Allee effect and Holling type - II response in a discrete fractional order prey - predator model

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Abstract. A gauss type prey - predator model is considered with Allee effect and Holling type II response. Fractional order two species system is discretized. The discretized system exhibits much richer and complex dynamics than its corresponding continuous version. Bifurcation types like flip, neimark sacker and chaos exist in the discretized system. Existence of the positive fixed points is established and local stability of discrete fractional order system is discussed with the variational matrix. It is also shown that the system allows a flip bifurcation and a neimark - sacker bifurcation. Rich dynamical nature of the system is established through time plots with phase trajectory, exciting invariant circles and periodic doubling bifurcation which leads to chaos.

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1. Background of the mathematical model
Many researchers have been attracted by differential equation models of fractional – order because of its applications in a lot of engineering fields as well as science [5]. Fractional – order differential equation is an magnificent tool for the description of memory and heredity effect of different processes [1]. Prey – predator model described by fractional order systems have been analysed by many researchers and many papers studied the stability and dynamical behavior of such systems [8, 9]. Moreover, some work obtained conditions for existence of bifurcations [3] and the stability of limit cycles [4].

Kolmogorov model is one of the general prey – predator models of continuous timed developed with first ordersystem of differential equations. Kolmogorov models can be classified into many types such as Gauss – type models, Lotka – Volterra models, Kuang and Freedman model, Hsu model, and Huang and Merrill model. Prey – predator models of gauss type have been used widely [14]. Several researchers have used Allee effect for system of multi species and have included the prey – predator model of gauss-type with allee effect [12]. In addition, functional responses are often associated with many population models [2], which refers to the predation rate as a functions of prey density and predator density. For example, Kanokrat et al.[10] examined the fractional order prey – predator model of gauss – type in the presence of Allee effect and functional response of Holling type – III.

This paper, we consider an ecosystem of two species interaction with Allee effect as well as functional response of Holling type II. The interaction of fractional order differential equations is described by:
\[ D'x(t) = r(1 - x(t))(x(t) - \mu)x(t) - \frac{\beta x(t)y(t)}{x(t) + \delta}; \]
\[ D'y(t) = \frac{\sigma x(t)y(t)}{x(t) + \delta} - \eta y(t) \]  

(1)

2. Discretization and existence of fixed points

It has been exhibited that the discretized system produces much richer complex dynamics than its continuous counterparts. Many bifurcation types like as neimark-sacker, flip and chaos exist for the discretized system. Applying the discretization process for the fractional-order system using piecewise constant arguments methods to (1), we obtain the discrete version of the form \([5, 6, 11]\)

\[
x(t+1) = x(t) + \frac{\rho^\gamma}{\Gamma(1+\gamma)} \left[ r(1 - x(t))(x(t) - \mu)x(t) - \frac{\beta x(t)y(t)}{x(t) + \delta} \right]
\]

\[
y(t+1) = y(t) + \frac{\rho^\gamma}{\Gamma(1+\gamma)} \left[ \frac{\sigma x(t)y(t)}{x(t) + \delta} - \eta y(t) \right] \]  

(2)

Here \(\gamma\) is fractional order and \(\rho\) is step size of the discretization. \(x(t)\) and \(y(t)\) denote prey and predator populations size at time \(t\). In Biological populations, all the parameters assume positive values. \(r(1 - x(t))(x(t) - \mu)x(t)\) is represented by Allee effect and \(\frac{\beta x(t)y(t)}{x(t) + \delta}\) is expressed as the Holling type – II. The meaning of parameters are provided in the following table:

| Symbols | Meaning | Symbols | Meaning |
|---------|---------|---------|---------|
| \(r\) | Growth rate of \(x(t)\) | \(\mu\) | Minimum viable population |
| \(\beta\) | Maximum reduction rate of \(x(t)\) due to \(y(t)\) | \(\delta\) | Amount of \(x(t)\) at which \(y(t)\) rate is maximal |
| \(\sigma\) | Maximum efficiency of consumption rate of \(x(t)\) due to \(y(t)\) | \(\eta\) | Natural death rate of \(y(t)\) |

System (2) has four fixed points in the \((x, y)\)-plane: \(F_r = (0, 0), F_{s} = (1, 0), F_{ab} = (\mu, 0)\) and

\[
F_i = \left[ \frac{\eta\delta}{\sigma - \eta}, \frac{\rho^\gamma}{\Gamma(1+\gamma)}\left( \frac{\mu}{\eta - \sigma} + \frac{\eta\delta}{\eta} \left( \eta\delta + (\eta - \sigma)(1 + \mu) \right) \right) \right] \]

respectively. The feasibility condition for the interior fixed point \(F_i\) is \(\mu > \frac{\eta\delta}{(\sigma - \eta)}\) if \(\sigma > \eta\).

