Theory of Weiss oscillations in the presence of small-angle impurity scattering.

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We calculate the magnetoresistivity of a two-dimensional electron gas in the presence of a periodic potential within classical transport theory, using realistic models of impurity scattering. The magnetoresistivity induced by geometric resonance of the cyclotron orbits in the periodic grating, known as Weiss oscillations, are shown to be affected strongly by the small-angle scattering processes dominant in conventional semiconductor heterostructures. Our results are in full agreement with experimental findings.

I. INTRODUCTION

In 1989, Weiss, von Klitzing, Ploog, and Weimann [1] discovered the striking effect of a periodic potential varying in one direction on the magnetoresistivity of a two-dimensional electron gas (2DEG). They observed that even a weak potential modulation (grating) with a wave vector \( \mathbf{q} \parallel \mathbf{e}_c \) induces very strong oscillations of the magnetoresistivity \( \rho_{xx}(B) \), while showing almost no effect on \( \rho_{yy}(B) \) and \( \rho_{xy}(B) \). The oscillation period corresponds to a change of the ratio \( 2R_c/a \) by unity, where \( R_c \) is the cyclotron radius, and \( a = 2\pi/q \) the grating wave length. These findings were explained by Gerhardts et al [2] and Winkler et al [3] in terms of the quantum-mechanical band structure induced by the modulation (see also [2] for explanation of the weak oscillations induced by the grating in \( \rho_{yy}(B) \) and \( \rho_{xy}(B) \) and [3] for discussion of the temperature dependence of the magnetoresistivity). Beenakker [4] presented a semiclassical theory of the effect. He showed that the geometric resonance of the cyclotron motion in the grating induces an extra contribution to the drift velocity of the guiding center, with a r.m.s. amplitude oscillating with period \( 2R_c/a \). These arguments were supported by an exact solution of the Boltzmann equation, where isotropic scattering was assumed. However, while accounting for the above experimental features, the theoretical results [2,3,6] disagree strongly with the experimental findings.

The problem of the damping of the Weiss oscillations was addressed in a number of publications. In Ref. [2] an empirical observation was made that inclusion of the factor \( \exp[-\pi/\omega_c \tau_s] \) in Eq. (1) with \( \tau^{-1} \) being the total relaxation rate (as opposed to the momentum relaxation rate \( \tau^{-1} \)) yields a reasonable description of the experimentally observed damping. The authors of Ref. [3] argued quite oppositely that only the transport time should enter the expression for the damping factor, and proposed an empirical form \( \exp[-4\pi^2 \gamma^2 (R_c/a)^2 (\omega_c \tau)^{-1}] \), with a fitting parameter \( \gamma \). Monte Carlo simulations [3], as well as numerical solution of the Boltzmann equation [6], showed that allowing for anisotropic scattering leads to a stronger damping of the oscillations. However, despite considerable interest in this problem, no quantitative theory of the Weiss oscillations for the experimentally relevant situation, when small angle scattering is dominant, has been available so far. The purpose of the present paper is to fill in this gap.

II. BOLTZMANN EQUATION FORMALISM

We begin by reminding the reader briefly about the Boltzmann equation approach to the problem [6]. One starts from the kinetic equation for the distribution function \( F(x,n) \) of electrons on the Fermi surface,

\[
\mathcal{L}F(x,n) = -e v(x) \mathbf{E} n, \\
\mathcal{L} = e v(x) n \frac{\partial}{\partial r} + \omega_c \frac{\partial}{\partial \phi} - \sin \phi v'(x) \frac{\partial}{\partial \phi} - C,
\]

where \( n = (\cos \phi, \sin \phi) \) is the direction of the electron velocity, \( v'(x) = (2m(E_F + eU(x))]^{1/2} \) is the magnitude of the velocity, \( -e \) is the electron charge, and \( C \) is the collision integral (to be discussed below). In the absence of the grating, the solution is given by

\[
\Delta \rho_{xx}(B) \propto B,
\]

\[
\frac{\Delta \rho_{xx}}{\rho_{xx}} \simeq \eta^2 \frac{l^2}{aR_c^2} \cos^2(qR_c - \pi/4)
\]

Here \( \eta \) is the relative amplitude of the potential modulation \( eU(x) = -\eta E_F \cos qx, \) \( l = v_F \tau \) is the mean free path, \( \omega_c = eB/m = v_F/R_c \) the cyclotron frequency, \( E_F \) the Fermi energy, and \( v_F \) the Fermi velocity. In contrast with this result, the experimental damping is much stronger, so that only few oscillations (6–7 in [3]) are observed.

\[
\rho_{xy}(B) \propto B,
\]

\[
\frac{\Delta \rho_{xy}}{\rho_{xy}} \simeq \eta^2 \frac{l^2}{aR_c^2} \cos^2(qR_c - \pi/4)
\]

\[
\frac{\Delta \rho_{yy}}{\rho_{yy}} \simeq \eta^2 \frac{l^2}{aR_c^2} \cos^2(qR_c - \pi/4)
\]

\[
\frac{\Delta \rho_{xy}}{\rho_{xy}} \simeq \eta^2 \frac{l^2}{aR_c^2} \cos^2(qR_c - \pi/4)
\]

\[
\frac{\Delta \rho_{yy}}{\rho_{yy}} \simeq \eta^2 \frac{l^2}{aR_c^2} \cos^2(qR_c - \pi/4)
\]
\[ F^{(0)}(\mathbf{n}) = -\frac{2}{\epsilon_0\nu F} n_i \sigma_{ij}^{(0)} E_j, \]  

where \( \sigma^{(0)} \) is the Drude conductivity tensor,

\[ \sigma^{(0)} = \frac{\sigma_0 C_s^2}{1 + S^2} \left( \begin{array}{cc} -S & 1 \\ S & 1 \end{array} \right); \quad \sigma_0 = e^2 \nu F \tau / 2, \]  

