New Construction of Optimal Interference-Free ZCZ Sequence Sets by Zak Transform

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Abstract

In this paper, a new construction of interference-free zero correlation zone (IF-ZCZ) sequence sets is proposed by well designed finite Zak transform lattice tessellation. Each set is characterized by the period of sequences $KM^2$, the set size $K$ and the length of zero correlation zone $M^2 - 1$, which is optimal with respect to the Tang-Fan-Matsufuji bound. In particular, all sequences in these sets have sparse and highly structured Zak and Fourier spectra, which can decrease the computational complexity of the implementation of the banks of matched filters. Moreover, for the parameters proposed in this paper, the new construction is essentially different from the general construction of optimal IF-ZCZ sequence sets given by Popović.

Index Terms. Sequences, Correlation, Zero correlation zone, Interference-free, Finite Zak transform, Fourier transform.

1 Introduction

Sequence set with good correlation is of considerable interest in many applications in communication and radar system. The ideal sequence set should have perfect impulse-like auto-correlation as well as all-zero cross-correlation for all pairs of sequences. Unfortunately, the famous Welch bound [23] implies that it is impossible to have impulse-like autocorrelation functions and all-zero cross-correlation functions simultaneously in a sequence set.

While the ideal sequence set is unattainable, an alternate compromise is to require that out-of-phase auto-correlation and cross-correlation of sequences are equal to zero in a finite zone. Such sequences are referred to as zero correlation zone (ZCZ) sequences which have found many applications in quasi-synchronous CDMA (QS-CDMA) system [7], ranging system [6], channel estimation [10] and spectrum-spreading [14]. Moreover, ZCZ sequences have been deployed as uplinked random access channel preambles in the fourth-generation cellular standard LTE [8].
Generally, an \((N, K, Z_{cz})\) ZCZ sequence set is characterized by the period of sequences \(N\), the set size \(K\) and the length of zero correlation zone \(Z_{cz}\). The Tang-Fan-Matsufuji bound \([21]\) implies that the parameters of a ZCZ sequence set must satisfy

\[ K(Z_{cz} + 1) \leq N. \]

This theoretical bound describes the tradeoff between the set size and the length of ZCZ for a fixed period of a sequence set. Increasing the length of ZCZ must be achieved at the expense of reducing the set size and vice versa. A ZCZ sequence set reaching the theoretical bound is called optimal.

It is remarkable that the above theoretical bound does not make any restrictions on the behavior of correlation functions outside ZCZ. We take into account ZCZ sequence sets with all-zero cross-correlations, which are referred to as interference-free ZCZ (IF-ZCZ) sequence sets in this paper. It’s possibly beneficial when IF-ZCZ sequences are deployed in some interference-limited systems, such as heterogeneous cellular networks (HetNet) \([19]\), multi-function radar system envisaged for autonomous cars \([18]\). For instance, an envisioned HetNet is composed of many low powered base stations (BS) within the coverage area of conventional BS. With the increasing density of BS in a HetNet, the signals from multiple BS in its neighborhood are superposed to the mobile terminal (MT) leading to severe interference-limited performance. MT may attempt to connect to the closest low powered BS but the strongest signal may come from a conventional BS. On account of some given scenarios of HetNet, the delays of conventional cell signals may lay in a correlation zone whose length is hard to predict. IF-ZCZ sequence set with proper parameters is a good candidate for these scenarios.

A series of papers are devoted to the design of IF-ZCZ sequence sets. In \([15, Th.4]\), Matsufuji et al. proposed optimal \((KM, K, M - 1)\) IF-ZCZ sequence sets with restriction that \(K\) and \(M\) must be relatively prime. In \([16, Th.4]\), Mow proposed optimal \((KM, K, M - 1)\) IF-ZCZ sequence set, where \(M\) must be a square-free integer. Recently in \([24, Construction 2]\), Zhang indicated that it is not necessary for \(M\) to be square-free. In \([2, Construction A]\), Brozik proposed suboptimal IF-ZCZ sequence sets through the theory of finite Zak transform (FZT). All sequences in these sets have period \(N = M^3\) and set size \(M\). If \(M\) is a prime, this construction is optimal with \(Z_{cz} = M^2 - 1\). If \(M\) is not a prime, \(Z_{cz} = TM - 1\), where \(T\) is an arbitrary nontrivial factor of \(M\). An extension of construction A, called construction A’, was also proposed in \([2, Construction A’]\). The IF-ZCZ sequence sets in construction A’ have parameters \((KM^2, K, TM - 1)\), where \(T\) is a nontrivial factor of \(M\). In \([3, Construction B]\), Brozik proposed an optimal construction of \((KM, K, M - 1)\) IF-ZCZ sequence sets by FZT. In \([17, Equ.2]\), Popović presented a general construction including all aforementioned constructions of optimal IF-ZCZ sequence sets.

The Zak transform is named by Zak, who studied it systematically for applications in solid-state physics \([25]\). After Janssen’s work \([11]\), it began to be used in digital signal processing. Meanwhile,
the Zak transform has been used for numerous applications in signal and echo analysis, ambiguity function [20], Weyl-Heisenberg expansions [12,13,22], and the design of sequence [11,14]. Moreover, the recently proposed orthogonal time frequency space (OTFS) modulation technique [9] multiplexes QAM information in Zak space.

Inspired by the excellent idea given in [2], we propose a new construction of optimal \((KM^2, K, M^2 - 1)\) IF-ZCZ sequence sets. The size and the period of sequences in new construction are both the same as those in [2]. Note that the Zak space constructions of IF-ZCZ sequence sets in [2] are not optimal except for a special case. The optimality of the proposed construction in this paper is guaranteed by a newly designed lattice tessellation in Zak space.

