Neutrino mixing in SO(10) GUTs with non-abelian flavor symmetry in the hidden sector

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The relation between the mixing matrices of leptons and quarks: \( U_{\text{PMNS}} \approx V^\dagger_{\text{CKM}} U_0 \), where \( U_0 \) is a matrix of special forms (e.g. BM, TBM), can be a clue for understanding the lepton mixing and neutrino masses. It may imply the Grand unification and existence of a hidden sector with certain symmetry which generates \( U_0 \) and leads to the smallness of neutrino masses. We apply the residual symmetry approach to obtain \( U_0 \). The residual symmetries of both the visible and hidden sectors are \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Their embedding in a unified flavor group is considered. We find that there are only several possible structures of \( U_0 \), including the BM mixing and matrices with elements determined by the golden ratio. Realization of the BM scenario based on the \( \text{SO}(10) \) GUT with the \( \text{S}_4 \) flavor group is presented. Generic features of this scenario are discussed, in particular, the prediction of CP phase \( 144^\circ \lesssim \delta_{\text{CP}} \lesssim 210^\circ \) in the minimal version.

I. INTRODUCTION

There is an appealing approximate relation between the mixing matrices of leptons, \( U_{\text{PMNS}} \), and quarks, \( V_{\text{CKM}} \) [1–5]:

\[
U_{\text{PMNS}} \simeq V^\dagger_{\text{CKM}} U_0,
\]

where \( U_0 \) is close to the bi-maximal (BM, \( U_{\text{BM}} \)) [6, 7], or tri-bi-maximal (TBM, \( U_{\text{TBM}} \)) [8, 9] mixing matrices. In particular, Eq. (1) gives a relation between the Cabibbo angle \( \theta_C \) and the leptonic mixing angles \( \theta_{13} \) and \( \theta_{23} \):

\[
\sin^2 \theta_{13} \simeq \sin^2 \theta_{23} \sin^2 \theta_C.
\]

The relation (1) is in a good agreement with available experimental results [10] and has been widely studied in the literature [11–26]. If not accidental, it can be the clue for understanding peculiar features of the lepton mixing, and eventually, the origins of neutrino masses. The following logical steps lead to a rather restricted scenario.

1. The relation (1) implies that leptons “know” about quarks. It should be a kind of quark-lepton unification, probably the Grand Unification at high energy scale. Indeed, the Grand Unification can ensure similarity of the Dirac mass matrices of quarks and leptons: \( m_{\nu_D} \sim m_{\text{up}_D} \) and \( m_{\nu_D} \sim m_{\text{down}_D} \). This leads to appearance of mixing \( \sim V_{\text{CKM}} \) in the lepton sector.\(^1\)

2. At the same time the difference between the quark and lepton mixing implies existence of some new physics responsible for generation of matrix \( U_0 \). The structure of this matrix indicates certain underlying symmetry which is difficult to extend to the quark sector.

3. It is natural to assume that the same new physics is responsible for \( U_0 \) and smallness of neutrino masses. In the Grand Unification framework the simplest way to get small neutrino masses is to invoke the high mass scale type I seesaw mechanism [27–31]:

\[
m_{\nu} = -m_D \frac{1}{M_R} m_D^T,
\]

where \( M_R \) is the mass matrix of right-handed (RH) neutrino components. Thus, in the seesaw mechanism, \( M_R \) with specific properties could be responsible for generation of \( U_0 \).

4. The latter, however, implies very strong (quadratic) hierarchy of masses of the neutrinos and enormous fine tuning which is very difficult (if possible) to justify in the usual seesaw mechanism. One way to solve this problem is to introduce the double seesaw mechanism in which the RH neutrinos themselves acquire masses via the seesaw mechanism [32]. This opens up a possibility to cancel the strong hierarchy as a result of certain symmetry [33, 34]. Furthermore, the structure of mass matrix of \( S \), governed by certain symmetry can eventually lead to the required mixing \( U_0 \).

These general arguments can be realized in the following scenario.

\(^1\) Alternatively, the relations between the mass matrices can be obtained as a consequence of common flavor symmetry in both sectors.
There is the Grand Unification based on $SO(10)$ gauge symmetry group [35, 36] with fermions in 16-plet representations which include also the RH neutrinos. This ensures similarity of the Dirac mass matrices of the quarks and leptons and also the coincidence of scales $M_{SN} \sim M_{GUT}$.

A hidden sector exists which consists of singlet fermions and bosons of $SO(10)$. This sector couples with the visible one via the RH neutrino portal. The fermions $S$ that participate in the double seesaw mechanism belong to this sector. Symmetries of the hidden sector lead to the matrix $U_0$ with required properties.

Information about mixing in the hidden sector should be communicated to the visible sector. The minimal possibility is to fix basis of states in all sectors (visible, portal, hidden), and this can be done by introducing the basis fixing symmetry [37–39]. In the case of three generations the simplest possibility is the $Z_2 \times Z_2$ symmetry [37]. In turn, such a symmetry can be a part of intrinsic symmetry of theory which is always present. The basis fixing symmetry ensures that all mass matrices of the visible sector and portal are diagonal. This symmetry is spontaneously broken in the hidden sector by interactions with flavons leading to another unbroken (intrinsic) $Z_2 \times Z_2$ and generating $U_0$. No such a structure exists in the quark sector.

Additional physics should be introduced to generate the CKM mixing.

In the visible sector $Z_2 \times Z_2$ is broken by another mechanism leading to the CKM mixing. The double seesaw mechanism allows to disentangle generation of CKM and $U_0$ mixings. Furthermore, introducing the hidden sector allows one to construct economical renormalizable theory with flavor symmetry.

In general, $Z_2 \times Z_2$ can lead to large mixing in $U_0$, but it does not produce specific structures such as BM or TBM. To this end, non-abelian symmetry should be introduced in the hidden sector. In other words, the basis symmetry $Z_2 \times Z_2$ should be promoted to non-abelian symmetry. In this paper we study such a possibility. We focus on the symmetry issues: the interplay between the gauge $SO(10)$ and discrete flavor symmetries. We also consider generation of $U_i \simeq V_{CKM}$.

The paper is organized as follows. In sect. II we describe the scenario in details. In sect. III we study the possibility to generate the matrix $U_0$ using the residual symmetry approach applied to the visible and hidden sectors. We find all possible structures of $U_0$. In sect. IV we present realization the residual symmetry mechanism which generates $U_0 = U_{BM}$ and is based on $S_4$ symmetry group. We explore a possibility to generate the CKM mixing and study the predictions for the PMNS mixing in sect. V. Conclusions are given in sect. VI.

## II. FRAMEWORK

Let us describe the main elements of the framework.

1. **Visible, portal and hidden sectors.** The visible sector includes 3 families of fermions accommodated in three 16-plets of $SO(10)$: $(\psi_1, \psi_2, \psi_3)$. The fermions get masses mainly via the Yukawa couplings with a 10-plet scalar fields $H^{(10)}$. Additional non-renormalizable interactions will be added to generate difference of masses of down quarks and charged leptons.

The hidden sector consists of fermions $S_i$ and bosons $\phi_i$, which are all singlets of $SO(10)$. In the simplest version three fermionic singlets are introduced. Connection between the visible and hidden sectors is established via the portal interaction of $S_i$ and $\psi_i$. For this the 16-plet of scalar fields, $H^{(16)}$, should be introduced.

Thus, the Yukawa interactions relevant for generation of fermion masses are

$$
\mathcal{L} \supset y^{\psi}_{ij} \psi_i \psi_j H^{(10)} + y^{S}_{ij} \psi_i S_j H^{(16)} + h_{ij} S_i S_j \phi_{ij},
$$

where $y^{\psi}_{ij}$, $y^{S}_{ij}$ and $h_{ij}$ are the Yukawa coupling constants of the visible, portal and hidden sectors correspondingly.

2. **Double seesaw.** After the scalar fields develop VEV’s, the visible sector interactions generate the Dirac mass matrices $m_D = y^{\psi}_{ij} \langle H^{(10)} \rangle$ at the electroweak (EW) scale. The portal interactions produce the matrix $M_{RS} = y^{S}_{ij} \langle H^{(16)} \rangle$ which mixes the RH neutrinos with the singlet fermions.\(^2\) Flavon VEV’s $\langle \phi_{ij} \rangle$ generate the mass matrix

\(^2\) A linear seesaw contribution $m_{LS}$ (see [37]) can also be generated, but in this framework its contribution is negligibly small.
of singlets $M_{S_{ij}} = h_{ij} \langle \phi_{ij} \rangle$. Consequently, the total mass matrix of neutral leptons in the basis $(\nu_L, \nu_R^c, S)$ (here $\nu_R^c \equiv (\nu_R)^c$) becomes

$$\mathcal{M} = -\frac{1}{2} \begin{pmatrix} 0 & m_D & 0 \\ m_D^{\dagger} & 0 & M_{RS} \\ 0 & M_{RS}^{\dagger} & M_S \end{pmatrix},$$

which is the mass matrix of the double seesaw mechanism [40]. It leads to the Majorana masses of the RH neutrinos $\nu_R$

$$M_R = -M_{RS}M_S^{-1}M_{RS}^{\dagger},$$

and the mass matrix of light neutrinos $\nu_L$

$$m_\nu = m_D (M_{RS}^{-1})^T M_SM_{RS}^{-1}m_D^T.$$

The Higgs multiplet $H^{(16)}$ breaks the GUT symmetry, so that the natural scale of the largest portal mass term is $M_{RS} \sim M_{GUT} = \mathcal{O}(10^{16} \text{ GeV})$. The singlets $S$, which are not protected by $SO(10)$, may have masses of higher scales, say the string-Planck scale, $M_S \sim M_{Pl} \sim (10^{18} - 10^{19}) \text{ GeV}$. In this case $M_R = M_{RS}^2 / M_{Pl} = \mathcal{O}(10^{14} \text{ GeV})$, which produces correct scale of light neutrino masses $m_\nu = \mathcal{O}(0.1 \text{ eV})$. This coincidence can be considered as another support of the framework.

