Recombination of dyons into calorons
in SU(2) lattice fields at low temperatures

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I. INTRODUCTION

At high temperatures, near but below the deconfinement temperature, classical solutions of Yang-Mills equations with nontrivial holonomy (KvB calorons \[3, 4, 5, 6\]) are seen on the lattice for SU(2) gauge theory to be frequently dissociated into dyons \[3, 4, 5, 6\]. This means that the distance between the dyons forming a caloron,

\[d = \frac{\pi \rho^2}{b},\]

is larger than the size of dyons (which is \(b/\pi\) for the case of maximally nontrivial holonomy). Here \(\rho\) is the instanton size parameter and \(b\) is the temporal periodicity interval.

These observations have been made by the use of cooling. Therefore, the question arises why such a substructure has not been observed by previous authors who have used the cooling method. In this paper we will argue that there is only a certain window of temperature or space-time asymmetry where it can be revealed by this method.

The interest in the existence of caloron constituents has increased since it has been demonstrated in Ref. \[8\] that a constituent substructure very reminiscent of the caloron solutions can also be identified without cooling, above and below the phase transition. This can be achieved by using the localization properties of the fermionic zero modes of a suitably chirally-improved Dirac operator. The similarity with the properties of a caloron solution is strikingly realized for a certain fraction of configurations with topological charge \(Q = \pm 1\), where the single zero mode is seen to change its localization when the periodicity of fermionic boundary conditions becomes modified. A systematic study \[9\] of the typical pattern of localization and delocalization followed by jumps of the zero mode has revealed that this pattern depends on the timelike holonomy exactly in a caloron-like way. Whereas the topological density has a much more complicated structure, the positions where the zero mode is undissociated actually show the signatures expected for caloron constituents \[8\]: they are local maxima of the topological density \(g(x)\) with a sign as required by the chirality of the mode. This suggests that (dissociated or non-dissociated) calorons might really form the semiclassical background of the gauge field near the phase transition.

Coming back to SU(2) calorons with their two constituents, it seems that the quantities \(d\) and \(\rho\) appearing in eq. \(1\) are impossible to be measured simultaneously: when \(d\) is seen by observation of separate positions no instanton-like profile (of topological density) is observed which could be used to define \(\rho\). When \(d\) goes to zero \((d \ll b)\) an instanton size parameter \(\rho\) can be measured by comparison with the instanton’s action density profile, but then no dyons are seen as separate objects. More precisely: the parameter \(d\) cannot be measured for all caloron configurations as the distance between constituents as long as only the action or topological charge densities are available as local observables to describe them.

The time periodicity parameter \(b\) defines the temperature \(T\): \(b = 1/T\). In order to demonstrate how the recombination of constituents depends on the temperature we can change the temporal extent \(b\) of the lattice.

We employ the standard relaxation cooling technique using the Wilson lattice action and concentrate on the investigation of lowest-action field configurations with \(Q = \pm 1\). Besides of the fact that solutions with \(|Q| = 1\) are not existing on a torus in a mathematically strict sense \[4\], also the cooling method has a limited relevance. It cannot be used for revealing the full topological structure of the QCD vacuum even if the latter is semi-classical.
to a certain extent. In particular, it considerably weakens the chiral condensate compared with its value for equilibrium fields \[ \rho \approx \frac{\pi}{g} \]. Moreover, it is well-known that the Wilson action depends on the instanton (or caloron) scale-size. Therefore, its minimization shrinks the localized solutions until they disappear ‘through the lattice meshes’. Thus, small excitations will be lost first. In the literature - besides fermionic methods - there are better techniques like improved cooling, smoothing, smearing etc. allowing to decipher the topological long-range structure of the gauge fields. But this task is not our concern in this paper. Here we ask the more modest question, what kinds of simple classical solutions can be found from equilibrium gauge fields at different temperatures by successively minimizing the action.

This question remains interesting for those who want to build or apply semi-classical-like approximations, i.e models like the instanton gas or liquid model \[ \frac{\pi}{g} \] being successful in many phenomenological applications for which chiral symmetry breaking (and not quark confinement) is playing the major rôle \[ \frac{\pi}{g} \]. Until now most of the cooling (or smoothing) results obtained at \( T < T_c \) have been interpreted in terms of BPST instanton and Harrington-Shepard (HS) caloron solutions, respectively, all exhibiting trivial asymptotic holonomy. Here we would like to convince the reader that the simplest caloron or instanton solutions seen after cooling have typically non-trivial holonomy, irrespectively of their possible dissociation into dyons pairs. Therefore, they cannot be correctly interpreted as BPST or HS solutions.

