Large Deviation Principle for Occupation Measures of Stochastic Generalized Burgers–Huxley Equation

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Abstract
The present work deals with the global solvability as well as asymptotic analysis of the stochastic generalized Burgers–Huxley (SGBH) equation perturbed by a white-in-time and correlated-in-space noise defined in a bounded interval of \( \mathbb{R} \). We first prove the existence of a unique mild as well as strong solution to the SGBH equation and then obtain the existence of an invariant measure. Later, we establish two major properties of the Markovian semigroup associated with the solutions of the SGBH equation, that is, irreducibility and the strong Feller property. These two properties guarantee the uniqueness of invariant measures and ergodicity also. Then, under further assumptions on the noise coefficient, we discuss the ergodic behavior of the solution of the SGBH equation by providing a large deviation principle for the occupation measure for large time (Donsker–Varadhan), which describes the exact rate of exponential convergence.

Keywords  Stochastic generalized Burgers–Huxley equation · Irreducibility · Strong Feller · Invariant measures · Large deviation principle · Occupation measures

Mathematics Subject Classification  49J20 · 35Q35 · 35B50

1 Introduction
The stochastic generalized Burgers–Huxley equation (SGBH) describes a prototype model for describing the interaction between reaction mechanisms, convection effects...
and diffusion transports (cf. [27]). We consider the generalized Burgers–Huxley equation perturbed by a random forcing, which is white-in-time and correlated-in-space noise, as

\[
\frac{\partial u(t, \xi)}{\partial t} = \nu \frac{\partial^2 u(t, \xi)}{\partial \xi^2} - \alpha u^\delta(t, \xi) \frac{\partial u(t, \xi)}{\partial \xi} + \beta u(t, \xi) (1 - u^\delta(t, \xi))(u^\delta(t, \xi) - \gamma) + G \frac{\partial^2 \tilde{W}(t, \xi)}{\partial \xi \partial t},
\]

(1.1)

for \((t, \xi) \in (0, T) \times (0, 1)\), where \(\nu > 0\) is the viscosity coefficient, \(\alpha > 0\) is the advection coefficient, \(\beta > 0\), \(\delta \geq 1\) and \(\gamma \in (0, 1)\) are parameters. The noise coefficient \(G : L^2(0, 1) \rightarrow L^2(0, 1)\) is a bounded linear operator, \(\tilde{W}(t, \xi), t \geq 0, x \in (0, 1)\) is a zero mean Gaussian process, whose covariance function is given by

\[
E[\tilde{W}(t, \xi)\tilde{W}(s, \zeta)] = (t \wedge s)(\xi \wedge \zeta), \quad t, s \geq 0, \xi, \zeta \in \mathbb{R}.
\]

On the other hand, one can consider a cylindrical Brownian process \(W(\cdot)\) by setting

\[
W(t) = \frac{\partial \tilde{W}(t)}{\partial \xi} = \sum_{k=1}^{\infty} e_k \beta_k(t),
\]

(1.2)

where \(\{e_k\}_{k=1}^{\infty}\) is an orthonormal basis of \(L^2(0, 1)\) and \(\{\beta_k\}_{k=1}^{\infty}\) is a sequence of independent real Brownian motions in a fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\) adapted to a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). It is well-known that the series (1.2) does not converge in \(L^2(0, 1)\), but it is convergent in any Hilbert space \(U\) such that the embedding \(L^2(0, 1) \subset U\) is Hilbert–Schimdt (cf. [8]). With the above formulation, we rewrite equation (1.1) as

\[
du(t) = \left(\nu \frac{\partial^2 u(t, \xi)}{\partial \xi^2} - \alpha u^\delta(t, \xi) \frac{\partial u(t, \xi)}{\partial \xi} + \beta u(t, \xi) (1 - u^\delta(t, \xi))(u^\delta(t, \xi) - \gamma)\right) dt + GdW(t).
\]

(1.3)

The equation (1.3) is supplemented by the Dirichlet boundary condition:

\[
u(0, \xi) = u(t, 1) = 0, \quad t \geq 0,
\]

(1.4)

and the initial condition

\[
u(0, \xi) = x(\xi).
\]

(1.5)

Let \(L^2(\mathcal{O}) := L^2(0, 1)\) and \(A := -\frac{\partial^2}{\partial \xi^2}\). In order to prove the existence of mild solution, for \(Q = GG^*\), we assume that

\[
D(A^\varepsilon) \subset \text{Im}(Q^\frac{1}{2}), \quad \text{for some } 0 < \varepsilon < 1,
\]

(1.6)
where $\text{Im}(Q^{\frac{1}{2}})$ denotes the range of the operator $Q^{\frac{1}{2}}$. It is equivalent to say that the range of the definition of $A^{-\frac{\delta}{2}}$ in $L^2(\mathcal{O})$ is contained in $\text{Im}(Q^{\frac{1}{2}})$. Under the assumption (1.6), for any $\nu, \alpha, \beta > 0$, the existence of a unique mild solution $u \in C([0, T]; L^p(\mathcal{O}))$ is established in [28]. For $1 \leq \delta < 2$, we prove the existence of an invariant measure for system (1.3)–(1.5). Under the following assumption:

$$D(A^{\frac{\delta}{2}}) \subset \text{Im}(Q^{\frac{1}{2}}), \quad \text{for some } \frac{1}{2} < \varepsilon < 1,$$

we prove the existence of a strong solution to the system (1.3)–(1.5), for any $\nu, \alpha, \beta > 0, \gamma \in (0, 1)$ and $1 \leq \delta < \infty$. For $\delta \in [1, 2]$, the uniqueness of strong solution is established for any $\nu, \alpha, \beta > 0, \gamma \in (0, 1)$ and for $2 < \delta < \infty$, the uniqueness is obtained for $\beta \nu > 2^{2(\delta-1)}\alpha^2$. For these cases, we prove the existence of an invariant measure for any real $\delta \geq 1$ as well as a LDP for the occupation measure for large time (Donsker–Varadhan) for $\delta \in [1, 2]$. The assumption (1.7) implies $\text{Tr}(GG^*) < \infty$, which tells us that the energy injected by the random force is finite. The condition (1.7) also indicates that the noise is not too degenerate.

The stochastic Burgers equation perturbed by cylindrical Gaussian noise is considered in the work [11], where the authors established the existence and uniqueness of mild solution, along with the existence of an invariant measure. The uniqueness of invariant measure is obtained in [9], by showing that the Markov semigroup associated with the solution is irreducible and strong Feller (Chapter 14, [9]). The existence and uniqueness of invariant measures for stochastic Burgers equations perturbed by multiplicative noise is established in [12]. For a sample literature on stochastic Burgers equations, the interested readers are referred to see [1, 3, 10, 11, 21], etc., and references therein. For a comprehensive study on ergodicity for infinite dimensional systems, one may refer to [9, 13], etc. The irreducibility of the semigroup corresponding to the solution of stochastic real Ginzburg–Landau equation driven by $\alpha$-stable noises is proved in [43]. The global solvability results (the existence and uniqueness of strong solutions) and asymptotic analysis (the existence and uniqueness of invariant measures) of stochastic Burgers–Huxley equation is carried out in the paper [30]. In the work [28], the author studied SGBH equation perturbed by space-time white noise and established the existence and uniqueness of mild solution with the help of fixed point and stopping time arguments. Ergodicity results for the stochastic real Ginzburg–Landau equation driven by $\alpha$-stable noises are available in [43, 47], etc.

The theory of large deviations, which provides asymptotic estimates for the probabilities of the rare events, is one of the important research topics in probability theory and received the required attention after the contributions of Varadhan. One can find the theory of large deviation along with its applications in [14, 16, 38, 41], etc. Several authors have established the Wentzell–Freidlin-type LDP for different classes of stochastic partial differential equations (SPDEs) (cf. [5, 25, 39], etc). By using a weak convergence approach, the Wentzell–Freidlin-type LDP for 2D stochastic Navier–Stokes equations (SNSE) perturbed by a small multiplicative noise has been obtained in [40]. Exponential estimates for exit from a ball of radius $R$ by time $T$ for solutions

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of the stochastic Burgers–Huxley equation in the context of Freidlin–Wentzell-type LDP is studied in [30].

An SPDE is ergodic means that the occupation measure of its solution converges to a unique invariant measure. A Donsker–Varadhan-type LDP provides an estimate on the probability of occupation measures deviations from the invariant measure (cf. [15, 17–19]). Thus, it is quite interesting to ask whether the occupation measures satisfy the Donsker–Varadhan-type LDP. Similar to Wentzell–Freidlin-type LDP, a good number of works are available in the literature regarding Donsker–Varadhan-type LDP (cf. [17–19, 21–24, 46], etc., and references therein). A criterion for LDP for occupation measures has been developed in the work [46], the so-called hyper-exponential recurrence for strong Feller and irreducible Markov process. However, the recurrence condition is very strong, and it is not easy to verify this condition for SPDEs. The author in [21] and [22] verified this recurrence condition and proved the LDP for occupation measures of solutions of stochastic Burgers equation and 2D SNSE in bounded domains. Recently, the authors in [44] verified the hyper-exponential recurrence and proved the LDP for occupation measures for a class of nonlinear monotone SPDEs including the stochastic porous medium equation, stochastic $p$-Laplace equation, stochastic fast-diffusion equation, etc. In the context of stochastic convective Brinkman–Forchheimer equations, the hyper-exponential recurrence is verified in [26], and the authors proved LDP for occupation measures. Authors in [24] established the LDP for occupation measures for a class of dissipative PDE’s perturbed by a bounded random kick force. LDP for occupation measures for a class of dissipative PDE’s perturbed by an unbounded kick force is studied in [23] and for stochastic reaction–diffusion equations driven by subordinate Brownian motions is established in [42].

In the present work, we first prove the existence and uniqueness of mild as well as strong solutions for the system (1.3)–(1.5) under assumptions (1.6) and (1.7), respectively. The existence of an invariant measure, strong Feller and topological irreducibility properties of the Markov semigroup corresponding to the solution of SGBH equation (1.1) and hence the uniqueness of invariant measure (Doob’s theorem) are also obtained (Table 1). Then, we discuss the ergodic behavior of SGBH equation by providing an LDP for occupation measures w.r.t. the stronger $\tau$-topology and an LDP of Donsker–Varadhan. To prove LDP w.r.t. the $\tau$-topology for SGBH equation, we establish the hyper-exponential recurrence given in [46]. Let us now summarize the results obtained in this work as a table to emphasize the dependence of noise and different parameters appearing in (1.1).

The article is organized as follows. In Sect. 2, we define linear and nonlinear operators, and the necessary function spaces needed to obtain the solvability and LDP results for our model. Then, we provide an abstract formulation of SGBH equation (see (2.8)) perturbed by the non-degenerate additive noise and discuss the existence and uniqueness of mild as well as strong solutions (Theorems 2.6 and 2.7). The existence and uniqueness of invariant measures for our model is discussed in Sect. 3. Under assumption (1.6) and $1 \leq \delta < 2$, we followed similar arguments as in [9] to establish the existence of an invariant measure (Theorem 3.3). Using energy equality (Itô’s formula), we obtained the existence of an invariant measure under the assumption (1.7) and $1 \leq \delta < \infty$ (Theorem 3.1) in the same section. Then, we discuss two
Table 1 Assumptions on noise and restrictions on δ.

| δ ∈ [1, ∞) | Mild solution (Assm. (1.6)) | Strong solution (Assm. (1.7)) |
|-------------|------------------------------|-------------------------------|
| δ ∈ [1, 2]  | Existence for any v, α, β > 0, γ ∈ (0, 1) | Existence for any v, α, β > 0, γ ∈ (0, 1) |
| δ ∈ (2, ∞), βν > 2^{(δ−1)α}a^2 | Existence and uniqueness | Existence and uniqueness |
| Invariant measure | Existence for δ ∈ [1, 2) | Existence for δ ∈ [1, ∞) with βν > 2^{(δ−1)α}a^2 for δ ∈ (2, ∞) |
| Irreducibility | δ ∈ [1, 2] | δ ∈ [1, 2] |
| Strong Feller | δ ∈ [1, ∞) with βν > 2^{(δ−1)α}a^2 for δ ∈ (2, ∞) | δ ∈ [1, ∞) with βν > 2^{(δ−1)α}a^2 for δ ∈ (2, ∞) |
| Uniqueness of invariant measure | δ ∈ [1, 2) | δ ∈ [1, 2] |
| LDP | – | δ ∈ [1, 2] |

properties of the Markov semigroup associated with the solutions of SGBH equation, that is, irreducibility and strong Feller property (Propositions 4.1 and 4.3). For the proof of strong Feller property, we followed [9], and for irreducibility, we borrowed ideas from [9] and [43]. We state our main result of Donsker–Varadhan-type LDP of occupation measures for the solution of SGBH equation in Sect. 5 (Theorem 5.1 and Corollary 5.2) with the help of exponential estimates for the strong solution of SGBH equation (Proposition 5.7) and the hyper-exponential recurrence given in [46].

2 Mathematical Formulation

This section provides the necessary function spaces needed to obtain the major results of this paper.

2.1 Function Spaces

Let us fix \( \mathcal{O} = (0, 1) \). Let \( C_0^\infty(\mathcal{O}) \) denote the space of all infinitely differentiable functions having compact support in \( \mathcal{O} \). The Lebesgue spaces are denoted by \( L^p(\mathcal{O}) \) for \( p \in [1, \infty) \), and the norm in \( L^p(\mathcal{O}) \) is denoted by \( \| \cdot \|_{L^p} \) and for \( p = 2 \), the inner product in \( L^2(\mathcal{O}) \) is denoted by \( (\cdot, \cdot) \). We denote the usual Sobolev spaces by \( W^{k,p}(\mathcal{O}), k \in \mathbb{N}, p \in [1, \infty] \) and Hilbertian Sobolev spaces by \( H^k(\mathcal{O}), k \in \mathbb{N} \). Let \( H^1_0(\mathcal{O}) \) denote the closure of \( C_0^\infty(\mathcal{O}) \) in the \( H^1 \)-norm. As we are working in \( \mathcal{O} \) (bounded domain), by using Poincaré inequality, the norm \( (\| \cdot \|_{L^2}^2 + \| \partial_\xi \cdot \|_{L^2}^2)^{\frac{1}{2}} \) is equivalent to the seminorm \( \| \partial_\xi \cdot \|_{L^2} \) and hence \( \| \partial_\xi \cdot \|_{L^2} \) defines a norm on \( H^1_0(\mathcal{O}) \). We have the continuous embedding \( H^1_0(\mathcal{O}) \subset L^2(\mathcal{O}) \subset H^{-1}(\mathcal{O}) \), where \( H^{-1}(\mathcal{O}) \) is the dual space of \( H^1_0(\mathcal{O}) \). For the bounded domain \( \mathcal{O} \), the embedding \( H^1_0(\mathcal{O}) \subset L^2(\mathcal{O}) \) is compact.
The duality pairing between $H^1_0(O)$ and its dual $H^{-1}(O)$, and $L^p(O)$ and its dual $L^{p^{-1}}(O)$ is denoted by $\langle \cdot, \cdot \rangle$. In one dimension, we have the continuous embedding: $H^1_0(O) \subset L^\infty(O) \subset L^p(O)$, for $p \in [1, \infty)$. Also the embedding of $H^\sigma(O) \subset L^p(O)$ is compact for any $\sigma > \frac{1}{2} - \frac{1}{p}$, for $p \geq 2$. The following interpolation inequality will be used frequently in the paper. Assume that $1 \leq p \leq r \leq q \leq \infty$ and $\frac{1}{r} = \frac{a}{p} + \frac{1-\theta}{q}$. For $u \in L^p(O) \cap L^q(O)$, we have $u \in L^r(O)$, and

$$\|u\|_{L^r} \leq \|u\|_{L^p}^\theta \|u\|_{L^q}^{1-\theta}.$$ 

The following fractional form of the Gagliardo–Nirenberg inequality (see [31] and [32]) is also used in the sequel. Fix $1 \leq q, l \leq \infty$ and $n = 1$. Suppose also that a real number $\theta$ and a nonnegative number $j$ are such that

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{l} - \frac{m}{n}\right)\theta + \frac{1-\theta}{q}, \quad \frac{j}{m} \leq \theta \leq 1,$$

then we have

$$\|D^j u\|_{L^p} \leq C\|D^m u\|_{L^l}^\theta \|u\|_{L^q}^{1-\theta},$$

for all $u \in W^{m,l}(O) \cap H^1_0(O)$.

### 2.2 Linear Operator

Let $A$ be the self-adjoint and unbounded operator on $L^2(O)$ defined by

$$Au := -\frac{\partial^2 u}{\partial \xi^2},$$

with domain $\text{Dom}(A) = H^2(O) \cap H^1_0(O) = \{u \in H^2(O) : u(0) = u(1) = 0\}$. The eigenvalues and the corresponding eigenfunctions of $A$ are given by

$$\lambda_k = k^2 \pi^2 \quad \text{and} \quad e_k(\xi) = \sqrt{\frac{2}{\pi}} \sin(k\pi x), \quad k = 1, 2, \ldots.$$ 

As we are working on the bounded domain $O$, the inverse of $A$, that is, $A^{-1}$ exists and is a compact operator on $L^2(O)$. Moreover, we can define the fractional powers of $A$ and

$$\|A^\frac{1}{2} u\|_{L^2}^2 = \sum_{j=1}^{\infty} |(u, e_j)|^2 \geq \lambda_1 \sum_{j=1}^{\infty} |(u, e_j)|^2 = \lambda_1 \|u\|_{L^2}^2 = \pi^2 \|u\|_{L^2}^2.$$ 

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which is the Poincaré inequality. An integration by parts yields

\[(Au, v) = (\partial_\xi u, \partial_\xi v) =: a(u, v), \quad \text{for all } v \in H^1_0(\mathcal{O}),\]

so that \(A : H^1_0(\mathcal{O}) \to H^{-1}(\mathcal{O})\). Let us define the operator \(A_p = -\frac{\partial^2}{\partial \xi^2}\) with \(D(A_p) = W^{1,p}_0(\mathcal{O}) \cap W^{2,p}(\mathcal{O})\), for \(1 < p < \infty\) and \(D(A_1) = \{ u \in W^{1,1}(\mathcal{O}) : u \in L^1(\mathcal{O}) \}\), for \(p = 1\). From Proposition 4.3, Chapter 1 [2] (see [33] also), we know that for \(1 \leq p < \infty\), \(A_p\) generates an analytic semigroup of contractions in \(L^p(\mathcal{O})\).

### 2.3 Nonlinear Operators

We define two nonlinear operators in this subsection.

#### 2.3.1 The Operator \(B(\cdot)\)

Let us define \(b : H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O}) \to \mathbb{R}\) as

\[b(u, v, w) = \int_0^1 (u(\xi))^{\delta} \frac{\partial v(\xi)}{\partial x} w(\xi) d\xi.\]

Using an integration by parts and boundary conditions, it is immediate that

\[b(u, u, u) = (u^{\delta} \partial_\xi u, u) = \int_0^1 (u(\xi))^{\delta} \frac{\partial u(\xi)}{\partial \xi} u(\xi) d\xi = \frac{1}{\delta + 2} \int_0^1 (u(\xi))^{\delta+2} d\xi = 0,\]

and

\[b(u, u, v) = -\frac{1}{\delta + 1} b(u, v, u),\]

for all \(u, v \in H^1_0(\mathcal{O})\). In general, using integration by parts, one can easily show that

\[b(u, u, |u|^{p-2} u) = (u^{\delta} \partial_\xi u, |u|^{p-2} u) = 0, \quad (2.1)\]

for all \(p \geq 2\) and \(u \in H^1_0(\mathcal{O})\). For \(w \in L^2(\mathcal{O})\), we define \(B(\cdot, \cdot) : H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O}) \to L^2(\mathcal{O})\) by

\[B(u, v, w) = b(u, v, w) \leq \|u\|^{\delta}_{L^\infty} \|\partial_\xi v\|_{L^2} \|w\|_{L^2} \leq \|u\|^{\delta}_{H^1_0} \|v\|_{H^1_0} \|w\|_{L^2},\]

so that \(\|B(u, v)\|_{L^2} \leq \|u\|_{H^1_0} \|v\|_{H^1_0}\). We set \(B(u) = B(u, u)\), so that we can easily obtain \(\|B(u)\|_{L^2} \leq \|u\|^{\delta+1}_{H^1_0}\). One can show that the operator \(B(\cdot)\) is a locally Lipschitz
operator, that is, (cf. [27])

\[ \|B(u) - B(v)\|_{L^2} \leq C\delta(1 + 2\delta)r^\delta\|u - v\|_{H_0^1}, \]  

(2.2)

for \(\|u\|_{H_0^1}, \|v\|_{H_0^1} \leq r\).

