Multidimensional SDE with anticipating initial process and reflection

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Abstract
In this paper, the strong solutions $(X, L)$ of multidimensional stochastic differential equations with reflecting boundary and possible anticipating initial random variables is established. The key is to obtain some substitution formula for Stratonovich integrals via a uniform convergence of the corresponding Riemann sums and to prove continuity of functionals of $(X, L)$.

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1 Introduction and main results

Let $O$ be a smooth bounded open set in $\mathbb{R}^d$. $n(x)$ denotes the cone of unit outward normal vectors to $\partial O$ at $x$, that is,

\begin{align}
(i) \quad & \exists C_0 \geq 0, \forall x \in \partial O, \forall x' \in \bar{O}, \exists k \in n(x) \\
& \implies (x - x', k) + C_0 |x - x'|^2 \geq 0, \quad (1.1)
\end{align}

\begin{align}
(ii) \quad & \forall x \in \partial O, \exists C \geq 0, \exists k \in \mathbb{R}^d, \forall x' \in \bar{O}, \\
& (x - x', k) + C |x - x'|^2 \geq 0, \implies k = \theta n(x) \quad (1.2)
\end{align}

for some $\theta \geq 0$, where $\partial O$ denotes the boundary of $O$, $\bar{O}$ denotes the closure of $O$. We assume that $B_t$ is an $\mathbb{R}^d$-valued $\mathcal{F}_t$- Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, \mathbb{P})$ satisfying the usual assumptions. We consider the following stochastic differential equations on domain $O$ with reflecting boundary conditions:

\begin{equation}
X_t(x) = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s(x)) \circ dB_s - L^x_t, \quad \forall \ t \in [0,1], \quad (1.3)
\end{equation}

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where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are continuous functions, $\circ$ denotes the Stratonovich integral. A pair $(X_t(x), L^x_t, t \in [0,1])$ of continuous and $\mathcal{F}_t$-adapted processes is called a solution to equations (1.3) if there exists a measurable set $\tilde{\Omega}$ with $P(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$

(i) for each $x \in \tilde{O}$ the function $s \mapsto L^x_s$ with values in $\mathbb{R}^d$ has bounded variation on any interval $[0,T]$ and $L^x_0 = 0$.

(ii) for all $t \geq 0$, $X_t(x) \in \tilde{O}$ and $(X_t(x), L^x_t, t \in [0,1])$ satisfies Eq.(1.3).

(iii) $|L^x_t| = \int_0^t I_{(X_s(x) \in \partial O)} d|L^x_s|$ and $L^x_t = \int_0^t \xi(X_s(x)) d|L^x_s|$ (1.4)

with $\xi(X_s(x)) \in \mathfrak{n}(X_s(x))$, where the $|L^x_t|$ denotes the total variation of $L^x$ on $[0,t]$.

Remark that (iii) implies that the support of $d|L^x|$ is included in $\{s : X_s(x) \in \partial O\}$ and the force $L^x$ keeps the process $X$ be in $\tilde{O}$.

This type of reflected stochastic differential equations has been studied notably by Skorohod[17], Tanaka[20], Lions and Sznitman[11], and Saisho[18], and also by Stroock and Varadhan[19] who used a submartingale problem formulation, and other authors. Moreover, such reflected diffusions can also be reduced to studying multivalued stochastic differential equations( see [4, 5, 22, 23] and references therein). It is well-known (see [11]) that for any given initial value $x \in \tilde{O}$ the Eq.(1.3) has a unique solution provided that $\|\sigma(\cdot)\|$ and $|\tilde{b}(\cdot)|$ are uniformly bounded real-valued functions on $\mathbb{R}^d$ and satisfy a uniform Lipschitz condition: $\exists c > 0$ such that

$$\|\sigma(y) - \sigma(z)\| \leq c|y - z|, \quad |\tilde{b}(y) - \tilde{b}(z)| \leq c|y - z|$$

for any $y, z \in \mathbb{R}^d$, where $\tilde{b}_i(x) = b_i(x) + \frac{1}{2} \sum_{k,j=1}^d \frac{\partial \sigma_{jk}(x)}{\partial x_k}\sigma_{kj}(x)$, $\|\sigma(y)\| := \sqrt{\sum_{i,j=1}^d \{\sigma_{ij}(y)\}^2}$ and $|\tilde{b}(y)| := \sqrt{\sum_{i=1}^d \{\tilde{b}_i(y)\}^2}$.

The natural question arises: does there still exist a pair $(X_t, L_t, t \in [0,1])$ of stochastic processes to solve Eq.(1.3) if the initial value is an arbitrary random variable $Z$ which belongs to $\tilde{O}$ with probability one and may depend on the whole Brownian paths?

On one hand, the answer is not immediately clear because one needs to deal with anticipating stochastic integration. On the other hand, on a given
financial market, different agents generally have different levels of information; besides the public information, some of them may possess privileged information, which leads them to make anticipations on some future realizations of functionals of the price process, therefore, for a financial corporation, the studying the problem of optimal dynamic risk control/dividends distribution has to face the question (see [3, 16, 1, 6] and references therein). The main aim of this paper is to give an affirmative answer to the question above. Let us describe now more precisely main results of this paper as follows.

**Theorem 1.1.** Assume that $\Omega$ is a smooth bounded open set in $\mathbb{R}^d$ and there exists a function $\phi \in C^2_b(\mathbb{R}^d)$ such that
\begin{equation}
\exists \alpha > 0, \forall \in \partial \Omega, \forall \zeta \in \mathfrak{n}(x), (\nabla \phi(x), \zeta) \leq -\alpha C_0,
\end{equation}
the functions $\sigma$ and $b$ satisfy that $\sigma, \tilde{b}$ and $\nabla \sigma$ are bounded, and the following
\begin{align}
|\tilde{b}(x) - \tilde{b}(y)| + |\sigma(x) - \sigma(y)| &+ \|((\nabla \sigma \cdot \sigma)(x) - (\nabla \sigma \cdot \sigma)(y))
\end{align}
\begin{align}
&+ \|((\nabla \sigma \cdot \nabla \sigma \cdot \sigma)(x) - (\nabla \sigma \cdot \nabla \sigma \cdot \sigma)(y)) + \|((\nabla \sigma \cdot \tilde{b})(x) - (\nabla \sigma \cdot \tilde{b})(y))
\end{align}
\begin{align}
&+ \|((\sigma^T \cdot \nabla^2 \sigma \cdot \sigma)(x) - (\sigma^T \cdot \nabla^2 \sigma \cdot \sigma)(y)) \leq k|x - y|
\end{align}
for some constant $k > 0$, where $C_0$ is given by (1.1), $\sigma^T$ denotes transpose of $\sigma$, $\nabla \sigma$ and $\nabla^2 \sigma$ denote $\sigma$’s derivatives of first and second order with respect to spatial variable $x$, respectively. Then for any random variable $Z$ with $\mathbb{P}\{Z \in \bar{\Omega}\} = 1$ the pair $(X_t(Z), L^Z_t, t \in [0, 1])$ solves the following stochastic differential equation on domain $\Omega$ with reflecting boundary conditions:
\begin{equation}
X_t(Z) = Z + \int_0^t b(X_s(Z))ds + \int_0^t \sigma(X_s(Z)) \circ dB_s - L^Z_t
\end{equation}
with $X_t(Z) \in \bar{\Omega}$, and satisfies
(1) the function $s \mapsto L^Z_s$ with values in $\mathbb{R}^d$ has bounded variation on any interval $[0, T]$ and $L^Z_0 = 0$.
(2)
\begin{equation}
|L^Z_t| = \int_0^t I(\xi(X_s(Z)) \in \partial \Omega) d|L^Z|_s \quad \text{and} \quad L^Z_t = \int_0^t \xi(X_s(Z)) d|L^Z|_s
\end{equation}
with $\xi(X_s(Z)) \in \mathfrak{n}(X_s(Z))$, where $(X_t(x), L^x_t, t \in [0, 1])$ is the unique solution of Eq.(1.3), the stochastic integral in Eq.(1.8) is interpreted as anticipating Stratonovich integral.