The paper is organized as follows. In section II we draw the attention to the static nature of configurations near the deconfining transition. Section III presents our results on semiclassical configurations at finite temperatures, pointing out the loss of “staticity” and the increasing importance of the Polyakov loop for detecting the non-trivial substructure at lower temperature. In section IV we extend the cooling studies to the symmetric torus. We emphasize that the KvB caloron solutions were constructed in an infinite spatial volume. When used at finite volume considerable deviations are to be expected when the spatial box size is no longer large compared to \( b \), since the typical size of the constituents is \( b/\pi \). Finally, in section V we discuss the consequences, also in the perspective of a twin paper by Gatteringer et al.

II. NON-STATICITY AND SEPARATION INTO CONSTITUENTS

It turns out that the possibility to observe the dyonic constituents of a KvB caloron as lumps of action depends on \( (\frac{\pi \rho}{b})^2 \). In \( SU(2) \) LGT at \( \beta \equiv 4/g_0^2 = 2.2 \) on a lattice \( 16^3 \times 4 \) the parameter \( \rho \) is concentrated near the value \( \rho \sim 2.5a \) (\( a \) is the lattice spacing) \[ \frac{\pi}{g} \]. With \( b = 4a \),

\[
\left(\frac{\pi \rho}{b}\right)^2 \approx 4 \gg 1.
\]

This means that dyons are well separated. The value of \( \rho \) in \[ \frac{\pi}{g} \] was determined by fitting the lattice caloron with the analytic KvB caloron, and formula \[ \frac{\pi}{g} \] was used. On the lattice \( 16^3 \times 6 \) (with \( b = 6a \)) and at the same \( \beta = 2.2 \) (i.e., at a temperature 1.5 times lower) the parameter \( (\pi \rho)^2 \) would be of the order \( O(1) \). Then, from this simple arithmetics, one would expect that caloron are not dissociated into dyonic lumps anymore.

The possibility to measure the distance between dyons inside a caloron just by detecting the peaks of the action density on the lattice is given only in the case of well-separated objects. Indirectly this distance can be measured by measuring a quantity that can be called non-staticity \[ \frac{\pi}{g} \]. Unfortunately, cooling yields metastable plateaus only for temperatures below the deconfining temperature. On the other hand, this allows us to restrict ourselves in the following to maximally non-trivial holonomy because the average Polyakov line vanishes. To describe for this special case the relation between distance \( d \) and non-staticity, we have considered analytical caloron solutions in continuous space-time. We divided the time interval \( b \) into \( N_t \) time slices and expressed the action in the \( i \)-th time slice, \( S_i = \sum_{\vec{x}} S_{\vec{x},i} \), in terms of the local action density \( s_{\vec{x},i} \). The non-staticity \( \delta_t \) is defined as

\[
\delta_t = \frac{\sum_{i=1}^{N_t} \sum_{\vec{x}} |s_{\vec{x},i+1} - s_{\vec{x},i}|}{\sum_{i=1}^{N_t} \sum_{\vec{x}} s_{\vec{x},i}}.
\]

Obviously, this definition depends on the number of time slices \( N_t \) such that one would have \( \delta_t \to 0 \) for \( N_t \to \infty \). In order to get an asymptotically \( N_t \) independent quantity we will modify the definition of \( \delta_t \) as follows

\[
\delta_t = \frac{\sum_{i=1}^{N_t} \sum_{\vec{x}} |s_{\vec{x},i+1} - s_{\vec{x},i}| N_t}{\sum_{i=1}^{N_t} \sum_{\vec{x}} s_{\vec{x},i}} 4.
\]

The factor 1/4 has been chosen such that for a lattice \( 16^3 \times 4 \), where all the simulations were initially done using \[ \frac{\pi}{g} \], the two definitions \[ \frac{\pi}{g} \] and \[ \frac{\pi}{g} \] agree.

