THE COSMIC GALOIS GROUP AS KOSZUL DUAL TO WALDHAUSEN’S $A(\ast)$

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Abstract. The world is so full of a number of things
I’m sure we shall all be as happy as kings.

Robert Louis Stevenson, A Child’s Garden of Verses

1. Basic Questions

1.1 Existence: Why is there something, rather than nothing?
This does not seem very accessible by current methods. A more realistic
goal may be

Classification: Given that there’s something, what could it be?
This suggests a

Program: If things fall into categories $(\mathcal{A}, \mathcal{B}, \ldots)$, hopefully small and
stable enough to be manageable, techniques from K-theory may be useful.

1.2 In fact Blumberg, Gepner, and Tabuada ([4], see also [10]) have con-
structed a Cartesian closed category $\mathsf{Cat}_{\infty}^{\text{perf}}$ of small stable $\infty$-categories,
egg $\mathcal{A}, \mathcal{B}, \mathsf{Fun}_{\text{ex}}^{\text{ex}}(\mathcal{A}, \mathcal{B}), \ldots$ and there is then a (similarly Cartesian closed) big
spectral category of pre-motives: with objects as above, and morphism
objects

$$\text{Hom}_{\mathsf{Mot}}(\mathcal{A}, \mathcal{B}) := K(\mathsf{Fun}_{\text{ex}}(\mathcal{A}, \mathcal{B})) \in K(\mathsf{S}) - \text{Mod}$$

enriched over Waldhausen’s $A$-theory spectrum. [The superscript ‘ex’ signi-
fies functors which preserve finite limits and colimits, and the objects of the
category are taken to be idempotent complete (ie, the category is suitably
localized with respect to Morita equivalence).]
Such a category has a functorial completion to a pre-triangulated category $\text{Mot}$ ([6 §4.5]: ie, whose homotopy category is triangulated); this involves enlarging the set of objects by adjoining suitable cofibers, generalizing the classical Karoubification in Grothendieck’s original construction of a category of pure motives.

1.3 Such ‘big’ categories allow comparisons between objects from quite different areas of mathematics (eg homotopy theory and algebraic geometry), and they raise a host of questions.

This posting summarizes a talk at the Hamburg 2011 conference on structured ring spectra

[http://www.math.uni-hamburg.de/home/richter/hh2011.html](http://www.math.uni-hamburg.de/home/richter/hh2011.html).

It is concerned with the motivic (Tannakian? Galois? descent?) groups of such categories as a tool for sorting out their relations. It is a report on work in progress with Andrew Blumberg and Kathryn Hess, without whose support it would not be even a fantasy. I also want to thank Michael Ching, Ralph Cohen, Bjorn Dundas, and Bill Dwyer for their help, and in particular for enduring more than their share of foolish questions. Finally, much of this work is based on ideas of Andrew Baker and Birgit Richter, and I owe them thanks for interesting conversations over many years, and in particular for putting together this remarkable meeting.

2. Some examples

(of things that live in this big world of motives):

2.1 If $X$ is an algebraic variety over a field $k$, and $\mathcal{A}_X = D_{\text{perf}}(\mathcal{O}_X)$ is the derived category of quasicoherent sheaves of $\mathcal{O}_X$-modules, then the class of $\mathcal{A}_X$ is a version of the classical motive of $X$. The subcategory generated by such things has Hom-objects naturally enriched over $K(k - \text{Mod})$; a cycle map associates to a subvariety $Z$ of $X \times Y$, a resolution of its defining sheaf $I_Z$ of functions, and thus a bimodule morphism from $X$ to $Y$ . . .

2.2 This example fits in the general framework of $\mathbb{A}^1$-homotopy theory, but over more general rings the subject is in flux. If $X$ is an arithmetic variety, eg over the spectrum of integers of a number field, Deligne and Goncharov [9] have constructed a good category of mixed Tate motives over $\text{Spec} \ Z$, with Hom objects enriched over $K(\mathbb{Z}) \otimes \mathbb{Q}$. The periods of algebraic varieties [12] define similar categories of motives.

