QUASIPOTENTIAL AND EXIT TIME FOR 2D STOCHASTIC NAVIER-STOKES EQUATIONS DRIVEN BY SPACE TIME WHITE NOISE

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Abstract. We are dealing with the Navier-Stokes equation in a bounded regular domain $D$ of $\mathbb{R}^2$, perturbed by an additive Gaussian noise $\partial u^\delta / \partial t$, which is white in time and colored in space. We assume that the correlation radius of the noise gets smaller and smaller as $\delta \searrow 0$, so that the noise converges to the white noise in space and time. For every $\delta > 0$ we introduce the large deviation action functional $S_{0,T}^\delta$ and the corresponding quasi-potential $U_\delta$ and, by using arguments from relaxation and $\Gamma$-convergence we show that $U_\delta$ converges to $U = U_0$, in spite of the fact that the Navier-Stokes equation has no meaning in the space of square integrable functions, when perturbed by space-time white noise. Moreover, in the case of periodic boundary conditions the limiting function $U$ is explicitly computed.

Finally, we apply these results to estimate of the asymptotics of the expected exit time of the solution of the stochastic Navier-Stokes equation from a basin of attraction of an asymptotically stable point for the unperturbed system.

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Date: January 27, 2014.
1. Introduction

Let $\mathcal{O}$ be a regular bounded open domain of $\mathbb{R}^2$. We consider here the 2-dimensional Navier-Stokes equation in $\mathcal{O}$, perturbed by a small Gaussian noise

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) - (u(t, x) \cdot \nabla u(t, x)) u(t, x) - \nabla p(t, x) + \sqrt{\varepsilon} \eta(t, x),$$

with the incompressibility condition

$$\text{div } u(t, x) = 0$$

and initial and boundary conditions

$$u(t, x) = 0, \quad x \in \partial \mathcal{O}, \quad u(0, x) = u_0(x).$$

Here $0 < \varepsilon << 1$ and $\eta(t, x)$ is a Gaussian random field, white in time and colored in space.

In what follows, for any $\alpha \in \mathbb{R}$ we shall denote by $V_\alpha$ the closure in the space $[H^\alpha(\mathcal{O})]^2$ of the set of infinitely differentiable 2-dimensional vector fields, having zero divergence and compact support on $\mathcal{O}$, and we shall set $H = V_0$ and $V = V_1$. We will also set

$$D(A) = [H^2(\mathcal{O})]^2 \cap V, \quad Ax = -\Delta x, \quad x \in D(A).$$

The operator $A$ is positive and self-adjoint, with compact resolvent, and $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\{e_k\}_{k \in \mathbb{N}}$ will denote the eigenvalues and the eigenfunctions of $A$. Moreover, we will define the bilinear operator $B : V \times V \to V_{-1}$ by setting

$$\langle B(u, v), z \rangle = \int_{\mathcal{O}} z(x) \cdot [(u(x) \cdot \nabla) v(x)] \, dx.$$

With these notations, if we apply to each term of the Navier-Stokes equation above the projection operator into the space of divergence free fields, we formally arrive to the abstract equation

$$du(t) + Au(t) + B(u(t), u(t)) = \sqrt{\varepsilon} \, dw^Q(t), \quad u(0) = u_0,$$

where the noise $w^Q(t)$ is assumed to be of the following form

$$w^Q(t) = \sum_{k=1}^{\infty} Q e_k \beta_k(t), \quad t \geq 0,$$

for some sequence of independent standard Brownian motions $\{\beta_k(t)\}_{k \in \mathbb{N}}$ defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and a linear operator $Q$ defined on $H$ (for all details see Section [2]).

As well known, white noise in space and time (that is $Q = I$) cannot be taken into consideration in order to study equation (1.1) in the space $H$. But if we assume that $Q$ is a compact operator satisfying suitable conditions, as for example $Q \sim A^{-\alpha}$, for some $\alpha > 0$, we have that for any $u_0 \in H$ and $T > 0$ equation (1.1) is well defined in $C([0, T]; H)$ and the validity of a large deviation principle and the problem of the exit of the solution of equation (1.1) from a domain can be studied.

As in our previous work [9], where a class of reaction-diffusion equations in any space dimension perturbed by multiplicative noise has been considered, in the present paper we want to see how we can describe the small noise asymptotics of equation (1.1), as if the noisy perturbation were given by a white noise in space and time. This means that, in spite of
the fact that equation (1.1) is not meaningful in $H$ when the noise is white in space, the
relevant quantities for the large deviations and the exit problems associated with it can be
approximated by the analogous quantities that one would get in the case of white noise in
space. In particular, when periodic boundary conditions are imposed, such quantities can be
explicitly computed and such approximation becomes particularly useful.
Thus, in what follows we shall consider a family of positive linear operators $\{Q_\delta\}_{\delta \in (0,1]}$ defined
on $H$, such that for any fixed $\delta \in (0,1]$, equation (1.1), with noise
$$w_\delta(t) = \sum_{k=1}^{\infty} Q_\delta e_k \beta_k(t), \quad t \geq 0,$$
is well defined in $C([0,T]; H)$, and $Q_\delta$ is strongly convergent to the identity operator in $H$, for
$\delta \searrow 0$. For each fixed $\delta \in (0,1]$, the family $\{L(u^\varepsilon_\delta)\}_{\varepsilon \in (0,1]}$ satisfies a large deviation principle
in $C([0,T]; H)$ with the action functional (sometimes called the rate function)
$$S_{0,T}^\delta(u) = \frac{1}{2} \int_0^T |Q_\delta^{-1}(u'(t) + Au(t) + B(u(t), u(t)))|^2_H \, dt,$$and the corresponding quasi-potential is defined by
$$U_\delta(x) = \inf \{S_{0,T}^\delta(u) : u \in C([0,T]; H), \ u(0) = 0, \ u(T) = x, \ T > 0\}.$$Our purpose here is to show that, in spite of the fact we cannot prove any limit for the solution $u^\varepsilon_\delta$ of equation (1.1), nevertheless for all $x \in H$ such that $U_\delta(x) < \infty$
$$\lim_{\delta \to 0} U_\delta(x) = U(x), \quad (1.3)$$where $U(x)$ is defined as $U_\delta(x)$, with the action functional $S_{0,T}^\delta$ replaced by
$$S_{0,T}(u) = \frac{1}{2} \int_0^T |u'(t) + Au(t) + B(u(t), u(t))|^2_H \, dt.$$To this purpose, the key idea consists in characterizing the quasi-potentials $U_\delta$ and $U$ as
$$U_\delta(x) = \min \{S_{-\infty,0}^\delta(u) : u \in \mathcal{X} \text{ and } u(0) = 0\}, \quad (1.4)$$and
$$U(x) = \min \{S_{-\infty,0}(u) : u \in \mathcal{X} \text{ and } u(0) = 0\}, \quad (1.5)$$where
$$\mathcal{X} = \{u \in C((-\infty,0]; H) : \lim_{t \to -\infty} |u(t)|_H = 0\}$$and the functionals $S_{-\infty,0}^\delta$ and $S_{-\infty,0}$ are defined on $\mathcal{X}$ in a natural way, see formulae (5.6) and (6.3) later on. In this way, in the definition of $U_\delta$ and $U$, the infimum with respect to time $T > 0$ has disappeared and we have only to take the infimum of suitable functionals in the space $\mathcal{X}_\varepsilon := \{u \in \mathcal{X} : u(0) = x\}$. In particular, the convergence of $U_\delta(x)$ to $U(x)$ becomes the convergence of the infima of $S_{-\infty}^\delta$ in $\mathcal{X}_\varepsilon$ to the infimum of $S_{-\infty}$ in $\mathcal{X}_\varepsilon$, so that (1.3) follows once we prove that $S_{-\infty}^\delta$ is Gamma-convergent to $S_{-\infty}$ in $\mathcal{X}_\varepsilon$, as $\delta \searrow 0$. Moreover, as a consequence of (1.5), in the case of the stochastic Navier-Stokes equations with periodic boundary conditions we can prove, see section 7, that
$$U(x) = |x|^2_V, \quad (1.6)$$
so that $U(x)$ can be explicitly computed and the use of (1.3) in applications becomes particularly relevant. Let us point out the a similar explicit formula for the quasipotential has been derived for linear SPDEs by Da Prato, Pritchard and Zabczyk in [13] and in the recent work by the second and third authors for stochastic reaction diffusion equations in [9].

The proofs of characterizations (1.4) and (1.5) and of the Gamma-convergence of $S_{\delta}^{\infty}$ to $S_{-\infty}$ are based on a thorough analysis of the Navier-Stokes equation with an external deterministic force in the domain of suitable fractional powers of the operator $A$.

The fundamental motivation for proving (1.3) is provided by the study of the expected exit time $\tau_{x,\delta}$ of the solution $u_{x,\delta}$ from a domain $D$, which is attracted to the zero function. Actually, in the second part of the paper we prove that if there exists $y_\delta \in \partial D$ such that

$$U_\delta(y_\delta) = \min_{y \in \partial D} U_\delta(y),$$

(and this is true for example if $D$ is an open ball in $H$) then, for any fixed $\delta > 0$

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}(\tau_{x,\delta}^x) = \inf_{y \in \partial D} U_\delta(y). \quad (1.7)$$

This means that, as in finite dimension, the expectation of $\tau_{x,\delta}^x$ can be described in terms of the quantity $U_\delta(x)$. Moreover, once we have (1.3), by a general argument introduced in [9] and based on Gamma-convergence, we can prove that if $D$ is a domain in $H$ such that any point $x \in V \cap \partial D$ can be approximated in $V$ by a sequence $\{x_n\}_{n \in N} \subset D(A^{1/2+\alpha}) \cap \partial D$ (think for example of $D$ as a ball in $H$), then

$$\lim_{\delta \to 0} \inf_{x \in \partial D} U_\delta(x) = \inf_{x \in \partial D} U(x).$$

According to (1.7), this implies that for $0 < \varepsilon << \delta << 1$

$$\mathbb{E} \tau_{\varepsilon,\delta}^x \sim \exp \left( \frac{1}{\varepsilon} \inf_{x \in \partial D} U(x) \right).$$

In particular, if $D$ is the ball of $H$ of radius $c$, in view of (1.6) for any $x \in D$ we get,

$$e^{-\frac{c^2x^2}{\varepsilon}} \mathbb{E} \tau_{\varepsilon,\delta}^x \sim 1, \quad 0 < \varepsilon << \delta << 1.$$
2. Notation and preliminaries

Let \( \mathcal{O} \subset \mathbb{R}^2 \) be an open and bounded set. We denote by \( \Gamma = \partial \mathcal{O} \) the boundary of \( \mathcal{O} \). We will always assume that the closure \( \overline{\mathcal{O}} \) of the set \( \mathcal{O} \) is a manifold with boundary of \( C^\infty \) class, whose boundary \( \partial \mathcal{O} \) is denoted by \( \Gamma \). Namely, we will assume that \( \mathcal{O} \) satisfies condition (7.10) from [25, chapter I], that is \( \Gamma \) is a 1-dimensional infinitely differentiable manifold, \( \mathcal{O} \) being locally on one side of \( \mathcal{O} \). Let us also denote by \( \nu \) the unit outer normal vector field to \( \Gamma \).

It is known that \( \mathcal{O} \) is a Poincaré domain, i.e. there exists a constant \( \lambda_1 > 0 \) such that the following Poincaré inequality is satisfied

\[
\lambda_1 \int_{\mathcal{O}} \varphi^2(x) \, dx \leq \int_{\mathcal{O}} |\nabla \varphi(x)|^2 \, dx, \quad \varphi \in H_0^1(\mathcal{O}). \tag{2.1}
\]

In order to formulate our problem in an abstract framework, let us recall the definition of the following functional spaces. First of all, let \( \mathcal{D}(\mathcal{O}) \) (resp. \( \mathcal{D}(\overline{\mathcal{O}}) \)) be the set of all \( C^\infty \) class vector fields \( u : \mathbb{R}^2 \to \mathbb{R}^2 \) with compact support contained in the set \( \mathcal{O} \) (resp. \( \overline{\mathcal{O}} \)). \( L^2(\mathcal{O}) = L^2(\mathcal{O}, \mathbb{R}^2) \), \( E(\mathcal{O}) = \{ u \in L^2(\mathcal{O}) : \text{div} \, u \in L^2(\mathcal{O}) \} \), \( H^k(\mathcal{O}) = H^{k,2}(\mathcal{O}, \mathbb{R}^2) \), \( k \in \mathbb{N} \), \( U = \{ u \in \mathcal{D}(\mathcal{O}) : \text{div} \, u = 0 \} \), \( H = \text{the closure of } \mathcal{V} \text{ in } L^2(\mathcal{O}) \), \( H^1_0(\mathcal{O}) = \text{the closure of } \mathcal{D}(\mathcal{O}) \text{ in } H^1(\mathcal{O}) \), \( V = \text{the closure of } \mathcal{V} \text{ in } H^1_0(\mathcal{O}) \).

The inner products in all the \( L^2 \) spaces will be denoted by \( (\cdot, \cdot) \). The space \( E(\mathcal{O}) \) is a Hilbert space with a scalar product \( (u, v)_{E(\mathcal{O})} := (u, v)_{L^2(\mathcal{O})} + (\text{div} \, u, \text{div} \, v)_{L^2(\mathcal{O})}. \tag{2.2} \)

We endow the set \( H \) with the inner product \( (\cdot, \cdot)_H \) and the norm \( |\cdot|_H \) induced by \( L^2(\mathcal{O}) \). Thus, we have

\[
(u, v)_H = \sum_{j=1}^2 \int_{\mathcal{O}} u_j(x)v_j(x) \, dx,
\]

The space \( H \) can also be characterised in the following way. Let \( H^{-\frac{1}{2}}(\Gamma) \) be the dual space of \( H^{1/2}(\Gamma) \), the image in \( L^2(\Gamma) \) of the trace operator \( \gamma_0 : H^{1,2}(\mathcal{O}) \to L^2(\Gamma) \) and let \( \gamma_\nu \) be the bounded linear map from \( E(\mathcal{O}) \) to \( H^{-\frac{1}{2}}(\Gamma) \) such that, see [36, Theorem I.1.2],

\[
\gamma_\nu(u) = \text{the restriction of } u \cdot \nu \text{ to } \Gamma, \quad \text{if } u \in \mathcal{D}(\overline{\mathcal{O}}). \tag{2.3}
\]

Then, see [36, Theorem I.1.4],

\[
H = \{ u \in E(\mathcal{O}) : \text{div} \, u = 0 \text{ and } \gamma_\nu(u) = 0 \},
\]

\[
H^1 = \{ u \in E(\mathcal{O}) : u = \nabla p, \ p \in H^{1,2}(\mathcal{O}) \}.
\]
Let us denote by $P : L^2(\mathcal{O}) \to H$ the orthogonal projection called usually the Leray-Helmholtz projection. It is known, see for instance [36, Remark I.1.6] that

$$Pu = u - \nabla (p + q), \quad u \in L^2(\mathcal{O}),$$

where, for $u \in L^2(\mathcal{O})$, $p$ is the unique solution of the following homogenous boundary Dirichlet problem for the Laplace equation

$$\Delta p = \text{div} u \in H^{-1,2}(\mathcal{O}), \quad p \in H^{1,2}_0(\mathcal{O})$$

and $q \in H^{1,2}(\mathcal{O})$ is the unique solution of the inhomogenous following Neumann boundary problem for the Laplace equation

$$\Delta q = 0, \quad \frac{\partial q}{\partial \nu} \bigg|_{\Gamma} = \gamma_{\nu}(u - \nabla p).$$

Note that the function $p$ above satisfies $\nabla p \in L^2(\mathcal{O})$ and $\text{div} (u - \nabla p) = 0$ and therefore $u - \nabla p \in E(\mathcal{O})$ so that $q$ is well defined.

It is proved in [36, Remark I.1.6] that $P$ maps continuously the Sobolev space $H^1(\mathcal{O})$ into itself. Below, we will discuss continuity of $P$ with respect to other topologies.

Since the set $\mathcal{O}$ is a Poincaré domain, the norms on the space $V$ induced by norms from the Sobolev spaces $H^1(\mathcal{O})$ and $H^1_0(\mathcal{O})$ are equivalent. The latter norm and the associated inner product will be denoted by $|\cdot|_V$ and $(\cdot, \cdot)_V$, respectively. They satisfy the following equality

$$(u, v)_V = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \, dx, \quad u, v \in H^1_0(\mathcal{O}).$$

Since the space $V$ is densely and continuously embedded into $H$, by identifying $H$ with its dual $H'$, we have the following embeddings

$$V \subset H \cong H' \subset V'.$$

Let us observe here that, in particular, the spaces $V$, $H$ and $V'$ form a Gelfand triple.

We will denote by $|\cdot|_V'$ and $(\cdot, \cdot)$ the norm in $V'$ and the duality pairing between $V$ and $V'$, respectively.

The presentation of the Stokes operator is standard and we follow here the one given in [7].

