ON THE EASIEST WAY TO CONNECT $k$ POINTS IN THE RANDOM INTERLACEMENTS PROCESS

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Abstract. We consider the random interlacements process with intensity $u$ on $\mathbb{Z}^d$, $d \geq 5$ (call it $I^u$), built from a Poisson point process on the space of doubly infinite nearest neighbor trajectories on $\mathbb{Z}^d$. For $k \geq 3$ we want to determine the minimal number of trajectories from the point process that is needed to link together $k$ points in $I^u$. Let

$$n(k, d) := \left\lfloor \frac{d}{2} (k - 1) \right\rfloor - (k - 2).$$

We prove that almost surely given any $k$ points $x_1, \ldots, x_k \in I^u$, there is a sequence of $n(k, d)$ trajectories $\gamma^1, \ldots, \gamma^{n(k, d)}$ from the underlying Poisson point process such that the union of their traces $\bigcup_{i=1}^{n(k, d)} \text{Tr}(\gamma^i)$ is a connected set containing $x_1, \ldots, x_k$. Moreover we show that this result is sharp, i.e. that a.s. one can find $x_1, \ldots, x_k \in I^u$ that cannot be linked together by $n(k, d) - 1$ trajectories.

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1. Introduction

The random interlacement set is the trace left by a Poisson point process on the space of doubly infinite nearest neighbor trajectories modulo time shift on $\mathbb{Z}^d$. This Poisson point process is governed by the intensity measure $uv$ where $u > 0$ and $v$ is a measure on the space of doubly infinite trajectories which was constructed by Sznitman in [Szn10], see (2.9) below. This measure essentially makes the trajectories in the Poisson point process look like double sided simple random walk paths. The interlacement set is a site percolation model that exhibits polynomially decaying infinite-range dependence which sometimes complicates analysis.

One of the motivations for introducing the random interlacements model was to use it as a tool for the study of the behavior of simple random walks on large but finite graphs. For instance, random interlacements describe the local picture left by the trace of a simple random walk on a discrete torus or a discrete cylinder, see [Win08] and [Szn09a] respectively. Recent works that also have used random interlacements to obtain results about simple random walks on large graphs are for example [Szn09c, Szn09a, TW11, Bela] and [Bela].

It is known that the interlacement set is always a connected set, see Corollary (2.3) in [Szn10]. Recently, in [RS10] and [PT11] a stronger result was shown: given any two points $x$ and $y$ in the interlacement set, one can find a path between $x$ and $y$ using the trace of at most $[d/2]$ trajectories. The proofs in [RS10] and [PT11] are very different; in [PT11] the concept of stochastic dimension from [BKPS04] is used, while in [RS10] the approach of the problem is based on estimating capacities of random sets constructed using random walks.

The result we present in this paper completes these works, giving a full picture of how a finite number of points are connected together within the interlacement set. Fix $k \geq 2$, and $d \geq 5$, given a realization $I^u$ of the random interlacement of intensity $u$ constructed from the Poisson point process $\omega_u$ on the space of double trajectories (see the next section for formal definition), a.s. for any sequence of points $x_1, \ldots, x_k \in I^u$, there is a sequence of $n(k, d)$ trajectories $\gamma^1, \ldots, \gamma^{n(k, d)} \in \omega_u$ such that
(a) $\bigcup_{i=1}^{n(k,d)} \text{Tr}(\gamma^{n(k,d)})$ is a connected set (where Tr denote the trace or image of a doubly infinite trajectory $\gamma: \mathbb{Z} \to \mathbb{Z}^d$),
(b) $x_j \in \bigcup_{i=1}^{n(k,d)} \text{Tr}(\gamma^{n(k,d)}) \quad \forall j \in [1,k]$.

In addition, this result is sharp: of course the $n(k,d)$ trajectories are not always needed to link the $k$ points (e.g. $x_1, \ldots, x_k$ might all lie on the trace of a common trajectory) but with probability one, there exist $y_1, \ldots, y_k \in \mathbb{Z}^n$ such that but there is no sequence of $n(k,d) - 1$ trajectories such that satisfies the two conditions (a) and (b) above.

The main results from [RS10] and [PT11] corresponds to the case $k = 2$. The proof of the upper bound for $n(k,d)$ pushes the techniques developed in [RS10] further, while the proof of the lower bound uses a more novel approach based on diagrammatic sums.

In the next section we give a rigorous definition of the random interlacement process and state our result in full detail.

2. Notation and results

2.1. Definition and construction of random interlacements. We consider the trajectory spaces $W$ and $W_+$ of doubly infinite and infinite transient nearest neighbor trajectories in $\mathbb{Z}^d$ (and $\mathcal{W}, \mathcal{W}_+$ the usual sigma algebras associated to them):

$$W := \{ \gamma : \mathbb{Z} \to \mathbb{Z}^d; |\gamma(n) - \gamma(n + 1)| = 1, \forall n \in \mathbb{Z}; |\{n; \gamma(n) = y\}| < \infty, \forall y \in \mathbb{Z}^d\},$$

$$W_+ := \{ \gamma : \mathbb{N} \to \mathbb{Z}^d; |\gamma(n) - \gamma(n + 1)| = 1, \forall n \in \mathbb{Z}; |\{n; \gamma(n) = y\}| < \infty, \forall y \in \mathbb{Z}^d\},$$

where we use the convention that $\mathbb{N}$ includes 0. For $\gamma \in W$, we define the trace of $\gamma$, $\text{Tr}(\gamma) = \{\gamma(n), n \in \mathbb{Z}\}$. For trajectories $\gamma, \gamma' \in W$, we write $\gamma \sim \gamma'$ if for some $k \in \mathbb{Z}$ we have $\gamma(\cdot) = \gamma'(\cdot + k)$.

The space of trajectories in $W$ modulo time shift will be denoted by $W^*$ and is defined as follows:

$$W^* := W/\sim.$$ 

As the trace is invariant modulo time-shift we can naturally extend the notion of trace to $W^*$.

For $K \subset \mathbb{Z}^d$ and $\gamma \in W_+$, we let $H_K(\gamma), \tilde{H}_K(\gamma)$ and $T_K(\gamma)$ denote the entrance time, hitting time and exit time of $K$ by $\gamma$ by:

$$H_K(\gamma) := \inf\{n \geq 0 : \gamma(n) \in K\},$$

$$\tilde{H}_K(\gamma) := \inf\{n \geq 1 : \gamma(n) \in K\},$$

$$T_K(\gamma) := \inf\{n \geq 0 : \gamma(n) \notin K\}. (2.3)$$

For $x \in \mathbb{Z}^d$, set $H_x := H_{\{x\}}$. Let $P_x$ be the law on $W_+$ which corresponds to a simple (i.e. nearest-neighbor symmetric) random walk on $\mathbb{Z}^d$ started at $x$. For $K \subset \mathbb{Z}^d$, let $P^K_x$ be the law of simple random walk started at $x$ conditioned on the event that the walk does not hit $K$:

$$P^K_x[\cdot] := P_x[\cdot | \tilde{H}_K = \infty].$$

For a finite $K \subset \mathbb{Z}^d$, we define the equilibrium measure

$$e_K(x) := \begin{cases} P_x[\tilde{H}_K = \infty], & x \in K, \\ 0, & x \notin K. \end{cases} (2.4)$$

