Solitons and admissible families of rational curves in twistor spaces

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Abstract

It is well known that twistor constructions can be used to analyse and to obtain solutions to a wide class of integrable systems. In this paper we express the standard twistor constructions in terms of the concept of an admissible family of rational curves in certain twistor spaces. Examples of such families can be obtained as subfamilies of a simple family of rational curves using standard operations of algebraic geometry. By examination of several examples, we give evidence that this construction is the basis of the construction of many of the most important solitonic and algebraic solutions to various integrable differential equations of mathematical physics. This is presented as evidence for a principle that, in some sense, all soliton-like solutions should be constructable in this way.

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1. Introduction

Penrose’s nonlinear graviton construction [24] realizes any four-dimensional conformal manifold with a anti-self-dual Weyl tensor as a family of rational curves in a complex three-manifold, T, known as the twistor space. There are many generalizations, most notably the extension by Ward to the anti-self-dual Yang–Mills equations [26], in which any anti-self-dual Yang–Mills field can be reconstructed directly from a family of rational curves in the total space of a holomorphic vector bundle over a region in twistor space⁴. Similar constructions apply

⁴ This construction is usually viewed in the context of finding trivializations of the holomorphic vector bundle over the lines in twistor space, but this, by Liouville’s theorem, is equivalent to lifting the lines in twistor space to the total space of the holomorphic vector bundle.
to a large family of integrable systems both by considering families of rational curves in other complex manifolds and by symmetry reduction of these two basic twistor correspondences, see [21] for a catalogue of such reductions and full details of the twistor correspondences.

In effect, the task of solving these equations reduces to that of the construction of such families of rational curves. A strategy proposed by one of us [9,11,12] is to consider admissible families of rational curves. An admissible family of curves is a local family of curves that admits an extension in some modification of the complex manifold of the same dimension to a family of compact rational curves that is a complete (in the sense of Kodaira) family of rational curves. To construct such families, one can consider a simple space such as the total space of some bundle $O(k_1) \oplus O(k_2) \oplus \cdots \oplus O(k_r) \rightarrow \mathbb{C}P^3$, $k_1 \geq k_2 \cdots \geq k_r \geq 0$ that has many rational curves; here the space of sections has dimension $\sum_{i=1}^r (k_i + 1)$. One can then restrict the family in such a way that the restricted family is an admissible family with the appropriate dimension and so can be regarded in its own right as a moduli space of rational curves in some (different) twistor space. There is a theorem due to Bernstein and Gindikin [2,10] to the effect that admissible families can only be obtained from some large family by requiring intersection with or tangency to some family of submanifolds. This, in turn, corresponds to the operation of taking a blowup of or branched cover over the original twistor space at the chosen submanifolds. These algebraic geometrical modifications of the twistor space can be thought of as finite nonlinear deformations of the twistor space in Penrose’s nonlinear graviton construction. In Penrose’s approach to the nonlinear graviton construction, we perturb the complex structure in a neighbourhood of a rational curve. In our construction, we make a birational transformation of a manifold with a large family of rational curves so that only an appropriately sized family of the curves survive after this modification.

Such a procedure will not give rise to the general solution as these methods use data on subspaces of codimension at least 1 to that required for the general solution. Thus these solutions are special. In this paper, our aim is to show that many of the most important soliton-like solutions in mathematical physics arise from this construction. In particular, standard procedures for obtaining special solutions by introducing a hierarchy and imposing hidden symmetries fit naturally into this construction. We present this as evidence for a principle that all interesting soliton-like solutions arise in this way.

In section 2 we outline the basic methodology associated with families of rational curves. In section 3 we explain how the imposition of a restriction on a simple but large admissible family of rational curves arises naturally in a situation in which hierarchies are considered and higher symmetries imposed. In the subsequent sections we show how various standard global solutions (solutions arising from the Ward ansatz such as instantons and monopoles in $\mathbb{R}^3$, Korteweg de Vries and nonlinear Schrödinger solitons and asymptotically locally Euclidean (ALE) gravitational instantons) fit into this framework. For the nonlinear graviton construction, the construction of families of rational curves is connected only with the conformal part of the problem. In particular, the problem of constructing a self-dual four-metric on $M$ is equivalent to constructing a pair of one-forms $\varphi(t), \psi(t)$ depending on a rational parameter $t$ such that the two-form $F(t) = \varphi(t) \wedge \psi(t)$ is quadratic in $t$ [9]. The conformal part of this problem is the integrability of the kernel distribution of $\varphi(t)$ and $\psi(t)$ for all $t$. This weaker problem can be reformulated as a problem of finding an $O(1) \oplus O(1)$-family of rational curves. As a result we need to modify the tangency–intersection construction in such a way that we obtain the solution to the complete problem, not just its conformal part. Thus, in section 5.2, on the construction of ALE solutions, we work with a special type of tangency–intersection condition connected with the lifting of rational curves on certain branched coverings. In the subsequent sections, we demonstrate that classes of three-dimensional Einstein–Weyl spaces, and ODEs satisfying a set of over-determined Wünschmann-type constraints, can be reconstructed from
admissible families. The basic facts about bundles over $\mathbb{CP}^1$ and the deformation theory are summarized in the appendix.

2. Admissible families of rational curves

In twistor theory, the data of a solution to an integrable equation are encoded into the complex structure of the twistor space $T$. The reconstruction of the solution from the complex structure on $T$ reduces to the construction of families of rational curves in $T$. The parameter space $\mathcal{M}$ of the family is either a space–time or some related space (i.e. the total space of a Yang–Mills bundle over space–time) and the full solution to the integrable equation can be obtained directly from the correspondence with the rational curves in $T$. In this paper, we study such families of rational curves locally in $T$, and the global condition that the curves extend to global rational curves in $T$ is replaced by the condition that the family be admissible as defined below.

A family of rational curves in $T$ is complete if it contains all small deformations in the family. We will give a local (in $T$) characterization of complete families of rational curves. Since we are working locally, this characterization will be birationally invariant. Roughly speaking, this will mean that, perhaps after some modification of $T$, it will be the family of all rational curves of some fixed topological type in some manifold $\tilde{T}$. However, after another birational map, it may cease to be a maximal in the new manifold (and in general, it will not be maximal in $T$). We will work in the complex holomorphic category throughout.

Initially, we will define families of local rational curves (which may not necessarily extend to become global). Let $T$ be a complex manifold of dimension $r+1$ in which a family of curves are embedded, and let $\mathcal{M}$ be a manifold of dimension $n$, parametrizing the curves with $m \in \mathcal{M}$ corresponding to the curve $C_m$ in $T$. Our considerations are local in the first instance, and we will assume that everything is in general position.

For $z \in T$, let $Z$ be the dual submanifold in $\mathcal{M}$, which parametrizes the curves $C$ that pass through $z$. Such a $Z$ will be called an $\alpha$-surface. Inside the tangent space $T_m \mathcal{M}$ at $m \in \mathcal{M}$, we define an $\alpha$-plane to be a tangent plane $\alpha(z)$ to some $Z$, for which $z \in C_m$. Let $v_m$ be the union of these $\alpha$-planes in $T_m$.

