Capture zones of the family of functions $\lambda z^m \exp(z)$.

Núria Fagella *
Dep. de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via de les Corts Catalanes, 585
08005 Barcelona, Spain
fagella@maia.ub.es

Antonio Garijo †
Dep. d’Eng. Informàtica i Matemàtiques
Universitat Rovira i Virgili
Av. Països Catalans, 26
43007 Tarragona, Spain
agarijo@etse.urv.es

Abstract

We consider the family of entire transcendental maps given by $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ where $m \geq 2$. All functions $F_{\lambda,m}$ have a superattracting fixed point at $z = 0$, and a critical point at $z = -m$. In the dynamical plane we study the topology of the basin of attraction of $z = 0$. In the parameter plane we focus on the capture behaviour, i.e., $\lambda$ values such that the critical point belongs to the basin of attraction of $z = 0$. In particular, we find a capture zone for which this basin has a unique connected component, whose boundary is then non-locally connected. However, there are parameter values for which the boundary of the immediate basin of $z = 0$ is a quasicircle.

1 Introduction

Our goal in this paper is to study some dynamical aspects of the families of entire transcendental maps

$$F_{\lambda,m}(z) = \lambda z^m \exp(z), \quad m \geq 2.$$ 

Observe that $m = 0$ corresponds to the exponential family $E_{\lambda}(z) = \lambda \exp(z)$, the simplest example of an entire transcendental map with a unique asymptotic value, $z = 0$, in analogy with the well known quadratic family of polynomials $z \to z^2 + c$. The exponential map has been thoroughly studied by many authors (see for example, [Devaney & Krych, 1984], [Devaney & Tangerman, 1986]).

The case $m = 1$ corresponds to $G_{\lambda}(z) = \lambda z \exp(z)$ which appeared for the first time in [Baker, 1970] as an example of an entire transcendental map whose Julia set is the whole plane (for an appropriate value of $\lambda$). Later on this family was studied in [Fagella, 1995] and [Geyer, 2001]. The asymptotic value $z = 0$ of $G_{\lambda}(z)$ is fixed and its multiplier is $G'_{\lambda}(0) = \lambda$.

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Hence its dynamical character depends on the parameter $\lambda$. Besides this point, the dynamical behavior of $G_\lambda$ is determined by the orbit of the critical point $z = -1$.

Some functions in the family $F_{\lambda, m} = \lambda z^m \exp(z)$ for $m \geq 2$ have been used in the literature as examples of certain dynamical phenomena (see for example [Bergweiler, 1995], for a Baker domain at a positive distance from any singular orbit). But, to our knowledge, no systematic study has been made before this work.

All functions of the form $F_{\lambda, m}$, with $m \geq 2$, have a superattracting fixed point at $z = 0$ of multiplicity $m$, which is also an asymptotic value. The only other critical point is $z = -m$.

The coexistence of a superattracting fixed point and a free critical point makes this family very much analogous to the family of cubic polynomials $C_a(z) = z^3 - 3a^2 z + 2a^3 + a$ which is described in [Milnor, 1991].

Let $f$ be a transcendental entire function. It is known that the dynamical plane can be decomposed into two invariant sets. The first one is an open set, namely the Fatou set or stable set, denoted by $F(f)$, and it is formed by points, $z_0 \in \mathbb{C}$ whose iterates $\{f^{n}\}$ form a normal family, in the sense of Montel, in some neighbourhood of $z_0$. The second one, namely the Julia or chaotic set, denoted by $J(f)$, is defined as the complement of the Fatou set, that is $J(f) = \mathbb{C} - F(f)$. Fatou ([Fatou, 1926]) showed that the Julia set of an entire transcendental function is a completely invariant, closed, nonempty and perfect set. As in the polynomial case, it may also be defined as the closure of the set of repelling periodic points.

We denote by $A(0) = A_{\lambda, m}(0)$ the basin of attraction of $z = 0$, i.e., the set of all $z$ such that $F_{\lambda, m}^n(z) \to 0$ as $n$ tends to $+\infty$. We also denote by $A^*(0) = A^*_{\lambda, m}(0)$ the connected component of $A(0)$ containing $z = 0$.

In Sec. 2 we concentrate on the dynamical plane and study the basin of attraction $A(0)$ (see Fig. 1). The skeleton of the main components of $A(0)$ is needed to study later the parameter planes. We summarize the main results with regard to $A(0)$ in the following theorem.

**Theorem A.**

1. There exists $\epsilon_0 = \epsilon_0(|\lambda|, m) > 0$, defined as the unique positive solution of $x^{m-1} e^x = 1/|\lambda|$, such that $A^*(0)$ contains the disk $D_{\epsilon_0} = \{z \in \mathbb{C}; |z| < \epsilon_0\}$.

2. There exist $x_0 = x_0(|\lambda|, m) < 0$ and a function $C(x) \geq 0$ such that the open set

$$H_{|\lambda|, m} = \left\{ z = x + yi \middle| \begin{array}{l} x \in (-\infty, x_0) \\ y \in (-C(x), C(x)) \end{array} \right\}$$

satisfies $F_{\lambda, m} (H_{|\lambda|, m}) \subset D_{\epsilon_0}$.

3. There exist infinitely many strips, denoted by $S^k_{\lambda, m}$, which are preimages of $H_{|\lambda|, m}$. These horizontal strips extend to $+\infty$, and they have asymptotic width equal to $\pi$.

Section 3 is dedicated to parameter planes. For some parameter values, the free critical point $z = -m$ belongs to the basin of attraction of $z = 0$, $A(0)$, in which case we say...
that it has been “captured”. The connected components of parameter space for which this phenomenon occurs are called capture zones, and they clearly do not exist for members of the family with $m < 2$ such as the exponential. Hence it is natural to study the set of parameters for which the orbit of $z = -m$ is bounded. That is, we define the sets

$$B_m = \{ \lambda \in \mathbb{C} \mid F_{\lambda,m}^n(-m) \not\to \infty \}. $$

In each of these sets, we may also distinguish between two different behaviours. Those parameter values for which $-m \in A^*(0)$ and those for which this does not occur. Let $\hat{B}_m$ denote the interior of $B_m$. We will study the sets

$$C_m^n = \{ \lambda \in \hat{B}_m \mid F_{\lambda,m}^n(-m) \in A^*(0) \text{ and } n \text{ is the smallest number with this property} \}.$$

Although each $B_m$ contains infinitely many different capture zones, there is one which is dynamically very different from all others. We define the main capture zone $C_m^0$ as the set of parameter values $\lambda$ for which the critical point $z = -m$ belongs to the immediate basin of 0. That is,

$$C_m^0 = \{ \lambda \in \hat{B}_m \mid -m \in A^*(0) \}.$$ 

As we shall see, this is a quite special component of $B_m$ since its boundary separates the parameter values for which $\partial A^*(0)$ is a Cantor bouquet from those for which it is a Jordan curve (also, this boundary separates the parameter values for which $\mathcal{F}(F_{\lambda,m})$ has one or infinitely many components). The detailed study of this boundary will be the object of a future paper. Our goal in Sec. 3 is to describe the main features of the parameter planes of the functions $F_{\lambda,m}$ and, in particular, the structure of the capture zones (Fig. 1). We summarize some of these facts in the following theorem.

Theorem B.
1. The critical point $-m$ belongs to $A^*(0)$ if and only if the critical value $F_{\lambda,m}(-m)$ belongs to $A^*(0)$. Hence $C_m^1 = \emptyset$.

2. The main capture zone $C_m^0$ is bounded.

3. The set $C_m^0$ contains the disk \( \{ \lambda \in \mathbb{C} \mid |\lambda| < \min\left(\frac{1}{e}, \left(\frac{e}{m}\right)^m\right) \} \).

4. If $\lambda \in C_m^0$ then $A(0) = A^*(0)$, i.e., the basin of attraction of $z = 0$ has a unique connected component and hence it is totally invariant. Moreover, the boundary of $A^*(0)$ (which equals the Julia set) is a Cantor bouquet and hence it is disconnected and non-locally connected.

