Hom-L-R-smash products, Hom-diagonal crossed products and the Drinfeld double of a Hom-Hopf algebra

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Abstract

We introduce the Hom-analogue of the L-R-smash product and use it to define the Hom-analogue of the diagonal crossed product. When $H$ is a finite dimensional Hom-Hopf algebra with bijective antipode and bijective structure map, we define the Drinfeld double of $H$; its algebra structure is a Hom-diagonal crossed product and it has all expected properties, namely it is quasitriangular and modules over it coincide with left-right Yetter-Drinfeld modules over $H$.

Introduction

Hom-type algebras appeared first in physical contexts, in connection with twisted, discretized or deformed derivatives and corresponding generalizations, discretizations and deformations of vector fields and differential calculus (see [1, 2, 11, 12, 13, 14, 18, 19, 20, 26, 27, 30]). These papers dealt mainly with $q$-deformations of Heisenberg algebras (oscillator algebras), the Virasoro algebra and quantum conformal algebras, applied in Physics within string theory, vertex operator models, quantum scattering, lattice models and other contexts.

In [24, 29] the authors showed that a new quasi-deformation scheme leads from Lie algebras to a broader class of quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras. The study of the class of Hom-Lie algebras, generalizing usual Lie algebras and where the Jacobi identity is twisted by a linear map, has become an active area of research. The corresponding associative algebras, called Hom-associative algebras, where introduced in [34] and it was shown that a commutator of a Hom-associative algebra gives rise to a Hom-Lie algebra. For further results see [3, 22, 23, 33, 35, 36, 37]. The coalgebra counterpart and the related notions of Hom-bialgebra and Hom-Hopf algebra were introduced in [36, 37] and some of their properties, extending properties of bialgebras and Hopf algebras, were described.

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The original definitions of Hom-bialgebra and Hom-Hopf algebra involve two different linear maps $\alpha$ and $\beta$, with $\alpha$ twisting the associativity condition and $\beta$ the coassociativity condition. Afterwards, two directions of study were developed, one considering the class such that $\beta = \alpha$, which are still called Hom-bialgebras and Hom-Hopf algebras (cf. [21, 35, 39, 41, 46, 47, 48]) and another one, initiated in [9], where the map $\alpha$ is assumed to be invertible and $\beta = \alpha^{-1}$ (these are called monoidal Hom-bialgebras and monoidal Hom-Hopf algebras). Yetter-Drinfeld modules, integrals, the Drinfeld double and Radford’s biproduct have been studied for monoidal Hom-bialgebras in [13, 16, 31]. Yetter-Drinfeld modules over Hom-bialgebras were studied in [38] and we will introduce the Drinfeld double in this paper. Since Hom-bialgebras and monoidal Hom-bialgebras are different concepts, it turns out that our definitions, formulae and results are also different from the ones in [13, 31].

One of the main tools to construct examples of Hom-type algebras is the so-called ”twisting principle” which was introduced by D. Yau for Hom-associative algebras and since then extended to various Hom-type algebras. It allows to construct a Hom-type algebra starting from a classical-type algebra and an algebra homomorphism.

The twisted tensor product $A \otimes_R B$ of two associative algebras $A$ and $B$ is a certain associative algebra structure on the vector space $A \otimes B$, defined in terms of a so-called twisting map $R : B \otimes A \rightarrow A \otimes B$, having the property that it coincides with the usual tensor product algebra $A \otimes B$ if $R$ is the usual flip map. This construction was introduced in [10, 12] and it may be regarded as a representative for the Cartesian product of noncommutative spaces, see [28, 32] for more on this subject. In [39] we generalized this construction to Hom-associative algebras: if $R$ is a linear map $R : B \otimes A \rightarrow A \otimes B$ between two Hom-associative algebras $A$ and $B$ satisfying some conditions (such an $R$ is called Hom-twisting map), we can construct the so-called Hom-twisted tensor product $A \otimes_R B$, which is a Hom-associative algebra.

The L-R-smash product over a cocommutative Hopf algebra was introduced and studied in a series of papers [4, 5, 6, 7], inspired by the theory of deformation quantization. This construction was substantially generalized in [40], as follows: if $H$ is a bialgebra or quasi-bialgebra and $D$ is an $H$-bimodule algebra, the L-R-smash product is a certain associative algebra structure on $D \otimes H$, denoted by $D \rhd \rhd H$. A common generalization of twisted tensor products of algebras and L-R-smash products over bialgebras was introduced in [17], under the name L-R-twisted tensor product of algebras.

The diagonal crossed product ([9, 25]) is a construction that associates to a Hopf or quasi-Hopf algebra $H$ with bijective antipode and to an $H$-bimodule algebra $D$ a certain associative algebra structure on $D \otimes H$, denoted by $D \bowtie H$. It was proved in [10] that we actually have an algebra isomorphism $D \bowtie H \simeq D \bowtie H$. The importance of the diagonal crossed product stems from the fact that, if $H$ is finite dimensional and $D = H^*$, then $H^* \bowtie H$ is the algebra structure of the Drinfeld double of $H$.

The ultimate aim of this paper is to construct the Drinfeld double of a finite dimensional Hom-Hopf algebra $H$ with bijective antipode and bijective structure map. We can expect from the beginning that its algebra structure has to be a Hom-analogue of a diagonal crossed product between $H^*$ and $H$, but it is not clear at all a priori how to define this Hom-analogue. So, we proceed as follows. We define first the Hom-analogue of the L-R-twisted tensor product of algebras, which is a natural generalization of the Hom-twisted tensor product. It is defined as follows: if $A$ and $B$ are two Hom-associative algebras and $R : B \otimes A \rightarrow A \otimes B$ and $Q : A \otimes B \rightarrow A \otimes B$ are linear maps satisfying some conditions, the Hom-L-R-twisted tensor product $A \otimes_R B$ is a certain Hom-associative algebra structure on $A \otimes B$. The key result is Proposition 2.3, saying that if $Q$ is bijective then the map $P := Q^{-1} \circ R : B \otimes A \rightarrow A \otimes B$ is a Hom-twisting map (and
we have an algebra isomorphism $A_Q \otimes_R B \simeq A \otimes_P B$). Now if $H$ is a Hom-bialgebra and $D$ is an $H$-bimodule Hom-algebra, we can define in a natural way a Hom-L-R-twisted tensor product $D_Q \otimes_R H$, which is denoted by $D \bowtie H$ and called the Hom-L-R-smash product. It turns out that, under some extra hypotheses (among them, the existence of a bijective antipode on $H$), the map $Q$ is bijective, so we have the Hom-twisted tensor product $D \otimes_P H$, where $P = Q^{-1} \circ R$; this will be the Hom-diagonal crossed product $D \bowtie H$ we are looking for. Moreover, if $H$ is a finite dimensional Hom-Hopf algebra with bijective antipode and bijective structure map, we can build such a Hom-diagonal crossed product $H^* \bowtie H$, and this will be the algebra structure of the Drinfeld double $D(H)$.

To find the rest of the structure of $D(H)$, we define left-right Yetter-Drinfeld modules over $H$, note that they form a braided monoidal category (we analyzed this in detail for left-left Yetter-Drinfeld modules in [38]), prove that the category of modules over $D(H)$ is isomorphic to the category of left-right Yetter-Drinfeld modules and then transfer all the structure from $H \otimes D^H$ to $D(H)$. It turns out that $D(H)$ is a quasitriangular Hom-Hopf algebra, as expected.

## 1 Preliminaries

We work over a base field $k$. All algebras, linear spaces etc... will be over $k$; unadorned $\otimes$ means $\otimes_k$. For a comultiplication $\Delta : C \to C \otimes C$ on a vector space $C$ we use a Sweedler-type notation $\Delta(c) = c_1 \otimes c_2$, for $c \in C$. Unless otherwise specified, the (co)algebras ((co)associative or not) that will appear in what follows are not supposed to be (co)unital, and a multiplication $\mu : V \otimes V \to V$ on a linear space $V$ is denoted by juxtaposition: $\mu(v \otimes v') = vv'$.

We recall some concepts and results, fixing the terminology to be used throughout the paper. For Hom-structures, we use terminology as in our previous papers [38], [39].

**Proposition 1.1** ([77]) Let $A$ and $B$ be two associative algebras and $R : B \otimes A \to A \otimes B$, $Q : A \otimes B \to A \otimes B$ two linear maps, for which we use a Sweedler-type notation $R(b \otimes a) = a_R \otimes_R b_R = a r \otimes r$, and $Q(a \otimes b) = a_Q \otimes_Q b_Q = a q \otimes q$, for all $a \in A, b \in B$, satisfying the following conditions, for all $a, a' \in A$ and $b, b' \in B$:

\[
\begin{align*}
(aa')_R \otimes_R b_R &= a_R a'_R \otimes_R (b_R)_r, \\
a_R \otimes_R (bb')_R &= (a_R)_r \otimes_R b_R b'_R, \\
(aa')_Q \otimes_Q b_Q &= a_Q a'_Q \otimes_Q (b_Q)_q, \\
(aQ) \otimes_Q (bb')_Q &= (aQ)_q \otimes_Q b_Q b'_Q, \\
b_R \otimes_R (aQ)_Q \otimes Q b'_Q &= b_R \otimes_R (aQ)_R \otimes Q b'_Q, \\
aR \otimes_R (bR)_Q \otimes Q a'_Q &= aR \otimes_Q (bQ)_R \otimes Q a'_Q.
\end{align*}
\]

If we define on $A \otimes B$ a multiplication by $(a \otimes b)(a' \otimes b') = a_Q a'_R \otimes_R b_R b'_Q$, then this multiplication is associative. This algebra structure will be denoted by $A_Q \otimes_R B$ and will be called the L-R-twisted tensor product of $A$ and $B$ afforded by the maps $R$ and $Q$. In the particular case $Q = id_{A \otimes B}$, the L-R-twisted tensor product $A_Q \otimes_R B$ reduces to the twisted tensor product $A \otimes_R B$ introduced in [10], [33], whose multiplication is defined by $(a \otimes b)(a' \otimes b') = a a'_R \otimes_R b_R b'$.

**Example 1.2** Let $H$ be a bialgebra and $D$ an $H$-bimodule algebra in the usual sense, with actions $H \otimes D \to D, h \otimes d \mapsto h \cdot d$ and $D \otimes H \to D, d \otimes h \mapsto d \cdot h$. Define the linear maps

\[R : H \otimes D \to D \otimes H, \quad R(h \otimes d) = h_1 \cdot d \otimes h_2,\]
Q : D ⊗ H → D ⊗ H, \ Q(d \otimes h) = d \cdot h_2 \otimes h_1.

Then we have an L-R-twisted tensor product \( D \otimes_R H \), which is denoted by \( D \sharp H \) and is called the L-R-smash product of \( D \) and \( H \) (cf. [40]). If we denote \( d \otimes h := d \sharp h \), for \( d \in D \), \( h \in H \), the multiplication of \( D \sharp H \) is given by

\[(d \sharp h)(d' \sharp h') = (d \cdot h_2')(h_1 \cdot d') \sharp h_1 h'.\]

If \( H \) is moreover a Hopf algebra with bijective antipode, we can define as well the so-called diagonal crossed product \( D \bowtie H \) (cf. [23], [8]), an associative algebra structure on \( D \otimes H \) whose multiplication is defined (we denote \( d \otimes h := d \bowtie h \)) by

\[(d \bowtie h)(d' \bowtie h') = d(h_1 \cdot d' \cdot S^{-1}(h_3)) \bowtie h_2 h'.\]

If the action of \( H \) on \( D \) is unitail, we have \( D \sharp H \simeq D \bowtie H \), cf. [40].

**Definition 1.3** (i) A Hom-associative algebra is a triple \((A, \mu, \alpha)\), in which \( A \) is a linear space, \( \alpha : A \to A \) and \( \mu : A \otimes A \to A \) are linear maps, with notation \( \mu(a \otimes a') = aa' \), such that

\[\alpha(aa') = \alpha(a)\alpha(a'), \quad \text{(multiplicativity)}\]

\[\alpha(a)(a'a'') = (aa')\alpha(a''), \quad \text{(Hom - associativity)}\]

for all \( a, a', a'' \in A \). We call \( \alpha \) the structure map of \( A \).

A morphism \( f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B) \) of Hom-associative algebras is a linear map \( f : A \to B \) such that \( \alpha_B \circ f = f \circ \alpha_A \) and \( f \circ \mu_A = \mu_B \circ (f \otimes f) \).

