Cohomology of projective schemes: From annihilators to vanishing

Marc Chardin

Introduction

This article comes from our quest for bounds on the Castelnuovo-Mumford regularity of schemes in terms of their “defining equations”, in the spirit of [BM], [BEL], [CP] or [CU]. The references [BS], [BM], [V] or [C] explains how this notion of regularity is a measure of the algebraic complexity of the scheme, and provides several computational motivations.

It was already remarked by several authors (see for instance [M], [MV], [NS1] or [NS2]) that one may bound the Castelnuovo-Mumford regularity of a Cohen-Macaulay projective scheme in terms of its \(a\)-invariant and the power of the maximal ideal that kills all but the top local cohomology modules. Such a connection is a particular case of Lemma 2.0, which shows the way but will not be used in the sequel, because some natural killers have stronger annihilation properties that leads to sharper estimates.

In connection with our previous joint work with Philippon, we introduce partial annihilators of modules (i.e. elements that annihilate in some degrees) and prove that uniform (partial)annihilators of Koszul homology modules give rise to (partial)annihilators of Čech cohomology modules of the quotient by a sequence of parameters (Proposition 2.1). Combined with Proposition 2.3, this leads to our key for passing from annihilators to vanishing: Proposition 2.4.

The third section gathers results on uniform annihilators that have two main sources: tight closure and liaison. For the applications to Castelnuovo-Mumford regularity, the key is to determine an annihilator of the cohomology modules on which we have a control in terms of degrees of generators. The Jacobian ideal, which kills phantom homology by a theorem of Hochster and Huneke, and the ideal we construct via liaison are the only ones that we are able to control today, it would be interesting to have such a control on other natural annihilators (e.g. the parameter test ideal).

The results on regularity are combinations of the two preceeding ingredients: control of annihilators and passing from annihilators to vanishing. They hold in any characteristic and, in positive characteristic, improves the ones of [CU] for the unmixed part (even if not completely comparable). Also, they do not rely on Kodaira vanishing. The main result of a new type is the following (Theorem 4.4),

**Theorem.** Let \( R \) be a polynomial ring over a field, \( I \subset R \) be a homogeneous \( R \)-ideal generated by forms of degrees \( d_1 \geq \cdots \geq d_s \). Consider \( r \leq s \) and \( J \subset R \), an intersection of isolated primary component of codimension \( r \) of \( I \). Set \( S := \text{Proj}(R/J) \), \( Z := \text{Proj}(R/I) \) and assume that \( Z \neq \emptyset \) and

1. \( Z \) have at most isolated singularities on \( S \),
2. \( S \) do not meet the other components of \( Z \).

Then

\[
\text{reg}(R/J) \leq \dim(R/J)(d_1 + \cdots + d_r - r - 1) + 1.
\]
It is a first step in the direction of bounding the regularity of all the isolated components of a scheme in terms of degrees of generators. It seems reasonable to hope that hypothesis (2) may be removed; it may even be that (1) and (2) are superfluous.

Most of the estimates we are aware of gives either huge bounds or only concerns the top dimensional component. There are two exceptions: our result with Philippon on the zero dimensional part (which is also crucial here) and our joint work with Ulrich [CU, 4.7 (b)], where we have an hypothesis on the singularities of the total scheme \( Z \) in place of the component \( S \).

\section{Notations}

Set \( R := k[X_0, \ldots, X_n] \) and \( \mathfrak{m} := (X_0, \ldots, X_n) \), where \( k \) is a field.

If \( \gamma = (\gamma_1, \ldots, \gamma_s) \) is a collection of elements of \( R \), and \( M \) is a \( R \)-module \( K_\bullet(\gamma; M) \) will denote the Koszul complex of \( \gamma \) on \( M \) and \( H_i(\gamma; M) \) its \( i \)-th homology module (and similar notations for Koszul cohomology). We denote by \( \mathcal{C}_\mathfrak{m}(M) \) the \( \check{\text{C}} \text{ech} \) complex

\[
0 \to M \to \oplus_i M_{X_i} \to \oplus_{i<j} M_{X_iX_j} \to \cdots \to M_{X_0 \cdots X_n} \to 0
\]

(starting at \( \mathcal{C}_0^\mathfrak{m}(M) = M \)) and by \( H_i^\mathfrak{m}(M) \) its \( i \)-th cohomology module.