\( S = \omega c \tau, \) and \( \nu \) is the density of states at the Fermi level. Using the Ansatz \( F = [v(x)/v_F] F^{(0)} + F^{(1)} \), one may transform Eq. (1) into the form

\[ \mathcal{L} F^{(1)}(x, \mathbf{n}) = v(x) v'(x) \frac{e\tau}{1 + S^2} (E_x - SE_y) \]  

(5)

This implies that the grating induced correction to the conductivity tensor, \( \delta \sigma = \sigma^{(0)} + \delta \sigma \), is given by

\[ \delta \sigma = \frac{\sigma_0 C_s^2}{1 + S^2} (G(x, \phi) v(x) \cos \phi) \left( \begin{array}{cc} -S & 1 \\ S & -S^2 \end{array} \right), \]  

(6)

where \( G(x, \phi) \) is the solution to the equation

\[ \mathcal{L} G = -2v(x) v'(x) / v_F', \equiv -2eU'(x) / mv_F^2 \]  

(7)

and the angular brackets \( \langle \ldots \rangle \) denote the averaging over both the velocity direction and a period of the modulation,

\[ \langle O(x, \phi) \rangle = \frac{1}{a} \int_0^a dx \int_0^{2\pi} d\phi \frac{1}{2\pi} O(x, \phi). \]  

(8)

Calculating now the resistivity tensor \( \rho = \sigma^{-1} \), one finds that \( \rho = \rho^{(0)} + \delta \rho \), where \( \rho^{(0)} = [\sigma^{(0)}]^{-1} \) and the only non-zero component of \( \delta \rho \) is \( \delta \rho_{xx} \) given by

\[ \delta \rho_{xx} = -\frac{1 + S^2}{\sigma_0} \frac{\langle Gv \cos \phi \rangle}{1 + \langle Gv \cos \phi \rangle} \]  

(9)

For a weak modulation, \( \eta \ll 1 \), the corrections \( \delta \rho \) and \( \delta \sigma \) are proportional to \( \eta^2 \). Therefore, to calculate the average \( \langle Gv \cos \phi \rangle \) in the leading order, one would have to solve the Boltzmann equation (1) up to the second order in \( \eta \). However, it was observed by Beenakker [3] that the following identity holds:

\[ \langle G(x, \phi)v(x) \cos \phi \rangle = \frac{e\tau}{m(1 + S^2)} \langle G(x, \phi)U'(x) \rangle. \]  

(10)

Since \( U(x) \propto \eta \), it is now sufficient to calculate \( G(x, \phi) \) to first order in \( \eta \), in order to find \( \delta \sigma \) or \( \delta \rho \) to the order \( \eta^2 \).

We note that, while only presented for the isotropic scattering case in Ref. [3], the identity [11] holds for a general collision integral \( C \) as well, since its derivation uses only the fact that \( C\{1\} = 0 \), \( C\{\cos \phi\} = -(1/\tau) \cos \phi \), and \( C\{\sin \phi\} = -(1/\tau) \sin \phi \).

To proceed further, one has to solve the kinetic equation [1]. This requires first specifying the collision integral. Its general form reads:

\[ C\{F\}(\phi) = \int d\phi' P(\phi - \phi') [F(\phi') - F(\phi)], \]  

(11)

where \( P(\phi - \phi') \) is the scattering cross-section. The collision integral can be most conveniently characterized by its eigenvalues

\[ C\{e^{im\phi}\} = -\lambda_m e^{im\phi}; \quad m = 0, \pm 1, \pm 2, \ldots \]  

(12)

Note that \( \lambda_0 = 0 \) reflects the particle number conservation; \( \lambda_1 = 1/\tau \) is the transport relaxation rate, while \( \lambda_\infty = 1/\tau_s \) is the total relaxation rate. Expanding the solution of Eq. (7) in the angular harmonics, \( G(x, \phi) = \sum_{m = -\infty}^{\infty} g_m e^{im\phi} + c.c. \), we get to the leading order in \( \eta \) the following coupled system of equations [10]:

\[ a_n g_n = g_{n+1} + g_{n-1} + c_n; \]  

\[ a_n = \frac{2i}{v_F q} (m\omega_c + \lambda_n); \quad c_n = -\frac{\eta}{v_F} \delta_{n0} \]  

According to Eqs. (10), (11), the correction to the resistivity tensor is determined by \( g_0 \), since

\[ \langle Gv \cos \phi \rangle = -\frac{\eta q E_F \tau}{m(1 + S^2)} \Im g_0. \]  

(14)