With the definition \[ \frac{\pi}{g} \] at hands we can calculate the non-staticity of an analytic KvB caloron. The non-staticity depends on the holonomy and on the distance between the constituents inside the caloron. For maximally nontrivial holonomy (which coincides with the average holonomy in the confinement phase) the constituent dyons have equal mass. For this simplified case we have determined a bifurcation value of the non-staticity, \( \delta_t^* = 0.27 \), choosing the distance between the constituents such that the two lumps of action density (dyons) merge into one lump of action density. This single lump is what we call a "recombined caloron". As already mentioned, the non-staticity uses the discreteness of a lattice configuration. The value \( \delta_t^* \) is obtained by inserting the analytic form of the action density of a continuum KvB caloron solution \[ \frac{\pi}{g} \], calculated exactly at the point of bifurcation, into the evaluation of \[ \frac{\pi}{g} \] using a grid with lattice spacing \( a = b/N_t \). In order to see how well-defined at finite \( N_t \) this "bifurcation value" \( \delta_t^* \)
can be, we have evaluated it for various (even and odd) $N_t \geq 4$. The function $\delta^*_t(N_t)$ that goes to $\delta^*_t = 0.27$ for $N_t \to \infty$ is presented in Fig. 1.

If the non-staticity is lower than the bifurcation value $\delta^*_t$ two dyons can be distinguished by the two maxima of the action density. If the non-staticity is bigger than $\delta^*_t$ two dyons appear recombined into a caloron with only one action density maximum.

III. RECOMBINATION OF DYONS INTO CALORONS WITH LOWERING TEMPERATURE

The main results of this paper are obtained from ensembles of $SU(2)$ gauge field configurations created by heat bath Monte Carlo at $\beta = 2.2$ with respect to the Wilson action $S_W$ on lattices $16^3 \times N_t$ with $N_t = 4$, $N_t = 5$ and $N_t = 6$. Each ensemble consisted of about 8000 independent configurations. Cooling of these configurations was performed using the fastest possible relaxation with respect to the Wilson action. For this method each link $U_{x,\mu}$ is immediately replaced by the projection to $SU(2)$ of the staples around it, $\tilde{U}_{x,\mu}$.

The cooled configurations studied in this paper have been identified when the cooling history finally has arrived at a quasi-stable plateau on the level of a single instanton action $S_W \approx S_{\text{inst}} = 2\pi^2 \beta$. Lateron, we describe also some results concerning higher plateaus of action, e.g. $S_W \approx 3 S_{\text{inst}}$ which are passed earlier in the cooling process. More precisely, on such plateaux we stopped cooling always at local minima of the violation of the lattice equations of motion \cite{3}. Here, the violation is defined as

$$ V = \sum_{x,\mu} \left( \frac{1}{2} \text{tr} \left[ U_{x,\mu} - \tilde{U}_{x,\mu} \right] \text{tr} \left[ U_{x,\mu} - \tilde{U}_{x,\mu} \right] \right)^{1/2} \quad (4) $$

In addition to that, the following conditions have been imposed for the automatized selection of the classical solutions:

- the action fits into the window $0.5 < S_W / S_{\text{inst}} < 1.25$,
- the decrease of action has slowed down to $|\Delta S_W| / S_{\text{inst}} < 0.05$ and
- the violation of the equations of motion should be sufficiently weak, $V < 25$.

In most of the cases for which the first and second conditions hold we find a small minimum violation $V < 20$. In a few cases there were also local minima with much larger $V$-values, which hardly could be interpreted as extended solutions of the lattice equations of motion. Thus, the third condition was implied in order to select "good" solutions. We have varied our stopping criteria \cite{21} and seen that the details of the ensemble of the 'frozen' objects slightly change. Our main observations reported below do not depend on the details of the procedure. The efficiency of the conditions was such that 80 \% (in the case $N_t = 4$), 60 \% ($N_t = 5$) and 55 \% ($N_t = 6$) of the equilibrium configurations ended up in a cooled configuration at the one-instanton action plateau. These cooled configurations include events with topological charge $Q = \pm 1$, i.e. real calorons, as well as configurations with topological charge $Q = 0$ which in our previous paper \cite{4} had been identified as static dyon-antidyon pairs ($DD$). The latter constitute 18 \% ($N_t = 4$), 14 \% ($N_t = 5$) and 8 \% ($N_t = 6$), respectively, of all cooled configurations at the one-instanton level. They have been discarded from the considerations in this paper.

