Telescopers for Rational and Algebraic Functions via Residues

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ABSTRACT
We show that the problem of constructing telescopers for functions of \( m \) variables is equivalent to the problem of constructing telescopers for algebraic functions of \( m - 1 \) variables and present a new algorithm to construct telescopers for algebraic functions of two variables. These considerations are based on analyzing the residues of the input. According to experiments, the resulting algorithm for rational functions of three variables is faster than known algorithms, at least in some examples of combinatorial interest. The algorithm for algebraic functions implies a new bound on the order of the telescopers.

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1. INTRODUCTION
The problem of creative telescoping is to find, for a given “function” \( f \) in several variables \( t_1, \ldots, t_n, x_1, \ldots, x_m \), linear differential operators \( L \) involving only the \( t_i \) and derivations with respect to the \( t_i \), and some other “functions” \( g_1, \ldots, g_m \) such that

\[
L(f) = D_{x_1}(g_1) + \cdots + D_{x_m}(g_m),
\]

where \( D_{x_j} \) denotes the derivative with respect to \( x_j \). The main motivation for computing such operators \( L \) (called “telescopers” for \( f \)) is that, under suitable technical assumptions on \( f \) and the domain \( \Omega \), these operators have the definite integral

\[
F(t_1, \ldots, t_n) = \int_{\Omega} f(t_1, \ldots, t_n, x_1, \ldots, x_m) \, dx_1 \cdots dx_m
\]
as a solution. Once differential operators for \( F \) have been found, other algorithms can next be used for determining possible closed forms, or asymptotic information, or recurrence equations for the series coefficients of \( F \).

There are general algorithms for computing telescopers when the input \( f \) is holonomic [25, 15, 24, 20, 9] as well as special-purpose algorithms designed for restricted input classes [25, 26, 5]. The focus in the present paper is on two such restricted input classes: rational and algebraic functions of several variables. Our first result is that an algorithm for computing telescopes for rational functions of \( m \) variables directly leads to an algorithm for computing telescopes for algebraic functions of \( m - 1 \) variables and vice versa (Section 2). Our second result is a new algorithm for creative telescoping of algebraic functions of two variables (Section 3), which, by the equivalence, also implies a new algorithm for creative telescoping of rational functions of three variables. The algorithm for algebraic functions is mainly interesting because it implies a new bound on the order of the telescope in this case (Theorem 15), while the implied algorithm for rational functions is mainly interesting because at least for some examples it provides an efficient alternative to other methods (Section 4).

For a precise problem description, let \( k \) be a field of characteristic zero, and \( k(t, x) \) be the field of rational functions in \( t \) and \( x = (x_1, \ldots, x_m) \) over \( k \). Let \( x_m \) denote the \( m - 1 \) variables \( x_1, \ldots, x_{m-1} \). The algebraic closure of a field \( K \) will be denoted by \( \overline{K} \). The usual derivations \( \partial/\partial t \) and \( \partial/\partial x_k \), are denoted by \( D_t \) and \( D_{x_k} \), respectively. Let \( k(t)(D_t) \) be the ring of linear differential operators in \( t \) with coefficients in \( k(t) \). Then we are interested in the following two problems:

Problem 1. Given \( f \in k(t, x) \), find a nonzero operator \( L \in k(t)(D_t) \) such that

\[
L(f) = D_{x_1}(g_1) + \cdots + D_{x_m}(g_m)
\]

for some \( g_j \in k(t, x) \). Such an \( L \) is called a \textit{telescopers} for \( f \), and the rational functions \( g_1, \ldots, g_m \) are called certificates of \( L \).
Problem 2. Given $\alpha \in k(t, \mathfrak{x}_m)$, find a nonzero operator $L \in k(t)(D_t)$ such that

$$L(\alpha) = D_{x_1}(\beta_1) + \cdots + D_{x_m}(\beta_m - 1) \text{ for some } \beta_j \in k(t, \mathfrak{x}_m).$$

Such an $L$ is called a telescope for $\alpha$, and the algebraic functions $\beta_1, \ldots, \beta_m - 1$ are called certificates of $L$.

Both the equivalence of these two problems and the new algorithm for Problem 2 (when $m = 2$) are based on the general idea of eliminating residues in the input. As an introduction to this approach, let us consider the problem of finding a telescope and certificate for a rational function in two variables, that is, given a rational function $f \in k(t, x)$, we want to find a nonzero $L \in k(t)(D_t)$ such that $L(f) = D_x(g)$ for some $g \in k(t, x)$. We may consider $f$ as an element of $K(x)$, where $K = k(t)$, and as such we may write

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(x - \beta_i)^j},$$

where $p \in K[x]$, the $\beta_i$ are the roots in $K$ of the denominator of $f$ and the $\alpha_{i,j}$ are in $K$. We refer to the element $\alpha_{1,1}$ as the residue of $f$ at $\beta_1$. Using Hermite reduction, one sees that a rational function $h \in K(x)$ is of the form $h = D_x(g)$ for some $g \in K(x)$ if and only if all residues of $h$ are zero. Therefore to find a telescope for $f$ it is enough to find a nonzero operator $L \in K(D_t)$ such that $L(f)$ has only zero residues. For example assume that $f$ has only simple poles, i.e., $f = \sum_{a,b} a \in K[x], \deg_a a < \deg_b b$ and squarefree.

We then know that the Rothstein-Trager resultant [23, 19]

$$R := \text{resultant}_{x}(a - zD_x(b), b) \in K[z]$$

is a polynomial whose roots are the residues at the poles of $f$. Given a squarefree polynomial in $K[z] = k(t)[z]$, differentiation with respect to $t$ and elimination allow one to construct a nonzero linear differential operator $L \in k(t)(D_t)$ such that $L$ annihilates the roots of this polynomial. Applying $L$ to each term of (1) one sees that $L(f)$ has zero residues at each of its poles. Applying Hermite reduction to $L(f)$ allows us to find a $g$ such that $L(f) = D_x(g)$.