2.3.2 The Operator \(c(\cdot)\)

Let us define the operator \(c : H_0^1(\mathcal{O}) \to L^2(\mathcal{O})\) by

\[ c(u) = u(1 - u^\delta)(u^\delta - \gamma). \]

It is easy to compute that

\[ (c(u), u) = (u(1 - u^\delta)(u^\delta - \gamma), u) = ((1 + \gamma)u^\delta + \gamma u - u^{2\delta + 1}, u) = (1 + \gamma)(u^{\delta + 1}, u) - \gamma\|u\|_{L^2}^2 - \|u\|_{L^2(\delta + 1)}^2, \]

for all \(u \in L^{2(\delta + 1)}(\mathcal{O}) \subset H_0^1(\mathcal{O})\). The operator \(c(\cdot)\) is also locally Lipschitz, that is, (cf. [27])

\[ \|c(u) - c(v)\|_{L^2} \leq C\pi ((1 + \gamma)(1 + \delta)2^\delta r^\delta + \gamma + (1 + 2\delta)2^{2\delta + 2\delta})\|u - v\|_{H_0^1}, \]  

(2.3)

for \(\|u\|_{H_0^1}, \|v\|_{H_0^1} \leq r\).

2.4 Linear Problem

Let us first consider the following stochastic heat equation:

\[
\begin{cases}
\frac{dz(t)}{dt} = -\nu Az(t) \, dt + GdW(t), & t \in (0, T), \\
z(0) = 0,
\end{cases}
\]  

(2.4)

where \(G\) satisfies the assumption (1.6). Then, from Chapter 5, [8], we infer that the solution of (2.4) is unique and it can be defined by the stochastic convolution

\[ z(t) = \int_0^t R(t - s)GdW(s), \]

where \(R(t) = e^{-tA}\). Also note that the process \(z\) is a Gaussian process and it is mean square continuous taking values in \(L^2(\mathcal{O})\) and \(z\) has a version, which has \(\mathbb{P}\)-a.s., \(\lambda\)-Hölder continuous paths w.r.t. \((t, \xi) \in [0, T] \times [0, 1]\) for any \(\lambda \in (0, \frac{1}{2}]\) (for more details see Theorem 5.22, [8]). Under the assumption (1.7), one can show that \(z \in \mathbb{C}(0, T; L^2(\mathcal{O})) \cap L^2(0, T; H_0^1(\mathcal{O})), \mathbb{P}\)-a.s.
Since \( R(t) = e^{-\nu t A} \) is an analytic semigroup, we infer that \( R(\cdot) : L^p(\mathcal{O}) \to L^q(\mathcal{O}) \) is a bounded map whenever \( 1 \leq p \leq q \leq \infty \) and \( t > 0 \), and there exists a constant \( C \) depending on \( p, q \) and \( \nu \) such that (see Lemma 3, Part I, [36])

\[
\| R(t) f \|_{L^q} \leq Ct^{-\frac{1}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) \| f \|_{L^p},
\]

for all \( t \in (0, T] \) and \( f \in L^p(\mathcal{O}) \). Moreover, we have

\[
\| A^\sigma R(t) \|_{L(L^p)} \leq \frac{C}{t^{\sigma}},
\]

for any \( \sigma \in (0, 1) \), \( p \geq 1 \) and \( t \in (0, T] \).

### 2.5 Mild Solution

In this subsection, we provide the definition of mild solution, and we state a result form [28], where the existence and uniqueness of a global mild solution to the SGBH equation (1.3) are established. One can rewrite the abstract formulation of the problem (1.3) as

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{du(t)}{dt} = \{ -\nu Au(t) - \alpha B(u(t)) + \beta c(u(t)) \} dt + GdW(t), \quad t \in (0, T), \\
u(0) = x,
\end{array} \right.
\end{align*}
\]

(2.8)

where \( x \in L^p(\mathcal{O}) \), for \( p \geq 2(\delta + 1) \).

**Definition 2.1** An \( L^p(\mathcal{O}) \)-valued and \( \mathcal{F}_t \)-adapted stochastic process \( u : [0, \infty) \times [0, 1] \times \Omega \to \mathbb{R} \) with \( \mathbb{P} \)-a.s continuous trajectories on \( t \in [0, T] \), is a mild solution to (2.8), if for any \( T > 0 \), \( u(t) := u(t, \cdot, \cdot) \) satisfies the following integral equation:

\[
\begin{align*}
u(t) &= R(t)x - \alpha \int_0^t R(t-s)B(u(s))ds + \beta \int_0^t R(t-s)c(u(s))ds \\
&\quad + \int_0^t R(t-s)GdW(s),
\end{align*}
\]

\( \mathbb{P} \)-a.s., for all \( t \in [0, T] \).

**Theorem 2.2** [28] Let the \( \mathcal{F}_0 \)-measurable initial data \( x \) be given and \( x \in L^p(\mathcal{O}) \), \( \mathbb{P} \)-a.s. Then, there exists a unique mild solution of (2.8), which belongs to \( C([0, T]; L^p(\mathcal{O})) \), for \( p \geq 2(\delta + 1) \), \( \mathbb{P} \)-a.s.

### 2.6 Strong Solution

Let us now discuss the existence and uniqueness of strong solution to the system under assumption (1.7).
**Definition 2.3** [Strong solution] Let $x \in L^2(\mathcal{O})$ be given. An $L^2(\mathcal{O})$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted stochastic process $u(\cdot)$ is called strong solution to the system (2.8) if

- the process

$$u \in L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O}))) \cap L^2(\Omega; L^2(0, T; H^1_0(\mathcal{O}))) \cap L^2(\delta+1)(\Omega; L^2(\delta+1)(0, T; L^2(\delta+1)(\mathcal{O})))$$

and $u \in C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H^1_0(\mathcal{O})) \cap L^2(\delta+1)(0, T; L^2(\delta+1)(\mathcal{O})), \mathbb{P}$-a.s.,

- the process $u$ satisfies

$$(u(t), \varphi) = (x, \varphi) + \int_0^t (-vAu(s) - \alpha B(u(s)) + \beta c(u(s)), \varphi)ds + \int_0^t (GdW(s), \varphi),$$

for all $t \in [0, T]$ and $\varphi \in H^1_0(\mathcal{O})$, $\mathbb{P}$-a.s.,

- the following energy equality is satisfied:

$$\|u(t)\|_L^2 + 2\gamma \int_0^t \|\partial_x u(s)\|_L^2 ds + 2\beta \gamma \int_0^t \|u(s)\|_L^2 ds + 2\beta \int_0^t \|u(s)\|_{L^2(\delta+1)}^2 ds$$

$$= \|x\|_L^2 + 2\beta(1 + \gamma) \int_0^t (u^{\delta+1}(s), u(s)) ds$$

$$+ \text{Tr}(GG^*)t + 2\int_0^t (GdW(s), u(s)) ds,$$

for all $t \in [0, T], \mathbb{P}$-a.s.

For $x \in L^2(\mathcal{O})$, in order to prove the existence of mild solution and strong solution under the assumptions (1.6) and (1.7), respectively, let us first set

$$v(t) := u(t) - z(t), \quad t \geq 0.$$

Then, $u(\cdot)$ is a solution of (2.8) if and only if $v(t)$ is a solution of

$$\begin{cases}
\frac{dv(t)}{dt} = -vAv(t) - \alpha B(v(t) + z(t)) + \beta c(v(t) + z(t)), \quad t \in (0, T),
\vspace{10pt}
v(0) = x,
\end{cases}$$

(2.10)

which is, for fixed $\omega \in \Omega$, a deterministic system. One can rewrite (2.10) in the mild form as

$$v(t) = R(t)x - \alpha \int_0^t R(t - s)B(v(s) + z(s))ds + \beta \int_0^t R(t - s)c(v(s) + z(s))ds.$$  

(2.11)

If $v \in C([0, T]; L^2(\mathcal{O}) \cap L^2(0, T; H^1_0(\mathcal{O})) \cap L^2(\delta+1)(0, T; L^2(\delta+1)(\mathcal{O}))$ satisfies (2.11), we say that it is a weak solution of (2.10). For each fixed $\omega \in \Omega$, the weak
form of (2.10) can be written as

\[
(v(t), \varphi) = (x, \varphi) + \int_{0}^{t} \langle -vA v(s) - \alpha B(v(s) + z(s)) + \beta c(v(s) + z(s)), \varphi \rangle ds,
\]

for all \( t \in [0, T] \) and \( \varphi \in H_{0}^{1}(\mathcal{O}) \). The next result provides the existence of weak solution to the system (2.10).

**Theorem 2.4** Let \( x \in L^{2}(\mathcal{O}) \) be the given initial data. Then, there exists a weak solution \( v \in C([0, T]; L^{2}(\mathcal{O})) \cap L^{2}(0, T; H_{0}^{1}(\mathcal{O})) \cap L^{2(\delta+1)}(0, T; L^{2(\delta+1)}(\mathcal{O})) \) to system (2.10).

For \( \delta \in [1, 2] \), the weak solution is unique for any \( \nu, \alpha, \beta > 0, \gamma \in (0, 1) \) and for \( 2 < \delta < \infty \), the weak solution is unique for \( \beta \nu > 2^{2(\delta-1)}\alpha^{2} \).

**Proof** The proof based on the standard Faedo–Galerkin approximation and compactness arguments. Let us start with the Faedo–Galerkin approximation:

**Step (1): Faedo–Galerkin approximation.** Let \( H_{n} = \text{span}\{e_{1}, \ldots, e_{n}\} \), where \( \{e_{1}, \ldots, e_{n}, \ldots\} \) be a complete orthogonal system of eigenfunctions of the Laplacian operator \( A = -\frac{\partial^{2}}{\partial x^{2}} \) and let \( \Pi_{n} : L^{2}(\mathcal{O}) \rightarrow H_{n} \) be the orthogonal projection operator, that is, \( x_{n} = \Pi_{n} x = \sum_{j=1}^{n} (x, e_{j}) \), for \( x \in L^{2}(\mathcal{O}) \). We define

\[
B_{n}(u) = \Pi_{n} B(\Pi_{n} u), \quad c_{n}(u) = \Pi_{n} c(\Pi_{n} u), \quad G_{n} = \Pi_{n} G \Pi_{n}.
\]

Let us define \( z_{n}(t) := \int_{0}^{t} R(t - s)G_{n}dW(s) \). Then, it has been shown in Lemma 3.3, [28] that

\[
z_{n} \rightarrow z \text{ in } C([0, T] \times [0, 1]).
\]

Let us first consider the following finite dimensional system:

\[
\begin{align*}
\frac{dv_{n}(t)}{dt} &= -vA v_{n}(t) - \alpha B(v_{n}(t) + z_{n}(t)) + \beta c(v_{n}(t) + z_{n}(t)), \quad t \in (0, T), \\
v_{n}(0) &= x_{n}.
\end{align*}
\]

(2.12)

Since \( B(\cdot) \) and \( c(\cdot) \) satisfy the locally Lipschitz conditions (see (2.2) and (2.3)), the above system has a unique local solution \( v_{n} \in C([0, T^{*}]; H_{n}) \) for some \( 0 < T^{*} < T \). Now, we show that the time \( T^{*} \) can be extended to \( T \) by deriving a uniform energy estimate.

**Step (2): L^{2}-energy estimate.** Let us take the inner product of (2.12) with \( v_{n} \) to get

\[
\frac{1}{2} \frac{d}{dt} \left\| v_{n}(t) \right\|_{L^{2}}^{2} + v \left\| \frac{\partial}{\partial t} v_{n}(t) \right\|_{L^{2}}^{2} = -\alpha \langle B(v_{n}(t) + z_{n}(t)), v_{n}(t) \rangle + \beta \langle c(v_{n}(t) + z_{n}(t)), v_{n}(t) \rangle,
\]

(2.13)
for a.e. \( t \in [0, T] \). Using integration by parts, Taylor’s formula, (2.1), Hölder’s and Young’s inequalities, we estimate the term \( -\alpha(B(v_n + z_n), v_n) \) as

\[
-\alpha(B(v_n + z_n), v_n) \\
= \frac{\alpha}{\delta + 1} (v_n^{\delta+1}, v_n) + \alpha(z_n(\theta_1 v_n + (1 - \theta_1)z_n)^\delta, \partial_\xi v_n) \\
= \alpha(z_n(\theta_1 v_n + (1 - \theta_1)z_n)^\delta, \partial_\xi v_n) \\
\leq \frac{\nu}{2} \| \partial_\xi v_n \|_{L^2}^2 + \frac{\alpha^2}{2\nu} \| z_n(\theta_1 v_n + (1 - \theta_1)z_n)^\delta \|_{L^2}^2 \\
\leq \frac{\nu}{2} \| \partial_\xi v_n \|_{L^2}^2 + 2^{(\delta-1)} \frac{\alpha^2}{2\nu} \| z_n \|_{L^2(\delta+1)}^2 \| v_n \|_{L^2(\delta+1)}^{2\delta} + 2^{(\delta-1)} \frac{\alpha^2}{2\nu} \| z_n \|_{L^2(\delta+1)}^{2(\delta+1)} \\
\leq \frac{\nu}{2} \| \partial_\xi v_n \|_{L^2}^2 + \frac{\beta}{4} \| v_n \|_{L^2(\delta+1)}^{2(\delta+1)} \left[ \frac{1}{\delta + 1} \left( \frac{4\delta}{\beta(\delta + 1)} \right)^\delta + \frac{\alpha^2 2^{2(\delta-1)}}{2\nu} \right] \| z_n \|_{L^2(\delta+1)}^{2(\delta+1)}.
\]

(2.14)

Let us now consider the second term of right-hand side of (2.13). We estimate using Taylor’s formula, Hölder’s and Young’s inequalities as

\[
\beta(c(v_n + z_n), v_n) \\
= \beta(1 + \gamma)(v_n^{\delta+1}, v_n) + \beta(1 + \gamma)(\delta + 1)(z_n(\theta_2 v_n + (1 - \theta_2)z_n)^\delta, v_n) \\
- \beta \gamma \| v_n \|_{L^2}^2 - \beta \gamma \| v_n \|_{L^2(\delta+1)}^{2(\delta+1)} - \beta(2\delta + 1)(z_n(\theta_3 v_n + (1 - \theta_3)z_n)^{2\delta}, v_n) \\
\leq \beta(1 + \gamma) \| v_n \|_{L^2(\delta+1)}^{\delta+1} \| v_n \|_{L^2} + \beta(1 + \gamma)(\delta + 1)2^{\delta-1} \| z_n \|_{L^2(\delta+1)} \| v_n \|_{L^2}^{\delta} \\
+ \beta(1 + \gamma)(\delta + 1)2^{\delta-1} \| z_n \|_{L^2(\delta+1)} \| v_n \|_{L^2} - \beta \gamma \| v_n \|_{L^2(\delta+1)}^{2\delta} - \beta \gamma \| v_n \|_{L^2}^{2\delta} \\
- \beta \| v_n \|_{L^2(\delta+1)}^{2(\delta+1)} - \beta(2\delta + 1)2^{\delta-1} \| z_n \|_{L^2(\delta+1)} \| v_n \|_{L^2(\delta+1)}^{2\delta} \\
+ \beta(2\delta + 1)2^{\delta-1} \| z_n \|_{L^2(\delta+1)}^{2(\delta+1)} \| v_n \|_{L^2(\delta+1)}^{2(\delta+1)} \\
\leq 2\beta(1 + \gamma)^2 \| v_n \|_{L^2}^2 + \frac{\beta}{\delta + 1} \left( \frac{8\delta}{\delta + 1} \right)^\delta [(1 + \gamma)^2(\delta + 1)2^{2\delta-3}]^{\delta+1} \| z_n \|_{L^2(\delta+1)}^{2(\delta+1)} \\
+ \beta \| v_n \|_{L^2}^2 + \beta(1 + \gamma)^2(\delta + 1)2^{\delta-3} \| z_n \|_{L^2(\delta+1)}^{2(\delta+1)} + \frac{\beta \gamma^2}{4} \| z_n \|_{L^2}^2 - \frac{\beta}{2} \| v_n \|_{L^2(\delta+1)}^{2(\delta+1)} \\
+ \frac{\beta}{\delta + 1} \left( \frac{8\delta}{\delta + 1} \right)^\delta [(2\delta + 1)2^{\delta-1}]^{\delta+1} \| z_n \|_{L^2(\delta+1)}^{2(\delta+1)} \\
+ \beta \left( \frac{2\delta + 1}{2(\delta + 1)} \right) \left( \frac{4}{\delta + 1} \right)^{2\delta+1} [(2\delta + 1)2^{\delta-1}]^{2(\delta+1)} \| z_n \|_{L^2(\delta+1)}^{2(\delta+1)}.
\]

(2.15)
Using estimates (2.14)–(2.15) in (2.13), we obtain

\[
\frac{d}{dt} \|v_n(t)\|_{L^2}^2 + v \|\partial_x v_n(t)\|_{L^2}^2 + \frac{\beta}{2} \|v_n(t)\|_{L^{2(\delta+1)}}^{2(\delta+1)} \\
\leq C(\beta, \gamma) \|v_n(t)\|_{L^2}^2 + C(\alpha, \beta, \gamma, \delta, \nu) \|z_n(t)\|_{L^{2(\delta+1)}}^{2(\delta+1)}
\]

\[
\leq C(\beta, \gamma) + \frac{\beta}{4} \|v_n(t)\|_{L^{2(\delta+1)}}^{2(\delta+1)} + C(\alpha, \beta, \gamma, \delta, \nu) \|z_n(t)\|_{L^{2(\delta+1)}}^{2(\delta+1)},
\]

for a.e. \( t \in [0, T] \). Integrating the above inequality from 0 to \( t \) and using the fact that \( \|x_n\|_{L^2} \leq \|x\|_{L^2} \), we get

\[
\|v_n(t)\|_{L^2}^2 + v \int_0^t \|\partial_x v_n(s)\|_{L^2}^2 ds + \frac{\beta}{2} \int_0^t \|v_n(s)\|_{L^{2(\delta+1)}}^{2(\delta+1)} ds \\
\leq C(\alpha, \beta, \gamma, \delta, \nu) \left( \|x\|_{L^2}^2 + \int_0^t \|z(s)\|_{L^{2(\delta+1)}}^{2(\delta+1)} ds \right),
\]

for all \( t \in [0, T] \).

