a marche. Alors ...

MINIMAL COHOMOLOGY CLASSES AND JACOBIANS

Olivier Debarre (*)

The main purpose of this article is to describe all effective algebraic cycles with minimal cohomology class in the Jacobian of a complex curve. More precisely, let \((A, \theta)\) be a complex principally polarized abelian variety of dimension \(g\). For \(0 \leq d \leq g\), the cohomology class \(\theta_d = \theta^d/d!\) is minimal, i.e. non-divisible, in \(H^{2d}(A, \mathbb{Z})\). When \((A, \theta)\) is isomorphic to the Jacobian \((JC, \theta)\) of a curve \(C\) of genus \(g\), the image of the symmetric product \(C^{(g-d)}\) by any Abel-Jacobi map is a subvariety \(W_{g-d}(C)\) of \(JC\) with class \(\theta_d\) ([ACGH], p. 25).

Our main result (theorem 5.1) implies that any effective algebraic cycle in \(JC\) with class \(\theta_d\) is a translate of either \(W_{g-d}(C)\) or \(-W_{g-d}(C)\).

For \(1 < d < g\), I know of only one other family of principally polarized abelian varieties with an effective algebraic cycle with minimal class: in the 5-dimensional intermediate Jacobian \(JT\) of a cubic threefold \(T\) in \(\mathbb{P}^4\), the image by any Abel-Jacobi map of the Fano surface of lines contained in \(T\) is a surface with class \(\theta_3\) ([CG], [B]).

I suspect that these should be the only examples of effective algebraic cycles with class \(\theta_d\) on abelian varieties of dimension \(g\), when \(1 < d < g\). This holds for any \(g\) and \(d = g - 1\) by Matsusaka’s criterion ([M]), and for \(g = 4\) and \(d = 2\) by a result of Ran ([R1]).

In a second part, the main theorem is used to prove a weak version of this conjecture: for \(1 < d < g\), the Jacobian locus (resp. the locus of intermediate Jacobians of cubic threefolds) is an irreducible component of the set of principally polarized abelian varieties of dimension \(g\) for which \(\theta_d\) (resp. \(\theta_3\)) is the class of an effective algebraic cycle. This result was first proved by Barton and Clemens ([BC]) for \(g = 4\) and \(d = 2\); a slightly weaker form of it appeared later in [R1] (corollary III.1), but the proof is incomplete.

Throughout this article, we will be working over the field of complex numbers. A variety will be a reduced projective (complex) scheme of finite type.

The author would like to thank the M.S.R.I., where this research was done, for its hospitality and support.

I. Subvarieties With Minimal Classes In Jacobians

(*) Partially supported by N.S.F. Grant DMS 92-03919 and the European Science Project “Geometry of Algebraic Varieties”, Contract no. SCI-0398-C (A).
Let \((JC, \theta)\) be the Jacobian of a smooth curve \(C\) of genus \(g\). The aim of this part is to prove that any effective algebraic cycle in \(JC\) with class \(\theta_d\) is a translate of either \(W_{g-d}(C)\) or \(-W_{g-d}(C)\).

We begin with a few preliminaries.

1. **Non-degenerate subvarieties**

   This notion was introduced by Ran ([R1], [R2]). Let \(A\) be an abelian variety of dimension \(g\) and let \(W\) be a subvariety of \(A\) of pure dimension \(d\). Let \(W_{\text{reg}}\) be the smooth part of \(W\). We say that \(W\) is non-degenerate if the restriction map \(H^0(A, \Omega^d_A) \rightarrow H^0(W_{\text{reg}}, \Omega^d_{W_{\text{reg}}})\) is injective. By [R1], lemma II.1, this is equivalent to each one of the following properties:

   - the cup-product map \(\cdot[W]: H^{d,0}(A) \rightarrow H^{g-g-d}(A)\) is injective, where \([W] \in H^{g-d,g-d}(A)\) is the cohomology class of \(W\),
   - the contraction map \(\cdot\{W\}: H^{g,d}(A) \rightarrow H^{g-d,0}(A)\) is injective, where \(\{W\} \in H_{d,d}(A)\) is the homology class of \(W\).

   This allows the extension of the definition to effective algebraic cycles. Note that on a principally polarized abelian variety, an effective algebraic cycle with class a multiple of a minimal class is non-degenerate.

2. **The property \((P)\)**

   Let \(V\) and \(W\) be two irreducible subvarieties of an abelian variety \(A\) and let \(f: V \times W \rightarrow A\) be the addition map. We will say that \(V\) has property \((P)\) with respect to \(W\) if, for \(v\) generic in \(V\), the only irreducible subvariety of \(f^{-1}(v + W)\) which dominates both \(v + W\) via \(f\), and \(W\) via the second projection, is \(\{v\} \times W\).

   \((2.1)\) Note that this implies that \(\{v\} \times W\) is a component of \(f^{-1}(v + W)\), hence that the latter has the same dimension as \(W\) at some point, hence that \(f\) is generically finite onto its image; in particular, \(\dim(V) + \dim(W) \leq \dim(A)\). Note also that if \(V\) has property \((P)\) with respect to \(W\), then any translate of \(V\) has property \((P)\) with respect to any translate of \(W\).

**Example 2.2.** If \(C\) is a curve of genus \(g\), it is not difficult to check that \(W_d(C)\) has property \((P)\) with respect to \(W_e(C)\) whenever \(d + e \leq g\).

**Lemma 2.3.** Let \(V\) and \(W\) be two irreducible subvarieties of an abelian variety \(A\) and let \(g: W \times W \rightarrow A\) be the subtraction map. Then, \(V\) has property \((P)\) with respect to \(W\) if and only if, for \(v\) generic in \(V\), the only irreducible subvariety of \(g^{-1}(V - v)\) which dominates both factors \(W\), is the diagonal.

