Elie Cartan’s Geometrical Vision or How to Avoid Expression Swell

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Abstract

The aim of the paper is to demonstrate the superiority of Cartan’s method over direct methods based on differential elimination for handling otherwise intractable equivalence problems. In this sense, using our implementation of Cartan’s method, we establish two new equivalence results. We establish when a system of second order ODE’s is equivalent to a flat system (second derivations are zero), and when a system of holomorphic PDE’s with two independent variables and one dependent variable is flat. We consider the problem of finding transformation that brings a given equation to the target one. We shall see that this problem becomes algebraic when the symmetry pseudogroup of the target equation is zerodimensional. We avoid the swelling of the expressions, by using non-commutative derivations adapted to the problem.

Key words: Cartan’s equivalence method, differential algebra

Introduction

Present ODE-solvers make use of a combination of symmetry methods and classification methods. Classification methods are used when the ODE matches a recognizable pattern (e.g. as listed in (Kamke, 1944)). Significant progress would be made if it was possible to compute in advance the differential invariants that allow to decide whether the equation to be solved is equivalent to one of the list by a change of coordinates. We will show that, for the computation of these invariants, the geometrical approach offers advantages over non geometrical approaches (e.g. Riquier, Ritt, Kolchin etc.)

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The change of coordinates that maps a given differential equation to a target one is a solution of certain PDE’s system. Differential algebra allows us to compute the integrability conditions of this system. Unfortunately, in practice one is often confronted with computer output consisting of several pages of intricate formulae. Even so, for more complicated problems with higher complexity such those treated in section 5, and many others coming from biology, physics, etc. differential algebra is not efficient due to expression swell.

In his equivalence method, Élie Cartan formulated the PDE’s system as a linear Pfaffian system. In this way, the integrability conditions are computed with a process called absorption of torsion leading to sparse structure equations. In addition, this computation is done by separately and symmetrically treating the linear Pfaffian system. This divides the number of variables by two.

Regarding ODE-solvers, Cartan’s equivalence method is complementary to symmetry method. In the case when the symmetry pseudogroup of the input equation is zerodimensional, actual DE-solvers are unusable. Thus, in such a case, one can map this equation to a known equation. We shall see that the change of coordinates realizing the equivalence can be computed without integrating differential equation.

The paper is organized as follows. In section 1, we define the equivalence problem of differential equations under a given pseudogroup of diffeomorphisms. In section 2, using Diffalg, we solve two examples which will serve for our argumentation and we finger out the limitations of such technique. We give a first optimization, to avoid the expression swell, by using non commutative derivations. The goal of section 3 is to introduce the reader to the calculation of the integrability conditions of linear Pfaffian systems. In section 4, Cartan’s method is applied to the equivalence problem presented in section 1. In section 5, we give new results on the equivalence of ODE’s and PDE’s systems with flat systems (Theorems 2 and 3). In section 6, we consider the problem of finding transformation that brings a given equation to the target one which is usually even harder than, to establish the equivalence. We shall see that this problem becomes algebraic when the symmetry pseudogroup is zerodimensional. We avoid the swelling of the expressions, by using non-commutative derivation adapted to the problem.

1 Formulation of local equivalence problems

An equivalence problem is the following data : a class of (systems of) differential equations $\mathcal{E}_f$ and a pseudogroup of transformations acting on this class. The two differential equations $\mathcal{E}_f$ and $\mathcal{E}_f$ are said to be equivalent under the
pseudogroup \( \Phi \) (we write \( \mathcal{E}_f \sim_\Phi \mathcal{E}_{\bar{f}} \)) if and only if there exists a local diffeomorphism \( \varphi \in \Phi \) which maps the solutions of \( \mathcal{E}_f \) to the solutions of \( \mathcal{E}_{\bar{f}} \). The change of coordinates \( \varphi \in \Phi \) is solution of a PDE’s system that we can generated by an algorithm. A program like Diffalg (Boulier et al., 1995; Boulier, 2006) or Rif (Reid et al., 1996) can compute the integrability conditions of such PDE’s system, then the existence of \( \varphi \) is decidable.

