LIMIT DISTRIBUTIONS FOR EUCLIDEAN RANDOM PERMUTATIONS

DOR ELBOIM AND RON PELED

Abstract. We study the length of cycles in the model of spatial random permutations in Euclidean space. In this model, for given length $L$, density $\rho$, dimension $d$ and jump density $\varphi$, one samples $\rho L^d$ particles in a $d$-dimensional torus of side length $L$, and a permutation $\pi$ of the particles, with probability density proportional to the product of values of $\varphi$ at the differences between a particle and its image under $\pi$. The distribution may be further weighted by a factor of $\theta$ to the number of cycles in $\pi$. Following Matsubara and Feynman, the emergence of macroscopic cycles in $\pi$ at high density $\rho$ has been related to the phenomenon of Bose-Einstein condensation. For each dimension $d \geq 1$, we identify sub-critical, critical and super-critical regimes for $\rho$ and find the limiting distribution of cycle lengths in these regimes. The results extend the work of Betz and Ueltschi. Our main technical tools are saddle-point and singularity analysis of suitable generating functions following the analysis by Bogachev and Zeindler of a related surrogate-spatial model.

1. Introduction

We study the model of spatial random permutations in Euclidean space, defined as follows. Let $d \geq 1, N \geq 1$ be integers and $L > 0, \theta > 0$ be real numbers. Let $X$ be a random variable taking values in $\mathbb{R}^d$. We assume that $X$ is absolutely continuous with respect to Lebesgue measure with a density which we denote by $\varphi$. Throughout the paper we make the following assumptions on $X$ and $\varphi$,

$$\mathbb{E}(X) = 0 \quad \text{and} \quad \varphi \text{ is a Schwartz function},$$

where we recall that a Schwartz function is a smooth function whose derivatives (of any order) decay faster than any polynomial (see Section 2).

Define the domain $\Lambda$ by

$$\Lambda := \{x = (x_1, \ldots, x_d) : 0 \leq x_j < L\} = [0, L)^d.$$

We ‘wrap’ $X$ in $\Lambda$ to obtain a periodic density function $\varphi_\Lambda$ defined by

$$\varphi_\Lambda(x) := \sum_{k \in \mathbb{Z}^d} \varphi(x + L \cdot k), \quad x \in \mathbb{R}^d.$$ 

The spatial random permutation model in $\Lambda$ (with periodic boundary conditions) is a joint probability distribution on $N$ particles in $\Lambda$ and a permutation in $\mathcal{S}_N$, the permutation group on $\{1, 2, \ldots, N\}$, in which the density of a pair $(x, \pi) \in \Lambda^N \times \mathcal{S}_N$, with respect to the Lebesgue

\textit{Date:} May 16, 2018.
Research of D.E. and R.P. supported by ISF grant 861/15 and by ERC starting grant 678520 (LocalOrder).
School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel.
Emails: dorelbom@mail.tau.ac.il, peledron@post.tau.ac.il.
measure on $\Lambda^N$ times the counting measure on $\mathcal{S}_N$, is proportional to

$$\theta^C(\pi), \prod_{i=1}^{N} \varphi(x_{\pi(i)} - x_i),$$

(3)

where $x = (x_1, \ldots, x_N)$ and where we write $C(\pi)$ for the total number of cycles in $\pi$. The model may be ill-defined for small values of $N$, as the expression (3) may equal zero for all pairs $(x, \pi) \in \Lambda^N \times \mathcal{S}_N$, but our proofs imply that this can only happen for finitely many values of $N$ (depending on $\varphi$). As we shall see, many properties of the model are governed by the density of particles in $\Lambda$, given by

$$\rho := \frac{N}{L^d},$$

(4)

Our main object of study is the limiting distribution of the cycle lengths in $\pi$, when the pair $(x, \pi)$ is sampled from the density (3) and $N$ and $L$ are taken to infinity. We will see that several asymptotic regimes arise according to the dimension $d$ and the limiting behavior of the density $\rho$. We consider separately the asymptotic behavior when $d = 1$, $d = 2$ and $d \geq 3$. In dimensions $d = 1, 2$ we allow $\rho$ to change with $N$ whereas in dimensions $d \geq 3$ we keep it fixed as $N$ increases.

We proceed to introduce the notation required for stating our main theorems. We order the cycles of $\pi$ according to the minimal index appearing in the cycle, so that the first cycle is the cycle containing 1, the second cycle (when $C(\pi) \geq 2$) is the cycle containing the minimal $2 \leq i \leq N$ which is not in the first cycle, and so on. Let $L_j(\pi)$, $1 \leq j \leq C(\pi)$, be the length of the $j$th cycle in this order and set $L_j(\pi) = 0$ for $j > C(\pi)$. We let $(\ell_j(\pi))_{j=1}^{\infty}$ be the sequence of cycle lengths $(L_j(\pi))_{j=1}^{\infty}$ arranged in non-increasing order. We often write $L_j$ and $\ell_j$ instead of $L_j(\pi)$ and $\ell_j(\pi)$ when $\pi$ is clear from the context.

We consider the limit $N \to \infty$, where we allow the density $\rho$ (and hence $L$) to vary with $N$ in a way to be prescribed, and define the limiting fraction of points in macroscopic cycles,

$$\nu := \lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{E}\left(\frac{1}{N} \sum_{j=1}^{C(\pi)} L_j(\pi) \cdot 1_{(L_j(\pi) \geq \epsilon N)}\right) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{P}(L_1(\pi) \geq \epsilon N),$$

(5)

where the pair $(x, \pi)$ is sampled from the density (3), we write $1_A$ for the indicator random variable of the event $A$ and where the second equality follows by symmetry (as the distribution of $\pi$ is invariant under relabeling of $\{1, \ldots, N\}$, see Proposition 3.1).

Denote by PD($\theta$) the Poisson-Dirichlet distribution with parameter $\theta$. This distribution is the limiting distribution of $\frac{1}{N}(\ell_1(\pi), \ell_2(\pi), \ldots)$ when $\pi$ has the Ewens distribution with parameter $\theta$ (that is, when $\mathbb{P}(\pi = \pi_0)$ is proportional to $\theta^{C(\pi_0)}$ for $\pi_0 \in \mathcal{S}_N$. The case $\theta = 1$ corresponds to a uniform permutation); see [23] for further background.

Denote by gamma($\alpha, \beta$) the gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$, whose density with respect to Lebesgue measure on $[0, \infty)$ is given by

$$\frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}, \quad r > 0,$$

where $\Gamma$ denotes the gamma function.

We write Var($X$) for the variance of $X$ in dimension $d = 1$, Cov($X$) for the covariance matrix of $X$ in dimensions $d \geq 1$ and det($A$) for the determinant of a matrix $A$. Finally, we write $\varphi^{*j}$ to denote the $j$-fold convolution $\varphi \ast \varphi \ast \cdots \ast \varphi$ (see Section 2).
The density function in the 1D critical regime: the limiting density of $L_1/N$ with $\theta = 1$ and $\rho/\sqrt{N} \to \alpha$ for $\alpha = 0.3, 0.8$ and $\alpha = 2$ respectively.

By (ii) and (iv) of Theorem 1.1, when $\rho/\sqrt{N} \to 0$ then $L_1/N \to 0$ and when $\rho/\sqrt{N} \to \infty$ then $L_1/N$ converges in distribution to $U[0, 1]$ (see Remark 3.5).

**Theorem 1.1** (One dimension). Let $(x, \pi)$ be randomly sampled from the density (3) with $d = 1$. Assume that $X$ satisfies the assumptions in (1). Then, as $N \to \infty$:

(i) (Sub-critical I) Suppose the density $\rho > 0$ is fixed as $N \to \infty$. Then $\nu = 0$ and

$$L_1 \xrightarrow{d} Y,$$

where $Y$ is the integer-valued random variable defined by

$$P(Y = j) = \theta \rho^{-1} \varphi^r_j(0) r^*_j, \quad j \geq 1$$

and

$$r_* is the unique number satisfying $0 < r_* < 1$ and $\sum_{j=1}^{\infty} \varphi^{r_j}(0) r^*_j = \rho \theta^{-1}$. (7)

(ii) (Sub-critical II) Suppose the density $\rho$ satisfies $\rho \to \infty$ and $\rho = o(\sqrt{N})$. Then $\nu = 0$ and

$$\frac{\theta^2 L_1}{2 \text{Var}(X) \rho^2} \xrightarrow{d} \text{gamma} \left( \frac{1}{2}, 1 \right).$$

(iii) (Critical) Suppose that the density $\rho$ satisfies $\frac{\rho}{\sqrt{N}} \to \alpha \in (0, \infty)$. Then $\nu = 1$ and

$$\frac{L_1}{N} \xrightarrow{d} \mu,$$

where $\mu$ is the probability measure on $(0, 1)$ whose density function is

$$\frac{1}{Z} \left( \sum_{m \in \mathbb{Z}} e^{-2\pi^2 \sigma^2 \alpha^2 m^2 x} \right) (1 - x)^{-\frac{3}{2}} \sum_{n=0}^{\infty} (-1)^n \left( \frac{-2\theta}{n} \right) (\theta + n) \exp \left( \frac{-((\theta + n)^2}{2\alpha^2 \sigma^2 (1 - x)} \right),$$

for $x \in (0, 1)$, where

$$Z = \frac{1}{\theta} \sum_{n=0}^{\infty} (-1)^n \left( \frac{-2\theta}{n} \right) (\theta + n) \exp \left( \frac{-(\theta + n)^2}{2\alpha^2 \sigma^2} \right), \quad \sigma^2 := \text{Var}(X),$$

and where we set $\binom{t}{n} := \frac{1}{n!} t(t-1) \cdots (t-n+1)$. See Figure 1.
(iv) (Super-critical) Suppose the density $\rho$ satisfies $\rho = \omega(\sqrt{N})$ and $\rho \leq N$. Then $\nu = 1$ and
\[
\frac{1}{N} (\ell_1, \ell_2, \ldots) \overset{d}{\to} \text{PD}(\theta).
\]

**Theorem 1.2** (Two dimensions). Let $(x, \pi)$ be randomly sampled from the density (3) with $d = 2$. Let
\[
\alpha_c := \frac{\theta}{2\pi \sqrt{\det(Cov(X))}}.
\]
Assume that $X$ satisfies the assumptions in (1). Then as $N \to \infty$:

(i) (Sub-critical I) Suppose the density $\rho > 0$ is fixed as $N \to \infty$. Then $\nu = 0$ and
\[
L_1 \overset{d}{\to} Y,
\]
where $Y$ is the integer-valued random variable given by (6) and (7).

(ii) (Sub-critical II, critical) Suppose the density $\rho$ satisfies $\rho \to \infty$ and $\frac{\rho}{\log N} \to \alpha \in [0, \alpha_c]$. Then $\nu = 0$ and
\[
\frac{\alpha_c \log L_1}{\rho} \overset{d}{\to} U[0,1].
\]

(iii) (Super-critical) Suppose the density $\rho$ satisfies $\frac{\rho}{\log N} \to \alpha \in (\alpha_c, \infty]$ and $\rho \leq N$. Then $\nu = 1 - \frac{\alpha_c}{\alpha}$,
\[
\frac{1}{\nu N} (\ell_1, \ell_2, \ldots) \overset{d}{\to} \text{PD}(\theta) \quad \text{and} \quad \frac{\log L_1}{\log N} \overset{d}{\to} \mu,
\]
where $\mu$ is a probability measure on $[0,1]$ with an atom of mass $1 - \frac{\alpha_c}{\alpha}$ at the point 1 and constant density on $(0,1)$, and we set $\frac{\alpha_c}{\alpha} := 0$ when $\alpha = \infty$.

**Theorem 1.3** (Dimension $d \geq 3$). Let $(x, \pi)$ be randomly sampled from the density (3) with $d \geq 3$. Suppose that the density $\rho > 0$ is fixed as $N \to \infty$. Let
\[
\rho_c := \theta \sum_{j=1}^{\infty} \varphi^j(0).
\]
Assume that $X$ satisfies the assumptions in (1). Then as $N \to \infty$:

(i) (Sub-critical) If $\rho < \rho_c$ then $\nu = 0$ and
\[
L_1 \overset{d}{\to} Y,
\]
where $Y$ is the integer-valued random variable given by (6) and (7).

(ii) (Critical) If $\rho = \rho_c$ then $\nu = 0$ and
\[
L_1 \overset{d}{\to} Y,
\]
where $Y$ is the integer-valued random variable given by (6) with $\rho = \rho_c$ and $r_* = 1$. Thus,
\[
\mathbb{P}(Y = j) \sim \frac{\theta}{\rho_c (2\pi)^{\frac{d}{2}} \sqrt{\det(Cov(X))}} \cdot j^{-\frac{d}{2}}, \quad j \to \infty,
\]
where $\sim$ denotes that the left-hand side is asymptotic to the right-hand side as $j \to \infty$. 
(iii) (Super-critical) If \( \rho > \rho_c \) then \( \nu = 1 - \frac{\rho}{\rho_c} \),

\[
\frac{1}{\nu N}(\ell_1, \ell_2, \ldots) \xrightarrow{d} \text{PD}(\theta)
\]

and

\[
P(L_1 = j) \rightarrow \theta \rho^{-1} \varphi^{*j}(0), \quad j \geq 1.
\] (12)

1.1. Extensions and remarks. The above theorems identify sub-critical, critical and super-critical regimes of density governing the asymptotic distribution of cycle lengths in \( \pi \) in each dimension \( d \geq 1 \). The limiting distribution of the length of the cycle containing 1 is determined in all regimes and a Poisson-Dirichlet limit law is proved for the joint distribution of cycle lengths in the super-critical regimes. We make several remarks concerning these statements and additional results which may be deduced with the techniques of this paper:

- **Integrability:** Our proofs hinge on an integrability property of the model, that the marginal probability of the permutation \( \pi \) has a representation as a product of weights depending only on the length of the cycles in \( \pi \). See Proposition 3.1 below for the exact statement. This fact was also central in previous works on the model (see, e.g., [9, Proposition 3.1]).

- **Joint distribution of cycle lengths:** In the sub-critical cases in all dimensions \( d \geq 1 \) and the critical cases in dimensions \( d \geq 2 \), one may extend the above limit laws to apply to the joint distribution of any fixed number of \( L_1, L_2, \ldots \), yielding that the cycle lengths become asymptotically independent and identically distributed (see Remark 5.6 and Remark 7.4). More precisely, when \( \rho \) is fixed as \( N \to \infty \), with \( \rho \leq \rho_c \) if \( d \geq 3 \), one obtains that for any fixed \( k \),

\[
(L_1, L_2, \ldots, L_k) \xrightarrow{d} (Y_1, Y_2, \ldots, Y_k)
\]

where the \((Y_j)\) are independent, each having the distribution of the corresponding \( Y \) variable in the above theorems. Similarly, in the sub-critical II regime in one dimension it holds that

\[
\left( \frac{\theta^2 L_1}{2 \text{Var}(X)\rho^2}, \frac{\theta^2 L_2}{2 \text{Var}(X)\rho^2}, \ldots, \frac{\theta^2 L_k}{2 \text{Var}(X)\rho^2} \right) \xrightarrow{d} (Y_1, Y_2, \ldots, Y_k)
\]

where the \((Y_j)\) are independent, each having the gamma \((\frac{1}{2}, 1)\) distribution. The analogous statement holds in the sub-critical II and critical regimes in two dimensions.

- **Joint distribution of \((x, \pi)\):** Examination of the joint density \((3)\) of \((x, \pi)\) reveals that, after conditioning on \( \pi \), the distribution of \( x \) becomes a collection of independent random walk bridges on the torus \( \Lambda \), with jump density \( \varphi \) and uniform starting points, whose lengths are the lengths of the cycles in \( \pi \). Thus, the marginal distribution of \( \pi \) determines the joint distribution of \((x, \pi)\) in a simple manner.

- **Number of cycles of given length:** Denote by \( C_j \) the number of cycles of length \( j \) in the random permutation \( \pi \). One can relate the moments of \( C_j \) and the joint distribution of the \((L_j)\), as discussed in Remark 3.5. For instance, \( \mathbb{E}(C_j/N) = \frac{1}{j} \mathbb{P}(L_1 = j) \).

Consequently \( \mathbb{E}(C(\pi)) = N \cdot \mathbb{E}(\frac{1}{L_1}) \).

Another simple consequence of these relations, deduced by bounding the variance of \( C_j \), is that in the regime where \( \rho \) and \( j \) are fixed as \( N \to \infty \), with \( \rho \leq \rho_c \) if \( d \geq 3 \), then \( C_j/N \) converges in probability to \( \mathbb{E}(C_j/N) \).

We do not study the distribution of the \((C_j)\) further in this work.
• Number of cycles: Our techniques provide further information on the distribution of the number of cycles $C(\pi)$. Specifically, they provide access to the asymptotics of $\mathbb{E}(t^{C(\pi)})$ as $N \to \infty$ in the above regimes, for most values of $t > 0$. This is further explained in Remark 3.6 but is not developed in this work.

• Density in critical regime in one dimension: Probability laws involving the Jacobi theta function have appeared in several works; see [13] for a survey. The limiting density (8) obtained in the critical regime in one dimension is of this kind, though we have not seen its exact expression in previous works (a similar expression is in [13, equation (3.11)]).

• Schwartz assumption: Our theorems are proved under the assumptions in (1). While the assumption that $\mathbb{E}(X) = 0$ is essential to the results, the assumption that $\phi$ is a Schwartz function may be weakened, requiring that $\phi$ has sufficiently many derivatives and these decay sufficiently fast. We have not attempted to keep track of the minimal assumptions used in the proof of each result.

1.2. Physics background, previous results and related models.

1.2.1. Background and previous results. One of the main motivations for studying the spatial random permutation model (3) comes from physics, where it was proposed by Matsubara [37] and Feynman [24] to express quantities arising in the interacting Bose gas as expectation values in a model of random permutations. It was observed that the ideal (non-interacting) Bose gas gives rise to the spatial random permutation model studied in this work with $\theta = 1$ and Gaussian $\varphi$ (this is sometimes called the Feynman-Kac representation of the ideal Bose gas). With this link, the phenomenon of Bose-Einstein condensation was related to the appearance of macroscopic cycles in the random permutation. Sütő [44, 45] further elucidated this link, combining exact calculations and certain approximations, by showing that Bose-Einstein condensation in the ideal Bose gas occurs exactly when macroscopic cycles arise in the corresponding spatial random permutation model and by deriving the limiting distribution of $L_1$ in both the sub-critical and super-critical regimes in dimensions $d \geq 3$.

A mathematical investigation of the spatial random permutation model (3) was conducted by Betz and Ueltschi [8, 9]. They studied the model for the class of jump densities $\phi$ having a non-negative Fourier transform, in particular, having $\varphi(x) = \varphi(-x)$ for all $x$, and satisfying certain additional technical assumptions. Their results include the formula (10) for $\rho_c$, a proof that the fraction of points in macroscopic cycles (equivalently, of ‘super-constant’ size) equals $\min(0, 1 - \frac{\omega}{\rho})$ at each fixed density $\rho$ and the Poisson-Dirichlet limit law for the length of macroscopic cycles in the super-critical regime. Their results apply in dimensions $d \geq 3$ and also in dimensions $d \in \{1, 2\}$ when the jump density $\varphi$ is such that $\rho_c$ is finite there (this can occur when $\varphi$ has heavy tails). Their analysis includes the parameter $\theta$ and, in fact, allows more general cycle weights converging to $\theta$.

Our results compare with those of Betz and Ueltschi as follows. Our assumption (1) yields a class of jump densities $\varphi$ which is neither wider nor narrower than that of [8, 9], as there exist slowly-decaying functions with non-negative Fourier transform (e.g., $(1 + |x|)^{-\gamma}$, $1 < \gamma < 2$, in dimension $d = 1$) and there exist Schwartz functions with zero center of mass having complex, or real but sometimes negative, Fourier transform (such as $\exp(-x^4)$ or $\exp(-(x-2)^2) + 2\exp(-(x+1)^2)$ in dimension $d = 1$). In the intersection of the two classes, the results of [8, 9] apply only in dimensions $d \geq 3$ and yield there the fraction of points in cycles of macroscopic size and the Poisson-Dirichlet limit law. Our work extends the analysis.
in dimensions \( d \geq 3 \) by further providing the limiting distribution of cycle lengths for all values of \( \rho \). We further analyze the distribution of cycle lengths in dimensions 1 and 2.

1.2.2. Surrogate-spatial model. Bogachev and Zeindler [16] studied a related model for random permutations which they term the surrogate-spatial model. In this model the probability measure on permutations is given by cycle weights, with the formula

\[
P(\pi = \sigma) = \frac{1}{Z} \prod_{j=1}^{N} (\theta_j + N \kappa_j)^{C_j(\sigma)}, \quad \sigma \in S_N,
\]

where \( C_j(\sigma) \) is the number of cycles of length \( j \) in \( \sigma \) and \((\theta_j), (\kappa_j)\) are sequences independent of \( N \) for which various behaviors are allowed. The spatial random permutations model also admits a similar form, see Proposition 3.1, in which, very roughly,

\[
\theta_j = \theta \quad \text{and} \quad \kappa_j = \frac{1}{\rho_j d/2},
\]

see Corollary 4.6 and Lemma 4.3. Indeed, the spatial random permutation model served as the inspiration for the surrogate-spatial model [16] and the two models are close relatives of each other with the choice [14] and when \( \rho \) is fixed. The impressive work of Bogachev and Zeindler provides a very detailed analysis of the surrogate-spatial permutation model including identifying sub-critical, critical and super-critical regimes, finding the limiting distributions of cycle lengths and establishing a Poisson-Dirichlet limit law in the super-critical regime. Their proofs rely on saddle point and singularity analysis of suitable generating functions. Our approach in this paper is inspired by their analysis, adapting the techniques to the model here and augmenting them with additional tools as necessary, for instance, in dimensions 1 and 2 when \( \rho \) tends to infinity.

1.2.3. Random permutations with cycle weights. There has been significant activity [11, 10, 19, 6, 46, 36, 40, 22, 34, 5, 20] in recent years in studying random permutations in which the probability of a permutation is proportional to a product over its cycles of a weight depending on the length of the cycle. The spatial random permutation model has this general form, see Proposition 3.1, but differs from the models studied in this literature in that the weight assigned to a cycle depends both on the length of the cycle and on the size of the permutation.

1.2.4. Spatial random permutations. The model studied in this work, of spatial random permutations in Euclidean space, may be thought of as a particular case of a wider class. Informally, one may think of a spatial random permutation as a random permutation which is biased towards the identity in some underlying geometry. This broad idea covers many specific cases including the well-studied interchange model [21, 17, 2, 42, 1, 30, 31, 14, 33, 15] and the Mallows model (defined in dimension 1) [35, 43, 17, 27, 28, 38, 12, 39, 26, 3]. The study of the cycle structure is of great interest in such models as well. In particular, the famous Tóth conjecture [47] states that for \( d \geq 3 \), infinite orbits arise in the interchange process on \( \mathbb{Z}^d \) run for a sufficiently long time, whereas for \( d \in \{1, 2\} \) no infinite orbits arise after any fixed amount of time. This is easy to establish in dimension 1 (where further quantitative results are announced by Kozma and Sidoravicius) but remains wide open in dimensions \( d \geq 2 \) (the \( d \geq 3 \) case is closely related to the long-standing open question of establishing a phase transition for the quantum Heisenberg ferromagnet [47]; see also Section 1.2.5 below). In this regard, the possibility of universality is especially intriguing, that the results of this paper, such as the limiting distributions obtained in dimensions 1 and 2 when \( \rho \) grows with
N or the general features of the sub-critical, critical and super-critical behavior in dimensions \(d \geq 3\), are valid also for the other spatial permutation models. We illustrate this possibility with specific analog statements on the interchange and Mallows models, after describing these models briefly.

Let \(G = (V,E)\) be a finite or infinite bounded-degree graph. The *interchange model* \([47]\) (also called the *stirring model* in some of the literature) gives a dynamics on permutations in \(S_V\), one-to-one and onto functions \(\pi : V \to V\), which is associated to the structure of the graph. Each edge of the graph is endowed with an independent Poisson process of rate 1. An edge is said to *ring* at time \(t\) if an event of its Poisson process occurs at that time. Starting from the identity permutation \(\pi_0 \in S_V\), the interchange process, introduced by Tóth \([47]\), is the permutation-valued stochastic process \((\pi_t)\) obtained by performing a transposition along each edge at each time that it rings. The interchange model on the subset \(\{0, \ldots, M-1\}^d \subset \mathbb{Z}^d\) equipped with periodic boundary conditions bears formal similarity with the model of Euclidean random permutations studied in this work, under a suitable translation of the parameters. In the interchange model, the location of a particle at time \(t \ll M^2\) has roughly a centered Gaussian distribution with covariance matrix \(2t\) times the identity. The two models are thus similar after rescaling space by a factor of \(\sqrt{t}\), taking \(L = M/\sqrt{t}\) and \(N = M^d\), so that \(\rho = N/L^d = t^{d/2}\).

The *Mallows distribution* \([35]\) on \(S_n\) with parameter \(q > 0\) gives probability proportional to \(q^{\text{inv}(\pi)}\) to each permutation \(\pi\), where \(\text{inv}(\pi) := |(s,t) : s < t \text{ and } \pi_s > \pi_t|\) counts the number of inversions in \(\pi\). For \(0 < q < 1\), the model may also be defined on the set of all integers \(\mathbb{Z}\) \([28]\). Focusing on the case \(0 < q < 1\), an important feature of the Mallows model is the typical displacement of points in the permutation, with \(|\pi(s) - s|\) typically being of order \(\min(q, n)\) \([18, 28, 12]\). Similarly to the interchange model, this suggests a similarity of the Mallows model with the Euclidean random permutation model in one dimension, when the parameters are chosen as \(L = n/\min\left(\frac{q}{1-q}, n\right)\) and \(N = n\), so that \(\rho = \min\left(\frac{q}{1-q}, n\right)\). The threshold for the emergence of macroscopic cycles, and the Poisson-Dirichlet distribution, in the Mallows model was found in \([26]\) and is in qualitative correspondence with the results of Theorem 1.1.

