Gδ SETS IN σ-IDEALS GENERATED BY COMPACT SETS

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Abstract. Given a compact Polish space E and the hyperspace of its compact subsets K(E), we consider the class of Gδ σ-ideals of compact subsets of E that can be represented via a compact subset of K(E). If we extend such an ideal I by considering Gδ (or analytic) sets that are covered by countable unions of sets in I, we show that the extended collection can still be represented via some compact subset of K(E).

1. Introduction

Let E be a compact Polish space and let K(E) denote the hyperspace of its compact subsets, equipped with the Vietoris topology. A nonempty set I ⊆ K(E) is an ideal of compact sets if it is closed under the operations of taking subsets and finite unions. An ideal I is a σ-ideal if it is also closed under countable unions whenever the union itself is compact. Such families arise commonly in analysis out of various notions of smallness.

It is now long established that the condition of being an ideal or σ-ideal of compact sets is strongly related to the complexity of I considered as a subset of K(E). By a result of Dougherty-Kechris and Louveau (see [2]), we know that if I is a Gδ ideal, it must in fact be a σ-ideal, and by results of Kechris–Louveau–Woodin proved in the seminal paper [3, section 1], we know that if a σ-ideal is co-analytic or analytic, it must be either complete co-analytic or simply Gδ. Having established this dichotomy [3] focused more on co-analytic ideals than on the class of Gδ ideals, however the latter includes rich examples and its theory is far from trivial.

In this article we consider Gδ σ-ideals of compact sets that also satisfy the following condition, formulated by Solecki in [7]: a collection of compact sets I ⊆ K(E) has property (*) if, for any sequence of sets (Kn)n∈N ⊆ I, there exists a Gδ set G such that ∪n Kn ⊆ G and K(G) ⊆ I. While property (*) is stronger than the condition of being a σ-ideal, it holds in all natural examples of Gδ σ-ideals, including the ideals of compact meager sets, measure-zero sets, sets of dimension ≤ n for fixed n ∈ N, and Z-sets. (See [7] for these and other examples and a discussion of property (⋆).) Solecki has shown that any such ideal is represented via the meager ideal of a closed subset of K(E), as follows. For A ⊆ E, we define A+ as the set of all compact subsets

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of $E$ that intersect $A$, i.e.,

$$A^* = \{ K \in \mathcal{K}(E) : K \cap A \neq \emptyset \}.$$ 

The representation theorem ([7, section 3]) says that, if $I \subseteq \mathcal{K}(E)$ is coanalytic and nonempty, then $I$ has property (*) if and only if there exists a closed set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that

$$\forall K \in \mathcal{K}(E), \ K \in I \iff K^* \text{ is meager in } \mathcal{F}.$$ 

Here when we say that $K^*$ is meager in $\mathcal{F}$, we mean that $K^* \cap \mathcal{F}$ is relatively meager in $\mathcal{F}$. (This representation is a ‘category analogue’ to a result of Choquet (see [1]) that establishes a correspondence between alternating capacities of order $\infty$ on $E$ and probability Borel measures on $\mathcal{K}(E)$.)

We have shown in [6] that as long as $I$ has no nonmeager sets the set $\mathcal{F}$ can be assumed to be upward closed, i.e., it contains all compact supersets of its members. The upward closedness of $\mathcal{F}$ ensures that the map $K \mapsto K^* \cap \mathcal{F}$, a fundamental function in this context, is continuous.

Motivated by results on $G_\delta$ ideals with property (*), Solecki has framed in [7, section 7] some natural further questions about the relationship of $G_\delta$ subsets of $E$ to a representing set $\mathcal{F}$. The present paper takes up the following: given a $G_\delta$ ideal $I$ with property (*), does $I$ have a representing set $\mathcal{F}$ for which, given a $G_\delta$ set $G \subseteq E$, $G^*$ will be meager in $\mathcal{F}$ precisely when $G$ is covered by countably many compact sets in $I$? We answer in the affirmative, and if the ideal under consideration contains only sets with empty interiors, we construct a representing set $\mathcal{F}$ that is also upward closed.

(The other questions remain open, including one about whether every thin $G_\delta$ ideal has property (*).)

**Theorem 1.** Let $E$ be a compact Polish space and let $I \subseteq \mathcal{K}(E)$ be a $G_\delta$ ideal of compact sets, with property (*). Then there exists a compact set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that any $G_\delta$ subset $G$ of $E$ is covered by countably many sets in $I$ if and only if $G^*$ is meager in $\mathcal{F}$.

Further, if $I$ contains only meager sets, then there exists a set $\mathcal{F}$ as above that is also upward closed.

In the statement of the theorem we can in fact replace $G_\delta$ sets by analytic sets by invoking Solecki’s theorem from [8]: for any family of compact subsets of a Polish space $E$, an analytic subset of $A$ of $E$ cannot be covered by countably many sets from the family if and only if $A$ has a $G_\delta$ subset that cannot be so covered. We give this stronger result as a corollary.

**Corollary 2.** Let $E$ be a compact Polish space and let $I \subseteq \mathcal{K}(E)$ be a $G_\delta$ ideal of compact sets, with property (*). Then there exists a compact set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that any analytic subset $A$ of $E$ is covered by countably many sets in $I$ if and only if $A^*$ is meager in $\mathcal{F}$.

Further, if $I$ contains only meager sets, then there exists a set $\mathcal{F}$ as above that is also upward closed.
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Note that the set $\mathcal{F}$ (of both the theorem and the corollary) will indeed be a representing set for $I$, i.e., for any compact $K \subseteq E$, the membership of $K$ in $I$ will be characterized by the meagerness of $K^*$ in $\mathcal{F}$. (The trivial proof of this is indicated in Proposition 3)

We remark that the condition that $I$ contain only meager sets is necessary if the set $\mathcal{F}$ of Theorem 1 is to be upward closed (indeed, if any representing set for $I$ is to be upward closed). For if $\mathcal{F} \subseteq \mathcal{K}(E)$ is nonempty and upward closed and $U \subseteq E$ is nonempty and open, the set $U^* \cap \mathcal{F}$, an open subset of $\mathcal{F}$, is automatically nonempty. Thus if any set $K \subseteq E$ has nonempty interior, $K^*$ will have nonempty interior in $\mathcal{F}$.

A final remark on a consequence of Theorem 1. In some cases, the notion of smallness that defines $I$ can also be applied in a natural way to $G_\delta$ sets. (For example, $I$ might consist of compact nullsets with respect to Lebesgue measure on $[0,1]$; here the notion of being a nullset applies to $G_\delta$ sets as well.) Theorem 1 implies that a representing set $\mathcal{F}$ that works to identify small compact sets $K$ (via the meagerness of $K^*$ in $\mathcal{F}$) need not work to identify small $G_\delta$ sets. (Continuing with the same example, the theorem gives us a set $\mathcal{F}$ such that, if $K \subseteq E$ is compact then $K^*$ is meager in $\mathcal{F}$ exactly when $K$ is a nullset — but given a $G_\delta$ set $G$, the meagerness of $G^*$ in $\mathcal{F}$ depends not at all on whether $G$ is null, but only on whether $G$ is covered by countably many compact nullsets.)

