Wasserstein Convergence Rate for
Empirical Measures of Markov Chains

Adrian Riekert∗
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Abstract

We consider a Markov chain on $\mathbb{R}^d$ with invariant measure $\mu$. We are interested in the rate of convergence of the empirical measures towards the invariant measure with respect to the 1-Wasserstein distance. The main result of this article is a new upper bound for the expected Wasserstein distance, which is proved by combining the Kantorovich dual formula with a Fourier expansion. In addition, we show how concentration inequalities around the mean can be obtained.

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1 Introduction and main results

1.1 Empirical measures

Let $X_0, X_1, X_2, \ldots$ be a Markov chain on $\mathbb{R}^d$ with invariant probability distribution $\mu$. For $n \in \mathbb{N}$ we define the empirical measure

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i},$$

a random probability measure on $\mathbb{R}^d$. Under suitable conditions these measures will converge to $\mu$ as $n \to \infty$. The purpose of this article is to quantify the rate of convergence with respect to the 1-Wasserstein distance given by

$$W_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \, d\pi(x, y),$$

(1)

where $\Pi(\mu, \nu)$ denotes the set of all couplings between $\mu$ and $\nu$, i.e., the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\nu$ [20]. This is a classical problem with numerous applications, including, e.g., clustering [14], density estimation [7], and Monte Carlo integration.

∗Faculty of Mathematics and Computer Science, University of Münster, Münster, Germany; e-mail: ariekert@uni-muenster.de
The special case where the $X_i$ are i.i.d. with common distribution $\mu$ has been studied extensively, see, e.g., [11, 1, 10, 6, 21, 15]. The most general and tight results [12, 16] state that $\mathbb{E}[W_1(\mu, \mu_n)]$ is of order $n^{-1/d}$ if $d \geq 3$ and $\mu$ has a finite $q$-th moment for some $q > \frac{d}{d-1}$.

Some of the proofs for the i.i.d. case can be adapted to Markov chains, but usually only under strong additional assumptions, such as absolute continuity of the initial distribution with respect to $\mu$ (see, e.g., [12] and [6]). This requires that one already has access to some approximation of $\mu$ to start the Markov chain with, which is not always the case in applications. We will instead follow the approach from [13], which does not require such an assumption on the initial distribution.

### 1.2 Contractive Markov chains

Let $\mathcal{P}(\mathbb{R}^d)$ denote the set of Borel probability measures on $\mathbb{R}^d$ and $\mathcal{P}_1(\mathbb{R}^d)$ the set of measures in $\mathcal{P}(\mathbb{R}^d)$ with a finite first moment. On this set the Wasserstein distance $W_1$ is finite and possesses the following dual formulation.

**Lemma 1.1 (Kantorovich duality).** If $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ then

$$W_1(\mu, \nu) = \sup_{f \in \text{Lip}_1(\mathbb{R}^d)} |\mu(f) - \nu(f)|.$$  

Here $\text{Lip}_1(\mathbb{R}^d)$ denotes the space of all Lipschitz continuous functions on $\mathbb{R}^d$ with Lipschitz constant at most 1. The proof can be found in [20].

Let $P(x, dy)$ be a Markov kernel on $\mathbb{R}^d$ and $X_0, X_1, X_2, \ldots$ the corresponding time-homogeneous Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with initial distribution $X_0 \sim \gamma_0 \in \mathcal{P}(\mathbb{R}^d)$. Denote by $P^n$ the $n$-fold iteration of $P$. As usual, we introduce the averaging operator

$$(P f)(x) = \int_{\mathbb{R}^d} f(y)P(x, dy)$$

for bounded or nonnegative measurable functions $f$, and similarly the action on measures $\nu$

$$(\nu P)(B) = \int_{\mathbb{R}^d} P(y, B)d\nu(y), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

We require the following assumption on the transition kernel.

**Assumption 1.** There are constants $D \geq 1$ and $\kappa \in (0, 1)$ such that

$$W_1(P^n(x, \cdot), P^n(y, \cdot)) \leq D\kappa^n|x - y|$$

for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$.

In particular, this means that $P^n(x, \cdot) \in \mathcal{P}_1(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. In the case $D = 1$, this assumption is equivalent to the Markov chain having uniformly positive Ricci curvature in the sense of Ollivier [18]. In this case, it is enough to require the condition for $n = 1$, then the estimate for general $n$ follows by iterated application. Intuitively, the assumption means that if we start two Markov chains at different points $x$ and $y$ then the chains can be coupled in such a way that they approach each other as $n \to \infty$. See [18] for an introduction to this topic on general metric spaces and several examples of Markov chains which satisfy the assumption. Often one can only ensure the Ricci curvature to be positive by choosing a different (equivalent) metric, which results in the additional factor $D$. This weaker condition is still sufficient for our proofs.

The assumption implies a similar relation for arbitrary initial distributions.
**Lemma 1.2** ([18], Proposition 20). If Assumption 1 holds then for any $\mu_0, \nu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and $n \in \mathbb{N}$ we have

$$W_1(\mu_0 P^n, \nu_0 P^n) \leq D \kappa^n W_1(\mu_0, \nu_0).$$

Since the space $\mathcal{P}_1(\mathbb{R}^d)$ is complete, the Banach fixed point theorem implies that there is a unique invariant probability measure $\mu \in \mathcal{P}_1(\mathbb{R}^d)$ for the Markov chain (i.e. $\mu P = \mu$). Moreover, for any initial distribution $\gamma_0 \in \mathcal{P}_1(\mathbb{R}^d)$ we have $\lim_{n \to \infty} W_1(\gamma_0 P^n, \mu) = 0$ exponentially fast.