For a graded \( R \)-modules \( M \), we will set \( a_i(M) := \sup \{ \mu \mid H_i^\mathfrak{m}(M)_\mu \neq 0 \} \) and \( b_i(M) := \sup \{ \mu \mid \text{Tor}^R_{i}(M, k)_\mu \neq 0 \} \). If \( M \) is finitely generated, \( a_i(M) \) is finite and therefore \( \text{reg}(M) := \max_i \{ a_i(M) + i \} = \max_i \{ b_i(M) - i \} \) also is. The \( a \)-invariant of \( M \) is \( a(M) := a_{\dim M}(M) \), and \( \text{Tor}^R_i(M, k) = H_i(X_0, \ldots, X_n; M) \).

If \( B \) is a standard graded algebra, \( H \) a graded \( B \)-module and \( S \subseteq \mathbb{Z} \), we set \( \text{Ann}_{\mathbb{Z}}^S(H) := \{ x \in B \mid (xH)_\nu = 0, \forall \nu \in S \} \).

We denote by \( \mathcal{Z}_M \) the set of tuples \( z \) of homogeneous elements such that \( \dim M/(z)M = 0 \), set \( d := \dim M \) and

\[
\mathcal{R}_M^\mu := \bigcap_{z \in \mathcal{Z}_M} \bigcap_i \text{Ann}_{R}^{\mu-i}(H^i(z; M)), \quad \hat{\mathcal{R}}_M^\mu := \bigcap_{z \in \mathcal{Z}_M} \bigcap_{i<d} \text{Ann}_{R}^{\mu-i}(H^i(z; M)),
\]

\[
\mathcal{C}^{i,\mu}_M := \text{Ann}_{R}^{\mu-i}(H^i_m(M)), \quad \mathcal{C}^{\mu}_M := \bigcap_i \mathcal{C}^{i,\mu}_M, \quad \hat{\mathcal{C}}^{\mu}_M := \bigcap_{i<d} \mathcal{C}^{i,\mu}_M.
\]

If \( M \) is Cohen-Macaulay and equidimensional, \( \hat{\mathcal{C}}^{\mu}_M = \hat{\mathcal{R}}^{\mu}_M = A \) for any \( \mu \). With these definitions,

\[
\text{reg}(M) = \min \{ \mu \mid \mathcal{C}^{\mu}_M = A \} = \min \{ \mu \mid \mathcal{R}^{\mu}_M = A \}.
\]

Following [CP] we define \( \mathfrak{b}\text{-reg}(M) := \min \{ \mu \mid \mathfrak{b} \subseteq \mathcal{R}^{\mu}_M \} \).
§2. Partial annihilators of Koszul and local cohomologies

Before investigating results that are better fitted to the kind of annihilators we are able to build in the next paragraph, let us state a basic result which is typical of how one passes from annihilators to vanishing,

**Lemma 2.0.** Let $M$ be a finitely generated graded $R$-module of dimension $d$, $\gamma = (\gamma_1, \ldots, \gamma_s)$ a collection of homogeneous elements of $R$ of degrees $\delta_1 \geq \cdots \geq \delta_s$ and $J$ the $R$-ideal they generate.

Let $N, i$ be integers and assume that,

1. $J \subseteq \text{Ann}_{R}^{\geq N}(H_{m}^{i}(M))$,
2. $a_{i+j}(H_{j}(\gamma ; M)) \leq N$, for every $j$,
3. $a_{i+k}(M) \leq N - (\delta_1 + \cdots + \delta_{k+1})$ for $0 < k < s$.

Then,

$$ a_i(M) \leq N. $$

To apply the result, note that if, for example, $M$ is Cohen-Macaulay on the punctured spectrum and $0 < i < d$, choosing for $\gamma$ a complete system of homogeneous parameters in $\cap_{i \neq d} \text{Ann}(H_{m}^{i}(M))$, (1) and (2) are void, and (3) is obtained by descending recursion on $i$ from an estimate of the $a$-invariant of $M$.

**Proof.** Consider the double complex, $C_{m}^{i}K_{\bullet}(\gamma ; M)$. It gives rise to two spectral sequences, whose first terms are $E_{1}^{pq} = C_{m}^{p}(H_{q}(\gamma ; M))$ and $E_{2}^{pq} = H_{m}^{p}(H_{q}(\gamma ; M))$, for the first one.

If $\mu > N$ and $p - q = i$, $(E_{2}^{pq})_{\mu} = 0$ by (2), and therefore $(E_{\infty}^{pq})_{\mu} = 0$.

On the other hand the second spectral sequence gives,

$$ E_{1}^{i0} = K_{q}(\gamma ; H_{m}^{p}(M)) = \bigoplus_{i_{1} \ll \cdots \ll i_{q}} H_{m}^{p}(M)[-\delta_{i_1} - \cdots - \delta_{i_q}]. $$

Note that $E_{1}^{i0} = H_{m}^{i}(M)$ and $E_{2}^{i0} = H_{m}^{i}(M)/JH_{m}^{i}(M)$ so that $(E_{2}^{i0})_{\mu} = H_{m}^{p}(M)_{\mu}$ for $\mu > N$ by (1).

