DIFFERENTIAL OPERATORS, SHIFTED PARTS, AND HOOK LENGTHS

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Abstract. We discuss Sekiguchi-type differential operators, their eigenvalues, and a generalization of Andrews-Goulden-Jackson formula. These will be applied to extract explicit formulae involving shifted partitions and hook lengths.

1. Differential operators.

The standard Jack symmetric polynomials $P_\lambda(y_1,\ldots,y_n;\alpha)$ (see Macdoland, Stanley [5, 10]) as well as their shifted counterparts (replace $\theta=1/\alpha$; see Okounkov-Olshanski [7] and references therein) have been studied. The former appear as eigenfunctions of the Sekiguchi differential operators

$$D(u;\theta) = a_\delta^{-1} \det \left( y_i^{n-j} \left( y_i \frac{\partial}{\partial y_i} + (n-j)\theta + u \right) \right)_{i,j=1}^n,$$

where $\delta := (n-1,n-2,\ldots,1,0)$ and $\lambda$ are partitions, $a_\delta = \prod_{1 \leq i < j \leq n} (y_i - y_j)$ is the Vandermonde determinant and $u$ is a free parameter.

Under a general result, S. Sahi proves [8, Theorem 5.2] the existence of a unique polynomial $P_\mu^*(y;\theta)$, now known as shifted Jack polynomials, satisfying a certain vanishing condition. In the special case $\theta=1$, Okounkov and Olshanski [6,7] relate $P_\mu^*(y;1)$ to the Schur functions $s_\lambda$ and call them shifted Schur polynomials.

One motivation for this paper is the result due to Andrews, Goulden and Jackson [1, Thm 2.1] stating that

**Theorem 1.1.** For $n,m \in \mathbb{N}$ and summing over all partitions $l(\lambda) \leq n$, we have

$$\sum_\lambda s_\lambda(y_1,\ldots,y_n)s_\lambda(w_1,\ldots,w_m) \prod_{i=1}^n (x - \lambda_i - n + i)$$

$$= \prod_{j=1}^n \prod_{k=1}^m (1 - y_j w_k)^{-1} \times \left[ t_1,\ldots,t_n \right] (1 + t_1 + \cdots + t_n)^x \prod_{k=1}^m \left( 1 - \sum_{j=1}^n \frac{t_j y_j w_k}{1 - y_j w_k} \right).$$
Remar 1.2: To put things in perspective with the above operator view point, let us change $x$ to $-u$ and multiply through by $(-1)^n$ to find

$$G := \sum_{\lambda} s_\lambda(y_1, \ldots, y_n)s_\lambda(w_1, \ldots, w_m) \prod_{i=1}^{n}(\lambda_i + (n - i) + u)$$

$$= (-1)^n \prod_{j=1}^{n} \prod_{k=1}^{m}(1 - y_j w_k)^{-1} \times [t_1, \ldots, t_n][(1 + t_1 + \cdots + t_n)^{-u} \prod_{k=1}^{m} \left(1 - \sum_{j=1}^{n} t_j y_j w_k \right)].$$

Observe that the differential operators in $D(u; \theta)$ do not commute with the Vandermonde determinants, so the action may be regraded as one-sided. On the other hand, a closer look at Theorem 1.1 (rather its reformulation Remark 1.2) shows its basic underpinning being, in a sense, the opposite end of the bracket for Sakiguchi when $\theta = 1$.

Definition 1.3. A left-Sakiguchi operator is the composed map $D'(u; 1) := a_\delta^{-1}L(u; 1) \circ \psi_\delta$, where

$$L(u; 1) = \prod_{i=1}^{n}(y_i \frac{\partial}{\partial y_i} + u), \quad \text{and} \quad \psi_\delta : F \rightarrow a_\delta F.$$

Proposition 1.4.
(a) The Schur polynomials are eigenfunctions to $D'(u; 1)$; meaning that

$$D'(u; 1)s_\lambda = \left(\prod_{i}(\lambda_i + (n - i) + u)\right)s_\lambda.$$

In particular, the action of $D'(u; 1)$ on symmetric polynomials is diagonalizable having distinct eigenvalues.

