Semiclassical models for uniform-density Cosmic Strings and Relativistic Stars

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Abstract

In this paper we show how quantum corrections, although perturbatively small, may play an important role in the analysis of the existence of some classical models. This, in fact, appears to be the case of static, uniform-density models of the interior metric of cosmic strings and neutron stars. We consider the fourth order semiclassical equations and first look for perturbative solutions in the coupling constants $\alpha$ and $\beta$ of the quadratic curvature terms in the effective gravitational Lagrangian. We find that there is not a consistent solution; neither for strings nor for spherical stars. We then look for non-perturbative solutions and find an explicit approximate metric for the case of straight cosmic strings. We finally analyse the contribution of the non-local terms to the renormalized energy-momentum tensor and the possibility of this terms to allow for a perturbative solution. We explicitly build up a particular renormalized energy-momentum tensor to fulfill that end. These state-dependent corrections are found by simple considerations of symmetry, conservation law and trace anomaly, and are chosen to compensate for the local terms. However, they are not only ad hoc, but have to depend on $\alpha$ and $\beta$, what is not expected to first perturbative order. We then conclude that non-perturbative solutions are valuable for describing certain physical

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I. INTRODUCTION

It is well-known that semiclassical quantization of matter fields in a curved background reveals the necessity to include higher order terms in the gravitational Lagrangian in order to absorb the divergent behavior coming from the one-loop expansion of the matter field (see Ref. [1] for a review). These terms can be expressed as quadratic combinations of the curvature tensor and of its derivatives. The resulting theory has field equations including fourth order derivatives of the metric.

The loop expansion to quantum gravity can be formally interpreted as an expansion in powers of \( l_{pl}^2 \) (i.e. in powers of \( \hbar \) in \( G = c = 1 \) units). One formally expands both a quantum metric operator \( \hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{\Psi}_{\mu\nu} \) and a quantum matter field operator \( \hat{\Phi}_{\mu\nu} \) (for simplicity a scalar field) about any fixed vacuum classical solution \( g_{\mu\nu}^{(0)} \). However, if we consider the full series in this expansion we discover that not only does not converge but is non-renormalizable. In the lowest (one-loop) order, the matter divergences can be absorbed in the constants of the theory. However, this finiteness cannot be attained at this one-loop order for gravity coupled to matter fields. This raises the well-known graviton loop problem since, there is no obvious mathematical nor physical justification for dropping the ‘one-loop’ of the graviton, because in this scheme the one-loop quantum effects of the metric are just as important as those of the matter field.

On the other hand, the alternative \( 1/N \) approach to semiclassical gravity was investigated in Ref. [22]. In this approach one considers the Einstein–Hilbert gravitational action plus matter \( L^N \), where in the matter Lagrangian \( L^N_m \) one assumes the presence of \( N \) identical and free (i.e. non-self-interacting) conformally invariant scalar fields, all of which are in the same quantum state. Then, instead of expanding the theory in powers of \( \hbar \), which results in the ordinary loop expansion, one expands in powers of \( 1/N \). In this expansion each field is now coupled to gravity with coupling proportional to \( 1/N \) and all matter contributions (classical and one-loop) are included already at the leading order in \( 1/N \). Note that one of the advantages of such an expansion is that it is a manifestly gauge-invariant expansion (\( N \) being a gauge invariant parameter). It was also observed [23] that it is easier to examine the theory in the limit \( N \to \infty \), where even the one-loop graviton operator and Faddeev-Popov ghosts contributions can be neglected with respect to the contribution coming from a large number \( N \) of matter fields. Thus, in this approximation there are no corrections from the quantized degrees of freedom of the gravitational field itself and gravity can be simply treated classically. Moreover, observe also that the \( N \) fields, although interacting with gravity, do not couple with another possibly present matter field. Note also that in the \( N \to \infty \) limit, the fluctuations in the expected stress–energy tensor of the \( N \) matter fields become negligible. The applicability of the semiclassical field equations, Eqs. (1.2) below, is now given by the higher order \( 1/N \) corrections, which suggest that these equations would (in principle) be valid even near the Planck scale, where effects which are non-perturbative in \( \hbar \) may be important and one should solve the theory exactly [14].

In this paper we adopt the \( 1/N \) approach to quantum gravity. We then consider the following quadratic gravitational Lagrangian in four dimensions

\[
I = I_G + I^{ren}_m = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left\{ -2\Lambda + R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + 16\pi G L^{ren}_m \right\},
\]

(1.1)
where we have made use of the Gauss–Bonnet invariant to eliminate the dependence on the Riemann tensor.

The semiclassical field equations derived by extremizing the action $I$ can be written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha (1) H_{\mu\nu} + \beta (2) H_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle_{\text{ren}},$$

(1.2)

where

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = -\frac{16\pi G \partial I_{\mu\nu}^{\text{ren}}}{\sqrt{-g}},$$

(1.3)

is the renormalized expectation value of the quantized matter source and where

$$(1) H_{\mu\nu} = -2 R_{;\mu\nu} + 2 g_{\mu\nu} \Box R - \frac{1}{2} g_{\mu\nu} R^2 + 2 R R_{\mu\nu},$$

(1.4)

and

$$(2) H_{\mu\nu} = -2 R_{;\alpha ;\mu\nu}^{\alpha} + \Box R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Box R + 2 R^{\alpha}_{\mu} R^{\alpha}_{\alpha\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} R^{\alpha\beta}.$$

(1.5)

The (renormalized) values of the coupling constants $\alpha$ and $\beta$ come from the regularization process and depend on the number and types of fields present as well as on the method of regularization. However, since the higher derivative terms must have very little effect on the low-frequency domain (in order to avoid conflict with observations), it is usual to consider them as free numerically small parameters of the order of $l_{\text{pl}}^2$ (or equivalently, $\hbar$, for $G = c = 1$).

It was observed in [2,3], and independently in [4], that one can develop a perturbative approach to that problem that allows one to write the perturbative field equations in terms of second order derivatives of the metric and of the energy–momentum tensor. The system is thus reduced to an effective Einsteinian one. This perturbative reduction procedure can be described as follows. In the perturbative approach [4] we consider coupling constants $\alpha$ and $\beta$ to be of the same order of $l_{\text{pl}}^2$, and curvatures small enough to ensure that the perturbative series makes sense

$$|\alpha R| \ll 1, \quad |\beta R_{\mu\nu}| \ll 1.$$

(1.6)

Note that these conditions arise naturally since the description provided by the semiclassical gravity breaks down in a domain where the curvature is large i.e. near $|R_{\mu\nu}| \approx c^3/(\hbar G) = 3.9 \times 10^{65}\text{cm}^{-2}$.

