HYPERBOLIC DISTRIBUTION PROBLEMS
ON SIEGEL 3-FOLDS
AND HILBERT MODULAR VARIETIES

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Abstract. We generalize to Hilbert modular varieties of arbitrary dimension the work of W. Duke [14] on the equidistribution of Heegner points and of primitive positively oriented closed geodesics in the Poincaré upper half plane, subject to certain subconvexity results. We also prove vanishing results for limits of cuspidal Weyl sums associated with analogous problems for the Siegel upper half space of degree 2. In particular, these Weyl sums are associated with families of Humbert surfaces in Siegel 3-folds and of modular curves in these Humbert surfaces.

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1. INTRODUCTION

In this paper, we use the Maass correspondence for special orthogonal groups $\text{SO}(p,q)$, with integers $p, q \geq 1$, $p + q = m$, together with “accidental” isomorphisms between these groups and certain modular groups in the case $m = 3, 4, 5$, to derive explicit formulae expressing cuspidal Weyl sums, associated to hyperbolic distribution problems in Siegel 3-folds and

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Hilbert modular varieties, in terms of Fourier coefficients for Maass and Hilbert-Maass forms of half-integral weight. The case $m = 3$, $(p, q) = (2, 1)$ with base field $\mathbb{Q}$ was studied in [14]. Convexity and sub-convexity results for these Fourier coefficients, combined with an analogous treatment of the eigenfunctions for the continuous part of the spectrum of the Laplace-Beltrami operator, imply the equidistribution properties stated in this section.

Let $Q$ be a non-degenerate integral quadratic form whose signature over $\mathbb{R}$ is $(p, q)$, $p + q = m$, $pq \neq 0$. Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and

$$W_\lambda = \{x \in \mathbb{R}^m : Q(x) = \lambda\}.$$

The group $G = \Omega(Q)$ of $m \times m$ matrices leaving $Q$ invariant is isomorphic to $\text{SO}(p, q)$ and $G(\mathbb{R})$ acts transitively on $W_\lambda$. The stabilizer of any $x \in W_\lambda$ is isomorphic to $\text{SO}(p - 1, q)$ if $\lambda > 0$ and to $\text{SO}(p, q - 1)$ if $\lambda < 0$. A choice of Haar measure on $G(\mathbb{R})$ determines an invariant volume form on the majorant space $\mathcal{H}_Q$ (see §2). Let $\Delta_Q$ be the Laplace-Beltrami operator on $\mathcal{H}_Q$ and $\Gamma$ an arithmetic subgroup of $G$, given by a congruence subgroup of a unit group of $Q$. In [29], Maass constructed a $\theta$-lift on the space of $\Gamma$-invariant $L^2$-integrable functions on $\mathcal{H}_Q$. This theta-lift converges on the $\Delta_Q$-eigenfunctions for the discrete spectrum and has image a corresponding Maass cusp form of half integral weight (see Proposition 2.1).

As mentioned briefly in [14], p84, the classical Maass correspondence in the cases $m = 4$, $m = 5$ leads to the study of other modular groups not treated in that paper. These modular groups arise from considering families of polarized abelian varieties of complex dimension 2. Recall that the complex points $V(\mathbb{C})$ of the Siegel 3-fold can be realized as the quotient of the Siegel upper half space $\mathcal{H}_2$ of degree 2 by the integer points $\text{Sp}(4, \mathbb{Z})$ of the symplectic group in real dimension 4,

$$V(\mathbb{C}) \simeq \text{Sp}(4, \mathbb{Z}) \backslash \mathcal{H}_2.$$

The underlying variety $V$ has the structure of a quasi-projective variety defined over $\mathbb{Q}$. Here,

$$(1.1) \quad \mathcal{H}_2 = \{z \in M_2(\mathbb{C}) : z = z^t, \text{Im}(z) > 0\}$$

and,

$$(1.2) \quad \text{Sp}(4, \mathbb{R}) = \{g \in \text{GL}_4(\mathbb{R}) : g^t J g = J\},$$

where

$$(1.3) \quad J = \begin{pmatrix} 0 & -12 \\ 12 & 0 \end{pmatrix}.$$
The projective symplectic group

\[ \text{PSp}(4, \mathbb{R}) = \text{Sp}(4, \mathbb{R}) / \{ \pm \text{Id}_2 \} \]

is the full group of analytic automorphisms of the complex domain \( \mathcal{H}_2 \). The matrix \( J \) defines a complex structure and a symplectic form \( E \) on \( \mathbb{R}^4 \).

To every point \( z \in \mathcal{H}_2 \) we can associate the complex torus

\[ A = \mathbb{C}^2 / \mathbb{Z}^2 + z \mathbb{Z}^2 \]

where \( L = \mathbb{Z}^2 + z \mathbb{Z}^2 \), and the Riemann form \( E \) determines an \( \mathbb{R} \)-linear non-degenerate alternating form on \( \mathbb{C}^2 \times \mathbb{C}^2 \) taking integer values on \( L \times L \) which gives a principal polarization of \( A \). The complex torus \( A \) has the structure of an abelian surface. In fact, the points of the complex variety \( V(\mathbb{C}) \) correspond bijectively with the complex isomorphism classes of principally polarized abelian surfaces.

For an abelian variety \( A \), we let \( \text{End}(A) \) be the endomorphism ring and we put \( \text{End}_0(A) = \text{End}(A) \otimes \mathbb{Z} \mathbb{Q} \). If \( A \) is simple, then \( \text{End}_0(A) \) is a division algebra over \( \mathbb{Q} \) with a positive involution induced by the polarization of \( A \).

Albert [1, 2, 3] classified the division algebras over \( \mathbb{Q} \) with positive involution. For the case of abelian surfaces \( \text{dim}(A) = 2 \), this gives the following (see for example [55] Proposition (1.2)).

**Proposition 1.1.** If \( A \) is a simple abelian surface then \( \text{End}(A) \) is one of the following:

1. the ring \( \mathbb{Z} \),
2. an order in a real quadratic field \( F \),
3. an order in an indefinite quaternion division algebra \( \mathbb{Q} \) over \( \mathbb{Q} \),
4. an order in a quartic CM field \( K \) (totally imaginary quadratic extension of a real quadratic field).

We restrict ourselves to families of simple abelian surfaces, the non-simple case being essentially covered by [14], as then the abelian surface is isogenous to a product of elliptic curves. In Proposition 1.1, case (1) is the generic case which will also not interest us here. Case (2) leads to considering families of Hilbert modular surfaces, case (3) to families of modular curves and case (4) to families of CM points. Case (4) will appear as a special case of Theorem 1.2.

The case \( m = 5 \), \( (p,q) = (3,2) \), together with a natural isomorphism between \( \text{Sp}(4, \mathbb{R}) \) and \( \text{SO}(3,2) \) enables us to apply the results of [2] to \( \mathcal{H}_2 \) acted on by the lattice \( \text{Sp}(4, \mathbb{Z}) \). Let \( \mathcal{R}_1 \) be a fundamental domain for the action of \( \text{Sp}(4, \mathbb{Z}) \) on \( \mathcal{H}_2 \) and normalize the invariant volume form \( \omega_1 \) on \( \mathcal{H}_2 \) so that \( \mathcal{R}_1 \) has volume \( \omega_1(\mathcal{R}_1) = 1 \). We describe in [2] sub-domains \( S_d \), \( d > 0 \) and \( E_d \), \( d < 0 \) arising from \( \lambda = d \) square-free with \( d \equiv 1 \mod 4 \). The complex surfaces \( S_d \) are related to case (2) in that they are the Humbert surfaces and parameterize abelian surfaces whose endomorphism ring contains an order in the real quadratic field \( F = \mathbb{Q}(\sqrt{d}) \). On the other hand \( E_d \) has the structure of a real variety of real dimension 3.
The case \( m = 4 \), \((p,q) = (2,2)\), together with a natural isomorphism between \( \text{SL}(2,F) \otimes \mathbb{R} \), where \( F = \mathbb{Q}(\sqrt{d}) \), and \( \text{SO}(2,2) \) leads to considering \( \mathcal{H}^2 \) acted on by the lattice \( \text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) \), where \( \mathcal{O} \) is the ring of integers of \( F \). Let \( \mathcal{R}_2 \) be a fundamental domain for the actions of \( \text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) \) on \( \mathcal{H}^2 \) and normalize the invariant volume form \( \omega_2 \) on \( \mathcal{H}^2 \) so that \( \omega_2(\mathcal{R}_2) = 1 \).

We describe in §4 subdomains \( X_{d,n} \), \( n > 0 \) arising from \( \lambda = n \) square-free with \( n \equiv N(\alpha)N(\mathcal{O}^\vee) \), some \( \alpha \in F \) (the case \( n < 0 \) leads to nothing new as \( \text{SO}(2,1) \simeq \text{SO}(1,2) \)). The complex curves \( X_{d,n} \) are related to case (3) in that they parameterize abelian surfaces whose endomorphism ring contains an order in the indefinite quaternion algebra \( \mathcal{Q}_{d,n} \) over \( \mathbb{Q} \) with parameters \( (d,-n/\delta d) \), for a certain \( \delta \in F \).

In §5 we derive in Proposition 5.1 vanishing results for limits of the cuspidal Weyl sums on the side of the orthogonal groups in the above situations, which then apply to the modular side by the discussion of §4. Although we apply our results to families of principally polarized abelian varieties, the same arguments go through without this polarization assumption. In this paper, we do not explore in these same situations the analytically more involved question of how to modify the classical Maass correspondence for the eigenfunctions of the continuous spectrum of \( \Delta_Q \). Alternatively, one may derive directly in the case \( m = 4,5 \) upper bounds for Weyl sums for eigenfunctions for the continuous spectrum, that is the analogues of the results for \( m = 3 \) of our §7. We hope to return to this in a later paper. In general, the cuspidal case we treat here is arithmetically more interesting. Together, such results give the following.

**Equidistribution in genus 2:**

(i) The family \( \{S_d\}_{d>0} \) of Humbert surfaces and the family \( \{E_d\}_{d<0} \) of real 3-folds, where \( d \) is square-free and \( d \equiv 1 \) mod 4, are equidistributed in \( \text{Sp}(4,\mathbb{Z}) \backslash \mathcal{H}_2 \). Namely, if \( \Omega_1 \) is a convex region with smooth boundary in \( \mathcal{R}_1 \) we have

\[
\lim_{d \rightarrow \infty} \frac{\text{Vol}(S_d \cap \Omega_1)}{\text{Vol}(S_d)} = \omega_1(\Omega_1),
\]

\[
\lim_{-d \rightarrow \infty} \frac{\text{Vol}(E_d \cap \Omega_1)}{\text{Vol}(E_d)} = \omega_1(\Omega_1),
\]

(1.4)

(ii) Let \( d \) be a positive square-free integer and \( \mathcal{O} \) the ring of integers of \( \mathbb{Q}(\sqrt{d}) \). The family \( \{X_{d,n}\}_{n>0} \) of modular curves, where \( n \) is congruent mod \( d \) to the norm of an ideal in the same class as the inverse different \( \mathcal{O}^\vee \) of \( \mathbb{Q}(\sqrt{d}) \), is equidistributed in \( \text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) \backslash \mathcal{H}^2 \). Namely, if \( \Omega_2 \) is a convex region with smooth boundary in \( \mathcal{R}_2 \) we have

\[
\lim_{n \rightarrow \infty} \frac{\text{Vol}(X_{d,n} \cap \Omega_2)}{\text{Vol}(X_{d,n})} = \omega_2(\Omega_2)
\]

(1.5)
The case $m = 3$, $(p, q) = (2, 1)$, together with a natural isomorphism between $\text{SL}(2, F) \otimes \mathbb{Q} \mathbb{R}$, where $F$ is a totally real field of degree $g$ over $\mathbb{Q}$, and $\text{SL}(2, \mathbb{R})^g$ leads to considering $\mathcal{H}^g$ acted on by the lattice $\text{SL}(\mathcal{O} \oplus \mathcal{A})$ where $\mathcal{A}$ is a fractional ideal in $F$. The case $g = 1$ was treated in [14]. However, for the case $g > 1$, we need to adapt the classical Maass correspondence to the Hilbert modular situation (see §3) and we need new (as yet unproven) subconvexity results (see §8), hence our assumption of the generalized Riemann hypothesis (GRH) in Theorem 1.2. In order to treat the continuous spectrum, in §7 we study directly the corresponding Eisenstein Weyl sums.

Let’s recall the associated families of abelian varieties. Let $A$ be a $g$-dimensional complex torus where now $g \geq 1$. Let $F$ be a totally real field with $[F : \mathbb{Q}] = g$. Then $A$ has real multiplication (RM) if $\text{End}(A)$ contains an order $\mathcal{O}$ in $F$. Such a complex torus always has the structure of an abelian variety (see [55], p207). We assume throughout that $\mathcal{O}$ is the ring of integers of $F$ and we denote the inverse different by $\mathcal{O}^\vee$. Let $A$ be a fractional ideal of $F$.

The group

$$\text{SL}(\mathcal{O} \oplus \mathcal{A}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, F) : \alpha, \delta \in \mathcal{O}, \beta \in \mathcal{A}^{-1}, \gamma \in \mathcal{A} \right\},$$

acts by fractional linear transformations on $\mathcal{H}^g$, via the $g$ Galois embeddings of $F$ into $\mathbb{R}$, and induces an action of $\text{PSL}(\mathcal{O} \oplus \mathcal{A}) = \text{SL}(\mathcal{O} \oplus \mathcal{A})/\{\pm \text{Id}_2\}$. The quotient space

$$\text{PSL}(\mathcal{O} \oplus \mathcal{A}) \backslash \mathcal{H}^g$$

is called a Hilbert modular variety and corresponds bijectively to the complex isomorphism classes of polarized $g$-dimensional abelian varieties $A$ with RM by $\mathcal{O}$. In particular, as a complex torus we may write

$$A(\mathbb{C}) = \mathbb{C}^g/(\mathcal{A} + z.\mathcal{O})$$

for some $z \in \mathcal{H}^g$ with $A + z.\mathcal{O}$ embedded in $\mathbb{C}^g$ using the Galois embedding of $F$ into $\mathbb{R}^g$. The abelian variety $A$ is principally polarized when $\mathcal{A} = \mathcal{O}^\vee$.

When $g = 2$, we recover an example of case (2) in Proposition 1.1. There is a natural modular embedding

$$\text{PSL}(\mathcal{O} \oplus \mathcal{O}^\vee) \backslash \mathcal{H}^2 \to \text{Sp}(4, \mathbb{Z}) \backslash \mathcal{H}_2,$$

which is described in detail for arbitrary $g$ in [55], Chapter IX, so that the Hilbert modular surfaces can be viewed as subsurfaces of the Siegel 3-fold. In this context, they are referred to as Humbert surfaces.

We consider in §10 the situation arising from $\lambda = \Delta \in \mathcal{O}$, $\Delta \neq 0$. When $\Delta$ is totally negative this leads to the set $\Lambda_\Delta$ of Heegner points (coming from $\text{SO}(2)^9$) and when $\Delta$ is totally positive to families $\mathcal{G}_\Delta$ of real $g$-dimensional varieties (coming from $\text{SO}(1, 1)^g$), which for $g = 1$ are primitive closed geodesics (see [14]). The Heegner points correspond to abelian varieties of dimension $g$ with complex multiplication by an order in $F(\sqrt{\Delta})$. 

and so the case \( g = 2 \) is related to case (4) in Proposition 1.1. Let \( \mathcal{R}_g \) be a fundamental domain for the action of \( \text{SL}(2, \mathcal{O}) \) on \( \mathcal{H}^g \) and normalize the invariant volume form \( \mu_g \) on \( \mathcal{H}^g \) so that \( \mu_g(\mathcal{R}_g) = 1 \).

