Interpolative gap bounds for nonautonomous integrals

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Abstract
For nonautonomous, nonuniformly elliptic integrals with so-called \((p, q)\)-growth conditions, we show a general interpolation property allowing to get basic higher integrability results for Hölder continuous minimizers under improved bounds for the gap \(q/p\). For this we introduce a new method, based on approximating the original, local functional, with mixed local/nonlocal functionals, and allowing for suitable estimates in fractional Sobolev spaces.

Keywords Regularity · Non-autonomous functionals · \((p, q)\)-growth

Mathematics Subject Classification 35J60 · 35J70

1 Introduction
In this paper we give new contributions to the regularity theory of minima of integral functionals of the type

\[
W_{1,1}^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto \mathcal{F}(w, \Omega) := \int_{\Omega} F(x, Dw) \, dx .
\] (1.1)

Here, as in the following, \(\Omega \subset \mathbb{R}^n\) will denote an open subset, where \(n \geq 2\) and \(N \geq 1\); the function \(F: \Omega \times \mathbb{R}^N \times n \to [0, \infty)\) will always be Carathéodory integrand.

To Vladimir Gilelevich Maz’ya, master of analysis.

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The definition of (local) minimizer we use here is standard in the literature and it is given by

**Definition 1** A map $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$ in (1.1) if, for every open subset $\tilde{\Omega} \subseteq \Omega$, we have $\mathcal{F}(u; \tilde{\Omega}) < \infty$ and $\mathcal{F}(u; \tilde{\Omega}) \leq \mathcal{F}(w; \tilde{\Omega})$ holds for every competitor $w \in u + W^{1,1}_{0}(\tilde{\Omega}; \mathbb{R}^N)$.

We will abbreviate local minimizer simply by minimizer. The main point in this paper is that the integrand $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^n \to [0, \infty)$ is both nonautonomous and nonuniformly elliptic. Specifically, following the notation used in [20,21], we are in the situation when the ellipticity ratio

$$\mathcal{R}_F(z, B) \equiv \mathcal{R}(z, B) := \frac{\sup_{x \in B} \text{highest eigenvalue of } \partial_{zz} F(x, z)}{\inf_{x \in B} \text{lowest eigenvalue of } \partial_{zz} F(x, z)} ,$$

(B $\subset \Omega$ is any ball), is such that $\mathcal{R}_F(z, B) \to \infty$ when $|z| \to \infty$ for at least one ball $B$. This happens for instance in the paramount example given by the double phase functional

$$w \mapsto \int_{\Omega} \left[ |Dw|^p + a(x)|Dw|^q \right] \, dx , \text{ where } 1 < p < q , \, 0 \leq a(\cdot) \in C^\alpha(\Omega) , \, \alpha \in (0, 1] .$$

Indeed, on a ball $B$ such that $B \cap \{a(\cdot) = 0\} \neq 0$, in the case of (1.3) we have

$$\mathcal{R}(z, B) \approx \|a\|_{L^{\infty}(B)}|z|^{q-p} + 1 .$$

The functional in (1.3) has been introduced by Zhikov [60–63] in the setting of Homogenization theory [63]. For minima of (1.3), there is by a now a rather complete regularity theory, initiated many years ago in [25]; see [3] for the most updated statements. In particular, it has been proved that the condition on the gap $q/p$

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n} ,$$

is necessary [25,27] and sufficient [3] for the local Hölder gradient continuity of minima. Failure of (1.5) implies that in general, minimizers, that by definition are $W^{1,p}$-regular, do not belong to $W^{1,q}$. Therefore even the initial integrability bootstrap fails. The bound in (1.5) reflects the delicate balance between the smallness of $a(\cdot)$ around $\{a(\cdot) = 0\} \cap B$, and the growth of $F(x, z)$ with respect to the gradient variable $z$, which is necessary to keep $\mathcal{R}(z, B)$ under control when proving a priori estimates for minima. We refer to [3,17,53] for a larger discussion.

Functionals of the type in (1.3) fall in the realm of so-called functionals with $(p, q)$-growth following Marcellini’s terminology [44–46], i.e. those satisfying unbalanced ellipticity conditions of the type (for $|z|$ large)

$$|z|^{p-2}I_d \lesssim \partial_{zz} F(\cdot, z) \lesssim |z|^{q-2}I_d \implies \mathcal{R}_F(z, B) \lesssim |z|^{q-p}$$
for \(1 < p \leq q\). We report in Sect. 1.3 a brief review on this kinds of problems. A distinctive feature of such functionals is that, in general, gap bounds of the type

\[
\frac{q}{p} < 1 + o(n), \quad o(n) \approx \frac{1}{n}
\]  

(1.7)

are necessary and sufficient for regularity of minima [25,45]. Note that the bound appearing in (1.5) is of the type in (1.7). This said, an interesting phenomenon of interpolative nature appears when considering a priori more regular minimizers. For instance, assuming that minima are bounded, leads to non-dimensional bounds on the distance \(q - p\), that can be made independent of \(n\); see for instance [3,14,16]. In particular, in [17] the authors have proved that assuming that minima are bounded allows to replace (1.5) by

\[
q \leq p + \alpha.
\]  

(1.8)

This is better than (1.5) provided \(p \leq n\) (that is when boundedness of minima is not automatically implied by Sobolev-Morrey embedding, and it is therefore a genuine assumption). The bound in (1.8) is optimal as shown by the counterexamples in [25,27]. A partial generalization of the result in [3] has been obtained in [20], under the same bound in (1.8), with strict inequality. More recently, in the specific case of the double phase functional (1.3), in [3] the authors have proved that, assuming a priori \(C^{0,\gamma}\)-regularity for \(0 < \gamma < 1\), minimizers are regular provided

\[
q < p + \frac{\alpha}{1 - \gamma},
\]  

(1.9)

thus providing a further weakened gap bound. This condition is sharp too, as recently proved in [2]. It gives back (1.8) for \(\gamma \to 0\). In particular, note that the asymptotic of (1.9) for \(\gamma \to 1\) is of the type

\[
q < p + O(\gamma), \quad O(\gamma) \approx \frac{1}{1 - \gamma}.
\]  

(1.10)

This is in accordance with the fact that the focal point in regularity for \((p, q)\)-problems is Lipschitz continuity of minima. Once this is achieved, the functional in question goes back to the realm of uniformly elliptic ones as growth conditions for large \(|z|\) become irrelevant; see also [20, Sect. 6]. Bounds of the type in (1.9) have an interpolative nature. In fact, in the scheme of Caccioppoli type inequalities coming up in regularity estimates, the a priori, assumed regularity on minima, allows for a better control of nonuniform ellipticity.

The above mentioned result in [3] holds in the scalar case \(N = 1\) and for the specific functional in (1.3); these facts play a crucial role in the analysis there. Therefore the question arises of whether or not bounds with asymptotics as in (1.10) imply regularity of \(C^{0,\gamma}\)-minima for general functionals with \((p, q)\)-growth. The question is open already in the autonomous case \(F(x, Dw) \equiv F(Dw)\). We are here able to give a first, positive answer by showing that bounds as in (1.10) are in general sufficient to prove
the basic step of improving the regularity of $C^{0,\gamma}$-minima from $W^{1,p}$ to $W^{1,q}$. See condition (1.18) below. This initial integrability bootstrap is usually the starting point from proving higher regularity—see [20,25]—and it is essentially the best possible result in the vectorial case considered here.

