Partial cubes with pre-hull number at most 1

Norbert Polat
I.A.E., Université Jean Moulin (Lyon 3)
6 cours Albert Thomas
69355 Lyon Cedex 08, France
norbert.polat@univ-lyon3.fr

Abstract

We prove that a connected bipartite graph $G$ is a partial cube if and only if the set of attaching points of any copoint of $G$ is convex. A consequence of this result is that any connected bipartite graph with pre-hull number at most 1 is a partial cube. We show that the class of partial cubes with pre-hull number at most 1 is closed under gated subgraphs, gated amalgams and cartesian products.

Keywords: Geodesic convexity; Copoint; Attaching point; Pre-hull number; Bipartite graph; Partial cube; Median graph, Netlike partial cube.

1 Introduction

The (geodesic) pre-hull number $ph(G)$ of a graph $G$ is a parameter which measures the intrinsic non-convexity of $V(G)$ in terms of the number of iterations of the pre-hull operator associated with the interval operator $I_G$ which are necessary, in the worst case, to reach the canonical minimal convex extension of copoints of $V(G)$ when they are extended by the adjunction of an attaching point. In [9], where this concept was introduced, the question whether any connected bipartite graph with pre-hull number at most 1 is a partial cube was considered, but only partial results were obtained [9, Sections 6 and 7]. Note that a connected bipartite graph with pre-hull number greater than 1 may or may not be a partial cube. The first part of the present paper deals with the research of a definitive answer to this question.

In [9] we proved that, for any copoint $K$ of a partial cube $G$, the set $Att(K)$ of all attaching points of $K$ is convex (Att-convexity of $G$). In the first part of this paper (Section 3), we show that Att-convexity is a necessary and sufficient condition for a (finite or infinite) connected bipartite graph to be a partial cube. The affirmative answer to the above question follows immediately: any connected bipartite graph with pre-hull number at most 1 is a partial cube.
The class of partial cubes with pre-hull number at most 1 contains most of the mainly studied partial cubes such as: median graphs, cellular bipartite graphs, benzenoid graphs and netlike partial cubes. We show that this class is closed under gated subgraphs (but not convex ones), gated amalgams and cartesian products.

## 2 Preliminaries

### 2.1 Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. If \( x \in V(G) \), the set \( N_G(x) := \{ y \in V(G) : xy \in E(G) \} \) is the neighborhood of \( x \) in \( G \). For a set \( S \) of vertices of a graph \( G \) we put \( N_G(S) := \bigcup_{x \in S} N_G(x) - S \), and we denote by \( \partial_G(S) \) the edge-boundary of \( S \) in \( G \), that is the set of all edges of \( G \) having exactly one end-vertex in \( S \). Moreover, \( G[S] \) is the subgraph of \( G \) induced by \( S \), and \( G - S := G[V(G) - S] \).

Paths are considered as subgraphs rather than as sequences of vertices. Thus an \((x, y)\)-path is also a \((y, x)\)-path. If \( u \) and \( v \) are two vertices of a path \( P \), then we denote by \( P[u, v] \) the segment of \( P \) whose end-vertices are \( u \) and \( v \).

Let \( G \) be a connected graph. The usual distance between two vertices \( x \) and \( y \), that is, the length of any \((x, y)\)-geodesic (= shortest \((x, y)\)-path) in \( G \), is denoted by \( d_G(x, y) \). A connected subgraph \( H \) of \( G \) is isometric in \( G \) if \( d_H(x, y) = d_G(x, y) \) for all vertices \( x \) and \( y \) of \( H \). The (geodesic) interval \( I_G(x, y) \) between two vertices \( x \) and \( y \) of \( G \) consists of the vertices of all \((x, y)\)-geodesics in \( G \).

### 2.2 Convexities

A convexity on a set \( X \) is an algebraic closure system \( \mathcal{C} \) on \( X \). The elements of \( \mathcal{C} \) are the convex sets and the pair \((X, \mathcal{C})\) is called a convex structure. See van de Vel [10] for a detailed study of abstract convex structures. Several kinds of graph convexities, that is, convexities on the vertex set of a graph \( G \), have already been investigated. We will principally work with the geodesic convexity, that is, the convexity on \( V(G) \) which is induced by the geodesic interval operator \( I_G \). In this convexity, a subset \( C \) of \( V(G) \) is convex provided it contains the geodesic interval \( I_G(x, y) \) for all \( x, y \in C \). The convex hull \( \text{co}_G(A) \) of a subset \( A \) of \( V(G) \) is the smallest convex set which contains \( A \). The convex hull of a finite set is called a polytope. A subset \( H \) of \( V(G) \) is a half-space if \( H \) and \( V(G) - H \) are convex.

A copoint at a point \( x \in X \) is a convex set \( C \) which is maximal with respect to the property that \( x \notin C \); \( x \) is an attaching point of \( K \). Note that \( \text{co}_G(K \cup \{x\}) = \text{co}_G(K \cup \{y\}) \) for any two attaching points \( x \) and \( y \) of \( K \). We denote by \( \text{Att}(K) \) the set of all attaching points of \( K \), i.e.,

\[
\text{Att}(K) := \text{co}_G(K \cup \{x\}) - K.
\]
We denote by $I_G$ the pre-hull operator of the geodesic convex structure of $G$, i.e. the self-map of $P(V(G))$ such that $I_G(A) := \bigcup_{x,y \in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $co_G(A) = \bigcup_{n \in \mathbb{N}} I_G^n(A)$. Furthermore we will say that a subgraph of a graph $G$ is convex if its vertex set is convex, and by the convex hull $co_G(H)$ of a subgraph $H$ of $G$ we will mean the smallest convex subgraph of $G$ containing $H$ as a subgraph, that is,

$$co_G(H) := G[co_G(V(H))].$$

2.3 Bipartite graphs and partial cubes

All graphs considered here are connected.

For an edge $ab$ of a graph $G$, let

$$W^G_{ab} := \{x \in V(G) : d_G(a, x) < d_G(b, x)\},$$

$$U^G_{ab} := \{x \in W_{ab} : x \text{ has a neighbor in } W_{ba}\}.$$

If no confusion is likely, we will simply denote $W^G_{ab}$ and $U^G_{ab}$ by $W_{ab}$ and $U_{ab}$, respectively. Note that the sets $W_{ab}$ and $W_{ba}$ are disjoint and that $V(G) = W_{ab} \cup W_{ba}$ if $G$ is bipartite and connected.