3. Intrinsic symmetries. Both the visible and hidden sector interactions (4) have the built-in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ flavor symmetries [41–44]. The symmetries are related to the Majorana character of interaction $(\bar{\psi}^T \cdot \psi, \quad S^T \cdot S)$ and obvious in the basis where the mass matrices are diagonalized. In a general basis, the visible interactions are invariant under the transformation

$$\psi \rightarrow T \psi, \quad T = U_\psi \text{diag} \left[ (-1)^m, (-1)^n, (-1)^k \right] U_\psi^\dagger,$$

where $m, n, k = 0, 1,$ and $U_\psi$ is a unitary matrix that diagonalizes $m_D$. Different choices of $m, n, k$ generate different $T$’s, including trivial cases $T = \pm \mathbb{I}$. These $T$’s form the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ group, which can be reduced to $G_V = \mathbb{Z}_2 \times \mathbb{Z}_2$ if generators with the overall negative sign are removed. Similarly, in the hidden sector the mass terms $1/2 (M_S)_{ij} S_i S_j$ are invariant under

$$S \rightarrow RS, \quad R = U_S \text{diag} \left[ (-1)^m, (-1)^n, (-1)^k \right] U_S^\dagger,$$

where $U_S$ is a unitary matrix that diagonalizes $M_S$. So, the hidden sector has another $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, which is denoted as $G_H$ henceforth.

4. Screening. The condition

$$m_D \propto M_{RS}^T$$

leads, as follows from Eq. (7), to

$$m_\nu \propto M_S,$$

i.e. to “screening” (cancellation) of the Dirac structures, and consequently, to the same structure of the mass matrices of light neutrinos and the heavy singlets [33]. Consequently the light neutrinos and the heavy singlets have the same mixing. The RGE effects do not destroy the cancellation [33].

The screening condition (10) can be a consequence of further unification, e.g. embedding of $\psi$ and $S$ into the 27-dimensional representation of $E_6$-multiplet. It can be a remnant of $E_6$ symmetry which is not fully realized. In this case $S$ can not be considered as belonging to the hidden sector. Another possibility is a common flavor symmetry acting in the visible and portal sectors. In fact, it is easy to get

$$m_D M_{RS}^{-1T} = d,$$

where $d$ is a diagonal matrix. The Klein symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the same charge assignment for $\phi_i$ and $S_i$ allow to achieve this. If the charges of three components $\psi_1, \psi_2, \psi_3$ are different, e.g. $(-, -) \quad (+, +)$ and scalar multiplets $H^{(10)}, \quad H^{(16)}$ have zero charges, the matrices $m_D$ and $M_{RS}$ are both diagonal [37]. However, additional symmetry should be introduced to make ratios of the element in $m_D$ and $M_{RS}$ to be equal, so that $d = \mathbb{I}$. A kind of
permutation symmetry $\psi \leftrightarrow S$ could be used.

5. **Basis fixing symmetry.** $G_b = \mathbb{Z}_2 \times \mathbb{Z}_2$ was introduced as the basis fixing symmetry in all the sectors, thus allowing to communicate information about mixing from the hidden sector to the visible one. This symmetry leads to diagonal structure of all Dirac mass matrices, even if several Higgs 10-plets (or other representations) with zero charges are introduced. This means that no CKM-mixing is generated: $U_{\text{CKM}} = \mathbb{I}$. Therefore generation of CKM mixing would require breaking of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. $G_b$ can be identified with $G_V$. To generate mixing $U_0$, $G_b$ should be broken in the hidden sector.

6. **Flavons.** The key element of the framework is that scalars of the hidden sector $\phi$ do carry non-trivial $G_b$ charges, in contrast to $H^{(10)}$ and $H^{(18)}$. When $\phi_i$ get VEVs, $G_b$ is spontaneously broken in the hidden sector. This leads to non-diagonal matrix $M_S$, and consequently, to mixing of singlets $S_i$. It is this mixing that generates the matrix $U_0$. The Klein (abelian) symmetry is not enough to obtain special form of $M_S$ and consequently, $U_0$ like TBM. For this non-abelian symmetry should be introduced in the hidden sector.

### III. $U_0$ FROM THE RESIDUAL SYMMETRIES OF VISIBLE AND HIDDEN SECTORS

In the case of complete screening, the portal interactions do not influence the mixing and one can immediately discuss the bases of the visible and hidden sectors. Then the mixing can be understood as a relation between the bases in which generators of $G_V$ and $G_H$ have diagonal forms. According to (8) and (9), these two basis should be connected by $U_0 = U_0^V U_S$. Thus, we start with common basis fixing symmetry $G_b$ in all the sectors and then to promote $G_b$ to a larger non-abelian group $G_f$ in the hidden sector. Then breaking of $G_f$ should be arranged in such a way that $G_b \in G_f$ is broken and another intrinsic unbroken $G_H = (\mathbb{Z}_2 \times \mathbb{Z}_2)^H$ symmetry is realized.

One can view this procedure as the residual symmetry approach: $G_H$ and $G_V$ are embedded into a unified flavor group

$$ G_f \supset G_H, G_V. $$

This embedding ensures that information about mixing from the hidden sector is transmitted to the visible sector. Then $G_f$ is broken explicitly down to $G_V$ in the visible (low mass scale) sector and it is broken down to $G_H$ spontaneously in the hidden sector.\(^3\)

This is similar to the usual residual symmetry approach \([41–44]\) when instead of mass matrices of the charged leptons and neutrinos we use the mass matrices of $\psi$ and $S$. Here the residual symmetries operate at different energy scales: the GUT-scale and the Planck scale.

An important feature is that in both sectors the residual symmetries are given by the Klein groups:

$$ G_V = (\mathbb{Z}_2 \times \mathbb{Z}_2)^V, \quad G_H = (\mathbb{Z}_2 \times \mathbb{Z}_2)^H. $$

(13)

Embedding of two Klein groups into a finite group and its consequences for mixing have been explored in \([39]\). The only difference is that in \([39]\) the results of embedding were applied to the relative matrix between interactions with Higgs 10-plet and Higgs 126-plet, while here we deal with the relative rotation between the mass basis generated by Higgs 10-plet (visible sector which coincides with matrix of portal) and the mass basis generated by Higgs singlets in the hidden sector. In what follows, we briefly remind the important points and present the main results.

In a 3-dimensional irreducible representation of $G_f$, the elements $T \in G_V$ and $R \in G_H$ in Eq. (8) and Eq. (9) with positive determinants can be written as

$$ T_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, $$

(14)

$$ R_1 = U_0 \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} U_0^\dagger, \quad R_2 = U_0 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} U_0^\dagger, \quad R_3 = U_0 \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} U_0^\dagger, $$

(15)

\(^3\) One can consider also spontaneous symmetry breaking in the visible and portal sectors, but this would introduce further complication of the model.
where we use the basis in which $T$ are diagonal. By definition the group $G_f$ should contain all these elements as well as their products. Furthermore, since $G_f$ is a finite group, any product of $T$ and $R$ should have a finite order:

$$W^p_{ij} \equiv (T_i R_j)^p = \left(T_i U_0 R_i^T U_0^\dagger\right)^p = I, \quad (i, j = 1, 2, 3),$$

where $p$ is a positive integer. This is the symmetry group condition [39] which determines the $i$-$j$ element of $U_0$:

$$|U_0|_{ij}^2 = \cos^2 \frac{n \pi}{p},$$

where $n$ and $p$ are integers. The detailed derivation of Eq. (17) is presented in the Appendix A. Using one generator $T_i \in G_V$ and another one $R_j \in G_H$ fixes the $i$-$j$ element of the matrix $U_0$. The indices $i, j = 1, 2, 3$ are identified by the positive diagonal elements in Eq. (14) and Eq. (15). For instance, $i = 1$ corresponds to generators with +1 in the 1-1 position, etc.

For each pair of $(i, j)$, it can be its own power $p_{ij}$, and the corresponding $n_{ij} < p_{ij}$. Furthermore, for fixed $p_{ij}$, several $n_{ij}$ can exist. Taking three different symmetry group relations given by one $T$ and three $R$ (or vice versa) we can fix three elements of row (column) of the mixing matrix and they should satisfy the unitarity condition. Using (17) we can write the unitarity condition, in general, as

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

where $\alpha$, $\beta$ and $\gamma$ are rational numbers of $\pi$.

As we will see, the unitarity condition in the form (18) already strongly restricts the number of possibilities even before further applications of the group theory constraints. Without loss of generality, we assume that $\cos \alpha \leq \cos \beta \leq \cos \gamma$ and $0 \leq \alpha, \beta, \gamma \leq 90^\circ$. Under these assumptions, we only need to consider two cases: $\cos \alpha = 0$ and $\cos \alpha \neq 0$.