(b) The operators $\{D'(u; 1) : u \in \mathbb{C}\}$ form a commutative algebra, say $\mathcal{D}'(n; 1)$. Moreover, if $\Lambda^1(n)$ is the algebra of polynomials $g(z_1, \ldots, z_n)$ that are symmetric in the variables $z_i - i$ then by part (a) we have the Harish-Chandra isomorphism

$$\mathcal{D}'(n; 1) \rightarrow \Lambda^1(n)$$

mapping an operator $D$ to a polynomial $d(\lambda)$ such that $D(s_\lambda) = d(\lambda)s_\lambda$.

Proof: Since $a_\delta s_\lambda = a_{\delta + \lambda}$ it suffices to check that

$$a_{\delta + \lambda} \prod_{i=1}^{n}(\lambda_i + (n - i) + u) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \cdot \prod_{i=1}^{n}(\lambda_i + (n - i) + u)_{\sigma(i)}^{\lambda_i + n - i} = L(u; 1)a_{\delta + \lambda}.$$

This however is straightforward. \Box

Now extend the left-Sakiguchi operation to $D'(u; \theta) := a_{\theta \delta}^{-1}L(u; 1) \circ \psi_{\theta \delta}$ acting on symmetric functions. Denote $s'_{\theta \delta + \lambda} = a_{\theta \delta}^{-1}a_{\theta \delta + \lambda}$ where $\theta \delta$ is a simple homothety map $(n - i) \rightarrow (n - i)\theta$. It is clear that

$$D'(u; \theta)s'_{\theta \delta + \lambda} = \left(\prod_{i}(\lambda_i + (n - 1)\theta + u)\right)s'_{\theta \delta + \lambda}.$$
Thus, both the Sakiguchi operators (1) and the full left-Sakiguchi (2) share the same eigenvalues. One concludes $D(u; \theta)$ and $D'(u; \theta)$ are similar transformations. Therefore, a parallel analysis regarding $D'(u; 1)$ can be carried out as in [4,6,7,8] but we defer such an undertaking to the interested reader. Instead our main focus will be on the operator $L(u; 1)$ and the eigenvalues $\phi_\lambda(u) := \prod_i (\lambda_i + n - i + u)$. Keep in mind that

$$L(u; 1)a_{\delta+\lambda} = \phi_\lambda(u)a_{\delta+\lambda}. $$

The paper is organized as follows.

2. Shifted parts.

A conjecture on Guoniu Han [3, Conj. 3.1] asserts that for all positive integers $k$, the expression

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \sum_{v \in \lambda} h_v^{2k}$$

is a polynomial function of $n$. Richard P. Stanley actually settles this conjecture in some generalized form [9, Theorem 4.3]. In the course of the proof he also shows several intermediate results. In this section, we provide an alternative proof for one of those statements [9, Lemma 3.1].

Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of $n$, adding as many 0’s at the tail as needed. Throughout this paper, $f_\lambda$ denotes the number of standard Young tableaux (SYT) of shape $\lambda$ and $H_\lambda = \prod_{v \in \lambda} h_v$ will be the product of the hook lengths $h_v$ where $v$ ranges over all squares in the Young diagram of $\lambda$. The content of the cell $v$ is denoted $c_v$. Define

$$A_\lambda(u) = \frac{\phi_\lambda(u)}{H_\lambda}. $$

In the sequel, the following well-established facts (see Frame, Robinson and Thrall [2]) shall be utilized repeatedly

(3) $f_\lambda = \frac{n!}{H_\lambda}, \quad \frac{1}{H_\lambda} = \det \left( \frac{1}{(\lambda_i - i + j)!} \right)_{i,j=1}^n.$

Lemma 2.1 [9, Lemma 3.1] Let $p_1 = y_1 + \cdots + y_n$ and $e_i$ the elementary symmetric functions. Then

$$\sum_{i=1}^n \binom{u+i-1}{i} p_i e_{n-i} = \sum_{\lambda \vdash n} A_\lambda(u) s_\lambda. $$

