Derivation of simplified model governing behavior of Mindlin plate with elastic support traversed by partially distributed moving load

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ABSTRACT
The practical importance of dynamic response of elements of structures such as plates when load moves on them cannot be overemphasized in both engineering and applied sciences. The dynamic behavior of an elastic plate resting on a subgrade and traversed by uniform partially distributed moving load is considered in this paper and its simplified governing equations derived. The elastic plate is Mindlin rectangular plate. In particular, the model governing such moving load problem is simplified analytically. The simplified governing model derived is easier to handle. Numerical methods can easily be applied to this simplified model and a lot of computational time is saved.

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1. Introduction

The investigation of moving load issue is by and large of reasonable significance in Engineering and Applied Sciences. Such reviews are significant while considering the unwavering quality, wellbeing and execution of present day structures over which loads like vehicles and train move (Gbadeyan and Agarana, 2014; Mindlin, 1951). The arrangement of such moving load issue under thought, requests the displaying of the mechanical conduct of the soil as flexible subgrade, and the type of collaboration between the plate and the soil. In this sort of framework, it is important to couple practical models of the foundation with investigation of the structure. A few foundation models have been accounted for in the writing and examinations on the static deflection; the dynamic reaction and the gravitational impacts of the moving load are taken into account (Mindlin, 1951; Boay, 1993; Fryba, 1972). The following assumptions are made:

- The plate is of constant cross-section.
- The moving load moves with a constant speed.
- The moving load is guided in such a way that it keeps contact with the plate throughout the motion.
- The plate is continuously supported by a Winkler foundation.
- The moving load is a partially distributed moving load.
- The rectangular Mindlin plate is elastic.

2. The governing equations

The set of dynamic equilibrium equations which govern the behaviour of Mindlin plate with elastic support and traversed by a partially distributed moving load can be written as (Gbadeyan and
where, \( \psi_x(x,y,T) \) and \( \psi_y(x,y,T) \) are local rotation in the \( x \) and \( y \) directions respectively, \( W(x,y,T) \) is the traversed displacement of the plate at time \( T \). \( B = B_x B_y \) such that

\[
B_x = \begin{cases} 
1 - H\left(x - \xi - \frac{\zeta}{2}\right), & 0 \leq T \leq \frac{\zeta}{u} \\
H\left(x - \xi - \frac{\zeta}{2}\right) - H\left(x - \xi - \frac{\zeta}{2}\right), & \frac{\zeta}{u} \leq T \leq \frac{\zeta + \varepsilon}{u} \\
H\left(x - \xi - \frac{\zeta}{2}\right), & \frac{\zeta + \varepsilon}{u} \leq T \leq \frac{\zeta + \varepsilon + \varepsilon}{u} \\
0, & T > \frac{\zeta + \varepsilon + \varepsilon}{u} 
\end{cases} 
\]

\[
B_y = \begin{cases} 
H\left(y - y_1 - \frac{\nu}{2}\right) - H\left(y - y_1 - \frac{\nu}{2}\right), & 0 \leq T \leq \frac{\nu}{u} \\
H\left(y - y_1 - \frac{\nu}{2}\right), & \frac{\nu}{u} \leq T \leq \frac{\nu + \varepsilon}{u} \\
0, & T > \frac{\nu + \varepsilon}{u} 
\end{cases} 
\]

\( H(x) \) is the Heaviside function defined as:

\[
H(x) = \begin{cases} 
1, & x > 0 \\
0.5, & x = 0 \\
0, & x < 0 
\end{cases} 
\]

\( U \) is the velocity of a load of rectangular dimension \( \varepsilon \) by \( \mu \) with one of its lines of symmetry moving along \( Y = Y_1 \).
\( A = \mu \varepsilon \), the area of the load in contact with the plate.

The plate is \( L_x \) by \( L_y \) in dimensions and \( = UT + \frac{\zeta}{2} \), \( h \) and \( k_1 \) are thickness of the plate and load respectively.
\( \rho \) and \( \rho_L \) are the densities of the plate and load respectively.

\( G \) is the modulus of the plate.
\( D \) is the flexural rigidity of the plate defined by:

\[
D = \frac{1}{2} E h^2 [(1 - \nu^2)] = Gh^3 / 6(1 - \nu) 
\]

\( k^2 \) is the shear correction factor.

\( v \) is the poisson’s ratio of the plate.
\( g \) is the acceleration due to gravity.
\( E \) is Young modulus of elasticity.
\( M_L \) is mass of the load.