The following statements are analogous to the sub-critical results of Theorem 1.1 and Theorem 1.2 (in the infinite-volume limit):

- **Let** \(L_1(t)\) **be the length of the cycle containing the origin in the interchange process on** \(\mathbb{Z}^d\) **at time** \(t\). **Then, there are constants** \(c_1, c_2 > 0\) **such that when** \(d = 1\),
  \[
  \frac{L_1(t)}{c_1 \cdot t} \xrightarrow{d} \text{gamma}\left(\frac{1}{2}, 1\right), \quad t \to \infty,
  \]
  **and when** \(d = 2\),
  \[
  \frac{\log L_1(t)}{c_2 \cdot t} \xrightarrow{d} U[0,1], \quad t \to \infty.
  \]

- **Let** \(0 < q < 1\) **and consider the Mallows model on** \(\mathbb{Z}\) **[28]**. **Let** \(L_1(q)\) **be the length of the cycle containing** \(0\). **Then there is a constant** \(c > 0\) **such that**
  \[
  c \cdot L_1(q)(1-q)^2 \xrightarrow{d} \text{gamma}\left(\frac{1}{2}, 1\right) \quad \text{as} \quad q \uparrow 1.
  \]

However, analytical calculations of Angel and Hutchcroft \([4]\) imply that the statement \((17)\) does not hold. This is also apparent in simulations, see Figure 2. On the positive side, the simulations seem to indicate that the limit in \((17)\) should at least exist and share some general
Figure 2. The empirical density of $0.3 \cdot L_1(q)(1 - q)^2$ as obtained from the simulation (in blue) for $q = 0.9$ and $q = 0.975$ respectively, and the density of gamma($\frac{1}{2}, 1$) distribution (in red). The simulations were sampled using the algorithm described in Section 3.5 of [26] with 100000 iterations.

features with the gamma distribution. Given that the Mallows statement (17) does not hold, we also do not expect the corresponding statement for the 1D interchange model to hold exactly as in (15), though, again, the limit there is still likely to exist and bear similarities with the gamma distribution. The situation for the 2D statement (16) is less clear. One may be more hopeful that it holds as written, as the presence of the logarithm on the left-hand side renders the statement more robust to small deviations in the distribution of $L_1(t)$ itself.

1.2.5. Quantum Heisenberg model in two dimensions. Continuing further with the analogy to the interchange process, one may also speculate that the relation (16) in $d = 2$ continues to hold also for the interchange model which is tilted by a factor of 2 to the power of the number of cycles (in analogy with setting $\theta = 2$ in for the model studied in this work). It is known from the work of Tóth [47] that this tilted model is in direct correspondence with the quantum Heisenberg model, at the temperature $T$ which is the inverse of the time $t$ to which the interchange model is run. Precisely, the spin-spin correlation between sites $x$ and $y$ in the quantum Heisenberg model equals a constant times the probability that $x$ and $y$ are in the same cycle [17] (see also [48]). Thus, the length of the cycle containing a given site $x$ equals the sum of spin-spin correlations between $x$ and the other sites. A relation of the form (16) (accompanied with appropriate integrability conditions) would then imply that the sum of spin-spin correlations is of the order $\exp(cT)$ for some constant $c > 0$. This is in accordance with the predicted behavior of the Heisenberg ferromagnet, which states that spin-spin correlations decay exponentially at every positive temperature, with a correlation length which is exponential in the inverse temperature. Such predictions, going back to Polyakov [41], remain wide open in the mathematical literature.

1.2.6. Band matrices. We mention that the results regarding the emergence of macroscopic cycles in one dimension bear formal similarity with a conjectured localization transition for random band matrices. This similarity is detailed in [26, Section 1.2.2] in the context of the Mallows measure on permutations. One may also define random band matrices in dimensions
where they are rather poorly understood. Indeed, even taking the bandwidth to be one (when the band matrix corresponds to a random Schrödinger operator) leads to many unsolved problems around the famous Anderson localization phenomenon (foremost among these is the question of delocalization of random Schrödinger operators at low disorder in dimensions \( d \geq 3 \). Also conjectured but unresolved is the fact that two-dimensional random band matrices exhibit localization for any fixed bandwidth). Does the similarity between the emergence of macroscopic cycles and the localization properties of random band matrices extend to dimensions \( d \geq 2 \)? If true, such similarity would predict that in two dimensions, the critical bandwidth for delocalization in a random band matrix on the box \([-L,L]^2\) is of order \( \sqrt{\log L} \).

1.3. **Acknowledgements.** We thank Daniel Ueltschi for helpful advice and for referring us to the paper of Bogachev and Zeindler [16]. We thank Volker Betz, Gady Kozma, Mikhail Sodin and Elad Zelingher for useful discussions. We thank Xiaolin Zeng for helpful comments on an earlier version of this work. We thank Omer Angel and Tom Hutchcroft for considering the validity of the statement (17) on the Mallows model and letting us know the conclusion of their calculations.

2. **Preliminaries**

**Constant policy:** Throughout the paper we regard the dimension \( d \geq 1 \), the density function \( \varphi \) (satisfying the assumptions (1)) and the real \( \theta > 0 \) as fixed and our emphasis is on the behavior of the various quantities of interest as the parameters \( N \) and \( L \) (or the density \( \rho = \frac{N}{L^d} \)) change. Constants such as \( C, c, \epsilon, \delta \) denote positive numerical values which may depend on \( d, \varphi \) and \( \theta \) but are independent of all other parameters (in particular, of \( N \) and \( L \)). When the constant depends on an additional parameter this will be noted explicitly, writing, for instance, \( C_n \) for a value which depends on \( d, \varphi, \theta \) and \( n \). The constants \( C, c \) or their counterparts depending on additional parameters, are regarded as generic constants in the sense that their value may change from one appearance to the next, with the value of \( C \) increasing and the value of \( c \) decreasing. However, constants labeled with a fixed number, such as \( C_0, c_0 \), have a fixed value throughout the section that they appear in.

**Oh notation:** For two functions \( f, g \), possibly depending on many parameters, we write \( f = O(g) \) to denote that \( \frac{|f|}{|g|} \leq C \) where \( C \) is as above, that is, independent of all parameters besides \( d, \varphi \) and \( \theta \). We use a similar notation when the constant may depend on additional parameters writing, for instance, \( f = O_n(g) \) to denote that \( \frac{|f|}{|g|} \leq C_n \). We write \( f = o(g) \) as \( k \to k_0 \) (where \( k_0 \) may be infinity) to denote that \( \lim_{k \to k_0} \frac{|f|}{|g|} = 0 \). If we write, in addition, that the little Oh is uniform in \( m \in I_k \) (where \( I_k \) may or may not depend on \( k \)) we mean that \( \lim_{k \to k_0} \sup_{m \in I_k} \frac{|f|}{|g|} = 0 \). The notation \( f \sim g \) as \( k \to k_0 \) means that \( f = (1 + o(1))g \) as \( k \to k_0 \) and we may again add a uniformity requirement.

**Notation:** We write \( \mathbb{N} := \{1, 2, 3, \ldots\} \) for the set of positive integers.

We use the Pochhammer symbol \((x)_n\) defined by

\[
(x)_n := x(x-1)\cdots(x-n+1), \quad x \in \mathbb{R}, \ n \in \mathbb{N}
\]

and \((x)_0 := 1\).
A smooth function $f : \mathbb{R}^d \to \mathbb{C}$ is called Schwartz if
\[
\sup_{x \in \mathbb{R}^d} \left| x_1^{\alpha_1} \cdots x_d^{\alpha_d} \frac{\partial^{\beta_1 + \cdots + \beta_d} f}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} \right| < \infty \quad \text{for any integers } \alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d \geq 0.
\]

We use the shorthand $f^{(n)}$ to denote the $n$-th derivative of a function $f$ and write $[z^n] f(z)$ to denote the coefficient of $z^n$ in the power series of $f(z)$. The convolution $f * g$ of integrable $f, g : \mathbb{R}^d \to \mathbb{C}$ is defined, as usual, by $(f * g)(x) := \int f(y)g(x-y)dy$. We write $f^{*j}$ to denote the $j$-fold convolution $f * f * \cdots * f$.

We denote the closure of a set $\Omega \subseteq \mathbb{C}$ by $\overline{\Omega}$ and let $\mathbb{D}$ denote the open unit disc,
\[
\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}.
\]

We use the standard branches of the argument, logarithm and power functions on the complex plane. That is, we take arg$(z) \in (-\pi, \pi]$ for $z \in \mathbb{C}$ and consider log $z := \log |z| + i \arg(z)$ and $z^\alpha := e^{\alpha \log z}$ for $z \in \mathbb{C} \setminus (-\infty, 0]$.

We write $\| v \|$ for the Euclidean norm of a vector $v \in \mathbb{R}^d$. Such vectors are thought of as column vectors for purposes of matrix multiplication and we write $v^T$ for the row vector obtained after transposition. Similarly, for a matrix $B$ we write $B^T$ for the transposed matrix.

We write $1_A$ for the indicator random variable of an event $A$.

We denote by $N(\mu, \Sigma)$ the multivariate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$.

In several places in the paper it is convenient to discuss the ‘square root’ matrix of the covariance matrix Cov$(X)$ of $X$. To this end we point out that Cov$(X)$ is (symmetric) positive definite as $X$ has a density. In the sequel we fix $A$ to be a (symmetric) positive definite matrix satisfying $A^2 = \text{Cov}(X)$. (18)

3. Exact expression for the distribution of $\pi$

In this section, following Betz and Ueltschi [9] and Bogachev and Zeindler [16], we start by proving that the marginal probability of the permutation in the spatial random permutation model has a representation as a product of cycle weights. We then find a convenient generating function for the partition functions arising in this representation and express the main statistics of interest to us (such as the distribution of $L_1$) in terms of the partition functions.

We remind that the integer $d \geq 1$, density function $\varphi$ (satisfying the assumptions (1)) and real $\theta > 0$ are fixed. Throughout the section we fix also the integer $N \geq 1$ and real $L > 0$ and let $(x, \pi)$ be randomly sampled from the density (3).

3.1. Marginal distribution of $\pi$ and generating function. We again denote by $C_j(\sigma)$ the number of cycles of length $j$ of a permutation $\sigma$.

**Proposition 3.1.** The marginal distribution of the permutation $\pi$ is given by
\[
\mathbb{P}(\pi = \sigma) = \frac{1}{N!H_N(L)} \prod_C W_{L,|C|} = \frac{1}{N!H_N(L)} \prod_{j=1}^N (W_{L,j})^{C_j(\sigma)}, \quad \sigma \in S_N
\]
where the first product runs over all cycles $C$ of $\sigma$, $|C|$ denotes the length of the cycle $C$,
\[
W_{L,j} := \theta L^d \sum_{k \in \mathbb{Z}^d} \varphi^{*j}(Lk)
\]
and

\[ H_N(L) := \frac{1}{N!} \sum_{\sigma \in S_N} \prod_{j=1}^{N} (W_{L,j})^{C_j(\sigma)} \]  

(21)

is the partition function.

We include the factor \( N! \) in (19) for consistency with the notation in [9] and as it simplifies some of the resulting generating function formulas below (e.g., equality (29)).

In the proof, we write \( x \pmod{LZ^d} \), where \( x \in \mathbb{R}^d \), to denote the unique point \( y \in \Lambda \) satisfying that \( x - y \in L\mathbb{Z}^d \). We also require the notion of the \( j \)-fold convolution of \( \varphi_\Lambda \) with itself on the torus \( \Lambda \). Define \( \varphi_\Lambda \ast^{(j+1)} y = \int_\Lambda \varphi_\Lambda \ast^j (x) \varphi_\Lambda (y - x)dx, \quad y \in \mathbb{R}^d. \) 

**Claim 3.2.** For any \( j \geq 1 \) we have

\[ \varphi_\Lambda \ast^j(y) = \sum_{k \in \mathbb{Z}^d} \varphi^*(y + Lk), \quad y \in \mathbb{R}^d. \]  

(23)

**Proof.** Intuitively, the identity (23) can be understood as follows, the left-hand side is the density of the sum on the torus \( \Lambda \) of \( j \) independent copies of \( X \pmod{L\mathbb{Z}^d} \), whereas the right-hand side is the density of the projection to the torus of the sum in \( \mathbb{R}^d \) of \( j \) independent copies of \( X \). We turn to prove (23) formally by induction. The case \( j = 1 \) is exactly the definition of \( \varphi_\Lambda \). Suppose (23) holds for some \( j \geq 1 \). Then

\[
\varphi_\Lambda \ast^{(j+1)}(y) = \int_\Lambda \varphi_\Lambda \ast^j (x) \varphi_\Lambda (y - x)dx
\]

\[
= \int_\Lambda \sum_{k_1, k_2 \in \mathbb{Z}^d} \varphi^*(x + Lk_1) \varphi(y - x + Lk_2)dx
\]

\[
= \sum_{k, k_1 \in \mathbb{Z}^d} \int_\Lambda \varphi^*(x + Lk_1) \varphi(y - x + Lk - Lk_1)dx
\]

\[
= \sum_{k \in \mathbb{Z}^d} \int_\Lambda \varphi^*(x) \varphi(y + Lk - x)dx = \sum_{k \in \mathbb{Z}^d} \varphi^{*(j+1)}(y + Lk). \quad \square
\]

**Proof of Proposition 3.1.** The marginal probability on permutations is

\[ \mathbb{P}(\pi = \sigma) = \frac{\theta^{C(\sigma)}}{Z_N} \prod_{\Lambda}^{N} \varphi_\Lambda \left(x_{\sigma(i)} - x_i \right) dx_1 \cdots dx_N, \quad \sigma \in S_N, \]  

(24)

where \( Z_N \) is the appropriate normalization factor. It is straightforward that the integral factorizes according to the cycles in the permutation \( \pi \). The contribution of each fixed point equals

\[ \theta \int_\Lambda \varphi_\Lambda (0)dy = \theta |\Lambda| \sum_{k \in \mathbb{Z}^d} \varphi(Lk) = W_{L,1}. \]
The contribution of each cycle \((y_1, \ldots, y_j)\) of length \(j \geq 2\) is (with the convention that \(y_{j+1}\) is \(y_1\))

\[
\theta \int_{\Lambda^j} \prod_{i=1}^{j} \varphi_\Lambda (y_{i+1} - y_i) \, dy_1 \cdots dy_j = \theta \int_{\Lambda} \left( \prod_{i=1}^{j} \varphi_\Lambda (y_{i+1} - y_i) \, dy_2 \cdots dy_j \right) dy_1. \tag{25}
\]

By considering the change of variables \(y_1 \mapsto y_1\) and \(y_i \mapsto y_i + y_1 \,(\text{mod} \, L\mathbb{Z}^d)\) for \(2 \leq i \leq j\) and using the fact that \(\varphi_\Lambda\) is periodic, we see that the inner integral does not depend on the value of \(y_1\). Thus, the expression (25) becomes

\[
\theta |\Lambda| \int_{\Lambda^{j-1}} \varphi_\Lambda (y_2) \varphi_\Lambda (y_3 - y_2) \cdots \varphi_\Lambda (y_j - y_{j-1}) \varphi_\Lambda (-y_j) \, dy_2 \cdots dy_j
\]

\[
= \theta |\Lambda| \varphi_\Lambda^{xy}(0) = \theta L^d \sum_{k \in \mathbb{Z}^d} \varphi_{\Lambda}^{*j} (Lk) = W_{L,j}, \tag{26}
\]

where in the first equality we used (22) and in the second equality we appealed to Claim 3.2.

The next well-known identity is a special case of the enumeration theorem of Pólya.

**Lemma 3.3.** Let \( (a_j)_{j \in \mathbb{N}} \) be a sequence of complex numbers. Then the following formal power series expansion holds

\[
\exp \left( \sum_{j=1}^{\infty} \frac{a_j}{j} z^j \right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^{n} a_j^{C_{j,\sigma}}. \tag{27}
\]

**Proof.** The proof is a relatively straightforward calculation, presented below with a probabilistic flavor.

We assume that \(a_j > 0\) and that \(\sum_{j=1}^{\infty} a_j < \infty\). This is without loss of generality since if (27) holds for such sequences then it holds in a formal sense for all sequences. Recall that a random variable \(Y\) has the Poisson(\(\lambda\)) distribution if \(P(Y = n) = \exp(-\lambda) \frac{\lambda^n}{n!}\) for integers \(n \geq 0\). Note that it has the probability generating function \(E(z^Y) = e^{\lambda(z-1)}\). Define an infinite sequence of independent Poisson random variables,

\[
X_j \sim \text{Poisson} \left( \frac{a_j}{j} \right), \quad j \geq 1
\]

and set

\[
X := \sum_{j=1}^{\infty} jX_j.
\]

Note that \(X\) is almost surely finite as it has finite expectation by our assumption on \((a_j)\).

The equality (27) follows by calculating \(E(z^X)\) in two ways. On the one hand,

\[
E(z^X) = E \left( \prod_{j=1}^{\infty} z^{jX_j} \right) = \prod_{j=1}^{\infty} E(z^{jX_j}) = \prod_{j=1}^{\infty} \exp \left( \frac{a_j}{j} \frac{(z^j - 1)}{j} \right) = \exp \left( \sum_{j=1}^{\infty} \frac{a_j}{j} \frac{(z^j - 1)}{j} \right),
\]
On the other hand,
\[
\mathbb{E}(z^X) = \sum_{n=0}^{\infty} z^n \cdot \mathbb{P}(X = n) = \sum_{n=0}^{\infty} z^n \sum_{(c_1, c_2, \ldots) \colon \sum_{j=1}^{\infty} j c_j = n} \mathbb{P}(\forall j, X_j = c_j)
\]
\[
= \exp \left( - \sum_{j=1}^{\infty} \frac{a_j}{j} \right) \sum_{n=0}^{\infty} z^n \prod_{j=1}^{\infty} \frac{a_j}{j!} = \exp \left( - \sum_{j=1}^{\infty} \frac{a_j}{j} \right) \sum_{n=0}^{\infty} \frac{z^n}{n!} \prod_{\sigma \in S_n} \prod_{j=1}^{n} a_j^{c_j(\sigma)},
\]
where the last equality follows as the number of permutations \( \sigma \in S_n \) with cycle structure \((c_1, \ldots, c_n)\) is \(\prod_{j=1}^{n!} j^{c_j}\). \( \square \)

Lemma 3.3 can be used to obtain a convenient expression for the normalization constant \(H_N(L)\) involved in the marginal probability distribution \(19\). To this end set \(H_0(L) := 1\) and define the function
\[
G_L(z) := \sum_{j=1}^{\infty} \frac{W_{L,j}}{j} z^j.
\]
By Lemma 3.3 (with \(a_j = W_{L,j}\)), the generating function of the sequence \((H_n(L))_{n \geq 0}\) is given by
\[
\sum_{n=0}^{\infty} H_n(L) z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \prod_{\sigma \in S_n} (W_{L,j})^{c_j(\sigma)} = \exp \left( \sum_{j=1}^{\infty} \frac{W_{L,j}}{j} z^j \right) = e^{G_L(z)}.
\]
The equality holds in a formal sense, as we relied on Lemma 3.3. However, one may check that the non-negative sequence \((W_{L,j})_j\) is bounded, whence the sum in (28) converges for \(|z| < 1\), implying the same for the Taylor series (29) of its exponential. The boundedness of \((W_{L,j})_j\) follows from its definition (20) as the density of a convolution on the torus (see (22) and (23)) and is also a consequence of our subsequent Corollary 4.6.

3.2. The distribution of the cycle lengths through the generating function. Recall the definitions of \(L_1, L_2, \ldots\) appearing before the \[.\]

**Lemma 3.4.** For any integers \(m \geq 1\) and \(j_1, \ldots, j_m \geq 1\) with \(j_1 + \cdots + j_m \leq N\) we have
\[
\mathbb{P}(L_1 = j_1, \ldots, L_m = j_m) = \frac{H_{N-j_1-\cdots-j_m}(L)}{H_N(L)} \cdot \prod_{k=1}^{m} \frac{W_{L,j_k}}{N - j_1 - \cdots - j_{k-1}}.
\]
In particular, for \(m = 1\),
\[
\mathbb{P}(L_1 = j) = \frac{W_{L,j}}{N} \cdot \frac{H_{N-j}(L)}{H_N(L)}, \quad 1 \leq j \leq N.
\]
**Proof.** Note that there are \((N-1) \cdots (N-j+1) = (N-1)_{j-1}\) possibilities for a cycle of length \(j\) containing the element 1. Thus, using \(19\) and \(21\), we get
\[
\mathbb{P}(L_1 = j) = \frac{(N-1)_{j-1}}{N! H_N(L)} \sum_{\sigma \in S_{N-j}} \prod_{j=1}^{N-j} (W_{L,j})^{c_j(\sigma)} = \frac{W_{L,j}}{N} \cdot \frac{H_{N-j}(L)}{H_N(L)},
\]
which proves the lemma for \( m = 1 \). Similarly, for \( m = 2 \),
\[
\mathbb{P}(L_1 = j_1, L_2 = j_2) = \frac{(N - 1)_{j_1-1} \cdot W_{L,j_1} \cdot (N - j_1 - 1)_{j_2-1} W_{L,j_2} \cdot (N - j_1 - j_2)! \cdot H_{N-j_1-j_2}(L)}{N! H_N(L)}
\]
\[
= \frac{W_{L,j_1} W_{L,j_2}}{N(N - j_1)} \frac{H_{N-j_1-j_2}(L)}{H_N(L)}.
\]

The general case \( m \geq 1 \) is handled in the same manner. \( \square \)

Remark 3.5. The relation between the distribution of \( L_1 \) and the \((C_j)\) is as follows,
\[
\mathbb{P}(L_1 = j) = \mathbb{E} \left( \frac{j C_j}{N} \right), \quad 1 \leq j \leq N, \tag{33}
\]

since
\[
\mathbb{P}(L_1 = j) = \sum_{k \geq 0 \text{ s.t. } \mathbb{P}(C_j = k) > 0} \mathbb{P}(L_1 = j \mid C_j = k) \cdot \mathbb{P}(C_j = k) = \sum_{k=0}^{\infty} \frac{j^k}{N} \cdot \mathbb{P}(C_j = k) = \mathbb{E} \left( \frac{j C_j}{N} \right),
\]

where the second equality follows as there are \( jk \) elements in cycles of length \( j \) and any \( 1 \leq i \leq N \) is equally likely to be one of them. In a similar manner one checks that
\[
\mathbb{P}(L_1 = j, L_2 = j) = \mathbb{E} \left( \frac{j C_j}{N} \cdot \frac{j(C_j - 1)}{N - j} \right), \quad 1 \leq j \leq N. \tag{34}
\]

One may similarly obtain expressions involving higher moments of \( C_j \). A useful expression for the variance of \( C_j \) is obtained by combining (33) and (34),
\[
\text{Var} \left( \frac{C_j}{N} \right) = \frac{N - j}{N} \cdot \frac{\mathbb{P}(L_1 = j, L_2 = j)}{j^2} + \frac{\mathbb{P}(L_1 = j)}{N j} - \left( \frac{\mathbb{P}(L_1 = j)}{j} \right)^2.
\]

Remark 3.6. Our analysis of the asymptotic distribution of cycle lengths proceeds by determining the asymptotics of the partition function \( H_n(L) \), in various asymptotic regimes of \( n \) and \( L \), and then applying Lemma 3.4. Further information on the random permutation \( \pi \) may be obtained by varying the various parameters it depends on. For instance, the number of cycles \( C(\pi) \) may be studied as follows. Let us write \( H_n^\theta(L) \) instead of \( H_n(L) \) to note explicitly the dependence of the partition function on \( \theta \). Then,
\[
\mathbb{E}(e^{C(\pi)}) = \frac{H_n^\theta(L)}{H_n^\theta(L)}
\]
as one immediately verifies using Proposition 3.1. Thus, the asymptotic behavior of the partition function can be used to determine the probability generating function of the number of cycles. This direction is not developed in this work.

4. Basic properties of the generating function

In this section we state and prove some basic properties of the generating function \( G_L \).
4.1. **Fourier transform.** We frequently use the Fourier transform \( \hat{f} \) (or \( F(f) \)) of a function \( f : \mathbb{R}^d \to \mathbb{C} \), defined by
\[
\hat{f}(t) = F(f)(t) := \int_{\mathbb{R}^d} f(x)e^{-2\pi ix \cdot t} \, dx, \quad t \in \mathbb{R}^d,
\]
where we write \( x \cdot t \) for the standard scalar product in \( \mathbb{R}^d \). In the following claims we collect some basic facts about the Fourier transform of a Schwartz function.

**Claim 4.1.** Let \( f : \mathbb{R}^d \to \mathbb{C} \) be a Schwartz function. Then:

(i) For any invertible linear transformation \( B : \mathbb{R}^d \to \mathbb{R}^d \),
\[
F(f \circ B) = \frac{1}{|\text{det} B|} \cdot \hat{f} \circ (B^T)^{-1}.
\]

(ii) The Fourier inversion theorem states that \( \hat{f}(x) = f(-x) \) for all \( x \in \mathbb{R}^d \).

(iii) For any \( j \in \mathbb{N} \), the functions \( f^j \) and \( \hat{f}^j \) are Schwartz functions and we have \( \hat{f}^j = \hat{f}^j \).

(iv) The Poisson summation formula holds:
\[
\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{m \in \mathbb{Z}^d} \hat{f}(m).
\]

In the following claim we restrict ourselves to the density function \( \varphi \).

**Claim 4.2.** The Fourier transform \( \hat{\varphi} \) of \( \varphi \) satisfies:

(i) For every \( t \in \mathbb{R}^d \), \( |\hat{\varphi}(t)| \leq 1 \) with equality if and only if \( t = 0 \).

(ii) We have
\[
\hat{\varphi}(t) = 1 - 2\pi^2 t^T \text{Cov}(X) t + O(||t||^3) = 1 - 2\pi^2 ||A t||^2 + O(||t||^3), \quad ||t|| \leq 1
\]
where \( A \) is given in (18), In particular, as \( A \) is positive definite, there is a \( c > 0 \) so that \( |\hat{\varphi}(t)| \leq 1 - c ||t||^2 \) when \( ||t|| \leq 1 \).

The results in Claim 4.1 and Claim 4.2 are standard and we refer the reader to [29] for the proofs.