The capacity of a finite set $K \subset \mathbb{Z}^d$ is defined as

$$\text{cap}(K) := \sum_{x \in \mathbb{Z}^d} e_K(x). (2.5)$$

and the normalized equilibrium measure of $K$ is given by

$$\bar{e}_K(\cdot) := e_K(\cdot)/\text{cap}(K). (2.6)$$
For \( x, y \in \mathbb{Z}^d \) we let \( |x - y| := \|x - y\|_1 \) denote the \( l_1 \) distance (which corresponds to the graph distance on \( \mathbb{Z}^d \)) between \( x \) and \( y \). The following bounds of hitting-probabilities are well-known, see Theorem 4.3.1 in [LL10]. For any \( x, y \in \mathbb{Z}^d \) with \( x \neq y \),
\[
  c|x - y|^{-(d-2)} \leq P_x[H_y < \infty] \leq c'|x - y|^{-(d-2)}.
\]

We are now ready to introduce a Poisson point process on \( W^* \times \mathbb{R}_+ \). For \( K \subset \mathbb{Z}^d \), let
\[
  W_K := \{ \gamma \in W : \gamma(Z) \cap K \neq \emptyset \}.
\]

Let \( \pi^* \) be the projection from \( W \) to \( W^* \) and let \( W_K^* := \pi^*(W_K) \) be the set of trajectories in \( W^* \) that enter \( K \). We denote by \( Q_K \) the finite measure on \( W_K^* \) such that for \( A, B \in W_+ \) and \( x \in \mathbb{Z}^d \),
\[
  Q_K[(X_n)_{n \geq 0} \in A; X_0 = x, (X_n)_{n \geq 0} \in B] = P^K_x[A]e_K(x)P_x[B].
\]

We let the measure \( \nu \) be the unique \( \sigma \)-finite measure such that
\[
  1_{W_K^*} \nu = \pi^* \circ Q_K, \text{ for all finite } K \subset \mathbb{Z}^d.
\]

Szniatman proved the existence and uniqueness of \( \nu \) in Theorem 1.1 of [Szn10]. We introduce the space of locally finite point measures in \( W^* \times \mathbb{R}_+ \):
\[
  \Omega := \left\{ \omega = \sum_{i=1}^{\infty} \delta_{(\gamma_i, u_i)} : \gamma_i \in W^*, u_i > 0, \omega(W^*_K \times [0, u]) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d \text{ and } u > 0 \right\},
\]

as well as the space of locally finite point measures on \( W^* \):
\[
  \bar{\Omega} := \left\{ \sigma = \sum_{i=1}^{\infty} \delta_{\gamma_i} : \sigma_i(W^*_K) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d \right\}.
\]

For \( 0 \leq u' \leq u \) the map \( \omega_{u', u} \) from \( \Omega \) into \( \bar{\Omega} \) is defined as
\[
  \omega_{u', u} := \sum_{i=1}^{\infty} \delta_{\gamma_i} 1\{u' < u_i \leq u\}, \text{ for } \omega = \sum_{i=1}^{\infty} \delta_{(\gamma_i, u_i)} \in \Omega.
\]

If \( u' = 0 \), we use the short-hand notation \( \omega_u \). For convenience reasons we often improperly consider \( \omega_u \) as a set of trajectories instead of a point measure.

On \( \Omega \) we consider \( \mathbb{P} \), the law of a Poisson point process with intensity measure \( \nu(d\gamma)dx \) (see Equation (1.42) in [Szn10] for a characterization of \( \mathbb{P} \)). It is easy to see that under \( \mathbb{P} \), the point process \( \omega_{u, u'} \) is a Poisson point process on \( \bar{\Omega} \) with intensity measure \( (u - u')\nu(d\sigma^*) \). Given \( \sigma \in \bar{\Omega} \), the set of points in \( \mathbb{Z}^d \) that is visited by at least one trajectory in \( \sigma \) is denoted by
\[
  \mathcal{I}(\sigma) := \bigcup_{\gamma \in \sigma} \text{Tr}(\gamma).
\]

For \( 0 \leq u' \leq u \), we define the random interlacement set between intensities \( u' \) and \( u \) as
\[
  \mathcal{I}^{u', u} := \mathcal{I}(\omega_{u', u}).
\]

In case \( u' = 0 \), we use the short-hand notation \( \mathcal{I}^{u} \). For a point process \( \sigma \) on \( \Omega \) or \( \bar{\Omega} \) we let \( \sigma|_A \) denote the restriction of \( \sigma \) to \( A \subset W^* \).

We conclude this section by stating the convention for the use of constants throughout the paper: The letters \( c, c', C, C', \ldots \) denote finite positive constants which are allowed to depend only on the dimension \( d \) and the intensity \( u \). Their values might change from line to line. Numbered constants \( c_i \) are finite positive, and supposed to be the same inside a certain neighborhood (for example a
Remark 1. Notice that when $d \geq 3$. Hence in what follows, we will only care about the case $d = 3$ or $d = 4$ the trace of each trajectory in $X^k$ intersects the traces of $X^k$. Let $\gamma \in W^*$, with a canonical element of its equivalence class $(\gamma_n)_{n \geq 0}$.

2.2. Main result. We say that the sequence of trajectories $(\gamma^i)^n_{i=1}$ connects the sequence of points $(x_i)^k_{i=1}$ if the union of their traces (or images) includes a connected subset that contains $x_1, ..., x_k$. We say that $(\gamma^i)^n_{i=1}$ connects strictly $(x_i)^k_{i=1}$ if it connects it and there is no strict subsequence of $(\gamma^i)^n_{i=1}$ that does. Note that if a sequence of trajectories connects points, one can extract from it a subsequence that connects them strictly.

**Theorem 2.1.** For every $k \geq 2$, for every $u > 0$, and for $\mathbb{P}$—almost every realization of the Poisson process $\omega_u$, the two following properties are satisfied:

(i) Given a sequence of $k$ points $(x_i)^k_{i=1}$ in $(\mathcal{I}^u)^k$, it is possible to find a sequence $(\gamma^i)^n_{i=1}$ in $(\omega_u)^{n(k,d)}$ that connects it.

(ii) It is possible to find $(x_i)^k_{i=1}$ in $(\mathcal{I}^u)^k$ such that there exists no sequence $(\gamma^i)^{n(k,d)-1}_{i=1} \in (\omega_u)^{n(k,d)}$ that connects it.

**Remark 2.2.** The result is restricted to $d \geq 5$ but this is not in fact a true restriction. Indeed if $d = 3$ or $4$ the trace of each trajectory in $\omega_u$ intersect the trace of all the others, so that Theorem 2.1 trivially holds with $n(k,3) = n(k,4) = k$.

The proofs of (i) and (ii) are quite independent and are found in Section 3 and Section 4 respectively. In what follows we say that a sequence of points $(x_i)^k_{i=1}$ is $n$-connected (in $(\mathcal{I}^u)$) if (i) occurs with $n(k,d)$ replaced by $n$.

3. Proof of (i) of Theorem 2.1

As will be seen later in this section, in order to prove that $n(k,d)$ trajectories are sufficient to connect $k$ points, it is essentially sufficient to prove this in the case $k = 2$ and $k = 3$. The case $k = 2$ having been proved in [RS10] and [PT11], we can focus on the case $k = 3$.

The first step is to reformulate the result.

**Proposition 3.1.** Let $d \geq 5$ and suppose $x_1, x_2, x_3$ in $\mathbb{Z}^d$. Let $X^1, X^2, X^3$ be three independent simple random walks on $\mathbb{Z}^d$ with starting points $x_1, x_2, x_3$ respectively. Consider also a random-interlacement process $\omega_u$ independent of the $X^i$’s.

For any choice of the $x_i$, for every $u > 0$, almost surely one can find $d - 4$ trajectories $(\gamma^i)^{d-4}_{i=1}$ in $(\omega_u)^d$ such that the union of the traces of the $\gamma^i$’s forms a connected subset that intersects the traces of $X^1, X^2$ and $X^3$.

