Rearrangement and Prekopa-Leindler type inequalities

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Abstract

We investigate the interactions of functional rearrangements with Prékopa-Leindler type inequalities. It is shown that that a general class of integral inequalities tighten on rearrangement to “isoperimetric” sets with respect to a relevant measure. Applications to the Borell-Brascamp-Lieb, Borell-Ehrhart, and the recent polar Prékopa-Leindler inequalities are demonstrated. It is also proven that an integrated form of the Gaussian log-Sobolev inequality decreases on half-space rearrangement.

1 Introduction

The Prékopa-Leindler inequality can be understood as a functional generalization of the dimension free statement of the Brunn-Minkowski inequality\(^1\) (BMI) on Euclidean space.

\begin{theorem}[Prékopa-Leindler]
For \(f, g, h : \mathbb{R}^d \to [0, \infty)\) Borel measurable satisfying for a fixed \(t \in (0, 1)\) and any \(x, y \in \mathbb{R}^d\)
\[
f((1-t)x + ty) \geq g^{1-t}(x)h^t(y),
\]
then
\[
\int_{\mathbb{R}^d} f(z)dz \geq \left( \int_{\mathbb{R}^d} g(z)dz \right)^{1-t} \left( \int_{\mathbb{R}^d} h(z)dz \right)^t.
\]
\end{theorem}

The BMI can be recovered by taking indicator functions of sets by \(f = \mathbb{1}_{(1-t)A+tB}, g = \mathbb{1}_A, \text{and} h = \mathbb{1}_B\). The Prékopa-Leindler inequality has become a useful tool in the study of log-concave distributions in probability and statistics, particularly in high dimension, and a point of interest and unification between probabilists and convex geometers.

In parallel, research of the last several decades has built intimate connections between the inequalities of information theory and convex geometry. Perhaps the most celebrated of several links between the two subjects is the entropy power inequality (EPI) of Information Theory as analog to the BMI of Convex Geometry. Although Brascamp and Lieb [10] gave a proof of BMI as a consequence of an optimal Young’s inequality in ’76 (the sharp constants in Young’s inequality due independently to Beckner see [4]) and Lieb [16] gave a proof of the EPI with the same machinery in ’78, it was Costa and Cover [13] who brought attention to this analogy in ’84 and it was not realized until Dembo Cover and Thomas [14] that the proofs of [10] (BMI) and [16] (EPI) could be unified.

\(^1\)To recall BMI, in a dimension free form, says that for \(A, B\) compact in \(\mathbb{R}^d\), then \(|(1-t)A+tB| \geq |A|^{1-t}|B|^t\), where we have used \(|\cdot|\) to denote the Lebesgue measure.
In the time since, the interface between subjects has grown, blurring the lines between probability, information theory, and convex geometry. For further background we direct the reader to [17]. What we present here, is intended to build on these connections. We show that rearrangement inequalities behave nicely with the Prekopa-Leindler and related inequalities in analogy with [19], where it is shown that spherically symmetric decreasing rearrangements decrease the Rényi entropy of independent sums.

An alternative motivation for this investigation is the Brascamp-Lieb-Barthe inequalities relationship to the Brascamp-Lieb-Luttinger rearrangement inequalities [12]. The Brascamp-Lieb inequality [11] enjoys the Brascamp-Lieb-Luttinger inequality [12] as a rearrangement analog. In [2] Barthe used an optimal transport argument to prove Brascamp-Lieb and simultaneously demonstrated a dual inequality that includes Prekopa-Leindler as a special case. It is natural to ask for a rearrangement inequality analog of Barthe’s result. We show that in the special case that the linear maps are scalar multiplication, that a strong rearrangement inequality exists.

The paper is organized in the following manner; in Section 2 we will give definitions and background on a notion of rearrangements, in Section 3 we give a rearrangement inequality for Prékopa-Leindler, before giving a general version in Section 4. In Section 5 we give applications of the theorem derived in Section 4 in special cases, Borell-Brascamp-Lieb, Borell-Ehrhart, and the recently developed Polar Prékopa-Leindler. In Section 6 we show that a similar argument to our main theorem, shows that the Gaussian log-Sobolev inequality, sharpens on a certain half-space rearrangement. Finally in Section 7 we discuss connections with the work of Barthe and Brascamp-Lieb-Luttinger, closing with an open problem.

2 Preliminaries

2.1 Spherically symmetric non-decreasing rearrangements

Given a nonempty measurable set $A \subseteq \mathbb{R}^d$ we define its symmetric rearrangement $A^*$ to be the origin centered ball of equal volume. Explicitly

$$A^* = \left\{ x : |x|_2 < \left( |A| / \omega_d \right)^{\frac{1}{d}} \right\},$$

where $\omega_d$ is the volume of the $d$-dimensional unit ball and with the understanding that $A^* = \{0\}$ in the case that $|A| = 0$ and $A^* = \mathbb{R}^d$ when $|A| = \infty$.

Via the layer-cake decomposition of a non-negative function $f$ as

$$f(x) = \int_0^{f(x)} 1 dt = \int_0^{\infty} 1_{\{y : f(y) > t\}}(x) dt$$

we can extend this notion of symmetrization to functions via the following definition.

