THE ξ-STABILITY ON THE AFFINE GRASSMANNIAN

ZONGBIN CHEN

Abstract. We introduce a notion of ξ-stability on the affine grassmannian $X$ for the classical groups. For the group $SL_d$, we calculate the Poincaré series of the quotient $X^\xi/T$ of the stable part $X^\xi$ by the maximal torus $T$ by a process analogue to the Harder-Narasimhan reduction.

Introduction

Let $k$ be an algebraically closed field, $F = k((\epsilon))$ the field of Laurent series with coefficients in $k$, $O = k[[\epsilon]]$ the ring of integers of $F$, $p = \epsilon k[[\epsilon]]$ the maximal ideal of $O$. Let $val : F^\times \to \mathbb{Z}$ be the discrete valuation normalized by $val(\epsilon) = 1$.

Let $G$ be a classical group over $k$, let $T$ be a maximal torus of $G$. Let $K = G(O)$ be the standard maximal compact subgroup of $G(F)$. Let $X^G = G(F)/K$ be the affine grassmannian associated to $G$. We simplify $X^G$ to $X$ when the context is clear. We introduce a notion of ξ-stability on the affine grassmannian $X$, which is a local version of the ξ-stability on the Hitchin space introduced by Chaudouard and Laumon. First of all, we show that the quotient $X^\xi/T$ of the stable part $X^\xi$ by the torus $T$ exists as an ind-$k$-scheme, by a calculation with the geometric invariant theory of Mumford. Then we introduce a reduction process which permits to reduce the non-ξ-stable parts onto the ξ-M-stable parts on the affine grassmannian associated to the Levi subgroups of $G$ containing $T$. Finally, for the group $SL_d$, we calculate the Poincaré series of $X^\xi/T$. There are two ingredients in the proof: the first one is the fact that $X^\xi/T$ is homologically smooth and hence satisfies the Poincaré duality, the second one is to give a lower bound on the codimension of the non-ξ-stable parts. The main result can be summarized in the following theorem.

Theorem 0.1. Let $G$ be a classical group over $k$, let $T$ be a maximal torus of $G$. The geometric quotient $X^\xi/T$ exists as an ind-$k$-scheme, and it satisfies the valuative criterion of properness. For $G = SL_d$, the Poincaré series of $X^\xi/T$ is

$$\frac{1}{(1 - t^2)^d - 1} \prod_{i=1}^{d-1} (1 - t^{2i})^{-1}.$$ 

Notations. Let $\Phi = \Phi(G, T)$ be the root system of $G$ with respect to $T$, let $W$ be the Weyl group of $G$ with respect to $T$, and let $\tilde{W}$ be the extended affine Weyl group. For any subgroup $H$ of $G$ which is stable under the conjugation of $T$, we note $\Phi(H, T)$ for the roots appearing in $\text{Lie}(H)$. We use the $(G, M)$ notation of Arthur. Let $F(T)$ be the
set of parabolic subgroups of $G$ containing $T$, let $\mathcal{L}(T)$ be the set of Levi subgroups of $G$ containing $T$. For every $M \in \mathcal{L}(T)$, we denote by $\mathcal{P}(M)$ the set of parabolic subgroups of $G$ whose Levi factor is $M$. Let $X^*(M) = \text{Hom}(M, \mathbb{G}_m)$ and $a^*_M = X^*(M) \otimes \mathbb{R}$. The restriction $X^*(M) \to X^*(T)$ induces an injection $a^*_M \hookrightarrow a^*_T$. Let $(a^*_T)^*$ be the subspace of $a^*_T$ generated by $\Phi(M, T)$. We have the decomposition in direct sums
\[ a^*_T = (a^*_T)^* \oplus a^*_M. \]

The canonical pairing
\[ X_*(T) \times X^*(T) \to \mathbb{Z} \]
can be extended linearly to $a_T \times a_T^* \to \mathbb{R}$, with $a_T = X_*(T) \otimes \mathbb{R}$. For $M \in \mathcal{L}(T)$, let $a^*_T \subset a_T$ be the subspace orthogonal to $a^*_M$, and $a_M \subset a_T$ be the subspace orthogonal to $(a^*_T)^*$, then we have the decomposition
\[ a_T = a_M \oplus a^*_T, \]

let $\pi_M, \pi^M$ be the projections to the two factors.

For $M \in \mathcal{L}(T)$, we use $\Lambda_M$ to denote the quotient of $X_*(T)$ by the coroot lattice of $M$ (the subgroup of $X_*(T)$ generated by the coroots of $T$ in $M$). We have a canonical homomorphism
\[ \text{ind}^M : M(F) \to \Lambda_M \]
such that $\chi(\text{ind}^M(m)) = \text{val}(\chi(m)), \forall \chi \in X^*(M)$. It is invariant under the right translation of $M(O)$, so it defines an application $\text{ind}^M : \mathcal{X}^M \to \Lambda_M$. Its fibers $\mathcal{X}^M(\lambda) := \text{ind}^{-1}(\lambda), \lambda \in \Lambda$ are the connected components of $\mathcal{X}^M$, they are all translations of the neutral connected component $\mathcal{X}^M(0)$. For $M = \text{GL}_d \times \cdots \times \text{GL}_d$, $\Lambda_M$ can be identified naturally with $\mathbb{Z}^r$, and the application $\text{ind}^M : \mathcal{X}^M \to \mathbb{Z}^r$ is nothing but
\[ \text{ind}^M((m_1, \ldots, m_r)) = (\text{val}(\text{det}(m_1)), \ldots, \text{val}(\text{det}(m_r))). \]

For any point $x \in \mathcal{X}^M$, we call $\text{ind}^M(x)$ the index of the lattice represented by $x$. For $M = \text{GL}_d$, we simplify $\text{ind}^{\text{GL}_d}$ to $\text{ind}$.

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1. **The ind-$k$-scheme $\mathcal{X}^\xi/T$**

1.1. **The notion of $\xi$-stability.** For $M \in \mathcal{L}(T)$, the natural inclusion of $M(F)$ in $G(F)$ induces a closed immersion of $\mathcal{X}^M$ in $\mathcal{X}^G$. For $P = MN \in \mathcal{F}(T)$, we have the retraction
\[ f_P : \mathcal{X} \to \mathcal{X}^M \]
which sends $gK = nmK$ to $mM(O)$, where $g = nmk, n \in N(F), m \in M(F), k \in K$ is the Iwasawa decomposition. More generally we can define $f^L_P : \mathcal{X}^L \to \mathcal{X}^M$ for $L \in \mathcal{L}(T), L \supset M$ and $P_L \in \mathcal{P}^L(M)$. These retractions satisfy the transition property: Suppose that $Q \in \mathcal{P}(L)$ satisfy $Q \supset P$, then
\[ f_P = f^L_P \circ f_Q. \]
We have the function $H_P : \mathcal{X} \to \mathfrak{a}_M^G = \mathfrak{a}_M / \mathfrak{a}_G$ which is the composition of $\text{ind}^M \circ f_P$ and the natural projection of $\Lambda_M$ to $\mathfrak{a}_M^G$.

