Yang-Mills and Supersymmetry Covariance Must Coexist

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ABSTRACT
Supersymmetry and Yang-Mills type gauge invariance are two of the essential properties of most, and possibly the most important models in fundamental physics. Supersymmetry is nearly trivial to prove in the (traditionally gauge-noncovariant) superfield formalism, whereas the gauge-covariant formalism makes gauge invariance manifest. In 3+1-dimensions, the transformation from one into the other is elementary and essentially unique. By contrast, this transformation turns out to be impossible in the most general 1+1-dimensional case. In fact, only the (manifestly) gauge- and supersymmetry-covariant formalism guarantees both universal gauge-invariance and supersymmetry.

There are cracks in every building;
that's how the light comes in.
— L. Cohen

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1. Introduction

Both global (rigid) and local (gauged) symmetry play an important rôle in physics. Local ‘internal’ (Yang-Mills gauge) symmetry provides for understanding of all fundamental interactions except gravity, while the supersymmetric extension of spacetime (‘external’) symmetries provides for the so essential stability under quantum fluctuations, and the only known link between ‘internal’ and ‘external’ symmetries. It is then vexing to find most of the existing literature fail to treat these two essential symmetries in a simultaneously covariant fashion [1,2,3].

The purpose of the present article is then to explore this, first briefly in the familiar $N=1$ supersymmetric 3+1-dimensional spacetime, and then more extensively in the (2,2)-supersymmetric 1+1-dimensional one. In both cases, the ‘misalignment’ between Yang-Mills and supersymmetry covariance provides a geometrical interpretation—indeed, a rigorous definition—of the prepotential superfields [1,2,3]. In the familiar $N=1$ supersymmetric 3+1-dimensional spacetime, the (de)covariantizing transformation between the two formalisms is well known and essentially unique [1,2]. By contrast, and perhaps not surprisingly [4,5,6], in the (2,2)-supersymmetric 1+1-dimensional case, we find two distinct (de)covariantizing transformations which toggle between the gauge-covariant and the more traditional simple (i.e., gauge-noncovariant) framework 1). One of them simplifies matters about gauge-covariantly (anti)chiral superfields but complicates dealing with non-minimal gauge-covariantly twisted-(anti)chiral superfields, while the other achieves precisely the opposite. Both decovariantizing transformations, however, complicate matters regarding gauge-covariantly unidexterous (lefton and righton) and all but one of the non-minimally (NM-)haploid superfields [1,2,3,7]. Therefore, it is impossible to turn all gauge-covariantly haploid superfields simultaneously into simply haploid superfields.

This article is organized as follows: Section 2 recalls the notion, notation and use of gauge-covariant superderivatives, and defines the various gauge superfields. The familiar case of $N=1$ supersymmetric 3+1-dimensional spacetime is briefly reviewed in Section 3, which also presents a swift and elementary derivation of the geometrical origin of the gauge prepotential superfield, complementing the literature [3]; see also §3.6.3 of Ref. [2]. Section 4 then explores this in (2,2)-supersymmetric 1+1-dimensional spacetime. In particular, we find it impossible to ensure universal gauge-invariance for every model constructed in the simple (gauge-noncovariant) formalism by the ‘standard’ judicious insertion of $e^{2V}$-like terms [2,4]. By contrast, it is straightforward to gauge-covariantize (and so render gauge-invariant) all of them by gauge-covariantizing all (super)derivatives [1].

2. Gauge-Covariant Superderivatives

Local (gauge) symmetry affects parallel transport and so changes all (super)derivatives into gauge-covariant (super)derivatives. This in turn modifies the supersymmetry algebra, and is the starting point for our analysis following §4.2.b of [1] and §3.6.4 of [2].

1) Throughout, ‘simple’ (‘simply’) will be used to mean ‘gauge-noncovariant(ly)’, as opposed to ‘gauge-covariantly’. Also, supersymmetry will be treated in a manifestly covariant fashion.
2.1. Universal (super)derivatives

Adjusting the notation so as to conform with Refs. [9,10], we follow Ref. [1] and start by defining the covariant (super)derivatives

\[
\nabla_\alpha = D_\alpha - i \Gamma_\alpha , \quad \bar{\nabla}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} - i \bar{\Gamma}_{\dot{\alpha}} , \quad \nabla_m = \partial_m - i \Gamma_m ,
\]

(2.1)

where \(\alpha (\dot{\alpha})\) label the components of the (co-)spinors, and \(m\) those of the vectors of the appropriate Lorentz symmetry group. The \(\nabla\)'s representing the geometrical concept of parallel transport, their definition (2.1) must be universal. That is, all (super)fields transforming under the same gauge symmetry must couple to the same gauge (super)fields appearing in the definitions (2.1), and in the same way. Equivalently, the coupling of gauge (super)fields to matter (super)fields only depends on the charges of the latter. The use of the term ‘universal gauge-covariance’ is herein meant to emphasize this universality in coupling of gauge and matter (super)fields.

The gauge (connexion form coefficient) superfields \(\Gamma, \bar{\Gamma}\) are gauge algebra valued. For nonsimple gauge groups, \(G = \prod_I G_I\), the \(\Gamma\)'s are linear combinations of gauge superfields each of which is valued in the gauge algebra of only one factor, \(G_I\). Gauge coupling parameters are suppressed and are easily reinserted by replacing \(\Gamma \rightarrow g \Gamma\).

Recall that the superderivatives already include the \(\frac{1}{2}\)-form connexions’, \(\partial_m - i \bar{\theta}^m \sigma_{a\dot{a}} \partial_m\) and \(+ i \theta^m \sigma^m_{a\dot{a}} \partial_m\), valued in the space-time translation group generators, \(\partial_m\). The (fermionic) ‘gauge superfields’ \(- i \bar{\theta}^m \sigma^m_{a\dot{a}}\) and \(+ i \theta^m \sigma^m_{a\dot{a}}\), however being constant in spacetime, supersymmetry is rigid (global).

2.2. Gauge transformations

Gauge transformations act linearly on ‘matter’ superfields

\[
X' = G X ,
\]

(2.3a)

implemented by the unitary operator superfield \(G\):

\[
X' = X \mathcal{G}^\dagger = X \mathcal{G}^{-1} .
\]

(2.3b)

It is with respect to this gauge transformation that the (super)derivatives (2.1) are covariant, so that (suppressing spacetime indices):

\[
\nabla' = G \nabla G^{-1} .
\]

(2.4)

Being a gauge group element, \(G\) can be written as

\[
G = e^{ie} , \quad \mathcal{E} \overset{\text{def}}{=} e' T_i , \quad [T_j, T_k] = i f_{jk}^l T_l ,
\]

