FIRST-ORDER DIFFERENTIAL EQUATIONS
FOR A PARTICLE WITH SPIN $S = 1$

A system of first-order differential equations for a particle with nonzero mass and spin $S = 1$
is constructed. As distinct from the Proca–Duffin–Kemmer (PDK) equations, the system has
the form of the dynamical equation $i\hbar \partial_t \hat{\Psi} = \hat{H} \hat{\Psi}$ (with constraints) with a Hamiltonian linear
in momentum. The six-component wave function yields the positive-definite probability density
$\hat{\Psi}^\dagger \hat{\Psi} \geq 0$. The system of equations has much in common with the Dirac and Maxwell equations.

Key words: first-order differential equations, nonzero mass, spin

1. Introduction

The explicitly covariant PDK equations for a particle with spin $S = 1$ cannot be presented in the form
$i\hbar \partial_t \hat{\Psi} = \hat{H} \hat{\Psi}$ with the Hamiltonian linear in derivatives. The wave function has the so-called extra components,
and the quantity $\hat{\Psi}^\dagger \hat{\Psi}$ has no fixed sign. The exclusion of extra components leads to a Hamiltonian
of the second order in derivatives [1]. In addition, the PDK equations do not have some symmetries characteristic of the Maxwell equations (describing a massless field with spin 1) such as, for example, the symmetry relative to the changes $\vec{E} \rightarrow \vec{H}$, $\vec{H} \rightarrow \vec{E}$, and $t \rightarrow -t$.

In the present work, a system of first-order differential equations for a particle with spin $S = 1$ is proposed. It is different from the PDK equations, but its structure is similar to those of the Maxwell and Dirac equations (with constraints). The equations take the form $i\hbar \partial_t \hat{\Psi} = \hat{H} \hat{\Psi}$ with the Hamiltonian linear in derivatives and with the wave function without extra components, which has the meaning of the probability amplitude.

2. New equations of the first order in derivatives for a particle with spin 1 and with nonzero mass

Let us recall that if the Klein–Gordon–Fock equation (here and below, $\hbar = 1$, and $c = 1$)

$$ (\Box - m^2) A_\mu = 0, $$

supplemented by the condition

$$ \partial_\mu A_\mu = 0, $$

is considered to be the initial equation for a vector massive field, then the Proca equations of the first order in derivatives follow from (1) and (2), if the antisymmetric tensor

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu $$

is introduced, and (1) is rewritten (with regard for (2)) as

$$ \partial^\mu F_{\mu\nu} + m^2 A_\nu = 0. $$

Equations (1) together with (3) are the ten Proca equations for ten components of the field (six nonzero

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components $F_{\mu\nu}$ and four ones $A_\mu$). They can be also represented in the matrix form (see, e.g., [2]), by using the Duffin–Kemmer matrices.

If one introduces the notations (by analogy with electrodynamics)

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t},$$  \hspace{1cm} (5)

$$\vec{H} = \text{rot} \vec{A},$$  \hspace{1cm} (6)

or (what is the same)

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix},$$  \hspace{1cm} (7)

then Eqs. (4) take the form

$$\frac{\partial \vec{E}}{\partial t} = \text{rot} \vec{H} + m^2 \vec{A},$$  \hspace{1cm} (8)

$$\text{div} \vec{E} = -m^2 \phi.$$  \hspace{1cm} (9)

Thus, the Proca equations (4) look as (5), (6) and (8), (9) in other notations.

It is noteworthy that these equations have no symmetry related to the change $\vec{E} \to \vec{H}, \vec{H} \to \vec{E}$, and $t \to -t$ inherent to the Maxwell equations. The mass appears in the equations as $m^2$ (though it would be natural for it to appear in the first power as the quantity with dimension of energy, together with the time derivative). The wave function has extra components [1]: their number is 10 instead of $2(2S + 1) = 6$. Despite the degeneracy of the Duffin–Kemmer matrices [2], the system of equations [3], [4], [8], and [9] can be presented in the form $i\hbar \partial_\tau \Psi = \hat{H} \Psi$ [1], but it can be done only after the exclusion of extra components, and the Hamiltonian will be of the second order in derivatives.

