1 Introduction.

The aim of analysis by discs is to see the possibility to reduce certain problems in several variables to problems in one variable.

We shall work in the unit ball $B$ of $\mathbb{C}^n$ and a disc in $B$ is a holomorphic mapping $\phi : \mathbb{D} \to B$. The set of all discs in $B$ will be denoted by $D(B)$.

The analysis by discs can be roughly stated the following way: suppose you have a several variables object $O$ and a property $P$ true for all discs $\phi$, for the one variable object $O \circ \phi$; is the corresponding property $P$ true for $O$?

In order to be more specific with examples, let me recall the definitions of Hardy and Bergman spaces in the ball.

**Definition 1.1** Let $k \in \mathbb{N}$ then:

\[
A_k^p(B) := \{ f \in \mathcal{O}(B) / \|f\|_{A_k^p(B)}^p := \int_B |f(z)|^p (1 - |z|^2)^k \, d\lambda(z) < \infty \}.
\]

\[
H^p(B) := \{ f \in \mathcal{O}(B) / \|f\|_{H^p(B)}^p := \sup_{r<1} \int_{\partial B} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty \}.
\]
In order to have a natural problem in $H^p(\mathbb{B})$, the first question is:
if $f \in H^p(\mathbb{B})$ in what space of analytic functions in the unit disc $f \circ \phi$ is ?
If $\phi$ is a ”flat” disc, e.g. $\forall \zeta \in \mathbb{D}$, $\phi(\zeta) := (\zeta, 0) \in \mathbb{B} \subset \mathbb{C}^2$, then the ”subordination lemma” [2] gives us that $f \circ \phi$ is in a Bergman space. Our main result generalizes this to any disc in $\mathbb{B}$.

**Theorem 1.2** Let $\phi(z) := (\phi_1(z), \ldots, \phi_n(z))$ be an analytic disc in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ such that $\phi(0) = 0$ and let $1 \leq p \leq \infty$, $f \in H^p(\mathbb{B})$ then:
$$
\int_{\mathbb{B}} |f \circ \phi(z)|^p (1 - |z|^2)^{n-2} d\lambda(z) \leq C \|f\|_p^p.
$$

This theorem allows us to give an example where the analysis by discs is possible for $H^p$.

* (H$^p$ Corona) Let $\mathcal{O}$ be $g_1, g_2$ in $H^\infty(\mathbb{B})$ and $\mathcal{P}$ the property that
  \[ \forall f \in H^p(\mathbb{B}), \exists f_1, f_2 \in H^p(\mathbb{B}) \text{ with } f = f_1g_1 + f_2g_2 \]
then one can easily prove that $|g_1|^2 + |g_2|^2 \geq \delta > 0$ and using the Corona theorem in one variable one has:
  \[ (C_p) \exists C > 0, \forall \phi \in \mathcal{D}(\mathbb{B}), \forall h \in A^p_{-2}(\mathbb{D}) \exists h_1, h_2 \in A^p_{-2}(\mathbb{D}) \]
such that $h_1g_1 \circ \phi + h_2g_2 \circ \phi = h$, $\|h_j\|_{A^p_{-2}(\mathbb{D})} \leq C$, $j = 1, 2$.

Just take $h_j := \chi_j h$ where $\chi_j \in H^\infty(\mathbb{D})$, $g_1 \circ \phi \chi_1 + g_2 \circ \phi \chi_2 = 1$. I.e. the property $\mathcal{P}$ is still true uniformly for all the discs.

The analysis by discs is then:
let $g_1, g_2 \in H^\infty(\mathbb{B})$ such that $(C_p)$ is true, if $f \in H^p(\mathbb{B})$ are there $f_1, f_2 \in H^p(\mathbb{B})$ with $f_1g_1 + f_2g_2 = f$ ?

The answer is yes, because one can prove that the condition $(C_p)$ implies that $|g_1|^2 + |g_2|^2 \geq \delta > 0$ in $\mathbb{B}$ and using the $H^p(\mathbb{B})$ corona [3], we have the answer yes.

So the same kind of question arises naturally for the interpolating sequences:

** (H$^p$ interpolation) Let $\mathcal{O}$ be $S = \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{B}$ a sequence of points in the unit ball $\mathbb{B} \subset \mathbb{C}^n$. The property $\mathcal{P}$ is that $S$ is interpolating for $H^p(\mathbb{B})$ (see definitions in secton [4]).

Our main theorem gives us a necessary condition on discs passing through $S$ for $S$ to be interpolating in $H^p(\mathbb{B})$:

If $S$ is $H^p(\mathbb{B})$ interpolating and if $(\sigma, \phi)$ is a disc passing through $S$, i.e. $\phi(\sigma) = S$, then $\sigma$ is ”almost” interpolating for a weighted Bergman class.