Now we recall the definition of the anticipating Stratonovich integral (see [13]). For any $t \in [0, 1]$, let $\pi$ denote an arbitrary partition of the
interval $[0, t]$ of the form: $\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. Let $\|\pi\| = \sup_{0 \leq k \leq n-1} \{(t_{k+1} - t_k)\}$ denote the norm of $\pi$. For an $\mathbb{R}^d \times \mathbb{R}^d$-valued stochastic process $f = \{f_s, \ s \in [0, 1]\}$, we define its Riemann sums $S_\pi(f, t)$ by

$$S_\pi(f, t) = \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \left( \int_{t_k}^{t_{k+1}} f_s \, ds \right) \cdot (B_{t_{k+1}} - B_{t_k}). \quad (1.10)$$

We have the following

**Definition 1.1.** We say that a stochastic process $f = \{f_s, \ s \in [0, 1]\}$ is Stratonovich integrable with respect to $B$ if the family $S_\pi(f, t)$ converges in probability as $\|\pi\| \to 0$. In such a case we denote the limit by $\int_0^t f_s \circ dB_s$.

Let us now describe our approach. To prove Theorem 1.1, the natural idea is to replace $x$ in (i), (ii) and (iii) of Eq.(1.3) by the initial random variable $Z$ and prove that the pair $(X_t(Z), L^Z_t)$ solves the Eq.(1.8). To achieve this, the key is to establish the following substitution formula

$$\int_0^t \sigma(X_s(x)) \circ dB_s |_{x=Z} = \int_0^t \sigma(X_s(Z)) \circ dB_s, \quad (1.11)$$

$$\int_0^t f(X_s(Z))d|L^Z|_s = 0, \quad L^Z_t = \int_0^t \xi(X_s(Z))d|L^Z|_s \quad (1.12)$$

for all $t \in [0, 1]$, where $f$ is a continuous function defined on $\mathbb{R}^d$ with compact support included in $\mathcal{O}$ and $\xi(X_s(Z)) \in n(X_s(Z))$.

The novelty and difficulty of this paper are anticipation, reflection and shape of domain $\mathcal{O}$. Since Lions and Sznitman’s result in [11] states that the solution $(X_t(x), L_t^x)$ is Hölder continuous of order being less than $1/2$ with respect to the initial value $x$, the regularity is not good enough to satisfy the required hypothesis of substitution formula in the literature (see [13]), it seems that we can not apply the existing substitution formula to prove (1.11). Moreover, because reflecting boundary conditions and shape of domain $\mathcal{O}$, it is also impossible to prove (1.11) by using Itô-Ventzell formula used by cone and Pardoux[14], Kohatsu-Higa and León[9]. Instead, we prove (1.11) by showing the uniform convergence (w.r.t.x) of the corresponding Riemann Sums $S_\pi(\sigma(X_s(x), t))$. The Garsia, Rodemich and Rumsey’s Lemma and moments estimates for one-point and two-point motions will play an important role. To prove (1.12) we need only to show that the functionals $F(t, x) := \int_0^t f(X_s(x))d|L^x|_s$, $L^x_t$ and $G(t, x) := \int_0^t \xi(X_s(x))d|L^x|_s$ are continuous in $(t, x)$, doing this depends on the shape function $\phi$ of domain $\mathcal{O}$ in (1.6).
Remark 1.1. It seems that this new approach can be used to study perturbed stochastic Skorohod equations with anticipating initial processes because the solutions of these SDEs are Hölder continuous with order being less than $\frac{1}{2}$ and not differentiable w.r.t. initial value $x$ (see [7, 22] for adapted case). We shall study it in forthcoming paper.

This paper is organized as follows. Firstly we study the regularity of the solution $(X_t(x), L^x_t)$. Secondly we devote to showing continuity of functionals $F(t, x) := \int_0^t f(X_s(x))d|L^x|_s$, $L^x_t$ and $G(t, x) := \int_0^t \xi(X_s(x))d|L^x|_s$. In Section 4 we prove the uniform convergence (w.r.t. $x$) of the Riemann Sums $S_n(\sigma(X_s(x), t))$. Finally we prove Theorem 1.1 in Section 6.

Throughout this paper we make the following convention: the letter $c$ or $c(p_1, p_2, p_3, \ldots, \cdot)$ depending only on $p_1, p_2, p_3, \ldots, \cdot$ will denote an unimportant positive constant, whose values may change from one line to another one.

2 Regularity of the solution $(X_t(x), L^x_t)$ of Eq.(1.3)

The main aim of this section is to study regularity of the solution $(X_t(x), L^x_t)$ w.r.t. $(t, x)$ via the shape function $\phi$ of domain $\mathcal{O}$ in (1.6).

Proposition 2.1. Assume that the smooth bounded open $\mathcal{O}$, the coefficients $\sigma$ and $b$ satisfy the same conditions as in Theorem 1.1. $(X_t(x), L^x_t)$ is a solution of Eq.(1.3). Then there is a constant $c$ such that

$$
\mathbb{E}\left\{ \sup_{0 \leq t \leq 1} |X_t(x) - X_t(y)|^p \right\} \leq c|x - y|^p, 
$$

(2.1)

$$
\mathbb{E}\left\{ \sup_{0 \leq t \leq 1} |L^x_t - L^y_t|^p \right\} \leq c|x - y|^p
$$

(2.2)

for any $x, y \in \mathcal{O}$ and $p \geq 1$.

Proof. By Hölder inequality, we need only to prove Proposition 2.1 for $p \geq 4$. Let $\tilde{b}(x) = b(x) + \frac{1}{2} \sum_{k,j=1}^{d} \frac{\partial \sigma_{kj}(x)}{\partial x_k}(x)\sigma_{kj}(x)$, that is, $\tilde{b}(x) = b(x) + \frac{1}{2}(\nabla \sigma \cdot \sigma)(x)$ for any $x \in \mathbb{R}^d$. We write solution $(X_t(x), L^x_t)$ of Eq.(1.3) in Itô’s form as follows: for $x \in \mathcal{O}$

$$
X_t(x) = x + \int_0^t \tilde{b}(X_s)ds + \int_0^t \sigma(X_s(x))dB_s - L^x_t, 
$$

(2.3)

$$
|L^x_t| = \int_0^t I_{(X_s(x) \in \partial \mathcal{O})}d|L^x|_s, 
$$

(2.4)

$$
L^x_t = \int_0^t \xi(X_s(x))d|L^x|_s \text{ with } \xi(X_s(x)) \in \mathbf{n}(X_s(x))
$$

(2.5)
and \((X_t(y), L^y_t)\) also satisfy the same equations above for \(y \in \mathcal{O}\).

Applying Itô’s formula to function \(\phi \in C^2_c(\mathbb{R}^d)\) satisfying (1.6) and stochastic process \(X_t(x)\), we have

\[
\phi(X_t(x)) = \phi(x) + \int_0^t (\nabla \phi^T \sigma)(X_s(x)) dB_s + \int_0^t (\nabla \phi^T \tilde{b})(X_s(x)) ds
\]

\[
- \int_0^t (\nabla \phi^T \xi)(X_s(x)) d|L^x_s| + \frac{1}{2} \int_0^t \text{tr}\{(\nabla^2 \phi \sigma^T)(X_s(x))\} ds,
\]

(2.6)

where \(\text{tr}(A)\) denote the trace of \(A\). Similarly, we have same expression for \(\phi(X_t(y))\).