In concluding remarks of [17], it concluded that the general construction in [17] is the “most general possible construction of the sequence set having jointly the properties of all-zero cross-correlation, zero autocorrelation zone, complementarity, and optimality”. The proposed construction in this paper shares the same properties, but it is different from the general construction [17] of optimal IF-ZCZ sequence sets from the perspective of FZT. For general construction [17], the non-zero values of Zak spectrum of any sequence are all in a certain row, which depends on the index of sequence. For our new construction, there is only one non-zero element in each column of Zak spectra, and the rows with non-zero elements are equidistantly positioned. Moreover, the sparse and highly structured Zak and Fourier spectra of the new construction can decrease the computational complexity of the implementation of the banks of matched filters by FZT algorithm [2] and DFT algorithm [17]. Besides, the alphabet size in the general construction [17] cannot be smaller than the period of the sequences, while the alphabet size of the new construction here can be a factor of the period of the sequences.

The rest of this paper is organized as follows. The basic definitions and essential notations will be introduced in Section 2. In Section 3 we propose a new construction of optimal IF-ZCZ sequence sets based on constant magnitude sequences, and give some properties of the new construction. In Section 4, we give the Zak spectra analysis of known constructions and the comparisons of the new construction with known constructions. Finally, Section 5 concludes the paper.

2 The Basic Definitions and Notations

In this section, we introduce some basic definitions and notations of IF-ZCZ sequence set and Zak transform. Throughout the paper, \(\omega_N := e^{-2\pi \sqrt{-1}/N}\) is the \(N\)-th root of unity.

2.1 ZCZ Sequence Set

Let \(\mathcal{S} = \{u_a : 1 \leq a \leq K\}\) be a sequence set with \(K\) complex sequences of period \(N\). We first define the correlation of sequences.
**Definition 1.** Let \( u_a = (u_a(0), u_a(1), \ldots, u_a(N-1)) \) and \( v_b = (u_b(0), u_b(1), \ldots, u_b(N-1)) \) be two complex sequences of length \( N \). The cross-correlation between \( u_a \) and \( u_b \) at shift \( n \) is defined by

\[
\theta_{a,b}(n) = \sum_{m=0}^{N-1} u_a(m) u_b^*(m-n), \quad 0 \leq n \leq N-1
\]

where \( m-n \) is taken modulo \( N \) and the symbol \( * \) denotes the complex conjugation. When \( a = b \), the auto-correlation of \( u_a \) at shift \( n \) is denoted by \( \theta_a(n) = \theta_{a,a}(n) \).

The length \( Z_{cz} \) of zero correlation zone (ZCZ) of the set \( S \) is defined by

\[
Z_{cz} = \max \{ T | \theta_{a,b}(n) = 0 \ (0 \leq |n| \leq T \text{ for } a \neq b \text{ and } 0 < |n| \leq T \text{ for } a = b \} \}
\]

\( S \) is referred to as an \((N, K, Z_{cz})\)-ZCZ sequence set. It is clear \( |\theta_{u,v}(-n)| = |\theta_{v,u}(n)| \). To determine the value of \( Z_{cz} \), it suffices to compute the correlation function of sequence set \( S \) with \( 0 \leq n \leq N-1 \), i.e.,

\[
Z_{cz} = \max \{ T | \theta_{a,b}(n) = 0 \ (0 \leq n \leq T \text{ for } a \neq b \text{ and } 0 < n \leq T \text{ for } a = b \} \}
\]

The parameters of a ZCZ sequence set must satisfy the following theoretical bound.

**Fact 1.** (Tang-Fan-Matsufuji bound [21]) For any \((N, K, Z_{cz})\)-ZCZ sequence set, we have

\[
Z_{cz} \leq \frac{N}{K} - 1.
\]

Furthermore, an \((N, K, Z_{cz})\)-ZCZ sequence set is said to be optimal if \( Z_{cz} = \frac{N}{K} - 1 \).

**Definition 2.** A sequence set \( S \) is referred to as an interference-free (IF) sequence set if the cross-correlation of any two distinct sequences in \( S \) is always zero.

An \((N, K, Z_{cz})\)-ZCZ sequence set \( S \) is referred to as \((N, K, Z_{cz})\) IF-ZCZ sequence set if the cross-correlation of any two distinct sequences in \( S \) is always zero. It is obvious that the parameters of an IF-ZCZ sequence set must also satisfy the above Tang-Fan-Matsufuji bound.

### 2.2 Finite Zak Transform

In addition to the traditional method to study the sequences in time and Fourier space, another approach to study the sequences in Zak space was proposed in [23].

Suppose for the remainder of this paper that \( N = LM \), where \( L \) and \( M \) are positive integers, and set \( n = rM + k \), \( m = iL + j \) for \( 0 \leq k, i \leq M - 1, 0 \leq r, j \leq L - 1 \).
Definition 3. The $L \times M$ finite Zak transform (FZT) [25] of sequence $u = (u(0), u(1), \cdots, u(N-1))$ is defined by

$$U(j, k) = \sum_{r=0}^{L-1} u(rM + k)\omega^r_L. \quad (3)$$

If we set $U(j, k)$ as the entry of an $L \times M$ matrix $U$ at row $j$ and column $k$, it is much simple to understand the FZT by the product of the matrices. The sequence $u$ can be re-expressed by an $L \times M$ matrix $A^u$, where the entry $A^u(r, k) = u(rM + k)$, i.e,

$$A^u = \begin{pmatrix}
  u(0) & u(1) & \cdots & u(M-1) \\
  u(M) & u(M+1) & \cdots & u(2M-1) \\
  \vdots & \vdots & \ddots & \vdots \\
  u(N-M) & u(N-M+1) & \cdots & u(N-1)
\end{pmatrix}.$$

Let $F$ be the DFT matrix of order $L$. It is straightforward that

$$U = F \cdot A^u.$$ 

If $L = N$ and $M = 1$, the FZT is identical to the DFT. If $L = 1$ and $M = N$, the FZT of $u$ is identical to the original sequence $u$. Thus, the FZT is a primary time-frequency representation which can concurrently encodes both the time and the frequency information about a sequence.