**Step (3): Weak and strong convergences along a subsequence and existence.** For the initial data \( x \in L^2(\mathcal{O}) \), using the estimate (2.17) and an application of Banach–Alaoglu theorem yields the existence of a subsequence \( \{v_{n_k}\} \) of \( \{v_n\} \) such that (for convenience, we still denote the index \( n_k \) by \( n \)):

\[
\begin{align*}
& v_n \overset{w^*}{\rightharpoonup} v \text{ in } L^\infty(0, T; L^2(\mathcal{O})), \\
& v_n \rightharpoonup v \text{ in } L^2(0, T; H^1_0(\mathcal{O})), \\
& v_n \rightharpoonup v \text{ in } L^{2(\delta+1)}(0, T; L^{2(\delta+1)}(\mathcal{O})),
\end{align*}
\]

as \( n \to \infty \). One can establish the following weak convergence along a subsequence (still denoting by the same notation)

\[
\frac{dv_n}{dt} \overset{w}{\rightharpoonup} \frac{dv}{dt} \text{ in } L^{2(\delta+1)}(0, T; H^{-1}(\mathcal{O})).
\]

Since the embedding \( H^1_0(\mathcal{O}) \subset L^2(\mathcal{O}) \) is compact, using Aubin–Lions compactness lemma (Theorem 1, [37]), we deduce the following strong convergence (along a subsequence):

\[
v_n \to v \text{ in } L^2(0, T; L^2(\mathcal{O})) \text{ as } n \to \infty.
\]

Using the convergence obtained in (2.18) and (2.19), one can also show that

\[
B_n(v_n + z_n) \overset{w}{\rightharpoonup} B(v + z) \text{ in } L^2(0, T; L^2(\mathcal{O})),
\]

and

\[
c_n(v_n + z_n) \overset{w}{\rightharpoonup} c(v + z) \text{ in } L^2(0, T; L^2(\mathcal{O})),
\]
as \( n \to \infty \). Thus, one can pass to limit in equation (2.12), and use the convergences (2.18), (2.20) and (2.21) to deduce that \( v(\cdot) \) is a weak solution. Moreover, we have \( v \in W^{1, 2^{(d+1) \over 2d+1}}(0, T; H^{-1}(\Omega)) \) and an application of Theorem 2, page 302, [20] implies \( v \in C[0, T]; H^{-1}(\Omega) \). Moreover, from Proposition 1.7.1, [6], we have \( v \in C_w([0, T]; L^2(\Omega)) \), where \( C_w([0, T]; L^2(\Omega)) \) is the space of functions \( v : [0, T] \to L^2(\Omega) \) which are weakly continuous. Since \( v \in L^2(0, T; H^\alpha(\Omega)) \cap L^2(0, T; L^2(\Omega)) \) and \( {d \over dt} v \in L^{2^{(d+1) \over 2d+1}}(0, T; H^{-1}(\Omega)) \subset L^2(0, T; H^{-1}(\Omega)) + L^{2^{(d+1) \over 2d+1}}(0, T; L^2(\Omega)), \) an application of Exercise 8.2, [35] yields that \( v \in C([0, T]; L^2(\Omega)) \) and it satisfies the energy equality

\[
\|v(t)\|^2_{L^2} + 2v \int_0^t \|\partial_x v(s)\|^2_{L^2} ds = \|x\|^2_{L^2} + 2 \int_0^t (-\alpha B(v(s) + z(s)) + \beta c(v(s) + z(s)), v(s))ds, \tag{2.22}
\]

for all \( t \in [0, T] \).

**Step (4): Uniqueness.** Let \( v_1 \) and \( v_2 \) be the two weak solutions of the system (2.12) and the initial data \( x \). Note that \( w = v_1 - v_2 \) satisfies:

\[
{d \over dt} v_1 - \nu A v_1 = -\alpha [(v_1 + z)^{\delta + 1} - (v_2 + z)^{\delta + 1}, \partial_x (v_1 + z)] + \beta [(v_1 + z)(1 - (v_1 + z)^{\delta}) \times ((v_1 + z)^{\delta} - \gamma) - (v_2 + z)(1 - (v_2 + z)^{\delta})(v_2 + z)^{\delta} - \gamma)],
\]

\[
w(0) = 0, \tag{2.23}
\]

in \( H^{-1}(\Omega) \), for a.e. \( t \in [0, T] \). Taking the inner product of first equation of above system (2.23) with \( w \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_{L^2} + \nu \|\partial_x w\|^2_{L^2} = \frac{\alpha}{\delta + 1} ((v_1 + z)^{\delta + 1} - (v_2 + z)^{\delta + 1}, \partial_x w) + \beta ((v_1 + z)(1 - (v_1 + z)^{\delta}) \times ((v_1 + z)^{\delta} - \gamma) - (v_2 + z)(1 - (v_2 + z)^{\delta})(v_2 + z)^{\delta} - \gamma), w). \tag{2.24}
\]

For the term \( \frac{\alpha}{\delta + 1} ((v_1 + z)^{\delta + 1} - (v_2 + z)^{\delta + 1}, \partial_x w) \), we use Taylor’s formula, Hölder’s inequality, interpolation inequality, Gagliardo–Nirenberg inequality and Young’s inequality to estimate as

\[
\frac{\alpha}{\delta + 1} ((v_1 + z)^{\delta + 1} - (v_2 + z)^{\delta + 1}, \partial_x w) \\
\leq \alpha^{\delta - 1} (\|v_1 + z\|^\delta_{L^2(\Omega)} + \|v_2 + z\|^\delta_{L^2(\Omega)}) \|w\|_{L^2(\Omega)} \|\partial_x w\|_{L^2} \\
\leq \alpha^{\delta - 1} (\|v_1 + z\|^\delta_{L^2(\Omega)} + \|v_2 + z\|^\delta_{L^2(\Omega)}) \|w\|_{L^2}^{\delta + 2 \over (\delta + 1)^2} \|\partial_x w\|_{L^2}^{3 \delta + 2 \over (\delta + 1)^2}.
\]
\[ \leq \frac{1}{2} \| \partial_x w \|_{L^2}^2 + c^2 \frac{4^3 \delta + 2}{(\delta + 1)^4} \left( \alpha^{2(\delta + 1)} \right)^{\frac{3 \delta + 2}{(\delta + 1)^2}} \times \left( \| v_1 + z \|_{L^2(\delta + 1)} + \| v_2 + z \|_{L^2(\delta + 1)} \right) \| w \|_{L^2}^2. \]  

(2.25)

Let us take the term \( \beta(1 + \gamma)((v_1 + z)^{\delta+1} - (v_2 + z)^{\delta+1}, w) \) and estimate it using Taylor’s formula, Hölder’s and Young’s inequalities as

\[ \beta(1 + \gamma)((v_1 + z)^{\delta+1} - (v_2 + z)^{\delta+1}, w) \]
\[ \leq \beta(1 + \gamma)(\delta + 1)2^{\delta-1} \left( \| (|v_1 + z|^\delta + |v_2 + z|^\delta) w \|_{L^2} \right) \| w \|_{L^2} \]
\[ \leq \frac{\beta}{4} \| v_1 + z \|_{L^2}^2 + \frac{\beta}{4} \| v_2 + z \|_{L^2}^2 + \beta 2^{2\delta-1}(1 + \delta)^2(1 + \gamma)^2 \| w \|_{L^2}^2. \]  

(2.26)

Also, we have

\[ -\beta \gamma ((v_1 + z) - (v_2 + z), w) = -\beta \gamma (w, w) = -\beta \gamma \| w \|_{L^2}^2. \]  

(2.27)

In order to estimate the final term of right-hand side of (2.24), we use the following formula (cf. [29]):

\[ (x|x|^{2\delta} - y|y|^{2\delta}, x - y) \geq \frac{1}{2} \| x \|_{L^2}^2 + \frac{1}{2} \| y \|_{L^2}^2. \]  

(2.28)

Let us take the term \( -\beta((v_1 + z)^{2\delta+1} - (v_2 + z)^{2\delta+1}, w) \) and using formula (2.28) to estimate it as

\[ -\beta((v_1 + z)^{2\delta+1} - (v_2 + z)^{2\delta+1}, w) \leq -\frac{\beta}{2} \| v_1 + z \|_{L^2}^2 - \frac{\beta}{2} \| v_2 + z \|_{L^2}^2. \]  

(2.29)

Combining (2.26), (2.27) and (2.29), we get

\[ \beta(((1 + \gamma)((v_1 + z)^{\delta+1} - (v_2 + z)^{\delta+1})) - \gamma ((v_1 + z) - (v_2 + z)) - ((v_1 + z)^{2\delta+1} - (v_2 + z)^{2\delta+1}), w) \]
\[ \leq -\frac{\beta}{4} \| v_1 + z \|_{L^2}^2 - \frac{\beta}{4} \| v_2 + z \|_{L^2}^2 \]
\[ - \beta \gamma \| w \|_{L^2}^2 + \beta 2^{2\delta-1}(1 + \delta)^2(1 + \gamma)^2 \| w \|_{L^2}^2. \]  

(2.30)
Using (2.25) and (2.30) in (2.24), we obtain
\[
\frac{d}{dt} \|w\|^2_{L^2} + v \| \partial_\xi w \|^2_{L^2} + \frac{\beta}{2} \|v_1 + z|\delta w\|^2_{L^2} + \frac{\alpha}{2} \|v_2 + z|\delta w\|^2_{L^2} + 2\beta \gamma \|w\|^2_{L^2} \\
\leq c2^{2\delta+1} \|\partial_\xi w\|^2_{L^2} + \left(\frac{\delta + 2}{4(\delta + 1)}\right)^{2\delta+2} \left(\frac{3\delta + 2}{2\nu(\delta + 1)}\right)^{2\delta+2} \left(\|v_1 + z\|^2_{L^2(\delta+1)} + \|v_2 + z\|^2_{L^2(\delta+1)}\right) \|w\|^2_{L^2} \\
+ \beta 2^{2\delta}(1 + \delta)^2(1 + \gamma)^2 \|w\|^2_{L^2}.
\]

(2.31)

An application of Gronwall’s inequality in (2.31) yields
\[
\|w(t)\|^2_{L^2} \leq \|w(0)\|^2_{L^2} \exp(2^\delta \beta(1 + \delta)^2(1 + \gamma)^2 T) \\
\times \exp\left\{C(\alpha, \delta, \nu) \int_0^T \left(\|v_1(t) + z(t)\|^2_{L^2(\delta+1)} + \|v_2(t) + z(t)\|^2_{L^2(\delta+1)}\right) \, dt\right\},
\]

(2.32)

for all \(t \in [0, T]\). For \(1 \leq \delta \leq 2\), the term appearing inside the exponential is finite, since \(v_1(\cdot)\) and \(v_2(\cdot)\) are weak solutions of the system. Since \(w(0) = 0\), the uniqueness follows from (2.32) for any \(\nu, \delta, \alpha\) and \(\delta \in [1, 2]\).

From (2.32), we obtain the uniqueness of the weak solution with a restriction on \(\delta \in [1, 2]\). To remove this restriction, we estimate the term \(\alpha \frac{(\delta + 1)}{\delta} ((v_1 + z)^{\delta+1} - (v_2 + z)^{\delta+1}, \partial_\xi w)\), with the help of Taylor’s formula, Hölder’s and Young’s inequalities as
\[
\alpha \frac{(\delta + 1)}{\delta} ((v_1 + z)^{\delta+1} - (v_2 + z)^{\delta+1}, \partial_\xi w) \\
\leq 2^{\delta-1} \alpha \|v_1 + z\|^2_{L^2} + \|v_2 + z\|^2_{L^2} \|\partial_\xi w\|^2_{L^2} \\
\leq \theta \nu \|\partial_\xi w\|^2_{L^2} + \frac{2^{2\delta-2} \alpha^2}{2\theta \nu} \|v_1 + z\|^2_{L^2} + \frac{2^{2\delta-2} \alpha^2}{2\theta \nu} \|v_2 + z\|^2_{L^2}.
\]

(2.33)

Using (2.26) and (2.32) in (2.24), we obtain
\[
\frac{d}{dt} \|w\|^2_{L^2} + v(1 - \theta) \|\partial_\xi w\|^2_{L^2} + \left(\beta(1 - \theta) + \frac{2^{2\delta-2} \alpha^2}{\theta \nu}\right) \\
\left\{\|v_1 + z\|^2_{L^2} + \|v_2 + z\|^2_{L^2}\right\} \\
+ 2\beta \gamma \|w\|^2_{L^2} \leq C(\alpha, \beta, \gamma, \delta, \nu) \|w\|^2_{L^2}.
\]

(2.34)

For \(\beta \nu > 2^{2\delta-2} \alpha^2\), an application of Gronwall’s inequality in (2.34) gives
\[
\|w(t)\|^2_{L^2} \leq \|w(0)\|^2_{L^2} \exp\{C(\alpha, \beta, \gamma, \delta, \nu) T\},
\]

(2.35)

for all \(t \in [0, T]\). Hence, the uniqueness follows from (2.35), provided \(\beta \nu > 2^{2\delta-1} \alpha^2\). □
Remark 2.5 Under additional regularity assumptions on \( z \) and \( x \in H^1_0(\mathcal{O}) \), one can prove the existence of a strong solution \( v \in L^\infty(0, T; H^1_0(\mathcal{O})) \cap L^2(0, T; H^2(\mathcal{O})) \) to problem (2.10) and hence from the estimate (2.32), it follows that the strong solution (in the deterministic sense) is unique for any \( v, \alpha, \beta > 0, \gamma \in (0, 1) \) and \( 1 \leq \delta < \infty \).

Since \( u = v + z \) and \( z \in C([0, T]; C([0, 1])) \), \( \mathbb{P} \)-a.s., we have the following results:

**Theorem 2.6** For \( x \in L^2(\mathcal{O}) \), under assumption (1.7), there exists a strong solution \( u \) to the equation (2.8) in the sense of Definition 2.3 such that

\[
  u \in C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; C([0, 1])) \cap L^{2(\delta+1)}(0, T; L^{2(\delta+1)}(\mathcal{O})), \quad \mathbb{P}\text{-a.s.}
\]

(2.36)

For \( \delta \in [1, 2] \), the strong solution is unique for any \( v, \alpha, \beta > 0, \gamma \in (0, 1) \) and for \( 2 < \delta < \infty \), the strong solution is unique for \( \beta v > 2^{(\delta-1)\alpha^2} \).

**Theorem 2.7** For \( x \in L^2(\mathcal{O}) \), under assumption (1.6), there exists a mild solution \( u \) to the equation (2.8) in the sense of Definition 2.1 such that (2.36) is satisfied.

### 3 Existence of an Invariant Measure

In this section, we discuss the existence of an invariant measures for solutions of equation (2.8). Let us first introduce some notations, which will be used in the upcoming sections. We denote the space of probability measures on \( L^2(\mathcal{O}) \) equipped with the Borel \( \sigma \)-field \( B \) by \( M_1(L^2(\mathcal{O})) \), the space of signed \( \sigma \)-additive measures of bounded variation on \( L^2(\mathcal{O}) \) by \( M_b(L^2(\mathcal{O})) \), the space of all bounded measurable functions on \( L^2(\mathcal{O}) \) by \( B_b(L^2(\mathcal{O})) \) and the space of all bounded continuous functions on \( L^2(\mathcal{O}) \) by \( C_b(L^2(\mathcal{O})) \). On the space \( M_b(L^2(\mathcal{O})) \), we consider \( \sigma(M_b(L^2(\mathcal{O}))), B_b(L^2(\mathcal{O})), \) the \( \tau \)-topology of convergence against measurable and bounded functions which is much stronger than the usual weak convergence topology \( \sigma(M_b(L^2(\mathcal{O})), C_b(L^2(\mathcal{O}))) \) ([17–19], Section 6.2, [7]). Let us denote \( \| \cdot \|_{\text{sup}} \) for the supremum norm in \( C_b \) (or \( B_b \)). We denote the duality relation between \( \varrho \in M_b(L^2(\mathcal{O})) \) and \( \psi \in B_b(L^2(\mathcal{O})) \) by \( \varrho(\psi) := \int \varrho \psi \, d\varrho \). In the sequel, we denote the law on \( C(\mathbb{R}^+; L^2(\mathcal{O})) \) of the Markov process with \( x \in L^2(\mathcal{O}) \) as initial state by \( \mathbb{P}_x \). We define \( \mathbb{P}_\varrho(\cdot) = \int \mathbb{P}_x \varrho(dx) \), where \( \varrho \) be any initial measure on \( L^2(\mathcal{O}) \).

Let \( E \) be any Borel subset of \( L^2(\mathcal{O}) \), and the transition probability measure \( P(t, x, \cdot) \) be defined by \( P(t, x, B) = \mathbb{P}\{u(t, x) \in B\} \), for all \( t > 0 \), \( x \in E \) and all Borelian sets \( B \in B(E) \), where \( u(t, x) \) is the solution of the SGBH equation (2.8) with the initial condition \( x \in L^2(\mathcal{O}) \). Such a process is shown to exists and Markovian. We define \( \{P_t\}_{t \geq 0} \), a Markov semigroup in the space \( C_b(E) \) corresponding to the strong solution of SGBH equation (2.8) as

\[
  (P_t \varphi)(x) = \mathbb{E}[\varphi(u(t, x))], \quad \text{for all } \varphi \in C_b(E).
\]

A Markov semigroup \( P_t, \ t \geq 0 \) is *Feller* if \( P_t : C_b(E) \to C_b(E) \) for arbitrary \( t > 0 \). Let us first consider the dual semigroup \( \{P_t^*\}_{t \geq 0} \) in the space \( M_1(E) \), which is defined
as
\[ \int_E \varphi d(P_t^* \rho) = \int_E P_t \varphi d\rho, \]
for all \( \varphi \in C_b(E) \) and \( \rho \in M_1(E) \). A measure \( \rho \in M_1(E) \) is called invariant if \( P_t^* \rho = \rho \) for all \( t \geq 0 \). Under assumptions (1.6) and (1.7), we prove the existence of an invariant measure for the SGBH equation (2.8) for the following two different cases:

(i) The noise coefficient has finite trace (assumption 1.7) and without any restriction on \( \delta (\beta \nu > 2^{(\delta-1)} \alpha^2 \text{ for } 2 < \delta < \infty) \).

(ii) The general case (assumption (1.6)) with the restriction on \( \delta \in [1, 2) \).

**Case (i):** \( \text{Tr}(GG^*) < \infty \). We state and prove the existence of the invariant measure for this case in the following Theorem:

**Theorem 3.1** Let us take the initial data \( x \in L^2(\mathcal{O}) \). Then, there exists an invariant measure for the system (2.8) with support in \( H^1_0(\mathcal{O}) \) (\( \beta \nu > 2^{(\delta-1)} \alpha^2 \), for \( 2 < \delta < \infty \)).