Bearing these conditions in mind, we observe that, up to first order in $\alpha$ and $\beta$, the two geometrical conserved tensors $(1) H_{\mu\nu}$ and $(2) H_{\mu\nu}$, can be computed in terms of any known classical zeroth order solution, $g^{(0)}_{\mu\nu}$, of Einstein field equations

$$R^{(0)}_{\mu\nu} - \frac{1}{2} R^{(0)} g^{(0)}_{\mu\nu} + \Lambda g^{(0)}_{\mu\nu} = \frac{8\pi G}{c^3} T^{(0)}_{\mu\nu},$$

(1.7)

where $T^{(0)}_{\mu\nu}$ is the stress energy momentum tensor of some matter fields, which can generally include both the source of an ordinary matter field $F^{(0)}_{\mu\nu}$ (e.g. electromagnetic field, perfect fluid, very energetic point–like particle, etc), given by $T^{(0)}_{\mu\nu} = T_{\mu\nu}(g^{(0)}_{\mu\nu}, F^{(0)}_{\mu\nu})$, and the source
of $N$ identical scalar fields not interacting with the precedent ordinary matter fields, given by $T_{\mu\nu}^{N\ (0)} = T_{\mu\nu}(g_{\mu\nu}^{(0)}, \phi^{(0)} j)$ with $j = 1, \ldots, N$. All matter fields are then coupled to $g_{\mu\nu}^{(0)}$, so that their only relevant back-reaction effects are on the gravitational field. Note that since we are interested in issues related to the back-reaction problem, the dynamics of the matter fields will not concern us here. Our purpose is to solve the equations of motion for gravity and to find the back-reaction of the matter fields on the gravitational field.

From the complete knowledge of the zeroth order classical system, i.e. $(g_{\mu\nu}^{(0)}, F_{\mu\nu}^{(0)}, \phi^{(0)} j)$, one can write the ‘reduced’ semiclassical equations to first-order (in $\alpha$ and $\beta$) as Einsteinian like ones

$$R_{\mu\nu}^{(1)} - \frac{1}{2} R^{(1)} g_{\mu\nu}^{(1)} + \Lambda g_{\mu\nu}^{(1)} = \frac{8\pi G}{c^3} \langle T_{\mu\nu}^{(1)} \rangle_{\text{ren}} - \alpha^{(1)} H_{\mu\nu}(g_{\mu\nu}^{(0)}) - \beta^{(2)} H_{\mu\nu}(g_{\mu\nu}^{(0)}) \equiv \frac{8\pi G}{c^3} T^{\text{eff\ (1)}}_{\mu\nu} ,$$

with an effective source $T_{\mu\nu}^{(1)\ \text{eff}}$.

Let us now consider the following convenient approximation where the renormalized energy-momentum tensor in the expression of $T_{\mu\nu}^{(1)\ \text{eff}}$ can be split into two additive parts,

$$\langle T_{\mu\nu}^{(1)} \rangle_{\text{ren}} = T_{\mu\nu}^{(1)} + \langle \Delta T_{\mu\nu}^{N\ (1)} \rangle ,$$

where the first term of the right hand side of this equation includes both $T_{\mu\nu}^{O\ (1)}$ and $T_{\mu\nu}^{N\ (1)}$, and the second term includes only the (state-dependent) expected value of the energy-momentum tensor of the $N$ identical free scalar fields, $\hat{\phi}^{(1)\ j}$, $\langle \Delta T_{\mu\nu}^{N\ (1)} \rangle = \langle \Delta T_{\mu\nu}(g_{\mu\nu}^{(1)}, \hat{\phi}^{(1)\ j}) \rangle = O(l_{pl}^2)$. A similar approximation can also be found in Ref. [6] in the linearized version of the one-loop approximation to quantum gravity. However, we note that the approximation (1.9) is more accurately justified in the context of $1/N$ expansion, where at least we have the following advantages:

- The scalar fields $\hat{\phi}^{(1)\ j}$ are free (i.e. non-self-interacting), so that only one-loop terms in these matter fields arise. The stress-energy momentum tensor of these fields is approximated in such a way that it can conveniently be split into a classical part, $T_{\mu\nu}^{N\ (1)}$, and into a quantum one, $\langle \Delta T_{\mu\nu}^{N\ (1)} \rangle$.

- We can use the same argument as that used for the one-loop graviton to neglect the one-loop (order $l_{pl}^2$) contribution of matter field operator $F_{\mu\nu}^{(1)}$, so that the ordinary matter, $T_{\mu\nu}^{O\ (1)} = T_{\mu\nu}(g_{\mu\nu}^{(1)}, F_{\mu\nu}^{(1)})$, can always be only treated classically.

- The state-independent (local) corrections are uniquely determined (up the ambiguity given by the value of the coupling constants $\alpha$ and $\beta$) by the two terms $^{(1)}H_{\mu\nu} = H_{\mu\nu}(g_{\mu\nu}^{(0)})$ and $^{(2)}H_{\mu\nu} = H_{\mu\nu}(g_{\mu\nu}^{(0)})$, which can always be built up from the zeroth-order classical metric $g_{\mu\nu}^{(0)}$.

A difficulty still remains. The state-dependent part $\langle \Delta T_{\mu\nu}^{N\ (1)} \rangle$ requires further and independent work to be determined and, although we consider here only very simple quantum matter fields, to compute this term it is in general a very difficult task. But things can extremely simplify when the quantum state is built up from the conformal vacuum and
fields and background metric are conformally invariant. In this situation, in fact, all the corrections can be expressed in term of local quantities.

Thus, once $T_{\mu\nu}^{(1)\,\text{eff}}$ is determined, one can try to solve the resulting reduced Einstein equations for the metric $g_{\mu\nu}^{(1)}$, which in turn will contain the first order corrections in $\alpha$ and $\beta$. Note that, formally, the iteration can be pursued up to the $n$-th order to find $T_{\mu\nu}^{(n)\,\text{eff}}$ and thus one might solve the $n$-th order Einstein type field equations to find $g_{\mu\nu}^{(n)}$, but since for many purposes it is enough and since explicit calculations are very complicated at order higher than the first we will stop the process to this order.

A useful procedure, even to the first order in the iteration, is to solve for only some of the components of the reduced field equations and to enforce the conservation of the effective energy–momentum tensor, i.e.

$$ T^{\text{eff}}_{\mu\nu} = 0 \quad (1.10) $$

This is the simplest way to ensure the existence of the solutions of the form $g_{\mu\nu}^{(1)}$, without having to solve the full set of field equations.