2. Preliminaries

We will say that a set $\mathcal{A} \subseteq \mathcal{K}(E)$ is downward closed if $\mathcal{A}$ contains all compact subsets of its members. We will use the notation $G_\delta(E)$ for the collection of all $G_\delta$ subsets of $E$.

The next proposition indicates the approach we will take in proving Theorem 1 if $\mathcal{F} \subseteq \mathcal{K}(E)$ is known to be a representing set for $I$, it suffices to consider $G_\delta$ sets all of whose nonempty open subsets have closure not in $I$.

**Proposition 3.** Let $E$ be a compact Polish space and let $I \subseteq \mathcal{K}(E)$ be a nonempty $G_\delta$ ideal of compact sets, with property (*). Let $\mathcal{F} \subseteq \mathcal{K}(E)$. Then the following holds

1. $\forall G \in G_\delta(E),$
   
   \[
   \left( G \subseteq \bigcup_{n \in \mathbb{N}} K_n \text{ for some } K_n \in I \right) \iff \left( G^* \text{ meager in } \mathcal{F} \right)
   \]

   if and only if the two conditions below hold:

2. $\forall K \in \mathcal{K}(E), \ K \in I \iff K^* \text{ is meager in } \mathcal{F},$

   and

3. $\forall \text{ nonempty } G \in G_\delta(E),$

   \[
   \left( \forall \text{ open } U, \ U \cap G \neq \emptyset \Rightarrow \overline{U \cap G} \notin I \right) \implies \left( G^* \text{ is nonmeager in } \mathcal{F} \right).
   \]
Proof. Let (1) hold. The forward implication of (2) is immediate because compact sets are $G_δ$; the converse is immediate in light of property (*).

To prove (3), suppose that $G$ is a nonempty $G_δ$ set such that for every nonempty relatively open subset of $G$ has closure not in $I$. By (1), to show that $G^*$ is nonmeager in $F$, it suffices to show that $G$ cannot be covered by countably many sets in $I$. So suppose $G \subseteq \bigcup_n F_n$ for some $F_n \in I$. Since $G$ is nonempty and Polish and each $F_n \cap G$ is relatively closed in $G$, for some $n \in \mathbb{N}$, $F_n \cap G$ contains a nonempty relatively open subset, say $U$, of $G$. Now $U \subseteq F_n \cap G \subseteq F_n \in I$, a contradiction.

Now let (2) and (3) hold. The implication from left to right in (1) is immediate because if $G \subseteq \bigcup_n K_n$ then $G^* \subseteq \bigcup_n K_n^*$. To prove the other direction, suppose $G$ is a $G_δ$ set not covered by the countable union of any sets in $I$. Set $G_0 = G$ and recursively define for successor ordinals $\alpha + 1$ and limit ordinals $\lambda$ the sets

$$G_{\alpha+1} = G_{\alpha} \setminus \bigcup \{ U : U \subseteq G_{\alpha} \text{ relatively open in } G_{\alpha} \text{ and } \overline{U} \in I \};$$

$$G_{\lambda} = \bigcap_{\alpha < \lambda} G_{\alpha}.$$

For each ordinal $\alpha$, $G_\alpha$ is a closed subset of $G$. So for some $\alpha_0 < \omega_1$, $G_{\alpha_0} = G_{\alpha_0 + 1}$. If $G_{\alpha_0}$ were empty, then $G$ would be covered by a countable number of sets $U$ with $\overline{U} \in I$, a contradiction. (Countability can be obtained by considering open sets only from some countable basis.) So $G_{\alpha_0}$ is nonempty, and the closure of every relatively open nonempty subset of it is outside $I$. Now by (3), $G_{\alpha_0}^*$ is nonmeager in $F$ and therefore so is $G^*$. □

We make a remark for later use. The transfinite process described in the proof above can be carried out on any closed subset $F$ of $E$. At each stage, the set we are removing from $F$ can be written as the countable union of closed sets in $I$. So we can write $F$ as

$$F = F' \cup \bigcup_{n \in \mathbb{N}} F_n,$$

where each $F_n$ is a compact set in $I$, and $F'$ is a closed subset of $F$, which, if nonempty, is everywhere big with respect to $I$.

In light of Proposition 3 to prove Theorem 1, we may find a compact $\mathcal{F} \subseteq \mathcal{K}(E)$ that serves as a representing set for $I$ and then simply show (3).

To examine the meagerness in $\mathcal{F}$ of some subset $A$ of $\mathcal{F}$, we will employ the Banach-Mazur game in $\mathcal{F}$ on $A$, which proceeds thus. Players I and II take turns playing nonempty open subsets of $\mathcal{F}$ as follows:

Player I

$V_0 \subseteq V_1 \subseteq \ldots$

Player II

$W_0 \subseteq W_1 \subseteq \ldots$

satisfying $V_0 \supseteq W_0 \supseteq V_1 \supseteq W_1 \supseteq \ldots$ Player II wins this run of the game if $\bigcap_n W_n \subseteq A$. The key fact about this game is that Player II has a winning strategy if and only if $A$ is comeager in $\mathcal{F}$; see, for example, Section 8.H of [4] for a proof.
We will say that a subset of $E$ is “everywhere big with respect to $I$” if all of its nonempty relatively open subsets have closure not in $I$.

We will make use of the basis for the Vietoris topology on $K(E)$ that consists of sets of the form 

$$\{ F \in K(E) : F \subseteq U_0, \ F \cap U_i \neq \emptyset \ \forall i = 1, \ldots, k \}$$

for some $k \in \mathbb{N}$ and open sets $U_0, \ldots, U_k \subseteq E$, with $U_i \subseteq U_0$ for each $i$. This topology is metrized by the Hausdorff metric, derived from the metric on $E$. Fix a complete metric on $E$ and for $x \in E$, let $B(x, r)$ denote the open ball of radius $r$ with respect to this metric. The Hausdorff distance between nonempty compact sets $F$ and $K$ is then the infimum of all $\epsilon$ such that $F \subseteq \bigcup_{x \in K} B(x, \epsilon)$ and $K \subseteq \bigcup_{x \in F} B(x, \epsilon)$.

Finally, we shall make use of the following equivalence proved by Solecki in [7, section 5]. Solecki has shown that for a nonempty set $I \subseteq K(E)$, $I$ is a $G_\delta$ set with property (*) if and only if there exists a sequence of open, downward closed sets $U_n \subseteq K(E)$, $n \in \mathbb{N}$, such that $I = \bigcap_n U_n$ and the sets $U_n$ satisfy the condition that

\[ \forall K \in U_n \ \exists m \in \mathbb{N} \ \forall L \in U_m \ K \cup L \in U_n. \]

For many $G_\delta$ ideals with property (*), the sets $U_n \subseteq K(E)$ refered to here present themselves naturally. For example, for the ideal of closed null sets for an outer regular Borel measure $\mu$, we may define the sets $U_n$ by setting

$$U_n = \{ K \in K(E) : K \subseteq U \text{ for some open } U \text{ such that } \mu(U) < \frac{1}{n} \}.$$ 

It is straightforward to see that the sets $U_n$ satisfy (4) and their intersection is $I$.