The main application in the case \( m = 3 \) is as follows.

**Theorem 1.2.** Let \( F \) be a totally real number field of degree \( g \geq 1 \) over \( \mathbb{Q} \) with ring of integers \( \mathcal{O} \). Assume that \( F \) has class number 1. Let \( \Delta \in \mathcal{O} \), \( \Delta \neq 0 \) be a generator of the relative discriminant of \( F(\sqrt{\Delta})/F \). Under GRH (or rather subconvexity), the families \( \{\Lambda_{\Delta}\}_{\Delta < 0} \) of Heegner points and \( \{G_{\Delta}\}_{\Delta \geq 0} \) of real \( g \)-dimensional subvarieties are equidistributed in \( \text{SL}(2, \mathcal{O})\backslash\mathcal{H}^g \). Namely, if \( \Omega_g \) is a region with smooth boundary in \( \mathcal{R}_g \) we have

\[
\lim_{N(\Delta) \to \infty, \Delta < 0} \frac{\text{Card} (\Lambda_{\Delta} \cap \Omega_g)}{\text{Card} (\Lambda_{\Delta})} = \mu_g(\Omega_g),
\]

\[
\lim_{N(\Delta) \to \infty, \Delta \geq 0} \frac{\sum_{\mathcal{C} \in G_{\Delta}} \text{Vol} (\mathcal{C} \cap \Omega_g)}{\sum_{\mathcal{C} \in G_{\Delta}} \text{Vol} (\mathcal{C})} = \mu_g(\Omega_g).
\]

(1.6)

We have made a number of simplifying assumptions which are not essential. In order to reduce the technicalities, we have assumed that \( \mathcal{A} = \mathcal{O}, \) that \( F \) has class number 1 and that \( \Delta \) generates a fundamental relative discriminant. The technicalities arising from arbitrary class number and arbitrary \( \mathcal{A} \) can be simplified by working in the adelic language. In \([7, 8]\) it was indicated how the fundamental discriminant assumption, appearing also in \([14]\), can be removed for the case \( g = 1 \) and those same ideas may be applicable here. See also \([18]\).

The present paper is organized as follows. In \([2]\) we recall the classical Maass correspondence of \([29]\) and derive formulae in Proposition 2.2 for Fourier coefficients of Maass forms of half-integral weight in terms of Weyl sums. In \([3]\) we prove new results that generalize, in Proposition 3.2 and Proposition 3.3 the Maass correspondence and the Fourier coefficient formulae to the case \( m = 3, (p,q) = (2,1) \) with base field \( F \) a totally real field of degree \( g \geq 1 \) and arbitrary class number. This extends results of \([14, 23, 29]\) that treat the case \( g = 1 \).

In \([4]\) we use the “accidental” isomorphisms to relate the results of \([2, 3]\) to Shimura varieties. The case \( m = 4, 5 \) leads to studying families of Humbert surfaces in Siegel 3-folds and of modular curves in these Humbert surfaces. The case \( m = 3 \) leads to studying families of Heegner points and of certain sub-domains of real dimension \( g \), which for \( g = 1 \) are primitive closed geodesics, in Hilbert modular varieties of complex dimension \( g \).

In \([5]\) we show how the subconvexity results of \([14]\) (in fact convexity results would suffice here) can be used to give vanishing of limits of cuspidal Weyl sums.

In \([6]\) we derive in Lemma 6.3 upper bounds for cuspidal Weyl sums in the Hilbert modular case. In \([7]\) we prove in Proposition 7.1 and Proposition 7.2 new results that extend classical formulae of Hecke \([22]\) expressing Eisenstein
Weyl sums, in the Hilbert modular case, in terms of central values of certain $L$-series. These results are of independent interest.

The results of §6 and §7 combined with subconvexity results for Fourier coefficients of Hilbert-Maass modular forms are then used to prove Theorem 1.2. The corresponding subconvexity results for the holomorphic case have been shown in [12]. We would need (in the notation of §3, where $\Delta$ is an integer of $F$ assumed square-free or a fundamental relative discriminant in the case of class number 1) the Fourier coefficients $\rho(\Delta, f)$ for $f$ a cusp form with $L^2$-norm 1 or an Eisenstein series, with eigenvalue $\lambda$ and half-integral weight $k$, to have an upper bound in the $\Delta$-aspect as good as $\rho(\Delta, f) \ll_{k, \epsilon} c(\lambda) |N_{E/\mathbb{Q}}(\Delta)|^{-1/4-\delta+\epsilon}$ for a fixed $\delta > 0$ and a positive explicit constant $c(\lambda)$. Partial progress towards subconvexity results in the Maass case have been made by Gergely Harcos [21], but the complete adaptation of the method of [12] to the Maass case remains elusive. Such results would follow however from GRH, so our Theorem 1.2 remains conditional. Although we do not pursue this here, from our methods we can estimate rates of convergence in the above equidistribution statements.

For compact maximal flats of $\text{SL}_n(\mathbb{Z})/\text{SL}_n(\mathbb{R})/\text{SO}(n)$, an equidistribution result has been obtained in [34], and this represents a different type of equidistribution result to that of [12], even in the $g = 1, n = 2$ case. An equidistribution result for Heegner points in Hilbert modular varieties using other methods has been announced by Zhang [61], assuming as yet unproven subconvexity results for Hilbert–Maass Fourier coefficients. In [33], subconvexity results are obtained for Rankin-Selberg $L$-functions which prove an equidistribution property for incomplete orbits of Heegner points over definite Shimura curves.

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Just after completing the write-up of this paper, we received a preprint of Clozel and Ullmo [10] where similar, and more general, equidistribution results are independently obtained. The methods and language used in their paper are quite different, even though in certain aspects a comparison with the present paper is likely implicit. They use, in particular, methods in ergodic theory due to Ratner [35], formulae of Waldspurger [57], and generalizations of Hecke’s formulae on Eisenstein series due to Wielonsky [58]. Their treatment of results analogous to our Theorem 1.2 also appeals to as yet unproven subconvexity results. In an earlier paper [9], these authors prove equidistribution results for certain families of Shimura subvarieties of

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1 As this paper goes to press, we have learnt that Akshay Venkatesh, in work in progress [56], claims the required subconvexity result using other methods.
positive dimension \(^2\). They use ergodic arguments, which do not give rates of convergence, in contrast to the methods used in the present paper.

2. **The classical Maass correspondence**

As in [14], §4, we exploit a construction of Maass forms as integrals of certain automorphic eigenfunctions for the ring of invariant differential operators, and in particular of the Laplace-Beltrami operator, against Siegel theta functions. We recall in outline this construction in order to fix notations, referring the reader to [13], [23], [29] for details.

Let \(Q\) be a symmetric \(m \times m\) matrix with half-integer off-diagonal elements and integer diagonal elements. Let \((p,q)\), with integers \(p, q \geq 0\) satisfying \(p + q = m\), be the signature of \(Q\). The majorant space \(\mathcal{H}_Q\) of \(Q\) is defined as

\[
\mathcal{H}_Q = \{ H \in M_m(\mathbb{R}) : H = {}^tH, H > 0, HQ^{-1}H = Q \},
\]

and is of real dimension \(pq\). It is the symmetric space attached to the group \(G = \Omega(Q)\) of all real \(m \times m\) matrices \(g\) such that

\[
Q[g] = {}^t gQg = Q,
\]

where \(A[B] = {}^tBAB\) for any matrices \(A, B\) for which this product makes sense. Indeed, the group \(G\) acts transitively on \(\mathcal{H}_Q\) by

\[
H \mapsto H[g], \quad H \in \mathcal{H}_Q, \quad g \in G.
\]

The isotropy group in \(G\) of any \(H \in \mathcal{H}_Q\) is a maximal compact subgroup \(K\). Analogous statements hold for the connected component of the identity of \(G\). An invariant metric on \(\mathcal{H}_Q\) is given by,

\[
ds^2 = \text{Trace}(H^{-1}dHH^{-1}dH).
\]

Let \(\Gamma\) be any group of finite index in the unit group

\[
\Gamma_Q = \text{SL}(m, \mathbb{Z}) \cap G
\]

and let \(\overline{\Gamma}\) be the quotient of \(\Gamma\) by \(\{ \pm \text{Id} \} \cap \Gamma\). Then \(\overline{\Gamma}\) acts discontinuously on \(\mathcal{H}_Q\) and is of finite covolume if \(Q\) is not a binary zero form, which we assume from now on. Let \(\Delta_Q\) be the Laplace-Beltrami operator on \(\mathcal{H}_Q\) and let \(d\nu\) be the invariant volume measure induced by \(ds^2\). Let \(\varphi = \varphi(H)\) be an eigenfunction of \(\Delta_Q\) on \(\overline{\Gamma}\backslash \mathcal{H}_Q\), with eigenvalue \(\lambda'\) defined by

\[
\Delta_Q \varphi + \lambda' \varphi = 0.
\]

For \(\varphi_1, \varphi_2\) functions on \(\overline{\Gamma}\backslash \mathcal{H}_Q\), define their inner product by

\[
<\varphi_1, \varphi_2> = \frac{1}{\text{Vol}(\overline{\Gamma}\backslash \mathcal{H}_Q)} \int_{\overline{\Gamma}\backslash \mathcal{H}_Q} \varphi_1 \overline{\varphi_2} d\nu.
\]

\(^2\)Note added in proof: there is a sequel to this paper by Ullmo [11]. There are also two new preprints of Zhang and Zhang-Jiang-Li [62], [63].
For \( z = u + iv \in \mathcal{H} \), the complex upper half plane, and \( H \in \mathcal{H}_Q \), let \( R = uQ + ivH \). Following Siegel \[53\], we define,

\[
\theta(z) = \theta(z, H) = \sum_{h \in \mathbb{Z}^m} \exp(2\pi i R[h]).
\]

From its definition it follows that, for each fixed \( z \in \mathcal{H} \), the function \( \theta(z, \cdot) \) on \( \mathcal{H}_Q \) is left \( \Gamma \)-invariant. Let the discriminant \( D \), the level \( N \) for \( Q \) and the definition of a Maass form of weight \( k \), discriminant \( D \) for level \( N \) be the same as in \[14\], §2, §4. We have the following result which is Theorem 4 of \[14\], except that we use \( \theta(z) \) instead of \( \theta(z) \) (which has the effect of exchanging \( p \) and \( q \)).

**Proposition 2.1.** Let \( \varphi \) be a non-constant eigenfunction of \( \Delta_Q \) on \( \Gamma \bs \mathcal{H}_Q \) with eigenvalue \( \lambda' \) and \( \langle \varphi, \varphi \rangle \) finite. Suppose,

\[
f(z) = v^{m/4} < \varphi, \theta(z) >
\]

is absolutely convergent for each \( z \in \mathcal{H} \). Then \( f(z) \) is a Maass cusp form of weight \( k = p - (m/2) \) and discriminant \( D \) for the congruence subgroup \( \Gamma_0(N) \) of \( \text{SL}(2, \mathbb{Z}) \) and it has eigenvalue \( \lambda = \frac{1}{4}(\lambda' + m - \frac{m^2}{4}) \).

In particular \( f(z) \) will satisfy

\[
(\Delta_k + \lambda)f = 0,
\]

where

\[
\Delta_k = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \frac{\partial}{\partial x}
\]

together with a transformation rule for \( \Gamma_0(N) \) with automorphy factor depending on \( k \) and \( D \), and a growth condition at the cusps.

We fix \( H_0 \in \mathcal{H}_Q \). As \( G \) acts transitively on \( \mathcal{H}_Q \), we can write \( H \in \mathcal{H}_Q \) as \( H = H_0[g^{-1}] \), \( g \in G \), and then,

\[
\theta(z, g) := \theta(z, H_0[g^{-1}]) = \sum_{h \in \mathbb{Z}^m} \exp(2\pi i uQ[h] - 2\pi vH_0[y^{-1}(h)]).
\]

As a function \( \theta(z, \cdot) \) on \( G \), it is left \( \Gamma \)-invariant and right \( K \)-invariant. Let \( \varphi(g) \) be the left \( \Gamma \)-invariant and right \( K \)-invariant function on \( G \) induced by the eigenfunction \( \varphi \) as in Proposition 2.1. Then, for an appropriate choice of Haar measure \( dg \) on \( G(\mathbb{R}) \), we have

\[
f(z) = v^{m/4} \sum_{h \in \mathbb{Z}^m} \exp(2\pi i uQ[h]) \int_{\Gamma \bs G} \varphi(g) \exp(-2\pi vH_0[\sqrt{v}g^{-1}(h)]) dg.
\]

On the other hand, we can write

\[
\lambda = s(1 - s) = \frac{1}{4} + \kappa^2, \quad s = \frac{1}{2} + i\kappa, \quad \text{Re}(s) \geq \frac{1}{2}.
\]
We know that \( f(z), z = u + iv, u, v \in \mathbb{R} \), has a Fourier expansion of the form

\[
(2.1)\quad f(z) = \rho(0)u^{\frac{1}{2}+i\kappa} + \rho'(0)u^{\frac{1}{2}-i\kappa} + \sum_{d \in \mathbb{Z}, d \neq 0} \rho(d)W_{\frac{2}{4}\text{sgn}(d),i\kappa}(4\pi|d|v)\exp(2\pi i du),
\]

where \( W_{\alpha,\beta}(\cdot) \) is the classical Whittaker function (see \cite{31}; in fact for \( f \) as in Proposition 2.1) we have \( \rho(0) = \rho'(0) = 0 \). Therefore, for \( d \neq 0 \),

\[
M_d(v) := \rho(d)W_{\frac{2}{4}\text{sgn}(d),i\kappa}(4\pi|d|v)
\]

\[
(2.2)\quad v^{m/4}\sum_{h \in \mathbb{Z}^m, Q[h] = d} \int_{\Gamma \backslash G} \varphi(g) \exp(-2\pi H_0[g^{-1}(\sqrt{v}h)]) \, dg.
\]

For every integer \( d \neq 0 \), it follows from general results of \cite{53} that the number of orbits, under the action of \( \Gamma \), of the solutions of \( Q[h] = d, \ h \in \mathbb{Z}^m \), is finite. Let the cardinality of this orbit be \( H(d) \): it is a generalized class number. Let \( \{h^{(1)}, \ldots, h^{(H(d))}\} \) be a set of representatives in \( \mathbb{Z}^m \) of these orbits and let \( \Gamma_j \) be the stabilizer of \( h^{(j)} \) in \( \Gamma \). Then,

\[
\begin{align*}
M_d(v) &= v^{-m/4}M_d(v) = \int_{\Gamma \backslash G} \sum_{h \in \mathbb{Z}^m, Q[h] = d} \exp(-2\pi H_0[g^{-1}(\sqrt{v}h)]) \varphi(g) \, dg \\
&= \sum_{j=1}^{H(d)} \int_{\Gamma_j \backslash G} \exp(-2\pi H_0[g^{-1}(\sqrt{v}h^{(j)})]) \varphi(g) \, dg.
\end{align*}
\]

We let,

\[
I_j = I_j(v) = \int_{\Gamma_j \backslash G} \exp(-2\pi H_0[g^{-1}(\sqrt{v}h^{(j)})]) \varphi(g) \, dg.
\]

We can compare directly with the discussion of \cite{29}, \S 5. In terms of our notations, the notations of that paper become: \( S = Q, u = \varphi, v = \frac{-(n-2)}{2} + 2i\kappa, x = 2u, y = 2v, \alpha = \frac{p}{2} - \frac{m}{4} + \frac{1}{2} + i\kappa, \beta = \frac{q}{2} - \frac{m}{4} + \frac{1}{2} + i\kappa, \) and \( t = d \).