Consider the two second order ODE \( \mathcal{E}_f \) and \( \mathcal{E}_{\bar{f}} \)

\[
\frac{d^2 y}{dx^2} = f \left( x, y, \frac{dy}{dx} \right) \quad \text{and} \quad \frac{d^2 \bar{y}}{d\bar{x}^2} = \bar{f} \left( \bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}} \right). \tag{1}
\]

Let \( \Phi \) denote the pseudogroup of local diffeomorphisms \( \varphi : \mathbb{C}^2 \to \mathbb{C}^2 \) defined by the Lie equations

\[
\frac{\partial \bar{x}}{\partial x} = 1, \quad \frac{\partial \bar{x}}{\partial y} = 0, \quad \frac{\partial \bar{y}}{\partial y} \neq 0.
\]

This gives \((\bar{x}, \bar{y}) = \varphi(x, y) = (x+C, \eta(x,y))\) where \(C\) is a constant and \(\eta(x,y)\) is an arbitrary function.

### 2 Equivalence problems and differential elimination

We shall consider equivalence problems with fixed (determined) target equation \( \mathcal{E}_{\bar{f}} \). The question is to find the explicit conditions on \( f \) such that \( \mathcal{E}_f \sim_\Phi \mathcal{E}_{\bar{f}} \).

For instance, consider the equivalence problem of 2\(^{nd}\) order ODE presented in the previous section. Let \((x, y, p = y', q = y'')\) denote a local coordinates system of the jet space \( J^2(\mathbb{C}, \mathbb{C}) \). Thus, the problem reads

\[
\exists \varphi \in \Phi, \quad \varphi^*(\bar{q} - \bar{f}(\bar{x}, \bar{y}, \bar{p})) = 0 \quad \text{mod} \quad q - f(x, y, p) = 0. \tag{2}
\]

The prolongation formulae (Olver, 1993) of \( \varphi \) are

\[
\varphi^* \bar{p} = D_x \eta = \eta_x + \eta_y p, \quad \varphi^* \bar{q} = D_x^2 \eta = \eta_{xx} + 2\eta_{xy} p + \eta_{yy} p^2 + \eta_y q
\]

where \( D_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + f(x, y, p) \frac{\partial}{\partial p} \) is the total derivative. The following examples explain how one can use differential elimination to solve such question.

**Example 1** Let us suppose that the target \( \bar{f} \) is identically zero. The equations (2) take the form of a polynomial PDE’s system

\[
\eta_{xx} + 2\eta_{xy} p + \eta_{yy} p^2 + \eta_y f = 0, \quad \eta_p = 0, \quad \eta_y \neq 0. \tag{3}
\]
By eliminating \( \eta \) in (3) using the ranking \( \eta \succ f \), we obtain the characteristic set

\[
\eta_{xx} = -\eta_y f + p \eta_y f_p - \frac{1}{2} \eta_y f_{pp}, \quad \eta_{xy} = \frac{1}{2} \eta_y f_p + \frac{1}{2} \eta_y f_{pp}, \\
\eta_{yy} = \frac{1}{2} \eta_y f_{pp}, \quad \eta_p = 0, \\
f_{ppp} = 0, \quad f_{xp} = -f_{pp} f + \frac{1}{2} f_p^2 - pf_{yp}.
\]

(4)

It follows that the 2nd order differential equation \( y'' = f(x, y, y') \) is reduced to

\[
\bar{y}'' = 0
\]

by a transformation \( \varphi \) of the form \( \varphi(x, y) = (x + C, \eta(x, y)) \) if and only if

\[
f_{ppp} = 0, \quad f_{xp} = -f_{pp} f + \frac{1}{2} f_p^2 - pf_{yp}. \tag{5}
\]

To obtain the change of coordinates \( \varphi \) we have to integrate the PDE's system given by the first four equations of (4).

The next example shows that, in favorable cases, we can determine the change of coordinates without any integration.

**Example 2** Suppose that the target equation is \((P_1)\), the Painlevé first equation \( \bar{y}'' = 6\bar{y}^2 + \bar{x} \). The problem formulation is as above and in this case DiffAlg returns the following characteristic set

