Non-isothermal Steady Flow of Non-Newtonian Fluid in an Axisymmetric Channel

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Abstract. Non-isothermal steady-state flow of a viscoplastic fluid in an axisymmetric channel is studied with account for viscous dissipation at a specified constant flow rate. The rheology of the medium is described by the Herschel-Bulkley law with a temperature dependence of yield stress and consistency defined by exponential law. On the solid wall, the no-slip boundary condition and the assigned temperature are used. The mathematical statement of the problem includes the dimensionless motion and energy equations and boundary conditions. The problem is solved numerically using a finite-difference approach. The difference equations are solved by sweep method. When applying a shock-capturing method for calculating the flow, the rheological model is regularized in order to eliminate stress singularity in the regions of zero shear rates. The steady-state distributions of velocity, temperature, viscosity, and dissipative function are obtained. A limiting value of pressure drop defining the existence domain of a steady solution is proved to exist. Two problem solutions are obtained at a specified pressure drop, which are referred to as high- and low-temperature flow regimes. As a result of computations, different flow structures with unyielded regions occurring near symmetry line and in the dead zone next to a solid wall are revealed.

1. Introduction
When processing the polymeric materials and metals in a liquid state, the steady-state flows in the plane and axisymmetric channels are frequently encountered in the technologies for various purposes. Moreover, the fluid medium behavior is characterized by non-Newtonian properties and temperature-dependent rheological parameters. In most cases, a successful solving of the problems of non-isothermal steady-state flows of non-Newtonian media in the channels is possible only by using numerical methods. Since the middle of the last century, the above mentioned flows have been attracting the attention of many researchers. One of the first efforts to consider one-dimensional non-isothermal flows of a viscoplastic fluid has been made in [1,2]. Stability of the steady-state solutions to one-dimensional problems and the associated phenomenon of hydrodynamic thermal explosion for the case of pseudoplastic fluid are discussed in papers [3-5]. In experimental studies [6,7], investigation of the viscoplastic fluid flows in a circular pipe at a specified flow rate is carried out for both laminar and turbulent regimes. The results of numerical studies on non-isothermal viscoplastic fluid flows in a flat channel, circular pipe, and concentric annular duct are presented in [8-12]. In these works, the heat flux on the solid wall is studied in details in terms of the governing parameters, but there is very scant data on the structure of the flow.

The purpose of this work is to study the flow structure occurred in the case of steady-state one-dimensional flow of a viscoplastic fluid with account for viscous dissipation and dependence of the rheological parameters on temperature.
2. Formulation of the Problem

A steady-state non-Newtonian incompressible fluid flow in an axisymmetric channel at a given constant flow rate is considered taking into account the effect of viscous dissipation. The fluid rheology is described by the Herschel-Bulkley law with a temperature dependence of the rheological parameters defined by exponential law. On the solid wall, the no-slip boundary condition is used, and the temperature is assigned as \( T_1 \). The flow is assumed to be one-dimensional and laminar. The solution domain and the coordinate system are presented in figure 1.

The flow is described by the motion and energy equations which are written with due regard to the above mentioned assumptions

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \eta \frac{\partial v}{\partial r} \right) = \frac{\partial p}{\partial x} + \frac{\lambda}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \eta \left( \frac{\partial v}{\partial r} \right)^2 = 0. \tag{1}
\]

Here, \( v \) is the axial velocity, \( T \) is the temperature, \( r \) is the radial coordinate, \( \frac{\partial p}{\partial x} \) is the axial component of the pressure gradient, and \( \lambda \) is the thermal conductivity.

System (1) is enclosed by the Herschel-Bulkley rheological law according to which the apparent viscosity \( \eta \) is defined as

\[
\eta(T, \dot{\gamma}) = \left( \tau_0 e^{\beta_1(T-T_0)} + k_0 \dot{\gamma}_0(T-T_0) \right) / \dot{\gamma}
\]

where \( \dot{\gamma} \) is the shear rate, \( \tau_0 \) is the yield stress at temperature \( T_0 \), \( k_0 \) is the consistency at temperature \( T_0 \), \( \beta_1 \) and \( \beta_2 \) are the constants in the rheological law, and \( n \) is the power-law index. The boundary conditions are assigned as follows:

\[
r = 0: \frac{\partial v}{\partial r} = 0, \frac{\partial T}{\partial r} = 0; \quad r = R: \quad v = 0, T = T_1.
\]