**Proof.** Let \(f: V \times W \rightarrow A\) be the addition map. Let \(v\) be generic in \(V\) and consider the automorphism \(h\) of \(A \times A\) defined by \(h(x, y) = (v + x - y, y)\). One checks that
\[ h(g^{-1}(V - v)) = f^{-1}(v + W). \] The proposition then follows from the fact that, if \( Z \) is a variety contained in \( g^{-1}(V - v) \), then \( f(h(Z)) = v + p_1(Z) \) and \( p_2(h(Z)) = p_2(Z). \)

**Proposition 2.4.** Let \( V \) and \( W \) be two irreducible subvarieties of an abelian variety \( A \). Assume that \( V \) has property \((P)\) with respect to \( W \). Then:

- the variety \((-V)\) has property \((P)\) with respect to \( W \),
- there exists a dense open set \( \Omega \) in \( V \) such that, if \( U \) is an irreducible subvariety of \( V \) which meets \( \Omega \), then \( U \) has property \((P)\) with respect to \( W \).

**Proof.** The first point follows from lemma 2.3. Let \( g : W \times W \to A \) be the difference map. Let \( \Omega \) be the open set of points \( v \) in \( V \) such that the only irreducible subvariety of \( g^{-1}(V - v) \) which dominates both factors is the diagonal. By the lemma, \( \Omega \) is dense in \( V \).

Take \( u \) in \( \Omega \cap U \); since \( U - u \subset V - u \), any irreducible subvariety of \( g^{-1}(U - u) \) which dominates both factors is also a subvariety of \( g^{-1}(V - u) \), hence is equal to the diagonal. By lemma 2.3, this proves the second point.

Although the following result can be checked directly, it is an easy consequence of lemma 2.3.

**Proposition 2.5.** Let \( C \) be a curve of genus \( g \) and let \( V \) be an irreducible subvariety of \( J_C \). Assume that \( V \) has property \((P)\) with respect to some \( W_d(C) \). Then \( V \) also has property \((P)\) with respect to any \( W_e(C) \) for \( 0 \leq e \leq d \).

**Proof.** Let \( g : W_d(C) \times W_d(C) \to J_C \) and \( h : W_e(C) \times W_e(C) \to J_C \) be the difference maps. Let \( v \) be generic in \( V \), and let \( Z \) be an irreducible subvariety of \( h^{-1}(v - V) \) which dominates both factors. Then \( \{ z + (w, w) \mid z \in Z, w \in W_{d-e}(C) \} \) is an irreducible subvariety of \( g^{-1}(v - V) \) which dominates both factors. By lemma 2.3, it is the diagonal of \( W_d(C) \times W_d(C) \), hence \( Z \) is the diagonal of \( W_e(C) \times W_e(C) \). By lemma 2.3 again, this proves that \( V \) has property \((P)\) with respect to \( W_e(C) \).

### 3. Ran’s theorem

Let \( V \) and \( W \) be two subvarieties of \( A \), of respective pure codimensions \( d \) and \( g - d \). Assume that \( s = V \cdot W > 0 \). The addition map \( V \times W \to A \) is then surjective, hence generically étale. It follows that for \( x \) generic in \( A \), the varieties \( V \) and \( x - W \) meet transversally at distinct smooth points \( v_1, \ldots, v_s \). Let \( P_i : T_0A \to T_0A \) be the projector with image \( T_{v_i}V \) and kernel \( T_{x-v_i}W \). Let \( c(V, W) \) be the endomorphism \( \sum_{i=1}^s \wedge^{g-d} P_i \) of \( \wedge^{g-d} T_0A \); Ran proves ([R1], theorem 2) that the transposed endomorphism \( ^t c(V, W) \) of \( \wedge^{g-d} T_0A \cong H^{g-d,0}(A) \) is equal to the composition:

\[
\begin{align*}
H^{g-d,0}(A) &\xrightarrow{-[V]} H^{g,d}(A) \xrightarrow{-[W]} H^{g-d,0}(A).
\end{align*}
\]

In particular, when \( V \) and \( W \) are non-degenerate, \( c(V, W) \) is an automorphism.
The following result of Ran on non-degenerate subvarieties with minimal intersection number motivates the somewhat abstruse definition of “property (P)”.

**Theorem 3.1.** (Ran) – If \( V \) and \( W \) are non-degenerate subvarieties of a \( g \)-dimensional principally polarized abelian variety \( (A, \theta) \), of respective pure codimensions \( d \) and \( g - d \), then \( V \cdot W \geq \left( \frac{g}{d} \right) \). If moreover \( W \) is irreducible and \( V \cdot W = \left( \frac{g}{d} \right) \), then \( V \) is irreducible and has property \( (P) \) with respect to \( W \).

**Proof.** Since \( c(V, W) \) is the sum of \( V \cdot W \) projectors of rank 1, we have:

\[
\left( \frac{g}{d} \right) = \text{rank}(c(V, W)) \leq V \cdot W.
\]

If there is equality, it follows from (7.4) in [R2] that \( W \) meets at most one component of \( V \). But by [R1], corollary II.6, \( W \) meets each component of \( V \). This implies that \( V \) is irreducible. To prove property \( (P) \), we keep the same notation as above and proceed as in the proof of [R1], theorem 5. If \( \alpha_i \) spans the line \( \bigwedge^{g-d} T_{v_i} V \) in \( \bigwedge^{g-d} T_0 A \), then \( \{\alpha_1, \ldots, \alpha_s\} \) (with \( s = \left( \frac{g}{d} \right) \)) is a basis for \( \bigwedge^{g-d} T_0 A \). The choice of an identification \( \bigwedge^n T_0 A \cong \mathbb{C} \) induces an isomorphism \( \bigwedge^d T_0 A \cong \bigwedge^{g-d} T_0^* A \). Let \( \beta_i \) be an element of \( \bigwedge^d T_{x-v_i} W \) such that \( \beta_i(\alpha_i) = 1 \). Then \( c(V, W) = \sum_{i=1}^s \alpha_i \otimes \beta_i \). In particular, for \( i \neq j \):

\[
\beta_i(\text{com}(V, W)^{-1}(\alpha_j)) = 0.
\]

Fix \( v = v_1 \) in \( V \) and let \( x \) vary in \( v + W \), so that \( v \in V \cap (x - W) \). We get:

\[
\bigwedge^d T_{x-v_i} W \wedge c(V, W)^{-1}(\bigwedge^{g-d} T_v V) = 0
\]

for \( i > 1 \). Since \( W \) is non-degenerate, the points \( x - v_1, \ldots, x - v_s \) must therefore describe a proper subvariety of \( W \). This proves the last part of the theorem, since \( p_2 f^{-1}(x) = W \cap (x - V) = \{x - v, x - v_1, \ldots, x - v_s\} \).