\[
\eta(x, y) = 1/12 f_y - 1/24 f_{x,p} - 1/24 f_{p,p} f + 1/48 f_p^2 - 1/24 p f_{y,p}, \\
C = -x + 1/16 p f_{y,y,p} f + 1/16 p^2 f_{p,p} f_{y,y,p} + 1/16 p f_{p,p} f_y \\
\vdots \quad \vdots \quad \vdots \\
f_{x,x,x,p} = -24 + 5/2 p f_{p,p} f_{y,p} - 4 f_x f_{x,y,p} + 2 f_{x,x,y} - 2 p f_{y,p} f_{f,x,p} \\
\vdots \\
f_{x,x,y,p} = -p f_{p,p} f_{y,y} + 3 p f_{y,y,p} f_y + f_{p,p} p^2 f_{y,y,p} - 3/2 p f_{p} f_{y,y,p} \\
\vdots \\
f_{x,y,y,p} = 2 f_{y,y} + f_y^2 - p f_{y,y,y,p} - f_{y,y,y,p} f - 2 f_{y,p} f_y - 1/2 f_{p,p} f_{y,y,p} + f_{p,y,y,p} - 1/2 f_{p,p} f_{x,y,p} - 1/2 f_{p,p} p f_{y,y,p} - 1/2 f_{p,p} f_y + 1/2 f_{p,p} f_{y,y,p}, \\
f_{x,p,p} = f_{y,p} - pf_{y,p,p}, \\
f_{p,p,p} = 0.
\]

The two first equations demonstrate that the change of variable \( \varphi \) is obtained without integrating differential equation and is unique. We shall see, in section 6, that this results from the fact that the symmetries pseudogroup of \((P_1)\)
is zerodimensional (in fact reduced to the identity). The other equations gives the requested conditions on $f$.

As the reader may have noticed, such explicit formulae consisting of several lines (pages) prove quite useless for practical application. In section 6, we shall see that the same formulae take a more compact form when they are written in term of the associated invariants.

More dramatically, the above brute-force method is rarely efficient due to expression swell involved by the use of commutative derivations. Indeed, the study of the equivalence of the 3rd order differential equation $y''' = f(x, y, y', y'')$ with $\bar{y}''' = 0$ under contact transformations $(\bar{x}, \bar{y}) = (\xi(x, y, y'), \eta(x, y, y'))$, requires the prolongation to $J^3 = (x, y, p = y', q = y'', r = y''')$, that is to find

\[
\bar{p} = \frac{D_x \eta}{D_x \xi},
\]

\[
\bar{q} = \frac{D_x^2 \eta D_x \xi - D_x \eta D_x^2 \xi}{D_x \xi^3},
\]

\[
\bar{r} = \frac{D_x^3 \eta D_x^2 \xi - 3D_x^2 \eta D_x^2 \xi D_x \xi + 3D_x \eta D_x^2 \xi^2 - D_x \eta D_x^3 \xi}{D_x \xi^5},
\]

where $D_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + q \frac{\partial}{\partial p} + f(x, y, p, q) \frac{\partial}{\partial q}$. The change of coordinates satisfies the PDE’s system $\{\bar{r} = 0, \ \frac{\partial \bar{y}}{\partial q} = 0, \ \xi_q = 0, \ \eta_q = 0\}$. Using commutative derivations, this system blows up and takes more than one hundreds of lines

\[
\bar{r} = 3p\eta_{xx}y\xi_x^3 - 12\eta_{xx}p^2\xi_{xy}y\xi_x - 6p^3\eta_y\xi_{xy}y\xi_x + 3p\eta_y\xi_{xx}^2\xi_x + \ldots
\]

100 lines of differential polynomials

and calculation (treatment by DIFFALG) do not finish!

The first optimization is to use the non commutative derivations $\{D_x, \frac{\partial}{\partial y}, \frac{\partial}{\partial p}, \frac{\partial}{\partial q}\}$ such in (Hubert, 2000) inspired by (Neut, 2003). A better alternative is to use the associated invariant derivations discussed later on.

3 The geometrical approach of the integrability conditions calculation

É. Cartan’s transforms an analytic PDE’s system into an equivalent linear Pfaffian system (with a condition that specifies the independent variables). He
gave a method to compute the integrability conditions of any analytic linear Pfaffian systems and therefore of any analytic PDE’s systems. This algorithm is based on the process of absorption of torsion which leads to sparse equations.

Recall that a Pfaffian system (with independence condition) on real analytic manifold \( M \) is a an exterior differential system of the form

\[
\begin{cases}
\omega^\alpha = 0, & (1 \leq \alpha \leq a) \\
\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \neq 0,
\end{cases}
\]

where \( \omega^\alpha \) and \( \theta^i \) are linearly independent differential 1-forms defined on \( M \) and \( a, n \in \mathbb{N} \). An integral manifold \( i : S \hookrightarrow M \) is a submanifold \( S \) of \( M \) such that \( i^\ast \omega^\alpha = 0 \) for all \( 1 \leq \alpha \leq a \) and \( i^\ast(\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n) \neq 0 \).