The main idea in the method described above is that nonzero residues are the obstruction to being the derivative of a rational function and one constructs a linear operator to remove this obstruction. Understanding how residues form an obstruction to integrability and constructing linear operators to remove this obstruction will be the guiding principal that motivates the results which follow.

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2. TELESCOPERS FOR RATIONAL FUNCTIONS

2.1 Rational and algebraic integrability

In this section, we give a criterion which decides whether or not $1$ is a telescope for a rational function in $k(t, x)$. Again, let $K = k(t)$. A rational function $f \in K(x)$ is said to be rational integrable with respect to $x$ if $f = \sum_{j=1}^{m} D_{x_j}(g_j)$ for some $g_j \in K(x)$. An algebraic function $\alpha \in K(\mathfrak{x}_m)$ is said to be algebraic integrable with respect to $\mathfrak{x}_m$ if $\alpha = \sum_{i=1}^{m} D_{x_i}(\beta_i)$ for some $\beta_i \in K(\mathfrak{x}_m)$. By taking traces, one can show that if $\alpha$ is algebraic integrable with respect to $\mathfrak{x}_m$, then an antiderivative of $\alpha$ already exists in the field $K(\mathfrak{x}_m)(\alpha)$.

For a rational function $f \in K(x)$, Hermite reduction with respect to $x_m$ decomposes $f$ into

$$f = D_{x_m}(r) + \frac{a}{b},$$

where $r \in K(x)$ and $a, b \in K(\mathfrak{x}_m)[x_m]$ such that $\deg_{x_m}(a) < \deg_{x_m}(b)$ and $b$ is squarefree with respect to $x$. It is clear that $f$ is rational integrable with respect to $x$ if and only if $a/b$ is (2) is rational integrable with respect to $x$. Over the field $K(\mathfrak{x}_m)$, one can write a rational function $f \in K(x)$ as

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(x_m - \beta_i)^j},$$

where $p \in K(\mathfrak{x}_m)[x_m]$ and the $\alpha_{i,j}, \beta_i$ are in $K(\mathfrak{x}_m)$. We call $\alpha_{i,j}$ the $x_m$-residue of $f$ at $\beta_i$, denoted by $\text{residuex}_m(f, \beta_i)$.

Proposition 3. Let $f \in K(x)$ and $\beta \in K(\mathfrak{x}_m)$. Then

(i) $\text{residuex}_m(f, \beta) = \alpha_i$ if $\beta = \beta_i$ or $\text{residuex}_m(f, \beta) = 0$ if $\beta \neq \beta_i$ for all $i = 1, \ldots, n$.

(ii) $D_{x_j}(\text{residuex}_m(f, \beta)) = \text{residuex}_m(D_{x_j}(f), \beta)$ for all $j$ with $1 \leq j \leq m - 1$.

Proof. The first assertion follows by observing the effect of $D_{x_m}$ on each term in the partial fraction decomposition of $g$. By Hermite reduction, we can decompose $f$ into

$$f = D_{x_m}(r) + \sum_{i=1}^{n} \frac{\alpha_i}{x_m - \beta_i}.$$
where \( r \in K(x) \), \( \alpha_i, \beta_i \in \overline{K(x_m)} \) and the \( \beta_i \) are pairwise distinct.

Suppose that all the \( x_m \)-residues \( \alpha_i \) of \( f \) are algebraic integrable with respect to \( x_m \), i.e., \( \alpha_i = \sum_{j=1}^{m-1} D_{x_j}(\gamma_{ij}) \) for some \( \gamma_{ij} \in K(\mathbf{x}_m)(\alpha_i) \). Note that for each \( j \) we have

\[
\frac{D_{x_j}(\gamma_{ij})}{x_m - \beta_i} = D_{x_j} \left( \frac{\gamma_{ij}}{x_m - \beta_i} \right) + D_{x_m} \left( \frac{\gamma_{ij} D_{x_j}(\beta_i)}{x_m - \beta_i} \right).
\]

Then we get

\[
\frac{\alpha_i}{x_m - \beta_i} = \sum_{j=1}^{m-1} D_{x_j}(\gamma_{ij}) + D_{x_m} \left( \sum_{j=1}^{m-1} \gamma_{ij} D_{x_j}(\beta_i) \right).
\]

Therefore, \( f \) is rational integrable with respect to \( x \) by taking

\[
g_j = \sum_{i=1}^{n} \frac{\gamma_{ij}}{x_m - \beta_i} \quad \text{and} \quad g_m = r + \sum_{i=1}^{n} \frac{\gamma_{ij} D_{x_j}(\beta_i)}{x_m - \beta_i}.
\]

Note that all the \( g_j \) and \( g_m \) are in \( K(x) \) because \( \gamma_{ij} \in K(\mathbf{x}_m)(\beta_i) \) and \( \beta_i \) are roots of a polynomial in \( K(\mathbf{x}_m)[x_m] \).

Suppose now that \( f \) is rational integrable with respect to \( x \), i.e., \( f = \sum_{j=1}^{\infty} D_{x_j}(g_j) \) for some \( g_j \in K(x) \). For any \( i \in \{1, 2, \ldots, n\} \), taking the \( x_m \)-residues of \( f \) and \( \sum_{j=1}^{m-1} D_{x_j}(g_j) \), respectively, and using Proposition 3 we get

\[
\text{resid}_{x_m}(f, \beta_i) = \alpha_i = \sum_{j=1}^{m-1} D_{x_j}(\text{resid}_{x_m}(g_j, \beta_i)),
\]

which implies that \( \alpha_i \) is algebraic integrable with respect to \( x_m \). 