Throughout this section we denote by \( \psi \) the density function of the Gaussian distribution \( N(0, \text{Cov}(X)) \), which is given by
\[
\psi(x) := \frac{1}{\sqrt{(2\pi)^d \det(\text{Cov}(X))}} e^{-\frac{1}{2}x^T \text{Cov}(X)^{-1}x} = \frac{1}{(2\pi)^{\frac{d}{2}} \det A} e^{-\frac{1}{2}||A^{-1}x||^2}, \quad x \in \mathbb{R}^d,
\]
where \( A \) is defined in (18). Observe that for any \( j \in \mathbb{N} \) we have
\[
\psi^j(x) = \frac{1}{(2\pi j)^{\frac{d}{2}} \det A} e^{-\frac{1}{2j}||A^{-1}x||^2}, \quad x \in \mathbb{R}^d
\]
and that
\[
\hat{\varphi}(t) = e^{-2\pi^2 ||A t||^2}, \quad t \in \mathbb{R}^d,
\]
where we used part (i) of Claim 4.1 and the fact that \( F(e^{-\frac{1}{2}x^T A^{-1}x}) = \pi^\frac{d}{2} e^{-\frac{1}{2}||A^{-1}x||^2} \).

For later reference we write explicitly the following consequence of (38): For any \( M \geq 0 \) there are \( C, c_M > 0 \) so that
\[
|\hat{\varphi}^j(t) - \hat{\varphi}^j(t)| \leq C j ||t||^3 e^{-c_M j ||t||^2}, \quad ||t|| \leq M, \ j \in \mathbb{N}.
\]
The inequality follows since when \( j \|t\|^3 \geq 1 \) and \( \|t\| \leq M \), using Claim 4.1,
\[
|\hat{\varphi}^j(t) - \hat{\psi}^j(t)| \leq |\hat{\varphi}^j(t)| + |\hat{\psi}^j(t)| \leq 2 \left( 1 - c_M \|t\|^2 \right)^j \leq 2 e^{-c_M \|t\|^2} \|t\|^2
\]
and when \( j \|t\|^3 \leq 1, \)
\[
\hat{\varphi}^j(t) = (1 - 2\pi^2 \|At\|^2 + O(\|t\|^3))^{j} = e^{-2\pi^2 j \|At\|^2 + O(j \|t\|^3)} = (1 + O(j \|t\|^3)) \hat{\psi}^j(t).
\]

4.2. Asymptotics. Recall that the weights \( W_{L,j} \) are given in (20) by the formula
\[
W_{L,j} = \theta L^d \sum_{k \in \mathbb{Z}^d} \varphi^{*j}(Lk) = \theta \sum_{m \in \frac{1}{L} \mathbb{Z}^d} \hat{\varphi}^j(m),
\]
where in the last equality we used the Poisson summation formula (see parts (i), (iii) and (iv) of Claim 4.1), and that the function \( G_L \) from (28) is given by
\[
G_L(z) = \sum_{j=1}^{\infty} \frac{W_{L,j} z^j}{j}.
\]
Define the function
\[
g(z) := \sum_{j=1}^{\infty} \frac{\theta \varphi^{*j}(0)}{j} z^j.
\]
We will show that \( W_{L,j} \) approximately equals \( \theta L^d \varphi^{*j}(0) \) for small \( j \) and therefore, \( G_L(z) \) approximately equals \( L^d g(z) \) for small values of \( |z| \). This will facilitate our analysis as the function \( g(z) \) is independent of \( L \). We will also see that \( g \) determines the critical density in dimensions \( d \geq 3 \) as
\[
g'(1) = \theta \sum_{j=1}^{\infty} \varphi^{*j}(0) = \rho_c.
\]
The following lemma, a multidimensional, local central limit theorem, determines the asymptotic behavior of \( \varphi^{*j}(0) \).

Lemma 4.3. For sufficiently large \( j \in \mathbb{N} \),
\[
\varphi^{*j}(0) = \left( 1 + O(j^{-\frac{1}{2}}) \right) \psi^{*j}(0) = \frac{1 + O(j^{-\frac{1}{2}})}{\det(\text{Cov}(X)) (2\pi)^d} \cdot j^{-\frac{d}{2}}.
\]

Proof. The second equality in (45) is by (40). We turn to prove the first one. Let \( j \in \mathbb{N} \). As \( \varphi \) and \( \psi \) are Schwartz functions, there exists a \( C_0 > 0 \) so that
\[
\max \left\{ |\hat{\varphi}(t)|, |\hat{\psi}(t)| \right\} \leq \frac{C_0}{\|t\|^{d+1}}, \quad t \in \mathbb{R}^d.
\]
Now by parts (ii) and (iii) of Claim 4.1 we have
\[
|\varphi^{*j}(0) - \psi^{*j}(0)| = \left| \int_{\mathbb{R}^d} \hat{\varphi}^j(t) dt - \int_{\mathbb{R}^d} \hat{\psi}^j(t) dt \right|
\leq \int_{\|t\| \leq 2C_0} |\hat{\varphi}^j(t) - \hat{\psi}^j(t)| dt + 2 \int_{\|t\| \geq 2C_0} \max \left\{ |\hat{\varphi}(t)|, |\hat{\psi}(t)| \right\} dt.
\]
We estimate each of the integrals separately. First, substituting (42) we obtain
\[
\int_{\|t\| \leq 2C_0} |\hat{\varphi}(t) - \hat{\psi}(t)| dt \leq C_j \int_{\mathbb{R}^d} \|t\|^3 e^{-c_j\|t\|^2} dt \leq C_j \frac{1}{d+1} \int_{\mathbb{R}^d} \|s\|^3 e^{-c\|s\|^2} ds \leq C_j \frac{1}{d+1},
\]
where in the second inequality we changed variables to \( s = \sqrt{j} t \). Second, using (46),
\[
\int_{\|t\| \geq 2C_0} \max\left\{|\hat{\varphi}(t)|, \hat{\psi}(t)\right\} dt \leq \int_{\|t\| \geq 2C_0} \left( \frac{C_0}{\|t\|^{d+1}} \right)^j dt \leq C e^{-c_j}.
\]
Substituting (48) and (49) in (47) yields the first equality in (45). \( \square \)

The next pair of lemmas determine the asymptotic behavior of \( (W_{L,j}) \) in all regimes of \( j \) and \( L \). A change in behavior takes place when \( j \) is approximately \( L^2 \).

**Lemma 4.4.** For any \( \epsilon > 0 \) there exists a \( C_\epsilon > 0 \) such that
\[
|W_{L,j} - \theta L^d \varphi^{*j}(0)| \leq C_\epsilon L^{-2}
\]
for all \( L \geq 1 \) and integer \( j \leq L^{2-\epsilon} \).

**Proof.** First recall that \( \varphi^{*j} \) is the density function of \( S_j := \sum_{i=1}^j X_i \) where \( (X_i), i \geq 1 \), are independent copies of \( X \). Intuitively, this implies that the sum \( S_j \) for \( j \leq L^{2-\epsilon} \) is unlikely to have \( \|S_j\| \geq L \). The lemma will follow from a quantitative estimate of this type together with control of the smoothness of \( \varphi^{*j} \).

Since \( \varphi \) is Schwartz,
\[
\sup_{j \geq 1} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial \varphi^{*j}}{\partial x_i}(x) \right| \leq \sup_{x \in \mathbb{R}^d} \left| \frac{\partial \varphi}{\partial x_i}(x) \right| \cdot \sup_{j \geq 1} \int_{\mathbb{R}^d} \varphi^{(j-1)}(t) dt < \infty, \quad 1 \leq i \leq d,
\]
as \( \varphi^{(j-1)} \) is a density function. Thus, there is a \( c > 0 \) so that for any \( k \in \mathbb{Z}^d \) and \( j \geq 1 \),
\[
\varphi^{*j}(x) \geq \frac{1}{2} \varphi^{*j}(Lk), \quad x \in B(Lk, c \varphi^{*j}(Lk)),
\]
where \( B(x, r) \subseteq \mathbb{R}^d \) is the ball of radius \( r \) centered at \( x \). We obtain that for every \( k \in \mathbb{Z}^d \setminus \{0\} \), on the one hand,
\[
P(\|S_j\| \geq L\|k\|) = \int_{\|x\| \geq L\|k\|} \varphi^{*j}(x) dx \geq \frac{1}{2} \varphi^{*j}(Lk) \cdot \text{Vol} \left[ B(Lk, c \varphi^{*j}(Lk)) \cap \{x \in \mathbb{R}^d : \|x\| \geq L\|k\| \} \right] \geq c(\varphi^{*j}(Lk))^{d+1}.
\]
On the other hand, by Markov’s inequality, for any even \( n \geq 2 \), there is a \( C_n > 0 \) so that,
\[
P(\|S_j\| \geq \|k\|\|L\|) \leq \frac{\mathbb{E} \|S_j\|^n}{\|k\|^n L^n} \leq \frac{C_n \|L\|^n}{\|k\|^n L^n} = \frac{C_n}{\|k\|^n L^{n-\frac{2}{n}}}.
\]
To see the second inequality in (52), note that if \( d = 1 \) then, as \( n \) is even, \( \|S_j\|^n = (X_1 + \cdots + X_j)^n \) and there are at most \( C_n j^{\frac{n}{2}} \) terms with non-zero expectation (and each term has expectation bounded by \( C_n \)). The case of higher dimensions may be reduced to the one-dimensional case by noting that, as \( n \) is even, \( \|S_j\|^n = (S_{j,1}^2 + \cdots + S_{j,d}^2)^{\frac{n}{2}} \leq C_n (S_{j,1}^n + \cdots + S_{j,d}^n) \), where we write \( S_j = (S_{j,1}, \ldots, S_{j,d}) \).
Thus, combining (51) and (52) and taking \( n \) sufficiently large as a function \( \epsilon \), we get
\[
\varphi^j(Lk) \leq \frac{C\epsilon}{L^{d+2}} \|k\|^{-d-1}, \quad k \in \mathbb{Z}^d \setminus \{0\},
\]
and therefore
\[
W_{L,j} - \theta L^d \varphi^j(0) = \theta L^d \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi^j(Lk) \leq C\epsilon L^{-2} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|k\|^{d+1}} \leq C\epsilon L^{-2}.
\]
□

**Lemma 4.5.** There are \( C, c > 0 \) such that for all \( L \geq 1 \) and \( j \in \mathbb{N} \),
\[
\left| W_{L,j} - \theta \sum_{m \in \mathbb{Z}^d} \hat{\psi}^j(m) \right| \leq C L^d j^{-\frac{d+1}{2}} e^{-c\frac{j}{L^2}}.
\]

**Proof.** The proof follows similar lines to the proof of Lemma 4.3. Let \( L \geq 1 \) and \( j \in \mathbb{N} \). Since \( \varphi \) and \( \psi \) are Schwartz, there exists a \( C_0 > 0 \) so that
\[
\max \left\{ \|\hat{\varphi}(t)|, \hat{\psi}(t) \right\} \leq \frac{C_0}{\|t\|^{d+1}}, \quad t \in \mathbb{R}^d.
\]
We have, by (43),
\[
\left| W_{L,j} - \theta \sum_{m \in \mathbb{Z}^d} \hat{\psi}^j(m) \right| \leq \theta \sum_{m \in \mathbb{Z}^d} |\hat{\varphi}^j(m) - \hat{\psi}^j(m)|
\leq \theta \sum_{m \in \mathbb{Z}^d} |\hat{\varphi}^j(m) - \hat{\psi}^j(m)| + 2\theta \sum_{m \in \mathbb{Z}^d} \max \left\{ \|\hat{\varphi}(m)|, \hat{\psi}(m) \right\}.
\]
We estimate each of the sums separately. First, by (42), we have
\[
\sum_{m \in \mathbb{Z}^d, \|m\| \leq 2C_0} |\hat{\varphi}^j(m) - \hat{\psi}^j(m)| \leq C j \sum_{m \in \mathbb{Z}^d, \|m\| \leq 2C_0} \|m\|^3 e^{-c\|m\|^2} = C j^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}^d} \|n\|^3 e^{-c\|n\|^2}.
\]
When \( j \geq L^2 \), one can easily see that the last expression is at most \( C j^{-\frac{1}{2}} e^{-c\frac{j}{L^2}} \leq C L^d j^{-\frac{d+1}{2}} e^{-c\frac{j}{L^2}} \).
When \( j \leq L^2 \) we have
\[
\sum_{n \in \mathbb{Z}^d} \|n\|^3 e^{-c\|n\|^2} \leq \sum_{i=1}^{\infty} \sum_{i-1 \leq \|n\| < i} i^3 e^{-c(i-1)^2} \leq C \left( \frac{L}{\sqrt{j}} \right)^d \sum_{i=1}^{\infty} i^{d+2} e^{-c(i-1)^2} \leq C \left( \frac{L}{\sqrt{j}} \right)^d.
\]
Second, substituting (54),
\[
\sum_{m \in \mathbb{Z}^d, \|m\| \geq 2C_0} \max \left\{ \|\hat{\varphi}(m)|, \hat{\psi}(m) \right\} \leq \sum_{m \in \mathbb{Z}^d, \|m\| \geq 2C_0} \left( \frac{C_0}{\|m\|^{d+1}} \right)^j \leq C L^d e^{-cj}.
\]
Inequality (53) follows by substituting all the bounds in (55). □

**Corollary 4.6.** There are \( C, c > 0 \) so that for any \( L \geq 1 \) and \( j \in \mathbb{N} \):

(i) If \( j \leq L^2 \)
\[
|W_{L,j} - \theta L^d \varphi^j(0)| \leq C \left( L^{-\frac{1}{2}} + e^{L^2 \frac{j}{L^2}} \right).
\]
(ii) If \( j \geq L^2 \)
\[
|W_{L,j} - \theta| \leq C e^{-c \frac{j^2}{L^2}}. \tag{57}
\]

**Proof.** Part (ii) is an immediate consequence of Lemma 4.5 as, when \( j \geq L^2 \),
\[
|W_{L,j} - \theta| \leq C e^{-c \frac{j^2}{L^2}} + \theta \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \hat{\psi}(m) \leq C e^{-c \frac{j^2}{L^2}} + \theta \sum_{m \in \mathbb{Z}^d \setminus \{0\}} e^{-c \|m\|^2} \leq C e^{-c \frac{j^2}{L^2}}.
\]

We turn to prove part (i). Fix \( \epsilon = \frac{1}{d+1} \). When \( j \leq L^{2-\epsilon} \) then (56) holds by Lemma 4.4. When \( L^{2-\epsilon} \leq j \leq L^2 \),
\[
|W_{L,j} - \theta L^d \varphi^{*j}(0)| \leq |W_{L,j} - \theta L^d \sum_{k \in \mathbb{Z}^d} \psi^{*j}(Lk)| + \theta L^d \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \psi^{*j}(Lk) + \theta L^d \left| \psi^{*j}(0) - \varphi^{*j}(0) \right|.
\]

We bound each term separately. First, as \( L^d \sum_{k \in \mathbb{Z}^d} \psi^{*j}(Lk) = \sum_{m \in \mathbb{Z}^d} \hat{\psi}(m) \) by the Poisson summation formula (as in (43)), the first term is bounded by \( CL^d j^{-\frac{d+1}{2}} \leq CL^{-\frac{d}{2}} \) by the choice of \( \epsilon \). Second, we use (40) to bound the second term,
\[
L^d \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \psi^{*j}(Lk) \leq C \left( \frac{L}{\sqrt{j}} \right)^d \sum_{k \in \mathbb{Z}^d \setminus \{0\}} e^{-c \frac{j^2}{L^2} \|k\|^2} \leq C \left( \frac{L}{\sqrt{j}} \right)^d e^{-c \frac{j^2}{L^2} \frac{1}{j}} \leq C e^{-c \frac{j^2}{L^2}}.
\]

Lastly, by Lemma 4.3, the third term is at most \( CL^d j^{-\frac{d+1}{2}} \leq CL^{-\frac{d}{2}} \) by the choice of \( \epsilon \). \( \square \)

**Corollary 4.7.** There is a \( C > 0 \) so that \( |W_{L,j} - \theta L^d \varphi^{*j}(0)| \leq C \) for any \( L \geq 1 \) and \( j \in \mathbb{N} \). Therefore, for any integer \( n \in \mathbb{N} \) there is a \( C_n > 0 \) for which
\[
\left| G^{(n)}_L(z) - L^d g^{(n)}(z) \right| \leq C_n (1 - |z|)^{-n}, \quad |z| < 1,
\]
where we recall that the functions \( G_L \) and \( g \) are defined in (28) and (44) respectively.

**Proof.** The bound \( |W_{L,j} - \theta L^d \varphi^{*j}(0)| \leq C \) follows in the case \( j \leq L^2 \) from Corollary 4.6 and in the case \( j \geq L^2 \) from Lemma 4.3 and Corollary 4.6. The bound on the generating functions is an immediate consequence, noting that \( \frac{(n-1)!}{(1-z)^n} = \sum_{j=1}^{\infty} (j-1) \cdots (j-n+1) z^{j-n} \) for \( n \in \mathbb{N} \) and \( |z| < 1 \). \( \square \)

### 4.3. Analytic continuation

We shall view the generating functions \( G_L \) and \( g \), given by the Taylor expansions (28) and (44), as functions of a complex variable. By Lemma 4.3 and Corollary 4.6, these functions are analytic in the open unit disk, as their Taylor expansions converge there. In this section we prove that they can be continued analytically to a larger domain \( \Delta_0 \).

**Definition 4.8.** For \( R > 1 \) and \( 0 < \beta < \frac{\pi}{2} \), define an open domain \( \Delta(R, \beta) \) in the complex plane by (see Figure 3)
\[
\Delta(R, \beta) := \{ z \in \mathbb{C} \setminus \{1\} : |z| < R, |\arg(z - 1)| > \beta \}.
\]

The following technical claim is required to derive the analytic continuation of \( G_L \) and \( g \).

**Claim 4.9.** There exists \( R_0 > 1 \) such that the following holds for the domain
\[
\Delta_0 := \Delta\left(R_0, \frac{\pi}{8}\right). \tag{58}
\]
We proceed to show that $G_{L}$ and $g$ can be continued analytically to $\Delta_{0}$.

**Lemma 4.10.** The generating functions $G_{L}$, for any $L \geq 1$, and $g$ extend to analytic functions in $\Delta_{0}$ (defined in (58)), where they are given by

$$G_{L}(z) = -\theta \sum_{m \in \mathbb{Z}^{d}} \log (1 - z \hat{\phi}(m)),$$

$$g(z) = -\theta \int_{\mathbb{R}^{d}} \log (1 - z \hat{\phi}(t)) \, dt,$$

with the sum and integral converging uniformly on compact subsets of $\Delta_{0}$.

We start with the proof of Claim 4.9.

**Proof of Claim 4.9.** We start with the first part. We assume that $t \neq 0$ as the case $t = 0$ is straightforward. By Claim 4.2 there exists a $\delta > 0$ such that for any $0 < ||t|| \leq \delta$,

$$|\arg(1 - \hat{\phi}(t))| \leq \frac{\pi}{24}, \quad |1 - \hat{\phi}(t)| \geq c ||t||^{2}, \quad |\arg \hat{\phi}(t)| \leq \frac{\pi}{24} \quad \text{and} \quad |\hat{\phi}(t)| \geq \frac{1}{2}.$$

Thus, for $||t|| \leq \delta$ and $z \in \mathbb{C} \setminus \{1\}$ with $\arg(z - 1) \geq \frac{\pi}{8}$ we have

$$|1 - z \hat{\phi}(t)| = |(1 - \hat{\phi}(t)) - (z - 1) \hat{\phi}(t)|$$

$$\geq c (||1 - \hat{\phi}(t)|| + |1 - z| \cdot |\hat{\phi}(t)|| \geq c (||t||^{2} + |1 - z|),$$

where $A$ is given in (18).
where in the first inequality we used the fact that for any $w_1, w_2 \in \mathbb{C}$ satisfying $|\arg w_1| \leq \frac{\pi}{24}$ and $|\arg w_2| \geq \frac{\pi}{12}$ we have

$$|w_1 - w_2| \geq \sin \frac{\pi}{24} \cdot \max (|w_1|, |w_2|) \geq c (|w_1| + |w_2|).$$

Again, by Claim 4.2, there exists some $\epsilon > 0$ such that $|\hat{\phi}(t)| \leq 1 - \epsilon$ for $\|t\| \geq \delta$. Thus, for $\|t\| \geq \delta$ and $|z| \leq 1 + \epsilon$,

$$|1 - z\hat{\phi}(t)| \geq 1 - |z| \cdot |\hat{\phi}(t)| \geq c. \quad (64)$$

The first part of the claim with $R_0 = 1 + \epsilon$ follows from (63) and (64).

We continue with the second part. Fix $R_0$ and $\Delta_0$ such that the first part holds. Using the Taylor expansion of $\hat{\phi}$ in (38), we obtain

$$|1 - z\hat{\phi}(t) - (1 - z + 2\pi^2 \|At\|^2)\hat{\phi}(t)| \leq \left| (1 - 2\pi^2 \|At\|^2 - \hat{\phi}(t)) + (z - 1)(1 - \hat{\phi}(t)) \right|$$

$$\leq C\|t\|^3 + C|1 - z| \cdot \|t\|^2 \leq C\|t\| \cdot |1 - z\hat{\phi}(t)|, \quad z \in \Delta_0, \quad \|t\| \leq 1, \quad (65)$$

where in the third inequality we used the first part of the claim. Dividing both sides by $|1 - z\hat{\phi}(t)|$ yields $60$.

We turn to prove Lemma 4.10.

**Proof of Lemma 4.10.** We will start by proving that the identities in (61) and (62) hold when $|z| < 1$. Then, we will show that the right-hand side of these identities define analytic functions in $\Delta_0$.

Fix $z$ with $|z| < 1$. By (28) and (43),

$$G_L(z) = \sum_{j=1}^{\infty} \frac{W_L j}{j} z^j = \theta \sum_{j=1}^{\infty} \sum_{m \in \mathbb{Z}^d} \frac{1}{j^2} \hat{\phi}^j(m) z^j$$

$$= \theta \sum_{m \in \mathbb{Z}^d} \sum_{j=1}^{\infty} \frac{1}{j} (z\hat{\phi}(m))^j = -\theta \sum_{m \in \mathbb{Z}^d} \log (1 - z\hat{\phi}(m)), \quad (66)$$

where the last equality holds as $|z\hat{\phi}(m)| \leq |z| < 1$. The change of order of summation in the third equality is justified as the sum converges absolutely. Indeed, since $\hat{\phi}$ is a Schwartz function (part (iii) of Claim 4.1) and $|\hat{\phi}(m)| \leq 1$,

$$\sum_{j=1}^{\infty} \sum_{m \in \mathbb{Z}^d} \frac{1}{j} |\hat{\phi}^j(m) z^j| \leq \sum_{j=1}^{\infty} |z|^j \sum_{m \in \mathbb{Z}^d} |\hat{\phi}(m)| \leq \frac{|z|}{1 - |z|} \sum_{m \in \mathbb{Z}^d} |\hat{\phi}(m)| < \infty. \quad (67)$$

We turn to prove the identity for $g$. Using Fourier inversion theorem (Part (iii) of Claim 4.1) we obtain,

$$g(z) = \sum_{j=1}^{\infty} \frac{\theta \phi^j(0)}{j} z^j = \theta \sum_{j=1}^{\infty} \frac{z^j}{j} \int_{\mathbb{R}^d} \hat{\phi}^j(t) dt = \theta \int_{\mathbb{R}^d} \sum_{j=1}^{\infty} \frac{1}{j} (z\hat{\phi}(t))^j dt = -\theta \int_{\mathbb{R}^d} \log (1 - z\hat{\phi}(t)) dt,$$

where the second inequality is justified by a calculation similar to (67).

Now, consider the right-hand side of the identities (61) and (62) for $z \in \Delta_0$. First note that the logarithms are well defined. Indeed, otherwise there are $z \in \Delta_0$ and $t \in \mathbb{R}^d$ so that $z\hat{\phi}(t) = R \geq 1$, but then $z' = \frac{z}{R} \in \Delta_0$ satisfies $1 - z'\hat{\phi}(t) = 0$ contradicting the bound in Claim 4.9.
It suffices to prove that the sum and integral in (61) and (62) converge uniformly on compact subsets of $\Delta_0$. Let $\Delta \subseteq \Delta_0$ be a compact subset. By the first part of Claim 4.9 for any $t \in \mathbb{R}^d$ and $z \in \Delta$, $|1 - z\hat{\phi}(t)| \geq c_\Delta$. Thus, considering separately the cases of small and large $|\hat{\phi}(t)|$, we obtain

$$|\log (1 - z\hat{\phi}(t))| \leq C_\Delta |\hat{\phi}(t)|.$$ 

The uniform convergence follows as $\hat{\phi}$ is a Schwartz function. □

In the following corollary we deduce explicit formulas for the derivatives of $G_L$ and $g$ in the domain $\Delta_0$ which will be of repeated use in the sequel.

**Corollary 4.11.** For any $L \geq 1$ and $n \in \mathbb{N}$,

$$G_L^{(n)}(z) = \theta (n - 1)! \sum_{m \in \mathbb{Z}^d} \frac{\hat{\phi}^n(m)}{(1 - z\hat{\phi}(m))^n}, \quad z \in \Delta_0.$$ \hspace{1cm} (68)

$$g^{(n)}(z) = \theta (n - 1)! \int_{\mathbb{R}^d} \frac{\hat{\phi}^n(t)}{(1 - z\hat{\phi}(t))^n} dt, \quad z \in \Delta_0.$$ \hspace{1cm} (69)

**Proof.** The formulas follow from Lemma 4.10 by differentiating the identities in (61) and (62) under the sum and integral signs, making use of the uniform convergence on compact subsets. □

4.4. **Near the singularity.** The following lemma determines the asymptotic behavior of the derivatives of the function $g$ near the potential singularity $z = 1$. The derivative of $g$ is closely related to the polylogarithm function. Indeed, they differ merely by a constant multiple when the density function $\phi$ is the Gaussian density $\psi$ (given by (39)). Our goal here is to show that, for general $\phi$, $g$ shares the asymptotic behavior of the polylogarithm function near $z = 1$.

**Lemma 4.12.** The following holds for any integer $n \geq 1$:

(i) If $d > 2n$ then the function $g^{(n)}$ may be extended to a continuous function in $\Delta_0$.

