APPROXIMATE CONTROLLABILITY OF DISCRETE SEMILINEAR SYSTEMS AND APPLICATIONS

Hugo Leiva∗
Louisiana State University
Department of Mathematics
Baton Rouge, LA 70803, USA

Jahnett Uzcategui
Universidad de Los Andes
Facualtad de Ciencias, Departamento de Matematica
Merida, 5101, Venezuela

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Abstract. In this paper we study the approximate controllability of the following semilinear difference equation
\[ z(n+1) = A(n)z(n) + B(n)u(n) + f(n, z(n), u(n)), \quad n \in \mathbb{N}^*, \]
where \( Z, U \) are Hilbert spaces, \( A \in l^\infty(\mathbb{N}, L(Z)), B \in l^\infty(\mathbb{N}, L(U, Z)), u \in l^2(\mathbb{N}, U), f \) is a suitable function. We prove that, under some conditions on the nonlinear term \( f \), the approximate controllability of the linear equation is preserved. Finally, we apply this result to a discrete version of the semilinear wave equation.

1. Introduction. One of the main sources for applications of discrete control systems methods are continuous control systems; that is to say, those models described by differential equations instead of difference equations. The reason for this is that while physical systems are modeled by differential equations, control laws are implemented often in a digital computer, whose inputs and outputs are sequences. A common approach to design controls in this case is to obtain a difference equation model that approximates the continuous system that will be controlled.

Considering this observation and using some ideas presented in [1]-[6] we will give sufficient conditions for the approximate controllability of the following semilinear difference equation
\[
\begin{aligned}
z(n+1) &= A(n)z(n) + B(n)u(n) + f(n, z(n), u(n)), \quad n \in \mathbb{N}^*, \\
z(0) &= z_0,
\end{aligned}
\]
where \( N^* = N \setminus \{0\}, z(n) \in Z, u(n) \in U, Z \) and \( U \) are Hilbert spaces, \( A \in l^\infty(\mathbb{N}, L(Z)), B \in l^\infty(\mathbb{N}, L(U, Z)), u \in l^2(\mathbb{N}, U), L(U, Z) \) denotes the space of all

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* Corresponding author: Hugo Leiva.
bounded linear operators from $U$ to $Z$ and $L(Z, Z) = L(Z)$. The nonlinear term $f : \mathbb{N} \times Z \times U \rightarrow Z$ satisfies the following conditions for $n \in \mathbb{N}, u \in l^2(\mathbb{N}, U), z \in Z$

$$\|\Phi(n, k)f(k-1, z, u(k-1))\| \leq M_k, \quad 1 \leq k \leq n,$$

$$\sum_{k=1}^{\infty} M_k < \infty,$$

and $\Phi = \{\Phi(n, m)\}_{(n, m) \in \Delta}$ with $\Delta = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n \geq m\}$ is the evolution operator associated to $A$, i.e.,

$$\Phi(n, m) = \begin{cases} A(n-1) \cdots A(m), & n > m, \\ I, & n = m, \end{cases}$$

where $I$ is the identity operator in the space of bounded and linear operator $L(Z)$. Then, for $z_0 \in Z$ the equation (1) has a unique solution given by

$$z(n) = \Phi(n, 0)z_0 + \sum_{k=1}^{n} \Phi(n, k)B(k-1)u(k-1)$$

$$+ \sum_{k=1}^{n} \Phi(n, k)f(k-1, z(k-1), u(k-1)), \quad n \in \mathbb{N}^*.$$ 

Corresponding to the nonlinear system (1) we consider also the linear system:

$$\begin{cases} z(n+1) = A(n)z(n) + B(n)u(n), & n > m \in \mathbb{N}^* , \\ z(m) = z_0. \end{cases}$$

From now on we shall use the following notation $[m, n_0]_\mathbb{N} = [m, n_0] \cap \mathbb{N}$ with $0 \leq m < n_0$.

**Definition 1.1. (Approximate Controllability of system (5))** The system (5) is said to be approximately controllable on $[m, n_0]_\mathbb{N}$ if for every $z_0, z_1 \in Z$ and $\epsilon > 0$ there exists $u \in l^2(\mathbb{N}, U)$ such that the corresponding solution of (5) satisfies $\|z(n_0) - z_1\| < \epsilon$.

