A MODEL CATEGORY STRUCTURE ON THE CATEGORY OF SIMPLICIAL MULTICATEGORIES

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Abstract. We establish a Quillen model structure on simplicial (symmetric) multicategories. It extends the model structure on simplicial categories due to J. Bergner [3]. We observe that our technique of proof enables us to prove a similar result for (symmetric) multicategories enriched over other monoidal model categories than simplicial sets.

1. Introduction

A multicategory can be thought of as a generalisation of the notion of category, to the amount that an arrow is allowed to have a source (or input) consisting of a (possibly empty) string of objects, whereas the target (or output) remains a single object. Composition of arrows is performed by inserting the output of an arrow into (one of) the input(s) of the other. Then a multifunctor is a structure preserving map between multicategories. For example, every multicategory has an underlying category obtained by considering only those arrows with source consisting of strings of length one (or, one input). At the same time, a multicategory can be thought of as an "operad with many objects". The relationship between all of these structures can be displayed in a diagram

\[ \text{Monoids} \xrightarrow{\text{Operads}} \xrightarrow{\text{Categories}} \xrightarrow{\text{Multicategories}} \]

in which the two composites agree and each horizontal (resp. vertical) inclusion is part of a coreflection (resp. reflection). (The above diagram can be extended downwards and to the right with similar diagrams.)

By allowing the symmetric groups to act on the various strings of objects of a multicategory, and consequently requiring that composition of arrows be compatible with these actions in a certain natural way, one obtains the concept of symmetric multicategory. We refer the reader to [17], [8] and [2] for the precise definitions, history and examples.

As there is a notion of category enriched over a symmetric monoidal category other than the category of sets, the same happens with multicategories. In this paper we mainly consider symmetric multicategories enriched over simplicial sets, simply called simplicial multicategories. Similarly, categories enriched over simplicial sets will be called simplicial categories. We shall assume that all simplicial (multi)categories in sight are small, that is, they have a set of objects rather than a class.

We are interested in doing homotopy theory in the category of simplicial multicategories and simplicial multifunctors between them. A first step is to put a (Quillen) model category structure on the aforementioned category; this is the central subject of this paper. In the first place one has to decide which class of arrows to invert, in other words one has to choose a class of weak equivalences. It turns
out that the right notion of weak equivalence of simplicial multicategories is what we call here \textit{multi DK-equivalence}. "DK" is for Dwyer and Kan. This notion has been previously defined in ([8], Def. 12.1), and is the obvious extension of the notion of Dwyer-Kan equivalence of simplicial categories [3], [6]. We recall below the definition.

Every simplicial multicategory has an underlying simplicial category, and this association is functorial. To every simplicial category \( C \) one can associate a genuine category \( \pi_0 C \), the category of connected components of \( C \). The objects of \( \pi_0 C \) are the objects of \( C \) and the hom set \( \pi_0 C(x, y) \) is \( \pi_0(C(x, y)) \). Now, a simplicial multifunctor \( f : M \to N \) is a multi DK-equivalence if \( \pi_0 f \) is essentially surjective and for every \( k \geq 0 \) and every \((k+1)\)-tuple \((a_1, \ldots, a_k; b)\) of objects of \( M \), the map \( M_k(a_1, \ldots, a_k; b) \to N_k(f(a_1), \ldots, f(a_k); f(b)) \) is a weak homotopy equivalence. Our first main result is

\textbf{Theorem.} (Theorem 4.5) The category of simplicial multicategories admits a Quillen model category structure with multi DK-equivalences as weak equivalences and fibrations defined in 4.3.2.

We call this model structure the \textit{Dwyer-Kan model structure} on simplicial multicategories. To prove this theorem we use the similar model structure on simplicial categories due to J. Bergner [3] together with a modification of some parts of Bergner’s original argument. In [19], I. Moerdijk conjectures that the homotopy coherent dendroidal nerve functor from simplicial multicategories to the category of dendroidal sets [20] is the right adjoint of a Quillen equivalence. This would extend A. Joyal’s result (unpublished) on the Quillen equivalence between the categories of simplicial categories and simplicial sets, where the latter category has the model structure for quasi-categories.

The technique used to prove the above theorem applies to other categories than simplicial sets as well. Thus, our second main result (Theorem 5.2) establishes the analogous Dwyer-Kan model structure on the category of small symmetric multicategories enriched over certain monoidal model categories \( \mathcal{V} \). Examples include small categories, simplicial abelian groups and compactly generated Hausdorff spaces. In the case of small categories, the result extends the work of S. Lack [15],[16]. The folklore case of symmetric multicategories can be recovered too. As expected, the Dwyer-Kan model structure on simplicial categories and on small symmetric multicategories enriched over compactly generated Hausdorff spaces will be Quillen equivalent. The technique can also be used to establish the analogous Dwyer-Kan model structure on the category of small categories enriched over certain monoidal model categories \( \mathcal{V} \); this will not be discussed here.

The paper is organised as follows. In sections 2 and 3 we review the notions and results from enriched (multi)category theory that we use. We work in full generality in both sections, in the sense that our (multi)categories are enriched over an arbitrary closed symmetric monoidal category. Section 4 contains the proof of the above theorem. In section 5 we prove the existence of a Dwyer-Kan model structure on symmetric multicategories enriched over other categories than simplicial sets, cf. above. We also prove that, under some conditions, "left Bousfield localization commutes with (multi)enrichment" (5.6). The idea is not new. Section 6 recalls two results from the general theory of model categories which we use to prove the main results. Section 7 generalises to enriched categories a result of R. Fritsch and D. M. Latch [9] about certain pushouts of small categories.
2. Review of $\mathcal{V}$-graphs and $\mathcal{V}$-categories

2.1. Let $\mathcal{V}$ be a complete and cocomplete closed symmetric monoidal category with initial object $\emptyset$ and unit $I$. The small $\mathcal{V}$-categories together with the $\mathcal{V}$-functors between them form a category written $\mathcal{V} \text{Cat}$. It is a closed symmetric monoidal category with unit the $\mathcal{V}$-category $I$ with a single object $*$ and $I(*,*) = I$. We denote by $\mathcal{V} \text{Graph}$ the category of small $\mathcal{V}$-graphs. A $\mathcal{V}$-graph is a $\mathcal{V}$-category without composition and unit maps. We denote by $\text{Ob}$ the functor sending a $\mathcal{V}$-category (or a $\mathcal{V}$-graph) to its set of objects. The functor $\text{Ob}$ is a Grothendieck bifibration. There is a free-forgetful adjunction

$$\mathcal{F} : \mathcal{V} \text{Graph} \rightleftharpoons \mathcal{V} \text{Cat} : \mathcal{U} \quad (1)$$

We write $\mathcal{V} \text{Graph}(S)$ (resp. $\mathcal{V} \text{Cat}(S)$) for the fibre category over a set $S$. The category $\mathcal{V} \text{Graph}(S)$ is a (nonsymmetric) monoidal category with monoidal product

$$X \square_S Y(a,b) = \prod_{c \in S} X(a,c) \otimes Y(c,b)$$

and unit

$$I_S(a,b) = \begin{cases} I, & \text{if } a = b \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\mathcal{V} \text{Cat}(S)$ is precisely the category of monoids in $\mathcal{V} \text{Graph}(S)$ with respect to $\square_S$.

2.2. Let $\mathcal{M}$ be a class of maps of $\mathcal{V}$. We recall that a $\mathcal{V}$-functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is locally in $\mathcal{M}$ if for each pair $x, y \in \mathcal{A}$ of objects, the map $f_{x,y} : \mathcal{A}(x,y) \rightarrow \mathcal{B}(f(x), f(y))$ is in $\mathcal{M}$. This definition makes obviously sense for morphisms of $\mathcal{V}$-graphs. When $\mathcal{M}$ is the class of isomorphisms of $\mathcal{V}$, a $\mathcal{V}$-functor which is locally an isomorphism is called full and faithful.