**Proof** Applying infinite dimensional Itô’s formula to the process \( \|u(\cdot)\|_{L^2}^2 \), we find

\[
\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_\xi u(s)\|_{L^2}^2 ds + \beta \int_0^t \|u(s)\|_{L^{2(\delta+1)}}^2 ds = \|x\|_{L^2}^2 + \text{Tr}(GG^*) t + \beta (1 + \gamma^2) \int_0^t \|u(s)\|_{L^2}^2 ds + 2 \int_0^t (GdW(s), u(s)),
\]

for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s., where we have used (2.1). Using the fact that \( L^{2(\delta+1)}(\mathcal{O}) \subset L^2(\mathcal{O}) \), Hölder’s and Young’s inequalities, we obtain

\[
\beta (1 + \gamma^2) \int_0^t \|u(s)\|_{L^2}^2 ds \leq \beta (1 + \gamma^2) \left( \int_0^t \|u(s)\|_{L^{2(\delta+1)}}^2 ds \right)^{\frac{1}{\delta+1}} \int_0^t \|u(s)\|_{L^{2(\delta+1)}}^{\delta+1} ds + C(\beta, \delta) t,
\]

where the constant \( C(\beta, \delta) = \left( \frac{2}{\beta(\delta+1)} \right)^{\frac{1}{\delta+1}} \). Using the above estimate in (3.1), we have

\[
\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_\xi u(s)\|_{L^2}^2 ds + \frac{\beta}{2} \int_0^t \|u(s)\|_{L^{2(\delta+1)}}^2 ds \leq \|x\|_{L^2}^2 + (\text{Tr}(GG^*) + C) t + 2 \int_0^t (GdW(s), u(s)),
\]

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for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s. Taking expectation in (3.2), we obtain
\[
\mathbb{E} \left[ \left\| u(t) \right\|_{L^2}^2 + 2v \int_{0}^{t} \left\| \partial_x u(s) \right\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_{0}^{t} \left( \left\| u(s) \right\|_{L^2(\delta+1)}^2 \right) \, ds \right] \leq \| x \|_{L^2}^2 + (\text{Tr}(\mathbb{G}^n) + C)t, \tag{3.3}
\]
where we have used the fact that the final term is a martingale. Thus, for all \( t > T_0 \), we have
\[
\frac{2v}{t} \mathbb{E} \left[ \int_{0}^{t} \left\| \partial_x u(s) \right\|_{L^2}^2 \, ds \right] \leq \frac{1}{T_0} \| x \|_{L^2}^2 + \text{Tr}(\mathbb{G}^n) + C. \tag{3.4}
\]
An application of Markov’s inequality yields
\[
\lim_{M \to \infty} \sup_{T > T_0} \left[ \frac{1}{T} \int_{0}^{T} \mathbb{P} \{ \left\| \partial_x u(t) \right\|_{L^2} > M \} \, dt \right] \\
\leq \lim_{M \to \infty} \sup_{T > T_0} \frac{1}{M^2} \mathbb{E} \left[ \frac{1}{T} \int_{0}^{T} \left\| \partial_x u(s) \right\|_{L^2}^2 \, ds \right] \\
\leq \lim_{M \to \infty} \sup_{T > T_0} \frac{1}{M^2} \left[ \frac{1}{T_0} \| x \|_{L^2}^2 + \text{Tr}(\mathbb{G}^n) + C \right] \\
= 0. \tag{3.5}
\]
Using estimate (3.5) and the compact embedding \( \mathbb{H}^1_0(\mathcal{O}) \subset L^2(\mathcal{O}) \), it is clear from the standard argument that the sequence of probability measures \( \mu_{t,x} (\cdot) = \frac{1}{t} \int_{0}^{t} \mathbb{P}_s(0, \cdot) \, ds \) is tight. That is, for each \( \epsilon > 0 \), there exist a compact subset \( K \subset L^2(\mathcal{O}) \) such that \( \mu_{t,x}(K^c) \leq \epsilon \) for all \( t > 0 \) and hence by Krylov–Bogoliubov theorem (see Theorem 2.2, [4] also), \( \mu_{t,x} \to \mu \), weakly for \( n \to \infty \) and the measure \( \mu \) is an invariant measure for the transition semigroup \( \{\mathbb{P}_t\}_{t \geq 0} \), and is defined as
\[
\mathbb{P}_t \varphi(x) = \mathbb{E}[\varphi(u(t, x))],
\]
for all \( \varphi \in C_b(L^2(\mathcal{O})) \), where \( u(\cdot) \) is the unique strong solution of (2.8) with the initial data \( x \in L^2(\mathcal{O}) \). If the distribution of \( x \in L^2(\mathcal{O}) \) is given by \( \mu \) (so is for \( u(\cdot) \) due to the invariance and it should noted that \( u \in \mathbb{H}^1_0(\mathcal{O}) \), a.s., for a.e. \( t \in [0, T] \), then the fact that \( \mathbb{H}^1_0(\mathcal{O}) \) is compactly embedded in \( L^2(\mathcal{O}) \), the estimates (3.4) and (3.5) along with Theorem 2.2, [4] imply that \( \mu \) has support in \( \mathbb{H}^1_0(\mathcal{O}) \). \( \square \)

**Case (ii): The general assumption (1.6).**

Let us now rewrite the problem in a different form. For any \( \kappa > 0 \), we set \( R_\kappa(t) = e^{-\kappa t} R(t), t \geq 0 \). Then, the mild solution of (2.8) with the initial data \( x = 0 \), is given by the integral form
\[
u(t) = -\alpha \int_{0}^{t} R_\kappa(t-s) B(u(s)) \, ds + \beta \int_{0}^{t} R_\kappa(t-s) c(u(s)) \, ds + z_\kappa(t), \tag{3.6}
\]
where \( z_\kappa(t) = \int_0^t R_\kappa(t - s)GdW(s) \), and \( \kappa > 0 \) will be fixed later. Now setting \( v(t) = u(t) - z_\kappa \) and transform the problem associated with (3.6) into the initial value problem

\[
\begin{cases}
    v'(t) = -vA v(t) - \alpha B(v(t) + z_\kappa(t)) + \beta c(v(t) + z_\kappa(t)) + \kappa z_\kappa(t), \\
    v(0) = 0.
\end{cases}
\]  

(3.7)

Also note that system (3.7) defines a transition semigroup \( P_t, t \geq 0 \), on \( B_b(L^2(\mathcal{O})) \), which holds the Feller property, since the solution of (3.7) depends continuously on the initial data. With the help of general theory developed in Chapter 6, [9], in order to prove the existence of an invariant measure for (3.7), it is sufficient to show that the family of measures

\[
\left\{ \frac{1}{T} \int_0^T P_s(0, \cdot) ds \right\},
\]

is tight on \( L^2(\mathcal{O}) \). Before going to prove the tightness of the family of measures defined above, we require the following Lemma.

**Lemma 3.2** For any \( \varepsilon > 0 \), there exists \( K_\varepsilon \) such that for all \( T > 0 \),

\[
\frac{1}{T} \int_0^T \mathbb{P}(\|v(s)\|_{L^2}^2 > K_\varepsilon) ds < \varepsilon.
\]  

(3.8)

**Proof** Let \( v \) be the solution of (3.7). Taking the inner product of first the equation of the system (3.7) with \( v(\cdot) \), we get

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|\partial_\xi v\|_{L^2}^2 = \frac{\alpha}{\delta + 1}((v + z_\kappa)^{\delta + 1}, \partial_\xi v) + \beta(1 + \gamma)((v + z_\kappa)^{\delta + 1}, v) - \beta \gamma (v + z_\kappa, v) - \beta ((v + z_\kappa)^{2\delta + 1}, v) + \kappa (z_\kappa, v).
\]  

(3.9)

A similar calculation as in (2.14) helps us to estimate the term \( \frac{\alpha}{\delta + 1}((v + z_\kappa)^{\delta + 1}, \partial_\xi v) \) as

\[
\frac{\alpha}{\delta + 1}((v + z_\kappa)^{\delta + 1}, \partial_\xi v) \leq \frac{v}{2} \|\partial_\xi v\|_{L^2}^2 + \frac{\beta}{8} \|v + z_\kappa\|_{L^2(\delta + 1)}^{2(\delta + 1)} + \frac{1}{\delta + 1} \left( \frac{\alpha^2}{2v} \right)^{\delta + 1} \left( \frac{8\delta}{\beta(\delta + 1)} \right)^{\delta} \|z_\kappa\|_{L^2(\delta + 1)}^{2(\delta + 1)}.
\]  

(3.10)

Next, we take the term \( \beta(1 + \gamma)((v + z_\kappa)^{\delta + 1}, v) - \beta \gamma (v + z_\kappa, v) - \beta ((v + z_\kappa)^{2\delta + 1}, v) \) and estimate it using the embedding \( L^2(\delta + 1)(\mathcal{O}) \subset L^2(\mathcal{O}) \), Hölder’s and Young’s
inequalities as
\[ (v + z_k)^{d+1}, v + z_k - z_k - \beta \gamma (v + z_k, v) - \beta ((v + z_k)^{2d+1}, v + z_k - z_k) \]
\[ \leq \beta (1 + \gamma) \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + \beta (v + z_k - z_k) \]
\[ \leq \beta (1 + \gamma) \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + \beta \gamma \| v \|^{\frac{d}{2}} L_2^{(d+1)} \]
\[ + \beta \| v + z_k \| L_2^{(d+1)} \leq \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} \]
\[ \leq - \frac{3\beta}{8} \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + 4\beta (1 + \gamma)^2 \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + \left( 4\beta (1 + \gamma)^2 + \frac{\beta \gamma}{2} \right) \| z_k \|^{2} L_2^{(d+1)} \]
\[ - \frac{\beta \gamma}{2} \| v \|^{\frac{d}{2}} L_2^{(d+1)} + \frac{1}{2} \left( \frac{2\beta + 1}{\beta (d+1)} \right) \| z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} \]
\[ \leq - \frac{\beta}{4} \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + \beta (1 + \gamma)^2 \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + \left( 4\beta (1 + \gamma)^2 + \beta \gamma \right) \| z_k \|^{2} L_2^{(d+1)} \]
\[ - \frac{\beta \gamma}{2} \| v \|^{\frac{d}{2}} L_2^{(d+1)} + \frac{1}{2} \left( \frac{2\beta + 1}{\beta (d+1)} \right) \| z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} \]
\[ \leq - \frac{\beta}{4} \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + \beta (1 + \gamma)^2 \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + \left( 4\beta (1 + \gamma)^2 + \frac{\beta \gamma}{2} \right) \| z_k \|^{2} L_2^{(d+1)} \]
(3.11)

Once again using the Cauchy–Schwarz inequality and Hölder’s inequality, we estimate the term \( \kappa (z_k, v) \) as
\[ \kappa (z_k, v) \leq \frac{\beta \gamma}{4} \| v \|^{\frac{d}{2}} L_2^{(d+1)} + \frac{\kappa^2}{\beta \gamma} \| z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} \]
(3.12)

Substituting (3.10)–(3.12) in (3.9), and using the Poincaré inequality \( \pi^2 \| v \|^{2} L_2^{(d+1)} \leq \| \partial_x v \|^{2} L_2^{(d+1)} \), we find
\[ \frac{d}{dr} \| v \|^{\frac{d}{2}} L_2^{(d+1)} + \left( v \pi^2 + \frac{\beta \gamma}{2} \right) \| v \|^{\frac{d}{2}} L_2^{(d+1)} + \frac{\beta}{4} \| v + z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} \]
\[ \leq C(\beta, \gamma, \kappa) \| z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + C(\nu, \alpha, \beta, \delta) \| z_k \|^{\frac{d+1}{2}} L_2^{(d+1)} + C(\beta, \gamma, \delta) \]
(3.13)

where \( C(\beta, \gamma, \kappa) = \left( 8\beta (1 + \gamma)^2 + \beta \gamma + \frac{\kappa^2}{\beta \gamma} \right) \),
\[ C(\beta, \gamma, \delta) = \frac{2\delta (4\beta (1 + \gamma)^2)^{\frac{\delta+1}{2}}}{\delta+1} \left( \frac{8}{\beta (\delta+1)} \right)^{\frac{1}{2}} \],
and
\[ C(\nu, \alpha, \beta, \delta) = \left( \frac{1}{\delta+1} \left( \frac{\alpha^2}{\nu} \right)^{\frac{\delta+1}{2}} \left( \frac{8}{\beta (\delta+1)} \right)^{\delta} + \frac{1}{\delta^2} \left( \frac{2\delta+1}{\beta (\delta+1)} \right)^{\frac{1}{2}} \right) \].

Finally, we proceed with a similar argument as in the work Chapter 14, [12]. We fix \( K > 1 \) and define
\[ \zeta (t) = \log (\| v (t) \|^{2} L_2^{(d+1)} \vee K) \].
Then, we have

$$
\zeta'(t) = \frac{1}{\|v(t)\|_L^2} \chi_{\{\|v(t)\|_L^2 \geq K\}} \frac{d}{dt} \|v(t)\|_L^2.
$$

On multiplying both sides of (3.13), with \(\frac{1}{\|v(t)\|_L^2} \chi_{\{\|v(t)\|_L^2 \geq K\}}\), we get

$$
\zeta'(t) + \left( v\pi^2 + \frac{\beta\gamma}{2} \right) \chi_{\{\|v(t)\|_L^2 \geq K\}} \leq \frac{1}{K} \left\{ C(\beta, \gamma, \kappa) \|z_\kappa\|_L^{2(\delta+1)} + C(\nu, \alpha, \beta, \delta) \|z_\kappa\|_L^{2(\delta+1)} + C(\beta, \gamma, \delta) \right\}.
$$

Integrating the above inequality from 0 to \(t\) and taking the expectation on both sides, we find

$$
\mathbb{E}[(\zeta(t) - \zeta(0)) + \left( v\pi^2 + \frac{\beta\gamma}{2} \right) \int_0^t \mathbb{P}(\|v(s)\|_L^2 \geq K) \, ds \leq \frac{1}{K} \int_0^t \left\{ C(\beta, \gamma, \kappa) \mathbb{E}(\|z_\kappa(s)\|_L^{2}) + C(\nu, \alpha, \beta, \delta) \mathbb{E}(\|z_\kappa(s)\|_L^{2(\delta+1)}) + C(\beta, \gamma, \delta) \right\} \, ds.
$$

Using the fact that \(\zeta(t) - \zeta(0) = \zeta(t) - \log K = \log \left( \frac{\|v(t)\|_L^2 \vee K}{K} \right) \geq 0\), we obtain

$$
\left( v\pi^2 + \frac{\beta\gamma}{2} \right) \frac{1}{t} \int_0^t \mathbb{P}(\|v(s)\|_L^2 \geq K) \, ds \leq \frac{1}{Kt} \int_0^t \left\{ C(\beta, \gamma, \kappa) \mathbb{E}(\|z_\kappa(s)\|_L^{2}) + C(\nu, \alpha, \beta, \delta) \mathbb{E}(\|z_\kappa(s)\|_L^{2(\delta+1)}) + C(\beta, \gamma, \delta) \right\} \, ds,
$$

hence (3.8) holds by choosing \(K\) sufficiently large.

Now, we state our main result of this section, that is, the existence of the invariant measure for the noise with the general assumption (1.6).

**Theorem 3.3** Let \(P_t, t \geq 0\) be the transition semigroup corresponding to the solutions of the system (2.8). Then, there exists an invariant measure for the semigroup \(P_t, t \geq 0\) for \(\delta \in [1, 2)\).

**Proof** First, we fix \(\kappa > 0\), for which Lemma 3.2 holds. We have the embedding \(D(A^\sigma) \subset L^2(O)\) is compact for any \(\sigma > 0\). In order to prove the tightness property, it is sufficient to show that for any \(\epsilon > 0\), there exists \(K > 0\), such that for all \(T > 1\)

$$
\frac{1}{T} \int_0^T \mathbb{P}(\|A^\sigma u(t)\|_L^2 \geq K) \, dt < \epsilon.
$$

\(\square\)
Now onward, we fix both \( \kappa > 0 \) and \( \sigma < \frac{2 - \delta}{4 \delta} \) for \( \delta \in [1, 2) \). For \( \sigma \in [0, \frac{1}{4}) \), it has been shown in Lemma 14.4.1, [9] that

\[
M = \sup_{t \geq 1} \mathbb{E}[\|A^\sigma z_\kappa(t)\|_{L^2}^2] < +\infty.
\]

An application of Markov’s inequality yields

\[
I_T = \frac{1}{T} \int_1^T \mathbb{P} \left\{ \|A^\sigma z_\kappa(t)\|_{L^2}^2 \geq K \right\} dt \\
\leq \frac{1}{TK} \int_1^T \mathbb{E} \left[ \|A^\sigma z_\kappa(t)\|_{L^2}^2 \right] dt \leq \frac{M}{TK} (T - 1).
\]

For the chosen sufficiently large \( K \), we can made \( I_T \) small uniformly for \( T \geq 1 \).

To prove (3.15), it is sufficient to show it for the process \( \hat{u}(\cdot) \) replaced by \( v(\cdot) \), since \( u(\cdot) = v(\cdot) + z(\cdot) \). This will be derived as did in Theorem 6.1.2, [9] by exploiting the regularizing effect of (2.8). For the mild solution \( v(\cdot) \) of (3.7) and any \( t > 0 \), we have

\[
A^\sigma v(t + 1) = A^\sigma R(1)v(t) - \alpha A^\sigma \int_t^{t+1} R(t + 1 - s)B(v(s) + z_\kappa(s))ds \\
+ \beta A^\sigma \int_t^{t+1} R(t + 1 - s)c(v(s) + z_\kappa(s))ds \\
+ \kappa A^\sigma \int_t^{t+1} R(t + 1 - s)z_\kappa(s)ds.
\]

Taking the \( L^2 \)-norm of above expression, we obtain

\[
\|A^\sigma v(t + 1)\|_{L^2} \leq \|A^\sigma R(1)v(t)\|_{L^2} + \alpha \left\| A^\sigma \int_t^{t+1} R(t + 1 - s)B(v(s) + z_\kappa(s))ds \right\|_{L^2} \\
+ \beta \left\| A^\sigma \int_t^{t+1} R(t + 1 - s)c(v(s) + z_\kappa(s))ds \right\|_{L^2} \\
+ \kappa \left\| A^\sigma \int_t^{t+1} R(t + 1 - s)z_\kappa(s)ds \right\|_{L^2}.
\]

(3.16)

Using semigroup property (see (2.6) and (2.7)), interpolation and Young’s inequalities, we estimate the term \( \left\| A^\sigma \int_0^t R(s)\partial_\xi u^{\delta + 1}(s)ds \right\|_{L^2} \) as

\[
\left\| A^\sigma \int_0^t R(s)\partial_\xi u^{\delta + 1}(s)ds \right\|_{L^2} \leq C \int_0^t s^{-(\sigma + \frac{3}{4})} \|u(s)\|_{L^{\delta+1}} ds
\]
In similar lines, we estimate the other terms in the right-hand side of (3.16) as

\[ \leq C \int_0^t s^{-(\sigma + \frac{3}{4})} \| u(s) \|_{L^2}^{\frac{\delta + 1}{\delta}} \| u(s) \|_{L^2(\delta + 1)}^{\frac{(\delta + 1)(\delta - 1)}{\delta}} \, ds \]

\[ \leq C\left( \int_0^t s^{-(\sigma + \frac{3}{4})} \| u(s) \|_{L^2}^{\frac{2\delta}{\delta + 1}} \, ds + \int_0^t \| u(s) \|_{L^2(\delta + 1)}^{2(\delta + 1)} \, ds \right) \]

\[ \leq C\left[ \int_0^t s^{-(\sigma + \frac{3}{4})} \| u(s) \|_{L^2}^{\frac{2\delta}{\delta + 1}} + \sup_{s \in [0, t]} \| u(s) \|_{L^2(\delta + 1)}^{2(\delta + 1)} \right]. \]

With the help of semigroup property (see (2.5) and (2.7)), the embedding $L^2(\delta + 1)(\mathcal{O}) \subset L^{\delta + 1}(\mathcal{O})$, interpolation and Young’s inequalities, we estimate the term $\| A^\sigma \int_0^t R(s)c(u(s))\, ds \|_{L^2}$ as

\[ \| A^\sigma \int_0^t R(s)c(u(s))\, ds \|_{L^2} \]

\[ \leq C \left( (1 + \gamma) \int_0^t s^{-(\sigma + \frac{3}{4})} \| u(s) \|_{L^2}^{\frac{\delta + 1}{\delta}} \, ds + \gamma \int_0^t s^{-\sigma} \| u(s) \|_{L^2} \, ds \right) \]

\[ + \int_0^t s^{-(\sigma + \frac{3}{4})} \| u(s) \|_{L^2(\delta + 1)}^{2(\delta + 1)} \, ds \]

\[ \leq C \left( \int_0^t s^{-(\sigma + \frac{3}{4})} \| u(s) \|_{L^2(\delta + 1)}^{\frac{\delta + 1}{\delta}} \, ds + \int_0^t s^{-(\sigma + \frac{3}{4})} \| u(s) \|_{L^2(\delta + 1)} \, ds \right) \]

\[ \leq C \left( t^{-(\sigma + \frac{3}{4})} + \int_0^t \| u(s) \|_{L^2(\delta + 1)}^{2(\delta + 1)} \, ds + (t + t^{-(\sigma + \frac{3}{4})}) \sup_{s \in [0, t]} \| u(s) \|_{L^2(\delta + 1)}^{2(\delta + 1)} \right). \]

