Simple Harish-Chandra modules over the super affine-Virasoro algebras

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Abstract
In this paper, we classify all simple Harish-Chandra modules over the super affine-Virasoro algebra $\hat{\mathcal{L}} = \mathcal{W} \ltimes (\mathfrak{g} \otimes \mathcal{A}) \oplus \mathbb{C}C$, where $\mathcal{A} = \mathbb{C}[t^{\pm 1}] \otimes \Lambda(1)$ is the tensor superalgebra of the Laurent polynomial algebra in even variable $t$ and the Grassmann algebra in odd variable $\xi$. $\mathcal{W}$ is the Lie superalgebra of superderivations of $\mathcal{A}$, and $\mathfrak{g}$ is a finite-dimensional perfect Lie superalgebra.

Keywords: super affine-Virasoro algebra, Witt superalgebra, weight module, cuspidal module

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1. Introduction
Throughout this paper, we denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ and $\mathbb{C}$ the sets of all integers, non-negative integers, positive integers and complex numbers, respectively. All vector spaces and algebras in this paper are over $\mathbb{C}$. A super vector space $V$ is a vector space endowed with a $\mathbb{Z}_2$-gradation $V = V_0 \oplus V_1$. The parity of a homogeneous element $v \in V_i$ is denoted by $|v| = i \in \mathbb{Z}_2$. When we write $|v|$ for an element $v \in V$, we will always assume that $v$ is a homogeneous element. We denote by $U(L)$ the universal enveloping algebra of the Lie (super)algebra $L$. Also, we denote by $\delta_{i,j}$ the Kronecker delta.

Let $\mathcal{A} = \mathbb{C}[t^{\pm 1}] \otimes \Lambda(1)$ be the tensor superalgebra of the Laurent polynomial algebra in even variable $t$ and the Grassmann algebra in odd variable $\xi$, and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional perfect Lie superalgebra (i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$). Then $\mathfrak{g} \otimes \mathcal{A}$ is a Lie superalgebra, which is called the super loop algebra (see [14], also named current Lie superalgebra in [19]), with $[x \otimes a, y \otimes b] = (-1)^{|a||b|}[x, y] \otimes ab$ for all $x, y \in \mathfrak{g}$, $a, b \in \mathcal{A}$. The universal central extensions of $\mathfrak{g} \otimes \mathcal{A}$ equipped with a nondegenerated homogeneous invariant supersymmetric bilinear form on $\mathfrak{g}$ were studied in [19, 20] recently.

Let $\mathcal{W}$ be the Witt superalgebra, i.e. the Lie superalgebra of superderivations of $\mathcal{A}$. Clearly $\mathfrak{g} \otimes \mathcal{A}$ becomes a $\mathcal{W}$-module by the usual actions of $\mathcal{W}$ on $\mathcal{A}$. So we can define a Lie
superalgebra $L$ associated to $\mathfrak{g}$ as $L = \mathcal{W} \ltimes (\mathfrak{g} \oplus A)$. By calculating the second cohomology group $H^2(L, \mathbb{C})$ of $L$, we obtain the universal central extension $\hat{L} = \mathcal{W} \ltimes (\mathfrak{g} \oplus A) \oplus \mathbb{C}C$, which is called the super affine-Virasoro algebra, also named super conformal current algebra in [14]. It can be viewed as a super version of the affine-Virasoro algebra defined in [12] (see also [16]) and it corresponds to a superconformal and chiral invariant 2-dimensional quantum field theory (see [14]).

The affine-Virasoro algebra is the semi-direct sum of the Virasoro algebra and the untwisted affine Lie algebra. There have been many researches on the representation theory of the affine-Virasoro algebras, see references [7, 9, 11, 16], and so on. However, the research concerning the representation theory of the super affine-Virasoro algebra is still seldom. All unitary irreducible representations for the subalgebra $\hat{\mathfrak{g}}_R = R \ltimes (\hat{\mathfrak{g}} \oplus A) \oplus \mathbb{C}C$ of $\hat{L}$ were constructed in [14], where $\hat{\mathfrak{g}}$ is a semisimple Lie algebra and $R$ is the centerless $N = 1$ superconformal algebra, a subalgebra of $\mathcal{W}$.

Based on the classification of simple jet modules introduced by Y.Billig in [1] (see also [8]), a complete classification of simple Harish-Chandra modules over Lie algebra of vector fields on a torus was given in [2] by the so-called $A$-cover theory. As we know, the classification of cuspidal modules is one of the most important steps in the classification of simple Harish-Chandra modules over various Lie (super)algebras. Using the $A$-cover theory, the classifications of simple Harish-Chandra modules over the Witt superalgebra ([22]) (also see [3]), the $N = 1$ superconformal algebra([4, 5]), the map (super)algebra related to the Virasoro algebra ([6]) were given. Certainly such researches for the $N = 1, 2$ superconformal algebras, the affine-Virasoro algebra were first given in [16, 17, 21] by other methods, respectively. Motivated by the above researches, we classify the simple Harish-Chandra modules over the super affine-Virasoro algebra $\hat{L}$ in this paper.

The paper is organized as follows. In Section 2, we give some definitions and preliminaries. In Section 3, we study the central extension of $L$ and get the super affine-Virasoro algebra $\hat{L}$. The $A\mathcal{L}$-modules are studied in Section 4. In Section 5, using the $A$-cover theory and the results of $A\mathcal{L}$-modules, we give the classification of simple cuspidal modules of $L$. Finally, we prove our main theorem in Section 6, see Theorem 6.4.

2. Preliminaries

In this section, we recall some necessary definitions and preliminary results.

2.1. The super affine-Virasoro algebra

Let $A = \mathbb{C}[t^{\pm 1}] \otimes \Lambda(1)$ be the tensor superalgebra of the Laurent polynomial algebra in even variable $t$ and the Grassmann algebra in odd variable $\xi$, and $\mathcal{W}$ be the Witt superalgebra. Denote by $d_i = t^{i+1} \frac{\partial}{\partial t}$, $h_i = t^i \xi \frac{\partial}{\partial \xi}$, $Q_i = t^i \frac{\partial}{\partial t}$, $G_i = t^{i+1} \xi \frac{\partial}{\partial \xi}$ for any $i \in \mathbb{Z}$, and $\Delta = \text{span}_{\mathbb{C}}\{t \frac{\partial}{\partial t}, \frac{\partial}{\partial \xi}\}$. Then $\mathcal{W} = A\Delta = \text{span}_{\mathbb{C}}\{d_i, h_i, Q_i, G_i \mid i \in \mathbb{Z}\}$ with brackets given by

$$[d_i, d_j] = (j - i)d_{i+j}, \quad [d_i, h_j] = j h_{i+j}, \quad [d_i, Q_j] = j Q_{i+j}, \quad [d_i, G_j] = (j - i)G_{i+j},$$

$$[h_i, Q_j] = -Q_{i+j}, \quad [h_i, G_j] = G_{i+j}, \quad [Q_i, G_j] = d_{i+j} + i h_{i+j}.$$

Obviously, $\mathcal{W} = \text{span}_{\mathbb{C}}\{d_i \mid i \in \mathbb{Z}\}$ is the Witt algebra. It is well known that $\mathcal{W}$ is isomorphic to the $N = 2$ (centerless) Ramond algebra (see [15, 17]).
Let \( g = g_0 \oplus g_1 \) be a finite-dimensional perfect Lie superalgebra (i.e. \( g = [g, g] \)). Then \( g \otimes \mathcal{A} \) becomes a \( W \)-module (resp. \( \mathfrak{W} \)-module) by the usual actions of \( \mathcal{W} \) (resp. \( \mathfrak{W} \)) on \( \mathcal{A} \). So we can define a Lie superalgebra \( \mathcal{L} \) (resp. \( \mathfrak{L} \)) associated to \( g \) as \( \mathcal{L} = \mathcal{W} \ltimes (g \otimes \mathcal{A}) \) (resp. \( \mathfrak{L} = \mathfrak{W} \ltimes (g \otimes \mathcal{A}) \)). It is easy to see that \( \mathcal{L} \) is a super subalgebra of \( \mathcal{L} \). In addition to the brackets on \( \mathcal{W} \), the rest of the nonzero brackets in \( \mathcal{L} \) are as follows:

\[
[d_i, x \otimes t^j] = jx \otimes t^{i+j}, \quad [d_i, x \otimes t^j] = jx \otimes t^{i+j},
\]

\[
[h_i, x \otimes t^j] = x \otimes t^{i+j}, \quad [Q_i, x \otimes t^j] = (-1)^{|x|} x \otimes t^{i+j},
\]

\[
[G_i, x \otimes t^j] = (-1)^{|x|} jx \otimes t^{i+j}, \quad [x \otimes t^j, y \otimes t^j] = [x, y] \otimes t^{i+j},
\]

where \( i, j \in \mathbb{Z} \) and \( x, y \in g \).