It is evident, from the structure of the process, that we can only generate perturbative developments of the solution, if it exists at all. The knowledge of $g_{\mu\nu}^{(1)}$ allow us to search for first quantum corrections to several physical quantities of interest.

It is also evident that for vacuum classical solutions the contribution of $(1) H^{(0)}_{\mu\nu}$ and $(2) H^{(0)}_{\mu\nu}$ terms vanishes and the only non–zero contribution will be given by the term $(\Delta T_{\mu\nu}^{N\,\text{eff}}(1))$. This happens, for example, in the spacetime of a Schwarzschild black hole. Note that from this, it is also clear now, that the first order solution to the exterior metric of a gauge string given in Ref. [4], Eq. (23), has the last line wrong. In fact it gives a $1/r^2$ term outside the core where $T_{\mu\nu}^{(0)} = 0$ and which is not present in the general relativistic solution. It is part of the motivations of this paper to correct this expression.

The rest of the paper is organized as follows: Sec. II A. deals with the constant density model for the core of straight local cosmic strings. We look, first of all, for static state–independent perturbative solutions to this problem and we find that they do not exist. We then study in Sec. II B. (local) non–perturbative corrections to the metric by a method developed in Ref. [7] that makes use of the properties of the fourth order field equations under conformal transformations (more generally Legendre transformations). Sec. II C. analyses the inclusion of some particular (non–local) quantum state term in the energy–momentum tensor that renders possible the existence of perturbative corrections to the metric.

In Sec. III we study the issue of the existence of equilibrium solutions with spherical symmetry. Here we also take a constant energy–density $\rho$ while we allow the pressure $p$ to be dependent on the radial coordinate $r$ to satisfy the Oppenheimer–Volkov equation. We again show the non consistency of this model and argue that non–local corrections to the renormalized energy–momentum tensor hardly can change here the situation. We end the paper with a further discussion of the perturbative approach to this uniform–density models.

### II. COSMIC STRINGS WITH UNIFORM–DENSITY CORES

In Refs. [2,3] the above perturbative method have been mainly applied to cosmological models. The main shortcoming here is the fact that we cannot trust the results in the most
interesting regime; near the initial singularity. The situation is less dramatic when dealing with black holes \cite{4,5}, since one is mainly interested in quantities evaluated outside or at the event horizon. The singularity at \( r = 0 \) being hidden to the outside world until the very last stages of the black hole evaporation.

On the other hand, we expect the perturbative approach to be very precise in the whole spacetime generated by a cosmic string (see conditions (1.6)), since (at GUT scale and below) curvature is moderate inside the string and vanishes outside.

A. Perturbative solution with local source

Let us consider a straight (gauge) string lying along the \( z \) axis. We have then a cylindrically symmetric system, Lorentz invariant in the \( z \) direction. In this case, a general enough metric can written as

\[
ds^2 = e^{2b(r)}(-dt^2 + dz^2) + dr^2 + r_0^2e^{2a(r)}d\phi^2,
\]

within the following Gaussian coordinates range: \(-\infty < t < +\infty, -\infty < z < +\infty, 0 \leq r \leq r_-, 0 \leq \phi < 2\pi\).

The non–vanishing components of the Einstein tensor for this metric read

\[
G_r^r = b'(r) \left(2a'(r) + b'(r)\right),
\]

\[
G_\phi^\phi = 3b'(r)^2 + 2b''(r),
\]

\[
G_t^t = a'(r)^2 + a'(r)b'(r) + b'(r)^2 + a''(r) + b''(r).
\]

Given the energy–momentum tensor (effective or not) and equating it to the Einstein tensor (over \( 8\pi \) since we choose units such that \( G = c = 1 \)) we get a set of equations whose solution determine the form of \( a(r) \) and \( b(r) \). These equations can be supplemented with the conservation law which in the case of an energy–momentum tensor diagonal and only dependent on the coordinate \( r \) (in its covariant–contravariant form) takes the following simplified form

\[
T^r_{\mu \rho} = \left[T^r_{\nu ,r} + 2b'(r)\left(T^\rho_r - T^\rho_t\right) + a'(r)\left(T^\rho_r - T^\rho_\phi\right)\right] \delta^r_\mu = 0.
\]

The \( \phi–\phi \) component of Einstein equations can be reduced to a Riccati equation upon the substitution \( y(r)\doteq b'(r) \). Thus,

\[
2y'(r) + 3y(r)^2 = 8\pi T^\phi_\phi.
\]

Once solved this equation for \( y(r) \) by making the sum \( G_r^r + G_\phi^\phi - 2G_t^t \) we also get a Riccati equation for \( a'(r)\doteq w(r) \)

\[
w'(r) + w(r)^2 = 8\pi \left[T_t^t - \frac{1}{2} \left(T^r_r + T^\phi_\phi\right)\right] - y(r)^2.
\]
These two last equations together with the conservation law \ref{2.3} are an equivalent set to Einstein equations \ref{2.2}–\ref{2.4} and have the advantage of being of lower differential order [bearing only first derivatives in variables \(y(r)\) and \(w(r)\)].

For the sake of simplicity we will now study the effect of the state–independent (local) semiclassical corrections to the uniform density model for the interior structure of straight cosmic strings. Thus the only non–vanishing components of \(T^{(0)}_{\mu\nu}\) are \[8–10\]

\[ T^t_t = T^z_z = \left( -\frac{1}{8\pi r_0^2} \right) \Theta(r_s - r), \tag{2.8} \]

where \(r_0\) is a constant that specifies the energy density, \(r_s\) is the “radius of the string”, and \(\Theta\) stands for the step function.

It is easy to verify that in this case Eqs. \ref{2.3}–\ref{2.7} are satisfied by \[8–10\]

\[ b(r) = 0, \tag{2.9} \]

\[
\exp[a(r)] = \begin{cases} 
\sin \left( \frac{r}{r_0} \right) & ; \ r \leq r_s \\
\left( \frac{r}{r_0} \right) \cos \left( \frac{r}{r_0} \right) & ; \ r \geq r_0 \tan \left( \frac{r_s}{r_0} \right)
\end{cases}, \tag{2.10}
\]

where we have imposed the equality of the intrinsic metric and the extrinsic curvature \[11\] on the matching surface located at \(r_- = r_s\) and \(r_+ = r_0 \tan(r_s/r_0)\). Thus \(r_s/r_0\) parametrize the family of exterior solutions while \(r_0\) gives the scale of the interior problem.

