Reanalysis of lattice QCD spectra leading to the $D_{s0}^*(2317)$ and $D_{s1}^*(2460)$

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Abstract: We perform a reanalysis of the energy levels obtained in a recent lattice QCD simulation, from where the existence of bound states of $K \bar{D}$ and $K \bar{D}^*$ are induced and identified with the narrow $D_{s0}^*(2317)$ and $D_{s1}^*(2460)$ resonances. The reanalysis is done in terms of an auxiliary potential, employing a single-channel basis $K \bar{D}$, and a two-channel basis $K \bar{D}^*$, $\eta \bar{D_s}^*$. By means of an extended Lüscher method we determine poles of the continuum $t$-matrix, bound by about 40 MeV with respect to the $K \bar{D}$ and $K \bar{D}^*$ thresholds, which we identify with the $D_{s0}^*(2317)$ and $D_{s1}^*(2460)$ resonances. Using a sum rule that generalizes Weinberg compositeness condition we can determine that the state $D_{s0}^*(2317)$ contains a $K \bar{D}$ component in an amount of about 70%, while the state $D_{s1}^*(2460)$ contains a similar amount of $K \bar{D}^*$. We argue that the present lattice simulation results do not still allow us to determine which are the missing channels in the bound state wave functions and we discuss the necessary information that can lead to answer this question.
1 Introduction

The scalar $D_{s0}^*(2317)$ and axial $D_{s1}^*(2460)$ mesons were experimentally found slightly below $K D$ and $K D^*$ thresholds [1–4]. These are one of the few shallow bound states in the meson sector, and therefore deserve special attention. The effect of thresholds was recently considered using lattice QCD for the first time in this system in [5, 6], where interpolators of $K D$ and $K D^*$ type have been employed in addition to $\bar{s}c$ ones. The $N_f = 2 + 1$ simulation obtained three energy levels for $m_\pi \simeq 156$ MeV in the $K D$ and $K D^*$ systems. The fact that these levels appear clean with the $K D$ and $K D^*$ interpolators, together with the observation that the lowest one appears below and not far from the corresponding $K D$ or $K D^*$ threshold, hint to a possible molecular structure for this state. The scattering length and the effective range were determined from the two lowest energy levels in [5, 6] and, using the effective range formula, bound states were found at about 40 MeV below the respective $K D$ and $K D^*$ thresholds. These were identified with the scalar $D_{s0}^*(2317)$ and axial $D_{s1}^*(2460)$ states respectively.

Actually, the two lower levels employed in the analysis of [5, 6] are separated by 130 MeV, which makes the use of the effective range formula a bit extreme, and the information of the upper level was disregarded. In the present work we perform a reanalysis of these lattice spectra which does not rely upon the effective range formula and takes advantage of the information of the three levels. The analysis is done using the auxiliary potential method [7], equivalent to the one of Lüscher [8] in single or coupled channels, but allowing also to obtain phase-shifts for arbitrary energies. The lattice simulations are particularly suited for this kind of study because, for the same value of $L$, they produce several energy
levels which provide information on the energy dependence of the potential needed to interpret the spectra.

We first perform a single channel analysis, with \(KD\) or \(KD^*\), which permits determining the two parameters of an energy dependent potential from a fit to the three energy levels of the box. This potential is then used in the continuum, leading to poles of the \(KD\) and \(KD^*\) scattering amplitudes, which lie about 40 MeV below the respective thresholds. A generalization of the Weinberg compositeness condition \([9, 10]\) is then used to determine the amount of \(KD\) and \(KD^*\) in the respective wave functions. A different method to learn about the amount of meson component, or equivalently the amount of non-meson component, \(Z\), in the wave function, is from the dependence of the spectrum on the twisting angle, imposing twisted boundary conditions on the fermion fields \([11]\).

The compositeness condition was generalized to a new sum rule in an arbitrary number of coupled channels \([12]\), which is reformulated in \([13–17]\) for the case of energy dependent potentials. The sum rule contains two terms (see Eq. (133) in \([16]\)), one involving the derivative of the two-particle loop function, which is identified with the probability of the state containing this particular two-particle component of the coupled channels. The second term involves the derivative of the potential with respect to the energy, which accounts for the probability of the state to be in other components not explicitly considered in the approach, for example omitted two-meson channels or \(\bar{q}q\). An illustrative example is given in \([18]\), where one starts from a two channel problem with energy independent potentials which generate dynamically a bound state. The problem is then reformulated in terms of one channel and an effective potential, which however becomes energy dependent. This allows one to see that the term in the sum rule involving the derivative of the loop function accounts for the probability of the channel retained, while the term involving the derivative of the potential accounts for the probability of the omitted channel.

Having this in mind, we repeat the analysis of the lattice results using a two channel basis, involving \(KD, \eta D_s\) for \(D_{s0}^*(2317)\) and \(KD^*, \eta D_s^*\) for \(D_{s1}^*(2460)\). The choice of channels relies on the results of coupled channels unitary approaches \([19–28]\), which found those to be the relevant ones. Alternative scenarios for a non-\(\bar{q}q\) structure of these states have been also given \([29–34]\). With two channels and three energy levels one is forced to treat the three components of the coupled-channel potential \((V_{11}, V_{12}, V_{22})\) as being energy independent. We observe that a fit to the energy levels is not possible in this case, indicating that these levels carry no information on the \(\eta D_s\) and \(\eta D_s^*\) channels. This can be explained since no interpolators of this type were used in \([5]\), while it was also found there that the levels obtained were tied to the interpolators used. Further lattice information will be needed in the future to make progress in this direction and learn more about the components that build up the \(D_{s0}^*(2317)\) and \(D_{s1}^*(2460)\) wave functions.

With the available limited lattice information, we can confirm that the bound states of \(KD\) and \(KD^*\) can be associated to the \(D_{s0}^*(2317)\) and \(D_{s1}^*(2460)\) states. We also confirm that these bound states are mostly of \(KD\) or \(KD^*\) nature, estimating about 70\% the probability of these components in their respective wave function.