We also need a similar result for the case of two trajectories, which is proved in [RS10] (with a different formulation).

**Proposition 3.2.** Let $d \geq 5$ and suppose $x_1, x_2$ in $\mathbb{Z}^d$. Let $X^1, X^2$ be two independent simple random walks on $\mathbb{Z}^d$ with starting points $x_1, x_2$ respectively. Consider also a random-interlacement process $\omega_u$ which is independent of the walks $X^1$ and $X^2$.

For every choice of $x_1, x_2$, for every $u > 0$, almost surely one can find $[d/2] - 2$ trajectories $(\gamma^i)^{[d/2]-2}_{i=1}$ in $(\omega_u)^{[d/2]-2}$ such that the union of the traces of the $\gamma^i$’s forms a connected subset that intersects the traces of $X^1$ and $X^2$.

**Remark 1.** Notice that when $d$ is even Proposition 3.1 can easily be deduced from Proposition 3.2. Hence in what follows, we will only care about the case $d$ odd.
Proof of Theorem 2.1 (i) from Proposition 3.1 and 3.2. First consider the case where \( k = 2p + 1 \) is odd. Let \( x_1, \ldots, x_k \in \mathbb{Z}^d \). We say that the sequence of points \( (x_1, \ldots, x_k) \) is well behaved for \( \omega_u \), and we will write \( WB \) if each point of the sequence belongs to the interlacement set and if there exists a sequence \( 0 < t_1 < \cdots < t_k \leq u \) such that for all \( i \) there exists \( (\gamma^i, t_i) \in \omega \) with \( x_i \in \gamma^i \).

An equivalent formulation of \((i)\) from Theorem 2.1 is

For all \( k \) and for all \( (x_i^k)_{i=1}^k \in (\mathbb{Z}^d)^k \) we have that

\[
P\left[ \exists (\gamma^i)_{i=1}^n(k,d), (\gamma^i)_{i=1}^n(k,d) \text{ connects } (x_i^k)_{i=1}^k \mid (x_i^k)_{i=1}^k \text{ } WB \right] = 1, \tag{3.1} \]
or alternatively

\[
\text{For all } k, P\left[ \forall (x_i^k)_{i=1}^k, (x_i^k)_{i=1}^k \text{ } WB \Rightarrow \exists (\gamma^i)_{i=1}^n(k,d), (\gamma^i)_{i=1}^n(k,d) \text{ connects } (x_i^k)_{i=1}^k \right] = 1. \tag{3.2} \]

Indeed clearly, if \((i)\) of Theorem 2.1 holds, so does \((3.1)\). We prove the other implication by contradiction: if \((i)\) from Theorem 2.1 is violated, with positive probability one can find \( k \) points in \( \mathbb{T}^u \) that cannot be connected by \( n(k,d) \) trajectories in \( \omega_u \). As these points are in \( \mathbb{T}^u \) one can by definition find a sequence \((\gamma^i)_{i=1}^n(k,d) \) of trajectories in \( \omega_u \) such that \( x_i \in \gamma^i \) for all \( i \). If all the \( \omega_i \) are distinct, a.s. after an eventual reordering of the sequence we get that \((x_1, \ldots, x_k)\) is well behaved so that \((3.1)\) cannot hold. If on the other hand if there are repetition in \((\gamma^i)_{i=1}^n(k,d)\), one can find one extracts a well behaved subsequence \((x_i')_{i=1}^{k'} \) of \((x_i^k)_{i=1}^k\) by deleting the points \( x_i \) for which

\[
\exists j < i, \gamma_i = \gamma_j, \tag{3.3} \]

and reordering the remaining subsequence. Then if \((3.1)\) holds then one can a.s. connect \((x_i')_{i=1}^{k'}\) with \( n(k',d) \) trajectories. Then using the definition \((3.3)\) one can link all the \((x_i^k)_{i=1}^k\) together with \( n(k',d) + (k - k') \leq n(k,d) \) trajectories (just by using the \( \gamma_j \) corresponding to the \( k' - k \) remaining points if necessary in addition to the trajectories that connect \((x_i')_{i=1}^{k'}\)) which yields a contradiction. Hence we can focus on proving \((3.1)\).

Let \( \tau_0 := 0 \) and for \( i = 1, \ldots, k \) let recursively

\[
\tau_i := \min\{s > \tau_{i-1} \mid x_i \in \mathbb{T}^s\}. \tag{3.4} \]

Note that by definition of \( \omega_u \), in \( \omega_{\tau_{i-1}, \tau_i} \), with probability one, there exists a unique trajectory \( \gamma^i \) which has \( x_i \) in its trace.

Furthermore, by the strong Markov property for Poisson, the law of \( \gamma^i \) is independent of that of \( \tau_i \) (and the \((\gamma^i)_{i=1}^n \) are independent) and if we parametrize the oriented trajectory \( \gamma^i \) as \((\gamma^i_n)_{n \in \mathbb{Z}} \) such that \( 0 \) is the first time \( w^i \) visits \( x_i \) is 0, then from the definition of the random interlacement (recall \((2.3)\))

\[
(\gamma^i_n)_{n \geq 0} \text{ is a simple random walk on } \mathbb{Z}^d \text{ started at } x_i. \tag{3.5} \]

Set \( \mathcal{T} := \max_{i \in [1, k]} \tau_i \). The event \( \{ (x_i^k)_{i=1}^k \text{ is well behaved} \} \) is equal to \( \{ \mathcal{T} \leq u \} \), and up to an event of zero-probability, it coincides with \( \{ c\mathcal{T} < u \} \).

Note that conditioned on \( \mathcal{T} \), the process \( \omega_{\mathcal{T}, u} \) is independent of \( \mathcal{T} \) and of the \( \gamma^i \)s. Hence, setting \( X^i := (\gamma^i_n)_{n \geq 0} \), we can apply Proposition 3.1 and for every \( j = 1, \ldots, p \) find a sequence of \((d - 4)\) trajectories \((\gamma^i_{k+ij(d-4)})_{i=1}^{k+ij(d-4)+1} \) in \( \omega_{\mathcal{T}, u} \) that connects together the traces of \( X^{2j-1}, X^{2j}, X^{2j+1} \).

One can then conclude by observing that \( k + p(d - 4) = n(k,d) \) for \( k \) odd and that \((\gamma^i)_{i=1}^{k+p(d-4)} \) is a set of trajectories in \( \omega_u \) that connects \((x_i^k)_{i=1}^k\).

The case \( k = 2p \) even is dealt similarly, the only difference being in the last step: we use Proposition 3.1 for \( i = 1, \ldots, p - 1 \) to connect together \( X^1, \ldots, X^{2p-1} \) and Proposition 3.2 to connect
$X_{2p-1}$ and $X_{2p}$ with the trajectories $(\gamma_{i,k+(p-1)(d-4)+[d/2]-2}^i)_{i=k+(p-1)(d-4)+1}^{d}$ from $\omega_{T,u}$, and conclude in a similar manner.