It is possible to propose a new version [3] of the equations for particles with spin $1$ of the first order in derivatives, where those questions do not arise. Let the field $A_\mu$ (generally speaking, a complex-valued one) satisfy Eq. (1) and condition (2). Keeping the previous notations for $\vec{E}$ and $\vec{H}$ (5) and (6), let us introduce the new fields $\vec{u}$ and $\vec{v}$ (it is the "third" stage of transition from the field $A_\mu$ to secondary ones):

$$\vec{u} = \frac{\partial \vec{H}}{\partial t} + im\vec{H},$$  \hspace{1cm} (10)

$$\vec{v} = \text{rot} \vec{H}.$$  \hspace{1cm} (11)

Since it follows directly from Eq. (1) that $\vec{H}$ also satisfies the equation

$$\left(\Box - m^2\right) \vec{H} = 0,$$  \hspace{1cm} (12)

and from relation (6) one has, as is known,

$$\text{div} \vec{H} = 0,$$  \hspace{1cm} (13)

we obtain for $\vec{u}$ and $\vec{v}$:

$$\left(\Box - m^2\right) \vec{u} = 0,$$  \hspace{1cm} (14)

$$\left(\Box - m^2\right) \vec{v} = 0,$$  \hspace{1cm} (15)

$$\text{div} \vec{u} = 0,$$  \hspace{1cm} (16)

$$\text{div} \vec{v} = 0.$$  \hspace{1cm} (17)

Instead of the equations of the second order in derivatives (14) and (15), we now get a system of equations of the first order. Let us differentiate relations (10) and (11) with respect to $t$ taking into account Eq. (12) and the fact that $\text{rot} \left(\text{rot} \vec{H}\right) = \vec{\nabla} \text{div} \vec{H} - \Delta \vec{H}$ (by virtue of (13)) = $-\Delta \vec{H}$. Then, passing on the right-hand sides from $\vec{H}$ to $\vec{u}$ and $\vec{v}$ according to (10) and (11), we get the first-order differential equations

$$\frac{\partial \vec{u}}{\partial t} = im\vec{u} - \text{rot} \vec{u}, \quad \frac{\partial \vec{v}}{\partial t} = -im\vec{v} + \text{rot} \vec{v}$$  \hspace{1cm} (18)

with the additional conditions (16) and (17). It is easy to see that, conversely, Eqs. (18), (16), and (17) satisfy Eq. (11) and condition (2). Keeping the previous notations for $\vec{E}$ and $\vec{H}$ (5) and (6), let us introduce the new fields $\vec{u}$ and $\vec{v}$ (it is the "third" stage of transition from the field $A_\mu$ to secondary ones):
yield the Klein–Gordon–Fock equations for each component of the field.

First, we note that the limit \( m \to 0 \) in (18), (16), and (17) results immediately in equations identical, by their form, to the Maxwell equations.

Second, the equations include the mass in the first power, along with derivatives.

Third, Eqs. (18), (16), and (17) do not change their form under the transformation \( \vec{u} \to \vec{v}, \vec{v} \to \vec{u} \), and \( t \to -t \).

Finally, the obtained equations have the form of dynamic ones,

\[
i \frac{\partial \hat{\Psi}}{\partial t} = \hat{\mathbb{H}} \hat{\Psi}
\]

with the Hamiltonian linear in derivatives and with the additional conditions (16) and (17). The wave function has the meaning of a probability amplitude (see below).

To write the Hamiltonian of a particle in an explicit form, it is convenient to give Eqs. (18) in the matrix form reminding the Dirac equations.

