A Normal Coordinate Expansion
for the Gauge Potential *

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Abstract

In this pedagogical note, I present a method for constructing a fully covariant normal coordinate expansion of the gauge potential on a curved space-time manifold. Although the content of this paper is elementary, the results may prove useful in some applications and have not, to the best of my knowledge, been discussed explicitly in the literature.

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1 Introduction

Riemannian normal coordinates are first introduced in the geometric interpretation of gravitation as a realization of the equivalence principal which requires the existence of an inertial reference frame at every point in space-time in which the effects of gravity can be locally neglected.

In addition to their axiomatic significance, normal coordinates have found a very useful place in perturbative quantum field theory on curved manifolds, and perhaps, in quantum gravity. Generally covariant Taylor-type expansions in normal coordinates have proved useful in the path-integral environment, for example, to perform loop calculations for the non-linear sigma model \cite{1, 2} and to analyze trace anomalies \cite{3}. Generally, one can use these normal coordinate expansions to perform the so-called proper time heat kernel expansion (sometimes known as the DeWitt expansion \cite{4}) both with \cite{5} and without \cite{6, 7} the benefit of worldline path-integral methods. Within the framework of the background field method, the latter expansions can be very useful for straightforward determinations of the ultraviolet behaviour of radiative corrections \cite{8}.

A good discussion of the generally covariant normal coordinate expansion for a tensor field can be found in reference \cite{1} where the expansion is given explicitly to fourth order. However, as far as the author is aware, very little has been published concerning the fully covariant expansion of a gauge field. The aim of this paper is to provide such a covariant expansion.

The following section reviews the construction of normal coordinates, and introduces the notation which will be used thereafter. In section \cite{3} the gauge field is introduced, along with the so-called radial gauge condition which turns out to fit very well in the normal coordinate system and can be used to construct a generally gauge-covariant normal coordinate expansion.
2 Normal Coordinates

Suppose we have some curved space-time manifold with a local coordinate system $q^\alpha$ defined in the neighbourhood of a fixed point $\phi$, and a corresponding metric with affine connection $\Gamma^\alpha_{\beta\gamma}(q)$. We would like to define what is meant by a normal coordinate system with $\phi$ at the origin.

For any given point $q^\alpha$ we construct a geodesic $\lambda^\alpha(q,t)$ which connects $q$ with $\phi$. Then $\lambda$ can be taken to satisfy the equation

$$\ddot{\lambda}^\alpha(q,t) + \Gamma^\alpha_{\beta\gamma}(q,t)\dot{\lambda}^\beta(q,t)\dot{\lambda}^\gamma(q,t) = 0,$$

for $t \in [0, 1]$, with end-points

$$\lambda^\alpha(q,0) = \phi^\alpha$$
$$\lambda^\alpha(q,1) = q^\alpha.$$

The normal coordinates of any point $q$ are defined to be the components of the tangent vector $\xi(q)$, at the origin $\phi$, of the geodesic ending at $q$, i.e.

$$\xi^\alpha(q) = \dot{\lambda}^\alpha(q,0).$$

Despite the suggestive notation, $\xi(q)$ is not a vector field since the right hand side of equation (2) is a tangent vector at the origin $\phi$, not at $q$.

3 The Gauge Field $A_\alpha = A^a_\alpha T^a$

We turn our attention to the task at hand, that of developing a fully covariant normal coordinate expansion for the gauge potential.

To start, it has been shown [7, 9] that an appropriate gauge condition for this type of expansion is the radial or synchronous gauge (a curved-space generalization of the Fock-Schwinger gauge [10, 11]) which fits very well in the normal coordinate construction. In the
basis of the normal coordinate system, the gauge condition is
\[ \xi^\alpha A_\alpha(q(\xi)) = 0. \]  
(3)

This condition fixes the gauge relative to a global gauge transformation.

Using either a method of integration along the geodesics (which is formally identical to the flat-space problem \[9, 10\]) or differential forms \[7\] one can show, in the normal coordinate system, that the radial gauge leads to gauge-covariant expression for the gauge potential,
\[ A_\gamma(q(\xi)) = \sum_{n=0}^{\infty} \left[ \xi^\beta \nabla_\beta(\phi) \right]^n \xi^\alpha F_{\alpha\gamma}(\phi), \]  
(4)

where \( \nabla_\beta = \partial_\beta + \text{adj} A_\beta \) is the gauge-covariant derivative in normal coordinates, and the covariant field strength tensor is \( F_{\alpha\gamma} = [\nabla_\alpha, \nabla_\gamma]. \)