We remind the reader that the recombination threshold $\delta^*_t = 0.27$, strictly speaking, reflects the recombination for maximally non-trivial holonomy only, i.e. with an asymptotic value of the Polyakov line $L_{as} = 0$. If one performs cooling without special restrictions concerning the holonomy, there is no guarantee that the asymptotic holonomy of the caloron configurations still coincides with the average Polyakov line of equilibrium configurations in confinement. For the purpose of defining an asymptotic holonomy $L_{as}$ for each cooled configuration, we have determined the average of $L_{\xi}$ over a 3D subvolume where the local 3D action density $s_\xi$ is low, for definiteness $s_\xi < 0.0001$. In Fig. 2 we present the distribution of cooled configurations over $L_{as}$ as a histogram (with bin size 0.1) for the three cases $N_t = 4$, $N_t = 5$ and $N_t = 6$. In the legend we show the respective volume fraction ($F \approx 0.15$) of the three-volume over which the "asymptotic" value $L_{as}$ is defined as an average, i.e. far from the humps of action and topological charge.

As explained above, the non-staticity $\delta_t$ is a measure which describes the distance from a perfectly (Euclidean) time independent configuration. In other words, the distributions of non-staticity of caloron events obtained by

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The dependence of non-staticity $\delta^*_t$, defined at the bifurcation point for analytic KvB caloron solutions, on the number of time slices $N_t$. The zigzag form of the curve can be explained by the qualitatively different arrangement of even and odd time points where the continuum KvB caloron solution has to be calculated in order to evaluate \cite{3}.}
\end{figure}
The distribution of holonomy $L_{as}$ for the three samples of cooled configurations corresponding to three temperatures below deconfinement at $N_t = 4$, $N_t = 5$ and $N_t = 6$.

cooling of equilibrium lattice configurations can be considered as a substitute for the distribution in dyon distances $d$. This quantity can be directly measured for cooled lattice gauge field configurations. We show in Fig. 4 the $\delta_t$ distributions for all our cooling products obtained at $\beta = 2.2$ on $16^3 \times N_t$ lattices.

In an attempt to make a fair comparison with calorons with non-trivial holonomy and to correct for the possible evolution of the asymptotic holonomy away from $L_{as} = 0$ during the cooling process, we defined a subsample by the requirement $|L_{as}| < 1/6$. One can see that the cut with respect to the asymptotic holonomy selects cooled configurations from the flat central part of the histogram shown in Fig. 2. On the other hand, we notice that a considerable fraction of cooled configurations has developed an asymptotic holonomy $|L_{as}| > 1/6$.

In Fig. 3 (a) we show the probability distribution over $\delta_t$ for cooled configurations with an action at the one-instanton plateau without the cut according to the asymptotic holonomy $|L_{as}|$. One can see that a relatively high fraction of configurations, obtained from the Monte Carlo equilibrium with $N_t = 4$, has $\delta_t < \delta_t^* = 0.27$. This means that they would be identifiable as consisting of two constituents by looking for the 3D action density on the lattice. For $N_t = 5$ it is only a minority of cooled configurations which falls below the threshold $\delta_t = 0.27$. No static (according to the non-staticity criterum) configurations have been found among cooling products at $N_t = 6$.

We have repeated the same analysis after applying the cut with respect to the asymptotic holonomy, $|L_{as}| < 1/6$. Then we get modified histograms in $\delta_t$ for the three temperatures. This is shown in Fig. 4 (b). The histograms got more pronounced peaks in $\delta_t$ which are positioned around 0.125, exactly around $\delta_t^* = 0.27$, and around 0.5 for $N_t = 4$, $N_t = 5$ and $N_t = 6$, respectively.

There are other criteria which could be used to characterize a more or less static configuration, for example, the presence of static Abelian monopoles emerging in the maximal Abelian projection. This criterium is not identical with the separation set by $\delta_t^*$. Thus, another subsample can be defined by the property that a pair of static Abelian monopoles has been found after fixing the cooled configuration to the maximal Abelian gauge and doing the Abelian projection. This subsample can be analyzed with respect to the three-dimensional distance $R$ between the Abelian monopoles.

