CHARACTERIZATION OF $n$-DIMENSIONAL NORMAL AFFINE SL$_n$-VARIETIES

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Abstract. We show that any normal irreducible affine $n$-dimensional SL$_n$-variety $X$ is determined by its automorphism group seen as an ind-group in the category of normal irreducible affine varieties. In other words, if $Y$ is an irreducible affine normal algebraic variety such that $\text{Aut}(Y) \cong \text{Aut}(X)$ as an ind-group, then $Y \cong X$ as a variety. If we drop the condition of normality on $Y$, then this statement fails. In case $n \geq 3$, the result above holds true if we replace $\text{Aut}(X)$ by $\mathcal{U}(X)$, where $\mathcal{U}(X)$ is the subgroup of $\text{Aut}(X)$ generated by all one-dimensional unipotent subgroups. In dimension 2 we have some interesting exceptions.

1. Introduction and main results

Our base field is the field of complex numbers $\mathbb{C}$. For an affine variety $X$, the automorphism group $\text{Aut}(X)$ has the structure of an ind-group. We will shortly recall the basic definitions and results in Section 2. The classical example is $\text{Aut}(\mathbb{A}^n)$, $n > 1$, the group of automorphisms of the affine $n$-space $\mathbb{A}^n$. Recently, Hanspeter Kraft proved the following result which shows that the affine $n$-space is determined by its automorphism group (see [Kr15]).

**Theorem 1.1.** Let $Y$ be a connected affine variety. If $\text{Aut}(Y) \cong \text{Aut}(\mathbb{A}^n)$ as ind-groups, then $Y \cong \mathbb{A}^n$ as varieties.

Note that this result was generalised in [CRX19] (see also [KRvS19] and in a similar spirit, see [LRU20, Thm. 1]) where the authors proved Theorem 1.1 under a weaker condition: namely, that groups $\text{Aut}(Y)$ and $\text{Aut}(\mathbb{A}^n)$ are isomorphic only as abstract groups. Moreover, recently Theorem 1.1 was generalized in [LRU18, Thm. 1.4] (see also [RvS21, Main Thm. 1]) where it was proved that an affine toric variety different from the algebraic torus is determined by its automorphism group seen as an ind-group in the category of normal affine irreducible varieties. If we drop the normality condition in [LRU18, Thm. 1.4], the situation changes. In this paper, we show that for “most” $n$-dimensional affine normal varieties $X$ endowed with a nontrivial regular SL$_n = \text{SL}_n(\mathbb{C})$-action, there are infinitely many

DOI: 10.1007/s00031-022-09701-3

*Supported by SNF (Schweizerischer Nationalfonds), project number P2BSP2_165390.

Received September 8, 2018. Accepted October 28, 2021.

Published online March 1, 2022.

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affine varieties $Y$ such that $\text{Aut}(Y) \simeq \text{Aut}(X)$ as an ind-group and we classify all such $Y$.

Let $d > 1$. Consider the action of $\mu_d = \{ \xi \in \mathbb{C}^* \mid \xi^d = 1 \}$ on $\mathbb{A}^n$ by scalar multiplication and denote by $A_{d,n}$ the quotient of $\mathbb{A}^n$ by $\mu_d$. Note that $A_{d,n}$ is normal. Denote also by $\pi : \mathbb{A}^n \to A_{d,n}$ the quotient map. This means that $A_{d,n}$ is an affine variety with coordinate ring

$$\mathcal{O}(A_{d,n}) = \mathbb{C}[x_1, \ldots, x_n]^{\mu_d} = \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{dk},$$

the algebra of invariants where $\mathbb{C}[x_1, \ldots, x_n]_{dk}$ denotes the homogeneous polynomials of degree $dk$. Note that $A_{d,n}$ is indeed an orbit space, because $\mu_d$ is finite. For $d > 1$, $A_{d,n}$ has an isolated singularity in $\mathbb{A}^n \setminus \{0\}$ and induces an étale covering $\mathbb{A}^n \setminus \{p(0)\}$ with Galois group $\mu_d$.

**Remark 1.2.** We will see in Lemma 6.1 and Proposition 5.4 that any affine normal variety endowed with a regular nontrivial $\text{SL}_n$-action is isomorphic to either $\text{SL}_2/T$, $\text{SL}_2/N$ or to $A_{d,n}$ for some $d \in \mathbb{N}$, where $T \subset \text{SL}_2$ is the standard subtorus and $N \subset \text{SL}_2$ is the normalizer of $T$. This implies that Theorem 1.3 and Theorem 1.4 below indeed provide the characterization of $n$-dimensional normal affine $\text{SL}_n$-varieties.

Consider the affine variety $A^s_{d,n}$ with coordinate ring

$$\mathcal{O}(A^s_{d,n}) = \mathbb{C} \oplus \bigoplus_{k=s}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{dk} \subset \mathcal{O}(A_{d,n}), \ s \geq 1.$$

Then the induced morphism $\eta : A_{d,n} \to A^s_{d,n}$ is the normalization and has the property that the induced map $\eta' : A_{d,n} \setminus \{\star\} \xrightarrow{\sim} A^s_{d,n} \setminus \{\star\}$ is an isomorphism, where $\star$ denotes the points corresponding to the homogeneous maximal ideals. In fact, $\eta$ is $\text{SL}_n$-equivariant, and $A_{d,n} \setminus \{\star\}$ is an $\text{SL}_n$-orbit. We prove the following result.

**Theorem 1.3.** Let $X$ be an irreducible affine variety. Then $\text{Aut}(X)$ and $\text{Aut}(A_{d,n})$ are isomorphic as ind-groups if and only if $X \simeq A^s_{d,n}$ as a variety for some $s \in \mathbb{N}$.

Theorem 1.3 and the following result shows that $\text{SL}_2/T$ and $\text{SL}_2/N$ are the only affine $n$-dimensional $\text{SL}_n$-varieties (except $\mathbb{A}^n$) that are determined by their automorphism groups in the category of affine irreducible varieties.

**Theorem 1.4.** Let $X$ be an irreducible variety such that $\text{Aut}(X) \simeq \text{Aut}(\text{SL}_2/T)$ respectively $\text{Aut}(X) \simeq \text{Aut}(\text{SL}_2/N)$ as ind-groups. Then $X \simeq \text{SL}_2/T$ respectively $X \simeq \text{SL}_2/N$ as varieties.

For an affine variety $X$ we denote by $U(X) \subset \text{Aut}(X)$ the subgroup generated by the one-dimensional unipotent subgroups. We do not know whether $U(X)$ has the structure of an ind-subgroup (i.e., whether $U(X) \subset \text{Aut}(X)$ is closed). That is why we introduce the definition of an algebraic isomorphism. This is an isomorphism $\phi : U(X) \xrightarrow{\sim} U(Y)$ such that for any subgroup $U \subset U(X)$, where $U$ is a closed one-dimensional unipotent subgroup of $\text{Aut}(X)$, the image $\phi(U) \subset \text{Aut}(Y)$ is a closed one-dimensional unipotent subgroup and $\phi|_U : U \xrightarrow{\sim} \phi(U)$ is an isomorphism of algebraic groups.
Theorem 1.5. Let $X$ be $A_{d,n}$, $SL_2/T$ or $SL_2/N$ and $Y$ be an irreducible affine variety. Assume that there is an algebraic isomorphism $\mathcal{U}(X) \cong \mathcal{U}(Y)$. Then

(a) if $X \simeq A_{2,2}$, then $Y \simeq SL_2/T$ or $Y \simeq A_{s,2}^s$ for some $s \in \mathbb{N}$,
(b) if $X \simeq SL_2/T$, then $Y \simeq SL_2/T$ or $Y \simeq A_{s,2}^s$ for some $s \in \mathbb{N}$,
(c) if $X \simeq A_{4,2}$, then $Y \simeq SL_2/N$ or $Y \simeq A_{4,2}^s$ for some $s \in \mathbb{N}$,
(d) if $X \simeq SL_2/N$, then $Y \simeq SL_2/N$ or $Y \simeq A_{4,2}^s$ for some $s \in \mathbb{N}$,
(e) if $X = A_{d,n}$, where $(d,n) \not\in \{(2,2),(2,4)\}$, then $Y \simeq A_{d,n}^s$ for some $s \geq 1$.

Acknowledgement. The author thanks both referees for important remarks and suggested improvements. The author would also like to thank Hanspeter Kraft for his support while writing this paper. In particular, Proposition 6.21 was pointed out to him by Hanspeter Kraft. Finally, the author is grateful to Michel Brion who suggested a number of improvements and Mikhail Zaidenberg for useful discussions.

2. Preliminaries

2.1. Ind-groups

The notion of an ind-group goes back to Shafarevich who called such objects *infinite dimensional groups* (see [Sh66]). We refer to [Kum02] and [FK18] for basic notions in this context.

Definition 2.1. By an *ind-variety* we mean a set $V$ together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ such that the following holds:

(a) $V = \bigcup_{k \in \mathbb{N}} V_k$,
(b) each $V_k$ has the structure of an affine algebraic variety,
(c) for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

A morphism from an ind-variety $V = \bigcup_{k \in \mathbb{N}} V_k$ to an ind-variety $W = \bigcup_{m \in \mathbb{N}} W_m$ is a map $\phi: V \to W$ such that for any $k$ there is an $m$ such that $\phi(V_k) \subset W_m$ and such that the induced map $V_k \to W_m$ is a morphism of algebraic varieties. *Isomorphisms* of ind-varieties are defined in the obvious way.

Two filtrations $V = \bigcup_{k \in \mathbb{N}} V_k$ and $V = \bigcup_{k \in \mathbb{N}} V'_k$ are called *equivalent* if for every $k$ there is an $m$ such that $V_k \subset V'_m$ is a closed subvariety as well as $V'_k \subset V_m$.

An ind-variety $V$ has a natural topology: a subset $S \subset V$ is open, (resp. closed), if $S_k = S \cap V_k \subset V_k$ is open, (resp. closed), for all $k$. A locally closed subset $S \subset V$ has the induced structure of an ind-variety. It is called an *ind-subvariety*. A subset $S \subset V$ that is a closed subset of some $V_k$ is called an *algebraic subset*.

The product of two ind-varieties is defined in the usual way. This allows to give the following definition.

Definition 2.2. An ind-variety $G$ is said to be an *ind-group* if the underlying set $G$ is a group such that the map $G \times G \to G$, $(g,h) \mapsto gh^{-1}$, is a morphism.

An ind-group $G$ is called *connected* if for every $g \in G$ there is an irreducible curve $C$ and a morphism $C \to G$ whose image contains the neutral element $e$ and $g$.

A closed subgroup $H$ of $G$ (i.e., $H$ is a subgroup of $G$ and is a closed subset) is again an ind-group under the closed ind-subvariety structure on $G$. A closed
subgroup $H$ of an ind-group $G$ is called an \textit{algebraic subgroup} if and only if $H$ is an algebraic subset of $G$.

**Proposition 2.3** ([FK18, Thm. 0.3.1]). Let $X$ be an affine variety. Then $\text{Aut}(X)$ has the structure of an ind-group such that for any algebraic group $G$, there is a correspondence between regular $G$-actions on $X$ and ind-group homomorphisms $G \to \text{Aut}(X)$.

If $G$ is an algebraic group acting regularly and faithfully on $X$, then by Proposition 2.3, we can consider $G$ as an algebraic subgroup of $\text{Aut}(X)$. We will often switch between these two points of view.

