The $M_L(z); C_L(z); W_L(z)$ associated Laguerre Polynomials

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2000 June 16

Abstract

In a previous paper we deformed Hermite polynomials to three associated polynomials .Here we apply the same deformation to Laguerre polynomials .

MSC:33C45;34A35
Keywords:Laguerre polynomials,deformation,associate polynomials.
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In recent papers we applied a deformation mechanism on Bessel and Neumann functions of integer orders $J_n(z)$, $N_n(z)$ [1] and Hermite polynomials $H_n(z)$ [2] to generate respectively Bessel and Neumann functions of real orders $J_{n+\lambda}(z), N_{n+\lambda}(z)$ ($\lambda$ real) and Hermite associated polynomials $M^s_{n\beta, H}(z); C^s_{n\beta, H}(z); W^s_{n\beta, H}(z)$. The structure underlying the deformation of Hermite polynomials has some characteristics.

- The associated polynomials come in triplicate and form the sequence $H_n(z) \to M^s_{n\alpha, H}(z) \to C^s_{n\alpha, H}(z) \leftarrow W^s_{n\alpha, H}(z) \leftarrow H_n(z)$

- When we try to define a measure $\mathcal{D}_s$ which ensures orthogonality of polynomials $M^s$ we find that $\mathcal{D}_s$ is not positive. It is rather a “charge” density. Moreover this measure has the form $\mathcal{D}_s \sim \mathcal{D}_0 H_s$ where $\mathcal{D}_0$ is the measure of Hermite polynomials

- The differential equation associated to $M$ is inhomogeneous whose homogeneous part is Hermite polynomials differential equation. The inhomogeneous term being of the form $\sim s \sum_{m=1}^n \alpha^m d_m^s \frac{d^m}{dz^m} M^s_{n\alpha, H}(z)$

The above structure seems to be general and not restricted to Hermite polynomials. We will see in this paper that the same structure appears when we deform Laguerre polynomials and that the above three characteristics remain valid where $H$ is replaced by $L$.

1. **The $M^s_{n\beta}(z)$ polynomials**

1.1 **Definition**

We define Polynomials $M^s_{n\beta\alpha}(z)$ as follows [3]

$$M^s_{n\beta\alpha}(z) = \exp\left[ s \sum_{m=1}^{\infty} \frac{\alpha^m}{m} \frac{d^m}{dz^m} \right] L_{n\beta}(z)$$

with $\alpha = \pm 1$ and $L_{n\beta}(z)$ is Laguerre polynomials which are defined as follows [3]

$$n! L_{n\beta}(z) = \sum_{m=0}^{n} (-1)^m \binom{n + \beta}{n - m} \frac{z^m}{m!}$$

\[1\text{We will use the short notation } M, C, W \text{ without the subscript } L, \text{ as only Laguerre deformations are involved in this paper. Also } d_m \text{ will stand for } d_m = \frac{d^m}{dz^m} \]

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Laguerre polynomials obey a functional relation which is at the origin of their deformation defined at \[1\]. It is
\[
\frac{d}{dz} L_{n\beta}(z) = -L_{(n-1)(\beta+1)}(z)
\] (2)

We can analyse polynomials \( M_{n\beta\alpha}(z) \) using the defining equation \[2\] or equivalently through their generating function which can also be used to infer the measure associated to them.

### 1.2 Generating function

The Laguerre polynomials generating function \( L(z, t, \beta) = \sum_{n=0}^{\infty} L_{n\beta}(z)t^n \ | t \) \((1\)

has a functional relation
\[
\frac{\partial}{\partial z} L(z, t, \beta) = -tL(z, t, \beta + 1)
\] (3)

