Levinson’s theorem for the Schrödinger equation in one dimension

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Levinson’s theorem for the one-dimensional Schrödinger equation with a symmetric potential, which decays at infinity faster than $x^{-2}$, is established by the Sturm-Liouville theorem. The critical case, where the Schrödinger equation has a finite zero-energy solution, is also analyzed. It is demonstrated that the number of bound states with even (odd) parity $n_+ (n_-)$ is related to the phase shift $\eta_+(0)[\eta_-(0)]$ of the scattering states with the same parity at zero momentum as

$$\eta_+(0) + \pi/2 = n_+ \pi, \quad \eta_-(0) = n_- \pi, \quad \text{for the non-critical case},$$

$$\eta_+(0) = n_+ \pi, \quad \eta_-(0) - \pi/2 = n_- \pi, \quad \text{for the critical case},$$

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I. INTRODUCTION

The Levinson theorem [1], an important theorem in the nonrelativistic quantum scattering theory, established the relation between the total number $n_\ell$ of bound states with angular momentum $\ell$ and the phase shift $\delta_\ell(0)$ of the scattering state at zero momentum for the Schrödinger equation with a spherically symmetric potential $V(r)$ in three dimensions:

$$\delta_\ell(0) - \delta_\ell(\infty) = \begin{cases} 
(n_\ell + 1/2) \pi & \text{when } \ell = 0 \text{ and a half-bound state occurs} \\
n_\ell \pi & \text{the remaining cases},
\end{cases}$$

(1)

where the potential $V(r)$ satisfies the following asymptotic conditions:

$$r^2 |V(r)|dr \rightarrow 0, \quad \text{at } r \rightarrow 0,$$

(2)

$$r^3 |V(r)|dr \rightarrow 0, \quad \text{at } r \rightarrow \infty.$$

(3)

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These two conditions are necessary for the nice behavior of the wave function at the origin and the analytic property of the Jost function, respectively. The first line in Eq. (1) was first expressed by Newton [2] for the case where a half-bound state of the $S$ wave occurs. The zero-energy solution to the Schrödinger equation is called a half-bound state provided that its wave function is finite, but does not decay fast enough at infinity to be square integrable.

During the past half-century, the Levinson theorem has been proved by several authors with the different methods and generalized to different fields [1-22]. Most of works mainly studied the Levinson theorem in the three-dimensional space. With the wide interest in lower-dimensional field theories recently, the two-dimensional Levinson theorem has been studied numerically [18] as well as theoretically [19-25]. With respect to the two-dimensional Schrödinger equation, the version of the Levinson theorem can be read as

\[
\eta_m(0) = \begin{cases} 
(n_m + 1)\pi & \text{when } m = 1 \text{ and a half-bound state occurs} \\
n_m\pi & \text{the remaining cases},
\end{cases}
\]

where $\eta_m(0)$ is the limit of the phase shifts at zero momentum for the $m$th partial wave, and $n_m$ is the total number of bound states with the given angular momentum $m\hbar$.

Due to the wide interest in lower-dimensional field theory recently, it may be worth studying the Levinson theorem in one dimension besides the study in two dimensions for completeness. In fact, it is a common knowledge that the one-dimensional quantum scattering describes many actual physical phenomena to a good approximation. For instance, the problem on the tunneling times has been discussed in [26]. Furthermore, the one-dimensional models are often applied to make the more complex higher-dimensional systems tractable. Consequently, it seems reasonable to study the one-dimensional Levinson theorem. This will be beneficial for understanding both the two-dimensional Levinson theorem and the three-dimensional one. Actually, it seems that the direct or implicit study of the one-dimensional Levinson theorem [16,27-34] has attracted much more attention than that of the two-dimensional one. Nevertheless, we attempt to approach this problem by the Sturm-Liouville theorem [35].

Generally speaking, there are several methods for studying the one-dimensional Levinson theorem for the nonrelativistic particle. One is based on the partial-wave analysis method [32, 33]. The second relies on the parity-eigenstate representation method [16, 34]. The third is to establish the Levinson theorem by the Jost function and the $S$ matrix method [29], which is essentially based on the orthogonality and completeness relation for the eigenfunctions of the total Hamiltonian, as was first noticed by Jauch [3].

