Research Article

Dynamics and Solutions’ Expressions of a Higher-Order Nonlinear Fractional Recursive Sequence

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The principle purpose of this article is to examine some stability properties for the fixed point of the below rational difference equation

\[ U_{n+1} = \xi U_n + (\epsilon U_{n-8}/(\mu U_n + \kappa U_{n-17})) \]

where \( \xi, \epsilon, \mu, \) and \( \kappa \) are arbitrary real numbers. Moreover, solutions for some special cases of the proposed difference equation are introduced.

1. Introduction

In recent years, many researchers have tended to use difference equations in mathematical models to explain the problems in different sciences since they have a lot of features such as they enable the scientists to introduce the predictions of their study and it gives more accurate results. In addition, there are various types of nonlinear difference equations that can be studied; one of the most commonly used is rational nonlinear difference equations. However, the research studies in the area of difference equations have two directions: first one is the analysis of the behavior of solutions. Therefore, there are a huge number of articles published to investigate the stability of the equilibrium points and the existence of the periodic solutions for the nonlinear difference equations (see, for example, [1–5]). The second direction is to obtain the expressions of the solution if it is possible since there is no explicit and enough methods to find the solution of nonlinear difference equations (see, for example, [6–11]).

Saleh and Farhat [12] investigated the stability properties and the period two solutions of all nonnegative solutions of the difference equation:

\[ V_{n+1} = \frac{a_1 V_n + a_2 V_{n-k}}{A + BV_{n-k}}. \]  

(1)

In [13], Jia studied the solutions’ behavior of the high-order fuzzy difference equation:

\[ V_{n+1} = \frac{A_1 V_{n-1} V_{n-2}}{B_2 + \sum_{i=3}^{k} D_i V_{n-i}}. \]  

(2)

Kerker et al. [14] investigated the global behavior of the rational difference equation:

\[ V_{n+1} = \frac{a_n + V_n}{a_n + V_{n-k}}. \]  

(3)

Khaliq and Elsayed [15] examined the dynamics behavior and existence of the periodic solution of the difference equation:

\[ V_{n+1} = \alpha_1 V_{n-2} + \frac{\alpha_2 V_{n-2}}{\gamma_1 V_{n-2} + \gamma_2 V_{n-5}}. \]  

(4)

In [16], Saleh et al. studied the properties’ stability for a nonlinear rational difference equation of a higher order:

\[ V_{n+1} = \beta_1 + \beta_2 V_n + \beta_3 V_{n-k} \]  

(5)

\[ \frac{B_1 V_n + B_2 V_{n-k}}{B_1 V_n + B_2 V_{n-k}}. \]

Sadiq and Kalim [17] obtained the solution behavior of the difference equation:
\[ V_{n+1} = a_1 V_{n} + \frac{a_2 V_{n-1}^2}{a_3 V_{n-1} + a_4 V_{n-1}} \]  

(6)

To see more related work on the nonlinear difference equation, refer to [18–43]. Our aim of this article is to investigate the dynamics of the solution for the below difference equation:

\[ U_{n+1} = \xi U_{n} + \frac{\epsilon U_{n-8}}{\mu U_{n-8} + \kappa U_{n-17}}, \]  

(7)

where \(\xi, \epsilon, \mu,\) and \(\kappa\) are arbitrary real numbers with initial conditions \(U_j\) for \(j = -17, -16, \ldots, 0.\)

This paper is collected as follows: in Section 2, the boundedness of the solution is presented, and we prove that the periodic solution of period two does not exist in the next section. Following that, we state the conditions of the local and global stability of the equilibrium point in Sections 4 and 5, respectively. Then, we introduce the solutions’ forms for some special cases in Section 6. Finally, we give some numerical examples in order to illustrate the behavior of the solutions.

2. Boundedness of Solution

**Theorem 1.** If the following condition

\[ (\xi + \frac{\epsilon}{\mu}) < 1, \]  

is true, then every solution of (7) is bounded.