**Definition 2.1.** For a measurable non-negative function $f$ define its non-decreasing symmetric rearrangement $f^*$ by

$$f^*(x) = \int_0^{\infty} 1_{\{y : f(y) > t\}}^*(x) dt$$

**Proposition 2.2.** $f^*$ is characterized by the equality

$$\{f^* > \lambda\} = \{f > \lambda\}^*.$$
Proof. By definition, \( x \in \{ f^* > \lambda \} \) implies
\[
\int_0^\infty 1_{\{y : f(y) > t\}^*}(x)dt > \lambda. (7)
\]
By the equality
\[
\int_0^\infty 1_{\{y : f(y) > t\}^*}(x)dt = \sup\{s : x \in \{ f > s \}^*\}, (8)
\]
there exists \( s_0 > \lambda \) such that \( x \in \{ f > s_0 \}^* \). Observe that since both sets are open origin symmetric balls that a volume inequality implies a containment. Thus since
\[
|\{ f > s_0 \}^*| = |\{ f > s_0 \}| \\
\leq |\{ f > \lambda \}| \\
= |\{ f > \lambda \}^*|,
\]
x \( \in \{ f > \lambda \}^* \). For the reverse containment, assume \( x \in \{ f > \lambda \}^* \), choose a sequence \( \lambda_n \) strictly decreasing to \( \lambda \), and consider the increasing sequence of sets \( \{ f > \lambda_n \} \). By the continuity of measure
\[
\lim_n |\{ f > \lambda_n \}| = |\bigcup_n \{ f > \lambda_n \}| = |\{ f > \lambda \}|. (9)
\]
This implies that \( \{ f > \lambda_n \}^* \) is a sequence of open origin symmetric balls whose volume increase to that of \( \{ f > \lambda \}^* \). Hence for large enough \( n \), we have \( x \in \{ f > \lambda_n \} \), and from this our result follows.

For the converse, if \( g \) satisfies \( \{ g > \lambda \} = \{ f > \lambda \}^* \)
\[
g(x) = \int_0^\infty 1_{\{g > t\}}(x)dt \\
= \int_0^\infty 1_{\{f > t\}}(x)dt \\
= f^*(x).
\]

**Corollary 2.3.** \( f^* \) is lower semi-continuous, spherically symmetric and non-increasing in the sense that \( |x| \leq |y| \) implies \( f^*(x) \geq f^*(y) \).

*Proof.*** \( f^* \) has open super level sets by equation (6), and is thus lower semi-continuous. To prove non-increasingness observe that using the characterization above \( f^*(y) > \lambda \) iff \( y \in \{ f > \lambda \}^* \) which implies by \( |x| \leq |y| \) that \( x \in \{ f > \lambda \}^* \), and thus \( f^*(x) > \lambda \). Applying this to \( \lambda_n \) increasing to \( f^*(y) \) yields our result. Observe that this implies spherical symmetry as we apply the preceding in either direction when \( |x| = |y| \).

\[
\int_0^\infty 1_{\{y : f(y) > t\}^*}(x)dt = \sup\{s : x \in \{ f > s \}^*\} \geq \lambda. (10)
\]

This notion and notation for rearrangements allow a particularly simple version of the classical Brunn-Minkowski inequality.

**Theorem 2.1.** *Brunn-Minkowski* For Borel \( A \) and \( B \),
\[
|A + B|_d \geq |A^* + B^*|_d
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3
2.2 More general rearrangements

One might observe that the rearrangement notions put forth above, can be extended naturally in broader contexts. For example if we replace the unit volume ball $\omega_d$ with any other open convex set $K$ with $|K|_d = 1$ containing the origin. Then we can define an analogous rearrangement $*_K$, by taking $A^*_K = |A|_d^TK$. This rearrangement notion can then be extended to functions by defining

$$f^*_K(x) = \int_0^\infty 1_{\{f>t\}}^*_K(x)dt. \quad (12)$$

Observe that the qualitative statement of Brunn-Minkowski, for Borel $A, B$

$$|A + B|_d \geq |A^*_K + B^*_K|_d, \quad (13)$$

is preserved.

As we will motivate in the coming sections we will give a more general rearrangement definition.

Definition 2.4. For Polish measure spaces with Borel $\sigma$-algebra $(M, \mathcal{B}(M), \mu)$ and $(N, \mathcal{B}(N), \alpha)$, we will call a set map

$$*: \mathcal{B}(M) \to \mathcal{B}(N) \quad (14)$$

a rearrangement when it satisfies the following.

- $*(A)$ is an open set satisfying $\alpha(*(A)) = \mu(A)$
- $\mu(A) \leq \mu(B)$ implies $*(A) \subseteq *(B)$
- For a sequence $A_i \subseteq A_{i+1}$, $*(\bigcup_{i=1}^\infty A_i) = \bigcup_{i=1}^\infty *(A_i)$.

For brevity of notation we will write $A^* = *(A)$. This again allows for an extension to functions.

Definition 2.5. For $f \in L^1(M, \mathcal{B}(M), \alpha)$ define

$$f^*(x) = \int_0^\infty 1_{\{f>t\}}^*(x)dt \quad (15)$$

Proposition 2.6. For non-negative $f \in L^1$,

$$\{f^* > \lambda\} = \{f > \lambda\}^*. \quad (16)$$

In particular $f^*$ is lower semi-continuous, and equi-measureable with $f$ in that $\mu\{f > \lambda\} = \alpha\{f^* > \lambda\}$.

Proof. Since $f^*(x) > \lambda$ implies $\int_0^\infty 1_{\{f>t\}}^*(x)dt > \lambda$, which in turn, by the monotonicity of $1_{\{f>t\}}^*$ implies the existence of $t > \lambda$ such that $x \in \{f > t\}^*$. From this it follows that

$$\{f^* > \lambda\} \subseteq \{f > \lambda\}^*. \quad (17)$$
Conversely, let us take for granted the proposition for simple functions, and take $s_n$ to be a sequence of increasing simple functions approximating $f$ pointwise, and uniformly on sets where $f$ is bounded. Then

$$\{f > \lambda\}^* = \left( \bigcup_{n=1}^{\infty} \{s_n > \lambda\} \right)^*$$

$$= \bigcup_{n=1}^{\infty} \{s_n > \lambda\}^*$$

$$= \bigcup_{n=1}^{\infty} \{s_n^* > \lambda\}^*.$$

Since $f_1 \leq f_2$, implies $f_1^* \leq f_2^*$ it follows that $\cup \{s_n^* > \lambda\} \subseteq \{f^* > \lambda\}$.

The fact that $f$ is lower semi-continuous follows from the assumption that $A^*$ is open. Equimeasurability follows from $\alpha\{f^* > \lambda\} = \alpha\{f > \lambda\}^* = \mu\{f > \lambda\}$.