### 3. Proving the theorem

The principal part of the proof of Theorem 1 is the construction of a suitable closed and upward closed set $F$ for the case where $I$ contains only meager sets. This construction requires $E$ to have infinitely many limit points, and also requires the existence of at least one meager compact set with at least three points that is everywhere big with respect to $I$. This necessitates the treatment of two relatively trivial special cases in a separate proposition, which we shall dispense with:

**Proposition 4.** Let $E$ be a compact Polish space and let $I \subseteq K(E)$ be a $G_\delta$ ideal of compact sets, with property (*), containing only meager sets. If either one of the following conditions holds:

1. $E$ has only finitely many limit points, or
2. every meager compact subset of $E$ that is everywhere big with respect to $I$ has at most two points,

then the conclusion of Theorem 1 holds.
Proof. Since $I$ has only meager sets, $I$ has some closed and upward closed set representing set $F \subseteq \mathcal{K}(E)$. Let $G \subseteq E$ be a $G_\delta$ set that is everywhere big with respect to $I$; we show that $G^*$ is nonmeager in $F$.

Suppose that $E$ has only finitely many limit points. If $G$ contains any isolated point, then $G$ has nonempty interior and so $G^*$ is nonmeager in $F$. On the other hand, if $G$ contains no isolated point of $E$, it must be a finite, and therefore closed, set (and it is nonempty). So we have $G = \overline{G} \notin I$, and again $G^*$ is nonmeager in $F$.

Coming to the other condition, suppose that every meager compact subset of $E$ that is everywhere big with respect to $I$ has at most two points. If $x \in E$ belongs to some such finite everywhere big meager set, then clearly $\{x\}$ is not in $I$. Note that there can be at most two points altogether, say $x_1$ and $x_2$, that show up in such sets. (If there were three distinct points making an appearance in finite everywhere big meager sets, then the three points together would comprise another such set.)

Take now our nonempty $G_\delta$ set $G$ which is everywhere big with respect to $I$. If $G$ contains one of the $x_i$, then $\{x_i\}^* \subseteq G^*$, and so $G^*$ is not meager in $F$. On the other hand, if $G$ contains no such $x_i$, then by replacing $G$ with a suitable nonempty relatively open subset of itself if necessary, we may assume that $\overline{G}$ contains no such $x_i$ either. Now write $\overline{G}$ as

$$\overline{G} = G \cup \bigcup_{n \in \mathbb{N}} F_n,$$

where each $F_n$ is a relatively closed and relatively meager subset of $\overline{G}$. Each $F_n$ is also a closed meager subset of $E$, and by the remark following Proposition 3, can be written as

$$F_n = F'_n \cup \bigcup_{m} F^n_{m},$$

where $F'_n$ is either empty or a meager compact subset of $E$ that is everywhere big with respect to $I$, and each $F^n_{m}$ is in $I$. Since $F'_n$ is disjoint from $\{x_1, x_2\}$, it must simply be empty. We now have

$$\overline{G} = G \cup \bigcup_{n,m} F^n_{m},$$

and so

$$\overline{G}^* = G^* \cup \bigcup_{n,m} F^n_{m}^*.$$

Since each $F^n_{m}^*$ is meager in $F$, but $\overline{G}^*$ is nonmeager in $F$, it must be the case that $G^*$ is nonmeager in $F$. □

Note that the preceding proposition has covered the case of the ideal of meager compact sets (denoted $\text{MGR}(E)$), because this ideal satisfies the second condition: there are no nonempty meager compact subsets of $E$ that are everywhere big with respect to $I$. 


We now turn to the construction of a set $F$ as in Theorem 1 in the case where $I$ has only meager set. The construction has two stages. First, we construct suitable representing sequences of closed and upwards closed sets for $I$. Second, we join these countably many sets into a single $F$ that will satisfy the conclusion of Theorem 1. For ease of exposition we separate the first stage into a proposition of its own. But first, a lemma:

**Lemma 5.** Let $F$ be a meager compact subset of a Polish space $E$. Then we may find a sequence of pairwise disjoint regular closed sets $(F_n)$ converging to and disjoint from $F$.

**Proof.** Since $F$ is meager, for each $\epsilon > 0$ we can find a finite nonempty set disjoint from $F$ whose Hausdorff distance from $F$ is less than $\epsilon$, and of course positive. This allows us to successively construct a sequence of pairwise disjoint finite sets $F_n$ converging to and disjoint from $F$. By successively replacing each finite set $F_n$ with a set of the form $\bigcup_{x \in F_n} B(x, \delta_n)$ for a suitably chosen positive $\delta_n$, we obtain a sequence satisfying the required conditions. □

**Proposition 6.** Let $E$ be a compact Polish space and let $I \subseteq \mathcal{K}(E)$ be a $G_\delta$ ideal of compact sets, with property $(\ast)$, containing only meager sets. Let $U$, $V$ be open subsets of $E$ with disjoint closures and suppose that $V$ contains a meager set $M$ that is everywhere big with respect to $I$. Then there exist closed and upward closed sets $F_m \subseteq \mathcal{K}(E)$, $m \in \mathbb{N}$, such that:

1. $\forall K \in \mathcal{K}(E), K \notin I \implies \exists m \ F_m \subseteq K^*$;
2. $\forall K \in \mathcal{K}(E), K \in I \implies \forall m \ K^*$ is meager in $F_m$;
3. $\forall$ nonempty $G \in G_\delta(U)$,
   
   $\left( \forall O \text{ open}, O \cap G \neq \emptyset \implies \overline{O \cap G} \notin I \right) \implies \left( \exists m \ G^* \text{ is comeager in } F_m \right)$.

Additionally,

4. $\forall m \in \mathbb{N}, F_m \subseteq \overline{V}$; and
5. $\forall m \in \mathbb{N}, \forall \text{ open } W \subseteq E, \overline{V \cup U} \subseteq W \implies \mathcal{K}(W) \cap F_m \neq \emptyset$.

**Proof.** Fix a sequence of open, downward closed sets $\mathcal{U}_n \subseteq \mathcal{K}(E)$ such that

$I = \bigcap_n \mathcal{U}_n$

and (4) holds for the sets $\mathcal{U}_n$. For $K \in \mathcal{K}(E)$ and $n \in \mathbb{N}$, we will say that “$K$ is $n$-small” if $K \in \mathcal{U}_n$, else, we will say that “$K$ is $n$-big”. Clearly $K \in I$ if and only if $K$ is $n$-small for all $n$, and as each $\mathcal{U}_n$ is downward closed, an $n$-big set cannot be contained in an $n$-small set.

Fix a countable basis $\mathcal{B}$ for $E$, closed under finite unions. By lemma (5), inside $V$ we may find a sequence of pairwise disjoint regular closed sets $(M_i)$, say, converging to and disjoint from $M$. Note that the union of finitely many
members of the sequence $(M_i)$ will have an open superset that is disjoint from all other members of the sequence. With each $B \in \mathcal{B}$ we associate a subsequence of $(M_i)$, whose members we shall denote $M(B, n)$, $n \in \mathbb{N}$, in such a way that for each $i$ the set $M_i$ appears in the sequence $(M(B, n))_n$ for exactly one set $B \in \mathcal{B}$.

