Phase coherent transmission through interacting mesoscopic systems

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Abstract

This is a review of the phase coherent transmission through interacting mesoscopic conductors. As a paradigm we study the transmission amplitude and the dephasing rate for electron transport through a quantum dot in the Coulomb blockade regime. We summarize experimental and theoretical work devoted to the phase of the transmission amplitude. It is shown that the evolution of the transmission phase may be dominated by non–universal features in the short–time dynamics of the quantum dot. The controlled dephasing in Coulomb coupled conductors is investigated. Examples comprise a single or multiple quantum dots in close vicinity to a quantum point contact. The current through the quantum point contact "measures" the state of the dots and causes dephasing. The dephasing rate is derived using widely different theoretical approaches. The Coulomb coupling between mesoscopic conductors may prove useful for future work on electron coherence and quantum computing.
I. INTRODUCTION

Electron transport on mesoscopic length scales has been intensively studied during the last fifteen years [1–4]. A variety of new phenomena has been discovered such as the weak localization, universal conductance fluctuations, Aharonov–Bohm oscillations in mesoscopic rings, or persistent currents. The origin of these phenomena is the quantum mechanical phase coherence of the electronic wave functions. The degree of coherence can be measured by the phase coherence length $L_\phi$ which is the typical length on which electrons travel without losing their phase coherence. In mesoscopic systems, $L_\phi$ exceeds the system size, and transport exhibits quantum interference effects.

How is mesoscopic transport modified by electron–electron interactions? Interactions generally play a minor role in good conductors where electron wave functions are delocalized over the whole system and screening is effective. Such systems, both in the presence and absence of disorder, have been successfully described by noninteracting electron models [5]. Interactions only lead to small corrections, e.g. they suppress the phase coherence of interfering electrons by coupling them to the bath of other electrons. In contrast to the transport in weakly interacting conductors, interactions strongly affect transport in high magnetic fields and in spatially confined geometries. Important examples are the fractional quantum Hall effect and the tunneling through small spatially confined electron islands, known as quantum dots. The electron interactions in a quantum dot give rise to a characteristic energy scale called the charging energy. Similar to the ionization energy of an atom, the charging energy is the energy necessary to remove or add a single electron to the quantum dot. The charging energy in state–of–the–art semiconductor quantum dots typically exceeds the single–particle level spacing by a factor 10 or more. Charging effects can drastically alter and nearly suppress the transport of charge through these systems at temperatures roughly below 1 K.

In this work we review the phase coherent transmission through strongly interacting systems. As a paradigm, we study the tunneling through quantum dots. The subject started in 1995 with an ingenious experiment by Yacoby et al. [6]. The authors utilized a novel interference device with a quantum dot embedded in one arm of an Aharonov–Bohm (AB) ring (see Fig. [a]). The measurement of flux–periodic current oscillations through the device proved for the first time that part of the tunneling current through a quantum dot is coherent. In a step beyond the demonstration of coherence, an improved version [7] of the experiment allowed to measure the phase of the transmission amplitude through the quantum dot. At sufficiently low temperatures, this phase is carried by a single resonant many–body state. The transmission phase displayed a number of unexpected properties. Most notably, virtually the same transmission phase was found for a whole sequence of resonant quantum dot states; between neighboring states the phase displayed a sharp phase
FIG. 1. Electron micrograph of the device used in the experiment [6]. The brighter regions indicate metallic gates; B is a metallic air bridge. Electrons flow between the regions labeled S and D through the left or the right arm of an Aharonov–Bohm ring. The quantum dot is inserted in the left arm. Taken from Ref. [6].

slip. These features have been addressed but not yet fully explained in a large number of theoretical papers.

The field expanded substantially in 1998 when it was realized that the coherence of quantum dot states can be controlled by external means. Controlled decoherence was achieved in a device with a quantum dot that was capacitively coupled to a quantum point contact in close vicinity. The quantum point contact acted as a measuring apparatus for the number of electrons on the quantum dot. Since number and phase are conjugate variables, the measurement caused the dephasing (decoherence) of electron states in the quantum dot. The loss of coherence was detected as a suppression of AB oscillations across the quantum dot. The experiments [8,9] for the first time allowed to study fundamental principles of quantum mechanics such as the number–phase or the particle–wave duality with solid state devices. New experiments in this direction, e.g. testing the suppression of tunneling due to frequent measurements, have been proposed [10,11].

The present work reviews the main theoretical and experimental results on phase coherent transport through quantum dots. We emphasize the theoretical developments. The most important formal tool is scattering theory. We use both the well–known scattering approach of Landauer to discuss aspects of noninteracting electrons and many–body scattering theory to address interaction effects. The scattering methods are supplemented by other techniques, most prominently by master equation approaches. Most of the theoretical work was motivated by and must be discussed in connection with the beautiful experiments [6–9] on quantum coherence done by researchers at the Weizmann Institute. An account of the relevant experimental results, therefore, precedes the theoretical discussions. It is as-
sumed that the reader is familiar with ballistic transport in mesoscopic systems. We assume basic knowledge of the Landauer formula, and of transport through Aharonov–Bohm rings, quantum point contacts and quantum dots. An overview about most of these concepts can be found in Refs. [1–4]. For a recent review on transport through quantum dots in the Coulomb blockade regime see Ref. [12]. Reference [13] gives a popular account of quantum dots and the Coulomb blockade.

In Sec. II we summarize the first experiments [6,7] on the phase coherent transmission through quantum dots in the Coulomb blockade regime. Both experiments aimed at measuring the phase of the transmission amplitude through a quantum dot. It turned out, that in the first experiment [6] this measurement was hampered by peculiar symmetries resulting from the two–terminal set–up used in that experiment. Section III is devoted to a discussion of these symmetries. We derive the two–terminal conductance through an AB ring with a quantum dot. The theory explains the sharp jump of the AB oscillation pattern found in the vicinity of transmission resonances. Section IV deals with the observed phase similarity of transmission peaks and with the sharp phase jump between the peaks. We show that these observations may be traced back to features of the short–time dynamics in the quantum dot that completely dominate the transport in the tunneling regime. In Sec. V we discuss the controlled dephasing of quantum dot levels due to the coupling to a mesoscopic conductors in the environment. Section VI deals with new theoretical proposals to study the dephasing of charge oscillations in coupled quantum dots. The proposed experiments could provide evidence for a fundamental quantum effect known as the quantum Zeno effect: the suppression of quantum transitions due to frequent observations with a measuring device. We conclude in Sec. VII.
II. MEASURING THE ELECTRON TRANSMISSION PHASE

The phase coherence of electronic wave functions lies at the heart of mesoscopic interference phenomena like the universal conductance fluctuations or weak–localization. In most cases, quantum interference has been studied with mesoscopic conductors that were connected to few or many conducting channels. Interaction effects are usually small in these systems due to good screening of the electron charge. Charging effects become important when the conductor is small and only weakly coupled to the external leads. A paradigm is a quantum dot in the Coulomb blockade regime.

The idea and the practical implementation of devices for measuring the electron transmission phase through a quantum dot was developed in a series of ingenious experiments [6,7] at the Weizmann Institute. The experiments utilized an Aharonov–Bohm (AB) ring with a quantum dot in one of its arms. The quantum dot was in the Coulomb blockade regime. The measurements of the electron phase revealed a number of unexpected results. The technique for measuring the electron phase and the main experimental results are described in this section.  

A. Experiment of Yacoby et al.

The first experiment addressing the electron phase through a quantum dot was realized by Yacoby et al. [6]. The experiment utilized a novel device (see Fig. 2) to measure the phase evolution through the dot against a fixed reference phase. The quantum dot was inserted in one arm of an Aharonov–Bohm ring. The basic idea was to extract the transmission phase through the quantum dot from the phase of the Aharonov–Bohm current across the ring. The device was defined by metallic gates on the top of a GaAs–AlGaAs heterostructure. The quantum dot was placed in the left arm of the ring, its conductance could be adjusted by two quantum point contacts. A third gate (the plunger gate P) controlled the area and the electrostatic potential at the dot. The dot was about 0.4μm × 0.5μm in size. A special lithographic process, invoking a metallic air bridge (B), enabled to contact the center metal gate that depleted the electrons underneath the ring’s center. Each of the arms of the Aharonov-Bohm ring supported a few conducting modes. The ring was connected to two external contacts, source (S) and drain (D), between which a small voltage was applied.

\[1\]

We do not discuss the experiment [14] that measured the phase and magnitude of the reflection coefficient of a quantum dot. This experiment [14] was performed in the integer quantum Hall regime and used a second quantum dot as an interferometer.
FIG. 2. Electron micrograph of the device used in the Yacoby–experiment. The brighter regions indicate the metal gates. Electrons flow between source and drain through the left or the right arm of the Aharonov–Bohm ring. The quantum dot is inserted in the left arm. The central metallic island is biased via an air bridge (B). Taken from Ref. [6].

The elastic mean free path in the two–dimensional electron gas was about 10 \( \mu m \) while the diameter of the Aharonov–Bohm ring was \( \approx 1 – 1.5 \mu m \). The phase coherence length \( L_\phi \) was larger than the ring’s circumference. The quantum dot had a resistance of 1M\( \Omega \) and a very small capacitance \( C \approx 160 aF \) corresponding to the charging energy \( e^2/2C \approx 0.5 \) meV. The dot contained around 200 electrons. Its average single–particle level spacing was \( \Delta \approx 40 \mu eV \). The experiment was performed at a temperature around 100 mK corresponding to the thermal energy \( kT \approx 9 \mu eV \). The intrinsic width \( \Gamma \) of the Coulomb peaks was estimated from the conductance peak height to \( \Gamma \approx 0.2 \mu eV \). These scales imply that the quantum dot was in the Coulomb blockade regime, and that the transmission at each Coulomb peak resulted from resonant tunneling through a single or few levels of the quantum dot.

The experiment [6] demonstrated for the first time that part of the tunneling current through a quantum dot is coherent. The experimental evidence is depicted in Fig. 3. Shown is the ring current vs. plunger voltage \( V_P \) for a fixed small source–drain voltage. The ring current was obtained by subtracting from the total current across the ring a large \( V_P \)–independent background due to the right arm. The Coulomb blockade in the dot creates sharp conductance peaks in the ring current for fixed magnetic field. Fixing the voltage \( V_P \) on the side of a current peak and sweeping the magnetic field, one observes periodic current oscillations. The period of the oscillations corresponds to one flux quantum threading the area of the ring, in agreement with the expected Aharonov–Bohm period. The oscillation contrast, defined as peak–to–peak current over the average current, was in the range 0.2 to 0.4.

In a next step, the current oscillations were investigated at different values of the plunger
FIG. 3. One of the ring’s current peaks as a function of the plunger gate voltage. The large current of the right arm has been subtracted. The top left part shows the Aharonov–Bohm oscillations of the current vs. magnetic field at fixed $V_P = V_m$. Inset: Oscillation contrast defined as peak–to–peak vs. average current through the dot. Taken from Ref. [6].

Voltage $V_P$. A change in $V_P$ was expected to modify both the magnitude and phase of the transmission amplitude through the dot. The experiment was motivated by the idea that the change in the transmission phase would be directly reflected in a shift of the Aharonov–Bohm oscillations which, in turn, could be seen experimentally. This one–to–one correspondence is suggested by the following argument: Suppose the ring and the leads support only one conducting channel. According to the Landauer formula, the zero–temperature current between the leads is proportional to the ring transmission coefficient $|t(E_F)|^2$ at the Fermi energy $E_F$. For fully coherent transport, $t = t_R \exp(2\pi i \Phi/\Phi_0) + t_L$, where $t_R$ and $t_L$ are the transmission amplitudes through the right and left arm, respectively, $\Phi$ is the flux through the ring and $\Phi_0$ the flux quantum. This yields the interference term

$$2\text{Re}\{t_L t_R^* \exp(-2\pi i \Phi/\Phi_0)\} = 2|t_R||t_L| \cos(\xi_L - \xi_R - 2\pi \Phi/\Phi_0),$$

(2.1)

where $\xi_R = \text{arg}(t_R)$, $\xi_L = \text{arg}(t_L)$. Any shift in the phase $\xi_L - \xi_R$ should thus be directly reflected in a shift of the Aharonov–Bohm oscillations.

The above argument turned out to be incorrect. It neglects multiple reflections through the ring. The argument fails in particular for a two–terminal geometry, i.e. a ring connected to two external leads. It was realized [15–19] shortly after the Yacoby–experiment that Onsager symmetries valid for a two–terminal device restrict the phase of the Aharonov–Bohm oscillations to either 0 or $\pi$, spoiling the correspondence between the Aharonov–Bohm phase and the transmission phase through the quantum dot. A discussion of this issue is presented.
in Sec. III. Despite of its failure for a two–terminal ring, the simple phase–argument catches the essential idea for measuring the electron phase as it was later realized in the multiple–terminal device of Schuster et al. [7].

Figure 4 shows the ring current and the Aharonov–Bohm oscillations measured at three successive peaks. The oscillations at similar points (A, B, and C in the figure) have the same Aharonov–Bohm phase. This repetition of the phase was found within the whole sequence of Coulomb peaks (comprising 12 peaks) investigated in the experiment. The evolution of the Aharonov–Bohm phase along a single Coulomb peak is displayed in Fig. 5. Four different interference patterns taken at the points 1, 2, 3, and 4 specified in Fig. 5(a) are shown in Fig. 5(b). The Aharonov–Bohm phase of the patterns shifts by $\pi$ as one sweeps through the peak. The shift happens rather abruptly between the points 2 and 3. This can be seen in Fig. 5(c) which summarizes the phase measurements along a Coulomb peak. The sharp jump in the measured Aharonov–Bohm phase is contrasted with the expected phase evolution of the transmission amplitude assuming resonant tunneling through a single level of the quantum dot. The latter phase increases smoothly on the scale of the peak width (which is of order $kT$). Theoretical arguments [15–19] proved that there is no scale associated with the rather sharp jump seen in the experiment.

B. Experiment of Schuster et al.

The Aharonov–Bohm phase in a two–terminal device is restricted to 0 or $\pi$. No such restriction exists in a multi-terminal probe. This motivated Schuster et al. [7] to perform an
FIG. 5. Evolution of the Aharonov–Bohm phase along a current peak. (a) Current vs. gate voltage at a current peak. (b) A series of interference patterns taken at the points specified in (a). The phase jumps between patterns 2 and 3. (c) Phase measured at two peaks (circles and triangles). The broken line is guide to the eye. The expected behavior of the quantum dot transmission phase in a 1D resonant tunneling model is shown by the solid line. Taken from Ref. [6].

interference experiment similar to the Yacoby–experiment [6] but with more than two leads connected to the interferometer. The electron micrograph of the device and a schematic description of the experiment are shown in Figs. 6, 7. The central element of the device is an Aharonov–Bohm ring with a quantum dot embedded in its right arm. Several contacts are connected to ring, namely the emitter (E), the collector (C) and a base region (B). The base contacts were held at zero chemical potential. Incorporated in the structure are additional barriers. They reflect diverging electrons into the two–slit device and towards the collector. The reflectors were necessary to enhance the collector signal that could otherwise not be measured. All contacts were defined by negatively biased gates on top of the heterostructure.

The quantum dot contained roughly 200 electrons with a mean single–particle level spacing around 55 $\mu$eV. The temperature of the two–dimensional electron gas was 100 $mK$. The intrinsic (zero–temperature) width $\Gamma$ of the Coulomb peaks was estimated to be of the order or even larger than $kT$. This was achieved by slightly opening the point contacts between the quantum dot and the ring. The large value of $\Gamma$ corresponds to a quantum dot resistance larger than the resistance quantum $R_K = h/e^2$ but much smaller than the dot resistance in the Yacoby–experiment. Working with a modest quantum dot resistance
FIG. 6. Scanning electron micrograph of the double–slit device used in the Schuster–experiment [7]. The grey areas are metallic gates on the top of the heterostructure. The quantum dot is inserted in the right slit. Taken from Ref. [7].

FIG. 7. Schematic of the device structure of the Schuster–experiment. An Aharonov–Bohm ring is coupled to an emitter (E), a collector (C) and a base region (B). Reflector gates reflect diverging electrons towards the collector. The quantum dot is defined by the central electrode and the three electrodes on the right hand side of the structure. Taken from Ref. [7].
enabled Schuster et al. to measure the Aharonov–Bohm oscillations not only at the Coulomb peaks but also in between.

Schuster et al. investigated the voltage drop $V_{CB}$ between collector and base for a fixed excitation voltage $V_{EB}$ applied between the emitter and the base. The Aharonov–Bohm interference in the transmission coefficient $T_{EC}$ leads to an oscillatory contribution to $V_{EC}$ from which the Aharonov–Bohm phase is extracted. Fig. 8(a) shows $V_{EC}$ measured as a function of the gate voltage $V_P$ for a fixed magnetic field. One observes pronounced resonance peaks as expected for a quantum dot in the Coulomb blockade regime. When the magnetic field is changed, the collector signal shows AB oscillations with the expected period $\Delta B = \Phi_0/A$ where A is the area of the AB ring. The observed oscillation patterns measured at the four points 1, 2, 3, and 4 close to a resonance are shown in Fig. 8(b). The oscillation pattern shifts smoothly as one moves through the resonance. Fig. 8(c) displays the phase and the squared magnitude of the AB signal at a resonance peak. The data points are represented by full circles. The phase shows the expected monotonic rise by $\pi$ over the width of the resonance.

Schuster et al. compared their data with a theoretical model for resonant transmission. They described the coherent part $t_{QD}$ of the transmission amplitude through the dot by the Breit–Wigner ansatz

$$t_{QD} = iC_N \frac{\Gamma_N/2}{E_F - E_N + i\Gamma_N/2}, \quad (2.2)$$

where $C_N$ is a complex amplitude, $E_F$ the energy of the electrons transmitted through the device, and $E_N$ and $\Gamma_N$ the energy and the width of the resonance in the quantum dot. Both the squared magnitude $|t_{QD}|^2$ and the phase $\arg t_{QD}$ are compared with experimental data in Fig. 8(c). Good agreement is found when $\Gamma_N \approx 4kT$ is used as a fit parameter.

The collector voltage $V_{CB}$, the magnitude, and the phase of the AB oscillations measured over a sequence of five peaks are shown in Fig. 9. The striking observation is that the phase is very similar at all peaks. The phase increases roughly by $\pi$ at each peak. Note that the peaks widen and start to overlap as the plunger voltage increases. At the same time, the overall variation of the phase is reduced. The likely origin of these effects is the electrostatic influence of the plunger on the point contacts at the quantum dot. They open slightly with increasing plunger voltage. A striking feature of the data is the sharp phase lapse by $\pi$ between the resonances. The phase lapse occurs when the magnitude of the AB oscillations vanishes. Unlike the phase jump at resonance found in the Yacoby–experiment, there is a scale associated with the phase lapse between the peaks. This becomes evident for increasing $V_P$ when the lapse smoothens and is resolved experimentally. While Schuster et al. were able to model the phase evolution at resonance, they could not explain the phase behavior between resonances. The origin of the phase lapse, the associated energy scale, and the similarity of the phase at subsequent peaks is addressed in Sec. IV.
FIG. 8. Conductance and phase evolution along a Coulomb peak. (a) Resonance peaks as a function of the plunger gate voltage. (b) A series of interference patterns taken at the points specified in (a). (c) Squared magnitude and phase of the Aharonov–Bohm oscillations (dots). The solid and dashed line are fits for the phase and the squared magnitude obtained with a Breit–Wigner model. Taken from Ref. [7].
FIG. 9. (a) A series of Coulomb peaks; (b) Magnitude of the Aharonov–Bohm oscillations; (c) Phase of the Aharonov–Bohm oscillations as a function of plunger gate voltage. The solid lines are guides to the eye. Taken from Ref. [7].

III. PHASE EVOLUTION IN A TWO–TERMINAL DEVICE

Electrons that tunnel through a quantum dot interact with many other particles including electrons in the dot and in the surrounding gates, phonons in the substrate, photons from the environment, etc. The interactions may spoil or even destroy the phase coherence of the transmitted electrons. The Yacoby–experiment [6] demonstrated for the first time that part of the tunnel current is coherent. As discussed in Sec. II, the experiment employed a quantum dot in an AB ring. Phase coherence was demonstrated by measuring AB oscillations of the ring current. The oscillations displayed a rigid phase, the phase taking on only two values 0 and $\pi$ with abrupt jumps between these values. In this section, we review theoretical explanations [15–19] of this observation. We show that the phase rigidity is imposed by the two–terminal character of the Yacoby–experiment (Sec. III A). Information about about the transmission phase through the quantum dot may still be obtained from the amplitude of the AB oscillations (Sec. III B).
A. Phase rigidity

The argument for the phase rigidity was presented by Levy Yeyati and Büttiker [15]. Their argument is based on reciprocity relations for the transmission amplitude derived by Büttiker [20,21]. These relations state that the two–terminal conductance is an even function of the flux through the ring:

\[ G(\Phi) = G(-\Phi). \] (3.1)

This relation holds for an arbitrary number of transverse channels in the leads. We briefly sketch a derivation of Eq. (3.1), valid for one channel in each lead: Time–reversal invariance implies \( t_{12}(\Phi) = t_{21}(-\Phi) \) for the transmission amplitude between the two leads labeled 1 and 2. The unitarity of the scattering matrix yields \( |t_{12}(\Phi)|^2 = |t_{21}(\Phi)|^2 \). Combining these relations, one finds \( |t_{12}(\Phi)|^2 = |t_{12}(-\Phi)|^2 \). Substitution in the Landauer formula yields the claimed symmetry \( G(\Phi) = G(-\Phi) \).

The conductance through the AB ring is a periodic function of \( \Phi \). Thus, \( G \) can be expanded in a Fourier series

\[ G(\Phi) = G_0 + G_1 \cos(2\pi \Phi/\Phi_0 + \delta_1) + \ldots, \] (3.2)

where the dots indicate the higher harmonics. Combining the expansion (3.2) with the symmetry (3.1), one finds that the phase \( \delta_1 \) can only be 0 or \( \pi \) (or equivalently an even or odd multiple of \( \pi \)). In the Yacoby–experiment, all expansion coefficients are a function of the plunger voltage \( V_P \) at the quantum dot. If the phase \( \delta_1 \) changes upon variation of \( V_P \), this change must be a sharp jump by \( \pi \).