5. If $\lambda \notin C_m^0$ then $A(0)$ has infinitely many components. Moreover, if $|\lambda| > \left(\frac{e}{m-1}\right)^{m-1}$, the boundary of $A^*(0)$ is a quasi-circle.

We also summarize some properties of the most obvious capture zones $C_m^2$ and $C_m^3$.

**Theorem C.**

1. The set $C_m^2$ contains an unbounded set to the left or to the right depending on the oddity of $m$. More precisely, there exists a real constant $D_0(m) > 0$, and a function $\alpha = \alpha(|\lambda|, m) \in (\pi/2, \pi)$, such that

   - for $m$ even, the set $C_m^2$ contains the open set $\{ \lambda \in \mathbb{C} \mid |\lambda| > D_0 \}$,
   - for $m$ odd, the set $C_m^2$ contains the open set $\{ \lambda \in \mathbb{C} \mid |\lambda| > D_0 \}$.

2. There exists infinitely many strips in $C_m^3$. If $m$ is even (resp. odd) then these horizontal strips extend to $+\infty$ (resp. $-\infty$) and they have an asymptotic width equal to $\left(\frac{e}{m}\right)^m \pi$.

![Figure 2: Sketch of capture zones contained in $B_m^0$. On the left hand side when $m$ is even and on the right hand side when $m$ is odd.](image-url)
2 The Dynamical Planes

Our goal in this Sec. is to describe the dynamical plane of the families of maps $F_{\lambda,m}$ given by the equation $F_{\lambda,m}(z) = \lambda z^m \exp(z)$, where $m \geq 2$.

The function $F_{\lambda,m}$ is a critically finite entire function, that is, it has a finite number of asymptotic values ($z = 0$), and critical values ($z = 0$ and $z = (-1)^m \lambda (\frac{m}{2})^m$). For this kind of functions there exists a characterization of the Julia set ([Devaney & Tangerman, 1986]), namely as the closure of the set of points whose orbits tend to $\infty$.

Using the characterization above we can plot an approximation of $J(F_{\lambda,m})$. Generally, orbits tend to $\infty$ in specific directions. In our case, if $\lim_{n \to \infty} |F_{\lambda,m}^n(z)| = +\infty$, then we have $\lim_{n \to \infty} \text{Re}(F_{\lambda,m}^n(z)) = +\infty$. Thus, an approximation of the Julia set is given by the set of points whose orbit contains a point with real part greater than, say, 50.

In Figs. 3-4, we display the Julia set of $F_{\lambda,m}$ for different values of $\lambda$ and $m$. The basin of attraction of $z = 0$ is shown in red, while the components of the Fatou set different from $A(0)$ are shown in blue. Points in the Julia set are shown in black.

![Figure 3](image3.png)  
(a) $\lambda = -2.1$  
(b) $\lambda = -8$  
(c) $\lambda = 6.9$

Figure 3: The Julia set for $F_{\lambda,2}$. Range $(-10,10) \times (-10,10)$.

In Fig. 3 we show the dynamical plane of function $F_{\lambda,2} = \lambda z^2 \exp(z)$, for three different values of $\lambda$. Apparently, the basin of 0 contains an infinite number of horizontal strips, that extend to $+\infty$ as their real parts tend to $+\infty$. Between these strips we find the well known structures, named Cantor Bouquets which are invariant sets of curves governed by some symbolic dynamics. The existence of this kind of structures in the Julia set are typical for critically finite entire transcendental functions ([Devaney & Tangerman, 1986]).

As we change the parameter $\lambda$ we observe that the relative position of these bands also changes, but not their width. Also, we can see the existence of an unbounded region that extends to $-\infty$ contained in $A(0)$.

In the next figure (Fig. 4) we show a mosaic of different dynamical planes for some values of $m$, specifically for $m = 2, 3, 4,$ and $5$. We choose $\lambda$ such that all these dynamical planes exhibit the same dynamical behaviour, or more precisely, so that the critical point $z = -m$,
is a superattracting fixed point. A simple computation gives \( \lambda = (-1)^{m-1}m\left(\frac{e}{m}\right)^m \). In these dynamical planes we see similar structures as in Fig. 3, even though the values of \( m \) are different.

Figure 4: The dynamical plane of \( F_{\lambda,m} \) for different values of \( m \). In every case \( \lambda = (-1)^{m-1}m\left(\frac{e}{m}\right)^m \). Range \((-10, 10) \times (-10, 10)\).

We start with the following general result regarding \( A(0) \).

**Lemma 2.1.** \( A(0) \) has either one or infinitely many connected components. Moreover, connected components different from \( A^*(0) \) are unbounded.

**Proof.** Using Sullivan’s theorem (Sullivan, 1985) we have that \( A^*(0) \) is the unique fixed connected component of \( A(0) \). For all other connected components of \( A(0) \) there exists a number \( i > 0 \) such that \( F_{\lambda,m}^i(U) = A^*(0) \), and \( i \) is the smallest number with this property. Suppose that there exist a finite number of connected components, and let \( U_0 = A^*(0) \), \( U_1 \), \( U_2 \), ..., \( U_N \) the connected components of \( A(0) \). We may choose the index \( i \) in the natural way so that \( F_{\lambda,m}^i(U_i) = A^*(0) \). Let \( z \in U_N \) such that is not exceptional; then, points in \( F_{\lambda,m}^{-1}(z) \) belong to \( A(0) \), but not to \( U_0 \cup U_1 \cup \ldots \cup U_N \), which is a contradiction.
Now suppose that $U$ is a connected component of $A(0)$ different from $A^*(0)$, and let $i > 0$ be the smallest number such that $F^i_{\lambda, m}(U) = A^*(0)$. Let $z \in U$, and denote by $\gamma$ a simple path in $A^*(0)$ that joins $F^i_{\lambda, m}(z)$ and 0. The preimage of $\gamma$ in $U$ must include a path $\gamma_1$ that joins $z$ and $\infty$, since 0 is an asymptotical value with no other finite preimage than itself. Thus we conclude that $U$ is unbounded.

2.1 Proof of Theorem A

In this Sec. we describe the basin of attraction of the superattracting fixed point $z = 0$. Since $z = 0$ is a superattracting fixed point, there exists $\epsilon_0 > 0$ such that the open disk $D_{\epsilon_0} = \{ z \in \mathbb{C}; |z| < \epsilon_0 \}$ is contained in the immediate basin of attraction of $z = 0$. First, we give an estimate of the size of the immediate basin of attraction, $A^*(0)$ (Proposition 2.3), which will prove the first part of theorem A. Secondly, we find the first preimage of $D_{\epsilon_0}$ (Proposition 2.5 and Proposition 2.6), proving the second part of theorem A. Finally, we find the second preimage of $D_{\epsilon_0}$ (Proposition 2.8), and prove the third part of theorem A.

Before proving Proposition 2.3 we first look at some properties of the real function $h(x) = x^m e^x$ where $m \geq 1$. In Fig. 5 we show the graph of this function. It has a relative extremum at $x = -m$, is a monotone function on $(-\infty, -m)$ and it satisfies that $|h(x)| \leq |h(-m)|$ for all $x \leq 0$. Also, it is an increasing function in $(0, +\infty)$.

![Graph of $h(x) = x^m \exp(x)$](image)

Figure 5: Graph of $h(x) = x^m \exp(x)$. The left hand side corresponds to $m$ even and the right hand side to $m$ odd.

Using these properties it is easy to prove the following auxiliary result.

**Lemma 2.2.** Given $r \in (0, |h(-m)|) = (0, (\frac{m}{e})^m]$, the equation $|h(x)| = r$ has a unique solution in $(-\infty, -m)$. Moreover, given $s > 0$, the equation $h(x) = s$ has a unique solution in $(0, +\infty)$.

We now turn to estimate the function $\epsilon_0(|\lambda|, m)$, and find its dependency on $\lambda \in \mathbb{C} - \{0\}$ and $m \geq 2$.