This is property \[2\] expressed in terms of the generating function. The action of the exponential deformation on \( L_n(z) \) is not straightforward (as property \[2\] involves the variation of both \( n \) and \( \beta \)), neither for \( L(z, t, \beta) \) (as property \[3\] involves the variation of \( \beta \)). We therefore look for another related function whose deformation is straightforward. The appropriate function is defined as follows. Put \( \beta = m + \lambda \) with \( m \) integer and \( \lambda \) a real number and define the function \( \Phi(z, t, \zeta, \lambda) \) formally as
\[
\Phi(z, t, \zeta, \lambda) = \sum_{m=-\infty}^{\infty} L(z, t, \beta)\zeta^m
\]

where \( L(z, t, \beta) \) coincides with the generating function for Laguerre polynomials when \( \text{Re} \beta - 1 \). The exact form of \( \Phi(z, t, \zeta, \lambda) \) for arbitrary \( \beta \) does not matter. This function has a simple property. It is an eigenstate of \( \frac{\partial}{\partial z} \)
\[
\frac{\partial}{\partial z} \Phi(z, t, \zeta, \lambda) = -t \frac{\zeta}{\zeta^m} \Phi(z, t, \zeta, \lambda)
\]

In fact
\[
\frac{\partial}{\partial z} \Phi(z, t, \zeta, \lambda) = \sum_{m=-\infty}^{\infty} \frac{\partial}{\partial z} L(z, t, \beta)\zeta^m = -t \sum_{m=-\infty}^{\infty} L(z, t, \beta + 1)\zeta^m
\]
\[
\begin{align*}
&= -\frac{t}{\zeta} \sum_{m=-\infty}^{\infty} L(z, t, \beta) \zeta^m \\
&= -\frac{t}{\zeta} \Phi(z, t, \zeta, \lambda)
\end{align*}
\]

It is the function \( \Phi(z, t, \zeta, \lambda) \) that is simple to deform as it is an eigenstate of the derivative operator. Applying the mapping operator in \( [1] \) we get

\[
\exp\left[s \sum_{m=1}^{\infty} \alpha^m \frac{d_m}{m} \Phi(z, t, \zeta, \lambda)\right] = \exp\left[-s \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (\alpha t \zeta)^m \Phi(z, t, \zeta, \lambda)\right]
\]

\[
= \exp\left[-s \ln(1 + \alpha t \zeta)\right] \Phi(z, t, \zeta, \lambda)
\]

\[
= (1 + \alpha t \zeta)^{-s} \Phi(z, t, \zeta, \lambda)
\]

The deformed generating function associated to \( L(z, t, \beta) \) is denoted \( M(z, t, \beta, \alpha) \) and is by definition

\[
(1 + \alpha t \zeta)^{-s} \Phi(z, t, \zeta, \lambda) = \sum_{m=-\infty}^{\infty} M(z, t, \beta, \alpha) \zeta^m
\]

Inserting the definition of \( \Phi(z, t, \zeta, \lambda) \) into \( [4] \) together with the expansion

\[
(1 + \alpha t \zeta)^{-s} = 1 - s \alpha t \zeta + \frac{s(s + 1)}{2!} \left(\frac{t}{\zeta}\right)^2 + \ldots
\]

we get the result

\[
(1 + \alpha t \zeta)^{-s} \Phi(z, t, \zeta, \lambda) = \sum_{m=-\infty}^{\infty} (1 + \alpha t \zeta)^{-s} L(z, t, \beta) \zeta^m
\]

\[
= \sum_{m=-\infty}^{\infty} L(z, t, \beta) \zeta^m
\]

\[
- s \alpha t \sum_{m=-\infty}^{\infty} L(z, t, \beta) \zeta^{m+1}
\]

\[
+ \frac{s(s + 1)}{2!} t^2 \sum_{m=-\infty}^{\infty} L(z, t, \beta) \zeta^{m+2}
\]

\[
+ \ldots
\]
Comparing with \( 4 \) we identify the generating function for \( M \) polynomials which reads

\[
M(z, t, \beta) = L(z, t, \beta) - s\alpha t L(z, t, \beta + 1) + s(s + 1) t^2 L(z, t, \beta + 2) + \ldots
\]