(2.5)

where \(T_i\) are the (hermitian) generators of the gauge group with structure constants \(f_{jk}^l\). \(\mathcal{E}\) is gauge algebra valued and must be Hermitian for \(G\) to be unitary.
The transposition in \((2.3)\) makes the action of gauge-covariant derivatives on \(X\) awkward: the derivative part of \(\nabla\) should act from the left as usual, but the gauge superfield (connexion form) part should act from the right. We therefore calculate (implicitly) using a double transposition \([2]\):
\[
(\mathcal{O} X) \overset{\text{def}}{=} (O^T X^T)^T = (\bar{O} X)^\dagger ,
\]
where \(\mathcal{O}\) denotes any gauge-covariant operator, \(O^T\) its transpose, and \(\bar{O} \equiv O^\dagger\) its Hermitian conjugate. In practice, and in cases when above ‘matrix’ notation would be ambiguous or confusing, we resort to the explicit gauge group index notation. With the matter fields forming a representation of the gauge group the elements of which are indexed by \(\alpha, \beta, \ldots\), Eqs. (2.3)–(2.4) and (2.6) become:
\[
X'_{\alpha} = G_{\alpha}^{\beta} X_{\beta}, \quad \bar{X}'_{\alpha} = \bar{X}_{\beta} G_{\beta}^{\dagger \alpha} = \bar{X}_{\beta} G^{-1}_{\beta}^{\alpha},
\]
\[
\nabla'_{\alpha} = G_{\gamma}^{\alpha} \nabla_{\gamma} [G^{-1}]_{\beta}^{\alpha},
\]
and
\[
(\mathcal{O} \bar{X})_{\alpha} \overset{\text{def}}{=} (O_{\alpha}^{\beta} \bar{X}_{\beta}) = (\bar{O}_{\beta}^{\alpha} X_{\beta})^\dagger ,
\]
respectively. This disentangles ordering issues and the ‘matrix’ action of the gauge fields on the matter fields: re-ordering now solely depends on the spin/statistics of the involved superfields and operators. In this article, explicit gauge group indices will be suppressed, hoping that the Reader will always be able to discern the implied meaning of the more compact ‘matrix’ notation.

2.3. Gauge superfields

Under the gauge transformation \((2.4)\), the gauge superfields, \(\Gamma\), transform inhomogeneously:
\[
\Gamma' = G \Gamma G^{-1} - iG^{-1}(DG) .
\]
Suitable choices of \(G\) then produce a cancellation on the right hand side for some of the component fields of \(\Gamma\). All such (partially fixed) choices of \(G\) are generally called Wess-Zumino gauges, and depend on the desired number and structure of component field cancellations on the right hand side of \(\Gamma\).

In contrast, field strength superfields and torsion superfields, \(F\) and \(T\), are defined by (anti)commutation of the covariant derivatives \((2.1)\), according the master formula \(2)\)
\[
[\nabla, \nabla] = T \cdot \nabla - iF ,
\]
which determines \(F, T\) in terms of the gauge superfields \(\Gamma\) upon using Eqs. \((2.1)\). By virtue of their definition \((2.11)\), field strength superfields, \(F\), and torsion superfields, \(T\), are covariant: they transform homogeneously with respect to the gauge transformation

\[\text{2) Here } [\ , \ ] \text{ denotes the (anti)commutator, as appropriate for the (anti)commuted quantities.} \]
G. Therefore, the vanishing of some of the component fields of a torsion or field strength superfield, T or F, is a gauge-invariant statement.

This then provides the only way to unambiguously constrain the superfield content in a supersymmetric model with gauge symmetry: constraints to this end involve only torsion and field strength superfields, and so are gauge-invariant statements. Finally, the use of (gauge-covariantly constrained) superfields guarantees the maintenance of both supersymmetry and gauge-covariance.

3. Yang-Mills Symmetry in (3, 1|1)-Superspacetime

The gauge-covariant superderivatives (2.1) close a modified supersymmetry algebra which has been discussed in the literature [1,2]:

\[ \{ \nabla_\alpha, \nabla_\alpha \} = 2i \sigma^m_{\alpha \dot{\alpha}} \nabla_m , \quad (3.1a) \]
\[ \{ \nabla_\alpha, \nabla_\beta \} = 0 = \{ \nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}} \}, \quad (3.1b) \]

stating, respectively, that \( T_{\alpha \dot{\alpha}} = 2i \sigma^m_{\alpha \dot{\alpha}} \) and \( F_{\alpha \dot{\alpha}} = 0 \), and that \( T_{\alpha \dot{\alpha}} = 0 = F_{\alpha \beta} \). The vanishing of the field strength, \( F_{\alpha \dot{\alpha}} \), on the right hand side of (3.1a) ensures no duplication of gauge fields per gauge transformation by enforcing

\[ \Gamma_m = i \sigma^m_{\alpha \dot{\alpha}} \left( \{ D_{\alpha}, \Gamma_{\dot{\alpha}} \} + \{ \Gamma_{\alpha}, D_{\dot{\alpha}} \} - i \{ \Gamma_{\alpha}, \Gamma_{\dot{\alpha}} \} \right). \quad (3.2) \]

The vanishing of (3.1d) ensures that gauge-covariantly chiral superfields (see below) can couple to the gauge superfields \( \Gamma \). Jacobi identities then further ensure that:

\[ [\nabla_{\dot{\alpha}}, \nabla_{\beta}] = \epsilon_{\dot{\alpha} \dot{\beta}} W_\beta , \quad \text{where} \quad \nabla_{\alpha \dot{\alpha}} \equiv \sigma^m_{\alpha \dot{\alpha}} \nabla_m , \quad (3.3) \]
\[ \nabla_{\dot{\alpha}} W_\alpha = 0 , \quad \text{and} \quad \nabla^\alpha W_\alpha + \nabla^i \nabla_{\dot{\alpha}} W_\alpha = 0 , \quad (3.4) \]
\[ F_{\alpha \dot{\alpha}, \beta \dot{\beta}} \equiv i [\nabla_{\alpha \dot{\alpha}}, \nabla_{\beta \dot{\beta}}] = (\epsilon_{\alpha \beta} \nabla_{(\dot{\alpha}} W_{\dot{\beta})} + \epsilon_{\dot{\alpha} \dot{\beta}} \nabla_{(\alpha} W_{\beta)}) , \quad (3.5) \]

where \( F_{\alpha \dot{\alpha}, \beta \dot{\beta}} = \sigma^m_{\alpha \dot{\alpha}} \sigma^n_{\beta \dot{\beta}} F_{mn} \), so that the gauge-covariantly chiral spin-\( \frac{1}{2} \) superfields \( W_\alpha, \nabla_{\dot{\alpha}} \) encode all the gauge field strengths.

