The dual of Brown representability for homotopy categories of complexes

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Abstract

We call product generator of an additive category a fixed object satisfying the property that every other object is a direct factor of a product of copies of it. In this paper we start with an additive category with products and images, e.g. a module category, and we are concerned with the homotopy category of complexes with entries in that additive category. We prove that the Brown representability theorem is valid for the dual of the homotopy category if and only if the initial additive category has a product generator.

Key words: Brown representability, Adjoint functors, Triangulated category, Homotopy category of complexes

1991 MSC: 18E30, 16D90, 55U35

Introduction

Brown representability is a key tool in the theory of triangulated categories. Recall that if $\mathcal{K}$ is a triangulated category with products then $\mathcal{K}^o$ is said to satisfy Brown representability if every homological product preserving functor $F : \mathcal{K} \to \mathcal{A}b$ is representable. Dually $\mathcal{K}$ satisfies Brown representability if every cohomological (contravariant) functor which sends coproducts into products $F : \mathcal{K} \to \mathcal{A}b$ is representable. Sometime we call these properties Brown representability for covariant, respectively contravariant functors.

The notion of well–generated triangulated category was introduced by Neeman in his influential book [16], where it is also shown that Brown representability holds for triangulated categories of this type. Prototypes for (algebraic)...
well–generated triangulated categories are derived categories and their local-
izations (see [19]). But until recently, only a very little was known about Brown
representability for homotopy categories of complexes. Only three papers [4],
[20] and [15] gave some information in this sense, the first for the homotopy
category over an abelian category without a generator and the other two for
homotopy category over module categories. The present work comes to com-
plete the picture started in [15]. To be precise, let \( R \) be a ring. We denote
by \( K(\text{Mod}(R)) \) the homotopy category of complexes of \( R \)-modules. In [15]
it is shown that \( K(\text{Mod}(R)) \) satisfies Brown representability if and only if
\( R \) is pure–semisimple. But for the dual \( K(\text{Mod}(R))^\circ \) only one direction was
shown: If \( K(\text{Mod}(R))^\circ \) satisfies Brown representability then \( \text{Mod}(R) \) must
have a product generator. Note that a product generator of an additive cate-
gory \( \mathcal{A} \) is defined to be an object \( G \) with the property that every object of \( \mathcal{A} \)
is a direct factor of a product of copies of \( G \). The module category over a pure
semi–simple ring \( R \) satisfies the dual property, namely \( \text{Mod}(R) = \text{Add}(G) \)
for some \( G \in \text{Mod}(R) \), where through \( \text{Add}(G) \) we understand the class of all
direct summands of direct sums of copies of \( G \). The main result in this paper
proves the equivalence between the conditions \( K(\text{Mod}(R))^\circ \) satisfies Brown
representability and \( \text{Mod}(R) \) has a product generator. Moreover our approach
may be easily dualized in order to give (a generalization of) results in [15]
about Brown representability for contravariant functors defined on homotopy
category of complexes.

The problem of Brown representability for covariant functors is difficult and
not completely solved even in the case of well–generated categories. For that
reason the method used to prove this kind of result deserves perhaps a few
words. In [14] we proved a generalization of Neeman’s variant of Brown repre-
sentability for contravariant functors defined on well–generated triangulated
categories. With this aim, we developed a technique, based on the fact that
every object of a well–generated category is the homotopy colimit of a tower
of objects which is constructed iteratively starting with a set. The whole con-
struction is analogous to the case of an object of an abelian category which is
filtered by a set (see [3, Definition 3.1.1]), but as usual, short exact sequences
are replaced by triangles. Naturally appeared the question if the construction
may be dualized in order to give some information about Brown representabil-
ity for covariant functors. Strictly in the setting of [14] the answer is probably
no, but we adapted here this method and we observed that if \( \text{Mod}(R) \) has a
product generator, then there is a set of complexes, such that every complex
in \( K(\text{Mod}(R)) \) is cofiltered by that set; roughly speaking, this means every
complex is isomorphic to the homotopy limit of an inverse tower constructed
iteratively starting with that set.

The paper is organized as follows: Section 1 contains a new proof of an old
(but seemingly not largely known) representability theorem due to Heller, for
functors \( F : \mathcal{K} \to \text{Ab} \), where \( \mathcal{K} \) is a triangulated category with products. Some
applications to much recent results are also indicated. Using this, we prove in
Section 2 a new representability theorem, supposing in addition that every
object of $\mathcal{K}$ is cofiltered by a set. Next Section 3 contains the main result
of this paper: If we consider an additive category $\mathcal{A}$ with split idempotents
and products, possessing images or kernels, then $\mathcal{K}(\mathcal{A})^o$ satisfies Brown repre-
sentability exactly if $\mathcal{A}$ has a product generator. In particular we apply this
for $\mathcal{A}$ being the module category over a ring $R$, thus $\mathcal{K}(\text{Mod}(R))^o$ satisfies
Brown representability if and only if $\text{Mod}(R)$ has a product generator.

