GENERALIZED DISPERSIVE EQUATIONS OF HIGHER ORDERS POSED ON BOUNDED INTERVALS: LOCAL THEORY

N. A. LARKIN & J. LUCHESI†

Abstract. Initial-boundary value problems for nonlinear dispersive equations of evolution of order \(2l + 1\), \(l \in \mathbb{N}\) with a convective term of the form \(u^k u_x\), \(k \in \mathbb{N}\) have been considered on intervals \((0, L)\), \(L \in (0, +\infty)\). The existence and uniqueness of local regular solutions have been established.

1. Introduction

Our goal in this paper is to study solvability of initial-boundary value problems for one-dimensional generalized dispersive equations of higher orders posed on a bounded interval

\[
u_t + \sum_{j=1}^{l} (-1)^{j+1} D_{x}^{2j+1} u + u^k D_x u = 0 \quad \text{in} \quad Q_T,\]

where \(x \in (0, L)\), \(Q_T = (0, T) \times (0, L)\); \(l, k \in \mathbb{N}\); \(T, L\) are real positive numbers. This equation includes as special cases classical dispersive equations: when \(l = k = 1\), we have the well-known Korteweg-de Vries (KdV) equation, see \([15, 20, 31]\), and when \(k = 1, l = 2\), we have the Kawahara equation \([5, 16, 23]\). For \(k = 1\), the Cauchy problem for dispersive equations of higher orders has been studied in \([3, 6, 11, 12, 17, 28, 32]\) and initial boundary value problems have been studied in \([4, 5, 7, 23, 30]\). Although dispersive equations were deduced for the whole real line, necessity to calculate numerically the Cauchy problem, approximating the real line by finite intervals \([4]\), implies to study initial-boundary value problems posed on bounded and unbounded intervals, \([5, 7, 19, 21, 23]\). What concerns \((1.1)\) with \(k > 1, l = 1\), called generalized Korteweg-de Vries equations, the Cauchy problem for \((1.1)\) has been studied in \([8, 9, 10, 18, 25, 26]\), where was proved that for \(k = 4\), called the critical case, the initial

Key words and phrases. Higher-order dispersive equations, local solutions, bounded domains.

2010 Mathematics Subject Classification: 35M20, 35Q72.

† Corresponding author.
problem is well-posed for small initial data, whereas for arbitrary initial data, solutions may blow-up in a finite time. The generalized KdV equation was intensively studied in order to understand the interaction between the dispersive term and nonlinearity in the context of the theory of nonlinear dispersive evolution equations \[13, 14, 29\]. In \[24\], an initial-boundary value problem for the generalized KdV equation with an internal damping posed on a bounded interval was studied in the critical case. Exponential decay of weak solutions for small initial data has been established. In \[2\], decay of weak solutions for \(l = 2, k = 2\) was established.

Our goal in this work is to prove the existence and uniqueness of local regular solutions for all \(k \in \mathbb{N}\) and for all finite positive \(L\).

Our paper has the following structure: Chapter 1 is Introduction. Chapter 2 contains notations and auxiliary facts. In Chapter 3, formulation of problems to be considered is given. In Chapter 4, we prove local existence and uniqueness of regular solutions.

2. Notations and auxiliary facts

Let \(x < (0, L)\), \(D^i = \frac{\partial^i}{\partial x^i}\), \(i \in \mathbb{N}\); \(D = D^1\). As in \[1\] p. 23, we denote for scalar functions \(f(x)\) the Banach space \(L^p(0, L)\), \(1 \leq p \leq +\infty\) with the norm:

\[
\|f\|_{L^p(0, L)} = \left( \int_0^L |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p \leq +\infty.
\]

For \(p = 2\), \(L^2(0, L)\) is a Hilbert space with the scalar product

\[
(u, v) = \int_0^L u(x)v(x) \, dx \quad \text{and the norm} \quad \|u\|^2 = \int_0^L |u(x)|^2 \, dx.
\]

The Sobolev space \(W^{m,p}(0, L)\), \(m \in \mathbb{N}\) is a Banach space with the norm:

\[
\|u\|_{W^{m,p}(0, L)} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(0, L)}, \quad 1 \leq p < +\infty.
\]

When \(p = 2\), \(W^{m,2}(0, L) = H^m(0, L)\) is a Hilbert space with the following scalar product and the norm:

\[
((u, v))_{H^m(0, L)} = \sum_{0 \leq |j| \leq m} \langle D^j u, D^j v \rangle, \quad \|u\|_{H^m(0, L)}^2 = \sum_{0 \leq |j| \leq m} \|D^j u\|^2.
\]

Let \(D(0, L)\) or \(D([0, L])\) be the space of \(C^\infty\) functions with compact support in \((0, L)\) or \([0, L]\). The closure of \(C^\infty\) functions in \(W^{m,p}(0, L)\) is denoted by \(W^{m,p}_0(0, L)\) and \((H^m_0(0, L)\) when \(p = 2\). For any space of functions, defined on an interval \((0, L)\), we omit the symbol \((0, L)\),
for example, $L^p = L^p(0, L)$, $H^m = H^m(0, L)$, $H^m_0 = H^m_0(0, L)$ etc. We use the following version of the Gagliardo-Nirenberg inequality:

**Lemma 2.1.** Let $u$ belong to $H^0_0(0, L)$, then the following inequality holds:

$$

\|u\|_\infty \leq \sqrt{2}\|D' u\|^{\frac{1}{2}}\|u\|^{1-\frac{1}{2}}.

(2.1)

Proof. Let $l = 1$, then for any $x \in (0, L)$

$$

u^2(x) = \int_0^x D[u^2(\xi)]d\xi \leq 2 \int_0^x |u(\xi)||D(\xi)|d\xi

\leq 2 \int_0^L |u||Du|dx \leq 2\|u\||Du|

$$

and $\|u\|_\infty \leq \sqrt{2}\|Du\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}$.

