FORMAL HODGE THEORY

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Abstract. We introduce formal (mixed) Hodge structures (of level \( \leq 1 \)) in such a way that the Hodge realization of Deligne’s 1-motives extends to a realization from Laumon’s 1-motives to formal Hodge structures (of level \( \leq 1 \)) providing an equivalence of categories.

Let \( \text{MHS}_{1}^{\text{fr}} \) denote the category of torsion free graded polarizable mixed Hodge structures of level \( \leq 1 \). We have a nice algebraic description of this category via \( \mathcal{M}_{1}^{\text{fr}} \), the category of Deligne’s 1-motives [5] (cf. also [3], including torsion, one obtains 1-motives with torsion describing \( \text{MHS}_{1} \)). Actually, Deligne’s Hodge realization provide an equivalence

\[
T_{\text{Hodge}} : \mathcal{M}_{1}^{\text{fr}} \xrightarrow{\sim} \text{MHS}_{1}^{\text{fr}}
\]

such that Cartier duality on \( \mathcal{M}_{1}^{\text{fr}} \) is transformed in \( \text{Hom}(-, \mathbb{Z}(1)) \) on \( \text{MHS}_{1}^{\text{fr}} \). Moreover, we have a natural generalization of Deligne’s 1-motives due to Laumon [6]. A Laumon 1-motive \( M := [F \rightarrow G] \) is a commutative formal group \( F = F^{0} \times F_{\text{et}} \), with torsion free étale part \( F_{\text{et}} \), a commutative connected algebraic group \( G \) and a map of abelian fppf-sheaves \( u : F \rightarrow G \). Let \( \mathcal{M}_{1}^{a,\text{fr}} \) denote the category of Laumon’s 1-motives and refer to its objects as 1-motives for short. Note that Cartier duality on \( \mathcal{M}_{1}^{\text{fr}} \) canonically extends to \( \mathcal{M}_{1}^{a,\text{fr}} \) (see [6]).

The purpose of this note is to introduce the abelian category \( \text{FHS}_{1}^{\text{fr}} \) of formal mixed Hodge structures (of level \( \leq 1 \)) in order to extend the Hodge realization \( T_{\text{Hodge}} \) of Deligne’s 1-motives \( \mathcal{M}_{1}^{\text{fr}} \) to a realization \( T_{\text{f}} \) from Laumon’s 1-motives \( \mathcal{M}_{1}^{a,\text{fr}} \) to \( \text{FHS}_{1}^{\text{fr}} \subset \text{FHS}_{1} \). We have that \( \text{MHS}_{1}^{\text{fr}} \subset \text{FHS}_{1}^{\text{fr}} \) in a canonical way, i.e., there is a fully faithful embedding such that the natural involution (Cartier duality) on \( \text{MHS}_{1}^{\text{fr}} \) extends to an involution on \( \text{FHS}_{1}^{\text{fr}} \).

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For the sake of exposition we here confine our study to level \( \leq 1 \) mixed Hodge structures. However, it is conceivable and suitable to consider formal mixed Hodge structures with arbitrary Hodge numbers: generalizing our definition below it’s not that difficult (we will treat such a matter nextly, cf. [1, 2.12] for the general setting). For example, enriched Hodge structures [4] (of level \( \leq 1 \)) can easily be recovered as “special” formal Hodge structures (see also [2] for details). In [2] we are also providing a “sharp” De Rham realization generalizing Deligne’s construction of De Rham realization in [5]. The main result of this paper can be summarized in the following way.

**Theorem.** There is an equivalence of categories with involution

\[
T_f : \mathcal{M}_{1,fr}^{a,fr} \cong \rightarrow \text{FHS}_{1,fr}
\]

between Laumon’s 1-motives and torsion free formal Hodge structures (of level \( \leq 1 \)) providing a diagram

\[
\begin{array}{ccc}
\mathcal{M}_{1,fr} & \cong & \text{MHS}_{1,fr} \\
\uparrow & & \uparrow \\
\mathcal{M}_{1,a,fr} & \cong & \text{FHS}_{1,fr}
\end{array}
\]

where

- \( \mathcal{M}_{1,fr} \hookrightarrow \mathcal{M}_{1,a,fr} \) and \( \text{MHS}_{1,fr} \hookrightarrow \text{FHS}_{1,fr} \) are canonical inclusions,
- \( \mathcal{M}_{1,a,fr} \rightarrow \mathcal{M}_{1,fr} \) and \( \text{FHS}_{1,fr} \rightarrow \text{MHS}_{1,fr} \) are “forgetful functors” denoted \((\_\_\_) \sim \rightarrow (\_\_)_{\text{\acute{e}t}}\), which are left inverses of the inclusions,
- \( T_f(M) \) coincide with \( T_{\text{Hodge}}(M) \) if \( M = M_{\text{\acute{e}t}} \) and, in general, we have a formula

\[
T_f(M)_{\text{\acute{e}t}} = T_{\text{Hodge}}(M_{\text{\acute{e}t}}).
\]

The plan of the paper is the following. In Section 1 we introduce the category FHS\(_1\). In Section 2 we construct \( T_f \) proving the theorem.