Proof: From Cauchy’s $\sum_{\lambda} s_\lambda(w)s_\lambda(y) = \prod_{i,j=1}^n (1 - w_i y_j)^{-1}$ we get $\sum_{\lambda \vdash n} f_\lambda s_\lambda(y) = p_1^n$. We already know that $a_{\lambda+\delta}\phi_\lambda(u) = L(u; 1)a_{\lambda+\delta}$. Therefore

$$\sum_{\lambda \vdash n} f_\lambda s_\lambda(y)\phi_\lambda(u) = a_{\delta}^{-1} \sum_{\lambda \vdash n} f_\lambda a_{\lambda+\delta}\phi_\lambda(u)$$

$$= a_{\delta}^{-1} L(u; 1) \sum_{\lambda \vdash n} f_\lambda a_{\lambda+\delta}$$

$$= a_{\delta}^{-1} L(u; 1) a_{\delta} \sum_{\lambda \vdash n} f_\lambda s_\lambda$$

$$= a_{\delta}^{-1} L(u; 1) a_{\delta} \cdot p_1^n.$$
A direct computation using Leibnitz’ Rule for multi-derivatives of products yields

\[ L(u;1)a_\delta p^n_1 = a_\delta \cdot n! \sum_{j=0}^{n} \binom{u+j-1}{u-1} p^j_1 e_{n-j}. \]

Consequently, we have

\[ \sum_{\lambda \vdash n} f_\lambda s_\lambda(\phi_\lambda(u)) = a_{\delta}^{-1} L(u;1)a_\delta p^n_1 = n! \sum_{i=0}^{n} \binom{u+i-1}{i} p^i_1 e_{n-i}. \]

By the hook length formula \( f_\lambda = n! H^{-1}_\lambda \) the proof is complete. □

**Corollary 2.2.** Let \( u \in \mathbb{N}, \lambda \vdash n, \mu := (\lambda^u_1, \lambda^u_2, \ldots, \lambda^u_n) \vdash (u+1)n \) and \( \lambda' = (\lambda_1', \ldots, \lambda'_n) \) the shape conjugate to \( \lambda \). Then the quantity \( A_\lambda(u) = \frac{H_{\lambda'}}{H_\lambda} \)

(a) enumerates the product of the hook lengths of the top row in \( \mu \) divided by \( H_\lambda \) or \( H_{\lambda'} \); symbolically

\[ A_\lambda(u) = \prod_{i=0}^{n} \binom{u+n-i-1}{n-i} f^{\lambda'/1}. \]

**Proof:** The symmetric polynomials \( p^{n-i}_1 e_i \) can be expanded in terms of the Schur basis \( \{s_\lambda\}_{\lambda \vdash n} \) as follows: \( p^{n-i}_1 e_i = \sum_{\lambda \vdash n} f^{\lambda'/1} s_\lambda \). Now apply the above lemma to obtain

\[ \sum_{\lambda \vdash n} A_\lambda(u)s_\lambda = \sum_{\lambda \vdash n} s_\lambda \sum_{i=0}^{n} \binom{u+n-i-1}{n-i} f^{\lambda'/1}. \]

Comparison of coefficients yields the identity of part (b). The assertion of part (a) is evident. □

**Remark 2.3.** Although \( H^{-1}_\lambda \prod_{i} (\lambda_i + (n-i) + u) \) is an integer, the quantity

\[ \frac{1}{H_\lambda} \prod_{i=1}^{n} (\lambda_i + (n-i) + u_i) \]

is not generally an integer even if all \( u_i \)'s are. Take for instance, \( n = 2, \lambda = (2,0), u_1 = 1 \) and \( u_2 = 2 \). Thus \( H^{-1}_\lambda(3 + u_1)u_2 = \frac{1}{2}(5)(1) \notin \mathbb{Z} \).