### 2.1. Boundary and initial conditions

For a complete formulation of the problem, a simply supported rectangular Mindlin plate is considered as an illustrative example. If the edge \( y = 0 \) of the plate is simply supported, it then follows that the deflection \( W \) along this edge must be zero. At the same time this edge can rotate freely with respect to the \( x \) – axis, i.e., there are no bending (\( M_y \)) along this edge. Therefore, the boundary conditions can be stated as follows (Gbadeyan and Dada, 2001; Nguyen-Thoi et al., 2013; Amiri et al., 2013; Agarana et al., 2016):

\[
W(x,y,T) = M_y(x,y,T) = 0, \text{ for } x = 0 \text{ and } x = a \\
W(x,y,T) = M_y(x,y,T) = 0, \text{ for } y = 0 \text{ and } y = b 
\]

the corresponding initial conditions are

\[
W(x,y,T) = 0 = \frac{\partial W}{\partial T} (x,y,0) 
\]

### 3. Simplification of governing equations

The acceleration \( \frac{\partial^2 W}{\partial T^2} \) is defined as follows (Agarana et al., 2016; Agarana and Gbadeyan, 2016; Agarana and Emetere, 2016):

\[
\frac{\partial^2 W}{\partial T^2} = \frac{\partial^2 W}{\partial x^2} (dx)^2 + \frac{\partial^2 W}{\partial y^2} (dy)^2 + \frac{\partial^2 W}{\partial x \partial y} (dx)(dy) 
\]

\[
\frac{\partial^2 W}{\partial x^2} = 2\frac{\partial^2 W}{\partial x \partial T} \frac{\partial T}{\partial t} + \frac{\partial W}{\partial x} \frac{\partial^2 x}{\partial T^2} + \frac{\partial^2 x}{\partial T^2} \frac{\partial W}{\partial x} + \frac{\partial^2 x}{\partial y \partial T} \frac{\partial W}{\partial x} + \frac{\partial^2 y}{\partial T^2} \frac{\partial W}{\partial x} 
\]

assuming that \( x = x(T) \) and \( y = y(T) \).

Furthermore, assuming the external load moves along a straight line parallel to \( x \)-axis with a constant velocity \( U \), then it follows respectively that (Agarana and Gbadeyan, 2016; Agarana and Emetere, 2016)

\[
\frac{dy}{dt} = 0 
\]

and

\[
\frac{dx}{dt} = 0. 
\]

Hence, Eq. 10 becomes

\[
\frac{\partial^2 W}{\partial T^2} = \frac{\partial^2 W}{\partial x^2} (dx)^2 + \frac{\partial^2 W}{\partial y^2} (dy)^2 + \frac{\partial^2 W}{\partial x \partial y} (dx)(dy) 
\]

similarly,

\[
\frac{\partial^2 \psi_x}{\partial T^2} = \frac{\partial^2 \psi_x}{\partial x^2} + 2U \frac{\partial^2 \psi_x}{\partial x \partial T} + U^2 \frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial^2 \psi_x}{\partial y \partial T} + \frac{\partial^2 \psi_x}{\partial y^2} 
\]

hence, the expression on the left-hand side of Eq. 1 finally reduces to

\[
-\frac{1}{\mu \varepsilon} (-M_y g - M_L \frac{\partial^2 W}{\partial T^2}) = P(x,y,T) 
\]

where \( P(x,y,T) \) is the moving load.

Substituting (11), (12) and (13) into (1), (2) and (3) respectively we have

\[
-\frac{1}{\mu \varepsilon} (-M_y g - M_L \frac{\partial^2 W}{\partial T^2}) = k^2 \frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial^2 \psi_x}{\partial x \partial T} + \frac{\partial^2 \psi_x}{\partial y \partial T} + \frac{\partial^2 \psi_x}{\partial y^2} - kW - M_f \frac{\partial^2 W}{\partial T^2} 
\]
\[ \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \]

(16)

\[ \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \]

(17)

The definitions for moments along x and y axes, twisting moment and shear deformation along x and y axes are given as follows respectively (Agarana and Etomere, 2016)

\[ M_x = -D \left( \frac{\partial \psi}{\partial x} + \gamma \frac{\partial w}{\partial x} \right) \]

(18)

\[ M_y = -D \left( \frac{\partial \psi}{\partial y} + \gamma \frac{\partial w}{\partial y} \right) \]

(19)

\[ M_{xy} = -D \left( \frac{1 - \nu}{2} \left( \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right) \]

(20)

\[ Q_x = -k^2 G_h \left( \psi_x - \frac{\partial w}{\partial x} \right) \]

(21)

\[ Q_y = -k^2 G_h \left( \psi_y - \frac{\partial w}{\partial y} \right) \]

(22)

Substituting Eqs. 17–21 into Eqs. 14–16, the simplified set of the governing equations can now be written as

\[ -P(x, y, T) = k^2 G_h \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial y^2} \right] + \rho h \frac{\partial^2 w}{\partial t^2} - k W - M \frac{\partial^2 w}{\partial t^2} \]

(23)

\[ \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = -k^2 G_h \left( \psi_x - \frac{\partial w}{\partial x} \right) \]