The above explanation of the phase rigidity relies on the symmetry relation (3.1). This relation remains valid even in the presence of inelastic scattering [15]. Inelastic processes can be included adopting a model suggested by Büttiker [21]: In addition to the two physical leads 1 and 2, the model includes a fictitious third lead that connects the AB ring with a phase–randomizing reservoir. The condition that no current flows through the third lead determines the chemical potential of the attached reservoir. The two–terminal conductance is found [21] to be

\[ G = \frac{2e^2}{\hbar} \left[ T_{12} + \frac{T_{23}T_{31}}{T_{32} + T_{31}} \right], \] (3.3)

where \( T_{ij} \) is the transmission coefficient from lead \( j \) to lead \( i \). The term \( T_{12} \) describes the direct elastic transmission between leads 1 and 2 while the second term on the right hand side

\[ ^2 \text{This symmetry was, implicitly, formulated in earlier work of Gefen, Imry, and Azbel} \] \[ ^2 \text{[22]} \]
FIG. 10. Amplitude of the AB harmonics vs. the phase \( \theta \). The quantum dot displays transmission peaks around \( \theta = 70 \) and \( \theta = 170 \). Taken from Ref. [18].

of Eq. (3.3) accounts for the inelastic transmission. Using the conservation of transmission probability \( \sum_n T_{mn} = 1 \) (for a single channel in each lead), one can write Eq. (3.3) in the form

\[
G = 2 \frac{e^2}{h} \left| 1 - T_{11} + \frac{T_{13}T_{31}}{1 - T_{33}} \right|.
\]  (3.4)

The reciprocity relations \( T_{mn}(\Phi) = T_{nm}(-\Phi) \) imply that \( G \) is an even function of flux. Consequently, the phase rigidity persists even in the presence of inelastic scattering. The only effect of inelastic processes is to reduce the amplitude of the AB oscillations by decreasing the direct elastic transmission \( T_{12} \).

The nature of the AB oscillations close to a phase jump was studied both theoretically and experimentally by Yacoby, Schuster and Heiblum [18]. These authors showed that the phase jump occurs at the point where the amplitude of the \( h/e \) periodic AB oscillations vanishes. At that point, the ring conductance is dominated by \( h/2e \) periodic oscillations. The theoretical analysis in Ref. [18] is based on a formula derived by Gefen, Imry and Azbel [22] for the two–terminal conductance across a single–channel AB ring with single scatterers in each arm,

\[
G = \frac{2e^2}{h} \left| \frac{2(Ae^{i\phi} + Be^{-i\phi})}{De^{2i\phi} + Ee^{-2i\phi} + C} \right|^2. 
\]  (3.5)

Here \( \phi = \pi \Phi/\Phi_0 \) is the dimensionless flux through the ring, and the constants \( A, \ldots, E \) defined in Ref. [22] are functions of the scattering amplitudes of the scatterers in the ring. The quantum dot in one of the arms is modeled as a 1D–resonant tunneling device. The reflection and transmission amplitude through the dot can be written in terms of the scattering
amplitudes of the adjacent tunneling barriers and a phase $\theta$ accumulated by the motion in the dot. A variation of the plunger voltage at the dot is simulated by a variation in $\theta$.

The predicted amplitudes of the $h/e$ and the $h/2e$ harmonics are shown in Fig. 10. The $h/e$ component grows as a conductance peak is approached from the left. Near the peak the $h/e$ component drops and vanishes for some value of $\theta$. The current oscillations are then dominated by the $h/2e$ oscillations. Upon further increase of $\theta$, the $h/e$ component changes sign and grows again. The change in the sign of the $h/e$ component is detected experimentally as a jump by $\pi$ in the AB phase.

Jacoby et al. [18] corroborated their theoretical arguments by a measurement of the AB oscillations through a ring with an artificial impurity. A small gate inserted in one arm of the ring creates a potential barrier for the electron motion. The potential at the gate can be varied by means of a metallic air bridge. The impurity replaces the quantum dot in the earlier experiment [6]. The AB oscillations of the ring resistance are displayed in Fig. 11 for increasing impurity strength. One observes that the $h/e$ component has a rigid phase. The $h/2e$ oscillations become strong when the $h/e$ oscillations vanish. The phase jump in the $h/e$ oscillations is thus associated with a qualitative change in the AB oscillations: The $h/e$ oscillations vanish and the AB oscillations show $h/2e$ periodicity.

The symmetry (3.1) of the two–terminal conductance strictly holds only in the linear response limit of infinitesimally small currents and voltages. Bruder et al. [17] investigated the nonlinear response regime using a tunneling Hamiltonian description of the quantum dot. They expressed the current through the AB ring in terms of Green functions of the quantum dot. This description includes charging effects in a non–perturbative way. Bruder
et al. recover a symmetric conductance in the linear response limit. In the nonlinear response regime they find derivations from the symmetry under sign change of the external flux. As a consequence, the phase of the AB oscillations can change continuously with the voltage at the quantum dot when a finite voltage difference is applied across the AB ring.

B. Aharonov–Bohm current

The phase of the AB oscillations in a two–terminal measurement is rigid and yields little information about the transmission phase through the quantum dot. The latter phase, however, strongly affects the amplitude of the AB oscillations as was first shown by Hackenbroich and Weidenmüller [16,19]. Their approach starts from a description of the dot and the AB ring in terms of a tunneling Hamiltonian. With little modification this Hamiltonian also serves as the starting point for the discussion in Sec. [V]. We introduce the model in Sec. [III B]. The derivation of the AB amplitude is sketched in Secs. [III B 2, III B 3]. The calculation is based on single–particle scattering theory and the Landauer formula. This restricts the approach to the regime near the conductance peaks and to temperatures \( kT \ll \Delta \), where \( \Delta \) is the single–particle level spacing of the quantum dot.

1. Tunneling Hamiltonian

Consider the system schematically represented in Fig. [12]. A conducting ring threaded by the magnetic flux \( \Phi \) is connected to two leads. The ring and the leads may support several transverse channels. A quantum dot is embedded in one arm of the ring. The quantum dot is weakly coupled to the ring by potential barriers on either side of the dot. The quantum dot is in the Coulomb blockade regime. The electrostatic potential at the dot is controlled by the gate voltage \( V_g \).

The system is described in terms of the tunneling Hamiltonian

\[
H = H_0 + H_T, \tag{3.6}
\]

where \( H_0 \) describes the isolated subsystems and \( H_T \) the couplings between these subsystems. Explicitly, \( H_0 \) is given by

\[
H_0 = \sum_{mr} \int dE E a_{mE}^{\dagger} a_{mE}^{\dagger} + \sum_{i} \epsilon_i d_i^{\dagger} d_i + \sum_{\lambda} \mathcal{E}_\lambda c_{\lambda}^{\dagger} c_{\lambda} + U, \tag{3.7}
\]

where \( r = 1, 2 \) labels the two leads, \( m \) the channels in either lead and \( i \) and \( \lambda \) the single–particle states in the AB ring and the quantum dot, respectively. The respective single–particle energies are denoted by \( E \) (the longitudinal energy in a channel), \( \epsilon_i \) and \( \mathcal{E}_\lambda \). The interaction \( U \) has the form
FIG. 12. An AB ring threaded by the magnetic flux $\Phi$ is connected to two external leads. A quantum dot is embedded in one arm of the ring. Tunnel barriers between the dot and the ring are modeled by matrix elements $V_{L,R}$. The coupling between the ring and the leads is described by the matrix elements $W_{1,2}$.

$$U = \frac{1}{2} U_0 (\hat{N}^2 - \hat{N}), \quad (3.8)$$

where $\hat{N} = \sum_\lambda c_\lambda^\dagger c_\lambda$ is the total number of electrons on the dot, $U_0 = e^2/C$ the charging energy, and $C$ the total capacitance between the dot and its surroundings. We use the standard picture of the Coulomb blockade [23] and assume that the energies $E_\lambda$ vary linearly with the gate voltage

$$E_\lambda = E_\lambda^0 + \alpha V_g, \quad (3.9)$$

where $\alpha$ is some function of the capacitance matrix of the system. Below we assume that the energies $E_\lambda$ are non-degenerate.

The couplings have the form

$$H_T = \left(\sum_{mri} \int dE W_{mi}^r (E) a_{mei}^r \dagger d_i + \text{H.c.}\right) + \left(\sum_{ip\lambda} V_{p\lambda i}^r c_\lambda^\dagger d_i + \text{H.c.}\right), \quad (3.10)$$

where the matrix elements $W$ couple states in the ring to states of the leads, $V$ provides the coupling between states in the ring and the dot, and $p = L, R$ labels either side of the dot. All matrix elements do not change appreciably on the scale of the charging energy. This weak energy dependence will be neglected below.

It follows from time-reversal invariance that for vanishing magnetic flux through the ring, all matrix elements can be taken real. To account for non-zero flux $\Phi$ through the AB ring, we attach to each matrix element $V_{\lambda i}^R$ a factor $\exp(i\phi)$, where $\phi \equiv 2\pi\Phi/\Phi_0$. We can thus write

$$V_{\lambda i}^L = V_{\lambda i}^{L*} = v_{\lambda i}^L, \quad (3.11)$$

$$V_{\lambda i}^R \exp(-i\phi) = V_{\lambda i}^{R*} \exp(i\phi) = v_{\lambda i}^R, \quad (3.12)$$
where $v^L_{\lambda i}$ and $v^R_{\lambda i}$ are real. This parameterization is adequate whenever the single–particle states both in the AB ring and in the dot do not change appreciably with flux, i.e. whenever the flux through each arm of the ring and through the dot is smaller than $\Phi_0$.

2. Scattering matrix

Near transmission resonances and for temperatures $kT$ much less than the single–particle level spacing, scattering through a quantum dot may be described in a single–particle picture. The scattering matrix may then be derived explicitly. We first sketch the calculation for noninteracting electrons (the limit of zero charging energy $U_0 \to 0$) and then indicated the changes resulting from nonzero $U_0$.

The scattering matrix $S = 1 - 2\pi i T$ at energy $E$ and (dimensionless) flux $\phi$ is obtained from the Lippmann–Schwinger equation

$$T = H_T + H_T (E - H_0 + i\eta)^{-1}T$$

(3.13)

for the transition operator $T$. Here, $\eta$ is positive infinitesimal. Iteration of the Lippmann–Schwinger equation yields the Born series

$$T = H_T + H_T \frac{1}{E - H_0 + i\eta} H_T + H_T \frac{1}{E - H_0 + i\eta} H_T \frac{1}{E - H_0 + i\eta} H_T + \cdots$$

(3.14)

For non–interacting electrons the right–hand–side reduces to a geometric series which can easily be resumed. For the $S$–matrix element $S_{mn}^{rs}$ connecting channel $n$ in lead $s$ with channel $m$ in lead $r$ one obtains the result

$$S_{mn}^{rs}(E, \phi) = \delta^{rs} \delta_{mn} - 2\pi i [W D^{-1}_{\text{ring}} W^\dagger]_{mn}^{rs} - i [\gamma D^{-1}_{\text{dot}} \bar{\gamma}]_{mn}^{rs},$$

(3.15)

where we used the shorthand $W$ and $V \equiv V^L + V^R$ for the coupling matrix elements and introduced the propagators

$$D_{\text{ring},ik} = (E - \epsilon_i)\delta_{ik} + i\pi [W^\dagger W]_{ik},$$

(3.16)

$$D_{\text{dot},\lambda\mu} = (E - \mathcal{E}_\lambda)\delta_{\lambda\mu} - [V D^{-1}_{\text{ring}} V^\dagger]_{\lambda\mu},$$

(3.17)

for the ring and the dot, respectively. The partial width amplitudes $\gamma^{s}_{m\lambda}$, $\bar{\gamma}^{s}_{m\lambda}$ are given by

$$\gamma^{r}_{m\lambda} = \sqrt{2\pi} [W (D^{-1}_{\text{ring}})^\dagger V^{s\dagger}]_{m\lambda},$$

(3.18)

$$\bar{\gamma}^{s}_{n\mu} = \sqrt{2\pi} [W (D^{-1}_{\text{ring}})^\dagger V^{s\dagger}]_{n\mu},$$

(3.19)

The terms on the right–hand–side of Eq. (3.15) have a simple physical interpretation. The first term involving the Kronecker deltas describes the reflection from channel $m$ back into the same channel. The second term is flux and voltage independent and accounts for the
scattering of electrons through the free arm of the ring. The last term describes the scattering of electrons through the dot. The entire flux and voltage dependence resides in this term (via both $D_{\text{dot}}$ and the partial width amplitudes). Note that the flux dependence does not reduce to a phase factor $\exp(i\phi)$ as one naively might have expected. The origin for the more complicated flux dependence is the multiple scattering of electrons through the ring which gives rise to higher harmonics of the current oscillations.

In the Coulomb blockade regime, the quantum dot is weakly coupled to the AB ring. The quantum dot resonances are then isolated i.e. their spacing is much larger than their width. In this case and for $E$ close to a resonance, one can approximate $D_{\text{dot}}$ by the diagonal matrix

$$
(D_{\text{dot}})_{\lambda\mu} \equiv (E - \mathcal{E}_\lambda - \Delta \mathcal{E}_\lambda + i\Gamma_\lambda/2)\delta_{\lambda\mu}.
$$

(3.20)

Here, $\Delta \mathcal{E}_\lambda$ is the energy shift (of the resonance position with respect to the bound state energy of the isolated system), and $\Gamma_\lambda$ the total width of the resonance with index $\lambda$. Using the unitarity of $S$, one can show that for isolated resonances

$$
\Gamma_\lambda = \sum_{mt} |\gamma^t_{m\lambda}|^2 = \sum_{mt} |\tilde{\gamma}^t_{m\lambda}|^2.
$$

(3.21)

Note that because of the oscillatory behavior of the partial width amplitudes, also $\Gamma_\lambda$ is an oscillating function of flux. We further note that the $S$-matrix (3.15) has the well-known symmetry properties

$$
S(\phi)S^\dagger(\phi) = S^\dagger(\phi)S(\phi) = 1,
$$

(3.22)

$$
S(\phi)S^*(\phi) = S^*(\phi)S(-\phi) = 1,
$$

(3.23)

that follow from current conservation and full time–reversal invariance (including a reversal of the magnetic field). Both symmetries together imply that the linear conductance is an even function of magnetic flux \[21\] and impose the phase rigidity as discussed in Sec. III A.

The $S$–matrix (3.15) for non–interacting electrons may be generalized to the case of nonzero charging energy using the Hartree–Fock approximation. For isolated quantum dot resonances, one finds \[19\] that the scattering matrix $S(E_F, \phi)$ at the Fermi energy has the matrix elements

$$
S_{rs}^{\alpha\beta}(E_F, \phi) = \delta^{rs}\delta_{mn} - 2\pi i[W_D^\dagger W^\dagger]_{rn}^{rs} - i \sum_\lambda (E_F - E_\lambda + i\Gamma_\lambda/2)^{-1}\bar{\gamma}^{t\alpha}_{n\lambda},
$$

(3.24)

where $\gamma_{m\lambda}$, $\bar{\gamma}^{t\alpha}_{n\lambda}$, and $\Gamma_\lambda$ are defined in Eqs. (3.18), (3.19), and (3.21), respectively. The $S$–matrix has the same form as in the non–interacting case (3.15). However, the single–particle energies $\mathcal{E}_\lambda$ have been replaced by the resonance energies $E_\lambda$ which are determined by the set of self–consistent equations
\[ E_\lambda = E_\lambda + \Delta E_\lambda + U_0 \sum_{\mu \neq \lambda} \langle n_\mu \rangle, \]  

where the average occupation probability \( \langle n_\lambda \rangle \) of the level \( \lambda \) is given by

\[ \langle n_\lambda \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} dE f(E-E_F) \frac{\Gamma_\lambda}{(E-E_\lambda+i\Gamma_\lambda/2)^2}. \]

Here \( f(E-E_F) = \frac{1}{[1+\exp((E-E_F)/kT)]} \) denotes the Fermi function. The solution to these equations is easily found when the single-particle levels are non-degenerate. Whenever one level becomes filled, all higher levels are shifted upwards by an amount equal to the charging energy. In effect, the Hartree–Fock approach reduces to a picture of isolated resonances with a ‘stretched’ level spacing due to the Coulomb interaction.

We note that the mean-field approach used here is only valid close to the resonances and at temperatures \( kT \) much less than the single-particle level spacing \( \Delta \) in the quantum dot. The conductance at higher temperatures \( kT \sim \Delta \) can no longer be reduced to a single-particle scattering problem. The sequential (incoherent) tunneling through a quantum dot in the regime \( kT \sim \Delta \) has been studied by Beenakker \[23\] and by Meir et al. \[34\].

3. Conductance

The dimensionless conductance \( g = (h/e^2)G \) of the AB ring with the quantum dot is obtained from the multi-channel Landauer formula

\[ g = 2 \int dE \left( -\frac{\partial f}{\partial E} \right) \sum_{m,n=1}^{N} |t_{mn}(E)|^2. \]

Here \( t_{mn}(E) = S_{mn}^{12}(E) \) is the transmission amplitude through the ring for an electron entering the ring via channel \( n \) in lead two, and leaving it via channel \( m \) in lead one. The derivative of the Fermi function \( f \) is given by \( -\frac{\partial f}{\partial E} = (4kT)^{-1} \cosh^{-2}\left((E-E_F)/2kT\right) \), and \( E_F \) is the Fermi energy in the leads. A factor two accounts for the spin degeneracy of the electron.

It is assumed that both the charging energy \( U_0 \) and \( kT \) are much larger than the resonance widths \( \Gamma_\lambda \). Moreover, \( kT \) shall be much smaller than the single-particle level spacing \( \Delta \). Then an appreciable current can pass the dot only if a resonance in the dot is close to the Fermi energy, \( E_\lambda \approx E_F \). The contribution of other resonances to the transmission amplitude can be neglected. To simplify notation, we suppress the index \( \lambda \) below. According to Eq. (3.24), the transmission amplitude has the form

\[ t_{mn} = t_{\text{ring},mn} - i \frac{\gamma_{mn}^{1,2*}}{E - E' + i\Gamma/2}, \]
where $E'$ is the resonance energy and $t_{\text{ring},mn} = -2i\pi \sum_{ik} W_{mi}^{1}(D^0)_{ik}^{-1} W_{nk}^{2*}$ is the transmission through the free arm of the ring. This contribution is independent of $\phi$ and $V_g$ and only weakly dependent on energy (this energy dependence will be neglected).

We now confine ourselves to a symmetric dot where $v_i^L = v_i^R \exp(i\chi) = v_i$. Due to time–reversal symmetry, $v_i$ can be chosen real, and $\chi$ can only take the values 0 and $\pi$. Substituting Eq. (3.28) into the Landauer formula, summing over the channels and integrating over energy, one finds that the conductance can be written as

$$g = g_{\text{ring}} + \left[1 + \cos(\phi - \chi)\right]x \frac{\Gamma}{kT} B \sin(\xi - \beta)$$

$$+ \left[1 + \cos(\phi - \chi)\right]y \frac{\pi \Gamma}{2kT} \cosh^{-2} \left(\frac{E_F - E'}{2kT}\right), \quad (3.29)$$

where $g_{\text{ring}}$, $x$, $y$ are positive coefficients, $\xi$ is a real phase shift, and $\Gamma = (1 + \cos[\phi - \chi])^{-1}\Gamma$ is the flux–independent width. We note that $g$ is an even function of the magnetic flux $\phi$ as required for a two–terminal measurement. Both the amplitude $B$ and the phase $\beta$ are functions of $(E_F - E')/(kT)$ (see Fig. 13). The phase $\beta$ may be identified with the transmission phase through the quantum dot. It takes the value $\pi/2$ at $E = E'$, while approaching 0 for $E \to -\infty$ and $\pi$ for $E \to \infty$, respectively.

Equation (3.29) allows us to discuss the current through the AB ring as a function of energy (voltage $V_g$) and magnetic flux. We refer to the first term on the right hand side of Eq. (3.29) as background term, to the second term as interference term, and to the third term as resonance term. The background term comprises the bulk part of the current. This term is independent of both energy and flux. It is due to transmission of electrons through the free arm of the AB ring. This term is subtracted in the experimental analysis from the total current through the device. The other terms are smaller than the background term by a factor $\Gamma/kT \ll 1$. The interference term is due to the interference of the background amplitude with the transmission amplitude for passage through the quantum dot. This term displays resonant behavior and depends explicitly on the geometry of the ring through the phase shift $\xi$. The amplitude of the resonance term has the usual temperature dependence known for thermally broadened resonances [23] of quantum dots directly coupled to leads. However, in the AB geometry this term is flux–modulated and contributes to the AB oscillations.

We now focus on the AB phase. As a function of the voltage at the dot, only the amplitude of the current oscillations is changed. The AB phase is unaffected unless the interference term changes sign. In this case, the AB phase jumps by $\pi$. The location of the phase jump depends on the system specific phase shift $\xi$. Eq. (3.29) predicts that all higher harmonics vanish identically. This is a consequence of keeping only the resonance closest to $E_F$ and neglecting the far away resonances. Such resonances as well as an asymmetric dot
produce higher (e.g. $\Phi_0/2$ periodic) harmonics which may be observable whenever the $\Phi_0$ periodic harmonic vanishes.

Inelastic scattering in the quantum dot destroys the unitarity of the $S$-matrix. Phenomenologically, this increases the widths of the transmission resonances $\Gamma$ by the addition of an inelastic width $[25,21]$ and thereby reduces the amplitudes of both the interference term and the resonance term. The fundamental property of the current to be an even function of the applied flux remains unchanged. As a result, with properly rescaled amplitudes, Eq. (3.29) accounts for the current through the system even in the presence of inelastic scattering. The basic conclusions concerning the temperature and flux dependence of the various terms remain valid.
IV. TRANSMISSION PHASE THROUGH A QUANTUM DOT

The experiment of Yacoby et al. [6] proved for the first time that part of the transmission through a quantum dot in the Coulomb blockade regime is coherent. The phase of the transmission amplitude through the dot could not be measured in this experiment due to the phase rigidity discussed in Sec. III. The problem of phase rigidity was solved by Schuster et al. [7] who replaced the two–terminal AB ring by a multiple–terminal device. The AB phase in the Schuster–experiment increased roughly by $\pi$ whenever the gate voltage at the quantum dot was swept through a transmission resonance, and the profile of the phase evolution was well described by a Breit–Wigner formula. At the same time, the phase displayed unexpected properties: (i) It was nearly identical for all resonances investigated in the experiment, and (ii) between each pair of adjacent resonances the phase slipped roughly by $\pi$ on a very small energy scale. Both observations (i) and (ii) were totally unexpected. The theoretical work addressing these observations is reviewed in this section.