For $F \in \mathcal{K}(E)$, we introduce some useful terms. For $B \in \mathcal{B}$, we will say that “$F$ allows $B$” if $F \cap M(B, n)$ has nonempty interior for every $n \in \mathbb{N}$. For $i \in \mathbb{N}$ we will say that “$F$ is blank in $M_i$” if $F \cap M_i = \emptyset$. We will say that “$F$ has $k$ blanks before $M_i$” if $k$ of the sets $F \cap M_1, F \cap M_2, \ldots, F \cap M_{i-1}$ are empty.

Now for $m \in \mathbb{N}$, we will define a set $A_m \subseteq \mathcal{K}(E)$ thus. For $F \in \mathcal{K}(E)$, we will say that $F \in A_m$ if the following conditions hold:

Cond. 1. There exists $r \in \mathbb{N}$ such that $F \cap M_i$ has nonempty interior for all $i \geq r$.

Cond. 2. The set $F \cap M$ is not in $I$.

Cond. 3. There exist $p \in \mathbb{N}$ and $B_1, B_2, \ldots, B_p \in \mathcal{B}$, arranged in the order in which the first terms of their associated subsequences occur within the sequence $(M_i)$ (so that, for example, $M(B_1, 1)$ occurs in $(M_i)$ earlier than $M(B_2, 1)$ does) such that:

3a) $F$ allows each $B_i$;
3b) the union of the sets $\overline{B_i}$ is $m$-small. i.e.,
$$\bigcup_{i=1}^{p} \overline{B_i} \in \mathcal{U}_m;$$
3c) for each $j = 1, \ldots, p$, if $F$ has $k_j$ blanks before $M(B_j, 1)$ then
$$\bigcup_{i=j}^{p} \overline{B_i} \in \mathcal{U}_{k_j};$$
3d) finally,
$$E \setminus F \cap \overline{U} \subseteq \bigcup_{i=1}^{p} B_i.$$

We will say that the tuple $(B_1, B_2, \ldots, B_p, r)$ witnesses the membership of $F$ in $A_m$. Note that Cond. 1 implies that $F$ allows all but finitely many basic sets. Note also that from Cond. 3d we have that $F$ and the sets $B_i$ cover $\overline{U}$.

It is easy to see that $A_m$ is upward closed. (If some set $F$ in $A_m$ has witness $(B_1, B_2, \ldots, B_p, r)$, then the same witness will work for any compact superset of $F$. Here we make use of the fact that the sequence $(\mathcal{U}_n)$ is decreasing.)

Set $\mathcal{F}_m = \overline{A_m}$. This is still an upward closed set.

Our goals are now to show that 5 – 9 hold. We start with the last two. Certainly 5 is immediate (each $F$ in $\mathcal{F}_m$ intersects $M$). To see 9,
fix $m \in \mathbb{N}$ and fix an open superset $W$ of $\bigvee U$. We want to show that $\mathcal{K}(W) \cap \mathcal{F}_m \neq \emptyset$. Let $W'$ be open such that $\bigvee U \subseteq W' \subseteq W$. It is clear that the set $W'$ satisfies Cond. 1 and Cond. 2, and that $W'$ allows all basic sets and has no blanks at all in $(M_i)$. Since

$$E \setminus W' \cap U = \emptyset,$$

we can simply take any basic set $B$ whose closure is in $U_m$ to get a witness $\langle B, 1 \rangle$ for the membership of $W'$ in $\mathcal{A}_m$. (Certainly $U_m$ will contain the closures of some basic sets, just because any member of the open set $U_m$ is the limit of a sequence of the closures of basic sets.)

We move on to (5). Fix a compact set $K \subseteq U$ not in $I$, and fix $m_0$ such that $K \notin U_{m_0}$. For $F \in \mathcal{A}_{m_0}$, we have $\overline{E \setminus F} \cap U \subseteq \bigcup B_i$ for some basic sets $B_i$ satisfying the condition that $\bigcup B_i$ is $m_0$-small. The sets $F$ and $\bigcup B_i$ cover $U$ and therefore cover $K$. Since $K$ is $m_0$-big it cannot fit inside $\bigcup B_i$, and so must intersect $F$. Thus $\mathcal{A}_{m_0} \subseteq K^*$; the latter being closed we also get $\mathcal{F}_{m_0} \subseteq K^*$.

Next, (6). Fix $K \in I, m \in \mathbb{N}$ and $\epsilon > 0$. Let $F \in \mathcal{A}_m$ and let the tuple $\langle B_1, B_2, \ldots, B_p, r \rangle$ witness the membership of $F$ in $\mathcal{A}_m$. We will find a set in $\mathcal{A}_m \setminus K^*$ within $\epsilon$ of $F$; this will suffice to show that $K^*$ is nowhere dense, and hence meager, in $\mathcal{F}_m$.

The sets $B_i$ satisfy the condition that

$$\bigcup_{i=1}^p B_i \in \mathcal{U}_m,$$

and they also satisfy the following $p$ many conditions: for each $j = 1, \ldots, p$, if $F$ has $k_j$ blanks before $M(B_j, 1)$ then

$$\bigcup_{i=j}^p B_i \in \mathcal{U}_{k_j}.$$

We may now pick a $q \in \mathbb{N}$ large enough so that:

- we can add a $q$-small set to $\bigcup B_i$ and stay in $\mathcal{U}_m$;
- for each $j = 1, \ldots, p$, we can add a $q$-small set to $\bigcup B_i$ and stay in $\mathcal{U}_{k_j}$;
- if $F$ has a total of $t$ blanks in the sequence $(M_i)$, then $q > t$.

Now, $F \cap M \notin K$ because $F \cap M \notin I$. Let $x \in F \cap M \setminus K$. Pick $\delta < \epsilon$ such that $B(x, \delta)$ is disjoint from $K$. We have $M \in B(x, \delta)^*$, an open set. Let $i_0$ be such that $\forall i \geq i_0, M_i \in B(x, \delta)^*$, i.e., $M_i \cap B(x, \delta) \neq \emptyset$. As the interior of each $M_i$ is dense in it, we have $\forall i \geq i_0, (M_i)^{\circ} \cap B(x, \delta) \neq \emptyset$.

Now consider the finitely many sets $M_i$ for $i < i_0$. If $(F \cap M_i)^{\circ}$ is nonempty, then $(F \cap M_i)^{\circ} \setminus K$ must be nonempty as $K$ is meager. For such $i$, let $V_i$ be a nonempty open set such that $V_i \subseteq (F \cap M_i)^{\circ} \setminus K$.

Consider a sequence of basic open sets containing $K$ whose closures converge to $K$. As $K$ belongs to the open set $U_q$, these closures eventually
lie in $\mathcal{U}_q$. Pick a member $B$ of this sequence such that all of the following conditions hold:

- $M(B, 1)$ occurs in the sequence $(M_i)$ after $M_{i_0}$ and after $M(B_p, 1)$;
- $\overline{B} \in \mathcal{U}_q$;
- $\overline{B}$ is disjoint from the finite union of the sets $\overline{V}_i$ just defined.