### 1.3 Rate of convergence in expectation

Our goal is to find estimates from above for the quantity

$$E[W_1(\mu_n, \mu)] = E\left[\sup_{f \in \text{Lip}_1(\mathbb{R}^d)} |\mu_n(f) - \mu(f)|\right].$$

Hence we need to obtain uniform bounds over $f \in \text{Lip}_1(\mathbb{R}^d)$. To deal with this problem, Boissard [5] uses approximations in terms of the covering numbers of the set $\mathcal{N}_d(\text{Lip}_1(K))$ for a compact subset $K \subset \mathbb{R}^d$. The proofs assume that the transition kernels satisfy a transportation inequality, which is equivalent to an exponential moment condition and therefore a rather strong assumption.

To obtain uniform bounds in $f$, we will instead follow the approach by Kloeckner [13] and use an approximation of $f$ by its Fourier series. For Lipschitz functions, the Fourier series converges uniformly on compact sets and reasonably fast. Kloeckner uses this to prove that if the Markov chain is supported on a compact set $K \subset \mathbb{R}^d$ and $d \geq 3$ then there is a constant $C$ depending on $K$, $d$, and $D$ such that for all $n$ large enough

$$E[W_1(\mu, \mu_n)] \leq C \frac{(\log n')^{d-2} + 1}{(n')^{1/d}},$$

where $n' = (1-\kappa)n$ [13, Theorem 1.1]. This rate of convergence is only a power of logarithm slower than the one for the independent case. For $d = 2$ one has $E[W_1(\mu, \mu_n)] \leq C \frac{\log n'}{(n')^{1/2}}$ and for $d = 1$ it holds that $E[W_1(\mu, \mu_n)] \leq C \frac{\log n'}{(n')^{1/2}}$ for $n$ large enough.

In the following we will generalize this result to Markov chains on the entire space $\mathbb{R}^d$. Since the Fourier series does not converge uniformly on $\mathbb{R}^d$, we will use a truncation argument. For this we need the following moment assumption.

**Assumption 2.** There exist some $q > 1$ and $M \in (0, \infty)$ such that $\sup_{n \in \mathbb{N}_0} (E|X_n|^q)^{1/q} \leq M$.

**Remark.** In the i.i.d. case $X_i \sim \mu$, this condition is equivalent to finiteness of the $q$-th moment for $\mu$. This assumption is also necessary if one wants to obtain meaningful estimates for the rate of convergence. If one only assumes finiteness of the first moment then the rate of convergence may be arbitrarily slow [4].

In particular we assume that the initial distribution $\gamma_0$ has a finite $q$-th moment. This is the only assumption on $\gamma_0,$ no absolute continuity or further regularity is needed. Observe that Assumption 2 also implies that the $q$-th moment of the invariant measure $\mu$ is bounded by $M$.

Our main result about the speed of convergence of $\mu_n$ to $\mu$ is the following. We will always write $\lesssim$ for inequalities which hold up to a constant depending on $d$ and $D$. 


Theorem 1.3. Suppose that Assumption 1 and Assumption 2 are satisfied. With \( n' = (1 - \kappa)n \) it holds for all large enough \( n \) that

\[
\mathbb{E}[W_1(\mu, \mu_n)] \lesssim \begin{cases} 
M \left( \frac{(\log n')^{d-2+1/4}}{(n')^{1/4}} \right)^{1-1/q}, & d \geq 3 \\
M \left( \frac{(\log n')^{1-1/q}}{(n')^{1/4}} \right), & d = 2 \\
M \left( \frac{(\log n')^{1/2}}{(n')^{1/2}} \right)^{1-1/q}, & d = 1.
\end{cases}
\]

In particular, for large \( q \) we almost obtain the result from the compact case. Moreover, the speed of convergence is proportional to \((1 - \kappa)\). Hence for \( \kappa \) close to 1 we need more samples of \( X_i \) to obtain a good approximation compared to the independent case.

For the proof, we will approximate a Lipschitz function \( f \in \text{Lip}_q(\mathbb{R}) \) by its Fourier series uniformly on a compact set \( K = [-R, R]^d \). The Lipschitz continuity allows us to bound the Fourier coefficients independently of \( f \). The integrals over the complement of \( K \) can be estimated by using the moment condition and the fact that any Lipschitz function is dominated by \(|x|^q\) since \( q > 1 \).

In order to verify Assumption 2 in applications, one can use the following simple criterion.

Proposition 1.4. Let \( f : \mathbb{R}^d \to [0, \infty) \) be a measurable function with \( \mathbb{E}f(X_0) < \infty \). Suppose that there are constants \( C < \infty \) and \( \gamma \in (0, 1) \) such that \( (Pf)(x) \leq \gamma f(x) + C \) for each \( x \in \mathbb{R}^d \). Then one has \( \sup_{n \in \mathbb{N}_0} \mathbb{E}f(X_n) < \infty \).

Proof. We may assume that \( \mathbb{E}f(X_0) \leq \frac{C}{1 - \gamma} \), otherwise one can replace \( C \) with a larger constant.

Then we show by induction that \( \mathbb{E}f(X_n) \leq \frac{C}{1 - \gamma} \) for each \( n \). By assumption this holds for \( n = 0 \), and moreover

\[
\mathbb{E}f(X_{n+1}) = \mathbb{E}[(Pf)(X_n)] \leq \gamma \mathbb{E}f(X_n) + C.
\]

Hence \( \mathbb{E}f(X_n) \leq \frac{C}{1 - \gamma} \) implies the same estimate for \( n + 1 \), which completes the proof.