It is shown there that

\[
\exp(2\pi dv) \sum_{j=1}^{H(d)} I_j(v)
\]

satisfies a second order differential equation (\cite{29}, (86)) and by looking at the behavior as \( v \to \infty \), one sees that it is a multiple of a standard solution of that equation related to the Whittaker function, which fits with (2.1), (2.2). Indeed, we have (\cite{29}, (91))

\[
(2.3)\quad M_d(v) = v^{n/4} \sum_{j=1}^{H(d)} I_j(v) = A(2\pi|d|)^{-m/4}W_{\frac{2}{4}\text{sgn}(d),i\kappa}(4\pi|d|v),
\]

for some \( A \neq 0 \) independent of \( v \).

We now describe this factor \( A \). From now on \( c_1, c_2, \ldots \) will be positive constants depending only on \( Q \) and the sign of \( d \); these constants can in fact be explicitly computed. The function \( \varphi = \varphi(g) \) on \( G \sim \text{SO}(p, q) \) is \( K \)-invariant on the right and is an eigenfunction of the appropriately normalized
Casimir operator on $G$. Fix a solution $E$ of $Q[E] = \text{sgn}(d)$. We can find $l_j \in G$ such that

$$l_j^{-1} h^{(j)} = \sqrt{|d|} E,$$

since $G$ acts transitively on the set

$$\{ x \in \mathbb{R}^m : Q[x] = d \neq 0 \}.$$

We then have,

$$I_j = \int_{\Gamma \setminus G} \exp(-2\pi \text{d}v H_0[a^{-1}(E)]) \varphi(l_j g) dg,$$

where

$$\Gamma' = l_j^{-1} \Gamma_j l_j$$

is the stabilizer of $E$ in $G(\mathbb{R})$. Let $d\gamma$ be a fixed Haar measure on $\Gamma'(\mathbb{R})$. Such a choice determines a Haar measure $da$ on $\Gamma'(\mathbb{R}) \setminus G(\mathbb{R})$ such that

$$dg = d\gamma da.$$

We have,

$$(2.4) \quad I_j = \int_{\Gamma'(\mathbb{R}) \setminus G(\mathbb{R})} \exp(-2\pi \text{d}v H_0[a^{-1}(E)]) \int_{\Gamma' \setminus \Gamma'} \varphi(l_j g) d\gamma da.$$

Let $\psi(g) = \varphi(l_j g)$; then $\psi(g)$ is also an eigenfunction of the normalized Casimir operator with the same eigenvalue as $\varphi(g)$. Let

$$J_j(a) = \int_{\Gamma' \setminus \Gamma'} \psi(g) d\gamma,$$

then

$$J_j(\gamma a k) = J_j(a), \quad \gamma \in \Gamma'(\mathbb{R}), k \in K,$$

so that $J_j(a)$ is uniquely determined by its value on $\Gamma'(\mathbb{R}) \setminus G(\mathbb{R}) / K$. In \[29\], this integral is rewritten in terms of the variable $w = H_0[a^{-1}(E)]$ and is shown to be a multiple of a standard function in $w$ by using (2.4) and comparing with (2.3). Alternatively, one may use the above discussion together with a uniqueness argument as done in \[23\], (3.7) and (3.23) for the case $(p, q) = (2, 1)$. This enables us to write, for $e$ the identity of $G$,

$$J_j(a) = J_j(e) V_{\lambda}(a)$$

where $V_{\lambda}(a)$ is determined by the condition $V_{\lambda}(e) = 1$. We then have,

$$I_j = J_j(e) \int_{\Gamma'(\mathbb{R}) \setminus G(\mathbb{R})} \exp(-2\pi \text{d}v H_0[a^{-1}(E)]) V_{\lambda}(a) da.$$

As in \[29\], (103), we can compare this directly with (2.3) to deduce that,

$$\rho(d) = c_1 |d|^{-m/4} \left\{ \sum_{j=1}^{H(d)} \int_{\Gamma' \setminus \Gamma'} \varphi(l_j g) d\gamma \right\}. $$

We have shown the following. The notation $d\gamma$ is used again, now to denote the induced Haar measure on $\Gamma_j(\mathbb{R})$. 

Proposition 2.2. We have

\[
\rho(d) = c_1 |d|^{-m/4} \left\{ \sum_{j=1}^{H(d)} \int_{\Gamma_j \backslash \Gamma} \varphi(\gamma) d\gamma \right\}.
\]

We can also check this against the formula given in [29], pp288–289. Namely,

\[
\rho(d) = (2\pi)^{-m/4} |d|^{-m/4} - \frac{m}{4} \alpha_d(Q, \varphi),
\]

where

\[
\alpha_d(Q, \varphi) = c_2 |d|^{-m/2+1} \sum_{j=1}^{H(d)} \int_{\Gamma \backslash \Gamma} \varphi(l_j \gamma) d\gamma.
\]

can be interpreted as Siegel’s mass [53] of the representation of \(d\) by \(Q\), weighted against \(\varphi\).

3. The Maass correspondence for the Hilbert modular case

In this section, we generalize the classical Maass correspondence in the case \((p, q) = (2, 1)\) to the Hilbert modular case. Let \(F\) be a totally real number field of degree \(g\) over \(\mathbb{Q}\). As in §1, we let \(\mathcal{O}\) be the ring of integers of \(F\) and \(\mathcal{A}\) be a fractional ideal of \(F\). We define \(\Gamma_{\mathcal{A}} = \text{SL}(\mathcal{O} \oplus \mathcal{A})\) to be the group of matrices of determinant 1 in the maximal order in \(M_2(F)\) given by

\[
\begin{pmatrix}
  0 & \mathcal{A}^{-1} \\
  \mathcal{A} & 0
\end{pmatrix}.
\]

For \(z = (z_1, \ldots, z_g) \in \mathbb{C}^g\), \(z_j = u_j + \sqrt{-1}v_j\), \(u_j, v_j \in \mathbb{R}\), and \(\alpha \in F\), let

\[
\alpha \cdot z = \alpha_1 z_1 + \cdots + \alpha_g z_g,
\]

with \(\alpha_j, j = 1, \ldots, g\) the Galois conjugates of \(\alpha\), and let

\[
N(v) = \prod_{j=1}^{g} v_j.
\]

Let \(S \in M_2(\mathbb{Z})^g\) have all its coordinates equal to the matrix

\[
Q = \begin{pmatrix}
  0 & 0 & -2 \\
  0 & 1 & 0 \\
-2 & 0 & 0
\end{pmatrix}
\]

which has signature \((2, 1)\). The majorant space \(H_Q\) is isomorphic to the upper half plane \(\mathcal{H}\). For \(z \in \mathcal{H}^g\) with coordinates \(z_j = u_j + iv_j\), \(v_j > 0\) and \(H \in H_Q^g\) with coordinates \(H_j \in H_Q\), let \(R\) have coordinates \(R_j = u_j Q + iv_j H_j\), \(j = 1, \ldots, g\). Let \(L\) be the lattice \(\mathcal{A}^{-1} \oplus \mathcal{O} \oplus \mathcal{A}\) in \(F^3\). We define the theta function

\[
\theta(z, H) := N(v)^{3/4} \sum_{h \in L} \exp(2\pi i (\mathcal{H} \cdot R \cdot h)),
\]

where \(\mathcal{H} \cdot R \cdot h\) denotes the action of \(\mathcal{H}\) on \(R\) followed by the action of \(h\) on \(R\).
where
\[ t^j h \cdot R \cdot h = \sum_{j=1}^{g} t^j h_j R_j h_j, \]
for \( h_j, j = 1, \ldots, g \) the Galois conjugates of \( h \in L \). Let \( H_0 \in H^g_Q \) have each coordinate equal to the majorant of \( Q \) given by
\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]
Then, there are matrices \( C_j \) in \( \Omega(Q) \) such that
\[ H^j = H_0 C_j, \quad j = 1, \ldots, g. \]
As \( \Omega(Q) \) is isomorphic to \( \text{SO}(2,1) \) which is in turn isomorphic to \( \text{SL}(2, \mathbb{R}) \), we may write
\[
\theta(z, H) \text{ as a function } \theta(z, g) \text{ with } g \in G = \Omega(Q)^g, \quad g \in \text{SL}(2, \mathbb{R}).
\]
Letting
\[
s(x_1, x_2, x_3) = \exp(-2\pi(2x_1^2 + x_2^2 + 2x_3^2)),
\]
we have from (3.1)
\[
\theta(z, g) = N(v)^{3/4} \sum_{h \in L} \exp(2\pi i((h^2_2 - 4h_1 h_3) \cdot u)) N(s(\sqrt{v} \cdot g^{-1}(h)),
\]
where
\[
N(s(\sqrt{v} \cdot g^{-1}(h))) = \prod_{j=1}^{g} s(\sqrt{v} \cdot g_j^{-1}(h)).
\]
By [26], §7 there is a congruence subgroup \( \Gamma_1 \), and a multiplier \( J \) such that for \( \gamma_1 \in \Gamma_1 \),
\[
\theta(\gamma_1 z, g) = J(\gamma_1, z) \theta(z, g),
\]
and for \( \gamma \in \Gamma_A, \ k \in K_\infty = K^g \), where \( K \) is the maximal compact of \( \text{SL}(2, \mathbb{R}) \), we have
\[
\theta(z, \gamma g k) = \theta(z, g).
\]
We may adapt the discussion of [23], §2 to our situation. For \( j = 1, \ldots, g \), let \( \Delta_{1/2}^{(j)} \) be the Laplacian in the variable \( z_j = u_j + iv_j \) given by
\[
\Delta_{1/2}^{(j)} = v_j^2 \left( \frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2} \right) - i \frac{v_j}{2} \frac{\partial}{\partial u_j}
\]
We have
\[
D_{g_j}^{(j)} \theta(z, g) = 4 \Delta_{1/2}^{(j)} \theta(z, g) + \frac{3}{4} \theta(z, g), \quad j = 1, \ldots, g.
\]
A Maass-Hilbert form \( \varphi \) may be viewed as a function on \( G \) which is \( K_\infty \)-invariant on the right and which is an eigenfunction of the Casimir operators \( D^{(j)} \), satisfying for \( r_j \in \mathbb{R}, \ j = 1, \ldots, g, \)

\[
D^{(j)} \varphi(g) = \left( -\frac{1}{4} - (2r_j)^2 \right) \varphi(g), \quad r_j \in \mathbb{R}.
\]

Let

\[
U = L^2_{cusp}(\Gamma_A \backslash \mathcal{H}^g) = \{ \varphi : \mathcal{H}^g \to \mathbb{C} : \varphi(\gamma z) = \varphi(z), \ \gamma \in \Gamma_A, \ \int_{\Gamma_A \backslash \mathcal{H}^g} |\varphi|^2 N(v)^{-\frac{1}{2}} N(du) N(dv) < \infty, \ \int_0^1 \cdots \int_0^1 \varphi(x, y) N(dx) = 0, \ \text{a.e. } y \}.
\]

We can make \( U \) into a Hilbert space using the natural inner product. This space is invariant under the action of the unique self-adjoint extensions of the \( g \) Laplacians

\[
\Delta^{(j)} = v_j^2 \left( \frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2} \right), \quad j = 1, \ldots, g,
\]

which provide a basis of the algebra of invariant differential operators on \( \mathcal{H}^g \). The Maass-Hilbert forms are (simultaneous) eigenfunctions for all \( g \) Laplacians \( \Delta^{(j)} \). Let \( \frac{1}{2} \in \mathbb{Q}^g \) be the vector with all its coordinates equal to \( \frac{1}{2} \) and \( \lambda = (\lambda_j)_{j=1}^g \in \mathbb{C}^g \). We may write \( \lambda_j = s_j(1 - s_j), \ \text{Re}(s_j) \geq 1/2. \)

A Hilbert-Maass form \( f \) for \( \Gamma_1 \) of weight \( \frac{1}{2} \) and eigenvalue \( \lambda \) is a function \( f : \mathcal{H}^g \to \mathbb{C} \) satisfying

\[
f(\gamma z) = J(\gamma, z) f(z), \quad \gamma \in \Gamma_1
\]

\[
\Delta^{(j)}_{1/2} f = \lambda_j f, \quad j = 1, \ldots, g,
\]

with polynomial growth at the cusps. Such a function of \( z = u + iv \in \mathcal{H}^g \), has a Fourier series development in terms of the classical Whittaker functions of the form

\[
f(u + iv) = \rho_0(v, f) + \sum_{\xi \in \mathcal{O}(f, \Gamma_1), \xi \neq 0} \rho(\xi, f) N \left( W_{\frac{1}{4} \text{sgn}(\xi), s - \frac{1}{2}}(4\pi |\xi| v) \right),
\]

where \( \mathcal{O}(f, \Gamma_1) \) is a certain ideal in \( F \) and

\[
N \left( W_{\frac{1}{4} \text{sgn}(\xi), s - \frac{1}{2}}(4\pi |\xi| v) \right) = \prod_{j=1}^g W_{\frac{1}{4} \text{sgn}(\xi_j), s_j - \frac{1}{2}}(4\pi |\xi_j| v_j),
\]

with \( \xi_1, \ldots, \xi_g \) the Galois conjugates of \( \xi \in F \). The form \( f \) is cuspidal if it vanishes at the cusps of \( \Gamma_1 \). Let

\[
V = L^2_{cusp}(\Gamma_1 \backslash \mathcal{H}^g) = \{ f : \mathcal{H}^g \to \mathbb{C} : f(\gamma z) = J(\gamma, z) f(z), \ \gamma \in \Gamma_1, \ \text{f cuspidal and square integrable} \}.
\]

Then, by exactly similar arguments to those of [23, Proposition 2.3], we may derive the analogue of Proposition 2.1.
Proposition 3.1. If \( \varphi \in U \), viewed as a function on \( G \), is an eigenfunction of the \( D_{g_j}^{(j)} \) with eigenvalues \(-\left(\frac{1}{4} + (2r_j)^2\right), j = 1, \ldots, n\) then

\[
f(z) = \int_{\Gamma A \setminus G} \varphi(g) \theta(z, g) dg
\]
is an element of \( V \) and is an eigenfunction of \( \Delta_{1/2}^{(j)} \), with eigenvalues

\[-\left(\frac{1}{4} + r_j^2\right), j = 1, \ldots, g.\]

Let \( \Delta \in \Theta \) be totally negative, and let

\[
(3.10) \quad M_\Delta(v) = \int_0^1 \cdots \int_0^1 \left( \int_{\Gamma A \setminus G} \varphi(g) \theta(u + iv, g) dg \right) N(\exp(-2\pi i (\Delta u))).
\]

Then,

\[
(3.11) \quad M_\Delta(v) = N(v)^{3/4} \int_{\Gamma A \setminus G} \sum_{h^2 - 4h_1h_3 = \Delta} N(s(\sqrt{v}g^{-1}(h))) \varphi(g) dg.
\]

Let \( h(\Delta) \) be the number of \( \Gamma A \)-orbits of vectors \( h \in L \) such that \( h_2^2 - 4h_1h_3 = \Delta \) and let \( h^{(i)} \) be a representative of the \( i \)-th orbit and \( \Gamma_i \) the stabilizer of \( h^{(i)} \). Then we may write,

\[
(3.12) \quad M_\Delta(v) = N(v)^{3/4} \sum_{i=1}^{h(\Delta)} \int_{\Gamma_i \setminus G} N(s(\sqrt{v}g^{-1}(h^{(i)}))) \varphi(g) dg.
\]

When \( g = 1 \), this corresponds to the situation considered in [23], (3.2).

