + \sigma_{\text{el}}^{\xi \neq 0} = \sigma_{\text{el}}^{\xi=0} - \xi \left[ \frac{\beta^2}{2\pi^2} \sum_i g_i q_i^2 \left| q_i B \right| \int dp_3 \frac{p_3^2}{\omega_i^3} \tau_i^B \left( \frac{f_i}{f_i^0} \right) \left( 1 - \sum_j g_j q_j^2 \left| q_j B \right| \int dp_3 \frac{p_3^2}{\omega_j^3} \tau_j^B \left( \frac{f_j}{f_j^0} \right) \right) \right] - \frac{\beta}{2\pi^2} \sum_i g_i q_i^2 \left| q_i B \right| \int dp_3 \frac{p_3^2}{\omega_i^3} \tau_i^B \left( \frac{f_i}{f_i^0} \right) \left( 1 - \sum_j g_j q_j^2 \left| q_j B \right| \int dp_3 \frac{p_3^2}{\omega_j^3} \tau_j^B \left( \frac{f_j}{f_j^0} \right) \right) \right] .
$$

(29)

Before analysing the results on the electrical conductivity in the presence of anisotropies arising either due to the expansion or due to the strong magnetic field, we wish to understand first how the distribution function in an isotropic medium gets affected in the presence of anisotropies because, in kinetic theory approach, the conductivities are mainly affected by the distribution function embodying the effects of anisotropy, the phase-space factor and the relaxation time. Therefore, we must understand how the ratios, $f_{\text{ex,aniso}}^{\text{B,iso}}$, $f_{\text{B,aniso}}^{\text{B,iso}}$, depend on the temperature at low and high momenta or vice-versa, which are numerically plotted in figures 4 and 5, respectively. The observations in the above figures can be readily understood by estimating an order of estimate from equations (3), (10) and (19), for $u$ quark. For weak-momentum anisotropy ($\xi \ll 1$), the ratios are: $f_{\text{ex,aniso}}^{\text{B,iso}} \sim e^{-cB}$ and $f_{\text{B,aniso}}^{\text{B,iso}} \sim e^{-c'B}$, in both low and high momentum limits, with the constant, $c < c' < 1$. The crucial negative and positive signs in exponentials arise
Figure 5: Variation of the ratio $f_{\text{aniso}}/f_{\text{iso}}$ with momentum in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion and strong magnetic field ($15 \, m_2^2$) at (a) low temperature and (b) high temperature, where current quark mass has been used.

due to the positive and negative anisotropic parameter, in expansion-driven and magnetic field-driven cases, respectively.

Let us start with the variation of $f_{\text{aniso}}^{\text{ex}}/f_{\text{iso}}$ with $T$ in low momentum regime (figure 4a). As $T$ increases, $p/T$ decreases, resulting an increase in $f_{\text{aniso}}^{\text{ex}}/f_{\text{iso}}$ due to the lesser Boltzmann damping and an obvious decrease in $f_{\text{aniso}}^{\text{B}}/f_{\text{iso}}$. The slower and relative faster variations are due to the smaller value of $c$ with respect to $c'$. For higher momentum the variations (in figure 4b) as well as the magnitudes of the ratios are more pronounced. The variations of the ratios with momentum at a fixed temperature (in figures 5a and 5b) are much easier to understand because the variable ($p$) in the exponential is proportional to $p/T$, hence the variations become just opposite to the variation with temperature in figure 4.

The above observations on the distribution functions facilitate to understand the results on the electrical conductivity for a thermal QCD medium with three flavors ($u$, $d$ and $s$) with their current masses in figure 6. For the isotropic medium (denoted by solid line), $\sigma_{\text{el}}$ increases with the increase of temperature, whereas due to insertion of weak-momentum anisotropy (labelled as dotted line), $\sigma_{\text{el}}$ decreases a little because the ratio $f_{\text{aniso}}^{\text{ex}}/f_{\text{iso}}$ is always less than 1 for the entire range of temperature (as in figure 4a). On the other hand, the relative magnitude of $\sigma_{\text{el}}$ in magnetic field-driven anisotropic medium (labelled as dashed-dotted line) becomes very large due to relatively large ratio, $f_{\text{aniso}}^{\text{B}}/f_{\text{iso}}$. 

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Figure 6: Variation of electrical conductivity with temperature in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion and strong magnetic field ($15 \, m_e^2$), where current quark masses have been used. (as in figure 4). However $\sigma_{el}$ increases with $T$, albeit the ratio, $f^{B}_{aniso}/f_{iso}$ decreases with temperature (as in figure 4). The decrease in $f^{B}_{aniso}/f_{iso}$ at high temperature becomes much slower and approaches towards unity, but the phase-space factor ($\sim |q_iB|$) and the relaxation time in SMF limit ($|q_iB| \gg T^2$) together compensate the minimal decrease in $f^{B}_{aniso}$ and gives an overall increasing trend in $\sigma_{el}$ in the presence of strong magnetic field (seen in figure 6).

Such large value of $\sigma_{el}$ in the presence of strong magnetic field arises due to the large relaxation-time ($\tau^B$), because it is inversely proportional to the square of the mass. The similar results are also recently found in [26], where $\sigma_{el}$ is calculated in the diagrammatic method in the strong magnetic field regime and its large value is due to the smaller value of current quark masses. This motivates us to recalculate it with quasiparticle masses in subsection 5.1.

3 Thermal conductivity

This section is devoted to the determination of the thermal conductivity of a hot QCD medium using the relativistic Boltzmann transport equation. In non-relativistic case, the heat equation is obtained by the validity of the first and second laws of thermodynamics, where the flow of heat is proportional to the temperature gradient and the proportionality factor is called the thermal conductivity. This implies that when two bodies at different
temperatures are set in thermal contact, heat flows from the hotter body to the colder body. The heat does not flow directly, but it diffuses, depending on the internal structure of the medium it travels through. Similarly for a relativistic QCD system also, the behavior of heat flow depends on the features of the medium. Thermal conductivity of a particular medium helps to describe the heat flow in that medium and it may leave significant effects on the hydrodynamic evolution of the systems with nonzero baryon chemical potential. To see how the heat flow gets affected, we have calculated thermal conductivity for both isotropic and anisotropic hot QCD mediums in subsections 3.1 and 3.2, respectively.