If $\cos \alpha$ is zero ($\alpha = \pi/2$), Eq. (18) reduces to $\cos^2 \beta + \cos^2 \gamma = 1$, from which one immediately obtains $\gamma = \pi/2 - \beta$. Since $\beta = \pi q/p$, we get infinite number of solutions for the angles:

$$(\alpha, \beta, \gamma) = \pi \left(\frac{1}{2}, \frac{q}{p}, \frac{1}{2} - \frac{q}{p}\right).$$

For $\cos \alpha \neq 0$ (all cosines are non-zero) a numerical search for all rationals $q/p$ with $p \leq 100$ gives only two solutions:

$$(\alpha, \beta, \gamma) = \pi \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}\right),$$

and

$$(\alpha, \beta, \gamma) = \pi \left(\frac{2}{5}, \frac{1}{3}, \frac{1}{5}\right).$$

The elements of $|U_0|$ which correspond to (19), (20) (21) are

$$v_1 \equiv \left(0, \cos \frac{q}{p} \pi, \sin \frac{q}{p} \pi\right), \quad v_2 \equiv \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right), \quad v_3 \equiv \left(\frac{\sqrt{5} + 1}{4}, \frac{1}{2}, \frac{\sqrt{5} - 1}{4}\right).$$

The last solution in Eq. (22) can be expressed in terms of the golden ratio,

$$\varphi \equiv \frac{1}{2} \left(1 + \sqrt{5}\right) \approx 1.618,$$

$$v_3 = \frac{1}{2} (\varphi, 1, \varphi^{-1})^T.$$ The first solution in (22) has two interesting possibilities:

$$v_{1a} = (1, 0, 0), \quad v_{1b} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right).$$

Now using the vectors in Eq. (22) as building blocks, we can construct complete mixing matrices. There is a freedom to take $v_i$ as rows or columns of the matrix and also to permute elements within $v_i$. Not all combinations are allowed by unitarity. If the unitarity is satisfied for column, one should arrange the elements in each column so that
it is satisfied for rows as well. Clearly, using three times the same column with permuted elements will automatically satisfy the unitarity condition for whole the matrix.

Let us consider first that at least one of the columns of \( |U_0| \) is in the form of \( v_1 \) in Eq. (22), which means that \( U_0 \) has at least one zero entry. In general, one can prove\(^4\) that the number of zero entries in a \( 3 \times 3 \) unitary matrix can only be 1 or 4 or 6. The matrix with 4 zeros constructed of \( v_1 \) is

\[
|U_0| = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \frac{2\pi}{p} & \sin \frac{2\pi}{p} \\
0 & \sin \frac{2\pi}{p} & \cos \frac{2\pi}{p}
\end{pmatrix},
\]

(25)

The case of 6 zeros corresponds to \( |U_0| = I \).

For the case of single zero, we take \( v_1 \) while the other columns can not contain zero entries, and therefore they have to be of the form \( v_2 \) or \( v_3 \). The conclusion about the columns of \( |U_0| \) also hold for the rows, which implies that the rows consist of one \( v_1^T \), and two \( v_2^T \) or \( v_3^T \). As a result, the two non-zero elements in \( v_1 \) have to be \( 1/\sqrt{2} \), \( \varphi/2 \), or \( \varphi^{-1}/2 \). The squared sum of the two non-zero elements should be 1, leaving only one option: \( (0,1/\sqrt{2},1/\sqrt{2}) \).

Therefore, in the case of 1 zero, we have:

\[
|U_0| = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & 1/2 & 1/\sqrt{2}
\end{pmatrix},
\]

(26)

which coincides with the BM mixing matrix \([7, 45–49]\). There is no other possibilities with \( v_1 \).

Next, let us consider \( |U_0| \) constructed from \( v_2 \), or \( v_3 \), or \( v_2 \) and \( v_3 \) together. There is only one solution in each case:

\[
|U_0| = \frac{1}{2} \begin{pmatrix}
\sqrt{2} & 1 & 1 \\
1 & \sqrt{2} & 1 \\
1 & 1 & \sqrt{2}
\end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix}
\varphi & 1 & \varphi^{-1} \\
\varphi^{-1} & 1 & \varphi \\
\varphi^{-1} & \varphi & 1
\end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix}
\varphi^{-1} & 1 & \varphi \\
1 & \sqrt{2} & 1 \\
1 & \varphi & \varphi^{-1}
\end{pmatrix}.
\]

(27)

The symmetry group condition gives the moduli of elements, \( |U_0| \). To reconstruct \( U_0 \) completely one needs to find the phases of elements which ensure orthogonality of the rows and columns in \( U_0 \). Without loss of generality, we assume that \( u_{11}, u_{12}, u_{13}, u_{23}, \) and \( u_{33} \) are real. Then the orthogonality of the columns gives

\[
u_{11}u_{13} + |u_{21}|e^{i\phi_{21}}u_{23} + |u_{31}|e^{i\phi_{31}}u_{33} = 0,
\]

(28)

\[
u_{12}u_{13} + |u_{22}|e^{i\phi_{22}}u_{23} + |u_{32}|e^{i\phi_{32}}u_{33} = 0,
\]

(29)

where \( \phi_{ij} = \text{arg}(u_{ij}) \). Using the graphic representation of the equalities (28, 29) (i.e., the unitarity triangles) we obtain:

\[
\cos \phi_{2j} = \frac{|u_{3j}|^2u_{23}^2 - u_{1j}^2u_{13}^2 - |u_{2j}|^2u_{23}^2}{2u_{1j}u_{13}|u_{2j}|u_{23}}, \quad \cos \phi_{3j} = \frac{|u_{2j}|^2u_{23}^2 - u_{1j}^2u_{13}^2 - |u_{3j}|^2u_{33}^2}{2u_{1j}u_{13}|u_{3j}|u_{33}} \quad (j = 1, 2).
\]

(30)

Consequently, the phases for the three matrices in Eq. (27) equal

\[
\text{arg}(U_0) = \begin{pmatrix}
0 & 0 & 0 \\
\arccos -\frac{1}{2\sqrt{2}} & -\arccos \frac{3}{\sqrt{2}} & 0 \\
\arccos -\frac{1}{2\sqrt{2}} & \arccos -\frac{3}{\sqrt{2}} & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
\pi & -\pi & 0 \\
0 & \pi & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
\frac{2\pi}{3} & -\arccos \left(\sqrt{\frac{3}{2}}\right) & 0 \\
-\frac{2\pi}{3} & \frac{2\pi}{3} & 0
\end{pmatrix}.
\]

The matrices in Eqs. (25) and (26), contain zero mixing angles and therefore the phases can be removed by rephasing. Orthogonality in Eq. (25) and Eq. (26) can be achieved by adding minus signs, e.g. to 2-1, 3-1, 2-3 elements in Eq. (26) and to 3-2 element in Eq. (25). The mixing matrices including the phases are summarized in the Table I.

Notice that till now we used only a general form of matrix elements as cosines of rational numbers of \( \pi \) (17) and the unitarity. The symmetry group condition is necessary but not sufficient one for embedding into the finite group. Still one should check that embedding is possible from the group theory point of view. Using \( U_0 \) given in the Table

\(^4\) The proof is straightforward enumeration. The number of zeros can not be larger than 6 because it implies that at most two elements of \( U_0 \) can be non-zero, which is impossible for a unitary matrix. If there are five zeros, i.e., four elements are non-zero, then the orthogonality of rows/columns requires one of the four elements to be zero. Likewise, one can check that \( U_0 \) with two or three zeros have the same problem.
I, we can find the corresponding generators $R_i$ according to Eq. (15), and check the group presentations. Then using the GAP program [50], we identify all the corresponding finite groups (see the last column of Table I). Notice that the block diagonal matrices $U_{q/p}$ are generated by a dihedral group $D_p$, of with the order $p$ determined by the denominator of the rational angle. The matrices $U_{BM}, U_{v2},$ and $U_{GR}$ are generated by the groups $S_4, PSL(3,2)$ and $A_5$, of orders 24, 168 and 60 correspondingly. It has been well known (see, e.g., [48, 49]) that the $S_4$ symmetry can be used to obtain $U_{BM}$. The last and most complicated matrix $U_{GR-v2}$ can be obtained in a 1080-order group, which is a non-split extension of $A_6$ by $C_3$, denoted as $C_3 \cdot A_6$ in the GAP classification system.

Reconstructing finite groups from the residual symmetries have been studied Ref. [51] using theorems on sums of roots of unity, which is technically similar to the trace approach formulated in Eq. (A4). Some of the finite groups presented in the Table I (e.g. $S_4, A_5$) are the same as those found in [51]. However, one should note that Ref. [51] sets a finite order of $TR^TR$, while in Eq. (16) we use order of $TR$. Consequently, we obtain some additional groups such as $PSL(3,2)$ and $C_3 \cdot A_6$.

In summary, the BM mixing matrix can be obtained for $U_0$ in our approach. Also the matrices $U_{v2}$ (constructed with columns $v_2, v_2, v_2$) and $U_{GR}(v_3, v_3, v_3)$ can be of the phenomenological interest once certain corrections are taken into account. The matrix $U_{q/p}$ can be considered for the 2-3 mixing if the 1-2 mixing is generated, e.g. from the portal interactions. Notice that the TBM mixing can not be obtained in this framework. This is because we require that the residual symmetries should be $Z_2 \times Z_2$ while TBM actually needs a $Z_3$ residual symmetry. Without the requirement of $Z_2 \times Z_2$, TBM may be obtained in $SO(10)$ frameworks—see e.g. [52].

