On homotopy invariants of finite degree

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Abstract
We prove that homotopy invariants of finite degree distinguish homotopy classes of maps of a connected compact CW-complex to a nilpotent connected CW-complex with finitely generated homotopy groups.

§ 1. Introduction

\( N = \{0, 1, \ldots\} \). Space means "pointed topological space". CW-complexes are also pointed (the basepoint being a vertex). Map means "basepoint preserving continuous map". Homotopies, the notation \([X,Y]\), etc. are to be understood in the pointed sense.

Invariants of finite degree. Let \( X \) and \( Y \) be spaces, \( V \) be an abelian group, and \( f: [X,Y] \to V \) be a function (a homotopy invariant). Let us define a number \( \text{Deg} f \in N \cup \{\infty\} \), the degree of \( f \). Given a map \( a: X \to Y \) and a number \( r \in N \), we have the map \( a^r: X^r \to Y^r \) (the Cartesian power), which induces the homomorphism \( C_0(a^r): C_0(X^r) \to C_0(Y^r) \) between the groups of (unreduced) zero-dimensional chains with the coefficients in \( \mathbb{Z} \). Let the inequality \( \text{Deg} f \leq r \) be equivalent to the existence of a homomorphism \( l: \text{Hom}(C_0(X^r), C_0(Y^r)) \to V \) such that \( f([a]) = l(C_0(a^r)) \) for all maps \( a: X \to Y \). As one easily sees, \( \text{Deg} f \) is well defined by this condition. Finite-degree invariants are those of finite degree.

Main results.

1.1. Theorem. Let \( X \) be a connected compact CW-complex, \( Y \) be a nilpotent connected CW-complex with finitely generated homotopy groups, and \( u_1, u_2 \in [X,Y] \) be distinct classes. Then, for some prime \( p \), there exists a finite-degree invariant \( f: [X,Y] \to \mathbb{Z}_p \) such that \( f(u_1) \neq f(u_2) \).

Related facts were known for certain cases where \([X,Y]\) is an abelian group [10, 11]. Theorem 1.1 follows (see § 11) from a result of Bousfield–Kan and Theorem 1.2.

We call a group \( p \)-finite (for a prime \( p \)) if it is finite and its order is a power of \( p \).

1.2. Theorem. Let \( p \) be a prime, \( X \) be a compact CW-complex, and \( Y \) be a connected CW-complex with \( p \)-finite homotopy groups. Then every invariant \( f: [X,Y] \to \mathbb{Z}_p \) has finite degree.
Probably, Theorem 1.2 can be deduced from Shipley’s convergence theorem [12], which we do not use. We use an (approximate) simplicial model of $Y$ that admits a harmonic (see § 6) embedding in a simplicial $\mathbb{Z}_p$-module.

**Non-nilpotent examples.** The following examples show the importance of the nilpotency assumption in Theorem 1.1. We consider finite-degree invariants on $\pi_n(Y) = [S^n, Y]$.

**1.3.** Let $Y$ be a space with $\pi_1(Y)$ perfect. Then, for any abelian group $V$, any finite-degree invariant $f: \pi_1(Y) \to V$ is constant.

This follows from Lemmas 12.2 and 3.6.

**1.4.** Take $n > 1$. Let $Y$ be a space such that $\pi_n(Y) \cong \mathbb{Z}^2$ and an element $g \in \pi_1(Y)$ induces an order 6 automorphism on $\pi_n(Y)$. Then, for any abelian group $V$, any finite-degree invariant $f: \pi_n(Y) \to V$ is constant.

This follows from Lemmas 12.2 and 12.3 and claim 3.7.

**An example: maps** $S^{n-1} \times S^n \to S^n_\mathbb{Q}$ (cf. [1, Example 4.6]). Take an even $n > 0$. Let $c: S^{n-1} \times S^n \to S^{2n-1}$ be a map of degree 1. Put $i = [id] \in \pi_n(S^n)$, $j = i \ast i \in \pi_{2n-1}(S^n)$ (the Whitehead square), and $u(q) = (qj) \circ [c] \in [S^{n-1} \times S^n, S^n]$, $q \in \mathbb{Z}$. Let $i: S^n \to S^n_\mathbb{Q}$ be the rationalization. Put $\bar{u}(q) = [l] \circ u(q) \in [S^{n-1} \times S^n, S^n_\mathbb{Q}]$. The classes $u(q), q \in \mathbb{Z}$, are pairwise distinct; moreover, the classes $\bar{u}(q), q \in \mathbb{Z}$, are pairwise distinct (the proof is omitted). Is it true that, under the assumptions of Theorem 1.1, there must exist an $r \in \mathbb{N}$ such that the elements of $[X, Y]$ are distinguished by invariants of degree at most $r$? No, as the following claim shows.

**1.5.** Let $V$ be an abelian group and $f: [S^{n-1} \times S^n, S^n] \to V$ be an invariant of degree at most $r \in \mathbb{N}$. Then $f(u(q)) = f(u(0))$ whenever $r! \mid q$.

The following claim shows the importance of the assumption of Theorem 1.1 that $Y$ has finitely generated homotopy groups.

**1.6.** Let $V$ be an abelian group and $f: [S^{n-1} \times S^n, S^n_\mathbb{Q}] \to V$ be an invariant of finite degree. Then $f(\bar{u}(q)) = f(\bar{u}(0)), q \in \mathbb{Z}$.

The following claim shows that, under the assumptions of Theorem 1.1, finite-degree invariants taking values in $\mathbb{Q}$ may not distinguish rationally distinct homotopy classes.

**1.7.** Let $f: [S^{n-1} \times S^n, S^n] \to \mathbb{Q}$ be an invariant of finite degree. Then $f(u(q)) = f(u(0)), q \in \mathbb{Z}$.

**Elusive elements of $H_0(Y^X)$.** The space of maps $X \to Y$ is denoted $Y^X$. An invariant $f: [X, Y] \to V$ gives rise to the homomorphism $+ f: H_0(Y^X) \to V$, $[u] \mapsto f(u)$ (here $[u]$ denotes the basic element corresponding to $u$). Is it true that, under the assumptions of Theorem 1.1, for any non-zero element $w \in H_0(Y^X)$ there exist an abelian group $V$ and a finite-degree invariant $f: [X, Y] \to V$ such that $+ f(w) \neq 0$? No, as the following claim shows.
1.8. Take $n > 1$. Let $Y$ be a space and $u_1, u_2 \in \pi_n(Y)$ be elements of coprime finite orders. Put $w = [u_1 + u_2] - [u_1] - [u_2] + [0]$. Let $V$ be an abelian group and $f : \pi_n(Y) \to V$ be an invariant of finite degree. Then $f(w) = 0$.

This follows from Lemmas 12.2 and 3.8. □

If the group $\pi_n(Y)$ is torsion and divisible, then the same is true for any elements $u_1, u_2 \in \pi_n(Y)$ (this follows from Lemmas 12.2 and 3.9). In this case, $\pi_n(Y)$ cannot be finitely generated (without being zero). In return, $Y$ can be $p$-local, e. g. $Y = K(P, n)$ (the Eilenberg–MacLane space) for $P = \mathbb{Z}[1/p]/\mathbb{Z}$.

§ 2. Preliminaries

We say crew for “pointed set” and archism for “basepoint preserving function”. We use the standard model structure on the category of simplicial crews (and archisms) [7, Corollary 3.6.6]. The words fibration, cofibration, etc. refer to it. A fibring simplicial archism is a fibration. An isotypical simplicial archism, or an isotypy, is a weak equivalence. Isotypic simplicial crews are weakly equivalent ones.

An abelian group is a crew (the basepoint being 0); a simplicial abelian group is a simplicial crew.

We call a simplicial crew $T$ compact if it is generated by a finite number of simplices, and gradual if the crews $T_q$, $q \in \mathbb{N}$, are finite.

For simplicial crews $K$ and $T$, define $\mathcal{R}$-homomorphisms

$$K^X_\mu : C_0(Y^X) \to \text{Hom}(C_0(X^r), C_0(Y^r)), \quad [a] \mapsto C_0(a^r),$$

$r \in \mathbb{N}$. We have the projection $\mathcal{R}$-homomorphism

$$X_\nu : C_0(Y^X) \to H_0(Y^X).$$

For simplicial crews $K$ and $T$, define $\mathcal{R}$-homomorphisms

$$K^T_\mu : C_0(T^K) \to \text{Hom}_0(C_*(K^r), C_*(T^r)), \quad [b] \mapsto C_*(b^r),$$

$r \in \mathbb{N}$. Here $[b]$ is the basic chain corresponding to a simplex $b \in (T^K)_0$, i. e. a simplicial archism $b : K \to T$; $b^r : K^r \to T^r$ is the Cartesian power; $C_*(b^r) : C_*(K^r) \to C_*(T^r)$ is the induced $\mathcal{R}$-homomorphism of graded $\mathcal{R}$-modules of chains; $\text{Hom}_0$ denotes the $\mathcal{R}$-module of grading-preserving $\mathcal{R}$-homomorphisms.