**Definition 1.2. (Approximate Controllability on Free Time of (1))** The system (1) is said to be approximately controllable on free time if for every $z_0, z_1 \in Z$ and $\epsilon > 0$ there exists $u \in l^2(\mathbb{N}, U)$ and $n_0 \in \mathbb{N}$ such that the corresponding solution of (1) satisfies $\|z(n_0) - z_1\| < \epsilon$.

We will prove the following statement: If conditions (2)-(3) hold and the linear system (5) is approximately controllable on $[m, n_0]_\mathbb{N}$, for all $0 \leq m < n_0$, then the semilinear system (1) is approximately controllable on free time. Moreover, we can find a sequence of controls steering the system from the initial state $z_0$ to an $\epsilon$-neighborhood of the final state $z_1$ on some time $n_0$.

There is a broad literature on the study of difference equations, but the study of controllability of such equations is still in effervescence, however there are good references in this respect. In [17], [18] the concept of approximate controllability for non-autonomous linear systems described by linear skew-products semiflows is characterized, and the authors showed that stabilizability implies approximate controllability for the case of systems associated to uniformly*-positive linear skew-products semiflows. In [19] and [20], it was shown that the stability of the linear difference equation implies the controllability of the system. In [11], the authors considered the approximate controllability of abstract discrete-time systems similar
to (1). They get results on approximate controllability of semilinear system (1) by perturbing the reachable set of linear system (5).

We have already obtained some results on exact and approximate controllability for linear and semilinear difference equations, which appear in [14], [15], [16].

2. Controllability of the linear equation. In this section we will present some characterization of the approximate controllability for the linear difference equation (5). To this end, we note that the solution of (5) is given by the discrete variation constant formula

\[ z(n) = \Phi(n, m)z_0 + \sum_{k=m+1}^{n} \Phi(n, k)B(k-1)u(k-1), \quad n > m. \]  

(6)

Definition 2.1. For the linear system (5) we define the following concepts:

a) The controllability map (for \( 0 < m < n_0 \in \mathbb{N} \)) is defined as follows

\[ B_{mn_0}^u = \sum_{k=m+1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1). \]  

(7)

b) The grammian map (for \( 0 < m < n_0 \in \mathbb{N} \)) is defined by

\[ L_{B_{mn_0}} = B_{mn_0}^u B_{mn_0}^{u*}. \]

The proof of the following Proposition can be seen in [14].

Proposition 2.1. The adjoint \( B_{mn_0}^{u*} \) of the operator \( B_{mn_0}^u \) is given by \( B_{mn_0}^{u*} : Z \rightarrow l^2(\mathbb{N}, U) \)

\[ (B_{mn_0}^{u*}z)(k-1) = \begin{cases} B^*(k-1)\Phi^*(n_0, k)z, & k \leq n_0, \\ 0, & k > n_0, \end{cases} \]  

(8)

and

\[ L_{B_{mn_0}} z = \sum_{k=m+1}^{n_0} \Phi(n_0, k)B(k-1)B^*(k-1)\Phi^*(n_0, k)z, \quad z \in Z. \]  

(9)

The following Lemma will be used to stablish our next Theorem and it can be founded in ([1], [2], [7], [8] and [16]).

Lemma 2.1. If \( W, Z \) are Hilbert spaces and \( G : W \rightarrow Z \) is a linear bounded operator, then the following statements are equivalent:

a) \( \text{Range}(G) = Z \).

b) \( \ker(G^*) = \{0\} \).

c) \( \langle GG^*z, z \rangle > 0, \quad z \not= 0 \text{ in } Z \).

d) \( \lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1}z = 0 \).

e) For all \( z \in Z \) we have \( Gu_\alpha = z - \alpha(\alpha I + GG^*)^{-1}z \), where

\[ u_\alpha = G^*(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0, 1]. \]  

So, \( \lim_{\alpha \to 0} Gu_\alpha = z \) and the error \( Ez \) of this approximation is given by the formula

\[ Ez = \alpha(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0, 1]. \]

f) Moreover, if we consider for each \( v \in L^2(\tau - \delta, \tau; U) \) the sequence of controls given by

\[ u_\alpha = G^*_{\tau\delta}(\alpha I + Q_{\tau\delta})^{-1}z + (v - G^*_{\tau\delta}(\alpha I + Q_{\tau\delta})^{-1}G_{\tau\delta}v), \quad \alpha \in (0, 1], \]
we get that:

\[ G_{\tau \delta} u_\alpha = z - \alpha (\alpha I + Q_{T\delta})^{-1}(z + G_{\tau \delta} v) \]

and

\[ \lim_{\alpha \to 0} G_{\tau \delta} u_\alpha = z, \]

with the error \( E_{\tau \delta} z \) of this approximation given by the formula

\[ E_{\tau \delta} z = \alpha (\alpha I + Q_{T\delta})^{-1}(z + G_{\tau \delta} v), \quad \alpha \in (0, 1]. \]