**Example 5.** Let \( f = 1/(x_1 + x_2) \). Then the \( x_2 \)-residue of \( f \) at \(-x_1\) is 1. Since \( 1 = D_{x_1}(x_1) \), \( f \) is rational integrable with respect to \( x_1 \) and \( x_2 \). More precisely,

\[
f = D_{x_2} \left( \frac{x_1}{x_1 + x_2} \right) + D_{x_2} \left( -\frac{x_1}{x_1 + x_2} \right).
\]

**Example 6.** Let \( f = 1/(x_1 x_2) \). Then the \( x_2 \)-residue of \( f \) at 0 is \( 1/x_1 \). Since \( 1/x_1 \) has no antiderivative in \( K(x_1) \), \( f \) is not rational integrable with respect to \( x_1 \) and \( x_2 \).

### 2.2 Equivalence

**Theorem 7.** Let \( f \in k(t, x) \). Then \( L \in k(t)[D_t] \) is a telescoper for \( f \) if and only if \( L \) is a telescoper for every \( x_m \)-residue of \( f \).

**Proof.** By a similar calculation as in the proof of Proposition 3, we have

\[
L(\text{resid}_{x_m}(f, \beta)) = \text{resid}_{x_m}(L(f, \beta)) \quad (3)
\]

for any \( L \in k(t)[D_t] \) and \( \beta \in \overline{k(t, x_m)} \). If \( L \in k(t)[D_t] \) is a telescoper for \( f \), then \( L(f) = \sum_{j=1}^{m-1} D_{x_j}(g_j) \) for some \( g_j \in k(t, x) \). By Proposition 3 and Equation (3), for the \( x_m \)-residue \( \alpha := \text{resid}_{x_m}(f, \beta) \) at any pole \( \beta \) of \( f \) with respect to \( x_m \), we have

\[
L(\alpha) = \sum_{j=1}^{m-1} D_{x_j}(\text{resid}_{x_m}(g_j, \beta)).
\]

So \( L \) is a telescoper for \( \alpha \). Conversely, assume that \( L \) is a telescoper for any \( x_m \)-residue of \( f \). Note that any \( x_m \)-residue of \( L(f) \) is of the form \( L(\text{resid}_{x_m}(f, \beta)) \), which is algebraic integrable by assumption. Then \( L(f) \) is rational integrable by Lemma 4. Therefore, \( L \) is a telescoper for \( f \).

Now we can present an explicit translation between the two telescoping problems by using Theorem 7.

If we can solve Problem 2, then for a rational function \( f \in k(t, x) \), first, we can perform Hermite reduction to decompose \( f \) into \( f = D_{x_m}(r) + q/b; \) second, we compute the resultant \( R := \text{resultant}_{x_m}(a - z D_{x_m}(b), b) \in k(t, \mathbf{x}_m)[z] \); finally, we get a telescoper for \( f \) by constructing telescopers for all the roots of \( R \) in \( k(t, \mathbf{x}_m) \) and taking their least common left multiple.

On the other hand, if we can solve Problem 1, then for an algebraic function \( \alpha \in k(t, \mathbf{x}_m) \) with minimal polynomial \( F \in k(t, \mathbf{x}_m) \), we compute a telescoper \( L \) for the rational function \( f = x_m D_{x_m}(F)/F \). Note that \( \alpha \) is the \( x_m \)-residue of \( f \) at \( \alpha \). Therefore, \( L \) is a telescoper for \( \alpha \).

**Example 8.** Consider the rational function

\[
f = \frac{2y(1-x)x(x+1)(x+2)(t+x)(xy-y-t^4)}{(1-x)(2-x + (x+1)(x+2)(t+x)(xy-y-t^4)).
\]

In order to find a telescoper for \( f \), we view \( f \) as a rational function in \( y \) with coefficients in \( k(t, x) \) and determine its residues in \( k(t, x). \) Write \( a \) and \( b \) for the numerator and denominator of \( f \). Since \( b \) is squarefree, the residues of \( f \) are precisely the roots of the the Rothstein-Trager resultant \( \text{resultant}_y(a - z D_y(b), b) \in k(t, x)[z] \). In the present example, these are

\[
\frac{t^4}{x-1} \pm \sqrt{x(x+1)(x+2)(t+x)}
\]

According to Theorem 7, it now suffices to find a telescoper for this algebraic function. This problem is discussed in the following section.

### 3. TELESCOPERS FOR ALGEBRAIC FUNCTIONS

We showed above how focusing on residues can yield a technique to find telescopers of rational functions by reducing this question to a similar one for algebraic functions. In this section we describe an algorithm to solve this latter problem for algebraic functions of two variables. In what follows, the term “algebraic function” will always refer to functions of two variables \( f \) and \( x \). When one tries to use residues to solve the problem of finding telescopers for algebraic functions one must deal with several complications.

The first is a technical complication. One does not have a global way of expressing a function similar to partial fractions and so must rely on local expansions. This forces one to look at differentials rather than functions in order to define the notion of residue in a manner that is independent of local coordinates. The second complication is a more substantial one. There are differentials \( dx \) having zero residues everywhere that are not of the form \( d\beta = D_{x_j}(\beta)dx \), i.e., \( \alpha \) is not the derivative of an algebraic function. Nonetheless, one knows that there must exist an operator \( L \in k(t)[D_t] \) of order equal to twice the genus of the curve associated to \( f \) such that \( L(\alpha)dx = d\beta \) for some algebraic \( \beta \). This will force us to add an additional step to find our desired telescoper.
In Section 3.1, we will gather some facts concerning differen-
tials in function fields of one variable that will be needed in
our algorithm. In Section 3.2 we describe the algorithm.

3.1 Derivations and Differentials

In this section we review some notation and facts concern-
ing function fields of one variable (cf. [2, 4, 8, 11, 16]). In
the previous section the results and calculations depended hea-
vely on the notion of the residue of a rational function of
y at an algebraic function βi of x. In the present section
we shall also need to use the notion of a residue but since
we are dealing with algebraic functions instead of rational
functions, the appropriate notion is that of a residue of a
differential ω at a place P of the associated function field E.
We will denote this by residueP ω and refer to the above
mentioned books for basic definitions and properties.
We note that when f ∈ E = K(x)(y), and βi ∈ K(x), then
residuey(f, βi) = residueP ω, where ω = fdx and P is the
place (y − βi) of E.