The system of equations (1) is written in terms of the dimensionless parameters as

\[
\frac{1}{\zeta} \frac{\partial}{\partial \zeta} \left( \zeta \eta \frac{\partial u}{\partial \zeta} \right) = Bn \delta p, \quad \frac{1}{\zeta} \frac{\partial}{\partial \zeta} \left( \zeta \frac{\partial \theta}{\partial \zeta} \right) + Br \eta \left( \frac{\partial u}{\partial \zeta} \right)^2 = 0, \tag{2}
\]

where \( \zeta \) is the dimensionless coordinate (\( \zeta = r/R \)), \( u \) and \( \theta \) are the dimensionless velocity and the temperature, respectively, \( Br = \frac{\beta_1 k_0 U^{n+1}}{\lambda L^{n+1}} \) is the Brinkman number, \( Bn = \frac{\tau_0 L^{n+1}}{\lambda U^{n+1}} \) is the Bingham number,
$\delta p$ is the dimensionless pressure drop per unit-length, $k_1$ is the consistency at wall temperature and $\tau_1$ is the yield stress at wall temperature. The typical scales for velocity and pressure are the average velocity $U$ and $\tau_1$, respectively. The dimensionless temperature is referred to as $\theta = \beta_2 (T - T_i)$. The dimensionless apparent viscosity is defined by formula

$$\eta = \left( Bn e^{-\beta_0} + e^{-\theta} \frac{du}{d\zeta} \right) \left[ \frac{du}{d\zeta} \right]^{-1}.$$  \hspace{1cm} (3)

The boundary conditions are written as

$$\zeta = 0: \frac{du}{d\zeta} = 0, \frac{d\theta}{d\zeta} = 0;$$

$$\zeta = 1: u = 0, \theta = 0.$$ \hspace{1cm} (4)

Here, $\beta = \beta_1 / \beta_2$ is the dimensionless parameter. The value $\delta p$ is taken in accordance with the equality of volume flow rate per unit-area to unity

$$2 \int_0^1 u \zeta d\zeta = 1.$$ \hspace{1cm} (5)

Consequently, the problem solving is reduced to the obtaining of velocity and temperature profiles which satisfy the system of equations (2),(3) and conditions (4),(5).

3. Method of Solving

The formulated problem is solved numerically. The solution domain is covered by a computational grid presented in figure 2. The velocity is calculated in the grid nodes denoted by integer indices, the viscosity and temperature, in the grid nodes denoted by fractional indices.

![Figure 2. Computational grid.](image)

The finite-difference analogs for system (2) are written in compliance with employment of a sweep method [13]

$$\zeta_{i+\frac{1}{2}, \frac{1}{2}} \eta_{i+\frac{1}{2}, \frac{1}{2}} u_{i+1} - \left( \zeta_{i+\frac{1}{2}, \frac{1}{2}} \eta_{i+\frac{1}{2}, \frac{1}{2}} + \zeta_{i+\frac{1}{2}, \frac{1}{2}} \eta_{i+\frac{1}{2}, \frac{1}{2}} \right) u_{i+1} + \zeta_{i+\frac{3}{2}, \frac{1}{2}} \eta_{i+\frac{3}{2}, \frac{1}{2}} u_{i+1} = Bn \delta p h^2 \zeta_i,$$

$$\zeta_{i+1} \eta_{i+1} - \left( \zeta_{i+1} + \zeta_i \right) \eta_{i+1} + \zeta_{i+1} \eta_{i+1} + \omega \eta_{i+1} \left( u_{i+1} - u_i \right)^2 = 0.$$ \hspace{1cm} (6)

The boundary conditions (4) in terms of the finite-differences are presented as follows:

$$u_0 = u_1 = \frac{Bn \delta p h^2}{4 \eta_{i+1}}, \eta_{i+1} = \eta_{0.5};$$

$$u_N = 0, \eta_{N+0.5} = -\eta_{N-0.5}.$$ \hspace{1cm} (7)

The Herschel-Bulkley model has a feature of “infinite” apparent viscosity in the regions where the stress level is less than yield stress. When using a shock-capturing method for calculating viscoplastic medium flows without explicit separating of the unyielded regions, a regularization of the rheological model is implemented to ensure the stability and accuracy of the calculations in the regions of low strain rate. In this work, a modified rheological model [14] is utilized which is characterized by apparent viscosity defined by formula

$$\eta = ...$$
\[ \eta = Bn e^{-\rho \theta} \left[ 1 - \exp \left( \frac{dU}{d\zeta} e^{-\theta} \right) \right] \left| \frac{dU}{d\zeta} \right|^{-1} + e^{\theta} \left| \frac{dU}{d\zeta} \right|^{n-1}, \]

where \( \varepsilon \) is the small parameter. In the region of shear flow, the viscosity values calculated using the initial and regularized models are almost coincide with each other; in the unyielded region, regularization provides a large but final viscosity value. The separation of zones with stress level less than yield stress is performed by means of the following condition:

\[ \eta \left| \frac{dU}{d\zeta} \right| < Bn e^{-\rho \theta}. \]

Difference analogues (6), (7) approximate differential equations (2) and conditions (4) with the second order of accuracy. The resulting systems of algebraic equations (6) are solved by sweep method. A simultaneous solving of the systems of difference equations (6) intended to obtain the steady-state fields of velocity and temperature requires an iterative process to be organized. A zero temperature is used as initial approximation for iterative process.