We first define the bilinear form $a : V \times V \to \mathbb{R}$ by setting

$$a(u, v) := (\nabla u, \nabla v)_H, \quad u, v \in V.$$

As obviously the bilinear form $a$ coincides with the scalar product in $V$, it is $V$-continuous, i.e. there exists some $C > 0$ such that

$$|a(u, u)| \leq C|u|_V^2, \quad u \in V.$$

Hence, by the Riesz Lemma, there exists a unique linear operator $\mathcal{A} : V \to V'$, such that $a(u, v) = \langle \mathcal{A}u, v \rangle$, for $u, v \in V$. Moreover, since $\mathcal{O}$ is a Poincaré domain, the form $a$ is $V$-coercive, i.e. it satisfies $a(u, u) \geq \alpha|u|_V^2$ for some $\alpha > 0$ and all $u \in V$. Therefore, in view of the Lax-Milgram theorem, see for instance Temam [36, Theorem II.2.1], the operator $\mathcal{A} : V \to V'$ is an isomorphism.
Next we define an unbounded linear operator $A$ in $H$ as follows

$$
\begin{align*}
D(A) &= \{ u \in V : Au \in H \} \\
Au &= Au, \ u \in D(A).
\end{align*}
$$

(2.9)

It is now well established that under suitable assumptions related to the regularity of the domain $\mathcal{O}$, the space $D(A)$ can be characterized in terms of the Sobolev spaces. For example, see [23], where only the 2-dimensional case is studied but the result is also valid in the 3-dimensional case, if $\mathcal{O} \subset \mathbb{R}^2$ is a uniform $C^2$-class Poincaré domain, then we have

$$
\begin{align*}
D(A) &= V \cap H^2(\mathcal{O}) = H \cap H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}), \\
Au &= -P\Delta u, \ u \in D(A).
\end{align*}
$$

(2.10)

It is also a classical result, see e.g. Cattabriga [11] or Temam [35, p. 56], that $A$ is a positive self adjoint operator in $H$ and

$$
(Au, u) \geq \lambda_1 |u|^2_H, \ u \in D(A).
$$

(2.11)

where the constant $\lambda_1 > 0$ is from the Poincaré inequality [24]. Moreover, it is well known, see for instance [35, p. 57] that $V = D(A^{1/2})$. Moreover, from [38, Theorem 1.15.3, p. 103] it follows that

$$
D(A^{\alpha/2}) = [H, D(A)]^\frac{\alpha}{2},
$$

where $[\cdot, \cdot]^\frac{\alpha}{2}$ is the complex interpolation functor of order $\frac{\alpha}{2}$, see e.g. [25], [38] and [32, Theorem 4.2]. Furthermore, as shown in [38, Section 4.4.3], for $\alpha \in (0, \frac{1}{2})$

$$
D(A^{\alpha/2}) = H \cap H^\alpha(\mathcal{O}).
$$

(2.12)

The above equality is responsible for the following result.

**Proposition 2.1.** Assume that $\alpha \in (0, \frac{1}{2})$. Then the Leray-Helmholtz projection $P$ is a well defined and continuous map from $H^\alpha(\mathcal{O})$ into $D(A^{\alpha/2})$.

**Proof.** Let us fix $\alpha \in (0, \frac{1}{2})$. Since, by its definition the range of $P$ is contained in $H$, it is sufficient to prove that for every $u \in H \cap H^\alpha(\mathcal{O})$, $Pu \in H^\alpha(\mathcal{O})$. For this aim, let us fix $u \in H \cap H^\alpha(\mathcal{O})$. Then $\text{div} \ u \in H^{\alpha-1,2}(\mathcal{O})$. Therefore, by the elliptic regularity we infer that the solution $p$ of the problem (2.15) belongs to the Sobolev space $H^{\alpha+1,2}(\mathcal{O}) \cap H_0^{1,2}(\mathcal{O})$ and therefore $\nabla p \in H^\alpha(\mathcal{O})$.

Since by [36, Theorem 1.1.2], $\gamma_\nu$ is a bounded linear map from $E(\mathcal{O})$ to $H^{-\frac{1}{2}}(\Gamma)$, and by, $\gamma_\nu$ is a bounded linear map from $E(\mathcal{O}) \cap H^1(\mathcal{O})$ to $H^{\frac{1}{2},2}(\Gamma)$, by the standard interpolation argument we infer that $\gamma_\nu$ is a bounded linear map from $E(\mathcal{O}) \cap H^\alpha(\mathcal{O})$ to $H^{-\frac{1}{2}+\alpha,2}(\Gamma)$. Thus we infer that $\gamma_\nu(u - \nabla p) \in H^{-\frac{1}{2}+\alpha,2}(\Gamma)$ and therefore, again by the elliptic regularity, we have that the solution $q$ of the problem (2.16) belongs to $H^{\alpha+1}(\mathcal{O})$ and therefore $\nabla q \in H^\alpha(\mathcal{O})$. This proves that $Pu \in H^\alpha(\mathcal{O})$ as required.

The proof is complete. 

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1These assumptions are satisfied in our case.
Let us finally recall that by a result of Fujiwara–Morimoto [20] that the projection $P$ extends to a bounded linear projection in the space $L^q(D)$, for any $q \in (1, \infty)$.

Now, consider the trilinear form $b$ on $V \times V \times V$ given by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad u, v, w \in V.$$  

Indeed, $b$ is a continuous trilinear form such that

$$b(u, v, w) = -b(u, w, v), \quad u, v, w \in H^1_0(\mathcal{O}),$$  \hspace{1cm} (2.13)

and

$$|b(u, v, w)| \leq C \begin{cases} 
|u|^{1/2}_H |\nabla u|^{1/2}_H |\nabla v|^{1/2}_H |\nabla w|^{1/2}_H |w|_H & u, v, w \in V, \quad u, v \in D(A), \quad w \in H \\
|u|^{1/2}_H |\nabla u|^{1/2}_H |\nabla v|_H |w|_H & u \in D(A), \quad v \in V, \quad w \in H \\
|u|_H |\nabla v|_H |w|_H & u \in H, \quad v \in V, \quad w \in D(A) \\
|u|^{1/2}_H |\nabla u|^{1/2}_H |\nabla v|_H |w|_H |\nabla w|_H & u, v, w \in V 
\end{cases}$$  \hspace{1cm} (2.14)

for some constant $C > 0$ (for a proof see for instance [36, Lemma 1.3, p.163] and [35]).

Define next the bilinear map $B : V \times V \to V'$ by setting

$$\langle B(u, v), w \rangle = b(u, v, w), \quad u, v, w \in V,$$

and the homogenous polynomial of second degree $B : V \to V'$ by

$$B(u) = B(u, u), \quad u \in V.$$  

Let us observe that if $v \in D(A)$, then $B(u, v) \in H$ and the following inequality follows directly from the first inequality in (2.14)

$$|B(u, v)|^2_H \leq C |u|_H |\nabla u|_H |\nabla v|_H |\nabla w|_H, \quad u \in V, \quad v \in D(A).$$  \hspace{1cm} (2.15)

Moreover, the following identity is a direct consequence of (2.13).

$$\langle B(u, v), v \rangle = 0, \quad u, v \in V.$$  \hspace{1cm} (2.16)

Let us also recall the following fact (see [4, Lemma 4.2]).

**Lemma 2.2.** The trilinear map $b : V \times V \times V \to \mathbb{R}$ has a unique extension to a bounded trilinear map from $L^4(\mathcal{O}) \times (L^4(\mathcal{O}) \cap H) \times V$ and from $L^4(\mathcal{O}) \times V \times L^4(\mathcal{O})$ into $\mathbb{R}$. Moreover, $B$ maps $L^4(\mathcal{O}) \cap H \ (\text{and so } V)$ into $V'$ and

$$|B(u)|_{V'} \leq C_1 |u|_L^4(\mathcal{O})^2 \leq 2^{1/2}C_1 |u|_H |\nabla u|_H \leq C_2 |u|_V^2, \quad u \in V.$$  \hspace{1cm} (2.17)

**Proof.** It it enough to observe that due to the Hölder inequality, the following inequality holds

$$|b(u, v, w)| \leq C |u|_L^4(\mathcal{O}) |\nabla v|_L^2(\mathcal{O}) |w|_L^4(\mathcal{O}), \quad u, v, w \in H^1_0(\mathcal{O}).$$  \hspace{1cm} (2.18)

Thus, our thesis follows from (2.13).  

Let us also recall the following well known result, see [36] for a proof.
Lemma 2.3. For any $T \in (0, \infty]$ and for any $u \in L^2(0, T; D(A))$ with $u' \in L^2(0, T; H)$, we have
\[
\int_0^T |B(u(t), u(t))|_H^2 dt < \infty.
\]

Proof. Our assumption implies that $u \in C([0, T]; V)$ (for a proof see for instance [40, Proposition I.3.1]). Then, we can conclude thanks to (2.15).

The restriction of the map $B$ to the space $D(A) \times D(A)$ has also the following representation
\[
B(u, v) = P(u \nabla v), \quad u, v \in D(A),
\]
where $P$ is the Leray-Helmholtz projection operator and $u \nabla v = \sum_{j=1}^2 u^j D_j v \in L^2(O)$. This representation together with Proposition 2.1 allows us to prove the following property of the map $B$.

Proposition 2.4. Assume that $\alpha \in (0, \frac{1}{2})$. Then for any $s \in (1, 2]$ there exists a constant $C > 0$ such that
\[
|B(u, v)|_{D(A^{\alpha/2})} \leq C |u|_{D(A^{s})} |v|_{D(A^{1+s\alpha/2})}, \quad u, v \in D(A).
\]

Proof. In view of equality (2.19), since the Leray-Helmholtz projection $P$ is a well defined and continuous map from $H^\alpha(O)$ into $D(A^{\alpha/2})$ and since the norms in the spaces $D(A^{\frac{1}{2}})$ are equivalent to norms in $H^s(O)$, it is enough to show that
\[
|u \nabla v|_{H^\alpha} \leq C |u|_{H^s} |v|_{H^{1+s\alpha}} , \quad u, v \in H^2(O).
\]
The last inequality is a consequence of the Marcinkiewicz Interpolation Theorem, the complex interpolation and the following two inequalities for scalar functions which can be proved by using Gagliado-Nirenberg inequalities
\[
|uv|_{L^2} \leq C |u|_{H^{s,2}} |v|_{L^2}, \quad u \in H^{s,2}, v \in L^2,
\]
\[
|uv|_{H^{1,2}} \leq C |u|_{H^{s,2}} |v|_{H^{1,2}}, \quad u \in H^{s,2}, v \in H^{1,2}.
\]

3. The skeleton equation

We are here dealing with the following functional version of the Navier-Stokes equation
\[
\begin{cases}
  u'(t) + \nu Au(t) + B(u(t), u(t)) = f(t), & t \in (0, T)
  \\
  u(0) = u_0,
\end{cases}
\]
where $T \in (0, \infty]$ and $\nu > 0$. Let us recall the following definition (see [36, Problem 2, section III.3])

\[2\text{Note that in the case } T = \infty \text{ one also has lim}_{t \to \infty} u(t) = 0 \text{ in } V.\]
Definition 3.1. Given \( f \in L^2(0, T; V') \) and \( u_0 \in H \), a solution to problem (3.1) is a function \( u \in L^2(0, T; V) \) such that \( u' \in L^2(0, T; V') \), \( u(0) = u_0 \) and (3.1) is fulfilled.

It is known (see e.g. [36 Theorem III.3.1/2]) that for every \( f \in L^2(0, T; V') \) and \( u_0 \in H \) there exists exactly one solution \( u \) to problem (3.1). Moreover, see [36 Theorem III.3.10], if \( f \in L^2(0, T; H) \) then this solution \( u \) has the following properties

\[
\sqrt{\cdot} u(\cdot) \in L^2(0, T; D(A)) \cap L^\infty(0, T; V), \quad \sqrt{\cdot} u'(\cdot) \in L^2(0, T; H).
\]

Moreover, there exists \( C > 0 \) such that for all \( T > 0 \)

\[
|\sqrt{\cdot} u|^2_{L^\infty(0, T; V)} + |\sqrt{\cdot} u|^2_{L^2(0, T; D(A))} \leq \exp \left[ C(|u_0|^2_H + |f|^2_{L^2(0, T; V')}) \right] \left( |u_0|^2_V + |f|^2_{L^2(0, T; H)} \right). \tag{3.2}
\]

Finally, if \( f \in L^2(0, T; H) \) and \( u_0 \in V \), then

\[
|u|^2_{C([0, T], V)} + |u|^2_{L^2(0, T; D(A))} \leq \exp \left[ C\left(|u_0|^2_H + |f|^2_{L^2(0, T; V')}\right) \right] \left( |u_0|^2_V + |f|^2_{L^2(0, T; H)} \right). \tag{3.3}
\]

and

\[
|u|^2_{L^2(0, T; D(A))} \leq \left( 1 + C^2 \left[ |u_0|^2_H + |f|^2_{L^2(0, T; V')} \right] \right) \exp \left[ C^2 \left[ |u_0|^2_H + |f|^2_{L^2(0, T; V')} \right] \right] \left( |u_0|^2_V + |f|^2_{L^2(0, T; H)} \right). \tag{3.4}
\]

Now we will formulate and prove some generalisations of the above results when the data \( u_0 \) and \( f \) are slightly more regular. Similar results in the case of integer order of the Sobolev spaces has been studied in [33] where some compatibility conditions are imposed.

Proposition 3.2. Assume that \( \alpha \in (0, \frac{1}{2}) \). If \( f \in L^2(0, T; D(A^{\frac{\alpha+1}{2}})) \) and \( u_0 \in D(A^{\frac{\alpha+1}{2}}) \), then the unique solution \( u \) to the problem (3.1) satisfies

\[
u \in L^2(0, T; D(A^{1+\frac{\alpha}{2}})) \cap C([0, T]; D(A^{\frac{\alpha+1}{2}})), \quad u'(\cdot) \in L^2(0, T; D(A^{\frac{\alpha}{2}})). \tag{3.5}
\]

Proof. Let us fix \( T > 0 \). Since by Proposition 2.1 \( B \) is a bilinear continuous map from \( D(A^{\frac{\alpha+1}{2}}) \times D(A^{\frac{\alpha+1}{2}}) \) to \( D(A^{\frac{\alpha}{2}}) \) it follows (see for instance [4] for the simplest argument) that for every \( R, \rho > 0 \) there exists \( T_0 = T_0(R, \rho) \in (0, T) \) such that for every \( u_0 \in D(A^{\frac{\alpha+1}{2}}) \) and \( f \in L^2(0, T; D(A^{\frac{\alpha}{2}})) \) such that

\[
|u_0|_{D(A^{\frac{\alpha+1}{2}})} \leq R, \quad |f|_{L^2(0, T; D(A^{\frac{\alpha}{2}}))} \leq \rho
\]

It is known, see for instance [36 Lemma III.1.2] that these two properties of \( u \) imply that there exists a unique \( \bar{u} \in C([0, T], H) \). When we write \( u(0) \) later we mean \( \bar{u}(0) \).
there exists a unique solution \( v \) to problem \((3.1)\) which satisfy conditions \((3.3)\) on the time interval \([0, T_\star]\). Since \( D(A^{1/2}) \subset V \) and \( L^2(0, T; D(A^{1/2})) \subset L^2(0, T; \mathcal{H}) \) with the embeddings being continuous, \( u_0 \in V \) and \( f \in L^2(0, T; \mathcal{H}) \). Therefore, there exists a unique solution \( u \) to problem \((3.1)\) on the whole real half-line \([0, \infty)\) which satisfies \((3.3)\) and \((3.4)\).

By the uniqueness, \( u = v \) on \([0, T_\star]\). Hence it is sufficient to show that the norm of \( u \) in \( L^2(0, T_\star; D(A^{1/2})) \cap C([0, T_\star]; D(A^{1/2})) \) and the norm of \( u' \) in \( L^2(0, T_\star; D(A^{1/2})) \) are bounded by a constant depending only on \( |A^{\frac{1}{2}}u_0|_H^2 \) and \( \int_0^{T_\star} |A^{\frac{\alpha}{2}}f(s)|_H^2 \, ds \).

For this aim, let us denote the right hand side of inequality \((3.4)\) by \( K(T) \). Calculating the derivative of \( |A^{\frac{1}{2}}u(t)|_H^2 \) and using inequality \((2.20)\), with \( s = 2 \), and the Gronwall Lemma we get

\[
|A^{\frac{1}{2}}u(t)|_H^2 + \int_0^t |A^{\frac{\alpha}{2}}u(s)|_H^2 \, ds \leq e^{K(T)}|A^{\frac{1}{2}}u_0|_H^2 + e^{K(T)} \int_0^t |A^{\frac{\omega}{2}}f(s)|_H^2 \, ds
\]

\[
\leq e^{K(T)}\left(|A^{\frac{1}{2}}u_0|_H^2 + \int_0^T |A^{\frac{\omega}{2}}f(s)|_H^2 \, ds\right), \quad t \in [0, T_\star].
\]

Finally, the bound for the norm of \( u' \) in \( L^2(0, T_\star; D(A^{1/2})) \) follows from the estimate we have for \( f \) and from the estimates we have for \( Au \) and \( B(u, u) \), as a consequence of the estimate we have just proved for \( u \) and of Proposition \((2.20)\). ★

Now, for any \(-\infty < a < b \leq \infty\) such that \( a < b \) and for any two reflexive Banach spaces \( X \) and \( Y \) such that \( X \hookrightarrow Y \) continuously, we denote by \( W^{1,2}(a, b; X, Y) \) the space of all \( u \in L^2(a, b; X) \) which are weakly differentiable as \( Y \)-valued functions and their weak derivative belongs to \( L^2(a, b; Y) \). The space \( W^{1,2}(a, b; X, Y) \) is a separable Banach space (and Hilbert if both \( X \) and \( Y \) are Hilbert spaces), with the natural norm

\[
|u|_{W^{1,2}(a,b;X,Y)} = |u|_{L^2(a,b;X)}^2 + |u'|_{L^2(a,b;Y)}^2, \quad u \in W^{1,2}(a, b; X, Y).
\]

Later on, we will use the shortcut notation

\[
W^{1,2}(a, b) = W^{1,2}(a, b; D(A), \mathcal{H}).
\]

We conclude this section with the statement of a couple of results which are obvious adaptations of deep results from \([26]\) to the 2-dimensional case. To this purpose, there is no need to mention that all what we have said about equation \((3.1)\) in the time interval \([0, T]\) applies to any time interval \([a, b]\), with \(-\infty < a < b < \infty\).

**Definition 3.3.** Assume that \(-\infty < a < b \leq \infty\) and \( f \in L^2_{\infty}((a, b); \mathcal{H}) \). A function \( u \in C^0((a, b); \mathcal{H}) \) is called a very weak solution to the Navier-Stokes equations \((5.1)\) on the interval

\footnote{Some authors, for instance Vishik and Fursikov, use the notation \( \mathcal{H}^{1,2}(a, b; X, Y) \). Our choice is motivated by the notation used in the monograph \([25]\), who however use notation \( W(a, b) \).}
Moreover, whenever this makes sense, we will denote $$S_{12} \ Z.$$ BRZEŹNIAK AND S. CERRAI AND M. FREIDLIN

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Proposition 3.4. Assume that $$-\infty \leq a < b \leq \infty$$ and $$f \in L^2_{\text{loc}}((a,b);H)$$. Suppose that the functions $$u,v \in C((a,b);H)$$ are very weak solutions to the Navier-Stokes equations (3.1) on the interval $$(a,b)$$, with $$u(t_0) = v(t_0)$$, for some $$t_0 \in (a,b)$$. Then $$u(t) = v(t)$$ for all $$t \geq t_0$$.