**1. Formal Hodge Structures**

1.1. **Paradigma.** Consider a commutative formal group \( H = H^0 \times H_Z \) over \( \mathbb{C} \) along with a mixed Hodge structure on the étale part \( H_Z \), i.e., say \( H_{\text{\acute{e}t}} := (H_Z, W_\ast, F^\ast_{\text{Hodge}}) \in \text{MHS}_1 \) for short. For the mixed Hodge structure \( H_{\text{\acute{e}t}} \in \text{MHS}_1 \) we here denote \( H_Z \) the finitely generated abelian underlying group, along with the weight filtration \( W_{-2} \subseteq W_{-1} \) of \( H_\mathbb{Q} := H_Z \otimes \mathbb{Q} \) and \( F^0_{\text{Hodge}} \subseteq H_\mathbb{C} := H_Z \otimes \mathbb{C} \) the Hodge filtration. We say that \( H \) is free if the étale part of the formal group is free, so that: \( H_Z = \mathbb{Z}^r \) and \( H^0 = \widehat{\mathbb{C}}^s \) non-canonically. (Note that here \( \widehat{\mathbb{C}} \) denotes the
connected formal additive group). For $H$ free we also denote by $W_*H_{\text{ét}}$ and $\text{gr}^wH_{\text{ét}}$ the corresponding objects of $\text{MHS}_1$.

1.1.1. Definition. Define a formal Hodge structure (of level $\leq 1$) as follows: (i) a formal group $H$ such that $H_{\text{ét}} \in \text{MHS}_1$, (ii) a finite dimensional $\mathbb{C}$-vector space $V$ with a two steps filtration $V^0 \subseteq V^1 \subseteq V$ by sub-spaces, (iii) a group homomorphism $v : H \to V$ and (iv) a $\mathbb{C}$-isomorphism $\sigma : H_{\mathbb{C}}/F^0_{\text{Hodge}} \cong V/V^0$ restricting to an isomorphism $W_{-2}H_{\mathbb{C}} \cong V^1/V^0$. We further assume that the following condition holds: if $v_Z : H_Z \to V$ is the induced map, $c : H_Z \to H_{\mathbb{C}}/F^0_{\text{Hodge}}$ is the canonical map and $\text{pr} : V \to V/V^0$ is the projection then the following diagram commutes.

\[
\begin{array}{ccc}
H_Z & \xrightarrow{v_Z} & V \\
\downarrow c & & \downarrow \text{pr} \\
H_{\mathbb{C}}/F^0_{\text{Hodge}} & \xrightarrow{\sigma} & V/V^0 \\
\end{array}
\]

commutes. Denote $(H, V)$ for short such a structure.

Define a morphism $\phi$ between $(H, V)$ and $(H', V')$ as follows. We let $\phi := (f, g)$ be a pair of maps in the following commutative square

\[
\begin{array}{ccc}
H & \xrightarrow{v} & V \\
\downarrow f & & \downarrow g \\
H' & \xrightarrow{v'} & V' \\
\end{array}
\]

where $f : H \to H'$ is a homomorphism of formal groups such that $f_{\text{ét}} : H_{\text{ét}} \to H'_{\text{ét}}$ is a map in $\text{MHS}_1$ and $g : V \to V'$ is a $\mathbb{C}$-homomorphism compatible with the filtrations, i.e., $g(V^i) \subseteq V'^i$ for $i = 0, 1$. We further assume that the following diagram commutes

\[
\begin{array}{ccc}
H_{\mathbb{C}}/F^0_{\text{Hodge}} & \xrightarrow{\sigma} & V/V^0 \\
\downarrow \mathcal{J} & & \downarrow \mathcal{F} \\
H'_{\mathbb{C}}/F^0_{\text{Hodge}} & \xrightarrow{\sigma'} & V'/V'^0 \\
\end{array}
\]

where $\mathcal{J}$ and $\mathcal{F}$ are the canonically induced maps.

1.1.2. Definition. Let $\text{FHS}_1$ denote the category whose objects are $(H, V)$, the morphisms are $\phi = (f, g)$ as above and the composition is given by gluing the squares (3) (the condition (3) is preserved by gluing). Let $\text{FHS}^\text{fr}_1 \subset \text{FHS}_1$ denote the full subcategory given by $(H, V)$ such that $H$ is free.

1.1.3. Proposition. The category $\text{FHS}_1$ is abelian. A short exact sequence

$$0 \to (H, V) \to (H', V') \to (H'', V'') \to 0$$
is given by an exact sequence on each component (formal groups and filtered vector spaces) so that

\[ 0 \to H_\text{ét} \to H'_\text{ét} \to H''_\text{ét} \to 0 \]

is exact in MHS$_1$.