In Fig. 4(a) we show the histogram over $\delta_t$ of all cooled configurations obtained from the Monte Carlo ensemble at $N_t = 4$ together with the histogram of those which explicitly exhibit the dyon-dyon structure in terms of Abelian monopoles. One can see that practically all cooled configurations below $\delta_t^*$ possess this structure, but above $\delta_t^*$ the fraction rapidly goes to zero. We show the same for $N_t = 6$, i.e. at lower temperature, in Fig. 4(b). In this case, at the peak value around $\delta_t = 0.5$ only 20% of the solutions still are characterized by a static dyon-dyon pair, whereas at higher non-staticity this is never the case. In these two distributions no cut with respect to the asymptotic holonomy has been applied.

Up to now we have two tentative definitions of the position of the constituents: one is the position of the two static Abelian monopoles (in MAG, as long as they are static) and the other is defined by the maxima of the 3D action density. In the case of well-separable maxima of the latter these maxima fall close to the positions of the MAG monopoles. In the other limiting cases of recombined maxima or imbalanced maxima (this corresponds to an asymptotic holonomy far from zero) of the 3D action density at least the absolute maximum of action density is still easy to find. It is either the single maximum or it takes the role of one of the maxima. For the case of $N_t = 4$, i.e. the temperature just below deconfinement, we show in Fig. 5(a) the histogram with respect to the local Polyakov loop at the two sorts of 3D constituent points, the loci of static monopoles or the maximum of action density. In both definitions the histogram peaks near to $L_{as} = \pm 1$. The peak is however more pronounced for the monopole locations, less pronounced for the maxima of the 3D action density.

The relation between the non-staticity $\delta_t$ and the distance $R$ between the static dyonic constituents emerging (in MAG as Abelian monopole) given in lattice units, is presented in Fig. 5(b) for the higher temperature, near the deconfinement temperature ($N_t = 4$). For this temperature such dyon-dyon events are clearly distinguishable among the cooled configurations as long as the non-staticity $\delta_t < 0.6$. For extremely low non-staticity $\delta_t$ (left from the peaks in Fig. 5(a) and (b)) we find an average distance $R \approx 7$, whereas near $\delta_t^*$ the average distance is $R \approx 4$. For higher $\delta_t$, the part of solutions which still possesses a clear dyon-dyon structure in terms of Abelian monopoles, has them localized at distances $R$ between
one and two lattice spacings.

In order to represent how the Polyakov line behaves inside a lump of action, we have chosen the absolute maximum of the 3D action density (denoted as central point $\vec{x}_0$) and have explored the Polyakov loop in its neighborhood. For this purpose, we have defined a locally summed-up Polyakov line $L_{tot}$ (summed over the central point $\vec{x}_0$ and its six nearest neighbors $\vec{x}_i$ ($i = 1, .., 6$):

$$L_{tot} = \sum_{i=0}^{6} L_{\vec{x}_i}$$

(5)

and a kind of Polyakov-line dipole moment over the same set of 3D lattice points with respect to the central point:

$$\vec{M}_{tot} = \sum_{i=1}^{6} L_{\vec{x}_i} (\vec{x}_i - \vec{x}_0).$$

(6)

The absolute value $|L_{tot}|$ of the first quantity tests the amount of local coherence of the Polyakov line. The absolute value $|\vec{M}_{tot}|$ of the second quantity tests the amount of presence of opposite-sign Polyakov lines rep-
of action density rises with increasing non-staticity $\delta$. The difference is that separable lumps of action (those with low $\delta_t$), they are characterized by a relatively smooth change of the Polyakov line inside. In the region of large $\delta_t$ where they are not separable according to the action density, the Polyakov line changes rapidly in the neighbourhood of the absolute maximum of action density. For the lower temperatures, $N_t = 5$ and $N_t = 6$, the relationship between these properties of an action cluster and $\delta_t$ is the same. The difference is that separable lumps of action (those with low $\delta_t$) become very rare. Fig. 5(b) shows how the "dipole moment" $|\vec{M}_{tot}|$ of the Polyakov line around a maximum of action density rises with increasing non-staticity $\delta_t$. In the region where one can separate the constituents according to the action density, the dipole moment is small, emphasizing again the homogeneity of the Polyakov line around the central point. In the region beyond $\delta_t^*$ the dipole moment gradually stabilizes around a value of 1.5.