In turn, \( g_0 \) is found from the system (13) as

\[ g_0 = \frac{1}{v_F} R_1 + R_{-1}, \]  

(15)

where \( R_1 = g_1/g_0 \) is determined by the set of (homogeneous) equations (13) with \( n \geq 1 \), and \( R_{-1} = R_1^{\omega_c \rightarrow -\omega_c} \). For a generic set of eigenvalues \( \lambda_n \), the solution \( R_1 \) can be presented as a continuous fraction,

\[ R_1 = \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots}}}. \]  

(16)

The result can be expressed in closed form in the case when \( \lambda_n \) depends linearly on \( n \), \( \lambda_n = \tau^{-1}[1 + 2p(1 - 1)] \) with an arbitrary \( p \), yielding [11]

\[ R_1 = -J_{-1}/J_{-}; \quad R_{-1} = J_{1+}/J_+; \]  

\[ J_{\pm} = J_{\pm}(1 - 2pi)/(S \pm 2pi) (Q/(1 \pm 2pi/S)); \]  

\[ J_{1\pm} = J_{1\pm}(1 - 2pi)/(S \pm 2pi) (Q/(1 \pm 2pi/S)); \]  

where \( Q = qR_c \). The two most important particular cases are \( p = 0 \), corresponding to a white-noise random potential (isotropic scattering), and \( p = 1 \), describing the scattering by a white-noise random magnetic field with the scattering rate \( P(\phi) = \tau^{-1} \cot^2(\phi/2) \). In the next section we investigate \( \delta \rho_{xx} \) in these two situations, exploiting the existence of the exact analytical solution [17]. This will give us a first example of how the small angle scattering (dominant in the random magnetic field case, when \( \tau_s^{-1} = \infty \)) modifies the magnetic field dependence of the amplitude of the Weiss oscillations.
III. WHITE-NOISE RANDOM POTENTIAL VERSUS RANDOM MAGNETIC FIELD

Using Eqs. (14), (15), and (17), we find for arbitrary values of the parameter $p$

$$\langle Gv \cos \phi \rangle = -\frac{\eta^2 q l}{2(1 + S^2)} \text{Im} \left( \frac{J_+ - J_-}{J_+ + J_-} \right)^{-1} \tag{18}$$

For the particular case of isotropic scattering, $p = 0$, the following identity can be used to simplify Eq. (18):

$$J_- J_+ - J_+ J_- = \frac{2i}{S Q} \left[ J_+ J_- - \frac{\sinh(\pi/S)}{\pi/S} \right] \tag{19}$$

This allows us to reduce Eq. (18) to the form

$$\langle Gv \cos \phi \rangle = -\frac{\eta^2 (q l)^2}{4(1 + S^2)} \frac{J_+ J_-}{\sinh(\pi/S) - J_+ J_-} \tag{20}$$

Substituting (20) in (9) and assuming that $\eta$ is small enough so that $\langle Gv \cos \phi \rangle \ll 1$ (this will be valid for all $B$, if $\eta^2 q l/4 \ll 1$), we finally get

$$\begin{align*}
\frac{\delta \rho_{xx}}{\rho_0} &= \frac{\eta^2}{4} (q l)^2 \frac{J_{1/S}(Q) J_{-1/S}(Q)}{\sinh(\pi/S) - J_{1/S}(Q) J_{-1/S}(Q)} \\
&= \frac{\eta^2}{4} (Q S)^2 \frac{J_{1/S}(Q) J_{-1/S}(Q)}{\sinh(\pi/S) - J_{1/S}(Q) J_{-1/S}(Q)} \
&\equiv \frac{\eta^2}{4} (Q S)^2 \frac{J_{1/S}(Q) J_{-1/S}(Q)}{\sinh(\pi/S) - J_{1/S}(Q) J_{-1/S}(Q)} \tag{21}
\end{align*}$$

where $\rho_0 = \sigma_0^{-1}$ is the Drude resistivity. Eq. (21) is completely equivalent to Beenakker’s result (Eq. (5) of Ref. [2]), which is, however, presented in a somewhat different form of an infinite series. The equivalence follows from the Bessel function identity

$$\sum_{n=-\infty}^{\infty} \frac{J_n(z)}{\nu^2 + n^2} = \frac{\pi}{\nu \sinh \pi \nu} J_{\nu} (z) J_{-\nu} (z) \tag{22}$$

We will assume that $q l \equiv Q S \gg 1$, which is a necessary condition for the existence of oscillations. At $Q \equiv q R_c \gg 1$ (i.e. in the oscillation region), the asymptotic behavior of $\delta \rho_{xx}/\rho_0$ reads according to Eq. (21)

$$\begin{align*}
\frac{\delta \rho_{xx}}{\rho_0} &= \frac{\eta^2}{2\pi} Q S^2 \cos^2(Q - \pi/4), \quad \pi/S \ll 1 \tag{23} \\
\frac{\delta \rho_{xx}}{\rho_0} &= \frac{\eta^2}{4} Q S [1 + 2e^{-\pi/S} \sin 2Q], \quad \pi/S \gg 1 \tag{24}
\end{align*}$$

Thus, the amplitude of oscillations depends linearly on magnetic field $B$ in the high-field regime $\omega_c \gg \pi/\tau$, and gets exponentially damped in $1/B$, $(\delta \rho_{xx}/\rho_0)_{osc} \propto e^{-\pi/\omega_c \tau}$, in the lowest fields, $\omega_c \ll \pi/\tau$. Note that for a typical value of the transport time $\tau$ in the GaAs heterostructures, $\tau \sim 100$ ps, the condition $\pi/\omega_c \tau \sim 1$ corresponds to very low fields, $B \sim 0.01$ T.

