COMPUTATION OF SOME TRANSCENDENTAL INTEGRALS
FROM PATH SIGNATURES

ANDREW URSITTI

Abstract. It is shown that if $\gamma$ is a path of finite $p$ variation ($1 \leq p < 2$) in a
Euclidean vector space and $f, g, h$ are Lipschitz functions on the trace of $\gamma$ then
$s \mapsto F(s) = \int_{\gamma} f^* gdh$ defines an entire holomorphic function provided the con-
vex hull of the image of $f$ does not contain zero. If in addition $|\log z| \leq \log 2$
on the convex hull of the image of $f$ then for any $s \in \mathbb{C}$, $F(s)$ can be computed
from the nonnegative integer values \{\(F(k)\)\}_{k \in \mathbb{N}}. If in addition to these hy-
potheses each of $f, g, h$ is a polynomial, then the values $F(k)$ are computable
directly from the signature of $\gamma$ thus all values of $F(s)$ are computable from
the signature. As a special case the winding number of a closed path $\gamma$ around an
affine submanifold of codimension two is computed from finitely many terms
of the signature provided certain estimates are satisfied.

1. Introduction

In this note we will show how certain transcendental integrals of the form
$\int_{\gamma} f^* gdh$ can be algorithmically recovered from the signature of the path $\gamma$. As
a special case we will give an alternate proof of a result due originally to P. Yam
[Yam08] concerning the recovery of the winding number of a path around a codi-
mension two affine submanifold from the signature of the path provided certain
estimates are satisfied. The signature of $\gamma$ is the infinite tensor (i.e. formal series)
$X_\gamma \in \bigoplus_{k \geq 0} V^\otimes k$ defined in degree zero to be 1 and in degree $k > 0$ by the iterated
integral of tensors
$$\int_{0 < t_1 < \cdots < t_k < T} \, d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_k}.$$ 

The signature is a homomorphism from the collection of all paths beginning at zero
of finite $p$ variation ($1 \leq p < 2$), viewed as a group under concatenation, into the
group of infinite tensors with 1 in degree zero. An orientation preserving change
of parameter does not change the signature, and an orientation reversing change of
parameter inverts the signature, as was proved by K.T. Chen [Che58] for piecewise
$C^1$ paths (the corresponding results are easily proved for paths of finite $p$ variation
with $1 \leq p < 2$ using the Young-Löeve integration theory [Yon36,FV10]). Choosing
a specific path and concatenating it with its inverse therefore produces a path with
trivial signature, and Chen later proved [Che58] that concatenations of such paths are
essentially the only paths with trivial signature. Specifically, Chen defined a path to be irreducible if it doesn’t contain any segments which consist of a path and its
inverse concatenated in succession, and then proved that two irreducible paths

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have the same signature if and only if they differ by a translation and an orientation preserving change of parameter. This was later generalized to paths of bounded variation by Hambly and Lyons [HL10], who defined the notion of tree-like paths and proved that two paths of bounded variation have the same signature if and only if the concatenation of one with the inverse of the other is a lipschitz tree-like path. Boedihardjo, Ni and Qian [BNQ14] proved that two simple paths of finite p variation (1 ≤ p < 2) in the plane have the same signature if and only if they differ by translation and orientation preserving change of parameter. These uniqueness results show that one should expect topological data such as the winding number to be contained in the signature.

It should be mentioned that in [BNQ14] the authors also proved that for a closed path in the plane with variation less than two, the moments of the winding number about any specific point can be recovered from the signature by first evaluating the signature on Lyndon words to recover any value in section 2 (Theorem 2.3). Under these hypotheses we shall prove in section 2 that (E, g, h) ↦ \int E \log(\gamma)gdh defines a C-trilinear map \theta(\mathbb{C}) × \text{Lip}(S_γ) × \text{Lip}(S_γ) → \mathbb{C} and give an explicit bound on its absolute value (Corollary 2.2). Choosing E = E_k,s ∈ \theta(\mathbb{C}) given by E_k,s(z) = z^k e^{sz} defines the integral \int E \log f^k f^*gdh and naturally one expects that varying the parameter s ∈ \mathbb{C} should produce an entire function with derivative \int E \log f^k f^*gdh interpolates the values of \{F(k)\}_{k \in \mathbb{N}}. In section 3 we use a general procedure developed by Boas and Buck [BB64,Buc47,Buc48] to recover any value F(s) from the nonnegative integer values \{F(k)\}_{k \in \mathbb{N}}, provided certain estimates are satisfied. Specifically, we shall prove:

1Here \mathbb{N} = \{0, 1, 2, 3, \ldots\}, i.e. \ 0 ∈ \mathbb{N}. 
Theorem 1.1. If $|\log z| < \log 2$ for all $z \in h(f(S_γ))$, then the series

$$
\sum_{0 \leq n} (-1)^n \binom{s}{n} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} \int_γ f^k gdh
$$

converges to $\int_γ f^* gdh$.