3 A Rearrangement theorem for Prekopa-Leindler

We will begin with a special case of our more general result to build some intuition for the abstractions to follow. For $f, g : \mathbb{R}^d \to [0, \infty)$ and $t \in [0, 1]$ define $f \Box g(z) = \sup_{(1-t)x + ty = z} f^{1-t}(x)g^t(y)$, and define the lower integral of a general non-negative function $h$,

**Theorem 3.1.** For $f, g : \mathbb{R}^d \to [0, \infty)$ Borel and $t \in (0, 1)$

$$\int_{\mathbb{R}^d} f \Box g(z)dz \geq \int_{\mathbb{R}^d} f^* \Box g^*(z)dz,$$

what is more, when $\psi$ is a non-negative and non-decreasing, Borel function,

$$\int_{\mathbb{R}^d} \psi(f \Box g)(z)dz \geq \int_{\mathbb{R}^d} \psi(f^* \Box g^*)(z)dz, \quad (19)$$

First, a convenient characterization of the super level sets of $f \Box g$. For $\lambda \in (0, \infty)$, define

$$S_0 = S_0(\lambda) = \{s \in \mathbb{Q}_+^2 : s_1^{1-t}s_2^t > \lambda\},$$

where we have used $\mathbb{Q}_+$ for the non-negative rationals. The subscript 0 is used to reference the geometric mean, we will have use for more general averaging in the sequel.

We will show $f \Box g$ to be universally measurable in the proof, from which the universal measurability of $\psi(f \Box g)$ will follow.

**Proof.** Now observe that for $z \in \cup_{s \in S_0}(1-t)\{f > s_1\} + t\{g > s_2\}$, clearly $f \Box g(z) > \lambda$. Conversely, if $f \Box g(z) > \lambda$, then there exists a pair of $x$ and $y$ such that $(1-t)x + ty = z$ and $f^{1-t}(x)g^t(y) > \lambda$. By the continuity of the map $(u, v) \mapsto u^{1-t}v^t$, there exists $(s_1, s_2)$ rational satisfying $s_1 < f(x)$, $s_2 < g(y)$, and $s_1^{1-t}s_2^t > \lambda$. Thus

$$\{f \Box g > \lambda\} = \bigcup_{s \in S_0(\lambda)} (1-t)\{f > s_1\} + t\{g > s_2\}. \quad (21)$$
Let us remark, that the sum of Borel sets is universally measurable, and hence \( \{f \square g > \lambda\} \) is as well. This shows we are well justified in our notation \( \int_{\mathbb{R}^d} f \square g(z)dz \). By Brunn-Minkowski and the characterizing property of rearrangements on super level sets

\[
|(1-t)\{f > s_1\} + t\{g > s_2\}| \geq |(1-t)\{f > s_1\} + t\{g > s_2\}|^*
\]

\[
= |(1-t)\{f^* > s_1\} + t\{g^* > s_2\}|.
\]

Now applying (21) to \( f^* \square g^* \) and observing that,

\[
(1-t)\{f^* > s_1\} + t\{g^* > s_2\}
\]

is an origin centered ball in \( \mathbb{R}^d \) for every \( s \in S_0(\lambda) \), we see that

\[
\{f^* \square g^* > \lambda\} = \bigcup_{s \in S_0(\lambda)} (1-t)\{f^* > s_1\} + t\{g^* > s_2\}
\]

But using (22), obviously

\[
|(1-t)\{f^* > s_1\} + t\{g^* > s_2\}| \leq \bigcup_{s \in S_0(\lambda)} (1-t)\{f > s_1\} + t\{g > s_2\}
\]

and a result stronger than our claim follows,

\[
|\{f \square g > \lambda\}| \geq |\{f^* \square g^* > \lambda\}|.
\]

Using the layer-cake decomposition of the integral

\[
\int_{\mathbb{R}^d} \psi(f \square g)(z)dz = \int_0^\infty |\{\psi(f \square g) > t\}|dt.
\]

Notice that by the non-decreasingness, \( \psi^{-1}(\lambda, \infty) \) is an interval of the form \( [\lambda, \infty) \) or \( (\lambda, \infty) \) for a non-negative \( x \), and from this, we can use (26) (and continuity of measure if the interval is closed) to obtain our result. \( \square \)

4 A Rearrangement theorem

Suppose the existence of \( m : M^n \rightarrow M \) and \( \eta : N^n \rightarrow N \) such that \( m(A_1, \ldots, A_n) = \{x = m(a_1, \ldots, a_n) : a_i \in A_i\} \) and \( \eta(B_1, \ldots, B_n) = \{y = \eta(b_1, \ldots, b_n) : b_i \in B_i\} \) are universally measurable for \( A_i \) and \( B_j \) Borel. Suppose further that \( \{\eta(A^n_1, \ldots, A^n_n)\}_A \) indexed on \( n \)-tuples of Borel sets is totally ordered in the sense that for any Borel \( A_1, \ldots, A_n \) and \( A'_1, \ldots, A'_n \) we have either

\[
\eta(A^n_1, \ldots, A^n_n) \subseteq \eta(A'^n_1, \ldots, A'^n_n)
\]

or

\[
\eta(A^n_1, \ldots, A^n_n) \supseteq \eta(A'^n_1, \ldots, A'^n_n)
\]

and finally that \( \mu(m(A_1, \ldots, A_n)) \geq \alpha(\eta(A^n_1, \ldots, A^n_n)) \).
Examples

1. When \((M, m, \mu) = (N, \eta, \alpha) = (\mathbb{R}^d, m_t, dx)\), where \(t = (t_1, \ldots, t_n) \in \mathbb{R}^n\), defines a map \(m_t\) by vector space operations,

\[ x = (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n t_i x_i. \]  

(30)

