Mechanism of the quantum speed-up

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Abstract

We explain the mechanism of the quantum speed-up — quantum algorithms requiring fewer computation steps than their classical equivalent — for a family of algorithms. Bob chooses a function and gives to Alice the black box that computes it. Alice, without knowing Bob’s choice, should find a character of the function (e.g., its period) by computing its value for different arguments. There is naturally correlation between Bob’s choice and the solution found by Alice. We show that, in quantum algorithms, this correlation becomes quantum. This highlights an overlooked measurement problem: sharing between two measurements the determination of correlated (thus redundant) measurement outcomes. All is like Alice, by reading the solution at the end of the algorithm, contributed to the initial choice of Bob, for half of it in quantum superposition for all the possible ways of taking this half. This contribution, back evolved to before running the algorithm, where Bob’s choice is located, becomes Alice knowing in advance half of this choice. The quantum algorithm is the quantum superposition of all the possible ways of taking half of Bob’s choice and, given the advanced knowledge of it, classically computing the missing half. This yields a speed-up with respect to the classical case where, initially, Bob’s choice is completely unknown to Alice.

1 Foreword

Our explanation of the speed-up relies on the interplay between the unitary part of the quantum algorithm and the initial and final measurement operations. Because of the interdisciplinary character of the work, we spend a few lines to introduce the language of quantum computation.

An algorithm is the computation that solves a problem. Quantum computation is the implementation of the algorithm at a fundamental physical level. A quantum algorithm yields a speed-up when it requires fewer computation steps than its classical equivalent, sometimes demonstrably fewer than classically possible.

We focus on a family of quantum algorithms that comprises the major speed-ups. Bob, the problem setter, chooses a function out of a known set of functions.

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and gives to Alice a black box that computes it. Alice, without knowing Bob’s choice, should find a character of the function by computing its value for different arguments.

The essential things of a quantum computation process are:

1. The register, which contains a number or a quantum superposition thereof.
   The register’s initial state is the input of the computation; e. g.: \( \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \).

2. The (reversible) computation, a unitary transformation \( U \) that sends the input into the output – the result of the computation or the solution of the problem. E. g.: \( U \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \).

3. The initial measurement, required to prepare the register in the desired initial state. In particular, we are interested in the preparation of the choice of the problem on the part of Bob.

4. The final measurement, performed by Alice in the output state of the register to read the solution of the problem.

The seminal speed-up was discovered by Deutsch [1] in 1985. The subsequent speed-ups can be seen as ingenious mathematical extrapolations of Deutsch’s algorithm. We provide an example of speed-up. Given a chest of four drawers. Bob hides a ball in one of the drawers, Alice should locate the ball by opening different drawers. Classically, to be sure of locating the ball, Alice should plan to open three drawers. Quantally, one drawer. This is the simplest instance of Grover’s [2] quantum search algorithm.

In 2001, Grover [3] called for a two-line explanation of the reason for the speed-up, one that does not enter into the mathematical detail of each quantum algorithm. It can be said that quantum computer science prevalingly explored the opposite direction, focusing on the mathematical gear of quantum algorithms and trying to unify it. In the context of the present work, it is important to note that such attempts mostly focus on the unitary transformation part of quantum algorithms. Apropos of this, it should be noted that many quantum algorithms are unitary transformations starting and ending with a sharp state of the register. This might have provided the feeling that the speed-up is essentially deterministic in character.

In 2009, Gross, Flammia, and Eisert [4] claimed that the exact reason for the quantum speed-up had never been explained.

Our explanation of the quantum speed-up, relying on the interplay between the unitary and the non-unitary part of the quantum algorithm, goes against the aforesaid trend. It also shows that the apparent determinism of the quantum speed-up is a sort of ”visual illusion”.

2 Extended summary

Our explanation of the speed-up starts with the obvious observation that there is correlation between the problem and its solution – e. g., one to one correlation
between the drawer number initially chosen by Bob and the drawer number eventually found by Alice. The explanation can be divided in three steps. The first step (Section 3.1) is showing that, in quantum algorithms, this correlation becomes quantum.

We divide the register into sub-registers – for short "registers" as well. In the four drawer case, we have a two-quantum-bit (qubit) register $B$, under the control of Bob, and a two-qubit register $A$, under the control of Alice. Let $b$ be the number of the drawer with the ball, $a$ that of the drawer opened by Alice. Bob writes his choice of the value of $b$, say $b = 00$, in register $B$. Alice writes the number of the drawer she wants to open in register $A$. Reading the content of a register amounts to measuring a corresponding observable. We call $\hat{B}$ the content of register $B$, of eigenvalues $b \in \{00, 01, 10, 11\}$, and $\hat{A}$ the content of register $A$, of eigenvalues $a \in \{00, 01, 10, 11\}$. We note that $\hat{A}$ and $\hat{B}$ commute.

We assume that register $B$ is initially in a maximally mixed state – so that the value of $b$ is completely undetermined. In fact we want to examine the entire process that leads to the determination of Bob’s choice.

In order to prepare register $B$ with the desired value of $b$, Bob should measure $\hat{B}$ in the maximally mixed state of register $B$. He obtains an eigenvalue at random, say $b = 01$. Then he changes the corresponding eigenstate into the desired one, by applying to $B$ a suitable permutation of the basis vectors, namely a unitary transformation $U_B$. At this point Alice runs the unitary part of the quantum algorithm, which eventually writes the solution in register $A$. Alice acquires the solution $a = b = 00$ by measuring $\hat{A}$.

A crucial point of our argument is noting that there is quantum correlation between the outcome of the initial measurement of $\hat{B}$ and that of the final measurement of $\hat{A}$. In fact, quantum correlation concerns repetitions of the same quantum experiment. Therefore, from the standpoint of it, the unitary transformation $U_B$ should be considered fixed. The fact that Bob chooses $U_B$ to always obtain the choice 00, independently of the outcome of measuring $\hat{B}$, belongs to a different film. Looking only the latter film originates the aforesaid "visual illusion" – the apparent determinism of quantum computation and the speed-up.

Instead, the quantum speed up essentially relies on (non-deterministic) quantum correlation. As the fixed permutation of a randomly selected value of $b$, Bob’s choice $b = 00$ should be considered random. Up to the fixed permutation $U_B$, there is quantum correlation between Bob’s choice $b = 00$ and Alice’s reading of the solution $a = b = 00$.

