Resonance and limit cycle in a noise driven Lorenz model

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The effect of an external noise on the Lorenz model is investigated near the onset of convection and near the Hopf bifurcation. We show the existence of a diverging time scale near the onset of convection and a resonance near the Hopf bifurcation. Our calculation provides an understanding of the noise induced stabilization of the limit cycle that had been observed numerically.

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The noise induced phenomena have an intense interest in several disciplines ranging from physics, to chemistry, and to biology [1]. Physical systems are usually not isolated from their environment, and the environmental influences on the system often appear as fluctuations. It is clear that the fluctuations are an integral part of the evolution of physical, chemical and biological systems and must be understood if we are to accurately and quantitatively describe the world around us, particularly at the small scale. In the past few years, it has become clear that fluctuations can actually be used constructively, by us or by nature, to produce organised behavior that is not possible in the absence of noise. Examples where noise leads to organised behaviour include stochastic resonance [2, 3], noise induced phase transitions [4, 5], noise induced pattern formation [6, 7], and noise induced transport [8, 9]. The constructive role of noise is only possible in the non-linear non-equilibrium systems and is entirely the result of the intricate interplay of noise and non-linearity away from equilibrium.

The effect of noise on dynamical systems include noise induced hopping between multiple stable attractors [10, 11] and noise induced stabilization of the Lorenz attractor [8, 12] near the threshold of its formation. We will focus on the second aspect here and as an example of the effect of noise on hydrodynamic instability. We consider the Lorenz model with an external noise source. Zippelius et. al. [12] first studied the correlations in this model. We will supplement their work by focusing specially on the situations where the control parameter is in the vicinity of an instability. This will lead to a noise induced resonance near the Hopf bifurcation point. We then focus on the effect that the noise can have on the limit cycle beyond the Hopf bifurcation point. Ordinarily the limit cycle is unstable, the numerical work of Gao et. al. [8] shows it can be stabilized by noise and here we explicitly show how such a stabilization is possible by drawing on the technique of McLaughlin and Martin [13].

We study the statistical properties of the following noise driven Lorenz model.

\[
\begin{align*}
\dot{x} &= \sigma(y-x) + \eta_1(t) \\
\dot{y} &= r x - y - xz + \eta_2(t) \\
\dot{z} &= -bz + xy + \eta_3(t)
\end{align*}
\]

where, \(\sigma\) is the Prandtl number, \(b\) is a geometric factor, and the Rayleigh number \(r\) is the control parameter in this model. The parameter \(\sigma = 10\) and \(b = 8/3\) are held fixed at their standard values, as was originally done by Lorenz. \(\eta_i(t)\) is an external white noise source with zero mean and \(\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij} 2D \delta(t-t')\), \(i, j = 1, 2, 3\). \(D\) is the noise strength. \(D = 0\) describes the unforced Lorenz model. When the noise is absent the Lorenz system [14, 15] shows remarkable change in behavior depending on the control parameter \(r\). In the conduction range, \(r < 1\), the trivial steady state solution is stable and loses stability to the other two describing steady convection through a bifurcation at \(r = 1\). Thus, for \(r > 1\), there is a pair of stable fixed points, \((\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1)\) and these in turn lose stability at \(r = r_T = \sigma (\sigma + b + 3)/(\sigma - b - 1) = 24.74\) through a Hopf-bifurcation. For \(r > r_T\), no stable steady state solution exists and the system has a strange attractor. Depending on initial conditions, the solution may settle down on any of these three attractors. The trajectories are non-periodic and wander around in the vicinity of strange set of attracting points for \(r > r_T\) and turbulence sets in.

Now we study the effect of noise on the Lorenz model in the conduction, convection and turbulent regime. We calculate the time dependent correlation functions and compare with the behavior of the unforced model.