**Definition 2.1.** We will say that $v_m$ satisfies the locally rational (LR) condition if it can be presented as the union of two-dimensional flat cones, i.e. cones sitting inside two-dimensional subspaces of $T_m \mathcal{M}$. These two-planes, called $\beta$-planes, will be denoted $\beta(s)$ and we will assume that they intersect only at 0 and are transversal to the planes $\alpha(x)$. Let $V_m$ be the union of these two-planes: clearly $v_m \subset V_m$.

**Lemma 2.2.** If the curves do in fact admit an extension so that they form a complete family of global rational curves, then the LR condition is satisfied.

**Proof.** In the case where the curves are global, the normal bundle $NC_m$ of a rational curve $C_m$ in $T$ will be $\oplus_{i=1}^r O(k_i)$ for some integers $k_i$ and we will assume that these integers are constant in the family. (Here $O(k)$ denotes the line bundle of Chern class $k$ on $\mathbb{CP}^1$.) By Kodaira’s theorem, $T_m \mathcal{M} = \Gamma(\mathbb{CP}^1, N)$. If we express $\mathbb{CP}^1$ as the projectivization $\mathbb{PS}$ of a two-dimensional complex vector space $S^*$, we obtain $T_m \mathcal{M} = \oplus_{i=1}^r \oplus k_i S$ (and so $n = \sum (k_i + 1)$).

Let $\pi \in S$; then the general tangent vector to the $\alpha$-plane corresponding to the point where $\pi$ vanishes on $\mathbb{PS}$ will lie in the image of the vector space $W = \oplus_{i=1}^r \oplus k_i S$ embedded into $T_m \mathcal{M}$ by symmetrization of each summand with $\pi$. Hence, each element of $\mathbb{P}(W)$ corresponds to the two-plane in $T_m \mathcal{M}$ spanned by the image of symmetrizing each summand with $S$. □

The structure of the incidence cones $v_m \subset T_m \mathcal{M}$ supplies the curves $C_m$ with the canonical projective structure of rational curves, induced by the projectivization of one of
the two-planes $\beta(s)$. This structure is independent of the choice of $s$. If we consider curves $C_m$ without parametrization, we can identify vectors in $T_mM$ with sections of the normal bundles $NC_m$. For manifolds of parametrized curves we can interpret them as sections of tangent bundles $TC_m$.

Even in the general local case, the cones $V_m$ must be linearly equivalent to cones for the case when $M$ is the manifold of sections of a vector-bundle $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r), k_1 \geq \cdots \geq k_r \geq 0$. These families of curves play the role of the flat model in this geometry. In the generic situation, $V_m$ for different $m \in M$ are linearly equivalent and correspond to some choice of $k_1 \geq \cdots \geq k_r \geq 0$.

This gives the manifold $M$ a generalized conformal structure $\{V_m\}$. It is natural to investigate when this structure uniquely defines the manifold $T$ with the curves $C_m$. The first step is to understand when the $\alpha$ and $\beta$ planes can be uniquely reconstructed. It turns out that this is possible if $V_m \neq \mathbb{C}^p$ for some $p$, which is equivalent to the condition $k_2 > 0$. Then the $\alpha$-surfaces $Z$ are defined as integral submanifolds whose tangent planes are $\alpha$-planes at every point. It turns out that with these conditions the $Z$, if they exist, are unique (although existence is not automatic). Then we can construct $T$ as the manifold of integral submanifolds of such $Z$, and the curves $C_m \subset T$ defined as the sets of solutions passing through $m$.

For the definition of $Z$, in general it is necessary to produce an equation of second order, or to consider over $M$ the fibering whose fibre over $m \in M$ parametrizes the set of $\alpha(z) \subset T_m$ and a Frobenius distribution whose projection at an $\alpha$-plane $Z$ to $M$ consists of the $\alpha$-plane $Z$. Under the condition $k_2 > 0$, this Frobenius distribution can be uniquely reconstructed from the integrability conditions.

**Proposition 2.3 (generalized Desargues theorem [2, 12]).** If the $V_m$ are subspaces of $T_mM$ which are different from $\mathbb{C}^p$, $p \leq m$ (or equivalently $k_2 > 0$), then the field $V_m$ determines at most one family of $\alpha$-surfaces parametrized by a manifold $\tilde{T}$. If $V_m \subset T_mM$ arose from a family of curves $C_m$ in a manifold $T$, then the manifold $T$ with curves $C_m$ can be reconstructed uniquely as a subset of $\tilde{T}$. All curves $C_m$ admit a canonical extension up to global rational curves (after appropriate extension of the manifold $T$ to $\tilde{T}$).

In the general case (when the condition of the above proposition is not satisfied) the extendibility of the curves $C$ to global rational curves requires an algebraic condition on the family of $\alpha$-planes. The simplest way to specify this is to use the (LR) condition as follows.

1. We first construct a family $\tilde{C}$ consisting of the family of all curves $C$ together with projective parametrizations that are compatible with a local projective structure on them corresponding to a decomposition of $V_m$ into $\beta$-planes. This gives an $n+3$-dimensional family $\tilde{M}$ of curves fibred over $M$ with fibres $\text{PSL}(2, \mathbb{C})$. (2) We define an $(R)$-structure on $M$ to be an (LR)-structure on some choice of $\tilde{M}$. One should think of this transition from $M$ to $\tilde{M}$ as analogous to the prolongation of a second order system of ODEs to a larger first order system. The essential point here is the existence of a canonical decomposition of the cones $V_m$ into $\alpha$-subspaces transversal to $\beta$-subspaces.

Let us remark that when $n = r + 1 = 2$ the condition $(R)$ is equivalent to the Cartan condition: curves $U(m)$ are defined by a differential equation of second order which is a polynomial of degree 3 in the first derivative, i.e. they define a projective structure.

A principal result of the theory of families of rational curves concerns ‘admissible’ subfamilies:

**Definition 2.4.** Given a (perhaps local) family of rational curves with the property $(R)$, a subfamily will be said to be admissible if the property $(R)$ is induced on it.

It emerges that only very special subfamilies have this property. Let us start from the generic case.
Proposition 2.5. Let $\mathcal{M}$ be a family of (global) rational curves on $T$. Let $T_1, \ldots, T_p$ be submanifolds in $T$ of codimensions more than 1 and $S_1, \ldots, S_q$ have codimension 1 and $s_1, \ldots, s_q$ be natural numbers. Let $\mathcal{M}(T, S, s)$ be the subfamily of curves $C_m$ which intersect all $T_i$ and have at their intersection with each $S_j$ tangency of order $s_j$. Then this subfamily is admissible and a generic admissible subfamily can be represented in such a form.

To obtain all admissible subfamilies, not just the generic ones, we need to replace the conditions of intersection with the $T_j$ by the condition that we take those curves that admit a lift to a tower $\tilde{T}$ of blowups of $T$. Then we consider the hypersurfaces $S_j$ to be in $\tilde{T}$ and add the condition of tangency of order $s_j$.