Let K be a differential field of characteristic zero with
derivation denoted by Dι (for example, K = k(t) with Dι as
above). Let x be transcendental over K and E = K(x, y) an
algebraic extension of K(x). We may extend the derivation
Dι to a derivation Dι x on K(x) by first letting Dι x(ω) = 0
and then taking the unique extension to E. We define a
derivation Dι x on K(x) by letting Dι x be zero on K, Dι x(x) = 1
and taking the unique extension of Dι x from K(x) to E.
We shall also assume that the constants E/Dι x = \{c ∈ E |
Dι x(c) = 0\} are precisely K. This is equivalent to saying
that the minimal polynomial of y over K(x) is absolutely
irreducible (cf. [10]). In [8], Chapter VI, §7, Chevalley says
that Dι x can be used to define a map (which we again
denote by Dι x) on differentials such that Dι x(dfx) = (Dι x(f))dx.
The map Dι x furthermore has the following properties:

1. Dι x(dg) = d(Dι x g) for any g ∈ E, and
2. for any place P of E and any differential ω,
   residueP(Dι x ω) = Dι x(residueP(ω)).

Given α ∈ E we will want to find an operator L ∈ K(Dι x)
and an element β ∈ E such that L(α) = Dι x(β). In terms of
differentials, this latter equation may be written as L(ω) =
dβ, where ω = αdx.

We shall have occasion to write our field E as E = K(\bar{x}, \bar{y})
for some other \bar{x} which is transcendental over K and \bar{y} alge-
braic over K(\bar{x}) and work with the derivation Dι x defined in
a similar manner as above. We will need to know that if we
can find a telescope with respect to the derivation Dι x then
we can convert this into a telescope with respect to Dι x.
The following lemma and proposition allow us to do this.

Lemma 9. Let x and \bar{x} be as above and let ω be a differential
of E. For any i = 1, 2, . . . there exists ui ∈ E such that

\begin{equation}
(Dι x i) (\bar{\omega}) = (Dι x i) (\omega) = dui.
\end{equation}

Proof. Write \omega = αdx. Lemma 1 of [16] (see also Lemma 3
in Chapter VI, §7 of [8]) implies that

\begin{equation}
Dι x (\omega) − Dι x (\bar{\omega}) = −d(αDι x (\bar{x})).
\end{equation}

Letting ui = −αDι x (\bar{x}), we have equation (5) for i = 1. One
can verify by induction that (5) holds for u_{i+1} = Dι x (ui) −
v_{i}Dι x (\bar{x}), where v_{i} = (Dι x i)(αDι x (\bar{x}) · Dι x (x)).

Proposition 10. Let α ∈ E, ω = αdx,

\begin{align*}
(Dι x i) (\omega) + a_{n−1}(Dι x i) (\omega) + · · · + a_{0} ∈ K(Dι x).
\end{align*}

and \bar{β} ∈ E such that

\begin{align*}
((Dι x i) (\omega) + a_{n−1}(Dι x i) (\omega) + · · · + a_{0})(\omega) = d\bar{β}.
\end{align*}

One can effectively find \beta ∈ E such that

\begin{align*}
((Dι x i) (\omega) + a_{n−1}(Dι x i) (\omega) + · · · + a_{0})(\omega)
= d(\bar{β} − u_{n} − u_{n−1}u_{n−1} − · · · − a_{i}u_{1}),
\end{align*}

which implies the conclusion of the proposition with \beta =
\bar{β} − u_{n} − u_{n−1}u_{n−1} − · · · − a_{1}u_{1}.

In the algorithm described in the next section, we will
consider a differential ω in E = K(x, y) and assume that

1. ω has no poles at any place above the place of K(x)
at infinity, and
2. the places where ω does have a pole are all unramified
above places of K(x).

We describe below an algorithm that allows one to select
an \bar{x} ∈ E such that E = K(\bar{x}, y) and such that ω satisfies
conditions 1. and 2. above with respect to K(\bar{x}). The
algorithm of Section 3.2 can be used to produce a telescope
with respect to Dι x and Proposition 10 allows one to con-
versthis telescope to a telescope with respect to Dι x.
In the following proposition, the proof that condition 2. can be
fulfilled was outlined to us by Barry Trager [21, 22].

Proposition 11. Let \omega be a differential in E = K(x, y).

One can effectively find an \bar{x} ∈ E such that E = K(\bar{x}, y)
and

1. ω has no poles at any place above the place of K(\bar{x})
at infinity, and
2. the places where ω does have a pole are all unramified
above places of K(\bar{x}).

Proof. If 1. does not hold, let c ∈ K be selected so that ω
has no poles above x = c, let

\begin{equation}
\bar{x} = \frac{cx}{x−c}.
\end{equation}

This change of variables interchanges c and the point at in-
finity, so 1. is now satisfied with respect to K(\bar{x}) and we
shall henceforth abuse notation and assume that 1. is sat-
fified with respect to K(x).

Let Ω be a nonsingular curve that is a model of E. The
elements of E can be considered as functions on Ω. As noted
in [21, p. 63], ramification occurs when the line of projection
from the curve down to the x-axis is tangent to the curve
and, for each pole of ω, there are only a finite number of
projection directions that are tangent to the curve at this
pole. Therefore for all but finitely many choices of an integer \(m\), if we let \(\bar{x} = x + my\), \(\omega\) will satisfy 2. with respect to \(K(\bar{x})\). One can refine this argument and produce a finite set of integers \(m\) that are to be avoided. This is done in the following way.