For $l \geq 2$ the proof will be done in several steps:

**Step 1:** $\|Du\| \leq \|D^2 u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}$, for all $u \in H^0_0(0, L)$.

Let $u \in H^0_0(0, L)$, then

$$

\|Du\|^2 = \int_0^L (Du)^2 dx = -\int_0^L uD^2 u dx \leq \int_0^L \|u\||D^2 u|dx \leq \|u\||D^2 u|,

$$

and Step 1 is proved.

**Step 2:** Let $u \in H^l_0(0, L)$, then $\|Du\| \leq \|D^l u\|^{\frac{1}{l+1}}\|u\|^{l+1-(\frac{1}{l+1})}$, for all $l \geq 2$.

We proceed by induction on $l$. The case $l = 2$ is proved in Step 1. Suppose the result is valid for $m > 2$, then if $u \in H^{m+1}_0(0, L)$, we have $u \in H^2_0(0, L)$ and $Du \in H^{m+1}_0(0, L)$. Consequently,

$$

\|Du\| \leq \|D^2 u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}

(2.2)

$$

and

$$

\|D^2 u\| \leq \|D^{m+1} u\|^{\frac{1}{m+1}}\|Du\|^{m+1-(\frac{1}{m+1})}.

(2.3)

$$

Substituting (2.3) into (2.2), we obtain

$$

\|Du\| \leq \|D^{m+1} u\|^{\frac{1}{m+1}}\|Du\|^{1-\frac{1}{m+1}}.

$$

Therefore $\|Du\| \leq \|D^l u\|^{\frac{1}{l+1}}\|u\|^{l+1-(\frac{1}{l+1})}$ for all $l \geq 2$.

**Step 3:** $\|u\|_\infty \leq \sqrt{2}\|D^l u\|^{\frac{1}{2}}\|u\|^{l+1-(\frac{1}{2})}$, for all $u \in H^l_0(0, L)$ and $l \geq 2$.

Let $l \geq 2$ and $u \in H^l_0(0, L)$. Then by the case $l = 1$ and Step 2,

$$

\|u\|_\infty \leq \sqrt{2}\|D^l u\|^{\frac{1}{2}}\|u\|^{l+(1-\frac{1}{2})}\|u\|^{\frac{1}{2}} = \sqrt{2}\|D^l u\|^{\frac{1}{2}}\|u\|^{l-\frac{l}{2}}.

$$

The proof of Lemma 2.1 is complete.

**Lemma 2.2.** (See [27], p. 125). Suppose $u$ and $D^m u$, $m \in \mathbb{N}$ belong to $L^2(0, L)$. Then for the derivatives $D_i u$, $0 \leq i < m$, the following inequality holds:

$$

\|D^i u\| \leq A_1\|D^m u\|^{\frac{i}{m}}\|u\|^{1-\frac{i}{m}} + A_2\|u\|,

(2.4)

$$
4. Local solutions

We start with the linearized version of (3.1)

\[ u_t + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} u = f \quad \text{in} \quad Q_T \]  

subject to initial-boundary conditions:

\[ u(0, x) = u_0(x), \quad x \in (0, L); \]  

\[ D^i u(t, 0) = D^i u(t, L) = D^l u(t, L) = 0, \quad i = 0, \ldots, l - 1, \]  

where \( l, k \in \mathbb{N} \) and \( u_0 \) is a given function.

Theorem 4.1. (See [22], Theorem 4.1.) Let \( g \in L^2(0, L) \). Then for all \( a > 0 \) the stationary equation: \( au + Au = g \) in \( (0, L) \), subject to boundary conditions: \( D^i u(0) = D^i u(L) = D^l u(L) = 0, \ i = 0, \ldots, l - 1 \) admits a unique regular solution \( u \in H^{2l+1}(0, L) \) such that

\[ \|u\|_{H^{2l+1}} \leq C\|g\| \]  

with the constant \( C \) depending only on \( L, l \) and \( a \).

Theorem 4.2. Let \( u_0 \in D(A) \) and \( f \in H^1(0, T; L^2(0, L)) \). Then for every \( T > 0 \), problem (4.1), (3.2), (3.3) has a unique solution \( u = u(t, x) \) such that

\[ u \in C([0, T]; D(A)) \cap C^1([0, T]; L^2(0, L)). \]
Proof. According to Theorem 4.1, the operator $aI + A$ is surjective for all $a > 0$. Moreover, if $u \in D(A)$ then

$$(Au, u) = \frac{1}{2}(D^atu(0))^2 \geq 0$$

and the result follows by the semigroup theory. (See [33], Lemma 2.2.3 and Corollary 2.4.2.) □

**Theorem 4.3.** Let $u_0 \in D(A)$. Then there exists a real $T_* \in (0, T]$ such that (3.1)–(3.3) has a unique regular solution $u = u(t, x)$:

$$u \in L^\infty(0, T_*; H^1_0(0, L) \cap H^2(0, L) \cap L^2(0, T_*; H^{2l+1}(0, L)), \quad u_t \in L^\infty(0, T_*; L^2(0, L)) \cap L^2(0, T_*; H^1_0(0, L)).$$

Proof. The proof will be done using the Banach fixed point theorem. First, let $v, v_t \in X = L^\infty(0, T; L^2(0, L)) \cap L^2(0, T; H^1_0(0, L)), \) then $v^kDv \in H^1(0, T; L^2(0, L))$. Indeed, since $v, v_t \in L^2(0, T; H^1_0(0, L))$, we have $v \in C\big([0, T]; H^1_0(0, L)\big)$ and by (2.1):

$$\|v^kDv\|_{L^2(Q_T)}^2 \leq \int_0^T \|v\|_{L^\infty}^{2k}(t)\|Dv\|^2(2)(t)dt \leq \int_0^T 2^k\|v\|_{H^1_0}^{2k}(t)\|v\|_{H^1_0}^2(t)dt$$

$$\leq 2^k\|v\|_{C([0, T]; H^1_0(0, L))}^{2k}\|v\|_{L^2(0, T; H^1_0(0, L))}^2.$$  

On the other hand, $(v^kDv)_t = kv^{k-1}v_tDv + v^kDv_t$. Hence

$$\|(v^kDv)_t\|_{L^2(Q_T)}^2 \leq 2k^2\|v^{k-1}v_tDv\|_{L^2(Q_T)}^2 + 2\|v^kDv_t\|_{L^2(Q_T)}^2.$$  