### IV. BIMAXIMAL MIXING FROM THE HIDDEN SECTOR

| Matrix | $|U_0|$ | $U_0$ | $(TR_i)^n = 1$ | Group |
|--------|--------|--------|----------------|--------|
| $U_{q/p}$ | $\cos \left( \frac{\pi}{2} \frac{q}{p} \pi - \frac{q}{p} \pi \right)$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{2\pi}{p} & \sin \frac{2\pi}{p} \\ 0 & -\sin \frac{2\pi}{p} & \cos \frac{2\pi}{p} \end{pmatrix}$ | $\begin{pmatrix} (T_1 R_1)^4 \\ (T_2 R_1)^2 \\ (T_2 R_2)^4 \\ (T_1 R_2)^3 \\ (T_2 R_1)^3 \\ (T_2 R_2)^3 \end{pmatrix}$ | $D_p$ |
| $U_{BM}$ | $\cos \left( 45^{\circ} 45^{\circ} 90^{\circ} \right)$ | $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/2 & 1/2 & -1/\sqrt{2} \\ -1/2 & 1/2 & 1/\sqrt{2} \end{pmatrix}$ | $\begin{pmatrix} (T_1 R_1)^3 \\ (T_2 R_1)^3 \\ (T_2 R_2)^3 \end{pmatrix}$ | $S_4$ |
| $U_{v2}$ | $\cos \left( 45^{\circ} 60^{\circ} 60^{\circ} \right)$ | $\begin{pmatrix} 1/\sqrt{3} & -\sqrt{3} + 1 & \sqrt{3} - 1 \\ \frac{1}{2} & 1/\sqrt{2} & -\sqrt{3} - 1 \\ -1/\sqrt{2} & \frac{1}{2} & \sqrt{3} + 1 \end{pmatrix}$ | $\begin{pmatrix} (T_1 R_1)^3 \\ (T_2 R_1)^3 \\ (T_2 R_2)^3 \end{pmatrix}$ | $PSL(3,2)$ |
| $U_{GR}$ | $\cos \left( 36^{\circ} 60^{\circ} 72^{\circ} \right)$ | $\begin{pmatrix} \varphi/2 & 1/2 & \varphi^{-1}/2 \\ 1/2 & -\varphi^{-1}/2 & -\varphi/2 \\ \varphi^{-1}/2 & -\varphi/2 & 1/2 \end{pmatrix}$ | $\begin{pmatrix} (T_1 R_1)^5 \\ (T_2 R_1)^5 \\ (T_2 R_2)^5 \end{pmatrix}$ | $A_5$ |
| $U_{GR-v2}$ | $\cos \left( 36^{\circ} 60^{\circ} 72^{\circ} \right)$ | $\begin{pmatrix} \varphi/2 & \frac{1}{2} & \varphi^{-1}/2 \\ \frac{1}{2} & -\frac{2+\sqrt{3}+\sqrt{5}}{4\varphi} & \frac{1}{4} \left( \sqrt{3} - 1 \right) \\ \varphi^{-1}/2 & \frac{1}{4} \left( \sqrt{3} + 1 \right) & -\frac{1}{4} \left( \sqrt{3} + 1 \right) \end{pmatrix}$ | $\begin{pmatrix} (T_1 R_1)^3 \\ (T_2 R_1)^3 \\ (T_2 R_2)^3 \end{pmatrix}$ | $C_3 \cdot A_6$ |

| Table II. Field content of the model and symmetry assignments. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| type            | $\psi$          | $S$             | $H^{(10)}$       | $H^{(16)}$       | $\eta$          | $\xi$           | $\phi$          |
| $SO(10)$        | $16$            | $1$             | $10$            | $16$            | $1$             | $1$             | $1$             |
| $S_4$           | $3$             | $3$             | $1$             | $1$             | $2$             | $3'$            |
For definiteness we consider generating the BM mixing from the $S_4$ embedding of the residual symmetries. Details of the group $S_4$, which has five irreducible representations $1, 1', 2, 3$ and $3'$, are given in Appendix B. All the fermions are assigned to the 3-dimensional representation $3$. The $SO(10)$ Higgs multiplets $H^{(10)}, H^{(16)}$ are flavor singlets. In contrast, the Higgs fields in the hidden sector have non-trivial $S_4$ assignments, and so the flavor symmetry is broken in this sector spontaneously. The symmetry assignments for the fields are given in the Table II.

We assume that in the visible and portal sectors, the $S_4$ symmetry is broken explicitly down to the residual symmetry $(Z_2 \times Z_2)^V$. The $(Z_2 \times Z_2)^V$ charges of $\psi$ and $S$ are

$$
\text{Fields : } \psi_1, S_1, \psi_2, S_2, \psi_3, S_3
$$

$$
(Z_2 \times Z_2)^V : (+, -) (-, +) (-, -)
$$

The visible and portal sectors are invariant under the transformation

$$
\psi \to T_i \psi, \quad S \to T_i S, \quad (i = 1, 2),
$$

where $T_i$ are defined in Eq. (14), and the transformations in Eq. (31) belong to a sub-group of $S_4$.

Due to the $(Z_2 \times Z_2)^V$ symmetry (31), the Yukawa interactions in the visible and portal sectors are flavor-diagonal, so that the Lagrangian (4) reduces to

$$
\mathcal{L}_\psi = \sum_{i=1}^{3} \left[ y_\psi^i \psi_i H^{(10)} + y_\psi^i \psi_i S H^{(16)} \right].
$$

In the hidden sector, the Yukawa interactions are

$$
\mathcal{L}_\text{hidden} = y^\phi_{ijk} S_i S_j \phi_k + y^\xi_{ijk} S_i S_j \xi_k + y^\eta S_i S_j \eta,
$$

where the Yukawa couplings $y^\phi_{ijk}$ and $y^\xi_{ijk}$ are determined by the $S_4$ symmetry. According to the CG coefficients of $S_4$ (see Appendix B), the products of these Yukawa couplings with the flavon fields (which eventually determine the mass matrix of $S$) can be expressed in the following matrix forms in the basis $(S_1, S_2, S_3)^T$:

$$
\sum_k y^\phi_{ijk} \phi_k = y^\phi \begin{pmatrix} 0 & \phi_2 - \phi_3 & -\phi_2 - \phi_3 \\ \phi_2 - \phi_3 & \sqrt{2} \phi_1 \\ -\phi_2 - \phi_3 & 0 & -\sqrt{2} \phi_1 \end{pmatrix},
$$

and

$$
\sum_k y^\xi_{ijk} \xi_k = y^\xi \begin{pmatrix} -e^{\pi i / 3} \xi_1 - \xi_2 / \sqrt{3} \\ 0 \\ \frac{1}{6} \left( 3 e^{\pi i / 3} \xi_1 + \sqrt{3} \xi_2 \right) \frac{1}{2} \left( e^{\pi i / 3} \xi_1 - \sqrt{3} \xi_2 \right) \\ 0 \frac{1}{2} \left( e^{\pi i / 3} \xi_1 - \sqrt{3} \xi_2 \right) \frac{1}{6} \left( 3 e^{\pi i / 3} \xi_1 + \sqrt{3} \xi_2 \right) \end{pmatrix}.
$$

To obtain non-trivial flavor structures, $S_4$ should be broken down to $(Z_2 \times Z_2)^H$ which differs from $(Z_2 \times Z_2)^V (Z_2 \times Z_2)^H$ is represented by the matrices $R$ Eq. (15) in $3$ of $S_4$ and by $R^{(1)}, R^{(1')}$, $R^{(2)}$, and $R^{(3)}$ in the representations $1, 1', 2,$ and $3'$, $(Z_2 \times Z_2)^H$ (see the appendix). Since the flavons $\phi$ and $\xi$ are assigned to $3'$ and $2$ and break $S_4$ down to $(Z_2 \times Z_2)^H$, their vacuum expectation values (VEVs) should be invariant under $(Z_2 \times Z_2)^H$, i.e.

$$
R^{(3')} \langle \phi \rangle = \langle \phi \rangle, \quad R^{(2')} \langle \xi \rangle = \langle \xi \rangle.
$$

This gives

$$
\langle \phi \rangle \propto (0, 0, 1)^T, \quad \langle \xi \rangle \propto (0, 1)^T,
$$

where we used explicit forms of $R^{(3')}$ and $R^{(2')}$ from (B4) (B5). The potentials which produce the vacuum alignment (37) can be easily constructed [48]. Finally from Eqs. (34), (35) and (37), we obtain the explicit form of $M_S$, and consequently, $m_\nu$:

$$
m_\nu \propto M_S = \begin{pmatrix} a - 2c & b & b \\ b & a + c & -3c \\ b & -3c & a + c \end{pmatrix},
$$
where
\[ a = y^0(\eta), \quad b = y^0(\phi), \quad c = \frac{y^k}{2}(\xi). \] (39)

The mass matrix in Eq. (38) is diagonalized by \( U_0 = U_{BM} \) with the eigenvalues
\[ U_0^T m_r U_0 = \begin{pmatrix} a - \sqrt{2}b - 2c & 0 & 0 \\ 0 & a + \sqrt{2}b - 2c & 0 \\ 0 & 0 & a + 4c \end{pmatrix}. \] (40)

The three parameters \( a, b, \) and \( c \) are sufficient to fit three light neutrino masses.

V. CKM MIXING AND PMNS MIXINGS

Generation of \( U_0 \) and screening in our framework require that the CKM-type mixing originates from the down components of the EW doublets. Therefore, to reproduce the relation (1), the mixing of the charged leptons should be approximately equal to the down quark mixing: \( U_l \approx V_{CKM}. \) This approximate equality of mixings should be reconciled with the difference of masses of down quarks and charged leptons in the second and the first generations. In fact, according to the two-loop RGE running in the Standard Model \[53\] we have at the GUT scale
\[ m_\mu \approx 3m_s \gg m_e, m_d. \] (41)

This problem was extensively discussed before in connection to the quark-lepton complementarity \[16\]. Actually, in the case of strong mass hierarchy the difference of mixings related to the difference of masses is not large and may be even needed to better fit of the data.