In the whole paper we will assume, without any loss of generality, that $$\nu = 1$$.

Definition 3.5. Assume that $$-\infty \leq a < b \leq \infty$$. Given a function $$u \in C((a,b);H)$$ we say that

$$u' + Au + B(u,u) \in L^2(a,b;H), \quad \text{(resp. } \in L^2_{\text{loc}}((a,b);H))$$

if there exists $$f \in L^2(a,b,H)$$, (resp. $$f \in L^2_{\text{loc}}((a,b);H)$$) such that $$u$$ is a very weak solution of the Navier-Stokes equations (3.1) on the interval $$(a,b)$$.

Obviously, the corresponding function $$f$$ is unique and we will denote it by $$H(u)$$, i.e.

$$[H(u)](t) := u'(t) + Au(t) + B(u(t),u(t)), \quad t \in (a,b). \quad (3.7)$$

If $$H(u) = u' + Au + B(u,u) \in L^2_{\text{loc}}((a,b);H)$$, then for any $$a < t_0 < t_1 < b$$ we define

$$S_{t_0,t_1}(u) := \frac{1}{2} \int_{t_0}^{t_1} |H(u)(t)|_H^2 \, dt. \quad (3.8)$$

Moreover, whenever this makes sense, we will denote $$S_{-T} := S_{-T,0}$$. In particular, when $$a = -\infty$$ and $$b \geq 0$$, we set

$$S_{-\infty}(u) := \frac{1}{2} \int_{-\infty}^{0} |H(u)(t)|_H^2 \, dt. \quad (3.9)$$

An obvious sufficient condition for the finiteness of $$S_{t_0,t_1}(u)$$ is that $$u', Au$$ and $$B(u,u)$$ all belong to $$L^2(t_0,t_1;H)$$. The next result shows that this is not so far from a necessary condition. This is the reason why we have presented the modified results from [26], see also [21].

Lemma 3.6. Suppose that $$T > 0$$ and $$u \in C([0,T];H)$$ is such that

$$u' + Au + B(u,u) \in L^2(0,T;H).$$

Then $$u(T) \in V$$ and $$u \in W^{1,2}(t_1,T)$$, for any $$t_1 \in (0,T)$$. Moreover, if $$u(0) \in V$$, then $$u \in W^{1,2}(0,T)$$.

Proof. Let us denote $$f = u' + Au + B(u,u)$$. Since $$L^2(0,T;H) \subset L^2(0,T;V')$$ and $$u \in C([0,T];H)$$, by the existence and the uniqueness results (see [36] Theorems III.3.1-2 and Proposition 3.4, respectively), we infer that $$u \in L^2(0,T;V)$$ and $$u' \in L^2(0,T;V')$$. Hence for every $$t_1 \in (0,T)$$ we can find $$t_0 \in (0,t_1)$$ such that $$u(t_0) \in V$$ and therefore by [36] Theorem III.3.10, $$u \in W^{1,2}(t_0,T)$$. In particular, $$u \in C([t_0,T];V)$$ and hence $$u(T) \in V$$. 

If the additional assumption that \( u(0) \in V \) is satisfied, then by what we have just seen ([36 Theorem III.3.10]) or by the maximal regularity and the uniqueness of solutions to 2D NSEs), we can conclude that \( u \in W^{1,2}(0, T) \).

\[ \text{Lemma 3.7.} \quad \text{Assume that } \alpha \in [0, 1/2] \text{ and suppose that } u \in C([0, T]; H), \text{ for some } T > 0, \text{ is such that } \]

\[ u' + Au + B(u, u) \in L^2(0, T; D(A^{\frac{\alpha+1}{2}})). \]

Then \( u(T) \in D(A^{\frac{\alpha+1}{2}}) \) and \( u \in W^{1,2}(t_0, T; D(A^{\frac{\alpha}{2}}), D(A^{\frac{\alpha}{2}})), \) for any \( t_0 \in (0, T) \). Moreover, if \( u(0) \in D(A^{\frac{\alpha+1}{2}}), \) then \( u \in W^{1,2}(0, T; D(A^{\frac{\alpha+1}{2}}), D(A^{\frac{\alpha}{2}})). \)

\[ \text{Proof.} \quad \text{Denote } f = u' + Au + B(u, u) \text{ and let us fix } t_0 \in (0, T) \text{ and some } t_1 \in (0, t_0). \text{ By Lemma 3.6 we infer that } u \in W^{1,2}(t_1, T). \text{ In particular, there exists } t_2 \in (t_1, t_0) \text{ such that } u(t_2) \in D(A) \subset D(A^{\frac{\alpha+1}{2}}). \text{ The last embedding holds since by assumptions } \alpha < \frac{1}{2}. \text{ Since by our assumption, } f \in L^2(0, T; D(A^{\frac{\alpha}{2}})), \text{ in view of Proposition 3.2 and Proposition 3.4 we infer that } u \in W^{1,2}(t_2, T; D(A^{\frac{\alpha+1}{2}}), D(A^{\frac{\alpha}{2}})). \text{ This implies that } u \in C([t_2, T]; D(A^{\frac{\alpha+1}{2}})) \text{ and in particular that } u(T) \in D(A^{\frac{\alpha+1}{2}}) \text{ and } u \in W^{1,2}(t_0, T; D(A^{\frac{\alpha}{2}}), D(A^{\frac{\alpha}{2}})) \text{ as in our first claim. The second claim follows from our last argument. The last claim is a trivial consequence of the first one.} \]

In what follows, for any \( c > 0 \) and \( \gamma \geq 0 \) we shall denote

\[ B_{c, \gamma} := \{ x \in D(A^\gamma) : |x|_{D(A^\gamma)} \leq c \}. \]

Moreover, for any \( x \in H, c > 0 \) and \( \gamma \geq 0 \) we shall denote

\[ t_{c, \gamma}^x := \inf \{ t \geq 0 : u^x(t; 0) \in B_{c, \gamma} \}, \]

where, for any \( 0 \leq s \leq t \), we denote by \( u^x(t; s) \) the solution of the problem

\[ \begin{cases}  
    u'(t) + Au(t) + B(u(t), u(t)) = 0, & t > s,  \\
    u(s) = x.  
\end{cases} \tag{3.10} \]

\[ \text{Proposition 3.8.} \quad \text{For any } c_1, c_2 > 0 \text{ and } \gamma \in [0, 3/4], \text{ there exists } T = T(\gamma, c_1, c_2) > 0 \text{ such that } \]

\[ \sup_{x \in B_{c_1, 0}} t_{c_2, \gamma}^x < T, \]
and

\[ u^x(t; 0) \in B_{\infty, \gamma}, \quad t > T. \]

**Proof.** Let \( x \in H \) be fixed. In view of (3.2), we have that

\[ \sqrt{t}u^x(\cdot; 0) \in L^2(0, \infty; D(A)) \cap L^\infty(0, \infty; V), \]

and

\[ |\sqrt{t}u^x|_{L^\infty(0, \infty; V)}^2 + |\sqrt{t}u^x|_{L^2(0, \infty; D(A))}^2 \leq \exp \left( C |x|^4_H \right) |x|^2_H. \]

In particular, there exists \( t_0 = t_0(x) \leq 1 \), such that \( u^x(t_0) \in D(A) \) and

\[ |u^x(t_0)|^2_V \leq 2 \exp \left( C |x|^4_H \right) |x|^2_H. \]

As a consequence of (3.3), this implies

\[ |u^x|_{C([t_0, +\infty); V)}^2 + |u^x|_{L^2([t_0, +\infty; D(A))}^2 \leq \exp \left( C |u(t_0)|_H^4 \right) |u(t_0)|_V^2 \]

\[ \leq \exp \left( C |x|^4_H \right) 2 \exp \left( C |x|^4_H \right) |x|^2_H. \]

Now, if take \( \alpha \in (0, 1/2) \) and differentiate \( |A_{1/2+}^t u(t)|_H^2 \), according to (2.20), for any \( s \in (1, 2) \) we get

\[ \frac{1}{2} \frac{d}{dt} |A_{1/2+}^t u(t)|_H^2 = -|A_{2/2+}^t u(t)|_H^2 - \langle A_{2/2+}^t u(t), A^{\frac{1}{2}} g B(u(t), u(t)) \rangle_H \]

\[ \leq -|A_{2+}^t u(t)|_H^2 + |A_{2+}^t u(t)|_H^2 |A_2^t u(t)|_H^2 |A_{1/2+}^t u(t)|_H^2. \]

Therefore, by using interpolation we get

\[ \frac{1}{2} \frac{d}{dt} |A_{1/2+}^t u(t)|_H^2 = -|A_{2+}^t u(t)|_H^2 + |A_{2+}^t u(t)|_H^2 |A_{1+}^t u(t)|_V^2 |u(t)|_{1+\alpha}^{2+\alpha}. \]

As \( s \in (1, 2) \), we have that \( (s + 2\alpha)/(1 + \alpha) < 2 \), then we can use the Young inequality and we get

\[ \frac{d}{dt} |A_{1/2+}^t u(t)|_H^2 + |A_{2+}^t u(t)|_H^2 \leq \kappa_1 |u(t)|_V^p \leq \kappa_2 |A_{2+}^t u(t)|_H^2 |u(t)|_V^{p-2}, \]

where

\[ p = \frac{3 + \alpha - s}{1 + \alpha} + \frac{2(1 + \alpha)}{2 - s} > 2. \]

In what follows, we shall need the following two results, whose proof is postponed to the end of this section.

**Lemma 3.9.** For any \( x \in H \) and any \( \kappa > 0 \), there exists \( t_1 = t_1(x, \kappa) \geq 0 \) such that

\[ |u^x(t)|_V \leq \kappa, \quad t \geq t_1. \]

Moreover, for any \( R > 0 \) and \( \kappa > 0 \)

\[ \sup_{|x|_H \leq R} t_1(x, \kappa) =: t_R(\kappa) < \infty. \]
Lemma 3.10. For any \( x \in H \), \( \alpha \in (0, 1/2) \) and \( t \geq 0 \), there exists \( t_2 = t_2(x, \alpha, t) \in [t, t+1] \) such that \( u^x(t_2) \in D(A^{1+\alpha}) \) and for any \( R > 0 \)
\[
\sup_{|x| \leq R} |u^x(t_2)|_{D(A^{1+\alpha})} =: \kappa_R < \infty.
\] (3.14)

In view of Lemma 3.9 there exists \( t_1 = t_1(x, (2\kappa_2)^{1/\alpha}) > t_0(x) \) such that
\[
|u^x(t)|_V \leq (2\kappa_2)^{1/\alpha}, \quad t \geq t_1.
\]

Then, from (3.12) we get
\[
\frac{d}{dt} A^{1+\alpha} u(t) |_{H} + \kappa_3 |A^{1+\alpha} u(t)|^2_H \leq 0, \quad t \geq t_1.
\] (3.15)

According to Lemma 3.10 there exists some \( t_2 = t_2(x, \alpha, t_1) \in [t_1(x), t_1(x) + 1] \) such that \( u^x(t_2) \in D(A^{1+\alpha}) \) and
\[
\sup_{x \in B_{c_1,0}} |u^x(t_2)|_{D(A^{1+\alpha})} = \kappa_{c_1},
\]
so that
\[
A^{1+\alpha} u(t) |_{H} \leq e^{-\kappa_3(t-t_2)|u^x(t_2)|} |A^{1+\alpha} u(t_2)|_{D(A^{1+\alpha})} \leq e^{-\kappa_3(t-t_2|x|_{A^{1+\alpha}})\kappa_{c_1}}, \quad t \geq t_2.
\]

This means that there exists \( t_3 = t_3(x, t_2) \geq t_2 \), such that
\[
|u^x(t)|_{D(A^{1+\alpha})} \leq c_2, \quad t \geq t_3.
\]

Now, if we fix \( \gamma \in (1/2, 3/4) \) we can find \( \alpha \in (0, 1/2) \) such that \( \gamma = (1 + \alpha)/2 \) and then, in view of what we have seen, we have
\[
t_{2,\gamma}^x \leq t_3(x, t_2(x, 2\gamma - 1, t_1)).
\]

Moreover, we can immediately check that
\[
\sup_{x \in B_{c_2,0}} t_3(x, t_2(x, 2\gamma - 1, t_1)) =: T(\gamma, c_1, c_2) < \infty,
\]
and
\[
u^x(t) \in B_{c_2,\gamma}, \quad t \geq T(\gamma, c_1, c_2).
\]

Finally, the case \( \gamma \in [0, 1/2] \) follows as \( D(A^{1\gamma}) \hookrightarrow D(A^{1\gamma_2}) \), if \( \gamma_1 \geq \gamma_2 \).

Proof of Lemma 3.9. In [36] Theorem III.3.12 it is already proved that if \( x \in V \), then
\[
\lim_{t_0 \to \infty} |u^x(t)|_V = 0.
\]

Here, we want to show that the limit above is uniform with respect to \( x \) in a ball of \( H \). For any \( x \in H \) and \( M > 0 \), we denote
\[
\tau_M^x := \inf \left\{ t \geq t_0(x) : |u(t)|_V^2 \leq M \right\}.
\]
As
\[ |u^x(t)|_H^2 + \int_{t_0(x)}^t |u^x(s)|_V^2 \, ds \leq |x|_H^2, \quad t > t_0(x), \]
this implies
\[ x \in B_{R,0} \implies \tau^x_M \leq t_0(x) + \frac{R}{M} \leq 1 + \frac{R}{M}. \]
As shown in [36, Remark III.3.9], it holds
\[ \frac{d}{dt} |u^x(t)|_V^2 + \kappa_1 |u^x(t)|_V^2 \leq \kappa_2 |u^x(t)|_V^2 |u^x(t)|_H |x|_H^2. \]
Hence, if we pick \( M_R = \kappa_1/(2R\kappa_2) \), for any \( x \in B_{R,0} \) we get
\[ \frac{d}{dt} |u^x(\tau^x_{M_R})|^2_V + \frac{\kappa_1}{2} |u^x(\tau^x_{M_R})|^2_V \leq 0. \]
This means that \( |u(t)|_V \) will decay after \( \tau^x_{M_R} \), so that
\[ \frac{d}{dt} |u^x(t)|_V^2 + \frac{\kappa_1}{2} |u^x(t)|_V^2 \leq 0, \quad t \geq \tau^x_{M_R}. \]
In particular, as a consequence of (3.11), for any \( x \in B_{R,0} \) we get
\[ |u^x(t)|_V^2 \leq |u^x(\tau^x_{M_R})|^2_V e^{(t-\tau^x_{M_R})\frac{\kappa_1}{2}} \leq C_R e^{(t-\tau^x_{M_R})\frac{\kappa_1}{2}}, \quad t \geq \tau^x_{M_R}. \]
Therefore, if we pick \( t_1 = t_1(R, \kappa) \) such that
\[ e^{(t_1-\tau^x_{M_R})\frac{\kappa_1}{2}} \leq \frac{\kappa}{C_R}, \]
our lemma follows.

**Proof of Lemma 3.10.** If we multiply both sides of (3.15) by \( (t-t_1) \), we have
\[ \frac{d}{dt} \left( (t-t_1)|A^{1+n/2}u(t)|_H^2 \right) + (t-t_1)|A^{2+n/2}u(t)|_H^2 \leq |A^{1+n/2}u(t)|_H^2. \]
Hence, if we integrate, we get
\[ (t-t_1)|A^{1+n/2}u(t)|_H^2 + \int_{t_1}^t (s-t_1)|A^{2+n/2}u(s)|_H^2 \, ds \leq \int_{t_1}^t |A^{1+n/2}u(s)|_H^2 \, ds \leq c |u|_{L^2(t_1, \infty; D(\Lambda))}. \]
According to (3.11), this implies (3.14).
4. SOME BASIC FACTS ON RELAXATION AND $\Gamma$-CONVERGENCE

Let us assume that $X$ is a topological space satisfying the first axiom of countability, i.e. every point in $X$ has a countable local base. For any $x \in X$ we shall denote by $\mathcal{N}(x)$ the set of all open neighborhoods of $x$ in $X$.

**Definition 4.1.** Let $F : X \to \mathbb{R}$ be a function.

1. The function $F$ is called lower semi-continuous if for any $t \in \mathbb{R}$, the set $\{ x \in X : F(x) \leq t \}$ is closed in $X$.
2. The function $F$ is called coercive if for any $t \in \mathbb{R}$, the closure of the set $\{ x \in X : F(x) \leq t \}$ is countably compact, i.e. every countable open cover has a finite subcover.

Now, let $\{ F_n \}_{n \in \mathbb{N}}$ be a sequence of functions all defined on $X$ with values in $\mathbb{R}$.

**Definition 4.2.** The sequence of functions $\{ F_n \}_{n \in \mathbb{N}}$ is called equi-coercive if for any $t \in \mathbb{R}$ there exists a closed countably compact set $K_t \subset X$ such that

$$\bigcup_{n \in \mathbb{N}} \{ x \in X : F_n(x) \leq t \} \subset K_t.$$ 

Let us note that if $Y$ is a closed subspace of $X$, then the restrictions to $Y$ of functions lower semi-continuous, coercive and equi-coercive, remain such on $Y$.

As proved in [11, Proposition 7.7], the following characterization of equi-coercive sequences holds.

**Proposition 4.3.** The sequence $\{ F_n \}_{n \in \mathbb{N}}$ is equi-coercive if and only if there exists a lower semi-continuous coercive function $\Psi : X \to \mathbb{R}$ such that

$$F_n(x) \geq \Psi(x), \quad x \in X, \quad n \in \mathbb{N}.$$ 

Now, we introduce the notion of relaxation of a function $F$.

**Definition 4.4.** The lower semi-continuous envelope, or the relaxed function, of a function $F : X \to \overline{\mathbb{R}}$ is defined by

$$(sc^{-}F)(x) = \sup \{ G(x) : G \in \mathcal{G}(F) \}, \quad x \in X,$$

where $\mathcal{G}(F)$ is the set of all lower semi-continuous functions $G : X \to \overline{\mathbb{R}}$ such that $G \leq F$.