Here $h(\cdot)$ denotes the convex hull. It should be noted that this is an iterated sum, and the order of summation should not be changed (however, the sum in $n$ is absolutely convergent once the inner sums in the parameter $k$ are computed). The inequality $|\log z| < \log 2$ is satisfied by at most one branch of the logarithm on $h(f(S_γ))$, since $\log 2$ is less than $2\pi$, and this is a crucial observation since superficially Theorem 1.1 implies that $F(s)$, which depends on the chosen branch of the logarithm, can be computed from the integer-exponent values $\{F(k)\}_{k \in \mathbb{N}}$, which do not depend on this choice.

If $f, g$ and $h$ are polynomials then $F(k) = \int_γ f^k gdh$ can be extracted directly from the signature of $γ$. Specifically, if $γ(0) = 0$ and if $x_1, \ldots, x_d$ is a basis for $V^*$ and $α, β \in \mathbb{N}^d$ are multi-indices, then

(1.1)

$$
\int_γ x^α d(x^β) = \sum_{1 \leq i \leq d} \beta_i \sum_{σ ∈ Σ_{|α|+|β|-1}} \langle σ((x_1^{α_1+β_1}) \cdot \cdots \cdot x_d^{α_d+β_d})/x_1 \rangle \otimes x_i, X_γ \rangle.
$$

In this expression, $(x_1^{α_1+β_1}) \cdot \cdots \cdot x_d^{α_d+β_d})/x_i$ indicates the tensor $(x_1^{α_1+β_1}) \otimes \cdots \otimes x_d^{α_d+β_d}$ with exactly one factor $x_i$ removed. If $β_i > 0$ (which is the only case that matters) at least one factor of $x_i$ appears inside of a consecutive list of such factors in $x_1^{α_1+β_1} \otimes \cdots \otimes x_d^{α_d+β_d}$, so this “division” operation is well defined. The entire argument $σ((x_1^{α_1+β_1}) \cdot \cdots \cdot x_d^{α_d+β_d})/x_i$ can be interpreted as follows: from $x_1^{α_1+β_1} \cdot \cdots \cdot x_d^{α_d+β_d}$, remove one factor $x_i$, let the permutation $σ$ permute the remaining $|α| + |β| - 1$ factors, then replace the factor $x_i$ on the right.

Integrals of the form $\int_γ PdQ$ where $P$ and $Q$ are polynomials can then be computed by splitting $P$ and $Q$ into monomials and using (1.1). With minor adjustments one can do away with the requirement $γ(0) = 0$.

In particular, if $f, g$ and $h$ are polynomials then the values $\{F(k)\}_{k \in \mathbb{N}}$ can be extracted directly from $X_γ$ so evidently Theorem 1.1 shows that if $|\log z| < \log 2$ on the convex hull of the trace of $f \circ γ$, then every value of the entire function $F(s) = \int_γ f^* gdh$ can be recovered from the signature of $γ$. In particular, if in addition we assume that $γ$ is a closed path then we can use this method to recover the winding number of $γ$ around the codimension two affine submanifold $\{x_1 = ξ_1, x_2 = ξ_2\}$ provided the standing hypothesis $1/2 < (x_1 \circ γ - ξ_1)^2 + (x_2 \circ γ - ξ_2)^2 < 2$ is satisfied. The $x_1 \wedge x_2$-oriented winding number of $γ$ around $\{x_1 = ξ_1, x_2 = ξ_2\}$ is given by

$$
W_γ(x_1 \wedge x_2; ξ_1, ξ_2) = \frac{1}{2π} \int_γ \frac{(x_1 - ξ_1)dx_2 - (x_2 - ξ_2)dx_1}{(x_1 - ξ_1)^2 + (x_2 - ξ_2)^2}.
$$

Thus, further specifying the parameters to $f = (x_1 - ξ_1)^2 + (x_2 - ξ_2)^2$, $g = x_1 - ξ_1$, $h = x_2 - ξ_2$, and then switching $g$ and $h$ for the second summand, we find that

$$
F(s) = \int_γ [(x_1 - ξ_1)^2 + (x_2 - ξ_2)^2]^s((x_1 - ξ_1)dx_2 - (x_2 - ξ_2)dx_1)
$$

This suggests the following algorithm for computing $F(s)$ and $F(k)$ for $k \geq 1$.
can be recovered from the signature by Theorem 1.1 provided that \(1/2 < (x_1 \circ \gamma - \xi)^2 + (x_2 \circ \gamma - \xi)^2 < 2\). In particular, the winding number \(\frac{1}{|\pi E(-1)|} = W_\gamma(x_1 \wedge x_2; \xi_1, \xi_2)\) can be found in this manner. Specifically, we prove:

**Theorem 1.2.** If in addition to the standing hypotheses \(\gamma\) is a closed path, then

\[
W_\gamma(x_1 \wedge x_2; \xi_1, \xi_2) = \frac{1}{2\pi} \sum_{0 \leq n \leq N} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} \sum_{k_1 + k_2 = k} \frac{k!}{k_1!k_2!} \sum_{j_1 + j_2 = 2k_1 j_1^1 j_2^1 = 2k_1 j_2^2 = 2k_2} (2k_1)! (2k_2)! \left\langle T_{j_1^1,j_2^1} T_{j_2^2}, X_\gamma \right\rangle
\]

where \(T_{j_1^1,j_2^1} \in V^* \otimes (j_1^1 + j_2^1 + 1) \otimes V^* \otimes (j_1^2 + j_2^2)\) is defined in (4.1).