Let us consider the final term $\| A^\sigma \int_0^t R(s)z_\kappa(s)\, ds \|_{L^2}$. We estimate using semigroup property (see (2.7)) and Young’s inequality as

\[ \| A^\sigma \int_0^t R(s)z_\kappa(s)\, ds \|_{L^2} \]

\[ \leq C \left[ \int_0^t \left( s^{-2\sigma} + \| z_\kappa(s) \|_{L^2}^2 \right) \, ds \right] \]

\[ \leq C \left( t^{-2\sigma + 1} + t \sup_{s \in [0, t]} \| z_\kappa(s) \|_{L^2}^2 \right). \]

In similar lines, we estimate the other terms in the right-hand side of (3.16) as

\[ \| A^\sigma R(1)v(t) \|_{L^2} \leq K_1 \| v(t) \|_{L^2}, \]

\[ \alpha \left\| A^\sigma \int_t^{t+1} R(t + 1 - s)B(v(s) + z_\kappa(s))\, ds \right\|_{L^2} \]

\[ \leq C \left( \sup_{s \in [0, 1]} \| (v + z_\kappa)(t + s) \|_{L^2}^2 + \int_t^{t+1} \| (v + z_\kappa)(s) \|_{L^2(\delta + 1)}^{2(\delta + 1)} \, ds \right), \]

\[ \beta \left\| A^\sigma \int_t^{t+1} R(t + 1 - s)c(v(s) + z_\kappa(s))\, ds \right\|_{L^2} \]
\[
\leq C \left(1 + \int_t^{t+1} \| (v + z_\kappa) (s) \|_{L^2(\delta+1)}^{2(\delta+1)} ds + \sup_{s \in [0,1]} \| (v + z_\kappa) (t + s) \|_{L^2}^2 \right),
\]
\[
\kappa \left\| A^\sigma \int_t^{t+1} R(t + 1 - s) z_\kappa (s) ds \right\|_{L^2} \leq C \left(1 + \sup_{s \in [0,1]} \| z_\kappa (t + s) \|_{L^2}^2 \right),
\]
and the right-hand sides of the above estimates are finite provided \( \sigma < \frac{2 - \delta}{4\delta} \) and \( \delta \in [1, 2) \). Using the above estimates in (3.16), we obtain
\[
\| A^\sigma v(t + 1) \|_{L^2} \leq K_1 \| v(t) \|_{L^2} + C \sup_{s \in [0,1]} \| v(t + s) \|_{L^2}^2 + K_3 \sup_{s \in [0,1]} \| z_\kappa (t + s) \|_{L^2}^2
\]
\[
+ C \int_t^{t+1} \| (v + z_\kappa) (s) \|_{L^2(\delta+1)}^{2(\delta+1)} ds + C,
\]
(3.17)
for \( \sigma < \frac{2 - \delta}{4\delta} \) and \( \delta \in [1, 2) \). Integrating (3.13) from \( t \) to \( t + s \) and taking supremum over \( s \in [0, 1] \), we find
\[
\sup_{s \in [0,1]} \| v(t + s) \|_{L^2}^2 \leq \| v(t) \|_{L^2}^2 + C(\beta, \gamma, \kappa) \int_t^{t+1} \| z_\kappa (r) \|_{L^2}^2 dr
\]
\[
+ C(v, \alpha, \beta, \delta) \int_t^{t+1} \| z_\kappa (r) \|_{L^2(\delta+1)}^{2(\delta+1)} dr + C(\beta, \gamma, \delta).
\]
Again integrating (3.13) from \( t \) to \( t + 1 \), we obtain
\[
\int_t^{t+1} \| (v + z_\kappa) (r) \|_{L^2(\delta+1)}^{2(\delta+1)} dr \leq \frac{4}{\beta} \left\{ \| v(t) \|_{L^2}^2 + C(\beta, \gamma, \kappa) \int_t^{t+1} \| z_\kappa (r) \|_{L^2}^2 dr
\]
\[
+ C(v, \alpha, \beta, \delta) \int_t^{t+1} \| z_\kappa (r) \|_{L^2(\delta+1)}^{2(\delta+1)} dr + C(\beta, \gamma, \delta) \right\}.
\]
Using the above two estimates in (3.17), we get
\[
\| A^\sigma v(t + 1) \|_{L^2} \leq K_1 \| v(t) \|_{L^2} + K_2 \| v(t) \|_{L^2}^2 + K_3 \sup_{s \in [0,1]} \| z_\kappa (t + s) \|_{L^2}^2
\]
\[
+ K_4 \int_t^{t+1} \| z_\kappa (r) \|_{L^2}^2 dr + K_5 \int_t^{t+1} \| z_\kappa (r) \|_{L^2(\delta+1)}^{2(\delta+1)} dr + K_6,
\]
(3.18)
where \( K_i > 0 \), for \( i \in \{1, \ldots, 6\} \) are constants. Using the fact \( \mathbb{P} \{ K_6 \geq \frac{K}{6} \} = 0 \), since \( K \) has been chosen sufficiently large so the left-hand cannot exceed to the right-hand
value. Finally, we have
\[
\frac{1}{T} \int_0^T \mathbb{P} \left\{ \| A^T v(t + 1) \|_{L^2} \geq K \right\} dt \\
\leq \frac{1}{T} \int_0^T \mathbb{P} \left\{ K_1 \| v(t) \|_{L^2} \geq \frac{K}{6} \right\} dt + \frac{1}{T} \int_0^T \mathbb{P} \left\{ K_2 \| v(t) \|_{L^2}^2 \geq \frac{K}{6} \right\} dt \\
+ \frac{1}{T} \int_0^T \mathbb{P} \left\{ K_3 \sup_{s \in [0, 1]} \| z_{\kappa}(t + s) \|_{L^2}^2 \geq \frac{K}{6} \right\} dt \\
+ \frac{1}{T} \int_0^T \mathbb{P} \left\{ K_4 \int_t^{t+1} \| z_{\kappa}(r) \|_{L^2}^2 dr \geq \frac{K}{6} \right\} dt \\
+ \frac{1}{T} \int_0^T \mathbb{P} \left\{ K_5 \int_t^{t+1} \| z_{\kappa}(r) \|_{L^2^{(\delta+1)}}^2 dr \geq \frac{K}{6} \right\} dt.
\]

The first two terms in the right-hand side of above inequality can be made arbitrarily small uniform in time $T$ with the help of Lemma 3.2 for sufficiently large $K$. For the remaining terms of the above inequality, we use Markov’s inequality and the fact that
\[
\sup_{t \geq 0} \mathbb{E} \left\{ \| z_{\kappa}(t) \|_{L^2}^2 + \| z_{\kappa}(t) \|_{L^2^{(\delta+1)}}^2 \right\} < \infty,
\]
and hence (3.15) follows.

\section{4 Irreducibility and Strong Feller}

In this section, we discuss two properties of the Markov semigroup $P_t, t \geq 0$ associated with the solution of the SGBH equation (2.8), namely irreducibility and strong Feller. For any Borel subset $E$ of $L^2(\mathcal{O})$, a Markov semigroup $P_t, t \geq 0$ is

- **irreducible** if $P(t, x, B) > 0$, for all $t > 0$, $x \in E$ and any non-empty open subset $B \subset E$,
- **strong Feller** if $P_t$ can be extended to the space $\mathcal{B}_b(E)$ for any $t > 0$, that is, $P_t \varphi$ is continuous and bounded in $E$ for all Borel bounded functions $\varphi$ in $E$.

The above properties are essentially related to the uniqueness of invariant measures.

\subsection{4.1 Irreducibility}

Let us first show that the Markov semigroup $P_t, t \geq 0$ is irreducible by using the ideas in [9, 34, 43], etc.

**Proposition 4.1** The transition semigroup $P_t, t \geq 0$ on the space $\mathcal{B}_b(L^2(\mathcal{O}))$ corresponding to the solution of SGBH equation (2.8) is irreducible for $\delta \in [1, 2]$.

**Proof** Let us prove the irreducibility of the transition semigroup $P_t, t \geq 0$ in the following steps:
**Step 1: Exact controllability result.** Let us fix $T > 0$, $a \in L^2(\mathcal{O})$ and $b \in H^1_0(\mathcal{O})$. We first show that there exist $u \in L^2(0, T; L^2(\mathcal{O}))$ such that for the solution $x(t)$, $t \in [0, T]$ of the control problem

$$
\begin{align*}
\partial_t x(t, \xi) &= -v Ax(t, \xi) - \alpha B(x(t, \xi)) + \beta c(x(t, \xi)) + u(t, \xi), \ t > 0, \ \xi \in \mathcal{O}, \\
x(t, 0) &= x(t, 1) = 0, \ t > 0, \\
x(0, \xi) &= a(\xi), \ \xi \in [0, 1],
\end{align*}
$$

one has

$$
x(T, \xi) = b(\xi), \ \xi \in [0, 1].
$$

In addition, assume that $a \in H^1_0(\mathcal{O})$. Then, there exists $v \in L^2(0, T; L^2(\mathcal{O}))$, such that for the solution $z(t)$, $t \in [0, T]$, of the linear problem associated with (4.1) (see Proposition 14.4.3, [9]),

$$
\begin{align*}
\partial_t z(t, \xi) &= -v Az(t, \xi) + v(t, \xi), \ t > 0, \ \xi \in \mathcal{O}, \\
z(t, 0) &= z(t, 1) = 0, \ t > 0, \\
z(0, \xi) &= a(\xi), \ \xi \in [0, 1],
\end{align*}
$$

one has

$$
z(T, \xi) = b(\xi), \ \xi \in [0, 1].
$$

Since $a \in H^1_0(\mathcal{O})$, it is immediate that $z \in L^\infty(0, T; H^1_0(\mathcal{O})) \cap L^2(0, T; D(A))$ and one can easily verify that $z \in C([0, T]; H^1_0(\mathcal{O}))$. Now, we define a control

$$
v(t) = \begin{cases} 
0, & \text{for } 0 \leq t \leq t_0, \\
\frac{b - z(t_0)}{T - t_0} + vA z(t), & \text{for } t_0 < t \leq T,
\end{cases}
$$

where $0 < t_0 < T$. Since $z \in L^2(0, T; D(A))$, we obtain $v \in L^2(0, T; L^2(\mathcal{O}))$. For the above control, the solution of the problem (4.2) is given by

$$
z(t) = \begin{cases} 
e^{-vA t} a, & \text{for } 0 \leq t \leq t_0, \\
\frac{t - t_0}{T - t_0} b + \frac{T - t}{T - t_0} z(t_0), & \text{for } t_0 < t \leq T,
\end{cases}
$$

and it can be easily seen that $z(T) = b$. Thus, the linear problem (4.2) is exactly controllable.

Let us now define a control $u(\cdot)$ as

$$
u(t) = \frac{-\alpha}{\delta + 1} \partial_\xi z^{\delta+1}(t) - \beta (1 + \gamma) z^{\delta+1}(t) + \beta_2 z(t) + \beta_2 z^{\delta+1}(t) + v(t),
$$

$$
= \alpha B(z(t)) - \beta c(z(t)) + v(t), \ \text{for a.e. } (t, \xi) \in [0, T] \times [0, 1].
$$
By a direct substitution, one can prove that \( z(\cdot) \) is the solution of (4.1).

Now, we show that the control \( u(\cdot) \in L^2(0, T; L^2(\mathcal{O})) \). Taking the \( L^2 \)-norm of the control \( u \), we get
\[
\|u\|_{L^2}^2 \leq C \left( \frac{\alpha^2}{(\delta + 1)^2} \|\partial_\xi z\|_{L^2}^{\delta + 1} + \beta^2(1 + \gamma)^2 \|z\|_{L^2(\delta + 1)}^2 \right.
+ \beta^2 \gamma^2 \|z\|_{L^2}^2 + \beta^2 \|z\|_{L^2(2\delta + 1)}^2 + \|u\|_{L^2}^2 \bigg). 
\]
(4.3)

Using Hölder’s inequality, we estimate the term \( \frac{\alpha^2}{(\delta + 1)^2} \|\partial_\xi z\|_{L^2}^{\delta + 1} \) as
\[
\frac{\alpha^2}{(\delta + 1)^2} \|\partial_\xi z\|_{L^2}^{\delta + 1} = \alpha^2(\delta \partial_\xi z, z^\delta \partial_\xi z) \leq \alpha^2 \|z\|_{L^\infty} \|\partial_\xi z\|_{L^2}^2.
\]
Substituting it in (4.3) and integrating the resultant from 0 to \( T \), we obtain
\[
\int_0^T \|u(s)\|_{L^2}^2 \, ds
\leq C \left( \alpha^2 \sup_{s \in [0, T]} \|z(s)\|_{L^2}^\delta \int_0^T \|\partial_\xi z(s)\|_{L^2}^2 \, ds + \beta^2(1 + \gamma)^2 \int_0^T \|z(s)\|_{L^2(\delta + 1)}^2 \, ds \right.
\]
\[
+ \beta^2 \gamma^2 \int_0^T \|z(s)\|_{L^2}^2 \, ds + \int_0^T \|u(s)\|_{L^2}^2 \, ds \bigg) < \infty,
\]
so that \( u \in L^2(0, T; L^2(\mathcal{O})) \). Thus, the exact controllability of system (4.1) follows for \( a \in H^1_0(\mathcal{O}) \). Let us now consider \( a \in L^2(\mathcal{O}) \). Since \( H^1_0(\mathcal{O}) \) is densely embedded in \( L^2(\mathcal{O}) \), for any \( a \in L^2(\mathcal{O}) \), we can find a sequence \( a_n \in H^1_0(\mathcal{O}) \) such that \( \|a_n - a\|_{L^2} \to 0 \) as \( n \to \infty \). Let \( x_n(\cdot) \) be the unique solution of (4.1) corresponding to the initial data \( a_n \in H^1_0(\mathcal{O}) \). Then by the continuous dependence on the initial data result (cf. (2.32)), we have
\[
\|x_n(t) - x(t)\|_{L^2} \leq C(\|a\|_{L^2}, \alpha, \beta, \gamma, \delta, \nu, T, \|a - a_n\|_{L^2}) \to 0 \text{ as } n \to \infty. 
\]
(4.4)

Since \( a_n \in H^1_0(\mathcal{O}) \), there exists a control \( u_n \in L^2(0, T; L^2(\mathcal{O})) \) such that \( x_n(T) = b \). Let us now consider
\[
\|x(T) - b\|_{L^2} \leq \|x(T) - x_n(T)\|_{L^2} + \|x_n(T) - b\|_{L^2} \leq C\|a_n - a\|_{L^2}, \quad (4.5)
\]
by using (4.4). Taking \( n \to \infty \), one obtains the required result.

Taking the inner product with \( x(\cdot) \) to the first equation in (4.1), we find
\[
\frac{1}{2} \frac{d}{dt} \|x(t)\|_{L^2}^2 + v \|\partial_\xi x(t)\|_{L^2}^2 + \beta \gamma \|x(t)\|_{L^2}^2 + \beta \|x(t)\|_{L^2(\delta + 1)}^2 = \beta(1 + \gamma)(x^{\delta + 1}(t), x(t)) + (x(t), v(t)).
\]
It follows that if \( v = 0 \), then for a.e. \( t \in [0, T] \), we have

\[
\frac{1}{2} \frac{d}{dt} \|x(t)\|_{L^2}^2 + v \|\partial_t x(t)\|_{L^2}^2 + \beta \frac{d}{dt} \|x(t)\|_{L^{2(\delta+1)}}^{2(\delta+1)} \leq \frac{2\beta(\delta+1)}{\delta} (1 + \eta) \frac{2(\delta+1)}{\delta+1} \left( \delta + 2 \right)^{\frac{\delta+2}{\delta}}.
\]

by an application of Hölder’s and Young’s inequalities. From the above estimate, it is immediate that for a.e. \( t \in [0, T] \), we get \( x(t) \in H_0^1(\mathcal{O}) \). Thus, it is enough to define \( v(t) = 0 \) in \([0, \tilde{t}]\), where \( \tilde{t} \in (0, T) \) is a moment such that \( x(\tilde{t}) \in H_0^1(\mathcal{O}) \) and for the remaining part of the interval \([\tilde{t}, T]\), we use the first part of the proof.

**Step 2: Irreducibility.** In order to prove our result, we need to estimate the \( L^2 \)-distance between the solution \( u(\cdot) \) to equation (2.8) with \( x(0) = a \in L^2(\mathcal{O}) \), and the function \( x(\cdot) \). From Theorems 2.6 and 2.7, for any \( a \in L^2(\mathcal{O}) \), we infer that \( u(t, a) \in H_0^1(\mathcal{O}) \), for a.e. \( t \in [0, T] \), \( \mathbb{P} \)-a.s. Since \( u(\cdot) \) is a Markov process in \( L^2(\mathcal{O}) \), for any \( b \in L^2(\mathcal{O}) \), \( T > 0 \), \( \eta > 0 \), (cf. Theorem 2.3, [43])

\[
\mathbb{P} \left\{ \|u(T, a) - b\|_{L^2} < \eta \right\} = \int_{H_0^1} \mathbb{P} \left\{ \|u(T, a) - b\|_{L^2} < \eta | u(t, a) = v \right\} \mathbb{P} \{ u(t, a) = dv \} \]
\[
= \int_{H_0^1} \mathbb{P} \left\{ \|u(T - t, a) - b\|_{L^2} < \eta \right\} \mathbb{P} \{ u(t, a) = dv \}, \quad (4.6)
\]

for a.e. \( t \in [0, T] \). In order to prove that \( \mathbb{P} \left\{ \|u(T, a) - b\|_{L^2} < \eta \right\} > 0 \), it is sufficient to show that \( \mathbb{P} \left\{ \|u(T, a) - b\|_{L^2} < \eta \right\} > 0 \), for any \( T > 0 \) and \( a \in H_0^1(\mathcal{O}) \) (see the controllability results also).