From the definition, one has immediately that $sc^{-}F$ is lower semi-continuous, $sc^{-}F \leq F$ and $sc^{-}F \geq G$, for any $G \in \mathcal{G}(F)$, so that $sc^{-}F$ can be regarded as the greatest lower semi-continuous function majorized by $F$. Moreover, it is possible to prove that

$$(sc^{-}F)(x) = \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} F(y), \quad x \in X,$$

(see [11, Proposition 3.3]).

The following result, whose proof can be found in [11, Proposition 3.6], provides a possible characterization of $sc^{-}F$ which we will use later on in the paper.

**Proposition 4.5.** For any function $F : X \to \overline{\mathbb{R}}$, its lower semi-continuous function $sc^{-}F$ is characterized by the following properties:

1. for any $x \in X$ and any sequence $\{ x_n \}_{n \in \mathbb{N}}$ convergent to $x$ in $X$, it holds

$$(sc^{-}F)(x) \leq \liminf_{n \to \infty} F(x_n);$$
(2) for any \( x \in X \) there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \) convergent to \( x \) in \( X \) such that
\[
(sc^{-} F)(x) \geq \limsup_{n \to \infty} F(x_n).
\]

Next, we introduce the notion of \( \Gamma \)-convergence for sequences of functions.

**Definition 4.6.** The \( \Gamma \)-lower limit and the \( \Gamma \)-upper limit of the sequence \( \{F_n\}_{n \in \mathbb{N}} \) are the functions from \( X \) into \( \mathbb{R} \) defined respectively by
\[
\Gamma - \liminf_{n \to \infty} F_n(x) = \sup_{U \in \mathcal{N}(x)} \liminf_{n \to \infty} \inf_{y \in U} F_n(y),
\]
\[
\Gamma - \limsup_{n \to \infty} F_n(x) = \sup_{U \in \mathcal{N}(x)} \limsup_{n \to \infty} \inf_{y \in U} F_n(y).
\]

If there exists a function \( F : X \to \mathbb{R} \) such that
\[
\Gamma - \liminf_{n \to \infty} F_n = \Gamma - \limsup_{n \to \infty} F_n = F,
\]
then we write
\[
F = \Gamma - \lim_{n \to \infty} F_n,
\]
and we say that the sequence \( \{F_n\}_{n \in \mathbb{N}} \) is \( \Gamma \)-convergent to \( F \).

In [11, Proposition 5.7] we can find the proof of the following result, which links \( \Gamma \)-convergence and relaxation of functions and provides a useful criterion for \( \Gamma \)-convergence.

**Proposition 4.7.** If \( \{F_n\}_{n \in \mathbb{N}} \) is a decreasing sequence converging to \( F \) pointwise, then \( \{F_n\}_{n \in \mathbb{N}} \) is \( \Gamma \)-convergent to \( sc^{-} F \).

We conclude by giving a criterion for convergence of minima for \( \Gamma \)-convergent sequences (for a proof see [11, Theorem 7.8]).

**Theorem 4.8.** Suppose that the sequence \( \{F_n\}_{n \in \mathbb{N}} \) is equi-coercive in \( X \) and \( \Gamma \)-converges to a function \( F \) in \( X \). Then, \( F \) is coercive and
\[
\min_{x \in X} F(x) = \lim_{n \to \infty} \inf_{x \in X} F_n(x).
\]

5. **The Large Deviation Action Functional**

For any fixed \( \varepsilon, \delta \in (0, 1] \) and \( u_0 \in \mathbb{H} \), we consider the problem
\[
du(t) + Au(t) + B(u(t), u(t)) = \sqrt{\varepsilon} dw_\delta(t), \quad u(0) = u_0,
\]
where
\[
w_\delta(t) = \sum_{k=1}^{\infty} Q_\delta e_k \beta_k(t), \quad t \geq 0,
\]
\( \{e_k\}_{k \in \mathbb{N}} \) is the basis which diagonalizes the operator \( A \), \( \{\beta_k\}_{k \in \mathbb{N}} \) is a sequence of independent Brownian motions all defined on the stochastic basis \( (\Omega, \mathcal{F}, F_t, \mathbb{P}) \) and \( Q_\delta \) is a positive bounded linear operator on \( \mathbb{H} \) for any \( \delta \in (0, 1] \).

In what follows, we shall assume that the family \( \{Q_\delta\}_{\delta \in (0, 1]} \) satisfies the following conditions.
Theorem 5.1. For any $\delta \in (0, 1]$, $Q_\delta$ is a positive linear operator on $H$, the operator $A^{-1}Q_\delta^2$ is trace class, and there exists some $\beta > 0$ such that $Q_\delta : H \to D(A^{\frac{\beta}{2}})$ is an isomorphism. Moreover,

$$\lim_{\delta \to 0} Q_\delta y = y, \quad y \in H, \quad \lim_{\delta \to 0} Q_\delta^{-1} y = y, \quad y \in D(A^{\frac{\beta}{2}}),$$

the limits above being in $H$, and for any $1 \geq \sigma \geq \delta \geq 0$

$$|Q_\sigma^{-1} y|_H \geq |Q_\delta^{-1} y|_H, \quad y \in D(A^{\frac{\beta}{2}}). \quad (5.2)$$

Remark 2. The RKHS of the Wener process $w_\delta$ is equal to $Q_\delta(H)$ and hence by Assumption 5.1 it is contained in the space $D(A^{\beta})$ for some $\beta > 0$ and hence the results from [7] are applicable.

Remark 3. It is easy to see that for the Navier-Stokes equations in a $d$-dimensional domain, $d \geq 2$, any number $\beta > \frac{d}{2} - 1$ and the operators

$$Q_\delta := (I + \delta A^{\beta/2})^{-1}$$

satisfy Assumption 5.1.

Now, for any $-\infty \leq t_0 < t_1 \leq \infty$, $\delta \in [0, 1]$ and $u \in C([t_0, t_1]; H)$, we define

$$S_{t_0, t_1}^\delta (u) := \frac{1}{2} \int_{t_0}^{t_1} |Q_\delta^{-1} (\mathcal{H}(u)(t))|_H^2 dt. \quad (5.3)$$

When $\delta = 0$, the superscript 0 will be omitted. So we put $S_{t_0, t_1} = S_{t_0, t_1}^0$. Note that a necessary condition for $S_{t_0, t_1}^\delta (u)$ to be finite is that $\mathcal{H}(u)(t)$ belongs to $D(A^{\beta/2})$, for a.a. $t \in (t_0, t_1)$.

For any $T > 0$, $\varepsilon, \delta \in (0, 1]$ and $u_0 \in H$, equation (5.1) admits a unique solution $u_{\varepsilon, \delta} \in C([0, T]; H)$ (for a proof see e.g. [17]). As shown in the next theorem, an immediate consequence of the contraction principle is that the family $\{\mathcal{L}(u_{\varepsilon, \delta})\}_{\varepsilon \in (0, 1]}$ satisfies a large deviation principle in $C([0, T]; H)$.

Theorem 5.2. For any $x \in H$ and $\delta \in (0, 1]$, the family $\{\mathcal{L}(u_{\varepsilon, \delta})\}_{\varepsilon \in (0, 1]}$ satisfies a large deviation principle in $C([0, T]; H)$, with a good action functional $S_{0,T}^\delta$.

Proof. For every $\varepsilon > 0$, we denote by $z_{\varepsilon, \delta}(t)$ the Ornstein-Uhlenbeck process associated with $A$ and $Q_\delta$, that is the solution of the linear problem

$$dz(t) + Az(t) = \sqrt{\varepsilon} Q_\delta dw(t), \quad z(0) = 0. \quad (5.4)$$

We have

$$z_{\varepsilon, \delta}(t) = \sqrt{\varepsilon} \int_0^t e^{-(t-s)A} Q_\delta dw(s), \quad t \geq 0.$$

As well known (see e.g. [41] Theorem 3), under our assumptions the family $\{\mathcal{L}(z_{\varepsilon, \delta})\}_{\varepsilon \in (0, 1]}$ satisfies a large deviation principle in $C([0, T]; L^4(D))$, with action functional

$$I_{0,T}^\delta (u) = \frac{1}{2} \int_0^T |Q_\delta^{-1}(u(t) + Au(t))|_H^2 dt.$$
Moreover, if we define the mapping $F_x : C([0, T]; L^4(D)) \to C([0, T]; H)$ which associate to every $\varphi \in C([0, T]; L^4(D))$ the solution $v \in C([0, T]; H)$ of the problem
\[ v'(t) + Av(t) + B(v(t) + \varphi(t), v(t) + \varphi(t)) = 0, \quad v(0) = x, \]
we have, $\mathbb{P}$-a.s.,
\[ u_{x,\varepsilon} = F_x(z_{\varepsilon,\delta}). \]
In [7, Theorem 4.6], it is proved that the mapping $F_x$ is continuous, so that by the contraction principle, the large deviation principle for $\{z_{\varepsilon,\delta}\}_{\varepsilon \in (0,1]}$ on $C([0, T]; L^4(D))$ with action functional $I_{0,T}$ may be transferred to a large deviation principle for $\{u_{x,\varepsilon}\}_{\varepsilon \in (0,1]}$ on $C([0, T]; H)$ with action functional $S_{0,T}^\delta$.

Next, for any $T > 0$ we set
\[ S_T^\delta := S_{0,T}^\delta, \quad S_{-T}^\delta := S_{-T,0}^\delta \]
and
\[ S_T := S_{0,T}, \quad S_{-T} := S_{-T,0}. \]
In particular,
\[ S_{-\infty}^\delta(u) := \frac{1}{2} \int_{-\infty}^{0} |Q_{\varepsilon}^{-1}(\mathcal{H}(u)(t))|_H^2 dt \]  
\[ \text{(5.5)} \]
and
\[ S_{-\infty}(u) := \frac{1}{2} \int_{-\infty}^{0} |\mathcal{H}(u)(t)|_H^2 dt. \]  
\[ \text{(5.6)} \]
We conclude the present section with the description of some relevant properties of $S_{-\infty}$ and $S_{-\infty}^\delta$.

Before proceeding we need to introduce the following two functional spaces.
\[ \mathcal{X} = \{ u \in C((-\infty, 0]; H) : \lim_{t \to -\infty} |u(t)|_H = 0 \}, \]  
\[ \text{(5.7)} \]
\[ \mathcal{X}_x = \{ u \in \mathcal{X} : u(0) = x \}, \]  
\[ \text{(5.8)} \]
and
\[ W^{1,2}(t_0, t_1) = W^{1,2}(t_0, t_1; D(A), H) = \{ u \in L^2(t_0, t_1; D(A)) : u' \in L^2(t_0, t_1; H) \} \]  
\[ \text{(5.9)} \]
We endow the space $\mathcal{X}_x$ with the topology of uniform convergence on compact intervals, i.e. the topology induced by the metric $\rho$ defined by
\[ \rho(u, v) := \sum_{n=1}^{\infty} 2^{-n} \left( \sup_{s \in [-n,0]} |u(s) - v(s)|_H \wedge 1 \right), \quad u, v \in \mathcal{X}. \]

The set $\mathcal{X}_x$ is a closed in $\mathcal{X}$ and we endow it with the trace topology induced by $\mathcal{X}$.

Let us note here, see for instance [40, Proposition I.3.1], that for $u \in W^{1,2}(t_0, \infty)$, then
\[ \lim_{t \to \infty} |u(t)|_V = 0. \]
Similarly, if \( u \in W^{1,2}(-\infty, t_1) \), then
\[
\lim_{t \to -\infty} |u(t)|_V = 0
\]
see Proposition A.3.

**Proposition 5.3.** The functionals \( S_{-\infty} \) and \( S_{-\infty}^\delta \), \( \delta \in (0, 1] \), are lower-semicontinuous in \( \mathcal{X} \).

**Proof.** In order to prove the lower semi-continuity of \( S_{-\infty} \) and \( S_{-\infty}^\delta \), it is sufficient to show that if a sequence \( \mathcal{X} \)-valued \( \{u_n\}_{n=1}^\infty \) is convergent in \( \mathcal{X} \) to a function \( u \in \mathcal{X} \), then
\[
\liminf_{n \to \infty} S_{-\infty}(u_n) \geq S_{-\infty}(u) \tag{5.10}
\]
and for any \( \delta \in (0, 1] \)
\[
\liminf_{n \to \infty} S_{-\infty}^\delta(u_n) \geq S_{-\infty}^\delta(u). \tag{5.11}
\]
We prove (5.11), as (5.10) is a particular case, with \( \delta = 0 \).

First we assume that \( u \in \mathcal{X} \) is such that \( S_{-\infty}(u) = \infty \). We want to show that
\[
\liminf_{n \to \infty} S_{-\infty}(u_n) = +\infty.
\]
Suppose by contradiction that \( \liminf_n S_{-\infty}^\delta(u_n) < \infty \). Then, after extracting a subsequence, we can find \( C > 0 \) such that
\[
|u_n' + Au_n + B(u_n, u_n)|_{L^2(-\infty, 0; D(A^{\frac{\delta}{2}}))} \leq C, \ n \in \mathbb{N}.
\]
By Proposition A.2 (Proposition A.1 if \( \delta = 0 \)), we have that the sequence \( \{u_n\} \) is bounded in \( W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})) \) and hence we can find \( \tilde{u} \in W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})) \) such that, after another extraction of a subsequence,
\[
u_n \to \tilde{u}, \ \text{as}, \ n \to \infty \text{ weakly in } W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})).
\]
By the uniqueness of the limit we infer that \( u = \tilde{u} \), so that \( u \in W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})) \) and \( S_{-\infty}(u) < \infty \), which contradicts our assumption.

Thus, assume that \( S_{-\infty}^\delta(u) < \infty \). In view of the last part of Lemma 3.7 (Lemma 3.6 if \( \delta = 0 \)) we have that \( u(0) \in D(A^{\frac{1+\delta}{2}}) \) and \( \{u_n\} \subset W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})) \).

Now, assume that (5.11) is not true. Then there exists \( \varepsilon > 0 \) such that, after the extraction of a subsequence,
\[
S_{-\infty}^\delta(u_n) < S_{-\infty}(u) - \varepsilon, \ n \in \mathbb{N}.
\]
Hence, if we set \( f_n = u_n' + Au_n + B(u_n, u_n) \), we have that the sequence \( \{f_n\} \) is bounded in \( L^2(-\infty, 0; D(A^{\frac{\delta}{2}})) \) and then, by Proposition A.2 (Proposition A.1 if \( \delta = 0 \)), we have that \( \{u_n\} \) is bounded in \( W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})) \). This implies that we can find \( \hat{u} \in W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})) \) such that, after a further extraction of a subsequence,
\[
u_n = \hat{u}, \ \text{as} \ n \to \infty, \text{ weakly in } W^{1,2}(-\infty, 0; D(A^{1+\frac{\delta}{2}}), D(A^{\frac{\delta}{2}})),
\]
and, by uniqueness of the limit, we infer that \( u = \hat{u} \). Moreover, as the sequence \( \{f_n\} \) is bounded in \( L^2(-\infty, 0; D(A^{\frac{\delta}{2}})) \), after another extraction of a subsequence, we can find \( \hat{f} \in \).
\( L^2(-\infty, 0; D(A^{\frac{\alpha}{2}})) \) such that \( f_n \) converges weakly to \( \tilde{f} \) in \( L^2(-\infty, 0; D(A^{\frac{\alpha}{2}})) \). By employing nowadays standard compactness argument, see for instance [7, section 5] we can show that 
\[
\tilde{f} = f = u' + Au + B(u, u).
\]
Thus, since the mapping 
\[
u \in L^2(-\infty, 0; D(A^{\frac{\alpha}{2}})) \mapsto \int_{-\infty}^0 |Q^{-1}_\delta u(t)|^2_H \, dt \in \mathbb{R}
\]
is convex and lower semi-continuous, it is also weakly lower semi-continuous, so that
\[
\liminf_n S_{-\infty}^\delta(u_n) = \frac{1}{2} \liminf_{n \to \infty} |Q^{-1}_\delta f_n|^2_{L^2(-\infty, 0; D(A^{\frac{\alpha}{2}}))} \geq \frac{1}{2} |Q^{-1}_\delta f|^2_{L^2(-\infty, 0; D(A^{\frac{\alpha}{2}}))} = S_{-\infty}^\delta(u).
\]

**Proposition 5.4.** The operators \( S_{-\infty} \) and \( S_{-\infty}^\delta \) have compact level sets in \( \mathcal{X} \). Moreover, the family \( \{S_{-\infty}^\delta\}_{\delta \in (0, 1]} \) is equi-coercive.

**Proof.** First, notice that we have only to prove the compactness of the level sets of \( S_{-\infty} \).

Actually, due to Assumption 5.1,
\[
S_{-\infty}^\delta \geq S_{-\infty}, \quad \delta \in (0, 1], \quad (5.12)
\]
and then, as \( S_{-\infty} \) is lower-semicontinuous, the compactness of the level sets of \( S_{-\infty} \) implies the compactness of the level sets of \( S_{-\infty}^\delta \). Moreover, in view of Proposition 4.3 (5.12) and the compactness of the level sets of \( S_{-\infty} \) imply the equi-coercivity of the family \( \{S_{-\infty}^\delta\}_{\delta \in (0, 1]} \).

Hence, let us prove that every sequence \( \{u_n\} \) in \( \mathcal{X} \) such that \( S_{-\infty}(u_n) \leq r \), for any \( n \in \mathbb{N} \), has a subsequence convergent in \( \mathcal{X} \) to some \( u \in \mathcal{X} \) such that \( S_{-\infty}(u) \leq r \).