But in contrast to the $H^p$ Corona case, we prove:

**Theorem 1.3** For any $p \in [1, \infty]$, there is a sequence $S$ in $\mathbb{B} \subset \mathbb{C}^2$ such that all disc $(\sigma, \phi)$ passing through $S$ is such that $\sigma$ is interpolating for $A^p(\mathbb{D})$ with a uniformly bounded constant but $S$ is not an interpolating sequence for $H^p(\mathbb{B})$.

The hypothesis on the sequence $S$ in the previous theorem is stronger than the necessary condition and the fact that nevertheless $S$ is not $H^p(\mathbb{B})$ interpolating proves that the analysis by discs is not pertinent in the case of $H^p(\mathbb{B})$ with a finite $p$.

These two questions, Corona and interpolating sequences, remain open for $H^\infty(\mathbb{B})$. Following the question asked by J. Globevnik [7], with P. Thomas [5] we already studied the interpolating sequences for $H^\infty(\mathbb{B})$ from this point of view, and we obtain partial answer.
This work is organized the following way:
in the first section we prove that the image by a disc \( \phi \) in the ball \( B \) of \( \mathbb{C}^n \) of the Bergman measure \( (1 - |z|^2)^n d\lambda(z) \) of the unit disc is a Carleson measure in \( B \).
In the second section we characterize the pairs \( (a_j, \alpha_j)_{j=1,\ldots,N} \) of points \( a_j \in B, \alpha_j \in \mathbb{D} \) such that there is an analytic disc \( \phi \) from \( \mathbb{D} \) to \( B \) sending \( \alpha_j \) to \( a_j \). This theorem is a generalization of the Pick-Nevanlinna theorem.
In section 3 we recall the definitions of interpolating sequences for Hardy and Bergman spaces and, because we shall need it, we recall that they are invariant by the automorphisms of the ball.
Finally in section 4 we build the counter-example, i.e. we prove theorem \ref{thm:counter-example}.

\section{A Carleson measure.}

The aim of this section is to prove that if \( \phi : \mathbb{D} \rightarrow B \) is an analytic disc in \( B \) such that \( \phi(0) = 0 \), then:

\begin{equation}
(*) \forall f \in H^p(B), \int_B |f \circ \phi(z)|^p (1 - |z|^2)^{n-2} d\lambda(z) \leq C \|f\|_p^p.
\end{equation}

Let \( \mu \) be the measure in the ball \( B \) image by the map \( \phi \) of the weighted Lebesgue measure \( \lambda \) on the unit disc \( \mathbb{D} \); this means precisely:
\[ \forall f \in C(B), \int_B f d\mu = \int_{\mathbb{D}} f \circ \phi(1 - |z|^2)^{n-2} d\lambda(z). \]

On the other hand if the measure \( \mu \) is Carleson in \( B \), then:
\[ \forall f \in H^p(B), \int_B |f|^p d\mu \leq \|\mu\|_C \|f\|_p^p, \]

where \( \|\mu\|_C \) is the Carleson norm of \( \mu \).

Hence in order to get (\(*\)) it suffices to prove that \( \mu \) is a Carleson measure in \( B \).

By the Carleson-Hörmander characterization of these measures, it is enough to show:
\[ \int_{Q(\zeta, \delta)} d\mu \leq C\delta^2, \]

where \( Q(\zeta, \delta) := \{ z \in B, |1 - \zeta \cdot \overline{z}| < \delta \} \) is the pseudo-ball centered in \( \zeta \in \partial B \) and of radius \( \delta \).

By invariance by complex rotation of the map \( \phi \), we can send \( \zeta \) to \( 1 = (1,0,\ldots,0) \) and in fact it remains to show that:
\[ \int_{Q(1, \delta)} d\mu \leq C\delta^2, \]

with \( Q(1, \delta) := \{ z \in B, |1 - z_1| < \delta \}. \)

Finally the definition of \( \mu \) leads to show:
\[ \int_{Q(1, \delta)} d\mu = \int_{\mathbb{D}} \chi_{d(1,\delta)}(t)(1 - |t|^2)^{n-2} d\lambda(t) \leq C\delta^2, \]

with \( \chi_{d(1,\delta)}(t) \) the indicatrix of the disc \( d(1,\delta) := \{ z \in \mathbb{D}, |1 - z| < \delta \}. \)

This will be achieved via a series of lemmas.

\textbf{Lemma 2.1} Let \( \phi : \mathbb{D} \rightarrow \mathbb{D} \) such that \( \phi(0) = 0 \) and suppose \( \phi \) inner, i.e. the boundary values \( \phi^* \) of \( \phi \) are of modulus one, then the image by \( \phi^* \) of the Lebesgue’s measure \( d\sigma \) on \( \mathbb{T} \) is the Lebesgue’s measure \( d\sigma \) on \( \mathbb{T} \).