(ii) A Hom-coassociative coalgebra is a triple \((C, \Delta, \alpha)\), in which \( C \) is a linear space, \( \alpha : C \to C \) and \( \Delta : C \to C \otimes C \) are linear maps (\( \alpha \) is called the structure map of \( C \)) such that

\[\alpha(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha, \quad \text{(comultiplicativity)}\]

\[\Delta(\alpha \otimes \alpha) = (\alpha \otimes \Delta) \circ \Delta. \quad \text{(Hom - coassociativity)}\]

A morphism \( g : (C, \Delta_C, \alpha_C) \to (D, \Delta_D, \alpha_D) \) of Hom-coassociative coalgebras is a linear map \( g : C \to D \) such that \( \alpha_D \circ g = g \circ \alpha_C \) and \( (g \otimes g) \circ \Delta_C = \Delta_D \circ g \).

**Remark 1.4** Assume that \((A, \mu_A, \alpha_A)\) and \((B, \mu_B, \alpha_B)\) are two Hom-associative algebras; then \((A \otimes B, \mu_{A\otimes B}, \alpha_A \otimes \alpha_B)\) is a Hom-associative algebra (called the tensor product of \( A \) and \( B \)), where \( \mu_{A\otimes B} \) is the usual multiplication: \((a \otimes b)(a' \otimes b') = aa' \otimes bb'\).

**Definition 1.5** Let \((A, \mu_A, \alpha_A)\) be a Hom-associative algebra, \( M \) a linear space and \( \alpha_M : M \to M \) a linear map.

(i) \((43), (48)\) A left \( A \)-module structure on \((M, \alpha_M)\) consists of a linear map \( A \otimes M \to M \), \( a \otimes m \mapsto a \cdot m \), satisfying the conditions (for all \( a, a' \in A \) and \( m \in M \))

\[\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m), \quad (1.7)\]

\[\alpha_A(a)(a' \cdot m) = (aa') \cdot \alpha_M(m). \quad (1.8)\]

(ii) \((39)\) A right \( A \)-module structure on \((M, \alpha_M)\) consists of a linear map \( M \otimes A \to M \), \( m \otimes a \mapsto m \cdot a \), satisfying the conditions (for all \( a, a' \in A \) and \( m \in M \))

\[\alpha_M(m \cdot a) = \alpha_M(m) \cdot \alpha_A(a), \quad (1.9)\]

\[(m \cdot a) \cdot \alpha_A(a') = \alpha_M(m) \cdot (aa'). \quad (1.10)\]

If \((M, \alpha_M)\), \((N, \alpha_N)\) are left (respectively right) \( A \)-modules (\( A \)-actions denoted by \( \cdot \)), a morphism of left (respectively right) \( A \)-modules \( f : M \to N \) is a linear map with \( \alpha_N \circ f = f \circ \alpha_M \) and \( f(a \cdot m) = a \cdot f(m) \) (respectively \( f(m \cdot a) = f(m) \cdot a \)), \( \forall a \in A, \ m \in M \).
Definition 1.6 ([39], [37]) A Hom-bialgebra is a quadruple \((H, \mu, \Delta, \alpha)\), in which \((H, \mu, \alpha)\) is a Hom-associative algebra, \((H, \Delta, \alpha)\) is a Hom-coassociative coalgebra and moreover \(\Delta\) is a morphism of Hom-associative algebras.

Thus, a Hom-bialgebra is a Hom-associative algebra \((H, \mu, \alpha)\) endowed with a linear map \(\Delta : H \to H \otimes H\), with notation \(\Delta(h) = h_1 \otimes h_2\), such that, for all \(h, h' \in H\), we have:

\[
\Delta(h_1) \otimes \alpha(h_2) = \alpha(h_1) \otimes \Delta(h_2),
\]

\[
\Delta(hh') = h_1 \otimes h'_2,
\]

\[
\Delta(\alpha(h)) = \alpha(h_1) \otimes \alpha(h_2).
\]

Proposition 1.7 ([37], [43]) (i) Let \((A, \mu)\) be an associative algebra and \(\alpha : A \to A\) an algebra endomorphism. Define a new multiplication \(\mu_\alpha := \alpha \circ \mu : A \otimes A \to A\). Then \((A, \mu_\alpha, \alpha)\) is a Hom-associative algebra, denoted by \(A_\alpha\).

(ii) Let \((C, \Delta)\) be a coassociative coalgebra and \(\alpha : C \to C\) a coalgebra endomorphism. Define a new comultiplication \(\Delta_\alpha := \Delta \circ \alpha : C \to C \otimes C\). Then \((C, \Delta_\alpha, \alpha)\) is a Hom-coassociative coalgebra, denoted by \(C_\alpha\).

(iii) Let \((H, \mu, \Delta)\) be a bialgebra and \(\alpha : H \to H\) a bialgebra endomorphism. If we consider the Hom-bialgebra \(\Delta(h) := \Delta(h_1) \otimes \alpha(h_2)\), for all \(h \in H\), then \(H_\alpha = (H, \mu_\alpha, \Delta_\alpha, \alpha)\) is a Hom-bialgebra.

Proposition 1.8 ([48]) Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra. If \((M, \alpha_M)\) and \((N, \alpha_N)\) are \(H\)-modules, then \((M \otimes N, \alpha_M \otimes \alpha_N)\) is also a \(H\)-module, with \(H\)-action defined by \(H \otimes (M \otimes N) \to M \otimes N, h \otimes (m \otimes n) \mapsto h \cdot (m \otimes n) := h_1 \cdot m \otimes h_2 \cdot n\).

Definition 1.9 ([43]) Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra. A Hom-associative algebra \((A, \mu_A, \alpha_A)\) is called a left \(H\)-module Hom-algebra if \((A, \alpha_A)\) is a left \(H\)-module, with action denoted by \(H \otimes A \to A, h \otimes a \mapsto h \cdot a\), such that the following condition is satisfied:

\[
\alpha^2_H(h) \cdot (aa') = (h_1 \cdot a)(h_2 \cdot a'), \quad \forall h \in H, a, a' \in A.
\]

Proposition 1.10 ([43]) Let \((H, \mu_H, \Delta_H)\) be a bialgebra and \((A, \mu_A)\) a left \(H\)-module algebra in the usual sense, with action denoted by \(H \otimes A \to A, h \otimes a \mapsto h \cdot a\). Let \(\alpha_H : H \to H\) be a bialgebra endomorphism and \(\alpha_A : A \to A\) an algebra endomorphism, such that \(\alpha_A(h \cdot a) = \alpha_H(h) \cdot \alpha_A(a)\), for all \(h \in H\) and \(a \in A\). If we consider the Hom-bialgebra \(H_{\alpha_H} = (H, \alpha_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H)\) and the Hom-associative algebra \(A_{\alpha_A} = (A, \alpha_A \circ \mu_A, \alpha_A)\), then \(A_{\alpha_A}\) is a left \(H_{\alpha_H}\)-module Hom-algebra in the above sense, with action denoted by \(H_{\alpha_H} \otimes A_{\alpha_A} \to A_{\alpha_A}, h \otimes a \mapsto h \circ a := \alpha_A(h \cdot a) = \alpha_H(h) \cdot \alpha_A(a)\).

Definition 1.11 ([39]) Assume that \((H, \mu_H, \Delta_H, \alpha_H)\) is a Hom-bialgebra. A Hom-associative algebra \((C, \mu_C, \alpha_C)\) is called a right \(H\)-module Hom-algebra if \((C, \alpha_C)\) is a right \(H\)-module, with action denoted by \(C \otimes H \to C, c \otimes h \mapsto c \cdot h\), such that the following condition is satisfied:

\[
(cc') \cdot \alpha^2_H(h) = (c \cdot h_1)(c' \cdot h_2), \quad \forall h \in H, c, c' \in C.
\]

Proposition 1.12 ([39]) Let \((H, \mu_H, \Delta_H)\) be a bialgebra and \((C, \mu_C)\) a right \(H\)-module algebra in the usual sense, with action denoted by \(C \otimes H \to C, c \otimes h \mapsto c \cdot h\). Let \(\alpha_H : H \to H\) be a bialgebra endomorphism and \(\alpha_C : C \to C\) an algebra endomorphism, such that \(\alpha_C(c \cdot h) = \alpha_C(c) \cdot \alpha_H(h)\), for all \(h \in H\) and \(c \in C\). Then the Hom-associative algebra \(C_{\alpha_C} = (C, \alpha_C \circ \mu_C, \alpha_C)\) becomes a right module Hom-algebra over the Hom-bialgebra \(H_{\alpha_H} = (H, \alpha_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H)\), with action defined by \(C_{\alpha_C} \otimes H_{\alpha_H} \to C_{\alpha_C}, c \otimes h \mapsto c \cdot h := \alpha_C(c \cdot h) = \alpha_C(c) \cdot \alpha_H(h)\).
Proposition 1.13 (39) Let \((D, \mu, \alpha)\) be a Hom-associative algebra and \(T : D \otimes D \to D \otimes D\) a linear map, with notation \(T(d \otimes d') = d^T \otimes d'^T\), for \(d, d' \in D\), satisfying the conditions
\[
(\alpha \otimes \alpha) \circ T = T \circ (\alpha \otimes \alpha),
\]
\[
T \circ (\alpha \otimes \mu) = (\alpha \otimes \mu) \circ T_{13} \circ T_{12},
\]
\[
T \circ (\mu \otimes \alpha) = (\mu \otimes \alpha) \circ T_{13} \circ T_{23},
\]
\[
T_{12} \circ T_{23} = T_{23} \circ T_{12},
\]
where we used a standard notation for the operators \(T_{ij}\), namely \(T_{12} = T \otimes \text{id}_D\), \(T_{23} = \text{id}_D \otimes T\) and \(T_{13}(d \otimes d' \otimes d'') = d^T \otimes d'^T \otimes d''^T\). Then \(D^T := (D, \mu \circ T, \alpha)\) is also a Hom-associative algebra. The map \(T\) is called a Hom-twistor.

Proposition 1.14 (39) Let \((A, \mu_A, \alpha_A)\) and \((B, \mu_B, \alpha_B)\) be two Hom-associative algebras and \(R : B \otimes A \to A \otimes B\) a linear map, with Sweedler-type notation \(R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r\), for \(a \in A, b \in B\). Assume that the following conditions are satisfied:
\[
\alpha_A(a_R) \otimes \alpha_B(b_R) = \alpha_A(a)_R \otimes \alpha_B(b)_R, \tag{1.20}
\]
\[
(aa')_R \otimes \alpha_B(b) = a_{R'} \otimes \alpha_B((b)_R), \tag{1.21}
\]
\[
\alpha_A(a)_R \otimes (bb')_R = \alpha_A((a)_R) \otimes b_r b'_r, \tag{1.22}
\]
for all \(a, a' \in A\) and \(b, b' \in B\) (such a map \(R\) is called a Hom-twisting map). If we define a new multiplication on \(A \otimes B\) by \((a \otimes b)(a' \otimes b') = aa'_{R'} \otimes b_r b'_r\), then \(A \otimes B\) becomes a Hom-associative algebra with structure map \(\alpha_A \otimes \alpha_B\), denoted by \(\alpha_{A \otimes B}\) and called the Hom-twisted tensor product of \(A\) and \(B\).

Theorem 1.15 (39) Let \((A, \mu_A, \alpha_A)\), \((B, \mu_B, \alpha_B)\) and \((C, \mu_C, \alpha_C)\) be three Hom-associative algebras and \(R_1 : B \otimes A \to A \otimes B\), \(R_2 : C \otimes B \to B \otimes C\), \(R_3 : C \otimes A \to A \otimes C\) three Hom-twisting maps, satisfying the braid condition
\[
(id_A \otimes R_2) \circ (R_3 \otimes id_B) \circ (id_C \otimes R_1) = (R_1 \otimes id_C) \circ (id_B \otimes R_3) \circ (R_2 \otimes id_A). \tag{1.23}
\]

Define the maps
\[
P_1 : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C, \quad P_1 = (id_A \otimes R_2) \circ (R_3 \otimes id_B),
\]
\[
P_2 : (B \otimes_{R_2} C) \otimes A \to A \otimes (B \otimes_{R_2} C), \quad P_2 = (R_1 \otimes id_C) \circ (id_B \otimes R_3).
\]
Then \(P_1\) is a Hom-twisting map between \(A \otimes_{R_1} B\) and \(C\), \(P_2\) is a Hom-twisting map between \(A\) and \(B \otimes_{R_2} C\), and the Hom-associative algebras \((A \otimes_{R_1} B) \otimes_{P_1} C\) and \(A \otimes_{P_2} (B \otimes_{R_2} C)\) coincide; this Hom-associative algebra will be denoted by \(A \otimes_{R_1} B \otimes_{R_2} C\) and will be called the iterated Hom-twisted tensor product of \(A, B, C\).

