Defining work done on electromagnetic field

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The problem of defining work done on electromagnetic field (EMF) via moving charges does not have a ready solution, because the standard Hamiltonian of EMF does not predict gauge-invariant energy changes. This limits applications of statistical mechanics to EMF. We obtained a new, explicitly gauge-invariant Hamiltonian for EMF that depends only on physical observables. This Hamiltonian allows to define thermodynamic work done on EMF and to formulate the second law for the considered situation. It also leads to a direct link between this law and the electrodynamic arrow of time, i.e. choosing retarded, and not advanced solutions of wave-equations. Measuring the thermodynamic work can give information on whether the photon mass is small but non-zero.

Introduction. Hamiltonian dynamics is essential for statistical mechanics and thermodynamics [1]. Basic distribution functions of statistical mechanics (e.g. canonical or microcanonical) are formulated in the phase-space and are based on the conservation of energy and of the phase-space volume (the Liouville’s theorem) [1–4]. Also the basic quantities of thermodynamics—energy, work and heat—are defined via the Hamiltonian of the system; e.g. the change of the time-dependent Hamiltonian for a thermally isolated system defines the work done externally [2–4]. The first law divides energy into work and heat [2], while the second law limits work-extraction via cyclic processes [3]. The third law studies work as a resource for cooling [5].

Our aim is to understand thermodynamic work done by moving charges (sources of work) on electromagnetic field (EMF); the research done on EMF from various angles (field-theoretic, quantum, statistical etc) is reflected e.g. in [6–10] [12]. We stress that thermodynamics and electrodynamics share at least two structural features. (1.) Both study systems with many degrees of freedom. (2.) Both need specific subsystems (work-sources) whose motion is prescribed in the sense that the back-reaction on them is partially neglected [14]. For thermodynamics these are e.g. vessels of a gas [2], while for EMF these are moving charges [6].

Given these similarities, the work done on EMF by moving charges is to be defined via the Hamiltonian of EMF. For stationary charges the Hamiltonian is conserved; hence there is no thermodynamic work. Now L refers to coordinates φ(x, t) and A_i(x, t) and velocities ̇A_i(x, t) that are parametrized by a continuous index x and discrete index i. Hence the Lagrange equations deduced from L have the usual form, but with variational derivatives

\[ \frac{d}{dt}[\delta L/\delta ̇A_j(y)] = \delta L/\delta A_j(y), \quad \delta L/\delta ̇\phi(y) = 0. \]  

(4)

Note that L does not contain ̇\phi; hence the last equation in (4). When working out (4) we standardly assume that ρ, J_i, E_i and B_i decay to zero at the spatial infinity, apply integration by parts, and employ known formulas of variational calculus, e.g. \[ \delta A_i(x)/\delta A_j(y) = \delta_{ij}(x-y) \] with Kronecker and Dirac’s deltas, respectively. Hence we get from (4) equations of motion:

\[ \partial_k ̇\phi - A_k = \Delta A_k - \partial_k(\partial_i A_i) + J_k, \]  

\[ \Delta ̇\phi = -\rho - \partial_i A_i, \]  

(5)

(6)

where Δ = ∂_k∂_k is the Laplace operator. Eqs. (2, 3) show that (5) and (6) become (resp.) the Maxwell’s equations

\[ ̇E_i = \epsilon_{ijk}\partial_j B_k - J_i, \quad \partial_i E_i = \rho. \]  

(7)
Eqs. (5, 6) also imply the conservation of charge:
\[ \dot{\rho} + \partial_i J_k = 0. \] (8)

The standard Hamiltonian of EMF is constructed from (1). One should note here that strictly speaking the EMF is a singular system, since \( L \) does not contain \( \phi \) [11, 13]. This singularity can be dealt with in various equivalent ways, also via the full Dirac’s formalism [11, 13]. But the simplest way is to carry out the Legendre transformation with respect to \( A_i \) only [11, 13]:
\[ H_D = \int \! d^3 x \, \mathcal{H}_D, \quad \mathcal{H}_D = p_i \dot{A}_i - \mathcal{L}, \] (9)
where the canonic momentum \( p_i \) is defined from
\[ \delta H_D / \delta \dot{A}_k(y) = 0 \quad \text{or} \quad \dot{A}_i = p_i - \partial_i \phi. \] (10)

Putting (10) into \( \mathcal{H}_D \), and making integration by parts we arrive at [11, 13]:
\[ \mathcal{H}_D = \frac{1}{2} \rho_i^2 + \frac{1}{2} B_i^2 - J_i A_i + \phi(\partial_i p_i + \rho), \] (11)
where \( \phi \) is now the Lagrange multiplier for the constraint \( \partial_i p_i + \rho = 0 \) (given also by (4, 6, 10)). Hamilton equations of motion are read from (11) with canonic coordinates \( A_i \), momenta \( p_i \) and the Lagrange factor \( \phi \) [11, 13]:
\[ \dot{A}_i = \delta H_D / \delta p_i, \quad \dot{p}_i = -\delta H_D / \delta A_i, \quad \delta H_D / \delta \phi = 0. \] (12)