According to the last part of Proposition A.1 there exists \( M > 0 \) such that
\[
|u_n|_{W^{1, 2}(-\infty, 0)} \leq M, \quad n \in \mathbb{N}.
\]
Hence by the Banach-Alaoglu Theorem, we can find \( u \in W^{1, 2}(-\infty, 0) \) and a subsequence that is weakly convergent to \( u \) in \( W^{1, 2}(-\infty, 0) \). Note that \( u \) being an element of \( W^{1, 2}(-\infty, 0) \) it must satisfy
\[
\lim_{t \to -\infty} |u(t)|_H = 0.
\]
Since the embedding \( D(A) \hookrightarrow H \) is compact we infer that for each \( T > 0 \) one can extract a subsequence strongly convergent in \( C([-T, 0], H) \). By the uniqueness of the limit we infer that the later limit is equal to the restriction of \( u \) to the interval \([-T, 0]\). In particular \( u(0) = x \) and therefore \( u \in \mathcal{X} \). Moreover, by employing the Helly’s diagonal procedure, we can find a subsequence of \( \{u_n\} \) which is convergent in \( \mathcal{X} \) to \( u \) and, as \( S_{-\infty} \) is lower semicontinuous, we have that \( S_{-\infty}(u) \leq r \). This completes the proof of the compactness of the level sets of \( S_{-\infty} \).
6. THE QUASI-POTENTIAL

We define, for $x \in H$, the following family of $[0, \infty]$-valued functions:

$$U(x) := \inf \{ S_{-T}(u) : T > 0, u \in C([-T, 0]; H), \text{ with } u(-T) = 0, u(0) = x \}, \quad (6.1)$$

and for any $\delta \in (0, 1]$

$$U_\delta(x) := \inf \{ S_{-T}^\delta(u) : T > 0, u \in C([-T, 0]; H), \text{ with } u(-T) = 0, u(0) = x \}. \quad (6.2)$$

Note that with our notation $U = U_0$.

As a consequence of Lemma 3.6 and Lemma 3.7, we have the following fact.

**Proposition 6.1.** We have

$$U(x) < \infty \iff x \in V. \quad (6.3)$$

Moreover, if Assumption 5.1 is satisfied for some $\beta \in (0, \frac{1}{2})$, then

$$U_\delta(x) < \infty, \text{ for some } \delta \in [0, 1] \iff U_\delta(x) < \infty \text{ for every } \delta \in [0, 1] \iff x \in D(A^{\frac{1+\beta}{2}}). \quad (6.4)$$

**Proof.** We prove (6.4), as (6.3) turns out to be a special case, corresponding to the case $\beta = 0$. Assume that $U_\delta(x) < \infty$. Then, according to (6.2) we can find $T > 0$ and $u \in C([-T, 0]; H)$ such that $u(-T) = 0$, $u(0) = x$ and

$$u' + Au + B(u, u) \in L^2(-T, 0; D(A^{\frac{\beta}{2}})).$$

Hence, by Lemma 3.7, we infer that $x \in D(A^{\frac{1+\beta}{2}})$.

Conversely, let us assume that $x \in D(A^{\frac{1+\beta}{2}})$ and $T > 0$. Since the map

$$W(-T, 0; D(A^{1+\frac{\beta}{2}}), D(A^{\frac{\beta}{2}})) \ni v \mapsto v(0) \in D(A^{\frac{1+\beta}{2}})$$

is surjective, see [25, Theorem 3.2, p.21 and Remark 3.3, p.22] for a proof, we can find $u_1 \in W(-T, 0; D(A^{1+\frac{\beta}{2}}), D(A^{\frac{\beta}{2}}))$ such that $u_1(0) = x$. By Proposition 2.20, we infer that $u_1' + Au_1 + B(u_1, u_1) \in L^2(-T, 0; D(A^{\frac{\beta}{2}}))$. Moreover, there exists $t_0 \in (-T, 0)$ such that $u_1(t_0) \in D(A^{1+\frac{\beta}{2}})$. This means that if we define

$$u_2(t) := \frac{t + T}{t_0 + T} u_1(t_0), \quad t \in [-T, t_0],$$

we have that $u_2(-T) = 0$, $u_2 \in W(-T, t_0; D(A^{1+\frac{\beta}{2}}), D(A^{\frac{\beta}{2}}))$ and $u_2' + Au_2 + B(u_2, u_2) \in L^2(-T, t_0; D(A^{\frac{\beta}{2}}))$. Finally, if we define

$$u(t) := \begin{cases} u_2(t), & t \in [-T, t_0] \\ u_1(t), & t \in [t_0, 0], \end{cases}$$

we can conclude that $S_{-T}^\delta(u) < \infty$. 

Now we can prove the following crucial characterization of the functionals $U_\delta$ and $U$. 


Theorem 6.2. For any $x \in V$ we have
\[ U(x) := \min \{ S_{-\infty}(u) : u \in \mathcal{X}_x \} . \]  
(6.5)

Analogously, if Assumption 5.1 is satisfied for some $\beta \in (0, \frac{1}{2})$, then for any $\delta \in (0, 1]$ and $x \in D(A^{1+\beta})$ we have
\[ U_\delta(x) := \min \{ S_{-\infty}^\delta(u) : u \in \mathcal{X}_x \} . \]  
(6.6)

Proof. We prove (6.6), as (6.5) is a special case, corresponding to $\beta = 0$ in Assumption 5.1.

Let us fix $T > 0$ and $u \in C([-T, 0]; H)$ such that $u(-T) = 0$, $u(0) = x$ and $S_{-T}^\delta(u) < \infty$ and let us define
\[ \bar{u}(t) := \begin{cases} u(t), & \text{if } t \in [-T, 0], \\ 0, & \text{if } t \in (-\infty, -T]. \end{cases} \]  
(6.7)

Obviously, $\bar{u} \in \mathcal{X}_x$. We will prove that
\[ S_{-\infty}^\delta(\bar{u}) = S_{-T}^\delta(u) \]  
(6.8)

Since $S_{-T}^\delta(u) < \infty$, the function $u$ satisfies the assumptions of Lemma 6.3. Therefore, $x \in D(A^{1+\beta})$ and $u$ belongs to $W^{1,2}(-T, 0; D(A^{1+\beta}), D(A^{\frac{\beta}{2}}))$. Since obviously the zero function is an elements of the space $W^{1,2}(-\infty, -T; D(A^{1+\beta}), D(A^{\frac{\beta}{2}}))$, we infer, see for instance [1], that
\[ \bar{u} \in W^{1,2}(-\infty, 0; D(A^{1+\beta}), D(A^{\frac{\beta}{2}})), \] and (6.8) holds. In particular, this implies that
\[ \inf \{ S_{-\infty}^\delta(u) : u \in \mathcal{X}_x \} \leq S_{-T}^\delta(u). \]

Taking now the infimum over all $u$ as above, in view of the definition of $U_\delta(x)$ we infer that
\[ \inf \{ S_{-\infty}^\delta(u) : u \in \mathcal{X}_x \} \leq U_\delta(x). \]

It remains to prove the converse inequality. To this purpose, we will need the following two results, whose proofs are postponed to the end of this section.

Lemma 6.3. For every $\delta \in [0, 1]$, $b > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that for any $y \in D(A^{1+\beta})$ such that $|y|_{D(A^{1+\beta})} < \eta$, we can find
\[ v \in W(0, b; D(A^{1+\beta}), D(A^{\frac{\beta}{2}})) \] with
\[ S_{0,b}^\delta(v) < \varepsilon, \quad v(0) = 0, \quad v(b) = y. \]

Recall that in the case $\delta = 0$, we have $S_{0,b}^\delta = S_{0,b}$ and we take $\beta = 0$.

Lemma 6.4. Assume that $u \in \mathcal{X}$. Then for each $\delta \in [0, 1]$ and $\varepsilon > 0$ we can find $T_\varepsilon > 0$ and $v_\varepsilon \in C([-T_\varepsilon, 0]; H)$ such that $v_\varepsilon(-T_\varepsilon) = 0$, $v_\varepsilon(0) = u(0)$ and
\[ S_{-T_\varepsilon}^\delta(v_\varepsilon) \leq S_{-\infty}^\delta(u) + \varepsilon. \]
Thus, let us prove
\[ U_\delta(x) \leq \inf \{ S^\delta_{-\infty}(u) : u \in \mathcal{X}_x \}. \tag{6.9} \]
Obviously, we may assume that the right hand side above is finite and so we can find \( u \in \mathcal{X}_x \)
such that \( S^\delta_{-\infty}(u) < \infty \). In view of Lemma 6.4, for any \( \varepsilon > 0 \),
\[
\inf \{ S^\delta_{-T}(v) : T > 0, v \in C([-T, 0], H), v(-T) = 0, v(0) = x \} \leq S^\delta_{-\infty}(u) + \varepsilon.
\]
This implies that \( U_\delta(x) \leq S^\delta_{-\infty}(u) + \varepsilon \). Thus, by taking the infimum over \( \varepsilon > 0 \) and then
over all admissible \( u \) we get (6.9). Finally, we remark that the infima are in fact minima, as
the level sets of \( S_{-\infty} \) and \( S^\delta_{-\infty} \) are compact (see Proposition 5.3).
This completes the proof of (6.6), provided we can prove Lemmas 6.3 and 6.4.

**Proof of Lemma 6.3.** Let us fix \( b > 0 \) and consider the mapping
\[
\Phi_{0,b} : W^{1,2}(0, b; D(A^{1+\beta}), D(A^\beta)) \ni v \mapsto v' + Av + B(v, v) \in L^2(0, b; D(A^\beta)).
\]
Due to Lemma 2.4, the mapping \( \Phi_{0,b} \) is well defined and continuous. Moreover
\[
S^\delta_{0,b}(u) \leq c |\Phi_{0,b}v|_{L^2(0, b; D(A^\beta))}, \quad v \in W^{1,2}(0, b; D(A^{1+\beta}), D(A^\beta)).
\tag{6.10}
\]
Now, by proceeding as in [25], Remark 3.3, p. 22] we can show that there exists a continuous linear map
\[ R : D(A^{1+\beta}) \to W^{1,2}(0, b; D(A^{1+\beta}), D(A^\beta)), \]
such that \([Ry](b) = y \) for every \( y \in D(A^{1+\beta}) \). Thus the map
\[
\Phi_{0,b} \circ R : D(A^{1+\beta}) \to L^2(0, b; D(A^\beta))
\]
is continuous and then for every \( \varepsilon > 0 \) we can find \( \eta > 0 \) such that
\[
|y|_{D(A^{1+\beta})} < \eta \implies |\Phi_{0,b}(Ry)|_{L^2(0, b; D(A^\beta))} < \frac{\varepsilon}{c}.
\]
Since \( v = Ry \) satisfies \( v \in W^{1,2}(0, b; D(A^{1+\beta}), D(A^\beta)) \), \( v(0) = 0 \) and \( v(b) = y \), due to (6.10)
the proof is complete.

**Proof of Lemma 6.4.** We give the proof here for \( \delta > 0 \), as \( \delta = 0 \) is a special case. Let us
assume that \( u \in \mathcal{X}_x \) for some \( x \in H \), and fix \( \varepsilon > 0 \). It is sufficient to assume that \( S^\delta_{-\infty}(u) < \infty \).
Then by (3.8) and (3.9) we can find \( T_\varepsilon > 0 \) such that
\[
S^\delta_{-\infty,-T_\varepsilon}(u) < \frac{\varepsilon}{3}.
\]
Moreover, the function \( u \) satisfies the assumptions of Lemma 3.7. Therefore, \( x \in D(A^{1+\beta}) \)
and \( u \) belongs to \( W^{1,2}_{\text{loc}}(-\infty, 0; D(A^{1+\beta}), D(A^\beta)) \). As a consequence of Proposition A.2, this implies
\[
\lim_{t \to -\infty} |u(t)|_{D(A^{1+\beta})} = 0.
\tag{6.11}
Then, $T_\varepsilon$ can be chosen in such a way that
\[ |u(-T_\varepsilon)|_{D(A^{1+\beta/2})} < \eta, \]
where we choose $\eta > 0$ as in Lemma 6.3, corresponding to $b = 1$ and $\xi$. Then by Lemma 6.3, we can find $w \in W^{1,2}(-T_\varepsilon - 1, -T_\varepsilon; D(A^{1+\beta/2}), D(A^{\beta/2}))$ such that
\[ S_{-T_\varepsilon - 1, -T_\varepsilon}(w) < \frac{\varepsilon}{3}, \quad w(-T_\varepsilon - 1) = 0, \quad w(-T_\varepsilon) = u(-T_\varepsilon). \]

Next, we define
\[ \bar{u}(t) := \begin{cases} u(t), & \text{if } t \in [-T_\varepsilon, 0], \\ w(t), & \text{if } t \in [-T_\varepsilon - 1, -T_\varepsilon]. \end{cases} \]

(6.12)

Obviously, $\bar{u}(0) = x$ and $\bar{u} \in C([-T_\varepsilon - 1, 0]; H)$ and, arguing as before (and hence using for instance [4]), we infer that $\bar{u} \in W^{1,2}(-T_\varepsilon - 1, 0; D(A^{1+\beta/2}), D(A^{\beta/2}))$. Moreover,
\[ S_{-T}(\bar{u}) = S_{-T_\varepsilon - 1, -T_\varepsilon}(w) + S_{-T_\varepsilon}(u) < \frac{\varepsilon}{3} + \left[ S_{-\infty}(u) - S_{-\infty, -T_\varepsilon}(u) \right] < \frac{\varepsilon}{3} + S_{-\infty}(u). \]

This concludes the proof of Lemma 6.4.

Next, we prove that both $U$ and $U^\delta$ have compact level sets.

**Proposition 6.5.** For any $r > 0$ and $\delta \in (0, 1]$, the sets
\[ K_r = \{ x \in H : U(x) \leq r \}, \quad K_r^\delta = \{ x \in H : U^\delta(x) \leq r \} \]
are compact in $H$. In particular, both functions $U$ and $U^\delta$ are lower semi-continuous in $H$.

**Proof.** Let $\{x_n\}$ be a sequence in $K_r$. In view of identity 6.5 Theorem 6.2, for any $n \in \mathbb{N}$ there exists $u_n \in \mathcal{A}$ such that
\[ S_{-\infty}(u_n) = U(x_n) + \frac{1}{n} \leq r + 1. \]

In particular,
\[ \{u_n\} \subset \{S_{-\infty} \leq r + 1\}, \]
so that, thanks to the compactness of the level sets of $S_{-\infty}$ proved in Proposition 5.4, we have that there exists $\{u_{n_k}\} \subset \{u_n\}$ and $\bar{u} \in C((-\infty, 0]; H)$ such that
\[ \lim_{k \to \infty} u_{n_k} = \bar{u}, \quad \text{in } C((-\infty, 0]; H). \]

This implies that
\[ \lim_{k \to \infty} u_{n_k}(0) = \bar{u}(0). \]

Now, due to the lower semi-continuity of $S_{-\infty}$ proved in Proposition 5.3, this yields
\[ S_{-\infty}(\bar{u}) \leq \liminf_{k \to \infty} S_{-\infty}(u_{n_k}) \leq r. \]

On the other hand, by the definition of $U$, $U(\bar{u}(0)) \leq S_{-\infty}(\bar{u})$. Hence we can conclude that $\bar{u}(0) \in K_r$, and the compactness of $K_r$ follows.
The compactness of the level sets of $U_\delta$ can be proved analogously.

We conclude this section by studying the continuity of $U$ in $V$.

** Proposition 6.6.** The mapping $U: V \to \mathbb{R}$ is continuous.

**Proof.** In the previous proposition we have seen that $U$ is lower semi-continuous in $H$. In particular, it is lower semi-continuous in $V$. Thus, we prove that $U$ is also upper semi-continuous in $V$, we can conclude that it is continuous on $V$.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $V$ converging to some $x$ in $V$. As $x \in V$, according to Proposition 6.3 and Theorem 6.2, there exists $u \in X_x \cap W^{1,2}(-\infty, 0)$ such that $U(x) = S_{-\infty}(u)$. Now, we define

$$u_n(t) = u(t) + e^{tA}(x_n - x), \quad t \leq 0.$$  

Clearly $u_n(0) = x_n$. Then, as $x_n - x \in V$, we have that $u_n \in X_{x_n} \cap W^{1,2}(-\infty, 0)$. Moreover, as $x_n$ converges to $x$ in $V$ and $V = (H, D(A))_{1,2}$, we infer that $u_n$ converges to $u$ in $W^{1,2}(-\infty, 0)$, so that

$$\lim_{n \to \infty} S_{-\infty}(u_n) = S_{-\infty}(u).$$

This allows to conclude that

$$U(x) = S_{-\infty}(u) = \lim_{n \to \infty} S_{-\infty}(u_n) \geq \limsup_{n \to \infty} U(x_n),$$

so that upper semi-continuity follows.

7. **Stochastic Navier Stokes equations with periodic boundary conditions**

All what we have discussed throughout the paper until now applies to the case when the Dirichlet boundary conditions are replaced by the periodic boundary conditions. In the latter case it is customary to study our problem in the 2-dimensional torus $\mathbb{T}^2$ (of fixed dimensions $L \times L$), instead of a regular bounded domain $O$. All the mathematical background can be found in the small book [34] by Temam. In particular, the space $H$ is equal to

$$H_\infty = \{ u \in L^2_0(\mathbb{T}^2, \mathbb{R}^2) : \text{div}(u) = 0 \ \text{and} \ \gamma_\nu(u)|_{\Gamma_j} = -\gamma_\nu(u)|_{\Gamma_{j+2}}, \ j = 1, 2, \}$$

where $L^2_0(\mathbb{T}^2, \mathbb{R}^2)$ is the Hilbert space consisting of those $u \in L^2(\mathbb{T}^2, \mathbb{R}^2)$ which satisfy $\int_{\mathbb{T}^2} u(x) \, dx = 0$ and $\Gamma_j$, $j = 1, \ldots, 4$ are the four (not disjoint) parts of the boundary of $\partial(\mathbb{T}^2)$ defined by

$$\Gamma_j = \{ x = (x_1, x_2) \in [0, L]^2 : x_j = 0 \}, \quad \Gamma_{j+2} = \{ x = (x_1, x_2) \in [0, L]^2 : x_j = L \}, \ j = 1, 2.$$

The Stokes operator $A$ can be defined in a natural way and it satisfies all the properties know in the bounded domain case, inclusive the positivity (2.11) (with $\lambda_1 = \frac{4\pi^2}{L^2}$) and the following one involving the nonlinear term $B$

$$\langle Au, B(u, u) \rangle_H = 0, \ u \in D(A),$$  

(7.1)
see [34, Lemma 3.1] for a proof. The Leray-Helmholtz projection operator $P$ has the following explicit formula using the Fourier series, see [34, (2.13)]

$$
[P(f)]_k = \frac{L^2}{4\pi^2} \left( f_k - \frac{(k \cdot f) f_k}{|k|^2} \right), \quad k \in \mathbb{Z}^2 \setminus \{0\}, \quad f = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} f_n e^{\frac{2\pi i n \cdot x}{L}} \in L^2_0(\mathbb{T}^2, \mathbb{R}^2).
$$

It follows from the above that $P$ is a bounded linear map from $D(A^\alpha)$ to itself for every $\alpha \geq 0$, compare with Proposition [2.1] in the bounded domain case.