Our understanding of the transmission phase is largely based on quantum mechanical scattering of non–interacting particles. The one–dimensional analogue of a quantum dot is a double–barrier well as depicted in Fig. 14(a). The tunneling barriers mimic negatively biased gates on either side of the dot. A change in the gate voltage at the quantum dot is modeled by a shift of the potential in the well. The calculation of the transmission amplitude $t$ through the double–barrier well is a textbook problem. The transmission coefficient $|t|^2$ and the transmission phase $\text{arg}(t)$ are shown in Fig. 14(b) as a function of the well potential. The phase increases smoothly by $\pi$ over the width of a transmission resonance and stays nearly constant between resonances. As a result, the phase of neighboring resonances always differs by $\pi$. We refer to such resonances as off–phase resonances below; resonances with similar transmission phase will be called in–phase resonances. The appearance of off–phase resonances is not a special property of the double–barrier quantum well but holds generally for scattering through time–reversal invariant strictly one–dimensional systems. In such systems, the wave functions can be chosen real. The wave functions representing neighboring resonant states differ by one additional node, and with each node the phase of the transmission amplitude increases by $\pi$. As a result, the phase evolution in the one–dimensional model is qualitatively different from the phase evolution measured in Refs. [6,7]. Thus two questions arise: First, how can in–phase resonances occur in the Schuster– (and Yacoby–) experiment, and why are all resonances in phase? Second, why does the phase slip between resonances and why so sharply?

Although we will focus on the transmission phase, we note that the underlying questions go much deeper and concern the nature of electron transport through quantum dots. Starting with the work of Jalabert, Stone and Alhassid [26] in the early 1990’s, the theoretical work on quantum dots has almost exclusively addressed universal aspects described by random
FIG. 14. (a): Double-barrier well potential. (b): Transmission coefficient $|t|^2$ (top) and transmission phase $\arg(t)$ (bottom) both as a function of $W$. Parameters are $V_0 = 10.0$, $E = 2.0$. All energies are given in units of $\hbar^2/2ma^2$.

matrix theory. Employing random matrix theory, quantities like the statistical distribution of the conductance peak heights [26] and parametric peak–height correlations [27,28] have been calculated and tested in various experiments. Because of the enormous success of these calculations random matrix theory has emerged as a paradigm for the theoretical description of quantum dots. However, in recent years a number of experiments showed clear deviations from random matrix behavior. Among these are transport experiments on small vertical quantum dots [29], conductance peak–height measurements on lateral quantum dots [30], and the Coulomb–blockade interference experiments [6,7]. We will show below that the failure of random matrix theory for the experiments [6,7] may be traced back to features of the short–time dynamics in the quantum dot. Under certain conditions such non–universal features can totally dominate the quantum transport in the tunneling regime and generate strong deviations from random matrix predictions.

The material covered in this section is organized systematically so that papers with related theoretical ideas are discussed together. This ordering displays the underlying physical concepts most clearly. Sometimes the systematic ordering could only be achieved at the expense of giving up a strictly chronological order. In Sec. IV A we summarize the scattering theory through a region of interacting electrons, and derive the transmission phase through a weakly coupled quantum dot. This section forms the basis for the arguments presented in the subsequent sections. In Sec. IV B, we discuss a general theorem known as the Friedel sum rule. The Friedel sum rule relates the amount of charge added to a conducting system to the sum of all scattering phase shifts. We clarify the relation between the scattering phase shifts that enter in the Friedel sum rule and the transmission phase measured in a quantum dot interference device. The phase lapse between resonances is discussed in Sec. IV C.
Mechanisms for in–phase resonances are summarized in Sec. [IVD]. We emphasize, that despite of the arguments for in–phase resonances, none of the theoretical studies excludes the possibility of out–off–phase resonances. Further experimental work is necessary to clarify whether the latter may or may not be found in quantum dots.

A. Transport theory

The mean–field theory for transport through a quantum dot described in Sec. [III B] is only valid near the conductance peaks and at temperatures much smaller than the mean level spacing $\Delta$. When these conditions are not met, transport through a quantum dot can no longer be reduced to a mean–field problem. Several techniques have been developed to calculate the current through interacting electron systems. Our presentation follows the approach of Refs. [31–33] which is based on earlier work of Meir, Wingreen, and Lee [34,24].

1. Generalized Landauer formula

We consider an Aharonov–Bohm ring with a quantum dot embedded in one arm. The arm containing the dot is modeled by the tunneling Hamiltonian

$$H = H_L + H_R + H_{QD} + H_T,$$  \hspace{1cm} (4.1)

$$H_{L(R)} = \sum_k \epsilon_k^{L(R)} a_k^{L(R)} \dagger a_k^{L(R)},$$  \hspace{1cm} (4.2)

$$H_{QD} = \sum_\lambda \mathcal{E}_\lambda c_\lambda^\dagger c_\lambda + \frac{1}{2} U_0 (\hat{N}_2^2 - \hat{N}),$$  \hspace{1cm} (4.3)

$$H_T = \sum_{k,\lambda} \left( V_{k,\lambda}^{L} c_k^\dagger c_\lambda^\dagger + \text{H.c.} \right) + L \leftrightarrow R.$$  \hspace{1cm} (4.4)

Here, $H_L$ and $H_R$, respectively, describe the regions to the left and right of the QD, $H_{QD}$ is the Hamiltonian of the isolated QD including the charging energy, and $H_T$ represents the tunneling of electrons in and out of the QD. All energies are counted from the Fermi level in the leads. For simplicity, we assumed only one transverse channel in the left and right region. In the absence of a magnetic field, the matrix elements $V_{k,\lambda}^{L,R}$ can be chosen real.

An exact formula for the current through the ring can be derived using the nonequilibrium Keldysh formalism [24]. The current is expressed in terms of the Fermi function in the leads and local properties of the interacting region. In general, the formula for the current includes inelastic scattering, spin flips, and even scattering processes of several electrons. However, in the linear response limit and for sufficiently low temperature only elastic processes are allowed by energy conservation. Then, the conductance through the Aharonov–Bohm ring reduces to the generalized Landauer–type formula $G_{\text{ring}} = (2e^2/h) T_{\text{ring}}$, where the total transmission probability $T_{\text{ring}}$ is given by
\[ T_{\text{ring}} = \int dE \left( -\frac{\partial f}{\partial E} \right) |t_0 + t(E) \exp[2\pi i \Phi/\Phi_0]|^2. \] (4.5)

Here \( t_0 \) and \( t(E) \) denote the transmission amplitude through the free arm and the arm with the quantum dot, respectively, \( \Phi \) is the magnetic flux through the ring, and \( \Phi_0 = h/e \) the elementary flux quantum. Note that we have neglected higher harmonics. From the interference term in Eq. (4.3) one can extract the amplitude of transmission through the arm with the quantum dot

\[ t_{\text{QD}} = \int dE \left( -\frac{\partial f}{\partial E} \right) t(E). \] (4.6)

We identify \( \theta_{\text{QD}} \equiv \arg(t_{\text{QD}}) \) with the transmission phase through the quantum dot. The transmission coefficient through the arm with the dot is given by

\[ T_{\text{QD}} \equiv \int dE \left( -\frac{\partial f}{\partial E} \right) |t(E)|^2. \] (4.7)

Both \( \theta_{\text{QD}} \) and \( T_{\text{QD}} \) can be measured in quantum dot interference experiments. The transmission amplitude \( t(E) \) can be expressed in terms of the retarded Green function \( G^r_{\lambda\mu} \) of the dot

\[ t(E) = \sum_{\lambda,\mu} V^L_\lambda(E) G^r_{\lambda\mu}(E) V^R_\mu(E), \] (4.8)

where we introduced the tunneling amplitudes \( V^{L,R}_\lambda(E) = [2\pi \rho^{L,R}(E)]^{1/2} V^{L,R}_{k(E),\lambda} \) with the density of states \( \rho^{L,R}(E) \) in lead \( L,R \). To simplify notation, we will drop the argument \( E \) and write \( V^{L,R}_\lambda \) below. We emphasize that the Green function \( G^r_{\lambda\mu} \) must be calculated in the presence of interactions and tunneling.

In the limit \( t_0 = 0 \) of vanishing transmission through the free arm, the ring conductance reduces to the two–terminal conductance \( G_{\text{QD}} = (2e^2/h)T_{\text{QD}} \) through a quantum dot coupled directly to reservoirs. We derive \( G_{\text{QD}} \) for the case when the Green function is diagonal in the single–particle basis of the dot. Substitution of Eq. (4.8) into Eq. (4.5) then yields a product of two Green functions which can be evaluated using the relation

\[ G^r_{\lambda\lambda} G^{r*}_{\lambda\lambda} = \frac{1}{\Sigma_{\lambda\lambda} - \Sigma^*_{\lambda\lambda}} (G^r_{\lambda\lambda} - G^{r*}_{\lambda\lambda}) = \frac{\Im G^r_{\lambda\lambda}}{\Im \Sigma_{\lambda\lambda}}. \] (4.9)

where \( \Sigma \) is the self–energy. The diagonal elements \( \Sigma_{\lambda\lambda} = i(\Gamma^L_\lambda + \Gamma^R_\lambda)/2 \) are given in terms of the partial widths \( \Gamma^{L,R}_\lambda = |V^{L,R}_\lambda|^2 \) for decay of the state \( \lambda \) into the left and right lead, respectively. The current takes the form

\[ G_{\text{QD}} = \frac{2e^2}{h} \int dE \left( \frac{\partial f}{\partial E} \right) \sum_{\lambda} \frac{\Gamma^L_\lambda \Gamma^R_\lambda}{\Gamma^L_\lambda + \Gamma^R_\lambda} \Im G^r_{\lambda\lambda}(E). \] (4.10)

This result was first obtained in Refs. [34,24]. Note that \( -(1/\pi)\Im G^r_{\lambda\lambda}(E) \) is the local level density in the presence of interactions and tunneling. The above derivation can be
generalized to include the electron spin. In this case the conductance takes a form similar to Eq. (4.10), but the factor 2 is replaced by an explicit summation over the electron spin. The resulting expression has been used to study the conductance in the Kondo regime [35,24].

2. Retarded Green function

To derive the dot Green function we employ the equations–of–motion method [34]. The derivation starts from the retarded Green function $G_{\lambda:\mu}(t)$, defined by

$$G_{\lambda:\mu}(t) = -i\theta(t)\langle\{c_{\lambda}(t), c_{\mu}^\dagger(0)\}\rangle,$$

(4.11)

where the curly brackets denote the anticommutator. The expectation value is the thermal average with respect to the Hamiltonian $H$. The operator $c_{\lambda}(t)$ is the solution of the Heisenberg equation

$$i\frac{\partial c_{\lambda}(t)}{\partial t} = [c_{\lambda}, H].$$

(4.12)

Differentiating Eq. (4.11) with respect to $t$ and substituting Eq. (4.12) one obtains the equation of motion

$$\frac{\partial}{\partial t} G_{\lambda:\mu}(t) = -i\delta(t)\delta_{\lambda:\mu} - \theta(t)\langle\{[c_{\lambda}, H], c_{\mu}^\dagger\}\rangle.$$

(4.13)

When $H$ is quadratic in the particle operators, one can express the right–hand–side of this equation in terms of two-particle Green functions. This yields a closed set of equations which is readily solved for the exact Green functions. Since the quantum dot Hamiltonian $H$ is not quadratic, one generates higher–order Green functions which must be approximated to obtain a closed set of equations for $G_{\lambda:\mu}(t)$. From the solution one finds the Green function $G_{\lambda:\mu}(E)$ upon Fourier transformation. It is illustrative to study $G_{\lambda:\mu}(E)$ in various limiting cases.

Isolated dot: This case is obtained for zero coupling $V_{\lambda L,R}^{L,R} = 0$. Since no electrons can tunnel into or out of the quantum dot, the total number of dot electrons $N$ is conserved. The Fock states of the dot can be separated into classes with $N$ electrons. The grand–canonical expectation value of an operator $\hat{A}$ can then be expressed in terms of canonical expectation values

$$\langle \hat{A} \rangle = \sum_{N=0}^{\infty} P_N \langle \hat{A} \rangle_N,$$

(4.14)

where the equilibrium probability $P_N$ to find $N$ electrons on the dot is given by

$$P_N = \frac{\text{tr}_N \exp(-\beta H)}{\sum_{M=0}^{\infty} \text{tr}_M \exp(-\beta H)}.$$
The retarded Green function follows upon substitution of \( \hat{A} = -i\theta(t)c_\lambda(t)c_\mu^\dagger(0) \) and \( \hat{A} = -i\theta(t)c_\mu^\dagger(0)c_\lambda(t) \). Fourier transformation yields the result

\[
G_{\lambda\mu}^r(E) = \sum_{N=0}^{\infty} P_N \delta_{\lambda\mu} \left[ \frac{1 - \langle \hat{n}_\lambda \rangle_N}{E - (\mathcal{E}_\lambda + N \cdot U_0) + i\delta} + \frac{\langle \hat{n}_\lambda \rangle_N}{E - (\mathcal{E}_\lambda + (N - 1) \cdot U_0 + i\delta)} \right],
\]

where \( \delta \to 0^+ \). Note that each level \( \lambda \) contributes twice: The first term on the right-hand-side of Eq. (4.16) accounts for electron propagation through the level \( \lambda \) while the second term describes the propagation of a hole. The particle–hole structure originates from the charging energy which generally assigns the addition or removal of electrons a different energy. We note that the particle–hole structure disappears for \( U_0 \to 0 \).

**Noninteracting electrons:** The limit \( U_0 \to 0 \) corresponds to noninteracting electrons. In this case the equations of motion close and one readily finds

\[
G_{\lambda\mu}^r = [E - \mathcal{E} + \Sigma(E)]^{-1}_{\lambda\mu}.
\]

The self–energy \( \Sigma(E) = \Sigma^L(E) + \Sigma^R(E) \) has two contributions resulting from the coupling to the left and right lead. They are given by

\[
\Sigma_{\lambda\mu}^{L,R}(E) = \sum_k V_{k\lambda}^{L,R} V_{k\mu}^{L,R} \frac{1}{E - \epsilon_k^{L,R} + i\delta}.
\]

Since the states in the leads are dense, one can replace the sum in Eq. (4.18) by an integral. The imaginary part of this integral yields the widths \( \Gamma_{\lambda}^{L,R} \) for decay of the state \( \lambda \) into the left and right lead, while the real part gives an energy shift. For weak coupling \( \Gamma_{\lambda}^{L,R} \ll \Delta \) and close to the resonances, the Green function can be approximated by the diagonal terms \( G_{\lambda\mu}^r = \delta_{\lambda\mu}G_{\lambda\mu}^r \) yielding

\[
G_{\lambda\mu}^r(E) = \delta_{\lambda\mu} \frac{1}{E - \mathcal{E}_\lambda + i\Gamma_\lambda/2},
\]

with the total width \( \Gamma_\lambda = \Gamma_\lambda^L + \Gamma_\lambda^R \). In Sec. [III] we arrived at the same result using single–particle scattering theory.

3. **Weak–coupling limit**

The weak–coupling regime is characterized by \( \Gamma_\lambda \ll kT, \Delta \). In this regime the Green function can be approximated by
\[ G_{\lambda\mu}^*(E) \approx \sum_{N=0}^{\infty} P_N \delta_{\lambda\mu} \left[ \frac{1 - \langle \hat{n}_\lambda \rangle_N}{E - (\varepsilon_\lambda + N \cdot U_0) + i \Gamma_\lambda / 2} + \frac{\langle \hat{n}_\lambda \rangle_N}{E - (\varepsilon_\lambda + (N - 1) \cdot U_0 + i \Gamma_\lambda / 2) + i} \right]. \] (4.20)

Note that due to the coupling all quantum dot states acquire a finite width. The widths are identical to the widths obtained in the non-interacting case. Corrections arise for \( \Gamma_\lambda \geq kT \) and lead to a normalization of the self-energy. In particular, the approximation (4.20) fails in the Kondo regime where higher iterations of the equations of motion become important.

In the weak–coupling limit both the transmission amplitude and the transmission coefficient can be obtained analytically. We first substitute the weak–coupling result for the Green function (4.20) into Eq. (4.8). The transmission amplitude (4.6) and the transmission coefficient (4.7) then reduce to sums of integrals which can be computed using contour integration. For the transmission amplitude, the integrals are of the type

\[ \int dE \left( \frac{\partial f}{\partial E} \right) \frac{1}{E - \varepsilon + i \Gamma / 2} = \frac{\beta}{2\pi i} \psi^{(1)} \left( \frac{1}{2} + \frac{\beta \Gamma}{4\pi} + i \frac{\beta \varepsilon}{2\pi} \right), \] (4.21)

and for the transmission coefficient they have the form

\[ \int dE \left( \frac{\partial f}{\partial E} \right) \text{Im} \frac{1}{E - \varepsilon + i \Gamma / 2} = -\pi \frac{\partial}{\partial \varepsilon} \text{Re} f(\varepsilon + i \Gamma / 2) - \frac{\beta}{(2\pi)^2} \sum_{\sigma = \pm} \text{Re} \psi^{(1)} \left( \frac{1}{2} + \frac{\sigma \beta \Gamma}{4\pi} + i \frac{\beta \varepsilon}{2\pi} \right). \] (4.22)

Here \( \beta = 1/kT \) is the inverse temperature, \( f = 1/(1 + \exp(\beta E)) \) is the Fermi function, and \( \psi^{(1)}(z) \) the trigamma function [36]. Combining results, the transmission amplitude reduces to a sum over energy levels and occupation numbers,

\[ t_{\text{QD}} = \frac{\beta}{2\pi i} \sum_{\lambda} \sum_{N=0}^{\infty} V_\lambda^L V_\lambda^R P_N \left[ \left[ 1 - \langle \hat{n}_\lambda \rangle \right] \psi^{(1)} \left( \frac{1}{2} + \frac{\beta \Gamma_\lambda}{4\pi} + i \frac{\beta (\varepsilon_\lambda - U_0 \cdot N)}{2\pi} \right) + \langle \hat{n}_\lambda \rangle \psi^{(1)} \left( \frac{1}{2} + \frac{\beta \Gamma_\lambda}{4\pi} + i \frac{\beta (\varepsilon_\lambda - U_0 \cdot (N - 1))}{2\pi} \right) \right]. \] (4.23)

The transmission coefficient \( T_{\text{QD}} \) and the conductance \( G_{\text{QD}} = (2e^2/h)T_{\text{QD}} \) reduce to similar sum. The result takes a simple form in two limiting cases. First, in the cotunneling regime deep in the transmission valley \( N \) we have \( P_M \approx \delta_{M,N} \). The first term in Eq. (4.22) vanishes exponentially, while the trigamma function can be approximated by its asymptotic expansion [36]. The conductance then simplifies to

\[ G_{\text{QD}} \approx 2\frac{e^2}{h} \sum_{\lambda} \Gamma_\lambda^L \Gamma_\lambda^R \left[ \frac{\langle \hat{n}_\lambda \rangle_N}{(\varepsilon_\lambda + U_0 \cdot (N_1))^2} + \frac{1 - \langle \hat{n}_\lambda \rangle_N}{(\varepsilon_\lambda + U_0 \cdot (N))^2} \right]. \] (4.24)
This is the standard result for the conductance in the cotunneling regime. Second, near the transmission resonances, we can neglect the terms involving the trigamma function which are smaller than the Fermi function by a factor $\beta \Gamma \lambda$. After a little algebra one finds

$$G_{\text{QD}} \simeq \frac{e^2}{kT} \sum_{\lambda} \sum_{N=0}^{\infty} \frac{\Gamma^L_R}{\Gamma^L_R + \Gamma^R_R} P_N [1 - \langle \hat{n}_\lambda \rangle_N] f(\mathcal{E}_\lambda + U_0 \cdot N).$$

(4.25)

This is the well-known result for the conductance in the regime of sequential tunneling that was obtained independently by Beenakker using a master equation approach and by Meir, Wingreen, and Lee using the generalized Landauer formula.

The results derived in this subsection solve the problem of electron transmission through a weakly coupled dot. They express the transmission phase in terms of the coupling matrix elements and the single-particle energies of the quantum dot. These quantities must be obtained from a dynamical model of the dot and its coupling to the leads. Various such models are studied below. Deviations from the weak-coupling regime are addressed in Secs. III D 4, III D 5.

B. Friedel sum rule

Levy Yeyati and Büttiker were the first authors to study the origin of the in-phase resonances. They discussed the subject in terms of the Friedel sum rule. The Friedel sum rule is a powerful relation that links the number of localized electron states in a conducting region to the sum of all scattering phase shifts at the Fermi energy. The sum rule holds even in the presence of interactions. There is an analogue of the Friedel sum rule in scattering theory. Known as Levinson’s theorem, it relates the number of bound states in a scattering region to the sum of the scattering phase shifts at zero energy. Employing the Friedel sum rule, Levy Yeyati and Büttiker argued that the addition of a charge $\Delta Q$ to a conducting region $\Omega$ should change the phase of the transmission amplitude $t$ through that region according to

$$\Delta Q/e = \Delta \text{arg}(t)/\pi.$$  

(4.26)

If the region $\Omega$ is chosen to include only the quantum dot, then the addition of one electron to the dot should cause a phase increase by $\pi$. Consequently, successive resonances would be off-phase by $\pi$ rather than in-phase as in the experiments. Levy Yeyati and Büttiker emphasized, however, that the charge $\Delta Q$ entering in Eq. (4.26) is not the additional charge on the quantum dot but rather the additional charge within the whole coherence volume. This volume includes the quantum dot and the AB ring around the quantum dot. A variation of the gate voltage at the quantum dot may also modify the electrostatic potential in the vicinity of the dot. This could result in the addition of extra charge to the AB ring outside the
quantum dot. Levy Yeyati and Büttiker studied a situation where with each electron added to the quantum dot, the extra charge $\delta e$ is added to the AB ring. In a model calculation for a two–terminal device using the values $\delta \sim 0.3$ and $\delta \sim 1$, sequences of two or three consecutive in–phase resonances were found.

The connection between the Friedel sum rule and in–phase resonances was reconsidered and clarified by Lee [41] and Taniguchi and Büttiker [42]. The Friedel sum rule makes a statement about the sum of all scattering phases. The sum rule can be stated in the form

$$\Delta Q/e = [\Delta \ln \text{Det}(S)]/2\pi i,$$

where $S$ is the scattering matrix. Writing the eigenvalues of $S$ in the form $e^{2i\alpha_i}$, the right hand side represents the change in the sum $\alpha/\pi = \sum_{i=1}^{N} \alpha_i/\pi$ of all eigenphases, where $N$ is the dimension of $S$. Thus,

$$\Delta Q/e = \Delta \alpha/\pi.$$

The transmission amplitude $t$, on the other hand, is a matrix element of $S$ in a basis of left and right moving scattering states. In general, the phase of $t$ is not simply related to the sum of the eigenphases $\Delta \alpha \neq \Delta \text{arg}(t)$. The relation (4.28) is thus different from the Friedel sum rule (4.27), (4.28). Generally, the expression (4.26) is not expected to be correct.

