CONNECTED COMPONENTS OF POSITIVE SOLUTIONS FOR A DIRICHLET PROBLEM INVOLVING THE MEAN CURVATURE OPERATOR IN MINKOWSKI SPACE

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ABSTRACT. In this paper we study global bifurcation phenomena for the Dirichlet problem associated with the prescribed mean curvature equation in Minkowski space

\[
\begin{cases}
-\text{div}\left( \frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) = \lambda f(x, u, \nabla u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Here \(\Omega\) is a bounded regular domain in \(\mathbb{R}^N\), the function \(f\) satisfies the Carathéodory conditions, and \(f\) is either superlinear or sublinear in \(u\) at 0. The proof of our main results are based upon bifurcation techniques.

1. Introduction. Hypersurfaces of prescribed mean curvature in Minkowski space \(\mathbb{L}^{N+1}\), with coordinates \((x_1, \cdots, x_N, t)\) and metric \(\sum_{j=1}^{N} dx_j^2 - dt^2\), are of interest in differential geometry and in general relativity. It is known that the study of spacelike submanifolds of codimension one in \(\mathbb{L}^{N+1}\) with prescribed mean extrinsic curvature leads to Dirichlet problems of the type

\[
\begin{cases}
-\text{div}\left( \frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) = \lambda f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\] (1.1)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), with a boundary \(\partial \Omega\) of class \(C^2\), and \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) satisfies the Carathéodory conditions.

By a solution of (1.1) we mean a function \(u \in W^{2,r}(\Omega)\), for some \(r > N\), with \(||\nabla u||_\infty < 1\), which satisfies the equation a.e. in \(\Omega\) and vanishes on \(\partial \Omega\). These are strong strictly spacelike solutions of (1.1) according to the terminology of, e.g., [1,7,14,22].

In [1,14] some general solvability results for (1.1) were proved under the assumption that the function \(f\) is globally bounded. Yet, as all spacelike solutions are uniformly bounded by the quantity \(\frac{1}{2}d(\Omega)\), with \(d(\Omega)\) the diameter of \(\Omega\), one can always reduce to that situation by truncation. Nevertheless it should be observed that if one already knows that problem (1.1) admits zero as a solution, the results in

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provide no further information. Therefore it may be interesting to investigate in such cases the existence of non-trivial, in particular positive solutions. We point out that while this topic has been largely discussed in the literature for the Dirichlet problem associated with various classes of semilinear elliptic equations, relatively little is known about the problem (1.1), at least when \( \Omega \) is a general domain in \( \mathbb{R}^N \).

For the case \( N = 1 \), the existence and multiplicity of positive solutions of the Dirichlet problem for the quasilinear ordinary differential equation have been extensively studied by Coelho et al. [9] via variational or topological methods. For the special case \( \Omega \) is a ball, the existence and multiplicity of positive solutions of (1.1) have been investigated by several authors, see Bereanu, Jebelean and Torres [3,4] by using upper and lower solutions, Leray-Schauder degree arguments and critical point theory for convex, lower semicontinuous perturbations of \( C^1 \)-functionals, Coelho, Corsato and Rivetti [10] by using variational methods, Ma, Gao and Lu [20] via bifurcation techniques.

Relatively little is known about the existence and multiplicity of positive solutions of (1.1) when \( \Omega \) is a general domain in \( \mathbb{R}^N \). Previous works on the problem

\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = \lambda \tilde{f}(x, u, \nabla u) & \text{in } \Omega, \\
u u^p & \text{in } B,
\end{cases}
\]

\( (1.2) \) include Corsato, Obersnel, Omari and Rivetti [13]. They used topological argument to obtain the following

**Theorem A.** (Corsato et al. [13, Theorem 3.1 (i)]) Assume

- \( (h_0) \) \( \lambda \geq 0 \).
- \( (h_1) \) \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), with a boundary \( \partial \Omega \) of class \( C^2 \).
- \( (h_2) \) \( \tilde{f} : \Omega \times [0, \frac{d(\Omega)}{2}] \times \bar{B}_1(0) \rightarrow \mathbb{R} \) satisfies the Carathéodory conditions and
  \[
  \text{esssup}_{\Omega \times [0, \frac{d(\Omega)}{2}] \times \bar{B}_1(0)} |\tilde{f}| < \infty.
  \]

- \( (a_1) \) There exist an open ball \( B \subseteq \Omega \), \( a_1 > 0 \) and \( p_1 \in (0, 1) \) such that
  \[
a_1 s^{p_1} \leq \tilde{f}(x, s, \xi) \quad \text{for a.e. } x \in B, \quad s \in [0, \frac{d(\Omega)}{2}], \quad \xi \in \bar{B}_1(0).
  \]

Then for every \( \lambda > 0 \) (1.2) has at least one positive solution.

Their proof of the above result heavily depends upon an existence result from Coelho, Corsato and Rivetti [10] for the radially symmetric problem

\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = \nu u^p & \text{in } B, \\
u u^p & \text{in } \partial B,
\end{cases}
\]

(1.3)

where \( B \) is an open ball in \( \mathbb{R}^N \) and \( \nu > 0 \), \( p > 0 \) are given. To use this result to construct lower solutions of (1.2), \( f \) has to satisfy the growth restriction \((a_1)\).

Of course, the natural question is what would happen if the growth restriction \((a_1)\) is violated.

It is the purpose of this paper to study the global behavior of positive solutions of (1.2) without the growth restriction \((a_1)\) via a variant of the global bifurcation
Assume that Theorem 1.2. via the same method. To wit, we have

Let \( \zeta \) \# theorem \[16,21\] and the connectivity properties of the superior limit of certain infinity collection of connected sets \[18,19\].

Let \( E := \{ u \in C^1(\Omega) \mid u(x) = 0 \text{ for } x \in \partial \Omega \} \) be the usual Banach space under the normal

\[
\| u \|_E := \max \{ |\nabla u(x)| \mid x \in \bar{\Omega} \}.
\]

Let

\[
S = \{(\lambda, u) \mid (\lambda, u) \in (0, \infty) \times E \text{ is a solution of (1.2), } u \geq 0 \text{ and } u(x) \neq 0 \text{ in } \Omega^3 \times E \}.
\]

The first main result is the following

**Theorem 1.1.** Let \((h_0)\) and \((h_1)\) hold. Assume that

\((A0)\) \( \bar{\Omega} \times [0, \frac{d(\Omega)}{2}] < B_1(0) \rightarrow \mathbb{R}^+ \) satisfies the Carathéodory conditions and

\[
\text{esssup}_{\Omega \times [0, \frac{d(\Omega)}{2}] \times B_1(0)} |\bar{f}| < \infty,
\]

and

\[
\text{essinf}_{(x,p) \in \Omega \times B_1(0)} \{ |\bar{f}(x, s, p)| \} > 0 \text{ for all } s > 0.
\]