The purpose of this paper is to demonstrate the one-dimensional Levinson theorem for the Schrödinger equation by the Sturm-Liouville theorem. We arrive at the final result read as

\[
\eta_+(0) + \pi/2 = n_+, \quad \eta_-(0) = n_-, \quad \text{for the non-critical case},
\]

\[
\eta_+(0) = n_+\pi, \quad \eta_-(0) - \pi/2 = n_-\pi, \quad \text{for the critical case}.
\]

where the $n_+$ and $n_-$ denote the number of bound states with even parity and odd parity, and the $\eta_+(0)$ and $\eta_-(0)$ denote the phase shift of the scattering states with the same parity at zero momentum, respectively. This conclusion coincides with that shown in [16].
It is readily to find from Eq. (5) that the Levinson theorem for the odd-parity case in one dimension is the same as that for the case $\ell = 0$ in three dimensions. However, the even-parity case has no counterpart compared to the three-dimensional Levinson theorem. This is a very interesting feature in one-dimensional Levinson theorem for the Schrödinger equation.

This paper is organized as follows. For simplicity, we first discuss the cutoff potential case, where the potential is vanishing beyond a sufficiently large distance $x_0$, and leave the discussion for the general case where the potential has a tail at infinity in the Sec. V. In Sec. II the logarithmic derivative of the wave function of the Schrödinger equation is chosen as the phase angle [35], which is proved to be monotonic with respect to the energy (the Sturm-Liouville theorem). In Sec. III, according to this monotonic property, the number of bound states is shown to be related with the logarithmic derivative of zero energy at $x_0$ when the potential changes from zero to the given value. It will be further shown that the logarithmic derivative of zero energy at $x_0$ also determines the limit of the phase shifts at zero momentum in Sec. IV, which leads to the establishment of the one-dimensional Levinson theorem. The critical case, where a zero-energy solution occurs, is also analyzed there.

II. NOTATIONS AND THE STURM-LIOUVILLE THEOREM

Throughout this paper the natural units $\hbar = 1$ and $2m = 1$ are employed. Let us consider the one-dimensional Schrödinger equation with a symmetric potential $V(x)$

$$\frac{d^2\psi(x)}{dx^2} + [E - V(x)]\psi(x) = 0, \quad V(-x) = V(x),$$

where $E$ denotes the energy of the particle. For simplicity, we first discuss the case with a cutoff potential:

$$V(x) = 0, \quad \text{when } x \geq x_0,$$

where $x_0$ is a sufficiently large distance. Introduce a parameter $\lambda$ for the potential $V(x)$:

$$V(x, \lambda) = \lambda V(x),$$

where the potential $V(x, \lambda)$ changes from zero to the given potential $V(x)$ as $\lambda$ increases from zero to one. After introducing the parameter $\lambda$, the one-dimensional Schrödinger equation can be modified as

$$\frac{\partial^2}{\partial x^2} \psi(x, \lambda) + [E - V(x, \lambda)] \psi(x, \lambda) = 0. \quad (8)$$

Since the potential is symmetric, the energy eigenfunctions can be combined into those with a definite parity, which satisfy the following boundary conditions at the origin:

$$\psi^{(o)}(x, \lambda) \bigg|_{x=0} = 0, \quad \text{for the odd – parity case,}$$

$$\frac{\partial \psi^{(e)}(x, \lambda)}{\partial x} \bigg|_{x=0} = 0, \quad \text{for the even – parity case,} \quad (9)$$
Therefore, in the course of studying the one-dimensional Levinson theorem we only need to discuss the wavefunction in the range \(0 \leq x < \infty\) with the given parities, even parity case and odd parity one, respectively.

Now, we are going to solve Eq. (8) in two ranges \([0, x_0]\) and \([x_0, \infty)\), and match two solutions at \(x_0\). Ignoring the effect of the normalization factor, which is irrelevant to our discussion, we only need one matching condition at \(x_0\), which is the condition for the logarithmic derivative of the wave function [35]:

\[
A(E, \lambda) \equiv \left\{ \frac{1}{\psi(x, \lambda)} \frac{\partial \psi(x, \lambda)}{\partial x} \right\}_{x=x_0^-} = \left\{ \frac{1}{\psi(x, \lambda)} \frac{\partial \psi(x, \lambda)}{\partial x} \right\}_{x=x_0^+} .
\] (10)

According to the condition (9), there exists only one solution near the origin. For example, for the free particle (\(\lambda = 0\)), the solution to Eq. (8) at the range \([0, x_0]\) is real and read as:

\[
\psi^{(e)}(x, 0) = \begin{cases} 
\cos(kx) & \text{when } E = k^2 > 0 \\
\cosh(kx) & \text{when } E = -\kappa^2 \leq 0,
\end{cases}
\] (11)

for the even-parity case, and

\[
\psi^{(o)}(x, 0) = \begin{cases} 
\sin(kx) & \text{when } E = k^2 > 0 \\
\sinh(kx) & \text{when } E = -\kappa^2 \leq 0,
\end{cases}
\] (12)

for the odd-parity case.

