Calculations of \( \eta \)-nuclear quasi-bound states in few-body systems

M. Schäfer\(^1,2\)\(^*\), N. Barnea\(^3\), E. Friedman\(^3\), A. Gal\(^3\), and J. Mareš\(^2\)

\( ^1 \)Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Břehová 7, 11519 Prague 1, Czech Republic
\( ^2 \)Nuclear Physics Institute, Academy of Sciences of Czech Republic, 250 69 Řež, Czech Republic
\( ^3 \)Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

Abstract. We report on our Stochastic Variational Method (SVM) calculations of \( \eta \)-nuclear quasi-bound states in s-shell nuclei as well as the very recent calculation of the p-shell nucleus \( ^6 \)Li. The \( \eta N \) potentials used were constructed from \( \eta N \) scattering amplitudes obtained within coupled-channel models that incorporate \( N^* (1535) \) resonance. We found that \( \eta^3 \)Li is bound in the \( \eta N \) interaction models that yield \( \text{Re} a_{\eta N} \geq 0.67 \) fm. Additional repulsion caused by the imaginary part of \( \eta N \) potentials shifts the onset of \( \eta \)-nuclear binding to \( \eta^4 \)He, yielding very likely no quasi-bound state in \( \eta^3 \)He.

1 Introduction

The current status of our theoretical studies of \( \eta \)-nuclear quasi-bound states, including discussion of the self-consistent treatment of the strong energy dependence of \( \eta N \) scattering amplitudes derived from coupled-channel meson-baryon interaction models have been discussed thoroughly in Refs. [1–3]. So far, few-body calculations of \( \eta \)-nuclear quasi-bound states have been restricted to s-shell nuclei up to \( \eta^4 \)He. In this contribution, we present our first SVM calculation of the \( \eta \)-nuclear quasi-bound state in the p-shell nuclear system \( \eta^6 \)Li, taking into account all possible spin-isospin configurations. Moreover, we focus on the effect of the imaginary part of the complex \( V_{\eta N} \) potential on the \( \eta \) binding energy \( B_\eta \). We show that the effect could be considerable in light \( \eta \)-nuclear systems and must be taken into account in the study of the onset of \( \eta \)-nuclear binding.

2 Theoretical approach

Properties of \( \eta \)-nuclear quasi-bound states are studied within the SVM with a correlated Gaussian basis [4]. This approach was successfully applied in our previous calculations of s-shell \( \eta \)-nuclei and proved itself as highly accurate method with straightforward extension to systems with the number of particles \( N \geq 5 \).

The wave function of an \( \eta \)-nuclear system with orbital momentum \( L = 0 \) is expanded as a linear combination of correlated Gaussians

\(* e-mail: m.schafer@ujf.cas.cz*
where $\chi^k_{SA}$ are corresponding spin (isospin) parts of a given spin (isospin) configuration. The matrix $A_k$ is symmetric positive definite and includes $N(N - 1)/2$ variational parameters. The SVM optimizes the variational basis step-by-step in a random trial and error procedure (details can be found in Ref. [5]).

SVM calculations of $\eta$-nuclear quasi-bound states in p-shell nuclei represent a rather challenging task. First, the computational complexity scales with $N!$, second, the amount of different spin-isospin configurations starts to increase quite rapidly. Preliminary results [6] showed that taking into account only one configuration underestimated binding of the nuclear core $^6\text{Li}$ by approximately 1.8 MeV. This led to development of a new high-performance SVM code which was used in the very recent fully self-consistent calculation of $\eta^6\text{Li}$, taking into account all possible spin-isospin configurations.

In our study of $\eta$-nuclear quasi-bound states we use the Minnesota NN central potential [7] which reproduces well properties of the ground states of s-shell and light p-shell nuclei. The interaction of the $\eta$ meson with nucleons is described by a complex two-body potential and is taken according to [1] as

$$V_{\eta N}(\delta \sqrt{s}, r) = -\frac{4\pi}{2\mu_{\eta N}} b(\delta \sqrt{s}) \rho_N(r), \quad \rho_N(r) = \left(\frac{\Lambda}{2} \sqrt{\pi}\right)^3 \exp\left(-\frac{\Lambda^2 r^2}{4}\right),$$

where $\mu_{\eta N}$ stands for the $\eta N$ reduced mass, $\delta \sqrt{s} = \sqrt{s} - \sqrt{s_{\eta}}$ is the energy shift with respect to the $\eta N$ threshold, $\Lambda$ is a scale parameter which is inversely proportional to the range of $V_{\eta N}$, and $b(\delta \sqrt{s})$ is an energy dependent complex amplitude.

The value of $\Lambda$ is connected to EFT momentum cut-off; its upper bound corresponds to vector-meson exchange $\Lambda \leq 3.9\text{ fm}^{-1}$ or more restrictively to $\Lambda \leq 3.0\text{ fm}^{-1}$ excluding $\rho N$ channel from dynamical generation of the $N^{*}(1535)$ resonance [3].