This can be best seen by virtually deferring the measurement of $\hat{B}$ at the end of the unitary part of the quantum algorithm – see Section 3.1 for details. The state of the two registers, before the measurement of either $\hat{B}$ or $\hat{A}$, is:

$$\frac{1}{2} \left( e^{i\varphi_0} |00\rangle_B |00\rangle_A + e^{i\varphi_1} |01\rangle_B |01\rangle_A + e^{i\varphi_2} |10\rangle_B |10\rangle_A + e^{i\varphi_3} |11\rangle_B |11\rangle_A \right).$$

(1)

The $\varphi_i$ are independent random phases, each with uniform distribution in $[0, 2\pi]$. We use the random-phase representation of a density operator to keep the ket
vector representation of the quantum algorithm. The density operator is the average over all $\phi_i$ of the product of the ket by the bra: $\langle \psi | \langle \psi \rangle_{\phi_i}$. The von Neumann entropy of this state is two bits. Measuring $\hat{B}$ or $\hat{A}$ projects (1) on:

$$|00\rangle_B |00\rangle_A .$$

Back evolving the projection of (1) on (2), by the inverse of the time forward unitary transformation, restores the projection of the initial state of register $B$ on $b = 01$.

Let us sum up the situation. We are dealing with two measurements – Bob’s measurement of $\hat{B}$ and Alice’s measurement of $\hat{A}$ – whose outcomes are completely (as in the four drawers case) or partly (more in general) correlated.

It is of course unquestionable that quantum measurement determines an eigenvalue of the measured observable, but when there are two measurements for two redundant or partly redundant eigenvalues, what do we have to say?

Interestingly, while quantum correlation has been the source of an enormous amount of research, the problem of fairly sharing between two measurements the determination of two completely (or partly) correlated eigenvalues – thus completely (or partly) redundant with one another – has been overlooked.

To analyze this problem, it is useful to introduce the reduced density operators of registers $B$ and $A$ in state (1), respectively $\rho_B$ and $\rho_A$. The usual way of solving this problem is thinking that the measurement performed first takes the lion’s share. If we assume that the measurement of $\hat{B}$ is performed first, we ascribe to it: the zeroing of the entropy of $\rho_B$ and the zeroing (or reduction) of the entropy of $\rho_A$. If we assume that the measurement of $\hat{A}$ is performed first, we ascribe to it: the zeroing of the entropy of $\rho_A$ and the zeroing (or reduction) of the entropy of $\rho_B$. We call these two perspectives “the lion’s share perspectives”.

The second step of our explanation (Section 3.2) relies on performing this sharing in a way that is not affected by the order of the two measurement, or the fact that the two measurement are performed simultaneously.

This is justified by the following consideration. Determination means reduction of entropy, due to the projection – associated with quantum measurement – of a state of higher entropy on one of lower entropy. However, while quantum measurements are localized in time, the corresponding projections are not, they can be back evolved along the unitary part of the quantum algorithm by the inverse of the time-forward unitary transformation. Therefore, there is no reason to ascribe the lion’s share to the measurement performed first, not to speak of the fact that the two measurements can be simultaneous.

To present ends, it suffices to focus on sharing – between Alice’s and Bob’s measurements – the projection on Bob’s choice. The physical meaning of such a sharing is of course to be found in the notion of partial measurement of $\hat{B}$. For example, we can think of measuring $B_0$, the content of the left cell of register $B$, in state (1). A-priori, this yields either $b_0 = 0$ or $b_0 = 1$ ($b_0$ being the left bit of $b$). In present assumptions, the overall measurement of $\hat{B}$ projects on Bob’s choice $b = 00$, we are in fact discussing how to share this projection. This
naturally implies the assumption that the measurement of $\hat{B}_0$ yields $b_0 = 0$.

Summing up, we should divide the projection on Bob’s choice into two projections. Each projection is associated with a partial measurement of $\hat{B}$, with outcome post-selected to match with Bob’s choice. One projection should be ascribed to Bob’s measurement of $\hat{B}$, the other to Alice’s measurement of $\hat{A}$.

We define our sharing rule as follows. To start with, we get rid of all redundancy by resorting to Occam’s razor, or law of parsimony. In Newton’s formulation [5], it states “We are to admit no more causes of natural things than such that are both true and sufficient to explain their appearances”. This law of parsimony requires (i) that the two projections completely determine Bob’s choice without any over-determination, namely without projecting twice on the same information.

It is reasonable to require that the two projections keep the common qualities of the two lion’s share perspectives. By this we mean that we require that each projection ”properly” reduces the entropies of both $\rho_B$ and $\rho_A$. Moreover, from a quantitative standpoint, we require that entropy reductions are shared in a way that mirrors the symmetry of the measurement situation. For example, in the four drawer case – Eq. (1) – this implies that the sharing of entropy reductions between Alice’s and Bob’s measurements is fifty-fifty. This is condition (ii).

Eventually, we require (iii) that the sharing of the projection on Bob’s choice is done in all the possible ways, compatible with the former conditions, in quantum superposition.

The above conditions (i) through (iii) are enough to univocally solve the ”sharing problem” in all the quantum algorithms examined in this paper.

The third step of our explanation (Section 3.3) is showing that Alice’s contribution, back evolved to before running the algorithm (by the inverse of the time-forward unitary transformation), where Bob’s choice is positioned, becomes Alice knowing half of Bob’s choice in advance, in all possible ways in quantum superposition.

Correspondingly, the quantum algorithm turns out to be the quantum superposition of all the possible ways of taking half of Bob’s choice and, given the advanced knowledge of this half, classically computing the missing half. This of course yields a speed-up with respect to the classical case where Bob’s choice, initially, is completely unknown to Alice.

This explanation of the quantum speed-up has two main implications:

(I) The quantum speed-up comes from comparing two classical algorithms, with and without advanced knowledge of half of Bob’s choice. This implication is thus a tool for finding the achievable speed-ups – a central problem in quantum computation. Moreover, this problem is brought to an entirely classical framework, an important simplification.

(II) It shows that the quantum speed-up hosts a special causality loop. In quantum search, Alice knows in advance 50% of the bits that specify the solution, in all possible ways in quantum superposition, and, given the advanced knowledge of these bits, reaches the solution with fewer computation steps. Each individual history contains a causality loop; in the four drawer case, Alice knows in advance that the ball is in one of two drawers, and this before opening
any drawer. Thus, opening any one of these two drawers allows Alice to locate
the ball. This would be impossible if there were only one isolated history. It
becomes possible because, in the superposition of all histories, all the draw-
ers are opened and there is cross-talk between histories because of quantum
interference.