3.1 Thermal conductivity for an isotropic thermal medium

Heat flow four-vector is defined as the difference between the energy diffusion and the enthalpy diffusion,

\[ Q_\mu = \Delta_\mu \alpha T^{\alpha \beta} u_\beta - h \Delta_\mu \alpha N_\alpha, \]  

(30)

where the projection operator is defined as \( \Delta_\mu \alpha = g_\mu \alpha - u_\mu u_\alpha \), \( h \) is the enthalpy per particle which in terms of energy density, pressure and particle number density is represented as \( h = (\varepsilon + P)/n \), \( T^{\alpha \beta} \) denotes the energy-momentum tensor and \( N_\alpha \) is the particle flow four-vector. \( N_\alpha \) and \( T^{\alpha \beta} \) are also known as the first and second moments of the distribution function with the following expressions

\[ N_\alpha = \sum_i 2 g_i \int \frac{d^3 p}{(2\pi)^3 \omega_i} p^\alpha f_i(x, p), \]  

(31)

\[ T^{\alpha \beta} = \sum_i 2 g_i \int \frac{d^3 p}{(2\pi)^3 \omega_i} p^\alpha p^\beta f_i(x, p). \]  

(32)

It is also possible to obtain the particle number density from eq. (31), the energy density and the pressure from eq. (32) as \( n = N_\alpha u_\alpha \), \( \varepsilon = u_\alpha T^{\alpha \beta} u_\beta \) and \( P = -\Delta_{\alpha \beta} T^{\alpha \beta}/3 \), respectively. From equations (30), (31) and (32), one can find that in the rest frame of the heat bath or fluid, heat flow four-vector is orthogonal to the fluid four-velocity, i.e.

\[ Q_\mu u^\mu = 0. \]  

(33)

Thus in the rest frame of the fluid, the heat flow is purely spatial and this component of heat flow due to the action of external disturbances can be written, in terms of the nonequilibrium part of the distribution function as

\[ Q = \sum_i 2 g_i \int \frac{d^3 p}{(2\pi)^3 \omega_i} \frac{P}{\omega_i - h_i} \delta f_i(x, p). \]  

(34)

In order to define the thermal conductivity for a system, the number of particles in that system must be conserved and therefore it requires the associated chemical potential to be
nonzero. In the beginning of the universe and also in the initial stage of ultrarelativistic heavy ion collision, the value of chemical potential ($\mu$) is small but nonzero. In the Navier-Stokes equation, the heat flow is related to the thermal potential ($U = \mu/T$) as

\[ Q_\mu = -\kappa \frac{nT^2}{\varepsilon + P} \nabla \mu U \]

\[ = \kappa \left[ \nabla_\mu T - \frac{T}{\varepsilon + P} \nabla_\mu P \right], \quad (35) \]

where the coefficient $\kappa$ is known as the thermal conductivity and $\nabla_\mu = \Delta_{\mu\nu} \partial^\nu$ is the four-gradient, which, in the rest frame of the heat bath, i.e. in the local rest frame, is replaced by $\partial_j$ (or $\partial/\partial x^j$). Thus, in the local rest frame, the spatial component of the heat flow is written as

\[ \mathbf{Q} = -\kappa \left[ \frac{\partial T}{\partial x} - \frac{T \partial P}{n \hbar} \right]. \quad (36) \]

The thermal conductivity ($\kappa$) can be determined by comparing equations (34) and (36), so we need to first find $\delta f_i$. In the local rest frame, the flow velocity and temperature depend on the spatial and temporal coordinates, so the distribution function can be expanded in terms of the gradients of flow velocity and temperature. Thus, the relativistic Boltzmann transport equation (4) takes the following form,

\[ p^\nu \partial_\mu T \frac{\partial f_i}{\partial T} + p^\nu \partial_\mu (p^\nu u_\nu) \frac{\partial f_i}{\partial p^\rho} + q_1 \left[ F^{0j} p_j \frac{\partial f_i}{\partial p^0} + F^{pj} p_0 \frac{\partial f_i}{\partial p^j} \right] = -\frac{p^\nu u_\nu}{\tau_i} \delta f_i, \quad (37) \]

where $f_i = f_i^{\text{iso}} + \delta f_i$ and $p_0 = \omega_i - \mu_i$, which, for very small value of $\mu_i$, can be approximated as $p_0 \approx \omega_i$. After dropping out the infinitesimal correction to the local equilibrium distribution function ($\delta f_i$) from the left hand side of eq. (37) and then using the following partial derivatives,

\[ \frac{\partial f_i^{\text{iso}}}{\partial T} = \frac{p_0}{T^2} f_i^{\text{iso}} (1 - f_i^{\text{iso}}), \quad (38) \]

\[ \frac{\partial f_i^{\text{iso}}}{\partial p^\rho} = -\frac{1}{T} f_i^{\text{iso}} (1 - f_i^{\text{iso}}), \quad (39) \]

\[ \frac{\partial f_i^{\text{iso}}}{\partial p^j} = -\frac{p^j}{T p_0} f_i^{\text{iso}} (1 - f_i^{\text{iso}}), \quad (40) \]
we solve eq. (37) to get the disturbance,
\[
\delta f_i = -\frac{\tau_i f_{i}^{iso}(1 - f_{i}^{iso})}{p_0} \left[ \frac{p_0}{T} \left\{ p^0 \partial_0 T + p^j \partial_j T \right\} - \frac{1}{T} \left\{ p^0 \partial_0 p_0 + p^j \partial_j p_0 \right\} \right.
\]
\[
- \frac{1}{T} \left\{ p^0 p' \partial_0 u_\nu + p^j p' \partial_j u_\nu \right\} - \frac{2q_i}{T} \mathbf{E} \cdot \mathbf{p} \left. \right]\]
\[
= -\frac{\tau_i f_{i}^{iso}(1 - f_{i}^{iso})}{T} \left[ \frac{p_0}{T} \partial_0 T + \frac{1}{T} p^j \partial_j T + T \partial_0 \left( \frac{\mu}{T} \right) + \frac{T}{p_0} p^j \partial_j \left( \frac{\mu}{T} \right) \right.
\]
\[
- p' \partial_0 u_\nu - \frac{p^j p' \partial_j u_\nu}{p_0} - \frac{2q_i}{p_0} \mathbf{E} \cdot \mathbf{p} \right].
\]
(41)

Substituting \( \partial_j \left( \frac{\mu}{T} \right) = -\frac{h}{T} \left( \partial_j T - \frac{T}{n h} \partial_j P \right) \) and using \( \partial_0 u_\nu = \nabla_\nu P/(n h) \) from the energy-momentum conservation, we get the final expression for \( \delta f_i \), after simplifying, as
\[
\delta f_i = -\frac{\tau_i f_{i}^{iso}(1 - f_{i}^{iso})}{T} \left[ \frac{p_0}{T} \partial_0 T + \left( \frac{p_0 - h}{p_0} \right) \frac{p^j}{T} \left( \partial_j T - \frac{T}{n h} \partial_j P \right) + T \partial_0 \left( \frac{\mu}{T} \right) \right.
\]
\[
- \frac{p^j p' \partial_j u_\nu}{p_0} - \frac{2q_i}{p_0} \mathbf{E} \cdot \mathbf{p} \right].
\]
(42)

After substituting the \( \delta f_i \) expression in eq. (34) and then comparing it with eq. (36), we get the thermal conductivity for the isotropic medium,
\[
\kappa_{iso} = \frac{\beta^2}{3 \pi^2} \sum_i g_i \int dp \frac{p^4}{\omega_i^2} (\omega_i - h_i)^2 \tau_i f_{i}^{iso}(1 - f_{i}^{iso}) .
\]
(43)

### 3.2 Thermal conductivity for an anisotropic thermal medium

In this subsection we will consider an anisotropic QCD medium, where the particle distribution is anisotropic in the momentum space and may be generated at the early stages of the ultrarelativistic heavy ion collisions. In this process we will first observe the effects due to the weak-momentum anisotropy on the thermal conductivity of hot QCD medium caused by the initial asymptotic expansion and then by the strong magnetic field as well.
3.2.1 Expansion-induced anisotropy