2.3. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a $\mathcal{V}$-functor and let $u = \text{Ob}(f)$. Then $f$ factors as $\mathcal{A} \xrightarrow{f^u} u^* \mathcal{B} \rightarrow \mathcal{B}$, where $f^u$ is a map in $\mathcal{V} \text{Cat}(\text{Ob}(\mathcal{A}))$. One has $u^* \mathcal{B}(a,a') = \mathcal{B}(f(a), f(a'))$ and $u^* \mathcal{B} \rightarrow \mathcal{B}$ is full and faithful.

2.4. For an object $A$ of $\mathcal{V}$, we denote by $(2, A)$ the $\mathcal{V}$-graph with two objects $0$ and $1$ and with $(2, A)(0,0) = (2, A)(1,1) = (2, A)(1,0) = \emptyset$ and $(2, A)(0,1) = A$.

2.5. Let $\mathcal{C}$ be a small category. We endow $\mathcal{V}^\mathcal{C}$ with the pointwise monoidal product. We have a full and faithful functor

$$\varphi : \mathcal{V}^\mathcal{C} \text{Cat} \longrightarrow (\mathcal{V} \text{Cat})^\mathcal{C},$$

given by $\text{Ob}(\varphi(A)(j)) = \text{Ob}(A)$ for all $j \in \mathcal{C}$ and $\varphi(A)(j)(a,b) = A(a,b)(j)$ for all $a,b \in \text{Ob}(A)$. The category $\mathcal{V}^\mathcal{C} \text{Cat}$ is coreflective in $(\mathcal{V} \text{Cat})^\mathcal{C}$, that is, the functor $\varphi$ has a right adjoint $G$, defined as follows. For $B \in (\mathcal{V} \text{Cat})^\mathcal{C}$ we put $\text{Ob}(G(B)) = \lim \text{Ob}(B(i))$ and $G(B)((a_i), (b_i))(j) = B(j)(a_j, b_j)$.

3. Review of symmetric $\mathcal{V}$-multigraphs and symmetric $\mathcal{V}$-multicategories

3.1. Let $\mathcal{V}$ be a complete and cocomplete closed symmetric monoidal category. For the notions of symmetric $\mathcal{V}$-multicategory and symmetric $\mathcal{V}$-multifunctor
we refer the reader to ([17], 2.2.21) and ([8], 2.1, 2.2). If \( M \) is a symmetric \( \mathcal{V} \)-multicategory, \( k \geq 0 \) is an integer and \( (a_1, \ldots, a_k; b) \) is a \((k + 1)\)-tuple of objects, we follow ([8], 2.1(2)) and denote by \( M_k(a_1, \ldots, a_k; b) \) the \( \mathcal{V} \)-object of "\( k \)-morphisms". When \( k = 0 \), the \( \mathcal{V} \)-object of 0-morphisms is denoted by \( M(\, ; b) \).

The small symmetric \( \mathcal{V} \)-multicategories together with the symmetric \( \mathcal{V} \)-multifunctors between them form a category written \( \mathcal{V} \text{SymMulticat} \). It is a symmetric monoidal category with tensor product defined pointwise. Precisely, if \( M, N \in \mathcal{V} \text{SymMulticat} \) then \( M \otimes N \) has \( \text{Ob}(M) \times \text{Ob}(N) \) as set of objects and

\[
(M \otimes N)_k((a_1, a'_1), \ldots, (a_k, a'_k); (b, b')) = M_k(a_1, \ldots, a_k; b) \otimes N_k(a'_1, \ldots, a'_k; b').
\]

The unit \( \text{Com} \) of this tensor product has a single object \(* \) and \( \text{Com}_k(*, \ldots, *; *) = I \).

When \( \mathcal{V} \) is the category \( \text{Set} \) of sets, symmetric \( \text{Set} \)-multicategories will be simply referred to as \( \text{multicategories} \), and the category will be denoted by \( \text{SymMulticat} \).

A symmetric \( \mathcal{V} \)-multigraph is by definition a symmetric \( \mathcal{V} \)-multicategory without composition and unit maps. We shall write \( \mathcal{V} \text{SymMultigraph} \) for the category of symmetric \( \mathcal{V} \)-multigraphs with the evident notion of arrow. When \( \mathcal{V} = \text{Set} \), the category is denoted by \( \text{SymMultigraph} \).

We denote by \( \text{Ob} \) the functor sending a symmetric \( \mathcal{V} \)-multicategory (or a symmetric \( \mathcal{V} \)-multigraph) to its set of objects. The functor \( \text{Ob} \) is a Grothendieck bifibration.

There is a free-forgetful adjunction

\[
\mathcal{F}_\text{Multi} : \mathcal{V} \text{SymMultigraph} \rightleftarrows \mathcal{V} \text{SymMulticat} : \mathcal{U}_\text{Multi} \quad (2)
\]

We write \( \mathcal{V} \text{SymMultigraph}(S) \) (resp. \( \mathcal{V} \text{SymMulticat}(S) \)) for the fibre category over a set \( S \). The category \( \mathcal{V} \text{SymMultigraph}(S) \) admits a (nonsymmetric) monoidal product which preserves filtered colimits in each variable. The category \( \mathcal{V} \text{SymMulticat}(S) \) is precisely the category of monoids in \( \mathcal{V} \text{SymMultigraph}(S) \) with respect to this monoidal product. From the general theory of limits and colimits in Grothendieck bifibrations it follows that \( \mathcal{V} \text{SymMulticat} \) is complete and cocomplete.

If \( \mathcal{V} \) is moreover accessible, it follows from the general theory ([18], Thm. 5.3.4) that \( \mathcal{V} \text{SymMulticat} \) is accessible.

\[3.2. \text{\( \mathcal{V} \)-categories and symmetric \( \mathcal{V} \)-multicategories can be related by the (fibred) adjunction}
\[
E : \mathcal{V} \text{Cat} \rightleftarrows \mathcal{V} \text{SymMulticat} : (\,)_1 \quad (3)
\]

where

\[
(EA)_n(a_1, \ldots, a_n; b) = \begin{cases} A(a_1, b) & \text{if } n = 1, \\ \emptyset & \text{otherwise}, \end{cases}
\]

and \( M_1(a, b) = M_1(a; b) \). The functor \( E \) is full and faithful.

\[3.3. \text{Let } \mathcal{M} \text{ be a class of maps of } \mathcal{V}. \text{ We say that a symmetric } \mathcal{V} \text{-multifunctor } f : M \to N \text{ is locally in } \mathcal{M} \text{ if for each integer } k \geq 0 \text{ and each } (k + 1)\text{-tuple of objects } (a_1, \ldots, a_k; b), \text{ the map } f : M_k(a_1, \ldots, a_k; b) \to N_k(f(a_1), \ldots, f(a_k); f(b)) \text{ is in } \mathcal{M}. \text{ When } \mathcal{M} \text{ is the class of isomorphisms of } \mathcal{V}, \text{ a symmetric } \mathcal{V} \text{-multifunctor which}
\]
is locally an isomorphism is called full and faithful.

3.4. We recall that a \(\mathcal{V}\)-multigraph \(M\) consists of a set of objects \(\text{Ob}(M)\) together with an object \(M_k(a_1, \ldots, a_k; b)\) of \(\mathcal{V}\) assigned to each integer \(k \geq 0\) and each \((k+1)\)-tuple of objects \((a_1, \ldots, a_k; b)\). We write \(\mathcal{V}\text{Multigraph}\) for the resulting category. In the case when \(\mathcal{V} = \text{Set}\), this category is denoted by \(\text{Multigraph}\) and its objects will be called multigraphs.