Let us rewrite the control problem (4.1) as

\[
\begin{aligned}
&d\mathbf{3}(t) + vA\mathbf{3}(t)dt = udt, \quad \mathbf{3}(0) = 0, \\
&d\mathbf{y}(t) + vAy(t)dt = \{-\alpha B(y(t) + \mathbf{3}(t)) + \beta c(y(t) + \mathbf{3}(t))\}dt, \quad y(0) = a,
\end{aligned}
\]  

and the stochastic problem (2.8) as

\[
\begin{aligned}
&d\mathbf{z}(t) + vAz(t)dt = GdW, \quad \mathbf{z}(0) = 0, \\
&dv(t) + vAv(t)dt = \{-\alpha B(v(t) + \mathbf{z}(t)) + \beta c(v(t) + \mathbf{z}(t))\}dt, \quad v(0) = a,
\end{aligned}
\]  

where we have set \( x(t) = y(t) + \mathbf{3}(t) \) and \( u(t) = v(t) + \mathbf{z}(t) \). Now, we subtract second equation of the system (4.7) from second equation of system (4.8) to obtain

\[
\frac{d}{dt}(v(t) - y(t)) + vA(v(t) - y(t)) = -\alpha(B(v(t) + \mathbf{z}(t)) - B(y(t) + \mathbf{3}(t)))
\]
\[
+ \beta(c(v(t) + \mathbf{z}(t)) - c(y(t) + \mathbf{3}(t))),(\text{4.9})
\]

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for a.e., $0 \leq t \leq T$ in $H^{-1}(\mathcal{O})$. Taking the inner product with $v(t) - y(t)$, we get

$$
\|v(t) - y(t)\|_{L^2}^2 + 2v \int_0^t \|\partial_\xi (v(s) - y(s))\|_{L^2}^2 ds
= -2\alpha \int_0^t (B(v(s) + z(s)) - B(y(s) + z(s)), v(s) - y(s)) ds
+ 2\beta \int_0^t (c(v(s) + z(s)) - c(y(s) + z(s)), v(s) - y(s)) ds.
$$

(4.9)

In a similar way, as we did in (2.25), one can estimate the penultimate term from the right-hand side of the expression (4.10) as

$$
-\alpha \int_0^t (B(v(s) + z(s)) - B(y(s) + z(s)), v(s) - y(s)) ds
\leq \frac{2v}{3} \int_0^t \|\partial_\xi (v(s) - y(s))\|_{L^2}^2 ds + \frac{\delta + 2}{4(\delta + 1)} \left( \frac{3(3\delta + 2)}{4v(\delta + 1)} \right) \frac{4\delta + 2}{4\delta + 2} \int_0^t \|v(s) + z(s)\|_{L^2(\delta + 1)}^2 ds
+ \|v(s) + z(s)\|_{L^2(\delta + 1)}^2 \|v(s) - y(s)\|_{L^2}^2 ds + \frac{2^2(\delta - 1)}{4} \int_0^t \|v(s) + z(s)\|_{L^2(\delta + 1)}^2 ds.
$$

(4.10)

Let us take the second term in the right-hand side of (4.9) and rewrite it as

$$
\beta \int_0^t (c(v(s) + z(s)) - c(y(s) + z(s)), v(s) - y(s)) ds
= \beta \int_0^t ((1 + \gamma)(v(s) + z(s))^{\delta + 1} - \gamma(v(s) + z(s)) - (v(s) + z(s))^{2\delta + 1}
- (1 + \gamma)(y(s) + z(s))^{\delta + 1} - \gamma(y(s) + z(s)) - (y(s) + z(s))^{2\delta + 1}, v(s) - y(s)) ds.
$$

(4.11)

To estimate the first term in the right-hand side of (2.26), one can use the similar calculation as in (2.26) to arrive at

$$
\beta(1 + \gamma) \int_0^t ((v(s) + z(s))^{\delta + 1} - (y(s) + z(s))^{\delta + 1}, v(s) - y(s)) ds
\leq \frac{\beta}{4} \int_0^t (\|v(s) + z(s)\|_{L^2}^2 (v(s) + z(s) - y(s) - z(s))\|_{L^2}^2
+ \|y(s) + z(s)\|_{L^2}^2 (v(s) + z(s) - y(s) - z(s))\|_{L^2}^2) ds + 2^2\beta(1 + \gamma)^2(\delta + 1)^2
\times \int_0^t (\|v(s) + z(s) - y(s) - z(s))\|_{L^2}^2 + \|z(s) - z(s))\|_{L^2}^2) ds.
$$

(4.12)
Let us consider the second term of the right-hand side of (4.11). Using Cauchy–Schwarz and Young’s inequalities, we obtain
\[
- \beta \gamma \int_0^t (v(s) + z(s) - y(s) - \tilde{z}(s), v(s) - y(s))ds \\
\leq \frac{3\beta \gamma}{2} \int_0^t \|v(s) - y(s)\|^2_{L^2}ds + \frac{2}{\beta \gamma} \int_0^t \|z(s) - \tilde{z}(s)\|^2_{L^2}ds.
\]
(4.13)

Using Taylor’s formula, (2.28), Hölder’s and Young’s inequalities, we estimate the term \(-\beta \int_0^t ((v(s) + z(s))^{2\delta + 1} - (y(s) + \tilde{z}(s))^{2\delta + 1}, v(s) - y(s))ds\) as

\[
- \beta \int_0^t (v(s) + z(s))^{2\delta + 1} - (y(s) + \tilde{z}(s))^{2\delta + 1}, v(s) + z(s) - y(s) - \tilde{z}(s) + \tilde{z}(s))ds \\
\leq - \frac{\beta}{2} \int_0^t \|v(s) + z(s)\|^{\delta} (v(s) + z(s) - y(s) - \tilde{z}(s))^{\delta}ds \\
- \frac{\beta}{2} \int_0^t \|y(s) + \tilde{z}(s)\|^{\delta} (v(s) + z(s) - y(s) - \tilde{z}(s))^{\delta}ds \\
+ 2^{2\delta - 1} \beta (2\delta + 1) \int_0^t \|v(s) + z(s) - y(s) - \tilde{z}(s)\|_{L^{2(\delta + 1)}} \|v(s) + z(s)\|_{L^{2(\delta + 1)}}^{2\delta} \\
+ \|y(s) + \tilde{z}(s)\|_{L^{2(\delta + 1)}}^{2\delta} \|z(s) - \tilde{z}(s)\|_{L^2}ds \\
\leq - \frac{\beta}{2} \int_0^t \|v(s) + z(s)\|^{\delta} (v(s) + z(s) - y(s) - \tilde{z}(s))^{\delta}ds \\
- \frac{\beta}{2} \int_0^t \|y(s) + \tilde{z}(s)\|^{\delta} (v(s) + z(s) - y(s) - \tilde{z}(s))^{\delta}ds \\
+ \frac{\beta}{2} \int_0^t \|v(s) + z(s) - y(s) - \tilde{z}(s)\|^{2(\delta + 1)}_{L^{2(\delta + 1)}} ds + \frac{2(\delta + 1)}{2\delta + 1} \left(2^{2\delta - 1} \beta (2\delta + 1)\right)_{L^{2(\delta + 1)}}^{2(\delta + 1)} \\
\times \frac{1}{(\beta(\delta + 1))^{2\delta + 1}} \int_0^t \left(\|v(s) + z(s)\|_{L^{2(\delta + 1)}}^{4\delta(\delta + 1)} + \|y(s) + \tilde{z}(s)\|_{L^{2(\delta + 1)}}^{4\delta(\delta + 1)}\right) \|z(s) - \tilde{z}(s)\|_{L^{2(\delta + 1)}}^{2(\delta + 1)}ds \\
\leq - \frac{\beta}{2} \int_0^t \|v(s) + z(s)\|^{\delta} (v(s) + z(s) - y(s) - \tilde{z}(s))^{\delta}ds \\
- \frac{\beta}{2} \int_0^t \|y(s) + \tilde{z}(s)\|^{\delta} (v(s) + z(s) - y(s) - \tilde{z}(s))^{\delta}ds \\
+ \frac{\beta}{2} \int_0^t \|v(s) + z(s) - y(s) - \tilde{z}(s)\|^{2(\delta + 1)}_{L^{2(\delta + 1)}} ds + \frac{\beta}{2} \int_0^t \|z(s) - \tilde{z}(s)\|^{2(\delta + 1)}_{L^{2(\delta + 1)}} ds \\
+ C(\beta, \delta) \int_0^t \left(\|v(s) + z(s)\|_{L^{2(\delta + 1)}}^{2(\delta + 1)} + \|y(s) + \tilde{z}(s)\|_{L^{2(\delta + 1)}}^{2(\delta + 1)}\right)ds.
\]
(4.14)

Using the estimates (4.10)–(4.14) in (4.9), we get

\[
\|v(t) - y(t)\|^2_{L^2} + \frac{2v}{3} \int_0^t \|\partial_x (v(s) - y(s))\|^2_{L^2}ds + \frac{\beta}{2} \int_0^t (\|v(s) + z(s)\|^{\delta} \\
\times (v(s) + z(s) - y(s) - \tilde{z}(s))\|^2_{L^2} + \|y(s) + \tilde{z}(s)\|^{\delta} (v(s) + z(s) - y(s) - \tilde{z}(s))\|^2_{L^2})ds.
\]
\[
\begin{align*}
&\leq \int_0^t \{ C(v, \delta) \|v(s) + z(s)\|_{L^2(\delta+1)}^2 + \|y(s) + \zeta(s)\|_{L^2(\delta+1)}^2 \} + C(\beta, \gamma, \delta) \|v(s) - y(s)\|_{L^2}^2 \, ds \\
&+ \int_0^t \{ C(\beta, \gamma, \delta) \|z(s) - \zeta(s)\|_{L^2}^2 + C(\beta) \|z(s) - \zeta(s)\|_{L^2(\delta+1)}^2 \} \, ds \\
&+ C(v, \alpha, \delta) \int_0^t (\|v(s) + z(s)\|_{L^2(\delta+1)}^2 + \|y(s) + \zeta(s)\|_{L^2(\delta+1)}^2) \|z(s) - \zeta(s)\|_{L^2(\delta+1)}^2 \, ds \\
&+ C(\beta, \delta) \int_0^t \|v(s) + z(s)\|_{L^2(\delta+1)}^2 + \|y(s) + \zeta(s)\|_{L^2(\delta+1)}^2 \, ds.
\end{align*}
\]

(4.15)

Assuming
\[
\sup_{s \in [0, T]} \|z(s)\|_{L^2(\delta+1)} \leq \Upsilon,
\]
and applying of Gronwall’s inequality in (4.15), we arrive at
\[
\|v(t) - y(t)\|_{L^2}^2 \leq C(v, \alpha, \beta, \gamma, \Upsilon, \delta, T) \left\{ \sup_{t \in [0, T]} \|z(t) - \zeta(t)\|_{L^2(\delta+1)}^2 + \sup_{s \in [0, T]} \|z(s) - \zeta(s)\|_{L^2(\delta+1)}^2 \right\},
\]

(4.16)

provided \(2\delta^2 \leq 2\delta + 4\), that is, \(\delta \in [1, 2]\).

Since \(A\) is an analytic semigroup satisfying (2.7), from Theorem 5.25, [8], for \(0 < \vartheta < \frac{\varepsilon}{4} + \frac{\delta - 1}{4(\delta+1)} < \frac{\delta}{2(\delta+1)}\), we infer that \(z \in C([0, T]; W^{0,2(\delta+1)}(\mathcal{O}))\), \(\mathbb{P}\)-a.s.

The embedding \(W^{0,2(\delta+1)}(\mathcal{O}) \subset H^{\frac{\delta}{2(\delta+1)}+\vartheta}(\mathcal{O})\) (Sobolev’s embedding) implies that \(z \in C([0, T]; H^{\frac{\delta}{2(\delta+1)}+\vartheta}(\mathcal{O}))\), \(\mathbb{P}\)-a.s., for \(0 < \vartheta < \frac{\varepsilon}{4} + \frac{\delta - 1}{4(\delta+1)}\). Since \(a \in H^1_0(\mathcal{O})\), it is immediate that \(a \in H^{\frac{\delta}{2(\delta+1)}+\vartheta}(\mathcal{O})\). Note that the support of the distribution of the processes \(z \in C([0, T]; H^{\frac{\delta}{2(\delta+1)}+\vartheta}(\mathcal{O}))\) is the closure of the set of functions \(\int_0^t R(t-s)w(s) \, ds, \ t \in [0, T] , \ w \in L^2(0, T; H^{\frac{\delta}{2(\delta+1)}+\vartheta}(\mathcal{O}))\). Since \(H^{\frac{\delta}{2(\delta+1)}+\vartheta}(\mathcal{O}) \subset L^{2(\delta+1)}(\mathcal{O})\), for arbitrary \(\eta > 0\), we have

\[
0 < \mathbb{P} \left\{ \sup_{t \in [0, T]} \|z(t) - \zeta(t)\|_{H^{\frac{\delta}{2(\delta+1)}+\vartheta}} < \eta \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} \|z(t) - \zeta(t)\|_{L^{2(\delta+1)}} < \eta \right\}.
\]

Let us fix
\[
\gamma = \eta + \sup_{t \in [0, T]} \|\zeta(t)\|_{L^{2(\delta+1)}}.
\]
Then, we have

\[
0 < \mathbb{P} \left\{ \sup_{t \in [0,T]} \| z(t) - \tilde{z}(t) \|_{L^2(\delta+1)} < \eta \right\} \\
\leq \mathbb{P} \left\{ \sup_{t \in [0,T]} \| z(t) - \tilde{z}(t) \|_{L^2(\delta+1)} < \eta \text{ and } \sup_{t \in [0,T]} \| z(t) \|_{L^2(\delta+1)} \leq \gamma \right\}.
\]

Let us now consider

\[
\mathbb{P} \left\{ \| u(T) - b \|_{L^2}^2 < \eta \right\} \\
= \mathbb{P} \left\{ \| v(T) + z(T) - y(T) + y(T) - \tilde{z}(t) + \tilde{z}(t) - b \|_{L^2}^2 < \eta \right\} \\
= \mathbb{P} \left\{ \| v(T) - y(T) + z(T) - \tilde{z}(t) + x(T) - b \|_{L^2}^2 < \eta \right\} \\
\geq \mathbb{P} \left\{ \| v(T) - y(T) \|_{L^2}^2 < \frac{\eta}{4}, \| z(T) - \tilde{z}(t) \|_{L^2}^2 < \frac{\eta}{4}, \| x(T) - b \|_{L^2}^2 < \frac{\eta}{2} \right\} \\
= \mathbb{P} \left\{ \| v(T) - y(T) \|_{L^2}^2 < \frac{\eta}{4}, \| z(T) - \tilde{z}(t) \|_{L^2}^2 < \frac{\eta}{4} \right\} \\
\geq \mathbb{P} \left\{ \sum_{j \in D} \sup_{t \in [0,T]} \| z(t) - \tilde{z}(t) \|_{L^2(\delta+1)}^j < C(\nu, \alpha, \beta, \gamma, \gamma, \delta, T, \eta) \right\} \\
\geq \mathbb{P} \left\{ \sum_{j \in D} \sup_{t \in [0,T]} \| z(t) - \tilde{z}(t) \|_{L^2(\delta+1)}^j < C(\nu, \alpha, \beta, \gamma, \gamma, \delta, T, \eta) \right\} \\
\text{and } \sup_{t \in [0,T]} \| z(t) \|_{L^2(\delta+1)} \leq \gamma \right\} \\
> 0,
\]

where \( j \in D = \{2, 2(\delta + 1)\} \), and since \( \eta > 0 \) can be chosen arbitrary so that the required result holds. Hence, the transition semigroup \( P_t, t \geq 0, \) is irreducible.

\section*{4.2 Strong Feller Property}

Let \( u(\cdot, x) \) be the mild solution of (2.8), which has been established in [28]. We show that the corresponding transition semigroup \( P_t, t \geq 0, \) has the strong Feller property on the space \( L^2(O) \). To prove this property, we take a modified version of SGBH equation. For this purpose, we define a cut-off function and a mollifier as, for any \( R > 0 \)

\[
\Phi(r) = \begin{cases} 
1 & \text{for } r \in [0, R], \\
0 & \text{for } r \in [R + 1, \infty),
\end{cases}
\]  

\section*{Acknowledgments}

which is a $C^1$ function defined on $[0, \infty)$. Now, we define a mollifier

$$M_R(x) = x \Phi(\|x\|_{L^2}), \ x \in L^2(\mathcal{O}). \quad (4.19)$$

Also note that for any $x \in L^2(\mathcal{O}), M_R \in C^1_b(\mathcal{L}^2(\mathcal{O}))$ and

$$D_x M_R(x) = \Phi(\|x\|_{L^2}) I + \frac{\Phi'(\|x\|_{L^2})}{\|x\|_{L^2}} (x \otimes x), \ x \in L^2(\mathcal{O}).$$

**Proposition 4.2** If $M_R$ is a mollifier defined by (4.19), then the modified SGBH equation

$$\begin{cases}
\frac{du}{dt} = -\nu Au - \alpha B(M_R(u)) + \beta c(M_R(u))dt + GdW(t), \\
u(0) = x,
\end{cases} \quad (4.20)$$

has a unique mild solution on the time interval $[0, T]$. Moreover, the transition semigroup $P^R_t, \ t \geq 0$ has the strong Feller property.

**Proof** We prove Proposition 4.2 in two steps. In step 1, we put a remark on the existence and uniqueness of the mild solution to the system (4.20). In step 2, we discuss the proof of strong Feller property of the corresponding transition semigroup.

**Step 1: The existence and uniqueness of mild solution:** The proof of existence and uniqueness of solutions to (4.20) is similar to that of Theorem 2.6, since one can derive an a-priori bound of the form (2.17) (see [28] also).

**Step 2: Strong Feller property:** For $0 < R < \infty$, we denote the directional derivative by $U^R(t)$ at $x$ in the direction of $h$ of the mapping $x \mapsto u^R(t, x)$ (where $u^R(\cdot)$ is the solution of the system (4.21)), that is,

$$U^R(t) = \left[D_x u^R(t, x)\right] \cdot h,$$

for given $x, h \in L^2(\mathcal{O})$. Note that it is also the derivative of the mapping $x \mapsto u^R(t, x) = v^R(t, x) + z(t)$. Thus, $U^R$ is the solution of the first variation equation associated with the system (4.20) and is given by

$$\begin{cases}
\frac{dU^R}{dt} = -\nu AU^R - \alpha \partial_\xi (M_R^\delta(u) M'_R(u) U^R) + \beta (1 + \gamma) (1 + \delta) M_R^\delta(u) M'_R(u) U^R \\
- \beta \gamma M'_R(u) U^R - \beta (2 + \delta) M_R^\delta(u) M'_R(u) U^R, \\
U^R(0) = h.
\end{cases} \quad (4.21)$$
We consider the mild solution of (4.21), and taking the $L^2$-norm to find

$$
\| U^R(t) \|_{L^2} \leq \| R(t) h \|_{L^2} + \alpha \int_0^t \| R(t-s) \partial \xi (M_R^\delta(u)M'_R(u)U^R) \|_{L^2} ds
$$

$$
+ \beta(1 + \gamma)(1 + \delta) \int_0^t \| R(t-s)M_R^\delta(u)M'_R(u)U^R \|_{L^2} ds
$$

$$
+ \beta \gamma \int_0^t \| R(t-s)M'_R(u)U^R \|_{L^2} ds
$$

$$
+ \beta(2\delta + 1) \int_0^t \| R(t-s)M'^\delta_R(u)M'_R(u)U^R \|_{L^2} ds.
$$

(4.22)

Applying the semigroup property (see (2.5) and (2.6)) on the terms of the right-hand side of (4.22), we get

$$
\| R(t-s) \partial \xi (M_R^\delta(u)M'_R(u)U^R) \|_{L^2} \leq C(t-s)^{-\frac{3}{2}} \| M_R^\delta(u)M'_R(u)U^R \|_{L^1},
$$

$$
\| R(t-s)M_R^\delta(u)M'_R(u)U^R \|_{L^2} \leq C(t-s)^{-\frac{1}{2}} \| M_R^\delta(u)M'_R(u)U^R \|_{L^1},
$$

$$
\| R(t-s)M'_R(u)U^R \|_{L^2} \leq C \| M'_R(u)U^R \|_{L^1},
$$

$$
\| R(t-s)M'^\delta_R(u)M'_R(u)U^R \|_{L^2} \leq C(t-s)^{-\frac{1}{2}} \| M'^\delta_R(u)M'_R(u)U^R \|_{L^1}.
$$

Using the above estimates in (4.22), we obtain

$$
\| U^R(t) \|_{L^2} \leq \| h \|_{L^2} + \alpha C \int_0^t (t-s)^{-\frac{3}{2}} \| M_R^\delta(u)M'_R(u)U^R \|_{L^1} ds
$$

$$
+ C \beta(1 + \gamma)(1 + \delta) \int_0^t (t-s)^{-\frac{1}{2}} \| M_R^\delta(u)M'_R(u)U^R \|_{L^1} ds
$$

$$
+ C \beta \gamma \int_0^t \| M'_R(u)U^R \|_{L^2} ds + \beta(2\delta + 1)
$$

$$
\int_0^t (t-s)^{-\frac{1}{2}} \| M'^\delta_R(u)M'_R(u)U^R \|_{L^1} ds.
$$

(4.23)

Now, using Hölder’s inequality, for $x, z \in L^2(\mathcal{O})$, we have

$$
\left\{
\begin{array}{l}
\| M'_R(x)z \|_{L^2} \leq C \| z \|_{L^2}, \\
\| M_R^\delta(x)M'_R(x)z \|_{L^1} \leq \| M_R^\delta(x) \|_{L^2} \| M'_R(x)z \|_{L^2} \leq C \| z \|_{L^2}, \\
\| M'^\delta_R(x)M'_R(x)z \|_{L^1} \leq \| M'^\delta_R(x) \|_{L^2} \| M'_R(x)z \|_{L^2} \leq C \| z \|_{L^2},
\end{array}
\right.
$$

(4.24)

where the constant $C$ depends on the choice of the cut-off function $\Phi_R$. Thus, one can conclude that the solution of (4.21) exists and belongs to $C([0, T]; L^2(\mathcal{O}))$, provided $T$ is sufficiently small. Substituting the bounds (4.24) in (4.23) and an application of
Gronwall’s lemma gives

$$\sup_{t \in [0, T]} \| U^R (t, x) \|_{L^2} \leq C_T \| h \|_{L^2}, \text{ for } h, x \in L^2(\mathcal{O}),$$

where $C_T$ is a non-random constant. From Bismut–Elworthy formula (Lemma 7.1.3, [9]) and Theorem 7.1.1, [9], we obtain that the strong Feller property holds for a short time and then with the help of semigroup property, we can extend that interval to $[0, +\infty)$. \qed

**Proposition 4.3** The transition semigroup $P_t, \ t \geq 0$ associated with the solution $u(\cdot)$ of (2.8) has the strong Feller property.