The universal central extension \( \hat{\mathcal{L}} \) (resp. \( \hat{\mathfrak{L}} \)) of \( \mathcal{L} \) (resp. \( \mathfrak{L} \)) is called a super affine-Virasoro algebra (or super conformal current algebra in \([14]\)).

Denoted by \( \mathcal{K} \) the associative superalgebra generated by \( \mathcal{A} \) and \( \bar{\mathcal{A}} \), which is called the super Weyl algebra. For any \( \lambda \in \mathbb{C} \), let \( \sigma_\lambda \) be the automorphism of \( \mathcal{K} \) with \( \sigma_\lambda(d_i) = d_i + \lambda, \sigma_\lambda(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t}, \sigma_\lambda(\lambda) = \text{id}_{\mathcal{A}} \). Denote \( \mathcal{A}(\lambda) := \mathcal{A}^{\sigma_\lambda} \). It is clear that \( \mathcal{A}(\lambda) \cong \mathcal{K}/\mathcal{I}_\lambda \), where \( \mathcal{I}_\lambda \) is the left ideal of \( \mathcal{K} \) generated by \( d_i - \lambda \) and \( \frac{\partial}{\partial t} \). We need the following lemmas.

**Lemma 2.1.** ([22], Lemma 3.5) 1. \( \mathcal{A}(\lambda) \) is a strictly simple \( \mathcal{K} \)-module.

2. Any simple weight \( \mathcal{K} \)-module is isomorphic to some \( \mathcal{A}(\lambda) \) for some \( \lambda \in \mathbb{C} \) up to a parity-change.

**Lemma 2.2.** For the Lie superalgebra \( \mathcal{L} \), we have the following relations:

\[
[(t - 1)^k d_i, (t - 1)^j d_j] = (l + k + j - i)(t - 1)^{k+l-1} d_{i+j},
\]

\[
[(t - 1)^k d_i, (t - 1)^j h_j] = (l + j)(t - 1)^{k+l-1} h_{i+j},
\]

\[
[(t - 1)^k Q_i, (t - 1)^j Q_j] = (l + j)(t - 1)^{k+l-1} Q_{i+j} + (l - 1)^{k+l-1} Q_{i+j},
\]

\[
[(t - 1)^k G_i, (t - 1)^j G_j] = (l + k + j - i)(t - 1)^{k+l-1} G_{i+j} + (l - 1)(t - 1)^{k+l-1} G_{i+j},
\]

\[
[(t - 1)^k h_i, (t - 1)^j Q_j] = -(t - 1)^{k+l} Q_{i+j},
\]

\[
[(t - 1)^k h_i, (t - 1)^j G_j] = (t - 1)^{k+l} G_{i+j},
\]

\[
[(t - 1)^k Q_i, (t - 1)^j G_j] = (t - 1)^{k+l} d_{i+j} + (l + k)(t - 1)^{k+l} h_{i+j} + (l - 1)(t - 1)^{k+l-1} h_{i+j},
\]

\[
[(t - 1)^k h_i, (t - 1)^j Q_j] = (t - 1)^{k+l} Q_{i+j} = 0,
\]

\[
[(t - 1)^k h_i, x \otimes (t - 1)^j t^j] = jx \otimes (t - 1)^{k+l} t^{i+j} + lx \otimes (t - 1)^{k+l-1} t^{i+j+1},
\]

\[
[(t - 1)^k d_i, x \otimes (t - 1)^j t^j] = jx \otimes (t - 1)^{k+l} t^{i+j},
\]

\[
[(t - 1)^k Q_i, x \otimes (t - 1)^j t^j] = (t - 1)^{k+l} t^{i+j+1},
\]

\[
[(t - 1)^k G_i, x \otimes (t - 1)^j t^j] = 0 = [(t - 1)^k G_i, x \otimes (t - 1)^j t^j]
\]

for all \( k, l \in \mathbb{Z}_+ \) and \( x \in g \).

**Proof.** The results follow from direct computations.\( \square \)
2.2. The central extension of Lie superalgebras

A central extension \( \widetilde{L} \) of the Lie superalgebra \( L \) is a short exact sequence of the Lie superalgebras

\[
0 \to \mathfrak{c} \overset{i}{\to} \widetilde{L} \overset{\tau}{\to} L \to 0,
\]

where \( \mathfrak{c} \) is a commutative Lie algebra over \( \mathbb{C} \), i.e. \( \mathfrak{c}_1 = 0 \) and \([\mathfrak{c}, \mathfrak{c}] = 0\). Sometimes we denote the above central extension by a pair \((\widetilde{L}, \tau)\) and call a central extension of \( L \) by \( \mathfrak{c} \). A central extension \((\widetilde{L}, \tau)\) is called universal if for any central extension \((\hat{L}, \iota)\) of \( L \) there exists a unique homomorphism \( \psi: \hat{L} \to L \) such that \( \iota \circ \psi = \tau \). A Lie superalgebra \( L \) has the universal central extension if and only if \( L \) is perfect.

It is well known that \( \hat{L} = L \oplus \mathbb{C} \) is a 1-dimensional central extension of Lie superalgebra \( L \) if and only if \( \hat{L} \) is the direct sum of \( L \) and \( \mathbb{C} \) as vector spaces and the bracket \([\cdot, \cdot]_1\) in \( \hat{L} \) is given by

\[
[x, y]_1 = [x, y] + \alpha(x, y)C, \quad [x, C]_1 = 0
\]

for all \( x, y \in L \), where \([ \cdot, \cdot ] \) is the bracket in \( L \) and \( \alpha: L \times L \to \mathbb{C} \) is a bilinear form on \( L \) satisfying the following conditions

\[
\alpha(x, y) = -(1)^{|x||y|}\alpha(y, x),
\]

\[
\alpha(x, [y, z]) = \alpha([x, y], z) + (-1)^{|x||y|}\alpha(y, [x, z])
\]

for \( x, y, z \in L \). The bilinear form \( \alpha \) is called a 2-cocycle on \( L \). A 2-cocycle is called a 2-coboundary if there is a linear function \( \rho \) from \( L \) to \( \mathbb{C} \) such that \( \alpha(x, y) = \rho([x, y]) \) for all \( x, y \in L \). The set of all 2-cocycles on \( L \) is a vector space, denoted by \( Z^2(L, \mathbb{C}) \). The set of all 2-coboundaries is a subspace of \( Z^2(L, \mathbb{C}) \), denoted by \( B^2(L, \mathbb{C}) \). From [10], the set of equivalence classes of such central extensions are known to be parameterized by the second cohomology group \( H^2(L, \mathbb{C}) = Z^2(L, \mathbb{C})/B^2(L, \mathbb{C}) \).

2.3. Weight modules

For the Lie superalgebra \( \mathcal{L} \) (resp. \( \mathfrak{L} \)) defined in Section 2.1, an \( \mathcal{L} \)-module (resp. \( \mathfrak{L} \)-module) \( V \) is called a weight module if the action of \( \mathfrak{d}_0 \) on \( V \) is diagonalizable, i.e. \( V = \bigoplus_{\lambda \in \mathbb{C}} V_{\lambda} \), where \( V_{\lambda} = \{ v \in V \mid \mathfrak{d}_0 v = \lambda v \} \). The support set of a weight module \( V \) is defined by \( \text{supp}(V) = \{ \lambda \in \mathbb{C} \mid V_{\lambda} \neq 0 \} \). A weight \( \mathcal{L} \)-module (resp. weight \( \mathfrak{L} \)-module) \( V \) is called Harish-Chandra if \( \dim V_{\lambda} < \infty, \forall \lambda \in \text{supp}(V) \), and is called cuspidal or uniformly bounded if there exists some \( N \in \mathbb{N} \) such that \( \dim V_{\lambda} \leq N, \forall \lambda \in \text{supp}(V) \).