**2.2. Locally nilpotent derivations and $\mathbb{G}_a$-actions**

Additive group actions on affine varieties can be described by a certain kind of derivations. We recall some of the basics here (see [Fre06] for details). Let $\lambda: \mathbb{G}_a \to \text{Aut}(X)$ be a $\mathbb{G}_a$-action on an affine variety $X$. Such an action induces a derivation on the level of regular functions $\mathcal{O}(X)$ by

$$\delta_\lambda: \mathcal{O}(X) \to \mathcal{O}(X), \quad f \mapsto \left[ \frac{d}{ds} \lambda(s)^*(f) \right]_{s=0},$$

where $\mathbb{G}_a = \text{Spec}(\mathbb{C}[s])$. We have that for every $f \in \mathcal{O}(X)$ there exists an $k \in \mathbb{N}$ with $\delta_\lambda^k(f) = 0$. Derivations that have such a property are called \textit{locally nilpotent}. Moreover, every $\mathbb{G}_a$-action on $X$ arises from a certain locally nilpotent derivation $\delta$ and the $\mathbb{G}_a$-action $\lambda_\delta: \mathbb{G}_a \times X \to X$ is obtained from $\delta$ via

$$(\alpha_\delta(s))^*: \mathcal{O}(X) \to \mathcal{O}(X)[s], \quad f \mapsto \exp(s\delta)(f) := \sum_{i=0}^{\infty} \frac{s^i \delta^i(f)}{i!}.$$

**3. Automorphisms**

**Proposition 3.1.** Let $\pi: \mathbb{A}^n \to A_{d,n}$. Then every automorphism of $A_{d,n}$ lifts to an automorphism of $\mathbb{A}^n$ which commutes with each element of $\mu_d$.

**Proof.** The quotient map $\pi: \mathbb{A}^n \to A_{d,n}$ induces a natural embedding of $\mathcal{O}(A_{d,n})$ into $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, \ldots, x_n]$. So, we assume that $\mathcal{O}(A_{d,n})$ is a subring of $\mathbb{C}[x_1, \ldots, x_n]$ and equals $\bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{dk}$, where $\mathbb{C}[x_1, \ldots, x_n]_{dk}$ denotes the homogeneous polynomials of degree $dk$. Let $\phi \in \text{Aut}(A_{d,n})$. First we claim that $p_i = \phi^*(x_i^d)$ and $p_j = \phi^*(x_j^d)$ are coprime in $\mathbb{C}[x_1, \ldots, x_n]$, where $i \neq j$ and $\phi^*$ is the pull-back of $\phi$. Let $p$ be a common factor of $p_i$ and $p_j$. Then $\tilde{p} \in \text{Aut}(A_{d,n})$. Hence, $(\phi^*)^{-1}(\tilde{p})$ is a common factor of $(\phi^*)^{-1}(p_i^d) = x_i^{d^2}$ and $(\phi^*)^{-1}(p_j^d) = x_j^{d^2}$. Therefore, $\tilde{p} \in \mathbb{C}$ and then, $p \in \mathbb{C}$.

We have

$$\phi^*(x_i^d)\phi^*((x_j^d)^{d-1}) = \phi^*(x_i^d x_j^{d(d-1)}) = \phi^*(x_i^{d-1} x_j^d),$$

which means that $p_i p_j^{d-1} = q^d$ for some $q \in \mathcal{O}(A_{d,n})$. Because $p_i$ is coprime with $p_j$, it follows that $p_i = q_i^d$ for some $q_i \in \mathbb{C}[x_1, \ldots, x_n]$. 274 ANDRIY REGETA
Since for an automorphism \( \phi: A_{d,n} \to A_{d,n} \) we have that \( \phi^*(x_i^d) = q_i^d \) for some \( q_i \in \mathbb{C}[x_1, \ldots, x_n] \), we define the morphism
\[
\tilde{\phi} = (q_1, \ldots, q_n): \mathbb{A}^n \to \mathbb{A}^n
\]
given by the map
\[
(x_1, \ldots, x_n) \mapsto (q_1, \ldots, q_n).
\]
If \( \phi \) is the identity automorphism, then the restriction of \( \tilde{\phi}^* \) to \( \mathcal{O}(A_{d,n}) \) is the identity and we have that \( q_i^d = x_i^d \), which implies that \( q_i = w_i x_i \) for some \( w_i \in \mathbb{C}^* \), \( w_i^d = 1 \). In this case \( \tilde{\phi} \) is an automorphism; i.e., the trivial automorphism of \( A_{d,n} \) lifts to an automorphism of \( \mathbb{A}^n \) which we denote by \( \Delta(w_1, \ldots, w_n) \).

Let now \( \theta: A_{d,n} \to A_{d,n} \) be the inverse automorphism of \( \phi \in \text{Aut}(A_{d,n}) \). Since \( \phi \circ \theta \) is the trivial automorphism of \( A_{d,n} \), it lifts to an automorphism \( \Delta(w_1, \ldots, w_n) \) of \( \mathbb{A}^n \) and so \( \tilde{\phi} \) is an automorphism with the inverse \( \tilde{\phi} \circ \Delta(w_1, \ldots, w_n)^{-1} \). To finish the proof, we need to show that \( \tilde{\phi} \) commutes with \( \mu_d \). Indeed, since \( \tilde{\phi}^* \) preserves \( \mathbb{C}[x_1, \ldots, x_n]^{\mu_d} \), we have that \( q_i^d(\xi x_1, \ldots, \xi x_n) = q_i^d(x_1, \ldots, x_n) \), where \( \xi \in \mu_d \).

Hence,
\[
q_i(\xi x_1, \ldots, \xi x_n) = \xi^l q_i(x_1, \ldots, x_n)
\]
for some \( l = 1, \ldots, d - 1 \). Since polynomial \( q_i \) has a linear summand it follows that \( l = 1 \). The proof follows. \( \square \)

Let \( X \) be an affine variety, \( H \) be a finite group that acts faithfully on \( X \), and let \( \pi: X \to X/H \) be the quotient morphism. Since \( H \) acts faithfully, \( H \) naturally embeds into \( \text{Aut}(X) \) and we identify \( H \) with its image in \( \text{Aut}(X) \). Denote by \( \text{Aut}^H(X) \subset \text{Aut}(X) \) the subgroup of all automorphisms of \( X \) which normalize \( H \): i.e., the subgroup of those automorphisms \( \phi \) such that \( \phi^{-1} \circ H \circ \phi = H \).

\textbf{Lemma 3.2.}

(a) There is a canonical homomorphism of groups \( \phi: \text{Aut}^H(X) \to \text{Aut}(X/H) \),

(b) if \( X \) is normal and contains only finitely many fixed points of \( H \) then every \( \mathbb{C}^+ \)-action on \( X/H \) lifts to a \( \mathbb{C}^+ \)-action on \( X \).

\textbf{Proof.} (a) Let \( h \in H, f \in \mathcal{O}(X)^H \) and \( \phi \in \text{Aut}^H(X) \). Then \( \phi^*: \mathcal{O}(X) \to \mathcal{O}(X) \) is an isomorphism and
\[
h(\phi^*(f)) = \phi^*((\phi^*)^{-1} \circ h \circ \phi^*)(f) = (\phi^* \circ h')(f) = \phi^*(f)
\]
for some \( h' \in H \). Therefore \( \phi^*(f) \in \mathcal{O}(X)^H \), which means that \( \phi \) induces an automorphism of \( X/H \).

(b) This follows from [MM09, Thm. 1.3]. \( \square \)

Let us recall that a closed algebraic subgroup \( U \) of \( \text{Aut}(X) \) is a \( 1 \)-dimensional unipotent subgroup if \( U \cong \mathbb{C}^+ \).

\textbf{Proposition 3.3.} The homomorphism \( \phi_d : \text{Aut}^{\mu_d}(\mathbb{A}^n) \to \text{Aut}(A_{d,n}) \) is surjective with kernel \( \mu_d \). Moreover, every \( 1 \)-dimensional unipotent subgroup of \( \text{Aut}(A_{d,n}) \) is the image of some \( 1 \)-dimensional unipotent subgroup of \( \text{Aut}^{\mu_d}(\mathbb{A}^n) \).
Proof. The surjectivity of $\phi_d$ follows from Proposition 3.1. The last claim of the statement follows from Lemma 3.2 (b). What remains is to compute the kernel of $d$.

It is clear that

$$\text{Aut}^d(A^n) = \left\{ f = (f_1, \ldots, f_n) \in \text{Aut}(A^n) \mid f_i \in \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{kd+1}, i = 1, \ldots, n \right\}.$$ 

Now let $f = (f_1, \ldots, f_n) \in \text{Aut}^d(A^n)$ be such that the map $f'$ induced by $f$ on $A^n/\mu_d$ is the identity. This means that $f'$ acts trivially on $O(A^n/\mu_d) = \mathbb{C} \oplus \bigoplus_{k \geq 1} \mathbb{C}[x_1, \ldots, x_n]_{kd}$.

Hence, $f'(x_i^d) = x_i^d$ for any $i$ which implies that $f = (\xi_1 x_1, \ldots, \xi_n x_n)$, where $\xi_i^d = 1$ for $i = 1, \ldots, n$. In particular, $f'(x_i^{d-1} x_j) = x_i^{d-1} x_j$ which implies that $\xi_i^{d-1} \xi_j = 1$ for any $i, j$. Because $\xi_i^{d-1} \xi_i = 1$, we conclude that $\xi_i = \xi_j$. The claim follows.

4. Root subgroups

Let $G$ be an ind-group, and let $T \subset G$ be a closed torus.

**Definition 4.1.** A closed subgroup $U \subset G$ isomorphic to $\mathbb{C}^+$ and normalized by $T$ is called a root subgroup with respect to $T$. The character of $T$ on $\text{Lie}U \simeq \mathbb{C}$ (i.e., the algebraic action of $T$ on $\text{Lie}U$) is called the weight character of $U$.

Let $X$ be an affine variety and consider a nontrivial algebraic action of $\mathbb{C}^+$ on $X$, given by $\lambda: \mathbb{C}^+ \rightarrow \text{Aut}(X)$. If $f \in O(X)$ is $\mathbb{C}^+$-invariant, then the modification $f \cdot \lambda$ of $\lambda$ is defined in as

$$(f \cdot \lambda)(s)x = \lambda(f(x)s)x$$

for $s \in \mathbb{C}$ and $x \in X$. It is easy to see that this is again a $\mathbb{C}^+$-action. In fact, the corresponding locally nilpotent derivation to $f \cdot \lambda$ is $f \delta_\lambda$, where $\delta_\lambda$ is the locally nilpotent derivation that correspond to $\lambda$ (see Section 2.2 for details). It is clear that if $X$ is irreducible and $f \neq 0$, then $f \cdot \lambda$ and $\lambda$ have the same invariants. If $U \subset \text{Aut}(X)$ is a closed subgroup isomorphic to $\mathbb{C}^+$ and if $f \in O(X)^U$ is a $U$-invariant, then in a similar way we define the modification $f \cdot U$ of $U$. Choose an isomorphism $\lambda: \mathbb{C}^+ \rightarrow U$ and set

$$f \cdot U = \{(f \cdot \lambda)(s) \mid s \in \mathbb{C}^+\}.$$ 

Note that $\text{Lie}(f \cdot U) = f \text{Lie}U \subset \text{Vec}(X)$, where $\text{Vec}(X)$ denotes the Lie algebra of (algebraic) vector fields on $X$, i.e., $\text{Vec}(X) = \text{Der}(O(X))$, the Lie algebra of derivations of $O(X)$.

If a torus $T$ acts linearly and rationally on a vector space $V$, then we call $V$ multiplicity-free if the weight spaces $V_\alpha$ are all of dimension less than or equal to 1.
Lemma 4.2 ([Kr15, Lem. 6.2]). Let $X$ be an irreducible affine variety and let $T \subset \text{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subset \text{Aut}(X)$ with respect to $T$ such that the $T$-module $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.