There are two known proofs of this theorem: one uses algebraic geometrical methods [2] and the other one uses geometrical methods of nonlinear differential equations [10]. The proof of this theorem splits into several steps: the condition of admissibility of a subfamily $\mathcal{M} \subset \mathcal{M}$ can be written down as an explicit nonlinear differential equation. It turns out that this equation can be integrated by a generalized Hamilton–Jacobi method with multidimensional bicharacteristics.

3. Hierarchies for the anti-self-duality equations and invariance under hidden symmetries

A key feature of soliton solutions to, for example, the KdV equations is that they arise as solutions that are invariant under one or more hidden symmetries. A hidden symmetry is usually understood in the context of hierarchies associated with the equation.

A hierarchy associated with an integrable system is an over-determined system of (completely integrable) partial differential equations on a higher-dimensional space, usually the Cartesian product of the space–time for the original integrable system with a space of higher times. The hierarchy equations restrict us to giving the original completely integrable system on each leaf of the foliation on which the ‘higher time’ variables are constant. The flows along higher time variables are know as hidden symmetries since they evolve solutions to the original equations into different solutions to the original equations.

The connection with the above theory is that the solution to the hierarchy will arise from some admissible family of rational curves, and each solution to the original system at fixed values of the higher times arises from an admissible subfamily. Proposition 2.5 implies that such admissible subfamilies must be obtained by intersection and tangency.

A standard strategy for obtaining soliton solutions is to require that a solution can be embedded into a solution to the hierarchy that is invariant under one or more higher flows. More generally, one can require that the solution to the hierarchy admit one or more symmetries that are not necessarily symmetries of the original system. Thus the solution to the hierarchy can be taken to arise from a simple admissible family, and the solitonic solution to the original system can be obtained by intersection and tangency as described earlier.

The twistor correspondences generalize straightforwardly to the hierarchy. In the case of the Bogomolny equations, the twistor space corresponding to a solution is the total space of a holomorphic vector bundle over ‘minitwistor space’, $\mathcal{O}(2)$ (the total space of the line bundle of Chern class 2 over the Riemann sphere, $\mathbb{C}P^1$). In [20] it was shown that the $SU(N)$ Bogomolny equations embed into a hierarchy, referred to as the Bogomolny hierarchy, for which the twistor space is the total space of a holomorphic vector bundle $\mathcal{O}(n)$ for some (arbitrarily large) $n > 2$. The rational curves in this space have a normal bundle that is a direct sum of the trivial $\mathbb{C}^N$ bundle with $\mathcal{O}(n)$. This yields the standard hierarchies for the KdV and the nonlinear Schrödinger equations under symmetry reduction. This was extended
in [21] to a correspondence for hierarchies for the SU(N) ASDYM equations, in which a solution to the hierarchy corresponds to a holomorphic vector bundle over the total space of $O(n) \oplus O(n) \to \mathbb{CP}^1$ so that the rational curves have a normal bundle given by the direct sum of $\mathbb{C}^N$ with $O(n) \oplus O(n)$. In [6, 7] the twistor correspondences were extended to give a hierarchy for the hyper-Kähler equations in which the twistor space is, as usual, a three-dimensional complex manifold fibred over $\mathbb{CP}^1$ and admits a Poisson structure on the fibres, but now the family of rational curves has normal bundle $O(n) \oplus O(n)$.

It is natural, therefore, to regard the geometry arising on the moduli space of rational curves with some arbitrary but fixed normal bundle, $O(k_1) \oplus O(k_2) \oplus \cdots \oplus O(k_r) \to \mathbb{CP}^1$, $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$, as the most general hierarchy associated with equations that admit a twistor correspondence. This can be thought of as a set of differential equations implied by the (LR)-condition on the generalized conformal structure formed by the family of incidence cones $V_m$ in each $T_mM$.

A solution to some version of the anti-self-duality equations will then extend to a hierarchy if it can be realized as arising from an admissible subfamily of the family of rational curves associated with that hierarchy. By proposition 2.5, if one is just given the solution to the hierarchy, admissible subfamilies are found by requiring intersection or tangency to submanifolds and all admissible subfamilies arise in this way.

A key application of hierarchies is performing symmetry reduction, but with respect to a ‘hidden’ symmetry, and in practice soliton solutions often arise in this way. This will mean that we will consider a solution that can be embedded into a hierarchy that admits at least one explicit symmetry. The hierarchy can admit many symmetries without the original solution admitting any at all.

If one wishes to look for such solutions, one can consider simple solutions to the hierarchy, perhaps even trivial ones corresponding to a constant conformal structure or flat connection, but then find non-trivial solutions by using intersection and tangency to find a non-trivial admissible subfamily.

4. Examples

We have seen then that solutions of many problems of mathematical physics that can be integrated by the inverse problem method require the construction of some family of rational curves. Using proposition 2.5 it is possible to produce such families as (admissible) subfamilies of some simple families of rational curves depending on a larger number of parameters, for example, the families of sections of a vector bundle on the projective line. Let us remember that these families play the role of flat objects in this geometry and we are interested in solutions that admit embedding in flat solutions of a bigger dimension. Of course, not all local solutions can be produced in such a way (such solutions depend on fewer functional parameters and will be partly algebraic) but we can expect that some ‘good’ global solutions can be included in this construction. They are in a sense quasi soliton solutions. We show here that it is indeed the case for several important problems: we will find intersection–tangency conditions in their solutions.

We believe that it is realistic to build ansatz for solutions for a number of problems starting from these ideas.

4.1. The Ward ansatz and intersection conditions

In the case of the Ward construction for solutions to the ASD Yang–Mills equations, the twistor space $T$ is the total space of a holomorphic vector bundle $E \to U$, where $U$ is some region in
$\mathbb{CP}^3$ and $E$ is assumed to be trivial on each real line in $U$ (i.e. on each line that is invariant under some anti-holomorphic conjugation $\sigma : U \rightarrow U$). Usually, the key step in the construction of a solution is the task of finding an explicit trivialization of $E$ over the (real) lines in $U$. This is equivalent to finding the rational curves in $E$ that are sections of $E$ over each line in $U$.

The process of finding a trivialization of $E$ over a line in $U$ requires, in the Cech description of the bundle, the solution to a Riemann–Hilbert problem and this is difficult to find explicitly. However, it can be done explicitly in the case where the patching matrix is upper triangular and this is the Ward ansatz. When $E$ has rank 2 with structure group $\text{SL}(2, \mathbb{C})$ (which we will assume from here on), this can be expressed as the requirement that $E$ contain a line subbundle $L^*$ and sits in a short exact sequence

$$0 \longrightarrow L^* \longrightarrow E \longrightarrow L \longrightarrow 0.$$ 

The line bundle $L$ must have non-negative degree $k \geq 0$ on each line if $E$ is to be trivial on each line in $U$ (a positive degree line subbundle $L^*$ would contradict triviality, but not a negative degree one).

