ON GENERALIZED MAASS RELATIONS FOR THE MIYAWAKI-IKEDA LIFT

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Abstract. Some generalizations of the Maass relation for Siegel modular forms of higher degrees have been obtained by several authors. In the present article we first give a new generalization of the Maass relation for Siegel-Eisenstein series of arbitrary degrees. Furthermore, we show that the Duke-Imamoglu-Ibukiyama-Ikeda lifts satisfy this generalized Maass relation with some modifications. As an application of the generalized Maass relation in the present article we give a new proof of the Miyawaki-Ikeda lifts of two elliptic modular forms. Namely, we compute the standard $L$-function of the Miyawaki-Ikeda lifts of two elliptic modular forms by using the generalized Maass relation.

1. Introduction

1.1. The Maass relation is a relation among Fourier coefficients of Siegel-Eisenstein series of degree two, and the Maass relation characterizes the Saito-Kurokawa lifts (cf. [E-Z 85].) In his article [Ya 86] Yamazaki has obtained a generalization of the Maass relation for Siegel-Eisenstein series of arbitrary degrees. Furthermore, in [Ya 89] Yamazaki obtained a relation among Jacobi-Eisenstein series of arbitrary degrees. Here the Jacobi-Eisenstein series is a Jacobi form which is constructed like a Siegel-Eisenstein series. This relation among Jacobi-Eisenstein series is necessary to obtain a new generalization of the Maass relation, which is different from the generalized Maass relation in [Ya 86]. However, the relation among Jacobi-Eisenstein series in [Ya 89] is not enough to obtain a new generalization of the Maass relation, because in [Ya 89] the Jacobi-Eisenstein series of index 1 is treated and we need the relation among the Jacobi-Eisenstein series of arbitrary index. One of the aim of the present article is to generalize the relation among Jacobi-Eisenstein series obtained in [Ya 89] for arbitrary index and to give a new generalization of the Maass relation for Siegel-Eisenstein series of general degrees.

On the other hand, a generalization of the Saito-Kurokawa lift for Siegel modular forms of even degrees was conjectured by Duke and Imamoglu, and by Ibukiyama, independently, and the conjecture was solved by Ikeda [Ik 01]. In the present article, we
call these lifts the Duke-Imamoglu-Ibukiyama-Ikeda lifts. It is known that the Duke-
Imamoglu-Ibukiyama-Ikeda lifts satisfy the generalized Maass relations in [Ya 86] by in-
serting the Satake parameters of the preimage of the Duke-Imamoglu-Ibukiyama-Ikeda
lift into the relation (cf. [Ha 11].)

By applying the Duke-Imamoglu-Ibukiyama-Ikeda lift, Ikeda [Ik 06] solved and gener-
alized one of the two conjectures posed by Miyawaki [Mi 92] under a certain assumption.
Namely, he obtained lifts from pairs of an elliptic modular form and a Siegel modular
form of degree $r$ to Siegel modular forms of degree $2n + r$ under the assumption that
the constructed Siegel modular form does not vanish identically. In the present article
we call these lifts the Miyawaki-Ikeda lifts. In [Ik 06] Ikeda obtained a conjecture about
the relation between the Petersson norm of the Miyawaki-Ikeda lift and a special value
of a certain $L$-function. For more details about the conjecture of non-vanishing of the
Miyawaki-Ikeda lift, we refer the reader to [Ik 06].

The purpose of the present article is as follows:

(1) we generalize the relation among Jacobi-Eisenstein series given in [Ya 89] for
arbitrary integer-indices and obtain a new generalization of the Maass relation
for the Siegel-Eisenstein series of arbitrary degrees (Theorem 1.1),
(2) we show a new generalization of the Maass relation for the Duke-Imamoglu-
Ibukiyama-Ikeda lifts (Theorem 1.2),
(3) we obtain a new proof for the Miyawaki-Ikeda lifts of two elliptic modular forms
by using the generalized Maass relations, namely, we give the expression of the
standard $L$-function of the Miyawaki-Ikeda lifts of two elliptic modular forms
(Theorem 1.3 and Corollary 1.4.)

As for generalization of the Maass relation, Kohnen [Ko 02] obtained another kind of
generalization of the Maass relation which is related to the Fourier-Jacobi coefficients of
matrix index of size $2n - 1$, while the generalization of the Maass relation in the present
article is related to the Fourier-Jacobi coefficients of integer index. It is known that the
generalized Maass relation in [Ko 02] characterizes the image of the Duke-Imamoglu-
Ibukiyama-Ikeda lifts (cf. Kohnen-Kojima [KK 05], Yamana [Ya 10].)

We remark that a certain identity of the spinor $L$-function of the Miyawaki-Ikeda lift
of two elliptic modular forms has been given by Heim [He 12] for the case of degree three
and weight twelve. This identity has been generalized in [Ha 12] for any odd degrees
$2n - 1$ and for any even weights $k$.

1.2. We explain our results more precisely. We denote by $\mathfrak{H}_n$ the Siegel upper-half
space of size $n$. For integers $n$ and $k > n + 2$, the Siegel-Eisenstein series of weight $k$ of
degree $n + 1$ is defined by

$$E_k^{(n+1)}(Z) := \sum_{M=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_{n+1,0} \setminus \Gamma_{n+1}} \det(CZ + D)^{-k},$$
where \( \tau \in \mathcal{H}_{n+1} \) and where \( \Gamma_{n+1} := \text{Sp}_{n+1}(\mathbb{Z}) \) is the symplectic group of size \( 2n + 2 \) with entries in \( \mathbb{Z} \) and we set \( \Gamma_{n+1,0} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{n+1} \mid C = 0 \right\} \). A Fourier-Jacobi expansion of \( E_k^{(n+1)} \) is given by
\[
E_k^{(n+1)} \left( \begin{pmatrix} \tau & z \\ \iota & \omega \end{pmatrix} \right) = \sum_{m=0}^{\infty} e_{k,m}^{(n)}(\tau, z)e^{2\pi i m \omega},
\]
where \( \tau \in \mathcal{H}_{n}, \omega \in \mathcal{H}_1 \) and \( z \in \mathbb{C}^n \). The form \( e_{k,m}^{(n)} \) is called the \( m \)-th Fourier-Jacobi coefficient of \( E_k^{(n+1)} \). We remark that \( e_{k,m}^{(n)} \) is a Jacobi form of weight \( k \) of index \( m \) of degree \( n \) (cf. Ziegler [Zi 89].)

We denote by \( J_{k,m}^{(n)} \) the space of Jacobi forms of weight \( k \) of index \( m \) of degree \( n \). For the definition of Jacobi forms of higher degree, we refer the reader to [Zi 89] or Section 2.2 in the present article. We define two kinds of index-shift maps:
\[
V_{l,n-1}(p^2) : J_{k,m}^{(n)} \rightarrow J_{k,mp^2}^{(n)},
\]
\[
U(p) : J_{k,m}^{(n)} \rightarrow J_{k,mp^2}^{(n)}.
\]
Here the index-shift map \( V_{l,n-1}(p^2) \) \((0 \leq l \leq n)\) is given by the action of the double coset \( \Gamma_n \text{diag}(1_l, p1_{n-1}, p^2 1_l, p1_{n-1}) \Gamma_n \). For the precise definition of \( V_{l,n-1}(p^2) \) see Section 2.4 and we define \((\phi | U(d))(\tau, z) := \phi(\tau, dz) \) for \( \phi \in J_{k,m}^{(n)} \) and for any natural number \( d \).

**Theorem 1.1.** Let \( e_{k,m}^{(n)} \) be the \( m \)-th Fourier-Jacobi coefficient of Siegel-Eisenstein series. Then we obtain the relation
\[
e_{k,m}^{(n)} \left( V_{0,n}(p^2), ..., V_{n,0}(p^2) \right) = \left( e_{k,m}^{(n)} \left| \frac{U(p^2)}{p^2} \right|, e_{k,m}^{(n)} |U(p)|, e_{k,m}^{(n)} |mp^2| \right) \begin{pmatrix} 0 & 1 \\ p^{-k} & p^{-k}(1 + p \delta_{p|m}) \end{pmatrix} A_{p^2,2n+1}^{p,k}.
\]

where the both sides of the above identity are vectors of functions and \( A_{p^2,2n+1}^{p,k} \) is a certain matrix with size \( 2 \times (n + 1) \) which depends only on \( p \) and \( k \), and where we regard \( e_{k,m}^{(n)} \) as identically \( 0 \) if \( p^2 \nmid m \). Here \( \delta_{p|m} \) is defined by \( 1 \) or \( 0 \), according as \( p|m \) or \( p \nmid m \).

For the precise definition of \( A_{p^2,2n+1}^{p,k} \), see Section 2.6.

The relation in Theorem 1.1 is a new generalization of the Maass relation for Siegel-Eisenstein series of arbitrary degrees. As for the function \( e_{k,m}^{(n)} |V_n(p) \), a similar identity has already been given in [Ya 86]. Here the operator \( V_n(p) \) is obtained from the double coset \( \Gamma_n \text{diag}(1_n, p1_n) \Gamma_n \).

Now we apply the relation in Theorem 1.1 to the Duke-Imamoglu-Ibukiyama-Ikeda lifts. We denote by \( S_k(\Gamma_n) \) the space of Siegel cusp forms of weight \( k \) of degree \( n \).
Let \( f \in S_{2k}(\Gamma_1) \) be a normalized Hecke eigenform and let \( F \in S_{k+n}(\Gamma_{2n}) \) be a Duke-Imamoglu-Ibukiyama-Ikeda lift of \( f \) (cf. Ikeda [Ik 06].) We remark that there is no canonical choice of \( F \), however \( F \) is determined up to constant multiple. We consider the Fourier-Jacobi expansion of \( F \):

\[
F\left(\begin{pmatrix} \tau & z \\ t & \omega \end{pmatrix}\right) = \sum_{m=1}^{\infty} \phi_m(\tau, z)e^{2\pi i m \omega},
\]

where \( \tau \in \mathcal{H}_n \), \( \omega \in \mathcal{H}_1 \) and \( z \in \mathbb{C}^n \). Then \( \phi_m \) is the \( m \)-th Fourier-Jacobi coefficient of \( F \) and is a Jacobi cusp form of weight \( k + n \) of index \( m \) of degree \( 2n - 1 \). We denote by \( J_{k,m}^{(n)\text{cusp}} \) the space of Jacobi cusp forms of weight \( k \) of index \( m \) of degree \( n \). The restriction of the maps \( V_{i,n-1}(p^2) \) and \( U(p) \) to \( J_{k,m}^{(n)\text{cusp}} \) gives the maps from \( J_{k,m}^{(n)\text{cusp}} \) to \( J_{k,mp^2}^{(n)\text{cusp}} \). Let \( \alpha_p^{\pm 1} \) be the complex numbers which satisfy

\[
(\alpha_p + \alpha_p^{-1})p^{k-\frac{1}{2}} = a(p),
\]

where \( a(p) \) is the \( p \)-th Fourier coefficient of \( f \).

The following theorem is a generalization of the Maass relation for the Duke-Imamoglu-Ibukiyama-Ikeda lifts, which is different from the ones in [Ko 02] and in [Ha 11].

**Theorem 1.2.** Let \( \phi_m \in J_{k+n,n,m}^{(2n-1)\text{cusp}} \) be the \( m \)-th Fourier-Jacobi coefficient of the Duke-Imamoglu-Ibukiyama-Ikeda lift \( F \) as the above. Then we have

\[
\phi_m | (V_{0,2n-1}(p^2), \ldots, V_{2n-1,0}(p^2)) = p^{-(n-1)(2k-1)} \left( \phi_{\frac{m}{p^2}} | U(p^2), \phi_m | U(p), \phi_{mp^2} \right) \begin{pmatrix} 0 & p^{-k-n} & 1 \\ p^{-k-n} & 0 & p^{-2k-2n+2} \\ 0 & p^{-k-n+1} & \delta_p m \end{pmatrix} A'_{2,2n}(\alpha_p),
\]

where \( A'_{2,2n}(\alpha_p) \) is a certain matrix with size \( 2 \times 2 \) which depends only on \( f \) and \( p \).

We regard the form \( \phi_{\frac{m}{p^2}} \) as identically zero if \( p^2 \nmid m \). The matrix \( A'_{2,2n}(\alpha_p) \) is obtained by substituting \( X_p = \alpha_p \) into a matrix \( A'_{2,2n}(X_p) \). For the precise definition of \( A'_{2,2n}(X_p) \), see Section 2.6.

Now we apply the relation in Theorem 1.2 to the Miyawaki-Ikeda lifts of two elliptic modular forms. Let \( f \) and \( F \) be as above. Let \( g \in S_{k+n}(\Gamma_1) \) be a normalized Hecke eigenform. Then one can construct a Siegel cusp form \( \mathcal{F}_{f,g} \) of weight \( k + n \) of degree \( 2n - 1 \):

\[
\mathcal{F}_{f,g}(\tau) := \int_{\Gamma_1 \backslash \mathcal{H}_1} F\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right) g(\omega) \text{Im}(\omega)^{k+n-2} d\omega.
\]

The form \( \mathcal{F}_{f,g} \) is the Miyawaki-Ikeda lift of \( g \) associated to \( f \). It is shown by Ikeda [Ik 06] that if \( \mathcal{F}_{f,g} \) is not identically zero, then \( \mathcal{F}_{f,g} \) is an eigenfunction for Hecke operators for the Hecke pair \((\Gamma_{2n-1}, \text{Sp}_{2n-1}(\mathbb{Q}))\). Furthermore, the standard \( L \)-function of \( \mathcal{F}_{f,g} \) is expressed
as a certain product of $L$-functions related to $f$ and $g$. Now by virtue of Theorem 1.2 we obtain a new proof of these facts by using the generalized Maass relations.