The forgetful functor from symmetric \(\mathcal{V}\)-multigraphs to \(\mathcal{V}\)-multigraphs has a left adjoint \(\text{Sym}\) defined by

\[
(S\text{ym}\mathcal{M})_k(a_1, \ldots, a_k; b) = \prod_{\sigma \in \Sigma_k} M_k(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(k)}; b),
\]

where \(\Sigma_k\) is the symmetric group on \(k\) elements.

3.5. For each integer \(k \geq 0\) we denote by \(k+1\) the set \(\{1, 2, \ldots, k, *\}\), where \(* \notin \{1, 2, \ldots, k\}\). We have a functor

\[
(k+1) : \mathcal{V} \to \mathcal{V}\text{Multigraph}
\]

given by \((k+1, A)_n(a_1, \ldots, a_n; b) = \emptyset\) unless \(n = k\) and \(a_i = i\) and \(b = *\), in which case we define it to be \(A\). To give a map of \(\mathcal{V}\)-multigraphs \((k+1, A) \to M\) is to give a map \(A \to M_k(a_1, \ldots, a_k; b)\).

3.6. Let \(\mathcal{C}\) be a small category. We endow \(\mathcal{V}^\mathcal{C}\) with the pointwise monoidal product. We have a full and faithful functor

\[
\varphi' : \mathcal{V}^\mathcal{C}\text{SymMulticat} \longrightarrow (\mathcal{V}\text{SymMulticat})^\mathcal{C},
\]

given by \(\text{Ob}(\varphi'(M)(j)) = \text{Ob}(M)\) for all \(j \in \mathcal{C}\) and \(\varphi'(M)(j)_k(a_1, \ldots, a_k; b) = M_k(a_1, \ldots, a_k; b)(j)\) for each \((k+1)\)-tuple of objects \((a_1, \ldots, a_k; b)\). The category \(\mathcal{V}^\mathcal{C}\text{SymMulticat}\) is coreflective in \((\mathcal{V}\text{SymMulticat})^\mathcal{C}\). The right adjoint to \(\varphi'\), denoted by \(G'\), can be defined as follows. For \(M \in (\mathcal{V}\text{SymMulticat})^\mathcal{C}\) we put \(\text{Ob}(G'(M)) = \text{lim\text{Ob}(M(i)})\) and

\[
G'(M)_k((a_i^1), \ldots, (a_i^k); (b_j))(j) = M(j)_k(a_i^1, \ldots, a_i^k; b_j).
\]

We have a commutative square of adjunctions

\[
\begin{array}{ccc}
\mathcal{V}^\mathcal{C}\text{Cat} & \xrightarrow{E} & (\mathcal{V}\text{SymMulticat})^\mathcal{C} \\
\downarrow G & & \downarrow G' \\
(\mathcal{V}\text{Cat})^\mathcal{C} & \xleftarrow{\varphi} & (\mathcal{V}\text{SymMulticat})^\mathcal{C}.
\end{array}
\]

4. THE DWYER-KAN MODEL STRUCTURE ON \(\text{SSymMulticat}\)

In this section we prove the theorem stated in the introduction. We first recall the analogous (Dwyer-Kan) model structure on simplicial categories.

We denote by \(\text{Cat}\) the category of small categories. It has a natural model structure in which a cofibration is a functor monic on objects, a weak equivalence is an equivalence of categories and a fibration is an isofibration [12]. We recall that a functor \(F : \mathcal{C} \to \mathcal{D}\) between small categories is an isofibration if for any object \(x\) of \(\mathcal{C}\), and any isomorphism \(v : F(x) \to y'\) in \(\mathcal{D}\), there exists an isomorphism \(u : x \to x'\) in \(\mathcal{C}\) such that \(F(u) = v\). The trivial fibrations are the equivalences surjective on objects.
Let \( S \) be the category of simplicial sets, regarded as having the classical model structure. Let \( \pi_0 : S \to \text{Set} \) be the set of connected components functor. By change of base it induces a functor \( \pi_0 : \text{SCat} \to \text{Cat} \) which is the identity on objects.

**Definition 4.1.** \([3]\) Let \( f : A \to B \) be a morphism in \( \text{SCat} \).

1. The morphism \( f \) is homotopy essentially surjective if the induced functor \( \pi_0 f : \pi_0 A \to \pi_0 B \) is essentially surjective.
2. The morphism \( f \) is a DK-equivalence if \( f \) is homotopy essentially surjective and locally a weak homotopy equivalence.
3. The morphism \( f \) is a DK-fibration if \( f \) is locally a Kan fibration and \( \pi_0 f \) is an isofibration.
4. The morphism \( f \) is a trivial fibration if \( f \) is a DK-equivalence and a DK-fibration.

A morphism is a trivial fibration iff it is surjective on objects and locally a trivial fibration.

**Theorem 4.2.** \([3]\) The category \( \text{SCat} \) of simplicial categories admits a cofibrantly generated model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. A generating set of trivial cofibrations consists of

(B1) \( \{ F(2, X) \xrightarrow{F(2,j)} F(2, Y) \} \), where \( j \) is a horn inclusion, and

(B2) inclusions \( I \xrightarrow{\delta_y} H \), where \( \{ H \} \) is a set of representatives for the isomorphism classes of simplicial categories on two objects which have countably many simplices in each function complex. Furthermore, each such \( H \) is required to be cofibrant and weakly contractible in \( \text{SCat}(\{x, y\}) \). Here \( \{x, y\} \) is the set with elements \( x \) and \( y \) and \( \delta_y \) omits \( y \).

**Definition 4.3.** Let \( f : M \to N \) be a morphism in \( \text{SSymMulticat} \).

1. The morphism \( f \) is a multi DK-equivalence if \( f \) is locally a weak homotopy equivalence and \( f_1 \) is homotopy essentially surjective.
2. The morphism \( f \) is a multi DK-fibration if \( f \) is locally a Kan fibration and \( \pi_0 f_1 \) is an isofibration.
3. The morphism \( f \) is a trivial fibration if \( f \) is a multi DK-equivalence and a multi DK-fibration.

**Remark.** A morphism is a trivial fibration iff it is surjective on objects and locally a trivial fibration. The class of simplicial multifunctors having the left lifting property with respect to the trivial fibrations, let us call them cofibrations, can be given an explicit description. To begin with, let \( f : M \to N \) be a simplicial multifunctor and let \( u = \text{Ob}(f) \). The map \( f \) factors as \( M \to u_1 M \xrightarrow{f_1} N \), where \( f_1 \) is a map in \( \text{SSymMulticat}(\text{Ob}(N)) \). Then \( f \) is a cofibration iff \( u \) is injective and \( f_1 \) is a cofibration in \( \text{SSymMulticat}(\text{Ob}(N)) \), cf. \([23]\). We recall that since \( u \) is injective, the object \( u_1 M \) is easily described. One has

\[
u_1 M(x_1, \ldots, x_k; y) = \begin{cases} M(a_1, \ldots, a_k; b), & \text{if } x_i = u(a_i), y = u(b) \\ I, & \text{if } k = 1, x_1 = y \notin \text{Im}(u) \\ \emptyset, & \text{otherwise} \end{cases}\]

**Lemma 4.4.** The functor \( E(3.2(3)) \) sends DK-equivalences to multi DK-equivalences. The functor \( (\cdot)_1 \) preserves trivial fibrations.