\textbf{Proof}:

let \( f \in A(\mathbb{D}) \) and set \( \mu := \phi^* \sigma \) the image of the measure \( \sigma \) by \( \phi^* \), then:
\[
\int_T f \, d\mu = \int_T f \circ \phi^* \, d\sigma = f \circ \phi(0),
\]
because \(\sigma\) represents 0 for \(A(\mathbb{D})\). But \(\phi(0) = 0\) hence we have:
\[
\int_T f \, d\mu = f(0),
\]
hence \(\mu\) also represents 0. By unicity of the representing measure of 0 on \(T\), we must have \(\mu = \sigma\). \(\Box\)

**Lemma 2.2** Let \(\mu\) be a probability measure on \(\overline{\mathbb{D}}\) which represents 0 i.e.
\[
\forall f \in A(\mathbb{D}), \int f(z) \, d\mu(z) = f(0);
\]
let \(\nu\) be its balayée by the Poisson kernel
\[
\forall g \in C(T), \int_T g(\zeta) \, d\nu(\zeta) := \int_T \tilde{g}(z) \, d\mu(z),
\]
where \(\tilde{g}(z) := \int_T P(\zeta; z) g(\zeta) \, d\sigma(\zeta)\) is the harmonic extension of \(g\) in \(\mathbb{D}\).
Then \(d\nu = d\sigma\).

**Proof:**
let \(f \in A(\mathbb{D})\) then \(\tilde{f}_T = f\) because \(f\) is harmonic in \(\mathbb{D}\), hence by definition of \(\nu\) we get:
\[
\int_T f(\zeta) \, d\nu(\zeta) = \int_T f(z) \, d\mu(z) = f(0);
\]
hence \(\nu\) is a representing measure of 0 supported by \(T\) and by unicity of the representing measure of 0 on \(T\) we have \(\nu\) is the normalized Lebesgue measure \(\sigma\) on \(T\). \(\Box\)

**Lemma 2.3** Let \(I := T \cap d(1, \delta)\) and \(\tilde{I}(z)\) the harmonic extension of the indicatrix \(\chi_I\) of \(I\), then the set
\[
A_\delta := \{z \in \mathbb{D} / \tilde{I}(z) > \frac{\pi}{4}\}
\]
contains the disc \(d(1, \delta)\).

**Proof:**
we shall do it in the half plane, the computations being easier. The Poisson kernel is \(P(z, t) = \frac{2y}{(x-t)^2 + y^2}\) with \(z = x + iy\); let \(I = ] - \delta, \delta[\) then
\[
\tilde{I}(z) = \int_{-\delta}^{\delta} \frac{2y}{(x-t)^2 + y^2} \, dt = \int_{x-\delta/y}^{x+\delta/y} \frac{du}{1 + u^2},
\]
by the change \(u = \frac{t-x}{y}\), so we get:
\[
\tilde{I}(z) = \arctan \left( \frac{x+\delta}{y} \right) - \arctan \left( \frac{x-\delta}{y} \right),
\]
hence if \(0 \leq x < \delta, 0 < y < \delta\) we get \(\tilde{I}(z) \geq \arctan(1) = \frac{\pi}{4}\). \(\Box\)

**Lemma 2.4** Let \(\phi \in H^\infty(\mathbb{D}), \phi(0) = 0\; for \; 0 \leq \rho < 1\; let \forall z \in \mathbb{D}, \phi_\rho(z) := \phi(\rho z),\) then:
\[
|\{\zeta \in T / \phi_\rho(\zeta) \in A_\delta\}| \leq 2 |I| = 4\delta.
\]

**Proof:**
let \(\mu_\rho\) be the image by \(\phi_\rho\) of the Lebesgue’s measure \(\sigma\) then we have:
\[ \int f(z) \, d\mu_\rho(z) = \int_T f \circ \phi_\rho(\zeta) \, d\sigma(\zeta) = f \circ \phi_\rho(0) = f(0). \]

Hence \( \mu \) represents 0; let \( \nu \) be the balayé by the Poisson kernel of \( \mu \), then by lemma 2.2 \( \nu = \sigma \).