**Proof.** Straightforward. \(\square\)

### 1.2. Étale structures.

We can recover mixed Hodge structures as follows.

**1.2.1. Definition.** Define \((H, V)_{\text{ét}} := (H_Z, V/V^0)\) where \((H_Z)_{\text{ét}} := H_\text{ét}, v_{\text{ét}} : H_Z \to V/V^0\) is the composition of pr and \(v_Z\) (cf. 1.1.1) and \((V/V^0)^0 := 0 \subseteq (V/V^0)^1 := V^1/V^0 \subseteq V/V^0\). Say that a formal Hodge structure is \(\text{étale}\) if \((H, V) = (H, V)_{\text{ét}}, \text{i.e., if } H^0 = V^0 = 0\).

**1.2.2. Lemma.** The full subcategory FHS$_1^{\text{ét}}$ of \(\text{étale}\) structures is equivalent to MHS$_1$ via \(c\) and the forgetful functor \((H, V) \mapsto H_\text{ét}\). The functor \(e : (H, V) \mapsto (H, V)_{\text{ét}}\) is a left inverse of the inclusion FHS$_1^{\text{ét}} \subset$ FHS$_1$ and, for \((H', V') \in$ FHS$_1^{\text{ét}}, \text{we have}

\[ \text{Hom}((H, V), (H', V')) \subseteq \text{Hom}((H, V)_{\text{ét}}, (H', V')) \]

where the equality holds if \(v(H^0) \subseteq V^0\) (cf. 1.3.1 below).

**Proof.** Actually, for the equivalence, we are easily left to show that if \((H, V)\) is \(\text{étale}\) then \(c(H_\text{ét}) := (H_Z, H_C/F_0^0 H_{\text{Hodge}}) \cong (H, V)\). The claimed isomorphism is \((1, \sigma)\) granted by (1) since \(V^0 = H^0 = 0\).

For the other claims, let \((H, V) \in$ FHS$_1$ and \((H', V') \in$ FHS$_1^{\text{ét}}$ and consider a map \(\phi = (f, g) : (H, V) \to (H', V')\) whence induced maps \(\overline{f}\) and \(\overline{g}\) and a diagram

\[
\begin{array}{ccc}
H & \overset{v}{\longrightarrow} & V \\
\uparrow & & \downarrow \\
H_Z & \overset{v_{\text{ét}}}{\longrightarrow} & V/V^0 \\
\overline{f} & & \overline{g} \\
H' & \overset{v'}{\longrightarrow} & V'
\end{array}
\]
In fact $H' = H'_{\text{et}}$ is étale thus $f(H^0) = 0$ and $f$ factors through $H_Z$ yielding $\overline{f}$ and, similarly, we get a filtered map $\overline{g} : V/V^0 \to V'$ since $V'^0 = 0$ and $g(V^0) = 0$. Now $\overline{\phi} := (\overline{f}, \overline{g})$ yields a map by diagram chase. Note that if $v(H^0) \subseteq V^0$ then $(H, V) \to (H, V)_{\text{et}}$ (cf. (3)) and we can lift back, by composition, any morphism $\phi' : (H, V)_{\text{et}} \to (H', V')$ as the condition (3) is tautological. \hfill $\square$

1.2.3. **Remark.** Note that under the equivalence we then get a canonical inclusion $c : \text{MHS}_1^{\text{fr}} \hookrightarrow \text{FHS}_1^{\text{fr}}$ such that $e : \text{FHS}_1^{\text{fr}} \to \text{MHS}_1^{\text{fr}}$ is a left inverse and for $H' \in \text{MHS}_1^{\text{fr}}$

\[
\text{Hom}_{\text{FHS}_1^{\text{fr}}}((H, V), (H'_Z, H'_C/F_{\text{Hodge}})) \subseteq \text{Hom}_{\text{MHS}_1^{\text{fr}}}(H_{\text{et}}, H')
\]

1.3. **Connected structures.** A $\mathbb{C}$-vector space $V$ will be regarded as an object $(0, V)$ of FHS$_1$ filtered as $V = V^1 = V^0$. Similarly, a formal group $H$ is regarded as an object $(H, 0)$ of FHS$_1$ so that $H = H^0 \times H_Z$ and $H_Z$ is pure of weight zero.

For $(H, V) \in \text{FHS}_1$ we have that $V^0$ is a substructure of $(H, V)$ and we can consider the quotient $(H, V)/V^0 = (H, V/V^0)$ in FHS$_1$. We can also regard $(H, V)_{\text{et}}$ as a substructure of $(H, V)/V^0$ and we obtain a canonical exact sequence

\[
0 \to (H, V)_{\text{et}} \to (H, V)/V^0 \to H^0 \to 0
\]

1.3.1. **Definition.** Say that $(H, V) \in \text{FHS}_1$ is **connected** if $H = H^0$ is connected, i.e., if $(H, V)_{\text{et}} = 0$. Denote $\pi(H, V) := (H^0, V)$ the connected structure given by $V = V^1 = V^0$ and the restriction of $v$ to $H^0 \subseteq H$. Let FHS$_1^0$ denote the full subcategory of FHS$_1$ determined by connected structures.