Our main result is

**Theorem 4.5.** The category \( \text{SSymMulticat} \) admits a cofibrantly generated model structure in which the weak equivalences are the multi DK-equivalences and the fibrations are the multi DK-fibrations. The model structure is right proper.
Proof. We shall use theorem 6.1. We take in loc. cit.:
- the set $I$ to be $E(∅ → I) \cup \{F_{\text{Multi} \text{Sym}}(k + 1, i)\}_{k \geq 0}$, where $i$ is a generating cofibration of $S$;
- the set $J$ to be $E(B2) \cup \{F_{\text{Multi} \text{Sym}}(k + 1, j)\}_{k \geq 0}$, where $j$ is a horn inclusion;
- the class $W$ to be the class of multi DK-equivalences.

It is enough to prove that $J - \text{cof} \subset W$ and that $W \cap J - \text{inj} = I - \text{inj}$. Notice that $I - \text{inj}$ is the class of trivial fibrations, and that by definition we have $W \cap J - \text{inj} = I - \text{inj}$. The next four lemmas complete the proof of the existence of the model structure. Right properness is standard, see e.g. ([3], Prop. 3.5). □

Lemma 4.6. Let $δ_y : I → H$ be a map belonging to the set $B2$ from theorem 4.2. Then in the pushout diagram

\[
\begin{array}{c}
E \times T \xrightarrow{x} M \\
\downarrow Eδ_y \\
EH \xrightarrow{j} N
\end{array}
\]

the map $M → N$ is a multi DK-equivalence.

Proof. We factor the map $δ_y$ as $I(δ_y)^u \xrightarrow{u} H → H$ where $u = Ob(δ_y)$ (2.3) and then we take consecutive pushouts:

\[
\begin{array}{c}
E \times T \xrightarrow{x} M \\
\downarrow E(δ_y)^u \downarrow j \\
EH \xrightarrow{j} M' \\
\downarrow \downarrow \\
EH \xrightarrow{j} N
\end{array}
\]

By lemma 4.7 the map $(δ_y)^u$ is a trivial cofibration in the category of simplicial monoids, therefore the map $j$ is a trivial cofibration in $\text{SSymMulticat}(Ob(M))$. We conclude by an application of the adjunction 3.6(4) to the bottom pushout diagram above, together with ([9], Prop. 5.2) (or prop. 7.1) and lemma 4.8. □

Lemma 4.7. Let $A$ be a cofibrant simplicial category. Then for each $a ∈ Ob(A)$ the simplicial monoid $a^*A$ (2.3) is cofibrant.

Proof. Let $S = Ob(A)$. $A$ is cofibrant if it is cofibrant as an object of $\text{SCat}(S)$. The cofibrant objects of $\text{SCat}(S)$ are characterised in ([7], 7.6): they are the retracts of free simplicial categories. Therefore it suffices to prove that if $A$ is a free simplicial category then $a^*A$ is a free simplicial category for all $a ∈ S$. Recall ([7], 4.5) that $A$ is a free simplicial category iff (i) for all $n ≥ 0$ the category $φ(A)_n$ (see 2.5 for its definition) is a free category on a graph $G_n$, and (ii) for all epimorphisms $α : [m] → [n]$ of $∆$, $α^* : φ(A)_n → φ(A)_m$ maps $G_n$ into $G_m$.

Let $a ∈ S$. The category $φ(a^*A)_n$ is a full subcategory of $φ(A)_n$ with object set $\{a\}$, hence it is free as well. A set $G_n^{a^*A}$ of generators can be described as follows. An element of $G_n^{a^*A}$ is a path from $a$ to $a$ in $φ(A)_n$ such that every arrow in the path belongs to $G_n$ and there is at most one arrow in the path with source and target $a$. Since every epimorphism $α : [m] → [n]$ of $∆$ has a section, $α^*$ maps $G_n^{a^*A}$ into $G_m^{a^*A}$. □
**Lemma 4.8.** Let $A$ and $B$ be two small categories and let $i: A \hookrightarrow B$ be a full and faithful inclusion. Let $M$ be a multicategory. Then in the pushout diagram

$$
\begin{array}{ccc}
EA & \xrightarrow{E_1} & EB \\
\downarrow & & \downarrow \\
M & \rightarrow & N
\end{array}
$$

the map $M \to N$ is a full and faithful inclusion.

*Proof.* Let $(B - A)^+$ be the preorder with objects all finite subsets $S \subseteq Ob(B) - Ob(A)$, ordered by inclusion. For $S \subseteq (B - A)^+$, let $A_S$ be the full subcategory of $B$ with objects $Ob(A) \cup S$. Then $B = \lim_{(B - A)^+} A_S$. On the other hand, a filtered colimit of full and faithful inclusions of multicategories is a full and faithful inclusion. This is because the forgetful functor from $\text{SymMulticat}$ to $\text{Multigraph}$ preserves filtered colimits and a filtered colimit of full and faithful inclusions of multigraphs is a full and faithful inclusion. Therefore one can assume that $Ob(B) = Ob(A) \cup \{q\}$, where $q \not\in Ob(A)$. Furthermore, by (3), Prop. 5.2 (or prop. 7.1) it is enough to consider the following situation. $M$ is a multicategory, $M_1$ its underlying category (3.2), $i: M_1 \hookrightarrow B$ is a full and faithful inclusion with $Ob(B) = Ob(M) \cup \{q\}$, and the pushout diagram is

$$
\begin{array}{ccc}
EM_1 & \xrightarrow{E_1} & EB \\
\downarrow & & \downarrow \\
M & \rightarrow & N
\end{array}
$$

where $\epsilon_M$ is the counit of the adjunction 3.2(3) (with $\mathcal{V} = \text{Set}$). But this follows by taking $\mathcal{V} = \text{Set}$ in the next lemma. \hfill \Box

**Lemma 4.9.** Let $\mathcal{V}$ be a cocomplete closed symmetric monoidal category. Let $M$ be a small symmetric $\mathcal{V}$-multicategory, $M_1$ its underlying $\mathcal{V}$ category (3.2), $B$ a small $\mathcal{V}$-category with $Ob(B) = Ob(M) \cup \{q\}$ and $i: M_1 \hookrightarrow B$ a full and faithful inclusion. Then in the pushout diagram

$$
\begin{array}{ccc}
EM_1 & \xrightarrow{E_1} & EB \\
\downarrow & & \downarrow \\
M & \rightarrow & N
\end{array}
$$

the map $M \to N$ is a full and faithful inclusion. Here $\epsilon_M$ is the counit of the adjunction 3.2(3).

*Proof.* Let $\otimes$ be the tensor product of $\mathcal{V}$. We shall explicitly describe the $\mathcal{V}$-morphisms of $k$-morphisms of $N$. For $k \geq 0$ and $(a_1, \ldots, a_k; a)$ a $(k + 1)$-tuple of objects with $a \in Ob(M)$ and $a_i \in Ob(M)$ ($i = 1, \ldots, k$), we put $N_k(a_1, \ldots, a_k; a) = M_k(a_1, \ldots, a_k; a)$. Then we set $N(\cdot; q) = \int_{x \in Ob(M)} B(x, q) \otimes M(\cdot; x)$ and

$$
N_k(a_1, \ldots, a_k; q) = \int_{x \in Ob(M)} B(x, q) \otimes M_k(a_1, \ldots, a_k; x) \text{ if } a_i \in Ob(M) \text{ (} i = 1, k).$$