1 A new proof for Heller’s representability theorem

Consider a preadditive category $\mathcal{K}$. We write $\mathcal{K}(K, K')$ for the abelian group
of morphisms between $K$ and $K'$ in $\mathcal{K}$. By a (right) $\mathcal{K}$-module we understand
a functor $X : \mathcal{K} \to \text{Ab}$. In this paper modules will always be at right, so for
dealing with a left $\mathcal{K}$-module we have to consider a right $\mathcal{K}^o$-module, that is
a functor $X : \mathcal{K} \to \text{Ab}$. A $\mathcal{K}$-module is called finitely presentable if there is an
exact sequence of functors

$$\mathcal{K}(-, K_1) \to \mathcal{K}(-, K_0) \to X \to 0$$

for some $K_0, K_1 \in \mathcal{K}$. We denote $\text{Hom}_\mathcal{K}(X, Y)$ the class of all natural trans-
formations between two $\mathcal{K}$-modules. Generally there is no reason for this class
to be a set. However, using Yoneda lemma, we know that $\text{Hom}_\mathcal{K}(X, Y)$ is actually
a set, provided that $X$ is finitely presentable. We consider the category
$\text{mod}(\mathcal{K})$ of all finitely presentable $\mathcal{K}$-modules, having $\text{Hom}_\mathcal{K}(X, Y)$ as mor-
phisms spaces, that is $\text{mod}(\mathcal{K})(X, Y) = \text{Hom}_\mathcal{K}(X, Y)$ for all $X, Y \in \text{mod}(\mathcal{K})$.

The Yoneda functor

$$H = H_\mathcal{K} : \mathcal{K} \to \text{mod}(\mathcal{K}^o)^o$$

is an embedding of $\mathcal{K}$ into $\text{mod}(\mathcal{K}^o)^o$, according to Yoneda lemma. Moreover
$\text{mod}(\mathcal{K}^o)^o$ has kernels. If, in addition, $\mathcal{K}$ has products then $\text{mod}(\mathcal{K}^o)^o$ is complete
and the Yoneda embedding preserves products. It is also well–known
(and easy to prove) that, if $F : \mathcal{K} \to \mathcal{A}$ is a functor into an additive category
with kernels, then there is a unique, up to a natural isomorphism, kernel pre-
serving functor $F^* : \text{mod}(\mathcal{K}^o)^o \to \mathcal{A}$, such that $F = F^*H_\mathcal{K}$ (see [11, Lemma
A.1]). Moreover, $F$ preserves products if and only if $F^*$ preserves limits.

Let $F : \mathcal{K} \to \text{Ab}$ be a functor. The category of elements of $F$, denoted by
$\mathcal{K}/F$, has as objects pairs of the form $(X, x)$ where $X \in \mathcal{K}$ and $x \in F(X)$, and
a map between $(X, x)$ and $(Y, y)$ in $\mathcal{K}/F$ is a map $f : X \to Y$ in $\mathcal{K}$ such that
$F(f)(x) = y$. Recall that the solution set condition for functors with values in
the category of abelian groups $F : \mathcal{K} \to \text{Ab}$ may be stated as follows: There
is a set $S$ of objects in $\mathcal{K}$, such that for any $K \in \mathcal{K}$ and any $y \in F(K)$ there are $S \in S$, $x \in F(S)$ and $f : S \to K$ satisfying $F(f)(x) = y$ (see [12, Chapter V, §6, Theorem 3]). We may reformulate this by saying that the category

$$S/F = \{(S, x) \mid S \in S, x \in F(S)\}$$

is weakly initial in $\mathcal{K}/F$, that is for every $(K, y) \in \mathcal{K}/F$ there exists a map $(S, x) \to (K, y)$ for some $(S, x) \in S/F$. Via Yoneda lemma, every object $(S, x) \in S/F$ corresponds to a natural transformation $\mathcal{K}(S, -) \to F$. In these terms, the existence of a solution set is further equivalent to the fact that there are objects $S_i \in \mathcal{K}$ indexed over a set $I$ and a functorial epimorphism

$$\bigoplus_{i \in I} \mathcal{K}(S_i, -) \to F \to 0.$$

We say that $F$ has a solution object provided that there is an object $S \in \mathcal{K}$ and a functorial epimorphism

$$\mathcal{K}(S, -) \to F \to 0,$$

or equivalently, the category $\mathcal{K}/F$ has a weakly initial object. Note that if there are arbitrary products in $\mathcal{K}$, and the functor $F$ preserves them, then the existence of a solution set is clearly equivalent to that of a solution object. Obviously if $F \cong \mathcal{K}(S, -)$ is representable, then $F$ has a solution object.