We estimate

$$I_1 \leq 2k^2 \int_0^T \left[ \|v\|_{L^\infty}^{2(k-1)}(t)\|v_t\|_{L^2(0, L)}^2(t)\|Dv\|_{L^2(0, L)}^2(t) \right]dt$$

$$\leq 2^k \int_0^T \left[ 2^{k-1}\|v\|_{H^1_0}^{2(k-1)}(t)2\|v_t\|_{H^1_0}^2(t)\|v\|_{H^1_0}^2(t) \right]dt$$

$$\leq 2^{k+1}k^2\|v\|_{C([0, T]; H^1_0(0, L))}^{2k}\|v_t\|_{L^2(0, T; H^1_0(0, L))}^2;$$

$$I_2 \leq 2 \int_0^T \|v\|_{L^\infty}^{2k}(t)\|Dv_t\|_{L^2(Q_T)}^2(t)dt \leq 2^{k+1}\|v\|_{C([0, T]; H^1_0(0, L))}^{2k}\|v_t\|_{L^2(0, T; H^1_0(0, L))}^2.$$  

Taking $f = -v^kDv$ in Theorem 4.2, define an operator $P$, related to (4.1), (3.2), (3.3) such that $v \mapsto u = Pv$, and the space:

$$V = \{v(t, x) : v, v_t \in X; \ v(0, \cdot) \equiv u_0\}.$$
with the norm

\[ \|v\|_V^2 = \text{ess sup}_{t \in (0,T)} \{ \|v\|^2(t) + \|v_t\|^2(t) \} + \int_0^T \sum_{j=1}^l \left[ \|D^j v\|^2(t) + \|D^j v_t\|^2(t) \right] dt. \]

Consider in \( V \) a ball

\[ B_R = \{ v \in V : \|v\|_V^2 \leq 8R^2 \}, \]

where \( R > 0 \) satisfies the inequality:

\[ (1 + L)(1 + l) \left( 2^k \|u_0\|_{H^2_0}^{2(k+1)} + \|u_0\|_{H^{2l+1}}^2 \right) \leq R^2. \]  

\textbf{Lemma 4.4.} There is a real \( 0 < T_0 \leq T \) such that the operator \( P \) maps \( B_R \) into itself.

\textit{Proof.} First, if \( v \in B_R \), then due to (4.3), we have

\[ \|Dv\|^2(t) \leq \|Du_0\|^2 + \int_0^T \left[ \|Dv\|^2(t) + \|Dv_t\|^2(t) \right] dt \leq 9R^2. \]  

(4.4)

We will need the following estimates:

\textbf{Estimate 1.} Multiplying (4.1) by 2\((1+x)u\) and integrating over (0, \( L \)), we obtain

\[ \frac{d}{dt} (1 + x, u^2)(t) + \sum_{j=1}^l (2j + 1) \|D^j u\|^2(t) + (D^j u(t,0))^2 = -2(v^k Dv, (1 + x)u)(t). \]  

(4.5)

Making use of (2.1) and (4.4), we estimate

\[ -2(v^k Dv, (1 + x)u)(t) \leq (1 + L) \|v^k Dv\|^2(t) + (1 + x, u^2)(t) \leq (1 + L) \|v\|_{H_0^2}^{2k}(t) \|Dv\|^2(t) + (1 + x, u^2)(t) \leq (1 + L)2^k \|v\|_{H_0^2}^{2k}(t) \|Dv\|^2(t) + (1 + x, u^2)(t) \leq (1 + L)9(34)^k R^{2k+2} + (1 + x, u^2)(t). \]

Then (4.5) becomes

\[ \frac{d}{dt} (1 + x, u^2)(t) + \sum_{j=1}^l (2j + 1) \|D^j u\|^2(t) + (D^j u(t,0))^2 \leq (1 + x, u^2)(t) + (1 + L)9(34)^k R^{2k+2}. \]  

(4.6)
By the Gronwall Lemma and (4.3),

\[(1 + x, u^2)(t) \leq e^T \left((1 + x, u^0_0)^2 + (1 + L)9(34)^k R^{2k+2} T\right)
\]

\[\leq e^T \left(\frac{R^2}{2} + (1 + L)9(34)^k R^{2k+2} T\right).\]

Choosing \(0 < T_1 \leq T\) such that \(e^{T_1} \leq 2\) and \((1 + L)9(34)^k R^{2k+2} T_1 \leq \frac{1}{2}\), we conclude

\[(1 + x, u^2)(t) \leq 2 \left(\frac{R^2}{2} + \frac{R^2}{2}\right) = 2R^2, \quad t \in (0, T_1]. \quad (4.7)\]

Substituting (4.7) into (4.6) and integrating over \((0, T_1]\), we get

\[3 \int_0^{T_1} \sum_{j=1}^l \|D^j u\|^2(t) dt \leq R^2 \left((1 + L)9(34)^k R^{2k+2} + 2\right) T_1 + (1 + x, u^0_0).\]

Taking \(0 < T_2 \leq T_1\) such that \((1 + L)9(34)^k R^{2k+2} T_2 \leq 2\) and making use of (4.3), we conclude

\[\int_0^{T_2} \sum_{j=1}^l \|D^j u\|^2(t) dt \leq \frac{2R^2 + R^2}{3} = R^2. \quad (4.8)\]