We turn now to the random magnetic field case, where $p = 1$. Analyzing Eq. (18) in this case, we find the following two asymptotic regimes:

$$\begin{align*}
\frac{\delta \rho_{xx}}{\rho_0} &= \frac{\eta^2}{8} S^2 \cos^2(Q - \pi/4), \quad 4Q/S \ll 1 \\
\frac{\delta \rho_{xx}}{\rho_0} &= \frac{\eta^2}{4} Q S [1 + 2e^{-4Q/S} \sin 2Q], \quad 4Q/S \gg 1
\end{align*} \tag{25} \tag{26}$$

Again Eq. (25) corresponds to a power-law behavior of the oscillation amplitude in the range of relatively high magnetic fields $B$, while Eq. (26) describes the exponential damping of the oscillations in low fields. There is however a number of important differences between the results for the isotropic scattering [Eqs. (23), (24)] and those for the random magnetic field scattering [Eqs. (25), (26)]:

i) the “high” field behavior (25) is $\delta \rho_{xx}/\rho_0 \propto B^2$, rather than $\delta \rho_{xx}/\rho_0 \propto B$ as in Eq. (23);

ii) the value of the magnetic field, at which the exponential damping starts, is much larger in the random magnetic field case ($\omega_c \tau \sim 2\sqrt{q l}$), than for the isotropic scattering ($\omega_c \tau \sim \pi$);

iii) the low-field damping is of a Gaussian form (with respect to $1/B$) in the random magnetic field case, $(\delta \rho_{xx}/\rho_0)_{osc} \propto \exp[-4q l/(\omega_c \tau)^2]$, rather than of a simple exponential, $\exp(-\pi/\omega_c \tau)$, as in Eq. (24).

All these features are consequences of the small-angle scattering, dominant in the random magnetic field case. In the next two sections we will consider the Weiss oscillations in the long-range correlated random potential, which also leads to small-angle scattering. Since in this case the exact solution cannot be expressed in a simple analytic form, we will use different methods to find the shape of the Weiss oscillations.

IV. LONG-RANGE RANDOM POTENTIAL: EXPONENTIAL DAMPING OF WEISS OSCILLATIONS

The random potential experienced by electrons of a 2DEG in the GaAs heterostructures is very smooth, since the impurities are separated from the 2DEG layer by a relatively large distance (spacer) $d_s > k_F^{-1}$ ($k_F$ being the Fermi wave vector). Assuming that the positions of ionized donors are uncorrelated, while their density is equal to that of the electrons, $n_e = k_F^2/2\pi$, and treating the screening in the random phase approximation, one finds for the random potential correlation function $W(|r - r'|) = \langle U(r) U(r') \rangle$ in momentum space

$$\hat{W}(q) = \frac{n_e}{(2\nu^2)} e^{-2\eta d_s}, \tag{27}$$
where $\nu = m/2\pi$ is the density of states per spin orientation. The scattering cross-section in the Born approximation is thus given by

$$P(\phi) = \nu \bar{W}(2k_F \sin \phi/2) = \frac{\pi n_e}{2m} e^{-4k_F d_s \sin \phi/2}.$$  \hspace{1cm} (28)

The total ($\tau_s^{-1}$) and the momentum ($\tau^{-1}$) relaxation rates are defined as

$$1/\tau_s = \int_0^{2\pi} d\phi P(\phi) = \frac{k_F}{4md_s};$$  \hspace{1cm} (29)

$$1/\tau = \int_0^{2\pi} d\phi P(\phi)(1 - \cos \phi) = \frac{1}{16mk_F d_s^2},$$  \hspace{1cm} (30)

so that their ratio is $\tau/\tau_s = (2k_F d_s)^2 \gg 1$. Both $\tau$ and $\tau_s$ are experimentally measurable: the former is found from the dc conductivity at $B = 0$, while the latter follows from the Dingle analysis of the damping of the Shubnikov–de Haas oscillations. For the structures of the type used in the Weiss oscillation experiment \cite{1} ($n_e \sim 3 \times 10^{11}\text{cm}^{-2}$, $d_s \sim 30\text{nm}$), their typical values are $\tau \sim 50\text{ps}$, $\tau_s \sim 3\text{ps}$, so that the ratio $\tau/\tau_s \sim 20$ is large, in agreement with the theory. Note, however, that the experimental values of $\tau$ and $\tau_s$ are usually somewhat different from the theoretical estimates \cite{29,30}. This was attributed to correlations between the positions of the scatterers \cite{22,33}. Below we will express our results for the damping of the Weiss oscillations in terms of $\tau$ and $\tau_s$, so that for comparison with experiment one should simply use the experimental values of these quantities.