In the rest of this Section the category $\mathcal{K}$ will be triangulated with split idempotents. For definition and basic properties of triangulated categories the standard reference is [16]. Note that $\mathcal{K}$ has split idempotents, provided that $\mathcal{K}$ has countable coproducts or products, according to [16, Proposition 1.6.8] or its dual. Recall that $\mathcal{K}$ is supposed to be additive. A functor $\mathcal{K} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ is called homological if it sends triangles into exact sequences. A contravariant functor $\mathcal{K} \to \mathcal{A}$ which is homological regarded as a functor $\mathcal{K}^o \to \mathcal{A}$ is called cohomological (see [16, Definition 1.1.7 and Remark 1.1.9]). An example of a homological functor is the Yoneda embedding $H_\mathcal{K} : \mathcal{K} \to \text{mod}(\mathcal{K}^o)^o$. We know that in this case $\text{mod}(\mathcal{K}^o)^o$ is equivalent to $\text{mod}(\mathcal{K})$ (see [16, Remark 5.1.19 and what follows]). Moreover it is an abelian category, and for every functor $F : \mathcal{K} \to \mathcal{A}$ into an abelian category, the unique left exact functor $F^* : \text{mod}(\mathcal{K}^o)^o \to \mathcal{A}$ extending $F$ is exact if and only if $F$ is homological, by the dual of [10, Lemma 2.1]. Note that this is the reason for which $\text{mod}(\mathcal{K}^o)^o$ (or often the equivalent category $\text{mod}(\mathcal{K})$) is called the abelianization of the triangulated category $\mathcal{K}$. By [16, Corollary 5.1.23], $\text{mod}(\mathcal{K}^o)^o$ is a Frobenius abelian category, with enough injectives and enough projectives, which are, up to isomorphism, exactly objects of the form $\mathcal{K}(K, -)$ for some $K \in \mathcal{K}$.

Observe that in the particular case when the codomain of the homological functor $F$ is the category $\text{Ab}$ of all abelian groups, then it may be easily seen
that $F^*(X) \cong \text{Hom}_{K^o}(X, F)$, naturally for all $X \in \text{mod}(K^o)^o$. Thus we obtain:

**Lemma 1** If $K$ is a triangulated category with split idempotents, then a homological functor $F : K \to \text{Ab}$ is representable if and only if its extension $F^* : \text{mod}(K^o)^o \to \text{Ab}$ is representable.

**PROOF.** As before $F^*(X) \cong \text{Hom}_{K^o}(X, F)$, for all $X \in \text{mod}(K^o)^o$. If $F$ is representable, then $F \in \text{mod}(K^o)^o$, so $F^*$ is represented by $F$. Conversely if $F^*$ is representable by an object in $\text{mod}(K^o)^o$ then this object must be isomorphic to $F$, therefore $F \in \text{mod}(K^o)^o$. Because $F^*$ is exact, $F$ must be projective, hence representable (see [16, Lemma 5.1.11]).

**Lemma 2** If $K$ is a triangulated category with split idempotents, then a cohomological functor $F : K \to \text{Ab}$ has a solution object if and only if $F^* : \text{mod}(K^o)^o \to \text{Ab}$ has a solution object.

**PROOF.** Suppose $F$ has a solution object, i.e. there is a functorial epimorphism $H(K) = \text{K}(K, -) \to F \to 0$, with $K \in K$. In order to show that $F^*$ has a solution object, it is enough to prove that the induced natural transformation

$$
\text{Hom}_{K^o}(-, H(K)) \to \text{Hom}_{K^o}(-, F) \cong F^*
$$

is an epimorphism. That is, we want to show that the map

$$
\text{Hom}_{K^o}(X, H(K)) \to \text{Hom}_{K^o}(X, F)
$$

is surjective, for all $X \in \text{mod}(K^o)^o$. According to [16, 5.1.23] every finitely presentable $K^o$–module $X$ admits an embedding $0 \to X \to H(U)$, that is an epimorphism from the projective object $H(U)$ to $X$ in the opposite category $\text{mod}(K^o)^o$. Since $H(K) \in \text{mod}(K^o)^o$ is projective–injective and $F^*$ is exact, we obtain a diagram with exact rows:

$$
\begin{array}{ccc}
\text{Hom}_{K^o}(H(U), H(K)) & \longrightarrow & \text{Hom}_{K^o}(X, H(K)) \\
\downarrow & & \downarrow \\
\text{Hom}_{K^o}(H(U), F) & \longrightarrow & \text{Hom}_{K^o}(X, F) \\
\end{array}\to
$$

By Yoneda lemma we know that the first vertical map is isomorphic to $K(K, U) \to F(U)$, hence it is surjective, thus the diagram above proves the direct implication.

Conversely if there is $X \in \text{mod}(K^o)^o$ and a natural epimorphism

$$
\text{Hom}_{K^o}(-, X) \to \text{Hom}_{K^o}(-, F) \to 0,
$$

...
then let $H(K) \to X \to 0$ be an epimorphism in $\text{mod}(K^o)$ (that is a monomorphism in the opposite direction in $\text{mod}(K^o)^o$), with $K \in \mathcal{K}$. Consider the composed map

$$\text{Hom}_{K^o}(-, H(K)) \to \text{Hom}_{K^o}(-, X) \to \text{Hom}_{K^o}(-, F).$$

Evaluating it at $H(U)$ for an arbitrary $U \in \mathcal{K}$, we obtain a surjective natural map $K(K, U) \to F(U)$, hence $F$ has a solution object.

**Theorem 3** [6, Theorem 1.4] If $\mathcal{K}$ is a triangulated category with products, then a homological products preserving functor $F: \mathcal{K} \to \text{Ab}$ is representable if and only if it has a solution object.