The compositeness of the \(D_{s0}^*(2317)\) based on indirect lattice data was first discussed in \([25]\), but employing a different method. The scattering lengths of other scattering chan-
nels, free from disconnected diagrams, were obtained on the lattice and used to determine
the parameters of their effective field theory, which was subsequently used to indirectly de-
termine the scattering parameters of $DK$ scattering and the pole position in this channel.
Similarly, the scattering lengths from the simulation of Ref. [25] were employed in [27, 28]
to fix the low-energy constants of a covariant chiral unitary theory, which was then used to
also identify, as composite states, the heavy-quark spin and flavour symmetry counterparts
of the $D^*_s$.  

As mentioned above, additional lattice information could help us improve our knowl-
edge on the additional building blocks that these states might have. Indeed, preliminary
spectra for these channels obtained including $KD$, $\bar{s}c$ as well as $\eta D_s$ interpolating fields
have been presented in [35]. Their plan is to perform a two-coupled channel analysis using
a parametrization of the scattering matrix on the energy. This strategy has recently lead
to the first results of the two-coupled channel system $K\pi - K\eta$ from lattice QCD; the
pole positions of the scattering matrix were subsequently found and related to the strange
mesons [36]. The approach presented here offers an alternative way to extract physical
information from the lattice spectra in the future.

2 Compositeness of states

We collect here the essential expressions relevant to interpret the nature of hadrons gen-
erated dynamically from a given meson-meson interaction. Let us take two mesons ($K$
and $D$ for example) and an interacting potential $V$. The Lippmann-Schwinger equation
produces the scattering amplitude $T$

$$ T = V + VGT, \quad (2.1) $$

where $G$ stands for the two meson propagator. We shall take relativistic propagators and
Eq. (2.1) will be the Bethe-Salpeter equation. The on-shell factorization of $V$ and $T$ allows
one to convert Eq. (2.1) into an algebraic equation with $G$ given by

$$ G = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_1^2 + i\epsilon} \frac{1}{(P - q)^2 - m_2^2 + i\epsilon}, \quad (2.2) $$

where $P$ is the total two meson momentum. This factorization was justified in Refs. [37, 38]
by using dispersion relations in which the smooth energy dependent contribution of the left-
hand-side cut was replaced by a constant in the region of interest. The energy dependence
was shown to be particularly weak in the case of the meson-baryon interaction [38] due to
the large baryon mass and, consequently, it will be even weaker in the present case due to
the larger mass of the $D$ and $D^*$ mesons. The neglect of the left hand cut is also inherent
in the Lüscher formalism, as we shall see in Sect. 3.2.

Upon integration of the $q^0$ variable the loop function becomes

$$ G = \int \frac{d^3q}{(2\pi)^3} I(q), \quad I(q) = \frac{\omega_1(q) + \omega_2(q)}{2\omega_1(q)\omega_2(q) \left[ P^2 - (\omega_1(q) + \omega_2(q))^2 + i\epsilon \right]}, \quad (2.3) $$
where \( \omega(q) \) is the meson on-shell energy. The loop function must be conveniently regularized with a cut-off \( q_{\text{max}} \), or employing dimensional regularization techniques.

Assume now the Bethe-Salpeter equation projected over \( S \)-wave and \( V \) an energy independent potential in one channel (say \( KD \)). We then have

\[
T(1-VG) = V, \quad T = \frac{V}{1-VG} = \frac{1}{V^{-1}-G}. \tag{2.4}
\]

Let us now assume that the interaction \( V \) produces a bound state, which we will refer to as a two meson composite state or a dynamically generated state. We shall see that the energy independent potential can not lead to a genuine state, for example a \( \bar{q}q \) state with a weak coupling to two mesons. In the case of one channel, the coupling \( g \) of the bound state is obtained by requiring that around the pole \( s = s_0 \) (with \( s = P^2 \) being a Mandelstam variable)

\[
T \sim \frac{g^2}{s-s_0}, \quad \text{hence:} \quad g^2 = \lim_{s \to s_0} (s-s_0)T. \tag{2.5}
\]

Since \( V^{-1} - G = 0 \) at the bound state pole, we find in the case of an energy independent potential using L'Hopital's rule

\[
g^2 = \frac{1}{\frac{\partial G}{\partial s}}, \quad -g^2 \frac{\partial G}{\partial s} = 1. \tag{2.6}
\]

The property of Eq. (2.6) can be generalized to coupled channels and, in the case of an energy independent potential (and two channels), one finds:

\[
V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix}, \quad G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}, \tag{2.7}
\]

\[
T = (1-VG)^{-1}V, \tag{2.8}
\]

\[
ge_i g_j = \lim_{s \to s_0} (s-s_0)T_{ij}, \quad \sum_i \left( -g^2_i \frac{\partial G_i}{\partial s} \right) = 1 \tag{2.9}
\]

Equation (2.6) is the generalization of the Weinberg compositeness condition [9], which applies to loosely bound states, to higher binding energies, while Eq. (2.9) is the extension to many coupled channels [12]. By solving the Schrödinger equation in momentum space in coupled channels and normalizing the wave function of the bound state to unity, it was found [12]

\[
\int d^3p \left| \langle p \mid \Psi_i \rangle \right|^2 = g^2_i \frac{\partial G_i}{\partial E}, \tag{2.10}
\]

with \( \mid \Psi_i \rangle \) being the \( i \) component of the bound state in the \( i \)th channel, so that each term of the sum in Eq. (2.9) represents the probability to have this channel in the wave function of the bound state:

\[
P_i = -g^2_i \frac{\partial G_i}{\partial s}, \tag{2.11}
\]

As discussed in [12] there is a different normalization of the amplitudes, and hence the couplings, between [12] and field theoretical approach used here, which leaves the probability to be expressed here as in Eq. (2.11)
and the sum of these probabilities saturates the wave function. Note that, by construction, in the case we are discussing here all the components of the composite state are of meson-meson type. We will elaborate more on these issues in Sect. 5.