Now we have:
\[ \int \tilde{I}(z) \, d\mu_\rho(z) = \int_T \chi_I(\zeta) \, d\nu(\zeta) = \int_T \chi_I(\zeta) \, d\sigma(\zeta) = |I| = 2\delta. \]

On the other hand by definition of \( \mu \) we have:
\[ |I| = \int_T \tilde{I}(z) \, d\mu_\rho(z) = \int_T \tilde{I} \circ \phi_\rho(\zeta) \, d\sigma(\zeta), \]

hence
\[ \int_{\{\tilde{I}(z) > \pi/4\}} \frac{\pi}{4} \, d\mu_\rho(z) \leq \int_{\{\tilde{I}(z) > \pi/4\}} \tilde{I}(z) \, d\mu_\rho(z) \leq |I|, \]

and \( \mu_\rho(A_\delta) \leq \frac{4}{\pi} |I| \).

But \( \mu_\rho(A_\delta) = \int \chi_{A_\delta}(z) \, d\mu_\rho(z) = \int_T \chi_{A_\delta} \circ \phi_\delta(\zeta) \, d\sigma(\zeta) = |\{\zeta \in T / \phi_\rho(\zeta) \in A_\delta\}|, \]

hence \( |\{\zeta \in T / \phi_\rho(\zeta) \in A_\delta\}| \leq \frac{4}{\pi} |I| \).

**Theorem 2.5** Let \( \phi(z) := (\phi_1(z), \ldots, \phi_n(z)) \) be an analytic disc in the unit ball \( B \) of \( \mathbb{C}^n \) such that \( \phi(0) = 0 \), then the image \( \mu \) in \( B \) of the Bergman measure \( (1 - |z|^2)^{n-2} \, d\lambda(z) \) on \( D \) is a Carleson measure in \( B \).

**Proof:**

it is enough to prove that, \( \mu(Q(1, \delta)) \lesssim \delta^2 \) by rotation, with \( Q(1, \delta) := \{z = (z_1, \ldots, z_n) \in B / |1 - z_1| < \delta\} \).

So
\[ \int Q(w) \, d\mu(w) = \int_D \chi_Q(\phi_1(z), \ldots, \phi_n(z)) \, d\lambda(z) \leq \int_D \chi_{A_\delta}(\phi_1(z)) \, d\lambda(z), \]

because \( Q \) involves only \( \phi_1 \) and \( A_\delta \) contains the disc \( d(1, \delta) \) by lemma 2.2

But \( \phi_1(0) = 0 \) which means by Schwarz lemma that \( \phi_1(z) = zg(z) \) still with \( g : D \rightarrow D \), hence in order to have that \( \phi_1(z) \in A_\delta \) we need to have:
\[ |z| \geq 1 - \delta \quad (2.1) \]

Now applying the previous lemma to \( \phi_1 \) we get:
\[ |\{\zeta \in T / \phi_\rho(\zeta) \in A_\delta\}| \leq \frac{8}{\pi} \delta. \]

We can compute the area of the set \( \{z \in D / \phi_1(z) \in A_\delta\} \) by cutting it with circles of radius \( \rho \) with \( 1 - \delta \leq \rho < 1 \) and we get:

\[ \lambda(\{z \in D / \phi_1(z) \in A_\delta\}) \leq |\{\zeta \in T / \phi_\rho(\zeta) \in A_\delta\}| \times |\{1 - \delta \leq \rho < 1\}| \leq \frac{8}{\pi} \delta^2, \]

and the theorem.

As pointed out by J. Bruna, the one dimensional part of the previous proof can be slightly shortened by use of the Littlewood’s subordination lemma ([4], p. 10).

**Corollary 2.6** Let \( \phi(z) := (\phi_1(z), \ldots, \phi_n(z)) \) be an analytic disc in the unit ball \( B \) of \( \mathbb{C}^n \) such that \( \phi(0) = 0 \) and let \( 1 \leq p \leq \infty \), \( f \in H^p(B) \) then:
\[ \int_D |f \circ \phi(z)|^p (1 - |z|^2)^{n-2} \, d\lambda(z) \leq C \|f\|_p^p, \]
Proof:
the image is a Carleson measure and we can apply the Carleson-Hörmander inequality. 

At this point one can ask for the converse of the previous corollary: suppose that $f \in H^\infty(\mathbb{B})$ and $\sup_{\phi \in \mathcal{D}(\mathbb{B})} \| f \circ \phi \|_{A_{p-2}(\mathbb{D})} \leq C$ is it true that $\| f \|_{H^p(\mathbb{B})} \leq KC$?

The answer is no and the following counter-example was suggested by A. Borichev.

In the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ take a inner function $f$ such that $f(0) = 0$. This means that $f \in H^\infty(\mathbb{B})$ and $|f^*| = 1$, a.e. on $\partial \mathbb{B}$, where $f^*$ are the boundary values of $f$. We have that $\forall k \in \mathbb{N}$, $\| f^k \|_p = 1$; on the other hand, $f \circ \phi$ is still bounded by 1 in $\mathbb{D}$ and $f \circ \phi(0) = 0$ hence by Schwarz lemma we have $|f \circ \phi(z)| \leq |z|$.

