FOCAL SURFACES OF FRONTS ASSOCIATED TO UNBOUNDED PRINCIPAL CURVATURES

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Abstract. We study focal surfaces of (wave) fronts associated to unbounded principal curvatures near non-degenerate singular points of initial fronts. We give characterizations of singularities of those focal surfaces in terms of types of singularities and geometrical properties of initial fronts. Moreover, we investigate behavior of the Gaussian curvature of the focal surface.

1. Introduction

Focal surfaces (or caustics) of regular surfaces in the Euclidean 3-space $\mathbb{R}^3$ can be characterized by several ways: loci of centers of principal curvature spheres of initial surfaces, singular value sets of the normal congruence, and the bifurcation set of the family of distance squared functions for instance (cf. [4,12,15,28]). Although initial surfaces have no singular points, their focal surfaces have singularities in general ([1,2,12,28,29,37]). It is known that types of (co)rank one singularities on focal surfaces correspond to geometric properties arising from principal curvatures of initial surfaces ([12,28,29]). Investigating singularities of focal surfaces, one might have new geometrical properties of surfaces ([3–5,12,23]).

On the other hand, there are classes of surfaces with singular points called frontals and (wave) fronts. These surfaces admit smooth unit normal vector field even at singular points. Owing to this property, frontals and fronts can be considered as a kind of generalization of regular surfaces (i.e., immersions) in $\mathbb{R}^3$. Recently, there are several studies on frontals and fronts from differential geometric viewpoint, and various geometric invariants at singular points are introduced ([6–11,16,20,21,25–27,34]). Moreover, relation among behavior of the curvatures of frontals and fronts and geometric invariants are considered ([11,21,32,34,36]). In particular, for a front, it is known that one principal curvature can be extended as a bounded $C^\infty$ function near a non-degenerate singular point, and another is unbounded near such a singular point ([22,24,36]). Although one principal curvature of a front is unbounded near a non-degenerate singular point, the radius function (i.e., the reciprocal) of it can be extended as a $C^\infty$ function near the singular point. This fact plays a crucial role to study focal surfaces of fronts with bounded Gaussian curvature because the bounded principal curvature vanishes on the set of singular points of such fronts, and hence we cannot define corresponding focal surfaces near such points (cf. [17,19,24,30]).

In this paper, we investigate singularities and geometrical properties of focal surfaces of fronts associated to unbounded principal curvatures. We note that characterizations of singularities on focal surfaces relative to bounded principal curvatures are given in [35]. Moreover, if the initial front has a cuspidal edge, then the focal surface associated...
to the unbounded principal curvature is regular at the corresponding point ([35]). Thus
we consider focal surfaces of fronts with singular points of the second kind, which are
classes of non-degenerate singular points of fronts. A swallowtail singularity is a typical
example. To give characterizations of singularities of the focal surface, we recall behavior
of principal curvatures and principal vectors around a non-degenerate singular point in
Section 3. By observing the unbounded principal curvature and corresponding principal
vector, we extend a concept of sub-parabolic points to singular points of the second kind
on fronts in Section 4. In particular, we show relation between behavior of the Gaussian
curvature and sub-parabolic points of a front (Propositions 4.3 and 4.4).

In Section 5, we study a focal surface of a front associated to the unbounded principal
curvature. We give characterizations of singularities on focal surfaces by geometrical
properties of initial fronts (Theorem 5.6 and Proposition 5.9). Furthermore, we study
contact between singular sets of the initial front and the focal surface associated to the
unbounded principal curvature. We characterize types of singularities of the focal surface
in terms of the contact order of these curves (Proposition 5.14). Finally, we observe that
the behavior of the Gaussian curvature of the focal surface. Especially, for the case of a
singular point of the second kind, we give characterization for the rational boundedness of
the Gaussian curvature of the focal surface by a certain geometrical property of the initial
front (Theorem 5.15).

2. Preliminaries

We recall some notions on fronts. For details, see [1, 2, 6, 12, 18, 34].

2.1. Fronts. Let \((\Sigma; u, v)\) be a domain in the \((u, v)\)-plane \(\mathbb{R}^2\). Let \(f: \Sigma \to \mathbb{R}^3\) be a
\(C^\infty\) map. Then \(f\) is said to be a frontal if there exists a \(C^\infty\) map \(\nu: \Sigma \to \mathbb{S}^2\) such that
\(\langle df_q(X), \nu(q) \rangle = 0\) for any \(q \in \Sigma\) and \(X \in T_q\Sigma\), where \(\mathbb{S}^2\) is the standard unit sphere in
\(\mathbb{R}^3\) and \(\langle \cdot, \cdot \rangle\) is the canonical inner product on \(\mathbb{R}^3\). Moreover, a frontal \(f\) is a front if the
pair \((f, \nu): \Sigma \to \mathbb{R}^3 \times \mathbb{S}^2\) gives an immersion. We call \(\nu\) a unit normal vector field or a
Gauss map of \(f\).

We fix a frontal \(f\). A point \(p \in \Sigma\) is called a singular point of \(f\) if \(f\) is not an immersion
at \(p\). We denote by \(S(f)\) the set of singular points of \(f\). Set a function \(\lambda: \Sigma \to \mathbb{R}\) by
\[\lambda(u, v) = \det(f_u, f_v, \nu)(u, v) \quad (f_u = \partial f/\partial u, \ f_v = \partial f/\partial v),\]
where \(\det\) is the determinant of \(3 \times 3\) matrices. Then one can check that \(\lambda^{-1}(0) = S(f)\)
holds by the definition. We call the function \(\lambda\) the signed area density function of \(f\).

Take a singular point \(p \in S(f)\) of a frontal \(f\). Then \(p\) is said to be non-degenerate (resp. degenerate) if \((\lambda_u(p), \lambda_v(p)) \neq (0, 0)\) (resp. \((\lambda_u(p), \lambda_v(p)) = (0, 0)\)) holds. For a
non-degenerate singular point \(p\) of \(f\), there exist an open neighborhood \(U(\subset \Sigma)\) of \(p\) and
a regular curve \(\gamma: (-\varepsilon, \varepsilon) \ni t \mapsto \gamma(t) \in U\) \((\varepsilon > 0)\) such that \(\gamma(0) = p\) and \(\lambda(\gamma(t)) = 0\)
on \(U\) by the implicit function theorem. Moreover, since a non-degenerate singular point
\(p\) is a corank one singular point (i.e., \(\text{rank } df_p = 1\)), there exists a never-vanishing vector
field \(\eta\) on \(U\) such that \(df_q(\eta_q) = 0\) for any \(q \in S(f) \cap U\) \((\eta_q \in T_qU)\). We call \(\gamma\) and \(\eta\) the
singular curve for \(f\) and a null vector field, respectively. We remark that one can take a
null vector field \(\eta\) of a front near a corank one singular point \(p\).

A non-degenerate singular point \(p\) is said to be of the first kind if \(\gamma' = d\gamma/dt\) and \(\eta\) are
linearly independent at \(p = \gamma(0)\). Otherwise, it is said to be of the second kind. Let \(p\) be
a singular point of the second kind of a frontal \(f\). Then \(p\) is said to be admissible if for
each open neighborhood \(U\) of \(p\), the intersection \(S(f) \cap U\) contains a singular point of the
first kind. Otherwise, we call \(p\) non-admissible. We say that a non-degenerate singular
point \(p\) is admissible if \(p\) is either of the first kind or the admissible second kind.
Definition 2.1. Let \( f : (\Sigma, p) \rightarrow \mathbb{R}^3 \) be a \( C^\infty \) map germ and \( p \) a singular point of \( f \). Then

1. \( f \) is a cuspidal edge at \( p \) if \( f \) is \( \mathcal{A} \)-equivalent to the germ \((u, v) \mapsto (u, v^2, v^3)\) at the origin.
2. \( f \) is a swallowtail at \( p \) if \( f \) is \( \mathcal{A} \)-equivalent to the germ \((u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uw^2)\) at the origin.
3. \( f \) is a cuspidal butterfly at \( p \) if \( f \) is \( \mathcal{A} \)-equivalent to the germ \((u, v) \mapsto (u, 5v^4 + 2uv, 4v^5 + uw^2)\) at the origin.
4. \( f \) is a cuspidal lips at \( p \) if \( f \) is \( \mathcal{A} \)-equivalent to the germ \((u, v) \mapsto (u, 3v^4 + 2u^2v^2, v^3 + u^2v)\) at the origin.
5. \( f \) is a cuspidal beaks at \( p \) if \( f \) is \( \mathcal{A} \)-equivalent to the germ \((u, v) \mapsto (u, 3v^4 - 2u^2v^2, v^3 - u^2v)\) at the origin.

Here, two map germs \( f_1, f_2 : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) are \( \mathcal{A} \)-equivalent if there exist diffeomorphism germs \( \varphi : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \) on the source and \( \Phi : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0) \) on the target such that \( f_2 = \Phi \circ f_1 \circ \varphi^{-1} \).

A cuspidal edge, a swallowtail and a cuspidal butterfly are non-degenerate front singular points. Moreover, a cuspidal edge and a swallowtail are generic singularities of fronts in \( \mathbb{R}^3 \), and a cuspidal lips/beaks and a cuspidal butterfly are generic corank one singularities of one-parameter bifurcation of fronts (cf. [2, 37]). Further, a cuspidal edge is a singular point of the first kind, and a swallowtail and a cuspidal butterfly are of the admissible second kind (cf. [21]). For these singular points, the following characterizations are known.

Fact 2.2 ([13, 14, 18, 33]). Let \( f : \Sigma \rightarrow \mathbb{R}^3 \) be a front and \( p \in \Sigma \) a corank one singular point of \( f \). Then we have the following.

1. \( f \) is a cuspidal edge at \( p \) if and only if \( \eta \lambda(p) \neq 0 \).
2. \( f \) is a swallowtail at \( p \) if and only if \( d\lambda(p) \neq 0, \eta \lambda(p) = 0 \) and \( \eta \eta \lambda(p) \neq 0 \).
3. \( f \) is a cuspidal butterfly at \( p \) if and only if \( d\lambda(p) \neq 0, \eta \lambda(p) = \eta \eta \lambda(p) = 0 \) and \( \eta \eta \eta \lambda(p) \neq 0 \).
4. \( f \) is a cuspidal lips at \( p \) if and only if \( d\lambda(p) = 0 \) and \( \det(\mathcal{H}_1)(p) > 0 \), that is, \( \lambda \) has a Morse type singularity of index zero or two at \( p \).
5. \( f \) is a cuspidal beaks at \( p \) if and only if \( d\lambda(p) = 0, \eta \eta \lambda(p) \neq 0 \) and \( \det(\mathcal{H}_1)(p) < 0 \), that is, \( \lambda \) has a Morse type singularity of index one and \( \eta \eta \lambda \neq 0 \) at \( p \).

Here, \( \lambda \) is the signed area density function of \( f \) as in (2.1), \( \eta \) is a null vector field, \( \eta \lambda \) means the directional derivative of \( \lambda \) in the direction \( \eta \) and \( \det(\mathcal{H}_1) \) is the Hessian of \( \lambda \).