(ii) If $d = 2n$ then for $z \in \Delta_0$ we have

$$g^{(n)}(z) = \frac{\theta}{(2\pi)^{\frac{d}{2}} \sqrt{\det(Cov(X))}} \log \left( \frac{1}{1 - z} \right) + O(1).$$

(iii) If $d < 2n$ then for $z \in \Delta_0$ we have

$$g^{(n)}(z) = \frac{\theta \cdot \Gamma \left( n - \frac{d}{2} \right)}{(2\pi)^{\frac{d}{2}} \sqrt{\det(Cov(X))}} (1 - z)^{\frac{d}{2} - n} + O_n \left( |1 - z|^{\frac{d}{2} - n + \frac{1}{2}} + |\log (1 - z)| \right),$$

where $\Gamma$ is the gamma function.

**Proof.** Inside the unit disc, the three parts follow in a straightforward manner from the definition of $g$ in (44) and the asymptotic behavior of $\varphi^{+j}(0)$ given in Lemma 4.3. However, since we need the asymptotics outside the radius of convergence as well, we will derive it using the analytic continuation representation of $g^{(n)}$ given in (69).

We start with part (i) When $d > 2n$, we have by (59),

$$\sup_{z \in \Delta_0} \int_{\mathbb{R}^d} \left| \frac{\hat{\phi}^n(t)}{(1 - z\hat{\phi}(t))^n} \right| dt \leq C_n \int_{||t|| \leq 1} \frac{1}{||t||^{2n}} + C_n \int_{||t|| \geq 1} |\hat{\phi}(t)| dt < \infty$$
as \( \hat{\varphi} \) is a Schwartz function. Thus, the integral on the right-hand side of (60) converges uniformly in \( z \in \Delta_0 \) and therefore it is continuous.

We turn to prove parts (ii) and (iii). Using (60), the Taylor expansion of \( \hat{\varphi} \) in (38) and the bounds (60) and (59) we get

\[
g^{(n)}(z) = O_n(1) + \theta(n - 1)! \int_{\|t\| \leq 1} \frac{1 + O_n(\|t\|)}{(1 - z + 2\pi^2\|A\|^2)^n} dt, \quad z \in \Delta_0.
\]

We develop the above integral using a change of variables,

\[
\int_{\|t\| \leq 1} \frac{1 + O_n(\|t\|)}{(1 - z + 2\pi^2\|A\|^2)^n} dt = O_n(1) + \frac{1}{\det(A)} \int_{\|x\| \leq 1} \frac{1 + O_n(\|x\|)}{(1 - z + 2\pi^2\|x\|^2)^n} dx
\]

\[
= O_n(1) + \frac{\text{Vol}(S^{d-1})}{\det A} \int_0^1 \frac{r^{d-1} (1 + O_n(r))}{(1 - z + 2\pi^2r^2)^n} dr = O_n(1) + \frac{2(2\pi)^{-\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{1}{\det A} \int_0^1 s^{d-1} (1 + O_n(s)) ds,
\]

where we denote by \( \text{Vol}(S^{d-1}) = 2\pi^{\frac{d}{2}} (\Gamma(\frac{d}{2}))^{-1} \) the surface area of the \((d - 1)\)-dimensional unit sphere and where an additional \( O_n(1) \) term is added due to the changes in the domain of integration. We set

\[
I_{n,d}(\xi) := \int_0^1 \frac{s^{d-1}}{s^2 + (\xi + s)^2}, \quad \xi \in \mathbb{C} \setminus \{0\}, \quad |\arg(\xi)| \leq \frac{7\pi}{8}.
\]

The last computation shows that, for \( z \in \Delta_0 \),

\[
\left| g^{(n)}(z) - \frac{2\theta(n - 1)! (2\pi)^{-\frac{d}{2}}}{\Gamma(\frac{d}{2})} \cdot I_{n,d}(1 - z) \right| \leq C_n \int_0^1 \frac{s^ds}{(1 - z + s^2)^n} \leq C_n I_{n,d+1}(|1 - z|). \quad (70)
\]

Thus is suffices to estimate \( I_{n,d}(\xi) \) as \( \xi \to 0 \). To this end, note that \( I_{n,d}(\xi) \) is bounded when \( d > 2n \) and that, using integration by parts,

\[
I_{n,d}(\xi) = \int_0^1 \frac{s^{d-1}}{(\xi + s)^2} ds = \int_0^1 \frac{s^d}{(\xi + s)^n} \left[ \frac{1}{d} \int_0^1 \frac{s^{d+1}}{(\xi + s)^{n+1}} ds \right]
\]

\[
= \frac{2n}{d} \int_0^1 \frac{s^{d+1}}{(\xi + s)^{n+1}} ds = \frac{2n}{d} I_{n+1,d+2}(\xi) + O_n(1). \quad (71)
\]

As a result, it suffices to estimate \( I_{k,1} \) and \( I_{k,2} \) for \( k \geq 1 \). For any \( \xi \in \mathbb{C} \setminus \{0\} \) with \( |\arg(\xi)| \leq \frac{7\pi}{8} \), we have

\[
I_{k,2}(\xi) = \int_0^1 \frac{s}{(\xi + s)^{k+1}} ds = \frac{1}{2} \int_0^1 \frac{dy}{(\xi + y)^k} = O_k(1) + \begin{cases} \frac{-\frac{1}{2} \log(\xi)}{2(k - 1)^{\frac{1}{2}}} \xi^{-(k-1)} & \text{for } k = 1 \\ \frac{\frac{1}{2} (k - 1)^{-\frac{1}{2}}}{2(k - 1)!} & \text{for } k > 1 \end{cases}
\]

and

\[
I_{k,1}(\xi) = \int_0^1 \frac{ds}{(\xi + s)^n} = O_k(1) + \frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{(\xi + s)^k} = \frac{\sqrt{\pi} \Gamma(\frac{1}{2} - \frac{1}{k}) \xi^{\frac{1}{2} - k}}{2(k - 1)!} + O_k(1),
\]

where the last equation follows using a standard residue argument. Indeed, it follows by integrating over the semi-circle contour noting that the integrand has a unique pole in \( \{ \text{Im}(z) > 0 \} \), located at \( i\sqrt{\xi} \) having order \( k \). The details are left to the reader.
We conclude that, when \( d = 2n \),
\[
I_{n,d}(\xi) = -\frac{1}{2} \log \xi + O(1)
\]
and when \( d < 2n \),
\[
I_{n,d}(\xi) = \frac{\Gamma \left( \frac{d}{2} \right) \Gamma \left( n - \frac{d}{2} \right)}{2(n-1)!} \xi^{\frac{d}{2} - n} + O_n(1)
\]
as one can check that the last expressions satisfy the recursion in (71) and the cases \( d = 1, 2 \) (note that \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \)). Substituting the last estimates in (70) finishes the proof of the lemma. \( \square \)

5. The sub-critical regime

In this section we establish parts (i) and (ii) of Theorem 1.1, part (i) and the sub-critical regime in part (ii) of Theorem 1.2 and part (i) of Theorem 1.3.

Let \( \rho = \rho(N) \) and consider several possibilities for the dimension \( d \) and the asymptotic regime of \( \rho(N) \) as \( N \to \infty \) corresponding to the various statements in the theorems. We give names to these cases to simplify later reference. By the name (Sub-Critical) we refer collectively to any of the following cases:

(SubConst1) : Dimension \( d = 1 \) and density \( \rho \) satisfying \( \rho \to \rho_* \in (0, \infty) \) as \( N \to \infty \).

(Sub1) : Dimension \( d = 1 \) and density \( \rho \) satisfying \( \rho \to \infty \) and \( \rho = o(\sqrt{N}) \).

(SubConst2) : Dimension \( d = 2 \) and density \( \rho \) satisfying \( \rho \to \rho_* \in (0, \infty) \) as \( N \to \infty \).

(Sub2) : Dimension \( d = 2 \) and density \( \rho \) satisfying \( \rho \to \infty \) and \( \frac{\rho}{\log N} \to \alpha \in [0, \alpha_c) \).

(Sub3) : Dimension \( d \geq 3 \) and density \( \rho \) satisfying \( \rho \to \rho_* \in (0, \rho_c) \) as \( N \to \infty \).

We remind the reader that the critical thresholds \( \alpha_c \) and \( \rho_c \) are defined in (9) and (10). We note explicitly that we allow \( \rho \) to vary with \( N \) in the cases (SubConst1), (SubConst2) and (Sub3), thus obtaining a somewhat stronger results than those stated in part (i) of the main theorems. This additional flexibility will be used in our analysis of the critical regime in dimensions \( d \geq 3 \) in Section 7.

Our strategy, following [16, Section 3.2], is to use saddle point analysis to estimate the Cauchy integral (recall (29)),
\[
H_n(L) = [z^n] e^{G_L(z)} = \frac{1}{2\pi i} \oint_C e^{G_L(z)} \frac{dz}{z^{n+1}}
\]
(72)
(see Theorem 5.1) and then apply the results of Section 3 to connect \( H_n(L) \) with the distribution of \( L_1 \) (Section 5.2).

5.1. Saddle point analysis. The integrand in (72) is \( \exp(G_L(z) - (n+1) \log(z)) \) and thus its saddle points, the critical points of the exponent, are the solutions \( z \) to \( zG'_L(z) = n + 1 \). Motivated by this and the fact that the values of \( n \) in our analysis will be close to \( N \), we define \( r_N = r_{N,L} \) to be the unique \( 0 < r < 1 \) satisfying
\[
rG'_L(r) = N.
\]
(73)
This solution exists as, by Corollary 4.6, \( G'_L(r) \to 1 \), for all \( L \geq 1 \). \( r_N \) is unique since \( G'_L \) has non-negative Taylor coefficients. The saddle point method suggests to take a contour of integration that passes through \( r_N \).
In the following theorem we find the asymptotic behavior of $H_{N-j}(L)$. In the theorem and what follows we set, for $N \geq 1$,

$$a_N := r_N G'(r_N) + r_N^2 G''(r_N) \quad \text{and} \quad b_N := r_N G'_L(r_N) + 3r_N^2 G''(r_N) + r_N^3 G'''(r_N) \quad (74)$$

and note that by (73) and the fact that the Taylor coefficients of $G_L$ are non-negative we have $a_N, b_N \geq 0$.

**Theorem 5.1.** Let $j_N$ be a sequence of integers such that $1 \leq j_N \leq N$ and $j_N^2 = o(a_N)$ as $N \to \infty$. Then, in each of the (Sub-Critical) asymptotic regimes,

$$H_{N-j}(L) = [z^{N-j}]e^{e_L(z)} \sim \frac{e^{G_L(r_N)}}{r_N^{N-j} \sqrt{2\pi a_N}}, \quad N \to \infty. \quad (75)$$

uniformly in $0 \leq j \leq j_N$.

The proof of the theorem relies on the following technical estimates.

**Proposition 5.2.** In each of the (Sub-Critical) cases:

(i) As $N \to \infty$,

$$b_N^2 = o(a_N^3). \quad (76)$$

(ii) There exists $c' > 0$ such that, for large $N$,

$$\text{Re} \left( G_L(r_Ne^{it}) \right) \leq G_L(r_N) - c' \cdot \frac{1}{a_N} \sqrt{|t|}, \quad a_N^{-1} \leq |t| \leq \pi, \quad (77)$$

where $c'$ may depend on $\rho_*$ in the cases (Sub3), (SubConst1) and (SubConst2) (in addition to the usual dependence on $d, \phi$ and $\theta$).

Let us first show how Theorem 5.1 follows from Proposition 5.2.

**Proof of Theorem 5.1.** Fix a sequence $(r_N)_{N=1}^{\infty} \subseteq (0, \pi)$ so that as $N \to \infty$,

$$a_N^2 \to \infty, \quad b_N^3 \to 0 \quad \text{and} \quad t_Nj_N \to 0. \quad (78)$$

This is possible by part (i) of Proposition 5.2 and the assumption on $j_N$, e.g., by setting $t_N := \min \{a_N^{-1/4}, j_N^{-1/2}, b_N^{-1/6}, a_N^{-1/4}, \pi \}$. By the Cauchy integral formula with the contour $\gamma$ parametrized as $\gamma(t) := r_Ne^{it}$ for $t \in [-\pi, \pi]$,

$$H_{N-j}(L) = [z^{N-j}]e^{G_L(z)} = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{G_L(z)}}{z^{N-j+1}} \, dz = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{G_L(r_Ne^{it})-itN+itj} \, dt = I_1 + I_2,$$

where $I_1$, $I_2$ are the integrals corresponding to $|t| \leq t_N$ and $t_N < |t| \leq \pi$. We start by estimating $I_1$ via a Taylor expansion of $G_L(r_Ne^{it})$ around $t = 0$. A straightforward calculation yields that

$$\frac{d}{dt} G_L(r_Ne^{it}) = ir_Ne^{it} G'_L(r_Ne^{it}),$$

$$\frac{d^2}{dt^2} G_L(r_Ne^{it}) = -r_Ne^{it} G''_L(r_Ne^{it}) - r_N^2 e^{2it} G'''_L(r_Ne^{it}),$$

$$\frac{d^3}{dt^3} G_L(r_Ne^{it}) = -ir_Ne^{it} G''_L(r_Ne^{it}) - 3i r_N^2 e^{2it} G'''_L(r_Ne^{it}) - i r_N^3 e^{3it} G''''_L(r_Ne^{it}),$$

and so forth.
from which we have, by the definition of \( r_N \) and as \( G_L \) has non-negative Taylor coefficients, that
\[
G_L \left( r_N e^{it} \right) = G_L(r_N) + itN - \frac{a_N}{2}t^2 + O(b_Nt^3), \quad |t| \leq t_N.
\]
Substituting this expansion in the definition of \( I_1 \) and using the relations (78),
\[
I_1 = \frac{e^{G_L(r_N)}}{2\pi r_N^{-\frac{1}{2}}} \int_{-t_N}^{t_N} e^{-\frac{2}{N}t^2 + o(1)} dt \sim \frac{e^{G_L(r_N)}}{\pi r_N^{-\frac{1}{2}} \sqrt{2} a_N} \int_{-\infty}^{\infty} e^{-s^2} ds = \frac{e^{G_L(r_N)}}{\pi r_N^{-\frac{1}{2}} \sqrt{2} a_N}, \quad N \to \infty.
\]
We turn to bound \( I_2 \). By part (ii) of Proposition 5.2, for large enough \( N \),
\[
|I_2| \leq \frac{C}{r_N^{-\frac{1}{2}}} \int_{t_N}^{\pi} e^{\Re(G_L(r_Ne^{it}))} dt \leq \frac{C \cdot e^{G_L(r_N)}}{r_N^{-\frac{1}{2}}} \int_{t_N}^{\infty} e^{-s} \sqrt{s} ds = \frac{C \cdot e^{G_L(r_N)}}{r_N^{-\frac{1}{2}} \sqrt{a_N \pi N}} \int_{t_N}^{\infty} e^{-s} \sqrt{s} ds,
\]
and the last expression is negligible compared to \( I_1 \) as \( \sqrt{a_N}t_N \to \infty \) by (78). \( \square \)

The rest of Subsection 5.1 is devoted to the proof of Proposition 5.2.

Recall the definition of \( r_N \) in (73) and that \( \rho = \frac{N}{r^2} \) is the density. In the next lemma we
find the asymptotic behavior of \( r_N \).

**Lemma 5.3.** As \( N \to \infty \) we have:

(i) In case (Sub1),
\[
1 - r_N \sim \frac{\theta^2}{2 \text{Var}(X)}, \quad \text{in particular} \quad 1 - r_N \to 0.
\]

(ii) In case (Sub2),
\[
\log \left( \frac{1}{1 - r_N} \right) \sim \frac{\rho}{\alpha_c}, \quad \text{in particular} \quad \frac{1}{1 - r_N} = N^{\frac{\rho}{\alpha_c} + o(1)}.
\]

(iii) In cases (Sub3), (SubConst1) and (SubConst2),
\[
r_N \to r_* < 1,
\]
where \( r_* \) is the unique solution of \( rg'(r) = \rho_c \) for \( r \in (0, 1) \).

**Proof.** We start with the case \( d = 1 \). Corollary 4.7 and Lemma 4.12 imply that
\[
\left| rG_L'(r) - \frac{r\theta L}{\sqrt{2 \text{Var}(X)(1 - r)}} \right| \leq C \left( \frac{r}{1 - r} + Lr \log \left( \frac{1}{1 - r} \right) \right), \quad r \in [0, 1), \ L \geq 1. \quad (79)
\]
We work in the asymptotic regime given by (Sub1). Denote \( I := [\frac{\rho^2}{8 \text{Var}(X)}, \frac{2\rho^2}{\text{Var}(X)^2}] \) and recall that \( \rho = \frac{N}{r^2} \to \infty \) as \( N \to \infty \). Having the terms in (79) in mind, we observe that as \( N \to \infty \), uniformly in \( 1 - r \in I \),
\[
\frac{r\theta L}{\sqrt{2 \text{Var}(X)(1 - r)}} \in \left[ \frac{1}{4} N, 2N \right] \quad \text{and} \quad \frac{r}{1 - r} + Lr \log \left( \frac{1}{1 - r} \right) = o(N),
\]
where we used that \( \rho^2 = o(N) \) and \( L \log \left( \frac{1}{1 - r} \right) \sim 2N \log(\rho) \). It follows, since \( r_N \) is the unique value in \( [0, 1) \) for which (73) holds and since \( rG_L'(r) \) is increasing in \( [0, 1) \) that
\[
N = r_NG_L'(r_N) \sim \frac{\theta L}{\sqrt{2 \text{Var}(X)(1 - r_N)}}, \quad N \to \infty,
\]
from which the required asymptotic formula follows.

We work in the asymptotic regime (Sub2), where \( \rho = \frac{N}{1^2} \to \infty \) and \( \frac{\rho}{\log N} \to \alpha \in [0, \alpha_c) \) as \( N \to \infty \). Fix \( 0 < \epsilon < \min \left\{ \frac{\alpha_c}{\alpha} - 1, 1 \right\} \). Denote \( I := \left[ \exp \left( -\frac{(1+\epsilon)\rho}{\alpha_c} \right), \exp \left( -\frac{\rho}{2\alpha_c} \right) \right] \). Then, as \( N \to \infty \), uniformly in \( 0 < r < \epsilon < 1 \), we have

\[
\left| rG_L'(r) - rL^2\alpha_c \log \left( \frac{1}{1-r} \right) \right| \leq C \left( \frac{r}{1-r} + L^2 \right), \quad r \in [0, 1), \; L \geq 1.
\]

We work in the asymptotic regime (Sub2), where \( \rho = \frac{N}{1^2} \to \infty \) and \( \frac{\rho}{\log N} \to \alpha \in [0, \alpha_c) \) as \( N \to \infty \). Fix \( 0 < \epsilon < \min \left\{ \frac{\alpha_c}{\alpha} - 1, 1 \right\} \). Denote \( I := \left[ \exp \left( -\frac{(1+\epsilon)\rho}{\alpha_c} \right), \exp \left( -\frac{\rho}{2\alpha_c} \right) \right] \). Then, as \( N \to \infty \), uniformly in \( 0 < r < \epsilon < 1 \), we have

\[
\left| rG_L'(r) - rL^2\alpha_c \log \left( \frac{1}{1-r} \right) \right| \leq C \left( \frac{r}{1-r} + L^2 \right), \quad r \in [0, 1), \; L \geq 1.
\]

In cases (Sub3), (SubConst1) and (SubConst2), for sufficiently large \( N \) we have

\[
N = r_N G_L'(r_N) \sim L^2\alpha_c \log \left( \frac{1}{1-r_N} \right), \quad N \to \infty,
\]

from which the required asymptotic formula follows.

In cases (Sub3), (SubConst1) and (SubConst2), for sufficiently large \( N \) we have

\[
\frac{r_* + 1}{2} G_L' \left( \frac{r_* + 1}{2} \right) \geq L^2 \frac{r_* + 1}{2} g' \left( \frac{r_* + 1}{2} \right) = N \rho^{-1} \frac{r_* + 1}{2} g' \left( \frac{r_* + 1}{2} \right) > N,
\]

where the first inequality is by comparison of the Taylor coefficients (see (20), (28) and (44)) and the last inequality holds as \( \rho \to \rho_* = r_* g'(r_*) \) and as \( r g'(r) \) is strictly increasing. As \( rG_L'(r) \) is increasing, we conclude that, for sufficiently large \( N, r_N \leq \frac{r_*+1}{2} < 1 \). Thus, by Corollary 4.7

\[
N = r_N G_L'(r_N) \sim N \rho^{-1} r_N g'(r_N) \sim N \rho_*^{-1} r_N g'(r_N)
\]

so that \( r_N g'(r_N) \to \rho_* \) and therefore \( r_N \to r_* \).

In the next lemma we find the asymptotic behavior of \( a_N \) and \( b_N \).

**Lemma 5.4.** As \( N \to \infty \) we have:

(i) In case (Sub1),

\[
a_N \sim \frac{\text{Var}(X) N \rho^2}{\theta^2} \quad \text{and} \quad b_N \sim \frac{3 \text{Var}(X)^2 N \rho^4}{\theta^4}.
\]

(ii) In case (Sub2),

\[
a_N \sim \frac{\alpha_c N}{\rho (1-r_N)} = N^{1+\frac{\alpha_c}{\alpha}+o(1)} \quad \text{and} \quad b_N \sim N^{1+\frac{2\alpha_c}{\alpha}+o(1)}.
\]

(iii) In cases (Sub3), (SubConst1) and (SubConst2),

\[
c_{\rho} N \leq a_N \leq C_{\rho} N \quad \text{and} \quad c_{\rho} N \leq b_N \leq C_{\rho} N.
\]

**Proof.** In (Sub1), by Corollary 4.7, Lemma 4.12 and Lemma 5.3 as \( N \to \infty \) we have

\[
G_L''(r_N) \sim L g''(r_N) \sim \frac{\theta L}{2\sqrt{2\pi} (1-r_N)^{\frac{3}{2}}} \sim \frac{\sigma^2 N \rho^2}{\theta^2},
\]

(81)
where we denoted $\sigma := \sqrt{\text{Var}(X)}$. In a similar manner,

$$
G''_L(r_N) \sim \frac{36L}{4\sqrt{2\pi}}(1-r_N)^{-\frac{3}{2}} \sim \frac{3\sigma^4N\rho^4}{\theta^4}.
$$

(82)

The last two asymptotic equalities together with the definitions of $a_N$ and $b_N$ from (74) and the definition of $r_N$ imply that $a_N \sim G''_L(r_N) \sim \frac{a^2N\rho^2}{\sigma^2}$ and $b_N \sim G''_L(r_N) \sim \frac{3\sigma^4N\rho^4}{\theta^4}$ as $N \to \infty$.

Similarly in (Sub2),

$$
a_N \sim G''_L(r_N) \sim \frac{\alpha_cN}{\rho(1-r_N)} = N^{1+\frac{\alpha_c}{\rho}+o(1)}, \quad b_N \sim G''_L(r_N) = N^{1+\frac{2\alpha_c}{\rho}+o(1)}.
$$

(83)

Similar asymptotic estimates apply also to the cases (Sub3), (SubConst1) and (SubConst2) and show that $G''_L(r_N)$ and $G''_L(r_N)$ have order of magnitude $N$ as $N \to \infty$. Thus, $a_N$ and $b_N$ are also of the same order of magnitude.

We now have everything needed to prove Proposition 5.2.

Proof of Proposition 5.2 Part (i) of the proposition follows immediately from Lemma 5.4.

We turn to prove part (ii). Let $j_0$ be a fixed large integer so that, by Lemma 4.3, $\varphi^{(1)}(0) > c_{j_0}^{-\frac{d}{2}} > 0$ for all $j \geq j_0$. Thus, using the definitions (28) of $G_L$ and (20) of $W_{L,j}$, we have

$$
G_L(r_N) - \text{Re} \left( G_L \left( r_N e^{it} \right) \right) = \sum_{j=1}^{\infty} \frac{W_{L,j} r_N^j}{j} \left( 1 - \text{Re} \left( e^{itj} \right) \right) 
\geq cN \sum_{j=1}^{\infty} \frac{\varphi^{(1)}(0)r_N^j}{j} (1 - \cos(tj)) \geq cN \sum_{j=j_0}^{\infty} \frac{r_N^j}{j^{\frac{3}{2}+1}} (1 - \cos(tj)) =: S.
$$

(84)

Our goal is to show that there exists $c' > 0$ (depending on $\rho$, in (Sub3), (SubConst1) and (SubConst2)) such that $S \geq c'a_N^{1/4}\sqrt{|t|}$ for large $N$, uniformly in $a_N^{-1/2} \leq |t| \leq \pi$. Examining the expression for $S$ reveals that it may be useful to compare the relative sizes of $\frac{N}{\rho}$ and $a_N$.

We record the following relations as $N \to \infty$, which follow directly from Lemma 5.4 and Lemma 5.3

$$
a_N^{1/4} \leq C \frac{N}{\rho} (1-r_N)^{-\frac{d}{4}+1} \quad \text{and} \quad a_N \leq C \frac{N}{\rho} (1-r_N)^{\frac{d}{2}-2}, \quad \text{in (Sub1), (Sub2)},
$$

(85)

Suppose first that $|t| \geq \frac{1}{2j_0}$. In this case $\max\{1 - \cos(t(j_0)), 1 - \cos(t(j_0) + 1)\} \geq c$. Thus, when $N$ is sufficiently large, by (85) and part (iii) of Lemma 5.4,

$$
S \geq cN\rho^{-1}r_N^{j_0+1} \geq c \cdot a_{N_0}^{\frac{1}{2}} \geq c \cdot a_N^{\frac{1}{2}} \sqrt{|t|}.
$$

Second, suppose that $|t| < \frac{1}{2j_0}$. Consider the cases (Sub3), (SubConst1) and (SubConst2). We bound the sum $S$ by the $j = j_0$ element and use the inequality $1 - \cos(x) \geq cx^2$ for $|x| \leq \frac{1}{2}$ and part (iii) of Lemma 5.4 to obtain that for $|t| \geq a_N^{-\frac{3}{2}}$ and large $N$, $S \geq cN\rho^{-1}r_N^{j_0+2} \geq c_{\rho} \cdot a_N t^2 \geq c_{\rho} \cdot a_N^{\frac{1}{2}} \sqrt{|t|}$.