This implies that $\| f^k \circ \phi \|_{A_p(\mathbb{D})} \leq \int_\mathbb{D} |z|^{kp} d\lambda(z) = \frac{1}{kp + 2}$; hence

$$\sup_{\phi \in \mathcal{D}(\mathbb{B})} \| f^k \circ \phi \|_{A_p(\mathbb{D})} \leq \left( \frac{1}{kp + 2} \right)^{1/p} \to 0 \text{ when } k \to \infty,$$

hence the constant $K$ cannot be finite.

3 A Pick-Nevanlinna theorem.

Modifying the proof of the classical theorem of Pick and Nevanlinna by Nagy and Foias, we get:

**Theorem 3.1** Let $\sigma = \{\alpha_k\}_{k=1,...,N} \subset \mathbb{D}$ and $v = \{v_k\}_{k=1,...,N} \subset \mathbb{B}$. There is a function $f : \mathbb{D} \to \mathbb{B}$ such that $\forall k = 1,...,N$, $f(\alpha_k) = v_k$ iff:

$$\forall k = 1,...,N, \forall h_k \in \mathbb{C}, \sum_{k,l=1}^N \overline{h_k} h_l \frac{1 - \overline{v_k} v_l}{1 - \overline{\alpha_k} \alpha_l} \geq 0.$$

If $\mathbb{B} = \mathbb{D}$ the unit disc, this is the Pick-Nevanlinna theorem.

Let $\alpha \in \mathbb{D}$ and $k_\alpha(z) = \frac{1}{1 - \overline{\alpha} z}$ the Cauchy kernel associated to it and:

$$E_\alpha := \text{span}\{k_\alpha, \alpha \in \sigma\} \subset L^2(\mathbb{D}).$$

To a function $f \in H^\infty(\mathbb{D})$ let us associate the operator:

$$\forall h \in E_\sigma, \pi_\sigma(f) h := P_{E_\sigma}(f h) \in H^2(\mathbb{D}).$$

This is clearly a anti-holomorphic representation of $H^\infty(\mathbb{D})$ in $\mathcal{L}(H^2(\mathbb{D}))$ and we have $\forall \alpha \in \sigma, \pi_\sigma(f) k_\alpha = \overline{f(\alpha)} k_\alpha$.

Now to $f = (f_1,...,f_n) : \mathbb{D} \to \mathbb{B}$ holomorphic, let us associate the operator:

$$\forall h \in E_\sigma, \pi_\sigma(f) h := (\pi_\sigma(f_1) h, ..., \pi_\sigma(f_n) h) \in \mathcal{H} := H^2(\mathbb{D}) \oplus \cdots \oplus H^2(\mathbb{D}).$$

**Lemma 3.2** If $f : \mathbb{D} \to \mathbb{B}$ then $\| \pi_\sigma(f) \|_{op} \leq 1$.

Proof:

we have $\pi_\sigma(f) h = P_{E_\sigma \oplus \cdots \oplus E_\sigma}(\overline{f_1} h, ..., \overline{f_n} h)$;

now the projection is a contraction and we have

$$\| (\overline{f_1} h, ..., \overline{f_n} h) \|_{L^2(\mathbb{T}) \oplus \cdots \oplus L^2(\mathbb{T})}^2 = \sum_{j=1}^n \int_\mathbb{T} |\overline{f_j}(\zeta) h(\zeta)|^2 d\sigma(\zeta),$$

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hence
\[ \| (f_1, \ldots, f_n) \|_{L^2(T) \oplus \cdots \oplus L^2(T)}^2 = \int_\mathbb{T} \sum_{j=1}^n |f_j(\zeta)|^2 |h(\zeta)|^2 \, d\sigma(\zeta) \leq \| h \|_{L^2(T)}^2. \]

So \( \| \sigma(f)h \|_2^2 \leq \| (f_1, \ldots, f_n) \|_{L^2(T) \oplus \cdots \oplus L^2(T)}^2 \leq \| h \|_2^2 \), and the lemma. \( \Box \)