**Theorem 1.3.** Let $F_{f,g} \in S_{k+n}(\Gamma_{2n-1})$ be the Miyawaki-Ikeda lift of $g$ associated to $f$. Then

$$F_{f,g}|(T_{0,2n-1}(p^2), ..., T_{2n-1,0}(p^2)) = p^{2nk+n-1} (p^{-k-n}, p^{-2k-2n+2}\lambda_g(p^2)) A'_{2,2n}(\alpha_p) F_{f,g},$$

where $T_{l,2n-1-l}(p^2)$ are Hecke operators (see Section 2.4) and $A'_{2,2n}(\alpha_p)$ is the same matrix in Theorem 1.2. Here $\lambda_g(p^2)$ is the eigenvalue of $g$ for $T_{1,0}(p^2)$.

We denote by $\beta_p^{\pm 1}$ the complex numbers which satisfy:

$$(\beta_p + \beta_p^{-1})p^{\frac{k+n-1}{2}} = b(p),$$

where $b(p)$ is the $p$-th Fourier coefficient of $g$. The adjoint $L$-function of $g$ is defined by

$$L(s, g, \text{Ad}) := \prod_p \left\{ (1 - p^{-s})(1 - \beta_p^2 p^{-s})(1 - \beta_p^{-2} p^{-s}) \right\}^{-1}.$$

**Corollary 1.4.** If $F_{f,g}$ is not identically zero, then the Satake parameter of $F_{f,g}$ at prime $p$ is

$$\{\mu_1^{\pm 1}, ..., \mu_{2n-1}^{\pm 1}\} = \{\beta_p^{\pm 2}, \alpha_p^{\pm 1} p^{-n+\frac{3}{2}}, \alpha_p^{\pm 1} p^{-n+\frac{5}{2}}, ..., \alpha_p^{\pm 1} p^{n-\frac{3}{2}}\}.$$

Furthermore, the standard $L$-function of $F_{f,g}$ is

$$L(s, F_{f,g}, \text{st}) = L(s, g, \text{Ad}) \prod_{i=1}^{2n-2} L(s + k + n - 1 - i, f),$$

where $L(s, f)$ is the Hecke $L$-function of $f$. (see Section 2.3 for the definition of the standard $L$-function.)

We remark that Theorem 1.3 follows from Corollary 1.4. And Corollary 1.4 has already been shown by Ikeda [Ik 01] for more general case, namely for Siegel modular form $g \in S_{k+n}(\Gamma_r)$. Here we obtained a new proof of Theorem 1.3 and Corollary 1.4 by using the generalized Maass relation.

Furthermore, we remark that a certain identity of the spinor $L$-function of $F_{f,g}$ has been obtained in [Ha 12] which is a generalization of the case $(n, k) = (2, 12)$ in [He 12].

This paper is organized as follows: In Section 2 we give a notation and review some operators for Jacobi forms, and in Section 3 we shall show a certain relation among Jacobi-Eisenstein series with respect to the index-shift maps. In Section 4 we shall prove Theorem 1.1 while we shall prove Theorem 1.2, Theorem 1.3 and Corollary 1.4 in Section 5.

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2. Operators on Jacobi forms

2.1. Symbols. We denote by $M_{i,j}(R)$ the set of all matrices with entries in the ring $R$ and put $M_n(R) := M_{n,n}(R)$. For any square matrix $A \in M_n(\mathbb{Z})$ we denote by $\text{rank}_p(A)$ the rank of $A$ in $M_n(\mathbb{Z}/p\mathbb{Z})$. For any two matrices $A \in M_n(\mathbb{Z})$ and $B \in M_{n,m}(\mathbb{Z})$ we write $A[B]$ for $^tBAB$. The set of all half-integral symmetric matrices of size $n$ is denoted by $\text{Sym}_n$.

We put $J_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and set

$$\text{GSp}_n^+(\mathbb{R}) := \left\{ M \in M_{2n}(\mathbb{R}) \mid MJ_n^tM = \nu(g)J_n, \nu(g) > 0 \right\},$$

where the number $\nu(g)$ is called the similitude of $g$.

We put $\Gamma_n := \text{Sp}_n(\mathbb{Z}) \subset \text{SL}_{2n}(\mathbb{Z})$. For any square matrix $x$ we set $e(x) := e^{2\pi i \text{tr}(x)}$, where $\text{tr}(x)$ denotes the trace of $x$. For any natural number $m$ we put $< m > := \frac{m(m+1)}{2}$.

The symbol $\mathfrak{H}_n$ denotes the Siegel upper-half space of size $n$. The action of $\text{GSp}_n^+(\mathbb{R})$ on $\mathfrak{H}_n$ is given by $g \cdot \tau := (A\tau + B)(C\tau + D)^{-1}$ for $g = (A B \ C D) \in \text{GSp}_n^+(\mathbb{R})$ and $\tau \in \mathfrak{H}_n$.

The symbol $\text{Hol}(\mathfrak{H}_n \rightarrow \mathbb{C})$ (resp. $\text{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \rightarrow \mathbb{C})$) denotes the space of all holomorphic function on $\mathfrak{H}_n$ (resp. $\mathfrak{H}_n \times \mathbb{C}^n$). For any integer $k$, we define the slash operator $|_k$:

$$(F|_k g)(\tau) := \det(C\tau + D)^{-k}F(g \cdot \tau),$$

where $F \in \text{Hol}(\mathfrak{H}_n \rightarrow \mathbb{C})$, $g = (A B \ C D) \in \text{GSp}_n^+(\mathbb{R})$ and $\tau \in \mathfrak{H}_n$. By this definition the group $\text{GSp}_n^+(\mathbb{R})$ acts on $\text{Hol}(\mathfrak{H}_n \rightarrow \mathbb{C})$.

2.2. Jacobi group. We define a subgroup of $\text{GSp}_{n+1}^+(\mathbb{R})$:

$$G_n^J := \left\{ \gamma \in \text{GSp}_{n+1}^+(\mathbb{R}) \mid \gamma = \begin{pmatrix} A & 0 & B^* \\ C & \nu(g) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}_n^+(\mathbb{R}) \right\}.$$

A bijective map from $\text{GSp}_n^+(\mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}$ to $G_n^J$ is given by

$$\left[ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu), \kappa \right] \mapsto \begin{pmatrix} A & 0 & B^* \\ C & \nu(g) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 \\ \lambda & 1 & \mu + \kappa \\ 0 & 0 & 1 \end{pmatrix},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}_n^+(\mathbb{R})$, $\lambda, \mu \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}$. We identify $\text{GSp}_n^+(\mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}$ and $G_n^J$. By this bijection the group $G_n^J$ can be regarded as a semi-direct product of $\text{GSp}_n^+(\mathbb{R})$ and $((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R})$, namely $G_n^J \cong \text{GSp}_n^+(\mathbb{R}) \times ((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R})$.

Let $k$ and $m$ be integers and let $\phi \in \text{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \rightarrow \mathbb{C})$ be a holomorphic function on $\mathfrak{H}_n \times \mathbb{C}^n$. We define the slash operator $|_{k,m}$:

$$(\phi|_{k,m}\gamma)(\tau, z) := ((\phi(\tau, z)e(m\omega))|_{k}\gamma)e(-\nu(g) m \omega),$$
where \( \left( \frac{\tau}{t}, \frac{z}{\omega} \right) \in \mathfrak{H}_{n+1}, \tau \in \mathfrak{H}_n, \omega \in \mathfrak{H}_1, z \in \mathbb{C}^n \) and \( \gamma \in G^J_n \). We remark that the RHS of the above definition does not depend on the choice of \( \omega \). By this definition, the group \( G^J_n \) acts on \( \text{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \to \mathbb{C}) \).

For \( \gamma = [g, (\lambda, \mu), \kappa] \in G^J_n \) we have

\[
(\phi|_{k,m}\gamma)(\tau, z) = \det(C\tau + D)^{-k} e(-\nu(g) m ((C\tau + D)^{-1}C)[z + \tau\lambda + \mu])
\]
\[
\times e(\nu(g) m (i\lambda_\tau \lambda + 2^i \lambda_z + 2^i \lambda_\mu + \kappa))
\]
\[
\times \phi(g \cdot \tau, \nu(g) ((C\tau + D)^{-1}(z + \tau\lambda + \mu)),
\]

where \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}^+_n(\mathbb{R}) \).

We put a discrete subgroup \( \Gamma^J_n \) of \( G^J_n \):

\[
\Gamma^J_n := \{ [M, (\lambda, \mu), \kappa] \in G^J_n | M \in \Gamma_n, (\lambda, \mu) \in \mathbb{Z}^n \times \mathbb{Z}^n, \kappa \in \mathbb{Z} \}.
\]

We denote by \( J^{(n)}_{k,m} \) the space of Jacobi forms of weight \( k \) of index \( m \) of degree \( n \) (cf. Ziegler [Zi 89].) For \( n > 1 \) the space \( J^{(n)}_{k,m} \) is defined by

\[
J^{(n)}_{k,m} := \{ \phi \in \text{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \to \mathbb{C}) | \phi|_{k,m}\gamma = \phi \text{ for any } \gamma \in \Gamma^J_n \}.
\]

2.3. The standard \( L \)-functions. Let \( F \in S_k(\Gamma_n) \) be a Siegel cusp form which is an eigenform for all Hecke operators. Let \( \{\mu_0,p, \mu_1,p, \ldots, \mu_n,p\} \) be the Satake parameter of \( F \) at prime \( p \). The standard \( L \)-function of \( F \) is defined by

\[
L(s, F, st) := \prod_p \left( 1 - p^{-s} \right) \prod_{i=1}^n \left( 1 - \mu_{i,p} p^{-s} \right) (1 - \mu_{i,1}^{-1} p^{-s}) \right)^{-1}.
\]

In our setting we have \( \mu_0^2 p \mu_1 p \cdots \mu_n p = p^{nk-n} \).

2.4. index-shift maps of Jacobi forms. For any function \( \phi \in J^{(n)}_{k,m} \) and for any matrix \( g \in \text{GSp}^+_n(\mathbb{R}) \cap M_{2n}(\mathbb{Z}) \) we define

\[
\phi| V(\Gamma_n g \Gamma_n) := \sum_i \phi|_{k,m} [g_i, (0,0), 0],
\]

where \( \Gamma_n g \Gamma_n = \bigcup_i \Gamma_n g_i \) is a coset decomposition. It is known that \( \phi| V(\Gamma_n g \Gamma_n) \) is well-defined and belongs to \( J^{(n)}_{k,\nu(g)m} \).

For any integer \( l \) (\( 0 \leq l \leq n \)), we define

\[
\phi| V_{l,n-l}(p^2) := \phi| V(\Gamma_n \text{diag}(1_l, p1_{n-l}, p^21_l, p1_{n-l}) \Gamma_n).
\]

For any non-negative integer \( d \) we define

\[
(\phi| U(d))(\tau, z) := \phi(\tau, dz).
\]
Then $\phi|V_{l,n-1}(p^2) \in J_{k,mp^2}^{(n)}$ and $\phi|U(d) \in J_{k,md^2}^{(n)}$.

Let $F$ be a Siegel modular form of weight $k$ of degree $n$. Let $g$ be an element of $\text{GSp}_n^+ (\mathbb{R}) \cap M_{2n}(\mathbb{Z})$. For any double coset $\Gamma_n g \Gamma_n$, the Hecke operator $T(\Gamma_n g \Gamma_n)$ is defined by

$$F|T(\Gamma_n g \Gamma_n) := \nu(g)^{nk-<n>} \sum_i F|_{k,gi},$$

where $\Gamma_n g \Gamma_n = \bigcup_i \Gamma_n g_i$ is a coset decomposition. For any integer $l$ ($0 \leq l \leq n$), we define

$$F|T_{l,n-1}(p^2) := F|T(\text{diag}(1_l, p1_{n-l}, p^21_l, p1_{n-l}) \Gamma_n).$$

For any Jacobi form $\phi \in J_{k,m}^{(n)}$, we define the function

$$W(\phi)(\tau) := \phi(\tau, 0)$$

for $\tau \in \mathcal{H}_n$. From the definition of Jacobi forms, it follows that $W(\phi)$ is a Siegel modular form of weight $k$ of degree $n$.

Furthermore, due to the straightforward calculation, we obtain

$$W(\phi)|T_k(\Gamma_n g \Gamma_n) = \nu(g)^{nk-<n>} W(\phi|V(\Gamma_n g \Gamma_n))$$

for any Jacobi form $\phi \in J_{k,m}^{(n)}$ and for any $g \in \text{GSp}_n^+ (\mathbb{R}) \cap M_{2n}(\mathbb{Z})$.

2.5. **Siegel $\Phi$-operator for Jacobi forms.** Let $\phi \in \text{Hol}(\mathcal{H}_n \times \mathbb{C}^n \to \mathbb{C})$ be a holomorphic function. We define the Siegel $\Phi$-operator:

$$\Phi(\phi)(\tau_1, z_1) := \lim_{t \to +\infty} \phi \left( \left( \begin{array}{cc} \tau_1 & 0 \\ 0 & it \end{array} \right), \left( \begin{array}{c} z_1 \\ 0 \end{array} \right) \right),$$

where $\tau_1 \in \mathcal{H}_{n-1}$ and $z_1 \in \mathbb{C}^{n-1}$.

