AN ESTIMATE FOR THE ANISOTROPIC MAXIMUM CURVATURE IN THE PLANAR CASE

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Abstract. We fix a Finsler norm $F$ and, using the anisotropic curvature flow, we prove that the anisotropic maximum curvature $k^F_{\text{max}}$ of a smooth Jordan curve is such that $k^F_{\text{max}}(\gamma) \geq \sqrt{\frac{\kappa}{A}}$, where $A$ is the area enclosed by $\gamma$ and $\kappa$ the area of the unitary Wulff shape associated to the anisotropy $F$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$, be a bounded, connected, $C^2$ smooth domain and let us set $\gamma := \partial \Omega$, so that $\gamma$ is a smooth Jordan curve. In [PI] the following inequality is proved:

$$k_{\max}(\gamma) \geq \sqrt{\frac{\pi}{A(\Omega)}},$$

where $k_{\max}(\gamma)$ is the maximum curvature of $\gamma$ and $A(\Omega)$ is the area enclosed by $\gamma$; moreover, equality holds if and only if $\gamma$ is a circle. Since the original work [PI] is hardly available, we refer the reader to [HT] for the proof of (1.1).

In [P] the author provides a new proof of the inequality (1.1) by means of the curve shortening flow. Recalling the definition, we have that a family $u : \mathbb{S}^1 \times [0, T] \to \mathbb{R}^2$ of smooth Jordan curves flows by anisotropic curvature flow if

$$\frac{\partial u(\theta, t)}{\partial t} = -k(\theta, t)\nu(\theta, t),$$

where $\nu(\theta, t)$ and $k(\theta, t)$ are respectively the outer unit normal and the curvature of the curve $u(\cdot, t)$ at the point $u(\theta, t)$. For some reference about the curve shortening flow and its properties see, for example, [GH, G].

The purpose of the present paper is to find the analogous result of (1.1) in the anisotropic case. More precisely, let $F : \mathbb{R}^2 \to [0, +\infty)$ be a Finsler norm; we denote by

$$\mathcal{W} = \{\xi \in \mathbb{R}^2 : F^\alpha(\xi) < 1\},$$

the unit Wulff shape centered at the origin and we set $\kappa := A(\mathcal{W})$. For every $x \in \partial \Omega$, the $F$-anisotropic curvature is defined as

$$k^F_{\partial \Omega}(x) = \text{div} \left(n^F_{\partial \Omega}(x)\right),$$
\[ n^F(x) = \nabla F(\nu_{\partial \Omega}(x)) \] is the anisotropic normal, while \( \nu_{\partial \Omega}(x) \) is the Euclidean outer unit normal. Then, we denote by \( k^F_{\text{max}}(\partial \Omega) \) the maximum curvature over \( \partial \Omega \), that is
\[
k^F_{\text{max}}(\partial \Omega) := ||k^F_{\partial \Omega}||_{L^\infty(\partial \Omega)}.
\]
The main result of this work is the following.

\textbf{Main Theorem.} Let \( \Omega \subseteq \mathbb{R}^2 \) such that \( \gamma := \partial \Omega \) is a smooth Jordan curve. Then,
\[
k^F_{\text{max}}(\partial \Omega) \geq k^F_{\text{max}}(\partial \mathcal{W}^*),
\]
where \( \mathcal{W}^* \) is a Wulff shape having the same area as \( \Omega \). Moreover, equality holds if and only if \( \Omega \) coincides with a Wulff shape.

Equivalently, the result in (1.3) can be restated in the following form:
\[
k^F_{\text{max}}(\gamma) \geq \sqrt{\frac{\kappa}{A(\Omega)}},
\]
(1.4)

Section 3 is dedicated to the proof of the Main Theorem. The scheme of the proof is close to the one used in [P] and it is based on the use of the anisotropic flow.

We recall that a family \( u : S^1 \times [0,T] \to \mathbb{R}^2 \) of smooth Jordan curves flows by anisotropic curvature flow if
\[
\frac{\partial u(\theta,t)}{\partial t} = \left( F(\nu(\theta,t)) k^F(\theta,t) \right) \nu(\theta,t),
\]
(1.5)
where \( k^F(\theta,t) \) is the anisotropic curvature of the curve \( u(\cdot,t) \) at the point \( u(\theta,t) \).