We have the projection $\mathcal{R}$-homomorphism

$$K^T_\nu : C_0(T^K) \to H_0(T^K).$$
§ 3. Group algebras and gentle functions

Let $\mathcal{R}[G]$ denote the group $\mathcal{R}$-algebra of a group $G$. An element $g \in G$ has the corresponding basic element $[g] \in \mathcal{R}[G]$. The augmentation ideal $\mathcal{R}[G] \subseteq \mathcal{R}[G]$ is the kernel of the $\mathcal{R}$-homomorphism $\mathcal{R}[G] \rightarrow \mathcal{R}$, $[g] \mapsto 1$. The ideal $\mathcal{R}[G]^s$ ($s > 0$) is $\mathcal{R}$-generated by elements of the form $(1 - g_1) \ldots (1 - g_s)$.

Let $V$ be an abelian group. A function $f: G \rightarrow V$ gives rise to the homomorphism $f^+: Z[\mathcal{R}[G]] \rightarrow V$, $[g] \mapsto f(g)$. We call $f$ $r$-gentle if $f^+|_{\mathcal{R}[G]}$ is $r$-gentle for some $r \in \mathbb{N}$ [9, Ch. V].

Let $p$ be a prime.

3.1. Lemma. Let $U$ be a finite $\mathbb{Z}_p$-module of dimension $m$. Then $|\mathbb{Z}_p[U]|^{(p-1)m+1} = 0$.

3.2. Corollary. Let $U$ and $V$ be $\mathbb{Z}_p$-modules. If $U$ is finite, then every function $f: U \rightarrow V$ is gentle.

3.3. Lemma [4, Proposition 1.2]. Let $U$, $V$, and $W$ be abelian groups, $f: U \rightarrow V$ be an $r$-gentle function, and $g: V \rightarrow W$ be an $s$-gentle one ($r, s \in \mathbb{N}$). Then the function $g \circ f: U \rightarrow W$ is $rs$-gentle.

This follows from [9, Ch. V, Theorem 2.1].

3.4. Corollary. Let $U$ and $V$ be abelian groups and $f: U \rightarrow V$ be an $r$-gentle ($r \in \mathbb{N}$) function. Then, for any $s \in \mathbb{N}$, the $\mathcal{R}$-homomorphism $f^+_\mathcal{R}: \mathcal{R}[U] \rightarrow \mathcal{R}[V]$, $[u] \mapsto [f(u)]$.

3.5. Lemma. Let $I$ be a set. For each $i \in I$, let $U_i$ and $V_i$ be abelian groups and $f_i: U_i \rightarrow V_i$ be an $r$-gentle ($r \in \mathbb{N}$) function. The the function

$$\prod_{i \in I} f_i: \prod_{i \in I} U_i \rightarrow \prod_{i \in I} V_i$$

is $r$-gentle.

The following claims are used only in discussion of the examples of § 1, not in the proof of the main results.

3.6. Lemma. Let $G$ be a perfect group and $V$ be an abelian group. Then any gentle function $f: G \rightarrow V$ is constant.

This follows from [9, Ch. III, Corollary 1.3].

3.7. Let $U$ be an abelian group isomorphic to $\mathbb{Z}^2$, $J: U \rightarrow U$ be an automorphism of order 6, $V$ be an abelian group, and $f: U \rightarrow V$ be a gentle function. Suppose that the function $\mathbb{Z} \times U \rightarrow V$, $(t, u) \mapsto f(J^6u - u)$, is gentle. Then $f$ is constant.
The proof is omitted.

3.8. Lemma. Let \( U \) and \( V \) be abelian groups, \( f : U \to V \) be a gentle function, and \( u_1, u_2 \in U \) be elements of coprime finite orders. Then \( f(u_1 + u_2) - f(u_1) - f(u_2) + f(0) = 0 \). 

3.9. Lemma. Let \( U \) be a divisible torsion abelian group, and \( V \) be an abelian group. Then every gentle function \( f : U \to V \) is \( 1 \)-gentle.

3.10. Lemma. Let \( G \) and \( H \) be groups. Then the ideal \( |G|H|H|^s \) \( (s > 1) \) is \( R \)-generated by elements of the form \( (1 - |a_1|) \ldots (1 - |a_s - q|)(1 - |b_1|) \ldots (1 - |b_q|) \), where \( 0 \leq q \leq s \), \( a_t \in G \times 1 \subseteq G \times H \), and \( b_t \in 1 \times H \subseteq G \times H \).

3.11. Lemma. A function \( F : \mathbb{Z} \to \mathbb{Q} \) is \( r \)-gentle (\( r \in \mathbb{N} \)) if and only if it is given by a polynomial of degree at most \( r \).

§ 4. Keys of a commutative square

Let \( E \) be a commutative ring. Consider the diagram of simplicial \( E \)-modules and \( E \)-homomorphisms

\[
\begin{array}{ccc}
V' & \xrightarrow{v'} & W \\
| & v' & | \\
U & \xrightarrow{f'} & V'' \\
& | & | \\
& g' & \downarrow \\
W & \xrightarrow{g'} & W \\
& | & | \\
& t' & \downarrow \\
U & \xrightarrow{f''} & V'' & \xrightarrow{t''} & W.
\end{array}
\]

where the square is commutative: \( f' \circ g' = f'' \circ g'' \). We call the quadruple \( (s', s'', t', t'') \) a key of this square if we have \( (-s', s'') \circ (-f', f'') + (g', g'') \circ (t', t'') = \text{id} \) in the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{(-f', f'')} & V' \oplus V'' & \xrightarrow{(g', g'')} & W. \\
& (s', s'') & | & (t', t'') & \\
& \downarrow & | & \downarrow & \\
& (-s', s'') \circ (-f', f'') + & (g', g'') \circ (t', t'') & = & \text{id}.
\end{array}
\]

The pair \( (t', t'') \) is called a half-key in this case.

4.1. Lemma. Let

\[
\begin{array}{ccc}
V' & \xrightarrow{v'} & W \\
| & v' & | \\
U & \xrightarrow{f'} & V'' & \xrightarrow{t''} & W \\
& | & | \\
& g' & \downarrow \\
W & \xrightarrow{g''} & W \\
& | & | \\
& t' & \downarrow \\
U & \xrightarrow{f''} & V''
\end{array}
\]

be a commutative square of simplicial \( E \)-modules and \( E \)-homomorphisms with a half-key, \( T \) be a simplicial crew, and \( k' : T \to V' \) and \( k'' : T \to V'' \) be simplicial
archisms such that \( f' \circ k' = f'' \circ k'' \). Consider the simplicial archism \( l = t' \circ k' + t'' \circ k'' : T \to W \). Then \( g' \circ l = k' \) and \( g'' \circ l = k'' \).

By a sector of a simplicial \( E \)-homomorphism \( h : \tilde{W} \to W \) we mean a simplicial \( E \)-homomorphism \( s : W \to \tilde{W} \) such that \( h \circ s = \text{id} \).

4.2. **Lemma.** Consider a commutative diagram of simplicial \( E \)-modules and \( E \)-homomorphisms

\[
\begin{array}{ccc}
0 & \to & \tilde{U} \\
\downarrow f & & \downarrow \tilde{p} \\
0 & \to & U \\
\end{array}
\quad
\begin{array}{ccc}
\tilde{V} & \to & \tilde{W} \\
\downarrow g & & \downarrow h \\
V & \to & W \\
\end{array}
\]

Suppose that its rows are split exact and \( h \) has a sector. Then the left-hand square has a key.

**Proof.** Let \( (k, l) \) and \( (\tilde{k}, \tilde{l}) \) (see the diagram below) be splittings:

\[
p \circ k = \text{id}, \quad l \circ q = \text{id}, \quad k \circ p + q \circ l = \text{id},
\]

\[
\tilde{p} \circ \tilde{k} = \text{id}, \quad \tilde{l} \circ \tilde{q} = \text{id}, \quad \tilde{k} \circ \tilde{p} + \tilde{q} \circ \tilde{l} = \text{id},
\]

and \( s \) be a sector: \( h \circ s = \text{id} \). Put \( r = \tilde{q} \circ s \circ l \) and \( \tilde{k} = \tilde{k} + r \circ (k \circ f - g \circ \tilde{k}) \). Then \( (0, k, \tilde{k}, r) \) is a key.