Proof of the theorem:
denote by \( Z \) the shift operator (multiplication by \( z \)) on \( H^2(\mathbb{D}) \) and also the shift on
\[ \mathcal{H} := H^2(\mathbb{D}) \oplus \cdots \oplus H^2(\mathbb{D}), \]
i.e. \( \forall h = (h_1, \ldots, h_n) \in \mathcal{H} \to (Zh)(z) = (zh_1(z), \ldots, zh_n(z)). \)
These operators are isometries and we get easily:
\[ \| \sigma(f)h \|_2^2 = \| \sigma(f) \|_2^2 \leq \| h \|_2^2 \]
because \( \mathcal{E} \) and suppose that \( \| Zh \|_2 = (\tilde{\sigma}(i), \ldots, \tilde{\sigma}(i)) \).
Now let any \( f \in H^\infty(\mathbb{D})^n \) be such that \( \forall k = 1, \ldots, N, \ f(\alpha_k) = v_k \) and set \( T := \sigma(f) \in \mathcal{L}(E_\sigma, \mathcal{H}) \), and suppose that \( \| T \|_{op} \leq 1 \) (which is equivalent to the condition in the theorem), then we have \( Z^*T = TZ^* \) on \( E_\sigma \).
We can apply the lifting of the commutant theorem: there is an operator \( \tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_n) \in \mathcal{L}(H^2(\mathbb{D}), \mathcal{H}) \) such that:
(i) \( \tilde{T}_{|E_\sigma} = T \);
(ii) \( \| \tilde{T} \| = \| T \| \);
(iii) \( Z^*\tilde{T} = \tilde{T}Z^* \).
As in \([4]\) (iii) implies that \( Z^*\tilde{T}_j = \tilde{T}_jZ^* \), we know that this is the condition for \( \tilde{T}_j \) to be associated to a function \( \tilde{f}_j \in H^\infty(\mathbb{D}) \), again as in \([4]\) and \( \tilde{T} \) is the adjoint of the operator \( T_f \) of multiplication by \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n) \).
By (ii) we get \( \| T_f \| = \| \tilde{T} \| \leq 1 \) hence \( \tilde{f} : \mathbb{D} \to \mathbb{B} \).
Finally by (i) we get that \( \forall k = 1, \ldots, N, \ \tilde{f}(\alpha_k) = v_k \), and the theorem. \( \Box \)

The analogue for the polydisc is straightforward with the usual PN theorem:

**Theorem 3.3** Let \( \sigma = \{\alpha_k\}_{k=1}^N \subset \mathbb{D} \) and \( v = \{v_k\}_{k=1}^N \subset \mathbb{D}^n \). There is a function \( f : \mathbb{D} \to \mathbb{D}^n \) such that \( \forall k = 1, \ldots, N, \ f(\alpha_k) = v_k \) iff:
\[ \forall m = 1, \ldots, n, \ \forall k = 1, \ldots, N, \ \forall h_k \in \mathbb{C}, \ \sum_{k,l=1}^N \frac{h_kh_l}{1 - \bar{\alpha}_k \alpha_l} \geq 0. \]

Proof:
just apply the PN theorem. \( \Box \)

### 4 Interpolating sequences and automorphisms.

**Definition 4.1** If \( S = \{a_k, \ k \in \mathbb{N}\} \) is a sequence of points in \( \mathbb{B} \), then we define:
\[ \ell^{p}_H(S) := \{\lambda = \{\lambda_k\}_{k \in \mathbb{N}} / \sum_{k \in \mathbb{N}} |\lambda_k|^p (1 - |a_k|^2)^n =: \|\lambda\|^{p}_H < \infty\}. \]

This sequence space depends on the sequence \( S \). The same for the weighted Bergman classes:
Definition 4.2 If $S = \{a_k, k \in \mathbb{N}\}$ is a sequence of points in $\mathbb{B}$, then we define:

$$\ell^p_{A_l}(S) := \{\lambda = \{\lambda_k\}_{k \in \mathbb{N}} / \sum_{k \in \mathbb{N}} |\lambda_k|^p (1 - |a_k|^2)^{n+l+1} =: \|\lambda\|_{p,A_l} < \infty\}.\$$

Now we can define the interpolating sequences for Hardy spaces:

Definition 4.3 We say that $S$ is interpolating for $H^p(\mathbb{B})$ with constant $C$ if:

$$\forall \lambda \in \ell^p_H(S), \; \exists f \in H^p(\mathbb{B}) / \forall k \in \mathbb{N} \; f(a_k) = \lambda_k \text{ and } \|f\|_p \leq C \|\lambda\|_{H,p}.\$$

The set of interpolating sequences for $H^p(\mathbb{B})$ will be denoted by $IH^p(\mathbb{B})$.

And for the Bergman classes:

Definition 4.4 We say that $S$ is interpolating for $A^p_l(\mathbb{B})$ with constant $C$ if:

$$\forall \lambda \in \ell^p_{A_l}(S), \; \exists f \in A^p_l(\mathbb{B}) / \forall k \in \mathbb{N} \; f(a_k) = \lambda_k \text{ and } \|f\|_{A^p_l(\mathbb{B})} \leq C \|\lambda\|_{A_l,p}.\$$

The set of interpolating sequences for $A^p_l(\mathbb{B})$ will be denoted by $IA^p_l(\mathbb{B})$.