For some reference see, for example, [A, BP, CZ, MNP]. In the proof we will reduce our study to the case of convex curves and we will use the so called Wulff- Gage inequality. This inequality, proved in [GO], states that, if \( K \subseteq \mathbb{R}^2 \) is a convex set,
\[
\int_{\partial K} (k^F_{\partial K}(x)^2 F(\nu_{\partial K}(x)) \, d\mathcal{H}^1(x)) \geq \frac{\kappa P_F(K)}{A(K)},
\]
(1.6)
where \( P_F(K) = \int_{\partial K} F(\nu_{\partial K}(x)) \, d\mathcal{H}^1(x) \) is the anisotropic perimeter of \( K \). The isotropic version of this inequality was proved in [GH] for convex sets of the plane and generalized in [BBH, FKN1, FKN2] for non convex sets, whose boundary is simply connected.

We point out that in [PP] the authors show that inequality (1.1) can be generalized in higher dimensions if we restrict to the class of sets which are star-shaped; in this case balls still achieve the minimal maximal mean curvature among domains with the same volume. However, if we remove the additional topological constraint of starshapedness and consider bounded smooth domains with a connected boundary the result, as showed in [FNT], is no longer true for \( n > 3 \).

Moreover in [PP] the problem of minimizing the maximal curvature is linked to an estimate of the Laplacian eigenvalue problem with Robin boundary conditions as the boundary parameter \( \alpha \) goes to \(-\infty\). Let \( \Omega \) be a bounded, open subset of
$\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary; its Robin eigenvalues related to the Laplacian are the real numbers $\lambda$ such that
\[
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \alpha u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]
(1.7)

admits non trivial $W^{1,2}(\Omega)$ solutions; $\alpha$ is an arbitrary real constant, which will be referred to as boundary parameter of the Robin problem. In particular, the first non trivial Robin eigenvalue of $\Omega$ is characterized by the expression
\[
\lambda_1(\alpha, \Omega) = \min_{u \in W^{1,2}(\Omega) \setminus \{0\}} \int_{\Omega} \frac{|Du|^2}{|u|^2} \, dx + \alpha \int_{\partial \Omega} |u|^2 \, dH^1 \int_{\Omega} |u|^2 \, dx.
\]

Let us now assume that $\alpha < 0$ and $\Omega \subset \mathbb{R}^n$ is a bounded and Lipschitz domain. If we put a constant function as a test function in the Rayleigh quotient above, we find out that the first eigenvalue is always strictly negative. In 1977 Bareket conjectured that the maximizer of the first Robin eigenvalue with negative parameter among sets with the same volume was a ball $[B]$. However in [FK] the authors disproved Bareket’s conjecture, showing that the first Robin-Laplacian computed on a spherical shell is asymptotically greater than the one computed on a ball with the same volume. In [PP] this was clarified by showing that for $\Omega \subseteq \mathbb{R}^n$ of class $C^{1,1}$, then the following two-terms asymptotics holds
\[
\lambda_1(\Omega, \alpha) = -\alpha^2 - (n - 1)\alpha \sup_{\partial \Omega} H + o(\alpha^{2/3}),
\]
(1.8)
as $\alpha \to -\infty$, where $H$ is the mean curvature of the boundary, that is a generalisation of the curvature in higher dimension. We recall that in [FK], it is proved that Bareket’s conjecture holds for $\alpha$ negative small enough in absolute value.

Our inequality can possibly have an application in the study of the anisotropic counterpart of the Robin problem, that is the problem
\[
\begin{aligned}
-\text{div} (F(Du)F_\xi(Du)) &= \lambda_F(\alpha, \Omega) u \quad \text{in } \Omega \\
\langle F(Du)F_\xi(Du), \nu_{\partial \Omega} \rangle + \alpha F(\nu_{\partial \Omega}) u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $F$ is a fixed Finsler norm, that has been studied for example in [DG, GT, PT], and with the following variational characterization of the first eigenvalue:
\[
\lambda_{1,F}(\alpha, \Omega) = \min_{u \in W^{1,2}(\Omega) \setminus \{0\}} \int_{\Omega} \frac{F^2(Du)}{|u|^2} \, dx + \alpha \int_{\partial \Omega} |u|^2 F(\nu_{\partial \Omega}) \, dH^1(x) \int_{\Omega} |u|^2 \, dx.
\]