For example, if \( C \) is a curve of genus \( g \), the theorem implies that \( W_d(C) \) has property \( (P) \) with respect to \( W_{g-d}(C) \) for \( d \leq g \). Together with proposition 2.5, this proves the claim of example 2.2.

4. An auxiliary result

Let \( C \) be a smooth curve of genus \( g \) and let \( n > 1 \). We define a subset \( T_n \) of \( C^{(n)} \) as follows: consider all surjective morphisms \( C \to C' \) of degree \( r > 1 \), where \( C' \) is a smooth irrational curve. Such a morphism induces a map \( \psi : C' \to C^{(r)} \); let \( T_n \) be the union of all \( \psi(C') + C^{(n-r)} \), for \( r \leq n \), obtained in this way, of the inverse image in \( C^{(n)} \) of \( W_n^1(C) \) and of the diagonal \( 2C + C^{(n-2)} \). For \( n \leq g \), it is a proper closed subvariety of \( C^{(n)} \).

**Proposition 4.1.** – Let \( (JC, \theta) \) be the Jacobian of a smooth curve \( C \) of genus \( g \geq 2 \) and let \( Z \) be a subvariety of \( C^{(n)} \) of pure codimension \( m \). Then, for any \( E \) in \( C^{(n-m)} \), the variety
Z meets $E + C^m$. Moreover, if there exists $E$ in $C^{(n-m)}$ such that $Z \cdot (E + C^m) = 1$, then either $Z$ is contained in $T_n$, or there exist points $c_1, \ldots, c_m$ of $C$ such that $Z = c_1 + \ldots + c_m + C^{(n-m)}$.

**Proof.** The first part follows from the fact that the cohomology class of $E + C^m$ is the $(n-m)$-fold self-intersection of the cohomology class of the ample divisor $C^{(n-1)}$ in $C^n$ ([ACGH], pp. 309, 310).

Assume now that $Z \cdot (E + C^m) = 1$. We first do the case when $Z$ is a curve. We may assume that $Z$ is not contained in $x + C^{(n-1)}$ for any $x$ in $C$. Since $Z \cdot (x + C^{(n-1)}) = 1$ for any $x$ in $C$, the curve $Z$ is smooth and there exists a morphism $\tau : C \to C^{(n-1)}$ such that $Z = \{x + \tau(x) \mid x \in C\}$. Let $\Gamma$ be the curve $\{(x, \tau(x)) \mid x \in C\}$ in $C \times C^{(n-1)}$. If the induced morphism $\phi : \Gamma \to Z$ is not birational, $Z$ is contained in $T_n$.

Since $Z$ is smooth, we are left with the case where $\phi$ is an isomorphism. If $\tau$ is constant, the proof is over. Assume therefore that $\tau(C)$ is a curve. If $n = 2$, we get $Z = \{x + \tau(x) \mid x \in C\}$, where $\tau$ is an involution of $C$. In particular, $Z$ is contained in $T_2$. We assume $n \geq 3$ and proceed by induction. Let us show that the curve $\tau(C)$ satisfies the same property as $Z$. Let $x$ be a point of $C$ and assume that $\tau(y) = x + D$ and $\tau(y') = x + D'$ are both in $x + C^{(n-1)}$ for some $y$ and $y'$ on $C$. Then, $y + x + D$ and $y' + x + D'$ are on $Z$ hence the hypothesis on $Z$ implies $y + D = y' + D'$. Therefore, either $y = y'$ and $D = D'$, or there exists an element $E$ of $C^{(n-3)}$ such that $D = y' + E$ and $D' = y + E$. Then, both $(y, \tau(y))$ and $(y', \tau(y'))$ are sent by addition to $x + y + y' + E$ hence, since $\phi$ is an isomorphism, we get again $y = y'$ and $D = D'$. It follows that $\tau(C)$ and $x + C^{(n-2)}$ have a single common point and one checks that the intersection is transverse. We can therefore apply the induction hypothesis to $\tau(C)$: since we have assumed that $Z$ is not contained in any $x + C^{(n-1)}$, the curve $\tau(C)$ is in $T_{n-1}$. It follows that $Z$ is contained in $C + T_{n-1}$, hence in $T_n$. This finishes the proof of the lemma when $Z$ is a curve.

We now do the general case, by induction on the dimension of $Z$. Let $Z$ of dimension $n - m > 1$ satisfy the hypothesis. For $x$ generic in $C$, the inverse image of $Z$ by the map $C^{(n-1)} \to C^n$ which sends $D$ to $x + D$ satisfies the same property. Therefore, either $Z$ is contained in $C + T_{n-1}$, hence in $T_n$, or there exists a morphism $\tau : C \to C^m$ such that:

$$Z = \{x + \tau(x) + C^{(n-m-1)} \mid x \in C\}.$$

But $\tau$ must then be constant. This finishes the proof of the proposition. \[\blacksquare\]

5. The main theorem

We now prove our main result.

**Theorem 5.1.** Let $(JC, \theta)$ be the Jacobian of a smooth curve $C$ of genus $g$ and let $V$ be an effective non-degenerate algebraic $(g - d)$-cycle in $JC$ such that $V \cdot W_d(C) = \binom{g}{d}$. Then $V$ is a translate of either $W_{g-d}(C)$ or $-W_{g-d}(C)$.
We refer to §1 for the definition of a non-degenerate cycle. Recall that any effective algebraic cycle with class $\theta_d$ satisfies the hypotheses of the theorem.