Now, this latter equation is not exactly of the desired form since the derivatives (\( \nabla_\beta \)) are not covariant under reparametrization of the manifold. However, using the methods of reference \[1\] it is straightforward to write these gauge-covariant derivatives at the origin in terms of the corresponding fully-covariant derivatives, denoted by indices trailing the semicolon (\( ; \)). For example, one can show that \[1\]
\[ \nabla_{\beta_1} F_{\beta_0\gamma} \doteq F_{\beta_0\gamma;\beta_1} \]  
(5)
\[ \nabla_{\beta_2} \nabla_{\beta_1} F_{\beta_0\gamma} \doteq F_{\beta_0\gamma;\beta_1\beta_2} + \frac{1}{2} R_{\beta_1\beta_2\gamma;\beta_3}^{\delta} F_{\beta_0\delta}, \]  
(6)
\[ \nabla_{\beta_3} \nabla_{\beta_2} \nabla_{\beta_1} F_{\beta_0\gamma} \doteq F_{\beta_0\gamma;\beta_1\beta_2\beta_3} + \frac{1}{3} R_{\beta_1\beta_2\gamma;\beta_3\beta_4}^{\delta} F_{\beta_0\delta} + F_{\beta_1\beta_2\gamma;\beta_3\beta_4}^{\delta} F_{\beta_0\delta;\beta_4}, \]  
(7)
\[ \nabla_{\beta_4} \nabla_{\beta_3} \nabla_{\beta_2} \nabla_{\beta_1} F_{\beta_0\gamma} \doteq F_{\beta_0\gamma;\beta_1\beta_2\beta_3\beta_4} + \frac{3}{5} R_{\beta_1\beta_2\gamma;\beta_3\beta_4\beta_5}^{\delta} F_{\beta_0\delta;\beta_4} + 2 F_{\beta_1\beta_2\gamma;\beta_3\beta_4}^{\delta} F_{\beta_0\delta;\beta_3\beta_4} + \frac{1}{5} R_{\beta_1\beta_2\gamma;\beta_3\beta_4\beta_5}^{\delta} R^{\epsilon}_{\beta_5\beta_4\delta} F_{\beta_0\epsilon}. \]  
(8)

where \( \doteq \) indicates equality at the origin only after symmetrization of the \( \beta_i \) indices. Substitution of equations (5)-(8) into eq. (4) yields the fully covariant normal coordinate expansion

\[ \text{1}\]The author suspects that the fourth derivative of a rank-two tensor implied in reference \[1\] is not entirely correct. The corresponding coefficients presented here for the field strength (equation (8)) should be reliable.
to fifth-order in the normal coordinates,

\[
A_\alpha(q) = \frac{1}{2} \left\{ F_{\beta\alpha} \right\} \xi^\beta + \frac{1}{3} \left\{ F_{\beta_0\gamma;\beta_1} \right\} \xi^\beta_0 \xi^\beta_1 + \frac{1}{8} \left\{ F_{\beta_0\gamma;\beta_1\beta_2} + \frac{1}{3} R^\delta_{\beta_1\beta_2\gamma} F_{\beta_0\delta} \right\} \xi^\beta_0 \xi^\beta_1 \xi^\beta_2 \\
+ \frac{1}{3!5} \left\{ F_{\beta_0\gamma;\beta_1\beta_2\beta_3} + \frac{1}{2} R^\delta_{\beta_1\beta_2\gamma;\beta_3} F_{\beta_0\delta} + R^\delta_{\beta_1\beta_2\gamma} F_{\beta_0\delta;\beta_3} \right\} \xi^\beta_0 \xi^\beta_1 \xi^\beta_2 \xi^\beta_3 \\
+ \frac{1}{4!6} \left\{ F_{\beta_0\gamma;\beta_1\beta_2\beta_3\beta_4} + \frac{3}{5} R^\delta_{\beta_1\beta_2\gamma;\beta_3\beta_4} F_{\beta_0\delta} + 2 R^\delta_{\beta_1\beta_2\gamma;\beta_3} F_{\beta_0\delta;\beta_4} \right\} \xi^\beta_0 \xi^\beta_1 \xi^\beta_2 \xi^\beta_3 \xi^\beta_4 \\
+ 2 R^\delta_{\beta_1\beta_2\gamma} F_{\beta_0\delta;\beta_3\beta_4} + \frac{1}{5} R^\delta_{\beta_1\beta_2\gamma} R^\epsilon_{\beta_3\beta_4\delta} F_{\beta_0\epsilon} \right\} \xi^\beta_0 \xi^\beta_1 \xi^\beta_2 \xi^\beta_3 \xi^\beta_4 \\
+ O\xi^6.
\]

All coefficients in braces \{\cdots\} are evaluated at the origin, where the basis vectors for the normal coordinate system coincide with those of the original system. Since potential on the left hand side of this equation is not a vector at the origin, its indices must refer to the normal coordinate basis. (This is also true of the expansions presented in reference [1]).

Higher order corrections to this expansion should be very straightforward to obtain by using the methods of ref. [1] along with equation (4).

4 Discussion

By fusing the ungauged developments of ref. [1] with the gauge expansion of refs. [7, 9], we have been able to show how a normal coordinate expansion can be constructed for the gauge field with fully covariant coefficients in curved space.

Although this type of expansion may have additional uses, the motivation for this work was in constructing a proper time (DeWitt) expansion for the gauged heat kernel in curved space-time. In that context, these results could contribute to a covariant analysis of the renormalization group in non-Abelian gauge theory. The results of the former study are forthcoming [5].

Finally, it should be pointed out that equation (4) agrees with a third-order normal coordinate expansion which can be found in ref. [3] (also in the context of proper time
expansions). Regrettably, the gauge field expansion falls outside of the main interests of that work, so all relevant details have been omitted. The author is not aware of anywhere else where this expansion has been constructed explicitly.

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