Gauge-covariantly chiral and antichiral superfields are then defined to satisfy

\[ \nabla_{\alpha} \Phi = 0 , \quad \nabla_{\dot{\alpha}} \Phi = 0 , \quad (3.6) \]

whereas gauge-covariantly complex linear superfields and their conjugates satisfy

\[ \nabla^i \nabla_{\dot{\alpha}} \Theta = 0 , \quad \nabla^\alpha \nabla_{\alpha} \bar{\Theta} = 0 , \quad (3.7) \]

respectively. From Eq. (3.6), it easily follows that

\[ 0 = \{ \nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}} \} \Phi = T_{\alpha \beta} \nabla \Phi + F_{\dot{\alpha} \dot{\beta}} \Phi . \quad (3.8) \]

This implies that \( F_{\dot{\alpha} \dot{\beta}} = 0 \) or \( \Phi \) would have to be chargeless. Also, \( T^{\alpha}_{\dot{\alpha} \dot{\beta}} = 0 \) and \( T^{m}_{\dot{\alpha} \dot{\beta}} = 0 \) so \( \Phi \) would not be forced to be a constant. Finally, the component fields of \( T_{\dot{\alpha} \dot{\beta} \gamma} \equiv T_{(\dot{\alpha} \dot{\beta}) \gamma} \) include spin-\( \frac{3}{2} \) fields (the traceless part) and the appearance of \( \nabla \Theta \) on the right hand side of the (3.8), as a special case of (2.11), would induce a further modification of Eqs. (2.1); \( T_{(\dot{\alpha} \dot{\beta}) \gamma} \) is therefore also set to zero, justifying (3.1).
3.1. Misalignment of gauge- and super-symmetry

In the gauge-covariant framework, the fermionic integration is expressed as an appropriate gauge-covariant superderivative, followed by the \( \theta, \bar{\theta} \to 0 \) projection. For example \([6]\),

\[
\text{D-term: } \int d^4 \theta \ K = \frac{1}{8} \text{Tr} \left[ \{ \nabla^\alpha \nabla_\alpha, \nabla^{\dot{\alpha}} \nabla_{\dot{\alpha}} \} K \right], \tag{3.9}
\]

and in similar vein:

\[
\text{F-term: } \int d^2 \theta \ W = \frac{1}{2} \text{Tr} \left[ \nabla^\alpha \nabla_\alpha W \right], \tag{3.10}
\]

where ‘\(|\)’ denotes setting \( \theta, \bar{\theta} = 0 \). The gauge-covariance \((2.3)-(2.4)\) of the constraints \((3.6)\) and \((3.7)\) makes the expansion of fermionic integral like \((3.9)\) and \((3.10)\) straightforward, if perhaps tedious to evaluate.

Proving supersymmetry of the fermionic integrals \((3.9)\) and \((3.10)\) is practically trivial. The (gauge-covariantized) supersymmetry transformation operator may be rewritten as

\[
\delta_{\epsilon, \bar{\epsilon}} \overset{\text{def}}{=} \epsilon \cdot Q + \bar{\epsilon} \cdot \bar{Q} = \epsilon \int d\theta + \bar{\epsilon} \int d\bar{\theta} + 4\xi^m \partial_m + \eta^i (\epsilon, \bar{\epsilon}; \Gamma, \bar{\Gamma}) T_i, \tag{3.11}
\]

where \(\xi^m = \Im (\epsilon \cdot \sigma^m \cdot \bar{\theta})\). The quantities \(\eta^i\) are linear combinations of \(\epsilon, \bar{\epsilon}\) and functions of \(\Gamma, \bar{\Gamma}\), the exact form and value of which depend on the precise choice of \(Q, \bar{Q}\). Notice that all the Lagrangian densities, such as \((3.9)\) and \((3.10)\), are gauge-invariant, and so must be annihilated by the gauge group generators, \(T_i\). Being \(\theta, \bar{\theta}\)-independent, they are also annihilated by the Berezin superintegration operators.

Therefore, the supersymmetry transformations of the Lagrangian densities \((3.9)\) and \((3.10)\) must be proportional, respectively, to the total derivatives

\[
\partial_m \left[ \{ \nabla^\alpha \nabla_\alpha, \nabla^{\dot{\alpha}} \nabla_{\dot{\alpha}} \} \xi^m \right] \quad \text{and} \quad \partial_m \left[ \nabla^\alpha \nabla_\alpha \xi^m W \right]. \tag{3.12}
\]

Finally: the spacetime integrals of these expressions (the supersymmetry variations of the corresponding terms in the action) vanish owing to the usual assumptions of sufficiently rapid vanishing at ‘infinity’ of all involved fields in open spacetimes, and by virtue of Stokes’ theorems in closed spacetimes.

3.2. The decovariantizing transformation

Since the gauge-covariant superderivatives \((2.1)\) transform covariantly \((2.4)\), there is of course no gauge in which the \(\nabla, \bar{\nabla}\)’s would become ‘ordinary’ superderivatives, \(D, D\). More precisely, no unitary operator can possibly transform the \(\nabla, \bar{\nabla}\)’s into the \(D, D\)’s.

There do exist, however, nonunitary operators \(\mathcal{H}\) such that

\[
\nabla_\alpha = \mathcal{H}^{-1} D_\alpha \mathcal{H}, \quad \text{and} \quad \bar{\nabla}_{\dot{\alpha}} = (\nabla_{\alpha})^\dagger = \mathcal{H} D_{\alpha} \mathcal{H}^{-1}. \tag{3.13}
\]

\(3)\ The gauge-covariantization, \(Q, \bar{Q}\), of the supercharges, \(Q, \bar{Q}\), is constrained by the requiring that \(\{ \nabla_\alpha, Q_\beta \} = 0 = \{ \bar{\nabla}_{\dot{\alpha}}, Q_{\dot{\beta}} \}\), and \(\{ Q_\alpha, \bar{Q}_{\dot{\alpha}} \} = -2i\sigma^m_{\alpha \dot{\alpha}} \nabla_m\). Therefore, if \(\Gamma, \bar{\Gamma}\) gauge (non)-abelian symmetries, the \(\eta^i\) will be (non)linear functions of \(\Gamma, \bar{\Gamma}\).
The expansion (2.1) provides the identification
\[ \Gamma_\alpha = +i\mathcal{H}^{-1}(D_\alpha \mathcal{H}) , \quad \Gamma_{\dot{\alpha}} = -i(D_{\dot{\alpha}} \mathcal{H})\mathcal{H}^{-1} . \] (3.14)
Through Eq. (3.2), this also determines \( \Gamma_m \). Therefore, all gauge superfields are expressible in terms of \( \mathcal{H} \) and its Hermitian conjugate. Being non-unitary, \( \mathcal{H} \) is an element of the complexified gauge group, \( \mathcal{G}^c \). Written as an exponential, \( \mathcal{H} \) has a complex exponent, the antihermitian part of which can be removed by a compensating gauge transformation, so that, in this ‘Hermitizing’ gauge,
\[ \mathcal{H} \rightarrow \mathcal{H}^g = e^V , \quad V^\dagger = V . \] (3.15)
Therefore, \( V \) generates the coset \( \mathcal{G}^c / \mathcal{G} \), locally at the particular (unitary, ‘Hermitizing’) gauge transformation which turned \( \mathcal{H} \) Hermitian.

According to Eq. (2.11), the gauge-covariant (dynamical and physically relevant!) field strengths are, through the substitution (3.13), determined as second (super)derivatives of \( \mathcal{H}, \mathcal{R} \) and their inverses. It then follows that the lowest and the next-to-lowest components of \( \mathcal{H}, \mathcal{R} \)—and so also \( \mathcal{V} \)—are unphysical, and are eliminated in suitable so-called Wess-Zumino gauge(s) \([1,9,2]\); see §2.3.