Eqs. (12) bring back (5, 6). On solutions of (5, 6), we have \( E_i = -p_i \) from (2, 10). Hence \( \mathcal{H}_D = \frac{1}{2} E_i^2 + \frac{1}{2} B_i^2 - J_i A_i \), for which \( J_i = 0 \) reduces to the well-known [6]
\[ H_{\text{free}} = \frac{1}{2} \int \! d^3 x \left[ E_i^2 + B_i^2 \right]. \] (13)

Now \( H_D \) is generally time-dependent due to \( \rho \) and \( J_i \). As for any time-dependent Hamiltonian, we have
\[ \dot{H}_D = \int \! d^3 x \left[ A_i \frac{\delta H_D}{\delta \dot{A}_i} + p_i \frac{\delta H_D}{\delta p_i} + \phi \frac{\delta H_D}{\delta \phi} \right] + \int \! d^3 x \left[ \dot{\phi} \dot{J}_i - \dot{J}_i A_i \right]. \] (14)

Now (14) nullifies due to (12), so \( \dot{H}_D \) is determined by (15). Hence \( H_D \) is conserved if \( \dot{\rho} = \dot{J}_i = 0 \), where the Lagrangian (1) is time-translation invariant. Eq. (15) could be guessed directly from (1).

But we cannot apply \( \dot{H}_D \) and (15) for calculating energy change. Recall that equations of motion (5, 6, 7) are invariant with respect to gauge change
\[ \phi \rightarrow \phi + \chi, \quad A_k \rightarrow A_k - \partial_k \chi, \] (16)
where \( \chi(x, t) \) is arbitrary. This invariance relates to the zero mass of EMF [9]. Due to (8), the Lagrangian (1) changes under (16) by a full time-derivate: \( L \) → \( L - \frac{d}{dt} \int d^3 x \rho \chi \). Eq. (15) also changes by a full time-derivate under the gauge-change (16)
\[ \dot{H}_D \rightarrow \dot{H}_D + \frac{d}{dt} \int d^3 x \rho \chi, \] (17)
where we used (8). For a Lagrangian a shift by a full time-derivatives is allowed [20], but for a Hamiltonian it is a problem, since it alters the energy change \( \int_{t_1}^{t_2} \dot{H}_D dt \) between \( t_1 \) and \( t_2 \). Now \( \dot{H}_D \) is gauge-invariant for a particular case \( \rho(x, t_1) = \rho(x, t_2) = 0 \) for all \( x \). This is too restrictive for the definition of the energy change and work. Indeed, in a standard task of thermodynamics a many-body system (e.g. EMF) is employed as an energy storage, i.e. the time-dependent parameters are driven by different sources that exchange work through the system. For such cases it is simply necessary to calculate the energy change up to a given time, because this is the work that goes to one of the work-sources.

The gauge-variant \( \dot{H}_D \) is not suitable for defining work.

Gauge-invariant Hamiltonian. We now assume that (together with \( E_i \) and \( B_i \) also \( \phi \) and \( A_i \) decay to zero for \( |x| \rightarrow \infty \)). This assumption implies a partial gauge-fixing [cf. (16)], but our final results will not depend on it. Now (6) is solved via the inverse Laplacian \( \Delta^{-1} \) as
\[ \phi = -\Delta^{-1}(\rho + \partial_i A_i), \] (18)
\[ \Delta^{-1} f(x) = -\frac{1}{4\pi} \int d^3 y \frac{f(y)}{|x - y|}, \] (19)
where we note that the solution of the homogeneous Laplace equation (that could appear in the RHS of (19)) nullifies due to assumed boundary conditions: \( \phi(x) \rightarrow 0 \) for \( |x| \rightarrow \infty \). Note in (18) that for \( \Delta^{-1}(\partial_i A_i) \) to be finite it is necessary that \( \dot{A}_i \) decays to zero at infinity, which we already assumed.

We put back (18) into \( L \) trying to find a Lagrangian for \( A_i \). In subsequent calculations we shall employ (8), (18), (19), \( \epsilon_{ijk} \epsilon_{lmn} = \delta_{ij} \delta_{km} - \delta_{ik} \delta_{mj} \), and a commutativity relation \( \partial_i \Delta^{-1} = \Delta^{-1} \partial_i \), which holds when acting on functions decaying at infinity; e.g.
\[ \partial_i \phi - \dot{A}_i = -\Delta^{-1}(\partial_i \rho + \epsilon_{ikl} \partial_k \dot{B}_l). \] (20)

We also neglect one full time-derivate (allowed for a Lagrangian), and also full space-derivatives, due to assumed boundary conditions. After some transformations, see section 1 of [26], we get a Lagrangian that instead of \( A_i \) depends directly on the magnetic field \( B_i \):
\[ L_B = \int d^3 x \mathcal{L}_B, \] (21)
\[ \mathcal{L}_B = \frac{1}{2} \rho \Delta^{-1} \rho - \frac{1}{2} \dot{B}_i \Delta^{-1} \dot{B}_i - \frac{1}{2} B_i B_i - B_i \Delta^{-1} (R_i), \] (22)
\[ R_i \equiv \epsilon_{ijk} \partial_j \partial_k. \] (23)

Eq. (21) comes with a constraint that follows from (3)
\[ \partial_j B_j = 0, \] (23)
and confirms that EMF has two independent coordinates.