2.2. Geometric invariants. We recall geometric invariants of fronts.

2.2.1. Geometric invariants of cuspidal edges. First we consider the case of cuspidal edges. Let \( f : \Sigma \rightarrow \mathbb{R}^3 \) be a front, \( v \) its Gauss map and \( p \) a cuspidal edge of \( f \). Let \( \gamma(t) \) be a singular curve passing through \( p \) and \( \eta \) a null vector field of \( f \). Then one can define the following geometric invariants: the singular curvature \( \kappa_s \) ([34]), the limiting normal curvature \( \kappa_v \) ([21, 34]), the cuspidal curvature \( \kappa_c \) ([21]) and the cuspidal torsion \( \kappa_t \) ([20]). We note that \( \kappa_s \) is an intrinsic invariant and its sign has a geometrical meaning (see [9, 34]). It is known that \( \kappa_c \) does not vanish when \( \gamma \) consists of cuspidal edges ([20, 21]). We remark that these invariants can be defined at singular points of the first kind for frontals but not fronts. In such cases, \( \kappa_v \) vanishes at non-front singular points (cf. [21, Proposition 3.11]).

The limiting normal curvature \( \kappa_v \) relates to the boundedness of the Gaussian curvature of a front with a cuspidal edge. In fact, the Gaussian curvature \( K \) of a front \( f \) is bounded on a sufficiently small neighborhood \( U \) of a cuspidal edge \( p \) if and only if \( \kappa_v \) vanishes along the singular curve \( \gamma \) through \( p \) ([21, Theorem 3.9]). In this case, \( K \) can be extended as a \( C^\infty \) function on \( U \). Moreover, the following property holds.
Fact 2.3 ([21, Remark 3.19], [7, Theorem 1.9]). Let $f$ be a front in $\mathbb{R}^3$ and $p$ a cuspidal edge. Let $K$ be the Gaussian curvature of $f$ defined on the set of regular points of $f$. Suppose that $K$ is bounded on a sufficiently small neighborhood $U$ of $p$. Then $K$ satisfies
\begin{equation}
4K = -4\kappa_1^2 - \kappa_2\kappa_3^2
\end{equation}
at $p$.

On the other hand, we can take a coordinate system $(U; u, v)$ around $p$ satisfying the following properties:
1. the $u$-axis is the singular curve,
2. $\partial_v$ gives a null vector field, and
3. there are no singular points other than the $u$-axis.

We call this local coordinate system $(U; u, v)$ adapted ([18, 21, 34]). Moreover, an adapted coordinate system $(U; u, v)$ is called special adapted if the frame $\{f_u, f_v, v\}$ gives an orthonormal frame along the $u$-axis in addition to the above conditions ([34, Lemma 3.2], [21]).

On an adapted coordinate system $(U; u, v)$, since $\eta = \partial_v, f_v(u, 0) = 0$ holds. Thus there exists a $C^\infty$ map $g: U \rightarrow \mathbb{R}^3$ such that $f_v = vg$. We note that $g$ does not vanish along the $u$-axis since $f_v = g$ holds along the $u$-axis. Therefore the pair $\{f_u, g, v\}$ gives a frame along $f$. Moreover, when we take a special adapted coordinate system $(U; u, v)$ around a cuspidal edge $p$, then $\{f_u, g, v\}$ gives an orthonormal frame along the $u$-axis. In particular, $v$ can be taken as $v = (f_u \times g)/|f_u \times g|$. Using these mappings, we define the following functions:
\begin{equation}\label{eq:2.3}
\begin{align*}
\overline{E} &= \langle f_u, f_u \rangle, & \overline{F} &= \langle f_u, g \rangle, & \overline{G} &= \langle g, g \rangle, \\
\overline{L} &= -\langle f_u, v_u \rangle, & \overline{M} &= -\langle g, v_u \rangle, & \overline{N} &= -\langle g, v_v \rangle.
\end{align*}
\end{equation}

We remark that $\overline{E}\overline{G} - \overline{F}^2 > 0$ on $U$. Relation between geometric invariants stated above and the functions in (2.3) are known ([9, 36]).

2.2.2. Geometric invariants at singular points of the second kind. We next consider geometric invariants at a singular point of the admissible second kind of a front. Let $p$ be a singular point of the second kind of a front $f$ in $\mathbb{R}^3$. Then one can take a local coordinate system $(U; u, v)$ centered at $p$ satisfying
1. $f_u(p) = 0$,
2. the $u$-axis is the singular curve on $U$, and
3. $|f_v(p)| = 1$.

We also call this local coordinate system adapted. Further, if an adapted coordinate system $(U; u, v)$ around a singular point of the second kind $p$ satisfies $\langle f_{uv}, f_v \rangle(p) = 0$, then $(U; u, v)$ is called a strongly adapted coordinate system (cf. [21, Definitions 4.1 and 4.6]).

Using an adapted coordinate system $(U; u, v)$ around a singular point of the second kind $p$, we define geometric invariants at $p$ as follows:
\begin{equation}
\kappa_v(p) = \lim_{u \to 0} \frac{\langle f_{uv}, v \rangle}{|f_u|^2}(u, 0), & \mu_c = \frac{-\langle f_{uv}, v_u \rangle}{|f_{uv} \times f_v|^2}(p).
\end{equation}

$\kappa_v(p)$ is the limiting normal curvature and $\mu_c$ is the normalized cuspidal curvature at $p$ (see [21, Proposition 2.9 and (4.7)]). These invariants are related to the boundedness of the Gaussian and the mean curvature near singular points of the second kind (see [21, Propositions 4.2 and 4.3, Theorem 4.4]).
Let us take an adapted coordinate system \((U; u, v)\) around a singular point of the (admissible) second kind \(p\). Then one can take a null vector field \(\eta\) as
\[
\eta = \partial_u + e(u)\partial_v,
\]
where \(e(u)\) is a \(C^\infty\) function with \(e(0) = 0\) ([21]). We note that there exists a positive integer \(l\) such that \(e(0) = e'(0) = \cdots = e^{(l-1)}(0) = 0\) and \(e^{(l)}(0) \neq 0\) if \(p\) is of the admissible (cf. [31, Lemma 2.2]).

Since \(df(\eta) = \eta f = f_u + e(u)f_v = 0\) along the \(u\)-axis, there exists a \(C^\infty\) map \(h: U \rightarrow \mathbb{R}^3\) such that \(\eta f = vh\), and hence \(f_u = vh - e(u)f_v\). Using this map \(h\), we have \(\lambda = \det(f_u, f_v, v) = v\det(h, f_u, v)\). By the non-degeneracy, \(\lambda_v(p) \neq 0\) holds. Thus we have \(\det(h, f_v, v)(p) \neq 0\). This implies that \(\{h, f_v, v\}\) gives a frame along \(f\) near \(p\). We take \(v\) satisfying \(\lambda_v(p) > 0\), so one may take \(v\) as
\[
v = \frac{h \times f_v}{|h \times f_v|}
\]
in what follows.

We define the following functions:
\[
\tilde{E} = \langle h, h \rangle, \quad \tilde{F} = \langle h, f_v \rangle, \quad \tilde{G} = \langle f_v, f_v \rangle, \\
\tilde{L} = -\langle h, v_u \rangle, \quad \tilde{M} = -\langle h, v_v \rangle, \quad \tilde{N} = -\langle f_v, v_v \rangle.
\]
We note that \(\tilde{E}\tilde{G} - \tilde{F}^2 > 0\) on \(U\). Using functions as in (2.5), differentials \(v_u\) and \(v_v\) of \(v\) can be written as follows.

**Lemma 2.4** ([36, Lemma 2.8]). On an adapted coordinate system \((U; u, v)\), \(v_u\) and \(v_v\) can be written as
\[
v_u = \frac{\tilde{F}(v\tilde{M} - e(u)\tilde{N}) - \tilde{G}\tilde{L}}{\tilde{E}\tilde{G} - \tilde{F}^2} h + \frac{\tilde{F}\tilde{L} - \tilde{E}(v\tilde{M} - e(u)\tilde{N})}{\tilde{E}\tilde{G} - \tilde{F}^2} f_v, \\
v_v = \frac{\tilde{F}\tilde{N} - \tilde{G}\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2} h + \frac{\tilde{F}\tilde{M} - \tilde{E}\tilde{N}}{\tilde{E}\tilde{G} - \tilde{F}^2} f_v.
\]

If \(f\) is a front at a singular point of the second kind \(p\), then \(\eta v \neq 0\) along the singular curve \(\gamma\). We rephrase this condition using functions as in (2.5). By
\[
\eta = \partial_u + e(u)\partial_v
\]
and Lemma 2.4, it follows that
\[
dv(\eta) = v_u + e(u)v_v = -\frac{\tilde{L} + e(u)\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2} (\tilde{G}h - \tilde{F}f_v)
\]
holds along the \(u\)-axis. Thus if \(f\) is a front around \(p\), then \(\tilde{L} + e(u)\tilde{M} \neq 0\) along the \(u\)-axis, in particular, \(\tilde{L}(p) \neq 0\). Moreover, we have the following.

**Lemma 2.5** ([21, 36]). Take a strongly adapted coordinate system \((U; u, v)\) around a singular point of the (admissible) second kind \(p\) of a front \(f\) in \(\mathbb{R}^3\). Then \(\kappa_v(p) = \tilde{N}(p)\) and \(\mu_v = \tilde{L}(p)/\tilde{E}(p)\) hold.

**Proof.** For \(\kappa_v\), it follows from [36, Lemma 2.9]. For \(\mu_v\), we have the expression by (2.5) and [21, (4.13)]. \(\square\)

2.2.3. **Rational boundedness of the Gaussian curvature.** Let \(f: \Sigma \rightarrow \mathbb{R}^3\) be a front and \(p \in \Sigma\) a non-degenerate singular point of \(f\). Then the Gaussian curvature \(K\) of \(f\) is unbounded near \(p\) in general. In [21], a notion of the rational boundedness for unbounded functions was introduced. (For precise definition and descriptions of the rational boundedness, see [21, Definition 3.4 and Page 260].) For the Gaussian curvature of a front, the following assertion holds.
Fact 2.6 ([21, Corollary C]). Let $f : \Sigma \to \mathbb{R}^3$ be a front and $p \in \Sigma$ a non-degenerate singular point of $f$, where $\Sigma$ is a domain in $\mathbb{R}^2$. Then the following statements are equivalent.

1. The Gaussian curvature of $f$ is rationally bounded at $p$.
2. The limiting normal curvature of $f$ vanishes at $p$.
3. The Gauss map of $f$ has a singularity at $p$.

We will characterize rational boundedness of the Gaussian curvature of the focal surface in terms of certain geometrical property of the initial front.