For the above \( \mathcal{A} \) and \( \mathfrak{g} \), set \( \tilde{\mathcal{L}} = \mathcal{W} \rtimes ((\mathfrak{g} \otimes \mathcal{A}) \oplus \mathcal{A}) \). An \( \mathcal{L} \)-module \( V \) is called an \( \mathcal{A} \mathcal{L} \)-module if \( \mathcal{A} \) acts associatively, i.e. \( t^0 v = v, f g v = f(gv), \forall f, g \in \mathcal{A}, v \in V \). Let \( V \) be a weight \( \mathcal{A} \mathcal{L} \)-module and \( V = \bigoplus_{\lambda \in \mathbb{C}} V_{\lambda} \). For any \( v \in V_{\lambda} \) and \( i \in \mathbb{Z} \), we have

\[
\begin{align*}
d_0 d_i v &= (\lambda + i)d_i v, & d_0 h_i v &= (\lambda + i)h_i v, \\
d_0 Q_i v &= (\lambda + i)Q_i v, & d_0 G_i v &= (\lambda + i)G_i v, \\
d_0(x \otimes t^i)v &= (\lambda + i)x \otimes t^i v, & d_0(x \otimes t^i)\xi v &= (\lambda + i)x \otimes t^i \xi v, \\
d_0 t^i v &= (\lambda + i)t^i v, & d_0 t^i \xi v &= (\lambda + i)t^i \xi v.
\end{align*}
\]

Thus if \( V \) is simple, then \( \text{supp}(V) = \lambda + \mathbb{Z} \) for some \( \lambda \in \mathbb{C} \).
2.4. Some useful results

A module $M$ over an associative superalgebra $B$ is called strictly simple if it is a simple module over the associative algebra $B$ (forgetting the $\mathbb{Z}_2$-gradation).

**Lemma 2.3.** ([22], Lemma 2.1, 2.2) Let $B, B'$ be unital associative superalgebras, and $M, M'$ be $B, B'$-modules, respectively.

1. $M \otimes M' \cong \Pi(M) \otimes \Pi(M')$ as $B \otimes B'$-modules.
2. If $B'$ has a countable basis and $M'$ is strictly simple, then
   (1) any $B \otimes B'$-submodule of $M \otimes M'$ is of the form $N \otimes M'$ for some $B$-submodule $N$ of $M$;
   (2) any simple quotient of the $B \otimes B'$-module $M \otimes M'$ is isomorphic to some $\overline{M} \otimes M'$ for some simple quotient $\overline{M}$ of $M$;
   (3) $M \otimes M'$ is a simple $B \otimes B'$-module if and only if $M$ is a simple $B$-module;
   (4) if $V$ is a simple $B \otimes B'$-module containing a strictly simple $B' = \mathbb{C} \otimes B'$-module $M'$, then $V \cong M \otimes M'$ for some simple $B$-module $M$.

3. The universal central extension of $\mathcal{L}$

In this section, we discuss the structure of the universal central extension of $\mathcal{L} = \mathcal{W} \ltimes (\mathfrak{g} \otimes \mathcal{A})$ by the 1-dimensional center, where $\mathfrak{g}$ is a finite-dimensional perfect Lie superalgebra.

Let $\alpha'$ be a 2-cocycle on $\mathcal{L}$ and $\rho$ be a linear function from $\mathcal{L}$ to $\mathbb{C}$. Set $\alpha = \alpha' + \alpha_\rho$, where $\alpha_\rho \in B^2(\mathcal{L}, \mathbb{C})$ and $\alpha_\rho(x, y) = \rho([x, y])$. Then $\alpha$ is a 2-cocycle on $\mathcal{L}$ which is equivalent to $\alpha'$. For a given $x \in \mathfrak{g}$, we define

$$\rho(x \otimes 1) = \alpha'(d_1, x \otimes t^{-1}), \quad \rho(x \otimes t^k) = -\frac{1}{k} \alpha'(d_0, x \otimes t^k), k \neq 0;$$
$$\rho(x \otimes \xi) = \alpha'(d_1, x \otimes t^{-1}\xi), \quad \rho(x \otimes t^k\xi) = -\frac{1}{k} \alpha'(d_0, x \otimes t^k\xi), k \neq 0.$$ 

So $\alpha(d_0, x \otimes t^k) = \alpha(d_0, x \otimes t^k\xi) = 0$ for $k \neq 0$.

By the definition of 2-cocycle, we have

$$\alpha(d_0, [d_i, x \otimes t^j]) = \alpha([d_0, d_i], x \otimes t^j) + \alpha(d_i, [d_0, x \otimes t^j]).$$

Because $\alpha(d_0, x \otimes t^k) = 0$, there is $(i+j) \alpha(d_i, x \otimes t^j) = 0$. Then $\alpha(d_i, x \otimes t^j) = 0, i+j \neq 0$.

Similarly, for $i+j \neq 0$, we get

$$\alpha(h_i, x \otimes t^j) = \alpha(Q_i, x \otimes t^j) = \alpha(G_i, x \otimes t^j) = 0,$$
$$\alpha(d_i, x \otimes t^j) = \alpha(h_i, x \otimes t^j) = \alpha(Q_i, x \otimes t^j) = \alpha(G_i, x \otimes t^j) = 0.$$

Let $p_x(i) = \alpha(d_i, x \otimes t^{-i})$. Since

$$\alpha(d_{i+j}, [d_{i+j}, x \otimes t^{-j}]) = \alpha([d_{i+j}, d_{i+j}], x \otimes t^{-j}) + \alpha(d_{i+j}, [d_{i+j}, x \otimes t^{-j}]),$$

we have $jp_x(i + j) = (2i + j)p_x(j) + jp_x(-i)$. By letting $i = -1$, there is $jp_x(j - 1) = (j - 2)p_x(j) + jp_x(1)$. Because $\rho(x \otimes 1) = \alpha'(d_1, x \otimes t^{-1}), p_x(1) = \alpha'(d_1, x \otimes t^{-1}) + \ldots$
\[ \rho([d_1, x \otimes t^{-1}]) = 0. \] Then \((j - 2)p_x(j) = jp_x(j - 1).\) And we get \(p_x(j) = \frac{j(j-1)}{2}p_x(2)\) by recursion. Hence
\[
\alpha(d_i, x \otimes t^j) = \frac{i(i-1)}{2}p_x(2)\delta_{i+j,0}.
\]
Similarly, we get
\[
\alpha(d_i, x \otimes t^j \xi) = \frac{i(i-1)}{2}p_x(2)\delta_{i+j,0},
\]
where \(p_x'(i) = \alpha(d_i, x \otimes t^{-i} \xi).\)
Let \(a_x(i) = \alpha(h_i, x \otimes t^{-i}).\) Since
\[
\alpha(h_{i+j}, [d_{-i}, x \otimes t^{-j}]) = \alpha([h_{i+j}, d_{-i}], x \otimes t^{-j}) + \alpha(d_{-i}, [h_{i+j}, x \otimes t^{-j}])),
\]
we have \(ja_x(i + j) = (i + j)a_x(j).\) By letting \(j = 1,\) there is \(a_x(i) = ia_x(1).\) Hence
\[
\alpha(h_i, x \otimes t^j) = ia_x(1)\delta_{i+j,0}.
\]
And similarly, we get
\[
\alpha(Q_1, x \otimes t^j) = ib_x(1)\delta_{i+j,0},
\]
where \(b_x(i) = \alpha(Q_i, x \otimes t^{-i}).\)
Let \(a_x'(i) = \alpha(h_i, x \otimes t^{-i} \xi).\) Since
\[
\alpha(h_{i+j}, [d_{-i}, x \otimes t^{-j} \xi]) = \alpha([h_{i+j}, d_{-i}], x \otimes t^{-j} \xi) + \alpha(d_{-i}, [h_{i+j}, x \otimes t^{-j} \xi]),
\]
we have
\[
ja_x'(i + j) = (i + j)a_x'(j) - p_x'(i).
\]
Putting \(j = 1\) in \(3.1,\) there is \(a_x'(1) = ia_x'(1) - \frac{i(i-1)}{2}p_x'(2).\) Putting \(i = -1\) in \(3.1,\) there is \(a_x'(j) = ja_x'(1)\) since \(p_x'(1) = 0.\) So \(p_x'(2) = 0\) and
\[
\alpha(h_i, x \otimes t^j \xi) = ia_x'(1)\delta_{i+j,0}.
\]
Similarly, we have \(p_x(2) = 0\) and
\[
\alpha(Q_1, x \otimes t^j \xi) = ib_x'(1)\delta_{i+j,0},
\]
where \(b_x'(i) = \alpha(Q_i, x \otimes t^{-i} \xi).\)
Let \(q_x(i) = \alpha(G_i, x \otimes t^{-i}).\) Since
\[
\alpha(G_{i+j}, [d_{-i}, x \otimes t^{-j}]) = \alpha([G_{i+j}, d_{-i}], x \otimes t^{-j}) + \alpha(d_{-i}, [G_{i+j}, x \otimes t^{-j}])),
\]
we have
\[
jq_x(i + j) = (2i + j)q_x(j).
\]
Putting \(j = 1\) in \(3.2,\) there is \(q_x(i) = (2i - 1)q_x(1).\) Putting \(i = 1\) in \(3.2,\) there is \(q_x(j) = \frac{j(j+1)}{2}q_x(1).\) So \(q_x(1) = 0\) and
\[
\alpha(G_i, x \otimes t^j) = 0.
\]
Similarly, we get
\[
\alpha(G_i, x \otimes t^j \xi) = 0.
\]
The Jacobi identity for the 2-cocycle $\alpha$ for the elements $h_{i+j}, Q_{-i}$ and $x \otimes t^{-j}$ gives $b_2(1) = 0$. The Jacobi identity for the 2-cocycle $\alpha$ for the elements $h_{i+j}, G_{-i}$ and $x \otimes t^{-j}$ gives $a'_2(1) = 0$. The Jacobi identity for the 2-cocycle $\alpha$ for the elements $h_{i+j}, Q_{-i}$ and $x \otimes t^{-j}\xi$ gives $b'_2(1) = (-1)^{|x|}a_x(1)$. Furthermore, since