Before the proof of Proposition 3.1 for $d$ odd, (in what follows we always consider that $d$ is odd) we must introduce additional notation in order to reformulate the statement. Introduce the number

$$k_d := \lfloor d/2 \rfloor - 2 = \frac{d-3}{2}.\quad(3.6)$$

For a finite set $A \subset \mathbb{Z}^d$ and $\sigma \in \tilde{\Omega}$, let $N_A(\sigma)$ be the number of trajectories in $\sigma$ that intersect $A$. Let $\gamma_1, \ldots, \gamma_{N_A(\sigma)}$ be the trajectories from $\sigma$ that intersect $A$ parameterized so that $\gamma_n \in A$ and $\gamma_n \notin A$ for all $n < 0$ and all $i \in \{1, \ldots, N_A(\sigma)\}$. For $\sigma \in \tilde{\Omega}$, $A \subset \mathbb{Z}^d$ and $R \in \mathbb{Z}_+$ we define the random set of vertices $\Psi(\sigma, A, R)$ as

$$\Psi(\sigma, A, R) := \bigcup_{i=1}^{N_A(\sigma)} \{\{\gamma_i(t) : 1 \leq t \leq R^2/8\} \cap B(\gamma_i(0), R/2)\}.\quad(3.7)$$

**Definition 3.1.** Let $r, R \in \mathbb{R}_+ \cup \{\infty\}$ with $r < R$. For $\sigma \in \tilde{\Omega}$, let $\sigma_R$ be the restriction of $\sigma$ to the trajectories that intersect $B(R)$. Let $\sigma_{r,R}$ be the restriction of $\sigma_R$ to the set of trajectories that do not intersect $B(r)$. Observe that $\sigma_r$ and $\sigma_{r,R}$ are supported on disjoint sets of trajectories and that

$$\sigma = \sigma_r + \sigma_{r,\infty}.\quad(3.8)$$

Let $(\sigma^{(i,j)})_{1 \leq i \leq 4, 1 \leq j \leq k_d}$ in $\tilde{\Omega}$ be a family of i.i.d. random interlacement processes with parameter $\tilde{u} := u/4k_d$ defined by

$$\sigma^{(i,j)} := \omega_{\tilde{u}((i-1)k_d+(j-1)),\tilde{u}((i-1)k_d+j)},\quad(3.9)$$

and let $(X^i)_{i=1}^3$ be three independent simple random walks starting from $x_1$, $x_2$ and $x_3$ respectively. Given $R$, let $T^i(B(R))$ be the first exit time of $X^i$ from $B(R)$ and $Y^i := (Y^i_{n,R})_{n \geq 0} = (X^i_{n+T^i(B(R))})_{n \geq 0}$ (and $Y^i = X^i$ when $R = \infty$). We call $P$ the probability measure governing all these processes.

We define sequences of random subsets of $\mathbb{Z}^d$. For $0 \leq r < R \leq \infty$, and $i = 1, 2, 3$ set

$$A^{(1)}_i(r, R) = A^{(i)}_i(r, R) := \{Y^i_{n,R} : 1 \leq n \leq R^2/8\} \cap B(Y^i_{0,R}, R/2),\quad 1 \leq i \leq 3.\quad(3.10)$$

Then recursively for $2 \leq s \leq k_d$ and with $r$, $R$, $j$ as above, define

$$A^{(j)}_i(r, R) := \Psi \left(\sigma^{(i,j)}_{r,\infty}, A^{(j-1)}_i(r, R), R\right) = \Psi \left(\sigma^{(i,j)}_{r,s,R}, A^{(j-1)}_i(r, R), R\right).\quad(3.11)$$

We simply write $A^{(j)}_i$ when $r = 0$ and $R = \infty$. Note that by construction if $y \in A^{(j)}_i(r, R)$ then there exists a sequence of $k_d - 1$ trajectories in $T^a$ linking it to the trace of $X^i$. Thus to prove proposition 3.1 it is in fact sufficient to prove (recall that $2(k_d - 1) + 1 = d - 4$),

**Lemma 3.3.** With probability one, one can find $\gamma \in \sigma^{(4,1)}$ that connects $A^{(k_d)}_1$, $A^{(k_d)}_2$ and $A^{(1)}_3$ together.

Inspired by [RS10], we prove Lemma 3.3 by combining Borel’s Lemma and

**Lemma 3.4.** Let $d \geq 5$ be odd. Given $x_1, x_2, x_3 \in \mathbb{Z}^d$. Let $R$ and $r$ be integers, such that $R > \max(|x_1|, |x_2|, |x_3|)$.

There exist constants $c(u, d) > 0$, $R_0(u, d) < \infty$ and $\varepsilon(u, d) > 0$, such that for any $r$ and $R$ with $R > R_0$ and $\varepsilon R \geq r^{d-2}$,
\[
\mathbb{P} \left[ \exists \gamma \in \sigma_{r,2R}^{(4,1)} : \gamma \text{ connects } A_1^{(k_d)}(r,R), A_2^{(k_d)}(r,R), \text{ and } A_3^{(1)}(r,R) \right] \geq c. \tag{3.12}
\]

We prove Lemma 3.4 by using a method based on the control of the capacity of the sets \(A_i^{(j)}(r,R)\) at the end of the section.

**Proof of Lemma 3.4 from Lemma 2.4.** For real numbers \(r < R\) such that \(x_1, x_2, x_3 \in B(R)\), set
\[
D(r,R) := \{ \exists \gamma \in \sigma^{(4,1)} : \gamma \text{ connects } A_1^{(k_d)}(r,R), A_2^{(k_d)}(r,R), \text{ and } A_3^{(1)}(r,R) \}. \tag{3.13}
\]

We choose \(\epsilon\) so that Lemma 3.4 applies. Let \(r_0 := \max(|x_1|, |x_2|, |x_3|)\) and \(R_0 := \epsilon^{-1} r_0^{-d-2}\). For \(k \geq 1\), we define recursively
\[
r_k := d R_{k-1}^2 \quad \text{and} \quad R_k := \epsilon^{-1} r_k^{-d-2}. \tag{3.14}
\]
We write \(D_k = D_k(X^1, X^2, X^3)\) (reasons for underlining only the dependence in \(X^i\) will become clear later) for \(D(r_k, R_k)\) and write \(t_k\) for \(\mathbb{1}_{D_k}\), the indicator function of \(D_k\). We want to show that
\[
\mathbb{P} [D_k \text{ occurs for infinitely many values of } k] = 1, \tag{3.15}
\]
which implies Lemma 3.3.

We will be done using Borel’s Lemma (it is cited as in Lemma 4.12 in [RS10]) if we can show that there is some \(c\) such that for all \(k \geq 1\) we have almost surely
\[
\mathbb{P}[D_k | \iota_1, \ldots, \iota_{k-1}] \geq c. \tag{3.16}
\]

Let \(I_k := (\iota_{i})_{i=1}^k\). It is measurable with respect to the \(\sigma\)-algebra generated by the following random objects: \(\{X_n^i : 1 \leq n \leq T_{B(R_{k-1})} + R_{k-1}^2/8\}\) and \(\{\sigma_{R_{k-1}(1+k_d)}^{(i,j)}\}_{i \leq 3, j \leq k_d}\).

On the other hand, the event \(D_k\) depends on \(\sigma_{R_{k-1}(1+k_d)}^{(i,j)}\) only through \(\sigma_{R_{k-1}(1+k_d)}^{(i,j)}\) \(i \leq 4, j \leq k_d\). Since \(R_{k-1}(1+k_d) < r_k\), the point measures \(\sigma_{R_{k-1}(1+k_d)}^{(i,j)}\) \(i \leq 4, j \leq k_d\) and \(\sigma_{R_{k-1}(1+k_d)}^{(i,j)}\) \(i \leq 4, j \leq k_d\) are independent.

Let \(\tilde{X}_i^n\) be defined by \(\tilde{X}_i^n := X_{n+B(R_{k-1})+R_{k-1}^2/8}^i\). By the strong Markov property, conditionally on \(\tilde{X}_i^n\), \(\tilde{X}_i\) is independent of \(X_i\) (and its law is the one of a simple random walk). Furthermore, as \(R_{k-1} + R_{k-1}^2/8 < r_k\), \(D_k\) depends on \(X^i\) only through \(\tilde{X}_i\).