(24)

\[ \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = -k^2 G_h \left( \psi_y - \frac{\partial w}{\partial y} \right) \]

(25)

Eq. 22 can be written as

\[ -k^2 G_h \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial y^2} \right) + \rho h \frac{\partial^2 w}{\partial t^2} + k W + M \frac{\partial^2 w}{\partial t^2} = P(x, y, T) \]

(26)

which can be expressed as

\[ \frac{\partial}{\partial x} \left[ -k^2 G_h \left( \frac{\partial w}{\partial x} + \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left[ -k^2 G_h \left( \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right] - \rho h \frac{\partial^2 w}{\partial t^2} + k W + M \frac{\partial^2 w}{\partial t^2} = P(x, y, T) \]

(27)

Now, substituting Eq. 20 and Eq. 21 into Eqs. 23, 24, and 26, we have:

\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho h \frac{\partial^2 w}{\partial t^2} + k W + M \frac{\partial^2 w}{\partial t^2} = P(x, y, T) \]

(28)

\[ \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = -k^2 G_h \left( \psi_x - \frac{\partial w}{\partial x} \right) \]

(29)

\[ \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = -k^2 G_h \left( \psi_y - \frac{\partial w}{\partial y} \right) \]

(30)

3.1. Further simplification

Eqs. 27, 28, and 29 can further be simplified. Firstly Eq. 11 can be written as:

\[ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + U \frac{\partial \psi}{\partial x} + U \frac{\partial^2 w}{\partial t \partial x} + U^2 \frac{\partial \psi}{\partial x} \]

(31)

but from Eq. 20

\[ \frac{\partial w}{\partial x} = \psi_x - \frac{Q_x}{\alpha h} \]

(32)

which leads to

\[ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + U \frac{\partial^2 w}{\partial t \partial x} + U \frac{\partial \psi}{\partial x} + U \frac{\partial^2 w}{\partial t \partial x} + U \frac{\partial w}{\partial x} \]

(33)

where \(-k^2\). Solving for \(\frac{\partial \psi}{\partial x}\) in Eq. 17 and Eq. 18 we have

\[ \frac{\partial \psi}{\partial x} = \frac{M_x - L \psi_x}{\alpha h} \]

(34)

substituting Eq. 34 into Eq. 33 yields

\[ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + U \frac{\partial^2 w}{\partial t \partial x} + U \frac{\partial \psi}{\partial x} + U \frac{\partial^2 w}{\partial t \partial x} + U \frac{\partial w}{\partial x} = \frac{M_x - L \psi_x}{\alpha h} \]

(35)

similarly, we have

\[ \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial t} + U \frac{\partial^2 \psi}{\partial x^2} + U \frac{\partial \psi}{\partial x} \]

(36)

by virtue of Eq. 34. Therefore,

\[ \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial t} + U \frac{\partial^2 \psi}{\partial x^2} + U \frac{\partial \psi}{\partial x} \]

(37)

similarly, Eq. 13 reduces to

\[ \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial t} + U \frac{\partial^2 \psi}{\partial x^2} + U \frac{\partial \psi}{\partial x} \]

(38)

recalling Eq. 14, we have

\[ P(x, y, T) = \frac{1}{\mu} \left[ M_{Lg} - M_D \frac{\partial^2 w}{\partial t^2} \right] B \]

(39)

which reduces to

\[ P(x, y, T) = \frac{M_D}{\mu} \left[ g + \frac{\partial^2 w}{\partial t^2} \right] B \]

(40)

where

\[ A = \mu \epsilon \]

substituting Eq. 35 into Eq. 41 we have

\[ P(x, y, T) = \frac{M_D}{\mu} \left[ g + \frac{\partial^2 w}{\partial t^2} \right] B \]

(42)

\[ P(x, y, T) = \frac{M_D}{\mu} \left[ g + \frac{\partial^2 w}{\partial t^2} \right] B \]

(43)

Therefore the simplified governing equations as derived from above are given by substituting Eq. 43 into Eq. 28 with Eq. 29 and Eq. 30.
4. Conclusion

Various versions of differential equation(s) governing the behaviour of plates under a moving load appear in literature. However, almost all of them are not easy to handle; a lot of computational time is required. Those that are relatively easy to handle are with many assumptions, like neglecting the effects of both rotatory inertia and shear deformation. Others assumed that the plate is not supported by any subgrade and the effect of damping neglected. The main contribution of this paper is to present a simplified set of partial differential equations modelling the dynamic behaviour of plate under a moving load. In contrast to the models of Gbadeyan and Dada (2006), this model is simple and was derived analytically. Additionally, the exact solution of this model can be sort for instead of an approximate solution. Finally, this simplified model should be considered as a more practical representation of real life situation that is easier to solve less computation time and high level of accuracy.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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