Two edges $xy$ and $uv$ are in the Djoković-Winkler relation $\Theta$ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

If $G$ is bipartite, the edges $xy$ and $uv$ are in relation $\Theta$ if and only if $d_G(x, u) = d_G(y, v)$ and $d_G(x, v) = d_G(y, u)$. The relation $\Theta$ is clearly reflexive and symmetric.

**Lemma 2.1.** Let $C$ be a convex set of a bipartite graph $G$. Then $C \subseteq W_{ab}$ for any edge $ab \in \partial_G(C)$ with $a \in C$.

**Proof.** Let $x \in C$ and $ab \in \partial_G(C)$ with $a \in C$. Suppose that $x \notin W_{ab}$. Then $b \in I_G(x, a)$, and thus $b \in C$ by the convexity of $C$, contrary to the fact that $ab \in \partial_G(C)$.

**Remark 2.2.** If $G$ is bipartite, then, by [4] Lemma 11.2, the notation can be chosen so that the edges $xy$ and $uv$ are in relation $\Theta$ if and only if

$$d_G(x, u) = d_G(y, v) = d_G(x, v) - 1 = d_G(y, u) - 1,$$

or equivalently if and only if

$$y \in I_G(x, v) \text{ and } x \in I_G(y, u).$$

From now on, we will always use this way of defining the relation $\Theta$. Note that, in this way, the edges $xy$ and $yx$ are not in relation $\Theta$ because $y \notin I_G(x, x)$ and $x \notin I_G(y, y)$. In other word, each time the relation $\Theta$ is used, the notation of an edge induces an orientation of this edge.
We recall the main characterizations of partial cubes, that is of isometric subgraphs of hypercubes (see [4]). Partial cubes are particular connected bipartite graphs.

**Theorem 2.3.** (Djoković [3, Theorem 1] and Winkler [11]) A connected bipartite graph \( G \) is a partial cube if and only if it has one of the following properties:

(i) For every edge \( ab \) of \( G \), the sets \( W_{ab} \) and \( W_{ba} \) are convex.

(ii) The relation \( \Theta \) is transitive.

It follows in particular that the half-spaces of a partial cube \( G \) are the sets \( W_{ab} \), \( ab \in E(G) \). Furthermore we can easily prove that the copoints of a partial cube are its half-spaces. The following technical lemma will be used later.

**Lemma 2.4.** Let \( G \) be a partial cube, \( F \) an isometric subgraph of \( G \), and \( ab \) an edge of \( F \). Then

- \( W_{ab}^F = W_{ab}^G \cap V(F) \) and \( W_{ba}^F = W_{ba}^G \cap V(F) \)
- \( U_{ab}^F \subseteq U_{ab}^G \cap V(F) \).

If moreover \( F \) is convex in \( G \), then

- \( U_{ab}^F = U_{ab}^G \cap V(F) \).

**Proof.** The first assertions are immediate consequences of the definitions of \( W_{ab} \) and \( U_{ab} \), and of the fact that \( F \) is isometric in \( G \). Assume now that \( F \) is convex in \( G \). Let \( x \in U_{ab}^G \cap V(F) \), and let \( y \) be the neighbor of \( x \) in \( U_{ba}^G \). Then \( y \in I_G(x,b) = I_F(x,b) \) since \( F \) is convex. Hence \( x \in U_{ab}^F \). Therefore \( U_{ab}^F \supseteq U_{ab}^G \cap V(F) \), and we are done by the above converse inclusion. \( \square \)

In the following lemma we list some well-known properties of partial cubes.

**Lemma 2.5.** Let \( G \) be a partial cube. We have the following properties:

(i) Each interval of \( G \) is finite and convex.

(ii) Each polytope of \( G \) is finite.

(iii) Let \( x, y \) be two vertices of \( G \), \( P \) an \((x,y)\)-geodesic and \( W \) an \((x,y)\)-path of \( G \). Then each edge of \( P \) is in relation \( \Theta \) with some edge of \( W \).

(iv) A path \( P \) in \( G \) is a geodesic if and only if no two distinct edges of \( P \) are \( \Theta \)-equivalent.

### 3 A characterization of partial cubes

**Definition 3.1.** A bipartite graph \( G \) is said to be \( \text{Att-convex} \) if for each copoint \( K \) of \( G \), the set \( \text{Att}(K) \) is convex.

We now state the main result of this section.

**Theorem 3.2.** A connected bipartite graph \( G \) is a partial cube if and only if it is \( \text{Att-convex} \).
To prove this theorem we will need several properties of expansions of a graph, a concept which was introduced by Mulder \[5\] to characterize median graphs and which was later generalized by Chepoi \[2\].

The cartesian product \(G_1 \square G_2\) of two graphs \(G_1\) and \(G_2\) is the graph with vertex set \(V(G_1) \times V(G_2)\), two vertices being adjacent if they have one coordinate adjacent and the other equal.

**Definition 3.3.** A pair \((V_0, V_1)\) of sets of vertices of a graph \(G\) is called a proper cover of \(G\) if it satisfies the following conditions:
- \(V_0 \cap V_1 \neq \emptyset\) and \(V_0 \cup V_1 = V(G)\);
- there is no edge between a vertex in \(V_0 - V_1\) and a vertex in \(V_1 - V_0\);
- \(G[V_0]\) and \(G[V_1]\) are isometric subgraphs of \(G\).

**Definition 3.4.** An expansion of a graph \(G\) with respect to a proper cover \((V_0, V_1)\) of \(G\) is the subgraph of \(G \square K_2\) induced by the vertex set \((V_0 \times \{0\}) \cup (V_1 \times \{1\})\) (where \(\{0, 1\}\) is the vertex set of \(K_2\)).

An expansion of a partial cube is a partial cube (see \[6\]). If \(G'\) is an expansion of a graph \(G\), then we say that \(G\) is a \(\Theta\)-contraction of \(G'\), because, as we can easily see, \(G\) is obtained from \(G'\) by contracting each element of some \(\Theta\)-class of edges of \(G'\). More precisely the natural surjection of \(G'\) onto \(G\) is a contraction, that is, an application which maps any two adjacent vertices to adjacent vertices or to a single vertex. A \(\Theta\)-contraction of a partial cube is a partial cube as well (see \[6\]).