We consider the two cases \( \Delta << 0 \), totally negative, and \( \Delta >> 0 \), totally positive.

**Case (i):** Let \( \Delta \) be totally negative. Let \( h^{(1)}, \ldots, h^{(h(\Delta))} \) be as above. Consider the integral

\[
I_i = \int_G N(s(\sqrt{v}g^{-1}(h^{(i)}))) \varphi(g) dg.
\]

Then (3.12) becomes

\[
(3.13) \quad M_\Delta(v) = N(v)^{3/4} \sum_{i=1}^{h(\Delta)} \frac{I_i}{|\Gamma_i|}.
\]

The group \( \text{SL}(2, \mathbb{R}) \) acts transitively on the \( g \) hyperboloids of \( x \in \mathbb{R}^3 \) with \( t^xQx = \Delta_j \). Therefore, we can find a \( \mathbf{g}^{(i)} = (\mathbf{g}_j^{(i)}) \) \( j = 1 \in G \) such that

\[
(\mathbf{g}_j^{(i)})^{-1}(h_j^{(i)}) = \frac{\sqrt{|\Delta_j|}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]
Let
\[ E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in (\mathbb{R}^3)^g, \]
and
\[ \psi(g) = \varphi(\bar{g}(t)g). \]

We have
\[ I_i = \int_G N \left( s \left( \sqrt{v/\Delta} \frac{E}{4}^{-1} \right) \right) \psi(g). \]

Using as in \[23\], after (3.6), the Cartan decomposition of \( SL(2, \mathbb{R}) \), we may write, as an integral over \( a = (a_j)_{j=1}^g \in \mathbb{R}^g \) with \( \delta(a) = \frac{a_j^2 - a_{-j}^2}{2} \),
\begin{equation}
I_i = \int_1^\infty \ldots \int_1^\infty N \left( t \left( \sqrt{v/\Delta} \begin{pmatrix} a_{-j}^2 & 0 \\ 0 & a_j^2 \end{pmatrix} \right) E \right) \times \left( \int_K \int_K \psi(k_1gk_2)dk_1dk_2 \right) N(\delta(a)) N\left( \frac{da}{a} \right).
\end{equation}

Now \( \psi(g) \) is an eigenfunction of the \( D_{g,j}^{(j)} \) with the same eigenvalues \( \lambda_j \) as \( \varphi \). As in \[23\], we may use uniqueness arguments to show that there is a standard spherical function \( \omega_j(g_j) \) with eigenvalue \( \lambda_j \) such that \( \omega_j(e) = 1 \) and
\begin{equation}
I_i = \varphi(\bar{g}(t)) N \left( Y_{\lambda_j} \left( \sqrt{v/\Delta} \right) \right),
\end{equation}

where
\begin{equation}
Y_{\lambda_j}(t) = \int_1^\infty s \left( t \begin{pmatrix} a_{-j}^2 & 0 \\ 0 & a_j^2 \end{pmatrix} \right) \omega_{\lambda_j} \left( \begin{pmatrix} a_j & 0 \\ 0 & a_{-j}^{-1} \end{pmatrix} \right) \delta(a_j) \frac{da_j}{a_j}.
\end{equation}

In conclusion,
\begin{equation}
M_\Delta(v) = N(v)^{3/4} \sum_{i=1}^{\text{h}(\Delta)} \frac{1}{|\Gamma_i|} \varphi(g(t)) N \left( Y_{\lambda_j} \left( \sqrt{v/\Delta} \right) \right).
\end{equation}

From \[23\], we have the asymptotic formula
\[ Y_{\lambda_j}(t) \sim \frac{\exp(-8\pi t^2)}{32\pi t^2}, \quad t \to \infty, \]
and therefore as \( v_j \to \infty, j = 1, \ldots, g, \)
\begin{equation}
N \left( Y_{\lambda_j} \left( \sqrt{v/\Delta} \right) \right) \sim \exp(-2\pi \sum_{j=1}^g v_j |\Delta_j|) \left( \prod_{j=1}^g 8\pi v_j |\Delta_j| \right)^{-1}.
\end{equation}
On the other hand, we write,

\[(3.19) \quad M_\Delta(v) = \rho(\Delta) N \left( W_{-\frac{1}{4},ir} (4\pi |\Delta| v) \right). \]

Then \( \rho(\Delta) \) is the “\( \Delta \)”-th Fourier coefficient of the function \( f(z) \) of Proposition 3.1.

We have the asymptotic formula as \( v_j \to \infty, \ j = 1, \ldots, g, \)

\[(3.20) \quad N \left( W_{-\frac{1}{4},ir} (4\pi |\Delta| v) \right) \sim \exp(-2\pi \sum_{j=1}^{g} v_j |\Delta_j|) \left( \prod_{j=1}^{g} 4\pi v_j |\Delta_j| \right)^{-1/4}. \]

From equations (3.18), (3.19) and (3.20) we deduce the following result.

**Proposition 3.2.** For \( \Delta \ll 0 \), the “\( \Delta \)”-th Fourier coefficient of the function \( f(z) \) of Proposition 3.1 is given by,

\[(3.21) \quad \rho(\Delta) = 2^{-g} (4\pi)^{-3g/4} |N(\Delta)|^{-3/4} \sum_{i=1}^{h(\Delta)} \frac{1}{|\Gamma_i|} \varphi(g(i)). \]

**Case (ii):** Let \( \Delta \) be totally positive. Let \( h^{(1)}, \ldots, h^{(h(\Delta))} \) be as above. Consider the integral

\[ I_i = \int_{\Gamma_i \setminus G} N \left( s(\sqrt{v}g^{-1}(h^{(i)})) \right) \varphi(g) dg. \]

Then (3.12) becomes

\[(3.22) \quad M_\Delta(v) = N(v)^{3/4} \sum_{i=1}^{h(\Delta)} I_i. \]

The group \( \text{SL}(2, \mathbb{R}) \) acts transitively on the \( \mathfrak{g} \) hyperboloids of \( x \in \mathbb{R}^3 \) with \( {}^t x Q x = \Delta_j \). Therefore, we can find an \( \ell^{(i)} = (\ell^{(i)}_j)_{j=1}^{g} \in G \) such that

\[ (\ell^{(i)}_j)^{-1}(h^{(i)}_j) = \sqrt{\Delta_j} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \]

Let

\[ E' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^g \in (\mathbb{R}^3)^g. \]

and

\[ \psi(g) = \varphi(\ell^{(i)}g). \]

We have

\[ I_i = \int_{\Gamma_i \setminus G} N \left( s \left( \sqrt{v} \Delta g^{-1} E' \right) \right) \psi(g) dg, \]

where

\[ \Gamma_i' = \ell^{(i)} \Gamma_i \ell^{(i)}. \]
Suppose from now on that $\Delta$ is not a square. The group $\Gamma'(\mathbb{R}) = \Gamma'_i(\mathbb{R})$ is the $g$-th power of the stabilizer of \[
abla = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \] and can be written as

$$\Gamma'(\mathbb{R}) = \prod_{j=1}^{g} \left\{ \pm \begin{pmatrix} p_{j}^{1/2} & 0 \\ 0 & p_{j}^{-1/2} \end{pmatrix}, \quad 0 < p_{j} < \infty \right\}.$$  

The group $\Gamma'_i$ is a discrete free abelian subgroup of $\Gamma'(\mathbb{R})$ of rank $g$ over $\mathbb{Z}$, see for example [16], Chapter 1, Section 5. We can decompose each component of $g = (g_{j})_{j=1}^{g} \in G$ as

$$g_{j} = \begin{pmatrix} 1 & \xi_{j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{j}^{1/2} & 0 \\ 0 & p_{j}^{-1/2} \end{pmatrix} k_{j}$$

$$= \begin{pmatrix} p_{j}^{1/2} & 0 \\ 0 & p_{j}^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & \xi_{j}/p_{j} \\ 0 & 1 \end{pmatrix} k_{j}, \quad k_{j} \in K, \ 0 < p_{j} < \infty, \ -\infty < \xi_{j} < \infty.$$

We have

$$g_{j}^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\xi_{j}/p_{j} \\ 1 \\ 0 \end{pmatrix}.$$ 

Let $t_{j} = \xi_{j}/p_{j}$, then $N(dt) := \prod_{j=1}^{g} dt_{j}$ is a Haar measure on $\Gamma'(\mathbb{R}) \backslash G(\mathbb{R})$ and $N \left( \frac{dp}{p} \right) := \prod_{j=1}^{g} \frac{dp_{j}}{p_{j}}$ is a Haar measure on $\Gamma'(\mathbb{R})$. We may assume that $dg = N(dt)N \left( \frac{dp}{p} \right)$.

With these notations, we may write

$$I_{i} = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp \left( -2\pi \sum_{j=1}^{g} v_{j} \Delta_{j} (2t_{j}^{2} + 1) \right) \times$$

$$\times \int_{\Gamma'_{i}} \psi \left( \begin{pmatrix} p_{j}^{1/2} & 0 \\ 0 & p_{j}^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) N \left( \frac{dp}{p} \right) N \left( dt \right).$$

Let

$$J_{i}(e) = \int_{\Gamma'_{i} \backslash \Gamma'(\mathbb{R})} \psi \left( \begin{pmatrix} p_{j}^{1/2} & 0 \\ 0 & p_{j}^{-1/2} \end{pmatrix} \right) N \left( \frac{dp}{p} \right) = \int_{\Gamma_{i} \backslash \Gamma_{i}(\mathbb{R})} \varphi (\gamma) d\gamma,$$

where $d\gamma$ is the invariant measure on $\Gamma_{i}(\mathbb{R})$ induced by $N \left( \frac{dp}{p} \right)$.

Arguing as in [23], Case (ii) (where $g = 1$), we can again use the Casimir operators to see that there is, for each $j = 1, \ldots, g$ a unique even function
V_{\lambda_j}(t_j) of \( t_j \in \mathbb{R} \) determined by the condition \( V_{\lambda_j}(0) = 1 \) and such that

\begin{equation}
I_i = J_i(e) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left( -2\pi \sum_{j=1}^{g} v_j \Delta_j (2t_j^2 + 1) \right) \times V_{\lambda_1}(t_1) \cdots V_{\lambda_g}(t_g) dt_1 \cdots dt_g.
\end{equation}

From the asymptotics in [23] we have, as \( v_j \to \infty, j = 1, \ldots, g \),

\[ I_i \sim J_i(e) 2^{-g} \prod_{i=1}^{g} \exp \left( -2\pi \sum_{j=1}^{g} v_j \Delta_j \right) (v_j \Delta_j)^{-1/2}. \]

Hence,

\begin{equation}
M_{\Delta}(v) \sim N(v)^{1/4} N(\Delta)^{-1/2} 2^{-g} \prod_{i=1}^{g} \exp \left( -2\pi \sum_{j=1}^{g} v_j \Delta_j \right) \sum_{i=1}^{\infty} J_i(e).
\end{equation}

We also have the asymptotic formula for \( j = 1, \ldots, g \),

\[ W_{4,ir_j}(4\pi \Delta_j v_j) \sim \exp(-2\pi v_j \Delta_j) (4\pi \Delta_j v_j)^{1/4}, \quad v_j \to \infty. \]

On the other hand, we write,

\begin{equation}
M_{\Delta}(v) = \rho(\Delta) N \left( W_{4,ir_j}(4\pi \Delta v) \right).
\end{equation}

Then \( \rho(\Delta) \) is the “\( \Delta \)”-th Fourier coefficient of the function \( f(z) \) of Proposition 3.1.

We have the asymptotic formula for \( v_j \to \infty \) and \( j = 1, \ldots, g \),

\begin{equation}
M_{\Delta}(v) \sim (4\pi)^{g/4} \rho(\Delta) \exp \left( -2\pi \sum_{j=1}^{g} v_j \Delta_j \right) N(\Delta)^{1/4} N(v)^{1/4}.
\end{equation}

From equations (3.25), (3.26), (3.27) we deduce the following result.

**Proposition 3.3.** For \( \Delta \gg 0 \), the “\( \Delta \)”-th Fourier coefficient of the function \( f(z) \) of Proposition 3.1 is given by,

\begin{equation}
\rho(\Delta) = 2^{-g} (4\pi)^{-g/4} N(\Delta)^{-3/4} \sum_{i=1}^{h(\Delta)} \int_{\Gamma_i \backslash \Gamma(1)} \varphi(\gamma) d\gamma.
\end{equation}

4. **Families of symmetric domains**

We describe the families of symmetric domains to which we will apply the Maass correspondence of §2 and §3. These will correspond in particular to subvarieties of the Siegel modular variety of genus 2 and to certain Heegner points in arbitrary genus, coming from Hilbert modular varieties.
We exploit a natural isomorphism between $\text{SO}(3, 2)$ and $\text{Sp}(4, \mathbb{R})$, following [52]. Let $Q$ be the quadratic form on $\mathbb{R}^5$ of signature $(3, 2)$ given by

\[(4.1)\quad Q(x) = Q(x_1, \ldots, x_5) = x_2^2 - 4x_3x_1 - 4x_4x_5, \quad x = \begin{pmatrix} x_1 & \cdots & x_5 \end{pmatrix} \in \mathbb{R}^5.\]

Then $\mathcal{H}_2$ is isomorphic to the space of vectors $Z = t(z_1, \ldots, z_5) \in \mathbb{C}^5$ with $z_5 = 1$ and

\[Z^t QZ = 0, \quad Z^t QZ < 0, \quad \text{Im}(z_1) > 0.\]

We recover the description of [11] by setting

\[z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix},\]

the lower Siegel half space of degree 2 corresponding to the condition $\text{Im}(z_1) < 0$.