**Estimate 2.** Differentiating (4.1) with respect to \(t\), multiplying the result by \(2(1 + x)\) and integrating over \((0, L)\), one gets

\[\frac{d}{dt}(1 + x, u^2_t)(t) + \sum_{j=1}^l (2j + 1)\|D^j u_t\|^2(t) + (D^j u_t(t, 0))^2
\]

\[= 2\left(-kv^{k-1}v_tDv, (1 + x)u_t(t)\right) + 2\left(-v^k Dv, (1 + x)u_t(t)\right). \quad (4.9)\]

Making use of (2.1) and (4.4), we estimate for an arbitrary \(\epsilon > 0:\)

\[I_1 \leq \epsilon(1 + L)k\|v^{k-1}v_tDv\|^2(t) + \frac{1}{\epsilon}(1 + x, u^2_t)(t)
\]

\[\leq \epsilon(1 + L)k\|v\|^{2(k-1)}(t)\|v_t\|^2(t)\|Dv\|^2(t) + \frac{1}{\epsilon}(1 + x, u^2_t)(t)
\]

\[\leq \epsilon(1 + L)k2^k\|v\|^{2(k-1)}(t)\|v_t\|^{2}(H^1_0(t))\|Dv\|^2(t) + \frac{1}{\epsilon}(1 + x, u^2_t)(t)
\]

\[\leq \epsilon(1 + L)k2^k(17R^2)^{k-1} \left(8R^2 + \|Dv_t\|^2(t)\right) 9R^2 + \frac{1}{\epsilon}(1 + x, u^2_t)(t),\]
\[ I_2 \leq \epsilon(1 + L)\|v^k Dv_t\|^2(t) + \frac{1}{\epsilon}(1 + x, u_t^2)(t) \leq \epsilon(1 + L)\|v\|_{\infty}^{2k}(t)\|Dv_t\|^2(t) + \frac{1}{\epsilon}(1 + x, u_t^2)(t) \]
\[ \leq \epsilon(1 + L)(34)^k R^{2k}\|Dv_t\|^2(t) + \frac{1}{\epsilon}(1 + x, u_t^2)(t). \]

Substituting \( I_1, I_2 \) into (4.9) and taking \( \epsilon = (16(1 + L)\alpha_k R^{2k})^{-1} \) with \( \alpha_k = 9k^{2k}(17)^{k-1} + (34)^k \), we reduce it to the inequality
\[ \frac{d}{dt}(1 + x, u_t^2)(t) + \sum_{j=1}^{l}(2j + 1)\|D^j u_t\|^2(t) + (D^l u_t(t, 0))^2 \leq \frac{1}{16}\|Dv_t\|^2(t) + 9k(34)^{k-1} R^{2k} \alpha_k^{-1} + (32)(1 + L)\alpha_k R^{2k}(1 + x, u_t^2)(t). \quad (4.10) \]

By the Gronwall Lemma,
\[ (1 + x, u_t^2)(t) \leq e^{(32)(1 + L)\alpha_k R^{2k}T}(1 + x, u_0^2)(0) + e^{(32)(1 + L)\alpha_k R^{2k}T}(9k(34)^{k-1} R^{2k} \alpha_k^{-1} + (32)(1 + L)\alpha_k R^{2k})(1 + x, u_0^2). \]

Due to (2.1),
\[ \|u_0^k Du_0\|^2 \leq \|u_0\|_{\infty}^{2k}\|Du_0\|^2 \leq 2^k \|u_0\|_{L^1}^{(1 - \frac{1}{k})2k}\|D^j u_0\|_{L^1}^{\frac{j}{k}}\|Du_0\|^2 \]
\[ \leq 2^k \|u_0\|_{L^1}^{(1 - \frac{1}{k})2k}\|u_0\|_{L^1}^{\frac{j}{k}}\|u_0\|_{L^1}^{2} = 2^k \|u_0\|_{L^1}^{2(k+1)}, \]
whence (4.3) implies
\[ (1 + x, u_t^2)(0) \leq (1 + L)(1 + l) \left( \|u_0^k Du_0\|^2 + \|u_0\|^2_{H^{2l+1}} \right) \]
\[ \leq (1 + L)(1 + l) \left( 2^k \|u_0\|^2_{H^{2l+1}} + \|u_0\|^2_{H^{2l+1}} \right) \leq R^2. \]

Taking \( 0 < T_3 \leq T \) such that \( e^{(32)(1 + L)\alpha_k R^{2k}T_3} \leq 2, 9k(34)^{k-1} \alpha_k^{-1} T_3 \leq \frac{1}{2} \), we get
\[ (1 + x, u_t^2)(t) \leq 2 \left( R^2 + \frac{R^2}{2} + \frac{R^2}{2} \right) = 4R^2, \quad t \in (0, T_3]. \quad (4.11) \]

Substituting (4.11) into (4.10) and integrating over \( (0, T_3) \), we obtain
\[ 3 \int_0^{T_3} \sum_{j=1}^{l} \|D^j u_t\|^2(t)dt \leq R^2 \left( (128)(1 + L)\alpha_k R^{2k} + 9k(34)^{k-1} \alpha_k^{-1} \right) T_3 \]
\[ + \frac{1}{16} \int_0^{T_3} \|Dv_t\|^2(t)dt + (1 + x, u_t^2)(0). \]
Let $0 < T_4 \leq T_3$ satisfy $(128)(1 + L)\alpha_k R^{2k} + 9k(34)^{k-1}\alpha_k^{-1}) T_4 \leq 2$, then

$$\int_0^{T_4} \sum_{j=1}^l \|D^j u_j\|^2(t) dt \leq \frac{1}{3} \left( 2R^2 + \frac{R^2}{2} + \frac{R^2}{2} \right) = R^2. \quad (4.12)$$

Choosing $T_0 = \min\{T_2, T_4\}$ and taking into account (4.7), (4.8), (4.11), (4.12), we complete the proof of Lemma 4.4.