It is known that if $\phi \in J_{k,m}^{(n)}$ is a Jacobi form, then the function $\Phi(\phi)$ is also a Jacobi form which belongs to $J_{k,m}^{(n-1)}$.

2.6. **The Satake isomorphism and the Siegel $\Phi$-operator.** Let $\mathcal{H}_p^n$ be the local Hecke ring with respect to the Hecke pair $(\Gamma_n, \text{GSp}_n^+ (\mathbb{R}) \cap M_{2n}(\mathbb{Z}[p^{-1}]))$. We denote by $\mathbb{C}[X_0^{\pm 1}, ..., X_n^{\pm 1}]^{W_n}$ the subring of the polynomial ring $\mathbb{C}[X_0^{\pm 1}, ..., X_n^{\pm 1}]$ which is invariant under the action of the Weyl group $W_n$ associated to the symplectic group. The Satake isomorphism $\varphi_n : \mathcal{H}_p^n \to \mathbb{C}[X_0^{\pm 1}, ..., X_n^{\pm 1}]^{W_n}$ is given by

$$\Gamma_n g \Gamma_n = \bigcup_i \Gamma_n \begin{pmatrix} p^l & D_i^{-1} \\ 0 & D_i \end{pmatrix} \mapsto X_0^{l} \sum_i \prod_j \left( \frac{X_j}{p^j} \right)^{l_{i,j}},$$

where $\nu(g) = p^l$ and $D_i = \begin{pmatrix} p^{l_{i,1}} & * & * \\ & \ddots & \ast \\ & & p^{l_{i,n}} \end{pmatrix}$ (cf. Andrianov [An 79].)
We write \( \varphi = \varphi_n \) for simplicity. In this article we consider the subring of \( \mathcal{H}_p^n \) which is generated by \( T_{0,n}(p^2)^{\pm 1} \) and \( T_{l,n-l}(p^2) \) \((l = 1, \ldots, n)\).

The following proposition follows from [Kr 86, Satz].

**Proposition 2.1.** If \( n \geq 2 \) we have

\[
\begin{align*}
\varphi(T_{n,0}(p^2)) &= X_n \left\{ (X_n^{-1} + (p-1)p^{-1} + X_n) \varphi(T_{n-1,0}(p^2)) + (p^2 - 1)p^{-1} \varphi(T_{n-2,1}(p^2)) \right\}, \\
\varphi(T_{1,n-1}(p^2)) &= X_n \left\{ p^{1-n} \varphi(T_{1,n-2}(p^2)) + (X_n^{-1} + (p-1)p^{-n} + X_n) \varphi(T_{0,n-1}(p^2)) \right\}, \\
\varphi(T_{0,n}(p^2)) &= X_n \left\{ p^{-n} \varphi(T_{0,n-1}(p^2)) \right\},
\end{align*}
\]

and for \( 1 < j < n \) we have

\[
\begin{align*}
\varphi(T_{j,n-j}(p^2)) &= X_n \left\{ p^{-n} \varphi(T_{j,n-j-1}(p^2)) + (X_n^{-1} + p^{j-n-1}(p-1) + X_n) \varphi(T_{j-1,n-j}(p^2)) + (p^{2n-2j+2} - 1)p^{j-n-1} \varphi(T_{j-2,n-j+1}(p^2)) \right\}.
\end{align*}
\]

**Proof.** We obtain this proposition by replacing \( p^{-r} \) in [Kr 86 Satz] by \( p^{-n}X_n \). For the detail the reader is referred to [Kr 86 Satz]. \( \square \)

Now for integers \( l (2 \leq l) \), \( t (0 \leq t \leq l) \), \( j (0 \leq j \leq l) \), we put

\[
b_{t,j} := b_{t,j,t,p}(X_l) = \begin{cases} 
(p^{2t-2j+2} - 1)p^{j-1-l}X_l & \text{if } t = j - 2, \\
1 + p^{j-1-l}(p-1)X_l + X_l^2 & \text{if } t = j - 1, \\
p^{-l+j}X_l & \text{if } t = j, \\
0 & \text{otherwise},
\end{cases}
\]

and we put a matrix

\[
B_{l,l+1}(X_l) := (b_{t,j})_{t=0, \ldots, l-1}^{j=0, \ldots, l} = \begin{pmatrix} b_{0,0} & \cdots & b_{0,l} \\
\vdots & \ddots & \vdots \\
b_{l-1,0} & \cdots & b_{l-1,l} \end{pmatrix}
\]

with entries in \( \mathbb{C}[X_l, X_l^{-1}] \). From Proposition 2.1 and from the definition of \( B_{l,l+1}(X_l) \), we have the identity:

\[
(\varphi(T_{0,l}(p^2)), \ldots, \varphi(T_{l,0}(p^2))) = (\varphi(T_{0,l-1}(p^2)), \ldots, \varphi(T_{l-1,0}(p^2)))B_{l,l+1}(X_l).
\]

For Jacobi forms we obtain the following lemma.

**Lemma 2.2.** Let \( \phi \in \mathcal{J}_{k,m}^{(l)} \) be a Jacobi form such that \( \Phi(\phi) \) is not identically zero. Then we have

\[
\Phi(\phi|(V_{0,l}(p^2), \ldots, V_{l,0}(p^2))) = (\Phi(\phi)|(V_{0,l-1}(p^2), \ldots, V_{l-1,0}(p^2)))B_{l,l+1}(p^{l-k}),
\]

where we put \( \phi|(V_{0,l}(p^2), \ldots, V_{l,0}(p^2)) := (\phi|V_{0,l}(p^2), \ldots, \phi|V_{l,0}(p^2)). \)
Proof. Let \( \gamma = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, (0, 0, 0) \in G' \) with \( A = \begin{pmatrix} A^* & 0 \\ a & a \end{pmatrix}, B = \begin{pmatrix} B^* & b_1 \\ b_2 & b \end{pmatrix}, D = \begin{pmatrix} D^* & \delta \\ 0 & d \end{pmatrix} \), where \( A^*, D^* \in \text{GL}_{l-1}(\mathbb{R}) \) and \( B^* \in M_{l-1}(\mathbb{R}) \). Then

\[
\Phi(\phi_{k,m} \gamma) = d^{-k} \Phi(\phi)|_{k,m} \gamma^*,
\]

where \( \gamma^* = \begin{pmatrix} (A^* & B^*) \\ 0 & D^* \end{pmatrix}, (0, 0, 0) \in G'_l \).

The rest of the proof of this lemma is the same to the case of Siegel modular forms (cf. [Kr 86, Satz].) Thus we conclude this lemma.

We define a matrix

\[
B_{2,n+1}(X_2, X_3, \ldots, X_n) := \prod_{l=2}^{n} B_{l,l+1}(X_l),
\]

which entries are in \( \mathbb{C}[X_2^\pm, \ldots, X_n^\pm] \). Then we have

\[
(\varphi(T_{0,n}(p^2)), \ldots, \varphi(T_{n,0}(p^2))) = (\varphi(T_{0,1}(p^2)), \varphi(T_{1,0}(p^2))) B_{2,n+1}(X_2, \ldots, X_n).
\]

The precise expression of \( \varphi(T_{l,n-1}(p^2)) \) by using the elementary symmetric polynomials has been given in [Kr 86, Korollar 2].

To explain our results we define two matrices \( A_{2,n+1}^{p,k} \) and \( A_{2,2n}(X_p) \). First we define a \( 2 \times (n + 1) \) matrix

\[
A_{2,n+1}^{p,k} := B_{2,n+1}(p^{2-k}, p^{3-k}, \ldots, p^{n-k}).
\]

We remark that the matrix \( A_{2,n+1}^{p,k} \) depends only on the prime \( p \) and the integer \( k > 0 \). We set a \( 2 \times 2n \) matrix

\[
B'_{2,2n}(X_2, \ldots, X_{2n-1}) := \left( \prod_{i=2}^{2n-1} X_i \right)^{-1} B_{2,2n}(X_2, \ldots, X_{2n-1}).
\]

From the definition of \( B_{2,2n}(X_2, \ldots, X_{2n-1}) \) it is not difficult to see that the entries in the matrix \( B'_{2,2n}(X_2, \ldots, X_{2n-1}) \) belong to \( \mathbb{C}[X_2 + X_2^{-1}, \ldots, X_{2n-1} + X_{2n-1}^{-1}] \). We define a \( 2 \times 2n \) matrix

\[
A'_{2,2n}(X_p) := B'_{2,2n}(p^{\frac{2}{3} - n} X_p, p^{\frac{5}{3} - n} X_p, \ldots, p^{-\frac{2}{3} + n} X_p).
\]

In Section [5.3] we will show \( A'_{2,2n}(X_p) = A'_{2,2n}(X_p^{-1}) \).

3. Jacobi-Eisenstein series

The goal of this section is to show a certain relation among Jacobi-Eisenstein series with respect to the index-shift maps \( V_{l,n-1}(p^2) \) (\( l = 0, \ldots, n \)).
3.1. **Definition of Jacobi-Eisenstein series.** For integers \( k, m \) and \( n \), we define the Jacobi-Eisenstein series of weight \( k \) of index \( m \) of degree \( n \) by

\[
E_{k,m}^{(n)}(\tau, z) := \sum_{\gamma \in \Gamma_{n,0} \backslash \Gamma_n^J} (1_{k,m} \gamma),
\]

where we put

\[
\Gamma_{n,0}^J := \left\{ \left( \begin{array}{cc} A & 0 \\ D & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1_n & 0 \\ D & 1_n \end{array} \right) \in \Gamma_n \mid \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right) \in \Gamma_n, \mu, \nu, \kappa \in \mathbb{Z} \right\}.
\]

It is known that if \( k > n+2 \), then \( E_{k,m}^{(n)} \) converges and belongs to \( J_{k,m}^{(n)} \) (cf. Ziegler [Zi 89].)

The purpose of this section is to show that \( E_{k,m}^{(n)} \mid V_{l,n-l}(p^2) \) is a linear combination of three forms \( E_{k,m}^{(n)} \mid U(p^2) \), \( E_{k,m}^{(n)} \mid U(p) \) and \( E_{k,mp}^{(n)} \).

**Lemma 3.1.** Let \( m \) and \( n \) be positive integers. Then the forms \( \left\{ E_{k,m}^{(n)} \mid U(d) \right\}_d \) are linearly independent, where \( d \) runs over all positive integers such that \( d^2 \mid m \).

**Proof.** Let \( \Phi \) be the Siegel \( \Phi \)-operator for Jacobi forms introduced in Section 2.5. It follows from the definition that \( \Phi(E_{k,m}^{(n)}) = E_{k,m}^{(n-1)} \). Hence it is enough to show that the forms \( \left\{ E_{k,m}^{(1)} \mid U(d) \right\}_d \) are linearly independent.

Let \( E_{k,m}^{(1)}(\tau, z) = \sum_{n',r} c(n',r) e(n' \tau + rz) \) be the Fourier expansion of \( E_{k,m}^{(1)} \). We call \( c(n',r) \) the \((n',r)\)-th Fourier coefficient of \( E_{k,m}^{(1)} \). Let \( n' > 0 \) and \( r \geq 0 \) be integers such that \( 4n'm - r^2 > 0 \). Then it is known that the \((n',r)\)-th Fourier coefficient of \( E_{k,m}^{(1)} \) is not zero (cf. Eichler-Zagier [E-Z 85, p.17–p.20].) On the other hand, for any \( d > 1 \) such that \( d^2 \mid m \), the \((n',r)\)-th Fourier coefficient of \( E_{k,m}^{(1)} \mid U(d) \) is zero unless \( d \mid r \). Therefore we obtain this lemma. \( \square \)

3.2. **Definition of a form \( K_{i,j}^{(n)} \).** We quote some symbols from [Ya 89]. For a fixed prime \( p \) and for \( 0 \leq i \leq j \leq n \), we put

\[
\delta_{i,j} := \text{diag}(1_i, p1_{j-i}, p^21_{n-j})
\]

and

\[
\delta_i := \delta_{i,n} = \text{diag}(1_i, p1_{n-i}).
\]

And for \( x = \text{diag}(0_i, x_{2,2}, 0_{n-j}) \) with \( x_{2,2} = t_{x_{2,2}} \in M_{j-i}(\mathbb{Z}) \) we set

\[
\delta_{i,j}(x) := \begin{pmatrix} p^2\delta_{i,j}^{-1} & x \\ 0_n & \delta_{i,j} \end{pmatrix}.
\]
We denote by $\Gamma_{n,0}$ the set of all matrices $\begin{pmatrix} A & B \\ 0_0 & D \end{pmatrix}$ in $\Gamma_n$. We set

$$
\Gamma(\delta_{i,j}) : = \left\{ \begin{pmatrix} A & B \\ 0_0 & D \end{pmatrix} \in \Gamma_{n,0} \left| A \in \delta_{i,j}\text{GL}_n(\mathbb{Z}) \delta_{i,j}^{-1} \right. \right\},
$$

$$
\Gamma(\delta_i) : = \left\{ \begin{pmatrix} A & B \\ 0_0 & D \end{pmatrix} \in \Gamma_{n,0} \left| A \in \delta_i\text{GL}_n(\mathbb{Z}) \delta_i^{-1} \right. \right\},
$$

and put a subgroup $\Gamma(\delta_{i,j}(x))$ of $\Gamma(\delta_{i,j})$:

$$
\Gamma(\delta_{i,j}(x)) := \Gamma_n \cap (\delta_{i,j}(x)^{-1}\Gamma_{n,0}\delta_{i,j}(x)).
$$

For $\lambda \in \mathbb{Z}^n$ and for $M \in \text{GSp}_n^+(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$ we put

$$
j(k, m; M, \lambda)(\tau, z) := (1_{k,m}[1_{2n}, (\lambda, 0), 0][M, (0, 0), 0])(\tau, z).
$$

For two matrices $x = \text{diag}(0_1, x_{2,2}, 0_{n-j})$ and $y = \text{diag}(0_j, y_{2,2}, 0_{n-j})$ such that $x_{2,2} = {}^tx_{2,2}$, $y_{2,2} = {}^ty_{2,2} \in M_{j-i}(\mathbb{Z})$, we say they are equivalent and write $[x] = [y]$ if there exists a matrix $u = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ p_2u_{2,1} & u_{2,2} & u_{2,3} \\ p^2u_{3,1} & pu_{3,2} & u_{3,3} \end{pmatrix} \in \delta_{i,j}\text{GL}_n(\mathbb{Z})\delta_{i,j}^{-1} \cap \text{GL}_n(\mathbb{Z})$ which satisfies $u_{2,2}x_{2,2}^tu_{2,2} \equiv y_{2,2} \mod p$, where $u_{2,2} \in M_{j-i}(\mathbb{Z})$, $u_{1,1} \in M_i(\mathbb{Z})$ and $u_{3,3} \in M_{n-j}(\mathbb{Z})$.