We may also introduce $Q$ as the quinary quadratic form on the space $V$ of alternating matrices of the form $M(x), \ x \in \mathbb{R}^5$, where

\[M(x) = \begin{pmatrix} 0 & -2x_4 & x_2 & -2x_1 \\ 2x_4 & 0 & 2x_3 & -x_2 \\ -x_2 & -2x_3 & 0 & 2x_5 \\ 2x_1 & x_2 & -2x_5 & 0 \end{pmatrix}.\]

With $J$ the standard symplectic $4 \times 4$ matrix as in [11], we have

\[t M(x) J M(x) = Q(x) \cdot J, \quad x \in \mathbb{R}^5,\]

and this defines $(V, Q)$. The isomorphism between $\text{SO}(3, 2)$ and $\text{Sp}(4, \mathbb{R})$ can then be seen via the action of $g \in \text{Sp}(4, \mathbb{R})$ on $M(x) \in V$ preserving $Q$ and given by

\[g : M(x) \rightarrow g M(x)^t g.\]

Fix $\lambda \in \mathbb{R}, \ \lambda \neq 0$. The group $\text{SO}(Q)$ acts transitively on the solutions $x \in \mathbb{R}^5$ of $Q(x) = \lambda$ and the isotropy group of any such $x$ is isomorphic to $\text{SO}(2, 2)$ if $\lambda > 0$ and to $\text{SO}(1, 3)$ if $\lambda < 0$. For $Q(x) > 0$, let

\[\mathcal{R}_x = \{ z \in \mathcal{H}_2 : \begin{pmatrix} z_1 \\ 12 \end{pmatrix} M(x)^t \begin{pmatrix} z_1 \\ 12 \end{pmatrix} = 0 \}.\]

For $Q(x) < 0$, let

\[\mathcal{R}_x^- = \{ z \in \mathcal{H}_2 : \begin{pmatrix} z_1 \\ 12 \end{pmatrix} M(x)^t \begin{pmatrix} z_1 \\ 12 \end{pmatrix} = 0 \}.\]

By checking at $z = \sqrt{-1}I_2 \in \mathcal{H}_2$ and using transitivity one sees that the domains $\mathcal{R}_x, \ Q(x) > 0$ are real isomorphic to the symmetric space for $\text{SO}_0(2, 2)$ and complex isomorphic to $\mathcal{H}^2$. On the other hand, the domains $\mathcal{R}_x^-, \ Q(x) < 0$, are real isomorphic to the symmetric space for $\text{SO}_0(3, 1)$.

For $d \in \mathbb{Z}$ let

\[W_d = \{ h \in \mathbb{Z}^5 : Q(h) = d \}.\]

Assume from now on that $d$ is square-free. When $d > 0$, let $S_d$ be the complex surface in $\text{Sp}(4, \mathbb{Z}) \setminus \mathcal{H}_2$ given by the union of the images of the $\mathcal{R}_h, \ h \in W_d$ (with $h$ primitive as $d$ is square-free). The surface $S_d$ will
be non-trivial if and only if $d \equiv 1 \mod 4$. Then $S_d$ is called the Humbert surface of invariant $d$. For a general reference on Humbert surfaces see [55], Chapter IX. The components of the surface $S_d$ are images of Hilbert modular surfaces, induced by the identification of the $\mathcal{O}$-module $\mathcal{O} \oplus \mathcal{O}^\vee$ (with the standard alternating form derived from the trace of $F = \mathbb{Q}(\sqrt{d})$ over $\mathbb{Q}$) with the $\mathbb{Z}$-lattice $\mathbb{Z}^4$ (with the standard symplectic form). This amounts to viewing Hilbert modular surfaces as sub-varieties of Siegel 3-folds. In particular, by [55], Chapter IX, Proposition (2.3), the abelian surface

$$A(\mathbb{C}) = \mathbb{C}^4 / \mathbb{Z}^4 + z \cdot \mathbb{Z}^4$$

has endomorphism ring $\text{End}(A)$ containing $\mathcal{O}$ if and only if $z \mod \text{Sp}(4, \mathbb{Z})$ is in $S_d$.

When $d < 0$, let $\mathcal{E}_d$ be the real 3-dimensional variety in $\text{Sp}(4, \mathbb{Z}) \backslash \mathcal{H}^2$ given by the union of the images of the $\mathcal{R}^-_{h^*}, h \in W_d$ (with $h$ primitive as $d$ is square-free). We have $\mathcal{E}_d$ non-trivial if and only if $-d \equiv 3 \mod 4$.

We now turn to studying the Hilbert modular surfaces $X_d$ with $X_d(\mathbb{C}) = \text{PSL}(\mathcal{O} \oplus \mathcal{O}^\vee) \backslash \mathcal{H}^2$ where $\mathcal{O}$ is the ring of integers of $F = \mathbb{Q}(\sqrt{d}), d > 0$ square-free. There is an isomorphism between $\text{SO}(2, 2)$ and $\text{SL}(2, F) \otimes_{\mathbb{Q}} \mathbb{R}$. Fix an integral ideal $\mathcal{A}$ in the same genus as $\mathcal{O}^\vee$ and let $\delta = \text{N}(\mathcal{A})$ be the norm of $\mathcal{A}$. Let $\sigma$ be the non-trivial Galois automorphism of $F$ determined by $\sigma : \sqrt{d} \mapsto -\sqrt{d}$. As in [27] and in [55], Chapter V (but with some minor differences in conventions) we let

$$Y_d = \{ M \in M_2(F) : M = \begin{pmatrix} a\sqrt{d} & \alpha \\ -\alpha^\sigma & b\sqrt{d} \end{pmatrix}, \alpha \in F, a, b \in \mathbb{Q} \}.$$ 

As a $\mathbb{Q}$-vector space $Y_d$ is isomorphic to $\mathbb{Q}^4$. Define $Q_d : Y_d \to \mathbb{Q}$ to be the quadratic form given by

$$Q_d[M] = \det M = abd + \alpha \alpha^\sigma.$$ 

Then $Q_d$ has signature $(2, 2)$ and we may embed $\text{SL}(2, F)$ into $\text{SO}(Q)$ by the action

$$g : M \mapsto g^\sigma M g^{-1}, \quad g \in \text{SL}(2, F).$$

This induces a representation

$$\text{SL}(2, F) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \to \text{SO}(Q) \simeq \text{SO}(2, 2)$$

and a corresponding isomorphism between $\mathcal{H}^2$ and the majorant space $\mathcal{H}_d = \mathcal{H}_{Q_d}$ of $Q_d$. In $Y_d$ we can define the lattice of “integral elements” given by

$$Y_d(\mathbb{Z}) = \{ M = \begin{pmatrix} a\sqrt{d} & \alpha \\ -\alpha^\sigma & b\sqrt{d}/\delta \end{pmatrix} : \alpha \in \mathcal{A}^{-1}, a, b \in \mathbb{Z} \}.$$ 

For $\lambda \in \mathbb{R}$, $\lambda \neq 0$ the group $\text{SO}(Q_d)$ acts transitively on the $M \in Y_d$ with $Q_d[M] = \lambda$ and the isotropy group of any such $M$ is isomorphic to $\text{SO}(2, 1)$. Therefore, one may assume that $\lambda > 0$. 


For $M \in Y_d$ with $Q_d[M] > 0$ let

$$\mathcal{H}_M = \{(z_1, z_2) \in \mathcal{H}^2 : (z_2, 1) M \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0\}.$$

By checking at $z = (\sqrt{-1}, \sqrt{-1})$ and using transitivity, we see that $\mathcal{H}_M$ is isomorphic to the symmetric space for $\text{SO}(2,1) \sim \text{SL}(2, \mathbb{R})$. Namely, it is the graph of a fractional linear fractional transformation and is therefore a copy of $\mathcal{H}$ embedded into $\mathcal{H}^2$.

Let $n$ be a positive square-free integer. For $M \in Y_d(\mathbb{Z})$ with $Q_d[M] = n$ let

$$\Gamma_M = \{ g \in \text{SL}(\mathbb{O} \oplus \mathbb{O}^\vee) : {}^t g^\sigma M g = M \}. $$

Let $\mathcal{X}_M$ be the image of the curve $\Gamma_M \setminus \mathcal{H}_M$ in $\text{SL}(\mathbb{O} \oplus \mathbb{O}^\vee) \setminus \mathcal{H}^2$. Finally $\mathcal{X}_{d,n}$ is defined as the curve given by the union of all the $\mathcal{X}_M$, $M \in Y_d(\mathbb{Z})$, $Q_d[M] = n$. It is called a modular curve and is non-trivial if and only if for some $\alpha \in F$,

$$ n \equiv N(\alpha) N(A) \mod d. $$

Moreover, from [55], p102, all irreducible components of $\mathcal{X}_{d,n}$ have the same volume. The curve $\mathcal{X}_{d,n}$ corresponds to abelian surfaces whose endomorphism ring contains an order in a quaternion algebra. Namely, let $Q_{d,n}$ be the quaternion algebra over $\mathbb{Q}$ with parameters $(d, -n/\delta d)$: it has basis elements 1, $i$, $j$, $k$ where

$$ i^2 = d, \quad j^2 = -\frac{n}{\delta d}, \quad k = ij = -ji. $$

For $M \in Y_d(\mathbb{Z})$ with $Q_d[M] = n$ the following algebra is isomorphic to $Q_{d,n}$ (see [55], Chapter V, Proposition (1.5)),

$$ Q_M = \{ g \in M_2(F) : {}^t g^\sigma M g = \det(g)M \} $$

and contains the order of discriminant $n^2$ given by

$$ \mathbb{O}_M = Q_M \cap \begin{pmatrix} \mathbb{O} & A^{-1} \\ A & \mathbb{O} \end{pmatrix}. $$

For an abelian surface $A$, we have $\text{End}(A)$ contains $\mathbb{O}_M$ if and only if

$$ A(\mathbb{C}) \simeq \mathbb{C}^2/\mathbb{O}^\vee + z.\mathbb{O} $$

with $z = (z_1, z_2) \in \mathcal{H}_M$.

We now turn to studying the case treated in [53] where $m = 3$, $(p,q) = (2,1)$ but we work over a totally real field $F$, with $[F : \mathbb{Q}] = g$ and so

$$ \text{SL}(2,F) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \text{SL}(2,\mathbb{R})^g \simeq \text{SO}(2,1)^g. $$

Let $\sigma_1, \ldots, \sigma_g$ be the Galois embeddings of $F$ into $\mathbb{R}$ and for $\alpha \in F$, let $\alpha^{(j)}$, $j = 1, \ldots, g$ denote its Galois conjugates. Let $A$ be a fractional ideal of $F$ and let $L = A^{-1} \oplus \mathbb{O} \oplus A$ in $F^3$. Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form

$$ Q(x) = x_2^2 - 4x_1x_3, \quad x = {}^t(x_1, x_2, x_3) \in \mathbb{R}^3. $$
Let $\Delta \in \mathcal{O}$, $\Delta \neq 0$, and consider the set

$$W_{\Delta} = \{h \in L : Q(h) = \Delta\}.$$ 

As in §3 let $h(\Delta)$ be the number of $\Gamma_{\mathcal{A}} = \text{SL}(\mathcal{O} \oplus \mathcal{A})$-orbits of vectors $h \in W_{\Delta}$. Let $h = (\alpha, \beta, \gamma)$ and consider the quadratic equations,

$$(4.2) \quad \alpha(j)z^{2} + \beta(j)z + \gamma(j) = 0, \quad j = 1, \ldots, g.$$ 

Then, if $\alpha \neq 0$, we associate to these equations the points

$$z_{j}^\pm = \frac{-\beta(j) \pm \sqrt{\Delta(j)}}{2\alpha(j)}, \quad j = 1, \ldots, g,$$

where on the right hand side of $(4.2)$ we choose $z_{j}^+ \in \mathcal{H}$ if $\Delta \ll 0$ and we choose $z_{j}^- \in \mathcal{H}$ if $\Delta \gg 0$ (these choices will depend on the sign of $\alpha(j)$). If $\alpha = 0$, let $z_{j}^+ = \sqrt{-1}\infty$ and $z_{j}^- = \gamma_{j}/\beta_{j}$. Let $z_{h} = (z_{j}^+)^{g}_{j=1} \in \mathbb{C}^{g}$ and let $\Gamma_{h}$ be the stabilizer in $\Gamma_{\mathcal{A}}$ of $z_{h}^+$. If $\Delta \ll 0$ (totally negative) let,

$$\Lambda_{\Delta} = \{z_{h} = (z_{j}^+)^{g}_{j=1} \in \mathcal{H}^{g} \mod \Gamma_{\mathcal{A}}, h \in W_{\Delta}\}.$$ 

We refer to this as the set of Heegner points associated to $\Delta$. It has cardinality $h(\Delta)$. If $\Delta \gg 0$ (totally positive), then $\Gamma_{h}$ is a discrete subgroup of rank $g$ of $\Gamma_{h}(\mathbb{R}) \simeq (\mathbb{R}_{>0})^{9}$, embedded into $G(\mathbb{R}) = \text{PSL}(2, \mathbb{R})^{g}$ using Galois embeddings. The image of $\Gamma_{h}(\mathbb{R})$ in $\Gamma \setminus G(\mathbb{R})$ determines $C_{h}$, the real $g$ dimensional variety in $\Gamma_{\mathcal{A}} \setminus \mathcal{H}^{g}$ obtained by reducing mod $\Gamma_{\mathcal{A}}$ the product of the $g$ semi-circle geodesics in $\mathcal{H}$ joining $z_{j}^-$ to $z_{j}^+$, $j = 1, \ldots, g$. These geodesics are given by the equations

$$2\alpha(j)|w_{j}|^{2} + \beta(j)(w_{j} + \overline{w}_{j}) + 2\gamma(j) = 0, \quad j = 1, \ldots, g.$$ 

Let $\mathcal{G}_{\Delta}$ denote the set of representatives of such $C_{h} \mod \Gamma_{\mathcal{A}}$. In the case $g = 1$, $\Delta = d > 0$, these are the primitive closed geodesics $C \in \Lambda_{d}$ considered in Theorem 1 of [14].

### 5. Cuspidal Weyl sums and Equidistribution in genus 2

As in §3, let $Q$ be a quadratic form in $m$ variables of signature $(p, q)$ where $p, q \geq 0$ and $p + q = m$ and let $\Gamma$ be a lattice in $\Omega(Q)$. Let $c_{1}, c_{2}, \ldots$ be constants depending only on $Q$ and $\Gamma$. Let $d \in \mathbb{Z}$, $d \neq 0$ be a square-free integer and $\Gamma_{j}$, $j = 1, \ldots, H(d)$ the stabilizers in $\Gamma$ of a set of representatives mod $\Gamma$ of the set

$$\{x \in \mathbb{R}^{m} : Q[x] = d\}.$$ 

Let $\varphi$ be a Maass cusp form of weight 0 on $\overline{\Gamma \setminus \mathcal{H}_{Q}}$ with eigenvalue $\lambda'$. Then we may apply Proposition 2.1 to

$$f(z) = v^{m/4} < \varphi, \theta(z)>.$$
Therefore $f(z)$ is a cuspidal Maass form of discriminant $D$ for $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$, where $D$ and $N$ are determined by $Q$ as in [14], p81, and of weight $k = p - \frac{m^2}{2}$ with eigenvalue $\lambda = \frac{1}{4}(\lambda' + m - \frac{m^2}{4})$. Let

$$W_{\text{cusp}}(d, \lambda) = \mu(d)^{-1} \left( \sum_{i=1}^{H(d)} \int_{\Gamma_i \backslash \Gamma(\mathbb{R})} \varphi(\gamma)d\gamma \right)$$

where we define

$$\mu(d) = \sum_{i=1}^{H(d)} \text{Vol}(\Gamma_i \backslash \Gamma(\mathbb{R})).$$

From Proposition 2.2 we have,

$$W_{\text{cusp}}(d, \lambda) = c_3 |d|^{m/4} \mu(d)^{-1} \rho(d).$$

By Siegel’s mass formula $\mu(d)$ is a product of local densities. For $m \geq 4$ we have the effective lower bound

$$\mu(d) \geq c_4 |d|^{m/2 - 1}.$$ 

Therefore,$$
W_{\text{cusp}}(d, \lambda) \leq c_5 |d|^{1 - \frac{m}{4}} \rho(d).
$$

By well-known bounds for Fourier coefficients of cusp forms of integral and half-integral weight (or using the stronger Theorem 5 of [14]), we know that for $m \geq 4$

$$\lim_{|d| \to \infty} |d|^{1 - \frac{m}{4}} \rho(d) = 0.$$

This implies the following result.