2.3 There is a great deal of interest in noncommutative motives over a field, perhaps also represented by suitable derived categories of perfect objects [1] . . .
but my concern in this talk is to ask how the most classical example of all,

2.4 topological spaces

might fit in this framework. In particular, in this new world of big motives, how does the ‘underlying space’ or ‘Betti’ functor

\[ X \in \text{Varieties over } \mathbb{Z} \mapsto X(\mathbb{C}) \in \text{Spaces} \]

behave? This reality check is the principal motivation for this talk.

3. Fiber functors and their motivic automorphism groups

3.1 There are dual approaches [3,5,11] to the study of spaces in this context, both involving categories of modules over ring-spectra:

\[ X \mapsto [\Omega X_+] = FX \in A_{\infty} \text{- algebras}, \]

and

\[ X \mapsto [X_+, \underline{\alpha}] = DX \text{ (Spanier – Whitehead dual)} \in E_{\infty} \text{- algebras}. \]

The first leads to Waldhausen’s \(A(X) = K(\Omega X_+ - \text{Mod})\), while the second leads to Williams’ \(\forall(X) = K(DX - \text{Mod})\); together these constructions generalize Grothendieck’s classical covariant and contravariant versions of K-theory.

Both \(DX\) and \(FX\) are supplemented \(\$\)-algebras, and in good cases (ie if \(X\) is both finite and simply-connected) then

\[ FX \cong \text{Hom}_{\$}(DX, DX), \quad DX \cong \text{Hom}_{\$}(FX, FY) \]

expresses a kind of ‘double centralizer’ duality.

3.2 Here I’ll work with the second of these alternatives, in the category with finite \(CW\)-spaces \(X, Y\) as objects, and morphisms

\[ \text{Hom}_{\text{Mot}}(X, Y) \sim K(DX \land DY^{\text{op}} - \text{Mod}) \]

defined by the K-theory spectra of \(\text{right-compact } DX - DY^{\text{op}}\)-bimodules [4 §2.16]. This category can then be made pre-triangulated, as above.

There are many technical variants of this construction: for example, BGT consider both Karoubi-Villamayor and Bass-Thomason K-theory. Later we will want to modify categories of this sort by completing their morphism objects in various ways, and eventually we will be interested in constructions based on THH and its relatives (TR, TC, …); then I’ll label the resulting categories by the functors defining their morphism objects.
For example, the cyclotomic trace defines a monoidal spectral functor
\[ \text{Mot}_K \to \text{Mot}_{TC} \]
of pre-triangulated categories (and hence a triangulated functor between their homotopy categories).

### 3.3 Tannakian analogs of Galois groups are a central topic in the usual theory of motives: complicated categories can sometimes be identified, via some kind of descent, with categories of representations of groups of automorphisms of interesting forgetful (monoidal, ‘fiber’) functors to simpler categories. Weil cohomologies (Hodge, étale, crystalline) are classical examples, but the following example may be more familiar here:

Ordinary cohomology (with coefficients in \( \mathbb{F}_2 \) and the grading neglected), viewed as a monoidal functor
\[ H : (\text{Spectra}) \ni X \mapsto H^*(X, \mathbb{F}_2) \in (\mathbb{F}_2 - \text{Vect}) , \]
has a group-valued functor
\[ \text{Aut}^H : (\mathbb{F}_2 - \text{Alg}) \ni A \mapsto \text{Aut}_A^A(H^*(-, A)) \]
of natural automorphisms, which is (co)represented by the dual Steenrod algebra:
\[ \text{Aut}_A^A(H^*(-, A)) \cong \text{Hom}_{\text{Alg}}(A^*, A) . \]
The vector-space valued functor \( H^* \) thus lifts to a functor taking values in representations of a proalgebraic groupscheme, or (in more familiar language), in the category of \( A^* \)-comodules.

Here I want to look at (pre-triangulated, spectral, monoidal) categories built by reducing the morphism objects in BGT-style categories modulo the kernel of the Dennis trace \( K(\$) \to \$ \) (much as we can consider the category obtained from chain complexes over \( \mathbb{Z} \) by reducing their internal Hom-objects modulo \( p \)).