If $k > 0$, $(E_{1}^{iq+k,k+1})_{\mu} = 0$ for $\mu > N$ by (3) (also note that $E_{0}^{pq} = 0$ for $q > s$). As a consequence, $(E_{2}^{i+1,k+1})_{\mu} = 0$ for $k > 0$ and $\mu > N$, so that $0 = (E_{\infty}^{i0})_{\mu} = (E_{2}^{i0})_{\mu} = H_{m}^{i}(M)_{\mu}$ if $\mu > N$.

We now turn to our main technical result on annihilators,

**Proposition 2.1.** Let $M$ be a finitely generated graded $R$-module of dimension $d$. Then $\hat{R}_{M}^{i} \subseteq C_{m}^{i}$ and $\hat{R}_{M}^{i} \subseteq C_{m}^{i}$. Furthermore, if $\gamma = (\gamma_1, \ldots, \gamma_s)$ is an homogeneous system of parameters, $\varepsilon := \dim H_{1}(\gamma ; M)$ and $\sigma := \sum_{i=1}^{s}(\deg \gamma_i - 1)$, then

(i) For $\varepsilon \leq i < d - s$, $\hat{R}_{M}^{i} \subseteq C_{M}^{i,\mu+\sigma \gamma ; M}.$

(ii) For $i \geq \varepsilon$, $\hat{R}_{M}^{i} \subseteq C_{M}^{i,\mu+\sigma \gamma ; M}.$

For the proof, we will need the following lemma, which is part of the “folklore” (see e. g. [V, 8.3.1] for a similar statement),
Lemma 2.2. Let \( B \) be a standard graded algebra, \( \mathfrak{m} := B_{>0}, x := (x_1, \ldots, x_d) \) forms in \( B_1 \) and set \( x^t := (x_1^t, \ldots, x_d^t) \). For every \( i \) and \( t > 0 \), and every finitely generated graded \( B \)-module \( M \) such that \( \dim(M/(x)M) = 0 \), there exists a homogeneous morphism of \( B \)-modules,
\[
\psi^t_i : H^i(x^t; M) \to H^i_m(M)
\]
which is functorial in \( M \) and such that the restriction \( \psi^t_{i,\mu} : H^i(x^t; M)_{\geq \mu} \to H^i_m(M)_{\geq \mu} \) is an isomorphism for \( t > \max_i \{a_i(M)\} - \mu \).

Note that the condition \( \dim(M/(x)M) = 0 \) is always satisfied if \( \dim(B/(x)) = 0 \).

Proof of Lemma 2.2. Consider the two spectral sequences arising from the double complex \( D^{**} := C^* \cdot \tilde{K}^*(x^t; M) \). As \( \dim(M/(x)M) = 0 \), one spectral sequence abouts at the first step and provides an isomorphism \( H^q(D^*_{tot}) \simeq 'E^{0q}_1 = H^q(x^t; M) \).

On the other hand, the second spectral sequence have as first terms:
\[
"E^{pq}_1 = H^p_m(M)[tq]^{(y)}
\]
and one has inclusions \( "E^{pq}_1 \supseteq "E^{pq}_2 \cdots \supseteq "E^{pq}_\infty \). We define \( \psi^t_i \) as the composed map,
\[
H^i(x^t; M) = H^i(D^*_{tot}) \longrightarrow "E^{pq}_\infty \longrightarrow "E^{pq}_1 = H^i_m(M).
\]

To see that \( \psi^t_{i,\mu} := (\psi^1_{i,\mu})_{\geq \mu} \) is an isomorphism for \( t > \max_i \{a_i(M)\} - \mu \), remark that for such \( t \) and \( \mu \) one has \( ("E^{pq}_1)_{\mu} = 0 \) for \( q > 0 \), so that all the above inclusions are equalities and \( H^i(D^*_{tot})_{\mu} \to ("E^{pq}_1)_{\mu} \) is an isomorphism. \( \square \)

Proof of Proposition 2.1. We may assume that \( k \) is infinite.

For the first statement, consider a sequence of parameters \( y_1, \ldots, y_d \) in \( A_1 \). By hypothesis \( [\hat{R}^\mu_M H^i(y_1, \ldots, y_d; M)]_{\geq \mu - i} = 0 \) for any \( i \) and \( t > 0 \); therefore \( [\hat{R}^\mu_M H^i_m(M)]_{\geq \mu - i} = 0 \) for any \( i \) by Lemma 2.2. Also \( [\hat{R}^\mu_M H^i(y_1, \ldots, y_d; M)]_{\geq \mu - i} = 0 \) for any \( i \neq d \) and \( t > 0 \) implies that \( [\hat{R}^\mu_M H^i_m(M)]_{\geq \mu - i} = 0 \) for \( i \neq d \).