The difference between Eqs. (4.26) and (4.28) can be explicitly demonstrated in the scattering through a quasi-1D system [41]. We assume a mirror reflection symmetry $x \rightarrow -x$ and the absence of a magnetic field. The scattering region at $|x| < R$ is connected to two single–channel leads at $x < -R$ and $x > R$, respectively. The scattering states can be decomposed into even and odd scattering states. For $|x| > R$ these states read

$$\psi_e(x) = e^{-ik|x|} + e^{2i\alpha e} e^{ik|x|}$$

and

$$\psi_o(x) = \text{sgn}(x)[e^{-ik|x|} + e^{2i\alpha_o e} e^{ik|x|}].$$

Note that there are
two scattering phases $\alpha_e$, $\alpha_o$ corresponding to two scattering channels, one in each lead. Rather than in a basis of even and odd scattering states, one can formulate the scattering problem in a basis of right and left moving scattering states, $\psi_l(x) = [\psi_e(x) - \psi_o(x)]/2$, $\psi_o(x) = [\psi_e(x) + \psi_o(x)]/2$. The scattering matrix in the new basis takes the form

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix},$$

(4.29)

where $t$, $t'$ are the transmission amplitudes and $r$, $r'$ the reflection amplitudes from the left and right, respectively. From the relations between the basis vectors and Eq. (4.29), one obtains the transmission amplitudes

$$t = t' = ie^{i\alpha} \sin \beta$$

(4.30)

in terms of the phases $\alpha \equiv \alpha_e + \alpha_o$ and $\beta \equiv \alpha_e - \alpha_o$. In true 1D systems, even and odd resonant states alternate in energy (see Fig. 15(a)) and the angle $\beta$ is restricted to the range $0 < \beta < \pi$. Then both the total phase $\alpha$ and the transmission phase $\arg(t)$ change by $\pi$ between neighboring resonances. In quasi-1D systems, however, even and odd states do not necessarily alternate in energy, and there may be sequences of states with the same parity (see Fig. 15(b)). Within such sequences, $\beta$ varies by more than $\pi$ and $\sin \beta$ changes sign from one resonance to the next. Neighboring resonances then have the same transmission phase $\arg(t)$ while the total phase $\alpha$ changes by $\pi$ in keeping with the Friedel sum rule (4.27), (4.28). Note that Eq. (4.30) predicts a transmission zero ($\sin \beta = 0$) between resonances with the same parity. A transmission zero corresponds to a singular point of the transmission phase. The phase jumps abruptly by $\pi$ when the system is swept through the transmission zero.

The arguments presented above demonstrate that in–phase resonances do not violate the Friedel sum rule. They also show that the spatial dimension of the scattering region is important: While neighboring resonances are always out–off–phase in strictly 1D systems, both out–off–phase and in–phase resonances can be found in higher dimensional systems.

C. Phase lapse

1. Breit–Wigner model

The sharp phase lapse between resonances was investigated by Hackenbroich and Weidenmüller [43] for the weak–coupling regime $\Gamma < kT < \Delta$. Similar to the calculation for the two–terminal case, cf. Sec. IIIB, the multi–terminal transmission amplitude was obtained using scattering theory and the diagonal approximation for the dot propagator. In this approximation the transmission amplitude reduces to a sum of Breit–Wigner resonances.
The internal dynamics of the dot and the coupling to the leads enter via the resonance energies $E_\lambda$ and the partial width amplitudes $V_{pm}^{\lambda}$ for the decay of the resonance $\lambda$ into channel $m$ of the lead $p$. Hackenbroich and Weidenmüller did not calculate the resonance parameters from a dynamical model. They rather assumed these parameters to be identical for all resonances and investigated the consequences of this assumption for the transmission phase: By assumption, all resonances showed in-phase behavior. At each resonance the phase increased by $\pi$ on a scale $\sim kT$. Between the resonances, the phase slipped by $\pi$ on the scale of the intrinsic resonance width $\Gamma$. The Breit–Wigner model \[43\] thus predicts a phase lapse between resonance peaks on the scale of the intrinsic resonance width $\Gamma$.

We note that the experiment \[7\] was not performed in the weak–coupling regime. Rather, the intrinsic resonance width $\Gamma$ in the experiment was estimated to be of the order $\Gamma \sim (3−4)kT$. The Breit–Wigner model of Hackenbroich and Weidenmüller therefore cannot explain the phase lapse observed in \[7\]. Moreover, there is theoretical evidence (see Sec. IV C 2) that the diagonal approximation for the quantum dot propagator fails between the resonances. As a result, the transmission phase may display an sharp lapse rather than a smooth decrease on the scale $\Gamma$ as predicted by the Breit–Wigner model.

2. Transmission zero

In 1998, a number of authors \[44–47\] studied models for the transmission through a quantum dot and found abrupt jumps of the transmission phase. The jumps occurred at singular points where both the real and the imaginary part of the complex transmission amplitude vanished. Transmission zeros were found in numerical studies exploring Fano resonances \[44–46\] and the interplay of multiple resonances \[47\]. In Refs. \[44–46\] the quantum dot was modeled as a quasi-1D or 2D region of regular shape. In all studies time–reversal invariance was assumed, the dot was connected to two single–channel leads and electron-electron interactions were neglected. In–phase resonances occurred either due to sequences of states with the same parity \[44–46\] or by assuming \[47\] state–independent coupling matrix elements to the leads. A more general approach to the connection between transmission zeros and in–phase resonances was presented by Lee \[41\] (see the discussion after Eq. (4.30)). He showed that the transmission always vanishes between neighboring in–phase resonances, provided (i) the scattering region is connected to two single–channel leads and (ii) the system is time–reversal invariant.

Why do the models \[44–47\] display an abrupt phase jump and not a smooth phase slip on the scale $\Gamma$ as predicted by the Breit–Wigner model \[43\]? This question was addressed by Sun and Lin \[47\] for a quantum dot with symmetric tunneling barriers. It turned out that the diagonal approximation for the quantum dot propagator used in Ref. \[43\] fails for time–reversal invariant systems coupled to single–channel leads. Here we present an
argument that proves this failure also for asymmetric barriers. Consider a quantum dot that is coupled to two single-channel leads, denoted $L$, $R$. We assume $kT \ll \Delta$ and include electron–electron interactions in a mean field way. The scattering matrix of the system can be derived with the methods of Sec. III B. For the scattering amplitude $S_{mn}$ at energy $E$ one has

$$S_{mn}(E) = \delta_{mn} - 2i\pi \sum_{\mu\nu} W_{m\mu}^*(D^{-1}(E))_{\mu\nu} W_{n\nu},$$

(4.31)

where $m$, $n$ refer to the physical channels. The indices $\mu$ and $\nu$ of the inverse propagator $D_{\mu\nu}$ refer to a complete set of quantum dot states, and $W_{m\mu}$ is the coupling matrix element between channel $m$ and the dot state $\mu$. The inverse propagator $D$ has the form

$$D_{\mu\nu}(E) = (E - E_\mu)\delta_{\mu\nu} + i\pi \sum_m W_{m\mu} W_{m\nu}^*,
$$

(4.32)

where the sum runs over $m = L, R$ and where $E_\mu$ includes an energy shift resulting from the coupling to the leads. First consider the system at zero magnetic field. Then the matrix elements can be chosen real. To simplify the argument, we restrict ourselves to two levels 1, 2 in the dot. The quantum dot propagator is given by

$$D^{-1}(E) = \frac{1}{\det D(E)} \begin{pmatrix}
E - E_2 + i\Gamma_{22}/2 & -i\Gamma_{12}/2 \\
-i\Gamma_{21}/2 & E - E_1 + i\Gamma_{11}/2
\end{pmatrix},
$$

(4.33)

with $\Gamma_{\mu\nu} \equiv 2\pi \sum_m W_{m\mu} W_{m\nu}^*$. Combining Eqs. (4.31) and (4.33), one obtains the transmission amplitude $t_{LR}(E)$ between left and right lead

$$t_{LR}(E) = -\frac{2i\pi}{\det D(E)} [\gamma_{11}(E - E_2) + \gamma_{22}(E - E_1) +
$$

$$\frac{i}{2}(\gamma_{11}\Gamma_{22} + \gamma_{22}\Gamma_{11} - \gamma_{12}\Gamma_{21} - \gamma_{21}\Gamma_{12})],
$$

(4.34)

where $\gamma_{\mu\nu} \equiv W_{L\mu}^* W_{R\nu}$. Substituting the definitions of $\Gamma_{\mu\nu}$ and $\gamma_{\mu\nu}$ into Eq. (4.34), one finds that the term in the round brackets on the right hand side vanishes identically. Therefore, if $\gamma_{11}$ and $\gamma_{22}$ have the same sign, the transmission amplitude $t_{LR}$ vanishes for some energy between $E_1$ and $E_2$. We thus obtain a transmission zero. Note, that for the above argument, it is crucial to keep the off-diagonal elements of the propagator. They were neglected in Ref. [43].

It turns out that both the condition of time–reversal invariance and of single–channel leads are necessary for a transmission zero. If one of these conditions is relaxed, one generally no longer finds a vanishing transmission. E.g. consider the case that time–reversal invariance is broken. The matrix elements $W_{m\mu}$ then can take complex values and the term in round brackets in Eq. (4.34) is no longer guaranteed to vanish. As a consequence, one generically finds a smooth phase slip and not an abrupt phase jump between resonances. However,
the phase slip may take place on an energy scale much smaller that the total width \( \Gamma \). This is suggested by the following observation. In the experiment, all resonances behave similarly (they display nearly the same phase and peak height). This suggests that different resonances have similar coupling matrix elements. For the simple two–level dot studied here, \( \gamma_{\mu\nu} \approx \gamma \) and \( \Gamma_{\mu\nu} \approx \Gamma \) independent of the level indices \( \mu, \nu \). This choice results in a near cancelation of the terms in the round bracket in Eq. (4.34), and the phase slip may get very sharp.

We finally discuss the effect of electron–electron interactions and finite temperature on the transmission zeros. Lee [41] argued that the analysis presented for the noninteracting case applies equally to interacting systems provided quasiparticle excitations remain well defined at \( E = E_F \). The argument is based on the Friedel sum rule which is known to hold even in the presence of interactions. Unfortunately, no numerical or analytical study has explicitly demonstrated transmission zeros in the presence of interactions. Finite temperature smears out the transmission zeros. The sharp phase jump is then replaced by a rapid but continuous lapse of the phase \([47]\). The precise energy scale of this lapse depends on the detailed electron dynamics in the dot. We note that the phase lapse observed in Ref. [7] is indeed continuous and has been resolved experimentally, cf. Sec. [11].

3. Disordered dot

The dots used in the experiments [6,7] were roughly 50 times smaller than the elastic mean free path. Under these conditions, transport through the dots is ballistic. Most of the theoretical studies of the transmission phase pertain to the ballistic regime, and neglected the influence of disorder. Baltin and Gefen [32,48] took a different approach and investigated the transmission amplitude through a disordered quantum dot. They identified a generic mechanism for phase correlations and formulated an approximate sum rule: According to this rule, the change in the transmission phase \( \Delta \theta \) between two consecutive transmission valleys is \( 0 \pmod{2\pi} \). The frequency of deviations from this rule is small in \( \Delta/U_0 \) where \( \Delta \) is the mean single particle level spacing and \( U_0 \) the charging energy of the quantum dot. The sum rule pertains to individual, disorder specific systems. Baltin and Gefen also calculated a disorder–averaged phase–phase correlation function and observed an enhancement of correlations with increasing interaction strength. Below we present the main arguments of the study [32].

The analysis of Baltin and Gefen proceeds from the scattering theory in the weak–coupling limit as presented in Sec. [4.3]. However, the basic argument can be understood from a simple toy model [32] based on two assumptions: (i) The single–particle level spacing is constant \( \mathcal{E}_\lambda = \lambda \Delta \) and (ii) the product of the coupling matrix elements is a random variable \( V^L_\lambda V^R_\lambda = V \eta_\lambda \) where \( \eta_\lambda \) takes the values \(+1\) and \( -1 \) with equal probability. These simple
assumptions model the fluctuations in the wave functions due to disorder. A more elaborate random–matrix calculation of the energies and the couplings confirms the predictions of the toy model described below. Consider first the noninteracting case \((U_0 = 0)\). The transmission amplitude then becomes

\[
t(E) = V \sum_\lambda \frac{\eta_\lambda}{E - \mathcal{E}_\lambda + i\Gamma_\lambda},
\]

where the system can be tuned in or out of resonance by the gate voltage \(V_g\) (we assume \(\mathcal{E}_\lambda = \mathcal{E}_\lambda^{(0)} - eV_g\)). At each resonance, the transmission phase increases by \(\pi\). The signs of \(\eta_\lambda\) govern the phase evolution between the resonances. If \(\eta_\lambda \cdot \eta_{\lambda+1} > 0\) there is a phase lapse by \(\pi\) between the resonances \(\lambda\) and \(\lambda + 1\); for \(\eta_\lambda \cdot \eta_{\lambda+1} < 0\) there is no phase lapse. In strictly one–dimensional systems \(\eta_\lambda\) alternate in sign implying no phase lapse as discussed in Sec. \[\text{IV B}\]. For a disordered noninteracting dot the phase lapses occur at random.

The situation is different in the presence of interactions. In the conductance valleys a large number of random terms \(\sim U_0/\Delta\) contribute to \(t(E)\). The terms in the \(N\)th valley and the \((N+1)\)th valley differ very little from each other (essentially by the contribution of one level). The transmission in the valleys is therefore determined by a background that varies little between neighboring valleys. Figure [\[\text{FIG. 16}\]] shows the evolution of \(\text{Re}(t)\) and \(\text{arg}(t)\) for \(kT = \Delta/12\) and specific series of couplings, both for the noninteracting case \(U_0 = 0\) (upper panel) and \(U_0 = 60\Delta\) (lower panel). For interacting electrons the change in phase \(\Delta\text{arg}(t) = 0\) as expected from the sum rule. Note, however, that the phase evolution as a function of voltage is different from what has been seen experimentally: The phase in the disordered dot decreases near the resonances and not in the valleys between resonances.
The crucial influence of the interaction is further illustrated by the disorder–averaged phase correlation function

\[ C_{\theta} \equiv \langle \cos \theta(x, N) \cos \theta(\bar{x}, N + \delta N) \rangle. \]  

(4.36)

Here, \( x \) and \( \bar{x} \) with \( 0 \leq x, \bar{x} \leq 1 \) specify the voltage in the \( N \)--th and \( (N + \delta N) \)--th valley. \( C_{\theta} \) decays slowly on a scale \( \delta N \sim U_0/\Delta \). Figure 17 shows \( C_{\theta} \) vs. \( \delta N \) for \( kT = \Delta/12 \). While there are no correlations for noninteracting electrons, one observes a slow decay of \( C_{\theta} \) in the presence of interactions. The decay is slower for stronger interactions as expected from the arguments given above.

The sum rule found for disordered dots is in accord with the observations made in the experiment [7], however, the evolution of the phase vs. gate voltage is in disagreement. We believe that the latter fact indicates a fundamental difference between the phase behavior found in disordered (or fully chaotic) systems and the phase behavior in ballistic dots with mixed dynamics.

D. In–phase resonances

Several mechanisms for peak–correlations and in–phase resonances have been proposed. These include the effect of finite temperature, deformations of the dot confining potential,
and energy shifts caused by the coupling between the dot and the leads. We summarize these mechanisms in Secs. [IV D 1]-[IV D 4]. The mechanisms rely on the assumption that the transmission is dominated by a few quantum states of the dot. The *dynamical origin* for this behavior is discussed in Secs. [IV D 3]-[IV D 5]. The investigation suggests that the in–phase resonances result from features of the short–time dynamics in the dot that completely dominate the quantum transport in the tunneling regime. Alternative proposals for peak correlations are reviewed in Sec. [IV D 6].

1. Finite temperature

Transport at finite temperatures differs qualitatively from transport at zero temperature. Close to a conductance peak electrons tunnel through a single level if \( kT \ll \Delta \). In contrast, few or many levels contribute to each conductance peak if \( kT \) is of the order \( \Delta \). Similar sets of levels contribute to neighboring peaks which generates peak correlations. Oreg and Gefen [49] studied the transmission phase correlations resulting from this effect. More recently, temperature induced correlations of the conductance peak–height in fully chaotic or diffusive dots have been calculated [50] using random matrix theory. A comparison with the peak–height correlations measured for chaotic dots has been given in Ref. [51].

Oreg and Gefen demonstrated in–phase behavior for a simple model with two levels in the dot. The calculation is an application of the formalism developed in Sec. [IV A]. Due to the small number of levels, all contributions to the transmission amplitude can be computed explicitly. The starting point is the two–level Hamiltonian for the quantum dot,

\[
H_{QD} = \mathcal{E}_a a^\dagger a + \mathcal{E}_b b^\dagger b + U_0 a^\dagger b^\dagger b, \tag{4.37}
\]

where \( U_0 \) denotes the strength of the electron–electron interaction. The dot is coupled via matrix elements \( V_{aL}, V_{aR}, V_{bL}, V_{bR} \) to two single–channel leads \( L \) and \( R \). As in Sec. [IV A] all energies are counted from the Fermi energy in the leads. To obtain the transmission amplitude (4.20) through the dot, we compute the occupation probabilities of the dot states. The quantum dot supports four many–body states. They can be labeled \( |0\rangle \) (no electron in the dot), \( |a\rangle = a^\dagger |0\rangle \), \( |b\rangle = b^\dagger |0\rangle \), and \( |ab\rangle = a^\dagger b^\dagger |0\rangle \). The respective energies are \( E_0 = 0 \), \( E_a = \mathcal{E}_a \), \( E_b = \mathcal{E}_b \), and \( E_{ab} = \mathcal{E}_a + \mathcal{E}_b + U_0 \). The equilibrium probabilities to find the dot in either of the states \( |i\rangle = |0\rangle, |a\rangle, |b\rangle, |ab\rangle \) are given by

\[
P_i = \exp(-\beta E_i) / \sum_i \exp(-\beta E_i). \tag{4.38}
\]

In terms of the \( P_i \), the matrix elements of the retarded Green function are

\[
G_{aa}(E) = \frac{1}{(P_0 + P_a)(E - \mathcal{E}_a + i\Gamma_a/2)}
\]
FIG. 18. Left: Occupation probability of the four many–body states as a function of $V_g$. Right: Argument of $t_{RL}$ (solid line) and its magnitude (dashed line) as a function of $V_g$. For the parameters $\Delta = 10\mu eV$, $kT = 20\mu eV$, $U_0 = 500\mu eV$. Taken from Ref. [49].

Two consecutive resonances with similar phase can arise in this model if one level is significantly stronger coupled to the leads than the other level. For definiteness, let $a$ be the strongly coupled level. At the first resonance, either an electron or a hole tunnel via level $a$ through the dot while level $b$ is empty. These processes are represented by the first term on the right hand side of Eq. (4.39). At the second resonance, again charge is transferred predominantly via level $a$, this time with level $b$ being occupied. The corresponding processes are described by the second term on the right hand side of Eq. (4.39). Two consecutive resonances are then dominated by the same single–particle level $a$. This scenario differs qualitatively from the transmission of independent electrons (which is recovered in the limit $U_0 \to 0$), where two consecutive resonances are associated with two different single–particle states.

We use the parameterization $\mathcal{E}_a = V_g$, $\mathcal{E}_b = V_g + \Delta$ of the single–particle energies in terms of the voltage $V_g$ at the dot. The occupation probabilities of the four many body states and the modulus and phase of $t_{QD}$ are displayed vs. $V_g$ in Fig. 18. For both plots, $\Delta = 0.5kT$, and $|V_b^L| = |V_b^R| = |0.3V_a^L| = |0.3V_a^R|$. The two resonances are in phase since both are dominated by the same level $a$. The phase lapse between resonances takes place on the scale $\Gamma_a$. This is expected since we neglected the off–diagonal elements of the dot Green function, cf. Sec. IV C.
2. Deformations

Transport experiments in the Coulomb blockade regime study the transmission through quantum dots as a function of a gate voltage $V_g$ applied to the dot. Most theoretical studies of the Coulomb blockade assumed that $V_g$ exclusively regulates the depth of potential well in the dot. Changes in $V_g$ then translate into an overall shift of the single–particle levels of the dot. This picture is clearly an oversimplification. In any experimental realization of a quantum dot, variation of $V_g$ modifies both the depth and the shape of the dot confining potential. The consequences for the Coulomb blockade were first investigated by Hackenbroich, Heiss, and Weidenmüller [52]. Shape variations lead to avoided crossings of single–particle levels. This can generate sequences of conductance resonances which carry essentially the same internal wave function. Within such sequences one may find in–phase resonances and/or strong correlations of the conductance peak heights. Vallejos, Lewenkopf, and Mucciolo [53] showed that shape deformations may change the spacing distribution of Coulomb blockade peaks. Stopa [54] demonstrated shape deformations in self–consistent calculations of the dot confining potential. We discuss shape deformations for a quantum dot with a parabolic confining potential. Then we consider more realistic confinement potentials with level repulsion and chaotic classical electron motion.

Consider a quantum dot with a parabolic confining potential. The potential is characterized by two oscillator frequencies $\omega_x$ and $\omega_y$. For simplicity, we keep $\omega_x = \omega$ fixed while $\omega_y = \omega(1 - \beta)$ depends on the deformation parameter $\beta$ with $0 \leq \beta < 1$. Fig. 19 shows part of the single-particle spectrum versus $\beta$. The levels form a network of intersecting straight lines. The single–particle state associate with each line is characterized by two non–negative integer quantum numbers ($\lambda_x, \lambda_y$). States with small (large) values of $\lambda_y$ have small (large) negative slopes. We show below that a pattern similar to Fig. 19 may also

FIG. 19. Energy levels of the two-dimensional harmonic oscillator with frequencies $\omega_x = \omega$ and $\omega_y = \omega(1 - \beta)$ as functions of $\beta$. Taken from Ref. [52].
be found for the spectrum of deformed non-integrable potentials, with one difference: The points of intersection disappear and are replaced by avoided crossings.

To illustrate the mechanism for peak correlations, we replace Fig. 19 by the idealized picture shown in Fig. 20. A set of equally spaced straight lines (A-levels) runs nearly parallel to the $\beta$-axis, a second set (B-levels) has large negative slope. To model the generic case, actual crossings have been replaced by avoided crossings. The distance between B-lines is denoted by $\delta_{\text{cross}}$. Wave functions retain their identity across avoided crossings. Therefore, the wave function on any A-level is nearly independent of $\beta$. A change of the gate potential and, therefore, of $\beta$ is slow on the scale of the characteristic times of the quantum dot. Therefore, each electron in an occupied level follows the deformation $\beta$ adiabatically. At each avoided crossing, the electron wave function changes from A–type to B–type or vice versa.