Proposition 1.16 (39) Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra, \((A, \mu_A, \alpha_A)\) a left \(H\)-module Hom-algebra and \((C, \mu_C, \alpha_C)\) a right \(H\)-module Hom-algebra, with actions denoted by \(H \otimes A \to A\), \(h \otimes a \mapsto h \cdot a\) and \(C \otimes H \to C\), \(c \otimes h \mapsto c \cdot h\), and assume that the structure maps \(\alpha_H, \alpha_A, \alpha_C\) are bijective. Then:
(i) We have the following Hom-twisting maps:
\[
R_1 : H \otimes A \to A \otimes H, \quad R_1(h \otimes a) = \alpha_H^2(h_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-1}(h_2),
\]
Thus, we can consider the Hom-associative algebras $A \otimes_{R_1} H$ and $H \otimes_{R_2} C$, which are denoted by $A\#H$ and respectively $H\#C$ and are called the left and respectively right Hom-smash products. If we denote $a \otimes h := a\#h$ and $h\#c := h \otimes c$, for $a \in A$, $h \in H$, $c \in C$, the multiplications of the smash products are given by

$$
(a\#h)(a'\#h') = a(\alpha_H^{-1}(h_1) \cdot \alpha_A^{-1}(a')) \# \alpha_H^{-1}(h_2)h',
$$

$$
(h\#c)(h'\#c') = h(\alpha_H^{-1}(h_1') \# \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h_2'))c',
$$

and the structure maps are $\alpha_A \otimes \alpha_H$ and respectively $\alpha_H \otimes \alpha_C$.

(i) Consider as well the trivial Hom-twisting map $R$ and the structure maps are $\alpha$ for all $a, a' \in A$. If we denote $R := \alpha \otimes R$ and is called the Hom-smash product. Its structure map is denoted by $A\#H\#C$ and is called the Hom-twisted tensor product. Its iterations are denoted by $A, B$, $R$, $Q$.

**Proposition 2.1** Let $(H, \mu, \Delta, \alpha)$ be a Hom-bialgebra and $R \in H \otimes H$ an element, with Sweedler-type notation $R = R^1 \otimes R^2 = r^1 \otimes r^2$. Then $(H, \mu, \Delta, \alpha, R)$ is called quasitriangular Hom-bialgebra if the following axioms are satisfied:

$$
(\Delta \otimes \alpha)(R) = \alpha(R^1) \otimes \alpha(r^1) \otimes R^2r^2, \tag{1.24}
$$

$$
(\alpha \otimes \Delta)(R) = R^1r^1 \otimes \alpha(r^2) \otimes \alpha(R^2), \tag{1.25}
$$

$$
\Delta^{\text{cop}}(h)R = R\Delta(h), \tag{1.26}
$$

for all $h \in H$, where we denoted as usual $\Delta^{\text{cop}}(h) = h_2 \otimes h_1$.

## 2 Hom-L-R-twisted tensor products of algebras

We introduce the Hom-analogue of Proposition 1.1.

**Proposition 2.1** Let $(A, \mu_A, \alpha_A)$ and $(B, \mu_B, \alpha_B)$ be two Hom-associative algebras and $R : B \otimes A \to A \otimes B$, $Q : A \otimes B \to A \otimes B$ two linear maps, with notation $R(a \otimes b) = a_AR \otimes b_R = a_r \otimes b_r$, and $Q(a \otimes b) = a_Q \otimes b_Q = a_q \otimes b_q$, for all $a \in A$, $b \in B$, satisfying the conditions:

$$
\alpha_A(a_R) \otimes \alpha_B(b_R) = \alpha_A(a)_R \otimes \alpha_B(b)_R, \tag{2.1}
$$

$$
\alpha_A(a_Q) \otimes \alpha_B(b_Q) = \alpha_A(a)_Q \otimes \alpha_B(b)_Q, \tag{2.2}
$$

$$
(a a')_R \otimes \alpha_B(b)_R = a_Ra'_R \otimes \alpha_B((b_R)_r), \tag{2.3}
$$

$$
\alpha_A(a)_R \otimes (b b')_R = \alpha_A(a)_r \otimes b_Rb'_R, \tag{2.4}
$$

$$
(a a')_Q \otimes \alpha_B(b)_Q = a_Qa'_Q \otimes \alpha_B((b_Q)_q), \tag{2.5}
$$

$$
\alpha_A(a)_Q \otimes (b b')_Q = \alpha_A(a)_q \otimes b_Qb'_Q, \tag{2.6}
$$

$$
b_R \otimes (a_R)_Q \otimes b'_Q = b_R \otimes (a_Q)_R \otimes b'_Q, \tag{2.7}
$$

$$
a_R \otimes (b_R)_Q \otimes a_Q = a_R \otimes (b_Q)_R \otimes a'_Q, \tag{2.8}
$$

for all $a, a' \in A$ and $b, b' \in B$. Define a new multiplication on $A \otimes B$ by $(a \otimes b)(a' \otimes b') = a_Qa'_R \otimes b_Rb'_Q$. Then $A \otimes B$ with this multiplication is a Hom-associative algebra with structure map $\alpha_A \otimes \alpha_B$, denoted by $A \otimes R_B$ and called the Hom-L-R-twisted tensor product of $A$ and $B$ afforded by the maps $R$ and $Q$. 

7
Proof. The fact that $\alpha_A \otimes \alpha_B$ is multiplicative follows immediately from (2.1) and (2.2). Now we compute (denoting $R = r = \mathcal{R}$ and $Q = q = \mathcal{Q}$):

$$(\alpha_A \otimes \alpha_B) (a \otimes b) [(a' \otimes b') (a'' \otimes b'')]$$

$$(\alpha_A \otimes \alpha_B) (a \otimes b) [(a' \otimes b') (a'' \otimes b'')]$$

$$= (\alpha_A(a) \otimes \alpha_B(b))(a'_{R}a''_{R} \otimes b'b'_{Q})$$

$$= \alpha_A(a)q(a'_{R}a''_{R}) \otimes \alpha_B(b)(b'b'_{Q})q$$

$$= \alpha_A((a'_{R})q)((a'_{R})_{R}(a''_{R})_{R}) \otimes \alpha_B((b'b'_{Q})_{R}(b'b'_{Q})_{R})$$

$$= \alpha_A((a'_{R})_{Q})\alpha_A((a''_{R})_{R}) \otimes ((b'b'_{Q})_{R}(b'b'_{Q})_{R})$$

$$= (a'_{R}a''_{R} \otimes b'b'_{Q}) (\alpha_A(a''_{R}) \otimes \alpha_B(b'b'_{Q}))$$

$$= [(a \otimes b)(a' \otimes b') (\alpha_A \otimes \alpha_B) (a'' \otimes b'')]$$

finishing the proof. \qed

Obviously, a Hom-twisted tensor product $A \otimes_{R} B$ is a particular case of a Hom-L-R-twisted tensor product, namely $A \otimes_{R} B = A \otimes_{Q} B$, where $Q = id_{A \otimes B}$. On the other hand, if $A$ and $B$ are Hom-associative algebras and $Q : A \otimes B \rightarrow A \otimes B$ is a linear map satisfying the conditions (2.2), (2.5) and (2.6), then the multiplication $(a \otimes b)(a' \otimes b') = aQa' \otimes bb'_{Q}$ defines a Hom-associative algebra structure on $A \otimes B$, denoted by $A \otimes_{Q} B$; this is a particular case of a Hom-L-R-twisted tensor product, namely $A \otimes_{Q} B = A \otimes_{R} B$, where $R$ is the flip map $b \otimes a \mapsto a \otimes b$. Also, if $A \otimes_{Q} B$ is a Hom-L-R-twisted tensor product, we can consider as well the Hom-associative algebras $A \otimes_{R} B$ and $A \otimes B$.

By using some computations similar to the ones performed in [39], Propositions 2.6 and 2.10, one can prove the following two results:

**Proposition 2.2** Let $(A, \mu_A, \alpha_A)$ and $(B, \mu_B, \alpha_B)$ be Hom-associative algebras and $A \otimes_{Q} B$ a Hom-L-R-twisted tensor product. Define the linear maps $T, U, V : (A \otimes B) \otimes (A \otimes B) \rightarrow (A \otimes B) \otimes (A \otimes B)$, by

$$T((a \otimes b) \otimes (a' \otimes b')) = (a \otimes b_{R}) \otimes (a' \otimes b'_{Q}),$$

$$U((a \otimes b) \otimes (a' \otimes b')) = (a \otimes b_{R}) \otimes (a' \otimes b'_{Q}),$$

$$V((a \otimes b) \otimes (a' \otimes b')) = (a \otimes b_{R}) \otimes (a' \otimes b').$$

Then $T$ is a Hom-twistor for $A \otimes B$, $U$ is a Hom-twistor for $A \otimes_{R} B$, $V$ is a Hom-twistor for $A \otimes_{Q} B$ and $A \otimes_{Q} B = (A \otimes B)^{T} = (A \otimes B)^{U} = (A \otimes B)^{V}$ as Hom-associative algebras.

**Proposition 2.3** Let $(A, \mu_A)$ and $(B, \mu_B)$ be two associative algebras, $\alpha_A : A \rightarrow A$ and $\alpha_B : B \rightarrow B$ algebra maps and $A \otimes_{Q} B$ an L-R-twisted tensor product such that $(\alpha_A \otimes \alpha_B) \circ R = R \circ (\alpha_B \otimes \alpha_A)$ and $(\alpha_A \otimes \alpha_B) \circ Q = \alpha_Q \circ (\alpha_A \otimes \alpha_B)$. Then we have a Hom-L-R-twisted tensor product $A_{\alpha_A} \otimes_{Q} B_{\alpha_B}$, which coincides with $(A \otimes_{Q} B)_{\alpha_A \otimes \alpha_B}$ as Hom-associative algebras.

The next result is the Hom-analogue of [17], Proposition 2.9.
Proposition 2.4 Let $A_\mathcal{Q} \otimes_R B$ be a Hom-L-R-twisted tensor product of the Hom-associative algebras $(A, \mu_A, \alpha_A)$ and $(B, \mu_B, \alpha_B)$ with bijective structure maps $\alpha_A$ and $\alpha_B$ and assume that $Q$ is bijective with inverse $Q^{-1}$. Then the map $P : B \otimes A \rightarrow A \otimes B$ defined by $P = Q^{-1} \circ R$ is a Hom-twisting map, and we have an isomorphism of Hom-associative algebras $Q : A \otimes_R B \simeq A_\mathcal{Q} \otimes_R B$.