**Proposition 1.1** (Arthur). Let $B', B'' \in \mathcal{P}(T)$ be two adjacent Borel subgroups, let $\alpha_{B', B''}^\vee$ be the coroot which is positive with respect to $B'$ and negative with respect to $B''$. Then for any $x \in \mathcal{X}$, we have

$$H_{B'}(x) - H_{B''}(x) = n(x, B', B'') \cdot \alpha_{B', B''}^\vee,$$

with $n(x, B', B'') \in \mathbb{N}$.

**Proof.** Let $P$ be the parabolic subgroup generated by $B'$ and $B''$, let $P = MN$ be the Levi factorization. The application $H_{B'}$ factor through $f_P$, i.e. we have commutative diagram

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f_P} & \mathcal{X}
\end{array}$$

and similarly for $H_{B''}$. Since $M$ has semisimple rank 1, the proposition is thus reduced to $G = \text{SL}_2$. In this case, let $T$ be the maximal torus of the diagonal matrices, $B' = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$, $B'' = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$, and we identify $\mathfrak{a}_T^G$ with the line $H = \{(x, -x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ in the usual way. By the Iwasawa decomposition, any point $x \in \mathcal{X}$ can be written as $x = \begin{pmatrix} a \\ b \\ d \end{pmatrix}$.

Let $m = \min\{\text{val}(a), \text{val}(b)\}$, $n = \text{val}(d)$, then $m + n \leq \text{val}(a) + \text{val}(d) = 0$ and

$$H_{B'}(x) = (-n, n), \quad H_{B''}(x) = (m, -m).$$

So

$$H_{B'}(x) - H_{B''}(x) = (-n + m, n + m) = -(n + m) \cdot \alpha_{B', B''}^\vee,$$

and the proposition follows. \qed

**Definition 1.1.** For any point $x \in \mathcal{X}$, we denote by $\text{Ec}(x)$ the convex envelope in $\mathfrak{a}_T^G$ of the $H_{B'}(x)$, $B' \in \mathcal{P}(T)$.

**Definition 1.2.** Let $\xi \in \mathfrak{a}_T^G$, it is said to be generic if $\alpha(\xi) \notin \mathbb{Z}$, $\forall \alpha \in \Phi(G, T)$.

In the following, we always suppose that $\xi$ is generic.

**Definition 1.3.** For any point $x \in \mathcal{X}$, we say that it is $\xi$-stable if $\xi \in \text{Ec}(x)$.

Let $\mathcal{X}^\xi$ be the open sub-ind-k-scheme of $\mathcal{X}$ of the $\xi$-stable points. It is invariant under the action of the maximal torus $T$. Since $\xi$ is generic, the action of $T/Z_G$ on $\mathcal{X}^\xi$ is free, where $Z_G$ is the center of $G$. For $\xi, \xi' \in \mathfrak{a}_T^G$, there exists $w \in \tilde{W}$ such that $\mathcal{X}^\xi = w \mathcal{X}^\xi'$. So the “quotient” $\mathcal{X}^\xi / T$ is independent of the choice of $\xi$.

The aim of the rest of this section is to establish the following proposition.
Proposition 1.2. The geometric quotient $\mathcal{X}^\xi / T$ of $\mathcal{X}^\xi$ by $T$ exists as an ind-$k$-scheme. In fact, it is the direct limit of a family of projective varieties over $k$.

Although the proof is a case by case analysis, the main idea rests the same. So we will give a detailed proof only for $\text{GL}_d$, and indicate the modifications for the other classical groups.

1.2. The group $\text{GL}_d$. Let $G = \text{GL}_d$, let $T$ be the maximal torus of the diagonal matrices. The affine grassmannian classifies the lattices in $F^d$, i.e.

$$\mathcal{X} = \{ L \subset F^d \mid L \text{ is an } O\text{-module of finite type such that } L \cdot F = F^d \}.$$

Since all the connected components of $\mathcal{X}$ are translations of the neutral connected component $\mathcal{X}^{(0)}$, it is enough to study $\mathcal{X}^{(0)}$. Let $\{e_i\}_{i=1}^d$ be the natural basis of $F^d$ over $F$.

Proposition 1.3. Let $\xi \in t$ be such that $\sum_{i=1}^d \xi_i = 0$. A lattice $L \in \mathcal{X}$ of index 0 is $\xi$-stable if and only if for any permutation $\tau \in \mathfrak{S}_d$, we have

$$\xi_{\tau(1)} + \cdots + \xi_{\tau(i)} \leq \text{ind}(L \cap (F e_{\tau(1)} \oplus \cdots \oplus F e_{\tau(i)})),$$

and the proposition follows. \hfill \Box

Let $L_0 = O^d$, let $\mathcal{X}_n$ be the closed sub-scheme of $\mathcal{X}^{(0)}$ defined by

$$\mathcal{X}_n = \{ L \in \mathcal{X}^{(0)} \mid L \supset e^n L_0 \}.$$

It is a $T$-invariant projective $k$-variety, and we have

$$\mathcal{X}^{(0)} = \lim_{\to n} \mathcal{X}_n.$$

We will prove the following result, which implies the proposition 1.2.

Proposition 1.4. The quotient $\mathcal{X}_n^\xi / T$ is a projective $k$-variety.

1.2.1. A non-standard quotient of the grassmannian. Let $E_1, \cdots, E_d$ be vector spaces over $k$ of dimension $\dim(E_i) = N_i$. Let $X$ be the grassmannian of sub vector spaces of dimension $n$ of $E_1 \oplus \cdots \oplus E_d$. We have the Plücker immersion $\iota : X \to \mathbf{P}^N$, $N = (N_1 + \cdots + N_d) - 1$, and the line bundle $L = \iota^* O_{\mathbf{P}^N}(1)$ is naturally endowed with a $\text{Aut}(E_1 \oplus \cdots \oplus E_d)$-linearization.

The torus $T = \mathbb{C}_m^d$ acts on $E_1 \oplus \cdots \oplus E_d$ with its $i$-th factor acts as homothetic on $E_i$; thus it acts on $X$. Given $r_i, s_i \in \mathbb{N}$, let $S \subset T$ be the sub torus of $T$ defined by

\begin{equation}
S = \left\{ \begin{bmatrix} t_1^{r_1} & \cdots & t_d^{r_d-1} \\ t_1^{s_1} & \cdots & t_d^{s_d-1} \end{bmatrix} \mid t_i \in k^\times \right\}.
\end{equation}
We will give a geometric description of the (semi-)stable points of $X$ under the action of $S$ with respect to the polarization given by the line bundle $L$, using the criteria of Hilbert-Mumford. Let $Z$ be a projective algebraic variety over $k$ endowed with the action of a reductive group $H$ over $k$, let $L$ be an ample $H$-equivariant line bundle over $Z$. Let $\lambda: \mathbb{G}_m \to H$ be a homomorphism of algebraic group, then $\mathbb{G}_m$ acts on $Z$ via the morphism $\lambda$.

For any point $x \in Z$, the point $x_0 := \lim_{t \to 0} \lambda(t)x$ exists since $Z$ is projective. Obviously $x_0 \in Z_{\mathbb{G}_m}$, thus $\mathbb{G}_m$ acts on the stalk $L_{x_0}$. The action is given by a character of $\mathbb{G}_m$, $\alpha: t \to t^r$, for some $r \in \mathbb{Z}$. Let $\mu^L(x, \lambda) = -r$.