In the next Theorem we will show that, in this case, an explicit representation of $U(x)$ can be given, for any $x \in V$.

Theorem 7.1. Assume that periodic boundary conditions hold. Then

$$
U(x) = \begin{cases} 
|x|^2_V, & x \in V, \\
+\infty, & x \in H \setminus V.
\end{cases}
$$

Proof. By Theorem 6.2 we have that

$$
U(x) = \inf \{ S_{-\infty}(u) : u \in X, \quad x \in H,
$$

and in Proposition 6.1 we have seen that $U(x) < \infty$ if and only if $x \in V$. Now, let us fix $x \in V$ and $u \in X$ such that $S_{-\infty}(u) < \infty$. In view of Proposition A.1, we have that

$$
u \in C((\infty, 0]; V) \cap L^2(\infty, 0; D(A)), \quad u' \in L^2(\infty, 0; H)
$$

and

$$
\lim_{t \to -\infty} |u(t)|_V = 0. \tag{7.2}
$$

We have

$$
|u'(t) + Au(t) + B(u(t), u(t))|_H^2 = |u'(t) - Au(t) + B(u(t), u(t))|_H^2
$$

$$
+ 4|Au(t)|_H^2 + 4 \langle u'(t) - Au(t) + B(u(t), u(t)), Au(t) \rangle_H.
$$

Then, thanks to (7.1) we get

$$
|u'(t) + Au(t) + B(u(t), u(t))|_H^2 = |u'(t) - Au(t) + B(u(t), u(t))|_H^2 + 4 \langle u'(t), Au(t) \rangle_H.
$$

According to (7.2), this means that

$$
S_{-\infty}(u) = \frac{1}{2} \int_{-\infty}^0 |u'(t) - Au(t) + B(u(t), u(t))|_H^2 dt + \int_{-\infty}^0 \frac{d}{dt}|u(t)|_V^2 dt
$$

$$
= \frac{1}{2} \int_{-\infty}^0 |u'(t) - Au(t) + B(u(t), u(t))|_H^2 dt + |u(0)|_V^2.
$$

In particular,

$$
U(x) \geq |x|^2_V.
$$

On the other hand, if we show that for any $x \in V$ there exists $\bar{u} \in W^{1,2}((\infty, 0) \cap X)$ such that $\bar{u}'(t) - A\bar{u}(t) + B(\bar{u}(t), \bar{u}(t)) = 0$, for $t \in (\infty, 0)$, we conclude that $U(x) = |x|^2_V$. 

As we have seen in Section 2, if $x \in V$ then the problem
\[
\begin{cases}
v'(t) + Av(t) - B(v(t), v(t)) = 0, & t > 0, \\
v(0) = x,
\end{cases}
\]
admits a unique solution $v \in L^2(0, +\infty; D(A)) \cap C([0, +\infty); V)$, with $v' \in L^2(0, +\infty; H)$, with
\[
\lim_{t \to \infty} |v(t)|_V^2 = 0.
\]
This means that if we define
\[
\bar{u}(t) = v(-t), \quad t \leq 0,
\]
we can conclude our proof, as $\bar{u} \in W^{1,2}(-\infty, 0) \cap \mathcal{X}$ and $\bar{u}'(t) - A\bar{u}(t) + B(\bar{u}(t), \bar{u}(t)) = 0$.

8. Convergence of $U_\delta$ to $U$

The aim of this section is to prove the following result.

**Theorem 8.1.** Under Assumption 5.1, we have
\[
\lim_{\delta \to 0} U_\delta(x) = U(x), \quad x \in D(A^{\beta+1}). \tag{8.1}
\]

**Proof.** Let us fix $x \in D(A^{\beta+1})$. In view of Theorem 6.2
\[
U(x) = \inf \{ S_{-\infty}(u) : u \in \mathcal{X} \}.
\]
and for any $\delta \in (0, 1]$
\[
U_\delta(x) = \inf \{ S^\delta_{-\infty}(u) : u \in \mathcal{X} \}.
\]
Thus, thanks to Theorem 4.8, our theorem is proved provided we show that for any $x \in D(A^{\beta+1})$ the family $\{ S^\delta_{-\infty} \}_{\delta \in (0, 1]}$ is equi-coercive in $\mathcal{X}$ and
\[
\Gamma - \lim_{\delta \to 0} S^\delta_{-\infty} = S_{-\infty}, \quad \text{in} \ \mathcal{X}.
\]

Before we formulate our next result let us introduce an auxiliary functional $\tilde{S}_{-\infty} : \mathcal{X} \to [0, \infty]$, where $x \in D(A^{\beta+1})$ is fixed, by the formula
\[
\tilde{S}_{-\infty}(v) := \begin{cases} S_{-\infty}(v), & \text{if } v \in \mathcal{X} \cap W^{1,2}(-\infty, 0; D(A^{\beta+1}), D(A^{\beta+1})), \\
\infty, & \text{if } v \in \mathcal{X} \setminus W^{1,2}(-\infty, 0; D(A^{\beta+1}), D(A^{\beta+1})).\end{cases} \tag{8.2}
\]
Proposition 8.2. Assume that \( x \in \mathbb{H} \) and take \( u \in \mathcal{X}_x \). Then there exists a sequence \( \{u_n\} \) in \( \mathcal{X}_x \subset W^{1,2}(\mathbb{R}, 0; D(A_\beta^{\frac{\alpha+1}{2}}), D(A_\beta^{\frac{\beta}{2}})) \) such that
\[
\lim_{n \to \infty} \sup_{t \in [\mathbb{R}, 0]} |u_n(t) - u(t)| = 0, \tag{8.3}
\]
and
\[
S_{-\infty}(u) \geq \limsup_{n \to \infty} S_{-\infty}(u_n). \tag{8.4}
\]

Proof. Assume that \( S_{-\infty}(u) < \infty \). Then, according to Proposition A.1, we have that \( u \in \mathcal{X}_x \subset W^{1,2}(\mathbb{R}, 0) \) and, by Lemma 3.6, \( x = u(0) \in \mathbb{V} \). Since \( W^{1,2}(\mathbb{R}, 0) \hookrightarrow C_b((\mathbb{R}, 0], H) \), it is enough to find a sequence \( \{u_n\} \) satisfying (8.4) and, instead of (8.3), the following stronger condition
\[
\lim_{n \to \infty} |u_n - u|_{W^{1,2}(-\infty, 0)} = 0. \tag{8.5}
\]
Suppose we have found a sequence \( \{u_n\} \subset \mathcal{X}_x \subset W^{1,2}(\mathbb{R}, 0; D(A_\beta^{\frac{\alpha+1}{2}}), D(A_\beta^{\frac{\beta}{2}})) \) satisfying (8.5). In view of Definition (8.2), \( \tilde{S}_{-\infty}(u_n) = S_{-\infty}(u_n) \) for every \( n \). Therefore, in view of (8.5), we obtain (8.4), as \( S_{-\infty} \) is a continuous functional on \( W^{1,2}(-\infty, 0) \). Let us finally observe that the existence of the required sequence is just a consequence of the density of the space \( \mathcal{X}_x \subset W^{1,2}(\mathbb{R}, 0; D(A_\beta^{\frac{\alpha+1}{2}}), D(A_\beta^{\frac{\beta}{2}})) \) in \( \mathcal{X}_x \subset W^{1,2}(-\infty, 0) \).

Next, we prove that the family \( \{S_{-\infty}^\delta\}_{\delta \in (0, 1]} \) is \( \Gamma \)-convergent on \( \mathcal{X}_x \) to \( S_{-\infty} \). For this aim, we first prove that the family \( \{S_{-\infty}^\delta\}_{\delta \in (0, 1]} \) is \( \Gamma \)-convergent to \( \text{sc}^-\tilde{S}_{-\infty} \) and then we will identify \( \text{sc}^-\tilde{S}_{-\infty} \) with \( S_{-\infty} \).

Proposition 8.3. Under Assumption 5.1, if \( x \in \mathbb{H} \), then
\[
\Gamma - \lim_{\delta \to 0} S_{-\infty}^\delta = \text{sc}^-\tilde{S}_{-\infty} \text{ in } \mathcal{X}_x. \tag{8.6}
\]

Proof. According to Proposition 4.7, the proof of (8.6) follows, once we show that for any \( u \in \mathcal{X}_x \) the function
\[
(0, 1] \ni \delta \mapsto S_{-\infty}^\delta(u)
\]
is decreasing and
\[
\lim_{\delta \to 0} S_{-\infty}^\delta(u) = \tilde{S}_{-\infty}(u), \ u \in \mathcal{X}_x. \tag{8.7}
\]
Let us fix a function \( u \in \mathcal{X}_x \). Then, in view of Assumption 5.1, for each \( y \in D(A_\beta^{\frac{\beta}{2}}) \), the function \( (0, 1] \ni \delta \mapsto Q^{-1}_\delta y_H^\beta \in \mathbb{R} \) is decreasing. This implies that for any fixed \( u \) the mapping \( (0, 1] \ni \delta \mapsto S_{-\infty}^\delta(u) \) is decreasing.
Now, in order to prove (8.7) we distinguish two cases, when $$\tilde{S}_{-\infty}(u) = \infty$$ and when $$\tilde{S}_{-\infty}(u) < \infty$$. In view of (8.2), if $$u \in \mathcal{X}_x \cap W^{1,2}(-\infty, 0; D(A^{\delta+1}, D(A^{\delta})))$$, then

$$\tilde{S}_{-\infty}(u) = S_{-\infty}(u) = \frac{1}{2} \int_{-\infty}^{0} |\mathcal{H}(u)(t)|^2_H dt.$$  

On the other hand, we have $$S_{-\infty}^{\delta}(u) = \frac{1}{2} \int_{-\infty}^{0} |Q^{-1}_{\delta}(\mathcal{H}(u)(t))|^2_H dt$$ and hence the result will follow by the Lebesgue dominated convergence theorem once we have observed that according to Assumption 5.1, for all $$y \in D(A^{\delta})$$ it holds $$Q^{-1}_{\delta}y \to y$$, as $$\delta \searrow 0$$, and $$|Q^{-1}_{\delta}y|_H \leq |Q^{-1}_1y|_H$$.

**Theorem 8.4.** If $$x \in V$$, then

$$\Gamma - \lim_{\delta \to 0} S_{-\infty}^{\delta} = S_{-\infty} \text{ on } \mathcal{X}_x.$$  

*Proof.* Let us fix $$x \in V$$. It remains to prove that on $$\mathcal{X}_x$$

$$\text{sc}^{-} \tilde{S}_{-\infty} = S_{-\infty}.$$  

In view of Proposition 4.3 this follows if we show that for every sequence $$\{u_n\}_n \subset \mathcal{X}_x$$ convergent to $$u$$ in $$\mathcal{X}_x$$ it holds

$$S_{-\infty}(u) \leq \liminf_n \tilde{S}_{-\infty}(u_n),$$  

and for some sequence $$\{u_n\}_n \subset \mathcal{X}_x$$ convergent to $$u$$ in $$\mathcal{X}_x$$ it holds

$$S_{-\infty}(u) \geq \limsup_n \tilde{S}_{-\infty}(u_n).$$  

But it is immediate to check that (8.9) follows from (5.10) and (8.10) is a consequence of Proposition 8.2.  

**9. Application to the exit problem**

Let us recall that a domain $$D \subset H$$ is said to be invariant and attracted to the asymptotically stable equilibrium 0 of the system

$$u'(t) + Au(t) + B(u(t), u(t)) = 0, \quad u(0) = x,$$  

iff, for any $$x \in D$$ and $$t \geq 0$$, $$u^x(t) \in D$$, where $$u^x(t)$$, $$t \geq 0$$, denotes the unique solution to (9.1), and

$$\lim_{t \to \infty} u^x(t) = 0.$$  

It is well known that

$$|u^x(t)|^2_H + \int_0^t |u^x(s)|^2_{V} ds \leq |x|^2_H,$$  

and by the Poincaré inequality (2.11) we infer that every ball in $$H$$ is invariant and attracted to 0.
Throughout this section, we will denote by \( D \) a bounded domain in \( H \) which is invariant and attracted to 0. For any \( x \in D \), \( \varepsilon > 0 \) and \( \delta \in (0,1] \) we will denote by \( \tau_{\varepsilon, \delta}^x \) the exit time of the solution \( u_{\varepsilon, \delta}^x \) of equation (5.1) from the domain \( D \), that is

\[
\tau_{\varepsilon, \delta}^x = \inf \{ t \geq 0 : u_{\varepsilon, \delta}^x(t) \notin D \}.
\]

Our purpose here is to prove the following exponential estimate for the expectation of \( \tau_{\varepsilon, \delta}^x \).

**Theorem 9.1.** Assume that \( \delta \in (0,1] \) and that there exists \( y_\delta \in \partial D \) such that

\[
\inf_{y \in \partial D} U_\delta(y) = U_\delta(y_\delta).
\]

Then, for any \( x \in D \)

\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \tau_{\varepsilon, \delta}^x = \inf_{y \in \partial D} U_\delta(y) = \min_{y \in \partial D} U_\delta(y).
\]

As we already pointed out in [9, Section 7], the proof of the previous result is based on the few lemmas, proofs of which are postponed till the Appendix B. Actually, the arguments used in finite dimension (see [15, proof of Theorem 5.7.11] and [16, proof of Theorem 4.1]), can be adapted to this infinite dimensional case, once the following preliminary results are proven.

**Lemma 9.2.** For any \( \eta > 0 \) and \( \mu > 0 \), there exists \( T = T(\eta, \mu) > 0 \) and \( h = h(\eta, \mu) > 0 \) such that for all \( x \in B_{\mu,0} \) there exist \( T(x) \leq T \) and \( v^x \in C([0,T(x)];H) \), with \( v^x(0) = x \), such that

\[
d_H(v^x(T(x)), \bar{D}) > h,
\]

and

\[
S_{0,T(x)}(v^x) \leq \inf_{y \in \partial D} U_\delta(y) + \eta.
\]

**Lemma 9.3.** There exists \( \mu_0 > 0 \) such that for any \( \eta > 0 \) and \( \mu \in (0, \mu_0] \) there exists \( T = T(\eta, \mu) > 0 \) such that

\[
\lim_{\varepsilon \to 0} \varepsilon \log \left( \inf_{x \in B_{\mu,0}} \mathbb{P} \left( \tau_{\varepsilon, \delta}^x \leq T \right) \right) > - \left( \inf_{x \in \partial D} U_\delta(x) + \eta \right).
\]

**Lemma 9.4.** For any \( x \in D \), \( \varepsilon > 0 \) and \( \mu > 0 \), such that \( B_{\mu,0} \subset D \), it holds

\[
\lim_{t \to \infty} \lim_{\varepsilon \to 0} \inf \log \mathbb{P} (\sigma_{\varepsilon, \delta, \mu} > t) = -\infty,
\]

where

\[
\sigma_{\varepsilon, \delta, \mu} := \inf \left\{ t \geq 0 : u_{\varepsilon, \delta}^x(t) \in B_{\mu,0} \cup \partial D \right\}.
\]

Moreover,

\[
\lim_{\varepsilon \to 0} \mathbb{P} (u_{\varepsilon, \delta}^x(\sigma_{\varepsilon, \delta, \mu}) \in B_{\mu,0}) = 1.
\]

**Lemma 9.5.** For every \( \rho, \lambda > 0 \) and \( x \in D \), there exist \( T = T(\rho, \lambda, x) < \infty \) such that

\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left( \sup_{t \in [0,T]} |u_{\varepsilon, \delta}^x(t) - x|_H \geq \rho \right) < -\lambda.
\]
In view of Lemmas 9.2 to 9.5, by proceeding as in [9, Theorem 7.7], we can conclude that the following approximation result holds.

**Theorem 9.6.** Suppose that Assumption 5.1 is satisfied. Then, if for any $x \in V \cap \partial D$ there exists a sequence $\{x_n\} \subset D(A^{\frac{1+\beta}{2}}) \cap \partial D$ such that

$$\lim_{n \to \infty} |x_n - x|_V = 0, \quad (9.4)$$

then

$$\lim_{\delta \to 0} \inf_{x \in \partial D} U_\delta(x) = \inf_{x \in \partial D} U(x). \quad (9.5)$$

In view of Theorem 9.1, this implies the following corollary.

**Corollary 9.7.** Under the assumptions of Theorem 9.6, for $0 < \varepsilon << \delta << 1$, the following asymptotic formula holds:

$$\mathbb{E} \tau_{x, \delta}^x \sim \exp \left( \frac{1}{\varepsilon} \inf_{x \in \partial D} U(x) \right).$$

**Remark 4.** (1) As in [9, Remark 7.8], we notice that if we take $D = B_H(r)$, for $r > 0$, then the approximation condition (9.4) assumed in Theorem 9.6 is fulfilled. Actually, as $D(A^{\frac{1+\beta}{2}})$ is dense in $V$, we can find a sequence $\{\hat{x}_n\} \subset D(A^{\frac{1+\beta}{2}})$ which is convergent to $x$ in $V$. Then, if we set $x_n = r\hat{x}_n/|\hat{x}_n|_H$, we conclude that $\{x_n\} \subset D(A^{\frac{1+\beta}{2}}) \cap \partial D$ and (9.4) holds.

(2) Limit (9.5) follows from Theorem 6.2 and (9.4) in virtue of a general argument based on $\Gamma$-convergence and relaxation, which applies to more general situations, and which has been introduced in [9]. Actually, we define

$$\tilde{U}(x) = \begin{cases} U(x), & x \in \Lambda_\beta, \\ +\infty, & x \in H \setminus \Lambda_\beta, \end{cases}$$

and for any $\delta \in (0, 1]$

$$\tilde{U}_\delta(x) = \begin{cases} U_\delta(x), & x \in \Lambda_\beta, \\ +\infty, & x \in H \setminus \Lambda_\beta, \end{cases}$$

where $\Lambda_\beta = D(A^{\frac{1+\beta}{2}}) \cap \partial D$. One can prove that

$$\Gamma - \lim_{\delta \to 0} \tilde{U}_\delta = \text{sc}^{-\tilde{U}}, \quad \text{in } H,$$

and then, by using (9.4) and the continuity of $U$ in the space $V$ proved in Proposition 6.6, one can show

$$\text{sc}^{-\tilde{U}}(x) = \begin{cases} U(x), & x \in \partial D, \\ +\infty, & x \in H \setminus \partial D. \end{cases}$$

This implies (9.5).
APPENDIX A. PROOFS OF SOME AUXILIARY RESULTS

Proposition A.1. Assume that $z \in \mathcal{X}$ is such that $S_{-\infty}(z) < \infty$. Then the following conditions are satisfied.