It is easy to visualize a genuine state that couples weakly to a meson-meson component by using a meson-meson potential of the type:

$$ V = \frac{b}{s - s_R}, \quad (2.12) $$

which we refer to as a CDD pole [39]. Now

$$ T = \frac{1}{s - s_R - G}, \quad g^2 = \frac{1}{b - \frac{\partial G}{\partial s}}, \quad (2.13) $$

and

$$ P = -g^2 \frac{\partial G}{\partial s} = 1 - g^2 \frac{1}{b}. \quad (2.14) $$

In the limit of $b \to 0$ (small coupling of the genuine state to meson-meson) we have $g^2 \to 0$ and the pole appears at $s = s_R$. Then the amount of meson-meson component, $-g^2 \frac{\partial G}{\partial s}$, goes to zero and we have a representation for a genuine state, or, in general, a state different from the explicit two meson state considered. It is interesting to note a distinct feature in the potential of Eq. (2.12), namely its energy dependence.

These ideas are generalized in Ref. [16], with the sum rule

$$ -\sum_i g^2_i \frac{\partial G_i}{\partial s} - \sum_{i,j} g_i g_j G_i \frac{\partial V_{i,j}}{\partial s} G_j = 1, \quad (2.15) $$

evaluated at the pole. The first term in Eq. (2.15) is associated in Ref. [16] to the composite part of the state (meson-meson in the present case) and the second term, involving the derivative of the potential, to the genuine part of the state. Actually, this second part accounts for the state components that have not been considered in the coupled channel problem. This is easily shown in the case of two channels in Ref. [18], where one channel is eliminated and its effects are accounted for by means of an effective potential in the remaining channel. Take $V_{22} = 0$, for simplicity, and consider $V_{ij}$ energy independent to saturate the state with the two channels in Eq. (2.7). It is then easy to obtain from Eq. (2.8),

$$ T_{11} = \frac{V_{11} + V_{12}^2 G_2}{1 - (V_{11} + V_{12}^2 G_2) G_1}, \quad (2.16) $$

making clear that solving a one-channel problem with the effective potential

$$ V_{\text{eff}} = V_{11} + V_{12}^2 G_2, \quad (2.17) $$
gives the same amplitude $T_{11}$ obtained in the two channel case. The novelty is that now $V_{\text{eff}}$ becomes energy dependent. Then, the term $-g^2_1 \frac{\partial G_1}{\partial s}$, which accounts for the probability of channel 1 in the state, is the same in both formulations and the second term in Eq. (2.15) is, by construction of $V_{\text{eff}}$, the probability of the second channel that has been eliminated. We are going to use these findings to analyze the lattice spectra of Ref. [5].
3 Analysis of the lattice spectra

The lattice simulation of Ref. [5] obtained three energy levels in the scalar channel using the $KD$ and $\bar{s}c$ interpolators, and three levels in the axial channel using the $KD^*$ and $\bar{s}c$ interpolators. Table 1 collects the levels of ensemble (2), with $N_f = 2 + 1$ and close-to-physical pion mass $m_\pi = 156$ MeV. The lattice spacing is $a = 0.0907 (13)$ fm and the box size $L = 2.90$ fm. The kaon with mass $m_K = 504(1)$ MeV obeys the usual relativistic dispersion relation $E_K(p) = (m^2_K + p^2)^{1/2}$.

Table 1. Energy levels for the scalar ($KD$) and axial ($KD^*$) channels found in the simulation Ref. [5]. The relative errors in the lattice spacing $a$ and in $aE$ have been added in quadrature. Only the energy differences, for example $E_{\text{lat}}^1 - \bar{m}_{D_s}^{\text{lat}}$ with $\bar{m}_{D_s}^{\text{lat}} = \frac{1}{4}(m_D + 4m_{D^*}) = 1.8407(6)$ MeV, can be compared to the experiment.

|                | $KD$ channel | $KD^*$ channel |
|----------------|--------------|----------------|
| $E_1$ (MeV)   | 2086 (34)    | 2232 (33)      |
| $E_2$ (MeV)   | 2218 (33)    | 2349 (34)      |
| $E_3$ (MeV)   | 2419 (36)    | 2528 (53)      |

The simulation [5, 6] treated the charm quark using the so-called Fermilab method, where the leading discretization errors related to the charm quark cancel in the energy differences (with respect to the reference mass of a meson containing the same number of charm quarks). We employ the dispersion $E(p)$ for $D$ and $D^*$ mesons determined in the simulation of Ref. [5]

$$E_{D(D^*)}(\vec{p}) = M_1 + \frac{\vec{p}^2}{2M_2} - \frac{(\vec{p}^2)^2}{8M_4^3}, \quad m_{D(D^*)} = M_1$$ (3.1)

where $M_1, M_2, M_4$ are given in Table 3.

Table 2. $M_i$ from the dispersion relation $E(p)$ (3.1) for $D$ and $D^*$ mesons. The rest energies, i.e. the masses $M_1$, can be compared to experiment via the difference $M_i^{\text{lat}} - m_{D_s}^{\text{lat}}$ with $m_{D_s}^{\text{lat}} = \frac{1}{4}(m_D + 4m_{D^*}) = 1.751(3)$ MeV [5].

|                | $D$ meson | $D^*$ meson |
|----------------|-----------|-------------|
| $M_1$ (MeV)   | 1639      | 1788        |
| $M_2$ (MeV)   | 1801      | 1969        |
| $M_4$ (MeV)   | 1936      | 2132        |

3.1 Analysis by means of the effective range formula

In Ref. [5] the scattering length and effective range for $KD$ and $KD^*$ scattering were obtained using only the two lowest energy levels of the lattice simulation and employing

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2The second level in the axial channel of Ref. [5] is attributed to the $D_{s1}(2536)$ resonance in $KD^*$ d-wave scattering and is therefore not used in the present paper which considers $KD^*$ scattering in s-wave.

3Results of set 2 in [5] are used in the axial channel.
Lüscher’s approach to extract the infinite volume phase shifts. In this section we analyze these results by means of an effective range formula to obtain the binding energy of the state and check the fulfillment of the sum-rule of Eq. (2.6).

The effective range approximation reads

\[ p \cot \delta = \frac{1}{a_0} + \frac{1}{2} r_0 p^2, \quad T = -\frac{8\pi E}{p \cot \delta - i p}. \] (3.2)

Below threshold, one writes \( p = i \tilde{p} \), and a pole of the \( T \) matrix is obtained for \( \cot \delta = i \).