In general, there are two approaches to keep the relation \( U_l \approx U_d \) for different masses. The one is to decouple completely the generation of masses from mixing, so that the mixing comes from certain relations between the elements of mass matrix, whereas the masses are determined by absolute values of the elements. This decoupling is difficult to obtain for small CKM mixing: simple discrete symmetries usually lead to large nonzero angles\[6\]. Hence this approach would require substantial complications of the model. Another possibility, which we will implement here, is that one of the Higgs multiplets dominates in the generation of charged fermion masses, so that \( M_d \) and \( M_l \) have roughly the same form, and thus approximately equal mixing. One can also add a mass matrix proportional to the unit matrix: this does not change the mixing but affects the mass ratio.

With one 10-plet, which conserves the basis fixing symmetry, we obtain at the GUT scale the diagonal mass matrices:
\[ M_{d}^{(10)} = M_{l}^{(10)} = \frac{v_d}{v_u} M_{u}^{(10)} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \] (42)

where \( v_d \) and \( v_u \) are the VEVs of the 10-plet which generate masses of the upper and bottom components of the EW doublets correspondingly. The difference of masses of up and down components is due to the difference of VEVs: \( v_d/v_u \approx m_b/m_t. \) Since \( k_1 : k_2 : k_3 = m_u : m_c : m_t, \) we obtain for \( k_3 \approx m_b \sim m_\tau \sim 1 \text{ GeV}, \) and the other masses \( k_2 \approx 3 \text{ MeV}, k_1 \approx 10^{-2} \text{ MeV}, \) which are much smaller than \( m_s \) and \( m_d \) correspondingly. So, additional sources of mass and mixing are needed. The simplest possibility is that the total matrices of the down-type quarks and charged leptons consist of
\[ M_d = M_{d}^{(10)} + M_x, \quad M_l = M_{d}^{(10)} + aM_x, \] (43)

where \( |a| \approx 3 \) is needed to reproduce (41), and maximal values of elements in \( M_x \) should be \( M_x^{max} \sim m_s \) not to destroy the \( b-\tau \) unification. The matrix \( M_x \) is non-diagonal, thus breaking the basis fixing symmetry or \( G_V, \) and producing the CKM mixing.

\[ \text{We take the following convention in the definitions of } U_{ud}, U_{ue}, U_e, \text{ and } U_{0}: \]
\[ M_u = U_{ud} \text{diag}(m_u, m_c, m_t) U_{ud}^T, \quad M_d = U_{ud} \text{diag}(m_d, m_s, m_b) U_{ud}^T, \]
\[ M_{e} = U_{e} \text{diag}(m_e, m_\mu, m_\tau) U_{e}^T \text{ and } M_{\nu} = U_{0} \text{diag}(m_\nu, m_\mu, m_\tau) U_{0}^T. \]

\[ \text{In this convention, the CKM and PMNS matrices should be } U_{CKM} = U_{ud} U_{d}, \text{ and } U_{PMNS} = U_{0}^T U_{0}. \]

\[ \text{There are, however, some finite groups contain small angles—see e.g. [54–56].} \]
Correct masses and mixing can be obtained provided that $M_x$ has the structure

$$M_x \approx \begin{pmatrix} d_1 & f & f' \\ f & d_2 & d' \\ f' & d' & d_3 \end{pmatrix},$$

with

$$f' \approx f, \quad d' \sim d_2 \sim d_3, \quad d_1 \ll d_2, d_3,$$

and

$$f \approx d_2 \sin \theta_C.$$  

That is, the mass matrix $M_x$ (44) has the dominant 2-3 block and the Cabibbo suppressed 12- and 13-elements. It is similar to the TBM or BM mass matrices. Therefore it is also similar to the structure of $M_S$, and consequently, to $m_\nu$, in the case of normal mass hierarchy. It is interesting to speculate that common Planck scale physics is responsible for the structure of $M_S$ and $M_x$.

Numerically we need to have

$$d_1 \sim 0.1v_{EW} \frac{M_{GUT}}{M_{Pl}} \sim (30-100) \text{ MeV},$$

comparable to the masses of second generation, i.e. muon and s-quark.

The total mass matrices of the down quarks and charged leptons (43) for $a = -3$ become

$$M_d \approx \begin{pmatrix} d_1 + k_1 & f & f' \\ f & d_2 + k_2 & d' \\ f' & d' & d_3 + k_3 \end{pmatrix}, \quad M_l \approx \begin{pmatrix} -3d_1 + k_1 & -3f & -3f' \\ -3f & -3d_2 + k_2 & -3d' \\ -3f' & -3d' & -3d_3 + k_3 \end{pmatrix},$$

where $d_1 \gg k_1$ and $d_2 \gg k_2$. So, for the second and the first generations the contributions from $M_x$ dominate. This leads to (i) about three times larger mass of muon than the mass of s quark, and (ii) to approximately the same 1-2 mixing of leptons and quarks. For the third generation, the contribution from the 10-plet dominates, thus ensuring the approximate $b-\tau$ unification. In general, $m_\tau = m_b + \mathcal{O}(4 m_\mu)$.

From (48) we obtain for the 2-3 quark mixing

$$V_{cb} \approx \frac{d'}{k_3} \approx \frac{d'}{m_b} \approx \frac{m_s}{m_b},$$

and the 1-3 quark mixing

$$V_{ub} \approx \frac{f'}{k_3} \sim \frac{f}{m_b} \sim \frac{m_s}{m_b} \sin \theta_C,$$

in agreement with observations. According to (49) and (50), $V_{ub} \sim V_{cb} V_{us}$.

The lepton mixing parameters are about 3 times larger:

$$U_{\mu 3} \approx -\frac{3d'}{m_\tau}, \quad U_{e 3} \approx -\frac{3f'}{m_\tau}.$$  

This corresponds to the angles $\theta_{23}^l \sim (4 - 5)^\circ$ and $\theta_{13}^l \sim 1^\circ$, which give a sizable deviation from maximal 2-3 mixing and observable corrections to the 1-3 mixing.

Decoupling of the third state produces small corrections to the 1-2 sub-matrix of $M_d$: The correction to the 1-1 element $f'^2/m_b \sim 0.05$ MeV, the relative corrections to other elements are of the order $d'/m_b \sim m_s/m_b \sim 3\%$ and can be neglected. The corresponding relative corrections in the lepton sector are 3 times larger; the correction to the 1-1 element which is an order of magnitude larger: $9f'^2/m_b \sim 0.45$ MeV can be important for the mass of electron.

In the 1-2 sector we can reproduce the Gatto-Sartori-Tonin relation: $\sin \theta_C \sim \sqrt{m_\mu/m_\tau}$.

Let us make few comments on possible origins of $M_x$. The straightforward way is to introduce a 126-plet which produces $a = -3$ in Eq. (43). This 126-plet should not contribute substantially to the masses of neutrinos, not to destroy the inverse seesaw with screening. For this, the VEVs of the $SU(2)$ singlet and triplet in 126-plet should be
zero or small. The mass of 126-plet can be at Planck scale to avoid the problem of perturbativity of the theory (see, e.g., the review [57]).

Another possibility [58] is to use composite 126-plet constructed from the product of two 16-plets. The coupling with fermions is given by non-renormalizable operators suppressed by the Planck scale $M_{Pl}$:

$$\mathcal{L} \supset \frac{1}{M_{Pl}} \psi \psi H^{(16)} H^{(16)}.$$  \hfill (52)

Here $H^{(16)}$ is new 16-plet of scalars with zero VEV of the $SU(2)$ triplet and singlet components. Similar operator with $H^{(16)} H^{(16)}$ can be forbidden by additional symmetry with respect to transformations $H^{(16)} \rightarrow i H^{(16)}$, $H^{(16)} \rightarrow -i H^{(16)}$, $S \rightarrow -i S$, $(\eta, \xi, \phi) \rightarrow -(\eta, \xi, \phi)$. Then one should assume that due to some Planck scale physics the down Higgs doublet in the composite 126-plet acquire the VEV

$$\langle (H^{(16)} H^{(16)})^i \rangle_d = v_d M_{GUT}.$$  \hfill (53)

For $v_d \sim 0.1 v_{EW}$, this reproduces Eq. (47).

Notice that instead of the non-renormalizable interaction (52) we can introduce

$$\mathcal{L} \supset \frac{1}{M_{Pl}} \psi \psi H^{(10)} H^{(45)},$$  \hfill (54)

where $H^{(45)}$ is the 45-plet responsible for the $SO(10)$ symmetry breaking [58]. The product $H^{(10)} H^{(45)}$ contains antisymmetric 120-plet, and therefore can remove the degeneracy of charge lepton and d-quark masses. However, the matrix (44) with diagonal elements cannot be reproduced.

In what follows, for simplicity we will consider mixing of the first two generations only. Inclusion of corrections from the 1-3 or 2-3 mixing changes the following results very little. The mass matrices (48) can be diagonalized by

$$U_d = \begin{pmatrix} c & \tilde{s} & 0 \\ -\tilde{s}^* & c & 0 \\ 0 & 0 & 1 \end{pmatrix} P_d, \quad U_l = \begin{pmatrix} c_l & \tilde{s}_l & 0 \\ -\tilde{s}_l^* & c_l & 0 \\ 0 & 0 & 1 \end{pmatrix} P_l,$$  \hfill (55)

where

$$c \equiv \cos \theta_C, \quad \tilde{s} \equiv \sin \theta_C e^{i \phi_C}, \quad c_l \equiv \cos \theta_l, \quad \tilde{s}_l \equiv \sin \theta_l e^{i \phi_l},$$

and $P_d$ and $P_l$ are diagonal matrices containing complex phases. Although all elements of the mass matrices in Eq. (48) are complex, for simplicity, we assume that only $f$ is complex. Then six real parameters $d_{1,2,3}$ and $k_{1,2,3}$ allow us to accommodate the six masses ($m_d, m_s, m_b, m_e, m_l, m_r$), while the complex $f$ generates the Cabibbo mixing, $\sin \theta_C$, with a complex phase $\phi_C$, and analogous mixing $\sin \theta_l$ and phase $\phi_l$ in the lepton sector.