\((A1)\) There exist \( \alpha \in C(\Omega \times [0, \frac{d(\Omega)}{2}]) \) and \( \beta \in C(\Omega \times [0, \frac{d(\Omega)}{2}] \times B_1(0)) \) such that

\[
\bar{f}(x, s, \xi) = \alpha(x, s) + \beta(x, s, \xi)s.
\]

\((A2)\) \( \alpha(x, s) > 0 \text{ for } (x, s) \in \Omega \times [0, \frac{d(\Omega)}{2}] \) and \( \alpha(x, 0) = 0 \text{ for } x \in \Omega. \)

\((A3)\)

\[
\lim_{s \to 0} \frac{\alpha(x, s)}{s} = \infty \text{ uniformly for } x \in \Omega.
\]

\((A4)\)

\[
\lim_{s \to 0} \beta(x, s, \xi) = 0 \text{ uniformly for } x \in \Omega \text{ and } \xi \in B_1(0).
\]

Then there exists a component \( \mathcal{C} \subset S \) such that

1. \( \mathcal{C} \cap ([0, \infty) \times \{0\}) = \{(0, \theta)\} \), where \( \theta \) is the zero element in \( E \);
2. \( \mathcal{C} \) joins \((0, \theta)\) with infinity in \( \lambda\)-direction.

We also deal with the case when \((A3)\) is replaced by the sublinear growth condition

\((A5)\)

\[
\lim_{s \to 0} \frac{\alpha(x, s)}{s} = 0 \text{ uniformly for } x \in \Omega,
\]

via the same method. To wit, we have

**Theorem 1.2.** Assume that \((h_0)\), \((h_1)\), \((A0)-(A2)\), \((A4)\) and \((A5)\) are fulfilled. Then there exists a component \( \zeta \subset S \) such that

1. \( \zeta \cap ([0, \infty) \times \{0\}) = \emptyset \), where \( \theta \) is the zero element in \( E \);
2. \( \mu_* = \inf \{ \lambda \mid (\lambda, u) \in \zeta > 0 \} \);
3. \( \lim_{\lambda \to \infty} \lambda \in \zeta \text{ with } \|u\|_E > 1/2 \|u\|_E = 1; \)
4. \( \zeta \) goes to infinity in \( \lambda\)-direction in \( \{ (\lambda, u) \mid \lambda \in [0, \infty) \times \{ u \in E \mid \|u\|_E \leq 1/2 \} \} \), and
5. \( \lim_{\lambda \to \infty} \lambda \in \zeta \text{ with } \|u\|_E < 1/2 \|u\|_\infty = 0. \)
Corollary 1. Assume \((h_0), (h_1), (A_0)\)-(A4) are fulfilled. Then (1.2) has at least one positive solution for every \(\lambda > 0\).

Corollary 2. Assume \((h_0), (h_1), (A_0)-(A2), (A4)\) and \((A5)\) are fulfilled. Then there exists \(\lambda^* > 0\) such that, for every \(\lambda \geq \lambda^*\), (1.1) has at least two positive solutions.

Remark 1. It is worth remarking that we don’t need the nonlinearity \(\tilde{f}\) satisfies growth condition of the form

\[ a_1 s^p \leq \tilde{f}(x, s, \xi) \text{ for a.e. } x \in B, s \in [0, \frac{d(\Omega)}{2}], \xi \in \overline{B}(0), \]

or

\[ b_1 s^q \leq b(x, s, \xi) \text{ for a.e. } x \in B, s \in [0, \frac{d(\Omega)}{2}], \xi \in \overline{B}(0), \]

see \((a_1)\) and \((b_1)\) in Corsato et al [13, Theorem 3.1].

Remark 2. Clearly, Theorem A and [13, Theorem 3.1] give no information on the interesting problem as to what happens to the norms of positive solutions of (1.2) as \(\lambda\) varies in \((0, \infty)\). However, the connected components in Theorems 1.1 and 1.2 are very useful for computing the numerical solutions of (1.2) as they can be used to guide the numerical work. For example, they can be used to estimate the \(u\)-interval in advance in applying the finite difference method, and they together with the fact

\[ |\nabla u(x)| < 1, \quad 0 \leq u(x) \leq \frac{1}{2} d(\Omega), \quad x \in \Omega, \]

can be used to restrict the range of initial values we need to consider in applying the shooting method.

We get the proof of Theorem 1.1 in three steps. In Section 2 we deal with the case that the nonlinearity is of finite non-zero slope at \(u = 0\), i.e. \(f_0 \in (0, \infty)\). In Section 3 we state some preliminaries about the connectivity properties of the superior limit of certain infinity collection of connected sets from Ma and An [18,19].

Section 4 is devoted to treating the case that the nonlinearity is of infinite slope \(f_0 = \infty\) at \(u = 0\) and prove our main result. Crucial to this approach is to construct a sequence of functions \(\{f^{[n]}\}\) which is asymptotic linear at 0 and satisfies

\[ f^{[n]} \rightarrow f, \quad (f^{[n]})_0 \rightarrow \infty. \]

By means of the corresponding auxiliary problems, we obtain a sequence of unbounded components \(\{C^{[n]}\}\) via global bifurcation theorem [2,19], and this enable us to find an unbounded component \(C\) satisfying

\[ (0, \theta) \in C \subset \limsup C^{[n]}. \]

Finally, in Section 5, we deal with the case that the nonlinearity is of infinite slope \(f_0 = 0\) at \(u = 0\) and prove Theorem 1.2 via the same argument.

For other results on the prescribed mean curvature equation in Minkowski space, see [2,11,12] and the references therein.

Notation. We list a few notations that will be used throughout this paper. We write \(s^+ = \max\{s, 0\}\), \(s^- = -\min\{s, 0\}\). We denote by \(B_\rho(x_0)\), or simply by \(B\) if no disambiguation is needed, the open ball in \(\mathbb{R}^N\) centered at \(x_0\) and having radius \(\rho\). For functions \(u, v : U \rightarrow \mathbb{R}\), with \(U\) a subset of \(\mathbb{R}^N\) having positive measure, we write \(u \leq v\) (in \(U\)) if \(u(x) \leq v(x)\) a.e. in \(U\), and \(u < v\) (in \(U\)) if \(u \leq v\) and \(u(x) < v(x)\) in a subset of \(U\) having positive measure. A function \(u\) such that \(u > 0\)
is called positive. Assume that $\mathcal{O}$ is an open bounded set with a boundary $\partial \mathcal{O}$ of class $C^1$, for functions $u, v \in C^1(\overline{\mathcal{O}})$, we write $u \ll v$ (in $\overline{\mathcal{O}}$) if $u(x) < v(x)$ for every $x \in \mathcal{O}$ and, if $u(x) = v(x)$ for some $x \in \partial \mathcal{O}$, then
\[
\frac{\partial u}{\partial \nu}(x) < \frac{\partial v}{\partial \nu}(x),
\]
where $\nu = \nu(x)$ denotes the unit outer normal to $\mathcal{O}$ at $x \in \partial \mathcal{O}$. A function $u$ such that $u \gg 0$ is called strictly positive.