Let \([I]\) and \([J]\) denote the exterior differential ideals respectively generated by \( I = (\omega^\alpha) \) and \( J = (\omega^\alpha, \theta^i) \) for \( 1 \leq \alpha \leq a \) and \( 1 \leq i \leq n \).

**Definition 1** A Pfaffian system \( I \subset J \subset \Omega^1 M \) is linear if and only if \( dI = 0 \mod [J] \).

One obtains a local basis \( (\omega^\alpha, \theta^i, \pi^\rho) \) of \( \Omega^1 M \) by completing the basis \( (\omega^\alpha, \theta^i) \) of \( J \) by the 1-forms \( \pi^\rho \in \Omega^1 M \) \( (1 \leq \rho \leq r) \). If the Pfaffian system is linear then there exist analytic functions \( A^\alpha_{\rho i} \) and \( T^\alpha_{jk} \) defined on \( M \) such that

\[
d\omega^\alpha = A^\alpha_{\rho i} \pi^\rho \wedge \theta^i + \frac{1}{2} T^\alpha_{jk} \theta^j \wedge \theta^k \mod [I], \quad (1 \leq \alpha \leq a).
\]

**Proposition 1** Given two analytic manifolds \( X \) and \( U \). Every \( q \)-order PDE’s system \( \mathcal{E}^q \subset J^q(X, U) \) is equivalent to a linear Pfaffian systems defined on \( \mathcal{E}^q \).

**Proof 1** Suppose that \( x = (x^i)_{1 \leq i \leq n} : \text{dim} X \) are local coordinates of \( X \) and \( u = (u^\alpha)_{1 \leq \alpha \leq m} : \text{dim} U \) are local coordinates of \( U \), then \( (x^i, u^\gamma) \), where \( I = i_1 i_2 \cdots i_\ell \) and \( 0 \leq \ell \leq q \), constitutes a local coordinates of \( J^q \). The contact system of \( J^q \) is \( \{du^\gamma_i - u^\gamma_j dx^i = 0, \ \ell < q, \ 1 \leq \alpha \leq m\} \) with \( dx^1 \wedge \cdots \wedge dx^n \neq 0 \) and these equations continue to hold when we restrict to \( \mathcal{E}^q \).

### 3.1 Essential elements of torsion

Since \( i^\ast(\theta^1) \wedge i^\ast(\theta^2) \wedge \cdots \wedge i^\ast(\theta^n) \neq 0 \), the forms \( i^\ast(\pi^\rho) \) are linear combinations of the forms \( i^\ast(\theta^i) \), i.e. there are coefficients \( \lambda^\rho_i \) such \( i^\ast(\pi^\rho) = \lambda^\rho_i i^\ast(\theta^i) \). Substituting into (8) leads to

\[
\sum_{1 \leq j < k \leq n} i^\ast \left( T^\alpha_{jk} - A^\alpha_{\rho j} \lambda^\rho_i + A^\alpha_{\rho k} \lambda^\rho_i \right) i^\ast(\theta^j \wedge \theta^k) = 0, \quad (1 \leq \alpha \leq a).
\]
These conditions are equivalent to the system (by omitting the pullback \( i^* \))

\[
T_{j}^{\alpha} = A_{\rho j}^{\alpha} \lambda_{k}^{\rho} - A_{\rho k}^{\alpha} \lambda_{j}^{\rho}, \quad (1 \leq \alpha \leq a; 1 \leq j < k \leq n)
\]  

which is linear in the unknown coefficients \( \lambda_{i}^{\rho} \). By eliminating the coefficients \( \lambda_{i}^{\rho} \) using the standard Gaussian elimination, one obtains linear combinations of the functions \( T_{j}^{\alpha} \), called essential torsion elements, which inevitably vanish. Reciprocally, this vanishing ensures the existence of the \( \lambda_{i}^{\rho} \).

**Theorem 1** The essential torsion elements are real-valued functions defined on \( M \) and vanishing on any integral manifold \( S \subset M \) of the linear Pfaffian system \( I \subset J \subset \Omega^1 M \). In other words, they are the integrability conditions.