The phase $\phi_C$ has no physical meaning for the $2 \times 2$ form of $U_d$. In contrast, as we will see, $\phi_l$ is directly related to the CP phase in the PMNS matrix. Introduction of small 1-3 and 2-3 mixing will make $\phi_C$ to be the origin of CP violation in the CKM mixing. But this will have little effect on the PMNS mixing.

Using the hierarchy $m_d \ll m_s$ and $m_c \ll m_\mu$ as well as the smallness of $\sin \theta_C \ll 1$, we obtain the following approximate relations (for more details, see Appendix C):

$$\frac{\phi_l}{\phi_C} = 1 + O \left( \frac{m_d}{m_s} \right),$$  \hfill (56)

$$\frac{\sin 2 \theta_l}{\sin 2 \theta_C} \approx \frac{3 (m_s + m_d \cos \phi_1)}{m_\mu + m_c \cos \phi_2} \approx 1,$$  \hfill (57)

where

$$\phi_1 \equiv \pi - \arcsin \left( \frac{s^2 m_s}{c^2 m_d} \sin 2 \phi_1 \right) - 2 \phi_l, \quad \phi_2 \equiv \arcsin \left( \frac{s^2 m_\mu}{c^2 m_c} \sin 2 \phi_1 \right) - 2 \phi_l.$$  \hfill (58)

Eq. (58) shows complicated dependence of the phases, on known quantities (fermion masses, the Cabibbo angle) and on $\phi_l$ which in turn is related to the leptonic CP phase. Values of $\phi_1$ and $\phi_2$ for two special values of $\phi_l$ can be obtained from (58),

$$(\phi_1, \phi_2) \approx \begin{cases} (0^\circ, 180^\circ) & \text{for } \phi_l = \pm 90^\circ \quad (i) \\ (180^\circ, 0^\circ) & \text{for } \phi_l = 0^\circ \text{ or } 180^\circ \quad (ii) \end{cases}.$$  \hfill (59)
For other values of $\phi_l$ results of numerical study will be presented later.

Eqs. (56) and (57) show that the charged leptons do have approximately the same mixing as the down-type quarks, $\theta_l \approx \theta_C$ and $\phi_l \approx \phi_C$. Recall that the factor 3 in Eq. (57) originates from the effective 126-plets. Taking the 1$\sigma$ range values of $m_e$, $m_\mu$, $m_u$, $m_s$ from Ref. [53], and using Eq. (57) we can evaluate the ratio of the angles for the two choices of phases [cf. Eq. (59)]:

\[
\frac{\theta_l}{\theta_C} = \begin{cases} 
0.871 - 1.22 & (i) \\
0.999 - 1.35 & (ii) 
\end{cases}
\]

(60)

For other values of the phases, one would get intermediate results between those in the cases (i) and (ii).

According to Eq. (55), the PMNS matrix should be

\[
U_{PMNS} = \begin{pmatrix}
c_l & -s_l^* & 0 \\
s_l & c_l & 0 \\
0 & 0 & 1
\end{pmatrix} U_{BM}
\]

(61)

or explicitly

\[
U_{PMNS} = \begin{pmatrix}
\frac{1}{2} (s_l^2 + \sqrt{2} c_l) & -\frac{s_l^2}{2} + \frac{c_l}{\sqrt{2}} & \frac{s_l^2}{\sqrt{2}} \\
\frac{s_l}{\sqrt{2}} - \frac{c_l}{\sqrt{2}} & \frac{1}{2} (\sqrt{2} s_l + c_l) - \frac{s_l}{\sqrt{2}} \\
-\frac{1}{2} & \frac{1}{2} (\sqrt{2} s_l + c_l) - \frac{s_l}{\sqrt{2}}
\end{pmatrix}.
\]

(62)

Notice that the matrix of phases $P_l$ does not appear here since it can be removed by rephasing of the fields. From (62) one finds mixing parameters in the standard parametrization,

\[
s_{13} = \frac{s_l}{\sqrt{2}},
\]

(63)

\[
s_{12}^2 = \frac{1}{2} - \frac{\sqrt{2} c_l s_l \cos \phi_l}{2 - s_l^2},
\]

(64)

\[
s_{23}^2 = \frac{c_l^2}{2 - s_l^2} \approx \frac{1}{2} \left( 1 - \frac{1}{2} s_l^2 \right),
\]

(65)

\[
\sin \delta_{CP} = -\sin \phi_l - s_l^2 \sin \phi_l \cos^2 \phi_l + O(s_l^4).
\]

(66)
The above results can be expressed in terms of known variables ($\theta_C$, $m_s$, $m_d$, $m_\mu$, $m_e$) and $\phi_l$. Although $\phi_l \approx \phi_C$, we cannot connect it to the CP violation in the CKM mixing without introducing 1-3 and 2-3 mixing. Using expression Eq. (C18) for the mixing parameter $s_{13}$ and replacing $\phi_l$ by $\delta_{\text{CP}}$, we obtain from (63)

$$s_{13} \approx 3 \sin \theta_C \sqrt{2} \left| \frac{m_s - m_d e^{-2 i \delta_{\text{CP}}} e^{-i \sin^{-1} \left( \frac{m_\mu}{m_d} \tan^2 \theta_C \sin 2 \delta_{\text{CP}} \right)}}{m_\mu + m_e e^{-2 i \delta_{\text{CP}}} e^{-i \sin^{-1} \left( \frac{m_\mu}{m_e} \tan^2 \theta_C \sin 2 \delta_{\text{CP}} \right)}} \right|. \quad (67)$$

From (63) and (64) we obtain the relation between observables:

$$s_{23}^2 \approx \frac{1}{2} + \frac{s_{13} \cos \delta_{\text{CP}}}{c_{13}^2}. \quad (68)$$

In Fig. 1 we show the mixing angles $\theta_{12}$ and $\theta_{13}$ as functions of the CP phase $\delta_{\text{CP}} \approx \phi_l$ according to Eqs. (64) and (63). We used $\sin \theta_C = 0.225$ and the values of $m_s$, $m_d$, $m_\mu$, $m_e$ from Ref. [53]. The angle $\theta_{23}$ is in the first octant: $\sin^2 \theta_{23} = 0.49$. Taking $1\sigma$ allowed interval for $\theta_{12}$ we obtain from (68) $\cos \delta_{\text{CP}} < -0.86$ or

$$\delta_{\text{CP}} \in (0.80 \pi, 1.16 \pi). \quad (69)$$

Including 2-3 mixing in $V_{\text{CKM}} \simeq U_1$ changes the prediction for $\delta_{\text{CP}}$ by a few degrees. The result (69) is in agreement with general phenomenological analysis [60] for the case of BM mixing receiving corrections from charge lepton mixing $U_l$. According to [60] $\cos \delta_{\text{CP}}$ should be in the range $[-1.00, -0.72]$ at $3\sigma$ confidence level.

The upper value of the interval (69) is in agreement with global fit results at about $1\sigma$ level. Notice that $\delta_{\text{CP}}$ is strongly restricted here by the requirement of maximal possible reduction of the 1-2 mixing from its BM value $\sin^2 \theta_{12} = 0.5$. So, that the best value would be $\cos \delta_{\text{CP}} = -1$. The only other parameter that enters the relation (68) is the 1-3 mixing which is measured very precisely. If we would use the TBM matrix $U_0 = U_{\text{TBM}}$ instead of BM, no large corrections from $U_l^T$ is required and $\cos \delta_{\text{CP}}$ should be close to zero. However, $U_0 = U_{\text{TBM}}$ can not be obtained in our residual symmetry approach. Thus, future measurements of $\delta_{\text{CP}}$ will test the scenario.

The values of mixing angles are subjects of the renormalization group (RG) corrections. The CKM mixing receives small corrections. For the PMNS mixing in our framework the RG corrections can be significant due to strong hierarchy of the right-handed neutrino masses. The strong hierarchy originates from the up-type quark mass spectrum, as the right-handed neutrino mass matrix is determined by $m_D^T m_D^{-1} m_D$ and in $SO(10)$ models, $m_D$ is the same as $M_u$ (or approximately the same in multi-Higgs variations). From this one obtains the masses of the order $(10^4, 10^9, 10^{14})$ GeV. Between the lightest and heaviest right-handed neutrinos, one or two of them are integrated out while the others remain in the RG equations. This is where RG running may have larger effects on the flavor structure [61].

The RG effects with such a strong hierarchy have been studied in [18]. In the SM extended by RH neutrinos, the corrections mainly depend on the lightest neutrino mass, Majorana phases and the mass ordering. The result in [18] shows that for $m_1 = 10^{-3}$ eV in the case of normal mass ordering, the correction to $\theta_{12}$ is in the range $(-1.5^\circ, 1.0^\circ)$. For smaller/larger $m_1$, the correction can be significantly suppressed/enhanced. For example, when $m_1 = 10^{-4}$ eV, the correction can be reduced down to $0.02^\circ$ (cf. Fig. 10 in [18]), independent of the Majorana phases; while for $m_1 = 10^{-2}$ eV with zero Majorana phases, it can reach $10^\circ$.

The other two mixing angles $\theta_{13}$ and $\theta_{23}$ are generally much more stable with respect to the RG corrections than $\theta_{12}$ [62–64]. In the scenario with strong hierarchy of right-handed neutrino masses, the correction to the 1-3 mixing is found to be always smaller than $0.3^\circ$ in the SM, which is negligible compared to the uncertainties caused by the fermion masses at the GUT scale—see Eq. (60).