Letting our rearrangement \(*\) be \(*K\) as in section (2), for \(K\) open, convex, and symmetric. Taking \(B_i = sgn(t_i)A_i\) so that \(t_1A_1 + \cdots + t_nA_n = |t_1|B_1 + \cdots + |t_n|B_n\). Using the symmetry and convexity of \(K\), and the definition of our rearrangement as a scaling of \(K\), it follows that

\[ t_1A_1^* + \cdots + t_nA_n^* = \left( \sum_{i=1}^n |t_i| |A_i|^{\frac{1}{n}} \right) K \]  

(31)

and hence that the images of \(m_t\) are totally ordered. Finally, since Brunn-Minkowski implies that

\[ ||t_1B_1 + \cdots + t_nB_n|| \geq ||t_1B_1^* + \cdots + t_nB_n^*||, \]

(32)

it follows that

\[ |t_1A_1 + \cdots + t_nA_n| \geq |t_1A_1^* + \cdots + A_n^*|. \]  

(33)

2. In the previous example if we restrict to \(t_i \geq 0\) we can extend the argument above to non-symmetric \(K\).

3. When \((M, m, \mu)\) is a Gaussian measure on a Banach Space \(M\) and \(m\) defined as \(x = (x_1, \ldots, x_n) \mapsto \sum_i t_i x_i\) for \(t_i \geq 0\), and \((N, \eta, \alpha)\) with \(N = \mathbb{R}, \eta\) defined by \(y \mapsto \sum_i t_i y_i\) and \(\alpha = \gamma\) be the standard Gaussian distribution. Explicitly, \(\gamma\) is given by the density function \(f(x) = e^{-x^2/2}(2\pi)^{-1/2}\). It is the content of the Borell-Ehrhart theorem, (as we will see below) that taking \(A^*\) to be the open half space interval, the above is satisfied.

Now let us generalize the geometric mean used in Prekopa-Leindler. Let \(\mathcal{M} : (0, \infty)^n \to (0, \infty)\) be continuous and increasing in the sense that \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) satisfying \(x_i > y_i\) implies \(\mathcal{M}(x) > \mathcal{M}(y)\). Extend \(\mathcal{M}\) to \([0, \infty)^n\) by \(\mathcal{M}(x) = 0\) for \(\prod_i x_i = 0\).

Examples

1. For \(t = (t_1, \ldots, t_n)\) with \(t_i > 0\) and \(p \in [-\infty, 0) \cup (0, \infty]\) take

\[ \mathcal{M}(u) = \mathcal{M}_p(t)(u) = (t_1 u_1^p + \cdots + t_n u_n^p)^{\frac{1}{p}}. \]  

(34)

with \(\mathcal{M}_{-\infty}(u) = \min_i u_i\) and \(\mathcal{M}_{\infty}(u) = \max_i u_i\)

2. Take \(\mathcal{M}_0(t) = \prod_i t_i^{1/i}\). Note that in the case that \(\sum_i t_i = 1\), \(\mathcal{M}_0\) is the limiting case of the previous example.

3. Define for \(t_i \geq 0\) and \(u \in (0, 1)^n\),

\[ \mathcal{M}_p(t)(u) = \Phi(t_1 \Phi^{-1}(u_1) + \cdots + t_n \Phi^{-1}(u_n)) \]

where \(\Phi(s) = \int_{-\infty}^{s} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx\).

\[ \]
Definition 4.1. For \( f = \{f_i\}_{i=1}^n \) with \( f_i : M \to (0, \infty) \) define
\[
\square_M f(z) = \sup_{m(x) = z} \mathcal{M}(f_1(x_1), \ldots, f_n(x_n)).
\] (36)

Let us further denote for \( f_* = \{f_i^*\}_{i=1}^n \). Analogously for \( g = \{g_i\}_{i=1}^n \) with \( g_i : N \to [0, \infty) \) take
\[
\square_N g(z) = \sup_{\eta(x) = z} \mathcal{M}(g_1(x_1), \ldots, g_n(x_n)).
\] (37)

Theorem 4.1. For \( f = \{f_i\}_{i=1}^n \), with \( f_i \) Borel measurable from \( M \) to \( [0, \infty) \), the assumed inequality
\[
\mu(m(A_1, \ldots, A_n)) \geq \alpha(\eta(A_1^*, \ldots, A_n^*))
\] (38)
can be extended to functions in the sense that for non-negative non-decreasing Borel \( \psi \)
\[
\int \psi(\square_M f) d\mu \geq \int \psi(\square_N f_*) d\alpha.
\] (39)

Let us define a set for \( M \) as above, and \( \lambda > 0 \)
\[
S_M = S_M(\lambda) = \{q \in \mathbb{Q}^n : \mathcal{M}(q) > \lambda\}
\] (40)

Proof. We will prove that for \( \lambda \in (0, \infty) \), \( \mu(\square_M f > \lambda) \geq \alpha(\square_N f_* > \lambda) \). We first observe that by arguments similar to Theorem 3.1
\[
\{\square_M f > \lambda\} = \bigcup_{q \in S_M(\lambda)} m(\{f_1 > q_1\}, \ldots, \{f_n > q_n\}).
\] (41)

Indeed, suppose \( \square_M f(z) > \lambda \). This implies the existence of some \( x \) such that \( m(x) = z \) and \( \mathcal{M}(f_1(x_1), \ldots, f_n(x_n)) > \lambda \). By the continuity of \( \mathcal{M} \) there exists \( q_i \in \mathbb{Q} \) such that \( \mathcal{M}(q_1, \ldots, q_n) > \lambda \) and \( f(x_i) > q_i \). The opposite direction is immediate. Analogously
\[
\{\square_N f_* > \lambda\} = \bigcup_{q \in \mathbb{Q}^n : \mathcal{M}(q) > \lambda} \eta(\{f_1^* > q_1\}, \ldots, \{f_n^* > q_n\}).
\] (42)