For the unweighted Bergman classes, i.e. $l = 0$, we shall set $\ell^p_{A_0}(S) = \ell^p_A(S)$

In a paper by Jevtic-Massaneda-Thomas, the authors prove the following theorem.

Theorem 4.5 Let $S = \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{B}$ be an interpolating sequence in $A^p_l(\mathbb{B})$ and $\Phi$ be an automorphism of $\mathbb{B}$ then $\Phi(S) := \{\Phi(a_k)\}_{k \in \mathbb{N}} \subset \mathbb{B}$ is still an interpolating sequence in $A^p_l(\mathbb{B})$ with the same constant.

The following theorem is in W. Rudin’s book (\cite{Rudin}, section 7.5):

Theorem 4.6 Let $\psi \in \text{Aut}(\mathbb{B})$, $a := \psi^{-1}(0)$ then the linear operator:

$$Tf(z) := \frac{(1 - |a|^2)^{n/p}}{(1 - \overline{a} \cdot z)^{2n/p}} f \circ \psi(z),$$

is an isometry, i.e. $\|Tf\|_{H^p(\mathbb{B})} = \|f\|_{H^p(\mathbb{B})}$.

Using it we get, exactly the same way as in the paper by Jevtic-Massaneda-Thomas, the authors prove the following theorem.

Theorem 4.7 Let $S = \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{B}$ be an interpolating sequence in $H^p(\mathbb{B})$ and $\Phi$ be an automorphism of $\mathbb{B}$ then $\Phi(S) := \{\Phi(a_k)\}_{k \in \mathbb{N}} \subset \mathbb{B}$ is still an interpolating sequence in $H^p(\mathbb{B})$ with the same constant.

In fact in \cite{Jevtic-Massaneda} I explicitly stated a unitary representation of the group Aut$(\mathbb{B})$ in $H^2(\mathbb{B})$ which gives as a corollary the fact that $H^2(\mathbb{B})$ interpolating sequences are invariant by automorphisms.
5 Interpolating sequences and discs.

In this section we shall give the counter example in the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ and for $1 \leq p < \infty$.

If $S = \{a_k, k \in \mathbb{N}\}$ is in $IH^p(\mathbb{B})$ and if $(\sigma, \phi)$ is a disc through $S$, $\sigma = \{\alpha_k, k \in \mathbb{N}\}$, using automorphisms of the ball and of the disc, we may suppose that $a_0 = 0$ and $\alpha_0 = 0$ without changing the interpolating constants.

We have:

$$\forall \lambda \in \ell^p_H(S), \exists f \in H^p(\mathbb{B}) \text{ s.t. } \forall k \in \mathbb{N}, f(a_k) = \lambda_k,$$

hence, for any $\lambda \in \ell^p_H(S)$, there is a $g \in A^p(\mathbb{D})$ such that all disc passing through $0, S$ and if $(\sigma, \phi)$ is a disc through $S$, $\sigma = \{\alpha_k, k \in \mathbb{N}\}$, sending $a_k, \alpha_k$.

So we have:

**Lemma 5.2** If $(\sigma, \phi)$ is a disc through $S$ with $a_0 = 0, \alpha_0 = 0$, we have that

$$\forall k \in \mathbb{N}, |a_k| \leq |\alpha_k| \text{ hence } 1 - |a_k|^2 \geq 1 - |\alpha_k|^2.$$

Proof:

$\phi$ sends $\mathbb{D}$ into $\mathbb{B}$ hence it decreases the Gleason distance:

$$d_G(a_0, a_k) \leq d_G(\alpha_0, \alpha_k);$$

because $d_G(a, b) := \sup\{||f(b)||; f \in H^\infty, ||f||_{\infty} \leq 1, f(a) = 0\}$ and if $f \in H^\infty(\mathbb{B})$ then

$$g := f \circ \phi \in H^\infty(\mathbb{D}) \text{ so }$$

$$d_G(\alpha, \beta) = \sup\{|g(\beta)|; g \in H^\infty(\mathbb{D}), ||g||_{\infty} \leq 1, g(a) = 0\} \geq d_G(a, b).$$

Because $a_0 = 0, \alpha_0 = 0$ we have $d_G(a_0, a_k) = |a_k| \leq |\alpha_k| = d_G(\alpha_0, \alpha_k)$ and the lemma. □

**Corollary 5.3** If $(\sigma, \phi) \in \mathcal{D}_S, \lambda \in \ell^p_H(S) \longrightarrow \lambda \in \ell^p_A(\sigma)$ and $\|\lambda\|_{A,p} \leq \|\lambda\|_{H,p}.$

Proof:

$$\|\lambda\|_{H,p}^p := \sum_{k \in \mathbb{N}} |\lambda_k|^p (1 - |a_k|^2)^2 \geq \sum_{k \in \mathbb{N}} |\lambda_k|^p (1 - |\alpha_k|^2)^2 = \|\lambda\|_{A,p}^p. \quad \square$$

Hence if $\sigma \subset \mathbb{D}$ is interpolating for $A^p(\mathbb{D})$ it is interpolating in $A^p(\mathbb{D})$ also for $\ell^p_H(S)$.