Next, let $(a_1, \ldots, a_k)$ be a $k$-tuple of objects of $M$. For each $1 \leq s \leq k$ let $\{i_1, \ldots, i_s\}$ be a nonempty subset of $\{1, \ldots, k\}$. We denote by $(a_1, \ldots, a_k)^{i_1, \ldots, i_s}$ the $k$-tuple of objects of $B$ obtained by inserting $q$ in the $k$-tuple $(a_1, \ldots, a_k)$ at the spot $i_j$ ($1 \leq j \leq s$). For each $1 \leq j \leq s$ and $x_i \in Ob(M)$ we denote by $(a_1, \ldots, a_k)^{i_1, \ldots, i_s}(x_{i_1}, \ldots, x_{i_s})$ the $k$-tuple of objects of $M$ obtained by inserting $x_{i_j}$ in the $k$-tuple $(a_1, \ldots, a_k)$ at the spot $i_j$. We put
\[ N_k((a_1, \ldots, a_k)^{q_1, \ldots, q_r}; a) = \]
\[ = \int_{x, q} \prod_{i=1}^{r} \int_{x_i}^{Ob(M)} M_k((a_1, \ldots, a_k)^{x_1, \ldots, x_r}; a) \otimes B(q, x_i) \otimes \cdots \otimes B(q, x_r), \]
if \( a \in Ob(M) \), and
\[ N_k((a_1, \ldots, a_k)^{q_1, \ldots, q_r}; q) = \]
\[ = \int_{x, q} \prod_{i=1}^{r} \int_{x_i}^{Ob(M)} B(x, q) \otimes M_k((a_1, \ldots, a_k)^{x_1, \ldots, x_r}; x) \otimes B(q, x_i) \otimes \cdots \otimes B(q, x_r). \]
This completes the definition of the \( V \)-objects of \( k \)-morphisms of \( N \). To prove that \( N \) is a symmetric \( V \)-multicategory is long and tedious. Once this is proved, the fact that it has the desired universal property follows. \( \square \)

5. THE DWYER-KAN MODEL STRUCTURE ON \( \mathcal{V} \text{SymMulticat} \), FOR CERTAIN \( \mathcal{V} \)'S

Using the same method of proof as for theorem 4.5, one can prove a similar result for other categories that simplicial sets. The precise statements follow after some definitions.

Let \( \mathcal{V} \) be a monoidal model category [21] with cofibrant unit \( I \). We denote by \( \mathcal{W} \) (resp. \( \mathfrak{fib} \)) the class of weak equivalences (resp. fibrations) of \( V \). We have a functor \( \gamma : \mathcal{V}\text{Cat} \rightarrow \text{Cat} \) obtained by change of base along the symmetric monoidal composite functor
\[ \mathcal{V} \xrightarrow{\gamma} \text{Ho}(\mathcal{V}) \xrightarrow{\text{Hom}_{\text{Ho}(\mathcal{V})}(I, \_)} \text{Set}. \]

**Definition 5.1.** Let \( f : M \rightarrow M' \) be a morphism in \( \mathcal{V}\text{SymMulticat} \).

1. The morphism \( f \) is a multi DK-equivalence if \( f \) is locally in \( \mathcal{W} \) and \( [f_1]_V \) is essentially surjective.
2. The morphism \( f \) is a multi DK-fibration if \( f \) is locally in \( \mathfrak{fib} \) and \( [f_1]_V \) is an isofibration.
3. The morphism \( f \) is a trivial fibration if \( f \) is a multi DK-equivalence and a multi DK-fibration.

A morphism is a trivial fibration iff it is surjective on objects and locally a trivial fibration.

To prove the next theorem we recall the following definition. An adjoint pair \( F : S \Rightarrow V : G \) is said to be a monoidal Quillen pair if it is a Quillen pair, the functors \( F \) and \( G \) are symmetric monoidal and the unit and counit of the adjunction are monoidal natural transformations.

**Theorem 5.2.** (a) Let \( \mathcal{V} \) be a cofibrantly generated monoidal model category with cofibrant unit and which satisfies the monoid axiom of [21]. Suppose furthermore that \( \mathcal{V} \) satisfies the following technical conditions:

(i) [11], Thm. 2.1 if \( I \) (resp. \( J \)) denotes a generating set of cofibrations (resp. trivial cofibrations) of \( \mathcal{V} \), then the domains of \( I \) (resp. \( J \)) are required to be small relative to \( \mathcal{V} \otimes I \)-cell (resp. \( \mathcal{V} \otimes J \)-cell);

(ii) any transfinite composition of weak equivalences of \( \mathcal{V} \) is a weak equivalence;

(iii) for any set \( S \), the category \( \mathcal{V}\text{SymMulticat}(S) \) admits a model structure with fibrations and weak equivalences defined pointwise.

Let
\[ F : S \Rightarrow V : G \]
be a monoidal Quillen pair such that \( G \) preserves the weak equivalences. Then the category \( \mathcal{V}\text{SymMulticat} \) admits a cofibrantly generated model structure in which the weak equivalences are the multi DK-equivalences and the fibrations are the multi DK-fibrations. The model structure is right proper if the model structure on \( \mathcal{V} \) is right proper.
(b) The category \textbf{CatSymMulticat} admits a cofibrantly generated model structure in which the weak equivalences are the multi DK-equivalences and the fibrations are the multi DK-fibrations. The model structure is right proper.

Proof. (a) The proof is very similar to the proof of 4.5. Sufficient conditions for \((iii)\) to hold are given in ([2], Thm. 2.1). We have a commutative square of adjunctions

\[
\begin{array}{ccc}
\text{SCat} & \overset{E}{\longrightarrow} & \text{SSymMulticat} \\
\text{G'} & \text{G'} & \text{G'} \\
\text{VCat} & \overset{E}{\longrightarrow} & \text{VSymMulticat} \\
\end{array}
\]

The pairs \((F', G')\) are induced by change of base. The functors \(G'\) preserve the trivial fibrations and the functors \(F'\) preserve the (multi) DK-equivalences. We take in 6.1:

- the set \(I\) to be \(E(\emptyset \to I) \cup \{F_{\text{MultiSym}(k+1,i)}\}_{k \geq 0}\), where \(i\) is a generating cofibration of \(\mathcal{V}\);

- the set \(J\) to be \(EF'(B2) \cup \{F_{\text{MultiSym}(k+1,j)}\}_{k \geq 0}\), where \(j\) is a generating trivial cofibration of \(\mathcal{V}\);

- the class \(W\) to be the class of multi DK-equivalences.

It is enough to prove that \(J \rightarrow \text{cof} \subset W\) and that \(W \cap J \rightarrow \text{inj} = I \rightarrow \text{inj}\). Notice that \(I \rightarrow \text{inj}\) is the class of trivial fibrations. We have a natural isomorphism of functors

\[
\eta : [\_sG' \cong [\_]_{\mathcal{V}} : \text{VCat} \rightarrow \text{Cat}
\]

such that for all \(A \in \text{VCat}\), \(\eta_A\) is the identity on objects. This and the hypothesis imply that \(W \cap J - \text{inj} = I - \text{inj}\). Thus, it remains to show that if \(\delta_y : I \rightarrow \mathcal{H}\) is a map belonging to the set \(B2\) and \(M\) is any symmetric \(\mathcal{V}\)-multicategory, then in the pushout diagram

\[
\begin{array}{ccc}
EF'I & \xrightarrow{x} & M \\
EF'(\delta_y) & \downarrow & \downarrow \\
EF'H & \rightarrow & N
\end{array}
\]

the map \(M \rightarrow N\) is a multi DK-equivalence. As in the proof of 4.6, we have pushout diagrams

\[
\begin{array}{ccc}
EF'I & \xrightarrow{x} & M \\
EF'(\delta_y)^u & \downarrow & \downarrow \\
EF'(u^*\mathcal{H}) & \rightarrow & M'
\end{array}
\]

\[
\begin{array}{ccc}
EF'H & \rightarrow & N',
\end{array}
\]

where \(F'(\delta_y)^u\) is a trivial cofibration of monoids in \(\mathcal{V}\) and \(F'(u^*\mathcal{H}) \rightarrow F'\mathcal{H}\) is a full and faithful inclusion in \(\text{VCat}\). It follows that \(j\) is a trivial cofibration in \(\text{VSymMulticat}(\text{Ob}(M))\) and therefore, using 7.1 and 4.9, that \(M \rightarrow N\) is a multi DK-equivalence. Right properness is standard.