2. The case of finite non-zero slope. Let us introduce the weighted eigenvalue problem
\[
\begin{cases}
-\Delta u = \lambda m(x) u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{2.1}
\]
where $m \in L^\infty(\Omega)$ satisfies
\begin{itemize}
\item[(A6)] $m(x) \geq 0$ in $\Omega$ and $m^+(x_0) > 0$ for some $x_0 \in \Omega$.
\end{itemize}

Denote by $K : L^\infty(\Omega) \to E$ the operator which sends any function $v \in L^\infty(\Omega)$ onto the unique solution $w \in W_2, r(\Omega)$ ($r \geq N$) of
\[
\begin{cases}
-\Delta u = v & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{2.2}
\]

Let $L : L^\infty(\Omega) \to E$ be defined by
\[
L(u) = K(mu).
\]
Both $K$ and $L$ are completely continuous and (2.1) is equivalent to
\[
u = \lambda Lu,
\]
so that the eigenvalues of (2.1) are precisely the characteristic values of $L$.

**Lemma 2.1.** ([5,15]) Assume (A6). Then the non-negative eigenvalues of problem (2.1) form a sequence $\{\lambda_n(m)\}_n$ such that
\[
0 < \lambda_1(m) < \lambda_2(m) \leq \lambda_3(m) \leq \cdots \to \infty.
\]
The eigenspace corresponding to the minimum eigenvalue $\lambda_1(m)$ is spanned by an eigenfunction $\varphi_1$ with $\varphi_1 \gg 0$. Moreover, the algebraic multiplicity of $\lambda_1(m)$ as a characteristic value of $L$ is 1. Finally, all eigenfunctions corresponding to any eigenvalue $\lambda_j(m)$ with $j > 1$ change sign.

In this section, we always assume the following
\begin{itemize}
\item[(C1)] there exists $m \in L^\infty(\Omega)$ such that
\[
\lim_{s \to 0^+} \frac{\tilde{f}(x, s, \xi)}{s} = m(x) \text{ uniformly for } x \in \Omega, \xi \in B_1(0). \tag{2.3}
\end{itemize}

**Theorem 2.2.** Assume $\text{(h0), (h1), (A6), (A0) and (C1).}$ Then there exists a connected component $\mathcal{C}$ of positive solutions of (1.2) which joins $(\lambda_1(m), 0)$ to infinity in $\lambda$-direction. Moreover, there exists $\lambda_* \in (0, \lambda_1(m)]$ such that, for all $\lambda \in (0, \lambda_*)$, the problem (1.2) has no positive solutions and, for all $\lambda > \lambda_1(m)$, it has at least one positive solution.
To prove Theorem 2.2, we need some preliminaries.

Let us define a function $f : \bar{\Omega} \times \mathbb{R} \times B_1(0) \rightarrow \mathbb{R}$ by setting, for a.e. $x \in \bar{\Omega}$ and $\xi \in B_1(0)$,

$$f(x, s, \xi) = \begin{cases} \tilde{f}(x, s, \xi), & \text{if } 0 \leq s \leq \frac{d(\Omega)}{2}, \\ 0, & \text{if } s \geq d(\Omega), \\ \text{linear}, & \text{if } \frac{d(\Omega)}{2} < s < d(\Omega), \\ \tilde{f}(x, 0, \xi), & \text{if } s < 0. \end{cases}$$  \hspace{1cm} (2.4)

We may extend $\alpha$ and $\beta$ to be defined in $\Omega \times \mathbb{R}$ and $\Omega \times \mathbb{R} \times B_1(0)$ by the same way, respectively. For the sake of simplicity in the notation, the modified functions will still be denoted by $\alpha$ and $\beta$.

It is easy to see that, within the context of positive solutions, problem (1.2) is equivalent to the same problem

$$\begin{aligned} -\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) &= \lambda f(x, u, \nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial \Omega, \end{aligned}$$  \hspace{1cm} (2.5)

where $f$ satisfies all the properties assumed in the statement of Theorem 2.2.

**Lemma 2.3.** All of solutions of (2.5) are nonnegative in $\Omega$.

**Proof.** In fact, it is an immediate consequence of the fact $f(x, s, \xi) \geq 0$ for $(x, s, \xi) \in \Omega \times (-\infty, 0) \times B_1(0)$ and [13, Lemma 2.1]. \hfill \square

**Lemma 2.4.** A function $u \in W^{2,r}(\Omega)$ is a positive solution of (2.5) if and only if it is a positive solution of the problem

$$\begin{aligned} -(1 - |\nabla u|^2)\Delta u - \sum_{i,j=1}^N \partial_x u \partial_x u \partial_{x_i,x_j} u &= \lambda h(|\nabla u|) f(x, u, \nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial \Omega, \end{aligned}$$  \hspace{1cm} (2.6)

where

$$h(z) = \begin{cases} (1 - z^2)^{3/2}, & \text{if } |z| \leq 1, \\ 0, & \text{if } |z| > 1. \end{cases}$$

**Proof.** It is clear that a positive solution $u \in W^{2,r}(\Omega)$ of (2.5) is a positive solution of (2.6) as well.

Conversely, suppose that $u \in W^{2,r}(\Omega)$ is a positive solution of (2.6). We aim to show that

$$||\nabla u||_{\infty} < 1.$$  \hspace{1cm} (2.7)

Assume by contradiction that this is not the case. Then

$S_1 := \{ x \in \bar{\Omega} | ||\nabla u(x)||_{\infty} \geq 1 \} \neq \emptyset, \quad S_0 := \{ x \in \Omega | ||\nabla u(x)||_{\infty} = 0 \} \neq \emptyset.$

Let $x^* \in S_0$ and $x_1 \in S_1$ be such that

$$||x^* - x_1||_{\infty} = \text{dist}(S_0, S_1).$$

Then, there exists an open set $V \subset \bar{\Omega}$ satisfying

1. $\partial V$ is of $C^2$;
2. $x^* \in V \subset \bar{\Omega}$;
By (A0), there exists $\gamma \in L^\infty(\Omega)$ such that
\[
|f(x, s, \xi)| \leq \gamma(x) \quad \text{for a.e. } x \in \Omega, \ |s| \leq ||u||_\infty, \ \xi \in B_1(0).
\]
In particular
\[
|f(x, s, \xi)| \leq \gamma(x) \quad \text{for a.e. } x \in V, \ |s| \leq ||u||_\infty, \ \xi \in B_1(0).
\]
Now, for
\[
v := f(x, u(x), \nabla u(x)),
\]
let us consider the problem
\[
\begin{aligned}
-\text{div}\left(\frac{\nabla w}{\sqrt{1 - |\nabla w|^2}}\right) &= v \quad \text{in } V, \\
w &= u \quad \text{on } \partial V.
\end{aligned}
\tag{2.8}
\]
By [13, Lemma 2.1], (2.8) has at most one solution.