Similar to the inverse DFT, we can define the inverse FZT, where

$$u(rM + k) = L^{-1} \sum_{j=0}^{L-1} U(j, k)\omega^{-rj}_L. \quad (4)$$

The sequence $u$ can be recovered from the inverse FZT.

2.3 FZT and Correlation of Sequences

The relationship between the Zak spectra and the Fourier spectra of the sequence $u$ can be given by the following formula.

$$\hat{u}(iL + j) = \sum_{k=0}^{M-1} U(j, k)\omega^{jk}_N\omega^{jk}_M. \quad (5)$$

For sequences $u_a$ and $u_b$ of length $N$, their cross-correlation $\theta_{a,b}(n)$ at shift $n$ can be calculated by Definition 1. We take $\{\theta_{a,b}(0), \theta_{a,b}(1), \cdots, \theta_{a,b}(N-1)\}$ as a sequence of length $N$, denoted by $\theta_{a,b}$. The relationship between the Zak spectra of $u_a$ and $u_b$ and the Zak spectra of their correlation sequence $\theta_{a,b}$ is given below.
Fact 2. (\[2-5\]) Let $U_a$, $U_b$ and $\Theta_{a,b}$ be the FZTs of sequences $u_a$, $u_b$, $\theta_{a,b}$, respectively. Then we have

$$\Theta_{a,b}(j,k) = \sum_{l=0}^{M-1} U(j,l)V^\ast(j,l-k). \quad (6)$$

Note that $l - k$ in formula (6) may be less than 0, which is not well defined. On the other hand, by extending the definition of the FZT, it is shown that FZT is quasi-periodic in the time variable \[2\], i.e,

$$U(j, k + M) = \omega_{L}^{-j} U(j,k). \quad (7)$$

It is much clear to re-express the formula (6) as following:

$$\Theta_{a,b}(j,k) = \omega_{L}^{-j} \sum_{l=0}^{k-1} U(j,l)V^\ast(j,l - k + M) + \sum_{l=k}^{M-1} U(j,l)V^\ast(j,l - k). \quad (8)$$

From Fact 2, the correlation of the sequences can be studied by the FZT. Several Zak space constructions of IF-ZCZ sequence sets were proposed in \[2\]\[3\], where the ideas are based on the following observations.

1. For the case $a \neq b$, if $\Theta_{a,b}$ is a zero matrix, $\theta_{a,b}$ must be a zero sequence, i.e., $u_a$ and $u_b$ must be interference-free.

2. For the case $a = b$, we use $\theta_a$ and $\Theta_a$ to denote the sequence of auto-correlation $\theta_{a,a}$ and its FZT $\Theta_{a,a}$ respectively. If the entries of $\Theta_a$ are all zeroes except the first column, the length of ZCZ of sequences $u$ must be equal to $TM - 1$ where $T$ is a positive integer. The optimality of IF-ZCZ sequence sets is depended on the value $T$.

Note that the Zak space constructions of IF-ZCZ sequence sets in \[2\] are not optimal except for a special case. We will propose a new construction of optimal IF-ZCZ sequence set, in which the size and the length of sequences are both the same as those in \[2\]. The optimality of the new construction is guaranteed by a newly designed lattice tessellation in Zak space.

3 Main Construction

By extending the excellent idea in \[2\], we propose a construction of optimal $(KM^2, K, M^2 - 1)$ IF-ZCZ sequence sets in this section. we first show the properties of sequences in the new construction, and then give the detailed proof.
3.1 Main results

For positive integers $K$ and $M$, $\mathcal{S}$ is a set containing $K$ sequences with period $N = KM^2$. Each sequence in $\mathcal{S}$ is based on the following constant magnitude sequence and permutation:

(1) $c_a = (c_a(0), c_a(1), \cdots, c_a(M-1))$ is a complex sequence with unit magnitude elements of period $M$ for $1 \leq a \leq K$.

(2) $\pi_a$ is an arbitrary permutation of the set $\{0, 1, \cdots, M-1\}$ for $1 \leq a \leq K$.

Let $u_a$ be the $a$th sequence of $\mathcal{S}$, whose $n$th ($n = rM + k$) element is defined as

$$u_a(rM + k) = c_a(k)\omega_{KM}^{r(K\pi_a(k)+a)}$$

for $0 \leq r \leq KM - 1$ and $0 \leq k \leq M - 1$.

**Theorem 1.** $\mathcal{S}$ is an optimal $(N, K, M^2 - 1)$ IF-ZCZ sequence set.

Zak spectra of sequences in $\mathcal{S}$ are given below.

**Theorem 2.** Sequence $u_a$ in set $\mathcal{S}$ has sparse and highly structured $KM \times M$ Zak spectra:

$$U_a(j, k) = \begin{cases} KMc_a(k), & \text{if } j = KM - K\pi_a(k) - a, \\ 0, & \text{else} \end{cases}$$

where $0 \leq j \leq MK - 1$, $0 \leq k \leq M - 1$.