We define a function $K_{i,j}^\alpha$ on $\Gamma_n \times \mathbb{C}$ by

$$
K_{i,j}^\alpha := K_{i,j,m,p}^\alpha(\tau, z) = \sum_{[x]} \sum_{M \in \Gamma(\delta_{i,j}(x)) \cap \Gamma_n} \sum_{\lambda \in \mathbb{Z}^n} j(k, m; \delta_{i,j}(x)M, \lambda)(\tau, z),
$$

where in the first summation in the RHS, $[x]$ runs over all equivalence classes which satisfy $\text{rank}_p(x) = \alpha$.

**Proposition 3.2** (Yamazaki [Ya 89]). The double coset $\Gamma_n \left( \begin{pmatrix} \delta_l & 0_n \\ 0_n & p^2\delta_l^{-1} \end{pmatrix} \right) \Gamma_n$ is a disjoint union

$$
\Gamma_n \left( \begin{pmatrix} \delta_l & 0_n \\ 0_n & p^2\delta_l^{-1} \end{pmatrix} \right) \Gamma_n = \bigcup_{0 \leq i, j \leq n} \bigcup_{[x]} \Gamma_{n,0}\delta_{i,j}(x)\Gamma_n,
$$

where in the last union in the RHS, $[x]$ runs over all equivalence classes which satisfy $\text{rank}_p(x) = l - n - i + j$.

**Proof.** This proposition has been shown in [Ya 89, Corollary 2.2]. \qed

**Lemma 3.3.** We obtain

$$
E_{k,m}^{(n)}|V_{n-l}(p^2) = \sum_{i,j} K_{i,j}^{l-i-n+j}.
$$
Proof. It follows from Proposition 3.2 and from the definitions of \( E^{(n)}_{k,m} \), \( V_{i,n-i}(p^2) \) and \( K_{i,j}^\alpha \).

Lemma 3.4. If \( p^2 \mid m \), then
\[
K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{x=\text{diag}(0, x_{2,2}, 0_{n-j})} \sum_{x_{2,2}=1}^{M \in \Gamma(\delta_{i,j}) \backslash \Gamma_n} \sum_{\text{rank}_p(x_{2,2})=\alpha} j\left(k, \frac{m}{p^2}; \left(\begin{array}{c} 1_n \, 0 \\ 0 \, 1_n \end{array}\right) M, \lambda\right)(\tau, p^2 z).
\]

If \( p^2 \nmid m \), then
\[
K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{x=\text{diag}(0, x_{2,2}, 0_{n-j})} \sum_{x_{2,2}=1}^{M \in \Gamma(\delta_{i,j}) \backslash \Gamma_n} \sum_{\text{rank}_p(x_{2,2})=\alpha} j\left(k, m; \left(\begin{array}{c} 1_n \, 0 \\ 0 \, 1_n \end{array}\right) M, \lambda\right)(\tau, p z).
\]

We remark that this lemma has been shown for the case \( m = 1 \) by Yamazaki [Ya 89].

Proof. The proof of this lemma is an analogue to [Ya 89] and straightforward. If \( p^2 \nmid m \), then the proof is similar to the case \( m = 1 \). Hence we assume \( p^2 \mid m \) and shall prove this lemma.

We put \( U := \{ \left(\begin{array}{c} 1_n \\ 0 \end{array}\right) \mid s = t \, s \in M_n(\mathbb{Z}) \} \). Then the set
\[
U' := \{ \left(\begin{array}{c} 1_n \\ 0 \end{array}\right) \mid s = \left(\begin{array}{c} 0 \\ 0 \end{array}\right) \mod p, s_{23} \in M_{j-i,n-j}(\mathbb{Z}), s_{33} = t \, s_{33} \in M_{n-j}(\mathbb{Z}) \}
\]
is a complete set of representatives of \( \Gamma(\delta_{i,j}(x)) \backslash \Gamma(\delta_{i,j}(x))U \). Thus
\[
\sum_{\text{rank}_p(x) = \alpha} j(k, m; \delta_{i,j}(x) M, \lambda)(\tau, z) = \sum_{\text{rank}_p(x) = \alpha} \sum_{M \in \Gamma(\delta_{i,j}(x)) \backslash \Gamma_n} j\left(k, m; \delta_{i,j}(x) M, \lambda\right)(\tau, z) \sum_{\left(\begin{array}{c} 1_n \\ 0 \end{array}\right) \in U'} e(p^2 m^t \lambda \delta_{i,j}^{-1} s \delta_{i,j}^{-1} \lambda)
\]
\[
= p^{j-i(n-j)+(n-j)(n-i+1)} \sum_{\text{rank}_p(x) = \alpha} \sum_{M \in \Gamma(\delta_{i,j}(x)) \backslash \Gamma_n} \sum_{\lambda \in \mathbb{Z}^n} j(k, m; \delta_{i,j}(x) M, \lambda)(\tau, z).
\]

We remark
\[
j(k, m; \delta_{i,j}(x), \lambda)(\tau, z) = p^{-k(2n-i-j)} e(m^t \lambda (p^2 \delta_{i,j}^{-1} \tau \delta_{i,j}^{-1} + p^{-1} x) \lambda + 2p^2 m^t \lambda \delta_{i,j}^{-1} z).
\]
Hence if we put $X = p^2 \delta_{ij}^{-1} \lambda$, then $X' \in (p^2 \mathbb{Z})^i \times (p\mathbb{Z})^{i-j} \times \mathbb{Z}^{n-j}$ and we have
\[
 j(k, m : \delta_{i,j}(x), \lambda)(\tau, z) = p^{-k(2n-i-j)} j(k, p^{-2}m : \left( \begin{array}{c} p^{-1}x \\ \alpha \end{array} \right), \lambda')(\tau, p^2z).
\]
Thus
\[
 K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{[x]} \sum_{\lambda' \in (p^2 \mathbb{Z})^i \times (p\mathbb{Z})^{i-j} \times \mathbb{Z}^{n-j}} j(k, p^{-2}m : \left( \begin{array}{c} p^{-1}x \\ \alpha \end{array} \right), \lambda')(\tau, p^2z)
\]
\[
 \times \sum_{\lambda' \in (p^2 \mathbb{Z})^i \times (p\mathbb{Z})^{i-j} \times \mathbb{Z}^{n-j}} j(k, p^{-2}m : \left( \begin{array}{c} p^{-1}x \\ \alpha \end{array} \right), \lambda')(\tau, p^2z).
\]
Here, the matrix $u$ in the above summation belongs to $\delta_{i,j} \text{GL}(n, \mathbb{Z})\delta_{i,j}^{-1} \cap \text{GL}(n, \mathbb{Z})$. Hence $t' u$ stabilizes the lattice $(p^2 \mathbb{Z})^i \times (p\mathbb{Z})^{i-j} \times \mathbb{Z}^{n-j}$. Furthermore, the summation over the equivalent class $[x]$ and the summation over the representatives of $\Gamma(\delta_{i,j}(x))U \backslash \Gamma(\delta_{i,j})$ turns into the summation over $x = \text{diag}(0, x_{22}, 0)$ such that $x_{22} = t' x_{22} \in M_{j-i}(\mathbb{Z})$ mod $p$ and $\text{rank}_p(x) = \alpha$. Therefore we conclude this lemma.

3.3. **Summation $G_j^n(m, \lambda)$**. We define
\[
g_p(n, i) := \begin{cases} \prod_{a=1}^i (p^{n-a+1} - 1)(p^a - 1)^{-1} & \text{if } 1 \leq i \leq n, \\ 1 & \text{if } i = 0, \\ 0 & \text{otherwise}. \end{cases}
\]
For any $\lambda \in \mathbb{Z}^n$ and for $0 \leq j \leq n$ we define
\[
 G_j^n(m, \lambda) := \sum_{x = t' x \in M_n(\mathbb{Z}/p\mathbb{Z})} e \left( \frac{m}{p} t' \lambda x \lambda \right).
\]

**Proposition 3.5.** For $m \in \mathbb{Z}$ and for $\lambda \in \mathbb{Z}^n$ we have
\[
 G_j^n(m, \lambda) = \begin{cases} p^{\frac{j}{p}}(\frac{j}{p}+1)g_p(n, j) \prod_{\alpha=1}^j (p^{\alpha} - 1) & \text{if } m \lambda \equiv 0 \mod p, \\ (-1)^j p^{\frac{j}{p}}(\frac{j}{p}+1)g_p(n-1, 2\frac{j}{p}) \prod_{\alpha=1}^{j-1} (p^{\alpha} - 1) & \text{if } m \lambda \not\equiv 0 \mod p. \end{cases}
\]
Proof. If \( p \mid m \), then \( G_j^n(m, \lambda) = G_j^n(1, 0) \). And if \( p \not\mid m \), then \( G_j^n(m, \lambda) = G_j^n(1, \lambda) \). Hence we need to calculate the case \( m = 1 \). The calculation of \( G_j^n(1, \lambda) \) has already been obtained by [Ya 89, Lemma 3.1]. \( \square \)

3.4. Some cardinalities. In this subsection we will give some lemmas to calculate \( K_{i,j}^n \).

For \( 0 \leq i \leq j \leq n \), we put

\[
H_i := \delta_i \text{GL}_n(\mathbb{Z}) \delta_j^{-1} \cap \text{GL}_n(\mathbb{Z}),
\]

\[
H_{i,j} := \delta_{i,j} \text{GL}_n(\mathbb{Z}) \delta_j^{-1} \cap \text{GL}_n(\mathbb{Z}).
\]



We define two sets

\[
S_i := \left\{ \begin{pmatrix} * & \ast \\ p^i & b \end{pmatrix}^{-1} \in \text{GL}_n(\mathbb{Z}) \mid b \in \mathbb{Z}^i \right\},
\]

\[
S_{i,j} := \left\{ \begin{pmatrix} * & * & \ast \\ p^i b_1 & p^j b_2 & \ast \end{pmatrix}^{-1} \in \text{GL}_n(\mathbb{Z}) \mid b_1 \in \mathbb{Z}^i, b_2 \in \mathbb{Z}^{j-i} \right\},
\]

where \( b, b_1 \) and \( b_2 \) in the above sets are column vectors.

Lemma 3.6. We have

\[
|H_i \setminus \text{GL}_n(\mathbb{Z})| = g_p(n, i),
\]

\[
|H_i \setminus S_i| = g_p(n - 1, i).
\]

Furthermore, we have

\[
|H_{i,j} \setminus \text{GL}_n(\mathbb{Z})| = p^{j(n-j)} g_p(n, j) g_p(j, i),
\]

\[
|H_{i,j} \setminus S_i| = p^{j(n-j)} g_p(n - 1, i) g_p(n - i, n - j),
\]

\[
|H_{i,j} \setminus S_{i,j}| = p^{j(n-j)} g_p(n - 1, j) g_p(j, i).
\]

Proof. These are elementary. We leave details to the reader. \( \square \)

Lemma 3.7. Let \( B(\lambda) \) be a function on \( \lambda \in \mathbb{Z}^n \). We put \( L_0 := (p^2 \mathbb{Z})^j \times (p \mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j} \).