We claim that any solutions $w$ of (2.8) satisfies
\[
||\nabla w||_\infty \leq 1 - \mathcal{V}
\tag{2.9}
\]
for some constant $\mathcal{V} = \mathcal{V}(V, ||\gamma||_\infty) \in (0, 1)$ which is only dependent of $\gamma$ and $V$.

We first assume that $v$ satisfies $v \in C^{0,1}(V)$.

In this case, we may modify the differential operator on the left of the equation in (2.8) in such a way that Lieberman [17, Theorem 1] applies. Thus, (2.8) has a unique solution $w$ satisfying $w(x) = u(x)$ in $x \in V$, and
\[
||u||_{C^{1,\alpha}} < c_1, \quad x \in V
\tag{2.10}
\]
for some constant $c_1 = c_1(V, ||\gamma||_\infty)$. By [1, Corollary 3.4, Theorem 3.5] there exists a constant $\mathcal{V} = \mathcal{V}(V, ||\gamma||_\infty) \in (0, 1)$, such that for any solution $u$ of (2.8), we have
\[
||\nabla u||_\infty \leq 1 - \mathcal{V}.
\tag{2.11}
\]
However, this contradicts the fact $x_1 \in S_1$.

The general case of a function $v \in L^\infty(V)$, with $||v||_\infty < ||\gamma||_\infty$, can be easily dealt with by approximation. In fact, fix $r > N$ and let $\{v_n\}$ be a sequence in $C^{0,1}(V)$ converging to $v$ in $L^r(V)$ and satisfying $||v_n||_\infty < ||\gamma||_\infty$ for all $n$. The corresponding solutions $\{u_n\}$ of (2.8) satisfy (2.10) and
\[
||u_n||_{C^{1,\alpha}} < c_2, \quad x \in V, \ n \in \mathbb{N}
\tag{2.12}
\]
for some constant $c_2 = c_2(V, ||\gamma||_\infty)$. We can extract a subsequence of $\{u_n\}$ which converges weakly in $W^{2,r}(V)$ to a solution $u$ of (2.8). Clearly, estimate (2.11) is valid. We get the desired contradiction again.

Therefore, (2.7) is valid, and $u$ is a positive solution of (2.5). \qed

Now, let us define the operator $\mathcal{M} : W^{2,r}(\Omega) \to L^\infty(\Omega)$ by
\[
\mathcal{M}u = -\left[1 - |\nabla u|^2\right] \Delta u + \sum_{i,j=1}^N \partial_{x_i} u \partial_{x_j} u \partial_{x_i, x_j} u,
\]
and let $F$ denote the Nemytskii operator associated with $hf$, i.e.
\[
F(u)(x) = h(|\nabla u|)f(x, u, \nabla u), \quad u \in W^{2,r}(\Omega).
\]
Then (2.5) (i.e. (2.6)) is equivalent to
\[
\mathcal{M}u = \lambda F(u),
\tag{2.13}
\]
i.e.
\[ u = \mathcal{M}^{-1}(\lambda F(u)) \]
(2.14)
in \( E \). From Corsato et al [13, Lemmas 2.2 and 2.3], the operator \( \mathcal{M}^{-1} : L^{\infty}(\Omega) \to E \) is well-defined and is completely continuous. In the following we shall apply the Leray-Schauder degree theory, mainly to the mapping \( \Phi_\lambda : E \to E \),
\[ \Phi_\lambda(u) = u - \mathcal{M}^{-1}(\lambda F(u)). \]
(2.15)
For \( R > 0 \), let \( \mathcal{B}_R = \{ u \in E : ||u||_E < R \} \), let deg(\( \Phi_\lambda, \mathcal{B}_R, 0 \)) denote the degree of \( \Phi_\lambda \) on \( \mathcal{B}_R \) with respect to 0, and let \( i(\Phi_\lambda, u_0, 0) \) be the index of the solution \( u_0 \) of the equation \( \Phi_\lambda(u) = 0 \).

**Lemma 2.5.** Let \( \Lambda \subset \mathbb{R}^+ \) be a compact interval such that \( \lambda_1(m) \notin \Lambda \). Then there exists \( \delta > 0 \) with the property that \( \Phi_\lambda(u) \neq 0 \), \( \forall \lambda \in \Lambda \), \( 0 < ||u||_E \leq \delta \).

**Proof.** Assume on the contrary that there exist \( \mu_n \in \Lambda \) and a sequence \( u_n \to 0 \) in \( E \) such that
\[ \Phi_{\mu_n}(u_n) = 0, \quad \forall n. \]
Then
\[ u_n = \mathcal{M}^{-1}[\mu_n F(u_n)], \]
which is equivalent to
\[
\begin{cases}
- (1 - |\nabla u_n|^2) \Delta u_n - \sum_{i,j=1}^N \partial_{x_i} u_n \partial_{x_j} u_n \partial_{x_i,x_j} u_n u_n = \mu_n h(|\nabla u_n|) f(x, u_n, \nabla u_n) & \text{in } \Omega, \\
\partial_{x_i} u_n \partial_{x_j} u_n \partial_{x_i,x_j} u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(2.16)
Let
\[ v_n = \frac{u_n}{||u_n||_E}. \]
We may assume that \( v_n \to v^* \) in \( E \). Then
\[
\begin{cases}
- (1 - |\nabla u_n|^2) \Delta v_n - \sum_{i,j=1}^N \partial_{x_i} u_n \partial_{x_j} u_n \partial_{x_i,x_j} v_n u_n = \mu_n h(|\nabla u_n|) f(x, u_n, \nabla u_n) v_n & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(2.17)
We claim that
\[ ||\partial_{x_i,x_j} v_n||_\infty \leq K \]
(2.18)
for some positive constant \( K \) which is independent of \( i, j \) and \( n \).
Since
\[ ||\mu_n h(|\nabla u_n|) f(x, u_n, \nabla u_n) v_n||_\infty \leq M_1 \]
for some \( M_1 \), independent of \( \mu_n \) and \( u_n \). By Corsato [13, Lemma 2.2],
\[ ||v_n||_{W^{2,r}} \leq c ||\mu_n h(|\nabla u_n|) f(x, u_n, \nabla u_n) v_n||_\infty \leq c M_1, \quad \forall r \in N, \]
and subsequently,
\[ ||\partial_{x_i,x_j} v_n||_{L^r} \leq c M_1, \quad \forall r \in N. \]
So, by Theorem 2.1.10 (3) in [6],
\[ \partial_{x_i,x_j} v_n \in L^\infty(\Omega) \]
Lemma 2.5, applied to the interval $\Lambda = \left[0, \delta > 0\right]$ and using (C1) and the standard method, we may deduce that
\[
-\Delta v^* = \mu m(x) v^* \quad \text{in } \Omega,
\]
\[
v^* = 0 \quad \text{on } \partial \Omega.
\]
(2.19)
This implies that $\hat{\mu} = \lambda_1(m)$. This contradicts the fact that $\lambda_1(m) \notin \Lambda$.