Given the above expression for the metric and Eq. \ref{2.8} we can compute the local part of the effective energy–momentum tensor from Eq. \ref{1.8}. Doing so, we obtain for \(r < r_s\)

\[ T^t_t^\text{eff} = T^z_z^\text{eff} = -\left( \frac{1}{8\pi r_0^2} \right) + \left( \frac{2\alpha + \beta}{8\pi r_0^4} \right), \tag{2.11} \]

\[ T^r_r^\text{eff} = T^\phi_\phi^\text{eff} = -\left( \frac{2\alpha + \beta}{8\pi r_0^4} \right). \tag{2.12} \]

Since the local part of effective energy–momentum tensor vanishes for \(r > r_s\), the general form of metric exterior to the string remain unchanged. Only the parameter \(r_s\) and the constant part of \(g_{\phi\phi}\) are expected a priory to vary. We will find, however, a somewhat unexpected result when we study the internal structure. In fact, since \(T^\phi_\phi^\text{eff} = \text{constant}\), Eq. \ref{2.6} can be easily integrated. The general solution reads

\[ b(r) = \frac{2}{3} \ln \left[ \cosh \left( \sqrt{6\pi T^\phi_\phi^\text{eff}} (r - C_1) \right) \right] + C_2. \tag{2.13} \]

The constants \(C_1\) and \(C_2\) can be determined from the matching conditions at \(r_s\) with the exterior metric. In particular one would obtain \(C_1 = r_s\) and \(C_2 = 0\). Note that since we have reduced the original fourth order set of equations \[11\] to an effective one of second order (Eq. \ref{1.8}), the matching conditions will be that of ordinary General Relativity \[11\].
\[
\begin{align*}
\left. b(r) \right|_{r_+}^{ext} &= b(r) \left. \right|_{r_-}^{int} \quad \text{and} \quad \left. a(r) \right|_{r_+}^{ext} = a(r) \left. \right|_{r_-}^{int} \\
\partial_r b(r) \left. \right|_{r_+}^{ext} &= \partial_r b(r) \left. \right|_{r_-}^{int} \quad \text{and} \quad \partial_r a(r) \left. \right|_{r_+}^{ext} = \partial_r a(r) \left. \right|_{r_-}^{int},
\end{align*}
\]

(2.14)

i. e. the metric and extrinsic curvature should match as the boundary is approached from each side.

One has also to keep in mind that since we are considering a perturbative approach, only up to terms linear in \( T^\phi_{\text{eff}} \) have to be retained in (2.13), i.e.

\[
b(r) \approx -\frac{(2\alpha + \beta)}{4r_0^4} (r - C_1)^2 + C_2.
\]

(2.15)

On the other hand, the conservation law (2.5) applied to \( T^\nu_{\mu \text{eff}} \) (see Eqs. (2.11)-(2.12)) leads to first order to

\[
\frac{1}{4\pi r_0^2} b'(r) \approx 0.
\]

(2.16)

These two last equations are obviously \textit{inconsistent} unless \( 2\alpha + \beta = 0 \), what, in our case, leads us back to Einstein theory. Equivalently, if \( b'(r) = 0 \), Eq. (2.13) is not satisfied for the non–vanishing source (2.11). What fails? Well, let us recall what our hypothesis have been. We have supposed that the semiclassical corrections to the gravitational theory are \textit{perturbative} and can be absorbed in an effective energy–momentum tensor (see Eq. (1.8)). We then supposed that the resulting metric could be described by Eq. (2.1), cylindrically symmetric, Lorentz invariant and \textit{static}. On this last point let us note that observe in Eq. (2.13), although we have equal and constant energy density and pressure in the \( z \)-direction, as in the general relativistic model that tend to balance each other, it also appears now a constant, but different from zero, radial and angular component of the pressure that would destroy the string since they are unbalanced.

But let us keep the condition of staticity and explore a possible alternative which could make stable this simple model.

**B. Non–perturbative solution with local source**

A second possibility is that the effects of the higher order terms in the Lagrangian (1.1) lead to local non–perturbative corrections. In that case, the method stated in Sec. I would give no result since by its procedure can only get perturbative corrections. In Ref. [7] it was developed a method to reduce the order of theory from fourth to second by introducing two auxiliary fields: A scalar field \( \chi \) and a massive spin two field \( \psi_{\mu \nu} \). These fields added to the usual graviton and account for the extra degrees of freedom of higher order theories. This method, although in a first approximation allows to compute non–perturbative corrections to the general relativistic metric. In fact, in Ref. [7] it was found that (for the sake of simplicity we take \( \beta = 0 \)) the metric solution to the higher order problem \( g^{Q}_{\mu \nu} \) is conformally related to the general relativistic metric \( g^E_{\mu \nu} \)

\[
g^{Q}_{\mu \nu} \simeq (1 + \chi(r)) g^E_{\mu \nu},
\]

(2.17)
where the field $\chi$ satisfies the equation

$$\Box - m_0^2 \chi = -\frac{8\pi G}{3} T^{(\text{Matter})}, \quad m_0^2 = \frac{1}{6\alpha} \tag{2.18}$$

with the $\Box$ operator taken with respect to $g_{\mu\nu}^E$.

The metric outside the core of the string can be easily found from the above formulae

$$\chi(r) = C K_0(m_0 r), \tag{2.19}$$

where $m_0$ is real from the no-tachyons constraints

$$3\alpha + \beta \geq 0, \quad \beta \leq 0, \tag{2.20}$$

and $C$ is a constant to be determined by the matching conditions with the internal metric.

To find the internal metric is a little more involved. We find the solution

$$\chi(r) = \chi_p(r) + \chi_h(r),$$

where $\chi_p(r) = \frac{2}{3} (r_0^2 m_0^2)$ is the particular solution to (2.18), and

$$\chi_h(r)'' + \frac{1}{r_0} \cot \left( \frac{r}{r_0} \right) \chi_h(r)' - m_0^2 \chi_h(r) = 0. \tag{2.21}$$

The solution to this equation can be written in terms of Hypergeometric functions

$$\chi_h(r) = C_1 F \left( a, b, 1, \cos^2 \left( \frac{r}{2r_0} \right) \right) + C_2 \left\{ F \left( a, b, 1, \cos^2 \left( \frac{r}{2r_0} \right) \right) \ln \left[ \cos^2 \left( \frac{r}{2r_0} \right) \right] \right. \right.$$

$$+ \sum_{k=1}^{\infty} \cos^{2k} \left( \frac{r}{2r_0} \right) \frac{(a)_k (b)_k}{k!} \left[ \psi(a + k) - \psi(a) + \psi(b + k) - \psi(b) \right.$$

$$- 2\psi(k + 1) + 2\psi(1) \right\} \tag{2.22}$$

where coefficients

$$a = \frac{1 + \sqrt{1 - 2r_0 \alpha}}{2}, \quad b = \frac{1 - \sqrt{1 - 2r_0 \alpha}}{2} \tag{2.23}$$

shows the essentially non-perturbative character of the solution. Note that this non-perturbative solution is always convergent (and without poles) in the range of values of the coordinate $r$ and, thus, well behaved at $r = 0$, what is in agreement with the hypothesis made in Ref. [7].