The non-unitary transformation of gauge-covariant derivatives (3.13) induces a corresponding transformation of gauge-covariantly constrained superfields. The chirality condition (3.6) becomes
\[ 0 = \nabla_\dot{\alpha} \Phi = \mathcal{H} \hat{D}_{\hat{\alpha}} \mathcal{H}^{-1} \Phi , \] (3.16)
which begs for the conjugate pair of definitions
\[ \check{\Phi} \overset{\text{def}}{=} \mathcal{H}^{-1} \Phi , \quad \check{\check{\Phi}} \overset{\text{def}}{=} \check{\Phi} \check{\mathcal{R}}^{-1} . \] (3.17)
Easily, \( \check{\Phi} \) and \( \check{\check{\Phi}} \) are chiral and antichiral with respect to the ‘ordinary’ superderivatives (2.2):
\[ 0 = \nabla_\dot{\alpha} \check{\Phi} = \mathcal{H} \hat{D}_{\hat{\alpha}} \check{\Phi} \Rightarrow \hat{D}_{\hat{\alpha}} \check{\Phi} = 0 , \] (3.18)
\[ 0 = \nabla_\alpha \check{\check{\Phi}}^\dagger = (\check{\mathcal{R}}^{-1} D_\alpha \check{\mathcal{R}})^\dagger \check{\check{\Phi}}^\dagger = \check{\mathcal{R}}^\dagger D_\alpha \check{\Phi}^\dagger \Rightarrow \hat{D}_\alpha \check{\Phi} = 0 , \]
Similarly, for gauge-covariantly complex linear superfields we define
\[ \check{\Theta} \overset{\text{def}}{=} \mathcal{H}^{-1} \Theta , \] (3.19)
where \( \check{\Theta} \) is complex linear with respect to the ‘ordinary’ superderivatives (2.2):
\[ 0 = \nabla^2 \check{\Theta} = \mathcal{H} \hat{D}^2 \check{\Theta} \Rightarrow \hat{D}^2 \check{\Theta} = 0 . \] (3.20)
The analogous is true of its Hermitian conjugate, \( \check{\Theta} \).

**Remark:** the condition \( \nabla_\dot{\alpha} = (\nabla_\alpha)^\dagger \), implicit in (2.4) and made explicit in (3.13), may be relaxed, so that \( \nabla_\dot{\alpha} \) (defining \( \check{\Phi} \)) and \( \nabla_\alpha \) (defining \( \check{\check{\Phi}} \); see below) are decovariantized independently and differently. This implies a separate decovariantizing transformation rule for the gauge-covariantly chiral and antichiral superfields, \( \Phi, \check{\Phi} \), and so leads to a group of transformations larger than \( \mathcal{G} \), or even \( \mathcal{G}^c \). The present discussion is straightforward albeit unnecessary to extend in this way, and we do not do so herein.
3.3. Lagrangian density

The decovariantizing transformation, of course, also changes Lagrangian density terms. For example,

\[ \text{Tr} \left[ \nabla^2 \nabla^2 \Phi \bar{\Phi} \right] = \text{Tr} \left[ \mathcal{H}^{-1} D^2 \mathcal{H} (\mathcal{H} D^2 \mathcal{H}^{-1})^\tau \mathcal{H} \Phi \bar{\Phi} \mathcal{H} \right], \]

\[ \rightarrow \text{Tr} \left[ \mathcal{H}^{-1} D^2 \mathcal{H}^{-1} D^2 \mathcal{H} \Phi \bar{\Phi} \mathcal{H} \right], \]

\[ = D^2 \mathcal{D}^2 \text{Tr} \left[ \bar{\Phi} e^{2V} \Phi \right]. \]  

(3.21)

Here the second line follows upon taking the Hermitizing gauge (3.15), and we used the cyclicity of the trace operation to obtain the last line. The same is true of the other half of the expression (3.9). In fact, the same result is obtained with a general D-term of the form \( \int d^4 \theta K \), as long as the real function \( K = K(\Phi, \bar{\Phi}, \Theta, \bar{\Theta}) \) transforms as \( K = \mathcal{H} \check{K} \mathcal{H} \), and where of course, \( \check{K} \) def = \( K(\check{\Phi}, \check{\bar{\Phi}}, \check{\Theta}, \check{\bar{\Theta}}) \). So, the (seemingly) ‘flat’ kinetic D-term in the gauge-covariant framework becomes the standard gauge-invariant kinetic D-term in the decovariantized framework and upon the Hermitizing gauge transformation (3.15).

This clearly identifies \( V \approx \log(\mathcal{H}) \), the local generator of the \( G^c/G \) coset, as the gauge prepotential superfield: Eqs. (3.14) become

\[ \Gamma_{\alpha} = +i(D_{\alpha} V), \quad \bar{\Gamma}_{\dot{\alpha}} = -i(D_{\dot{\alpha}} V). \]  

(3.22)

See Ref. [8] and § 3.6.3 of Ref. [2] for a derivation of these facts from the ‘opposite’ vantage point.

As well known, the F-term \( \text{Tr} \left[ \nabla^2 W \right] \) is gauge-invariant only if the superpotential \( W \) is. Then \( [T_i, W] = 0 \) and also \( [G, W] = 0 = [H, W] \), so that

\[ \text{Tr} \left[ \nabla^2 W \right] = \text{Tr} \left[ H^{-1} D^2 H W \right] = D^2 \text{Tr}[\check{W}], \]

(3.23)

using the cyclicity of the trace operation, and the gauge invariance of \( W = \check{W} \). Notice that the trace operator may well be omitted in the end expressions of (3.21) and (3.23).

In all cases, supersymmetry of the expressions in Eqs. (3.21) and (3.23) is straightforward, by virtue of the simple proof given in §3.1, involving Eqs. (3.11) and (3.12). Thus, the supersymmetry of the ‘gauge-decovariantized’ Lagrangian density terms (3.21) and (3.23) is still manifest, whereas gauge-covariance is marred by the use of the simple superderivatives (2.2), both in Berezin integration (3.21) and (3.23) and the definition of constrained superfields (3.17) and (3.19).

At present, in the \( N=1 \) supersymmetric 3+1-dimensional spacetime, this ‘misalignment’ may seem a matter of (in)convenience, although it does identify the gauge prepotential superfield, \( V \), as a local generator of the \( G^c/G \) coset.

4. Yang-Mills Symmetry in \( (1, 1 \mid 2, 2) \)-Superspacetime

Turn now to the \( (2, 2) \)-supersymmetric 1+1-dimensional spacetime. Let \( \sigma = \frac{1}{2}(\sigma^0 - \sigma^3) \) and \( \sigma^+ = \frac{1}{2}(\sigma^0 + \sigma^3) \) be the light-cone characteristic (bosonic) coordinates, and \( \zeta^\pm, \bar{\zeta}^\mp \) the
fermionic coordinates. The spinor indices \( \alpha, \dot{\alpha} = -, + \) in fact denote spin: \( \psi^- = \psi_+ \) has spin \(-\frac{1}{2}\)h, whereas \( \psi^+ = - \psi_- \) has spin \(+\frac{1}{2}\)h. Using \( \sigma^0 = -1 \) and the usual Pauli matrices for \( \sigma^1, \sigma^2, \sigma^3 \), Eqs. (2.2) become

\[
D_- \overset{\text{def}}{=} \partial_- - i \zeta^- \partial_- , \quad D_- \overset{\text{def}}{=} - \bar{\partial}_- + i \zeta^- \partial_- , \\
D_+ \overset{\text{def}}{=} \partial_+ - i \zeta^+ \partial_+ , \quad D_+ \overset{\text{def}}{=} - \bar{\partial}_+ + i \zeta^+ \partial_+ ,
\]

(4.1)

where \( \partial_{\pm} \overset{\text{def}}{=} \partial_{\sigma \mp} \) and \( \partial_{= \sigma} \overset{\text{def}}{=} \partial_{\sigma =} \). The remaining definitions of \( \delta \) carry over almost verbatim.