Suppose that for some $\beta$ the $N$ lowest single-particle levels of the dot are occupied and that the highest occupied level is an A-level. What happens as $\beta$ is increased by $\delta_{\text{cross}}$? The last occupied level moves adiabatically down at the avoided crossing. The associated wave function switches from A–type to B–type. The A-level that was the last occupied level before the avoided crossing becomes empty. Upon further increase of $\beta$, this very same A-level can be occupied by another electron (at deformation $\beta_{N+1}$). This process can repeat itself for a second, third, etc. time if $\delta_{\text{cross}} \approx \delta_e$, the change of $\beta$ needed to pull another electron into the dot. Then, subsequent conductance peaks would not be independent, but rather be manifestations of essentially the same single-particle state. Strong correlations of the conductance peak heights and the transmission phases are expected for such sequences.

How does the idealized picture of Figs. 19, 20 change for a more realistic spectrum with level repulsion? As a generic example consider the Hamiltonian

![Energy levels diagram](image-url)
\[ H = \frac{p^2}{2m} + \frac{m\omega^2}{2} (x^2 + (1 - \beta)^2 y^2) - \rho\hbar\omega L^2, \]  

where \( L \) is the dimensionless \( z \)-component of the angular momentum operator. The three-dimensional analogue of \( H \) is known as the Nilsson model and has been quite successful in explaining the spectra of deformed nuclei [55]. For \( \beta > 0 \) and \( \rho \neq 0 \), \( H \) is not integrable and displays level repulsion.

Figure 21 shows part of the spectrum of \( H \) versus \( \beta \). The overall pattern is quite similar to Fig. 19, however all crossings have been replaced by avoided crossings (mostly not visible in Fig. 21). One observes a set of nearly flat lines. Numerical inspection shows that the associated wave functions retain their identity over a large range of \( \beta \). The intensity of these wave functions is concentrated along periodic orbits which oscillate in the \( x \)-direction, with little or no motion in the \( y \)-direction. One expects that the bulk part of the current through the dot is carried by such orbits. The numerical observation of stable periodic orbits in the Nilsson model of Ref. [52] provided the first evidence for the dynamical origin of strongly coupled quantum states. We note that Fig. 21 also reveals steep levels. The corresponding wave functions change strongly with deformation as expected for parametric variations in chaotic systems.

### 3. Integrable dot

In the preceding sections we presented qualitative arguments for peak correlations induced either by temperature or deformations. A synthesis of both approaches was given by Baltin et al. [31]. It was shown that a quantum dot of the shape of a deformed harmonic os-
cillator can support sequences of up to 30 conductance resonances with similar transmission phase and similar peak height. All resonances within such a sequence are dominated by a single strongly coupled eigenstate of the dot. The study [31] pertains to an integrable dot in the regime of ballistic transport and is therefore restricted in generality and universality. However, aspects of the study are also relevant for ballistic dots of more general shape provided these dots support short orbits that strongly couple to the leads.

Consider the tunneling Hamiltonian $H$ defined in Sec. IV A and assume that the dot confining potential is an anisotropic harmonic oscillator potential. The states of the dot can be labeled by two quantum numbers $\lambda_x, \lambda_y$ and the energy eigenvalues $E_\lambda$ for $\lambda = (\lambda_x, \lambda_y)$ are given by

$$E_\lambda = \hbar \omega_x (\lambda_x + \frac{1}{2}) + \hbar \omega_y (V_g)(\lambda_y + \frac{1}{2}) - \alpha V_g + E_0. \quad (4.42)$$

To describe the deformation, we assume that the oscillator frequency $\omega_y (V_g) = \omega_x (1 - \gamma (V_g - V_0))$ in the transverse direction $y$ depends linearly on the gate voltage $V_g$ while the frequency $\omega_x$ in the longitudinal $x$–direction is held fixed. The parameter $\alpha$ relates the overall depth of the dot potential to the gate voltage. The constants $E_0$ and $V_0$ determine the number of electrons on the dot at zero deformation.

The matrix elements $V_{k,\lambda}^L, V_{k,\lambda}^R$ for tunneling from the left and right lead to the quantum dot are given [56] by the integrals

$$V_{k,\lambda}^{L(R)} = \frac{\hbar^2}{2m_B} \int_B dy \left[ \psi^*_k(x, y) \frac{\partial \Phi_{\lambda_x,\lambda_y}(x, y)}{\partial x} - \Phi_{\lambda_x,\lambda_y} \frac{\partial \psi^*_k(x, y)}{\partial x} \right]_{x=x_B}, \quad (4.43)$$

where $\psi^L_R$ denotes the wave function with wave vector $k$ in the left (right) lead, and $\Phi_{\lambda_x,\lambda_y}$ the wave function in the dot. The integration extends in the $y$–direction and $x_B$ is arbitrary but must be located within the barrier. We restrict ourselves to the case of a single transverse channel in each lead. The nodes of the wave functions of flat (steep) levels with large $\lambda_x$ ($\lambda_y$) are predominantly carried by the $x$–component ($y$–component, respectively).

Thus, the wave functions of flat levels extend much further into the barrier region and have considerably larger matrix elements $V_{k,\lambda}^{L(R)}$ than those of the steep levels. This important property is illustrated in Fig. 22. Due to their stronger coupling, flat levels will carry larger current than steep levels.

According to the scenario discussed in Sec. IV D 2 a flat level of the dot gives rise to a sequence of Coulomb peaks when it undergoes a single level crossing between every two successive peaks. At finite temperature, this condition need not to be met exactly, but must hold on average for a sufficiently large number of peaks. Within our model, we can estimate the number $\Delta N$ of correlated resonances. For $U_0 \gg \Delta$, the distance between neighboring resonances is $\delta V_g = U_0 / \alpha$. The number of intersection points of a flat level ($\lambda_x \neq 0, \lambda_y = 0$)
with steep levels is found using Eq. (4.42). For the total number of crossings in the voltage range \( V_g - V_0 \) one finds

\[
N_c = \frac{\lambda_x^2}{2} \frac{\gamma(V_g - V_0)}{1 - \gamma(V_g - V_0)}.
\]  

(4.44)

The number of steep levels around a given energy increases with deformation. This yields an increase of \( N_c \), and causes the divergence for the unphysical case of extreme deformation \( \omega_y \sim 1 - \gamma(V_g - V_0) \rightarrow 0 \). One crossing within the interval \( \delta V_g \) occurs for \( (\partial N_c)/\partial V_g = \alpha/U_0 \). This condition yields the voltage \( V_g^* \) where maximal correlations of the Coulomb peaks are found; \( V_g^* \) is used below. The number \( \Delta N \) of correlated peaks can be estimated as the number of resonances for which the flat level \((\lambda_x, 0)\) stays within the energy interval \( \Delta \) around \( E_F \). This yields

\[
\Delta N \simeq 2 \sqrt{\lambda_x} \sqrt{\frac{\alpha}{2\gamma U_0}},
\]  

(4.45)

with the deformation \( \omega_y/\omega_x = \sqrt{(\gamma U_0)/(2\alpha)} \lambda_x \). The result (4.45) explicitly relates the number of correlated levels to our model parameters. We note that \( \Delta N \) sets an upper bound for the number of correlated peaks. If \( V_g \) is substantially different from \( V_g^* \), the number of level crossings no longer matches the number of Coulomb peaks. Then the sequences of correlated peaks are shorter than \( \Delta N \) and the correlations are weaker.

The two–terminal conductance \( G \) and the transmission phase through the quantum dot are calculated using the formulas derived in Sec. [IV A 3]. In Fig. 23 we show \( G \) as a function of the gate voltage \( V_g \) for two values of \( kT \). The results are obtained for \( \alpha = 1 \), \( \gamma = 0.005 \), \( E_0 = -11 \) and \( V_0 = 90 \) where the energies and voltages are measured in units of \( U_0 \). The single–particle level spacing is \( \Delta = 0.03U_0 \) which roughly corresponds to the experiments [30,7]. About 100 Coulomb blockade resonances occur in the interval \( 100 < V_g < 200 \). In

FIG. 22. The thin solid line shows a cross section of the potential in longitudinal direction, the two barriers lying at opposite ends. The overlap of the dot wave functions (probability shown as dashed lines) with the lead wave function (probability shown as a solid line on the left) increases strongly with the quantum number \( \lambda_x \). Taken from Ref. [31].
both plots strong peaks with similar peak heights appear whenever a flat level is close to the Fermi energy. The peak–height correlations are more pronounced at higher temperature (case (b)). In the regions $V_g < 130$ and $V_g > 160$, there is no the flat level close to the Fermi energy and $G$ is small. On the scale of Fig. 23 some of the conductance peaks are not visible.

The phase $\theta$ of the transmission amplitude is shown vs. $V_g$ in Fig. 24. For the parameters chosen, the transmission is dominated by the flat level $\lambda = (14, 0)$; the width of this level is $\Gamma_\lambda = \Delta/15$. The solid lines at the bottom of the plot show the conductance peaks and help to identify the resonance positions. One observes a strikingly similar behavior of the phase at all resonances. This behavior is found not only within the $V_g$ interval shown but for the entire interval $130 < V_g < 160$ comprising 30 resonances. The phase increases by $\pi$ at each resonance and displays a sharp lapse by $\pi$ between adjacent resonances. The increase at resonance occurs on the scale $kT$ (we assumed $kT > \Gamma$) and the phase lapse between resonances on the scale $\Gamma$ (see the discussion in Sec. IV C).

The identical behavior of $\theta$ at all resonances reflects the fact that at each resonance, the transmission through the dot is dominated by a strongly coupled level $\lambda$. Similarly, the phase lapse between adjacent resonances is caused by the dominant level: At finite temperature the level $\lambda$ has a finite probability of being either occupied or empty and, thus, contributes to both an electron–like and a hole–like cotunneling process. The contribution of both processes to the transmission amplitude is

\[
t_{QD} = V^L_{\lambda}V^R_{\lambda} \left[ \frac{1 - \langle \hat{n}_\lambda \rangle_N}{E - (E_\lambda + N \cdot U_0) + i\Gamma_\lambda/2} \right]
\]
FIG. 24. Phase $\theta$ of the transmission amplitude versus gate voltage $V_g$ at $kT = \Delta/5$. The solid lines at the bottom of the plots display the conductance peaks. The flat level $\lambda_x = 14, \lambda_y = 0$ is at or near the Fermi energy. Taken from Ref. [31].

where the first (second) term represents the electron (hole) contribution, respectively. As the gate voltage $V_g$ scans the $N^{th}$ conductance valley, the sign of $\text{Re}(t_{QD})$ reverses, leading to a lapse in the transmission phase.

We emphasize that the phase lapse between resonances is a genuine interaction effect. For vanishing charging energy, the cotunneling amplitude (4.46) would reduce to a single, temperature-independent term. The phases of the transmission amplitude in consecutive valleys would not be correlated, and there would be no systematic phase lapse between resonances. We also note that the systematic phase slip only occurs at finite temperature. At zero temperature a flat level contributes to either particle-like or to hole-like cotunneling and no phase lapse is expected within the present model.

4. Energy shift

Silvestrov and Imry [58] demonstrated a mechanism for in-phase behavior that requires neither temperature nor dot deformations. They showed that the energy shift resulting from the coupling between the dot and the reservoirs plays an important role and may cause in-phase resonances. Energy shifts become relevant when they are of the order of the mean level spacing. This requires a coupling of the order $\Gamma \ln(U_0/\Gamma) \sim 2\pi \Delta$. As a result, the scenario of Silvestrov and Imry cannot explain the phase correlation in the Yacoby–experiment (where
Γ ≪ ∆) but is possibly relevant for the strong–coupling regime explored in the Schuster–experiment. The number of peaks that may be correlated due to energy shifts is estimated to be $\sim \frac{\Gamma}{(2\pi \Delta) \ln\left(\frac{U_0}{\Gamma}\right)}$.

Slivestrov and Imry illustrated their idea for a quantum dot of linear dimension $l$ and (dimensionless) confining potential

$$V(x, y) = -4x^2 \left(1 - \frac{x}{l}\right)^2 + \left(y + \frac{x^2}{4l}\right)^2 \left(1 + 2 \left(\frac{x}{l} - 1\right)^2\right).$$

The dynamics in the dot is non–integrable but the potential is approximately symmetric similarly to the dot used in the experiments. The dot is connected to a single lead attached at $x = 0$ and extending in negative $x$–direction with a parabolic confining potential in transverse direction. The Schrödinger equation in the dot and the lead can be solved numerically on a lattice. The kinetic energy is modeled by a standard nearest neighbor hopping term. Within the energy interval $1.5 < \epsilon < 4.7$ used below only a single propagating mode exists in the lead. The energy range corresponds to above–barrier scattering through the dot.

The numerical solution of the single–particle scattering problem reveals scattering resonances with very different width. Slivestrov and Imry extracted the widths by fitting the numerical results with a sum of Breit–Wigner resonances. This yields broad resonances with $\Gamma$ of order $(5 - 7)\Delta$ as well as very narrow resonances with width much less that $\Delta$. The projection of the scattering wave function corresponding to one broad resonance is shown in Fig. 25. It has large probability near the contact to the lead, and may be understood as the superposition of two contributions each quantized on a short classical trajectory.

Motivated by their numerical findings for the single–particle problem, Slivestrov and Imry then investigated many–particle effects. They studied a model with only one level (denoted $\mu$) coupled strongly to the lead. The width of this level was assumed to be $\Gamma \gg \Delta,$
the width of all other levels is much smaller than Δ. Even though one level was strongly transmitting, the standard charging model was employed to model the electron–electron interactions. It was assumed that the charging energy $U_0 \gg \Gamma$. To understand the origin of the energy shift calculated below, it is useful to decompose the total occupation number $\hat{N} = \hat{N'} + \hat{n}_\mu$ in two part, the first part $\hat{N'} = \sum_{\lambda \neq \mu} \hat{n}_\lambda$ describing the total occupation of the sharp levels and the part $\hat{n}_\mu$ the occupation of the broad level. The charging energy then reads

$$\hat{U} = \frac{1}{2} U_0 (\hat{N}^2 - \hat{N}) = \frac{1}{2} U_0 (\hat{N'}^2 - \hat{N'}) + U_0 \hat{N'} \hat{n}_\mu.$$ (4.48)

For fixed $N'$, the charging energy contributes a term $\sim \hat{n}_\mu$ to the Hamiltonian. This term can be combined with the single–particle term $\hat{E}_\mu \hat{n}_\mu$ and yields the effective energy $\hat{E}^{(N')}_{\mu} = \hat{E}_\mu + U_0 \hat{N'}$ which depends on the occupation number $N'$. Here and below we use the superscript to indicate the number of electrons in the sharp levels. For fixed $N'$ and vanishing coupling of the sharp levels, the scattering problem through the dot reduces to the well-known problem of a single state interacting with a continuum. The exact solution of this problem is known [55,58]. The coupling between the state and the continuum reduces the total energy of the system below its value found for the decoupled case. The energy shift introduced by the coupling is also known exactly [55,58].

We now investigate the ground state of the dot as a function of the gate voltage. Assume that the dot is initially filled with $N$ electrons and that the gate voltage is then tuned beyond the charge degeneracy point for $N$ and $N + 1$ electrons. The states in the dot are labeled in the order of increasing energy, and it is assumed that $\mu > N + 1$. Neglecting the coupling to the continuum, the ground state of the dot with $N + 1$ electrons is obtained by filling the lowest $N + 1$ levels and leaving the broad level $\mu$ empty. When the coupling is included, the total energy of this configuration is given by

$$E^{(N+1)}_{\text{tot}} = \sum_{\lambda=1}^{N+1} \mathcal{E}_\lambda + \frac{1}{2} U_0 [(N + 1)^2 - (N + 1)] - \frac{\Gamma}{2\pi} \ln \left( \frac{4E_F}{\mathcal{E}_{\mu}^{(N+1)}} \right),$$ (4.49)

where we have omitted the energy of the electrons on the leads (energies are counted from the Fermi level). The last term on the right hand side is the energy shift resulting from the coupling between the dot and the lead. The expression is valid provided $\mathcal{E}_{\mu}^{(N+1)} \gg \Gamma$. The corrections for $|\mathcal{E}_{\mu}^{(N+1)}| \leq \Gamma$ can be found in Ref. [58]. The energy (4.49) may be compared with the energy

$$E^{(N)}_{\text{tot}} = \sum_{\lambda=1}^{N} \mathcal{E}_\lambda + \mathcal{E}_\mu + \frac{1}{2} U_0 [(N + 1)^2 - (N + 1)] - \frac{\Gamma}{2\pi} \ln \left( \frac{4E_F}{\mathcal{E}_{\mu}^{(N)}} \right).$$ (4.50)

of the configuration where the broad level and $N' = N$ sharp levels are occupied. In the weak–coupling limit, $E^{(N+1)}_{\text{tot}}$ is always less than $E^{(N)}_{\text{tot}}$, and the transmission at the $N + 1$ Coulomb peak proceed via level $N + 1$. However, when $\Gamma$ is large enough so that
the configuration for which level $\mu$ is occupied has lowest energy. In this case, one finds a broad Coulomb peak caused by the transmission through the level $\mu$.

When the voltage is further increased, the two functions $E_{\text{tot}}^{(N)}$ and $E_{\text{tot}}^{(N+1)}$ cross. The voltage at the crossing point is determined by the equation

$$E_\mu(V_g) = U_0 N - \frac{U_0}{\exp[2\pi(E_\mu - E_{N+1})/\Gamma] + 1}$$

At the crossing point, the ground state jumps onto the branch $E_{\text{tot}}^{(N+1)}$ and the current–transmitting level $\mu$ is again empty. At zero temperature, the jump from one electron configuration to the other is accompanied by a sharp jump by $\pi$ of the transmission phase. The process of filling the broad level and jumping to another electron configuration can repeat itself. The number of consecutive correlated resonances caused by this process is $\sim \Gamma/(2\pi\Delta) \ln(U_0/\Gamma)$. We note that the analysis described above pertains to the ground state of the quantum dot. A generalization to finite temperature is required for a quantitative comparison with the experiment (e.g. to explain the finite scale observed for the phase lapse).

5. Bouncing–ball tunneling

In Secs. [IV D2][IV D3] we showed that quantum dots with integrable or mixed classical dynamics may support a subset of quantum states with exceptionally strong coupling to the leads. These states are quantized on short periodic orbits that connect the contacts to the leads. The dynamical origin for strong coupling was further clarified by Hackenbroich and Mendez [59]. Using a tunneling Hamiltonian approach, they showed that the relative coupling strength of the dot states strongly depends on the transverse width of the tunneling region between the dot and the leads. Exceptionally strong coupling to few dot states is only found for sufficiently wide leads, $ka_{\text{eff}} > 1$. Here, $k$ is the Fermi wave number and $a_{\text{eff}}$ the effective transverse width of the tunneling barrier. The transport under these conditions may be termed bouncing–ball tunneling (BBT), as the strongly conducting states are quantized on classical trajectories bouncing between the contacts to the leads. A unique fingerprint of BBT is found in the regime of strong coupling $\Gamma > kT$: Then the peaks in the tail of a sequence of correlated Coulomb peaks develop a characteristic line–shape asymmetry. The origin of this asymmetry is the breaking of the particle–hole symmetry as the strongly coupled bouncing–ball state moves away from the Fermi energy.

The crucial role of the transverse barrier width $a_{\text{eff}}$ can be understood from the following simple argument: Confinement in a lead of width $a_{\text{eff}}$ yields the transverse momentum spread $\hbar/a_{\text{eff}}$ for electrons injected in the quantum dot. Wide leads therefore result in near
normal injection and provide exceptionally strong coupling to the bouncing–ball states. In the tunneling Hamiltonian, $a_{\text{eff}}$ enters in the the matrix element $V^l_\lambda$ for tunneling between the lead $l$ and the state $\lambda$ in the quantum dot. For sufficiently high barriers, $V^l_\lambda$ is given by

$$V^l_\lambda = \left(\frac{\hbar^2}{m^*}\right) \int ds \psi_l(s, z) \partial_z \psi^*_\lambda(s, z) \bigg|_{z=0},$$

(4.53)

where the integration is performed along the edge between the potential barrier and the quantum dot ($\partial_z$ denotes the derivative normal to the barrier). The wave function $\psi_\lambda$ corresponds to Dirichlet boundary conditions in the dot, while the barrier tunneling is fully included in the lead wave function $\psi_l$. The transverse potential in the tunneling region can be taken quadratic [60] yielding $\psi_l \sim c_l \exp[-(s - s_l)^2/2a_{\text{eff}}^2]$, where $s$ is the transverse coordinate, $s_l$ the center of the constriction and $a_{\text{eff}}$ its effective width. One can restrict the calculation to the lowest transverse mode since higher modes are suppressed by the barrier penetration factor (included in $c_l$).

The central result of Ref. [59] is presented in Fig. 26. Shown is the coupling strength $g_\lambda \equiv \Gamma^U_\lambda \Gamma^D_\lambda / (\Gamma^U_\lambda + \Gamma^D_\lambda)$ over a sequence of 100 quantum states labeled by the index $\lambda$. Note that $g_\lambda$ is proportional to the conductance peak height $G_\lambda = (e^2/h)(\pi/2kT)g_\lambda$ measured in low temperature Coulomb blockade experiments [12]. The dot is described by a hard-wall confining potential at the boundary, parameterized in polar coordinates by $R(\phi) = R_0[1 + \epsilon \cos(2\phi)]$. Here, $\epsilon$ measures the quadrupolar deformation out of circular shape. A nonzero value for $\epsilon$ mimics the dots used in Refs. [6,7]. As in the experiments, the leads are attached opposite to each other at the boundary points closest to the origin (the points with $\phi = \pm \pi/2$). For the value $\epsilon = 0.2$ used for Fig. 26 the classical dynamics in the dot is almost completely chaotic except for two large islands associated with stable bouncing–ball motion between the contacts to the leads.

For narrow leads (Fig. 26(a)) one observes many peaks with widely different peak height. In striking contrast, the results for wide leads (Fig. 26(b)) show a few isolated large peaks, separated by 15 – 25 levels with much smaller peak height (not visible on the scale of Fig. 26(b)). All large peaks are associated with states quantized on stable bouncing–ball orbits. This is illustrated in the inset for the state $\lambda = 337$. The height of the small peaks not resolved in Fig. 26(b) is typically two or more orders of magnitude smaller than the maximum peak height. Such tiny peaks are difficult to resolve in Coulomb blockade interference experiments. Interference experiments with wide leads are therefore only sensitive to the strongly coupled bouncing–ball modes.