If the condition \( Pf \leq \gamma f + C \) holds for \( f(x) = |x|^q \), then Assumption 2 of the theorem will be satisfied.

1.4 Concentration

In addition to estimating the expectation of \( W_1(\mu, \mu_n) \), it is also of interest how well the Wasserstein distance concentrates around its expected value. In this section the Markov chain can be supported on an arbitrary Polish metric space \( \mathcal{X} \). If the state space is bounded, one can use standard bounded difference methods to obtain the following concentration inequality.

Theorem 1.5. Suppose that \( (X_n) \) is an exponentially contracting Markov chain in the sense of Assumption 1, with constants \( D = 1 \) and \( \kappa < 1 \), taking values in a metric space \( \mathcal{X} \) with \( \text{diam} (\mathcal{X}) \leq 1 \). Then for all \( t \geq 0 \) one has

\[
\mathbb{P}(W_1(\mu, \mu_n) \geq \mathbb{E}[W_1(\mu, \mu_n)] + t) \leq \exp\left(-2(1 - \kappa)^2 \cdot nt^2\right).
\]

For the case of i.i.d. random variables taking values in a metric space with diameter at most 1, Weed and Bach [21] showed that \( \mathbb{P}(W_1(\mu, \mu_n) \geq \mathbb{E}[W_1(\mu, \mu_n)] + t) \leq \exp\left(-2nt^2\right) \). We obtain the same sub-Gaussian concentration rate, up to the factor \((1 - \kappa)^2\). Observe that the rate of concentration does not depend on the dimension or any other specific properties of the state space – the result
holds for an arbitrary bounded metric space. It should be noted that Theorem 1.5 improves on [13, Theorem 5.4] by a factor of 4 in the exponent.

In the noncompact case, strong moment assumptions are needed in order to obtain useful concentration inequalities. We say that a measure $\mu \in \mathcal{P}(X)$ satisfies a transportation inequality $T_1(C)$ [3, 8] if for all $\nu \in \mathcal{P}(X)$ with $\nu \ll \mu$

$$W_1(\mu, \nu) \leq \sqrt{2CH(\nu \mid \mu)},$$

where $H(\nu \mid \mu)$ is the relative entropy. Using the Lipschitz properties of the Wasserstein metric, this allows us to prove a concentration inequality for $W_1(\mu, \mu_n)$.

**Theorem 1.6.** Let $(X_n)$ be an exponentially contracting Markov chain on a metric space $X$ as in Assumption 1, with $D = 1$ and $\kappa < 1$. Suppose that both the initial distribution and the transition kernels $P(x, \cdot)$ satisfy $T_1(C)$ for all $x \in X$. Then we have for all $t \geq 0$ and $n \in \mathbb{N}$ that

$$\mathbb{P}(W_1(\mu, \mu_n) \geq \mathbb{E}[W_1(\mu, \mu_n)] + t) \leq \exp\left(-\frac{1}{2}\kappa n^2(1 - \kappa)^2\right).$$

This is the same sub-Gaussian rate of concentration as in the bounded case.

## 2 Proofs

### 2.1 Upper bound for the expectation

Suppose that the assumptions 1 and 2 are satisfied. We may assume that $M = 1$, since otherwise the metric can be multiplied by $\frac{1}{M}$. We first prove some auxiliary statements. The first lemma we use is adapted from [13], where it is only formulated for Markov chains on compact sets.

**Lemma 2.1.** Let $\alpha \in (0, 1]$ and $f : \mathbb{R}^d \to \mathbb{C}$ be a bounded, $\alpha$-Hölder continuous function. Then for all $m, n \in \mathbb{N}$ one has

$$|\mathbb{E}f(X_n) - \mu(f)| \leq 2D^\alpha \kappa^\alpha \text{Hol}_\alpha(f)$$

and

$$|\text{Cov}(f(X_n), f(X_m))| \leq 8D^\alpha \kappa^\alpha|m-n|\|f\|_\infty \text{Hol}_\alpha(f),$$

where $\text{Hol}_\alpha(f)$ denotes the $\alpha$-Hölder constant of $f$.

In the case of a compact state space, any Hölder-continuous function is automatically bounded by a value depending on the Hölder constant. But the compactness assumption can be removed if one assumes additionally that $f$ is bounded and combines this with the moment condition.

**Proof.** Let $W_\alpha$ be the 1-Wasserstein distance with respect to the modified metric $|x - y|^n$. We first claim that

$$W_\alpha(\mu_0P^n, \nu_0P^n) \leq D^\alpha \kappa^\alpha W_\alpha(\mu_0, \nu_0)$$

(2)

for any $n \in \mathbb{N}$ and probability measures $\mu_0, \nu_0 \in \mathcal{P}_1$. Indeed, if $\mu_0 = \delta_x$ and $\nu_0 = \delta_y$ are Dirac measures then by Jensen’s inequality,

$$W_\alpha(\delta_xP^n, \delta_yP^n) \leq (W_1(\delta_xP^n, \delta_yP^n))^\alpha \leq D^\alpha \kappa^\alpha W_1(\delta_x, \delta_y)^\alpha$$

$$= D^\alpha \kappa^\alpha |x - y|^\alpha = D^\alpha \kappa^\alpha W_\alpha(\delta_x, \delta_y),$$

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using the assumption that $P$ is a contraction in $W_1$. Then we can use the same argument as in the proof of [18, Proposition 20] to obtain the estimate in the general case.