$$\alpha(x \otimes t^{i+j}, [h_{-i}, y \otimes t^{-j}]) = \alpha((x \otimes t^{i+j}, h_{-i}, y \otimes t^{-j}) + \alpha(h_{-i}, [x \otimes t^{i+j}, y \otimes t^{-j}])$$

and $g = [g, g]$, we get $a_x(1) = 0$. Therefore,

$$\alpha(d_i, x \otimes t^j) = \alpha(h_i, x \otimes t^j) = \alpha(Q_i, x \otimes t^j) = \alpha(G_i, x \otimes t^j) = 0,$$

$$\alpha(d_i, x \otimes t^{j+1}) + \alpha(d_i, x \otimes t^{j+1}) = \alpha(h_i, x \otimes t^{j+1}) = \alpha(Q_i, x \otimes t^{j+1}) = \alpha(G_i, x \otimes t^{j+1}) = 0$$

for any $x \in g$ and $i, j \in \mathbb{Z}$.

As above, it is easy to get

$$\alpha(x \otimes t^i, y \otimes t^j) = \alpha(x \otimes t^i, y \otimes t^j) = \alpha(x \otimes t^i, y \otimes t^j) = 0$$

for any $x, y \in g$ and $i + j \neq 0$. Since

$$\alpha(x \otimes t^{i+j}, [d_{-i}, y \otimes t^{-j}]) = \alpha([x \otimes t^{i+j}, d_{-i}], y \otimes t^{-j}) + \alpha(d_{-i}, [x \otimes t^{i+j}, y \otimes t^{-j}]),$$

we have $j\alpha(x \otimes t^{i+j}, y \otimes t^{-i+j}) = (i + j)\alpha(x \otimes t^i, y \otimes t^{-j})$. By letting $j = 1$, there is

$$\alpha(x \otimes t^i, y \otimes t^{-1}) = i\alpha(x \otimes t, y \otimes t^{-1}).$$

So

$$\alpha(x \otimes t^i, y \otimes t^j) = i\alpha(x \otimes t, y \otimes t^{-1}) \delta_{i+j, 0}.$$

Similarly, we get

$$\alpha(x \otimes t^i, y \otimes t^{-j}) = i\alpha(x \otimes t, y \otimes t^{-1}) \delta_{i+j, 0},$$

$$\alpha(x \otimes t^i, y \otimes t^{-j}) = i\alpha(x \otimes t, y \otimes t^{-1}) \delta_{i+j, 0}.$$

Since

$$\alpha(x \otimes t^{i+j}, [Q_{-i}, y \otimes t^{-j}]) = \alpha([x \otimes t^{i+j}, Q_{-i}], y \otimes t^{-j}) + \alpha(Q_{-i}, [x \otimes t^{i+j}, y \otimes t^{-j}]),$$

we get $(i + j)\alpha(x \otimes t, y \otimes t^{-1}) = 0$. Then $\alpha(x \otimes t, y \otimes t^{-1}) = 0$. Moreover, we get $\alpha(x \otimes t^i, y \otimes t^{-1}) = \alpha(x \otimes t^i, y \otimes t^{-1}) = 0$. Therefore,

$$\alpha(x \otimes t^i, y \otimes t^j) = \alpha(x \otimes t^i, y \otimes t^j) = \alpha(x \otimes t^i, y \otimes t^j) = 0$$

for any $x, y \in g$ and $i, j \in \mathbb{Z}$.

Now, we know that any nontrivial 2-cocycle $\alpha$ on $\mathcal{L}$ can be induced by a nontrivial 2-cocycle on $W$. According to the conclusion of [15], we get

$$\alpha(d_i, h_j) = -i^2 \delta_{i+j, 0}, \quad \alpha(h_i, h_j) = 2i^2 \delta_{i+j, 0}, \quad \alpha(Q_i, G_j) = -i^2 \delta_{i+j, 0},$$

and zero in all other cases. In addition, since $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$, we obtain the universal central extension of $\mathcal{L}$, which is denoted by $\hat{\mathcal{L}}$. 

7
Theorem 3.1. The super affine-Virasoro algebra $\hat{\mathcal{L}} = \mathcal{L} \oplus CC$ and the nonzero brackets of $\hat{\mathcal{L}}$ are as follows:

\[
\begin{align*}
[d_i, d_j] &= (j-i)d_{i+j}, \quad [d_i, h_j] = jh_{i+j} - i^2\delta_{i+j,0}C, \\
[d_i, Q_j] &= jQ_{i+j}, \quad [d_i, G_j] = (j-i)G_{i+j}, \quad [h_i, Q_j] = -Q_{i+j}, \quad [h_i, G_j] = G_{i+j}, \\
[h_i, h_j] &= 2i\delta_{i+j,0}C, \quad [G_i, G_j] = d_{i+j} + ih_{i+j} - i^2\delta_{i+j,0}C, \\
[h_i, x \otimes t^j\xi] &= jx \otimes t^{i+j}\xi, \quad [d_i, x \otimes t^j\xi] = jx \otimes t^{i+j}\xi, \\
[h_i, x \otimes t^j\xi] &= x \otimes t^{i+j}\xi, \quad [Q_i, x \otimes t^j\xi] = (-1)^{|x|x}x \otimes t^{i+j}, \\
[G_i, x \otimes t^j\xi] &= (-1)^{|x|}jx \otimes t^{i+j}\xi, \quad [x \otimes t^j\xi, y \otimes t^j\xi] = [x, y] \otimes t^{i+j},
\end{align*}
\]

where $i, j \in \mathbb{Z}$ and $x, y \in g$.

Remark 3.2. From the above we see that $\dim H^2(\mathcal{L}, \mathbb{C}) = 1$, which is different from that of Lie algebras case (see [16]), is also different from that of the current Lie superalgebra $g \otimes A$ (see [19]). The key point is the role of the Witt superalgebra $W$, see (3.3). In fact, if we consider the Lie superalgebra $\mathcal{L} = W \otimes (g \otimes A)$ for a finite-dimensional basic classical simple Lie superalgebra $g$, then from the similar proof of Theorem 3.1 in this paper and Proposition 3.14 in [19] we can obtain that $\dim H^2(\mathcal{L}, \mathbb{C}) = 2$.

It is clear that $\hat{\mathcal{L}}$ has a $\mathbb{Z}$-grading by the eigenvalues of the adjoint action of $d_0$. Then

$$\hat{\mathcal{L}} = \bigoplus_{n \in \mathbb{Z}} \hat{\mathcal{L}}_n = \hat{\mathcal{L}}^+ \oplus \hat{\mathcal{L}}_0 \oplus \hat{\mathcal{L}}^-,$$

where

$$\hat{\mathcal{L}}_{\pm} = \bigoplus_{n \in \mathbb{N}} \hat{\mathcal{L}}_{\pm n}, \quad \hat{\mathcal{L}}_0 = g \otimes CC + Cd_0 + Ch_0 + CQ_0 + CG_0.$$

Let $h$ be the Cartan subalgebra of Lie superalgebra $g$. Then $\hat{h} = h \oplus CC + Cd_0 + Ch_0$ be the cartan subalgebra of $\hat{\mathcal{L}}$. A highest weight module over $\hat{\mathcal{L}}$ is characterized by its highest weight $\Lambda \in \hat{h}^*$ and highest weight vector $v_0$ such that $(\hat{\mathcal{L}}^+ \oplus g_+)v_0 = 0$ and $hv_0 = \Lambda(h)v_0$, $\forall h \in h$.