For given $\Lambda$, $b(\delta \sqrt{s})$ is fitted to the phase shifts derived from subthreshold $\delta \sqrt{s} < 0$ scattering amplitude of the corresponding $\eta N$ interaction model. See Ref. [3] for details.

The energy dependence of $V_{\eta N}$ is treated self-consistently: we search for a SVM solution that fulfills $\delta \sqrt{s_{\eta}} = \langle \delta \sqrt{s}\rangle$ where $\delta \sqrt{s}$ enters $V_{\eta N}$ and $\langle \delta \sqrt{s}\rangle$ is obtained from the SVM solution for a given value of $\delta \sqrt{s}$ [3]:

$$\langle \delta \sqrt{s}\rangle = -\frac{B}{A} - \xi_N \frac{1}{A} \langle T_N \rangle + \frac{A - 1}{A} E_{\eta} - \xi_A \xi_N \left(\frac{A - 1}{A}\right)^2 \langle T_{\eta}\rangle,$$

where $B$ is the total binding energy, $T_N$ ($T_{\eta}$) denotes the kinetic energy of nucleons ($\eta$), and $A$ is the number of nucleons. The energy $E_{\eta} = \langle \psi|H - H_N|\psi\rangle$ where $H_N$ is Hamiltonian of the nuclear core, $\xi_N = m_{N(\eta)}/(m_N + m_\eta)$, and $\xi_A = A m_N/(A m_N + m_\eta)$.

The imaginary part of $V_{\eta N}$ is significantly smaller than its real part. This allows to calculate the width $\Gamma_{\eta}$ perturbatively [1]. The SVM $\eta$-nuclear calculations are thus performed only for the real part of the $\eta N$ potential and $\Gamma_{\eta}$ is evaluated using the expression

$$\Gamma_{\eta} = -2 \langle \Psi_{g.s.}|\text{Im} V_{\eta N}|\Psi_{g.s.}\rangle,$$
where $|\Psi_{gs,\eta}\rangle$ is the SVM solution for the $\eta$-nuclear ground state corresponding to $\text{Re}V_{\eta N}$. Another possible way how to calculate $\Gamma_\eta$ is to solve a generalized eigenvalue problem for complex Hamiltonian (including $\text{Im}V_{\eta N}$) using variationally determined SVM basis states for $\text{Re}V_{\eta N}$. This approach, already used in SVM calculations of kaonic nuclei [10], yields complex eigenenergy of the ground state $E = \text{Re}(E) + i\text{Im}(E)$ and consequently the width as $\Gamma_\eta = -2\text{Im}(E)$. This method takes into account the effect of the non-zero imaginary part of $V_{\eta N}$ on the $\eta$ binding energy. Namely, $\text{Im}V_{\eta N}$ acts as repulsion and thus makes the $\eta$ meson less bound in the nucleus.

3 Results

Results of our SVM calculations of the $\eta$ binding energies $B_\eta$ and widths $\Gamma_\eta$ in $\eta^3\text{He}$, $\eta^4\text{He}$, and $\eta^6\text{Li}$ are summarized in Fig. 1. The calculations were performed using the GW and CS models and the parameter $\Lambda = 2$ and $4 \text{ fm}^{-1}$. In the GW model, $\eta^6\text{Li}$ is rather comfortably bound for both values of $\Lambda$. On the other hand, the CS model yields $\eta$-nuclear quasi-bound state only for $\Lambda = 4 \text{ fm}^{-1}$, with $B_\eta = 0.68 \text{ MeV}$. 

![Figure 1](image_url)

**Figure 1.** SVM calculations of the binding energy $B_\eta$ and width $\Gamma_\eta$ in $\eta^3\text{He}$, $\eta^4\text{He}$, and $\eta^6\text{Li}$ using the Minnesota $NN$ potential with Coulomb force included and two $\eta N$ interaction models - GW and CS.

| $\eta^3\text{He}$ | $B_\eta$ [MeV] | $\Gamma_\eta$ [MeV] | $\delta \sqrt{s_{nc}}$ [MeV] |
|------------------|----------------|---------------------|---------------------|
| $\Lambda = 2 \text{ fm}^{-1}$ (Eq. 4) | 0.11 | 1.37 | -9.23 |
| $\Lambda = 2 \text{ fm}^{-1}$ (cmplx) | -0.25 | 1.32 | -8.87 |
| $\Lambda = 4 \text{ fm}^{-1}$ (Eq. 4) | 1.01 | 3.32 | -13.18 |
| $\Lambda = 4 \text{ fm}^{-1}$ (cmplx) | 0.36 | 3.44 | -12.72 |