**PROOF.** Under the hypotheses imposed on $\mathcal{K}$ and $F$, the abelian category $\text{mod}(\mathcal{K}^o)^o$ is complete and the induced functor $F^*: \text{mod}(\mathcal{K}^o)^o \to \text{Ab}$ preserves limits. Therefore it is representable if and only if it has a solution object, by Freyd’s Adjoint Functor Theorem. Thus the conclusion follows by combining Lemmas 1 and 2.

**Remark 4** Theorem 3 says more than the Neeman’s Freyd style representability theorem [17, Theorem 1.3]. Indeed the cited result states that if every cohomological functor which sends coproducts into products has a solution objects, then every such a functor is representable, whereas our result involves a fixed functor. However the result is known: It already appeared in Heller’s paper [6]. We have just proved the dual version because our argument is different from Heller’s one, and it shows us explicitly how the result follows from Freyd’s celebrated Adjoint Functor Theorem.

In the same sense in which Theorem 3 above is an improvement of [17, Theorem 1.3], we may improve [14, Theorem 3.7], which is the main result there and which uses Neeman’s result [17, Theorem 1.3] (for the unexplained terms see [14]):

**Corollary 5** Let $\mathcal{K}$ be a triangulated category with coproducts which is $\aleph_1$-perfectly generated by a projective class $\mathcal{P}$. If $F: \mathcal{K} \to \text{Ab}$ is a cohomological functor which sends coproducts to products, such that $\mathcal{P}^n/F$ has a weak terminal object for all $n \in \mathbb{N}$, then $F$ is representable.

In a particular case, namely in the presence of products, we may derive from the above results the dual of [18, Proposition 1.4]. In order to state this, recall that if $\mathcal{T}$ is a full subcategory of $\mathcal{T}$ then a $\mathcal{K}$-preenvelope of $T \in \mathcal{T}$ is a morphism $T \to X_T$ with $X_T \in \mathcal{K}$ such that the induced map $\mathcal{T}(X_T, X) \to \mathcal{T}(T, X)$ is surjective for all $X \in \mathcal{K}$. Dually we define the concept of precover. The subcategory $\mathcal{K}$ is called preenveloping is every object in $\mathcal{T}$ admits a $\mathcal{K}$-preenvelope.
Corollary 6 Let $\mathcal{T}$ be a triangulated category with products, and let $\mathcal{K}$ be a colocalizing subcategory. The following are equivalent:

(i) The inclusion $\mathcal{K} \to \mathcal{T}$ has a left adjoint.
(ii) Every object in $\mathcal{T}$ admits a $\mathcal{K}$–preenvelope.

PROOF. Since the implication (i)$\Rightarrow$(ii) follows from the general theory of adjoint functors, we only need to show the converse. But this follows immediately from Theorem 3 since, if $I : \mathcal{K} \to \mathcal{T}$ is the inclusion functor, then for every $T \in \mathcal{T}$ the functor $\mathcal{T}(T, I(\cdot)) : \mathcal{K} \to \text{Ab}$ is homological, preserves products and has a solution object, given by the functorial epimorphism $\mathcal{K}(X_T, \cdot) \to \mathcal{T}(T, I(\cdot))$, where $T \to X_T$ is a $\mathcal{K}$–preenvelope of $X$.

2 Cofiltered objects in triangulated categories

As before we denote by $\mathcal{K}$ a triangulated category with products. Let $\mathcal{S} \subseteq \mathcal{K}$ be a set of objects. We denote $\text{Prod}(\mathcal{S})$ the full subcategory of $\mathcal{K}$ consisting of all direct factors of products of objects in $\mathcal{S}$. We define inductively $\text{Prod}_0(\mathcal{S}) = \text{Prod}(\mathcal{S})$ and $\text{Prod}_n(\mathcal{S})$ is the full subcategory of $\mathcal{K}$ which consists of all objects $Y$ lying in a triangle $X \to Y \to Z \to X[1]$ with $X \in \text{Prod}_0(\mathcal{S})$ and $Y \in \text{Prod}_n(\mathcal{S})$. We suppose that $\mathcal{S}$ is closed under suspensions and desuspensions, so the same is true for $\text{Prod}_n(\mathcal{S})$, by [17] Remark 07. Moreover the same [17] Remark 07 tells us that if $X \to Y \to Z \to X[1]$ is a triangle with $X \in \text{Prod}_n(\mathcal{S})$ and $Y \in \text{Prod}_m(\mathcal{S})$ then $Z \in \text{Prod}_{n+m}(\mathcal{S})$. We say that an object $X \in \mathcal{K}$ is $\mathcal{S}$-cofiltered if it may be written as a homotopy limit $X \cong \text{holim} X_n$ of an inverse tower, with $X_0 \in \text{Prod}_0(\mathcal{S})$, and $X_{n+1}$ lying in a triangle $P_n \to X_{n+1} \to X_n \to P_n[1]$, for some $P_n \in \text{Prod}_0(\mathcal{S})$. Inductively we have $X_n \in \text{Prod}_n(\mathcal{S})$, for all $n \in \mathbb{N}$.