Using the Taylor series expansion of the anisotropic distribution function \(f^\text{aniso}_{\text{ex},i}\) up to the first order in \(\xi\), the following partial derivatives have been calculated as

\[
\frac{\partial f^\text{aniso}_{\text{ex},i}}{\partial T} = \frac{p_0 f_i^\text{iso}(1 - f_i^\text{iso})}{T^2} - \frac{\xi(p \cdot n)^2 f_i^\text{iso}(1 - f_i^\text{iso})}{2T^2p_0} \left[\frac{p_0}{T} - 1 - \frac{2p_0 f_i^\text{iso}}{T}\right],
\]

\[
\frac{\partial f^\text{aniso}_{\text{ex},i}}{\partial p^i} = -\frac{f_i^\text{iso}(1 - f_i^\text{iso})}{T} + \frac{\xi(p \cdot n)^2 f_i^\text{iso}(1 - f_i^\text{iso})}{2T^2p_0} \left[\frac{p_0}{T} + 1 - \frac{2p_0 f_i^\text{iso}}{T}\right],
\]

\[
\frac{\partial f^\text{aniso}_{\text{ex},i}}{\partial \rho^i} = -\frac{p_i f_i^\text{iso}(1 - f_i^\text{iso})}{Tp_0} - \frac{\xi p_i c(\alpha, \theta, \phi) f_i^\text{iso}(1 - f_i^\text{iso})}{2T^2p_0} \times \left[2 - \frac{p_i^2}{p_0^2} - \frac{p_i^2}{T^2p_0} + \frac{2p_i f_i^\text{iso}}{Tp_0}\right],
\]

which are then substituted in eq. (37) to obtain \(\delta f_i\),

\[
\delta f_i = -\frac{\tau_i f_i^\text{iso}(1 - f_i^\text{iso})}{T} \left[1 - \frac{\xi(p \cdot n)^2}{2p_0T} + \frac{\xi(p \cdot n)^2 f_i^\text{iso}}{p_0T}\right]
\]

\[
\times \left[p_0 \frac{\partial \mu}{\partial T} + \left(\frac{p_0 - h_i}{p_0}\right)^2 \frac{p^j}{T} \left(\partial_j T - \frac{T}{nh_i} \partial_j P\right) + T \partial \frac{\partial \mu}{\partial T} - \frac{p^j p^\nu}{p_0} \partial_j u^\nu\right]
\]

\[
- \frac{\tau_i f_i^\text{iso}(1 - f_i^\text{iso})}{T} \frac{\xi(p \cdot n)^2 f_i^\text{iso}}{2T^2p_0} \left[p_0 \frac{\partial \mu}{\partial T} + \left(\frac{p_0 + h_i}{p_0}\right)^2 \frac{p^j}{T} \left(\partial_j T - \frac{T}{nh_i} \partial_j P\right) - T \partial \frac{\partial \mu}{\partial T}\right]
\]

\[
+ \frac{2p_i \tau_i}{nh_i} E \cdot p_i f_i^\text{iso}(1 - f_i^\text{iso})
\]

\[
\times \left[1 + \frac{\xi(p \cdot n)^2}{2p_0^2} \left\{\frac{p_0^2}{T} - 1 - \frac{p_0}{T} + \frac{2p_0 f_i^\text{iso}}{T}\right\}\right].
\]

Now substituting the value of \(\delta f_i\) in eq. (34), we find the thermal conductivity in an expansion-driven anisotropic thermal QCD medium,

\[
\kappa^\text{aniso}_{\text{ex}} = \frac{\beta^2}{3\pi^2} \sum_g \int dp \frac{p^4}{\omega_i^4} \left[\left(\omega_i - h_i\right)^2 \tau_i f_i^\text{iso}(1 - f_i^\text{iso}) + \frac{\xi \beta^2}{18\pi^2} \sum_g \int dp \frac{p^6}{\omega_i^6} \left(\omega_i^2 - h_i^2\right)^2 \tau_i f_i^\text{iso}(1 - f_i^\text{iso})
\]

\[
- \frac{\xi \beta^3}{18\pi^2} \sum_g \int dp \frac{p^6}{\omega_i^6} \left(\omega_i - h_i\right)^2 \tau_i f_i^\text{iso}(1 - 2f_i^\text{iso})(1 - f_i^\text{iso})\right],
\]

where the first expression in R.H.S. is the thermal conductivity for the isotropic thermal QCD medium. Thus one can write \(\kappa^\text{aniso}_{\text{ex}}\) in terms of \(\kappa^\text{iso}\) as

\[
\kappa^\text{aniso}_{\text{ex}} = \kappa^\text{iso} + \frac{\beta^2}{3\pi^2} \sum_g \int dp \frac{p^4}{\omega_i^4} \left[\left(\omega_i - h_i\right)^2 \tau_i f_i^\text{iso}(1 - f_i^\text{iso}) + \frac{\xi \beta^2}{18\pi^2} \sum_g \int dp \frac{p^6}{\omega_i^6} \left(\omega_i^2 - h_i^2\right)^2 \tau_i f_i^\text{iso}(1 - f_i^\text{iso})
\]

\[
- \frac{\beta^3}{18\pi^2} \sum_g \int dp \frac{p^6}{\omega_i^6} \left(\omega_i - h_i\right)^2 \tau_i f_i^\text{iso}(1 - 2f_i^\text{iso})(1 - f_i^\text{iso})\right].
\]
We are now going to see how the thermal conductivity of the hot QCD medium gets modified due to the anisotropy developed by the strong magnetic field.

3.2.2 Magnetic field-induced anisotropy

The strong magnetic field restricts the dynamics of quarks to one spatial dimension \(i.e.\) along the direction of magnetic field. So there will be no conduction of heat by the quarks in the transverse directions. In this strong magnetic field scenario, eq. (34) is thus modified into

\[
Q_3 = \sum_i g_i |q_i B| \int dp_3 \frac{p_3}{\omega_i} (\omega_i - h_i B) \delta f_i(\tilde{x}, \tilde{p}) .
\]  

(50)

Similarly eq. (36) takes the following form,

\[
Q_3 = -\kappa \left[ \frac{\partial T}{\partial x^3} - \frac{T}{n h^B} \frac{\partial P}{\partial x^3} \right] \kappa \left[ \partial_3 T - \frac{T}{n h^B} \partial_3 P \right] ,
\]  

(51)

where \(h^B = (\varepsilon + P)/n \) represents the enthalpy per particle in a strong magnetic field. For the charged particles in the SMF limit, the particle number density \((n)\) is obtained from the following particle flow four-vector,

\[
N^\mu = \sum_i g_i |q_i B| \int dp_3 \frac{\tilde{p}^\mu}{\omega_i} f_i(\tilde{x}, \tilde{p}) ,
\]  

(52)

similarly the energy density \((\varepsilon)\) and the pressure \((P)\) are obtained from the following energy-momentum tensor,

\[
T^{\mu\nu} = \sum_i g_i |q_i B| \int dp_3 \frac{\tilde{p}^\mu \tilde{p}^\nu}{\omega_i} f_i(\tilde{x}, \tilde{p}) .
\]  

(53)