### 3.2 Absorption of torsion

Elie Cartan was used to calculate the integrability conditions provided by the preceding theorem using a process, called today absorption of torsion. In the structure equations (8), let us replace the \( \pi^{\rho} \) by the general linear combination \( \pi^{\rho} := \pi^{\rho} + \lambda_{i}^{\rho} \theta^{i} \). This yields

\[
\overline{T}_{j}^{\alpha} = A_{\rho j}^{\alpha}, \quad \overline{T}_{j}^{\alpha} = T_{j}^{\alpha} + A_{\rho j}^{\alpha} \lambda_{k}^{\rho} - A_{\rho k}^{\alpha} \lambda_{j}^{\rho}.
\]

The process of absorption of torsion consists in calculating the \( \lambda_{i}^{\rho} \) so that to fix the maximum of \( \overline{T}_{j}^{\alpha} \) to zero. After the absorption, the torsion elements remaining non zero form a basis of the essential torsion elements. The absorbed structure equations take now a simple (sparse) form which makes the calculations easier.

Before going to the complete algorithm (see Fig.1) let us sketch two basic concepts which are the involution and the prolongation (see Bryant et al. (1991) for rigorous exposition). By saying that the structure equations are in involution we simply mean that by any point of \( M \) there passes at least one ordinary integral manifold. In practice, involution is checked using Cartan characters.

What about prolongation? In the formalism of PDE’s systems, a prolongation consists in differentiating each equation w.r.t. each independent variable. Thus, one passes from a system \( \mathcal{E}^{q} \subset J^{q} \) to a new system \( \mathcal{E}^{q+1} \subset J^{q+1} \) having the same solutions. In the formalism of linear Pfaffian systems (and more generally, exterior differential systems) the manifold \( M \) is replaced by the grassmannian of the ordinary integral plans. The coordinates of \( M \) with the \( \lambda_{i}^{\rho} \), which remained arbitrary after the process of the absorption of torsion, constitute a system of local coordinates of this grassmannian.

Cartan-Kuranishi’ theorem (Kuranishi, 1957) guarantees that the algorithm
of the figure 1 stops. In other words, after a finite number of steps the system is either impossible or in involution.

4 Cartan’s method of equivalence

É. Cartan recasts the problem of local equivalence $\mathcal{E}_f \sim_\Phi \mathcal{E}_{\bar{f}}$ into the calculation of the integrability conditions of a linear Pfaffian system

$$\begin{align} 
\theta^i_f(\bar{a}, \bar{x}) &= \theta^i_f(a, x), \quad (1 \leq i \leq m) \\
\theta_1^f \wedge \cdots \wedge \theta^m_f &\neq 0, 
\end{align}$$

(11)

deﬁned on certain manifold $M$ with local coordinates $(a, x)$. The change of coordinates $\varphi \in \Phi$ is solution of this linear Pfaffian system where the two set of variables $(\bar{a}, \bar{x})$ and $(a, x)$ play a symmetrical role. In this setting, the integrability conditions appear under the symmetric form

$$I_f(\bar{a}, \bar{x}) = I_f(a, x)$$

(12)
The generic function $I_f$ is called \textit{fundamental invariant}. Every algorithm like \textsc{Diffalg} based on the notion of “ranking” breaks this symmetry. Cartan’s method, which is an application of the algorithm of the previous section, computes the integrability conditions (12) by \textit{separately} and \textit{symmetrically} treating the 1-forms $\theta_f$ and $\theta_{\bar{f}}$. This divides the number of variables by two.

To fix the ideas, let us return to the equivalence problem of 2\textsuperscript{nd} order ODE introduced in the first section. The differential equation $y'' = f(x, y, y')$ is equivalent to the Pfaffian system $(\Sigma_f)$

$$dp - f(x, y, p) \, dx = 0, \quad dy - p \, dx = 0 \quad \text{with} \quad dx \wedge dy \wedge dp \neq 0. \quad (13)$$

In the same way, the equation $\bar{y}'' = \bar{f}(\bar{x}, \bar{y}, \bar{y}')$ is equivalent to the Pfaffian system $(\Sigma_{\bar{f}})$. Now, since the first prolongation preserves the module of the contact forms, $(\Sigma_f)$ and $(\Sigma_{\bar{f}})$ are equivalent under $\varphi \in \Phi$ if and only if there exist functions $a = (a_1, a_2, a_3)$ such that

$$\begin{pmatrix}
\frac{d \bar{p} - \bar{f}(\bar{x}, \bar{y}, \bar{p}) \, d\bar{x}}{\omega_{\bar{f}}}
\frac{d \bar{y} - \bar{p} \, d\bar{x}}{d\bar{x}}
\frac{d \bar{\bar{x}}}{d\bar{x}}
\end{pmatrix} = \begin{pmatrix}
\begin{bmatrix}
a_1 & a_2 & 0 \\
0 & a_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
dp - f(x, y, p) \, dx \\
dy - p \, dx
\end{pmatrix}
\end{pmatrix}.\quad (13)$$