2.3. Second order derivatives of fronts. Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ its unit normal vector and $p$ a singular point of the second kind of $f$. Take an adapted coordinate system $(U; u, v)$ around $p$. We then prepare expressions of $h_u$, $h_v$ and $f_{uv}$ by the frame $\{h, f_u, \nu\}$ along $f$, where $h$ is a $C^\infty$ map defined on $U$ satisfying $df(\eta) = vh$. These shall play important role to analyze some geometric properties of focal surfaces of fronts.

Lemma 2.7. Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ its unit normal vector and $p \in \Sigma$ a singular point of the second kind. Take an adapted coordinate system $(U; u, v)$ centered at $p$. Then we have the following:

$$
\begin{align*}
    h_u &= \frac{\widehat{E}_u \widehat{G} - 2\widehat{F} A}{2(\widehat{E}\widehat{G} - \widehat{F}^2)} h + \frac{2\widehat{E} A - \widehat{E}_u \widehat{F}}{2(\widehat{E}\widehat{G} - \widehat{F}^2)} f_v + \widehat{L} v, \\
    h_v &= \frac{\widehat{E}_v \widehat{G} - 2\widehat{F} B}{2(\widehat{E}\widehat{G} - \widehat{F}^2)} h + \frac{2\widehat{E} B - \widehat{E}_v \widehat{F}}{2(\widehat{E}\widehat{G} - \widehat{F}^2)} f_v + \widehat{M} v, \\
    f_{uv} &= \frac{2\widehat{G} (\widehat{F}_u - B) - \widehat{F}_u \widehat{G}}{2(\widehat{E}\widehat{G} - \widehat{F}^2)} h + \frac{\widehat{E} \widehat{G}_v - 2\widehat{F}_v \widehat{F} + 2\widehat{F} B}{2(\widehat{E}\widehat{G} - \widehat{F}^2)} f_v + \widehat{N} v,
\end{align*}
$$

where $A = \langle h_u, f_v \rangle$ and $B = \langle h_v, f_v \rangle$.

Proof. Since $\{h, f_u, \nu\}$ is a moving frame along the front $f$, there exist $C^\infty$ functions $A_i, B_i, C_i : U \to \mathbb{R}$ ($i = 1, 2, 3$) such that

$$
    h_u = A_1 h + A_2 f_v + A_3 \nu, \quad h_v = B_1 h + B_2 f_v + B_3 \nu, \quad f_{uv} = C_1 h + C_2 f_v + C_3 \nu.
$$

We first consider for $h_u$. Taking inner products $\langle h_u, h \rangle (= \widehat{E}_u/2)$ and $\langle h_u, f_v \rangle (= A)$, we have

$$
\begin{pmatrix}
    \frac{\widehat{E}_u}{2} \\
    A
\end{pmatrix}
= \begin{pmatrix}
    \widehat{E} \\
    \widehat{F}
\end{pmatrix}
\begin{pmatrix}
    A_1 \\
    A_2
\end{pmatrix}
$$

Solving this equation for $A_1$ and $A_2$, we have

$$
A_1 = \frac{\widehat{E}_u \widehat{G} - 2\widehat{F} A}{2(\widehat{E}\widehat{G} - \widehat{F}^2)}, \quad A_2 = \frac{2\widehat{E} A - \widehat{E}_u \widehat{F}}{2(\widehat{E}\widehat{G} - \widehat{F}^2)}.
$$

Moreover, since $\langle h, \nu \rangle = 0$, we have $\langle h_u, \nu \rangle + \langle h, \nu_v \rangle = \langle h_u, \nu \rangle - \widehat{L} = 0$, and hence $A_3 = \langle h_u, \nu \rangle = \widehat{L}$. Thus we have the assertion for $h_u$. For $h_v$ and $f_{uv}$, we can show similarly by using $\widehat{F}_v = \langle h, f_v \rangle \nu = \langle h_v, f_v \rangle + \langle h, f_{uv} \rangle = B + \langle h, f_{uv} \rangle$. □

On an adapted coordinate system $(U; u, v)$, since $f_u = vh - e(u) f_v$, $f_{uu}$ and $f_{uv}$ can be written as $f_{uu} = vh_u - e'(u) f_v - e(u) f_{uv}$ and $f_{uv} = h + vh_v - e(u) f_{uv}$, respectively. Thus we can write $f_{uu}$ and $f_{uv}$ as

$$
\begin{align*}
    f_{uu} &= (v A_1 - e(u)(1 + v B_1 - e(u) C_1)) h \\
         &+ (v A_2 - e'(u) - e(u)(v B_2 - e(u) C_2)) f_v + (v \widehat{L} - e(u) \widehat{M} + e(u)^2 \widehat{N}) v, \\
    f_{uv} &= (1 + v B_1 - e(u) C_1) h + (v B_2 - e(u) C_2) f_v + (v \widehat{M} - e(u) \widehat{N}) v,
\end{align*}
$$

as in

$$
\begin{align*}
    f_{uu} &= (v A_1 - e(u)(1 + v B_1 - e(u) C_1)) h \\
         &+ (v A_2 - e'(u) - e(u)(v B_2 - e(u) C_2)) f_v + (v \widehat{L} - e(u) \widehat{M} + e(u)^2 \widehat{N}) v, \\
    f_{uv} &= (1 + v B_1 - e(u) C_1) h + (v B_2 - e(u) C_2) f_v + (v \widehat{M} - e(u) \widehat{N}) v,
\end{align*}
$$

as in

$$
\begin{align*}
    f_{uu} &= (v A_1 - e(u)(1 + v B_1 - e(u) C_1)) h \\
         &+ (v A_2 - e'(u) - e(u)(v B_2 - e(u) C_2)) f_v + (v \widehat{L} - e(u) \widehat{M} + e(u)^2 \widehat{N}) v, \\
    f_{uv} &= (1 + v B_1 - e(u) C_1) h + (v B_2 - e(u) C_2) f_v + (v \widehat{M} - e(u) \widehat{N}) v,
\end{align*}
$$

as in
where $A_i, B_i, C_i$ ($i = 1, 2$) correspond to the coefficient functions in (2.8). By (2.9), it follows that $f_{uu}(p) = 0$ and $f_{uv}(p) = h(p)$ hold.

For functions $A$ and $B$, the following relation holds.

**Lemma 2.8.** Under the above setting, we have

$$A + e(u)B = -\widehat{E} + \widehat{F}_u + e(u)\widehat{F}_v - \frac{v\widehat{E}_v}{2},$$

where $A = \langle h_u, f_v \rangle$ and $B = \langle h_v, f_v \rangle$.

**Proof.** Since $\widehat{F} = \langle h, f_v \rangle$, we have $\widehat{F}_u = \langle h_u, f_v \rangle + \langle h, f_{uv} \rangle = A + \langle h, f_{uv} \rangle$. By (2.8) and (2.9), it holds that

$$\langle h, f_{uv} \rangle = \widehat{E} + e(u)(B - \widehat{F}_v) + \frac{v\widehat{E}_v}{2}.$$ 

Thus we have the assertion. $\square$

3. **Principal curvatures and principal vectors**

We recall principal curvatures and principal vectors. We focus on a singular point of the second kind in this section. For the case of cuspidal edge, we can consider similar properties (see [35, 36]).

Let $f : \Sigma \to \mathbb{R}^3$ be a front and $p \in \Sigma$ a singular point of the second kind. Take a strongly adapted coordinate system $(U; u, v)$ around $p$. Then we set functions $\kappa_1$ and $\kappa_2$ on the set of regular points of $f$ by

$$\kappa_1 = \frac{\hat{k}_1 + \hat{k}_2}{2v(\widehat{E}\widehat{G} - \widehat{F}^2)}, \quad \kappa_2 = \frac{\hat{k}_1 - \hat{k}_2}{2v(\widehat{E}\widehat{G} - \widehat{F}^2)},$$

where

$$\hat{k}_1 = \widehat{G}(\widehat{L} + e(u)\widehat{M}) - 2v\widehat{F}\widehat{M} + v\widehat{E}\widehat{N},$$

$$\hat{k}_2 = \sqrt{\hat{k}_1^2 - 4v(\widehat{E}\widehat{G} - \widehat{F}^2)(\widehat{N}(\widehat{L} + e(u)\widehat{M}) - v\widehat{M}\widehat{N})}.$$ 

Since $\widehat{L} + e(u)\widehat{M} \neq 0$ on the $u$-axis, $\hat{k}_1$ and $\hat{k}_2$ are $C^\infty$ functions near $p$. Moreover, by direct calculations, we have $\kappa_1 \kappa_2 = K$ and $\kappa_1 + \kappa_2 = 2H$ on the set of regular points of $f$, where $K$ is the Gaussian curvature and $H$ is the mean curvature of $f$ written as

$$(3.2) \quad K = \frac{\widehat{N}(\widehat{L} + e(u)\widehat{M}) - v\widehat{M}^2}{v(\widehat{E}\widehat{G} - \widehat{F}^2)}, \quad H = \frac{\widehat{G}(\widehat{L} + e(u)\widehat{M}) - 2v\widehat{F}\widehat{M} + v\widehat{E}\widehat{N}}{2v(\widehat{E}\widehat{G} - \widehat{F}^2)}$$

(see [21, Pages 268 and 270]). Thus one can consider $\kappa_1$ and $\kappa_2$ as principal curvatures of $f$. We note that $H$ is unbounded near $p$ ([34, Corollary 3.5]), and hence at least one principal curvature may diverge near $p$. Indeed, the following holds.

**Proposition 3.1** (cf. [24, 36]). Let $f$ be a front in $\mathbb{R}^3$ and $p$ a non-degenerate singular point of $f$. Then one principal curvature can be extended as a bounded $C^\infty$ function at $p$, and another is unbounded near $p$.

We note that Medina-Tejeda [22] shows boundedness of principal curvatures of fronts with other singularities.

Let us denote by $\kappa$ and $\tilde{\kappa}$ the bounded principal curvature and the unbounded principal curvature, respectively. Then $\kappa(p) = \kappa_\nu(p)$ holds at an admissible singular point $p$ ([36, Theorem 3.1]). Moreover, setting $\tilde{\kappa} = \lambda \kappa$, where $\lambda$ is the signed area density function of $f$ in (2.1), $\kappa$ is bounded $C^\infty$ function on $U$, in particular, $\kappa(p) \neq 0$ ([36, Remark 3.2]). When $p$ is of the second kind, the relation $\tilde{\kappa}(p) = \lambda_\nu(p)\mu_\nu$ holds. Furthermore, we note
that the curvature radius function relative to \( \kappa \) can be extended as a \( C^\infty \) function on \( U \) because it can be written as \( 1/\kappa = \lambda/\hat{\kappa} \). We denote by \( \hat{\rho} \) the curvature radius function \( \lambda/\hat{\kappa} \) of the unbounded principal curvature.