In frequency space equation [11] reads
\(-i\omega + \sigma)x(\omega) - \sigma y(\omega) = \eta_1(\omega)\) \hspace{1cm} (2)

\[-r x(\omega) + (-i\omega + 1)y(\omega) + \sum_{\omega_i} x(\omega_1)z(\omega - \omega_1) = \eta_2(\omega)\]

\[-i\omega + b)z(\omega) - \sum_{\omega_i} x(\omega_2)y(\omega - \omega_2) = \eta_3(\omega)\]

Linearizing around the steady state \(x = y = z = 0\) i.e. neglecting the non-linear terms in (2), we can calculate the correlation function by solving the linearized equation. The correlation of the \(x\) variable reads

\[C_{xx}(\omega) = \langle x(\omega)x(-\omega) \rangle = \frac{(\omega^2 + 1) + \sigma^2}{(\omega^2 + (r - 1)\sigma^2 + (\sigma + 1)^2\omega^2)}\] \hspace{1cm} (3)

The time dependent correlation function \(C_{xx}(t)\) can be written as

\[C_{xx}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t}C_{xx}(\omega)\] \hspace{1cm} (4)

and is the sum of two exponential functions which are independent of the strength \(D\) of the fluctuating force. Clearly, the actual relaxation time of the non-linear system depends on the strength of the fluctuating force. The correlation function of \(x\) gets damped faster with increasing noise strength \(D\). The correlation functions \(C_{yy}(t)\) and \(C_{zz}(t)\) follow the same behavior in the regime \(r < 1\).

Near, \(r = 1\), the correlation time goes as

\[\tau \propto \frac{1}{r - 1}\] \hspace{1cm} (5)

i.e. the relaxation time becomes infinitely big as \(r = 1\) is approached - a sign of critical slowing down.

Now, for \(r > 1\), Fourier analysis of the Lorenz model breaks down in its present form. This is very similar to what happens when one enters a symmetry breaking phase in critical phenomena. Accordingly we need to go to the shifted variable \(u_1, u_2, u_3\) defined as \(u_1 = x - x_0\), \(u_2 = y - y_0\) and \(u_3 = z - z_0\).

The Lorenz equation takes the following form

\[\begin{align*}
(-i\omega + \sigma)u_1(\omega) - \sigma u_2(\omega) &= \eta_1(\omega) \\
-u_1(\omega) + (-i\omega + 1)u_2(\omega) + \sigma^2 u_3(\omega) &= \eta_2(\omega) \\
y_0u_1(\omega) - x_0u_2(\omega) + (-i\omega + b)u_3(\omega) &= \eta_3(\omega)
\end{align*}\] \hspace{1cm} (6)

Now, the correlation function takes the following form

\[\begin{align*}
\langle u_1(\omega)u_1(-\omega) \rangle &= \frac{(-\omega^2 + b\sigma)^2 + 2\sigma b(r^2 + \omega^2) + \sigma^2 b(r - 1)}{\omega^2[\omega^2 - b(r + \sigma)]^2 + [\omega^2(\sigma + 1 + b) - 2\sigma b(r - 1)]^2} \\
\end{align*}\]

The time dependent correlation function can be calculated from \(C_{u_1u_1}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t}C_{u_1u_1}(\omega)\). Their time dependence is determined by the three poles in the complex frequency plane. One is purely imaginary reflecting the exponential decay in the correlation function, the other two have finite real parts of the opposite sign, reflecting exponential decay of the correlations caused by the crossing between stable fixed points. In the absence of random force, the system is attracted in general to one of the stable fixed points \((\pm x_0, \pm y_0, z_0)\), depending on its initial condition. If the random force is applied, the trajectories are no longer confined to one of the steady state points. The exponential decay shows the motion from one fixed point to the other. The oscillatory motion slows down the decay of correlation function i.e. the memory effect of the initial state.