4. The $\mathcal{A}\mathcal{L}$-modules

Let $\mathcal{U} = U(\hat{\mathcal{L}})$ and $\mathcal{I}$ be the left ideal of $\mathcal{U}$ generated by $t^i, t^j - t^{i+j}, t^0 - 1, t^i \cdot \xi - t^i\xi$ and $\xi \cdot \xi$ for all $i, j \in \mathbb{Z}$. Then it is clear that $\mathcal{I}$ is an ideal of $\mathcal{U}$. Now we have the quotient algebra $\overline{\mathcal{U}} = \mathcal{U}/\mathcal{I} = (U(\mathcal{L})U(\mathcal{A}))/\mathcal{I}$. From PBW Theorem, we may identify $\mathcal{A}$, $\mathcal{L}$ with their images in $\overline{\mathcal{U}}$. Thus $\overline{\mathcal{U}} = \mathcal{A} \cdot U(\mathcal{L})$. Then the category of $\mathcal{A}\mathcal{L}$-modules is equivalent to the category of $\overline{\mathcal{U}}$-modules.

For $i \in \mathbb{Z}\setminus\{0\}$, let $\mathcal{T}$ be a subspace of $\overline{\mathcal{U}}$ spanned by $\{t^{-i} \cdot d_i - d_0, \ t^{-i} \cdot Qi - Q_0, \ t^{-i}\xi, \ d_i - t^{-i} \cdot G_i, \ t^{-i}\xi \cdot Qi - t^{-i} \cdot h_i, \ t^{-i} \cdot (x \otimes t^i), \ t^{-i}\xi \cdot (x \otimes t^i) - (-1)^{|x|}t^{-i} \cdot (x \otimes t^i)\}$. 

Lemma 4.1. 1. $[\mathcal{T}, d_0] = [\mathcal{T}, Q_0] = [\mathcal{T}, \mathcal{A}] = 0$;

2. $\mathcal{T}$ is a Lie super subalgebra of $\overline{\mathcal{U}}$.
Lemma 4.3. \[ \text{Lemma 4.4.} \]

2.2 Lemma 4.5.

Proof. Let \( U(T) \) be an associative subalgebra of \( U \). Define the map \( \phi : K \otimes U(T) \to U \) by \( \phi(x \otimes y) = x \cdot y, \forall x \in K, y \in U(T) \). Then the restrictions of \( \phi \) on \( K \) and \( U(T) \) are well-defined homomorphisms of associative superalgebras. Note that \( [T, d_0] = [T, Q_0] = [T, A] = 0, \phi \) is well-defined homomorphism of associative superalgebras. From

\[
\phi(t^i \otimes (t^{-i} \cdot d_0 + t^i d_0 \otimes 1)) = d_i, \ \phi(t^i \otimes (t^{-i} \cdot Q_0 + t^i Q_0 \otimes 1)) = Q_i, \ \phi(t^i \otimes (t^{-i} \cdot t, h_i)) = h_i, \\
\phi(t^i \otimes (t^{-i} \cdot t, t_i)) = t \otimes t,
\]

we can see that \( \phi \) is surjective.

By PBW theorem, we know that \( U \) has a basis consisting of monomials in variables \( \{d_i, h_j, Q_i, G_j, x \otimes t, x \otimes t^i \xi | i \in \mathbb{Z}\setminus\{0\}, j \in \mathbb{Z}, x \in g\} \) over \( K \). Therefore \( U \) has a \( K \)-basis consisting of monomials in the variables \( \{t^{-i} \cdot d_0 + t^i Q_0 + t^{-i} \cdot t, h_i, t^{-i} \cdot t, t^{-i} \cdot (x \otimes t), t^{-i} \cdot (x \otimes t^i) - (-1)^{|x|} t^{-i} \cdot (x \otimes t^i) | i \in \mathbb{Z}\setminus\{0\}, j \in \mathbb{Z}, x \in g\} \). So \( \phi \) is a surjective map and hence an isomorphism.

Since the generators of \( T \) are complex, we hope to find a simpler expression of \( T \). Let \( m \) be the maximal ideal of \( A \) generated by \( t - 1 \) and \( \xi \). Then \( m \Delta \) is a Lie super subalgebra of \( W = A \Delta \) (see Section 1). Note that \( m^{k+1} \Delta \) is spanned by the set

\[
\{(t - 1)^k d_i, (t - 1)^k Q_i, (t - 1)^k h_i, (t - 1)^k G_i | i \in \mathbb{Z}, k \in \mathbb{Z}_+\}
\]

and \( m^k A \) is spanned by the set

\[
\{(t - 1)^k t^i, (t - 1)^k t^i \xi, (t - 1)^k t^i \xi^k | i \in \mathbb{Z}, k \in \mathbb{Z}_+\}.
\]

By Lemma 2.2, it is easy to get the following lemma.

**Lemma 4.3.** For \( k \in \mathbb{Z}_+ \), let \( a_k = m^{k+1} \Delta \times (g \oplus m^k A) \). Then

1. \( a_0 \) is a Lie super subalgebra of \( L \);
2. \( a_{k+1} \) is an ideal of \( a_k \);
3. \( \{a_1, a_k \} \subseteq a_{k+1} \);
4. \( \{a_0, a_0 \} \supseteq a_1 \);
5. The ideal of \( a_0 \) generated by \( m^k \Delta \) contains \( a_k \).

**Lemma 4.4.** ([22], Lemma 3.7) \( m \Delta / m^2 \Delta \cong gl(1, 1) \).

**Lemma 4.5.** Any finite-dimensional \( m \Delta \)-module is annihilated by \( m^k \Delta \) for large \( k \).

Proof. Let \( V \) be any finite-dimensional \( m \Delta \)-module. Let \( \Delta' = \text{span}_C \{1, \frac{d}{dt}, t \Delta \} = \text{span}_C \{d_{-1}, Q_0\} \) and \( \Delta^+ = \mathbb{C}[t, \xi] \). Let \( d = (t - 1)d_{-1} + \xi Q_0 \) and \( m^+ = m \cap \Delta^+ \). Then \( m^+ = \oplus_{i=0}^{\infty} m_i^+ \) with

\[
m_i^+ = \{x \in m^+ | [d, x] = ix\} = \text{span}\{(t - 1)^i, (t - 1)^i \xi\}
\]
and \( m^+\Delta' \) is a Lie super subalgebra of \( W = A\Delta' \). Let \( f(\lambda) \) be the characteristic polynomial of \( d \) as an operator on \( V \). Then there exists some \( p \in \mathbb{N} \) such that \( (f(\lambda - l), f(\lambda)) = 1 \) for all \( l \geq p \). Since

\[
[d, (t - 1)^{l+1}d_{-1}] = l(t - 1)^{l+1}d_{-1}, \quad [d, (t - 1)^{l}h_0] = l(t - 1)^{l}h_0, \\
[d, (t - 1)^{l+1}Q_0] = l(t - 1)^{l+1}Q_0, \quad [d, (t - 1)^{l}G_{-1}] = l(t - 1)^{l}G_{-1},
\]

and \( m^+\Delta' = \text{span}\{(t - 1)^{l+1}d_{-1}, (t - 1)^{l}h_0, (t - 1)^{l+1}Q_0, (t - 1)^{l}G_{-1}\} \), we have \( [d, d'] = ld', \forall d' \in m^+_{l+1}\Delta' \). So \( f(d - l)d'v = df(d)v = 0 \), which implies that \( d'v = 0, \forall d' \in m^+_{l+1}\Delta', v \in V \). That is \( (m^+)^{p+1}\Delta'V = 0 \).