We next consider principal vectors \( \mathbf{V} \) and \( \tilde{\mathbf{V}} \) associated to \( \kappa \) and \( \hat{\kappa} \), respectively. Let \( \tilde{\mathbf{I}} \) and \( \hat{\mathbf{I}} \) be 2 \( \times \) 2 matrices given by

\[
\tilde{\mathbf{I}} = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_v, f_u \rangle & \langle f_v, f_v \rangle \end{pmatrix} = \begin{pmatrix} v^2 \hat{\mathbf{F}} - 2\nu e(u) \hat{\mathbf{F}} + e(u)^2 \hat{\mathbf{G}} & v \hat{\mathbf{F}} - e(u) \hat{\mathbf{G}} \\ v \hat{\mathbf{F}} - e(u) \hat{\mathbf{G}} & \hat{\mathbf{G}} \end{pmatrix},
\]

\[
\hat{\mathbf{I}} = \begin{pmatrix} -\langle f_u, v_u \rangle - \langle f_u, v_v \rangle \\ -\langle f_v, v_u \rangle - \langle f_v, v_v \rangle \end{pmatrix} = \begin{pmatrix} v \hat{\mathbf{L}} - e(u)(v \hat{\mathbf{M}} - e(u) \hat{\mathbf{N}}) & v \hat{\mathbf{M}} - e(u) \hat{\mathbf{N}} \\ v \hat{\mathbf{M}} - e(u) \hat{\mathbf{N}} & \hat{\mathbf{N}} \end{pmatrix}.
\]

When \( \mathbf{V} \) and \( \tilde{\mathbf{V}} \) are principal vectors relative to \( \kappa \) and \( \hat{\kappa} \) on \( U \setminus \{v = 0\} \), they satisfy \((\hat{\mathbf{I}} - \kappa \tilde{\mathbf{I}})\mathbf{V} = 0 \) and \((\tilde{\mathbf{I}} - \hat{\kappa} \hat{\mathbf{I}})\tilde{\mathbf{V}} = 0 \), respectively. Solving the equation \((\tilde{\mathbf{I}} - \hat{\kappa} \hat{\mathbf{I}})\tilde{\mathbf{V}} = 0 \), we have

\[
(3.3) \quad \mathbf{V} = (-\hat{\mathbf{M}} + \kappa \hat{\mathbf{F}}, \hat{\mathbf{L}} - \kappa (v \hat{\mathbf{E}} - e(u) \hat{\mathbf{F}})).
\]

Since \( \kappa \) is bounded on \( U \) and \( \tilde{\mathbf{L}}(p) \neq 0 \), \( \mathbf{V} \) can be defined on \( U \) ([36, (3.2)]).

On the other hand, we consider the equation \((\tilde{\mathbf{I}} - \kappa \tilde{\mathbf{I}})\tilde{\mathbf{V}} = 0 \). This equation can be modified as

\[
(3.4) \quad \begin{pmatrix} \lambda \hat{\mathbf{L}} - \hat{\kappa}(v \hat{\mathbf{E}} - e(u) \hat{\mathbf{F}}) \\ v(\lambda \hat{\mathbf{M}} - \hat{\kappa} \hat{\mathbf{F}}) - e(u)(\lambda \hat{\mathbf{N}} - \hat{\kappa} \hat{\mathbf{G}}) \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{V}}_1 \\ \hat{\mathbf{V}}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

by multiplying the equation by \( \lambda \) and factoring out \( v \) from the first row. We note that \( \hat{\kappa} \) does not vanish at \( p \), and hence we may take \( \tilde{\mathbf{V}} \) on \( U \) as

\[
(3.5) \quad \tilde{\mathbf{V}} = (\lambda \hat{\mathbf{N}} - \hat{\kappa} \hat{\mathbf{G}}, -\nu(\lambda \hat{\mathbf{M}} - \hat{\kappa} \hat{\mathbf{F}}) + e(u)(\lambda \hat{\mathbf{N}} - \hat{\kappa} \hat{\mathbf{G}})).
\]

Remark that \( \mathbf{V} \) and \( \tilde{\mathbf{V}} \) as in (3.3) and (3.5) are linearly independent on \( U \). Indeed, when we identify \( \tilde{\mathbf{V}} = (\tilde{\mathbf{V}}_1, \tilde{\mathbf{V}}_2) = (\hat{\mathbf{V}}_1, -\nu \hat{\mathbf{V}}_2 + e(u) \hat{\mathbf{V}}_1) \) with \( \mathbf{V} = \hat{\mathbf{V}}_1 \partial_u + (-\nu \hat{\mathbf{V}}_2 + e(u) \hat{\mathbf{V}}_1) \partial_v \), where we set \( \hat{\mathbf{V}}_1 = \lambda \hat{\mathbf{N}} - \hat{\kappa} \hat{\mathbf{G}} \) and \( \hat{\mathbf{V}}_2 = \lambda \hat{\mathbf{M}} - \hat{\kappa} \hat{\mathbf{F}} \), it holds that

\[
(3.6) \quad \tilde{\mathbf{V}} = \hat{\mathbf{V}}_1 \eta - \nu \hat{\mathbf{V}}_2 \partial_v.
\]

Since \( \mathbf{V} \) is not parallel to \( \eta \) along the \( u \)-axis, we have that \( \mathbf{V} \) and \( \tilde{\mathbf{V}} \) are linearly independent. Moreover, by \( d\hat{f}(\eta)(\nu \hat{\mathbf{F}}) = \partial_h \), we have \( d\hat{f}(\tilde{\mathbf{V}}) = \nu\{v(\lambda \hat{\mathbf{N}} - \hat{\kappa} \hat{\mathbf{G}})h - (\lambda \hat{\mathbf{M}} - \hat{\kappa} \hat{\mathbf{F}})f_v\} \).

We now define \( C^\infty \) maps \( x, y : U \rightarrow \mathbb{R}^3 \) by

\[
(3.7) \quad x = -\nu(\lambda \hat{\mathbf{M}} - \hat{\kappa} \hat{\mathbf{F}})h + (\hat{\mathbf{L}} + e(u) \hat{\mathbf{M}}) - \nu \hat{\mathbf{F}})f_v, \quad y = (\lambda \hat{\mathbf{N}} - \hat{\kappa} \hat{\mathbf{G}})h - (\lambda \hat{\mathbf{M}} - \hat{\kappa} \hat{\mathbf{F}})f_v.
\]

We note that \( x \) and \( y \) satisfy \( x = df(\tilde{\mathbf{V}}) \) and \( dy(\tilde{\mathbf{V}}) = \nu y \), respectively.

**Lemma 3.2.** Under the above setting, \( x \) and \( y \) given by (3.7) are perpendicular to each other.

**Proof.** By a straightforward calculation with relations \( 2\lambda H = \lambda \kappa + \hat{\kappa} \) and \( \lambda K = \kappa \hat{\kappa} \), where \( K \) and \( H \) are the Gaussian and the mean curvature, we have \( \langle x, y \rangle = 0 \). \( \square \)

By this lemma, \( \{x, y, \nu\} \) gives an orthogonal frame along \( f \). Moreover, when we set \( e_1 = x/|x| \) and \( e_2 = y/|y| \), then \( \{e_1, e_2, \nu\} \) is an orthonormal frame along \( f \). Particularly, the existence of curvature line coordinate systems for fronts around non-degenerate singular points is known [24, Lemma 1.3]. We note that a different approach to study singular surfaces using moving frame is known ([8]).
Remark 3.3. If \( p \) is a cuspidal edge of a front \( f \), then principal vectors are given by
\[
V = (\tilde{N} - \nu k \tilde{G}, -\tilde{M} + k \tilde{E}), \quad \tilde{V} = (v(\lambda \tilde{M} - k \tilde{F}), -\lambda \tilde{L} + k \tilde{E})
\]
by taking a special adapted coordinate system \((U; u, v)\) and using functions as in (2.3) ([35, 36]), where \( \kappa \) is the bounded principal curvature and \( \tilde{k} = \lambda \kappa \). In this case, we see that
\[
df(V) = (\tilde{N} - \nu k \tilde{G}) f_u + v(-\tilde{M} + k \tilde{E}) g, \quad df(\tilde{V}) = v\left((\lambda \tilde{M} - k \tilde{F}) f_u + (-\lambda \tilde{L} + k \tilde{E}) g\right).
\]

Then setting \( x, y : U \to \mathbb{R}^3 \setminus \{0\} \) by \( x = df(V) \) and \( y = (\lambda \tilde{M} - k \tilde{F}) f_u + (-\lambda \tilde{L} + k \tilde{E}) g \), one can see \( df(\tilde{V}) = vy \) and it holds that \( \langle x, y \rangle = 0 \) by a similar calculation as the proof of Lemma 3.2. Thus we can take an orthonormal frame \( \{e_1, e_2, v\} \) along \( f \) by setting \( e_1 = x / |x| \) and \( e_2 = y / |y| \).

Remark 3.4. The pair of principal vectors \( V \) and \( \tilde{V} \) is a curvature line frame generator of a front (cf. [32]). Moreover, the pair \( \{e_1, e_2\} \) is a curvature line frame corresponding to \( \{V, \tilde{V}\} \).

For the unit normal vector \( v \) of a front \( f \), the following properties hold.

**Lemma 3.5.** Let \( f : \Sigma \to \mathbb{R}^3 \) be a front, \( v \) its unit normal vector and \( p \) a singular point of the second kind. Let \( \kappa \) and \( \tilde{k} \) be the bounded and unbounded principal curvature on a neighborhood \( U \) of \( p \), respectively. Let \( V \) and \( \tilde{V} \) the corresponding principal vectors. Then we have
\[
dv(V) = -\kappa df(V), \quad df(\tilde{V}) = -\tilde{\rho} dv(\tilde{V}),
\]
where \( \tilde{\rho} = \lambda / \tilde{k} \).

**Proof.** Let us take an adapted coordinate system \((U; u, v)\) centered at \( p \). We first show the relation between \( df(V) \) and \( dv(V) \). Denote by \( V = V_1 \partial_u + V_2 \partial_v \) the principal vector relative to the bounded principal curvature \( \kappa \) (see (3.3)). Then \( dv(V) \) can be written as
\[
dv(V) = V_1 v_u + V_2 v_v = (V_1 X_1 + V_2 Y_1) h + (V_1 X_2 + V_2 Y_2) f_v
\]
by Lemma 2.4, where \( X_i, Y_i (i = 1, 2) \) are functions in (2.6). By direct computations, we have \( V_1 X_1 + V_2 Y_1 = \nu k (\tilde{M} - \kappa \tilde{F}) \) and \( V_1 X_2 + V_2 Y_2 = -\kappa ((\tilde{L} + e(u) \tilde{M}) - \nu k \tilde{E}) \). Thus it holds that
\[
dv(V) = \kappa \left(v(\tilde{M} - \kappa \tilde{F}) - ((\tilde{L} + e(u) \tilde{M}) - \nu k \tilde{E}) f_u\right) = -\kappa df(V) (= -\kappa x)
\]
by (3.7).