We assume that the sum \( \sum_{A \in H_{i,j} \setminus \text{GL}_n(\mathbb{Z})} \sum_{\lambda \in L_0} B(\lambda) \) converges absolutely. Then we have

\[
\sum_{A \in H_{i,j} \setminus \text{GL}_n(\mathbb{Z})} \sum_{\lambda \in L_0} B(\lambda) = a_0 \sum_{\lambda \in \mathbb{Z}^n} B(\lambda) + a_1 \sum_{\lambda \in \mathbb{Z}^n} B(p\lambda) + a_2 \sum_{\lambda \in \mathbb{Z}^n} B(p^2\lambda),
\]

where \( a_0, a_1 \) and \( a_2 \) are integers which satisfy

\[
a_0 + a_1 + a_2 = |H_{i,j} \setminus \text{GL}_n(\mathbb{Z})|, \quad a_0 + a_1 = |H_{i,j} \setminus S_i| \quad \text{and} \quad a_0 = |H_{i,j} \setminus S_{i,j}|.
\]

Proof. For \( \lambda \in \mathbb{Z}^n \) we denote by \( \gcd(\lambda) \) the greatest common divisor of all entries in \( \lambda \). Let \( X \) be a complete set of representatives of \( H_{i,j} \setminus \text{GL}_n(\mathbb{Z}) \). For \( \lambda \in \mathbb{Z}^n \) we define

\[
N(\lambda) := \left| \{ A \in X \mid \lambda \in \mathcal{A}L_0 \} \right|.
\]
We remark that $N(\lambda)$ does not depend on the choice of $X$. To show this lemma, it is enough to calculate $N(\lambda)$ for given $\lambda \in \mathbb{Z}^n$.

By the definition of $S_{i,j}$ and $S_i$, we have

$$S_{i,j} = \left\{ A \in \text{GL}_n(\mathbb{Z}) \mid t(0, \ldots, 0, 1) \in tAL_0 \right\},$$

$$S_i = \left\{ A \in \text{GL}_n(\mathbb{Z}) \mid t(0, \ldots, 0, p) \in tAL_0 \right\}.$$

Hence we have $N(t(0, \ldots, 0, 1)) = |H_{i,j} \backslash S_{i,j}|$ and $N(t(0, \ldots, 0, p)) = |H_{i,j} \backslash S_i|$. Furthermore, we have $N(t(0, \ldots, 0, p^2)) = |H_{i,j} \backslash \text{GL}_n(\mathbb{Z})|$.

For any $\lambda \in \mathbb{Z}^n$, there exists a matrix $B \in \text{GL}_n(\mathbb{Z})$ such that $tB\lambda = \text{gcd}(\lambda)^t(0, \ldots, 0, 1)$. Thus we have $N(\lambda) = N(\text{gcd}(\lambda)^t(0, \ldots, 0, 1))$. Hence $N(\lambda)$ equals to $|H_{i,j} \backslash S_{i,j}|$, $|H_{i,j} \backslash S_i|$ or $|H_{i,j} \backslash \text{GL}_n(\mathbb{Z})|$, according as $\text{gcd}(p^2, \text{gcd}(\lambda)) = 1$, $p$ or $p^2$. Therefore we obtain this lemma. \qed

3.5. Calculation of the function $K_{i,j}^\alpha$. For simplicity we define

$$G_{ij}^\alpha(m) := G_{ij}^\alpha(m, \lambda),$$

where $\lambda \in \mathbb{Z}^n$ is an vector which satisfy $\lambda \not\equiv 0 \mod p$. Due to Proposition 3.3, the value $G_{ij}^\alpha(m)$ does not depend on the choice of $\lambda$.

**Lemma 3.8.** If $p^2 | m$, then we have

$$K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)}G_{ij}^{j-i}(0) \times \left\{ a_0E_{k,p^2}^{(n)}(\tau, p^2z) + a_1E_{k,m}^{(n)}(\tau, pz) + a_2E_{k,mp^2}^{(n)}(\tau, z) \right\},$$

where

$$a_0 + a_1 + a_2 = |H_{i,j} \backslash \text{GL}_n(\mathbb{Z})|, \quad a_0 + a_1 = |H_{i,j} \backslash S_i| \quad \text{and} \quad a_0 = |H_{i,j} \backslash S_{i,j}|.$$

If $p^2 \nmid m$, then we have

$$K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)} \left\{ (G_{ij}^{j-i}(0) - G_{ij}^{j-i}(m)) \left[ \Gamma(\delta_j); \Gamma(\delta_{i,j}) \right] \right. \times \left\{ g_p(n-1, j)E_{k,m}^{(n)}(\tau, pz) + p^{n-j}g_p(n-1, j-1)E_{k,mp^2}^{(n)}(\tau, z) \right\} \left. + G_{ij}^{j-i}(m) \left[ \Gamma(\delta_i); \Gamma(\delta_{i,j}) \right] \right. \times \left\{ g_p(n-1, i)E_{k,m}^{(n)}(\tau, pz) + p^{n-i}g_p(n-1, i-1)E_{k,mp^2}^{(n)}(\tau, z) \right\} \right\},$$

where $\Gamma(\delta_{i,j})$ and $\Gamma(\delta_i)$ are groups denoted in Section 3.2.

In particular, the function $K_{i,j}^\alpha$ is a linear combination of $E_{k,p^2}^{(n)}|U(p^2)$, $E_{k,m}^{(n)}|U(p)$ and $E_{k,mp^2}^{(n)}$. 

Proof. First we assume $p^2 | m$. In this case the sum $G^{j-i}_\alpha(m, \lambda')$ equals to $G^{j-i}_\alpha(0)$ for any $\lambda' \in \mathbb{Z}^{j-i}$. Hence due to Lemma 3.4 we obtain

$$K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)}G^{j-i}_\alpha(0) \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda \in (p^2 \mathbb{Z})^i \times (p \mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j}} j\left(k, \frac{m}{p^2}; M, \lambda\right)(\tau, p^2 z).$$

If $\{A_t\}_t$ is a complete set of representatives of $H_{i,j} \setminus \text{GL}_n(\mathbb{Z})$, then the set $\left\{\begin{pmatrix} A_t & 0 \\ 0 & t^{-1}A_t \end{pmatrix}\right\}_t$ is a complete set of representatives of $\Gamma(\delta_{i,j}) \setminus \Gamma_{n,0}$. Hence we have

$$K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)}G^{j-i}_\alpha(0) \sum_{M \in \Gamma_{n,0} \setminus \Gamma_n} \sum_{A \in H_{i,j} \setminus \text{GL}_n(\mathbb{Z})} \sum_{\lambda \in (p^2 \mathbb{Z})^i \times (p \mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j}} j\left(k, \frac{m}{p^2}; M, tA\lambda\right)(\tau, p^2 z).$$

From Lemma 3.7 we obtain

$$K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)}G^{j-i}_\alpha(0) \sum_{M \in \Gamma_{n,0} \setminus \Gamma_n} \left\{ a_0 \sum_{\lambda \in \mathbb{Z}^n} j\left(k, \frac{m}{p^2}; M, \lambda\right)(\tau, p^2 z) + a_1 \sum_{\lambda \in \mathbb{Z}^n} j\left(k, \frac{m}{p^2}; M, p\lambda\right)(\tau, p^2 z) + a_2 \sum_{\lambda \in \mathbb{Z}^n} j\left(k, \frac{m}{p^2}; M, p^2 \lambda\right)(\tau, p^2 z) \right\}.$$

Due to the two identities

$$j\left(k, \frac{m}{p^2}; M, p\lambda\right)(\tau, p^2 z) = j(k, m; M, \lambda)(\tau, pz)$$

and

$$j\left(k, \frac{m}{p^2}; M, p^2 \lambda\right)(\tau, p^2 z) = j(k, mp^2; M, \lambda)(\tau, z),$$

we have

$$K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)}G^{j-i}_\alpha(0) \times \left\{ a_0 E^{(n)}_{k,m}(\tau, p^2 z) + a_1 E^{(n)}_{k,m}(\tau, pz) + a_2 E^{(n)}_{k,mp^2}(\tau, z) \right\}.$$

Thus we showed this lemma for the case $p^2 | m$. 
We now assume $p^2 \not| m$. In this case the sum $G_{\alpha}^{j-i}(m, \lambda')$ equals to $G_{\alpha}^{j-i}(0)$ or $G_{\alpha}^{j-i}(m)$, according as $\lambda' \in p\mathbb{Z}^{j-i}$ or $\lambda' \notin p\mathbb{Z}^{j-i}$. Thus due to Lemma 3.4 we have

\[ K_{i,j}^{\alpha} = p^{-k(2n-i)+j(n-j)(n-i+1)} \left\{ (G_{\alpha}^{j-i}(0) - G_{\alpha}^{j-i}(m)) \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda \in (p\mathbb{Z})^j \times \mathbb{Z}^{n-j}} \times j(k, m; M, \lambda)(\tau, pz) + G_{\alpha}^{j-i}(m) \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda \in (p\mathbb{Z})^j \times \mathbb{Z}^{n-j}} j(k, m; M, \lambda)(\tau, pz) \right\}. \]

Here we have

\[ \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda \in (p\mathbb{Z})^j \times \mathbb{Z}^{n-j}} j(k, m; M, \lambda)(\tau, pz) = [\Gamma(\delta_{i,j}) \setminus \Gamma_n] \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda \in (p\mathbb{Z})^j \times \mathbb{Z}^{n-j}} j(k, m; M, \lambda)(\tau, pz) = [\Gamma(\delta_{i,j}) \setminus \Gamma_n] \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda \in (p\mathbb{Z})^j \times \mathbb{Z}^{n-j}} j(k, m; M, \lambda)(\tau, pz) \]

and

\[ \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda \in (p\mathbb{Z})^j \times \mathbb{Z}^{n-i}} j(k, m; M, \lambda)(\tau, pz) = [\Gamma(\delta_{i,j}) \setminus \Gamma_n] \left\{ g_p(n - 1, i) E_{k,m}^{(n)}(\tau, pz) + p^{n-j} g_p(n - 1, j - 1) E_{k,m}^{(n)}(\tau, z) \right\}. \]

Hence we showed this lemma also for the case $p^2 \not| m$. \hfill \Box

The following proposition has been shown by Yamazaki [Ya 89, Theorem 3.3] for the case $m = 1$. We generalize it for any positive-integer $m$.

**Proposition 3.9.** For any natural number $l$ ($0 \leq l \leq n$), the form $E_{l,m}^{(n)}|_{V_l, n-l}(p^2)$ is a linear combination of $E_{k,m}^{(n)}|U(p^2)$, $E_{k,m}^{(n)}|U(p)$ and $E_{k,m}^{(n)}$ over $\mathbb{C}$.

**Proof.** This proposition follows from Lemma 3.3 and Lemma 3.8 \hfill \Box

3.6. **Relation among Jacobi-Eisenstein series.** Now we shall calculate the coefficients in the linear combinations in Proposition 3.9. This calculation can be directly done by using the values of $G_{\alpha}^{j-i}(m)$ and $g_p(a, b)$. However, we will here use the Siegel $\Phi$-operators for simplicity.
We set
\[
\begin{pmatrix}
  a_{0,m,p,k} \\
  a_{1,m,p,k} \\
  a_{2,m,p,k}
\end{pmatrix} :=
\begin{cases}
  \frac{p^{-2k+2}}{p^{-k}(p-1)} & \text{if } p^2|m, \\
  0 & \text{if } p^2 \not|m \text{ and } p|m, \\
  \frac{p^{-2k+2} + p^{-k+1} - p^{-k}}{1} & \text{if } p^2 \not|m, \\
  \frac{p^{-2k+2} - p^{-k}}{p^{-k+1} + 1} & \text{if } p|m.
\end{cases}
\]

Lemma 3.10. For the Jacobi-Eisenstein series \( E_{k,m}^{(1)} \) of degree 1, we have the identity
\[
E_{k,m}^{(1)}|_{V_{0,1}(p^2), V_{1,0}(p^2)} = \left( E_{k,m}^{(1)}|_{U(p^2), E_{k,m}^{(1)}|_{U(p)}, E_{k,m}^{(1)}|_{U(p^2)}} \right) \begin{pmatrix}
  0 & a_{0,m,p,k} \\
  p^{-k} & a_{1,m,p,k} \\
  0 & a_{2,m,p,k}
\end{pmatrix}.
\]

Proof. Since \( \Gamma_1(p^2) \Gamma_1 = \Gamma_1(p^2) \), the relation \( E_{k,m}^{(1)}|_{V_{0,1}(p^2)} = p^{-k} E_{k,m}^{(1)}|_{U(p)} \) is obvious.

From Lemma 3.3 we obtain
\[
E_{k,m}^{(1)}|_{V_{1,0}(p^2)} = K_{0,0} + K_{0,1} + K_{1,1}.
\]

Due to Lemma 3.6 and Lemma 3.8 we have
\[
K_{0,0} = \begin{cases}
  \frac{p^{-2k+2}}{p^{-k}(p-1)} E_{k,m}^{(1)}(\tau, p^2 z) & \text{if } p^2|m, \\
  \frac{p^{-2k+2}}{p^{-k}(p-1)} E_{k,m}^{(1)}(\tau, p z) & \text{if } p^2 \not|m,
\end{cases}
\]
\[
K_{0,1} = \begin{cases}
  p^{-k}(p-1) E_{k,m}^{(1)}(\tau, p z) & \text{if } p|m, \\
  p^{-k+1} E_{k,m}^{(1)}(\tau, z) - p^{-k} E_{k,m}^{(1)}(\tau, p z) & \text{if } p \not|m,
\end{cases}
\]
\[
K_{1,1} = E_{k,m}^{(1)}(\tau, z).
\]

Therefore this lemma follows. \( \square \)

Let \( B_{l,t+1}(X_l), B_{2,n+1}(X_2, ..., X_n) \) and \( A_{2,n+1}^{p,k} \) be matrices introduced in Section 2.6. We recall \( A_{2,n+1}^{p,k} = B_{2,n+1}(p^{2-k}, p^{2-k}, ..., p^{n-k}) \) and the matrix \( A_{2,n+1}^{p,k} \) has the size \( 2 \times (n + 1) \).