Thus by conditional independence
\[
\mathbb{P}[D_k | I_k, (\tilde{X}_i^n)_{i=1}^3] = \mathbb{P}[D_k(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) | (\tilde{X}_i^n)_{i=1}^3] \geq c, \tag{3.17}
\]
where the last inequality follows from Lemma 3.4 with \((x_1, x_2, x_3)\) replaced by \((\tilde{X}_i^n)_{i=1}^3\).

We can now focus on the proof of Lemma 3.4. Before starting we cite results from [RS10] that give estimates on the capacities of the sets \(A_i^{(j)}(r,R)\).

**Lemma 3.5.** [RS10, Lemmata 4.7, 4.8] Let \(d \geq 5\) and let \(j\) be a positive integer. There exist constants \(C_s = C_s(u,d)\) and \(\epsilon_s = \epsilon_s(u,d)\) such that for any positive integers \(r\) and \(R\) with \(r^{d-2} \leq \epsilon_s R\) and if \(x_i \in B(R)\), \(i \leq 3\) we have
\[
\mathbb{E} [\text{cap}(A_i^{(j)}(r,R))] \geq C_s R^{\min(d-2, 2s)}. \tag{3.18}
\]
Moreover, under the same condition there exist positive finite constants \(c_s = c_s(u,d)\),
\[
\mathbb{E} [\text{cap}(A_i^{(s)}(r,R))] \leq c_s R^{\min(d-2, 2s)}, \tag{3.19}
\]
and
\[
\mathbb{E} [\text{cap}(A_i^{(s)}(r,R))]^2 \leq c_s R^{2 \min(d-2, 2s)}. \tag{3.20}
\]
As a consequence (using Chebychev inequality and changing the value of $c_s$ if needed),

$$P[\text{cap}(A_1^{(s)}(r, R)) \geq c_s R^{\min(d-2,2s)}] \geq c_s. \tag{3.21}$$

**Proof of Lemma 3.4.** We choose the constants $\epsilon_s$ from Lemma 3.5 and assume that $r$ and $R$ are such that Lemma 3.5 applies. We consider the two following events

$$E_1 := \{ \exists \gamma \in \sigma^{(4,1)}_{2R} : \gamma \text{ connects } A_1^{(k_d)}(r, R), A_2^{(k_d)}(r, R), \text{ and } A_3^{(1)}(r, R) \},$$

$$E_2 := \{ \exists \gamma \in \sigma^{(4,1)}_r : \gamma \text{ intersects } A_3^{(1)}(r, R) \}. \tag{3.22}$$

Note that

$$\{ \exists \gamma \in \sigma^{(4,1)}_{r,2R} : \gamma \text{ connects } A_1^{(k_d)}(r, R), A_2^{(k_d)}(r, R), \text{ and } A_3^{(1)}(r, R) \} \supset E_1 \setminus E_2. \tag{3.23}$$

Let $P^{(4,1)}$ denote the law of $\sigma^{(4,1)}$. Our main task is to prove that there exists a universal constant $c$ such that

$$P^{(4,1)}(E_1) \geq 1 - \exp(-c R^{d-2d} \text{ cap}(A_1^{(k_d)}(r, R)) \cap \text{cap}(A_2^{(k_d)}(r, R)) \cap \text{cap}(A_3^{(1)}(r, R))), \tag{3.24}$$

and

$$P^{(4,1)}(E_2) \leq u \cdot \text{cap}(A_3^{(1)}(r, R))/(R - r)^{d-2}. \tag{3.25}$$

According to (3.21) (and independence), choosing $c$ small enough one has with positive probability larger than $c$

$$\text{cap}(A_1^{(k_d)}(r, R)) \geq c R^{2k_d},$$

$$\text{cap}(A_2^{(k_d)}(r, R)) \geq c R^{2k_d},$$

$$\text{cap}(A_3^{(1)}(r, R)) \geq c R^2. \tag{3.26}$$

Hence (3.24), (3.25) and (3.19) implies (recall that $2k_d = d - 3$)

$$P[E_1] \geq c(1 - \exp(-c^4)) \text{ and } P[E_2] \leq P[E_1]/2, \tag{3.27}$$

provided that $R$ is large enough. This together with (3.23), is enough to conclude. From now on, we write $A_1, A_2$ and $A_3$ for $A_1^{(k_d)}(r, R), A_2^{(k_d)}(r, R)$ and $A_3^{(1)}(r, R)$. In order to prove (3.24) and (3.25) one considers the following construction of $\sigma^{(1,4)}_{W_K}$:

- Let $N$ be a Poisson variable of mean $\bar{u} \cdot \text{cap}(A_3)$.
- Conditionally on $N$, let $(\gamma^i)_{i=1}^N$ be a sequence of independent (and independent of $N$) of $N$ doubly-infinite trajectory with distribution $\pi^* \circ Q_{A_3},$ where $Q_{\bar{K}}(\cdot) := Q_K(\cdot)/Q_K(W_K)$ is the renormalized version of the measure defined in (2.8)

Note that from this construction one has

$$P^{(4,1)}[E_1 | N] = 1 - [1 - \bar{Q}_{A_3}(\gamma \text{ hits } A_1 \text{ and } A_2)]^N, \tag{3.28}$$

$$P^{(4,1)}[E_2 | N] = 1 - [1 - \bar{Q}_{A_3}(\gamma \text{ hits } B(r))]^N,$$

where $(\gamma_n)_{n \in \mathbb{Z}}$ is a trajectory distributed according to $\bar{Q}_{A_3}$. Let $P_x$ be the law of the simple random walk $Y$ starting from $x$ and $T_1$ and $T_2$ the hitting times of $A_1$ and $A_2$ respectively. From the definition of $\bar{Q}_{A_3}$ we have

$$\bar{Q}_{A_3}(\gamma \text{ hits } A_1 \text{ and } A_2) \geq \bar{Q}_{A_3}(\exists n_2 \geq n_1 \geq 0, \gamma_{n_1} \in A_1, \gamma_{n_2} \in A_2) \geq \min_{x \in A_3} P_x(T_1 \leq T_2 < \infty). \tag{3.29}$$
Moreover using the strong Markov property and the identity
\[ P_x[T_1 < \infty] = \sum_{z \in A_1} g(x, z)e_{A_1}(z), \]
we get
\[
P_x(T_1 \leq T_2 < \infty) \geq \left( \sum_{z \in A_1} g(x, z)e_{A_1}(z) \right) \left( \inf_{y \in A_1} \sum_{z \in A_2} g(y, z)e_{A_2}(z) \right)
\geq \left( \min_{y, z \in B((k+1)R)} g(y, z) \right)^2 \left( \sum_{z \in A_1} e_{A_1} \right) \left( \sum_{z \in A_2} e_{A_2} \right) \geq cR^{4-2d} \text{cap}(A_1) \text{cap}(A_2),
\]
(to get the last inequality recall (2.5) and (2.7)) and hence
\[ \bar{Q}_{A_1}(\gamma \text{ hits } A_1 \text{ and } A_2) \geq cR^{4-2d} \text{cap}(A_1) \text{cap}(A_2). \]
Together with the first line of (3.28) and averaging with respect to \( \mathcal{N} \), this proves (3.32).