**Proof.** Assume that \(\{U_n\}_{n=-12}^{\infty}\) is a solution of (7). Then, from (7), we have

\[ U_{n+1} = \xi U_{n} + \frac{\epsilon U_{n-8}}{\mu U_{n-8} + \kappa U_{n-17}} \leq \xi U_{n} + \frac{\epsilon U_{n-8}}{\mu U_{n-8} + \kappa U_{n-17}} \left(\xi + \frac{\epsilon}{\mu}\right) U_{n-8}. \]  

(9)

Hence,

\[ U_{n+1} \leq U_{n}, \quad \forall n \geq 0. \]  

(10)

Implies that the subsequences \(\{U_{9n-6}\}_{n=1}^{\infty}, \{U_{9n-5}\}_{n=1}^{\infty}, \{U_{9n-4}\}_{n=1}^{\infty}, \{U_{9n-3}\}_{n=1}^{\infty}, \{U_{9n-2}\}_{n=1}^{\infty}, \{U_{9n-1}\}_{n=1}^{\infty},\) and \(\{U_{9n}\}_{n=1}^{\infty}\) are nonincreasing. Thus, they are bounded from above by \(U_{\text{max}}\), where

\[ U_{\text{max}} = \text{max}\{U_{-17}, U_{-16}, U_{-15}, U_{-14}, U_{-13}, U_{-12}, U_{-11}, U_{-10}, U_{-9}, U_{-8}, U_{-7}, U_{-6}, U_{-5}, U_{-4}, U_{-3}, U_{-2}, U_{-1}, U_{0}\}. \]

3. Periodicity of the Solution

**Theorem 2.** For nonlinear difference equation (7), there is no periodic solution of period two.

**Proof.** To prove Theorem 2, suppose that (7) has a positive prime period two solutions presented as \(\ldots, \epsilon, f, e, f, \ldots\). Then,

\[ e = \frac{\xi f + \frac{\epsilon f^2}{\mu f + \kappa e}}{\mu f + \kappa e}, \]

(11)

\[ (\xi \mu + \epsilon) f^2 = (\mu - \xi \kappa) \epsilon f + \kappa e. \]

Similarly,

\[ f = \frac{\xi e + \frac{\epsilon e^2}{\mu e + \kappa f}}{\mu e + \kappa f}, \]

(12)

\[ (\xi \mu + \epsilon) e^2 = (\mu - \xi \kappa) \epsilon e + \kappa f e. \]

Subtracting (11) from (12), we get

\[ \kappa (e^2 - f^2) + (\xi \mu + \epsilon) (e^2 - f^2) = 0, \]

(13)

\[ (\kappa + \xi \mu + \epsilon) (e^2 - f^2) = 0. \]

Since \((\kappa + \xi \mu + \epsilon) \neq 0\), thus \(e = f\), and this contradicts the fact that \(e \neq f\). \(\square\)

4. The Equilibrium Point and Local Stability

The fixed points of (7) are given by

\[ U = \xi U + \frac{\epsilon U^2}{\mu U + \kappa U} \]

\[ (1 - \xi) U = \frac{\epsilon U^2}{(\mu - \kappa) U}. \]  

(14)

\[ ((1 - \xi) (\mu - \kappa) - \epsilon) U^2 = 0. \]

If \((1 - \xi) (\mu + \kappa) \neq \epsilon\), then (7) has only one equilibrium point which is \(U = 0\).

Assume \(g: (0, \infty)^2 \rightarrow (0, \infty)\) is a continuously differentiable function defined by

\[ g(v, w) = \xi v + \frac{\epsilon v^2}{\mu v + \kappa w}. \]

(15)

Therefore,

\[ \frac{\partial g}{\partial v} = \xi + \frac{\epsilon v^2 + 2\kappa \mu \nu w}{(\mu v + \kappa w)^2}, \]

(16)

\[ \frac{\partial g}{\partial w} = \frac{-\kappa v^2}{(\mu v + \kappa w)^2}. \]

Then,
Theorem 3. The fixed point \( \mathcal{U} = 0 \) is said to be a locally asymptotically stable if the relation
\[
\varepsilon (\mu + 3k) < (1 - \xi)(\mu + k)^2,
\]
is satisfied.

Proof. From Theorem 5.10 in [44], it follows that \( \mathcal{U} \) is asymptotically stable if
\[
|P_0| + |P_1| < 1,
\]
where \( P_0 = \xi + \left( (\varepsilon \mu + 2\varepsilon k)/(\mu + k)^2 \right) \) and \( P_1 = (-\varepsilon k)/(\mu + k)^2 \). Then,
\[
\left| \xi + \frac{\varepsilon \mu + 2\varepsilon k}{(\mu + k)^2} \right| + \left| \frac{-\varepsilon k}{(\mu + k)^2} \right| < 1,
\]
and
\[
\xi + \frac{3\varepsilon k}{(\mu + k)^2} < 1.
\]