Next we consider the case of \( df(\tilde{V}) \). By (3.5), we have
\[
dv(\tilde{V}) = (\lambda \tilde{N} - \tilde{k} \tilde{G})(v_u + e(\nu) v_v) - v(\lambda \tilde{M} - \kappa \tilde{F}) v_v.
\]
By (2.6), it follows that
\[
v_u + e(\nu) v_v = \frac{v \tilde{M} - \tilde{G}(\tilde{L} + e(\nu) \tilde{M})}{E \tilde{G} - E \tilde{F}^2} h - \frac{v \tilde{E} \tilde{M} - \tilde{F}(\tilde{L} + e(\nu) \tilde{M})}{E \tilde{G} - E \tilde{F}^2} f_v
\]
holds. Thus on the set of regular points \( U \setminus \{v = 0\} \), we have
\[
dv(\tilde{V}) = v \tilde{k} \left(-((\tilde{N} - \tilde{k} \tilde{G}) h + (\tilde{M} - \kappa \tilde{F}) f_u)\right).
\]
By multiplying \( \lambda \), we obtain
\[
\lambda dv(\tilde{V}) = v \tilde{k} \left(-((\tilde{N} - \tilde{k} \tilde{G}) h + (\tilde{M} - \kappa \tilde{F}) f_u\right) = -v \tilde{k} y = -\tilde{\rho} dv(\tilde{V}).
\]
Since \( \tilde{k} \neq 0 \) on \( U \), we have
\[
df(\tilde{V}) = -\tilde{\rho} dv(\tilde{V})
\]
on \( U \). \( \square \)
This lemma seems to correspond to the Rodrigues’ Theorem (cf. [29, Theorem 10.2]). We note that if \( f \) is a front at a singular point of the second kind \( p \), \( d\nu(\vec{V}) = \vec{V}_1 d\nu(\eta) \neq 0 \) holds at \( p \).

**Remark 3.6.** If we take an adapted coordinate system \((U; u, v)\) around a singular point of the second kind \( p \) of a front \( f \), then the signed area density \( \lambda \) of \( f \) satisfies \( \lambda(u, 0) = 0 \). Thus there exists a \( C^\infty \) function \( \hat{\lambda} \) on \( U \) such that \( \lambda = v \hat{\lambda} \). By the non-degeneracy, \( \lambda_v(p) \neq 0 \), and hence \( \hat{\lambda}(p) \neq 0 \). Thus we have

\[
d\nu(\vec{V}) = -\hat{k} y
\]
on \( U \). Moreover, it holds that

\[
d\nu(\bar{V}) = -\kappa x = -\kappa|x|e_1, \quad d\nu(\vec{V}) = -\hat{k} y = -\hat{k}|y|e_2,
\]

where \( e_1 = x/|x| \) and \( e_2 = y/|y| \).

4. **SUB-PARABOLIC POINTS**

For a regular surface, one can define a ridge point and a sub-parabolic point of the surface by using principal curvatures and corresponding principal vectors. They are related to singularities and geometrical properties of focal surfaces (cf. [5, 23, 28]). In this section, we investigate a sub-parabolic point of a front. First, we give definitions of a ridge point and a sub-parabolic point for fronts using the principal curvatures and corresponding principal vectors.

**Definition 4.1.** Let \( \bar{V} \) (resp. \( \bar{V} \)) be a principal vector with respect to the unbounded principal curvature \( \bar{k} \) (resp. the bounded principal curvature \( \kappa \)) of a front \( f \) with a non-degenerate singular point \( p \). Then \( p \) is a sub-parabolic point (resp. a ridge point) of \( f \) with respect to \( \kappa \) if the directional derivative \( \bar{V}_k \) (resp. \( \bar{V}_\kappa \)) of \( \kappa \) in the direction \( \bar{V} \) (resp. \( \bar{V} \)) vanishes at \( p \).

By the property of the principal vector relative to the unbounded principal curvature, the following characterization for a sub-parabolic point holds.

**Proposition 4.2.** Let \( f : \Sigma \rightarrow \mathbb{R}^3 \) be a front and \( p \in \Sigma \) a non-degenerate singular point. Then \( p \) is also a sub-parabolic point of \( f \) if and only if \( \eta \kappa = 0 \) at \( p \), where \( \eta \kappa \) means the directional derivative of the bounded principal curvature \( \kappa \) of \( f \) in the direction a null vector field \( \eta \).

**Proof.** Let us take an adapted coordinate system \((U; u, v)\) around \( p \). Then the principal vector \( \bar{V} \) associated to the unbounded principal curvature is parallel to \( \eta \) along the \( u \)-axis by (3.6) and Remark 3.3. Thus we have the assertion by the definition of a sub-parabolic point.

For the case of a cuspidal edge \( p \) of a front \( f \), the condition for \( p \) being a sub-parabolic point is characterized by the geometric invariants (see [35, Proposition 2.8]). On the other hand, for the case of a singular point of the second kind, we have the following characterization relating the behavior of the Gaussian curvature of a front.

**Proposition 4.3.** Let \( f : \Sigma \rightarrow \mathbb{R}^3 \) be a front and \( p \in \Sigma \) a singular point of the second kind. If the Gaussian curvature \( K \) of \( f \) is rationally continuous, then \( p \) is a sub-parabolic point.

For a notion of the rational continuity, see [21, Definition 3.4].
Proof. Let us take an adapted coordinate system $(U; u, v)$ centered at $p$. Then we define a co-vector $\omega_\nu(p)$ by $\omega_\nu(p) = \kappa_\nu'(p)du \in T_\nu^* \Sigma$, where $\kappa_\nu'(p) = d\kappa_\nu(u)/du|_{u=0}$. We note that $\kappa_\nu'(p)$ depends on the parameter $u$ of the singular curve, but $\omega_\nu$ does not depend on $u$ (see [21, Page 270]). Since $\kappa_\nu(p) = \kappa_\nu'(p)$ holds, the co-vector $\omega_\nu(p)$ may be written as $\omega_\nu(p) = \kappa_\nu(p)du$. Moreover, by Proposition 4.2 and $\eta = \partial_\nu$ at $p$, $\omega_\nu(p) = 0$ if and only if $p$ is a sub-parabolic point of $f$. On the other hand, it is known that the Gaussian curvature $K$ of $f$ is rationally continuous at $p$ if and only if $\kappa_\nu(p) = \omega_\nu(p) = 0$ ([21, Theorem 4.4]). Thus we have the conclusion. □

We next consider the case that the Gaussian curvature $K$ of a front $f$ is bounded near a non-degenerate singular point $p$. In such a case, we obtain the following.

**Proposition 4.4.** Let $f: \Sigma \to \mathbb{R}^3$ be a front and $p \in \Sigma$ a non-degenerate singular point. Suppose that the Gaussian curvature $K$ of $f$ is bounded on a sufficiently small neighborhood $U$ of $p$.

1. When $p$ is a cuspidal edge, $p$ is a sub-parabolic point of $f$ if and only if $K(p) = 0$.
2. When $p$ is of the second kind, then $p$ is a sub-parabolic point of $f$.

Proof. First we show the case of a cuspidal edge. In this case, it is known that $p$ is a sub-parabolic point of $f$ if and only if $4\kappa_1^2 + \kappa_2\kappa_3 = 0$ holds at $p$ ([35, Proposition 2.8]). On the other hand, the Gaussian curvature satisfies $4K = -(4\kappa_1^2 + \kappa_2\kappa_3)$ at $p$ by Fact 2.3. This shows the first assertion.

We next consider the second assertion. Let us take an adapted coordinate system $(u, v)$ on $U$. Since $K$ is bounded on $U$, the bounded principal curvature $\kappa$ satisfies $\kappa(u, 0) = \kappa_\nu(0) = 0$. Thus there exists a $C^\infty$ function $k: U \to \mathbb{R}$ such that $\kappa = vk$. Since the principal vector $\hat{V}$ relative to the unbounded principal curvature can be written as $\hat{V} = \hat{V}_1\eta - v\hat{V}_2\partial_v$ (see (3.6)), it holds that $\hat{V}_1 = \hat{V}_1\eta(vk) - v\hat{V}_2(k + vk) = \hat{V}_1(\eta(vk) + e(u)(k + vk)) - v\hat{V}_2(k + vk)$. Therefore we have $\hat{V}_1 = 0$ at $p$ since $e(0) = 0$. Thus we get the conclusion. □

For the case of a front with a cuspidal edge $p$, it is known that the Gaussian curvature of the focal surface associated to the unbounded principal curvature vanishes at $p$ if and only if $p$ is a sub-parabolic point ([35, Corollary 3.9]). For the case of a singular point of the second kind, we will show relation between behavior of the Gaussian curvature of the focal surface associated to the unbounded principal curvature and a sub-parabolic point of the initial front in the next section.

### 5. Focal surfaces of fronts

We consider singularities and geometric properties of focal surfaces of fronts associated to unbounded principal curvatures. In particular, we focus on the case that the initial front has a singular points of the second kind.

Let $f: \Sigma \to \mathbb{R}^3$ be a front, $v$ its Gauss map and $p \in \Sigma$ a non-degenerate singular point of $f$. Take an adapted coordinate system $(U; u, v)$ centered at $p$. Then we consider a map $\mathcal{F}: U \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\mathcal{F}(u, v, w) = f(u, v) + wv(u, v).$$

The map $\mathcal{F}$ is called a normal congruence of $f$ (cf. [15]). We consider the singular set and the singular value set of $\mathcal{F}$. By direct calculations using the Weingarten formula, we have

$$\det(\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_w) = (1 - wk)(\lambda - k),$$

Thus the set of singular points $S(\mathcal{F})$ of $\mathcal{F}$ is the union of $S_1(\mathcal{F}) = \{(u, v, w) \mid 1 - \kappa(u, v)w = 0\}$ and $S_2(\mathcal{F}) = \{(u, v, w) \mid \lambda(u, v) - \kappa(u, v)w = 0\}$. Hence the image $\mathcal{F}(S(\mathcal{F}))$ of $S(\mathcal{F})$
Proof. Let us take a strongly adapted coordinate system $\mathcal{F}$ is
\[
\mathcal{F}(S(\mathcal{F})) = \{ f(u,v) + \rho(u,v)\nu(u,v) \mid (u,v) \in U, \ w = \rho(u,v) \} \\
\cup \{ f(u,v) + \hat{\rho}(u,v)\nu(u,v) \mid (u,v) \in U, \ w = \hat{\rho}(u,v) \},
\]
where $\rho = 1/\kappa$ and $\hat{\rho} = \lambda/\hat{\kappa}$. We define $C^\infty$ maps $C, \tilde{C}: U \to \mathbb{R}^3$ by
\[
(5.1) \quad C = f + \rho \nu, \quad \tilde{C} = f + \hat{\rho} \nu.
\]
Then one can notice immediately that the singular value set of $\mathcal{F}$ coincides with the union of the images of $C$ and $\tilde{C}$: $\mathcal{F}(S(\mathcal{F})) = \text{Im} C \cup \text{Im} \tilde{C}$. We call $C$ and $\tilde{C}$ focal surfaces (or caustics) of $f$. In particular, we call $\tilde{C}$ the focal surface associated to the unbounded principal curvature of $f$. We remark that $C$ cannot be defined near a singular point when the Gaussian curvature of an original front is bounded near the singular point. However, we can define and consider $\tilde{C}$ even if the Gaussian curvature is bounded. We note that singularities of $C$ for a front with non-degenerate singular points are studied in [35]. Thus we focus on $\tilde{C}$ of a front $f$ in the following of this paper.