At the Hopf bifurcation point, i.e. \(r = \frac{\sigma(b + \sigma + 3)}{\sigma - b - 1}\), the correlation function takes the following form

\[\begin{align*}
\langle u_1(\omega)u_1(-\omega) \rangle &= \frac{(-\omega^2 + b\sigma)^2 + 2\sigma b(r^2 + \omega^2) + \sigma^2 b(r - 1)}{[\omega^2 - \omega_0^2]^2[\omega^2 + (\sigma + 1 + b)]}
\end{align*}\]

where, \(\omega_0 = b(\sigma + \frac{\sigma(b + \sigma + 3)}{\sigma - b - 1})\). Now, the real time correlation function \(C_{u_1u_1}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t}C_{u_1u_1}(\omega)\) exists as a principle value. Hence, the real time correlation function goes as

\[C_{u_1u_1}(t) \propto Re \text{e}^{i\omega_0 t}\] \hspace{1cm} (9)

This is a consequence of the fluctuating force. It is similar in appearance to the resonance in a simple harmonic oscillator subjected to a sinusoidal force.

Approach to this resonance is of the following

\[\langle u_1 u_1 \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{i\omega_0 t}\] \hspace{1cm} (10)

when, \(r = \frac{\sigma(b + \sigma + 3)}{\sigma - b - 1} - \epsilon\).

In the absence of noise there is no periodic state above the Hopf bifurcation point. This is because the limit cycle is unstable. In this case however, a periodic state was observed when the stochastic force was turned on. We will try to understand this on the basis of perturbation theory in terms of small noise strength \(\epsilon\). We return to Eq.(8). Writing \(r = r_0 + \Delta r\), \(x_0 = x_{00} + \hat{c}\), where \(x_{00} = \sqrt{b(r - 1)}\), \(\hat{c} = \frac{\Delta r}{2}\sqrt{\frac{b}{r - 1}}\), we find in real time

\[L \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} + \hat{c} \begin{pmatrix} 0 \\ -u_1 u_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_1 u_2 \\ 0 \end{pmatrix} \] \hspace{1cm} (11)
The driving terms on the right hand side need to be analysed. The second term is time periodic with frequency \( w_0 \) and has an amplitude proportional to \((\Delta r)A\). In the first term, we need to analyse a typical term \((u_{11}u_{30})\) e.g. to understand the different time dependent components of the drive. The term \(u_{30}\) is periodic with frequency \( w_0 \). Two of the components of \(u_{11}\) as shown above are constant in time and periodic with frequency \(2w_0\). The product \(u_{30}u_{11}\) then has terms with frequency \(w_0\) and the amplitude of these terms are proportional to \(A^3\). The operator \(L\) has a zero mode at frequency \(w_0\), consequently in order to allow a finite solution of Eq. (17), the right hand side has to be orthogonal to the left eigenvector of \(L\) with zero eigenvalue. In the absence of the stochastic drive, this is the entire calculation and the result is that \(\Delta r = -\beta^2 A^2\), where \(\beta\) is a constant. Thus there is no real value of \(A\) if \(\Delta r\) is positive and the limit cycle of the Lorenz model cannot be seen.

In the presence of stochastic term, things change because \(u_{30}\) (Eq. (14)) has a term proportional to \(\eta\) and \(u_{11}\) has the two terms which are listed in (ii) and (iv) above. If we consider \(u_{30}u_{11}\) then there are two terms whose solution is \(\text{Amplitude}_{u_{30}u_{11}}\). Since we are in the presence of stochastic terms we can only talk about averages over \(\eta\). Conse-
sequently, we first need to average Eq. (17) over $\eta$ and only terms involving product of two $\eta$’s will survive. After the averaging the term with structure $A\eta e^{i\omega_0 t}$ will acquire the structure $A e^{i\omega_0 t}$ and will be a part of the dangerous term on the right hand side. The orthogonality condition now leads to an equation $\Delta r + N = -\beta^2 A^2$, where $N$ is the extra contribution coming from the stochastic term. In this particular case $N$ is negative and hence for $\Delta r < N$, we can see the limit cycle stabilized and this is the mechanism which allows the time series to be periodic in a region of $r$, where there is no stable limit cycle.

In closing, we have studied the statistical properties of noise driven Lorenz model. We have seen critical slowing down at stationary bifurcation point. At $r > 1$, the Fourier analysis breaks down due to lack of time translational invariance in present form of Lorenz model. An interesting resonance appears at the Hopf-bifurcation point, which shows the the noise induced stability. We also analysed the noise induced stability of limit cycle in the Lorenz model.

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