Proof. Let $a, a' \in A$ and $b, b' \in B$; we denote $Q^{-1}(a \otimes b) = a_{Q^{-1}} \otimes b_{Q^{-1}} = a_{q^{-1}} \otimes b_{q^{-1}}$, $P(b \otimes a) = a_{P} \otimes b_{P} = a_{q} \otimes b_{q}$, $Q = q = \overline{Q} = \overline{q}$. The relation (1.20) for $P$ follows immediately from (2.1) and (2.2). Now we compute:

\[
Q(a_P a'_P \otimes \alpha_B((b_P)_P)) = ((a_R)_R^{-1}(a'_R)_R^{-1})_Q \otimes \alpha_B(((b_R)_R^{-1})_r)_Q
\]

\[
= ((a_R)_R^{-1}(a'_R)_R^{-1})_Q \otimes \alpha_B(((b_R)_R^{-1})_r)_Q)
\]

\[
= ((a_R)_R^{-1}Qa'_P \otimes \alpha_B(((b_R)_R^{-1})_r)_Q)
\]

\[
= (a_R)_R^{-1}Qa'_P \otimes \alpha_B((b_R)_r)
\]

\[
= (a_R)_R^{-1} \otimes \alpha_B(b)_R = R(\alpha_B(b) \otimes a).
\]

By applying $Q^{-1}$ to this equality we obtain $P(a_B(b) \otimes a') = a_P a'_P \otimes \alpha_B((b_P)_P)$, which is (1.21) for $P$. Similarly one can prove (1.22) for $P$, so $P$ is indeed a Hom-twisting map. Since $Q$ satisfies (2.2), the only thing left to prove is that $Q : A \otimes_R B \rightarrow A_\mathcal{Q} \otimes_R B$ is multiplicative. We compute:

\[
Q((a \otimes b)(a' \otimes b'))
\]

\[
= (a'_P)_Q \otimes (b_P)_Q
\]

\[
= (a'_P)_Q \otimes \alpha_B(a^{-1}_B(b_P)_Q)
\]

\[
= a_{Q}(a'_P)_Q \otimes \alpha_B(a^{-1}_B(b_P)_Q)
\]

\[
= a_{Q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q))_Q
\]

\[
= q \alpha_B((a^{-1}_B(b_P)_Q))_Q
\]

\[
= q \alpha_B((a^{-1}_B(b_P)_Q)_Q)
\]

\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
\]

\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
\]

\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
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= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
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\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
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\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
\]

\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
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= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
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\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
\]

\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
\]

\[
= a_{q}(a'_P)_Q \otimes \alpha_B((a^{-1}_B(b_P)_Q)_Q)
\]
Proposition 2.5 Let \( A \otimes R_1 \otimes R_2 \otimes C \) be an iterated Hom-twisted tensor product of the Hom-associative algebras \((A, \mu_A, \alpha_A)\), \((B, \mu_B, \alpha_B)\), \((C, \mu_C, \alpha_C)\) for which the map \( R_3 : C \otimes A \rightarrow A \otimes C \) is the flip map \( c \otimes a \mapsto a \otimes c \). Define the linear maps

\[
R : B \otimes (A \otimes C) \rightarrow (A \otimes C) \otimes B, \quad R(b \otimes (a \otimes c)) = (a_{R_1} \otimes c) \otimes b_{R_1},
\]
\[
Q : (A \otimes C) \otimes B \rightarrow (A \otimes C) \otimes B, \quad Q((a \otimes c) \otimes b) = (a \otimes c_{R_2}) \otimes b_{R_2}.
\]

Then we have a Hom-L-R-twisted tensor product \((A \otimes C) \otimes_R B\), and an isomorphism of Hom-associative algebras \( A \otimes R_1 \otimes R_2 \otimes C \simeq (A \otimes C) \otimes_R B\), \( a \otimes b \otimes c \mapsto (a \otimes c) \otimes b\).

Proof. One has to prove the relations (2.1)-(2.8) for \( R \) and \( Q \). The relations (2.1)-(2.6) are easy consequences of the fact that \( R_1 \) and \( R_2 \) are Hom-twisting maps, (2.7) is trivially satisfied while (2.8) is a consequence of the braid relation, which in our situation \((R_3 \text{ is the flip})\) boils down to

\[
a_{R_1} \otimes (b_{R_1})_{R_2} \otimes c_{R_2} = a_{R_1} \otimes (b_{R_1})_{R_2} \otimes c_{R_2}, \quad \text{for all} \quad a \in A, b \in B, c \in C.
\]

So indeed \((A \otimes C) \otimes_R B\) is a Hom-L-R-twisted tensor product, with multiplication

\[
((a \otimes c) \otimes b)((a' \otimes c') \otimes b') = (a a'_{R_1} \otimes c_{R_2} c') \otimes b_{R_1} b'_{R_2}.
\]

Again because \( R_3 \) is the flip, the multiplication in the iterated Hom-twisted tensor product \( A \otimes R_1 \otimes R_2 \otimes C \) is given by \((a \otimes b \otimes c)(a' \otimes b' \otimes c') = a a'_{R_1} \otimes b_{R_1} b'_{R_2} \otimes c_{R_2} c'\), so obviously we have

\[
A \otimes R_1 \otimes R_2 \otimes C \simeq (A \otimes C) \otimes_R B.
\]

\( \square \)

3 Hom-L-R-smash product

Definition 3.1 Let \((A, \mu_A, \alpha_A)\) be a Hom-associative algebra, \( M \) a linear space and \( \alpha_M : M \rightarrow M \) a linear map. Assume that \((M, \alpha_M)\) is both a left and a right \( A \)-module (with actions denoted by \( A \otimes M \rightarrow M \), \( a \otimes m \mapsto a \cdot m \) and \( M \otimes A \rightarrow M \), \( m \otimes a \mapsto m \cdot a \)). We call \((M, \alpha_M)\) an \( A \)-bimodule if the following condition is satisfied, for all \( a, a' \in A, m \in M \):

\[
\alpha_A(a) \cdot (m \cdot a') = (a \cdot m) \cdot \alpha_A(a').
\]

(3.1)

Remark 3.2 Obviously, \((A, \alpha_A)\) is an \( A \)-bimodule.

We prove now that this is indeed the "appropriate" concept of bimodule for the class of Hom-associative algebras. Recall first from [31] the following concept. Let \( C \) be a class of (not necessarily associative) algebras, \( A \in C \) and \( M \) a linear space with two linear actions \( a \otimes m \mapsto a \cdot m \) and \( m \otimes a \mapsto m \cdot a \) of \( A \) on \( M \). On the direct sum \( A \oplus M \) one can introduce an algebra structure (called the semidirect sum or split null extension) by defining a multiplication in \( A \oplus M \) by

\[
(a, m)(a', m') = (aa', m \cdot a' + a \cdot m'),
\]

for all \( a, a' \in A \) and \( m, m' \in M \). Then, if \( A \oplus M \) with this algebra structure is in \( C \), we say that \( M \) is an \( A \)-bimodule with respect to \( C \). If \( C \) is the class of all associative algebras or of all Lie algebras, one obtains the usual concepts of bimodule for these types of algebras. We have then the following result:
Proposition 3.3 Let \((A, \mu_A, \alpha_A)\) be a Hom-associative algebra, \(M\) a linear space, \(\alpha_M : M \to M\) a linear map and two linear actions \(a \otimes m \mapsto a \cdot m\) and \(m \otimes a \mapsto m \cdot a\) of \(A\) on \(M\). Then the split null extension \(B = A \oplus M\) is a Hom-associative algebra with structure map \(\alpha_B\) defined by \(\alpha_B((a, m)) = (\alpha_A(a), \alpha_M(m))\) if and only if \((M, \alpha_M)\) is an \(A\)-bimodule as in Definition 3.1.

Proof. It is easy to see that
\[
\alpha_B((a, m))[m', m''](a', m'') = (\alpha_A(a)(a' a''), \alpha_M(m) \cdot (a' a'') + \alpha_A(a) \cdot (m' \cdot a'')) \\
+ \alpha_A(a) \cdot (a' \cdot m''),
\]

so the multiplication on \(B\) is Hom-associative if and only if
\[
\alpha_M(m) \cdot (a' a'') + \alpha_A(a) \cdot (m' \cdot a'') + \alpha_A(a) \cdot (a' \cdot m'') = (m' \cdot a'') \cdot \alpha_A(a'') + (a' \cdot m'') \cdot \alpha_A(a''') + (aa') \cdot \alpha_M(m'')
\]
for all \(a, a', a'' \in A\) and \(m, m', m'' \in M\). Also, clearly \(\alpha_B\) is multiplicative if and only if
\[
\alpha_M(m \cdot a') + \alpha_M(a \cdot m') = \alpha_M(m \cdot a') + \alpha_A(a) \cdot \alpha_M(m')
\]
for all \(a, a' \in A\) and \(m, m' \in M\). It is then obvious that if \((M, \alpha_M)\) is an \(A\)-bimodule then \(B\) is Hom-associative. Conversely, assuming the two relations above, we take \(m' = m'' = 0\), then \(m = m' = 0\), then \(m = m'' = 0\) in the first relation and \(m = 0\), then \(m' = 0\) in the second relation and we obtain the five relations saying that \((M, \alpha_M)\) is an \(A\)-bimodule. \(\square\)

Similarly to Theorem 4.5 in [48], one can prove the following result:

Proposition 3.4 Let \((A, \mu_A)\) be an associative algebra, \(\alpha_A : A \to A\) an algebra endomorphism, \(M\) an \(A\)-bimodule in the usual sense with actions \(A \otimes M \to M\), \(a \otimes m \mapsto a \cdot m\) and \(M \otimes A \to M\), \(m \otimes a \mapsto m \cdot a\), and \(\alpha_M : M \to M\) a linear map satisfying the conditions \(\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m)\) and \(\alpha_M(m \cdot a) = \alpha_M(m) \cdot \alpha_A(a)\) for all \(a \in A, m \in M\). Then \((M, \alpha_M)\) becomes a bimodule over the Hom-associative algebra \(A_{\alpha_A}\) with actions \(A_{\alpha_A} \otimes M \to M\), \(a \otimes m \mapsto a \cdot m := \alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m)\) and \(M \otimes A_{\alpha_A} \to M\), \(m \otimes a \mapsto m \cdot a := \alpha_M(m \cdot a) = \alpha_M(m) \cdot \alpha_A(a)\).

Definition 3.5 Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra. An \(H\)-bimodule Hom-algebra is a Hom-associative algebra \((D, \mu_D, \alpha_D)\) that is both a left and a right \(H\)-module Hom-algebra and such that \((D, \alpha_D)\) is an \(H\)-bimodule.

We can introduce now the Hom-analogue of the L-R-smash product.

Theorem 3.6 Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra, \((D, \mu_D, \alpha_D)\) an \(H\)-bimodule Hom-algebra, with actions denoted by \(H \otimes D \to D\), \(h \otimes d \mapsto h \cdot d\) and \(D \otimes H \to D\), \(d \otimes h \mapsto d \cdot h\), and assume that the structure maps \(\alpha_D\) and \(\alpha_H\) are both bijective. Define the linear maps
\[
R : H \otimes D \to D \otimes H, \quad R(h \otimes d) = \alpha_H^{-2}(h_1) \cdot \alpha_D^{-1}(d) \otimes \alpha_H^{-1}(h_2),
\]
\[
Q : D \otimes H \to D \otimes H, \quad Q(d \otimes h) = \alpha_D^{-1}(d) \cdot \alpha_H^{-2}(h_2) \otimes \alpha_H^{-1}(h_1),
\]
for all \(d \in D, h \in H\). Then we have a Hom-L-R-twisted tensor product \(D \otimes_R H\), which will be denoted by \(D \triangleright H\) (we denote \(d \otimes h := d \triangleright h\) for \(d \in D, h \in H\)) and called the Hom-L-R-smash product of \(D\) and \(H\). The structure map of \(D \triangleright H\) is \(\alpha_D \otimes \alpha_H\) and its multiplication is
\[
(d \triangleright h)(d' \triangleright h') = [\alpha_D^{-1}(d) \cdot \alpha_H^{-2}(h_2)][\alpha_H^{-2}(h_1) \cdot \alpha_D^{-1}(d')] \triangleright \alpha_H^{-1}(h_2 h_1).
\]
Proof. Note first that $D \# H$, so $R$ satisfies the conditions (2.1), (2.3). With a proof similar to the one in [39], one can prove that the map $D$ satisfies the conditions (2.2), (2.5), (2.6).

Proof of (2.7):

$$h_R \otimes (d_R)Q \otimes h'_Q = \alpha_H^{-1}(h_2) \otimes (\alpha_H^{-2}(h_1) \cdot \alpha_D^{-1}(d))Q \otimes h'_Q$$

$$= \alpha_H^{-1}(h_2) \otimes \alpha_D^{-1}(\alpha_H^{-2}(h_1) \cdot \alpha_D^{-1}(d)) \cdot \alpha_H^{-1}(h'_1)$$

$$= \alpha_H^{-1}(h_2) \otimes (\alpha_H^{-2}(h_1) \cdot \alpha_D^{-2}(d)) \cdot \alpha_H^{-2}(h'_2) \otimes \alpha_H^{-1}(h'_1)$$

(1.7)

$$= \alpha_H^{-1}(h_2) \otimes \alpha_H^{-2}(h_1) \cdot (\alpha_D^{-1}(d) \cdot \alpha_H^{-2}(h'_2)) \otimes \alpha_H^{-1}(h'_1)$$

(1.9)

$$= h_R \otimes (\alpha_D^{-1}(d) \cdot \alpha_H^{-2}(h'_2))R \otimes \alpha_H^{-1}(h'_1)$$

$$= h_R \otimes (d_Q)R \otimes h'_Q.$$  

finishing the proof.