**Proposition 5.1.** For $m \geq 4$,

$$\lim_{|d| \to \infty} W_{\text{cusp}}(d, \lambda) = 0.$$

The Equidistribution in genus 2 statement of §1 is a direct corollary of Proposition 5.1 and the discussion of §4 once we treat in a similar way the eigenfunctions of the continuous spectrum of $\Delta_Q$ and show vanishing results for their Weyl sums. We hope to return to this in a later paper.

### 6. CUSPIDAL WEYL SUMS IN THE HILBERT MODULAR CASE

In [4] the integral quadratic form $Q$ is in $m = 3$ variables and is of signature $(p, q) = (2, 1)$, and we work over a totally real field $F$ of degree $g$ over $\mathbb{Q}$ and with the lattice $\Gamma_A$ in $G = \text{SL}(2, \mathbb{R})^g$. Recall that $\mathcal{A}$ is a fractional ideal in $F$ and that we denote by $\mathcal{O}$ the ring of integers of $F$ and by $L$ the lattice $\mathcal{A}^{-1} \oplus \mathcal{O} \oplus \mathcal{A}$ in $F^3$. Let $\Delta \in \mathcal{O}$, $\Delta \neq 0$ and let $\Gamma_i$, $i = 1, \ldots, h(\Delta)$, be the stabilizers in $\Gamma_A$ of a set of representatives mod $\Gamma_A$ of the set

$$\{ h \in L : Q[h] = \Delta \}. $$

Let \( \varphi \in U \) be an eigenfunction of \( \Delta_{0}^{(j)} \) with eigenvalue \( \lambda_{j}, \ j = 1, \ldots, g \). Then we may apply Proposition 3.1 to

\[
f(z) = \int_{\Gamma_{A}\setminus G} \varphi(g) \theta(z, g) dg.
\]

Suppose \( \Delta \ll 0 \). With the notations of \( \S \) Case (i), let

\[
W_{\text{cusp}}(\Delta, \lambda) = h(\Delta)^{-1} \sum_{i=1}^{h(\Delta)} \frac{1}{|\Gamma_{i}|} \varphi(\overline{g}^{(i)}).
\]

From the discussion of \( \S 5 \), we have

\[
W_{\text{cusp}}(\Delta, \lambda) = h(\Delta)^{-1} \sum_{z \in \Lambda_{\Delta}} \frac{1}{|\Gamma_{z}|} \varphi(z)
\]
where \( \Gamma_{z} \) is the stabilizer of \( z \) in \( \Gamma_{A} \) and \( h(\Delta) = \text{Card}(\Lambda_{\Delta}) \). We may take

\[
\overline{g}^{(i)} = \left( \begin{array}{cc} 1 & h_{2}^{(i)}/2h_{1}^{(i)} \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \sqrt{|\Delta_{j}|}/2h_{1}^{(i)} & 0 \\ 0 & \sqrt{|\Delta_{j}|}/2h_{1}^{(i)} \end{array} \right)^{1/2}.
\]

Then, from Proposition 3.2, we have the following,

**Proposition 6.1.** For \( \Delta \in \emptyset, \Delta \ll 0 \) we have

\[
W_{\text{cusp}}(\Delta, \lambda) = 2^{g}(4\pi)^{3g/4}|N_{F/\mathbb{Q}}(\Delta)|^{3/4} h(\Delta)^{-1} \rho(\Delta).
\]

Suppose \( \Delta \gg 0 \). With notations as in \( \S 3 \) let

\[
W_{\text{cusp}}(\Delta, \lambda) = h(\Delta)^{-1} \sum_{i=1}^{h(\Delta)} \int_{\Gamma_{i}\setminus \Gamma_{i}(\mathbb{R})} \varphi(\gamma) d\gamma,
\]
where

\[
\mu(\Delta) = \sum_{i=1}^{h(\Delta)} \int_{\Gamma_{i}\setminus \Gamma_{i}(\mathbb{R})} d\gamma.
\]

From the discussion of \( \S 5 \), we have

\[
W_{\text{cusp}}(\Delta, \lambda) = \mu(\Delta)^{-1} \sum_{C \in G_{\Delta}} \int_{C} \varphi(z) ds,
\]
where

\[
\mu(\Delta) = \sum_{C \in G_{\Delta}} \text{Vol}(C).
\]

To explain this last quantity, we consider the Euclidean hyperbolic distance in each copy of \( H_{\mathbb{R}} \) in \( H^{g} \) given by

\[
ds_{j}^{2} = y_{j}^{-2} \left( (dx_{j})^{2} + (dy_{j})^{2} \right), \quad j = 1, \ldots, g.
\]
Then we have the real $g$-form

$$dS = \prod_{j=1}^{g} ds_j$$

and

$$\text{Vol}(C) = \int_C dS.$$

By (3.28) we have the following,

**Proposition 6.2.** For $\Delta \in \mathcal{O}$, $\Delta \gg 0$ we have

$$W_{\text{cusp}}(\Delta, \lambda) = 2^g(4\pi)^{g/4}N_{F/Q}(\Delta)^{3/4}\mu(\Delta)^{-1}\rho(\Delta).$$

These results enable us bound the cuspidal Weyl sums from above in terms of the Fourier coefficients of Maass cusp eigenforms of weight $1/2$ and level 4. From Propositions 6.1 and Proposition 6.2 we deduce directly the following.

**Lemma 6.3.** For $\Delta \ll 0$ and as $|N_{F/Q}(\Delta)| \to \infty$ we have,

$$|W_{\text{cusp}}(\Delta, \lambda)| \ll \frac{|N_{F/Q}(\Delta)|^{3/4}}{h(\Delta)}\rho(\Delta).$$

For $\Delta \gg 0$ and as $N_{F/Q}(\Delta) \to \infty$ we have,

$$|W_{\text{cusp}}(\Delta, \lambda)| \ll \frac{N_{F/Q}(\Delta)^{3/4}}{\mu(\Delta)}\rho(\Delta).$$

The implied constants depend only on the field $F$ (and in fact can be bounded above explicitly by a function of $g$ only).

7. Eisenstein Weyl sums: the case of the Hilbert modular group for class number 1

We continue with the notations of §§3–5. The methods of Maass, in particular Proposition 2.1 and Proposition 3.1, do not apply to the eigenfunctions of the continuous spectrum of the Laplacian, which is non-trivial in all cases considered in this paper as the group actions are not co-compact. These eigenfunctions are furnished by the Eisenstein series. We are therefore led to consider averages or Weyl sums as in §5 with cusp forms replaced by Eisenstein series.

We shall restrict ourselves to the case $m = 3$, $(p, q) = (2, 1)$ and to the Hilbert modular group $\Gamma = \Gamma_A = \text{PSL}(2, \mathcal{O})$ where $\mathcal{O} = \mathcal{O}_F$ is the ring of integers of a totally real field $F$ of degree $g \geq 1$ with class number 1. We repeatedly use facts about Eisenstein series from [16], [19], [16], [17], and [18]. The generalization to arbitrary class number and to arbitrary $\Gamma_A$ should be straightforward if somewhat technical, with some necessary material available in [51]. In order to extend to the case $g > 1$ the arguments of [12] on Eisenstein Weyl sums, we will generalize in this section some
classical arguments due to Hecke [22] and Kronecker [26]. It would be of interest to consider also the cases $m = 4$, $(p, q) = (2, 2)$ and $m = 5$, $(p, q) = (3, 2)$, in order, for example, to deduce equidistribution results on the non-cuspidal part of the corresponding $L^2$-spaces.

The Eisenstein series are $\Gamma$-automorphic eigenfunctions of the Laplacians $\Delta_j \Gamma$ corresponding to the continuous part of the spectrum. As we shall see shortly, they are functions of $(z, s) \in \mathcal{H} \times \mathbb{C}$, and $m \in \mathbb{Z}^g - 1$. Let,

$$L^2(\mathcal{H}^g) = \{ \varphi : \mathcal{H}^g \to \mathbb{C} : \varphi(\gamma z) = \varphi(z), \gamma \in \Gamma, \int_{\Gamma \backslash \mathcal{H}^g} |\varphi|^2 N(v)^{-2} N(du) N(dv) < \infty \}.$$ 

Using general results and more specifically those of [16] and [47], we have a decomposition,

$$L^2(\mathcal{H}^g) = L^2_{\text{cusp}}(\mathcal{H}^g) \oplus \mathcal{R} \oplus \mathcal{E},$$

where $\mathcal{E}$ is generated in an appropriate $L^2$-sense by the Eisenstein series evaluated at $s = \frac{1}{2} + it$, $t \in \mathbb{R}$, and $\mathcal{R}$ is generated by the residues of their finitely many poles in $s \in (\frac{1}{2}, 1]$ (the Eisenstein series are not themselves $L^2$-integrable). For $g \geq 1$, the only such pole occurs at $s = 1$ and has residue given by the volume of the fundamental region of $\Gamma$. The cuspidal part of the $L^2$-decomposition is given by (3.6).

As $F$ is assumed to have class number 1, the group $\Gamma$ has 1 cusp at infinity with stabilizer $\Gamma_\infty$. The Eisenstein series are of the form

$$(7.2) \quad E(z, s, m) = \sum_{\gamma \in \Gamma \backslash \Gamma} y^s(\gamma(z)) \lambda_m(y(\gamma(z))), \quad \text{Re}(s) > 1.$$ 

Here $z = (z_j)_{j=1}^g \in \mathbb{C}^g$ with $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$ and $y(\gamma(z)) = \prod_{j=1}^g \text{Im}(\gamma(j)(z_j))$ where $\gamma(j)$ is obtained from $\gamma$ by applying the $j$-th Galois embedding to its entries, for a given ordering of the Galois embeddings of $F$ into $\mathbb{R}$, starting with the identity embedding. Moreover, we have $s \in \mathbb{C}$ and $m = (m_q)_{q=1}^{g-1} \in \mathbb{Z}^{g-1}$ with $\lambda_m$ an exponential sum similar to a Grössencharakter. Namely, there are parameters $e_j^{(q)} \in \mathbb{R}$, $q = 1, \ldots, g - 1$, $j = 1, \ldots, g$, determined by the choice of basis of the unit group $\mathcal{O}^*$ of $\mathbb{O}$, such that

$$\lambda_m(z) = \prod_{j=1}^g \prod_{q=1}^{g-1} |z_j|^{\pi i m_q e_j^{(q)}}.$$ 

Moreover, if $\varepsilon_q = (\varepsilon_q^{(1)}, \ldots, \varepsilon_q^{(g)})$, $q = 1, \ldots, g - 1$, are totally positive generators of the group $\mathcal{O}^*$ embedded in $(\mathbb{R}^*)^g$ by the Galois embeddings of $F$ into $\mathbb{R}$, then the $e_j^{(q)}$ are determined by the equations

$$\sum_{j=1}^g e_j^{(q)} = 0, \quad q = 1, \ldots, g - 1,$$
and
\[ \sum_{j=1}^{g} e_j^{(q)} \log \varepsilon_j^{(q)} = \delta_{r,q}, \quad r, q = 1, \ldots, g - 1. \]

These conditions ensure, in particular, that for \( u = (u^{(1)}, \ldots, u^{(g)}) \) in \( \mathcal{O}^{*} \) we have,

\[ (7.3) \quad \prod_{j=1}^{g} \prod_{q=1}^{g-1} |u^{(j)}|^{\pi im_j e_j^{(q)}} = 1. \]

The series \( E(z, s, m) \) has a meromorphic continuation to all of \( s \in \mathbb{C} \). Full details can be found in [16], Chapter II. We shall often perform formal manipulations with the series definition of \( E(z, s, m) \) without specifying each time the domain of convergence.

We have, using [16], p47,

\[ (7.4) \quad E(z, s, m) = \sum_{\{c, d\}, (c, d) = 1} \prod_{j=1}^{g} \frac{y_j^{s_j}}{|c^{(j)} z_j + d^{(j)}|^{2s_j}}, \]

where the summation is over \( c, d \in \mathcal{O} \), with \( \{c, d\} \) meaning that pairs differing by multiplication by an element of \( \mathcal{O}^{*} \) are identified and \( (c, d) = 1 \) means that \( c, d \) generate \( \mathcal{O} \). As usual \( c^{(j)}, d^{(j)} \) are the \( j \)-th Galois conjugates of \( c, d \). Furthermore,

\[ (7.5) \quad s_j = s + \pi i \sum_{q=1}^{g-1} m_q e_j^{(q)}, \quad j = 1, \ldots, g. \]

In that same reference, it is shown that if

\[ (7.6) \quad F(z, s, m) = \sum_{\{c, d\}} \prod_{j=1}^{g} \frac{y_j^{s_j}}{|c^{(j)} z_j + d^{(j)}|^{2s_j}} \]

then,

\[ (7.7) \quad F(z, s, m) = L(2s, \lambda_{-2m}) E(z, s, m), \]

where \( L(s, \lambda_m) \) is the Hecke zeta function given by

\[ (7.8) \quad L(s, \lambda_m) = \sum_{(b)} \frac{\lambda_m(b)}{|N_{F/\mathbb{Q}}(b)|^s}. \]

Here, the sum is over the (principal) integral ideals \( (b) \) of \( F \) and

\[ \lambda_m(b) = \prod_{j=1}^{g} \prod_{q=1}^{g-1} |y_j^{(j)}|^{\pi im_j e_j^{(q)}}. \]

This last expression is well-defined thanks to (7.3).
With the notation of §4 and §5, the Eisenstein Weyl sums are given, for $\Delta \ll 0$, by

$$W_{\text{Ei}}(\Delta, t, m) = \frac{1}{h(\Delta)} \sum_{z_h \in \Lambda_{\Delta}} E(z_h, \frac{1}{2} + it, m),$$

where $h(\Delta)$ is the cardinality of $\Lambda_{\Delta}$ and, for $\Delta \gg 0$, by

$$W_{\text{Ei}}(\Delta, t, m) = \frac{1}{\mu(\Delta)} \sum_{C \in G_{\Delta}} \int_C E(z, \frac{1}{2} + it, m) \, dz,$$

where

$$\mu(\Delta) = \sum_{C \in G_{\Delta}} \text{Vol}(C).$$

In order to relate these Weyl sums to Fourier coefficients of half-integral weight Hilbert-Maass Eisenstein series, we need to generalize for $g > 1$ some classical arguments due to Hecke [22] and Kronecker [26] that apply to the special case $g = 1$ as in [14]. In particular, the formulae of Proposition 7.1 and Proposition 7.2 at the end of this section generalize these classical results.

From now on, we assume that $\Delta \in \mathcal{O}$ generates the ideal given by the relative discriminant of $L = F(\sqrt{\Delta})$ over $F$. This replaces the fundamental discriminant assumption in [14], Theorem 1. As $F$ has class number 1, the ideals of $L$ are free $\mathcal{O}$-modules of rank 2. Let $\rho \to \overline{\rho}$, $\rho \in L$, denote the non-trivial automorphism of $L$ over $F$. We may choose a relative basis $\{1, \Omega\}$ of $\mathcal{O}_L$ over $\mathcal{O}$ such that $\Delta = (\Omega - \overline{\Omega})^2$. Let $\mathcal{O}_L^*$ denote the units of $\mathcal{O}_L$ and $\chi_{L/F}$ denote the relative field character for $L$ over $F$.