4.3. **Lemma.** Let \( L \) and \( M \) be simplicial crews, \( j : L \to M \) be an isotypical cofibration, and \( Q \) be a fibrant simplicial crew. Then \( Q^j : Q^M \to Q^L \) is an isotypical fibration.
4.4. Lemma. Let $Q$ and $R$ be simplicial crews, $c: Q \rightarrow R$ be a fibration, and $N$ be a simplicial crew isotypic to a point. Then $c^N: Q^N \rightarrow R^N$ is an isotypical fibration.

4.5. Lemma. Suppose that $E$ is a field. Let $V$ and $W$ be simplicial $E$-modules and $f: W \rightarrow V$ be an isotypical fibring simplicial $E$-homomorphism. Then $f$ has a sector.

4.6. Lemma. Suppose that $E$ is a field. Let $L$ and $M$ be simplicial crews, $j: L \rightarrow M$ be an isotypical cofibration, $Q$ and $R$ be simplicial $E$-modules, and $c: Q \rightarrow R$ be a fibring simplicial $E$-homomorphism. Then the commutative square

\[
\begin{array}{ccc}
Q^L & \xrightarrow{Q^j} & Q^M \\
\downarrow{c^L} & & \downarrow{c^M} \\
R^L & \xrightarrow{R^j} & R^M
\end{array}
\]

has a key.

Proof. Consider the (strictly) cofibration sequence

\[
L \xrightarrow{j} M \xrightarrow{k} N.
\]

Since $j$ is isotypical, the simplicial crew $N$ is isotypic to a point. We have the following diagram of simplicial $E$-modules and $E$-homomorphisms:

\[
\begin{array}{cccccc}
0 & \xrightarrow{\cdot} & Q^L & \xrightarrow{Q^j} & Q^M & \xrightarrow{Q^k} & Q^N & \xrightarrow{\cdot} & 0 \\
\downarrow{c^L} & & \downarrow{c^M} & & \downarrow{c^N} & & \\
0 & \xrightarrow{\cdot} & R^L & \xrightarrow{R^j} & R^M & \xrightarrow{R^k} & R^N & \xrightarrow{\cdot} & 0.
\end{array}
\]

We show that the rows are split exact. Consider the upper row. Obviously, it is exact in the middle and the right-hand terms. $Q$ is fibrant since it is a simplicial abelian group. By Lemma 4.3, $Q^j$ is an isotypical fibration. By Lemma 4.5, $Q^j$ has a sector. Therefore, the upper row is split exact. The same is true for the lower row. By Lemma 4.4, $c^N$ is an isotypical fibration. By Lemma 4.5, $c^N$ has a sector. By Lemma 4.2, the desired key exists.

§ 5. Quasisimplicial archisms

A quasisimplicial archism $f: K \rightarrow L$ between simplicial crews $K$ and $L$ is a sequence of archisms $f_q: K_q \rightarrow L_q$, $q \in \mathbf{N}$. Let $\hat{\text{Ar}}(K, L)$ denote the crew of quasisimplicial archisms and $\text{sAr}(K, L)$ denote the subcrew of simplicial ones.

A quasisimplicial archism $f: U \rightarrow V$ between simplicial abelian groups is $r$-gentle if the archisms $f_q: U_q \rightarrow V_q$ are $r$-gentle.
Let $T$ be a simplicial crew. For $m, q \in \mathbb{N}$, let $[m/q]$ be the set of non-strictly increasing functions $[m] \to [q]$ (where $[q] = \{0, \ldots, q\}$) and consider the archism

$$T(m, q) = (T(h))_{h \in [m/q]} : T_q \to T_{m/q}.$$  

We call $T$ $m$-soluble if, for any $q$, the archism $T(m, q)$ is injective.

Let $p$ be a prime.

5.1. Lemma. Let $T$ be a gradual simplicial crew, $U$ be a gradual simplicial $\mathbb{Z}_p$-module, $R$ be an $m$-soluble ($m \in \mathbb{N}$) simplicial $\mathbb{Z}_p$-module, $d : T \to U$ be a cofibration, and $k : T \to R$ be a simplicial archism. Then, for some $r \in \mathbb{N}$, there exists an $r$-gentle quasisimplicial archism $w : U \to R$ such that $w \circ d = k$.

Proof. Since $d_m : T_m \to U_m$ is injective, there exists an archism $v : U_m \to R_m$ such that $v \circ d_m = k_m$. By Corollary 3.2, $v$ is $r$-gentle for some $r \in \mathbb{N}$. Take $q \in \mathbb{N}$. We have the commutative diagram

$$\begin{array}{ccc}
U_q & \xrightarrow{d_q} & T_q \\
| & & \downarrow \scriptstyle{T(m,q)} \\
U_{m/q} & \xrightarrow{d_{m/q}} & T_{m/q} \\
| & & \downarrow \scriptstyle{k_m} \\
U_{m/q} & \xrightarrow{w} & R_{m/q} \\
| & & \downarrow \scriptstyle{k_{m/q}} \\
R_{m/q} & \xrightarrow{R(m,q)} & R_q
\end{array}$$

By Lemma 3.5, the archism $v_{m/q}$ is $r$-gentle. Since the $\mathbb{Z}_p$-homomorphism $R(m, q)$ is injective, there exists a $\mathbb{Z}_p$-homomorphism $f : R_{m/q} \to R_q$ such that $f \circ R(m, q) = \text{id}$. Consider the $r$-gentle archism

$$w_q : U_q \xrightarrow{w} U_{m/q} \xrightarrow{v_{m/q}} R_{m/q} \xrightarrow{f} R_q.$$  

Using the diagram, we get $w_q \circ d_q = k_q$. □

5.2. Lemma. Let $M$ be a simplicial crew, $U$ and $V$ be simplicial abelian groups, and $t : U \to V$ be an $r$-gentle ($r \in \mathbb{N}$) quasisimplicial archism. Then the archism $t_{\#} : \check{\text{Ar}}(M, U) \to \check{\text{Ar}}(M, V)$, $f \mapsto t \circ f$, is $r$-gentle.

Proof. This follows from Lemma 3.5 because of the commutative diagram

$$\begin{array}{ccc}
\prod_{q \in \mathbb{N}, k \in M_q^x} U_q & \xrightarrow{t_{\#}} & \prod_{q \in \mathbb{N}, k \in M_q^x} V_q \\
| & & | \\
\prod_{q \in \mathbb{N}, k \in M_q^x} \check{\text{Ar}}(M, U) & \xrightarrow{t_{\#}} & \prod_{q \in \mathbb{N}, k \in M_q^x} \check{\text{Ar}}(M, V)
\end{array}$$

where $M_q^x = M_q \setminus \{\text{basepoint}\}$. □
5.3. Lemma. Let $M$ and $T$ be simplicial crews, $U$ and $R$ be simplicial $\mathbb{Z}_p$-modules, $d: T \to U$ and $k: T \to R$ be simplicial archisms, and $w: U \to R$ be an $r$-gentle ($r \in \mathbb{N}$) quasisimplicial archism such that $w \circ d = k$. Then there exists an $r$-gentle quasisimplicial archism $z: U^M \to R^M$ such that $z \circ d^M = k^M$.

Proof. Take $q \in \mathbb{N}$. We have the commutative diagram

\[
\begin{array}{cccccc}
(U^M)_q & \xrightarrow{(d^M)_q} & (T^M)_q & \xrightarrow{(k^M)_q} & (R^M)_q \\
\downarrow & & \downarrow & & \downarrow \\
\hat{s}\text{Ar}(\Delta^q_+ \wedge M, U) & \xrightarrow{\hat{w}_q} & \hat{s}\text{Ar}(\Delta^q_+ \wedge M, R),
\end{array}
\]

where the $\mathbb{Z}_p$-homomorphism $i: (U^M)_q = \hat{s}\text{Ar}(\Delta^q_+ \wedge M, U) \to \hat{s}\text{Ar}(\Delta^q_+ \wedge M, U)$ is the inclusion and $j$ is analogous. By Lemma 5.2, the archism $\hat{w}_q$ is $r$-gentle. There is a $\mathbb{Z}_p$-homomorphism $f: \hat{s}\text{Ar}(\Delta^q_+ \wedge M, R) \to (R^M)_q$ such that $f \circ j = \text{id}$. Consider the $r$-gentle archism

\[
z_q: (U^M)_q \xrightarrow{i} \hat{s}\text{Ar}(\Delta^q_+ \wedge M, U) \xrightarrow{\hat{w}_q} \hat{s}\text{Ar}(\Delta^q_+ \wedge M, R) \xrightarrow{f} (R^M)_q.
\]

Using the diagram, we get $z_q \circ (d^M)_q = (k^M)_q$. \qed

§ 6. Harmonic cofibrations

Let $T$ be a simplicial crew and $U$ be a simplicial abelian group. A cofibration $d: T \to U$ is called $r$-harmonic ($r \in \mathbb{N}$) if, for any compact simplicial crews $L$ and $M$ and any isotypical cofibration $j: L \to M$, there exist a simplicial archism $x: T^L \to T^M$ and an $r$-gentle quasisimplicial archism $y: U^L \to U^M$ such that $d^M \circ x = y \circ d^L$ and $T^j \circ x = \text{id}$.

\[
\begin{array}{ccc}
T^L & \xrightarrow{x} & T^M \\
\downarrow & & \downarrow \\
T^j & \xrightarrow{d^M} & T^M \\
\end{array}
\begin{array}{ccc}
U^L & \xrightarrow{y} & U^M \\
\downarrow & & \downarrow \\
U^j & \xrightarrow{d^M} & U^M \\
\end{array}
\]

A cofibration is harmonic if it is $r$-harmonic for some $r \in \mathbb{N}$.