Define \(f(x) := |x|^p, x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d\). Then

\[
\nabla f(x) = p|x|^{p-2} x, \quad \nabla^2 f(x) = p|x|^{p-2} I_{d \times d} + p(p-2)|x|^{p-4} xx^T.
\]

(2.7)

Let \(m_t = X_t(x) - X_t(y), D_t = \phi(X_t(x)) + \phi(X_t(y))\) and \(N_t = \exp\{-\frac{p}{\alpha} D_t\}\). By Itô’s formula and (2.5),

\[
df(m_t) = p|m_t|^{p-2} m_t^T (\tilde{b}(X_t(x)) - \tilde{b}(X_t(y))) dt
\]

\[
+ p|m_t|^{p-2} m_t^T (\sigma(X_t(x)) - \sigma(X_t(y))) dB_t
\]

\[
- p|m_t|^{p-2} m_t^T \xi(X_t(x)) d|L^x_t|
\]

\[
+ p|m_t|^{p-2} m_t^T \xi(X_t(y)) d|L^y_t|
\]

\[
+ \frac{1}{2} \text{tr}\{(\nabla^2 f(m_t)) (\sigma(X_t(x)) - \sigma(X_t(y)))(\sigma(X_t(x)) - \sigma(X_t(y)))^T\} dt
\]

(2.8)

\[
dN_t = -\frac{p}{\alpha} N_t [(\nabla \phi^T \sigma)(X_t(x)) + (\nabla \phi^T \sigma)(X_t(y))] dB_t
\]

\[
-\frac{p}{\alpha} N_t [(\nabla \phi^T \tilde{b})(X_t(x)) + (\nabla \phi^T \tilde{b})(X_t(y))] dt
\]

\[
+\frac{p}{\alpha} N_t (\nabla \phi^T \xi)(X_t(x)) d|L^x_t|
\]

\[
+\frac{p}{\alpha} N_t (\nabla \phi^T \xi)(X_t(y)) d|L^y_t|
\]

\[
-\frac{p}{2\alpha} N_t \text{tr}\{(\nabla^2 \phi \sigma^T)(X_t(x)) + (\nabla^2 \phi \sigma^T)(X_t(y))\} dt
\]

\[
+\frac{p^2}{2\alpha^2} N_t \text{tr}\{[(\nabla \phi^T \sigma)(X_t(x)) + (\nabla \phi^T \sigma)(X_t(y))]^T
\]

\[
\times[(\nabla \phi^T \sigma)(X_t(x)) + (\nabla \phi^T \sigma)(X_t(y))]\} dt
\]

(2.9)

and the stochastic contraction \(df(m_t) \cdot dN_t\) is given by

\[
df(m_t) \cdot dN_t = -\frac{p^2}{\alpha} N_t m_t|^{p-2} \text{tr}\left\{m_t^T (\sigma(X_t(x)) - \sigma(X_t(y)))^T
\]

\[
\times[(\nabla \phi^T \sigma)(X_t(x)) + (\nabla \phi^T \sigma)(X_t(y))]^T\right\} dt.
\]

(2.10)
Therefore, by Itô's formula again and (2.8)-(2.10),

\[
N_t f(m_t) = \exp\left\{ -\frac{p}{\alpha} [\phi(x) + \phi(y)] \right\} |x - y|^p + \int_0^t N_s df(m_s) \\
+ \int_0^t f(m_s) dN_s + \int_0^t df(m_s) \cdot dN_s \\
= \exp\left\{ -\frac{p}{\alpha} [\phi(x) + \phi(y)] \right\} |x - y|^p \\
+ p \int_0^t N_s |m_s|^{p-2} m_s^T (\bar{b}(X_s(x)) - \bar{b}(X_s(y))) ds \\
+ p \int_0^t N_s |m_s|^{p-2} m_s^T (\sigma(X_s(x)) - \sigma(X_s(y))) dB_s \\
- p \int_0^t N_s |m_s|^{p-2} (m_s, \xi(X_t(x))) d|L^x|_s \\
+ p \int_0^t N_s |m_s|^{p-2} (m_s, \xi(X_t(y))) d|L^y|_s \\
+ \frac{1}{2} \int_0^t N_s \text{tr} \{ \nabla^2 f(m_s)(\sigma(X_s(x)) - \sigma(X_s(y)))(\sigma(X_s(x))) \\
- \sigma(X_s(y)))^T \} ds \\
- \frac{p}{\alpha} \int_0^t N_s f(m_s) [ (\nabla \phi^T \sigma)(X_s(x)) + (\nabla \phi^T \sigma)(X_s(y)) ] dB_s \\
- \frac{p}{\alpha} \int_0^t N_s f(m_s) [ (\nabla \phi^T \bar{b})(X_s(x)) + (\nabla \phi^T \bar{b})(X_s(y)) ] ds \\
+ \frac{p}{\alpha} \int_0^t N_s |m_s|^{p-2} |m_s|^2 (\nabla \phi(X_s(x)), \xi(X_s(x))) d|L^x|_s \\
+ \frac{p}{\alpha} \int_0^t N_s |m_s|^{p-2} |m_s|^2 (\nabla \phi(X_s(y)), \xi(X_s(y))) d|L^y|_s \\
- \frac{p}{2\alpha} \int_0^t N_s f(m_s) \text{tr} \{ [ (\nabla^2 \phi \sigma^T)(X_s(x)) + (\nabla^2 \phi \sigma^T)(X_s(y)) ] ds \\
+ \frac{p^2}{2\alpha^2} \int_0^t N_s f(m_s) \text{tr} \left\{ [ (\nabla \phi^T \sigma)(X_s(x)) + (\nabla \phi^T \sigma)(X_s(y)) ]^T \\
\times [ (\nabla \phi^T \sigma)(X_s(x)) + (\nabla \phi^T \sigma)(X_s(y)) ] \right\} ds \\
- \frac{p^2}{\alpha} \int_0^t N_s |m_s|^{p-2} \text{tr} \left\{ [ m_s^T (\sigma(X_s(x)) - \sigma(X_s(y))) ]^T \\
\times [ (\nabla \phi^T \sigma)(X_s(x)) + (\nabla \phi^T \sigma)(X_s(y)) ]^T \right\} ds \\
:= \sum_{i=1}^{13} a_i(t). \quad (2.11)
\]
Similarly, since \( \phi \)

Using equality,

By condition (1.6),

Hence

Using \( \phi \in C^2(\mathbb{R}^d) \) and (2.11)-(2.12),

where \( a_i(t) = \sup_{s \in [0,t]} \{ a_i(t) \} \), \( m^*(t) = \sup_{s \in [0,t]} \{ \|m(t)\| \} \).

Since \( \phi \) is bounded, by Burkholder inequality (see \([2]\)) and (1.7), we have

Similarly, since \( \phi \) and \( \nabla \phi \sigma \) are bounded on \( \bar{O} \), we also have

Using \( \phi \) and \( \nabla \phi \sigma \) are bounded on \( \bar{O} \), the condition (1.7) and Hölder inequality,

Using \( \phi \) and \( \nabla \phi \sigma \) are bounded on \( \bar{O} \), the condition (1.7) and Hölder inequality,
By the same way as in (2.16)
\[ E\{(s_i^3(t))^2\} \leq c \int_0^t E\{|m^*(s)|^{2p}\}ds \]  
(2.17)
for \( i = 2, 6, 8, 11, 12. \)

Putting the above inequalities (2.13)-(2.17) together implies that
\[ E\{(m^*(t))^{2p}\} \leq c|x-y|^{2p} + c \int_0^t E\{(m^*(s))^{2p}\}ds, \]  
(2.18)
By Gronwall inequality,
\[ E\{\sup_{t \in [0,1]} |X_t(x) - X_t(y)|^{2p}\} \leq c|x-y|^{2p}. \]  
(2.19)

So proof of (2.1) has been done by Hölder inequality. Using
\[ |L_s^x - L_t^y| \leq |x-y| + |X_t(x) - X_t(y)| + |\int_0^t \tilde{b}(X_u(x)) - \tilde{b}(X_u(y))|du|ds| \]
\[ + |\int_0^t (\sigma(X_u(x)) - \sigma(X_u(y)))dB_u|, \]
(2.2) is a direct consequence of (2.1). Thus we complete proof of Proposition 2.1.