In equations of motion \( \frac{d}{dt} \frac{\delta L_B}{\delta \dot{B}_k(y)} = \frac{\delta L_B}{\delta B_i(y)} \) we use

\[
\delta L_B / \delta \dot{B}_k(y) = -\Delta^{-1}(\dot{B}_k)(y). \tag{24}
\]

This leads to an autonomous equations for \( B_i \) that can be also derived from the Maxwell’s equations (7)

\[
\dot{B}_i - \Delta B_i = R_i = 0. \tag{25}
\]

Using (24) we introduce the canonical momentum Hamiltonian via the usual Legendre transformation

\[
\mathcal{H}_B = \int \frac{d^3 x}{2} \mathcal{H}_B(x) = \int d^3 x \left[ \Pi_k \dot{B}_k - L_B \right], \tag{26}
\]

\[
\mathcal{H}_B = -\frac{\rho \Delta^{-1} \rho}{2} - \frac{\Pi_i \Delta \Pi_i}{2} + \frac{B_i B_i}{2} + B_i \Delta^{-1}(R_i). \tag{27}
\]

where constraint (23) is implied.

(i) Eq. (25) can be reproduced from (27) via Hamilton equations \( \Pi_k = \frac{\delta H}{\delta \dot{B}_k} \) and \( B_k = -\frac{\delta H}{\delta B_k} \).

(ii) Though (27) depends only on the magnetic field \( B_i \) and its derivatives, it is consistent with (13): apply \( \epsilon_{i j k} \partial_j E_k = -\partial_i \) (deduced from (2, 3)) and employ there (7). Then we can express \( E_i \) via \( B_k \) and \( \partial_i \rho \):

\[
E_i = \Delta^{-1}(\partial_i \rho + \epsilon_{i j k} \partial_j \dot{B}_k). \tag{28}
\]

We put (28) into (13) and integrate by parts:

\[
H_{\text{free}} = -\frac{1}{2} \int d^3 x \left[ -\rho \Delta^{-1} \rho - \dot{B}_i \Delta^{-1}(\dot{B}_i) + B_i^2 \right], \tag{29}
\]

i.e. (27) for \( R_i = 0 \) agrees with (13). In particular, (29) includes the case of free (and generically space-localized) EMF fields.

(iii) We define work done by charges via the standard formula accepted in statistical mechanics [2]:

\[
\dot{H}_B = \int d^3 x \left[ -\frac{1}{2} \frac{d}{dt}(\rho \Delta^{-1} \rho) + B_i \Delta^{-1}(R_i) \right], \tag{30}
\]

where we employed the same method as in (14). Now \( \dot{H}_B \) consists of two parts: the electrostatic due to \( \rho \Delta^{-1} \rho \) (see section 2 of [26]) and vortical due to \( \Delta^{-1}R_i \); cf. (22). We stress that the electrostatic contribution does not depend on fields, it depends only on the externally controlled \( \rho(x, t) \). But we keep it, e.g. because it allows (27) to agree with a well-accepted expression (13). Section 4 of [26] shows that \( \dot{H}_B \) is conserved if \( J_k = 0 \) and \( \rho \) is demanded to be bounded for all times.

To avoid confusions note that the work done on EMF according to (30) does not directly relate to radiation, e.g. (30) is zero for a rectilinear motion of charges [cf. (22)], where we do expect radiation if this motion is accelerated [6]. Indeed, in the considered set-up, where fields nullify at infinity, the radiation is always a part of fields, the one that has a specific asymptotics far from charges [6]. On the other hand, if (30) is non-zero, then there is certainly acceleration and hence radiation. Note that we always deal with the full energy (space-integrated) energy of EMF. The localization of this energy is not studied; this is another (and more difficult) problem [24].

(iv) What if photon has a small but non-zero mass \( m \)? Due to its foundational importance, this question ponders in physics for decades [9, 10]. Experiments put stringent bounds on \( m \) [9], but they cannot show that \( m = 0 \). Even within such bounds \( m > 0 \) can be relevant e.g. in cosmology [23]. We show that \( m > 0 \) leads to a different definition of work. Recall that massive electrodynamics is a consistent theory [9, 10] (see section 3.1 of [26]) that amounts to adding to \( L \) in (1) the massive term \( m^2(\phi^2 - A_k^2) \). This changes equations of motion (5, 6) by adding \(-m^2 A_k \) to the RHS of (5) and \( m^2 \phi \) to the RHS of (6). New equations produce \( \dot{\phi} + \partial_t A_k = m^2(\phi + \partial_t A_k) \).