Now consider the cases (Sub1) and (Sub2). Observe that when $j \leq \min\{|t|^{-1}, (1-r_N)^{-1}\}$ we have the bounds $r_N^j \geq c$ and $1 - \cos(tj) \geq ct^2j^2$. Suppose $N$ is large. If $a_N^{-\frac{3}{2}} \leq |t| \leq 1-r_N$
then, by (85),
\[ S \geq c \frac{N}{\rho} t^2 \sum_{j=j_0}^{(1-r_N)^{-1}} j^1-\frac{d}{2} \geq c \frac{N}{\rho} t^2 (1-r_N)^{1-2} \geq c \cdot a_N t^2 \geq c \cdot a_N^{2} \sqrt{|t|}, \]
whereas if $|t| \geq 1 - r_N$ then, by (85),
\[ S \geq c \frac{N}{\rho} |t|^{1-1} \sum_{j=j_0} |t|^{1-2} \geq c \frac{N}{\rho} |t|^{1/2} \geq c \cdot a_N^{1/2} \sqrt{|t|}. \]

Putting all of the above cases together proves the required lower bound on $S$ and finishes the proof of the proposition. □

5.2. Proof of the sub-critical parts of the main theorems. The next corollary restates the sub-critical parts of Theorem 1.1, Theorem 1.2 and Theorem 1.3 in the notation of this section.

Corollary 5.5. In the (Sub-Critical) cases $\nu = 0$ and the following holds as $N \to \infty$:

(i) In case (Sub1),
\[ \frac{\theta^2 L_1}{2 \text{Var}(X) \rho^2} \xrightarrow{d} \text{gamma} \left( \frac{1}{2}, 1 \right). \]

(ii) In case (Sub2),
\[ \frac{\alpha_c \log L_1}{\rho} \xrightarrow{d} U[0, 1]. \]

(iii) In cases (Sub3), (SubConst1) and (SubConst2),
\[ L_1 \xrightarrow{d} Y, \]
where $Y$ is the integer-valued random variable defined by
\[ P(Y = j) = \theta r_*^{j} \phi*(0) r_*^{j}, \quad j \geq 1 \]
and
\[ r_* \text{ is the unique number satisfying } 0 < r_* < 1 \text{ and } \sum_{j=1}^{\infty} \phi*(0) r_*^{j} = \rho \theta^{-1}. \] (86)

Proof. The corollary follows easily by substituting the estimates given in Theorem 5.1. Let $j_N$ be a sequence of integers satisfying $1 \leq j_N \leq N$ and
\[ j_N = o \left( \min \left( \sqrt{a_N}, L^2 \right) \right), \quad N \to \infty. \] (87)

By Lemma 3.4, theorem 5.1 and Corollary 4.6, uniformly in $1 \leq j \leq j_N$ with $\phi*(0) \neq 0$,
\[ P(L_1 = j) = \frac{W_{L,j}}{N} \cdot \frac{H_{N-j}(L)}{H_N(L)} \sim \frac{\theta L^d \phi*(0)}{N} \cdot r_N^{j}, \quad r_N^{j} = \theta \rho^{-1} \phi*(0) r_N^{j}, \quad N \to \infty. \] (88)

Note that, by Lemma 4.3, there are only finitely many $j$ for which $\phi*(0) = 0$, and that $L^d \phi*(0) \geq c$ when $\phi*(0) \neq 0$ and $j \leq L^2$. The above arguments also adapt to show that $P(L_1 = j) \to 0$ when $\phi*(0) = 0$.

In cases (Sub3), (SubConst1) and (SubConst2), one first verifies using (44) that the definition of $r_*$ given by (86) coincides with the one given in Lemma 5.3. It then follows from (88) with fixed $j$ and Lemma 5.3 that $L_1 \xrightarrow{d} Y$, proving part (iii).
Consider the case (Sub1). Denote $\sigma := \sqrt{\text{Var}(X)}$. Let $0 < a < b < \infty$ and set $j_N = \lfloor \frac{2\sigma^2 \rho^2 b}{a^2} \rfloor$. Observe that $j_N$ satisfies (87) by Lemma 5.4. Thus, we may use (88) to obtain

$$\mathbb{P} \left( a \leq \frac{\theta^2 L_1}{2\sigma^2 \rho^2} \leq b \right) = \sum_{j=\lfloor \frac{2\sigma^2 \rho^2 a}{a^2} \rfloor}^{\lfloor \frac{2\sigma^2 \rho^2 b}{a^2} \rfloor} \mathbb{P} (L_1 = j) = \frac{\theta^2}{2 \sqrt{\pi \sigma^2 \rho^2}} \sum_{j=\lfloor \frac{2\sigma^2 \rho^2 a}{a^2} \rfloor}^{\lfloor \frac{2\sigma^2 \rho^2 b}{a^2} \rfloor} \left( \frac{\theta^2 j}{2\sigma^2 \rho^2} \right)^{-\frac{1}{2}} \exp \left( - \frac{\theta^2 j}{2\sigma^2 \rho^2} \right) \to \frac{1}{\sqrt{\pi}} \int_a^b x^{-\frac{1}{2}} e^{-x} dx \quad \text{(89)}$$

where in the second asymptotic equality we used Lemma 4.3, Lemma 5.3, and $1 - x = e^{-x + o(x^2)}$ as $x \to 0$ and the final limit is obtained by convergence of the Riemann sum to the integral. This finishes the proof of part (i).

Consider the case (Sub2). Let $0 < a < b < \infty$ and set $j_N = \frac{bN}{\rho}$. Note that we have $j_N = o \left( (1 - r_N)^{-1} \right)$ by Lemma 5.3 and that $j_N$ satisfies (87) by Lemma 5.4. Thus, we may use (88) to obtain

$$\mathbb{P} \left( a \leq \frac{\alpha \log L_1}{\rho} \leq b \right) = \sum_{j=\lfloor \frac{bN}{\rho} \rfloor}^{\lfloor \frac{bN}{\rho} \rfloor} \mathbb{P} (L_1 = j) = \frac{\theta}{\rho} \sum_{j=\lfloor \frac{bN}{\rho} \rfloor}^{\lfloor \frac{bN}{\rho} \rfloor} \varphi^{*j}(0) r_N^j \sim \alpha \rho^{-1} \sum_{j=\lfloor \frac{bN}{\rho} \rfloor}^{\lfloor \frac{bN}{\rho} \rfloor} \frac{1}{j} \to b - a,$$

where the second asymptotic equality follows from Lemma 4.3 and from $r_N^j \to 1$. This finishes the proof of part (ii).

The fact that $\nu = 0$ follows from the above results as, for any $\epsilon > 0$, $\mathbb{P} (L_1 \geq \epsilon N) \to 0$, as $N \to \infty$. \hfill \Box

Remark 5.6. The analysis in Corollary 5.5 extends to the study of the joint distribution of $L_1, L_2, \ldots$. The key fact is that an analog of (88) remains valid. By Lemma 3.4, for any fixed $m$ as $N \to \infty$,

$$\mathbb{P} (L_1 = j_1, \ldots, L_m = j_m) = \frac{H_{N-j_1-\cdots-j_{k-1}}(L)}{H_N(L)} \cdot \prod_{k=1}^m W_{L,j_k} \sim \mathbb{P} (L_1 = j_1) \cdots \mathbb{P} (L_m = j_m),$$

uniformly in $1 \leq j_1 + \cdots + j_m \leq j_N$ with $\varphi^{*j_k}(0) \neq 0$ for all $k$, as follows from theorem 5.1 and Corollary 4.6 in the same manner as in (88). One may then follow the analogous steps to the analysis in Corollary 5.5 and deduce that the $(L_k)$ become asymptotically independent and identically distributed, in the sense explained in the remark following Theorem 1.3.

6. The Supercritical Case

In this section we prove part (iv) in Theorem 1.1, part (iii) in Theorem 1.2, and part (iii) in Theorem 1.3. In particular we show the convergence to the Poisson-Dirichlet distribution.

Let $\rho = \rho(N)$ and consider several possibilities for the dimension $d$ and the asymptotic regime of $\rho(N)$ as $N \to \infty$ corresponding to the various statements in the theorems. We give names to these cases to simplify later reference. By the name (Super-Critical) we refer collectively to any of the following cases:
(Super1) : Dimension $d = 1$ and density $\rho$ satisfying $\rho = \omega\left(\sqrt{N}\right)$ and $\rho \leq N$.
(Super2) : Dimension $d = 2$ and density $\rho$ satisfying $\frac{\rho}{\log N} \to \alpha \in (\alpha_c, \infty)$.
(Hyper2) : Dimension $d = 2$ and density $\rho$ satisfying $\rho = \omega(\log N)$ and $\rho \leq N$.
(Super3) : Dimension $d \geq 3$ and density $\rho > \rho_c$, fixed as $N \to \infty$.

Our assumption that $\rho \leq N$ in all cases is equivalent to taking the side length $L$ of $\Lambda$ to be at least 1, as we assumed in the estimates of Section 4. Define the auxiliary parameter $\tau = \tau_{d,\rho}$ in the cases above:

$$\tau = \tau_{d,\rho} := \begin{cases} 0 & \text{in cases (Super1) and (Hyper2)} \\ \frac{\alpha_c}{\alpha} & \text{in case (Super2)} \\ \frac{\rho_c}{\rho} & \text{in case (Super3)} \end{cases} \quad (90)$$

We will show that, in all of the (Super-Critical) cases, $\tau$ is the fraction of points in non-macroscopic cycles, or equivalently $\tau = 1 - \nu$ where $\nu$ is defined in (5).

As in the sub-critical case, our strategy is to apply the Cauchy integral formula to obtain the asymptotic behavior of $H_n(L) = \left[z^n\right] e^{GL(z)}$ and then use the results of Section 3 which connect $H_n(L)$ with the distribution of the cycle lengths $(L_1, L_2, \ldots)$.

### 6.1. Singularity analysis

In the following theorem we find the asymptotic behavior of $H_{N-j}(L)$. We first introduce an analytic function $F_L$ defined by

$$F_L(z) := G_L(z) + \theta \log (1 - z) = -\theta \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \log (1 - z \hat{\varphi}(m)), \quad z \in \Delta_0, \quad (91)$$

where the second equality uses (61) and where we recall from (58) that

$$\Delta_0 = \left\{ z \in \mathbb{C} \setminus \{1\} : |z| < R_0, \ |\arg(z - 1)| > \frac{\pi}{8}\right\},$$

with $R_0 > 1$ determined by Claim 4.9. For every $L \geq 1$, the sum in (91) converges uniformly in $\Delta_0$, by Claim 4.9 and the fact that $\hat{\varphi}$ is a Schwartz function, and therefore $F_L$ is continuous at $z = 1$. Moreover, by the definition of $G_L$ in (28), in the closed unit disc $F_L$ is given by the power series

$$F_L(z) = \sum_{j=1}^{\infty} \frac{W_{L,j} - \theta}{j} z^j, \quad z \in \mathbb{D}, \quad (92)$$

where the sum converges absolutely in $\mathbb{D}$ by Corollary 4.6. In particular $F_L(1)$ is real.

**Theorem 6.1.** In the (Super-Critical) cases, for every $\epsilon > 0$,

$$H_{N-j}(L) = \left[z^{N-j}\right] e^{GL(z)} \sim \frac{e^{F_L(1)} N^{\theta-1} \left(1 - \frac{j}{N}\right)^{\theta-1}}{\Gamma(\theta)}, \quad N \to \infty,$$

uniformly in $j \leq (1 - \tau - \epsilon) N$.

The proof of Theorem 6.1 following [10], proceeds by expressing $H_{N-j}(L)$ as a contour integral and changing the contour of integration to a ‘pacman-shaped’ contour $\gamma$ defined as
follows (see Figure 4): \( \gamma = \gamma (\eta, \beta, N) = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \), where + denotes concatenation of contours,

\[
\begin{align*}
\gamma_1(t) &= 1 - te^{-i\beta}, \quad t \in \left[-\eta, -\frac{1}{N}\right] \\
\gamma_2(t) &= 1 + \frac{1}{N}e^{-it}, \quad t \in [\beta, 2\pi - \beta] \\
\gamma_3(t) &= 1 + te^{i\beta}, \quad t \in \left[\frac{1}{N}, \eta\right] \\
\gamma_4(t) &= R_1e^{it}, \quad t \in [\beta', 2\pi - \beta']
\end{align*}
\]

and \( R_1 = |1 + \eta e^{i\beta}|, \quad \beta' = \arg(1 + \eta e^{i\beta}) \). We will see that the main contribution to the Cauchy integral comes from a small neighborhood of 1. In the proof of Theorem 6.1 we estimate this contribution using the following proposition that identifies the behavior of \( G_L \) close to 1 and on the rest of \( \gamma \).

**Proposition 6.2.** In the (Super-Critical) cases, for any \( \epsilon > 0 \) there exists \( 0 < \eta < \frac{1}{10} \) and \( N_0 > 0 \) such that \( \gamma = \gamma (\eta, \frac{\pi}{3}, N) \) satisfies \( \gamma \subseteq \Delta_0 \) for \( N \geq N_0 \) and the following holds:

1. There exists a sequence \( t_N \to \infty \), \( t_N = o(\sqrt{N}) \), for which
   
   \[
   G_L(z) = -\theta \log (1 - z) + F_L(1) + \tau N(z - 1) + o(1), \quad N \to \infty,
   \]
   uniformly in \( z \in \Delta_0 \) such that \( \frac{1}{N} \leq |1 - z| \leq \frac{t_N}{N} \).

2. For any \( z \in \gamma_1 \cup \gamma_3 \) and \( N \geq N_0 \),
   
   \[
   \operatorname{Re}(G_L(z)) \leq -\theta \log |1 - z| + F_L(1) + (\tau + \epsilon) N \operatorname{Re}(z - 1).
   \]

3. For any \( z \in \gamma_4 \) and \( N \geq N_0 \),
   
   \[
   \operatorname{Re}(G_L(z)) \leq F_L(1) + (\tau + \epsilon) N \log |z|.
   \]

We start by proving Theorem 6.1 given the proposition. We require also the following well-known Hankel integral [32, Theorem 8.4b]: For \( \beta \in (0, \pi) \), define the contour \( \gamma' = \gamma'(\beta) = \)
Figure 5. The contour $\gamma'(\pi/3)$.

\[ \gamma_1 + \gamma_2 + \gamma_3 \text{ by} \]

\[ \gamma_1'(t) := -te^{-i\beta}, \quad t \in (-\infty, -1), \]

\[ \gamma_2'(t) := e^{-it}, \quad t \in [\beta, 2\pi - \beta], \quad \text{(94)} \]

\[ \gamma_3'(t) := te^{i\beta}, \quad t \in [1, \infty), \]

see Figure 5. Then, if $\beta \in (0, \pi/2)$, we have

\[ \frac{1}{2\pi i} \int_{\gamma'} (\omega)^{-s}e^{-\omega}d\omega = \frac{1}{\Gamma(s)}, \quad s \in \mathbb{C}. \]  

Proof of Theorem 6.1. Fix $0 < \epsilon < \frac{1}{2\pi}$ where $\tau$ is given in (90). Let $\eta$ and $N_0$ be the numbers and $(t_N)$ be the sequence from Proposition 6.2 corresponding to $\epsilon$. Set $\gamma := \gamma(\eta, \pi/3, N)$ where $N \geq N_0$ is taken large enough for the following calculations. We assume throughout this proof that $0 \leq j \leq (1 - \tau - 2\epsilon)N$.

By the Cauchy integral formula

\[ \left[ z^{N-j} \right] e^{G_L(z)} = \frac{1}{2\pi i} \oint_{\gamma} \frac{e^{G_L(z)}}{z^{N+1-j}}dz = I_1 + I_2 + I_3, \]

where $I_1$, $I_2$ and $I_3$ are the corresponding integrals over $\gamma \cap \{ |1-\epsilon| \leq \frac{t_N}{N} \}$, $(\gamma_1 \cup \gamma_3) \cap \{ |1-\epsilon| \geq \frac{t_N}{N} \}$ and $\gamma_4$ respectively. We estimate separately the three parts.

We start with $I_1$. The following holds as $N \to \infty$, uniformly in $0 \leq j \leq (1 - \tau - 2\epsilon)N$ and $z \in \Delta_0 \cap \{ |1-\epsilon| \leq \frac{t_N}{N} \}$. We have

\[ z^{-(N+1-j)} = \exp \left[ - (N + 1 - j) \log z \right] = \exp \left[ - (N - j) (z - 1) + o(1) \right], \]

where we used that $t_N = o(\sqrt{N})$. Substituting this and part (i) of Proposition 6.2 in $I_1$ we obtain that

\[ I_1 = \frac{e^{F_L(1)}}{2\pi i} \int_{\gamma \cap \{ |1-\epsilon| \leq \frac{t_N}{N} \}} (1 - z)^{-\theta} \exp \left[ (\tau N - N + j) (z - 1) + o(1) \right]dz, \]  

(96)
We make the change of variables \( z = 1 + \frac{\omega}{N(1-\tau)} \), so that \( (\tau N - N + j)(z - 1) = -\omega \), to obtain
\[
I_1 = \frac{e^{F_L(1)}(N(1-\tau)-j)^{\theta-1}}{2\pi i} \int_{\tilde{\gamma}} (-\omega)^{-\theta} e^{-\omega + o(1)} d\omega,
\]
where \( \tilde{\gamma} \) is the image of \( \gamma \cap \{ |1-z| \leq \frac{t_N}{N} \} \) under the change of variables, which is a modification of the \( \gamma'(\frac{\pi}{3}) \) from (94) having the circular arc at radius \( 1-\tau - \frac{4}{N} \geq 2\epsilon \) and having finite ‘arms’, terminating at radius \( t_N(1-\tau - \frac{4}{N}) \geq 2\epsilon t_N \to \infty \). As the integral in (95) with \( s = \theta \) and \( \beta = \frac{\tau}{3} \) converge to a non-zero quantity, we conclude that
\[
I_1 \sim \frac{e^{F_L(1)}N^{\theta-1} (1-\tau - \frac{4}{N})^{\theta-1}}{\Gamma(\theta)} \quad (97).
\]

We turn to bound \( I_2 \). A Taylor expansion shows that
\[
\left| \text{Re}(\log z) - \text{Re}(z - 1) + \frac{1}{2} \text{Re}[(z - 1)^2] \right| \leq \frac{1}{8} |z - 1|^3, \quad |z - 1| \leq \frac{1}{10}. \quad (98)
\]
Thus, for \( z \in \gamma_1 \cup \gamma_3 \),
\[
\text{Re}(\log z) \geq \text{Re} z - 1 > 0.
\]
Thus, for such \( z \) and using our assumption that \( 0 \leq j \leq (1-\tau - 2\epsilon)N \),
\[
|z^{-(N-j+1)}| \leq \exp \left[ - (\tau + 2\epsilon) N (\text{Re} z - 1) \right].
\]
Substituting this and part (ii) of Proposition 6.2 in \( I_2 \) we obtain that
\[
|I_2| \leq e^{F_L(1)} \int_{(\gamma_1 \cup \gamma_3) \cap \{ |1-z| \geq \frac{t_N}{N} \}} |1-z|^{-\theta} \exp \left[ -\epsilon \cdot N(\text{Re} z - 1) \right] |dz|
\]
\[
\leq e^{F_L(1)} N^{\theta-1} \int_{\gamma' \cap \{ |\omega| \geq t_N \}} |\omega|^{-\theta} e^{-\epsilon \text{Re} \omega} = o(I_1), \quad N \to \infty, \quad (99)
\]
where in the second inequality we made the change of variables \( z = 1 + \frac{\omega}{N} \) and used the contour \( \gamma' \) defined in (94) and the final estimate follows from (97) and as the integral in (99) tends to 0. We bound \( I_3 \) using part (iii) of Proposition 6.2 by
\[
|I_3| \leq e^{F_L(1)} \int_{\gamma_4} |z|^{(\tau+\epsilon-1)N+j-1} |dz| \leq e^{F_L(1)} \int_{\gamma_4} |z|^{-\epsilon N} |dz| \leq 2\pi e^{F_L(1)} R_1^{-\epsilon N},
\]
which is exponentially smaller than \( I_1 \) by (97).
\[\square\]

The rest of Section 6.1 is devoted to the proof of Proposition 6.2

6.1.1. Near \( z = 1 \). In this section we prove parts (i) and (ii) of Proposition 6.2.

We first note that, by the definition (91) of \( F_L \),
\[
G_L(z) = -\theta \log (1-z) + F_L(z) = -\theta \log (1-z) + F_L(1) + \int_{1}^{z} F'_L(w) dw, \quad z \in \Delta_0. \quad (100)
\]
This representation will allow us to prove parts \((i)\) and \((ii)\) by suitably estimating \(F'_L\). We start by writing \(F'_L\) as a series, using again the definition \((91)\),

\[
F'_L(z) = \theta \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \frac{\hat{\varphi}(m)}{1 - z \hat{\varphi}(m)}, \quad z \in \Delta_0.
\]

We start with the following bound on \(F'_L(z)\),

\[
|F'_L(z)| \leq \theta \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \left| \frac{\hat{\varphi}(m)}{1 - z \hat{\varphi}(m)} \right| + \theta \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \left| \frac{\hat{\varphi}(m)}{1 - z \hat{\varphi}(m)} \right| \leq C \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \frac{1}{\|m\|^2} + C \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \frac{\|m\|^2}{1 - \|m\|^2} \leq CL^d \int_{\frac{1}{L} \leq \|t\| \leq 1} \frac{1}{\|t\|^2} + CL^d,
\]

where in the second inequality we used Claim \(4.9\). Now it is relatively straightforward to deduce parts \((i)\) and \((ii)\) of Proposition \(6.2\) in the cases where \(\tau = 0\). Indeed, the last expression in \((\text{Super1})\) is \(O(L^2) = O\left(\frac{N^2}{\rho^2}\right) = o(N)\) and in \((\text{Hyper2})\) it is \(O(L^2 \log L) = O\left(\frac{N \log N}{\rho}\right) = o(N)\) as \(N \to \infty\). Therefore, by \((100)\), we get that in both these cases,

\[
G_L(z) = -\theta \log (1 - z) + F_L(1) + o(N) \cdot |1 - z|, \quad N \to \infty,
\]

uniformly in \(z \in \Delta_0\). This implies parts \((i)\) and \((ii)\) of the proposition in these cases (the fact that the \(o(N)\) term is uniform in \(z \in \Delta_0\) implies the existence of a sequence \(t_N \to \infty\), \(t_N = o(\sqrt{N})\) satisfying that \(\frac{t_N}{N}\) times the \(o(N)\) still tends to 0).

We turn to prove parts \((i)\) and \((ii)\) in the cases \((\text{Super2})\) and \((\text{Super3})\). In these asymptotic regimes, it turns out that \(F'_L\) is well-approximated by \(L^d g'\), as the next lemma makes precise.

**Lemma 6.3.** There exists a \(C > 0\) so that for any \(L \geq 1\) and \(z \in \Delta_0\) the following holds:

\((i)\) If \(d = 2\) then

\[
|F'_L(z) - L^2 g'(z)| \leq CL^2 \max \left\{ 1, \log \left( \frac{L^2}{|1 - z|} \right) \right\}.
\]

\((ii)\) If \(d \geq 3\) then

\[
|F'_L(z) - L^d g'(z)| \leq C \left( L^{d-1} + L^2 \log L \right).
\]

**Proof.** We approximate the Riemann sum \(L^{-d} F'_L(z)\) by the corresponding integral (see \((101)\) and \((69)\)). Decomposing the domain of integration to cubes around each \(m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}\) we obtain

\[
|F'_L(z) - L^d g'(z)| = \left| \theta \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \frac{\hat{\varphi}(m)}{1 - z \hat{\varphi}(m)} - \theta L^d \int_{\mathbb{R}^d} \frac{\hat{\varphi}(t)}{1 - z \hat{\varphi}(t)} dt \right|
\leq \theta L^d \int_{Q_L} \frac{|\hat{\varphi}(t)| dt}{|1 - z \hat{\varphi}(t)|} + \theta L^d \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \int_{m + Q_L} \frac{|\hat{\varphi}(m) - \hat{\varphi}(t)| dt}{|1 - z \hat{\varphi}(m)| \cdot |1 - z \hat{\varphi}(t)|}
\]

\((105)\).
where we set $Q_L := [-2L^{-1}, 2L^{-1}]^d \subseteq \mathbb{R}^d$. We bound each term in (105) separately and start with the second one. For any $m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}$ and $t \in m + Q_L$ we have
\[
|\hat{\phi}(m) - \hat{\phi}(t)| \leq C L^{-1} \max_{x \in m + Q_L} \max_i \left| \frac{\partial \hat{\phi}}{\partial t_i}(x) \right| \leq C L^{-1} \min\{\|t\|, \|t\|^{-d-1}\},
\] (106)
where the second inequality follows as $\frac{\partial \hat{\phi}}{\partial t_i}$ decays fast, $\frac{\partial \hat{\phi}}{\partial t_i}(0) = 0$ and $\frac{\partial^2 \hat{\phi}}{\partial t_i^2}$ is bounded. Thus, by part (i) of Claim 4.9 we obtain
\[
L^d \sum_{m \in \frac{1}{L}\mathbb{Z}^d \setminus \{0\}} \int_{m + Q_L} \frac{|\hat{\phi}(m) - \hat{\phi}(t)|}{|1 - z\hat{\phi}(m)| \cdot |1 - z\hat{\phi}(t)|} dt \leq C L^{d-1} \int_{L^{-1} \leq \|t\| \leq 1} \frac{\|t\|^{-3} dt + C L^{d-1} \int_{\|t\| \geq 1} \|t\|^{-d-1} dt}{\|t\|^{-1}}
\leq C L^{d-1} \int_{L^{-1}}^{1} r^{-4} dr + C L^{d-1}
\]
and we check that the last expression is bounded by the right-hand side of (103) when $d = 2$ and by the right-hand side of (104) when $d \geq 3$.

Next, we bound the first term on the right-hand side of (105), again using Claim 4.9 and using also that $|\hat{\phi}(t)| \leq 1$. When $d = 2$,
\[
\int_{Q_L} \frac{|\hat{\phi}(t)|}{|1 - z\hat{\phi}(t)|} dt \leq C \int_{\|t\| \leq C L^{-1}} \frac{dt}{|1 - z| + \|t\|^2} \leq C \int_{0}^{C L^{-2}} \frac{dr}{|1 - z| + r} \leq C \log \left(\frac{|1 - z| + C L^{-2}}{|1 - z|}\right).
\]
When $d \geq 3$,
\[
\int_{Q_L} \frac{|\hat{\phi}(t)|}{|1 - z\hat{\phi}(t)|} dt \leq C \int_{\|t\| \leq C L^{-1}} \frac{1}{\|t\|^2} dt \leq C \int_{0}^{C L^{-1}} r^{-3} dr \leq C \frac{1}{L^{d-2}}.
\]
Substituting these bounds in (105) finishes the proof of the lemma.