From Zak spectra in Theorem 2, the correlation of the sequences can be determined by the following results.

**Corollary 1.** For sequence $u_a$ and $u_b$ in $\mathcal{S}$ and shift $n$, we have the cross-correlation

$$\theta_{a,b}(n) = 0$$

for $\forall n$, and the auto-correlation

$$\theta_a(n) = \begin{cases} N\omega_K^{\frac{an}{M}}, & \text{if } M^2 \mid n, \\ 0, & \text{else}. \end{cases}$$

Fourier spectra of sequence $u_a$ can be obtained by Zak spectra in Theorem 2 and formula (5).

**Corollary 2.** Fourier spectra of sequence $u_a$ in set $\mathcal{S}$ are determined as following.

$$\hat{u}_a(iKM + j) = \begin{cases} KMc_a(\pi_a^{-1}(M - \frac{i+a}{K})\omega_N^{(iKM+j)\pi_a^{-1}(M\frac{j+a}{K})}), & \text{if } K \mid (j + a), \\ 0, & \text{else} \end{cases}$$

where $0 \leq i \leq M - 1$ and $0 \leq j \leq MK - 1$. 


Remark 1. If we choose the sequence $c_a$ such that $c_a(k) = 1$ for all $a$ and $k$, then the $n$th element of sequence $u_a$ in $S$ can be written in the form

$$u_a(n = rM + k) = \omega^{r(K\pi_a(k) + a)}_{KM}$$

for $0 \leq r \leq KM - 1$ and $0 \leq k \leq M - 1$. In this case, the alphabet size of sequences in $S$ is $KM$ and every sequence in $S$ has binary Zak spectra. The sparse and binary support of the Zak transform facilitates sequence storage.

From Theorem 2 and Corollary 2, it can be easily seen that both the Zak and Fourier spectra of sequences in $S$ are sparse and with certain structure, but the expression of the non-zero Zak spectra is much simpler than that of Fourier spectra. This is the reason why FZT is employed to study the IF-ZCZ sequence set, instead of DFT. We use the following example to illustrate the Zak spectra of sequences in our construction.

Example 1. Let $K = 2, M = 4, N = 32$, $c_1 = c_2 = (1, 1, 1, 1)$, $\pi_1(k) = k$ and $\pi_2(0, 1, 2, 3) = (1, 3, 2, 0)$. There are two sequences in the set $S$:

$$u_1 = (\omega^0_8, \omega^0_8, \omega^2_8, \omega^2_8, \omega^0_8, \omega^0_8, \omega^3_8, \omega^3_8, \omega^0_8, \omega^0_8, \omega^2_8, \omega^2_8, \omega^0_8, \omega^0_8, \omega^1_8, \omega^1_8, \omega^7_8, \omega^7_8),$$

$$u_2 = (\omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^0_4, \omega^1_4, \omega^1_4, \omega^1_4, \omega^1_4, \omega^1_4, \omega^1_4, \omega^1_4, \omega^1_4).$$

From Theorem 2, their Zak spectra can be respectively given below.

$$U_1 = 8 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix}^T, \quad U_2 = 8 \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

The FZTs of cross-correlation $\theta_{1,2}$ and auto-correlation $\theta_1$ and $\theta_2$ are given as following.

$$\Theta_{1,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad \Theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad \Theta_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$
By the inverse FZT, we can obtain the cross-correlation $\theta_{1,2}$ and the auto-correlation $\theta_1$ and $\theta_2$ from $\Theta_{1,2}$, $\Theta_1$, and $\Theta_2$, respectively:

$$\theta_{1,2} = 0,$$

$$\theta_1 = 32(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$

$$\theta_2 = 32(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0).$$

It is clear that the sequence set $\{u_a \mid 1 \leq a \leq 2\}$ is an optimal $(32,2,15)$ IF-ZCZ sequence set.

**Remark 2.** Note that the IF-ZCZ set proposed [2, Example 3] cannot reach the Tang-Fan-Matsufuji bound with the sequence length $N = 32$ and set size $K = 2$, while the sequence set in Example 2 from our construction is optimal with the same length and size parameters.

Similar to the IF-ZCZ sequence set given in [2,17], the construction proposed here is also a periodically complementary sequence set.

**Corollary 3.** $S$ is a periodically complementary sequence set, i.e.,

$$\sum_{a=1}^{K} \theta_2(n) = 0$$

for $0 < n < N$.

### 3.2 Proofs for the Main results

Let $L = KM$ in this subsection. We first compute the $L \times M$ FZT of the sequence $u_a$ in our construction.

**Proof of Theorem 2.** By the definition of $L \times M$ FZT, we have

$$U_a(j,k) = \sum_{r=0}^{L-1} s_a(rM+k)\omega_L^{rj} = c_a(k)\sum_{r=0}^{L-1}\omega_L^{r(K\pi_a(k)+a+j)}$$

$$= Lc_a(k)\delta(j + K\pi_a(k) + a - L).$$
for $0 \leq j \leq L - 1, 0 \leq k \leq M - 1$, where $\delta(\cdot)$ is the delta function such that $\delta(0) = 1$ and $\delta(j) = 0$ for $j \neq 0$. \hfill \Box$

Note that for sequence $u_a$, the positions $(j, k)$ of the non-zero elements in matrix $U_a$ must satisfy $j + K\pi_a(k) + a = L$. In other words, the entries of $U_a(j, k)$ are all zeroes except $U_a(j, \pi_a^{-1}(M - \frac{j + a}{K}))$ for $K \mid (j + a)$ and $0 \leq j \leq L - 1$.