This gives
\[
\mu(\square_M f > \lambda) = \mu \left( \bigcup_{q \in \mathbb{Q}^n : \mathcal{M}(q) > \lambda} m(\{f_1 > q_1\}, \ldots, \{f_n > q_n\}) \right)
\]\[
\geq \sup_{q \in \mathbb{Q}^n : \mathcal{M}(q) > \lambda} \mu(m(\{f_1 > q_1\}, \ldots, \{f_n > q_n\}))
\]\[
\geq \sup_{q \in \mathbb{Q}^n : \mathcal{M}(q) > \lambda} \alpha(\eta(\{f_1^* > q_1\}, \ldots, \{f_n^* > q_n\}))
\]\[
= \alpha \left( \bigcup_{q \in \mathbb{Q}^n : \mathcal{M}(q) > \lambda} \eta(\{f_1^* > q_1\}, \ldots, \{f_n^* > q_n\}) \right)
\]\[
= \alpha(\square_N f_* > \lambda)
\]
where the first inequality is obvious, the second is by the assumption on the rearrangement sets, and the next equality is by the assumption of total orderedness.

\[\square\] 

Definition 4.2. For \( t \in (0, 1) \), and non-negative \( f, g : \mathbb{R}^d \to \mathbb{R} \) we define \( f \square_t g : \mathbb{R}^d \to \mathbb{R} \) via
\[
f \square g(z) = f \square t g(z) = \sup \{\mathcal{M}_{\alpha}(f(x), g(y)) : (1 - t)x + ty = z\}.
\]
5 Applications

5.1 Borell-Brascamp-Lieb type inequalities

In the case that $\lambda \in (0, 1)$ and $-\infty \leq p \leq \infty$ and setting

$$M(u, v) = M^\lambda_p(u, v) = \begin{cases} ((1 - \lambda)a^p + \lambda b^p)^{\frac{1}{p}} & \text{if } uv \neq 0 \\ 0 & \text{if } uv = 0 \end{cases} \quad (43)$$

The Borell-Brascamp-Lieb inequality, generalizes the Prékopa-Leindler inequality (with the understanding that $M^\lambda_0(u, v) = u^{1-\lambda}v^\lambda$) and can be stated as follows.

**Theorem 5.1.** [8, 11] For Borel functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\int f \Box M_p g dx \geq M_p/((np+1) \left( \int f, \int g \right)$$

when $p \geq -1/n$.

We can thus provide the following sharpening,

**Theorem 5.2.** For Borel functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\int f \Box M_p g dx \geq \int f^* \Box M_p^* g^* dx$$

$$\geq M_p/((np+1) \left( \int f, \int g \right)$$

when $p \geq -1/n$.

5.2 The Gaussian case

Again on $\mathbb{R}^d$ let $\gamma$ denote the standard Gaussian measure,

$$\gamma_d(A) = \int_A e^{-|x|^2/2} / (2\pi)^{d/2} dx. \quad (44)$$

When $d = 1$, we will suppress the subscript and just right $\gamma$. As is customary will call any affine pushforward of $\gamma$ Gaussian. The notion of a Gaussian measure can be extended to a Banach Space $(M, \mathcal{B}(M), \mu)$ when the affine pushforward of $\mu$ to a finite dimensional space is Gaussian.

Define a rearrangement $*$ from the Gaussian measure space $(M, \mathcal{B}(M), \mu)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma_1)$, by

$$A^* = \{ x \in \mathbb{R} : x < t \} \quad (45)$$

where $t$ is chosen to satisfy $\mu(A) = \gamma(A^*)$. Explicitly, $t = \Phi^{-1}(\mu(A))$, where we recall $\Phi(x) = \int_{-\infty}^x \gamma(y)dy$. A functional half-space rearrangement by

$$f^{*h}(x) = \int_0^\infty 1_{\{f > t\}}^{*h}(x)dt \quad (46)$$
The Borell-Ehrhart’s inequality [15] [9] is usually stated as the following. That for a Gaussian measure with \( t \in (0, 1) \)

For \( A, B \) Borel subsets of a Gaussian measure space \((M, \mathcal{B}(M), \mu)\) and \( t \in (0, 1) \) then

\[
\mu((1-t)A + tB) \geq \Phi((1-t)\Phi^{-1}(\mu(A)) + t\Phi^{-1}(\mu(B)))
\]

(47)

can be equivalent formulated in our terminology and notation as

**Theorem 5.3.** [15][9] For \( A, B \) Borel subsets of a Gaussian measure space \((M, \mu)\), and \( \ast \) our halfspace rearrangement to \( \mathbb{R} \)

\[
\mu((1-t)A + tB) \geq \gamma((1-t)A^\ast + tB^\ast).
\]

(48)

Letting \( \alpha = \gamma \) be the standard Gaussian measure on \( \mathbb{N} = \mathbb{R} \), then by a direct application of Theorem 4.1 we have an immediate extension. Taking \( m(x, y) = (1-t)x + ty \) and \( \eta(x, y) = (1-t)x + ty \) and

\[
\mathcal{M}(u, v) = \Phi((1-t)\Phi^{-1}(u) + t\Phi^{-1}(v))
\]

(49)

**Theorem 5.4.** For Borel measurable \( f, g : M \to [0, 1] \),

\[
\int f\Box_Mgd\mu \geq \int f^\ast\Box_Ng^\ast d\gamma.
\]

(50)

Observe that contains Theorem 5.3 by taking \( f = \mathbb{1}_A \) and \( g = \mathbb{1}_B \). This is not the first functional extension of the Borell-Ehrhart inequality, in fact Borell was able to achieve the first full proof of the result by lifting to a functional setting and using a semi-group argument. This technique was streamlined by Barthe and Huet who gave the following generalization.

Fix a set \( I \subseteq \{1, 2, \ldots, n\} \) and set of positive numbers \( \lambda_1, \ldots, \lambda_n \) satisfying \( \sum \lambda_i \geq 1 \) and

\[
\lambda_j - \sum_{i \neq j} \lambda_i \leq 1
\]

(51)

for \( j \notin I \), and take \( \Phi \) to be the distribution of function of a standard Gaussian random variable. Define

\[
\mathcal{M}(u_1, \ldots, u_n) = \mathcal{M}_\Phi(u_1, \ldots, u_n) = \Phi(t_1\Phi^{-1}(u_1) + \cdots + t_n\Phi^{-1}(u_n))
\]

(52)

We can cast Barthe and Huet’s extension of the Borell-Ehrhart inequality as the following

**Theorem 5.5.** [3] For Borel \( f_1, \ldots, f_n \) from \( \mathbb{R}^d \) to \([0, 1]\) such that \( \Phi^{-1} \circ f_i \) is concave for \( i \in I \),

\[
\int \Box df \geq \mathcal{M}\left(\int f_1d\gamma, \ldots, \int f_n d\gamma\right).
\]

(53)

We can present the following which slightly loosens the hypothesis and sharpens the conclusions.