Note that \( \pi^* \circ \bar{Q}_{A_2} \) is invariant under change of orientation of the trajectories (see Theorem 1.1 of [Szn10]) so that if \( \bar{T} := \max\{n|\gamma_n \in A_3\}, (\gamma_n)_{n \geq 0} \) and \( (\bar{\gamma}_{T-n})_{n \geq 0} \) have the same law. Hence
\[ \bar{Q}_{A_3}(\gamma \text{ hits } B(r)) \leq 2\bar{Q}_{A_3}((\gamma_n)_{n \geq 0} \text{ hits } B(r)). \]
Moreover (recall (3.30))
\[ \bar{Q}_{A_3}((\gamma_n)_{n \geq 0} \text{ hits } B(r)) \leq \max_{|x| \geq R/2} P_x(H_{B_r} < \infty) = \max_{|x| \geq R/2} \sum_{z \in A_1} g(x, z)e_{B(r)}(z) \leq C(r/(R-r))^{d-2}. \]

All of this combined gives
\[ \bar{Q}_{A_3}(\gamma \text{ hits } B(r)) \leq C(r/(R-r))^{2-d}. \]
Combining with (3.28) and averaging with respect to \( \mathcal{N} \) gives
\[ \mathbb{P}^{(4.1)}[E_2] \leq u \text{cap}(A_3)(r/(R-r))^{2-d}. \]
\( \square \)

### 4. Proof of (ii) in Theorem 2.1

The aim of this Section is to prove that if one selects \( k \) points very distant from each another in the random interlacement, they are really unlikely to be connected by less than \( n(k, d) \) trajectories (together with a quantitative upper-bound on the probability).

**Proposition 4.1.** Given \( \varepsilon > 0 \), for any \( x_1, \ldots, x_k \in \mathbb{Z}^d \) and for any \( n < n(k, d) \) one has
\[ \mathbb{P}[\text{\( k \) points \( i = 1, \ldots, k \) are \( n \)-connected}] \leq C(d, k, \varepsilon) \max_{i \neq j} |x_i - x_j|^{-1+\varepsilon}. \]

Whereas it is quite intuitive that Proposition 4.1 implies the second half of Theorem 2.1, the proof is not completely straight-forward so we write it in full details.
Proof of Theorem \[2.1\] (ii) from Proposition \[4.1\] Set \( n < n(k,d) \). For \( i = 1, \ldots, k \) denote by \( B^i_R \) the Euclidean ball of center \( R e_1 \) (with \( e_1 = (1,0,\ldots,0) \in \mathbb{Z}^d \)) and of radius \( R \). We want to show that the the probability of the event

\[
A_R := \{ \exists (x_i^k)_{i=1}^k \in \prod_{i=1}^k B^i_R, \ (x_i^k)_{i=1}^k \text{ is not } n\text{-connected} , \forall i \in [1,k], x_i \in \mathcal{I}_y \}. \tag{4.2}
\]

tends to one when \( R \) tends to infinity, so that \( \mathbf{P} \left[ \bigcup_{R \geq 1} A_R \right] = 1 \) (which implies Theorem \[2.1\] (ii)). According to Proposition \[4.1\] using a union bound, one has for \( R \) large enough

\[
\mathbf{P}(E_R^1) = \mathbf{P} \left[ \exists (x_i^k)_{i=1}^k \in \prod_{i=1}^k B^i_R, \ (x_i^k)_{i=1}^k \text{ is } n\text{-connected} \right] \leq C R^{kd} e^{R^2/2}. \tag{4.3}
\]

Moreover from the definition of random interlacement (in particular of the measure \( \nu \) in equation \[2.9\]) we have

\[
\mathbf{P}(E_R^2) = \mathbf{P} \left[ \forall i \in [1,k], \mathcal{I}_y \cap B^i_R = \emptyset \right] \leq k e^{-\text{cap}(B^i_R)} \leq k e^{-c R^{d-2}}. \tag{4.4}
\]

Hence we conclude that the probability of \( A_R = (E_R^1 \cup E_R^2)^c \) tends to one as \( R \to \infty \). \( \square \)

We prove the result by induction on \( k \). The strategy that we use is the following: first we encode the way the \( k \) points are connected by some tree scheme \( \mathcal{T} \). This is done in Proposition \[4.2\]. Then we bound from above the probability that \( k \) points are connected together using a given scheme by a diagraphmatic sum (Lemma \[4.4\]). Finally we prove an upper-bound on this sum (Proposition \[4.5\]). For some tree-schemes the multi-index sum given by Lemma \[4.4\] is infinite and those to be treated separately. However they are easily dealt with by using the induction hypothesis.

**Proposition 4.2.** Assume there is a sequence of distinct trajectories \((\gamma^i)_{i=1}^n\) \((\gamma^i \neq \gamma^j \text{ for } i \neq j)\) that connects strictly \((x_i^k)_{i=1}^k\).

Then one can construct:

(a) a sequence \((y_i^m)_{i=1}^m \in (\mathbb{Z}^d)^m\), with \( m = n + k - 1 \) and \( y_i = x_i \) for \( i \leq k \),

(b) a tree \( \mathcal{T} \) with \( m \) labeled vertices \( A_1, \ldots, A_m \), and \( m - 1 \) oriented edges whose set we call \( \mathcal{E} \),

(c) a function \( t : \mathcal{E} \to \{1, \ldots, n\} \), that to each edge associates a type,

that satisfies the following properties:

(i) The set of oriented edges that share the same label forms an (oriented) path in the tree.

(ii) For all indices \( i \leq k \) the edges connected to the vertex \( A_i \) (ignoring their orientation) are all of the same type (hence those vertices have at most degree 2). For \( i \geq k + 1 \) the edges connected to the vertex \( A_i \) are of two different types (exactly).

(iii) If \( A_{a_1} A_{a_2} \ldots A_{a_l} \), \( l \geq 2 \) is the path of vertices linked by edges of type \( h \) and \((\gamma^h_{a_{i}})_{i=0}^{n} \) is a time parametrization of \( \gamma^h \), then there exists a non-decreasing sequence \( b_1, \ldots, b_l \in \mathbb{Z} \) such that \( w_{b_i} = y_{a_i} \) for all \( i \in [1,l] \).

Given \((\mathcal{T}, \mathcal{E}, t)\) satisfying (i) – (ii) we say that \((x_i^k)_{i=1}^k\) is connected with scheme \((\mathcal{T}, \mathcal{E}, t)\) (or \( \mathcal{T} \) to simplify notation), if there exists \((y_i^m)_{i=k+1}^m \in (\mathbb{Z}^d)^m\), \((\gamma^i)_{i=1}^n \in (\omega^n)\), \( \gamma^i \neq \gamma^j \) for \( i \neq j \), such that (iii) holds. Furthermore if this holds with \((y_i^m)_{i=k+1}^m\) fixed, we say that \((x_i^k)_{i=1}^k\) is connected with scheme \((\mathcal{T}, \mathcal{E}, t)\) using \((y_i^m)_{i=k+1}^m\).

**Remark 4.3.** Remark that we allow repetition in the sequence \( y_1, \ldots, y_m \) and that the choice of the tree may not be unique. Moreover it can easily be checked by the reader that if a sequence of points is connected with scheme \((\mathcal{T}, \mathcal{E}, t)\), then the sequence is \( n \)-connected. An example for the construction of \( \mathcal{T} \) together with the type function is given in figure \[4.3\].
Here for a reason that will become clear soon).

We are now ready to construct the tree \( T \). First we construct a path

\[ A_{n'+k}A_{n'+k+1} \ldots A_{n+k+1}A_k \]

composed of \( n' - n \) edges of different types \((n' + 1) to n\), just as one did for the \( k = 2 \) case.

Then one plugs \( A_{n'+k} \) into the old tree \( T' \) as follows. Let \( A_{a_1}, \ldots, A_{a_l} \), \( l \geq 2 \) be the path of vertices linked by edges of type \( n' \). By \((iii)\) of the induction hypothesis, there exists a non-decreasing sequence in \( \mathbb{Z} \), \( b_1, \ldots, b_l \) such that \( \gamma_{b_i}^{n'} = y_{a_i} \) for all \( i \in [1, l] \). By definition \( y_{n'+k} = \gamma_{b_i}^{n'} \) for some \( b_i \in \mathbb{Z} \).

