Parametric Phase Tracking
via Expectation Propagation

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Abstract

In this work we propose simple algorithms for signal detection in a single-carrier transmission corrupted by a strong phase noise. The proposed phase tracking algorithms are formulated within the framework of a parametric message passing (MP) which reduces the complexity of the Bayesian inference by using distributions from a predefined family; here, of Tikhonov distributions. This stays in line with previous works mainly inspired by the well-known Colavolpe–Barbieri–Caire (CBC) algorithm which gained popularity due to its simplicity and possibility for decoder-aided operation. In our work we mainly focus on practically relevant case of one-shot phase tracking that does not require decoder’ feedback. Applying the principles of the expectation propagation (EP), we notably improve the performance of the phase tracking before the decoder’s feedback can be even considered. The EP algorithms can be also integrated in the decoding loop in the spirit of joint decoding and phase tracking.

Index Terms

  circular moment matching, expectation propagation, optical communications, wireless backhaul, phase noise, phase tracking, Tikhonov distributions

I. INTRODUCTION

In this work we propose and analyze simple algorithms for signal detection in the presence of the phase noise.

We consider the single-carrier transmission which is often used for high speed communication in frequency non-selective channels e.g., in satellite channels [1], wireless backhaul [2], or optical communications.
communications [3], [4]. The limiting factor in achieving a high spectral efficiency is then not only due to the presence of the additive white Gaussian noise (AWGN) but also due to the phase-noise which is caused by the instability of the phase reference (such as a laser or an RF oscillator).

Although sometimes the concept of “strong” phase noise is used, it merely depends on the constellation size: with sufficiently large modulation, the phase noise becomes always a limiting factor for reliable communications. It is especially true for the wireless backhaul and optical channel where the modulation size is aggressively increased [2]–[4]. The wireless backhaul, for example, calls for the use of constellation which may contain thousands of points [2].

Since the phase noise is a random process with memory, the term “phase tracking” is often used and a common strategy is to use pilots symbols which, providing reliable reference, facilitate the tracking of the phase for payload symbols. Increasing the density of the pilots, usually improves the performance at the cost of decreased spectral efficiency. Therefore, the challenge of the phase tracking is not to eliminate the pilots altogether but rather to attain desirable performance (e.g., as measured by the errors rates) with the limited number of pilots.

The problem of phase tracking can be conveniently formulated using the graph relating all the involved variables [1], [5], which also leads to optimal solution defined by the message passing (MP) defined over the graph. It allows us to track the distribution of the phase at each symbol but, to represent accurately the distributions, a large number of samples may be required [1], [6] even if simplifications may be sought by truncating the support of the distributions [2].

Therefore, many works looked into the possibility of parametric phase tracking, where the distributions of the phase is assumed to belong to a predefined family; Tikhonov distributions [1], [6] or Gaussians [4], [7] are the most popular choices. Then, instead of estimating the samples of the distributions, only a few parameters need to be tracked.

The distributions in the MP are naturally mixtures (result of averaging the contribution of the unknown payload symbols over the entire constellation), with potentially large number of elements. The mixture reduction is thus at the heart of any parametric phase tracking, and consists in replacing the mixture with a member from the adopted family of distributions. In this work we opt for the Tikhonov family of distributions which appear naturally in the context of the phase tracking. This allows us to focus on the principles of the MP; we discuss this issue in more details in Sec. [IV].
In most works the mixture reduction consists in replacing the mixture with one distribution. The notable difference is [6] which proposes to replace the mixture with another mixture but which contains a small number of elements. The improvements are thus obtained by making the parametrization more involved both in terms of the number of required parameters and of their estimation.

This is a general feature of the parametric/approximate MP algorithms: the challenge resides in finding a right balance between their performance and the implementation complexity.

With that regard, the Colavolpe–Barbieri–Caire (CBC) algorithm [1] makes uttermost simplification: the mixture reduction is performed before the MP recursion is invoked. This yields a very simple MP algorithm which, due to its simplicity and ensuing popularity, should be considered “canonical”. However, the cost is paid with relatively poor performance which must be improved by leveraging the presence of the decoder as only in this way the CBC algorithm can exploit the information about the modulation constellation. The resulting joint phase tracking and decoding relies thus on the iterative exchange of information between the phase tracking and the decoder. This approach has been often reused, e.g., [4], [8].

It is not without the pitfalls, however, and it was shown in [7] that, for large coding rates the CBC algorithm introduces an error floor in high signal-to-noise ratio (SNR). In our work we will also implement the CBC algorithm, observe the error floor effect, and remove it by a scaling down of the unreliable logarithmic likelihood ratios (LLRs) delivered by the decoder.

Even with this improvement, the CBC algorithm is failing to approach the limits defined by the discretized message passing (DMP) algorithm with moderate number of decoding iterations. In fact, the original work, [1], considered hundreds of decoding iterations, which is not the common solution in the current industrial practice where ten(s) of iterations are rather preferred.

To remedy the poor performance of the CBC, the mixture reduction may be done within the recursive equations of the MP. This is the idea of the algorithms proposed in [6], [7] which, even in the absence of the decoder’s feedback are able to exploit the form of the constellation; this improves the performance at the cost of more complex MP.

What the CBC algorithm and many works on parametric MP, e.g., [6], [7], have in common is that they rely heavily on the decoder to improve the estimation of the phase via joint decoding - phase tracking. And while it is clear that such a joint operation improves the performance, much less attention was paid to the performance of “one-shot” phase tracking (meaning that
the latter is decoupled from the decoder’s outcome). This is a relevant issue mainly because it is compatible with a current industrial practice but also because efficient one-shot algorithm should yield larger improvements when operating jointly with the decoder.

The above considerations explain the main motivation behind our work: we want to exploit the form of the constellation without relying on the decoder’s feedback.

We first note that the CBC algorithm as well as those proposed in [6], [7], relying on approximations (due to the mixture reduction), are suboptimal by nature. Our main idea is to improve them iteratively using the expectation propagation (EP) [9] which is a general framework for iterative refinement of the approximations in parametric MPs.

We show that the EP improves significantly the performance of one-shot receivers closing notably the performance gap to the DMP. This advantage also materializes in the iterative versions of the proposed algorithms.

The rest of the paper is organized as follows. In Sec. II we introduce the adopted system model, in Sec. III we outline the fundamentals of the Bayesian phase tracking while its parametric formulation is explained in Sec. IV.

The main contribution lies in Sec. V which explains how the EP framework may be used to derive a new phase tracking algorithms which, as shown in numerical examples, improve the performance without any help from the decoder. We also show that, when combined with the decoder, the new algorithms approach closely the DMP limits. The conclusions are drawn in Sec. VII while the appendices show the details of the operations on the circular distributions and provide new approximations required in the latter.

II. System Model

We consider transmission over a channel corrupted by the additive noise and phased noise

\[ y_n = x_n e^{j\theta_n} + v_n, \quad n = 0, \ldots, N \]  

(1)

where \( x_n \in \mathcal{A} \) is the transmitted complex symbol drawn from the \( M \)-ary constellation \( \mathcal{A} \), i.e., \( |\mathcal{A}| = M \), \( y_n \) are samples of the received signal, \( v_n \) is the additive noise, and \( \theta_n \) is the phase noise.

\(^1\)To our best knowledge, the phase tracking algorithms used in industrial products do not use decoder’s feedback.

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We assume the constellation $\mathcal{A}$ is zero mean and energy normalized so, modelling $x_n$ as random variables obtained by uniform sampling of $\mathcal{A}$, we have $E[x_n] = 0$ and $E[x_n^2] = 1$. We model $v_n$ as complex independent, identically distributed (i.i.d.) Gaussian variables with zero mean and variance $N_0$ (AWGN model); $\theta_n$ is modelled as a Wiener process

$$\theta_n = \theta_{n-1} + w_n, \quad n = 1, \ldots, N,$$

(2)

where $w_n$ are i.i.d. zero-mean, real Gaussian variables with variance $\sigma_w^2$; the initial value $\theta_0$ is modelled as uniformly distributed over the interval $(-\pi, \pi)$.

This model is popular in wireless and optical communications, e.g., [1], [4]. The variance of the additive noise $v_n$ is determined by the thermal/optical noise at the receiver and the attenuation on the propagation path; the SNR is defined as $\text{SNR} = \frac{1}{N_0}$. The variance of the phase noise $w_n$ reflects the (in)stability of the oscillator used in the demodulation process. We suppose that both, $\sigma_w^2$ and SNR, are known at the receiver.

The transmitted symbols $x_n$ are obtained via bit-interleaved coded modulation (BICM) [10, Chap. 1.4] in two steps: 1) the information bits $\{b_n\}_{n=1}^{N_b}$ are encoded using the binary encoder of rate $r$ into the coded bits $\{c_n\}_{n=1}^{N_c}$, where $N_b = rN_c$ (in the numerical examples we use the low-density parity-check (LDPC) codes, and 2) the coded bits $\{c_n\}$ are regrouped into length-$m$ labels $\tilde{c}_n = [c_n, 1, \ldots, c_n, m], \, n = 1, \ldots, N_s$, where $N_s m = N_c$; they are next mapped onto the symbols from the constellation $\mathcal{A}$

$$s_n = \Phi[\tilde{c}_n], \quad n = 1, \ldots, N_s,$$

(3)

where $\Phi[\cdot] : \{0, 1\}^m \mapsto \mathcal{A}$ defines the mapping; in this work, $\mathcal{A}$ is $M$-ary quadrature amplitude modulation (QAM) and $\Phi[\cdot]$ is the Gray mapping [10, Sec. 2.5.2]. Finally, reference symbols (pilots) are interleaved with the payload $\{s_n\}_{n=1}^{N_s}$ so the transmitted sequence can be presented as

$$\{x_n\}_{n=1}^{N} = \{x_0, x_1, \ldots, x_{L-1}, x_L, x_{L+1}, \ldots, x_N\},$$

(4)

where we indicate with arrows the pilot symbols $x_n, n \in \mathbb{N}_{\text{pilots}}$, while $x_n, n \in \mathbb{N}_{\text{payload}}$ are
payload symbols with the sets of indices to the payload symbols and to the pilots defined as

\[ N_{\text{pilots}} = \{0, L, 2L, \ldots, FL\} \]  
\[ N_{\text{payload}} = \{1, \ldots, N\} \setminus N_{\text{pilots}}. \]  

We will use the values of \( L \) which satisfy \( N_s = F(L - 1) \) for integer \( F \). We also use the mapping \( n' = n'(n) \) which allows us to index the symbols in \( \{x_n\} \) by skipping the pilots, that is, \( x_{n'} = s_n, n = 1, \ldots, N_s \).

Similarly, the BICM decoding is carried out in two steps [10, Chap. 1.4]

1) Demodulation: consists in finding the marginal conditional probability of the coded bits

\[ \Pr \{c_{n,k} = c | \mathbf{y}\} \]  
\[ \propto \sum_{a \in A_c} \Pr \{s_n = a | \mathbf{y}\}, \]  

where \( \mathbf{y} = \{y_n\}_{n=1}^N \) gathers all the channel outcomes and

\[ A_{k,c} = \{\Phi[c], c = [c_1, \ldots, c_{k-1}, c, c_{k+1}, \ldots, c_m]\} \]

is a sub-constellation comprising only the symbols labeled by the bit with value \( c \in \{0, 1\} \) at the position \( k = 1, \ldots, m \); \( \propto \) will be used in the text to indicate that the distributions are defined up to a multiplicative factor which is independent of the distribution argument (here \( c \) or \( a \)).