For \( i \) and \( i' \), we choose \( x := (x_{s+1}, \ldots, x_d) \) with \( x_i \in A_1 \) so that \( (\gamma, x) \) is a complete system of parameters. Let us consider the double complex \( K^* \cdot (x^t; K^*(\gamma; M)) \) whose corresponding total complex is \( K^*(x^t; \gamma; M) \).

Let \( \alpha \in H^i_m(M/(\gamma)M) \) be an element of degree \( h \). For proving (i) (resp. (ii)), we have to show that if \( \tau \in \hat{R}^\mu_M \) (resp. \( \tau \in \hat{R}^\mu_M \)) is such that \( \deg \tau > \mu - i + \sigma - h \) then \( \tau \alpha = 0 \in H^i_m(M/(\gamma)M) \) for \( \varepsilon \leq i < d - s \) (resp. for \( \varepsilon \leq i \leq d - s \)).

For \( t := \max_{i,j} \{a_i(H^j(\gamma; M))\} + \sigma + s - h + 1 \) and \( \ell \geq h - \sigma - s \), we consider the spectral sequence
\[
(E^{pq}_2)_{\geq \ell} = [H^p(x^t; H^q(\gamma; M))]_{\geq \ell} \Rightarrow H^{p+q}(x^t, \gamma; M)_{\geq \ell}.
\]

Now \( t > \max_i \{a_i(H^q(\gamma; M))\} - \ell \) so that Lemma 2.2 and the isomorphism \( H^q(\gamma; M) \simeq H_{s-q}(\gamma; M)[\sigma + s] \) provides a sequence of isomorphisms,
\[
[H^p(x^t; H^q(\gamma; M))]_{\geq \ell} \xrightarrow{\psi_{t}^{p,q}} H^p_m(H^q(\gamma; M))_{\geq \ell} \xrightarrow{\theta_{t}^{p,q}} H^p_m(H_{s-q}(\gamma; M))_{\geq \ell + \sigma + s}.
\]
As $H^p_m(H_{s-q}(\gamma ; M)) = 0$ for $p > \varepsilon$ and $q \neq s$ (notice that $\dim H_j(\gamma ; M) \leq \varepsilon$ for $j > 0$), we obtain that $(E^{pq}_2)_{\geq \ell} = 0$ for these values of $p$ and $q$, so that for $i \geq \varepsilon$ we also have an isomorphism

$$H^{i+s}(x^i, \gamma ; M)_{\geq \ell} \xrightarrow{\phi^{i+s}_{\ell, \ell}} [H^i(x^i ; H^s(\gamma ; M))]_{\geq \ell} .$$

Now for $i \geq \varepsilon$ the isomorphisms $\eta^{i,s}_{\ell, \ell} := \theta^{i,s}_{\ell, \ell} \circ \psi^{i,s}_{\ell, \ell} \circ \phi^{i,s}_{\ell, \ell}$ provides a commutative diagram

$$
\begin{array}{ccc}
H^{i+s}(x^i, \gamma ; M)_{\geq h-s} & \xrightarrow{\eta^{i,s}_{h-s}} & [H^i_m(M/(\gamma)M)]_{\geq h} \\
\downarrow \times \tau & & \downarrow \times \tau \\
H^{i+s}(x^i, \gamma ; M)_{\geq \nu-s} & \xrightarrow{\eta^{i,s}_{\nu-s}} & [H^i_m(M/(\gamma)M)]_{\geq \nu}
\end{array}
$$

where $\nu := h + \deg \tau \geq h$, so that $\nu - \sigma - s = \deg \tau + h - \sigma - s > \mu - (i + s)$.

By hypothesis $[\tau H^{i+s}(x^i, \gamma ; M)]_{\geq \mu-(i+s)} = 0$ for $i \neq d - s$ (resp. for any $i$), so the vertical maps are 0, and in particular $\tau \alpha = 0$. $\Box$

Proposition 2.3. Let $M$ be a finitely generated graded $R$-module, $x \in R$ an homogeneous element. Assume that $[xH^i_m(M)]_\mu = 0$ and $\dim(H^i_1(x ; M)) \leq i$. Then

$$H^i_m(M)_\mu \subseteq H^i_m(M/xM)_{\mu} .$$

Proof. Set $K := H_1(x ; M) = \ker(M[-\delta] \xrightarrow{\times x} M))$ with $\delta := \deg x$. One has an exact sequence

$$0 \rightarrow K \rightarrow M[-\delta] \xrightarrow{\times x} M \rightarrow M/xM \rightarrow 0 .$$