The assumption that $E$ admits such a line bundle (or that the patching matrix can be expressed in upper triangular form) is known as the Ward ansatz and has been very fruitful in constructing solutions to the anti-self-dual Yang–Mills equations and its reductions, see Ward [27]. In particular, all instanton solutions can be obtained in this way [1], all monopoles [13] and solitons for the nonlinear Schrödinger equations [20]. The following discussion is a paraphrase of Hitchin’s discussion of monopoles [13].

When the solution to the anti-self-dual Yang–Mills equations on space–time has gauge group $\text{SU}(2)$ or $\text{SU}(1, 1)$ on a real slice, the reality conditions imply that the anti-holomorphic involution $\sigma : U \rightarrow U$ lifts to give an isomorphism between $\sigma^* E$ and $E$ (see Atiyah [1] or Mason and Woodhouse [21] for a full discussion).

We assume that $\tilde{L}^* := \sigma^* L^*$ is generically a linearly independent line subbundle to $L^*$. In this case the intersection/tangency ideas of the previous section can be brought into play since we have a map

$$\rho : E \hookrightarrow L \oplus \tilde{L}$$

and this will be a fibrewise vector space isomorphism except on the codimension-1 set $T \subset U$ on which $L^*$ and $\tilde{L}^*$ coincide as line subbundles of $E$. Denote the image of $\rho$ over $T$ by $\tilde{T}$. (Since on $T$, $\rho$ is onto both $L$ and $\tilde{L}$ separately, $\tilde{V}$ is the graph of an invertible map from $L$ to $\tilde{L}$, or alternatively a trivialization $e$ of $L \otimes \tilde{L}^*$.)

The data of $L$, $\tilde{L}$ and $\tilde{V}$ are sufficient for reconstructing the original solution. Sections of $E$ over lines in $U$ correspond precisely to sections of $L \oplus \tilde{L}$ that pass through $\tilde{T}$. Since the line bundles $L$ and $\tilde{L}$ have degree $k$, there will be $2k + 2$ sections of $L \oplus \tilde{L}$ over each line in $U$. The submanifold $T$ must therefore have degree $2k$ so that there are $2k$ conditions on the $2k + 2$ sections, reducing the number of sections of $E$ to 2 as required by triviality on lines.

We therefore have the following proposition.

**Proposition 4.1.** Suppose $E \rightarrow U$ is a rank 2 holomorphic vector bundle such that $\bar{\sigma}^* E = E$ and there is a line subbundle $L^* \subset E$ such that $\tilde{L}^* := \sigma^* L^*$ is generically linearly independent of $L^*$. Then the admissible family of rational curves consisting of sections of $E$ over lines in $U$ is equivalent to the admissible subfamily of sections of $L \oplus \tilde{L}$ over lines in $U$ that intersect the codimension-2 subset $\tilde{T}$ of $L \oplus \tilde{L}$ as defined earlier.

4.2. The monopole solutions

Monopoles are solutions to the ASD Yang–Mills equations on $\mathbb{R}^4$ with a single translation symmetry and with a finite energy condition. In [13] Hitchin showed that monopoles were, via
the Ward correspondence, in a correspondence with holomorphic vector bundles on $T \mathbb{C}P^1$ that are constructed from a certain ’spectral curve’ $T$ in $T \mathbb{C}P^1$. The construction involves the use of homogeneous line bundles $L(n) \to T \mathbb{C}P^1$ that can be described as follows. Introduce affine coordinate $\lambda$ on $\mathbb{C}P^1$ and corresponding fibre coordinate $\eta$ on $T \mathbb{C}P^1$ so that $(\eta, \lambda) \leftrightarrow \eta \partial/\partial \lambda$.

We can define $L(n)$ by the transition function $\lambda^{-n} \exp(\mu/\lambda)$ with respect to the open covering $U_0 = \{ \lambda \neq \infty \}$ and $U_\infty = \{ \lambda \neq 0 \}$. $(O(n)$ is the pullback of the $-n$th power of the tautological bundle on $\mathbb{C}P^1$ and $L(0)$ is the exponential of the class obtained by pulling back the ‘unit element’ of $H^1(\mathbb{C}P^1, T^* \mathbb{C}P^1)$ and contracting it with the vector $\eta \partial/\partial \zeta$, and $L(n) = L \otimes O(n)$.)

Hitchin shows that if $E \to T \mathbb{C}P^1$ is the Ward transform of a monopole solution, then we can express $E$ as an extension in two different ways,

$$0 \to L(-k) \overset{\iota^*}{\to} E \overset{\sigma}{\to} L^*(k) \to 0, \quad 0 \to L^*(-k) \overset{\iota^*}{\to} E \overset{\sigma}{\to} L(k) \to 0,$$

where $L^*(k) = (L(0))^* \otimes O(k)$. The map $\sigma_\mu, \nu$ determines a section $\psi$ of $O(2k)$. The zero set of $\psi$ is the spectral curve $T$. Such spectral curves determine $E$ and are characterized by the conditions that (i) they are compact, (ii) invariant under the real structure $\tau : (\eta, \lambda) \to (-\eta/\lambda^2, -1/\lambda)$ and (iii) the line bundle $L(0)^2$ is trivial on restriction to $T$.

This construction fits into the previous discussion by virtue of the fact that, in order to reverse the Ward construction to reconstruct the solution to the Bogomolny equations, one must first find a holomorphic trivialization for $E$ over each section $\sigma_x : \mathbb{C}P^1 \to T \mathbb{C}P^1$ that is invariant under the real structure $\tau$, for $x \in \mathbb{R}^3$. This is equivalent to finding the appropriate family of holomorphically embedded rational curves in the total space of $E$ that cover $\sigma_x$. By projection in each of the two short exact sequences above, these curves are a subset of those in $\tilde{E} = L(k) \oplus L^*(k)$ over $\sigma_x$. They can be characterized as the curves that intersect the codimension-2 subset of $\tilde{E}$ consisting of the line subbundle $\tilde{T}$ of $\tilde{E}|_T$ defined by the trivialization of $L(0)^2$ over $T$.

It is worth noting that it is straightforward to find the trivialization of $\tilde{E}$ over each $\sigma_x$. This is set out in [13] for $L(0)$ and one can simply multiply that section by a pair of polynomials of degree $k$ in $\xi$ leading to $2k + 1$ sections. Given the spectral curve, $T$ and the trivialization $e$ of $L^2(0)$ over $T$, the incidence condition with $\tilde{T}$ is $2k$ conditions on the $2k + 1$ unknowns and leads to the desired two-dimensional space of sections of $E$.

The implementation of this as a strategy for writing down exact solutions of monopoles is hard. The principal difficulty is in choosing the spectral curve $S$ (the simplest examples, the axisymmetric solutions, are known, but in general the triviality of $L(0)^2$ on $T$ is a hard condition to impose explicitly). Even then, one must solve for the trivialization of $L(0)^2$ on $T$ and then find the intersection points of $T$ with a generic $\sigma_x$ in order to impose the incidence condition with $\tilde{T}$.