4.1. Gauge-covariant extension of supersymmetry

We now turn to determine the analogues of Eqs. (3.1)-(3.5). To begin with, we generalize the standard supersymmetry algebra by inserting the so far unrestricted field strength superfields (choosing numerical coefficients for later convenience) \( \delta \):

\[
\{ \nabla_-, \nabla_+ \} \overset{\text{def}}{=} \mathbf{A} , \quad \{ \nabla_-, \nabla_+ \} \overset{\text{def}}{=} \mathbf{A} , \\
\{ \nabla_- , \nabla_+ \} \overset{\text{def}}{=} \mathbf{B} , \quad \{ \nabla_- , \nabla_+ \} \overset{\text{def}}{=} \mathbf{B} , \\
\{ \nabla_- , \nabla_- \} \overset{\text{def}}{=} 2 \mathbf{C}_- , \quad \{ \nabla_+ , \nabla_+ \} \overset{\text{def}}{=} 2 \mathbf{C}_+ , \\
\{ \nabla_- , \nabla_- \} \overset{\text{def}}{=} 2 i \nabla_+ , \quad \{ \nabla_+ , \nabla_- \} \overset{\text{def}}{=} 2 i \nabla_-, \\
[ \nabla_-, \nabla_- ] \overset{\text{def}}{=} - \mathbf{W} , \quad [ \nabla_-, \nabla_- ] \overset{\text{def}}{=} - \mathbf{W} , \\
[ \nabla_+ , \nabla_- ] \overset{\text{def}}{=} - \mathbf{W} , \quad [ \nabla_+ , \nabla_- ] \overset{\text{def}}{=} - \mathbf{W} , \\
[ \nabla_- , \nabla_+] \overset{\text{def}}{=} + \mathbf{W} , \quad [ \nabla_- , \nabla_+] \overset{\text{def}}{=} + \mathbf{W} , \\
[ \nabla_+ , \nabla_+] \overset{\text{def}}{=} + \mathbf{W} , \quad [ \nabla_+ , \nabla_+] \overset{\text{def}}{=} + \mathbf{W} , \\
[ \nabla_- , \nabla_+] \overset{\text{def}}{=} - i \mathbf{F} , \quad [ \nabla_- , \nabla_+] \overset{\text{def}}{=} + 2 \mathbf{D} ,
\]

where \( \square \) is the gauge-covariant d’Alembertian (wave operator). Notice that the field strengths \( \mathbf{A}, \mathbf{B}, \mathbf{C}_-, \mathbf{C}_+ \) and the various \( \mathbf{W} \)’s are all complex \(^4\), the \( \mathbf{C} \)’s being defined by the conjugates of Eqs. (4.2d). No new field strength has been introduced in Eqs. (4.2), as it would merely redefine the gauge superfields \( \Gamma_-, \Gamma_+ \). Also, all torsion except \( T^- = 2i = T^+_++ \) vanishes, just as in the limit of vanishing gauge coupling. This is because the only derivative-valued connection 1-forms are those within the superderivatives, \( D_-, \bar{D}_-, \) and so the torsion content is unchanged by the covariantization (2.1).

Unlike the 3+1-dimensional case, various consistency conditions now allow four ‘pure’ types of Yang-Mills symmetry gauging \(^{3,5}\):

\(^4\) Being first order bosonic derivatives, the \( \nabla_-, \nabla_+ \) are antihermitean; the above definitions of the \( \mathbf{W} \)’s then ensure them to be the hermitean conjugates of the \( \mathbf{W} \)’s.

\(^5\) Only the first two of these have been studied in the literature \(^{3,4}\).
1. Type-A, where $A, \tilde{A} \neq 0$, but $B, \tilde{B}, C, \bar{C}, C_\pm, \bar{C}_\pm = 0$;
2. Type-B, where $B, \tilde{B} \neq 0$, but $A, \tilde{A}, C, \bar{C}, C_\pm, \bar{C}_\pm = 0$;
3. Type-C, where $C, \bar{C} \neq 0$, but $A, \tilde{A}, B, \tilde{B}, C_\pm, \bar{C}_\pm = 0$;
4. Type-C, where $C_\pm, \bar{C}_\pm \neq 0$, but $A, \tilde{A}, B, \tilde{B}, C, \bar{C} = 0$.

Equivalently, the gauge group is $G = G_A \times G_B \times G_C \times G_\pm$ and the different gauging types project on the respective factors in $G$.

Mixed types of symmetry gauging, where more than one conjugate pair among the $A, \tilde{A}, B, \tilde{B}, C, \bar{C}$’s is nonzero, are plagued by duplication of gauge fields per gauge symmetry [3], and are avoided following standard wisdom [1]. In all cases, however, the field strength superfields, $W, \tilde{W}$ and $F$ are completely determined in terms of the nonzero superfields among the $A, \tilde{A}, B, \tilde{B}, C, \bar{C}$ [3]. Throughout this note, only the ‘pure’ gauging types (listed as 1.–4. above) are considered.

4.2. Gauge-covariantly constrained superfields

Adapting from Ref. [7], we recall the definition of the minimal (first-order constrained) gauge-covariantly haploid superfields:

1. Chiral: \( (\nabla_+ \Phi) = 0 = (\nabla_- \Phi) \), \hspace{1cm} (4.3a)
2. Antichiral: \( (\nabla_+ \bar{\Phi}) = 0 = (\nabla_- \bar{\Phi}) \), \hspace{1cm} (4.3b)
3. Twisted-chiral: \( (\nabla_+ \bar{\Xi}) = 0 = (\nabla_- \bar{\Xi}) \), \hspace{1cm} (4.3c)
4. Twisted-antichiral: \( (\nabla_+ \bar{\Xi}) = 0 = (\nabla_- \bar{\Xi}) \), \hspace{1cm} (4.3d)
5. Lefton: \( (\nabla_- \Lambda) = 0 = (\nabla_- \Lambda) \), \hspace{1cm} (4.3e)
6. Righton: \( (\nabla_+ \Upsilon) = 0 = (\nabla_+ \Upsilon) \). \hspace{1cm} (4.3f)

Similarly,

1. NM-Chiral: \( ([\nabla_+, \nabla_-] \Theta) = 0 \), \hspace{1cm} (4.4a)
2. NM-Antichiral: \( ([\nabla_-, \nabla_+] \bar{\Theta}) = 0 \), \hspace{1cm} (4.4b)
3. NM-Twisted-chiral: \( ([\nabla_+, \nabla_-] \Pi) = 0 \), \hspace{1cm} (4.4c)
4. NM-Twisted-antichiral: \( ([\nabla_-, \nabla_+] \bar{\Pi}) = 0 \), \hspace{1cm} (4.4d)
5. NM-(Almost)-Lefton: \( ([\nabla_-, \nabla_-] A) = 0 \), \hspace{1cm} (4.4e)
6. NM-(Almost)-Righton: \( ([\nabla_+, \nabla_+] U) = 0 \), \hspace{1cm} (4.4f)

are the non-minimal (second-order constrained) gauge-covariantly haploid superfields.