In Lemmas 3.5–3.10, \(G\) will be a connected bipartite graph and \(G'\) an expansion of \(G\) with respect to a proper cover \((V_0, V_1)\) of \(G\). The following notation will be used.

- For \(i = 0, 1\) denote by \(\psi_i : V_i \rightarrow V(G')\) the natural injection \(\psi_i : x \mapsto (x, i), x \in V_i\), and let \(V'_i := \psi_i(V_i)\). Note that \(V'_0\) and \(V'_1\) are complementary half-spaces of \(G'\). It follows in particular that these sets are copoints of \(G'\).
- For any vertex \(x\) of \(G\) (resp. \(G'\)), denote by \(i(x)\) an element of \(\{0, 1\}\) such that \(x\) belongs to \(V_{i(x)}\) (resp. \(V'_{i(x)}\)). If \(x \in V(G')\) and also if \(x \in V(G) - (V_0 \cap V_1)\), then \(i(x)\) is unique; if \(x \in V_0 \cap V_1\) it may be 0 or 1.
- For \(A \subseteq V(G)\) put
  \[
  \psi(A) := \psi_0(A \cap V_0) \cup \psi_1(A \cap V_1).
  \]

Note that in the opposite direction we have that for any \(A' \subseteq V(G')\),

\[
\text{pr}(A') = \psi_0^{-1}(A' \cap V'_0) \cup \psi_1^{-1}(A' \cap V'_1),
\]

where \(\text{pr} : G \square K_2 \rightarrow G\) is the projection \((x, i) \mapsto x\).

The following lemma is a restatement with more precisions of \[7, Lemma 4.5\].

**Lemma 3.5.** Let \(G\) be a connected bipartite graph and \(G'\) an expansion of \(G\) with respect to a proper cover \((V_0, V_1)\) of \(G\), and let \(P = (x_0, \ldots, x_n)\) be a path in \(G\). We have the following properties:
Proof. We have 
\[ S \subseteq \psi(\text{pr}(S)) \subseteq \psi(\text{co}_{G'}(\text{pr}(S))). \]
By Corollary 3.6, \( \psi(\text{co}_{G'}(\text{pr}(S))) \) is convex in \( G' \). Hence \( \text{co}_{G'}(S) \subseteq \psi(\text{co}_{G'}(\text{pr}(S))). \) Therefore \( \text{pr}(\text{co}_{G'}(S)) \subseteq \psi(\text{co}_{G'}(\text{pr}(S))). \)

Lemma 3.8. Let \( K' \) be a convex set of \( G' \) which meets both \( V_0' \) and \( V_1' \). Then \( K := \text{pr}(K') \) is a convex set of \( G \).

Proof. Let \( u, v \in K \). If \( i(u) \neq i(v) \), then \( I_G(u, v) = \text{pr}(I_{G'}(u', v')) \) by Lemma 3.5 and hence \( I_G(u, v) \subseteq K \).

Now assume that \( i(u) = i(v) \), say \( i(u) = i(v) = 0 \). Let \( P = \langle x_0, \ldots, x_n \rangle \) be a \((u, v)\)-geodesic in \( G \) with \( x_0 = u \) and \( x_n = v \). In general, not all \( P \) is contained in \( G[V_0] \). Let \( 0 = i_0 < i_1 < \ldots < i_{2p+1} = n \) be subscripts such that the segments \( P[x_{i_{2k}}, x_{i_{2k+1}}] \) are alternatively contained in \( G[V_0] \) and \( G[V_1] \). Thus \( x_{i_0}, \ldots, x_{i_{2p}} \in V_0 \cap V_1 \). Since \( G[V_0] \) is isometric in \( G \) there is an \((x_{i_{2k-1}}, x_{i_{2k+1}})\)-segment of \( P \) in \( G[V_0] \), \( h = 1, \ldots, p \). Replacing each \((x_{i_{2k-1}}, x_{i_{2k}})\)-segment of \( P \) by the corresponding \( P_h \) one obtains a new \((u, v)\)-geodesic \( P_0 \) with \( V(P_0) \subseteq V_0 \). Hence \( \psi_0(P_0) \) is a \((u', v')\)-geodesic in \( G' \), and therefore \( V(P_0) \subseteq K \).

It follows in particular that \( \psi_0(x_{i_k}) \in K' \cap V_0', k = 1, \ldots, 2p \). By hypothesis there exists a vertex \( w \in K' \cap V_1' \). From the construction of \( G' \) it then follows that \( y_k := \psi_1(x_{i_{2k}}) \in I_{G'}(\psi_0(x_{i_k}), w) \), and hence \( y_k \in K' \). Since \( G[V_1] \) is an isometric subgraph of \( G \) we deduce that \( \psi_1(P[x_{i_{2k-1}}, x_{i_{2k}}]) \) is a \((y_{2k-1}, y_{2k})\)-geodesic. Hence \( V(P[x_{i_{2k-1}}, x_{i_{2k}}]) \subseteq K \), and therefore \( V(P) \subseteq K \).
Lemma 3.9. If $K'$ is a copoint of $G'$ which meets both $V'_0$ and $V'_1$, then $K := \text{pr}(K')$ is a copoint of $G$ such that $\text{Att}(K) = \text{pr}(\text{Att'}(K'))$ (where Att and Att' denote the sets of attaching points in $G$ and $G'$, respectively).

Proof. Let $u \in \text{Att}'(K')$ and abbreviate $i(u)$ by $i$. Thus $u \in V'_i$. By Lemma 3.8 $K$ is a convex set of $G$. Moreover $x := \text{pr}(u) \notin K$. Suppose that $K$ is not a copoint at $x$. Then $G$ contains a convex set $K_0$ with $x \notin K_0$ and $K \subseteq K_0$. By Corollary 3.6, $\psi(K_0)$ is a convex set of $G'$ which strictly contains $K'$. Hence $u \in \psi(K_0)$ because $K'$ is a copoint at $u$, contrary to the fact that $x \notin K_0$. Consequently $K$ is a copoint at $x$.