4.3. Korteweg de Vries and nonlinear Schrödinger solitons

In [20] it is shown that NLS solitons can be obtained from the Ward ansatz also, and so can be obtained from imposing intersection conditions on some simple admissible family. Here we can see that the construction can be made very explicit.

The KdV and NLS equations arise as symmetry reductions of the $\text{SL}(2, \mathbb{C})$ Bogomolny equations in a $2+1$ signature under a null translation. The corresponding holomorphic vector bundles over regions in $O(2)$ have rank 2 and admit a lift of the symmetry $K = \partial_\mu$, where $(\lambda, \mu)$ are coordinates on $O(2)$, with $\lambda$ being an affine coordinate on the base $\mathbb{C}P^1$ and $\mu$ the fibre coordinate on $O(2)$. Since $O(2) = T \mathbb{C}P^1$, the choice of $\lambda$ determines the trivialization in which $\mu$ corresponds to the tangent vector $\mu \partial/\partial \lambda$. 
Over a neighbourhood of \( \lambda = \infty \), we use coordinates \((\lambda', \mu') = (1/\lambda, \mu/\lambda^2)\). In these coordinates \( V = \lambda'^2 \partial/\partial \mu' \) and so \( K \) fixes the fibre at \( \infty \) to second order. The symmetry reductions to the KdV and NLS equations are distinguished by the action of the lift \( \bar{K} \) of \( K \) to the bundle \( E \) at \( \lambda = \infty \) to second order: we have \( \bar{K} = \Lambda + O(1/\lambda^2) \), where

\[
\text{for NLS } \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and } \quad \text{for KdV } \Lambda = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{\lambda} & 0 \end{pmatrix}.
\]

The trivial solutions correspond to bundles \( E_0 \) with \( \bar{K} \) given as above on a neighbourhood of \( \lambda = \infty \) and patching function \( \exp(\mu \Lambda) \) to a trivialization that extends over \( \lambda \neq \infty \) in which \( \bar{K} \) has trivial lift.

The solitons can be obtained from the trivial solutions by considering first \( E_0(k) = O(k) \otimes E_0 \). This bundle has \( 2k + 2 \) sections over each conic in \( O(2) \). We choose an admissible subfamily by choosing first \( k \) points \( \{\lambda_1, \ldots, \lambda_k\} \) in the upper half plane in \( \mathbb{C} \), and then an invariant section \( \gamma_i \) of \( \mathbb{P}(E^*_0) \) over each fibre \( \lambda = \lambda_i \) of \( O(2) \). We require that \( \gamma_i \) not lie in an eigenspace for \( \Lambda \). These data determine \( 2k \) codimension-2 submanifolds of \( E_0(k) \), the kernel of \( \gamma_i \) over \( \lambda = \lambda_i \) and the kernel of \( \bar{\gamma}_i \) over \( \lambda = \bar{\lambda}_i \). (Note that \( \bar{\gamma}_i \) needs to be interpreted appropriately according to the reality condition that we wish the final solution to satisfy).

In order to see explicitly that this yields the standard formulae for the appropriate admissible subfamily of the space of sections of \( E_0(k) \) over a given conic \( C \) in \( O(2) \), we first represent the sections of \( E_0(k) \) over \( C \) as sections of \( E_0 \) that have simple poles at each \( \lambda_i \). The residues must lie in the kernel of the corresponding \( \gamma_i \), and \( k \) of the remaining \( k + 2 \) coefficients are fixed by the \( k \) conditions of lying in the kernels of the \( \bar{\gamma}_i \) at \( \bar{\lambda}_i \). See example 9.3.3 and section 12.4 from [21] for further details of such solutions.

### 4.4. The ALE solutions

In the previous subsections we have seen that we can express many of the most familiar soliton/instanton solutions in terms of subfamilies of some simple admissible family by imposing intersection conditions. In this subsection we see that the ALE hyper-Kähler solutions (gravitational instantons) can be expressed as an admissible subfamily of a simple family by imposing tangency conditions.

Hyper-Kähler manifolds \((\mathcal{M}, g)\) that have the topology of \( \mathbb{R}^4 \) at infinity and approach the flat Euclidean metric \( \eta = dx_1^2 + \cdots + dx_4^2 \) sufficiently fast, in the sense that

\[
g_{ab} = \eta_{ab} + O(r^{-4}), \quad (\partial_a)^p (g_{bc}) = O(r^{-4-p}), \quad r^2 = x_1^2 + \cdots + x_4^2,
\]

have to be flat. A weaker asymptotic condition one can impose is that \( g \) should be asymptotically locally Euclidean (ALE).

The ALE spaces are non-compact, complete hyper-Kähler manifolds which satisfy the above condition only locally for \( r \to \infty \). Globally the neighbourhood of infinity must look like \( S^3/\Gamma \times \mathbb{R} \), where \( \Gamma \) is a finite group of isometries acting freely on \( S^3 \) (a Kleinian group). These manifolds belong to the class of gravitational instantons because their curvature is localized in a ‘finite region’ of a space–time.

Finite subgroups of \( \Gamma \subset SU(2) \) correspond to Platonic solids in \( \mathbb{R}^3 \). They are the cyclic groups and the binary dihedral, tetrahedral, octahedral and icosahedral groups (one can think of the last three as Möbius transformations of \( S^2 = \mathbb{C}P^1 \) which leave the points corresponding to vertices of a given Platonic solid fixed). Each of them can be related to a Dynkin diagram of a simple Lie algebra. All Kleinian groups act on \( \mathbb{C}^2 \), and the ‘infinity’ \( S^3 \subset \mathbb{C}^2 \). Let \((z_1, z_2) \in \mathbb{C}^2 \). For each \( \Gamma \) there exist three invariants \( x, y, z \) which are
polynomials in \((z_1, z_2)\) invariant under \(\Gamma\). These invariants satisfy some algebraic relations which we list below:

| Group          | Dynkin diagram | Relation \(F_\Gamma(x, y, z) = 0\) |
|----------------|----------------|-----------------------------------|
| Cyclic         | \(A_k\)        | \(xy - z^k = 0\),                |
| Dihedral       | \(D_{k-1}\)    | \(x^2 + y^2z + z^k = 0\),        |
| Tetrahedral    | \(E_6\)        | \(x^2 + y^3 + z^2 = 0\),         |
| Octahedral     | \(E_7\)        | \(x^2 + y^3 + yz^3 = 0\),        |
| Icosahedral    | \(E_8\)        | \(x^2 + y^3 + z^5 = 0\).         |

In each case

\[ \mathbb{C}^2 / \Gamma \subset \mathbb{C}^3 = \{(x, y, z) \in \mathbb{C}^3, \ F_\Gamma(x, y, z) = 0\}. \]

The manifold \(M\) on which an ALE metric is defined is obtained by minimally resolving the singularity at the origin of \(\mathbb{C}^2 / \Gamma\). This desingularization is achieved by taking \(M\) to be the zero set of

\[ \tilde{F}_\Gamma(x, y, z, \lambda) = F_\Gamma(x, y, z) + \sum_{i=1}^r a_i(\lambda) f_i(x, y, z), \]

where \(f_i\) span the ring of polynomials in \((x, y, z)\) divided by the ideal generated by \(\langle \partial_x F_\Gamma, \partial_y F_\Gamma, \partial_z F_\Gamma \rangle\).