\textbf{Corollary 3.} For $\lambda \in [0, \lambda_1(m))$, $\lambda(\Phi_0, 0, 0) = 1$.

\textbf{Proof.} Lemma 2.5, applied to the interval $\Lambda = [0, \lambda]$, guarantees the existence of $\delta > 0$ such that
\[
u - \mathcal{M}^{-1}(t\lambda F(u)) \neq 0, \quad \forall t \in [0, 1], \quad 0 < ||u||_E \leq \delta.
\]
Therefore, for $\epsilon \in (0, \delta]$
\[
\deg(\Phi_\lambda, B_\epsilon, 0) = \deg(\Phi_0, B_\epsilon, 0) = \deg(I, B_\epsilon, 0) = 1.
\]

\textbf{Lemma 2.6.} Let $\lambda > \lambda_1(m)$. Then there exists $\delta > 0$ such that for any $u \in E$ with $0 < ||u||_E \leq \delta$, for any $\tau \geq 0$, $\Phi_\lambda(u) \neq \tau \varphi_1$.

\textbf{Proof.} We assume again on the contrary that there exist $\tau_n \geq 0$ and a sequence $u_n \to 0$ in $E$ with $||u_n||_E > 0$, such that
\[
\Phi_\lambda(u_n) = \tau_n \varphi_1, \quad \forall n.
\]
Then
\[
u_n = \mathcal{M}^{-1}(\lambda F(u_n)) + \tau_n \varphi_1.
\]
Because $\mathcal{M}^{-1}(\lambda F(\cdot)) : E \to E$ is completely continuous, we get that
\[
\tau_n \to 0, \quad \text{as } n \to \infty.
\]
(2.20)
From
\[
\mathcal{M}(u_n - \tau_n \varphi_1) = \lambda F(u_n),
\]
and the fact that $F(u_n) \geq 0$, we conclude that $u_n - \tau_n \varphi_1 \geq 0$ in $\Omega$, and subsequently, $u_n \geq 0$ in $\Omega$. Combining this with the decomposition $u_n = w_n + s_n \varphi_1$ and $||u_n||_E > 0$, it concludes that
\[
s_n = \int_\Omega u_n m \varphi_1 - \int_\Omega w_n m \varphi_1 > 0.
\]
(2.21)
Now,
\[
-\text{div} \left( \frac{\nabla (u_n - \tau_n \varphi_1)}{\sqrt{1 - |\nabla u_n - \tau_n \varphi_1|^2}} \right) = \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n),
\]
which implies that
\[
\int_\Omega \nabla (u_n - \tau_n \varphi_1) \nabla \varphi_1 \sqrt{1 - |\nabla u_n - \tau_n \varphi_1|^2} dx = \int_\Omega \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n) \varphi_1 dx.
\]
Since $\tau_n \to 0$ and $u_n \to 0$ in $E$, it follows that for arbitrary $\eta > 0$, there exists $N_1 > 0$, such that
\[
\frac{1}{\sqrt{1 - |\nabla u_n - \tau_n \varphi_1|^2}} \leq 1 + \eta.
\]
This implies that
\[(1 + \eta) \int_{\Omega} \nabla (u_n - \tau_n \varphi_1) \nabla \varphi_1 \, dx \geq \int_{\Omega} \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n) \varphi_1 \, dx,\]
\[(1 + \eta) \int_{\Omega} \nabla u_n \nabla \varphi_1 \, dx - (1 + \eta) \tau_n \int_{\Omega} |\nabla \varphi_1|^2 \, dx \geq \int_{\Omega} \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n) \varphi_1 \, dx,\]
\[(1 + \eta) \int_{\Omega} u_n (-\Delta \varphi_1) \, dx - (1 + \eta) \tau_n \int_{\Omega} |\nabla \varphi_1|^2 \, dx \geq \int_{\Omega} \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n) \varphi_1 \, dx,\]
\[(1 + \eta) \int_{\Omega} u_n m \varphi_1 \, dx \geq \int_{\Omega} \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n) \varphi_1 \, dx, (2.22)\]
\[(1 + \eta) \int_{\Omega} u_n |\tau_n \varphi_1| \, dx \geq \int_{\Omega} \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n) \varphi_1 \, dx, (2.23)\]
\[(1 + \eta) \lambda_1(m) \int_{\Omega} u_n m \varphi_1 \, dx \geq \int_{\Omega} \lambda \frac{h(|\nabla u_n|)}{h(|\nabla u_n - \tau_n \varphi_1|)} f(u_n) \varphi_1 \, dx, (2.24)\]
\[(1 + \eta) \lambda_1(m) s_n \geq \int_{\Omega} \lambda (1 - \sigma)(m - \sigma) u_n \varphi_1 \, dx, \quad n \geq \max\{N_1, N_2\}.\]

Since \(\sigma\) and \(\eta\) are arbitrary positive constants and (A6), it deduces
\[\lambda_1(m) \geq \lambda.\]
However, this contradicts the fact that \(\lambda > \lambda_1(m)\).

\[\square\]

**Corollary 4.** For \(\lambda \in (\lambda_1(m), \infty), i(\Phi_\lambda, 0, 0) = 0.\)

**Proof.** Let \(0 < \epsilon < \delta,\) where \(\delta\) is the number asserted in Lemma 2.6. As \(\Phi_\lambda\) is bounded on \(\mathcal{B}_\epsilon,\) there exists \(a > 0\) such that
\[\Phi_\lambda(u) \neq a \varphi_1, \quad \forall \ u \in \mathcal{B}_\epsilon.\]

By Lemma 2.6,
\[\Phi_\lambda(u) \neq t a \varphi_1, \quad \forall \ u \in \partial \mathcal{B}_\epsilon, \quad \forall \ t \in [0,1].\]

Hence,
\[\deg(\Phi_\lambda, \mathcal{B}_\epsilon, 0) = \deg(\Phi_0 - a \varphi_1, \mathcal{B}_\epsilon, 0) = 0.\]

\[\square\]

**Lemma 2.7.** Assume that there exists a sequence \((\mu_n, u_n) \in (0, \infty) \times E\) with \(u_n > 0\) such that
\[\Phi_{\mu_n}(u_n) = 0, \quad \forall \ n,\]
and \(\mu_n \to \bar{\mu}\) and \(u_n \to 0\) in \(E.\) Then \(\bar{\mu} = \lambda_1(m)\).
Proof. Let

\[ v_n = \frac{u_n}{\|u_n\|_E}. \]

By the same method to get (2.19), with obvious changes, we may get

\[ \begin{cases} -\Delta v^* = \bar{\mu}(x) v^* & \text{in } \Omega, \\ v^* = 0 & \text{on } \partial \Omega. \end{cases} \]

This implies that \( \bar{\mu} = \lambda_1(m) \).