Moreover, for each column $k$, there is a unique $j$ such that

$$j = L - K\pi_a(k) - a,$$

so there is only one non-zero element in each column of matrix $U_a$. For each row $j$, there is a unique non-zero element in this row if and only if $K \mid (j + a)$. Otherwise, the elements in row $j$ must be all zeroes.

**Proof of Corollary 1.** The FZT of the cross-correlation of sequences $u_a$ and $u_b$ can be calculated by applying the formula (8).

The row $j$ with non-zero element of Zak spectra matrix $U_a$ must satisfy $K \mid (j + a)$. If $a \neq b$, it is clear $\{j | j + a \equiv 0 \pmod{K}\} \cap \{j | j + b \equiv 0 \pmod{K}\} = \emptyset$, we obtain

$$\Theta_{a,b}(j, k) = 0$$

for $0 \leq j \leq L - 1, 0 \leq k \leq M - 1$. By applying the inverse FZT in (4), we obtain the cross-correlation of $u_a$ and $u_b$:

$$\theta_{a,b} = 0.$$

Thus the sequence set $S$ must be interference free.

If $a = b$, we have

$$\Theta_a(j, k) = \sum_{l=0}^{M-1} U_a(j, l)U_a^*(j, l - k)$$

by Fact 2.

If $K \nmid (j + a)$, we immediately have $\Theta_a(j, k) = 0$, since $U_a(j, l) = 0$ for $\forall l$.

If $K \mid (j + a)$ and $k \neq 0$, there is only one non-zero element in $j$th row, so the product of $U_a(j, l)$ and $U_a(j, l - k)$ must be zero for $k \neq 0$. Thus we have

$$\Theta_a(j, k) = 0.$$

If $K \mid (j + a)$ and $k = 0$, by applying formula (14), we have

$$\Theta_a(j, 0) = \sum_{l=0}^{M-1} S_a(j, l)S_a^*(j, l) = L^2.$$
Now we can calculate the periodic autocorrelation of $u_a$ by inverse FZT in (4), i.e.,

$$\theta_a(rM + k) = L^{-1} \sum_{j=0}^{L-1} \Theta_a(j, k)\omega^{-rj}. \tag{11}$$

For $k \neq 0$, the fact $\Theta_a(j, k) = 0$ for $\forall j$ leads to

$$\theta_a(rM + k) = 0.$$

For $k = 0$, since $\Theta_a(j, 0)$ equals $L^2$ if $j = (l + 1)K - a$ for $0 \leq l \leq M - 1$ and 0 otherwise, we have

$$\theta_a(rM + 0) = L^{-1} \sum_{j=0}^{L-1} \Theta_a(j, 0)\omega^{-rj}$$

$$= L \sum_{l=0}^{M-1} \omega^{-r(lK+K-a)}$$

$$= L\omega^{ra} \sum_{l=1}^{M} \omega^{-rl}. \tag{13}$$

Thus,

$$\theta_a(rM + k) = \begin{cases} N\omega^{ra}, & \text{if } k = 0 \text{ and } M \mid r, \\ 0, & \text{else}. \end{cases}$$

which complete the proof. \hfill \Box

From Corollary 1, we know the length of ZCZ in $S$ is $M^2 - 1$, which is optimal with respect to the Tang-Fan-Matsufuji bound, so Theorem 1 is proved.

**Proof of Corollary 2** Fourier spectra of sequences in $S$ can be determined by their Zak spectra in (13) and the relationship between the Zak and Fourier spectra in (5):

$$\hat{s}_a(iL + j) = \sum_{k=0}^{M-1} S_a(j, k)\omega_N^{jk}\omega_M^{ik}$$

$$= \sum_{k=0}^{M-1} Lc_a(k)\delta(j + K\pi_a(k) + a - L)\omega_N^{jk}\omega_M^{ik}. \tag{15}$$

If $K \nmid (j + a)$, we have $\delta(j + K\pi_a(k) + a - L) = 0$ for $\forall k$, so it is clear

$$\hat{s}_a(iL + j) = 0.$$

If $K \mid (j + a)$, we have $\delta(j + K\pi_a(k) + a - L) = 1$ if and only if $\pi_a(k) = M - \frac{j + a}{K}$. Then we obtain the non-zero Fourier spectra:

$$\hat{s}_a(iL + j) = Lc_a(\pi_a^{-1}(M - \frac{j + a}{K}))\omega_N^{((iL+j)\pi_a^{-1}(M - \frac{j + a}{K})}. \tag{16}$$
which complete the proof. □

Proof of Corollary 5. Let \( r = r'M \) if \( M \mid r \). By applying formula (13), we have

\[
\sum_{a=0}^{K-1} \theta_a(rM + k) = N \delta(k) \sum_{r'=0}^{K-1} \omega_L^{(r'M)a}
\]

\[
= N \delta(k) \sum_{r'=0}^{K-1} \omega_K^{r'a}
\]

\[
= NK \delta(k) \delta(r')
\]

which concludes that \( S \) is a periodically complementary sequence set. □

4 Zak Spectra Analysis of Known Constructions and Comparisons

There were several constructions of the optimal IF-ZCZ sequence sets. In particular, Popović [17] proposed a general construction including all the known optimal \((KM, K, M - 1)\) IF-ZCZ sequence sets introduced in [3, 15, 16, 24]. We will give Zak spectra analysis of the sequences in [17] in this section. The results show that the lattice tessellations in Zak space of the known general construction is different from our new construction.