**Theorem 1.5 (Hilbert-Mumford).** Let $Z$ be a projective algebraic variety over $k$ endowed with the action of a reductive group $H$ over $k$, let $L$ be an ample $H$-equivariant line bundle over $Z$. Let $Z^{ss}$ (resp. $Z^{st}$) be the open subvariety of $Z$ of the semi-stable (resp. stable) points. Then for any geometric point $x \in Z$, we have

1. $x \in Z^{ss} \iff \mu^L(x, \lambda) \geq 0, \forall \lambda \in \text{Hom}(\mathbb{G}_m, H),$
2. $x \in Z^{st} \iff \mu^L(x, \lambda) > 0, \forall \lambda \in \text{Hom}(\mathbb{G}_m, H).$

For the proof, the reader can consult [M], page 49-54.

**Lemma 1.6.** We have

$$V \in X^S \iff V = V_1 \oplus \cdots \oplus V_d,$$

where $V_i \subset E_i$ is a sub vector space.

For $n = (n_1, \cdots, n_{d-1}) \in \mathbb{Z}^{d-1}$ such that the cocharacter $\lambda_n \in X_*(S)$ defined by

$$\lambda_n(t) = \begin{bmatrix} t^{n_1 r_1} & \cdots & t^{n_{d-1} r_{d-1}} \\ & \cdots & \\ & & t^{-\sum_{i=1}^{d-1} n_i s_i} \end{bmatrix}$$

is regular. The stability condition is equivalent to the condition that $-\mu^L(V, \lambda_n) < 0$ for all such $n \in \mathbb{Z}^{d-1}$. Up to conjugation, we can suppose that

$$n_1 r_1 < \cdots < n_i r_i < -\sum_{j=1}^{d-1} n_j s_j < n_{i+1} r_{i+1} < \cdots < n_{d-1} r_{d-1}.$$

Let $\lim_{t \to 0} \lambda_n(t)V = V_1 \oplus \cdots \oplus V_d$, where $V_i \subset E_i$ is a sub vector space of dimension $a_i$. We have the relation
(3) \[ a_1 = \dim(V) - \dim(V \cap (E_2 \oplus \cdots \oplus E_d)), \]
\[ \vdots \]
\[ a_i = \dim(V \cap (E_i \oplus \cdots \oplus E_d)) - \dim(V \cap (E_{i+1} \oplus \cdots \oplus E_d)), \]
\[ a_d = \dim(V \cap (E_{d-1} \oplus \cdots \oplus E_d)) - \dim(V \cap (E_{d+1} \oplus \cdots \oplus E_d)) \]
\[ \vdots \]
\[ a_{d-1} = \dim(V \cap E_{d-1}). \]

So the stability condition can be written as

\[
- \mu^L(V, \lambda_n) = d - 1 \sum_{i=1}^{d-1} a_i r_i \quad - \sum_{j=1}^{d-1} n_j s_j
\]

(4) \[
= d - 1 \sum_{i=2}^{d-1} a_i (n_i - n_1) - a_d \left( n_1 + \sum_{j=1}^{d-1} n_j s_j \right) + n n_1 r_1 < 0,
\]

(We use the relation \( n = a_1 + \cdots + a_d \) in the second equality.)

The equality (4) is a question of maximal value of a linear functional on a convex region, so it suffices to look at the condition at the boundary, i.e.

(5) \[
n_1 r_1 = \cdots = n_i r_i \quad < \quad n_{i+1} r_{i+1} = \cdots = - \sum_{j=1}^{d-1} n_j s_j
\]
\[ = \cdots = n_d r_d, \quad 1 \leq i_0 \leq i; \]

and

(6) \[
n_1 r_1 = \cdots = - \sum_{j=1}^{d-1} n_j s_j = \cdots = n_i r_i \quad < \quad n_{i+1} r_{i+1}
\]
\[ = \cdots = n_d r_d, \quad i + 1 \leq i_0 \leq d - 1. \]

The inequality (5) gives

\[
n_1 < 0, \quad n_{i_0+1} r_{i_0+1} (1 + \sum_{j=i_0+1}^{d-1} s_j/r_j) = -n_1 r_1 \sum_{j=1}^{i_0} s_j/r_j,
\]

so the inequality (4) implies
The inequality (6) gives
\[ n_1 < 0, \quad n_{i_0+1} r_{i_0+1} + \cdots + n_d < n(1 + \sum_{j=i_0+1}^{d-1} s_j/r_j), \quad 1 \leq i_0 \leq i. \]
so the inequality (4) implies
\[ a_{i_0+1} + \cdots + a_d < \frac{n(\sum_{j=i_0+1}^{d-1} s_j/r_j)}{1 + \sum_{j=1}^{d-1} s_j/r_j}, \quad i + 1 \leq i_0 \leq d - 1. \]

We can express the inequalities (7) and (8) as inequalities in \( \dim(V \cap (E_{i_1} \oplus \cdots \oplus E_{i_r})) \) with the help of the dimension relation (3).
Let \( x = (x_1, \cdots, x_d) \in t \) with
\[ x_d = \frac{1}{1 + \sum_{i=1}^{d-1} s_i/r_i}, \quad x_i = \frac{s_i/r_i}{1 + \sum_{i=1}^{d-1} s_i/r_i}, \quad i = 1, \cdots, d - 1. \]
We remark that \( \sum_{i=1}^{d} x_i = 1 \). The above calculations can be reformulated as follows:

**Proposition 1.7.** A sub vector space \( V \subset E_1 \oplus \cdots \oplus E_d \) is \( S \)-stable if and only if for any permutation \( \tau \in S_d \), we have
\[ \dim(V)(x_{\tau(1)} + \cdots + x_{\tau(i)}) < \dim(V \cap (E_{\tau(1)} \oplus \cdots \oplus E_{\tau(i)})), \quad i = 1, \cdots, d - 1. \]

The same result holds for the semi-stable points with “\(<\)" replaced by “\(\leq\)".

**1.2.2. Comparison of two notions of stability.** First of all, we embed \( \mathcal{X}_n \) as a closed subvariety of some grassmannian.

**Lemma 1.8.** The algebraic variety \( \mathcal{X}_n \) is a Springer fiber.

**Proof.** For \( L \in \mathcal{X}_n \), we have automatically \( L \subset e^{(1-d)n} L_0 \). Let \( Gr_{nd,nd^2} \) be the grassmannian of sub vector spaces of dimension \( nd \) in \( k^{nd^2} \). We embed \( \mathcal{X}_n \) in \( Gr_{nd,nd^2} \) by the injective morphism \( g_n : \mathcal{X}_n \to Gr_{nd,nd^2} \) defined by
\[ g_n(L) = L/e^n L_0 \subset e^{(1-d)n} L_0/e^n L_0. \]
The image of \( g_n \) is the Springer fiber
\[ \mathcal{Y}_n = \{ V \in Gr_{nd,nd^2} \mid NV \subset V \}, \]
where \( N \in \text{End}(e^{(1-d)n} L_0/e^n L_0) \) is the endomorphism defined by the multiplication by \( e \). \( \square \)
For the reason of dimension, we use another embedding $\varphi_n^\perp : \mathcal{X}_n \to \Gr_{nd(d-1), nd^2}$. We define an inner product on the vector space $k^{nd^2} = e^{(1-d)n}L_0/e^nL_0$ by linearly expanding the relation
\[
(e^{(1-d)n+i}e_j, e^{n-1+i'}e_{j'}) = \delta_{i,i'}\delta_{j,j'}, \quad i, i' = 0, \cdots, dn - 1; \quad j, j' = 1, \cdots, d,
\]
For a sub vector space $V \subset k^{nd^2}$, let $V^\perp$ be the orthogonal complement of $V$ with respect to this inner product. It induces an isomorphism $\perp : \Gr_{nd, nd^2} \to \Gr_{nd(d-1), nd^2}$. Let $g_n^\perp(L) = g_n(L)^\perp \forall L \in \mathcal{X}_n$, and let $Y_n$ denote again the image of $\mathcal{X}_n$ in $\Gr_{nd(d-1), nd^2}$ under $g_n^\perp$.