(i) $z(0) \in V$,

(ii) $z(t)$ converges to 0 in $V$ as $t \to -\infty$, i.e.

$$\lim_{t \to -\infty} |z(t)|_V = 0.$$  \hspace{1cm} (A.1)

(iii) $z \in W^{1,2}(-\infty, 0)$, i.e.

$$\int_{-\infty}^{0} |Az(t)|_H^2 dt + \int_{-\infty}^{0} |z'(t)|_H^2 dt < \infty.$$  \hspace{1cm} (A.2)

Moreover, there exists a continuous and strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and if $z \in \mathcal{X}$ is a solution to the problem

$$z'(t) + Az(t) + B(z(t), z(t)) = f(t), \quad t \leq 0,$$

with $f$ being an element of $L^2(-\infty, 0)$, then $z \in W^{1,2}(-\infty, 0)$, $z(0) \in V$ and

$$|z(0)|_V^2 + \int_{-\infty}^{0} |Az(t)|_H^2 dt + \int_{-\infty}^{0} |z'(t)|_H^2 dt \leq \varphi(\int_{-\infty}^{0} |f(t)|_H^2 dt),$$

i.e.

$$|z(0)|_V^2 + |z|_{W^{1,2}(-\infty, 0)}^2 \leq \varphi(\|f\|_{L^2(-\infty, 0)}^2)$$

or

$$|z(0)|_V^2 + |z|_{W^{1,2}(-\infty, 0)}^2 \leq \varphi(S_{-\infty}(z)).$$

Proof. The argument below is a bit informal but it can easily be made fully rigorous. We will be careful with the constants as we want to prove the last part of the Proposition as well.

Claim (i) follows from Lemma 3.6. Next, we will be proving Claim (ii).

In view of Lemma 3.6, we can assume that $z \in W^{1,2}_{loc}(-\infty, 0)$). Since $S_{-\infty}(z) < \infty$, if we set

$$f(t) = z'(t) + Az(t) + B(z(t), z(t)), \quad t \leq 0,$$

we have that $f \in L^2(-\infty, 0; H)$. If we multiply equation (A.3) by $z$ and use equality (2.10), we get

$$\frac{1}{2} \frac{d}{dt} |z(t)|_H^2 + |z(t)|_V^2 = (f, z)_H \leq \frac{1}{2} |z(t)|_V^2 + \frac{1}{2\lambda_1} |f(t)|_H^2, \quad t < 0,$$

where $\lambda_1$ is the Poincaré constant of the domain $\mathcal{O}$. Hence,

$$|z(t)|_H^2 + \int_s^t |z(r)|_V^2 dr \leq |z(s)|_H^2 + \frac{1}{\lambda_1} \int_s^t |f(r)|_H^2 dr, \quad -\infty < s \leq t \leq 0.$$  \hspace{1cm} (A.4)

As

$$\lim_{t \to -\infty} |z(t)|_H = 0,$$
we infer that
\[
|z(t)|_H^2 + \int_{-\infty}^t |z(s)|_V^2 \, ds \leq \frac{1}{\lambda_1} \int_{-\infty}^t |f(r)|_H^2 \, dr, \quad -\infty < t \leq 0.
\]

This implies
\[
|z(t)|_H^2 \leq \frac{1}{\lambda_1} \int_{-\infty}^t |f(r)|_H^2 \, dr \leq \frac{1}{\lambda_1} \int_{-\infty}^0 |f(r)|_H^2 \, dr, \quad t \leq 0, \quad (A.5)
\]
and
\[
\int_{-\infty}^0 |z(s)|_V^2 \, ds \leq \frac{1}{\lambda_1} \int_{-\infty}^0 |f(r)|_H^2 \, dr. \quad (A.6)
\]

The latter inequality means that \( z \in L^2((-\infty, 0], V) \), which implies that we can find a decreasing sequence \( \{s_n\} \) such that \( s_n \to -\infty \) and
\[
\lim_{n \to \infty} |z(s_n)|_V = 0. \quad (A.7)
\]

Next we multiply equation (\( A.3 \)) by \( Az(t) \). Thanks to (2.15) and to the Young inequality, we get
\[
\frac{1}{2} \frac{d}{dt} |z(t)|_V^2 + |Az(t)|_H^2 = -(B(z(t), z(t)), Az(t))_H + (f(t), Az(t))_H \quad (A.8)
\]
\[
\leq \frac{1}{4} |Az(t)|_H^2 + \frac{C_2}{2} |z(t)|_H^4 |z(t)|_V^4 + \frac{1}{4} |Az(t)|_H^2 + |f(t)|_H^2.
\]

where \( C_2 = \frac{2\pi}{4} C^2 \) and \( C \) is the constant from inequality (2.15). Applying next the Poincaré inequality (2.1), we get,
\[
\frac{d}{dt} |z(t)|_V^2 + \lambda_1 |z(t)|_V^2 \leq C_2 \left[ |z(t)|_H^4 |z(t)|_V^2 \right] |z(t)|_V^2 + 2 |f(t)|_H^2. \quad (A.9)
\]

Hence, since \( \lambda_1 \geq 0 \), we have
\[
\frac{d}{dt} |z(t)|_V^2 \leq C_2 \left[ |z(t)|_H^4 |z(t)|_V^2 \right] |z(t)|_V^2 + 2 |f(t)|_H^2, \quad (A.10)
\]
and so, by the Gronwall Lemma, for any \(-\infty < s \leq t \leq 0\) we get
\[
|z(t)|_V^2 \leq |z(s)|_V^2 \exp \left( C_2 \int_s^t |z(r)|_H^2 |z(r)|_V^2 \, dr \right) \quad (A.11)
\]
\[
+ 2 \int_s^t |f(r)|_H^2 \exp \left( C_2 \int_r^t |z(\rho)|_H^2 |z(\rho)|_V^2 \, d\rho \right) \, dr.
\]

Using the above with \( s = s_n \) from (A.7) and then taking the limit as \( n \to \infty \) we infer that
\[
|z(t)|_V^2 \leq 2 \int_{-\infty}^t |f(r)|_H^2 \exp \left( C_2 \int_r^t |z(\rho)|_H^2 |z(\rho)|_V^2 \, d\rho \right) \, dr, \quad t \leq 0. \quad (A.12)
\]
Of course, for the above to be correct we need to show that the sequence
\[ \left\{ \int_{s_n}^t |z(r)|^2_H |z(r)|^2_V \, dr \right\}_{n \geq 1} \]
is bounded from above. But in view of estimates (A.5) and (A.6) we have
\[ \int_{-\infty}^0 |z(\rho)|^2_H |z(\rho)|^2_V \, d\rho \leq \frac{1}{\lambda_1^2} |f|_{L^2(-\infty,0,H)}^4 = \frac{1}{\lambda_1^2} |f|^4 < \infty \quad (A.13) \]
(note that here and in the rest of this proof, for the sake of brevity, we shall write $|f|$ instead of $|f|_{L^2(-\infty,0,H)}$). Therefore, since
\[ \int_r^t |z(\rho)|^2_H |z(\rho)|^2_V \, d\rho \leq \int_{-\infty}^0 |z(\rho)|^2_H |z(\rho)|^2_V \, d\rho, \quad -\infty < r \leq t \leq 0, \]
we can conclude that
\[ \sup_{t \leq 0} |z(t)|^2_V \leq 2 \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) \sup_{t \leq 0} \int_{-\infty}^t |f(r)|^2_H \, dr \leq 2 \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^2. \quad (A.14) \]
Moreover, as
\[ \int_{-\infty}^0 |f(r)|^2_H \exp \left( C_2 \int_r^t |z(\rho)|^2_H |z(\rho)|^2_V \, d\rho \right) \, dr < \infty, \]
we have
\[ \lim_{t \to -\infty} \int_{-\infty}^t |f(r)|^2_H \exp \left( C_2 \int_r^t |z(\rho)|^2_H |z(\rho)|^2_V \, d\rho \right) \, dr = 0, \]
so that from (A.12) we conclude that (A.1) holds.
Now, to prove the second part of this Proposition, we observe that from (A.8) we also have
\[ |z(0)|^2_V + \int_{-\infty}^0 |Az(t)|^2_H \, dt \leq C_2 \int_{-\infty}^0 \left[ |z(t)|^2_H |z(t)|^2_V \right] |z(t)|^2_V \, dt + 2 \int_{-\infty}^0 |f(t)|^2_H \, dt, \]
where we have used (A.1). Since by (A.5) and (A.14),
\[ \sup_{t \in (-\infty,0]} |z(t)|^2_H |z(t)|^2_V \leq \frac{2}{\lambda_1} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^4 < \infty, \]
we infer that
\[ |z(0)|^2_V + \int_{-\infty}^0 |Az(t)|^2_H \, dt \leq \frac{2 C_2}{\lambda_1} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^4 \int_{-\infty}^0 |z(t)|^2_V \, dt + 2 \int_{-\infty}^0 |f(t)|^2_H \, dt. \quad (A.15) \]
Hence, in view of (A.6), we infer that
\[ |z(0)|^2_V + \int_{-\infty}^0 |Az(t)|^2_H \, dt \leq \frac{2 C_2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2 |f|^2, \quad (A.16) \]
and this concludes the proof of the first part of (A.2).
In order to prove the second part, it is enough to show that
\[ \int_{-\infty}^{0} |B(z(t), z(t))|^2 \, dt < \infty. \]

Indeed, by the Minkowski inequality we have
\[ |z'|_{L^2(-\infty, 0; H)} \leq |Az|_{L^2(-\infty, 0; H)} + |B(z, z)|_{L^2(-\infty, 0; H)} + |f|_{L^2(-\infty, 0; H)}, \quad (A.17) \]

According to inequalities (A.5), (A.12) and (A.16) and to inequality (2.15), we have
\[ \int_{-\infty}^{0} |B(z(t), z(t))|^2 \, dt \leq C \int_{-\infty}^{0} |z(t)|_H |z(t)|^2 \, dt 
  \leq C \sup_{t \leq 0} |z(t)|_H |z(t)|_V \left( \int_{-\infty}^{0} |z(t)|^2 \, dt \right)^{\frac{1}{2}} \left( \int_{-\infty}^{0} |Az(t)|^2 \, dt \right)^{\frac{1}{2}} 
  \leq \frac{2}{\lambda_1} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^4 \frac{1}{\sqrt{\lambda_1}} |f| \left( \frac{2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2 |f|^2 \right)^{\frac{1}{2}} < \infty. \]

The final statement follows from inequalities (A.16), (A.17) and (A.18).

**Remark 5.**

(1) Our proof of Proposition [A.1] has been inspired by [7].

(2) Roughly speaking, the above result says that the following two equalities hold
\[ \{ z \in \mathcal{X} : S_{-\infty}(z) < \infty \} = \mathcal{X} \cap W^{1,2}(-\infty, 0) \]
\[ \{ z \in \mathcal{X} : u(0) = x \text{ and } S_{-\infty}(z) < \infty \} = \mathcal{X}_x \cap W^{1,2}(-\infty, 0), \quad x \in \mathcal{V}. \]

Next Proposition generalizes Proposition [A.1] to \( S_{-\infty}^\delta \).

**Proposition A.2.** Assume that \( \alpha \in (0, 1/2) \) and \( z \in \mathcal{X} \) is such that \( S_{-\infty}^\delta(z) < \infty \). Let \( f \in L^2(-\infty, 0; D(A_{\alpha^+}^\delta)) \) be defined as
\[ z'(t) + Az(t) + B(z(t), z(t)) = f(t), \quad t \leq 0. \quad (A.19) \]

Then \( z(0) \in D(A_{\alpha^+}^\delta) \),
\[ \lim_{t \to -\infty} |z(t)|_{D(A_{\alpha^+}^\delta)} = 0, \quad (A.20) \]

and \( z \in W^{1,2}(-\infty, 0; D(A_{\alpha^+}^\delta), D(A_{\frac{\alpha+1}{2}}^\delta)), \) i.e.
\[ \int_{-\infty}^{0} |A_{\alpha^+}^\delta z(t)|_H^2 \, dt + \int_{-\infty}^{0} |A_{\frac{\alpha+1}{2}}^\delta z'(t)|_H^2 \, dt < \infty. \quad (A.21) \]

Moreover, there exists a continuous and strictly increasing function \( \tilde{\varphi} : [0, \infty) \to [0, \infty) \) such that \( \tilde{\varphi}(0) = 0 \) and if \( z \in \mathcal{X} \) is a solution to the problem
\[ z'(t) + Az(t) + B(z(t), z(t)) = f(t), \quad t \leq 0, \]
with \( f \) being an element of \( L^2(-\infty, 0; D(A_\alpha^0)) \), then \( z \in W^{1,2}(-\infty, 0; D(A_{\alpha+1}^0), D(A_\alpha^0)) \), \( z(0) \in D(A_{\alpha+1}^0) \) and
\[
|z(0)|_{D(A_{\alpha+1}^0)}^2 + |z|_{W^{1,2}(-\infty,0;D(A_{\alpha+1}^0),D(A_\alpha^0)}^2 \leq \varphi(|f|_{L^2(-\infty,0)}^2)
\]

**Proof.** Following the methods from the proof of Proposition \( A.1 \) it is sufficient to prove the first part of Proposition \( A.2 \).

Let us fix \( \alpha \in (0, 1/2) \), \( x \in D(A_{\alpha+1}^0) \) and \( z \in \mathcal{X}_x \) such that \( S_{-\infty}^x(z) < \infty \). Let us define \( f \in L^2(-\infty, 0; D(A_{\alpha}^0)) \) by \( (A.19) \). Since the assumptions in the present proposition are stronger than the assumptions of Proposition \( A.1 \) we can freely use the results from the proof of the latter.

So, firstly, let us notice that by inequality \( (A.15) \) we can find a decreasing sequence \( \{s_n\} \) such that \( s_n \downarrow -\infty \) and
\[
\lim_{n \to \infty} |A_{\frac{n}{2}}^0 z(s_n)|_H = 0.
\] (A.22)

Arguing as in the proof of Proposition \( 3.2 \) if we calculate the derivative of \( |A_{\alpha+1}^0 u(t)|_H^2 \) and use inequality \( (2.20) \), with \( s = 2 \), to get the following generalisation of \( (A.8) \)
\[
\frac{1}{2} \frac{d}{dt} |A_{\alpha+1}^0 z(t)|_H^2 + |A_{\alpha+1}^0 z(t)|_H^2 = -(B(z(t), z(t)), A_{\alpha+1}^0 z(t))_H + (f(t), A_{\alpha+1}^0 z(t))_H \leq \frac{1}{2} |A_{\alpha+1}^0 z(t)|_H^2 + C|Az(t)|_H^2 + C |A_\alpha^0 f(t)|_H^2.
\] (A.23)

Let us note that contrary to \( (A.8) \), the highest power of \( z \) on the RHS of \( (A.23) \) is 4. Hence, we infer that
\[
\frac{d}{dt} |A_{\alpha+1}^0 z(t)|_H^2 + |A_{\alpha+1}^0 z(t)|_H^2 \leq C|Az(t)|_H^2 |A_{\alpha+1}^0 z(t)|_H^2 + 2|A_{\alpha}^0 f(t)|_H^2.
\] (A.24)

Therefore, by the Gronwall Lemma, for any \( -\infty < s \leq t \leq 0 \) we get
\[
|A_{\alpha+1}^0 z(t)|_H^2 \leq |A_{\alpha+1}^0 z(s)|_H^2 \exp \left( C \int_s^t |Az(r)|_H^2 dr \right) + 2 \int_s^t |A_{\alpha}^0 f(r)|_H^2 \exp \left( C \int_r^t |Az(\rho)|_H^2 d\rho \right) dr.
\] (A.25)

Using the above with \( s = s_n \) from \( (A.22) \) and then taking the limit as \( n \to \infty \), we infer that
\[
|A_{\alpha+1}^0 z(t)|_H^2 \leq 2 \int_{-\infty}^t |A_{\alpha}^0 f(r)|_H^2 \exp \left( C \int_r^t |Az(\rho)|_H^2 d\rho \right) dr, \quad t \leq 0.
\] (A.26)

As in the proof of the previous Proposition \( A.1 \) the above is true because now by inequality \( (A.16) \) the sequence
\[
\left\{ \int_{s_n}^t |Az(r)|_H^2 dr \right\}_{n \geq 1}
\]
is bounded from above by \( \frac{2C_2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2|f|^2 \), where \( |f| \) denotes \( |f|_{L^2(-\infty,0,\Pi)} \). Therefore, we can conclude that
\[
\sup_{t \leq 0} |A^{\frac{4+s}{2}} z(t)| \leq 2 \int_{-\infty}^{0} |A^{\frac{4+s}{2}} f(r)|^2_{H} dr \exp \left( \frac{2C_2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2|f|^2 \right).
\] (A.27)

Moreover, as
\[
\int_{-\infty}^{0} |A^{\frac{s}{2}} f(r)|^2_{H} \exp \left( C \int_{r}^{t} |Az(\rho)|^2_{H} d\rho \right) dr < \infty,
\]
we have that
\[
\lim_{t \to -\infty} \int_{-\infty}^{t} |A^{\frac{s}{2}} f(r)|^2_{H} \exp \left( C \int_{r}^{t} |Az(\rho)|^2_{H} d\rho \right) dr = 0.
\]

Hence (A.20) follows from (A.26).