In addition to this, it is shown that \(W_\gamma(x_1 \wedge x_2; \xi_1, \xi_2)\) can be computed from only finitely many terms in the signature, and an estimate on how many terms are necessary is given. All of this is detailed in section 4.

2. **Regularity of \(I_\gamma\)**

To condense notation, define

\[
M(E, f, g) = \|g\|_{\text{Lip}(S_\gamma)} \max_{f(S_\gamma)} |E \circ \log | f(S_\gamma)| + \|f\|_{\text{Lip}(S_\gamma)} \max_{S_\gamma} |g| \max_{h(S_\gamma)} |E' \circ \log | f(S_\gamma)| \frac{\text{dist}(0, h(f(S_\gamma)))}{\text{dist}(0, h(f(S_\gamma)))}
\]

Regarding \(\text{dist}(0, h(f(S_\gamma)))\), we note that \(f(S_\gamma)\) is compact so \(h(f(S_\gamma))\) is closed and since zero is not in \(h(f(S_\gamma))\) by hypothesis, \(\text{dist}(0, h(f(S_\gamma)))\) is positive.

**Lemma 2.1.** The map \((E \circ \log \circ f)g : S_\gamma \to C\) is Lipschitz on \(S_\gamma\) and satisfies \(\|(E \circ \log \circ f)g\|_{\text{Lip}(S_\gamma)} \leq M(E, f, g)\).

**Proof.** Let \(w_1, w_2 \in C, z_1, z_2 \in h(f(S_\gamma))\) and let \(l_{z_1}^{z_2} \subset h(f(S_\gamma))\) denote the oriented line segment connecting \(z_1\) to \(z_2\) so that

\[
|E(\log z_2)w_2 - E(\log z_1)w_1| = |w_2(E(\log z_2) - E(\log z_1)) + E(\log z_1)(w_2 - w_1)|
\leq |w_2||E(\log z_2) - E(\log z_1)| + |E(\log z_1)||w_2 - w_1|
= |w_2| \int_{l_{z_1}^{z_2}} E'(\log z) z^{-1} dz + |E(\log z_1)||w_2 - w_1|
\leq |w_2| \left(\max_{z \in l_{z_1}^{z_2}} |E'(\log z) z^{-1}|\right) |z_2 - z_1| + |E(\log z_1)||w_2 - w_1|
\]
For any $t_1, t_2 \in [0, T]$, we can use this estimate with $w_1 = g(\gamma(t_1))$, $w_2 = g(\gamma(t_2))$, $z_1 = f(\gamma(t_1))$ and $z_2 = f(\gamma(t_2))$ to write

$$|E(\log f(\gamma(t_2)))g(\gamma(t_2)) - E(\log f(\gamma(t_1)))g(\gamma(t_1))|$$

$$\leq |g(\gamma(t_2))| \left( \max_{z \in f'(\gamma(t_1))} |E'(\log z)| z^{-1} \right) |f(\gamma(t_2)) - f(\gamma(t_1))|$$

$$+ |E(\log f(\gamma(t_1)))||g(\gamma(t_2)) - g(\gamma(t_1))||$$

$$\leq \max_{S, g} |g| \max_{h(f(S_1))} |(E' \circ \log(\cdot))(\cdot)^{-1}| \|f\|_{\text{Lip}(S_1)} |\gamma(t_2) - \gamma(t_1)|$$

$$+ \max_{f(S_1)} |E \circ \log(\cdot)||g||_{\text{Lip}(S_1)} |\gamma(t_2) - \gamma(t_1)|$$

$$\leq \max_{S, g} |g| \max_{h(f(S_1))} |E' \circ \log(\cdot)||f||_{\text{Lip}(S_1)}$$

$$\times \text{dist}(0, h(f(S_1)))^{-1} |\gamma(t_2) - \gamma(t_1)|$$

$$+ \max_{f(S_1)} |E \circ \log(\cdot)||g||_{\text{Lip}(S_1)} |\gamma(t_2) - \gamma(t_1)|$$

$$= M(E, f, g) |\gamma(t_2) - \gamma(t_1)|.$$

Combining Lemma 2.1 with the Young-Léve integration theory [You36], we have the following corollary:

**Corollary 2.2.** The expression

$$I^f_1(E, g, h) = \int_\gamma (E \circ \log \circ f)gdh$$

defines a $C$-trilinear map $I^f_1(\cdot, \cdot, \cdot) : \mathcal{O}(C) \times \text{Lip}(S_1) \times \text{Lip}(S_2) \to C$ which satisfies the estimate

$$|I^f_1(E, g, h)| \leq \frac{1}{1 - 2^{1-2/p}} M(E, f, g)||h||_{\text{Lip}(S_1)} |\gamma|_{\mathbb{P};[0, T]}^2$$

$$+ |E \circ \log \circ f \circ \gamma(0)||h \circ \gamma(T) - h \circ \gamma(0)|.$$

**Proof.** From Young’s estimate (or rather a variation thereof presented in [FV10], e.g.),

$$|I^f_1(E, g, h)| \leq \frac{1}{1 - 2^{1-2/p}} \|(E \circ \log \circ f \circ \gamma)(g \circ \gamma)\|_{\mathbb{P};[0, T]} |h \circ \gamma(T) - h \circ \gamma(0)|$$

$$+ |E \circ \log \circ f \circ \gamma(0)||g \circ \gamma(0)||h \circ \gamma(T) - h \circ \gamma(0)|$$

$$\leq \frac{1}{1 - 2^{1-2/p}} M(E, f, g)||h||_{\text{Lip}(S_1)} |\gamma|_{\mathbb{P};[0, T]}^2$$

$$+ |E \circ \log \circ f \circ \gamma(0)||g \circ \gamma(0)||h \circ \gamma(T) - h \circ \gamma(0)|.$$

Now we would like to consider simultaneously the family of entire functions $\{E_k, s : k \in \mathbb{N}, s \in \mathbb{C}\}$ given by $E_k, s(z) = z^k e^{zs}$, thus producing the integrals

$$I^f_1(E_{k, s}, g, h) = \int_\gamma (E_{k, s} \circ \log \circ f)gdh = \int_\gamma (\log f)^k e^{s \log f} gdh = \int_\gamma (\log f)^k f^s gdh.$$

**Theorem 2.3.** For any $k \in \mathbb{N}$, $s \mapsto I^f_1(E_{k, s}, g, h)$ defines an entire function.
Proof. First, it must be proved that \( I'_I(E_{k,(1)}, g, h) \) is differentiable. The natural guess for the derivative is of course \( I'_I(E_{k+1,(1)}, g, h) \) so we attempt to verify the asymptotic equality
\[
I'_I(E_{k,s}, g, h) = I'_I(E_{k,s_o}, g, h) + (s - s_o)I'_I(E_{k+1,s_o}, g, h) + o(|s - s_o|)
\]
for every \( s_o \in \mathbb{C} \). However, since \( E \rightarrow I'_I(E, g, h) \) is \( \mathbb{C} \)-linear this is implied by
\[
|I'_I(E_{k,s}, g, h)| = o(|s - s_o|)
\]
for every \( s_o \in \mathbb{C} \) where \( I'_I(E_{k,s,s_o}, g, h) \) is \( \mathcal{O}(\mathbb{C}) \) is given by
\[
E_{k,s,s_o}(z) = z^k e^{s z} - z^k e^{s_o z} + (s - s_o)z^{k+1} e^{s_o z} = z^k e^{s_o z}(e^{(s-s_o)z} - 1) - (s - s_o).z.
\]
The rightmost expression shows that \( |E_{k,s,s_o}(z)| = O(|s - s_o|^2) \) pointwise for every \( z \) and uniformly for \( z \) in any bounded subset of \( \mathbb{C} \). In particular
\[
|\nabla h f(\gamma(0))| = o(|s - s_o|)
\]
and since \( \log \) must map the compact set \( f(S_\gamma) \) into another compact set,
\[
\max_{z \in f(S_\gamma)} |E_{k,s,s_o}(\log z)| = o(|s - s_o|).
\]
Also,
\[
E'_{k,s,s_o}(z) = kz^{k-1} e^{s_o z}(e^{(s-s_o)z} - 1) - (s - s_o)z
\]
\[
+ z^k e^{s_o z}(e^{(s-s_o)z} - 1) - (s - s_o) + z^k e^{s_o z}(s - s_o)(e^{(s-s_o)z} - 1)
\]
thus \( |E'_{k,s,s_o}(z)| = O(|s - s_o|^2) \) pointwise and uniformly in bounded subsets. In particular
\[
\max_{z \in f(S_\gamma)} |E'_{k,s,s_o}(\log z)| = o(|s - s_o|).
\]
On combining (2.1), (2.2) and (2.3), \( I'_I(E_{k,s,s_o}, g, h) = o(|s - s_o|) \) by the estimate given in Corollary 2.2. This proves that \( I'_I(E_{k,(1)}, g, h) \in \mathcal{O}(\mathbb{C}) \) with derivative \( I'_I(E_{k+1,(1)}, g, h) \).