**Theorem 2.1.** The linear system (5) is approximately controllable on \([m, n_0]\) if, and only if, one of the following statements holds:

a) \( \text{Range}(B^{mn_0}) = Z \).

b) \( \ker(B^{mn_0 \ast}) = \{0\} \).

c) \( \langle LB^{mn_0} z, z \rangle > 0, \ Z \neq 0 \text{ in } Z \).

d) \( B^*(k-1)\Phi^*(n_0, k)z = 0, \quad k \leq n_0 \Rightarrow z = 0 \).

e) \( \lim_{\alpha \to 0^+} \alpha (\alpha I + LB^{mn_0})^{-1} z = 0 \).

f) For all \( z \in Z \) we have \( B^{mn_0} u_\alpha = z - \alpha (\alpha I + LB^{mn_0})^{-1} z \), where

\[ u_\alpha = u_\alpha^{mn_0} = B^{mn_0 \ast} (\alpha I + LB^{mn_0})^{-1} z \in l^2(\mathbb{N}, U), \quad \alpha \in (0, 1]. \]

So, \( \lim_{\alpha \to 0^+} B^{mn_0} u_\alpha = z \) and the error \( E_\alpha z \) of this approximation is given by

\[ E_\alpha z = \alpha (\alpha I + LB^{mn_0})^{-1} z, \quad \alpha \in (0, 1]. \]

**Remark 2.1.** The foregoing theorem implies that the family of operators \( \Gamma_{\alpha mn_0} : Z \to l^2(\mathbb{N}, U) \) defined by

\[ \Gamma_{\alpha mn_0} z = B^{mn_0 \ast} (\alpha I + LB^{mn_0})^{-1} z, \quad \alpha \in (0, 1], \]

is an approximate right inverse of the operator \( B^{mn_0} \), i.e.,

\[ \lim_{\alpha \to 0^+} B^{mn_0} \Gamma_{\alpha mn_0} = I. \]

**Lemma 2.2.** If linear system (5) is approximately controllable on \([m, n_0]\), a sequence of controls steering the initial state \( z_0 \) to a \( \varepsilon \)-neighborhood of a final state \( z_1 \) in time \( n_0 \) is given by

\[ u_\alpha = B^{mn_0 \ast} (\alpha I + LB^{mn_0})^{-1} (z_1 - \Phi(n_0, m)z_0) \in l^2(\mathbb{N}, U), \]

and corresponding solutions \( y(n) = y(n, m, y_0, u_\alpha) \) of the initial value problem

\[
\begin{align*}
    y(n+1) &= A(n)y(n) + B(n)u_\alpha(n), \quad n > m \in \mathbb{N}^+, \\
    y(m) &= y_0,
\end{align*}
\]

satisfy

\[
\lim_{\alpha \to 0^+} y(n, m, y_0, u_\alpha) = z_1.
\]

i.e.,

\[
\lim_{\alpha \to 0^+} y(n) = \lim_{\alpha \to 0^+} \left\{ \Phi(n, m)z(m) + \sum_{k=m+1}^{n} \Phi(n, k)B(k-1)u_\alpha(k-1) \right\} = z_1.
\]
3. Controllability of the nonlinear equation. In this section we shall study the approximate controllability on free time of the nonlinear difference equation (1). To this end, we note that for each \( z \in Z \) and a control \( u \in L^2(\mathbb{N}, U) \) the equation (1) has a unique solution given by

\[
z(n) = \Phi(n, 0)z_0 + \sum_{k=1}^{n} \Phi(n, k)[B(k-1)u(k-1) + f(k-1, z(k-1), u(k-1))],
\]

for all \( n \in \mathbb{N}^* \). Now, we are ready to present and prove the main result of this paper, which is the approximate controllability of semilinear equation (1) on free time.

**Theorem 3.1.** Under conditions (2) and (3), if the system (5) is approximately controllable on \([m, n_0][n]\), for all \( 0 \leq m < n_0 \), then the system (1) is approximately controllable on free time.