In the following Sections, first we develop our argument in detail for Grover’s
algorithm, then we show that it holds unaltered for the very diverse quantum
algorithms that yield an exponential speed up.

With respect to Ref. [6], we bring to a fundamental physical level the prob-
lem of sharing between Alice’s and Bob’s measurements the determina-
tion of Bob’s choice. This allows us to extend, to Deutsch and Jozsa’s algorithm,
Simon’s algorithm and the Abelian hidden subgroup algorithm, the detailed
explanation of the mechanism of the speed-up already developed for Grover’s
algorithm.

3 Grover’s algorithm

Section 3.1 highlights quantum problem-solution correlation in Grover’s algo-

rithm. In Sections 3.2 through 3.4, we develop the notion of advanced knowl-
edge.

3.1 Quantum problem-solution correlation

We formalize the four drawer example. Bob chooses a two-bit number \( b \equiv b_0 b_1 \); Alice should find the value of \( b \) by computing the Kronecker function \( \delta (b, a) \) for various values of \( a \equiv a_0 a_1 \). The value of \( \delta (b, a) \) is 1 if \( a = b \), 0 otherwise, which tells Alice whether the ball is in drawer \( a \).

Usually, the value of \( b \) chosen by Bob is thought to be hard-wired inside
the black box that computes \( \delta (b, a) \). To highlight quantum correlation, we
add to the usual description of the quantum algorithm an imaginary quantum
register \( B \) that contains the hard-wired value.\(^1\) In Section 3.3, this imaginary
register will serve to represent the usual quantum algorithm (with the hard-
wired value) “with respect” to the observer Alice in the sense of relational
quantum mechanics.

In view of this representation, we should consider the entire process of de-
termination of the value of \( b \). Thus, we assume that register \( B \) is initially in a
maximally mixed state. Register \( A \) contains the value of \( a \) tried by Alice, the
one-qubit register \( V \) (like "value") is meant to contain the result of the compu-
tation of \( \delta (b, a) \), modulo 2 added to its former content for logical reversibility.

\(^1\)We have taken the expression “imaginary register” from Ref. [7], which highlights the
problem-solution symmetry of Grover’s and the phase estimation algorithms.
The two latter registers are initially prepared as required by Grover’s algorithm:

$$
|\psi\rangle = \frac{1}{4\sqrt{2}} (e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B ) \\
(|00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A ) (|0\rangle_V - |1\rangle_V ) . \quad (3)
$$

To prepare \( B \), Bob needs to know its state. To this end, he measures \( \hat{B} \) in state (3). This randomly selects a value of \( b \), say 01, projecting (3) on:

$$
P_\alpha |\psi\rangle = \frac{1}{2\sqrt{2}} |01\rangle_B (|00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A ) (|0\rangle_V - |1\rangle_V ) . \quad (4)
$$

we denote projection operators by the letter \( P \). The two-bit entropy of the quantum state goes to zero with the determination of the value of \( b \).

Then he applies to register \( B \) a permutation of the values of \( b \) – a unitary transformation \( U_B \) – that changes the randomly selected value of \( b \) into the desired one, say 00:

$$
U_B P_\alpha |\psi\rangle = \frac{1}{2\sqrt{2}} |00\rangle_B (|00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A ) (|0\rangle_V - |1\rangle_V ) . \quad (5)
$$

We note that Bob can choose the desired value off-line in any way, for example random with whatever probability distribution.

The reversible computation of \( \delta(b, a) \), represented by the unitary transformation \( U_f \) (f like "function evaluation"), sends (5) into:

$$
U_f U_B P_\alpha |\psi\rangle = \frac{1}{2\sqrt{2}} |00\rangle_B (- |00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A ) (|0\rangle_V - |1\rangle_V ) . \quad (6)
$$

A non-computational operation, the rotation \( U_A \) of the measurement basis of register \( A \) (the so called "inversion about the mean") yields:

$$
U_A U_f U_B P_\alpha |\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle_B |00\rangle_A (|0\rangle_V - |1\rangle_V ) . \quad (7)
$$

In (7), register \( A \) contains the solution of the problem – the value of \( b \) chosen by Bob. Alice acquires it by measuring \( \hat{A} \), which by the way leaves state (7) unaltered.

Of course, there is a one-to-one correlation between the value of \( b \) chosen by Bob and the solution found by Alice. Up to the permutation introduced by \( U_B \), this corresponds to the quantum correlation between the outcome of measuring \( \hat{B} \) in (3) and that of measuring \( \hat{A} \) in (7). As anticipated in the Extended summary, from the standpoint of quantum correlation, which concerns repetitions of the same quantum experiment, the permutation \( U_B \) should be considered fixed (the fact that Bob chooses it to obtain the desired value of \( b \) belongs to a different film).

With a fixed \( U_B \), all is like the value of \( b \) chosen by Bob was randomly selected – this value becomes the fixed permutation of the randomly selected
value. Moreover, we can virtually defer the measurement of \( \hat{B} \) at the end of the algorithm (in Section 3.3, we will show that this yields the quantum algorithm "relativized" to the observer Alice). The previous input-output sequence of ket vectors becomes:

\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B \right)
\]

\[
= (|00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A) (|0\rangle_V - |1\rangle_V),
\]

(8)

\[
U_B |\psi\rangle = |\psi\rangle
\]

(9)

\[
U_f U_B |\psi\rangle = \frac{1}{2\sqrt{2}} \left( e^{i\varphi_0} |00\rangle_B |0\rangle_A + e^{i\varphi_1} |01\rangle_B |0\rangle_A + e^{i\varphi_2} |10\rangle_B |10\rangle_A + e^{i\varphi_3} |11\rangle_B |11\rangle_A \right)
\]

\[
= (|0\rangle_V - |1\rangle_V),
\]

(10)

\[
P_a U_A U_f U_B |\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle_B |00\rangle_A (|0\rangle_V - |1\rangle_V).
\]

(12)

In sending (8) into (9), \( U_B \) should permute the suffixes of the \( \varphi_i \). We do not take this into account since it is irrelevant, given that the \( \varphi_i \) are independent random phases. The computation of \( \delta (b, a) \), namely \( U_f \), maximally entangles the content of register \( B \) with that of register \( A \). In (10), four orthogonal states of \( B \), each a value of \( b \), are correlated with four orthogonal states of \( A \), which means that the information about the value of \( b \) has propagated to register \( A \). The rotation of the measurement basis of \( A \) makes this information readable: entanglement also becomes correlation between measurement outcomes. We can see why Bob’s measurement can be virtually deferred at the end: the projection of (11) on (12), back evolved by \( U_B^\dagger U_f^\dagger A \), becomes the projection of (3) on (4).