**Theorem 5.6.** For Borel \( f_1, \ldots, f_n \) from \( \mathbb{R}^d \) to \([0, 1]\) such that \( \Phi^{-1} \circ f_i^\ast \) is concave for \( i \in I \) and

\[
\int \Box df \geq \int \Box f^\ast d\gamma
\]

\[
\geq \mathcal{M}\left(\int f_1^\ast d\gamma, \ldots, \int f_n^\ast d\gamma\right)
\]

\[
= \mathcal{M}\left(\int f_1d\gamma, \ldots, \int f_n d\gamma\right).
\]
The first inequality is a consequence, the equality is immediate as well following from our definition of rearrangement. Thus to prove the result we need only justify the second inequality, which follows from Theorem 5.5 once we know that rearrangement preserves $\Phi^{-1}$-concavity. For this we prove a general result.

**Proposition 5.1.** Suppose that $f$ is $\Psi$-concave in the sense that $t \in (0, 1)$ implies

$$f((1 - t)x_1 + tx_2) \geq \Psi(f(x_1), f(x_2))$$

where $\Psi$ is a continuous strictly increasing function, in the sense that $x_i > y_i$ implies $\Psi(x_1, x_2) > \Psi(y_1, y_2)$, and that the class of rearranged sets is stable under convex Minkowski combination. Then, $f$ being $\Psi$-concave implies that $f^*$ is $\Psi$-concave as well.

The proof depends on a set theoretic description of concavity, that $f$ is $\Psi$-concave can be equivalently stated as $\lambda_i \in \mathbb{R}$ implies

$$(1 - t)\{f > \lambda_1\} + t\{f > \lambda_2\} \subseteq \{f > \Psi(\lambda_1, \lambda_2)\},$$

which can be easily verified. Notice that the rearrangement $*$ is general, so long as a compatible measure exists, the theorem holds.

**Proof.** By the set theoretic representation we wish to show that $(1 - t)\{f^* > \lambda_1\} + t\{f^* > \lambda_2\} \subseteq \{f^* > \Psi(\lambda_1, \lambda_2)\}$, or equivalently

$$(1 - t)\{f > \lambda_1\}^* + t\{f > \lambda_2\}^* \subseteq \{f > \Psi(\lambda_1, \lambda_2)\}^*.$$ 

Since we have assumed that the rearrangement is stable under convex Minkowski summation, and that the set of all rearrangements is totally ordered, it is enough to show

$$\mu((1 - t)\{f > \lambda_1\}^* + t\{f > \lambda_2\}^*) \leq \mu(\{f > \Psi(\lambda_1, \lambda_2)\}^*),$$

But

$$\mu((1 - t)\{f > \lambda_1\}^* + t\{f > \lambda_2\}^*) \leq \alpha((1 - t)\{f > \lambda_1\} + t\{f > \lambda_2\})$$

by assumption, and

$$\alpha(\{f > \Psi(\lambda_1, \lambda_2)\}) = \mu(\{f > \Psi(\lambda_1, \lambda_2)\}^*)$$

by definition. From this the result follows.

This gives Theorem 5.6, let us also remark that it delivers the following for so called $s$-concave measures.

**Corollary 5.2.** The space of $s$-concave measures on a finite dimensional space is stable under rearrangement.

The notion of $s$-concavity is due to Borell [6]. A Radon measure $\mu$ is $s$-concave when it satisfies,

$$\mu((1 - t)A + tB) \geq ((1 - t)\mu^s(A) + t\mu^s(B))^\frac{1}{s}$$
for all compact sets $A, B$ and $t \in (0, 1)$ and $s \in [-\infty, \infty]$. When the support of $\mu$ has non-empty interior on a $d$-dimensional space this is equivalent to the existence of an $s' = s/(1-sd)$-concave density. That is, with $f = \frac{d\mu}{dx}$, with $dx$ the Lebesgue measure on the space,

$$f((1-t)x + ty) \geq \left((1-t)f'(x) + tf'(y)\right)^{\frac{1}{s'}},$$

(61)

for $x, y$ vectors and $t \in (0, 1)$. Taking $\Psi(u, v) = \left((1-t)u + tv\right)^{\frac{1}{s'}}$, we see that $f^*$ is $s'$-concave and hence $\mu^*$ is $s$-concave.

5.3 Polar Prekopa-Leindler

For $t, \lambda \in (0, 1)$, we take

$$M(u, v) = M_{t, \lambda}^{-\infty}(u, v) = \min\left\{u^{\frac{t}{1-t}}, v^{\frac{\lambda}{1-\lambda}}\right\}$$

Then we can state the recent polar analog of Prekopa-Leindler due to Artstein-Avidan, Florentin, and Segal as

**Theorem 5.7.** \cite{1} For $f, g : \mathbb{R}^d \to [0, \infty)$ Borel, and $\mu$ log-concave

$$\int f \Box_{M} g(x) d\mu(x) \geq M_{t, \lambda}^{-1}\left(\int f(x) d\mu(x), \int g(x) d\mu(x)\right).$$

(62)