**Proof of proposition** \[1.4\]. Since the quotient $\mathcal{X}/T$ doesn’t depend on the choice of $\xi$, we can suppose that $\sum_{i=1}^{d-1} \xi_i = 0$, $\xi_i \in \mathbb{Q}$ is positive and small enough for $i = 1, \cdots, d - 1$.

Let $E_i = p^{(1-d)n}e_i/p^ne_i$, let $V = g_n^\perp(L) \subset Y_n \subset \Gr_{nd(d-1), nd^2}$. For $\tau \in \mathfrak{S}_d$, we have the equality

\[
\dim(V \cap (E_{\tau(1)} \oplus \cdots \oplus E_{\tau(i)})) = \dim(L \cap (Fe_{\tau(1)} \oplus \cdots \oplus Fe_{\tau(i)})) + n(d - 1)i.
\]

Let $x_i = \xi_{i+n(d-1)}/nd(d-1)$. The hypothesis on $\xi$ implies that $x_i \in \mathbb{Q}$, $x_i > 0$ and $\sum_{i=1}^d x_i = 1$.

Take $r_i, s_i \in \mathbb{N}$ such that
\[
\begin{align*}
x_d &= \frac{1}{1 + \sum_{i=1}^{d-1} s_i/r_i}, & x_i &= \frac{s_i/r_i}{1 + \sum_{i=1}^{d-1} s_i/r_i}, & i &= 1, \cdots, d - 1.
\end{align*}
\]

Let $S_n \subset T$ be the torus defined in \[1\] for the above $r_i, s_i$. The equality \[9\] implies that
\[
\xi_{\tau(1)} + \cdots + \xi_{\tau(i)} < \dim(L \cap (Fe_{\tau(1)} \oplus \cdots \oplus Fe_{\tau(i)}))
\]
if and only if

\[
\dim(V)(x_{\tau(1)} + \cdots + x_{\tau(i)}) < \dim(V \cap (E_{\tau(1)} \oplus \cdots \oplus E_{\tau(i)})).
\]

Combining the proposition \[1.3\] and the proposition \[1.7\] we get
\[
L \in \mathcal{X}_n^\xi \iff V \in Y_n^{st}.
\]

Since $\xi$ is supposed to be generic and $\xi_1$ are positive and small enough for $i = 1, \cdots, d - 1$, the “$<$” in the inequality \[10\] is the same as “$\leq$”. That is to say that $Y_n^{ss} = Y_n^{st}$ and so the quotient $Y_n^{ss}/S_n = Y_n^{st}/S_n$ is a projective $k$-variety by the geometric invariant theory of Mumford. The following lemma shows that $\mathcal{X}_n^\xi/T = \mathcal{X}_n^\xi/S_n \cong Y_n^{ss}/S_n$, since $T/ZG$ acts freely on $\mathcal{X}_n^\xi$. So $\mathcal{X}_n^\xi/T$ is a projective $k$-variety. \[\square\]

**Lemma 1.9.** The morphism $S_n \to T/ZG$ is an isogeny.

**Proof.** It is equivalent to show that the induced morphism of character groups $X^*(T/ZG) \to X^*(S_n)$ has non zero determinant. By a direct calculation, this determinant is
\[
(1 + \sum_{i=1}^{d-1} s_i/r_i) \prod_{i=1}^{d-1} r_i.
\]
which is non zero by the choice of \( r_i, s_i \).

1.3. The groups \( \text{Sp}_{2d} \) and \( \text{SO}_{2d} \). Let \( (k^{2d}, \langle \cdot, \cdot \rangle) \) be the standard symplectic vector space over \( k \) such that \( \langle e_i, e_{2d+1-i} \rangle = \delta_{i,j}, i, j = 1, \ldots, d \). Let \( \text{Sp}_{2d} \) be the symplectic group associated to it, let \( T \) be the maximal torus of \( \text{Sp}_{2d} \) consisting of the diagonal matrices. Let \( (F^{2d}, \langle \cdot, \cdot \rangle) \) be the scalar extension of \( (k^{2d}, \langle \cdot, \cdot \rangle) \) to \( F \). For a lattice \( L \) in \( F^{2d} \), let

\[
L' = \{ x \in F^{2d} \mid \langle x, L \rangle \subset \mathcal{O} \}.
\]

The affine grassmannian associated to \( \text{Sp}_{2d} \) classifies the lattices \( L \) in \( F^{2d} \) such that \( L = L' \). Let

\[
\mathcal{X}_n = \{ L \in \mathcal{X} \mid \epsilon^n L_0 \subset L \subset \epsilon^{-n} L_0 \}.
\]

It is a \( T \)-invariant projective \( k \)-variety and we have \( \mathcal{X} = \lim_{n \to +\infty} \mathcal{X}_n \). Let \( \rho_n : \mathcal{X}_n \to \text{Gr}_{2nd, 4nd} \) be the injective \( T \)-equivariant morphism defined by

\[
\rho_n(L) = L/\epsilon^n L_0 \subset \epsilon^{-n} L_0/\epsilon^n L_0.
\]

Let \( Y_n \) be its image, it is isomorphic to \( \mathcal{X}_n \). Let \( \iota : \text{Gr}_{2nd, 4nd} \to \mathbb{P}^N, N = (4nd) - 1, \) be the Plücker embedding. Let \( L = (\iota \circ \rho_n)^* \mathcal{O}_{\mathbb{P}^N}(1) \), it is an ample \( T \)-equivariant line bundle on \( Y_n \).