If $[xH^i_m(M)]_\mu = 0$ it gives rise to a surjection,

$$
\ker(H^{i+1}_m(K)_\mu \xrightarrow{\text{can}} H^{i+1}_m(M)_\mu) \rightarrow \ker(H^i_m(M)_\mu \xrightarrow{\text{can}} H^i_m(M/xM)_\mu)
$$

which shows our claim. $\Box$

Proposition 2.4. Let $M$ be a finitely generated graded $R$-module and $\gamma = (\gamma_1, \ldots, \gamma_t)$ be an homogeneous system of parameters that is a regular sequence on $M$ outside $V(J)$ with $\dim M/JM \leq \varepsilon$.

Let $0 \leq \ell < t$, set $\sigma_\ell := \deg \gamma_1 + \cdots + \deg \gamma_\ell - \ell$ and assume either that

(i) $\varepsilon \leq \ell < d - \ell$ and $\gamma_j \in \widehat{R}_M^\mu$ for $j \leq \ell + 1$; or

(ii) $\ell \geq \varepsilon$ and $\gamma_j \in \widehat{R}_{sM}^\mu$ for $j \leq \ell + 1$.

Then,

$$H^i_m(M/(\gamma_1, \ldots, \gamma_\ell)_\nu \subseteq H^i_m(M/(\gamma_1, \ldots, \gamma_{\ell+1}))_{\nu}, \ \forall \nu > \mu - i + \sigma_\ell .$$

In particular, if $\gamma_j \in \widehat{R}_M^\mu$ for all $j$ and $\varepsilon \leq i \leq d - t$,

$$H^i_m(M)_\nu \subseteq H^i_m(M/(\gamma_1, \ldots, \gamma_t))_\nu, \ \forall \nu > \mu - i + \sigma_{t-1} .$$
Proof. For (i), 2.1 (i) implies that $[\gamma_{\ell+1}H^i_m(M/(\gamma_1,\ldots,\gamma_\ell))]_\nu = 0$ for $\nu > \mu - i + \sigma_\ell$ so the result follows from 2.3. The proof of (ii) is similar, replacing 2.1 (i) by 2.1 (ii). □

§3. Uniform annihilators of homologies

Notation. If $B$ is a homogeneous quotient of $R$ and $M$ a finitely generated graded $R$-module,

- $J_B$ is the jacobian ideal of $B$,
- $T_B$ is the test ideal of $B$ (see [HH3, §3] for details on the characteristic zero case),
- $\mathfrak{c}^*_M := \prod_{i \neq d} \mathfrak{c}^{i,-\infty}.$

Proposition 3.1. If $B$ is a homogeneous quotient of $R$ and $M$ a finitely generated graded $R$-module,

1. $T_B \subseteq \widehat{R}^{-\infty},$
2. $\mathfrak{c}^*_M \subseteq \widehat{R}^{-\infty} \subseteq \widehat{E}^{-\infty},$
3. If further $B$ is geometrically reduced and equidimensional, $J_B \subseteq T_B.$

Proof. The inclusion $T_B \subseteq \widehat{R}^{-\infty}$ directly follows from the phantom acyclicity criterion [HH1, 9.8]. The inclusion $\mathfrak{c}^*_M \subseteq \widehat{R}^{-\infty}$ is now standard (see [BH, 8.1.3 a]) and $\widehat{R}^{-\infty} \subseteq \widehat{E}^{-\infty}$ for $i \neq d$ by 2.1. Finally, (iii) is a special case of [HH3, 3.4]. □

Proposition 3.2. Let $A$ be a Gorenstein homogeneous ring and $I \subseteq A$ an ideal generated by forms of degrees $d_1 \geq \cdots \geq d_s$. Assume that $J$ is an intersection of primary components of $I$ of codimension $r := \text{codim}(I)$. Set $S := \text{Proj}(A/J)$ and $Z := \text{Proj}(A/I)$.

Then there exists an homogeneous ideal $\mathfrak{b}$ of $A$, containing $I$, generated in degree at most $\sigma := \text{reg}A + d_1 + \cdots + d_r - r$, such that $\mathfrak{b}$-reg$(A/J) \leq \sigma$ and $V(\mathfrak{b}) \subseteq W \cup (Z - S)$, where $W$ is the closed locus of points of $S$ where $Z$ is not locally a complete intersection $F$-rational scheme.