1.1 Statements of the results

In order to quantify ellipticity, we will use assumptions that are more general of those using the Hessian of $F(\cdot)$, as in (1.6), following the approach in [20,25]. In fact, rather than prescribing ellipticity of $F(\cdot)$, we will only require a quantified form of monotonicity, as in (1.11). No growth assumption from above will be required of $\partial_{zz}F(\cdot)$; the one on $F(\cdot)$ will be sufficient. Specifically, we assume that $F: \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is $C^1$-regular in the gradient variable $z$ and we require that

$$
\begin{align*}
\nu |z|^p &\leq F(x, z) \leq L(|z|^q + 1) \\
\nu (|z_1|^2 + |z_2|^2 + \mu^2)^{(p-2)/2} z_1 - z_2|^2 &\leq (\partial_z F(x, z_1) - \partial_z F(x, z_2)) \cdot (z_1 - z_2) \\
|\partial_z F(x, z) - \partial_z F(y, z)| &\leq L|x - y|^\alpha |z|^{q-1} + 1
\end{align*}
$$

(1.11)

hold whenever $x, y \in \Omega$, $z, z_1, z_2 \in \mathbb{R}^{N \times n}$, where $1 < p \leq q$, $\mu \in [0, 1]$, $\alpha \in (0, 1]$ and $0 < \nu \leq 1 \leq L$ are fixed constants. Note that (1.11)$_1$ implies that minimizers of $\mathcal{F}$ are automatically locally $W^{1,p}$-regular. As we are dealing with a nonautonomous functional, the so-called Lavrentiev phenomenon naturally comes into the play [43,60–62]. Its nonoccurrence is a necessary condition for regularity of minima. Therefore we are led to consider a functional, called Lavrentiev gap, providing a quantitative measure of such a phenomenon. We refer the reader to [1,20,25] for more information. Since we are in fact interested in the regularity of a priori Hölder continuous minima of the functional (1.1), we will adopt suitable definitions of relaxed and gap functionals aiming at exploiting this fact. For this, let us consider a functional of the type in (1.1), where the integrand $F(\cdot)$ satisfies (1.11), and let us fix numbers numbers $H > 0$ and $\gamma \in (0, 1)$. Given a ball $B \subset \Omega$, the natural relaxation of $\mathcal{F}$ we consider here is

$$
\tilde{\mathcal{F}}_{H,\gamma}(w, B) := \inf_{\{w_j\} \in \mathcal{C}_{H,\gamma}(w, B)} \left\{ \liminf_{j \to \infty} \int_B F(x, Dw_j) \, dx \right\},
$$

(1.12)

defined for any $w \in W^{1,1}(B, \mathbb{R}^n)$, where (see (2.1) below for the notation)

$$
\mathcal{C}_{H,\gamma}(w, B) := \left\{ \{w_j\} \subset W^{1,\infty}(B, \mathbb{R}^N): w_j \to w \text{ in } W^{1,p}(B, \mathbb{R}^N), \right. \left. \sup_{j} |w_j|_{0,\gamma; B} \leq H \right\}.
$$

(1.13)

Accordingly, as in [1,20,25], we consider the Lavrentiev gap functional

$$
\mathcal{L}_{F, H, \gamma}(w, B) := \tilde{\mathcal{F}}_{H,\gamma}(w, B) - \mathcal{F}(w, B),
$$

(1.14)
defined for every \( w \in W^{1,1}(B) \) such that \( \mathcal{F}(w, B) \) is finite; we set \( \mathcal{L}_{F, H, \gamma}(w, B) = 0 \) otherwise.

**Remark 1** We here collect a few immediate consequences of the above definitions.

- The convexity of \( F(\cdot) \), implied by (1.11)_2, provides that the functional \( \mathcal{F} \) in (1.1) is lower semicontinuous with respect to the weak convergence of \( W^{1,p} \). It follows that \( \mathcal{L}_{F, H, \gamma}(\cdot, B) \geq 0 \) holds whenever \( w \in W^{1,1}(B, \mathbb{R}^N) \) and \( B \Subset \Omega \) is a ball.
- Consider a map \( w \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap C_{\text{loc}}^{0,\gamma}(\Omega, \mathbb{R}^N) \). A simple mollification argument then shows that \( \mathcal{C}_{H, \gamma}(w, B) \) is non empty whenever \( B \Subset \Omega \) for \( H \approx \|w\|_{C^{0,\gamma}(B)} \). For this see also the proofs in Sect. 3.11 below. This anticipates that, when considering a \( C^{0,\gamma} \)-regular minimizer \( u \) of the functional \( \mathcal{F} \), it will happen that we will use \( \mathcal{L}_{F, H, \gamma}(u, B) \) with \( H \approx \|u\|_{C^{0,\gamma}(B)} \) (see for instance Theorem 2 below, and compare (1.22) with (1.19)).
- A straightforward consequence of (1.12)–(1.14) is

**Proposition 1** Let \( w \in W^{1,1}(B, \mathbb{R}^N) \) be such that \( \mathcal{F}(w, B) \) is finite, where \( B \subset \Omega \) is a fixed ball. Then \( \mathcal{F}(w, B) = \tilde{\mathcal{F}}_{H, \gamma}(w, B) \) for some \( H > 0 \), if and only if there exists \( \{w_j\} \in C_{H, \gamma}(w, B) \) such that

\[
\mathcal{F}(w_j, B) \to \mathcal{F}(w, B). \tag{1.15}
\]

- Accordingly, if \( \mathcal{C}_{H, \gamma}(w, B) \) is empty, then \( \tilde{\mathcal{F}}_{H, \gamma}(w, B) = \infty \); moreover, again in this case, if \( \mathcal{F}(w, B) \) is finite, then \( \tilde{\mathcal{F}}_{H, \gamma}(w, B) = \mathcal{L}_{F, H, \gamma}(w, B) = \infty \). This is in accordance with the fact that when \( w \) is not \( C^{0,\gamma} \)-regular, it is in general impossible to build a sequence from \( \mathcal{C}_{H, \gamma}(w, B) \), for any \( H > 0 \), approximating \( w \) in energy in the sense of (1.15) below. Such approximations are in fact usually built via smooth convolutions when \( w \) is already Hölder continuous.
- Conversely, if \( \mathcal{C}_{H, \gamma}(w, B) \) is non empty, then \( w \in C^{0,\gamma}(B) \) and \( \|w\|_{0,\gamma;B} \leq H \).
- If \( w \) is a locally \( W^{1,q} \cap C^{0,\gamma} \)-regular map, then a density and convolution argument gives that \( \mathcal{L}_{F, H, \gamma}(w, B) = 0 \) holds for every ball \( B \Subset \Omega \), with \( H \approx \|w\|_{C^{0,\gamma}(B)} + 1 \). This last condition is therefore in a sense necessary to prove the local \( W^{1,q} \)-regularity of \( C^{0,\gamma} \)-regular minima of the original functional \( \mathcal{F} \).

Accordingly to the last bullet in Remark 1, the main result of this paper is now

**Theorem 1** Let \( u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \) be a minimizer of functional (1.1), under assumptions (1.11) and

\[
q < p + \frac{\min\{\alpha, 2\gamma\}}{\vartheta(1-\gamma)}, \quad \text{where} \quad \vartheta := \begin{cases} 
1 & \text{if } p \geq 2 \\
\frac{2}{p} & \text{if } 1 < p < 2, \quad 0 < \gamma < 1.
\end{cases} \tag{1.16}
\]

Assume also that

\[
\mathcal{L}_{F, H, \gamma}(u, B_r) = 0 \tag{1.17}
\]
holds for a ball $B_r \subseteq \Omega$ with $r \leq 1$, and for some $H > 0$. If $q$ is a number such that

$$q \leq q < p + \frac{\min\{\alpha, 2\gamma\}}{\vartheta(1 - \gamma)}$$

and $B_q \subseteq B_r$ is a ball concentric to $B_r$, then

$$\|Du\|_{L^q(B_q)} \leq \frac{c}{(r - \rho)^{\kappa_1}} \left([\mathcal{F}(u, B_r)]^{1/p} + H + 1\right)^{\kappa_2}$$

holds for constants $c = c(n, p, q, v, L, \alpha, \gamma, q)$ and $\kappa_1, \kappa_2 \equiv \kappa_1, \kappa_2(n, p, q, \alpha, \gamma, q)$. In particular, if (1.17) holds for every ball $B_r \subseteq \Omega$, then $u \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N)$.