**Proof.** By replacing any multiple component of $V$ by a sum of translates, we may assume that $V$ is a subvariety. Ran’s theorem 3.1 implies that $V$ is irreducible and has property $(P)$ with respect to $W_d(C)$. We first prove by induction on $g - d$ that any $(g - d)$–dimensional irreducible subvariety $V$ of $JC$ which has property $(P)$ with respect to $W_d(C)$ is a translate of some $W_r(C) - W_{g-d-r}(C)$, with $0 \leq r \leq g - d$. This is obvious for $g - d = 0$, hence we assume $g - d > 0$.

For any positive integer $e$, we define the addition map $f^+_e : V \times W_e(C) \to JC$ and the subtraction map $f^-_e : V \times W_e(C) \to JC$. When $e \leq d$, proposition 2.5 implies that $V$ has property $(P)$ with respect to $W_e(C)$, and so does $-V$ (proposition 2.4); it follows from (2.1) that $f^+_e$ and $f^-_e$ are finite onto their images.

**Proposition 5.2.** Let $V$ be an irreducible subvariety of $JC$ of codimension $d < g$ having property $(P)$ with respect to $W_d(C)$. Then there exists an irreducible subvariety $U$ of $JC$ such that either $V = U + C$ or $V = U - C$.

**Proof.** The maps $f^+_d$ are not birational ([R1], lemma II.15). Let $n$ be the smallest integer such that either $f^+_n$ or $f^-_n$ is not birational onto its image. Since both the hypotheses and conclusion of the lemma hold for $V$ if and only if they do for $-V$ (proposition 2.4), we may assume that $f^+_n$ is not birational onto its image. For $v$ generic in $V$, there exists an irreducible component $\Gamma$ of $(f^+_n)^{-1}(v + W_n(C))$, distinct from $\{v\} \times W_n(C)$, which dominates $v + W_n(C)$. Since $V$ has property $(P)$ with respect to $W_n(C)$ (proposition 2.5), the projection $p_2(\Gamma)$ is a subscheme $G$ of $W_n(C)$ of dimension $m < n$. For $D$ generic in $G$, there exists a subvariety $S_D$ of $V$ of dimension $n - m$ such that:

$$S_D + D \subset v + W_n(C).$$

Let $E$ be generic in $W_m(C)$. By proposition 4.1, the subvariety $E + W_{n-m}(C)$ of $W_n(C)$ meets $G$ at some point $D = E + E'$. For the same reason, the subvariety $S_D + D - v$ of $W_n(C)$ meets $E' + W_m(C)$ at $r$ points $E' + E_1, \ldots, E' + E_r$ with $r > 0$, i.e. there exist points $v_1, \ldots, v_r$ of $S_D \subset V$ such that:

$$v_i + (E + E') - v = E' + E_i, \quad \text{i.e.} \quad v_i - E_i = v - E.$$

Since $m < n$, the map $f^-_m : V \times W_m(C) \to JC$ is birational onto its image. Therefore, since $v$ and $E$ are generic, we must have $E_i = E$ for all $i$ and $v \in S_D$. In other words:

$$(S_D + D - v) \cap (E' + W_m(C)) = \{D\}$$

as sets. This is actually a scheme-theoretic equality; if $\epsilon$ is a tangent vector such that:

$$\epsilon \in T_D(S_D + D - v) = T_vS_D \subset T_vV,$$
$$\epsilon \in T_D(E' + W_m(C)) = T_EW_m(C),$$

6
then \((\epsilon, \epsilon)\) is in the kernel of the differential of \(f_m^-\) at \((v, E)\), hence is 0.

By proposition 4.1, \(E' + W_m(C)\) meets each component of \(S_D + D - v\), which must therefore be irreducible. Recall that since \(f_n^+(\Gamma) = v + W_n(C)\), the subvarieties \(S_D + D - v\) cover \(W_n(C)\) as \(D\) varies in \(G\). In particular, we may assume that \(S_D + D - v\) is not contained in the image of \(\mathcal{T}_n\) (defined in §4) in \(W_n(C)\). It then follows from proposition 4.1 applied to the strict transform of \(S_D + D - v\) in \(C^{(n)}\) that there exists an effective divisor \(E_D\) of degree \(m\) such that:

\[S_D + D - v = W_{n-m}(C) + E_D.\]

Moreover, since \(v \in S_D\), the divisor \(E_D' = D - E_D\) is effective. If \(c\) is in the support of \(E_D'\) and if \(x\) is any point of \(C\), the point \(v - c + x\) is in \(S_D\) hence in \(V\). Since \(v\) is generic, this implies that some irreducible component \(T\) of the scheme:

\[
\{(v, c) \in V \times C \mid v - c + C \subset V\}
\]
dominates \(V\). Note that \(U = f_1^-(T)\) satisfies \(U + C \subset V\). In particular, \(\dim U < \dim V \leq \dim T\). In general, if \(f_1^-\) contracts a subvariety \(F\) of \(V \times C\) to a point, the projection \(F \to C\) is an isomorphism. This implies that \(U = f_1^-(T)\) has dimension \(\dim T - 1\), and that \(V = p_1(T) \subset U + C\), hence \(V = U + C\). ■

Replacing \(V\) with \(-V\) if necessary, we may assume that \(V = U + C\). The following proposition shows that \(U\) satisfies the induction hypothesis.

**Proposition 5.3.** Let \(V\) be an irreducible subvariety of \(JC\) of codimension \(d < g\) having property \((P)\) with respect to \(W_d(C)\). Assume that \(V = U + C\). Then \(U\) has property \((P)\) with respect to \(W_{d+1}(C)\).

**Proof.** By proposition 2.4, \(U\) has property \((P)\) with respect to \(W_d(C)\), hence also with respect to \(C\) (proposition 2.5).