The following proposition has been shown by Yamazaki [Ya 89, Theorem 4.1] for the case \( m = 1 \). We generalize it for any positive-integer \( m \).
Lemma 3.1, we obtain we have For any Jacobi-Eisenstein series Proposition 3.11.

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introduced in Section 2.5. From Lemma 2.2 and from the fact that Φ(n)

identity holds Hence by using Siegel Φ-operator

Thus this proposition follows for any positive-integer m.

⊓⊔

4. Generalized Maass relation for Siegel–Eisenstein series

The purpose of this section is to prove Theorem 1.1 Let e(n)

Jacobi coefficient of Siegel-Eisenstein series E(n)

In this section we write \( \sum_{d|m} \) for \( \sum_{d>m} \), and \( \sum_{d^2|m} \) for \( \sum_{d^2>m} \), for simplicity.
4.1. Fourier-Jacobi coefficients. We define an arithmetic function

\[ g_k(m) := \sum_{d^2|m} \mu(d) \sigma_{k-1} \left( \frac{m}{d^2} \right), \]

where \( \mu(d) \) is the Möbius function and we put \( \sigma_{k-1}(m) := \sum_{d|m} d^{k-1} \) as usual.

Lemma 4.1. We obtain

\[ g_k(mp) = \begin{cases} 
(p^{k-1} + 1) g_k(m) & \text{if } p \nmid m, \\
p^{k-1} g_k(m) & \text{if } p \mid m.
\end{cases} \]

Proof. The function \( g_k(m) \) is a multiplicative function, namely \( g_k(ml) = g_k(m)g_k(l) \) if \( \gcd(m, l) = 1 \). Hence we obtain the identity \( g_k(m) = m^{k-1} \prod_{q \text{ prime}} \left( 1 + \frac{1}{q^{k-1}} \right) \). This lemma follows from this identity. \( \square \)

The following proposition is a special case of a result in [Bo 83, Satz 7].

Proposition 4.2 (Boecherer [Bo 83]). We have

\[ e_{k,m}^{(n)} = \sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) E_k^{(n)} \left( \frac{m}{d^2} \right) |U(d)|. \]

Proof. For the proof of this proposition, the reader is referred to [Ya 86, Theorem 5.5]. \( \square \)

Proposition 4.3. For any \( n > 0 \) and for any \( m > 0 \) we have the identity

\[ \sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E_k^{(n)} \left( \frac{m}{d^2} \right) |U(p^2d)|, E_k^{(n)} \left( \frac{m}{d^2} \right) |U(pd)|, E_k^{(n)} \left( \frac{m}{d^2} \right) |U(d)| \right) \begin{pmatrix} a_0, m/p, p, k \\ a_1, m/p, p, k \\ a_2, m/p, p, k \end{pmatrix} = \begin{pmatrix} e_{k,p}^{(n)} |U(p^2)|, e_{k,m}^{(n)} |U(p)|, e_{k,mp^2}^{(n)} \end{pmatrix} \begin{pmatrix} p^{-k}(-1 + p \delta_{p|m}) \\ p^{-2k+2} \end{pmatrix}, \]

where \( \delta_{p|m} \) is 1 or 0, according as \( p|m \) or \( p \nmid m \).
Proof. Due to Proposition 4.2 and Lemma 4.1, this proposition is obtained by straightforward calculation as follows. We set nine functions

\[ E_{g1} := \sum_{\frac{m}{d^2} \equiv 0 (p^2)} p^{-2k+2} E_{k, \frac{m}{d^2}}^{(n)} |U(p^2 d) g\left(\frac{m}{d^2}\right), \]

\[ E_{g2} := \sum_{\frac{m}{d^2} \equiv 0 (p^2)} p^{-k} (p - 1) E_{k, \frac{m}{d^2}}^{(n)} |U(pd) g\left(\frac{m}{d^2}\right), \]

\[ E_{g3} := \sum_{\frac{m}{d^2} \equiv 0 (p^2)} E_{k, \frac{m^2}{d^2}}^{(n)} |U(d) g\left(\frac{m}{d^2}\right), \]

\[ E_{g4} := \sum_{\frac{m}{d^2} \equiv 0 (p)} \sum_{\frac{m}{d^2} \equiv 0 (p^2)} p^{-2k+2} E_{k, \frac{m}{d^2}}^{(n)} |U(pd) g\left(\frac{m}{d^2}\right), \]

\[ E_{g5} := \sum_{\frac{m}{d^2} \equiv 0 (p)} \sum_{\frac{m}{d^2} \equiv 0 (p^2)} (p^{-k+1} - p^{-k}) E_{k, \frac{m}{d^2}}^{(n)} |U(pd) g\left(\frac{m}{d^2}\right), \]

\[ E_{g6} := \sum_{\frac{m}{d^2} \equiv 0 (p)} \sum_{\frac{m}{d^2} \equiv 0 (p^2)} E_{k, \frac{m^2}{d^2}}^{(n)} |U(d) g\left(\frac{m}{d^2}\right), \]

\[ E_{g7} := \sum_{\frac{m}{d^2} \equiv 0 (p)} \sum_{\frac{m}{d^2} \equiv 0 (p^2)} p^{-2k+2} E_{k, \frac{m}{d^2}}^{(n)} |U(pd) g\left(\frac{m}{d^2}\right), \]

\[ E_{g8} := \sum_{\frac{m}{d^2} \equiv 0 (p)} \sum_{\frac{m}{d^2} \equiv 0 (p^2)} (-p^{-k}) E_{k, \frac{m}{d^2}}^{(n)} |U(pd) g\left(\frac{m}{d^2}\right), \]

and

\[ E_{g9} := \sum_{\frac{m}{d^2} \equiv 0 (p)} \sum_{\frac{m}{d^2} \equiv 0 (p^2)} (p^{-k+1} + 1) E_{k, \frac{m^2}{d^2}}^{(n)} |U(d) g\left(\frac{m}{d^2}\right). \]
If \( \text{ord}_m \equiv 1 \pmod{2} \), then

\[
\sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E^{(n)}_{k, \frac{m}{d^2}} |U(p^2)\right) = \sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E^{(n)}_{k, \frac{m}{d^2}} |U(pd)\right) = \sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E^{(n)}_{k, \frac{m}{d^2}} |U(d)\right)
\]

and

\[
Eg_1 = E^{(n)}_{k, \frac{m}{d^2}} |U(p^2)\),
Eg_2 + Eg_5 = p^{-k}(p - 1)E^{(n)}_{k, \frac{m}{d^2}} |U(p)\),
Eg_3 + Eg_4 + Eg_6 = p^{-2k+2}E^{(n)}_{k, \frac{m}{d^2}} |U(d)\).
\]

Because of the assumption \( \text{ord}_m \equiv 1 \pmod{2} \), we have \( \delta_{p|m} = 1 \).

Hence this proposition follows for the case \( \text{ord}_m \equiv 1 \pmod{2} \).

If \( \text{ord}_m \equiv 0 \pmod{2} \), then

\[
\sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E^{(n)}_{k, \frac{m}{d^2}} |U(p^2)\right) = \sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E^{(n)}_{k, \frac{m}{d^2}} |U(pd)\right) = \sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E^{(n)}_{k, \frac{m}{d^2}} |U(d)\right)
\]

and

\[
Eg_1 = \delta_{p^2|m} \left\{ E^{(n)}_{k, \frac{m}{d^2}} |U(p^2)\right) + p^{-k+1} \sum_{d^2|m} E^{(n)}_{k, \frac{m}{d^2}} |U(pd)\right) g \left( \frac{m}{d^2} \right) \right\},
Eg_2 + Eg_8 = \delta_{p^2|m} \left\{ p^{-k+1} E^{(n)}_{k, \frac{m}{d^2}} |U(p)\right) - p^{-k+1} \sum_{d^2|m} E^{(n)}_{k, \frac{m}{d^2}} |U(pd)\right) g \left( \frac{m}{d^2} \right) \right\}
\]

\[
Eg_3 + Eg_7 + Eg_9 = p^{-2k+2}E^{(n)}_{k, \frac{m}{d^2}} |U(d)\).
\]

Here \( \delta_{p^2|m} \) is defined by 1 or 0, according as \( p^2|m \) or \( p^2 \not|m \). Because of the assumption \( \text{ord}_m \equiv 0 \pmod{2} \), we have \( \delta_{p^2|m} = \delta_{p|m} \).

Therefore this proposition follows also for the case \( \text{ord}_m \equiv 0 \pmod{2} \). \( \square \)
4.2. **Proof of Theorem 1.1.** Now we shall prove Theorem 1.1. For any prime $p$ and for any positive-integer $d$, the operators $V_{l,n}(p^2)$ and $U(d)$ are compatible. Hence from Proposition 4.2, Proposition 3.11 and Proposition 4.3, we have

\[
\sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E_{k,m}^{(n)} | (V_{l,n}(p^2), ..., V_{l,n}(p^2)) \right) |U(d)\)
\[
= \sum_{d^2|m} g_k \left( \frac{m}{d^2} \right) \left( E_{k,m}^{(n)} | U(p^2d), E_{k,m}^{(n)} | U(pd), E_{k,m}^{(n)} | U(d) \right) \begin{pmatrix} 0 & a_0 & a_{1/p}\ a_1 & a_{2/p} & a_{3/p}\ 0 & a_{2/p} & a_{3/p} \end{pmatrix} A_{d^2,n+1}^{p,k}
\]
\[
= \left( e_{k,m}^{(n)} | U(p^2), e_{k,m}^{(n)} | U(p), e_{k,m}^{(n)} | U(d) \right) \begin{pmatrix} 0 & p^{-k} & 1\ p^{-k} & 0 & p^{-2k+2}\ 0 & p^{-2k+2} & 0 \end{pmatrix} A_{d^2,n+1}^{p,k}.
\]

Thus we obtain Theorem 1.1.

5. **Generalized Maass relation for the Miyawaki-Ikeda lifts**

In this section we shall show Theorem 1.2, Theorem 1.3 and Corollary 1.4. Let $\phi_m \in J_{k+n,m}^{(2n-1)}$ be the $m$-th Fourier-Jacobi coefficient of the Duke-Imamoglu-Ibukiyama-Ikeda lift $F$ stated in Theorem 1.2.

In this section the letters $p$ and $q$ are reserved for prime numbers. For example, the symbol $\prod_{p|N}$ denotes the product over primes $p$ such that $p|N$.

5.1. **Fourier coefficients of $\phi_m$.** We take the Fourier expansion of $\phi_m$:

\[
\phi_m(\tau, z) = \sum_{N,R} C_m(N,R)e(N\tau)e(i^nRz),
\]

where in the summation $N \in \text{Sym}^{*}_{2n-1}$ and $R \in \mathbb{Z}^{2n-1}$ run over all elements which satisfy $4N - R^tR > 0$. We set $M = \left( \begin{array}{c} N \\ \frac{1}{2}R \end{array} \right) m$. We denote by $D_M$ and by $f_M$ the integers such that $(-1)^n \det(2M) = D_M f_M$, where $D_M$ is a fundamental discriminant and $f_M$ is a positive integer. Then the $(N,R)$-th Fourier coefficient $C_m(N,R)$ of $\phi_m$ is

\[
C_m(N,R) = C(|D_M|) f_M^{k-\frac{1}{2}} \prod_{p|f_M} \tilde{F}_p(M; \alpha_p),
\]

where $C(|D_M|)$ is the $|D_M|$-th Fourier coefficient of $h$ which corresponds to $g$ by Shimura correspondence, and $\tilde{F}_p(M; X_p) \in \mathbb{C}[X_p + X_p^{-1}]$ is a certain Laurent polynomial introduced in [Ik 01, §1].
5.2. **Matrix** $A_{2,2n}^{p,k+n}$. Let $A_{2,2n+1}^p$ and $A_{2,2n}(X_p)$ be the matrices introduced in Section 2.6.

**Lemma 5.1.** For any even integer $k$ we obtain

$$A_{2,2n}^{p,k+n} = p^{-(n-1)(2k-1)} A_{2,2n}'(p^{-(k-\frac{1}{2})}).$$

**Proof.** From the definition of $A_{2,2n}^{p,k}$,

$$A_{2,2n}^{p,k+n} = B_{2,2n}(p^{2-n-k}, p^{3-n-k}, \ldots, p^{n-1-k})$$

$$= \left( \prod_{i=2}^{2n-1} p^{i-n-k} \right) B_{2,2n}'(p^{\frac{3}{2}-n-(k-\frac{1}{2})}, p^{\frac{5}{2}-n-(k-\frac{1}{2})}, \ldots, p^{\frac{3}{2}+n-(k-\frac{1}{2})})$$

$$= p^{-(n-1)(2k-1)} A_{2,2n}'(p^{-(k-\frac{1}{2})}).$$

$\square$

5.3. **Proof of Theorem 1.2.** Let $g \in \text{GSp}_{2n-1}(\mathbb{R}) \cap M_{4n-2}(\mathbb{Z})$ be a matrix such that the similitude of $g$ is $\nu(g) = p^2$. We write the coset decomposition $\Gamma_{2n-1} g \Gamma_{2n-1} = \bigcup_i \Gamma_n g_i$ with $g_i = \begin{pmatrix} A_i & B_i \\ 0_{2n-1} & D_i \end{pmatrix}$. We take the Fourier expansion of $\phi_m|V(\Gamma_{2n-1} g \Gamma_{2n-1})$:

$$(\phi_m|V(\Gamma_{2n-1} g \Gamma_{2n-1}))(\tau, z) = \sum_{N,R} C_m(g; N, R) e(N\tau) e(i R z),$$

where in the summation $N \in \text{Sym}_{2n-1}^*$ and $R \in \mathbb{Z}^{2n-1}$ run over all elements which satisfy $4Nmp^2 - R'R > 0$.