A possible future direction of investigation could be the generalization of inequality (1.8) for the study of the anisotropic Robin problem.
2. Preliminaries

2.1. Notation. In the following we will denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean scalar product and by $| \cdot |$ the Euclidean norm in $\mathbb{R}^2$. We denote by $\mathcal{H}^1$ the 1–dimensional Hausdorff measure in $\mathbb{R}^2$. If $\Omega$ is a set of $\mathbb{R}^2$ with Lipschitz boundary, for $\mathcal{H}^1$–almost every $x \in \partial \Omega$, $\nu_{\partial \Omega}(x)$ is the outward unit Euclidean normal to $\partial \Omega$ at $x$. Moreover, $A(\Omega)$ is the area of the set $\Omega$, i.e. its Lebesgue measure in $\mathbb{R}^2$.

2.2. Finsler norm: definitions and some properties. Let $F : \mathbb{R}^2 \to [0, +\infty)$ be a convex function such that for some constant $a > 0$

$$a|\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^2$$

(2.1)
and

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^2.$$  

(2.2)

These hypotheses on $F$ imply that there exists $b \geq a$ such that

$$F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^2.$$  

Moreover, throughout this paper, we will assume that $F^2$ is strongly convex, that is $F \in C^2(\mathbb{R}^2 \setminus \{0\})$ and that the Hessian matrix $\nabla^2 F^2$ is positive definite in $\mathbb{R}^2 \setminus \{0\}$. Under these assumptions, $F$ is called an elliptic norm. The polar function $F^o : \mathbb{R}^2 \to [0, +\infty]$ of $F$ is defined as

$$F^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F(\xi)}$$

and it is easy to verify that also $F^o$ is a convex function that satisfies properties (2.2) and (2.1). $F$ and $F^o$ are usually called Finsler norm. Furthermore, it holds

$$F(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F^o(\xi)},$$

which implies the following anisotropic version of the Cauchy Schwartz inequality

$$|\langle \xi, \eta \rangle| \leq F(\xi)F^o(\eta), \quad \forall \xi, \eta \in \mathbb{R}^2.$$  

We introduce now the following notations:

$$\mathcal{W} = \{\xi \in \mathbb{R}^2 : F^o(\xi) < 1\}$$

is called the unit Wulff shape centered at the origin and we put $\kappa = V(\mathcal{W})$. Moreover, we denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered at $x_0$ with measure $\kappa r^2$, so that $\mathcal{W}_r(0) = \mathcal{W}_r$. We observe that the strong convexity of $F^2$ implies that $\mathcal{W}$ is strictly convex and this ensures that $F^o \in C^1(\mathbb{R}^2 \setminus \{0\})$. More precisely, we have that the strict convexity of the level sets of $F$ is equivalent
to the continuous differentiability of $F^o$ in $\mathbb{R}^2 \setminus \{0\}$; for more details see [S]. We conclude this paragraph recalling some useful properties of $F$ and $F^o$:

\[ \langle \nabla F(\xi), \xi \rangle = F(\xi), \quad F(\nabla F^o(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}; \]

\[ F^o(\nabla F^o(\xi)) = F^o(\nabla F(\xi)) = F(\xi) \nabla F^o(\nabla F(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}. \]

2.3. Anisotropic perimeter. In the following we are fixing a Finsler norm $F$.

**Definition 2.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary, the anisotropic perimeter of $\Omega$ is defined as

\[ P_F(\Omega) = \int_{\partial \Omega} F(\nu_{\partial \Omega}(x)) d\mathcal{H}^1(x), \]

where $\nu_{\partial \Omega}$ is the Euclidean outer normal to $\partial \Omega$ defined almost everywhere.

Clearly, the anisotropic perimeter of $\Omega$ is finite if and only if the Euclidean perimeter of $\Omega$, that we denote by $P(\Omega)$, is finite. Indeed, by the quoted properties of $F$ we have that

\[ aP(\Omega) \leq P_F(\Omega) \leq bP(\Omega). \]

Moreover the following isoperimetric inequality is proved for the anisotropic perimeter, see for instance [AFLT, Bu, DG, DP, FM].