Let \(u\) be generic in \(U\) and let \(\phi_{d+1}^+ : U \times W_{d+1}(C) \to JC\) be the addition map. Let \(\Gamma_0\) be an irreducible subvariety of \(U \times W_{d+1}(C)\) such that \(\phi_{d+1}^+(\Gamma_0) = u + W_{d+1}(C)\) and \(p_2(\Gamma_0) = W_{d+1}(C)\). Let \(\Gamma\) be an irreducible subvariety of \(U \times C^{(d+1)}\) which maps onto \(\Gamma_0\) by the natural map:

\[\Pi_U = \text{Id}_U \times \pi_{d+1} : U \times C^{(d+1)} \to U \times W_{d+1}(C).\]

We have \(\phi_{d+1}^+ \Pi_U(\Gamma) = u + W_{d+1}(C)\) and \(p_2(\Gamma) = C^{(d+1)}\). Moreover, since \(\phi_{d+1}^+ \Pi_U\) is generically finite onto its image, the dimension of \(\Gamma\) is \(d + 1\). We need to show that \(\Gamma = \{u\} \times C^{(d+1)}\).

For any \(c\) in \(C\), each component of \(\Gamma \cap (\phi_{d+1}^+ \Pi_U)^{-1}(u + c + W_d(C))\) that dominates \(u + c + W_d(C)\) has dimension \(d\). Since \(p_2(\Gamma) = C^{(d+1)}\), at least one of these components, say \(\Gamma_c\), projects onto a divisor \(Z_c\) in \(C^{(d+1)}\). We use the elementary:
Lemma 5.4. - Let $Z$ be an irreducible divisor in $C^{(d+1)}$ and let $g : C \times C^{(d)} \to C^{(d+1)}$ be the natural map. Then, there exists an irreducible divisor $Z'$ in $C \times C^{(d)}$ such that $g(Z') = Z$ and $p_2(Z') = C^{(d)}$.

Proof. By proposition 4.1, $g^*Z \cdot (C \times \{D\}) = Z \cdot (C + D)$ is non-zero for any point $D$ in $C^{(d)}$, hence $p_2(g^{-1}(Z)) = C^{(d)}$. Some component $Z'$ of $g^{-1}(Z)$ must satisfy the same property. Its dimension is then $d$, hence it must map onto $Z$ by the finite map $g$.

Pick such a divisor $Z'_c$ for $Z_c$ and let $\Gamma'_c$ be a component of $Z'_c \times Z_c \Gamma_c$ that projects onto $Z'_c$. The situation is the following:

$$
\begin{array}{c}
\Gamma'_c \subset U \times C \times C^{(d)} \xrightarrow{Id_U \times g} U \times C^{(d+1)} \supset \Gamma_c \\
\downarrow \quad \quad \downarrow p_{23} \quad \quad \downarrow p_2 \\
Z'_c \subset C \times C^{(d)} \xrightarrow{g} C^{(d+1)} \supset Z_c \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow p_2 \\
C^{(d)} = C^{(d)}
\end{array}
$$

Since $Id_U \times g$ is finite, it maps $\Gamma'_c$ onto $\Gamma_c$. Set $\phi^+_{1,d} = \phi^+_1 \times Id_{C^{(d)}}$ and consider:

$$
U \times C \times C^{(d)} \xrightarrow{\phi^+_{1,d}} V \times C^{(d)} \xrightarrow{\Pi_V} V \times W_d(C) \xrightarrow{\phi^+_d} JC.
$$

Then:

$$\phi^+_d \Pi_V \phi^+_{1,d}(\Gamma'_c) = \phi^+_{d+1} \Pi_U(\Gamma_c) = u + c + W_d(C)$$

$$p_2 \Pi_V \phi^+_{d}(\Gamma'_c) = \pi_d p_3(\Gamma'_c) = W_d(C).$$

Since $V$ has property $(P)$ with respect to $W_d(C)$, this implies:

$$\Pi_V \phi^+_{1,d}(\Gamma'_c) = \{u + c\} \times W_d(C)$$

for $c$ generic. Therefore:

$$\Gamma_c = \{u'\} \times (c' + C^{(d)}) \quad \text{and} \quad Z_c = c' + C^{(d)},$$

for some points $u'$ of $U$ and $c'$ of $C$ such that $u' + c' = u + c$. The union as $c$ varies in $C$ of all $Z_c$'s must be $C^{(d+1)}$. It follows that $c'$ must describe the whole of $C$. Recall that $U$ has property $P$ with respect to $C$, hence that the only component of $(\phi^+_1)^{-1}(u + C)$ which dominates both $u + C$ via $\phi^+_1$ and $C$ via $p_2$ is $\{u\} \times C$. This implies $\Gamma_c = \{u\} \times (c + C^{(d)})$ for $c$ generic, hence the proposition. ■
It then follows from our induction hypothesis that $U$ is a translate of some $W_{r-1}(C) - W_{g-d-r}(C)$, hence that $V$ is a translate of $W_r(C) - W_{g-d-r}(C)$.

When $C$ is hyperelliptic, $-C$ is a translate of $C$, hence $-W_{g-d-r}(C)$ is a translate of $W_{g-d-r}(C)$, and $V$ is a translate of $W_{g-d}(C)$. To conclude the proof of the theorem, we may therefore assume that $C$ is non-hyperelliptic.

Lemma 5.5. – Take $r, d \geq 0$ with $d < g$ and $d + r \leq g$, and assume that $C$ is non-hyperelliptic. Then:

$$(W_r(C) - W_{g-d-r}(C)) \cdot W_d(C) = \binom{g-d}{r} \binom{g}{d}.$$  

Proof. The subtraction map $W_r(C) \times W_{g-d-r}(C) \to JC$ is birational onto its image. In fact, if $(D, E)$ and $(D', E')$ are two distinct elements of $W_r(C) \times W_{g-d-r}(C)$ such that $D - E \equiv D' - E'$, then:

- either $h^0(C, D + E') = 1$, in which case $(D, E)$ belongs to the divisor \{ $(D, E) \mid \text{Supp}(D) \cap \text{Supp}(E) \neq \emptyset$ \},
- or $h^0(C, D + E') > 1$, in which case $(D, E)$ belongs to:

$$\{ (D, E) \mid \exists L \in W^1_{g-d}(C) \quad D, E \leq |L| \}.$$  

This locus has dimension $\max_{a>0} \left( \dim W^a_{g-d}(C) + 2a \right)$, which is less than $g - d$ if $C$ is non-hyperelliptic (Martens' theorem, [ACGH], p. 191).