It follows that $\text{pr}(\text{Att'}(K')) \subseteq \text{Att}(K)$. On the other hand, by Lemma 3.8 $\text{pr}(\text{co}_{G'}(\{u\} \cup K'))$ is a convex set of $G$ containing $\{x\} \cup K$. Hence $\text{Att}(K) \subseteq \text{pr}(\text{Att'}(K'))$.

Going from $G$ to $G'$ we have:

Lemma 3.10. If $K$ is a copoint of $G$ which meets $V_0 \cap V_1$, then $K' := \psi(K)$ is a copoint of $G'$ such that $\text{Att'}(K') = \psi(\text{Att}(K))$.

Proof. Let $x \in \text{Att}(K)$. By Corollary 3.6 $\psi(K)$ is a convex set of $G'$ such that $\psi_i(x)(x) \notin \psi(K)$. Let $K'$ be a copoint of $G'$ at $\psi_i(x)(x)$ which contains $\psi(K)$. Then $K' \cap V'_1 \neq \emptyset$ for $i = 0, 1$. By Lemma 3.9 $\text{pr}(K')$ is a copoint of $G$ at $x$ which contains $K$, and thus is equal to $K$. Hence $K' = \psi(K)$. Now $\text{Att'}(\psi(K)) \subseteq \psi(\text{Att}(K))$ by Lemma 3.9 and moreover $(u, 0), (u, 1) \in \psi(K)$ for each $u \in K \cap V_0 \cap V_1$. It follows that $\text{Att'}(\psi(K)) = \psi(\text{Att}(K))$.

Lemma 3.11. $G$ is Att-convex if so is $G'$.

Proof. Assume that $G'$ is Att-convex. Let $K$ be a copoint of $G$. We will show that $\text{Att}(K)$ is convex. We distinguish two cases.

Case 1. $K \cap V_i = \emptyset$ for some $i \in \{0, 1\}$.

Say $i = 0$. Hence $K \subseteq V_1 - V_0$. Then $K = \psi(K)$ is convex in $G'$ by Corollary 3.6. Let $A := \psi(\text{Att}(K))$. Then $K \cup A = \psi(\text{Att}(K))$, and thus $K \cup A$ is convex in $G'$ by Corollary 6.6 since $K \cup \text{Att}(K)$ is convex in $G$.

Let $u \in A \cap V'_1$, $u' = \psi_1(u)$, and let $K'$ be a copoint at $u'$ in $G'$ containing $K$ or equal to $K$. Suppose that $K' \cap V'_0 \neq \emptyset$. Then, by Lemma 3.9 $\text{pr}(K')$ is a copoint at $u$ in $G$ with $K \subseteq K'$, contrary to the fact that $K$ is a copoint at $u$. Therefore

$$K' \subseteq V'_1. \quad (1)$$

Suppose that $\text{Att'}(K') \cap V'_0 \neq \emptyset$. Then, because $K' \subseteq V'_1$ by (1) and since $V'_1$ is convex, there exists a vertex $x_0 \in \text{Att'}(K') \cap V'_0 \cap N_{G'}(K')$. Let $x_1$ be the neighbor of $x_0$ in $V'_1$. Then $x_1 \in K'_1$ and $x_1 \in I_{G'}(u', x_0)$ by Lemma 2.4, contrary to the fact that $\text{Att'}(K')$ is convex since $G'$ is Att-convex by assumption. Therefore

$$\text{Att'}(K') \subseteq V'_1. \quad (2)$$
Suppose that \( A \cap K' \neq \emptyset \), and let \( x \in \text{Att}(K) \) be such that \( \psi_1(x) \in K' \). Because \( K \cup \text{Att}(K) = \text{co}_G(K \cup \{x\}) \) since \( K \) is a copoint at \( x \), it follows that \( u \in \text{co}_G(K \cup \{x\}) \). Hence \( u' \in \text{co}_G(K \cup \{\psi_1(x)\}) \) is \( K' \), contrary to the facts that \( K' \) is a copoint at \( u' \). Therefore

\[
A \cap V' \subseteq \text{Att}'(K').
\]  

We distinguish two subcases.

Subcase 1.1. \((K \cup \text{Att}(K)) \cap V_0 = \emptyset\).

Then \( \text{Att}(K) = \text{pr}(A) = A \subseteq \text{Att}'(K') \) by (3). Hence \( \text{co}_G(\text{Att}(K)) \cap K = \emptyset \) since \( \text{Att}'(K') \) is convex and disjoint from \( K' \), and thus from \( K \). Therefore \( \text{Att}(K) \) is convex since so is \( K \cup \text{Att}(K) \).

Subcase 1.2. \((K \cup \text{Att}(K)) \cap V_0 \neq \emptyset\).

Then \( \text{Att}(K) \cap V_0 \cap V_1 \neq \emptyset \), and thus \( A \cap V_i' \neq \emptyset \) for \( i = 0, 1 \). The set \( A \cap V_0' \) is equal to \((K \cup A) \cap V_0'\), and the set \((K \cup A) \cap V_1'\) are convex since so are the sets \( K \cup A \). \( V_0' \) and \( V_1' \). By (3) and the fact that \( \text{Att}'(K') \) is convex since \( G' \) is \( \text{Att} \)-convex by assumption, we infer that \( \text{co}_G'(A \cap V_0') \subseteq \text{Att}'(K') \), and thus \( \text{co}_G'(A \cap V_0') \cap K = \emptyset \) since \( K' \subseteq K' \). Because \((K \cup A) \cap V_1' = K \cup (A \cap V_1')\) is convex, it follows that \( A \cap V_1' \) is also convex. Hence \( A \), which is equal to the union of the two convex sets \( A \cap V_0' \) and \( A \cap V_1' \), is convex by Lemma 3.5. Therefore \( \text{Att}(K) = \text{pr}(A) \) is convex by Lemma 8 since \( A \cap V_i' \neq \emptyset \) for \( i = 0, 1 \).

Case 2. \( K \cap V_0 \cap V_1 \neq \emptyset \).