By the height of a 0-connected space $Y$ we mean the supremum of those $q \in \mathbb{N}$ for which $\pi_q(Y) \neq 1$ (the supremum of the empty set is 0).

6.1. Lemma. Let $p$ be a prime and $Y$ be a connected CW-complex of finite height with $p$-finite homotopy groups. Then there exist a gradual simplicial crew $T$ with $|T| \simeq Y$, a gradual simplicial $\mathbb{Z}_p$-module $U$, and a harmonic cofibration $d: T \to U$.  

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Proof. (Induction along the Postnikov decomposition of $Y$ with fibres of the form $\mathcal{K}(\mathbb{Z}_p, q)$..) Let $n$ be the height of $Y$. If $n = 0$, then $Y$ is contractible, we put $T = U = 0$ and that is all. Otherwise, choose an order $p$ element $e \in \pi_n(Y)$ fixed by the canonical action of $\pi_1(Y)$. Its existence follows from the well-known congruence $|\text{Fix}_G X| \equiv |X|$ (mod $p$) for an action of a $p$-finite group $G$ on a finite set $X$ (cf. the remark in [2, Ch. II, Example 5.2(iv)]). We attach cells to $Y$ to get a map $Y \to \tilde{Y}$ inducing isomorphisms on $\pi_q$, $q \neq n$, and an epimorphism with the kernel generated by $e$ on $\pi_n$. The space $Y$ is homotopy equivalent to the homotopy fibre of some map $\tilde{Y} \to \mathcal{K}(\mathbb{Z}_p, n + 1)$ [6, Lemma 4.70].

We assume (as an induction hypothesis) that there are gradual simplicial crew $\tilde{T}$ with $|\tilde{T}| \simeq \tilde{Y}$, gradual simplicial $\mathbb{Z}_p$-module $\tilde{U}$, and $r$-harmonic ($r \geq 1$) cofibration $d: \tilde{T} \to \tilde{U}$.

Let $R$ be a gradual $(n+1)$-soluble simplicial $\mathbb{Z}_p$-module with $|R| \simeq \mathcal{K}(\mathbb{Z}_p, n + 1)$, $Q$ be a gradual simplicial $\mathbb{Z}_p$-module isotypic to a point, and $c: Q \to R$ be a fibring simplicial $\mathbb{Z}_p$-homomorphism (see [3]). There is a Cartesian square of simplicial crews and archisms

$$
\begin{array}{ccc}
T & \overset{h}{\longrightarrow} & Q \\
\downarrow{f} & & \downarrow{c} \\
\tilde{T} & \overset{k}{\longrightarrow} & R,
\end{array}
$$

where $|T| \simeq Y$. Put $U = \tilde{U} \times Q$. Let $\mathbb{Z}_p$-homomorphisms $a: U \to \tilde{U}$ and $b: U \to Q$ be the projections. Let $d: T \to U$ be the simplicial archism given by the conditions $a \circ d = \tilde{d} \circ f$ and $b \circ d = h$. Obviously, $d$ is a cofibration.

By Lemma 5.1, for some $s \geq 1$ there is an $s$-gentle quasisimplicial archism $w: \tilde{U} \to R$ such that $w \circ \tilde{d} = k$.

We show that $d$ is $r$-s-harmonic. Take compact simplicial crews $L$ and $M$ and an isotypical cofibration $j: L \to M$. We need a simplicial archism $x: T^L \to T^M$ and an $r$-s-harmonic quasisimplicial archism $y: U^L \to U^M$ such that $d^M \circ x = y \circ d^L$ and $T^j \circ x = \text{id}$. Since $d$ is $r$-harmonic, there are a simplicial archism $\tilde{x}: \tilde{T}^L \to \tilde{T}^M$ and an $r$-gentle quasisimplicial archism $\tilde{y}: \tilde{U}^L \to \tilde{U}^M$ such that $d^M \circ \tilde{x} = \tilde{y} \circ d^L$ and $T^j \circ \tilde{x} = \text{id}$.

We have the commutative square of simplicial $\mathbb{Z}_p$-modules and $\mathbb{Z}_p$-homomorphisms with a half-key

$$
\begin{array}{ccc}
Q^L & \overset{t'}{\longrightarrow} & Q^M \\
\downarrow{c^L} & & \downarrow{c^M} \\
R^L & \overset{t''}{\longrightarrow} & R^M
\end{array} = \begin{array}{ccc}
Q^L & \overset{Q^L}{\longrightarrow} & Q^M \\
\downarrow{c^L} & & \downarrow{c^M} \\
R^L & \overset{R^L}{\longrightarrow} & R^M
\end{array}
$$

(the half-key exists by Lemma 4.6). We have the simplicial archism

$$
u = t' \circ h^L + t'' \circ k^M \circ \tilde{x} \circ f^L: T^L \to Q^M.$$
We have \( c^L \circ h^L = k^L \circ f^L = k^L \circ T^j \circ \bar{x} \circ f^L = R^i \circ k^M \circ \bar{x} \circ f^L \). Therefore, by Lemma 4.1, \( Q^j \circ u = h^L \) and \( c^M \circ u = k^M \circ \bar{x} \circ f^L \).

Define the desired \( x \) by the conditions \( f^M \circ x = \bar{x} \circ f^L \) and \( h^M \circ x = u \):

\[
\begin{array}{c}
\xymatrix{T^M \ar[r]^{h^M} \ar[d]_{f^M} & Q^M \ar[d]_{c^M} \\
T^L \ar[r]_{k^M} \ar[u]_{\bar{x} \circ f^L} & R^M.
\end{array}
\]

This is possible because the square is Cartesian and the conditions are compatible: \( k^M \circ \bar{x} \circ f^L = c^M \circ u \). We have \( T^j \circ x = \text{id} \) because \( f^L \circ T^j \circ x = T^j \circ f^M \circ x = T^j \circ \bar{x} \circ f^L = f^L \) and \( h^L \circ T^j \circ x = Q^j \circ h^M \circ x = Q^j \circ u = h^L \).

By Lemma 5.3, there is an \( s \)-gentle quasisimplicial archism \( z: U^M \to R^M \) such that \( z \circ \bar{d}^M = k^M \). We have the quasisimplicial archism \( v = t' \circ b^L + t'' \circ \bar{y} \circ a^L: U^L \to Q^M \).

By Lemma 3.3, it is \( rs \)-gentle.