Hence,
\[
\varepsilon (\mu + 3k) < (1 - \xi)(\mu + k)^2.
\]

Finally, the proof is done. \( \square \)

5. Global Attractivity of the Fixed Point

Theorem 4. The fixed point \( \mathcal{U} \) of (7) has to be a global attracting when
\[
\mu(1 - \xi) \neq \varepsilon.
\]

Proof. From (16), we see that the function \( g(v, w) \), which defined in (15), is increasing in \( v \) and decreasing in \( w \). Let \((\rho, \tau)\) be a solution of the system:
\[
\begin{align*}
\tau &= g(\tau, \rho), \\
\rho &= g(\rho, \tau), \\
\tau &= \xi \tau + \frac{\varepsilon \tau^2}{\mu \tau + \kappa \rho}, \\
\rho &= \xi \rho + \frac{\varepsilon \rho^2}{\mu \rho + \kappa \tau}.
\end{align*}
\]

Therefore,
\[
\mu (1 - \xi)^2 + \kappa (1 - \xi) \tau \rho = \varepsilon \tau^2,
\]
\[
\mu (1 - \xi) \rho^2 + \kappa (1 - \xi) \tau \rho = \varepsilon \rho^2.
\]

Subtracting (25) from (26), we get
\[
(\mu (1 - \xi) - \varepsilon)(\tau^2 - \rho^2) = 0,
\]
and then, \( \rho = \tau \) if \( \mu (1 - \xi) \neq \varepsilon \). Thus, from Theorem 5.20 in [44], we observe that there exists only one solution for (7) and it is a global attractor if \( \mu (1 - \xi) \neq \varepsilon \). \( \square \)

6. Special Cases

Now, we present the solutions' expressions for special cases of (7):
\[
U_{n+1} = U_{n-8} \pm \frac{U_{n-8}^2}{U_{n-8} \pm U_{n-17}},
\]
where the initial conditions are
\[
U_{-17}, U_{-16}, U_{-15}, U_{-14}, U_{-13}, U_{-12}, U_{-11}, U_{-10}, U_{-9}, U_{-8}, U_{-7}, U_{-6}, U_{-5}, U_{-4}, U_{-3}, U_{-2}, U_{-1},
\]
and \( U_0 \) are arbitrary real numbers.

6.1. First Equation. We solve the equation
Theorem 5. Assume \( \{U_n\}_{n=-17}^{\infty} \) is a solution of (30); thus, for \( n = 0, 1, \ldots \),

\[
U_{9n-8} = U_{-8} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-8} + F_{2i} U_{-12}}{F_{2i} U_{-8} + F_{2i-1} U_{-12}} \right),
\]

\[
U_{9n-4} = U_{-4} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-4} + F_2 U_{-13}}{F_2 U_{-4} + F_{2i-1} U_{-13}} \right),
\]

\[
U_{9n-7} = U_{-7} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-7} + F_{2i} U_{-16}}{F_{2i} U_{-7} + F_{2i-1} U_{-16}} \right),
\]

\[
U_{9n-3} = U_{-3} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-3} + F_{2i} U_{-12}}{F_{2i} U_{-3} + F_{2i-1} U_{-12}} \right),
\]

\[
U_{9n-6} = U_{-6} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-6} + F_{2i} U_{-15}}{F_{2i} U_{-6} + F_{2i-1} U_{-15}} \right),
\]

\[
U_{9n-2} = U_{-2} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-2} + F_{2i} U_{-11}}{F_{2i} U_{-2} + F_{2i-1} U_{-11}} \right),
\]

\[
U_{9n-5} = U_{-5} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-5} + F_{2i} U_{-14}}{F_{2i} U_{-5} + F_{2i-1} U_{-14}} \right),
\]

\[
U_{9n-1} = U_{-1} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-1} + F_{2i} U_{-10}}{F_{2i} U_{-1} + F_{2i-1} U_{-10}} \right),
\]

\[
U_{9n} = U_{0} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{0} + F_{2i} U_{-9}}{F_{2i} U_{0} + F_{2i-1} U_{-9}} \right),
\]  

(31)

where \( \{F_{i}\}_{i=1}^{\infty} = \{1, 1, 2, 3, 5, \ldots \} \) is the Fibonacci sequence.