Let \(M\) be an indeterminate and consider the field \(E_1 = E(M) = k_1(\bar{x}, y)\), where \(k_1 = K(M)\) and \(\bar{x} = x + My\). Let \(\sigma = K[M]\) and assume that (after a possible change of \(y\)) \(y\) satisfies a monic polynomial over \(\sigma[\bar{x}]\). The behavior of various objects in \(E_1\) when one reduces \(\sigma\) modulo a prime ideal of \(\sigma\) is considered in [11, Chapter III, §6]. We shall be interested in reducing modulo ideals of the form \((M - m)\), where \(m\) is an integer. One can effectively calculate an integral basis \(\{w_i(M)\}\) of the integral closure of \(k_1[\bar{x}]\) in \(E_1\) (cf. [12, 21]) and from this a complementary basis \(\{w'_i(M)\}\) (2, Chapter 5, §2), [4, §22]). In Chapter III §6.2 of [11], Eichler gives a method that will produce a finite set \(S \subset \mathbb{Z}\) such that for \(m \notin S\), the set \(\{w_i(m)\}\) is again an integral basis of the integral closure of \(K[\bar{x}]\) in \(E\). This method can be refined (and the set \(S\) slightly increased if need be) so that \(\{w'_i(m)\}\) is a complementary basis. Expressing \(\omega\) in terms of this complementary basis,

\[
\omega = \frac{1}{b(\bar{x})} \sum_{i=1}^n p_i(M, \bar{x})w'_i(M)d\bar{x},
\]

one sees that \(\omega\) will have poles precisely at the zeroes of \(b(\bar{x})\).

If one selects \(m \in \mathbb{Z}\) such that \(b(\bar{x})\) is relatively prime to \(D(\bar{x})\), the discriminant of the integral basis \(\{w_i(m)\}\), then \(\omega\) will not have poles at ramification points. The finitely many values of \(m\) that do not satisfy this latter condition are roots of

\[
S(M) = \text{resultant}_X(\text{resultant}_Y(\beta(b(X + MY)), F(X, Y)), \text{resultant}_Y(D(X + MY), F(X, Y))),
\]

where \(F \in K[X, Y]\) is the minimal polynomial of \(y\) over \(K(x)\).

### 3.2 An Algorithm to Calculate Telescopes for Algebraic Functions

We assume we are given a function field of one variable \(E = K(x, y)\) and a differential \(\omega\) in \(E\). We shall furthermore assume that \(\omega\) satisfies conditions 1. and 2. of Proposition 11. We will describe an algorithm to find \(a_0, \ldots, a_n \in K\), not all zero, and \(\beta \in E\) such that

\[
(a_n(D_\beta^x)^n + a_{n-1}(D_\beta^x)^{n-1} + \ldots + a_0)(\omega) = d\beta.
\]

If \(\omega = adx\), then \(L = a_n(D_\beta^x)^n + a_{n-1}(D_\beta^x)^{n-1} + \ldots + a_0\) is a telescope for \(a\) with certificate \(\beta\). The algorithm has two steps. The first step finds an operator \(L_1\) such that applying this operator to \(\omega\) results in a differential \(L_1(\omega)\) with only zero residues. The second step finds an operator \(L_2\) of order at most twice the genus of \(E\) and an element \(\beta \in E\) such that \(L_2(L_1(\omega)) = d\beta\).

**Step 1.** We will describe two methods for constructing an operator that annihilates the residues of \(\omega\). The first one requires one to calculate in algebraic extensions of \(K\) while the second only requires calculations in \(K\). Throughout, let \(F(x, Y) \in K[x, y]\) be a minimal polynomial of \(y\) over \(K(x)\) and let

\[
\omega = adx = \frac{A}{B}dx
\]

for some \(A \in K[x, y]\) with no finite poles and \(B \in K[x]\).

**Method 1.** We make no assumptions concerning ramification at the poles but for convenience we do assume that the poles of \(\omega\) only occur at finite points. Let \(a \in \mathbb{R}\) be a root of \(B\). For any branch of \(F(x, Y) = 0\) at \(x = a\), we may write

\[
\omega = p_a(z)dz,
\]

where \(z = (x - a)^{1/m}\) for some positive integer \(m\) and \(p_a\) is a Laurent series in \(z\) with coefficients in \(K\). One can calculate the coefficient of \(1/z\) in \(p_a\) and this will be the residue of \(\omega\) at this place. In this way, one can calculate the possible residues \(\{r_1, \ldots, r_s\}\) of \(\omega\). Let \(K_1\) be a Galois extension of \(K\) containing \(\{r_1, \ldots, r_s\}\). Let \(C\) be the field of \(D\)-constants in \(K_1\) and \(\{\tilde{r}_1, \ldots, \tilde{r}_s\}\) be a \(C\)-basis of \(C_{r_1} + \ldots + C_{r_s}\). Let \(L(Y) = w(Y, \tilde{r}_1, \ldots, \tilde{r}_s)\) where \(w(\ldots)\) is the Wronskian determinant. One sees that \(L(Y)\) is a nonzero linear differential polynomial with coefficients in \(K_1\) such that \(L_1(r_i) = 0\) for \(i = 1, \ldots, s\). Define

\[
L_1(Y) = \text{lcm}\{L^\sigma(Y) \mid \sigma \in G\},
\]

where \(G\) is the Galois group of \(K_1\) over \(K\). \(L^\sigma(Y)\) denotes the linear differential polynomial resulting from applying \(\sigma\) to each coefficient of \(L\) and \(\text{lcm}\) denotes the least common left multiple. We then have that \(L_1(Y)\) has coefficients in \(K\) and annihilates the residues of \(\omega\).

**Method 2.** We now assume that \(\omega\) has poles only at finite places and that there is no ramification at the poles. This implies that at any place corresponding to a pole, we may write \(\omega = \sum_{i \geq 0} \alpha_i(x - x_0)^i\) for some \(\alpha_i \in K\). Therefore the residue of \(\omega\) at this place is

\[
\alpha_{-1} = \left(\frac{D_\beta((x - x_0)^{-i-1}a_i)}{x = x_0}\right).
\]

This is the key to the following, parts of which in a slightly different form appear in [7].