Consider first the case $\Delta \ll 0$. By (7.11) we have

$$L(2s, \lambda_{-2m}) \sum_{z_h \in \Lambda_{\Delta}} E(z_h, s, m) = \sum_{z_h \in \Lambda_{\Delta}} \sum_{\langle c, d \rangle \in \mathcal{O}} \prod_{j=1}^g Q_h^{(j)}(c^{(j)}, d^{(j)})^{-s_j},$$

where

$$Q_h^{(j)}(c^{(j)}, d^{(j)}) = \frac{|c^{(j)}z_j + d^{(j)}|^2}{y_j}, \quad j = 1, \ldots, g.$$

In the notation of §4 (except we denote $z_j^+$ by $z_j$), for $h = (\alpha, \beta, \gamma) \in \mathcal{O}^3$, we have

$$Q_h^{(j)}(c^{(j)}, d^{(j)}) = \frac{2}{\sqrt{|\Delta^{(j)}|}} q_h^{(j)}(-d^{(j)}, c^{(j)}),$$

where

$$q_h^{(j)}(x, y) = \alpha^{(j)}x^2 + \beta^{(j)}xy + \gamma^{(j)}y^2 = \alpha^{(j)}(x - z_jy)(x - \overline{z}_jy).$$

Therefore,

$$\prod_{j=1}^g Q_h^{(j)}(c^{(j)}, d^{(j)})^{-s_j} = \prod_{j=1}^g |\Delta^{(j)}/4|^{s_j/2} \prod_{j=1}^g q_h^{(j)}(-d^{(j)}, c^{(j)})^{-s_j}.$$
There is a bijection between the ideal classes of $\mathcal{O}_L$ and the points $z_h \in \Lambda_\Delta$. To $h = (\alpha, \beta, \gamma) \in W_\Delta$, we associate the ideal $\mathcal{A}_h$ with basis $\{\alpha, \beta + \sqrt{\Delta}\}$. The relative norm over $F$ of $\mathcal{A}_h$ is generated by $\alpha$. The relative norms of the integral ideals in the ideal class $\text{Cl}(\mathcal{A}_h)$ of $\mathcal{A}_h$ are generated by $q_h(-d, c)$, for $c, d \in \mathcal{O}$. Therefore,

$$
(7.16) \quad \sum_{\{c,d\}}^{g} \prod_{j=1}^{g} Q_h^{(j)}(c^{(j)}, d^{(j)})^{-s_j} = 2^{-sg} |N_{F/Q}(\Delta)|^{s/2} \lambda_{m/2}(\Delta/4) \times
$$

$$
\times \sum_{\mathcal{A} \in \text{Cl}(\mathcal{A}_h)} \frac{\lambda_{m}(N_{L/F}(\mathcal{A}))}{N_{L/Q}(\mathcal{A})^s}.
$$

Combining (7.11) and (7.16) we conclude that

$$
(7.17) \quad 2^{sg} L(2s, \lambda_{-2m}) \sum_{z_h \in \Lambda_\Delta} E(z_h, s, m) = |N_{F/Q}(\Delta)|^{s/2} \lambda_{m/2}(\Delta/4) \times
$$

$$
\times L(s, \lambda_{-m}, L),
$$

where

$$
(7.18) \quad L(s, \lambda_{-m}, L) = \sum_\mathcal{A} \frac{\lambda_{m}(N_{L/F}(\mathcal{A}))}{N_{L/Q}(\mathcal{A})^s} = \sum_\mathcal{A} \sum_{\mathcal{A} \in \text{Cl}(\mathcal{A}_h)} \frac{\lambda_{m}(N_{L/F}(\mathcal{A}))}{N_{L/Q}(\mathcal{A})^s},
$$

with $\mathcal{A}$ ranging over the non-zero integral ideals of $L$ and $\mathcal{A}$ ranging over the ideal classes of $L$. From (7.17) and (7.18) we deduce,

$$
(7.19) \quad 2^{sg} L(2s, \lambda_{-2m}) \sum_{z_h \in \Lambda_\Delta} E(z_h, s, m) = |N_{F/Q}(\Delta)|^{s/2} \lambda_{m/2}(\Delta/4) \times
$$

$$
\times L(s, \lambda_{-m}) L(s, \chi_{L/F} \lambda_{-m}),
$$

where

$$
(7.20) \quad L(s, \chi_{L/F} \lambda_{-m}) = \sum_{(b)} \frac{\chi_{L/F}(b) \lambda_{m}(b)}{|N_{F/Q}(b)|^s},
$$

the sum ranging over the ideals $(b)$ of $F$. We deduce finally the following result.

**Proposition 7.1.** For $\Delta \ll 0$ we have,

$$
(7.21) \quad W_Eis(\Delta, t, m) = 2^{-sg} \frac{L\left(\frac{1}{2} + it, \lambda_{-m}\right)}{L(1 + it, \lambda_{-2m})} \times
$$

$$
\times \frac{|N_{F/Q}(\Delta)|^{\frac{s}{2} + it} \lambda_{m/2}(\Delta/4)}{h(\Delta)} \frac{1}{\lambda_{m/2}(\Delta/4)} L\left(\frac{1}{2} + it, \chi_{F(\sqrt{\Delta})/F} \lambda_{-m}\right).
$$

Now consider the case $\Delta \gg 0$. Combining (7.6) and (7.10) we have

$$
(7.22) \quad L(2s, \lambda_{-2m}) \sum_{C_h \in \mathcal{G}_\Delta} \int_{C_h} E(z, s, m) \, ds = \sum_{C_h \in \mathcal{G}_\Delta} \int_{C_h} \sum_{\{c,d\}}^{g} \prod_{j=1}^{g} \frac{y_j^{s_j}}{|c^{(j)} + d^{(j)}|^{2s_j}} \, ds.
$$
We adapt some ideas of Hecke \cite{22} for the case $g = 1$, as they are explained in \cite{59}, \cite{60}, to the general case $g \geq 1$. Let $\mathcal{A}$ be a fixed ideal class of $L = F(\sqrt{\Delta})$ and $\mathcal{B}$ a fixed element of $\mathcal{A}^{-1}$. We have a correspondence $\mathcal{A} \mapsto \mathcal{A}\mathcal{B} = (\eta)$, which is a bijection between the set of ideals of $\mathcal{A}$ and the set of principal ideals with $\eta \in \mathcal{B}$.

For a fixed ideal class $\mathcal{A}$, define

\begin{equation}
L(s, \lambda_m, \mathcal{A}) = \sum_{\mathcal{A} \in \mathcal{A}} \frac{\lambda_m(N_{L/F}(\mathcal{A}))}{N_{L/Q}(\mathcal{A})^s}.
\end{equation}

Define as in \cite{7.18},

\begin{equation}
L(s, \lambda_m, L) = \sum_{\mathcal{A}} \frac{\lambda_m(N_{L/F}(\mathcal{A}))}{N_{L/Q}(\mathcal{A})^s} = \sum_{\mathcal{A}} L(s, \lambda_m, \mathcal{A}),
\end{equation}

with $\mathcal{A}$ ranging over the non-zero integral ideals of $L$ and $\mathcal{A}$ ranging over the ideal classes of $L$. For $\mathcal{B}$ a fixed element of $\mathcal{A}^{-1}$, we have

\begin{equation}
L(s, \lambda_m, \mathcal{A}) = N_{L/Q}(\mathcal{B})^s \lambda_m(N_{L/F}(\mathcal{B})) \sum_{\mathcal{A} \in \mathcal{A}} \frac{\lambda_m(N_{L/F}(\mathcal{A}))}{N_{L/Q}(\mathcal{A})^s}.
\end{equation}

Two numbers $\eta_1$, $\eta_2 \in \mathcal{B}$ define the same principal ideal if and only if $\eta_1 = \epsilon\eta_2$, for $\epsilon \in \mathcal{O}_L^\times$. Hence,

\begin{equation}
L(s, \lambda_m, \mathcal{A}) = N_{L/Q}(\mathcal{B})^s \lambda_m(N_{L/F}(\mathcal{B})) \sum'_{\eta \in \mathcal{B}/\mathcal{O}_{L}^\times} \frac{\lambda_m(N_{L/F}(\eta))}{|N_{L/Q}(\eta)|^s},
\end{equation}

with $\sum'$ denoting a sum over non-zero elements. We have an exact sequence

\begin{equation}
1 \to \mathcal{O}_{L,1}^\times \to \mathcal{O}_L^\times \to \mathcal{O}^\times,
\end{equation}

where the right most arrow is given by the reduced norm from $L$ to $F$ and $\mathcal{O}_{L,1}^\times$ is the group of units of $\mathcal{O}_L$ of reduced norm 1. The image $N_{L/F}(\mathcal{O}_L^\times)$ is of finite index in $\mathcal{O}^\times$. Moreover, $\mathcal{O}_{L,1}^\times$ is a free abelian group of rank $g$ (as remarked already in \cite{3}). Notice that $\mathcal{O}_{L,1}^\times \cap \mathcal{O}^\times = \{ \pm 1 \}$ and that the group $\mathcal{O}_{L,1}^\times \mathcal{O}^\times$ is of finite index $i$ in $\mathcal{O}_L^\times$. We have therefore from \cite{7.26},

\begin{equation}
L(s, \lambda_m, \mathcal{A}) = N_{L/Q}(\mathcal{B})^s \lambda_m(N_{L/F}(\mathcal{B}))i^{-1}S(s, \lambda_m, \mathcal{B}),
\end{equation}

where

\begin{equation}
S(s, \lambda_m, \mathcal{B}) = \sum'_{\eta \in \mathcal{B}/\mathcal{O}_{L,1}^\times \mathcal{O}^\times} \frac{\lambda_m(N_{L/F}(\eta))}{|N_{L/Q}(\eta)|^s}.
\end{equation}

Let $\varepsilon^{(i)}$, $i = 1, \ldots, g$, be generators of $\mathcal{O}_{L,1}^\times/\{ \pm 1 \}$. Let $\xi_j$ be the extension to $L$ of the $j$-th Galois embedding of $F$ into $\mathbb{R}$ chosen so that $\xi_j(\sqrt{\Delta}) = \sqrt{\Delta^{(j)}} > 0$, $j = 1, \ldots, g$. Let $\eta_j = \xi_j(\eta)$ for $\eta \in L$. We may suppose that $\varepsilon_j^{(i)} > 0$ for $i, j = 1, \ldots, g$. We have

\begin{equation}
S(s, \lambda_m, \mathcal{B}) = \sum'_{\eta \in \mathcal{B}/\mathcal{O}^\times} \frac{\lambda_m(N_{L/F}(\eta))}{|N_{L/Q}(\eta)|^s} \left| \eta_1 \eta_2 \cdots \eta_g \eta_i \right|^{-s_1} \cdots \left| \eta_2 \eta_3 \cdots \eta_g \right|^{-s_g}.
\end{equation}
Hecke observed the following identity (see [59], p161): for \( a, b \in \mathbb{R}, a, b \neq 0, \)

\[
\int_{-\infty}^{\infty} \frac{dv}{(a^2v^2 + b^2e^{-v})^s} = \frac{c(s)}{|ab|^s},
\]

where

\[
c(s) = \int_{-\infty}^{\infty} \frac{dv}{(e^v + e^{-v})^s}.
\]

Let \( N(c(s)) = \prod_{j=1}^{g} c(s_j) \). We deduce that,

\[
N(c(s))S(s, \lambda_m, B) = \sum'_{\eta \in B/\mathcal{O}^*} \int_{\mathcal{L}_g \setminus \mathbb{R}^g} \frac{N(dv)}{\prod_{j=1}^{g} (\eta_j^2 e^{v_j} + \eta_j^{-2} e^{-v_j})^{s_j}},
\]

where \( N(dv) = \prod_{j=1}^{g} dv_j \).

Now suppose the ideal \( \mathcal{B} \) has basis \( \{1, w\} \) (with \( w > \overline{w} \)). Then \( \eta = cw + d \) for \( c, d \in \mathcal{O} \) and \( \eta_j = c^{(j)} w_j + d^{(j)} \) and,

\[
\eta_j^2 e^{v_j} + \eta_j^{-2} e^{-v_j} = (c^{(j)} w_j + d^{(j)})^2 e^{v_j} + (c^{(j)} \overline{w}_j + d^{(j)})^2 e^{-v_j}.
\]

Let \( w_j^+ = \max(w_j, \overline{w}_j) \) and \( w_j^- = \min(w_j, \overline{w}_j), j = 1, \ldots, g \). Make the change of variables,

\[
z_j = \frac{w_j^+ - \sqrt{-1} e^{v_j} + w_j^-}{\sqrt{-1} e^{v_j} + 1}, \quad j = 1, \ldots, g.
\]

Then as \( v_j \) ranges from \(-\infty\) to \( \infty \) the variable \( z_j \) runs over the geodesic in \( \mathcal{H} \) joining \( w_j^- \) to \( w_j^+ \). A direct calculation shows, with \( y_j = \text{Im}(z_j) \), that

\[
y_j |c^{(j)} z_j + d^{(j)}|^{-2} = (w_j^+ - w_j^-) \left\{ e^{v_j} (c^{(j)} w_j + d^{(j)})^2 + e^{-v_j} (c^{(j)} \overline{w}_j + d^{(j)})^2 \right\}^{-1}.
\]

In the notation of [44] let \( h = (\alpha, \beta, \gamma) \in W_{\Delta} \) and let \( w_j^+ = z_j^+, w_j^- = z_j^- \). Then \( \mathcal{B} = \mathcal{B}_h = \mathcal{O} + O z_j^+ \) and \( w_j^+ - w_j^- = \sqrt{\Delta_{(j)}/|\alpha^{(j)}|} \) if \( \alpha \neq 0 \). Under the change of variables (7.36), the quotient \( \mathcal{L}_g \setminus \mathbb{R}^g \) becomes the quotient
\( \Gamma_h \backslash \Gamma_h(\mathbb{R}) \) realized as \( \mathcal{C}_h \). From (7.33) and (7.37) we deduce

\[
(7.38) \quad N(c(s))S(s, \lambda_{-m}, \mathcal{B}) = \prod_{j=1}^{g} \left( \sqrt{|\Delta(j)|/|\alpha(j)|} \right)^{-s_j} \times 
\left\{ \sum_{\{c,d\}} \int_{\mathcal{C}_h} \prod_{j=1}^{g} y_j^{s_j} |c(j)z_j + d(j)|^{-2s_j} ds \right\}.
\]

Now,

\[
(7.39) \quad \frac{N_{L/Q}(\mathcal{B})^{s} \lambda_{m}(N_{L/F}(\mathcal{B}))}{(N_{L/F}(\mathcal{B}))^{s}} = \prod_{j=1}^{g} |\alpha(j)|^{-s_j}.
\]

Let \( A_h \) denote the ideal class of \( \mathcal{B}_h \). Combining (7.6), (7.7), (7.28), (7.29), (7.38), and (7.39) we deduce that,

\[
(7.40) \quad L(2s, \lambda_{-2m}) \int_{\mathcal{C}_h} E(z, s, m) ds = N(c(s))iN_{F/Q}(\Delta)^{s/2} \lambda_{m/2}(\Delta) L(s, \lambda_{-m}, A_h).
\]

There is a bijection between the ideal classes of \( \mathcal{O}_L \) and the representatives of \( \mathcal{G}_\Delta \) with \( \mathcal{C}_h \) corresponding to \( A_h, h \in W_\Delta \). We conclude that

\[
(7.41) \quad L(2s, \lambda_{-2m}) \sum_{\mathcal{C}_h \in \mathcal{G}_\Delta} \int_{\mathcal{C}_h} E(z, s, m) ds = N(c(s))iN_{F/Q}(\Delta)^{s/2} \lambda_{m/2}(\Delta) L(s, \lambda_{-m}, L),
\]

and finally that

\[
(7.42) \quad L(2s, \lambda_{-2m}) \sum_{\mathcal{C}_h \in \mathcal{G}_\Delta} \int_{\mathcal{C}_h} E(z, s, m) ds = N(c(s))iN_{F/Q}(\Delta)^{s/2} \lambda_{m/2}(\Delta) \times 
\left( L(s, \lambda_{-m}) L(s, \chi_{L/F} \lambda_m) \right).
\]

We deduce finally the following result.