Now in terms of the gradients of flow velocity and temperature, the RBTE (25) in the presence of a strong magnetic field can be written as

\[
\tilde{p}^\mu \frac{\partial T}{\partial \tilde{x}^\mu} \frac{\partial f_i}{\partial T} + \tilde{p}^\mu \frac{\partial (\tilde{p}^\nu u_\nu)}{\partial \tilde{x}^\mu} \frac{\partial f_i}{\partial p^0} + q_i \left[ F^{03} p_3 \frac{\partial f_i}{\partial p^0} + F^{30} p_3 \frac{\partial f_i}{\partial p^3} \right] = \frac{\tilde{p}^\nu u_\nu}{\tau_i^B} \delta f_i ,
\]  

(54)
where the variables, $\hat{\tilde{p}}^\mu = (p^0, 0, 0, p^3)$ and $\hat{\tilde{x}}^\mu = (x^0, 0, 0, x^3)$ are suited to the strong magnetic field calculation. Using the following partial derivatives,

$$
\frac{\partial f_{\text{aniso}}^{\xi=0}}{\partial T} = \frac{p_0 f_{i}^{\xi=0} (1 - f_{i}^{\xi=0})}{T^2} - \frac{\xi p_3 f_{i}^{\xi=0} (1 - f_{i}^{\xi=0})}{2T^3 p_0} \left[ \frac{p_0}{T} - 1 - \frac{2p_0 f_{i}^{\xi=0}}{T} \right],
$$

(55)

$$
\frac{\partial f_{\text{aniso}}^{\xi=0}}{\partial p_0} = -\frac{f_{i}^{\xi=0} (1 - f_{i}^{\xi=0})}{T} + \frac{\xi p_3 f_{i}^{\xi=0} (1 - f_{i}^{\xi=0})}{2T^2 p_0} \left[ \frac{p_0}{T} + 1 - \frac{2p_0 f_{i}^{\xi=0}}{T} \right],
$$

(56)

$$
\frac{\partial f_{\text{aniso}}^{\xi=0}}{\partial p^3} = -\frac{p^3 f_{i}^{\xi=0} (1 - f_{i}^{\xi=0})}{T p_0} - \frac{\xi p^3 f_{i}^{\xi=0} (1 - f_{i}^{\xi=0})}{2T p_0}
$$

$$
\times \left[ 2 - \frac{p_3^2}{p_0^2} - \frac{p_3^2}{T p_0} + \frac{2p^3 f_{i}^{\xi=0}}{T p_0} \right],
$$

(57)

in eq. (54), we obtain $\delta f_i$ as

$$
\delta f_i = -\frac{x^B f_i^{\xi=0} (1 - f_i^{\xi=0})}{T} \left[ 1 - \frac{\xi p_3^2}{2p_0 T} + \frac{\xi p_3^2 f_i^{\xi=0}}{p_0 T} \right]
$$

$$
\times \left[ \frac{p_0}{T} \partial_T + \left( \frac{p_0 - h_i^B}{p_0} \right) \frac{p^3}{T} \left( \partial_T - \frac{T}{nh_i^B} \partial_3 P \right) \right. + T \partial_0 \left( \frac{\mu}{T} \right) - \frac{p^3 \tilde{p}^\nu}{p_0} \partial_3 u_\nu \left[ \frac{p_0}{T} \partial_T + \left( \frac{p_0 + h_i^B}{p_0} \right) \frac{p^3}{T} \left( \partial_T - \frac{T}{nh_i^B} \partial_3 P \right) \right. - T \partial_0 \left( \frac{\mu}{T} \right)
$$

$$
+ \frac{2p^3}{nh_i^B} \partial_3 P + \frac{p^3 \tilde{p}^\nu}{p_0} \partial_3 u_\nu \right) + 2q_i \tau_i^B \frac{p_0}{p_0 T} E_3 p_3 f_i^{\xi=0} (1 - f_i^{\xi=0})
$$

$$
\times \left[ 1 + \frac{\xi p_3^2}{2p_0^2} \left\{ \frac{p_0}{p_3^2} - 1 - \frac{p_0}{T} + \frac{2p_0 f_{i}^{\xi=0}}{T} \right\} \right],
$$

(59)

which is then substituted in eq. (50) to give the value of thermal conductivity in a strong magnetic field,

$$
\kappa_{\text{aniso}}^B = \frac{\beta^2}{2\pi^2} \sum_i g_i |q_i B| \int dp_3 \frac{p_3^2}{\omega_i^2} (\omega_i - h_i^B)^2 \tau_i^B f_i^{\xi=0} (1 - f_i^{\xi=0})
$$

$$
+ \frac{\xi \beta^2}{4\pi^2} \sum_i g_i |q_i B| \int dp_3 \frac{p_3^4}{\omega_i^4} (\omega_i - h_i^B)^2 \tau_i^B f_i^{\xi=0} (1 - f_i^{\xi=0})
$$

$$
- \frac{\xi \beta^3}{4\pi^2} \sum_i g_i |q_i B| \int dp_3 \frac{p_3^4}{\omega_i^4} (\omega_i - h_i^B)^2 \tau_i^B f_i^{\xi=0} (1 - 2f_i^{\xi=0})(1 - f_i^{\xi=0}),
$$

(60)
Figure 7: Variation of thermal conductivity with temperature in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion and strong magnetic field (15 $m_T^2$), where current quark masses have been used.

where the first expression in R.H.S. is the $\xi = 0$ part of the thermal conductivity. Thus $\kappa_B^{\text{aniso}}$ can be rewritten in terms of $\xi$-dependent and independent parts as

$$
\kappa_B^{\text{aniso}} = \kappa^{\xi=0} + \kappa^{\xi \neq 0}
$$

$$
= \kappa^{\xi=0} + \xi \left[ \frac{\beta^2}{4\pi^2} \sum_i g_i |q_i|B| \int dp_3 \frac{p_3^4}{\omega_i^2} (\omega_i^2 - h_i^2) \tau_i^B f_i^{\xi=0} (1 - f_i^{\xi=0}) \right. \\
\left. - \frac{\beta^2}{4\pi^2} \sum_i g_i |q_i|B| \int dp_3 \frac{p_3^4}{\omega_i^2} (\omega_i^2 - h_i^2)^2 \tau_i^B f_i^{\xi=0} (1 - 2 f_i^{\xi=0}) (1 - f_i^{\xi=0}) \right]. \quad (61)
$$

Figure 7 depicts how the thermal conductivity varies with temperature for the isotropic medium and for the anisotropic mediums due to expansion-driven anisotropy and strong magnetic field-driven anisotropy. We have observed that $\kappa$ for the isotropic medium increases with the increase in temperature. Similar increasing behavior of $\kappa$ is also noticed for the expansion-driven anisotropic medium, however its magnitude decreases. If the origin of anisotropy is strong magnetic field, then the magnitude of $\kappa$ jumps to a higher value, but its increase with temperature becomes relatively faster than in isotropic medium and expansion-driven anisotropic medium as well. The above observations on the thermal conductivity could similarly be attributed to the behaviors of respective distribution functions, the phase-space factor and the relaxation time scale, where the last two factors are severely affected by the strong magnetic field only. This again validates the use of quasiparticle masses for the thermal conductivity in subsection 5.2.
4 Applications

This section is devoted to study how the above behaviors observed in the electrical and thermal conductivities will help to understand some specific properties of medium. In subsection 4.1, we will observe how the interplay between the conductivities through the Wiedemann-Franz law gets modified in a thermal QCD medium in the presence of anisotropies arising due to different causes. In subsection 4.2, we will calculate the Knudsen number to have a say whether the thermal QCD medium is still in local equilibrium even in the presence of different anisotropies discussed hereinabove.