We now fix $N \in \text{Sym}_{2n-1}^*$ and $R \in \mathbb{Z}^{2n-1}$ such that $4Nmp^2 - R'R > 0$. And we set $M_1 = \begin{pmatrix} N & \frac{1}{2p} R' \\ \frac{1}{2p} R & m \end{pmatrix}$.

**Lemma 5.2.** The $(N, R)$-th Fourier coefficient $C_m(g; N, R)$ of $\phi_m|V(\Gamma_{2n-1} g \Gamma_{2n-1})$ is

$$C_m(g; N, R) = p^{-(2n-1)(k-\frac{1}{2})} C(|D_{M_1}|) J_{M_1}^{k-\frac{1}{2}} \sum_i \det D_i^{-(k+n)}$$

$$\times \prod_{q|f_{M_1}[\text{diag}(p^{-1l} D_i, 1)]} \widetilde{F}_q \left( M_1[\text{diag}(p^{-1l} D_i, 1)]; \alpha_q \right).$$

Here we regard $\widetilde{F}_q (M_1[\text{diag}(p^{-1l} D_i, 1)]; X_q)$ as 0, if $M_1[\text{diag}(p^{-1l} D_i, 1)] \not\in \text{Sym}_{2n}^*$.

**Proof.** From the definition of $V(\Gamma_{2n-1} g \Gamma_{2n-1})$ the $(N, R)$-th Fourier coefficient of the form $\phi_m|V(\Gamma_{2n-1} g \Gamma_{2n-1})$ is

$$\sum_i \det D_i^{-(k+n)} C(|D_{M_1}|) J_{M_1}^{k-\frac{1}{2}} \prod_{q|f_{M_1}[i]} \widetilde{F}_q (M_1[i]; \alpha_q),$$
where $M_{1,i} := M_1[\text{diag}(p^{-1t}D_i, 1)]$. Thus this lemma follows from the fact that if $M_{1,i}$ is a half-integral symmetric matrix, then $D_{M_{1,i}} = D_{M_1}$ and $f_{M_{1,i}} = p^{-(2n-1}(\det D_i)f_{M_1}$.

Now we shall prove Theorem 1.2. In the same manner as in Lemma 5.2 we obtain the fact that the $(N, R)$-th Fourier coefficient of $e_k^{(2n-1)}|V(\Gamma_{2n-1}g\Gamma_{2n-1})$ is

$$p^{-(2n-1)(k-\frac{1}{2})}h_{k+\frac{1}{2}}(|D_{M_1}|) f_{M_1}^{k-\frac{1}{2}} \sum_i \det D_i^{-n-\frac{1}{2}} \prod_{q \mid f_{M_1}[\text{diag}(p^{-1t}D_i, 1)]} \mathcal{F}_q \left( M_1[\text{diag}(p^{-1t}D_i, 1)]; q^{k-\frac{1}{2}} \right),$$

where $h_{k+\frac{1}{2}}(|D_{M_1}|)$ is the $|D_{M_1}|$-th Fourier coefficient of the Cohen type Eisenstein series of weight $k + \frac{1}{2}$ which corresponds to the Eisenstein series of weight $2k$ by the Shimura correspondence.

By the virtue of Theorem 1.1, the form $e_k^{(2n-1)}|V(\Gamma_{2n-1}g\Gamma_{2n-1})$ is a linear combination of $e_k^{(2n-1)}|U(p^2)$, $e_k^{(2n-1)}|U(p)$ and $e_k^{(2n-1)}|U(p^2)$, $e_k^{(2n-1)}|U(p)$ and $e_k^{(2n-1)}|U(p^2)$. Hence there exists constants $u_0$, $u_1$ and $u_2$, such that

$$e_k^{(2n-1)}|V(\Gamma_{2n-1}g\Gamma_{2n-1}) = u_0 e_k^{(2n-1)}|U(p^2) + u_1 e_k^{(2n-1)}|U(p) + u_2 e_k^{(2n-1)}|U(p^2).$$

We remark that the constants $u_0$, $u_1$ and $u_2$ depend on the choices of $p$, $k$, $m$ and $n$. The $(N, R)$-th Fourier coefficient of the form of the above RHS is

$$u_0 h_{k+\frac{1}{2}}(|D_{M_1}|) p^{-(2n-1)(k-\frac{1}{2})} f_{M_1}^{k-\frac{1}{2}} \prod_{q \mid f_{M_0}} \mathcal{F}_q \left( M_0; q^{k-\frac{1}{2}} \right)$$

$$+ u_1 h_{k+\frac{1}{2}}(|D_{M_1}|) f_{M_1}^{k-\frac{1}{2}} \prod_{q \mid f_{M_1}} \mathcal{F}_q \left( M_1; q^{k-\frac{1}{2}} \right)$$

$$+ u_2 h_{k+\frac{1}{2}}(|D_{M_1}|) p^{-(2n-1)(k-\frac{1}{2})} f_{M_1}^{k-\frac{1}{2}} \prod_{q \mid f_{M_2}} \mathcal{F}_q \left( M_2; q^{k-\frac{1}{2}} \right),$$

where $M_0 = \left( \frac{N}{2p^t R}, \frac{1}{2p^t} \right)$ and $M_2 = \left( \frac{N}{2p^t R}, \frac{1}{2p^t} \right)$. Because $h_{k+\frac{1}{2}}(|D_{M_1}|) \neq 0$, we obtain

$$p^{-(2n-1)(k-\frac{1}{2})} \sum_i \det D_i^{-n-\frac{1}{2}} \prod_{q \mid f_{M_1}[\text{diag}(p^{-1t}D_i, 1)]} \mathcal{F}_q \left( M_1[\text{diag}(p^{-1t}D_i, 1)]; q^{k-\frac{1}{2}} \right)$$

$$= u_0 p^{-(2n-1)(k-\frac{1}{2})} \prod_{q \mid f_{M_0}} \mathcal{F}_q \left( M_0; q^{k-\frac{1}{2}} \right) + u_1 \prod_{q \mid f_{M_1}} \mathcal{F}_q \left( M_1; q^{k-\frac{1}{2}} \right) + u_2 p^{-(2n-1)(k-\frac{1}{2})} \prod_{q \mid f_{M_2}} \mathcal{F}_q \left( M_2; q^{k-\frac{1}{2}} \right).$$
We denote by \( c_0(N, R) \), \( c_1(N, R) \) and \( c_2(N, R) \) the \((N, R)\)-th Fourier coefficients of \( e^{(2n-1) \frac{j}{2} R} U(p^2) \), \( e^{(2n-1) \frac{j}{2} R} U(p) \) and \( e^{(2n-1) \frac{j}{2} R} U(p^m p^2) \), respectively. We remark \( c_0(N, R) = 0 \) if \( p^2 \nmid m \).

Furthermore, we remark that \( c_0(N, R) = 0 \) if \( R \notin pZ^{2n-1} \), and \( c_1(N, R) = 0 \) if \( R \notin pZ^{2n-1} \).

Because the forms in the set \( \{ F^{(2n-1)}_k U(d) \} \), where \( d \) runs over all positive-integers such that \( d^2 \mid m \), are linearly independent (see Lemma 3.1) and because of Proposition 4.2 three forms \( e^{(2n-1) \frac{j}{2} R} U(p^2) \), \( e^{(2n-1) \frac{j}{2} R} U(p) \) and \( e^{(2n-1) \frac{j}{2} R} U(p^m p^2) \) are linearly independent.

From now on we assume \( p^2 \mid m \) for simplicity. The proof of Theorem 1.2 for the case \( p^2 \nmid m \) is similar to the case \( p^2 \mid m \).

There exist pairs of matrices \((N_j, R_j)\) \((j = 1, 2, 3)\) such that
\[
\det \left( \begin{pmatrix} c_0(N_1, R_1) & c_1(N_1, R_1) & c_2(N_1, R_1) \\ c_0(N_2, R_2) & c_1(N_2, R_2) & c_2(N_2, R_2) \\ c_0(N_3, R_3) & c_1(N_3, R_3) & c_2(N_3, R_3) \end{pmatrix} \right) \neq 0.
\]

For \( j = 1, 2, 3 \), we define
\[
M_0^{(j)} := \left( \begin{array}{ccc} N_j & \frac{1}{2} R_j \\ \frac{1}{2} p^2 R_j & m \\ \frac{1}{2} p^2 R_j & p^2 \end{array} \right), \quad M_1^{(j)} := \left( \begin{array}{ccc} N_j & \frac{1}{2} R_j \\ \frac{1}{2} p R_j & m \end{array} \right), \quad M_2^{(j)} := \left( \begin{array}{ccc} N_j & \frac{1}{2} R_j \\ \frac{1}{2} R_j & mp^2 \end{array} \right),
\]
and we put a \(3 \times 3\) matrix
\[
C\left( \{(N_j, R_j)\}_j; \{X_q\}_q \text{prime} \right) := \left( \prod_{q \mid f_{M_i^{(j)}}} \tilde{F}_q \left( M_i^{(j)}; X_q \right) \right)_{j=1,2,3, \ i=0,1,2}.
\]

Then from the identity (5.1) we have
\[
p^{-(2n-1)(k-\frac{1}{2})} \sum_i (\det D_i)^{-n-\frac{1}{2}}
\times \left( \prod_{q \mid f_{M_1^{(1)}}} \tilde{F}_q \left( M_1^{(1)}[\text{diag}(p^{1-t} D_i, 1)]; q^{k-\frac{1}{2}} \right) \right)
\times \left( \prod_{q \mid f_{M_1^{(2)}}} \tilde{F}_q \left( M_1^{(2)}[\text{diag}(p^{1-t} D_i, 1)]; q^{k-\frac{1}{2}} \right) \right)
\times \left( \prod_{q \mid f_{M_1^{(3)}}} \tilde{F}_q \left( M_1^{(3)}[\text{diag}(p^{1-t} D_i, 1)]; q^{k-\frac{1}{2}} \right) \right)
= C \left( \{(N_j, R_j)\}_j; \{q^{k-\frac{1}{2}}\}_q \right) \begin{pmatrix} u_0 p^{k-\frac{1}{2}} \\ u_1 \\ u_2 p^{-k+\frac{1}{2}} \end{pmatrix}.
\]
Hence we obtain

\[
\sum_i (\det D_i)^{-n-\frac{3}{2}} C\left(\{(N_j, R_j)_j; \{q^{k-\frac{1}{2}}\}_q\right) \left(\prod_{q|f_{M_1^{(1)}}[\text{diag}(p^{-1}t D_i, 1)]} \tilde{F}_q \left(M_1^{(1)}[\text{diag}(p^{-1}t D_i, 1)]; q^{k-\frac{1}{2}}\right) \right) 
\times \left(\prod_{q|f_{M_1^{(2)}}[\text{diag}(p^{-1}t D_i, 1)]} \tilde{F}_q \left(M_1^{(2)}[\text{diag}(p^{-1}t D_i, 1)]; q^{k-\frac{1}{2}}\right) \right) 
\times \left(\prod_{q|f_{M_1^{(3)}}[\text{diag}(p^{-1}t D_i, 1)]} \tilde{F}_q \left(M_1^{(3)}[\text{diag}(p^{-1}t D_i, 1)]; q^{k-\frac{1}{2}}\right) \right) 
= p^{(2n-1)(k-\frac{1}{2})} \begin{pmatrix} u_0 p^{k-\frac{1}{2}} \\ u_1 \\ u_2 p^{-k+\frac{1}{2}} \end{pmatrix}.
\]

The RHS of the above identity does not depend on the choices of \((N_j, R_j)\) \((j = 1, 2, 3)\). Furthermore, the above identity holds for infinitely many integer \(k\). Therefore there exist Laurent polynomials \(\Phi_i(X_p) \in \mathbb{C}[X_p + X_p^{-1}] \ (i = 0, 1, 2)\) which are independent of the choices of \((N_j, R_j)\) \((j = 1, 2, 3)\), such that

\[
\sum_i (\det D_i)^{-n-\frac{1}{2}} C\left(\{(N_j, R_j)_j; \{X_q\}_q\right) \left(\prod_{q|f_{M_1^{(1)}}[\text{diag}(p^{-1}t D_i, 1)]} \tilde{F}_q \left(M_1^{(1)}[\text{diag}(p^{-1}t D_i, 1)]; X_q\right) \right) 
\times \left(\prod_{q|f_{M_1^{(2)}}[\text{diag}(p^{-1}t D_i, 1)]} \tilde{F}_q \left(M_1^{(2)}[\text{diag}(p^{-1}t D_i, 1)]; X_q\right) \right) 
\times \left(\prod_{q|f_{M_1^{(3)}}[\text{diag}(p^{-1}t D_i, 1)]} \tilde{F}_q \left(M_1^{(3)}[\text{diag}(p^{-1}t D_i, 1)]; X_q\right) \right) 
= \begin{pmatrix} \Phi_0(X_p) \\ \Phi_1(X_p) \\ \Phi_2(X_p) \end{pmatrix}.
\]