**Remark 2** The reader might of course wonder where the a priori $C^{0,\gamma}$-regularity of the minimizer $u$ is assumed in Theorem 1. This is hidden in assumption 1.17. In fact, as $F(\cdot, Du) \in L^1_{\text{loc}}(\Omega)$ by minimality, it follows from the fourth and the fifth bullet of Remark 1 that $C_{H,\gamma}(u, B_r)$ is non-empty and therefore $u \in C^{0,\gamma}(B_r, \mathbb{R}^N)$ (with $[u]_{0,\gamma; B_r} \leq H$). This said, the bound in (1.18) is exactly of the type in (1.10). Let us now consider the case $p \geq 2$. When $\alpha \leq 2\gamma$ the bound in (1.16) coincides with (1.9), that is the one considered in [3] for the specific functional (1.3).

As mentioned above, assumption (1.17) is in a sense necessary to prove local $W^{1,q}_{\text{loc}}$-regularity of minimizers of $\mathcal{F}$. As a matter of fact, condition (1.17) is always satisfied in a large number of situations. A very relevant one is when the integrand is autonomous, i.e., $F(x, Du) \equiv F(Du)$. In such a case, the Lavrentiev gap disappears due to basic convexity arguments, and we have:

**Theorem 2** Let $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap C^{0,\gamma}_{\text{loc}}(\Omega, \mathbb{R}^N)$ be a minimizer of functional (1.1), where $0 < \gamma < 1$, under assumptions (1.11) with $F(x, z) \equiv F(z)$ and

$$q < p + \frac{\min\{1, 2\gamma\}}{\vartheta(1 - \gamma)},$$

where $\vartheta$ is as in (1.16). If $q$ is a number such that

$$q \leq q < p + \frac{\min\{1, 2\gamma\}}{\vartheta(1 - \gamma)}$$

and $B_q \subseteq B_r \subseteq \Omega$ are concentric balls, then

$$\|Du\|_{L^q(B_q)} \leq \frac{c}{(r - \rho)^{\kappa_1}} \left([\mathcal{F}(u, B_r)]^{1/p} + [u]_{0,\gamma; B_r} + 1\right)^{\kappa_2}$$

holds with $c, \kappa_1, \kappa_2$ as in (1.19). In particular, $u \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N)$.

Another situation when (1.17) can be automatically satisfied, is when the integrand $F(\cdot)$ is equivalent to a convex function $G: \mathbb{R}^{N \times n} \to [0, \infty)$ modulo a multiplicative
factor, i.e.,
\[ b(x)G(z) \lesssim F(x, z) \lesssim b(x)G(z) + 1, \quad 0 \leq b(\cdot), 1/b(\cdot) \in L^\infty(\Omega). \tag{1.23} \]

**Corollary 1** Let \( u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap C^{0,\gamma}_{\text{loc}}(\Omega, \mathbb{R}^N) \) be a minimizer of functional (1.1) under the assumptions (1.11) and (1.16). Furthermore, assume that (1.23) is satisfied too. Then (1.22) holds whenever \( B_Q \Subset B_r \Subset \Omega \) are concentric balls and for the range of exponents in (1.18).

Back to the full nonautonomous case, a relevant example in this setting is given by [3, Theorem 4]. This deals with functionals modelled on the double phase functional in (1.3), i.e., growth conditions as
\[ |z|^p + a(x)|z|^q \lesssim F(x, z) \lesssim |z|^p + a(x)|z|^q + 1, \quad 0 \leq a(\cdot) \in C^{0,\alpha}(\Omega), \tag{1.24} \]
are assumed for every \((x, z) \in \Omega \times \mathbb{R}^{N \times n}\), no matter (1.11) are satisfied or not. Then (1.9) guarantees that the approximation in energy (1.15) holds for a sequence of \( W^{1,\infty}\)-regular maps \( \{w_j\} \), provided \( w \in C^{0,\gamma} \) holds and the bound in (1.9) is in force. This fact allows to draw another consequence from Theorem 1, that is

**Corollary 2** Let \( u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap C^{0,\gamma}_{\text{loc}}(\Omega, \mathbb{R}^N) \) be a minimizer of functional (1.1) under the assumptions (1.11) and (1.16). Furthermore, assume that (1.24) is satisfied too. Then (1.22) holds for the range of exponents displayed in (1.18).

More general cases of double sided bounds as in (1.24) for which similar corollaries hold can be found in [25, Sect. 5]. We refer to this last paper also for a larger discussion on the use of Lavrentiev gap functionals in this setting.

### 1.2 Novelties and techniques

The technique leading to the proof of Theorem 1 makes use of three main ingredients. The first one is a method aimed at approximating, on a fixed ball \( B_r \Subset \Omega \), the original minimizer \( u \) of \( \mathcal{F} \) with a sequence \( \{u_j\} \) of \( W^{1,q}\)-regular solutions to a different kind of variational problems. This is necessary, as the starting lack of \( W^{1,q}\)-integrability of \( u \) does not allow to use the Euler–Lagrange system of \( \mathcal{F} \). The possibility of this approximation relies on assumption (1.17), that can be used, in a sense, to find good boundary values to build the approximating minimizers \( \{u_j\} \). Here we encounter a first difficulty as, in order to get uniform a priori integrability estimates on \( \{Du_j\} \), we would need that the sequence \( \{u_j\} \) is bounded in \( C^{0,\gamma} \), exactly as \( u \) (see Remark 2). We will actually build the sequence \( \{u_j\} \) in a way that it is bounded in a fractional Sobolev–Slobodevsky space \( W^{s,2d} \). For our purposes this is actually sufficient, as these spaces are in a sense a good approximation of \( C^{0,\gamma} \), once proper choices of \( s \) close to \( \gamma \) and \( d \) large enough, are made. To find \( \{u_j\} \), we employ a novel approximation using additional nonlocal terms, i.e., adding a suitable truncated Gagliardo-type seminorm
term to the original functional $F$, together with a more standard $L^2$-penalization term. Specifically, we consider minimizers $\{u_j\}$ of perturbed functionals of the type

$$w \mapsto F(w, B_r) + \varepsilon \int_{B_r} |Dw|^{2d} \, dx + \int_{B_r} (|w|^2 - M_0^2)_{+}^d \, dx \quad (1.25)$$

for $\varepsilon \equiv \varepsilon_j$ small, $s$ close to $\gamma$ and suitably large $d, M_0$ and $M$. The last two constants depend on $\|u\|_{C^{0,\gamma}}$, which is finite by assumption. Here, following a standard notation, we are denoting

$$(t - k)_+ := \max\{t - k, 0\}, \quad \text{fort, } k \in \mathbb{R}. \quad (1.26)$$

Functionals of mixed local/nonlocal type, in the quadratic/linear case $F(x, z) \equiv |z|^2$, have been recently studied in [6,7] under special boundary conditions. As far as we know, this is the first paper where nonlinear functionals of the type in (1.25) are considered, and a priori estimates are presented. Let us mention that in the setting of functionals with $(p, q)$-growth conditions, purely local approximations aimed at using the $L^\infty$-information, have been considered for the first time in [14] in the autonomous case; see also [20] for the nonautonomous case under assumptions (1.11). These approximations are made using only the first line in (1.25), and are not suitable to preserve the initial $C^{0,\gamma}$-regularity information, or at least some part of it. Adding the last line in (1.25) therefore turns out to be crucial. On the other hand this turns out to generate different a priori estimates. The second ingredient, which is taken from [20,25], is a suitable use of the difference quotients techniques in the setting of Fractional Sobolev spaces. At this stage, the Hölder continuity of $\partial_{\varepsilon} F(\cdot)$ in (1.11) is automatically read as a fractional differentiability and allows to get uniform estimates in Nikolski spaces for $Du_j$. We finally use the last and third ingredient. The assumed Hölder continuity of $u$ implies that $\{u_j\}$ is uniformly bounded in $W^{s,2d}$, as mentioned above. This allows to use a Gagliardo-Nirenberg type interpolation inequality on $Du_j$, which improves the exponents intervening when using fractional Sobolev embedding theorem in the setting of Caccioppoli type inequalities; see Lemma 3 below. Ultimately, we get uniform higher integrability estimates on the sequence $\{Du_j\}$, that finally imply the higher integrability of $Du$ in (1.19).