Proof. We show that the expressions in (31) are solutions of (30) by applying mathematical induction. First, the results hold for \( n = 0 \). Second, we suppose that the forms are satisfied for \( n - 1 \) and \( n - 2 \). Now, we prove that the results are satisfied for \( n \):

\[
U_{9n-17} = U_{-17} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-6} + F_{2i} U_{-15}}{F_{2i} U_{-6} + F_{2i-1} U_{-15}} \right),
\]

\[
U_{9n-13} = U_{-13} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-2} + F_{2i} U_{-11}}{F_{2i} U_{-2} + F_{2i-1} U_{-11}} \right),
\]

\[
U_{9n-16} = U_{-16} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-2} + F_{2i} U_{-16}}{F_{2i} U_{-2} + F_{2i-1} U_{-16}} \right),
\]

\[
U_{9n-12} = U_{-12} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-6} + F_{2i} U_{-12}}{F_{2i} U_{-6} + F_{2i-1} U_{-12}} \right),
\]

\[
U_{9n-15} = U_{-15} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-6} + F_{2i} U_{-15}}{F_{2i} U_{-6} + F_{2i-1} U_{-15}} \right),
\]

\[
U_{9n-11} = U_{-11} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-2} + F_{2i} U_{-11}}{F_{2i} U_{-2} + F_{2i-1} U_{-11}} \right),
\]

\[
U_{9n-14} = U_{-14} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-2} + F_{2i} U_{-14}}{F_{2i} U_{-2} + F_{2i-1} U_{-14}} \right),
\]

\[
U_{9n-10} = U_{-10} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-1} + F_{2i} U_{-10}}{F_{2i} U_{-1} + F_{2i-1} U_{-10}} \right),
\]

\[
U_{9n-9} = U_{-9} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{0} + F_{2i} U_{-9}}{F_{2i} U_{0} + F_{2i-1} U_{-9}} \right),
\]

\[
U_{9n-18} = U_{-18} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{0} + F_{2i} U_{-9}}{F_{2i} U_{0} + F_{2i-1} U_{-9}} \right),
\]

\[
U_{9n-26} = U_{-26} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-8} + F_{2i} U_{-17}}{F_{2i} U_{-8} + F_{2i-1} U_{-17}} \right),
\]

\[
U_{9n-22} = U_{-22} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-4} + F_{2i} U_{-13}}{F_{2i} U_{-4} + F_{2i-1} U_{-13}} \right),
\]

\[
U_{9n-25} = U_{-25} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-2} + F_{2i} U_{-16}}{F_{2i} U_{-2} + F_{2i-1} U_{-16}} \right),
\]

\[
U_{9n-21} = U_{-21} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-3} + F_{2i} U_{-12}}{F_{2i} U_{-3} + F_{2i-1} U_{-12}} \right),
\]

\[
U_{9n-20} = U_{-20} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-2} + F_{2i} U_{-11}}{F_{2i} U_{-2} + F_{2i-1} U_{-11}} \right),
\]

\[
U_{9n-24} = U_{-24} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-6} + F_{2i} U_{-15}}{F_{2i} U_{-6} + F_{2i-1} U_{-15}} \right),
\]

\[
U_{9n-23} = U_{-23} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-5} + F_{2i} U_{-14}}{F_{2i} U_{-5} + F_{2i-1} U_{-14}} \right),
\]

\[
U_{9n-19} = U_{-19} \prod_{i=1}^{n} \left( \frac{F_{2i+1} U_{-1} + F_{2i} U_{-10}}{F_{2i} U_{-1} + F_{2i-1} U_{-10}} \right),
\]

(32)

From (30), it follows that
Similarly, one can investigate other expressions. The proof is done.

Theorem 6. Let \( \{U_n\}_{n=0}^{\infty} \) be a solution of (35); then, for \( n = 0, 1, \ldots \),

\[
U_{n+1} = U_{n-8} + \frac{U_{n-8}^2}{U_{n-8} - U_{n-17}}.
\]
Theorem 7. Let \( \{U_n\}_{n=1}^{\infty} \) be a solution of (37); then, for \( n = 0, 1, \ldots \),

\[
U_{n-8} = \frac{U_{-8} - U_{-9}}{U_{n-8} + U_{n-17}},
\]

\[
U_{n-4} = \frac{U_{-4} - U_{-5}}{U_{n-4} + U_{n+1}}
\]

\[
U_{n-7} = \frac{U_{-7} - U_{-8}}{U_{n+1} - U_{n+1}}
\]

\[
U_{n-3} = \frac{U_{-3} - U_{-4}}{U_{n+1} - U_{n+1}}
\]

where \( \{F_i\}_{i=0}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, \ldots \} \) is the Fibonacci sequence.