**Proposition 2.2.** Assume that the smooth bounded open \( \mathcal{O} \), the coefficients \( \sigma \) and \( b \) satisfy the same conditions as in Theorem 1.1. \((X_t, L_t^x)\) is a solution of Eq. (1.3). Then there is a constant \( c \), which is independent of \( x \), such that for any \( p \geq 2 \)
\[ \sup_{x \in \mathcal{O}} E\{|X_t(x) - X_s(x)|^{2p}\} \leq c|t-s|^{\frac{p}{2}}, \]  
(2.20)
\[ \sup_{x \in \mathcal{O}} E\{|L_t^x - L_s^x|^{2p}\} \leq c|t-s|^{\frac{p}{2}}. \]  
(2.21)

**Proof.** By Hölder inequality, we need only to prove Proposition 2.2 for \( p \geq 4 \). For \( t \geq s \geq 0 \), similar to that of Proposition 2.1, we define \( m_t, D_t, N_t \) and \( f \) here by
\[
m_t(x) = X_t(x) - X_s(x) \\
= \int_s^t \tilde{b}(X_u(x))du + \int_s^t \sigma(X_u(x))dB_u - \int_s^t \xi(X_u(x))d|L_u^x| \]
with \( \xi(X_s(x)) \in n(X_s(x)) \),
\[
D_t = \phi(X_t(x)),
N_t = \exp\{-\frac{p}{\alpha}D_t\},
G_t = N_t^{-1},
\]
\[
f(x) = |x|^2, x = (x_1, x_2, \cdots, x_d)^T \in \mathbb{R}^d. \]
By the same way as in (2.11),

\[
|X_t(x) - X_s(x)|^2 = 2G_t \int_s^t N_u m_u^T b(X_u(x)) du \\
+ 2G_t \int_s^t N_u m_u^T \sigma(X_u(x)) dB_u \\
- 2G_t \int_s^t N_u (m_u, \xi(X_u(x))) d|L^x|_u \\
+ G_t \int_s^t N_u \text{tr}\{(\sigma \sigma^T)(X_u(x))\} du \\
- \frac{2}{\alpha} G_t \int_s^t N_u f(m_u) [(\nabla \phi^T \sigma)(X_u(x))] dB_u \\
- \frac{2}{\alpha} G_t \int_s^t N_u f(m_u)(\nabla \phi^T \tilde{b})(X_u(x)) du \\
+ \frac{2}{\alpha} G_t \int_s^t N_u m_u^2 (\nabla \phi(X_u(x)), \xi(X_u(x))) d|L^x|_u \\
- \frac{1}{\alpha} G_t \int_s^t N_u f(m_u) \text{tr}\{(\nabla^2 \phi \sigma \sigma^T)(X_u(x))\} du \\
+ \frac{2}{\alpha^2} G_t \int_s^t N_u f(m_u) \text{tr}\\left\{[(\nabla \phi^T \sigma)(X_u(x))]^T[(\nabla \phi^T \sigma)(X_u(x))]\right\} du \\
- \frac{4}{\alpha} G_t \int_s^t N_u \text{tr}\\left\{(m_u^T \sigma(X_u(x)))^T((\nabla \phi^T \sigma)(X_u(x)))^T\right\} du \\
:= \sum_{i=1}^{10} d_i(t).
\]

By condition (1.6),

\[d_3(t) + d_7(t) \leq 0. \tag{2.22}\]

Therefore

\[
\mathbb{E}\{|X_t(x) - X_s(x)|^{2p}\} \leq c(p) \sum_{i=1}^{2} \mathbb{E}\{|d_i(t)|^p\} + c(p) \sum_{i=4}^{6} \mathbb{E}\{|d_i(t)|^p\} \\
+ c(p) \sum_{i=7}^{10} \mathbb{E}\{|d_i(t)|^p\}. \tag{2.23}
\]

Since \(\sigma, N_t\) and \(G_t\) are uniformly bounded, by Burkholder (see [2]) and Hölder inequalities and Young’s inequality: for any real positive \(x, y, \eta, p, q\)
with \( p^{-1} + q^{-1} = 1 \) there exists \( c < +\infty \) such that \( xy \leq \eta x^p + cy^q \), we have
\[
E\{|d_2(t)|^p\} \leq c E\{| \int_s^t N_u m_u^T \sigma(X_u(x)) dB_u |^p \}
\leq c E\{| \int_s^t N_u^2 \text{tr} \{ \sigma(X_u(x)) m_u m_u^T \sigma(X_u(x)) \} du |^p \}
\leq c E\{| \int_s^t |m_u|^2 du |^p \}
\leq c |t - s|^{\frac{p}{2}} + c \int_s^t E\{|m_u|^{2p} \} du.
\tag{2.24}
\]
Similarly,
\[
E\{|d_5(t)|^p\} \leq c \int_s^t E\{|m_u|^{2p} \} du.
\tag{2.25}
\]
Since \( \sigma, \nabla \phi \sigma, N_t \) and \( G_t \) are uniformly bounded, by Hölder inequalities and Young’s inequality, we have
\[
E\{|d_{10}(t)|^p\} \leq c E\{| \int_s^t N_u \text{tr} \{ m_u^T (\sigma(X_u(x)))^T (\nabla \phi^T \sigma)(X_u(x)) \} du |^p \}
\leq c E\{| \int_s^t |m_u| du |^p \}
\leq c |t - s|^p + c \int_s^t E\{|m_u|^{2p} \} du.
\tag{2.26}
\]
Similarly,
\[
E\{|d_1(t)|^p\} \leq c |t - s|^p + c \int_s^t E\{|m_u|^{2p} \} du,
\tag{2.27}
E\{|d_4(t)|^p\} \leq c |t - s|^p,
\tag{2.28}
E\{|d_i(t)|^p\} \leq c \int_s^t E\{|m_u|^{2p} \} du, \quad i = 6, 8, 9.
\tag{2.29}
\]
Putting the above inequalities (2.23)-(2.29) together, we obtain
\[
E\{|X_t(x) - X_s(x)|^{2p}\} \leq c |t - s|^\frac{p}{2} + c \int_s^t E\{|X_u(x) - X_u(x)|^{2p}\} du.
\]
The Gronwall-Bellman inequality (see [15] for Theorem 1.3.1) implies that
\[
E\{|X_t(x) - X_s(x)|^{2p}\} \leq c |t - s|^{\frac{p}{2}}.
\]
Therefore the proof of (2.20) has been done. Using
\[
|L_t^x - L_s^x| \leq |X_t(x) - X_s(x)| + | \int_s^t \tilde{b}(X_u(x)) dB_u | + | \int_s^t \sigma(X_u(x)) dB_u |,
\]

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(2.21) is a direct consequence of (2.20). Thus we complete proof of Proposition 2.2. \(\square\)

Since the domain \(\mathcal{O}\) is bounded, the following follows immediately from Proposition 2.1 and Hölder’s inequality.

**Proposition 2.3.** Assume that the smooth bounded open \(\mathcal{O}\), the coefficients \(\sigma\) and \(b\) satisfy the same conditions as in Theorem 1.1. \((X_t(x), L^x_t)\) is a solution of Eq.(1.3). Then there is a constant \(c\), which is independent of \(x\), such that

\[
E\{\sup_{0 \leq t \leq 1} |X_t(x)|^p \} \leq c(1 + |x|)^p, \tag{2.30}
\]

\[
E\{\sup_{0 \leq t \leq 1} |L^x_t|^p \} \leq c(1 + |x|)^p, \tag{2.31}
\]

for any \(x \in \mathcal{O}\) and \(p \geq 1\).