Since \( s_n = x_{n'} \), to obtain (7) we have to calculate the conditional distribution of the symbols \( x_n \)

\[ P_n(a) \triangleq \Pr \{x_n = a | \mathbf{y}\}, \quad a \in A; \]  

this operation involves marginalization over the phase \( \theta_n \) and the transmitted symbols \( x_l, l \neq n \), and is a “phase tracking” since, as a byproduct of (10), we will obtain the distribution of the phase \( f(\theta_n | \{y_l\}_{l=1}^N) \).
2) **Soft-input decoding**: using (10) and (7) we calculate the LLRs for the coded bits $c_{n,k}$, i.e.,

$$
\lambda_{n,k} = \log \frac{\Pr \{c_{n,k} = 1 | y \}}{\Pr \{c_{n,k} = 0 | y \}}
\approx \max_{a \in A_{k,1}} \hat{P}_n'(a) - \max_{a \in A_{k,0}} \hat{P}_n'(a),
$$

where we applied the max-log simplification using the log-probability, $\hat{P}_n(a) = \log P_n(a)$; the logarithmic domain eases implementation and will appear in all parametric derivations; we use again the mapping $n' = n'(n)$.

The LLRs, $\lambda_{n,k}$, are fed to the binary decoder which operates in abstraction of how they were calculated. This very separation of the operation of the decoder and the demodulator is the distinctive feature of the BICM which made it, de facto, a standard approach to design spectrally efficient transceivers. The obvious advantage is that both, the demodulator (comprising the phase-tracking) and the decoder may be designed and implemented independently of each other. Such an operation is characteristic of “one-shot” demodulators which are the most common solutions in the BICM transceivers.

On the other hand, the inherent simplicity of one-shot demodulators is a source of performance limitation which may be overcome by forcing the two-way exchange between the demodulator and the decoder. We also consider this option and we will assume that the decoder provides the demodulator with the prior LLRs, $\lambda^a_{n,k}$, for the bits $c_{n,k}$ from which the prior symbol probabilities $P^a_n(a) = \Pr \{x_n = a \}$ are calculated as

$$
\hat{P}^a_n(a) = \log P^a_n(a) \propto \sum_{k=1}^{m} \lambda^a_{n,k} \text{bit}_k(a)
$$

where $\text{bit}_k(a)$ is the value of the $k$-th bit in the label of the symbol $a$.

If we opt for such an iterative demodulation/phase-tracking, we will nevertheless preserve the LLR calculation in (11) which does not take advantage of the LLRs, $\lambda^a_{n,k}$.

### III. Phase Tracking

Our goal now is to find $P_n(a)$, $a \in \mathcal{A}$ exploiting the relationships between the involved random variables.

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2 It is possible, however. This approach is known to provides little gain in the case of the Gray-mapped constellations we use here [11] but yields gains with appropriately designed mapping [12].
As in [1], [7], we show in Fig. 1 a graph which captures these relationships: the squares represent the functions taking as arguments the variables which, in turn, are represented by the directly connected circles. In particular

\[ g_n(\theta_n, x_n) \propto f(y_n|x_n, \theta_n) \]

\[ \propto \exp\left( -\text{SNR} \left| y_n - x_n e^{j\theta_n} \right|^2 \right) \]

(13)

(14)

corresponds to the relationship defined in (1), where we use \( f(\cdot | \cdot) \) to denote the conditional probability density function (PDF); the model (2) yields

\[ q(\theta_n, \theta_{n+1}) \propto f(\theta_n | \theta_{n+1}) \propto f(\theta_{n+1} | \theta_n) \]

\[ = \omega(\theta_n - \theta_{n+1}), \]

(15)

(16)

with

\[ \omega(\theta) = \frac{1}{\sqrt{2\pi\sigma^2_w}} \sum_{k=-\infty}^{\infty} \exp \left( -\frac{(\theta - k2\pi)^2}{2\sigma^2_w} \right) \]

(17)
being a zero-mean, wrapped Gaussian distribution defined over any interval of length $2\pi$.

In Fig. 1 we also show a shaded rectangle labeled as “modulation/coding graph”; it contains a graph which describes the relationship $\{x_n\} \rightarrow \{s_n\} \rightarrow \{c_n\} \rightarrow \{b_n\}$, and its knowledge is used for the demodulation (to implement (11)) and for the decoding.

If the phase tracking has to be carried out and no decoding was not yet executed, the payload symbols $x_n$ are assumed to be i.i.d., with non-informative a priori distribution over the transmitted symbols, i.e., $P^a_n(a) \propto 1$. In a case we would like to perform the joint decoding/demodulation/phase tracking, as done, e.g., in [1], [4], [7], we use $P^a_n(a)$ obtained from the decoding results.

Thus, from the point of view of the phase tracking, the relationships induced by the modulation/coding graph are summarized by $P^a_n(a)$. As a direct consequence, the graph we deal with in Fig. 1 is a tree, i.e., contain no loops. Therefore, the efficient marginalization can be done exactly by the MP algorithm as follows [1], [6], [7]

$$P_n(a) \propto \int_{-\pi}^{\pi} \alpha_n(\theta)\beta_n(\theta)g_n(\theta, a) \, d\theta, \quad a \in A,$$

where

$$\alpha_n(\theta) \propto (\alpha_{n-1}(\theta)\gamma_{n-1}(\theta)) \ast \omega(\theta), \quad (19)$$

$$\beta_n(\theta) \propto (\beta_{n+1}(\theta)\gamma_{n+1}(\theta)) \ast \omega(\theta), \quad (20)$$

$$\gamma_n(\theta) \propto \sum_{a \in A} P^a_n(a)g_n(\theta, a), \quad (21)$$

are (proportional to) the marginal distributions of the phase $\theta_n$ conditioned, respectively, on $y_0, \ldots, y_{n-1}$, on $y_{n+1}, \ldots, y_N$, and on $y_n$. All the distributions of the phase are periodic and so is the convolution in (19) and (20).

$^3$That is,

$$\alpha_n(\theta) \propto f(\theta_n|y_0, \ldots, y_{n-1}) \quad (22)$$

$$\beta_n(\theta) \propto f(\theta_n|y_{n+1}, \ldots, y_N) \quad (23)$$

$$g_n(\theta, a) \propto f(\theta_n|y_n, a) \quad (24)$$

and thus the integrand in (18) is the posterior distribution of the phase

$$f(\theta_n|y, a) \propto \alpha_n(\theta)\beta_n(\theta)g_n(\theta, a) \quad (25)$$

while $P_n(a)$ is given by (10).
IV. PARAMETRIC MESSAGE PASSING

The operations (18)-(21) which allow us to calculate $P_n(a)$ may be implemented on the discretized versions of the distributions $\alpha_n(\theta)$, $\beta_n(\theta)$, and $\gamma_n(\theta)$ defined over the samples of $\theta \in (-\pi, \pi]$. In such a discretized message passing (DMP), the multiplications and additions are straightforward to implemented and the convolution is efficiently implemented via fast Fourier transform (FFT).

However, to ensure sufficient precision of the discretization, a large number of samples, $N_\theta$, may be needed. Since this implies a large computational complexity as well as a storage requirements, many previous works have gone in the direction of parametric representation of the involved periodic distributions. Then, instead of tracking the samples in DMP, we only need to track (a few) parameters which represent the distributions.

We thus should find a family of the distributions, $\mathcal{F}$, on which the operations defined in (18)-(21) are easily dealt with. In particular, we have to handle multiplication of the periodic functions as well as, the convolution in (19) and (20); the integration in (18), when done on the parametric forms of distributions, is usually straightforward.

In this work we consider a family of Tikhonov distributions which were already used, e.g., in [1], [6], [14]. Here, to simplify the notation we use its unnormalized version

$$\mathcal{T}(\theta; z) = e^{\Re[z e^{-j\theta}]}$$ \hspace{1cm} (26)

where $\Re[\cdot]$ denotes the real part and $z$ is a complex parameter whose phase is the circular mean of the underlying variable and the value of $|z|$ is, approximatively, a measure of precision [15].

The literature often used $N_\theta = 8M$ e.g., [1], [6], [7], [13] but finding the smallest $N_\theta$ which ensures the sufficient accuracy of the MP, (19)-(21), is difficult.

The general rule is that all the distributions involved in the MP should be represented without “significant” loss of accuracy so the sampling step should decrease when the spectral support of the distributions increases.

Then, by approximating the distributions by Gaussians, we know that their spectral support is proportional to their precision (inverse of the variance). Thus, observing that the precision of $g_\mu(\theta, a)$ increases with the SNR, see also [30]-[31], and the precision of $\omega(\theta)$ decreases with $\sigma^2_w$, we may conclude that the number of samples will grow when we increase the SNR or when we decrease the phase-noise level. These operational conditions are also required to transmit reliably high-order constellations. So, indeed, we should expect that the $N_\theta$ will grow with $M$ but we are not aware of any formal study backing up the heuristics $N_\theta = 8M$ (that turned out unnecessarily conservative in our case, see Footnote 12 in Sec. VI).

The normalized distribution is obtained as

$$\mathcal{T}_{\text{norm}}(\theta; z) = \frac{\mathcal{T}(\theta; z)}{\int_0^{2\pi} \mathcal{T}(\theta; z) d\theta} = \frac{e^{\Re[z e^{-j\theta}]} 2\pi I_0(|z|)}{2\pi I_0(|z|)}$$ \hspace{1cm} (27)

where $I_0(\cdot)$ is the zero-th order modified Bessel function.
The multiplications are trivial over $\mathcal{F}$

$$\mathcal{T}(\theta; z_1)\mathcal{T}(\theta; z_2) = \mathcal{T}(\theta; z_1 + z_2), \quad (28)$$

while the convolution with the Gaussian, appearing in (19) and (20), must be treated via approximations as done before [1, Appendix] and also explained using the concept of a circular moment matching (CMM) in Appendix C

$$\mathcal{T}(\theta; z) * \omega(\theta) \propto \mathcal{T}(\theta; \frac{z}{1 + \sigma_w^2 |z|}). \quad (29)$$

We also note that $g_n(\theta, a) \in \mathcal{F}$, so (13) can be rewritten as

$$g_n(\theta, a) \propto e^{-\text{SNR}|a|^2} \mathcal{T}(\theta; z_{g,n}(a)) \quad (30)$$

$$z_{g,n}(a) = 2\text{SNR} y_n a^*. \quad (31)$$

Instead of Tikhonov family we might opt for the wrapped Gaussians (17) but

- The family of circular Gaussian distributions is not closed under multiplications [15, Sec. II.B] so approximations are needed. This aspect is not always emphasized in the literature: in some works, e.g., in [7], the Gaussian distributions are used without explicit wrapping. Ignoring the circularity, it produces an apparent closeness under multiplication but may lead to interpretation errors when “locking” on the wrong phase, see example in [15, Fig. 1];

- We have to start using approximations already for $g_n(\theta, a)$; it can be done, of course, e.g., as in [7], but requires the introduction of additional layer of approximations before the MP is derived. This stands in contrast with a natural representation of $g_n(\theta, a)$ by a Tikhonov distribution in (30), and

- Comparing to the Tikhonov distributions, Gaussians are less well suited to represent the uniform or almost-uniform distributions of the phase (in Tikhonov family this is obtained by setting $z \approx 0$, while in Gaussian the variance must tend to infinity).