To prove the first part of the lemma, we use the dual representation for $W_\alpha$, noting that the Hölder constant Hol$_\alpha(f)$ is the Lipschitz seminorm of $f$ with respect to the metric $|x-y|$.$^\alpha$. The fact that $\mu$ is the stationary measure together with (2) now implies

\[
\|E f(X_n) - \mu(f)\| = |\gamma_0(P^n f) - \mu(f)| = |\gamma_0(P^n f) - \mu(P^n f)| \\
\leq Hol_\alpha(f) W_\alpha(\gamma_0 P^n, \mu P^n) \leq Hol_\alpha(f) D^\alpha \kappa^\alpha (W_\alpha(\delta_0, \mu) + W_\alpha(\delta_0, \gamma_0)) \\
\leq 2D^\alpha \kappa^\alpha Hol_\alpha(f).
\]

Here we used that

\[
W_\alpha(\delta_0, \mu) = \int |x|^\alpha \, d\mu(x) \leq \left( \int |x|^q \, d\mu(x) \right)^{\alpha/q} \leq 1
\]

by the moment condition and Jensen’s inequality, applied to the convex function $u \mapsto u^{\alpha/q}$ on $[0, \infty)$. Analogously, this estimate holds for $W_\alpha(\delta_0, \gamma_0)$, since the initial distribution $\gamma_0$ satisfies the same moment condition. This proves the first assertion.

For the second one, note first that if the process starts in an arbitrary point $x \in \mathbb{R}^d$ then

\[
| \langle P^n f \rangle(x) - \mu(f) \rangle | \leq Hol_\alpha(f) D^\alpha \kappa^\alpha W_\alpha(\delta_2, \mu) \leq Hol_\alpha(f) D^\alpha \kappa^\alpha (|x|^\alpha + 1),
\]

by the triangle inequality and the same argument as before. After translating $f$ we may assume that $\mu(f) = 0$. In particular $f$ then takes positive and negative values, so its $L^\infty$-norm increases by a factor of at most 2. We also assume that $n \geq m$ and write $n = m + t$. Clearly $|f(X_m)| \leq \|f\|_\infty$, and by the first part we get

\[
\| (E f(X_n))(E f(X_m)) \| \leq 2\|f\|_\infty D^\alpha \kappa^\alpha Hol_\alpha(f).
\]

For the term $E[f(X_n) f(X_m)]$, we invoke the Markov property to obtain

\[
\| E[f(X_n) f(X_m)] \| = \| E[f(X_m) E[f(X_{m+t}) | X_m]] \| = \| E[f(X_m)(P^t f)(X_m)] \| \\
\leq Hol_\alpha(f) D^\alpha \kappa^\alpha (E[f(X_m)] + E[|f(X_m)| \cdot |X_m|^\alpha]) \\
\leq 2\|f\|_\infty D^\alpha \kappa^\alpha Hol_\alpha(f),
\]

where we used again that $E[X_m|^\alpha \leq 1$ by the moment condition. Combining the two estimates yields the result (the additional factor of 2 comes from translating $f$).

Since we want to use a Fourier approximation, we need estimates in terms of the basis functions $e_k(x) = \exp(\pi i k \cdot x)$ for $k \in \mathbb{Z}^d$. As the Lipschitz constant of $e_k$ grows too rapidly as $\|k\|_\infty \to \infty$, we instead apply Lemma 2.1 to obtain bounds in terms of the $\alpha$-Hölder constant of $e_k$, for some parameter $\alpha$ to be specified later. The Hölder constant can be bounded as follows.

**Lemma 2.2 ([13, Lemma 4.2]).** For $\alpha \in (0, 1]$, $k \in \mathbb{Z}^d$ the $\alpha$-Hölder constant of $e_k$ satisfies

\[
Hol_\alpha(e_k) \leq 2^{1-\alpha} \pi^{\|k\|_\infty^2} |k|^{\alpha}.
\]

**Proof.** Since $|\nabla e_k| = \pi |k|$, the function $e_k$ is Lipschitz with constant $\text{Lip}(e_k) \leq \pi |k| \leq \pi \sqrt{d} \|k\|_\infty$. On the other hand, one has $\|e_k\|_\infty = 1$, and thus we obtain for $x \neq y$

\[
\frac{|e_k(x) - e_k(y)|}{|x-y|^\alpha} \leq \min \left( \frac{2}{|x-y|^\alpha}, \pi \sqrt{d} \cdot \|k\|_\infty |x-y|^{1-\alpha} \right).
\]
If |x - y| ≤ 2( πd∥k∥∞)−1, then the second term does not exceed 21−απα∥d∥α/2∥k∥∞, and otherwise the first term is not larger than this bound. The claim follows.

These two lemmas enable us to prove the following useful estimate for |μn(e_k) − μ(e_k)|.

**Lemma 2.3.** For all α ∈ (0, 1], k ∈ ℤd, n ≥ (1 − κα)−1 one has

\[
E|μ_n(e_k) - μ(e_k)|^2 ≤ \frac{∥k∥_{∞}^{2α}}{n(1 - κα)}.
\]

**Proof.** Note that

\[
E|μ_n(e_k) - μ(e_k)|^2 = E \left( \left( \frac{1}{n} \sum_{j=1}^{n} e_k(X_j) - μ(e_k) \right)^2 \right)
\]

\[
= \frac{1}{n^2} \left( \sum_{j=1}^{n} (E[e_k(X_j)] - μ(e_k)) \right)^2 + \frac{1}{n^2} \sum_{1 \leq j,l \leq n} Cov(e_k(X_j), e_k(X_l))
\]

Using Lemma 2.1 and Lemma 2.2, we obtain

\[
\frac{1}{n^2} \sum_{1 \leq j,l \leq n} |Cov(e_k(X_j), e_k(X_l))| \leq \frac{16D^2 \pi^α dα/2∥k∥_{∞}^α}{n^2} \sum_{1 \leq j,l \leq n} κ^α |j-l| \leq \frac{∥k∥_{∞}^α}{n(1 - κα)}.
\]

Similarly, for the second term we get

\[
\frac{1}{n^2} \left( \sum_{j=1}^{n} (E[e_k(X_j)] - μ(e_k)) \right)^2 \leq \frac{∥k∥_{∞}^{2α}}{n^2} \left( \sum_{j=1}^{n} 1 \right)^2 \leq \frac{∥k∥_{∞}^{2α}}{n^2(1 - κα)^2} \leq \frac{∥k∥_{∞}^{2α}}{n(1 - κα)}
\]

where we used the assumption on n in the last step. The proof is complete.