4.1 Wiedemann-Franz law

According to the Wiedemann-Franz law, the ratio of charged particle contribution of thermal conductivity to electrical conductivity is proportional to the temperature,

$$\frac{\kappa}{\sigma_{el}} = LT,$$

(62)

where the proportionality factor $L$ is called the Lorenz number. This law is perfectly satisfied by the matter which are good thermal and electrical conductors, such as metals. However for different cases, the deviation of the Wiedemann-Franz law has been observed, such as for the thermally populated electron-hole plasma in graphene, describing the signature of a Dirac fluid [39], for the two-flavor quark matter in the Nambu-Jona-Lasinio (NJL) model [40] and for the strongly interacting QGP medium [41]. In this work we intend to see how the Lorenz number for the thermal QCD matter varies by observing the ratio $(\kappa/\sigma_{el})$ as a function of temperature in the presence of expansion-driven and strong magnetic-field driven anisotropies in figure 8.

In the isotropic medium, the ratio is found to increase linearly with temperature. When the isotropic medium is subjected to an expansion-driven anisotropy, $\kappa/\sigma_{el}$ shows almost same increasing behavior with temperature like in isotropic case, but its magnitude and the slope of the linear increase get enhanced. If the origin of anisotropy is strong magnetic field, then both the magnitude and the slope of the linear increase of the ratio with the temperature become smaller than the former descriptions. Thus in two different types of anisotropies we have observed nearly opposite behavior of $\kappa/\sigma_{el}$, which can also be understood from the opposite behavior in electrical and thermal conductivities for the two aforesaid anisotropic mediums. This observation thus implies different Lorenz numbers $(\kappa/(\sigma_{el}T))$ at the same temperature, thus violates the Wiedemann-Franz law.
Figure 8: Variation of the ratio of thermal conductivity to electrical conductivity with temperature in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion and strong magnetic field ($15 \, m_\pi^2$), where current quark masses have been used.

4.2 Knudsen number

The Knudsen number is required to be small for small deviation from the equilibrium in the hydrodynamic regime. The Knudsen number ($\Omega$) is defined as

$$\Omega = \frac{\lambda}{L},$$

where $\lambda$ denotes the mean free path and $L$ is the characteristic length scale. One can calculate the mean free path by using the thermal conductivity ($\kappa$) of the medium,

$$\lambda = \frac{3\kappa}{vC_V},$$

where $v$ is the relative speed and $C_V$ is the specific heat. Therefore the Knudsen number can be recast in terms of the thermal conductivity as

$$\Omega = \frac{3\kappa}{LvC_V},$$

where we take $v \simeq 1$, $L = 3 \, \text{fm}$, and $C_V$ is evaluated from the energy density.

In an isotropic medium, the Knudsen number decreases with the increase of temperature, which explains that the mean free path becomes much smaller than the characteristic length scale of the system. As a result the medium approaches the equilibrium faster. When the medium exhibits a weak-momentum anisotropy due to the asymptotic expansion initially, the Knudsen number does not deviate considerably from its value in the
Figure 9: Variation of the Knudsen number with temperature in the presence of momentum anisotropies \((\xi = 0.6)\) both due to asymptotic expansion (left panel) and strong magnetic field \((15 m_\pi^2)\) (right panel), where current quark masses have been used.

isotropic medium (seen in the left panel of figure 9). However if the origin of anisotropy is the strong magnetic field \((eB = 15 m_\pi^2)\), a significant deviation from the isotropic one can be seen, where the Knudsen number has a larger magnitude (denoted as dashed-dotted line in the right panel of figure 9), which defies physical interpretation and has fortunately been cured in the quasiparticle model (seen in figure 15).

5 Quasiparticle description of hot QCD matter

Till now we, in fact, have not incorporated any interactions among quarks and gluons in a thermal QCD medium either in the presence or absence of strong magnetic field. As a matter of fact, the magnitude and the variation of the electrical conductivity, thermal conductivity and Knudsen number become unrealistic. Hence we must resort to the quasiparticle description of particles, known as QPM, where different flavors acquire the medium generated masses, in addition to their current masses. The thermal mass is generated due to the interaction of quark with the other particles of the medium, thus the quasiparticle model properly describes the collective properties of the medium. Earlier this model was explained in different approaches such as the Nambu-Jona-Lasinio and PNJL based quasiparticle models \([62–65]\), quasiparticle model based on Gribov-Zwanziger quantization \([66, 67]\) and thermodynamically consistent quasiparticle models \([68, 69]\). However, for our calculation, the effective mass (squared) of \(i\)-th flavor in a pure thermal
medium is taken from [70],

\[ m_i^2 = m_{i0}^2 + \sqrt{2} m_{i0} m_{iT} + m_{iT}^2, \]  

(66)

where \( m_{i0} \) is the current mass and \( m_{iT} \) is the thermal mass of \( i \)-th flavor, which is given [71, 72] by

\[ m_{iT}^2 = \frac{g'^2 T^2}{6}, \]  

(67)

where \( g' \) is the running coupling that runs with the temperature of the medium (we have taken its one-loop result).

Now, for a thermal medium in the presence of a strong magnetic field, the above effective mass (squared) can be generalized as

\[ m_i^2 = m_{i0}^2 + \sqrt{2} m_{i0} m_{iTB} + m_{iTB}^2, \]  

(68)

where \( m_{iTB} \) is the mass of \( i \)-th flavor at finite temperature and strong magnetic field.

Let us revisit first, how the medium generated mass can be calculated in the thermal medium only. The medium generated mass of the particle can be determined from the pole of effective propagator, which is defined as

\[ S(p) = \frac{1}{\gamma^\mu p_\mu - m_i - \Sigma(p)}, \]  

(69)

where the quark self-energy, \( \Sigma(p) \) is of the order of \( gT \).