1.3 Brief review on nonuniformly elliptic functionals

Functionals with $(p, q)$-growth fall in the larger realm of nonuniformly elliptic problems. In the case of autonomous integrals of the type

$$w \mapsto \int_{B} F(Dw) \, dx, \quad (1.27)$$
the nonuniform ellipticity of \( F(\cdot) \) amounts to assert that the ellipticity ratio

\[
\sup_z \mathcal{R}_F(z) := \frac{\text{highest eigenvalue of } \partial_{zz} F(z)}{\text{lowest eigenvalue of } \partial_{zz} F(z)}
\]  

(1.28)

becomes unbounded when \( |z| \to \infty \). This is in accordance to the analogous, nonautonomous definition in (1.2), that in fact reduces to (1.28) in the autonomous case. We refer to [36,40,57] for this definition. As for equations, nonuniform ellipticity originally stems from some remarkable models, as for instance the minimal surface equation. It is today a classical topic of its own in pde theory. Early work in this line include those of Ladyzhenskaya and Uraltseva [40,41], Hartman and Stampacchia [29], Trudinger [57,58], Ivanov [34,35], Ivočkina and A.P. Oskolkov [37]. In particular, Serrin’s landmark paper [56] has established important solvability and regularity results, and Ivanov’s monograph [36] contains a thoroughly exposition of the theory until the beginning of the eighties. These works deal with elliptic equations, also in nondivergence form, and feature a number of a priori estimates and existence theorems under various structural conditions.

The variational theory is probably the best setting where to frame nonuniform ellipticity, without knowing additional strucutures. In fact, when considering elliptic equations, there is always a problem in identifying the correct set of test functions one can use. The well-known difference between distributional and energy solutions becomes wider. In particular, when considering an elliptic equation of the type \( \text{div } a(Du) = 0 \), it is crucial to have the possibility to test the distributional form with functions that are proportional to the solution \( u \). In the case of \((p,q)\)-growth conditions, this is possible only when \( u \in W^{1,q} \), but this is not verified in general. This leads, when dealing with equations, to establish, simultaneously, existence and regularity theorems for solutions, as for instance shown in [4,45]. This ambiguity, widely discussed by Zhikov [60,61], is not present in the case of minimizers, where minimality allows to work out approximation procedures. These are of the type also considered in this paper. These approximations schemes are able to select all minimizers in the autonomous case (and all minimizers of the relaxed functional in nonautonomous one). Once combined with suitable a priori estimates, such approximation schemes lead to regularity results for minimizers.

After a first pioneering paper of Uraltseva and Urdaletova [59], functionals with \((p,q)\)-growth have been systematically investigated by Marcellini [44–46], whose work started what is a by now rather vast literature on the subject. For this, we refer to the surveys [48,49,52,53] for a reasonable overview. Typical examples of functionals with \((p,q)\)-growth are

\[
w \mapsto \int_\Omega \left[ |Dw|^p + \sum_{i=1}^n a_i(x)|D_i w|^{p_i} \right] \ dx ,
\]
where $1 \leq a_i(x) \leq L$ and $1 < p = p_1 \leq p_2 \leq \cdots \leq p_n = q$,

\[
\begin{align*}
  w \mapsto & \int_\Omega |Dw|^{p(x)} \, dx, \quad 1 < p \leq p(x) \leq q, \\
  w \mapsto & \int_\Omega |Dw|^{p(x)B(|Dw|)} \, dx, \quad 1 < p \leq p(x) \leq q - \gamma, \; 1 \leq B(|z|) \leq \gamma, \\
  w \mapsto & \int_\Omega |Dw|^{p(x)B(|Dw|)} \, dx,
\end{align*}
\]

where $1 < p \leq p(x) \leq q - q_1$ and $1 \leq B(|z|) \leq L(1 + |z|^{q_1})$. Functionals of the type in the above displays typically come up in applied settings, where standard growth conditions are not sufficient to catch important aspects in modelling. We refer for instance to applications in the theory of Homogenization [60–63], Image segmentation [31], Elasticity [43] and non-Newtonian Fluid mechanics. For more on applications we refer to [52, 53]. Moreover, connections emerge with abstract theory of functions spaces and Harmonic Analysis. It that case it happens that assumptions implying regularity of minimizers naturally connect with those implying good functional theoretic properties such as density of smooth functions and boundedness of maximal and integral operators. See [32] and [53, Sect. 6].

As emphasized above, the gap $q/p$ is a crucial quantity to measure the rate of nonuniform ellipticity, that is the growth of $\mathcal{R}_F(z)$ in (1.28). In this case we have growth conditions as in (1.4) and therefore we have polynomial nonuniform ellipticity. Gap bounds of the type in (1.7) are always essential, also in the parabolic case [9]. It is important to remark that such gap bound conditions play a basic role both in the scalar case $N = 1$, and in the vectorial one $N > 1$. In the latter, partial regularity comes into the play. This means that minimizers are proved to be regular only outside a closed, negligible subset, with (sometimes) the possibility to estimate the Hausdorff dimension of the singular set. Results of this type are available both in the convex case [15] and in the quasiconvex one [55]; see also the recent [54]. We refer to these last three papers for a list of references and again to [52] for an overview on partial regularity and singular set dimension reduction results. We note that a great deal of work has been devoted to find better bounds on $q/p$, amongst those of the type (1.7), starting with [44, 45]. Bounds under nonautonomous structures have been found and/or used in [8, 13, 18, 19, 30–32, 38]. Recent, interesting work using adapted cut-off functions has been realised in [5, 33, 54], allowing to get sharp bounds in the case one is interested in finding conditions implying boundedness of $u$ rather than that of $Du$. Better bounds for gradient $L^\infty$-estimates can be obtained as well. Improved bounds also occur under special structures [10].

Polynomial nonuniform ellipticity is not the only type of nonuniform ellipticity. Functionals with faster growth emerged over the years several times in the literature. A model prototype is provided by

\[
\begin{align*}
  w \mapsto & \int_\Omega \exp(|Dw|^p) \, dx, \quad p \geq 1.
\end{align*}
\]
Exponential growth functionals as in (1.29) are classical in the Calculus of Variations starting by the work of Duc and Eells [24], Lieberman [42], and Marcellini [46]. They are treated for instance in the setting of weak KAM-theory in [26]. Even faster growth conditions can be considered by allowing arbitrary compositions of exponentials, i.e.,

\[
w \mapsto \int_\Omega \exp(\exp(\ldots \exp(c(x)|Dw|^p)\ldots)) \, dx, \quad p \geq 1, \quad 0 < v \leq c(\cdot) \leq L.
\]

(1.30)

When considering the functional in (1.30) in the autonomous case \(c(\cdot) \equiv 1\), we have that [4, (6.13)]

\[
R_F(z) \lesssim t^{p-1} \exp(\exp(\ldots \exp(|z|^p)\ldots)) + 1
\]

(1.31)

where, if \(k \geq 1\) is the number of composing exponentials involved in (1.30), the number in (1.31) is \(k - 1\) (it is zero in the case of (1.29)). Regularity results for functionals as in (1.29) can be found in [4,46,47], to which we refer for further references. For the nonautonomous case, i.e., when \(c(\cdot)\) in (1.30) is not a constant, more delicate conditions emerge. A typical result in this setting claims that minimizers of (1.30) are locally Lipschitz provided \(c(\cdot) \in W^{1,d}\), for some \(d > n\). For this kind of theorems, we refer to [21].