As was shown in \cite{3}, the physical origin of the oscillations can be traced to the additional contribution to the drift velocity of the guiding center,

$$v_{dr} = \frac{1}{2\pi B} \int_0^{2\pi} d\phi E(X + R_c \cos \phi),$$  \hspace{1cm} (31)

where $X$ is the $x$-coordinate of the guiding center and $E(x) = -U''(x) = -\left(\frac{e}{\hbar e^2 \epsilon} \sin qx\right)$ is the electric field induced by the grating. Averaging $v_{dr}^2$ over $X$, one finds \cite{3}:

$$\langle v_{dr}^2 \rangle = \frac{1}{2} \left(\frac{\eta E_F q}{2\pi e B}\right)^2 \text{Re} \int_0^{2\pi} d\phi_1 d\phi_2 e^{iqR_c (\sin \phi_1 - \sin \phi_2)}.$$  \hspace{1cm} (32)

For $qR_c \gg 1$, the oscillations are determined by the interference of the contributions from the vicinities of the saddle points $\phi_1 = 0$, $\phi_2 = \pi$ (and vice versa), leading to

$$\langle v_{dr}^2 \rangle_{osc} \propto \cos(2qR_c - \pi/2)$$  \hspace{1cm} (33)

(in this section we do not keep power-law prefactors, since we calculate the exponential damping factor only).

Impurity scattering leads to a change $\Delta X$ of the guiding center position $X$ during the time an electron moves along the cyclotron orbit from $\phi = 0$ to $\phi = \pi$. This introduces in Eq.(33) a damping factor

$$D = \langle e^{iz\Delta X} \rangle$$  \hspace{1cm} (34)

For isotropic scattering, any scattering event kicks an electron far away from its original cyclotron orbit, so that $D$ is determined simply by the probability for an electron to remain unscattered during the half cyclotron period time $\pi/\omega_c$, yielding $D \approx e^{-\pi/\omega_c \tau}$, in agreement with Eq.(24). The situation is, however, different when the scattering is of small-angle character, so that even after one or several scattering events an electron remains close to its original cyclotron orbit and contributes to the oscillations.

We first consider the random magnetic field scattering. Since in this case the exact solution has been found (Eq.(29)), this will be an additional check for the correctness of our method of calculation in this section.

In this case, the probability for an electron to be scattered by an angle $\delta \phi$ in a small time interval $\delta t$ is given by \cite{14}

$$P(\delta \phi) = \frac{1}{\pi} \frac{2\delta t/\tau}{(2\delta t/\tau)^2 + (\delta \phi)^2}$$  \hspace{1cm} (35)

Scattering by the angle $\delta \phi$ changes the position $X$ of the guiding center by $R_c \sin \delta \phi$, so that

$$\Delta X = R_c \sum_k \sin \phi_k \delta \phi_k,$$  \hspace{1cm} (36)

where $\phi_k \equiv \phi(t_k) = \omega_c t_k$; $t_k = k\pi/N \omega_c$; $k = 1, 2, \ldots, N$; $N \to \infty$. According to Eq.(33),

$$\langle e^{iz\delta \phi} \rangle = \int d(\delta \phi) P(\delta \phi) e^{iz\delta \phi} = e^{-2z\delta t/\tau}.$$  \hspace{1cm} (37)

Therefore, the damping factor $D$ is equal to

$$D = \prod_k \langle e^{i\eta R_c \sin \phi_k \delta \phi_k} \rangle = \prod_k e^{-2qR_c \sin \phi_k \delta \phi_k/\tau} = \exp \left\{ -\frac{2qR_c}{\omega_c \tau} \int_0^{\pi} d\phi \sin \phi \right\} = \exp \left\{ -\frac{4qR_c}{\omega_c \tau} \right\},$$  \hspace{1cm} (38)

in precise agreement with the exact solution, Eq.(29).

We turn now to the long-range random potential. In this case, the scattering cross-section \cite{28} implies that the distribution function of the scattering angles $\delta \phi$ in a time interval $\delta t$ is given by

$$P(\delta \phi) = \left(1 - \frac{\delta t}{\tau_s}\right) \delta(\delta \phi) + \delta t P(\delta \phi).$$  \hspace{1cm} (39)

Therefore,

$$\langle e^{iz\delta \phi} \rangle = 1 - \frac{\delta t}{\tau_s} \frac{z^2}{\tau_s^2 + (2k_F d_s)^2}$$  \hspace{1cm} (40)
This leads to the following result for the damping factor
\[ D = \prod_k \langle e^{iqR_c \sin \phi_k} \delta \phi_k \rangle \]
\[ = \exp \left\{ -\frac{(qR_c)^2}{\omega_c \tau_s} \int_0^\pi \frac{d\phi \sin^2 \phi}{(qR_c \sin \phi)^2 + (2k_F d_s)^2} \right\} \]
\[ = \exp \left\{ -\frac{\pi}{\omega_c \tau_s} \left[ 1 - \frac{1}{1 + \left( \tau_s / \tau \right)^2 (qR_c)^2} \right] \right\}. \]  
(41)

As is seen from Eq.(11), the region of exponentially damped oscillations can be further subdivided into two regimes: \( D \simeq \exp\{-\pi(qR_c)^2/2\omega_c \tau\} \) at \( (qR_c)^2 \ll \tau / \tau_s \) and \( D \simeq \exp\{-\pi/\omega_c \tau_s\} \) at \( (qR_c)^2 \gg \tau / \tau_s \). In the former the leading contribution to \( D \) comes from multiply scattered carriers, while in the latter \( D \) is governed by those electrons, which remain unscattered within the time \( \pi / \omega_c \).