To sum up the above series we replace the generating function of Laguerre polynomials by its expression \( L(z, t, \beta) = e^{-zt}(1 - t)^{-\beta + 1} \) and get

\[
M(z, t, \alpha) = e^{\frac{-zt}{1-t}} \left( 1 - \frac{s\alpha t}{1-t} + s(s + 1) \left(\frac{\alpha t}{1-t}\right)^2 + \ldots \right)
\]

\[
= e^{\frac{-zt}{1-t}} \left( 1 - t(1 - \alpha) \right)^{-s}
\]

Polynomials \( M \), them, are by definition

\[
\frac{e^{\frac{-zt}{1-t}}}{(1 - t)^{\beta + 1} (1 - t(1 - \alpha))^{-s}} = \sum_{n=0}^{\infty} \frac{M_{n\beta\alpha}(z)}{n!} t^n
\]

At this point we may distinguish two different cases. The case \( \alpha = 1 \) does not yield a new generating function but the generating function of Laguerre polynomials shifted by \( s \) i.e. \( \beta \rightarrow \beta - s \). This is however an interesting result as it allows us to connect Laguerre polynomials with different value \( 2 \) of the parameter \( \beta \)

\[
L_{n(\beta-s)}(z) = \exp[s \sum_{n=1}^{\infty} \frac{d_n}{m^2} L_n(z)]
\]

The second more interesting case \( \alpha = -1 \) will lead to a new family of Laguerre related polynomials. They are obtained via the relation (denote \( M_{n\beta(-1)}^s(z) = M_{n\beta}^s(z) \))

\[\text{The parameter } s \text{ can be taken to be real and of any sign in the case } \alpha = 1 \]. We will see that in the more interesting case \( \alpha = -1 \), partial-orthogonality will require \( s \) to be only integer and positive
\[
\frac{e^{-zt}}{(1-t)^{\beta-s+1}} (1-2t)^{-s} = \sum_{n=0}^{\infty} \frac{M_{n\beta}^s(z)}{n!} t^n
\]
\[(5)\]

\[M_{n\beta}^s(z) = \exp[s \sum_{m=1}^{\infty} (-1)^m \frac{d_m}{m}] L_{n\beta}(z)\]

Inverting \[(5)\] by contour integration

\[
\frac{M_{n\beta}^s(z)}{n!} = \frac{1}{2\pi i} \oint \frac{e^{-zt}}{(1-t)^{\beta-s+1} t^{n+1}} (1-2t)^{-s} \, dz
\]

we get an expression of \(M\) polynomials with a useful rearrangement of terms. In fact, expanding the power series

\[(1-2t)^{-s} = 1 + 2st + \frac{s(s+1)}{2!} (2t)^2 + \cdots \frac{s(s+1) \cdots (s+k-1)}{k!} (2t)^k + \cdots\]

we get

\[
M_{n\beta}^s(z) = L_{n(\beta-s)} + L_{(n-1)(\beta-s)} 2s \binom{n}{1} \cdots + L_{(n-k)(\beta-s)} 2^k s(s+1) \cdots (s+k-1) \binom{n}{k} \cdots 2^n s(s+1) \cdots (s+n-1)
\]

1.3 Properties

In the following, unless necessary, we will use the compact notation \(\exp(s \sum)\) to mean \(\exp[s \sum_{m=1}^{\infty} (-1)^m \frac{d_m}{m}]\)

- connection to \(L\)

\[
\exp(-s \sum) M_{n\beta}^s(z) = L_{n\beta}(z)
\]
\[
\exp(s' \sum) M_{n\beta}^s(z) = M_{n\beta}^{s+s'}(z)
\]
• Various combinations

We may combine the $\alpha = \pm 1$ deformations and get

\[
\exp\left[s \sum_{m=1}^{\infty} ((-1)^m + 1) \frac{d_m}{m} \right] L_{n\beta}(z) = M^s_{n(\beta-s)}(z)
\]

\[
\exp\left[s \sum_{m=1}^{\infty} ((-1)^m - 1) \frac{d_m}{m} \right] L_{n\beta}(z) = M^s_{n(\beta+s)}(z)
\]