Now take an arbitrary function f ∈ Lip_1(ℤ^d). To estimate |μ_n(f) − μ(f)|, we can assume f(0) = 0. The main difference of this work to [13] is the following truncation argument. For some radius R > 0 to be determined later, let K := [−R, R]^d. The idea will be to approximate f on K by its Fourier series g, and to use the moment condition to obtain bounds for the integral over ℤ^d \ K = K^c. More precisely, we will use the following lemma, which is a direct consequence of the triangle inequality.

**Lemma 2.4.** If g : ℤ^d → ℂ is any bounded measurable function, then

\[
|μ_n(f) - μ(f)| \leq 2∥f - g∥_{∞(K)} + |μ_n(g) - μ(g)| + \int_{K^c} (|f| + |g|) \, dμ_n + \int_{K^c} (|f| + |g|) \, dμ.
\]

Now we can clearly find a 1-Lipschitz function f̂ : [−2R, 2R]^d → ℂ with periodic boundary conditions such that f|_K = f̂|_K. Then the function g : [−1, 1]^d → ℂ defined by g(x) = f(2Rx) is 2R-Lipschitz and has periodic boundary conditions. Let F^g(x) = \sum_{k∈ℤ^d} g_k e^{πi k·x} be the Fourier series of g, and for given J ∈ ℤ

\[
F^g_J(x) = \sum_{k∈ℤ^d, ||k||_{∞} ≤ J} g_k e^{πi k·x}
\]

the approximation of order J. Then Theorem 4.4 in [19] implies that

\[
||F^g_J - g||_{∞([−1,1]^d)} \leq C_dR \frac{(log J)^d}{J}
\]
where \( C_d \leq C_d 2^d \) for an absolute constant \( C \). Hence if we define \( F_f^f(x) := F_f^f(x/2R) \), then
\[
\|F_f^f - f\|_{L^\infty([-R, R]^{d})} \leq \|F_f^f - \tilde{f}\|_{L^\infty([-2R, 2R]^{d})} \lesssim R \left( \frac{\log J}{J} \right)^d.
\]
Now we apply Lemma 2.4 for \( g = F_f^f \). To estimate the tail terms for \( f \), we will use the moment condition. Since \( f \) is 1-Lipschitz with \( f(0) = 0 \), we have \( |f(x)| \leq |x| \) for any \( x \in \mathbb{R}^d \) and therefore, by Hölder’s inequality,
\[
\int_{K^c} |f| \, d\mu \leq \int_{K^c} |x| \, d\mu \leq \left( \int_{K^c} |x|^q \, d\mu \right)^{1/q} (\mu(K^c))^{1-1/q} \leq (\mu(K^c))^{1-1/q}.
\]
On the other hand, since \( |x| \geq R \) for \( x \in K^c \) the Markov inequality gives \( \mu(K^c) \leq R^{-q} \). Thus we obtain
\[
\int_{K^c} |f| \, d\mu \leq R^{1-q}.
\]
Analogously, for the measure \( \mu_n \) we have
\[
\int_{K^c} |f| \, d\mu_n \leq \int_{K^c} |x| \, d\mu_n.
\]
To estimate the tail integrals over \( F_f^f \), note that by periodicity one has
\[
\sup_{x \in K^c} |F_f^f(x)| = \|F_f^f\|_{L^\infty([-2R, 2R]^{d})} \lesssim R \left( \frac{\log J}{J} \right)^d + \|\tilde{f}\|_{L^\infty([-2R, 2R]^{d})} \lesssim R \left( \frac{\log J}{J} \right)^d + R \sqrt{d},
\]
since \( \tilde{f} \) is by assumption 1-Lipschitz. This yields
\[
\int_{K^c} |F_f^f| \, d\mu \lesssim (\mu(K^c)) \left( R \frac{\left( \frac{\log J}{J} \right)^d}{J} + R \sqrt{d} \right) \leq R \frac{\left( \frac{\log J}{J} \right)^d}{J} + R^{1-q} \sqrt{d}.\]
Analogously, we obtain for \( \int_{K^c} |F_f^f| \, d\mu_n \) the upper bound
\[
\int_{K^c} |F_f^f| \, d\mu_n \lesssim R \frac{\left( \frac{\log J}{J} \right)^d}{J} + R \sqrt{d} \cdot \mu_n(K^c).
\]
Hence these terms are of the same order as the corresponding expressions for \(|f|\).
It remains to estimate the term \(|\mu(F_f^f) - \mu_n(F_f^f)|\) from above. Writing \( e_k^f(x) := e^{2\pi i k \cdot x} \), we obtain
\[
|\mu(F_f^f) - \mu_n(F_f^f)| \leq \sum_{0 < \|k\| \leq J} |\hat{g}_k| |e_k^f - \hat{g}_k^f|.
\]
We use the Cauchy-Schwarz inequality and the fact that \( g \) is 2R-Lipschitz, which leads to the bound for the first factor. Moreover, the term for \( k = 0 \) corresponds to a constant function and can therefore be ignored. Combining all the estimates, we obtain the following lemma.
Lemma 2.5. For all $f \in \text{Lip}_1(\mathbb{R}^d)$ and all $R > 0$, $J \in \mathbb{N}$ it holds that

$$|\mu_n(f) - \mu(f)| \lesssim R \left( \frac{\log J}{J} \right)^d + R^{1-q} + \int_{K^c} |x| \, d\mu_n + R\mu_n(K^c) + R \left( \sum_{0 < \|k\|_\infty \leq J} \frac{|\mu(e_k^R) - \mu_n(e_k^R)|^2}{\|k\|_\infty^2} \right)^{1/2}.$$ 