**Proposition 2.3.** If we define $\epsilon_0$ as the unique positive solution of the real equation

$$x^{m-1} e^x = \frac{1}{|\lambda|},$$

then $D_{\epsilon_0} = \{ z \in \mathbb{C}; |z| < \epsilon_0 \} \subset A^*(0)$. Moreover, if $\lambda \in \mathbb{R}^+$ we have that $\epsilon_0$ lies in $\partial A^*(0)$.
Proof. In order to prove that $D_{\epsilon_0}$ is contained in $A^*(0)$, we use Schwartz’s lemma. That is, it suffices to prove that if $|z| \leq \epsilon_0$ then $|F_{\lambda,m}(z)| \leq \epsilon_0$.

Suppose $|z| \leq \epsilon_0$, we have

$$|F_{\lambda,m}(z)| = |\lambda z^m e^z| = |\lambda||z|^m e^{Re(z)} \leq |z||\lambda||z|^{m-1} e^{|z|}.$$

Since $|z| < \epsilon_0$ it follows that $|z|^{m-1} e^{|z|} < \epsilon_0^{m-1} e^{\epsilon_0}$, and using that $\epsilon_0^{m-1} e^{\epsilon_0} = 1/|\lambda|$, we conclude

$$|F_{\lambda,m}(z)| \leq |z||\lambda||z|^{m-1} e^{|z|} \leq |z| \leq \epsilon_0.$$

If $\lambda \in \mathbb{R}^+$, we have that $\lambda \epsilon_0^m \exp(\epsilon_0) = \epsilon_0$, i.e., $\epsilon_0$ is a fixed point. The multiplier of this fixed point is $\epsilon_0 + m > 1$, and hence $\epsilon_0$ lies in the Julia set. By definition we have that $D_{\epsilon_0} \subset A^*(0)$, then $\epsilon_0$ lies in the boundary of $A^*(0)$.

In the following auxiliary result we find a lower bound for $\epsilon_0$, which will be used in the next Sec..

Lemma 2.4. The value of $\epsilon_0$ is always larger or equal than $\min\{1, (\frac{1}{|\lambda|e})^{\frac{1}{m-1}}\}$

Proof. Suppose $|\lambda| \geq 1/e$. This condition is equivalent to $\frac{1}{|\lambda|e} \leq 1$, hence we must prove that $\epsilon_0 \geq (\frac{1}{|\lambda|e})^{1/(m-1)}$. Using that $x^{m-1}e^x$ is an increasing function on $(0, +\infty)$, this condition is equivalent to

$$\epsilon_0^{m-1} e^{\epsilon_0} \geq \frac{1}{|\lambda|e} \ e^{\left(\frac{1}{|\lambda|e}\right)^{\frac{1}{m-1}}}.$$

By definition we have that $\epsilon_0^{m-1} e^{\epsilon_0} = \frac{1}{|\lambda|}$, then

$$\frac{1}{|\lambda|} \geq |\lambda|e^{\left(\frac{1}{|\lambda|e}\right)^{\frac{1}{m-1}}}$$

or equivalently

$$e \geq e^{\left(\frac{1}{|\lambda|e}\right)^{\frac{1}{m-1}}}$$

and this follows if $|\lambda| \geq 1/e$.

If $|\lambda| \leq 1/e$, we must prove that $\epsilon_0 \geq 1$. Using the same argument, i.e., that $x^{m-1}e^x$ is an increasing function, it follows that this condition is equivalent to

$$\epsilon_0^{m-1} e^{\epsilon_0} \geq e$$

and this follows if $|\lambda| \leq 1/e$.\qed
Next we want to find an open set \( H_{|\lambda|, m} \subset \mathbb{C} \) such that \( F_{\lambda,m}(H_{|\lambda|, m}) \subset D_{\epsilon_0} \) (Fig. 6). To that end, we first obtain a value in \( \mathbb{R}^- \), namely \( x_0 = x_0(|\lambda|, m) \), such that for all \( x \in \mathbb{R}^- \) with \( x \leq x_0 \) we have that \( F_{\lambda,m}(x) \in D_{\epsilon_0} \) (Proposition 2.5). After finding \( x_0 \), we will look for an upper bound \( C(x) \geq 0 \), such that if \( z = x + yi \), with \( x < x_0 \) and \( |y| \leq C(x) \), then \( F_{\lambda,m}(z) \in D_{\epsilon_0} \) (Proposition 2.6).

![Figure 6: Sketch of \( H_{|\lambda|, m} \) satisfying \( F_{\lambda,m}(H_{|\lambda|, m}) \subset D_{\epsilon_0} \subset A^*(0) \).](image)

**Proposition 2.5.** For all \( \lambda \in \mathbb{C} \) and \( m \geq 2 \), there exists \( x_0 \in (-\infty, -m] \), such that for all \( x \leq x_0 \), we have \( F_{\lambda,m}(x) \in D_{\epsilon_0} \).

**Proof.** We suppose that \( z = x + 0i \) and we impose \( |F_{\lambda,m}(z)| = \epsilon_0 \), that is

\[
|F_{\lambda,m}(z)| = |\lambda||x|^m e^x = \epsilon_0
\]

or equivalently

\[
|h(x)| = |x|^m e^x = \frac{\epsilon_0}{|\lambda|},
\]

where \( h(x) \) is the auxiliary function defined above.

If \( |h(-m)| \leq \frac{\epsilon_0}{|\lambda|} \), then we take \( x_0 = -m \), and for all \( x \in (-\infty, x_0) \), we have \( |h(x)| \leq |h(-m)| \leq \frac{\epsilon_0}{|\lambda|} \). On the other hand, if \( |h(-m)| > \frac{\epsilon_0}{|\lambda|} > 0 \), we define \( x_0 \) as the unique solution of equation \( |h(x)| = \frac{\epsilon_0}{|\lambda|} \) in the interval \((-\infty, -m)\). Since \( |h(x)| \) is an increasing function in \((-\infty, -m)\), it follows that for all \( x \in (-\infty, x_0) \), then \( |h(x)| \leq |h(x_0)| = \frac{\epsilon_0}{|\lambda|} \).

**Proposition 2.6.** Let \( x_0 = x_0(|\lambda|, m) \) be as in Proposition 2.5. There exists \( C(x) \geq 0 \) such that the open set (Fig. 6)

\[
H_{|\lambda|, m} = \left\{ z = x + yi \left| \begin{array}{l}
x \in (-\infty, x_0) \\
y \in (-C(x), C(x))
\end{array} \right. \right\}
\]

satisfies \( F_{\lambda,m}(H_{|\lambda|, m}) \subset D_{\epsilon_0} \).
We want to calculate an upper bound $C(x) \geq 0$, such that if $z = x + yi$, with $x \in (-\infty, x_0)$ and $|y| \leq C(x)$, then $F_{\lambda,m}(z) \in D_{\epsilon_0}$. If we require that $F_{\lambda,m}(z) \in D_{\epsilon_0}$, we obtain the definition of $C(x)$. Let $z = x \pm yi$, with $x \in (-\infty, x_0)$. Then

$$|F_{\lambda,m}(z)| = |\lambda||z|^m \exp(Re(z)) = |\lambda|\left[\sqrt{x^2 + y^2}\right]^m \exp(x) = |\lambda|(x^2 + y^2)^{m/2} \exp(x).$$

Using the expression above, and requiring $|F_{\lambda,m}(z)| \leq \epsilon_0$, we obtain

$$|y| \leq \sqrt{\epsilon_0^{2/m} \exp(-x^2/m) - x^2}.$$

Thus, the right hand side of this inequality gives an analytic expression for the function $C(x)$.

The next lemma gives a simpler condition to assure that a point $z \in \mathbb{C}$ lies in $H_{|\lambda|,m}$. See Fig. 7.

**Lemma 2.7.** The point $z = x + yi$ lies in $H_{|\lambda|,m}$ if there exists $k \geq 1$ and $A \geq 0$, such that $|y| \leq A|x|^k$ and $|x|$ is large enough.