Say that $(H, V) \in \text{FHS}_1$ is **special** if $v(H^0) \subseteq V^0$, i.e., if $v : H \to V$ restricts to $v^0 : H^0 \to V^0$. Denote FHS$_1^s$ the full subcategory of special structures and $(H, V)^0 := (H^0, V^0) \in \text{FHS}_1^0$ the connected structure determined by $(H, V) \in \text{FHS}_1^s$.

1.3.2. **Lemma.** The functor $(H, V) \mapsto \pi(H, V)$ is a left inverse of the inclusion $v : \text{FHS}_1^0 \subset \text{FHS}_1$. The category FHS$_1^0$ is equivalent to the category of linear mappings between finite dimensional $\mathbb{C}$-vector spaces. For $(H', V') \in \text{FHS}_1^0$ and $(H, V) \in \text{FHS}_1^s$

\[
\text{Hom}((H', V'), (H, V)) \cong \text{Hom}((H', V'), (H, V)^0)
\]

**Proof.** The first claim is clear. Moreover, the equivalence is provided by $(H, V) \mapsto \text{Lie}(H) \to V$. Finally, a map from $(H', V')$ connected to
(H, V) special is given by a commutative square

\[
\begin{array}{ccc}
H' & \longrightarrow & V' \\
f \downarrow & & \downarrow g \\
H & \longrightarrow & V
\end{array}
\]

such that \( f(H') \subseteq H^0 \) and \( g(V') \subseteq V^0 \).

1.3.3. Remark. Note that \((H, V)\) with \(H_{\text{et}}\) pure of weight zero exists if and only if \(V = V^1 = V^0\). Thus if \((H, V)\) is special then \((H, V)^0\) is the largest connected formal substructure of \((H, V)\) and we have a non canonical extension

\[(5) \quad 0 \rightarrow (H^0, V^0) \rightarrow (H, V) \rightarrow (H, V)_{\text{et}} \rightarrow 0\]

From lemmas 1.2.2 and 1.3.2 it follows that the functors \((H, V) \mapsto (H, V)^0\) and \((H, V) \mapsto (H, V)_{\text{et}}\) are, respectively, a right adjoint of \(\text{FHS}^0_1 \subset \text{FHS}_1\) and a left adjoint of \(\text{FHS}^0_{\text{et}} \subset \text{FHS}_1\). However, special structures do have disadvantages, see 2.2.5 and 2.3.2.

1.3.4. Proposition. The category \(\text{FHS}^0_1\) forms a Serre abelian subcategory of \(\text{FHS}_1\) yielding the extension

\[0 \rightarrow \text{FHS}^0_1 \xrightarrow{\pi} \text{FHS}_1 \xrightarrow{\epsilon} \text{MHS}_1 \rightarrow 0\]

where \(\pi e = 1\) and \(ec = 1\).

Proof. It follows from the lemmas 1.2.2, 1.3.2 and (4). In fact, it is clear (cf. 1.1.3) that \(\text{FHS}^0_1\) forms a Serre subcategory. Since \(e(\text{FHS}^0_1) = 0\) we have a factorisation \(\pi : \text{FHS}_1/\text{FHS}^0_1 \rightarrow \text{MHS}_1\) via the canonical projection \(t : \text{FHS}_1 \rightarrow \text{FHS}_1/\text{FHS}^0_1\) and the equivalence \(\text{FHS}^0_{\text{et}} \cong \text{MHS}_1\). Since \(e = et\) and \(ec = 1\) then \(etc = 1\). We also have \(tetc = 1\) since applying \(t\) to (4) for \((H, V) \in \text{FHS}_1\) we get a natural isomorphism

\[tc(H_{\text{et}}) \cong t(H, V)_{\text{et}} \cong t(H, V)\]

1.4. Construction. We provide a Laumon 1-motive out of a free formal mixed Hodge structure (of level \(\leq 1\)). The construction is similar to [5, p. 55-56].

For \((H, V) \in \text{FHS}^\text{fr}_1\) the Laumon 1-motive \(\overline{(H, V)} := [F \xrightarrow{\psi} G]\) functorially associated to \((H, V)\) is given as follows. Set \(F := H^0 \times \text{gr}_W^0(H_Z)\). Since (1) holds true \(W_{-1}(H_Z)\) injects in \(V\) via \(\nu_Z : H_Z \rightarrow V\) in such a way that \(W_{-1}(H_Z) \cap V^0 = 0\). Set \(G(\mathbb{C}) := V/W_{-1}(H_Z)\) obtaining a
we get the algebraic group $G$ and set $M$ a field $\mathbb{P}$.