The matrices $g(a)$ form a matrix Lie group, called the \textit{structural} group. Now, multiplying the two sides by $g(a)$ gives

$$\begin{pmatrix}
\begin{bmatrix}
a_1 & a_2 & 0 \\
0 & a_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
dp - f(x, y, p) \, dx \\
dy - p \, dx
\end{pmatrix}
\end{pmatrix}.$$
involving the fundamental invariants
\[ I_1(a, x) = -\frac{1}{4}(f_p)^2 - f_y + \frac{1}{2}D_x f_p, \quad I_2(a, x) = \frac{f_{ppp}}{2a_3^2}, \quad I_3(a, x) = \frac{f_{yp} - D_x f_{pp}}{2a_3}. \]

The final invariant 1-forms are
\[ \theta^1 = a_3 \left( \frac{1}{2} f_p d_p - f \right) dx - \frac{1}{2} f_p dy + dp, \quad \theta^2 = a_3 \left( dy - p dx \right), \]
\[ \theta^3 = dx, \quad \theta^4 = \left( \frac{1}{2} f_p - \frac{1}{2} f_{pp} p \right) dx + \frac{1}{2} f_{pp} dy + \frac{1}{a_3} dp. \]

Dual to these forms are the invariant derivations
\[ X_1 = \frac{1}{a_3} \partial / \partial p, \quad X_2 = \frac{1}{a_3} \partial / \partial y + \frac{1}{2} a_3 \frac{\partial}{\partial p} - \frac{1}{2} f_{pp} \frac{\partial}{\partial a_3}, \]
\[ X_3 = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + f \frac{\partial}{\partial p} - \frac{1}{2} f_{pp} a_3 \frac{\partial}{\partial a_3}, \quad X_4 = a_3 \frac{\partial}{\partial a_3}. \]

Thus, the differential of any function \( H(x, y, p, a_3) \) can re-expressed as \( dH = \sum_{i=1}^{4} X_i(H) \theta^i \).

5 Differential relations between the fundamental invariants

The algebra of invariants associated to a given equivalence problem is a differential algebra generated by the fundamental invariants and closed under the invariant derivations. These derivations, which generally do not commute, allow us to compute a complete system of invariants from fundamental invariants. One obtains most of the syzygies i.e. the differential relations between the fundamental invariants using Poincaré lemma \( d^2 = 0 \) where \( d \) denotes the exterior derivation. This low cost computation does not require the expression of the invariants in local coordinates or any elimination which is particularly useful since the invariants can be very big (1.1 Mo in the case of ODE’s and PDE’s systems below).

For the problem (1), the relations provided by Poincaré lemma are
\[ X_1 I_1 = -I_3, \quad X_4 I_1 = 0, \quad X_4 I_2 = -2I_2, \quad X_1 I_3 = -X_3 I_2, \quad X_4 I_3 = -I_3. \]

In the particular case when \( \bar{f} = 0 \), all invariants vanish and thus, according to (12), the corresponding invariants of \( y'' = f(x, y, p) \) must vanish too. Now, since \( I_1 \) and \( I_2 \) form a basis of the differential ideal generated by the three fundamental invariants, one finds the same conditions (3).
5.1 Second order ODE's systems

Given the following ODE's system \((S_F)\):

\[
\begin{align*}
\ddot{x}^1 &= F^1(t, x, \dot{x}), \\
\ddot{x}^2 &= F^2(t, x, \dot{x}),
\end{align*}
\]

where \(\dot{x} = (\dot{x}^1, \dot{x}^2)\) denotes the derivative of \(x^1\) and \(x^2\) according to \(t\). Two systems \((S_F)\) and \((S')\) are said to be equivalent under a point transformation if and only if there exist functions \(a_1, \ldots, a_{15}\) on \(\mathbb{C}^3\) in \(\mathbb{C}\) such that

\[
\begin{pmatrix}
\dot{x}^1 - F^1(t, x, \dot{x})dt \\
\dot{x}^2 - F^2(t, x, \dot{x})dt \\
\dot{x}^1 - \dot{x}^1 dt \\
\dot{x}^2 - \dot{x}^2 dt \\
dt
\end{pmatrix} =
\begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & 0 \\
a_5 & a_6 & a_7 & a_8 & 0 \\
0 & 0 & a_9 & a_{10} & 0 \\
0 & 0 & a_{11} & a_{12} & 0 \\
0 & 0 & a_{13} & a_{14} & a_{15}
\end{pmatrix}
\begin{pmatrix}
\dot{x}^1 - F^1(t, x, \dot{x})dt \\
\dot{x}^2 - F^2(t, x, \dot{x})dt \\
x^1 - \dot{x}^1 dt \\
x^2 - \dot{x}^2 dt \\
dt
\end{pmatrix}
\]