### 3 Continuity of functionals of local times

**Proposition 3.1.** Assume that the smooth bounded open \(\mathcal{O}\), the coefficients \(\sigma\) and \(b\) satisfy the same conditions as in Theorem 1.1. \((X_t(x), L^x_t)\) is a solution of Eq.(1.3). Then the functions \(X_t(x)\) and \(L^x_t\), \(F(t, x)\) and \(G(t, x)\) are jointly continuous in \((t, x)\) on \([0, 1] \times \mathcal{O}\), where \(F(t, x) := \int_0^t f(X_s(x))d|L^x|_s\) and \(G(t, x) := \int_0^t \xi(X_s(x))d|L^x|_s\), \(f\) is a continuous function defined on \(\mathbb{R}^d\) with compact support included in \(\mathcal{O}\) and \(\xi(X_s(x)) \in \mathbb{n}(X_s(x))\).

**Proof.** By Kolmogorov’s continuity criterion of random fields (see [10] for Theorem 1.4.1), Proposition 2.1 and 2.2, the functions \(X_t(x)\) and \(L^x_t\) are Hölder continuous in \((t, x)\). Since proof of continuity of \(G(t, x)\) w.r.t.\((t, x)\) is similar to that of \(G(t, x)\), we need only deal with the proof of \(F(t, x)\). Remarking that

\[
|F(t, x) - F(s, x)| \leq \sup_{y \in \mathcal{O}}(|f(y)|)|L^x_t - L^x_s|, \tag{3.1}
\]

the function \(F(t, x)\) is continuous in \(t\) uniformly with respect to \(x\) in compact set \(\mathcal{O}\) by Proposition 2.2 and Kolmogorov’s continuity criterion (see Theorem 1.4.1 in [10]). Thus, it suffices to show the continuity of \(F(t, x)\) w.r.t.\(x\) for any fixed \(t\). Let \(x_n, x \in \mathcal{O}\) with \(x_n \to x\) as \(n \to +\infty\). By Propositions 2.1-2.2, and \(X_t(x)\) and \(L^x_t\) are Hölder continuous in \((t, x) \in [0, 1] \times \mathcal{O}\), we have

\[
L^x_{t_n} \to L^x_t, \quad X_t(x_n) \to X_t(x), \tag{3.2}
\]
uniformly in $t$, as $n \to +\infty$. Therefore, there exist constants $C_1, C \geq 1$ such that for all $n \geq 1$

$$|L^x_n| \leq C + |L^x| \leq C + |L^x|_1 \leq C + C_1 \quad (3.3)$$

due to bound of total variation of $L^x$ on $[0,1]$. Since the function $f(x)$ is bounded and continuous, by (3.2) and (3.3),

$$\left| \int_0^t [f(X_s(x_n)) - f(X_s(x))] dL^x_n \right| \to 0 \quad (3.4)$$
as $n \to +\infty$. Because $L^x_n$ and $L^x$ are continuous processes with bounded variation, by (3.2), the sequence of finite sign measures $dL^x_n$ on $[0,1]$ converges weakly to the finite sign measure $dL^x$ on $[0,1]$. Therefore, for bounded continuous function $f(X_s(x))$ on $[0,1]$, we have

$$\lim_{n \to \infty} \int_0^t f(X_s(x)) dL^x_n = \int_0^t f(X_s(x)) dL^x. \quad (3.5)$$

The proof of Proposition 3.1 follows from (3.4) and (3.5). $\square$

As a direct consequence of Proposition 3.1, we have the following.

**Proposition 3.2.** Assume that the smooth bounded open $\Omega$, the coefficients $\sigma$ and $b$ satisfy the same conditions as in Theorem 1.1. $(X_t(x), L^x_t)$ is a solution of Eq.(1.3). Then there exists a set $\tilde{\Omega} \in \mathcal{F}$ with $P(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$

$$\left| \int_0^t b(X_s(x)) ds \right|_{x=Z} = \int_0^t b(X_s(Z)) ds, \quad (3.6)$$

$$\left| \int_0^t f(X_s(Z)) d|L^Z|_s \right| = 0, \quad L^Z_t = \int_0^t \xi(X_s(Z)) d|L^Z|_s \quad (3.7)$$

for all $t \in [0,1]$, where $f$ is a continuous function defined on $\mathbb{R}^d$ with compact support included in $\Omega$ and $\xi(X_s(Z)) \in \mathfrak{n}(X_s(Z))$.

### 4 Moments estimates for one-point and two-point motions

For any $R > 0$ and $x \in [-R,R]^d \cap \bar{\Omega}$, let $(X_t(x), L^x_t)$ be a solution of Eq.(1.3). We define $S_n(t, x)$ and $I(t, x)$ by

$$S_n(t, x) := S_x(\sigma(X_s(x)), t),$$

$$I(t, x) := \int_0^t \sigma(X_s(x)) \circ dB_s$$

$$= \int_0^t \sigma(X_s(x)) dB_s + \frac{1}{2} \int_0^t (\nabla \cdot \sigma)(X_s(x)) ds.$$
Write

\[ S_\pi(t, x) = \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \left( \int_{t_k}^{t_{k+1}} \sigma(X_s(x)) \, ds \right) (B_{t_{k+1}} - B_{t_k}) \]

\[ = \sum_{k=0}^{n-1} \sigma(X_{t_k}(x)) (B_{t_{k+1}} - B_{t_k}) + \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \left( \int_{t_k}^{t_{k+1}} (\sigma(X_s(x)) - \sigma(X_{t_k}(x))) \, ds \right) (B_{t_{k+1}} - B_{t_k}) \]

(4.1)

By Ito’s formula and (1.4), for \( s \geq t_k \),

\[ \sigma_{ij}(X_s(x)) - \sigma_{ij}(X_{t_k}(x)) = \int_{t_k}^{s} (\nabla \sigma_{ij} \cdot \sigma)(X_u(x)) \, dB_u \]

\[ + \int_{t_k}^{s} (\nabla \sigma_{ij} \cdot b)(X_u(x)) \, du \]

\[ - \int_{t_k}^{s} (\nabla \sigma_{ij} \cdot \xi)(X_u(x)) \, d|L^x|_u \]

\[ + \frac{1}{2} \int_{t_k}^{s} \left\{ (\nabla \sigma_{ij} \cdot (\nabla \sigma \cdot \sigma))(X_u(x)) \right\} du \]

\[ + \frac{1}{2} \int_{t_k}^{s} \text{tr} \left\{ (\nabla^2 \sigma_{ij} \cdot \sigma \sigma^T)(X_u(x)) \right\} du. \]

So we informally write \( \sigma(X_s(x)) - \sigma(X_{t_k}(x)) \) as follows:

\[ \sigma(X_s(x)) - \sigma(X_{t_k}(x)) = \int_{t_k}^{s} (\nabla \sigma \cdot \sigma)(X_u(x)) \, dB_u \]

\[ + \int_{t_k}^{s} (\nabla \sigma \cdot b)(X_u(x)) \, du \]

\[ - \int_{t_k}^{s} (\nabla \sigma \cdot \xi)(X_u(x)) \, d|L^x|_u \]

\[ + \frac{1}{2} \int_{t_k}^{s} \left\{ (\nabla \sigma \cdot (\nabla \sigma \cdot \sigma))(X_u(x)) \right\} du \]

\[ + \frac{1}{2} \int_{t_k}^{s} \text{tr} \left\{ (\nabla^2 \sigma \cdot \sigma \sigma^T)(X_u(x)) \right\} du. \]

(4.2)

Thus we can write \( S_\pi(t, x) - I(t, x) \) as follows:

\[ S_\pi(t, x) - I(t, x) = \sum_{i=1}^{6} A_{ir}, \]