Lemma 4.5. There is a real $0 < T_* \leq T_0$ such that the mapping $P$ is a contraction in $B_R$.

Proof. For $v_1, v_2 \in B_R$, denote $u_i = P v_i$, $i = 1, 2$, $w = v_1 - v_2$ and $z = u_1 - u_2$. Then $z$ satisfies the equation

$$z_t + \sum_{j=1}^l (-1)^{j+1} D^{2j+1} z = -v_1^k Dw - (v_1^k - v_2^k) Dv_2 \text{ in } Q_{T_0} \quad (4.13)$$

and homogeneous boundary conditions (3.2) and initial data $z(0, \cdot) \equiv 0$.

Estimate 3. Multiplying (4.13) by $2(1 + x) z$ and integrating over $(0, L)$, we obtain

$$\frac{d}{dt} (1 + x, z^2)(t) + \sum_{j=1}^l (2j + 1) \|D^j z\|^2(t) + (D^j z(t, 0))^2$$

$$= 2(-v_1^k Dw, (1 + x)z)(t) + 2(-v_1^k - v_2^k) Dv_2, (1 + x)(t). \quad (4.14)$$

Making use of (2.1) and (4.4), we estimate for an arbitrary $\epsilon > 0$:

$$I_1 \leq \epsilon (1 + L) \|v_1^k Dw\|^2(t) + \frac{1}{\epsilon} (1 + x, z^2)(t)$$

$$\leq \epsilon (1 + L)(34)^k R^{2k} \|Dw\|^2(t) + \frac{1}{\epsilon} (1 + x, z^2)(t).$$

In order to estimate $I_2$, we need the following inequality: (See details in [22].)

$$|v_1^k - v_2^k| \leq k2^{(k-1)}(|v_1|^{(k-1)} + |v_2|^{(k-1)})|w|. \quad (4.15)$$
Then

$$I_2 \leq \epsilon(1 + L)\|v_k^1 - v_k^2\|Dv_2\|^2(t) + \frac{1}{\epsilon}(1 + x, z^2)(t)$$

$$\leq \epsilon(1 + L)k^22^{2k-1}\sum_{i=1}^{2}\|v_i\|_{\infty}^{2(k-1)}(t)(|w|^2, |Dv_2|^2)(t) + \frac{1}{\epsilon}(1 + x, z^2)(t)$$

$$\leq \epsilon(1 + L)k^22^{3k-1}(17)^{-k-2}\|w\|_{\infty}^2(t)\|Dv_2\|^2(t) + \frac{1}{\epsilon}(1 + x, z^2)(t)$$

$$\leq \epsilon(1 + L)9k^22^{3k}(17)^{-k-1}R^2\|w\|_{H_0^1}(t) + \frac{1}{\epsilon}(1 + x, z^2)(t).$$

Taking $$\epsilon = (16(1 + L)\beta_kR^{2k})^{-1}$$ with $$\beta_k = 9k^22^{3k}(17)^{-k-1} + (34)^k$$, we write (4.14) as

$$\frac{d}{dt}(1 + x, z^2)(t) + \sum_{j=1}^{l}(2j + 1)\|D^jz\|^2(t) + (D^l z(t, 0))^2$$

$$\leq (32)(1 + L)\beta_kR^{2k}(1 + x, z^2)(t) + \frac{1}{16}\|w\|_{H_0^1}^2(t). \quad (4.16)$$

By the Gronwall Lemma,

$$(1 + x, z^2)(t) \leq e^{(32)(1 + L)\beta_kR^{2k}T_0}(1 + x, z^2)(0) + \frac{1}{16} \int_0^{T_0}\|w\|_{H_0^1}^2(t)dt.$$

Choosing $$0 < T_5 \leq T_0$$ such that $$e^{(32)(1 + L)\beta_kR^{2k}T_5} \leq 2$$, we conclude

$$(1 + x, z^2)(t) \leq \frac{1}{8}\|w\|_{V}^2, \quad t \in (0, T_5]. \quad (4.17)$$

Substituting (4.17) into (4.16) and integrating over $$(0, T_5)$$, we find

$$3\int_0^{T_5}\sum_{j=1}^{l}\|D^jz\|^2(t)dt \leq 4(1 + L)\beta_kR^{2k}\|w\|_{V}^2T_5 + \frac{1}{16}\|w\|_{V}^2.$$

Taking $$0 < T_6 \leq T_5$$ such that $$4(1 + L)\beta_kR^{2k}T_6 \leq \frac{5}{16}$$, we get

$$\int_0^{T_6}\sum_{j=1}^{l}\|D^jz\|^2(t)dt \leq \frac{1}{3}\left(\frac{5}{16} + \frac{1}{16}\right)\|w\|_{V}^2 = \frac{1}{8}\|w\|_{V}^2. \quad (4.18)$$
**Estimate 4.** Differentiating (4.13) with respect to $t$, multiplying the result by $2(1 + x)z_t$ and integrating over $(0, L)$, we obtain

\[
\begin{align*}
\frac{d}{dt}(1 + x, z_t^2)(t) + \sum_{j=1}^l (2j + 1)\|D^j z_t\|^2(t) + (D^l z_t(t, 0))^2 & \\
\leq 2(-kv_1^{k-1}v_{1t} Dw, (1 + x)z_t)(t) + 2(-v_1^k Dw_t, (1 + x)z_t)(t) + \\
2(-kv_1^{k-1}v_{1t} Dv_2, (1 + x)z_t)(t) + 2(-kv_1^{k-1} - v_2^{k-1})v_{2t} Dv_2, (1 + x)z_t)(t) + \\
2(-(v_1^k - v_2^k)Dv_{2t}, (1 + x)z_t)(t) & .
\end{align*}
\]

(4.19)