4.1 Zak Spectra Analysis of Known Constructions

The general construction [17] can be re-expressed in the following manner. For positive integer \( K \) and \( M \), each sequence \( v_a \) is based on the following perfect sequence and functions:

1. \( h_a = (h_a(0), \ldots, h_a(M - 1)) \) is an arbitrary perfect sequence of length \( M \) for \( 0 \leq a \leq K - 1 \).
2. \( \pi \) is an arbitrary permutation of the set \( \{0, 1, \ldots, K - 1\} \).
3. \( f \) is a function from the set \( \mathbb{Z}_K \) to the set \( \mathbb{Z}_N \) such that \( f(a) \equiv \pi(a) \ (\text{mod } K) \).

The \( n \)th element of sequence \( v_a \) is defined as

\[
v_a(n = rM + k) = h_a(k) \omega_N^{f(a)k + rM\pi(a)}, \tag{14}
\]

for \( 0 \leq k \leq M - 1 \) and \( 0 \leq a, r \leq K - 1 \).
Remark 3. If we set \( f(a) = ea + t \) for \( \gcd(e, K) = 1 \) and \( \pi(a) \equiv f(a) \pmod{K} \), where \( \gcd \) denotes the greatest common divisor, (14) can be re-expressed by

\[
v_a(n) = h_a(n \mod M)\omega_N^{(ea+t)n},
\]

which is exactly the expression of sequences in [17].

Theorem 3. Sequence \( v_a \) has sparse and highly structured \( K \times M \) Zak spectra:

\[
V_a(j, k) = \begin{cases} 
K h_a(k) \omega_N^{f(a)k}, & \text{if } K \mid (\pi(a) + j), \\
0, & \text{else}
\end{cases}
\]

for \( 0 \leq j \leq K - 1, 0 \leq k \leq M - 1 \).

Proof. By the definition of \( K \times M \) FZT, we have

\[
V_a(j, k) = \sum_{r=0}^{K-1} v_a(rM + k)\omega_K^r \\
= \sum_{r=0}^{K-1} h_a(k)\omega_N^{f(a)k+rM}\omega_K^r \\
= h_a(k)\omega_N^{f(a)k} \sum_{r=0}^{K-1} \omega_K^r(\pi(a)+j).
\]

The result follows that \( \sum_{r=0}^{K-1} \omega_K^r(\pi(a)+j) \) equals \( K \) if \( K \mid (\pi(a) + j) \) and 0 otherwise.

From Theorem 3, the elements of \( j \)th row in Zak spectra matrix of sequence \( v_a \) are all non-zeroes if \( K \mid (\pi(a) + j) \), and the elements are all zeroes for other rows. Thus the lattice tessellations in Zak space of the known general construction are different from our new construction.

Similar to our new construction, the properties of sequences \( v_a \) (\( 0 \leq a \leq K - 1 \)) can be well analyzed by their Zak spectra and Fourier spectra. Since they have been well studied in [17], we just list the properties without the proof as follows.

Fourier spectra of sequence \( v_a \):

\[
\hat{v}_a(iK + j) = \begin{cases} 
K \hat{\varphi}_a(i), & \text{if } K \mid (\pi(a) + j), \\
0, & \text{else}
\end{cases}
\]

where \( 0 \leq j \leq K - 1, 0 \leq i \leq M - 1 \), \( \varphi_a(k) = h_a(k)\omega_M^{(\pi(a) + j)k} \) is also a perfect sequence of length \( M \), and \( \hat{\varphi} \) is the DFT of \( \varphi \).
Cross-correlation of sequence \( v_a \):

\[
\theta_{v_a}(n) = \begin{cases} 
N \omega_{M}^{\frac{an}{M}} & \text{if } M \mid n, \\
0 & \text{else}.
\end{cases}
\]

**Remark 4.** It is clear that the set \( \{ v_{\pi(a)} \} \) is identical to the set \( \{ v_a \} \) for \( 0 \leq a \leq K - 1 \), so we can always set \( \pi(a) = a \) and \( f(a) = t_{a}K + a \) in (14) by the following expression.

\[
v_a(n = rM + k) = h_a(k)\omega_{N}^{(t_{a}K+a)k+rMa} = (h_a(k)\omega_{M}^{t_{a}k})\omega_{N}^{an}.
\]

Since \( \{ h_a(k)\omega_{M}^{t_{a}k} \} \) is also a perfect sequence of length \( M \), sequences in general construction [17] can be simplified by

\[
v_a(n = rM + k) = h_a(k)\omega_{N}^{an}.
\]

The following example is used to illustrate the Zak spectra of sequences \( v_a \) \( (0 \leq a \leq K - 1) \).