Lemma 7 Let $\mathcal{K}$ be a triangulated category and let $\mathcal{S} \subseteq \mathcal{K}$ be a set closed under suspensions and desuspensions. Suppose that every $X \in \mathcal{K}$ is $\mathcal{S}$-cofiltered. Then every homological product preserving functor $F : \mathcal{K} \to \text{Ab}$ has a solution object.

PROOF. We shall prove a statement equivalent to the conclusion, namely that the category of elements $\mathcal{T}/F$ has a weakly initial object. In order to do this, we shall apply the dual of the argument used in the proof of [14] Proposition 3.6. Since the hypotheses are slightly modified, we sketch here this argument (in the dual form appropriate to the present approach).
By [17] Lemma 2.3, we know that the category $\text{Prod}_n(\mathcal{S})/F$ has a weakly initial object denoted $(T_n, t_n)$, for all $n \in \mathbb{N}$. Let $I$ be the non-empty set of all inverse towers of the form

$$T_0 \leftarrow u_0 T_1 \leftarrow u_1 T_2 \leftarrow \cdots$$

with $F(w_n)(t_{n+1}) = t_n$, for all $n \in \mathbb{N}$, and denote by $T(i)$ the homotopy limit of the tower $i \in I$. By [1] Lemma 5.8(2), there is an exact sequence

$$0 \rightarrow \lim_{\leftarrow} F(T_n[1]) \rightarrow F(\lim_{\leftarrow} T_n) \rightarrow \lim_{\leftarrow} F(T_n) \rightarrow 0.$$  

Clearly $(t_n)_{n \in \mathbb{N}} \in \lim_{\leftarrow} F(T_n)$, thus there exists $t(i) \in F(T(i)) = F(\lim_{\leftarrow} T_n)$ which maps in $(t_n)_{n \in \mathbb{N}}$ via the surjective morphism above. Putting $T = \prod_{i \in I} T(i)$ and $t = (t(i))_{i \in I}$ we claim that $(T, t)$ is a weakly initial object in $\mathcal{K}/F$. In order to prove the claim, consider an object $X \in \mathcal{K}$. By hypothesis, there is an inverse tower

$$X_0 \leftarrow u_0 X_1 \leftarrow u_1 X_2 \leftarrow \cdots$$

whose homotopy limit is $X$ such that $X_0 \in \text{Prod}_0(\mathcal{S})$, and every $X_{n+1}$ lies in a triangle $P_n \rightarrow X_{n+1} \xrightarrow{a_n} X_n \rightarrow P_n[1]$, for some $P_n \in \text{Prod}_0(\mathcal{S})$. We use again [1] Lemma 5.8(2) for constructing the commutative diagram with exact rows:

$$
\begin{array}{c}
0 \rightarrow \lim_{\leftarrow} \mathcal{K}(T, X_n[-1]) \\
\downarrow \\
0 \rightarrow \lim_{\leftarrow} F(X_n[-1])
\end{array}
\begin{array}{c}
\mathcal{K}(T, \lim_{\leftarrow} X_n) \\
\downarrow \\
F(\lim_{\leftarrow} X_n)
\end{array}
\begin{array}{c}
\lim_{\leftarrow} \mathcal{K}(T, X_n) \\
\downarrow \\
\lim_{\leftarrow} F(X_n)
\end{array}
\rightarrow 0
$$

whose columns are induced by the natural transformations which correspond to $t \in F(T)$ under Yoneda Lemma. If we show that the two extreme vertical arrows are surjective, the same is true for the middle arrow too, and we are done. But for the first vertical map this follows by the commutative diagram:

$$\Pi_{n \in \mathbb{N}} \mathcal{K}(T, X_n[-1]) \rightarrow \lim_{\leftarrow} \mathcal{K}(T, X_n[-1])$$

$$\Pi_{n \in \mathbb{N}} F(X_n[-1]) \rightarrow \lim_{\leftarrow} F(X_n[-1])$$

whose arrows connected with the south-west corner are both surjective.

In order to prove that the third vertical map above is surjective, we consider an element $x \in \lim_{\leftarrow} F(X_n)$, that is $x = (x_n)_{n \in \mathbb{N}} \in \prod F(X_n)$ such that $F(u_n)(x_{n+1}) = x_n$, for all $n \in \mathbb{N}$. Next we construct a commutative diagram

$$
\begin{array}{cccccccc}
T_0 \xleftarrow{w_0} & T_1 \xleftarrow{w_1} & T_2 & \cdots \\
\downarrow{f_0} & \downarrow{f_1} & \downarrow{f_2} & \\
X_0 \xleftarrow{u_0} & X_1 \xleftarrow{u_1} & X_2 & \cdots
\end{array}
$$
whose upper line is a tower in $I$, and satisfying $F(f_n)(t_n) = x_n$ for all $n \in \mathbb{N}$. This construction is performed inductively as follows: $f_0$ comes from the fact that $(T_0, t_0)$ is weakly initial in $\text{Prod}_0(S)/F$. Suppose the first $n$ steps are done. We construct the following commutative diagram whose rows are triangles and the middle square is homotopy pull-back (see [16, Definition 1.4.1]):

\[
\begin{array}{ccc}
P_n & \rightarrow & Y_{n+1} \\
\downarrow & & \downarrow \\
X_{n+1} & \rightarrow & X_n \\
\uparrow & & \uparrow \\
P_n & \rightarrow & X_n \\
\end{array}
\]