We now turn to the calculation of the transmission coefficient $T_{\text{QD}}$ and the phase $\theta_{\text{QD}}$. We assume wide leads and $kT < \Delta$. The case of a weakly coupled dot was studied in Sec. [IV A]. Here, we consider a more open dot characterized by $\Gamma_\mu \sim \Delta$. We identify $\mu$ with the
FIG. 26. Coupling strength $g_\lambda$ for a sequence of 100 states in the interval $290 \leq \lambda \leq 390$. Results are for a quantum dot with quadrupolar shape and deformation $\epsilon = 0.2$. (a) Narrow leads: $ka_{\text{eff}} = 0.1$. (b) Wide leads: $ka_{\text{eff}} = 5.0$. Here $k = (2m^*E_\lambda)^{1/2}/\hbar$ is the wave number associated with the bouncing–ball state $\lambda = 337$. Inset: Real–space projection of the state $\lambda = 337$.

bouncing–ball state closest to the Fermi energy. All other states $\lambda \neq \mu$ in the vicinity of $E_F$ have a much smaller width $\Gamma_\lambda \ll \Delta$. The Green function for this case may be obtained using the equations of motion method (see Sec. IV A 2). It is diagonal up to small off–diagonal corrections $O(\sqrt{\Gamma_\lambda \Gamma_\mu/\Delta})$, and given by

$$G_{\mu\mu}(E) = \sum_{N' = 0}^{\infty} \frac{P_{N'}}{E - (E_\mu + U_0 \cdot N') + i\Gamma_\mu/2}. \quad (4.54)$$

Here $N'$ counts the total number of electrons in all dot levels except for the level $\mu$ and $P_{N'}$ is the respective occupation probability. The transmission through the states $\lambda \neq \mu$ is negligible. Equation (4.54) is the generalization of the weak–coupling result derived in Sec. IV A 2 to the case of a single strongly conducting quantum state. The probability $P_{N'} = Z^{-1}\exp[-\Omega(N')/kT]$ with $Z = \sum_{N'} \exp[-\Omega(N')/kT]$ is related to the thermodynamic potential $\Omega(N')$ of the dot. To evaluate $P_{N'}$, we replace $\Omega_N$ by $\Omega_{N'} + [E_\mu + U_0 \cdot N']\langle n_\mu \rangle_{N'}$, where $\Omega_0$ is calculated for the dot with level $\mu$ excluded from the spectrum, and

$$\langle n_\mu \rangle_{N'} = -\frac{1}{\pi} \int dE \text{Im} \frac{f(E)}{E - (E_\mu + U_0 \cdot N') + i\Gamma_\mu/2} \quad (4.55)$$

is the canonical occupation probability of level $\mu$.

In Fig. 27 we show the transmission coefficient $T_{QD}$ and the transmission phase $\theta_{QD}$ vs. gate voltage $V_g$ (we assumed $E_\mu = E_\mu^{(0)} - eV_g$). All peaks shown result from transmission
FIG. 27. (a) Transmission coefficient $T_{QD}$ and (b) phase $\theta_{QD}$ as a function of gate voltage $V_g$ evaluated for $kT = 0.2\Delta$, $\Gamma_\mu = 1.5\Delta$, and $U_0 = 12\Delta$. The peak asymmetry in (a) is a unique fingerprint of BBT.

Through the level $\mu = 337$. The peaks have comparable height and similar phase in qualitative agreement with the experiments [6,7]. The central peak has a Lorentzian shape of width $\Gamma_\mu$. Note that the transmission peaks develop a peculiar asymmetry as the conducting level moves away from the Fermi energy: Each peak to the left and to the right of the central peak falls off more rapidly on the side facing the central peak. This pattern extends over the whole sequence and becomes more pronounced for the peaks in the tails. The asymmetry results from the breaking of the particle–hole symmetry in the transmission through the bouncing–ball state and is a unique fingerprint of BBT. Inspection of the data of the experiment [7] reveals the same asymmetry, providing strong evidence that the peak correlations in this experiment are due to BBT.

Note that the magnitude of the conductance peaks in Fig. 27(a) decreases in the tails of the sequence. This is in contrast to the experiment [6] where similar peak–heights were observed. This discrepancy may have several reasons: First, the billiard model used here completely neglects deformations of the dot shape. As discussed in Sec. IV D 2, IV D 3 deformations can enhance peak–height correlations by “pinning” the conducting levels close to the Fermi energy. The model further neglects the electrostatic influence of the plunger gate on the point contacts defining the tunneling barriers. In most experiment, this influence is significant: Increasing the plunger voltage, inevitably opens the dot. To model this effect one may reduce the barrier height with increasing voltage. In turn, this would result in enhanced peak heights on one side of a sequence. This voltage induced enhancement and
the suppression by temperature may result in a number of peaks with similar peak height.

6. Other mechanisms

In two recent papers [61, 62] Wu et al. and Kang proposed that the in–phase resonances reflect an interference effect in the AB ring rather than a property of the quantum dot. The authors investigate the transmission of non–interacting electrons through a single–channel AB ring. A two–terminal AB ring is studied in Ref. [61] and both a two–terminal and a four–terminal ring in Ref. [62]. The quantum dot is modeled in both studies as a symmetric 1D double barrier well (identical barriers on either side). Full coherence is assumed through the dot and the AB ring. For an integer value of flux quanta threading the ring, a set of conductance peaks is found upon variation of the dot potential. All peaks are in phase. A new set of in–phase peaks appears at half–integer flux, while the previously found peaks disappear. Each peak of the new set is located between two peaks of the previous set. The explanation of the effect is straightforward and follows from the scattering theory presented in Sec. III B. There it was shown that the oscillatory part of the conductance close to a resonance of the quantum dot is of the form

\[ G_{AB} \propto 1 + \cos(\phi - \chi), \] (4.56)

where \( \phi = 2\pi \Phi / \Phi_0 \) is the dimensional flux and where \( \chi \) takes the values 0 and \( \pi \) for even and odd states, respectively, of the quantum dot (cf. Eq. (3.29)). As a result, only even states contribute to the AB current if an integer number of flux quanta are threading the ring. Clearly, all even states are in phase. Conversely, for half integer number of flux quanta, only odd states contribute and, again, all with the same phase. Based on Eq. (4.56) one expects contributions of both even and odd states for intermediate values of the flux. Precisely this has been reported by Kang [62] for \( \Phi = \Phi_0 / 4 \) and \( \Phi = 3\Phi_0 / 4 \).

The ideas presented in Refs. [61, 62] cannot account for the experimentally observed in–phase behavior. First, Refs. [61, 62] find in–phase resonances only for integer or half–integer number of flux quanta threading the ring. For generic values of the flux, neighboring resonances are always out of phase by \( \pi \). Second, the destructive interference responsible for the in–phase resonances in [61, 62] is only complete if the transport through the AB ring is fully coherent. In the experiments [6, 7], however, most of the current is incoherent. The coherent component comprises only about 10 or 20 percent of the total current. The incoherent ring current displays resonant behavior regardless of the value of the external flux or the parity of the dot state. Hence, experimentally both even and odd resonances are observed for all values of flux, and each current resonance is associated with the charging of the dot by one additional electron. The prediction [61, 62] of missing resonances and
the addition of the charge $2e$ between neighboring resonances is in contradiction to the experiments.
V. CONTROLLED DEPHASING

In 1996 Gurvitz \[63\] realized that mesoscopic electron devices could be utilized to study and control decoherence. He investigated the resonant tunneling through two capacitively coupled quantum dots. Gurvitz showed that the second dot could serve as a detector for the charge accumulated in the first dot. Solving the equations of motion of the entire system and then tracing over the detector variables, Gurvitz obtained the reduced density matrix of the measured dot. The coupling to the detector dot lead to a damping of the off–diagonal elements and thus caused decoherence.

Gurvitz’ paper has triggered new experimental and theoretical developments exploring the controlled decoherence of quantum transport by the environment. Since the destruction of coherence is not necessarily related to energy relaxation, the new subject is commonly referred to as controlled dephasing. The first demonstrations of controlled dephasing were achieved in beautiful which–path experiments of Buks et al. \[8\] and Sprinzak et al. \[9\]. The experiments employed the transport through a single \[8\] or through two \[9\] tunnel–coupled quantum dots. In close proximity and capacitively coupled to one dot was a quantum point contact (QPC). Electrons passing through the quantum dot interact with electrons in the QPC. This modifies the transmission through the QPC, so that the QPC serves as a detector for the charge of the dot. It was found that a current flowing through the QPC leads to dephasing in the quantum dot. In the experiment \[8\] dephasing was detected as a reduction of AB oscillations using a device with the quantum dot embedded in an AB ring. The set–up can be viewed as a which–path interferometer, the archetype of a position measurement apparatus in a double–slit device. We discuss the which–path experiments in Sec. VA. The theoretical work devoted to dephasing in mesoscopic electron structures is summarized in Sec. VB.

A. Which–path experiments

1. Experiment of Buks et al.

Fig. 28 shows the scanning electron micrograph and a schematic description of the set–up used in the which–path experiment of Buks et al. \[8\]. Part of the device, including the multi–terminal AB ring and the quantum dot embedded in its right arm is identical with the setup used in the Schuster–experiment (see Sec. II). The new element is the QPC to the right of the quantum dot. It is visible as a small constriction in the two–dimensional electron gas. The width of the QPC and hence its transmission coefficient is controlled by the gate voltage \(V_g\). The central gate is contacted with a metallic air bridge. This gate at the potential \(V_P\) depletes the area between the two arms of the ring and, at the same
time, serves as plunger gate of the quantum dot. The dot is tuned to the Coulomb blockade regime with the resistance of the tunneling barriers at either side of the dot being greater than $h/2e^2$.

The collector current $I_c$ measured for a fixed a.c. emitter voltage $V_E = 10\mu V$ displays AB oscillations as depicted in Fig. 29. The solid line shows the AB oscillations with vanishing drain–source voltage $V_d = 0$ across the QPC. The dotted line is measured with $V_d = 100\mu V$. The reduction of the visibility reflects the dephasing introduced by the which–path measurement.

To study the dephasing quantitatively as a function of the QPC characteristics, Buks et al. investigated the calibration device shown in the inset of Fig. 30(a). The device contains a QD and a QPC similar to these in the which–path interferometer. The conductance of the QD is scanned through a series of Coulomb peaks by changing the plunger voltage $V_P$. Due to the proximity of QD and QPC, the transmission coefficient $T_d$ of the QPC is also affected by the potential of the QD. This gives rise to the smooth increase of $T_d$ with increasing plunger voltage $V_P$ (see Fig. 30(a)). However, whenever a conductance peak is being scanned and an additional electron is being added to the QD, $T_d$ displays a faster and opposite change with amplitude $\Delta T_d$ on the scale of the peak width. This decrease of transmission reflects the change in the QD potential due to the additional electron. Fig. 30(b) shows $\Delta T_d$ averaged over several Coulomb peaks as a function of the transmission $T_d$. The reduced value of $\Delta T_d$ near $T_d = 0$ and $T_d = 1$ is a consequence of approaching the conductance plateaus. At the plateaus, changes in the QPC potential have very little effect on the QPC transmission.

Fig. 31(b) summarizes the main result of the experiment. The visibility of the AB oscillations is shown vs. the QPC voltage $V_g$. The visibility is obtained upon dividing the peak–to–valley value of $I_C$ by the average $I_C$. The measurement was done when the QD was tuned to a conductance peak using the central metal island as a plunger gate. Fig. 31(a) shows that the detector transmission increases from 0 to 1 as $V_g$ is being changed. We note that any modulation of the visibility reflects the dephasing introduced by the detector; a small visibility corresponds to strong dephasing. For small detector voltage $V_d = 10\mu V$, the visibility is practically constant with $V_g$. For $V_d = 100\mu V$, one observes a pronounced structure: The visibility peaks near the conductance plateaus $T_d = 0, 1$, and in between for $T_d = 0.5$. It turns out that the weak dephasing at these points is due the small amount of which–path information in the detector, cf. Sec. V B 1. The detector is inefficient both at the conductance plateaus and at $T_d = 0.5$ where the QPC current is very noisy. The theoretical approaches summarized in Sec. V B obtain for the visibility the result $\nu = \nu_0\nu_d$, where $\nu_0$ is the intrinsic visibility of the AB ring and $\nu_d$ the reduction of the visibility due to the detector. At zero temperature, $\nu_d$ is calculated to

$$\nu_d = 1 - \frac{eV_d}{4\pi T} \frac{(\Delta T_d)^2}{8T_d(1 - T_d)}, \quad (5.1)$$
FIG. 28. Top: Scanning electron micrograph of the which-path device. Bright regions indicate metallic gates. A quantum dot is defined in the right arm of a multi-terminal AB ring. A quantum point contact to the right of the quantum dot serves as a which-path detector. Bottom: Schematic description of the electrodes and contacts in the which-path experiment. The interferometer is composed of three different regions, collector C, emitter E, and base regions B. The quantum point contact is to the right of the quantum dot. A finite voltage $V_d$ is applied across the quantum point contact. Taken from Ref. [8].
FIG. 29. AB oscillations of the collector current $I_C$. The solid line is measured with QPC drain source voltage $V_d = 0 \, \mu$V. The dotted line, with reduced visibility, is measured with $V_d = 100 \, \mu$V. Taken from Ref. [8].

FIG. 30. Conduction characteristics of the calibration device shown in the inset of (a). (a) Conductance through the quantum dot and transmission through the QPC both as a function of the plunger gate voltage $V_p$. (b) Induced average change $\Delta T_d$ in the transmission coefficient of the QPC due to adding an electron to the quantum dot as a function of $T_d$. Each data point is obtained by averaging over several Coulomb peaks. Taken from Ref. [8].
FIG. 31. (a) QPC transmission coefficient $\tau_d$ as a function of QPC gate voltage $V_g$. (b) Measured visibility of the AB oscillations as a function of $V_g$ for two values of the QPC drain source voltage $V_d$. The visibility is defined as the peak–to–peak signal divided by the average signal. Error bars indicate the fluctuations in visibility due to fluctuations of the device’s properties. Taken from Ref. [8].

FIG. 32. Visibility as a function of QPC drain–source voltage $V_d$ for fixed QPC transmission coefficient $\tau_d = 0.2$. The visibility decreases linearly for $eV_d > kT$ and saturates for low $V_d$. Taken from Ref. [8].
where $\Gamma$ is the intrinsic (zero-temperature) width of the QD resonance. The solid line in Fig. 31(b) shows the prediction of Eq. (5.1), where $\Gamma = 0.5 \mu eV$ was used as a fitting parameter. The quantity $\Delta T_d$ was taken from the calibration device, cf. Fig. 30.

The dependence of the visibility on the drain source voltage $V_d$ is illustrated in Fig. 32. For $eV_d \gg kT$, the visibility drops linearly as expected from Eq. (5.1). A deviation from the linear dependence occurs near $eV_d \approx kT$.

2. Experiment of Sprinzak et al.

The which–path information in the setup of Buks et al. is encoded in the current flowing through the QPC–detector. Sprinzak et al. [9] used a different approach and devised a setup in which the which–path information enters only as a quantum mechanical phase. The setup is shown in Fig. 33: A QPC serves as the detector for the oscillation of charge in a double quantum dot (DQD) interferometer. The device is subjected to a high magnetic field corresponding to two filled Landau levels of the two-dimensional electron gas. Under these conditions most current flows in a chiral motion along the the edges of the sample. Scattering between the edge states only occurs at the QPC constriction where the two edges of the sample are close together. The nearby DQD interferometer leaves the current through the QPC unaffected but does change the phase of the transmitted electron mode. The phase of the transmitted electrons therefore contains information about the state of the DQD. Note that the DQD is not inserted in a standard two–path interferometer since the presence of a high magnetic field would prevent electrons from choosing either path with nearly equal probability. The experiment, therefore, does not probe the path of electrons through the interferometer but the state of the interferometer. In keeping with Ref. [9] we still refer to the setup as a which–path experiment.

The dephasing rate of the interferometer can be extracted from the conductance through the DQD. The DQD is being tuned to resonance by means of the two plunger gate voltages $V_{p1}$ and $V_{p2}$. For the two quantum dots in series, resonances are found when resonant levels in both dots are degenerate. The resulting Coulomb peaks are located on a hexagonal lattice and each peak has the width $2\Gamma$ where $\Gamma$ is the intrinsic resonance width of a single dot [64,12]. In the presence of the detector one expects to observe a reduction of the peak height and a broadening of the peak width to $2(\Gamma + \hbar/t_d)$. A single resonance peak with its contour at half maximum is shown in Fig. 34. Due to the asymmetry of the peak shape, the area in this contour is taken as a measure of the dephasing rate.

Figure 35 shows the dependence of the contour area and the peak height as a function of the QPC transmission probability $T_d$. Both plots were obtained for a fixed QPC voltage $V_d$. The measured area (Fig. 35a) yields a dephasing rate proportional $T_d(1 - T_d)$. Similarly, the peak height (Fig. 35b) has an inverse dependence on that expression. Note that the result
FIG. 33. Schematic illustration of the DQD interferometer and the QPC detector. The device is studied in strong perpendicular magnetic field (5–10 Tesla) that imposes chiral electron motion near the edges of the sample. The edge states are partly transmitted (|t⟩) and partly reflected (|r⟩) by the QPC. The DQD is weakly coupled to its own leads and is tuned to resonance by the two plunger gate voltages $V_{p1}$, $V_{p2}$. Taken from Ref. [9].

FIG. 34. View of one Coulomb peak as a function of the two plunger gate voltages $V_{p1}$ and $V_{p2}$. The dashed line is a contour drawn at half maximum of the peak height. The area enclosed by this contour is being used as a measure of the peak width. Taken from Ref. [9].
FIG. 35. Top: The area of the contour at half peak height as a function of the transmission probability $T_d$ for two values of the applied bias $V_d = 0$ and 2 mV. The dependence agrees qualitatively with the expected $T_d(1 - T_d)$. Inset: The dependence of the contour area on the applied voltage $V_d$ for $T_d = 0.7$. Bottom: The peak height (in units of $e^2/h$) as a function of the transmission probability $T_d$ for two values of applied bias $V_d = 0$ and 2 mV. Taken from Ref. [9].

for the dephasing rate differs from the rate $\propto (\Delta T_d)^2/[T_d(1 - T_d)]$ found in the experiment of Ref. [8]. We show in Sec. V B that the difference is due the different nature of the which–path information: In the present device this information resides in the transmission phase while in the case of the experiment [8] it is encoded in the transmission probability of the QPC. The inset of Fig. 35 shows that the contour area and hence the dephasing rate grow nearly linearly with the detector voltage $V_d$.

The measurements show that a phase change in the detector leads to dephasing of the interferometer even though no interference experiment is being performed in the detector. The question arises onto whether the detector must be phase coherent in order to dephase the interferometer. To investigate this question, Sprinzak et al. inserted an artificial dephaser (a floating Ohmic contact) in the path of the transmitted electrons before they reached the DQD. Even though phase coherence was destroyed, dephasing persisted (however, a small decrease of dephasing was observed and traced back to the finite capacitance of the Ohmic contact which effectively eliminated high frequency components of the shot noise).
In the past, experimental investigations of decoherence (dephasing) have been executed with photons, cooled atoms, and neutrons. The which–path experiments of Refs. [8,9] for the first time allowed to investigate controlled dephasing with mesoscopic electron structures. Widely different methods have been used [65,66,10,8,11,67–69,93] to investigate the problem theoretically. Conceptually, some of the authors investigated the influence of the interferometer on the state of the detector [65,8,9] while other authors [66,10,11,67] analyzed how the detector affects the interferometer. Both descriptions lead to equivalent results as is known since the work of Stern et al. [70].

In Sec. V B 1 we present a simple argument for the dephasing rate in the which–path interferometer. The argument displays the intimate link between the dephasing rate and the efficiency with which the QPC detector measures the state of the interferometer. Rigorous derivations of the dephasing rate and the suppression of AB oscillations in a which–path interferometer are given in Sec. V B 2-V B 5.

1. Heuristic argument

A qualitative argument for the dephasing of quantum dot states due to the coupling to a QPC detector was given by Aleiner et al. [65] and Buks et al. [8]. Their argument relates the dephasing rate to the which–path information obtained in the detector. The original argument only considers the information contained in the current through the QPC. Below we generalize the argument to include, in addition, the information encoded in the phase of the transmission amplitude through the QPC.

An electron in the quantum dot interacts electrostatically with electrons in the QPC–detector. The interaction generally modifies both the modulus and the phase of the transmission amplitude through the QPC. The change in the transmission probability induced by the presence of one electron on the quantum dot is denoted by $\Delta T_d$; the corresponding change of the transmission phase is $\Delta \phi$. In experiment, these changes cannot be detected with certainty due to current and phase fluctuations. The current fluctuations are due to the well–known quantum shot noise (for reviews on shot noise see Refs. [71–73]): Let $N = (2eV_d/h)t_T$ be the number of electrons that probe the QPC biased with the voltage $V_d$ during a time $t_T$. Then the number of transmitted electrons $N_T$ is a binomial random variable with the expectation value $\langle N_T \rangle = N T_d$ and the standard deviation $\delta N_T = \sqrt{N T_d (1 - T_d)}$. From the uncertainty $\delta N_T$ one finds the uncertainty $\delta T_d = \sqrt{T_d (1 - T_d)}/N$ in $T_d$. Detection of the electron on the quantum dot requires that $\delta T_d$ is less than or equal to $\Delta T_d$. The equality yields the dephasing rate
reflecting the which-path information encoded in the detector current. A similar reasoning applies to the transmission phase $\phi$: The uncertainty in $\phi$ is given by $\delta \phi \sim 1/\delta N_{\phi}$ where $\delta N_{\phi}$ is the uncertainty in the number of electrons transmitted in the time $t_\phi$. The condition that $\delta \phi$ equals the phase change $\Delta \phi$ yields the dephasing rate

$$\frac{1}{t_\phi} \propto \frac{eV_d}{\hbar} \frac{(\Delta T_d)^2}{T_d(1 - T_d)}.$$  \hfill (5.2)

Adding the dephasing rates due to the change in the current and the phase, one finds that the total dephasing rate has the form

$$\frac{1}{t_d} = C_T \frac{eV_d}{\hbar} \frac{(\Delta T_d)^2}{T_d(1 - T_d)} + C_\phi \frac{eV_d}{\hbar} \frac{T_d(1 - T_d)(\Delta \phi)^2}{},$$  \hfill (5.3)

where $C_T$ and $C_\phi$ are coefficients of order 1. The rigorous calculations presented in Sec. V B support the simple estimate (5.4) and yield expressions for the coefficients $C_T$ and $C_\phi$.