### 3.4 Hess’s theory of homotopical descent [14] provides us with the needed technology: a cofibrant replacement

\[ K(\$) \xrightarrow{\text{tr}} \$ \xrightarrow{\rho} Q(\$) \]
(of the sphere spectrum \( \$ \) as \( K(\$) \)-algebra\(^1\) with \( \tau \) a cofibration, and \( \rho \) a weak equivalence) associates a ‘Hessian’ co-ring spectrum
\[ Q(\$) \wedge_{K(\$)} Q(\$) (= \text{THH}_{K(\$)}(\$) ) \]

\(^1\)Note that \( K(\mathbb{Z}) \) is not similarly supplemented over \( \mathbb{Z} \! \)!
(analogous to a Hopf-Galois object in the sense of Rognes [17]) to the Dennis trace.

Similarly,

\[ \mathcal{S} \to HF_2 \]

produces the dual Steenrod algebra

\[ Q(HF_2) \wedge_{\mathcal{S}} Q(HF_2) \sim A^*. \]

The resulting theory of descent relates a \( K(\mathcal{S}) \)-module spectrum \( V \) to a

\[ \text{THH}_{K(\mathcal{S})}(\mathcal{S}) := \mathcal{S} \wedge_{K(\mathcal{S})} V = \text{THH}(K(\mathcal{S}), V) , \]

and

\[ K(DX \wedge D Y^{op}) \to K(DX \wedge D Y^{op}) \wedge_{K(\mathcal{S})} := K(\wedge_{K(\mathcal{S})} DX \wedge D Y^{op}) \]

defines a monoidal functor

\[ \omega_{K(\mathcal{S})} : \text{Mot}_{K(\mathcal{S})} \to \text{Mot}_{K(\mathcal{S})} , \]

the latter category being enriched over spectra with an \( \mathcal{S} \wedge_{K(\mathcal{S})} \) - comodule

action (the analog of representations of \( \text{Aut}(\omega_{K(\mathcal{S})}) \)).

We expect a more careful version of this construction to provide effective homotopical descent for a version of \( \text{Mot}_{\mathcal{S}} \) with suitably completed morphism objects [14 §4, §5.5].

3.5 The notation above is unsatisfactory: it reflects similar difficulties with notation for Koszul duality. In the classical case of a morphism \( A \to B \) of algebras over a field \( k \), the covariant functor

\[ V \mapsto V \wedge_A^L B := V_B : D(A - \text{Mod}) \to D(A \wedge_B - \text{Comod}) \]

has a contravariant \( k \)-vector-space dual

\[ V \mapsto V_B^\dagger := (V_B)^* \cong \text{RHom}_A(V, B) \]

with values in some derived category of \( \text{RHom}_A(B, B) := A_B^L \)-modules [Cartan-Eilenberg VI §5], which is in good cases a (Koszul) duality. In the formulation above,

\[ \mathcal{S} \wedge_{K(\mathcal{S})} \mathcal{S} := \mathcal{S} \wedge_{K(\mathcal{S})} \mathcal{S} = \text{THH}(K(\mathcal{S}), \mathcal{S}) \]

is the analog of the algebra of functions on a group object, while

\[ \mathcal{S} \wedge_{K(\mathcal{S})} \mathcal{S} := \text{RHom}(K(\mathcal{S}), \mathcal{S}, \mathcal{S}) \]

is the analog of its (convolution, \( L^1 \)) group algebra.
4. CYCLOTOMIC VARIANTS

4.1 The constructions above have a straightforward analog

$$\text{Mot}_{\text{TC}} \to \text{Mot}_{\text{TC}^\dagger}$$

built from topological cyclic homology; where now

$$\text{TC}^\dagger(-) := \text{THH}_{\text{TC}(\$)}(\text{TC}(-)) \in \text{THH}_{\text{TC}(\$)}(\$) := \$\text{TC}^\dagger - \text{Comod}$$

(with profinite completions implicit but suppressed).[^1]