• Recursion formulas

\[
\frac{d}{dz} M^s_{n\beta}(z) = -M^s_{(n-1)(\beta+1)}(z)
\]

\[
M^s_{n\beta}'(z) = M^s_{n(\beta-s'+s)} + M^s_{(n-1)(\beta-s'+s)} 2(s' - s) \binom{n}{1} \cdots \\
+ M^s_{(n-k)(\beta-s'+s)} 2^k(s' - s)(s' - s + 1) \cdots (s' - s + k - 1) \binom{n}{k} \cdots \\
2^n(s' - s)(s' - s + 1) \cdots (s' - s + n - 1)
\]

The first relation is trivial. The second relation is obtained as follows

\[
M^s_{n\beta}'(z) = \exp(s \sum) \exp((s' - s) \sum) L_{n\beta}(z)
\]

\[
= \exp(s \sum) M^s_{n\beta}'-s(z)
\]

\[
= \exp(s \sum) [L_{n(\beta-s'+s)} + L_{(n-1)(\beta-s'+s)} 2(s' - s) \binom{n}{1} \cdots \\
+ L_{(n-k)(\beta-s'+s)} 2^k(s' - s)(s' - s + 1) \cdots (s' - s + k - 1) \binom{n}{k} \cdots \\
2^n(s' - s)(s' - s + 1) \cdots (s' - s + n - 1)]
\]

And applying the exponential operator on each Laguerre polynomial above we obtain the desired result
1.4 Differential equation

The differential equation obeyed by Laguerre polynomials is of the form

\[(z \frac{d^2}{dz^2} + (\beta - z + 1) \frac{d}{dz} + n)L_n(z) = 0\]

To find the differential equation obeyed by \(M_{n\beta\alpha}^s(z)\) we apply the deformation operator on both sides of the above equation and get

\[(z \frac{d^2}{dz^2} + (\beta - z + 1) \frac{d}{dz} + n)M_{n\beta\alpha}^s(z) = -\left[\exp(s \sum_{\alpha} \cdot z) \left(\frac{d^2}{dz^2} - \frac{d}{dz}\right)L_{n\beta}(z)\right]
= -s \sum_{m=1}^{n} \alpha^m \left(\frac{d^{m+1}}{dz^{m+1}} - \frac{d^{m}}{dz^{m}}\right)M_{n\beta\alpha}^s(z)\]

In the above we use the identity

\[\left[\exp(s \sum_{\alpha} \cdot z)\right] = s \sum_{m=1}^{\infty} \alpha^m \frac{d^{m-1}}{dz^{m-1}} \exp(s \sum_{\alpha})\]

(which has a form appropriate to reproduce a differential equation) and limit the sum to \(n\) as \(M\) is identically vanishing for negative \(n\). In the case \(\alpha = 1\) the right hand side reduces to \(s \frac{d}{dz}M_{n\beta(1)}^s(z)\) as the terms are alternating and hence compensating each other except the first derivative, but this is the differential equation of Laguerre polynomials with the parameter \(\beta\) shifted \(\beta \rightarrow \beta - s\) and this we know from subsection 1.3. The case of interest is \(\alpha = -1\)

\[(z \frac{d^2}{dz^2} + (\beta + s - z + 1) \frac{d}{dz} + n)M_{n\beta}^s(z) = 2s \sum_{m=1}^{n} (-1)^m \frac{d^m}{dz^m}M_{n\beta}^s(z)\]

\[= 2s \sum_{m=1}^{n} M_{(n-m)(\beta+m)}^s(z)\]

where we use the evident recursion relation \(\frac{d^m}{dz^m}M_{n\beta}^s(z) = (-1)^m M_{(n-m)(\beta+m)}^s(z)\).