Hence the charge-conservation (8) and \( m > 0 \) lead to the Lorenz gauge \( \phi + \partial_t A_k = 0 \) [9]. Then (15) still applies for the change of the total Hamiltonian, but now no gauge-change (16) can be made. Hence (15) is consistent for \( m > 0 \). Moreover, for \( m > 0 \) the method of (18–20) can be generalized, but it does not lead to a Lagrangian (or Hamiltonian) description of \( A_i \); see section 3.2 of [26].

Hence (15) and (30) provide consistent and different definitions of work for (resp.) \( m > 0 \) and \( m = 0 \). If the work done on EMF can be measured independently, this will show whether or not the photon has a mass.

Arrows of time. We apply (30) to the second law. We assume that \( R_i \) is switched on at some initial time, and there were no free fields before that time: \( R_i(x, t) = B_i(x, t) = 0 \) for \( t \leq 0 \). Given these initial conditions, (25) shows that \( B_i(x, t) \) for \( t > 0 \) can be related to \( R_i \) via the retarded solution [6]

\[
B_i(x, t) = \frac{1}{4\pi} \int \frac{dy}{|x - y|} R_i(y, t - |x - y|). \tag{31}
\]

We get from (30, 31)

\[
\dot{W} = \int d^3 x B_i \Delta^{-1}(\dot{R}_i)(x, t) = -\frac{1}{4\pi} \int d^3 x \times \int d^3 y R_i(y, t - |x - y|) \int \frac{d^3 z}{|x - z|} \dot{R}_i(z, t). \tag{32}
\]

We calculate (32) in the non-relativistic limit [6]; cf. section 5.2 of [26]. It assumes that \( R_i(y, t) \) as a function of \( y \) is well-localized in the vicinity of \( y = 0 \); e.g. \( R_i(y, t) \approx f_i(t) \delta(y) \). Using this in the RHS of (32), and going to spherical coordinates in \( d^3 x \), we end up with

\[
\dot{W} = -\int_0^\infty \frac{4\pi}{4\pi} \int d^3 y R_i(y, t - r) \int d^3 z \dot{R}_i(z, t). \tag{33}
\]

Recall that all fields vanish at infinity and that \( R_i(y, t) =
0 for \( t \leq 0 \). We get from (33):
\[
\dot{W} = -\frac{\chi_i(t)\dot{\chi}_i(t)}{4\pi}, \quad \chi_i(t) \equiv \int d^3 x \int_0^t ds \, R_i(x,s),
\]
(34)

Now (30), (34) imply for the energy change:
\[
\{H_B\}^t_0 = \{E_S\}^t_0 + \int_0^t \frac{ds}{4\pi} [\dot{\chi}_i(s)]^2 - \frac{1}{4\pi} \chi_i(t)\dot{\chi}_i(t),
\]
(35)

where \( E_S = -\frac{1}{4\pi} \int d^3 x \rho \Delta^{-1} \rho \) is the electrostatic energy and \( \{X\}^t_0 = X(t) - X(0) \). Eq. (35) makes a thermodynamic sense: \( \{H_B\}^t_0 \) is the work, which for the considered thermally isolated system (EMF) is defined via its Hamiltonian. \( \{E_S\}^t_0 \) is the part of energy that depends only on the state of the system at times 0 and \( t \), but does not depend on the trajectory. Hence it accounts for the reversible work. If we impose the cyclicity condition, assuming that besides \( R_i(y,t) = 0 \) for \( t \leq 0 \), it also holds \( R_i(y,\tau) = 0 \), then the last terms in (35) vanishes at \( t = \tau \) due to \( \chi_i(\tau) = 0 \). Hence we get the statement of the second law: the irreversible work \( \{H_B - E_S\}^\tau_0 = \{W\}^\tau_0 \) is non-negative, i.e. the energy is put into EMF. Here the validity of this statement relates to the definition (30) and our assumption on localized \( R_i(x,t) \). We stress that such a statement is not be deduced from (15), even if we assume one of standard gauges (e.g. the Lorenz gauge); see section 5 of [26].

Above derivation was done assuming initial conditions. Alternatively, we can employ final conditions assuming that \( R_i(x,t) = B_i(x,t) = 0 \) for \( t > \tau \). Then the connection between \( R_i(x,t) \) and \( B_i(x,t) = 0 \) for \( t < \tau \) is to be given via the advanced solution of (25):
\[
P_i^{[\text{adv}]}(x,t) = \frac{1}{4\pi} \int |x-y| \, R_i(y,t + |x-y|).
\]
(36)

The fact that normally one employs retarded solution (31) via initial conditions, and not the advanced solution (36) via final conditions amounts to the electrodynamic arrow of time [16–19].

Repeating the above steps and imposing the cyclicity condition \( R_i(x,t) = 0 \) for \( t < 0 \), we get [cf. (34)]
\[
\{H_B - E_S\}^\tau_0 = -\int_0^\tau \frac{ds}{4\pi} [\dot{\chi}_i(s)]^2.
\]
(37)

Now instead of the second law we got its opposite: the energy is extracted from EMF. This links the thermodynamic arrow of time (second law or putting work into the many-body system) and the electrodynamic arrow. Relations between the cosmological and thermodynamical arrows were recently explored in [25].