We thus see that, irrespectively of the adopted family $\mathcal{F}$, approximations are necessary (to implement multiplication if $\mathcal{F}$ is a Gaussian family, or to implement the convolution if $\mathcal{F}$ is a Tikhonov family) but this is not the main issue in the parametric MP. The principal difficulty stems from the presence of $\gamma_n(\theta)$ which is a mixture of Tikhonov distributions which we rewrite
as
\[
\gamma_n(\theta) \propto \sum_{a \in \mathcal{A}} e^{\eta_{g,n}(a)} T(\theta; z_{g,n}(a)),
\]
(32)
\[
\eta_{g,n}(a) = -\text{SNR} |a|^2 + \hat{P}_n^a(a),
\]
(33)
where we again use \( \hat{P}_n^a(a) = \log P_n^a(a) \) because the logarithmic representation simplifies the implementation.

There are essentially two approaches that have been used in the literature to deal with this issues and we compare them in the following.

**A. Out-of recursion projection: CBC algorithm**

Taking advantage of the fact that \( \gamma_n(\theta) \) is not obtained recursively, its approximation, \( \tilde{\gamma}_n(\theta) \) may be obtained before the recursive steps of MP (19)-(20) are executed. This may be seen as the following approximation of the right-hand side (r.h.s.) of (19):

\[
\alpha_n(\theta) \propto F\left[ (\alpha_{n-1}(\theta) \gamma_{n-1}(\theta)) \ast \omega(\theta) \right]
\]
(34)
\[
\approx F[\alpha_{n-1}(\theta) \gamma_{n-1}(\theta)] \ast \omega(\theta)
\]
(35)
\[
\approx (\alpha_{n-1}(\theta) F[\gamma_{n-1}(\theta)]) \ast \omega(\theta)
\]
(36)
\[
= (\alpha_{n-1}(\theta) \tilde{\gamma}_{n-1}(\theta)) \ast \omega(\theta)
\]
(37)

where,
\[
\tilde{\gamma}_n(\theta) = F[\gamma_n(\theta)] = T(\theta; z_{\tilde{\gamma},n})
\]
(38)
\[
= F\left[ \sum_{a \in \mathcal{A}} e^{\eta_{g,n}(a)} T(\theta; z_{g,n}(a)) \right],
\]
(39)
and, with a slight abuse of notation we use \( F[\cdot] \) to denote the operator which projects \( \gamma_n(\theta) \) onto the Tikhonov family \( F \). To pass from (34) to (35) we used (approximate) closeness of \( F \) with respect to convolution and this approximation holds well. On the other hand, to pass from (35) to (36) we used the approximation \( F[\alpha_{n-1}(\theta) \gamma_{n-1}(\theta)] \approx \alpha_{n-1}(\theta) F[\gamma_{n-1}(\theta)] \). Since \( \gamma_n(\theta) \) is a mixture of Tikhonov distributions, this approximation is very crude but it allows us to precalculate \( \tilde{\gamma}_n(\theta) \) before the MP recursion starts. This idea underlies the Colavolpe–Barbieri–
Caire (CBC) algorithm introduced in [1].

Since all the distributions belong then to \( F \), i.e.,
\[
\alpha_n(\theta) = \mathcal{T}(\theta; z_{\alpha,n}) \quad \text{and} \quad \beta_n(\theta) = \mathcal{T}(\theta; z_{\beta,n}),
\]
replacing \( \gamma_n(\theta) \) with \( \tilde{\gamma}_n(\theta) \) (as defined in (38)) and applying (28) and (29), we obtain the following parametric MP:

\[
z_{\alpha,0} = 0, \quad (40)
\]
\[
z_{\alpha,n} = \frac{z_{\alpha,n-1} + z_{\tilde{\gamma},n-1}}{1 + |z_{\alpha,n-1} + z_{\tilde{\gamma},n-1}|^{2}}, \quad n = 1, \ldots, N - 1 \quad (41)
\]
\[
z_{\beta,N} = 0, \quad (42)
\]
\[
z_{\beta,n} = \frac{z_{\beta,n+1} + z_{\tilde{\gamma},n+1}}{1 + |z_{\beta,n+1} + z_{\tilde{\gamma},n+1}|^{2}}, \quad n = N - 1, \ldots, 1, \quad (43)
\]

which is exactly equivalent to the CBC algorithm defined in [1, Eq. (36)-(37)] and is provided for completeness and as a starting point for the discussion.

Using (28) and (30)-(31) in (18), the “extrinsic” symbol probabilities are also calculated as [1, Eq. (35)]

\[
P_n(a) \propto \int_{0}^{2\pi} \mathcal{T}(\theta; z_{\Sigma,n}(a)) e^{-\text{SNR}|a|^2} d\theta, \quad (44)
\]
\[
\hat{P}_n(a) = \log P_n(a) \quad (45)
\]
\[
\propto -\text{SNR}|a|^2 + \hat{I}_0(|z_{\Sigma,n}(a)|), \quad (46)
\]

where

\[
z_{\Sigma,n}(a) = z_{\alpha,n} + z_{\beta,n} + z_{g,n}(a), \quad (47)
\]

and we avoid numerical issues due to the exponential grows of \( I_0(\cdot) \) by using its log-version

\[
\hat{I}_0(x) = \log I_0(x) = x + \Delta(x), \quad (48)
\]

with \( \Delta(x) \) being the log-domain corrective factor.\(^6\)

Further, the log-probability \( \hat{P}_n(a) \) should be used in (11).

Due to its simplicity the CBC algorithm (40)-(43), should be treated as a “canonical” solution

\(^6\)The approximation \( \Delta(x) \approx -\frac{1}{2} \log(2\pi x)|x|^{1/2} \) is tight for large \( x \) [6, Eq. (96)] [8, Eq. (18)]; we truncate it for small arguments, \( x < \frac{1}{2\pi} \), because we know that \( \Delta(0) = 0 \). For better accuracy, \( \Delta(x) \) may be implemented via lookup table. It is also possible to simply set \( \Delta(x) \approx 0. \)
to the problem of phase tracking: it is explicit and requires no fine-tuning. This, and its historical
precedence, explain the considerable attention it received up to now.

The remaining issue is the projection in (38); [1] proposed to approximate \( \gamma_n(\theta) \) with \( \tilde{\gamma}_n(\theta) \in \mathcal{F} \) by, first matching a Gaussian to \( f(y_n|\theta_n, a) \propto g_n(\theta, a) \) from which a Tikhonov distribution was derived as [1, Sec. IV.B]

\[
\tilde{\gamma}_n(\theta) = \mathcal{T}(\theta; z_{\gamma,n}) \tag{49}
\]

\[
z_{\gamma,n} = \begin{cases} 
    z_{g,n}(x_n) & \text{if } n \in \mathbb{N}_{\text{pilots}} \\
    \frac{2 \nu_n m^*}{\lambda_0 + \nu_n} & \text{if } n \in \mathbb{N}_{\text{payload}}
\end{cases} \tag{50}
\]

where

\[
m_n = \sum_{a \in \mathcal{A}} a P^a_n(a), \quad \nu_n = \sum_{a \in \mathcal{A}} |a|^2 P^a_n(a) - |m_n|^2, \tag{51}
\]

are the mean and the variance of \( x_n \).

The approximation is needed only for the payload positions, \( n \in \mathbb{N}_{\text{payload}} \). For the pilots \( x_n, n \in \mathbb{N}_{\text{pilots}} \), we have \( \gamma_n(\theta) = g_n(\theta, x_n) \) so we can use directly the parameter of the Tikhonov distribution \( z_{\gamma,n} = z_{g,n}(x_n) \).

This Gaussian approximation (GA) approach was often reused in the literature, e.g., [4], [8], [14] as the integral part of the CBC algorithm.

On the other hand, recognizing that the distribution (32) is circular, instead of the GA we may apply the CMM. It relies on minimization of the Kullback-Leibler (KL) distance between \( \tilde{\gamma}_n(\theta) \) and the mixture \( \gamma_n(\theta) \), and changes the way the parameter \( z_{\gamma,n} \) is calculated, which we express as follows

\[
z_{\gamma,n} = \begin{cases} 
    z_{g,n}(x_n) & \text{if } n \in \mathbb{N}_{\text{pilots}} \\
    \text{CMM}\left[\{\eta_{g,n}(a), z_{g,n}(a)\}_{a \in \mathcal{A}}\right] & \text{if } n \in \mathbb{N}_{\text{payload}}
\end{cases} \tag{52}
\]

where the dependence of \( z_{\gamma,n} \) on \( \eta_{g,n}(a) \) and \( z_{g,n}(a) \), indexed by \( a \in \mathcal{A} \) is defined in closed-form by a function \( \text{CMM}[\cdot] \) shown in Appendix \([A]\).

Although the CMM was already used in the context of the phase tracking by [6], we are not aware of it being applied as a part of the CBC algorithm. And while the implementation
of the function CMM[:,] is slightly less straightforward than the GA shown in (49)-(51), it is worthwhile to evaluate the advantage of the former.

An important observation, already made e.g., in [6, Sec. I] or [8, Sec. IV], is that, using a non-informative prior \( P^a(a) \propto 0 \), the approximation \( \tilde{\gamma}_n(\theta) \) is also non-informative for the payload symbols; that is, \( z_{\tilde{\gamma},n} = 0, n \in \mathbb{N}_{\text{payload}} \). Then, the only useful information is obtained from the non-zero \( z_{\tilde{\gamma},n} \) (i.e., from the pilots).

Thus, the MP in (40)-(43) may be then seen as a *pilot-only* based recursive estimation of the phase. In fact, it is independent of the form of the constellation \( A \) used to modulate the payload symbols and this is why, in order to exploit the knowledge of \( A \), the CBC phase tracking must be placed in the “decoding-loop”. Known also as the joint phase tracking and decoding, it consists in alternate execution of i) the MP (40)-(43) and ii) the demodulation/decoding; the latter, providing an informative prior \( \hat{P}_n^a(a) \) on the payload symbols, improves the phase tracking, see e.g., [1], [4], [14].

**B. Intra-recursion projection: SKR algorithm**

As we said, the disadvantage of the pre-MP projection from Sec. IV-A is that, without the decoder’s feedback, the structure of the constellation \( A \) is ignored. In order to exploit the form of \( A \), Shayovitz and Raphaeli [6], as well as, Kreimer and Raphaeli [7] proposed to perform the projection on \( F \) after the multiplications in (19)-(20) are executed. That is, the projections are carried out inside the MP recursion.