Let \( b \) be the ideal of \( m\Delta' = m\Delta \) generated by \( (m^+)^{p+1}\Delta' \). Then \( bV = 0 \). For any \( x \in (m^+)^{p+1}, y \in m^+_{1} \) and \( i \in \mathbb{Z} \), since \( xd \in b, yxd \in b \) and

\[
[yt^d, xd] - [t^d, yxd] = [y, xd]t^d + y[t^d, xd] - [t^d, y]xd - y[t^d, xd] = -2t^iyxd,
\]

we have \( m^{p+2}d \subseteq b \). So

\[
y\partial = [zd, y\partial] + (-1)^{(\|y\|+|\partial|)}i\|y\|\partial, z]d \in b
\]

for any \( z \in m^{p+2}, y \in m^+_{1} \) and \( \partial \in \Delta' \). Therefore, \( m^{p+2}\Delta' \subseteq b \). By letting \( k = p + 3 \), we get \( m^k\Delta'V = 0 \). □

**Proposition 4.6.** The Lie superalgebras \( T \) and \( a_0 \) are isomorphic.

**Proof.** Obviously, \( a_0 = m\Delta \times (g \otimes A) \) is spanned by the set \( \{d_i - d_0, h_i, Q_i - Q_0, G_i, x \otimes t^i, x \otimes t^i \xi | i \in \mathbb{Z}, x \in g \} \). It is easy to verify that the linear map \( \varphi : T \to a_0 \) defined by

\[
\varphi(t^{-i} \cdot d_i - d_0) = d_i - d_0, \quad \varphi(t^{-i} \cdot Q_i - t^{-i} \cdot h_i) = -h_i, \\
\varphi(t^{-i} \cdot Q_i - Q_0) = Q_i - Q_0, \quad \varphi(t^{-i} \cdot d_i - t^{-i} \cdot G_i) = -G_i, \\
\varphi(t^{-i} \cdot (x \otimes t^i)) = x \otimes t^i, \quad \varphi(t^{-i} \xi \cdot (x \otimes t^i) - (-1)^{\|x\|}t^{-i} \cdot (x \otimes t^i)\xi) = (-1)^{\|x\|}x \otimes t^i \xi
\]

is a Lie superalgebra isomorphism. □

For any \( a_0 \)-module \( V \), we have the \( \mathcal{A} \)-module \( \Gamma(\lambda, V) := (A(\lambda \otimes V)^{\xi_1}, \) where \( \varphi_1 : \mathcal{T} \simeq K \otimes U(T) \rightarrow K \otimes U(a_0) \). More precisely, \( \Gamma(\lambda, V) = A \otimes V \) with actions

\[
t^i\xi \circ (a \otimes v) = t^i\xi a \otimes v, \\
d_i \circ (a \otimes v) = t^i a \otimes (d_i - d_0) \cdot v + t^i(\lambda a + d_0(a)) \otimes v, \\
h_i \circ (a \otimes v) = t^i a \otimes h_i \cdot v + \xi(Q_i(a)) \otimes v, \\
Q_i \circ (a \otimes v) = (-1)^{|a|}t^i a \otimes (Q_i - Q_0) \cdot v + t^i(Q_0(a)) \otimes v, \\
G_i \circ (a \otimes v) = (-1)^{|a|}t^i a \otimes G_i \cdot v + \xi(\lambda a + d_i(a)) \otimes v, \\
(x \otimes t^i) \circ (a \otimes v) = (-1)^{|a|}t^i a \otimes (x \otimes t^i) \cdot v, \\
(x \otimes t^i \xi) \circ (a \otimes v) = (-1)^{|a|(|\xi|+1)}t^i \xi a \otimes (x \otimes t^i) \cdot v + (-1)^{|a|(|\xi|+1)}t^{-i} a \otimes (x \otimes t^i \xi) \cdot v
\]

for any \( a \in \mathcal{A}, v \in V \).
Lemma 4.7. 1. For any $\lambda \in \mathbb{C}$ and any simple $a_0$-module $V$, $\Gamma(\lambda, V)$ is a simple weight $\mathcal{A}\mathcal{L}$-module.

2. Any simple weight $\mathcal{A}\mathcal{L}$-module $M$ is isomorphic to some $\Gamma(\lambda, V)$ for some $\lambda \in \supp(M)$ and some $a_0$-module $V$.

Proof. 1. From Lemmas 2.3 and 2.1, we know that $\mathcal{A}(\lambda) \otimes V$ is a simple $\mathcal{K} \otimes U(T)$-module for any $\lambda \in \mathbb{C}$ and any simple $a_0$-module $V$. From the definition of $\Gamma(\lambda, V)$, we have the first statement.

2. Let $M$ be any simple weight $\mathcal{A}\mathcal{L}$-module with $\lambda \in \supp(M)$. Then $M^{\varphi_{i_1}^{-1}}$ is a simple $\mathcal{K} \otimes U(a_0)$-module. Fix a nonzero homogeneous element $v \in (M^{\varphi_{i_1}^{-1}})_\lambda$. Since $V' = \bigcap_{\lambda} v_{\lambda}$ is a finite-dimensional super subspace with $\frac{d}{dx}$ acting nilpotently, we may find a nonzero homogeneous element $v' \in V'$ with $z_x v' = 0$. Clearly, $\mathcal{K} v'$ is isomorphic to $\mathcal{A}(\lambda)$ or $\Pi(\mathcal{A}(\lambda))$. From Lemma 2.3, there exists a simple $U(a_0)$-module $V$ such that $M^{\varphi_{i_1}^{-1}} \cong \mathcal{A}(\lambda) \otimes V$ or $M^{\varphi_{i_1}^{-1}} \cong \Pi(\mathcal{A}(\lambda)) \otimes V$. Furthermore, there is $\Pi(\mathcal{A}(\lambda)) \otimes V \cong \mathcal{A}(\lambda) \otimes \Pi(V')$ by Lemma 2.1. So this conclusion holds.

Now, to classify all simple weight $\mathcal{A}\mathcal{L}$-modules, it suffices to classify all simple $a_0$-modules. In particular, to classify all simple cuspidal $\mathcal{A}\mathcal{L}$-modules, it suffices to classify all simple finite-dimensional $a_0$-modules. Now we introduce a known conclusion that we are going to use.

Lemma 4.8. ([18], Theorem 2.1, Engel’s Theorem for Lie superalgebras) Let $V$ be a finite-dimensional module for the Lie superalgebra $L = L_0 \oplus L_1$ such that the elements of $L_0$ and $L_1$ respectively are nilpotent endmorphisms of $V$. Then there exists a nonzero $x \in V$ such that $xV = 0$ for all $x \in L$.

Lemma 4.9. 1. Let $V$ be any finite-dimensional $a_0$-module. Then there exists $k \in \mathbb{N}$ such that $a_k V = 0$.

2. Let $V$ be any simple finite-dimensional $a_0$-module. Then $a_1 V = 0$.

Proof. 1. Let $V$ be any finite-dimensional $a_0$-module. Then $V$ is a finite-dimensional $\mathfrak{m} \Delta$-module. So the first statement follows from Lemmas 4.3 and 4.5.

2. Let $V$ be any simple finite-dimensional $a_0$-module. Then $V$ is a simple finite-dimensional $a_0/\text{ann}(V)$-module, where $\text{ann}(V)$ is the ideal of $a_0$ that annihilates $V$ and $a_k \subseteq \text{ann}(V)$ for some $k \in \mathbb{N}$. So $V$ is a finite-dimensional module for $(a_0)_0 + \text{ann}(V)$. By Lemma 4.3, we have

\[(a_1)_{0} + \text{ann}(V))^{k-1} \subseteq (a_k)_{0} + \text{ann}(V) = \text{ann}(V),\]

which implies that $(a_1)_{0} + \text{ann}(V)$ acts nilpotently on $V$. Since $[x, x] \in (a_1)_{0} + \text{ann}(V)$ for any $x \in (a_1)_{1} + \text{ann}(V)$, every element in $(a_1)_{1} + \text{ann}(V)$ acts nilpotently on $V$. Hence, by Lemma 4.8, there is a nonzero element $v \in V$ annihilated by $a_1 + \text{ann}(V)$.

Let $V' = \{v \in V | xv = 0, \forall x \in a_1\}$. So $V' \neq \emptyset$. For any $y \in a_0, x \in a_1$ and $v \in V'$, there is $xyv = (-1)^{|x||y|} y xv + [x, y] v = 0$, which implies that $y v \in V'$. Hence $V' = V$ by the simplicity of $V$. And therefore $a_1 V = 0$.