Proof. For each point $x \in S$ where $Z$ is locally a complete intersection there exists forms $f_1^x, \ldots, f_r^x$ of degrees $d_1, \ldots, d_r$ that defines a complete intersection $S \cup S^x$ such that $x \not\in S^x$. By [CU, 1.7 (iii)], if $S$ is $F$-rational at $x$ there exists a form $h_f^x$ of degree $\sigma$ such that $h_f^x(x) \neq 0$ and $I_S = \langle f_1^x, \ldots, f_r^x \rangle : (h_f^x)$. By [CP, Prop. 2], $A/I_S$ is $(\sigma, (h^x))$-regular because $A/(f_1^x, \ldots, f_r^x)$ is $\sigma$-regular. It follows that $A/I_S$ is $(\sigma, \mathfrak{b})$-regular with $\mathfrak{b} := I + \sum_{x} f_x^x(\langle f_1^x, \ldots, f_r^x, h_f^x \rangle)$, where the sum is taken over $x$ and $f^x := (f_1^x, \ldots, f_r^x)$ as above. Notice that $\mathfrak{b}$ is generated in degree at most $\sigma$ unless $\sigma < d_1$, in which case $d_2 = 1$ and $A$ is a polynomial ring, so that choosing $\mathfrak{b} := A$ gives the result in this exceptional case. The zero set of $\mathfrak{b}$ is contained by construction in the complement of the points of $S$ where $Z$ is locally a complete intersection and $S$ (or equivalently $Z$) is $F$-rational. □

§4. Castelnuovo-Mumford regularity

First we note that the Theorem 1 of [CP] readily extends to standard graded Cohen-Macaulay algebras,
Theorem 4.0. Let $k$ be a field and $A$ a standard graded Cohen-Macaulay $k$-algebra of dimension $n + 1$. If $f_1, \ldots, f_t$ are forms in $A$ of degrees $d_1 \geq \cdots \geq d_t$ and $S$ is a subscheme of the zero dimensional component of $\text{Proj}(A/(f_1, \ldots, f_t))$, then

$$\text{reg}(S) \leq \text{reg}(A) + d_1 + \cdots + d_n - n.$$ 

Proof. See [CP], the proof of Theorem 1. □

We will now extend this result to the components of a scheme of positive dimension. We have three main results in this direction.

Proposition 4.1. Let $k$ be a field and $A$ a standard graded Gorenstein $k$-algebra, $f_1, \ldots, f_t$ be forms in $A$ of degrees $d_1 \geq \cdots \geq d_t \geq 2$, $I$ the ideal they generate and $Z := \text{Proj}(A/I)$. Let $J$ be an intersection of primary components of $I$ of codimension $r := \text{codim}(I)$, $S := \text{Proj}(A/J)$ and assume that

1. $Z$ is a complete intersection $F$-rational scheme locally on $S$ outside finitely many points,
2. $S \cap \overline{Z - S} = \emptyset$.

Then,

$$\text{reg}(S) \leq (\dim S + 1)(\text{reg}(A) + d_1 + \cdots d_r - r - 1) + 1.$$ 

Proof. Consider $b$ as given by 3.2. As $\dim(A/(J + b)) \leq 1$ by (1), we may choose elements $\gamma_1, \ldots, \gamma_t \in b \subseteq R^*_A/J$ (with $\sigma = \text{reg}A + d_1 + \cdots + d_r - r$) that forms a sequence of parameters in $A/J$ with $t := \dim A/J - 1$ and $\deg \gamma_i = \sigma$.

By 2.4, $H^i_m(A/J) \mu \subseteq H^i_m(A/J + (\gamma)) \mu$ for $\mu > \sigma + (t - 1)(\sigma - 1)$ and $i \geq 1$. Now $H^1_m(A/J + (\gamma)) = 0$ for $i \geq 2$ and (2) implies that $\text{Proj}(A/J + (\gamma))$ is an isolated component of $\text{Proj}(A/I + (\gamma))$ so that by 4.0, $H^1_m(A/J + (\gamma)) \mu = 0$ for $\mu > \sigma + t(\sigma - 1)$. The claim follows. □

Theorem 4.2. Let $k$ be a field and $A$ a standard graded Gorenstein $k$-algebra, $f_1, \ldots, f_t$ be forms in $A$ of degrees $d_1 \geq \cdots \geq d_t \geq 2$, $I$ the ideal they generate and $Z := \text{Proj}(A/I)$.

Let $J$ be an intersection of primary components of $I$ of codimension $r := \text{codim}(I)$ and $S := \text{Proj}(A/J)$. Assume that $\dim S$ is positive and

1. $S$ is Cohen-Macaulay,
2. $Z$ is a complete intersection locally on $S$ outside finitely many points,
3. $S$ is $F$-rational outside a scheme of dimension at most one,
4. $\dim(Z - S) \leq 0$ or $\dim(Z - S) = 1$ and the one dimensional component of $Z - S$ is geometrically reduced.