In the case that $\mu$ is Lebesgue or Gaussian, this can be sharpened to

**Theorem 5.8.** For $f, g : \mathbb{R}^d \to [0, \infty)$ Borel, and $\mu$ either Gaussian, with $\ast$ the half space rearrangement, or Lebesgue with $\ast$ a convex set rearrangement, then

$$\int f \Box_{M} g d\mu \geq \int f^* \Box_{M} g^* d\mu$$

$$\geq M_{t, \lambda}^{-1}\left(\int f d\mu, \int g d\mu\right).$$

6 Gaussian Log-Sobolev inequality

The Prekopa-Leindler inequality is sometimes referred to as a reverse of Hölder’s inequality. Let us recall that Hölder’s inequality $\|fg\|_1 \leq \|f\|_p \|g\|_q$ when specialized to a single function gives a convexity result for $L_p$-norms. When Prekopa-Leindler is specialized to a single function, the resultant inequality is less transparent, but still rather interesting. Recall the Gaussian Prekopa Leindler inequality,

**Theorem 6.1** (Gaussian Prekopa-Leindler). For $u, v, w : \mathbb{R}^d \to [0, \infty]$ such that,

$$u((1-t)x + ty) \geq e^{-t(1-t)|x-y|^2/2}v^{1-t}(x)w^{t}(y)$$

(63)

then with $\gamma$ denoting the standard Gaussian measure in $\mathbb{R}^d$,

$$\int u d\gamma \geq \left(\int v d\gamma\right)^{1-t} \left(\int w d\gamma\right)^{t}$$

(64)
The proof of the theorem is a consequence of the usual PLI combined with the fact that the Hessian of the Gaussian potential is the identity matrix. This property of being strongly log-concave is equivalent to satisfying an improved PLI above. We proceed, following arguments of Bobkov-Ledoux [5] that connect this strengthened PLI to the log-Sobolev inequality. For a fixed $p > 1$ and $f$, take $w = f^p v = 1$ and $t = \frac{1}{p}$ then for any $u$, satisfying
\[
u((1 - t)x + ty) \geq e^{-t(1-t)|x-y|^2/2} f(y) \tag{65}
\]
we have an upper bound on the $L_p(\gamma)$ norm of $f$ from 6.1
\[
\int u \geq \|f\|_p \tag{66}
\]
With the interest of determining the optimal such $u$ achievable through the methods of PLI, we define the operator
\[
Q_t f(z) = \sup_{\{(x,y):(1-t)x + ty = z\}} e^{-t(1-t)|x-y|^2/2} f(y), \tag{67}
\]
and let $*$ denote the half-space rearrangement of a set under the standard Gaussian measure $\gamma$.

**Theorem 6.2.** For $f$ such that $Q_t f$ is Lebesgue measurable and $\lambda > 0$,
\[
\gamma(\{Q_t f > \lambda\}) \geq \gamma(\{Q_t f^* > \lambda\}) \tag{68}
\]
where $f^*$ is the half-space rearrangement of $f$.

**Proof.** Taking $\lambda = (1-t)/t$ and $w = z - x$, we get an expression that will be useful to us later
\[
Q_t f(z) = \sup_w e^{-\lambda|w|^2/2} f(z + \lambda w). \tag{69}
\]
Further, taking $a = z + \lambda w$ we get an expression will aid the description of the super-level sets of $Q_t f$,
\[
Q_t f(z) = \sup_a e^{-|z-a|^2/2\lambda} f(a). \tag{70}
\]
we will first express $\{Q_t f > \lambda\}$ as the union of simpler sets. To this denote set $S = S(\lambda, q_1, q_2) = \{q = (q_1, q_2) \in \mathbb{Q}_d^2 : q_1 q_2 > \lambda\}$ and it is straight forward to verify
\[
\{Q_t f > \lambda\} = \bigcup_{q \in S} \left( \{f > q_1\} + \left\{ w \in \mathbb{R}^d : |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \right). \tag{71}
\]
Indeed, if $z = y + w$ with $f(y) > q_1$ and $|w| < a(q_2)$, then taking $x = (y + \frac{w}{r})$, then $z = (1-t)x + ty$ and
\[
e^{-t(1-t)|x-y|^2/2} f(x) = e^{-(1-t)|w|^2/2t} f(x) \geq q_2 q_1 > \lambda. \tag{72}
\]
Conversely, for $Q_t f(z) > \lambda$, there exists by definition $x, y$ such that $z = (1-t)x + ty$,
\[
e^{-t(1-t)|x-y|^2/2} f(x) > \lambda. \tag{73}
\]
Taking \( w = t(y - x) \), \( z = x + w \), the above inequality is

\[
E^{-(1-t)|w|^2/2t} f(x) > \lambda.
\]

(74)

By continuity there exists rational \( q_i \) such that \( q_1 < f(x) \) and \( q_2 < e^{-(1-t)|w|^2/2t} \) such that \( q_2q_1 > \lambda \). But this is \( |w| < a(q_2) \) and hence \( z \in S(\lambda, q_1, q_2) \). Applying the Gaussian isoperimetric inequality [18, 7], which in our preferred formulation states that \( \gamma(A + B) \geq \gamma(A entrenched) \) where \( B \) is a Euclidean ball, to the above inequality we have,

\[
\gamma(\{Q_t f > \lambda\}) = \gamma \left( \bigcup_{q \in S} \{ f > q_1 \} + \left\{ w \in \mathbb{R}^d : |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \right)
\]

\[
\geq \sup_{q \in S} \gamma \left( \{ f > q_1 \} + \left\{ w \in \mathbb{R}^d : |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \right)
\]

\[
\geq \sup_{q \in S} \gamma \left( \{ f > q_1 \}^* + \left\{ w \in \mathbb{R}^d : |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \right). \tag{75}
\]

But \( \{ f > q_1 \}^* \) is \( \{ f^* > q_1 \} \) is a half space and hence the family of \( \{ f^* > q_1 \} + \left\{ |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \) indexed by \( S(\lambda, q_1, q_2) \) is a family of totally ordered sets, hence

\[
\sup_{q \in S} \gamma(\{ f > q_1 \}^* + \left\{ |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\}) = \gamma \left( \bigcup_{q \in S} \{ f^* > q_1 \} + \left\{ |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \right) \tag{76}
\]

Applying (71) we have

\[
\gamma \left( \bigcup_{q \in S} \{ f^* > q_1 \} + \left\{ |w| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \right) = \gamma(\{Q_t f^* > \lambda\}), \tag{76}
\]

and our theorem follows. \( \square \)

Combining Theorems 6.1 and 6.2 we have the following.

