DOES EVERY CONTRACTIVE ANALYTIC FUNCTION IN A POLYDISK HAVE A DISSIPATIVE N-DIMENSIONAL SCATTERING REALIZATION?

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Abstract. No.

The title question was posed by D. Kalyuzhnyi-Verbovetskyi [1, Problem 1.3]. Let \( L(H, K) \) denote the set of all bounded linear operators between a pair of Hilbert spaces \( H, K \), and let \( \mathbb{D}^n \) and \( \mathbb{T}^n \) denote the open unit polydisk, and the unit \( n \)-torus, respectively.

Definition 1. An dissipative \( n \)D scattering system is a tuple

\[
\alpha = (n; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})
\]

where:

i) \( n \geq 1 \) is an integer;

ii) \( \mathcal{X}, \mathcal{U}, \mathcal{Y} \) are Hilbert spaces;

iii) \( A, B, C, D \) are \( n \)-tuples of operators (so \( A = (A_1, \ldots, A_n) \), etc.) with

\[
A_k \in L(\mathcal{X}, \mathcal{X}), \quad B_k \in L(\mathcal{U}, \mathcal{X}), \quad C_k \in L(\mathcal{X}, \mathcal{Y}), \quad D_k \in L(\mathcal{U}, \mathcal{Y});
\]

iv) The operator \( \zeta G \in L(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y}) \) is contractive for all \( \zeta \) in the unit \( n \)-torus \( \mathbb{T}^n \), where

\[
\zeta G := \sum_{k=1}^{n} \zeta_k G_k
\]

and the \( G_k \) are the \( 2 \times 2 \) block operators

\[
G_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}
\]

Given such a system, its transfer function is the \( L(\mathcal{U}, \mathcal{Y}) \)-valued function

\[
\theta_\alpha(z) = zD + zC(I_X - zA)^{-1}zB.
\]

defined for all \( z \in \mathbb{D}^n \). It is shown in [3] that the transfer function \( \theta_\alpha \) is a contractive operator function; that is, it is analytic in the unit polydisk \( \mathbb{D}^n \) and satisfies

\[
\| \theta_\alpha(z) \|_{L(\mathcal{U}, \mathcal{Y})} \leq 1
\]

for all \( z \in \mathbb{D}^n \). The question is then whether every contractive operator function in \( \mathbb{D}^n \), vanishing at the origin, is such a transfer function. The answer is known to be “yes” when \( n = 1 \) or \( 2 \), and in fact a stronger result is true: \( G \) can be chosen so that \( \zeta G \) is unitary (that is, the scattering system is conservative). It was also known that when \( n = 3 \), there exist contractive operator functions which do not have conservative realizations; this is due to the failure of von Neumann’s inequality in three variables. (See [1,3] for a discussion.) In this note we show the answer is still “no” in the dissipative case when \( n = 3 \), and give an explicit counterexample (in the scalar case \( \mathcal{U} = \mathcal{Y} = \mathbb{C} \)).

We first show that any polynomial with a dissipative realization must satisfy a restricted form of von Neumann’s inequality. Let \( \mathcal{T} \) denote the set of all \( n \)-tuples of commuting operators \( T = \ldots \)

\[\text{Date: December 21, 2013.}\]
\[\text{Research partially supported by NSF grant DMS 1101134.}\]
(T_1, \ldots, T_n) on Hilbert space satisfying the following condition: whenever X = (X_1, \ldots, X_n) is an n-tuple of operators satisfying

\[ \| \sum_{k=1}^{n} z_k X_k \| \leq 1 \]

for all \( z = (z_1, \ldots, z_n) \in \mathbb{D}^n \), then

\[ \| \sum_{k=1}^{n} T_k \otimes X_k \|_{L(H \otimes K)} \leq 1 \]

where the \( T_k \) act on \( H \) and the \( X_k \) act on \( K \).

It is easy to see that the \( T \) satisfying this condition must be commuting contractions, but when \( n \geq 3 \) it is known that not every \( n \)-tuple of commuting contractions belongs to \( \mathcal{T} \).

**Theorem 2.** If \( p \) is a polynomial which can be realized as the transfer function of a dissipative \( nD \) scattering system, then

\[ \| p(T) \| \leq 1 \]

for all \( T \in \mathcal{T} \).

We will say such \( p \) satisfy the *restricted von Neumann inequality*.

**Proof of Theorem.** Suppose \( p \) is a polynomial vanishing at 0 and \( p = \theta_{\alpha} \) for some \( \alpha \) as in Definition \([\text{??}]\) we work only in the scalar case \( \mathcal{U} = \mathcal{V} = \mathbb{C} \). First note that since analytic functions in the polydisk satisfy a maximum principle relative to \( \mathbb{T}^n \), the dissipativity condition (iv) implies

\[ \| zG \| := \| \sum_{k=1}^{n} z_k G_k \| \leq 1 \]

for all \( z \in \mathbb{D}^n \). Then by definition, if \( T \in \mathcal{T} \), we have

\[ \| \sum_{k=1}^{n} T_k \otimes G_k \| \leq 1. \]