As a consequence of Proposition 2.3 we immediately obtain the following result:

**Proposition 3.7** Let $(H, \mu_H, \Delta_H)$ be a bialgebra and $(D, \mu_D)$ an $H$-bimodule algebra in the usual sense, with actions denoted by $H \otimes D \rightarrow D$, $h \otimes d \mapsto h \cdot d$ and $D \otimes H \rightarrow D$, $d \otimes h \mapsto d \cdot h$. Let $\alpha_H : H \rightarrow H$ be a bialgebra endomorphism and $\alpha_D : D \rightarrow D$ an algebra endomorphism such that $\alpha_D(h \cdot d) = \alpha_H(h) \cdot \alpha_D(d)$ and $\alpha_D(d \cdot h) = \alpha_D(d) \cdot \alpha_H(h)$ for all $d \in D$, $h \in H$. If we consider the Hom-bialgebra $H_{\alpha_H} = (H, \alpha_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H)$ and the Hom-associative algebra $D_{\alpha_D} = (D, \alpha_D \circ \mu_D, \alpha_D)$, then $D_{\alpha_D}$ is an $H_{\alpha_H}$-bimodule Hom-algebra with actions $H_{\alpha_H} \otimes D_{\alpha_D} \rightarrow D_{\alpha_D}$, $h \otimes d \mapsto h \triangleright d := \alpha_D(h \cdot d) = \alpha_H(h) \cdot \alpha_D(d)$ and $D_{\alpha_D} \otimes H_{\alpha_H} \rightarrow D_{\alpha_D}$, $d \otimes h \mapsto d \triangleright h := \alpha_D(d \cdot h) = \alpha_D(d) \cdot \alpha_H(h)$. If we assume that moreover the maps $\alpha_H$ and $\alpha_D$ are bijective, if we denote by $D_{\alpha_H} \# H$ the L-R-smash product between $D$ and $H$, then $\alpha_D \otimes \alpha_H$ is an algebra endomorphism of $D_{\alpha_H} \# H$ and the Hom-associative algebras $(D_{\alpha_H} \# H)_{\alpha_D} \otimes \alpha_H$ and $D_{\alpha_D} \# H_{\alpha_H}$ coincide.

**Example 3.8** Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra such that $\alpha_H$ is bijective. The vector space $H^*$ becomes a Hom-associative algebra with multiplication and structure map defined by:

$$(f \bullet g)(h) = f(\alpha^{-2}_H(h_1))g(\alpha^{2}_H(h_2)).$$
for all \( f, g \in H^* \) and \( h \in H \). Then for any \( p, q \in \mathbb{Z} \), this Hom-associative algebra \( H^* \) can be organized as an \( H \)-bimodule Hom-algebra (denoted by \( H^*_{p,q} \)) with actions defined as follows:

\[
\begin{align*}
\rightarrow &: H \otimes H^* \to H^*, \quad (h \to f)(h') = f(\alpha_H^{-2}(h')\alpha_H^{p}(h)), \\
\leftarrow &: H^* \otimes H \to H^*, \quad (f \leftarrow h)(h') = f(\alpha_H^{q}(h)\alpha_H^{-2}(h')).
\end{align*}
\]

for all \( h, h' \in H \) and \( f \in H^* \). Note that \( \beta \) is bijective and \( \beta^{-1} = \alpha_H^* \), the transpose of \( \alpha_H \).

So, we can consider the Hom-L-R-smash product \( H^*_{p,q} \triangleright H \), whose structure map is \( \beta \otimes \alpha_H \) and whose multiplication is

\[
(f \triangleright h)(f' \triangleright h') = [\alpha_H^*(f) \leftarrow \alpha_H^{-2}(h'_2)] \bullet [\alpha_H^{-2}(h_1) \rightarrow \alpha_H^*(f')] \triangleright \alpha_H^{-1}(h_2h'_1).
\]

**Example 3.9** Let \( (H, \mu_H, \Delta_H, \alpha_H) \) be a Hom-bialgebra, \( (A, \mu_A, \alpha_A) \) a left \( H \)-module Hom-algebra and \( (C, \mu_C, \alpha_C) \) a right \( H \)-module Hom-algebra, with actions denoted by \( H \otimes A \to A \), \( h \otimes a \mapsto h \cdot a \) and \( C \otimes H \to C \), \( c \otimes h \mapsto c \cdot h \). Define \( D := A \otimes C \) as the tensor product Hom-associative algebra. Define the linear maps

\[
\begin{align*}
H \otimes (A \otimes C) &\to A \otimes C, \quad h \cdot (a \otimes c) = h \cdot a \otimes \alpha_C(c), \\
(A \otimes C) \otimes H &\to A \otimes C, \quad (a \otimes c) \cdot h = \alpha_A(a) \otimes c \cdot h.
\end{align*}
\]

Then one can easily check that \( A \otimes C \) with these actions becomes an \( H \)-bimodule Hom-algebra.

Assume that moreover the structure maps \( \alpha_H, \alpha_A, \alpha_C \) are bijective, so we can consider the Hom-L-R-smash product \( (A \otimes C) \triangleright H \). Then, by writing down the formula for the multiplication in \( (A \otimes C) \triangleright H \), and then by applying Proposition 2.5, we obtain that \( (A \otimes C) \triangleright H \) is isomorphic to the two-sided Hom-smash product \( A\#H\#C \).

**4 Hom-diagonal crossed product**

**Definition 4.1** (i) If \( (A, \mu, \alpha) \) is a Hom-associative algebra, we say that \( A \) is unital if there exists an element \( 1_A \in A \) such that

\[
\begin{align*}
\alpha(1_A) &= 1_A, \\
1_Aa &= a1_A = \alpha(a), \quad \forall \ a \in A.
\end{align*}
\]

If \( f : A \to B \) is a morphism of Hom-associative algebras, we say that \( f \) is unital if \( f(1_A) = 1_B \).

(ii) If \( (C, \Delta, \alpha) \) is a Hom-coassociative coalgebra, we say that \( C \) is counital if there exists a linear map \( \varepsilon_C : C \to k \) such that

\[
\begin{align*}
\varepsilon_C \circ \alpha &= \varepsilon_C, \\
\varepsilon_C(c_1)c_2 &= c_1\varepsilon_C(c_2) = \alpha(c), \quad \forall \ c \in C.
\end{align*}
\]

If \( f : C \to D \) is a morphism of Hom-coassociative coalgebras, we say that \( f \) is counital if \( \varepsilon_D \circ f = \varepsilon_C \).

**Definition 4.2** ([36], [37]) A Hom-Hopf algebra \( (H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, S) \) is a Hom-bialgebra such that \( (H, \mu_H, \alpha_H, 1_H) \) is a unital Hom-associative algebra, \( (H, \Delta_H, \alpha_H, \varepsilon_H) \) is a counital Hom-coassociative coalgebra, and \( S : H \to H \) is a linear map (called the antipode) such that

\[
\Delta_H(1_H) = 1_H \otimes 1_H.
\]

---

13
Proposition 4.3 Let \((H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, S)\) be a Hom-Hopf algebra with bijective antipode and \((D, \mu_D, \alpha_D)\) an \(H\)-bimodule Hom-algebra, with actions \(H \otimes D \to D\), \(h \otimes d \mapsto h \cdot d\) and \(D \otimes H \to D\), \(d \otimes h \mapsto d \cdot h\), such that \(\alpha_D\) and \(\alpha_H\) are both bijective and

\[
1_H \cdot d = d \cdot 1_H = \alpha_D(d), \quad \forall \, d \in D.
\]  \hfill (4.15)

Then the map \(Q : D \otimes H \to D \otimes H\), \(Q(d \otimes h) = \alpha_D^{-1}(d) \cdot \alpha_H^{-2}(h_2) \otimes \alpha_H^{-1}(h_1)\) is bijective, with inverse \(Q^{-1} : D \otimes H \to D \otimes H\), \(Q^{-1}(d \otimes h) = \alpha_D^{-1}(d) \cdot \alpha_H^{-2}(S^{-1}(h_2)) \otimes \alpha_H^{-1}(h_1)\).

Proof. We check only that \(Q^{-1} \circ Q = id\), the proof for \(Q \circ Q^{-1} = id\) is similar and left to the reader. We compute:

\[
(Q^{-1} \circ Q)(d \otimes h) = Q^{-1}(\alpha_D^{-1}(d) \cdot \alpha_H^{-2}(h_2) \otimes \alpha_H^{-1}(h_1))
\]

\[
= \alpha_D^{-1}(\alpha_D^{-1}(d) \cdot \alpha_H^{-2}(h_2)) \cdot \alpha_H^{-2}(S^{-1}(\alpha_H^{-1}(h_1)_2)) \otimes \alpha_H^{-1}(\alpha_H^{-1}(h_1)_1)
\]

\[
= \alpha_D^{-2}(d) \cdot \alpha_H^{-3}(h_2) \cdot \alpha_H^{-2}(S^{-1}(\alpha_H^{-1}(h_1)_2)) \otimes \alpha_H^{-1}(h_1)_1
\]

\[
= \alpha_D^{-2}(d) \cdot \alpha_H^{-3}(h_2) \cdot \alpha_H^{-3}(S^{-1}(h_1)_2) \otimes \alpha_H^{-1}(h_1)
\]

\[
= \alpha_D^{-1}(d) \cdot (\alpha_H^{-4}(h_2) \alpha_H^{-4}(S^{-1}(h_1)_2)) \otimes \alpha_H^{-1}(h_1)
\]

\[
= \alpha_D^{-1}(d) \cdot (\alpha_H^{-4}(h_2) \alpha_H^{-4}(S^{-1}(h_1)_2)) \otimes \alpha_H^{-1}(h_1)
\]

\[
= \alpha_D^{-1}(d) \cdot (\alpha_H^{-4}(h_2) \alpha_H^{-4}(S^{-1}(h_1)_2)) \otimes \alpha_H^{-1}(h_1)
\]

\[
= \alpha_D^{-1}(d) \cdot 1_H \otimes \alpha_H^{-1}(h_1 \varepsilon_H(h_2))
\]

\[
= d \otimes h,
\]

finishing the proof. \(\square\)

Assume now that we are in the hypotheses of Proposition 4.3 and consider the Hom-L-R-smash product \(D \triangleright H = D \circ Q \otimes_R H\). Since the map \(Q\) is bijective, we can apply Proposition 2.4 and we obtain that the map \(P : H \otimes D \to D \otimes H\), \(P = Q^{-1} \circ R\) is a Hom-twisting map and we have an isomorphism of Hom-associative algebras \(Q : D \otimes_H H \simeq D \triangleright H\).
Let \( \alpha_H \in \text{Hom}_H \text{Aut}(H) \) be a Hom-twisting map, so the multiplication of \( D \triangleleft \triangleright H \) is defined (denoting \( d \otimes h := d \triangleright h \)) by

\[
(d \triangleright h)(d' \triangleright h') = d[(\alpha_H^{-3}(h_1) \cdot \alpha_D^{-2}(d')) \cdot \alpha_H^{-3}(S^{-1}(h_2)_2)] \triangleright \alpha_H^{-2}(h_2)_1 h'.
\]

By a direct computation, one can check that the "twisting principle" holds also for diagonal crossed products, namely:

**Proposition 4.5** Assume that we are in the hypotheses and notation of Proposition 3.7, assuming moreover that \( \alpha_H \) and \( \alpha_D \) are bijective and \((H, \mu_H, \Delta_H, 1_H, \varepsilon_H)\) is a unital and counital Hopf algebra with bijective antipode \( S \) and we have \( \alpha_H(1_H) = 1_H, \varepsilon_H \circ \alpha_H = \varepsilon_H, S \circ \alpha_H = \alpha_H \circ S \). Then \( H \alpha_H = (H, \mu_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H, 1_H, \varepsilon_H, S) \) is a Hom-Hopf algebra, the \( H \alpha_H \)-bimodule Hom-algebra \( D \alpha_D = (D, \mu_D \circ \mu_D, \Delta_D) \) satisfies the hypotheses of Proposition 3.3, the map \( \alpha_D \otimes \alpha_H \) is an algebra endomorphism of the diagonal crossed product \( D \triangleleft \triangleright H \) and the Hom-associative algebras \((D \triangleleft \triangleright H)_{\alpha_D \otimes \alpha_H}\) and \( D \alpha_D \triangleleft \triangleright H \alpha_H \) coincide.

We need to characterize (left) modules over a Hom-diagonal crossed product, and we obtain first a characterization of (left) modules over a Hom-twisted tensor product. We begin with some definitions.