By Lemma 3.10 \( K' := \psi(K) \) is a copoint of \( G' \) such that \( \text{Att}'(K') = \psi(\text{Att}(K)) \). The set \( \text{Att}'(K') \) is convex because \( G' \) is \text{Att}-convex by assumption. Furthermore \( K' \cap V_i' \neq \emptyset \) for \( i = 0, 1 \). Hence, by Lemma 3.9 \( \text{pr}(K') \) is a copoint of \( G \) such that \( \text{Att}(\text{pr}(K')) = \text{pr}(\text{Att}'(K')) \). Because \( \text{Att}(K) \subseteq \text{pr}(\text{Att}'(K')) \) and since \( K \) is a copoint, it follows that \( K = \text{pr}(K') \) and \( \text{Att}(K) = \text{pr}(\text{Att}'(K')) \).

If \( \text{Att}'(K') \cap V_i' \neq \emptyset \) for \( i = 0, 1 \), then \( \text{Att}(K) \) is convex by Lemma 3.5 since \( \text{Att}'(K') \) is convex. Suppose that \( \text{Att}'(K') \subseteq V_i' - V_{i-1}' \) for some \( i = 0 \) or \( 1 \). Then \( \text{Att}(K) = \psi^{-1}(\text{Att}'(K')) \subseteq V_i' - V_{i-1}' \). It follows that \( \text{Att}(K) = \text{Att}'(K') \). Therefore \( \text{Att}(K) \) is convex since so is \( \text{Att}'(K') \).

Consequently \( G \) is \text{Att}-convex.

\[ \square \]

Lemma 3.12. A bipartite graph \( G \) is a partial cube if and only if every polytope of \( G \) induces a partial cube.

Proof. We only have to prove the sufficiency. Let \( ab \) be an edge of \( G \), and let \( cd \) and \( ef \) be two other edges of \( G \) such that each of them is in relation \( \Theta \) with \( ab \). Then the polytope \( A := \text{co}_G(a, b, c, d, e, f) \) induces a partial cube \( F \) by hypothesis. Because \( F \) is a convex subgraph of \( G \), it follows that both the edges \( cd \) and \( ef \) are in relation \( \Theta \) with \( ab \) in \( F \). Because \( F \) is a partial cube, we infer from Theorem 2.3 that these edges are in relation \( \Theta \) in \( F \), and thus in \( G \). Consequently the relation \( \Theta \) in \( G \) is transitive, which proves that \( G \) is a partial cube by Theorem 2.3.

\[ \square \]
Lemma 3.13. Any convex subgraph of an Att-convex graph is also Att-convex.

Proof. Let $H$ be a convex subgraph of an Att-convex graph $G$, and let $K$ be a copoint at a vertex $x$ of $H$. Then $K$ is convex in $G$, and thus it is contained in a copoint $K'$ at $x$ in $G$. Clearly $K = K' \cap V(H)$. Moreover $\text{Att}(K) \subseteq \text{Att}(K') \cap V(H)$. Because $V(H)$ and $\text{Att}(K')$ are convex in $G$, it follows that $\text{Att}(K)$ is contained in a convex subset of $V(H)$ which does not meet $K$. It follows that $\text{Att}(K)$ is convex because so is $K \cup \text{Att}(K)$.

Proof of Theorem 3.2 We only have to prove the sufficiency because of [9, Theorem 6.7] which in particular states that a connected bipartite graph $G$ is a partial cube if and only if it is Att-convex and $N_G(K) \subseteq \text{Att}(K)$ for each copoint $K$ of $G$.

Case 1. $G$ is finite.

The proof will be by induction on the order of $G$. This is obvious if $G$ has one or two vertices since $K_1$ and $K_2$ are hypercubes. Let $n \geq 2$. Suppose that every connected bipartite graph whose order is at most $n$ and which is Att-convex is a partial cube. Let $G$ be an Att-convex connected bipartite graph whose order is $n + 1$.

Because $G$ is finite, there exists a copoint $K$ of $G$ which is maximal with respect to inclusion. Then $\text{Att}(K) = V(G) - K$, since otherwise there would exist a copoint at some vertex $x \not\in K \cup \text{Att}(K)$ strictly containing $K$, contrary to the maximality of $K$. Because $G$ is Att-convex, $\text{Att}(K)$ is convex and thus $K$ is a half-space. Therefore the edges in $\partial_G(K)$ are pairwise in relation $\Theta$.

Let $F$ be the graph obtained from $G$ by identifying, for each edge between $K$ and $V(G) - K$, the endvertices of this edge. Clearly $G$ is an expansion of $F$. Note that $F$ is a bipartite graph whose order is at most $n$, and that it is Att-convex by Lemma 3.11. Hence $F$ is a partial cube by the induction hypothesis. Therefore $G$ is also a partial cube by the properties of expansions.

Case 2. $G$ is infinite.

We denote by $\mathcal{C}$ the class of all Att-convex connected bipartite graphs whose vertex set is a polytope. Let $H \in \mathcal{C}$. A subset $S$ of $V(H)$ such that $V(H) = \text{co}_H(S)$ is called a spanning set of $H$. We define:

$$d(S) := \sum_{x,y \in S} d_H(x,y)$$

$$d(H) := \min \{d(S) : S \text{ is a finite spanning set of } H\}.$$

Claim. Any $H \in \mathcal{C}$ is a finite partial cube.

We first prove by induction on $d(H)$ that any $H \in \mathcal{C}$ is finite. This is obvious if $d(H) = 0$ since $H = K_1$. Let $n$ be a non-negative integer. Suppose that any $H \in \mathcal{C}$ such $d(H) \leq n$ is finite. Let $H \in \mathcal{C}$ be such that $d(H) = n + 1$, and let $S$ be a finite spanning set of $H$ such that $d(S) = d(H)$. By [10], $V(H)$ cannot be the union of a non-empty chain of proper convex subsets. Hence $V(H)$ contains a maximal convex subset $K$. Then $K$ is a copoint of any element of $V(H) - K$, i.e. $\text{Att}(K) = V(H) - K$. It follows that $K$ is a half-space since $\text{Att}(K)$ is
convex because $H$ is $\text{Att}$-convex by hypothesis. Therefore the edges in $\partial_H(K)$ are pairwise in relation $\Theta$.