The constants $C, C_1$ and $C_2$ have to be determined with the matching conditions

$$\chi(r) \bigg|_{r_+}^{\text{ext}} = \chi(r) \bigg|_{r_-}^{\text{int}} \quad \text{and} \quad \partial_r \chi(r) \bigg|_{r_+}^{\text{ext}} = \partial_r \chi(r) \bigg|_{r_-}^{\text{int}}, \tag{2.24}$$

but since both these external and internal non-perturbative solutions are already of complicate forms we do not write, here, explicitly their values.

We finally observe, that although, in Ref. [3], it was emphasized the idea that a “self-consistent” method should be the only valid one to find “physical” solutions of the semiclassical field equations, in Ref. [4] (where we developed a closely related approach) we never considered the perturbative method as an absolute physical prescription for “throwing out” all the non-perturbative solutions. Counterexamples to Ref. [3] claim can, in fact, be found in recent papers [13,14].
C. Perturbative solution with non–local source

Other possibility we have to discuss is to include in the effective energy–momentum tensor a state–dependent (non–local) piece. The local part of the total mean value is the one we considered in (1.8).

It is not difficult to compute \( \langle \Delta T_{\mu \nu} \rangle \) outside the string since the space is locally flat but topologically and, thus, globally different from Minkowski spacetime. In fact this non–local part will be the whole correction since the local part vanishes. One can perform the full quantum field theoretical computation \[15–17\] or make simpler symmetry and conservation arguments \[18\] to arrive to

\[
\langle \Delta T_{\mu \nu} \rangle = \frac{A \bar{h}}{r^4} \text{diag}(1, 1, 1, -3),
\]

where the constant \( A \) depends on the number of conformal massless free fields we consider and we will keep its value generic [In Refs. \[15–17\] it was found that for a single scalar field \( A = (1440\pi^2)^{-1}[\cos^{-4}(r_s/r_0) - 1]$].

Taking expression (2.25) for the source of the semiclassical Einstein equations outside the core of the string and integrating expressions (2.6) and (2.7), one gets for the corrected metric up to linear terms in \( \bar{h} \)

\[
ds^2 = \left(1 - \frac{4\pi \bar{h} A}{r^2}\right) (-dt^2 + dz^2) + dr^2 + \left(1 + \frac{16\pi \bar{h} A}{r^2}\right) \cos\left(\frac{r_s}{r_0}\right)^2 r^2 d\phi^2.
\]

with coordinate range: \(-\infty < t < +\infty, -\infty < z < +\infty, r_+ \leq r < +\infty, 0 \leq \phi < 2\pi,$
and where now the boundary reads \( r_+ = r_0 \tan(r_s/r_0) + 2\pi A \bar{h} \cot(r_s/r_0)/r_0 \). The spacetime possess an \( r \)–dependent deficit angle

\[
D = 8\pi \mu - 16\pi^2 \frac{A h}{r^2}(1 - 4\mu)
\]

Now, in the interior of the cosmic string (\( r < r_s \)), we shall reverse the problem. Using similar arguments as in Ref. \[18\], we will find the form of the \( \langle \Delta T_{\mu \nu} \rangle \) inside the string such that a static solution for the (classical) uniform–density source exists. Note that although in any static spacetime there exists a wide class of Hadamard states \[19\] for which this expectation value makes physical sense, in our approximation (1.9), we chose the \( N \) scalar fields to be in the static vacuum state (i. e. \( \langle \Delta T_{\mu \nu} \rangle \equiv \langle 0|T^N_{\mu \nu}|0 \rangle \)), so that these fields do not contribute to the classical matter source (2.8).

Let us first note that when we include the non–local part of the energy–momentum tensor in the source of Einstein equations, and consider first order correction in \( \alpha, \beta, \) and \( \bar{h} \), a formal solution for the metric coefficients in terms of the source can be easily found. In fact, from (2.9) and (2.13) we obtain

\[
b(r) \cong C_2 - \frac{(2\alpha + \beta)}{4r_0^4}(r - C_1)^2 + 4\pi \int_r^{r_+} dr' \int_{r'}^{r_+} dr'' \langle \Delta T^\phi_{\phi} \rangle,
\]

while from Eq. (2.7) we find
\[ a(r) \cong \ln \left[ \sin \left( \frac{r}{r_0} \right) \right] + \frac{C_3}{r_0} - \left[ C_4 r_0 + \frac{(2\alpha + \beta) r}{2r_0^3} \right] \cot \left( \frac{r}{r_0} \right) \]

\[ -4\pi \int \frac{dr'}{\sin^2 \left( \frac{r}{r_0} \right)} \int dr'' \sin^2 \left( \frac{r}{r_0} \right) \left( \langle \Delta T_r \rangle + \langle \Delta T^{\phi}_\phi \rangle - 2\langle \Delta T^t_t \rangle \right), \]  

(2.29)

where here \( \cong \) stands for up to first order terms in \( \alpha, \beta \) and \( \bar{h} \). And the constants \( C_i \) have to be determined upon matching this metric with the exterior one (2.26).

Again we find that the conservation law (2.4) proves to be very useful in analyzing the different possibilities. Up to first order corrections and in the background of the constant density string (2.11) one gets

\[ \langle \Delta T_r \rangle, r + \frac{\cot \left( \frac{r}{r_0} \right)}{r_0} \left( \langle \Delta T_r \rangle - \langle \Delta T^{\phi}_\phi \rangle \right) + \frac{b(r)}{4\pi r_0^2} = 0. \]  

(2.30)

Two cases can be straightforwardly analyzed:

\textit{a. Non–constant radial pressure:} \( \langle \Delta T_r \rangle, r \neq 0 \) and \( \langle \Delta T_r \rangle = \langle \Delta T^{\phi}_\phi \rangle \). [Note that this last equality of components of the energy–momentum tensor is already fulfilled (c.f. (2.12)) by the classical and local one–loop parts of the \( T_{\mu \nu} \).]