Now let $B'$ be an open set such that $K \subseteq B' \subseteq \overline{B'} \subseteq B$. We now claim that the set $F' = (F \setminus B') \cup \overline{B(x, \delta)}$ is in $A_m$.

Cond. 1 is satisfied as $\overline{B(x, \delta)} \cap M_i$ has nonempty interior for all $i \geq i_0$. Cond. 2 is satisfied as $\overline{B(x, \delta)} \cap M$ contains $\overline{B(x, \delta)} \cap M$, which is not in $I$ as $M$ is everywhere big with respect to $I$.

The tuple $(B_1, B_2, \ldots, B_p, B, i_0)$ witnesses the membership of $F'$ in $A_{n,m}$: To see this, first note that $F' \cap M_i$ has nonempty interior for every $i$ where $F' \cap M_i$ had nonempty interior, so $F'$ allows any basic set that $F$ allowed, including each $B_j$. The basic set $B$ is allowed as the entire sequence $(M(B, n))_n$ lies in the $i_0$-tail of $(M_i)$. Hence Cond. (3a) holds. Cond. (3b) holds for this witness by choice of $q$. Cond. (3c) now consists of $p+1$ conditions. For $j = 1, \ldots, p$, if $F'$ has $k_j'$ blanks before $M(B_j, 1)$, then $k_j' \leq k_j$, and by choice of $q$ we have $\bigcup_{i=1}^p \overline{B_i} \cup \overline{B} \in \mathcal{U}_{k_j} \subseteq \mathcal{U}_{k_j'}$. The $p+1$ condition is that we must have $\overline{B} \in \mathcal{U}_k$, where $k$ is the number of blanks that $F'$ has before $M(B, 1)$. Now, $F'$ can have no more than $t$ blanks before $M(B, 1)$ because it has no more than $t$ blanks anywhere, and thus $\overline{B} \in \mathcal{U}_q \subseteq \mathcal{U}_t \subseteq \mathcal{U}_k$.

So Cond. (3d) holds. Finally, for Cond. (3e) note that

$$
\mathcal{U} \cap E \setminus F' = \left[ \overline{B'} \cap \mathcal{U} \right] \cup \left[ \mathcal{U} \cap E \setminus \overline{F} \right] \subseteq \overline{B'} \cup \bigcup_{i=1}^p B_i \subseteq B \cup \bigcup_{i=1}^p B_i.
$$

Thus we have shown that $K^*$ is meager in $F_m$.

Having established that the sets $F_m$ determine membership of compact sets $K$ in $I$, we now address (7):

Let $G \subseteq U$ be a nonempty $G_\delta$ set that is everywhere big with respect to $I$. Let $(H_n)$ be an increasing sequence of closed sets that are relatively meager in $G$ such that $G = G \setminus \bigcup_n H_n$.

Fix $m_0$ such that $G \notin \mathcal{U}_{m_0}$, so that $F_{m_0} \subseteq \overline{G^*}$. In order to show that $G^*$ is comeager in $F_{m_0}$, we will play the Banach-Mazur game in $F_{m_0}$ on the set $G^*$; we will describe a winning strategy for Player II, i.e., if Player I is playing sets $V_n$ and Player II is playing sets $W_n$ (all open subsets of $F_{m_0}$ satisfying the inclusions of the game) we will show that $\bigcap_n W_n \subseteq G^*$.
Player I starts the game by playing some $\mathcal{V}_1$. If $F$ is a set in $\mathcal{V}_1 \cap \mathcal{A}_{m_0}$ with witness $\langle B_1, B_2, \ldots, B_p, r \rangle$, then we know that $D \setminus \bigcup_{j=1}^{p} B_j$ is nonempty. (In fact, this is why we know that $D$ intersects $F$.) If Player II can construct $\mathcal{W}_1$, a further nonempty open subset of $\mathcal{F}_{m_0}$, and $D_1$, a nonempty relatively open subset of $\overline{G}$, such that

- $D_1 \cap H_1 = \emptyset$, and
- $\mathcal{W}_1 \subseteq D_1$,

and if Player II can keep going in this way, producing with each successive $\mathcal{W}_n$ that it plays, an open subset $D_n$ of $\overline{G}$, such that

- $D_n \subseteq D_{n-1}$ for $n > 1$,
- $D_n \cap H_n = \emptyset$, and
- $\mathcal{W}_n \subseteq D_n$,

then Player II will have a winning strategy.

Therefore it suffices to show the following.

**Claim:** Let $D$ and $H$ be nonempty subsets of $\overline{G}$, with $D$ relatively open in $\overline{G}$, and $H$ relatively meager and closed in $\overline{G}$. Suppose that $\mathcal{V}$ is a nonempty relatively open subset of $\mathcal{F}_{m_0}$ such that for any $F \in \mathcal{V} \cap \mathcal{A}_{m_0}$, if $\langle B_1, B_2, \ldots, B_p, r \rangle$ witnesses the membership of $F$ in $\mathcal{A}_{m_0}$, then

$$D \setminus \bigcup_{j=1}^{p} B_j \neq \emptyset.$$ 

Then there exists a further nonempty relatively open subset $\mathcal{V}'$ of $\mathcal{F}_{m_0}$, contained in $\mathcal{V}$, and a further nonempty relatively open subset $D'$ of $\overline{G}$, such that $\overline{D'} \subseteq D \setminus H$ and, for any $F \in \mathcal{V}' \cap \mathcal{A}_{m_0}$, if $\langle B_1, B_2, \ldots, B_p, r \rangle$ witnesses the membership of $F$ in $\mathcal{A}_{m_0}$, then

$$\overline{D'} \setminus \bigcup_{j=1}^{p} B_j \neq \emptyset.$$ 

(This then implies that $F \in \overline{D'}^\circ$, and thus we have $\mathcal{V}' \subseteq \overline{D'}^\circ$.)

We prove the claim.

Without loss of generality, we may assume that $\mathcal{V}$ has the form

$$\mathcal{V} = \{ K \in \mathcal{F}_{m_0} : K \subseteq V_0, \ K \cap V_i \neq \emptyset \ \forall i = 1, \ldots, l \},$$

where the sets $V_1, \ldots, V_l$ are nonempty open subsets of the open set $V_0$. We can further assume that each $V_i$ is an $\epsilon$-ball for some fixed $\epsilon$.

Fix $F \in \mathcal{V} \cap \mathcal{A}_{m_0}$, with witness $\langle B_1, B_2, \ldots, B_p, r_F \rangle$, say. Since the open set

$$E \setminus \bigcup_{j=1}^{p} B_j$$
intersects $D$, it intersects $D$. As $H$ is meager in $G$, within $D$ we can find a further open subset of $G$, say $D'$, such that
\[ D' \subseteq D \setminus \left( \bigcup_{j=1}^{p} B_j \cup H \right). \]

Since $G$ is everywhere big with respect to $I$, we know that $D' \notin I$. Fix a $k$ such that $D' \notin \mathcal{U}_k$.