In the range \([x_0, \infty)\), we have \(V(x) = 0\). For \(E > 0\), there exist two oscillatory solutions to Eq. (8) whose combination can always satisfy the matching condition (10), so that there is a continuous spectrum for \(E > 0\). Assuming that the phase shifts \(\eta_{\pm}(k, \lambda)\) are zero for the free particles (\(\lambda = 0\)), we have

\[
\psi(x, \lambda) = \begin{cases} 
\cos(kx + \eta_+(k, \lambda)), & \text{for the even-parity case} \\
\sin(kx + \eta_-(k, \lambda)), & \text{for the odd-parity case},
\end{cases}
\] (13)

\[
\eta_{\pm}(k, 0) = 0, \quad \text{when } k > 0. \quad (14)
\]

We would like to make some remarks here. First, at the first sight, the wavefunction in Eq. (13) seems not to have a definite parity. As a matter of fact, the solutions (13) are only suitable in the region \([x_0, \infty)\). The corresponding solutions in the region \((-\infty, -x_0]\) can be calculated according to the parity of the solution. For example, in the odd-parity case, the solution in the region \((-\infty, -x_0]\) is

\[-\sin(k|x| + \eta_- (k, \lambda)) = \sin(kx - \eta_- (k, \lambda)).\]

Second, the solution (13) for the even-parity case can be rewritten as

\[
\sin(kx + \eta_+(k, \lambda) + \pi/2),
\] (15)

\(\eta_+(k, \lambda) + \pi/2\) plays the same role in the even-parity case as \(\eta_- (k, \lambda)\) in the odd-parity case. Therefore, we only need to establish the Levinson theorem for the odd-parity case, and the Levinson theorem for the even-parity case can be obtained by replacing \(\eta_- (k, \lambda)\) with \(\eta_+(k, \lambda) + \pi/2\).
At last, in the region \([x_0, \infty)\), the potential \(V(x, \lambda)\) is vanishing and does not depend on \(\lambda\). However, the phase shifts \(\eta_{\pm}(k, \lambda)\) depend on \(\lambda\) through the matching condition (10):

\[
\tan \eta_-(k, \lambda) = -\tan(kx_0) \frac{A(E, \lambda) - k \cot(kx_0)}{A(E, \lambda) + k \tan(kx_0)},
\]

for the odd-parity case, and the similar formula for the even-parity case can be obtained by replacing \(\eta_-(k, \lambda)\) with \(\eta_+(k, \lambda) + \pi/2\).

The phase shifts \(\eta_-(k, \lambda)\) are determined from Eq. (16) up to a multiple of \(\pi\) due to the period of the tangent function. In our convention (14), the phase shift \(\eta_-(k, \lambda)\), \(k > 0\), changes continuously as \(\lambda\) increases from zero to one. In other words, the phase shift \(\eta_-(k, \lambda)\) is determined completely in our convention, so is \(\eta_+(k, \lambda)\). For simplicity we define

\[
\eta_{\pm}(k) = \eta_{\pm}(k, 1).
\] (17)

Since there is only one finite solution at infinity for \(E \leq 0\), both for the even-parity case and for the odd-parity case:

\[
\psi(x, \lambda) = \exp(-\kappa x), \quad \text{when} \quad x_0 \leq x < \infty.
\] (18)

The solution satisfying the matching condition (10) will not always exist for \(E \leq 0\). Except for \(E = 0\), if and only if there exists a solution of energy \(E\) satisfying the matching condition (10), a bound state appears at this energy. Therefore, there is a discrete spectrum for \(E \leq 0\). The finite solution for \(E = 0\) is a constant one. It decays not fast enough to be square integrable such that it is not a bound state if the matching condition (10) is satisfied.