**Proof.** Given an initial state \( z_0 \), a final state \( z_1 \) and \( \varepsilon > 0 \), we want to find a control \( u_\alpha^m \in L^2(\mathbb{N}, U) \) steering the system from \( z_0 \) to an \( \varepsilon \)-neighborhood of \( z_1 \) on time \( n_0 \). Specifically,

\[
\lim_{\alpha \to 0^+} \left\{ \Phi(n_0, 0)z_0 + \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u_\alpha^m(k-1) + \sum_{k=1}^{n_0} \Phi(n_0, k)f(k-1, z_\alpha^m(k-1), u_\alpha^m(k-1)) \right\} = z_1.
\]

Consider any \( u \in L^2(\mathbb{N}, U) \) and the corresponding solution \( z(n) = z(n, 0, z_0, u) \) of initial value problem (1). For \( \alpha \in (0, 1] \), we define the control \( u_\alpha^m \in L^2(\mathbb{N}, U) \) as follows:

\[
u_\alpha^m(n) = \begin{cases} 
  u(n), & \text{if } 0 < n \leq m, n \in \mathbb{N}, \\
  u_\alpha(n), & \text{if } m < n \leq n_0, n \in \mathbb{N},
\end{cases}
\]

where

\[
u_\alpha = \mathcal{B}^{m,n_0}\left(\alpha I + L_{\mathcal{B}^{m,n_0}}\right)^{-1}(z_1 - \Phi(n_0, m)z_0) \in L^2(\mathbb{N}, U).
\]

Now, assume that \( 0 < m < n_0 \) and \( n_0 > 1 \). Then the corresponding solution \( z_\alpha^m(n) = z(n, 0, z_0, u_\alpha^m) \) of initial value problem (1) at time \( n_0 \) can be written as follows:

\[
z_\alpha^m(n_0) = \Phi(n_0, 0)z_0 + \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u_\alpha^m(k-1)
+ \sum_{k=1}^{n_0} \Phi(n_0, k)f(k-1, z_\alpha^m(k-1), u_\alpha^m(k-1))
+ \Phi(n_0, m) \left\{ \Phi(m, 0)z_0 + \sum_{k=1}^{m} \Phi(m, k)B(k-1)u_\alpha^m(k-1)
+ \sum_{k=1}^{m} \Phi(m, k)f(k-1, z_\alpha^m(k-1), u_\alpha^m(k-1)) \right\}
+ \sum_{k=m+1}^{n_0} \Phi(n_0, k)B(k-1)u_\alpha^m(k-1)
+ \sum_{k=m+1}^{n_0} \Phi(n_0, k)f(k-1, z_\alpha^m(k-1), u_\alpha^m(k-1)).
\]
Therefore, from the above two inequalities we get the following estimate which completes the proof of the theorem.

Therefore, the solution $z^m_\alpha(n) = z(n,0,z_0,u^m_\alpha)$ of initial value problem (1) at time $n_0$ can be written as follows:

$$z^m_\alpha(n_0) = \Phi(n_0,m)z(m) + \sum_{k=m+1}^{n_0} \Phi(n_0,k)B(k-1)u_\alpha(k-1) + \sum_{k=m+1}^{n_0} \Phi(n_0,k)f(k-1,z^m_\alpha(k-1),u_\alpha(k-1)).$$

The corresponding solution $y^m_\alpha(n) = y(n,m,z(m),u_\alpha)$ of initial value problem (10) at time $n_0$ is given by

$$y(n_0) = \Phi(n_0,m)z(m) + \sum_{k=m+1}^{n_0} \Phi(n_0,k)B(k-1)u_\alpha(k-1).$$

Hence, for $m, n_0 \in \mathbb{N}$ big enough, with $0 < m < n_0$, we obtain that

$$\|z^m_\alpha(n_0) - y^m_\alpha(n_0)\| \leq \sum_{k=m+1}^{n_0} \|\Phi(n_0,k)f(k-1,z^m_\alpha(k-1),u^m_\alpha(k-1))\| \leq \sum_{k=m+1}^{n_0} M_k < \frac{\varepsilon}{2} \text{ for a big enough } m.$$

On the other hand, from Lemma 2.2, there exists $\alpha > 0$ such that

$$\|y^m_\alpha(n_0) - z_1\| \leq \frac{\varepsilon}{2}.$$

Therefore, from the above two inequalities we get the following estimate

$$\|z^m_\alpha(n_0) - z_1\| \leq \|z^m_\alpha(n_0) - y^m_\alpha(n_0)\| + \|y^m_\alpha(n_0) - z_1\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which completes the proof of the theorem.