In sum, we found a new gauge-invariant Hamiltonian for electromagnetic field (EMF) that holds all desiderata for defining work. In particular, it leads to the second law (in contrast to other definitions), relates it with the electrodynamic arrow of time, and differs from the Hamiltonian obtained in the limit of vanishing photon mass. Elsewhere, we shall quantize this Hamiltonian and explore its consequences for quantum electrodynamics.

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SUPPLEMENTARY MATERIAL

1. Derivation of $L_B$

Let us recall the Lagrangian of EMF with sources

$$L = \frac{1}{2} (\partial_i \phi + \hat{A}_i)^2 - \frac{1}{2} B_i^2 - \rho \phi + J_i A_i,$$

(38)

as well as the relation between $\phi$ and $A_i$ via the inverse Laplacian [cf. (1–3) and (18, 19) of the main text]:

$$\phi = -\Delta^{-1}(\rho + \partial_i \hat{A}_i),$$

(39)

$$(\Delta^{-1} f)(x) = -\int \frac{d^3y}{4\pi \|x - y\|} f(y),$$

(40)

We recall that derivatives commute with the inverse Laplacian:

$$\frac{\partial}{\partial x_i} (\Delta^{-1} f)(x) = -\int \frac{d^3y}{4\pi \|x - y\|} \frac{\partial f(y)}{\partial y_i},$$

(41)

where it is assumed that $f(x) \to 0$ for $|x| \to 0$. We shall write (41) as

$$\partial_i \Delta^{-1} = \Delta^{-1} \partial_i,$$

(42)

and employ it freely.

We integrate (38) by parts and write it as

$$\int d^3x L = \int d^3x \left[ -\frac{1}{2} \phi \Delta \phi - \rho \phi - \phi \partial_i \hat{A}_i + \frac{1}{2} \hat{A}_i^2 - \frac{1}{2} B_i^2 + J_i A_i \right].$$

(43)

Now the first three terms in the RHS of (43) are to be transformed via (39), and

$$\int d^3x a(x)(\Delta^{-1} b)(x) = \int d^3x b(x)(\Delta^{-1} a)(x).$$

(45)

We get

$$\int d^3x \hat{L} = \int d^3x \left[ \frac{1}{2} (\partial_i \hat{A}_i) \Delta^{-1} (\partial_k \hat{A}_k) + (\partial_i \hat{A}_i) \Delta^{-1}(\rho) - \frac{1}{2} \rho \Gamma^{-1} \rho + \frac{1}{2} \hat{A}_i^2 - \frac{1}{2} B_i^2 + J_i A_i \right].$$

(46)

Now we transform $(\partial_i \hat{A}_i)\Delta^{-1}(\rho)$. We write it as

$$\frac{d}{dt} [ (\partial_i \hat{A}_i) \Delta^{-1}(\rho) ] = (\partial_i \hat{A}_i) \Delta^{-1}(\rho) \frac{d\rho}{dt} + \partial_k \hat{A}_k \Delta^{-1}(\rho) \frac{d\rho}{dt},$$

(47)

neglect the full time-derivative, employ the charge conservation $\rho + \partial_k J_k = 0$ and replace $(\partial_i \hat{A}_i)\Delta^{-1}(\rho)$ by $(\partial_i \hat{A}_i)\Delta^{-1}(\partial_k J_k)$. Hence the transformed Lagrangian reads:

$$\int d^3x L_B = \int d^3x \left[ -\frac{1}{2} \rho \Delta^{-1} \rho + \frac{1}{2} (\partial_i \hat{A}_i) \Delta^{-1}(\partial_k \hat{A}_k) + (\partial_i \hat{A}_i) \Delta^{-1}(\partial_k J_k) + \frac{1}{2} \hat{A}_i^2 - \frac{1}{2} B_i^2 + J_i A_i \right].$$

(48)

Now recall definitions of the magnetic field $B_i$ and of the vorticity of the charge flow $R_i$:

$$B_i = \epsilon_{ijk} \partial_j A_k, \quad \epsilon_{ijk} B_i = \partial_i A_k - \partial_k A_i, \quad R_i = \epsilon_{ijk} \partial_j J_k.$$

(49)

(50)

The following relations are deduced via integration by parts, (41) and (49, 50):

$$-\frac{1}{2} \int d^3x B_i \Delta^{-1} \dot{B}_i = \frac{1}{2} \int d^3x \left[ (\partial_k \hat{A}_k) \Delta^{-1}(\partial_i \hat{A}_i) + \hat{A}_k^2 \right],$$

(51)

$$\int d^3x (\partial_i \hat{A}_i) \Delta^{-1}(\partial_k J_k) = -\int d^3x A_i \Delta^{-1}(\partial_k [\partial_j \hat{J}_k - \partial_k \hat{J}_i + \partial_k \hat{J}_j])$$

$$= -\int d^3x [A_i J_i + B_i \Delta^{-1} R_i].$$

(52)

Putting (51, 52) into (48) we see that $\int d^3x L_B$ can be expressed only via $B_i$ and $R_i$. Then we are back to the expression for $\int d^3x L_B$ used in the main text; cf. (21, 22) of the main text.