2.1. \textbf{Paradigma.} For a Laumon 1-motive $M = \left[F \xrightarrow{u} G\right] \in \mathcal{M}_1^{\text{fr}}$ over a field $k$ (algebraically closed of characteristic zero) we here denote $F = F^0 \times F_{\text{ét}}$ the formal group where $F_{\text{ét}}$ is further assumed torsion free. Denote $V(G) := \mathbb{G}^a \subseteq G$ the additive factor and display the connected algebraic group $G$ as an extension

\begin{equation}
0 \to V(G) \to G \to G_x \to 0
\end{equation}

where $G_x$ is the semi-abelian quotient. The algebraic group $G_x$ is an extension of an abelian variety $A$ by a torus $T$.

2.1.1. \textbf{Definition.} For $M = \left[F \xrightarrow{u} G\right] \in \mathcal{M}_1^{\text{fr}}$ set $M_{\text{ét}} := \left[F_{\text{ét}} \xrightarrow{u_{\text{ét}}} G_x\right] \in \mathcal{M}_1^{\text{fr}}$. Say that $M$ is \textit{étale} if $M = M_{\text{ét}}$, \textit{i.e.}, it is a Deligne 1-motive. Say that $M$ is \textit{connected} if $M_{\text{ét}} = 0$, \textit{i.e.}, $F = F^0$ is connected and $G = V(G)$ is a vector group. Say that $M$ is \textit{special} if $u(F^0) \subseteq V(G)$ and set $M^0 := [F^0 \to V(G)]$. 

\begin{equation}
0 \to W_{-1}(H_{\mathbb{Z}}) \to H \to F \to 0
\end{equation}

where $u$ is just induced by $v$. Regarding the complex group $G(\mathbb{C})$ we then have it in a diagram

\begin{equation}
\begin{array}{ccccccc}
0 & \to & W_{-1}(H_{\mathbb{Z}}) & \xrightarrow{v} & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V^0 & \xrightarrow{=} & V^0 & & V^0 & \to & V^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & W_{-1}(H_{\mathbb{Z}}) & \xrightarrow{v_G} & V & \to & G(\mathbb{C}) \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & W_{-1}(H_{\mathbb{Z}}) & \xrightarrow{\epsilon} & H_{\mathbb{C}} / F^0_{\text{Hodge}} & \to & J(W_{-1}(H_{\text{ét}})) \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & & 0 & \to & 0 \\
\end{array}
\end{equation}

obtained \textit{via} $\sigma$ and $\left[\begin{array}{c} \Phi \end{array}\right]$. This is showing that $G(\mathbb{C})$ is an extension of the complex torus $J(W_{-1}(H_{\text{ét}}))$ by a $\mathbb{C}$-vector group. Thus, by G.A.G.A., we get the algebraic group $G$. 

2. Formal Hodge realization

\begin{equation}
(7)
0 \to V(G) \to G \to G_x \to 0
\end{equation}
2.1.2. **Lemma.** The functor $M \mapsto M_{\text{et}}$ is a left inverse of the inclusion $\mathcal{M}_1^{\text{fr}} \subset \mathcal{M}_1^{a, \text{fr}}$ of Deligne’s 1-motives and for $M' \in \mathcal{M}_1^{\text{fr}}$ we have

$$\text{Hom}(M, M') \subseteq \text{Hom}(M_{\text{et}}, M')$$

If $M$ is special we then get an extension

$$0 \to M^0 \to M \to M_{\text{et}} \to 0 \tag{8}$$

such that if $M'$ is étale then $\text{Hom}(M_{\text{et}}, M') \cong \text{Hom}(M, M')$ and if $M'$ is connected then $\text{Hom}(M', M^0) \cong \text{Hom}(M', M)$.

**Proof.** Let $M = [F \xrightarrow{u} G] \in \mathcal{M}_1^{a, \text{fr}}, M' = [F' \xrightarrow{u'} G'] \in \mathcal{M}_1^{\text{fr}}$. Let $(f, g) : M \to M'$ be a map. Then get a diagram (cf. the proof of 1.2.2)

$$
\begin{array}{ccc}
F & \xrightarrow{u} & G \\
\uparrow & & \downarrow \\
F_{\text{et}} & \xrightarrow{u_{\text{et}}} & G_x \\
\uparrow f & & \downarrow g \\
F' & \xrightarrow{u'} & G'
\end{array}
$$

where $f$ and $g$ are the induced maps since $M'$ is étale, yielding a map $(f, g) : M_{\text{et}} \to M'$. In fact, $\text{Hom}(F, F') = \text{Hom}(F_{\text{et}}, F')$ because $F'$ is étale and $F^0$ is mapped to zero and $\text{Hom}(G, G') = \text{Hom}(G_x, G')$ because $\text{Hom}(\mathbb{G}_a, \mathbb{G}_m) = \text{Hom}(\mathbb{G}_a, A) = 0$ and $G'$ is semi-abelian. Moreover, $M \to M_{\text{et}}$ if $M$ is special, yielding (8). For the isomorphisms then note that $\text{Hom}(M^0, M') = 0$ if $M'$ is étale and, equivalently, $\text{Hom}(M', M_{\text{et}}) = 0$ if $M'$ is connected. \hfill \Box