It is now easy to prove that the minimal haploid superfields (1.3) couple, selectively, to only some of the gauge superfields [3,11]. For example [6], applying $\nabla_-$ and $\nabla_+$ to the first and the second equation in (1.3a) and adding the results produces:

\[
\begin{align*}
(\nabla_- \Phi) = 0 \\
(\nabla_+ \Phi) = 0 \\
\end{align*}
\Rightarrow 0 = (\nabla_- \nabla_+ \Phi) \equiv (B \Phi). \hspace{1cm} (4.5a)
\]
For both $B, \Phi$ to be nonzero, it must be the gauge group $G_B$ generators, in which the $B$’s take values, annihilate the gauge-covariantly (anti)chiral superfields. And, since the generators of a Lie group are Hermitian, the $B$’s are valued in the same generators as are the $B$’s, and we obtain:

$$ (B \Phi) = 0 = (\bar{B} \Phi) , \quad \text{and} \quad (B \Phi) = 0 = (\bar{B} \Phi) . \quad (4.5b) $$

That is, type-B gauge (super)fields cannot couple to gauge-covariantly (anti)chiral superfields, whose type-B charges therefore must be zero. A similar argument shows that $\Phi, \bar{\Phi}$ cannot couple to type-C gauge (super)fields either. There appears however no such restriction with regard to the coupling of $\Phi, \bar{\Phi}$ to type-A gauge (super)fields: $\Phi, \bar{\Phi}$ can have only type-A charges. Indeed, upon dimensional reduction from 3+1- to 1+1-dimensional spacetime, we identify $A = 2(\Gamma_2 + i \Gamma_3)$, and see that Eqs. (3.1) are consistent with Eqs. (4.2–d) upon taking this selectivity in coupling into account.

Following through in this fashion, each minimal gauge-covariantly haploid superfields (4.3) is found [3,11] to be able to couple to at least one type of gauge (super)fields, and that each type of gauge (super)fields can couple to at least one minimal gauge-covariantly haploid superfield $^6)$. This selectivity is charted in Table 1.

|        | $A, \bar{A}$ | $B, \bar{B}$ | $C_\pm, \bar{C}_\mp$ | $C_=, \bar{C}_=$ |
|--------|-------------|-------------|----------------------|------------------|
| $\Phi, \bar{\Phi}$ | $\checkmark$ | $-$ | $-$ | $-$ |
| $\Xi, \bar{\Xi}$ | $-$ | $\checkmark$ | $-$ | $-$ |
| $\Lambda, \bar{\Lambda}$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $-$ |
| $\Upsilon, \bar{\Upsilon}$ | $\checkmark$ | $\checkmark$ | $-$ | $\checkmark$ |

**Table 1:** The minimal coupling of gauge (super)fields (4.2) to gauge-covariantly haploid ‘matter’ superfields (4.3) is highly selective: the entry ‘$\checkmark$’ indicates that the minimal coupling type interaction is possible, and ‘$-$’ that it is impossible.

By contrast, all non-minimal gauge-covariantly haploid superfields (4.4) can couple indiscriminately to all gauge (super)fields; supersymmetry induces no integrability restrictions for them [3,11].

### 4.3. The chiral (type-A) decovariantization

Just as the gauge-covariantly (anti)chiral and the type-A gauge superfields descend from the 3+1-dimensional case through dimensional reduction, so does the decovariantizing transformation (3.13):

$$ \nabla_\pm = \mathcal{H}^{-1} D_\pm \mathcal{H} , \quad \text{and} \quad \nabla_\mp = (\nabla_\pm) ^\dagger = \mathcal{H} D_\mp \mathcal{H}^{-1} . \quad (4.6) $$

$^6) \text{However, only the } \Phi, \bar{\Phi} \text{ coupling to type-A gauge superfields may be regarded as stemming from dimensional reduction from 3+1-dimensional spacetime. Other haploid superfields and gauging types do not have an } N=1 \text{ supersymmetric 3+1-dimensional analogue.} $
In the ‘Hermitizing’ gauge, $\mathcal{H} \to e^V$, just as before:

$$\Gamma = i\mathcal{H}^{-1}(D\mathcal{H}) = i(D\alpha V),$$

and $\Gamma_m$ through Eq. (3.2), are all expressed in terms of $\mathcal{H}$ and its Hermitian conjugate.

The chirality condition (4.3a) becomes

$$0 = \nabla\Phi = \mathcal{H}D\Phi, \quad \Phi \equiv \mathcal{H}^{-1}\Phi,$$

and similarly for its conjugate, so $\check{\Phi}$ ($\check{\Phi}$) are (anti)chiral with respect to the ‘ordinary’ superderivatives (4.1). The gauge-covariantly NM-chiral superfields fare similarly:

$$0 = \nabla^2 \Theta \equiv \tfrac{1}{2} [\nabla_+, \nabla_-] \Theta = \mathcal{H}\nabla^2 \check{\Theta}, \quad \check{\Theta} \equiv \mathcal{H}^{-1}\Theta,$$

whereupon $\nabla^2 \check{\Theta} = 0$, and analogously for $\check{\Theta}$ ($\check{\Theta}$) are (anti)chiral with respect to the ‘ordinary’ superderivatives (4.1). The gauge-covariantly unidexterous haploid superfields (4.11) to their non-minimal counterparts (4.12)

$$0 = [\nabla_-, \nabla_-] A = [\mathcal{H}^{-1}D_\mathcal{H} \mathcal{H}D_\mathcal{H}^{-1} - \mathcal{H}D_\mathcal{H} \mathcal{H}^{-1} \mathcal{H}^{-1} D_\mathcal{H}] A,$$

and—owing to the indiscriminate coupling of nonminimal haploid superfields—neither to the non-minimal twisted-chiral superfields (4.11) to their non-minimal counterparts (4.12)

$$0 = [\nabla_+, \nabla_-] \Pi = [\mathcal{H}^{-1}D_\mathcal{H} \mathcal{H}D_\mathcal{H}^{-1} - \mathcal{H}D_\mathcal{H} \mathcal{H}^{-1} \mathcal{H}^{-1} D_\mathcal{H}] \Pi,$$

or their Hermitian or parity-conjugates. No superfield redefinition can bring either of these to their corresponding simple (gauge-noncovariant) form.