The upper triangle shows that $Y_{n+1} \in \text{Prod}_{n+1}(S)$ where $(T_{n+1}, t_{n+1})$ is weakly initial, hence we find a map $(T_{n+1}, t_{n+1}) \rightarrow (Y_{n+1}, y_{n+1})$ in $\text{Prod}_{n+1}(S)/F$. Now $Y_{n+1}$ is obtained via a triangle

\[ Y_{n+1} \rightarrow T_n \times X_{n+1} \xrightarrow{(f_n, -u_n)} X_n \rightarrow Y_{n+1}[1]. \]

Applying the homological functor $F$ we get an exact sequence:

\[ F(Y_{n+1}) \rightarrow F(T_n) \times F(X_{n+1}) \xrightarrow{(F(f_n), F(-u_n))} F(X_n). \]

Since $F(f_n)(t_n) - F(u_n)(x_{n+1}) = x_n - x_n = 0$ we get an element $y_{n+1} \in Y_{n+1}$, which maps in $(t_n, x_{n+1})$, via the first morphism of the exact sequence above. The morphism $f_{n+1}$ is the composition $T_{n+1} \rightarrow Y_{n+1} \rightarrow X_{n+1}$. The upper row above is, as noticed, an inverse tower in $I$, and let denote it by $i$. Finally the element $t \in T$ maps to $(x_n)_{n \in \mathbb{N}} \in \lim_{\leftarrow} F(X_n)$, via the map $F(T) \rightarrow F(T(i)) \rightarrow \lim_{\leftarrow} F(T_n) \rightarrow \lim_{\leftarrow} F(X_n)$, proving that the map $\lim_{\leftarrow} \mathcal{K}(T, X_n) \rightarrow \lim_{\leftarrow} F(X_n)$ is surjective.

Combining Theorem 3 and Lemma 7 we obtain:

**Theorem 8** Let $\mathcal{K}$ be a triangulated category. Suppose there is a set $S \subseteq \mathcal{K}$ closed under suspensions and desuspensions, such that every $X \in \mathcal{K}$ is $S$-cofiltered. Then every homological product preserving functor $F : \mathcal{K} \rightarrow \text{Ab}$ is representable, therefore $\mathcal{K}^\circ$ satisfies Brown representability.

### 3 Brown representability for the dual of a homotopy category

Throughout this section $\mathcal{A}$, will denote an additive category, that is preadditive, with zero object and finite biproducts; we suppose also that $\mathcal{A}$ has split idempotents. We consider categories $\mathcal{C}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ called the category of complexes respectively the homotopy category of complexes over $\mathcal{A}$, both of
them having as objects complexes of objects in $\mathcal{A}$, that is a chain of objects and morphisms (called *differentials*) in $\mathcal{A}$ of the form

$$X = \cdots \to X^{n-1} \xrightarrow{d_{X}^{n-1}} X^{n} \xrightarrow{d_{X}^{n}} X^{n+1} \to \cdots,$$

such that $d_{X}^{n}d_{X}^{n-1} = 0$ for all $n \in \mathbb{Z}$. The morphisms in the category $\mathbf{C}(\mathcal{A})$ are families $(f_{n})_{n \in \mathbb{Z}}$ of morphisms in $\mathcal{A}$ commuting with differentials, and

$$\mathbf{K}(\mathcal{A})(X, Y) = \mathbf{C}(\mathcal{A})(X, Y)/ \sim$$

where $\sim$ is an equivalence relation called *homotopy*, defined as follows: two maps of complexes $(f_{n})_{n \in \mathbb{Z}}, (g_{n})_{n \in \mathbb{Z}} : X \to Y$ are homotopically equivalent if there is $s_{n} : X^{n} \to Y^{n-1}$, for all $n \in \mathbb{Z}$ such that $f_{n} - g_{n} = d_{Y}^{n-1}s_{n} + s_{n+1}d_{X}^{n}$. Note that $\mathbf{C}(\mathcal{A})$ is an exact category (in the sense of [8, Section 4]) with respect to all short exact sequences which split in each degree (see [8, Example 4.3]), and $\mathbf{K}(\mathcal{A})$ may be constructed as the stable category of this exact category by [8, Example 6.1]. Hence $\mathbf{K}(\mathcal{A})$ is a triangulated category. Note that the structure of triangulated category comes with a translation functor denoted by $[1]$, where $X[1]^{n} = X^{n+1}$ and $d_{X[1]}^{n} = -d_{X}^{n+1}$. It is well–known that $\mathbf{K}(\mathcal{A})$ has (co)products provided that $\mathcal{A}$ does the same. Considering every object in $\mathcal{A}$ as a complex concentrated in degree 0, the category $\mathcal{A}$ may be embedded in $\mathbf{K}(\mathcal{A})$.