**Proof.** We will prove that $A|x|^k \leq C(x)$ as $x \to -\infty$. Using the definition of $C(x)$, this is equivalent to showing

$$A|x|^k < \sqrt{\left[\frac{\epsilon_0}{|\lambda|}\right]^{m/2} \exp(-x^2/m) - x^2},$$

or

$$\left[A^2|x|^{2k} + x^2\right] \exp(x^2/m) < \left[\frac{\epsilon_0}{|\lambda|}\right]^{m/2}.$$

The left hand side of this inequality is a function that tends to zero as $x$ tends to $-\infty$, whereas the right hand side is positive.

We proceed now to the third iterate, by proving the existence of some strips in dynamical plane, such that the image of these open sets under $F_{\lambda,m}$ is contained in $H_{|\lambda|,m}$ (see Fig. 7).

Before calculating the preimage of the set $H_{|\lambda|,m}$, we first find the preimage of the negative real axis under the function $F_{\lambda,m}$. Hereafter, we denote by $Arg(.) \in (-\pi, \pi]$ the argument. Using the definition of $F_{\lambda,m}$ it is easy to see that
Figure 7: Relation between $H_{|\lambda|,m}$ and $y = A|x|^k$ for $k = 1, 2$

$$\text{Arg}(F_{\lambda,m}(z)) = \text{Arg}(\lambda) + m\text{Arg}(z) + \text{Im}(z) \pmod{2\pi}. $$

Finding the preimages of $\mathbb{R}^-$ is equivalent to solving

$$\text{Arg}(F_{\lambda,m}(z)) = \pi. $$

We denote $r = |z|$ and $\alpha = \text{Arg}(z)$. Then the equation above is equivalent to

$$\text{Arg}(\lambda) + m\alpha + r\sin(\alpha) = (2k + 1)\pi \quad k \in \mathbb{Z}. $$

Hence, we obtain

$$r = \rho(\alpha) = \frac{(2k + 1)\pi - m\alpha - \text{Arg}(\lambda)}{\sin(\alpha)} \quad \alpha \in (-\pi, \pi).$$

We denote each of these curves by $\sigma_k = \sigma_k(\lambda, m)$, where the possible values of the argument depend on $k$. More precisely,

if $m = 2j$ for $j \in \mathbb{Z}$

$$\sigma_k = \rho(\alpha)e^{i\alpha} \quad \text{where} \quad \begin{cases} 0 < \alpha < \pi & \text{if } k \geq j \\ 0 < \alpha < \frac{(2k+1)\pi - \text{Arg}(\lambda)}{m} & \text{if } 0 \leq k \leq j - 1 \\ \frac{(2k+1)\pi - \text{Arg}(\lambda)}{m} < \alpha < 0 & \text{if } -j \leq k \leq 0 \\ -\pi < \alpha < 0 & \text{if } k \leq -(j + 1) \end{cases}$$

if $m = 2j + 1$ for $j \in \mathbb{Z}$

$$\sigma_k = \rho(\alpha)e^{i\alpha} \quad \text{where} \quad \begin{cases} 0 < \alpha < \pi & \text{if } k \geq j + 1 \\ 0 < \alpha < \frac{(2k+1)\pi - \text{Arg}(\lambda)}{m} & \text{if } 0 \leq k \leq j \\ \frac{(2k+1)\pi - \text{Arg}(\lambda)}{m} < \alpha < 0 & \text{if } -j \leq k \leq 0 \\ -\pi < \alpha < 0 & \text{if } k \leq -(j + 1) \end{cases}$$
In Fig. 8 we show some of these curves for $m = 5$. As their real parts tend to $+\infty$, the $\sigma_k$’s are asymptotic to the lines $\text{Im}(z) = (2k + 1)\pi - \text{Arg}(\lambda)$. There are $m$ of these curves, that start at the origin and tend to $+\infty$. The others are asymptotic to the lines $\text{Im}(z) = (2k + 1)\pi - \text{Arg}(\lambda)$ when $k < 0$, or $\text{Im}(z) = 2k\pi - \text{Arg}(\lambda)$ when $k > 0$.

![Graph of $\sigma_k$ for $\lambda = 0.45 + 0.35i$ and $m = 5$.](image1)

![The julia set of $F_{0.45+0.35i,5}$.](image2)

**Figure 8:** Strips in the dynamical plane.

Now we may find preimages of the open set $H_{|\lambda|,m}$. First we find preimages of the interval $(-\infty, x_0)$. There exists one preimage of this interval on each curve $\sigma_k$. Moreover, the preimage of a real number tending to $-\infty$, is a complex number on $\sigma_k$ whose real part tends to $+\infty$. Hence, the preimages of the set $H_{|\lambda|,m}$ contain some strips, namely $S_{|\lambda|,m}^k$, around $\sigma_k$. (See Fig. 8).

Let $z \in \sigma_k$. If we evaluate $F_{\lambda,m}(z)$, we have

$$F_{\lambda,m}(z) = |\lambda||z|^m e^{Re(z)} e^{[\text{Arg}(\lambda)+m\text{Arg}(z)+\text{Im}(z)]i} = -|\lambda||z|^m e^{Re(z)}$$

since each $\sigma_k$ is a preimage of the negative real axis. The expression above shows that if we keep $Re(z)$ constant, and we increase the index $k$ of the curve $\sigma_k$, we obtain values tending to $-\infty$, since $|z|$ increases. Hence, if we denote by $q_0(k)$ the preimage of $x_0$ on $\sigma_k$, its real part decreases as $|k|$ increases (at least after a certain point). This fact explains the apparent arrangement of the strips in dynamical plane (Fig. 8).

We now prove that these strips have an asymptotic width equal to $\pi$. We fix a value $k \in \mathbb{Z}$, and we recall that $\sigma_k$ tends asymptotically to the line $\text{Im}(z) = (2k + 1)\pi - \text{Arg}(\lambda)$, as its real part tends to $+\infty$.

**Proposition 2.8.** Given any $y \in ((2k + 1)\pi - \text{Arg}(\lambda) - \frac{\pi}{2}, (2k + 1)\pi - \text{Arg}(\lambda) + \frac{\pi}{2})$, there exists a real number $x_*$ such that for all $x \geq x_*$, $F_{\lambda,m}(x+yi) \in H_{|\lambda|,m}$. 

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Proof. Let $z = x + iy$, where $y \in ((2k + 1)\pi - \text{Arg}(\lambda) - \frac{\pi}{2}, (2k + 1)\pi - \text{Arg}(\lambda) + \frac{\pi}{2})$. If we write the parameter $\lambda$ in polar coordinates, $\lambda = se^{i\beta}$, then

$$F_{\lambda,m}(z) = \lambda z^m \exp(z) = se^{i\beta} (x + yi)^m e^y e^{iy} = s(x + yi)^m e^y e^{i(y+\beta)} = \{P(x) + Q(x)i\} e^y \{\cos(y + \beta) + \sin(y + \beta)i\}$$

where

$$\begin{align*}
P(x) &= Re(s(x + yi)^m) = sx^m + O(x^{m-2}) \\
Q(x) &= Im(s(x + yi)^m) = msyx^{m-1} + O(x^{m-3}).
\end{align*}$$

Using the expressions above in $F_{\lambda,m}(z)$, we obtain

$$F_{\lambda,m}(z) = \{P(x) \cos(y + \beta) - Q(x) \sin(y + \beta)\} ex + \{P(x) \sin(y + \beta) + Q(x) \cos(y + \beta)\} ei = \{sx^m \cos(y + \beta) + O(x^{m-1})\} e^x + \{sx^m \sin(y + \beta) + O(x^{m-1})\} ei. \quad (1)$$

We recall that $y \in (\frac{\pi}{2} + 2k\pi - \beta, \frac{3\pi}{2} + 2k\pi - \beta)$, or equivalently $y + \beta \in (\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi)$ which implies that $-1 \leq \cos(y + \beta) < 0$

Since $Re(F_{\lambda,m}(z)) = \{sx^m \cos(y + \beta) + O(x^{m-1})\} e^x$, it follows that

$$\lim_{x \to +\infty} Re(F_{\lambda,m}(z)) = \lim_{x \to +\infty} \{\cos(y + \beta)x^m + O(x^{m-1})\} e^x = -\infty.$$ 

Hence, there exists $x_0$ large enough, such that $Re(F_{\lambda,m}(z)) < x_0$ for $x \geq x_0$.