5.1. Singularities of $\tilde{C}$. We first investigate singularities of $\tilde{C}$. When a front $f$ has a cuspidal edge $p$, then $\tilde{C}$ is regular at $p$ ([35, Proposition 3.7]). Thus we treat the case that a front $f$ has a singular point of the second kind $p$ in the following of this subsection.

Lemma 5.1. The map $y$ as in (3.7) is a normal vector to the focal surface $\tilde{C}$.

Proof. Let us take a strongly adapted coordinate system $(U; u, v)$ around $p$. By direct calculations, we have
\[
\tilde{C}_u = f_u + \hat{\rho} \nu_u + \hat{\rho}_u \nu, \quad \tilde{C}_v = f_v + \hat{\rho} \nu_v + \hat{\rho}_v \nu.
\]
On the other hand, we get
\[
\langle y, f_u \rangle = (\lambda \tilde{N} - \hat{\kappa} \tilde{G})(v \tilde{E} - e(u) \tilde{F}) - (\lambda \tilde{M} - \hat{\kappa} \tilde{F})(v \tilde{F} - e(u) \tilde{G}), \\
\langle y, f_v \rangle = (\lambda \tilde{N} - \hat{\kappa} \tilde{G}) \tilde{F} - (\lambda \tilde{M} - \hat{\kappa} \tilde{F}) \tilde{G}, \\
\langle y, \nu_u \rangle = -(\lambda \tilde{N} - \hat{\kappa} \tilde{G}) \hat{L} + (\lambda \tilde{M} - \hat{\kappa} \tilde{F})(v \tilde{M} - e(u) \tilde{N}), \\
\langle y, \nu_v \rangle = -(\lambda \tilde{N} - \hat{\kappa} \tilde{G}) \hat{M} + (\lambda \tilde{M} - \hat{\kappa} \tilde{F}) \hat{N}.
\]
Therefore by $\hat{\rho} \hat{\kappa} = \lambda$ and $\langle y, \nu \rangle = 0$, it follows that
\[
\langle \tilde{C}_u, y \rangle = -v(\tilde{E} \tilde{G} - \hat{L}^2)(\hat{\kappa} - 2\lambda H + \lambda \hat{\rho} K) = -v(\tilde{E} \tilde{G} - \hat{L}^2)(\hat{\kappa} - (\lambda \kappa + \hat{\kappa}) + \lambda \kappa) = 0, \\
\langle \tilde{C}_v, y \rangle = \lambda(\tilde{N} \tilde{F} - \hat{M} \tilde{G}) + \hat{\rho}(\tilde{M} \tilde{G} - \tilde{N} \tilde{F}) = 0
\]
hold. Thus we have the assertion. \hfill $\Box$

Since $y$ does not vanish near $p$, $e_2 = y/|y|$ is a unit normal vector field of $\tilde{C}$ by Lemma 5.1. Thus $\tilde{C}$ is a frontal near $p$. We remark that for the case of cuspidal edge, we have similar properties using the map $y$ defined in Remark 3.3.

Lemma 5.2. Let $f$ be a front in $\mathbb{R}^3$ and $p$ a singular point of the second kind of $f$. Then the set of singular points $S(\tilde{C})$ of $\tilde{C}$ in (5.1) coincides with the zero set of $\tilde{V} \tilde{\rho}$. In particular, $p$ is also a singular point of $\tilde{C}$.

Proof. Let us take a strongly adapted coordinate system $(U; u, v)$ around $p$. We denote by $\lambda \tilde{C}$ the signed area density function of $\tilde{C}$ given by $\lambda \tilde{C} = \det(\tilde{C}_u, \tilde{C}_v, e_2) = \langle \tilde{C}_u \times \tilde{C}_v, e_2 \rangle$. 
We calculate it explicitly. Since \( \nu \) can be taken as \( \nu = (h \times f_v)/|h \times f_v| \) and \( e_2 \) is perpendicular to \( \nu \), we need

\[
\overline{C}_u \times \overline{C}_v \equiv \hat{\rho}_v f_u \times \nu + \hat{\rho}_\nu v_u \times \nu + \hat{\rho}_u v_n \times f_v + \hat{\rho}_\nu u \times \nu \mod \nu.
\]

Here for two \( C^\infty \) maps \( \alpha, \beta : U \to \mathbb{R}^3, \alpha \equiv \beta \mod \nu \) implies that there exists a \( C^\infty \) function \( \delta : U \to \mathbb{R} \) such that \( \alpha - \beta = \delta \nu \) holds. By the vector triple product and Lemma 2.4, we have

\[
f_u \times \nu = \frac{(v \hat{F} - e(u)\hat{G})h + (e(u)\hat{F} - v \hat{E})f_v}{|h \times f_v|}, \quad f_v \times \nu = \frac{\hat{G} h - \hat{F} f_v}{|h \times f_v|},
\]

\[
v_u \times \nu = \frac{(e(u)\hat{N} - v \hat{M})h + \hat{L} f_v}{|h \times f_v|}, \quad v_v \times \nu = \frac{-\hat{N} h + \hat{M} f_v}{|h \times f_v|}.
\]

Therefore, one can see that

\[
\overline{C}_u \times \overline{C}_v \equiv \frac{\hat{\rho}_u y}{\hat{k}|h \times f_v|} \left( (\lambda \hat{N} - \hat{k} \hat{G})h - (\lambda \hat{M} - \hat{k} \hat{F})f_v \right)
\]

\[
+ \frac{\hat{\rho}_v}{\hat{k}|h \times f_v|} \left( (-v(\lambda \hat{M} - \hat{k} \hat{F}) + e(u)(\lambda \hat{N} - \hat{k} \hat{G}))h
\]

\[
+ (\lambda \hat{L} - \hat{k}(v \hat{E} - e(u)\hat{F}))f_v \right) \mod \nu.
\]

On the other hand, for \( \overline{V} = (\overline{V}_1, \overline{V}_2) \) as in (3.5), we note that

\[
(\lambda \hat{L} - \hat{k}(v \hat{E} - e(u)\hat{F}))\overline{V}_1 + (\lambda \hat{M} - \hat{k} \hat{F})\overline{V}_2 = 0
\]

holds by (3.4). Since \( \overline{V}_1 \neq 0 \) near \( p \), we have

\[
(\lambda \hat{L} - \hat{k}(v \hat{E} - e(u)\hat{F})) = - (\lambda \hat{M} - \hat{k} \hat{F})\frac{\overline{V}_2}{\overline{V}_1}.
\]

Hence it follows that

\[
\overline{C}_u \times \overline{C}_v \equiv \frac{\hat{\rho}_u y}{\hat{k}|h \times f_v|} + \frac{\hat{\rho}_v}{\hat{k}|h \times f_v|} \left( \overline{V}_2 h - (\lambda \hat{M} - \hat{k} \hat{F})\frac{\overline{V}_2}{\overline{V}_1} f_v \right)
\]

\[
\equiv \frac{\hat{\rho}_u y}{\hat{k}|h \times f_v|} + \frac{\overline{V}_2 \hat{\rho}_v}{\overline{V}_1 \hat{k}|h \times f_v|} (\overline{V}_1 h - (\lambda \hat{M} - \hat{k} \hat{F})f_v)
\]

\[
\equiv \frac{(\overline{V} \hat{\rho}) y}{\overline{V}_1 \hat{k}|h \times f_v|} \mod \nu.
\]

Summing up, we obtain

\[
(5.2) \quad \lambda \overline{C} = \frac{(\overline{V} \hat{\rho}) y}{\overline{V}_1 \hat{k}|h \times f_v|}.
\]

Thus \( S(\overline{C}) = (\overline{V} \hat{\rho})^{-1}(0) \) holds. Further, since \( \hat{\rho}_u(p) = \overline{V}_2(p) = 0 \),

\[
\overline{V} \hat{\rho} = \overline{V}_1 \hat{\rho}_u + \overline{V}_2 \hat{\rho}_v = 0
\]

holds at \( p \). Hence \( p \) is also a singular point of \( \overline{C} \). \( \square \)

**Lemma 5.3.** Let \( p \) be a singular point of the second kind of a front \( f \) in \( \mathbb{R}^3 \). Then the focal surface \( \overline{C} \) satisfies \( \text{rank} \, d\overline{C}_p = 1 \). Moreover, \( \overline{V} \) can be taken as a null vector field \( \eta \overline{C} \) of \( \overline{C} \) near \( p \).
Proof. Take a strongly adapted coordinate system \((U; u, v)\) centered at \(p\). By a direct calculation, we see that
\[
\hat{C}_u = 0, \quad \hat{C}_v = f_v + \hat{\rho}_v v = f_v + \frac{\lambda_v}{k} v = f_v + \frac{1}{\mu_c} v \neq 0
\]
hold at \(p\). Thus we have the first assertion.

We next show the second assertion. By Lemma 3.5, we have
\[
d\hat{C}(\hat{V}) = df(\hat{V}) + \hat{\rho} dv(\hat{V}) + (\hat{V}\hat{\rho}) v = (\hat{V}\hat{\rho}) v.
\]
Since \(S(\hat{C}) = (\hat{V}\hat{\rho})^{-1}(0)\), \(d\hat{C}(\hat{V})\) vanishes on \(S(\hat{C})\). This implies that \(\hat{V}\) is a null vector field of \(\hat{C}\). \(\square\)

Lemma 5.4. Let \(p\) be a singular point of the second kind of a front \(f: \Sigma \to \mathbb{R}^3\). Then we have the following:

1. When \(p\) is a swallowtail of \(f\), then \(p\) is a non-degenerate singular point of \(\hat{C}\).
2. When \(p\) is not a swallowtail of \(f\), then \(p\) is a non-degenerate singular point of \(\hat{C}\) if and only if \(V(\hat{V}\hat{\rho}) \neq 0\) at \(p\), where \(V\) is a principal vector associated to the bounded principal curvature \(\kappa\).

Proof. Taking a strongly adapted coordinate system \((U; u, v)\) centered at \(p\) with \(\lambda_v(p) > 0\), we have
\[
(\hat{V}\hat{\rho})_u = (\hat{V}_1)_u \hat{\rho}_u + \hat{V}_1 \hat{\rho}_{uu} + (\hat{V}_2)_u \hat{\rho}_v + \hat{V}_2 \hat{\rho}_{uv} = (\hat{V}_2)_u \hat{\rho}_v = -e'(0)k \frac{\lambda_v}{k} = -e'(0)\lambda_v,
\]
\[
(\hat{V}\hat{\rho})_v = (\hat{V}_1)_v \hat{\rho}_u + \hat{V}_1 \hat{\rho}_{uv} + (\hat{V}_2)_v \hat{\rho}_v + \hat{V}_2 \hat{\rho}_{vv} = \hat{V}_1 \hat{\rho}_{uv} = -\hat{k} \hat{\rho}_{uv}
\]
at \(p\) by (3.5). Thus by the definition, \(p\) is a non-degenerate singular point of \(\hat{C}\) if and only if \((e'(0), \hat{\rho}_{uv}) \neq (0, 0)\) at \(p\) because \(\lambda_v(p) \neq 0\) and \(\hat{k}(p) \neq 0\). In particular, the condition \(e'(0) \neq 0\) holds when \(p\) is a swallowtail of the initial front \(f\) by Fact 2.2. Thus the first assertion holds.