The fermion singlets $S$ from the hidden sector may produce further corrections. Since the mass scale of $S$ is not much higher than the GUT scale, we expect small RG corrections from, e.g. $10^{16}$ GeV to $10^{18}$ GeV. At the GUT scale or below, the singlets can be integrated out, generating the heavy Majorana masses of right-handed neutrinos. Therefore, we can assume that the SM plus type I seesaw is valid up to the GUT scale and in this range, one can adopt the conclusions from [18]. Above the GUT scale, up to the mass scale of $S$ we expect small RG corrections due to small interval or running. Furthermore, the mixing of these singlets with active neutrinos is strongly suppressed.

Variations of the CP phase predictions may be possible. Essentially the result (69) is obtained in assumption of negligible RG corrections. Large RG corrections ($\sim 10^3$) to the 1-2 mixing can be obtained for the degenerate pair of $\nu_1$ and $\nu_2$. This can be realized for $m_1 \sim 10^{-2}$ eV or inverted mass ordering. In this case RG corrections can reduce $\theta_{12}$ down to $35^\circ$ and large corrections from $U_l$ are not needed. Consequently, $\cos \delta_{\text{CP}}$ can be small.

One can use some other matrices from the Table I, e.g. $U_{GR}$. This however, will require large corrections from $U_l^T$ and its substantial deviation from $V_{\text{CKM}}$. Another possibility is to produce some mixing from the portal interactions. Finally one can abandon the residual symmetry approach and fix structure of $M_S$ using other symmetries or principles.

Finally let us comment on the viability of leptogenesis in this framework. As previously mentioned, the typical RH neutrino masses are $(10^4, 10^9, 10^{14})$ GeV, which is very hierarchical. The lightest RH neutrino in this scenario is too
light to produce the observed baryon asymmetry [65]. However, the second RH neutrino is in the proper mass range and may realise successful $N_2$-leptogenesis (see, e.g., [66, 67]).

VI. CONCLUSIONS

1. The relation between the lepton and quark mixings [cf. Eq. (1)] can imply the Grand Unification and existence of the hidden sector which is connected to the visible sector via the RH neutrino portal. The Grand Unification ensures the approximate equality $U_l \approx V_d \approx V_{CKM}$, and consequently, $U_{PMNS} \approx V_{CKM}^\dagger U_0$, whereas the hidden sector with certain symmetries generates $U_0$ and produces the smallness of neutrino masses.

2. We focus on the symmetry aspects of this scenario—the interplay of discrete flavor symmetries and the SO(10) gauge symmetry. We develop the residual symmetry approach to generate $U_0$ which connects the visible and hidden sectors. The $Z_2 \times Z_2$ residual symmetries of the visible and hidden sectors are intrinsic symmetries of the SO(10) Yukawa interactions.

3. Embedding of these residual symmetries into a unified finite flavor group fixes the moduli of mixing matrix elements in the form of cosines of rational multiples of $\pi$. Imposing the unitarity condition results in only a few forms of $U_0$ which include the BM matrix. Using these matrices we reconstructed the group presentations and thus identified the corresponding symmetry groups. The flavor symmetry is broken in the hidden sector spontaneously. In the visible sector (at lower energy scales) the breaking can be explicit. In the latter case only the basis symmetry in the hidden sector is promoted to a larger non-abelian symmetry. We considered a specific model which realizes the BM mixing for $U_0$.

4. We use the 126-plet with the Planck scale mass or composite 126-plet originating from the Plank-scale physics to generate the CKM mixing and $U_l \sim V_{CKM}$ as well as differences of quark and lepton masses of the second and the first generations. Thus, generation of CKM mixing and the mass differences of the down quarks and charge leptons of the first the second generations are connected. Interestingly, the CKM mixing and the corresponding $U_l$ matrix can be reproduced with additional contribution from the Planck-scale physics with the flavor structure similar to the one for $S$ and consequently, the light neutrinos.

5. Assuming that the RGE corrections are small we expect the leptonic CP violation phase to be in the range $144^\circ \lesssim \delta_{CP} \lesssim 210^\circ$. Future measurements of $\delta_{CP}$ in accelerator neutrino experiments such as T2K, NOVA and DUNE will be an important test of this scenario.

6. Coupling of the hidden sector with visible one realizes the double seesaw mechanism. It allows to disentangle generation of the $U_0$ mixing and the CKM mixing. The latter is related to the Planck scale suppressed non-renormalizable interactions.

7. An important feature of this scenario is a very strong hierarchy of masses of the RH neutrinos. This can lead to significant renormalization group effects which correct, in particular, the 1-2 mixing. The lightest RH neutrino with mass $10^4$ GeV has mixing with active neutrinos of the order $10^{-7}$. So, it can not be observed at colliders, but could play some role in leptogenesis. The $N_2$ leptogenesis can be realized.

8. Future precise measurements of the CP phase, establishing mass ordering and absolute scale (degeneracy) of masses will provide important tests of the scenario. In particular, establishing strong normal mass hierarchy and substantial CP violation would exclude the simplest realization based on the BM mixing from the hidden sector.

No new physics related to the neutrino mass generation should be observed at LHC and other future collider experiments. Proton decay might be detected at some level.

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Appendix A: Symmetry group condition

All the $T$’s and $R$’s in Eq. (14) and Eq. (15) are $SU(3)$ matrices, i.e. $\det T = \det R = 1$, which implies that the product $T_i R_j$ is also an $SU(3)$ matrix. One can use properties of $SU(3)$ matrices to derive the relations below.
Introducing the eigenvalues of $W_{ij} \equiv T_i R_j$ ($\lambda_1$, $\lambda_2$, $\lambda_3$) we can represent this matrix as

$$W_{ij} = T_i R_j = U_{TS} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} U_{TS}^\dagger. \quad (A1)$$

Then according to (16), $\lambda_i^p = 1$. Keeping in mind that the eigenvalues satisfy the relations $|\lambda_1|^2 = |\lambda_2|^2 = |\lambda_3|^2 = \lambda_1 \lambda_2 \lambda_3 = 1$ (the latter follows from $\det W_{ij} = 1$), we can parametrize them as

$$\lambda_1 = e^{-i(\theta_2 + \theta_3)}, \quad \lambda_2 = e^{i\theta_2}, \quad \lambda_3 = e^{i\theta_3} \quad (A2)$$

with

$$\theta_i = \frac{2\pi n_i}{p}. \quad (A3)$$

Since unitary transformations do not change the trace of a matrix, we have

$$\text{tr} (T_i R_j) = \sum_i \lambda_i = e^{-i(\theta_2 + \theta_3)} + e^{i\theta_2} + e^{i\theta_3}. \quad (A4)$$

For a given $p$ this sum has discrete sets of values.

On the other hand, we compute the trace of $T_i R_j$ from Eq. (14) and Eq. (15):

$$\text{tr} (T_i R_j) = 4(U_0)_{ij} (U_0)^*_{ij} - 1 = 4 |(U_0)_{ij}|^2 - 1, \quad (A5)$$

which is a real number. Therefore, Eq. (A4) must be a real number too, which requires that

$$\sin \theta_2 + \sin \theta_3 - \sin(\theta_2 + \theta_3) = 0. \quad (A6)$$

This equation has only three solutions in the range $[0, 2\pi]$:

$$\theta_2 = 0, \quad \theta_3 = 0, \quad \theta_2 + \theta_3 = 0, \quad (A7)$$

or equivalently, $n_1 = 0$, $n_2 = 0$, $n_3 = 0$. Consequently, in any of these three cases, the eigenvalues of $T_i R_j$ can be taken as $(1, e^{2\pi n_i/p}, e^{-2\pi n_i/p})$, and thus the trace equals

$$\text{tr} (T_i R_j) = 1 + 2 \cos \frac{2\pi n_{ij}}{p_{ij}} = 4 \cos^2 \frac{\pi n_{ij}}{p_{ij}} - 1. \quad (A8)$$

Using Eq. (A5) and Eq. (A8) we immediately obtain equality (17).

**Appendix B: The $S_4$ group and its representations**

The group $S_4$ is the permutation group of four objects. It can be defined by four generators $r_1$, $r_2$, $t_1$ and $t_2$ with the following relations:

$$r_1^2 = r_2^2 = t_1^2 = t_2^2 = 1, \quad (B1)$$

$$(t_1 r_1)^4 = (t_1 r_2)^4 = (t_2 r_1)^3 = (t_2 r_2)^3 = 1. \quad (B2)$$

The group has five irreducible representations, denoted as $1$, $1'$, $2$, $3$ and $3'$, in which the generators are represented by the following matrices:

$$R_1^{(3)} = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}, \quad R_2^{(3)} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_1^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_2^{(3)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (B3)$$
\[ R_1^{(3')} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2^{(3')} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_1^{(3')} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_2^{(3')} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \]

\[ R_1^{(2)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_1^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2^{(2)} = \begin{pmatrix} \frac{1}{\sqrt{3}} e^{\frac{i\pi}{6}} \sqrt{\frac{3}{2}} e^{-\frac{i\pi}{6}} \end{pmatrix}; \]

where \( \omega \equiv \exp\left(\frac{2\pi i}{3}\right) \), and

\[ R_1^{(1')} = -1, \quad R_2^{(1')} = 1, \quad T_1^{(1')} = 1, \quad T_2^{(1')} = -1; \]

\[ R_1^{(1)} = 1, \quad R_2^{(1)} = 1, \quad T_1^{(1)} = 1, \quad T_2^{(1)} = 1. \]