Finally, we use lemma 2.7 to conclude the proof. From equation (1) we have

$$\lim_{x \to +\infty} \frac{Re(F_{\lambda,m}(z))}{Im(F_{\lambda,m}(z))} = \lim_{x \to +\infty} \frac{sx^m \cos(y + \beta) + O(x^{m-1})}{sx^m \sin(y + \beta) + O(x^{m-1})} = \frac{\cos(y + \beta)}{\sin(y + \beta)}.$$ 

Hence, if $\sin(y + \beta) \neq 0$, it follows that $Im(F_{\lambda,m}(z)) \approx \frac{\sin(y + \beta)}{\cos(y + \beta)} Re(F_{\lambda,m}(z))$, and we can apply lemma 2.7 (with $A = \frac{\sin(y + \beta)}{\cos(y + \beta)}$ and $k = 1$). Using this lemma, we conclude that $F_{\lambda,m}(z)$ lies in $H_{|\lambda|,m}$ if $x$ is large enough.

On the other hand, if $\sin(y + \beta) = 0$, then $\cos(y + \beta) = -1$, and we obtain

$$F_{\lambda,m}(z) = \{-P(x) - Q(x)i\} e^x = \{-sx^m + O(x^{m-1})\} e^x + \{msyx^{m-1} + O(x^{m-2})\} ie^x.$$
Hence
\[
\lim_{x \to +\infty} \frac{\text{Re}(F_{\lambda,m}(z))}{\text{Im}(F_{\lambda,m}(z))} = \lim_{x \to +\infty} \frac{-sx^m + O(x^{m-1})}{-msyx^{m-1} + O(x^{m-2})} = +\infty.
\]

If \(x\) is large enough, there exists \(K > 0\) such that
\[
\left| \frac{\text{Re}(F_{\lambda,m}(z))}{\text{Im}(F_{\lambda,m}(z))} \right| > K
\]
that is, \(|\text{Im}(F_{\lambda,m}(z))| \leq \frac{1}{K}|\text{Re}(F_{\lambda,m}(z))|\). Hence, using lemma 2.7 \((A = \frac{1}{K}, \ k = 1)\), we also obtain that \(F_{\lambda,m}(z)\) lies in \(H_{|\lambda|,m}\).

\[
\square
\]

3 The Parameter Planes

The orbit of the free critical point \(z = -m\), determines in large measure the dynamics of \(F_{\lambda,m}\). Indeed, the functions \(F_{\lambda,m}(z) = \lambda z^m \exp(z)\) are entire maps with a finite number of critical and asymptotic values. These kind of functions do not have wandering domains nor Baker domains. By the Sullivan classification, we know that if the orbit of \(z = -m\) tends to \(\infty\) then the Fatou set must coincide with the basin of 0, i.e., \(\mathcal{F}(F_{\lambda,m}) = A(0)\), since no other Fatou components can exist besides those that belong to \(A(0)\). The set \(B_m\) is defined as before as
\[
B_m = \{\lambda \in \mathbb{C} | F_{\lambda,m}^n(-m) \not\to \infty\}.
\]

In each of these sets, we may also distinguish between two different behaviours: those parameters values for which \(-m \in A(0)\) and those for which this does not occur. Let \(B_m^\circ\) denote the interior of \(B_m\).

**Definition.** Let \(U\) be a connected component of \(B_m^\circ\). We say that \(U\) is a **capture zone** if for all \(\lambda\) in \(U\) it is true that \(\lim_{n \to +\infty} F_{\lambda,m}^n(-m) = 0\), or in other words, \(-m \in A(0)\). We then say that the orbit of the critical point is captured by the basin of attraction of the superattracting fixed point \(z = 0\).

In Figs. 9-10, we show a numerical approximation of the set \(B_m\) for different values of \(m\). The capture zones are shown in red, while other components of \(B_m\) are shown in blue. The parameter values for which the orbit of the free critical point tends to \(\infty\) are shown in black. In these sets we can see a countable quantity of horizontal strips. If \(m\) is even these strips extend to \(+\infty\) as the real part of \(\lambda\) tends to \(+\infty\), whereas if \(m\) is odd these strips extend to \(-\infty\) as the real part of \(\lambda\) tends to \(-\infty\). Not surprisingly, the distribution of these capture zones in the parameter plane (Fig. 2) appears to be similar to the distribution of \(A(0)\) in the dynamical plane (Fig. 1).

We start with the following simple facts.
Proposition 3.1. Let $U$ be a capture zone of $B_m$ and let $\lambda \in U$. Then $F(F_{\lambda,m}) = A(0)$ and $\mathcal{J}(F_{\lambda,m}) = \partial A(0)$.

Proof. As in the case of the critical value tending to $\infty$, since the only free critical point of $F_{\lambda,m}$ lies in the basin of 0, no other components different from those in $A(0)$ can exist in $\mathcal{F}(F_{\lambda,m})$. Let $\mathcal{D}$ be the union of all the components of the Fatou set, then $\mathcal{J}(F_{\lambda,m}) = \partial \mathcal{D}$ ([Carleson & Gamelin, 1993]). In our case, if $\lambda \in U$ and $U$ is a capture zone, then $\mathcal{D} = A(0)$.

The main objective of this Sec. is to describe the most obvious capture zones contained

Figure 9: Parameter plane for $F_{\lambda,2}$. Color codes are explained in the text.
in $B_m$, as well as to describe the dynamical plane for parameter values that belong to such components.

### 3.1 Proof of Theorem B

In this Sec. we describe the main capture zone $C^0_m$. We recall that

$$ C^0_m = \{ \lambda \in B_m \mid F_{\lambda,m}^n(-m) \in A^*(0) \text{ and } n \text{ is the smallest number with this property} \} $$

We prove each statement of theorem B in a different proposition.

**Proposition 3.2.** The critical point $-m$ belongs to $A^*(0)$ if and only if the critical value $F_{\lambda,m}(-m)$ belongs to $A^*(0)$. Hence $C^1_m = \emptyset$. 

---

![Figure 10](image-url): Parameter plane for $F_{\lambda,m}$, for different values of $m$. 

(a) $B_3$. Range $(-12, 12) \times (-12, 12)$

(b) $B_4$. Range $(-4, 2) \times (-3, 3)$

(c) $B_5$. Range $(-0.8, 0.8) \times (-0.8, 0.8)$

(d) $B_6$. Range $(-0.15, 0.15) \times (-0.15, 0.15)$
Prove. Suppose that $F_{λ,m}(-m) \in A^*(0)$. Let $γ$ be a simple path in $A^*(0)$ that joins $F_{λ,m}(-m)$ and 0. The set of preimages of $γ$ must include a path $γ_1$ that joins $−∞$ with $−m$, and also a path $γ_2$ that joins $−m$ and 0 (since $−m$ is a critical point and 0 is a fixed point and asymptotic value). Hence $γ_1 \cup γ_2 \subset A^*(0)$ and so does $−m$. Conversely, if $−m \in A^*(0)$ we have that $F_{λ,m}(-m) \in A^*(0)$. \qed

**Proposition 3.3.** The set $C^0_m$ contains the disk $\{ λ \in \mathbb{C}; |λ| < \min(\frac{1}{e}, (\frac{e}{m})^m)\}$.

**Proof.** We denote $D_m$ the open disk $\{ λ \in \mathbb{C}; |λ| < \min(\frac{1}{e}, (\frac{e}{m})^m)\}$. Let $λ \in D_m$, we will prove that $F_{λ,m}(-m)$ lies in $D_{ε_0}$ which we know belongs to $A^*(0)$. In order to do so, we use that $ε_0 \geq \min(1, (\frac{1}{|λ|})^{1/(m-1)})$ (lemma 2.4). We choose $λ \in D_m$. Then $|λ| < \frac{1}{e}$, and hence $ε_0 \geq 1$. The condition $λ \in D_m$ also implies that $|λ| < (\frac{e}{m})^m$. Hence

$$|F_{λ,m}(-m)| = |λ||(−m)^m e^{-m}| = |λ| \left(\frac{m}{e}\right)^m < 1 \leq ε_0,$$

and $F_{λ,m}(-m)$ lies in $A^*(0)$. \qed

**Proposition 3.4.** The set $C^0_m$ is bounded. In fact it is contained in the closed disk $\{ λ \in \mathbb{C}; |λ| \leq (\frac{e}{m-1})^{m-1}\}$.