This is a differential equation we can interpret equivalently as an equation of order \(n\) with varying coefficients or a linear system of second order differential equations or, if we plug the explicit expression for the polynomials \(M\) into the \(\sum\), as an inhomogeneous differential equation whose homogeneous part is the differential equation of Laguerre polynomials.
2 $C_{n\beta}^s(z), W_n^s(z)$ Polynomials

To find the measure we assume that Polynomials $M$ are a priori partial\-orthogonal with respect to the measure $\mathcal{D}_{s\beta}(z)$ to be determined, which means that we assume

$$\int_0^{\infty} M_{n\beta}^s(z) M_{0\beta}^s(z) \mathcal{D}_{s\beta}(z) \, dz = \delta_{n0} \Gamma(\beta + 1)$$

Multiply both sides of $\mathcal{D}_{s\beta}(z)$ by $M_{n\beta}^s(z) = L_{0\beta}(z) = 1$ and then integrate with measure $\mathcal{D}_{s\beta}(z)$ we should get

$$\int_0^{\infty} \frac{e^{-zt}}{(1-t)^{\beta-s+1}} (1 - 2t)^{-s} \mathcal{D}_{s\beta}(z) \, dz = \Gamma(\beta + 1)$$

Let us check that the following expression of the measure is the correct one

$$\mathcal{D}_{s\beta}(z) = (-1)^s e^z \frac{d^s}{dz^s}(e^{-2z} z^\beta)$$

In fact we have

$$\int_0^{\infty} \frac{e^{-zt}}{(1-t)^{\beta+1}} \left( \frac{1 - 2t}{1-t} \right)^{-s} (1 - 2t)^{-s} e^z \frac{d^s}{dz^s}(e^{-2z} z^\beta) \, dz$$

$$= \int_0^{\infty} \frac{1 - 2t}{1-t} \frac{1}{(1-t)^{\beta+1}} \, dz \frac{d^s}{dz^s} \left( e^{\left(\frac{1-2\beta}{1-t}\right)z} \right) e^{-2z} z^\beta \, dz$$

$$= \int_0^{\infty} \frac{1}{(1-t)^{\beta+1}} e^{-z} e^{-z} z^\beta \, dz$$

$$= \Gamma(\beta + 1)$$

Where the last integral is the partial \-orthogonality of Laguerre polynomials.

As for the case of $M_{n\alpha, H}^s$ polynomials( which are the result of the action of the same deformation on Hermite polynomials ) whose measure is $\mathcal{D}_{s\beta,H}(z) = \frac{(-\alpha)^s}{\sqrt{\pi}} \exp(-z^2) H_s(z - \frac{\alpha}{2})$, it is possible, surprisingly, to rewrite the measure in terms of Laguerre polynomials. After some algebra we get

$$\mathcal{D}_{s\beta}(z) = (-1)^s z^{\beta-s} e^{-z} L_{s(\beta-s)}(2z)$$

where we have used the definition of laguerre polynomials in terms of higher derivatives $L_{n\beta}(z) = z^{-\beta} e^z \frac{d^n}{dz^n}(z^{\beta+n} e^{-z})$.
This is a real function of the real variable $z$ but not positive. It is not a measure but rather a density with no definite sign. We will however continue to call it measure. Out of this measure, we can form orthogonal polynomials whose existence is related to the finiteness of the integral

$$\int_0^\infty z^n \mathcal{D}_{s\beta}(z)\,dz$$

for $n = 1, 2, \ldots$. In fact we can compute it for any value $n$, in principle. To proceed we can reexpress the quantity $z^n \mathcal{D}_{s\beta}(z)$ as a linear combination of $\mathcal{D}_{p\beta}(z)$ with $s - n \leq p \leq s$

$$z^n \mathcal{D}_{s\beta}(z) = \sum_{p=0}^{n} \binom{n}{n-p} s(s-1) \cdots (s+1-p) \mathcal{D}_{(s-p)(\beta+n-p)}(z) \quad (7)$$

To see how the above expression is evaluated we worked out various components and guessed the general pattern. Here we just show the first component calculation

\[ z \mathcal{D}_{s\beta}(z) = (-1)^s z^{\beta+1-s} e^{-z} L_{s(\beta-s)} = (-1)^s z^{\beta+1-s} e^{-z} (L_{s(\beta+1-s)} - sL_{(s-1)(\beta+1-s)}) = (-1)^s z^{\beta+1-s} e^{-z} L_{s(\beta+1-s)} + s(-1)^{s-1} z^{\beta-(s-1)} e^{-z} L_{(s-1)(\beta-(s-1))} = s \mathcal{D}_{(s-1)\beta}(z) + \mathcal{D}_{s(\beta+1)}(z) \]