3. Proof of theorem 1.1

Recall from the introduction that \( \gamma : [0, T] \rightarrow V \) is a path of \( p \) variation \((1 \leq p < 2)\) taking values in the euclidean vector space \( V \), with signature \( X_\gamma \in \bigoplus_{k \geq 0} V^{\otimes k} \), as described in the introduction. The functions \( f, g, h : S_\gamma \rightarrow \mathbb{C} \) are Lipschitz and the convex hull of the image of \( f \) does not contain zero. Our task in this section is to prove Theorem 1.1 so we will herein assume that \( \log : h(f(S_\gamma)) \rightarrow \mathbb{C} \), if it exists, is the unique branch of the logarithm such that \( |\log z| < \log 2 \) on \( h(f(S_\gamma)) \). Such a unique logarithm exists, for instance, if \( f \) is positive and satisfies \( 1/2 < f < 2 \) on \( S_\gamma \), for then \( h(f(S_\gamma)) = f(S_\gamma) \) is a compact subinterval of \((1/2, 2)\) whence \( |\log f| < \log 2 \).

Thus, with \( F(s) = I'_I(E_{0,s}, g, h) = \int_s f^s gdh \) as in the introduction, we are tasked with computing the value \( F(s) \) from the known values \( \{F(k)\}_{k \in \mathbb{N}} \) which come directly from the signature. There is a general procedure developed by Boas and Buck \[BB64,Buc47,Buc48\] which can accomplish this task, provided that \( F \) satisfies certain growth conditions at infinity. It seems appropriate to briefly describe the procedure rather than simply quoting the relevant results. If \( H \in \mathcal{O}(\mathbb{C}) \) is a generic entire function which satisfies an estimate of the form \( |H(z)| \leq Ae^{B|z|} \) for \( z \in (1, \infty) \) then its Laplace transform \( \mathcal{L}H(w) = \int_0^\infty H(z)e^{-wz}dz \) defines a
holomorphic function in the region \( \{ \Re w > B \} \), and it is natural to ask if \( \mathcal{L}H \) extends to a holomorphic function in the neighborhood of infinity defined by \( \{|w| > B\} \). If this is the case then it is easy to deduce what the Taylor coefficients at infinity must be if \( w \in (B, \infty) \) then

\[
\mathcal{L}H(w) = \int_0^\infty H(z) e^{-wz} dz = \sum_{n \geq 0} \frac{H(n)(0)}{n!} \int_0^m z^n e^{-wz} dz + \int_m^\infty H(z) e^{-wz} dz.
\]

By letting \( m \to \infty \) the remainder tends to zero and we recognize \( \Gamma(n+1) = n! \) in each term so that \( \mathcal{L}H(w) = \sum_{n \geq 0} H(n)(0)/w^{n+1} \) on \((B, \infty)\). Therefore, \( \mathcal{L}H \) will extend to the region \( \{|w| > B\} \) provided that \( \limsup_{n \to \infty} |H(n)(0)|^{1/n} < B \) for then

\[
\limsup_{n \to \infty} \left| \frac{H(n)(0)}{w^{n+1}} \right|^{1/n} = \frac{1}{|w|} \limsup_{n \to \infty} \frac{|H(n)(0)|^{1/n}}{|w|^{1/n}} < 1.
\]

This will be the case if the estimate \( |H(z)| < Ae^{B|z|} \) holds for all \( z \) and not only for \( z \in (1, \infty) \), for then by Cauchy’s estimate \( |H(n)(0)| < n!Ae^{B/r}/n^r \) for all \( r > 0 \) and this is minimized at \( r = n/B \) so that \( |H(n)(0)| < n!AB^n e^n/n^n \). Therefore

\[
|H(n)(0)|^{1/n} < A^{1/n} B \frac{e(n)/n}{n} = (2\pi n)^{1/2n} A^{1/n} B \frac{e(n/2)/n}{n}.
\]

so that \( \limsup_{n \to \infty} |H(n)(0)|^{1/n} < B \), by Stirling’s estimate. Thus, the power series \( \mathcal{B}H(w) = \sum_{n \geq 0} H(n)(0)/w^{n+1} \) converges absolutely to an analytic function, uniformly on compact subsets of the region \( \{|w| > B\} \), and therefore defines a holomorphic function in a neighborhood of \( \infty \in \mathbb{P}_\mathbb{C} \), taking the value 0 at \( \infty \) and extending \( \mathcal{L}H \). The extension \( \mathcal{B}H \) of \( \mathcal{L}H \) is usually referred to as the Borel transform of \( H \).