(4.3)
where
\[
A_{1\pi}(x) := \sum_{i=0}^{n-1} \sigma(X_{t_i}(x))(B_{t_{i+1}} - B_{t_i}) - \int_0^t \sigma(X_s(x))dB_s,
\]
\[
A_{2\pi}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^s (\nabla \sigma \cdot \sigma)(X_u(x))dB_u \right\}(B_{t_{i+1}} - B_{t_i})
\]
\[
- \frac{1}{2} \int_0^t (\nabla \sigma \cdot \sigma)(X_s(x))ds,
\]
\[
A_{3\pi}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^s (\nabla \xi \cdot \sigma)(X_u(x))d[L^x_x] \right\}(B_{t_{i+1}} - B_{t_i})
\]
\[
A_{4\pi}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^s (\nabla \sigma \cdot b)(X_u(x))du \right\}(B_{t_{i+1}} - B_{t_i})
\]
\[
A_{5\pi}(x) := \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^s (\nabla \sigma \cdot \nabla \sigma)(X_u(x))du \right\}(B_{t_{i+1}} - B_{t_i})
\]
\[
A_{6\pi}(x) := \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^s \text{tr}\{((\nabla^2 \sigma \cdot \sigma)^T)(X_u(x))\}du \right\}(B_{t_{i+1}} - B_{t_i}).
\]

**Proposition 4.1.** Assume that the smooth bounded open \( \mathcal{O} \), the coefficients \( \sigma \) and \( b \) satisfy the same conditions as in Theorem 1.1. \((X_t(x), L^x_t)\) is a solution of Eq.(1.3). Then for any \( p \geq 2 \) and \( R > 0 \) there exist constant \( c(p,R) \), which is independent of \( t \) and \( \pi \), and \( \beta_0 \in (0,1) \) such that
\[
\sup_{x \in [-R,R]^d \cap \mathcal{O}} E\{ |S_{\pi}(t,x) - I(t,x)|^{2p} \} \leq c(p,R)\|\pi\|^{\beta_0 p}.
\] (4.4)

**Proof.** By Burkholder-Davis-Gundy and Hölder inequalities, we have
\[
\{ E\{ |A_{1\pi}(x)|^{2p} \} \}^{\frac{1}{p}} \leq c(p) \left[ E\left( \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |(\sigma(X_s(x)) - \sigma(X_{t_i}(x)))|^2 ds \right) \right]^{\frac{1}{p}}
\]
\[
\leq c(p) \sum_{i=0}^{n-1} \left\{ E\left( \int_{t_i}^{t_{i+1}} |(\sigma(X_s(x)) - \sigma(X_{t_i}(x)))|^2 ds \right)^p \right\}^{\frac{1}{p}}
\]
\[
\leq c(p) \sum_{i=0}^{n-1} (t_{i+1} - t_i)^{-\frac{1}{p}} \left[ \int_{t_i}^{t_{i+1}} E\|\sigma(X_s(x)) - \sigma(X_{t_i}(x))\|^{2p} ds \right]^{\frac{1}{p}}
\]
\[
\leq c(p)\|\pi\|^{\frac{\beta_0}{2}},
\]
where we have used Proposition 2.2 and the condition (1.7). Thus
\[
E\{ |A_{1\pi}(x)|^{2p} \} \leq c\|\pi\|^{\frac{\beta_0}{2}}.
\] (4.5)

Using Fubini Theorem, \( A_{2\pi} \) can be further written as
\[
A_{2\pi}(x) = A_{2\pi}^{(1)}(x) + A_{2\pi}^{(2)}(x) + A_{2\pi}^{(3)}(x),
\] (4.6)
where
\[
A^{(1)}_{2\pi}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (t_{i+1} - u)((\nabla \sigma \cdot \sigma)(X_u(x)) - (\nabla \sigma \cdot \sigma)(X_{t_i}(x))) du,
\]
\[
A^{(2)}_{2\pi}(x) := -\frac{1}{2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ((\nabla \sigma \cdot \sigma)(X_u(x)) - (\nabla \sigma \cdot \sigma)(X_{t_i}(x))) du,
\]
\[
A^{(3)}_{2\pi}(x) := \sum_{i=0}^{n-1} \left\{ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (t_{i+1} - u)(\nabla \sigma \cdot \sigma)(X_u(x)) dB_u(B_{t_{i+1}} - B_{t_i}) \right. \\
\left. - \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (t_{i+1} - u)(\nabla \sigma \cdot \sigma)(X_u(x)) du \right\}
\]
\[
:= \sum_{i=0}^{n-1} (A_i + B_i).
\]

It follows from (1.7) and Proposition 2.2 that
\[
\{ E\{ |A^{(1)}_{2\pi}(x)|^p \} \}^{\frac{1}{p}} \leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \{ E\{ (|\nabla \sigma \cdot \sigma|(X_u(x)) - (\nabla \sigma \cdot \sigma)(X_{t_i}(x)))^p \} \}^{\frac{1}{p}} du
\]
\[
\leq c \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \{ E\{ |X_u(x) - X_{t_i}(x)|^p \} \}^{\frac{1}{p}} du
\]
\[
\leq c \| \pi \|^{\frac{1}{p}}. \tag{4.7}
\]

Similar arguments lead to
\[
\{ E\{ |A^{(2)}_{2\pi}(x)|^p \} \}^{\frac{1}{p}} \leq c \| \pi \|^{\frac{1}{p}}. \tag{4.8}
\]

Since $A^{(3)}_{2\pi}$ is a martingale and $\nabla \sigma \cdot \sigma$ is bounded on $\Omega$, using Burkholder-Davis-Gundy and Hölder inequalities, we obtain that
\[
\{ E\{ |A^{(3)}_{2\pi}(x)|^{2p} \} \}^{\frac{1}{p}} \leq c \left\{ \sum_{i=0}^{n-1} |A_i + B_i|^2 \right\}^{\frac{1}{p}}
\]
\[
\leq c \sum_{i=0}^{n-1} (E|A_i|^{2p})^{\frac{1}{p}} + c \sum_{i=0}^{n-1} (E|B_i|^{2p})^{\frac{1}{p}}
\]
\[
= c(p) \sum_{i=0}^{n-1} \left\{ (E\left[ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (t_{i+1} - u)(\nabla \sigma \cdot \sigma)(X_u(x)) dB_u \right) \right.
\times (B_{t_{i+1}} - B_{t_i})^{2p})^{\frac{1}{p}}
\left. + (E\left[ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (t_{i+1} - u)(\nabla \sigma \cdot \sigma)(X_u(x)) du \right]^{2p})^{\frac{1}{p}} \right\}
\leq c(p) \| \pi \|. \tag{4.9}
\]
So we deduce from (4.7)-(4.9) that
\[
\mathbb{E}\{|A_{2\pi}(x)|^{2p}\} \leq c\|\pi\|^p. \tag{4.10}
\]