**Proposition 7.2.** For \( \Delta \gg 0 \) we have,

\[
(7.43) \quad W_{\text{Eis}}(\Delta, t, m) = N(c(s))iL(\frac{1}{2} + it, \lambda_{-m}) L(1 + it, \lambda_{-2m}) \times 
\left( \frac{N_{F/Q}(\Delta)^{t + it \lambda_{m/2}(\Delta)}}{\mu(\Delta)} L(\frac{1}{2} + it, \chi_{F(\sqrt{\Delta})/F} \lambda_{-m}) \right).
\]

From Propositions 7.1 and Proposition 7.2 we deduce directly the following.
Lemma 7.3. For $\Delta \ll 0$ and as $|N_{F/Q}(\Delta)| \to \infty$ we have,

$$|W_{Eis}(\Delta, t, m)| \ll \frac{|N_{F/Q}(\Delta)|^{\frac{1}{2}}}{h(\Delta)} L\left(\frac{1}{2} + it, \chi_{F(\sqrt{\Delta})/F^{\lambda}}\right).$$

For $\Delta \gg 0$ and as $N_{F/Q}(\Delta) \to \infty$ we have,

$$|W_{Eis}(\Delta, t, m)| \ll \frac{N_{F/Q}(\Delta)^{\frac{1}{2}}}{\mu(\Delta)} L\left(\frac{1}{2} + it, \chi_{F(\sqrt{\Delta})/F^{\lambda}}\right).$$

The implied constants depend on $t$ and on the field $F$.

In [14], the Eisenstein Weyl sums for $g = 1$ are shown to be proportional to the Fourier coefficients of Eisenstein series of half-integral weight and level 4 using explicit formulae for these coefficients derived in [20]. In [46], general formulae for Fourier coefficients of Eisenstein series of half-integral weight and level dividing 4 for the group $\text{Sp}(m, F)$ are obtained, where $F$ is a totally real algebraic number field. The case $m = 1$ gives generalizations of the formulae of [20] to the Hilbert modular case. In the notations of [46], Theorem 6.1, the product $L(s, \lambda - m)L(s, \chi_{F(\sqrt{\Delta})/F^{\lambda}})$ occurring (at $s = \frac{1}{2} + it$) in Proposition 7.1 and Proposition 7.2 is proportional to the product of the first Fourier coefficient and the $\Delta$-th Fourier coefficient $c_f(\Delta, s)$ of the Eisenstein series $E' = E'(z, s, 1/2, 0, \lambda - m, 4)$ of level 4 and weight 1/2. The $c_f(\Delta, s)$ of [46] correspond to $|N_{F/Q}(\Delta)|^{1/2} \rho(\Delta, E')$ with the conventions that we adopt in §4 (3.8).

These results enable us to bound the Eisenstein Weyl sums from above in terms of the Fourier coefficients of Eisenstein series of weight 1/2 and level 4, or alternatively central values of $L$-functions.

8. Expected subconvexity results and proof of Theorem 1.2

We continue with the assumptions and notations of §6 and §7. The equidistribution results of Theorem 1.2 would follow, without GRH, from an unconditional proof of

$$\lim_{|N_{F/Q}(\Delta)| \to \infty} W_{\text{cusp}}(\Delta, \lambda) = 0,$$

and

$$\lim_{|N_{F/Q}(\Delta)| \to \infty} W_{Eis}(\Delta, t, m) = 0.$$

As $\Delta$ is a fundamental (relative) discriminant which is totally definite, the number $h(\Delta)$ can be replaced by the class number of $\mathcal{O}_L$, $L = F(\sqrt{\Delta})$ (see for example [13], Chapter 7, [32]). Moreover, for $\Delta \gg 0$, we have by [16], p36, that $\mu(\Delta) = h(\Delta)R$ where $R$ is a regulator associated to $\mathcal{O}^*_{L,1}$ and given by

$$R = \det\left(2 \log e_j^{(i)}\right)_{i,j=1}^g.$$
The results of \([13]\) and \([17]\) show that the (ineffective) lower bounds
\[
h(\Delta) \gg_{\varepsilon} |N_{F/Q}(\Delta)|^{1/2-\varepsilon}, \quad \text{as } |N_{F/Q}(\Delta)| \to \infty,
\]
and
\[
h(\Delta)R \gg_{\varepsilon} N_{F/Q}(\Delta)^{1/2-\varepsilon}, \quad \text{as } N_{F/Q}(\Delta) \to \infty,
\]
provided by the Brauer-Siegel Theorem \([6]\), \([51]\) together with generalizations to the case \(g > 1\) of the subconvexity results for \(g = 1\) in \([14]\). Theorem 5 would imply (8.1) and (8.2). The corresponding subconvexity results for the holomorphic case have been shown in \([12]\). We would need the Fourier coefficients \(\rho(\Delta, f)\) for \(f\) a cusp form with \(L^2\)-norm 1 or an Eisenstein series, with eigenvalue \(\lambda\) and half-integral weight \(k\), to have an upper bound in the \(\Delta\)-aspect as good as \(\rho(\Delta, f) \ll_{k,\varepsilon} c(\lambda)|N_{F/Q}(\Delta)|^{-1/4-\delta+\varepsilon}\) for a fixed \(\delta > 0\) and a positive explicit constant \(c(\lambda)\). From Lemma 7.3, the desired result for Eisenstein series would follow from a subconvexity result for \(L\)-functions of \(GL(1, F)\). Partial progress towards subconvexity results in the Maass case have been made by Gergely Harcos \([21]\), but the complete adaptation of the \(GL(2, F)\) methods of \([12]\) to the Maass case remains elusive. Such results would follow however from GRH, so our Theorem 1.2 remains conditional.

**Note added in Proof:** As remarked to us by Emmanuel Ullmo, some comments must be added about the dependence on the parameter \(m \in \mathbb{Z}^{g-1}\) in the upper bounds for Eisenstein series. We will add such a comment shortly.
References

[1] A.A. Albert, On the construction of Riemann matrices, I, II, *Annals of Math.* 35 (1934), 1–28; 36 (1935), 367–394.
[2] A.A. Albert, A solution of the principal problem in the theory of Riemann matrices, *Annals of Math.* 35 (1934), 500–515.
[3] A.A. Albert, Involutorial simple algebras and real Riemann matrices, *Annals of Math.* 36 (1935), 886–964.
[4] W.L. Baily, Jr., On the Theory of Hilbert Modular Functions I. Arithmetic Groups and Eisenstein Series, *Journal of Algebra* 90 (1984), 567–605.
[5] W.L. Baily, Jr., Arithmetic Hilbert Modular Functions, II, *Revista Matemática Iberoamericana* 1, No.1 (1985), 85–119.
[6] R. Brauer, On the Zeta-function of algebraic number fields I, II, *American J. Math.* 1 69 (1947) 243–250, II *ibid* 72 (1950) 739–746.
[7] L. Clozel, Hee Oh, E. Ullmo, Hecke operators and equidistribution of Hecke points, *Invent. Math.* 1 44, 327–351 (2001).
[8] L. Clozel, E. Ullmo, Equidistribution des points de Hecke, to appear in Contributions to Automorphic Forms, Geometry and Arithmetic, volume in honor of J. Shalika, Johns Hopkins Univ. Press, editors Hida, Ramakrishnan, Shahidi.
[9] L. Clozel, E. Ullmo, Equidistribution de sous-variétés spéciales, to appear in Annals of Math.
[10] L. Clozel, E. Ullmo, Equidistribution de mesures algébriques, to appear in Compositio.
[11] E. Ullmo, Equidistribution de sous-variétés spéciales II, preprint.
[12] J.W. Cogdell, I.I. Piatetski-Shapiro, P. Sarnak, Estimates on the critical line for Hilbert modular L-functions and applications, in preparation.
[13] H. Cohen, *Advanced topics in computational number theory*, GTM 193, Springer-Verlag, 2000.
[14] W. Duke, Hyperbolic distribution problems and half-integral weight Maass forms, *Inv. Math.* 92 (1988), 73-90.
[15] W. Duke, Z. Rudnick, P. Sarnak, Density of integer points on affine homogeneous varieties, *Duke Math. J.* 71, No. 1 (1993), 143–179.
[16] I.Y Efrat, The Selberg Trace Formula for PSL$_2(\mathbb{R})^n$, *Memoirs of the AMS* 65, No. 359 (1987).
[17] I.Y. Efrat, Cusp forms in higher rank, Thesis, NYU, 1983
[18] A. Eskin, H. Oh, Integer points on a family of homogeneous varities and unipotent flows, preprint 2002.
[19] A. Eskin, Z. Rudnick, P. Sarnak, A proof of Siegel’s weight formula, *International Mathematics Research Notices* 5 (1991) 65–69.
[20] D. Goldfeld, J. Hoffstein, Eisenstein series of $\frac{1}{2}$-integral weight and the mean value of real Dirichlet L-series, *Invent. math.* 80 (1985), 185–208.
[21] G. Harcos, Doctoral Dissertation, Princeton University 2003.
[22] E. Hecke, Über die Kroneckersche Grenzformel für reelle quadratische Körper und die Klassenzahl relativ-abelscher Körper, *Verhandl. d. Naturforschenden Gesell. i. Basel* 28, 363–372 (1917).
[23] S. Katok, P. Sarnak, Heegner points, cycles and Maass forms, *Israel J. Math.* 84 (1993), 193–227.
[24] W. Kohnen, Special quadratic forms, Siegel modular groups and Siegel modular varieties, *International J. Math.* 1, No 4 (1990), 397–429.
[25] K. Khuri-Makdisi, On the Fourier coefficients of nonholomorphic Hilbert modular forms of half-integral weight, *Duke Math. J.*, 84, No 2 (1996), 399–452.
[26] L. Kronecker, Zur Theorie des elliptischen Modulfunktionen, Werke 4, 347–495 and 5, 1–132, Leipzig, 1929.
S.S. Kudla, Relations between automorphic forms produced by theta-functions, in International Summer School on Modular Functions Bonn 1976 à compléter!!.

S.S. Kudla, On modular forms of $\frac{1}{2}$-integral weight and Siegel modular forms of genus 2, unpublished notes.

H. Maass, Über die räumliche Verteilung der Punkte in Gittern mit indefiner Metrik, Math. Ann. 138 (1959) 287–315.

H. Maass, Über eine neue Art von nichtanalytische automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann. 121 (1949), 141–183.

W. Magnus, F. Oberhettinger, Formulas and theorems for the functions of mathematical physics, Berlin Heidelberg New York: Springer 1966.

M.W. Mastropietro, Quadratic forms and Relative Quadratic Extensions, dissertation Doctor of Philosophy, UCSD, 2000.

P. Michel, The subconvexity problem for Rankin-Selberg $L$-functions and equidistribution of Heegner points, Annals of Math., to appear.

Hee Oh, Hecke orbits of compact maximal flats in $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})/SO(n)$, preprint 2002.

M. Ratner, Interactions between ergodic theory, Lie groups and number theory, in Proceedings of the ICM, Zürich, Switzerland 1991, Birkhäuser Verlag, Basel, 1995, 157–182.

P. Sarnak, Diophantine Problems and Linear Groups, in Proceedings of the International Congress of Mathematicians, Kyoto, Japan, 1990, The Mathematical Society of Japan, 1991, 459–471.

P. Sarnak, Integrals of Products of Eigenfunctions, IMRN, No.6 (1994), 251–260.

P. Sarnak, Estimates for Rankin-Selberg $L$-functions and quantum unique ergodicity, J. of Functional Analysis, 184 (2001), 419–453.

P. Sarnak, M. Wakayama, Equidistribution of holonomy about closed geodesics, Duke Mathematical Journal, 100, No. 1 (1999).

I. Satake, Algebraic Structures of Symmetric Domains Pub. Math. Soc. Japan 14, Shoten Pub. and Princeton Uni. Press 1980.

G. Shimura, On analytic families of polarized abelian varieties and automorphic functions, Ann. of Math 78 (1963), 149–192.

G. Shimura, Moduli of Abelian Varieties and Number Theory, Proceedings of Symposia in Pure Math., IX, AMS, (1966), 312–332.

G. Shimura, On modular forms of half-integral weight, Ann. Math. 97 (1973), 440–481.

G. Shimura, On the holomophy of certain Dirichlet series, Proc London Math. Soc. 31 (1975), 79–96.

G. Shimura, On Eisenstein series, Duke Math. J. 50, No 2 (1983), 417–476.

G. Shimura, On Eisenstein series of half-integral weight, Duke Math. J. 52 (1985), 281–314.

G. Shimura, On the Fourier coefficients of Hilbert modular forms of half-integral weight, Duke Math. J. 71, No2, (1983), 501–557.

G. Shimura, On the Eisenstein series of Hilbert modular groups, Revista Matemática Iberoamericana 1, No. 3 (1985), 1–42.

G. Shimura, An exact mass formula for orthogonal groups, Duke Math. J. 97, No.1 (1999), 1–66.

G. Shimura, The number of representations of an integer by a quadratic form, Duke Math. J. 100, No.1 (1999) 59–92.

C.L. Siegel, Über die Classenzahl quadratischen Zahlkörpern, Acta Arith. 1 (1935), 83–86., Ges. Abh. I, 406–408, Springer 1966.

C.L. Siegel, Symplectic geometry, Amer. J. Math. 65 (1943), 1–86.
[53] C.L. Siegel, On the theory of indefinite quadratic forms, *Annals of Math.* 45 (1944), 577–622.

[54] C.M. Sorensen, Fourier expansion of Eisenstein series on the Hilbert modular group and Hilbert class fields, Trans. A.M.S. 354, Number 12, 4847–4869.

[55] G. Van der Geer, *Hilbert Modular Surfaces*, Springer-Verlag, (1980).

[56] A. Venkatesh, *Sparse equidistribution problems, period bounds, and subconvexity*, preprint in progress (private communication).

[57] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Comp. Math. 54 (1985), 173–242.

[58] Séries d’Eisenstein, intégrales toroïdales et une formule de Hecke, *Enseign. Math.* (2) 31 (1985), 93–135.

[59] D. Zagier, A Kronecker Limit Formula for Real Quadratic Fields, *Math. Ann.* 213 (1975), 153–184.

[60] D. Zagier, Eisenstein series an the Riemann zeta-function, in Automorphic Forms, Representation Theory and Arithmetic, Papers presented at the Bombay Colloquium 1979, Springer-Verlag, 1981, 275–302.

[61] S. Zhang, Gross-Zagier formula for GL2, *Asian J. Math.*, 5, no 2, (2001), 183-290.

[62] S. Zhang, Equidistribution of CM-points on quaternion Shimura varieties, preprint.

[63] S. Zhang, D. Jiang, J. Li, Periods and distribution of cycles on Hilbert modular varieties, preprint.

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