We now turn to the Sturm-Liouville theorem. Denote by \(\overline{\psi}(x, \lambda)\) the solution to Eq. (8) corresponding to the energy \(\overline{E}\)

\[
\frac{\partial^2}{\partial x^2} \overline{\psi}(x, \lambda) + \left[\overline{E} - V(x, \lambda)\right] \overline{\psi}(x, \lambda) = 0.
\] (19)

Multiplying Eq. (8) and Eq. (19) by \(\overline{\psi}(x, \lambda)\) and \(\psi(x, \lambda)\), respectively, and calculating their difference, we obtain

\[
\frac{\partial}{\partial x} \left\{ \psi(x, \lambda) \frac{\partial \overline{\psi}(x, \lambda)}{\partial x} - \overline{\psi}(x, \lambda) \frac{\partial \psi(x, \lambda)}{\partial x} \right\} = -\left(\overline{E} - E\right) \overline{\psi}(x, \lambda) \psi(x, \lambda).
\] (20)

According to the boundary condition (9), the derivative of the wavefunction for the even-parity case and the wavefunction for the odd-parity case are vanishing at the origin, respectively. Therefore, integrating (20) in the range \(0 \leq x \leq x_0\), we obtain

\[
\frac{1}{E - \overline{E}} \left\{ \psi(x, \lambda) \frac{\partial \overline{\psi}(x, \lambda)}{\partial x} - \overline{\psi}(x) \frac{\partial \psi(x, \lambda)}{\partial x} \right\}_{x = x_0^{-}} = -\int_{0}^{x_0} \overline{\psi}(x, \lambda) \psi(x, \lambda) dx.
\]

Taking the limit, we arrive at

\[
\frac{\partial A(E, \lambda)}{\partial E} = \frac{\partial}{\partial E} \left( \frac{1}{\psi(x, \lambda)} \frac{\partial \psi(x, \lambda)}{\partial x} \right)_{x = x_0^{-}} = -\psi(x_0, \lambda)^{-2} \int_{0}^{x_0} \psi(x, \lambda)^2 dx \leq 0.
\] (21)
Similarly, from the boundary condition that when $E < 0$ the function $\psi(x, \lambda)$ tends to zero at infinity, and when $E = 0$ the derivative of the function is equal to to zero at infinity, we have

$$\frac{\partial}{\partial E} \left( \frac{1}{\psi(x, \lambda)} \frac{\partial \psi(x, \lambda)}{\partial x} \right)_{x=x_0^+} = \psi(x_0, \lambda)^{-2} \int_{x_0}^{\infty} \psi(x, \lambda)^2 dx > 0. \quad (22)$$

Therefore, when $E \leq 0$, it is evident that both sides of Eq. (10) are monotonic with respect to the energy $E$: as the energy increases, the logarithmic derivative of the wave function at $x_0^-$ decreases monotonically, but that at $x_0^+$ increases monotonically. This is the essence of the Sturm-Liouville theorem.

### III. THE NUMBER OF BOUND STATES

In this section we will establish the relation between the number of bound states and the logarithmic derivative $A(0, \lambda)$ of the wave function at $x = x_0^-$ for zero energy when the potential changes, in terms of the monotonic property of the logarithmic derivative of the wave function with respect to the energy $E$.

For $E \leq 0$, we obtain the logarithmic derivative at $x = x_0^+$ from Eq. (18):

$$\left( \frac{1}{\psi(x, \lambda)} \frac{\partial \psi(x, \lambda)}{\partial x} \right)_{x=x_0^+} = \begin{cases} 0 & \text{when } E \sim 0 \\ -\kappa \sim -\infty & \text{when } E \to -\infty. \end{cases} \quad (23)$$

On the other hand, when $\lambda = 0$, the logarithmic derivative at $x = x_0^-$ can be calculated from Eqs. (11) and (12) for $E \leq 0$:

$$A(E, 0) = \left( \frac{1}{\psi(x, 0)} \frac{\partial \psi(x, 0)}{\partial x} \right)_{x=x_0^-} = \kappa \tanh(\kappa x_0) = \begin{cases} 0 & \text{when } E \sim 0 \\ \kappa \sim \infty & \text{when } E \to -\infty. \end{cases} \quad (24)$$

for the even-parity case, and

$$A(E, 0) = \left( \frac{1}{\psi(x, 0)} \frac{\partial \psi(x, 0)}{\partial x} \right)_{x=x_0^-} = \kappa \coth(\kappa x_0) = \begin{cases} \frac{1}{x_0} & \text{when } E \sim 0 \\ \kappa \sim \infty & \text{when } E \to -\infty. \end{cases} \quad (25)$$

for the odd-parity case.