We proceed to deduce parts (i) and (ii) of Proposition 6.2 for the cases (Super3) and (Super2). Consider first the case (Super3). By the definitions of $\tau$, $g$ and $\rho$ in (90), (44) and (4) respectively, we have that $\tau N = L^d g'(1)$. Thus, using Lemma 6.3 and Lemma 4.12 we get that for any $z \in \Delta_0$,
\[
\left| \int_{1}^{z} F_L'(w) dw - \tau N(z - 1) \right| \leq \int_{1}^{z} \left| F_L'(w) dw - \int_{1}^{z} L^d g'(1) dw \right| \leq \int_{1}^{z} \left| F_L'(w) - L^d g'(w) \right| \cdot |dw| + L^d \int_{1}^{z} |g'(w) - g'(1)| \cdot |dw| \leq C L^{d-1} \log L \cdot |z - 1| + C L^d |z - 1|^{3/2}.
\]

Parts (i) and (ii) of the proposition follows in case (Super3) by substituting the last result in (100) and taking $t_N = \sqrt{L}$ and sufficiently small $\eta > 0$. 

\[\square\]
Finally, consider the case (Super2). We estimate the integral on the right-hand side of (100) as follows. Let \( z \in \Delta_0 \) with \( |1 - z| \geq N^{-1} \) and let \( x \in \Delta_0 \) be the unique point on the line segment connecting 1 and \( z \) satisfying \( |1 - x| = (N \log N)^{-1} \). We have,

\[
\int_1^x F'_L(w) dw + \alpha_c L^2(z - 1) \log (1 - z) \leq \int_1^x F'_L(w) dw + \alpha_c L^2 \log (1-w) dw + CL^2 |1 - z|
\]

\[
\leq \int_1^x |F'_L(w)| dw + CL^2 \int_1^x |\log (1-w)| dw + \int_1^x |F'_L(w) + \alpha_c L^2 \log (1-w)| dw + CL^2 |1 - z|,
\]

where \( \alpha_c \) is defined in (9). We bound the first and the third integrals separately. To bound the first integral, note that for any \( w \in \Delta_0 \), we have by (102), \( |F'_L(w)| \leq CL^2 \log L \leq CN \). To bound the third integral observe that by Lemma 6.3 and Lemma 4.12, for \( w \in \Delta_0 \) with \( |1 - w| \geq (N \log N)^{-1} \) and large enough \( N \), we have

\[
|F'_L(w) + \alpha_c L^2 \log (1-w)| \leq |F'_L(w) - L^2 g'(w)| + L^2 |g'(w) + \alpha_c \log (1-w)| \leq \frac{N}{\sqrt{\log N}}.
\]

Therefore, for \( z \in \Delta_0 \) with \( |1 - z| \geq N^{-1} \) and sufficiently large \( N \),

\[
\int_1^x F'_L(w) dw + \alpha_c L^2(z - 1) \log (1 - z) \leq \frac{N}{\sqrt{\log N}} |1 - z|.
\]

Part (i) of the proposition follows since, as \( N \to \infty \),

\[-\alpha_c L^2(z - 1) \log (1 - z) = \alpha_c L^2 \log N(z - 1) + o(N)|1 - z| = \tau N(z - 1) + o(N)|1 - z|,
\]

uniformly in \( z \in \Delta_0 \) with \( \frac{1}{N} \leq |1 - z| \leq \frac{\log N}{N} \), where in the last asymptotic equality we used that \( \frac{\theta}{\log N} \to \alpha \) as \( N \to \infty \) and that \( \tau = \frac{\alpha c}{\alpha} \). Part (ii) of the proposition follows as, for large enough \( N \) and \( z \in \gamma_1 \cup \gamma_2 \),

\[
\Re (-\alpha_c L^2(z - 1) \log (1 - z)) \leq \alpha_c L^2 \log N \Re(z - 1) + CL^2 |1 - z| \leq (\tau + \epsilon)N \Re(z - 1).
\]

6.1.2 Far from \( z = 1 \). In this section we prove the third part of Proposition 6.2.

We start with dimensions 1 and 2 whose treatment is relatively straightforward. We claim that

\[
|F_L(z)| \leq CL^d, \quad z \in \overline{\Delta_0}.
\]

This implies part (iii) of the proposition in cases (Super1), (Super2) and (Hyper2) as, for any \( z \in \gamma_4 \),

\[
\Re G_L(z) \leq |G_L(z)| = |\theta \log (1 - z) + F_L(z)| \leq CL^d \leq F_L(1) \leq CL^d \leq F_L(1) + \epsilon N \log |z|,
\]

where the last inequality holds for large enough \( N \) since \( L^d = o(N) \) in these cases.

To see (108), fix \( C_0 > 0 \) such that

\[
|\hat{\varphi}(t)| \leq \frac{C_0}{R_0 \|t\|^{d+1}}, \quad t \in \mathbb{R}^d,
\]

where \( R_0 \) is the radius from the definition of \( \Delta_0 \) in (58). We have, for \( z \in \overline{\Delta_0} \),

\[
|F_L(z)| \leq \sum_{m \in \frac{1}{2} \mathbb{Z}^d \setminus \{0\}} |\log (1 - z \hat{\varphi}(m))| + C \sum_{m \in \frac{1}{2} \mathbb{Z}^d, 0 < \|m\| \leq 2C_0} (1 + |\log (\|m\|)|) + \sum_{m \in \frac{1}{2} \mathbb{Z}^d, \|m\| \geq 2C_0} \frac{C}{\|m\|^{d+1}},
\]
where in the second inequality we used that \( c\|m\|^2 \leq |1 - z\hat{\varphi}(m)| \leq C \) for \( \|m\| \leq 2C_0 \) by Claim 4.9 and that \(|\log(1 + x)| \leq C|x|\) for \(|x| \leq \frac{1}{2}\). One can easily check that the last expression is at most \( CL^d \) which finishes the proof of part (iii) of Proposition 6.2 in the cases (Super1), (Super2) and (Hyper2).

We turn to prove part (iii) of Proposition 6.2 in the remaining case (Super3). We start with the following lemma which bounds the real part of the function \( g \).

**Lemma 6.4.** Let \( d \geq 3 \). There is an \( \eta > 0 \) such that for any \( z \in \gamma_4 = \gamma_4(\eta, \frac{\pi}{3}) \) we have

\[
\Re g(z) \leq g(1) + g'(1) \log |z| = g(1) + g'(1) \log R_1.
\]  

(109)

**Proof.** We use different arguments for \( z \) close to 1 and far from 1. First we claim that there exists a \( \delta > 0 \) such that the inequality (109) holds in \( \Omega_\delta = \left\{ z \in \Delta_0 \mid |1 - z| \leq \delta, |\arg(z - 1)| \geq \frac{\pi}{3}, |z| \geq 1 \right\} \).

For this we expand \( g(z) \) around \( z = 1 \). We have

\[
g(z) = g(1) + g'(1)(z - 1) + \int_1^z \int_1^w g''(x)dx\,dw
\]

and therefore, by Lemma 4.12, we have the following asymptotics as \( z \to 1 \) according to the dimension \( d \),

- \( g(z) = g(1) + g'(1)(z - 1) + c_1(1 - z)^{\frac{3}{2}}(1 + o(1)), \quad d = 3 \),
- \( g(z) = g(1) + g'(1)(z - 1) - c_2 \log(1 - z)(z - 1)^2(1 + o(1)), \quad d = 4 \),
- \( g(z) = g(1) + g'(1)(z - 1) + c_3(z - 1)^2(1 + o(1)), \quad d \geq 5 \),

for some positive constants \( c_1, c_2 \) and \( c_3 \). Thus, for sufficiently small \( \delta > 0 \) and \( z \in \Omega_\delta \) we have

\[
\Re g(z) \leq g(1) + g'(1)(\Re z - 1) \leq g(1) + g'(1) \log |z|,
\]

where in the second inequality we use that

\[
\log |z| = \Re(\log z) = \Re(z - 1) - \frac{1}{2} \Re(z - 1)^2 + O(z - 1)^3, \quad z \in \Delta_0.
\]

Fix \( \delta > 0 \) such that (109) holds in \( \Omega_\delta \). As \( g \) has non-negative Taylor coefficients (see (44)), positive with finitely many exceptions by Lemma 4.3 it follows that for any \( z \neq 1 \) with \( |z| \leq 1 \),

\[
\Re g(z) < g(1).
\]  

(110)

Thus, by continuity and compactness arguments, there is \( R_\delta > 1 \) such that the inequality (110) holds in

\[
D := \left\{ z \in \Delta_0 \mid |z| \leq R_\delta, |1 - z| \geq \delta \right\}.
\]

Now, the fact that for sufficiently small \( \eta > 0 \),

\[
\gamma_4 = \gamma_4(\eta, \frac{\pi}{3}) \subseteq \Omega_\delta \cup D,
\]

completes the proof of the lemma. \( \square \)
We can now finish the proof of Proposition 6.2 for the case (Super3). We note that $F_L(0) = G_L(0) = g(0) = 0$. By Lemma 6.3 for large enough $N$ and $z \in \Delta_0$, we have

$$|F_L(z) - L^d g(z)| \leq \int_0^z |F_L'(w) - L^d g'(w)| \cdot |dw| \leq C(L^{d-1} + L^2 \log L) < \frac{\epsilon}{3} \log(R_1) N. \tag{111}$$

Thus, recalling that $\tau N = L^d g'(1)$, we get that for large enough $N$ and $z \in \gamma_4$,

$$\operatorname{Re} G_L(z) \leq \operatorname{Re} F_L(z) + C \leq L^d \operatorname{Re} g(z) + \frac{2 \epsilon}{3} \log(R_1) N \leq L^d g(1) + L^d g'(1) \log |z| + \frac{2 \epsilon}{3} \log(R_1) N \leq F_L(1) + (\tau + \epsilon) N \log |z|, \tag{112}$$

where in the second and fourth inequalities we used (111) and in the third one we used Lemma 6.4. This implies the third part of Proposition 6.2 in the case (Super3) and finishes the proof of all parts of the proposition.

6.2. Convergence to Poisson-Dirichlet. In this section we establish the convergence of the cycle length to the Poisson-Dirichlet distribution in the (Super-Critical) cases. Recall that $\tau$ is defined in (90). We denote by beta$(\alpha, \beta)$ the beta distribution with shape parameters $\alpha$ and $\beta$ whose density with respect to Lebesgue measure on $[0,1]$ is given by

$$\frac{\Gamma(\alpha + \beta) x^{\alpha - 1} (1 - x)^{\beta - 1}}{\Gamma(\alpha) \Gamma(\beta)} , \quad x \in [0,1].$$

**Definition 6.5** (Modified stick-breaking process). We define a sequence of random variables $(X_1, X_2, \ldots)$ inductively: Let $Y_1, Y_2, \ldots$ be independent random variables with distribution beta$(1, \theta)$ and let $U_1, U_2, \ldots$ be independent random variables with distribution $U[0,1]$ which are independent of $Y_1, Y_2, \ldots$. Set $S_0 := 0$ and, inductively for $k \geq 0$,

$$A_{k+1} = \left\{ U_{k+1} < 1 - \frac{\tau}{1 - S_k} \right\} , \quad X_{k+1} := (1 - S_k - \tau) Y_{k+1} 1_{A_{k+1}}, \quad S_{k+1} := \sum_{j=1}^{k+1} X_j.$$

In other words, $X_1$ equals 0 with probability $\tau$ and is otherwise distributed as a beta$(1, \theta)$ fraction of $1 - \tau$. Conditioned on $X_1$, $X_2$ equals 0 with probability $\frac{\tau}{1 - X_1}$ and is otherwise distributed as a beta$(1, \theta)$ fraction of $1 - X_1 - \tau$, etc.

The name of the process arises from the intuitive idea of taking a stick (interval) of length 1 for which the sub-interval of length $\tau$ in its beginning is deemed the ‘unbreakable’ part. Then, one iteratively samples a point uniformly in the remaining part of the stick and, if the point does not fall in the unbreakable part, one removes (breaks) the part of the stick from the sampled point until the end of the stick. The existence of the unbreakable part is the reason that the process is called a ‘modified’ stick breaking process.

Note that $X_k \in \sigma(Y_1, \ldots, Y_k, U_1, \ldots, U_k)$ and that $\sum_{k=1}^{\infty} X_k = 1 - \tau$, a.s.

**Claim 6.6.** The sequence $\frac{1}{\tau^k} (X_1, X_2, \ldots)$ rearranged in decreasing order (i.e. the largest element of the sequence, the second largest element and so on) has the PD$(\theta)$ distribution.

The reader unfamiliar with the PD$(\theta)$ distribution may take the above as its definition, in which case the content of Claim 6.6 is that the distribution of the rearranged sequence does not depend on $\tau$ (the proof of the claim does not use the specific value of $\tau$ given by (90)).
**Theorem 6.7.** In all the (Super-Critical) cases,

\[ \frac{1}{N} (L_1, L_2, \ldots) \xrightarrow{d} (X_1, X_2, \ldots). \]

The theorem together with Claim 6.6 imply the statements on convergence to the Poisson-Dirichlet distribution in the (Super-critical) parts of our main theorems. This is a consequence of the fact that for any \( \epsilon > 0 \),

\[ \lim_{k_0 \to \infty} \lim_{N \to \infty} P(\text{there exists } k \geq k_0 \text{ for which } L_k \geq \epsilon N) = 0, \]

where \( \rho \) can depend on \( N \) in an arbitrary fashion in the limit. This can be proved using similar arguments to those used in Remark 3.5. A similar argument appears in [16, Theorem 5.9].

In order to prove Theorem 6.7, we prove the convergence in distribution of the finite-dimensional marginals by induction. We assume that

\[ \frac{1}{N} (L_1, \ldots, L_k) \xrightarrow{d} (X_1, \ldots, X_k), \quad (113) \]

and prove that

\[ \frac{1}{N} (L_1, \ldots, L_{k+1}) \xrightarrow{d} (X_1, \ldots, X_{k+1}). \quad (114) \]

We need the following lemma in order to prove (114). Before the lemma let us define

\[ S_N^k := \frac{1}{N} \sum_{j=1}^{k} L_j, \]

and observe that, by the induction hypothesis (113), \( S_N^k \xrightarrow{d} S_k \).

**Lemma 6.8.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded and continuous function. Then

(i) There is a bounded and continuous function \( \psi = \psi_f : \mathbb{R} \to \mathbb{R} \) so that

\[ \mathbb{E} f(X_{k+1}) \mid X_1, \ldots, X_k) = \psi(S_k), \quad \text{a.s.} \quad (115) \]

(ii) There are functions \( \psi_N = \psi_{N,f} \) so that

\[ \mathbb{E} f \left( \frac{L_{k+1}}{N} \right) \mid L_1, \ldots, L_k \) = \psi_N(S^N_k), \quad \text{a.s.} \quad (116) \]

(iii) The functions \( \psi \) and \( \psi_N \) from the previous parts satisfy

\[ \mathbb{E} |\psi_N(S^N_k) - \psi(S^N_k)| \to 0, \quad N \to \infty. \]

We first explain how Lemma 6.8 implies (114), and therefore Theorem 6.7.

**Proof of Theorem 6.7.** In order to prove (114), it suffices to show that for any bounded and continuous functions \( f_1, \ldots, f_{k+1} : \mathbb{R} \to \mathbb{R} \),

\[ \mathbb{E} \left( f_1 \left( \frac{L_1}{N} \right) \cdots f_{k+1} \left( \frac{L_{k+1}}{N} \right) \right) \to \mathbb{E} (f_1(X_1) \cdots f_{k+1}(X_{k+1})), \quad N \to \infty. \quad (117) \]

Using Lemma 6.8 with \( f = f_{k+1} \) we get as \( N \to \infty \) that

\[ \mathbb{E} \left( f_1 \left( \frac{L_1}{N} \right) \cdots f_{k+1} \left( \frac{L_{k+1}}{N} \right) \right) = \mathbb{E} \left( \mathbb{E} \left( f_1 \left( \frac{L_1}{N} \right) \cdots f_{k+1} \left( \frac{L_{k+1}}{N} \right) \mid L_1, \ldots, L_k \right) \right) \]

\[ = \mathbb{E} \left( f_1 \left( \frac{L_1}{N} \right) \cdots f_k \left( \frac{L_k}{N} \right) \psi_N(S^N_k) \right) = \mathbb{E} \left( f_1 \left( \frac{L_1}{N} \right) \cdots f_k \left( \frac{L_k}{N} \right) \psi(S^N_k) \right) + o(1), \quad (118) \]
where in the last equation we also used that $f_i$, $1 \leq i \leq k$, are bounded. The function,

$$(y_1, \ldots, y_k) \mapsto f_1(y_1) \cdots f_k(y_k) \psi \left( \sum_{j=1}^{k} y_j \right)$$

is bounded and continuous and therefore, by the induction hypothesis (113) and Lemma 6.8 as $N \to \infty$,

$$
\mathbb{E} \left( f_1 \left( \frac{L_1}{N} \right) \cdots f_k \left( \frac{L_k}{N} \right) \psi(S_k^N) \right) \to \mathbb{E} \left( f_1 (X_1) \cdots f_k (X_k) \psi(S_k) \right).
$$

This implies (117) and therefore the theorem. □

We turn to prove Lemma 6.8. The following claim proves, in particular, the first part of the lemma.

**Claim 6.9.** For a bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$, define $\psi = \psi_f$ on $[0, 1 - \tau)$ by

$$
\psi(s) = \frac{\tau}{1 - s} f(0) + \frac{\theta}{1 - s} \int_{0}^{1-s-\tau} f(x) \left( 1 - \frac{x}{1 - s - \tau} \right)^{\theta-1} dx
$$

Then, $\psi$ extends to a continuous and bounded function on $\mathbb{R}$ and we have

$$
\mathbb{E} \left( f \left( X_{k+1} \right) | X_1, \ldots, X_k \right) = \psi(S_k), \quad a.s.
$$

The proof follows directly from the definition of $X_k$ and we leave it to the reader. Note that $\psi$ can be extended to a continuous and bounded function on $\mathbb{R}$ since the limit as $s \to 1 - \tau$ exists.

We need the following claim for the proof of the third part of Lemma 6.8. Before the claim we fix a sequence of integers $(a_N)$ such that as $N \to \infty$, $\max \left( L_2^2, \frac{N}{\log N} \right) = o(a_N)$ and $a_N = o(N)$ (note this is possible in all of the (Super-Critical) cases).

**Claim 6.10.** In all of the (Super-Critical) cases we have,

$$
\liminf_{N \to \infty} \theta \rho^{-1} \sum_{j=1}^{a_N} \varphi^s j(0) \geq \tau.
$$

**Proof.** The claim holds trivially when $\tau = 0$. Thus it remains to prove the claim in cases (Super2) and (Super3).

In case (Super3),

$$
\lim_{N \to \infty} \theta \rho^{-1} \sum_{j=1}^{a_N} \varphi^s j(0) = \theta \rho^{-1} \sum_{j=1}^{\infty} \varphi^s j(0) = \frac{g'(1)}{\rho} = \frac{\rho_c}{\rho} = \tau.
$$

(121)
In case (Super2), by Lemma 4.3, for any $\delta > 0$,
\[
\liminf_{N \to \infty} \theta \rho \sum_{j=1}^{a_N} \varphi^*(0) \geq \liminf_{N \to \infty} \theta \rho \sum_{j=L^\delta}^{L^2} \varphi^*(0) = \lim_{N \to \infty} \frac{1}{\alpha \log N} \sum_{j=L^\delta}^{L^2} \frac{\alpha_c}{j} \geq \lim_{N \to \infty} \frac{\alpha_c (\log (L^2) - \log L^\delta)}{\alpha \log N} = \tau - \frac{\delta}{2\tau}. \tag{122}
\]

The claim follows as we can take arbitrarily small $\delta$. \hfill \Box

**Proof of Lemma 6.8.** Denote by $\text{Supp}(S^N_k)$ the support of the discrete random variable $S^N_k$ defined by
\[
\text{Supp}(S^N_k) := \{ s \in \mathbb{R} \mid P(S^N_k = s) > 0 \}.
\]
Define
\[
\psi_N(s) := \sum_{l=1}^{N-sN} f \left( \frac{l}{N} \right) \cdot \frac{W_{L,l}}{N(1-s)} \cdot \frac{H_{N-sN-l}(L)}{H_{N-sN}(L)}, \quad s \in \text{Supp}(S^N_k)
\]
and note that (116) holds by Lemma 3.4. Let
\[
A_{N,\epsilon} = \text{Supp}(S^N_k) \cap \{ s \in \mathbb{R} \mid s < 1 - \tau - \epsilon \}.
\]
It suffices to prove that for any $\epsilon > 0$,
\[
\max_{s \in A_{N,\epsilon}} |\psi_N(s) - \psi(s)| \to 0, \quad N \to \infty, \tag{123}
\]
since, using the induction hypothesis (113) and the fact that $\psi_N, \psi$ are bounded, we get
\[
\limsup_{N \to \infty} E \left( \left| \psi_N \left( S^N_k \right) - \psi \left( S^N_k \right) \right| \mathbb{1}_{\{S^N_k \geq 1 - \tau - \epsilon\}} \right) \leq \limsup_{N \to \infty} C \cdot P \left( S^N_k \geq 1 - \tau - \epsilon \right) = C \cdot P( S_k \geq 1 - \tau - \epsilon) \xrightarrow{\epsilon \to 0} 0. \tag{124}
\]
Let us first prove that
\[
\max_{s \in A_{N,\epsilon}} \left| \sum_{l=1}^{a_N} \frac{W_{L,l}}{N(1-s)} \cdot \frac{H_{N-sN-l}(L)}{H_{N-sN}(L)} - \frac{\tau}{1-s} \right| \to 0, \quad N \to \infty. \tag{125}
\]
We start with an upper bound on the sum in (125). For any positive $\delta < \epsilon$ and $s \in \text{Supp}(S^N_k)$ we have
\[
\sum_{l=1}^{a_N} \frac{W_{L,l}}{N(1-s)} \cdot \frac{H_{N-sN-l}(L)}{H_{N-sN}(L)} \leq 1 - \sum_{l=a_N+1}^{N-S^N_k} \frac{W_{L,l}}{N(1-s)} \cdot \frac{H_{N-sN-l}(L)}{H_{N-sN}(L)} \tag{126}
\]
since
\[
\sum_{l=1}^{N-S^N_k} \frac{W_{L,l}}{N(1-S^N_k)} \cdot \frac{H_{N-S^N_k-l}(L)}{H_{N-S^N_k}(L)} = \mathbb{E} \left( 1 | L_1, \ldots, L_k \right) = 1, \quad a.s.
\]
Using Theorem 6.1 (with $\delta$ as $\epsilon$) and substituting $W_{L,l} \sim \theta$ (which follows from Corollary 4.6 for $l \geq a_N$) we obtain as $N \to \infty$,

$$
(1)^{\{1-s-\tau-\delta\}N} \sum_{l=a_N+1}^{a_N} \frac{W_{L,l}}{N (1 - s)} \cdot \frac{H_{N-sN-L}(L)}{H_{N-sN}(L)} = \theta + o(1) \frac{1}{1-s} \sum_{l=a_N+1}^{a_N} \frac{1}{N} \frac{1}{1 - \frac{l}{N}} \frac{1}{1 - s - \tau}^{\theta-1}
$$

$$
= \theta + o(1) \frac{1}{1-s} \int_0^1 \left(1 - \frac{x}{1 - s - \tau}\right)^{\theta-1} dx = 1 - s - \tau \frac{1}{1-s} - \frac{(1 - s - \tau)^{1-\theta} \delta^\theta}{1-s} + o(1),
$$

where all the estimates hold uniformly in $s \in A_{N,\epsilon}$. Substituting the last estimate in (126) and choosing $\delta > 0$ arbitrarily small, we get

$$
\sum_{l=1}^{a_N} \frac{W_{L,l}}{N (1 - s)} \cdot \frac{H_{N-sN-L}(L)}{H_{N-sN}(L)} \leq \frac{\tau}{1-s} + o(1), \quad N \to \infty,
$$

uniformly in $s \in A_{N,\epsilon}$. This completes the proof of (125).

From the last equation and (127) we also get that

$$
\limsup_{N \to \infty} \sup_{s \in A_{N,\epsilon}} \max_{l \in \{1-s-\tau-\delta\}N} \frac{W_{L,l}}{N (1-s)} \cdot \frac{H_{N-sN-L}(L)}{H_{N-sN}(L)} \leq C_\epsilon \delta^\theta.
$$

(128)

Now, for $s \in A_{N,\epsilon}$ and $\delta > 0$, we write

$$
\psi_N(s) = \sum_{l=1}^{a_N} f \left( \frac{l}{N} \right) \cdot \frac{W_{L,l}}{N (1 - s)} \cdot \frac{H_{N-sN-L}(L)}{H_{N-sN}(L)} = S_1 + S_2 + S_3,
$$

where $S_1, S_2$ and $S_3$ are the sums corresponding to $l \leq a_N$, $a_N < l \leq (1 - s - \tau - \delta) N$ and $(1 - s - \tau - \delta) N < l \leq N - sN$ respectively. By the continuity of $f$ and (125),

$$
\max_{s \in A_{N,\epsilon}} \left| S_1 - \frac{\tau}{1-s} f(0) \right| \to 0, \quad N \to \infty.
$$

By the same arguments as in (127),

$$
\limsup_{N \to \infty} \sup_{s \in A_{N,\epsilon}} \left| S_2 - \frac{\theta}{1-s} \int_0^{1-s-\tau-\delta} f (x) \left(1 - \frac{x}{1 - s - \tau}\right)^{\theta-1} dx \right| \leq C_{\epsilon,f} \delta^\theta.
$$

Finally, as $f$ is bounded and by (128),

$$
\limsup_{N \to \infty} \sup_{s \in A_{N,\epsilon}} \left( |S_3| + \left| \frac{\theta}{1-s} \int_{1-s-\tau-\delta}^{1-s-\tau} f (x) \left(1 - \frac{x}{1 - s - \tau}\right)^{\theta-1} dx \right| \right) \leq C_{\epsilon,f} \delta^\theta.
$$

Equation (123) follows using the triangle inequality and taking arbitrarily small $\delta$. \qed
6.3. The length of a cycle containing a given point. To finish the proof of our main theorems in the (Super-critical) cases it remains to establish the second limit law in part (iii) of Theorem 1.2 and the limit (12) in part (iii) of Theorem 1.3 (as convergence to the Poisson-Dirichlet law follows from Theorem 6.7). We start with Theorem 1.2. The proof that the limiting distribution has a constant density on \((0, 1)\) is almost identical to the proof of part (ii) of Corollary 5.5. Indeed, using Lemma 3.4, Theorem 6.1, Corollary 4.6 and Lemma 4.3, we get that for any \(0 < a < b < 1\),

\[
P\left( a \leq \frac{\log L_1}{\log N} \leq b \right) = \sum_{j=[N^a]}^{[N^b]} \frac{W_{L,j}}{N} \cdot \frac{H_{N-j}(L)}{H_N(L)} \sim \frac{\alpha_c}{\rho} \sum_{j=[N^a]}^{[N^b]} \frac{1}{j} \rightarrow \frac{\alpha_c}{\alpha} (b-a), \quad N \to \infty,
\]

where in the last limit we used that \(\frac{\rho}{\log N} \to \alpha\). On the other hand, by Theorem 6.7, we have that \(\frac{L_1}{N} \to X_1\) and therefore for any \(\epsilon > 0\),

\[
\liminf_{N \to \infty} P\left( \left| \frac{\log L_1}{\log N} - 1 \right| \leq \epsilon \right) \geq P(X_1 > 0) = 1 - \tau = 1 - \frac{\alpha_c}{\alpha},
\]

from which the limit law follows.