**Theorem 2.2.** Let $\Omega$ be a subset of $\mathbb{R}^2$ with finite perimeter. Then,

\[ P_F(\Omega) \geq 2\kappa^2 A(\Omega)^{\frac{1}{2}}, \]

where $A(\Omega)$ is the area of $\Omega$. Equality holds if and only if $\Omega$ is homothetic to a Wulff shape.

2.4. Anisotropic curvature. Since the main result of this paper concerns set of $\mathbb{R}^2$ with $C^2$ boundary, from now on we will restrict our study to this class of sets.

**Definition 2.3.** Let $\Omega$ be an open, bounded subset of $\mathbb{R}^2$ with $C^2$ boundary. At each point of $\partial \Omega$, we define the $F$-normal vector:

\[ n_{\partial \Omega}(x) = \nabla F(\nu_{\partial \Omega}(x)), \]

sometimes called the Cahn-Hoffman field.

In particular, we observe that, by the properties of $F$, we have that

\[ F^o(n_{\partial \Omega}) = 1. \] (2.3)

We now give the definitions of anisotropic curvature and of anisotropic maximum curvature.
Definition 2.4. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set with $C^2$ boundary. For every $x \in \partial \Omega$, we define the $F$-anisotropic curvature as

$$ k^F_{\partial \Omega}(x) = \text{div} \left( n^F_{\partial \Omega}(x) \right). $$

Moreover, we denote by $k^F_{\text{max}}(\partial \Omega)$ its maximum over $\partial \Omega$, that is

$$ k^F_{\text{max}}(\partial \Omega) := \|k^F_{\partial \Omega}\|_{L^\infty(\partial \Omega)}. $$

We recall that for a Wulff shape of the form $\frac{1}{\lambda}W \subset \mathbb{R}^2$, with $\lambda > 0$, we have that (for the details of the computation see [BP]), for every $x \in \partial \left( \frac{1}{\lambda}W \right)$,

$$ k^F_{\partial K}(x) = \lambda. $$

Finally, in order to prove our main theorem, we will need the following result related to the anisotropic curvature of a convex set, whose proof can be found in [GO] (see Theorem 0.7).

Proposition 2.5 (Wulff-Gage inequality). Let $K \subset \mathbb{R}^2$ be a bounded convex set with $C^2$ boundary. Then,

$$ \int_{\partial K} (k^F_{\partial K}(x)^2 F(\nu_{\partial K}(x))) d\mathcal{H}^1(x) \geq \frac{\kappa P_F(K)}{A(K)} $$

(2.4)

and there is equality if and only if $K$ is a Wulff shape.

2.5. Anisotropic curvature flow. Throughout this paper, we will use the following notations. We consider a family of closed curves $u = u(s,t) : S^1 \times [0,T] \to \mathbb{R}^2$, where $s$ the arc-length parameter and we use the conventional notation $\partial_s(u(s,t)) = u_s(s,t)$. Moreover, $\tau(s,t) = u_s(s,t) = (\sin(\theta(s,t)), -\cos(\theta(s,t)))$ will be the unit tangent and $\nu(s,t) = (\cos(\theta(s,t)), \sin(\theta(s,t)))$ the unit normal of $u$; $\theta = \theta(s,t)$ is called the normal angle (determined modulo $2\pi$) and we may use it to parametrize the curve $u(\cdot, t)$. The classical Frenet formulas assert that

$$ u_{ss}(s,t) = \tau_s(s,t) = k(s,t)\nu(s,t), \quad (2.5) $$

$$ \nu_s(s,t) = -k(s,t)\tau(s,t), \quad (2.6) $$

where $k$ is the scalar curvature. Another useful relation is the following

$$ k(s,t) = \theta_s(s,t). \quad (2.7) $$

Finally, we recall the definition of support function (see for instance [S]). Let $\gamma : S^1 \to \mathbb{R}^2$ be a smooth Jordan curve, let us take the normal angle $\theta$ as parameter for $\gamma$ and let us denote its components by $\gamma(\theta) = (x(\theta), y(\theta))$. The support function associated to $\gamma$ is defined as

$$ h(\theta) := \langle (x(\theta), y(\theta)), (\cos \theta, \sin \theta) \rangle. $$