From the perspective we adopted, the resulting Shayovitz–Kreimer–Raphaeli (SKR) algorithm is obtained implementing the approximations (35)

\[
\begin{align*}
\alpha_n(\theta) &\propto F[\alpha_{n-1}(\theta)\gamma_{n-1}(\theta)] \ast \omega(\theta) \quad (53) \\
&\approx \tilde{\alpha}_{n-1}(\theta) \ast \omega(\theta), \\
\beta_n(\theta) &\propto F[\beta_{n+1}(\theta)\gamma_{n+1}(\theta)] \ast \omega(\theta) \quad (55) \\
&\approx \tilde{\beta}_{n+1}(\theta) \ast \omega(\theta).
\end{align*}
\]

7Approximations of non-linear functions are involved, see Appendix A and Appendix B

8If the GA (49)-(51), is used, this is true, if the constellation \( A \) is zero mean, which is obvious from (51). On the other hand, if the CMM approach is used, this is true if the constellations \( A \) may be decomposed into constant modulus, zero-mean sub-constellations; the demonstration is easy and omitted for sake of space. For the most popular constellations such as \( M \)-QAM, both conditions hold.
where we define the “auxiliary” distributions obtained via projections

\[
\tilde{\alpha}_n(\theta) = \mathcal{T}(\theta; z_{\tilde{\alpha},n}) = \mathcal{F} \left[ \sum_{a \in A} P_n^a(a) \alpha_n(\theta) g_n(\theta, a) \right] \\
(57)
\]

\[
z_{\tilde{\alpha},n} = \text{CMM} \left[ \{ \eta_{g,n}(a), z_{\alpha,n} + z_{g,n}(a) \} \}_{a \in A} \right], \quad (58)
\]

and

\[
\tilde{\beta}_n(\theta) = \mathcal{T}(\theta; z_{\tilde{\beta},n}) = \mathcal{F} \left[ \sum_{a \in A} P_n^a(a) \beta_n(\theta) g_n(\theta, a) \right], \quad (59)
\]

\[
z_{\tilde{\beta},n} = \text{CMM} \left[ \{ \eta_{g,n}(a), z_{\beta,n} + z_{g,n}(a) \} \}_{a \in A} \right]. \quad (60)
\]

Using (29) in (53) and (55) we obtain the following parametric MP:

\[
z_{\alpha,0} = 0, \quad z_{\tilde{\alpha},0} = z_{g,0}(x_0) \quad (61)
\]

\[
z_{\alpha,n} = \frac{z_{\tilde{\alpha},n-1}}{1 + |z_{\tilde{\alpha},n-1}| \sigma_w^2}, \quad n = 1, \ldots, N - 1 \quad (62)
\]

\[
z_{\tilde{\alpha},n} = \text{CMM} \left[ \{ \eta_{g,n}(a), z_{\alpha,n} + z_{g,n}(a) \} \}_{a \in A} \right] \quad (63)
\]

\[
z_{\beta,N} = 0, \quad z_{\tilde{\beta},N} = z_{g,N}(x_N) \quad (64)
\]

\[
z_{\beta,n} = \frac{z_{\tilde{\beta},n+1}}{1 + |z_{\tilde{\beta},n+1}| \sigma_w^2}, \quad n = N - 1, \ldots, 1 \quad (65)
\]

\[
z_{\tilde{\beta},n} = \text{CMM} \left[ \{ \eta_{g,n}(a), z_{\beta,n} + z_{g,n}(a) \} \}_{a \in A} \right]. \quad (66)
\]

Comparing to the CBC algorithm from Sec. [IV-A], the Shayovitz–Kreimer–Raphaeli (SKR) algorithms invokes the CMM function twice for each time \( n \) and this, even if no feedback is obtained from the decoder. This increases the computational burden comparing to the CBC algorithm, which invokes the CMM function only once per \( n \) (and does not need it at all in the case the decoder’s feedback is not available). The hope is that, introducing approximations (projection) inside the MP recursion will minimize the approximation errors.

Again, we can apply the SKR algorithm iteratively, using the decoder’s feedback which is taken into account via \( P_n^a(a) \).
C. SKR and [6], [7]

The SKR algorithm should be seen as a “canonical” representation of the algorithms shown in [6], [7]: it relies on the Tikhonov distribution as suggested by [6], while the projections defined in (53) and (55) are the essence of the algorithm proposed in [7].

On the other hand, the choice of the family $\mathcal{F}$ in [7] and in [6] leads to a certain ambiguity of implementation of the algorithms. In particular,

- In [6]: because $\mathcal{F}$ which is a mixture of Tikhonov distributions, the projection requires clustering of the elements of the mixture. This is computationally complex procedure even for moderate-size problems.\(^9\) The heuristics are thus required and depend very much on the implementation details (such as, for example, enumeration order of the elements of the mixture or the thresholds used in the clustering). Further, despite the heuristics, a non-negligible complexity is required by the algorithms proposed in [6]. In the follow-up work [7] (by the same author as in [6]) the mixture was replaced by single (Gaussian) distribution.\(^10\)
- In [7]: using the Gaussian family $\mathcal{F}$, the approximations must start already when representing $g_{n}(\theta, a)$. In fact, [7] describes in details different strategies for such approximations but there is no unique answer to how this should be done.

In fact, the SKR algorithm is a particular version of the algorithm shown in [6] (obtained if we force the family $\mathcal{F}$ to contain one Tikhonov distribution, which may be seen as a “degenerate” mixture).

However, implementing the SKR algorithm using the CMM projection formulas derived in [6, Appendix A] leads to significant error, see Fig. 4. This is because the last approximation step, expressed in [6, Eq. (102)], assumes that all terms of the mixture have a similar circular mean and this assumption does not materialize in practice. On the other hand, implementing the approximation step defined in [6, Eq. (101)] yields the results similar to those obtained by the SKR algorithm we defined in this work.

\(^9\)This is because even if the order of the reduced mixture is fixed to $L < M$ clusters, there is a formidably large number of possible assignments of $M$ elements into $L$ groups and it is usually unfeasible to test all possibilities.

\(^10\)In fact, [7] says about [6] “However, good results for high order constellations require much higher complexity and were not demonstrated.” This suggest that the excessive complexity is the very reason why the author of [7] refrained from showing the results of his own algorithm developed in [6] even for $M = 16$. 
This observation is anecdotal but confirms the importance of testing the approximations in the projection formulas; this is the goal of Appendix A where the derivation are approximations-free.

V. EXPECTATION PROPAGATION: SELF-ITERATIONS IN THE PHASE-TRACKING

As we will see in Sec. VI, the performance of the one-shot DMP indicates that satisfactory solutions may be obtained without relying on the decoders’ feedback. Therefore, our goal will be to develop a reliable parametric one-shot phase tracking by exploiting the form of the constellation $\mathcal{A}$.

To explain why this can be done we note that, although the phase tracking MP is defined on the tree (graph) that is known to yield the optimal solution in one run (backward-forward processing), this is only true for the DMP which tracks the entire (discretized) distributions. On the other hand, the MP in the CBC and the SKR algorithm is based on approximated distributions $\tilde{\gamma}_n(\theta)$ (in CBC) or $\tilde{\alpha}_n(\theta)$ and $\tilde{\beta}_n(\theta)$ (in SKR), therefore, a guarantee of optimality does not exist.

To remedy the resulting sub-optimality, we will improve the approximations iteratively; this will be done with “self-iterations”, that is, without any help from the decoder: the priors $\hat{P}_a(a)$ will not change during the self-iterations.

We follow here the EP idea [9, Ch. 3.2] which addresses the very problem of using approximate distributions in the MP. We will apply the projection $\mathcal{F}[:]$ many times (iteratively) taking into account the results obtained in previous iterations. Since most of the resulting operations are almost identical to those we already defined in the CBC and the SKR algorithms, we will use the parenthesized superscript $(i)$ to denote the variables/functions obtained in the $i$-th iteration, where $i = 1, \ldots, I_{ep}$ and $I_{ep}$ is the maximum number of EP iterations.

A. Expectation propagation in CBC algorithm

The EP applied to our problem relies on the following idea: in the iteration $i$, to find the approximation $\tilde{\gamma}_n^{(i)}(\theta)$ we will rely on the distributions $\tilde{\alpha}_n^{(i-1)}(\theta)$ and $\tilde{\beta}_n^{(i-1)}(\theta)$ calculated in the previous iteration. This is done by exploiting the posterior distributions of $\theta_n$ after the iteration $i-1$ which is a mixture given by

$$f_n^{(i-1)}(\theta) \propto \gamma_n(\theta) \left( \alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta) \right).$$

\hspace{1cm} (67)
Our objective is to find $\tilde{\gamma}^{(i)}(\theta) \in \mathcal{F}$ which generates the approximate posterior

$$f_n^{(i-1,i)}(\theta) \propto \tilde{\gamma}^{(i)}(\theta) \left( \alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta) \right)$$  \hspace{1cm} (68)

close to $f_n^{(i-1)}(\theta)$, that is, we require

$$\tilde{\gamma}^{(i)}(\theta) \left( \alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta) \right) \approx \gamma_n(\theta) \left( \alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta) \right).$$  \hspace{1cm} (69)

To satisfy (69) we apply the projection, $\mathcal{F}[\cdot]$ to its both sides. Since the left-hand side (l.h.s.) contains the three terms from the family $\mathcal{F}$, it is not affected by the projection (remember, the product of the Tikhonov distributions is also a Tikhonov distribution), we obtain

$$\tilde{\gamma}^{(i)}(\theta) \propto \mathcal{F}\left[ \gamma_n(\theta) \left( \alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta) \right) \right] \frac{\alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta)}{\alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta)}.$$  \hspace{1cm} (70)

Of course, for $i = 1$, we deal with non-informative distributions $\alpha^{(0)}(\theta) \propto 1$ and $\beta^{(0)}(\theta) \propto 1$, which reduces (70) to $\tilde{\gamma}^{(1)}(\theta) = \mathcal{F}[\gamma_n(\theta)]$; this is what was done in Sec. IV, i.e., $\tilde{\gamma}^{(1)}(\theta)$ is the same as $\tilde{\gamma}_n(\theta)$ obtained in (38)\(^{11}\).

For $i > 1$, the informative distributions, $\alpha_n^{(i-1)}(\theta)$ and $\beta_n^{(i-1)}(\theta)$ are available so we have to calculate the r.h.s. of (69)

$$\gamma_n(\theta) \left( \alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta) \right) = \sum_{a \in \mathcal{A}} e^{\eta_{g,a}(a)} \mathcal{T}(\theta; z^{(i-1)}_{\Sigma,n}(a)), $$  \hspace{1cm} (71)

where

$$z^{(i-1)}_{\Sigma,n}(a) = z^{(i-1)}_{\alpha,n} + z^{(i-1)}_{\beta,n} + z_{g,n}(a)$$  \hspace{1cm} (72)

and we see that $z^{(1)}_{\Sigma,n}(a)$ is the same as (47).