Therefore, the pole appears for the value of \( \tilde{p} \) that satisfies

\[ \frac{1}{2} r_0 \tilde{p}^2 - \tilde{p} - \frac{1}{a_0} = 0. \] (3.3)

Taking the values for \( a_0 \) and \( r_0 \) determined by the lattice simulation [5], quoted in Table 3.1, we obtain the bound momentum \( \tilde{p} \) and the binding energy

\[ B = -\frac{\tilde{p}^2}{2\mu}, \quad \mu = \frac{m_K m_{D/D^*}}{m_K + m_{D/D^*}}, \] (3.4)

also quoted in the table. A bound state of \( KD \) with a binding energy of 38(9) MeV is associated to the \( D_{s0}^{*}(2317) \). We note that the unitary coupled-channel approach of [21] generates such a state from the interaction of the \( KD \) and \( \eta D_s \) channels mostly. The \( D_{s1}^{*}(2460) \) is related to a bound \( KD^* \) state with binding energy 44(6) MeV. In the unitary coupled-channel approach it is mainly composed of \( KD^* \) and \( \eta D_{s}^{*} \) [22].

| Channel | \( a_0 \) [fm] | \( r_0 \) [fm] | \( B \) [MeV] | \( |g| \) [GeV] | \(-g^2 \partial G/\partial s\) |
|---------|----------------|----------------|-------------|---------------|----------------|
| \( KD \) | -1.33(20) | 0.27(17) | 38(9) | 12.6(1.5) | 1.14(0.15) |
| \( KD^* \) | -1.11(0.11) | 0.10(0.10) | 44(6) | 12.6(0.7) | 0.96(0.06) |

Table 3. Binding energy \( B \), meson-meson coupling \( |g| \) and sum-rule [Eq. (2.6)], for the bound states obtained in the lattice QCD simulation of Ref. [5], analyzed using an effective range formula.

It is interesting to test the sum rule of Eq. (2.6) for the states obtained. The \( g^2 \) at the pole can be expressed as

\[ g^2 = \frac{16\pi s \tilde{p}}{\mu(1 - r_0 \tilde{p})}, \] (3.5)

and listed in Table 3.1. Since \( \partial G/\partial s \) is convergent, we obtain the sum rules quoted in the last column, which, within errors, are all compatible with unity. The coupling to the \( KD \) channel is \( g_{KD} = 12.6 \) GeV, which is of the order of the one obtained in the chiral unitary approach in Ref. [21], \( g_{KD} = 10.21 \) GeV. Note, however, that this smaller value would provide a probability for the \( KD \) channel of about 60 – 70%, leaving room for the other channels considered in the unitary coupled-channel approach. Similarly, in the \( KD^* \) channel, we find a coupling \( g_{KD^*} = 12.6 \) GeV, compared to the value of around 10 GeV quoted in Ref. [22], also leaving room for the additional meson-meson components considered in that work.
Although the results obtained with the effective range formula are qualitatively reasonable, and the existence of the bound state emerges as a solid statement, one can see that the approximation has its limitations when one is demanding more information from it. There is also the fact that the first two levels are separated by 132 MeV, which makes this approximation a bit extreme. Furthermore, the information of the third level is not used, and, as shown in Ref. [5], this level cannot be accounted for by means of the effective range formula. All these reasons advise a new reanalysis which we offer in the next subsection.

3.2 Analysis of lattice spectra by means of an auxiliary potential

First, we are going to make the analysis with only one channel. Anticipating that the $\eta D_s$ and $\eta D^*_s$ channels also play a role in the $D_s^0(2317)$ and $D_{s1}(2460)$ resonances, as found in Refs. [21] and [22], we shall leave room for these and possible $\bar{q}q$ components, by using an energy dependent potential. As a first step we take a potential linear in $s$,

$$V = \alpha + \beta (s - s_{th}), \quad (3.6)$$

with $s_{th} = (M_{D^*} + M_K)^2$, since only the derivative of the potential is needed to obtain the sum rule. Later on we shall also use another type of potential.

In the finite box, the $T$ matrix of Eqs. (2.1) and (2.4) is replaced by

$$\tilde{T} = \frac{1}{V^{-1} - \tilde{G}}, \quad (3.7)$$

where $\tilde{G}$ is the two meson loop function in the box given by [40]

$$\tilde{G} = G + \lim_{q_{\max} \to \infty} \left[ \frac{1}{L^3} \sum_{\vec{q}_i} I(\vec{q}_i) - \int_{q < q_{\max}} \frac{d^3q}{(2\pi)^3} I(\vec{q}) \right]; \quad \vec{q}_i = \frac{2\pi}{L} \vec{n}_i, \quad \vec{n}_i \in \mathbb{Z}^3. \quad (3.8)$$

The $G$ in the continuum, Eq. (2.3), can be regularized with a cut-off $q'_{\max}$ or employing dimensional regularization. The latter choice, followed in Ref. [40], cannot be applied here because we employ the dispersion relation of Eq. (3.1). For this reason we adopt the cut-off method, with a cut-off value that gives equivalent results to those of the chiral unitary approach of Refs. [21, 22]. Any value of $q'_{\max}$ can, in principle, be taken since changes in $G$ can be accommodated by changes in $V^{-1}$ when we require that $\tilde{T}$ has poles at the energies of the lattice spectra by demanding that $V^{-1} - \tilde{G} = 0$. Note, in addition, that we are interested finally in results for the continuum. Hence, at the energies of the lattice spectra we have $V^{-1} = \tilde{G}$, and then the continuum $T$ matrix is

$$T = \frac{1}{V^{-1} - G} = \frac{1}{\tilde{G} - G} = \lim_{q_{\max} \to \infty} \left[ \frac{1}{L^3} \sum_{\vec{q}_i} I(\vec{q}_i) - \int_{q < q_{\max}} \frac{d^3q}{(2\pi)^3} I(\vec{q}) \right], \quad (3.9)$$

which is then independent of the cut-off $q'_{\max}$ employed to regularize $G$. However, in the transfer of strength from $G$ to $V^{-1}$ one will be introducing some energy dependence in $V^{-1}$ that would change the probability $Z$ of not having the main meson-meson component.
Figure 1. Fits to the lattice data of Ref. [5] for the $K D$ system using the potential of Eq. (3.6).

considered. We shall come back to this issue in section 5 where systematic uncertainties are studied.