**3. Determinants.**

In this section, we compile a few determinantal evaluations some of which are well-known and others are residues from the previous sections. The first statement is valid for arbitrary parameters \( z_i \)'s and the proof is tailor-made for Dodgson’s method of condensation (for a charming proof, see [11]).

**Proposition 3.1.** Given the variables \( z = (z_1, \ldots, z_n) \), there holds

\[ \det \left( \binom{z_i + n - i}{n-j} \right)_{i,j=1}^{n} = \prod_{1 \leq i < j \leq n} \frac{z_i - z_j + j - i}{j - i}. \]
Proof: Introduce two integral parameters $b, c$ to generalize the claim further as
\[
\det \left( \frac{z_{i+b} + n - (i + b)}{n - (j + c)} \right)_{i,j=1}^n = \prod_{1 \leq i < j \leq n} \frac{z_{i+b} - z_{c+j} + (j + c) - (i + b)}{(j + c) - (i + b)}.
\]

The rest is an automatic application of Dodgson. □

Corollary 3.2. Let $\lambda \vdash n$ with length $l = l(\lambda)$ and $\omega_i(u) = \lambda_i + (n - i) + u$.

(a) \[
\det \left( \frac{1}{(\lambda_i - i + j)!} \right)_{i,j=1}^n = \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^n (\lambda_i + n - i)!} = \frac{1}{H_\lambda}.
\]

(b) \[
\det \left( \frac{i}{(\lambda_i - i + j)!} \right)_{i,j=1}^n = \frac{n!}{H_\lambda} = f_\lambda \in \mathbb{Z}.
\]

(c) \[
\det \left( \frac{\omega_i(u)}{\omega_i(j - n)!} \right)_{i,j=1}^n = \prod_{i=1}^n \frac{\omega_i(u)}{\omega_i(0)!} \prod_{i<j} \omega_i(u) - \omega_j(u) = A_\lambda(u) \in \mathbb{Z};
\]

with the convention that $\frac{1}{y} := 0$ whenever $y < 0$.

(d) \[
\det \left( \frac{\lambda_i + l - i}{l - i} \right)_{i,j=1}^l = \frac{1}{H_\lambda} \prod_{i=1}^l \frac{\lambda_i + l - i)!}{(l - i)!} = \prod_{v \in \lambda} \frac{l + cu}{h_u} \in \mathbb{Z}.
\]

Proof: In Prop. 3.1, divide by $(z_i + n - i)! (n - j)!^{-1}$ then replace $\lambda_i$ for $z_i$. This together with (3) prove part(a). The remaining parts follow directly from Prop. 3.1 and Cor. 2.2. □

4. Andrews-Goulden-Jackson formula.

We begin with an extraction of certain coefficients from Theorem 1.1 (or equivalently Remark 1.2). Notation: given a function $F(x, y, \ldots)$, the coefficient of the monomial $x^a y^b \cdots$ is designated by $[x^a y^b \cdots]F$. Also $s(n, k)$ and $c(n, k)$ stand for the signed and unsigned stirling numbers of the first kind, respectively. The falling factorials $(x)_k = x(x - 1) \cdots (x - k + 1)$ generate
\[
(x)_k = \sum_{i=0}^k s(k, i) x^i.
\]

Lemma 4.1. The coefficient of the falling factorial $(x)_{n-k}$ in Remark 1.2 is
\[
\prod_{i=1}^n \prod_{j=1}^m (1 - y_i w_j)^{-1} = \sum_{1 \leq j_1 < \cdots < j_k} \prod_{r=1}^k y_{j_1}^{w_{i_r}}.
\]

Moreover, if we extract the coefficient of $[w_1, \ldots, w_n]$ we get
\[
(n)_k \cdot (y_1 + \cdots + y_n)^{n-k} \cdot \sum_{j_1 < \cdots < j_k} \prod_{r=1}^k y_{j_1} = (n)_k \cdot (y_1 + \cdots + y_n)^{n-k} \cdot e_k(y_1, \cdots, y_n).
\]
Corollary 4.2. Let \( m = n \) and fix \( \beta \). Then the following coefficients are polynomials in \( n \), of degree \( 2\beta \).