The dimension \(r\) of this ring is equal to the number of non-trivial conjugacy classes of \(\Gamma\) which is \(k - 1, k + 1, 6, 7\) and 8, respectively [3]. Kronheimer [17,18] proved that for each \(\Gamma\) a unique hyper-Kähler metric exists on a minimal resolution \(M\), and that this metric is precisely the ALE metric with \(\mathbb{R}^4 / \Gamma\) as its infinity. His construction was a combination of the hyper-Kähler quotient [15] with the twistor theory.

The degrees \(p, q\) and \(r\) are such that \(\tilde{F}_\Gamma(x, y, z, \lambda)\) is a function homogeneous of some degree \(s\). Therefore

\[ \tilde{F}_\Gamma : \mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r) \rightarrow \mathcal{O}(s). \]

To determine the integers \(p, q, r, s\), we require that the normal bundle to a section of \(PT \rightarrow \mathbb{CP}^1\) should have first Chern class 2. To impose this we restrict the Jacobian of the above map to the normal bundle of the curve, and not that the first Chern class is \(p + q + r - s\), which we therefore require to be 2. This gives us the following:

\[
\begin{align*}
A_k & \quad T = \{(x, y, z, \lambda) \in \mathcal{O}(k) \oplus \mathcal{O}(k) \oplus \mathcal{O}(2) \rightarrow \mathbb{CP}^1, x y - z^k - a_1 z^{k-2} - \cdots - a_{k-1} = 0\}, \\
D_{k-1} & \quad T = \{(x, y, z, \lambda) \in \mathcal{O}(2k) \oplus \mathcal{O}(2k - 2) \oplus \mathcal{O}(4) \rightarrow \mathbb{CP}^1, x^2 + y^2z + z^k + a_1 y^2 + a_2 y + a_3 z^{k-2} + \cdots + a_k z + a_{k+1} = 0\}, \\
E_6 & \quad T = \{(x, y, z, \lambda) \in \mathcal{O}(12) \oplus \mathcal{O}(8) \oplus \mathcal{O}(6) \rightarrow \mathbb{CP}^1, x^2 + y^3 + z^4 + y(a_1 z^2 + a_2 z + a_3) + a_4 z^2 + a_5 z + a_6 = 0\}, \\
E_7 & \quad T = \{(x, y, z, \lambda) \in \mathcal{O}(18) \oplus \mathcal{O}(12) \oplus \mathcal{O}(8) \rightarrow \mathbb{CP}^1, x^2 + y^3 + yz^3 + y^2(a_1 z + a_2) + y(a_3 z + a_4) + a_5 z^2 + a_6 z + a_7 = 0\}, \\
E_8 & \quad T = \{(x, y, z, \lambda) \in \mathcal{O}(30) \oplus \mathcal{O}(20) \oplus \mathcal{O}(12) \rightarrow \mathbb{CP}^1, x^2 + y^3 + z^5 + y(a_1 z^3 + a_2 z^2 + a_3 z + a_4) + a_5 z^3 + a_6 z^2 + a_7 z + a_8 = 0\}.
\end{align*}
\]

In each case the twistor space is the three-dimensional hypersurface \(\tilde{F}_\Gamma(x, y, z) = 0\) in the rank 3 bundle \(\mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r) \rightarrow \mathbb{CP}^1\), where now \(x(\lambda), y(\lambda), z(\lambda)\) are coordinates of
the fibres of $\mathcal{O}(p), \mathcal{O}(q), \mathcal{O}(r)$, respectively, $f_p = f_1(x, y, z)$, and $a_i = a_i(\lambda)$ is a global section of the appropriate power of $\mathcal{O}(1)$ to make $\tilde{F}$ homogeneous. Therefore we have projections

$$f_p : \mathcal{PT} \rightarrow \mathcal{O}(p), \quad f_q : \mathcal{PT} \rightarrow \mathcal{O}(q), \quad f_r : \mathcal{PT} \rightarrow \mathcal{O}(r)$$

and we can, for example, express $\mathcal{PT}$ as a branched cover of $\mathcal{O}(p) \oplus \mathcal{O}(q)$ branched over the singular locus of $f_p \oplus f_q$. Rational curves in $\mathcal{PT}$ project to give rational curves in $\mathcal{O}(p) \oplus \mathcal{O}(q)$ tangent to the singular locus of $f_p \oplus f_q$ and the condition that a rational curve (i.e. a section) in $\mathcal{O}(p) \oplus \mathcal{O}(q)$ admits a lift to $\mathcal{PT}$ is that it should be tangent to the singular locus of $f_p \oplus f_q$. This is an admissible subfamily by 2.5. Thus, in particular, we see that the ALE spaces can be realized as admissible subfamilies of the spaces of sections of $\mathcal{O}(p) \oplus \mathcal{O}(q)$, in fact in three different ways.

4.5. Three-dimensional Einstein–Weyl structures

Here we consider classes of Einstein–Weyl spaces of dimension 3. Such spaces arise as the generic geometry on the space of rational curves lying in a surface with normal bundle $\mathcal{O}(2)$. In this section we see how examples of three-dimensional Einstein–Weyl spaces can be constructed by considering admissible subfamilies of relatively simple higher-dimensional families of rational curves on surfaces by imposing intersection and tangency.

Let $\mathcal{M}$ be a three-dimensional manifold with a torsion-free connection $D$ and a conformal structure $[h]$ which is compatible with $D$ in the sense that $Dh = \omega \otimes h$ for some one-form $\omega$. Here $h \in \mathcal{M}$ is a representative metric in a conformal class. If we change this representative by $h \rightarrow \psi^2 h$, then $\omega \rightarrow \omega + 2d \ln \psi$, where $\psi$ is a non-vanishing function on $\mathcal{M}$. The space of oriented $D$-geodesics in $\mathcal{M}$ is a manifold $T$ of dimension 4. There exists a fixed point free map $\tau : T \rightarrow T$ which reverses an orientation of each geodesic. Let $\gamma$ be an oriented geodesic in $\mathcal{M}$, and let $U$ be a vector field tangent to $\gamma$.

The almost-complex structure on $T$ defined by

$$J(V) = \frac{U \times V}{\sqrt{h(U, U)}}$$

is integrable if for any choice of $h \in [h]$ the symmetrized Ricci tensor of $D$ is proportional to $h$. This is the conformally invariant Einstein–Weyl condition. Hitchin [13] has demonstrated the one-to-one correspondence between local solutions to the Einstein–Weyl equations, and complex surfaces (twistor spaces) $\mathcal{T}$ equipped with a fixed-point free anti-holomorphic involution $\tau$, and a $\tau$-invariant rational curve with a normal bundle $\mathcal{O}(2)$.