One then constructs \( T \) from \( T' \) by adding a new edge of type \( n_2 \) to include \( A_{k+n_2} \) in the tree in the following manner.

(a) if \( b \leq b_1 \), one adds an edge \( A_{k+n_2}A_{a_1} \) (and the path \( A_{n'+k}A_{n'+k+1} \ldots A_{n+k-1}A_k \) previously constructed),

(b) if \( b \in (b_i, b_{i+1}] \) then one replaces the edge \( A_{a_i}A_{a_{i+1}} \) by two edges \( A_{a_i}A_{n_2+k} \) and \( A_{n_2+k}A_{a_{i+1}} \),

(c) if \( b > b_l \) then one adds an edge \( A_{a_l}A_{n_2+k} \).

\[ Figure 4.1.: \] Examples of the process of tree creation when \( k = 3 \) and \( n_1 = 4 \). On the left, the \( n_1 \) oriented trajectories are represented together with the \( x \)s and the points of intersection of the trajectories. On the right this is encoded in the corresponding tree.
We prove equation (4.8) in two steps. First we show that given subsets (the last equality being obtained using independence. Secondly we show that for any choice of points

$$\Pr \left[ (x_i)_{i=1}^k \text{ is connected with scheme } \mathcal{T} \right] \leq C \max_{i \neq j} |x_i - x_j|^{-1+\varepsilon}. \quad (4.6)$$

For this purpose we will use the following Lemma that estimates the l.h.s. of (4.6).

**Lemma 4.4.** Let $\mathcal{E}$ denote the set of edges of $\mathcal{T}$, a tree with $n+k-1$ vertices. Then

$$\Pr \left[ (x_i)_{i=1}^k \text{ is connected with scheme } \mathcal{T} \right] \leq C \sum_{(y_i)_{i=k+1}^n} \prod_{i \neq j} \left( |y_i - y_j| + 1 \right)^{2-d}. \quad (4.7)$$

**Proof.** By a simple union bound it is sufficient to prove that

$$\Pr \left[ x_1, \ldots, x_k \text{ are connected with scheme } \mathcal{T} \text{ using } (y_i)_{i=k+1}^n \right] \leq C \prod_{A_i \not\in \mathcal{E}} \left( |y_i - y_j| + 1 \right)^{2-d}. \quad (4.8)$$

We prove equation (4.8) in two steps. First we show that given subsets $E_1, \ldots, E_n$ of $W^*$ with finite $\nu$-measure, one has

$$\Pr \left[ \exists (\gamma^i)_{i=1}^n \in (\omega_u)^n, \forall i \neq j, \gamma^i \neq \gamma^j, \forall i, \gamma^i \in E_i \right] \leq u^n \prod_{i=1}^n \nu(E_i). \quad (4.9)$$

Indeed let $\omega_{dt} = \omega_{u(t+d)}$ denote infinitesimal division of the Poisson process. One has

$$\Pr \left[ \exists (\gamma^i)_{i=1}^n \in (\omega_u)^n, \forall i \neq j, \gamma^i \neq \gamma^j, \forall i, \gamma^i \in E_i \right] \leq \int_{\{(t_i)_{i=1}^n \in [0,u]^n | \forall i \neq j, t_i \neq t_j\}} \Pr[\forall i \in [1,n] \omega_{dt_i} \cap E_i \neq \emptyset]$$

$$= \int_{\{(t_i)_{i=1}^n \in [0,u]^n | \forall i \neq j, t_i \neq t_j\}} \prod_{i=1}^n \nu(E_i) dt_i, \quad (4.10)$$

the last equality being obtained using independence. Secondly we show that for any choice of points $(z_i)_{i=1}^m$ one has

$$\nu(\{ \gamma : \gamma \text{ visits } z_1, z_2, \ldots, z_m \text{ in that order } \}) \leq C_m \prod_{i=1}^{m-1} \frac{1}{(|z_{i+1} - z_i| + 1)^{d-2}}. \quad (4.11)$$

Parameterizing $\gamma = (\gamma_n)_{n \geq 0}$ so that $0$ is the first time of visit of $z_1$ and using the definition of $\nu$ given by (2.28)-(2.29) one has

$$\nu(\{ \gamma : \gamma \text{ visits } z_1, z_2, \ldots, z_m \text{ in that order } \}) = P_0(H_0 = \infty) P_{z_1}(\exists n_2 \leq n_3 \leq \cdots \leq n_m, \forall i \in [2,m], X_{n_i} = y_{n_i})$$

$$= P_0(H_0 = \infty) \prod_{i=1}^{m-1} P_{z_i}(H_{z_{i+1}} < \infty), \quad (4.12)$$

When $n' = n$ the procedure is exactly the same except that $y_{n'+k}$ is replaced by $x_k$ (and $A_{n'+k}$ by $A_k$) and that only the second stage is needed (the paths to be plugged is only the single point $A_k$ in this case). We let the reader check that assumptions (i) – (iii) are satisfied by $\mathcal{T}$. \hfill \Box
where the last inequality follows by multiple application of the Markov property at the successive stopping times $H_{z_i}$. Then (4.11) is deduced by using (2.7).

Combining (4.11) with (4.9) used for the events $E_i := \{ \gamma^i \text{ visits successively } y_{a_1}^i, \ldots, y_{a_{m_i}}^i \}$ where $A_{a_1}^i A_{a_2}^i \ldots A_{a_{m_i}}^i$ are the paths corresponding to oriented edges of type $i$ in $T$ we get (4.8).

\[ \square \]

Our problem is that for some schemes in $\mathbb{T}_n$, the r.h.s. of (4.7) diverges. Therefore, we must first identify which are the bad trees for which that happens and prove (4.6) for them without using (4.7). Afterwards, we use the following proposition that gives an upper-bound for the r.h.s. of (4.7) for the good trees, and allow us to conclude.

**Proposition 4.5.** Given a labeled tree $T$ with $k$ leaves $A_1, \ldots, A_k$ and $m$ nodes $A_{k+1}, \ldots, A_{k+m}$ and edges $E$, we associate to each edge a length $l(e) \in [0, d)$. Suppose that the lengths of the edges are such that:

(i) The total length of the tree $l(T) = \sum_{e \in E} l(e)$ is strictly smaller than $d(k-1)$.

(ii) The length of any (strict) subtree containing at least $k_1$ of the original leaves $A_i$ is at least $d(k_1-1)$.

Then for any $\varepsilon > 0$ there exists a $C_\varepsilon$ such that, for every $x_1, \ldots, x_k$

\[
\sum_{(y_i)_{i=1}^{k+m} \in (\mathbb{Z}^d)^m} \prod_{i \leq j} (|y_i - y_j| + 1)^{(A_i A_j)_{i \neq j} - d} \leq C_\varepsilon \max_{i \neq j} |x_i - x_j|^{d(k-1) - l(T) + \varepsilon}.
\]

(4.13)

where we use the convention that $y_i = x_i$ for $i \leq k$.

The proof is postponed to the end of the section.

**Proof of Proposition 4.1**. The statement is proved by induction on $k$. The case $k = 2$ can easily be proved using Proposition 4.5. So we only need to focus on the induction step. It is necessary to prove (4.6) for all trees with $k + n - 1$ vertices.

First consider the trees where there exists $i \leq k$ such that $A_i$ is not a leaf (after permutation of the indices we can consider that $A_1$ is not a leaf). In that case $A_1$ has degree two and the tree $T$ can be split into two trees, each of them linking $k_1$ and $k_2$ of the $A_i$s together, and using respectively $n_1$ and $n_2$ types of edges respectively, with $k_1 + k_2 = k + 1$ and $n_1 + n_2 = n + 1$ (recall that the two edges getting out of $A_1$ are of the same type).