\[
rac{1}{n!} [y_1 \cdots y_n, w_1 \cdots w_n, x^{n-\beta}] G = \frac{1}{n!} \sum_{\lambda \vdash n} f^2_{\lambda} e_{\beta} (\{ \lambda_i + n - i : 1 \leq i \leq n \}) = \sum_{\alpha = n - \beta}^{n} c(\alpha, n - \beta) \binom{n}{\alpha}.
\]

Examples 4.3: We list the first few polynomials \( \frac{1}{n!} \sum_{\lambda \vdash n} f^2_{\lambda} e_{\beta} \).

a) for \( \beta = 0 \) we have \( \binom{n}{0} \).

b) for \( \beta = 1 \) we have \( \binom{n+1}{2} \).

c) for \( \beta = 2 \) we have \( -(\binom{n}{2}) - (\binom{n+1}{3}) + 3(\binom{n+2}{4}) \).

d) for \( \beta = 3 \) we have \( \binom{n}{3} - 5(\binom{n+1}{4}) - 10(\binom{n+2}{5}) + 15(\binom{n+3}{6}) \).

e) for \( \beta = 4 \) we have \( 2(\binom{n}{4}) + 19(\binom{n+1}{5}) - 20(\binom{n+2}{6}) - 105(\binom{n+3}{7}) + 105(\binom{n+4}{8}) \).

5. Main Results: Towards a generalization.

Let \( y = (y_1, \ldots, y_n) \), \( w = (w_1, \ldots, w_n) \). Recall the eigenvalues and eigenfunctions of \( D'(u; 1) \):

\[
\phi_{\lambda}(u) = \prod_i (\lambda_i + (n - i) + u), \quad L(u; 1) = \prod_i (y_i \frac{\partial}{\partial y_i} + u).
\]

For brevity, write \( L(u) \) instead of \( L(u; 1) \). Next we proceed to compute

\[
\text{(STAN-N)} \quad \sum_{\lambda} s_{\lambda}(w) s_{\lambda}(y) \prod_{j=1}^{N} \phi_{\lambda}(u_j) = \sum_{\lambda} s_{\lambda}(w) s_{\lambda}(y) \prod_{j=1}^{N} \prod_{i=1}^{n} (\lambda_i + (n - i) + u_j).
\]

Note once again that

\[
a_{\delta + \lambda} \prod_{j=1}^{N} \phi_{\lambda}(u_j) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{N} \phi_{\lambda}(u_j) y_{\sigma(i)}^{\lambda_i + n - i} = \prod_{j=1}^{N} L(u_j) a_{\delta + \lambda}.
\]

This in turn implies

\[
\sum_{\lambda} s_{\lambda}(w) s_{\lambda}(y) \prod_{j=1}^{N} \phi_{\lambda}(u_j) = a_{\delta}^{-1} \left( \prod_{j=1}^{N} L(u_j) \right) \sum_{\lambda} s_{\lambda}(w) a_{\delta + \lambda} \left( \prod_{j=1}^{N} \phi_{\lambda}(u_j) \right)
= a_{\delta}^{-1} \left( \prod_{j=1}^{N} L(u_j) \right) \sum_{\lambda} s_{\lambda}(w) a_{\delta + \lambda} \left( \prod_{j=1}^{N} \phi_{\lambda}(u_j) \right)
= a_{\delta}^{-1} \left( \prod_{j=1}^{N} L(u_j) \right) a_{\delta}(y) \sum_{\lambda} s_{\lambda}(w) s_{\lambda}(y)
= a_{\delta}^{-1} \left( \prod_{j=1}^{N} L(u_j) \right) a_{\delta}(y) \prod_{i,k=1}^{n} (1 - y_i w_k)^{-1},
\]
where the last equality uses Cauchy’s summation formula.