Thinking that all measurements are performed in the maximally entangled state (11) makes it more clear that the value of \( b \) is randomly selected by either Bob’s or Alice’s measurement; by the way, since \( A \) and \( B \) commute, the order of these two measurements (which can also be simultaneous) in state (11) is irrelevant. Either measurement projects state (11) on the solution eigenstate (12), where both registers contain the selected value of \( b \); correspondingly, the two-bit entropy of the quantum state goes to zero.

In view of what will follow, it is useful to introduce the reduced density operator (in the random phase representation) of register \( B \) in state (11):

\[
\rho_B = \frac{1}{2} \left( e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B \right).
\]

(13)
Incidentally, we note that $\rho_B$ is the same in states (8) through (11), up to the irrelevant permutation of random phases; the algorithm is in fact the identity on the reduced density operator of the control register. The projection of state (11) on the solution eigenstate (12) implies of course the projection of $\rho_B$ on $|00\rangle_B$; we will also say “on $b \in \{00\}$”. By the way, considering also $\rho_A$ would lead to completely redundant considerations.

3.2 Sharing the projection on Bob’s choice

Given that now the value of $b$ can be determined by measuring either $\hat{A}$ in state (11) or $\hat{B}$ in the same or any former state, we share the projection of state (11) on (12) into two projections, one ascribed to Alice’s measurement the other to Bob’s, in such a way that conditions (i) through (iii) of the sharing rule described in the Extended summary are satisfied.

In the case of Grover’s algorithm, the (in general) $n$ bits that specify the value of $b$ are independently selected in a random way (as the fixed permutation of a similar selection). Thus, condition (i) states that the projection on $p$ of these bits ($0 \leq p \leq n$) should be ascribed to Alice’s measurement, that on the other $n - p$ bits to Bob’s. This also means ascribing an entropy reduction of $p$ bits to Alice’s measurements, of $n - p$ bits to Bob’s. Thus, condition (ii) implies $p = n/2$.

As also anticipated in the Extended summary, the physical meaning of sharing the projection on the value of $b$ is related to the notion of partial measurement of $\hat{B}$. For example, going back to $n = 2$, we can think of measuring $\hat{B}_0$, the content of the left cell of register $B$, in state (11). This yields either $b_0 = 0$ or $b_0 = 1$, projecting $\rho_B$ on either $\frac{1}{\sqrt{2}} (e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B)$ or, respectively, $\frac{1}{\sqrt{2}} (e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B)$. In the present assumption, the overall measurement of $\hat{B}$ projects $\rho_B$ on $b \in \{00\}$, we are in fact discussing how to share this projection. This naturally implies the assumption that the measurement of $\hat{B}_0$ projects $\rho_B$ on $\frac{1}{\sqrt{2}} (e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B)$; we also say “on $b \in \{00, 01\}$”.

Under the same assumption, measuring $\hat{B}_1$, the content of the right cell of $B$, projects $\rho_B$ on $b \in \{00, 10\}$. There is still one partial measurement that yields one bit of information about the value of $b$, that of $\hat{B}_X$, the exclusive or of the contents of the two cells. Measuring $\hat{B}_X$ projects – always under the same assumption – on $b \in \{00, 11\}$. Summing up, a possible way of dividing the projection of $\rho_B$ on $b \in \{00\}$ is to split it into any two of the following three projections, on: $b \in \{00,01\}$, $b \in \{00,10\}$, and $b \in \{00,11\}$. In the four drawers visualization, this means of course pairing drawer 00 with another drawer in all possible ways.

This example is not fortuitous. In fact, it is the only way of sharing (between Alice’s and Bob’s measurements) the projection on $b \in \{00\}$ that satisfies conditions (i) and (ii). One can readily see that the measurement of any pair of observables among $\hat{B}_0$, $\hat{B}_1$, and $\hat{B}_X$, selects a value of $b$ without projecting twice on any bit of this value. Furthermore, the entropy drop associated with either one of the two measurements is the same. One can also see that there is
no other way of satisfying the above conditions; condition (iii) will be addressed further below.

We provide an example for \( n = 4 \). The projection on, say, \( b \in \{0000\} \) can be shared, say, in the projection on \( b \in \{0000,0001,0010,0011\} \) and that on \( b \in \{0000,0100,1000,1100\} \). The former projection corresponds to measuring \( \hat{B}_0 \) and \( \hat{B}_1 \) and finding \( b_0 = b_1 = 0 \), the latter to measuring \( \hat{B}_2 \) and \( \hat{B}_3 \) and and finding \( b_2 = b_3 = 0 \).

### 3.3 Advanced knowledge

We show that ascribing to Alice part of Bob’s choice, implies that she knows in advance, before running the algorithm, that part of the choice.

To this end, we introduce the notion of relativized quantum algorithm, in the sense of relational quantum mechanics [8]. We note that states (9) through (12) are the original quantum algorithm – we mean states (3) through (7) (to be counted twice, before and after the measurement of \( \hat{A} \)) – but with the quantum state relativized to the observer Alice. By definition, initially Alice does not know the content of register \( B \). To her, register \( B \) is in a maximally mixed state even if Bob has already chosen the value of \( b \). The two-bit entropy of state (9) represents Alice’s complete ignorance of the value of \( b \). When Alice measures \( \hat{A} \) at the end of the algorithm, the quantum state (11) is projected on the solution eigenstate (12). This projection is random to Alice, it is actually on the value of \( b \) chosen by Bob. The entropy of the quantum state goes to zero and Alice acquires full knowledge of the value of \( b \). Thus, the entropy of the relativized quantum state gauges Alice’s ignorance of the value of \( b \) throughout the execution of the algorithm.