The exponential damping found in the above is efficient at low enough magnetic fields, \( \omega_c \tau < \left( \pi/2 \right) \langle q \rangle^2 \). At stronger magnetic fields, the \( B \)-dependence of \( \delta \rho_{xx} \) is of a power-law nature. This will be considered in the next section using a different method.

V. LONG-RANGE RANDOM POTENTIAL: HIGH MAGNETIC FIELDS

To find the grating-induced correction to the resistivity at relatively strong fields, we return to the Boltzmann equation approach of Sec.III. For the scattering cross-section (28) the eigenvalues \( \lambda_l \) of the collision integral are equal to
\[ \lambda_l = \frac{n_e}{m} \int_0^\pi d\phi e^{-4k_F d_s \sin \phi/2} (1 - \cos \ell \phi) \]
\[ \simeq \frac{n_e}{m} \int_0^\pi d\phi e^{-2k_F d_s \phi} (1 - \cos \ell \phi) \]
\[ = \tau^{-1} \frac{l^2}{(\tau_s / \tau)^2 + 1} \]  
(42)

The approximation we will make at this point will be to consider the limit \( \tau_s / \tau \rightarrow 0 \), which reduces Eq.(12) to \( \lambda_l = l^2 / \tau \). This approximation is valid for not too small magnetic fields, such that \( \omega_c \gg qv_F (\tau_s / \tau)^{1/2} \), where the total relaxation rate \( \tau_s \) is irrelevant, and only the transport one, \( \tau \), is important. This will allow us to find \( \delta \rho_{xx} \) in the high-field regime (where the \( B \)-dependence of the oscillation amplitude is of the power-law nature), which was outside the regime of validity of the consideration in Sec.IV, where only the exponential damping factor was studied. Furthermore, we will also reproduce the result of Sec.IV for the intermediate regime \( (2/\pi) \omega_c \tau \ll (qR_c)^2 \ll \tau / \tau_s \), where the damping is already exponential, but \( \tau_s \) is still irrelevant.

Within the above approximation, the collision integral \( C\{G\} \) is replaced by the differential operator \( \tau^{-1} \partial^2 G / \partial^2 \phi^2 \), and Eq.(48) with \( G(x, \phi) = g(\phi) e^{iqx} + c.c. \) takes the form
\[ \frac{\partial g}{\partial \phi} + i q R_c \cos \phi g - \frac{1}{\omega_c \tau} \frac{\partial^2 g}{\partial \phi^2} = i \eta q / 2 \omega_c \]  
(43)

For \( \omega_c \tau \gg 1, q \ell \gg 1 \), we can solve this equation iteratively considering the \( \partial^2 g / \partial \phi^2 \) term as a perturbation. Dropping this term in (43) and differentiating with respect to \( \phi \) we find
\[ \frac{\partial^2 g}{\partial \phi^2} \simeq i q R_c \left( g \sin \phi - \frac{\partial g}{\partial \phi} \cos \phi \right) \]  
(44)

Substituting this back to Eq.(43), we get a first order differential equation, the solution of which reads
\[ g(\phi) \simeq \frac{i \eta q}{2 \omega_c} \int_{-\infty}^\phi d\phi' \]
\[ \times \exp \left\{ -i Q (\sin \phi - \sin \phi') - \frac{Q^2}{2 S} (\phi - \phi') \right\}. \]  
(45)

Here we introduced again the compact notations \( Q = q R_c, S = \omega_c \tau \). Substituting now the solution (45) into Eqs. (8), (15), we finally get
\[ \frac{\delta \rho_{xx}}{\rho_0} = \frac{\eta^2}{2} S^2 \frac{\pi Q^2 / 2 S}{\sinh(\pi Q^2 / 2 S)} I_{i Q^2 / 2 S}(Q) J_{i Q^2 / 2 S}(Q) \]  
(46)

Combining the results of this and the preceding section, we get a complete description of \( \delta \rho_{xx} \) in the full range of magnetic fields \( B \), which will be analyzed in Sec.VI.

VI. LONG-RANGE RANDOM POTENTIAL: ANALYSIS OF THE RESULTS

According to the results of Secs. III, IV, we find in general the following three regions of the magnetic field \( B \) with different behavior of the oscillation amplitude:

i) high field domain, \( Q \ll Q_1 \), with \( Q_1 = (2q \pi / \ell)^{1/3} \).

