Comment on “What does the Letelier-Gal’tsov metric describe?”

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We show that the Letelier-Gal’tsov (LG) metric describing multiple crossed strings in relative motion \cite{1} does solve the Einstein equations, in spite of the discontinuity uncovered recently by Krasnikov \cite{2}, provided the strings are straight and moving with constant velocities.

PACS numbers:

In a recent note \cite{2}, Krasnikov raised an interesting question about the continuity of the LG metric. He showed, on the example of a (parallel) two-cosmic string LG metric, that the metric component $g_{ty}$ (where $y$ is the coordinate orthogonal to strings and their relative displacement) is generically discontinuous across the cuts associated with the positions of strings. Arguing that the metric must be continuous, he concluded that the strings should be at rest with respect to each other.

Before proceeding with a more elaborate analysis, we note it would be too strong to require the metric to be everywhere continuous. To define a spacetime, one needs a set of overlapping, diffeomorphic charts. We will show that, provided the string motion is geodesic and the strings do not collide, the Riemann tensor computed from the discontinuous metric vanishes outside the strings. It follows that there is a set of overlapping charts such that the metric is continuous in each.

Let us first briefly reformulate Krasnikov’s argument. For simplicity, we shall consider here only the case of moving parallel strings, which can be reduced to that of moving conical singularities in 2+1 dimensions. The reduced LG metric can be written in the ADM form,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

with

$$N = 1, \quad h_{\zeta\zeta} = \frac{1}{2} Z_{,\zeta} Z_{,\zeta}, \quad N_\zeta = h_{\zeta\zeta} N_{\bar{\zeta}} = \frac{1}{2} Z_{,\zeta} Z_{,\bar{\zeta}}$$

($\zeta = x + iy$), where, in the case of two strings,

$$Z(t, \zeta) = \int_{\zeta_0}^\zeta \psi(t, \xi) d\xi, \quad \psi(t, \xi) = (\xi - \alpha_1(t))^{\mu_1} (\xi - \alpha_2(t))^{\mu_2}$$

($\mu_1 > -1$). This is defined only in the cut complex plane with two line cuts extending from the two conical singularities to infinity. We choose as the first cut the horizontal half-axis $|\alpha_1 + \infty|$. The second cut, starting in $\alpha_2$, will be assumed not to intersect the first cut (in this choice we differ from \cite{2}). Following \cite{2}, we also choose a gauge such that $\alpha_1(t) = 0$, and $\alpha_2(t)$ and $\zeta_0$ real, with $\alpha_2 < \zeta_0 < 0$.

We choose $\zeta$ to be on the first cut, and wish to evaluate at a given time $t$ the discontinuity of

$$N_\zeta = \int_{\zeta_0}^\zeta \frac{\psi(\xi)}{Z_{,\zeta} Z_{,\bar{\zeta}}} d\xi.$$

First,

$$N_\zeta(\zeta \pm i0) = \frac{1}{2} \psi(\zeta) e^{i \pi (1 + 1)} Z_{,\zeta}(\zeta \pm i0),$$

with

$$Z_{,\zeta}(\zeta \pm i0) = -\mu_2 \alpha_2 \int_{\zeta_0}^\zeta \frac{\psi(\xi)}{\xi - \alpha_2} d\xi.$$
Choosing \( \Gamma_+ (\Gamma_-) \) to go first along an arbitrary path from \( \zeta_0 \) (a fixed point anywhere in the \( \zeta \) plane) to \( \alpha_1 = 0 \), then (after a small upper (lower) half-circle around the origin) to follow the upper (lower) bank \( \gamma_+ (\gamma_-) \) of the cut from 0 to \( \zeta \), we obtain

\[
N_\zeta (\zeta \pm i0) = \frac{1}{2} \left( \overline{\chi (\zeta)} \psi (\zeta) + Ve^{\mp i\pi \mu_1} \psi (\zeta) \right),
\]

where

\[
V(t) = e^{-i\pi \mu_1} \dot{Z}(0)
\]

is the velocity of the first string (real in our gauge), and

\[
\chi (\zeta) = -\dot{\alpha}_2 \int_0^\zeta \frac{\psi (\xi)}{\xi - \alpha_2} d\xi.
\]

Extending (10) to \( y = \text{Im}(\zeta) \neq 0 \), we obtain, in the vicinity of the cut,

\[
N_\zeta (\zeta) = \frac{1}{2} \left( \overline{\chi (\zeta)} \psi (\zeta) + Ve^{-i\pi \mu_1} \psi (\zeta) \right) = \frac{1}{2} \left( \overline{\chi (\zeta)} + V \cos (\pi \mu_1) \right) \psi (\zeta) - \frac{i}{2} \beta \psi (\zeta) \epsilon (y),
\]

where \( \epsilon (y) \) is the sign function, and \( \beta = V \sin (\pi \mu_1) \). So, while the lapse \( N \) and the two-metric \( h_{ij} \) are by construction single-valued and thus continuous, the shift \( N_\zeta \) is not continuous across the cut, except in the static case \( V(t) = 0 \). A priori this discontinuity will lead to delta and gradient of delta contributions to the Einstein tensor. We shall see that the gradient of delta contribution vanishes identically. However the delta contribution remains, so that generically there must be a matter source along the cut. We shall now show that this delta contribution vanishes, iff the motion is geodesic, \( \beta (t) = 0 \). The Riemann tensor (completely determined in 2+1 dimensions by the Einstein tensor) then vanishes outside the point sources \( \zeta = \alpha_i (t) \).