In this case we have \( G_r - G^{\phi }_{\phi } = 0 \). From Eqs. (2.2) and (2.3) we get an equation for \( b(r) \) which can be immediately integrated to give

\[ b(r) = -D_1 \cos \left( \frac{r}{r_0} \right) + D_2. \]  

(2.31)

The constants \( D_1 \) and \( D_2 \) can be determined by the matching conditions with the exterior metric (2.26) at \( r = r_s \). Doing so, we obtain

\[ D_1 = \frac{4\pi Ah \cot^3 \left( \frac{r_s}{r_0} \right)}{r_0^2 \sin \left( \frac{r_s}{r_0} \right)}, \]

\[ D_2 = \frac{2\pi Ah}{r_0^2} \cot^2 \left( \frac{r_s}{r_0} \right) \left[ 2 \cot^2 \left( \frac{r_s}{r_0} \right) - 1 \right]. \]  

(2.32)

Now, from Eqs. (2.6) and (2.12) we find

\[ \langle \Delta T^r_r \rangle = \langle \Delta T^{\phi}_\phi \rangle = \frac{(2\alpha + \beta)}{8\pi r_0^3} + \frac{D_1}{8\pi r_0^3} \cos \left( \frac{r}{r_0} \right). \]  

(2.33)

From Eq. (2.29) we see that to find the metric coefficient \( a(r) \) we need also to know \( \langle \Delta T^t_t \rangle \). We can find it making use of the expression for the trace anomaly of a massless, conformal–ally coupled field (see [1] for a review on the subject)

\[ \langle \Delta T \rangle = \frac{\hbar}{2880\pi^2} \left[ AC_{\mu \nu \lambda \rho} C^{\mu \nu \lambda \rho} + B \left( R_{\mu \nu} R^{\mu \nu} - \frac{1}{3} R^2 \right) + C \square R \right] \]

\[ = \frac{\hbar}{2880\pi^2} \left[ \frac{4}{3r_0^4} + \frac{B}{3r_0^4} \right] = \frac{\hbar B}{1440\pi^2 r_0^4}. \]  

(2.34)
where \( A \) and \( B \) depend on the type and number of fields considered, and the last equality defined \( B \).

We then find the other two components of the non–local part of the energy–momentum tensor

\[
\langle \Delta T^t_t \rangle = \langle \Delta T^{z^2}_t \rangle = \frac{1}{2} \langle \Delta T \rangle - \langle \Delta T^r_r \rangle.
\] (2.35)

We can now explicitly perform the integral in Eq. (2.29)

\[
a(r) \equiv \ln \left[ \sin \left( \frac{r}{r_0} \right) + \frac{D_3}{r_0} + \frac{2}{3} D_1 \cos \left( \frac{r}{r_0} \right) \right]
- \left[ D_4 r_0 - \frac{(2\alpha + \beta) r}{2r_0^2} + \frac{h B r}{720\pi r_0^3} \right] \cot \left( \frac{r}{r_0} \right),
\] (2.36)

where the constants \( D_3 \) and \( D_4 \) can be found from the matching conditions with the exterior metric (2.26) at \( r = r_s \),

\[
D_3 = -\frac{2r_0D_1}{3} \cos \left( \frac{r}{r_0} \right) + \left[ D_4 r_0^2 - \frac{(2\alpha + \beta) r_s}{2r_0^2} + \frac{h B r_s}{720\pi r_0^3} \right] \cot \left( \frac{r}{r_0} \right) + \frac{8\pi A h}{r_0} \cot^2 \left( \frac{r}{r_0} \right),
\] (2.37)

\[
D_4 = \frac{(2\alpha + \beta) r_s}{2r_0^2} - \frac{h B r_s}{720\pi r_0^3} \cos^2 \left( \frac{r}{r_0} \right) - \frac{2D_1 \sin \left( \frac{r}{r_0} \right)}{2r_0} - \frac{16\pi A h}{r_0} \cot^3 \left( \frac{r_s}{r_0} \right)
+ \left( \frac{h B r_s}{720\pi r_0^3} - \frac{(2\alpha + \beta) r_s}{2r_0^4} \right) \cot \left( \frac{r}{r_0} \right),
\] (2.38)

\( b. \) Constant radial pressure: Let us consider the complementary condition \( \langle \Delta T^r_r \rangle \neq \langle \Delta T^\phi_\phi \rangle \) in the simplifying case \( \langle \Delta T^r_r \rangle, r = 0 \).

From the conservation law (2.35) we find that to first order in \( h \)

\[
y(r) \equiv b'(r) = -4\pi r_0 \cos \left( \frac{r}{r_0} \right) \left( \langle \Delta T^r_r \rangle - \langle \Delta T^\phi_\phi \rangle \right).
\] (2.39)

Plugging this equation into (2.40) we obtain a first order differential equation for \( \langle \Delta T^\phi_\phi \rangle \), which can be readily integrated to produce

\[
\langle \Delta T^\phi_\phi \rangle = \frac{\langle \Delta T^r_r \rangle}{\cos^2 \left( \frac{r}{r_0} \right)} + E_1 \frac{\sin \left( \frac{r}{r_0} \right)}{\cos^2 \left( \frac{r}{r_0} \right)} - \frac{(2\alpha + \beta)}{8\pi r_0^4} \tan \left( \frac{r}{r_0} \right),
\] (2.40)

where \( E_1 \) is a constant of integration.

Now, with this expression we go back to Eq. (2.40) which allows us to find

\[
b(r) = E_2 - \frac{(2\alpha + \beta) r}{2r_0^3} - 4\pi r_0^2 \left\{ \langle \Delta T^r_r \rangle \ln \left[ \cos \left( \frac{r}{r_0} \right) \right] + E_1 \ln \left[ \frac{\cos \left( \frac{r}{2r_0} \right) + \sin \left( \frac{r}{2r_0} \right)}{\cos \left( \frac{r}{2r_0} \right) - \sin \left( \frac{r}{2r_0} \right)} \right] \right\},
\] (2.41)
where $E_1$ and $E_2$ can be completely determined by the matching conditions.