3. Comparison with the Dirac equations

Let us introduce a 6-component wave function

\[
\hat{\Psi} = \frac{1}{\sqrt{2}} \begin{pmatrix} u_x \\ u_y \\ u_z \\ v_x \\ v_y \\ v_z \end{pmatrix},
\]

and the matrices

\[
\hat{a}_k = \hat{\sigma}_2 \otimes \hat{S}_k, \quad k = 1, 2, 3,
\]

\[
\hat{b} = \hat{\sigma}_3 \otimes \hat{I}.
\]

Here, \( \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \( \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) are the corresponding Pauli matrices, and \( \hat{S}_k \) are the matrices for spin 1:

\[
\hat{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},
\]

\[
\hat{S}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

It is easy to verify that, in the above notations, Eqs. (18) take the form

\[
i \frac{\partial \hat{\Psi}}{\partial t} = \hat{\mathbb{H}} \hat{\Psi} \equiv \left( \vec{a} \cdot \vec{p} \right) \hat{\Psi} + \hat{m} \hat{\Psi}.
\]

The Dirac equations differ from Eqs. (24) by the change of matrices

\[
\hat{a}_k \to \hat{a}_k = \hat{\sigma}_2 \otimes \hat{S}_k, \quad \hat{b} \to \hat{\beta} = \hat{\sigma}_3 \otimes \hat{I}
\]

(\( \hat{\alpha} \) and \( \hat{\beta} \) belong to one of the representations of the Dirac matrices). The comparison of relations (25) with (21), (22), and (23) indicates that the difference between our equations (24) and Dirac ones consists formally in the difference between spin matrices: \( \hat{S}_k \) for the spin \( S = 1 \) instead of \( \hat{\sigma}_k \) for the spin \( S = \frac{1}{2} \), which is natural.

However, the nonformal difference is more profound. Let us recall that the Maxwell equations

\[
\partial_i \vec{E} = \vec{\text{rot}} \vec{H}, \quad \partial_i \vec{H} = -\vec{\text{rot}} \vec{E}
\]

(in the matrix form, they look like Eqs. (24) at \( m = 0 \)) are not reduced to the equation \( \Box \hat{\Psi} = 0 \) for each of the components, if the additional conditions \( \text{div} \vec{E} = 0 \) and \( \text{div} \vec{H} = 0 \) are ignored. Similarly, the Klein–Gordon–Fock equation does not follow from Eqs. (24) (for \( m \neq 0 \)), if the additional conditions imposed on the wave function (\( \text{div} \vec{u} = 0 \) and \( \text{div} \vec{v} = 0 \)) are not taken into account. This is easily seen also from Eqs. (18) with conditions (16) and (17). After the second differentiation with respect to the time. In notations of Eqs. (24), this looks as follows.

Let us differentiate Eqs. (24) with respect to \( i \frac{\partial}{\partial t} \):

\[
- \partial^2 \frac{\partial}{\partial t^2} \hat{\Psi} = \hat{\mathbb{H}}^2 \hat{\Psi} = \left( \hat{\vec{a}} \cdot \hat{\vec{p}} \right) \hat{\Psi} + m \left( \hat{\vec{a}} \hat{\vec{a}} \cdot \hat{\vec{p}} \right) \hat{\Psi} + m^2 \hat{\vec{b}} \hat{\vec{b}} \hat{\Psi}.
\]

By virtue of the identities

\[
\hat{b}^2 = \hat{I}, \quad \hat{a}_k \hat{b} + \hat{b} \hat{a}_k = 0, \quad k = 1, 2, 3,
\]

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following from definitions (21) – (23), it remains to require that
\[ (\hat{a} \cdot \vec{p}) (\hat{a} \cdot \vec{p}) \hat{\Psi} = -\Delta \hat{\Psi}, \]  
(28)
and then Eq. (24) appear to be the Klein–Gordon–Fock equation. However, Eq. (25) is valid only together with the wave function \( \Psi \) satisfying the additional conditions (16) and (17) for its components, but the relation (28) is not valid in the operator sense. In other words, as distinct from the Dirac equation, we have
\[ (\hat{a} \cdot \vec{p})^2 \neq -\Delta. \]  
(29)