Equation (3.9) is the formulation employed in the approach of Ref. [7], where it is shown that Lüscher formula is recovered if some terms of $I(q)$, which are exponentially suppressed, are eliminated. These terms can be relevant in the case of relativistic particles and small volumes [41, 42], which is not the case here. However, we cannot use the standard Lüscher approach either, based on the relativistic relationship $\omega(q) = (m^2 + q^2)^{1/2}$, since we are forced to employ the dispersion relation of Eq. (3.1). In this case, Eq. (3.9) gives the appropriate extension of the Lüscher formalism.

There is another approximation inherent in our approach (or the one of Lüscher) when we assume that the potential is volume independent. Within the framework of the chiral unitary approach such effects were investigated in [43, 44] in the $\pi \pi$ scattering in the scalar sector and the $\rho$ sector and it was concluded that for values of $L m_\pi > 1.5$ they could be safely neglected. In the present case, given the large masses involved, loops in the t-channel, which originate this volume dependence, are even less relevant.

With the formalism exposed above, a best fit is carried to the three lattice levels obtained in [5], demanding that the $\tilde{T}$ derived from Eq. (3.7) using the potential of Eq. (3.6) has poles at the three energies. In order to find the desired magnitudes and associated statistical errors, we perform a series of fits to different sets of three energies, generated with a Gaussian weight within the errors of the lattice levels. With the parameters obtained in each fit we evaluate the different magnitudes. From the results obtained in the different fits, we then determine the central values and statistical errors of these magnitudes.

We show in Figs. 3.2 and 3.2 the results obtained from the fits to the levels for the $K D$ and $K D^*$ systems, respectively. The procedure outlined above gives us a pole for the
$KD$ system with binding energy

$$B(KD) = m_D + m_K - E_B(KD) = 46 \pm 21 \text{ MeV},$$  \hspace{1cm} (3.10)

to be compared to the value $36.6(16.6)(0.5)$ MeV obtained with the effective range formula and to the $45$ MeV binding in the physical case. For the $KD^*$ system we get the binding energy

$$B(KD^*) = m_{D^*} + m_K - E_B(KD^*) = 52 \pm 22 \text{ MeV}.$$  \hspace{1cm} (3.11)

The probabilities for the $KD$, $KD^*$ components, obtained from Eqs. (2.11), (2.5), are:

$$P(KD) = 76 \pm 12 \%, \text{ for the } D_{s0}^*(2317);$$ \hspace{1cm} (3.12)

$$P(KD^*) = 53 \pm 17 \%, \text{ for the } D_{s1}^*(2460).$$ \hspace{1cm} (3.13)

This means that there is a large amount of $KD$ and $KD^*$ components in the corresponding bound states, leaving about $30\%$ and almost $50\%$ of strength for other channels, respectively.

### 3.3 Fit with a CDD pole

One near-threshold level was found in [5, 6] when only $\bar{c}c$ interpolators were used$^4$, and one wonders what is the $\bar{c}c$ component in the meson states at hand. We therefore explore whether there could be an admixture of some genuine component in the bound state by refitting the lattice levels adding a CDD pole to the potential of Eq. (3.6):

$$V = \alpha + \beta(s - s_{th}) + \frac{\gamma^2}{s - M_{CDD}^2};$$ \hspace{1cm} (3.14)

$^4$Its energy however changes when $D^{(*)}K$ interpolators were used in addition to $\bar{c}c$ ones.
which, as seen in section 2, is suited to accommodate a genuine state. This has been shown to be the proper way to account for genuine components in different works [18, 37, 45, 46] in the continuum. An analysis of “synthetic” lattice spectra in terms of this potential was done in [40]. It was also recently employed to analyze lattice spectra with the \( \pi K \) and \( \eta K \) channels in [36].

Since we have four parameters (\( \alpha, \beta, \gamma \) and \( M_{\text{CDD}} \)) and three energy levels, we can obtain solutions with many sets of parameters which are, obviously, correlated. However, the values of the parameters do not have a particular significance and what matters is the value of the magnitudes derived from the different fits. The statistics of the obtained fits shows a clear preference for solutions with a \( M_{\text{CDD}} \) value that lies far away (more than 300 MeV) from the \( KD \) and \( KD^* \) thresholds, such that it effectively provides a linear dependence in \((s - s_{\text{th}})\) at the energies where the poles are found. This is an indication that the lattice energies do not favour a CDD component, or at least not a significant one. Obviously, future lattice results with more accuracy and different volumes will allow one to be more precise on this issue.

With the potential of Eq. (3.14) we obtain the following binding energies

\[
B(KD) = 29 \pm 15 \text{ MeV}, \quad (3.15)
\]
\[
B(KD^*) = 37 \pm 23 \text{ MeV}, \quad (3.16)
\]

and probabilities

\[
P(KD) = 67 \pm 14 \%, \text{ for the } D_{20}^*(2317), \quad (3.17)
\]
\[
P(KD^*) = 61 \pm 26 \%, \text{ for the } D_{31}^*(2460), \quad (3.18)
\]

which are compatible within errors with those of Eqs. (3.10)–(3.13), obtained with the linear potential.

### 3.4 Two channel analysis

After this exercise we perform a two channel analysis including the \( \eta D_s \) channel for the \( D_{20}^*(2317) \) state and the \( \eta D_s^1 \) channel for the \( D_{31}^*(2460) \), which were found also relevant in Refs. [21, 22].

Since we only have three energy levels we use an energy independent potential, Eq. (2.7), which has three parameters, \( V_{11}, V_{12}, V_{22} \). By doing so, we would force the states to saturate with the \( KD^*(s) \), \( \eta D_s^1(s) \) components. The comparison of the two procedures would allow us to make statements about the amount of each channel in the respective states.