If $\ast$ denotes the Pontryagin product on $H^\bullet(JC, \mathbb{Q})$, it follows that the cohomology class of $(W_r(C) - W_{g-d-r}(C))$ is:

$$\theta_{g-r} \ast \theta_{d+r} = \binom{g-d}{r} \theta_d.$$  

Its intersection number with $W_d(C)$ is therefore:

$$\binom{g-d}{r} \theta_d \cdot \theta_{g-d} = \binom{g-d}{r} \binom{g}{d}.$$  

This proves the lemma. □

But this intersection number is equal to $V \cdot W_d(C)$, hence to $\binom{g}{d}$. It follows that either $r = g - d$, or $r = 0$, hence either $V$ is a translate of $W_{g-d}(C)$, or it is a translate of $-W_{g-d}(C)$. □
6. Preliminaries

For any positive integer \( g \), let \( \mathcal{A}_g \) be the moduli space of complex principally polarized abelian varieties of dimension \( g \), let \( J_g \) be the closure in \( \mathcal{A}_g \) of the subvariety which corresponds to Jacobians of smooth curves of genus \( g \), and let \( \mathcal{CT}_5 \) be the closure in \( \mathcal{A}_5 \) of the subvariety which corresponds to intermediate Jacobians of smooth cubic threefolds. For \( 0 < d \leq g \), the subset \( \mathcal{C}_{g,d} \) of \( \mathcal{A}_g \) which corresponds to principally polarized abelian varieties \( (A, \theta) \) for which \( \theta_d \) is the class of an effective algebraic cycle (so that \( \mathcal{C}_{g,1} = \mathcal{C}_{g,g} = \mathcal{A}_g \)), is closed in \( \mathcal{A}_g \). Indeed, let \( \mathcal{X} \to S \) be a versal family of abelian varieties of dimension \( g \) with a relatively ample line bundle \( L \) on \( \mathcal{X} \) which induces principal polarizations on the fibers, such that the classification morphism \( S \to \mathcal{A}_g \) is a finite cover. Fix an embedding of \( \mathcal{X} \) in some projective space \( \mathbb{P}_S^N \), using for example the sections of \( L \otimes 3 \). Chow coordinates show that the family of effective cycles in \( \mathcal{X} \) whose fibers over \( S \) have codimension \( d \) and degree \( \theta_d(3\theta)^{g-d} \), is projective over \( S \). In particular, so is its closed subset which parametrizes effective cycles with class \( \theta_d \) in the fibers. Consequently, its image in \( S \) is also closed, hence so is its image \( \mathcal{C}_{g,d} \) in \( \mathcal{A}_g \).

7. Degenerations of abelian varieties

Let \( \Delta = \{ t \in \mathbb{C} \mid |t| < 1 \} \). A degeneration of principally polarized abelian varieties of dimension \( g + 1 \) will be a proper family \( \mathcal{X} \to \Delta \), with \( \mathcal{X} \) smooth, whose fibers over \( \Delta^* \) are principally polarized abelian varieties of dimension \( g + 1 \) and whose central fiber \( X \) is a projective variety whose normalization is a \( \mathbb{P}^1 \)-bundle \( P \) over a principally polarized abelian variety \( (A, \theta) \) of dimension \( g \). More precisely, there exists an element \( a \) of \( A \) such that \( P = P(\mathcal{O}_A \oplus \mathcal{O}_A(\bar{a})) \), where \( \bar{a} \) is the image of \( a \) by the canonical isomorphism \( \phi_a : A \to \text{Pic}^0(A) \). The bundle \( P : P \to A \) has two disjoint sections \( P_0 \) and \( P_\infty \), and \( X \) is obtained from \( P \) by identifying any point \( x \) on \( P_0 \) with the point \( x - a \) on \( P_\infty \). The singular locus \( X_s \) of \( X \) will always be identified with \( A \) via the isomorphism \( X_s \to P_0 \to A \).

By [N], theorem 16.1, we may also assume the existence of a relatively ample line bundle \( L \) on \( \mathcal{X} \) which induces the principal polarization on the smooth fibers and which restricts on \( X \) to an ample line bundle whose pull-back to \( P \) is \( \mathcal{O}_P(p^*\Theta + P_0) \simeq \mathcal{O}_P(p^*\Theta_a + P_\infty) \), for a suitable representative \( \Theta \) of the polarization \( \theta \). Another (more accessible) reference for the existence of \( L \) is [HW], proposition 4.1.3 (the proof is given in the surface case, but works in general).

The projective varieties \( X \) with their polarizations correspond to the boundary points of \( \mathcal{A}_{g+1} \) in a suitable partial compactification \( \overline{\mathcal{A}}_{g+1} \), where they form a divisor \( \partial \mathcal{A}_{g+1} \). There is a surjective map \( q : \partial \mathcal{A}_{g+1} \to \mathcal{A}_g \) which, in the above notation, sends \( X \) with its polarization to \( (A, \theta) \), and whose generic fiber is isomorphic to \( A/\pm 1 \). For any subvariety
Proposition 7.1. – For $1 < d < g + 1$, the image by $q$ of the boundary $\partial \mathcal{C}_{g+1,d}$ is contained in $\mathcal{C}_{g,d} \cap \mathcal{C}_{g,d-1}$.

Proof. Keeping the same notation, we assume moreover that the smooth fibers of $\mathcal{X} \to \Delta$ correspond to elements of $\mathcal{C}_{g+1,d}$. Then, there exists a subvariety $Z$ of $\mathcal{X}$ such that the cycles associated with the fibers of $Z \to \Delta$ over $\Delta^*$ are sums of effective cycles, each with class $\theta_d$. Over the open interval $(0,1)$, the scheme $Z$ contains a subscheme whose fibers have class $\theta_d$. Its topological closure meets the central fiber $X$ in a union $Z = Z_1 \cup \cdots \cup Z_s$ of irreducible algebraic subvarieties of codimension $d$. For $i = 1, \ldots, s$, let $Y_i$ be the closure in $\mathbf{P}$ of the inverse image of $Z_i$ by the bijection $\mathbf{P} - \mathbf{P}_\infty \to X$. Then:

$$
(7.2) \quad \sum_{i=1}^s m_i[Y_i] = \left(p^*\theta + [\mathbf{P}_0]\right)^d/d! = p^*\theta_d + [\mathbf{P}_0] : p^*\theta_{d-1},
$$

for some positive integers $m_1, \ldots, m_s$. It follows that the image of $\sum_{i=1}^s m_i Y_i$ in $\mathbf{A}$ has class $\theta_{d-1}$, and that its intersection with $\mathbf{P}_\infty$ has class $\theta_d$. This proves the proposition. ■

8. Proof of the theorem

Theorem 8.1. – For $1 < d < g$, the locus $\mathcal{J}_g$ is a component of $\mathcal{C}_{g,d}$. Moreover, $\mathcal{C}T_5$ is a component of $\mathcal{C}_{5,3}$ and is not contained in $\mathcal{C}_{5,2}$.