2. Electrostatic energy for point charges

Here we recall how to calculate the change of the electrostatic energy:

$$\frac{dE_e}{dt} = -\frac{1}{2} \frac{d }{dt \int d^3x \rho \Delta^{-1} \rho}$$

(53)

$$= \frac{1}{2} \frac{d }{dt \int d^3x d^3x \frac{\rho(x,t)\rho(y,t)}{|x-y|}},$$

(54)

for point charges. The point here is that for point charges (53) is infinite, but its change in time is finite, since infinities cancel out.

The charge density for $N$ points with charges $e_\alpha$ and coordinate vectors $r_\alpha$ reads

$$\rho(x,t) = \sum_{\alpha=1}^N e_\alpha \delta(x - r_\alpha(t)).$$

(55)

We shall temporarily move from (55) to regularized $\delta$-functions $\hat{\delta}(x - r_\alpha(t))$, where $\hat{\delta}(0)$ is finite:

$$\rho(x,t) = \sum_{\alpha=1}^N e_\alpha \hat{\delta}(x - r_\alpha(t)).$$

(56)
Putting (56) into (54) we get
\[
\frac{dE_s}{dt} = \sum_{\alpha} \frac{1}{2} \frac{d}{dt} \int \frac{d^3 x \, d^3 x}{4\pi} \frac{e_\alpha^2 \delta(x - r_\alpha(t)) \delta(y - r_\alpha(t))}{|x - y|} \partial \rho(y) \frac{\partial}{\partial y} \partial \rho(t)
\] (57)
\[
+ \sum_{\alpha \neq \beta} \frac{1}{2} \frac{d}{dt} \int \frac{d^3 x \, d^3 x}{4\pi} \frac{e_\alpha e_\beta \delta(x - r_\alpha(t)) \delta(y - r_\beta(t))}{|x - y|} \partial \rho(y) \frac{\partial}{\partial y} \partial \rho(t)
\] (58)
It is seen that (57) nullifies, since the integral is finite and does not depend on time (via \(r_\alpha(t)\)). In (58) we can take the regularization out:
\[
\frac{dE_s}{dt} = \frac{1}{2} \frac{d}{dt} \sum_{\alpha \neq \beta} \frac{e_\alpha e_\beta}{|r_\alpha(t) - r_\beta(t)|} \partial \rho(y) \frac{\partial}{\partial y} \partial \rho(t)
\] (59)
which is the final and finite result.

3. Massive electrodynamics

3.1 Equations of motion and Hamiltonian

The Proca Lagrangian \(\int d^3 x \mathcal{L}\) of electrodynamics with mass \(m\) reads [9, 10]
\[
\mathcal{L}_m = \frac{1}{2} (\partial_i \phi + \dot{A}_i)^2 - \frac{1}{2} B_i^2 + \frac{m^2}{2} (\phi^2 - A_i^2) - \rho \phi + 3 J_i A_i
\] (60)
This theory is not gauge-invariant. Equations of motion read
\[
\partial_k (\partial \phi + \dot{A}_k) = \Delta A_k - \partial_k (\partial_t A_i) + J_k - m^2 A_k,
\] (61)
\[
\Delta \phi - m^2 \phi = -\rho - \partial_t \dot{A}_i.
\] (62)
Apply \(\partial_t\) to (62) and \(\partial_k\) to (61). Together these lead to
\[
\dot{\rho} + \partial_k J_k = m^2 \phi + \partial_k A_k.
\] (63)
Hence if we impose the charge conservation \(\dot{\rho} + \partial_k J_k = 0\), then (63) together with \(m > 0\) leads to the Lorenz gauge [9, 10]
\[
\dot{\phi} + \partial_k A_k = 0.
\] (64)
Using (64) we present (62) and (61) as
\[
\Delta \phi - m^2 \phi = -\rho + \dot{\phi},
\] (65)
\[
\Delta A_k - m^2 A_k = -J_k + \dot{A}_k,
\] (66)
i.e. \(\phi\) and \(A_k\) are decoupled up to (64).
Let us turn to Hamiltonizing (60). We write
\[
\mathcal{H}_m = p_i \dot{A}_i - \mathcal{L}_m,
\] (67)
and via the Legendre transform:
\[
\frac{\delta}{\delta A_k(y)} \int d^3 x \, \mathcal{H}_m = 0, \quad \frac{\delta}{\delta \phi(y)} \int d^3 x \, \mathcal{H}_m = 0.
\] (68)
exclude four variables \(\dot{A}_i\) and \(\phi\) in favor of three momenta \(p_i\) [13]:
\[
\phi = \frac{1}{m^2} (\rho + \partial_t p_i), \quad p_i = \dot{A}_i + \frac{1}{m^2} \partial_k (\rho + \partial_t p_i).
\] (69)
(70)
Putting (69, 70) into (67) we get:
\[
\mathcal{H}_m = \frac{1}{2} p_i^2 + \frac{1}{2} B_i^2 + \frac{m^2}{2} A_i^2 + \frac{1}{2m^2} (\rho + \partial_t p_i)^2 - J_i A_i,
\] (71)
where \(p_i\) and \(A_i\) are (resp.) independent canonical momenta and coordinates:
\[
\dot{p}_j = \Delta A_j - \partial_j \partial_k A_k - m^2 A_j + J_j,
\] (72)
\[
\dot{A}_j = p_j - \frac{1}{m^2} \partial_j (\rho + \partial_t p_i).
\] (73)
For the time-derivative we get
\[
\frac{d}{dt} \int d^3 x \, \mathcal{H}_m = \int d^3 x [\frac{\dot{\phi}}{m^2} (\rho + \partial_t p_i) - \dot{J}_i A_i],
\] (74)
where we employed (70). Now (74) is finite and well-defined for \(m \to 0\) [cf. (65, 66)], where it amounts to using the Lorenz gauge.