For each $j = 1, \ldots, p$, we have that $F$ allows $B_j$. In the remainder of the proof, the idea is to find an open neighbourhood of $F$ in $\mathcal{K}(E)$ within which the specific sets $B_1, \ldots, B_p$ remain allowed, but basic sets $B$ other than these are not allowed unless the closure of their union is $k$-small. This will force any member of $A_{m_0}$ in this neighbourhood to intersect $D'$, which is $k$-big.

Recall that $B_p$ is the last basic set in the witnessing tuple. Pick $r$ such that
- $M_r$ appears in $(M(B_p, n))_n$, the subsequence of $(M_i)$ associated with $B_p$;
- $r$ is large enough so that for $i \geq r$, the Hausdorff distance between $M_i$ and $M$ (the limit of $(M_i)$) is less than $\epsilon$.

There are up to $r$ many basic sets $B$ whose associated subsequences have appeared within the initial $r$ terms of the sequence $(M_i)$. These include the sets $B_1, \ldots, B_p$. For each of these finitely many basic sets $B$ other than $B_1, \ldots, B_p$, find one member of the associated sequence that occurs after $M_r$ and tag it. Now again find $r' \in \mathbb{N}$ such that $M_{r'}$ lies within the subsequence $(M(B_p, n))_n$ and also $M_{r'}$ occurs after all the up to $r-p$ tagged sets. Ensure also, by choosing a larger $r'$ if necessary, that there are at least $k$ sets $M_i$ with $r < i < r'$ that do not occur in $(M(B_j, n))_n$ for any $j = 1, \ldots, p$. Now let $R$ be the union of all the sets $M_i$ for $r < i < r'$ except for those that are associated with the specific basic sets $B_1, \ldots, B_p$, i.e.,
\[ R = \bigcup \{ M_i : r < i < r', M_i \text{ does not occur in } (M(B_j, n))_n \text{ for } 1 \leq j \leq p \}. \]

$R$ is a closed set that contains all the tagged sets and it is a finite union of at least $k$ many members of $(M_i)$. It is disjoint from all members of $(M_i)$ that it does not contain, and also from $M$. We also know that $R$ cannot completely contain any $\epsilon$-ball. (Any such ball that it did contain would be forced to intersect $M$ as well as be disjoint from $M$.) In particular, $R$ cannot completely contain any of the sets $V_i$. For $1 \leq i \leq l$, fix $y_i$ in $V_i \setminus R$.

Set
\[ \mathcal{V}' = \mathcal{K}(V_0 \setminus R) \cap \mathcal{V}. \]

We show that the sets $D'$, $\mathcal{V}'$ satisfy all the required conditions. We already have $D' \subseteq D \setminus H$. We now show that $\mathcal{V}' \neq \emptyset$. Since $F \in \mathcal{V}$ we have that $F \subseteq V_0$. Let $O$ be an open superset of $R$ so chosen that it does not contain any of the points $y_i$ for $i = 1, \ldots, l$ and also so that it does not intersect any
set $M_j$ that is disjoint from $R$. We may also ensure that $O \cap U = \emptyset$, as $R$ is a closed set disjoint from $\overline{U}$. Now let

$$F' = (F \setminus O) \cup \{y_i : i = 1, \ldots, l\}.$$  

It is clear that $F' \subseteq V_0 \setminus R$ and $F' \cap V_i \neq \emptyset$ for $1 \leq i \leq l$. To show that $F' \in A_{m_0}$, it suffices to show that $F \setminus O \in A_{m_0}$, since the addition of points does not affect membership in $A_{m_0}$. First note that since $O$ meets only finitely many of the sets $M_i$, the sequence $(M_i \cap (F \setminus O))_i$ still has a tail with nonempty interiors (starting at say $r_{F'}$), so that Cond. (1) holds. Next, note that $F \setminus O$ allows each of $B_1, \ldots, B_p$, which already satisfy Cond. (3b) and Cond. (3c), and that

$$E \setminus (F \setminus O) \cap U = E \setminus F \cap U.$$  

Therefore the tuple $\langle B_1, B_2, \ldots, B_p, r_{F'} \rangle$ witnesses the membership of $F \setminus O$ in $A_{m_0}$. Thus $\nu'$ is nonempty.  

We now show that for any $L \in \mathcal{V}' \cap A_{m_0}$, if the membership of $L$ in $A_{m_0}$ is witnessed by $\langle B_1', B_2', \ldots, B_p', r_L \rangle$, then

$$\overline{D_n} \setminus \bigcup_{j=1}^{p'} \overline{B_j} \neq \emptyset.$$  

So let $L \in \mathcal{V}' \cap A_{m_0}$. What are the basic sets $B$ that $L$ can allow? (This is the dénouement!) If $B$ is not one of the original $B_i$ obtained from the set $F$, then $M(B, 1)$ cannot occur before $M_r$. (If it did, some member of its associated sequence is a tagged set included in $R$, but $L$ is disjoint from $R$.) $M(B, 1)$ cannot occur between $M_r$ and $M_{r'}$ as in this range the only $M_i$ remaining in $V_0 \setminus R$ were part of the sequences associated with the original $B_i$. Therefore the sequence associated with $B$ is entirely contained in the $r'$-tail of $(M_i)$.  

With this in mind, any witness of the membership of $L$ in $A_{m_0}$ must have the form $\langle B_{i_1}, B_{i_2}, \ldots, B_{i_s}, B_1', B_2', \ldots, B_p', r_L \rangle$, where the sets $B_{i_1}, B_{i_2}, \ldots, B_{i_s}$ are taken from $B_1, B_2, \ldots, B_p$ and $M(B_1', 1)$ lies in the $r'$-tail of $(M_i)$.  

Note also that $L$, missing the whole of $R$, has at least $k$ many blanks occurring before $M(B_1', 1)$ in the sequence $(M_i)$. Therefore by Cond. (3c),

$$\bigcup_{j=1}^{s'} \overline{B_j'} \in \mathcal{U}_k,$$

while $\overline{D'} \notin \mathcal{U}_k$. We also know that the whole of $\overline{D'}$ lies outside $\bigcup_{j=1}^{p'} \overline{B_j}$. So

$$\overline{D'} \setminus \left( \bigcup_{j=1}^{s} \overline{B_j} \cup \bigcup_{j=1}^{s'} \overline{B_j'} \right) = \overline{D'} \setminus \bigcup_{j=1}^{s'} \overline{B_j'} \neq \emptyset.$$  

Thus the sets $\mathcal{V}'$ and $D'$ satisfy all the required conditions, and the Claim holds, completing the proof. \qed
We now wish to construct a single upward closed set $F$ satisfying Theorem 1 for ideals that contain only meager sets. We first find pairs of disjoint open sets $U_n, V_n$ such that the sets $U_n$ cover $E$ and for each pair $U_n, V_n$, we can apply Proposition (6) to get representing sequences $(F^m_n)$. We then want to obtain a single representing set $F$ that does the job for the whole space $E$. The idea is that we form $F$ by taking some manner of union of all the $F^m_n$, in such a way that for a fixed $n, m$ we are able to find an open subset of $K(E)$ within which all the members of $F$ come only from this particular $F^m_n$. The essence of the construction of $F$ is that we marry each $F^m_n$ to a particular ‘permission set’ before throwing them all into a common union, in such a way that permission sets can be excluded or included from our desired open set as needed.