4. **Application.** In general, given a controlled evolution equation

$$z' = Az + Bu, \quad z \in Z, u \in U, t > 0,$$

where $z \in Z$, $u \in U$, $Z$ and $U$ are Hilbert spaces, $A$ is the infinitesimal generator of a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, one can consider a discretization on its flow, the same that is used in [9] and [10] to study the exponential dichotomy of evolution equation. That is to say

$$z(n+1) = T(n)z(n) + Bu(n), \quad n \in \mathbb{N}^*,$$

where the control $u = \{u(n)\}_{n \geq 1}$ belongs to $l^2(\mathbb{N},U)$ and $z(n) \in Z$.

As an application of the main result of this paper we shall consider a discretization on flow of the controlled nonlinear wave equation.

$$\begin{cases}
y_{tt} - \Delta y = u(t,x), & x \in \Omega, \\
y = 0 & \text{on } \mathbb{R} \times \partial \Omega, \\
y(0,x) = y_0(x), & y_t(0,x) = y_1(x), & x \in \Omega,
\end{cases}$$

(18)
where $\Omega$ is a bounded domain in $\mathbb{R}^n$, the distributed control $u \in L^2(0, \tau; L^2(\Omega))$, $y_0 \in H^2(\Omega) \cap H_0^1$, $y_1 \in L^2(\Omega)$. The system (18) can be written as an abstract second order equation in the Hilbert space $X = L^2[0, 1]$ as follows:

$$\begin{align*}
&y'' = -Ay + u(t), \\
y(0) = y_0, \quad y'(0) = y_1,
\end{align*}$$

(19)

where the operator $A$ is given by $A\phi = -\Delta \phi$ with domain $D(A) = H^2(\Omega, \mathbb{R}) \cap H_0^1(\Omega, \mathbb{R})$.

The operator $A$ has the following properties: the spectrum of $A$ consists only eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$, each one with multiplicity $\gamma_n$ equal to the dimension of the corresponding eigenspace.

a) There exists an orthonormal and complete set $\{\phi_n\}$ of eigenvectors of $A$.

b) For all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n \xi,$$

(20)

where $< \cdot, \cdot >$ is the inner product in $X$ and

$$E_n x = \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k}.$$  

(21)

So, $\{E_n\}$ is an orthonormal and complete family of projections in $X$ and $x = \sum_{n=1}^{\infty} E_n x, \quad x \in X$.

c) $-A$ generates an analytical semigroup $\{e^{-At}\}$ given by

$$e^{-At} x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x.$$  

(22)

d) The spaces of fractional powers $X^r$ are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x.$$  

(23)

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert space with the norm

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|w\|^2 + \|v\|^2.$$

Now, using the change of variables $y' = v$, the second order equation (19) can be written as a first order system of ordinary differential equations in the Hilbert space $Z = X^{1/2} \times X$ as

$$\begin{align*}
&z' = Az + Bu(t), \quad z \in Z, \\
z(0) = z_0,
\end{align*}$$

(24)
where
\[ z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \tag{25} \]

\( A \) is an unbounded linear operator with domain \( D(A) = D(A) \times X, \ u \in L^2(0, \tau, U) \) with \( U = X \).

The proof of the following Theorem follows directly from Lemma 2.1 in [13].

**Theorem 4.1.** The operator \( A \) given by (25), is the infinitesimal generator of a strongly continuous group \( \{T(t)\}_{t \in \mathbb{R}} \) given by
\[ T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, z \in Z, t \geq 0, \tag{26} \]
where \( \{P_j\}_{j \geq 1} \) is a complete family of orthogonal projections in the Hilbert space \( Z \) given by
\[ P_j = \text{diag}[E_j, E_j], n \geq 1 \tag{27} \]
and
\[ A_j = R_j P_j, \quad R_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}, j \geq 1. \tag{28} \]

Note that
\[ R_j^* = \begin{bmatrix} 0 & -1 \\ \lambda_j & 0 \end{bmatrix}, \quad A_j = R_j P_j, \quad A_j^* = R_j^* P_j, \quad j \geq 1, \]
and there exist \( M > 1 \) such that \( \|T(t)\| \leq M \).