We proceed to prove (12) in part (iii) of Theorem 1.3. Under the corresponding assumptions, by Lemma 3.4, Theorem 6.1 and Corollary 4.6 for any fixed \(j \in \mathbb{N}\),

\[
P(L_1 = j) = \frac{W_{L,j}}{N} \cdot \frac{H_{N-j}(L)}{H_N(L)} \to \theta \rho^{-1} \varphi^j(0), \quad N \to \infty.
\]

7. The critical case in dimensions \(d \geq 2\)

In this section we find the asymptotic distribution of \(L_1\) in the critical regimes in dimensions \(d \geq 2\), establishing the case \(\alpha = \alpha_c\) in part (iii) of Theorem 1.2 and proving part (iii) in Theorem 1.3. The results are deduced as a consequence of the estimates in the sub-critical regime, by proving a type of monotonicity statement as the density increases.

An approach via contour integrals, as done in the sub-critical and super-critical cases, is also possible, though the estimates seem quite involved in dimensions \(d \in \{2, 3, 4\}\) where \(g''(1) = \infty\). Although it is not used in the proof of our main theorems, we also follow this route and find the asymptotics of the partition function in dimensions \(d \geq 5\) (see Theorem 7.5 below) as it may be useful for obtaining further information on the random permutation (e.g., the probability-generating function of the number of cycles as discussed in Remark 3.6).

7.1. Distribution of \(L_1\). We proceed to find the limiting distribution of \(L_1\) in the critical regimes in dimensions \(d \geq 2\). In the following claim we prove monotonicity of the cycle weights.

**Claim 7.1.** For each \(0 < \epsilon < 1\) there is \(j_0(\epsilon)\) such that for any \(j \geq j_0(\epsilon)\) and \(L \geq 1\),

\[
W_{L,j} \leq (1 + \epsilon)W_{L,j-\frac{j}{2}}, \quad 0 \leq i \leq \frac{j}{2}.
\]  

**Proof.** First, we claim that for sufficiently large \(j_0\),

\[
\inf_{L \geq 1, j \geq j_0} W_{L,j} > 0.
\]  

When \(j_0 \leq j \leq L^2\), this inequality follows as \(W_{L,j} \geq \theta L^d \varphi^j(0)\) and by Lemma 4.3. When \(j \geq \max\{L^2, j_0\}\) it follows from Lemma 4.5 noting that \(\hat{\psi}(0) = 1\).
We turn to prove \(129\). We consider separately the cases \(j \leq L^{2 - \frac{1}{d}}\) and \(j > L^{2 - \frac{1}{d}}\). In the first case the inequality follows from part (i) of Corollary 4.6 and Lemma 4.3, whereas in the second case it follows from Lemma 4.5. Indeed, when \(j > L^{2 - \frac{1}{d}}\),

\[
W_{L,j} \leq \theta \sum_{m \in \mathbb{Z}^d} \hat{\psi}^j(m) + C L^{-\frac{1}{2}} e^{-c_1 j^2} \leq \theta \sum_{m \in \mathbb{Z}^d} \hat{\psi}^{j-i}(m) + C L^{-\frac{1}{2}} e^{-c_1 j^2}
\]

\[
\leq W_{L,j-i} + C L^{-\frac{1}{2}} e^{-c_1 j^2} \leq (1 + \epsilon)W_{L,j-i},
\]

where in the last inequality we used \(130\).

The following lemma is the main step in the proof of the results of this section.

**Lemma 7.2.** For each \(0 < \epsilon < 1\) there is \(N_0(\epsilon)\) such that:

(i) Suppose that \(d \geq 3\) and \(\rho \to \rho_c\) as \(N \to \infty\). Then, for \(N \geq N_0(\epsilon)\)

\[
\frac{H_{N-1}(L)}{H_N(L)} \geq 1 - \epsilon
\]

(ii) Suppose that \(d = 2\) and that \(\frac{\rho}{\log N} \to \alpha_c\) as \(N \to \infty\). Then, for \(N \geq N_0(\epsilon)\)

\[
\frac{H_{N-j}(L)}{H_N(L)} \geq 1 - \epsilon \quad \text{for all} \quad 0 \leq j \leq N^{1-\epsilon}.
\]

**Proof.** Using \(130\), fix some \(j_0 > 1\) for which \(\inf_{L \geq j_1 \geq j_0} W_{L,j} > 0\). Let \(L \geq 1\) be sufficiently large for the following arguments (as a function of \(\epsilon\)).

We start with the first part. Let \(r_1 := (1 + \epsilon)^{-1}\) and \(\rho_1 := r_1 g'(r_1)\). Note that \(\rho_1 < g'(1) = \rho_c\). Let \(N_1 := \lceil \rho_1 L^d \rceil\) and \(j_1 := \lceil N_1^{\frac{1}{d}} \rceil\). We have that \(r_{N_1,L} \to r_1\) as \(L \to \infty\) by Lemma 5.3, where \(r_{N_1,L}\) is defined in \(73\). Therefore, using Theorem 5.1 and the bounds on \(a_N\) in Lemma 5.4 we get that for sufficiently large \(L\),

\[
H_{i+1}(L) \leq (1 + 2\epsilon)H_i(L), \quad N_1 - j_1 \leq i \leq N_1 - 1
\]

and

\[
H_{N_1-j_1}(L) \leq (1 - c_1)^{j_1-j_0} H_{N_1-j_0}(L).
\]

We proceed by proving that

\[
H_K(L) \leq \left(1 + 2\epsilon + Ke^{-c_1 N_1^{\frac{1}{d}}}\right) H_{K-1}(L)
\]

using induction on \(N_1 \leq K \leq 2\rho_c L^d\). Note that \(131\) easily follows from this for sufficiently large \(N\) as \(\epsilon\) is arbitrary. We already proved the case \(K = N_1\) in \(133\). We make the induction hypothesis that \(135\) holds for all \(N_1 \leq K' < K\). Using Lemma 3.4 we get that

\[
\sum_{j=1}^{K} \frac{W_{L,j}}{K}. H_{K-j}(L) = \mathbb{P}(1 \leq L_1 \leq K) = 1,
\]

and therefore

\[
H_K(L) = \sum_{i=0}^{K-1} \frac{W_{L,K-i}}{K}. H_i(L).
\]
Now we break the sum in (136) to three parts and estimate each one separately. First, observe that by Claim 7.1
\[ \sum_{i=0}^{N_1-j_1-1} \frac{W_{L,K^{-i}}}{K} \cdot H_i(L) \leq (1 + \epsilon) \sum_{i=0}^{N_1-j_1-1} \frac{W_{L,K^{-i}}}{K-1} \cdot H_i(L). \]

Second, by (134) and the bounds in Corollary 4.6 and Lemma 4.3
\[ \frac{W_{L,K^{-N_1-j_1}}}{K} \cdot H_{N_1-j_1}(L) \leq C(1 - c_\epsilon)^{j_1-j_0} H_{N_1-j_0}(L) \leq e^{-c_\epsilon N_1^3} H_{K-1}(L), \]
where in here we used that
\[ H_{K-1}(L) = \sum_{i=0}^{K-2} \frac{W_{L,K^{-i}}}{K-1} \cdot H_i(L) \geq \frac{W_{L,K^{-N_1-j_0}}}{K-1} \cdot H_{N_1-j_0}(L) \geq \frac{c}{K} H_{N_1-j_0}(L). \] (137)

Third, by (133) and the induction hypothesis
\[ \sum_{i=N_1-j_1+1}^{K-1} \frac{W_{L,K^{-i}}}{K} \cdot H_i(L) = \sum_{i=N_1-j_1}^{K-2} \frac{W_{L,K^{-i}}}{K} \cdot H_{i+1}(L) \leq \left( 1 + 2\epsilon + (K-1)e^{-c_\epsilon N_1^3} \right) \sum_{i=N_1-j_1}^{K-2} \frac{W_{L,K^{-i}}}{K-1} \cdot H_i(L). \] (138)

Adding up the contributions and using (136) we obtain
\[ H_K(L) \leq e^{-c_\epsilon N_1^3} H_{K-1}(L) + \left( 1 + 2\epsilon + (K-1)e^{-c_\epsilon N_1^3} \right) \sum_{i=0}^{K-2} \frac{W_{L,K^{-i}}}{K-1} \cdot H_i(L) \]
\[ = \left( 1 + 2\epsilon + Ke^{-c_\epsilon N_1^3} \right) H_{K-1}(L) \]
establishing (135).

We turn to prove the second part. Let \( \alpha_1 := (1 - 5\epsilon)\alpha_{c_\epsilon}, N_1 = N_1(L) \) be the smallest integer for which \( \frac{N_1}{\log N_1} \geq \alpha_1 L^2 \) and \( j_1 := \lceil N_1^{1-3\epsilon} \rceil \). By Lemma 5.3 for sufficiently large \( L \),
\[ 1 - N_1^{-1-6\epsilon} = r_{N_1,L} < 1 - N_1^{-1-4\epsilon}. \] (139)
Thus, by Theorem 5.1 (with \( N_1 \) replacing \( N \)) and the estimate for \( a_N \) in part (ii) of Lemma 5.4 we have for sufficiently large \( L \) and any \( 0 \leq j \leq N_1^{1-7\epsilon} \),
\[ H_{i+j}(L) \leq (1 - N_1^{6\epsilon-1})^{-j} H_i(L) \leq (1 + \epsilon) H_i(L), \quad N_1 - j_1 \leq i \leq N_1 - j \] (140)
and
\[ H_i(L) \leq (1 - N_1^{4\epsilon-1})^\frac{1}{2} H_{N_1-j-j_0}(L) \leq e^{-\frac{1}{2} N_1^2} H_{N_1-j-j_0}(L), \quad N_1 - j_1 \leq i \leq N_1 - \frac{j_1}{2}, \] (141)
where we recall that \( j_0 \) was fixed in the beginning of the proof. We proceed by proving that for any \( 0 \leq j \leq N_1^{1-7\epsilon} \),
\[ H_K(L) \leq \left( 1 + \epsilon + K e^{-\frac{1}{2} N_1^2} \right) H_{K-j}(L), \] (142)
using induction on \( N_1 \leq K \leq N_2 \), where \( N_2 \) is the largest integer for which \( \frac{N_2}{\log N_2} \leq 2\alpha_\epsilon L^2 \). Note that (132) easily follows from this as \( \epsilon \) is arbitrary. The case \( K = N_1 \) is given by (140).
Suppose that (142) holds for \( N_1 \leq K' < K \) and fix \( 0 \leq j \leq N_1^{1-\epsilon} \). We bound the sum in (136) as before. First, by Claim 7.1 using that \( j \leq N_1^{1-\epsilon} \leq \frac{1}{2} j_1 \) for sufficiently large \( L \),

\[
\sum_{i=0}^{N_1-j_1-1} \frac{W_{L,K-i}}{K} \cdot H_i(L) \leq (1 + \epsilon) \sum_{i=0}^{N_1-j_1-1} \frac{W_{L,K-j-i}}{K-j} \cdot H_i(L).
\]

Second, by the bounds in Corollary 4.6 and (141),

\[
\sum_{i=N_1-j_1}^{N_1-j_1+j-1} \frac{W_{L,K-i}}{K} \cdot H_i(L) \leq C L^d e^{-\frac{1}{4} N_1 \epsilon} H_{N_1-j} \leq e^{-\frac{1}{4} N_1 \epsilon} H_{K-j}(L),
\]

where in the last inequality we use the same arguments as in (137). Third, by the induction hypothesis and (140),

\[
\sum_{i=N_1-j_1}^{K-1} \frac{W_{L,K-i}}{K} \cdot H_i(L) = \sum_{i=N_1-j_1}^{K-j-1} \frac{W_{L,K-j-i}}{K} \cdot H_{i+j}(L)
\]

\[
\leq (1 + \epsilon + (K - 1) e^{-c N_1}) \sum_{i=N_1-j_1}^{K-j-1} \frac{W_{L,K-j-i}}{K-j} \cdot H_i(L).
\]

Adding up the contributions and using (136) yields (142). \( \square \)

**Corollary 7.3.** We have the following limit laws as \( N \to \infty \):

(i) Suppose that \( d \geq 3 \) and \( \rho = \rho_c \) is fixed as \( N \to \infty \). Then

\[
L_1 \overset{d}{\to} Y,
\]

where \( Y \) is the integer-valued random variable defined by

\[
\Pr(Y = j) = \frac{\theta_{\rho_c}^j(0)}{\rho_c}, \quad j \in \mathbb{N}.
\]

(ii) Suppose that \( d = 2 \) and \( \frac{\rho}{\log N} \to \alpha_c \) as \( N \to \infty \). Then

\[
\frac{\log L_1}{\log N} \overset{d}{\to} U[0,1].
\]

**Proof.** We start with part (i). Fix \( j \in \mathbb{N} \). By Lemma 3.4 and Lemma 7.2 for any \( \epsilon > 0 \) we have for sufficiently large \( N \),

\[
\Pr(L_1 = j) = \frac{W_{L,j} H_{N-j}(L)}{N H_N(L)} \geq (1 - \epsilon)^j \frac{W_{L,j} H_{N}(L)}{N} \geq (1 - \epsilon)^j \frac{\theta_{\rho_c}^j(0)}{\rho_c}
\]

where the last inequality follows from the definition of \( W_{L,j} \) in (20), using that \( N = \rho_c L^d \). Thus, as \( \epsilon \) is arbitrary,

\[
\liminf_{N \to \infty} \Pr(L_1 = j) \geq \frac{\theta_{\rho_c}^j(0)}{\rho_c} = \Pr(Y = j).
\]

Since (146) holds for any \( j \in \mathbb{N} \), we conclude that \( L_1 \overset{d}{\to} Y \) as \( N \to \infty \).
Asymptotics of the partition function in dimensions explained in the remark following Theorem 1.3. Indeed, starting with Lemma 3.4, one obtains that when \( \epsilon > 0 \) for any \( \epsilon > 0 \), the limit law follows from this as \( \epsilon \) is arbitrary.

**Remark 7.4.** Similarly to the sub-critical cases (see Remark 5.6), the analysis in Corollary 7.3 extends in a straightforward manner to the study of the joint distribution of \( L_1, L_2, \ldots \). Indeed, starting with Lemma 3.4 one obtains that when \( d \geq 3 \) and \( \rho = \rho_c \),

\[
\mathbb{P}(L_1 = j_1, \ldots, L_m = j_m) = \frac{H_{N-j_1-\cdots-j_m}(L)}{H_N(L)} \cdot \prod_{k=1}^{m} \frac{W_{L,j_k}}{N - j_1 - \cdots - j_{k-1}} \geq (1 - \epsilon) \prod_{k=1}^{m} \frac{\theta_{j_k}(0)}{\rho_c},
\]

for any \( \epsilon > 0, j_1, \ldots, j_m \in \mathbb{N} \) and sufficiently large \( N \) (depending on \( \epsilon, j_1, \ldots, j_m \)). And that, when \( d = 2 \) and \( \frac{\rho}{\log N} \to \alpha_c \),

\[
\mathbb{P}(L_1 = j_1, \ldots, L_m = j_m) = \frac{H_{N-j_1-\cdots-j_m}(L)}{H_N(L)} \cdot \prod_{k=1}^{m} \frac{W_{L,j_k}}{N - j_1 - \cdots - j_{k-1}} \geq (1 - \epsilon) \prod_{k=1}^{m} \frac{\alpha_j}{\rho \cdot j_k},
\]

for any \( \epsilon > 0, N^\epsilon \leq j_1, \ldots, j_m \leq N^{1-\epsilon} \) and sufficiently large \( N \) (depending on \( \epsilon \)).

One may then follow the analogous steps to the analysis in Corollary 7.3 and deduce that the \( (L_k) \) become asymptotically independent and identically distributed, in the sense explained in the remark following Theorem 1.3.

### 7.2. Asymptotics of the partition function in dimensions \( d \geq 5 \).

**Theorem 7.5.** Suppose that \( d \geq 5 \) and that \( \rho = \rho_c \) is fixed as \( N \to \infty \). Then, uniformly in \( j \leq N^{\frac{1}{d}} \), we have

\[
H_{N-j}(L) = [z^{N-j}] e^{G_{L}(z)} \sim \frac{e^{F_{L}(1)} N^{\frac{d-1}{2}} \left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right)^{\frac{d-1}{2}}}{2 \Gamma \left( \frac{d+1}{2} \right)}, \quad N \to \infty.
\]

In order to prove Theorem 7.5, we use the contour

\[
\gamma = \gamma \left( \eta, \beta(\eta), \sqrt{N} \right),
\]

where \( \beta(\eta, N) \) is defined in (93), \( \eta \) is sufficiently small and \( \beta(\eta) > \frac{\pi}{2} \) is the corresponding angle such that \( \gamma_4 \) is contained in the unit circle i.e., \( 1 + \eta e^{i\beta(\eta)} = 1 \) (see Figure 6). Note that, unlike the super-critical case, the distance from the singularity \( z = 1 \) in this case is \( N^{-\frac{1}{d}} \) and \( \gamma \) is contained in the closed unit disc \( \overline{D} \).

**Proposition 7.6.** Suppose that \( d \geq 5 \) and that \( \rho = \rho_c \) is fixed as \( N \to \infty \). Then there is \( 0 < \eta < \frac{1}{10} \) such that the following holds for the contour \( \gamma = \gamma \left( \eta, \beta(\eta), \sqrt{N} \right) \):

...
(i) As \( N \to \infty \), uniformly in \( z \in D \) with \(|1 - z| \leq N^{-\frac{1}{2}}\), we have

\[
G_L(z) = -\theta \log(1 - z) + F_L(1) + N(z - 1) + \frac{g''(1)}{2g'(1)}N(z - 1)^2 + o(1).
\]

(ii) There is \( c > 0 \) such that for any \( z \in \gamma_1 \cup \gamma_3 \) and large enough \( N \),

\[
\text{Re}(G_L(z)) \leq -\theta \log|1 - z| + F_L(1) + N \text{Re}(z - 1) + cN \text{Re}(z - 1)^2.
\]

(iii) There is \( c > 0 \) such that for any \( z \in \gamma_4 \) and large enough \( N \),

\[
\text{Re}(G_L(z)) \leq F_L(1) - cN.
\]

**Proof.** First we bound the difference between \( F'_L \) and \( Ld\hat{g}' \). Substituting the Taylor expansions of \( F'_L \) and \( g' \) that follow from (92) and (44) respectively, we obtain

\[
\left| F'_L(z) - Ld\hat{g}'(z) \right| \leq \sum_{j=1}^{\infty} \left| W_{L,j} - \theta - \theta Ld\varphi^*(0) \right| \leq \sum_{j=1}^{L^2} C + C \sum_{j=L^2}^{\infty} (e^{-c\frac{L^2}{j}} + L^2 j^{-\frac{3}{2}}) \leq CL^2, \quad z \in \overline{D},
\]

where in the second inequality we used Corollary 4.6, Corollary 4.7 and Lemma 4.3. Moreover, by Lemma 4.12 when \( d \geq 5 \) we have

\[
g'(z) = g'(1) + g''(1)(z - 1) + O(|z - 1|^{\frac{3}{2}}), \quad z \to 1.
\]

Now, using (149) and (150) and noting that \( N = \rho_cLd = g'(1)Ld \) we get

\[
\left| \int_{1}^{z} F_L(w)dw - N(z - 1) - \frac{g''(1)}{2g'(1)}N(z - 1)^2 \right| = \left| \int_{1}^{z} F_L(w)dw - Ld \int_{1}^{z} g'(1) - g''(1)(w - 1)dw \right|
\]

\[
\leq \int_{1}^{z} |F_L(w) - Ld\hat{g}(w)||dw| + Ld \int_{1}^{z} |g'(w) - g'(1) - g''(1)(w - 1)||dw|
\]

\[
\leq CL^2|z - 1| + CN|z - 1|^{\frac{3}{2}}, \quad z \in \overline{D}.
\]

The first and second parts of the proposition follow by substituting the last equation in (100).
We turn to prove the third part. Take $0 < \eta < \frac{1}{10}$ so that the first and second parts of the proposition hold. There is $c_\eta > 0$ such that for any $z \in \gamma_4$ (that is $|z| = 1$) and large enough $N$,

$$\text{Re} \ G_L(z) \leq F_L(z) + C_\eta \leq L^d \text{Re} \ g(z) + CL^{d-1} \leq L^d g(1) - c_\eta N \leq F_L(1) - c_\eta N,$$

where in the second and fourth inequalities we used (111) and the third inequality follows from the fact that $g$ has non-negative Taylor coefficients which are positive with finitely many exceptions. \hfill \Box

Now, we turn to prove Theorem 7.5.

**Proof of Theorem 7.5.** Let $\gamma$ be the contour from (148) with $0 < \eta < \frac{1}{10}$ such that Proposition 6.2 holds. Note that we also have $\frac{\pi}{2} < \beta(\eta) < \frac{\pi}{8}$. By the Cauchy integral formula

$$[z^{N-j}] e^{G_L(z)} = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{G_L(z)}}{z^{N-j+1}} \, dz = I_1 + I_2 + I_3,$$

where $I_1$, $I_2$ and $I_3$ are the corresponding integrals over $\gamma \cap \{ |1-z| \leq N^{-\frac{1}{4}} \}$, $(\gamma_1 \cup \gamma_3) \cap \{ |1-z| \geq N^{-\frac{1}{4}} \}$ and $\gamma_4$ respectively.

We start by estimating $I_1$. The following holds as $N \to \infty$, uniformly in $0 \leq j \leq N^\frac{1}{2}$ and $z \in \mathbb{D} \cap \{ N^{-\frac{1}{2}} \leq |1-z| \leq N^{-\frac{1}{4}} \}$. Expanding $\log z$, we obtain

$$z^{-(N-j+1)} = \exp \left[ -N (z-1) + \frac{N}{2} (z-1)^2 + O \left( N |z-1|^3 + N^{\frac{1}{2}} |1-z| \right) \right] = \exp \left[ -N (z-1) + \frac{N}{2} (z-1)^2 + o(1) \right].$$

Therefore, by the first part of Proposition 7.6

$$I_1 = \frac{e^{F_L(1)}}{2\pi i} \int_{\gamma \cap \{ |1-z| \leq N^{-\frac{1}{4}} \}} (1-z)^{-\eta} \exp \left[ \left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right) N (z-1)^2 + o(1) \right] \, dz.$$

We make the change of variables

$$z = 1 - \left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right)^{-\frac{1}{2}} N^{-\frac{1}{2}} \sqrt{-\omega}, \quad \omega = -\left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right) N (z-1)^2,$$

to obtain

$$I_1 = \frac{e^{F_L(1)}}{2\pi i} \frac{\left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right)^{\frac{\theta+1}{2}} N^{\frac{\theta+1}{2}}}{4\pi i} \int_{\tilde{\gamma}} \left( -\omega \right)^{-\frac{\theta-1}{2}} e^{-\omega + o(1)} \, d\omega,$$

where $\tilde{\gamma}$ is the image of $\gamma \cap \{ |1-z| \leq N^{-\frac{1}{4}} \}$ under the change of variables, which is a modification of the $\gamma' (2\beta(\eta) - \pi)$ from (94) having the circular arc at radius $\left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right)$ and having finite ‘arms’, terminating at radius $\left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right) N^\frac{1}{4} \to \infty$. As the integral in (95) with $s = \frac{\theta+1}{2}$ and $\beta = 2\beta(\eta) - \pi < \frac{\pi}{2}$ converge to a non-zero quantity, we conclude that

$$I_1 \sim \frac{e^{F_L(1)} N^{\frac{\theta-1}{2}} \left( \frac{g''(1)}{2g'(1)} + \frac{1}{2} \right)^{\frac{\theta+1}{2}}}{2\Gamma \left( \frac{\theta+1}{2} \right)}.$$

(152)
We turn to bound the rest of the integral. As \( \eta < \frac{1}{10} \), and by the same arguments as in (98), we have
\[
\left| z^{-N+j-1} \right| \leq |z|^{-N-1} \leq Ce^{-N \text{Re}(\log z)} \leq Ce^{-N \text{Re}(z-1)}, \quad z \in \gamma_1 \cup \gamma_3.
\]
(153)
Thus, by the second part of Proposition 7.6,
\[
|I_2| \leq Ce^{F_{\gamma_1}} \int \left( \gamma_1 \cup \gamma_3 \right) \cap \{|1-z| \geq N^{-\frac{1}{2}}\} |1-z|^{-\theta} e^{cN \text{Re}(z-1)^2} |dz| \leq C e^{F_{\gamma_1}} N^{\theta+1} \int_{\gamma(2\beta(\eta)-\pi) \cap \{|\omega| \geq N^{\frac{1}{2}}\}} |\omega|^{-\theta} e^{-c \text{Re} \omega} |d\omega| = o(I_1), \quad N \to \infty,
\]
(154)
where in the second inequality we changed the variables by \( \omega = -N(z-1)^2 \).