In general, we can regard $M_{\text{et}}$ as a sub-1-motive of $M/V(G)$ and we obtain (cf. (4)) a canonical exact sequence

$$0 \to M_{\text{et}} \to M/V(G) \to F^0[1] \to 0 \tag{9}$$

Denote $M_{\text{et}}^\sharp = [F_{\text{et}} \xrightarrow{u^\sharp} G^\sharp] \in \mathcal{M}_1^{a, \text{fr}}$ (cf. 5) the universal $\mathbb{G}_a$-extension of $M_{\text{et}}$. The algebraic group $G^\sharp$ can be represented by an extension

$$0 \to \text{Ext}(M_{\text{et}}, \mathbb{G}_a)^\vee \to G^\sharp \to G_x \to 0 \tag{10}$$

where $\text{Ext}(M_{\text{et}}, \mathbb{G}_a)^\vee$ is given by the dual vector space of $\mathbb{G}_a$-extensions of $M_{\text{et}}$. The map $u^\sharp : F_{\text{et}} \to G^\sharp$ is a canonical lifting of $u_{\text{et}} : F_{\text{et}} \to G_x$.

Set $k = \mathbb{C}$. Recall that Deligne’s Hodge realization (see 5)

$$T_{\text{Hodge}}(M_{\text{et}}) := (H_Z, W_*, F^0_{\text{Hodge}})$$
of $M_{\text{ét}}$ is given by the pull-back

$$
\begin{align*}
H_Z & \xrightarrow{v^\natural} \text{Lie}(G_{\times}) \\
\downarrow & \downarrow \exp \\
F_{\text{ét}} & \xrightarrow{u_{\text{ét}}} G_{\times}
\end{align*}
$$

Here $W_{-1} := H_1(G_{\times})$, $W_{-2} := H_1(T)$ and

$$
F^0_{\text{Hodge}} := \text{Ker}(H_{\mathbb{C}} \to \text{Lie}(G_{\times}))
$$

2.1.3. **Lemma.** ([5, 10.1]) For $k = \mathbb{C}$ we have an isomorphism

$$
M_{\text{ét}}^\natural \cong [H_Z/W_{-1} \xrightarrow{\overline{t}} H_{\mathbb{C}}/W_{-1}]
$$

here $\overline{t}$ is the induced map $t : H_Z \to H_{\mathbb{C}} \mod W_{-1}(H_Z)$.

Actually (see [5, 10.1.8]) we have a bifiltered isomorphism (i.e., “periods”)

$$
\tau : \text{Lie}(G^\natural) \xrightarrow{\sim} H_{\mathbb{C}}
$$

such that

$$
\begin{align*}
H_Z & \xrightarrow{v^\natural} \text{Lie}(G^\natural) \xrightarrow{\tau} H_{\mathbb{C}} \\
\downarrow & \downarrow \\
H_Z^0 & \xrightarrow{\overline{\tau}} \text{Lie}(G_{\times}) \xrightarrow{\tau} H_{\mathbb{C}}/F^0_{\text{Hodge}}
\end{align*}
$$

(11)

commutes. Here $t = \tau v^\natural$ where $v^\natural$ is the canonical map induced by $u^\natural$, $\text{Lie}(G^\natural)$ is the pullback of (10) along $\exp : \text{Lie}(G_{\text{ét}}) \to G_{\times}$ and $H^1_{\text{DR}}(X\text{an}, \mathbb{C})$ is the cohomological Hodge realization over $k = \mathbb{C}$.

2.1.4. **Example.** (cf. [1, 1.1 & 3.3]) For $X$ proper over a field $k$, $\text{char}(k) = 0$, set $G := \text{Pic}^0_{X/k}$ and let $M = [0 \to G]$ be the corresponding 1-motive. Here $G_{\times} \cong \text{Pic}^0_{X/\text{ét}}$ and $G^\natural \cong \text{Pic}^\natural_{X/\text{ét}}$ are given by simplicial Pic and $\natural - \text{Pic}$ functors of a smooth proper hypercovering $X$. Thus $H_Z = H^1(X_{\text{an}}, \mathbb{Z})$, $\text{Lie}(G^\natural) = H^1_{\text{DR}}(X)$ and $\tau : H^1_{\text{DR}}(X) \cong H^1(X_{\text{an}}, \mathbb{C})$ by cohomological descent over $k = \mathbb{C}$.