Now, in additional presence of strong magnetic field along the \( z \)-direction (\( |q_i B| \gg T^2 \)), the transverse motion of quark ceases to exist (\( p_\perp \approx 0 \)), as a result, the effective quark propagator becomes a function of the parallel component of the quark momentum only,

\[ S(p_\parallel) = \frac{1}{\gamma^\mu p_\mu - m_i - \Sigma(p_\parallel)}, \]  

(70)

where \( \gamma^\mu p_\mu = \gamma^0 p_0 - \gamma^3 p_z \) with the notations of \( p_\parallel \equiv (p_0, 0, 0, p_z) \), \( \gamma^\mu_\parallel \equiv (\gamma^0, 0, 0, \gamma^3) \) and \( g^{\mu\nu}_\parallel = \text{diag}(1,0,0,-1) \) \(^2\). The following notations are also required to be known,

\[
\begin{align*}
p_\perp^i & \equiv (0, p_x, p_y, 0), \quad \gamma^\mu_\perp \equiv (0, \gamma^1, \gamma^2, 0), \\
g^{\mu\nu}_\perp & = \text{diag}(0,-1,-1,0), \quad g^{\mu\nu}_\parallel = g^{\mu\nu}_\parallel + g^{\mu\nu}_\perp, \\
p_\perp^2 & = p_0^2 - p_z^2, \quad p_\parallel^2 = p_x^2 + p_y^2.
\end{align*}
\]

The quark self-energy, \( \Sigma(p_\parallel) \) will now give the mass correction \( (m_{iTB}) \), due to thermal QCD medium in the ambience of strong magnetic field. In terms of the quark \( (S(k)) \) and gluon \( (D^{\mu\nu}(p - k)) \) propagators, the quark self-energy is written by the Feynman rules,

\[ \Sigma(p) = -\frac{4}{3} g'^2 i \int \frac{d^4k}{(2\pi)^4} \{ [\gamma_\mu, S(k)] \gamma^\mu D(p - k) \}, \]  

(71)

\(^2\)It is to be noted that \( p_\perp \) and \( p_\parallel \) are the same in our earlier notation of momentum in magnetic field-driven anisotropy.
where $4/3$ is the Casimir factor and $g$ is the running coupling that runs mainly with the magnetic field [73, 74], because magnetic field is the largest energy scale in the strong magnetic field regime. The quark propagator gets modified in the strong magnetic field [56, 75, 76] as

$$S(k) = ie^{-\frac{e^2}{\alpha m} \left( \gamma^0 k_0 - \gamma^3 k_z + m_i \right)} \left( 1 - \gamma^0 \gamma^3 \gamma^5 \right),$$  \hspace{1cm} (72)

however, the gluon propagator remains its form as in the pure thermal medium, which is given by

$$D^\mu\nu(p - k) = \frac{ig^{\mu\nu}}{(p - k)^2}.$$  \hspace{1cm} (73)

In the strong magnetic field limit, since the external quark momentum $p_\perp \approx 0$ [56], so, $(p - k)^2 \approx (p_0 - k_0)^2 - (p_z - k_z)^2 - k_z^2$, here we may also drop $k_z^2$ for $k_z^2 \ll T^2 \ll |q_i B|$, which largely simplifies the calculation. Substituting the expressions of quark and gluon propagators in eq. (71), then using the trace of gamma matrices, $\gamma_\mu (\gamma^0 k_0 - \gamma^3 k_z + m_i) (1 - \gamma_0 \gamma^3 \gamma^5) \gamma^\mu = -2 [(1 + \gamma^0 \gamma^3 \gamma^5) (\gamma^0 k_0 - \gamma^3 k_z) - 2m_i]$ and the discretized Matsubara frequency sum in the SMF limit, eq. (71) takes the following form,

$$\Sigma(p_\|) = \frac{2g^2 |q_i B| T}{3\pi^2} \sum_n \int dk_z \left[ \frac{(1 + \gamma^0 \gamma^3 \gamma^5) (\gamma^0 k_0 - \gamma^3 k_z) - 2m_i}{\left( k_z^2 - \omega_k^2 \right) \left[ (p_0 - k_0)^2 - \omega_{pk}^2 \right]} \right],$$  \hspace{1cm} (74)

where $\omega_k^2 = k_z^2 + m_i^2$, $\omega_{pk}^2 = (p_z - k_z)^2$ and $L^1$ and $L^2$ are two frequency sums,

$$L^1 = T \sum_n \frac{1}{k_z^2 - \omega_k^2} \frac{1}{\left[ (p_0 - k_0)^2 - \omega_{pk}^2 \right]}$$  \hspace{1cm} (75)

$$L^2 = T \sum_n \frac{1}{k_z^2 - \omega_k^2} \frac{1}{\left[ (p_0 - k_0)^2 - \omega_{pk}^2 \right]}.$$  \hspace{1cm} (76)

After calculating the frequency sums using hard thermal loop approximation [18, 77] and then substituting their values in eq. (74), we obtain

$$\Sigma(p_\|) = \frac{2g^2 |q_i B|}{3\pi^2} \int dk_z \frac{1}{4\omega_k} \left[ \frac{1}{e^{\beta \omega_k} - 1} + \frac{1}{e^{\beta \omega_k} + 1} \right] \left[ (\gamma^0 + \gamma^3 \gamma^5) \left\{ \frac{1}{p_0 + p_z} + \frac{1}{p_0 - p_z} \right\} + (\gamma^3 + \gamma^0 \gamma^5) \left\{ \frac{1}{p_0 - p_z} - \frac{1}{p_0 + p_z} \right\} \right]$$

$$= \frac{g^2 |q_i B|}{3\pi^2} \left( I_{k_z}^1 + I_{k_z}^2 \right) \left[ \frac{\gamma^0 p_0 + \gamma^3 p_z}{p_\|^2} + \frac{\gamma^0 \gamma^5 p_z + \gamma^3 \gamma^5 p_0}{p_\|^2} \right].$$  \hspace{1cm} (77)
The two $k_z$ integrations, $I^{1}_{k_z}$ and $I^{2}_{k_z}$ are respectively performed [78] to get

$$I^{1}_{k_z} = \int dk_z \frac{1}{\omega_k (e^{\beta \omega_k} - 1)} = \frac{\pi T}{2m_i} + \frac{1}{2} \ln \left( \frac{m_i}{4\pi T} \right) + \frac{\gamma E}{2},$$  

(78)

$$I^{2}_{k_z} = \int dk_z \frac{1}{\omega_k (e^{\beta \omega_k} + 1)} = -\frac{1}{2} \ln \left( \frac{m_i}{\pi T} \right) - \frac{\gamma E}{2},$$  

(79)

which are then substituted in eq. (77) to get

$$\Sigma(p_\parallel) = m_{iT}^2 B \left[ \gamma^0 p_0 + \gamma^3 p_z \right] \frac{2}{p_\parallel^2} + \gamma^0 \gamma^5 p_z \right] \frac{2}{p_\parallel^2},$$

where $m_{iT}^2$ is introduced as the mass (squared) at finite temperature and strong magnetic field with the following value,

$$m_{iT}^2 = \frac{g^2 |q_i B|}{3\pi^2} \left[ \frac{\pi T}{2m_i} - \ln(2) \right],$$  

(80)

which depends on both temperature and magnetic field.

In the quasiparticle description of particles, the distribution function now contains the effective masses of the particles. Therefore, the distribution functions in isotropic medium as well as in expansion-driven anisotropic medium depend only on temperature, whereas the distribution function in magnetic field-driven anisotropic medium depends on both temperature and magnetic field. So, from figures 10 and 11, we noticed that the behaviors of ratios ($f^{ex}_\text{aniso}/f_\text{iso}$ and $f^{B}_\text{aniso}/f_\text{iso}$) get flipped in comparison to their respective behavior in ideal case (as in figures 4 and 5). As the transport coefficients such as electrical conductivity and thermal conductivity are expressed in terms of the distribution function at finite temperature and/or magnetic field, so the knowledge about the behavior of distribution function in the QPM description is useful in understanding the transport properties of the hot QCD medium.

In the coming subsections we are going to discuss the results on the electrical conductivity, thermal conductivity and their applications using the quasiparticle model with three flavors ($u$, $d$ and $s$).