In this region \( \pi Q^2 / 2 S \ll 1 \), and Eq.(46) yields
\[ \frac{\delta \rho_{xx}}{\rho_0} = \frac{\eta^2}{\pi Q} S^2 \cos^2 (Q - \pi/4) \]  
(47)

The amplitude of the oscillations is proportional to \( B^3 \) in this regime, in contrast to the linear behavior, Eq.(23), for the isotropic random potential, and to the \( B^2 \) behavior, Eq.(25) in the case of random magnetic field scattering.

ii) intermediate domain \( Q_1 \ll Q \ll Q_2 \), with \( Q_2 = \sqrt{2 \pi / \tau_s} \). The oscillations get exponentially damped:
\[ \frac{\delta \rho_{xx}}{\rho_0} = \frac{\eta^2}{4} Q S^2 [1 + 2 e^{-\pi Q^2 / 2 S} \sin 2Q]. \]  
(48)
iii) low-field region, $Q \gg Q_2$. The exponential damping factor changes its form:

$$\frac{\delta \rho_{xx}}{\rho_0} = \frac{\eta^2}{4} Q S \left[ 1 + 2 e^{-\pi/S_s \sin 2Q} \right],$$

(49)

where $S_s = \omega_c \tau_s$. Note that Eq. (49) differs from the low-field behavior for the isotropic scattering case, Eq. (34), only in that $\tau$ is replaced by $\tau_s$ in the exponential damping factor.

The two regimes (i) and (ii) are jointly described by

$$\frac{\delta \rho_{xx}}{\rho_0} = \frac{\eta^2}{4} Q S \frac{\pi}{\sin \pi \mu J_{s\mu}(Q) J_{-s\mu}(Q)},$$

(50)

with $\mu = Q^2/2S$, while both the regimes (ii) and (iii) are given by

$$\frac{\delta \rho_{xx}}{\rho_0} = \frac{\eta^2}{4} Q S \left[ 1 + 2 \exp \left( -\frac{\pi}{S_s} \right) \sin 2Q \right].$$

(51)

Finally all the three regimes can be described by a single formula of the form (50) with

$$\mu = \frac{1}{S_s} \left[ 1 - \left( 1 + \frac{\tau_s}{\tau} Q^2 \right)^{-1/2} \right]^{-1/2}$$

(52)

Let us also note that the behavior of $\delta \rho_{xx}$ at $Q \ll 1$ (i.e. for magnetic fields stronger than those generating the oscillations) is the same for all the types of scattering,

$$\frac{\delta \rho_{xx}}{\rho_0} = \frac{\eta^2}{4} S^2,$$

(53)

as is easily found from Eqs. (18), (21), (48). In reality, in most of the Weiss oscillation experiments the magnetoresistivity at such strong fields is strongly affected by the Shubnikov-de Haas oscillations and by the quantum Hall effect. However, the giant quadratic magnetoresistance [53] was observed in the experiment [54], where the measurements were performed at somewhat higher temperatures.

Now we evaluate the boundaries $Q_1$, $Q_2$ of the regions with different behavior of the oscillation amplitude in a typical experimental situation. We find from the parameters of the sample of Ref. [1] ($q = 2\pi/382\text{nm}$, $n_e = 3.16 \times 10^{11} \text{cm}^{-2}$, $l = 12\mu\text{m}$) the transport time $\tau = 52\text{ps}$. Furthermore, a typical value of the total relaxation rate for such structures is $\tau_s \approx 3\text{ps}$ (as we will see below, with this value of $\tau_s$ our results perfectly describe the experimentally observed damping of the oscillations in low fields). This gives the following crossover values: $Q_1 \approx 5.0$, $Q_2 \approx 6.0$. For a higher mobility sample studied in Ref. [48] (the Weiss oscillation data are presented there in the inset of Fig. 1) these values are

somewhat larger: $Q_1 \approx 8.5$, $Q_2 \approx 11$. We conclude that although parametrically $Q_2/Q_1 \propto (k_F/q)^{1/3}$ is large at $k_F \gg q$, for realistic parameters $Q_2$ and $Q_1$ are numerically close to each other. Therefore, the regime (ii) corresponds in reality to a rather narrow intermediate domain between the regime (i) describing the fast ($\propto B^3$) drop of $\delta \rho_{xx}$ in high fields, and the regime (iii) corresponding to the exp($-\pi/\omega_c \tau_s$) damping of the oscillatory part of $\delta \rho_{xx}$ in low fields.

In Fig. 1 we compare our theoretical results with experimental data of Weiss et al. [1]. The sample parameters are $q = 2\pi/382\text{nm}$, $n_e = 3.16 \times 10^{11} \text{cm}^{-2}$, $\tau = 52\text{ps}$, $\tau_s = 3\text{ps}$, $\eta = 0.065$ we get a very good description of the experimentally observed magnetoresistivity [1].

![FIG. 1. Experimental data of Ref. [1] (dash-dotted line) compared with our theoretical results for the long-range potential scattering (full line). Parameters are: $q = 2\pi/382\text{nm}$, $n_e = 3.16 \times 10^{11} \text{cm}^{-2}$, $\tau = 52\text{ps}$, $\tau_s = 3\text{ps}$, $\eta = 0.065$](image)