Proof. By using mathematical induction, we prove that (38) are solutions of (37). First, the results for \( n = 0 \) are true. Second, assume that the assumption holds for \( n - 2 \) and \( n - 1 \).
Now, from (37), we have

\[
U_{gn-1} = U_{gn-10} - \frac{U_{9n-10}^2}{U_{9n-10} + U_{9n-19}},
\]

\[
= U_{9n-10} \left( 1 - \frac{U_{9n-10}}{U_{9n-10} + U_{9n-19}} \right),
\]

\[
= U_{9n-10} \left( 1 - \frac{(U_{9n-1}U_{9n-10} + F_{n-1}U_{9n-10})}{(U_{9n-1}U_{9n-10} + F_{n-1}U_{9n-10}) + (U_{9n-1}U_{9n-10} + F_{n-1}U_{9n-10})} \right),
\]

\[
= U_{9n-10} \left( 1 - \frac{(U_{9n-1}U_{9n-10} + F_{n-1}U_{9n-10})}{(U_{9n-1}U_{9n-10} + F_{n-1}U_{9n-10})(1 + (F_{n-1}U_{9n-10})/F_{n-1}U_{9n-10})} \right),
\]

\[
= U_{9n-10} \left( 1 - \frac{1}{1 + (F_{n-1}U_{9n-10})/F_{n-1}U_{9n-10}} \right),
\]

\[
= U_{9n-10} \left( 1 - \frac{F_{n-2}U_{9n-10} + F_{n-1}U_{9n-10}}{F_{n-2}U_{9n-10} + F_{n-1}U_{9n-10} + F_{n-1}U_{9n-10}} \right),
\]

\[
= U_{9n-10} \left( 1 - \frac{(F_{n} - F_{n-2})U_{9n-10} + (F_{n+1} - F_{n-1})U_{9n-10}}{F_{n}U_{9n-10} + F_{n+1}U_{9n-10}} \right),
\]

\[
= U_{9n-10} \left( \frac{F_{n}U_{9n-10} + F_{n+1}U_{9n-10}}{F_{n}U_{9n-10} + F_{n+1}U_{9n-10}} \right),
\]

\[
= \frac{U_{9n-10}}{F_{n}U_{9n-10} + F_{n+1}U_{9n-10}}.
\]
Thus,
\[ U_{n+1} = \frac{U_{n-1}U_{n-2}}{F_nU_{n-1} + F_{n+1}U_{n-10}}. \quad (41) \]

Similarly, one can see that the other forms are true. The proof is complete. □

6.4. Fourth Equation. We study the following equation:
\[ \{U_{17}, U_{16}, U_{15}, U_{14}, U_{13}, U_{12}, U_{11}, U_{10}, U_9, U_8, U_7, \]
\[ U_{6}, U_5, U_4, U_3, U_2, U_1, U_0, U_{-4}U_{-17}, U_{-7}U_{-16}, U_{-6}U_{-15}, U_{-5}U_{-14}, \]
\[ -U_{-4}U_{-13} - U_{-3}U_{-12} - U_{-2}U_{-11} - U_{-1}U_{-10} - U_0U_{-9}, U_{-17}, -U_{-16}, \]
\[ -U_{-15}, -U_{-14}, -U_{-13}, -U_{-12}, -U_{-11}, -U_{-10}, \]
\[ -U_{-9}, -U_{-8}, -U_{-7}, -U_{-6}, -U_{-5}, -U_{-4}, -U_{-3}, -U_{-2}, -U_{-1}, -U_0, \]
\[ U_{-8}U_{-17}, U_{-6}U_{-16}, U_{-5}U_{-15}, U_{-4}U_{-14}, U_{-3}U_{-13}, U_{-2}U_{-12}, U_{-1}U_{-11}, U_{-10}, \]
\[ U_0U_{-9}, U_{-17}, U_{-16}, U_{-15}, U_{-14}, U_{-13}, U_{-12}, U_{-11}, U_{-10}, U_{-9}, U_{-8}, U_{-7}, U_{-6}, U_{-5}, U_{-4}, U_{-3}, U_{-2}, U_{-1}, U_0, \ldots \}. \quad (43) \]

**Proof.** The proof of this case will be the same as the proof presented for Theorem 7 and will be omitted therefore. □

7. Numerical Examples

To illustrate the solution behavior of (7) for various cases, we present some numerical examples.