Let \( \text{GSp}_{2d} \) be the reductive group over \( k \) such that for any \( k \)-algebra \( R \),

\[
\text{GSp}_{2d}(R) = \{ g \in \text{GL}_{2d}(R) \mid \langle g v, g v' \rangle = \lambda(g) \langle v, v' \rangle, \lambda(g) \in R^\times, \forall v, v' \in R^{2d} \}.
\]

We have an exact sequence

\[
0 \to \text{Sp}_{2d} \to \text{GSp}_{2d} \overset{\lambda}{\to} \mathbb{G}_m \to 0,
\]

from which it follows that \( \mathcal{X} \) is the neutral connected component of \( \mathcal{X}^{\text{GSp}_{2d}} \). Let

\[
\tilde{T} = \left\{ \begin{bmatrix} t_{t_1} & & & & \ldots & & t_{t_d} & & & \ldots & & \end{bmatrix} ; \ t, \ t_i \in k^\times \right\}.
\]

It is a maximal torus of \( \text{GSp}_{2d} \). Let \( \mathbb{G}_m \) be the center of \( \text{GSp}_{2d} \), then \( \tilde{T} / \mathbb{G}_m \) acts freely on \( \mathcal{X} \) and we have

\[
\mathcal{X}^{\xi} / \tilde{T} = \mathcal{X}^{\xi} / T.
\]

Given a generic element \( \xi = (\xi_1, \ldots, \xi_d, -\xi_d, \ldots, -\xi_1) \in t \) such that \( \xi_i \in \mathbb{Q} \), we can find \( r_i, s_i \in \mathbb{Z}, i = 1, \ldots, d \), such that

\[
\xi_i = nd \frac{s_i/r_i}{2 + \sum_{i=1}^d s_i/r_i}, \quad i = 1, \ldots, d.
\]
Consider the sub torus $S_n$ of $\tilde{T}$ defined by

$$S_n = \left\{ \begin{bmatrix} t_1^{s_1} \cdots t_d^{s_d} t_1^{r_1} \\ \vdots \\ t_1^{s_1} \cdots t_d^{s_d} t_d^{-r_d} \\ \vdots \\ t_1^{s_1} \cdots t_d^{s_d} t_1^{-r_1} \end{bmatrix} ; t_i \in k^\times \right\}.$$ 

**Lemma 1.10.** The morphism $S_n \to \tilde{T}/G_m$ is an isogeny.

**Proof.** As before, we need to calculate the determinant of the morphism of character groups $X^*(\tilde{T}/G_m) \to X^*(S_n)$, it is

$$(2 + \sum_{i=1}^d s_i/r_i) \prod_{i=1}^d r_i,$$

which is non zero by the definition of $r_i, s_i$. \qed

As we have done for $GL_d$, we can calculate the $S_n$-(semi)-stable points $Y^s_n$ (resp. $Y^{ss}_n$) on $Y_n$ with respect to the polarization given by the line bundle $L$, and obtain the following comparison result, which implies the proposition 1.2.

**Proposition 1.11.** Under the above setting, a lattice $L \in \mathcal{X}_n$ is $\xi$-stable if and only if $\rho_n(L) \in Y^{ss}_n = Y^s_n$. In particular, $\mathcal{X}_n^{\xi}/\mathcal{T} = \mathcal{X}_n^{\xi}/(\tilde{T}/G_m) \cong Y^{ss}_n/S_n$ is a projective $k$-variety.

For the group $SO_{2d}$, the strategy is totally the same, unless we need to use the standard quadratic space $(k^{2d}, \langle , \rangle)$ over $k$ such that $\langle e_i, e_{2d+1-i} \rangle = \delta_{i,j}, i, j = 1, \ldots, 2d$, instead of the standard symplectic vector space.

### 1.4. The group $SO_{2d+1}$

Let $(k^{2d+1}, \langle , \rangle)$ be the standard quadratic space over $k$ such that $\langle e_i, e_{2d+2-i} \rangle = \delta_{i,j}, i, j = 1, \ldots, 2d + 1$. Let $SO_{2d+1}$ be the orthogonal group associated to it, and let $T$ be the maximal torus of $SO_{2d+1}$ consisting of the diagonal matrices. Let $(F^{2d+1}, \langle , \rangle)$ be the scalar extension of $(k^{2d+1}, \langle , \rangle)$ to $F$. For a lattice $L$ in $F^{2d+1}$, let

$$L^\vee = \{ x \in F^{2d+1} \mid \langle x, L \rangle \subset \mathcal{O} \}.$$ 

The affine Grassmannian associated to $SO_{2d+1}$ classifies the lattices $L$ in $F^{2d+1}$ such that $L = L^\vee$. Let

$$\mathcal{X}_n = \{ L \in \mathcal{X} \mid e^n L_0 \subset L \subset e^{-n} L_0 \}.$$ 

It is a $T$-invariant projective $k$-variety and

$$\mathcal{X} = \lim_{n \to +\infty} \mathcal{X}_n.$$ 

Let $\rho_n : \mathcal{X}_n \to \text{Gr}_{n(2d+1), 2n(2d+1)}$ be the injective $T$-equivariant morphism defined by

$$\rho_n(L) = L/e^n L_0 \subset e^{-n} L_0/e^n L_0.$$
Let $Y_n$ be its image, it is isomorphic to $\mathcal{X}_n$. Let $\mathcal{L}$ again be the $T$-equivariant line bundle on $Y_n$ induced by the Plücker embedding of $\text{Gr}_{n(2d+1),2n(2d+1)}$.

Let $\text{GO}_{2d+1}$ be the reductive group over $k$ such that for any $k$-algebra $R$,

$$\text{GO}_{2d+1}(R) = \{ g \in \text{GL}_{2d+1}(R) \mid \langle gv, gv' \rangle = \lambda(g) \langle v, v' \rangle, \lambda(g) \in R^\times, \forall \ v, \ v' \in R^{2d+1} \}.$$ 

We have an exact sequence

$$0 \to \text{SO}_{2d+1} \to \text{GO}_{2d+1} \xrightarrow{\lambda} \mathbb{G}_m \to 0,$$

from which it follows that $\mathcal{X}$ is the neutral connected component of $\mathcal{X}^{\text{GO}_{2d+1}}$. Let

$$\tilde{T} = \left\{ \begin{bmatrix} t^2t_1 & & & \cdots & & t^2t_d \\ & t & & & \cdots & & t \\ & & t^{-1}_d & & & \cdots & & t^{-1} \\ & & & \ddots & & & & \vdots \\ & & & & & & t^{-1}_1 \\ \end{bmatrix} ; \ t, t_i \in k^\times \right\}.$$ 

It is a maximal torus of $\text{GO}_{2d+1}$. Let $\mathbb{G}_m$ be the center of $\text{GO}_{2d+1}$, then $\tilde{T}/\mathbb{G}_m$ acts freely on $\mathcal{X}$ and we have

$$\mathcal{X}/\tilde{T} = \mathcal{X}^{\text{GO}_{2d+1}}/T.$$

Given a generic element $\xi = (\xi_1, \cdots, \xi_d, 0, -\xi_d, \cdots, -\xi_1) \in t$ such that $\xi_i \in \mathbb{Q}$, we can find $r_i, s_i \in \mathbb{Z}, \ i = 1, \cdots, d$, such that

$$\xi_i = n(d + \frac{1}{2}) \frac{s_i/r_i}{1 + \sum_{i=1}^{d} s_i/r_i}, \ i = 1, \cdots, d$$

Consider the sub torus $S_n$ of $\tilde{T}$ defined by

$$S_n = \left\{ \begin{bmatrix} \cdots & & t_1^{2s_1} \cdots t_d^{2s_d} t_1^{r_1} \\ & \ddots & & \cdots & & t_1^{2s_1} \cdots t_d^{2s_d} t_1^{r_1} \\ & & \ddots & & & \cdots & & t_1^{2s_1} \cdots t_d^{2s_d} t_1^{r_1} \\ & & & \ddots & & & & \vdots \\ & & & & \ddots & & & & \vdots \\ & & & & & \ddots & & & & \vdots \\ & & & & & & \cdots & & \cdots & & t_1^{-r_1} \\ \end{bmatrix} ; \ t_i \in k^\times \right\}.$$ 

**Lemma 1.12.** The morphism $S_n \to \tilde{T}/\mathbb{G}_m$ is an isogeny.

**Proof.** As before, we need to calculate the determinant of the morphism of character groups $X^*(\tilde{T}/\mathbb{G}_m) \to X^*(S_n)$, it is

$$(1 + \sum_{i=1}^{d} s_i/r_i) \prod_{i=1}^{d} r_i,$$
which is non zero by the definition of $r_i, s_i$. □

As before, we can calculate the $S_n$-(semi)-stable points $Y_n^{st}$ (resp. $Y_n^{ss}$) on $Y_n$ with respect to the polarization given by the line bundle $L$, and obtain the following comparison result. It implies the proposition 1.2.