By Propositions 2.1 and 2.2, it follows from Kolmogorov’s continuity criterion (see Theorem 1.4.1 in [10]) that there exist a random variable $K$ with $\mathbb{E}|K(\omega)|^p < +\infty$ and a positive constant $\beta \in (0, 1)$ such that
\[
\sup_{x \in [-R, R]^d \cap \tilde{O}} |L_t^x - L_s^x| \leq K(\omega)|t - s|^\beta. \tag{4.11}
\]

\[
\mathbb{E}\{|A_{3\pi}(x)|^{2p}\} \leq \mathbb{E}\left(\sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \int_t^s (\nabla \sigma \cdot \xi)(X_u(x))d|L_x^x|^{2p}\right)
\times \left(\sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^{2p}\right)
\leq c\mathbb{E}\left(\sum_{i=0}^{n-1} (|L_x^x|_{t_{i+1}} - |L_x^x|_{t_i})^{2p}\right) \times \left(\sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^{2p}\right)
\leq c\mathbb{E}\left(\left(\sup_{i} |L_x^x|_{t_{i+1}} - L_x^x|_{t_i}\right)^{3p}\right) \times \left(\sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^{2p}\right)
\leq c(\mathbb{E}\left(\left(\sup_{i} |L_x^x|_{t_{i+1}} - L_x^x|_{t_i}\right)^{3p}\right))^{\frac{2}{3}} (\mathbb{E}\left(\left|L_x^x|_{t_i}\right|^{3p}\right))^{\frac{1}{3}}
\times \left(\sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^{2p}\right)
\leq c\|\pi\|^{2p} \tag{4.12}
\]
due to the inequality (4.11) and $\nabla \sigma$ is bounded on $\tilde{O}$. For $p \geq 1$, by Hölder inequality and $\nabla^2 \sigma \cdot \sigma \cdot \sigma^T$ is bounded on $\tilde{O},$

\[
\mathbb{E}\{|A_{6\pi}(x)|^{2p}\}
\leq \mathbb{E}\left(\sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \int_t^s \text{Tr}\left\{(\nabla^2 \sigma \cdot \sigma \cdot \sigma^T)(X_u(x))\right\}d|u|^2\right)\]
\times \left(\sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^{2p}\right)
\leq c\|\pi\|^p \sup_{\pi} \mathbb{E}\left\{\left(\sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^{2p}\right)\right\}
\leq c\|\pi\|^p. \tag{4.13}
\]

Similarly,
\[
\mathbb{E}\{|A_{i\pi}(x)|^{2p}\} \leq c\|\pi\|^p, \quad \text{for } i = 4, 5. \tag{4.14}
\]
Putting the above estimates (4.5), (4.10) and (4.12)-(4.14) for \( A_i \pi \) \((i = 1, \cdots, 6)\) together, we deduce that
\[
\sup_{x \in [-R, R]^d \cap \mathcal{O}} \mathbb{E}\{ |S_{\pi}(t, x) - I(t, x)|^{2p} \} \leq c(p, R)\|\pi\|^\beta_0 p,
\]
where \( \beta_0 = \min\{1, \beta\} \). The proof of Proposition 4.1 is complete. \( \square \)

Next result is the moment estimates for the two point motions.

**Proposition 4.2.** Assume that the smooth bounded open \( \mathcal{O} \), the coefficients \( \sigma \) and \( b \) satisfy the same conditions as in Theorem 1.1. \( (X_t(x), L_t^x) \) is a solution of Eq.(1.3). Then for any \( p \geq 2 \) and \( R > 0 \) there exists constant \( c(p, R) \), which is independent of \( t \) and \( \pi \), such that
\[
\mathbb{E}\{ \sup_{t \in [0, 1]} |S_{\pi}(t, x) - S_{\pi}(t, y)|^p \} \leq c(p, R)|x - y|^p, \tag{4.15}
\]
for all \( x, y \in [-R, R]^d \cap \mathcal{O} \).

**Proof.** Similarly as (4.3),
\[
S_{\pi}(t, x) - S_{\pi}(t, y) = \sum_{i=1}^6 A_{i\pi}(x, y), \tag{4.16}
\]
where
\[
A_{1\pi}(x, y) := \sum_{i=0}^{n-1} \mathbb{E}\left\{ \left[ \sigma(X_{t_i}(x)) - \sigma(X_{t_i}(y)) \right] (B_{t_{i+1}} - B_{t_i}) \right\},
\]
\[
A_{2\pi}(x, y) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_s^{t_i} \left[ (\nabla \sigma \cdot \sigma)(X_u(x)) - (\nabla \sigma \cdot \sigma)(X_u(y)) \right] dB_u \right\} (B_{t_{i+1}} - B_{t_i}),
\]
\[
A_{3\pi}(x, y) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_s^{t_i} (\nabla \sigma \cdot \xi)(X_u(x)) d|L^x|_u - \int_s^{t_i} (\nabla \sigma \cdot \xi)(X_u(y)) d|L^y|_u \right\} (B_{t_{i+1}} - B_{t_i}),
\]
\[
A_{4\pi}(x, y) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_s^{t_i} \left[ (\nabla \sigma \cdot b)(X_u(x)) - (\nabla \sigma \cdot b)(X_u(y)) \right] du \right\} (B_{t_{i+1}} - B_{t_i}).
\]
Let \( |L^x - L^y|_t \) denote the total variation of \( L^x - L^y \) on \([0, t]\) for any \( t \in [0, 1] \), by (1.4), we have for any \( s \in [t_i, t_{i+1}] \)

\[
\begin{align*}
&\left| \int_{t_i}^{s} (\nabla \sigma \cdot \xi)(X_u(x))d|L^x|_u - \int_{t_i}^{s} (\nabla \sigma \cdot \xi)(X_u(y))d|L^y|_u \right|^2 \\
= &\left| \int_{t_i}^{s} (\nabla \sigma)(X_u(x))dL^x_u - \int_{t_i}^{s} (\nabla \sigma)(X_u(y))dL^y_u \right|^2 \\
\leq & c \left\{ |X_s(x) - X_s(y)| \right\} \{ |L^x|_{t_{i+1}} - |L^x|_{t_i} \} + c\{ |L^x - L^y|_{t_{i+1}} - |L^x - L^y|_{t_i} \}.
\end{align*}
\]

So

\[
\begin{align*}
(A_{3\pi}(x, y))^2 \leq & c \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \left\{ \left| \int_{t_i}^{s} (\nabla \sigma \cdot \xi)(X_u(x))d|L^x|_u \right| - \int_{t_i}^{s} (\nabla \sigma \cdot \xi)(X_u(y))d|L^y|_u \right\}ds \times \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^2 \\
\leq & \left[ 2c^2 \sup_{s \in [0, 1]} \{ |X_s(x) - X_s(y)| \} |L^x|_1^2 \right] \\
&+ 2c^2 \sup_{s \in [0, 1]} \{ |L^x_s - L^y_s| \} (|L^x|_1 + |L^y|_1) \\
&\times \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^2.
\end{align*}
\]
It follows from Propositions 2.1-2.3 and Hölder’s inequality that

\[
\begin{align*}
\mathbb{E}\{ \sup_{t \in [0,1]} |A_3 (x, y)|^{2p} \} & \leq c \left( \mathbb{E}\{ \sup_{0 \leq s \leq 1} |X_s (x) - X_s (y)|^{6p} \} \right)^{\frac{1}{6}} \\
& \times \left( \mathbb{E}\{ \left( \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^2 \right)^{3p} \} \right)^{\frac{1}{3}} \\
& + c \left( \mathbb{E}\{ \sup_{0 \leq s \leq 1} |L_s^x - L_s^y|^{3p} \} \right)^{\frac{1}{3}} \left( \mathbb{E}\{ |L_{t_{i+1}}^x|^{3p} + |L_{t_i}^y|^{3p} \} \right)^{\frac{1}{3}} \\
& \times \left( \mathbb{E}\{ \left( \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^2 \right)^{3p} \} \right)^{\frac{1}{3}} \\
& \leq c |x - y|^p.
\end{align*}
\]  

(4.17)

By Burkholder-Davis-Gundy inequalities, the condition (1.7) and Proposition 2.1, it follows easily that

\[
\mathbb{E}\{ \sup_{t \in [0,1]} |A_1 (x, y)|^p \} \leq C(p, R)|x - y|^p.
\]  

(4.18)