From the expression for the trace anomaly (2.34) we find the other two components of the non–local part of the energy–momentum tensor

$$\langle \Delta T^t_t \rangle = \langle \Delta T^z_z \rangle = \frac{1}{2} \left[ \langle \Delta T \rangle - \langle \Delta T^r_r \rangle - \langle \Delta T^\phi_\phi \rangle \right].$$

(2.42)

We can now explicitly perform the integral in Eq. (2.29)

$$a(r) \sim \ln \left[ \sin \left( \frac{r}{r_0} \right) \right] + E_2 - \frac{(2\alpha + \beta)r}{2r_0^3} + \frac{(2\alpha + \beta)}{2r_0^2} \ln \left[ \cos \left( \frac{r}{r_0} \right) \right] + \frac{E_3}{r_0}$$

$$- \left[ E_4 r_0 - \frac{8\pi E_0}{r_0} + \frac{(2\alpha + \beta)r}{2r_0^3} + \frac{\hbar r}{360\pi r_0^3} \right] \cot \left( \frac{r}{r_0} \right)$$

$$- 8\pi E_1 \ln \left[ \frac{\cos \left( \frac{r}{r_0} \right) + \sin \left( \frac{r}{2r_0} \right)}{\cos \left( \frac{r}{2r_0} \right) - \sin \left( \frac{r}{2r_0} \right)} \right] + \frac{16\pi E_1 r_0^2}{\sin \left( \frac{r}{r_0} \right)}$$

(2.43)

The constants $E_3$ and $E_4$ can be then found from the matching conditions with the exterior metric (2.26) at $r = r_s$.

To summarize, we have found the explicit form of the expectation value of the stress–energy–momentum tensor that allow a consistent static solution for the interior of the straight cosmic string. More explicitly, we have imposed the semiclassical field equations and solved them after applying the reduction of order procedure to find that the existence of a perturbative solution imply that the expectation value of the stress–energy–momentum tensor depends on $\alpha$ and $\beta$. Now, we really do not expect the first order non–local part to depend on $\alpha$ and $\beta$, since it should be computed on the zeroth order background. This can be explicitly verified by an independent computation of the renormalized energy–momentum tensor using field theory methods, but this is beyond the scope of the present work. The form of the expectation value, here, have been chosen ad hoc to satisfy the perturbative static field equations. In fact, the dependence on the coupling constants comes from the imposition of the matching conditions that allow to determine the integration constants left in the resolution of the reduced field equations (2.28)–(2.30). They contain terms that compensate for the local contributions to the effective energy–momentum tensor. Clearly, this is an inconsistency that leaves us with the only option of the non–perturbative dependence on $\alpha$ and $\beta$ of the straight cosmic string solution.

### III. SPHERICAL STARS WITH CONSTANT ENERGY DENSITY

In order to further understand this non existence of perturbative solutions and see how general this phenomenon might be, we will study another system with uniform–density, but this time possessing spherical symmetry instead of a cylindrical one.

We will represent the spherically symmetric metric by

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

(3.1)

The non–vanishing components of the Einstein tensor for the above metric are
\[ G_t^t = -\frac{1}{r^2} \left[ r(1 - e^{-2\Lambda}) \right]' , \quad (3.2) \]

\[ G_r^r = -\frac{1}{r^2} \left( 1 - e^{-2\Lambda} \right) + \frac{2}{r} \Phi' , \quad (3.3) \]

\[ G^\theta_\theta = e^{-2\Lambda} \left[ \Phi'' + \left( \Phi' \right)^2 + \frac{\Phi'}{r} - \Phi' \Lambda' - \frac{\Lambda'}{r} \right] , \quad (3.4) \]

\[ G^\varphi_\varphi = G^\theta_\theta . \quad (3.5) \]

The Einstein equation derived from equating (3.2) to \( T_t^t \) can be formally integrated

\[ e^{-2\Lambda(r)} = 1 + \frac{8\pi G}{r} \int_0^r r'^2 T_t^t(r') dr' , \quad (3.6) \]

where we have imposed the condition \( \Lambda(0) = 0 \) anticipating that we are going to consider systems that are will be regular at the origin of coordinates.

The difference \( G_r^r - G_t^t = 8\pi G(T_r^r - T_t^t) \) leads to a formal integral of \( \Phi(r) \)

\[ \Phi(r) = \Phi(0) - \Lambda(r) + 4\pi G \int_0^r r' e^{2\Lambda(r')} \left( T_r^r(r') - T_t^t(r') \right) dr' . \quad (3.7) \]

Instead of using the third independent Einstein equation above, we will consider the conservation of the energy–momentum tensor equation (which can be derived from the three independent Einstein equations anyway). The only relevant conservation equation will be the radial one

\[ T_{r,r} + \frac{2}{r} \left( T_r^r - T_\theta^\theta \right) + \Phi' = 0 . \quad (3.8) \]

Combining this last equation with \( G_r^r = 8\pi GT_r^r \) to eliminate \( \Phi' \) we obtain a constraint on the energy–momentum components

\[ T_{r,r} + \frac{2}{r} \left( T_r^r - T_\theta^\theta \right) + \frac{e^{2\Lambda}}{2r} \left[ 1 - e^{-2\Lambda} + 8\pi GT_r^r r^2 \right] \left( T_r^r - T_t^t \right) = 0 . \quad (3.9) \]

This is the, so called, Oppenheimer–Volkov equation.

Let us next consider a static perfect fluid described by the following energy–momentum tensor

\[ T_t^t = -\rho(r) , \quad T_r^r = T_\theta^\theta = T_\varphi^\varphi = p(r) , \quad (3.10) \]

where \( \rho \) is the energy density and \( p \) the pressure of an infinitesimal volume of matter.

An exact solution can be found \[20\] when \( \rho = \) constant. In fact, Eq. (3.4) can be trivially integrated to give

\[ e^{-2\Lambda(r)} = 1 - \frac{8\pi G r^2}{3} , \quad (3.11) \]
It is also easy to integrate the resulting Oppenheimer–Volkov equation (3.9)

\[ p(r) = \rho \left[ \frac{(\rho + 3pc)\sqrt{1 - \frac{8\pi G \rho r^2}{3}} - \rho - pc}{3\rho + 3pc - (\rho + 3pc)\sqrt{1 - \frac{8\pi G \rho r^2}{3}}} \right], \tag{3.12} \]

where \( p_c = p(0) \). The radial coordinate of the star \( R \) is defined as \( p(R) = 0 \), and from the above expression can be found to be given by

\[ R = \sqrt{\frac{3}{8\pi G \rho} \left[ 1 - \frac{\rho + pc}{\rho + 3pc} \right]}, \tag{3.13} \]

and thus the total mass \( M \) of the star will be given by \( M = 4\pi G \rho R^3/3 \).