Define the desired \( y \) by the conditions \( a^M \circ y = \bar{y} \circ a^L \) and \( b^M \circ y = v \):

\[
\begin{array}{c}
\xymatrix{U^M \ar[r]^{h^M} \ar[d]_{a^M} & Q^M \\
\bar{U}^M \ar[u]_{\bar{y} \circ a^L} \ar[r]_{a^M} & \bar{Q}^M.
\end{array}
\]

This is possible because \((a^M, b^M): U^M \to \bar{U}^M \times Q^M\) is an isomorphism. Obviously, \( y \) is \( rs \)-gentle. We have \( d^M \circ x = y \circ d^L \) because \( a^M \circ d^M \circ x = d^M \circ f^M \circ x = \bar{y} \circ d^L \circ f^L = \bar{y} \circ a^L \circ d^L = a^M \circ y \circ d^L \) and \( b^M \circ d^M \circ x = h^M \circ x = u = t' \circ h^L + t'' \circ k^M \circ \bar{x} \circ f^L = t' \circ h^L + t'' \circ \bar{x} \circ f^L = t' \circ h^L + t'' \circ \bar{y} \circ d^L \circ f^L = t' \circ b^L \circ d^L + t'' \circ \bar{y} \circ a^L \circ d^L = v \circ d^L = b^M \circ y \circ d^L \).
§ 7. Two filtrations of the module $C_0(U^K)$

7.1. Lemma. Let $U_i, i \in I$, be a finite collection of abelian groups. Put

$$U_J = \bigoplus_{i \in J} U_i, \quad J \subseteq I,$$

and $U = U_I$. Let $p_J : U \to U_J$ be the projections. Then for any $r \in \mathbb{N}$

$$\bigcap_{J \subseteq I : |J| \leq r} \ker(p_J)_R \subseteq \mathcal{R}[U]^{r+1}.\]$$

in the $\mathcal{R}$-algebra $\mathcal{R}[U]$.\]

Proof. Let $s_J : U_J \to U$ be the canonical embeddings. Put $q_J = s_J \circ p_J : U \to U$. We assume $|I| > r$ (otherwise, the assertion is trivial). For $u \in U$, we have (cf.
\[ \sum_{J \subseteq I: |J| \leq r} (-1)^{r-|J|} \binom{|J| - |I| - 1}{r - |J|} |q_J(u)| = \sum_{J \subseteq I} \sum_{M \subseteq I: M \supseteq J, |M| > r} (-1)^{|M| - |J|} |q_J(u)| = \sum_{M \subseteq I: |M| > r} \left( \sum_{J \subseteq M} (-1)^{|M| - |J|} |q_J(u)| \right) = \sum_{M \subseteq I: |M| > r} \prod_{i \in M} (|q_i(u)| - 1) \in R[U]^{|r+1|}. \]

It follows that for \( w \in R[U] \) we have
\[ w - \sum_{J \subseteq I: |J| \leq r} (-1)^{r-|J|} \binom{|J| - |I| - 1}{r - |J|} (q_J)_R(w) \in R[U]^{|r+1|}. \]

If
\[ w \in \bigcap_{J \subseteq I: |J| \leq r} \ker(p_J)_R, \]
then, using that \( \ker(p_J)_R = \ker(q_J)_R \), we get \( w \in R[U]^{|r+1|} \).

For a simplicial abelian group \( V \), the module \( C_0(V) = R[V_0] \) has the filtration \( C_0^s(V) = R[V_0]^s, s \in \mathbb{N} \).

**7.2. Corollary.** Let \( K \) be a compact simplicial crew, \( E \) be a field, \( U \) be a simplicial \( E \)-module, and \( r \in \mathbb{N} \) be a number. Consider the \( R \)-homomorphism
\[ C_0(U^K) \xrightarrow{K_{\mu_r}} \text{Hom}_0(C_\ast(K^r), C_\ast(U^r)). \]

Then \( \ker\left( K_{\mu_r} \right) \subseteq C_0^{|r+1|}(U^K) \).

**Proof.** Take an element \( B \in \ker\left( K_{\mu_r} \right) \). We show that \( B \in C_0^{|r+1|}(U^K) \).

There is \( n \in \mathbb{N} \) such that the simplicial crew \( K \) is generated by a finite collection of \( n \)-simplices: \( g_i \in K_n, i \in I \). We have the \( E \)-homomorphism \( h: (U^K)_0 \to U_n^1, b \mapsto (b(g_i))_{i \in I} \). It is injective. Therefore, there is an \( E \)-homomorphism \( f: U_n^1 \to (U^K)_0 \) such that \( f \circ h = \text{id} \). It suffices to show that \( h_R(B) \in R[U_n^1]^{|r+1|} \). Indeed, then \( B = f_R(h_R(B)) \in R[(U^K)_0]^{|r+1|} = C_0^{|r+1|}(U^K) \).

For \( J \subseteq I \), let \( p_J: U_n^1 \to U_{n_J}^I \) be the projection. Take \( J \subseteq I \) with \( |J| \leq r \).

By Lemma 7.1, it suffices to verify that \( (p_J)_R(h_R(B)) = 0 \).

Choose a function \( t: J \to \{1, \ldots, r\} \) and a simplex \( k = (k_1, \ldots, k_r) \in K_n^r \) such that \( k_{t(i)} = g_i, i \in J \). We have the \( E \)-homomorphism \( U_n^1: U_{n}^J \to U_{n_J}^I \), the
\( R \)-homomorphism \((U^r_n)_R : C_n(U^r) = R[U^r_n] \to R[J^r_n] \), and the commutative diagram

\[
\begin{array}{ccc}
R[(U^K)_0] & \xrightarrow{h_R} & R[J^r_n] \\
\downarrow^K_{U^r} & & \downarrow^{(p_J)_R} \\
\text{Hom}_0(C_*(K^r), C_*(U^r)) & \xrightarrow{\text{m} \to (U^r_n)_R(\text{v}([k]))} & R[J^r_n].
\end{array}
\]

Since \( K^r_{U^r}(B) = 0 \), we get \( (p_J)_R(h_R(B)) = 0 \).

\[ \square \]

§ 8. Simplicial approximation

8.1. Lemma. Let \( K \) be a compact simplicial crew, \( W \) be a simplicial crew, and \( f : |K| \to |W| \) be a map. Then there exist a compact simplicial crew \( L \), an isotypy \( e : L \to K \), and a simplicial archism \( g : L \to W \) such that \( f \circ |e| \sim |g| \).

See [8, Corollary 4.8].

8.2. Lemma. Let \( K \) be a compact simplicial crew, \( T \) be a simplicial crew, and \( r \in \mathbb{N} \) be a number. Then, for any \( A \in \ker |K|\mu_r \), there exist a compact simplicial crew \( L \), an isotypy \( e : L \to K \), and an element \( B \in \ker T\mu_r \) such that

\[
\begin{align*}
\text{Hom}_0(C_*(L^r), C_*(T^r)) & \xrightarrow{L\mu_r} C_0(T^L) \\
& \xrightarrow{\text{v} \to (U^r_n)_R(\text{v}([k]))} H_0(T^L) \\
\text{Hom}_0(|K|^{r}, C_0(|T|^{|K|})) & \xrightarrow{|K|\mu_r} C_0(|T|^{|K|}) \\
& \xrightarrow{\text{v} \to (U^r_n)_R(\text{v}([k]))} H_0(|T|^{|K|}).
\end{align*}
\]

Proof. We have

\[
A = \sum_{i=1}^m v_i a_i,
\]

where \( m \in \mathbb{N} \), \( v_i \in \mathbb{R} \), and \( a_i \in |T|^{|K|} \). For \( x \in |K| \), define an equivalence (relation) \( c(x) \) on the set \( I = \{1, \ldots, m\} : c(x) = \{(i,j) : a_i(x) = a_j(x)\} \). Put \( E = \{c(x) : x \in |K|\} \).

We call an equivalence on \( I \) neutral if

\[
\sum_{i \in J} v_i = 0
\]
for all its classes $J \subseteq I$. We show that for any $h_1, \ldots, h_r \in E$ the equivalence $h = h_1 \cap \ldots \cap h_r$ is neutral. For each $s = 1, \ldots, r$, there is a point $x_s \in |K|$ such that $h_s = c(x_s)$. Put $x = (x_1, \ldots, x_r) \in |K|^r$. In $C_0(|T|^r)$, we have
\[ \sum_{i \in I} v_i [a_i^e(x)] = \frac{|K|}{|T|^r} \mu_r(A) = 0. \]

It follows that $h$ is neutral because
\[ a_i^e(x) = a_j^e(x) \iff (i, j) \in h \]
for $i, j \in I$.