Now, the discretization of (24) on flow is given by
\[ \begin{cases} z(n + 1) = T(n)z(n) + B(n)u(n), z \in Z, \\ z(m) = z_0. \end{cases} \tag{29} \]
where
\[ u \in l^2(\mathbb{N}, U), \quad B : U \rightarrow Z, \quad Bu = \begin{bmatrix} 0 \\ u \end{bmatrix}. \]

In this case, the evolution operator associated to \( T(\cdot) \), is given by
\[ \Phi(m, n) = T(m-1)T(m-2) \ldots T(n), \quad n < m, \]
and \( \Phi(m, m) = I \).

Note that \( \Phi(m, n) = T(\Theta(m, n)) \) where \( \Theta(m, n) = \frac{m^2 - n^2 + n - m}{2} \in \mathbb{N}, m > n \).

First, we will show that (29) is approximately controllable on \([m, n_0]_\mathbb{N}\). In this case, we have
\[ \mathcal{B}^{mno} : l^2(\mathbb{N}, U) \rightarrow Z, \quad \mathcal{B}^{mno} u = \sum_{k=m+1}^{n_0} T(\Theta(n_0, k)) Bu(k-1) \]
and
\[ L_{\mathcal{B}^{mno}} : \mathcal{B}^{mno} \rightarrow Z, \quad L_{\mathcal{B}^{mno}} = \mathcal{B}^{mno} \mathcal{B}^{mno \ast}. \]

The verification that
\[ P_j BB^* = BB^* P_j \tag{30} \]
and \( T^*(t) = T(-t) \) is trivial.
Then
\[
L_{B^{m_0}} z = \sum_{k=m+1}^{n_0} T(\Theta(n_0, k)) BB^* T^*(\Theta(n_0, k)) z \\
= \sum_{k=m+1}^{n_0} \sum_{j=1}^{\infty} e^{A_j \Theta(n_0, k)} P_j BB^* \sum_{i=1}^{\infty} e^{-A_j \Theta(n_0, k)} P_j z \\
= \sum_{j=1}^{\infty} \sum_{k=m+1}^{\infty} e^{A_j \Theta(n_0, k)} BB^* e^{-A_j \Theta(n_0, k)} P_j z \\
= \sum_{j=1}^{\infty} L_{B_j^{m_0}} P_j z,
\]

where \( L_{B_j^{m_0}} = B_j^{m_0} E_j^{m_0} = \sum_{k=m+1}^{n_0} e^{A_j \Theta(n_0, k)} BB^* e^{-A_j \Theta(n_0, k)} \).

Hence, \( L_{B^{m_0}} = \sum_{j=1}^{\infty} L_{B_j^{m_0}} \).

Let \( z = [z_1, z_2]^T \) in \( Z \). It is not difficult to verify that
\[
L_{B_j^{m_0}} P_j z = \sum_{k=m+1}^{n_0} (n_0 - m)[0, E_j z_2]^T.
\]

Then
\[
\langle L_{B_j^{m_0}} P_j z, P_j z \rangle = \langle (n_0 - m)[0, E_j z_2]^T, [E_j z_1, E_j z_2]^T \rangle = (n_0 - m)\|E_j z_2\|^2 > 0, \forall j.
\]

Hence, using (30), we have, for \( z \neq 0 \) in \( Z \), that
\[
\langle L_{B^{m_0}} z, z \rangle = \sum_{j=1}^{\infty} L_{B_j^{m_0}} P_j z, \sum_{j=1}^{\infty} P_j z \rangle = \sum_{j=1}^{\infty} \langle L_{B_j^{m_0}} P_j z, P_j z \rangle \\
= (n_0 - m) \sum_{j=1}^{\infty} \|P_j z\|^2 = (n_0 - m) \|z\|^2 > 0.
\]

In consequence, by Theorem 2.1 part (c), the equation (27) is approximately controllable on \([m, n_0] \).

Finally, if we consider a perturbation of the equation (29), say
\[
\begin{cases}
  z(n + 1) = T(n)z(n) + B(n)u(n) + f(n, z(n), u(n)), z \in Z \\
  z(0) = z_0
\end{cases}
\]

where the nonlinear term \( f : N \times Z \times U \rightarrow Z \) is a suitable function, then from the results obtained in section 3, we have that (32) is approximately controllable on free time.

In particular one can consider \( f(n, z, u) = \left[ \frac{z^2}{z^2 + 1} \right] \left[ \frac{1}{n^2 + 1} \right] \sin u^2. \)

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E-mail address: hleiva@lsu.edu, hleiva@ula.ve
E-mail address: jahnettu@ula.ve