Proof of Theorem 2.2. By Corollaries 3-4 and a variant of the global bifurcation theorem of Rabinowitz [21], or index jump principle of Zeidler [24], yields the existence of a maximal closed connected set \( C \) in \( S \) such that \((\lambda_1(m), 0) \in C \) and at least one of the following conditions holds:

(i) \( C \) is unbounded in \( \mathbb{R} \times E \);

(ii) there exists a characteristic value \( \hat{\lambda}(m) \) of \( L \), with \( \hat{\lambda}(m) \neq \lambda_1(m) \), such that \((\hat{\lambda}(m), 0) \in C \).

Furthermore, Lemma 2.7 guarantees that the second case can not occur.

3. Superior limit and component.

Definition 3.1. ([23]) Let \( X \) be a Banach space and \( \{C_n \mid n = 1, 2, \cdots\} \) be a family of subsets of \( X \). Then the the superior limit \( \mathcal{D} \) of \( \{C_n\} \) is defined by

\[ \mathcal{D} := \limsup_{n \to \infty} C_n = \{ x \in X \mid \text{there exist } \{n_i \} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x \}. \]

Lemma 3.2. ([18, Lemma 2.4], [19, Lemma 2.2]) Assume that

(i) there exist \( z_n \in C_{n_i}, n = 1, 2, \cdots \), and \( z^* \in X \), such that \( z_n \to z^* \);

(ii) \( \lim_{n \to \infty} r_n = \infty \), where \( r_n = \sup\{\|x\| \mid x \in C_{n_i}\} \);

(iii) for every \( R > 0 \), \( \bigcup_{n=1}^{\infty} C_n \cap B_R \) is a relative compact set of \( X \), where \( B_R = \{x \in X \mid \|x\| \leq R\} \).

Then there exists an unbounded component \( C \) in \( \mathcal{D} \) with \( z^* \in C \).

4. Global behavior of positive solutions in the case \( f_0 = \infty \). To prove Theorem 1.1, we define \( \alpha^{[n]} : \Omega \times \mathbb{R} \to \mathbb{R} \) by

\[ \alpha^{[n]}(x, s) = \begin{cases} \alpha(x, s), & |s| \in \left(\frac{1}{n}, \infty\right), \\ n\alpha(x, \frac{1}{n}) s, & |s| \in \left[0, \frac{1}{n}\right], \end{cases} \]

and

\[ f^{[n]}(x, s, \xi) = \alpha^{[n]}(x, s) + \beta(x, s, \xi)s. \]

Let us define a function \( g^{[n]} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) by setting, for a.e. \( x \in \Omega \),

\[ g^{[n]}(x, s, \xi) = \begin{cases} f^{[n]}(x, s, \xi), & \text{if } |\xi| \leq 1, \\ f^{[n]}(x, s, \frac{\xi}{|\xi|}), & \text{if } |\xi| > 1. \end{cases} \]

Obviously,

\[ (g^{[n]})_0 = \lim_{s \to 0} \frac{g^{[n]}(x, s, \xi)}{s} = n\alpha(x, \frac{1}{n}) > 0 \quad \text{uniformly for } x \in \Omega \text{ and } \xi \in \mathbb{R}^N. \]
This together with (A3) implies that
\[
\lim_{n \to \infty} (g^{[n]})_0 = \infty. \tag{4.5}
\]

**Proof of Theorem 1.1.** Let us consider the auxiliary family of the problems
\[
\begin{cases}
-\left(1 - |\nabla u|^2\right)\Delta u - \sum_{i,j=1}^N \partial_{x_i} u \partial_{x_j} u \partial_{x_i,x_j} u = \lambda h(|\nabla u|)g^{[n]}(x, u, \nabla u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \tag{4.6}
\]

From (4.6) and Theorem 2.2, there exists a component \(C^{[n]}\) in the set \(S_n\)
\[
S_n = \{(\lambda, u) \mid \lambda \in [0, \infty) \times E, \ u \text{ is a positive solution of } (4.6)\}
\]
which joins \((\lambda_1(na(x, \frac{1}{n})), 0)\) to infinity in \(\lambda\)-direction.

Combining this and (4.4) and (4.5) and using Lemma 3.2, it follows that there exists a connected component \(C\) in \(\limsup C^{[n]}\) which joins \((0, \theta)\) to infinity in \(\lambda\)-direction.

Finally, we show that
\[
(\lambda^*, 0) \in C \Rightarrow \lambda^* = 0. \tag{4.7}
\]

Suppose on the contrary that there is a sequence \(\{(\mu_n, u_n)\} \subset C\) such that \(\mu_n \to \mu^*\) with some \(\mu^* > 0\) and \(||u_n||_E > 0\), and \(u_n \to 0\) in \(E\). Then
\[
\begin{cases}
-\left(1 - |\nabla u_n|^2\right)\Delta u_n - \sum_{i,j=1}^N \partial_{x_i} u_n \partial_{x_j} u_n \partial_{x_i,x_j} u_n = \mu_n h(|\nabla u_n|)f(x, u_n, \nabla u_n) & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega.
\end{cases} \tag{4.8}
\]

Divide both sides of (4.8) by \(||u_n||_E\). Since \(y_n := \frac{u_n}{||u_n||_E}\) is bounded in \(W^{2,r}(\Omega)\), after taking a subsequence if necessary, we have that \(y_n \to \hat{y}\) for some \(\hat{y} \in E\) with \(\hat{y} \gg 0\) and \(||\hat{y}||_E = 1\).

Let \(B\) be a given ball in \(\Omega\). Let \(\lambda_1(B)\) be the first eigenvalue of the linear eigenvalue problem
\[
\begin{cases}
-\Delta z = \rho \mu^* z & \text{in } B, \\
z = 0 & \text{on } \partial B,
\end{cases} \tag{4.9}
\]
and let \(\varphi\) be the corresponding eigenfunction with \(\varphi \gg 0\). Then for \(\rho_0 := \frac{\lambda_1(B) \mu^*}{\mu^*} + 1 > 0\), it follows from (4.8) and the fact that \(f_0 = \infty\) that
\[
\begin{cases}
-\Delta \hat{y} \geq \mu^* \rho_0 \hat{y} & \text{in } B, \\
\hat{y} \geq 0 & \text{on } \partial B. \tag{4.10}
\end{cases}
\]

Since
\[
-\int_B \varphi \Delta \hat{y} dx = \int_B \nabla \hat{y} \nabla \varphi dx - \int_{\partial B} \varphi \cdot \frac{\partial \hat{y}}{\partial n} dS = \int_B \nabla \hat{y} \nabla \varphi dx, \tag{4.11}
\]
\[
-\int_B \hat{y} \Delta \varphi dx = \int_B \nabla \hat{y} \nabla \varphi dx - \int_{\partial B} \hat{y} \cdot \frac{\partial \varphi}{\partial n} dS. \tag{4.12}
\]

Combining (4.11) and (4.12) and using (4.9) and the fact $\frac{\partial \varphi}{\partial n} < 0$ on $\partial B$, it follows that

$$-\int_B \varphi \Delta y dx = -\int_B y \Delta \varphi dx + \int_{\partial B} y \cdot \frac{\partial \varphi}{\partial n} dS = \lambda_1(B) \int_B y \varphi dx + \int_{\partial B} y \cdot \frac{\partial \varphi}{\partial n} dS \leq \lambda_1(B) \int_B y \varphi dx,$$

which together with (4.10) implies that

$$\lambda_1(B) \geq \mu^* \rho_0.$$

However, this is a contradiction. Therefore, (4.7) holds.