Proof. The proof of the first part is by induction on $g$. Since this property is empty for $g = 2$, we assume that it holds for some $g \geq 2$ and show that it then holds in dimension $g + 1$. Let $1 < d < g + 1$ and let $\mathcal{F}$ be a component of $\mathcal{C}_{g+1,d}$ which contains either $\mathcal{J}_{g+1}$ or $\mathcal{C}T_5$ (if $g = 4$). It is known that $q(\partial \mathcal{J}_{g+1}) = \mathcal{J}_g$ and $q(\partial \mathcal{C}T_5) = \mathcal{J}_4$. It follows from proposition 7.1 and our induction hypothesis that $\mathcal{J}_g$ is a component of $q(\partial \mathcal{F})$. With the notation of the proof of proposition 7.1, we may therefore assume that $(\mathbf{A}, \theta)$ is the Jacobian $(\mathcal{J}, \theta)$ of a generic smooth curve $C$ of genus $g$. Then, $\partial \mathcal{J}_g \cap q^{-1}\{(\mathcal{J}, \theta)\}$ corresponds to the image of the surface $C - C$ in $\mathcal{J}/\pm 1$. For $g = 4$, the curve $C$ has two pencils $g_3^1$ and $h_3^1$ of degree 3, and $\partial \mathcal{C}T_5 \cap q^{-1}\{(\mathcal{J}, \theta)\}$ corresponds to the point $\pm(g_3^1 - h_3^1)$ of $\mathcal{J}/\pm 1 (|C|)$.

We will show that $\partial \mathcal{C}_{g+1,d} \cap q^{-1}\{(\mathcal{J}, \theta)\}$ is equal to $(C - C) \cup \{\pm(g_3^1 - h_3^1)\}$ for $g = 4$ and $d = 3$, and to $C - C$ otherwise.

Keeping the notation of the proof of proposition 7.1, the image in $\mathcal{J}C$ of the effective cycle $Y = \sum_{i=1}^s m_i Y_i$ has class $\theta_{d-1}$. By theorem 5.1, it is a translate of $\pm W_{g-d+1}(C)$ which, possibly after translation and action of $(-1)_X$, we may assume to be $W_{g-d+1}(C)$. Consequently, after reindexing if necessary, we have $m_1 = 1$, the morphism $Y_1 \to p(Y_1)$ is birational, and $Y_i = p^{-1}(p(Y_i))$ for $i > 1$. Identity (7.2) and theorem 5.1 then imply that:
Case 1: either $Y = Y_1$ maps birationally onto $W_{g-d+1}(C)$ and meets $P_0$ and $P_\infty$ along subvarieties which are translates of $\pm W_{g-d}(C)$.

Case 2: or $Y = Y_1 + Y_2$, where $Y_1$ has class $[P_0] \cdot p^* \theta_{d-1}$ and maps isomorphically onto $W_{g-d+1}(C)$, and $Y_2$ is the pull-back by $p$ of a translate $V$ of $\pm W_{g-d}(C)$. In this case, either $Y_1$ is contained in $P_0$ (which means that $Z$ has a component contained in the singular locus $X_s$ of $X$, which may well happen, contrary to what is asserted in [BC] on p. 64, or it meets neither $P_0$ nor $P_\infty$. In the latter case, the line bundle $O_P(P_0 - P_\infty) = p^*O_{JC}(a)$ is trivial on $Y_1$. Since the restriction $\text{Pic}^0(JC) \to \text{Pic}^0(W_{g-d+1}(C))$ is injective, this implies that $a$ is 0 hence is in $C - C$. We may therefore assume that $Y_1$ is contained in $P_0$.

The following result makes the structure of the limit cycle more precise.

**Lemma 8.2.** Let $z$ be a point of $Z \cap X_s$. Then either $z$ belongs to a component of $Z$ contained in $X_s$, or each of the two branches of $X$ contains a local component of $Z$ at $z$.

**Proof.** Let $\varepsilon : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up of the smooth subscheme $X_s$. The strict transform $\tilde{X}$ of $X$ can be identified with $P$. Let $\tilde{Z}$ be the strict transform of $Z$. If no component of $Z$ through $z$ is contained in $X_s$, the scheme $Z \cap X_s = Z \cap X_s$ has pure codimension 2 in $Z$ at $z$. It follows that its ideal in $Z$ is not invertible at $z$, hence that the rational curve $\varepsilon^{-1}(z)$ is contained in $\tilde{Z}$. This curve meets $\tilde{X}$ at two points (one on $P_0$, the other on $P_\infty$) which correspond to the two branches of $X$ at $z$. It follows that the intersection of $\tilde{Z}$ with the Cartier divisor $\tilde{X}$ is non-empty, hence has dimension $\dim \tilde{Z} - 1 = g + 1 - d$, at each of these two points. Consequently, the intersection of $Z$ with each of the two branches of $X$ has dimension $g + 1 - d$ at $z$, and this proves the lemma.

In case 1, this means that $V = Y \cap P_0$ and $Y \cap P_\infty$ must be identified through the gluing process, hence that $Y \cap P_\infty = V - a$. In particular, both $V$ and $V - a$ are contained in $W_{g-d+1}(C)$.

In case 2, both $V$ and $V - a$ must be again contained in $W_{g-d+1}(C)$.