The next result is going to be of use in the sequel and can be found in [Lie11, Thm. 1]. We denote by $U(\mathbb{A}^n)$ the subgroup of $\text{Aut}(\mathbb{A}^n)$ of the following form

$$f = (f_1, \ldots, f_n) \in \text{Aut}(\mathbb{A}^n) \mid \text{jac}(f) = \det \left[ \frac{\partial f_i}{\partial x_j} \right]_{i,j} = 1$$

and by $T'_n$ a maximal subtorus of $U(\mathbb{A}^n)$ of the form

$$\{(t_1 x_1, \ldots, t_n x_n) \mid t_i \in \mathbb{C}^*, t_1 \cdots t_n = 1\}.$$

Lemma 4.3. Let $U \subset U(\mathbb{A}^n)$ be a one-dimensional unipotent subgroup. Then $U$ is a root subgroup with respect to $T'_n$ if and only if $U = U_\lambda = \{(x_1, \ldots, x_i + cm_i, \ldots, x_n) \mid c \in \mathbb{C}\}$, where $m_i = x_1^{\lambda_1} \cdots x_{i-1}^{\lambda_{i-1}} x_{i+1}^{\lambda_{i+1}} \cdots x_n^{\lambda_n}$. The character $\xi_\lambda$ corresponding to the root subgroup $U$ is the following:

$$\xi_\lambda: T'_n \rightarrow \mathbb{C}^*, \quad t = (t_1, \ldots, t_n) \mapsto t_1 t_1^{-\lambda_1} \cdots t_i t_i^{-\lambda_i} \cdots t_n^{-\lambda_n}.$$

Remark 4.4. The last lemma can also be expressed in the following way (see [KS13, Remark 2]): there is a bijective correspondence between the $T'_n$-stable one-dimensional unipotent subgroups $U \subset \text{Aut}(\mathbb{A}^n)$ and the characters of $T'_n$ of the form $\lambda = \sum_j \lambda_j \epsilon_j$ where one $\lambda_i$ equals 1 and the others are $\leq 0$. We will denote this set of characters by $X_u(T'_n)$:

$$X_u(T'_n) = \left\{ \lambda = \sum \lambda_j \epsilon_j \mid \lambda_i = 1 \text{ and } \lambda_j \leq 0 \text{ for } j \neq i \right\}.$$

If $\lambda \in X_u(T'_n)$, then $U_\lambda$ denotes the corresponding one-dimensional unipotent subgroup normalized by $T'_n$.

5. A special subgroup of $\text{Aut}(X)$

For any affine variety $X$ consider the normal subgroup $U(X)$ of $\text{Aut}(X)$ generated by closed one-dimensional unipotent subgroups. The group $U(X)$ was introduced and studied in [AFK13], where the authors called it the group of special automorphisms of $X$. Following [Kr15], we introduce the notion of an algebraic homomorphism between these groups.

Definition 5.1. A homomorphism $\phi: U(X) \rightarrow U(Y)$ is algebraic if for any subgroup $U \subset U(X)$ such that $U \subset \text{Aut}(X)$ is closed, $U \simeq \mathbb{C}^+$, the image $\phi(U) \subset \text{Aut}(Y)$ is closed, and $\phi|_U: U \rightarrow \phi(U)$ is a homomorphism of algebraic groups. We say that $\phi$ is an algebraic isomorphism if $\phi$ is an isomorphism of groups and $\phi|_U: U \simeq \phi(U)$ is an isomorphism of algebraic groups.

A subgroup $G \subset U(X)$ is called algebraic if $G \subset \text{Aut}(X)$ is a closed algebraic subgroup. The next lemma can be found in [Kr15, Lem. 4.2].
Lemma 5.2. Let \( \phi : \mathcal{U}(X) \to \mathcal{U}(Y) \) be an algebraic homomorphism. Then, for any algebraic subgroup \( G \subset \mathcal{U}(X) \) generated by one-dimensional unipotent subgroups of \( \text{Aut}(X) \), the image \( \phi(G) \) is an algebraic subgroup of \( \mathcal{U}(Y) \) and \( \phi|_G : G \to \phi(G) \) is a homomorphism of algebraic groups.

Let \( X \) be an affine variety and let \( \eta : \tilde{X} \to X \) be a normalization map. It is well known that any automorphism of \( X \) lifts uniquely to an automorphism of \( \tilde{X} \). Indeed, for a given automorphism \( \phi : X \to X \), the composition \( \phi \circ \eta : \tilde{X} \to X \) is a morphism, which by the universal property of normalization factors through a morphism \( \tilde{\phi} : \tilde{X} \to \tilde{X} \) such that \( \phi \circ \eta = \eta \circ \tilde{\phi} \). It remains to argue that \( \tilde{\phi} \) is an automorphism. But for the same reason, the inverse \( \phi^{-1} \) lifts to an automorphism \( \psi : \tilde{X} \to \tilde{X} \). Since \( \eta : \tilde{X} \to X \) is birational, the compositions \( \psi \circ \tilde{\phi} \) and \( \tilde{\phi} \circ \psi \) are equal to the identity on a dense open subset of an irreducible variety, hence everywhere. This shows that \( \eta \) induces a well-defined injective homomorphism \( \tilde{\eta} : \text{Aut}(X) \hookrightarrow \text{Aut}(\tilde{X}) \). Moreover, in [FK18, Prop. 12.1.1] it is proved that \( \tilde{\eta} \) is a closed immersion of ind-groups: i.e., \( \tilde{\eta}(\text{Aut}(X)) \subset \text{Aut}(\tilde{X}) \) is a closed subgroup and \( \tilde{\eta} \) induces the isomorphism of ind-groups \( \text{Aut}(X) \cong \tilde{\eta}(\text{Aut}(X)) \). Hence, we have the following statement.

Lemma 5.3. Let \( X \) be an irreducible affine variety, and let \( \eta : \tilde{X} \to X \) be its normalization. Then every automorphism of \( \tilde{X} \) lifts uniquely to an automorphism of \( \tilde{X} \) and the induced map \( \tilde{\eta} : \text{Aut}(X) \hookrightarrow \text{Aut}(\tilde{X}) \) is a closed immersion of ind-groups.

Proposition 5.4. Let \( n \geq 3 \) and let \( X \) be an \( n \)-dimensional irreducible affine variety endowed with a nontrivial \( \text{SL}_n \)-action. Then

\[
\mathcal{O}(X) = \sum_{i=1}^{d} \bigoplus_{k \geq 0} \mathbb{C}[x_1, \ldots, x_n]_{kd}
\]

for some \( d_1, \ldots, d_l \in \mathbb{N} \), where \( (d_1, \ldots, d_l) = d \) and the normalization of \( X \) is isomorphic to \( A_{d,n} \). The same holds when \( n = 2 \) and the normalization of \( X \) is \( A_{d,2} \) for some \( d \in \mathbb{N} \).

Proof. First, let \( n \geq 3 \). If \( X \) is normal, then from [KRZ20, Thm. 1.6 and Prop. 4.4(2); see also Example 4.5] it follows that \( X \simeq A_{d,n} \) for some \( d \in \mathbb{N} \). It is well known that

\[
\mathcal{O}(A_{d,n}) = \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{kd}
\]

is a direct sum of irreducible pairwise nonisomorphic \( \text{SL}_n \)-modules \( \mathbb{C}[x_1, \ldots, x_n]_{kd} \).

Now, consider any \( n \)-dimensional irreducible affine variety \( X \) endowed with a nontrivial \( \text{SL}_n \)-action and a normalization morphism \( \eta : A_{d,n} \to X \). Since any \( \text{SL}_n \)-action on \( \mathcal{O}(X) \) lifts to a \( \text{SL}_n \)-action on \( \mathcal{O}(A_{d,n}) \), it follows that \( \mathcal{O}(X) \) is a \( \text{SL}_n \)-submodule of \( \mathcal{O}(A_{d,n}) \) and therefore

\[
\mathcal{O}(X) = \bigoplus_{k \in \Omega} \mathbb{C}[x_1, \ldots, x_n]_{kd},
\]

where \( \Omega \) is a submonoid of \( \mathbb{N} \) under addition. Since \( \mathcal{O}(X) \) is finitely generated, \( \Omega \subset \mathbb{N} \) is a finitely generated submonoid: i.e., there exist \( d_1, \ldots, d_l \in \mathbb{N} \) such that \( \Omega = d_1 \mathbb{N} + \cdots + d_l \mathbb{N} \). The claim follows. \( \square \)
6. 2-dimensional case

6.1. Two dimensional normal affine \(\text{SL}_2\)-surfaces

The next result can be found in [Pop73], §3 (see also [Giz71] and [Kr84], §4).

Lemma 6.1. Let \(X\) be an affine normal irreducible variety of dimension two endowed with a nontrivial \(\text{SL}_2\)-action. Then \(X\) is \(\text{SL}_2\)-equivariantly isomorphic to one of the following varieties:

(a) \(A_{d,2}\) for some \(d \in \mathbb{N}\), where the \(\text{SL}_2\)-action on \(A_{d,2}\) is induced by the standard \(\text{SL}_2\)-action on \(\mathbb{A}^2\),
(b) \(\text{SL}_2 / T\), where \(T\) is the standard subtorus of \(\text{SL}_2\) and \(\text{SL}_2\) acts on \(\text{SL}_2 / T\) by left multiplication,
(c) \(\text{SL}_2 / N\), where \(N\) is the normalizer of \(T\) and \(\text{SL}_2\) acts on \(\text{SL}_2 / N\) by left multiplication.

The \(\text{SL}_2\)-action on \(\text{SL}_2 / T\) and on \(\text{SL}_2 / N\) from the Lemma above is transitive. The \(\text{SL}_2\)-variety \(A_{d,2}\) is the union of a fixed point and the orbit \((\mathbb{A}^2 \setminus \{0\})/\mu_d \cong \text{SL}_2 / U_d\), where \(\mu_d\) acts by scalar multiplication on \(\mathbb{A}^2 \setminus \{0\}\) and

\[
U_d = \left\{ \begin{bmatrix} \xi & t \\ 0 & \xi^{-1} \end{bmatrix} \mid t \in \mathbb{C}, \xi \in \mathbb{C}^*, \xi^d = 1 \right\}.
\]

Moreover, any closed subgroup of \(\text{SL}_2\) of codimension less than or equal to 2 is conjugate to either \(T\), or \(N\), or \(U_d\) for some \(d \geq 1\), or

\[
B = \left\{ \begin{bmatrix} a & t \\ 0 & a^{-1} \end{bmatrix} \mid t \in \mathbb{C}, a \in \mathbb{C}^* \right\}
\]
(see, for example, [Pop73, p. 803]).

The next result can be found, for example, in [Kr84, III.2.5, Folgerung 3].

Proposition 6.2. If a reductive group \(G\) acts on an affine variety \(X\) and if the stabilizer of a point \(x \in X\) contains a maximal torus, then the orbit \(Gx\) is closed.

Proposition 6.3. Let \(X\) be a two-dimensional affine \(\text{SL}_2\)-variety and let \(O = \text{SL}_2 x\) be the orbit of \(x \in X\). Then we are in one of the following cases:

(a) \(x\) is a fixed point,
(b) the orbit \(O\) is closed and \(\text{SL}_2\)-isomorphic to \(\text{SL}_2 / T\) or \(\text{SL}_2 / N\),
(c) \(\overline{O} = O \cup \{x_0\}\), where \(\overline{O}\) is the closure of the orbit \(O\) and \(x_0\) is a fixed point.

Moreover, either \(\overline{O} \cong \mathbb{A}^2\) or \(x_0\) is an isolated singular point.

Proof. If the stabilizer of \(x\) contains a maximal torus, then we are in case (a) or (b) by Proposition 6.2 and Lemma 6.1. Otherwise, from the classification of closed subgroups of \(\text{SL}_2\), it follows that the stabilizer of \(x\) coincides with \(U_d\) for some \(d \geq 1\) and \(\overline{O}\) does not contain orbits of dimension one. Hence, \(\overline{O} = O \cup \{x_0\}\). It is clear that if \(\overline{O}\) is singular, then \(x_0\) is an isolated singular point. If \(\overline{O}\) is smooth, then from Lemma 6.1 it follows that \(\overline{O}\) is isomorphic to \(\mathbb{A}^2\).