Making use of (2.1), (4.4), (4.15), we estimate for an arbitrary $\epsilon > 0$:

\[
I_1 \leq \epsilon(1 + L)k2^{k+3}(17)^{k-1}R^{2k}\|Dw\|^2(t) + \frac{1}{\epsilon}(1 + x, z_t^2)(t) \\
+ \epsilon(1 + L)k2^k(17)^{k-1}R^{2k-2}\|D_{1t}\|^2(t)\|Dw\|^2(t),
\]

\[
I_2 \leq \epsilon(1 + L)(34)^k R^{2k}\|Dw_t\|^2(t) + \frac{1}{\epsilon}(1 + x, z_t^2)(t),
\]

\[
I_3 \leq \epsilon(1 + L)9k2^k(17)^{k-1}R^{2k}\|w_t\|^2_{H^1_0}(t) + \frac{1}{\epsilon}(1 + x, z_t^2)(t),
\]

\[
I_4 \leq \epsilon(1 + L)9k(k - 1)^22^3k(17)^{k-2}R^{2k}\|w\|^2_{H^1_0}(t) + \frac{1}{\epsilon}(1 + x, z_t^2)(t) \\
+ \epsilon(1 + L)9k(k - 1)^22^3(k-1)(17)^{k-2}R^{2k-2}\|Dv_{2t}\|^2(t)\|w\|^2_{H^1_0}(t),
\]

\[
I_5 \leq \epsilon(1 + L)k^22^3k^{-1}(17)^{k-1}R^{2k-2}\|Dv_{2t}\|^2(t)\|w\|^2_{H^1_0}(t) + \frac{1}{\epsilon}(1 + x, z_t^2)(t).
\]

Taking $\epsilon = (16(1 + L)\gamma_k R^{2k})^{-1}$, where

\[
\gamma_k = k2^k(17)^{k-1}(25 + 18(k - 1)^24^k(17)^{-1} + k^{-1}(17) + k2^{2k+2}),
\]
Substituting (4.21) into (4.20) and integrating over \((0, T]\), we find that

\[
(4.19) \text{ becomes }
\]

\[
\frac{d}{dt} (1 + x, z_t^2)(t) + \sum_{j=1}^{l} (2j + 1)\|D^j z_t\|^2(t) + (D^l z_t(t, 0))^2 \\
\leq (16\gamma_k)^{-1} (k^{2+3k} (17)^{k-1} + k2^k (17)^{k-1} R^{-2}\|Dv_{1t}\|^2(t)) \|Dw\|^2(t) \\
+ (16\gamma_k)^{-1} ((34)^k + 9k2^k (17)^{k-1}) \|w_t\|^2_{H^0_0}(t) \\
+ (16\gamma_k)^{-1} 9k(k-1)^{2} 2^{3k} (17)^{k-2} \|w\|^2_{H^0_0}(t) \\
+ (16\gamma_k)^{-1} 9k(k-1)^{2} 2^{3(k-1)} (17)^{k-2} R^{-2}\|Dv_{2t}\|^2(t) \|w\|^2_{H^0_0}(t) \\
+ (16\gamma_k)^{-1} k^{2} 2^{3k-1} (17)^{k-1} R^{-2}\|Dv_{2t}\|^2(t) \|w\|^2_{H^0_0}(t) \\
+ 5(16)(1 + L)\gamma_k R^{2k}(1 + x, z_t^2)(t).
\]

(4.20)

By the Gronwall Lemma,

\[
(1 + x, z_t^2)(t) \leq e^{5(16)(1 + L)\gamma_k R^{2k} T_0} ((16\gamma_k)^{-1} \gamma_k) \|w\|^2_{\mathcal{V}}.
\]

Choosing \(0 < T_7 \leq T_0\) such that \(e^{5(16)(1 + L)\gamma_k R^{2k} T_7} \leq 2\), we conclude

\[
(1 + x, z_t^2)(t) \leq \frac{1}{8} \|w\|^2_{\mathcal{V}}, \quad t \in (0, T_7].
\]

(4.21)

Substituting (4.21) into (4.20) and integrating over \((0, T_7]\), we find

\[
3 \int_0^{T_7} \sum_{j=1}^{l} \|D^j z_t\|^2(t) dt \leq (10)(1 + L)\gamma_k R^{2k} \|w\|^2_{\mathcal{V}} T_7 + \frac{1}{16} \|w\|^2_{\mathcal{V}}.
\]

Choosing \(0 < T_8 \leq T_7\) such that \((10)(1 + L)\gamma_k R^{2k} T_8 \leq \frac{5}{16}\), we obtain

\[
\int_0^{T_8} \sum_{j=1}^{l} \|D^j z_t\|^2(t) dt \leq \frac{1}{3} \left( \frac{5}{16} + \frac{1}{16} \right) \|w\|^2_{\mathcal{V}} = \frac{1}{8} \|w\|^2_{\mathcal{V}}.
\]

(4.22)

Taking \(T_s = \min\{T_0, T_8\}\) and making use of (4.17), (4.18), (4.21), (4.22), we find that \(\|z\|^2_{\mathcal{V}} \leq \frac{1}{2} \|w\|^2_{\mathcal{V}}\). Hence \(P\) is a contraction in \(B_R\) and Lemma 4.5 is thereby proved. \(\square\)

According to Lemmas 4.3 and 4.4 and the contraction principle, problem (3.1)-(3.3) has a unique generalized solution \(u = u(t, x)\):

\[
u, u_t \in L^\infty(0, T_s; L^2(0, L)) \cap L^2(0, T_s; H^0_0(0, L)).
\]

(4.23)

Write (3.1) in the form

\[
u + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} u = u - u_t - u^k Du = F(t, x).
\]

(4.24)
Since \( u, u_t \) satisfy (4.23), it follows that \( u^k D u \in H^1(0, T_*; L^2(0, L)) \), for all \( k \in \mathbb{N} \), hence \( F \in L^\infty(0, T_*; L^2(0, L)) \). Due to (4.2), we get
\[
    u \in L^\infty(0, T_*; H_0^1(0, L) \cap H^{2l+1}(0, L)).
\] (4.25)