It is evident to see from Eqs. (23) and (25) that there is no overlap between two variant ranges of two logarithmic derivatives for the odd-parity case, namely there is no bound state for the free particle in the odd-parity case. However, there is one point overlap from Eqs. (23) and (24). It means that there is a finite solution at $E = 0$ when $\lambda = 0$ for the even-parity case. It is nothing but a constant solution. This solution is finite but does not decay fast enough at infinity to be square integrable. It is not a bound state, and called a half bound state. We will discuss the cases with a half bound state later.

Now, both for the even-parity case and for the odd-parity case, if $A(0, \lambda)$ decreases across the value zero as $\lambda$ increases, an overlap between the variant ranges of two logarithmic derivatives of two sides of $x = x_0$ appears. Since the logarithmic derivative of the wave function at $x_0^-$ decreases monotonically as the energy increases, and that at $x_0^+$ increases monotonically, the overlap means that there must exist one and only one energy for which the matching condition (10) is satisfied, that is, a bound state.
appears. From the viewpoint of node theory, when $A(0, \lambda)$ decreases across the value zero, a node for the zero-energy solution to the Schrödinger equation comes inwards from the infinity, namely, a scattering state changes to a bound state.

As $\lambda$ increases again, $A(0, \lambda)$ may decreases to $-\infty$, jumps to $\infty$, and then decreases again across the value zero, so that another overlap occurs and another bound state appears. Note that when the zero point in the zero-energy solution $\psi(x, \lambda)$ comes to $x = x_0$, $A(0, \lambda)$ goes to infinity. It is not a singularity.

Each time $A(0, \lambda)$ decreases across the value zero, a new overlap between the variant ranges of two logarithmic derivatives appears such that a scattering state changes to a bound state. At the same time, a new node comes inwards from infinity in the zero-energy solution to the Schrödinger equation. Conversely, each time $A(0, \lambda)$ increases across the value zero, an overlap between those two variant ranges disappears so that a bound state changes back to a scattering state, and simultaneously, a node goes outward and disappears in the zero-energy solution. The number of bound states $n_{\pm}$ is equal to the times that $A(0, \lambda)$ decreases across the value zero as $\lambda$ increases from zero to one, subtracted by the times that $A(0, \lambda)$ increases across the value zero. It is also equal to the number of nodes in the zero-energy solution. In the next section we will show that this number is nothing but the phase shift at zero momentum divided by $\pi$, i.e., $\eta_{-}(0)/\pi$ or $\eta_{+}(0)/\pi + 1/2$.

We should pay some attention to the critical case where $A(0, 1) = 0$. A finite zero-energy solution $\psi(x, 1) = c$ at $[x_0, \infty)$ will satisfy the matching condition (10) with the zero $A(0, 1)$. Note that when $A(0, 1) = 0$, the wave function at $x_0 -$, $\psi(x_0, 1)$, must be nonvanishing for the non-trivial solution. The constant $c$ is nothing but the non-vanishing value $\psi(x_0, 1)$. The constant solution is not square integrable so that it is not a bound state, and called a half bound state. As $\lambda$ increases from a number near and smaller than one and finally to reach one, if $A(0, \lambda)$ decreases and finally reaches the value zero, a scattering state becomes a half bound state, and no new bound state appears. Conversely, as $\lambda$ increases to reach one, if $A(0, \lambda)$ increases and finally reaches the value zero, a bound state becomes a half bound state, namely, a bound state disappears. This conclusion holds for both the even-parity case and the odd-parity case.

### IV. LEVINSON’S THEOREM

When $\lambda = 0$, the phase shifts $\eta_{\pm}(k, 0)$ are defined to be zero. As $\lambda$ increases from zero to one, and $\eta_{\pm}(k, 0)$ for $k > 0$ change continuously.