On the other hand, \(V(\hat{V}\hat{\rho})\) is calculated as
\[
V(\hat{V}\hat{\rho}) = V_1(\hat{V}\hat{\rho})_u + V_2(\hat{V}\hat{\rho})_v = e'(0)\lambda_v \hat{M} - \hat{k} \hat{\rho}_{uv} \hat{L}
\]
at \(p\) by (3.3). Thus \(V(\hat{V}\hat{\rho}) \neq 0\) if and only if \(\hat{\rho}_{uv} \neq 0\) at \(p\) when \(p\) is not a swallowtail, that is, \(e'(0) = 0\). Hence we obtain the second assertion. \(\square\)

When the initial front \(f\) has a cuspidal butterfly at \(p\), the focal surface \(\hat{C}\) may have a degenerate singularity at \(p\) by this lemma.

Proposition 5.5. Let \(f: \Sigma \to \mathbb{R}^3\) be a front with a singular point of the second kind \(p\). Then the focal surface \(\hat{C}\) as in (5.1) is a front at \(p\).

Proof. We take a strongly adapted coordinate system \((U; u, v)\) centered at \(p\). Since \(\hat{C}_u = 0\) at \(p\) and \(p\) is a corank one singular point of \(\hat{C}\), \(\hat{C}\) is a front at \(p\) if and only if \((e_2)_u(p) \neq 0\), where \(e_2 = y/|y|\) and \(y\) is a map given by (3.7). We note that this condition is equivalent to \(de_2(\hat{V}) \neq 0\) at \(p\) because \(\hat{V}\) is parallel to \(\hat{\partial}_u\) at \(p\) (see (3.5)). By a direct computation, it holds that
\[
(e_2)_u = \frac{y_u |y|^2 - y \langle y_u, y \rangle}{|y|^3}.
\]
We consider the expression of \(y_u\) at \(p\). Since \(\lambda(p) = \hat{F}(p) = \hat{G}_u(p) = 0\) and \(\hat{G}(p) = 1\), we get
\[
y_u = -\hat{k} h_u - \hat{k} h_u + \hat{G}_u f_v
\]
at \( p \), where \( h_u \) can be written as in (2.8). By Lemma 2.8, \( A = \langle h_u, f_\nu \rangle = \tilde{f}_u - \tilde{E} \) holds at \( p \). Thus \( y_u \) satisfies

\[
y_u = -\left( \tilde{k}_e + \frac{\tilde{k} \tilde{E}_u}{2E} \right) h + \tilde{k} \tilde{E} (f_\nu - \mu_e \nu)
\]

at \( p \) since \( \mu_e = (\tilde{L}/\tilde{E})(p) \) holds by Lemma 2.5. Since \( y = -\tilde{k} h \) at \( p \), we have

\[
y_u | y|^2 - y \langle y, y_u \rangle = \tilde{k}^2 \tilde{E}(\tilde{E} f_\nu - \tilde{L} \nu) = \tilde{k}^2 \tilde{E}^2 (f_\nu - \mu_e \nu) \neq 0
\]

at \( p \). Therefore \( (e_2)_u \neq 0 \) at \( p \), and hence \( \hat{C} \) is a front at \( p \). \( \square \)

If a front \( f \) is a cuspidal edge at \( p \), then \( \hat{C} \) is regular at \( p \), and hence we can also consider \( \hat{C} \) as a front near \( p \).

Summarizing above results and using the criteria for singularities given by Fact 2.2, we have the following characterizations.

**Theorem 5.6.** Let \( f : \Sigma \to \mathbb{R}^3 \) be a front and \( p \in \Sigma \) a singular point of the second kind. Then we have the following.

1. The focal surface \( \hat{C} \) given by (5.1) is a cuspidal edge at \( p \) if and only if \( V(\hat{V} \rho) \neq 0 \) and \( \nabla V(\hat{V} \rho) \neq 0 \) at \( p \). In particular, \( p \) is a swallowtail of \( f \).
2. The focal surface \( \hat{C} \) is a swallowtail at \( p \) if and only if \( V(\hat{V} \rho) \neq 0 \), \( \hat{V}(\hat{V} \rho) = 0 \) and \( \nabla \nabla V(\hat{V} \rho) \neq 0 \) at \( p \). In particular, \( p \) is a cuspidal butterfly of \( f \).

**Proof.** Let us take a strongly adapted coordinate system \( (U; u, v) \) centered at \( p \). By Lemma 5.3, \( \hat{V} \) as in (3.5) can be taken as a null vector field of \( \hat{C} \). Moreover, the signed area density function \( \lambda \hat{C} \) is proportional to \( V(\hat{V} \rho) \) by (5.2). Thus \( \hat{V} \lambda \hat{C} \neq 0 \) (resp. \( \hat{V} \lambda \hat{C} = 0 \) and \( \nabla \hat{V} \lambda \hat{C} \neq 0 \)) at \( p \) is equivalent to \( \tilde{V}(\tilde{V} \rho) \neq 0 \) (resp. \( \tilde{V}(\tilde{V} \rho) = 0 \) and \( \nabla \tilde{V}(\tilde{V} \rho) \neq 0 \)) at \( p \). By direct calculations, we see that \( \tilde{V}(\tilde{V} \rho)(p) = e'(0) \hat{k}(p) \lambda_e(p) = e'(0) \mu_e \) and \( \nabla \tilde{V}(\tilde{V} \rho)(p) = -3e'(0) \hat{k}(p) \tilde{V}_1(p)((\tilde{V}_1)_{u}(p) \hat{\rho}_v(p) + \tilde{V}_1(p) \hat{\rho}_{uv}(p)) - e''(0) \hat{k}(p) \tilde{V}_1(p)^2 \hat{\rho}_v(p) \) hold. Thus we have the assertions by Fact 2.2, Lemma 5.4 and Proposition 5.5. \( \square \)

**Example 5.7.** Let \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) be a map given by

\[
f(u, v) = \left( \frac{u^2}{2} - v, -\frac{u^3}{3} + uv, -\frac{u^4}{8} + \frac{u^2 v}{2} \right).
\]

Then the set of singular points of \( f \) is the \( u \)-axis, and \( f \) has a swallowtail at the origin (see Figure 1, left). Moreover, the null vector field \( \eta \) of \( f \) can be written as \( \eta = \partial_u - u \partial_v \). The image of focal surface \( \hat{C} \) is shown in the center of Figure 1. One can verify that \( \hat{C} \) has a cuspidal edge at the origin.

![Figure 1](image-url)
Example 5.8. Let \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) be a \( C^\infty \) map defined by
\[
f(u, v) = \left( \frac{1}{6} (u^3 - 6v), \frac{1}{8} - \frac{u^4}{3} + uv + v, \frac{1}{360} \left( -5u^6 - 18u^5 + 60u^3v + 180u^2v - 180v^2 \right) \right).
\]
The origin is a cuspidal butterfly of \( f \) and \( S(f) = \{ v = 0 \} \). By a direct calculation, we have \( (V\hat{\rho})_u = 0 \) and \( (V\hat{\rho})_v = 2(\neq 0) \) at the origin. This implies that \( V(V\hat{\rho}) \neq 0 \) holds at the origin, and hence the origin is a non-degenerate singular point of the focal surface \( \hat{C} \) of \( f \). Moreover, one can see that \( V(V\hat{\rho}) = 0 \) and \( VV(V\hat{\rho}) = -4 \neq 0 \) hold at the origin. Thus \( \hat{C} \) has a swallowtail at the origin (see Figure 2).

![Figure 2. Images of \( f \) (left) and its focal surface \( \hat{C} \) (center) of Example 5.8. The figure in the right-hand side shows the singular sets of both \( f \) (dashed) and \( \hat{C} \) around the origin.](image)

As mentioned above, the focal surface \( \hat{C} \) given by (5.1) of a front \( f \) might have a degenerate singularity at \( p \) when \( p \) is neither a cuspidal edge nor a swallowtail. By Theorem 5.6, when \( p \) is a cuspidal butterfly of \( f \), then \( \hat{C} \) may have a cuspidal lips/beaks at \( p \), which are degenerate singular points of fronts. Applying Fact 2.2 for degenerate singularities, we have the following.

Proposition 5.9. Let \( f : \Sigma \to \mathbb{R}^3 \) be a front and \( p \) a cuspidal butterfly of \( f \). Suppose that \( V(V\hat{\rho}) = 0 \) holds at \( p \). Then the focal surface \( \hat{C} \) as in (5.1) is a cuspidal lips (resp. cuspidal beaks) at \( p \) if and only if \( V\hat{\rho} \) has a Morse type singularity of index zero or two (resp. a Morse type singularity of index one) at \( p \).

Proof. By Lemma 5.4, \( p \) is a degenerate singular point of \( \hat{C} \). Moreover, by Theorem 5.6, the condition \( VV(V\hat{\rho}) \neq 0 \) holds at \( p \) when \( p \) is a cuspidal butterfly of the initial front \( f \). Thus we have the assertion by Fact 2.2. \( \square \)

Remark 5.10. Let \( p \) be a singular point of the second kind of a front \( f \) neither a swallowtail nor a cuspidal butterfly. Assume that the focal surface \( \hat{C} \) has a degenerate singularity at \( p \). Then \( \hat{C} \) cannot have a cuspidal lips/beaks at \( p \) because \( VV(V\hat{\rho}) = 0 \) holds at \( p \) automatically.

Example 5.11. Let \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) be a \( C^\infty \) map given by
\[
f(u, v) = \left( \frac{1}{6} (u^3 - 6v), \frac{1}{72} \left( u^6 - 9u^4 + 12u^3v + 72uv + 36w^2 \right), -\frac{1}{20} u^2 \left( u^3 - 10v \right) \right).
\]
Then the set of singular points of \( f \) is \( S(f) = \{ v = 0 \} \). Moreover, \( f \) has a cuspidal butterfly at the origin. By direct calculations, we see that \( (V\hat{\rho})_u = (V\hat{\rho})_v = 0 \) hold at the origin. Further, it follows that \( (V\hat{\rho})_{uu} = -1, (V\hat{\rho})_{uv} = 0 \) and \( (V\hat{\rho})_{vw} = 6 \) at the origin.
This implies that $\tilde{V}\hat{\rho}$ has a Morse type singularity of index one at the origin. Thus the focal surface $\tilde{C}$ of $f$ has a cuspidal beaks at the origin (see Figure 3).