\(^{11}\)Similarly, if we want to apply the projection operator, $\mathcal{F}[\cdot]$, to each of the terms under multiplication in the numerator in (70), the result will be the same as the original CBC algorithm: the projection $\mathcal{F}[\alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta)] = \alpha_n^{(i-1)}(\theta) \beta_n^{(i-1)}(\theta)$ will cancel out with the denominator and iterative improvement will not be possible. This “transparency” to the presence of $\alpha_n(\theta)$ and $\beta_n(\theta)$ is thus nothing but the conventional projection characteristic of the CBC algorithm and, although it may provide satisfactory results in other context, e.g., [16], here it is not useful.
Using (71) in the numerator of (70) yields

\[ \tilde{\gamma}_n(i) = T(\theta; z_{\tilde{\gamma}_n}) \]  
(73)

\[ \propto F[ \sum_{a \in A} e^{\eta g,n(a)} T(\theta; z_{\tilde{\gamma}_n}^{(i-1)}(a)) ] / T(\theta; z_{\alpha,n} + z_{\beta,n}) \]  
(74)

\[ = \frac{T(\theta; z_{\tilde{\gamma}_n})}{T(\theta; z_{\alpha,n} + z_{\beta,n})} \]  
(75)

where

\[ z_{\tilde{\gamma}_n} = \text{CMM}\left\{ \eta g,n(a), z_{\tilde{\gamma}_n}^{(i-1)}(a) \right\}_{a \in A} \]  
(76)

and thus

\[ z_{\tilde{\gamma}_n}^{(i)} = z_{\tilde{\gamma}_n}^{(i)} - (z_{\alpha,n}^{(i-1)} + z_{\beta,n}^{(i-1)}) \]  
(77)

1) Pitfalls and solutions: The elegant formulation of the EP is indeed appealing but occasionally runs into difficulty due to the division of the distributions in (75), or equivalently, due to subtraction of the coefficients in (77). The existence of a similar problem has been acknowledged in [9, Sec. 3.2] in the context of the Gaussian EP, where the division of the distributions may yield a “Gaussian” with negative variance, see also [17, Sec. IV.B] [18, Sec. IV.A]. Such results are usually uninterpretable so a workaround is needed.

In the context of the EP based on Tikhonov distributions \( T(\theta; z) \) which are defined by complex parameters \( z \), the purely numerical issue of invalid parameters (such as a negative variance in the Gaussian case) is avoided but the problem remains. To understand intuitively its source and devise a solution, we may look at the scenario where the problems materialize.

Example 1 (Pitfalls of EP): Assume that

- The circular means of the approximate posterior distribution, \( T(\theta; z_{\text{post},n}^{(i)}) \) and of the “extrinsic” distribution (defined by the product \( \alpha_n^{(i-1)}(\theta)\beta_n^{(i-1)}(\theta) \)) coincide, that is, \( \angle z_{\text{post},n}^{(i)} = \angle (z_{\alpha,n}^{(i-1)} + z_{\beta,n}^{(i-1)}) \); (remember, the phase \( \angle z_{\text{post},n}^{(i)} \) defines the circular mean); and that

- The variance of the posterior distribution is larger than the variance of the extrinsic distribution, i.e., \( |z_{\text{post},n}^{(i)}| < |z_{\alpha,n}^{(i-1)} + z_{\beta,n}^{(i-1)}| \); this may occur because the distribution in the argument of \( F[\cdot] \) in (74) is obtained multiplying the extrinsic distribution \( T(\theta; z_{\alpha,n}^{(i-1)} + z_{\beta,n}^{(i-1)}) \) with...
a mixture $\gamma_n(\theta)$; the variance of the resulting mixture (and thus of its projection result as well) may be larger than the variance of the extrinsic distribution.

Then, carrying out the subtraction in (77) we will obtain

$$z_{\tilde{\gamma},n}^{(i)} = (|z_{\text{post},n}^{(i)}| - |z_{\alpha,n}^{(i-1)} + z_{\beta,n}^{(i-1)}|) e^{j\angle z_{\text{post},n}^{(i)}}$$

(78)

$$= |z_{\tilde{\gamma},n}^{(i-1)}| e^{j(\angle z_{\text{post},n}^{(i)} + \pi)}.$$  

(79)

That is, the circular mean of the new distribution $\mathcal{T}(\theta; z_{\tilde{\gamma},n}^{(i)})$ will be in disagreement (by the largest possible value of $\pi$) with the mean of the posterior and extrinsic distributions obtained from the previous iteration. Such a result, being an artefact of the way the EP is defined [9, Sec. 3.2] is clearly counterintuitive and simply wrong.

This problem, characteristic of the EP will affect first $\tilde{\gamma}_n(\theta)$ and, propagating via the MP, will have detrimental affects on $\alpha_n(\theta)$ and $\beta_n(\theta)$.

Heuristics were devised for the Gaussian EP where the problem is clearly identified, i.e., the variance of the distributions after division becomes negative. For example, [9, Sec. 3.2] constrains the variance to be positive, while other proposed to identify the problematic cases (i.e., the negative variance) and then, if necessary i) eliminate the division of the distributions [17, Sec. IV.B], which here would mean $z_{\tilde{\gamma},n}^{(i)} = z_{\text{post},n}^{(i)}$, or ii) remove the update [18, Sec. IV.A], i.e., $z_{\tilde{\gamma},n}^{(i)} = z_{\tilde{\gamma},n}^{(i-1)}$.

On the other hand, instead of testing for the compliance with our prior requirements (that are not necessarily obvious to define), we might “smooth” the obtained parameters via recursive filter [18, Sec. IV.A], which would mean replacing (77) with

$$z_{\tilde{\gamma},n}^{(i)} = \zeta \left( z_{\text{post},n}^{(i)} - z_{\alpha,n}^{(i-1)} - z_{\beta,n}^{(i-1)} \right) + (1 - \zeta) z_{\tilde{\gamma},n}^{(i-1)}$$

(80)

where the smoothing parameter, $\zeta < 1$, must be chosen heuristically to strike a balance between the new solution and the history accumulated in the previous estimate. In this way we avoid rapid changes in the parameters $z_{\tilde{\gamma},n}^{(i)}$, regularizing the final solution. Note that the smoothing is not necessary in the first iteration, $i = 1$, which is based solely on the information obtained from the pilots (and from the decoder if we allows for it).
2) **Summary of the algorithm:** The proposed CBC+EP phase tracking algorithm is now summarized as Algorithm 1 where we integrated the notation so that the CBC algorithm is naturally the first iteration \((i = 1)\) of the CBC+EP algorithm, i.e., \(z_{\Sigma,n}^{(0)}(a) = z_{g,n}(a)\). This first iteration is also distinct: the smoothing \((80)\) is not applied for \(i = 1\), see lines 15-20 of the algorithm. Note that the notation with the iteration index \((i)\) may be removed (and the in-place calculation carried out) but we kept it for compatibility with the equations in the paper.

Last but not least, and at the risk of stating the obvious, we want to emphasize what the CBC+EP algorithm is not doing. Namely, it is not reusing the output “extrinsic” log-probabilities \(\hat{P}_n(a)\) as if they were newly calculated prior log-probabilities \(\hat{P}_n^a(a)\). This is clearly seen in the description of the algorithm: the prior probability \(\hat{P}_n^a(a)\) affects the weighting factors \(\eta_{g,n}(a)\) in line 7 which remain unaltered throughout the CBC+EP iterations; the CBC+EP algorithm rather changes the parameters of the Tikhnov distributions \(z_{\Sigma,n}^{(i)}(a)\).

**B. Expectation propagation in SKR algorithm**

The reasoning behind the EP we used to obtain the CBC+EP algorithm will be now applied to enhance the SKR algorithm. Armed with the previous considerations, we already know how to proceed: in the iteration \(i\) we first project on \(F\) the mixture which represents the posterior distribution which is calculated using results obtained in the iteration \(i - 1\). The latter are next removed from the projection results.

The difference with the previous section is that we are not seeking to improve \(\gamma_n(\theta)\) as we did in the case of the CBC algorithm, but rather we want to obtain improved estimates of \(\hat{\alpha}_n(\theta)\) and \(\hat{\beta}_n(\theta)\). Thus, we have to transform \((57)-(60)\) as follows:

\[
\hat{\alpha}_n^{(i)}(\theta) = \frac{\mathcal{F} \left[ \sum_{a \in A} P_n^a(a) \alpha_n^{(i)}(\theta) g_n(\theta, a) \beta_n^{(i-1)}(\theta) \right]}{\beta_n^{(i-1)}(\theta)}, \tag{81}
\]

and

\[
\hat{\beta}_n^{(i)}(\theta) = \frac{\mathcal{F} \left[ \sum_{a \in A} P_n^a(a) \beta_n^{(i)}(\theta) g_n(\theta, a) \alpha_n^{(i-1)}(\theta) \right]}{\alpha_n^{(i-1)}(\theta)}, \tag{82}
\]

where again we used indexing with \((i)\) to denote the results from the iteration \(i\).

The resulting SKR+EP algorithm is obtained applying this iteration-indexing in \((61)-(66)\) and...
Algorithm 1 CBC+EP phase tracking

1: Inputs:
2: \( y_n, n \in \{0, \ldots, N \} \) \hspace{1cm} \triangleright \text{Received signal}
3: \( x_n, n \in \mathbb{N}_{\text{pilots}} \) \hspace{1cm} \triangleright \text{Pilot symbols}
4: \( \hat{P}_n^a(a) \) \hspace{1cm} \triangleright \text{Log-probabilities from the decoder}
5: Initialization:
6: \( z_{(i)}^{n,n}, n \in \mathbb{N}_{\text{pilots}}, i = 1, \ldots, I_{\text{ep}} \)
7: \( \eta_{g,n}(a) \leftarrow -\text{SNR} \left| a \right|^2 + \hat{P}_n^a(a) \), \( a \in \mathcal{A}, n \in \mathbb{N}_{\text{payload}} \)
8: \( z_{g,n}(a) \leftarrow 2\text{SNR} y_n a^*, \ a \in \mathcal{A}, n \in \mathbb{N}_{\text{payload}} \)
9: \( z_{(0)}^{(0)}(a) \leftarrow z_{g,n}(a), \ n \in \mathbb{N}_{\text{payload}} \)
10: EP iterations:
11: for \( i \leftarrow 1, \ldots, I_{\text{ep}} \) do
12: Pre-MP : Finds \( \tilde{\gamma}^{(i)}_{n}(\theta) \)
13: for \( n \leftarrow \mathbb{N}_{\text{payload}} \) do
14: \( z_{(i)}^{n,n} \leftarrow \text{CMM} \left[ \eta_{g,n}(a), \{ z_{(i-1)}^{(i-1)}(a) \} \right]_{a \in \mathcal{A}} \)
15: if \( i = 1 \) then
16: \( z_{(i)}^{n,n} \leftarrow z_{\text{post},n}^{(i)} \)
17: else
18: \( z_{(i)}^{n,n} \leftarrow \zeta \left( z_{\text{post},n}^{(i)} - z_{(i-1)}^{\alpha,n} - z_{(i-1)}^{\beta,n} \right) + (1 - \zeta) z_{(i-1)}^{(i-1)} \)
19: end if
20: end for
21: MP recursive calculation:
22: for \( n \leftarrow 1, \ldots, N \) do
23: \( z_{(i)}^{\alpha,0} \leftarrow 0 \)
24: \( z_{(i)}^{(i)} \leftarrow \frac{z_{(i-1)}^{\alpha,n} + z_{(i-1)}^{\beta,n}}{1 + |z_{(i-1)}^{\alpha,n} + z_{(i-1)}^{\beta,n}|^2} \sigma_w^2 \)
25: end for
26: \( z_{(i)}^{\beta,N} \leftarrow 0 \)
27: for \( n \leftarrow N - 1, \ldots, 0 \) do
28: \( z_{(i)}^{(i)} \leftarrow \frac{z_{(i-1)}^{\beta,n+1} + z_{(i-1)}^{\beta,n}}{1 + |z_{(i-1)}^{\beta,n+1} + z_{(i-1)}^{\beta,n}|^2} \sigma_w^2 \)
29: end for
30: for \( n \leftarrow \mathbb{N}_{\text{payload}} \) do
31: \( z_{(i)}_{\Sigma,n}^{(i)}(a) \leftarrow z_{g,n}(a) + z_{(i)}^{(i)} + z_{(i)}^{(i)} \)
32: end for
33: Output: log-probabilities to be used in (11)
34: for \( n \leftarrow \mathbb{N}_{\text{payload}} \) do
35: \( \hat{P}_n^a(a) \leftarrow -\text{SNR} \left| a \right|^2 + \hat{I}_0 \left( |z_{(i_{\text{ep}})}_{\Sigma,n}^{(i_{\text{ep}})}(a)| \right) \)
36: end for
using \((81)-(82)\)