If we denote by $'$ the derivative with respect to $\theta$, we have that

$$ h'(\theta) = -x(\theta)\sin(\theta) + y(\theta)\cos(\theta). $$
Therefore, \( \gamma \) can be recovered from \( h \) by
\[
\begin{align*}
x(\theta) &= h(\theta) \cos(\theta) - h'(\theta) \sin(\theta), \\
y(\theta) &= h(\theta) \sin(\theta) + h'(\theta) \cos(\theta).
\end{align*}
\]
We now give the definition of the anisotropic flow; for more details and for the proofs of the properties below see for instance [CZ]. In the following, whenever no confusion is possible, we shall write \( \tau, \nu \) and \( k \) as referred to \( u \), using a notation that will not account for the choice of the curve, otherwise we will specify the curve to which they are referred.

**Definition 2.6.** The family \( u : \mathbb{S}^1 \times [0,T] \rightarrow \mathbb{R}^2 \) of smooth Jordan curves flows by anisotropic curvature flow if
\[
\frac{\partial u(s,t)}{\partial t} = (F(\nu(s,t)) k^F(s,t)) \nu(s,t).
\]

In the following two remarks we recall some important properties of the anisotropic curvature flow that we will use later.

**Remark 2.7.** We observe that, since the curve \( u \) is smooth and the anisotropy \( F \) is elliptic, we can write the anisotropic curvature as
\[
k^F(s,t) = \left( \nabla^2 F(\nu(s,t)) \tau(s,t) \cdot \tau(s,t) \right) k(s,t).
\]
Consequently, we have that the anisotropic curvature is controlled from above and from below by the Euclidean curvature.

**Remark 2.8.** If we consider a family of curves \( u(\cdot,t) \) flowing by anisotropic curvature flow, we have that the limiting shape is a round point and that there exists a time \( \bar{t} \in [0,T) \) such that \( u(\cdot,t) \) is convex for \( t \in [\bar{t},T) \), even though the initial curve is not convex. For a proof of this fact see, for instance, [CZ, CZ2, GL].

As observed in [MNP], we can rewrite the anisotropic flow as follows. For simplicity of notation, in the following formulas, we will not mention the dependence from \( s \) and \( t \). So, let us define
\[
\phi(\theta) := F(\nu) = F(\cos \theta, \sin \theta)
\]
and let us observe that, by the divergence theorem, \( k^F = \left( \nabla^2 \left(F^o(\nu)\right) \tau \cdot \tau \right) k. \)
Since we have \( F^o(\theta) + (F^o(\theta))'' = \nabla^2 \left(F^o(\nu)\right) \tau \cdot \tau \), then
\[
u_t = \psi(\theta) k \nu,
\]
where
\[
\psi(\theta) := \phi(\theta) \left( \phi(\theta) + \phi''(\theta) \right).
\]
In particular, the proof of the following result can be found in [MNP] (proof of Proposition 1).
Proposition 2.9. It holds
\[ (\partial_t - \psi \partial_{ss}) \left( \frac{(kF)^2}{2} \right) \leq (3kh\phi' + h'k\phi) \partial_s (kF)^2 + (kF)^4, \] (2.12)
where \( h = \phi + \phi'' \).

In [CZ] can be found the computation of the first derivative of the area enclosed by a family of curves that flows by the anisotropic curvature flow (see the following Proposition). More precisely, the first derivative is proved integrating by parts the formula that gives the area enclosed by a curve \( \gamma \), that is
\[ A(\gamma) = -\frac{1}{2} \int_\gamma \langle \gamma, \nu \rangle \, ds. \]

Proposition 2.10. Let \( u : \mathbb{S}^1 \times [0,T] \to \mathbb{R}^2 \) a family of smooth Jordan curves satisfying (2.8). If we denote by \( u_t(\cdot) := u(\cdot, t) \) and by \( A(t) \) the area enclosed by \( u_t \), we have
\[ \frac{dA(t)}{dt} = -\int_{u_t} F(\nu_{u_t}(s,t))k_{u_t}^F(s,t)ds, \] (2.13)
where \( \nu_{u_t} \) and \( k_{u_t}^F \) are respectively the unit normal and the anisotropic curvature of the curve \( u_t \).