With this result, we go back to our aim. We work on an example. We share the projection of state (11) on (12), namely the selection of \( b = 00 \), by ascribing to Alice’s measurement the projection of (11) on:

\[
\frac{1}{2} \left( e^{i\varphi_0} |0\rangle_B |0\rangle_A + e^{i\varphi_1} |0\rangle_B |1\rangle_A \right) (|0\rangle_V - |1\rangle_V),
\]

namely the selection of \( a_0 = b_0 = 0 \). This selection is like it was randomly generated at the time and location of Alice’s measurement. To become a contribution to Bob’s choice (itself the fixed permutation of a randomly generated value), it must propagate to the time and location of Bob’s choice – in particular to before running the algorithm and immediately after applying \( U_B \). Therefore, we should back evolve the corresponding projection by applying \( U_B^\dagger U_f^\dagger U_A^\dagger \) to the two ends of it, namely to states (11) and (14). This yields the projection of the initial state (9) – or, identically, (8) – on:

\[
\frac{1}{4} \left( e^{i\varphi_0} |0\rangle_B + e^{i\varphi_1} |0\rangle_B \right) (|0\rangle_A + |0\rangle_A + |10\rangle_A + |11\rangle_A) (|0\rangle_V - |1\rangle_V).
\]

The entropy of the initial state of register \( B \) is halved. Since this entropy represents Alice’s initial ignorance of Bob’s choice, this means that Alice, before
running the algorithm, knows \( n/2 \) of the bits of Bob’s choice, here one bit – in fact \( b_0 = 0 \).

We are at the level of elementary logical operations, where knowing means doing. Alice knows half of Bob’s choice (the value of \( b_0 \)) by acting like she knew it, namely by using it to identify classically the missing half (the value of \( b_1 \)) with a single computation of \( \delta (b, a) \). Correspondingly, as we showed in \([6]\), the quantum algorithm is the superposition of all the possible ways of taking one bit of information about Bob’s choice and, given the advanced knowledge of this bit, classically identifying the missing bit with a single computation of \( \delta (b, a) \) – see also Section 3.4. This explains the speed-up from three to one computation. This also satisfies condition (iii) of our sharing rule, namely that the projection on the value of \( b \) is shared between Alice’s and Bob’s measurements in all possible ways (compatibly with the other conditions) in quantum superposition.

This explanation of the mechanism of the speed-up generalizes to \( b \) any number of bits – see Ref. [6].

### 3.4 Superposition of classical computation histories

We show in which sense the quantum algorithm can be seen as a superposition of classical computations. As we have seen, in the assumption that Bob’s choice is \( b = 00 \), Alice’s advanced knowledge can be: \( b \in \{00, 01\} \), or \( b \in \{00, 10\} \), or \( b \in \{00, 11\} \). We start with the first possibility. Given the advanced knowledge of \( b \in \{00, 01\} \), to identify the value of \( b \) Alice should compute \( \delta (b, a) \) for either \( a = 00 \) or \( a = 01 \). Let us assume it is for \( a = 00 \). The outcome of the computation is \( \delta = 1 \). This originates two classical computation histories, depending on whether the initial state of register \( V \) is \( |0\rangle_V \) or \( |1\rangle_V \). Each classical history is represented as a sequence of sharp quantum states, as follows. The initial state of history 1 is \( e^{i\varphi_0} |00\rangle_B |00\rangle_A |0\rangle_V \) (the ket \( |00\rangle_B \) means that \( b = 00 \), the ket \( |00\rangle_A \) that the input of the computation of \( \delta \) is \( a = 00 \); the state after the computation of \( \delta \) is \( e^{i\varphi_0} |00\rangle_B |00\rangle_A |1\rangle_V \) (the result of the computation is modulo 2 added to the former content of register \( V \)). We are using the history phases that reconstruct the quantum algorithm: our present aim is to show that the quantum algorithm is a superposition of classical computation histories\(^2\). In history 2, the states before/after the computation of \( \delta \) are \( -e^{i\varphi_0} |00\rangle_B |00\rangle_A |1\rangle_V \to -e^{i\varphi_0} |00\rangle_B |00\rangle_A |0\rangle_V \). In the case that Alice computes \( \delta (b, a) \) for \( a = 01 \) instead, she obtains \( \delta = 0 \), which of course tells her again that \( b = 00 \). This originates other two histories. History 3: \( e^{i\varphi_0} |00\rangle_B |01\rangle_A |0\rangle_V \to e^{i\varphi_0} |00\rangle_B |01\rangle_A |0\rangle_V \); history 4: \( -e^{i\varphi_0} |00\rangle_B |01\rangle_A |1\rangle_V \to -e^{i\varphi_0} |00\rangle_B |01\rangle_A |1\rangle_V \). We develop in a similar way the other possibilities, also for all the possible choices of the value of \( b \). The computation step of Grover’s algorithm, namely the transformation of \( |0\rangle \) into \( |10\rangle \), is the superposition of all these histories.

By the way, this also explains quantum parallel computation. In fact, in the initial state of the quantum algorithm and in the superposition of all classical

---

\(^2\)History phases can also be found from scratch by maximizing entanglement – see Ref. [6].
computation histories, each $|ij\rangle_B$ is multiplied by:

$$\frac{1}{2\sqrt{2}}(|00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A)(|0\rangle_V - |1\rangle_V),$$

(16)
as one can readily check.

As one can see, the computation step we are dealing with is the identity on the reduced density operator of register $B$ (the control register) and entangles this register with register $A$ (the target register). The information contained in $B$ leaks to $A$ – see state (10). At this point we perform a non-computational step: a suitable rotation $U_A$ of the basis of register $A$ (the so called "inversion about the mean"). This branches each history into four histories; such branches interfere with one another to give state (11). Entanglement also becomes correlation between the possible measurement outcomes. By the way, this implicitly defines $U_A$ (inversion about the mean) as the rotation of the basis of register $A$ that maximizes correlation between possible measurement outcomes.

Summing up, Grover’s algorithm can be decomposed into a superposition of histories, which start from Alice’s advanced knowledge and whose computational part is entirely classical. This result also applies to $n > 2$, by iterating the sequence of the two steps (computational and non computational) $O(2^{n/2})$ times. By the way, this means a "quadratic" speed-up with respect to a classical algorithm that requires $O(2^n)$ computations.