**Proof** For $0 < R < \infty$, let $P^R_t, \ t \geq 0$, be the associated semigroup to the solution $u^R(t, x)$ of the system (4.20). Let us define

$$\tau^R_x = \inf \{ t \geq 0 : \| u^R(t, x) \|_{L^2} > R \}.$$

It is clear from the definition of cut-off function (4.18) and (4.19) that

$$u(t, x) = u^R(t, x), \text{ for all } t \leq \tau^R_x, x \in L^2(\mathcal{O}).$$

Let $\psi$ be any arbitrary function in $B_b(L^2(\mathcal{O}))$, then

$$|P^R_t \psi(x) - P_t \psi(x)| = |E[\psi(u^R(t, x))] - E[\psi(u(t, x))]|$$

$$= |E[\psi(u^R(t, x)) - \psi(u(t, x))] 1_{\{\tau^R_x \leq t\}}|$$

$$\leq 2 \sup_{z \in L^2(\mathcal{O})} |\psi(z)| Pr\{\tau^R_x \leq t\}.$$

We know that the functions $P^R_t \psi$ are continuous for all $R > 0$ and $t > 0$. Therefore, it is enough to prove that for any $M > 0$ and $t > 0$,

$$\lim_{R \to \infty} \sup_{\| x \|_{L^2} \leq M} Pr\{\tau^R_x \leq t\} = 0. \quad (4.25)$$

We have already set in Sect. 2 that $u(t) = v(t) + z(t), \ t \geq 0$, and a similar formulation corresponding to the system (4.20) gives $u^R(t) = v^R(t) + z(t), \ t \geq 0$ and

$$\sup_{0 \leq s \leq t} \| u^R(s, x) \|_{L^2}^2 \leq 2 \sup_{0 \leq s \leq t} \| v^R(s, x) \|_{L^2}^2 + 2 \sup_{0 \leq s \leq t} \| z(s, x) \|_{L^2}^2.$$

Note that estimate (2.17) is valid for the process $u^R(\cdot)$ also. Thus, if $\| x \|_{L^2} \leq M$, then we get

$$\| u^R(s, x) \|_{L^2}^2 \leq C(\alpha, \beta, \gamma, \delta, \nu) \left( M^2 + \int_0^t \| z(s) \|_{L^{2(\delta+1)}}^2 (\delta+1) ds \right) + 2 \sup_{0 \leq s \leq t} \| z(s, x) \|_{L^2}^2.$$
Since the right-hand side of the above inequality is finite and independent of \( x \), the equality (4.25) holds.

**Theorem 4.4** There exists a unique invariant measure \( \mu \) for the transition semigroup \( P_t, \ t \geq 0 \), corresponding to solutions of (2.8) (\( \delta \in [1, 2] \) under assumption (1.6) and \( \delta \in [1, 2] \) under assumption (1.7)). Moreover, \( \mu \) is ergodic and strongly mixing.

**Proof** From Theorems 3.1 and 3.3, we infer the existence of an invariant measure \( \mu \) for the transition semigroup \( P_t, \ t \geq 0 \), corresponding to solutions of (2.8). Since the semigroup is strong Feller (Proposition 4.3) and irreducible (Proposition 4.1), the uniqueness of invariant measures follows by Doob’s theorem (Theorem 4.2.1, [9]). Since \( \mu \) is a unique invariant measure, ergodicity follows from Theorem 3.2.6, [9] and strongly mixing is an immediate consequence of Theorem 4.2.1, [9].

**5 Large Deviation Principle**

In this section, we prove the LDP w.r.t. the topology \( \tau \) and Donsker–Varadhan LDP of the occupation measure for the SGBH equation (1.3) with \( x \in L^2(\Omega) \) under the assumption (1.7). Our goal is to derive the LDP of the occupation measure \( L_T \) of the solution \( u(\cdot) \) to system (1.3), where the occupation measure is defined as

\[
L_T(A) := \frac{1}{T} \int_0^T \delta_{u(s)}(A) \, ds, \quad \text{for all } A \in B(L^2(\Omega)),
\]

where \( \delta_a \) denotes the Dirac measure concentrated at point \( a \), and \( B(L^2(\Omega)) \) represents the Borelian \( \sigma \)-field in \( L^2(\Omega) \). The Donsker–Varadhan LDP for stochastic Burgers equation, 2D SNSE, stochastic convective Brinkman–Forchheimer equations are obtained in [21, 22, 26], respectively, and we follow these works to obtain Donsker–Varadhan LDP for our problem. All the results obtained in this section are true for \( \delta \in [1, 2] \) with any \( \nu, \alpha, \beta > 0, \gamma \in (0, 1) \) (see Theorem 2.6). Let us state our main result of this section:

**Theorem 5.1** Assume that \( \text{Tr}(GG^*) < \infty \) and (1.7) holds. Let \( 0 < \lambda_0 < \frac{\pi^2 \nu}{2\|Q\|_{L(L^2(\Omega))}} \), where \( \|Q\|_{L(L^2(\Omega))} \) is the norm of \( Q := GG^* \) as an operator in \( L^2(\Omega) \) and

\[
\Psi(x) = e^{\lambda_0 \|x\|_{L^2}^2}, \quad M_{\lambda_0, R} := \left\{ \varrho \in M_1(L^2(\Omega)) : \int_{L^2(\Omega)} \Psi(x) \varrho(dx) \leq R \right\}. \tag{5.1}
\]

The family \( \mathbb{P}_\varrho(L_T \in \cdot) \) as \( T \to +\infty \) satisfies the LDP w.r.t. the topology \( \tau \), with speed \( T \) and rate function \( J \) uniformly for any initial measure \( \varrho \) in \( M_{\lambda_0, R} \), where \( R > 1 \) is any fixed number. Here, the rate function \( J : M_1(L^2(\Omega)) \to [0, +\infty] \) is the level-2 entropy of Donsker–Varadhan (defined in (5.7)). Moreover, we have

(i) \( J \) is a good rate function on \( M_1(L^2(\Omega)) \) equipped with the topology \( \tau \) of the convergence against bounded and Borelian functions, that is, \( [J \leq a] \) is \( \tau \)-compact for every \( a \in \mathbb{R}^+ \).
(ii) For all open sets \( G \) in \( M_1(L^2(\mathcal{O})) \) w.r.t. the topology \( \tau \),

\[
\liminf_{T \to \infty} \frac{1}{T} \log \inf_{\varrho \in \mathcal{M}_{\lambda_0,R}} \mathbb{P}_\varrho \{ L_T \in G \} \geq - \inf_{G} J.
\] (5.2)

(iii) For all closed sets \( F \) in \( M_1(L^2(\mathcal{O})) \) w.r.t. the topology \( \tau \),

\[
\limsup_{T \to \infty} \frac{1}{T} \log \sup_{\varrho \in \mathcal{M}_{\lambda_0,R}} \mathbb{P}_\varrho \{ L_T \in F \} \leq - \inf_{F} J.
\] (5.3)

Furthermore, we have

\[
J(\varrho) < +\infty \Rightarrow \varrho \ll \mu, \quad \varrho(H^1_0(\mathcal{O})) = 1 \text{ and } \int_{H^1_0} \| \partial_x x \|^2_{L^2} d\varrho < +\infty, (5.4)
\]

where \( \mu \) is the unique invariant probability measure of \( u(t, \cdot) \).

**Corollary 5.2** Let \((B, \| \cdot \|_B)\) be a separable Banach space, and \( \psi : H^1_0(\mathcal{O}) \to B \) be a measurable function, bounded on balls \( \{ x : \| A^{\frac{1}{2}} x \|_{L^2} \leq R \} \) and satisfying

\[
\lim_{\| A^{\frac{1}{2}} x \|_{L^2} \to \infty} \frac{\| \psi(x) \|_B}{\| A^{\frac{1}{2}} x \|_{L^2}^2} = 0.
\] (5.5)

Then, \( \mathbb{P}_\varrho (L_T(\psi) \in \cdot) \) satisfies the Donsker–Varadhan LDP on \( B \) with speed \( T \) and the rate function \( I_\psi \) given by

\[
I_\psi(y) = \inf \{ J(\varrho) : J(\varrho) < +\infty, \varrho(\psi) = y \}, \text{ for all } y \in B,
\]

uniformly over initial distributions \( \varrho \in \mathcal{M}_{\lambda_0,R} \) (for any \( R > 1 \)).

**Example 5.3** Let us provide some examples of the assumptions \( \text{Tr}(Q) < \infty \) and (1.7) (cf. [21, 22]).

(i) We know that an \( L^{2}(\mathcal{O}) \)-valued cylindrical Wiener process \( \{ W(t) : 0 \leq t \leq T \} \) on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \) can be expressed as \( W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k \), where \( \beta_k(t), k \in \mathbb{N} \) are independent, one-dimensional Brownian motions on the space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \) (cf. [8]). Let us define \( Ge_k = \sigma_k e_k \), for \( k = 1, 2, \ldots \), so that

\[
GW(t) = \sum_{k=1}^{\infty} \sigma_k \beta_k(t)e_k.
\]

We also know that the eigenvalues of the Laplacian \( \lambda_k \sim k^2 \). Thus, the condition given in (1.7) becomes

\[
\frac{c}{k} \leq \sigma_k \leq \frac{C}{k^{1+\epsilon}},
\]

\( \varrho \) Springer
for any two positive constants $c$ and $C$ and some $\varepsilon > 0$.

(ii) An another example of noise for which our assumptions hold is $G := A^{-\beta}F$, where $F$ is any linear bounded and invertible operator on $L^2(\mathcal{O})$ and $\frac{1}{4} < \beta < \frac{1}{2}$.

Remark 5.4 1. The class (5.1) of permissible initial distributions for the uniform LDP is sufficiently rich. For example, choosing $R$ large enough, it accommodates all the Dirac probability measure $\delta_x$ with $x$ in any ball of $L^2(\mathcal{O})$.

2. The LDP w.r.t. the topology $\tau$ is stronger than that w.r.t. the usual weak convergence topology as in Donsker–Varadhan [17–19].

3. The assumption (1.7) plays an important role in Theorem 5.1. If the noise acts only a finite number of modes (that is, $\sigma_k = 0$ after a finite number $N$), in the first part of Example 5.3 as in Kolmogorov’s turbulence theory, we believe that the LDP w.r.t. the $\tau$-topology is false.

The existence of mild solution of our problem (2.8) is proved in Theorem 2.7 and strong solution is established in Theorem 2.6. We have already defined the transition semigroup corresponding to the solution of (2.8) by

$$P_t \psi(x) = E[\psi(u(t, x))] = E^x[\psi(u(t))], \text{ for all } \psi \in B_b(L^2(\mathcal{O})).$$

We have already proved that the transition semigroup $P_t$, $t \geq 0$ is irreducible, satisfies the strong Feller property (Propositions 4.1, 4.3), and it admits a unique invariant measure $\mu$ (Theorems 3.1, 4.4).

5.1 General Results on LDP

In this section, we provide some necessary notations, basic definitions and give some results from [45] on large deviations for the Markov process. Consider the $L^2(\mathcal{O})$-valued continuous Markov process,

$$(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, \{u_t\}_{t \geq 0}, \{P^x\}_{x \in L^2(\mathcal{O})}),$$

whose semigroup of Markov transition kernel is denoted by $\{P_t(x, dy)\}_{t \geq 0}$, where

- $\Omega = C(\mathbb{R}^+; L^2(\mathcal{O}))$ is the space of continuous functions from $\mathbb{R}^+$ to $L^2(\mathcal{O})$ equipped with the compact convergence topology,
- the natural filtration is $\mathcal{F}_t = \sigma \{u(s) : 0 \leq s \leq t\}$ for any $t \geq 0$ and $\mathcal{F} = \sigma \{u(s) : 0 \leq s\}$,
- $P^x_x\{u(0) = x\} = 1$.

As usual, we denote the law of Markov process with the initial state $x \in L^2(\mathcal{O})$ by $P^x$, and for any initial measure $\rho$ on $L^2(\mathcal{O})$, we define $P^\rho_x(\cdot) = \int_{L^2(\mathcal{O})} P^x_x(\cdot) \rho(dx)$. The empirical measure of level-3 is given by

$$R_t := \frac{1}{t} \int_0^t \delta_{\theta_s u} \, ds,$$
where \((\theta_s u)(t) = u(s + t)\), for all \(t, s \geq 0\) are the shifts on \(\Omega\). Therefore, \(R_t\) is a random element of \(M_1(\Omega)\), the space of probability measures on \(\Omega\). The level-3 entropy functional of Donsker–Varadhan \(H : M_1(\Omega) \to [0, +\infty]\) is defined by

\[
H(Q) := \begin{cases} 
\mathbb{E}\tilde{Q}_s^{\mathcal{F}_0} \left( \tilde{Q}_{\omega,(-\infty,0)}; \mathbb{P}_{\omega(0)} \right), & \text{if } Q \in M_1(\Omega), \\
+\infty, & \text{otherwise},
\end{cases}
\]

(5.6)

where

- \(M_1^s(\Omega)\) is the subspace of \(M_1(\Omega)\), whose elements are moreover stationary;
- \(\tilde{Q}\) is the unique stationary extension of \(Q \in M_1(\Omega)\) to \(\tilde{\Omega} := C(\mathbb{R}; L^2(\mathcal{O}))\);
- \(\mathcal{F}_t^s = \sigma\{u(r) : s \leq r \leq t\}\), for all \(s, t \in \mathbb{R}\), \(s \leq t\);
- \(\tilde{Q}_{\omega,(-\infty,t]}\) is the regular conditional distribution of \(\tilde{Q}\) knowing \(\mathcal{F}_t^{-\infty}\);
- \(h_G(\varrho, \mu)\) is the usual relative entropy or Kullback information of \(\varrho\) w.r.t. \(\mu\) restricted to the \(\sigma\)-field \(G\), is given by

\[
h_G(\varrho; \mu) := \begin{cases} 
\int \frac{d\varrho}{d\mu} |_G \log \left( \frac{d\varrho}{d\mu} |_G \right) d\mu, & \text{if } \varrho \ll \mu \text{ on } G, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

The level-2 entropy functional \(J : M_1(L^2(\mathcal{O})) \to [0, \infty]\), which governs the LDP in our main Theorem 5.1 is

\[
J(\mu) = \inf \{ H(Q) : Q \in M_1^s(\Omega) \text{ and } Q_0 = \mu \}, \quad \mu \in M_1(L^2(\mathcal{O})),
\]

(5.7)

where \(Q_0(\cdot) = Q(u(0) \in \cdot)\) is the marginal law at \(t = 0\). As introduced in [45], we define the restriction of the Donsker–Varadhan entropy to the \(\mu\) component, by

\[
H_\mu(Q) := \begin{cases} 
H(Q), & \text{if } Q_0 \ll \mu, \\
\infty, & \text{otherwise}.
\end{cases}
\]

and for level-2 entropy functional

\[
J_\mu(\varrho) := \begin{cases} 
J(\varrho), & \text{if } \varrho \ll \mu, \\
\infty, & \text{otherwise}.
\end{cases}
\]

A proof of the following result is available as Lemma 3.1, [21], and hence we omit it here.

**Lemma 5.5** For our system, \(J(\varrho) < \infty \Rightarrow \varrho \ll \mu\). Moreover, \(J = J_\mu\) on \(M_1(L^2(\mathcal{O}))\) and \(\{J = 0\} = \{\mu\}\).

### 5.2 Exponential Estimates for the Solution

In this subsection, we prove a crucial exponential estimate for the solution \(u(\cdot)\) to the SGBBH equation, which will be helpful to establish the LDP results. We need
the following result to discuss the proof of Proposition 5.7. First, recall the finite dimensional Galerkin approximation, that is,

\[
\begin{aligned}
&\begin{cases}
\frac{du_n(t)}{dt} = \{-\nu Au_n(t) - \alpha B_n(u_n(t)) + \beta c_n(u_n(t))\}dt + G_ndW(t), \quad t \in (0, T), \\
u_n(0) = x_n := \Pi_n x,
\end{cases}
\tag{5.8}
\end{aligned}
\]

and it satisfies the following apriori energy estimate:

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u_n(t)\|^2_{L^2} + \nu \int_0^T \|\partial_x u_n(s)\|^2_{L^2} ds + \beta \int_0^T \|u_n(s)\|^{2(\delta+1)}_{L^{2(\delta+1)}} ds \right] \\
\leq C(\|x\|^2_{L^2} + \text{Tr}(GG^*)T).
\tag{5.9}
\]

**Lemma 5.6** Let \(u_n(\cdot)\) and \(u(\cdot)\) be the solutions of the systems (5.8) and (2.8), respectively. Then, we have

\[
\|u_n(t)\|^2_{L^2} + 2\nu \int_0^t \|\partial_x u_n(s)\|^2_{L^2} ds + \beta \int_0^t \|u_n(s)\|^{2(\delta+1)}_{L^{2(\delta+1)}} ds \\
\overset{a.s.}{\longrightarrow} \|u(t)\|^2_{L^2} + 2\nu \int_0^t \|\partial_x u(s)\|^2_{L^2} ds + \beta \int_0^t \|u(s)\|^{2(\delta+1)}_{L^{2(\delta+1)}} ds,
\tag{5.10}
\]

for all \(t \in [0, T]\).