The cyclotomic trace

$$K(\$) \to \text{TC}(\$) \sim \$ \vee \Sigma \mathbb{C}P^\infty_1$$

(again mod completion) identifies the K-theory spectrum with \$ \vee \Sigma \mathbb{H}P^\infty_+ at regular odd primes [15, 18]. The cofibration

$$S^{-1} \to \Sigma \mathbb{C}P^\infty_1 \to \Sigma \mathbb{C}P^\infty_+$$

suggests that the Koszul dual of \(\text{THH}_{\text{TC}(\$)}\$ should be close to the tensor \$ \text{-algebra} \$\Omega \Sigma \mathbb{C}P^\infty_+ on \mathbb{C}P^\infty_+ [2]. In any case, \(\$\text{THH}^\dagger(\$) \otimes \mathbb{Q}\) can be identified with the algebra of quasisymmetric functions over \(\mathbb{Q}\), ie the algebra of functions on a pro-unipotent group with free Lie algebra. The cyclic structure on THH endows this Lie algebra with a \(T\)-action and thus with a grading, placing one generator in each odd degree [7].

This is very similar to Deligne’s motivic group for the category of mixed Tate motives, itself modeled on Shafarevich’s conjectured description of the absolute Galois group of \(\mathbb{Q}\) as a profree profinite extension of \(\hat{\mathbb{Z}}^\times\). It leads to the appearance of odd zeta-values in differential topology, systematically parallel to the appearance of even zeta-values (ie, Bernoulli numbers) in homotopy theory.

4.2 One concern with these constructions is that neither \(K\) nor \(\text{TC}\) is linear, in the sense of the calculus of functors.

\(\text{THH}_\$\(DX\)) is the realization of a cyclic object

$$n \mapsto (DX)^\wedge(n+1) \sim D(X^{n+1})$$

\(S\)-dual to the totalization of a (cocyclic) cosimplicial space modelling the free loop space \(LX\) (cf [13]; thanks to WD for this reference!). My hope is that the homotopy fixed points \(\text{THH}_\$\(DX\)^\text{ht}\) can be identified as something like

$$[ET^\dagger_+, [LX^\dagger_+, \$]]^\text{ht} = [LX^\text{ht}^\dagger_+, \$] = [LX^\dagger_+, [ET^\dagger_+, \$]]^\text{ht}$$

and that consequently \(TC(DX)\) will be accessible as a homotopy limit of things like \([LX^\dagger_+, \text{THH}_\$\(\$\)]^\text{ht}\).

[^1]: Another interesting variant can be built from THH, regarded as a \(T\)-equivariant spectrum.
This suggests that the inclusion $X \to LX$ of fixed points defines a kind of coassembly [20] map

$$TC(DX) \to [X_+, TC(\mathbb{S})]$$

as a $TC(\text{holim}) \to \text{holim}(TC)$ interchange. [The classical assembly map defines a composition

$$\text{Hom}_K(\mathbb{S}(\Omega X), \mathbb{S}) \to \text{Hom}_K(\mathbb{S}(\Omega X), \mathbb{S}) \sim DX \ldots]$$

4.3 If this is so, then we can add a third step

$$\text{Mot}_{TC} \to \text{Mot}_{TC} \to \text{Motlin}_{TC},$$

to the sequence of pre-triangulated monoidal functors above, with

$$\text{Homlin}_{TC}(X, Y) = \text{THH}_{TC}(\mathbb{S}(\mathbb{S}(\Omega X), \mathbb{S}), \text{holim}(TC(\mathbb{S}))) \in \mathbb{S}_{\text{holim}} - \text{Comod}.$$ Note that

$$\text{Homlin}_{TC}(X, Y) \otimes \mathbb{Q} = \text{HH}_{TC}(\mathbb{S}(\mathbb{S}(\Omega X), \mathbb{S}), H^*(Y \wedge DX))$$

$$= H^*(Y \wedge DX, \mathbb{Q}) = [Y, X]_{\mathbb{Q}},$$

so the rationalization of $\text{Motlin}_{TC}$ reduces to the (rationalized) category of finite spectra, (conjecturally!) reconciling the motive of an algebraic variety with the stable homotopy type of its underlying space. More generally,

$$[X, K(\mathbb{S})]_{K(\mathbb{S})} \sim [X, \mathbb{S}] \ldots$$

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