4 Deutsch&Jozsa’s algorithm

In Deutsch&Jozsa’s [9] algorithm, the set of functions known to both Bob and Alice is all the constant and "balanced" functions (with an even number of zeroes and ones) $f_b : \{0,1\}^n \rightarrow \{0,1\}$. Array (17) gives this set for $n = 2$. The string $b \equiv b_0, b_1, ..., b_{2^n-1}$ is both the suffix and the table of the function – the sequence of function values for increasing values of the argument. Specifying the choice of the function by means of the table of the function simplifies the discussion.

| $a$ | $f_{0000}(a)$ | $f_{1111}(a)$ | $f_{0011}(a)$ | $f_{1100}(a)$ | $f_{0101}(a)$ | $f_{1010}(a)$ | $f_{0110}(a)$ | $f_{1001}(a)$ |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 00  | 0             | 1             | 0             | 1             | 0             | 1             | 0             | 1             |
| 01  | 0             | 1             | 0             | 1             | 1             | 0             | 1             | 0             |
| 10  | 0             | 1             | 1             | 0             | 0             | 1             | 1             | 0             |
| 11  | 0             | 1             | 1             | 0             | 1             | 0             | 0             | 1             |

(17)

Alice should find whether the function selected by Bob is balanced or constant by computing $f_b(a) \equiv f(b,a)$ for appropriate values of $a$. In the classical case this requires, in the worst case, a number of computations of $f(b,a)$ exponential in $n$; in the quantum case one computation.

We give the relativized states before and after the unitary part of the algorithm, namely $U_AU_f$ ($U_B$, $U_f$, and $U_A$ play the same role as before but are of
course specific to the algorithm):

\[
U_B |\psi\rangle = \frac{1}{8} (e^{i\phi_0} |0000\rangle_B + e^{i\phi_1} |1111\rangle_B + e^{i\phi_2} |0011\rangle_B + e^{i\phi_3} |1100\rangle_B + \ldots) \\
(|00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A) (|0\rangle_V - |1\rangle_V) \quad (18)
\]

\[
U_AU_fU_B |\psi\rangle = \frac{1}{4} [(e^{i\phi_0} |0000\rangle_B - e^{i\phi_1} |1111\rangle_B) |00\rangle_A + (e^{i\phi_2} |0011\rangle_B - e^{i\phi_3} |1100\rangle_B) |10\rangle_A + \ldots]
\]

\[
(|0\rangle_V - |1\rangle_V). \quad (19)
\]

The entangled state (19) is reached with a single computation of \(f(b, a)\). Measuring \(\hat{A}\) in (19) yields the solution: all zeros if the function is constant, not so if it is balanced.

This time entanglement is a-symmetric and we should consider the reduced density operators of both registers \(B\) and \(A\) in state (19):

\[
\rho_B = \frac{1}{2\sqrt{2}} (e^{i\phi_0} |0000\rangle_B + e^{i\phi_1} |1111\rangle_B + e^{i\phi_2} |0011\rangle_B + e^{i\phi_3} |1100\rangle_B + \ldots),
\]

\[
\rho_A = \frac{1}{2} (e^{i\theta_0} |00\rangle_A + e^{i\theta_1} |01\rangle_A + e^{i\theta_2} |10\rangle_B + e^{i\theta_3} |11\rangle_A), \quad (20)
\]

\[
\text{the } \theta_i \text{ are independent random phases with uniform distribution in } [0, 2\pi] \text{ as well. This time the entropies of } \rho_B \text{ and } \rho_A \text{ are different, 3 bits and 2 bits respectively. Incidentally, we note that } \rho_B \text{ remains unaltered throughout the unitary transformation } U_AU_fU_B.
\]

We lay down the two lion’s share perspectives. If we assume that the measurement of \(\hat{B}\) is performed first, we ascribe to it: the zeroing of the entropy of \(\rho_B\) and the zeroing of the entropy of \(\rho_A\). If we assume that the measurement of \(\hat{A}\) is performed first, we ascribe to it: the zeroing of the entropy of \(\rho_A\) and the reduction of the entropy of \(\rho_B\).

Condition (ii) of the sharing rule becomes that each one of ”the two projections” (as defined in the sharing rule) properly reduces both entropies. This is enough to univocally solve the sharing problem.

To perform the sharing, in the first place we should identify the ”elementary” partial projections. A natural choice is considering the projections associated with measuring \(\hat{B}_0, \hat{B}_1, \ldots\), of course selecting the measurement outcomes that match with Bob’s choice. To fix ideas, we assume that Bob’s choice is \(b = 0011\). This means considering the projections on the single bits of the bit string \(b = 0011\) or, in equivalent terms, on the single rows of the table of \(f_b\) – see the third column of array (17). As we will see, this provides sufficient resolution to build the history superposition picture of the quantum algorithm; considering also Boolean functions of the bit string 0011 would generate repeated histories here.

Summing up, we should select two shares of the table of the function (the projection on one should be ascribed to Alice’s measurement, that on the other to Bob’s) in such a way that, together, they project on the value of \(b\) without
over-projecting on any part of it and, individually, reduce the entropies of both $\rho_B$ and $\rho_A$.

This implies that no share contains different values of the function or, if all the values are the same, more than 50% of the rows. Otherwise, the projection on that share would already tell that the function is balanced, or constant. For the no over-projection condition, this would mean ascribing to only one projection the entire reduction of the entropy of $\rho_A$, against condition (ii).

Conditions (i) and (ii) are satisfied if the two shares of Bob’s choice $b = 0011$ are respectively $f_b(00) = 0$, $f_b(01) = 0$ and $f_b(10) = 1$, $f_b(11) = 1$ – see array (17). One can easily check that any deviation from this sharing violates the aforesaid conditions. For example, if the two shares were instead $f_b(00) = 0$ and $f_b(11) = 1$, projecting on them would not determine Bob’s choice, thus violating condition (i). If they were $f_b(00) = 0$, $f_b(01) = 0$ and $f_b(11) = 1$, this would determine Bob’s choice, but the projection on the latter share would not reduce the entropy of $\rho_A$, thus violating condition (ii). Etc. We call either one of the two shares of the table a good half table.

Incidentally, nothing a-priori requires that we split the entire table of the function into two shares. In the quantum part of Shor’s [10] factorization algorithm – finding the period $R$ of a periodic function – conditions (i) and (ii) dictate that one share of the table is a set of $R$ consecutive rows, the other share a similar set with arguments displaced by a multiple of $R$ (the two sets should be taken in all possible ways in quantum superposition). Splitting the entire table into two shares, if the domain of the function spans more than two periods, would imply over-projection.

Back to Deutsch and Jozsa’s algorithm, besides Alice’s contribution to Bob’s choice, a good half table represents Alice’s advanced knowledge of this choice. In fact, since $\rho_B$ remains unaltered throughout the unitary part of the quantum algorithm, also the projection of $\rho_B$ on a good half table (on the superposition of the values of $b$ that match with it) remains unaltered. At the end of the relativized quantum algorithm, this projection represents Alice’s contribution to Bob’s choice. At the beginning, it changes Alice’s complete ignorance of Bob’s choice into knowledge of the good half table.