2. Green function

Controlled dephasing and the suppression of AB oscillations due to the interaction with a QPC have been investigated by Aleiner, Wingreen, and Meir [65]. The model consists of a QD in the Coulomb blockade regime which is embedded in an AB ring and capacitively coupled to a QPC (Fig. 36). It is assumed that $kT$ is much smaller than the single-particle level spacing $\Delta$. At a transmission peak two charging states of the dot with $N$ and $N + 1$ electron are degenerate. Electrons pass through a single level in the dot. The amplitude $t_{\text{QD}}(\epsilon)$ can be expressed in terms of the retarded Green function of this level,

$$t_{\text{QD}}(\epsilon) = -i\sqrt{4\Gamma_L \Gamma_R} \int dt e^{i\epsilon t} G_{\text{QD}}(t),$$  \hfill (5.5)

where $\Gamma_L, R$ are the partial width for decay of the level through the left and right tunneling barrier, respectively (we put $\hbar = 1$).

Electrons in the QD interact with electrons in the QPC. This interaction modifies the local potential in the QPC. We use the standard description of the QPC as a 1D system of noninteracting electrons. The Hamiltonian $H_N$ of the QPC when exactly $N$ electrons occupy the QD is written in terms of exact scattering states,

$$H_N = \int \frac{dk}{2\pi} \left[ \psi_L^\dagger(k) \psi_L(k) + \psi_R^\dagger(k) \psi_R(k) \right],$$  \hfill (5.6)

where $\psi_{L,R}$ are the destruction operators for left and right moving scattering states, respectively. A summation over spin indices is implied. The presence on the $N + 1$ electron changes the QPC Hamiltonian to $H_{N+1} = H_N + V$, where
The wire with the QPC is connected to two reservoirs between which the finite bias $eV_d$ is applied.

The Green function of the resonant level in the dot interacting with the wire is of the form

$$G_{QD}(t) = -i\Theta(t)e^{-iE_0t-\Gamma t}[P_{N+1}A_-(t) + P_NA_+(t)],$$

where $E_0$ is the energy of the level in the absence of the QPC, and $P_{N(N+1)}$ the probability of the charging state $N(N+1)$ of the dot, $P_N + P_{N+1} = 1$. The total width is $\Gamma = \Gamma_L + \Gamma_R$.

The coherence factors $A_{\pm}(t)$ describe the response of the wire to the addition (removal) of an electron from the dot,

$$A_+(t) = \langle e^{iH_N t} e^{-iH_{N+1} t} \rangle_{H_N},$$

$$A_-(t) = \langle e^{iH_N t} e^{-iH_{N+1} t} \rangle_{H_{N+1}}.$$  

The expectation values are taken with respect to an equilibrium ensemble in the wire with the Hamiltonian $H_N$ and $H_{N+1}$, respectively.

An interpretation of dephasing follows directly from these expressions. They can be understood as the scalar product of two states of the environment (the QPC) at time $t$. These states are identical at $t = 0$, and thus $A_\pm(t = 0) = 1$. Then, one state evolves with
the Hamiltonian $H_N$, the other state with $H_{N+1}$. Since the environment is not observed in the experiment, its coordinate is being integrated over. Quantum coherence is lost once the states of the environment have become almost orthogonal. Then the scalar product becomes almost zero and the environment has identified the state of the QD.

The operators $V_{LL}$ and $V_{RR}$ mix scattering states that propagate in the same direction. This only changes the phase of the transmission amplitude through the QPC. The mixing of scattering states moving in opposite directions, described by $V_{LR}$, modifies the transmission coefficient. Note that only the latter contribution is affected by the voltage drop in the wire. Hence, the nonequilibrium (finite $V_d$) part of the dephasing rate is entirely due to this latter contribution.

Aleiner et al. [65] obtained $A_\pm$ for arbitrary $\lambda$ using known results for the orthogonality catastrophe, i.e. the response of an equilibrium noninteracting electron system to a sudden perturbation. The nonequilibrium dephasing can be calculated in an expansion to second order in $\lambda_{LR}$. The result has the form

$$A_+(t) = \left(\frac{i\pi kT}{\xi_0 \sinh \pi kTt}\right)^{\alpha+\gamma} e^{-t/t_d + \gamma h(t,kT,eV_d)}.$$  \hspace{1cm} (5.13)

Here, $\xi_0$ is a high–energy cutoff, the smaller of the Fermi energy in the wire or the inverse rise time of the perturbation in the wire. The definitions of the exponents $\alpha$, $\gamma$ and the crossover function $h$ can be found in Ref. [65]. It suffices to note that $h(t)/t \to 0$ for large $t$. For a symmetric dot, the dephasing rate is given by

$$\frac{1}{t_d} = \frac{eV_d}{h} \frac{(\Delta T_d)^2}{4T_d(1 - T_d)},$$  \hspace{1cm} (5.14)

which agrees up to a constant factor with the estimate in Eq. (5.2).

The calculation of the coherence factor $A_- (t)$ yields $A_- (t) = A_+ (t)^*$. At the transmission peak, $P_N = P_{N+1} = 1/2$, and the transmission amplitude $t_{\text{QD}}$ is the Fourier transform of $G_{\text{QD}}(t)$. The final result can be approximated by [65],

$$t_{\text{QD}} \simeq \frac{2\pi \sqrt{\Gamma_L \Gamma_R}}{4kT + \pi \Gamma_{\text{tot}}} \left(\frac{kT + \Gamma_{\text{tot}}}{\xi_0}\right) \left(\frac{kT + \Gamma_{\text{tot}} + eV_d}{\xi_0}\right)^\gamma,$$  \hspace{1cm} (5.15)

where the total width $\Gamma_{\text{tot}} = \Gamma + \Gamma_d$ is given in terms of the intrinsic width $\Gamma$ and the width $\Gamma_d = \hbar/t_d$ introduced by the dephasing. For the which–path experiment [8] the exponents $\alpha$, $\gamma$ appear to be rather small so that the last two terms on the right hand side of Eq. (5.15) yield a factor 1. Moreover, $\Gamma, \Gamma_d \ll kT$, so that we can expand $t_{\text{QD}}$ to find

$$t_{\text{QD}} \simeq \frac{\pi \sqrt{\Gamma_L \Gamma_R}}{2kT} \left(1 - \frac{eV_d}{kT} \frac{(\Delta T_d)^2}{32 T_d(1 - T_d)}\right).$$  \hspace{1cm} (5.16)

The term in brackets on the right–hand–side represents the reduction of the visibility due to the interaction with the QPC.
3. Influence functional

A conceptually different approach to the which–path interferometer was presented by Levinson [66]. He studied an isolated quantum dot (not connected to leads) with a single level at energy $E_0$ and Hamiltonian $H_{\text{QD}} = E_0c^\dagger c$. The QPC and the interaction between QPC and QD are modeled precisely as in Sec. VB2, so that the total Hamiltonian takes the form

$$H = H_{\text{QD}} + H_N + c^\dagger cV,$$

where $H_N$ and $V$ are defined in Eqs. (5.6) and (5.7)–(5.9), respectively. Levinson investigated the effect of the environment on the QD. The charge transmitted through the QPC creates a fluctuating potential at the QD that modulates the electron states in the QD. This modulation dephases the states in the QD.

The coherence of the QD is quantitatively described by the expectation value $\langle c(t) \rangle$ taken with respect to the equilibrium ensemble with the Hamiltonian $H$. Here, $c(t) = e^{iHt}ce^{-iHt}$ is the amplitude in the Heisenberg representation. The QD has a coherent part if $\langle c(t) \rangle \neq 0$, while the case $\langle c(t) \rangle = 0$ and $\langle c^\dagger c(t) \rangle \neq 0$ corresponds to a totally incoherent quantum dot. The equation of motion for $c(t)$ reads

$$\frac{dc(t)}{dt} = i(E_0 + V(t))c(t),$$

where $V(t) = c^{iHt}V e^{-iHt}$. According to Eq. (5.18), $V(t)$ may be interpreted as a time dependent modulation of the energy $E_0$. Solving this equation and averaging, one finds

$$\langle c(t) \rangle = \langle c(0)e^{-iE_0t}T_1e^{-i\int_0^t V(t_1)dt_1} \rangle,$$

with the time ordering operator $T_1$. Note that the time ordered exponential on the right hand side of this equation is identical with the product $\exp(iH_Nt)\exp(-iH_{N+1}t)$ that enters in the coherence factors $A_\pm$ defined in the previous section. This proves the equivalence of the present approach with the one described in Sec. VB2. An expectation value of the type of that on the right hand side of Eq. (5.19) is known as the Feynman–Vernon influence functional.

The average in Eq. (5.19) is calculated assuming that the process described by $V(t)$ is a Gaussian process. This amounts to the approximation $\langle c(t) \rangle = \langle c(0) \rangle \times \exp(-iE_0t)\exp[-(1/2)\Phi(t)]$, where $\Phi(t) = \int_0^t dt' \int_0^t dt''K(t' - t'')$ with

$$K(t) = \frac{1}{2}[\langle V(t)V(0) \rangle + \langle V(0)V(t) \rangle].$$

The correlator $K(t)$ decays in time on a characteristic time scale $t_d$. For large $t \gg t_d$, one finds $\langle c(t) \rangle = \langle c(0) \rangle \exp(-iE_0t)\exp(-t/t_d)$ where the dephasing rate is given by...
\[
\frac{1}{t_d} = \frac{1}{2} \int_{-\infty}^{\infty} dt K(t). \tag{5.21}
\]

The dephasing rate may be found using known results for the current correlation functions in a QPC. For a symmetric dot one finds the results [66]

\[
\frac{1}{t_d} \simeq \mathcal{A} eV_d / \hbar \quad \text{for high bias } eV_d \gg kT, \tag{5.22}
\]
\[
\frac{1}{t_d} \simeq \mathcal{A} (eV_d)^2 / (\hbar kT) \quad \text{for low bias } eV_d \ll kT, \tag{5.23}
\]

with \( \mathcal{A} = (\Delta T_d)^2 / [8\pi T_d(1-T_d)] \). The result for the high bias limit is identical with Eq. (5.14) obtained by Aleiner et al. [65].

4. Master equation

Density matrix approaches to dephasing in coupled quantum dots were devised by Gurvitz [10] and Hackenbroich et al. [11]. The approaches include the dephasing in the which–path interferometers as a special case. Gurvitz’ work is reviewed in Sec. VI. The derivation given below is similar in spirit to the approach of Ref. [11].

We consider a quantum dot with a single energy level. The QD is coupled to a QPC but otherwise isolated from the environment. For the Hilbert space of the QD we choose the basis \( |a\rangle, |b\rangle \), representing an empty and an occupied dot level, respectively. We assume that the wire with the QPC supports only a single transverse mode so that the scattering matrix \( S_{\text{QPC}} \) through the QPC is a \( 2 \times 2 \) matrix that depends on the occupation of the QD,

\[
S_{\text{QPC}} = \begin{cases} 
S_a, & \text{if the QD is empty}, \\
S_b, & \text{if the QD is occupied}.
\end{cases} \tag{5.24}
\]

The relations (5.24) may be combined to the two–particle scattering matrix

\[
S_{\sigma\sigma'} = \delta_{\sigma\sigma'}[\delta_{\sigma a} S_a + \delta_{\sigma b} S_b], \tag{5.25}
\]

where both \( \sigma \) and \( \sigma' \) can take the values \( a, b \). Let \( \rho_{\text{tot}} = \rho \otimes \rho_{\text{QPC}} \) be the density matrix of the total system prior to the passage of an electron through the QPC. We choose \( \rho_{\text{QPC}} = \text{diag}[1,0] \) representing an incoming particle from one side of the QPC. The density matrix after scattering is \( \rho'_{\text{tot}} = S \rho_{\text{tot}} S^\dagger \). The reduced density matrix \( \rho' \) of the QD is obtained by tracing over the QPC variables

\[
\rho' = \text{Tr}_{\text{QPC}}[S \rho_{\text{tot}} S^\dagger]. \tag{5.26}
\]

In the basis of incoming and outgoing scattering states we can parametrize the scattering matrix

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\[ S_a = \begin{pmatrix} r_a & t_a \\ t_a & r'_a \end{pmatrix} = \begin{pmatrix} \cos \theta_a \exp[i\phi_{ra}] & i \sin \theta_a \exp[i\phi_{ra}] \\ i \sin \theta_a \exp[i\phi_{ta}] & \cos \theta_a \exp[i\phi_{ra}] \end{pmatrix}. \] (5.27)

in terms of one angle \( \theta_a \) and four phases \( \phi_a \). As a consequence of the unitarity of \( S_a \), the phases are not independent from each other but obey the relation \( \exp[i(\phi_{ra} - \phi_{ta})] = -\exp[-i(\phi_{ra} - \phi_{ta})] \). A parameterization similar to Eq. (5.27) but in terms of an angle \( \theta_b \) and phases \( \phi_b \) may be used for the scattering matrix \( S_b \). Combining Eqs. (5.25), (5.27) and substituting the result into Eq. (5.26) one finds that the diagonal elements of \( \rho \) do not change upon scattering through the QPC. The off-diagonal elements change according to

\[ \rho'_{ab} = \mathcal{P} \rho_{ab}, \] (5.28)

where

\[ \mathcal{P} = r_a^* r_b + t_a^* t_b. \] (5.29)

Note that \( \rho'_{ba} = \rho'^*_{ab} \) since \( \rho' \) is hermitian.

We will assume that the scattering through the QPC takes place on a time scale much shorter than the relevant time scales of the quantum dot. In particular, the scattering time shall be much shorter than the decoherence time \( t_d \). This assumption allows us to use the Markov approximation and neglect memory effects in the QPC. Then the density operator after the scattering of \( n \) electrons through the QPC is given by

\[ \rho_{ab}(t) = \mathcal{P}^n e^{i\epsilon n \Delta t} \rho_{ab}(0), \] (5.30)

where \( \Delta t = \hbar/(2eV_d) \) is the average time between two scattering events, \( t = n \Delta t \), and \( E_{a,b} \) are the energies of the empty and occupied dot level, respectively. The term involving \( \epsilon = (E_b - E_a)/\hbar \) accounts for the free evolution of the QD. Note that \( \mathcal{P} \) is generally a complex number. From its modulus we can read off the dephasing rate

\[ \frac{1}{t_d} = -\frac{1}{\Delta t} \ln |\mathcal{P}|, \] (5.31)

while the phase gives rise to the energy shift

\[ \Delta E = \frac{\hbar}{\Delta t} \arg \mathcal{P}. \] (5.32)

Both the dephasing rate and the energy shift can be obtained explicitly in the weak–coupling limit \( \mathcal{P} \approx 1 \). Then \( \mathcal{P} \) may be expanded in \( \Delta \theta = \theta_b - \theta_a \), \( \Delta \phi_r = \phi_{rb} - \phi_{ra} \), and \( \Delta \phi_t = \phi_{tb} - \phi_{ta} \). Using the parameterization (5.27), we can express \( \Delta \theta \) in terms of the transmission coefficients \( T_{b,a} = |t_{b,a}|^2 \). This yields \( (\Delta \theta)^2 = (\Delta T_d)^2/[4T_d(1-T_d)] \) where \( \Delta T_d = T_b - T_a \) and \( T_d = (T_b + T_a)/2 \). Substitution into Eqs. (5.31), (5.32) yields the dephasing rate
\[
\frac{1}{t_d} = \frac{eV_d}{\hbar} \frac{(\Delta T_d)^2}{4T_d(1-T_d)} + \frac{eV_d}{\hbar} T_d(1-T_d)(\Delta \phi)^2,
\] (5.33)

and the energy shift
\[
\Delta E = \frac{eV_d}{\pi} (1-T_d) \Delta \phi_r + \frac{eV_d}{\pi} T_d \Delta \phi_t,
\] (5.34)

where \(\Delta \phi = \Delta \phi_r - \Delta \phi_t\). The evolution of \(\rho\) in the weak–coupling case may be cast in the form of a master equation
\[
\frac{d\rho}{dt} = \left[ -\frac{1}{t_d} + i \frac{\Delta E}{\hbar} \right] \begin{pmatrix} 0 & \rho_{ab} \\ \rho_{ba} & 0 \end{pmatrix}.
\] (5.35)

The diagonal elements of \(\rho\) do not change in time. Thus the occupation probability of the quantum dot does not change in time. By contrast, the off–diagonal elements decay with the dephasing rate \(1/t_d\). The evolution of \(\rho\) thus indeed reflects dephasing (decoherence) and not energy relaxation.

The dephasing rate (5.33) reduces to the results derived in Secs. V B 2, V B 3 when the phase change \(\Delta \phi\) is negligible. The effect of non–zero \(\Delta \phi\) has been investigated in a paper by Stodolsky [67]. His result for the dephasing rate \((\propto 1 - \text{Re} P)\) differs from Eq. (5.34) and does not reproduce the dependence \(1/t_d \propto T_d(1-T_d)\) observed experimentally [9].

5. Charge relaxation

An new approach to the dephasing problem was developed by Büttiker and coworkers [75,76,68,69]. The authors consider the charge and potential fluctuations that result from the Coulomb coupling between mesoscopic conductors. These fluctuations govern the dephasing process and thus the dephasing rate. Based on earlier theoretical work on charge relaxation in interacting conductors [75,76], Büttiker and Martin [68,69] calculated the dephasing rate for a QD coupled to a QPC. The method applies to dephasing both from changes in the transmission probability and from changes in the transmission phase. The results obtained in both cases coincide with the formulae given earlier. Büttiker and Martin generalized their results to the case of more than one channel in the QPC and to a device that includes a phase randomizing voltage probe between the QPC and the QD (cf. the experiment of Ref. 9). It was found that the dephasing rate is unaffected by the voltage probe if there is only one channel in the QPC.
VI. QUANTUM ZENO EFFECT

The frequent repetition of a decohering measurement leads to a striking phenomenon known as the quantum Zeno effect \[77,78\]: The suppression of transitions between quantum states. The standard example is a two-level system with a tunneling transition between the two levels. Assume that the system at \( t = 0 \) is in prepared in the state \( a \). For small times \( t \), the probability to tunnel out of that state is \( P_{a \rightarrow b}(t) = \frac{1}{\hbar^2} |\langle b | \Omega_0 | a \rangle|^2 t^2 \), where \( \Omega_0/2 \) is the tunneling matrix element, and \( \omega_0 = \Omega_0/\hbar \) the tunneling frequency. However, if the interval \( t \) is split into \( N \) subintervals each with the length \( t/N \), and the system is measured at the end of each subinterval, the probability for tunneling is reduced to \( NP_{a \rightarrow b}(t/N) = (1/N)P_{a \rightarrow b}(t) \). In the limit of arbitrary dense measurements, \( N \rightarrow \infty \), the system is frozen in the initial state. Repeated measurements can thus completely hinder the natural evolution of a quantum system.

Despite considerable theoretical work on the quantum Zeno effect there is only little experimental proof for it. An experimental test using an induced hyperfine transition of Be ions \[79\] has been reported. Experiments on optical transitions \[80\] or atomic Bragg scattering \[81\] have been proposed. Further experimental evidence for the existence of the quantum Zeno effect is clearly desirable. Gurvitz \[10\] first pointed out that the quantum Zeno effect may be observed in semiconductor microstructures. He studied theoretically the tunneling of electric charge between two weakly coupled quantum dots. A QPC located in the vicinity of one of the dots served as a non-invasive detector for the charge on that dot (see Fig. 37). As expected from the quantum Zeno effect, Gurvitz found that the coupling to the QPC suppresses the tunneling oscillations between the two dots. New aspects of the problem were identified by Hackenbroich et al. \[11\]. It was found that an ac voltage in the QPC leads to parametric resonance and to a strong reduction of dephasing in the quantum dots. The resonance occurs when the frequency \( \omega \) of the ac signal equals twice the frequency \( \omega_0 \) of the internal charge oscillations in the double-dot system. The spectral density of the detector current was studied by Korotkov \[82,83\]. When the detector is weakly coupled to the dots the spectral density has a peak close to the frequency \( \omega_0 \) reflecting the oscillation of charge between the two quantum dots. For strong coupling the peak disappears and the spectral density develops a Lorentzian shape. In this regime charge transfer takes place via random jumps rather than by periodic oscillations.

In Sec. VI.A we investigate electron tunneling between two quantum dots in the presence of a QPC detector. The problem is studied in terms of the reduced density matrix of the coupled dots. The spectral density of the detector current is analyzed in Sec. VI.B. In Sec. VI.C we modify the system by connecting the dots to external electron reservoirs. We show that the dc current through the dots is affected by the detector and provides an indirect signature of the quantum Zeno effect. In Sec. VI.D we analyze the decay of an
FIG. 37. Mesoscopic device with two quantum dots and a QPC. Electric charge can tunnel coherently between the dots. The QPC measures the charge accumulated in one of the dots.

unstable quantum system, and investigate whether continuous measurements slow down its decay rate.

A. Charge oscillations

1. Damping

Consider electric charge oscillating between two quantum dots in the presence of a QPC in close vicinity (Fig. 37). The coupled dots may be viewed as the mesoscopic realization of a double–well potential. We assume that there is only one energy level in each dot and that the dots are occupied with a single electron. Let the states $|a\rangle$ and $|b\rangle$ with energies $E_a$ and $E_b$ represent the electron in the left and right dot, respectively. We first study the case that the interaction with the QPC is negligibly small. Then the dynamics of the two–dot system is governed by the tunneling Hamiltonian $(\Omega_0/2)(|a\rangle\langle b| + |b\rangle\langle a|)$. The elements $\rho_{ij}$ of the density matrix satisfy the Bloch equations,

$$\dot{\rho}_{aa} = i\frac{\omega_0}{2}(\rho_{ab} - \rho_{ba}),$$

$$\dot{\rho}_{bb} = i\frac{\omega_0}{2}(\rho_{ba} - \rho_{ab}),$$

$$\dot{\rho}_{ab} = i\epsilon\rho_{ab} + i\frac{\omega_0}{2}(\rho_{aa} - \rho_{bb}),$$

with the frequencies $\epsilon = (E_b - E_a)/\hbar$ and $\omega_0 = \Omega_0/\hbar$. The diagonal elements $\rho_{aa}(t)$ and $\rho_{bb}(t)$, respectively, are the probabilities to find the electron in the left and right dot. The off–diagonal elements satisfy $\rho_{ab}(t) = \rho_{ba}^*(t)$ since $\rho$ is hermitian. Solving the Bloch equations for the initial conditions $\rho = \text{diag}[1, 0]$ one finds

$$\rho_{aa}(t) = \frac{\omega_0^2 \cos^2(\omega t/2) + \epsilon^2}{\omega_0^2 + \epsilon^2},$$

(6.4)
where \( \omega = \sqrt{\omega_0^2 + \epsilon^2} \). An electron located initially in the left dot oscillates between the dots with the frequency \( \omega \). The amplitude of these oscillations is \( \omega_0^2 / (\omega_0^2 + \epsilon^2) \). Thus, if \( \epsilon \gg \omega_0 \), the electron essentially remains localized in the left dot.