Let $F$ be the graph obtained from $H$ by identifying, for each edge between $K$ and $V(H) - K$, the endvertices of this edge. Clearly $H$ is an expansion of $F$. By Lemma 3.11 $F$ is $\text{Att}$-convex. Let $S$ be a finite spanning set of $H$. By Lemma 3.7 we have

$$V(F) = \text{pr}(V(H)) = \text{pr}(\text{co}_H(S)) \subseteq \text{co}_F(\text{pr}(S)) \subseteq V(F).$$

Hence $V(F) = \text{co}_F(\text{pr}(S))$, i.e. $\text{pr}(S)$ is a finite spanning set of $F$. It follows that $F \in \mathcal{C}$. On the other hand, because $S$ is a finite spanning set of $H$, and because $K$ is a half-space, it follows that $K$ and $V(H) - K$ have non-empty intersections with $S$. Therefore $d(F) \leq d(\text{pr}(S)) < d(S) = d(H) = n + 1$. Hence $d(F) \leq n$, and thus, by the induction hypothesis, $F$ is finite. It follows that $H$, which is an expansion of $F$, is also finite.

$H$ is then a finite connected bipartite graph which is $\text{Att}$-convex. We then deduce, by Case 1 of this proof, that $H$ is a partial cube, which completes the proof of the claim.

Now, let $G$ be an infinite $\text{Att}$-convex bipartite graph. By Lemma 3.13 each polytope of $G$ is $\text{Att}$-convex, and thus is a partial cube by the above claim. Consequently $G$ is itself a partial cube by Lemma 3.12.

**Proposition 3.14.** Let $G$ be a connected bipartite graph. The following assertions are equivalent:

(i) $G$ is a partial cube.

(ii) $G$ is $\text{Att}$-convex.

(iii) For every convex subgraph $F$ of $G$, any maximal proper convex subset of $V(F)$ is a half-space of $F$.

**Proof.** (i) $\Rightarrow$ (iii): Let $F$ be a convex subgraph of a partial cube $G$. Then $F$ itself is a partial cube. Let $K$ be a maximal proper convex subset of $V(F)$. Then $K = W_{ab}$ for some edge $ab \in \partial_F(K)$ with $a \in K$. Hence $V(F) - K = W_{ba}$, which proves that $K$ is a half-space of $F$ by Theorem 2.3.

(iii) $\Rightarrow$ (ii): Assume that $G$ satisfies (iii), and let $K$ be a copoint of $G$. Then $X := K \cup \text{Att}(K)$ is convex, and $K$ is a maximal proper convex subset of $X$. Hence $K$ is a half-space of $G[X]$ by (iii). Therefore $\text{Att}(K)$ is convex in $G[X]$, and thus in $G$.

(ii) $\iff$ (i) is Theorem 3.2.

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**4 Partial cubes with pre-hull number at most 1**

We begin by recalling some definitions and results from [9]. In that paper we introduced and studied the concept of pre-hull number of a convexity. We recall its definition in the particular case of the geodesic convexity of a graph.
Definition 4.1. Let $G$ be a graph. The least non-negative integer $n$ (if it exists) such that $\text{co}_G(C \cup \{x\}) = \mathcal{I}_G^n(C \cup \{x\})$ for each vertex $x$ of $G$ and each copoint $C$ at $x$, is called the pre-hull number of a graph $G$ and is denoted by $\text{ph}(G)$. If there is no such $n$ we put $\text{ph}(G) := \infty$.

Proposition 4.2. (Polat and Sabidussi [9, Corollary 3.8]) The pre-hull number of a connected bipartite graph $G$ is zero if and only if $G$ is a tree.

Definition 4.3. (Polat and Sabidussi [9, Definition 7.1]) Call a set $A$ of vertices of a graph $G$ ph-stable if any two vertices $u, v \in \mathcal{I}_G(A)$ lie on a geodesic joining two vertices in $A$.

The condition of Definition 4.3 which is symmetric in $u$ and $v$, can be replaced by the formally “one-sided” condition: for any two vertices $u, v \in \mathcal{I}_G(A)$ there is a $w \in A$ such that $v \subseteq \mathcal{I}_G(u, w)$.

Proposition 4.4. (Polat and Sabidussi [9, Theorem 7.4]) Let $G$ be a bipartite graph. Then $\text{ph}(G) \leq 1$ if and only if, for every copoint $K$ of $G$, the set $\text{Att}(K)$ is convex and $\mathcal{N}_G(K) \cap \text{Att}(K)$ is ph-stable.

The following result follows immediately from the above proposition.

Proposition 4.5. (Polat and Sabidussi [9, Theorem 7.5]) Let $G$ be a partial cube. Then $\text{ph}(G) \leq 1$ if and only if $U_{ab}$ and $U_{ba}$ are ph-stable for every edge $ab$ of $G$.

From Theorem 3.2 and Proposition 4.4 we infer the second main result of this paper.

Theorem 4.6. Any connected bipartite graph $G$ such that $\text{ph}(G) \leq 1$ is a partial cube.

Note that a bipartite graph whose pre-hull number is greater than 1 may or may not be a partial cube. For example, 2 is the pre-hull number of both $K_{2,3}$, which is the smallest connected bipartite graph which is not a partial cube, and of the partial cube $Q_5^-$ (i.e. the 3-cube $Q_3$ minus a vertex). A lot of well-known partial cubes have a pre-hull number equal to 1: median graphs, benzenoid graphs, cellular bipartite graphs and more generally netlike partial cubes.

We will now study some properties of partial cubes whose pre-hull number is at most 1, with in particular the closure of the class of these graphs under usual operations of partial cubes.

Proposition 4.7. Let $G$ be a partial cube such that any finite subgraph of $G$ is contained in a finite convex subgraph of $G$ whose pre-hull number is at most 1. Then $\text{ph}(G) \leq 1$.

Proof. Let $ab \in E(G)$ and $u, v \in \mathcal{I}_G(U_{ab})$. Let $P_u$ and $P_v$ be geodesics joining vertices in $U_{ab}$ on which lie $u$ and $v$, respectively. Then $(a, b) \cup P_u \cup P_v$ is contained in a finite convex subgraph $F$ of $G$ such that $\text{ph}(G) \leq 1$. The set $U_{ab}^F$ is ph-stable since $\text{ph}(F) \leq 1$, and thus $u, v$ lie on an $(x, y)$-geodesic $R$ for some
Figure 1: $M_{4,1}$ with a copy of $Q_3^-$ as a convex subgraph.