### 5.1 Electrical conductivity

With the quasiparticle description as input, we have now recomputed the electrical conductivity for isotropic and anisotropic mediums by substituting the effective quark masses from eq. (66) in equations (9), (15), and from eq. (68) in eq. (29). We have replotted $\sigma_{el}$ as a function of temperature in figure 12 and found that there is an overall decrease in $\sigma_{el}$. Interestingly, for a magnetic field-driven weak-momentum anisotropy (denoted by dashed-dotted line), the magnitude of $\sigma_{el}$ now becomes smaller, which is at par with the
Figure 10: Variation of the ratio $f_{\text{aniso}}/f_{\text{iso}}$ with temperature in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion and strong magnetic field $(15 \, m_\pi^2)$ at (a) low momentum and (b) high momentum, where effective quark mass has been used.

Figure 11: Variation of the ratio $f_{\text{aniso}}/f_{\text{iso}}$ with momentum in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion and strong magnetic field $(15 \, m_\pi^2)$ at (a) low temperature and (b) high temperature, where effective quark mass has been used.
counterparts in isotropic and expansion-driven anisotropic mediums. However, $\sigma_{el}$ for this magnetic field-driven anisotropic medium, now decreases with the temperature, which is opposite to its variation in the expansion-driven anisotropy. The above observations on $\sigma_{el}$ fully resonate with the distributions seen in figures 10 and 11. We are now convinced that the quasiparticle description of particles tames the unusually large value of $\sigma_{el}$ in the strong magnetic field.

5.2 Thermal conductivity

We have also calculated the thermal conductivity for isotropic and anisotropic mediums with the quasiparticle description by substituting the effective quark masses from eq. (66) in equations (43), (49), and from eq. (68) in eq. (61). Figure 13 plots the variation of $\kappa$ with temperature for the isotropic medium, expansion- and strong magnetic field-driven anisotropic mediums with the quasiparticle description. The effects of quasiparticle description on the thermal conductivity can again be understood through the distribution functions with quasiparticle masses in figures 10 and 11. For the isotropic as well as expansion-driven anisotropic mediums, $\kappa$ is found to increase with temperature as in ideal case. The only noticeable finding is that, although the magnitude of $\kappa$ for the strong magnetic field-driven anisotropic medium is still larger than in isotropic medium but it has now become smaller and comparable with the value in isotropic medium at very large temperature.
5.3 Wiedemann-Franz law

Wiedemann-Franz law makes us understand the relation between the charge transport and the heat transport in a system. Here we have revisited the law in quasiparticle description of particles, unlike using the ideal description of particles earlier in previous subsection 4.1. From figure 14 we found that the magnitude of the ratio, $\kappa/\sigma_{el}$ for isotropic and expansion-driven anisotropic mediums now becomes smaller whereas for the magnetic field-driven anisotropic medium it becomes larger as compared to their respective values in the ideal case (figure 8).

5.4 Knudsen number

We have seen earlier that for a strong magnetic field-driven anisotropic medium, the Knudsen number ($\Omega$) in the ideal case (seen in figure 9) was very large. As a result, the thermal medium in the presence of strong magnetic field deviates much away from its equilibrium which is, however, not desirable. This is exactly circumvented here in quasiparticle description in figure 15, where we have found that $\Omega$ has now been decreased drastically in the presence of strong magnetic field at per with the estimates for $B = 0$ cases. However, there is an overall decrease of Knudsen number for all cases. Thus in the quasiparticle description, the probability of finding the system to be in local equilibrium...
Figure 14: Variation of the ratio of thermal conductivity to electrical conductivity with temperature in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion and strong magnetic field ($15 \, m_\pi^2$), where effective quark masses have been used.

is higher, due to the smaller value of Knudsen number.

6 Conclusions

In this work, we have studied the effect of strong magnetic field-driven anisotropy on the transport coefficients such as electrical conductivity and thermal conductivity of the hot QCD matter and compared them with their behavior in the expansion-driven anisotropy. In order to find these conductivities we have solved the relativistic Boltzmann transport equation in relaxation-time approximation. First we revisited the formulation of electrical and thermal conductivities for the isotropic thermal medium and then calculated these for the expansion-induced anisotropic thermal medium. Using the value of electrical conductivity we have then observed the variation of magnetic field with time and this explains that the lifetime of magnetic field becomes larger for an electrically conducting medium as compared to the vacuum, hence the strong magnetic field is expected to affect the charge transport and heat transport in the QCD medium and this motivated us to derive the aforesaid conductivities for a thermal medium in the presence of strong magnetic field-induced anisotropy. We have observed that both the electrical and thermal conductivities have noticeably larger values in the presence of strong magnetic field-driven anisotropy as compared to their respective values in the isotropic medium, however if the
Figure 15: Variation of the Knudsen number with temperature in the presence of momentum anisotropies ($\xi = 0.6$) both due to asymptotic expansion (left panel) and strong magnetic field ($15 m_\pi^2$) (right panel), where effective quark masses have been used.

Anisotropy is induced due to asymptotic expansion, then the values of the conductivities are seen to get marginally lowered than their values in the isotropic medium. So, in the two different types of anisotropic mediums, we noticed nearly opposite behavior of conductivities. However the large values of conductivities in a strong magnetic field are avoided using the quasiparticle masses. Next, as applications of electrical conductivity and thermal conductivity, we have studied the Wiedemann-Franz law to see the relative behavior of these conductivities, where this law is found to be violated in the presence of strong magnetic field. Then we have calculated the Knudsen number to observe whether the system is still in equilibrium in the presence of weak-momentum anisotropy which may be caused by either sources. We found that, in the quasiparticle description, the Knudsen number becomes less than one, thus the medium may remain in local equilibrium.