2 Preliminaries

2.1 Notation

In the rest of the paper, we denote by \(\Omega \subset \mathbb{R}^n\) an open subset, \(n \geq 2\). We denote by \(c\) a general constant larger than one. Different occurrences from line to line will be still denoted by \(c\). Special occurrences will be denoted by \(c_1, c_2, \tilde{c}\) or likewise. When a relevant dependence on parameters occurs, this will be emphasized by putting the correspondent parameters in parentheses. Finally, the symbol \(\lesssim\) denotes inequalities where absolute constants are involved. As usual, we denote by \(B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}\), the open ball with center \(x_0\) and radius \(r > 0\); when it is clear from the context, we omit denoting the center, i.e., \(B_r \equiv B_r(x_0)\). When not otherwise stated, different balls in the same context will share the same center. We will also denote \(B_1 = B_1(0)\) if not differently specified. Finally, with \(B\) being a given ball with radius \(r\) and \(\sigma\) being a positive number, we denote by \(\sigma B\) the concentric ball with radius \(\sigma r\). In denoting several function spaces like \(L^p(\Omega), W^{1,p}(\Omega)\), we will denote the vector valued version by \(L^p(\Omega, \mathbb{R}^k), W^{1,p}(\Omega, \mathbb{R}^k)\) in the case the maps considered take values in \(\mathbb{R}^k, k \in \mathbb{N}\). Sometimes we will abbreviate \(L^p(\Omega, \mathbb{R}^k) \equiv L^p(\Omega), W^{1,p}(\Omega, \mathbb{R}^k) \equiv W^{1,p}(\Omega)\). With \(\mathcal{B} \subset \mathbb{R}^n\) being a measurable subset with bounded positive measure \(0 < |\mathcal{B}| < \infty\), and with \(w: \mathcal{B} \to \mathbb{R}^k\), being a measurable
map, we will denote the integral average of \( w \) over \( B \) by

\[
(w)_B \equiv \int_B w(x) \, dx := \frac{1}{|B|} \int_B w(x) \, dx.
\]

Given \( z, \xi \in \mathbb{R}^{N \times n} \), their Frobenius product is defined as \( z \cdot \xi = z_i^\alpha \xi_i^\alpha \); it follows that \( \xi \cdot \xi = |\xi|^2 \) and in the rest of the paper we will use the classical Frobenius norm for matrices. We will use a similar notation for the scalar product in similar operator acting on maps \( \omega \mapsto \int \omega \). We will denote the integral average of \( w \) defined as

\[
\int_{\Omega_1} \| \xi \|_p \, dx =: \| \omega \|_{L^p(A)} + [w]_{0, \gamma; B}.
\]

whenever \( B \subset \mathbb{R}^n \) is a subset, \( \gamma \in (0, 1) \) and \( w: B \to \mathbb{R}^k \). Accordingly, the \( C^{0, \gamma} \)-norm of \( w \) is defined by \( \| w \|_{C^{0, \gamma}(B)} := \| w \|_{L^\infty(A)} + [w]_{0, \gamma; B} \).

### 2.2 Fractional spaces and interpolation inequalities

We collect here some basic facts about fractional Sobolev spaces. We refer to \([23, 25]\) for more results. For a map \( w: \Omega \to \mathbb{R}^k \) and a vector \( h \in \mathbb{R}^n \), we denote by \( \tau_h : L^1(\Omega, \mathbb{R}^k) \to L^1(\Omega|_h, \mathbb{R}^k) \) the standard finite difference operator defined as

\[
(\tau_h w)(x) \equiv \tau_h w(x) := w(x + h) - w(x),
\]

whenever \( \Omega|_h := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > |h| \} \) is not empty. We will consider a similar operator acting on maps \( \Psi: \Omega \times \Omega \to \mathbb{R}^k \), this time defined by

\[
(\tilde{\tau}_h w)(x) \equiv \tilde{\tau}_h \Psi(x, y) := \Psi(x + h, y + h) - \Psi(x, y).
\]

**Definition 2** Let \( \alpha_0 \in (0, \infty) \setminus \mathbb{N}, p \in [1, \infty), k \in \mathbb{N}, n \geq 2, \) and let \( \Omega \subset \mathbb{R}^n \) be an open subset.

- If \( \alpha_0 \in (0, 1) \), the fractional Sobolev space \( W^{\alpha_0, p}(\Omega, \mathbb{R}^k) \) consists of those maps \( w: \Omega \to \mathbb{R}^k \) such that the following Gagliardo type norm is finite:

\[
\| w \|_{W^{\alpha_0, p}(\Omega)} := \left\| w \right\|_{L^p(\Omega)} + \left( \int_\Omega \int_\Omega \frac{|w(x) - w(y)|^p}{|x - y|^{n + \alpha_0 p}} \, dx \, dy \right)^{1/p}.
\]

In the case \( \alpha_0 = [\alpha_0] + \{\alpha_0\} \in \mathbb{N} + (0, 1) > 1 \), we have \( w \in W^{\alpha_0, p}(\Omega, \mathbb{R}^k) \) iff

\[
\| w \|_{W^{\alpha_0, p}(\Omega)} := \left\| w \right\|_{W^{[\alpha_0], p}(\Omega)} + \left[ D^{[\alpha_0]} w \right]_{[\alpha_0], p; \Omega}.
\]
is finite. The local variant $W^{\alpha_0, p}_{\text{loc}}(\Omega, \mathbb{R}^k)$ is defined by requiring that $w \in W^{\alpha_0, p}_{\text{loc}}(\Omega, \mathbb{R}^k)$ if $w \in W^{\alpha_0, p}(\Omega, \mathbb{R}^k)$ for every open subset $\bar{\Omega} \Subset \Omega$.

- For $\alpha_0 \in (0, 1]$, the Nikol’skii space $N^{\alpha_0, p}(\Omega, \mathbb{R}^k)$ is defined by prescribing that $w \in N^{\alpha_0, p}(\Omega, \mathbb{R}^k)$ if and only if

$$\|w\|_{N^{\alpha_0, p}(\Omega, \mathbb{R}^k)} := \|w\|_{L^p(\Omega, \mathbb{R}^k)} + \left( \sup_{|h| \neq 0} \int_{\Omega_\beta} \frac{|w(x + h) - w(x)|^p}{|h|^\alpha_0} \, dx \right)^{1/p}.$$

The local variant $N^{\alpha_0, p}_{\text{loc}}(\Omega, \mathbb{R}^k)$ is defined analogously to $W^{\alpha_0, p}_{\text{loc}}(\Omega, \mathbb{R}^k)$.

We have that $W^{\alpha_0, p}(\Omega, \mathbb{R}^k) \subseteq N^{\alpha_0, p}(\Omega, \mathbb{R}^k) \subseteq W^{\beta, p}(\Omega, \mathbb{R}^k)$, for every $\beta < \alpha_0$, hold for sufficiently domains $\Omega$. These inclusions are somehow quantified in the following lemma

**Lemma 1** Let $w \in L^p(\Omega)$, $p \geq 1$, and assume that for $\alpha_0 \in (0, 1]$, $S \geq 0$ and an open and bounded set $\bar{\Omega} \Subset \Omega$ we have that $\|\tau_h w\|_{L^p(\bar{\Omega})} \leq S|\alpha_0| \alpha_0$ holds for every $h \in \mathbb{R}^n$ satisfying $0 < |h| \leq d$, where $0 < d \leq \text{dist}(\bar{\Omega}, \partial \Omega)$. Then $w \in W^{\beta, p}(\bar{\Omega}, \mathbb{R}^k)$ for every $\beta \in (0, \alpha_0)$, and the estimate

$$\|w\|_{W^{\beta, p}(\bar{\Omega})} \leq c \left( \frac{d(\alpha_0 - \beta)}{(\alpha_0 - \beta)^{1/p}} + \frac{\|w\|_{L^p(\bar{\Omega})}}{\min\{d^{n/p + \beta}, 1\}} \right) \quad (2.5)$$

holds with $c \equiv c(n, p)$. In particular, let $B_2 \Subset B_r \subset \mathbb{R}^n$ be concentric balls with $r \leq 1$, $w \in L^p(B_r, \mathbb{R}^k)$, $p > 1$ and assume that, for $\alpha_0 \in (0, 1]$, $S \geq 1$, there holds

$$\|\tau_h w\|_{L^p(B_2, \mathbb{R}^k)} \leq S|\alpha_0| \alpha_0 \text{ for every } h \in \mathbb{R}^n \text{ with } 0 < |h| \leq \frac{r - Q}{K}, \text{ where } K \geq 1 \quad (2.6)$$

Then it holds that

$$\|w\|_{W^{\beta, p}(B_2, \mathbb{R}^k)} \leq \frac{c}{(\alpha_0 - \beta)^{1/p}} \left( \frac{r - Q}{K} \right)^{\alpha_0 - \beta} S + c \left( \frac{K}{r - Q} \right)^{n/p + \beta} \|w\|_{L^p(B_r, \mathbb{R}^k)} \quad (2.7)$$

where $c \equiv c(n, p)$.