In the same way as described so far, we have analyzed configurations obtained by cooling at higher action plateaux. As an example we show in Fig. 6 the histogram of non-staticity $\delta_t$ for the same three temperatures represented by $N_t = 4$, $N_t = 5$ and $N_t = 6$. Although the precise border between static and non-static has not the clear meaning as for the one-caloron case, the trend is the same: at lower temperature the lumps of action tend to be more localized also in Euclidean time ("instanton-like"). Compared with the one-caloron case, the histograms are more smeared.

So far we have investigated the outcome from cooling for varying temporal lattice extent and fixed spatial volume. What happens when changing $\beta$ at fixed lattice size and fixed lattice asymmetry? In Figs. 8 we again show non-staticity $\delta_t$ histograms without applying cuts for $|L_{as}|$. We compare (a) for $N_t = 4$ the cases $\beta = 0.0$, $\beta = 2.04$ and $\beta = 2.20$ with each other, and (b) for $N_t = 6$ the case $\beta = 2.20$ with $\beta = 2.36$. The statistics for the new $\beta$-values (0.0, 2.04 and 2.36) is of order $O(200)$ field configurations each. Contrary to what one might have expected, we see that among the cases (a) and the cases (b), respectively, there are no qualitative differences. This means that the ratio of probabilities for the occurrence of dissociated calorons (or separate static dyon pairs) vs. non-dissociated (single lump) calorons is not a function of the temperature describing the original equilibrium ensemble. Instead, it is mainly determined by the geometry (the aspect ratio) of the lattice. The larger the temporal lattice extent is in comparison with the spatial extent the lower is the probability to find separated lumps of more or less static objects after cooling. In other words, once there is only one topological $Q \neq 0$ object left, everything is classical. As a rule, the object is smooth, such that only the size of the box matters or other "infrared forces" (e.g. those forbidding single instantons) play a role. This observation clearly shows the limitation of the cooling method applied to the lowest action plateaux.

![Diagram](image_url)
FIG. 6: The Polyakov line in the neighborhood of maxima of the action density: (a) modulus of the average summed-up Polyakov line $|L_{tot}|$ and (b) modulus of the corresponding "dipole moment" $|\vec{M}_{tot}|$ vs. non-staticity $\delta_t$, for the subsample with asymptotic holonomy near zero. For details of the definition see the text. The thick vertical line marks the non-staticity $\delta_t^*$ where the caloron recombines.

FIG. 7: The same as in Fig. 3 (a) for $S = 3S_{\text{inst}}$ plateaux. No cut in the asymptotic holonomy has been applied. The thick vertical line marks the non-staticity $\delta_t^*$ where the caloron recombines.

IV. INSTANTONS OR CALORONS ON A SYMMETRIC 4-TORUS

With low statistics (100 configurations) we have also cooled equilibrium configurations generated with $\beta = 2.2$ on symmetric lattices ($16^4$ representing "zero" temperature). In this case we have found for the classical configurations at the plateau $S_W \approx S_{\text{inst}}$ a broad distribution of non-staticity with a maximum around $\delta_t \approx 2$ and with a tail extending beyond 3 (see Fig. 9). These are obviously configurations with an action (topological charge) density well-localized in all four Euclidean directions. There is a non-trivial behaviour of the Polyakov line inside these non-dissociated, instanton-like objects resembling the Polyakov line associated with the time direction in the finite temperature case for non-dissociated calorons. We have mapped the cooled lattice configuration with the help of all 4 possible definitions of the Polyakov line, which are now, on a symmetric lattice, physically equivalent to each other. Fig. 10 shows the profiles of action density, topological charge, and of the Polyakov lines (for four possible definitions) as they are seen in appropriate planes intersecting the lump through the centrum. The latter is defined as the maximum of the 4D action density. For all types of Polyakov lines the characteristic double structure is seen exactly when the "asymptotic" value of the respective Polyakov line is not close to $\pm 1$.