\[
\begin{align*}
\hat{z}_{\alpha,0}^{(i)} &= 0, \quad \hat{z}_{\beta,0}^{(i)} = z_{g,0}(x_0), \\
\hat{z}_{\alpha,n}^{(i)} &= \zeta \frac{\hat{z}_{\alpha,n-1}^{(i)}}{1 + |\hat{z}_{\alpha,n-1}^{(i)}| \sigma_w^2} + (1 - \zeta) \hat{z}_{\alpha,n}^{(i-1)}, \\
\hat{z}_{\beta,n}^{(i)} &= \zeta \frac{\hat{z}_{\beta,n+1}^{(i)}}{1 + |\hat{z}_{\beta,n+1}^{(i)}| \sigma_w^2} + (1 - \zeta) \hat{z}_{\beta,n}^{(i-1)}, \\
\hat{z}_{\alpha,n}^{(i)} &= \text{CMM}\left[\eta_{g,n}(a), \hat{z}_{\alpha,n}^{(i)} + z_{g,n}(a) + \hat{z}_{\beta,n}^{(i-1)}\right]_{a \in A}, \\
\hat{z}_{\beta,n}^{(i)} &= \text{CMM}\left[\eta_{g,n}(a), \hat{z}_{\beta,n}^{(i)} + z_{g,n}(a) + \hat{z}_{\alpha,n}^{(i-1)}\right]_{a \in A},
\end{align*}
\]  

\((83)\) \((84)\) \((85)\) \((86)\) \((87)\) \((88)\)

where, the division by \(\beta_{n-1}(\theta)\) and by \(\alpha_{n-1}(\theta)\) appearing, respectively in \((81)\) and \((82)\) translates into subtraction of \(z_{\beta,n}^{(i-1)}\) and of \(z_{\alpha,n}^{(i-1)}\) given, respectively by \((85)\) and \((88)\).

In \((84)\) and \((87)\) we also introduce the smoothing via weighting with \(\zeta\) akin to the operation shown in \((80)\). For the sake of space we refrain from explicating all operations as we did for the CBC+EP algorithm.

\[C. \text{ Hybrid algorithm: CBC+SKR}\]

We note that both algorithms, CBC+EP as well as SKR+EP, calculate in each iteration the distributions \(\alpha_{n}^{(i)}(\theta)\) and \(\beta_{n}^{(i)}(\theta)\) which are then used in the next iteration according to the corresponding formulas. However, in the iteration \(i\), we place no restriction on how the distributions \(\alpha_{n}^{(i-1)}(\theta)\) or \(\beta_{n}^{(i-1)}(\theta)\) were obtained. This suggest that we might reuse the results obtained from different algorithms.

In particular, we might carry out the first iteration via the CBC algorithm and then, in the second iteration use the obtained estimates of \(\alpha_{n}^{(1)}(\theta)\) or \(\beta_{n}^{(1)}(\theta)\) in the SKR algorithm. We denote such a hybrid approach by CBC+SKR.

Noting that the CBC algorithm is merely the pilot-based phase estimation, the CBC+SKR algorithm provides an interesting insight into the formal use of pilots in two-stage phase esti-
In fact, a similar two-stage approach can be found in the literature. For example, in [7], the second stage consists of a regular parametric MP implemented using a Gaussian family $F$. But, in order to approximate $g_n(\theta, a)$ with a Gaussian distribution, we need to know which values of $\theta$ are the most relevant. This is done in the first stage: pilots are used to obtain “raw” estimates of the phase $\hat{\theta}_n$, and the approximations are found for $\theta \approx \hat{\theta}_n$.

Thus, in [7], the pilot-based phase estimation (first stage) is related to the approximation problems rather than to the MP formulation itself (but, of course, better approximation improves the results of the MP). This is different from the CBC+SKR algorithm where we are not concerned at all with approximation issues but rather exploit the probabilistic model of the pilot-based phase estimates (obtained by the CBC algorithm) to modify the MP in the SKR algorithm.

We show an example of the results obtained using CBC+SKR algorithm in Sec. [VI].

D. Comments on complexity

The complexity of decoding and demodulation (LLRs calculation) is considered to be common in all algorithms so we are concerned only with the complexity of the phase tracking algorithms per se, where the main complexity resides in the CMM and more precisely in the calculation of the circular moment as defined by (98) and the operation (97): we need to carry out $M$ non-linear operations (function $B(x)/x$), $M$ absolute value calculations, $M$ multiplications, $M$ complex multiplication and $M$ additions.

Thus the “CMM complexity” is a unit may be the basis for comparison.

The CBC algorithm is notably simple because we do not need the CMM at all; remember, the projection without decoder’s feedback yields $z_{\tilde{\gamma},n} = 0$ (for payload) and $z_{\tilde{\gamma},n} = z_{g,n}(a)$ (for pilots). So its CMM complexity is zero.

The algorithm CBC+EP($I_{ep}$) has the complexity of $I_{ep} - 1$ (of CMM complexity) units, the SKR+EP($I_{ep}$) algorithm has the complexity equal to $2I_{ep}$ units, while the hybrid algorithm CBC+SKR displays the complexity of two (2) units.

In this perspective, the hybrid algorithm, CBC+SKR, has the same complexity as the algorithm SKR. This is not exactly true because we need one addition (e.g., to calculate $z_{\alpha,n}^{(i)} + z_{\beta,n}^{(i-1)}$ before
running the CMM in (85)) and another one (e.g., to subtract $z_{\beta,n}^{(i-1)}$ after running the CMM in (85)), but this two additions can be neglected comparing with the unit cost of the CMM.

VI. Numerical Examples

All examples use $M$-QAM constellation with Gray mapping [10, Sec. 2.5.2]; the proprietary LDPC encoder with the rate $r = \frac{7}{8}$ produces the block of $N_c = 4032$ bits. We consider two modulation/phase-noise setups following [7, Fig. 8] i) 16-QAM with $\sigma_w = 0.1$ rad $\approx 5.7$ deg, and ii) 64-QAM with $\sigma_w = 0.05$ rad $\approx 2.9$ deg. The pilot symbols are pseudo-randomly drawn from a 4-QAM constellation; the pilot spacing is set to $L \in \{17, 25\}$ which allows us to place $L - 1 \in \{16, 24\}$ payload symbols between pilots, see (4), so dummy symbols are not needed to fill the frame of transmitted symbols $\{x_n\}$.

The decoding is based on the min-sum algorithm with scaling of the check-nodes messages by a constant factor equal to $\rho = 0.7$, see [19]; then, a fraction of dB loss was observed when comparing to the sum-product algorithm in the AWGN channel. The total number of decoding iterations equals $I_{\text{dec}} = 10$.

The DMP is implemented using $N_\theta = 64$ samples and its packet error rate (PER) curve is the performance limit for all phase tracking algorithms.

The “AWGN” curve corresponds to a hypothetical scenario when the phase-noise is removed and we only deal with the AWGN. Such results are unattainable no matter how sophisticated the phase-tracking and decoding strategy are, and we show them to illustrate the impact of the phase noise.

The packet error rate (PER) is estimated after transmitting $10^5$ blocks or after occurrence of 100 blocks in error, whichever comes first.

The demodulation/phase-tracking algorithms which use the decoder’s feedback are referred to as “iterative” and those which don’t – as “one-shot” algorithms.

A. One-shot receivers

We analyze first one-shot receivers which means that the MP phase tracking algorithm is run only once and has no prior information about the symbols, i.e., $\hat{P}_n(a) \propto 0$. In the case of the

12The numerical results obtained for 16-QAM and $N_\theta \in \{32, 64, 128\}$ were practically indistinguishable while a significant deterioration of the results was observed for $N_\theta = 16$. This indicates that the heuristics $N_\theta = 8M$ used e.g., in [1], [7], [13], may be too conservative but this issue is out of scope here.
CBC algorithm, it means that the phase reference for all the symbols is obtained using only the pilots.

1) CBC+EP: The smoothing parameter: We start showing in Fig. 2 the results produced by the CBC+EP($I_{ep}$) algorithm for $I_{ep} = 2$ when using different values of the smoothing parameter $\zeta$, see (80). The best performance is obtained for $\zeta = 0.4$ which will be used in all the remaining examples. Similar results were obtained for CBC+EP(3).

These results confirm the existence of the problems we delineated in Sec. V-A.1 and for illustration, we show in Fig. 3 an example of the PDFs

$$f_n(\theta) \propto \alpha_n(\theta) \beta_n(\theta)$$

$$\propto f(\theta_n|y_0, \ldots, y_{n-1}, y_{n+1}, \ldots, y_N)$$

obtained using the DMP, CBC and CBC+EP(2) algorithms.

We also show the true value of the phase $\theta_n \approx -38\text{deg}$ (a black circle on the axis) and we can see that while the DMP algorithms provides a very reliable probabilistic estimation of the latter (its mode/mean being close to $\theta_n$), the CBC algorithm, being based on pilots only is much less accurate. What is important here is to note that using the EP algorithm with $\zeta = 1$ yields the distribution $f_n(\theta)$ which also considerably misses the actual value of the phase: the difference

Fig. 2. PER vs. SNR for one-shot algorithm in 16-QAM transmission: CBC+EP(2), i.e., $I_{ep} = 2$, different values or the smoothing parameter $\zeta$. The thick lines indicate the results obtained for $\zeta = 0.4$ which is retained for further analysis. The results of the DMP and CBC algorithms are shown for reference.
Fig. 3. Example of the PDFs obtained using one-shot CBC+EP(2) phase tracking algorithms; $L = 25$, SNR = 16dB, $M = 16$; the true value of the phase noise, $\theta_n \approx -38\, \text{deg}$, is indicated by the position of a circle on the axis of $\theta$.

between the circular mean $\angle(z_{\alpha,n}^{(2)} + z_{\beta,n}^{(2)})$ and $\theta_n$ is close to 40deg. This is what [6] calls a “phase-slip”.

We may also appreciate that, thanks to the smoothing operation in (80), the CBC+EP(2) algorithms with $\zeta = 0.4$ mediates between $\zeta = 0$ (equivalent to the CBC algorithm) and $\zeta = 1$ (the second EP iteration without smoothing). So, while the first iteration of the EP algorithm being based on the pilots only is not very accurate, it is robust and may be used to regularize the results from the subsequent iterations: this is the role of smoothing (80).