**Proof.** We will prove that $−m \notin A^*(0)$ for all $λ \in \mathbb{C}$ such that $|λ| > (\frac{e}{m-1})^{m-1}$. Let $D$ the disk centered at 0 of radius $m − 1$. If we calculate the image of its boundary, $\{|z| = m − 1\}$, we obtain

$$|F_{λ,m}(z)| = |λ||z|^m e^{Re(z)} \geq |λ|(m − 1)^m e^{−(m−1)} > m − 1$$

where the inequality is obtained using $|λ| > (\frac{e}{m-1})^{m-1}$. This shows that $D \subset F_{λ,m}(D)$, and hence $A^*(0) \subset D$. Since $−m \notin D$, the proposition follows. \qed

**Proposition 3.5.** If $λ \in C^0_m$ then $A(0) = A^*(0)$, i.e., the basin of attraction of $z = 0$ has a unique connected component and hence it is totally invariant. Moreover, the boundary of $A^*(0)$ (which equals the Julia set) is a Cantor bouquet and hence it is disconnected and non-locally connected.

**Proof.** Let $λ \in C^0_m$. As in proposition 3.2, let $γ$ be a simple path in $A^*(0)$ that joins $F_{λ,m}(-m)$ and 0. The preimage of $γ$ must include a path $γ$ contained in $A^*(0)$ that joins $−∞$ with 0 passing through $−m$ ($γ$ maps 2-1 to $γ$). Since $H_{|λ|,m}$ intersects $γ$ so it follows that $H_{|λ|,m} \subset A^*(0)$. All preimages of $γ$, are contained in $A^*(0)$ as well, since they all intersect $H_{|λ|,m}$. In fact, we have that $A(0) = A^*(0)$ since any preimage of $D_0$ must contain points of $H_{|λ|,m}$. Hence $A(0)$ has a unique connected component. In fact, from [Devaney & Goldberg, 1987], [Baker & Domínguez, 2000], it follows that the Julia set has an uncountable number of connected components and it is not locally connected at any point.
Using \cite{Devaney & Tangerman, 1986}, one can show that the Julia set contains a Cantor bouquet tending to $\infty$ in the direction of the positive real axis. To see this, it is sufficient to construct a hyperbolic exponential tract on which $F_{\lambda,m}$ has asymptotic direction $\theta^*$. Let $B_r$ an open disk containing $F_{\lambda,m}(-m)$, the preimage of this set is an open set similar to $H|\lambda|,m$. Let $D$ the complement of this set. We have that $F_{\lambda,m}$ maps $D$ onto the exterior of $B_r$, then $D$ is an exponential tract for $F_{\lambda,m}$. We may choose the negative real axis to define the fundamental domains in $D$. Since the curves $\sigma_k$ for $k \in \mathbb{Z}$ are mapped by $F_{\lambda,m}$ onto this axis, it follows that $D$ has asymptotic direction $\theta^* = 0$. Furthermore, since $F_{\lambda,m}(z) = \lambda z^m \exp(z)$, one may check readily that $D$ is a hyperbolic exponential tract.

**Proposition 3.6.** If $\lambda \notin C^0_m$ then $A(0)$ has infinitely many components. Moreover, if $|\lambda| > \left(\frac{e}{m-1}\right)^{m-1}$, the boundary of $A^*(0)$ is a quasi-circle.

**Proof.** Using lemma \ref{lem:main} we have that $A(0)$ has either one or infinitely many connected components. If we suppose that $A(0)$ has only one connected component, then $A(0)$ is a completely invariant component of the Fatou set. We have that all the critical values of $F_{\lambda,m}$ are in $A(0)$ (see \cite{Baker, 1984}), and hence we conclude that $-m$ belongs to $A(0)$. However, is impossible if $\lambda \notin C^0_m$.

Let $\lambda \notin C^0_m$ such that $|\lambda| > \left(\frac{e}{m-1}\right)^{m-1}$. The main idea of this proof is the same as that used by Bergweiler in (\cite{Bergweiler, 1995}). We will show that $F_{\lambda,m}$ is a polynomial-like of degree $m$ in a neighbourhood of $0$, which includes the whole immediate basin. From the proof of proposition \ref{prop:basin} the disc $D$ centered at 0 of radius $m - 1$ satisfies $\overline{D} \subset F_{\lambda,m}(D)$, and hence $A^*(0) \subset D$.

Let $W$ be the component of $F_{\lambda,m}^{-1}(D)$ that contains the origin. It is clear that $\overline{W} \subset D$ and $A^*(0) \subset W$. Moreover, $F_{\lambda,m}$ is a proper function of degree $m$ from $W$ onto $D$, (see Fig. \[1\]). In the terminology of polynomial-like mappings, developed by Douady and Hubbard (\cite{Douady & Hubbard, 1985}), the triple $(F_{\lambda,m}; W, D)$ is a polynomial-like mapping of degree $m$. By the Straightening theorem, there exists a quasiconformal mapping, $\phi$, that conjugates $F_{\lambda,m}$ to a polynomial $P$ of degree $m$, on the set $W$. That is $(\phi^{-1} \circ F_{\lambda,m} \circ \phi)(z) = P(z)$ for all $z \in W$. Since $z = 0$ is superattracting for $F_{\lambda,m}$ and $\phi$ is a conjugacy, we have that $z = 0$ is superattracting for $P$. Hence, after perhaps a holomorphic change of variables, we may assume that $P(z) = z^m$.

Hence, $\partial A^*(0) = \phi(\mathbb{T})$, and the theorem follows. \hfill $\Box$

**Remark 3.7.** The reason to ask for $|\lambda| > \left(\frac{e}{m-1}\right)^{m-1}$ as a condition is as follows. We want to find a value $K > 0$ such that if $|z| = K$ then $|F_{\lambda,m}(z)| > K$. This condition is equivalent to

$$|F_{\lambda,m}(z)| \geq |\lambda||z|^m e^{-|z|} = |\lambda|(K)^m e^{-K} > K$$

or equivalently

$$|\lambda| > K^{1-m} e^K.$$
Figure 11: $F_{\lambda,m}$ is a polynomial-like mapping of degree $m$ near the origin.

We want to use this argument for the largest possible region of values of $\lambda$. Hence, we choose $K > 0$, such that $K^{1-m}e^K$ is minimum. This minimum value is reached exactly at $K = m - 1$.

### 3.2 Proof of Theorem C

In this Sec. we describe the capture zones $C_{m}^{2}$ (Proposition 3.8 and Proposition 3.9) and $C_{m}^{3}$ (Proposition 3.10).

We will construct the open set $C_{m}^{2}$ in two steps. In the first (Proposition 3.8) we obtain an unbounded interval of real numbers $I$, such that for all $\lambda \in I$, $F^{2}_{\lambda,m}(-m)$ lies in $A^{*}(0)$. In the second (Proposition 3.9), we will extend this construction to $\lambda$ in $\mathbb{C}$. We denote $\lambda = \lambda_1 + \lambda_2 i$, where $\lambda_1$ and $\lambda_2$, are the real and imaginary parts of $\lambda$.

**Proposition 3.8.** There exists an unbounded interval, $I$, such that for all real numbers $\lambda_1 \in I$, we have that $F^{2}_{\lambda_1,m}(-m) \in D_{\epsilon_0} \subset A^{*}(0)$.

**Proof.** Hereafter we denote $r_m = (\frac{\lambda_1}{m})^{m}$. We take $\lambda_1 \in \mathbb{R}$, and we impose that $F_{\lambda_1,m}(-m) \in H_{|\lambda_1|,m}$.