It is interesting to compare the pattern of the Polyakov line of these "caloron candidates" with the analytic KvB caloron formally constructed on a $16^4$ lattice corresponding to maximally non-trivial holonomy with respect to what has been chosen as "time" direction. For this construction, the two constituents have been placed along the $z$-direction, separated by 8 lattice spacings. The lattice caloron is obtained calculating link by link from the continuum gauge field $A_\mu$. Of course, such a constructed lattice caloron has irregularities at the boundary if the lattice action is evaluated under the assumption of periodicity. After some cooling the configuration turns into an (approximate) solution of the lattice equations of motion on the torus. By that time the boundary arte-
FIG. 8: The distribution of non-staticity $\delta_t$ after cooling for fixed $N_t$ but varying $\beta$-values. No cut in the asymptotic holonomy is made. The spatial lattice size is always $16^3$. (a) $N_t = 4$: $\beta = 0.0, 2.04, 2.20$, (b) $N_t = 6$: $\beta = 2.20, 2.36$.

FIG. 9: The distribution of non-staticity $\delta_t$ after cooling for $\beta = 2.2$ on the symmetric lattice $16^4$, obtained from 42 configurations with unit topological charge.

facts have disappeared (see Fig. 11), and the Wilson action has become $S_{\text{inst}} = 2\pi^2\beta$. The KvB caloron is now adjusted to periodic boundary condition also in the $x, y, z$ - directions. Despite the large separation the constituents formally have, judging according to the action density it is a non-dissociated caloron. It is seen that the cooling does not influence significantly the structure of the Polyakov lines of the KvB caloron witnessed by the time-directed Polyakov line ($p_{lt}$) showing the double peak structure mentioned above. The space-directed Polyakov lines ($p_{lx}, p_{ly}, p_{lz}$) have a simple structure characteristic for trivial holonomy. It is analytically clear that for caloron solutions asymptotically $p_{lx}, p_{ly}, p_{lz} \rightarrow \pm 1$. This is in contrast to the would-be ”caloron candidate” obtained by cooling from confining equilibrium lattice configurations at zero temperature (Fig. 10) where the double peak structure is present for all $t, x, y, z$ - directions and, generically, the asymptotic holonomy in all directions is non-trivial.

V. CONCLUSIONS AND PERSPECTIVES

In the present investigation we have subjected equilibrium lattice gauge fields corresponding to various temperatures to ordinary relaxation (usually called ”cooling”) in order to obtain an ensemble of classical solutions for further study. In this way we have extracted lowest-action classical solutions of unit topological charge typical for the given equilibrium ensemble. Notice that, at least as long this technique is used, this possibility is restricted to the confinement phase. We do not claim to find the real and complete topological structure hidden under the quantum fluctuations and consisting of topological lumps of compensating sign. What we wanted to see were the simplest solutions suitable as building blocks for a semi-classical modelling of the Yang-Mills path integral. In conclusion we can say, that cooling of equilibrium lattice fields in $SU(2)$ lattice gauge theory shows that there exist topological objects with a dyonic substructure, that we are able to resolve only in the confined phase. With increasing temporal lattice extent it becomes more probable that the observed dyons recombine into calorons such that it becomes impossible to perceive the substructure looking exclusively at the distribution of action and topological charge. Strictly speaking, we have found that the
FIG. 10: Profiles of the action density ($s$), the topological charge density ($q$), and of the Polyakov lines ($plt$, $plx$, $ply$, $plz$) calculated along all straight line paths parallel to the four axes for a $16^4$ lattice caloron found by cooling a Monte Carlo generated equilibrium gauge field down to the one-instanton action plateau. The center of the caloron (at the maximum of its action density) was found at the site $(x, y, z, t) = (7, 8, 8, 14)$. The planes shown in the figures cross just this point.
FIG. 11: Profiles of the action density ($s$), the topological charge density ($q$), and of the Polyakov lines ($\text{plt,plx,ply,plz}$) as in Fig. [10] but for a $16^4$ lattice caloron obtained from a discretized KvB solution corrected by some cooling to the one-instanton action plateau (300 cooling steps). The center of the caloron (at the maximum of its action density) was placed to the site $(x, y, z, t) = (8, 8, 8, 1)$. The planes shown in the figures cross this point.
fraction of dissociated calorons among all single-calorons events does not depend on the physical temperature of the equilibrium fields but on the geometric aspect ratio of the box, i.e. the asymmetry of the lattice. However, all calorons have a nontrivial holonomy which is mapped out by the behavior of the Polyakov line inside and outside these configurations.