On the other hand, \( u(t, \cdot) \in H^{2l+1}(0, L) \) implies \( D u(t, \cdot) \in H^l(0, L) \), for all \( l \in \mathbb{N} \). Then, there exists a constant \( K_* \) depending on \( l, L \), such that: (See [1], Theorem 4.39.)
\[
    \int_0^{T_*} \|u^k D u\|_{H^l(t)}^2 dt \leq K_* \int_0^{T_*} \|u\|_{H^l(t)}^2 \|D u\|_{H^l(t)}^2 dt 
\]
\[
    \leq K_* T_* \|u\|_{L^\infty(0, T_*; H^{2l+1}(0, L))}^{2(k+1)} < +\infty,
\]
therefore
\[
    u^k D u \in L^2(0, T_*; H^l(0, L)) \text{ for all } k \in \mathbb{N}.
\] (4.26)

Returning to (2.4) and taking into account (4.23), (4.26), we find
\[
    \sum_{j=1}^l (-1)^{j+1} D^{2j+1} u = -u_t - u^k D u \in L^2(0, T_*; H^l(0, L)).
\] (2.7)

Differentiating (4.27) \( l \) times with respect to \( x \), we obtain
\[
    \sum_{l<2j<2l} (-1)^{j+1} D^{2j+1} u = \sum_{2j\leq l} (-1)^{j+1} D^{(2j+1)+l} u - D^l[u_t + u^k D u].
\]

We estimate
\[
    \|D^{(2l+1)+l} u\|_t \leq \sum_{l<2j<2l} \|D^{(2j+1)+l} u\|_t + \sum_{2j\leq l} \|D^{(2j+1)+l} u\|_t + \|D^l[u_t + u^k D u]\|_t.
\] (2.8)

For \( l = 1 \), we have \( \sum_{l<2j<2l} \|D^{(2j+1)+l} u\|_t = 0 \). For \( l \geq 2 \), due to (2.4), there are constants \( A_i^l \), \( B_i^l \) \( (l < 2j < 2l) \) depending only on \( L, l \), such that
\[
    \|D^{(2j+1)+l} u\|_t \leq A_i^j \|D^{(2l+1)+l} u\|^{\alpha_i^j} \|u\|^{1-\alpha_i^j} \|t\| + A_i^j \|u\| \|t\,
\]
where \( \alpha_i^j = \frac{(2j+1)+l}{(2l+1)+l} \). Making use of the Young inequality with \( p^i = \frac{1}{\alpha_i^j} \), \( q^i = \frac{1}{1-\alpha_i^j} \) and arbitrary \( \epsilon > 0 \), we get
\[
    \|D^{(2j+1)+l} u\|_t \leq \epsilon \|D^{(2l+1)+l} u\|_t + C_j(\epsilon) \|u\|_t + A_i^j \|u\|_t.
\]
where \(C_j(\epsilon) = \left[q^j \left(\frac{p \chi}{(A_j)^p} \right) \frac{q_j}{p^j}\right]^{-1}\). Summing over \(l < 2j < 2l\), we find
\[
\sum_{l<2j<2l} \|D^{(2j)+l}u\|(t) \leq le\|D^{(2l+1)+l}u\|(t)
\]
\[+
\sum_{l<2j<2l} (C_j(\epsilon) + A^j_{2l})\|u\|(t). \tag{4.29}
\]
Substituting (4.29) into (4.28) and taking \(\epsilon = \frac{1}{2l}\), we get
\[
\|D^{(2l+1)+l}u\|^2(t) \leq C \left(\|u\|_{H^2(0,T_\ast)}^2(t) + \|u_t + u^kDu\|_{H^1(t)}^2\right), \tag{4.30}
\]
where \(C\) is a constant depending only on \(L, l\). By (4.23), (4.25), (4.26), it follows that \(D^{(2l+1)+l}u \in L^2(0,T_\ast;L^2(0,L))\). Again by (2.4), we obtain for the intermediate derivatives \(D^iu, i = (2l+1)+1, \ldots, (2l+1)+l-1:\)
\[
\|D^iu\|^2(t) \leq C_i \left(\|D^{(2l+1)+l}u\|^2(t) + \|u\|_{H^1(t)}^2\right), \tag{4.31}
\]
with the constants \(C_i\) depending only on \(L, l\). Substituting (4.30) into (4.31) and taking into account (4.23), (4.25), (4.26), we conclude
\[
u \in L^2(0,T_\ast;H^{(2l+1)+l}(0,L)). \tag{4.32}
\]
Combining (4.23), (4.25), (4.32), we complete the proof of Theorem 4.3.

**Conclusions.** Making use of the Contraction principle, we obtain local existence and uniqueness of a regular solution for all \(k \in \mathbb{N}\). We must mention a smoothing effect, first proved in [15] for the KdV equation. Roughly speaking, we proved that if \(u_0 \in H^{2l+1}(0,L)\), then \(u(t, \cdot) \in H^{(2l+1)+l}(0,L), t > 0\).

**References**

[1] R. Adams, J. Fournier, Sobolev Spaces, Second Edition (2003), Elsevier Science Ltd.

[2] Araruna F. D., Capistrano-Filho R. A., Doronin G. G.: Energy decay for the modified Kawahara equation posed in a bounded domain. J. Math. Anal. Appl. 385, 743–756 (2012).

[3] Biagioni H. A., Linares F: On the Benney - Lin and Kawahara equations. J. Math. Anal. Appl. 211, 131–152 (1997).

[4] J. Ceballos, M. Sepulveda, O. Villagran; The Korteweg-de Vries- Kawahara equation in a bounded domain and some numerical results, Appl. Math. Comput., 190 (2007) 912–936.