The EW space can be completely reconstructed from the twistor data; since $H^0(\mathbb{C}P^3, \mathcal{O}(2)) = \mathbb{C}$ and $H^1(\mathbb{C}P^3, \mathcal{O}(2)) = 0$, we can use Kodaira’s theorem A.1. The EW space is a space of those $\mathcal{O}(2)$ curves which are $\tau$-invariant. The family of such curves passing through a given point (and its conjugate) is a geodesic of a Weyl connection of $D$. To construct a conformal structure $[\mathcal{h}]$ consider a point on a $\tau$-invariant $\mathcal{O}(2)$ curve $C_m$. This point represents a point in a sphere of directions $(T_m \mathcal{M} = 0)/\mathbb{R}^4$, and the conformal structure on $C_m$ induces a quadratic conformal structure in $\mathcal{M}$.

One class of solutions can be constructed by taking an $n$-fold covering of a neighbourhood of a $(1, n)$ curve $\zeta = P(\lambda)/Q(\lambda)$ in the quadric $\mathbb{C}P^1 \times \mathbb{C}P^1$. Here $\zeta$ and $\lambda$ are affine coordinates on $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $P, Q$ are polynomials of degree $n$ in $\lambda$. The curve has a normal bundle $\mathcal{O}(2n)$, and the space of such curves is parametrized by $\mathbb{C}P^{2n+1}$ minus the hypersurface where the resultant of $P$ and $Q$ vanishes. In [22] Pedersen considered an $n$-fold cover $\mathcal{T}$ of $\mathbb{C}P^1 \times \mathbb{C}P^1$ branched along a fixed curve $\zeta = \lambda^n$. The $(1, n)$ curves which meet the fixed curve to the $n$th
order give rise to curves with a normal bundle \( \mathcal{O}(2) \) in \( T \) satisfying the condition
\[
\frac{P(\lambda)}{Q(\lambda)} = \lambda^n - \frac{(a\lambda^2 + b\lambda + c)^n}{Q(\lambda)}.
\]
Here \( a, b, c \) are complex coordinates on the resulting EW space. More work is required to impose Euclidean or Lorentzian reality conditions.

Another class of EW spaces could be constructed by blowing up a point on the quadric, and considering all \((1, n)\) curves passing through this point. The resulting curves in the blown up surfaces have normal bundle \( \mathcal{O}(2n - 1) \). This process may also be combined with taking the branched covering.

The explicit forms of resulting EW structures were determined only for \( n = 2 \) and \( n = 3 \) [22, 23]. They are quite complicated, but components of \( h \) and \( \omega \) are algebraic expressions in local coordinates \((a, b, c)\). According to our proposal these solutions should be regarded as ‘solitons’ of the EW geometry. Further analysis of the corresponding conformal invariants is required for justifying this claim.

Yet another class of examples arises from a rational curve in \( \mathbb{CP}^2 \) of degree \( d > 1 \) whose singularities are \( D = (d - 1)(d - 2)/2 \) distinct nodes. Let \( T \) be a surface obtained from \( \mathbb{CP}^2 \) by blowing up points on \( S \cup N \), where \( S \) is a set of \( s \) non-singular points and \( N \) is the set of \( n \) nodes on the curve. The resulting curve in the blown up space will have a normal bundle \( \mathcal{O}(k) \) where \( k \) depends on \( n \) and \( s \).

## 4.6. Generalized Wünschmann conditions

As a final application of the above ideas, related to the previous subsection, we consider an \( n \)-dimensional family of curves in a surface \( \Sigma \). Such curves have a natural lift to the bundle of \((n - 2)\)-jets of such curves \( T = J^{n-2}\Sigma \) over \( \Sigma \). This family of curves will be expressed as a family of solutions to an ordinary differential equation of order \( n \) below and will be considered as a family of curves in \( T \). We then ask whether we can characterize those differential equations that give rise to an admissible family of rational curves in \( T \). This turns out to be given by what we will refer to as the Wünschmann condition [28] and its generalizations on the coefficients of the equations. One can ask whether the projection of this family to a families of curves in \( J^r\Sigma, r < n - 2 \), is admissible, and this will lead to further generalizations of the Wünschmann conditions.

Consider a relation of the form
\[
\Psi(x, y, m) = 0
\]
between the complex variables \( m = (m_1, m_2, \ldots, m_n) \) (local coordinates on an \( n \)-dimensional manifold \( \mathcal{M} \)) and \((x, y)\) (complex local coordinates on a two-dimensional manifold \( \Sigma \)). For each fixed choice of \((x, y)\) the relation defines an \( \alpha \)-surface in \( \mathcal{M} \). Conversely each choice of \( m \) defines a curve \( C_m \) in \( \Sigma \). We can apply the implicit function theorem to \( \Psi = 0 \), and regard \( C_m \) as a graph \( x \rightarrow (x, y = Z(x, m)) \). Consider a system of equations consisting of \( y = Z(x, m) \) and the first \((n - 1)\) derivatives with respect to \( x \). Solving this system for \( m \) and differentiating once more with respect to \( x \) yields the ODE
\[
y^{(n)} := \frac{d^n y}{dx^n} = F(x, y, y', \ldots, y^{(n-1)}),
\]
where the explicit form of \( F \) is completely determined by \( \Psi \).

Asking that the \( \alpha \)-surfaces in \( \mathcal{M} \) arise from specific geometric structures on \( \mathcal{M} \) (which from now on will be identified with the space of solutions to the ODE) imposes additional constraints on \( F \). This idea goes back to Cartan [5], and his programme of ‘geometrizing’ ODEs.
A different approach based on twistor theory was suggested by Hitchin [14] and LeBrun [19]. In this approach the relation $\Psi = 0$ represents part of a rational curve in $\Sigma = T$ with a prescribed normal bundle. The local differential geometry of $\mathcal{M}$ is encoded in the global complex structure of $\mathcal{T}$, and the globality of the curve implies that $\alpha$-curves are the geodesics of a projective structure. The ODE does not explicitly appear in the correspondence between $\mathcal{M}$ and $\mathcal{T}$. The details of the Hitchin–LeBrun construction and its connection with the ODE approach have been worked out fully only for $n = 2$. In this case there exists an embedding of a rational curve with a normal bundle $\mathcal{O}(1)$ in $\mathcal{T}$ if and only if

$$
\frac{d^2}{dx^2} F_{11} - 4 \frac{d}{dx} F_{01} - F_1 \frac{d}{dx} F_{11} + 4 F_1 F_{01} - 3 F_0 F_{11} + 6 F_{00} = 0,
$$

where $F_0 = \partial F/\partial y$, $F_1 = \partial F/\partial y'$ and $d/dx = \partial/\partial x + y'\partial/\partial y + F\partial/\partial y'$. The two-dimensional moduli space $\mathcal{M}$ of $\mathcal{O}(1)$ curves (the space of solutions to the ODE is in this case equipped with a projective structure, in the sense that the $\alpha$-surfaces (here curves) of constant $(x, y)$ are the geodesics of a torsion-free projective connection. Conversely, given a projective structure on $\mathcal{M}$, one defines $\mathcal{T}$ as a quotient space of the foliation of $P(T\mathcal{M})$ by the orbits of the geodesic flow. Each projective tangent space $P(T_m\mathcal{M})$ maps to a rational curve with self-intersection number 1 in $\mathcal{T}$.