Fix the additive category $\mathcal{A}$ as before. For an object $G \in \mathcal{A}$ we denote by $\text{Prod}(G)$ respectively $\text{Add}(G)$ the full subcategory consisting of direct factors (or equivalently, direct summands) of a product (respectively coproduct) of copies of $G$ (assuming that the requested products or coproducts exist). We say that $\mathcal{A}$ has a *product generator* if there is an object $G \in \mathcal{A}$ such that $\mathcal{A} = \text{Prod}(G)$. For the dual situation when $\mathcal{A} = \text{Add}(G)$ we use the more standard terminology $\mathcal{A}$ is *pure–semisimple* (see [20, Definition 2.1 and Proposition 2.2]).

**Lemma 9** Let $\mathcal{A}$ be an additive category with split idempotents and products, which possesses a product generator $G$. Denote $\mathcal{S} = \{G[n] \mid n \in \mathbb{Z}\}$ the closure of $G$ under suspensions and desuspensions in $\mathbf{K}(\mathcal{A})$.

a) If given two composable maps $X \to Y \to Z$ whose composition is 0 in $\mathcal{A}$, then $X \to Y$ factors through a subobject $Y' \leq Y$ such that the composed map $Y' \to Y \to Z$ vanishes, then $\mathbf{K}(\mathcal{A})$ is $\mathcal{S}$-cofiltered.

b) If $\mathcal{A}$ has images or kernels, then $\mathbf{K}(\mathcal{A})$ is $\mathcal{S}$-cofiltered.

**PROOF.** a) We will show inductively that a bounded complex with less than $n + 1$ non–zero entries is in $\text{Prod}_{n}(\mathcal{S})$, where $n$ runs over all positive integers. This is clear for $n = 0$, since $G$ is a product generator of $\mathcal{A}$. Now we suppose the property true for any complex with $\leq n$ non–zero entries. Let

$$\cdots \to 0 \to X^{0} \to \cdots \to X^{n} \to 0 \to \cdots$$

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be a bounded complex. The diagram

\[
\begin{array}{cccccccc}
\cdots & 0 & 0 & \longrightarrow & X^n & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & & \\
\cdots & 0 & X^0 & \longrightarrow & X^{n-1} & X^n & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & & \\
\cdots & 0 & X_0 & \longrightarrow & X^{n-1} & 0 & \longrightarrow & 0 \\
\end{array}
\]

is an exact sequence of complexes which splits in each degree. According to [8, Example 6.1] it leads to a triangle proving the induction step.

Finally consider an infinite complex

\[X = \cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots.\]

By hypothesis, the map \(d^{n-1}\) factors through a subobject \(Y^n \leq X^n\), such that \(Y^n \longrightarrow X^n \xrightarrow{d^n} X^{n+1}\) vanishes, for all \(n \in \mathbb{Z}\). For all \(i \in \mathbb{N}\), consider the bounded complex

\[X(i) = \cdots \longrightarrow 0 \longrightarrow Y^{-i} \longrightarrow X^{-i} \longrightarrow X^{-i+1} \longrightarrow \cdots \longrightarrow X^{i-1} \longrightarrow X^i \longrightarrow 0 \longrightarrow \cdots,
\]

and the map of complexes \(\epsilon(i) : X(i + 1) \rightarrow X(i)\) as in the following diagram:

\[
\begin{array}{cccccccc}
\cdots & 0 & \longrightarrow & Y^{-i} & \longrightarrow & X^{-i} & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & & \\
\cdots & Y^{-i-1} & \longrightarrow & X^{-i-1} & \longrightarrow & X^{-i} & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & & \\
\cdots & X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots \\
\end{array}
\]

Applying [7, Lemma 2.6] we infer that \(X\) is isomorphic in \(K(A)\) to the homotopy limit of a the chain of bounded complexes

\[\cdots \longrightarrow X(2) \xrightarrow{\epsilon(1)} X(1) \xrightarrow{\epsilon(0)} X(0),\]

thus \(X\) is \(S\)-cofiltered.

b) We apply a) with \(Y^n = \text{im } d^{n-1}\) or \(Y^n = \ker d^n\), for all \(n \in \mathbb{Z}\).

**Theorem 10** Let \(A\) be an additive category with products and split idempotents, possessing also images or kernels. Then \(K(A)^o\) satisfies Brown representability if and only if \(A\) has a product generator. In particular, if \(R\) is a ring then \(K(\text{Mod}(R))^o\) satisfies Brown representability if and only if \(\text{Mod}(R)\) has a product generator.