Remark 6.4. Note that \(\text{SL}_2 / T \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta\), where \(\Delta\) is the diagonal, and \(\text{SL}_2 / N \cong \mathbb{P}^2 \setminus C\), where \(C\) is a smooth conic (see [Pop73, Lem. 2]).
6.2. The structure of \( \text{Aut}(\text{SL}_2/T) \)

The variety \( \text{SL}_2/T \) is isomorphic to the following so-called Danielewski surface: i.e., the smooth 2-dimensional affine quadric \( V(xz - y^2 + y) \subset \mathbb{A}^3 \) (see [DP09]) and the quotient map \( \pi: \text{SL}_2 \to \text{SL}_2/T \) is given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ab, ad, cd) \). It is not difficult to see that \( X = V(xz + y^2 - 1) \simeq V(xz - y^2 + y) \subset \mathbb{A}^3 \). From now on and until end of Section 6, we identify \( \text{SL}_2/T \) with \( X = V(xz + y^2 - 1) \).

Consider the orthogonal group \( \text{O}_3 = \text{O}(3, \mathbb{C}) \) associated with the quadratic form \( y^2 + xz \) generated by \( \tau: \mathbb{A}^3 \to \mathbb{A}^3 \) given by the map \((x, y, z) \mapsto (-x, -y, -z)\) and by the group \( \text{SO}_3 = \text{SO}(3, \mathbb{C}) \) that is composed of the matrices

\[
\frac{1}{ad - bc} \begin{pmatrix}
    a^2 & 2ab & -b^2 \\
    ac & ad + bc & -bd \\
    -c^2 & -2cd & d^2
\end{pmatrix}
\quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2.
\]

Following [Lam05, Thm. 6] (see also [MM90]), \( \text{Aut}(X) \) is the amalgamated product of the orthogonal group \( \text{O}_3 = \text{SO}_3 \times \langle \tau \rangle \) and \( J \rtimes \langle \tau \rangle \) along their intersection \( C \), where \( J \) is the subgroup of \( \text{Aut}(X) \) of the automorphisms of the form

\[
(x, y, z) \mapsto \left( \alpha x + 2\alpha yP(z) - \alpha zP^2(z), y - zP(z), \frac{1}{\alpha} z \right); \quad \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z].
\]

Note that \( \text{SO}_3 \) is generated by \( \mathbb{C}^+ \)-actions. Define \( \tilde{J} \) to be the subgroup of \( J \) generated by \( \mathbb{C}^+ \)-actions. The subgroup of \( \text{Aut}(X) \) generated by \( \text{SO}_3 \) and \( \tilde{J} \) coincides with the subgroup of \( \text{Aut}(X) \) generated by \( \text{SO}_3 \) and \( J \) and is a subgroup of \( \mathcal{U}(X) \). Moreover, because \( \tau \) normalizes \( \langle J, \text{SO}_3 \rangle \), we have that \( \text{Aut}(X) = \langle J, \text{SO}_3 \rangle \rtimes \langle \tau \rangle \). This implies that \( \text{Aut}(X) \) is not connected, and \( \tau \notin \text{Aut}^\circ(X) \). Since the closure of \( \mathcal{U}(X) \subset \text{Aut}(X) \) is connected as \( \mathcal{U}(X) \) is generated by connected subgroups and since \( \langle J, \text{SO}_3 \rangle \subset \mathcal{U}(X) \) we have that \( \mathcal{U}(X) \subset \text{Aut}(X) \) coincides with \( \text{Aut}^\circ(X) \) and hence is closed. Moreover, \( \text{Aut}(X) = \mathcal{U}(X) \rtimes \langle \tau \rangle \).

The following proposition is going to be of use later (see [Neu48, Cor. 8.11]).

**Proposition 6.5.** In the amalgamated product \( G = A \ast_C B \) with the unified subgroup \( C = A \cap B \), consider two subgroups \( \tilde{A} \subset A \) and \( \tilde{B} \subset B \), and let \( \tilde{G} = \langle \tilde{A}, \tilde{B} \rangle \). Assume that \( \tilde{A} \cap C = \tilde{C} = \tilde{B} \cap C \). Then \( \tilde{G} = \tilde{A} \ast_{\tilde{C}} \tilde{B} \).

**Lemma 6.6.** The group \( \mathcal{U}(X) \) is the amalgamated product of \( \text{SO}_3 \) and \( J \) along their intersection.

**Proof.** We know that \( \text{Aut}(X) \) is the amalgamated product of \( \text{O}_3 = \text{SO}_3 \times \langle \tau \rangle \) and \( J \rtimes \langle \tau \rangle \) along their intersection \( C \). Moreover, since \( \text{SO}_3 \cap C = J \cap \text{SO}_3 = J \cap C \), we have by Proposition 6.5 that \( \langle J, \text{SO}_3 \rangle \subset \text{Aut}(X) \) is the amalgamated product of \( J \) and \( \text{SO}_3 \) along their intersection. As \( \mathcal{U}(X) = \langle J, \text{SO}_3 \rangle \) the claim follows. \( \square \)

**Lemma 6.7.** The subgroup \( \text{Aut}_{\text{SO}_3}(X) \subset \text{Aut}(X) \) of those automorphisms that commute with \( \text{SO}_3 \) is \( \langle \tau \rangle \). Moreover, \( X/\langle \tau \rangle \simeq \text{SL}_2/N \).

**Proof.** Let \( \varphi \in \text{Aut}_{\text{SO}_3}(X) \). Since \( \text{Aut}(X) \) is the amalgamated product of \( \text{O}_3 \) and \( J \) along their intersection, we can write \( \varphi \) as the product \( a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \),
where $a_i \in O_3$ and $b_i \in J$. Since $a = \varphi \circ a \circ \varphi^{-1}$ for any $a \in SO_3$, we have that $b_k \in O_3 \cap J$ and $\varphi$ can be written as $\tilde{a}_1 \circ \tilde{b}_1 \circ \cdots \circ \tilde{a}_k$ for some $\tilde{a}_i \in O_3$ and $\tilde{b}_i \in J$. Assume first that $k > 1$. Hence, the expression
$$\tilde{a}_1 \circ \tilde{b}_1 \circ \cdots \circ \tilde{a}_k \circ a \circ (\tilde{a}_1 \circ \tilde{b}_1 \circ \cdots \circ \tilde{a}_k)^{-1} = a$$
implies that $\tilde{a}_k \circ a \circ \tilde{a}_k^{-1} \in O_3 \cap J$. Since this should hold for any $a \in SO_3$, we get a contradiction. Therefore, $k = 1$ and hence $\varphi \in O_3$. This implies that $\text{Aut}_{SO_3}(X) = \langle \varphi \rangle$ as the centralizer of $SO_3$ in $O_3$ is $\langle \varphi \rangle$.

To finish the proof we need to argue that $X/\langle \varphi \rangle \simeq SL_2/N$. We first note that $X/\langle \varphi \rangle$ is normal and since $SL_2$-action on $X$ is transitive, it follows that the induced action of $SL_2$ on $X/\langle \varphi \rangle$ is transitive too. Hence, from Lemma 6.1 it follows that $X/\langle \varphi \rangle \simeq SL_2/N$ as $X \not\subseteq X/\langle \varphi \rangle$.

Note that the subgroup $\mathcal{U}(X) = \text{Aut}^\circ(X) \subset \text{Aut}(X)$ is closed (see [Kr15, Lem. 6.3]), where $\text{Aut}^\circ(X)$ is the neutral component of $\text{Aut}(X)$. Hence, $\mathcal{U}(X)$ is an ind-group.

**Proposition 6.8.** We have the following properties.

(a) All closed subgroups $S \subset \text{Aut}(X)$ isomorphic to $PSL_2$ are conjugate.

(b) The root subgroups with respect to a maximal torus $\bar{T}$ of any $S \simeq PSL_2$ are multiplicity-free with weights $1, 2, 3, \ldots$ up to an automorphism of $\bar{T}$.

**Proof.** (a) Since $\text{Aut}(X)$ is the amalgamated product of $O_3$ and $J$ over their intersection, we have that by [Sr80] any closed subgroup $S \subset \text{Aut}(X)$ isomorphic to $S$ is conjugate to one of the factors $O_3$ or $J$. Since all unipotent subgroups of $J$ commute, $S$ can not be embedded into $J$ and hence $S$ is conjugate to a subgroup of $O_3$: i.e., to $SO_3$. The claim follows.

Now we are going to prove (b). Let $U \subset \text{Aut}(X)$ be a root subgroup with respect to $\bar{T}$. This means that $\bar{T} \times U$ is an algebraic subgroup of $\text{Aut}(X)$ and by [Sr80], $\bar{T} \times U$ is conjugate to a subgroup of either $O_3$ or $J$. If $\bar{T} \times U$ is conjugate to a subgroup of $O_3$, then the weight of $U$ with respect to $\bar{T}$ is either $1$ or $-1$, i.e., up to an automorphism of $\bar{T}$ we can assume that the weight is $1$. If $\bar{T} \times U$ is conjugate to a subgroup of $J$, then without loss of generality we can assume that $\bar{T} \times U$ is an algebraic subgroup of $J_{\leq k}$ generated by elements of the form
$$(x, y, z) \mapsto \left(\alpha x + 2\alpha y P(z) - \alpha z P^2(z), (y - z P(z)), \frac{1}{\alpha} z\right); \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z]_{\leq k}$$
for some natural $k$, where $\mathbb{C}[z]_{\leq k}$ denotes the polynomials of degree less or equal than $k$. Moreover, since all tori in $J_{\leq k}$ are conjugate we can assume that
$$\bar{T} = \{(tx, y, t^{-1}z) \mid t \in \mathbb{C}^*\}.$$  
By the following computation,
$$(tx, y, t^{-1}z) \circ (x + 2y P(z) - z P^2(z), (y - z P(z)), z) \circ (t^{-1}x, y, t z) = (x + 2yt P(tz) - zt P^2(tz), (y - zt P(tz)), z),$$
it is easy to see that a root subgroup $U \subset J_{\leq k}$ should have the form

$$U_i = \{ (x + 2cyP_i(z) - c^2zP_i^2(z), (y - czP_i(z)), z) \mid c \in \mathbb{C}, P_i(z) = z^i \}$$

for some natural $i \leq k$. Note that the root subgroup $U_i$ with respect to $\bar{T}$ has the weight $i + 1$. The claim follows.  

6.3. The structure of $\text{Aut}(\text{SL}_2/N)$

By Lemma 6.7, there exists an automorphism $\tau \in \text{Aut}_{\text{SO}_3}(X)$ such that the quotient $Y = X/\langle \tau \rangle$ is isomorphic to $\text{SL}_2/N$. In particular, $\mathcal{O}(Y) = \mathcal{O}(X)^\tau$. An automorphism $\phi$ of $X$ descends to an automorphism on $Y$ if and only if $\phi$ sends $\langle \tau \rangle$-orbits to $\langle \tau \rangle$-orbits. In fact, such an automorphism induces the automorphism of $\mathcal{O}(X)$ that sends $\langle \tau \rangle$-invariant functions of $\mathcal{O}(X)$ to $\langle \tau \rangle$-invariant functions of $\mathcal{O}(X)$. This condition for $\phi$ is equivalent to the condition that $\phi$ normalizes $\langle \tau \rangle$. Moreover, since $\tau$ has order two, $\phi$ commutes with $\tau$. Recall that by $\text{Aut}(\langle \tau \rangle)(X)$ we denote the subgroup of those elements of $\text{Aut}(X)$ that normalize $\langle \tau \rangle$, but in this particular case $\text{Aut}(\langle \tau \rangle)(X)$ is even the subgroup of those automorphisms of $\text{Aut}(X)$ that commute with $\tau$.