When applied, Cartan’s method yields 88 fundamental invariants. Without any need to the explicit expressions of the 88 invariants (over 1 M bytes of memory) Poincaré lemma shows that two invariants form a basis the differential ideal generated by the 88 invariants. If the functions \(F^1\) and \(F^2\) are identically zero, this two invariants vanish.

**Theorem 2 (Neut (2003))** The system \((S_F)\) is equivalent to the system \(\{\ddot{x}^1 = 0, \ddot{x}^2 = 0\}\) under a point transformations if and only if

\[
\begin{align*}
F^2_{x^1_x^1 x^2} &= 0, & F^1_{x^2 x^2 x^2} &= 0, & F^2_{x^2 x^2 x^2} &= 0, & F^1_{x^1 x^1 x^1} - 3F^2_{x^1 x^1 x^2} &= 0, & F^1_{x^1 x^1 x^2} - 3F^2_{x^1 x^1 x^2} &= 0, \\
F^1_{x^1 x^1 x^2} - F^2_{x^1 x^1 x^2} &= 0, & 2D_t F^1_{x^1} - F^1_{x^1} - F^2_{x^1} - 4F^1_{x^2} &= 0, & (F^2_{x^2})^2 - 2D_t F^1_{x^2} - 4F^2_{x^2} + 4F^1_{x^2} + 2D_t F^2_{x^2} + (F^1_{x^2})^2 &= 0, & -2D_t F^2_{x^2} + F^2_{x^2} F^2_{x^1} + F^1_{x^2} F^2_{x^1} &= 0
\end{align*}
\]

where \(D_t = \frac{\partial}{\partial t} + \dot{x}^1 \frac{\partial}{\partial x^1} + \dot{x}^2 \frac{\partial}{\partial x^2} + F^1 \frac{\partial}{\partial x^1} + F^2 \frac{\partial}{\partial x^2}\).

5.2 PDE's systems

Given a system of holomorphic PDE with two independent variables \((x^1, x^2) \in \mathbb{C}^2\) and one dependent variable \(u \in \mathbb{C}\)

\[
(S_f) : \, \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} = f_{\alpha \beta} \left( x, u, \frac{\partial u}{\partial x} \right), \quad f_{\alpha \beta} = f_{\beta \alpha} \text{ for } \alpha, \beta = 1 \ldots 2,
\]

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If we use the notation \( u^\prime := (u_\alpha)_{1 \leq \alpha \leq 2} \) and \( u^\prime\prime := (u_{\alpha\beta})_{1 \leq \alpha \leq \beta \leq 2} \) then \((S_f)\) reads \( u^\prime\prime = f(x, u, u^\prime) \). When \( f \equiv 0 \) the system is denoted by \((S_0)\). \((S_f)\) and \((S_\bar{f})\) are said to be locally equivalent under a bi-holomorphic transformation if and only if

\[
\begin{pmatrix}
\frac{dx}{dx} & \frac{d\bar{u}}{d\bar{x}} - \bar{f}d\bar{x} \\
\frac{d\bar{x}}{dx}
\end{pmatrix} = \begin{pmatrix}
a & 0 & 0 \\
A & M & 0 \\
B & 0 & N
\end{pmatrix}
\begin{pmatrix}
\frac{du}{dx} - u^\prime d\bar{x} \\
\frac{du'}{dx}
\end{pmatrix}
\]

where \( a \in \mathbb{C}^* \), \( A, B \in \mathbb{C}^2 \), \( M, N \in \text{GL}(2, \mathbb{C}) \). By applying Cartan’s method in S. S. Chern (1975) way (see also (Fels, 1995)), one obtains 15 structure equations involving 8 big invariants. For the system \((S_0)\), these invariants vanish.

**Theorem 3** The following propositions are equivalent

(i) The system \((S_f)\) is equivalent to the system \((S_0)\) under bi-holomorphic transformations.