Let $V$ be any simple finite-dimensional $a_0$-module. From Lemmas 4.4 and 4.9, we know that $V$ is a simple finite-dimensional module for $a_0/a_1 \cong \mathfrak{gl}(1, 1) \oplus \mathfrak{g}$. So, to classify all simple cuspidal $\mathcal{A}\mathcal{L}$-modules, it suffices to classify all simple finite-dimensional module for $\mathfrak{gl}(1, 1) \oplus \mathfrak{g}$. 

11
5. Cuspidal modules

In this section, we will classify all simple cuspidal modules for $L$ by using the $A$-cover theory. Let $I = g \otimes A$ and $i = \delta_{1M,0}W + (1 - \delta_{1M,0})I$. Consider $L$ as the adjoint $L$-module. For an $L$-module $M$, we can make the tensor product $L$-module $i \otimes M$ into an $AL$-module by defining

$$a \cdot (\omega \otimes v) = (a\omega) \otimes v, \forall a \in A, \omega \in i.$$ 

Denote $K(M) = \{ \sum_{i=1}^{k} \omega_i \otimes v_i \in i \otimes M | \sum_{i=1}^{k} (a\omega_i)v_i = 0, \forall a \in A \}$. Then it is easy to see that $K(M)$ is an $AL$-submodule of $i \otimes M$. Then we have the $AL$-module $\hat{M} = (i \otimes M)/K(M)$.

As in [2], we call $\hat{M}$ as the cover of $M$ if $iM = M$.

Clearly, the linear map

$$\pi: \hat{M} \rightarrow iM, \quad w \otimes y + K(M) \mapsto wy, \quad \forall w \in i, y \in M$$

is an $L$-module epimorphism.

Recall that in [2], the authors show that every cuspidal $W$-module is annihilated by the operators $\Omega^{(m)}_{k,s}$ for $m$ large enough.

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**Lemma 5.1.** ([2], Corollary 3.7) For every $l \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $k, s \in \mathbb{Z}$ the differentiators $\Omega^{(m)}_{k,s} = \sum_{i=0}^{m} (-1)^{i}\binom{m}{i}d_{k-i}d_{s+i}$ annihilate every cuspidal $W$-module with a composition series of length $l$.

For $L = W \ltimes (g \otimes A)$, we also want to find some operators belonging to $U(L)$ that can annihilate a given cuspidal $L$-module $M$. Obviously, $M$ is a cuspidal $W$-module and hence there exists $m \in \mathbb{N}$ such that $\Omega^{(m)}_{k,p}M = 0, \forall k, p \in \mathbb{Z}$. Therefore, $[\Omega^{(m)}_{k,p}, x \otimes t^j]M = 0, \forall j, k, p \in \mathbb{Z}, x \in g$. From Lemma 4.4 in [6], the authors show that

$$\sum_{i=0}^{m+2} (-1)^i \binom{m+2}{i}x \otimes t^{i+k+1-i}d_{p-1+i} = 0.$$ 

Similarly, from $[\Omega^{(m)}_{k,p}, x \otimes t^j]\xi = 0$ and $[d_i, x \otimes t^k\xi] = kx \otimes t^{i+k}\xi, \forall i, j, k, p \in \mathbb{Z}, x \in g$, we have

$$\sum_{i=0}^{m+2} (-1)^i \binom{m+2}{i}x \otimes t^{i+k+1-i}\xi d_{p-1+i} = 0.$$ 

Thus, we have the following lemma.

**Lemma 5.2.** Let $M$ be a cuspidal module over $L$. Then there exists $m \in \mathbb{N}$ such that for all $j, p \in \mathbb{Z}$ and $x \in g$, the operators $\sum_{i=0}^{m} (-1)^i \binom{m}{i}yd_{p+i}$ annihilate $M$, where $y \in \{x \otimes t^{-i}, x \otimes t^{-i}\xi\}$. 


Lemma 5.3. Suppose \( \mathfrak{g} \) is finite-dimensional. Let \( M \) be a cuspidal module for the Lie superalgebra \( \mathcal{L} \). Then its \( \mathcal{A} \)-cover \( \hat{M} \) is cuspidal.

Proof. The case of \( IM = 0 \) is proved in [22]. Now suppose \( IM \neq 0 \), so \( i = 1 \). Since \( \hat{M} \) is an \( \mathcal{A} \)-module, it is sufficient to show that one of its weight spaces is finite-dimensional, then all other weight spaces will have the same dimension. Fix a weight \( \alpha + p, p \in \mathbb{Z} \) and let us prove that \( \hat{M}_{\alpha + p} = \text{Span}\{(x \otimes t^{p-k}) \otimes M_{\alpha+k}, (x \otimes t^{p-k} \xi) \otimes M_{\alpha+k} \mid k \in \mathbb{Z}, x \in \mathfrak{g}\} \) is finite-dimensional. Assume that \( \alpha = 0 \) in the case that \( \alpha + \mathbb{Z} = \mathbb{Z} \), which means that \( \alpha + p \neq 0, \forall p \in \mathbb{Z} \).

We will prove by induction on \( |q| \) for \( q \in \mathbb{Z} \) and for all \( u \in M_{\alpha+q} \),

\[
(x \otimes t^{p-q}) \otimes u, (x \otimes t^{p-q} \xi) \otimes u \in \sum_{|k| \leq \frac{q}{2}} (x \otimes t^{p-k}) \otimes M_{\alpha+k} + (x \otimes t^{p-k} \xi) \otimes M_{\alpha+k} + K(M).
\]

If \( |q| \leq \frac{m}{2} \), the claim holds. If \( |q| > \frac{m}{2} \), we may assume \( q < -\frac{m}{2} \). The proof for \( q > \frac{m}{2} \) is similar. Since \( d_0 \) acts on \( M_{\alpha+q} \) with a nonzero scalar, we can write \( u = d_0 v \) for some \( v \in M_{\alpha+q} \). Then

\[
(x \otimes t^{p-q}) \otimes d_0 v = \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} (x \otimes t^{p-q-i}) \otimes d_i v - \sum_{i=1}^{m} (-1)^{i} \binom{m}{i} (x \otimes t^{p-q-i}) \otimes d_i v.
\]

From Lemma 5.2, there exists \( m \in \mathbb{N} \) such that \( \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} x \otimes t^{i} \otimes d_{p+i} v = 0 \) and \( \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} x \otimes t^{-i} \otimes d_{p+i} v = 0 \) for all \( j, p \in \mathbb{Z}, x \in \mathfrak{g} \) and \( v \in M \). Note that \( I \) has a natural module structure over the commutative superalgebra \( \mathcal{A} \)

\[
t^i (x \otimes t^j) = x \otimes t^{i+j}, t^i (x \otimes t^{j} \xi) = x \otimes t^{i+j} \xi, t^{i} \xi (x \otimes t^j) = x \otimes t^{i+j} \xi, t^{i} \xi (x \otimes t^{j} \xi) = 0
\]

for \( i, j \in \mathbb{Z} \) and \( x \in \mathfrak{g} \). Hence, we have

\[
\sum_{i=0}^{m} (-1)^{i} \binom{m}{i} (x \otimes t^{i-j}) \otimes d_{p+i} v, \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} (x \otimes t^{-i} \xi) \otimes d_{p+i} v \in K(M)
\]

for all \( j, p \in \mathbb{Z}, x \in \mathfrak{g} \) and \( v \in M \). Therefore,

\[
(x \otimes t^{p-q}) \otimes d_0 v \in \sum_{|k| \leq \frac{q}{2}} (x \otimes t^{p-k}) \otimes M_{\alpha+k} + (x \otimes t^{p-k} \xi) \otimes M_{\alpha+k} + K(M).
\]

Similarly, we have

\[
(x \otimes t^{p-q} \xi) \otimes d_0 v = \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} (x \otimes t^{p-q-i} \xi) \otimes d_i v - \sum_{i=1}^{m} (-1)^{i} \binom{m}{i} (x \otimes t^{p-q-i} \xi) \otimes d_i v
\]

\[
\in \sum_{|k| \leq \frac{q}{2}} (x \otimes t^{p-k}) \otimes M_{\alpha+k} + (x \otimes t^{p-k} \xi) \otimes M_{\alpha+k} + K(M).
\]

So the lemma follows from the fact that \( \dim M_{\alpha+k} < \infty \) for any fixed \( k \) and \( \mathfrak{g} \) is finite-dimensional.
Theorem 5.4. Any simple cuspidal \( \mathcal{L} \)-module is isomorphic to a simple quotient of a tensor module \( \Gamma(\lambda, V) \) for some simple finite-dimensional \( a_0 \)-module \( V \) and some \( \lambda \in \mathbb{C} \).