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References

[1] R. S. Bhalerao and J. Y. Ollitrault, Phys. Lett. B 641, 260 (2006).
[2] B. Alver et al., Phys. Rev. C 77, 014906 (2008).
[3] S. A. Voloshin, A. M. Poskanzer, A. Tang, G. Wang, Phys. Lett. B 659, 537 (2008).
[4] X. N. Wang and M. Gyulassy, Phys. Rev. Lett. 68, 1480 (1992).
[5] K. Adcox et al., Phys. Rev. Lett. 88, 022301 (2002).
[6] S. Chatrchyan et al., Phys. Rev. C 84, 024906 (2011).
[7] E. L. Feinberg, Nuovo Cim. A 34, 391 (1976).
[8] E. V. Shuryak, Phys. Lett. B 78, 150 (1978).
[9] J. I. Kapusta, P. Lichard and D. Seibert, Phys. Rev. D 44, 2774 (1991).
[10] J. P. Blaizot and J. Y. Ollitrault, Phys. Rev. Lett. 77, 1703 (1996).
[11] Helmut Satz, Nucl. Phys. A 783, 249 (2007).
[12] R. Rapp, D. Blaschke and P. Crochet, Prog. Part. Nucl. Phys. 65, 209 (2010).
[13] V. Skokov, A. Illarionov, and V. Toneev, Int. J. Mod. Phys. A 24, 5925 (2009).
[14] P. F. Kolb and R. Rapp, Phys. Rev. C 67, 044903 (2003).
[15] P. F. Kolb and U. W. Heinz, In Hwa, R.C. (ed.) et al., Quark gluon plasma, 634 (2003).
[16] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D 78, 074033 (2008).
[17] Y. Hirono, M. Hongo and T. Hirano, Phys. Rev. C 90, 021903 (2014).
[18] J. I. Kapusta and C. Gale, “Finite Temperature Field Theory Principles and Applications”, Cambridge University Press, 2006.
[19] S. Turbide, R. Rapp and C. Gale, Phys. Rev. C 69, 014903 (2004).
[20] O. Linnyk, W. Cassing and E. L. Bratkovskaya, Phys. Rev. C 89, 034908 (2014).
[21] P. V. Buividovich, M. N. Chernodub, D. E. Kharzeev, T. Kalaydzhyan, E. V. Luschevskaya and M. I. Polikarpov, Phys. Rev. Lett. 105, 132001 (2010).
[22] Seung-il Nam, Phys. Rev. D 86, 033014 (2012).
[23] D. E. Kharzeev, Prog. Part. Nucl. Phys. 75, 133 (2014).
[24] D. Satow, Phys. Rev. D 90, 034018 (2014).
[25] S. Pu, S. Y. Wu and D. L. Yang, Phys. Rev. D 91, 025011 (2015).
[26] K. Hattori and D. Satow, Phys. Rev. D 94, 114032 (2016).
[27] M. Kurian and V. Chandra, Phys. Rev. D 96, 114026 (2017).
[28] Adrian Dumitru, Yun Guo, Ágnes Mócsy and Michael Strickland, Phys. Rev. D 79, 054019 (2009).
[29] A. Dumitru, Y. Guo and M. Strickland, Phys. Lett. B 662, 37 (2008).
[30] M. Martinez and M. Strickland, Phys. Rev. C 78, 034917 (2008).
[31] R. Ryblewski and M. Strickland, Phys. Rev. D 92, 025026 (2015).
[32] A. Mukherjee, M. Mandal and P. Roy, Eur. Phys. J. A 53, 81 (2017).
[33] L. Bhattacharya, R. Ryblewski and M. Strickland, Phys. Rev. D 93, 065005 (2016).
[34] L. Thakur, N. Haque, U. Kakade and B. K. Patra, Phys. Rev. D 88, 054022 (2013).
[35] P. K. Srivastava, L. Thakur and B. K. Patra, Phys. Rev. C 91, 044903 (2015).
[36] L. Thakur, P. K. Srivastava, G. P. Kadam, M. George and H. Mishra, Phys. Rev. D 95, 096009 (2017).
[37] R. Mahajan, M. Barkeshli and S. A. Hartnoll, Phys. Rev. B 88, 125107 (2013).
[38] C. Proust, K. Behnia, R. Bel, D. Maude and S. I. Vedeneev, Phys. Rev. B 72, 214511 (2005).
[39] J. Crossno et al., Science 351, 1058 (2016).
[40] A. Harutyunyan, D. H. Rischke and A. Sedrakian, Phys. Rev. D 95, 114021 (2017).
[41] S. Mitra and V. Chandra, Phys. Rev. D 96, 094003 (2017).
[42] A. Muronga, Phys. Rev. C 76, 014910 (2007).
[43] A. Puglisi, S. Plumari and V. Greco, Phys. Rev. D 90, 114009 (2014).
[44] S. Yasui and S. Ozaki, Phys. Rev. D 96, 114027 (2017).
[45] S. Mitra and V. Chandra, Phys. Rev. D 94, 034025 (2016).
[46] M. Greif, I. Bouras, C. Greiner and Z. Xu, Phys. Rev. D 90, 094014 (2014).
[47] B. Feng, Phys. Rev. D 96, 036009 (2017).
[48] S. Gupta, Phys. Lett. B 597, 57 (2004).
[49] H.-T. Ding, A. Francis, O. Kaczmarek, F. Karsch, E. Laermann and W. Soeldner, Phys. Rev. D 83, 034504 (2011).
[50] G. Aarts, C. Allton, A. Amato, P. Giudice, S. Hands and J.-I. Skullerud, JHEP 1502, 186 (2015).
[51] H.-T. Ding, O. Kaczmarek and F. Meyer, Phys. Rev. D 94, 034504 (2016).

[52] C. Crecignani and G. M. Kremer, “The Relativistic Boltzmann Equation: Theory and Applications” (Boston, Birkhäuser, 2002).

[53] A. Hosoya and K. Kajantie, Nucl. Phys. B 250, 666 (1985).

[54] P. Romatschke and M. Strickland, Phys. Rev. D 68, 036004 (2003).

[55] K. Tuchin, Adv. High Energy Phys. 2013, 490495 (2013).

[56] V. P. Gusynin and A. V. Smilga, Phys. Lett. B 450, 267 (1999).

[57] V. P. Gusynin, V. A. Miransky and I. A. Shovkovy, Nucl. Phys. B 462, 249 (1996).

[58] F. Bruckmann, G. Endrödi, M. Giordano, S. D. Katz, T. G. Kovács, F. Pittler and J. Wellnhofer, Phys. Rev. D 96, 074506 (2017).

[59] K. Tuchin, Phys. Rev. C 83, 017901 (2011).

[60] Koichi Hattori, Shiyong Li, Daisuke Satow and Ho-Ung Yee, Phys. Rev. D 95, 076008 (2017).

[61] M. Greif, F. Reining, I. Bouras, G. S. Denicol, Z. Xu and C. Greiner, Phys. Rev. E 87, 033019 (2013).

[62] A. Dumitru and R. D. Pisarski, Phys. Lett. B 525, 95 (2002).

[63] K. Fukushima, Phys. Lett. B 591, 277 (2004).

[64] S. K. Ghosh, T. K. Mukherjee, M. G. Mustafa and R. Ray, Phys. Rev. D 73, 114007 (2006).

[65] H. Abuki and K. Fukushima, Phys. Lett. B 676, 57 (2009).

[66] N. Su and K. Tywoniuk, Phys. Rev. Lett. 114, 161601 (2015).

[67] W. Florkowski, R. Ryblewski, N. Su and K. Tywoniuk, Phys. Rev. C 94, 044904 (2016).

[68] V. M. Bannur, Phys. Rev. C 75, 044905 (2007).

[69] P. K. Srivastava, S. K. Tiwari and C. P. Singh, Phys. Rev. D 82, 014023 (2010).

[70] V. M. Bannur, JHEP 0709, 046 (2007).

[71] E. Braaten and R. D. Pisarski, Phys. Rev. D 45, R1827 (1992).

[72] A. Peshier, B. Kämpfer and G. Soff, Phys. Rev. D 66, 094003 (2002).

[73] E. J. Ferrer, V. de la Incera and X. J. Wen, Phys. Rev. D 91, 054006 (2015).
[74] M. A. Andreichikov, V. D. Orlovsky and Yu. A. Simonov, Phys. Rev. Lett. 110, 162002 (2013).

[75] S. Rath and B. K. Patra, JHEP 1712, 098 (2017).

[76] S. Rath and B. K. Patra, arXiv:1806.03008 [hep-th].

[77] M. L. Bellac, “Thermal Field Theory”, Cambridge University Press, 1996.

[78] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).