**Proposition 12.** Given \(\omega\) as above, one can compute a polynomial \(R \in K[Z]\) of degree

\[
m := \deg_Z(R) \leq \deg_Y(F) \deg_x(B^n),
\]

with \(B^*\) the square free part of \(B\), such that if \(a\) is a nonzero residue of \(\omega\) then \(R(a) = 0\). Furthermore, one can compute a nonzero operator \(L_1 = a_n(D_\beta^x)^n + a_{n-1}(D_\beta^x)^{n-1} + \ldots + a_0 \in K(D_\beta^x)\) such that \(\omega := L_1(\omega)\) has residue zero at all places.

**Proof.** We may write

\[
adx = \frac{A}{B}dx = \frac{A_1}{B_1}dx + \frac{A_2}{B_2}dx + \ldots + \frac{A_i}{B_i}dx,
\]

where the \(A, A_i \in K(x, y)\) are regular at finite places and \(B = B_1B_2 \cdots B_i \in K[x]\) is the squarefree decomposition of \(B\). To achieve our goal it is therefore enough to prove the claim for a differential of the form \(adx = \frac{A}{B}dx\), where \(A \in K(x, y)\) is regular at finite places and \(B \in K[x]\) is squarefree. Following [7], we let \(u\) be a differential indeterminate and let

\[
h = \frac{(A_n)^{n-1}}{(n-1)!} \in K(x, y)(u),
\]

where \(K(x, y)(u)\) is the ring of differential polynomials in \(u\) with coefficients in \(K(x, y)\) and \((\ldots)^{(l)}\) denotes \(l\)-fold differentiation with respect to \(x\). Let \(\mathcal{P}\) be a place where \(a\) has a pole and let \(a\) and \(b\) denote the values of \(x\) and \(y\) at the place. We note that since \(A\) is regular at \(\mathcal{P}\) and \(\mathcal{P}\) is not
ramified, any derivative of $A$ is also regular at $P$ (one needs the hypothesis that these places are unramified to make this claim). Taking into account the rules of differentiation, we see that
\[ h = \frac{p(x, y, u, u', \ldots, u^{(n-1)})}{q(x)u^n}, \]
where $p(x, Y, z_0, z_1, \ldots, z_{n-1}) \in K[x, Y, z_0, z_1, \ldots, z_{n-1}]$, $t$ is some positive integer and $q(x) \in K[x]$ does not vanish at $P$, i.e. $q(a) \neq 0$. Let
\[ \tilde{p} = p(x, Y, B', 1/2 B'', 1/4 B'''', \ldots, 1/n B^{(n)}) \in K[x, Y] \]
and
\[ \tilde{q} = q(x)(B')^i \in K[x]. \]
One then shows, as in [7], that $\tilde{p}(a, b)/\tilde{q}(a)$ is the residue of $\frac{\alpha dx}{\omega}$ at $P$.

The above argument shows that the polynomial
\[ R = \text{resultant}_t(\tilde{p} - Z\tilde{q}, F, B) \in K[Z] \]
vanishes at the residues of $\alpha dx$. The degree estimate for $R$ follows from the general degree estimate for resultant functions which states for any $S, T \in K[u, v]$ that $\deg_{u}(\text{resultant}_t(S, T))$ is at most
\[ \deg_u(S)\deg_v(T) + \deg_u(S)\deg_v(T). \]
This implies first that the inner resultant in the definition of $R$ has $Z$-degree at most $\deg_y(F)$. (Note that no degree estimates for $\tilde{p}$ and $\tilde{q}$ are needed because $\deg_2(F) = 0$.) Applying the rule again to the outer resultant gives the desired bound $\deg_y(F)\deg_u(B)$.

Let $R \in K[Z]$ be the polynomial above. If necessary, we may replace $R$ by a squarefree polynomial having the same nonzero roots so we shall assume that $R$ is squarefree and of degree $m$. Using the fact that $R$ and $\frac{\alpha dx}{\omega}$ are relatively prime, there exist polynomials $R_i \in K[Z]$ of degree at most $m - 1$ such that if $\gamma$ is a root of $R_i$, then $D_i(\gamma) = R_i(\gamma)$ for $i = 0, 1, \ldots, n$. Since each $R_i$ has degree at most $m - 1$, there exist $a_m, \ldots, a_0 \in K$, not all zero, such that $a_m(D_i)^m + a_{m-1}(D_i)^{m-1} + \ldots + a_0(\gamma) = 0$ for any root $\gamma$ of $R$. Using the fact that $\text{res}_{d\beta}(D_i^t(\omega)) = D_i^t(\text{res}_{d\beta}(\omega))$ for any place $A$, one sees that for $L_i = a_m(D_i)^m + a_{m-1}(D_i)^{m-1} + \ldots + a_0$, $\tilde{\omega} = L_1(\omega)$ has zero residue at any place.

**Remark.** Although Method 2 does not require calculations in an algebraic extension of $K$, one needs the condition on ramification to prove that it is correct. This condition is painful to verify and although Propositions 10 and 11 imply that we can make a transformation, if necessary, to guarantee that the differential has poles at places that are not ramified, making such a transformation can increase the complexity of the data. In practice, one should calculate the operator $L_1$ above without testing if the places at poles are ramified, calculate the operator $L_2$ as in step 2 below (which requires no assumption concerning ramification) and then test to see if the resulting operator $L_2 \circ L_1$ is a telescoper by checking if the identity $L_2(L_1(\omega)) = D_2(\beta)$ holds, a simple calculation in $K(x, y)$. If this equality does not hold, then one can make a change of variable $\tilde{x} := x + my$ for a random $m$ and try again. Proposition 11 guarantees that after a finite number of trials one will succeed.