**Proof.** The direct implication is [15, Theorem 2], whereas the converse follows by Lemma [9, b) and Theorem [8]. Finally note that the category \(\text{Mod}(R)\) is additive with products and has both images and kernels.
Remark 11 If the ring $R$ is pure–semisimple, then $\text{Mod}(R) = \text{Add}(G)$ for some $G \in \text{Mod}(R)$ (in fact $G$ is the direct sum of a family of representatives of all isomorphism classes of finitely presentable modules). In this case, $\text{Add}(G)$ is closed under products, so $G$ is product–complete hence $\text{Add}(G) = \text{Prod}(G)$ (see [9, Theorem 6.7]). Consequently $\mathcal{K}(\text{Mod}(R))^\circ$ satisfies Brown representability, by Theorem above. This was already known since $\text{Mod}(R)$ is a pure–semisimple finitely presentable category which is closed under products, so it is compactly generated by [20, Theorem 5.2]. It would be therefore interesting to characterize the class of rings $R$ for which the module category $\text{Mod}(R)$ has a product generator. If we could indicate a non pure–semisimple ring belonging to this class, then we would produce an example of a triangulated category with products and coproducts, namely $\mathcal{K} = \mathcal{K}(\text{Mod}(R))$ such that $\mathcal{K}^\circ$, but not $\mathcal{K}$, satisfies Brown representability. To the best of our knowledge, such an example is yet unknown. Note added in proof: It seems that a ring $R$ for which $\text{Mod}(R)$ has a product generator is pure–semisimple (see [2]), therefore Brown representability and its dual are equivalent for $\mathcal{K}(\text{Mod}(R))$.

Remark 12 There is an isomorphism of categories $\mathcal{K}(\mathcal{A})^\circ \rightarrow \mathcal{K}(\mathcal{A}^\circ)$, which is easy to establish (for example, this is written down in [13, Theorem 2.1.1]). Applying this isomorphism of categories, we may dualize all results in this section. Thus we may conclude that if $\mathcal{A}$ is an additive category with split idempotents and coproducts, possessing also images or cokernels, then $\mathcal{K}(\mathcal{A})$ satisfies Brown representability theorem if and only if $\mathcal{A}$ is pure–semisimple. Note that this statement is already known for $\mathcal{A} = \text{Mod}(R)$, or more generally for a finitely accessible category with coproducts $\mathcal{A}$, as we may see by a combination between [15, Theorem 1] and [20, Proposition 2.6]. However the results in [15] and [20] may not be dualized in order to obtain Theorem 10 since the argument used there for showing that $\mathcal{K}(\mathcal{A})$ satisfies Brown representability, where $\mathcal{A}$ is a pure–semisimple, finitely accessible additive category with coproducts goes as follows: If $\mathcal{A}$ enjoys all these properties, then $\mathcal{K}(\mathcal{A})$ is well generated by [20, Theorem 5.2], therefore it satisfies Brown representability by [16, Theorem 8.3.3 and proposition 8.4.2]. But none of the notions “module category”, “finitely accessible category” and “well generated triangulated category” is self–dual.

Remark 13 Let $R$ be a ring with $\text{gldim} \ R \leq 1$. Then the category $\text{Inj-}R$ of all injective modules is additive, closed under products, idempotents and images and every injective cogenerator of $\text{Mod}(R)$ is a product generator for $\text{Inj-}R$. Thus Theorem 10 gives another proof for the fact that $\mathcal{K}(\text{Inj-}R)^\circ$ satisfies Brown representability. This fact is already known since $\mathcal{K}(\text{Inj-}R)$ is equivalent to the derived category which is compactly generated.

Example 14 In the Introduction we said that this paper completes the picture in [14]. Note that [14, Theorem 3] gives an example of a triangulated coproduct preserving functor which has no right adjoint, namely the inclusion.
functor $\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}^0)$, where $\mathcal{A}$ is the full subcategory of all flat Mittag-Leffler abelian groups. Using the equivalence of categories $\mathbf{K}(\mathcal{A})^0 \cong \mathbf{K}(\mathcal{A}^0)$ from Remark 12, we obtain a triangulated product preserving functor which has no left adjoint.

Here we will provide another example of this kind, which holds only in an extension of ZFC. More precisely, assume there are no measurable cardinals. For every cardinal $\lambda$ let us denote by $\mathbb{Z}^\lambda$ the product of $\lambda$-copies of $\mathbb{Z}$ and by $\mathbb{Z}^{<\lambda}$ its subgroup consisting of sequences with support (i.e. the set of non-zero entries) of cardinality smaller then $\lambda$. Let $\mathcal{A} \subseteq \mathcal{A}^0$ be the closure under products and direct factors of the class of all abelian groups of the form $\mathbb{Z}^\lambda/\mathbb{Z}^{<\lambda}$, where $\lambda$ runs over all regular cardinals. The inclusion functor $\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}^0)$ is triangulated and preserves products. If we suppose that it has a left adjoint then $\mathbf{K}(\mathcal{A})$ must be preenveloping in $\mathbf{K}(\mathcal{A}^0)$ by Corollary 6. For $A \in \mathcal{A}$, the complex having $X$ in degree 0 and 0 elsewhere must have an $\mathbf{K}(\mathcal{A})$-preenvelope, which is a complex $X$ with entries in $\mathcal{A}$. It is not hard to see that $X \to X^0$ is an $\mathcal{A}$-preenvelope on $A$. But this contradicts [3, Proposition 2.5], where it is shown that, under the hypothesis of nonexistence of measurable cardinals, the class $\mathcal{A}$ is not preenveloping in $\mathcal{A}^0$.

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