**Proposition 1.13.** Under the above setting, a lattice $L \in X_n$ is $\xi$-stable if and only if $\rho_n(L) \in Y_n^{ss} = Y_n^{st}$. In particular, $X_n^\xi / T = X_n^\xi / (\tilde{T} / G_m) \cong Y_n^{ss} / S_n$ is a projective $k$-variety.

## 2. Reduction of Arthur-Kottwitz

We will introduce an analogue of the Harder-Narasimhan reduction on the affine grassmannian, which we will name the reduction of Arthur-Kottwitz.

For $P \in \mathcal{F}(T)$, let $P = MN$ be the standard Levi factorization. Let $\Phi_P(G, M)$ be the image of $\Phi(N, T)$ in $(\mathfrak{a}_M^G)^*$. For any point $a \in \mathfrak{a}_M^G$, we define a cone in $\mathfrak{a}_M^G$,

$$D_P(a) = \{ y \in \mathfrak{a}_M^G | \alpha(y - a) \geq 0, \forall \alpha \in \Phi_P(G, M) \}.$$

**Definition 2.1.** For any geometric point $x \in \mathcal{X}$, we define a semi-cylinder $C_P(x)$ in $a_T^G$ by

$$C_P(x) = \pi_M^{-1}(\text{Ec}(f_P(x))) \cap \pi_M^{-1}(D_P(H_P(x))).$$

By definition, we get a partition

$$a_T^G = \text{Ec}(x) \cup \bigcup_{P \in \mathcal{F}(T)} C_P(x),$$

such that the interior of any two members doesn’t intersect. The figure gives an idea of this partition for GL$_3$.

So for any $x \notin \mathcal{X}^\xi$, there exists a unique parabolic subgroup $P \in \mathcal{F}(T)$ such that $\xi \in C_P(x)$ since $\xi$ is generic. In this case, $f_P(x) \in \mathcal{X}^M$ is $\xi^M$-stable, where $\xi^M = \pi_M(\xi) \in a_T^M$.

We define

$$S_P = \{ x \in \mathcal{X} | \xi \in C_P(x) \}.$$

**Lemma 2.1.** We have a stratification of the affine grassmannian

$$\mathcal{X} = \mathcal{X}^\xi \sqcup \bigcup_{P \in \mathcal{F}(T), P \neq G} S_P.$$

For $P \in \mathcal{P}(M)$, let $P^-$ be the parabolic subgroup opposite to $P$ with respect to $M$. Let $\Lambda_{M,P} = D_{P^-}(\xi_M) \cap \Lambda_M$, where $\xi_M = \pi_M(\xi) \in a_M^G$. We have the disjoint partition

$$\Lambda_M = \bigcup_{P \in \mathcal{P}(M)} \Lambda_{M,P}.$$

Let

$$\mathcal{X}_P^M = \text{ind}^{M, -1}(\Lambda_{M,P}), \quad \mathcal{X}_P^M,\xi^M = \mathcal{X}_P^M,\xi^M \cap \mathcal{X}_P^M.$$
We remind that $\mathcal{X}^M$ is embedded naturally in $\mathcal{X}$ by the inclusion $M(F) \to G(F)$. In this way, $\mathcal{X}^M_{\mathcal{P}}$ is an embedding naturally in $S_P$.

**Proposition 2.2.** We have $S_P = N(F)\mathcal{X}^M_{\mathcal{P}}$.

**Proof.** By definition, we have $f_P(S_P) \subset \mathcal{X}^M_{\mathcal{P}}$, so $S_P \subset N(F)\mathcal{X}^M_{\mathcal{P}}$.

For the inverse inclusion, let $x \in \mathcal{X}^M_{\mathcal{P}} \subset S_P$, $u \in N(F)$, then $f_P(ux) = f_P(x)$ and so $H_P(ux) = H_P(x)$. They imply that $C_P(ux) = C_P(x)$, so $ux \in S_P$.

\[\square\]

The lemma 2.1 and the proposition 2.2 enable us to reduce the affine grassmannian $\mathcal{X}$ into the $\xi^M$-stable parts of $\mathcal{X}^M$, $M \in \mathcal{L}(T)$. This process is called the reduction of Arthur-Kottwitz.

**3. Poincaré series of $\mathcal{X}^{\xi}/T$ for SL$_d$**

Let $k = \mathbb{F}_p$, let $l$ be a prime number different from $p$. Let $G = \text{SL}_d$, let $T$ be the maximal torus of $G$ consisting of the diagonal matrices, let $B$ be the Borel subgroup of $G$ consisting of the upper triangular matrices. Let $X^+_\ast(T)$ be the cone of dominant cocharacters $\mu$ of $T$ with respect to $B$.

**3.1. Poincaré series of the affine grassmannian.** Let $V$ be a separated scheme of finite type over $k$, we use the notation:

$$H_i(V) = (H^i(V, \overline{\mathbb{Q}}_l))^*, \quad H_{i,c}(V) = (H^i_{\mathcal{C}}(V, \overline{\mathbb{Q}}_l))^*.$$
Its Poincaré polynomial is defined to be
\[ P_V(t) = \sum_{i=0}^{\dim(V)} \dim(H^i(V)) t^i. \]
For an ind-$k$-scheme $V = \lim_{n \to +\infty} V_n$, we define
\[ H^i(V) = \lim_{n \to +\infty} H^i(V_n), \quad H^i_c(V) = \lim_{n \to +\infty} H^i_c(V_n). \]
If $\dim(H^i(V)) < +\infty$ for all $i \in \mathbb{N}$, we define the Poincaré series of $V$ to be
\[ P_V(t) = \sum_{i=0}^{+\infty} \dim(H^i(V)) t^i. \]

**Proposition 3.1** (Bott). The Poincaré series of the affine grassmannian is
\[ P_{\mathcal{X}}(t) = \prod_{i=1}^{d-1} (1 - t^{2i})^{-1}. \]

The reader can find in [B] a topological proof, and in [IM] a combinatorial proof.

### 3.2. $\mathcal{X}^\xi/T$ is homologically smooth

The aim of this section is to prove the following result:

**Lemma 3.2.** For any $n \in \mathbb{N}$, the algebraic variety $\mathcal{X}_n^\xi/T$ is homologically smooth. In particular, it satisfies the Poincaré duality.

The proof is based on an observation of Lusztig in [L], later generalized by Mirkovic andVybornov in [MVy], which says that the affine grassmannian has the same singularity as the nilpotent cone.