If we calculate $F_{\lambda_1,m}(-m)$ we obtain

$$F_{\lambda_1,m}(-m) = \lambda_1 (-m)^{m} \exp(-m) = (-1)^{m} \frac{\lambda_1}{r_m}.$$  

This real value lies in $H_{|\lambda_1|,m}$, if and only if

$$|h(F_{\lambda_1,m}(-m))| < \frac{\epsilon_0}{|\lambda_1|}.$$
Recall that $h(x) = x^m \exp(x)$, and $\epsilon_0$ only depends on $|\lambda_1|$ and $m$. This condition is equivalent to

$$\frac{|\lambda_1|^{m+1}}{r_m^m} \exp((-1)^m \frac{\lambda_1}{r_m}) < \epsilon_0.$$ 

Using lemma 2.4 we have that $\epsilon_0 \geq \min\{1, \left(\frac{1}{|\lambda_1|e} \right)^{1/(m-1)}\}$. If we use this explicit lower bound we may impose

$$\frac{|\lambda_1|^{m+1}}{r_m^m} \exp((-1)^m \frac{\lambda_1}{r_m}) < \min\{1, \left(\frac{1}{|\lambda_1|e} \right)^{1/(m-1)}\}.$$ 

We define the auxiliary function

$$l(\lambda_1) = \begin{cases} 
|\lambda_1|^{m+1} + \frac{1}{m-1} \exp((-1)^m \frac{\lambda_1}{r_m}) \exp(1/(m-1)) & \text{if } |\lambda_1| > 1/e \\
|\lambda_1|^{m+1} \exp((-1)^m \frac{\lambda_1}{r_m}) & \text{if } |\lambda_1| \leq 1/e 
\end{cases}$$

and the above inequality is transformed into $l(\lambda_1) < r_m^m$.

Using some elementary properties of function $l(\lambda_1)$, one can see that

$$\begin{align*}
lim_{\lambda_1 \to -\infty} l(\lambda_1) &= 0 & \text{if } m \text{ is even} \\
lim_{\lambda_1 \to +\infty} l(\lambda_1) &= 0 & \text{if } m \text{ is odd.}
\end{align*}$$

Since $l(\lambda_1)$ is continuous and positive and it has a finite number of relative maxima and minima, we can find an unbounded interval of real numbers such that $l(\lambda_1) < r_m^m$.

If $m$ is even, we define $I = (-\infty, -D_0)$, where $-D_0 = -D_0(m) \leq 0$, is the smallest of the values such that $l(\lambda_1) = r_m^m$. If $m$ is odd, we choose $I = (D_0, +\infty)$, with $D_0 = D_0(m) \geq 0$, such that $D_0$ is the largest of the values for which $l(\lambda_1) = r_m^m$.

**Proposition 3.9.** Let $D_0(m) > 0$ be as in Proposition 3.8. There exists a function $\alpha = \alpha(|\lambda|, m) \in (\pi/2, \pi)$, such that

- for $m$ even, the set $C_2^2$ contains the open set
  $$\left\{ \lambda \in \mathbb{C} \left| \begin{array}{c}
|\lambda| > D_0 \\
|\text{Arg}(\lambda)| > \alpha
\end{array} \right. \right\}$$
- for $m$ odd, the set $C_2^2$ contains the open set
  $$\left\{ \lambda \in \mathbb{C} \left| \begin{array}{c}
|\lambda| > D_0 \\
|\text{Arg}(\lambda)| < \pi - \alpha
\end{array} \right. \right\}$$

**Proof.** Given $\lambda_1^* \in I$, we denote by $S$ the circle of radius $|\lambda_1^*|$ and centered at the origin. We will find all complex numbers $\lambda$ in $S$, such that $F_{\lambda,m}(-m) \in H_{|\lambda|,m}$.

All complex numbers $\lambda \in S$ have the same $H_{|\lambda|,m}$ set, since this set only depends on $|\lambda|$ and $m$. We denote it by $H_S$. 

When \( \lambda \) belongs to \( S \), the image of the critical point, \( F_{\lambda,m}(-m) = (-1)^m \frac{\lambda}{r_m} \), belongs to another circle, namely \( \tilde{S} \), and its argument verifies

\[
\arg(F_{\lambda,m}(-m)) = \begin{cases} 
\arg(\lambda) & \text{if } m \text{ is even} \\
\arg(\lambda) + \pi & \text{if } m \text{ is odd}
\end{cases}
\]

This circle is concentric with respect to \( S \) and its radius is equal to \( |\lambda_1^*| \) (see Fig. 12), which is larger than the radius of \( S \) if \( m = 2 \), and smaller if \( m \geq 3 \).

We take \( \lambda_1^* \) on \( I \). Using the construction above of the interval \( I \), we obtain that \( F_{\lambda_1^*,m}(-m) \in H_{|\lambda_1^*|,m} = H_S \). This fact assures a non-empty intersection of \( \partial H_S \) with \( \tilde{S} \).

Using the analytic definition of \( H_S \) (proof of Proposition 2.6), we can calculate \( \partial H_S \cap \tilde{S} \).

We find this intersection by solving:

\[
\begin{cases} 
\sqrt{\lambda_1^2 + \lambda_2^2} = \frac{|\lambda_1^*|}{r_m} \\
\lambda_2 = +C(\lambda_1) = \sqrt{\left[ \frac{\epsilon_0}{|\lambda_1^*|^2} \right]^{2/m} \exp(-\lambda_1 \frac{2}{m}) - \lambda_1^2} \\
\lambda_1 + i\lambda_2 \in \tilde{S} \\
\lambda_1 + i\lambda_2 \in \partial H_S
\end{cases}
\]

It is not difficult to show that this system has two conjugate solutions namely \( \zeta \) and \( \bar{\zeta} \).

If we write \( \zeta = \bar{\lambda}_1 + i\bar{\lambda}_2 \), then

\[
\bar{\lambda}_1 = \ln \frac{\epsilon_0^{m} m}{|\lambda_1^*|^{m+1}} \quad \bar{\lambda}_2 = +C(\bar{\lambda}_1)
\]

Let \( \alpha = \alpha(|\lambda_1^*|, m) \in (\pi/2, \pi) \) be the argument of \( \zeta \) (see Fig. 12).

If \( m \) is even then for all complex numbers with modulus equal to \( |\lambda_1^*| \), where \( \lambda_1^* \) lies in \( I \), and argument greater than \( \alpha \) in absolute value, it is verified that \( F_{\lambda,m}(-m) \in H_{|\lambda_1^*|,m} \). If \( m \) is odd, the same is true for all complex numbers with modulus equal to \( |\lambda_1^*| \), where \( \lambda_1^* \) lies in \( I \), and argument, in absolute value, smaller than \( \pi - \alpha \).

Parallel to the construction in dynamical plane we will now show the existence of a countable number of horizontal bands, all of which are also capture zones. See Fig. 9–10.

Apparently, when \( m \) is even, these strips extend to \( +\infty \), while if \( m \) is odd they extend to \( -\infty \). It also seems that their width decreases as \( m \) increases.