As was shown in [9, Remark 8.1], the class of all partial cubes whose pre-hull number is at most 1 is not closed under convex subgraphs. The graph $M_{n,1}$, $n \geq 4$, i.e. the cube $Q_n$ from which a pair of antipodal vertices has been removed, has a pre-hull number equal to 1. On the other hand $M_{n,1}$ contains copies of $Q_{n-1}^-$ (the cube $Q_{n-1}$ with only one vertex deleted) as convex subgraphs (see Figure 1 for $n = 4$, where $Q_3^-$ is depicted by the big points and the thick lines), and $\text{ph}(Q_{n-1}) = 2$ by [9, Theorem 5.8]. It was also shown in [8, Remark 3.3] that $Q_3^-$ is a retract of $M_{1,1}$, which proves that the class of all partial cubes whose pre-hull number is at most 1 is not closed under retracts. However, we will see that it is closed under gated subgraphs.

A set $A$ of vertices of a graph $G$ is said to be gated if, for each $x \in V(G)$, there exists a vertex $y$ (the gate of $x$) in $A$ such that $y \in I_G(x,z)$ for every $z \in A$. Any gated set is convex. Moreover the set of gated sets of a graph with the addition of the empty set is a convexity, and thus is closed under any intersections. We will say that a subgraph of a graph $G$ is gated if its vertex set is gated.

**Lemma 4.8.** Let $G$ be a partial cube, $F$ a gated subgraph of $G$, and $ab$ an edge of $F$. Then the gate in $F$ of any $x \in U_{ab}^G$ belongs to $U_{ab}^F$.

**Proof.** This is trivial if $x \in V(F)$. Assume that $x \in V(G - F)$, and let $y$ be the neighbor of $x$ in $U_{ba}^G$. Clearly, by Lemma 3.4

\begin{align*}
W_{ab}^F &\subseteq W_{ab}^G \quad \text{and} \quad W_{ba}^F \subseteq W_{ba}^G \\
U_{ab}^F &\subseteq U_{ab}^G \quad \text{and} \quad U_{ba}^F \subseteq U_{ba}^G
\end{align*}

since $F$ is convex in $G$. 

\end{proof}
Denote by \( g(x) \) and \( g(y) \) the gates in \( F \) of \( x \) and \( y \), respectively. Then \( g(x) \in I_G(x,a) \) and \( g(y) \in I_G(y,b) \). Hence \( g(x) \in W^F_{ab} \) and \( g(y) \in W^F_{ba} \). On the other hand, \( g(x) \in I_G(x,g((y))) \) and \( g(y) \in I_G(y,g(x)) \). It easily follows that the vertices \( g(x) \) and \( g(y) \) are adjacent. Therefore \( g(x) \in U^F_{ab} \) and \( g(y) \in U^F_{ba} \).

\[\square\]

**Theorem 4.9.** Let \( F \) be a gated subgraph of a partial cube \( G \) such that \( ph(G) \leq 1 \). Then \( ph(F) \leq 1 \).

**Proof.** Let \( u \) be an edge of \( F \). By Lemma 4.6, we have \( U^F_{ab} \subseteq U^G_{ab} \) and \( U^F_{ba} \subseteq U^G_{ba} \) since \( F \) is convex in \( G \). We will show that \( U^F_{ab} \) is \( ph \)-stable.

Let \( x, y \in I_F(U^F_{ab}) \). Because \( I_F(U^F_{ab}) \subseteq I_G(U^G_{ab}) \), and since \( U^G_{ab} \) is \( ph \)-stable by Proposition 4.5, it follows that \( y \in I_G(x,z) \) for some \( z \in U^G_{ab} \). By Lemma 4.8, the gate \( g(z) \) of \( z \) in \( F \) belongs to \( U^F_{ab} \). Moreover \( y \in I_F(x,g(z)) \) since \( g(z) \in I_G(y,z) \). Consequently \( U^F_{ab} \) is \( ph \)-stable.

In the same way we can prove that \( U^F_{ba} \) is \( ph \)-stable. We infer that \( ph(F) \leq 1 \) from Proposition 4.5.

We recall that a graph \( G \) is the *gated amalgam* of two graphs \( G_0 \) and \( G_1 \) if \( G_0 \) and \( G_1 \) are isomorphic to two intersecting gated subgraphs \( G_0' \) and \( G_1' \) of \( G \) whose union is \( G \). More precisely we also say that \( G \) is the gated amalgam of \( G_0 \) and \( G_1 \) *along* \( G_0' \cap G_1' \). The gated amalgam of two partial cubes is clearly a partial cube.

**Theorem 4.10.** Let \( G \) be the gated amalgam of two partial cubes \( G_0 \) and \( G_1 \). Then \( ph(G) \leq 1 \) if and only if \( ph(G_i) \leq 1 \) for \( i = 0, 1 \).

**Proof.** The necessity is clear by Theorem 4.9 since \( G_0 \) and \( G_1 \) are isomorphic to two gated subgraphs of \( G \). Conversely, assume that \( G = G_0 \cup G_1 \) where, for \( i = 0, 1 \), \( G_i \) is a gated subgraph of \( G \) such that \( ph(G_i) \leq 1 \). The subgraph \( G_01 := G_0 \cap G_1 \) is also gated in \( G \) as an intersection of gated subgraphs. Let \( u \) be an edge of \( G \). We will show that \( U^G_{ab} \) is \( ph \)-stable. We distinguish two cases.

**Case 1.** \( U^G_{ab} = U^{G_i}_{ab} \) for some \( i = 0 \) or 1.

Then \( U^G_{ab} \) is \( ph \)-stable since it is \( ph \)-stable by Proposition 4.5.