It is immediate to check that the quantum algorithm requires the number of function evaluations of a classical algorithm that knows in advance a good half table. In fact, the value of $b$, and thus the solution, are always identified by computing $f_b(a)$ for only one value of $a$ (anyone) outside the half table – see array (17). Thus, both the quantum algorithm and the advanced knowledge classical algorithm require just one function evaluation.

Now we go to the history superposition picture. It is convenient to group the histories with the same value of $b$. Starting with $b = 0011$, we assume that Alice’s advanced knowledge is the good half table $f(b,00) = 0$, $f(b,01) = 0$. As this is common to $b = 0000$ and $b = 0011$, in order to find the value of $b$ and thus the character of the function, Alice should perform function evaluation for either $a = 10$ or $a = 11$. We assume it is for $a = 10$. Since we are under the assumption $b = 0011$, the result of the computation is 1. This originates two classical computation histories, each consisting of a state before and one after function
evaluation. History 1: $e^{i\phi} |0011 \rangle_B |10\rangle_A |0\rangle_V \rightarrow e^{i\phi} |0011 \rangle_B |10\rangle_A |1\rangle_V$; history 2: $-e^{i\phi} |0011 \rangle_B |10\rangle_A |1\rangle_V \rightarrow -e^{i\phi} |0011 \rangle_B |10\rangle_A |0\rangle_V$. If she performs function evaluation for $a = 11$ instead, this originates other two histories, etc.

In this way, in the state before function evaluation, the value of $b$ is multiplied by the superposition of all the possible combinations of values of $a$ and contents of register $V$. This explains quantum parallel computation. Performing a single computation of $f(b, a)$ entangles registers $B$ and $A$.

Until now we have seen the function evaluation stage of the quantum algorithm. Before going further, it can be useful to summarize the overall picture. Alice knows in advance half of the value of $b$. In order to find the solution, a function of $b$, she should identify the entire value of $b$, by performing function evaluation for only one value of $a$ (anyone) outside the half table. This is done in all possible ways in quantum superposition. Function evaluation is the identity on $\rho_B$; it entangles registers $B$ and $A$. Information contained in the control register $B$ leaks to the target register $A$. Eventually, applying the Hadamard transform to register $A$ yields state (19); entanglement also becomes correlation between the possible measurement outcomes.

By the way, the fact that Alice, in each individual history, knows half of the value of $b$ and computes the missing half in order to find the solution, agrees with the fact that Alice cannot know the precise value of $b$ by measuring $\hat{A}$ in state (19). In fact this depends on the special form of state (19), where each eigenstate of $\hat{A}$ multiplies the superposition of two eigenstates of $\hat{B}$; this form emerges in the very superposition of the individual histories.

It is easy to see that the present analysis, like the notion of good-half-table, holds unaltered for $n > 2$.

5 Simon’s and the hidden subgroup algorithms

In Simon’s [11] algorithm, the set of functions is all the $f_b : \{0, 1\}^n \rightarrow \{0, 1\}^{n-1}$ such that $f_b(a) = f_b(c)$ if and only if $a = c$ or $a = c \oplus h(b)$; $\oplus$ denotes bitwise modulo 2 addition; the bit string $h(b)$, depending on $b$ and belonging to $\{0, 1\}^n$ excluded the all zeroes string, is a sort of period of the function. Array (22) gives the set of functions for $n = 2$. The bit string $b$ is both the suffix and the table of the function. Since $h(b) \oplus h(b) = 0$, each value of the function appears exactly twice in the table, thus 50% of the rows plus one surely identify $h(b)$.

|      | $h^{(0011)} = 01$ | $h^{(1100)} = 01$ | $h^{(1101)} = 10$ | $h^{(1010)} = 10$ | $h^{(0110)} = 11$ | $h^{(1001)} = 11$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $a$  | $f_{0011}(a)$   | $f_{1100}(a)$   | $f_{0101}(a)$   | $f_{1010}(a)$   | $f_{0110}(a)$   | $f_{1001}(a)$   |
| 00   | 0               | 1               | 0               | 1               | 0               | 1               |
| 01   | 0               | 1               | 1               | 0               | 1               | 0               |
| 10   | 1               | 0               | 0               | 1               | 1               | 0               |
| 11   | 1               | 0               | 1               | 0               | 0               | 1               |

Bob selects a value of $b$. Alice’s problem is finding the value of $h(b)$, ”hidden” in $f_b(a)$, by computing $f_b(a) = f(b, a)$ for different values of $a$. In
present knowledge, a classical algorithm requires a number of computations of \( f(b, a) \) exponential in \( n \). The quantum algorithm solves the hard part of this problem, namely finding a string \( s_j^{(b)} \) orthogonal to \( h^{(b)} \), with one computation of \( f(b, a) \). There are \( 2^{n-1} \) such strings. Running the quantum algorithm yields one of these strings at random (see further below). The quantum algorithm is iterated until finding \( n - 1 \) different strings. This allows us to find \( h^{(b)} \) by solving a system of modulo 2 linear equations.

We give the relativized states before and after the unitary part of the algorithm:

\[
U_B |\psi\rangle = \frac{1}{2\sqrt{6}} \left( e^{i\varphi_1} |0011\rangle_B + e^{i\varphi_1} |1100\rangle_B + e^{i\varphi_2} |0101\rangle_B + e^{i\varphi_3} |1010\rangle_B + \ldots \right) \\
\left( |00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A \right) |0\rangle_V.
\]

(23)

\[
U_A U_B U_B |\psi\rangle = \frac{1}{2\sqrt{6}} \left\{ (e^{i\varphi_1} |0011\rangle_B + e^{i\varphi_1} |1100\rangle_B) \left( |00\rangle_A + |10\rangle_A \right) |0\rangle_V + (|00\rangle_A - |10\rangle_A) |1\rangle_V \right\} + \\
\left( e^{i\varphi_2} |0101\rangle_B + e^{i\varphi_3} |1010\rangle_B \right) \left( |01\rangle_A + |11\rangle_A \right) |0\rangle_V + (|00\rangle_A - |01\rangle_A) |1\rangle_V + \ldots ,
\]

(24)

In (23), register \( V \) is prepared in the all zeros string (just one zero for \( n = 2 \)). State (24) is reached with a single computation of \( f(b, a) \). In (24), for each value of \( b \), register \( A \) (no matter the content of \( V \)) hosts even weighted superpositions of the \( 2^{n-1} \) strings \( s_j^{(b)} \) orthogonal to \( h^{(b)} \). By measuring \( A \) in this state, Alice obtains at random one of the \( s_j^{(b)} \). Then we iterate the "right part" of the algorithm (preparation of registers \( A \) and \( V \), computation of \( f(b, a) \), and measurement of \( A \)) until obtaining \( n - 1 \) different \( s_j^{(b)} \).