To complete the solution, \( \Phi \) can be found from Eq. (3.7)

\[ \Phi(r) = \ln \left[ \frac{3}{2} \sqrt{1 - \frac{8\pi G \rho R^2}{3}} - \frac{1}{2} \sqrt{1 - \frac{8\pi G \rho r^2}{3}} \right]. \tag{3.14} \]

Now, the above solution will represent our zeroth order solution in the perturbative scheme of Sec. I. In building up the effective energy–momentum tensor we first observe that the interior solution we have just described is conformally flat, i.e. its Weyl tensor vanishes. In this case tensors \((1)H_{\mu\nu}\) and \((2)H_{\mu\nu}\) are no longer independent but related to each other by \((1)H_{\mu\nu} = 3(2)H_{\mu\nu}\) and a new geometrical, object can be defined in a four dimensional spacetime, which is conserved only in the conformally flat case

\[ (3)H_{\mu\nu} = -R_\rho^\lambda R_{\nu\lambda} + \frac{2}{3} RR_{\mu\nu} + \frac{1}{2} R_{\lambda\sigma} R^{\lambda\sigma} g_{\mu\nu} - \frac{1}{4} R^2 g_{\mu\nu}, \tag{3.15} \]

For the background metric (3.11) and (3.14) one obtains \((3)\) the first order correction

\[ (3)H^r_r = (3)H^\theta_\theta = (3)H^\phi_\phi = \frac{1}{3}(8\pi G)^2 \rho(\rho + 2p) \frac{1}{\gamma} \Delta p(r), \tag{3.16} \]

\[ (3)H^t_t = -\frac{1}{3}(8\pi G)^2 \rho^2 \frac{1}{\gamma} \Delta \rho, \tag{3.17} \]

where \( \gamma \) stands for the coupling constant associated to \((3)H_{\mu\nu}\) in the Lagrangian, playing a similar rôle to \( \alpha \) and \( \beta \).

We first observe that the \((3)H_{\mu\nu}\) correction can be written in a perfect fluid form with the above first order modifications to the pressure \( \Delta p(r) \) and energy density \( \Delta \rho \) (which is still constant throughout the interior!). However, the bad news for this model are that these expressions for \( \rho + \Delta \rho \) and \( p(r) + \Delta p(r) \) do not satisfy the Oppenheimer–Volkov constraint (3.9), which for a perfect fluid takes the following simplified form

\[ \frac{dp}{dr} = -(\rho + p)\frac{e^{2\Lambda}}{2r} \left[ 1 - e^{-2\Lambda} + 8\pi G pr^2 \right]. \tag{3.18} \]

We observe that again we find an incompatibility when we suppose that perturbative corrections to the uniform–density model exist. The key point now have been that the perturbative method determines the form of the corrected energy–density and the \( r \)–dependence
of the corrected pressure. We thus no longer have the freedom to choose \( p(r) \) such to fit the Oppenheimer–Volkov equation. One can actually check\(^1\) that neither the terms coming from \( (\alpha + \beta/3) H_{\mu\nu} \) corrections are able to satisfy Eq. (3.9). In passing we note that this terms do not generate corrections of the form of a perfect fluid as did those generated by \( H_{\mu\nu} \).

Here the non existence of equilibrium solution seems even more grave than in the case of cosmic strings since we do not expect the non–local terms in the renormalized energy–momentum tensor to play any relevant rôle. In fact, the state–dependent part of the vacuum expectation value of the source vanishes if we suppose that, in addition the classical fluid, only massless conformally invariant free fields (such as photons and massless neutrinos) are present. In this case, in fact, since we are in a conformally flat spacetime there is no particle production and the Green’s functions can be found by making a conformal transformation to Minkowski spacetime, finding the appropriate Green’s functions there, and transforming back to the original spacetime. More complicated coupling can indeed be considered, but, again, it remains the question of its physical plausibility.

IV. CONCLUSION

Throughout this paper, we have followed the suggestions coming from the derivation of an effective action in the \( 1/N \) approximation to quantum gravity in the large \( N \) limit. We have, thus, assumed that there can exist interesting regimes of applicability of a semiclassical theory of gravity. We estimated that in some physical systems the quantum nature of matter is significant compared to the quantum nature of gravity which might remain negligible. In fact, although the semiclassical approximation breaks down at the regime of high (Planck scale \( l_{pl} \)) curvature, such as in the final stages of an evaporating black hole or at very early times in the evolution of Universe, the effective theory should be valid in many interesting cases, i. e. when the curvature approach, but always remains (significantly) less than the Planck scale. This is the case of cosmic string and spherical stars where, although semiclassical corrections are always small, it may happen that these corrections do not allow the interior gravitational field to be perturbative in the coupling constants \( \alpha \) and \( \beta \). In fact, we have seen that if we consider in the back–reaction problem only local contributions to the effective source of the semiclassical equations for uniform–density models of the core of cosmic strings and relativistic stars, we obtain that the interior metric acquires only non–perturbative corrections in the coupling constants \( \alpha \) and \( \beta \), even if the source depends linearly on them. This shows that one cannot always simply neglect them or truncate the solutions to the first perturbative order \(^3\). Note the different situation here with respect to the case of the “instability of the Minkowski space”. While to render Minkowski stable in Ref. \(^3\) it was considered the truncation of the solutions to the perturbative ones and them to first order; here, we have that such static perturbations do not exist for the uniform–density models with cylindrical or spherical symmetry.

\(^1\)These results have been verified by use of the program of analytic manipulation “Cartan”, working within “Mathematica” \(^2\) .
We here recall that in the $1/N$ approach to the effective semiclassical action it is consistent to consider solutions beyond the first perturbative order \cite{22,23}. In fact, in the case of large $N$ limit, Eqs. (1.2) are exact, and therefore it make perfectly sense to look at solutions which are either perturbatively expandable in powers of $\alpha$ and $\beta$ (to all the orders), as we did for the charged black hole case in Ref. \cite{5} or not–perturbative in those parameters, as in the present paper. In this case, it makes also sense to look for exact solutions to the field equations (See Ref. \cite{13}) since they are obtained from the $1/N$ approximation to quantum gravity and they are exact to the leading order (in the case of large $N$ limit). Note also that in the full $1/N$ approximation self–consistency only requires that these solutions should be of the same order of the $1/N$ leading order field equations.

One can then consider the contribution of perturbative, but very particular non–local terms. This method (plus considerations of symmetry and the expression of the trace anomaly) allowed us to determine the state–dependent terms in the renormalized energy–momentum tensor that make the static perturbative solution to exist. What these states correspond to goes beyond the scope of this paper and should be determined independently by use of the standard methods of quantum field theory in curved spacetime. However, one can argue that this states appear to depend on $\alpha$ and $\beta$ already at the first non–trivial order, while we would expect they only to depend on the zeroth order background metric.

Finally, we know that the uniform–density model is a first approximation to the more involved situation inside the core of strings and neutron stars. In such case of energy density and pressure dependent on $r$ the analysis should be possibly made numerically.

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