For the odd-parity case, the phase shift $\eta_{-}(k, \lambda)$ is calculated by Eq. (16). It is easy to see that the phase shift $\eta_{\pm}(k, \lambda)$ increases monotonically as the logarithmic derivative $A(E, \lambda)$ decreases:

$$
\frac{\partial \eta_{-}(k, \lambda)}{\partial A(E, \lambda)} \bigg|_k = \frac{-k \cos^2 \eta_{-}(k, \lambda)}{(A \cos(kx) + k \sin(kx))^2} \leq 0, \quad (26)
$$

The phase shift $\eta_{-}(0, \lambda)$ is the limit of the phase shift $\eta_{-}(k, \lambda)$ as $k$ tends to zero. Therefore, we are only interested in the phase shift $\eta_{-}(k, \lambda)$ at a sufficiently small momentum $k$, $k \ll 1/x_0$. For the small
momentum we obtain from Eq. (16)
\[
\tan \eta_-(k, \lambda) \sim -(kx_0) \frac{A(0, \lambda) - c^2k^2 - x_0^{-1} + k^2x_0/3}{A(0, \lambda) - c^2k^2 + k^2x_0}, \tag{27}
\]
where the expansion of \(A(E, \lambda)\) for the small \(k\) is used
\[
A(E, \lambda) \sim A(0, \lambda) - c^2k^2, \quad c^2 \geq 0, \tag{28}
\]
which is calculated from the sturm-Liouville theorem (21). In both the numerator and the denominator of Eq. (27) we included the next leading term, which is only useful for the critical cases where the leading terms are canceled each other.

First, it can be seen from Eq. (27) that, except for the special point where \(A(0, \lambda) = 0\), \(\tan \eta_-(k, \lambda)\) tends to zero as \(k\) goes to zero, namely, \(\eta_-(0, \lambda)\) is always equal to the multiple of \(\pi\) except for zero \(A(0, \lambda)\). In other words, if the phase shift \(\eta_-(k, \lambda)\) for a sufficiently small \(k\) is expressed as a positive or negative acute angle plus \(n\pi\), its limit \(\eta_-(0, \lambda)\) is equal to \(n\pi\), where \(n\) is an integer. It means that \(\eta_-(0, \lambda)\) changes discontinuously. When \(A(0, \lambda) = 0\), the limit \(\eta_-(0, \lambda)\) of the phase shift \(\eta_-(k, \lambda)\) is equal to \((n + 1/2)\pi\). It is not important for our discussion except for \(A(0, 1) = 0\), which we call the critical case and will discuss the critical case later.

Second, for a sufficiently small \(k\), if \(A(E, \lambda)\) decreases as \(\lambda\) increases, \(\eta_-(k, \lambda)\) increases monotonically. Assume that in the variant process \(A(E, \lambda)\) may decreases through the value zero, but does not stop at this value. As \(A(E, \lambda)\) decreases, each times \(\tan \eta_-(k, \lambda)\) for the sufficiently small \(k\) changes sign from positive to negative, \(\eta_-(0, \lambda)\) jumps by \(\pi\). However, each times \(\tan \eta_-(k, \lambda)\) changes sign from negative to positive, \(\eta_-(0, \lambda)\) remains invariant. Conversely, if \(A(E, \lambda)\) increases as \(\lambda\) increases, \(\eta_-(k, \lambda)\) decreases monotonically. As \(A(E, \lambda)\) increases, each time \(\tan \eta_-(k, \lambda)\) changes sign from negative to positive, \(\eta_-(0, \lambda)\) jumps by \(-\pi\), and each time \(\tan \eta_-(k, \lambda)\) changes sign from positive to negative, \(\eta_-(0, \lambda)\) remains invariant.

Third, as \(\lambda\) increases from zero to one, \(V(x, \lambda)\) changes from zero to the given potential \(V(x)\) continuously. Each time the \(A(0, \lambda)\) decreases from near and larger than the value zero to smaller than that value, the denominator in Eq. (27) changes sign from positive to negative and the remaining factor remains positive, such that the phase shift at zero momentum \(\eta_-(0, \lambda)\) jumps by \(\pi\). Conversely, each time the \(A(0, \lambda)\) increases across the value zero, the phase shift at zero momentum \(\eta_-(0, \lambda)\) jumps by \(-\pi\). Each time the \(A(0, \lambda)\) decreases from near and larger than the value \(x_0^{-1}\) to smaller than that value, the numerator in Eq. (27) changes sign from positive to negative, but the remaining factor remains negative, such that the phase shift at zero momentum \(\eta_-(0, \lambda)\) does not jump. Conversely, each time the \(A(0, \lambda)\) increases across the value \(x_0^{-1}\), the phase shift at zero momentum \(\eta_-(0, \lambda)\) does not jump, either.