If, for example, \(\mathcal{V} = \text{CGHaus}\) (that is, the category of compactly generated Hausdorff spaces with the Quillen model structure), then one can easily show that the model categories \(\text{SSymMulticat}\) and \(\text{CGHausSymMulticat}\) are Quillen equivalent.

(b) One follows again the proof of 4.5, using now theorem 5.3, lemma 5.4 instead of lemma 4.7, together with 7.1 and 4.9. \qed
Theorem 5.3. \[15, 16\] The category \( \text{CatCat}(\equiv 2\text{Cat}) \) of 2-categories admits a cofibrantly generated model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. A generating set of trivial cofibrations consists of

(L1) \( \{F(2, j)\} \), where \( j \) is the single generating trivial cofibration of \( \text{Cat} \) (\[15\], example 1.1), and

(L2) the map \( I \overset{\delta_y}{\to} E' \), where \( E' \) is the 2-category on two objects \( x \) and \( y \) described in the last paragraph of \((\ref{16})\), page 197) and \( \delta_y \) picks out the object \( x \); a formal description of \( E' \) is as follows: let \( G \) be the graph

\[
\begin{array}{ccc}
x & \overset{u}{\Rightarrow} & y
\end{array}
\]

and let \( \iota : \text{Set} \to \text{Cat} \) be the indiscrete category functor. Then \( \iota \) induces a functor \( \iota_2 : \text{Cat} \to 2\text{Cat} \) and \( E' \) is \( \iota_2 \) applied to the free category on the graph \( G \).

Lemma 5.4. Let \( I \overset{(\delta_y)^u}{\to} u^*E' \to E' \) be the factorisation \((2.3)\) of the map \( \delta_y \) in theorem 5.3. Then \( u^*E'(x, x) \) is cofibrant as a monoid in \( \text{Cat} \).

Proof. We show that the map \( 1 \to u^*E'(x, x) \) has the left lifting property with respect to the trivial fibrations of monoids in \( \text{Cat} \). Let \( (S, \cdot, 1) \) be a monoid in \( \text{Set} \) and let \( (B, \mu, e) \) be a monoid in \( \text{Cat} \), that is, a small strict monoidal category.

To give a strict monoidal functor \( F : \iota(S) \to B \) is to give, for each pair \( (r, s) \) of elements of \( S \), arrows \( f_{r,s} : Fr \to Fs \) of \( B \) such that \( f_{r,r} = id_{Fr} \) and \( \mu(f_{r,r}, f_{s,s}) = f_{r,s} \). Moreover, one must have \( F1 = e \) and \( (Fr)(Fs) = F(r \cdot s) \).

Since \( u^*E'(x, x) = \iota(\{1, vu, (vu)^2, (vu)^3, \ldots\}) \), the result follows.

Remark. The cofibrations of the model structures in 5.2 admit the same description as those of \( \text{SSymMulticat} \) (see the remark following 4.3). In the case of \( \text{CatSymMulticat} \), there is a characterisation of the cofibrations entirely parallel to Lack’s description of the cofibrations of \( 2\text{Cat} \) \([15\], lemma 4.1). More precisely, let \( \text{Ob} : \text{Cat} \to \text{Set} \) be the set of objects functor. It has a left adjoint \( D \), the discrete category functor, and a right adjoint \( \iota \), the indiscrete category functor. \( \text{Ob} \) induces by change of base a functor \( \text{Ob}' : \text{CatSymMulticat} \to \text{SymMulticat} \)

which has a left adjoint \( D' \) and a right adjoint \( \iota' \), both induced by change of base.

Call a multifunctor surjective if it is surjective on objects and locally surjective. Then a symmetric \( \text{Cat} \)-multifunctor \( f \) is a cofibration iff \( \text{Ob}'(f) \) has the left lifting property with respect to the surjective multifunctors. The proof proceeds as in \textit{loc. cit.}.

For later purposes we recall the following facts. A functor \( F : C \to D \) between small categories is a Morita equivalence if it is fully faithful and every object of \( D \) is a retract of an object in the image of \( F \). Let \( (\_)^{Kar} : \text{Cat} \to \text{Cat} \) be the idempotent completion functor. For a small category \( C \), \( C^{Kar} \) has objects \( (X, e) \), where \( X \in C \) and \( e : X \to X \) is an idempotent of \( X \). A morphism \( h : (X, e) \to (X', e') \) is a morphism \( h : X \to X' \) of \( C \) such that \( e'h = h \). There is a natural map \( \alpha_C : C \to C^{Kar} \) given by \( \alpha_C(X) = (X, id_X) \) and \( \alpha_C(h) = h \). The map \( \alpha_C \) is a Morita equivalence. It is an equivalence of categories iff the idempotents split in \( C \). If the idempotents split in a small category \( D \), then they split in any functor category \( D^C \). A functor \( F : C \to D \) between small categories is a Morita equivalence iff \( F^{Kar} \) is an equivalence of categories. The category \( \text{Cat} \) admits a cartesian model category structure with the class of Morita equivalences as weak equivalences and the class of injective on objects functors as cofibrations. We shall denote this model structure by \( \text{Cat}_M \). Let \( e \) be the monoid freely generated by one idempotent \( e \in e \).

Then \( e^{Kar} \) is the category with two objects 0 and 1, and arrows \( 0 \to 1 \) and \( 1 \to 0 \)
such that the composite $0 \to 1 \to 0$ is the identity. One can take $\alpha_e$ to be the single generating trivial cofibration of $\text{Cat}_M$. The model structure $\text{Cat}_M$ is not right proper.

**Corollary 5.5.** The category $\text{Cat}_M\text{SymMulticat}$ admits a cofibrantly generated model structure in which the weak equivalences are the multi DK-equivalences. The fibrant objects are the symmetric $\text{Cat}$-multicategories which locally are categories in which every idempotent splits. The fibrations between the fibrant objects are the multi $\text{DK}$-fibrations.

**Proof.** We shall apply 6.2 to the model structure on $\text{CatSymMulticat}$ obtained in 5.2(b). The cartesian functor $(\cdot)^{K\text{ar}}$ induces by change of base a functor $Q : \text{CatSymMulticat} \to \text{CatSymMulticat}$. Since $(\cdot)^{K\text{ar}}$ preserves filtered colimits, it follows from the construction of filtered colimits in $\text{CatSymMulticat}$ that $Q$ preserves filtered colimits. A symmetric $\text{Cat}$-multifunctor $f$ is a multi $\text{DK}$-equivalence (where we took $V = \text{Cat}_M$ in definition 5.1) iff $f$ is a $Q$-equivalence. This uses the fact that there is a (cartesian) natural isomorphism

$$\eta : \text{Hom}_{\text{Ho}(\text{Cat}_M)}(1, -) \gamma \Rightarrow \text{Hom}_{\text{Ho}(\text{Cat})}(1, -) \gamma (\cdot)^{K\text{ar}} : \text{Cat} \to \text{Set}.$$ 

$\eta$ is a natural isomorphism since $\text{Hom}_{\text{Ho}(\text{Cat})}(1, C) \cong \text{Hom}_{\text{Ho}(\text{Cat}_M)}(1, C)$ for $C$ $\text{Cat}_M$-fibrant. Conditions (A1) and (A2) of 6.2 are clear. □

**Remark.** To prove corollary 5.5 we have appealed to 6.2 since we were not able to check condition (A3) of Bousfield’s theorem ([5], Thm. 9.3 and Remark 9.5) when applied to the pair $(\text{CatSymMulticat}, Q)$ above.