For each equivalence $h$ on $I$, there is the corresponding simplicial subcrew $V(h) \subseteq T^m$ (the diagonal):
\[ V(h)_q = \{ (t_1, \ldots, t_m) \in T^m : t_i = t_j \text{ for all } (i, j) \in h \}. \]

Put
\[ W = \bigcup_{h \in E} V(h) \subseteq T^m. \]

We have the maps $a = (a_1, \ldots, a_m) : |K| \to |T|^m$ and $\tilde{a} = d^{-1} \circ a : |K| \to |T^m|$, where $d : |T^m| \to |T|^m$ is the canonical bijective map. For $x \in |K|$, we have $\tilde{a}(x) \in V(c(x))$. Therefore $\text{im} \tilde{a} \subseteq |W|$. Using Lemma 8.1, we find a compact simplicial crew $L$, an isotypy $e : L \to K$, and a simplicial archism $b = (b_1, \ldots, b_m) : L \to T^m$ such that $\text{im} b \subseteq W$ and $\tilde{a} \circ e \approx |b|$. Put
\[ B = \sum_{i=1}^m v_i [b_i]. \]

We have $a_i \circ e \approx |b_i|$. Therefore $H_0(|T|^e)(\frac{|K|}{|T|^r} \mu(A)) = \| f : \mu(B) \|$. We show that $K_{\mu_r}(B) = 0$. For $k = (k_1, \ldots, k_r) \in K_r^r$ ($q \in \mathbb{N}$), we have
\[ K_{\mu_r}(B)(\sum_{i=1}^m v_i [b_i^r(k)]) = \sum_{i=1}^m v_i [b_i^r(k)]. \]

Take $s = 1, \ldots, r$. Since $b \subseteq W$, there is $h_s \in E$ such that $b(k_s) \in V(h_s)$. Therefore, the function $i \mapsto b_i(k_s)$ is subordinate to (i.e. constant on the classes of) the equivalence $h_s$. Since $b_i^r(k) = (b_i(k_1), \ldots, b_i(k_r))$, the function $i \mapsto b_i^r(k)$ is subordinate to the equivalence $h = h_1 \cap \ldots \cap h_r$. Since $h$ is neutral, we get $K_{\mu_r}(B)(\sum_{i=1}^m v_i [b_i^r(k)]) = 0$. \hfill \Box

§ 9. The inclusion $\ker X_{\mu_r} \subseteq \ker Y_{\nu}$ for large $r$

9.1. Lemma. Let $X, Y, \hat{X}$, and $\hat{Y}$ be spaces. Suppose that $X \simeq \hat{X}$ and $Y \simeq \hat{Y}$. Then, for any $r \in \mathbb{N}$, we have
\[ \ker X_{\mu_r} \subseteq \ker Y_{\nu} \iff \ker \hat{X}_{\mu_r} \subseteq \ker \hat{Y}_{\nu}. \]
Proof. There are homotopy equivalences \( k: X \to \tilde{X} \) and \( h: \tilde{Y} \to Y \). We have the commutative diagram of \( R \)-modules and \( R \)-homomorphisms:

\[
\begin{array}{cccccc}
\text{Hom}(C_0(\tilde{X}^r), C_0(\tilde{Y}^r)) & \xrightarrow{\tilde{X}_r^\mu} & C_0(\tilde{Y}^X) & \xrightarrow{\tilde{X}_r^\nu} & H_0(\tilde{Y}^X) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}(C_0(X^r), C_0(Y^r)) & \xrightarrow{X_r^\mu} & C_0(Y^X) & \xrightarrow{X_r^\nu} & H_0(Y^X), \\
\end{array}
\]

where the vertical arrows are induced by \( k \) and \( h \). Since \( H_0(h^k) \) is an isomorphism, we get the implication \( \Rightarrow \). The implication \( \Leftarrow \) is analogous. 

Let \( p \) be a prime. Assume \( R = \mathbb{Z}_p \).

9.2. Let \( X \) be a compact CW-complex and \( Y \) be a connected CW-complex of finite height with \( p \)-finite homotopy groups. Then, for any sufficiently large \( r \in \mathbb{N} \), we have \( \ker \tilde{X}_r^\mu \subseteq \ker \tilde{X}_r^\nu \) in the diagram

\[
\begin{array}{cccccc}
\text{Hom}(C_0(\tilde{X}^r), C_0(\tilde{Y}^r)) & \xrightarrow{\tilde{X}_r^\mu} & C_0(\tilde{Y}^X) & \xrightarrow{\tilde{X}_r^\nu} & H_0(\tilde{Y}^X) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}(C_0(X^r), C_0(Y^r)) & \xrightarrow{X_r^\mu} & C_0(Y^X) & \xrightarrow{X_r^\nu} & H_0(Y^X), \\
\end{array}
\]

Proof. By Lemma 6.1, for some \( s \in \mathbb{N} \), there are a gradual simplicial crew \( T \) with \( |T| \simeq Y \), a gradual simplicial \( \mathbb{Z}_p \)-module \( U \), and an \( s \)-harmonic cofibration \( d: T \to U \). We have \( X \simeq |K| \) for some compact simplicial crew \( K \). Obviously, \((U^K)_0\) is a finite \( \mathbb{Z}_p \)-module. By Lemma 3.1, \( C_0^{t+1}(U^K) = 0 \) for some \( t \in \mathbb{N} \).

Take \( r \geq st \). We show that \( \ker \tilde{X}_r^\mu \subseteq \ker \tilde{X}_r^\nu \) in the diagram

\[
\begin{array}{cccccc}
\text{Hom}(C_0(|K|^r), C_0(|T|^r)) & \xrightarrow{|K|^r_\mu} & C_0(|T|^{|K|}) & \xrightarrow{|K|^r_\nu} & H_0(|T|^{|K|}) \\
\end{array}
\]

This will suffice by Lemma 9.1.

Take an element \( A \in \ker \tilde{X}_r^\mu \). We show that \( A \in \ker \tilde{K}_r^\mu \). By Lemma 8.2, there are a compact simplicial crew \( L \), an isotypy \( e: L \to K \), and an element \( B \in \ker \tilde{T}_r^\nu \) such that \( H_0(|T|^e)(\tilde{X}_r^\mu(A)) = \|\tilde{T}_r^\nu(B)\| \). Since \( |e| \) is a homotopy equivalence, \( H_0(|T|^e) \) is an isomorphism. Therefore it suffices to show that \( \tilde{T}_r^\nu(B) = 0 \).

Let a simplicial crew \( M \) be the (reduced) cylinder of \( e \). We have the homotopy commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{e} & L \\
\downarrow & & \downarrow \quad j \\
\uparrow i & & M \\
\end{array}
\]
where \( i \) and \( j \) are the canonical cofibrations. By the definition of a cylinder, \( i \) is an isotypy. Since \( e \) is an isotypy, \( j \) is an isotypy too. Since \( d \) is \( s \)-harmonic, there is the commutative diagram

\[
\begin{array}{ccc}
T^L & \xrightarrow{id} & T^L \\
\downarrow{d^L} & & \downarrow{d^M} \\
U^L & \xrightarrow{x} & U^M,
\end{array}
\]

where \( x \) is a simplicial archism and \( y \) is an \( s \)-gentle quasisimplicial archism. We have the commutative diagram of \( \mathbb{Z}_p \)-homomorphisms:

\[
\begin{array}{cccccc}
\text{Hom}_0(C_*(L'), C_*(T')) & \xrightarrow{L_{\mu'}} & C_0(T^L) & \xrightarrow{B} & C_0(T^L) & \xrightarrow{B_1} & C_0(T^L) \\
\downarrow{C_0(d_1)} & & \downarrow{C_0(d_1)} & & \downarrow{C_0(d_1)} & & \downarrow{C_0(d_1)} \\
\text{Hom}_0(C_*(L'), C_*(U')) & \xrightarrow{B_2} & C_0(U^L) & \xrightarrow{B_2} & C_0(U^L) & \xrightarrow{B_2} & C_0(U^L),
\end{array}
\]

where the vertical arrows are induced by the cofibration \( d \): \( B_1, \ldots, B_2' \) are the images of \( B \) in the corresponding modules. Since \( L_{\mu'}(B') = 0 \), we have \( L_{\mu'}(B') = 0 \). By Corollary 7.2, \( B' \in C_0^{t+1}(U^L) \). Since \( r \geq s \) and the archism \( y_0 \) is \( s \)-gentle, we have, by Corollary 3.4, \( B_1' \in C_0^{t+1}(U^M) \). Since \( (U^t)_0 \) is a homomorphism, \( B_2' \in C_0^{t+1}(U^K) \). We have \( C_0^{t+1}(U^K) = 0 \). It follows that \( B_2' = 0 \). Since \( d \) is a cofibration, \( C_0(d_1) \) is injective. Therefore \( B_2 = 0 \).

We have the commutative diagram of \( \mathbb{Z}_p \)-homomorphisms

\[
\begin{array}{cccccc}
C_0(T^L) & \xrightarrow{id} & C_0(T^L) & \xrightarrow{B} & C_0(T^L) & \xrightarrow{L_{\nu'}} & H_0(T^L) \\
\downarrow{C_0(x)} & & \downarrow{C_0(x)} & & \downarrow{C_0(x)} & & \downarrow{C_0(x)} \\
C_0(T^M) & \xrightarrow{B_1} & C_0(T^M) & \xrightarrow{M_{\nu'}} & H_0(T^M) & \xrightarrow{H_0(T^M)} & H_0(T^M) \\
\downarrow{C_0(T^M)} & & \downarrow{C_0(T^M)} & & \downarrow{C_0(T^M)} & & \downarrow{C_0(T^M)} \\
C_0(T^K) & \xrightarrow{B_2} & C_0(T^K) & \xrightarrow{K_{\nu'}} & H_0(T^K) & \xrightarrow{H_0(T^K)} & H_0(T^K).
\end{array}
\]

Since \( B_2 = 0 \), we get \( L_{\nu'}(B) = 0 \).