We proceed.

**Theorem 7.** Let $E$ be a compact Polish space and let $I \subseteq K(E)$ be a $G_\delta$ ideal of compact sets, with property ($\ast$), containing only meager sets. Then there exists a compact set $F \subseteq K(E)$ that is also upward closed, such that any $G_\delta$ subset $G$ of $E$ is covered by countably many sets in $I$ if and only if $G^*$ is meager in $F$.

**Proof.** By Proposition 8, it suffices to construct a set $F \subseteq K(E)$, closed and upward closed, such that

1. $\forall K \in K(E), \ K \notin I \implies K^*$ is nonmeager in $F$;
2. $\forall K \in K(E), \ K \in I \implies K^*$ is meager in $F$;
3. $\forall$ nonempty $G \in G_\delta(E)$, $G^*$ is nonmeager in $F$.

By Proposition 4, we may assume that $E$ has infinitely many limit points, and $E$ has some meager compact subset with at least three points that is everywhere big with respect to $I$. We first construct nonempty open subsets $U_1, U_2, U_3$ and $V_1, V_2, V_3$ of $E$ such that

1. the sets $U_1, U_2, U_3$ form an open cover of $E$;
2. the following strict inclusions hold: $V_1 \subseteq U_2 \setminus (U_3 \cup U_1)$, $V_2 \subseteq U_3 \setminus (U_1 \cup U_2)$, $V_3 \subseteq U_1 \setminus (U_2 \cup U_3)$, and, in each of the three set inclusions above, if the smaller set is removed from the larger one then at least one limit point of $E$ is left behind;
3. for each $n = 1, 2, 3$, there exists a sequence of closed and upward closed sets $(F^m_n)_{m \in \mathbb{N}}$, such that the sets $U = U_n, V = V_n$ and $F = F^m_n$ all satisfy (5), (6), (7), (8), and (9).

To do this, fix a meager compact subset $M$ of $E$, with at least three points, that is everywhere big with respect to $I$. We may assume that $E \setminus M$ contains at least three limit points of $E$. (Why? If $M$ contains all but perhaps two
of the infinitely many limit points of $E$, we can pick a suitable relatively open subset $U$ of $M$ such that $\overline{U}$ contains at least three of these points and omits another three. The set $\overline{U}$ is still both meager and everywhere big with respect to $I$, so we can replace $M$ with $\overline{U}$.) Let us say that $a_1$, $a_2$, and $a_3$ are limit points of $E$ not in $M$.

Since $M$ contains at least three points, we may find open sets $V_1, V_2, V_3$ with disjoint closures, all of which intersect $M$, and none of which contains any point $a_i$. Now let $U_1$ and $U_2$ be open sets with disjoint closures such that

$$U_1 \supseteq \overline{V_3} \cup \{a_3\}, \quad U_2 \supseteq \overline{V_1} \cup \{a_1\}, \quad \text{and} \quad (\overline{U_1} \cup \overline{U_2}) \cap (\overline{V_2} \cup \{a_2\}) = \emptyset,$$

and let $U_3$ be an open set such that

$$\overline{U_3} \cap (\overline{V_1} \cup \overline{V_3} \cup \{a_1, a_3\}) = \emptyset \quad \text{and} \quad U_1 \cup U_2 \cup U_3 = E.$$

For each $n$, since $V_n$ intersects $M$, which is everywhere big with respect to $I$, we may find a set $M_n \subseteq V_n$ such that $M_n$ is a meager compact set that is everywhere big with respect to $I$. For each $n$, the sets $U = U_n$, $V = V_n$, and $M = M_n$ now satisfy the hypothesis of Proposition 6, which we apply to get a sequence $(F_m^n)_m$ as required.

Having obtained the sets $U_i$ and $V_i$, now fix open sets $W_1, W_2, W_3$, each containing a limit point, such that

$$W_3 \subseteq U_2 \setminus (\overline{U_3} \cup \overline{U_1} \cup \overline{V_1}), \quad W_1 \subseteq U_3 \setminus (\overline{U_1} \cup \overline{U_2} \cup \overline{V_2}), \quad W_2 \subseteq U_1 \setminus (\overline{U_2} \cup \overline{U_3} \cup \overline{V_3}).$$

Note that we have ensured that if $\{i, j, k\} = \{1, 2, 3\}$, the set $\overline{U_i} \cup \overline{V_i} \cup \overline{W_i}$ is disjoint from either $\overline{V_j} \cup \overline{U_k}$ or from $\overline{V_k} \cup \overline{U_j}$.

We now build ‘permission sets’ inside the sets $W_n$, thus. Inside each open set $W_n$ we may find a sequence of nonempty open sets $(P^n_i)_i$, such that the sets $P^n_i$ are pairwise disjoint, and for any $j$, it is possible to find an open superset of $P^n_j$ that is disjoint from $P^n_i$ for all $i \neq j$. (For example, we may take a convergent sequence of distinct points in $W_n$ and put suitable open balls around the points to get the sets $P^n_i$.)

This completes the set-up for our definition of $\mathcal{F}$.

For $n = 1, 2, 3$ and $m \in \mathbb{N}$, define

$$\mathcal{B}_m^n = \mathcal{F}_m \cap P_m^n.$$

Note that the sets $\mathcal{B}_m^n$ are upward closed, and any set in $\mathcal{B}_m^n$ must meet both $V_n$ and $P_m^n$, which is contained in $W_n$. Now define

$$\mathcal{F} = \bigcup_{n,m} \mathcal{B}_m^n,$$

which is also an upward closed set.

Claim: For any fixed $n, m$, there exists an open set $O \subseteq E$ such that $\emptyset \neq \mathcal{K}(O) \cap \mathcal{F} \subseteq F_m^n$.

Proof of claim: Fix $n \leq 3$ and $m \in \mathbb{N}$. For the appropriate distinct $i$ and $j$, both different from $n$, $\overline{U_i} \cup \overline{V_i} \cup \overline{W_i}$ does not intersect $\overline{W_j}$ or $\overline{V_j}$. Let $O$
be an open superset of $\overline{U_n} \cup \overline{V_n} \cup \overline{P_m}$ that is disjoint from both $\overline{W_i}$ and from $\overline{V_j}$, and also disjoint from $\overline{P_k}$ for $k \neq m$.

To see that this $O$ is as required, first note that (9) holds for the sets $U = U_n, V = V_n$ and $\mathcal{F}_m = \mathcal{F}_m^n$, so there exists some $F \in \mathcal{K}(O) \cap \mathcal{F}_m^n$. The set $F \cup P_m^n$ is now a member of $\mathcal{K}(O) \cap \mathcal{B}_m^n$, so that $\mathcal{K}(O) \cap F \neq \emptyset$. Now suppose that there is some $F \in \mathcal{K}(O) \cap \mathcal{F}$ that is not in $\mathcal{F}_m^n$. Since $F \in \mathcal{F} = \bigcup_{i,j} B_{ij}^n$, the open set $\mathcal{K}(O) \setminus \mathcal{F}_m^n$ must contain a set $F'$ in $\mathcal{B}_m^{n_1}$ for some $n_1, m_1$. Any set in $\mathcal{B}_m^{n_1}$ meets $\overline{V_{n_1}}$, and $P_{m_1}^n$. However, since $F' \subseteq O$, the only possibility is that $n_1 = n$ and $m_1 = m$; all other possibilities have been excluded from $O$. This is a contradiction. So the claim holds.