Then,

$$\text{reg}(S) \leq \dim S(\text{reg}(A) + d_1 + \cdots d_r - r - 1) + 1.$$
Proof. Consider $\mathfrak{b}$ as given by 3.2. As $\mathfrak{b} \supseteq I$ and $\dim(A/\mathfrak{b}) \leq 2$ by (2), (3) and (4), we may choose elements $\gamma_1, \ldots, \gamma_t \in \mathfrak{b} \subseteq R^\sigma_{A/J}$ with $t := \dim A/J - 2$ and $\deg \gamma_i = \sigma$, that forms a sequence of parameters in $A/J$ and that are such that $\dim(A/(I + (\gamma))) \leq 2$ and the residual of $S' := \text{Proj}(A/J + (\gamma))$ in the top dimensional component of $Z' := \text{Proj}(A/I + (\gamma))$ is geometrically reduced.

By 2.4, $H^1_m(A/J) \mu \subseteq H^1_m(A/J + (\gamma))_\mu$ for $\mu > \sigma + (t - 1)(\sigma - 1)$. Now (1) and (4) implies that $S'$ is a union of isolated components of $Z'$ of maximal dimension whose residual is reduced so that by ([C, Theorem 35] or [CU, 4.7 (a)(ii)]) $H^1_m((A/J + (\gamma))_\mu = 0$ for $\mu > \sigma + t(\sigma - 1)$. The claim follows. \hfill \Box

The previous propositions only concerns the top dimensional component of the scheme defined by the given equations. We will now extend our results to the other isolated components, this time in the polynomial ring $R$.

Remark 4.3. Let $I \subset R$ be a homogeneous ideal, $J$ an intersection of isolated primary components of $I$ of codimension $r$. Set $Z := \text{Proj}(R/I)$ and $S := \text{Proj}(R/J)$. The following conditions are equivalent,

(i) $Z$ is smooth locally in codimension $c$ on $S$,
(ii) $S$ is smooth outside a scheme of codimension $c + 1$, and $p \in \text{Ass}(R/I) - \text{Ass}(R/J) \Rightarrow \text{codim}(R/(p + J)) > c$,
(iii) $\dim(R/I_r(Jac_R(I)) + J) > c$.

Theorem 4.4. Let $I \subset R$ be a homogeneous $R$-ideal generated by forms of degrees $d_1 \geq \cdots \geq d_s$. Consider $B := R/J$, where $J$ is an intersection of isolated primary component of codimension $r$ of $I$. Set $S := \text{Proj}(B)$ and assume that,

(1) $Z := \text{Proj}(R/I)$ have at most isolated singularities on $S$.
(2) $S$ do not meet the other components of $Z$.

Then

$$\text{reg}(S) \leq (\dim S + 1)(d_1 + \cdots + d_r - r - 1) + 1.$$ 

Proof. By 4.0 the proposition is true if $\dim S = 0$. Set $B := R/J$ and assume that $d := \dim B = \dim S + 1 \geq 2$. Note that by Remark 4.3, (1) implies that there exists $\gamma_1, \ldots, \gamma_{d - 1} \in I_r(Jac_R(I)) \subseteq J_B$ that forms a system of parameters. As $I_r(Jac_R(I))$ is generated in degree at most $\sigma := d_1 + \cdots + d_r - r$, we may choose $\gamma_i$ such that $\deg \gamma_i \leq \sigma$. By 3.1 (i) and (iii), $\gamma_i \in \widehat{B}^{-\infty}$. The $\gamma_i$'s forms a regular sequence on the smooth locus of $B$, so that by 2.4

$$H^1_m(B) \subseteq H^1_m(B/(\gamma_1, \ldots, \gamma_{d - i}))$$

for $1 \leq i < d$ and $1 \leq j \leq i$.