Such intuitive understanding of the observed phenomenon cannot entirely explain the mechanism behind this regularization which is rather involved due to the MP recursion. Moreover, while choosing $\zeta = 0.4$ improves the overall performance, there is no definite answer which of the solutions ($\zeta = 1$ or $\zeta = 0$) is closer to the exact distribution (DMP) for each $n$. Better regularization strategies are possible but this issue is a matter for separate investigation.

2) **SKR:** Here, we repeat the experience from the previous section for the algorithm SKR+EP(2) and show the results in Fig. 4. Without smoothing (i.e., with $\zeta = 1$) the EP deteriorates the results but even with appropriately chosen $\zeta = 0.8$ the gains are much less notable than in the case of the CBC+EP algorithm and this, despite the complexity of four (4) CMM complexity units, see Sec. [V-D] Quantitatively similar results are obtained for different pilot spacings, $L = 25$ and $L = 17$. 

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Fig. 4. PER vs. SNR for one-shot algorithms in 16-QAM transmission: DMP, SKR, SKR+EP(2) with different values of the smoothing parameter $\zeta$ (thick line for $\zeta = 0.8$; $L = 25$ (solid line) and $L = 17$ (dashed lines). The results “SKR+ [6]” correspond to the SKR algorithm based on the CMM defined by [6, Eq. (102)].

We thus only show the results of the SKR algorithm in the remaining figures.

We also show the results obtained using the SKR algorithm but where the CMM is implemented using the formulas from [6, Appendix A] (indicated by the “SKR+ [6]” curve). The poor performance is due to oversimplification of the formula [6, Eq. (102)] which should be avoided. On the other hand, the approximation [6, Eq. (101)] produces results equivalent to the SKR algorithm we show.

3) CBC, SKR, and Hybrid algorithms: The results obtained using the algorithms DMP, CBC, SKR, and CBC+SKR are now shown in Fig. 5 where we observe that

1) The performance decreases notoriously with the increased pilot spacing for all the algorithm but for the DMP, where the effect is much less notable. We may thus infer that it is indeed the projection on $F$ which is the principal source of degradation of the parametric phase tracking algorithm.

2) The PER curves show a tendency to error floor (i.e., increasing the SNR yields small PER gains) that is especially notable for a large pilot spacing, $L = 25$. This is because, for high SNR, the phase noise is a dominating distortion, however, the reasons behind error floor are not exactly the same in all algorithms. In particular

\[^{13}\text{Similar tendency is observed for the DMP below } \text{PER} = 10^{-4} \text{ as will be shown later in Fig. 6}\]
• In one-shot CBC algorithm, the mixture reduction for the payload symbols is trivial (i.e., $z_{\gamma,n} = 0$) so the uncertainty about the phase depends only on the pilots.

• The mixture reduction in the SKR, SKR+EP, and the CBC+EP algorithms relies on non-trivial operations based on the CMM which are prone to errors: the projection may over-emphasize one of the components $g_n(\theta, a)$ which does not represent the actual phase distribution (that is, when the component $g_n(\theta, a)$ dominating the mixture is the one with $a \neq x_n$); this leads to the phase-slip.
This algorithmically-produced phase-slip also explains why the SKR algorithm which, at each time \( n \), exploits the form of the constellation \( A \), may be outperformed by the CBC algorithm (for \( L = 17 \) in Fig. 5) which is merely based on the pilots and ignores the form of \( A \): the respective PER curves cross for \( \text{SNR} \approx 18.5 \text{dB} \) (when \( M = 16 \)) and \( \text{SNR} \approx 23.5 \text{dB} \) (when \( M = 64 \)).

In fact, this very dependence on the pilots only is the source of the robustness of the CBC algorithm: that eventual discrepancy between the estimated phase distribution and the actual one is only tributary to the noise at the pilots. Then, as we see, in suitable conditions (e.g., small \( L \)) the one-shot CBC algorithm may outperform more sophisticated algorithms such as SKR.

3) The one-shot CBC+EP algorithms outperform the SKR phase-tracking and again, the explanation should be sought in the different propensity to the phase-slip phenomenon, which we conjecture is due to the following:

- The SKR algorithm obtains \( \alpha_n(\theta) \) (or \( \beta_n(\theta) \)) using only the past message \( \alpha_{n-1}(\theta) \) see (58) (or future ones, \( \beta_{n+1}(\theta) \), see (60)), and does not use any of the messages \( \beta_n(\theta) \) (or \( \alpha_n(\theta) \)). On the other hand, the CBC+EP algorithm, to calculate \( \gamma_n(\theta) \), relies both on \( \alpha_{i-1}(\theta) \) and on \( \beta_{n-1}(\theta) \), see (70). Using for each \( n \), the future and the past observations increases the robustness of the CBC+EP algorithm.

- In the SKR algorithm, any phase slip in \( \alpha_n(\theta) \) will propagate to \( \alpha_{n+1}(\theta) \) and \( \alpha_{n+2}(\theta) \), etc. (the same reasoning applies to the messages \( \beta_n(\theta) \), \( \beta_{n-1}(\theta) \), etc.). In contrast, the EP algorithm finds the projections \( \tilde{\gamma}_{n}^{(i)}(\theta) \) independently for all \( n \). Of course, there is some dependence between the projections because they are based on the same estimates \( \alpha_{n}^{(i-1)}(\theta) \) and \( \beta_{n}^{(i-1)}(\theta) \) obtained in the previous iteration but it does not lead to the propagation of the errors as in the SKR algorithm.

In fact, the propensity of the SKR algorithm to produce the phase-slip should be seen as a reason behind the moderate improvements due to the EP as observed in the case of SKR+EP(2) in Sec. VI-A.2.

4) In the hybrid algorithm CBC+SKR both, the past and the future are exploited at each \( n \) and this removes the inefficiencies of the SKR algorithm we mentioned above: for \( L = 25 \), the CBC+SKR algorithm performs well both for high and low SNR behaving similarly as
the best of CBC+EP(2) and CBC+EP(3). For \( L = 17 \), the results of CBC+SKR algorithm are the same as those obtained by means of the algorithm CBC+EP(3). Note that the complexity of the algorithms CBC+EP(3) and CBC+SKR is also virtually the same.

5) The CBC+EP and CBC+SKR algorithms behave similarly for 16-QAM and 64-QAM. On the other hand, the poor performance of the SKR algorithm we observe for 16-QAM, disappears for 64-QAM transmission above \( \text{PER} = 10^{-2} \) (when \( L = 25 \)) and above \( \text{PER} = 10^{-3} \) (\( L = 17 \)). The propensity of the SKR algorithm to produce the phase-slips manifests for \( \text{SNR} > 21 \text{dB} \).

Explaining formally for such a behaviour requires more investigation. Nevertheless, it should be noted that keeping the coding rate constant (\( r = \frac{7}{8} \)) and increasing the constellation size (from \( M = 16 \) to \( M = 64 \)) provides a larger margin between the theoretical value of the mutual information (MI) required for decoding, i.e., \( r_m \), and the maximum theoretical value, \( m \), that the MI can achieve in the transmission of the \( M \)-ary constellation. We may then hypothesize that larger MI margin translates into smaller probability of decoding error (e.g., caused by the phase slips) and thus the differences between the algorithms appear for low values of the PER.

B. Iterative receivers

In the iterative algorithms we limit our considerations to a particular form of scheduling of operations between the demodulation/phase-tracking and the decoder: each demodulation/phase-tracking operation (where, the EP phase tracking is iterative in itself) is followed by one decoding iteration.

This is not necessarily the best solution but was extensively used in the literature, e.g., [1], [6], [7] and thus constitutes a well-accepted basis for comparison. This means that the iterative receiver has to run the demodulation/phase tracking \( I_{\text{dec}} \) times. The decoder’s state (captured by the internal messages from the check to variables nodes) is preserved between iterations.

As a reference for the iterative receivers we also can use the curve “All-pilots”. It is obtained assuming that, during the demodulation of the symbol \( x_l \), all the symbols in \( \{x_l\}_{l=1}^{N} \) but \( x_n \) are pilots. This corresponds to a hypothetical situation when the output of the decoder provides highly reliable information about the bits which gives certainty about the symbols \( x \) (that is, \( P_{n}^{a}(a) \approx 1 \) for \( a = x_n \)); these, in turn, become de facto pilots allowing for the best possible
phase tracking.

Then, when calculating $\hat{P}_n(a)$, the only uncertainty about the symbol $x_n$ is caused by the AWGN as well as by the residual phase noise – this part of it which cannot be estimated from other symbols $x_l, l \neq n$. The “All-pilots” curve is a performance limit for all joint phase-tracking and demodulation/decoding algorithms we study here and is the same irrespective of the pilot spacing $L$. If the iterative DMP is too complex to implement, the “All-pilots” curve may be treated as its proxy.

We only consider an example of 16-QAM and start with a short analysis of the results obtained using the canonical CBC algorithm. They are shown in Fig. 6, where two ways of carrying out the projection are considered: the GA (results denoted by CBC+GA) and the CMM (CBC+CMM).

While the difference between the CBC+GA and the CBC+CMM algorithms is well notable (which indicates the importance of the approximations/projection we make in the phase tracking), the qualitative behaviour of the CBC algorithms is the same and the gap to the DMP remains large.

More importantly, the error floor appears in the PER curve, a phenomenon which was already observed in [7, Fig. 8]. In fact, with increasing SNR we observed even a slight deterioration of the PER curve.

The error floor may be also observed for $L = 17$; here we show the results for extended scale of the PER and they should be read with caution because we only simulate the transmission of $10^5$ blocks; despite somewhat erratic behaviour of the PER curve, the error floor is visible.

Its origin should be sought in unreliable phase approximations due to the parametric MP. The decoding is then based on the LLRs calculated from unreliable phase estimates and will produce erroneous estimates of the coded bits; this may be seen as overconfident decoding of a wrong codeword.

In the iterative receivers this effect is amplified because the extrinsic LLRs of the decoder become prior LLRs, $\lambda_{n,k}^a$, defined for all coded bits $c_{n,k}$. This phenomenon does not need to occur systematically in each transmission and is not equally detrimental in all phase-tracking algorithms. In fact, the frequency of its occurrence will be measured by the level of the error floor: PER $\approx 10^{-3}$ for $L = 25$ and PER $\approx 10^{-5}$ for $L = 17$; this effect also occurs slightly

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14More precisely, for all those algorithms which do not use the decoder’s feedback in the LLR calculation in (11); see comments at the end of Sec. II.
Fig. 6. The iterative CBC algorithms for 16-QAM transmission and two different projection methods, the Gaussian approximation (GA) and the circular moment matching (CMM). Both exhibit error floor which is removed when the LLRs are scaled down via (90) with \( \psi = 0.7 \); \( L = 17 \) (dashed lines) and \( L = 25 \) (solid lines).

more often when the GA is used rather than the CMM; this indicates that the approximation \( \tilde{\gamma}_n(\theta) \) obtained by means of the CMM represents \( \gamma_n(\theta) \) more faithfully than does the GA.

To deal with incorrectly calculated LLRs, a well-known approach relies on scaling them down as

\[
\tilde{\lambda}_{n,k} = \psi \lambda_{n,k}^a,
\]

and next using them to calculate the prior symbol log-probabilities \( \hat{P}_n^a(a) \) via (12).