As we have mentioned above, $\phi \in \text{Aut}(X)$ induces an automorphism of $Y \simeq \text{SL}_2/N$ if and only if $\phi \in \text{Aut}(\langle \tau \rangle)(X)$. On the other hand, since $X \simeq \text{SL}_2/T$ is simply connected and the quotient map $\pi: X \to X/\langle \tau \rangle = Y$ is an étale covering, every automorphism $\varphi$ of $Y$ can be lifted to a continuous analytical automorphism of $X$ and hence by [Sr58, Prop. 20], $\varphi$ can be lifted to an automorphism $\bar{\varphi}$ of $X$: i.e., $\bar{\varphi} \in \text{Aut}(\langle \tau \rangle)(X)$. Hence, we have the surjective homomorphism $\text{Aut}(\langle \tau \rangle)(X) \to \text{Aut}(Y)$ with the kernel $\langle \tau \rangle$ and so

$$\text{Aut}(Y) \simeq \text{Aut}(\langle \tau \rangle)(X)/\langle \tau \rangle.$$  

(1)

Observe that $\text{SO}_3 \times \langle \tau \rangle$ is the subgroup of $\text{Aut}(\langle \tau \rangle)(X)$. Define the subgroup $J^{\langle \tau \rangle} \subset \text{Aut}(X)$ of those automorphisms from $J$ which normalize $\langle \tau \rangle$. It is not difficult to see that $J^{\langle \tau \rangle}$ is comprised of the following automorphisms:

$$\{ (x, y, z) \mapsto \left( \alpha x + 2\alpha yP(z) - \alpha zP^2(z), y - zP(z); \frac{1}{\alpha}z \right) \mid \alpha \in \mathbb{C}^*, P \in \bigoplus_{l=0}^{\infty} \mathbb{C}z^{2l} \}.$$  

We have the following statement.

Lemma 6.9. The subgroup $\text{Aut}(\langle \tau \rangle)(X) \subset \text{Aut}(X)$ is the direct product of $\langle \tau \rangle$ and the amalgamated product of $\text{SO}_3$ and $J^{\langle \tau \rangle}$ along their intersection.

Proof. As we have mentioned above, $\text{Aut}(\langle \tau \rangle)(X)$ is the subgroup of those automorphisms of $\text{Aut}(X)$ that commute with $\tau$. Assume $\phi \in \text{Aut}(X)$ commutes with $\tau$. Since $\text{Aut}(X)$ is the amalgamated product of $\text{SO}_3 \times \langle \tau \rangle$ and $J \rtimes \langle \tau \rangle$, one can write $\phi$ as a product $a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k$, where $a_i \in \text{SO}_3 \times \langle \tau \rangle$ and $b_i \in J \rtimes \langle \tau \rangle$. Further, because $\phi$ commutes with $\tau$, $\tau \phi \tau = \phi$ or equivalently,

$$\tau \circ a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \circ \tau = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k.$$
Since \( \tau \) commutes with \( \SO_3 \), one can rewrite this equation as follows:

\[
a_1 \circ (\tau \circ b_1 \circ \tau) \circ \cdots \circ a_k \circ (\tau \circ b_k \circ \tau) = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k.
\]

From the amalgamated product structure of \( \Aut(X) \), it follows that \( \tau \circ b_i \circ \tau = c_i \circ b_i \) for some \( c_i \in (\SO_3 \times \langle \tau \rangle) \cap (J \times \langle \tau \rangle) \). But this can happen only if \( b_i \) commutes with \( \tau \): i.e., \( b_i \in J^{(\tau)} \times \langle \tau \rangle \). Therefore, \( \Aut^{(\tau)}(X) \) is generated by \( \SO_3 \times \langle \tau \rangle \) and \( J^{(\tau)} \times \langle \tau \rangle \). Moreover, \( \Aut^{(\tau)}(X) = \langle \SO_3, J^{(\tau)} \rangle \times \langle \tau \rangle \) and by Proposition 6.5, \( \langle \SO_3, J^{(\tau)} \rangle \) is the amalgamated product of \( \SO_3 \) and \( J^{(\tau)} \) over their intersection. \( \square \)

From Lemma 6.9 and (1) we have the following statement.

**Lemma 6.10.** The automorphism group \( \Aut(Y) \) is isomorphic to the amalgamated product of \( \SO_3 \) and \( J^{(\tau)} \). In particular, \( \Aut(Y) = U(Y) \).

**Remark 6.11.** Lemma 6.10 can also be retrieved from [KPZ17, Rem. 3.9] and Remark 6.4 (see also [DG77, (2.4.3; l)]).

**Corollary 6.12.** We have the following properties.

(a) All closed subgroups \( S \subset \Aut(Y) \) isomorphic to \( \PSL_2 \) are conjugate.

(b) The root subgroups of \( \Aut(Y) \) with respect to a maximal torus \( \widetilde{T} \) of any \( S \simeq \PSL_2 \) are multiplicity-free with weights \( 1, 3, 5, \ldots \) up to an automorphism of \( \widetilde{T} \). In particular, \( U(\SL_2 / N) \) and \( U(\SL_2 / T) \) are not algebraically isomorphic.

**Proof.** (a) By Lemma 6.10, \( \Aut(Y) \) is the amalgamated product of \( \SO_3 \) and \( J^{(\tau)} \) and by [Sr80] any algebraic subgroup of the amalgamated product is conjugate to one of the factors. Since \( J^{(\tau)} \) does not contain a copy of \( \PSL_2 \), it follows that \( S \) is conjugate to a subgroup of \( \SO_3 \); i.e., to \( \SO_3 \) itself.

(b) Without loss of generality, we can assume that \( S \) equals \( \SO_3 \) and \( \widetilde{T} \subset \SO_3 \) is the subtorus of the form

\[
\{(t x, y, t^{-1}z) \mid t \in \mathbb{C}^*\}.
\]

Any root subgroup of \( \Aut(Y) \) with respect to \( \widetilde{T} \) lifts to a root subgroup \( U \) of \( \Aut^{(\tau)}(X) \) (see Lemma 3.2) with respect to the subtorus \( p^{-1}(\widetilde{T})^o \subset \Aut^{(\tau)}(X) \).

As it follows from the proof of Proposition 6.8(b), \( U \) coincides with

\[
U_{2i} = \{(x + 2y P_i(z) - z P_i^2(z), y - z P_i(z)), z \mid P_i(z) = z^{2i}\}
\]

for some \( i \in \mathbb{N} \cup \{0\} \). The weight of the root subgroup \( U_{2i} \subset \Aut^{(\tau)}(X) \) with respect to \( p^{-1}(\widetilde{T})^o \) is \( 2i + 1 \). Since the kernel of \( p^{-1}(\widetilde{T}) \to \widetilde{T} \) is trivial, we have that the set of weights of root subgroups of \( \Aut(Y) \) with respect to \( \widetilde{T} \) is \( \{2i + 1 \mid i \in \mathbb{N}\} \). This proves the first part of the statement. The second part follows because if there is an algebraic isomorphism \( \varphi : U(X) \to U(Y) \), then \( \varphi \) maps root subgroups of \( U(X) \) with respect to a subtorus \( \widetilde{T} \subset U(X) \) to root subgroups of \( U(Y) \) with respect to \( \varphi(\widetilde{T}) \) that have the same weights. But as follows from Proposition 6.8 and the first part of this proof, it is not the case. \( \square \)
Remark 6.13. Analogously as in the proof of Lemma 6.7, using amalgamated product structure of $\text{Aut}(Y)$ described in Lemma 6.10, we can show that the subgroup $\text{Aut}_{SO_3}(X) \subset \text{Aut}(X)$ of those automorphisms that commute with $SO_3$ is trivial.

6.4. On the automorphism group of $A_{d,2}$

Recall that by Proposition 3.3, there is a surjective homomorphism $\phi_d : \text{Aut}^d(\mathbb{A}^n) \to \text{Aut}(A_{d,n})$ of groups. Consider now the maximal subtorus

$$T_n = \{(t_1 x_1, \ldots, t_n x_n) \mid t_i \in \mathbb{C}^* \} \subset \text{Aut}(\mathbb{A}^n)$$

and recall that by $T'_n$ we denote the subtorus of the form

$$\{(t_1 x_1, \ldots, t_n x_n) \mid t_i \in \mathbb{C}^*, t_1 \cdots t_n = 1 \} \subset \text{Aut}(\mathbb{A}^n)$$

that has dimension $n - 1$. Then $T'_{d,n} = \phi_d(T'_n)$ is a maximal subtorus of $\text{U}(A_{d,n}) \subset \text{Aut}(A_{d,n})$.

Lemma 6.14. Let $U \subset \text{Aut}(A_{d,n})$ be a root subgroup with respect to $T'_{d,n}$ which has a character $\chi$. Then $U$ lifts to a root subgroup $\tilde{U} = (\phi_d^{-1}(U))^\circ \subset \text{Aut}^d(\mathbb{A}^n)$ with respect to $T'_n = (\phi_d^{-1}(T'_{d,n}))^\circ$ with character $\tilde{\chi} = \psi^*(\chi)$ such that the following diagram

$$1 \longrightarrow \mu_{(n,d)} \longrightarrow T'_n \longrightarrow \chi$$

commutes, where $\psi = \phi_d|_{T'_n}$ and $\psi^*(\chi)$ is the pull-back of $\chi$.

Proof. From Proposition 3.1, it follows that any root subgroup $U$ of $\text{Aut}(A_{d,n})$ with respect to $T'_{d,n}$ lifts to a unipotent subgroup $\tilde{U} = (\phi_d^{-1}(U))^\circ \subset \text{Aut}^d(\mathbb{A}^n)$. Moreover, $\tilde{U}$ is normalized by $(\phi_d^{-1}(T'_{d,n}))^\circ = T'_n$. Now, let $\tilde{u} \in \tilde{U}$ be a nontrivial element and $u = \phi_d(\tilde{u}) \in U$. We have group isomorphisms

$$\mathbb{C}^+ \xrightarrow{\simeq} \tilde{U}, \ s \mapsto \tilde{u}(s)$$

and

$$\mathbb{C}^+ \xrightarrow{\simeq} U, \ s \mapsto u(s).$$

Now the proof follows from the formula

$$\phi_d(\tilde{u}(\chi \circ \psi(t)s)) = u(\chi(t)s).$$

Observe that the homomorphism $\phi_d : \text{Aut}^d(\mathbb{A}^n) \to \text{Aut}(A_{d,n})$ induces the homomorphism $\tilde{\phi}_d : U^{\mu_d}(\mathbb{A}^n) \to U(A_{d,n})$ which has the kernel $\mu_{(n,d)}$, where $U^{\mu_d}(\mathbb{A}^n) \subset \text{Aut}^d(\mathbb{A}^n)$ is a subgroup generated by $\mathbb{C}^+$-actions.

In [BH03], it is proved that any faithful action of an $(n - 1)$-dimensional torus on an $n$-dimensional toric $T_Z$-variety $Z$ is conjugate to a subtorus of the big torus $T_Z$. This result is used in order to prove the following lemma.
Lemma 6.15. Let $T$ be an algebraic subtorus of $\mathcal{U}(A_{d,n})$ of dimension $(n - 1)$. Then there exists an algebraic isomorphism $F: \mathcal{U}(A_{d,n}) \cong \mathcal{U}(A_{d,n})$ such that $F(T) = T'_{d,n}$.

Proof. Since $T \subset \mathcal{U}(A_{d,n}) \subset \text{Aut}(A_{d,n})$ is an algebraic subtorus of dimension $n - 1$, and since $A_{d,n}$ is toric, by [BH03, Thm. p. 2] there exists $\varphi \in \text{Aut}(A_{d,n})$ such that $\varphi \circ T \circ \varphi^{-1} \subset T'_{d,n}$. Moreover, since $\mathcal{U}(A_{d,n})$ is a normal subgroup of $\text{Aut}(A_{d,n})$, $\varphi \circ T \circ \varphi^{-1} \subset T'_{d,n}$ and hence since $\dim T'_{d,n} = n - 1$, $\varphi \circ T \circ \varphi^{-1} = T'_{d,n}$. This proves that an algebraic isomorphism $F: \mathcal{U}(A_{d,n}) \rightarrow \mathcal{U}(A_{d,n})$, $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$ maps $T$ to $T'_{d,n}$. \hfill \Box

Let $Z$ be an irreducible affine variety of dimension $n \geq 2$ and $\psi: \mathcal{U}(Z) \cong \mathcal{U}(A_{d,n})$ be an algebraic isomorphism. Let $T$ be an $(n - 1)$-dimensional algebraic subtorus of $\mathcal{U}(Z)$. Then, after composing $\psi$ with a suitable algebraic isomorphism $F: \mathcal{U}(A_{d,n}) \rightarrow \mathcal{U}(A_{d,n})$ (see Lemma 6.15), we can assume that $\psi(T) = T'_{d,n}$.