We can invert this procedure as follows. If \( r > B \) then for fixed \( r \) both power series \( \mathcal{B}H(w) = \sum_{n \geq 0} H(n)(0)/w^{n+1} \) and \( e^{zw} = \sum_{n \geq 0} z^n w^n/n! \) converge absolutely and uniformly on the circle \( \{|w| = r\} \) and therefore

\[
\int_{|w|=r} \mathcal{B}H(w) e^{zw} dw = \sum_{n \geq 0} \left( \sum_{k \geq 0} \frac{H(k)(0)}{n!} \int_{|w|=r} w^{n-k-1} dw \right) z^n,
\]

but \( \int_{|w|=r} w^{n-k-1} dw \) is nonzero only if \( n = k \) so evidently

\[
(3.1) \quad H(z) = \frac{1}{2\pi i} \int_{|w|=r} \mathcal{B}H(w) e^{zw} dw.
\]

This is called the Pólya representation of \( H \), it is valid not only for the contour \( \{|w| = r\} \) but any simple closed contour contained in \( \{|w| > B\} \) and it suggests a generalization, due to Buck, which will allow us to compute any value \( H(s) \) from \( \{H(k)\}_{k \in \mathbb{N}} \) and thus prove Theorem 1.1 by substituting \( F(s) = \int g dh \) for \( H \). The essence of Buck’s method is that rather than settling only for the series expansion \( e^{zw} = \sum_{n \geq 0} z^n w^n/n! \), we can choose to write \( e^{zw} \) in any of a
number of different ways. In particular we will be interested in the binomial series expansion:

\[ e^z = (e^w)^z = (e^w - 1 + 1)^z = \sum_{n \geq 0} \begin{pmatrix} z \\ n \end{pmatrix} (e^w - 1)^n, \]

which is valid in the region defined by \(|e^w - 1| < 1\).

**Lemma 3.1.** The inclusion \(\{|w| = r\} \subset \{|e^w - 1| < 1\}\) holds if and only if \(r < \log 2\).

**Proof.** If \(w = x + iy\) then \(|e^w - 1|^2 = e^{2x} - 2e^x \cos y + 1\), so the first observation to be made is that if \(|w| = r\) and \(|e^w - 1| < 1\) then \(r < \pi/2\) necessarily, for otherwise the circle \(\{|x + iy| = r\}\) contains points with \(\cos y < 0\) which would imply \(|e^w - 1|^2 = e^{2x} - 2e^x \cos y + 1 > 1\). Having reduced consideration to \(r < \pi/2\), we observe that \((x, y) \mapsto e^{2x} - 2e^x \cos y + 1\) can achieve a maximum at a point \((x, y)\) in the circle \(x^2 + y^2 = r^2\) only if its gradient is orthogonal to \((y, -x)\), or in other words only if \(ye^x - y \cos y - x \sin y = 0\). Since \((\pm r, 0)\) can be checked individually we only care about the case \(0 < |y| < \pi/2\) and the necessary condition in this case reduces to \(e^x - \cos y - (\sin y/y)x = 0\) with \(\cos y, \sin y/y > 0\) so that the minimum value of \(x \mapsto e^x - \cos y - (\sin y/y)x\) is \((\sin y/y) - \cos y - (\sin y/y) \log(\sin y/y), \) but this is positive for \(y \in (-\pi/2, 0) \cup (0, \pi/2)\) and so \(ye^x - y \cos y - x \sin y = 0\) and \(|y| < \pi/2\) imply \(y = 0\). Thus, the extremal values of \((x, y) \mapsto e^{2x} - 2e^x \cos y + 1\) on the circle \(\{|x + iy| = r\}\) must occur at \((\pm r, 0)\). The maximum and minimum values are therefore \(e^{2r} - 2e^r + 1\) and \(e^{-2r} - 2e^{-r} + 1\) respectively and one finds \(r = \log 2\) as the threshold value for the inclusion of sets stated in the lemma. \(\square\)

So, if \(H\) is such that \(|H(z)| \leq Ae^{B|z|}\) with \(B < \log 2\) then \(r\) can be chosen such that \(\{|w| = r\}\) lies in both the region of absolute convergence of the Borel transform \(B\) and the region of absolute convergence of the series \(e^{zw} = \sum_{n \geq 0} \begin{pmatrix} z \\ n \end{pmatrix} (e^w - 1)^n\) and therefore

\[
H(z) = \frac{1}{2\pi i} \int_{|w| = r} B H(w)e^{zw} dw \\
= \sum_{n \geq 0} \begin{pmatrix} z \\ n \end{pmatrix} \frac{1}{2\pi i} \int_{|w| = r} B H(w)(e^w - 1)^n dw \\
= \sum_{n \geq 0} \begin{pmatrix} z \\ n \end{pmatrix} \sum_{0 \leq k \leq n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} \frac{1}{2\pi i} \int_{|w| = r} B H(w)e^{kw} dw \\
= \sum_{n \geq 0} \begin{pmatrix} z \\ n \end{pmatrix} \sum_{0 \leq k \leq n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} H(k)
\]

and therefore

\[
(3.2) \quad H(z) = \sum_{0 \leq n} (-1)^n \begin{pmatrix} z \\ n \end{pmatrix} \sum_{0 \leq k \leq n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} H(k).
\]