For this purpose we use the (1 + 2)-dimensional vacuum Einstein equations written in the ADM form

\[
\mathcal{H} \equiv -h^{1/2} R(h) + h^{-1/2} (\pi^{ij} \pi_{ij} - \pi^2) = 0,
\]

\[
\mathcal{H}' \equiv -2\pi^{ij,j} = 0
\]

(constraint equations), and

\[
\partial_t h_{ij} = 2h^{-1/2} (\pi_{ij} - h_{ij} \pi) + N_{ij,j} + N_{ij},
\]

\[
\partial_t \pi^{ij} = \frac{1}{2} h^{-1/2} \left( 2h^{ij} (\pi^{kl} \pi_{kl} - \pi^2) - 4 (\pi^{ik} \pi_{jk} - \pi^{ij} \pi) \right)
\]

\[
+ (\pi^{ij} N^k)_{,k} - N^i_{,k} \pi^{kj} - N^j_{,k} \pi^{ki}
\]

(evolution equations), where \( h_{ij} \) is the two-dimensional metric, \( N = 1 \), \( \pi = \pi^k k \), and \( \pi^{ij} \) are the momenta conjugate to the \( h_{ij} \), related to the extrinsic curvature by

\[
\pi_{ij} = h^{1/2} (Kh_{ij} - K_{ij}).
\]

Using the result \([11]\) and the non-vanishing Christoffel symbols \( \Gamma^\zeta_{\zeta \zeta} = \psi \zeta / \psi \) (together with its complex conjugate), we obtain in terms of the real spatial coordinates \( x, y \)

\[
\pi^{xx} = \beta \psi^{-1} (x) \delta (y), \quad \pi^{xy} = 0, \quad \pi^{yy} = 0.
\]

The component \( \pi^{xx} \) vanishes only in the static case \( \beta = 0 \), reflecting the fact that in this case the two-dimensional metric \( \xi \) is flat outside the conical singularities \( \zeta = \alpha_i \).

Inserting this result into the constraint equations, we find that the Hamiltonian constraint \([11]\) is identically satisfied (using the fact that \( R(h) = 0 \) for the metric \( \xi \)), while the momentum constraint \([12]\) is satisfied on account of \( \partial_y (|\psi|^2) |_{y=0} = 0 \). The evolution equations \([13]\) have already been used to compute the momenta \( \pi^{ij} \). There remain only the evolution equations \([14]\), which may be rewritten

\[
\partial_t \pi^{ij} = (\pi^{ij} N^k)_{,k} - N^i_{,k} \pi^{kj} - N^j_{,k} \pi^{ki}.
\]
Both sides of the \((yy)\) equation vanish identically. The \((xy)\) equation is satisfied on account of \(\partial_x N_y|_{y=0} = 0\). There only remains the potentially dangerous \((xx)\) equation. For its left-hand side we obtain

\[
\partial_t \pi^{xx} = \left( \dot{\beta} + \frac{\beta \mu_2 \dot{\alpha}_2}{x - \alpha_2} \right) \psi^{-1}(x) \delta(y).
\]  

(18)

For the right-hand side, using

\[
\partial_x N^x = -\frac{\mu_2 \dot{\alpha}_2}{x - \alpha_2} - \psi^{-1} \psi_x N^x \quad \text{for} \quad y = 0,
\]  

(19)

we find

\[
N^x \partial_x \pi^{xx} - \pi^{xx} \partial_x N^x = \frac{\beta \mu_2 \dot{\alpha}_2}{x - \alpha_2} \psi^{-1}(x) \delta(y).
\]  

(20)

Comparison of (18) and (20) shows that the \((xx)\) component of the evolution equation (17) is satisfied iff

\[
\dot{\beta} = 0.
\]  

(21)

The argument can be generalized to show that time derivatives

\[
\dot{Z}[t, z, \zeta = \alpha_i(t, z)] = Z_{,t}(\alpha_i) + Z_{,\zeta}(\alpha_i) \dot{\alpha}_i = \text{const.}
\]  

(22)

for each string, i.e. the strings must move with constant velocities. The extension to the case of non-parallel strings is straightforward and leads to the conclusion that the derivatives with respect to \(z\) must satisfy

\[
Z'[t, z, \zeta = \alpha_i(t, z)] = \text{const.},
\]  

(23)

meaning that the strings are straight. Therefore the complex trajectories \(Z[\alpha_i]\) must be linear functions of \(t\) and \(z\), or, in geometric terms, the world-sheets of the strings must be totally geodesic submanifolds. We have shown here that this well-known property of self-gravitating cosmic strings [3, 4] is the necessary and sufficient condition for the LG metric to represent a system of crossed straight cosmic strings moving in otherwise empty spacetime.

D.G. thanks LAPTH (Annecy) for hospitality while this note was being written. His work was also supported in part by the RFBR grant 02-04-16949.

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