To illustrate the difference between the long-range potential scattering and the isotropic scattering, we plotted in Fig. 2 the corresponding results for the modulation-induced resistivity correction at the same value of the transport time $\tau$. It is seen that the results for these two different scattering mechanism differ drastically not only in low magnetic fields (where the small-angle scattering leads to the exponential damping of oscillations), but also in high fields (due to the different power-law dependence of $\delta \rho_{xx}$ on $B$). In particular, already at the first maximum of $\delta \rho_{xx}$, the value of the correction is smaller by factor of a $\sim 5$ for the long-range random potential scattering, as compared to the isotropic scattering case, at the same value of the grating strength $\eta$. For the second maximum this ratio is already a factor of $\sim 10$, etc. As a consequence, one generates a considerable error when
This rapid fall-off is determined by the quadratic increase than in a white-noise random potential, where $\delta \rho_B$ falls of as the amplitude of the modulation induced correction $\delta \rho_B$ following features. In relatively strong magnetic fields $\delta \rho$ relaxation rates ($\tau$ $\propto$ $l^2$, in units of $\eta^2/4$ for the isotropic potential scattering ($\tau$ $= 52$ ps, full line) and the long-range random potential scattering ($\tau$ $= 52$ ps, $\tau_s$ $= 3$ ps, dash-dotted line). The sample parameters ($q = 2\pi/382$ nm, $n_e$ $= 3.16 \times 10^{11}$ cm$^{-2}$) are the same as in Ref. [5] and in Fig.1.

![Graph showing the modulation-induced correction to the 2DEG resistivity, $\delta \rho_{xx}/\rho_0$, in units of $\eta^2/4$ for the isotropic potential scattering VS magnetic field B.](image)

**FIG. 2.** Grating-induced correction to the 2DEG resistivity, $\delta \rho_{xx}/\rho_0$, in units of $\eta^2/4$ for the isotropic potential scattering ($\tau$ $= 52$ $\text{ps}$, full line) and the long-range random potential scattering ($\tau$ $= 52$ $\text{ps}$, $\tau_s$ $= 3$ $\text{ps}$, dash-dotted line). The sample parameters ($q = 2\pi/382$ nm, $n_e$ $= 3.16 \times 10^{11}$ cm$^{-2}$) are the same as in Ref. [5] and in Fig.1.

**VII. CONCLUSIONS**

In this paper, we have presented a theory of the Weiss oscillations of the magnetoresistivity in a 2DEG, for the case when the impurity scattering is predominantly of the small-angle nature. While this is precisely the experimental situation in currently fabricated semiconductor heterostructures, no quantitative theoretical description of this case has been available so far. Previous theories [2,3] assumed the scattering to be isotropic. We have shown that the small-angle scattering strongly affects the shape and the damping of the Weiss oscillations. This was demonstrated both for the scattering by a white-noise random magnetic field (linearly increasing eigenvalues $\lambda_l = \tau^{-1}(2l - 1)$ of the collision integral) and for long-range random potential scattering ($\lambda_l \approx \tau^{-1}l^2/[\tau_s/\tau l^2 + 1]$). In the latter case, which was of primary interest for us, the shape of the magnetoresistivity is determined by both the momentum and the total relaxation rates ($\tau^{-1}$ and $\tau_s^{-1}$, respectively), and has the following features. In relatively strong magnetic fields $B$, the amplitude of the modulation induced correction $\delta \rho_{xx}$ falls of as $B^3$ with decreasing $B$, i.e. much more rapidly, than in a white-noise random potential, where $\delta \rho_{xx} \propto B$. This rapid fall-off is determined by the quadratic increase of the eigenvalues $\lambda_l \propto l^2$ for not too large $l$. In low magnetic fields, the oscillatory part of the magnetoresistivity is found to be exponentially damped, much more efficiently than in the case of isotropic scattering with the same $\tau$. The low-field damping factor is of the form $e^{-\pi\omega_c/\tau_s}$, familiar from the theory of the Shubnikov-de Haas oscillations. Let us stress, however, that while the Shubnikov-de Haas oscillations are of purely quantum nature, the Weiss oscillations represent a classical effect. Nevertheless, due to the interplay of the finite modulation wave vector $q$ and the cyclotron radius $R_c$, the resistivity correction $\delta \rho_{xx}$ is determined by the whole spectrum of eigenvalues $\lambda_l$ of the collision integral (and, in particular, by $\lambda_\infty = 1/\tau_s$), rather than by $\lambda_1 = 1/\tau$ only (governing conventional transport in the absence of modulation). Finally, our results describe nicely the shape of the experimentally observed magnetoresistance and the dependence of the oscillation amplitude on the magnetic field.

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Note that the positive magnetoresistance in the very low fields (below 0.03T) is related to the existence of open (channeled) orbits in this regime, see P. H. Beeton, E. S. Alves, P. C. Main, L. Eaves, M. W. Dellow, M. Henini, O. H. Hughes, S. P. Beaumont, and C. D. W. Wilkinson, Phys. Rev. B 42, 9229 (1990) and Ref. [9]. This effect is beyond the perturbative-in-$\eta$ calculation employed in the present paper. Theoretically, the condition for the existence of the channeled orbits is $eB < \eta q k_F/2$, which defines a very narrow region of magnetic fields for small $\eta$. This regime is well separated from the Weiss oscillation regime if $\eta \ll a/v_F \tau_s$. For the experiment of Ref. [1] we find $a/v_F \tau_s \simeq 0.6$, and the condition is well satisfied, in agreement with experimental data of [1] presented in Fig. 1. At high fields, $B > 0.35T$ the experimental data show the onset of the Shubnikov-de Haas oscillations, which are also not included in our (essentially classical) theory.

Note that the r.m.s. amplitude $\epsilon$ of the density modulation used by Beenakker [6] is related to our parameter $\eta$ as $\epsilon = \eta/\sqrt{2}$. 