**Example 5.12.** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a $C^\infty$ map given by

$$f(u, v) = \left( \frac{1}{6} (u^3 - 6v), \frac{1}{72} (-u^6 - 9u^4 + 12u^3v + 72uv^2 - 36v^3), -\frac{1}{20} u^2(u^3 - 10v) \right).$$

This map $f$ is a front and the origin is a cuspidal butterfly. By direct computations, it holds that $\hat{e}V\hat{\rho}u = \hat{e}V\hat{\rho}v = 0, \hat{e}V\hat{\rho}uu = -1, \hat{e}V\hat{\rho}uv = 0$ and $\hat{e}V\hat{\rho}vv = -6$ at the origin. This means that $\tilde{V}\hat{\rho}$ has a Morse type singularity of index two at the origin. Thus the focal surface $\tilde{C}$ has a cuspidal lips at the origin (see Figure 4).

**5.2. Contact between singular curves.** We now consider contact properties of singular curves of the initial front and its focal surface. First we start with the definition of contact between regular curves on the plane (cf. [12, Page 74]).

**Definition 5.13.** Let $\alpha : I \ni t \mapsto \alpha(t) \in \mathbb{R}^2$ be a regular plane curve. Let $\beta$ be another plane curve defined by the zero set of a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$. We say that $\alpha$ has $(k+1)$-point contact at $t_0 \in I$ with $\beta$ if the composite function $c(t) = F(\alpha(t))$ satisfies

$$c(t_0) = c'(t_0) = \cdots = c^{(k)}(t_0) = 0, \quad c^{(k+1)}(t_0) \neq 0,$$

where $c^{(i)} = d^i c/dt^i$ $(1 \leq i \leq k + 1)$. Moreover, $\alpha$ has at least $(k + 1)$-point contact at $t_0$ with $\beta$ if the function $c(t) = F(\alpha(t))$ satisfies

$$c(t_0) = c'(t_0) = \cdots = c^{(k)}(t_0) = 0.$$

In this case, we call the integer $k$ the order of contact.

**Proposition 5.14.** Let $f : \Sigma \to \mathbb{R}^3$ be a front and $p$ a singular point of the second kind. Assume that the curve given by $(\tilde{V}\hat{\rho})^{-1}(0)$ is regular at $p$. Then the singular curve $\gamma$ for $f$ through $\gamma(0) = p$ has 1-point contact (resp. 2-point contact) at $p$ with $(\tilde{V}\hat{\rho})^{-1}(0)$ if and only if $p$ is a swallowtail (resp. cuspidal butterfly) of $f$. Moreover, $\tilde{C}$ has a cuspidal edge (resp. swallowtail) at $p$. 

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**Figure 3.** Images of $f$ (left) and its focal surface $\tilde{C}$ (center) of Example 5.11. The figure in the right-hand side shows the singular set of $\tilde{C}$ around the origin.

**Figure 4.** The initial front $f$ (left) and its focal surface $\tilde{C}$ (right) of Example 5.12.
Proof. Let us take a strongly adapted coordinate system \((U; u, v)\) centered at \(p\). Define the function \(c = \hat{V} \hat{\rho} \circ \gamma: (-\varepsilon, \varepsilon) \to \mathbb{R}\). Since \(\hat{V} = \hat{V}_1(\partial_u + e(u)\partial_v)\) and \(\hat{\rho} = 0\) on the \(u\)-axis, the function \(c\) can be written as
\[
c(u) = e(u)\hat{V}_1(0,0)\hat{\rho}_v(u,0) = e(u)\psi(u) \quad (\psi(u) = \hat{V}_1(0,0)\hat{\rho}_v(u,0)).
\]
By the Leibniz rule, one can see that
\[
(c^{(n)}(u) = \sum_{k=0}^{n} \binom{n}{k} e^{(k)}(u)\psi^{(n-k)}(u),
\]
where \(\binom{n}{k}\) are binomial coefficients. Thus we have the assertions from Theorem 5.6. 

The right-hand side of Figure 2 shows both the sets of singular points of \(f\) and \(\hat{C}\) of Example 5.8. We notice that these curves have 2-point contact at the origin. This point actually corresponds to the swallowtail of \(\hat{C}\).

5.3. Behavior of the Gaussian curvature of \(\hat{C}\). Let \(p\) be a singular point of the second kind of a front \(f\). Then the focal surface \(\hat{C}\) given by (5.1) has a front-type singularity at \(p\) by Theorem 5.6. Thus the Gaussian curvature \(K_{\hat{C}}\) of \(\hat{C}\) may diverge at \(p\) ([21, 34]). However, \(K_{\hat{C}}\) might be rationally bounded at \(p\). Hence we consider the condition for \(K_{\hat{C}}\) to be rationally bounded.

Theorem 5.15. Let \(f: \Sigma \to \mathbb{R}^3\) be a front and \(p\) a singular point of the second kind, where \(\Sigma\) is a domain in \(\mathbb{R}^2\). Let \(\hat{C}\) be the focal surface of \(f\) associated to the unbounded principal curvature given by (5.1). Suppose that \(\hat{C}\) has a non-degenerate singularity at \(p\). Then the Gaussian curvature \(K_{\hat{C}}\) of \(\hat{C}\) is rationally bounded at \(p\) if and only if \(p\) is a sub-parabolic point of \(f\). Moreover, the limiting normal curvature of \(\hat{C}\) vanishes at \(p\).

Proof. Let us take a strongly adapted coordinate system \((U; u, v)\) centered at \(p\). Then we consider the condition that the Gauss map of \(\hat{C}\) has a singularity at \(p\). The Gauss map of \(\hat{C}\) is given by \(e_2 = y/|y|\), where \(y\) is defined by (3.7). Define a function \(\Lambda_{\hat{C}}: U \to \mathbb{R}\) by
\[
\Lambda_{\hat{C}}(u, v) = \det((e_2)_{u}, (e_2)_{v}, (e_2)(u, v)).
\]
Then zeros of \(\Lambda_{\hat{C}}\) correspond to singular points of \(e_2\). By a direct calculation, \(\Lambda_{\hat{C}}(p) = 0\) is equivalent to \(\det(y_u, y_v, y)(p) = 0\). Thus we consider this condition. By a direct calculation, we have
\[
y = -\hat{k}h, \quad y_u = *_1h - \hat{k}h_u + \hat{k}\hat{F}_uf_v, \quad y_v = *_2h - \hat{k}h_v - (\lambda_v\hat{M} - \hat{k}\hat{F}_v)f_v
\]
at \(p\), where \(*_i\ (i = 1, 2)\) are some constants. Therefore one can see that
\[
\det(y_u, y_v, y) = -\hat{k} \left(\hat{k}^2\det(h_u, h_v, h) + \hat{k}(\lambda_v\hat{M} - \hat{k}\widehat{F}_v)\det(h_u, f_v, h) - \hat{k}^2\hat{F}_v\det(f_v, h_v, h)\right)
\]
holds at \(p\). By Lemmas 2.7 and 2.8, we see that
\[
h_u = \frac{\hat{E}_u + \hat{F}_u}{2\hat{E}} h + (\hat{F}_u - \hat{E}) f_v + \hat{L}, \quad h_v = \frac{\hat{E}_v + B f_u + \hat{M}v}{2\hat{E}} h + B f_v + \hat{M}v
\]
holds at \(p\), where \(B = (h_v, f_v)\). Since \(\hat{F} = \langle f_v, h \rangle\), we have \(\hat{F}_v = \langle f_{uv}, h \rangle + B\). On the other hand, since \(\hat{N} = \langle f_{vv}, v \rangle\), it follows that \(\hat{N}_u = \langle f_{uuv}, v \rangle + \langle f_{uvv}, v_u \rangle\). By \(f_u = vh - e(u) f_v\), one can see that \(f_{uuv} = 2h_u\) holds at \(p\). Moreover, \(v_u = -\mu_v h\) holds at \(p\) by Lemmas 2.4 and 2.5. Thus we obtain \(\hat{N}_u = 2\hat{M} - \mu_v \langle f_{uv}, h \rangle\) at \(p\), where we used the relation \(\langle h_v, v \rangle = \hat{M}\). Hence we get
\[
B = \langle f_v, h_v \rangle = \hat{F}_v - \frac{2\hat{M} - \hat{N}_u}{\mu_v}.
\]
at $p$. Therefore, by $\mu_c = (\tilde{L}/\tilde{E})(p)$, it follows that
\[
\det(h_u, h_v, h) = \det(\tilde{f}_u \tilde{M} - \tilde{f}_v \tilde{L}) + \tilde{E}(\tilde{M} - \tilde{N})\lambda_v, \quad \det(h_u, f_v, h) = -\tilde{L}\lambda_v, \quad \det(f_v, h_v, h) = \tilde{M}\lambda_v
\]
hold at $p$. By the above calculations and $k = \mu_c \lambda_v$ at $p$, we have
\[
\det(y_u, y_v, y) = -k^3 \tilde{E}\lambda_v \tilde{N}_u
\]
at $p$.

On the other hand, the directional derivative $\tilde{V}_\kappa$ of the bounded principal curvature $\kappa$ in the direction $\tilde{V}$ is calculated as
\[
\tilde{V}_\kappa = \kappa \tilde{N}_u = -\lambda_c \mu_c \tilde{N}_u
\]
at $p$. Thus $\lambda_c \tilde{N}_u = -\tilde{V}_\kappa / \mu_c$ holds at $p$, and hence we have
\[
\det(y_u, y_v, y) = \lambda_c^5 \mu_c^2 (\tilde{V}_\kappa)
\]
at $p$ by $\lambda_c(p)^2 = \tilde{E}(p)$. Therefore the Gauss map $e_2$ of $\tilde{C}$ has a singularity at $p$ if and only if $p$ is a sub-parabolic point of $f$. Thus by Fact 2.6, we have the conclusion.

Theorem 5.15 gives geometrical meanings of a sub-parabolic point of a front with a singular point of the second kind. As a corollary of this theorem, we have the following relation between behavior of the Gaussian curvature of the initial front and of the focal surface.

**Corollary 5.16.** Let $f : \Sigma \to \mathbb{R}^3$ be a front and $p$ a singular point of the second kind of $f$. Suppose that the Gaussian curvature $K$ is either bounded on a sufficiently small neighborhood of $p$ or rationally continuous at $p$. Then the Gaussian curvature $K^{\tilde{C}}$ of $\tilde{C}$ is rationally bounded at $p$.

**Proof.** First we assume that $K$ is bounded near $p$. By Proposition 4.4, $p$ is a sub-parabolic point of $f$. Thus by Theorem 5.15, $K^{\tilde{C}}$ is rationally bounded at $p$.

Next we suppose that $K$ is rationally continuous at $p$. By Proposition 4.3, $p$ is a sub-parabolic point of $f$. Thus it holds that the Gaussian curvature $K^{\tilde{C}}$ of $\tilde{C}$ is rationally bounded at $p$ by Theorem 5.15 again. Therefore we have the assertion.

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