Let \( \rho_B \) and \( \rho_A \) be the reduced density operators of respectively \( B \) and \( A \) in state (24). We lay down the two lion's share perspectives in the assumption that we always throw away the projections on \( |00\rangle_A \), which do not reduce the entropy of either \( \rho_B \) or \( \rho_A \) (tell nothing about the value of \( b \) or the solution). If we assume that the measurement of \( B \) is performed first, we ascribe to it: the zeroing of the entropy of \( \rho_B \) and the reduction (zeroing in the case \( n = 2 \)) of the entropy of \( \rho_A \). If we assume that the measurement of \( A \) is performed first, we ascribe to it: the zeroing of the entropy of \( \rho_A \) and the reduction of the entropy of \( \rho_B \). This can readily be checked by looking at the form of state (24).

Condition (ii) of the sharing rule becomes that each one of "the two projections" properly reduces both entropies. The analysis of the former section – about how to share the projection on the value of \( b \) between Alice’s and Bob’s measurements – still holds. Now half of Bob’s choice is any half table that does not contain the same value of the function twice, which would already specify the value of \( h^{(b)} \) and thus of all the \( s_j^{(b)} \).

We check that the quantum algorithm requires the number of function evaluations of a classical algorithm that knows in advance a good half table. In fact, the solution is always identified by computing \( f(b, a) \) for only one value of \( a \) (anyone) outside the half table. The new value of the function is necessarily

\^3The modulo 2 addition of the bits of the bitwise product of the two strings should be zero.
a value already present in the half table, which identifies $h^{(b)}$ and thus all the $s_i^{(b)}$. Thus, both the quantum algorithm and the advanced knowledge classical algorithm require just one function evaluation.

We go to the history superposition picture. Let us assume that Bob choose $b = 0011$. Alice’s advanced knowledge is either $f(b, 01) = 0$, $f(b, 10) = 1$ or $f(b, 00) = 0$, $f(b, 11) = 1$. Let us start with the former possibility. As this half table is common to $b = 0011$ and $b = 1010$, in order to find the value of $b$ and thus the character of the function, Alice should perform function evaluation for either $a = 00$ or $a = 11$. We assume that it is for $a = 00$. The result of the computation is 0. This originates two classical computation histories, each consisting of two states, before and after function evaluation. History 1: $e^{i\phi_0} |0011\rangle_B |00\rangle_A |0\rangle_V \rightarrow e^{i\phi_0} |0011\rangle_B |00\rangle_A |0\rangle_V$; history 2: $-e^{i\phi_0} |0011\rangle_B |00\rangle_A |1\rangle_V \rightarrow -e^{i\phi_0} |0011\rangle_B |00\rangle_A |1\rangle_V$. If she performs function evaluation for $a = 11$ instead, the result of the computation is 1. This originates other two histories, etc. The sum of all histories is the function evaluation stage of the quantum algorithm. After function evaluation, we should apply the Hadamard transform to register $A$. Each history branches into four histories; branches interfere with one another to yield state

$\rho$.

The present analysis holds unaltered for $n > 2$. It also applies to the generalized Simon’s problem and to the Abelian hidden subgroup problem. In fact the corresponding algorithms are essentially the same as the algorithm that solves Simon’s problem. In the hidden subgroup problem, the set of functions $f_b : G \rightarrow W$ map a group $G$ to some finite set $W$ with the property that there exists some subgroup $S \leq G$ such that for any $a, c \in G$, $f_b(a) = f_b(c)$ if and only if $a + S = c + S$. The problem is to find the hidden subgroup $S$ by computing $f_b(a)$ for various values of $a$. Now, a large variety of problems solvable with a quantum speed-up can be re-formulated in terms of the hidden subgroup problem [12, 13]. Among these we find: the seminal Deutsch’s problem, finding orders, finding the period of a function (thus the problem solved by the quantum part of Shor’s factorization algorithm), discrete logarithms in any group, hidden linear functions, self shift equivalent polynomials, Abelian stabilizer problem, graph automorphism problem.

6 Summary and conclusions

We summarize and position the results obtained. The present explanation of the speed-up:

(i) Relies on solving a fundamental measurement problem: how to share between redundant measurements the determination of correlated eigenvalues.

(ii) Shows that the apparent determinism of quantum algorithms is a visual illusion.

(iii) Applies to both quadratic and exponential speed-ups. The unifications achieved so far, which focus on the unitary part of the quantum algorithm, do not capture both kinds of speed-up.
(iv) Highlights the existence of a common scheme for the family of quantum algorithms examined: (a) Given the advanced knowledge of half of Bob’s choice, Alice finds the missing half through function evaluations; each function evaluation is the identity on the reduced density operator of the control register $B$ and increases the entanglement between this register and the target register $A$; information about Bob’s choice flows from the former to the latter register – this is of course a “character” of the function chosen by Bob. (b) By changing the basis of the target register after each function evaluation, entanglement also becomes correlation between the possible measurement outcomes (the application, after function evaluation, of the ”inversion about the mean”, or the Hadamard transform, or the quantum Fourier transform in the case of Shor’s algorithm, maximizes this correlation). (c) Steps (a) and (b) should be repeated the number of times required to classically find the missing half of Bob’s choice.

(v) Allows us to derive new quantum algorithms. Given a set of functions, we go through steps (a), (b), and (c), and see what is the character of the function obtained. Then we design the problem (finding that character of the function) around this result. Ref. [6] provides an example in point.

(vi) Provides a tool for ascertaining the quantum speed-up achievable in solving a problem – a central issue in quantum computation. The speed-up comes from comparing two classical algorithms, with and without advanced knowledge of half of Bob’s choice.

(vii) Highlights the existence of special causality loops. Each individual history contains such a loop: Alice knowing in advance half of Bob’s choice without computing it. This is possible because the missing computation is performed in other histories and quantum interference provides cross-talk between histories. The causality loop remains fully there in the superposition of all histories.

Possible future work is trying and extend the present explanation to other quantum algorithms and further investigating what the explanation means at a fundamental physical level. One could expect cross fertilization between these two prospects.

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