**Example 1.** To show the stability of (7), we set two groups for the values of the coefficients: (i) \( \xi = 0.5, \varepsilon = 0.1, \mu = 1.6, \) and \( \kappa = 0.2 \) and (ii) \( \xi = 0.5, \varepsilon = 5, \mu = 10, \) and \( \kappa = 0.001, \) and the initial conditions are
\[ U_{-17} = 0.1, \]
\[ U_{-16} = 0.2, \]
\[ U_{-15} = 0.3, \]
\[ U_{-14} = 0.4, \]
\[ U_{-13} = 0.5, \]
\[ U_{-12} = 0.6, \]
\[ U_{-11} = 0.7, \]
\[ U_{-10} = 0.8, \]
\[ U_{-9} = 0.9, \]
\[ U_{-8} = 1.2, \]
\[ U_{-7} = 1.5, \]
\[ U_{-6} = 2.2, \]
\[ U_{-5} = 2.3, \]
\[ U_{-4} = 2.5, \]
\[ U_{-3} = 4.2, \]
\[ U_{-2} = 4.6, \]
\[ U_{-1} = 4.8, \]
and \( U_0 = 5.2. \) The result is obtained in Figure 1. It is clear that (i) condition (23) is satisfied, which implies that the solution tends to the fixed point \( \bar{U} = 0, \) while the solution moves away from the fixed point for (ii) since condition (23) failed.

The following examples have explained the solutions of special case equations (30)–(42).

**Example 2.** We choose the initial conditions as
Example 3. In Figure 3, we set the initial conditions:

\[
\begin{align*}
U_{-17} &= 0.01, \\
U_{-16} &= 0.02, \\
U_{-15} &= 0.03, \\
U_{-14} &= 0.04, \\
U_{-13} &= 0.05, \\
U_{-12} &= 0.06, \\
U_{-11} &= 0.07, \\
U_{-10} &= 0.08, \\
U_{-9} &= 0.09, \\
U_{-8} &= 1.02, \\
U_{-7} &= 1.05, \\
U_{-6} &= 2.02, \\
U_{-5} &= 2.03, \\
U_{-4} &= 2.05, \\
U_{-3} &= 4.02, \\
U_{-2} &= 4.06, \\
U_{-1} &= 4.08,
\end{align*}
\]

and \( U_0 = 5.02 \). The solution is given in Figure 2.

Example 4. For (37), we choose the initial conditions as
Un – 6 – 4 – 2 0 1.5 2.5 3.5 4
–2
–4
–6
0 50 100 150
Un

and then, the result is shown in Figure 4.

Example 5. We set the values

\[ U_{-17} = 2, \]
\[ U_{-16} = 2.1, \]
\[ U_{-15} = 2.2, \]
\[ U_{-14} = 2.3, \]
\[ U_{-13} = 2.4, \]
\[ U_{-12} = 2.5, \]
\[ U_{-11} = 2.6, \]
\[ U_{-10} = 2.7, \]
\[ U_{-9} = 2.8, \]
\[ U_{-8} = 3, \]
\[ U_{-7} = 3.1, \]
\[ U_{-6} = 3.2, \]
\[ U_{-5} = 3.3, \]
\[ U_{-4} = 3.4, \]
\[ U_{-3} = 3.5, \]
\[ U_{-2} = 3.6, \]
\[ U_{-1} = 3.7, \]
\[ U_{0} = 3.8, \]
\[ U_{1} = 3.9, \]
\[ U_{2} = 4, \]
\[ U_{3} = 4.1, \]
\[ U_{4} = 4.2, \]
\[ U_{5} = 4.3, \]
\[ U_{6} = 4.4, \]
\[ U_{7} = 4.5, \]
\[ U_{8} = 4.6, \]
\[ U_{9} = 4.7, \]
\[ U_{10} = 4.8, \]
\[ U_{11} = 4.9, \]
\[ U_{12} = 5, \]
\[ U_{13} = 5.1, \]
\[ U_{14} = 5.2, \]
\[ U_{15} = 5.3, \]
\[ U_{16} = 5.4, \]
\[ U_{17} = 5.5. \]

Figure 4: The solution behavior of \( U_{n+1} = U_{n-8} - (U_{n-8}/U_{n-17}) \).

Figure 5: Plotting the solution of \( U_{n+1} = U_{n-8} - (U_{n-8}/U_{n-17}) \).

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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