In the limiting case of zero temperature (i.e. on a symmetric lattice) topological lumps with \( Q = \pm 1 \) obtained by cooling look instanton-like and, at the same time, have the characteristic double peak structure of the Polyakov line in all \( t, x, y, z \) directions. This distinguishes them from the real KvB solutions which possess this structure only with respect to a distinguished "time" direction.

Strictly speaking, an analytic solution of the Euclidean equations of motion with \( Q = \pm 1 \) is impossible on the 4D torus. Nevertheless, on the lattice quasistable solutions of this kind exist. If considered only as lumps of action and topological charge, there is no contradiction with the previous observations from "instanton searches". As our analysis shows, for temperatures much lower than the deconfinement temperature rotationally symmetric (in 4D) and (anti)selfdual lumps seem to be preferred under cooling.

At finite temperature, they can be subsumed under the general class of KvB solutions. For zero temperature, however, a so far unknown parametrization with nontrivial asymptotic holonomy has yet to be found.

After having completed the present investigation we were informed by Ch. Gattringer and R. Pullirsch about their paper "Topological lumps and Dirac zero modes in SU(3) lattice gauge theory on the torus" prior to publication. In this paper the authors concentrate on and more systematically continue the inspection of the low-lying modes of the chirally improved Dirac operator on the 4-torus, in the background of equilibrium lattice gauge fields at \( T = 0 \). It is extremely interesting that they find, for a certain fraction of Monte Carlo configurations in the \( |Q| = 1 \) sector, a similar pattern of hopping zero-modes as a function of varying fermionic boundary conditions as for \( T \neq 0 \). Moreover, the change of localization happens independently of which of the four directions is chosen as "imaginary time" direction along which periodicity can be purposely changed. The concurrent zero mode positions (two or three as in the finite temperature case) are consistent with being randomly distributed in the 4D periodic box. The authors argue that this observation hints at the existence of a semiclassical background consisting of localized (in 4D) instanton constituents. Whereas other (monopole ?) properties of the constituents are less obvious, a non-integer topological charge of the constituents has been hypothetically assumed in analogy with the dissociated KvB solutions. In fact, this should not be too difficult to be established.

It is in particular this last interpretation that has to pass further tests. In case it becomes confirmed then it would be difficult to reconcile this with our observations based on cooling. Our findings are consistent with a picture in which (at finite temperature) the temporal size of the box determines (inversely) the size of the background solutions. In contrast, the scenario of Ref. seems to imply a complete dissolution (within the available 4-volume) of some constituents which still span a coherent semiclassical background. This would mean that there is no scale of coherence which is dynamically generated and decoupled from the overall size of the box. If the latter picture can be confirmed we would have to blame cooling (or the cooling action we have used) for artificially driving all semiclassical configurations into integer-charged topological lumps at lower temperatures whereas separated monopole constituents are correctly reproduced by cooling only at higher temperatures. This would explain why previous studies of the topological properties using various cooling (smoothing) techniques (which actually missed the relevant temperature range) were not suitable to discover the non-trivial holonomy accompanying all topological changes.

There is still an ongoing debate on the question in as far a semi-classical interpretation of the QCD vacuum is valid at all. For the time being we have nothing to add to this discussion. But our results point to the fact that the instanton gas or liquid model (as well as other models) have to be reconsidered by taking into account the non-trivial holonomy structure. We believe that this will improve the bad performance of (trivial holonomy) instantons in comparison with lattice smoothing results as reported in.

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See Ref. [8] where it has been defined in a slightly differ-
ent way.

The criteria have been tested for $N_t = 4$ and applied also
for $N_t = 5$ and 6 (at the same $\beta$).

For the case of analytical caloron solutions, the relation
between the constituent locations, the locations of the maxima
of the action density and the locations where
$|P| = 1$ has been compared in Ref. [14].

Note that according to asymptotic scaling $\beta = 2.20$, $N_t = 4$ and $\beta = 2.36$, $N_t = 6$ would correspond approximately
to the same physical temperature