Note that the right-hand side does not depend on $f$ anymore. Hence the same bound also holds for $\sup_{f \in \text{Lip}_1(\mathbb{R}^d)} |\mu_n(f) - \mu(f)| = W_1(\mu_n, \mu)$. In the next step we take the expectation on both sides. Note that

$$E \left[ \int_{K^c} |x| \, d\mu_n \right] = \frac{1}{n} \sum_{i=1}^n E |X_i\, |X_i \in K^c| \leq R^{1-q},$$

since the distribution of each $X_i$ satisfies by assumption the same moment condition as $\mu$. Analogously,

$$E[\mu_n(K^c)] = \frac{1}{n} \sum_{i=1}^n P(X_i \in K^c) \leq R^{-q},$$

Hence we have shown the following.

Corollary 2.6. Under the assumptions of Theorem 1.3, for all $J \in \mathbb{N}$ and $R > 0$ one has

$$E[W_1(\mu_n, \mu)] \lesssim R \left( \frac{\log J}{J} \right)^d + R^{1-q} + R \left[ \sum_{0 < \|k\|_\infty \leq J} \frac{|\mu(e_k^R) - \mu_n(e_k^R)|^2}{\|k\|_\infty^2} \right]^{1/2}.$$ 

To prove the theorem, it remains to apply the estimate from Lemma 2.3 and then choose $\alpha$, $R$, and $J$ appropriately. The Hölder constant of $e_k^R$ is the constant for $e_k$ divided by $(2R)^\alpha$, but this factor is at most 1 for $R > 1/2$, so it can be ignored. By concavity of the square root function, for any $\alpha \in (0, 1)$ and $n$ large enough we have

$$E \left[ \left( \sum_{0 < \|k\|_\infty \leq J} \frac{|\mu(e_k^R) - \mu_n(e_k^R)|^2}{\|k\|_\infty^2} \right)^{1/2} \right] \leq \left( \sum_{0 < \|k\|_\infty \leq J} \frac{|\mu(e_k^R) - \mu_n(e_k^R)|^2}{\|k\|_\infty^2} \right)^{1/2} \leq \left( \sum_{j=1}^J \frac{j^{d+2\alpha-3}}{n(1-\kappa^\alpha)} \right)^{1/2},$$

using that the number of points $k \in \mathbb{Z}^d$ with $\|k\|_\infty = j$ is of order $j^{d-1}$. Now we choose $\alpha = 1/\log_2 J$, which gives $j^{2\alpha} \leq j^{2\alpha} = 4$ for all $1 \leq j \leq J$. Together with $1 - \kappa^\alpha \geq \alpha(1 - \kappa)$ this implies

$$E \left[ \left( \sum_{\|k\|_\infty \leq J} \frac{|\mu(e_k^R) - \mu_n(e_k^R)|^2}{\|k\|_\infty^2} \right)^{1/2} \right] \lesssim \sqrt{\frac{\log J}{n(1-\kappa)}} \left( \sum_{j=1}^J j^{d-3} \right)^{1/2} \lesssim J^{d/2-1} \sqrt{\frac{\log J}{n(1-\kappa)}},$$

since $\sum_{j=1}^J j^{d-3} = O(J^{d-2})$ if $d \geq 3$. Setting $n' = n(1-\kappa)$, we obtain

$$E[W_1(\mu_n, \mu)] \lesssim R \left( \frac{(\log J)^d}{J} + J^{d/2-1} \frac{\log J}{n'} \right) + R^{1-q}.$$ (3)
for $n$ large enough.

It remains to choose the parameters $R$ and $J$ in such a way that the right-hand side of this estimate is as small as possible. We first minimize this expression in $R$. The optimal value is given by

$$ R = (q - 1)^{1/q} \left( \left( \frac{\log J}{J} \right)^{d/2 - 1} \sqrt{\frac{\log J}{n'}} \right)^{-1/q}, $$

whence (3) yields

$$ \mathbb{E}[W_1(\mu_n, \mu)] \lesssim \left( \left( \frac{\log J}{J} \right)^{d/2 - 1} \sqrt{\frac{\log J}{n'}} \right)^{1 - 1/q}. $$

This is, up to the exponent $1 - 1/q$, the same bound Kloeckner obtained. In particular, the optimal choice of $J$ is given by

$$ J = \lfloor (\log n')^{2 - 1/d} (n')^{1/d} \rfloor. $$

This value can be obtained using the ansatz $J = (n')^\beta$ and ignoring terms of lower order, which leads to the optimal exponent of $n'$ if $\beta = \frac{1}{2}$. Then the estimate can be refined by setting $J = (n')^{1/d} (\log n')^\gamma$, and the optimal power of $\log n'$ is obtained for $\gamma = 2 - \frac{1}{d}$.