The case $n = 3$ which goes back to Cartan [5] was recently revisited by Tod [25]. The conformal structure on $\mathcal{M}$ is defined by demanding that hypersurfaces $z \subset \mathcal{M}$ corresponding to points in $\mathcal{T}$ are null (it could also be defined by declaring the curves in $\mathcal{M}$ that correspond to points of $\mathcal{T} J^1 \Sigma$ to be null geodesics). This conformal structure does not depend on $(x, y) \in \Sigma$ if $F(x, y, y', y'')$ satisfies a second order differential constraint

$$
\frac{1}{3} F_2 \frac{d}{dx} F_3 - \frac{1}{6} \frac{d^2}{dx^2} F_2 + \frac{1}{2} \frac{d}{dx} F_1 - \frac{2}{27} (F_3)^3 - \frac{1}{3} F_2 F_1 - F_0 = 0.
$$

This constraint has already appeared in a work by Wünschmann [28].

The only other case which has attracted some attention is $n = 4$. Bryant [4] has shown that there exists a correspondence between a class of fourth order ODEs and exotic non-metric holonomies in dimension 4. The conditions on $F$ are only implicit in Bryant’s work.

We shall say that $F$ satisfies the generalized Wünschmann conditions if there exists an $SL(2, \mathbb{C})$ invariant paraconformal structure

$$
\mathcal{T} \mathcal{M} \cong \mathbb{C}^2 \odot \mathbb{C}^2 \odot \cdots \odot \mathbb{C}^2 = S^{n-1}(\mathbb{C}^2).
$$

The explicit form of the generalized Wünschmann conditions has been worked out recursively in [8]. For example if $n = 4$ one gets

$$
\frac{11}{1600} (F_3)^3 - \frac{9}{50} (F_3)^2 \frac{d}{dx} F_3 = \frac{1}{200} (F_3)^2 F_2 + \frac{21}{100} \left( \frac{d}{dx} F_3 \right)^2 + \frac{1}{50} \left( \frac{d}{dx} F_3 \right) F_2
$$

$$
- \frac{9}{100} (F_2)^2 F_3 + \frac{7}{20} F_3(x) \frac{d^2}{dx^2} F_3 - \frac{1}{5} \frac{d^3}{dx^3} F_3 + \frac{3}{10} \frac{d^2}{dx^2} F_2 - \frac{1}{4} F_3 \frac{d}{dx} F_2 - F_0 = 0.
$$

If $F$ satisfies these conditions, then the space $\mathcal{M}$ of solutions to the corresponding ODE is equipped with a torsion-free connection with holonomy $G_3$ in the terminology of [4].

In general $F$ has to satisfy an over-determined system of $n - 2$ PDEs, and a priori it is not clear that any solutions exist. It can however be verified that the method of admissible curves provides (some) solutions to all the constraints. For example one can consider the blowups described in the last section and find simple solutions $F = (4/3)(y'')^2/y''$, or $F = (ay'' + b)^{3/2}$ when $n = 4$. 

Families of rational curves in twistor spaces 555
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Appendix: rational curves and their embeddings

Let $C^2$ be a vector space with coordinates $\pi = (\pi_0, \pi_1)$. Remove $\pi = (0, 0)$ and use $[\pi]$ as homogeneous coordinates on $\mathbb{C}P^1$. We shall also use the affine coordinate $\lambda = \pi_0/\pi_1$.

Holomorphic functions on $C^2 - 0$ extend to holomorphic functions on $C^2$ (Hartog’s theorem). Therefore homogeneous functions on $\mathbb{C}P^1$ are polynomials. In particular, holomorphic functions homogeneous of degree 0 are constant (Liouville theorem). Let us summarize some facts about holomorphic line bundles over $\mathbb{C}P^1$. First define a tautological line bundle $O(-1) = \{(\lambda, (\pi_0, \pi_1)) \in \mathbb{C}P^1 \times C^2 | \lambda = \frac{\pi_0}{\pi_1}\}$.

Other line bundles can be obtained from $O(-1)$ by algebraic operations:

- $O(-n) = O(-1)^{\otimes n}$,
- $O(n) = O(-n)^*$,
- $O = O(-1) \otimes O(1)$, $n \in \mathbb{N}$.

Equivalently $O(n)$ denotes the line bundle over $\mathbb{C}P^1$ with transition functions $\lambda^{-n}$ from the set $\lambda \neq \infty$ to $\lambda \neq 0$ (i.e. Chern class $n$). Its sections are given by functions homogeneous of degree $n$ in the sense that $f(\xi \pi) = \xi^n f(\pi)$. These are polynomials in $\lambda$ of degree $n$ with complex coefficients. The theorem of Grothendick states that all holomorphic line bundles over a rational curve are equivalent to $O(n)$ for some $n$. The spaces of global sections and the first cohomology groups are

$$H^0(\mathbb{C}P^1, O(n)) = \begin{cases} \mathbb{C} & \text{for } n < 0, \\ \mathbb{C}^{n+1} & \text{for } n \geq 0. \end{cases}$$

$$H^1(\mathbb{C}P^1, O(-n)) = \begin{cases} 0 & \text{for } n < 2, \\ \mathbb{C}^{n-1} & \text{for } n \geq 2. \end{cases}$$

The following result of Kodaira underlies the twistor approach to curved geometries. Let $T$ be a complex manifold of dimension $d + r$. A pair $(E, M)$ is called a complete analytic family of compact submanifolds of $T$ of dimension $d$ if

- $E$ is a complex analytic submanifold of $T \times M$ of codimension $r$ with the property that for each $m \in M$ the intersection $C_m := E \cap (T \times m)$ is a compact submanifold of $T \times m$ of dimension $d$.
- There exists an isomorphism

$$T_m M \simeq H^0(C_m, NC_m),$$

where $NC_m \rightarrow C_m$ is the normal bundle of $C_m$ in $T$.

**Theorem A.1 (Kodaira [16]).** Let $E$ be a complex compact submanifold of $T$ of dimension $d$, and let $NE$ be the normal bundle of $E$ in $T$. If $H^1(E, NE) = 0$, then there exists a complete analytic family $(E, M)$ such that $E = E(m_0)$ for some $m_0 \in M$.

We will apply the above theorem to the situation where $T$ is a twistor space and $E = \mathbb{C}P^1$. Roughly speaking, the moduli space $M$ is the ‘arena’ of differential geometry and integrable systems.
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