It remains to bound the integral over \( \gamma_4 \). By the third part of Proposition 7.6,
\[
|I_3| \leq \int_{\gamma_4} \left| e^{G_L(z)} \right| \left| z^{-N+j+1} \right| |dz| = \int_{\gamma_4} e^{\text{Re}(G_L(z))} |dz| \leq C e^{F_{\gamma_1}} e^{-\epsilon N},
\]
which is exponentially smaller than \( I_1 \) by (152).

8. The critical case in dimension 1

In this section we prove part (iii) in Theorem 1.1. Thus, throughout this section we assume that \( d = 1 \) and \( \frac{\sigma}{\sqrt{N}} \to \alpha \in (0, \infty) \) as \( N \to \infty \).

In the following theorem we find the asymptotic behavior of \( H_{N-j}(L) \). In the theorem and what follows we set \( \sigma^2 = \text{Var}(X) \). Recall that we use the standard branches of the functions \( \log(z), \sqrt{z} \) (see Section 2).

**Theorem 8.1.** For any \( \epsilon > 0 \) we have
\[
H_{N-j}(L) \sim \frac{C_0 e^{G_L(1-L^{-2})}}{N} \left( 1 - \frac{j}{N} \right)^{-\frac{3}{2}} \sum_{n=0}^{\infty} (-1)^n \left( \frac{-2\theta}{n} \right) (\theta + n) \exp \left( \frac{-(\theta + n)^2}{2\alpha^2 \sigma^2 (1 - \frac{j}{N})} \right),
\]
as \( N \to \infty \) uniformly in \( 0 \leq j \leq (1 - \epsilon) N \), where
\[
C_0 := \frac{e^{\sqrt{2}\sigma^{-1}\theta} \left( 1 - e^{-\sqrt{2}\sigma^{-1}} \right)^{2\theta}}{\sqrt{2\pi \alpha \sigma}}.
\]

In order to prove Theorem 8.1 we need the following integral calculation.

**Claim 8.2.** For any \( a > 0 \),
\[
\int_{\gamma'(\beta)} \frac{\omega e^{\omega^2 + 2\theta \omega}}{(1 - e^{2\omega})^{2\theta}} d\omega = -i \frac{\sqrt{\pi}}{\alpha^3} \sum_{n=0}^{\infty} (-1)^n \left( \frac{-2\theta}{n} \right) (\theta + n) e^{-\left(\frac{\theta + n}{\alpha} \right)^2},
\]
where the contour \( \gamma'(\beta) \) is defined in (94).
Lemma 8.3. We have
\[ \int_{\gamma(\frac{2\pi}{3})} \omega e^{\omega^2 + 2\theta \omega} (1 - e^{2\omega})^{2\theta} d\omega = \sum_{n=0}^{\infty} (-1)^n \left( -\frac{2\theta}{n} \right) \int_{\gamma(\frac{2\pi}{3})} \omega e^{\omega^2 + 2(\theta+n)\omega} d\omega, \]
where in here we used that
\[ \int_{\gamma(\frac{2\pi}{3})} \left| \omega e^{\omega^2 + 2\theta \omega} \right| \sum_{n=0}^{\infty} |(-1)^n \left( -\frac{2\theta}{n} \right) e^{2n\omega}| |d\omega| = \int_{\gamma(\frac{2\pi}{3})} \frac{|\omega e^{\omega^2 + 2\theta \omega}|}{(1 - |e^{2\omega}|)^{2\theta}} |d\omega| < \infty. \]
Now, by deforming the contour, we see that it suffices to evaluate
\[ \int_{\gamma(\alpha)} \omega e^{\omega^2 + 2(\theta+n)\omega} d\omega = ie^{-\frac{(\theta+n)^2}{a}} \int_{\mathbb{R}} \left( -\frac{\theta + n}{a} + it \right) e^{-at^2} dt = -\frac{i\sqrt{\pi}}{a^2} (\theta + n) e^{-\frac{(\theta+n)^2}{a}}, \]
where \( \gamma(n)(t) := -\frac{\theta+n}{a} + it \) for \( t \in \mathbb{R}. \)

For the proof of Theorem 8.1 we also require the following lemma.

Lemma 8.3. We have

\[ G_L(z) = c_0 + G_L(1 - L^2) - \sqrt{2} \sigma^{-1} \theta L \sqrt{1 - z} - 2\theta \log \left( 1 - e^{-\sqrt{2} \sigma^{-1} L \sqrt{1-z}} \right) + O \left( |1 - L^2 - z| L \log L \right), \quad z \in \Delta_0, \]

where \( c_0 = \sqrt{2} \sigma^{-1} \theta + 2\theta \log \left( 1 - e^{-\sqrt{2} \sigma^{-1}} \right) \). In particular,

\[ \text{Re}(G_L(z)) \leq C + G_L(1 - L^{-2}) + C |1 - z| L \log L, \quad z \in \Delta_0, \quad |1 - z| \geq L^{-2}. \]

Proof. We start be proving that

\[ \left| G'_L(z) - \frac{\theta L}{\sqrt{2} \sigma \sqrt{1 - z}} \coth \left( L \frac{\sqrt{1 - z}}{\sqrt{2} \sigma} \right) \right| \leq CL \log L, \quad z \in \Delta_0. \]

Using (68), we get that for any \( z \in \Delta_0, \)

\[
\begin{align*}
G'_L(z) &- \sum_{m \in \mathbb{Z}} \frac{\theta}{1 - z + 2\pi^2 \sigma^2 m^2} \\
&\leq \sum_{m \in \mathbb{Z}} \frac{\theta \hat{\varphi}(m)}{1 - z + 2\pi^2 \sigma^2 m^2} - \frac{\theta}{1 - z + 2\pi^2 \sigma^2 m^2} + \sum_{m \in \mathbb{Z}} \frac{\theta \hat{\varphi}(m)}{1 - z + 2\pi^2 \sigma^2 m^2} + \sum_{m \in \mathbb{Z}} \frac{\theta \hat{\varphi}(m)}{1 - z + 2\pi^2 \sigma^2 m^2} \\
&\leq C \sum_{m \in \mathbb{Z}} \frac{|m|}{1 - z + 2\pi^2 \sigma^2 m^2} + C \sum_{m \in \mathbb{Z}} \frac{|m|}{1 - z + 2\pi^2 \sigma^2 m^2} + C \sum_{m \in \mathbb{Z}} \frac{1}{m^2} \\
&\leq CL \sum_{m=1}^{L} \frac{m}{m^2} + CL \leq CL \log L,
\end{align*}
\]
where in the second inequality we used (60) and (38) to bound the first sum and (59) to bound the second sum, and in the last inequality we used that \( \hat{\phi} \) is Schwartz. Equation (157) follows from the last bound and the identity (see [25, page 351, example (2.1)]),

\[
\sum_{k \in \mathbb{Z}} \frac{1}{w^2 + k^2} = \pi \coth(\pi w), \quad w \in \mathbb{C} \setminus \{ik : k \in \mathbb{Z}\}
\]

(158)

with \( w = \frac{L \sqrt{1 - z}}{\sqrt{2} \pi \sigma} \).

Next, by a straightforward calculation we have, for any \( z \in \Delta_0 \),

\[
\frac{d}{dz} \left( \sqrt{2} \sigma^{-1} \theta L \sqrt{1 - z} + 2 \theta \log \left( 1 - e^{-\sqrt{2} \sigma^{-1} L \sqrt{1 - z}} \right) \right) = \frac{-\theta L}{\sqrt{2} \sigma \sqrt{1 - z}} \coth \left( \frac{L \sqrt{1 - z}}{\sqrt{2} \sigma} \right).
\]

Thus, using (157), we get that for any \( z \in \Delta_0 \),

\[
G_L(z) = G_L(1 - L^{-2}) + \int_{1-L^{-2}}^{z} G_L'(w)dw = c_0 + G_L(1 - L^{-2})
\]

\[- \sqrt{2} \sigma^{-1} \theta L \sqrt{1 - z} + 2 \theta \log \left( 1 - e^{-\sqrt{2} \sigma^{-1} L \sqrt{1 - z}} \right) + O\left( |1 - L^{-2} - z| \log L \right).
\]

**Proof of Theorem 8.1.** Fix \( 0 < \epsilon < 1 \) and assume throughout the proof that \( 0 \leq j \leq (1 - \epsilon)N \) and that \( N \) is large enough. Take \( 0 < \eta < \frac{1}{10} \) such that \( \gamma := \gamma(\eta, \pi, \frac{L}{2\sigma}, \frac{L^2}{2\sigma^2}) \subseteq \Delta_0 \), where \( \gamma(\eta, \beta, N) \) is defined in (93). We choose a sequence \( t_N \leq N^{\frac{3}{2}} \) such that as \( N \to \infty \) we have \( t_N \to \infty \) and \( t_N \left( \frac{N}{L^2} - \sigma^2 \right) \to 0 \). By the Cauchy integral formula

\[
H_{N-j}(L) = \frac{1}{2\pi i} \int_{\gamma} e^{G_L(z)} \frac{dz}{z^{N-j+1}} = I_1 + I_2 + I_3,
\]

where \( I_1, I_2 \) and \( I_3 \) are the corresponding integrals over \( \gamma \cap \{|1 - z| \leq \frac{t_N}{N}\} \), \( \gamma \cap \{|1 - z| \leq N^{-\frac{3}{2}}\} \) and \( \gamma \cap \{|1 - z| \geq N^{-\frac{3}{2}}\} \).

We start by evaluating \( I_1 \). The following holds as \( N \to \infty \), uniformly in \( 0 \leq j \leq (1 - \epsilon)N \) and \( z \in \Delta_0 \) such that \( |1 - z| \leq \frac{t_N}{N} \). First, by Lemma 8.3 we have that

\[
e^{G_L(z)} = e^{c_0 + G_L(1 - L^{-2})}e^{2\theta \omega} \left( 1 - e^{2\omega} \right)^{-2\theta} (1 + o(1)),
\]

(159)

where \( \omega := -2^{-\frac{1}{2}} \sigma^{-1} L \sqrt{1 - z} \). Second, expanding \( \log z \) at \( z = 1 \) we get

\[
z^{-(N-j+1)} = \exp \left[ -N \left( 1 - \frac{j}{N} \right) \left( z - 1 \right) + o(1) \right] = \exp \left[ 2\alpha^2 \sigma^2 \left( 1 - \frac{j}{N} \right) \omega^2 + o(1) \right],
\]

(160)

where the second equality relies on the fact that \( t_N \left( \frac{N}{L^2} - \sigma^2 \right) \to 0 \). Substituting (159) and (160) in \( I_1 \) and changing variables from \( z \) to \( \omega \) we obtain that

\[
I_1 = -\frac{2\alpha^2 c_0 + G_L(1 - L^{-2})}{\pi i L^2} \int_{\tilde{\gamma}} \frac{\omega e^{2\theta \omega}}{(1 - e^{2\omega})^{2\theta}} \exp \left( 2\alpha^2 \sigma^2 \left( 1 - \frac{j}{N} \right) \omega^2 + o(1) \right) d\omega,
\]

where \( \tilde{\gamma} \) is the image of \( \gamma \cap \{|1 - z| \leq \frac{t_N}{N}\} \) under the change of variables, which is a modification of the contour \( \gamma(\frac{2\pi}{3}) \) from Claim 8.2 having finite ‘arms’, terminating at radius \( \frac{L}{\sqrt{2} \sigma N} \sqrt{t_N} \to \).
\[ I_1 \sim \frac{C_0 e^{G_L(1-L^{-2})}}{N} \left(1 - \frac{j}{N}\right) \frac{-3}{2} \sum_{n=0}^{\infty} (-1)^n \left(-\frac{2\theta}{n}\right) (\theta + n) \exp \left(\frac{-(\theta + n)^2}{2\sigma^2 (1 - \frac{j}{N})}\right) \] (161)

uniformly in \(0 \leq j \leq (1 - \epsilon)N\).

We turn to bound \(I_2\). By the same arguments as in (98), we have

\[ |z^{-(N-j+1)}| = e^{-(N-j+1) \text{Re}(\log z)} \leq e^{-\epsilon N (\text{Re} z - 1)} \leq e^{-c_N |z-1|}, \quad z \in \gamma, \quad \frac{2\sigma^2}{L^2} \leq |1 - z| \leq \eta. \]

Therefore, using (156), we get as \(N \to \infty\),

\[ |I_2| \leq C e^{G_L(1-L^{-2})} \int_{\gamma \cap \{\frac{N}{2} \leq |1-z| \leq N^{-\frac{1}{3}}\}} e^{-c_N |z-1|} |dz| \leq \frac{C e^{G_L(1-L^{-2})}}{N} \int_{\gamma' \cap \{|s| \geq \epsilon N\}} e^{-c_3 |s|} |ds| = o(I_1), \]

where \(\gamma'(\beta)\) is defined in (94) and where in the second inequality we made the change of variables \(z = 1 + N^{-1} s\).

Finally, we bound \(I_3\) by

\[ |I_3| \leq \int_{\gamma \cap \{|1-z| \geq N^{-\frac{1}{3}}\}} e^{G_L(z)} \left|\frac{z}{z^{N-j+1}}\right| |dz| \leq C e^{CL(1 + cN^{-\frac{1}{3}})} \frac{-\epsilon N}{\sigma^2 L^2} \leq C e^{CL(1 + cN^{-\frac{1}{3}})} \frac{-\epsilon N}{\sigma^2 L^2} = o(I_1), \quad N \to \infty, \]

where in the second inequality and in the final estimate we used that \(|G_L(z)| \leq CL\) for any \(z \in \Delta_0\) with \(|1-z| \geq L^{-2}\). This fact follows, for example, from (108) and the definition of \(F_L\) in (91).

\[ \Box \]

**Corollary 8.4.** We have

\[ \frac{L_1}{N} \xrightarrow{d} \mu, \quad N \to \infty, \]

where \(\mu\) is a continuous probability measure supported on \((0, 1)\) with density function proportional to

\[ \left(\sum_{m \in \mathbb{Z}} e^{-2\pi^2 \sigma^2 \alpha^2 m^2 x}\right) \left(1 - x\right)^{\frac{-1}{2}} \sum_{n=0}^{\infty} (-1)^n \left(-\frac{2\theta}{n}\right) (\theta + n) \exp \left(\frac{-(\theta + n)^2}{2\sigma^2 (1 - x)}\right), \quad x \in (0, 1). \]

**Proof.** Let \(0 < a < b < 1\). By Lemma 4.5 and (41) we have that

\[ W_{L,j} \sim \theta \sum_{m \in \mathbb{Z}} e^{-2\pi^2 \sigma^2 \alpha^2 m^2 x} / L^2 \sim \theta \sum_{m \in \mathbb{Z}} e^{-2\pi^2 \sigma^2 \alpha^2 m^2 x} / L^2, \quad N \to \infty \]

uniformly in \(aN \leq j \leq bN\), where the second estimate follows as the function \(\sum_{m \in \mathbb{Z}} e^{-2\pi^2 \sigma^2 \alpha^2 m^2 x}\) is continuously differentiable on \([a, b]\) and as \(N / L^2 \to \alpha^2\). Thus, by Lemma 3.4 and Theorem 8.1
we have as $N \to \infty$,

$$\mathbb{P}
\left(a \leq \frac{L_{1}}{N} \leq b\right) = \sum_{j=\lceil aN \rceil}^{\lfloor bN \rfloor} \frac{W_{L,j} H_{N-j}(L)}{N} \frac{H_{N}(L)}{H_{N}(N)}$$

\[
\approx \frac{1}{Z} \sum_{j=\lceil aN \rceil}^{\lfloor bN \rfloor} \frac{1}{N} \left( \sum_{m \in \mathbb{Z}} e^{-2\pi^{2}j^{2}m^{2}} \right) \left( 1 - \frac{j}{N} \right)^{-\frac{3}{2}} \sum_{n=0}^{\infty} (-1)^{n} \left( -\frac{2\theta}{n} \right) (\theta + n) \exp \left( \frac{-(\theta + n)^{2}}{2\alpha^{2}\sigma^{2}(1 - \frac{j}{N})} \right)
\]

\[
\to \frac{1}{Z} \int_{a}^{b} \left( \sum_{m \in \mathbb{Z}} e^{-2\pi^{2}j^{2}m^{2}x} \right) (1 - x)^{-\frac{3}{2}} \sum_{n=0}^{\infty} (-1)^{n} \left( -\frac{2\theta}{n} \right) (\theta + n) \exp \left( \frac{-(\theta + n)^{2}}{2\alpha^{2}\sigma^{2}(1 - x)} \right) dx,
\]

where

$$Z = \frac{1}{\theta} \sum_{n=0}^{\infty} (-1)^{n} \left( -\frac{2\theta}{n} \right) (\theta + n) \exp \left( \frac{-(\theta + n)^{2}}{2\alpha^{2}\sigma^{2}} \right)$$

and where the last limit follows as the integrand is continuous on $[a, b]$. Thus, it suffices to show that

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{P} \left( \epsilon \leq \frac{L_{1}}{N} \leq (1 - \epsilon) \right) = 1.$$ 

First, note that by the bounds given in Corollary 4.6 and the asymptotics in Lemma 4.3, we have

$$W_{L,j} \leq C + C \frac{L}{\sqrt{j}}, \quad L \geq 1, \quad j \in \mathbb{N}. \quad (162)$$

Thus, using Theorem 8.1 we obtain that

$$\mathbb{P}(L_{1} \leq \epsilon N) = \sum_{j=1}^{\lfloor \epsilon N \rfloor} \frac{W_{L,j} H_{N-j}(L)}{N} \frac{H_{N}(L)}{H_{N}(N)} \leq \frac{C}{N} \sum_{j=1}^{\lfloor \epsilon N \rfloor} \left( 1 + \frac{L}{\sqrt{j}} \right) \leq C \epsilon + C \alpha \sqrt{\epsilon} \xrightarrow{\epsilon \to 0} 0. \quad (163)$$

Next, by (29) we have for large enough $N$

$$\sum_{n=0}^{\lfloor \epsilon N \rfloor} H_{n}(L) \leq C \sum_{n=0}^{\infty} H_{n}(L) \left( 1 - \frac{1}{\epsilon N} \right) = C e^{G_{L}(1 - \frac{1}{\epsilon N})}. \quad (164)$$

Therefore, by Theorem 8.1 with $j = 0$ and (162), we have

$$\mathbb{P}(L_{1} \geq (1 - \epsilon) N) = \sum_{j=\lfloor (1 - \epsilon) N \rfloor}^{N} \frac{W_{L,j} H_{N-j}(L)}{N} \frac{H_{N}(L)}{H_{N}(N)} \leq \frac{C_{\alpha}}{N H_{N}(L)} \sum_{n=0}^{\lfloor \epsilon N \rfloor} H_{n}(L)$$

\[
\leq C_{\alpha} e^{G_{L}(1 - \frac{1}{\epsilon N})} G_{L}(1 - L^{-2}) \xrightarrow{N \to \infty} C_{\alpha} e^{\sigma_{0} - \frac{\sqrt{2}q}{\sigma^{2} \sqrt{\epsilon}}} \left( 1 - e^{-\frac{\sqrt{2}q}{\sigma^{2} \sqrt{\epsilon}}} \right)^{-2\theta} \xrightarrow{\epsilon \to 0} 0,
\]

where the limit as $N \to \infty$ follows from Lemma 8.3.
5. Andrew D Barbour and Boris L Granovsky, *Random combinatorial structures: the convergent case*, Journal of Combinatorial Theory, Series A **109** (2005), no. 2, 203–220.
6. Florent Benaych-Georges, *Cycles of random permutations with restricted cycle lengths*, arXiv preprint arXiv:0712.1903 (2007).
7. Nathanaël Berestycki et al., *Emergence of giant cycles and slowdown transition in random transpositions and k-cycles*, Electronic Journal of Probability **16** (2011), 152–173.
8. Volker Betz and Daniel Ueltschi, *Spatial random permutations and infinite cycles*, Communications in mathematical physics **285** (2009), no. 2, 469–501.
9. _, *Spatial random permutations and poisson-dirichlet law of cycle lengths*, Electronic Journal of Probability **16** (2011), 1173–1192.
10. _, *Spatial random permutations with small cycle weights*, Probability theory and related fields **149** (2011), no. 1, 191–222.
11. Volker Betz, Daniel Ueltschi, Yvan Velenik, et al., *Random permutations with cycle weights*, The Annals of Applied Probability **21** (2011), no. 1, 312–331.
12. Nayantara Bhatnagar and Ron Peled, *Lengths of monotone subsequences in a mallows permutation*, Probability Theory and Related Fields **161** (2015), no. 3-4, 719–780.
13. Philippe Biane, Jim Pitman, and Marc Yor, *Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions*, Bulletin of the American Mathematical Society **38** (2001), no. 4, 435–465.
14. Marek Biskup, Thomas Richthammer, et al., *Gibbs measures on permutations over one-dimensional discrete point sets*, The Annals of Applied Probability **25** (2015), no. 2, 898–929.
15. Jakob Björnberg et al., *Large cycles in random permutations related to the heisenberg model*, Electronic Communications in Probability **20** (2015).
16. Leonid V Bogachev and Dirk Zeindler, *Asymptotic statistics of cycles in surrogate-spatial permutations*, Communications in Mathematical Physics **334** (2015), no. 1, 39–116.
17. Alexei Borodin, Persi Diaconis, and Jason Fulman, *On adding a list of numbers (and other one-dependent deterministic processes)*, Bulletin of the American Mathematical Society **47** (2010), no. 4, 639–670.
18. Mark Braverman and Elchanan Mossel, *Sorting from noisy information*, arXiv preprint arXiv:0910.1191 (2009).
19. Alessandra Cipriani and Dirk Zeindler, *The limit shape of random permutations with polynomially growing cycle weights*, arXiv preprint arXiv:1312.3517 (2013).
20. Steffen Dereich and Peter Mörters, *Cycle length distributions in random permutations with diverging cycle weights*, Random Structures & Algorithms **46** (2015), no. 4, 635–650.
21. Persi Diaconis and Mehrdad Shahshahani, *Generating a random permutation with random transpositions*, Probability Theory and Related Fields **57** (1981), no. 2, 159–179.
22. Nicholas M Ercolani and Daniel Ueltschi, *Cycle structure of random permutations with cycle weights*, Random Structures & Algorithms **44** (2014), no. 1, 109–133.
23. Shui Feng, *The Poisson-Dirichlet distribution and related topics: models and asymptotic behaviors*, Springer Science & Business Media, 2010.
24. Richard Phillips Feynman, *Atomic theory of the λ transition in Helium*, Physical Review **91** (1953), no. 6, 1291.
25. Theodore Gamelin, *Complex analysis*, Springer Science & Business Media, 2003.
26. Alexey Gladkich and Ron Peled, *On the cycle structure of mallows permutations*, arXiv preprint arXiv:1601.06991 (2016).
27. Alexander Gnedin and Grigori Olshanski, *q-exchangeability via quasi-invariance*, The Annals of Probability (2010), 2103–2135.
28. _, *The two-sided infinite extension of the mallows model for random permutations*, Advances in Applied Mathematics **48** (2012), no. 5, 615–639.
29. Loukas Grafakos, *Classical and modern fourier analysis*, Prentice Hall, 2004.
30. Alun Hammond, *Infinite cycles in the random stirring model on trees*, Bull. Inst. Math. Acad. Sin. (N.S.) **8** (2013), no. 1, 85–104.
31. _, *Sharp phase transition in the random stirring model on trees*, Probability Theory and Related Fields **161** (2015), no. 3-4, 429–448.
32. Peter Henrici, *Applied and computational complex analysis, volume 2: special functions, integral transforms, asymptotics, continued fractions*, vol. 2, John Wiley & Sons, 1991.
33. Roman Kotecký, Piotr Miłoś, and Daniel Ueltschi, *The random interchange process on the hypercube*, arXiv preprint arXiv:1509.02067 (2015).
34. Michael Lugo, *Profiles of permutations*, the electronic journal of combinatorics 16 (2009), no. 1, R99.
35. Colin L Mallows, *Non-null ranking models. i*, Biometrika 44 (1957), no. 1/2, 114–130.
36. Kenneth Maples, Dirk Zeindler, and Ashkan Nikeghbali, *On the number of cycles in a random permutation*, Electronic Communications in Probability 17 (2012).
37. Takeo Matsubara, *Quantum-statistical theory of liquid Helium*, Progress of Theoretical Physics 6 (1951), no. 5, 714–730.
38. Carl Mueller and Shannon Starr, *The length of the longest increasing subsequence of a random mallows permutation*, Journal of Theoretical Probability (2013), 1–27.
39. Sumit Mukherjee et al., *Fixed points and cycle structure of random permutations*, Electronic Journal of Probability 21 (2016).
40. Ashkan Nikeghbali, Dirk Zeindler, et al., *The generalized weighted probability measure on the symmetric group and the asymptotic behavior of the cycles*, Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, vol. 49, Institut Henri Poincaré, 2013, pp. 961–981.
41. A Polyakov, *Interaction of goldstone particles in two dimensions. Applications to ferromagnets and massive Yang-Mills fields*, Physics Letters B 59 (1975), no. 1, 79–81.
42. Oded Schramm, *Compositions of random transpositions*, Israel Journal of Mathematics 147 (2005), no. 1, 221–243.
43. Shannon Starr, *Thermodynamic limit for the mallows model on s n*, Journal of Mathematical Physics 50 (2009), no. 9, 095208.
44. András Sári, *Percolation transition in the bose gas*, Journal of physics. A, mathematical and general 26 (1993), no. 18, 4689–4710.
45. ______, *Percolation transition in the bose gas: Ir*, Journal of Physics A: Mathematical and General 35 (2002), no. 33, 6995.
46. AN Timashov, *Random permutations with cycle lengths in a given finite set*, Discrete Mathematics and Applications 18 (2008), no. 1, 25–39.
47. Bálint Tóth, *Improved lower bound on the thermodynamic pressure of the spin 1/2 Heisenberg ferromagnet*, letters in mathematical physics 28 (1993), no. 1, 75–84.
48. Daniel Ueltschi, *Random loop representations for quantum spin systems*, Journal of Mathematical Physics 54 (2013), no. 8, 083301.

Dor Elboim
Tel Aviv University, School of Mathematical Sciences
E-mail address: dorelbom@mail.tau.ac.il

Ron Peled
Tel Aviv University, School of Mathematical Sciences, Tel Aviv, 69978, Israel.
E-mail address: peledron@post.tau.ac.il
URL: http://www.math.tau.ac.il/~peledron