As $n' \to \infty$, we get $J \to \infty$ and $\alpha \to 0$. Moreover, up to logarithmic terms the term $J^{d/2 - 1} (n')^{-1/2}$ is of order $n'^{-1/d}$ and therefore $R \to \infty$, hence the above estimates are indeed valid. One can also check that $(2R)^\alpha \to 1$ as $n \to \infty$, hence we do not lose anything by estimating $1/(2R)^\alpha \leq 1$.

We need to verify that the requirement $n' \geq (1 - \kappa)^{-1}$ is satisfied if $n$ is large enough. Since $\kappa = \kappa_{1/\log J} \leq k_{d/\log n'}$, it suffices to show that $\kappa^d \leq (1 - 1/n') \log n'$ if $n$ is large enough. But this is true, as the right-hand side converges to 1 for $n \to \infty$. Hence for these $n$ we finally obtain

$$ \mathbb{E}[W_1(\mu_n, \mu)] \lesssim \left( \left( \frac{n'}{n'} \right)^{d/2 - 1/d} \log J \right)^{1 - 1/q}, $$

completing the proof of the theorem for $d \geq 3$.

Analogously, if $d = 2$ then $\sum_{j=1}^J j^{d-3} = \mathcal{O}(\log J)$, thus we obtain the bound

$$ \mathbb{E}[W_1(\mu_n, \mu)] \lesssim \left( \frac{\log J^2}{J} + \frac{\log J}{\sqrt{n'}} \right)^{1 - 1/q}. $$

This time the optimal value for $J$ is $J = \lfloor \sqrt{n'} \log n' \rfloor$, which leads to the claimed estimate. Finally, if $d = 1$ then $\sum_{j=1}^d j^{d-3} = \mathcal{O}(1)$, and thus we have

$$ \mathbb{E}[W_1(\mu_n, \mu)] \lesssim \left( \frac{\log J}{J} + \sqrt{\frac{\log J}{n'}} \right)^{1 - 1/q}. $$

Here the optimal choice for $J$ is $J = \lfloor \sqrt{n'} \log n' \rfloor$, and then the term in brackets is of order $\sqrt{\frac{\log n'}{n'}}$. This completes the proof of Theorem 1.3.
2.2 Concentration inequalities

To prove Theorem 1.5 we will use standard bounded difference arguments. We first formulate a general result concerning concentration of functions of \( n \) random variables. Let \( Y_1, \ldots, Y_n \) be random variables taking values in \( \mathcal{X} \) and let \( f: \mathcal{X}^n \to \mathbb{R} \) be a bounded function. For \( 1 \leq i \leq j \leq n \) denote the vector \( (Y_i, Y_{i+1}, \ldots, Y_j) =: Y_i^j \). We will also write \( Y = Y_1^n = (Y_1, \ldots, Y_n) \). For given elements \( y_i \in \mathcal{X} \) define

\[
\Delta_k(y_k^i) := \mathbb{E} \left[ f(Y) \mid Y_1^k = y_k^i \right] - \mathbb{E} \left[ f(Y) \mid Y_1^{k-1} = y_1^{k-1} \right], \quad 1 \leq k \leq n.
\]

That is, \( \Delta_k \) denotes how much the expectation of \( f(Y) \) changes under the additional information that \( Y_k \) takes the value \( y_k \). Next, we set

\[
D_k(y_1^{k-1}) = \sup_{x,y \in \mathcal{X}} \| \Delta_k(y_1^{k-1}, x) - \Delta_k(y_1^{k-1}, y) \|_{\infty},
\]

where \( (y_1^{k-1}, x) \) denotes the vector \( (y_1, \ldots, y_{k-1}, x) \). Finally, define

\[
\mathcal{C} := \sup \{ \sum_{k=1}^{n} |D_k(y_1^{k-1})|^2 \mid (y_1, \ldots, y_n) \in \mathcal{X}^n \}.
\]

Then the following result due to McDiarmid holds.

**Lemma 2.7** ([17, Theorem 3.7]). Under the above conditions, for any \( t \geq 0 \) one has

\[
\mathbb{P}(f(Y) \geq \mathbb{E}f(Y) + t) \leq \exp \left( -\frac{2t^2}{2\mathcal{C}} \right).
\]

In order to apply this to the Wasserstein distance \( W_1(\mu, \mu_n) \), we use that the function \( f: \mathcal{X}^n \to \mathbb{R} \) defined by \( f(x_1, \ldots, x_n) := W_1(\mu, \mu_n) \) is Lipschitz with constant \( \frac{1}{n} \). That is, for all \( x_1, \ldots, x_n, x_1', \ldots, x_n' \in \mathcal{X} \) we have

\[
|f(x_1, \ldots, x_n) - f(x_1', \ldots, x_n')| \leq \frac{1}{n} \sum_{k=1}^{n} d(x_k, x_k') =: d_n^{(1)}((x_1, \ldots, x_n), (x_1', \ldots, x_n')).
\]

This follows directly from the triangle inequality for the distance \( W_1 \).

To apply Lemma 2.7 to the function \( f \), we first need a general estimate for Lipschitz functions of the Markov chain.

**Lemma 2.8.** Let \( n \in \mathbb{N} \) and \( f: \mathcal{X}^{n+1} \to \mathbb{R} \) be a function which is Lipschitz with respect to the metric \( d_{n+1}^{(1)} \) with constant \( L \). Then the function \( F: \mathcal{X} \to \mathbb{R} \) defined by

\[
F(x) = \mathbb{E}^x f(X_0, X_1, \ldots, X_n)
\]

is Lipschitz with constant \( L \sum_{j=0}^{n} \kappa^j \leq \frac{L}{1-\kappa} \).