Next, we recall the classical fact that if

\[ F = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \]

is a block operator and \( \| F \| < 1 \), then the linear fractional operator

\[ Z + Y(I - W)^{-1}X \]

is contractive. Now apply this to the block operator

\[ rT \cdot G = \begin{pmatrix} rT \cdot A & rT \cdot B \\ rT \cdot C & rT \cdot D \end{pmatrix} \]

where \( T \cdot A := \sum_{k=1}^{n} T_k \otimes A_k, \) etc., and \( r < 1 \). We conclude that if \( T \in \mathcal{T} \), then the linear fractional operator

\[ rT \cdot D + rT \cdot C(I_{H \otimes X} - rT \cdot A)^{-1}rT \cdot B \]

is contractive for all \( r < 1 \). But it is straightforward to check that, since \( p \) is assumed to be given by the transfer function realization

\[ p(z) = zD + zC(I_X - zA)^{-1}zB, \]

\[ 2 \]
The expression (15) is equal to \( p(rT) \) (This can be done by expanding (16) in a power series, substituting \( rT \) for \( z \), and comparing coefficients with the expansion of (15) in powers of \( rT_1, \ldots rT_n \).) But then \( \|p(rT)\| \leq 1 \) for all \( T \in \mathcal{T} \) and \( r < 1 \), which suffices to establish the theorem. \( \square \)

It follows that any contractive polynomial which fails the restricted von Neumann inequality will fail to have a dissipative realization. In fact, the counterexample to the classical von Neumann inequality produced by Kaijser and Varopoulos is, it turns out, also a counterexample to the restricted inequality, as we now show. The computations are taken from a closely related example considered in [2].

Let \( e_1, \ldots e_5 \) denote the standard basis of \( \mathbb{C}^5 \). Consider the unit vectors

\[
\begin{align*}
v_1 &= \frac{1}{\sqrt{3}}(-e_2 + e_3 + e_4) \\
v_2 &= \frac{1}{\sqrt{3}}(e_2 - e_3 + e_4) \\
v_3 &= \frac{1}{\sqrt{3}}(e_2 + e_3 - e_4)
\end{align*}
\]

The Kaijser-Varopoulos contractions are the commuting \( 5 \times 5 \) matrices \( T_1, T_2, T_3 \) defined by

\[
T_j = e_{j+1} \otimes e_1 + e_5 \otimes v_j
\]

If \( p \) is the polynomial

(17) \[
p(z_1, z_2, z_3) = \frac{1}{5}(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3)
\]

then it is known that \( \sup_{\zeta \in \mathcal{T}} |p(\zeta)| = 1 \) but

(18) \[
\|p(T)\| = \frac{3\sqrt{3}}{5} > 1,
\]

so \( p \) fails the classical von Neumann inequality [5]. To show that this \( p \) fails the restricted von Neumann inequality, we show that already this \( T \) belongs to \( \mathcal{T} \); that is, if \( X_1, X_2, X_3 \) are operators which satisfy

(19) \[
\|z_1X_1 + z_2X_2 + z_3X_3\| \leq 1
\]

for all \( z \in \mathbb{D}^n \), then \( \| \sum_{k=1}^3 T_k \otimes X_k \| \leq 1 \). To see this, we compute and find

(20) \[
T_1 \otimes X_1 + T_2 \otimes X_2 + T_3 \otimes X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & 0 \\ X_2 & 0 & 0 & 0 & 0 \\ X_3 & 0 & 0 & 0 & 0 \\ 0 & Y_1 & Y_2 & Y_3 & 0 \end{pmatrix}
\]

where

\[
Y_1 = \frac{1}{\sqrt{3}}(-X_1 + X_2 + X_3)
\]

\[
Y_2 = \frac{1}{\sqrt{3}}(X_1 - X_2 + X_3)
\]

\[
Y_3 = \frac{1}{\sqrt{3}}(X_1 + X_2 - X_3)
\]
The norm of the matrix (20) is equal to the maximum of the norms of the first column and the last row. By (19), we have \( \| \pm X_1 \pm X_2 \pm X_3 \| \leq 1 \) for all choices of signs, so the last row of (20) has norm at most 1. To say that the first column has norm at most 1 amounts to saying that

\[
I - \sum_{k=1}^{n} X_k^* X_k \geq 0.
\]

This may be seen by averaging: by (19), the matrix valued function

\[
I - \sum_{i,j=1}^{n} \zeta_i \overline{\zeta_j} X_j^* X_i
\]

is positive semidefinite on \( \mathbb{T}^n \). Integrating against normalized Lebesgue measure on \( \mathbb{T}^n \) gives (21).

There is a general principle that transfer function realizations should be equivalent to von Neumann-type inequalities. Some recent, general results in this direction may be found in [2, 4].

References

[1] Vincent D. Blondel and Alexandre Megretski, editors. *Unsolved problems in mathematical systems and control theory*. Princeton University Press, Princeton, NJ, 2004.

[2] Michael T. Jury. Universal commutative operator algebras and transfer function realizations of polynomials. [http://arxiv.org/abs/1009.6219](http://arxiv.org/abs/1009.6219).

[3] Dmitriy S. Kalyuzhniy. Multiparametric dissipative linear stationary dynamical scattering systems: discrete case. *J. Operator Theory*, 43(2):427–460, 2000.

[4] Meghna Mittal and Vern I. Paulsen. Operator algebras of functions. *J. Funct. Anal.*, 258(9):3195–3225, 2010.

[5] N. Th. Varopoulos. On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory. *J. Functional Analysis*, 16:83–100, 1974.

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