The previous proof shows that $\text{Cat}$ is an example when ”left Bousfield localisation commutes with (multi)enrichment”. By this assertion we mean the following. Let $V$ be a monoidal model category with cofibrant unit $I$ such that $\mathcal{V}\text{SymMulticat}$ admits a model structure with multi $\text{DK}$-equivalences as weak equivalences and with trivial fibrations defined as in 5.1.3. Let $L(V)$ be a left Bousfield localisation of $V$ which is a monoidal model category. One can ask whether $L(V)\text{SymMulticat}$ admits a model structure with multi $\text{DK}$-equivalences as weak equivalences and with trivial fibrations defined as in 5.1.3. (In 5.5 we took $V = \text{Cat}_M$ and $L(\text{Cat}_M) = \text{Cat}_M$.) The next result shows that, under some conditions, the answer is affirmative; it’s proof, more general than that of 5.5, implies that

$$L(V)\text{SymMulticat} = L(V)\mathcal{V}\text{SymMulticat},$$ 

where $L(V)\mathcal{V}\text{SymMulticat}$ is a certain left Bousfield localisation of $\mathcal{V}\text{SymMulticat}$. The previous formula represents the meaning of ”left Bousfield localisation commutes with (multi)enrichment”.

**Theorem 5.6.** Let $V$ be a combinatorial monoidal model category with cofibrant unit $I$ such that $\mathcal{V}\text{SymMulticat}$ admits a model structure with multi $\text{DK}$-equivalences as weak equivalences and with trivial fibrations defined as in 5.1.3. Furthermore, suppose that

(i) $V$ admits a left Bousfield localisation $L(V)$ which is a cofibrantly generated monoidal model category;

(ii) for any set $S$, the category $L(V)\text{SymMulticat}(S)$ admits a model structure with fibrations and weak equivalences defined pointwise.

Then $L(V)\text{SymMulticat}$ admits a model structure with multi $\text{DK}$-equivalences as weak equivalences and with trivial fibrations defined as in 5.1.3. If, moreover, the fibrant objects of $\mathcal{V}\text{SymMulticat}$ are the locally fibrant ones and the fibrations between the fibrant objects are the multi $\text{DK}$-fibrations, then the same is true for $L(V)\text{SymMulticat}$. 


Proof. We shall use 6.2. We first remark that $\mathcal{V}\text{SymMulticat}$ has a weak factorisation system in which the left class is the class of maps which are bijective on objects and locally a trivial cofibration of $L(V)$, the right class being the class of maps which are locally a fibration of $L(V)$, cf. [23]. Let $A$ be the set of generating trivial cofibrations of $L(V)$. We apply the small object argument at the set $\{F_{\text{Multi}k\text{Sym}}(k + 1, j)\}_{k \geq 0, j \in A}$ to each morphism $M \to 1$ of $\mathcal{V}\text{Cat}$, where $1$ is the terminal symmetric $\mathcal{V}$-multicategory. We obtain, see e.g. ([1], Prop. 1.3), a functor $Q : \mathcal{V}\text{SymMulticat} \to \mathcal{V}\text{SymMulticat}$ and a natural transformation $\alpha : \text{Id} \Rightarrow Q$. The functor $Q$ is accessible by construction. The map $\alpha_M$ is bijective on objects and locally a trivial cofibration of $L(V)$. It follows that $Q$ satisfies condition (A1) of 6.2. If $X$ is $L(V)$-fibrant then $\text{Hom}_{\text{Ho}(V)}(I, X) \cong \text{Hom}_{\text{Ho}(L(V))}(I, X)$, and so a symmetric $\mathcal{V}$-multifunctor $f$ is a $Q$-equivalence iff $f$ is a multi $\text{DK}$-equivalence of $L(V)\text{SymMulticat}$. Condition (A2) of 6.2 is clear. The last part is standard. □

We end this section with two remarks on multicategories. The first is that the folklore model structure on $\text{SymMulticat}$ can be recovered using the method of proof of theorems 4.5 and 5.2(b). Indeed, let $\text{Set}$ have the the minimal model structure, in which the weak equivalences are the isomorphisms and all maps are cofibrations as well as fibrations. With $\mathcal{V} = \text{Set}$, definition 5.1 then reads: a multifunctor $f$ is a multi-equivalence (resp. a multi-fibration) if it is full and faithful and $f_1$ is essentially surjective (resp. $f_1$ is an isofibration). One takes in 6.1:

- the set $I$ to be $E(\emptyset \to 1) \cup \{F_{\text{Multi}k\text{Sym}}(k + 1, i)\}_{k \geq 0}$, where $i \in \{\emptyset \to *, ** \to *\}$ and $**$ is the two point set;
- the set $J$ to consist of $E(\delta_y)$, where $\delta_y : 1 \to [x, y]$ is the single generating trivial cofibration of $\text{Cat}$; we recall that $[x, y]$ is the category with two objects $x$ and $y$ and nonidentity arrows $s : x \to y$ and $s^{-1} : y \to x$ such that $ss^{-1} = \text{id}_y$ and $s^{-1}s = \text{id}_x$, and the functor $\delta_y$ omits $y$;
- the class $W$ to be the class of multi-equivalences.

The cofibrations of $\text{SymMulticat}$ are the multifunctors monic on objects.

The second remark is that the folklore model structure on $\text{SymMulticat}$ can be localised. We say that a multifunctor $f : M \to N$ is a 
Morita multi-equivalence if $f$ is full and faithful and every object of $N_1$ is a retract of an object in the image of $f_1$. The idempotent completion of $M$ is defined to be the pushout $E(M_1) \leftarrow E(M_1^{K\text{ar}}) \rightarrow M^{K\text{ar}}$. We obtain a functor $(_\alpha)^{K\text{ar}} : \text{SymMulticat} \to \text{SymMulticat}$ and a natural transformation $\alpha : \text{Id} \Rightarrow (_\alpha)^{K\text{ar}}$. Then $f$ is a Morita multi-equivalence iff $f^{K\text{ar}}$ is a multi-equivalence. The category $\text{SymMulticat}$ admits a cofibrantly generated model structure with Morita multi-equivalences as weak equivalences and the multifunctors monic on objects as cofibrations.

6. Model categories

We recall two results from the general theory of model categories.

Theorem 6.1. ([10], Thm. 11.3.1) Let $\mathcal{C}$ be a complete and cocomplete category and let $W$ be a class of maps of $\mathcal{C}$ that is closed under retracts and satisfies the two out three axiom. Let $J$ and $I$ be sets of maps of $\mathcal{C}$ such that

- both $I$ and $J$ permit the small object argument,
- $J \cap \text{cof} \subset W \cap I \cap \text{cof}$,
\[ \text{W} \cap \text{J} - \text{inj} = \text{I} - \text{inj}. \]

Then \( \mathcal{C} \) admits a cofibrantly generated model structure in which \( \text{W} \) is the class of weak equivalences, \( \text{I} \) is a set of generating cofibrations and \( \text{J} \) is a set of generating trivial cofibrations.

6.2. ([24], Remark 2.6) Let \( \mathcal{C} \) be a combinatorial model category. Suppose that there are an accessible functor \( Q: \mathcal{C} \to \mathcal{C} \) and a natural transformation \( \alpha: \text{Id} \Rightarrow Q \) satisfying the following properties:

(A1) the functor \( Q \) preserves weak equivalences;

(A2) for each \( X \in \mathcal{C} \), the maps \( Q(\alpha_X) \) and \( \alpha_{Q(X)} \) are weak equivalences.

Then \( \mathcal{C} \) admits a left Bousfield localisation with the class of \( Q \)-equivalences as weak equivalences. (A map \( f \) of \( \mathcal{C} \) is said to be a \( Q \)-equivalence if \( Q(f) \) is a weak equivalence.)