The value of parameter \( \delta \rho \), which ensures the fulfillment of condition (5) with a desired accuracy, is selected by bisection method [15]. When calculating the steady-state fields of velocity and temperature, the following conditions are used as convergence criteria

\[ \max_i \left| U^{k+1} - U^k \right| < \varepsilon_u, \quad \max_i \left| \theta^{k+1} - \theta^k \right| < \varepsilon_{\theta}, \quad 0 \leq i \leq N - 1, \]

where the superscript number corresponds to the iteration index, and the subscript number, to the computational node index.

The convergence of the algorithm for calculating \( \delta \rho \) is verified in accordance with condition

\[ \left| 1 - 2\sum_{i=1}^{N} \frac{U^i \zeta_{i+1} + U^i \zeta_i}{2} \right| < \varepsilon_u. \]

Testing the computational method, a benchmark problem of isothermal steady-state flow of a viscoplastic Herschel-Bulkley fluid in a circular pipe is solved. The corresponding analytical solution is written as [6]

\[ u = \begin{cases} u_0, & 0 \leq \zeta \leq \zeta_0; \\ u_0 \left[ 1 - \left( \frac{\zeta - \zeta_0}{1 - \zeta_0} \right)^{\frac{n}{n+1}} \right], & \zeta_0 < \zeta \leq 1. \end{cases} \]

Here, \( \zeta_0 = -2/\delta \rho \) is the radial coordinate of unyielded region boundary, \( u_0 = \left( -\frac{\delta \rho Bn}{2} \right) \frac{1}{n} \left[ \frac{n}{n+1} (1 - \zeta_0)^{\frac{n}{n+1}} \right] \) is the velocity value in the unyielded region. The pressure drop value ensuring the flow rate equal to unity is determined by equation

\[ \left( 1 - \zeta_0 \right)^{\frac{3n+1}{n}} - n \left( 1 - \zeta_0 \right)^{\frac{2n+1}{n}} + \frac{3n^2 + 5n + 1}{2n^2} \left( 1 - \zeta_0 \right)^{\frac{n+1}{n}} = 6n^3 + 11n^2 + 6n + 1 \left( 1 - \zeta_0 \right)^{\frac{1}{n}}. \]

| Table 1. The values of \( u_0, \delta \rho, \) and \( \zeta_0 \) versus grid step \( h \) |
|---|---|---|---|---|---|
| \( h \) | 1/10 | 1/20 | 1/50 | 1/100 | 1/200 |
| \( u_0 \) | 1.4562 | 1.4434 | 1.4399 | 1.4394 | 1.4393 | 1.4389 |
| \( -\delta \rho \) | 6.9629 | 6.9360 | 6.9285 | 6.9274 | 6.9272 | 6.9271 |
Table 2. The values of $u_0$, $\delta p$, and $\zeta_0$ versus regularization parameter

| $\varepsilon$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | Analytical solution |
|---------------|-----------|-----------|-----------|-----------|-----------|--------------------|
| $u_0$         | 1.3713    | 1.3415    | 1.3384    | 1.3381    | 1.3380    | 1.3379             |
| $-\delta p$   | 4.6453    | 4.6526    | 4.6532    | 4.6532    | 4.6532    | 4.6532             |
| $\zeta_0$     | 0.4305    | 0.4299    | 0.4298    | 0.4218    | 0.4123    | 0.4298             |

Table 3. The values of $u_0$, $\theta_b$, $\delta p$, and $\zeta_0$ versus grid step $h$

| $h$          | $1/10$ | $1/20$ | $1/50$ | $1/100$ | $1/200$ |
|--------------|--------|--------|--------|---------|---------|
| $u_0$        | 2.5484 | 2.5407 | 2.5384 | 2.5381  | 2.5380  |
| $\theta_b$   | 1.3863 | 1.3892 | 1.3900 | 1.3901  | 1.3901  |
| $-\delta p$  | 3.4395 | 3.4500 | 3.4528 | 3.4533  | 3.4534  |
| $\zeta_0$    | 0.1454 | 0.1445 | 0.1443 | 0.1442  | 0.1442  |

The approximation convergence of the computational algorithm developed for non-isothermal case is shown in table 3 where $\theta_b$ is the temperature on the symmetry axis.