**Definition 4.6** Let \( A \otimes_R B \) be a Hom-twisted tensor product of the unital Hom-associative algebras \((A, \mu_A, \alpha_A, 1_A)\) and \((B, \mu_B, \alpha_B, 1_B)\). We say that \( R \) is a unital Hom-twisting map if, for all \( a \in A, b \in B \), we have \( R(1_B \otimes a) = a \otimes 1_B \) and \( R(b \otimes 1_A) = 1_A \otimes b \). If this is the case, then \( A \otimes_R B \) is unital with unit \( 1_A \otimes 1_B \), the maps \( A \to A \otimes_R B, a \mapsto a \otimes 1_B \) and \( B \to A \otimes_R B, b \mapsto 1_A \otimes b \) are unital morphisms of Hom-associative algebras and for all \( a \in A, b \in B \) we have

\[
(a \otimes 1_B)(1_A \otimes b) = \alpha_A(a) \otimes \alpha_B(b).
\]

**Remark 4.7** Let \( D \triangleleft \triangleright H = D \otimes_R H \) be a Hom-diagonal crossed product such that \( D \) is a unital Hom-associative algebra and \( h \cdot 1_D = 1_D \cdot h = \varepsilon_H(h)1_D \), for all \( h \in H \). Then \( P \) is a unital Hom-twisting map, so \( D \triangleleft \triangleright H \) is unital with unit \( 1_D \triangleleft \triangleright 1_H \).

**Remark 4.8** If \( H \) is a Hom-Hopf algebra with bijective antipode and such that \( \alpha_H \) is bijective, we consider the \( H \)-bimodule Hom-algebra \( H_{p,q}^* \) defined in Example 3.3. It is easy to see that \( H_{p,q}^* \) is unital with unit \( \varepsilon_H \), its structure map \( \beta \) is bijective, we have \( 1_H \mapsto f = f \mapsto 1_H = \beta(f) \), for all \( f \in H^* \), and \( h \mapsto \varepsilon_H = \varepsilon_H \mapsto h = \varepsilon_H(h) \varepsilon_H \), for all \( h \in H \). Consequently, the Hom-diagonal crossed product \( H_{p,q}^* \triangleleft \triangleright H \) is unital with unit \( \varepsilon_H \triangleleft \triangleright 1_H \).

**Definition 4.9** If \((A, \mu_A, \alpha_A, 1_A)\) is a unital Hom-associative algebra and \((M, \alpha_M)\) is a left \( A \)-module, we say that \( M \) is unital if \( 1_A \cdot m = \alpha_M(m), \forall m \in M \). If \( \alpha_A \) is bijective, we denote by \( A \text{-} M \) the category whose objects are unital left \( A \)-modules \((M, \alpha_M)\) with \( \alpha_M \) bijective, the morphisms being morphisms of left \( A \)-modules.

**Proposition 4.10** Let \( R: B \otimes A \to A \otimes B \) be a unital Hom-twisting map between the unital Hom-associative algebras \((A, \mu_A, \alpha_A, 1_A)\) and \((B, \mu_B, \alpha_B, 1_B)\). Let \( M \) be a linear space and \( \alpha_M: M \to M \) a linear map, and assume that \( \alpha_A, \alpha_B, \alpha_M \) are bijective. Then \( (M, \alpha_M) \) is a
We prove (1.7):

\[ \alpha_B(b) \cdot (a \cdot m) = \alpha_A(a_R) \cdot (b_R \cdot m), \quad \forall a \in A, b \in B, m \in M. \quad (4.17) \]

If this is the case, the left \( A \otimes_R B \)-module structure on \( M \) is given by

\[ (a \otimes b) \cdot m = a \cdot (\alpha_B^{-1}(b) \cdot \alpha_M^{-1}(m)), \quad \forall a \in A, b \in B, m \in M. \quad (4.18) \]

**Proof.** If \( (M, \alpha_M) \) is a unital left \( A \otimes_R B \)-module (with action denoted by \( \cdot \)), define actions of \( A \) and \( B \) on \( M \) by \( a \cdot m = (a \otimes 1_B) \cdot m \) and \( b \cdot m = (1_A \otimes b) \cdot m \). Obviously we have \( 1_A \cdot m = 1_B \cdot m = \alpha_M(m) \), and the conditions (1.7) and (1.8) for the actions of \( A \otimes_R B \) follow immediately from the ones corresponding to the action of \( A \otimes_R B \). We need to prove (4.17). We compute:

\[
((1_A \otimes b)(a \otimes 1_B)) \cdot \alpha_M(m) = \begin{align*}
&= (1_A a_R \otimes b_R 1_B) \cdot \alpha_M(m) \\
&= (\alpha_A(a_R) \otimes \alpha_B(b_R)) \cdot \alpha_M(m) \\
&= \alpha_A(a_R) \cdot (\alpha_B^{-1}(b) \cdot \alpha_M^{-1}(m)).
\end{align*}
\]

On the other hand, by using (1.8), we have

\[
((1_A \otimes b)(a \otimes 1_B)) \cdot \alpha_M(m) = (1_A \otimes \alpha_B(b)) \cdot ((a \otimes 1_B) \cdot m) = \alpha_B(b) \cdot (a \cdot m),
\]

\[
((a_R \otimes 1_B)(1_A \otimes b_R)) \cdot \alpha_M(m) = \begin{align*}
&= (\alpha_A(a_R) \otimes 1_B) \cdot ((1_A \otimes b_R) \cdot m) \\
&= \alpha_A(a_R) \cdot (b_R \cdot m),
\end{align*}
\]

so we obtain \( \alpha_B(b) \cdot (a \cdot m) = \alpha_A(a_R) \cdot (b_R \cdot m) \). Finally, to prove (4.18), we compute:

\[
a \cdot (\alpha_B^{-1}(b) \cdot \alpha_M^{-1}(m)) = \begin{align*}
&= (a \otimes 1_B) \cdot ((1_A \otimes \alpha_B^{-1}(b)) \cdot \alpha_M^{-1}(m)) \\
&= (\alpha_A^{-1}(a) \otimes 1_B)(1_A \otimes \alpha_B^{-1}(b)) \cdot m \\
&= (a \otimes b) \cdot m.
\end{align*}
\]

Conversely, assume that \( (M, \alpha_M) \) is a unital left \( A \)-module and a unital left \( B \)-module and (4.17) holds, and define an action of \( A \otimes_R B \) on \( M \) by \( (a \otimes b) \cdot m = a \cdot (\alpha_B^{-1}(b) \cdot \alpha_M^{-1}(m)) \). We have

\[
(1_A \otimes 1_B) \cdot m = 1_A \cdot (1_B \cdot \alpha_M^{-1}(m)) = 1_A \cdot m = \alpha_M(m).
\]

We prove (4.7):

\[
\alpha_M((a \otimes b) \cdot m) = \begin{align*}
&= \alpha_M(a \cdot \alpha_M^{-1}(b \cdot m)) \\
&= \alpha_A(a) \cdot (b \cdot m) \\
&= \alpha_A(a) \cdot (\alpha_B^{-1}(b \cdot \alpha_M^{-1}(m))) \\
&= \alpha_A(a) \cdot (\alpha_B(b) \cdot \alpha_M(m)) \\
&= \alpha_A(a) \cdot \alpha_B(b) \cdot \alpha_M(m). 
\end{align*}
\]
Now we prove (1.8):
\[(a \otimes b)(a' \otimes b') \cdot \alpha_M(m) = (aa'_R \otimes b_R b') \cdot \alpha_M(m) = (aa'_R) \cdot ([\alpha_B^{-1}(b_R)\alpha_B^{-1}(b')] \cdot m) = (aa'_R) \cdot (b_R \cdot (\alpha_B^{-1}(b') \cdot \alpha_M^{-1}(m))) \]
\[(1.8) \alpha_A(a) \cdot (a'_R \cdot \alpha_M^{-1}(b_R \cdot (\alpha_B^{-1}(b') \cdot \alpha_M^{-1}(m)))) \]
\[(1.7) \alpha_A(a) \cdot (b \cdot (\alpha_A^{-1}(a') \cdot (\alpha_B^{-1}(b') \cdot \alpha_M^{-1}(m)))) \]
\[(1.7) \alpha_A(a) \cdot (b \cdot \alpha_M^{-1}(a' \cdot (\alpha_B^{-1}(b') \cdot \alpha_M^{-1}(m))) = (\alpha_A(a) \otimes \alpha_B(b)) \cdot (a' \cdot (\alpha_B^{-1}(b') \cdot \alpha_M^{-1}(m))) = (\alpha_A \otimes \alpha_B)(a \otimes b) \cdot ((a' \otimes b') \cdot m), \]
finishing the proof. \(\square\)

**Corollary 4.11** Let \(D \bowtie H\) be a Hom-diagonal crossed product such that \(D\) is unital and \(h \cdot 1_D = 1_D \cdot h = \varepsilon_H(h)1_D, \forall h \in H\). If \(M\) is a linear space and \(\alpha_M : M \rightarrow M\) a bijective linear map, then \((M, \alpha_M)\) is a unital left \(D \bowtie H\)-module if and only if \((M, \alpha_M)\) is a unital left \(D\)-module and a unital left \(H\)-module (actions denoted by \(\cdot\)) such that, \(\forall h \in H, d \in D, m \in M\):
\[
\alpha_H(h) \cdot (d \cdot m) = [\alpha_H^{-2}(h_1) \cdot \alpha_D^{-1}(d) \cdot \alpha_H^{-2}(S^{-1}((h_2)_2))] \cdot (\alpha_H^{-2}((h_2)_1) \cdot m), \quad (4.19)
\]
and if this is the case then we have
\[
(d \bowtie h) \cdot m = d \cdot (\alpha_H^{-1}(h) \cdot \alpha_M^{-1}(m)), \quad \forall d \in D, h \in H, m \in M. \quad (4.20)
\]

## 5 Left-right Yetter-Drinfeld modules

**Definition 5.1** \([39]\) Let \((C, \Delta_C, \alpha_C)\) be a Hom-coassociative coalgebra, \(M\) a linear space and \(\alpha_M : M \rightarrow M\) a linear map. A right \(C\)-comodule structure on \((M, \alpha_M)\) consists of a linear map \(\rho : M \rightarrow M \otimes C\) satisfying the following conditions:
\[
\begin{align*}
(\alpha_M \otimes \alpha_C) \circ \rho &= \rho \circ \alpha_M, \\
(\alpha_M \otimes \Delta_C) \circ \rho &= (\rho \otimes \alpha_C) \circ \rho.
\end{align*}
\]
We usually denote \(\rho(m) = m_{(0)} \otimes m_{(1)}\). If \(C\) is counital, then \((M, \alpha_M)\) is called counital if \(\varepsilon_C(m_{(1)})m_{(0)} = \alpha_M(m)\), for all \(m \in M\). If \((M, \alpha_M)\) and \((N, \alpha_N)\) are right \(C\)-comodules, a morphism of right \(C\)-comodules \(f : M \rightarrow N\) is a linear map with \(\alpha_N \circ f = f \circ \alpha_M\) and \(f(m)_{(0)} \otimes f(m)_{(1)} = f(m_{(0)}) \otimes m_{(1)}\), for all \(m \in M\).

The concept of left-left Yetter-Drinfeld module over a Hom-bialgebra was introduced in \([38]\). Similarly one can introduce left-right Yetter-Drinfeld modules. Since we will be interested here to work over Hom-Hopf algebras, we will impose unitality conditions and bijectivity of structure maps in the definition.
Definition 5.2 Let \((H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, S)\) be a Hom-Hopf algebra with bijective antipode and bijective \(\alpha_H\). Let \(M\) be a linear space and \(\alpha_M : M \rightarrow M\) a bijective linear map. Then \((M, \alpha_M)\) is called a left-right Yetter-Drinfeld module over \(H\) if \((M, \alpha_M)\) is a unital left \(H\)-module (action denoted by \(\cdot\)) and a counital right \(H\)-comodule (coaction denoted by \(m \mapsto m_{(0)} \otimes m_{(1)} \in M \otimes H\)) satisfying the following compatibility condition, for all \(h \in H, m \in M\):

\[
\alpha_H(h_1) \cdot m_{(0)} \otimes \alpha_H^2(h_2) \alpha_H(m_{(1)}) = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} \alpha_H^2(h_1).
\] (5.3)

We denote by \(HYD^H\) the category whose objects are left-right Yetter-Drinfeld modules over \(H\), morphisms being linear maps that are morphisms of left \(H\)-modules and right \(H\)-comodules.