**Case** $N = 1$: When $m = n$ this is exactly Remark 1.2 that will be re-formulated as:

\[ a_\delta^{-1}(y)L(u_1)a_\delta(y) \prod_{i,k=1}^n (1 - y_iw_k)^{-1} \]

\[ = (-1)^n \prod_{j=1}^n \prod_{k=1}^n (1 - y_jw_k)^{-1} \times [t_1, \ldots, t_n](1 + t_1 + \cdots + t_n)^{-u_1} \prod_{k=1}^n \left(1 - \sum_{j=1}^n \frac{t_jy_jw_k}{1 - y_jw_k}\right). \]

If we denote the right-hand side by $\text{RHS}(u_1)$ then

\[ \sum_{\lambda} s_\lambda(w)s_\lambda(y) \prod_{j=1}^N \phi_\lambda(u_j) = a_\delta^{-1}(y)L(u_N) \cdots L(u_2)a_\delta \left(a_\delta^{-1}L(u_1)a_\delta \prod_{i,k=1}^n (1 - y_iw_k)^{-1}\right) \]

\[ = a_\delta^{-1}(y)L(u_N) \cdots L(u_2)a_\delta \text{RHS}(u_1). \]

**Case** $N = 2$: We opt to proceed where we left-off in Lemma 4.1, i.e. we work on the coefficients of $[w_1 \cdots w_n](u_1)_{n-k}$. The advantage is two-fold: the notational amount is reduced and it is irrelevant to maintain the extra family of parameters $w_i$.

\[ (n)_kL(u_2)a_\delta(y_1 + \cdots + y_n)^{n-k}e_k(y_1, \ldots, y_n). \]

Since $(n - k)![s^k][z^n]e^{zp_1(y)} \prod_{j=1}^n (1 + y_jsz) = p_1^{n-k}e_k(y)$, where $p_1(y) = y_1 + \cdots + y_n$, one may consider instead

\[ (n)_kL(u_2)a_\delta F(y_1) \cdots F(y_n), \quad \text{with} \quad F(\alpha) = e^{z\alpha}(1 + \alpha sz). \]

At this point we invoke [1, Theorem 4.3] of Andrews, Goulden and Jackson asserting that

\[ L(u_2)a_\delta F(y_1) \cdots F(y_n) = (-1)^n a_\delta F(y_1) \cdots F(y_n)[t_1 \cdots t_n](1 + t_1 + \cdots + t_n)^{-u_2} \exp \left\{ \alpha \frac{\partial}{\partial \alpha} \log F(\alpha) \right\} \log \left(1 - \sum_{j=1}^n \frac{t_jy_j\alpha}{1 - y_j\alpha}\right). \]

Here $\sum_{i \geq 0} a_i z^i = \sum_{i \geq 0} b_i z^i = \sum_{i \geq 0} a_i b_i$. Note distributivity $(a + b) * c = a * c + b * c$. For our choice of the function $F(\alpha)$ it is then clear that

\[ \alpha \frac{\partial}{\partial \alpha} \log F(\alpha) = z\alpha + \frac{\alpha sz}{1 + \alpha sz}, \]

\[ \exp \left\{ (z\alpha) \log \left(1 - \sum_{j=1}^n \frac{t_jy_j\alpha}{1 - y_j\alpha}\right)\right\} = \exp \left\{-z \sum_{j=1}^n t_jy_j\right\}, \]

\[ \exp \left\{ \frac{\alpha sz}{1 + \alpha sz} \log \left(1 - \sum_{j=1}^n \frac{t_jy_j\alpha}{1 - y_j\alpha}\right)\right\} = \left(1 + \sum_{j=1}^n \frac{t_jy_jsz}{1 + y_jsz}\right)^{-1}. \]
Therefore
\[ a^{-1}_k(n)_k L(u_2) a \delta e^{z p_1(y)} \prod_{i=1}^{n} (1 + y_i s z) = (-1)^n (n)_k e^{z p_1(y)} \prod_{i=1}^{n} (1 + y_i s z) [t_1 \cdots t_n] (1 + t_1 + \cdots + t_n)^{-u_2} \]
(5)
\[ e^{-z p_1(t y)} \sum_{j=0}^{n} (-1)^j j! (s z)^j e_j(t y); \]
where \( p_1(t y) = t_1 y_1 + \cdots + t_n y_n \) and \( e_j(t y) = e_j(t_1 y_1, \cdots, t_n y_n) \).