The Clebsch-Gordan (CG) coefficients are given by

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} x_1y_1 + x_2y_2 + x_3y_3 \\ \sqrt{3} \end{pmatrix}^{(1)} \begin{pmatrix} \frac{1}{6\sqrt{2}} \left(6x_1y_1 - 3(x_2 + x_3)(y_2 + y_3)\right) \sqrt{3} \end{pmatrix}^{(2)} \]

\[ \begin{pmatrix} x_1y_1 - x_2y_2 \\ x_2y_1 - x_3y_3 \\ x_3y_2 - x_1y_3 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \left(x_2y_2 - x_3y_3\right) \\ x_2y_1 - x_3y_1 \\ x_3y_2 - x_1y_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \left(x_2y_1 - x_3y_1 + x_1(y_2 - y_3)\right) \\ - (x_2y_1 + x_3y_1 + x_1(y_2 + y_3)) \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix}, \]

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 3' \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} 3' \end{pmatrix} = \begin{pmatrix} x_1y_1 + x_2y_2 + x_3y_3 \\ \sqrt{3} \end{pmatrix}^{(1)} \begin{pmatrix} \frac{1}{\sqrt{2}} \left(x_1y_1 - x_2y_2\right) \sqrt{3} \end{pmatrix}^{(2)} \]

\[ \begin{pmatrix} \sqrt{2} (-x_3y_2 + x_2y_1) \\ (x_2y_1 + x_3y_1 - x_1(y_2 + y_3)) \\ -(x_2y_1 + x_3y_1 + x_1(y_2 - y_3)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \left(x_3y_2 + x_2y_3\right) \end{pmatrix}^{(3)} \]

\[ \begin{pmatrix} x_1y_2 - x_2y_1 \\ x_2y_1 - x_3y_3 \\ x_3y_2 - x_1y_3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_2y_1 + x_3y_1 + x_1(y_2 + y_3) \\ -(x_2y_1 + x_3y_1 + x_1(y_2 + y_3)) \end{pmatrix} \begin{pmatrix} \frac{1}{2} \end{pmatrix} \]

\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} 2 \end{pmatrix} = \begin{pmatrix} x_1y_1 - \omega x_2y_2 \\ \sqrt{2} \end{pmatrix}^{(1)} \begin{pmatrix} \frac{1}{\sqrt{2}} \left(x_1y_2 - x_2y_1\right) \sqrt{2} \end{pmatrix}^{(2)} \]

\[ \begin{pmatrix} x_2y_1 + x_3y_2 \\ -3d_1 + k_1 \\ -3d_2 + k_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \end{pmatrix} \]

\[ \begin{pmatrix} \tilde{c}_e (\tilde{c}_e \tilde{m}_{e} - \tilde{m}_e \tilde{s}_e) \\ \tilde{c}_e (\tilde{c}_e \tilde{m}_{e} - \tilde{m}_e \tilde{s}_e) \end{pmatrix} = \begin{pmatrix} -3d_1 + k_1 \\ -3d_2 + k_2 \end{pmatrix}, \]

\[ \tilde{c}_e (\tilde{c}_e \tilde{m}_{e} - \tilde{m}_e \tilde{s}_e) = \begin{pmatrix} -3d_1 + k_1 \\ -3d_2 + k_2 \end{pmatrix}. \]

Because \( \tilde{m}_d \tilde{s}_e^2 \ll \tilde{m}_e \tilde{c}_e^2 \) and \( \tilde{m}_e \tilde{s}_e^2 \ll \tilde{m}_e \tilde{c}_e^2 \), we neglect \( \tilde{m}_d \tilde{s}_e^2 \) and \( \tilde{m}_e \tilde{s}_e^2 \) below. Equating the corresponding elements of the matrices on the left-hand side and right-hand side we obtain expressions for \( d_{1,2} \) and \( k_{1,2} \):

\[ k_1 = \frac{1}{4} (3c^2 \tilde{m}_d + c^2 \tilde{m}_e + \tilde{m}_\mu \tilde{s}_l^2 + 3\tilde{m}_s \tilde{s}_e^2), \quad k_2 = \frac{1}{4} (3c^2 \tilde{m}_s + c^2 \tilde{m}_\mu), \]
\[ d_1 = \frac{1}{4} \left( e^2 \tilde{m}_d - c^2 \tilde{m}_e - \tilde{m}_\mu \tilde{m}_\mu' + \tilde{m}_s \tilde{s}^2 \right), \quad d_2 = \frac{1}{4} \left( e^2 \tilde{m}_s - c^2 \tilde{m}_\mu \right). \] (C4)

Since \(-3d_2 + k_2\) and \(d_2 + k_2\) in Eq. (C2) and Eq. (C1) are real under our assumptions, \(\tilde{m}_s\) and \(\tilde{m}_\mu\) should be approximately real, either positive or negative. Because \(d_2\) is dominant, we take positive \(\tilde{m}_s\) and negative \(\tilde{m}_\mu\), i.e. \(\tilde{m}_s \approx m_s\) and \(\tilde{m}_\mu \approx -m_\mu\). Furthermore, the equalities \(\text{Im}(d_1 + k_1) = \text{Im}(-3d_1 + k_1) = 0\) give

\[ e^2 m_d \sin \alpha_d + s^2 m_s \sin 2\phi_C = e^2 m_e \sin \alpha_e - s^2 m_\mu \sin 2\phi_I = 0, \] (C5)

which lead to

\[ \sin \alpha_d = -\frac{s^2 m_s}{e^2 m_d} \sin 2\phi_C, \quad \sin \alpha_e = \frac{s^2 m_\mu}{e^2 m_e} \sin 2\phi_I. \] (C6)

From the equality of the off-diagonal elements of (C1) (C2), we obtain

\[ f e^{i\phi_I} = c(s m_s - \tilde{m}_d \tilde{s}^* ) = \frac{1}{3} c_i (\tilde{s} m_\mu + \tilde{m}_e \tilde{s}^*_c), \] (C7)

or

\[ 2 f e^{i\phi_I} \approx 2 \theta_{C}(m_s e^{i\phi_{C}} - \tilde{m}_d e^{-i\phi_{C}}) = \frac{1}{3} \sin 2\theta_c (m_\mu e^{i\phi_I} + \tilde{m}_e e^{-i\phi_I}). \] (C8)

Because \(m_s \gg m_d\) and \(m_\mu \gg m_e\), one can immediately see from (C8) that the phases \(\phi_{C}\) and \(\phi_I\) should be approximately equal to \(\phi_I\):

\[ \phi_{C} = \phi_I + O \left( \frac{m_d}{m_s} \right), \quad \phi_{I} = \phi_I + O \left( \frac{m_e}{m_\mu} \right). \] (C9)

This reproduces the result in Eq. (56). Eq. (C8) also gives

\[ \frac{\sin 2\theta_I}{\sin 2\theta_{C}} = \frac{3(m_s e^{i\phi_{C}} - \tilde{m}_d e^{-i\phi_{C}})}{m_\mu e^{i\phi_{C}} + \tilde{m}_e e^{-i\phi_{C}}} \approx \frac{3(m_s + m_d e^{i\phi_{C}})}{m_\mu + m_e e^{i\phi_{C}}}, \] (C10)

where we have taken the approximation \(\phi_{C} \approx \phi_I \approx \phi_e\).

Due to the relations \(m_s \gg m_d\) and \(m_\mu \gg m_e\), in Eq. (C10) the imaginary parts in \(m_d e^{i\phi_{C}}\) and \(m_e e^{i\phi_{C}}\) can be neglected, which leads to Eq. (57).

Finally, we express \(k_{1,2}, k_{1,2} \) and \(f\) in terms of \((m_d, m_s, m_e, m_\mu, \theta_{C}, \phi_{I})\) and \(\theta_I\) inserting results of Eqs. (C9), and (C6) into Eqs. (C3), (C4) and (C7):

\[ k_1 = \frac{1}{4} \left( 3 \sqrt{c^4 m_d^2 - s^4 m_s^2} \sin^2 2\phi_I + \sqrt{c^4 m_e^2 - s^4 m_\mu^2} \sin^2 2\phi_I + \cos 2\phi_I \left( 3 s^2 m_s - s^2 m_\mu \right) \right), \] (C12)

\[ d_1 = \frac{1}{4} \left( \sqrt{c^4 m_d^2 - s^4 m_s^2} \sin^2 2\phi_I - \sqrt{c^4 m_e^2 - s^4 m_\mu^2} \sin^2 2\phi_I + \cos 2\phi_I \left( s^2 m_\mu + s^2 m_s \right) \right), \] (C13)

\[ k_2 = \frac{1}{4} \left( 3 e^2 m_s - e^2 m_\mu \right), \] (C14)

\[ d_2 = \frac{1}{4} \left( e^2 m_s + e^2 m_\mu \right), \] (C15)

\[ f e^{i\phi_I} = s c [m_s e^{i\phi_{C}} - m_d \exp(-i\alpha_d - i\phi_{C})]. \] (C16)

Here

\[ \alpha_d = -\arcsin \left( \frac{s^2 m_s}{c^2 m_d} \sin 2\phi_{C} \right). \] (C17)
The lepton mixing can be obtained from (C10)

\[
\sin 2\theta_1 \approx 3 \sin 2\theta_C \left| \frac{m_s - e^{-2i\phi_1} e^{-i\sin^{-1} \left( \frac{\sin(2\phi_1) m_d}{\sin^2 \phi_1} \right)} m_d}{m_\mu + e^{-2i\phi_1} e^{i\sin^{-1} \left( \frac{\sin(2\phi_1) m_\mu}{\sin^2 \phi_1} \right)} m_e} \right|,
\]

where the right-hand side still contains \(\theta_1\) but it only appears in the negligibly small term proportional to \(m_e\). So one can simply replace \(\theta_1\) by \(\theta_C\) in this term.

We have checked that our analytic results agree with numerical computations up to an order of \(10^{-4}\).

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