**Example 13** (continuing Ex. 8). Let $F = y^2 - x(x+1)(x+2)(x+t)$ and consider
\[ \omega = \left( \frac{t^4}{x-1} + \frac{1}{y} \right) dx = \frac{u}{v} dx, \]
where $u = (x-1)y + t^4 x(x+1)(x+2)(t+x)$ and $v = x(x+1)(x+2)(x+t)(x-1)$. The only pole of $\omega$ is a simple pole at $x = 1$, so the residues of $\omega$ are the roots of
\[ \text{res}_u(\text{res}_v(u - zD_2(v), F, v)) = (\ldots)(z - t^4)x^8, \]
where $(\ldots)$ stands for some factors which are free of $z$ and therefore irrelevant here. The only nonzero residue $\tilde{t}^4$ is annihilated by $L_1 := tD_1 - 4$, so
\[ \tilde{\omega} = (tD_1 - 4)(\omega) = -\frac{(9t + 8x)y}{2x(x+1)(x+2)(x+t)^2} dx \]
has no nonzero residues.

**Remark.** 1. In [21], Trager develops a Hermite reduction method for algebraic functions which, when applied to the differential $\omega$ above, shows how one can write $\omega = (D_2(g_1) + g_2) dx$, where $g_1, g_2 \in E$ and $g_2$ has only simple poles at finite points. Regrettably, $g_2$ may have poles (of higher order) at infinity. Nonetheless, it would be interesting to see if Trager’s procedure can be used to increase efficiency in our algorithm.

2. The above argument strongly relies on the fact that we are assuming that the places where $\omega$ has poles are not ramified above places in $K(x)$. It would be of interest to give a method to calculate an operator $L_1$ satisfying the conclusion of Proposition 12 without this assumption.

**Step 2.** Let $\tilde{\omega}$ be as in the conclusion of Proposition 12. Again using the fact that $\text{res}_{d\beta}(D_i^t(\tilde{\omega})) = D_i^t(\text{res}_{d\beta}(\tilde{\omega}))$ for any place $P$, we have for all $i \in Z$ that $(D_i^t)^i(\tilde{\omega})$ is again a differential with zero residues at all places. Such a differential is called a differential of the second kind ([8], p. 50) and a differential of the form $d\gamma, \gamma \in E$ is called an exact differential. Note that any exact differential is a differential of the second kind. Corollary 1 of ([8], p. 130) states that the factor space of the space of differentials of the first kind by the space of exact differentials is a $K$-vector space of dimension equal to $2G$, where $G$ is the genus of $E$. Therefore, there exist $a_2G, \ldots, a_0 \in K$ not all zero, such that for $L_2 = a_2G(D_i^t)^{2G} + a_{2G-1}(D_i^t)^{2G-1} + \ldots + a_0$, $L_2(\omega) = d\beta$ for some $\beta \in E$. Such $L_2$ and $\beta$ can be found as follows.

Let $\tilde{\omega} = \hat{\alpha} dx$ and let $[E : K(x)] = m$. For each $i \geq 0$, there exist $\alpha_{i,0}, \ldots, \alpha_{i,m-1} \in K(x)$ such that
\[ (D_i^t)^i(\hat{\alpha}) = (y, \ldots, y^{m-1}) \left( \begin{array}{c} \alpha_{i,0} \\ \vdots \\ \alpha_{i,m-1} \end{array} \right). \]
In addition, there exists an $m \times m$ matrix $A$ with entries in $K(x)$ such that
\[ (D_i(y), \ldots, D_i(y^{m-1})) = (y, \ldots, y^{m-1}) A. \]
Let $a_0, \ldots, a_2G$ be elements of $K$ and $\beta_0, \ldots, \beta_{m-1}$ elements of $K(x)$. Letting $\beta = \beta_0 + \beta_1 y + \ldots + \beta_{m-1} y^{m-1}$, the equation
\[ d\beta = (a_2G(D_i^t)^{2G}(\hat{\alpha}) + \ldots + a_0 \hat{\alpha}) dx \]
is equivalent to
\[
D_x \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{m-1} \end{pmatrix} + A \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{m-1} \end{pmatrix} \\
= a_{2G,0} \begin{pmatrix} \alpha_{2G,0} \\ \vdots \\ \alpha_{2G,m-1} \end{pmatrix} + \ldots + a_0 \begin{pmatrix} \alpha_{0,0} \\ \vdots \\ \alpha_{0,m-1} \end{pmatrix}.
\]
(6)

In [3], Barkatou describes a decision procedure for deciding if there exist nontrivial \(\beta_0, \ldots, \beta_{m-1} \in K(x)\) and \(a_0, \ldots, a_{2G} \in K\) satisfying (6) when \(K\) is a computable field (i.e., the arithmetic operations and derivation are computable and one has an algorithm to factor polynomials over \(K\)). Therefore one can apply this to \(K = k(t)\), where \(k\) is a computable field of characteristic zero to produce a desired \(L_2\) and \(\beta\).

**Example 14** (continuing Ex. 13). Let again \(F = y^2 - x(x + 1)(x + 2)(x + t)\) and consider the differential
\[
\omega = -\frac{(9t + 8x)y}{2x(x + 1)(x + 2)(x + t)^2} dx.
\]
Since the field \(E\) has genus 1 and \(\omega\) has only zero residues, there exists a telescoper for \(\omega\) of order 2. Indeed, the algorithm outlined above finds that \(L(\omega) = \beta \omega\), where
\[
L_2 = 4(99t^2 - 540t^4 + 1055t^6 - 870t^2 + 256t)D_t^2 \\
+ 4(297t^4 - 1269t^6 + 1900t^2 - 1152t + 256)D_t \\
+ 3(99t^3 - 306t^2 + 307t - 96)
\]
and
\[
\beta = \frac{3(429t^3 + 330t^2 x - 891t^2 - 648x + 384t + 256x)}{(t + x)^3}.
\]
For the differential \(\omega\) from Example 13, it follows that we have \(L\omega = d\beta\) with
\[
L = L_2 \circ (D_t - 4) = 4(t - 2)(t - 1)^2(99t^2 - 243t + 128)D_t^3 \\
+ 4t(99t^2 - 189t^2 - 810t^2 + 588t^2 - 256)D_t^2 \\
- 3(1089t^2 - 4770t^3 + 7293t^2 - 4512t + 1024)D_t \\
- 12(99t^3 - 306t^2 + 307t - 96).
\]

By **Theorem 7**, this operator \(L\) is also a telescoper for the trivariate rational function \(f\) from Example 8. Certificates \(g, h\) with
\[
L(f) = D_x(g) + D_y(h)
\]
are defined from \(\beta\) following the calculations in the proof of **Lemma 4**. They are however too long to be printed here.