Since $\mathcal{H}$ is not unitary, $\mathcal{H}^{-1} \neq \mathcal{H}$, no superfield redefinition will turn $\Lambda$ into a simple lefton. For example:

$$\hat{\Lambda}^{(1)} \equiv \mathcal{H}^{-1}\Lambda \quad \text{satisfies:} \quad \begin{cases} (D_\mathcal{H} \hat{\Lambda}^{(1)}) = 0, \\ (D_- \hat{\Lambda}^{(1)}) = 0, \end{cases}$$

$$\hat{\Lambda}^{(2)} \equiv \mathcal{H} \Lambda \quad \text{satisfies:} \quad \begin{cases} (D_- \mathcal{H}^{-1} \Lambda^{(1)}) = 0, \\ (D_- \Lambda^{(1)}) = 0. \end{cases}$$
The fate of the (non-)minimal gauge-covariant (almost)rightons is precisely analogous, respectively, to their lefton counterparts (4.11), and (4.12).

Thus, the chiral decovariantizing transformation (4.6) does define a ‘decovariantization’ of the gauge-covariant twisted chiral superfield, ˚Ξ (satisfying the simple, non-covariant constraints) but only owing to the vanishing type-A charges of Ξ= ˚Ξ. The remaining gauge-covariantly haploid superfields (4.3e-7) and (4.4c-7) cannot become ‘decovariantized’ by the chiral decovariantizing transformation (4.6).

On the flip-side: one may start with ‘ordinary’ chiral superfields, satisfying $D^2 ˚Φ = 0$ and their conjugates, and couple them to a gauge superfield by inserting $e^{2V}$ in the ‘ordinary’ D-term:

$$\int d^4ς ˚Φ e^{2V} ˚Φ = \frac{1}{8} \left[ \{D^2, D^2\} ˚Φ e^{2V} ˚Φ \right],$$

(4.14)

Through the ‘covariantizing’ transformation, the inverse of (4.6), this becomes

$$\int d^4ς ˚Φ e^{2V} ˚Φ \rightarrow \int d^4ς ˚Φ ˚Φ = \frac{1}{8} \text{Tr} \left[ \{∇^2, ∇^2\} ˚Φ ˚Φ \right],$$

(4.15)

where the cyclicity of the trace operation has been used to simultaneously keep both $Φ$ and ˚$Φ$ under the scope of action of the superderivatives and maintain the matrix notation for gauge-variant quantities.

But then, a minimal simple lefton superfield, $D_- ˚Λ = 0 = \overline{D}_- ˚Λ$, does not become a gauge-covariant lefton (4.3e) upon (re)covariantizing by the inverse of (4.6):

$$\overline{H}_- ∇_- ∇^{-1}_- ˚Λ = 0, \quad \Rightarrow \quad [∇_+ - (∇_- ∇^{-1}_-)] ˚Λ = 0,$$

$$H^{-1}_- ∇_- H ˚Λ = 0, \quad \Rightarrow \quad [∇_- + H^{-1}_- (∇_- H)] ˚Λ = 0.$$ 

(4.16)

This clearly modifies the type-A connection, i.e., the type-A gauge field coupling to ˚Λ. Therefore, ˚Λ couples to type-A gauge fields differently than, say, the chiral superfield does, although they may have precisely the same type-A charge(s). This violates the universality of the minimal coupling to gauge fields as required in §2.1.

A similarly awkward result follows for the minimal gauge-covariant rightons (4.3e, 7), and all non-minimal haploid superfields (4.3e-7).

4.4. The twisted-chiral (type-B) decovariantization

Perhaps not surprisingly, there exists another decovariantizing transformation:

$$∇_- = \mathcal{K}^{-1} D_- \mathcal{K}, \quad \text{and} \quad ∇_+ = ∇_+^\dagger = \mathcal{K} D_+ \mathcal{K}^{-1};$$

$$∇_+ = \mathcal{K} D_+ \mathcal{K}^{-1}, \quad \text{and} \quad ∇_+ = ∇_+^\dagger = \mathcal{K}^{-1} D_+ \mathcal{K}.$$ 

(4.17)

Then, ˚Ξ $\overset{\text{def}}{=} \mathcal{K}^{-1} Ξ$ is simply twisted-chiral:

$$D_+ ˚Ξ = 0 = \overline{D}_- ˚Ξ,$$ 

(4.18)
and $\hat{\Xi} \equiv \Xi K^{-1}$ is simply twisted-antichiral. Similarly, $\hat{\Pi} \equiv K^{-1} \Pi$ and $\hat{\Pi} \equiv \Pi K^{-1}$ are non-minimal simply twisted-chiral and twisted-antichiral superfields, respectively.

As before, in view of their non-trivial action on twisted chiral superfields and owing to the selectivity of type-B gauge coupling (see Table 1), the operators $\mathcal{K}, \mathcal{K}$ and their inverses must be type-B, i.e., $\mathcal{G}_B$-valued. Therefore, the type-B gauge-covariantly (anti)chiral superfields are in fact simply (anti)chiral:

$$0 = \nabla_- \Phi = \bar{D}_- \Phi, \quad 0 = \nabla_+ \Phi = \bar{D}_+ \Phi.$$  \hspace{1cm} (4.19)

Since also $K^{-1} \Phi = 1 \Phi$, we may trivially set $\hat{\Phi} \equiv K^{-1} \Phi = \Phi$. However, the decovariantizing transformation (4.17) merely complicates matters with non-minimal gauge-covariantly (anti)chiral superfields (4.4a,b), the minimal gauge-covariantly unidexterous superfields (4.3e,f) or their non-minimal counterparts (4.4e,f). The gauge-covariant constraint equations for these superfields do not become simple constrains, and vice versa.

Just as a judicious type-A gauge transformation ‘Hermitizes’ $\mathcal{H} \rightarrow e^\mathcal{V}$, an analogously judicious type-B gauge transformation ‘Hermitizes’ $\mathcal{K} \rightarrow e^\mathcal{W}$, $\mathcal{W}^\dagger = \mathcal{W}$. While $\mathcal{V} \equiv \log \mathcal{H}$ parametrizes, locally, the coset $\mathcal{G}_A/\mathcal{G}_A$ and is the gauge prepotential superfield for $\mathcal{G}_A$, $\mathcal{W} \equiv \log \mathcal{K}$ similarly parametrizes the coset $\mathcal{G}_B/\mathcal{G}_B$ and is the gauge prepotential superfield for $\mathcal{G}_B$. Since $\mathcal{G}_A$ and $\mathcal{G}_B$ commute, so do the two Hermitizations, whence $\mathcal{H}$ and $\mathcal{K}$ are simultaneously Hermitizable ($\mathcal{V}$ and $\mathcal{W}$ can coexist).

Finally, both the $\mathcal{H}$- and the $\mathcal{K}$-decovariantization commute with Hermitian conjugation, i.e., both Eqs. (4.4) and (4.17) are invariant under Hermitian conjugation. On the other hand, while the chiral, $\mathcal{H}$-decovariantization (4.6) commutes with parity, the twisted-chiral, $\mathcal{K}$-transformation does not: Eqs. (4.17) are not invariant under parity. Depending on the intended importance of parity in model building, this poses further restrictions on the use of the twisted-chiral, $\mathcal{K}$-decovariantization.