**Proof.** Let \( M \) be a simple cuspidal \( \mathcal{L} \)-module. If \( M \) is a trivial module of \( \mathcal{L} \), then \( M \) is a simple quotient of the simple cuspidal \( \mathcal{A} \mathcal{L} \)-module \( \mathbb{A} \otimes \mathbb{C} \) with \( \mathbb{C} \) a trivial module for \( \mathcal{L} \). Now suppose \( M \) is a non-trivial simple cuspidal \( \mathcal{L} \)-module. If \( IM = 0 \), then \( M \) is a simple cuspidal \( \mathcal{W} \)-module. So \( M \) is a simple quotient of a simple cuspidal \( \mathcal{A} \mathcal{W} \)-module by Theorem 3.11 in [22]. Since \( I \) is an ideal, any \( \mathcal{A} \mathcal{W} \)-module is naturally an \( \mathcal{A} \mathcal{L} \)-module with trivial \( I \) action. If \( IM \neq 0 \), then \( IM = M \) since \( M \) is simple. So there is an epimorphism \( \pi : \hat{M} \to M \). From Lemma 5.3, \( \hat{M} \) is cuspidal. Hence \( \hat{M} \) has a composition series of \( \mathcal{A} \mathcal{L} \)-submodules

\[
0 = \hat{M}^{(1)} \subset \hat{M}^{(2)} \subset \cdots \subset \hat{M}^{(s)} = \hat{M}
\]

with \( \hat{M}^{(i)}/\hat{M}^{(i-1)} \) being simple \( \mathcal{A} \mathcal{L} \)-modules. Let \( l \) be the minimal integer such that \( \pi(\hat{M}^{(l)}) \neq 0 \). Since \( M \) is simple \( \mathcal{L} \)-module, we have \( \pi(\hat{M}^{(l)}) = M \) and \( \pi(\hat{M}^{(l-1)}) = 0 \). This gives us an epimorphism of \( \mathcal{L} \)-modules from \( \hat{M}^{(l)}/\hat{M}^{(l-1)} \) to \( M \). From Lemma 4.7, we have \( \hat{M}^{(l)}/\hat{M}^{(l-1)} \) is isomorphic to a tensor module \( \Gamma(\lambda, V) \) for some simple finite-dimensional \( a_0 \)-module \( V \) and some \( \lambda \in \mathbb{C} \). This completes the proof. \( \square \)

Remark 5.5. For Lie superalgebra \( \mathfrak{g} = \mathfrak{g} \otimes \mathbf{A} \), by letting \( \mathfrak{d} = \mathfrak{g} \otimes \Lambda(1) \), we get \( \Sigma = \mathfrak{g} \otimes \mathbf{C}[t^{\pm 1}] \). This shows that \( \Sigma \) is exactly the superalgebra studied in [6]. Therefore, the conclusions on the cuspidal modules over \( \Sigma \), even Harish-Chandra modules over \( \Sigma \) can be directly obtained from [6].

6. Simple Harish-Chandra modules over super affine-Virasoro algebras

In this section, we will classify all simple Harish-Chandra modules over super affine-Virasoro algebras \( \hat{\mathcal{L}} \). Let \( \{x_s \mid s = 1, 2, \cdots, l\} \) be a basis of Lie superalgebra \( \mathfrak{g} = g_0 \oplus g_1 \). Then \( \dim \hat{\mathcal{L}}_n = 4 + 2l \) for \( n \in \mathbb{Z} \) and \( n \neq 0 \). Let \( M \) be a simple Harish-Chandra module over \( \hat{\mathcal{L}} \). By Schur’s Lemma, we may assume that the central element \( C \) acts on \( M \) by scalar \( c \).

**Lemma 6.1.** Suppose that \( M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda \) is a simple cuspidal module over \( \hat{\mathcal{L}} \). Then the action of central element \( C \) on \( M \) is trivial.

**Proof.** Let \( \tilde{d}_i = d_i + \frac{1}{2} i f_i \) for \( i \in \mathbb{Z} \). Then

\[
[d_i, d_j] = (j - i)\tilde{d}_{i+j} + \frac{1}{2} j^3 \delta_{i+j, 0} C.
\]

So \( \mathfrak{D} = \text{span}\{d_i, C \mid i \in \mathbb{Z}\} \), which is isomorphic to the Virasoro algebra, is a subalgebra of \( \hat{\mathcal{L}} \). Note that \( M \) is a cuspidal module over the Virasoro algebra \( \mathfrak{D} \). By [13], we have \( c = 0 \). \( \square \)

From Lemma 6.1, we know that the category of simple cuspidal \( \hat{\mathcal{L}} \)-modules is naturally equivalent to the category of simple cuspidal \( \mathcal{L} \)-modules. Thus, it remains to classify all simple Harish-Chandra modules over \( \hat{\mathcal{L}} \) which is not cuspidal. The following result is well known.
6.2 Lemma 6.2. Let $V$ be a weight module with finite-dimensional weight spaces for the Witt algebra $\mathfrak{W}$ with $\text{supp}(V) \subseteq \lambda + \mathbb{Z}$. If for any $v \in V$, there exists $N(v) \in \mathbb{N}$ such that $d_i v = 0, \forall i \geq N(v)$, then $\text{supp}(V)$ is upper bounded.

6.3 Lemma 6.3. Suppose $M$ is a simple Harish-Chandra module over $\hat{\mathcal{L}}$ which is not cuspidal, then $M$ is a highest (or lowest) weight module.

Proof. For a fixed $\lambda \in \text{supp}(M)$, there is a $k \in \mathbb{Z}$ such that $\dim M_{\lambda - k} > (4 + 2l)\dim M_{\lambda} + 4\dim M_{\lambda + 1}$ since $M$ is not cuspidal. Without loss of generality, we may assume that $k \in \mathbb{N}$. Then there exists a nonzero element $\omega \in M_{\lambda - k}$ such that

$$d_k \omega = d_{k+1} \omega = h_k \omega = h_{k+1} \omega = Q_k \omega = Q_{k+1} \omega = G_k \omega = G_{k+1} \omega = 0$$

and

$$x_s \otimes t^k \omega = x_s \otimes t^k \xi \omega = 0,$$

where $s \in \{1, 2, \ldots, l\}$. Therefore, we get $d_p \omega = h_p \omega = Q_p \omega = G_p \omega = x_s \otimes t^p \omega = x_s \otimes t^p \xi \omega = 0$ for all $p \geq k^2$ since $[\hat{\mathcal{L}}_i, \hat{\mathcal{L}}_j] = \hat{\mathcal{L}}_{i+j}$ for $j \neq 0$.

It is easy to see that $M' = \{v \in M \mid \dim \hat{\mathcal{L}}_+ v < \infty\}$ is a nonzero submodule of $M$. So $M = M'$ by the simplicity of $M$. Since $M$ is also the $d_0$-weight module over the Witt algebra $\mathfrak{W}$, we have $\text{supp}(M)$ is upper bounded by Lemma 6.2, that is $M$ is a highest weight module.

Combining with Lemma 4.7, Theorem 5.4 and Lemma 6.3, we have the following result.

Theorem 6.4. Let $M$ be a simple Harish-Chandra module over $\hat{\mathcal{L}}$. Then $M$ is a highest weight module, a lowest weight module, or a simple quotient of a tensor module $\Gamma(\lambda, V)$ for some simple finite-dimensional $\mathfrak{a}_0$-module $V$ and some $\lambda \in \mathbb{C}$.

Let $\mathfrak{g}$ be a finite-dimensional basic classical simple Lie superalgebra. From Remark 3.2, we see that $\dim H^2(\mathfrak{L}, \mathbb{C}) = 2$. By the similar proof as Lemma 3.2 in [16] and the conclusions in [6], we can get a similar result for the Lie superalgebra $\hat{\mathcal{L}}$.

Remark 6.5. Let $\mathfrak{g}$ be a finite-dimensional basic classical simple Lie superalgebra, and $M$ be a simple Harish-Chandra module over $\hat{\mathcal{L}}$. Then $M$ is a highest weight module, a lowest weight module, or a simple quotient of a tensor module $\Gamma(\lambda, V)$ for some simple finite-dimensional $\mathfrak{b}_0$-module $V$ and some $\lambda \in \mathbb{C}$, where $\mathfrak{b}_0 = (t - 1)\mathfrak{W} \ltimes (\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$.

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