**Remark**. Telescopers and certificates for holomorphic differentials arise in Manin’s solution of Mordell’s Conjecture [16, 17] and Step 2 of our procedure is just an effective version of considerations that appear in these papers. Telescopers for holomorphic differentials are also referred to as **Gauss-Manin Connections**.

Combining the estimates on the order of the operators computed in steps 1 and 2 gives the following bound on the order of telescopers for algebraic functions. It can be viewed as a generalization of Corollary 14 in [5], which says that for every rational function \(f = A/B \in K(x)\) there exists a telescoper of order at most \(\deg_x B^*\), where \(B^*\) is the square free part of \(B\).

**Theorem 15**. Let \(E\) be an algebraic extension of \(K(x)\), \(\alpha = A/B \in E\) so that \(A\) is regular at finite places and \(B \in K[x]\). Let \(B^*\) be the square free part of \(B\). Then there exists \(\beta \in E\) and a nonzero operator \(L \in K(D_t)\) with \(L(\alpha) = D_t(\beta)\) and
\[
\deg_{D_t}(L) \leq \lfloor E : K(x) \rfloor \deg_x(B^*) + 2\text{genus}(E).
\]

4. IMPLEMENTATION AND OTHER EXAMPLES

We have produced a prototype implementation of the algorithms described above on top of Koutschan’s Mathematica package “HolonomicFunctions.m” [13] and compared the performance to the built-in creative telescoping implementations of this package. In order to make the comparison as fair as possible, we have tried to reuse as much code from Koutschan’s package as possible, so that the timings will not implicitly compare two different implementations of some subroutine but reflect as closely as possible the speed-up (or slow-down) offered by the ideas presented above.

Five different methods to solve the creative telescoping problem for a rational function \(f \in K(t, x, y)\) were considered: **(CC)** first use Chyzak’s algorithm [9] to find a holonomic system \(S\) of operators in \(k(t, x, y)/(D_t, D_x)\) such that for all \(L \in S\) there exists a rational function \(g \in k(t, x, y)\) with \(L(f) = D_y(g)\), afterwards apply the same algorithm to \(S\) to obtain a telescoper \(L \in k(t)/(D_t)\) for \(f\); **(CK)** first compute \(S \subseteq k(t, x)/(D_t, D_x)\) as in variant **(CC)**, then apply Koutschan’s ansatz [14] to \(S\) to obtain a telescoper \(L\) for \(f\); **(K)** compute a telescoper for \(f\) directly with Koutschan’s ansatz; **(EC)** use the reduction from Section 2, then apply Chyzak’s algorithm to the resulting algebraic functions, and then take the least common left multiple of the results; **(EA)** use the reduction from Section 2, then apply the algorithm from Section 3 to the resulting algebraic functions, and then take the least common left multiple of the results.

Table 1 shows the performance of these five approaches for the following examples.

1. The rational function \(f\) from Example 8 above. This example is not representative but was particularly designed to be easy for our algorithms and difficult for the known ones.

2. Here \(f := \frac{1}{xy} h\left(\frac{1}{xy}, x, y\right)\) with \(h(t, x, y) = \left(1 - \frac{t}{x^2} - \frac{x}{t} - \frac{x}{y} - \frac{y}{x} - \frac{y}{t} - \frac{y}{x^2}\right)^{-1}\). This is the problem of enumerating diagonal 3D-Queens walks raised in [6]. Our calculation confirms the correctness of the telescoper conjectured there.

3. Let now \(h(t, x, y) = \left(1 - \frac{t}{x^2} - \frac{x}{t} - \frac{y}{x} - \frac{y}{t} - \frac{y}{x^2}\right)^{-1}\) and \(f = \frac{1}{xy} h\left(\frac{1}{xy}, x, y\right)\). This is a variation of the previous problem, with the points \((2n, n, n)\) replacing the diagonal and now allowing steps along the axes.

4. The rational function \(h(t, x, y) = 2t^2/(1 - (t - 3)(x + y + t + xy + xt + y^2 + 3xy^2))\) appears in [18] as the generating function for the probability of certain structures in random groves. See [18] for details on the combinatorial background. Here we compute the diagonal series coefficients of \(f\) by applying creative telescoping to \(f = \frac{1}{xy} h\left(\frac{1}{xy}, x, y\right)\). As can be seen in this example, our algorithms are in some cases not superior.
5. With $h$ as before, we now consider $f = \frac{1}{x+y} h(\frac{1}{x+y}, x, y)$. Note the large difference between CC and CK.

We have put timings for a number of additional examples on the website [1]. Also our code and the certificates for Example 13 can be found there. The examples we tested suggest that the reduction from rational functions to algebraic functions can cause a decent speed-up. It does seem to depend on whether the Rothstein-Trager resultant of the input factors into several small factors or not. If it does, it is advantageous because solving several small instances of Problem 2 is cheaper than solving a single big one. Whether after the reduction, the algorithm of Section 3 or some other method is applied to the resulting algebraic functions, makes usually not much of a difference. Our algorithm tends to be faster when Step 1 in Section 3.2 already finds a great part of the telescope, leaving only a small coupled differential system to be solved in Step 2.

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