Here \( \mathbb{E}^x \) denotes the expectation for the Markov chain starting in \( X_0 = x \), and \( d_{n+1}^{(1)} \) is given by (4).

**Proof.** We show the statement by induction on \( n \). If \( n = 1 \) we have for \( x, y \in \mathcal{X} \) that

\[
|F(x) - F(y)| = |\mathbb{E}^x f(x, X_1) - \mathbb{E}^y f(y, X_1)|
\]

\[
\leq |\mathbb{E}^x [f(x, X_1) - f(y, X_1)]| + |\mathbb{E}^x f(y, X_1) - \mathbb{E}^y f(y, X_1)|
\]

\[
\leq Ld(x, y) + \left| \int f(y, z) P(x, dz) - \int f(y, z) P(y, dz) \right|
\]

\[
\leq L(1 + \kappa)d(x, y),
\]

Here \( P(x, dz) \) denotes the transition probability of the Markov chain from state \( x \) to state \( z \).
where we used that $W_1(P(x, \cdot), P(y, \cdot)) \leq \kappa d(x, y)$ and the dual formulation Lemma 1.1.

Now suppose that the statement holds for some $n$, and let $F : \mathcal{X}^{n+1} \rightarrow \mathbb{R}$ be Lipschitz with constant $L$. For each $x \in \mathcal{X}$ define $F_n'(z) := \mathbb{E}^z f(x, X_0, \ldots, X_n)$, then by the induction hypothesis $z \mapsto F_n'(z)$ is Lipschitz with constant $L \sum_{j=0}^n \kappa^j$. The Markov property now implies that for $x, y \in \mathcal{X}$

\[
|F(x) - F(y)| \leq |\mathbb{E}^x [f(x, X_1, \ldots, X_{n+1}) - f(y, X_1, \ldots, X_{n+1})]| \\
+ |\mathbb{E}^x [f(y, X_1, \ldots, X_{n+1}) - \mathbb{E}^y f(y, X_1, \ldots, X_{n+1})]| \\
\leq L d(x, y) + \left| \int F_{n+1}'(z) P(x, dz) - \int F_{n+1}'(z) P(y, dz) \right| \\
\leq \left( L + \kappa L \sum_{j=0}^n \kappa^j \right) d(x, y) = L \sum_{j=0}^{n+1} \kappa^j d(x, y),
\]

where we again applied the duality. This completes the proof. \hfill \Box

**Remark.** Similar ideas have been used in [9] to estimate the concentration for separately Lipschitz functions of Markov chains. Here we really need to assume the contractivity with $D = 1$, since otherwise the inductive proof would not be possible.

**Proof of Theorem 1.5.** For $1 \leq i \leq j$ we set $X_i^j := (X_i, X_{i+1}, \ldots, X_j)$. As in Lemma 2.7, for given $x_1, \ldots, x_k \in \mathcal{X}$ we define

\[
\Delta_k(x_1^k) = \mathbb{E}[f(X_1^n) \mid X_1 = x_1, \ldots, X_k = x_k] - \mathbb{E}[f(X_1^n) \mid X_1 = x_1, \ldots, X_{k-1} = x_{k-1}].
\]

Now for all $x, y \in \mathcal{X}$ the Markov property implies

\[
\Delta_k(x_1^{k-1}, x) - \Delta_k(x_1^{k-1}, y) = \mathbb{E}[f(x_1^{k-1}, X_k^n) \mid X_k = x] - \mathbb{E}[f(x_1^{k-1}, X_k^n) \mid X_k = y] \\
= \mathbb{E}^x f(x_1, \ldots, x_{k-1}, x, X_1, \ldots, X_{n-k}) - \mathbb{E}^y f(x_1, \ldots, x_{k-1}, y, X_1, \ldots, X_{n-k}).
\]

Next, we apply Lemma 2.8, which leads to

\[
|\Delta_k(x_1^{k-1}, x) - \Delta_k(x_1^{k-1}, y)| \leq \frac{1}{n} \sum_{j=0}^{n-k} \kappa^j d(x, y) \leq \frac{1}{n(1 - \kappa)} d(x, y).
\]

Since the space $\mathcal{X}$ is bounded by assumption, we have $d(x, y) \leq 1$. Hence Lemma 2.7 can be applied with $C = \frac{1}{n(1 - \kappa)}$ to obtain the result. \hfill \Box

It remains to prove the concentration inequality under transportation assumptions. For this we shall use the well-known fact that a transportation inequality implies sub-Gaussian measure concentration [3].

**Proposition 2.9.** If $\mu$ satisfies $T_1(C)$ then it holds for all $f \in \text{Lip}(\mathcal{X})$ and $t \geq 0$ that

\[
\mu(f \geq \mu(f) + t) \leq \exp \left( - \frac{t^2}{2C \|f\|_{\text{Lip}}} \right).
\]

**Proof of Theorem 1.6.** We know that the function

\[
f : \mathcal{X}^n \rightarrow \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto W_1(\mu, \mu_n), \quad \mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{x_k}
\]

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is $\frac{1}{n}$-Lipschitz with respect to the metric $d^{(1)}_n$ on $\mathcal{X}^n$. Furthermore, [2, Theorem 1.1] implies that the distribution of $(X_1, X_2, \ldots, X_n)$ satisfies $T_1(C_n)$ with respect to the metric $d^{(1)}_n$ on $\mathcal{X}^n$, where

$$C_n = C \sum_{m=1}^{n} \left( \sum_{k=0}^{m-1} \kappa^k \right)^2 \leq C n \left( \sum_{k=0}^{\infty} \kappa^k \right)^2 = \frac{C n}{(1-n)^2}.$$  

Combining this with Proposition 2.9 completes the proof.

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