Thus, the example of Eq. (24) leads to the following generalization of the Dirac method to obtain the equations of the first order in derivatives from the Klein–Gordon–Fock equation: the expression \( \hat{H} \hat{\Psi} \) (26) must be equal to \( (\Delta + m^2) \hat{\Psi} \) with the solution included, and the solution must satisfy, if necessary, some additional conditions. But the above relation should not hold exceptionally as the operator equality. This leads to the more general relations for matrices \( \hat{a}_k, \hat{b} \) as compared with the well-known ones defining the Dirac matrices.

4. Spin operator, the total angular momentum as an integral of motion. Continuity equation for the current

Consider the operator
\[ \hat{J} = \hat{L} + \hat{\Sigma} \equiv [\hat{r} \times \vec{p}] + \hat{\Sigma}, \]  
(30)
where \( \hat{L} \) is the operator of orbital angular momentum, and the operator \( \hat{\Sigma} \) is, by definition,
\[ \hat{\Sigma}_1 = -i (\hat{a}_2 \hat{a}_3 - \hat{a}_3 \hat{a}_2) = \hat{I} \otimes \hat{S}_1, \]
\[ \hat{\Sigma}_2 = -i (\hat{a}_3 \hat{a}_1 - \hat{a}_1 \hat{a}_3) = \hat{I} \otimes \hat{S}_2, \]  
(31)
\[ \hat{\Sigma}_3 = -i (\hat{a}_1 \hat{a}_2 - \hat{a}_2 \hat{a}_1) = \hat{I} \otimes \hat{S}_3. \]
Using the explicit form of the Hamiltonian from Eq. (24),
\[ \hat{H} = (\hat{a} \cdot \vec{p}) + m \hat{b}, \]  
(32)
one can directly verify that operator (30) commutes with the Hamiltonian:
\[ [\hat{J}, \hat{H}] = 0, \]  
(33)
i.e., it is an integral of motion.

It is obvious that \( \hat{J} \) has the meaning of the operator of total angular momentum, and \( \hat{\Sigma} \) is the operator of spin of the particle. Since
\[ (\hat{\Sigma})^2 = \hat{\Sigma}_1^2 + \hat{\Sigma}_2^2 + \hat{\Sigma}_3^2 = 2\hat{I}, \]  
(34)
and, on the other hand, \( (\hat{\Sigma})^2 = S(S + 1)\hat{I} \), we find that \( S = 1 \), as it should be.

The equation of continuity can be obtained in usual way, by taking the scalar products of the first and second equations (13) with \( \vec{u}^* \) and \( \vec{v}^* \), respectively, and adding the results with the corresponding complex conjugate equalities. We get
\[ \frac{\partial}{\partial t} \rho + \text{div} \vec{j} = 0, \]  
(35)
where
\[ \rho = \frac{1}{2} \{ (\vec{u}^* \cdot \vec{u}) + (\vec{v}^* \cdot \vec{v}) \}, \]  
(36)
\[ \vec{j} = \frac{1}{2} \{ [\vec{u}^* \times \vec{u}] + [\vec{v} \times \vec{u}^*] \}. \]  
(37)

Here, we used the factor 1/2 for convenience of the comparison with electrodynamics: for \( m = 0 \) and real field \( A_\mu \), we get the field energy density instead of (36) and the Umov–Poynting vector instead of (37).