3. Main result and its proof

Theorem 3.1. Let \( \Omega \subseteq \mathbb{R}^2 \) such that \( \gamma := \partial \Omega \) is a smooth Jordan curve. Then,
\[ k_{\max}^F(\gamma) \geq \sqrt{\frac{\kappa}{A(\Omega)}} \] (3.1)
and there is equality if and only if \( \Omega \) coincides with a Wulff shape.

Proof. Step 1: Uniqueness. Using a standard argument we prove that, if inequality (3.1) is proved, then equality holds only for Wulff shapes. Let assume that (3.1) is true and, by contradiction, that the equality holds for a curve \( \gamma \) that is not the boundary of a Wulff shape. Thus, there exists a point \( x \in \gamma \) such that \( k_{\partial K}^F(x) \leq k_{\max}^F(\gamma) \). By a small local deformation around \( x \), we can construct a smooth Jordan curve \( \gamma' \) such that the following two conditions hold
- \( k_{\max}^F(\gamma') = k_{\max}^F(\gamma) \),
- the area \( A' \) enclosed by \( \gamma' \) is strictly smaller than the area \( A \) enclosed by \( \gamma \).
In this way we have a contradiction, since
\[ k_{\max}^F(\gamma') < \sqrt{\kappa/A'}. \]
Step 2: The inequality holds for convex curves

Let us assume that \( \gamma \) is a Jordan curve that is convex. Using inequality (2.4), we obtain
\[
\frac{\kappa}{A(K)} P_F(K) \leq \int_{\partial K} (k_{\partial K}^F(x))^2 F(\nu_{\partial K}(x)) \, dH^1(x) \leq \left( k_{\text{max}}^F(\partial K) \right)^2 P_F(K) \tag{3.2}
\]
and so inequality (3.1) immediately follows.

Step 3: The inequality holds for general curves

Using the anisotropic curvature flow, the case of general curves will be reduced to the case of convex curves, in the same spirit of [P]. We set \( A_0 := A(K) \) and we prove that \( k_{\text{max}}^F(\gamma) \geq \sqrt{A_0/\kappa} := C \), for every admissible \( \gamma \). By contradiction, there exists a smooth Jordan curve \( \gamma \) (not convex) such that \( k_{\text{max}}^F(\gamma) < C \). (3.3)

Let \( u(\cdot, t) \), with \( t \in [0, T] \), be the family of curves evolving by anisotropic curvature flow with \( u(\cdot, 0) = \gamma(\cdot) \); so that at time \( t = T \) the area enclosed by \( u(\cdot, T) \) is 0. We consider the family
\[
U(\cdot, t) := f(t) u(\cdot, t),
\]
where \( f \) is a non-negative function chosen in such a way that every curve of the family \( U(\cdot, t) \) encloses constant area. Therefore,
\[
f(t) = \sqrt{\frac{A_0}{A(t)}},
\]
where \( A(t) \) is the area enclosed by \( u_t(\cdot) := u(\cdot, t) \). Moreover, we observe that
\[
k_{U}^F = \left( \frac{1}{f} \right) k_u^F. \tag{3.4}
\]

Recalling that we denote by \( ' \) the derivative with respect to \( \theta \), using (3.4) and (2.12), we obtain
\[
(\partial_t - \psi \partial_{ss}) \left( \frac{k_U^F}{2} \right)^2 = (\partial_t - \psi \partial_{ss}) \left[ \frac{A(t) \left( \frac{k_u^F}{2} \right)^2}{2A_0} \right] =
\]
\[
= A'(t) \left( \frac{k_U^F}{2A_0} \right)^2 + \frac{A(t)}{A_0} (\partial_t - \psi \partial_{ss}) \left( k_u^F \right)^2 \leq
\]
\[
\leq A'(t) \left( \frac{k_U^F}{2A_0} \right)^2 + \frac{A(t)}{A_0} \left[ (3k_u h \phi' + h' k_u \phi) \partial_s (k_u^F)^2 + (k_u^F)^4 \right] =
\]
\[
= A'(t) \left( \frac{k_U^F}{2A_0} \right)^2 + \frac{A(t)}{A_0} \left( \frac{k_u^F}{2A_0} \right)^4 + \frac{A(t)}{A_0} \left[ (3k_u h \phi' + h' k_u \phi) \partial_s (k_u^F)^2 \right] =
\]
\[
= \frac{A'(t)}{A(t)} \left( k_U^F \right)^2 + \frac{A_0}{A(t)} \left( k_U^F \right)^4 + \frac{A(t)}{A_0} \left[ (3k_u h \phi' + h' k_u \phi) \partial_s (k_u^F)^2 \right]. \tag{3.5}
\]
At this point let us introduce some useful notations; we set \( k_u^F(\theta, t) := k_u^F(\theta) \) and \( k_U^F(\theta, t) := k_U^F(\theta) \). Now, by (3.3), there exists \( M \in (0, C) \) such that \( k_U^F(\theta) < M \) for every \( \theta \in \mathbb{S}^1 \) and we want to show that for every \( \theta \in \mathbb{S}^1 \) and for every \( t \)