Consider the filtration of the complex \( C_*(Y^X) \) formed by the kernels of the \( \mathbb{Z}_p \)-homomorphisms

\[
\begin{array}{cccccc}
C_q(Y^X) & \xrightarrow{i_q} & C_0(Y^{\Delta_q^+ \wedge X}) & \xrightarrow{\Delta_q^+ \wedge X \times Y_{\mu'}} & \text{Hom}(C_0((\Delta_q^+ \wedge X)^t), C_0(Y^t)),
\end{array}
\]

where \( i_q \) are the obvious isomorphisms. Does this filtration converge?
\[ \text{\textsection 10. Deducing Theorem 1.2 from claim 9.2} \]

10.1. Lemma. Let \( X, Y, \overline{X}, \) and \( \overline{Y} \) be spaces, \( k: X \to \overline{X} \) and \( h: \overline{Y} \to Y \) be maps, \( V \) be an abelian group, and \( f: [X, Y] \to V \) be an invariant. Consider the invariant \( \tilde{f}: [\overline{X}, \overline{Y}] \to V, \tilde{u} \mapsto f([h \circ \tilde{u} \circ [k]]) \). Then \( \deg \tilde{f} \leq \deg f \).

Proof. Take \( r \in \mathbb{N} \). The maps \( k \) and \( h \) induce a homomorphism
\[
t: \text{Hom}(C_0(\overline{X}^r), C_0(\overline{Y}^r)) \to \text{Hom}(C_0(X^r), C_0(Y^r)).
\]
We have \( t(C_0(\tilde{a}^r)) = C_0((h \circ \tilde{a} \circ k)^r) \), \( \tilde{a} \in \overline{Y}^X \). Assume that \( \deg f \leq r \). There is a homomorphism \( l: \text{Hom}(C_0(X^r), C_0(Y^r)) \to V \) such that \( f([a]) = l(C_0(a^r)) \) for all \( a \in Y^X \). Consider the homomorphism \( \tilde{l} = l \circ t: \text{Hom}(C_0(\overline{X}^r), C_0(\overline{Y}^r)) \to V \).

Proof of Theorem 1.2. (1) Case of \( Y \) of finite height. It suffices to show that the “universal” invariant \( F: [X, Y] \to H_0(Y^X; \mathbb{Z}_p), u \mapsto [u], \) has finite degree. For \( r \in \mathbb{N} \) we have the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathbb{Z}(C_0(X^r; \mathbb{Z}), C_0(Y^r; \mathbb{Z})) & \xrightarrow{\tilde{X}Y_{\mu r}} & \text{Hom}_\mathbb{Z}(C_0(Y^X; \mathbb{Z}), C_0(X^r; \mathbb{Z})) \\
m' \downarrow & & \downarrow m \\
\text{Hom}_\mathbb{Z}(C_0(X^r; \mathbb{Z}_p), C_0(Y^r; \mathbb{Z}_p)) & \xrightarrow{\tilde{X}Y_{\mu r}} & \text{Hom}_\mathbb{Z}(C_0(Y^X; \mathbb{Z}_p), C_0(X^r; \mathbb{Z}_p)) \\
\end{array}
\]

where \( m \) and \( m' \) are the homomorphisms of reduction modulo \( p \); the tilde over \( \mu \) in the upper row means “over \( \mathbb{Z} \)”. By claim 9.2, we have \( \ker \tilde{X}_{\mu r} \subseteq \ker \tilde{Y}_{\nu} \) for sufficiently large \( r \). Then there is a \( \mathbb{Z}_p \)-homomorphism \( t: \text{Hom}_\mathbb{Z}(C_0(X^r; \mathbb{Z}_p), C_0(Y^r; \mathbb{Z}_p)) \to H_0(Y^X; \mathbb{Z}_p) \) such that \( t \circ \tilde{X}_{\mu r} = \tilde{Y}_{\nu} \). For \( a \in Y^X \), we have \( F([a]) = (\tilde{X}_{\nu} \circ m)([a]) = (t \circ m') \circ \tilde{X}_{\mu r})([a]) = (t \circ m')(C_0(a^r; \mathbb{Z}_p)). \) Therefore \( \deg F \leq r \).

(2) General case. There are a connected CW-complex \( Y \) of finite height with \( p \)-finite homotopy groups and a \((\dim X + 1)\)-connected map \( h: Y \to \overline{Y} \) (\( \overline{Y} \) is obtained from \( Y \) by attaching cells of high dimensions). The induced function \( h_\#: [X, Y] \to [X, \overline{Y}] \) is bijective. Consider the invariant \( \tilde{f} = f \circ h_\#^{-1}: [X, Y] \to \mathbb{Z}_p \). By Lemma 10.1, \( \deg \tilde{f} \leq \deg \tilde{f} \). By (1), \( \deg \tilde{f} < \infty \).

\[ \text{\textsection 11. Deducing Theorem 1.1 from Theorem 1.2} \]

11.1. Lemma [2, Ch. VI, Proposition 8.6]. Let \( X \) be a connected compact CW-complex, \( Y \) be a nilpotent connected CW-complex with finitely generated homotopy groups, and \( u_1, u_2 \in [X, Y] \) be distinct classes. Then, for some prime \( p \), there exist a connected CW-complex \( \overline{Y} \) with \( p \)-finite homotopy groups and a map \( h: Y \to \overline{Y} \) such that \( [h] \circ u_1 \neq [h] \circ u_2 \) in \([X, \overline{Y}]\). □

Proof of Theorem 1.1. By Lemma 11.1, for some prime \( p \) there are a connected CW-complex \( Y \) with \( p \)-finite homotopy groups, and a map \( h: Y \to \overline{Y} \)
such that the classes \( \bar{u}_i = [h] \circ u_i, \ i = 1, 2, \) are distinct. There is an invariant \( f: [X, Y] \to \mathbb{Z}_p \) such that \( f(\bar{u}_1) \neq f(\bar{u}_2). \) By Theorem 1.2, \( \text{Deg} f < \infty. \) Consider the invariant \( f = f \circ h_\#: [X, Y] \to \mathbb{Z}_p. \) By Lemma 10.1, \( \text{Deg} f < \infty. \) We have \( f(u_1) = f(\bar{u}_1) \neq f(\bar{u}_2) = f(u_2). \)

\[\text{§ 12. Properties of finite-degree invariants}\]

Put \( E = \{0, 1\} \subseteq \mathbb{Z}. \) For \( e = (e_1, \ldots, e_n) \in E^n, \) put \( |e| = e_1 + \ldots + e_n. \)

Consider a wedge of spaces \( W = T_1 \vee \ldots \vee T_n. \) Let \( \text{in}_k^W: T_k \to W \) be the inclusions. For \( e \in E^n, \) put \( M_e^W = m_1 \vee \ldots \vee m_n: W \to W, \) where \( m_k : T_k \to T_k \) is: the identity if \( e_k = 1, \) and the constant map otherwise.

12.1. **Lemma.** Let \( X \) and \( Y \) be spaces, \( V \) be an abelian group, \( f: [X, Y] \to V \) be an invariant of degree at most \( r \in \mathbb{N}, \ W = T_1 \vee \ldots \vee T_{r+1} \) be a wedge of spaces, and \( k: X \to W \) and \( h: W \to Y \) be maps. Then

\[
\sum_{e \in E^{r+1}} (-1)^{|e|} f([h \circ M_e^W \circ k]) = 0.
\]

**Proof.** Consider the invariant \( \tilde{f}: [W, W] \to V, \tilde{u} \mapsto f([h] \circ \tilde{u} \circ [k]). \) We show that

\[
\sum_{e \in E^{r+1}} (-1)^{|e|} \tilde{f}([M_e^W]) = 0.
\]

By Lemma 10.1, \( \text{Deg} \tilde{f} \leq r, \) i. e. there is a homomorphism \( \iota: \text{Hom}(C_0(W^r), C_0(W^r)) \to V \) such that \( \tilde{f}([\tilde{u}]) = \iota(C_0(\tilde{u}^r)) \) for all \( \tilde{u} \in W^W \) (hereafter, \( R = \mathbb{Z}. \)) Therefore it suffices to show that

\[
\sum_{e \in E^{r+1}} (-1)^{|e|} C_0((M_e^W)^r) = 0.
\]

Take a point \( w = (w_1, \ldots, w_r) \in W^r. \) There is \( s \in \{1, \ldots, r+1\} \) such that \( \{w_1, \ldots, w_s\} \cap T_s \subseteq \{\text{basepoint}\}. \) The point \( (M_e^W)^r(w) \in W^r \) does not depend on the \( s \)th component of \( e. \) Since \( C_0((M_e^W)^r(|w|)) = [(M_e^W)^r(w)], \) it follows that

\[
\sum_{e \in E^{r+1}} (-1)^{|e|} C_0((M_e^W)^r(|w|)) = 0.
\]

\[\square\]

**Maps** \( S^n \to Y. \) In this subsection, we use multiplicative notation for homotopy groups.

12.2. **Lemma.** Let \( n \geq 1 \) be a number, \( Y \) be a space, \( V \) be an abelian group, and \( f: \pi_n(Y) \to V \) be an invariant of degree at most \( r \in \mathbb{N}. \) Then \( f \) is \( r \)-gentle.