Now consider the charge oscillations when the QPC detector is active. The transmission through the QPC depends on the electron position in the dot. We first assume that the electron position only affects the transmission probability and not the transmission phase. Let \( T_a \) (\( T_b \)) be the transmission probability if the electron is in the left (right) dot. Bloch equations for this case were derived by Gurvitz \[10\]. The resulting equations for the diagonal elements coincide with Eqs. (6.1), (6.2). Equation (6.3) is replaced by

\[
\dot{\rho}_{ab} = i\epsilon \rho_{ab} + i\frac{\omega_0}{2} (\rho_{aa} - \rho_{bb}) - 2\kappa_d \rho_{ab},
\]

(6.5)

with the dephasing rate \( \kappa_d = eV_d(\Delta T_d)^2/[8hT_d] \). Dephasing causes an exponential damping of the nondiagonal matrix element. Combination of Eqs. (6.1), (6.2), and (6.3) determines the time evolution of \( \rho \). For weak damping \( \omega_0 > \kappa_d \) and \( \epsilon = 0 \), one finds damped oscillations

\[
\rho_{aa}(t) - 1/2 \sim \exp(-\kappa_d t) \cos \sqrt{\omega_0^2 - \kappa_d^2} t.
\]

(6.6)

Note that the damping causes a redshift of the frequency \( \sqrt{\omega_0^2 - \kappa_d^2} \) and, hence, slows down the charge oscillations between the two dots. This suppression of transitions due to measuring with the QPC is a manifestation of the quantum Zeno effect. The off–diagonal elements of \( \rho \) vanish for large times in agreement with the standard picture of decoherence, and the density matrix reduces to the random statistical ensemble \( \rho \rightarrow \text{diag}[1/2, 1/2] \).

Gurvitz’ derivation of the rate equation implicitly assumes \( T_d \ll 1 \). The generalization to arbitrary value \( 0 < T_d < 1 \) was obtained by Hackenbroich et al. \[11\] using a two-particle scattering approach similar to the calculation of Sec. V B 4. The resulting Master equation is identical with Eqs. (5.1), (5.2), (5.3) but yields the dephasing rate

\[
\kappa_d = \frac{eV_d(\Delta T_d)^2}{h} \frac{(1 - T_d)}{8T_d(1 - T_d)}
\]

(6.7)

that differs from Gurvitz’ result by a factor \( 1/(1 - T_d) \). The result (6.7) holds provided the quantum dots only affect the transmission probability through the QPC. The result may easily be generalized to include a change in the transmission phase through the detector. One then finds \( \kappa_d = 1/(2t_d) \) where \( 1/t_d \) is the dephasing rate (5.33) of a single quantum dot computed in Sec. V B. [We note that the problem considered here reduces to the dephasing in a single quantum dot in the limiting case \( \omega_0 = 0 \).]

2. Parametric resonance

The charge oscillations between the two quantum dots may display a parametric resonance if the voltage drop \( V_d \) has an ac–component \[11\]. Consider \( V_d(t) = V_0 - V_1 \sin \omega_1 t \) with
FIG. 38. The occupation probability of one quantum dot as a function of time. The curves correspond to different values of the dephasing rate: $\kappa_d$ (dashed), $\kappa_d = 4\omega_0$ (dot–dashed), and $\kappa_d = 16\omega_0$ (solid). The figure is obtained for $\epsilon = 0$. Taken from Ref. [10].

FIG. 39. Parametric resonance in the double-dot system coupled to a QPC–detector. The upper part shows the oscillations of $\rho_{aa}$ for constant voltage (dashed curve) and for a time-dependent voltage as shown in the lower part (full curve). Damping of the oscillations is reduced by a factor two. Taken from Ref. [11].

$V_0, V_1 \geq 0$ and $V_1 \leq V_0$. Substitution into the equation of motion for $\rho_{aa}$ yields a damped harmonic oscillation with an oscillatory damping constant. Using a simple ansatz for $\rho_{aa}(t)$ and neglecting terms of order $O(V_d^2)$, one is led to an equation of the Mathieu type which is known to display parametric resonance close to the frequencies $\omega_1 = 2\omega/n$ where $n$ is a positive integer. Parametric resonance is most pronounced for $\omega_1 \approx 2\omega$. The damping near the resonance is strongly reduced, $\kappa_d = (\Delta T_d)^2/[8hT_d(1 - T_d)]e(V_0 - V_1/2)$. The resulting time evolution of $\rho_{aa}$ near resonance is illustrated in Fig. [39] and compared with the case where $V_d$ is time–independent.

B. Detector current

While the reduced density matrix $\rho$ of the two quantum dots is a convenient tool for the theoretical investigation of the quantum Zeno effect, one cannot directly measure $\rho$
in experiments. However, information about the dynamics in the quantum dots may be obtained from the current passing through the detector. The power spectrum of the detector current was studied by Hackenbroich et al. [11] and by Korotkov [82,83].

Korotkov’s approach [82,83] is based on the description of decoherence in terms of a stochastic wave function. This approach is well-known in quantum optics [84]. Variants of the approach are known as quantum trajectory or quantum jump approach. The method describes the evolution of the quantum system under investigation (the two quantum dots) conditioned on a particular measurement of the detector system (the QPC). For the conditional evolution of the quantum dot density matrix given the detector current \( I(t) \) one finds

\[
\begin{align*}
\dot{\rho}_{aa} &= i\frac{\omega_0}{2} (\rho_{ab} - \rho_{ba}) - \frac{2\Delta I}{S_0} \rho_{aa} \rho_{bb} [I(t) - I_0], \\
\dot{\rho}_{ab} &= i\epsilon \rho_{ab} + i\frac{\omega_0}{2} (\rho_{aa} - \rho_{bb}) + \frac{\Delta I}{S_0} (\rho_{aa} - \rho_{bb}) [I(t) - I_0] \rho_{ab} - \gamma \rho_{ab},
\end{align*}
\]

where \( S_0 \) is the low–frequency spectral power of the detector shot noise, and \( I_0 = (I_a + I_b)/2 \), \( \Delta I = I_b - I_a \) where \( I_a \) and \( I_b \) are the average currents when the electron localized on the left and right dot, respectively. The damping rate \( \gamma \) accounts for dephasing due to interaction with some environment not included in the detector. Eqs. (6.8), (6.9) are supplemented by the equation

\[
I(t) - I_0 = \Delta I (\rho_{aa} - \rho_{bb})/2 + \xi(t)
\]

for the detector current. Here, \( \xi(t) \) describes a Gaussian random process with zero average and spectral density \( S_0 \). Equations (6.8)-(6.10) describe the evolution of the system for a single realization of the random process \( \xi(t) \). Averaging over \( \xi \) reduces these equations to the deterministic Master equation for the evolution of the ensemble average derived in Sec. VI A.

The evolution of \( I(t) \) and \( \rho_{aa} \) are illustrated in Fig. 40 for two different values of the coupling \( \alpha \equiv \kappa_d/\omega_0 \). Note that the results hold for a particular realization of the random process \( \xi(t) \). Figure 40(a) is obtained for \( \alpha = 0.1 \) representing a weakly coupled detector. Charge can oscillate several times between the two dots while generating only weak modulations of the detector current. Superimposed on the modulations is the detector shot noise. Fig. 40(a) shows the detector current averaged over different time intervals. For small intervals the signal is noisy, while for long intervals individual oscillations cannot be resolved. As a result, charge oscillations are difficult to observe in the weak–coupling limit. The situation is different in the strong coupling case \( \alpha > 1 \) (Fig. 40(b)). The strong influence of the detector suppresses the quantum oscillations, so that the charge performs random jumps between two localized states. In this case, the properly averaged detector current follows the evolution of the quantum dots.
FIG. 40. Evolution of $\rho_{aa}$ (thick line) and the corresponding detector current $I(t)$ (thin slid, dotted, dashed line) averaged using rectangular time windows of size $\tau_a$. (a) Weak–coupling $\alpha = 0.1$, (b) strong–coupling $\alpha = 5.0$. Taken from Ref. [83].

FIG. 41. The detector current spectral density $S(\omega)$ for different coupling strength $\alpha$. Taken from Ref. [83].
The detector current may be further characterized by its spectral power. For a symmetric double-dot ($\epsilon = 0$), the spectral power can be obtained analytically yielding
\[
S(\omega) = S_0 + 4S_0 \frac{\alpha^2 \omega_0^4}{(\omega^2 - \omega_0^2)^2 + \alpha^2 \omega_0^2 \omega_0^2}.
\]
(6.11)
Figure 41 shows $S(\omega)$ for different values of the coupling strength $\alpha$. For weak coupling, the spectral power displays a peak close to the frequency $\omega_0$. The peak height equals four times the noise background. The full width at half maximum is given by $2\kappa_d$ and the peak is centered at the shifted frequency $\sqrt{\omega_0^2 - \kappa_d^2}$ [11]. The frequency shift with respect to the frequency $\omega_0$ is a consequence of the quantum Zeno effect. The peak in $S(\omega)$ gradually disappears with increasing coupling. In the strong coupling limit, $S(\omega)$ has a Lorentzian shape known from the classical theory of telegraph noise.

C. Current through the quantum dots

Due to high–frequency noise, a frequency resolved measurement of the QPC current is difficult to perform. To avoid the problems of high–frequency measurements several authors [10,11] investigated manifestations of the quantum Zeno effect in dc transport. The dc current through two quantum dots with a QPC in the vicinity was studied both for the dots in series and with the dots in a parallel circuit. The current displays a characteristic dependence on the detector efficiency which provides an indirect signature of the quantum Zeno effect.

1. Dots in a series

Consider the transmission through a double–well structure consisting of two coupled quantum dot in a series. The dots are modeled as in Sec. VIII A in terms of two levels with energies $E_a, E_b$, and coupling matrix element $\Omega_0/2$. Dot $a$ is coupled to an electron reservoir $a$ and dot $b$ to a reservoir $b$. The coupling strength is described by the partial widths $\Gamma_{a(b)}$ for the decay of level $a(b)$ into the reservoir $a(b)$. The Fermi energies in the reservoirs are chosen such that $E_a F \ll E_b, E_b F \ll E_a F$. The current $I_s$ through the double–dot system can be calculated using rate equations and is given by
\[
I_s = \frac{e}{\hbar} \frac{\Gamma_b + \hbar \kappa_d) \Omega_0^2}{4(E_a - E_b)^2 + (\Gamma_b + \hbar \kappa_d)^2 + \Omega_0^2(\Gamma_b + \hbar \kappa_d)(\frac{2}{\Gamma_b} + \frac{1}{\Gamma_a})}.
\]
(6.12)
For small energy difference $|E_a - E_b|$ the current decreases with $\kappa_d$. However, for large $|E_a - E_b|$ the current increases with $\kappa_d$. This may be understood as resulting from the delocalization of the electron due to a continuous measurement of the charge in one of the dots.
FIG. 42. Two coupled quantum dots form a double well potential and interact with a QPC measuring the position of an excess electron in the two dots. A current is driven through the dots via four external leads. The transmission and reflection coefficients of the leads are denoted by $T_u$, $T_l$, $R_l$.

### 2. Parallel quantum dots

We now discuss the quantum Zeno effect in a set-up for dc transport across two parallel quantum dots. The arrangement, proposed by Hackenbroich et al. [11], is shown in Fig. 42. Two dots are coupled via a tunneling barrier. Each dot is connected with two external leads that allow for the transport of current to and out of the dot. With current flowing into the lower dot, we calculate the branching ratio of the current transmitted through the upper and the lower dot, respectively.

We assume that both dots are in the Coulomb blockade regime, and that $kT$ is much smaller than the single-particle level spacing in the dots. Then we need to consider only a single energy level in each dot. Both levels are degenerate with energy $E_0 - i/2\Gamma$. The width $\Gamma$ is due to the coupling to the leads. A QPC measures the charge in the lower quantum dot. The measurement modifies the transmission through the two-dot system. This modification is found by calculating the two-particle scattering matrix for an electron passing through the QPC and another electron oscillating in the two-dot system.

We model the QPC in terms of plane waves with energy $\epsilon_k$, mean density $\rho_F$, and Hamiltonian

$$H_{\text{QPC}} + V = \sum_k \epsilon_k b_k^\dagger b_k + \sum_{k,k'} (W_{kk'} + V_{kk'}d_l^\dagger d_l) b_k^\dagger b_{k'}^\dagger .$$

(6.13)

Here, $V$ describes the interaction between the QPC and the lower QD. The potential $W_{kk'}$ mimics the constriction in the wire with the QPC while $d_l^\dagger$ ($b_k^\dagger$) denotes the creation operator for the state on the lower QD (for the QPC plane wave states).

In each of the leads, we consider only one transverse channel labeled by $c = \sigma, \mu$, with $\sigma = l, u$ denoting the lower (l) and upper (u) lead, and $\mu = +, -$ the left and right lead.
The two–particle scattering matrix can be obtained from the Lippmann–Schwinger equation yielding the scattering amplitudes

\[ S_{cc',kk'} = \delta_{cc'} \delta_{kk'} - 2\pi i \gamma_c \gamma_{c'} G_{\sigma\sigma',kk'}. \]  

(6.14)

Here, \( \gamma_c \) is the matrix element for tunneling from channel \( c \) to the adjacent dot. The two–particle Green function \( G \) for the joint transition between dots and QPC is given in terms of its inverse

\[ (G^{-1})_{\sigma\sigma',kk'} = \begin{pmatrix} G_0^{-1}\delta_{kk'} - W_{kk'} & \Omega_0/2 \delta_{kk'} \\ \Omega_0/2 \delta_{kk'} & G_0^{-1}\delta_{kk'} - W_{kk'} - V_{kk'} \end{pmatrix}. \]  

(6.15)

On the right hand side, we have explicitly displayed the matrix in \( \sigma \)-space. Here \( G_0^{-1} = E - E_0 - \epsilon_k + i \Gamma/2 \) is the inverse propagator of the single Breit–Wigner resonance, and \( E \) is the sum energy of the two incoming particles. We have assumed that all \( \gamma_c \equiv \gamma \) are identical and used the relation \( \Gamma = 4\pi \gamma \gamma^* \). For \( V = 0 \), we can diagonalize the matrix \( \epsilon_k \delta_{kk'} + W_{kk'} \) by a unitary transformation in \( k \)-space, and \( G \) reduces to the product of the unit matrix in \( k \)-space and the two \( k \)-dependent coupled Breit–Wigner resonances for the double–dot system. Then, all scattering processes are elastic, and the branching ratio for the transmission through the upper and lower lead is \( T_u^{(0)}/T_l^{(0)} = \Omega_0^2/\Gamma^2 \).

The full two–particle scattering matrix (6.14) allows for energy exchange between dots and detector: In contrast to the sum energy of the two incoming particles, the energy of electrons in the QPC is not conserved in the scattering process. Such inelastic processes are essential to ensure the unitarity of the S–matrix. Physically, the energy exchange allows for a position measurement of the dot electron without violation of the Heisenberg uncertainty relation.

To calculate the transmission and reflection coefficients through the double–dot system, we restrict ourselves to constant scattering potentials \( W_{kk'} \equiv W \) and \( V_{kk'} \equiv V \). We expand \( G \) in Eq. (6.15) in powers of \( V \) and resum the resulting series. We obtain two contributions to \( G \). The first is independent of \( V \) and describes independent elastic scattering through the QPC and the dots. This term obviously does not contribute to the quantum Zeno effect. The second contribution \( \tilde{G} \) involves energy exchange \( \Omega = \epsilon_k - \epsilon_{k'} \) between dots and QPC. For fixed incident energy \( E = E_0 + \epsilon_k \),

\[ \tilde{G}_{\sigma\sigma',kk'} = A(\Omega) \begin{pmatrix} \Omega_0^2 & -i\Omega_0\Gamma \\ -i\Omega_0(2\Omega + i\Gamma) & i\Gamma(2\Omega + i\Gamma) \end{pmatrix}, \]  

(6.16)

with the amplitude

\[ A(\Omega) = -\frac{4V}{F_{W} F_{W+V} (\Gamma^2 + \Omega_0^2)(2\Omega + i\Gamma)^2 - \Omega_0^2}. \]  

(6.17)
Here, $F_W = 1 + 2\pi i W \rho_F$. The transmission and reflection coefficients are calculated by adding the two–particle scattering probabilities and tracing over the degrees of freedom of the QPC. The resulting expressions are further simplified by using a weak–coupling expansion to second order in $V$. In this limit the application of a drain source voltage $V_d$ across the QPC is equivalent to the simultaneous scattering of $2eV_d \rho_F$ particles in different longitudinal QPC modes. The total effect of these particles is obtained by multiplying the result for one QPC–particle with the number of longitudinal modes.

The $V$–dependent correction to the branching ratio arises both from coherent (elastic) and incoherent (inelastic) scattering. We find that measurements with the QPC–detector have a twofold effect: (i) They suppress tunneling from the feeding lead into the lower dot and (ii) they suppress tunneling from the lower into the upper dot. Observation (i) is reflected in the increase of the reflection coefficient, while (ii) follows from the decrease of the branching ratio $T_u T_l = T(0)_u T(0)_l \left[ 1 - \frac{eV_d}{\pi \Gamma} \frac{(\Delta T_d)^2}{4 T_d(1 - T_d)} \right]$. (6.18)

Here, the ratio $T(0)_u / T(0)_l$ was given above, and $\Delta T_d$ and $T_d$ were defined in Sec. V. Both effects (i), (ii) have an obvious interpretation as manifestations of the quantum Zeno effect. The second term in the square bracket is up to a factor $\Gamma / (2\hbar)$ equal to the dephasing rate found for the isolated double–dot system.

D. Decay rate

In early work on the quantum Zeno effect it was argued [77] that an unstable quantum system slows down its decay under frequent or continuous observations. However, despite of further theoretical work [80] the suppression of decay remained controversial. Indeed, a simple argument suggests that the decay rate should not be influenced by observation: Consider the exponential decay from a discrete initial state into a continuum of final states. The probability for decay during a small time $t$ is linear in $t$ and given by $P(t) = \Gamma t / \hbar$. For $N$ measurements the decay probability is $NP(t/N) = P(t)$ and hence identical to the decay probability in the absence of any measurement.

Elattari and Gurvitz [87,88] addressed the problem for the system shown in Fig. 43. A quantum dot with a single energy level $E_0$ is coupled to an electron reservoir. The states $E_\alpha$ in the reservoir are dense. The dot is placed near a QPC connected to two separate reservoirs filled up to the Fermi energies $\mu_L$ and $\mu_R$, respectively. The transmission probability through the QPC depends on the occupation of the dot level. Thus, the QPC continuously monitors the dot occupation. The dynamics of the entire system is determined by the Schrödinger equation $i\hbar \partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle$, where the Hamiltonian $H$ comprises the quantum dot, the
QPC, their mutual interaction and the electron reservoirs. As the initial state $|\Psi(0)\rangle$ we choose an occupied quantum dot level and electron reservoirs filled up the Fermi levels $\mu_L$ and $\mu_R$.

The time evolution of the system can be expressed in the Bloch–type equations [87]

\[
\dot{\sigma}_{00} = -\frac{\Gamma}{\hbar} \sigma_{00}, \tag{6.19}
\]

\[
\dot{\sigma}_{\alpha\alpha} = i\frac{\Omega_{\alpha}}{\hbar} (\sigma_{0\alpha} - \sigma_{\alpha0}), \tag{6.20}
\]

\[
\dot{\sigma}_{\alpha0} = i\frac{E_0 - E_{\alpha}}{\hbar} \sigma_{0\alpha} - i\frac{\Omega_{\alpha}}{\hbar} \sigma_{00} - \frac{\Gamma + \hbar \kappa_d}{2\hbar} \sigma_{\alpha0}. \tag{6.21}
\]

Here, $\sigma_{00}$ is the probability for the electron to occupy the dot level, $\sigma_{\alpha\alpha}$ is the probability for a transition into level $\alpha$ in the continuum, and $\kappa_d$ is the decoherence rate induced by the detector. The coupling matrix element between the dot level and the level $\alpha$ is denoted by $\Omega_{\alpha}$, and $\Gamma = 2\pi |\Omega_{\alpha}|^2 \rho$ is the total width for decay into the continuum, where $\rho$ is the density of states in the reservoir coupled to the quantum dot. From Eq. (6.19) one immediately finds

\[
\sigma_{00}(t) = \exp(-\Gamma t/\hbar). \tag{6.22}
\]

Hence, continuous monitoring of the unstable system does not slow down its exponential decay. In contrast, the energy distribution of the electron in the continuum, $P(E_{\alpha}) \equiv \sigma_{\alpha\alpha}(t \to \infty)$ is affected by the detector. From the solution of Eqs. (6.19)-(6.21) in the limit $t \to \infty$, one finds a Lorentzian distribution centered around $E_{\alpha} = E_0$,

\[
P(E_{\alpha}) \propto \frac{\Gamma + \hbar \kappa_d}{(E_0 - E_{\alpha})^2 + (\Gamma + \hbar \kappa_d)^2/4}. \tag{6.23}
\]

The measurement results in a broadening of the linewidth from $\Gamma$ to $\Gamma + \hbar \kappa_d$. The broadening does not affect the decay rate $\Gamma$. 
FIG. 43. Schematic illustration of a QPC–detector near a quantum dot that is coupled to a reservoir. The energy level $E_0$ of the dot is occupied by an electron which can decay into the states $E_\alpha$ of the reservoir. Taken from Ref. [87].

VII. CONCLUSION

Semiconductor quantum dots allow to measure and control current at the single–electron level. Multiple quantum dots, wires and quantum point contacts can be integrated on a single chip into more complicated structures. Such integrated microstructures not only offer control over the number of transmitted electrons, they also allow to measure and manipulate the quantum state of these electrons.

We demonstrated that devices employing a quantum dot in an Aharonov–Bohm ring allow to measure the phase of the transmission amplitude through the quantum dot. Controlled dephasing of quantum dot states can be achieved by means of a quantum point contact in close proximity to the dot. It is a formidable task to extend these experiments to quantum dots in the Kondo regime. Theoretical predictions for the phase shift in this regime have been made [89]. Novel experiments testing the dephasing of charge tunneling between coupled quantum dots have been proposed [10,11]. The research in this field allows for the first time to test fundamental principles of quantum mechanics, such as the particle–wave duality, with solid state devices. Questions that previously could only be addressed in atomic physics or quantum optics may now be amenable to solid state physics. Further question may be investigated, e.g. exploring the connection of dephasing and the Fermi statistics [91]. These considerations promise interesting future research on electron coherence in a wide variety of mesoscopic systems.
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