4. Results and Discussion

A non-isothermal case with account for dependence of the rheological parameters on temperature is considered. A dependency diagram of the dimensionless pressure drop versus Brinkman number is shown in figure 3, while the Bingham number is obtained by formula

$$\text{BrBn}^2 = C.$$

In accordance with this equality, an increase in Br may be interpreted as an increase in the dimensional flow rate under otherwise equal conditions. Variation in the constant $C$ on the right side of the formula leads to a change in the quantitative characteristics of the curve in figure 3. The plot presented in figure 3 demonstrates the relation between pressure drop and average flow rate at $C = 25$. 
Figure 3. Dependence of $|\delta p|$ on $Br$ ($n = 1$ and $\beta = 1$).

It is notable that the value of $\delta p$ is limited over the whole range of flow rate variation. If the flow with a specified pressure drop is considered, there is no steady-state solution at $|\delta p| > 2.497$ which means that the heat generated due to viscous dissipation is not able to be removed from the pipe through the walls in time, and a phenomenon, which is referred to as a hydrodynamic thermal explosion (an analogue of the thermal explosion), occurs [16,17].

In the region of $|\delta p| < 2.497$, two regimes are possible for a specified pressure drop: low- and high-temperature regimes at a flow rate equal to unity. For example, the case when the governing parameters take the values corresponding to points 1 and 2 in figure 3. For these two regimes, the velocity and temperature profiles are presented in figures 4(a) and 4(b), respectively. The flow region can be separated into unyielded regions near the axis (painted in gray) and a shear flow in the rest part. The temperature and velocity both have constant values in the unyielded regions.

Figure 4. Velocity (solid line) and temperature (dashed line) profiles at $n = 1$, $\beta = 1$: (a) $Br = 0.003$, $Bn = 91.3$ and (b) $Br = 0.2$, $Bn = 11.2$.

With an increase in $Br$ (starting with point B, figure 3), a high-temperature flow regime with an unyielded region near the axis and a dead zone near the wall is observed. The profiles of unknown variables corresponding to $Br$ and $Bn$ numbers at point 3 (figure 3) are shown in figure 5. The fluid is at rest, and the temperature decreases linearly in the wall-adjacent zone. The dissipative function (figure 5(b), solid line) is equal to zero both in the unyielded region and in the wall-adjacent zone. The viscosity
behavior in the shear flow region is demonstrated in figure 5(b) (dashed line). The pressure drop at point A (figure 3) corresponds to a minimum value at which the flow is possible, i.e., the stress level is higher than the yield stress.

Figure 5. Profiles of the (a) velocity and temperature and (b) dissipative function and viscosity at \( n = 1, \beta = 1, \text{Br} = 1, \text{and Bn} = 5. \)

Figure 6 illustrates the plots of the coordinates of unyielded region boundaries and stagnant zone versus Bn number for three values of Br parameter. Dissipative effects are weak at Br = 0.1, and the temperature has little impact on the viscosity. As a result, the influence of the Bn number in the considered range is similar to that in the isothermal case: with an increase in Bn number, the central unyielded region expands, and the stagnant zone does not occur. Increase in Br up to 0.5 leads to the qualitative and quantitative changes in the flow pattern. On the one hand, large Bn numbers entail high viscous stresses in the shear flow region and, as a consequence, high dissipative function values are achieved. On the other hand, dissipation leads to an increase in temperature and to a decrease in viscosity. As a result, the flow regime with central unyielded region and dead zone occurs in the pipe at \( \text{Bn} > 6.65. \) At the same time, with an increase in Bn, the size of unyielded region decreases, while the dead zone increases. The flow regime with a dead zone is observed for Br = 2.5 at Bn > 2.

Figure 6. Coordinates of unyielded flow regions (\( n = 1 \) and \( \beta = 1. \))
Figure 7. Coordinates of unyielded flow regions (Br = 1: (a) n = 1 and (b) β = 1).

The effect of parameter β on the flow structure is shown in figure 7(a). The flow is characterized by the presence of unyielded region in the central part at β = 0. Increasing this parameter, the size of the unyielded region decreases, and the regime with both unyielded region and dead zone is observed. The central unyielded region does not occur at Bn = 5 and β > 2.7, and only a dead zone is distinguished in the flow. Investigation of the flow structure as a function of the power-law index n is presented in figure 7(b). In the considered range of variation in n, which covers dilatant and pseudoplastic regions, it is possible to obtain the above mentioned regimes depending on the governing parameters.

5. Conclusion
The mathematical statement of the problem of a steady-state non-isothermal flow of a viscoplastic Herschel-Bulkley fluid in a circular pipe is formulated in a one-dimensional approximation with account for viscous dissipation and dependence of the yield stress and consistency of the medium on temperature. The numerical algorithm based on the finite-difference method is developed to solve a stated problem. A set of test calculations is carried out. The kinematic and thermophysical characteristics of the flow are presented. The flow structure is studied in terms of the governing parameters. The flow regimes characterized by the presence of unyielded region in the vicinity of symmetry axis and a dead zone are revealed.

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