To finish the proof of Theorem 1.1 we need only to observe that the estimate \(|\log f| < \log 2\) implies the required growth condition \(|F(s)| \leq Ae^{B|s|}\) with \(B < \log 2\). This is a simple consequence of the results of section 2, specifically Corollary 2.2.
4. The winding number

Recall from the introduction that if in addition to the standing hypotheses we assume that $\gamma$ is a closed path, then

$$W_{\gamma}(x_1 \wedge x_2; \xi_1, \xi_2) = \frac{1}{2\pi} \int_{\gamma} \frac{(x_1 - \xi_1)dx_2 - (x_2 - \xi_2)dx_1}{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}$$

is the $x_1 \wedge x_2$-oriented winding number around the codimension two affine submanifold $\{x_1 = \xi_1, x_2 = \xi_2\}$ and it can be computed using Theorem 1.1 and the signature $X_{\gamma}$ since $W_{\gamma}(x_1 \wedge x_2; \xi_1, \xi_2) = \frac{1}{2\pi} F(-1)$ where $F$ is the entire function defined by

$$F(s) = \int_{\gamma} [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]^s ((x_1 - \xi_1)dx_2 - (x_2 - \xi_2)dx_1).$$

Our first task in this section is to prove Theorem 1.2. By (3.2),

This completes the proof of Theorem 1.2.

Provided the standing hypothesis $1/2 < (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 < 2$ is satisfied. Now the values $F(k)$ for $k \in \mathbb{N}$ can be computed from the signature in a rather explicit fashion using (1.1):

$$F(k) = \int_{\gamma} [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]^k ((x_1 - \xi_1)dx_2 - (x_2 - \xi_2)dx_1)$$

$$= \sum_{k_1 + k_2 = k} \frac{k!}{k_1!k_2!} \int_{\gamma} (x_1 - \xi_1)^{2k_1} (x_2 - \xi_2)^{2k_2} ((x_1 - \xi_1)dx_2 - (x_2 - \xi_2)dx_1)$$

$$= \sum_{k_1 + k_2 = k} \frac{k!}{k_1!k_2!} \sum_{j_1 + j_2 = 2k} \sum_{j_2 + 2j_2 = 2k_2} \frac{(2k_1)!}{j_1!j_1!} \frac{(2k_2)!}{j_2!j_2!} \langle T_{j_1,j_2}^{j_1,j_2}, \gamma \rangle$$

where $T_{j_1,j_2}^{j_1,j_2} \in V^{* \otimes (j_1 + j_2 + 1)} \bigoplus V^{* \otimes (j_1 + j_2 + 2)}$ is the dual tensor

$$T_{j_1,j_2}^{j_1,j_2} = (x_1 \circ \gamma(0) - \xi_1)^{j_1} (x_2 \circ \gamma(0) - \xi_2)^{j_2}$$

$$\times \left( \sum_{\sigma \in S_{j_1 + j_2 + 1}} \sigma [x_1^{(j_1 + 1)} \otimes x_2^{j_2}] \otimes x_1 - \sigma [x_1^{j_1} \otimes x_2^{(j_2 + 1)}] \otimes x_1 \right.$$}

$$+ (x_1 \circ \gamma(0) - \xi_1) \sum_{\sigma \in S_{j_1 + j_2}} \sigma [x_1^{j_1} \otimes x_2^{j_2}] \otimes x_2$$

$$- (x_2 \circ \gamma(0) - \xi_2) \sum_{\sigma \in S_{j_1 + j_2}} \sigma [x_1^{j_1} \otimes x_2^{j_2}] \otimes x_1 \right)$$

This completes the proof of Theorem 1.2.

For computational purposes one should exploit the fact that the winding number is an integer, and as such it is known once it is known within an error strictly less
than 1/2. Specifically, if \(- \log 2 < -r < \log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2] < r < \log 2\) then for any \(N\),

\[
2\pi W_\gamma(x_1 \land x_2; \xi_1, \xi_2) - \sum_{0 \leq n \leq N} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} F(k)
\]

\[
= \left| \sum_{N+1 \leq n \leq N} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} F(k) \right|
\]

\[
= \left| \sum_{N+1 \leq n \leq N} \frac{(-1)^n}{2\pi i} \int_{|w|=r} \mathcal{B} F(w)(e^w - 1)^n dw \right|
\]

\[
\leq \frac{1}{2\pi} \| \mathcal{B} F \|_{L^1(|w|=r)} \sum_{N+1 \leq n} \| e^{(\cdot)} - 1 \|_{L^\infty(|w|=r)} \frac{(e^{2r} - 2e^r + 1)^{N+1}}{2e^r - e^{2r}}
\]

by Lemma 3.1. We have proved:

**Corollary 4.1.** If \(N \in \mathbb{N}\), \(- \log 2 < -r < \log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2] < r < \log 2\) and

\[
\| \mathcal{B} F \|_{L^1(|w|=r)} \frac{(e^{2r} - 2e^r + 1)^{N+1}}{2e^r - e^{2r}} < 2\pi^2
\]

then \(W_\gamma(x_1 \land x_2; \xi_1, \xi_2)\) is equal to the integer nearest the finite sum

\[
\frac{1}{2\pi} \sum_{0 \leq n \leq N} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} \frac{k!}{k_1! k_2!} \sum_{j_1 + j_2 = k} \frac{(2k_1)! (2k_2)!}{j_1! j_2! j_1^2 j_2^2} \langle T^{j_1, j_2}_{j_1^2, j_2^2}, X_\gamma \rangle
\]

where \(T^{j_1, j_2}_{j_1^2, j_2^2} \in V^* \otimes (j_1^2 + j_2^2 + 1) \bigoplus V^* \otimes (j_1 + j_2^2 + 2)\) is defined in (4.7).

The winding number is therefore computable from only finitely many terms in the signature, and an estimate on the number of terms needed can be computed directly from an estimate of \(\| \mathcal{B} F \|_{L^1(|w|=r)}\). Such an estimate can be obtained in the general case \(1 \leq p < 2\) from Corollary 2.2 but we will only state the result precisely for the bounded variation case.

If \(\log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2] \leq \rho\) then uniformly on \(\gamma([0, T])\),

\[
|x_1 - \xi_1| \leq \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2} \leq e^{\rho/2}
\]

and likewise \(|x_2 - \xi_2| \leq e^{\rho/2}\). Therefore, if in addition to the standing hypotheses we also assume that \(\gamma\) is of bounded variation then

\[
|F^{(n)}(0)| = \left| \int_\gamma (\log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2])^n ((x_1 - \xi_1)dx_2 - (x_2 - \xi_2)dx_1) \right|
\]

\[
\leq \rho^n e^{\rho/2} (\text{len}(x_2 \circ \gamma) + \text{len}(x_1 \circ \gamma))
\]

provided that the lower bound \(-\rho \leq \log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]\) holds as well (so that \(\log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]\) is bounded in absolute value by \(\rho\)). Thus, if \(\rho < |w|\)
then
\[ |BF(w)| = \left| \sum_{n \geq 0} F^{(n)}(0)/w^{n+1} \right| \leq \sum_{n \geq 0} \rho^ne^{n/2} (\text{len}(x_2 \circ \gamma) + \text{len}(x_1 \circ \gamma)) |w|^{n+1} = e^{|\gamma/2|} (\text{len}(x_2 \circ \gamma) + \text{len}(x_1 \circ \gamma)) \]
and therefore if \(-\log 2 < r < \rho \leq \log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2] \leq \rho < r < \log 2\) then
\[ 2\pi W_\gamma(x_1 \land x_2; \xi_1, \xi_2) = \sum_{0 \leq n \leq N} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} F(k) \leq \frac{1}{2\pi} \|BF\|_{L^1(|w| = r)} \|e^{2r} - e^{2\gamma} + 1\|^{N+1} \frac{2e^r - e^{2\gamma}}{r - \rho} \leq \frac{e^{|\gamma/2|} (\text{len}(x_2 \circ \gamma) + \text{len}(x_1 \circ \gamma)) (e^{2r} - e^{2\gamma} + 1)^{N+1}}{2e^r - e^{2\gamma}}. \]

We have proved:

Corollary 4.2. If in addition to the standing hypotheses, we also assume that \(\gamma\) is of bounded variation, and if \(N \in \mathbb{N}\) and \(\rho, r > 0\) are chosen so that
\[-\log 2 < r < -\rho \leq \log[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2] \leq \rho < r < \log 2\]
and
\[ r \frac{e^{|\gamma/2|} (\text{len}(x_2 \circ \gamma) + \text{len}(x_1 \circ \gamma)) (e^{2r} - e^{2\gamma} + 1)^{N+1}}{2e^r - e^{2\gamma}} < \pi \]
then \(W_\gamma(x_1 \land x_2; \xi_1, \xi_2)\) is equal to the integer nearest the finite sum
\[ \frac{1}{2\pi} \sum_{0 \leq n \leq N} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} \frac{k!}{k_1!k_2!} \sum_{j_1 + j_2 = k_1} (2k_1)!(2k_2)! \sum_{j_1 + j_2 + 2 = k_2} (2j_1)!(2j_2)!(2j_1 + 2j_2 + 2) \langle T_{j_1, j_2}^{j_1, j_2, j_1 + j_2 + 1} V^{t \otimes j_1 + j_2 + 2} \rangle \]
where \(T_{j_1, j_2}^{j_1, j_2, j_1 + j_2 + 1} \in V^{t \otimes j_1 + j_2 + 2} \otimes V^{t \otimes j_1 + j_2 + 2} \) is defined in (4.7).

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Department of Mathematics, Purdue University

E-mail address: aursitti@math.purdue.edu