**Example 2.** Let \( K = M = 4, N = 16, f(a) = \pi(a) = a, \) and \( h_a = (1, 1, 1, -1) \) for \( 0 \leq a \leq 3 \). There are four sequences in the set:

\[
\begin{align*}
v_0 & = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\
v_1 & = (0_1^{16}, 1_1^{16}, 2_1^{16}, 3_1^{16}, 4_1^{16}, 5_1^{16}, 6_1^{16}, 7_1^{16}, 8_1^{16}, 1_1^{16}, 2_1^{16}, 3_1^{16}, 4_1^{16}, 5_1^{16}, 6_1^{16}), \\
v_2 & = (0_8^{16}, 1_8^{16}, 2_8^{16}, 3_8^{16}, 4_8^{16}, 5_8^{16}, 6_8^{16}, 7_8^{16}, 8_8^{16}, 1_8^{16}, 2_8^{16}, 3_8^{16}, 4_8^{16}, 5_8^{16}, 6_8^{16}), \\
v_3 & = (0_8^{16}, 1_8^{16}, 2_8^{16}, 3_8^{16}, 4_8^{16}, 5_8^{16}, 6_8^{16}, 7_8^{16}, 8_8^{16}, 1_8^{16}, 2_8^{16}, 3_8^{16}, 4_8^{16}, 5_8^{16}).
\end{align*}
\]

From Theorem 3, Zak spectra of the above sequences can be respectively given below.

\[
V_0 = 4 \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad V_1 = 4 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & \omega_1^{16} & \omega_2^{16} & \omega_3^{16}
\end{bmatrix},
\]

\[
V_2 = 4 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \omega_1^{8} & \omega_2^{8} & \omega_3^{8} \\
1 & \omega_1^{8} & \omega_2^{8} & \omega_3^{8}
\end{bmatrix}, \quad V_3 = 4 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \omega_1^{3} & \omega_2^{6} & \omega_3^{14} \\
1 & \omega_1^{16} & \omega_2^{16} & \omega_3^{16} \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
The FZT of auto-correlation of the above sequences can be respectively given by formula (8):

\[
\Theta_0 = 64 \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \Theta_1 = 64 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

\[
\Theta_2 = 64 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \Theta_3 = 64 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

### 4.2 Comparisons with Known Constructions

There are two methods to construct optimal IF-ZCZ sequence sets in the literature.

The first method focuses on sequences in time and Fourier space \([15 - 17, 24]\). It has been shown that all these constructions are special cases of the optimal \((KM, K, M - 1)\) IF-ZCZ sequence sets given in \([17]\).

Another method considers the sequences in Zak space \([2, 3]\). It has been shown that all the sequence sets in \([3]\) were also special cases of the construction in \([17]\). The construction A in \([2]\) gave an \((M^3, M, TM - 1)\) IF-ZCZ sequence set where \(T = M\) if and only if \(M\) is prime, so this construction is optimal if and only if \(M\) is a prime. An extension of construction A, called construction A’, was also proposed in \([2]\). The IF-ZCZ sequence set in construction A’ have parameters \((KM^2, K, TM - 1)\) where \(T\) is a nontrivial factor of \(M\).

Comparisons on the expression, period, set size, length of the ZCZ and the optimality of the sequences constructed in \([2, 17]\) and this paper are shown in Table 1.

It is clear that new construction in this paper is optimal, while both the constructions A and A’ are suboptimal in \([2]\), since \(T\) is a nontrivial factor of \(M\). Note that the sequences in this paper are directly given in the explicit form, while the constructions in \([2]\) were shown in Zak space. It can be easily shown in Zak space that the optimal case in \([2]\) with parameters \((M^3, M, M^2 - 1)\) for \(M\) prime is actually a special case of our new construction.

From the expressions of sequences and the FZT lattice tessellation shown in the previous section, it can be seen that the construction in this paper is essentially different from the general construction in \([17]\). In addition, the alphabet size in general construction \([17]\) cannot be smaller than the period of the sequences. We have shown in Remark \([1]\) that the alphabet size of the new construction can be \(KM\) while the length of the sequence is \(KM^2\). We have also explained in Remark \([1]\) that the sparse and binary support of the Zak spectra facilitates sequence storage for our new construction.
Table 1: The Comparisons With Known IF-ZCZ Sequence Sets

| Set $s_a(n)$ | Construction A in [2] | Construction A’ in [2] | New Construction |
|--------------|------------------------|------------------------|------------------|
| $h_a(k)\omega_N^{\alpha n} + 1$ | **                     | **                     | $c_a(k)\omega_{KM}^{r(K\pi_a(k) + a)}$ |
| Period $N$   | $KM$                   | $M^3$                  | $KM^2$           |
| Size         | $K$                    | $M$                    | $K$              |
| ZCZ          | $M - 1$                | $TM - 1^{(*)}$, $M^2 - 1$, $M$ nonprime | $TM - 1^{(*)}$, $M^2 - 1$, $M$ prime |
| IF           | Yes                    | Yes                    | Yes              |
| Optimal      | Yes                    | No                     | Yes              |

- *: $T$ is a nontrivial factor of $M$.
- **: Both the construction A and A’ were shown in Zak space. The explicit form of the sequences were not available.

5 Concluding Remarks

In this paper, we proposed a new construction of optimal IF-ZCZ sequence sets with parameters $(KM^2, K, M^2 - 1)$. FZT was employed to study both the new construction and the general construction [17]. Different FZT lattice tessellations in Zak space make a distinction between these two constructions.

The theory of FZT was initially applied to the design of chirp sequences by Brozik [1, 4, 5]. Two Zak space constructions of IF-ZCZ sequence set [2, 3] were given by Brozik recently. Although it has been shown that the results in [3, Construction B] are special cases of the general construction [17] and, the construction in [2] is suboptimal, the idea shown in [2] is important to construct new optimal IF-ZCZ sequence set.

We believe FZT is a powerful tool to study sequences with ZCZ. Our new construction was obtained by choosing a well designed FZT lattice tessellation. And the general construction [17] can also be obtained in the same manner. Intuitively, new sequence sets with ZCZ may be constructed by a deep study of sequences in Zak space.

References

[1] M. An, A. K. Brodzik, and R. Tolimieri, “Ideal sequence design in time frequency space, applications to radar, sonar and communication systems,” in Applied and Numerical Harmonic Analysis, Boston, MA, USA: Birkhäuser, 2008.