The key is now to extract the coefficient of \((n - k)! [y_1 \cdots y_n] [s^k] [z^n] \) from both sides of equation (5), and the matter rests on what we find on the right-hand side. The procedure has four stages.

**Coefficients \([s^k] \)**:
\[ [s^k] \prod_{i=1}^{n} (1 + y_i s z) \cdot \sum_{j=0}^{n} (-1)^j j! (s z)^j e_j(t y) = z^k \sum_{\alpha=0}^{k} (-1)^{\alpha} \alpha! e_\alpha(t y) e_{k-\alpha}(y). \]

**Coefficients \([z^n] \)**: Since there is already a gain of \( z^k \), actually we must read off \([z^{n-k}] \). So,
\[ [z^{n-k}] e^{z p_1(y)} e^{-z p_1(t y)} = \sum_{\beta=0}^{n-k} \frac{(-1)^{\beta}}{\beta! (n - k - \beta)!} p_1(y)^{n-k-\beta} p_1(t y)^\beta. \]

**Coefficients \([t_1 \cdots t_n] \)**:
\[ [t_1 \cdots t_n] (1 + t_1 + \cdots + t_n)^{-u_2} p_1(t y)^\beta e_{\alpha}(t y) = \frac{(\alpha + \beta)!}{\alpha!} e_{\alpha+\beta}(y) (-u_2)^{n-\alpha-\beta}. \]

**Coefficients \([y_1 \cdots y_n] \)**:
\[ [y_1 \cdots y_n] p_1(y)^{n-k-\beta} e_{k-\alpha}(y) e_{\alpha+\beta}(y) = (n - k - \beta)! \binom{n}{\alpha + \beta} \binom{n - \alpha - \beta}{k - \alpha}. \]

It is now time to combine all these into
\[ (n - k)! (n)_k \sum_{\alpha=0}^{k} \sum_{\beta=0}^{n-k} \frac{(-1)^{\alpha+\beta} \alpha! (n - k - \beta)!}{\beta! (n - k - \beta)! \alpha!} \binom{n}{\alpha + \beta} \binom{n - \alpha - \beta}{k - \alpha} (-u_2)^{n-\alpha-\beta} \]
\[ = n! \sum_{\alpha=0}^{k} \sum_{\beta=0}^{n-k} \frac{(-1)^{\alpha+\beta} (\alpha + \beta)!}{\beta!} \binom{n}{\alpha + \beta} \binom{n - \alpha - \beta}{k - \alpha} (-u_2)^{n-\alpha-\beta} \]
\[ = n! \sum_{\gamma=0}^{n-k} (-1)^{\gamma} \gamma! \binom{n}{\gamma} (-u_2)^{n-\gamma} \sum_{\alpha=0}^{k} \frac{1}{(\gamma - \alpha)!} \binom{n - \gamma}{k - \alpha}. \]

**Lemma 5.1 (STAN \( N = 2 \))**. Fix \( \alpha, \beta \in \mathbb{P} \). Then, the coefficient of \( u_1^{n-\alpha} u_2^{n-\beta} \) is given by
\[ \frac{1}{n!} \sum_{k=n-\alpha}^{n} \sum_{m=n-\beta}^{n} j^2 \chi_{\alpha}(\{\lambda_i + n - i : 1 \leq i \leq n\}) e_\beta(\{\lambda_j + n - j : 1 \leq j \leq n\}) \]
\[ = \sum_{k=n-\alpha}^{n} \sum_{m=n-\beta}^{n} c(k, n - \alpha) \cdot c(m, n - \beta) \cdot \binom{n}{m} \sum_{j=0}^{\min\{m, k\}} \binom{n - m}{j} \binom{m}{n - k - j}. \]
Here, the index \( j \) runs through its natural limits, i.e. in the sense that if \( Y > X \geq 0 \) then \( (X^Y) = 0 \).