Condition (2) implies that the unmixed part of $Z_i := \text{Proj}(B/(\gamma_1, \ldots, \gamma_{d - i}))$ is an isolated component of $\text{Proj}(R/(I + (\gamma_1, \ldots, \gamma_{d - i})))$, therefore the $a$-invariant of $Z_i$ is at most $(d - i)\sigma + d_1 + \cdots + d_r - n - 1$ by [CP, corollaire 2] (note that $\sigma \geq d_1$, unless $d_2 = 1$ where $R/I$ is Cohen-Macaulay and the result obvious). Therefore

$$a_i(B) \leq a(B/(\gamma_1, \ldots, \gamma_{d - i})) \leq (d - i)\sigma + (\sigma + r) - n - 1.$$
So that $a_i(B) + i \leq (d - i + 1)\sigma + i - d$, and

$$\operatorname{reg}(S) = \max_{i \geq 0} \{a_i(B) + i\} \leq d(\sigma - 1) + 1.$$ 

\[ \square \]

**Corollary 4.5.** Let $S \subseteq \mathbb{P}_n$ be a projective equidimensional scheme of dimension $d$ over a field. Assume that $S$ have at most isolated non smooth points and is of local embedding dimension at most $e$ and set $\kappa := \min\{n - d, \max\{e - d, d + 1\}\}$. Then,

$$\operatorname{reg}(S) \leq \kappa(d + 1)(\deg S - 1).$$

If in addition $S$ is Cohen-Macaulay,

$$\operatorname{reg}(S) \leq \kappa d(\deg S - 1).$$

In particular, if $S$ is smooth, $\kappa = \min\{n - d, d + 1\}$ and

$$\operatorname{reg}(S) \leq \kappa d(\deg S - 1).$$

Note that a better result for smooth schemes derives from Kodaira vanishing in characteristic zero ([BM], [BEL] or [CU]). Also note that the (global) embedding dimension of $S$ is at most $\deg S + d - 1$, so that a scheme with isolated non smooth points satisfies $\operatorname{reg}(S) \leq (\dim S + 1)(\deg S - 1)^2$.

**Proof.** As $S$ have isolated singularities of embedding dimension at most $e$, a general linear projection to $\mathbb{P}_m(k)$ is an isomorphism for $m = \kappa + d$. It follows that $\operatorname{reg}(S) \leq \operatorname{reg}(\pi(S))$ (and the only possibility for the inequality to be strict comes from the loss of global sections, reflected in $H^1_m$). Now $\pi(S)$ is scheme defined, up to points, by equations of degrees at most $\deg \pi(S) = \deg S$ (namely by the cones over $\pi(S)$), and the bound follows from 4.1 or 4.4 for the first result, and from 4.2 for the Cohen-Macaulay case. \[ \square \]

**Acknowledgements.** I am very grateful to Patrice Philippon and Bernd Ulrich for many stimulating discussions.
References

[BH] W. Bruns, J. Herzog, *Cohen-Macaulay rings*. Cambridge Stud. in Adv. Math., 39. Cambridge Univ. Press, Cambridge, 1993.

[BM] D. Bayer, D. Mumford, *What can be computed in algebraic geometry? Computational algebraic geometry and commutative algebra* (Cortona, 1991), 1–48. Sympos. Math. XXXIV, Cambridge Univ. Press, Cambridge, 1993.

[BS] D. Bayer, M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. 87 (1987), 1–11.

[BEL] A. Bertram, L. Ein, R. Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties*, J. Amer. Math. Soc. 4 (1991), 587–602.

[C] M. Chardin, *Applications of some properties of the canonical module in computational projective algebraic geometry*, J. Symbolic Comput. 29 (2000), 527–544.

[CP] M. Chardin, P. Philippon, *Régularité et interpolation*, J. Algebraic Geom. 8 (1999), 471–481.

See also the erratum at the address http://www.math.jussieu.fr/~chardin/textes.html.

[CU] M. Chardin, B. Ulrich, *Liaison and Castelnuovo-Mumford regularity*, Amer. J. Math., to appear.

[HH1] M. Hochster, C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. 3 (1990), 31–116.

[HH2] M. Hochster, C. Huneke, *Tight closure in equal characteristic zero*, preprint (1999).

[HH3] M. Hochster, C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147 (2002), 349–369.

[M] C. Miyazaki, *Graded Buchsbaum algebras and Segre products*, Tokyo J. Math. 12 (1989), 1–20.

[MV] C. Miyazaki, W. Vogel, *Bounds on cohomology and Castelnuovo-Mumford regularity*, J. Algebra 185 (1996), 626–642.

[NS1] U. Nagel, P. Schenzel, *Cohomological annihilators and Castelnuovo-Mumford regularity*, Contemp. Math. 159 (1994), 307–328.

[NS2] U. Nagel, P. Schenzel, *Degree bounds for generators of cohomology modules and Castelnuovo-Mumford regularity*, Nagoya Math. J. 152 (1998), 153–174.

[V] W. Vasconcelos, *Computational methods in commutative algebra and algebraic geometry*. Algorithms and Computation in Mathematics, 2. Springer-Verlag, Berlin, 1998.

Marc Chardin, Institut de Mathématiques, CNRS & Université Paris 6, 4, place Jussieu, F–75252 Paris CEDEX 05, France chardin@math.jussieu.fr