\[
k_u^F(\theta, t) < M < C.
\]

In order to prove (3.6), we proceed again by contradiction, assuming that there exists \( t^* \in (0, T) \) for which it is possible to find a \( \theta^* \) such that \( k_U^F(\theta^*, t^*) = M \). This means that \( \theta^* \) is a maximum for \( k_U^F(\cdot, t^*) \) and, as a consequence, it is a maximum also for \( k_u^F(\cdot, t^*) \). So, taking into account that at a maximal point \( \partial_s (k_u^F) \) vanishes and \( (k_u^F)_{ss} (\theta^*, t^*) \) is non-positive, from (3.5) we obtain that

\[
(\partial_t - \psi \partial_{ss}) \left( \frac{(k_u^F(\theta^*, t^*))^2}{2} \right) \leq \frac{M^2}{A(t^*)} \left( \frac{A'(t^*)}{2} + A_0 M^2 \right).
\]

Using (2.13), we have that

\[
A'(t^*) = - \int_{u_{t^*}} F(\nu_{u_{t^*}}(s, t^*)) k_{u_{t^*}}(s, t^*) ds = - \int_{\partial \Omega_{t^*}} F(\nu_{u_{t^*}}(x)) k_{u_{t^*}}(x) d\mathcal{H}^1(x)
\]

\[
\leq - aD \int_{u_{t^*}} k_{u_{t^*}}(x) d\mathcal{H}^1(x) = -2\pi aD,
\]

where \( \Omega_{t^*} \) is the set enclosed by \( u_{t^*} \). In the last inequality we have used the following facts: that, for every unit vector \( v \), \( F(v) \geq a \), the fact that the anisotropic curvature is controlled from above by the classical curvature since \( F \) is elliptic (see Remark 2.7), and, finally, the Gauss-Bonnet theorem. As a consequence,

\[
(\partial_t - \psi \partial_{ss}) \left( \frac{(k_u^F(\theta^*, t^*))^2}{2} \right) \leq - \frac{A_0 M^2}{\frac{A(t^*)}{A_0}} \left( \frac{\pi aD}{A_0} - M^2 \right) < 0,
\]

since we can assume, using a suitable scaling, that \( A_0 \) is such that \( \frac{\pi aD}{A_0} = C \). Now, having \( \partial_{ss} (k_u^F(\theta^*, t^*))^2 / 2 < 0 \), from (3.3), we have that

\[
\partial_t (k_u^F(\theta^*, t^*)) < 0,
\]

and so

\[
\partial_t (k_U^F(\theta^*, t^*)) < 0.
\]

It follows that \( k_U^F(\theta^*, t^* - \epsilon) > M \), for \( \epsilon > 0 \) small enough, which contradicts the choice of \( t^* \). In this way we have proved (3.6).

Now, for the properties of the anisotropic curvature flow (see Remark 2.8 and the reference therein), we know that for some \( \tau > 0 \) the curve \( U(\cdot, \tau) \) is convex and therefore, thanks to Step 2, we have that for some \( \theta \in [0, 2\pi] \)

\[
k_F(\theta, \tau) \geq C,
\]

that contradicts (3.6), concluding the proof.

\[\square\]
AN ESTIMATE FOR THE ANISOTROPIC MAXIMUM CURVATURE IN THE PLANAR CASE

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