The same relations can be obtained from Eq. (24) in the matrix form. Like in the case of Dirac equations, it is possible to obtain Eq. (36) from Eq. (24) in the standard way, with the probability density
\[ \rho = \hat{\Psi}^\dagger \hat{\Psi}, \]  
(38)
and the probability flow density
\[ \vec{j} = \hat{\Psi}^\dagger \hat{a} \hat{\Psi}. \]  
(39)
It is easy to verify that definitions (38) and (39) coincide with (36) and (37), respectively, if normalization (20) is used. (Of course, it is possible to normalize \( \rho \) and \( \vec{j} \) in another way, e.g., to the charge density and the current density, multiplying them by the quantity proportional to the charge of an electron.) It is obvious that the probability density (36) (or (38)) is positive-definite.

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5. Some generalizations

The equations for \( \vec{u}, \vec{v} \) (or, in the matrix form, Eq. (24)) together with the additional conditions (10) and (17) can be postulated as the initial (primary) ones, without mentioning the connection with the field \( \vec{H} \) given by (10) and (11).

Indeed, Eqs. (15), (10), and (17) are closed. They yield the Klein–Gordon–Fock equation for each component of the wave function. An analogous situation is characteristic of components of the wave function of the Dirac equations. Therefore, the "interpretation" of the wave function components (10) and (11) is not necessary. Moreover, other variants of the transition from (1) and (2) to (18), (16), and (17) are possible. For example, we can verify that the new quantities

\[
\vec{u} = \text{rot} \vec{E},
\]

\[
\vec{v} = \partial_t \vec{E} - im\vec{E} + m^2 e^{-imt} \partial_t^{-1} e^{imt} \vec{\nabla} \varphi
\]  

(40)

satisfy the same equations (18) with conditions (10) and (17). (Here, the symbol \( \partial_t^{-1} \) means the integration, for which one has \( \partial_t^{-1} \partial_t \chi = \partial_t \partial_t \chi = \chi \). In particular, if the field \( \chi \) decreases rapidly enough at \( t \to -\infty \), we may assume that \( \partial_t^{-1} \chi (t) \equiv \int_{-\infty}^{t} \chi (\tau) d\tau \).)

It is clear that, instead of (10) or (10) and (11), it is possible to take linear combinations of these definitions and to obtain again Eqs. (18), (10), and (17).

Another variant: instead of the two-stage transition \((A_\mu \to \vec{E}, \vec{H} \to \vec{u}, \vec{v})\), we can introduce

\[
\vec{H} = \text{rot} \vec{A},
\]

\[
\vec{E} = -\vec{\nabla} \varphi - \partial_t \vec{A} + im \left( \vec{A} + e^{-imt} \partial_t^{-1} e^{imt} \vec{\nabla} \varphi \right)
\]  

(41)

and verify that the same equations (18) with conditions (10) and (17) are obtained (where \( \vec{H} \) will be present instead of \( \vec{u} \), as well as \( \vec{E} \) instead of \( \vec{v} \)). In this case, the transition to electrodynamics \((m \to 0)\) becomes very simple already at the level of definitions (11).

At the same time, it is worth to pay attention to the fact that the relativistic covariance of Eqs. (18) with additional conditions (10) and (17) is satisfied automatically in the case of definitions like (10) and (11), where we start from relativistic equations (3) and (4).

We note also that the components of the wave function can be replaced by their linear combinations. Though Eqs. (18) with conditions (10) and (17) will be trivially changed in this case, this corresponds only to the transition to a new representation of the matrices in Eq. (24).

6. Equations for a particle in an external electromagnetic field

Let the particle possess a charge \( e \). As known, the natural way to introduce the interaction of a charged particle with an external electromagnetic field \( A_\mu = (\Phi, \vec{A}) \) is the "extension" of derivatives

\[
\partial_\mu \to \partial_\mu - ieA_\mu
\]  

(42)

in the equations of motion and in the additional conditions. In particular, if the definition

\[
\hat{\Pi}_A \equiv (\vec{a} \cdot (\vec{p} - e\vec{A})) + \hat{b}m
\]  