For $\mu \in X_+^\xi(T)$, let $\text{Sch}(\mu) = \overline{Ke^\mu K/K}$. We have the stratification in $K$-orbits
\[ \text{Sch}(\mu) = \bigcup_{\lambda \prec \mu, \lambda \in X_+^\xi(T)} Ke^\lambda K/K, \]
where $\lambda \prec \mu$ means:
\[ \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i, \quad i = 1, \ldots, d. \]

So $\text{Sch}(\mu)$ is equisingular along $Ke^\lambda K/K$, hence the local singularity of $\text{Sch}(\mu)$ along $Ke^\lambda K/K$ is the same as that of a transversal slice to $Ke^\lambda K/K$.

Let $L^{\leq 0}G = \{ g \in \text{GL}_d(k[[\epsilon^{-1}]]), \ g \equiv 1 \mod \epsilon^{-1} \}$.

**Lemma 3.3.** For $\lambda \in X_+^\xi(T)$, the orbit $L^{\leq 0}G \cdot e^\lambda$ is a transversal slice to $Ke^\lambda K/K$ passing the point $e^\lambda$ in the affine grassmannian $\mathcal{X}$.

The reader can consult [BL] for a proof.

For $N \in \mathbb{N}$, let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{gl}_N$.

**Theorem 3.4** (Borho-Macpherson). The nilpotent cone $\mathcal{N}$ is homologically smooth.
The reader can consult [BM] for a proof.

Let \( p(N) \) be the set of partitions of \( N \). By a partition of \( N \), we mean a tuple of numbers \( (a_1, \ldots, a_n) \in \mathbb{N}, a_1 \geq \cdots \geq a_n \geq 1 \) such that \( a_1 + \cdots + a_n = N \). For \( \lambda \in p(N) \), let \( u_\lambda \) be the Jordan matrix of type \( \lambda \) and let \( O_\lambda \) be the orbit of \( u_\lambda \) under the conjugation action of \( GL_N \). For \( \mu \in p(N) \), we have the stratification in \( GL_N \)-orbits

\[
O_\mu = \bigsqcup_{\lambda \in p(N) \atop \lambda \prec \mu} O_\lambda,
\]

where \( \lambda \prec \mu \) means that for any \( i \), we have

\[
\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i.
\]

Thus \( O_\mu \) is equisingular along \( O_\lambda \), and so the local singularity of \( O_\mu \) along \( O_\lambda \) is the same as that of a transversal slice to \( O_\lambda \) in \( O_\mu \).

In [MVY], Mirkovic and Vybornov construct a transversal slice to \( O_\lambda \) in \( O_\mu \). We review briefly their construction. Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \), let \( \{e_{i,j}, i = 1, \ldots, \lambda_j; j = 1, \ldots, r\} \) be the standard basis of \( k^N \) such that

\[
u_{\lambda}e_{1,j} = 0, \quad \text{and} \quad u_\lambda e_{i,j} = e_{i-1,j}, \quad i = 2, \ldots, \lambda_j; \quad j = 1, \ldots, r.
\]

Let \( V_j = \bigoplus_{i=1}^{\lambda_j} ke_{i,j}, \quad j = 1, \ldots, r \). With respect to this basis, we identify

\[
\mathfrak{gl}_N = \bigoplus_{i,j=1,\ldots,r} \operatorname{Hom}(V_i, V_j).
\]

Let \( u_i = u_\lambda|_{V_i} \), and

\[
C = \bigoplus_{i,j=1,\ldots,r} \operatorname{Hom}(\ker(u_i^{\lambda_j}), \ker(u_j^i)),
\]

where \( u_j^i \) is the transposition of \( u_j \).

**Lemma 3.5** (Mirkovic-Vybornov). The sub-variety \( (u_\lambda + C) \cap \overline{O_\mu} \) is a transversal slice to \( O_\lambda \) passing through \( u_\lambda \) in \( \overline{O_\mu} \).

Take any \( m \in \mathbb{N}, m \geq \lambda_1 \), let \( m - \lambda = (m - \lambda_d, \ldots, m - \lambda_1) \), it is a partition of \( dm \).

**Theorem 3.6** (Mirkovic-Vybornov). For \( \lambda, \mu \in X^+_T \), \( \lambda \prec \mu \), there exists an isomorphism

\[
(u_m - \lambda + C) \cap \overline{O_m - \mu} \cong (L^{<0}G \cdot \epsilon^\lambda) \cap \operatorname{Sch}(\mu).
\]

So the singularity of \( \operatorname{Sch}(\mu) \) along \( K\epsilon^\lambda K/K \) is the same as that of \( \overline{O_m - \mu} \) along \( O_m - \lambda \). In particular, we have

**Corollary 3.7.** For \( n \in \mathbb{N} \), the algebraic variety \( \mathcal{X}_n \) is homologically smooth.

**Proof.** Take \( \nu = (n, \ldots, n, -(d-1)n) \in X^+_T \), we have \( \mathcal{X}_n = \operatorname{Sch}(\nu) \). Now that \( \overline{O_n - \nu} \) is the nilpotent cone in \( \mathfrak{gl}_{dn} \), the corollary follows from theorem 3.4. \( \square \)
Proof of the lemma 3.3. Since $\mathcal{X}_n^\xi$ is open in $\mathcal{X}_n$, $\mathcal{X}_n^\xi$ is homologically smooth. It is a $T/G_m$-torsor over $\mathcal{X}_n^\xi/T$ since the action of $T/G_m$ is free. In particular, the natural projection $\mathcal{X}_n^\xi \to \mathcal{X}_n^\xi/T$ is a surjective smooth morphism. According to [G], corollary 17.16.3, locally it admits an étale section. So the torsor is locally trivial for the étale topology. This implies that $\mathcal{X}_n^\xi/T$ is homologically smooth since $\mathcal{X}_n^\xi$ is. □

3.3. Calculation of the Poincaré series.

Lemma 3.8. Let $\xi \in a_T^G$ be such that $0 < \alpha(\xi) < 1$, $\forall \alpha \in \Phi_B(G, T)$. We have $\dim(\mathcal{X}_n) = nd(d - 1)$, and the closed sub-variety $\mathcal{X}_n \setminus \mathcal{X}_n^\xi$ have dimension at most $n(d - 1)^2$, i.e. its codimension is at least $(d - 1)n$ in $\mathcal{X}_n$.

Proof. Let $\nu = (n, \cdots , n, (1 - d)n) \in X_*(T)$, then $\dim(\mathcal{X}_n) = \dim(\mathcal{I}e^\nu K/K) = nd(d - 1)$ since $\mathcal{X}_n = \overline{\mathcal{I}e^\nu K/K}$. Here $\mathcal{I}$ is the standard Iwahori subgroup of $G(F)$, i.e. it is the inverse image of $B$ under the reduction morphism $G(O) \to G(k)$.

The dimension of the closed sub-variety $\mathcal{X}_n \setminus \mathcal{X}_n^\xi$ is

$$\max\{\dim(S_P \cap \mathcal{X}_n), P \in \mathcal{F}(T), P \neq G\}.$$ 

An easy induction reduces the situation to the case where $P$ is a standard maximal parabolic subgroup, here standard means that $B \subset P$. Suppose that $P$ is of type $(r, d - r)$, $1 \leq r \leq d - 1$, i.e. its Levi factor is $M = \text{SL}(k^r \oplus k^{d-r})$.