(ii) The system \((S_f)\) admits a 15-dimensional point symmetries Lie group.

(iii) The functions \( f_{\alpha\beta} \) for \( \alpha, \beta = 1 \ldots 2 \) satisfy

\[
\frac{\partial^2 f_{11}}{\partial u_2 \partial u_2} = 0, \quad \frac{\partial^2 f_{22}}{\partial u_1 \partial u_1} = 0, \quad \frac{\partial^2 f_{12}}{\partial u_2 \partial u_1} \quad \frac{\partial^2 f_{11}}{\partial u_1 \partial u_2} = 0, \\
\frac{\partial^2 f_{22}}{\partial u_1 \partial u_2} = 0, \quad \frac{\partial^2 f_{11}}{\partial u_1 \partial u_1} - 4 \frac{\partial^2 f_{12}}{\partial u_1 \partial u_2} + \frac{\partial^2 f_{22}}{\partial u_2 \partial u_2} = 0.
\]

6 Change of coordinates calculation

One obtains the transformation \( \varphi \) without integrating any differential equation when the symmetry pseudogroup \( \mathcal{S}_f \subset \Phi \) of the target equation \( \mathcal{E}_f \) is zero-dimensional. Indeed, if the function \( \varphi \) is the general solution of a differential system (of non zero order) then it depends on, at least, one arbitrary constant and thus (the figure below) the symmetry pseudogroup \( \mathcal{S}_f \) is not zero-dimensional.

\[
\begin{array}{ccc}
(x, \mathcal{E}_f) & \xrightarrow{\varphi} & (\bar{x}, \mathcal{E}_f) \\
\downarrow \varphi & & \downarrow \sigma \in \mathcal{S}_f \\
(\bar{x}, \mathcal{E}_{\bar{f}}) & & (x, \mathcal{E}_f)
\end{array}
\]

**Example 3** Let us go back to the equivalence with the first Painlevé equation \((P_1)\) under transformations \( \varphi(x, y) = (\bar{x}, \bar{y}) = (x + C, \eta(x, y)) \). We refer the reader to the end of section 4 for the expressions of the fundamental invariants and the invariant derivations.
The specialization of these invariants on the Painlevé equation gives
\[ \bar{I}_1 = 12\bar{y}, \quad \bar{I}_2 = \bar{I}_3 = 0. \]

According to the equality of the invariants (12), we deduce that \( I_2 = I_3 = 0 \) which give the two last equations of (6). Also, we have
\[ 12\bar{y} = \bar{I}_1 = I_1 = -\frac{1}{4} f_p^2 - f_y + \frac{1}{2} D_x f_p \]
and this gives \( \eta(x,y) \), that is the first part of \( \varphi \). To find \( C = \bar{x} - x \), we have
\[ \bar{X}_3^2 \bar{I}_1 = -72\bar{y}^2 - 12\bar{x} \]
and according again to the equality of the invariants we obtain
\[ C = \bar{x} - x = -\frac{1}{24} I_1^2 - \frac{1}{12} X_3^2 I_1 - x. \]

As byproduct, the conditions (6) on the function \( f \) can be obtained by expressing that \( C \) is constant i.e. \( X_i(C) = 0 \) for \( 0 \leq i \leq 4 \)
\[ X_1 X_3^2 I_1 = 0, \quad I_1 X_2 I_1 + X_2 X_3^2 I_1 = 0, \quad I_1 X_3 I_1 + X_3^3 I_1 - 1 = 0 \]

The voluminous formulae (6) take now a more compact form, expressed in terms of differential invariants.

7 Conclusion

In this paper we demonstrated the superiority of Cartan’s method over direct methods based on differential elimination for handling equivalence problems. Indeed, we have seen that the use of the invariant derivations and the coding of the expressions in terms of invariants significantly reduce the size of these expressions. Moreover, in Cartan’s method, the frame is dynamically adapted (during the computation) using the absorption of torsion process. This leads to sparse structure equations and makes calculations easier. In addition, this computation is done by separately and symmetrically treating the considered linear Pfaffian system. This divides the number of variables by two. Also, we have seen that almost of the syzygies between the fundamental invariants are obtained using Poincaré lemma without any need of the expression of these invariants in local coordinates (which can take 1 Mo of memory). We have gave new equivalence results, using our software which is available at \texttt{www.lifl.fr/~neut/logiciels}. 13
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