Recall that in both cases, $V$ is a translate of $\pm W_{g-d}(C)$.

**Lemma 8.3.** Let $C$ be a smooth curve of genus $g$ and let $a$ be an element of $JC$. Assume that some translate of $W_{m+1}(C)$ contains $\varepsilon W_m(C)$ and $\varepsilon W_m(C) - a$, for some $\varepsilon = \pm 1$ and $0 \leq m \leq g - 2$. Then either $a \in C - C$, or $\varepsilon = -1$, $g = 4$ and $a$ is the difference of two $g_3^1$'s.

**Proof.** Let $L_a$ be the line bundle of degree 0 on $C$ associated with $a$. If $\varepsilon = 1$, there exists a line bundle $L$ on $C$ of degree 1 such that:

$$h^0(C, L(c_1 + \cdots + c_m)) > 0 \quad \text{and} \quad h^0(C, L \otimes L_a^{-1}(c_1 + \cdots + c_m)) > 0,$$

for any points $c_1, \ldots, c_m$ of $C$. Riemann-Roch implies that both $L$ and $L \otimes L_a^{-1}$ have a non-zero-section, hence that $a \in C - C$. 

12
If \( \epsilon = -1 \), we may assume that \( C \) is non-hyperelliptic and \( 0 < m \leq g - 3 \) (otherwise, \( -W_{m+1}(C) \) is a translate of \( W_{m+1}(C) \)). There exists a line bundle \( L \) on \( C \) of degree \( 2m + 1 \) such that:

\[
h^0\left( C, L(-c_1 - \cdots - c_m) \right) > 0 \quad \text{and} \quad h^0\left( C, L \otimes L^{-1}_{\alpha}(-c_1 - \cdots - c_m) \right) > 0 ,
\]

for all points \( c_1, \ldots, c_m \) of \( C \). Riemann-Roch implies that both \( L \) and \( L \otimes L^{-1}_{\alpha} \) are \( g_{m+1}^{2m+1} \)'s. By [ACGH], exercise B-7, p. 138, this implies under our assumptions that \( m = g - 3 \), and that the residual series of both \( L \) and \( L \otimes L^{-1}_{\alpha} \) are \( g_3^1 \)'s. If \( g > 4 \), there is at most one \( g_3^1 \) and \( a = 0 \). If \( g = 4 \), there are at most two \( g_3^1 \)'s and \( a \) is either 0 or their difference.

This proves that \( \partial C_{g+1,d} \) coincides over \( J_g \) with \( \partial (J_{g+1} \cup C T_5) \) for \( g = 4 \) and \( d = 3 \), and with \( \partial J_{g+1} \) otherwise. In particular:

\[
\dim \partial F \leq 2 + \dim J_g = 3g - 1 \quad \text{if} \quad F \supset J_{g+1} ,
\]

\[
\dim \partial F \leq \dim J_4 = \dim C T_5 - 1 \quad \text{if} \quad F \supset C T_5 .
\]

Since \( \partial F \) is the intersection of \( F \) with the Cartier divisor \( \partial A_{g+1} \), we have \( \dim F \leq \dim \partial F + 1 \). It follows that either \( F = J_{g+1} \) or \( F = C T_5 \), which finishes the proof of the induction step.

To conclude, note that if \( C \) is non-hyperelliptic of genus 4, then \( g_3^1 - h_3^1 \) is not in \( C - C \) if it is non-zero. Therefore, \( C T_5 \) is not contained in \( C_{5,2} \).

Note that we only needed for \( C - C \) to be a component of \( \partial C_{g+1,d} \cap q^{-1}\{(J C, \theta)\} \) for the proof. Our more precise result, combined with the fact that \( J_4 = C_{4,3} = C_{4,2} \), proves:

\[
\partial C_{5,2} = \partial J_5 \quad \text{and} \quad \partial C_{5,3} = \partial (J_5 \cup C T_5) .
\]

This makes the following conjecture plausible, at least in dimension 5:

**Conjecture** – For \( 1 < d < g \) and \( (g,d) \neq (5,3) \), then \( C_{g,d} = J_g \); furthermore, \( C_{5,3} = J_5 \cup C T_5 \).

A more tractable version would be the weaker:

**Conjecture’** – For \( 0 < d < g \), then \( C_{g,d} \cap C_{g,g-d} = J_g \).
References

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J., Geometry of algebraic curves. I. Grundlehren 267, Springer-Verlag, New York, 1985.

[BC] Barton, C., Clemens, C.H., A result on the integral Chow ring of a generic principally polarized complex Abelian variety of dimension four, Comp. Math. 34 (1977), 49–67.

[B] Beauville, A., Sous-variétés spéciales des variétés de Prym, Comp. Math. 45 (1982), 357–383.

[CG] Clemens, C.H., Griffiths, P.A., The Intermediate Jacobian of the Cubic Threefold, Ann. of Math. 95 (1972), 281–356.

[C] Collino, A., A cheap proof of the irrationality of most cubic threefolds, Boll. Un. Mat. Ital. 16 (1979), 451–465.

[HW] Hulek, K., Weintraub, S., The Principal Degenerations of Abelian Surfaces and Their Polarizations, Math. Ann. 286 (1990), 281–307.

[M] Matsusaka, T., On a characterization of a Jacobian Variety, Mem. Coll. Sc. Kyoto, Ser. A, 23 (1959), 1–19.

[N] Namikawa, Y., A New Compactification of the Siegel Space and Degeneration of Abelian Varieties, I, II, Math. Ann. 221 (1976), 97–141, 201–241.

[R1] Ran, Z., On subvarieties of abelian varieties, Invent. Math. 62 (1981), 459–479.

[R2] Ran, Z., A Characterization of Five-Dimensional Jacobian Varieties, Invent. Math. 67 (1982), 395–422.

Olivier Debarre
URA D 0752 du CNRS
Mathématique – Bâtiment 425
91405 Orsay Cedex – France
debarre@matups.matups.fr

Department of Mathematics
The University of Iowa
Iowa City IA 52242 – U.S.A.
debarre@math.uiowa.edu