We thus fit the \( V_{ij} \) parameters using

\[
\tilde{T} = (1 - V\tilde{G})^{-1}V, \quad (3.19)
\]

in two channels, looking for the poles of \( \tilde{T} \) and associating the first three levels to those of the lattice simulation.

We do not find any suitable fit to the data, which is an enlightening result. One could interpret it as an evidence that the energy levels obtained in [5] do not contain information
on the \( \eta D \) or \( \eta D^* \) channels. This seems to be the case because the three energies obtained there were tied to the use of \( q \bar{q} \) and meson-meson interpolators of \( KD \) or \( KD^* \) type. No interpolator was used containing information on the \( \eta D \) and \( \eta D^* \) channels, and no energy level was found which would be tied to these channels. It is indeed a common experience of lattice practitioners that a given two-hadron eigenstate is most often not seen unless explicitly implemented in the basis of interpolating fields. Although all states with a given quantum number are in principle expected in a dynamical simulation, a poor basis of interpolating fields is insufficient to render them in practice. The reason is that one would have to wait much time till these components show up in the time evolution of the state and this could happen in the region where the ratio of noise to signal is large, preventing any signal to be seen [47]. This also gives us some idea on how to proceed in the future if one wishes to make progress on determining the components of the \( D_{s0}^*(2317) \) and \( D_{s1}(2460) \) wave functions. The relevant fraction of the wave function that went to the \( \eta D \) and \( \eta D^* \) channels in chiral unitary studies [21, 22], of the order of 20\%, makes it advisable to include interpolators for the \( \eta D \) and \( \eta D^* \) channels in future lattice simulations. Such a simulation is underway and preliminary spectra have been presented in [35].

4 Scattering length and effective range

We can also obtain the scattering length and the effective range in each of the cases explored. For this we use Eq. (3.2), finding

\[
p \cot \delta = \text{Re} \left\{ -\frac{8\pi E}{T} \right\} \simeq \frac{1}{a_0} + \frac{1}{2} r_0 p^2.
\] (4.1)

Relating \( E \) to \( p \) via the dispersion relation of Eq. (3.1)

\[
E = \sqrt{m_K^2 + p^2 + E_{D(D^*)}(p)},
\] (4.2)

we obtain

\[
a_0 = -1.2 \pm 0.6 \text{ fm}, \quad r_0 = 0.04 \pm 0.16 \text{ fm for } KD,
\] (4.3)

\[
a_0 = -0.9 \pm 0.3 \text{ fm}, \quad r_0 = -0.3 \pm 0.4 \text{ fm for } KD^*
\] (4.4)

in the case the lattice data is analyzed using a single channel potential (3.6).

When we use the CDD potential of Eq. (3.14) we find

\[
a_0 = -1.4 \pm 0.4 \text{ fm}, \quad r_0 = -0.2 \pm 0.4 \text{ fm for } KD,
\] (4.5)

\[
a_0 = -1.3 \pm 0.6 \text{ fm}, \quad r_0 = -0.1 \pm 0.2 \text{ fm for } KD^*
\] (4.6)

The values for the scattering length and effective range obtained with the different methods are remarkably similar.

The values obtained also agree qualitatively with those obtained in Ref. [5] but should be considered more accurate. As we have discussed, we do not use the effective range formula to correlate the results. Our method allows us to cover a wider span of energies and we can make use of the three energy levels obtained in [5], while only the information of the lowest two was used in [5].
5 Evaluation of systematic uncertainties

In [25] the lowest lattice level obtained for the channels $D\bar{K}(I = 1)$, $D\bar{K}(I = 0)$, $D_s K$, $D\pi(I = 3/2)$, $D_s \pi$, free from disconnected diagrams, were employed to obtain, via the Lüscher formalism [9], the phase shifts in the continuum at the eigenenergies of the lattice box. The scattering length was then derived from the relationship $p \cot \delta(p) = 1/a_0$, disregarding the effective range term. The low energy constants of a chiral lagrangian were fitted to the scattering lengths of those channels employing a unitary approach. With these values of the coefficients, the coupled $K D, \eta D_s$ channels system was studied, from where the existence of a bound state associated to the $D_{s0}^*(2317)$ was established and the $K D$ scattering length was obtained. A $K D$ probability, $1 - Z$, in the $D_{s0}^*(2317)$ wave function of around 70% was found, where the value of $Z$ was determined from the scattering length via the relation [9, 10]

$$a_0 = -2 \frac{(1 - Z)}{(2 - Z)} \frac{1}{\sqrt{2} \mu \epsilon} \left[ 1 + \mathcal{O}\left(\sqrt{2} \mu \epsilon / \beta\right) \right]$$

(5.1)

where $\mu$ and $\epsilon$ are the reduced mass and binding energy, respectively, and $1/\beta$ accounts basically for the range of the interaction ($1/q_{\text{max}}$ in our approach). The term $\mathcal{O}(\sqrt{2} \mu \epsilon / \beta)$ was neglected, as usually done when using the Weinberg compositeness condition, which is valid for small binding energies.

In the present case $\sqrt{2} \mu \epsilon / \beta$ is of the order of 0.22 if we take $\beta = q_{\text{max}} = M_V = 780$ MeV, and the correcting terms can be rather relevant. Indeed, let us comment on the sensitivity of Eq. (5.1) in obtaining $Z$ from the value of $a_0$. Note that if $-2/\sqrt{2} \mu \epsilon < a_0 < -1/\sqrt{2} \mu \epsilon$, the resulting $Z$ would have unphysical negative values. This condition would obviously not be a problem for sufficiently small binding energies where Eq. (5.1) is applicable but, for the $K D$ state analyzed here, the value of the factor $-1/\sqrt{2} \mu \epsilon$ is $-1.12$ fm, close to the typical values found for the scattering lengths, and the extraction of $Z$ from $a_0$ using Eq. (5.1) is certainly problematic. Note that Ref. [25] obtained $a_0 \sim -0.85$ fm, from which, using Eq. (5.1), a probability $P_{KD} \sim 70\%$ was extracted, similar to the result obtained here in spite of the fact that we have a different value of the scattering length\(^5\).