**Theorem 6.3.** For \( t \in (0, 1) \) and \( p = \frac{1}{t} \),

\[
\|f\|_p = \|f^*\|_p \leq \int Q_t f d\gamma \leq \int Q_t f^* d\gamma. \tag{77}
\]

Let us sketch how Theorem 6.3 is related to the usual Gaussian log-Sobolev Inequality.

**Corollary 6.1.** For nice \( f \), and half-space rearrangement \( f^* \), with \( H_\gamma(f) \) defined to be \( \int f \log f d\gamma - \int f d\gamma \log \int f d\gamma \),

\[
H_\gamma(f) = H_\gamma(f^*) \leq \frac{1}{2} \int_{\{f^* > 0\}} |\nabla f^*|^2 f^{*} d\gamma \leq \frac{1}{2} \int_{\{f > 0\}} |\nabla f|^2 f d\gamma \tag{78}
\]

**Sketch.** By Theorem 6.3,

\[
\|f\|_p \leq \int Q_t f^* d\gamma \leq \int Q_t f d\gamma \tag{79}
\]
We can expand $\|f\|_p$ to obtain
\begin{equation}
\|f\|_p = \|f\|_1 + (p - 1)H_\gamma(f) + o(p - 1). \tag{80}
\end{equation}
and then from $(1 - t)x + ty = z$, writing $\lambda = (1 - t)/t$ and $w = x - z$, we have
\begin{equation}
Q_tf(z) = f(z + \lambda w)e^{-\lambda|w|^2/2} \tag{81}
\end{equation}
and investigating for small lambda, the Taylor expansion
\begin{equation}
f(z + \lambda w)e^{-\lambda|w|^2/2} = f(z) + \lambda \left(\nabla f(z) \cdot w - f(z)\frac{|w|^2}{2}\right) + o(\lambda) \tag{82}
\end{equation}
suggests
\begin{align*}
Q_tf(z) &= f(z) + \sup_w \lambda \left(\nabla f(z) \cdot w - f(z)\frac{|w|^2}{2}\right) + o(\lambda) \\
&= f(z) + \frac{\lambda |\nabla f(z)|^2}{2f(z)} + o(\lambda).
\end{align*}
Observing that $\lambda = p - 1$
\begin{equation}
\int Q_fd\gamma = \|f\|_1 + \frac{p - 1}{2} \int \frac{|
abla f|^2}{f}d\gamma + o(p - 1) \tag{83}
\end{equation}
Using the expansions in (83) and (80) in (79), and we achieve our inequality with $p \to 1$ after algebraic cancellation.

7 Barthe-Brascamp-Lieb and Rearrangement

The Brascamp-Lieb inequality is the following.

**Theorem 7.1.** \cite{10} For natural numbers $n$, $m$, and $\{n_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ a sequence of positive numbers such that $\sum_{i=1}^m c_i n_i = n$ then for surjective linear maps $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ then the inequality
\begin{equation}
\int_{\mathbb{R}^n} \prod_{i=1}^m f^{c_i}(B_i x) dx \leq C^{-1/2} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f \right)^{c_i} \tag{84}
\end{equation}
for $f : \mathbb{R}^{n_i} \to [0, \infty)$ integrable, and
\begin{equation}
C = \inf \left\{ \frac{\det(\sum_{i=1}^m c_i B'_i A_i B_i)}{\prod_i \det A_i} : A_i \text{ positive definite} \right\} \tag{85}
\end{equation}
The theorem enjoys a qualitative analog in the case that $n_i = d$, so that $n = md$ and $x \in \mathbb{R}^n$ can be expressed as $x = (x_1, \ldots, x_m)$ for $x_i \in \mathbb{R}^d$ and $B_i$ are of the form
\begin{equation}
B_i x = \sum_{j=1}^m B_{ij} x_j \tag{86}
\end{equation}
then the rearrangement theorem due to Brascamp-Lieb-Luttinger is what follows
Theorem 7.2. [12] For $B_i$ satisfying (86)
\[
\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(B_i x) dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i^*(B_i x) dx,
\] (87)
where $*$ represents the spherically symmetric decreasing rearrangement.

Notice that when Theorem 7.2 applies, it gives an intermediary inequality to Theorem 7.1. Indeed since $(f^c_i)^* = (f^*)^c_i$, applying Theorem 7.2 and then 7.1 gives
\[
\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i^c(B_i x) dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^{m} (f^*)^c_i(B_i x) dx \\
\leq C^{-1/2} \prod_{i=1}^{m} \left( \int_{\mathbb{R}^{n_i}} f \right)^{c_i}.
\]

Two decades ago Barthe gave the following reversal of Brascamp-Lieb, that serves as sort of dual inequality.

Theorem 7.3. [2] For natural numbers $n, m,$ and $\{n_i\}_{i=1}^{m}$ and $\{c_i\}_{i=1}^{m}$ a sequence of positive numbers such that $\sum_{i=1}^{m} c_i n_i = n$ then for surjective linear maps $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ then the inequality
\[
C^{1/2} \prod_{i=1}^{m} \left( \int_{\mathbb{R}^{n_i}} f \right)^{c_i} \leq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f_i^c(y_i) : \sum_{i} c_i B_i' y_i = x \right\} dx
\] (88)
for $f_i : \mathbb{R}^{n_i} \to [0, \infty)$ integrable, and $C$ defined in Theorem 7.1.

Question 7.1. Suppose that $B_i$ are of the form (86), and $f_i$ belong to $L_1(\mathbb{R}^d)$, when is it true that
\[
\int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f(y_i) : \sum_{i} B_i' y_i = x \right\} dx \geq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f^*(y_i) : \sum_{i} B_i' y_i = x \right\} dx
\] (89)

The results presented here verify the question in the special case that $B_i$ are scalar.

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