**Proof** A main point in Lemma 1, is the precise quantitative linkage between the size of $|h|$ appearing in (2.6), and the dependence on the constants appearing in (2.7). This will be crucial in the applications we will make of it. See Sects. 3.9 and 3.10 below. For this reason we decide to report the easy proof, since it does not appear in the literature.
but is often reported as folklore. Fubini’s theorem yields
\[
\int_\Omega \int_{\Omega \cap \{|x-y|<d\}} \frac{|w(x) - w(y)|^p}{|x-y|^{n+p\beta}} \, dx \, dy \
\leq \sup_{0<|h|\leq d} \int_{\Omega} \frac{|w(x+h) - w(x)|^p}{|h|^{\alpha_0 p}} \, dx \int_{B_d(0)} \frac{dh}{|h|^{n+(\beta-\alpha_0) p}} 
\leq c S^p \int_0^d \frac{dt}{t^{1+(\beta-\alpha_0) p}} \leq \frac{cd^{(\alpha_0-\beta)p} S^p}{(\alpha_0-\beta)p}.
\]

On the other hand we have that
\[
\int_\Omega \int_{\Omega \cap \{|x-y|\geq d\}} \frac{|w(x) - w(y)|^p}{|x-y|^{n+p\beta}} \, dx \, dy 
\leq \frac{2^{p-1}}{d^{n+p\beta}} \int_{B_d(0)} \left( |w(x)|^p + |w(y)|^p \right) \, dx \, dy = \frac{2^p \|w\|_{L^p(\Omega)}^{p}}{d^{n+p\beta}}.
\]

Connecting the estimates in the last two displays yields
\[
\int_\Omega \int_{\Omega} \frac{|w(x) - w(y)|^p}{|x-y|^{n+p\beta}} \, dx \, dy 
\leq c \left( d^{(\alpha_0-\beta)q} S^p \frac{\|w\|_{L^p(\Omega)}^{p}}{d^{n+p\beta}} + \frac{\|w\|_{L^p(\Omega)}^{p}}{d^{n+p\beta}} \right)
\]
with \(c \equiv c(n)\), from which the full inequality in (2.5) immediately follows. \(\Box\)

Recalling notations (2.1) and (2.4), we now report a suitable version of the fractional Sobolev-Morrey embedding.

**Lemma 2** Let \(w \in W^{s,t}(\mathbb{R}^n, \mathbb{R}^k)\), with \(t \geq 1\), \(s \in (0, 1)\) such that \(st > n\). If \(B \subset \mathbb{R}^n\) is a ball, then \(w \in C^{0,s-n/t}(B, \mathbb{R}^k)\) and the inequality
\[
[w]_{0,s-n/t;B} \leq c[w]_{s,t;\mathbb{R}^n}, \tag{2.8}
\]
holds for a constant \(c\), depending only on \(n, s, t\).

**Proof** This easily follows from the standard proofs in the literature; just a comment on the dependence of the constant \(c\). First observe that one can reduce the proof of (2.8) to the case when \(B \equiv B_1(0)\) via a standard scaling argument; i.e., considering \(\tilde{u}(x) := R^{-s+n/t} u(x_0 + Rx)\), where \(B = B_R(x_0)\). Once the scaling is done, we can use [23, (8.8)] and the fractional Poincaré inequality to get (2.8) with an absolute constant depending only on \(n, s, t\). \(\Box\)

The following is a Gagliardo-Nirenberg type inequality in fractional Sobolev spaces, taken from [17, Sect. 2.3]. The proof relies on a localization argument combined with the global inequalities from [11, Lemma 1, pag. 329]; see also [12,50]. To get the explicit dependence of the constants by the radii \(1/(r - \varrho)^k\) below, it is sufficient to trace back the dependence on the various constants in [17, Lemma 2.6].
Lemma 3 Let \( B_\varrho \Subset B_r \Subset \mathbb{R}^n \) be concentric balls with \( r \leq 1 \). Let \( 0 \leq s_1 < 1 < s_2 < 2 \), \( 1 < a, t < \infty \), \( \tilde{p} > 1 \) and \( \theta \in (0, 1) \) be such that

\[
1 = \theta s_1 + (1 - \theta) s_2, \quad \frac{1}{\tilde{p}} = \frac{\theta}{a} + \frac{1 - \theta}{t}.
\]

Then every function \( w \in W^{s_1,a}(B_r) \cap W^{s_2,t}(B_r) \) belongs to \( W^{1,\tilde{p}}(B_\varrho) \) and the inequality

\[
\|Dw\|_{L^{\tilde{p}}(B_\varrho)} \leq \frac{c}{(r - \varrho) \kappa} \|w\|_{s_1,a;B_r}^{\theta} \|Dw\|_{W^{s_2-1,t}(B_r)}^{1-\theta},
\]

holds for constants \( c, \kappa \equiv c, \kappa(n, s_1, s_2, a, t) \).

Next, a classical iteration lemma of [28, Lemma 6.1], that is

Lemma 4 Let \( \mathcal{Z} : [\varrho, r] \to \mathbb{R} \) be a nonnegative and bounded function, and let \( \theta \in (0, 1) \) and \( A \geq 0, \gamma_1 \geq 0 \) be numbers. Assume that

\[
\mathcal{Z}(t) \leq \theta \mathcal{Z}(s) + \frac{A}{(s - t)^{\gamma}}
\]

holds for \( \varrho \leq t < s \leq r \). Then the following inequality holds with \( c \equiv c(\theta, \gamma_1) \):

\[
\mathcal{Z}(\varrho_0) \leq \frac{cA}{(r - \varrho)^{\gamma}}.
\]

3 Proof of Theorems 1, 2 and Corollaries 1, 2

These proofs take twelve different steps. The first ten are dedicated to the proof of Theorem 1. In Step 11 we deal with Corollaries 1, 2 and Step 12 is dedicated to Theorem 2.

3.1 Step 1: Choice of parameters

Fix \( q \) as in (1.18). We start considering parameters \( s, d \) and \( \beta \) initially satisfying

\[
0 \leq s < \gamma, \quad 2d > \max\{q, n\}, \quad 0 < \beta < \min\{\alpha, 2\gamma\}.
\]

Accordingly, we define the function \( \tilde{p} \equiv \tilde{p}(s, d, \beta) \) as

\[
\tilde{p} := \frac{2d [p(1-s)+\beta]}{\beta+2d(1-s)} \text{ if } p \geq 2 \quad \text{and} \quad \tilde{p} := \frac{2dp[2(1-s)+\beta]}{p\beta+4d(1-s)} \text{ if } 1 < p < 2.
\]
Note that the inequality $d > p/2$, which is true by (3.1), in particular implies

$$\tilde{p} < 2d.$$  \hfill (3.3)