[5] Doronin G. G., Larkin N. A.: Kawahara equation in a bounded domain. Discrete Contin. Dynam. Syst., Ser B 10 (2008), 783-799.

[6] A.V. Faminskii, Cauchy problem for quasilinear equation of odd order, Mat. Sb., 180 (1989), 1183-1210. Transl. in Math. USSR-Sb., 68 (1991), 31-59. 1979.
[7] Faminskii A. V., Larkin N. A.: Initial-boundary value problems for quasilinear dispersive equations posed on a bounded interval. Electron. J. Differ. Equations. 1–20 (2010).

[8] Farah L. G., Linares F., Pastor A.: The supercritical generalized KDV equation: global well-posedness in the energy space and below. Math. Res. Lett. 18, no. 02, 357–377 (2011).

[9] G. Fonseca, F. Linares, G. Ponce, Global well-posedness for the modified Korteweg-de Vries equation, Comm. Part. Diff. Equats, 24 (3,4) (1999), 683-705.

[10] G. Fonseca, F. Linares, G. Ponce, Global existence for the critical generalized KDV equation, Proc. of the AMS, vol. 131, Number 6 (2002), 1847-1855.

[11] Huo Z., Jia Y.: Well-posedness for the fifth-order shallow water equations. Journal of Differential Equations. 246, 2448–2467 (2009).

[12] P. Isaza, F. Linares and G. Ponce, Decay properties for solutions of fifth order nonlinear dispersive equations, J Differ Equats. 258 (2015) 764–795.

[13] Jeffrey A. and Kakutani T.: Weak nonlinear dispersive waves: a discussion centered around the Korteweg-de Vries equation. SIAM Review, vol 14 no 4, pp. 582–643 (1972).

[14] Kakutani T. and Ono H.: Weak non linear hydromagnetic waves in a cold collision free plasma. J. Phys. Soc. Japan, 26, 1305–1318 (1969).

[15] Kato T.: On the Cauchy problem for the (generalized) Korteweg-de Vries equations. Advances in Mathematics Supplementary Studies, Stud. Appl. Math. 8, 92–128 (1983).

[16] Kawahara T.: Oscillatory solitary waves in dispersive media. J. Phys. Soc. Japan. 33, 260–264 (1972).

[17] Kenig C. E., Ponce G. and Vega L.: Higher-order nonlinear dispersive equations, Proc. Amer. Math. Soc., 122 (1), 157–166 (1994).

[18] Kenig C.E., Ponce G. and Vega L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation and the contraction principle. Commun. Pure Appl. Math. 46 No 4, 527–620 (1993).

[19] Kuvshinov R. V., Faminskii A. V.: A mixed problem in a half-strip for the Kawahara equation. (Russian) Differ. Uravn. 45, N. 3, 391-402 (2009); translation in Differ. Equ. 45 N. 3, 404–415 (2009).

[20] N. A. Larkin; Correct initial boundary value problems for dispersive equations, J. Math. Anal. Appl. 344 (2008) 1079–1092.

[21] Larkin N. A.: Korteweg-de Vries and Kuramoto-Sivashinsky Equations in Bounded Domains. J. Math. Anal. Appl. 297, 169–185 (2004).

[22] N. A. Larkin and J. Luchesi, Higher-order stationary dispersive equations on bounded intervals: a relation between the order of an equation and the growth of its convective term. arXiv:1806.08803v1 [math.AP] 22 Jun 2018.

[23] Larkin N. A., Simões M. H.: The Kawahara equation on bounded intervals and on a half-line. Nonlinear Analysis. 127, pp. 397–412 (2015).

[24] F. Linares and A. Pazoto; On the exponential decay of the critical generalized Korteweg-de Vries equation with localized damping. Proc. Amer. Math. Soc., 135, 1 (2007) 1515-1522.

[25] Y. Martel and F. Merle; Instability of solutions for the critical generalized Korteweg-de Vries equation. Geometrical and Funct. Analysis, 11 (2001) 74-123.
[26] F. Merle; *Existence of blow up solutions in the energy space for the critical generalized KdV equation.* J. Amer. Math. Soc., 14 (2001) 555-578.

[27] Nirenberg L: *On elliptic partial differential equations,* Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3 serie, tome 13, n 2 (1959), p. 115-162.

[28] Pilod D.: *On the Cauchy problem for higher-order nonlinear dispersive equations.* Journal of Differential Equations. 245, 2055–2077 (2008).

[29] L. Rosier and Bing-Yu Zhang, Global stabilization of the generalized Korteweg-de Vries equation on a finite domain, SIAM J. Control Optim. vol. 45, No. 3 (2006), 927-956.

[30] K. Sangare and A.V. Faminskii, Weak solutions of a mixed problem in a half-strip for a generalized Kawahara equation, Mathematical Notes, vol. 85 (2009), 90-100.

[31] Saut J.-C.: *Sur quelques généralisations de l’équation de Korteweg-de Vries.* J. Math. Pures Appl. 58, 21–61 (1979).

[32] Tao S. P., Cui S.B., *The local and global existence of the solution of the Cauchy problem for the seven-order nonlinear equation,* Acta Mathematica Sinica, 25 A, (4) 451-460 (2005).

[33] Zheng S.: *Nonlinear evolution equations,* Chapman Hill/CRC, 2004.

Nikolai A. Larkin
DEPARTAMENTO DE Matemática, UNIVERSIDADE ESTADUAL DE MARINGÁ, AV. COLOMBO 5790: AGÊNCIA UEM, 87020-900, MARINGÁ, PR, BRAZIL
E-mail address: nlarkine@uem.br

Jackson Luchesi
DEPARTAMENTO DE Matemática, UNIVERSIDADE TECNOLÓGICA FEDERAL DO PARANÁ - CÂMPUS PATO BRANCO, VIA DO CONHECIMENTO Km 1, 85503-390, PATO BRANCO, PR, BRAZIL
E-mail address: jacksonluchesi@utfpr.edu.br