Now (10) is immediate: let $K$ be a compact set not in $I$. Since $I$ is closed under countable union, and the $F_n$ sets $U_n$ form a cover of $E$, we may simply assume that $K$ is contained in one of the sets $U_n$, say $U_{n_0}$. Take $m_0$ such that $\mathcal{F}_{m_0}^{n_0} \subseteq K^*$. To establish (10), just take an open set $O$ such that $\emptyset \neq \mathcal{K}(O) \cap \mathcal{F} \subseteq \mathcal{F}_{m_0}^{n_0} \subseteq K^*$.

To see (11), let $K \in I$, let $F \in \mathcal{B}_m^n$ for some $n, m$, and let $\epsilon > 0$. Since $F \in \mathcal{F}_m^n$ and $K^*$ is meager in $\mathcal{B}_m^n$, there is some $F'$ in $\mathcal{F}_m^n \setminus K^*$ within $\epsilon$ of $F$. If $F' \cap P_m^n \neq \emptyset$, we are done. If $F' \cap P_m^n = \emptyset$, the addition to $F'$ of a suitable point in $P_m^n \setminus K$ (using the fact that $K$ is meager and $F$ intersects $P_m^n$) gives a set still within $\epsilon$ of $F$ and in $\mathcal{B}_m^n$, and we are done.

Finally, we turn to (12). Let $G \in G_\delta(E)$ be a set that is everywhere big with respect to $I$. Again, since the sets $U_n$ form a cover of $E$, by replacing $G$ with a suitable relatively open subset of itself, we may assume that $G$ is contained in one of the sets $U_n$, say $U_{n_0}$. Now (since (7) holds with $\mathcal{F}_m = \mathcal{F}_m^{n_0}$ and $U = U_{n_0}$) there exists $m_0$ such that $G^*$ is comeager in $\mathcal{F}_{m_0}^{n_0}$. Again take an open set $O \subseteq E$ such that $\emptyset \neq \mathcal{K}(O) \cap \mathcal{F} \subseteq \mathcal{F}_{m_0}^{n_0}$. The set $G^*$ is comeager in $\mathcal{K}(O) \cap \mathcal{F}$. To see this precisely, let $G$ be a $G_\delta$ subset of $\mathcal{K}(E)$ such that $G \cap \mathcal{F}_{m_0}^{n_0}$ is dense in $\mathcal{F}_{m_0}^{n_0}$ and contained in $G^*$. Now the set $G \cap \mathcal{K}(O) \cap \mathcal{F}$ is a $G_\delta$ subset of $\mathcal{F}$, it is dense in $\mathcal{K}(O) \cap \mathcal{F}$ as anything in $\mathcal{K}(O) \cap \mathcal{F}$ is actually from $\mathcal{F}_{m_0}^{n_0}$, and for the same reason, $G \cap \mathcal{K}(O) \cap \mathcal{F}$ is contained in $G^*$. This proves (12). \qed

It remains to prove the theorem for ideals that may contain non-meager sets.

**Theorem 8.** Let $E$ be a compact Polish space and let $I \subseteq \mathcal{K}(E)$ be a $G_\delta$ ideal of compact sets, with property (\star). Then there exists a compact set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that any $G_\delta$ subset $G$ of $E$ is covered by countably many sets in $I$ if and only if $G^*$ is meager in $\mathcal{F}$.

**Proof.** Starting with the whole space $E$, we once again carry out the transfinite procedure of Proposition 3 to obtain $E$ as the disjoint union of two sets:

$$E = U \cup E'$$

where $E'$ is a closed subset of $E$ that satisfies the condition that

$$\forall F \in \mathcal{K}(E'), \quad F \in I \implies F \text{ is relatively meager in } E',$$

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and $U$ is an open set that may be written as

$$U = \bigcup_{n} F_n,$$

where each $F_n$ is in $I$. This form of $U$ together with the fact that $I$ is closed under countable union makes it immediate that $\mathcal{K}(U) \subseteq I$. It is also immediate that any compact subset $K$ of $E$ is in $I$ if and only if $K \setminus U$ is in $I$.

Now let $I' = I \cap \mathcal{K}(E')$. It is easily checked that $I'$ is a $G_\delta$ ideal with property (*), both when $E$ and when $E'$ are considered as the underlying space. For a set $A \subseteq E'$, let

$$A^* = \{ K \in \mathcal{K}(E) : K \cap A \neq \emptyset \}$$

as usual and let

$$A'^* = \{ K \in \mathcal{K}(E') : K \cap A \neq \emptyset \}.$$

Since $I'$ contains only meager subsets of $E'$, the space $E'$ and the ideal $I'$ satisfy the hypothesis of Theorem 7. So there exists a nonempty compact set $F \subseteq \mathcal{K}(E')$ such that

(13) $\forall$ nonempty $G \in G_\delta(E')$, $\left( G \subseteq \bigcup_{n \in \mathbb{N}} K_n \text{ for some } K_n \in I' \right) \iff \left( G'^* \text{ meager in } F \right).$

We remark that Theorem 7 has given us the upward closedness of $F$ only as a subset of $\mathcal{K}(E')$. In any case, we have

$$F \subseteq \mathcal{K}(E') \subseteq \mathcal{K}(E).$$

We now claim that:

(14) $\forall$ nonempty $G \in G_\delta(E)$, $\left( G \subseteq \bigcup_{n \in \mathbb{N}} K_n \text{ for some } K_n \in I \right) \iff \left( G^* \text{ meager in } F \right).$

So let $G$ be a nonempty $G_\delta$ subset of $E$. Suppose $G^*$ is meager in $F$. Since $(G \cap E')^* \subseteq G^*$, we must have that $(G \cap E')^*$ is meager in $F$. So by (13),

$$G \cap E' \subseteq \bigcup_{n \in \mathbb{N}} K_n \text{ for some } K_n \in I',$$

and now

$$G = (G \cap E') \cup (G \cap U) \subseteq \bigcup_{n \in \mathbb{N}} K_n \cup \bigcup_{n \in \mathbb{N}} F_n,$$

where each $K_n$ and each $F_n$ is in $I$.

To prove the converse of (14), suppose that $G \subseteq \bigcup_{n \in \mathbb{N}} K_n$ for some $K_n \in I$. For each $n$, since $K_n \in I$ we have $(K_n \cap E') \in I'$. Now,

$$G^* \cap F \subseteq \bigcup_{n \in \mathbb{N}} (K_n^* \cap F) = \bigcup_{n \in \mathbb{N}} \left( (K_n \cap E')^* \cap F \right).$$
(The last equality comes from the fact that every set in $\mathcal{F}$ is wholly contained in $E'$ already.) Now since $(K_n \cap E')^*'$ is meager in $\mathcal{F}$ for every $n$, we have that $G^*$ is meager in $\mathcal{F}$. This concludes the proof. □

Theorems 7 and 8 together give Theorem 1.

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