Remark 5.3 Similarly to what happens for left-left Yetter-Drinfeld modules (see [38]), left-right Yetter-Drinfeld modules over Hopf algebras become, via the "twisting procedure", left-right Yetter-Drinfeld modules over Hom-Hopf algebras.

Remark 5.4 Similarly to what happens for Hopf algebras, one can prove that condition (5.3) is equivalent to (for all \(h \in H, m \in M\))

\[
(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \alpha_H^{-1}((h_2)_1) \cdot m_{(0)} \otimes [\alpha_H^{-2}((h_2)_2) \alpha_H^{-1}(m_{(1)})] S^{-1}(h_1).
\] (5.4)

Similarly to what we have proved in [38] for left-left Yetter-Drinfeld modules, one can prove the following result:

Proposition 5.5 Let \((H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, S)\) be a Hom-Hopf algebra with bijective antipode and bijective \(\alpha_H\).

(i) If \((M, \alpha_M)\) and \((N, \alpha_N)\) are objects in \(HYD^H\), then \((M \otimes N, \alpha_M \otimes \alpha_N)\) becomes an object in \(HYD^H\) (denoted in what follows by \(\hat{M} \otimes \hat{N}\)) with structures

\[
H \otimes (M \otimes N) \rightarrow M \otimes N, \quad h \otimes (m \otimes n) \mapsto h_1 \cdot m \otimes h_2 \cdot n,
\]

\[
M \otimes N \rightarrow (M \otimes N) \otimes H, \quad m \otimes n \mapsto (m_{(0)} \otimes n_{(0)}) \otimes \alpha_M^{-2}(n_{(1)} m_{(1)}).
\]

(ii) \((k, id_k)\) is an object in \(HYD^H\), with action and coaction defined by \(h \cdot \lambda = \varepsilon_H(h) \lambda\) and \(\lambda_{(0)} \otimes \lambda_{(1)} = \lambda \otimes 1_H\), for all \(\lambda \in k\).

(iii) \(HYD^H\) is a braided monoidal category, with tensor product \(\hat{\otimes}\), unit \((k, id_k)\), associativity constraints, unit constraints and braiding and its inverse defined (for all \((M, \alpha_M)\), \((N, \alpha_N)\), \((P, \alpha_P)\) in \(HYD^H\) and \(m \in M, n \in N, p \in P, \lambda \in k\)) by

\[
a_{M,N,P} : (M \hat{\otimes} N) \hat{\otimes} P \rightarrow M \hat{\otimes} (N \hat{\otimes} P), \quad a_{M,N,P}((m \otimes n) \otimes p) = \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p)),
\]

\[
l_M : k \hat{\otimes} M \rightarrow M, \quad l_M(\lambda \otimes m) = \lambda \alpha_M^{-1}(m),
\]

\[
r_M : M \hat{\otimes} k \rightarrow M, \quad r_M(m \otimes \lambda) = \lambda \alpha_M^{-1}(m),
\]

\[
c_{M,N} : M \hat{\otimes} N \rightarrow N \hat{\otimes} M, \quad c_{M,N}(m \otimes n) = \alpha_N^{-1}(n_{(0)}) \otimes \alpha_M^{-1}(\alpha_H^{-1}(n_{(1)}) \cdot m),
\]

\[
c_{M,N}^{-1} : N \hat{\otimes} M \rightarrow M \hat{\otimes} N, \quad c_{M,N}^{-1}(n \otimes m) = \alpha_N^{-1}(\alpha_H^{-1}(S(n_{(1)}) \cdot m) \otimes \alpha_N^{-1}(n_{(0)})).
\]

The proof of the following result is straightforward and is left to the reader.

Proposition 5.6 Let \((H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, S)\) be a Hom-Hopf algebra with bijective antipode and bijective \(\alpha_H\). Consider the unital Hom-associative algebra \(H^*\), with multiplication and structure map defined by

\[
(f \cdot g)(h) = f(\alpha_H^{-2}(h_1)) g(\alpha_H^{-2}(h_2)), \quad \forall f, g \in H^*, \; h \in H,
\]
\[ \beta : H^* \to H^* , \quad \beta(f)(h) = f(\alpha_H^{-1}(h)), \quad \forall f \in H^*, \quad h \in H. \]

(i) If \((M, \alpha_M)\) is a counital right \(H\)-comodule, with coaction \(m \mapsto m(0) \otimes m(1)\), then \((M, \alpha_M)\) becomes a unital left \(H^*\)-module, with action \(f \cdot m = f(m(1))m(0)\), for all \(f \in H^*, \ m \in M\).

(ii) Assume that \(H\) is moreover finite dimensional. If \((M, \alpha_M)\) is a unital left \(H^*\)-module (action denoted by \(\cdot\)), then \((M, \alpha_M)\) becomes a counital right \(H\)-comodule, with coaction defined by \(M \to M \otimes H, \ m \mapsto \sum_e e^* \cdot m \otimes e\), where \(\{e_i\}\), \(\{e^i\}\) is a pair of dual bases in \(H\) and \(H^*\) (of course, the coaction does not depend on the choice of the dual bases).

Let again \((H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, S)\) be a Hom-Hopf algebra with bijective antipode and bijective \(\alpha_H\). From now on, we will denote by \(H^*\) the unital \(H\)-bimodule Hom-algebra \(H_{0,0}^*\) (notation as in Example 3.3), whose unit is \(\varepsilon_H\), multiplication \(\cdot\), structure map \(\beta\) and \(H\)-actions

\[
\rightarrow : H \otimes H^* \to H^*, \quad (h \mapsto f)(h') = f(\alpha_H^{-2}(h')h),
\]

\[
\leftarrow : H^* \otimes H \to H^*, \quad (f \mapsto h)(h') = f(h\alpha_H^{-2}(h')).
\]

For \(f \in H^*\) and \(h \in H\), we will also denote \(f(h) = (f, h)\).

To simplify notation in the proofs of the next results, we will use the following form of Sweedler-type notation:

\[
\begin{align*}
&h_1 \otimes (h_2)_1 \otimes (h_2)_2 = h_1 \otimes h_{21} \otimes h_{22}, \\
&h_1 \otimes ((h_2)_1)_1 \otimes ((h_2)_1)_2 \otimes (h_2)_2 = h_1 \otimes h_{211} \otimes h_{212} \otimes h_{22}, \quad \text{etc}...
\end{align*}
\]

**Proposition 5.7** We have a functor \(F : \mathcal{H}_YD^H \to H^* \bowtie H^*\mathcal{M}\), given by \(F((M, \alpha_M)) = (M, \alpha_M)\) at the linear level, with \(H^* \bowtie H\)-action defined by

\[
(f \bowtie h) \cdot m = (f, (\alpha_H^{-1}(h) \cdot \alpha_M^{-1}(m))(1))/(\alpha_H^{-1}(h) \cdot \alpha_M^{-1}(m))(0),
\]

for all \(f \in H^*, \ h \in H, \ m \in M\). On morphisms, \(F\) acts as identity.

**Proof.** Let \((M, \alpha_M) \in \mathcal{H}_YD^H\). Since \(M\) is a unital right \(H\)-comodule, it becomes a unital left \(H^*\)-module by Corollary 4.11. The only thing we need to prove in order to have \((M, \alpha_M)\) a unital left \(H^* \bowtie H\)-module with the prescribed action is the compatibility condition

\[
\alpha_H(h) \cdot (f(m(1))m(0)) = \langle (\alpha_H^{-2}(h_1) \mapsto \alpha_H^{-2}(S^{-1}(h_{22})), (\alpha_H^{-2}(h_{21}) \cdot m(1)) \mapsto (\alpha_H^{-2}(h_{21}) \cdot m)(0),
\]

for all \(f \in H^*, \ h \in H, \ m \in M\). We compute the right hand side as follows:

\[
\begin{align*}
\text{RHS} &= (\alpha_H^{-2}(h_1) \mapsto \alpha_H^{-2}(S^{-1}(h_{22})))\alpha_H^{-2}((\alpha_H^{-2}(h_{21}) \cdot m)(1))\alpha_H^{-2}(h_{21}) \cdot m)(0) \\
&= (\alpha_H^{-4}(S^{-1}(h_{22}))(\alpha_H^{-2}(h_{21}) \cdot m)(1))\alpha_H^{-2}(h_{21}) \cdot m)(0) \\
&= (f, \alpha_H^{-3}(S^{-1}(h_{22}))(\alpha_H^{-2}(h_{21}) \cdot m)(1))\alpha_H^{-1}(h_{21})\alpha_H^{-2}(h_{21}) \cdot m)(0).
\end{align*}
\]

By replacing \(h\) with \(\alpha_H^2(h)\), it turns out that we need to prove the following relation:

\[
\alpha_H^3(h) \cdot (f(m(1))m(0)) = (f, [\alpha_H^{-1}(S^{-1}(h_{22})))\alpha_H^{-3}(h_{21}) \cdot m)(1)]\alpha_H(h_{21}) \cdot m)(0).
\]

Note first that, by repeatedly applying (1.11), we obtain

\[
h_1 \otimes h_{211} \otimes h_{212} \otimes h_{22} = h_1 \otimes \alpha_H(h_{21}) \otimes \alpha_H(h_{221}) \otimes h_{222} \otimes \alpha_H^{-2}(h_{222}). (5.5)
\]
Now we can compute:

\[
\langle f, [\alpha^{-1}_H(S^{-1}(h_{22}))\alpha^{-3}_H((h_{21}\cdot m)_{(1)})]\alpha_H(h_1)\rangle (h_{21}\cdot m)_{(0)}
\]

\[\equiv \langle f, \{\alpha^{-1}_H(S^{-1}(h_{22}))[(\alpha^{-5}_H(h_{2122})\alpha^{-4}_H(m_{(1)}))\alpha^{-3}_H(S^{-1}(h_{211}))]\}\alpha_H(h_1)\rangle \alpha^{-1}_H(h_{2121})\cdot m_{(0)}\]

\[\underline{Hom-assoc.} \equiv \langle f, \{[\alpha^{-2}_H(S^{-1}(h_{22}))(\alpha^{-5}_H(h_{2122})\alpha^{-4}_H(m_{(1)}))\}\alpha^{-2}_H(S^{-1}(h_{211}))\}\alpha_H(h_1)\rangle \alpha^{-1}_H(h_{2121})\cdot m_{(0)}\]

\[\underline{Hom-assoc.} \equiv \langle f, \{[(\alpha^{-5}_H(S^{-1}(h_{2222}))\alpha^{-5}_H(h_{2221}))\alpha^{-3}_H(m_{(1)})]\alpha^{-1}_H(S^{-1}(h_{21}))\}\alpha_H(h_1)\rangle \cdot h_{221}\cdot m_{(0)}\]

\[\equiv \langle f, \{[\varepsilon_H(h_{222})]_1\alpha^{-1}_H(S^{-1}(h_{21}))\}\alpha_H(h_1)\rangle h_{221}\cdot m_{(0)}\]

\[= \langle f, \alpha^{-2}_H(m_{(1)})\alpha^{-1}_H(S^{-1}(h_{21}))\}\alpha_H(h_1)\alpha_H(h_{22})\cdot m_{(0)}\]

\[\equiv \langle f, \alpha^{-1}_H(m_{(1)})\{\alpha^{-1}_H(S^{-1}(h_{12}))\}\alpha_H(h_{2})\}\alpha_H(h_2)\cdot m_{(0)}\]

\[\underline{Hom-assoc.} \equiv \langle f, \alpha^{-1}_H(m_{(1)})\varepsilon_H(h_{1})\alpha_H(h_2)\}\alpha_H(h_2)\cdot m_{(0)}\]

\[= \langle f, \alpha^{-1}_H(m_{(1)})\varepsilon_H(h_{1})\rangle \alpha_H(h_2)\cdot m_{(0)}\]

\[\underline{Hom-assoc.} \equiv \langle f, \alpha^{-1}_H(m_{(1)})\varepsilon_H(h_{1})\rangle \alpha_H(h_2)\cdot m_{(0)}\]

and this is exactly what we wanted to prove. The fact that morphisms in \(H \triangleright \triangleright H M\) become morphisms in \(H^* \triangleright H M\) is easy to prove and is left to the reader. \(\square\)

**Proposition 5.8** If \(H\) is finite dimensional, then we have a functor \(G : H^{\triangleright \triangleright} H M \rightarrow H \triangleright \triangleright H^*\), given by \(G((M, \alpha_M)) = (M, \alpha_M)\) at the linear level, and \(H^*-\)action and \(H^*\)-coaction on \(M\):

\[h \cdot m = (\varepsilon_H \otimes h) \cdot m,\]

\[M \rightarrow M \otimes H, \quad m \mapsto (e^i \otimes 1_H) \cdot m \otimes e_i := m_{(0)} \otimes m_{(1)},\]

where \(\{e_i\}, \{e^i\}\) is a pair of dual bases in \(H\) and \(H^*\). On morphisms, \(G\) acts as identity.