The function $\tilde{p}$ is increasing in all its variables (we again use that $d \geq p/2$ for this), with

$$\lim_{s \to \gamma, d \to \infty, \beta \to \min\{\alpha, 2\gamma\}} \tilde{p}(s, d, \beta) = p + \frac{\min\{\alpha, 2\gamma\}}{\vartheta(1 - \gamma)},$$  \hfill (3.4)

where $\vartheta$ has been defined in (1.16). In the following, we always take $d$ such that

$$d \geq \max\{q, n\}$$

In view of (3.4) we further increase both $s$ and $d$, while still keeping (3.5), and find $\beta$ such that

$$\beta < \alpha_0 := \min\{\alpha, 2\beta_0\} < \min\{\alpha, 2\gamma\}$$ \hfill (3.6)

and

$$q \leq q < \tilde{p}(s, d, \beta) < p + \frac{\min\{\alpha, 2\gamma\}}{\vartheta(1 - \gamma)}. \hfill (3.7)$$

Note that, by further increasing $s, d, \beta$ still in the range fixed in (3.1), conditions (3.5)–(3.7) still hold (as noted before, $\tilde{p}(s, d, \beta)$ is increasing with respect to all its variables). Keeping this in mind, we next distinguish two cases. When $p \geq 2$, note that

$$\lim_{s \to \gamma, d \to \infty, \beta \to \min\{\alpha, 2\gamma\}} \frac{\tilde{p}(s, d, \beta)(q - p)}{\tilde{p}(s, d, \beta) - p} = \frac{1 - s}{p(1 - s) + \beta} = \lim_{s \to \gamma, d \to \infty, \beta \to \min\{\alpha, 2\gamma\}} \frac{2d(q - p)(1 - s)}{\beta(2d - p)} = \frac{(q - p)(1 - \gamma)}{\min\{\alpha, 2\gamma\}}. \hfill (1.16)$$

Therefore, we finally again increase $s, d$ and $\beta$ again, in order to have

$$\frac{\tilde{p}(s, d, \beta)(q - p)}{\tilde{p}(s, d, \beta) - p} = \frac{1 - s}{p(1 - s) + \beta} < 1.$$ \hfill (3.8)

When instead $1 < p < 2$, we note that

$$\lim_{s \to \gamma, d \to \infty, \beta \to \min\{\alpha, 2\gamma\}} \frac{\tilde{p}(s, d, \beta)(q - p)}{\tilde{p}(s, d, \beta) - p} = \frac{2(1 - s)}{p(2(1 - s) + \beta)}$$

$$\lim_{s \to \gamma, d \to \infty, \beta \to \min\{\alpha, 2\gamma\}} \frac{4d(q - p)(1 - s)}{p\beta(2d - p)} = \frac{2(q - p)(1 - \gamma)}{p\min\{\alpha, 2\gamma\}} < 1.$$

Note that the inequality $d > p/2$, which is true by (3.1), in particular implies

$$\tilde{p} < 2d.$$  \hfill (3.3)
and therefore we again find $s$, $d$, and $\beta$ such that

$$\frac{\tilde{p}(s, d, \beta)(q - p)}{\tilde{p}(s, d, \beta) - p} \frac{2(1 - s)}{p[2(1 - s) + \beta]} < 1.$$  \hspace{0.5cm} (3.9)

From now on we will always consider this final choice of $s$, $d$ and $\beta$, and therefore we will use (3.2)–(3.9) for the rest of the proof. In view of this, from now on we will express any dependency from $(s, d, \beta)$ as a dependence on $(p, q, \alpha, \gamma, q)$. In the following, in order to shorten the notation, we will denote

$$\text{data} \equiv (n, N, p, q, \nu, L, \alpha, \gamma, q), \quad \text{data}_e \equiv (n, p, q, \alpha, \gamma, q). \hspace{0.5cm} (10.10)$$

### 3.2 Step 2: Hölder and Sobolev extensions

Let us note that in the definition 1.12 we can replace $C_{H, \gamma}(w, B)$ in (1.13) by

$$\{w_j \in W^{1,\infty}(B, \mathbb{R}^N); (w_j)_B = 0, w_j \rightharpoonup w \text{ in } W^{1,p}(B, \mathbb{R}^N), \sup \|w_j\|_{0,\gamma; B} \leq H \}.$$  \hspace{0.5cm} (11.1)

Such a replacement leaves the values of $\tilde{F}_{H, \gamma}$ and $\mathcal{L}_{F, H, \gamma}$ unaltered. Keeping this fact in mind, we fix a ball $B_r \subset B_{2r} \Subset \Omega$ with $r \leq 1$ and such that (1.17) holds, and, by Proposition 1, we can find a sequence

$$\{\tilde{u}_j \} \subset W^{1,\infty}(B_r, \mathbb{R}^N) \cap C^{0,\gamma}(B_r, \mathbb{R}^N), \quad (\tilde{u}_j)_{B_r} = 0 \hspace{0.5cm} (11.11)$$

such that, eventually passing to a not relabelled subsequence, it holds that

$$\begin{align*}
\tilde{u}_j &\rightharpoonup u \text{ weakly in } W^{1,p}(B_r, \mathbb{R}^N), \quad \tilde{u}_j \rightarrow u \text{ strongly in } L^p(B_r, \mathbb{R}^N) \\
\mathcal{F}(\tilde{u}_j, B_r) &\rightarrow \mathcal{F}(u, B_r) \hspace{0.5cm} (12.1)
\end{align*}$$

$$\begin{align*}
\nu \|[\tilde{u}_j]_{L^{\infty}(B_r)} &+ [\tilde{u}_j]_{0,\gamma; B_r} + r^{-\gamma-n/p} \|\tilde{u}_j\|_{L^p(B_r)} \leq cH \\
\nu \|D\tilde{u}_j\|_{L^p(B_r)} &\leq \mathcal{F}(\tilde{u}_j, B_r) \leq \mathcal{F}(u, B_r) + 1
\end{align*}$$

with the last two lines that hold for every $j \geq 1$, and where $c \equiv c(n, p)$ is an absolute constant. All the facts from (11.11) are directly coming from Proposition 1 but (12.3), that maybe deserves a few words; there only $[\tilde{u}_j]_{0,\gamma; B_r} \leq H$ directly comes from the definition of the Lavrentiev gap. For this, note that, if $x \in B_r$, then by (11.11) we find, by Jensen’s inequality, that $|\tilde{u}_j(x)| = |\tilde{u}_j(x) - (\tilde{u}_j)_{B_r}| \leq c\gamma [\tilde{u}_j]_{0,\gamma; B_r} \leq c\gamma H$. This implies (12.3). Now, by [51, Theorem 2], we can extend the $\tilde{u}_j$’s to the whole $\mathbb{R}^n$ by determining maps $\{\tilde{u}_j\}$ such that

$$[\tilde{u}_j]_{0,\gamma; \mathbb{R}^n} \leq c[\tilde{u}_j]_{0,\gamma; B_r} \leq cH \hspace{0.5cm} (13.3)$$

for an absolute constant $c$ which is independent of $j$. Let $\bar{\eta} \in C^1_0(B_{3r/2})$ be such that

$$1_{\bar{B}_r} \leq \bar{\eta} \leq 1_{\bar{B}_{3r/2}} \quad \text{and} \quad |D\bar{\eta}| \lesssim 1/r \hspace{0.5cm} (14.3)$$
and set \( \tilde{v}_j := \tilde{u}_j \eta \). By this very definition, (3.13) and (3.14) we have that

\[
\begin{aligned}
&\left\{ \begin{array}{l}
  r^{-\gamma} \| \tilde{v}_j \|_{L^\infty(\mathbb{R}^n)} \leq c_n H \\
  [\tilde{v}_j]_{0,\gamma;\mathbb{R}^n} \leq c_n H \\
  \tilde{v}_j \equiv \tilde{u}_j \equiv \tilde{u}_j \text{ in } B_r \\
  \text{supp } \tilde{v}_j \subset B_{3r/2} \\
  \tilde{v}_j \in W^{1,2d}(B_r, \mathbb{R}^N) \\
  r^2 [\tilde{v}_j]_{s,2d;\mathbb{R}^n} \leq c_n r^{\gamma+n/(2d)} H
\end{array} \right. \\
(3.15)
\end{aligned}
\]