**Case 2.** \( U^G_{ab} \neq U^{G_i}_{ab} \) for \( i = 0, 1 \).

Then, for \( i = 0, 1 \), \( G_i \) has an edge which is \( \Theta \)-equivalent to \( u \). Hence \( G_01 \), which is gated in \( G \), also has an edge \( \Theta \)-equivalent to \( u \). Then, without loss of generality, we can suppose that \( ab \in E(G_01) \). For any \( x \in V(G) \) and \( i = 0, 1 \), we denote by \( g_i(x) \) the gate of \( x \) in \( G_i \). Clearly

\[ W^G_{ab} = W^G_{ab0} \cup W^G_{ab1} \quad \text{and} \quad W^G_{ba} = W^G_{ba0} \cup W^G_{ba1} \quad (4) \]

\[ U^G_{ab} = U^G_{ab0} \cup U^G_{ab1} \quad \text{and} \quad U^G_{ba} = U^G_{ba0} \cup U^G_{ba1} \quad (5) \]

\[ I_G(U^G_{ab0}) \cup I_G(U^G_{ab1}) \subseteq I_G(U^G_{ab}) \quad (6) \]

Let \( u, v \in I_G(U^G_{ab}) \). If \( u, v \in I_G(U^G_{ab_i}) \) for some \( i = 0 \) or 1, then \( v \in I_G(u,w) \) for some \( w \in U_G(ab) \). Hence we are done because \( v \in I_G(u,w) \) by Lemma 3.4 and \( w \in U_G(ab) \) by 3.3.
Suppose that \( u \in V(G_0) - V(G_1) \) and \( v \in V(G_1) - V(G_0) \). We first show that \( u \in I_{G_0}(U_{ab}^{G_0}) \). Because \( u \in V(G_0) - V(G_1) \), we can suppose that \( u \in I_G(x,y) \) for some \( x \in U_{ab}^{G_0} \) and \( y \in U_{ab}^{G_1} \). Then \( g_0(y) \in U_{ab}^{G_0} \) by Lemma 4.8 and thus \( u \in I_{G_0}(x,g_0(y)) \) since \( g_0(y) \in I_{G_0}(u,y) \). It follows that \( g_1(u) \in I_{G_1}(g_1(x),g_0(y)) \subseteq I_{G_1}(U_{ab}^{G_1}) \). Analogously \( v \in I_{G_1}(U_{ab}^{G_1}) \). Hence \( v \in I_{G_1}(g_1(u),w) \) for some \( w \in U_{ab}^{G_1} \) because \( U_{ab}^{G_1} \) is ph-stable by Proposition 4.5. We infer that \( v \in I_G(u,w) \), which proves that \( U_{ab}^{G_1} \) is ph-stable.

In the same way we can prove that \( U_{ba}^{G_1} \) is ph-stable. Consequently \( ph(G) \leq 1 \) by Proposition 4.5.

We recall below three well-known properties of the cartesian product that we will use in the proof of the next theorem. The cartesian product of two partial cubes is clearly a partial cube.

**Proposition 4.11.** Let \( G = G_0 \square G_1 \) be the cartesian product of two connected graphs. We have the following properties:

- Distance Property: \( d_G(x,y) = d_{G_0}(pr_0(x),pr_0(y)) + d_{G_1}(pr_1(x),pr_1(y)) \) for any \( x, y \in V(G) \).
- Interval Property: \( I_G(x,y) = I_{G_0}(pr_0(x),pr_0(y)) \times I_{G_1}(pr_1(x),pr_1(y)) \) for any \( x, y \in V(G) \).
- Convex Subgraph Property: A subgraph \( F \) of \( G \) is convex if and only if \( F = pr_0(F) \square pr_1(F) \), where both \( pr_0(F) \) and \( pr_1(F) \) are convex.

**Theorem 4.12.** Let \( G = G_0 \square G_1 \) be the cartesian product of two partial cubes \( G_0 \) and \( G_1 \). Then \( ph(G) \leq 1 \) if and only if \( ph(G_i) \leq 1 \) for \( i = 0, 1 \).

**Proof.** Assume that \( ph(G) \leq 1 \). Let \( F_i \) be a \( G_i \)-fiber of \( G \) for some \( i = 0 \) or 1. Then \( F_i \) is a gated subgraph of \( G \). Indeed, by the Distance Property of the cartesian product, the projection onto \( F_i \) of any vertex \( x \) of \( G \) is the gate of \( x \) in \( F_i \). Therefore, by Theorem 4.9 \( F_i \), and thus \( G_i \), has a pre-hull number which is at most 1.

Conversely, assume that \( ph(G_i) \leq 1 \) for \( i = 0, 1 \). For any \( x \in V(G) \), we denote by \( x_0 \) and \( x_1 \) the projections of \( x \) onto \( G_0 \) and \( G_1 \), respectively, i.e. \( x = (x_0,x_1) \). Let \( ab \in E(G) \). Then \( a_i = b_i \) for exactly one \( i \), say \( i = 1 \). We will show that \( U_{ab}^{G} \) is ph-stable.

Clearly, for any \( cd \) of \( G \) is \( \Theta \)-equivalent to \( ab \) if and only if \( c_1 = d_1 \\) and \( c_0 d_0 \) is \( \Theta \)-equivalent to \( a_0 b_0 \). Hence

\[
U_{ab}^G = U_{a_0b_0}^{G_0} \times V(G_1). \tag{7}
\]

Let \( u, v \in I_G(U_{ab}^G) \). By the Interval Property of the cartesian product, \( u_0, v_0 \in I_{G_0}(U_{ab}^{G_0}) \). Then, because \( U_{a_0b_0}^{G_0} \) is ph-stable by Proposition 4.5, it follows that \( v_0 \in I_{G_0}(u_0,w_0) \) for some \( w_0 \in U_{a_0b_0}^{G_0} \). In the case where \( u_0 = v_0 \), we can choose \( w_0 \) as any element of \( U_{a_0b_0}^{G_0} \). Let \( w := (w_0,v_1) \). Then \( w \in U_{ab}^G \) by (7), and \( v \in I_G(u,w) \) by the Distance Property of the cartesian product. This proves that \( U_{ab}^{G_1} \) is ph-stable.

In the same way we can prove that \( U_{ba}^{G_1} \) is ph-stable. Consequently \( ph(G) \leq 1 \) by Proposition 4.5. \qed
From the above theorems we infer the following result:

**Corollary 4.13.** The class of all partial cubes whose pre-hull number is at most 1 is closed under gated subgraphs, gated amalgams and cartesian products.

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