Lemma 6.16. Root subgroups $U$ and $\psi(U)$ have the same weight characters with respect to $T$ and $\psi(T) = T'_{d,n}$, respectively: i.e., if $\chi: T'_{d,n} \rightarrow \mathbb{C}^*$ is the weight of $\psi(U)$, then the weight of $U$ is $\chi \circ \psi$.

Proof. Let $U$ be a root subgroup of $\mathcal{U}(Z)$ with respect to $T$ and $\text{Lie} U = C\nu$, where $\nu$ is a generator. Then $\psi(U)$ is the root subgroup of $\mathcal{U}(A_{d,n})$ with respect to $T'_{d,n}$. The algebraic isomorphism $\psi$ induces an isomorphism $d\psi_c^U: \text{Lie} U \rightarrow \text{Lie} (\psi(U))$. Note that the action of $T$ on $U$ induces the action of $T$ on Lie $U$. Then

$$
\psi(t) \circ d\psi_c^U(\nu) \circ \psi(t^{-1}) = \chi(\psi(t))d\psi_c^U(\nu) = d\psi_c^U(\chi(\psi(t))\nu) = d\psi_c^U(\chi \circ \psi(t)\nu),
$$

where $t \in T$. On the other hand,

$$
\psi(t) \circ d\psi_c^U(\nu) \circ \psi(t^{-1}) = d\psi_c^U(t \circ \nu \circ t^{-1}).
$$

The claim follows. \hfill \Box

Lemma 6.17. Let $d$ be even. Then the set of weights of root subgroups of $\text{Aut}(A_{d,2})$ with respect to $T'_{d,2}$ is $\{(kd + 2)/2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of $T'_{d,2}$.

Proof. By Proposition 3.3, any root subgroup of $\text{Aut}(A_{d,2})$ with respect to $T'_{d,2}$ lifts to a root subgroup of $\text{Aut}^u(A^2)$ with respect to $\phi^{-1}_d(T'_{d,2})^c = T'_2$. By Lemma 4.3, any root subgroup of $\text{Aut}(A^2)$ with respect to $T'_2$ is equal either to

$$
U_s = \{(x + cy^s, y) \mid c \in \mathbb{C}\}
$$

or to

$$
\tilde{U}_l = \{(x, y + cx^l) \mid c \in \mathbb{C}\}
$$

for some $s, l \in \mathbb{N} \cup \{0\}$. Root subgroups $U_s$ and $\tilde{U}_l$ belong to $\text{Aut}^u(A^2)$ if and only if $s, l \in d\mathbb{N} + 1$. The weight of the action of $T'_2 = \{(cx, c^{-1}y) \mid c \in \mathbb{C}^*\}$ on $U_s$ by $t \circ u \circ t^{-1}$, $t \in T'_2$ and $u \in U_s$ equals $s + 1$. Analogously, the weight of $T'_2$-action on $\tilde{U}_l$ is $-l - 1$. Therefore, the set of weights of root subgroups of $\text{Aut}^u(A^2)$ with respect to $T'_2$ is $\{kd + 2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of $T'_2$. Moreover, since the kernel of the map $\phi_d: T'_2 \rightarrow T_{d,2}$ is $\mu_2$ as $d$ is even, the statement follows from Lemma 6.14. \hfill \Box
By the Jung-Van der Kulk Theorem (see [Ju42] and [Kul53]), \( \text{Aut}(\mathbb{A}^2) = \text{Aff}_2 \ast_C J \), where \( \text{Aff}_2 \) is the group of affine transformations of \( \mathbb{A}^2 \) and

\[
J = \{(ax + c, by + f(x)) \mid a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \mathbb{C}[x]\}
\]
and \( C = \text{Aff}_2 \cap J \). Now, the subgroup \( \text{Aut}^{\mu_d}(\mathbb{A}^2) \subset \text{Aut}(\mathbb{A}^2) \) contains the standard \( \text{GL}_2 \subset \text{Aut}(\mathbb{A}^2) \) and

\[
J_d = \{(ax + c, by + f(x)) \mid a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \bigoplus_{l \geq 0} \mathbb{C}x^{ld+1}\}.
\]

By [AZ13, Thm. 4.2], \( \text{Aut}(A_{d,2}) \simeq \text{Aut}^{\mu_d}(\mathbb{A}^2)/\mu_d \) is the amalgamated product of \( \text{GL}_2/\mu_d \) and \( J_d/\mu_d \) along their intersection. Moreover, as we will see in Lemma 6.18 below, such an amalgamated product structure induces the amalgamated product structure of \( \mathcal{U}(A_{d,2}) \). Denote by \( \tilde{J}_d \) the subgroup of \( J_d \) of the form

\[
\left\{(ax + c, a^{-1}y + f(x)) \mid a \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \bigoplus_{l \geq 0} \mathbb{C}x^{ld+1}\right\}
\]
and by \( \tilde{T}_{d,2} \) the one-dimensional subtorus of \( \text{Aut}(A_{d,2}) \) induced by the \( \mathbb{C}^* \)-action on \( \mathbb{A}^2 \) given by the maps \( \{(x, y) \mapsto (cx, y) \mid c \in \mathbb{C}^*\} \). We have the following statement.

**Lemma 6.18.** The group \( \text{Aut}(A_{d,2}) \) is the semidirect product \( \mathcal{U}(A_{d,2}) \rtimes \tilde{T}_{d,2} \). Moreover, \( \mathcal{U}(A_{d,2}) \) is the amalgamated product of \( \text{SL}_2/(\mu_d \cap \text{SL}_2) \) and \( \tilde{J}_d/(\mu_d \cap \tilde{J}_d) \) along their intersection.

**Proof.** As \( \text{Aut}(A_{d,2}) \simeq \text{Aut}^{\mu_d}(\mathbb{A}^2)/\mu_d \), it is clear that \( \text{Aut}(A_{d,2}) \) is generated by \( \mathcal{U}(A_{d,2}) \) and \( \tilde{T}_{d,2} \). Further, the subgroup \( \mathcal{U}(A_{d,2}) \subset \text{Aut}(A_{d,2}) \) is normal and the subgroups \( \mathcal{U}(A_{d,2}) \) and \( \tilde{T}_{d,2} \) do not intersect. Indeed, if the subgroups \( \tilde{T}_{d,2} \) and \( \mathcal{U}(A_{d,2}) \) have a nontrivial intersection, then as \( \text{Aut}(A_{d,2}) \simeq \text{Aut}^{\mu_d}(\mathbb{A}^2)/\mu_d \), the subgroups \( \{(x, y) \mapsto (cx, y) \mid c \in \mathbb{C}^*\} \) and \( \text{Aut}(\mathbb{A}^2) \) of \( \text{Aut}(\mathbb{A}^2) \) also have a nontrivial intersection, which is not the case. Hence, we conclude that \( \text{Aut}(A_{d,2}) = \mathcal{U}(A_{d,2}) \rtimes \tilde{T}_{d,2} \).

Recall that \( \text{Aut}(A_{d,2}) \) is the amalgamated product of \( \text{GL}_2/\mu_d \) and \( J_d/\mu_d \) along their intersection and \( \text{GL}_2/\mu_d = \text{SL}_2/(\mu_d \cap \text{SL}_2) \times \tilde{T}_{d,2} \) and \( J_d/\mu_d = \tilde{J}_d/(\mu_d \cap \tilde{J}_d) \times \tilde{T}_{d,2} \). Since \( \tilde{T}_{d,2} \) is contained in the intersection \( \text{GL}_2/\mu_d \cap J_d/\mu_d \), it follows that

\[
\text{Aut}(A_{d,2}) = \tilde{T}_{d,2} \rtimes (\text{SL}_2/(\mu_d \cap \text{SL}_2) \ast_C \tilde{J}_d/(\mu_d \cap \tilde{J}_d)),
\]
where \( C \) is the intersection of \( A = \text{SL}_2/(\mu_d \cap \text{SL}_2) \) and \( B = \tilde{J}_d/(\mu_d \cap \tilde{J}_d) \). As both \( A \) and \( B \) are generated by unipotent subgroups it follows that \( A \ast_C B \subset \mathcal{U}(A_{d,2}) \). The other inclusion follows from (2). The claim follows. \( \square \)

**Remark 6.19.** Define the homomorphism of abstract groups

\[
\text{Aut}(A_{d,2}) = \tilde{T}_{d,2} \ltimes \mathcal{U}(A_{d,2}) \rightarrow \tilde{T}_{d,2}
\]
by projection onto the first factor. Such a homomorphism is a morphism of indgroups which implies that \( \mathcal{U}(A_{d,2}) \subset \text{Aut}(A_{d,2}) \) is a closed subgroup.
Remark 6.20. By Lemma 6.18, \( U(A_{d,2}) \) is the amalgamated product of the groups \( SL_2 / (\mu_d \cap SL_2) \) and \( \tilde{J}_d / (\tilde{J}_d \cap \mu_d) \) along their intersection. Note that if \( d \) is even, then \( SL_2 / (\mu_d \cap SL_2) \) is isomorphic to PSL, and if \( d \) is odd, then \( SL_2 / (\mu_d \cap SL_2) \) is isomorphic to SL.

The following result was pointed out to me by Hanspeter Kraft.

**Proposition 6.21.** Let \( Z \) be an irreducible affine normal variety of dimension 2.

(a) The groups \( U(SL_2 / T) \) and \( U(Z) \) are algebraically isomorphic if and only if \( Z \simeq SL_2 / T \) or \( Z \simeq A_{2,2} \).

(b) The groups \( U(SL_2 / N) \) and \( U(Z) \) are algebraically isomorphic if and only if \( Z \simeq SL_2 / N \) or \( Z \simeq A_{4,2} \).

**Proof.** Let \( X \) be isomorphic either to \( SL_2 / T \) or to \( SL_2 / N \). Then \( U(X) \) contains a copy of PSL (see Lemma 6.6 and Lemma 6.10, respectively). Hence, by Lemma 6.1, \( Z \) is isomorphic either to \( SL_2 / T \) to \( SL_2 / N \), or to \( A_{d,2} \) for some \( d \in \mathbb{N} \). We claim that \( Z \) can be isomorphic to \( A_{d,2} \) only if \( d \) is even. Indeed, assume that \( Aut(A_{d,2}) \) contains an algebraic subgroup \( S \) isomorphic to PSL. Since \( S \subset U(A_{d,2}) \) it follows from Lemma 6.18 that \( S \) is conjugate either to a subgroup of \( SL_2 / (\mu_d \cap SL_2) \) or \( \tilde{J}_d / (\mu_d \cap \tilde{J}_d) \) (see [Sr80]). Moreover, since \( \tilde{J}_d / (\mu_d \cap \tilde{J}_d) \) does not contain a copy of PSL, \( S \) should be conjugate to a subgroup of \( SL_2 / (\mu_d \cap SL_2) \). Hence, by Remark 6.20 we conclude that \( d \) is even.