In the second line we used the known relation $L_{s\beta} = L_{s(\beta+1)} - sL_{(s-1)(\beta+1)}$. With the integral $\int_0^\infty \mathcal{D}_{s\beta}(z)\,dz = \Gamma(\beta + 1)$ and it is property that it does not depend on the index $s$ and formula, the above integral can be computed trivially

$$\int_0^\infty z^n \mathcal{D}_{s\beta}(z)\,dz = \int_0^\infty \sum_{p=0}^{n} \binom{n}{n-p} s(s-1) \cdots (s+1-p) \mathcal{D}_{(s-p)(\beta+n-p)}(z)\,dz = \sum_{p=0}^{n} \binom{n}{n-p} s(s-1) \cdots (s+1-p) \Gamma(\beta + n - p + 1)$$

The most general form of $C_{n\beta}^s(z)$ polynomials ensuring partial-orthogonality is a linear combination of $M_{n\beta}^s(z)$ polynomials i.e.
\[ C_{0\beta}^s(z) = M_0^s(z) = 1 \]
\[ C_{n\beta}^s(z) = \sum_{i=0}^{n-1} w^n_i M_{(n-i)\beta}^s(z) \]

where the coefficients \( w^n_i \equiv w_{p}^{(n)} \) are to be determined using the orthogonality relation. We normalize polynomials \( C \) above such that \( w^0_n = 1 \). Coefficients \( w^n_i \) are obtained as follows. Define the determinant \( \Delta_n \) and \( \Delta'_n \) where we use the simpler notation \( M_n M_m \equiv \int_0^\infty M_n(z) M_m(z) \mathcal{D}_{s\alpha}(z) dz \) and where \( i \) means insertion at the \( i^{th} \) column \( 1 \leq i \leq n \)

\[
\Delta_n = \begin{vmatrix}
M_1 M_n & \cdots & \cdots & M_1 M_2 & M_1 M_1 \\
M_2 M_n & \cdots & \cdots & M_2 M_2 & M_2 M_1 \\
\vdots & \cdots & \cdots & \vdots & \vdots \\
M_n M_n & \cdots & \cdots & M_n M_2 & M_n M_1
\end{vmatrix}
\]

\[
\Delta'_n = -\begin{vmatrix}
M_1 M_n & \cdots & \cdots & M_{n+1} M_1 & \cdots & M_1 M_2 & M_1 M_1 \\
M_2 M_n & \cdots & \cdots & M_{n+1} M_2 & \cdots & M_2 M_2 & M_2 M_1 \\
\vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
M_n M_n & \cdots & \cdots & M_{n+1} M_n & \cdots & M_n M_2 & M_n M_1
\end{vmatrix}
\]

the coefficients are then given by the formula
\[
w^n_i = \frac{\Delta'_n}{\Delta_n} \Delta'^{i}_{n-1}
\]

Polynomials \( W_{n\beta}^s(z) \) are defined as
\[
C_{n\beta}^s(z) = \exp(s \sum) W_{n\beta}^s(z)
\]

They are the polynomials whose deformation leads to polynomials \( C_{n\beta}^s(z) \). Due to formula 8, they are related to Laguerre polynomials

\[
W_{n\beta}^s(z) = \sum_{i=0}^{n-1} w^n_i L_{(n-i)\beta}(z)
\]
References

[1] M.Mekhfi  “Unification of Bessel functions of different orders” .hep-th/9512159. Int ’ Journal of Theoretical Physics Vol.39, No.4, (2000) see also “ Mapping integer order Neumann functions to real orders” .math-ph/0007041

[2] M.Mekhfi  “A deformation of Hermite polynomials” math-ph/0010003 submitted.

[3] V.I.Smirnov  A Course On Higher Mathematics , Vol III , Part 2, Pergamon Press (1964).
I.S.Gradshteyn and M.Ryzhik  Table Of Integrals , Series and Products Academic Press ( 1980 )