2.2. **Construction.** Extending Deligne’s Hodge realization for a given Laumon 1-motive $M = [F \xrightarrow{u} G]$ over $\mathbb{C}$ consider the pull-back $T_{\overline{\jmath}}(F)$ of $u : F \to G$ along $\exp : \text{Lie}(G) \to G$, i.e.,

$$
\begin{align*}
T_{\overline{\jmath}}(F) & \xrightarrow{v} \text{Lie}(G) \\
\downarrow & \downarrow \exp \\
F & \xrightarrow{u} G
\end{align*}
$$
Here $T_{\mathfrak{f}}(F)$ is a formal group and we get a natural group homomorphism $v : T_{\mathfrak{f}}(F) \to \text{Lie}(G)$. We are going to show that

$$T_{\mathfrak{f}}(M) := (T_{\mathfrak{f}}(F), \text{Lie}(G)) \in \text{FHS}_1^{\mathfrak{f}}$$

is a formal Hodge structure. Note that if $M$ is connected then $T_{\mathfrak{f}}(M) = M$.

2.2.1. Remark. The additional data coming from Lie is really needed if we allow additive factors! For example, let $\hat{W} \to V$ be a linear map between $\mathbb{C}$-vector spaces, and let $M = [\hat{W} \xrightarrow{u} V]$ be the induced 1-motive where $\hat{W}$ is the formal completion at the origin (cf. [6, 5.2.5]). Note that all connected 1-motives are obtained in this way (see 1.3.2). For any embedding $V \subsetneq V'$ of vector spaces, we obtain another 1-motive $M' = [\hat{W} \xrightarrow{u'} V']$ such that $M \subsetneq M'$. For both $M$ and $M'$ then $T_{\mathfrak{f}}(\hat{W})$ is the infinitesimal group $\hat{W}$, $\text{Ker}(u) = \text{Ker}(u')$ and we cannot distinguish $M$ by $M'$ out of the formal group only.

2.2.2. Lemma. We have that $T_{\mathfrak{f}}(F)$ is the formal group $F^0 \times H_{\mathbb{Z}}$ such that $H_{\mathbb{Z}}$ is the above extension of $F_{\text{et}}$ by $H_1(G_\times)$.

Proof. Since formal groups are closed under extensions (cf. [6, 4.3.1]) $T_{\mathfrak{f}}(F)$ is a formal group, i.e., it is, by construction, an extension of $F$ by $H_1(G)$. Observe that [17] yields $\text{Lie}(G)$ as the pullback of $\text{Lie}(G_\times)$ along $\exp$ and $H_1(G) \cong H_1(G_\times)$. We then get a natural identification of $H_{\mathbb{Z}}$ with the étale part of $T_{\mathfrak{f}}(F)$, i.e., with the pullback of $F_{\text{et}} \to F$ along $T_{\mathfrak{f}}(F) \to F$. $\Box$

2.2.3. Lemma. If $\sigma := \overline{\tau}^{-1} : H_{\mathbb{C}}/F^0_{\text{Hodge}} \xrightarrow{\cong} \text{Lie}(G_\times)$ is the isomorphism induced by (11) then $\sigma$ restricts to $W_2(H_{\mathbb{C}}) \cong \text{Lie}(T)$ and the following

$$\begin{array}{ccc}
H_{\mathbb{Z}} & \xrightarrow{v_{\mathbb{Z}}} & \text{Lie}(G) \\
\downarrow & & \downarrow \text{pr} \\
H_{\mathbb{C}}/F^0_{\text{Hodge}} & \xrightarrow{\sigma} & \text{Lie}(G_\times)
\end{array}$$

commutes (here $v_{\mathbb{Z}}$ is the restriction of $v$ and $c$ is the canonical map cf. (11)).

Proof. Note that $c = \overline{\tau} \overline{v_{\mathbb{Z}}}$ in (11) and $v_{\mathbb{Z}} = \text{pr} \circ v_{\mathbb{Z}}$ by Lemma 2.2.2. $\Box$

2.2.4. Definition. Denote $T_{\mathfrak{f}}(M)$ the formal Hodge structure $(H, V) \in \text{FHS}_1^{\mathfrak{f}}$ where (i) $H := T_{\mathfrak{f}}(F) = F^0 \times H_{\mathbb{Z}}$, $H_{\text{et}} := T_{\text{Hodge}}(M_{\text{et}})$, granted by Lemma 2.2.2, (ii) $V := \text{Lie}(G)$, $V^1 := \text{Lie}(T) + V(G)$ and $V^0 := V(G)$, (iii) the map $v : T_{\mathfrak{f}}(F) \to \text{Lie}(G)$ defined above, and (iv) the isomorphism $\sigma := \overline{\tau}^{-1} : H_{\mathbb{C}}/F^0_{\text{Hodge}} \xrightarrow{\cong} \text{Lie}(G_\times)$ providing (11) by Lemma 2.2.3.
We then have $T_f(M)_{\text{ét}} = T_{\text{Hodge}}(M_{\text{ét}}) \in \text{MHS}^\text{fr}_1$ and the construction is clearly functorial (since the diagram (11) is natural) providing a functor

$$T_f : \mathcal{M}_1^a,\text{fr} \longrightarrow \text{FHS}^\text{fr}_1$$

such that $T_f(M) = T_{\text{Hodge}}(M)$ if $M$ is étale (via [2.2.2]) and $T_f(M) = M$ if $M$ is connected.