**Proof.** It is obvious that, for \((M, \alpha_M) \in H^{\triangleright \triangleright} H M\), \(G(M)\) is a unital left \(H\)-module and a counital right \(H\)-comodule (the coaction is obtained from the left \(H^*\)-action, which in turn is obtained by restricting the \(H^* \triangleright H^*\)-action). We need to prove the Yetter-Drinfeld compatibility condition \([5.3]\). Note first that by \([1.11]\) we have (for all \(h \in H\))

\[h_{11} \otimes h_{121} \otimes h_{122} \otimes h_2 = \alpha_H(h_{11}) \otimes \alpha_H(h_{21}) \otimes h_{221} \otimes \alpha^{-2}_H(h_{222}). \tag{5.6}\]

Note also that, by applying on an element in \(H\) on the first tensor component, one can see that

\[\beta((h \rightarrow \alpha^*_{H^2}(e^i)) \mapsto g) \otimes e_i = \beta(e^i) \otimes (g\alpha^{-2}_H(e_i))\alpha^2_H(h), \tag{5.7}\]

for all \(h, g \in H\). Now we compute:
\[
\begin{align*}
\alpha_H(h_1) \cdot m(0) & \otimes \alpha_H^2(h_2) \alpha_H(m(1)) \\
& = (\varepsilon \triangleright \alpha_H(h_1)) \cdot ((e^i \triangleright 1_H) \cdot m) \otimes \alpha_H^2(h_2) \alpha_H(e_i) \\
& \overset{\text{(1.8)}}{=} ((\varepsilon \triangleright h_1)(e^i \triangleright 1_H)) \cdot \alpha_M(m) \otimes \alpha_H^2(h_2) \alpha_H(e_i) \\
& = \{\beta(\alpha_H^2(h_1) \rightarrow \alpha_H^2(e^i)) \leftarrow \alpha_H^{-3}(S^{-1}(h_{12})) \triangleright \alpha_H^{-1}(h_{121})\} \cdot \alpha_M(m) \otimes \alpha_H^2(h_2) \alpha_H(e_i) \\
& \overset{\text{5.7}}{=} (\beta(e^i) \triangleright \alpha_H^{-1}(h_{121})) \cdot \alpha_M(m) \otimes \alpha_H^2(h_2) \{[\alpha_H^{-2}(S^{-1}(h_{122})) \triangleright \alpha_H^{-1}(e_i)]h_{11}\} \\
& \overset{\text{Hom-associ.}}{=} (\beta(e^i) \triangleright \alpha_H^{-1}(h_{121})) \cdot \alpha_M(m) \otimes \{\alpha_H(h_2)[\alpha_H^{-2}(S^{-1}(h_{122}))] \triangleright \alpha_H^{-1}(e_i)\} \alpha_H(h_{11}) \\
& \overset{\text{Hom-associ.}}{=} (\beta(e^i) \triangleright h_{21}) \cdot \alpha_M(m) \otimes \{[\alpha_H^{-2}(h_{222}) \alpha_H^{-2}(S^{-1}(h_{211})) \triangleright e_i] \alpha_H(h_1)\} \\
& \overset{\text{4.14}}{=} (\beta(e^i) \triangleright h_{21}) \cdot \alpha_M(m) \otimes (\varepsilon(h_{22})h_{1}e_i) \alpha_H(h_1) \\
& = (\beta(e^i) \triangleright \alpha_H(h_2)) \cdot \alpha_M(m) \otimes \alpha_H(e_i) \alpha_H^2(h_1) \\
& \overset{\text{4.20}}{=} \beta(e^i) \cdot (h_2 \cdot m) \otimes \alpha_H(e_i) \alpha_H^2(h_1) \\
& = (h_2 \cdot m(0) \otimes (h_2 \cdot m)(1)) \alpha_H^2(h_1),
\end{align*}
\]

where for the last equality we used the fact that \{\alpha_H(e_i)\} and \{\beta(e^i)\} is also a pair of dual bases. So indeed \(M \in \mathcal{H}YD^H\). We leave to the reader to prove that morphisms in \(\mathcal{H} \triangleright \mathcal{H} \mathcal{M}\) become morphisms in \(\mathcal{H}YD^H\). \(\square\)

Since it is obvious that the functors \(F\) and \(G\) are inverse to each other, we obtain:

**Theorem 5.9** If \(H\) is a finite dimensional Hom-Hopf algebra with bijective antipode and bijective structure map, the categories \(\mathcal{H} \triangleright \mathcal{H} \mathcal{M}\) and \(\mathcal{H}YD^H\) are isomorphic.

6 The Drinfeld double

We recall first a variation of a result in \([38]\):

**Theorem 6.1** \((38)\) Let \((H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, R)\) be a unital and counital quasitriangular Hom-bialgebra such that \(\alpha_H\) is bijective and \((\alpha_H \otimes \alpha_H)(R) = R\). Then \(\mathcal{H} \mathcal{M}\) is a prebraided monoidal category, with tensor product defined as in Proposition \((3.8)\) unit \((k, id_k)\) with action \(h \cdot \lambda = \varepsilon_H(h)\lambda\) for all \(h \in H, \lambda \in k\), associativity constraints defined by the same formula as the ones of the category \(\mathcal{H}YD^H\), i.e. \(a_{M,N,P} = \alpha_M^{-1} \otimes id_N \otimes \alpha_P\), for \(M,N,P \in \mathcal{H} \mathcal{M}\), and prebraiding defined by \(c_{M,N} : M \otimes N \to N \otimes M\), \(c_{M,N}(m \otimes n) = \alpha_N^{-1}(R^2 \cdot n) \otimes \alpha_M^{-1}(R^1 \cdot m)\), for all \(M,N \in \mathcal{H} \mathcal{M}\).

Let now \((H, \mu_H, \Delta_H, \alpha_H, 1_H, \varepsilon_H, S)\) be a finite dimensional Hom-Hopf algebra with bijective antipode and bijective \(\alpha_H\). We will construct the Drinfeld double \(D(H)\) of \(H\), which will be a quasitriangular Hom-Hopf algebra.

As a Hom-associative algebra, \(D(H)\) is the Hom-diagonal crossed product \(H^* \triangleright H\). So, its unit is \(\varepsilon_H \triangleright 1_H\), its structure map is \(\beta \otimes \alpha_H\) and its multiplication is defined by

\[
(f \triangleright h)(f' \triangleright h') = f \cdot [(\alpha^{-3}_H(h_1) \rightarrow \alpha^2_H(f')) \leftarrow \alpha^{-3}_H(S^{-1}(h_{22}))] \triangleright \alpha^2_H(h_{21})h',
\]
for all $f, f' \in H^*$ and $h, h' \in H$, where $\beta = \alpha_H^{-1}$ and

$$(f \cdot g)(h) = f(\alpha_H^{-2}(h_1))g(\alpha_H^{-2}(h_2)),
\rightarrow: H \otimes H^* \to H^*, \quad (h \vdash f)(h') = f(\alpha_H^{-2}(h')h),
\leftarrow: H^* \otimes H \to H^*, \quad (f \leftarrow h)(h') = f(h\alpha_H^{-2}(h')).$$

By Theorem 5.9, the category $D(H)_M$ is isomorphic to $H\mathcal{YD}^H$, which is a braided monoidal category. We transfer the structure from $H\mathcal{YD}^H$ to $D(H)_M$ and then to $D(H)$. We obtain thus the following result:

**Theorem 6.2** $D(H)$ is a quasitriangular Hom-Hopf algebra and we have an isomorphism of braided monoidal categories $D(H)_M \simeq H\mathcal{YD}^H$. The structure of $D(H)$ is the following.

Its counit is $\varepsilon(f \triangleright h) = f(1_H)\varepsilon_H(h)$, for all $f \in H^*$, $h \in H$.

Its comultiplication is defined by

$$\Delta : D(H) \to D(H) \otimes D(H), \quad \Delta(f \triangleright h) = (f_2 \circ \alpha_H^{-2} \triangleright h_1) \otimes (f_1 \circ \alpha_H^{-2} \triangleright h_2),$$

where we denoted $\mu_H^* : H^* \to H^* \otimes H^*$, the dual of $\mu_H$, defined by $\mu_H^*(f) = f_1 \otimes f_2$ if and only if $f(hh') = f_1(h)f_2(h')$, for all $h, h' \in H$.

The quasitriangular structure is the element

$$R = \sum_i (\varepsilon_H \triangleright \alpha_H^{-1}(e_i)) \otimes (e^i \triangleright 1_H) \in D(H) \otimes D(H),$$

where $\{e_i\}$, $\{e^i\}$ is a pair of dual bases in $H$ and $H^*$. It satisfies the extra condition $((\beta \otimes \alpha_H) \otimes (\beta \otimes \alpha_H))(R) = R$.

The antipode of $D(H)$ is given by the formula

$$S_{D(H)}(f \triangleright h) = (\varepsilon_H \triangleright S(\alpha_H^{-1}(h)))(f \circ \alpha_H \circ S^{-1} \triangleright 1_H), \quad \forall f \in H^*, h \in H.$$

**Proof.** We leave most of the details to the reader. Let us note that in order to prove (1.24), one has to prove first that $\Delta^{op}(\varepsilon_H \triangleright h)R = R\Delta(\varepsilon_H \triangleright h)$ and $\Delta^{op}(f \triangleright 1_H)R = R\Delta(f \triangleright 1_H)$, for all $f \in H^*$, $h \in H$. Let us prove one of the two properties of the antipode, namely

$$(f \triangleright h)S_{D(H)}((f \triangleright h)2) = f(1_H)\varepsilon_H(h)\varepsilon_H \triangleright 1_H.$$ 

Note that as a consequence of the Hom-associativity of $H$ we have

$$(ab)(cd) = \alpha_H(a)(\alpha_H^{-1}(bc)d), \quad \forall a, b, c, d \in H. \quad (6.1)$$

Now we compute:

$$(f \triangleright h)S_{D(H)}((f \triangleright h)2)$$

$$= (f_2 \circ \alpha_H^{-2} \triangleright h_1)S_{D(H)}(f_1 \circ \alpha_H^{-2} \triangleright h_2)$$

$$= [(f_2 \circ \alpha_H^{-1} \triangleright 1_H)(\varepsilon_H \triangleright \alpha_H^{-1}(h_1))](\varepsilon_H \triangleright S(\alpha_H^{-1}(h_2)))(f_1 \circ \alpha_H^{-1} \circ S^{-1} \triangleright 1_H)$$

$$\overset{\text{(6.1)}}{=} (f_2 \circ \alpha_H^{-2} \triangleright 1_H)([\beta^{-1} \circ \alpha_H^{-1}](\varepsilon_H \triangleright \alpha_H^{-1}(h_1)))(\varepsilon_H \triangleright S(\alpha_H^{-1}(h_2)))$$

$$\quad (f_1 \circ \alpha_H^{-1} \circ S^{-1} \triangleright 1_H)$$

$$= (f_2 \circ \alpha_H^{-2} \triangleright 1_H)(\beta^{-1} \circ \alpha_H^{-1})(h_1S(h_2))(f_1 \circ \alpha_H^{-1} \circ S^{-1} \triangleright 1_H).$$
\[ (4.8) \]
\[
\varepsilon_H(h)(f_2 \circ \alpha_H^{-2} \bowtie 1_H)(f_1 \circ \alpha_H^{-2} \circ S^{-1} \bowtie 1_H)
\]
\[
= \varepsilon_H(h)((f_2 \circ \alpha_H^{-2}) \bullet (f_1 \circ \alpha_H^{-2} \circ S^{-1}) \bowtie 1_H)
\]
\[
\equiv f(1_H)\varepsilon_H(h)\varepsilon_H \bowtie 1_H,
\]
finishing the proof. □

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