We identify $\Lambda_M$ with $\{(\lambda, -\lambda) \mid \lambda \in \mathbb{Z}\}$. For $(\lambda, -\lambda) \in \Lambda_M, P$, we have $\lambda \leq 0$. Let $X^\lambda = H_P^{-1}((\lambda, -\lambda)) \cap \mathcal{X}_n$, then the $S_P^\lambda := S_P \cap X^\lambda$ are the connected components of $S_P \cap \mathcal{X}_n$. So it is enough to bound the dimension of $X^\lambda$.

Lemma 3.9. We have the affine paving

$$\mathcal{X}_n = \bigcup_{\epsilon^\mu \in \mathcal{X}_n^T} \mathcal{X}_n \cap Be^\mu K/K,$$

where

$$\mathcal{X}_n \cap Be^\mu K/K = \begin{bmatrix} 1 & \cdots & p^{a_{i,j}} \end{bmatrix}^{-1} e^\mu K/K$$

with $a_{i,j} = \mu_i - n$, is isomorphic to an affine space of dimension

$$\sum_{i=2}^d (i-1)(n - \mu_i).$$

For the proof, the reader can refer to [G]. Prop. 2.2 and Cor. 2.5. For $\epsilon^\mu \in \mathcal{X}_n^T$, let $C(\mu) = \mathcal{X}_n \cap Be^\mu K/K$, it is of dimension

$$\sum_{i=2}^d (i-1)(n - \mu_i) = \sum_{i=2}^r (i-1)(n - \mu_i) + \sum_{i=r+1}^d (i-1)(n - \mu_i).$$
Since $B \subset P$, we have

$$X^\lambda = \bigsqcup_{e^\mu \in (X^\lambda)^T} C(\mu).$$

So the question is to bound the dimension of $C(\mu)$ under the condition that $\mu_i \leq n$ and that

$$\sum_{i=1}^r \mu_i = \lambda \leq 0, \quad \sum_{i=r+1}^d \mu_i = -\lambda \geq 0.$$

It takes the maximal value when

$$\mu_1 = \cdots = \mu_{r-1} = n, \mu_r = \lambda - (r-1)n;$$
$$\mu_{r+1} = \cdots = \mu_{d-1} = n, \mu_d = -\lambda - (d-r-1)n.$$

So we have

$$\dim(X^\lambda) \leq (r-1)(n + (r-1)n - \lambda) + (d-1)(n + (d-r-1)n + \lambda)$$
$$= (r-1)rn + (d-1)(d-r)n + (d-r)\lambda \leq (d-1)^2 n,$$

and then

$$\text{Codim}(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \geq nd(d-1) - (d-1)^2 n = (d-1)n.$$

□

Remark 3.1. We can also use the dimension formula in theorem 3.2 of [MV1] to obtain the same estimation.

For $n \in \mathbb{N}$, let $\tau_n$ be the truncation operator on $k[[t]]$ defined by

$$\tau_n \left( \sum_{i=10}^{+\infty} a_i t^{i} \right) = \sum_{i=10}^{n} a_i t^{i}.$$

Theorem 3.10. The Poincaré series of $\mathcal{X}^\xi/T$ is

$$P_{\mathcal{X}^\xi/T}(t) = \frac{1}{(1 - t^2)^{d-1}} \prod_{i=1}^{d-1} (1 - i^{2i})^{-1}.$$

Further more, we have

$$H_{2i+1}(\mathcal{X}^\xi/T) = 0,$$

and the Frobenius acts on $H_{2i}(\mathcal{X}^\xi/T)$ by $q^{-i}, \forall i \geq 0.$

Proof. We have the exact sequence

$$\cdots \to H_i(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \to H_i(\mathcal{X}_n) \to H_{i,c}(\mathcal{X}_n^\xi) \to H_{i-1}(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \to \cdots$$

By lemma 3.8

$$\dim(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \leq n(d-1)^2,$$
so
\[ H_i(\mathcal{F}_n \setminus \mathcal{F}_n^\xi) = 0, \quad i \geq 2n(d-1)^2 + 1, \]
and
\[ H_{i,c}(\mathcal{F}_n^\xi) = H_i(\mathcal{F}_n), \quad i \geq 2n(d-1)^2 + 2. \]

Since \( \mathcal{F}_n \) and \( \mathcal{F}_n^\xi \) are homologically smooth, they satisfy the Poincaré duality, which implies that

(11) \[ H_i(\mathcal{F}_n^\xi) = H_i(\mathcal{F}_n), \quad 0 \leq i \leq 2n(d-1) - 2, \]
since \( \dim(\mathcal{F}_n) = \dim(\mathcal{F}_n^\xi) = nd(d-1) \).

Because \( T/\mathbb{G}_m \) acts freely on \( \mathcal{F}_n^\xi \), we have
\[ H_i(\mathcal{F}_n^\xi / T) = H_{i,T/\mathbb{G}_m}(\mathcal{F}_n^\xi) = \bigoplus_{i_1 + i_2 = i} H_{i_1}(\mathcal{F}_n^\xi) \otimes H_{i_2}(B(T/\mathbb{G}_m)), \]
where \( B(T/\mathbb{G}_m) \) is the classifying space of \( T/\mathbb{G}_m \)-torsors.

Combining the equalities (11) and (12), we get

(13) \[ \tau_{2(d-1)n-2}[P_{\mathcal{F}_n^\xi / T}(t)] = \tau_{2(d-1)n-2}[(1 - t^2)^{1-d}P_{\mathcal{F}_n}(t)], \]
and
\[ H_{2i+1}(\mathcal{F}_n^\xi / T) = 0, \quad 0 \leq i \leq (d-1)n - 2, \]
and the Frobenius acts on \( H_{2i}(\mathcal{F}_n^\xi / T) \) by \( q^{-i} \), \( 0 \leq i \leq (d-1)n - 1 \), because it acts thus on \( H_s(\mathcal{F}_n) \) and \( H_s(B(T/\mathbb{G}_m)) \).

Since
\[ \lim_{n \to +\infty} \tau_{2(d-1)n}(P_{\mathcal{F}_n}(t)) = P_{\mathcal{F}}(t) = \prod_{i=1}^{d-1} (1 - t^{2i})^{-1}, \]
we can take limits of the two sides of (13), and get:
\[ P_{\mathcal{F}_n^\xi / T}(t) = \frac{1}{(1 - t^2)^{d-1}} \prod_{i=1}^{d-1} (1 - t^{2i})^{-1}. \]

This implies that we can also take limits of the two sides of (11) and (12), and obtain
\[ H_i(\mathcal{F}_n^\xi / T) = \bigoplus_{i_1 + i_2 = i} H_{i_1}(\mathcal{F}) \otimes H_{i_2}(B(T/\mathbb{G}_m)), \]
and the second part of the theorem follows. \( \square \)
Remark 3.2. The rotation torus $G_m$ acts on the quotient $\mathcal{X}^\xi/T$. It gives an affine paving of $\mathcal{X}^\xi/T$ for $SL_2$. But for $SL_d$, $d \geq 3$, the fixed points $(\mathcal{X}^\xi/T)^{G_m}$ are not discrete.

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Département de mathématiques, Bât. 425, Université Paris-sud 11, 91405 Orsay-Cedex, France

E-mail address: zongbin.chen@gmail.com