As $n < n(k, d)$, one has either $n_1 < n(k_1, d)$ or $n_2 < n(k_2, d)$. Suppose without loss of generality that $n_1 < n(k_1, d)$. In that case a subset of $k_1 < k$ vertices is connected by $n_1 < n(k_1, d)$ trajectories (see Remark 4.3) and one can use the induction hypothesis to get (4.6). In the rest of the proof we consider only trees for which all the $A_i$, $i \leq k$ are leaves.

A connected subgraph of $T$ which is a tree and whose leafs are leafs of $T$ is said to be a proper subtree of $T$. We consider now the trees $T$ with $k + n - 1$ vertices that have a proper subtree with $k_1$ vertices and that uses only edges of $n_1$ different types with $n_1 < n(k_1, d)$. Then according to Remark 4.3 a subset of $k_1 < k$ vertices is connected by $n_1 < n(k_1, d)$ trajectories and again one can prove (4.6) using the induction hypothesis.

Now suppose that $T$ is a tree for which all subtrees with $k_1 < k$ vertices use at least $n(k_1, d)$ type of edges. To each edge of the tree, we associate an edge-length 2, and apply Proposition 4.5 to conclude. Assumption (i) of the proposition is satisfied since $n < n(k, d)$ and the total number of edges $n + k - 2$ is given by Proposition 4.2. Assumption (ii) is satisfied because of our assumption on proper subtrees, indeed the reader can check that if a proper subtree with $k_1$ vertices uses $n_1$ type of edges, it must have at least $n_1 + k_1 - 2$ edges: this is because vertices in the tree have degree
at most 4 and that on vertices of degree 3 two of the incident edges have the same type, and on vertices of degree 4, one has two pairs of incident edges with the same type (by (ii) of Proposition 4.2).

Proof of Proposition 4.5. We perform the proof by induction on $k$. When $k = 2$, it is easy to show that the sum is equal to $O(|x_1 - x_2|^{|T| - d} \log |x_1 - x_2| \#\{\text{edges of length 0}\})$ where $l(T)$ is the length of the tree.

When $k \geq 3$ our strategy is to bound the r.h.s of (4.13) by sums corresponding to trees with $k - 1$ vertices and then conclude by using the induction hypothesis.

We remark that if $T$ includes two edges $e$ and $e'$ linked to a common vertex of degree two, one can replace it by a unique edge of length $l(e) + l(e') + \delta$ (see Figure 4.2). Indeed as long as $l(e) + l(e') < d$ we have

$$\sum_{y \in \mathbb{Z}^d} (|x - y| + 1)^{l(e)} (|y - z| + 1)^{l(e')} = O((1 - |x - z|)^{l(e) + l(e') + \delta - d}). \quad (4.14)$$

So if one calls $T_1$ the tree obtained after this change (relabeling the vertices of $T_1$ from $A_1, \ldots, A_{k+m-1}$, calling $E_1$ the corresponding edge set and for simplicity denote by $l$ the length of the edges on the new tree) one get that there exists a constant $C$ such that

$$\sum_{(y_i)_{i=k+1}^{k+m} \in (\mathbb{Z}^d)^m} \prod_{A_i \in E} (|y_i - y_j| + 1)^{(A_i, A_j)} \leq C \sum_{(y_i)_{i=k+1}^{k+m} \in (\mathbb{Z}^d)^m} \prod_{A_i \in E_1} (|y_i - y_j| + 1)^{(A_i, A_j)}. \quad (4.15)$$

Note that adding the $\delta$ is only necessary if one of the edges has length zero in order to avoid having a log term. Also note that one can choose the $\delta$ small enough so that after this transformation $l(T_1) \leq d(k - 1)$. In particular, this implies that all the edges are still of length smaller than $d$.

![Figure 4.2](image-url) Illustration of the two stages of the tree reduction procedure

Then after having reduced all consecutive edge in this manner we obtain (what we call the first stage of the reduction) a tree $T'$ with $k + m'$ vertices ($m' \leq m$) and $k$ leaves, no vertices of degree...
2, and satisfying
\[
\sum_{(y_i)_{i=k+1} \in (\mathbb{Z}^d)_m} \prod_{A_i, A_j \in E} (|y_i - y_j| + 1)^{(A_i A_j) - d} \leq C \sum_{(y_i)_{i=k+1} \in (\mathbb{Z}^d)_m} \prod_{A_i, A_j \in E'} (|y_i - y_j| + 1)^{(A_i A_j) - d}.
\]  \tag{4.16}

We can choose the δ small enough so that \( l(T_1) \leq l(T) + \varepsilon/2 \).

After the first stage of the reduction, it is possible to find in \( T' \) two leaves at graph distance 2 of each other (i.e. separated by only two edges): say without loss of generality that \( A_k \) and \( A_{k-1} \) are linked to \( A_{k+1} \) with edges \( A_k A_{k+1} \) and \( A_{k+1} A_{k-1} \) of length \( l_1 \) resp. \( l_2 \). We consider the inequality
\[
(x_k - y_{k+1})^{l_1-d} (x_{k+1} - y_k)^{l_2-d} \leq (x_k - y_{k+1})^{l_1+d} (x_{k+1} - y_k)^{l_2-d} + (x_{k-1} - y_{k+1})^{l_1+d} (x_{k+1} - y_{k-1})^{l_2-d}.
\]  \tag{4.17}

Let \( T''_1 \) and \( T''_2 \) be trees with \( k-1 \) leaves, obtained by replacing the edges \( e \) and \( e' \) in \( T' \) by a unique edge \( e'' \) of length \( l_1 + l_2 - d \geq 0 \) linking \( A_{k+1} \) and \( A_k \) resp. \( A_{k+1} \) and \( A_{k-1} \) and deleting the vertex left alone (\( A_{k-1} \) resp. \( A_k \)). Indeed using (4.17) one gets that
\[
\sum_{(y_i)_{i=k+1} \in (\mathbb{Z}^d)_m'} \prod_{A_i, A_j \in E'} (|y_i - y_j| + 1)^{(A_i A_j) - d} \leq \sum_{(y_i)_{i=k+1} \in (\mathbb{Z}^d)_m} \prod_{A_i, A_j \in E'} (|y_i - y_j| + 1)^{(A_i A_j) - d}
\]
\[
+ \sum_{(y_i)_{i=k+1} \in (\mathbb{Z}^d)_m} \prod_{A_i, A_j \in E'} (|y_i - y_j| + 1)^{(A_i A_j) - d}.
\]  \tag{4.18}

where \( E''_1 \) and \( E''_2 \) denote the edge sets of \( T''_1 \) and \( T''_2 \) respectively.

Note that for \( i = 1, 2 \), \( l(T''_i) = l(T') - d \leq l(T) - d + \varepsilon/2 \) so that condition (i) is satisfied if \( \varepsilon \) is small enough (the new tree has one less leaf). Note that any proper subtree of \( T'' \) that does not contain \( e'' \) is also a proper subtree of \( T' \) and any proper subtree \( \tau \) of \( T'' \) that contains \( e'' \) can be associated to a subtree \( \tau' \) of \( T'' \) by replacing \( e'' \) by \( e \) and \( e' \) (the inverse of the above transformation) such that \( l(\tau') = l(\tau) + d \) and \( \tau' \) has one more leaf than \( \tau \). Hence if condition (ii) is satisfied for \( T'' \) it is also satisfied for \( T' \) so that one can apply the induction hypothesis (with \( \varepsilon/2 \)) on the trees \( T''_1 \) and \( T''_2 \) to conclude.

\[\square\]

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