Therefore, the phase shift \(\eta_-(0)/\pi\) is just equal to the times \(A(0, \lambda)\) decreases across the value zero as \(\lambda\) increases from zero to one, subtracted by the times \(A(0, \lambda)\) increases across that value. As discussed in the previous section, we have proved that the difference of the two times is nothing but the number of bound states \(n_-\), namely, for the non-critical cases, the Levinson theorem for the one-dimensional Schrödinger equation in the odd-parity case is
\[
\eta_-(0) = n_-\pi. \tag{29}
\]
Fourth, we now turn to discuss the critical case where the logarithmic derivative $A(0, 1)$ ($\lambda = 1$) is equal to zero. In the critical case, the constant solution $\psi(x) = c$ ($c \neq 0$) in the range $[x_0, \infty)$ for zero energy will match this $A(0, 1)$ at $x_0$. In the critical case, it is obvious that there exists a half-bound state both for the even-parity case and for the odd-parity case. A half-bound state is not a bound state, because its wave function is finite but not square integrable. As $\lambda$ increases from a number near and less than one and finally reaches one, if the logarithmic derivative $A(0, \lambda)$ decreases and finally reaches, but not across, the value zero, according to the discussion in the previous section, a scattering state becomes a half bound state when $\lambda = 1$. On the other hand, the denominator in Eq. (27) is proportional to $k^2$ such that $\tan \eta_-(k, 1)$ tends to infinity. Namely, the phase shift $\eta_-(0, 1)$ jumps by $\pi/2$. Therefore, for the critical case the Levinson theorem becomes

$$\eta_-(0) - \pi/2 = n_- \pi. \tag{30}$$

Conversely, as $\lambda$ increases and reaches one, if the logarithmic derivative $A(0, \lambda)$ increases and finally reaches the value zero, a bound state becomes a half bound state when $\lambda = 1$, and the phase shift $\eta_-(0, 1)$ jumps by $-\pi/2$. In this situation, the Levinson theorem (30) still holds.

At last, for the even-parity case, the only change is to replace the phase shift $\eta_-(0)$ with the phase shift $\eta_+(0) + \pi/2$. Therefore, the Levinson theorem for the one-dimensional Schrödinger equation in the even-parity case is

$$\eta_+(0) + \pi/2 = n_+ \pi, \quad \text{for the non-critical cases},$$

$$\eta_+(0) = n_+ \pi, \quad \text{for the critical cases}. \tag{31}$$

Note that for the free particle in the even-parity case, there is a half bound state at $E = 0$. It is the critical case where $\eta_+(0) = 0$ and $n_+ = 0$. Combining Eqs. (29-31), we obtain the Levinson theorem for the one-dimensional Schrödinger equation as Eq. (5).

\section*{V. DISCUSSIONS}

Now, we discuss the general case where the potential $V(x)$ has a tail at $x \geq x_0$. First, we assume that

$$V(x) = bx^{-2}, \quad x \geq x_0. \tag{32}$$

It is obvious that when $b < -1/4$ there is an infinite number of bound states for the Schrödinger equation (8) such that the Levinson theorem (5) is violated. When $b \geq -1/4$, let

$$j(j + 1) = b, \quad j = -1/2 + (b + 1/4)^{1/2} \geq -1/2. \tag{33}$$

The Schrödinger equation (8) becomes the same as the radial equation in three dimensions except that the phase shift is $\eta_-(k, \lambda) - j\pi/2$ now. Repeating the proof in our previous paper [6], we obtain the
modified Levinson theorem for the Schrödinger equation (8) with the potential (32) in the non-critical cases:

\[ \eta_-(0) - j\pi/2 = n_\pi, \quad \eta_+(0) + (1-j)\pi/2 = n_+\pi. \] (34)

In other words, the Levinson theorem (5) is violated. It is obvious that the Levinson theorem will be violated more seriously if the potential tail decays at infinity slower than the potential tail (32). On the other hand, if the potential tail decays at infinity faster than the potential tail (32), for an arbitrarily given small positive number \( \epsilon \), there always exists a larger enough number \( x_0 \) such that

\[ (-\epsilon)(-\epsilon + 1)x^{-2} < V(x) < \epsilon(\epsilon + 1)x^{-2}, \quad x \geq x_0. \] (35)

Since \( \epsilon \) is arbitrarily small, no modification is needed to the Levinson theorem (5).

In conclusion, we establish the one-dimensional Levinson theorem (5) for the Schrödinger equation in one dimension with the potential satisfying

\[ V(-x) = V(x), \quad \lim_{x \to \infty} x^2V(x) = 0. \] (36)

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