By Corollary 6.12 we have \( U(SL_2 / T) \not\simeq U(SL_2 / N) \). Hence, to prove (a) we first need to show that an algebraic isomorphism \( \phi : U(A_{d,2}) \simeq U(SL_2 / T) \) implies that \( d = 2 \). By Lemma 6.17, the set of weights of root subgroups of \( U(A_{d,2}) \) with respect to \( T'_{d,2} \) is \( \{(kd + 2) / 2 \mid k \in \mathbb{N} \cup \{0\}\} \) up to an automorphism of \( T'_{d,2} \). Since \( T_{d,2} \) is a subgroup of some \( S \subset U(A_{d,2}) \) isomorphic to PSL we have by Proposition 6.8 that the set of weights of root subgroups of \( U(X \simeq SL_2 / T) \) with respect to \( \phi(T'_{d,2}) \) is \( \{1, 2, 3, \ldots \} \) up to an automorphism of \( \phi(T_{d,2}) \). By Lemma 6.16, the set of weights of root subgroups of \( U(A_{d,2}) \) with respect to \( T_{d,2} \) and of \( U(SL_2 / T) \) with respect to \( \phi(T_{d,2}) \) are equal. Therefore, \( d \) indeed equals 2. To finish the proof of (a) we need to show that \( U(A_{2,2}) \) and \( U(X \simeq SL_2 / T) \) are algebraically isomorphic. To do so, we first note that by Lemma 6.18 and Lemma 6.6 the first factors \( SL_2 / \mu_2 \) and \( SO_3 \) from the amalgamated product structure of \( U(A_{2,2}) \) and of \( U(X) \) respectively are isomorphic to PSL. Moreover, \( \tilde{J}_2 \) and \( J \) are algebraically isomorphic, as both \( \tilde{J}_2 \) and \( J \) are direct limits of isomorphic algebraic groups. Finally, the intersections \( SL_2 / \mu_2 \cap \tilde{J}_2 \subset Aut(A_{2,2}) \) and \( SO_3 \cap J \subset Aut(X) \) are also isomorphic as algebraic groups as they are both isomorphic to a Borel subgroup of PSL.

Define a homomorphism \( \varphi : U(A_{d,2}) \rightarrow U(X) \) that sends isomorphically the first factor \( SL_2 / \mu_2 \) of the amalgamated product of \( U(A_{d,2}) \) to the first factor \( SO_3 \) of the amalgamated product of \( U(X) \) in a way that \( \varphi(SL_2 / \mu_2 \cap \tilde{J}_2) = SO_3 \cap J \subset Aut(X) \) and the second factor \( \tilde{J}_2 \) of the amalgamated product of \( U(A_{d,2}) \) to the second factor \( J \) of the amalgamated product of \( U(X) \). Such a map is well defined and is an isomorphism as follows from the amalgamated product structure of \( U(A_{2,2}) \) and \( U(X \simeq SL_2 / T) \). The proof of (a) follows.

To prove (b) we first need to show that an algebraic isomorphism \( \phi : U(A_{d,2}) \simeq U(SL_2 / N) \) implies that \( d = 4 \). As we have already mentioned above in this proof,
the set of weights of root subgroups of $U(A_{d,2})$ with respect to $T'_d$ is $\{(kd + 2)/2 \mid k \in \mathbb{N}\}$ up to an automorphism of $T'_d$ (see Lemma 6.17). Further, analogously as in the first part of the proof we have that the set of weights of root subgroups of $U(Y \simeq \text{SL}_2 / N)$ with respect to $\phi(T'_d)$ is $\{1, 3, 5, \ldots \}$ up to an automorphism of $\phi(T'_d)$ (see Corollary 6.12). By Lemma 6.16, the set of weights of root subgroups of $U(A_{d,2})$ with respect to $T_d'$ and of $U(Y)$ with respect to $\phi(T'_d)$ coincide, which implies that $d = 4$. To finish the proof of (b) we need to show that groups $U(A_{d,2})$ and $U(Y \simeq \text{SL}_2 / N)$ are algebraically isomorphic. This follows analogously as in the previous paragraph in the case of groups $U(A_{2,2})$ and $U(X \simeq \text{SL}_2 / T)$. \hfill \Box

7. Higher-dimensional case

Consider the action of $\text{SL}_n$ on $A_{d,n}$ induced by the standard $\text{SL}_n$-action on $\mathbb{A}^n$. Denote by $S_{d,n} \subset \text{Aut}(A_{d,n})$ the image of $\text{SL}_n$ under the natural homomorphism $\text{SL}_n \to \text{Aut}(A_{d,n})$.

**Lemma 7.1.** We have an isomorphism $S_{d,n} \simeq \text{SL}_n / \mu_{(d,n)}$, where $(d,n)$ denotes the greatest common divisor of $d$ and $n$. Moreover, $S_{d,n} \subset U(A_{d,n})$.

**Proof.** By Proposition 3.3, there is a surjective homomorphism $\phi_d : \text{Aut}^d(\mathbb{A}^n) \to \text{Aut}(A_{d,n})$ of groups with ker $\phi_d = \mu_d$. Hence, $\text{Aut}(A_{d,n}) \simeq \text{Aut}^d(\mathbb{A}^n)/\mu_d$, which shows that $S_{d,n} \simeq \text{SL}_n / (\mu_d \cap \text{SL}_n) \simeq \text{SL}_n / \mu_{(d,n)}$. The second claim is clear since $S_{d,n}$ is generated by unipotent subgroups. \hfill \Box

**Lemma 7.2.** If there is an injective algebraic homomorphism

$$\varphi : S_{d,n} = \text{SL}_n / \mu_{(n,d)} \hookrightarrow U(A_{l,n}),$$

then $(n,d) = (n,l)$. In particular, if $U(A_{d,n})$ and $U(A_{l,n})$ are algebraically isomorphic, then $(d,n) = (l,n)$.

**Proof.** Applying Lemma 6.15, we can assume $\varphi(T_{d,n}) = T_{l,n}$. Hence, intersection $\varphi(S_{d,n}) \cap S_{l,n}$ contains $T_{l,n}$. We claim that $\varphi(S_{d,n}) = S_{l,n}$. To show this, we first note that the subgroup $T_{l,n} \subset \varphi(S_{d,n})$ lifts to $T'_n$ and by Proposition 3.3, each root subgroup of $\varphi(S_{d,n})$ with respect to $T_{l,n}$ lifts to a one-dimensional unipotent subgroup of $\text{Aut}(\mathbb{A}^n)$. Moreover, the subgroup $G$ of $\text{Aut}(\mathbb{A}^n)$ generated by all one-dimensional unipotent subgroups $U_i$ lifted from root subgroups of $\varphi(S_{d,n})$ with respect to $T_{l,n}$ is an algebraic subgroup of $\text{Aut}(\mathbb{A}^n)$. Indeed, if $G$ is not algebraic, then $G$ can not be written as a finite product of $U_i$. In contrast, $\varphi(S_{d,n})$ can be written as a finite product of root subgroups of $\varphi(S_{d,n})$ with respect to $T_{l,n}$. Moreover, $\phi_d$ induces a homomorphism of groups $G \to \varphi(S_{d,n})$ with a kernel $\mu_{(l,n)}$.

Hence, a homomorphism of groups $G \to \varphi(S_{d,n})$ is a homomorphism of algebraic groups with the kernel $\mu_{(l,n)}$ and $G$ is isomorphic to $\text{SL}_n$ that contains $T'_n$ as a maximal subtorus. It follows from [KRZ20, Thm. 1.1] that all subgroups of $\text{Aut}(\mathbb{A}^n)$ isomorphic to $\text{SL}_n$ are conjugate. Therefore, $G$ is conjugate to the standard $\text{SL}_n \subset \text{Aut}(\mathbb{A}^n)$: i.e., there exists $\psi \in \text{Aut}(\mathbb{A}^n)$ such that $\psi^{-1} \circ G \circ \psi = \text{SL}_n$. Since $T'_n \subset G$ we have that $\psi^{-1} \circ T'_n \circ \psi$ is a subtorus in $\text{SL}_n$ which implies that $\psi$ is a linear map that moreover belongs to $\text{GL}_n$. Now it is easy to see that $G$ coincides with $\text{SL}_n$. Therefore, $\varphi(S_{d,n}) = S_{l,n}$. 

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Therefore, $S_{d,n}$ is isomorphic to $S_{l,n}$ as an algebraic group. Hence, from Lemma 7.1 it follows that $(d, n) = (l, n)$. The second part of the statement follows from the first one directly since $U(A_{d,n})$ contains a copy of $S_{d,n}$. □

**Proposition 7.3.** Let $X$ be $A_{d,n}$, $SL_2 / T$ or $SL_2 / N$ and $Y$ be an irreducible affine variety. Assume that there is an algebraic isomorphism $U(X) \cong U(Y)$. Then $\dim Y \leq \dim X$. Moreover, if additionally $Y$ is normal, then

(a) if $X \simeq SL_2 / T$, then $Y \simeq A_{2,2}$ or $Y \simeq SL_2 / T$,
(b) if $X \simeq A_{2,2}$, then $Y \simeq A_{2,2}$ or $Y \simeq SL_2 / T$,
(c) if $X \simeq SL_2 / N$, then $Y \simeq A_{4,2}$ or $Y \simeq SL_2 / N$,
(d) if $X \simeq A_{4,2}$, then $Y \simeq A_{4,2}$ or $Y \simeq SL_2 / N$,
(e) if $X = A_{d,n}$, where $(d, n) \notin \{(2, 2), (2, 4)\}$, $Y \simeq X$.

**Proof.** Fix an algebraic isomorphism $\psi: U(X) \cong U(Y)$ and denote by $T'$ the image of $T_{d,n}'$ if $X = A_{2,2}$ or the image of a maximal subtorus $T$ of $U(X)$ if $X = SL_2 / T$ or $SL_2 / N$. By Lemma 6.14, Proposition 6.8, and Corollary 6.12, all root subgroups $U \subset U(Y)$ with respect to $T'$ have different weights. In particular, the root subgroups $O(Y)^U. U \subset U(Y)$ have different weights, which implies that $O(Y)^U$ is multiplicity-free because the map $O(Y)^U \rightarrow O(Y)^U. U$ is injective. Hence, by Lemma 4.2 we have that

$$\dim Y \leq \dim T' + 1 = n,$$

which proves the first part of the proposition.

Now (a), (b), (c) and (d) follow from Proposition 6.21.

To prove (e), we note that $U(A_{d,n})$ contains a copy of $SL_n / \mu_{(n,d)}$, which implies that $SL_n$ acts nontrivially on $Y$ and thus by Proposition 5.4, Lemma 6.1 and Proposition 7.3(a)-(d), $Y \simeq A_{l,n}$ for some $l \in \mathbb{N}$. Hence, $\psi: U(A_{d,n}) \cong U(A_{l,n})$.

By Lemma 6.15, there exists an algebraic isomorphism $F: U(A_{l,n}) \cong U(A_{l,n})$ such that $F(\psi(T_{d,n}')) = T_{l,n}'$. Therefore, we can assume that $\psi(T_{d,n}') = T_{l,n}'$.

Consider the $\mathbb{C}^+$-action

$$\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}$$
on $\mathbb{A}^n$.

It induces the $\mathbb{C}^+$-action $U$ on $A_{d,n}$ which is normalized by $T_{d,n}'$. Hence, $\psi(U) \subset U(A_{l,n})$ is a root subgroup with respect to $\psi(T_{d,n}') = T_{l,n}'$. By Proposition 3.3, $\psi(U)$ lifts to a $\mathbb{C}^+$-action on $\mathbb{A}^n$ normalized by $T_{l,n}'$. Since $(n, d) = (n, l)$ by Lemma 7.2 and because $U$ and $\psi(U)$ have the same weight characters with respect to $T_{d,n}'$ and $T_{l,n}'$ respectively (see Lemma 6.16), Lemma 6.14 implies that $\psi(U)$ lifts to a root subgroup

$$\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}$$
of $\text{Aut}(\mathbb{A}^n)$ with respect to $T_{l,n}'$. Therefore, $l \leq d$. Analogously, $d \leq l$, i.e., $d = l$. The proof follows. □

**Proof of Theorem 1.5.** Let $\psi: U(X) \cong U(Y)$ be an algebraic isomorphism. Proposition 7.3 implies that $\dim Y \leq \dim X$. Since $SL_n$ acts regularly and nontrivially on $X$, $SL_n$ also acts nontrivially and regularly on $Y$. 
First, let \( X \) be isomorphic to \( A_{d,n} \). Then by Lemma 6.1 and by Proposition 5.4, the normalization of \( Y \), which we denote by \( \tilde{Y} \), is isomorphic to \( SL_2/T \), \( SL_2/N \) or \( A_{l,n} \) for some \( l \geq 1 \). First, assume that \( \tilde{Y} \simeq A_{l,n} \). Hence Proposition 5.4 implies that

\[
O(Y) = \sum_{i=1}^{d} \bigoplus_{k \geq 0} \mathbb{C}[x_1, \ldots, x_n]_{kd_i}
\]

for some \( d_1, \ldots, d_l \in \mathbb{N} \), where \( (d_1, \ldots, d_l) = l \).