Using Burkholder-Davis-Gundy and Hölder’s inequalities, the condition (1.7) and Proposition 2.1,

\[
\left( \mathbb{E}\{ \sup_{t \in [0,1]} |A_2 (x, y)|^p \} \right)^{\frac{1}{p}} \leq \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \mathbb{E}\{ \left| \int_{t_i}^{s} \left[ (\nabla \sigma \cdot \sigma) (X_u (x)) \right. \right. \right. \\
- (\nabla \sigma \cdot \sigma) (X_u (y)) |dB_u \} (B_{t_{i+1}} - B_{t_i}) |^p \right\}^{\frac{1}{p}} \\
\leq \sum_{i=0}^{n-1} \left\{ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( \mathbb{E}\{ | \int_{t_i}^{s} \left[ (\nabla \sigma \cdot \sigma) (X_u (x)) \right. \right. \right. \\
- (\nabla \sigma \cdot \sigma) (X_u (y)) |dB_u |^{2p} \} \right)^{\frac{1}{2p}} \left( \mathbb{E}\{ |(B_{t_{i+1}} - B_{t_i})|^{2p} \} \right)^{\frac{1}{2p}} \right\}^{\frac{1}{p}} \\
\leq \sum_{i=0}^{n-1} \left\{ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( \int_{t_i}^{s} \mathbb{E}\{ |X_u (x) - X_u (y)|^{2p} \} \right)^{\frac{1}{2p}} du \right\}^{\frac{1}{2p}} \\
\times \int_{t_i}^{t_{i+1}} \mathbb{E}\{ |X_u (x) - X_u (y)|^{2p} \} \right\}^{\frac{1}{2p}} du \right\}^{\frac{1}{2p}} \\
\leq c(p)|x - y|.
\]  

(4.19)
Similarly,
\[
\left( \mathbb{E}\left\{ \sup_{t \in [0,1]} |A_{4\pi}(x,y) + A_{5\pi}(x,y)|^p \right\} \right)^{\frac{1}{p}} \leq c(p)|x-y|, \quad (4.20)
\]
\[
\left( \mathbb{E}\left\{ \sup_{t \in [0,1]} |A_{6\pi}(x,y)|^p \right\} \right)^{\frac{1}{p}} \leq c(p)|x-y|. \quad (4.21)
\]
Thus, combining above estimates (4.16)-(4.21) together, we complete the proof. \(\Box\)

The following result can be proved similarly as Proposition 4.2, but its proof is very easy, we omit it here.

**Proposition 4.3.** Assume that the smooth bounded open \(\mathcal{O}\), the coefficients \(\sigma\) and \(b\) satisfy the same conditions as in Theorem 1.1. \((X_t(x), L^x_t)\) is a solution of Eq.(1.3). Then for any \(p \geq 2\) and \(R > 0\) there exists constant \(c(p, R)\), which is independent of \(t\) and \(\pi\), such that
\[
\mathbb{E}\left\{ \sup_{x,y \in [0,1]} |I(t,x) - I(t,y)|^p \right\} \leq C(p, R)|x-y|^p, \quad (4.22)
\]
for all \(x, y \in [-R, R]^d \cap \overline{\mathcal{O}}\).

### 5 Uniform convergence of the Riemann sums

Let \(R > 0\) and \(p > 1\) be given. Define \(G_R := [-R, R]^d \cap \overline{\mathcal{O}}\). Then the following is a direct consequence of Garsia-Rodemich and Rumsey's Lemma (cf.\[22, 8, 4\]).

**Lemma 5.1.** Let \(f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n\) be a measurable stochastic field taking values in \(\mathbb{R}^n\) which is continuous, \(\mathbb{P}\)-a.s., \(p > 1\). Then there exists a constant \(c(p, R)\) such that
\[
\mathbb{E}\left\{ \sup_{x,y \in G_R} \rho(f(x), f(y))^p \right\} \leq c(p, R) \int \int_{G_R \times G_R} \mathbb{E}\left\{ \rho(f(x), f(y))^p | \rho(x, y)|^p \right\} I_{x \neq y} \ dx \ dy, \quad (5.1)
\]
where \((\mathbb{R}^m, d)\) and \((\mathbb{R}^n, \rho)\) are metric spaces.

Now we prove the main result of this section.

**Theorem 5.1.** Assume that the smooth bounded open \(\mathcal{O}\), the coefficients \(\sigma\) and \(b\) satisfy the same conditions as in Theorem 1.1. \((X_t(x), L^x_t)\) is a solution of Eq.(1.3). Then for any \(p \geq 2\) and \(R > 0\),
\[
\lim_{\|\pi\| \to 0} \mathbb{E}\left\{ \sup_{x \in G_R} |S_\pi(t,x) - I(t,x)|^{2p} \right\} = 0. \quad (5.2)
\]
Proof. Since sup_{x \in G_R} \{|f(x)|\} \leq sup_{x,y \in G_R} \{|f(x) - f(y)| + |f(x_0)| for any x_0 \in G_R and function f on \mathbb{R}^d, we have

\[ \mathbb{E}\left\{ \sup_{x \in G_R} |S_\pi(t, x) - I(t, x)|^{2p} \right\} \leq c(p) \mathbb{E}\left\{ \sup_{x,y \in G_R} |S_\pi(t, x) - S_\pi(t, y) - I(t, x) + I(t, y)|^{2p} \right\} + c(p) \mathbb{E}\left\{ |S_\pi(t, x) - I(t, x)|^{2p} \right\} \]

\[ \equiv B_{1\pi} + B_{2\pi}. \]  

(5.3)

By Lemma 5.1,

\[ B_{1\pi} \leq c(p, R) \int_{G_R \times G_R} \mathbb{E}\left\{ \frac{|S_\pi(t, x) - S_\pi(t, y) - I(t, x) + I(t, y)|^{2p}}{|x - y|^{2p}} \right\} I_{\{x \neq y\}} \, dxdy, \]  

(5.4)

By Propositions 4.1,

\[ \mathbb{E}\left\{ \frac{|S_\pi(t, x) - S_\pi(t, y) - I(t, x) + I(t, y)|^{2p}}{|x - y|^{2p}} \right\} I_{\{x \neq y\}} \leq \frac{c\|\pi\|^{2p}}{|x - y|^{2p}} \to 0 \]

as \|\pi\| \to 0.

Therefore, by Propositions 4.2, dominated convergence theorem and (5.4), we have

\[ B_{1\pi} \to 0, \text{ as } \pi \to 0. \]  

(5.5)

By Proposition 4.1,

\[ B_{2\pi} \to 0, \text{ as } \pi \to 0. \]  

(5.6)

Thus we complete the proof by (5.3), (5.5) and (5.6). \qed

6 Proof of Theorem 1.1

We will prove that \((X_t(Z), L_t^Z)\) solves the anticipating reflected SDE (1.8).

Since \((X_t(x), L_t^x)\) is a solution of Eq.(1.3), by Proposition 3.1, we need only to prove (1.11). By Theorem 5.1, the following holds almost surely on \{\omega; Z(\omega) \in G_M\}

\[ \int_0^t \sigma(X_s(x)) \circ dB_s|_{x=Z(\omega; Z(\omega) \in G_M)} = \lim_{\|\pi\| \to 0} S_\pi(t, x)|_{x=Z(\omega; Z(\omega) \in G_M)} \]
\[ = \int_0^t \sigma(X_s(Z)) \circ dB_s \chi_{\{\omega; Z(\omega) \in G_M\}}. \]

Letting \( M \rightarrow \infty \), we obtain the substitution formula (1.11), and therefore prove the Theorem. \( \square \)

**Remark 6.1.** By checking carefully the proof of Theorem 1.1 and using Theorem 3.1 proved by Lions and Sznitman (see [11]), the conditions on \( \mathcal{O} \) in Theorem 1.1 can be weaken, that is, if \( \mathcal{O} \) satisfies the admissibility condition (see [11], page 521) and the following condition: there exists a function \( \phi \) in \( C^2_b(\mathbb{R}^d) \) such that \( \exists \alpha > 0, \forall x \in \partial \mathcal{O}, \forall y \in \bar{\mathcal{O}}, \forall \xi \in \mathbf{n}(x) \Rightarrow \frac{1}{\alpha} (\nabla \phi(x), \xi)|y-x|^2 - (y-x, \xi) \leq 0 \), Theorem 1.1 also holds.

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