3.2 Exclusion of \(\phi\)

By analogy to the massless situation we can attempt to exclude \(\phi\) in the Lagrangian via (62):
\[
\phi = -\Gamma^{-1} (\rho + \partial_t A_i),
\] (75)
\[
\Gamma \equiv \Delta - m^2,
\] (76)
\[
(\Gamma^{-1} f)(x) = - \int \frac{d^3 y}{4\pi} \frac{f(y)}{|x - y|} e^{-m|x - y|}.
\] (77)
To this end we integrate (60) by parts and write it as
\[
\int d^3 x \, \mathcal{L}_m = \int d^3 x \left[ -\frac{1}{2} \delta \Gamma \phi - \rho \phi - \phi \partial_i \dot{A}_i \right]
\] (78)
\[
+ \frac{1}{2} A_i^2 - \frac{1}{2} B_i^2 - \frac{m^2}{2} A_i^2 + J_i A_i.
\] (79)
Now the first three terms in the RHS of (78) are to be transformed via (75), and
\[
\int d^3 x \, a(x) \Gamma^{-1} b(x) = \int d^3 x \, b(x) \Gamma^{-1} a(x).
\] (80)
We get
\[
\int d^3 x \, \mathcal{L}_m = \int d^3 x \left[ \frac{1}{2} (\partial_i \dot{A}_i) \Gamma^{-1} (\partial_k A_k) + (\partial_i \dot{A}_i) \Gamma^{-1} (\rho) \right]
\] (81)
Now we transform the term \((\partial_t \dot{A}_i)\Gamma^{-1}(\rho)\), since eventually we aim at reproducing the autonomous dynamics of \(A_i\) given by (66) that does not contain \(\rho\). We neglect the full time-derivative, employ the charge conservation \(\dot{\rho} + \partial_t J_k = 0\) and replace \((\partial_t \dot{A}_i)\Gamma^{-1}(\rho)\) by \((\partial_t A_i)\Gamma^{-1}(\partial_t J_k)\). Hence the transformed Lagrangian reads:
\[
\int d^3x L'_m = \int d^3x \left[ -\frac{1}{2} \rho \Gamma^{-1}(\rho) + \frac{1}{2} (\partial_t \dot{A}_i)\Gamma^{-1}(\partial_t \dot{A}_k) + \frac{1}{2} A_i^2 - \frac{1}{2} B_i^2 - \frac{m^2}{2} A_i^2 + J_i A_i \right].
\]
Note a relation
\[
-\int d^3x \dot{B}_i \Gamma^{-1} \dot{B}_i = \int d^3x \left[ (\partial_t \dot{A}_k)\Gamma^{-1}(\partial_t \dot{A}_i) + A_i^2 + m^2 \dot{A}_k \Gamma^{-1} \dot{A}_k \right] \tag{82}
\]
whose analogue was employed by us for the \(m > 0\) situation. Now employing (83) is not going to be useful, since the autonomous description for the \(m > 0\) case is provided directly by \(A_i\) and not by \(B_i\).
At any rate (82) contains only \(A_i, \dot{A}_i\) and \(J_i\) and we can look for Lagrange equations generated by it treating \(A_i, \dot{A}_i\) and \(J_i\) as (resp.) coordinates, velocities and external fields [cf. (61)]:
\[
\ddot{A}_i - \Delta A_i + m^2 A_i = J_i - \Gamma^{-1}(\partial_t \partial_t \dot{A}_k) - \Gamma^{-1}(\partial_t \partial_t J_i) - \partial_t \partial_t A_k. \tag{84}
\]
This is an integro-differential equation. Its LHS compares with (66), but its RHS is generally not zero. We conclude that (82) does not correspond to the autonomous description of \(A_i\) that is given by (66). While such a Lagrangian can be written down on the basis of (66), it does not relate to the original Lagrangian (60).