| $\eta^4\text{He}$ | $B_\eta$ [MeV] | $\Gamma_\eta$ [MeV] | $\delta \sqrt{s_{nc}}$ [MeV] |
|------------------|----------------|---------------------|---------------------|
| $\Lambda = 2 \text{ fm}^{-1}$ (Eq. 4) | 0.97 | 2.17 | -19.64 |
| $\Lambda = 2 \text{ fm}^{-1}$ (cmplx) | 0.77 | 2.22 | -19.50 |
| $\Lambda = 4 \text{ fm}^{-1}$ (Eq. 4) | 4.62 | 4.38 | -29.73 |
| $\Lambda = 4 \text{ fm}^{-1}$ (cmplx) | 4.40 | 4.41 | -29.60 |
In Table 1, we compare two approaches to evaluation of the width $\Gamma_\eta$ introduced in the previous section: the mean-value approach (Eq. 4) and the complex eigenvalue problem (cmplx) approach. Calculations of $\eta^3$He and $\eta^4$He were performed within the GW model with $\Lambda = 2$ and 4 fm$^{-1}$. It is apparent that the effect of the imaginary part of $V_{\eta N}$ on $B_\eta$, which is included in the cmplx approach, is quite significant in $\eta^3$He (with $\delta \sqrt{s}$ close to threshold) and decreases in $\eta^4$He with larger energy shift with respect to threshold. For the CS model (not shown in the table) the $\eta^3$He is not bound while in $\eta^4$He the effect of $\text{Im} V_{\eta N}$ is smaller (few tens of keV) due to the lower value of $\text{Im} V_{\eta N}$ than in the GW model. Table 1 illustrates that the size of the changes of $B_\eta$ caused by $\text{Im} V_{\eta N}$ decreases with the magnitude of the subthreshold energy shift $\delta \sqrt{s}$. Namely, the strength of $\text{Im} V_{\eta N}$ has for both CS and GW interaction models maximum close to threshold and decreases with $\sqrt{s}$, as shown in Figure 2 of Ref. [3]. Moreover, the cmplx method confirms the estimate of $\Gamma_\eta$ within the mean-value approach (Eq. 4), giving practically the same widths in all considered cases.

4 Summary

We performed few-body calculations of $\eta$-nuclear quasi-bound states in s-shell nuclei as well as in the p-shell nucleus $^6$Li within our newly developed high-performance SVM code. We considered the Minnesota $NN$ potential and two $\eta N$ interaction models - GW and CS. Calculations of $\eta^6$Li within the GW model yield the binding energy $B_\eta$ and corresponding width consistent with previous RMF calculations [11]. The CS model gives quasi-bound state only for $\Lambda = 4$ fm$^{-1}$. This suggests that to bind $\eta^6$Li, the real part of the $\eta N$ scattering length should be greater than $\text{Re} a_{\eta N} = 0.67$ fm, predicted by the CS model.

Next, we repeated our previous study of the onset of $\eta$-nuclear binding in He isotopes taking into account the effect of $\text{Im} V_{\eta N}$ on the binding energy $B_{\eta}$. We observed considerable decrease of $B_{\eta}$ in $^3\eta$He and rather negligible effects in $^4\eta$He as well as in $^6$Li. The $\eta$ meson is barely bound in $^3\eta$He even for the larger value of the cut-off parameter $\Lambda = 4$ fm$^{-1}$. This indicates that in order to study the $\eta^3$He system, one has to explore the resonance region as well, e.g., using the complex rotation method [12].

M. Schäfer acknowledges financial support from the CTU-SGS Grant No. SGS16/243/OHK4/3T/14 and from the organizers of the MESON2018 conference.

References

[1] N. Barnea, E. Friedman, A. Gal, Phys. Lett. B 747, 345 (2015)
[2] N. Barnea, B. Bazak, E. Friedman, A. Gal, Phys. Lett. B 771, 297 (2017); Erratum in Phys. Lett. B 775, 364 (2017)
[3] N. Barnea, E. Friedman, A. Gal, Nucl. Phys. A 968, 35 (2017)
[4] Y. Suzuki, K. Varga, Phys. Rev. C 52, 2885 (1995)
[5] Y. Suzuki, K. Varga, Stochastic Variational Approach to Quantum-Mechanical Few-Body Problems (Springer, Heidelberg, 1998)
[6] A. Cieplý, et al., PoS Hadron2017, 203 (2017)
[7] D. R. Thompson, M. LeMere, Y. C. Tang, Nucl. Phys. A 286, 53 (1977)
[8] A. M. Green, S. Wycech, Phys. Rev. C 71, 014001 (2005)
[9] A. Cieplý, J. Smejkal, Nucl. Phys. A 919, 334 (2013)
[10] S. Ohnishi et al., Phys. Rev. C 95, 065202 (2017)
[11] A. Cieplý, E. Friedman, A. Gal, J. Mareš, Nucl. Phys. A 925, 126 (2014)
[12] J. Nutall, H. L. Cohen, Phys. Rev. 188, 1542 (1969)