5. **Global behavior of positive solutions in the case $f_0 = 0$.** To prove Theorem 1.2, we define $\alpha^{[n]} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\alpha^{[n]}(x, s) = \begin{cases} \alpha(x, s), & |s| \in \left(\frac{1}{n}, \infty\right), \\ n\alpha(x, \frac{1}{n}) s, & |s| \in \left[0, \frac{1}{n}\right], \end{cases} \quad (5.1)$$

and

$$f^{[n]}(x, s, \xi) = \alpha^{[n]}(x, s) + \beta(x, s, \xi) s. \quad (5.2)$$

Let us define a function $g^{[n]} : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by setting, for a.e. $x \in \bar{\Omega}$,

$$g^{[n]}(x, s, \xi) = \begin{cases} f^{[n]}(x, s, \xi), & |\xi| \leq 1, \\ f^{[n]}(x, s, \frac{\xi}{|\xi|}), & |\xi| > 1. \end{cases} \quad (5.3)$$

Obviously,

$$(g^{[n]})_0 = \lim_{s \to 0} \frac{g^{[n]}(x, s, \xi)}{s} = n\alpha(x, \frac{1}{n}) > 0 \quad \text{uniformly for } x \in \Omega \text{ and } \xi \in \mathbb{R}^N. \quad (5.4)$$

This together with (A5) implies that

$$\lim_{n \to \infty} (g^{[n]})_0 = 0. \quad (5.5)$$

Now let us consider the auxiliary family of the problems

$$\begin{cases} -\Delta u - \sum_{i,j=1}^N \partial_{x_i} u \partial_{x_j} u \partial_{x_i,x_j} u = \lambda h(|\nabla u|)g^{[n]}(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \quad (5.6)$$

From (5.6) and Theorem 2.2, there exists a component $C^{[n]}$ in the set $\mathcal{S}_n$

$$\mathcal{S}_n = \{ (\lambda, u) | (\lambda, u) \in [0, \infty) \times E, u \text{ is a positive solution of } (5.6) \} \subset \mathbb{R} \times E$$

which joins $(\lambda_1(n\alpha(x, \frac{1}{n})), 0)$ to infinity in $\lambda$-direction.
Lemma 5.1. Let \( r \in (0, 1) \) be given. Let \( \{ (\mu_k, y_k) \} \subset C^n \) with
\[
\mu_k \to \infty \quad \text{as } k \to \infty, \tag{5.7}
\]
and
\[
||\nabla y_k||_\infty = r. \tag{5.8}
\]
Then there exists positive constant \( \sigma = \sigma(r) > 0 \) (independent of \( n \)) such that
\[
\mu_k h(||\nabla y_k||)g^{[n]}(x, y_k, \nabla y_k) \geq \sigma > 0. \tag{5.9}
\]
Proof. Assume on the contrary that (5.9) is not valid. Then, after taking a subsequence and relabeling, if necessary, it follows that
\[
\mu_k h(||\nabla y_k||)g^{[n]}(x, y_k, \nabla y_k) \to 0 \quad \text{as } k \to \infty. \tag{5.10}
\]
This together with (5.6) implies that
\[
||y_k||_E \to 0 \quad \text{as } k \to \infty.
\]
However, this is impossible.

Lemma 5.2. Let \( r \in (0, 1) \) be given. Then there exists positive constant \( \Lambda = \Lambda(r) > 0 \) (independent of \( n \)) such that
\[
C^n \cap \{ (\lambda, u) | |\lambda| \geq \Lambda(r), ||u||_E = r \} = \emptyset, \quad \forall n \in \mathbb{N}. \tag{5.11}
\]
Proof. Assume on the contrary that there exists a sequence \( \{ (\mu_k, y_k) \} \subset C^n \) such that
\[
\mu_k \to \infty \quad \text{as } k \to \infty, \tag{5.12}
\]
and
\[
||\nabla y_k||_\infty = r. \tag{5.13}
\]
Let \( B_\rho \) be a given ball with \( \bar{B} \subset \Omega \). Then it follows from Lemma 5.1 that
\[
\mathcal{M} y_k \geq \sigma, \quad x \in B_\rho,
\]
\[
y_k \geq 0, \quad x \in \partial B_\rho,
\]
where \( \sigma \) is as in Lemma 5.1.

Let \( \psi_1 \in E \) be the unique solution of the problem
\[
\mathcal{M} w = \sigma, \quad x \in B_\rho,
\]
\[
w = 0, \quad x \in \partial B_\rho.
\]
Then, from the comparison principle of [13, Lemma 2.1], we have
\[
y_k(x) \geq \psi_1(x), \quad x \in B_\rho,
\]
and accordingly
\[
y_k(x) \geq \beta, \quad x \in B_\rho.
\]
By the divergence theorem,
\[
-\int_{\partial B_\rho} \nabla y_k \cdot \nu ds = -\int_{B_\rho} \operatorname{div} \left( \frac{\nabla y_k}{\sqrt{1 - |\nabla y_k|^2}} \right) dx
\]
\[
= \int_{B_\rho} \mu_k h(r)g^{[n]}(x, y_k, \nabla y_k) dx \to +\infty,
\]
where \( \nu \) denotes the unit outward normal to \( B_\rho \). This means that
\[
\min \{ |1 - |\nabla y_k|^2| | x \in \partial B_\rho \} = 0.
\]
However, this is impossible since \( ||\nabla y_k||_\infty = r < 1 \).
Lemma 5.3. Let $n_0 \in \mathbb{N}$ and $\sigma_* \in (0, 1)$ be given. Then for any sequence $\{ \mu_n, u_n \}$ in the set
\[ \mathcal{C}^{(n_0)} \setminus \{ (\mu, y) \mid \mu \in (0, \infty) \times \{ y \in E \mid ||y||_E \leq \sigma_* \} \} \] (5.14)
with $\mu_n \to \infty$, we have
\[ \lim_{n \to \infty} ||u_n||_E = 1. \] (5.15)