Let \( \eta: A_{l,n} \to Y \) be the normalization morphism, which by Lemma 5.3 induces the algebraic homomorphism \( \tilde{\eta}: \mathcal{U}(Y) \to \mathcal{U}(A_{l,n}) \). Note that \( SL_n/\mu(n,d) \) acts faithfully on \( X \). Then \( SL_n/\mu(n,d) \) also acts faithfully on \( Y \) and therefore on \( A_{l,n} \). Hence, by Lemma 7.2 we have that \( (n,d) = (n,l) \).

Consider the \( \mathbb{C}^+ \)-action

\[
\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}
\]

on \( \mathbb{A}^n \). It induces the \( \mathbb{C}^+ \)-action \( U \) on \( A_{d,n} \) which is normalized by \( T_{d,n} \). Hence, \( \psi(U) \subset \mathcal{U}(Y) \) is a root subgroup with respect to \( \psi(T_{d,n}') \). By Lemma 6.15, there is an algebraic isomorphism \( \mathcal{U}(A_{d,n}) \cong \mathcal{U}(A_{l,n}) \) that maps \( T_{d,n}' \) to \( \psi^{-1}(\tilde{\eta}^{-1}(T_{l,n}')) \) and so there is an isomorphism \( \mathcal{U}(Y) \cong \mathcal{U}(Y) \) that maps \( \psi(T_{d,n}') \) to \( \tilde{\eta}^{-1}(T_{l,n}') \). Hence, we can assume that \( \psi(U) \) is a root subgroup with respect to \( \tilde{\eta}^{-1}(T_{l,n}') \). By Lemma 5.3, \( \psi(U) \) lifts to a \( \mathbb{C}^+ \)-action on \( A_{l,n} \), which is normalized by \( T_{l,n}' \) and then by Lemma 3.2(c), \( \psi(U) \) lifts to a \( \mathbb{C}^+ \)-action on \( \mathbb{A}^n \) normalized by \( T_{l,n}' \). Since \( (n,d) = (n,l) \) and because \( U \) and \( \psi(U) \) have the same weight characters with respect to \( \tilde{\eta}^{-1}(T_{l,n}') \) and \( T_{l,n}' \) respectively (i.e., if \( \chi: T_{l,n}' \to \mathbb{C}^* \) is the weight of \( \psi(U) \), then the weight of \( U \) is \( \chi \circ \psi \)), Lemma 6.14 implies that \( \psi(U) \) lifts to a root subgroup

\[
\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}
\]

of \( \text{Aut}(\mathbb{A}^n) \) with respect to \( T_{l,n}' \). Hence

\[
\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}
\]

induces an action on \( O(\mathbb{A}^n)^{\mu_i} \), which implies that \( l \mid d \). Moreover, \( \{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\} \) induces an action on \( O(Y) \). This implies that

\[
d + l_i \in Nl_1 + \cdots + Nl_s
\]

for any \( i \).

The \( \mathbb{C}^+ \) action on \( \mathbb{A}^n \) of the form \( \{(x_1 + cx_2^{l_i+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\} \) induces an action on \( O(Y) \). Since \( (d,n) = (l,n) \) it follows that

\[
\{(x_1 + cx_2^{l_i+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}
\]

induces an action on \( O(\mathbb{A}^n)^{\mu_i} \).

Hence, \( d \mid l_i \) for any \( i \) and then \( d \mid (l_1, \ldots, l_s) = l \). Therefore, \( d = l \). Now, because \( d \mid l_i \) for any \( i \), \( d + l_i \in Nl_1 + \cdots + Nl_s \) implies that

\[
Nl_1 + \cdots + Nl_s = N_{\geq \min_{i \mid l_i} \{l_i\}_{i=1}^d}d,
\]
where \( N \geq k = \{ m \in \mathbb{N} \mid m \geq k \} \).

Now assume that \( \tilde{Y} \) is isomorphic to \( \text{SL}_2/T \) or to \( \text{SL}_2/N \), then by Proposition 6.3, \( Y = \tilde{Y} \). Then (e) follows from Proposition 6.21.

Let now \( X \cong \text{SL}_2/T \). Then by Lemma 6.1, \( \tilde{Y} \) can only be isomorphic to \( \text{SL}_2/T \), \( \text{SL}_2/N \) or \( A_{2,2} \). By Proposition 6.21, \( \tilde{Y} \) is isomorphic to \( \text{SL}_2/T \) or to \( A_{2,2} \). If \( \tilde{Y} \cong \text{SL}_2/T \), from Proposition 6.3, it follows that \( Y = \tilde{Y} \). Hence, (b) follows from the first part of the proof. Statement (d) follows analogously. \( \square \)

**Proof of Theorem 1.3.** The isomorphism \( \text{Aut}(X) \cong \text{Aut}(A_{d,n}) \) induces an algebraic isomorphism \( U(X) \cong U(A_{d,n}) \). Note that \( X \) admits a torus action of dimension \( n \). From Theorem 1.5, it follows that \( X \) can only be isomorphic to \( A_{d,n}^s \). On the other hand, since normalization of \( A_{d,n}^s \) is equal to \( A_{d,n} \), it follows from Lemma 5.3 that there is a closed embedding \( \text{Aut}(A_{d,n}^s) \hookrightarrow \text{Aut}(A_{d,n}) \) of ind-groups. Now the proof follows from [RvS21, Prop. 9.1(3)]. \( \square \)

**Proof of Theorem 1.4.** Let \( Z \) be isomorphic either to \( \text{SL}_2/T \) or to \( \text{SL}_2/N \). Then an isomorphism \( \text{Aut}(X) \cong \text{Aut}(Z) \) induces an algebraic isomorphism \( U(X) \cong U(Z) \). By Theorem 1.5, \( X \) is isomorphic either to \( Z \) or to \( A_{2k,2}^s \) for some \( s \in \mathbb{N} \) and \( k \in \{1, 2\} \). To finish the proof we need to show that \( \text{Aut}(Z) \) can not be isomorphic to \( \text{Aut}(A_{2k,2}^s) \). But this is clear as all \( A_{2k,2}^s \) admit an action of a two-dimensional torus and varieties \( \text{SL}_2/T \) and \( \text{SL}_2/N \) do not admit such an action. \( \square \)

**References**

[AFK13] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, M. Zaidenberg, *Flexible varieties and automorphism groups*, Duke Math. J. 162 (2013), no. 4, 767–823.

[AZ13] I. Arzhantsev, M. Zaidenberg, *Acyclic curves and group actions on affine toric surfaces*, in: Affine Algebraic Geometry, Osaka, March 3–6, 2011, World Scientific, Singapore, 2013, pp. 1–41.

[BH03] F. Berchtold, J. Hausen, *Demushkin’s theorem in codimension one*, Math. Z. 244 (2003), no. 4, 697–703.

[CRX19] S. Cantat, A. Regeta, J. Xie, *Families of commuting automorphisms, and a characterization of the affine space*, arXiv:1912.01567 (2019).

[DG77] V. I. Danilov, M. H. Gizatullin, *Автоморфизмов аффинных поверхностей II*, Izv. AN CCCP, Сер. матем. 41 (1977), вып. 1, 54–103. Engl. transl.: V. I. Danilov, M. H. Gizatullin, *Automorphisms of affine surfaces II*, Math. USSR Izv. 11 (1977), no. 1, 51–98.

[DP09] A. Dubouloz, P.-M. Poloni, *On a class of Danielewski surfaces in affine 3-space*, J. Algebra 321 (2009), no. 7, 1797–1812.

[Fre06] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopaedia of Mathematical Sciences, Vol. 136, Subseries Invariant Theory and Algebraic Transformation Groups, Vol. VII, Springer-Verlag, Berlin, 2006.

[FK18] J.-P. Furter, H. Kraft, *On the geometry of the automorphism groups of affine varieties*, arXiv:1809.04175 (2018).

[Giz71] M. Gizatullin, *Аффинные поверхности, квазисимметричные относительно алгебраической группы*, Изв. AN CCCP, Сер. матем. 35 (1971), вып.
Engl. transl.: M. Gizatullin, *Affine surfaces which are quasihomo-
geneous with respect to an algebraic group*, Math. USSR Izv. 5 (1971), no. 4,
754–769.

[Ju42] H. W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine
Angew. Math. 184 (1942), 161–174.

[KPZ17] S. Kovalenko, A. Perepechko, M. Zaidenberg, *On automorphism groups of
affine surfaces*, in: *Algebraic Varieties and Automorphism Groups*, Adv. Stud.
Pure Math., Vol. 75, Math. Soc. of Japan, 2017, pp. 207–286.

[Kr15] H. Kraft, *Automorphism groups of affine varieties and a characterization of
affine n-space*, Trans. Moscow Math. Soc. 78 (2017), 171–186.

[Kr84] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspekte der
Mathematik, Vol. D1, Vieweg Verlag, Braunschweig, 1984.

[KRvS19] H. Kraft, A. Regeta, I. van Santen, *Is the affine space determined by its
automorphism group?*, Int. Math. Res. Not. 2021, no. 6, 4280–4300.

[KRZ20] H. Kraft, A. Regeta, S. Zimmermann, *Small G-varieties*, arXiv:2009.05559
(2020).

[KS13] H. Kraft, I. Stampfli, *Automorphisms of the affine Cremona group*, Annales
de l’Institut Fourier 63 (2013), no. 3, 1137–1148.

[Kul53] W. van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wiskunde
(3) 1 (1953), 33–41.

[Kum02] S. Kumar, *Kac–Moody Groups, Their Flag Varieties and Representation
Theory*, Progress in Mathematics, Vol. 204, Birkhäuser Boston Inc., Boston,
MA, 2002.

[Lam05] S. Lamy, *Sur la structure du groupe automorphismes de certaines surfaces
affines*, Publ. Mat. 49 (2005), no. 1, 3–20.

[Lie11] A. Liendo, *Roots of the affine Cremona group*, Transform. Groups 16 (2011),
no. 4, 81–105.

[LRU18] A. Liendo, A. Regeta, C. Urech, *Characterization of affine surfaces with a
torus action by their automorphism groups*, to appear in Ann. Scuola Normale
Sup. Pisa, arXiv:1805.03991 (2020).

[LRU20] A. Liendo, A. Regeta, C. Urech, *On the characterization of Danielewski
surfaces by their automorphism groups*, Transform. Groups, DOI:10.1007/
S00031-020-09606-z (2020).

[MM09] K. Masuda, M. Miyanishi, *Lifting of the additive group scheme actions*, Tohoku
Math. J. 61 (2009), no. 2, 267–286.

[MM90] L. Makar-Limanov, *On groups of automorphisms of a class of surfaces*, Israel
J. Math. 69 (1990), 250–256.

[Neu48] H. Neumann, *Generalized free products with amalgamated subgroups*, Amer. J.
Math. 70 (1948), 590–625.

[Pop73] V. L. Popov, *Kвазиоднородные аффинные алгебраические многообразия
gруппы SL(2)*, Izv. AN CCCP. Сер. матем. 37 (1973), вып. 4, 792–832.
Engl. transl.: V. L. Popov, *Quasihomogeneous affine algebraic varieties of the
group SL(2)*, Math. USSR-Izv. 7 (1973), no. 4, 793–831.

[RvS21] A. Regeta, I. van Santen (born Stampfli), *Characterizing smooth affine sphé-
rical varieties via the automorphism group*, J. de l’École Polytechnique —
Mathématiques 8 (2021), 379–414.
[Sh66] I. R. Shafarevich, On some infinite-dimensional groups, Rend. Mat. e Appl. (5) 25 (1966), no. 1-2, 208–212.

[Sr58] J. P. Serre, Espaces fibrés algébriques, Sminaire Claude Chevalley 3 (1958), no. 1, 1–37.

[Sr80] J. P. Serre, Trees, Springer, Berlin, 1980.

Funding Information Open Access funding enabled and organized by Projekt DEAL.

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