Now, to prove the second part of Proposition A.2 i.e. the first inequality in (A.20), we observe that from (A.24) we also have
\[
|A^{\frac{4+s}{2}} z(0)|^2_{H} + \int_{-\infty}^{0} |A^{\frac{4+s}{2}} z(t)|^2_{H} dt \leq C \int_{-\infty}^{0} |Az(t)|^2_{H} |A^{\frac{4+s}{2}} z(t)|^2_{H} dt + 2 \int_{-\infty}^{0} |A^{\frac{s}{2}} f(t)|^2_{H} dt.
\]

Taking into account inequalities (A.27) and (A.16) we infer that
\[
|A^{\frac{4+s}{2}} z(0)|^2_{H} + \int_{-\infty}^{0} |A^{\frac{4+s}{2}} z(t)|^2_{H} dt \leq C \left[ \frac{2C_2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2|f|^2 \right]
\]
\[
\left[ \int_{-\infty}^{0} |A^{\frac{s}{2}} f(r)|^2_{H} dr \exp \left( \frac{2C_2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2|f|^2 \right) \right] + 2 \int_{-\infty}^{0} |A^{\frac{s}{2}} f(t)|^2_{H} dt,
\]
and this concludes the proof of the first part of inequality (A.20).

As in the proof of the previous Proposition, in order to prove the third part of Proposition A.2 i.e. the second inequality in (A.20), it is enough to show that
\[
\int_{-\infty}^{0} |A^{\frac{s}{2}} B(z(t), z(t))|^2_{H} dt < \infty.
\]

According to inequalities (2.20) (with \( s = 2 \)), (A.16) and (A.27) we have
\[
\int_{-\infty}^{0} |A^{\frac{s}{2}} B(z(t), z(t))|^2_{H} dt \leq C \int_{-\infty}^{0} |Az(t)|^2_{H} |A^{\frac{4+s}{2}} z(t)|^2_{H} dt
\]
\[
\leq C \sup_{t \leq 0} |A^{\frac{4+s}{2}} z(t)|^2_{H} \int_{-\infty}^{0} |Az(t)|^2_{H} dt \leq \left[ \frac{4C_2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2|f|^2 \right]
\]
\[
\int_{-\infty}^{0} |A^{\frac{s}{2}} f(r)|^2_{H} \exp \left( \frac{2C_2}{\lambda_1^2} \exp \left( \frac{C_2}{\lambda_1^2} |f|^4 \right) |f|^6 + 2|f|^2 \right).
\]

The proof is now complete.
Thus, if we set \( x \). In particular, in view of Proposition 6.1, we have that \( \inf_{t \to -\infty} |z(t)|_V = 0. \)

Proof. In view of [25, Theorem 2.2, p. 13] it is enough to consider the case \( b = +\infty \). So let us take \( z \in W^{1,2}(-\infty, \infty) \). Then, since \( V = [D(A), H]_{1/2} \), according to [25, Theorem 3.1, p. 19], \( z : \mathbb{R} \to V \) is a bounded and continuous function. Moreover, by equivalence (2.27) p. 16. \( u(t) \to 0 \) in \( H \) as \( t \to \pm \infty \). Hence the result follows by applying [A.1].

Proof of Lemma 9.2. Assume that \( \rho, h > 0 \) for some \( \kappa > 0 \). In view of [25, Theorem 2.2, p. 13] it is enough to consider the case \( b = +\infty \). So let us take \( z \in W^{1,2}(-\infty, \infty) \). Then, since \( V = [D(A), H]_{1/2} \), according to [25, Theorem 3.1, p. 19], \( z : \mathbb{R} \to V \) is a bounded and continuous function. Moreover, by equivalence (2.27) p. 16. \( u(t) \to 0 \) in \( H \) as \( t \to \pm \infty \). Hence the result follows by applying [A.1].

Appendix B. Proofs of Lemmas in Section 9

Proof of Lemma 9.2. If \( \inf_{x \in \partial D} U_\delta(x) = +\infty \) there is nothing to prove. Thus, in what follows we can assume that \( \inf_{x \in \partial D} U_\delta(x) < +\infty \). This implies that there exists \( x_\delta \in \partial D \) such that

\[
U_\delta(x_\delta) < \inf_{x \in \partial D} U_\delta(x) + \frac{\eta}{2}.
\]

In particular, in view of Proposition 6.1 we have that \( x_\delta \in D(A^{1+\beta}) \), so that we can fix \( \bar{x}_\delta \in D(A^{1+\beta}) \), such that

\[
|x_\delta - \bar{x}_\delta|_{D(A^{1+\beta})} < \rho, \quad d_H(\bar{x}_\delta, D) > h,
\]

for some \( \rho, h > 0 \) to be chosen later on.

In view of Proposition 3.3 for any \( \kappa > 0 \) there exists \( T(\beta, \kappa, \mu) > 0 \) such that

\[
u^\kappa(t;0) \in B_{\kappa, \frac{\mu+1}{2}}, \quad x \in B_{\mu,0}, \quad t > T(\beta, \kappa, \mu).
\]

Thus, if we set \( T_1 = T(\beta, \kappa, \mu) + 1 \) and

\[
z_1(t) = \nu^\kappa(t;0), \quad t \in [0, T_1],
\]

we have that \( z_1(0) = x, \quad z_1(T_1) \in D(A^{1+\alpha}) \) and

\[
S_{0,T_1}^{\delta}(z_1) = 0.
\]

(B.2)

Now, we define

\[
z_2(t) = (T_1 + 1 - t)e^{-(t-T_1)A}z_1(T_1), \quad t \in [T_1, T_1 + 1].
\]

We have \( z_2(T_1) = z_1(T_1) \) and \( z_1(T_1 + 1) = 0 \). Moreover,

\[
\mathcal{H}(z_2)(t) = -e^{-(t-T_1)A}z_1(T_1) + B(z_2(t), z_2(t)),
\]

so that, according to Assumption 5.4 we have

\[
S_{T_1,T_1+1}^{\delta}(z_2) \leq c \int_{T_1}^{T} \left| e^{-(t-T_1)A}z_1(T_1) \right|^2_{D(A^{\frac{\beta}{2}})} dt
\]

\[
+ c \int_{T_1}^{T} \left| B(e^{-(t-T_1)A}z_1(T_1), e^{-(t-T_1)A}z_1(T_1)) \right|^2_{D(A^{\frac{\beta}{2}})} dt.
\]

Now, thanks to [22, 20], with \( s = 1 + \beta \), we have

\[
\left| B(e^{-(t-T_1)A}z_1(T_1), e^{-(t-T_1)A}z_1(T_1)) \right|_{D(A^{\frac{\beta}{2}})} \leq \left| e^{-(t-T_1)A}z_1(T_1) \right|^2_{D(A^{\frac{\beta}{2}})} \leq c \kappa^2,
\]

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so that
\[ S_{T_1,T_1+1}^\delta(z_2) \leq c \kappa^2 + c \kappa^4. \]
Therefore, we fix \( \kappa > 0 \) small enough such that
\[ S_{T_1,T_1+1}^\delta(z_2) \leq c \kappa^2 + c \kappa^4 < \frac{\eta}{6}, \]
and in correspondence of \( \kappa \) we fix \( T_1 = T(\beta, \kappa, \mu). \)
Next, in view of the definition of \( U_\delta \), we can fix \( T_2 > 1 \) and \( z_3 \in C([T_1 + 1, T_1 + T_2]; H) \) such that \( z_3(T_1 + 1) = 0 \), \( z_3(T_1 + T_2) = x_\delta \) and
\[ S_{T_1+1,T_1+T_2}^\delta(z_3) \leq U_\delta(x_\delta) + \frac{\eta}{6}. \]
Finally, we define
\[
z_4(t) = e^{-(t-T_1+T_2)A} x_\delta + \frac{t - (T_1 + T_2)}{T - (T_1 + T_2)} (e^{-(T-t)A} x_\delta - e^{-(t-T_1+T_2)A} x_\delta), \quad t \in [T_1 + T_2, T],
\]
for some \( T > T_1 + T_2 \) to be determined. Clearly \( z_4 \in C([T_1 + T_2, T]; H) \), with \( z_4(T_1 + T_2) = x_\delta \) and \( z_4(T) = \bar{x}_\delta \). Moreover,
\[
\mathcal{H}(z_4)(t) = \frac{1}{T - (T_1 + T_2)} \left( e^{-(T-t)A} \bar{x}_\delta - e^{-(t-(T_1+T_2))A} x_\delta \right)
\]
\[
+ 2 \left( \frac{t - (T_1 + T_2)}{T - (T_1 + T_2)} \right) A e^{-(T-t)A} \bar{x}_\delta + B(z_4(t), z_4(t)).
\]
As we did for \( z_2 \), thanks to Assumption 5.1 we get
\[
S_{T_1+T_2,T}^\delta(z_4) \leq \frac{c}{(T - (T_1 + T_2))^2} \int_{T_1+T_2}^T \| e^{-(t-t)A} \bar{x}_\delta - e^{-(t-(T_1+T_2))A} x_\delta \|^2_{D(A^\beta)} dt
\]
\[
+ c \int_{T_1+T_2}^T \| e^{-(T-t)A} \bar{x}_\delta \|^2_{D(A^{\beta} \frac{\partial}{\partial x})} dt + c \int_{T_1+T_2}^T \| B(z_4(t), z_4(t)) \|^2_{D(A^{\beta})} dt = I_1(T) + I_2(T) + I_3(T).
\]
Thanks to (B.1), we have
\[
\int_{T_1+T_2}^T \| e^{-(t-t)A} \bar{x}_\delta - e^{-(t-(T_1+T_2))A} x_\delta \|^2_{D(A^\beta)} dt
\]
\[
\leq 2 \int_{T_1+T_2}^T \| e^{-(T-t)A} - e^{-(t-(T_1+T_2))A} \| \| x_\delta \|^2_{D(A^{\beta} \frac{\partial}{\partial x})} dt + 2 \int_{T_1+T_2}^T \| e^{-(T-t)A} \| \| \bar{x}_\delta - x_\delta \|^2_{D(A^\beta)} dt
\]
\[
\leq 2 \int_{T_1+T_2}^T \| e^{-(T-t)A} - e^{-(t-(T_1+T_2))A} \| \| x_\delta \|^2_{D(A^{\beta} \frac{\partial}{\partial x})} dt + c (T - (T_1 + T_2)) \beta^2.
\]
Now, it is possible to show that
\[
\int_{T_1 + T_2}^{T} \left| e^{-(T-t)A} - e^{-(t-(T_1 + T_2))A} \right|_{D(A^{\frac{\beta}{2}})}^2 \, dt
\]
\[
\leq c (T - (T_1 + T_2))^2 \left| e^{-(T_1+T_2)A} - e^{-T_1 A} A^{\frac{\beta+1}{2}} x_\delta \right|_{H}^2
\]
and hence
\[
I_1(T) \leq c \left| e^{-(T_1+T_2)A} - e^{-T_1 A} A^{\frac{\beta+1}{2}} x_\delta \right|_{H}^2 + \frac{1}{T - (T_1 + T_2)^2} \rho^2.
\]
Moreover, we have
\[
I_2(T) \leq c (T - (T_1 + T_2)) \left| x_\delta \right|_{D(A^{1+\frac{\beta}{2}})}^2 \leq c (T - (T_1 + T_2)) \left( \left| x_\delta \right|_{D(A^{1+\frac{\beta}{2}})}^2 + \rho^2 \right).
\]
Finally, as we have
\[
|z_4(t)|_{D(A^{1+\frac{\beta}{2}})} \leq c \left( \left| x_\delta \right|_{D(A^{1+\frac{\beta}{2}})} + |\bar{x}_\delta|_{D(A^{1+\frac{\beta}{2}})} \right) \leq c \left( \left| x_\delta \right|_{D(A^{1+\frac{\beta}{2}})} + |x_\delta - \bar{x}_\delta|_{D(A^{1+\frac{\beta}{2}})} \right),
\]
according to (2.20) and (B.1), we get
\[
I_3(T) \leq c (T - (T_1 + T_2)) \left( \left| x_\delta \right|_{D(A^{1+\frac{\beta}{2}})}^2 + \rho^2 \right).
\]
Therefore, if we take \( T = T_1 + T_2 + T_3 \), for some \( T_3 \) to be determined, from (B.5), (B.6) and (B.7), we obtain
\[
S_{T_1+T_2,T_1+T_2+T_3}^\delta(z_4) \leq c \left[ \sqrt{I - e^{-T_3 A}} A^{\frac{\beta+1}{2}} x_\delta \right]_{H}^2 + \frac{1}{T_3} \rho^2 + c T_3 \left( \left| x_\delta \right|_{D(A^{1+\frac{\beta}{2}})}^2 + 1 \right) < \frac{\eta}{12},
\]
Now, if we take \( T_3 > 0 \) small enough such that
\[
c \left[ \sqrt{I - e^{-T_3 A}} A^{\frac{\beta+1}{2}} x_\delta \right]_{H}^2 + c T_3 \left( \left| x_\delta \right|_{D(A^{1+\frac{\beta}{2}})}^2 + 1 \right) < \frac{\eta}{12},
\]
and, in correspondence of such \( T_3, \rho < 1 \) such that
\[
\frac{1}{T_3} \rho^2 < \frac{\eta}{12},
\]
we conclude that
\[
S_{T_1+T_2,T_1+T_2+T_3}^\delta(z_4) \leq \frac{\eta}{6}.
\]
Therefore, if we define
\[
z(t) = \begin{cases} 
z_1(t), & t \in [0, T_1], 
z_2(t), & t \in [T_1, T_1 + 1], 
z_3(t), & t \in [T_1 + 1, T_1 + T_2], 
z_4(t), & t \in [T_1 + T_2, T_1 + T_2 + T_3],
\end{cases}
\]
we get $z \in C([0, T_1 + T_2 + T_3]; H)$, with $z(0) = x$ and $z(T_1 + T_2 + T_3) = \bar{x}_\delta$ and

$$S^\delta_{0, T_1 + T_2 + T_3}(z) \leq U_\delta(x) + \frac{\eta}{2} < \inf_{x \in \partial D} U_\delta(x) + \eta.$$ 

**Proof of Lemma 9.3 and lemma 9.4.** The proof of these two lemmas is analogous to the proof of [9, Lemmas 7.3, 7.4 and 7.5] and is based on the validity of a large deviation principle for the 2-D Navier-Stokes equation perturbed by additive noise, as proved in Theorem 5.2, and on Lemma 9.2. The arguments used in [9] are an adaptation to an infinite dimensional setting of the methods used in [15, Chapter 5]. □

**Proof of Lemma 9.5.** Let us fix $x \in H$ and $\delta \in (0, 1]$. For $\varepsilon > 0$, let us now denote by $z^\varepsilon$ the Ornstein-Uhlenbeck process defined by equation (5.4) and by $u^\varepsilon$ the solution to the stochastic Navier-Stokes Equation (5.1). By applying from Theorem 1.2 [8] (with $\xi(t)$ being the $\gamma$-radonifying natural embedding operator from $Q_{\delta}(H)$ to $H \cap L^4(O)$) we infer that there exists a constant $C > 0$ such that

$$\varepsilon \log \mathbb{P}(\|z^\varepsilon\|_{C([0, T_0]; L^4(O))} \geq R) \leq -\frac{R^2}{CT_0}, \quad R > 0, \quad \varepsilon > 0.$$ (B.9)

Let us now fix $\rho > 0$ and $\lambda > 0$. By the above inequality there exists $T_0$ such that

$$\varepsilon \log \mathbb{P}(\|z^\varepsilon\|_{C([0, T_0]; L^4(O))} \geq \frac{\rho}{3}) \leq -\frac{\lambda}{2} = \varepsilon > 0.$$ (B.10)

For a given $z \in H$ we denote by $v^z$ the unique solution to the problem

$$\frac{dv^z(t)}{dt} + Av^z(t) + B(v^z(t) + z(t), v^z(t) + z(t)) = 0, \quad t \in [0, T_0], \quad v^z(0) = x.$$ (B.11)

Note that $v^0$ is the unique solution to the deterministic NSEs with the initial data $v^0(0) = x$. Hence $v^0 \in C([0, T], H)$ and so by choosing $T_0$ smaller than before we can assume that

$$\|v^0 - x\|_{C([0, T_0]; H)} < \frac{\rho}{3}.$$ (B.12)

By [7, Theorem 4.6] we infer that there exists $\beta > 0$ such that

$$\|z\|_{C([0, T_0]; L^4(O))} < \beta \implies \|v^z - v^0\|_{C([0, T_0]; H)} < \frac{\rho}{3}.$$ (B.13)

By a simple uniqueness argument, the above holds with the same constant for all $T \in (0, T_0]$, i.e. there exists $\beta > 0$ such that for every $T \in (0, T_0]$,

$$\|z\|_{C([0, T]; L^4(O))} < \beta \implies \|v^z - v^0\|_{C([0, T]; H)} < \frac{\rho}{3},$$

$$\|v^0 - x\|_{C([0, T]; H)} < \frac{\rho}{3}.$$ (B.14)

Since, see [7], $u^\varepsilon - x = z^\varepsilon + v^z - v^0 + v^0 - x$, we infer that for every $\varepsilon > 0$,
$$\varepsilon \log P\left( |u^\varepsilon - x|_{C([0,T];H)} \geq \rho \right) \leq \varepsilon \log P\left( |z^\varepsilon|_{C([0,T];H)} \geq \frac{\rho}{3} \right)$$

$$+ \varepsilon \log P\left( |v^z - v^0|_{C([0,T];H)} \geq \frac{\rho}{3} \right) + \varepsilon \log P\left( |v^0 - x|_{C([0,T];H)} \geq \frac{\rho}{3} \right).$$

Let us note that by the second part of (B.15), the last term on the RHS of inequality (B.15) is equal to 0.

In order to estimate the first term on the RHS of inequality (B.15) let us choose $T \leq T_0$ such that

$$\frac{\beta^2}{CT} \geq \frac{\lambda}{2}$$

and then apply inequality (B.9) with $R = \beta$. We get that $\varepsilon \log P\left( |z^\varepsilon|_{C([0,T];H)} \geq \frac{\rho}{3} \right) \leq -\frac{\lambda}{2}$. In order to estimate the second term on the RHS of inequality (B.15) we use inequalities (B.14) and (B.10). Thus we deduce that

$$\varepsilon \log P\left( |u^\varepsilon - x|_{C([0,T];H)} \geq \rho \right) \leq \varepsilon \log P\left( |z^\varepsilon|_{C([0,T];H)} \geq \frac{\rho}{3} \right)$$

$$+ \varepsilon \log P\left( |z^\varepsilon|_{C([0,T_0];L^1(\mathcal{O}))} \geq \beta \right) \leq -\frac{\lambda}{2} - \frac{\lambda}{2} = -\lambda.$$

This completes the proof.

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