Proof. Assume on the contrary that there exists a sequence $\{ (\mu_n, u_n) \}$ in the set
\[ \mathcal{C}^{(n_0)} \setminus \{ (\mu, y) \mid \mu \in (0, \infty) \times \{ y \in E \mid ||y||_E \leq \sigma_* \} \} \] such that $\mu_n \to \infty$, $||u_n||_E \leq 1 - V$ for some constant $V \in (0, 1)$. Then $u_n$ is a positive eigenfunction corresponding to the first eigenvalue $\mu_n$ of the linear problem
\[ \begin{cases} -(1 - |\nabla u_n|^2) \Delta w - \sum_{i,j=1}^{N} \partial_{x_i} u_n \partial_{x_j} u_n \partial_{x_i, x_j} w = \mu_n \gamma_n w & \text{in } \Omega, \\ w = 0 & \text{on } \partial \Omega, \end{cases} \] (5.16)
where
\[ \gamma_n(x) = \begin{cases} h(|\nabla u_n|) \frac{g^{[n]}(x, u_n, \nabla u_n)}{u_n}, & u_n(x) \neq 0, \\ \alpha(x, 1/n), & u_n(x) = 0. \end{cases} \]
Combining the fact
\[ \lim_{n \to \infty} \gamma_n(x) = \alpha(x, 1/n) \quad \text{uniformly for } x \in \Omega, \]
and using Chow and Hale [8, Theorem 3.1 in Chapter 14], it deduces that
\[ \lim_{n \to \infty} \mu_n = \lambda_1(\alpha(\cdot, 1/n)), \] (5.17)
where $\lambda_1(\alpha(\cdot, 1/n))$ is the first eigenvalue of the linear problem
\[ \begin{cases} -\Delta u(x) = \mu_n \alpha(x, 1/n) u(x), & x \in \Omega, \\ u = 0, & x \in \partial \Omega. \end{cases} \] (5.18)
However, this contradicts the fact $\mu_n \to \infty$. Therefore, (5.15) is valid. \qed

Let
\[ Q = \{ (\lambda, u) \in [0, \infty) \times E \mid \lambda \geq \max \{ \Lambda(1/3), \Lambda(2/3) \}, \ u \geq 0, \ 1/3 < ||u|| < 2/3 \}. \]
Then, it follows from Lemma 5.2 and Lemma 5.3 that
\[ \mathcal{C}^{[n]} \cap Q = \emptyset, \quad \forall \ n \in \mathbb{N}. \]
Notice that the set
\[ \mathcal{Q} = \{ (\lambda, u) \in \mathcal{C}^{[n]} \mid ||u||_E = 1/2, \ n \in \mathbb{N} \} \]
is compact in $\mathbb{R} \times E$. Using this and Lemma 3.2 and Lemma 5.2 and Lemma 5.3 and the same argument used in the proof of Ma and An [19, Theorem 4.1], we may obtain
Lemma 5.4. There exists a connected component $\zeta$ in $S$ which satisfies

1. $\zeta \cap Q = \emptyset$;  

$$\lim_{(\lambda, u) \in \zeta \text{ with } ||u||_E > 1/2} \frac{||u||_E}{\lambda} = 1.$$  \hspace{1cm} (5.19)

2. $\zeta$ goes to infinity in $\lambda$-direction in $\{(\lambda, u) | (\lambda, u) \in [0, \infty) \times \{y \in E| ||y||_E \leq 1/2\}\}$, and  

$$\lim_{\lambda \to \infty, (\lambda, u) \in \zeta \text{ with } ||u||_E < 1/2} ||u||_\infty = 0.$$  \hspace{1cm} (5.20)

Proof. We only need to show (5.20).

Assume on the contrary that there exists a sequence $\{(\mu_n, u_n)\} \subset \zeta$ with  

$$\mu_n \to \infty, \quad ||u_n||_\infty \geq \sigma_1 > 0.$$  

Let  

$$C_2 := \inf\{f(x, s, p) | x \in \Omega, \quad \sigma_1 \leq s \leq d(\Omega)/2, \quad p \in B_1(0)\}.$$  

Then $C_2 > 0$, and subsequently,  

$$-\mathcal{M}(u_n) = \mu_n h(|\nabla u_n|) f(x, u_n, \nabla u_n) \geq \mu_n h(1/2) C_2 \to \infty,$$  

which implies  

$$||\nabla u_n||_\infty = 1.$$  

This contradicts the fact $||\nabla u_n||_\infty \leq 1/2$. \hfill \Box

Set  

$$\text{Proj}_R \zeta = \{\lambda | (\lambda, u) \in \zeta\}.$$  

Lemma 5.5. For any $\mu \in \text{Proj}_R \zeta$, there exists a constant $\delta(\mu) > 0$ such that  

$$||y||_E \geq \delta(\mu) \quad \text{for all } y \in \{u| (\mu, u) \in \zeta\}.$$  

Proof. Assume on the contrary that there exists $\{(\tilde{\mu}, y_n)\} \subset \zeta$ such that  

$$||y_n||_E \to 0, \quad n \to \infty.$$  \hspace{1cm} (5.21)

Let $v_n := \frac{y_n}{||y_n||_E}$. Then it follows from (5.16) and the standard method that there exists $v^* \in E$ with $||v^*||_E = 1$, and  

$$\begin{cases} -\Delta v^* = \tilde{\mu} 0 v^* & \text{in } \Omega, \\ v^* = 0 & \text{on } \partial\Omega, \end{cases}$$  

which implies $v^* = 0$. However, this contradicts the fact $||v^*||_E = 1$. \hfill \Box

Lemma 5.6.  

$$\inf \{\lambda | (\lambda, u) \in \zeta\} > 0.$$  

Proof. Assume on the contrary that there exists $\{(\mu_n, u_n)\} \subset \zeta$ such that  

$$\mu_n \to 0, \quad n \to \infty.$$  \hspace{1cm} (5.22)

Since $||u_n||_E < 1$ and $||u_n||_\infty \leq \frac{d(\Omega)}{2}$, it follows from \cite[Lemma 2.2]{13} that  

$$||u_n||_{C^{1, \alpha}} \leq C,$$  

where $C$ is a constant independent of $n$. So we may assume that $u_n \to u^*$ in $E$. From Lemma 5.4, we know that  

$$u^* \neq 0.$$  \hspace{1cm} (5.23)
It follows from
\[
\begin{aligned}
M(u_n) &= \mu_n h(|\nabla u_n|) f(x, u_n, \nabla u_n) \quad \text{in } \Omega, \\
u_n &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
that
\[
\begin{aligned}
-\Delta u^* &= 0 h(|\nabla u^*|) f(x, u^*, \nabla u^*) \quad \text{in } \Omega, \\
u^* &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
which implies that \( u^* = 0 \). However this contradicts (5.23).
\( \square \)

**Proof of Theorem 1.2.** It is an immediate consequence of Lemmas 5.4-5.6. \( \square \)

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