Incidentally, one could have evaluated $P = 1 - Z$ directly from the coupling also in the Weinberg approach using Eq. (24) from Ref. [9], which is equivalent to Eq. (2.11) used here but neglecting the $\mathcal{O}(\sqrt{2} \mu \epsilon / q_{\text{max}})$ terms in $(\partial G / \partial s)$. In this case, it is easy to evaluate the correcting terms to the Weinberg formula. Using, for simplicity, the nonrelativistic

\(^{5}\)In a follow up of the work of Ref. [25], the $K D$ scattering length of $-1.33(20)$ fm from the lattice work of Ref. [6] is used as a further constraint on the parameters of the chiral theory. No reanalysis of the $K D$ probability in the $D_{s0}^*(2317)$ state is done.
approach of [12] [see Eqns. (27), (29) there] one finds

\[
\frac{\partial G}{\partial E} = \frac{1}{\gamma} 8\pi \mu^2 \left[ \arctan \left( \frac{q_{\text{max}}}{\gamma} \right) - \frac{\gamma q_{\text{max}}}{\gamma^2 + q_{\text{max}}^2} \right] = (5.2)
\]

\[
= \frac{1}{\gamma} 8\pi \mu^2 \left[ \frac{\pi}{2} - 2 \left( \frac{\gamma}{q_{\text{max}}} \right) + \frac{4}{3} \left( \frac{\gamma}{q_{\text{max}}} \right)^3 + \ldots \right] = (5.3)
\]

\[
= \frac{1}{\gamma} 4\pi^2 \mu^2 \left[ 1 - \frac{4}{\pi} \left( \frac{\gamma}{q_{\text{max}}} \right) + \frac{8}{3\pi} \left( \frac{\gamma}{q_{\text{max}}} \right)^3 + \ldots \right]. (5.4)
\]

Hence, in the nonrelativistic expression

\[
1 - Z = g^2 \frac{\partial G}{\partial E}, (5.5)
\]

analogous to Eq. (2.11), the correcting factor to the Weinberg formula is\(^6\):

\[
F = \left[ 1 - \frac{4}{\pi} \left( \frac{\gamma}{q_{\text{max}}} \right) + \frac{8}{3\pi} \left( \frac{\gamma}{q_{\text{max}}} \right)^3 + \ldots \right]. (5.6)
\]

The correction to unity in the problem analyzed here amounts to 28%, meaning that obtaining 1 – Z from Eq. (5.5) in the Weinberg limit would give, for the same value of \(g^2\), a 28% larger value of 1 – Z than if finite range effects were considered. We note that, in a case like the present one, having a sizable finite range correction, it appears more accurate to obtain 1 – Z in the Weinberg limit from Eq. (5.5) than from Eq. (5.1).

The former discussion indicates the convenience of extending the Weinberg condition when the binding is not small. This is precisely what Eqs. (2.11) and (2.15) do, which are derived in many different ways including a method based on the generalized Ward identity [13–17]. Since \(G\) depends on the range, 1 – Z will have some dependence on \(q_{\text{max}}\), albeit weak because \((\partial G/\partial s)\) is convergent when \(q_{\text{max}} \to \infty\) as seen from Eq. (5.4). In order to estimate the uncertainties inherent to the method for not too small binding energies, like in the present case, we perform the fit to the lattice energies employing three different values of \(q_{\text{max}}\), 875 MeV, 1075 MeV, 1275 MeV, and the auxiliary potential linear in \(s\) of Eq. (3.6). This will inform us on the size of systematic uncertainties coming from this source. In order not to be confused by the statistical uncertainties, the fit for each value of \(q_{\text{max}}\) will be done to the central values of the lattice energies. Our results, shown in Table 4 for the KD system and in Table 5 for the KD\(^*\) one, confirm that the systematic uncertainties tied to the range are small and well within the statistical uncertainties.

There is another source of systematic uncertainty that we also face here. Unlike in [25], the lattice spectrum used here is calculated with a realistic value of \(m_\pi\) around 156 MeV. However, the \(D\) and \(D^*\) masses of the lattice simulation are smaller than the physical ones, which is related to the Fermilab method employed (see \(M_1\) in Table 3). This is the reason why we did not quote absolute values of the energies obtained, but the binding energies with respect to the thresholds. We can attempt to do an extrapolation of the

\(^{6}\)The normalizations for \(g\) in [12] and here are different. In [12], or in the Weinberg notation, \(\partial G/\partial E\) is used instead of \(\partial G/\partial s\), but the range correcting factor, \(F\), is the same.
Table 4. Dependence of the properties of the $KD$ bound state on $q_{\text{max}}$

| $q_{\text{max}}$ (MeV) | 875  | 1075 | 1275 | Average       |
|------------------------|------|------|------|---------------|
| $B$ (MeV)              | 36.6 | 35.5 | 35.5 | 35.9 ± 0.5    |
| $| g |$ (GeV)              | 10.6 | 10.37| 10.41| 10.46 ± 0.10  |
| $P$ (%)                | 82.15| 84.09| 87.16| 85 ± 2        |
| $a_0$ (fm)             | -1.2446| -1.2453| -1.249| -1.246 ± 0.002 |
| $r_0$ (fm)             | 0.22 | 0.19 | 0.19 | 0.20 ± 0.01   |

Table 5. Dependence of the properties of the $KD^*$ bound state on $q_{\text{max}}$

| $q_{\text{max}}$ (MeV) | 875  | 1075 | 1275 | Average       |
|------------------------|------|------|------|---------------|
| $B$ (MeV)              | 45.6 | 44.9 | 44.2 | 44.9 ± 0.6    |
| $| g |$ (GeV)              | 10.15| 10.32| 10.31| 10.26 ± 0.08  |
| $P$ (%)                | 57.42| 63.33| 66.10| 62 ± 4        |
| $a_0$ (fm)             | -0.967| -0.980| -0.986| -0.978 ± 0.008 |
| $r_0$ (fm)             | -0.03| -0.04| -0.06| -0.043 ± 0.013 |

results to physical masses. For this purpose we assume that the potential obtained can also be considered in absolute terms. Then we use this potential with the realistic masses and obtain the results shown in Tables 6 and 7.