4.5. No unidexterous (type-C) decovariantization

Finally, there is no decovariantizing transformation, akin to (4.6) or (4.17), which would turn either of the gauge-covariantly unidexterous superconstraints (4.3e,f) into their simple counterparts. To see this, simply note that the two superconstraints in Eqs. (4.3e) and also in Eqs. (4.3f) involve conjugate operators. Thus:

$$\nabla_- = \mathcal{C}^{-1} D_- \mathcal{C}, \quad \Rightarrow \quad \nabla_- = \nabla_-^\dagger = c D_- \mathcal{C}^{-1},$$  \hspace{1cm} (4.20)

upon which Eqs. (4.3f) become

$$\begin{cases} 0 = \nabla_- \Lambda, \\ 0 = \nabla_- \Lambda; \end{cases} \quad \Rightarrow \quad \begin{cases} 0 = \mathcal{C}^{-1} D_- \mathcal{C} \Lambda, \\ 0 = c D_- \mathcal{C}^{-1} \Lambda. \end{cases}$$  \hspace{1cm} (4.21)

Since $\mathcal{C}^{-1} \neq \mathcal{C}$, there is no redefinition of the gauge-covariant lefton which would turn both superconstraints into the simple lefton constraints,

$$D_- \Lambda = 0 = \bar{D}_- \Lambda,$$  \hspace{1cm} (4.22)

\footnote{Were $\mathcal{C}$ unitary, it would implement a proper gauge transformation, whereupon the type-$C_\pm$ gauge superfields in $\nabla_-, \nabla_-^\dagger$ would become ‘pure gauge’, for which the field strengths $\mathcal{C}_\pm, \mathcal{C}_\pm$ would have to vanish—contrary to the assumption for type-$C_\pm$ gauging; see §4.1.}
—and vice versa! The analogous holds for gauge-covariant rightons. There being no type-C decovariantizing operators, there is also no type-C analogue of $e^V$, $e^W$, and so no type-C prepotential superfield. For type-C gauge couplings the gauge superfields $C, \bar{C}$ and the related $W$’s provide the only consistent description, which is perhaps why they have not been uncovered in previous studies [5,6].

5. Summary and Conclusions

While the expressions (3.9) and (3.10) capture the most general $N=1$ supersymmetric Lagrangian in 3+1-dimensions, Ref. [7] finds many additional possible terms in the 1+1-dimensional case, each of which is straightforward to gauge-covariantize through the ‘minimal coupling’ substitution, $D, \bar{D} \to \nabla, \bar{\nabla}$ Refs. [1,3]. They are all manifestly gauge-invariant, and are manifestly supersymmetric as shown in §3.1. On the other hand, only a small subset of these models admits universal coupling to gauge fields through the traditional process of judicious insertion of the ‘standard’ $e^{2V}$-like terms [2,9].

The framework presented here (‘vector respresentation’ in Ref. [1]) differs from the ‘standard’ one [2,5,9] (‘chiral representation’ in Ref. [1]) as follows:

1. the gauge transformation, $G$, is unitary (2.3), and is generated by a Hermitian generator, $\mathcal{E} = -i \log G$;

2. the universal gauge-covariant (super)derivatives are uniquely defined (2.1);

3. the universal gauge-covariant (super)derivatives introduce the universal ‘minimal’ coupling of gauge fields to the matter fields, the latter of which are usually defined as constrained superfields;

4. these defining constraints, (4.3) and (4.4), all involve the universal gauge-covariant (super)derivatives, ensuring a gauge-invariant meaning to these superconstraints;

5. the Hermitian conjugate gauge-covariantly constrained superfields (4.3) and (4.4) transform with respect to all symmetry transformations in a Hermitian conjugate fashion;

6. all Lagrangian density terms found in Ref. [7] are straightforwardly gauge-covariantizable by simply gauge-covariantizing the (super)derivatives (2.1).

Unlike in 3+1-dimension, there are two (de)covariantizing transformations in the (2,2)-supersymmetric 1+1-dimensional spacetime: $\mathcal{H}$ (4.6), and $\mathcal{K}$ (4.17), neither of which is universal, whence there exists no universal gauge vector prepotential superfield. This is summarized in Table 2.

| $\Phi, \bar{\Phi}$ | $\Xi, \bar{\Xi}$ | $\Lambda$ | $\Upsilon$ | $\Theta, \bar{\Theta}$ | $\Pi, \bar{\Pi}$ | $A$ | $U$ |
|-------------------|------------------|--------|--------|------------------|------------------|-----|-----|
| $\mathcal{H}$      | $\checkmark$     | $\times$ | $\times$ | $\checkmark$     | $\times$         | $\times$ | $\times$ |
| $\mathcal{K}$      | $\checkmark$     | $\times$ | $\times$ | $\times$         | $\checkmark$     | $\times$ | $\times$ |

**Table 2:** The partial success of decovariantizing the superconstraint equations (4.3) and (4.4)—or covariantizing the corresponding *simple* superconstraints—using the chiral, $\mathcal{H}$, and the twisted-chiral transformations, $\mathcal{K}$: the entry ‘✓’ indicates successful (de)covariantization, ‘×’ the lack thereof.
Therefore, only models which involve either
1. (anti)chiral, twisted-(anti)chiral and NM-(anti)chiral, or
2. (anti)chiral, twisted-(anti)chiral and NM-twisted-(anti)chiral
superfields may be constructed equivalently in the gauge-covariant formalism (using the \( \nabla \)'s) or the simple formalism (using the \( D \)'s). That is, all models involving any combination of gauge-covariantly haploid superfields \((4.3)\) and \((4.4)\) other than the two subsets listed above cannot be completely decovariantized.

Conversely, models involving any combination of the simple (gauge-noncovariant) counterparts of \((4.3)\) and \((4.4)\) other than the two subsets listed above cannot be made gauge-invariant with universal couplings to gauge fields. This is because the involved defining simple superconstraints cannot be simultaneously made gauge-covariant, and so cannot represent gauge-invariant statements. The Reader should have no difficulty deriving the converse of the transformations \((4.12)\), proving that the re-covariantizations of the simple versions of, say, a chiral \((4.3a)\) and a NM-twisted-chiral \((4.4a)\) superfield are not a gauge-covariantly chiral and a gauge-covariantly NM-twisted-chiral superfield with respect to the same gauge-covariant (super)derivatives. Hence, either the gauge transformation or the gauge-covariant (super)derivatives have to be modified in a nonuniversal fashion, depending on the superfields on which they act \([2,4,9]\). If at all possible, this leads to a non-universal coupling of gauge (super)fields to matter (super)fields, contradicting the requirement of universality specified in §2.1.

Thus, the choice of the gauge-covariant formalism \([1]\) vs. the simple formalism \([2,9]\) is a matter of convenience and convention in 3+1-dimensional spacetime, even for the most general of models with Yang-Mills type gauge symmetry.

By contrast, in 1+1-dimensional spacetime, this is no longer an issue of aesthetics: the simple formalism can describe gauge-invariant models with universal coupling to gauge fields only upon a severe restriction in superfield content! On the other hand, the gauge-covariant formalism \([1,3]\) applies to all models of Ref. \([7]\), and ensures a universal minimal coupling of all matter in each of those models to the appropriate Yang-Mills type gauge superfields, as specified in Table 1.

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