3. Local stability and bifurcation conditions

Now we focus on the study of local stability of the interior fixed point \(F_i\). Local stability of the system (2) is determined by the absolute value of the characteristic roots from the variational matrix at the fixed point \(F_i\) is

\[
V(F_i) = \begin{bmatrix} \frac{SrA}{\sigma(\sigma - \eta)} - \frac{S\beta\eta}{\sigma} & 1 \\ \frac{SrB}{\beta} & 1 \end{bmatrix}
\]

where \(S = \frac{\rho^\gamma}{\Gamma(1+\gamma)}, A = \eta\delta(1 + \mu)(\sigma + \eta) - \mu(\sigma - \eta) - \eta\delta^2(\eta + 2\sigma)\) \begin{small}(\sigma - \eta)\end{small} and \(B = \mu(\eta - \sigma) + \frac{\delta\eta}{(\eta - \sigma)}\left[ \eta\delta + (1 + \mu)(\eta - \sigma) \right]\). Let the characteristic polynomial of \(V(x, y)\) is \(m^2 - \text{Trace}[V(F_i)]m + \text{det}[V(F_i)]\).
+Det[V(F_i)]=0. Here Trace[V(F_i)]=2+ \frac{SrA}{\sigma(\sigma-\eta)} and Det[V(F_i)]=1+ \frac{SrA}{\sigma(\sigma-\eta)} + \frac{S^2r\eta B}{\sigma}. Using Jury’s criterion [7, 15], we obtain the condition for local stability of the fixed point $F_i$.

3.1. Proposition. When $\mu > \frac{\eta \delta}{(\sigma-\eta)}, \sigma > \eta$, then $F_i$ is a positive fixed point of system (2) and

1. it is sink if $r > \frac{4\sigma[\eta-\sigma]}{2SA+S^2\eta B(\sigma-\eta)}$ and $S < \frac{A}{\eta B(\eta-\sigma)}$.
2. it is source if $r < \frac{4\sigma[\eta-\sigma]}{2SA+S^2\eta B(\sigma-\eta)}$ and $S > \frac{A}{\eta B(\eta-\sigma)}$.
3. it is saddle if $S \neq \frac{A}{\eta B(\eta-\sigma)}$.
4. it is non hyperbolic if one of the following condition holds
   
   (a) $r = \frac{4\sigma[\eta-\sigma]}{2SA+S^2\eta B(\sigma-\eta)}$.
   
   Following Jury’s criterion [13], when (4.(a)) of proposition - (3.1) holds, the eigenvalues of $V(F_i)$ are either $-1$ and 1 or $-1$ and $-1$. Therefore, the flip bifurcation of $F_i$ appear if $r$ varies in the small neighborhood of $FB_{F_i}$ where $FB_{F_i} = \{(r, \rho, \gamma, \mu, \beta, \delta, \eta, \sigma); r = \frac{4\sigma[\eta-\sigma]}{2SA+S^2\eta B(\sigma-\eta)}, \Delta \geq 0; r, \rho, \gamma, \mu, \beta, \delta, \eta, \sigma \geq 0\}$. Similarly, when (4.(b)) of proposition - (3.1) holds, the eigenvalues of $V(F_i)$ are a pair of conjugate complex numbers with modulus one. Hence the Neimark - Sacker Bifurcation of the fixed point $V(F_i)$ will arise if the parameter $S$ varies in the small neighborhood of $NSB_{F_i}$ where $NSB_{F_i} = \{(r, \rho, \gamma, \mu, \beta, \delta, \eta, \sigma); S = \frac{A}{\eta B(\eta-\sigma)}, \Delta < 0; r, \rho, \gamma, \mu, \beta, \delta, \eta, \sigma \geq 0\}$ where $\Delta = 8+\frac{SrA}{\sigma(\sigma-\eta)} \left[8+\frac{SrA}{\sigma(\sigma-\eta)}\right] + \frac{S^2r\eta B}{\sigma}$.

4. Numerical examples

This section presents the time line of the solutions, phase portraits and Bifurcation diagrams for (2) around the interior fixed point $F_i$ to exhibit the conditions arrived in the previous section. The purpose of the numerical examples are to study the analysis of suitable parameters and variation of the fractional order $\gamma$ on the system (2). The process of these effects is partitioned into the following cases:

For case (i.) Phase portrait for the system (2) is given in figures 1(A-B). Fixing the parameter values $\gamma = 0.65; \rho = 1.75; r = 2.99; \mu = 0.19; \delta = 0.99; \beta = 1.8; \eta = 0.7; \sigma = 1.675$ and the initial point $x = 0.75$ and $y = 0.45$, the interior point is $F_i = (0.7108, 0.4256)$ and $1 + \text{Trace}[V(F_i)] + \text{Det}[V(F_i)] = 3.4975 > 0$ and $\text{Det}[V(F_i)] = 0.9833 < 1$, which satisfy the Jury’s criteria and the eigenvalues are $\left|\lambda_{1,2}\right| = 0.9916 < 1$. Hence the system attains stability.
For case (ii.) Phase portrait the system (2) is presented in figures 2(A–B). Considering the parameters $\gamma = 0.85; \rho = 1.75; r = 2.99; \mu = 0.19; \delta = 0.99; \beta = 0.9; \eta = 0.7; \sigma = 1.675$ with $x = 0.75; y = 0.45$, we obtain $1 + \text{Trace}[V(F_1)] + \text{Det}[V(F_1)] = 3.4971 > 0$ and $\text{Det}[V(F_1)] = 1.0143 > 1$. Jury criteria are satisfied and the eigenvalues are $|\lambda_{1,2}| = 1.0143 > 1$. Hence the system is unstable.

Figure 1. (A) Prey - Predator asymptotic stability; (B) Phase trajectory of the system showing that $F_1$ is locally asymptotically stable.

Figure 2. (A) Oscillatory behavior of prey & predator populations; (B) Phase trajectory of the periodic orbit near $F_1$.

For case (iii.) In figure 3(A), When the growth rate $r$ varies from 1.5 and 2.75 with $\gamma = 0.7; \rho = 1.75; \mu = 0.101; \delta = 0.99; \beta = 0.79; \eta = 0.7; \sigma = 1.7$, system (2) is stable and the trajectories spiral clockwise / anti-clockwise towards the fixed point $F_i$. 
In figure 3(B), various phase portraits are given when varying the fractional derivative $\gamma = 0.5 - 0.99$ and the other values $\rho = 1.75; \mu = 0.101; \delta = 0.99; \beta = 0.79; \eta = 0.7; \sigma = 1.7$ are fixed. Also notice that the system (2) has both stable and unstable behavior for changing different values of $\gamma$.

A bifurcation diagram shows the sudden change in the nature of the equilibrium and periodic states of the system. Equilibrium states may appear or disappear, change their stability behavior and sometimes move into periodic states. The structural change of a system with respect to a particular parameter can be investigated with the help of a bifurcation diagram. Here the bifurcation for the system (2) in $(\gamma - x - y)$ space, in $(\gamma - x)$ and in $(\gamma - y)$ plane are given in figures 4 (A-B-C), when varying $\gamma \in [0.5, 2]$ with other parameters $\rho = 1.75; \mu = 0.101; \delta = 0.99; \beta = 0.79; \eta = 0.7; \sigma = 1.7$.

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**Figure 3.** (A) Phase trajectory with varying growth rate of prey; (B) Phase trajectory of the system varying the fractional order values.

**Figure 4.** Structure of Bifurcation (A) in $\gamma - x - y$ space; (B) in $\gamma - x$ plane; (C) in $\gamma - y$ plane.
Similarly, We choose the growth rate of prey $r$ in the range $1.2 \leq r \leq 2.4$ with $\gamma = 0.9; \rho = 1.75; \mu = 0.1; \delta = 0.99; \beta = 0.9; \eta = 0.8; \sigma = 0.7$. The bifurcation diagram in $(r-x-y)$ space and $(r-x)$ plane are shown in figures 5(A–B). Now we take the Allee constant $\mu$ in the range $1 \leq \mu \leq 2$ and fixing the other values as $\gamma = 0.7; \rho = 1.75; r = 1.9; \delta = 0.99; \beta = 0.9; \eta = 0.7; \sigma = 0.8$, we obtain figures 6(A–B). Also the bifurcation in $(\mu-x-y)$ space and $(\mu-x)$ plane are shown.

Figure 5. Bifurcation structure(A) in growth rate of prey – $x-y$ plane ; (B) in growth rate of prey – $x$ plane; (C) Periodic oscillation of prey varying the growth rate of prey

Figure 6. (A) Bifurcation structure in Allee effect constant – $x-y$ plane; (B) Bifurcation structure in Allee effect constant – $x$ plane;
5. Conclusion
In this work, discrete fractional order prey predator model of gauss type in the presence of Allee effect as well as Holling type II functional response proposed by (2). It is exhibited that the discretized system produces much richer dynamics. Also it is showed that the system has an interior fixed point in the closed first quadrant $R^2_+$. Local stability of system (2) for the interior positive fixed point is discussed and the stability conditions are obtained. The system (2) produces flip and Neimark - Sacker bifurcation in $R^2_+$. The analytical results have been verified with various numerical examples. Time line trajectories, phase plane portraits and period doubling bifurcation leading to chaos are presented.

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