Table 6. Extrapolation of the bound state properties to the physical mass of the $D$ meson, using $q_{\text{max}} = 1275$ MeV.

| $M_1$ (MeV) | 1631 Ref. [5] | 1867 Physical |
|-------------|----------------|---------------|
| $B$ (MeV)   | 35.5           | 31.9          |
| $| g |$ (GeV)   | 10.4           | 11.3          |
| $P$ (%)     | 87.2           | 88.3          |
| $a_0$ (fm)  | -1.25          | -1.33         |
| $r_0$ (fm)  | 0.19           | 0.14          |

A third source of systematic uncertainties comes from the use of one type or another of the potentials, Eqs. (3.6) or (3.14), that we have already discussed in Sects. 3.2 and 3.3, respectively. Comparing the values given in Eqs. (3.10)-(3.13) with those of Eqs. (3.15)-
Table 7. Extrapolation of the bound state properties to the physical mass of the $D^*$ meson, using $q_{\text{max}} = 1275$ MeV.

| $M_1$ (MeV) | 1788 Ref. [5] | 2008 Physical |
|-------------|---------------|---------------|
| $B$(MeV)    | 44            | 96            |
| $|g|$ (GeV)   | 10.3          | 14.2          |
| $P$ (%)     | 66.1          | 60.6          |
| $a_0$ (fm)  | -0.99         | -0.72         |
| $r_0$ (fm)  | -0.060        | -0.002        |

(3.18), we find that the systematic errors associated to the use of different potentials are:

\[
\begin{align*}
\delta B(KD) &= 8.5 \text{ MeV}, \\
\delta B(KD^*) &= 7.5 \text{ MeV}, \\
\delta P(KD) &= 4.5 \% , \\
\delta P(KD^*) &= 4.0 \% , \\
\delta a(KD) &= 0.1 \text{ fm} , \\
\delta a(KD^*) &= 0.2 \text{ fm} , \\
\delta r_0(KD) &= 0.1 \text{ fm} , \\
\delta r_0(KD^*) &= 0.1 \text{ fm} .
\end{align*}
\]

Altogether, summing these systematic errors in quadrature to those of Tables 4–7, we finally obtain the results:

\[
\begin{align*}
B(KD) &= 38 \pm 18 \pm 9 \text{ MeV} , \\
B(KD^*) &= 44 \pm 22 \pm 26 \text{ MeV} , \\
P(KD) &= 72 \pm 13 \pm 5 \% , \\
P(KD^*) &= 57 \pm 21 \pm 6 \% , \\
a(KD) &= -1.3 \pm 0.5 \pm 0.1 \text{ fm} , \\
a(KD^*) &= -1.1 \pm 0.5 \pm 0.2 \text{ fm} , \\
r_0(KD) &= -0.1 \pm 0.3 \pm 0.1 \text{ fm} , \\
r_0(KD^*) &= -0.2 \pm 0.3 \pm 0.1 \text{ fm} ,
\end{align*}
\]

where the first error is statistical and the second systematic, which should also add in quadrature.

6 Conclusions

In this work we have done a reanalysis of the lattice spectra obtained in [5, 6] for s-wave scattering channels $KD$ and $KD^*$, where bound states were identified with the $D_{s0}^*(2317)$
and $D_{s1}^*(2460)$ states. The analysis of [5, 6] derived the scattering length and the effective range from two of the energy levels. The information of the third level was not used. Here we have done a reanalysis of the lattice spectra that takes into account the information of the three levels. The essence of the new method was the use of an auxiliary potential which was allowed to be energy dependent in the case of considering only one channel. This is demanded to take into account the fact that the single channels will most probably not saturate the states. We found a bound state for both $KD$ and $KD^*$ scattering, which we associated to the $D_{s0}^*(2317)$ and $D_{s1}^*(2460)$ states.

In order to find out the most likely missing channels we were guided by the results of the chiral unitary approach which determines the $\eta D_s$ and $\eta D_s^*$ channels as the additional most important ones to saturate the wave function. However, the limited information from the lattice spectra drove us to use an energy independent potential with the consequence that the two channels chosen would saturate the wave function. With this restriction we found no solution, indicating that the lattice spectra does not contain information on the $\eta D_s$ and $\eta D_s^*$ channels. This seems to be the case since the levels found in [5] are largely tied to the interpolators used, and no interpolators accounting for $\eta D_s$ and $\eta D_s^*$ states were included.

We analyzed the lattice spectra considering only one channel and two energy dependent potentials. One potential is taken linear in $s$ and another one contains a CDD pole accounting for possible genuine $\bar{c}s$ components. The results with both methods were compatible within errors. We also studied systematic uncertainties from other sources, which were found, in all cases but one, reasonably smaller than the statistical errors. Our analysis confirmed the existence of bound states for the $KD$ and $KD^*$ channels with a binding of the order of 40 MeV, which we associated to the $D_{s0}^*(2317)$ and $D_{s1}^*(2460)$ states. We could also determine the scattering length and effective range for $KD$ and $KD^*$ scattering, improving on the previous results of [5] based on the information of the lowest two levels only and relying upon the effective range formula. Finally, we could determine within errors that the states found are mostly of meson-meson nature and, using a sum rule which generalizes the test of compositeness condition of Weinberg, we established the probability to find $KD$ and $KD^*$ in those states in an amount of about 70 % and 60 %, respectively. We discussed that, in order to be more precise on these numbers and obtain information on the channels that fill the rest of the probability, one must improve on the precision of the energy spectra and must include further interpolators that allow one to include the $\eta D_s$ and $\eta D_s^*$ channels in the analysis.

The exercise done shows the power of the method and the valuable information contained in the lattice spectra. The errors obtained here can be improved by having extra accuracy in the lattice spectra, additional levels, or more easy perhaps, spectra calculated for other lattice sizes. In any case, it has become clear that the information provided by the lattice spectra, and the flexibility to use different box sizes to obtain a rich spectrum of energies, is most useful when it comes to determine the energy dependence of the auxiliary potentials, which is essential to determine probabilities of meson meson components (or hadron hadron components in general) via the generalized sum rule.
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