2.2.5. Remark. Note that by applying $T_f$ to (9) we get (4), the extension (8) yields (5) and $M$ is special $\iff T_f(M)$ is special.

2.3. Conclusion. Summarizing up, see also 1.2 and 1.4, the theorem is proven, e.g., in order to show that $T_f$ yields an equivalence of categories we can argue as in [10.1.3]. For $(H, V) \in \text{FHS}^\text{fr}_1$ we have constructed, in 1.4, a 1-motive

$$\overrightarrow{(H, V)} := [H^0 \times \text{gr}^W_0(H_Z) \to V/W_{-1}(H_Z)]$$

It is clear that $T_f(\overrightarrow{(H, V)}) \cong (H, V)$, see (10), which is natural in $(H, V)$. Conversely, for $M = [F \to G]$ we have $T_f(M) := (T_f(F), \text{Lie}(G))$ such that $T_f(M) \cong M$ functorially in $M$ by construction. One obtains a duality on $\text{FHS}^\text{fr}_1$ after Cartier duality on $\mathcal{M}_1^a,\text{fr}$ by defining

$$T_f(M)'^\vee := T_f(M'^\vee)$$

The lemmas [2.2.2] and [2.1.2] further explain the diagram of the main theorem and the remaining claims.

2.3.1. Example. (cf. 2.1.4) For $X$ proper over $\mathbb{C}$ and $M = [0 \to \text{Pic}^0_{X/\mathbb{C}}]$ we have $T_f(M) = (H^1(X_{\text{an}}, \mathbb{Z}(1)), H^1(X, \mathcal{O}_X))$. Here we have $M_{\text{ét}} = [0 \to \text{Pic}^0_{X, /\mathbb{C}}]$ and a projection

$$\begin{array}{ccc}
\text{Lie} \text{Pic}^0_{X/\mathbb{C}} & \xrightarrow{\sim} & H^1(X, \mathcal{O}_X) \\
\downarrow & & \downarrow \\
\text{Lie} \text{Pic}^0_{X, /\mathbb{C}} & \xrightarrow{\sim} & \mathbb{H}^1(X, \mathcal{O}_X)
\end{array}$$

with kernel the additive factor of $\text{Pic}^0_{X/\mathbb{C}}$. Further considering $M^2_{\text{ét}} = [0 \to \text{Pic}^2_{X, /\mathbb{C}}]$ and $T_f(M^2_{\text{ét}}) = (H^1(X_{\text{an}}, \mathbb{Z}(1)), H^1_{\text{DR}}(X))$ we get the extension

$$0 \to F^0_{\text{Hodge}} \to T_f(M^2_{\text{ét}}) \to T_{\text{Hodge}}(M_{\text{ét}}) \to 0$$

2.3.2. Remark. Note that in 2.3.1 $M$ is special but the dual $M'^\vee$ is not special! Another more striking example is given by taking an abelian
variety $X$ and looking at the special 1-motive $[0 \to \text{Pic}_{X/\mathbb{C}}^\pm]$ which is the universal extension of the dual of $X$. The Cartier dual

$$[0 \to \text{Pic}_{X/\mathbb{C}}^\pm]^\vee = [\hat{X} \to X]$$

is not special. Actually, in general, the Cartier dual of a connected 1-motive is connected and the dual of étale is étale but the Cartier dual of $M$ special just fits in an extension

$$0 \to M_\text{ét}^\vee \to M^\vee \to (M^0)^\vee \to 0$$

dual to (8).

REFERENCES

[1] L. Barbieri-Viale: On the theory of 1-motives, a contribution to the Proceedings of the Workshop “Algebraic Cycles and Motives” on the occasion of the 75th birthday of J.P. Murre (2004, Lorentz Center, Leiden). Preprint http://arxiv.org/abs/math.AG/0502476

[2] L. Barbieri-Viale and A. Bertapelle: Sharp De Rham realization, in preparation.

[3] L. Barbieri-Viale, A. Rosenschon and M. Saito: Deligne’s conjecture on 1-motives, Annals of Math. 158 N. 2 (2003) 593-633.

[4] S. Bloch and V. Srinivas: Enriched Hodge Structures, in “Algebra, arithmetic and geometry. Part I, II.” Papers from the International Colloquium held in Mumbai, January 4–12, 2000. Edited by R. Parimala. Tata Institute of Fundamental Research Studies in Mathematics, 16, 171-184.

[5] P. Deligne: Théorie de Hodge III Publ. Math. IHES 44 (1974) 5–78.

[6] G. Laumon: Transformation de Fourier generalisee, http://arxiv.org/abs/alg-geom/9603004 - Preprint IHES (Transformation de Fourier geometrique, IHES/85/M/52) 47 pages.

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