[2] A. K. Brodzik, “On certain sets of polyphase sequences with sparse and highly structured Zak and Fourier transforms,” IEEE Trans. Inf. Theory., vol. 59, no. 10, pp. 6907-6916, Oct. 2013.
[3] A. K. Brodzik, “Polyphase Golay sequences with semi-polyphase Fourier transform and all-zero crosscorrelation: Construction B,” in Excursions in Harmonic Analysis, vol. 3. Basel, Switzerland: Birkhäuser, 2015, pp. 211-229. [Online]. Available: http://www.springer.com/gp/book/9783319132297.

[4] A. K. Brodzik and R. Tolimieri, “Bat chirps with good properties: Zak space construction of perfect polyphase sequences,” IEEE Trans. Inf. Theory., vol. 55, no. 4, pp. 1804-1814, Apr. 2009.

[5] A. K. Brodzik, “Characterization of Zak space support of the discrete chirp,” IEEE Trans. Inf. Theory., vol. 53, no. 6, pp. 2190-2203, Jun. 2007.

[6] J. D. Coker and A. H. Tewfik, “Simplified ranging systems using discrete wavelet decomposition,” IEEE Trans. Signal Process., vol. 58, no. 2, pp. 575-582, Feb. 2010.

[7] P. Z. Fan, “Spreading sequence design and theoretical limits for quasisynchronous CDMA systems,” EURASIP J. Wireless Commun. Netw., vol. 2004, no. 1, pp. 19-31, 2004.

[8] 3rd generation partnership project, technical specification group radio access network, Evolved Universal Terrestrial Radio Access (E-UTRA), Physical Channels and Modulation, document 3GPP TS 36.211, v14.3.0, Jun. 2017, sec. 5.7.2, p. 69.

[9] R. Hadani, and A. Monk, “OTFS: A new generation of modulation addressing the challenges of 5G,” OTFS PhysicsWhite Paper, Cohere Technologies, 7 Feb. 2018. Available online: https://arxiv.org/pdf/1802.02623.pdf.

[10] K. M. Z. Islam, T. Y. Al-Naffouri, and N. Al-Dhahir, “On optimum pilot design for comb-type OFDM transmission over doubly-selective channels,” IEEE Trans. Commun., vol. 59, no. 4, pp. 930-935, Apr. 2011.

[11] A. J. E. M. Janssen, “The Zak transform: A signal transform for sampled time-continuous signals,” Philips J. Res., vol. 43, pp. 23-69, 1988.

[12] A. J. E. M. Janssen, H. Feichtinger and T. Strohmer, Eds., “Zak transforms with few zeros and the tie,” Advances in Gabor Analysis pp. 31-70, 2002.

[13] A. J. E. M. Janssen, “On generating tight Gabor frames at critical density,” J. Fourier Anal. Applicat, vol. 9, pp. 175-214, 2003.

[14] X. Y. Jiang, “Code hopping communications for anti-interception with real-valued QZCZ sequences,” IEEE Trans. Commun., vol. 59, no. 3, pp. 680-685, Mar. 2011.
[15] S. Matsufuji, N. Kuroyanagi, N. Suehiro, and P. Z. Fan, “Two types of polyphase sequence sets for approximately synchronized CDMA systems,” IEICE TRANS. Fundam. Electron, Commun. Comput. Sci., vol. E86-A, no. 1, pp. 229-234, 2003.

[16] W. H. Mow, “A new unified construction of perfect root-of-unity sequences,” in Proc. IEEE 4th Int. Symp. Spread Spectr. Techn. Appl. (ISSSTA), Sep. 1996, pp. 955-959.

[17] B. M. Popović, “Optimum sets of interference-free sequences with zero autocorrelation zones”, IEEE Trans. Inf. Theory., vol. 64, no. 4, pp. 1406-1409, Apr 2018.

[18] S. M. Patole, M. Torlak, D. Wang, and M. Ali, “Automotive radars: A review of signal processing techniques,” IEEE Signal Process. Mag., vol. 34, no. 2, pp. 22-35, Mar. 2017.

[19] N. Rajamohan and A. P. Kannu, “Downlink synchronization techniques for heterogeneous cellular networks,” IEEE Trans. Commun., vol. 63, no. 11, pp. 4448-4460, Nov. 2015.

[20] R. Tolimieri and S. Winograd, “Computing the ambiguity surface,” IEEE Trans. Signal Processing., vol. ASSP-33, no. 4, pp. 1239-1245, Oct. 1985.

[21] X. H. Tang, P. Z. Fan, and S. Matsufuji, “Lower bounds on the maximum correlation of sequence set with low or zero correlation zone,” Electron. Lett., vol. 36, pp. 551-552, Mar. 2000.

[22] A. Joseph, A. K. Brodzik, and R. Tolimieri, “Under-sampled Weyl-Heisenberg expansions by orthogonal projections in Zak space,” Signal Processing, vol. 81, no. 11, pp. 2383-2402, Nov. 2001.

[23] L. R. Welch, “Lower bounds on the maximum cross correlation of signals,” IEEE Trans. Inf. Theory., vol. 20, no. 3, pp. 397-399, May 1974.

[24] Dan Zhang, “Zero correlation zone sequences from a unified construction of perfect polyphase sequences,” [Online]. Available: https://2019.ieee-isit.org/Papers/ViewPaper.asp?PaperNum=1374.

[25] J. Zak, “Finite translations in solid state physics,” Phys. Rev. Lett., vol. 19, pp. 1385-1397, 1967.