(43)

is introduced, then Eq. (24) transforms into

\[
(i \frac{\partial}{\partial t} - e\Phi) \hat{\Psi} = \hat{\Pi}_A \hat{\Psi}
\]  

(44)

under the additional conditions

\[
\left( \vec{p} - e\vec{A} \right) \cdot \vec{u} = 0, \quad \left( \vec{p} - e\vec{A} \right) \cdot \vec{v} = 0.
\]  

(45)

Let us obtain an equation of the second order in derivatives starting from Eq. (44). Acting by the operator \((i \frac{\partial}{\partial t} - e\Phi)\) on both sides of Eq. (44) and calculating its commutator with \( \hat{\Pi}_A \), we get

\[
\left( i \frac{\partial}{\partial t} - e\Phi \right)^2 \hat{\Psi} = \left( i \frac{\partial}{\partial t} - e\Phi \right) \hat{\Pi}_A \hat{\Psi} = \hat{\Pi}_A \left( i \frac{\partial}{\partial t} - e\Phi \right) \hat{\Psi} + ie \left( \vec{a} \cdot \vec{\epsilon} \right) \hat{\Psi},
\]  

(46)

where \( \vec{\epsilon} = -\frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi \) is the vector of external electric field intensity. Substituting \( \hat{\Pi}_A \hat{\Psi} \) instead of
\[
(i \frac{\partial}{\partial t} - e\Phi) \hat{\Psi} \text{ into (46) according to (44) and using the identity}
\]

\[
\hat{H}_A^2 = \left( \hat{a} \cdot (\hat{p} - e\hat{A}) \right)^2 + m^2,
\]

which is valid due to (27), we have

\[
\left( i \frac{\partial}{\partial t} - e\Phi \right)^2 \hat{\Psi} = \]

\[
\left( \hat{a} \cdot (\hat{p} - e\hat{A}) \right)^2 \hat{\Psi} + m^2 \hat{\Psi} +\]

\[
e e \left( \hat{a} \cdot \hat{E} \right) \hat{\Psi}, \tag{48}
\]

instead of (46).

We now use the following equality valid only with the wave function satisfying the additional conditions (45):

\[
\left( \hat{a} \cdot (\hat{p} - e\hat{A}) \right)^2 \hat{\Psi} = \]

\[
\left( \hat{p} - e\hat{A} \right)^2 \hat{\Psi} - e \left( \hat{\Sigma} \cdot \hat{H} \right) \hat{\Psi}, \tag{49}
\]

where \( \hat{H} = \text{rot}\hat{A} \) is the external magnetic field intensity, and \( \hat{\Sigma} \) is the operator of spin (31), (For brevity, we omit the proof of relation (49).)

As a result, we get the second-order equation

\[
\left( \left( i \frac{\partial}{\partial t} - e\Phi \right)^2 - \left( \hat{p} - e\hat{A} \right)^2 - m^2 \right) \hat{\Psi} = \]

\[
-e \left( \hat{\Sigma} \cdot \hat{H} \right) - i \left( \hat{a} \cdot \hat{E} \right) \hat{\Psi}, \tag{50}
\]

which differs from the Klein–Gordon–Fock equation (with the "extended" derivatives (42) ) by terms on the right-hand side. Relation (50) is very similar to the analogous equation for a particle with spin 1/2, though the term proportional to a magnetic field arises in (49) and (50) not only due to the the commutation relations of matrices, but also due to the properties of the wave function (45).

Equation (50) implies that a particle with spin \( S = 1 \) within the proposed approach has a normal magnetic moment equal to the Bohr magneton (without regard for the electromagnetic corrections).

7. Conclusion

A system of equations of the first order in derivatives, which differs from the well-known ones, is constructed for a massive particle with spin \( S = 1 \). By their structure, the equations have much in common with the Maxwell and Dirac equations (with constraints). The considered case generalizes the Dirac procedure of constructing the equations of the first order in derivatives, starting from the Klein–Gordon–Fock equation.

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