Now a natural question is:

if $S$ is a sequence in $\mathbb{B}$ containing $0$, such that any disc $(\sigma, \phi)$ passing through $S$ is interpolating for $A^p(\mathbb{D})$ then $S \in IH^p(\mathbb{B})$ ?

We notice that this is stronger, by the previous corollary, that the necessary condition which is:

$\sigma$ is interpolating in $A^p(\mathbb{D})$ for $\ell^p_H(S)$.

Despite this stronger condition the answer is no:

**Theorem 5.4** For any $p \in [1, \infty], there is a sequence $S$ in $\mathbb{B}$ such that all disc passing through $S$ is interpolating for $A^p(\mathbb{D})$ with a uniformly bounded constant but $S$ is not in $IH^p(\mathbb{B})$. 


Proof:
take \(1 \leq p < \infty\), we know that if a sequence \(\sigma\) in \(\mathbb{D}\) is enough separated, i.e.
\[\forall (\alpha, \beta) \in S^2, \alpha \neq \beta, \quad d_G(\alpha, \beta) \geq \delta,\]
for a certain \(\delta = \delta_p > 0\) then \(\sigma\) is in \(IA^p(\mathbb{D})\) with an interpolating constant depending only on \(\delta\) \[2\]. In \[10\], K. Seip characterized the interpolating sequences for \(A^p(\mathbb{D})\) but here it suffices to use the result in \[2\] which is true also in several variables.

Take a sequence \(S\) in \(\mathbb{B}\) such that \(S\) cannot be contained in the zero set of a \(H^p(\mathbb{B})\) function, but such that the points in \(S\) are separated by \(\delta\).

This is easy to do: as in \[2\] just take for \(S\) a full net of points in \(\mathbb{B}\) separated by \(\delta\). The following lemma tells us that \(S\) cannot be the zero set of any \(H^p(\mathbb{B})\) function.

Lemma 5.5 Suppose that the sequence \(S\) is such that for any point \(\zeta \in \partial \mathbb{B}\) the admissible convergence region \(\Gamma(\zeta, \alpha) := \{ z \in \mathbb{B} / |1 - \overline{\zeta} \cdot z| < \alpha(1 - |z|^2)\}\) of aperture \(\alpha\) contains an infinite number of points of \(S\), then \(S\) cannot be contained in a non-trivial \(H^p(\mathbb{B})\) zero set.

Proof of the lemma:
suppose that there is an \(H^p(\mathbb{B})\) function \(f\) which is zero on \(S\). We know that the Fatou convergence theorem is valid, which means that \(f\) admits admissible boundary values \(f^*\) almost everywhere on \(\partial \mathbb{B}\) and \(f\) is the Poisson integral of \(f^*\) (\[9\] §5.4, p72).

Hence we have \(f^*(\zeta) = \lim_{a \in S \cap \Gamma(\zeta, \alpha), a \rightarrow \zeta} f(a), a.e. \zeta \in \partial \mathbb{B}\); thus \(f^* = 0\) a.e. hence \(f = 0\). \(\square\)

Taking a full net of points in \(\mathbb{B}\) separated by \(\delta\), one sees easily that, with \(\alpha\) big enough with respect to \(\delta\), the condition of the lemma is satisfied.

Then for any disc \((\sigma, \phi)\) passing through \(S\), the points of \(\sigma\) are separated by at least \(\delta\), hence \(\sigma \in IA^p(\mathbb{D})\) with a uniform constant.

But \(S\) cannot be in \(IH^p(\mathbb{B})\) because \(S\) is not contained in a non-trivial \(H^p(\mathbb{B})\) zero set and this is a necessary condition:

if \(S\) is \(H^p(\mathbb{B})\) interpolating, then there is a function \(f \in H^p(\mathbb{B})\) such that \(f(a_1) = 1, \forall k > 1, f(a_k) = 0\). Hence \(f\) is zero on \(S \setminus \{a_1\}\) and of course \(f\) is not identically 0. So the function \(g(z) := \overline{a}_1 \cdot (z - a_1) \overline{1 - \overline{a}_1 \cdot z} f(z)\) is still in \(H^p(\mathbb{B})\), still not identically 0, and is zero on \(S\), which proves that \(S\) must be contained in the zero set of a \(H^p(\mathbb{B})\) function. \(\square\)

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