In particular, we have

\[
\sum_i \det D_i^{-n-\frac{1}{2}} \prod_{q|f_{M_1}[\text{diag}(p^{-1}t D_i, 1)]} \tilde{F}_q \left(M_1[\text{diag}(p^{-1}t D_i, 1)]; X_q\right) 
= \Phi_0(X_p) \prod_{q|f_{M_0}} \tilde{F}_q \left(M_0; X_q\right) + \Phi_1(X_p) \prod_{q|f_{M_1}} \tilde{F}_q \left(M_1; X_q\right) + \Phi_2(X_p) \prod_{q|f_{M_2}} \tilde{F}_q \left(M_2; X_q\right).
\]
Therefore, by substituting $X_q = \alpha_q$ in the above identity and by using the relations $p f_M = f_{M_1} = p^{-1} f_{M_2}$ and $D_M = D_{M_1} = D_{M_2}$, we obtain

\[
p^{-2n-1)k-\frac{1}{2}} C (|D_M|) \frac{k^{-\frac{1}{2}}}{f_{M_1}} \sum_i \det D_i^{-\frac{n}{2}} \prod_{q \neq f_{M_1}[\text{diag}(p^{-1} D_i, 1)]} \tilde{F}_q (M_1 \text{[diag]}(p^{-11} D_i, 1); \alpha_q)
\]

\[= p^{-2n-1)(k-\frac{1}{2})} \left\{ \Phi_0(\alpha_p) p^{k^{\frac{1}{2}}} C (|D_M|) \frac{k^{-\frac{1}{2}}}{f_{M_1}} \prod_{q \neq f_{M_1}} \tilde{F}_q (M_0; \alpha_q) \right. \]
\[+ \Phi_1(\alpha_p) C (|D_M|) \frac{k^{-\frac{1}{2}}}{f_{M_1}} \prod_{q \neq f_{M_1}} \tilde{F}_q (M_1; \alpha_q) \]
\[+ \Phi_2(\alpha_p) p^{k^{\frac{1}{2}}} C (|D_M|) \frac{k^{-\frac{1}{2}}}{f_{M_1}} \prod_{q \neq f_{M_1}} \tilde{F}_q (M_2; \alpha_q) \right\}.
\]

Thus

\[
\phi_m |V(\Gamma_{2n-1} \Gamma_{2n-1}) = \phi_m |V(\Gamma_{2n-1} \Gamma_{2n-1}) (p^2, \ldots, p^2) \quad \Phi_0(\alpha_p) \quad \Phi_1(\alpha_p) \quad \Phi_2(\alpha_p).
\]

Hence there exist Laurent polynomials $\Phi_{j,l}(X_p) \in \mathbb{C}[X_p + X_p^{-1}]$ ($j = 0, 1, 2$, $l = 0, \ldots, 2n - 1$) which satisfy

\[(5.2)\]
\[
\phi_m |(V_{0,2n-1} \Gamma_{2n-1}(p^2, \ldots, V_{2n-1,0}(p^2)) = \phi_m |(V_{0,2n-1}(p^2, \ldots, V_{2n-1,0}(p^2)) \quad \Phi_0(\alpha_p) \quad \cdots \quad \Phi_0(\alpha_p) \quad \Phi_1(\alpha_p) \quad \cdots \quad \Phi_1(\alpha_p) \quad \Phi_2(\alpha_p) \quad \cdots \quad \Phi_2(\alpha_p).
\]

Here the polynomials $\Phi_{j,l}(X_p)$ depend on the choices of $p$ and $m$, but not on the choice of $f$ which is the preimage of the Duke-Imamoglu-Ibukiyama-Ikeda lift $F$. The $m$-th Fourier-Jacobi coefficient $e_{k+l,m}(2n-1)$ of Siegel-Eisenstein series satisfies also the identity (5.2). Thus, because of Theorem 1.1 and of Lemma 5.1 we obtain

\[
\begin{pmatrix}
\Phi_{0,0}(p^{k^{\frac{1}{2}}}p^{\frac{m}{2}}) & \cdots & \Phi_{0,2^n-1}(p^{k^{\frac{1}{2}}}p^{\frac{m}{2}}) \\
\Phi_{1,0}(p^{k^{\frac{1}{2}}}p^{\frac{m}{2}}) & \cdots & \Phi_{1,2^n-1}(p^{k^{\frac{1}{2}}}p^{\frac{m}{2}}) \\
\Phi_{2,0}(p^{k^{\frac{1}{2}}}p^{\frac{m}{2}}) & \cdots & \Phi_{2,2^n-1}(p^{k^{\frac{1}{2}}}p^{\frac{m}{2}})
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
p^{-n^{\frac{1}{2}}} & p^{-n^{\frac{1}{2}}}(1 + p \delta_{p|m}) \\
p^{-n^{\frac{1}{2}}} & p^{-n^{\frac{1}{2}}}(1 + p \delta_{p|m})
\end{pmatrix} A_{2n}(p^{-(2n-1)^{\frac{1}{2}}}p^{-(2n-1)^{\frac{1}{2}}}).
\]
Furthermore, this identity holds for infinitely many $k$. Hence we obtain

$$
\begin{pmatrix}
\Phi_{0,0}(X_p) \\
\Phi_{1,0}(X_p) \\
\Phi_{2,0}(X_p)
\end{pmatrix}
\cdots
\begin{pmatrix}
\Phi_{0,2n-1}(X_p) \\
\Phi_{1,2n-1}(X_p) \\
\Phi_{2,2n-1}(X_p)
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
p^{-n-\frac{1}{2}} & p^{-n-\frac{1}{2}}(-1 + p \delta_{p|m}) \\
0 & p^{-2n+1}
\end{pmatrix}
A'_{2,2n}(X_p^{-1}).
$$

In particular, we get $A'_{2,2n}(X_p) = A'_{2,2n}(X_p^{-1})$. Due to the identities (5.2) and (5.3), we thus obtain Theorem 1.2.

5.4. Proof of Theorem 1.3: From the identity (2.1) in Section 2.4 and from Theorem 1.2, we obtain

$$
\phi_m(\tau, 0) |(T_{0,2n-1}(p^2), \ldots, T_{2n-1,0}(p^2))
= p^{2nk+n-1} \left(\frac{\phi_m}{p^2}(\tau, 0), \phi_m(\tau, 0), \phi_{mp^2}(\tau, 0)\right)
\begin{pmatrix}
0 & 1 \\
p^{-k-n} & p^{-k-n}(-1 + p \delta_{p|m}) \\
0 & p^{-2k-2n+2}
\end{pmatrix}
A'_{2,2n}(\alpha_p).
$$

Due to the identity $F \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right) = \sum_{m>0} \phi_m(\tau, 0)e(m\omega)$, we have

$$
\sum_{m>0} \left\{ \phi_m(\tau, 0) + p^{-k-n}(-1 + p \delta_{p|m})\phi_m(\tau, 0) + p^{-2k-2n+2}\phi_{mp^2}(\tau, 0) \right\} e(m\omega)
= p^{-2k-2n+2} F \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)|_{T_{1,0}(p^2)},
$$

where in the RHS we regard that the Hecke operator $T_{1,0}(p^2)$ acts on $F \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)$ as a function of $\omega \in \mathcal{H}_1$ for a fixed $\tau \in \mathcal{H}_1$. Therefore

$$
\sum_{m>0} \phi_m(\tau, 0) |(T_{0,2n-1}(p^2), \ldots, T_{2n-1,0}(p^2))e(m\omega)
= p^{2nk+n-1} \left(\frac{\phi_m}{p^2}(\tau, 0), p^{-2k-2n+2} F \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)|_{T_{1,0}(p^2)} \right) A'_{2,2n}(\alpha_p).
$$

We denote by $\langle h_1(\omega), h_2(\omega) \rangle_{\omega}$ the Petersson inner product of two elliptic modular forms $h_1, h_2$. The symbol $\lambda_g(p^2)$ denotes the eigenvalue of $g$ for $T_{1,0}(p^2)$.

Because $\left\langle F \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right), g(\omega) \right\rangle_{\omega} = \mathcal{F}_{f,g}(\tau)$ and because

$$
\left\langle F \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)|_{T_{1,0}(p^2)}, g(\omega) \right\rangle_{\omega} = \lambda_g(p^2)\mathcal{F}_{f,g}(\tau),
$$
we obtain

\[ \mathcal{F}_{f,g}(T_{0,2n-1}(p^2), ..., T_{2n-1,0}(p^2)) \]

\[ = \left( \sum_{m>0} \phi_m(\tau,0) \right| (T_{0,2n-1}(p^2), ..., T_{2n-1,0}(p^2)) c(m\omega), g(\omega) \right)_\omega \]

\[ = p^{2nk+n-1}(p^{-k-n}\mathcal{F}_{f,g}, p^{-2k-2n+2}\lambda_g(p^2)\mathcal{F}_{f,g}) A'_{2,2n}(\alpha_p). \]

Therefore we proved Theorem 1.3

5.5. Proof of Corollary 1.4. Let \{\mu_0, \mu_1, ..., \mu_{2n-1}\} be the Satake parameter of \mathcal{F}_{f,g} at a prime \(p\). We recall

\[ A'_{2,2n}(X_p) = B_{2,2n}(p^{\frac{1}{2}+n}X_p, p^{\frac{1}{2}+n}X_p, ..., p^{\frac{1}{2}+n}X_p), \]

where the matrices \(A_{2,2n}\) and \(B_{2,2n}\) are defined in Section 2.6. Because of the construction of \(A'_{2,2n}(X_p)\), the matrix \(A'_{2,2n}(\alpha_p)\) determines a Satake parameter \{\mu_2, ..., \mu_{2n-1}\} up to the action of the Weyl group \(W_{2n-1}\). Hence we can take

\[ \{\mu_2, ..., \mu_{2n-1}\} = \{p^{\frac{1}{2}+n}\alpha_p, p^{\frac{1}{2}+n}\alpha_p, ..., p^{\frac{1}{2}+n}\alpha_p\}. \]

Now, from Section 2.6 and Section 2.3, we recall

\[ (\varphi(T_{0,2n-1}(p^2)), \varphi(T_{1,2n-2}(p^2)), ..., \varphi(T_{2n-1,0}(p^2))) \]

\[ = \left( \prod_{i=2}^{2n-1} X_i \right) (\varphi(T_{0,1}(p^2)), \varphi(T_{1,0}(p^2))) B_{2,2n}(X_2, ..., X_{2n-1}) \]

and \(\mu_0^2 \mu_1 \cdots \mu_{2n-1} = p^{(2n-1)k}\), where \(\varphi\) is the Satake isomorphism denoted in Section 2.6 and where \(T_{1,l}(p^2) (l = 0, ..., 2n)\) is the Hecke operator denoted in 2.4. Furthermore, from a straightforward calculation we have

\[ \varphi(T_{0,1}(p^2)) = p^{-1}X_0X_1, \]

\[ \varphi(T_{1,0}(p^2)) = p^{-1}X_0X_1(pX_1^{-1} + (p - 1) + pX_1). \]

From Theorem 1.3 and the above relations, we have

\[ p^{2nk+n-1}(p^{-k-n}, p^{-2k-2n+2}\lambda_g(p^2)) A'_{2,2n}(\alpha_p) \]

\[ = \left( \prod_{i=2}^{2n-1} \mu_i \right) (p^{-1}\mu_0^2 \mu_1, p^{-1}\mu_0^2 \mu_1(p\mu_1^{-1} + (p - 1) + p\mu_1)) B_{2,2n}(\mu_2, ..., \mu_{2n-1}). \]

Hence, from the fact that the rank of the matrix \(A'_{2,2n}(\alpha_p)\) is two, we obtain

\[ p\mu_1^{-1} + (p - 1) + p\mu_1 = p^{-k-n+2}\lambda_g(p^2). \]

On the other hand, we have \(\lambda_g(p^2) = p^{k+n-2}(p\beta_p^2 + (p - 1) + p\beta_p^2)\). Thus we can take \(\mu_1 = \beta_p^2\). Hence we obtain

\[ \{\mu_1, \mu_2, \mu_3, ..., \mu_{2n-1}\} = \{\beta_p^2, p^{\frac{1}{2}+n}\alpha_p, p^{\frac{1}{2}+n}\alpha_p, ..., p^{\frac{1}{2}+n}\alpha_p\}. \]
up to the action of the Weyl group $W_{2n-1}$.

The standard $L$-function of $\mathcal{F}_{f,g}$ is

$$L(s, \mathcal{F}_{f,g}, \text{st}) = \prod_p \left( 1 - p^{-s} \right)^{2n-1} \prod_{i=1}^{2n-2} \left( 1 - \mu_i p^{-s} \right) \left( 1 - \mu_i^{-1} p^{-s} \right)^{-1}$$

$$= \prod_p \left( 1 - p^{-s} \right)^{2n-2} \prod_{i=1}^{2n-2} \left( 1 - \alpha_i p^{i+\frac{1}{2} - n-s} \right) \left( 1 - \alpha_i^{-1} p^{-1} p^{i+\frac{1}{2} - n-s} \right)^{-1}$$

$$= L(s, g, \text{Ad}) \prod_p \prod_{i=1}^{2n-2} \left( 1 - \alpha_i p^{i+\frac{1}{2} - n-s} \right) \left( 1 - \alpha_i^{-1} p^{i+\frac{1}{2} - n-s} \right)^{-1}.$$

Because $L(s, f) = \prod_p \left( 1 - \alpha_p p^{k-\frac{s}{2}} \right) \left( 1 - \alpha_p^{-1} p^{k-\frac{s}{2}} \right)^{-1}$, we obtain Corollary 1.4.

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