4. Natural configurations of charge density and current

As for any vector, one can apply the Helmholtz’s theorem (obtained e.g. via the Fourier representation) for representing the current \(J_k\) as
\[
J_i = J_i^\perp + J_i^\parallel, \tag{85}
\]
\[
\partial_i J_i^\perp = \epsilon_{ijk} \partial_j J_k^\parallel = 0. \tag{86}
\]
Let us now assume that \(J_i^\parallel\) does not depend on time: \(\dot{J}_k^\parallel = 0\). Using the continuity equation \(\dot{\rho} + \partial_t J_k = \dot{\rho} + \partial_t J_k^\parallel = 0\) and (86), we obtain
\[
\rho(x, t) = \rho(x, 0) - t \partial_k J_k^\parallel(x). \tag{87}
\]
If we demand that all involved charge densities stay bounded for any time \(t\), then (87) leads to \(\dot{\rho} = 0\), i.e. to \(\partial_k J_k^\parallel(x) = 0\) and hence to \(J_k^\parallel(x) = 0\).

Thus, under a natural additional condition, we conclude that stationary currents are vortical: \(J_i^\parallel = 0\) leads to \(J_k^\parallel(x) = 0\).

5. Calculation of \(\dot{H}_D\) in the Lorenz gauge

5.1 The Lorenz gauge

We return to the rate of the standard EMF Hamiltonian
\[
\dot{H}_D = \int d^3x \left[ \phi \dot{\rho} - J_i A_i \right], \tag{88}
\]
which is gauge-variant. We shall calculate (88) in the Lorenz gauge
\[
\dot{\phi} + \partial_t A_k = 0, \tag{89}
\]
and for slow and space-localized sources \(\rho\) and \(J_k\). The Lorenz gauge is selected, because it emerges in the limit \(m \to 0\) of the massive electrodynamics and because it is relativistically covariant. The purpose of the calculation is to check whether our conclusions on the second law can be seen on the level of (88, 89).

Here are the known retarded-potential solutions that hold (89):
\[
\phi(x, t) = \frac{1}{4\pi} \int \frac{d^3y}{|x - y|} \rho(y, t - |x - y|), \tag{90}
\]
\[
A_i(x, t) = \frac{1}{4\pi} \int \frac{d^3y}{|x - y|} J_i(y, t - |x - y|). \tag{91}
\]

5.2 The space-localized and slow (non-relativistic) approximation.

The space-localized approximation amounts to the following points elucidated on the example of (91). First, \(J_i(y, t)\) as a function of \(y\) is well-localized in the vicinity of (say) \(y = 0\). Hence the main contribution to the \(\int d^3x\) integral in (91) comes from \(|x| > 1\). We can put in the RHS of (91): \(|x - z| \approx |x|\) and \(|x - y| \approx |x|\). The second assumption is that \(J_i(y, t)\) is a slow function of \(t\), hence \(J_i(y, t - |x - y|) \approx J_i(y, t - |x|)\). Hence
\[
\phi(x, t) \simeq \frac{1}{4\pi |x|} \int d^3y \rho(y, t - |x|) \tag{92}
\]
\[
= \frac{1}{4\pi |x|} \int d^3y \rho(y, t), \tag{93}
\]
\[
A_i(x, t) \simeq \frac{1}{4\pi |x|} \int d^3y J_i(y, t - |x|) \tag{94}
\]
where in (93) we additionally used the charge conservation. It is now seen that within this approximation we
can put
\[ \int d^3 x \, \phi \dot{\rho} \simeq \int d^3 x \, \frac{\dot{\rho}(x, t)}{4\pi|x|} \int d^3 y \, \rho(y, t) \]
\[ \simeq \int d^3 x \, d^3 y \, \frac{\dot{\rho}(x, t) \rho(y, t)}{4\pi|x - y|} = -\frac{1}{2} \frac{d}{dr} \int d^3 x \, \rho \Delta^{-1} \rho. \] (95)

Hence the factor \( \phi \dot{\rho} \) in (88) approximately recovers the electrostatic energy change seen also for \( H_B \); see (26, 27) of the main text.

We now turn to (94) and the term \( A_i \dot{J}_i \) in (88):
\[ \int d^3 x \, \dot{J}_i A_i = \int_0^\infty \frac{u^2 du}{4\pi} \int_0^t r \, dr \, \dot{f}_i(r, t) \, f_i(u, t - r), \] (96)
\[ f_i(r, t) \equiv \int d\Omega \, J_i(r, \Omega, t), \] (97)

where we went to spherical variables, \( \int d\Omega \) is the integration over the spherical angles, and where we assumed
\[ J_i(x, t) = 0 \quad \text{for} \quad t < 0. \] (98)

We now check the sign of (96) under an additional condition
\[ J_i(x, \tau) = 0. \] (99)

Eqs. (98, 99) amount to a cyclic change. Now (96) does not have a definite sign, as can be seen e.g. by taking \( f_i(r, t) = \delta_{i1} a(r)b(t) \), where \( a(r) \) is a function well-localized at \( r \simeq r_0 \), and where \( b(t) = \sin t, \tau = 2\pi \). Then (96) changes its sign as a function of \( r_0 \) for \( r_0 < 2\pi \).