**Proof:** Derivation seen above. \( \square \)

Let \( K = (k_{ij}) \) be an infinite lower triangular matrix of non-negative integral entries, and define the row-sum and differences by

\[
\bar{r}_i = k_{i1} + \cdots + k_{ii}, \quad \text{and} \quad \Delta \bar{r}_{i-1} = \bar{r}_{i-1} - \bar{r}_i + k_{ii}.
\]

**Theorem 5.2.** For any integer \( N \geq 1 \), we have

\[
\frac{1}{n!} \sum_{\lambda \vdash n} s_{\lambda} f_{\lambda} \prod_{j=1}^{N} \phi_{\lambda}(u_j) = \sum_{\epsilon=1}^{N} \sum_{k_{\epsilon\epsilon} = 0}^{n} (n - k_{\epsilon\epsilon})! \left( u_{\epsilon} + n - k_{\epsilon\epsilon} - 1 \right) \times \sum_{\alpha = 2}^{N} \sum_{\Delta \bar{r}_{\alpha-1} \geq 0} \frac{n - k_{NN}}{\Delta \bar{r}_{N} - k_{NN}} \Delta \bar{r}_{N} \left( \frac{k_{\alpha \alpha}}{\Delta \bar{r}_{\alpha-1}} \right) \Delta \bar{r}_{\alpha-1} p_{\alpha}^{n-\bar{r}_{N}} \prod_{i=1}^{N} e_{k_{\alpha, i}}.
\]

**Proof:** A successive application of the argument that produces Lemma 5.1. \( \square \)

Another one of Stanley’s results [9, Theorem 3.3] proves polynomiality of the expression

\[
(6) \quad \Psi_n(W) = \frac{1}{n!} \sum_{\lambda \vdash n} f_{\lambda}^{2} W(\{\lambda_i + n - i : 1 \leq i \leq n\}; \{c_v : v \in \lambda\})
\]

where \( W(x, y) \) is a given power series over \( \mathbb{Q} \) of bounded degree that is symmetric in the variables \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) separately. In fact, the main kernel of our motivation for this paper emanates from studying \( \Psi_n(W) \).

Our main result, the corollary below, only deals with the shifted parts \( \{\lambda_i + n - i : 1 \leq i \leq n\} \) that appear in equation (6) and not the contents \( c_v \). However the output is indeed an explicit evaluation, the method of proof is different and the conclusion of polynomiality is preserved.

**Corollary 5.3 (STAN-N).** Given the \( N \)-tuple diagonal \( (k_{11}, \ldots, k_{NN}) \) of \( K \), we have

\[
\frac{1}{n!} \sum_{\lambda \vdash n} f_{\lambda}^{2} \prod_{j=1}^{N} e_{k_{jj}}(\{\lambda_i + n - i : 1 \leq i \leq n\})
\]

\[
= \sum_{\gamma = 1}^{N} \sum_{j, \gamma = n - k_{\gamma\gamma}}^{n} c(j, \gamma, n - k_{\gamma\gamma}) \sum_{\alpha = 2}^{N} \sum_{\Delta \bar{r}_{\alpha-1} \geq 0} (k_{\alpha \alpha})^{\Delta \bar{r}_{\alpha-1}} \left( k_{NN}, \ldots, k_{NN}, n - \bar{r}_{N} \right).
\]

If \( W(x) \) is a power series over \( \mathbb{Q} \) of bounded degree and symmetric in \( x = (x_1, x_2, \ldots) \), then

\[
\frac{1}{n!} \sum_{\lambda} f_{\lambda}^{2} W(\{\lambda_i + n - i : 1 \leq i \leq n\})
\]

is a polynomial function of \( n \).

**Proof:** Apply Theorem 5.2 and the generating function \( (x)_k = \sum_{i=0}^{k} s(k, i)x^i \) for stirling numbers, to get the formula. In particular polynomiality follows. \( \square \)

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