In the Sec. above we constructed similar strips around the curves \( \sigma_k \). These curves were preimages of the negative real axis. In this case, we define

\[
\Gamma_k = \{ \lambda \in \mathbb{C} \text{ such that } F_{\lambda,m}(-m) \in \sigma_k \}.
\]

To find an expression for the curves \( \Gamma_k \) we first write

\[
F_{\lambda,m}(-m) = \lambda(-m)^m \exp(-m) = (-1)^m \frac{\lambda}{r_m}
\]
Figure 12: Sketch of construction of the set $H_m$, $m = 2$.

where $r_m = \left( \frac{r}{m} \right)^m$. Recall that $z \in \sigma_k$ when

$$\text{Arg}(\lambda) + m\text{Arg}(z) + \text{Im}(z) = (2k + 1)\pi.$$ 

Hence we need that

$$\text{Arg}(\lambda) + m\text{Arg}(F_{\lambda,-m}) + \text{Im}(F_{\lambda,-m}) = (2k + 1)\pi.$$ 

If $m$ is even then $\text{Arg}(F_{\lambda,-m}) = \text{Arg}(\lambda)$, while if $m$ is odd then $\text{Arg}(F_{\lambda,-m}) = \text{Arg}(\lambda) + \pi$; thus we obtain the condition for $F_{\lambda,-m} \in \sigma_k$

\[
\begin{cases}
|\lambda| = \phi(\text{Arg}(\lambda)) = r_m \frac{(2k+1)\pi - (m+1)\text{Arg}(\lambda)}{-\sin(\text{Arg}(\lambda))} & \text{if } m \text{ is even} \\
|\lambda| = \phi(\text{Arg}(\lambda)) = r_m \frac{(2k+1-m)\pi - (m+1)\text{Arg}(\lambda)}{-\sin(\text{Arg}(\lambda))} & \text{if } m \text{ is odd}
\end{cases}
\]

As in the Sec. above, we need to impose $\phi(\text{Arg}(\lambda)) \geq 0$. If we denote $\theta = \text{Arg}(\lambda)$, we have:

if $m = 2j$ for $j \in \mathbb{Z}$

$$\Gamma_k = \phi(\theta)e^{i\theta} \begin{cases}
0 < \theta < \pi & \text{if } k \geq j + 1 \\
0 < \theta < \frac{(2k+1)\pi}{m+1} & \text{if } 0 \leq k \leq j \\
\frac{(2k+1)\pi}{m+1} < \theta < 0 & \text{if } -(j+1) \leq k \leq 0 \\
-\pi < \theta < 0 & \text{if } k \leq -(j+2)
\end{cases}$$
if \( m = 2j + 1 \) for \( j \in \mathbb{Z} \)

\[
\Gamma_k = \phi(\theta) e^{i\theta} = \begin{cases}
0 < \theta < \pi & \text{if } k \geq m \\
-\pi < \theta < 0 \cup \frac{2k+1-m}{m+1} < \theta < \pi & \text{if } j + 1 \leq k \leq m - 1 \\
-\pi < \theta < \pi & \text{if } k = j \\
-\pi < \theta < \frac{2k+1-m}{m+1} \cup 0 < \theta < \pi & \text{if } j - 1 \leq k \leq 0 \\
-\pi < \theta < 0 & \text{if } k \leq -1
\end{cases}
\]

In Fig. 13 we show some of these curves for some values of \( m \). If we suppose that \( m = 2j \) is even, then each \( \Gamma_k \) tends asymptotically to the line \( Im(z) = (2j+1)\pi r_m \) as its real part tends to \(+\infty\). We can classify these curves in three types. The first one is formed by curves whose real part runs from \(-\infty\) to \(+\infty\). There are two curves of the second kind, \( \Gamma_j \) and \( \Gamma_{-(j+1)} \), with real part in \([- (m + 1)r_m, \infty]\). The third group is formed by \( m \) curves, starting at the origin and tending to \(+\infty\). These \( m \) curves have indexes between \( j - 1 \) and \(-j\).

Figure 13: Strips in the parameter plane.

If we take \( m \) an odd index \( (m = 2j + 1) \), these curves tend asymptotically to the lines \( Im(z) = 2(k - m)\pi r_m \) as their real part tend to \(-\infty\). As above, we can classify these curves
in three types. The first one is formed by curves that extend from \(-\infty\) to \(+\infty\). The second one is formed by the curve \(\Gamma_j\), has a horseshoe shape, and cuts the real axis at the point \((m+1)r_m\). The third one is formed by \(m-1\) curves, starting at the origin and tending to \(-\infty\). These \(m\) curves have indexes between 0 and \(m-1\), except for \(\Gamma_j\).

Hence we have obtained some curves \(\Gamma_k\) such that if \(\lambda \in \Gamma_k\), then \(F_{\lambda,m}(-m) \in \sigma_k\). Hence, choosing \(\lambda \in \Gamma_k\) with \(\text{Re}(F_{\lambda,m}(-m))\) large enough, we obtain that \(F_{\lambda,m}(F_{\lambda,m}(-m)) \in H_{|\lambda|,m}\). Since \(\text{Re}(F_{\lambda,m}(-m)) = \text{Re}(\lambda)(-1)^mr_m\), this corresponds to taking \(\text{Re}(\lambda)\) or \(-\text{Re}(\lambda)\) large enough depending on \(m\) being even or odd.

By construction, the half curves we just defined belong each to \(C_m^3\). We will now show that a neighbourhood of \(\Gamma_k\) of asymptotic width equal to \(r_m \pi\) is also part of \(C_m^3\). We fix a value \(k \in \mathbb{Z}\) and we suppose that \(\lambda = \lambda_1 + i\lambda_2\), where \(\lambda_1\) and \(\lambda_2\) are real numbers. We will prove the following result.

**Proposition 3.10.** If \(m\) is even, for all \(\lambda_2 \in (r_m(\frac{\pi}{2} + 2k\pi), r_m(\frac{3\pi}{2} + 2k\pi))\) there exists \(\lambda^*_1\) such that, for all \(\lambda_1 > \lambda^*_1\) then \(F_{\lambda,m}^3(-m) \in A^x(0)\).

If \(m\) is odd, for all \(\lambda_2 \in (r_m(\frac{\pi}{2} + 2k\pi), r_m(\frac{3\pi}{2} + 2k\pi))\) there exists \(\lambda^*_1\) such that, for all \(\lambda_1 < \lambda^*_1\) then \(F_{\lambda,m}^3(-m) \in A^x(0)\).

**Proof.** Assume that \(m\) is even (the odd case is completely symmetric), and let \(\lambda_2 \in (r_m(\frac{\pi}{2} + 2k\pi), r_m(\frac{3\pi}{2} + 2k\pi))\). We recall that proposition 2.8 assures that, for all \(y \in ((2k+1)\pi - \text{Arg}(\lambda) - \frac{\pi}{2}, (2k+1)\pi - \text{Arg}(\lambda) + \frac{\pi}{2})\), there exists \(x \in \mathbb{R}\) such that, for all \(x \geq x_s\), the point \(F_{\lambda,m}(x+yi) \in H_{|\lambda|,m}\).

Using that \(F_{\lambda,m}(-m) = \frac{\lambda_1}{r_m} + \frac{\lambda_2}{r_m}i\), it suffices to prove that

\[
\text{Im}(F_{\lambda,m}(-m)) = \frac{\lambda_2}{r_m} \in \left(\frac{\pi}{2} + 2k\pi - \text{Arg}(\lambda), \frac{3\pi}{2} + 2k\pi - \text{Arg}(\lambda)\right).
\]

Choosing

\[
\text{Re}(F_{\lambda,m}(-m)) = \frac{\lambda_1}{r_m} > x^*.
\]

The first condition is equivalent to \(\text{Arg}(\lambda) \in (\alpha_1, \alpha_2)\), where \(\alpha_1 = \frac{\pi}{2} + 2k\pi - \frac{\lambda_2}{r_m} < 0\) and \(\alpha_2 = \frac{3\pi}{2} + 2k\pi - \frac{\lambda_2}{r_m} > 0\) (Fig. [4]).

Suppose that \(k > 0\). We denote by \(r\) the line through the origin with slope \(\tan(\alpha_2)\), and let \(s\) be the horizontal line through \(\frac{\lambda_1}{r_m}i\). We also denote by \(\tilde{\lambda}_1\) the abscissa of the intersection point between the lines \(r\) and \(s\) (Fig. [4]). All values of \(\lambda\) on \(s\), with abscissa greater than \(\tilde{\lambda}_1\) verify that \(0 < \text{arg}(\lambda) < \alpha_2\). Finally, we define \(\lambda^*_1 = \max\{r_m x_s, r_m \tilde{\lambda}_1\}\), and for this value both conditions are verified. Therefore, \(F_{\lambda,m}^2(-m) \in H_{|\lambda|,m}\), and it follows that \(F_{\lambda,m}^3(-m) \in D_{e_0} \subset A^x(0)\). If \(k \leq 0\), we replace the line of slope \(\tan(\alpha_2)\), by a line with slope \(\tan(\alpha_1)\).

\[\square\]

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Figure 14: Construction of the value $\tilde{\lambda}_1$

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