**Proof.** Take elements \( u_1, \ldots, u_{r+1} \in \pi_n(Y). \) We show that \( + f((1 - |u_1|) \ldots (1 - |u_{r+1}|)) = 0. \) Put \( W = S^n \vee \ldots \vee S^n \) \( (r + 1 \) summands). Let \( k: S^n \to W \) be
a map with $[k] = [\text{in}^W_1] \ldots [\text{in}^W_{r+1}]$ in $\pi_n(W)$, and $h: W \to Y$ be a map with $[h \circ \text{in}^W_s] = u_s$ in $\pi_n(Y)$. By Lemma 12.1,

$$
\sum_{c \in E^{r+1}} (-1)^{\hat{c}} f([h \circ M^W_c \circ k]) = 0.
$$

This is what we need because $[h \circ M^W_c \circ k] = u_1^{e_1} \ldots u_{r+1}^{e_{r+1}}$ in $\pi_n(Y)$. $\blacksquare$

We denote the Whitehead product by the sign $\ast$.

12.3. Lemma. Let $m, n \geq 1$ be numbers, $Y$ be a space, and $f: \pi_{m+n-1}(Y) \to V$ be an invariant of degree at most $r \in \mathbb{N}$. Then the function $b: \pi_m(Y) \times \pi_n(Y) \to V$, $(u, v) \mapsto f(u \ast v)$, is $r$-gentle.

Proof. Assume $r > 0$ (otherwise, the claim is trivial). Take elements $u_1, \ldots, u_p \in \pi_m(Y)$ and $v_1, \ldots, v_q \in \pi_n(Y)$, where $p, q \geq 0$ and $p + q = r + 1$. By Lemma 3.10, it suffices to show that $\hat{b}(1 - [\hat{u}_1]) \ldots (1 - [\hat{u}_p])(1 - [\hat{v}_1]) \ldots (1 - [\hat{v}_q]) = 0$.

Put $W = S^m \vee \ldots \vee S^m \vee S^n \vee \ldots \vee S^n$ ($p$ times $S^m$ and $q$ times $S^n$). Let $k: S^{m+n-1} \to W$ be a map with $[k] = ([\text{in}^W_1] \ldots [\text{in}^W_p]) \ast ([\text{in}^W_{p+1}] \ldots [\text{in}^W_{r+1}])$ in $\pi_{m+n-1}(W)$ and $h: W \to Y$ be a map with $[h \circ \text{in}^W_s] = u_s$ in $\pi_n(Y)$ for $s = 1, \ldots, p$ and $[h \circ \text{in}^W_s] = v_t$ in $\pi_n(Y)$ for $t = 1, \ldots, q$. By Lemma 12.1,

$$
\sum_{c \in E^{r+1}} (-1)^{\hat{c}} f([h \circ M^W_c \circ k]) = 0.
$$

This is what we need because $[h \circ M^W_c \circ k] = (u_1^{e_1} \ldots u_p^{e_p}) \ast (v_1^{e_{p+1}} \ldots v_{r+1}^{e_{r+1}})$ in $\pi_{m+n-1}(Y)$ and, consequently, $f([h \circ M^W_c \circ k]) = b(u_1^{e_1} \ldots u_p^{e_p}, v_1^{e_{p+1}} \ldots v_{r+1}^{e_{r+1}}) = b(\hat{u}_1^{e_1} \ldots \hat{u}_p^{e_p}, \hat{v}_1^{e_{p+1}} \ldots \hat{v}_{r+1}^{e_{r+1}})$.

Maps $S^{n-1} \times S^n \to S^{n}_{(q)}$. In this subsection, we prove claims 1.5–1.7 and use the objects defined in the corresponding subsection of §1. For $u \in \pi_n(Y)$ and $v \in \pi_{n}(Y)$, the class $(u, v) \in [S^{n} \vee S^{n}, Y]$ is defined in the obvious way.

Let $x: S^n \vee S^{2n-1} \to S^n \times S^{2n-1}$ be the canonical embedding of a wedge in the product. Consider the map $(pr_2, c): S^{n-1} \times S^n \to S^n \times S^{2n-1}$, where $pr_2: S^{n-1} \times S^n \to S^n$ is the projection and $c: S^{n-1} \times S^n \to S^{2n-1}$ is the map defined in §1. There exists a (unique up to homotopy) map $b: S^{n-1} \times S^n \to S^n \times S^{2n-1}$ such that $x \circ b \sim (pr_2, c)$. For $p, q \in \mathbb{Z}$, we have the homotopy classes

$$
v(p, q): S^{n-1} \times S^n \xrightarrow{[b]} S^n \vee S^{2n-1} \xrightarrow{(p, q)} S^n
$$

(wavy arrows present homotopy classes) and $\hat{v}(p, q) = [l] \circ v(p, q) \in [S^{n-1} \times S^n, S^n]$. Obviously, $v(0, q) = u(q)$ and $\hat{v}(0, q) = \hat{u}(q)$. We have $v(p, q) = v(p, 0)$ if $p \mid q$ (the proof is omitted) and $\hat{v}(p, q) = \hat{v}(p, 0)$ if $p \neq 0$ [1, Example 4.6].

Proof of 1.5. Take $q \in \mathbb{Z}$. Put $W = S^n \vee \ldots \vee S^n \vee S^{2n-1}$ ($r$ times $S^n$). Let $d: S^n \vee S^{2n-1} \to W$ be a map with $[d] = ([\text{in}^W_1] \ldots [\text{in}^W_r]) \ast ([\text{in}^W_{r+1}])$. Put

20
Let \( h: W \to S^n \) be a map with \([h] = (i, \ldots, i, qj)\). By Lemma 12.1,

\[
\sum_{e \in E^{r+1}} (-1)^{|e|} f([h \circ M_e^W \circ k]) = 0.
\]

Since \([h \circ M_e^W \circ k] = v(e_1 + \ldots + e_r, e_{r+1}q)\), we have

\[
\sum_{e' \in E^r} (-1)^{|e'|} \sum_{e'' \in E} (-1)^{|e''|} f(v(|e'|, e''q)) = 0.
\]

Assume \( r! \mid q \). If \( e' \neq (0, \ldots, 0) \), the inner sum vanishes because then \(|e'| \mid q\) and, consequently, the class \( v(|e'|, e''q) \) does not depend on \( e'' \). We get \( f(v(0,0)) - f(v(0,q)) = 0 \), i.e., \( f(u(q)) = f(u(0)) \).

Proof of 1.6. Assume \( \text{Deg } f \leq r \in \mathbb{N} \). Take \( q \in \mathbb{Z} \). As in the proof of 1.5, we get

\[
\sum_{e' \in E^r} (-1)^{|e'|} \sum_{e'' \in E} (-1)^{|e''|} f(v(|e'|, e''q)) = 0.
\]

If \( e' \neq (0, \ldots, 0) \), the class \( \bar{v}(|e'|, e''q) \) does not depend on \( e'' \). As in the proof of 1.5, we get \( f(\bar{u}(q)) = f(\bar{u}(0)) \).

Proof of 1.7. Assume \( \text{Deg } f \leq r \in \mathbb{N} \). Consider the invariant \( \tilde{f}: \pi_{2n-1}(S^n) \to Q, \bar{u} \mapsto f(\bar{u} \circ [c]) \). By Lemma 10.1, \( \text{Deg } \tilde{f} \leq r \). By Lemma 12.2, \( \tilde{f} \) is gentle. Consider the function \( F: \mathbb{Z} \to Q, q \mapsto f(u(q)) \). We have \( F(q) = \tilde{f}(qj) \). Therefore \( F \) is gentle, i.e., by Lemma 3.11, is given by a polynomial. By 1.5, \( F(q) = F(0) \) if \( r! \mid q \). It follows that \( F \) is constant.

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