7. APPENDIX: ON CERTAIN PUSHOUTS OF ENRICHED CATEGORIES

Let \( \mathcal{V} \) be a cocomplete closed symmetric monoidal category with tensor product \( \otimes \) and unit \( I \). We recall (2.2) that a \( \mathcal{V} \)-functor which is locally an isomorphism is called full and faithful.

**Proposition 7.1.** ([9], Prop. 5.2) Let \( A, B \) and \( C \) be three small \( \mathcal{V} \)-categories and let \( i: A \hookrightarrow B \) be a full and faithful inclusion. Then in the pushout diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{i'} & D
\end{array}
\]

the map \( i': C \to D \) is a full and faithful inclusion.

**Proof.** We shall construct \( D \) explicitly, as was done in the proof of ([9], Prop. 5.2). We put \( \text{Ob}(D) = \text{Ob}(C) \cup (\text{Ob}(B) - \text{Ob}(A)) \) and \( D(p, q) = C(p, q) \) if \( p, q \in \text{Ob}(C) \).

For \( p \in \text{Ob}(C) \) and \( q \in (\text{Ob}(B) - \text{Ob}(A)) \) we define

\[ D(p, q) = \int_{x \in \text{Ob}(A)} B(x, q) \otimes C(p, f(x)). \]

For \( p \in (\text{Ob}(B) - \text{Ob}(A)) \) and \( q \in \text{Ob}(C) \) we define

\[ D(p, q) = \int_{x \in \text{Ob}(A)} C(f(x), q) \otimes B(p, x). \]

For \( p, q \in (\text{Ob}(B) - \text{Ob}(A)) \) we define \( D(p, q) \) to be the pushout

\[
\begin{array}{ccc}
\int_{x \in \text{Ob}(A)} B(x, q) \otimes B(p, x) & \xrightarrow{\int_{x \in \text{Ob}(A)} f \in \text{Ob}(A)} & \int_{x \in \text{Ob}(A)} \int_{y \in \text{Ob}(A)} B(x, q) \otimes C(f(y), f(x)) \otimes B(p, y) \\
\downarrow & & \downarrow \\
B(p, q) & \xrightarrow{\int_{x \in \text{Ob}(A)} f \in \text{Ob}(A)} & D(p, q).
\end{array}
\]

We shall describe a way to see that, with the above definition, \( D \) is indeed a \( \mathcal{V} \)-category.

Let \( (B - A)^+ \) be the preorder with objects all finite subsets \( S \subseteq \text{Ob}(B) - \text{Ob}(A) \), ordered by inclusion. For \( S \in (B - A)^+ \), let \( \mathcal{A}_S \) be the full sub-\( \mathcal{V} \)-category of \( B \) with objects \( \text{Ob}(A) \cup S \). Then \( B = \lim_{(S - A)^+} \mathcal{A}_S \). On the other hand, a filtered colimit of full and faithful inclusions of \( \mathcal{V} \)-categories is a full and faithful inclusion. This is because the forgetful functor to \( \mathcal{V} \text{-Graph} \) preserves filtered colimits ([14], Cor. 3.4) and a filtered colimit of full and faithful inclusions of \( \mathcal{V} \)-graphs is a full and faithful
inclusion. Therefore one can assume from the beginning that $\text{Ob}(B) = \text{Ob}(A) \cup \{q\}$, where $q \not\in \text{Ob}(A)$.

Case 1: $f$ is full and faithful. In this case the pushout giving $D(q, q)$ is simply $B(q, q)$, all the other formulas remain unchanged. Then to show that $D$ is a $V$-category is straightforward.

Case 2: $f$ is the identity on objects. The map $i$ induces an adjoint pair

$$i_* : \mathcal{V}\text{-Cat}(\text{Ob}(A)) \rightleftarrows \mathcal{V}\text{-Cat}(\text{Ob}(B)) : i^*.$$  

One has

$$i_*^* A(a, a') = \begin{cases} A(a, a'), & \text{if } a, a' \in \text{Ob}(A), \\ \emptyset, & \text{otherwise}, \\ I, & \text{if } a = a' = q, \end{cases}$$

and $i$ factors as $A \to i_* A \to B$, where $i_* A \to B$ is the obvious map in $\mathcal{V}\text{-Cat}(\text{Ob}(B))$.

Then the original pushout can be computed using the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{i f} & i_* C \end{array}$$

where the square on the right is a pushout in $\mathcal{V}\text{-Cat}(\text{Ob}(B))$.

We claim that $D$ can be calculated as the pushout, in the category $\mathcal{B} \text{ Mod}_B$ of $(\mathcal{B}, \mathcal{B})$-bimodules in $(\mathcal{V}\text{Graph}(\text{Ob}(B)), \square_{\text{Ob}(B)}, \square_{\text{Ob}(B)})$, of the diagram

$$\begin{array}{ccc} B \square_{i_* A} B & \xrightarrow{B \square_{i_* A} f \square_{i_* A} B} & B \square_{i_* A} i_* C \square_{i_* A} B \\
\downarrow & & \downarrow m \\
B & \xrightarrow{m} & D. \end{array}$$

For this we have to show that $D$ is a monoid in $\mathcal{B} \text{ Mod}_B$. We first show that $B \square_{i_* A} i_* C \square_{i_* A} B$ is a monoid in $\mathcal{B} \text{ Mod}_B$. There is a canonical isomorphism

$$i_* C \square_{i_* A} i_* C \cong i_* C \square_{i_* A} B \square_{i_* A} i_* C$$

of $(i_* A, i_* A)$-bimodules which is best seen pointwise, using coends. This provides a multiplication for $B \square_{i_* A} i_* C \square_{i_* A} B$ which is again best seen to be associative by working pointwise, using coends. To define a multiplication for $D$ consider the cube diagrams

$$\begin{array}{c} B \cdot i_* A \cdot B \cdot B \cdot i_* A \cdot B \\
\downarrow \quad \downarrow \quad \downarrow \\
B \cdot i_* C \cdot B \cdot B \cdot i_* A \cdot B \\
\downarrow \quad \downarrow \quad \downarrow \\
D \cdot B \cdot i_* A \cdot B \\
\downarrow \quad \downarrow \quad \downarrow \\
B \cdot B \cdot B \cdot i_* C \cdot B \\
\downarrow \quad \downarrow \\
D \cdot B \cdot B \cdot i_* C \cdot B \\
\downarrow \quad \downarrow \\
B \cdot B \cdot B \cdot B \cdot i_* C \cdot B \\
\downarrow \quad \downarrow \\
D \cdot B \cdot B \cdot B \cdot i_* C \cdot B \end{array}$$
and

For space considerations we have suppressed tensors (always over $i_i A$, unless explicitly indicated) from notation. The right face of the first cube is the same as the left face of the latter cube. Let $PO_1$ (resp. $PO_2$) be the pushout of the left (resp. right) face of the first cube diagram. Let $PO_3$ be the pushout of the right face of the second cube diagram. We have pushout diagrams

Using these pushouts and the fact that $B \square_i A \square_i B C$ is a monoid one can define in a canonical way a map $\mu : D \cdot_B B \to D$. We omit the long verification that $\mu$ gives $D$ the structure of a monoid. The map $\mu$ was constructed in such a way that $m$ becomes a morphism of monoids. The fact that $D$ has the universal property of the pushout in the category $\mathcal{V} \mathbf{Cat}(\text{Ob}(B))$ follows from its definition.

Case 3: $f$ is arbitrary. Let $u = \text{Ob}(f)$. We factor $f$ as $A \xrightarrow{f^u} u^* C \to C$ (2.3) and take consecutive pushouts

Then apply cases 2 and 1.

Acknowledgements. We are deeply indebted to Professors André Joyal and Michael Makkai for many useful discussions and suggestions. The main results of this paper were obtained during the author’s stay at the CRM Barcelona. We would like to thank CRM for support and warm hospitality.

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