**Proof** Using (5.9) and the Banach–Alaoglu theorem, we can extract a subsequence \(\{u_n\}\) (for simplicity still denoting by \(u_n\)) such that

\[
\begin{aligned}
u_n &\overset{w^*}{\rightharpoonup} u \quad \text{in} \ L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O}))), \\
u_n &\overset{w}{\rightharpoonup} u \quad \text{in} \ L^2(\Omega; L^2(0, T; H^1_0(\mathcal{O}))), \\
u_n &\overset{w}{\rightharpoonup} u \quad \text{in} \ L^{2(\delta+1)}(\Omega; L^{2(\delta+1)}(0, T; L^{(\delta+1)}(\mathcal{O}))),
\end{aligned}
\tag{5.11}
\]

where \(u(\cdot)\) denotes the strong solution of the system (2.8). Since the solution of (5.8) is unique, the whole sequence converges to \(u(\cdot)\). A calculation similar to (3.3) yields

\[
\begin{aligned}
\mathbb{E}\left[ \|u_n(t)\|^2_{L^2} + 2\nu \int_0^t \|\partial_x u_n(s)\|^2_{L^2} ds + \beta \int_0^t \|u_n(s)\|^{2(\delta+1)}_{L^{2(\delta+1)}} ds \right] \\
\leq \|x_n\|^2_{L^2} + (\text{Tr}(G_nG_n^*) + C(\beta, \delta))t,
\tag{5.12}
\end{aligned}
\]

where the constant \(C(\beta, \delta) = \left(\frac{2}{\beta(\delta+1)}\right)^{\frac{\delta}{\delta+1}}\). Similarly, for the strong solution \(u(\cdot)\) of system (2.8), we get

\[
\begin{aligned}
\mathbb{E}\left[ \|u(t)\|^2_{L^2} + 2\nu \int_0^t \|\partial_x u(s)\|^2_{L^2} ds + \beta \int_0^t \|u(s)\|^{2(\delta+1)}_{L^{2(\delta+1)}} ds \right] \\
\leq \|x\|^2_{L^2} + (\text{Tr}(GG^*) + C(\beta, \delta))t,
\tag{5.13}
\end{aligned}
\]
for all \( t \in [0, T] \). From (5.12) and (5.13), we obtain

\[
\mathbb{E} \left[ \left\| u_n(t) \right\|_{L^2}^2 + 2v \int_0^t \left\| \partial_x u_n(s) \right\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \left\| u_n(s) \right\|_{L^2(\delta+1)}^2 \, ds \right] \\
- \mathbb{E} \left[ \left\| u_n(t) \right\|_{L^2}^2 + 2v \int_0^t \left\| \partial_x u_n(s) \right\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \left\| u_n(s) \right\|_{L^2(\delta+1)}^2 \, ds \right]
\]

\[
\leq \left\| \left\| x_n \right\|_{L^2}^2 - \| x \|_{L^2}^2 + \text{Tr}(G_nG_n^*) \cdot \text{Tr}(GG^*) \right\|
\]

\[
\leq \left\| \left\| x_n \right\|_{L^2}^2 - \| x \|_{L^2}^2 \right\| \left\| \left\| x_n \right\|_{L^2} + \| x \|_{L^2} \right\| + \left| \text{Tr}(G_nG_n^*) \cdot \text{Tr}(GG^*) \right|. \quad (5.14)
\]

It is easy to deduce that

\[
\left\| \left\| x_n \right\|_{L^2} - \| x \|_{L^2} \right\| \leq \left\| x_n - x \right\|_{L^2} = \left( \sum_{j=n+1}^{\infty} |(x, e_j)|^2 \right)^{1/2} \to 0 \text{ as } n \to \infty,
\]

and

\[
\text{Tr}(G_nG_n^*) - \text{Tr}(GG^*) = \text{Tr}(\Pi_n GG^* - GG^*) \leq \left\| \Pi_n - I \right\|_{L^2(\mathcal{Q})} \text{Tr}(GG^*) \to 0 \text{ as } n \to \infty.
\]

Passing \( n \to \infty \) in (5.14) by using the above convergences, we find

\[
\mathbb{E} \left[ \left\| u(t) \right\|_{L^2}^2 + 2v \int_0^t \left\| \partial_x u(s) \right\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \left\| u(s) \right\|_{L^2(\delta+1)}^2 \, ds \right] \\
= \mathbb{E} \left[ \left\| u(t) \right\|_{L^2}^2 + 2v \int_0^t \left\| \partial_x u(s) \right\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \left\| u(s) \right\|_{L^2(\delta+1)}^2 \, ds \right],
\]

as \( n \to \infty \). From the above convergence, one can extract an a.s. convergent subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \), that is,

\[
\left\| u_{n_k}(t) \right\|_{L^2}^2 + 2v \int_0^t \left\| \partial_x u_{n_k}(s) \right\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \left\| u_{n_k}(s) \right\|_{L^2(\delta+1)}^2 \, ds
\]

\[
\overset{a.s.}{\longrightarrow} \left\| u(t) \right\|_{L^2}^2 + 2v \int_0^t \left\| \partial_x u(s) \right\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \left\| u(s) \right\|_{L^2(\delta+1)}^2 \, ds \text{ as } k \to \infty,
\]

for all \( t \in [0, T] \). The above convergence holds for the original sequence \( \{u_n(\cdot)\} \), since \( u_n(\cdot), u(\cdot) \) are the unique solutions of (5.8) and (2.8), respectively. \( \square \)

**Proposition 5.7** For any \( 0 < \lambda_0 < \frac{\pi^2 v}{2\|Q\|_{L^2(\mathcal{Q})}} \), where \( \|Q\|_{L^2(\mathcal{Q})} \) is the norm of \( Q \) as an operator in \( L^2(\mathcal{Q}) \), and for any \( x \in L^2(\mathcal{Q}) \), the process \( u(\cdot) \) satisfies the
Following estimates:

\[
E^x \left\{ \exp \left( \lambda_0 \| u(t) \|_{L^2}^2 + \lambda_0 \nu \int_0^t \| \partial_x u(s) \|_{L^2}^2 \, ds + \frac{\lambda_0 \beta}{2} \int_0^t \| u(s) \|_{L^2(\delta+1)}^2 \, ds \right) \right\} \\
\leq e^{\lambda_0 \| x_0 \|_{L^2}^2 + \lambda_0 t (\text{Tr}(Q) + C)} ,
\]

where the constant \( C = \left( \frac{2}{B(\delta+1)} \right)^{\frac{1}{\delta+1}} \). In particular, the following estimates hold

\[
E^x \left\{ \exp \left( \lambda_0 \| u(t) \|_{L^2}^2 \right) \right\} \leq e^{\lambda_0 \| x_0 \|_{L^2}^2 + \lambda_0 t (\text{Tr}(Q) + C)} ,
\]
\[
E^x \left\{ \exp \left( \lambda_0 \nu \int_0^t \| \partial_x u(s) \|_{L^2}^2 \, ds \right) \right\} \leq e^{\lambda_0 \| x_0 \|_{L^2}^2 + \lambda_0 t (\text{Tr}(Q) + C)} ,
\]
\[
E^x \left\{ \exp \left( \lambda_0 \beta \int_0^t \| u(s) \|_{L^2(\delta+1)}^2 \, ds \right) \right\} \leq e^{\lambda_0 \| x_0 \|_{L^2}^2 + \lambda_0 t (\text{Tr}(Q) + C)} .
\]

**Proof** First we establish the result for the finite dimensional system (5.8), and then we pass the limit as \( n \to \infty \). From (3.2), we have

\[
\| u_n(t) \|_{L^2}^2 + 2 \nu \int_0^t \| \partial_x u_n(s) \|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \| u_n(s) \|_{L^2(\delta+1)}^2 \, ds \\
\leq \| x_n \|_{L^2}^2 + (\text{Tr}(G_n G_n^*) + C) t + 2 \int_0^t (G_n dW(s), u_n(s)) ,
\]

\( \mathbb{P} \)-a.s. Let us define

\[
Z_n(t) := \| u_n(t) \|_{L^2}^2 + \nu \int_0^t \| \partial_x u_n(s) \|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \| u_n(s) \|_{L^2(\delta+1)}^2 \, ds .
\]

Then from (5.19), we find

\[
Z_n(t) \leq \| x_n \|_{L^2}^2 - \nu \int_0^t \| \partial_x u_n(s) \|_{L^2}^2 + (\text{Tr}(G_n G_n^*) + C) t + 2 \int_0^t (G_n dW(s), u_n(s))
\]

\( \mathbb{P} \)-a.s., for all \( t \in [0, T] \). Applying Itô’s formula to the process \( e^{\lambda_0 Z_n(t)} \), using the chain rule, we obtain

\[
d(e^{\lambda_0 Z_n(t)}) = e^{\lambda_0 Z_n(t)} \left[ \lambda_0 dZ_n(t) + \frac{\lambda_0^2}{2} d[Z_n, Z_n] \right] \\
\leq \lambda_0 e^{\lambda_0 Z_n(t)} \left[ -\nu \| \partial_x u_n(t) \|_{L^2}^2 + \text{Tr}(Q_n) + C + 2\lambda_0 \| G_n^* u_n(t) \|_{L^2}^2 \right] dt \\
+ 2\lambda_0 e^{\lambda_0 Z_n(t)} (G_n dW(t), u_n(t)).
\]

The following inequalities are easy to obtain:

\[
\text{Tr}(Q_n) \leq \text{Tr}(Q), \quad \| x_n \|_{L^2} \leq \| x \|_{L^2}.
\]
Also we have
\[ \|G^*_n u_n\|_{L^2}^2 \leq \|G_n\|_{L(L^2(\mathcal{O}))}^2 \|u_n(t)\|_{L^2}^2 \leq \|Q\|_{L(L^2(\mathcal{O}))} \|u_n(t)\|_{L^2}^2, \] (5.22)
and by Poincaré’s inequality, we get
\[ \|u_n(t)\|_{L^2}^2 \leq \frac{1}{\pi^2} \|\partial_\xi u_n(t)\|_{L^2}^2. \] (5.23)

Using (5.21)–(5.23) in (5.20), we obtain
\[
\begin{align*}
\frac{d}{dt} \lambda_0 Z_n(t) &\leq \lambda_0 e^{\lambda_0 Z_n(t)} \left[ -\nu \|\partial_\xi u_n(t)\|_{L^2}^2 + \text{Tr}(Q) + C + \frac{2\lambda_0}{\pi^2} \|Q\|_{L(L^2(\mathcal{O}))} \|\partial_\xi u_n(t)\|_{L^2}^2 \right] \\
&\quad + 2\lambda_0 e^{\lambda_0 Z_n(t)} (G_n dW(t), u_n(t)).
\end{align*}
\] (5.24)
Integrating the above inequality from 0 to \( t \), we get
\[
\begin{align*}
\lambda_0 Z_n(t) &\leq \lambda_0 \|x\|_{L^2} \int_0^t e^{\lambda_0 Z_n(s)} \frac{d}{ds} \left( -\nu \|\partial_\xi u_n(t)\|_{L^2}^2 + \text{Tr}(Q) + C + \frac{2\lambda_0}{\pi^2} \|Q\|_{L(L^2(\mathcal{O}))} \|\partial_\xi u_n(t)\|_{L^2}^2 \right) ds \\
&\quad + 2\lambda_0 e^{\lambda_0 Z_n(t)} (G_n dW(t), u_n(t)).
\end{align*}
\] (5.24)
Taking expectation on both sides, we deduce
\[
\begin{align*}
\mathbb{E}^x \left( e^{\lambda_0 Z_n(t)} \right) &\leq e^{\lambda_0 \|x\|_{L^2}^2} \int_0^t \mathbb{E}^x \left( e^{\lambda_0 Z_n(s)} \right) ds \\
&\quad + \lambda_0 \mathbb{E}^x \left[ e^{\lambda_0 Z_n(t)} \int_0^t \left( \frac{2\lambda_0}{\pi^2} \|Q\|_{L(L^2(\mathcal{O}))} - \nu \right) \|\partial_\xi u_n(t)\|_{L^2}^2 ds \right],
\end{align*}
\] since the final term in (5.24) is a local martingale, hence the expectation is zero. Let us now choose \( 0 < \lambda_0 < \frac{\pi^2 \nu}{2\|Q\|_{L(L^2(\mathcal{O}))}} \), so that the third term in the right-hand side of the above inequality is negative, and we get
\[
\mathbb{E}^x \left( e^{\lambda_0 Z_n(t)} \right) \leq e^{\lambda_0 \|x\|_{L^2}^2} \int_0^t \mathbb{E}^x \left( e^{\lambda_0 Z_n(s)} \right) ds.
\]
An application of Gronwall’s lemma in the above inequality provides
\[
\mathbb{E}^x \left( e^{\lambda_0 Z_n(t)} \right) \leq e^{\lambda_0 \|x\|_{L^2}^2 \lambda_0 t (\text{Tr}(Q)+C)}.
\]
Letting $n \to \infty$ in the above inequality for $0 < \lambda_0 < \frac{\pi^2 \nu}{\|Q\|_{L^2(O)}}$, with the help of Lemma 5.6, we obtain for any $t \in [0, T]$,  
\[ \mathbb{E}^x(e^{\lambda_0 Z(t)}) \leq e^{\lambda_0\|x\|^2_{L^2}} e^{\lambda_0 t (\text{Tr}(Q) + C)}, \]
and the proof is completed. \hfill \Box

### 5.3 Uniform Upper Bound for the $\tau$-Topology

To prove the upper bound (5.3) in our main Theorem 5.1, we use the criterion of hyper-exponential recurrence established in Theorem 2.1, [46], for a general polish space $E$. The following result is a slight extension of the result in [46], to a uniform LDP over a non-empty family of initial measures (cf. [22]). To use this result we require two properties of the associated semigroup, strong Feller and irreducibility.

**Lemma 5.8** (Theorem 2.1, [46], or Lemma 6.1, [21]) For a subset $K$ in $L^2(O)$, let us define $\tau_k := \inf\{t \geq 0 : u(t) \in K\}$ and $\tau_k^{(1)} := \inf\{t \geq 1; u(t) \in K\}$. If for any $\lambda > 0$, there exists a compact subset $K$ in $L^2(O)$ such that  
\[ \sup_{\varrho \in M_{\lambda_0, R}} \mathbb{E}^\varrho[e^{\lambda \tau_k}] < \infty, \quad (5.25) \]
and  
\[ \sup_{x \in K} \mathbb{E}^x[e^{\lambda \tau_k^{(1)}}] < \infty, \quad (5.26) \]
then $[J \leq a]$ is $\tau$-compact for every $a \in \mathbb{R}^+$, and the upper bound (5.3) is uniform on $M_{\lambda_0, R}$ for the $\tau$-topology holds true.

To prove the upper bound (5.3), it is enough to show the estimates (5.25) and (5.26) holds for our model. For that we choose a compact subset $K \subset L^2(O)$ given by  
\[ K := \{x \in H^1_0(O); \|\partial_\xi x\|_{L^2} \leq M\}, \quad (5.27) \]
where $M$ is the finite real number, which will be fixed later. Using the definition of occupation measure for $n \geq 2$, we obtain  
\[ \mathbb{P}_e \left\{ \tau_k^{(1)} > n \right\} \leq \mathbb{P}_e \left\{ L_n(K) \leq \frac{1}{n} \right\} = \mathbb{P}_e \left\{ L_n(K^c) \geq 1 - \frac{1}{n} \right\}. \]

For the set $K$ defined in (5.27), an application of Markov’s inequality yields  
\[ L_n(K^c) \leq \frac{1}{M^2} L_n \left( \|\partial_\xi x\|_{L^2(O)}^2 \right). \]
For any fixed real number $\lambda_0$ such that $0 < \lambda_0 < \frac{\pi^2}{\|Q\|_{L^2(O)}}^2$, using Markov’s inequality, we obtain

$$\mathbb{P}_\rho \left\{ \tau^{(1)}_K \geq n \right\} \leq \mathbb{P}_\rho \left\{ L_n \left( \| \partial_\xi x \|_{L^2}^2 \right) \geq M^2 \left( 1 - \frac{1}{n} \right) \right\} \leq \mathbb{P}_\rho \left\{ \frac{\nu \lambda_0}{n} \int_0^n \| \partial_\xi u(s) \|_{L^2}^2 \, ds \geq \nu \lambda_0 M^2 \left( 1 - \frac{1}{n} \right) \right\} \leq \exp \left( - n \nu \lambda_0 M^2 \left( 1 - \frac{1}{n} \right) \right)\mathbb{P}_\rho \left\{ \exp \left( \nu \lambda_0 \int_0^n \| \partial_\xi u(s) \|_{L^2}^2 \, ds \right) \right\}. \tag{5.28}$$

For any initial measure $\rho \in M_1(L^2(O))$, integrating the exponential estimate (5.17) w.r.t. $\rho(dx)$, we find

$$\mathbb{E}_\rho \left\{ \exp \left( \nu \lambda_0 \int_0^n \| \partial_\xi u(s) \|_{L^2}^2 \, ds \right) \right\} \leq e^{\lambda \rho(\text{Tr}(Q)+C)} \rho(e^{\lambda_0 \| \cdot \|_{L^2}^2}).$$

Substituting the above inequality in (5.28), we get

$$\mathbb{P}_\rho (\tau^{(1)}_K \geq n) \leq \rho(e^{\lambda_0 \| \cdot \|_{L^2}^2}) e^{-n C_1 \lambda_0}, \text{ for all } n \geq 2, \tag{5.29}$$

where the constant $C_1 := \frac{M^2}{2} - \text{Tr}(Q) - C(\beta, \delta)$ and $C(\beta, \delta) = \left( \frac{2}{\beta(\delta+1)} \right)^{\frac{1}{2}} \delta^{\frac{1}{\delta+1}}$. Fix $\lambda > 0$. Using integration by parts formula and (5.29), we deduce

$$\mathbb{E}_\rho [e^{\lambda \tau^{(1)}_K}] = 1 + \int_0^{+\infty} \lambda e^{\lambda t} \mathbb{P}_\rho (\tau^{(1)}_K > t) \, dt \leq 1 + \sum_{n=0}^{\infty} \lambda e^{\lambda n} \mathbb{P}_\rho (\tau^{(1)}_K > n) \leq e^{2\lambda} + \sum_{n \geq 2} \lambda e^{\lambda(n+1)} \mathbb{P}_\rho (\tau^{(1)}_K > n) \leq e^{2\lambda} \left( 1 + \lambda \rho(e^{\lambda_0 \| \cdot \|_{L^2}^2}) \sum_{n \geq 2} e^{-n(\lambda_0 C_1 - \lambda)} \right).$$

Using definition (5.27) of the subset $K$, we can choose the constant $M$ appearing in the definition (5.27) of $K$ such that $\lambda_0 C_1 - \lambda \geq 1$. Also note that for any $x \in K$, we can use Poincaré inequality as $\|x\|_{L^2}^2 \leq \frac{\| \partial_\xi x \|_{L^2}^2}{\pi^2} \leq \frac{M^2}{\pi^2}$. Taking supremum over the set $\{ \rho = \delta_x x \in K \}$, we find

$$\sup_{x \in K} \mathbb{E}_\rho [e^{\lambda \tau^{(1)}_K}] \leq e^{2\lambda} \left( 1 + \lambda e^{\lambda_0 \frac{M^2}{\pi^2}} \sum_{n \geq 2} e^{-n(\lambda_0 C_1 - \lambda)} \right) < \infty,$$
and hence (5.27) holds. We can obtain (5.27) by the same procedure. From the definition of $\tau_k$, we have $\tau_k \leq \tau_k^{(1)}$, and hence one can compute that

$$
\sup_{\varrho \in \mathcal{M}_{\lambda_0, R}} \mathbb{E}^\varrho [e^{\lambda \tau_k}] \leq \sup_{\varrho \in \mathcal{M}_{\lambda_0, R}} \mathbb{E}^\varrho [e^{\lambda \tau_k^{(1)}}] \leq e^{2\lambda} \left( 1 + \lambda R \sum_{n \geq 2} e^{-n(\lambda_0 C_1 - \lambda)} \right) < \infty,
$$

which finishes the proof.

**Proof of Theorem 5.1** We have proved (5.25) and (5.26) of Lemma 5.8, which provide the good uniform upper and lower bounds of the large deviations, that is, part (i) to (iii) of Theorem 5.1. The first part of (5.4), that is, $J(\varrho) < \infty \implies \varrho \ll \mu$ is given in Lemma 5.5. The second part in (5.4), that is, for $\varrho \in \mathcal{M}_1(L^2(\mathcal{O}))$ with $J(\varrho) < \infty$, $\varrho(\|\partial_x^2 \varrho\|_{L^2}^2) < \infty$ can be established in a similar way as in the proof of Theorem 1.1, [21].

**Proof of Corollary 5.2** The exponential estimate in Proposition 5.7 is sufficient to extend the LDP of Theorem 5.1, for the unbounded functionals and its consequences. The proof will be on the similar lines as in the works [21, 22], etc.

**Remark 5.9** As discussed in [12], one can consider the following SGBH equation perturbed by multiplicative (or correlated) random force also:

$$
du(t) = \left\{ -\nu A u(t) - \alpha B(u(t)) + \beta c(u(t)) \right\} dt + g(u(t)) dW(t), \quad t \in (0, T),
$$

$$
u(0) = x, \quad (5.30)
$$

where $g : L^2(\mathcal{O}) \to [a, b]$ is Lipschitz continuous, $0 < a < b < \infty$. The analysis of such problems will be carried out in a future work. One can also consider SGBH equation perturbed by $\alpha$-stable noise and establish ergodicity results as discussed in [43, 47], etc. This problem will also be considered in a future work.

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