As shown in Fig. 6, this strategy efficiently removes the error floor. We used \( \psi = 0.7 \) but very similar outcomes were obtained for \( \psi \in (0.6, 0.9) \); this indicates that the operational condition of the CBC are set at the limit of reliable decoding; then, scaling down by any “reasonably” value \( \psi \) offsets the negative effect of unreliable phase estimates and removes the error floor.

The difference between the GA and the CMM results is very small after this modification yet well notable especially for \( L = 25 \). More importantly, the results are still far from the performance of one-shot DMP and thus we focus in Fig. 7 on the iterative phase tracking algorithms CBC+EP\((I_{aq})\), SKR, and CBC+SKR. The results obtained are in line with those previously reported in the literature, e.g., [6], [7]: the performance is notably improved approaching the

\[\text{References:}\]

[15] Akin to the scaling of the LLRs in the max-log decoders [19].
limits of the iterative DMP where only the SKR algorithm shows the tendency to error floor for $L = 25$.

However, the differences in the results, being measured by fractions of dB are of little practical importance thus, if we decide to use the decoder’s feedback, all algorithms are practically equally suitable for implementation. The choice should be merely guided by the complexity which is the lowest for the CBC+EP(2) algorithm.

VII. Conclusions

In this work we dealt with the problem of phase tracking in single-carrier transmission. This problem was often solved in the literature via message passing (MP) on the graph which describes the relationship between all involved random variables; the performance of the discretized message passing (DMP) is then considered a reference for the performance.

The algorithms from the literature, inspired mostly by the Colavolpe–Barbieri–Caire algorithm [1] rely heavily on the feedback from the decoder. This is also the case for the Shayovitz–Kreimer–Raphaeli algorithm we defined here and which succinctly summarizes the works of [6] and [7].

With that regard our work is different as we focused on the signal processing approach to the phase-tracking that does not require decoder’s feedback. This approach is much more in
line with the current industrial practice, which separates the operation of the modulation and decoding.

We proposed to improve the CBC and SKR algorithms iteratively, via the expectation propagation (EP). Using numerical examples, we have shown that the algorithms we obtained outperform notably the CBC and SKR algorithm before any feedback from the decoder is available. Moreover, the probabilistic framework allows us to design new algorithms via hybridization; and example of such algorithms (dubbed CBC+SKR) is given and shown to perform particularly well.

Finally, we have also shown that the new algorithms result in efficient phase tracking when placed in the decoding loop.

**APPENDIX**

A. **Circular moment matching**

The first circular moment of the circular distribution $g(\theta)$ is defined as

$$
\mu = M[g(\theta)] = \int_0^{2\pi} g(\theta) e^{i\theta} d\theta
$$

and for the normalized Tikhonov distribution we obtain

$$
M \left[ \frac{\mathcal{T}(\theta; z)}{2\pi I_0(|z|)} \right] = B(|z|) e^{i\angle z} = \frac{B(|z|)}{|z|} z,
$$

where

$$
B(x) = \frac{I_1(x)}{I_0(x)},
$$

$I_1(x)$ is the first-order Bessel function, and the last equation in (92) conveniently eliminates the need for calculation of the angle $\angle z$ (so non-linear functions are not needed).

Since $B(x)$ (or rather $B(x)/x$) will appear frequently and is not available in closed-form, we propose a simple approximation in Appendix B.

**Definition 1:** For the mixture of Tikhonov distributions

$$
g(\theta) = \sum_{a \in \mathcal{A}} e^{\eta(a)} \mathcal{T}(\theta; z(a)),
$$
appearing e.g., in (32), (58), (58), (71), the CMM projection on the space, \( \mathcal{F} \), of Tikhonov distributions

\[
\mathcal{T}(\theta; z_{\hat{g}}) = \mathcal{F} \left[ \sum_{a \in A} e^{\eta(a)} \mathcal{T}(\theta; z(a)) \right],
\]

is given by

\[
z_{\hat{g}} = \text{CMM} \left[ \{ \eta(a), z(a) \}_{a \in A} \right] = B^{-1}(|\mu_g|) \mu_g,
\]

where

\[
\mu_g = M[g(\theta)] = \sum_{a \in A} \xi(a) \frac{B(|z(a)|)}{|z(a)|} z(a)
\]

and

\[
\xi(a) = \frac{e^{\eta(a) - \tilde{\eta}_{\max}}}{\sum_{a \in A} e^{\eta(a) - \tilde{\eta}_{\max}}},
\]

\[
\tilde{\eta}(a) = \eta(a) + \hat{I}_0(|z(a)|),
\]

\[
\tilde{\eta}_{\max} = \max_{a \in A} \tilde{\eta}(a).
\]

To understand the derivation, we remind that the CMM consists in finding a (normalized) Tikhonov distribution \( \hat{g}(\theta; z_{\hat{g}}) = \mathcal{T}(\theta; z_{\hat{g}})/(2\pi I_0(|z_{\hat{g}}|)) \) closest, in the sense of the KL distance, to a distribution \( g(\theta) \). This amounts to solving the following optimization problem:

\[
z_{\hat{g}} = \arg\min_z \int_0^{2\pi} g(\theta) \log \frac{g(\theta)}{g(\theta; z)} \, d\theta
\]

\[
= \arg\max_z \int_0^{2\pi} g(\theta) \Re[z e^{-j\theta}] \, d\theta - \log I_0(|z|)
\]

\[
= \arg\max_z \Re[z \mu_g^*] - \log I_0(|z|)
\]

\[
= B^{-1}(|\mu_g|) e^{j\angle \mu_g} = B^{-1}(|\mu_g|) \frac{B^{-1}(|\mu_g|)}{|\mu_g|} \mu_g,
\]

where \( B^{-1}(\cdot) \) is the inverse of \( B(\cdot) \) and, again we preferred the angle-free formulation in (105).

Thus, (105) explains (97).

We then need to calculate the circular moment, \( \mu_g \), of \( g(\theta) \) so the latter must be represented
as a sum of normalized Tikhnonov distributions

\[ g(\theta) \propto \sum_{a \in A} e^{\tilde{\eta}(a)} \frac{T(\theta; z(a))}{2\pi I_0(|z(a)|)}, \]  

(106)

where \( \tilde{\eta}(a) \) is given by (100). And since \( g(\theta) \) must be normalized, we require that

\[ g(\theta) = \sum_{a \in A} \xi(a) \frac{T(\theta; z(a))}{2\pi I_0(|z(a)|)}, \]  

(107)

where \( \sum_{a \in A} \xi(a) = 1 \), and this explains (99) in which where we also introduced \( \tilde{\eta}_{\text{max}} \) to avoid overflows in numerical implementations.

Using simple algebra we find that, for a particular case of the large-argument approximation used in of \( B(\cdot) \) and \( B^{-1}(\cdot) \), see (108) and (110), the expressions (97) and (98) are equivalent to those shown in [6, Eq. (100) and Eq. (101)].

On the other hand, our expressions, being explicitly based on \( B(\cdot) \) and \( B^{-1}(\cdot) \) are valid for any argument (not only large values) so using them is much safer as it eliminates potential errors due to violation of approximation conditions. For example, it allowed us to find that the approximation [6, Eq. (102)] is the source of deteriorated performance, see Sec. VI-A.2.

B. Approximation of the function \( B(x) = \frac{I_1(x)}{I_0(x)} \)

We approximate \( B(x) \) as

\[ B(x) \approx \begin{cases} \frac{1}{2}x & \text{if } x \leq 1 \\ 1 - \frac{1}{2x} & \text{if } x > 1 \end{cases}. \]  

(108)

The large-argument approximation in (108) (case \( x > 1 \)) is known, e.g., [6, Appendix A] but becomes negative for \( x < 0.5 \). So, while most often \( x \) is assumed large, e.g., \( x > 2 \) [6, Appendix A] – in which case the large-argument approximation is quite precise, see Fig. X it is not uncommon to obtain \( x \approx 0 \). In such a case the large-argument approximation leads to approximation errors which are quite pernicious: even if approximating \( B(x) \) with a negative number \( B(x) \approx 1 - \frac{1}{2x} \) makes no sense, such an error is difficult to spot in numerical implementation as it may appear in the sum of complex numbers: see, for example (98) in the moment-matching procedure described in Appendix A.

In our simulation, such errors were relatively rare and did not lead to significant differences.
in performance, nevertheless, we believe it is much more sound to use the approximation which covers the entire range of the admissible arguments. This issue motivates us to introduce the small-argument approximation (for $x \leq 1$) which is obtained via Taylor series development of $B(x)$ around $x = 0$. Both, the large-, and the small-argument approximations meet at $x = 1$ which is the threshold for using one approximation or another in (108).

Of course, for the purpose of calculation of the circular moment, we rather need

$$B(x) \approx \begin{cases} \frac{1}{2} & \text{if } x \leq 1 \\ \frac{1}{x}(1 - \frac{1}{2x}) & \text{if } x > 1 \end{cases}$$

(109)

and

$$B^{-1}(y) \approx \begin{cases} 2 & \text{if } y \leq \frac{1}{2} \\ \frac{1}{2y(1-y)} & \text{if } y > \frac{1}{2} \end{cases}$$

(110)

obtained inverting (108).

We show in Fig. 8 the function $B(x)$ and its approximation (108) which gives us an idea about the approximation error. More importantly, and unlike the previously used large-argument approximation which was meaningful (i.e., non-negative) only for $x > \frac{1}{2}$, we cover all range of the argument $x$ using simple functions.

C. Convolving Tikhonov and Gaussian distributions

The convolution of the Tikhonov and the circular Gaussian distributions may be obtained by moment matching principle [15, Lemma 3]: the result of the convolution should have the same moment as the product of the moments of the convolved distributions, i.e.,

$$\mathcal{T}(\theta; \tilde{z}) \approx \mathcal{T}(\theta; z) * \omega(\theta)$$

(111)

$$M \left[ \frac{\mathcal{T}(\theta; \tilde{z})}{2\pi I_0(|\tilde{z}|)} \right] = M \left[ \frac{\mathcal{T}(\theta; z)}{2\pi I_0(|z|)} \right] M[\omega(\theta)]$$

(112)

Since $M[\omega(\theta)] = e^{-\frac{\sigma_w^2}{2}} \approx (1 - \frac{\sigma_w^2}{2})$, where the approximation is valid for small $\sigma_w^2$, using (92)
we can write

\[ B(|\tilde{z}|)e^{\jmath \tilde{z}} \approx B(|z|)e^{\jmath z} \left(1 - \frac{\sigma_w^2}{2}\right) \]  
(113)

\[ \tilde{z} \approx B^{-1} \left( B(|z|) \left(1 - \frac{\sigma_w^2}{2}\right) \right) e^{\jmath z} \]  
(114)

\[ \approx \begin{cases} 
\frac{z}{|z|\sigma^2 + 1} & \text{if } |z| > 1 \\
 z(1 - \frac{\sigma^2}{2}) & \text{if } |z| \leq 1,
\end{cases} \]  
(115)

where (115) is obtained using large- and small-argument approximations of \( B(x) \), shown in (108).

We note that the large-argument approximation (\(|z| > 1\)) is known, see [1] [14] [6], and using it did not affect (significantly) the results in our simulations but, again, rigorous approximations, valid for the entire range of input argument are, in general, more sound and useful.

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