hold for every \( j \in \mathbb{N} \), an absolute constant \( c_n \equiv c_n(n, d, \gamma, s) \equiv c_n(\text{data}_c) \), which is independent of \( j \). We confine ourselves to sketch the simple proofs of (3.15) and of (3.16), the other assertions being a direct consequence of the definition of \( \tilde{v}_j \) (recall also (3.11)). Let us first remark that, triangle inequality, (3.13) and finally (3.12), imply

\[
\| \tilde{u}_j \|_{L^\infty(B_{2r})} \leq c [\tilde{u}_j]_{0,\gamma;B_r} r^{\gamma} + \| \bar{u}_j \|_{L^\infty(B_r)} \leq c H r^{\gamma},
\]

that is (3.15)\(_1\). In order to prove (3.15)\(_2\), it is sufficient to note that if \( x, y \in B_{3r/2} \), then, using (3.13) and (3.16) it follows

\[
\begin{aligned}
|\tilde{v}_j(y) - \tilde{v}_j(x)| &= |\tilde{u}_j(y)\eta(y) - \tilde{u}_j(x)\eta(x)| \\
&\leq |\tilde{u}_j(y) - \tilde{u}_j(x)| + |\eta(y) - \eta(x)| \| \tilde{u}_j \|_{L^\infty(B_{2r})} \\
&\leq c [\tilde{u}_j]_{0,\gamma;B_{2r}} |x - y| + |D\eta| \| L^\infty(B_r) \| \| \tilde{u}_j \|_{L^\infty(B_{2r})} |x - y| \\
&\leq c H |x - y| + c \| \tilde{u}_j \|_{L^\infty(B_{2r})} |x - y|^{\gamma} \\
&\leq c_n H |x - y|^{\gamma}.
\end{aligned}
\]

As it is \( v_j \equiv 0 \) outside \( B_{3r/2} \), this is sufficient to conclude with (3.15)\(_2\). Finally, the proof of (3.15)\(_6\). By (3.15)\(_1,2,3,4\), we have

\[
\begin{aligned}
[\tilde{v}_j]_{s,2d;\mathbb{R}^n}^2 &= 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} \frac{|\tilde{v}_j(x) - \tilde{v}_j(y)|^{2d}}{|x - y|^{n+2sd}} \, dx \, dy \\
&+ \int_{B_{2r}} \int_{B_{2r}} \frac{|\tilde{v}_j(x) - \tilde{v}_j(y)|^{2d}}{|x - y|^{n+2sd}} \, dx \, dy \\
&= 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{3r/2}} \frac{|\tilde{v}_j(x)|^{2d}}{|x - y|^{n+2sd}} \, dx \, dy \\
&+ \int_{B_{2r}} \int_{B_{2r}} \frac{|\tilde{v}_j(x) - \tilde{v}_j(y)|^{2d}}{|x - y|^{n+2sd}} \, dx \, dy \\
&\leq c \int_{\mathbb{R}^n \setminus B_{3r/2}} \frac{dz}{|z|^{n+2sd}} \| \tilde{v}_j \|_{L^2d(B_{3r/2})}^{2d} dx \, dy \\
&+ c [\tilde{v}_j]_{0,\gamma;B_{2r}} \int_{B_{2r}} \int_{B_{2r}} \frac{dx \, dy}{|x - y|^{n+2sd(\gamma - \gamma)}}.
\end{aligned}
\]
\[
\leq \frac{cr^{n+2d(\gamma-s)}H^{2d}}{s(\gamma-s)} \equiv cr^{n+2d(\gamma-s)}H^{2d} \quad (3.17)
\]

for \( c \equiv c(data) \), and the proof of (3.15)_6 follows.

### 3.3 Step 3: Approximation via nonlocal functionals

We introduce the nonlocal Dirichlet class
\[
\mathbb{X}(\bar{v}_j, B_r) := \left\{ v \in \left( \bar{v}_j + W^{1.2d}_0(B_r, \mathbb{R}^N) \right) \cap W^{s,2d}(\mathbb{R}^n, \mathbb{R}^N) : v \equiv \bar{v}_j \text{ on } \mathbb{R}^n \setminus B_r \right\}.
\]

This is a convex, closed subset of \( W^{1.2d}_0(B_r, \mathbb{R}^N) \cap W^{s,2d}(\mathbb{R}^n, \mathbb{R}^N) \), and it is non-empty, as \( \bar{v}_j \in \mathbb{X}(\bar{v}_j, B_r) \) by (3.15). Next, we define \( u_j \in \mathbb{X}(\bar{v}_j, B_r) \) as the solution to
\[
u_j \mapsto \min_{w \in \mathbb{X}(\bar{v}_j, B_r)} \mathcal{F}_j(w, B_r), \quad (3.18)
\]
where, keeping in mind the notation in (1.26), it is
\[
\mathcal{F}_j(w, B_r) := \mathcal{F}(w, B_r) + \varepsilon_j \int_{B_r} (|Dw|^2 + \mu^2)^d \, dx
+ \int_{B_r} (|w|^2 - M_0^2)^d_+ \, dx
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(|w(x) - w(y)|^2 - M^2|x - y|^{2d})^d}{|x - y|^{n+2sd}} \, dx \, dy \quad (3.19)
\]
with
\[
\begin{align*}
\varepsilon_j &:= \frac{1}{\left( \|D\bar{v}_j\|_{L^{2d}(B_r)}^4 + j + 1 \right)} \\
M_0 &:= 16c_* r^{\gamma} H, \quad M := 16c_* H. \quad (3.20)
\end{align*}
\]

In (3.20) \( c_* \equiv c_*(data) \) is the same (absolute) constant has been defined in (3.15). Let us briefly point out how Direct Methods of the Calculus of Variations apply here to get the existence of \( u_j \) in (3.18); the main point is essentially to prove that the functional is coercive on \( \mathbb{X}(\bar{v}_j, B_r) \). With \( j \in \mathbb{N} \) being fixed, let us consider a minimizing sequence \( \{w_{j,k}\}_k \subset \mathbb{X}(\bar{v}_j, B_r) \), i.e., such that
\[
\lim_{k} \mathcal{F}_j(w_{j,k}, B_r) = \inf_{w \in \mathbb{X}(\bar{v}_j, B_r)} \mathcal{F}_j(w, B_r). \quad (3.21)
\]
We now have (recall that \( w_{j,k} \equiv \tilde{v}_j \) outside \( B_r \), \( \tilde{v}_j \equiv 0 \) outside \( B_{3r/2} \) and \( \tilde{v}_j \equiv \tilde{u}_j \) in \( B_r \))

\[
\| w_{j,k} \|_{L^2(\mathbb{R}^n)}^{2d} \leq \| \tilde{v}_j \|_{L^2(\mathbb{R}^n)}^{2d} + \| w_{j,k} \|_{L^2(B_r)}^{2d} \\
\leq c r^{n+2d} \gamma^{2d} + \| w_{j,k} - \tilde{v}_j \|_{L^2(\mathbb{R}^n)}^{2d} \\
\leq c r^{n+2d} \gamma^{2d} + \| D w_{j,k} \|_{L^2(B_r)}^{2d} + \| D \tilde{v}_j \|_{L^2(\mathbb{R}^n)}^{2d} \\
\leq \frac{c r^{2d} |\mathcal{F}_j(w_{j,k}, B_r)|}{\varepsilon_j} + c r^{2d} \| D \tilde{u}_j \|_{L^2(B_r)}^{2d} + c r^{n+2d \gamma} \gamma^{2d} \quad (3.22)
\]
for \( c \equiv c(\text{data}_0) \); in the second estimate in the above display we have used also (3.15)\(_1\) and in the last line the very definition of \( \mathcal{F}_j \) from (3.19). Furthermore, triangle inequality implies

\[
[w_{j,k}]_{2d; B_{2r}}^{2d} \leq c \int_{B_{2r}} \int_{B_{2r}} \frac{|w_{j,k}(x) - w_{j,k}(y)|^2 - M^2 |x - y|^{2d}}{|x - y|^{n+2sd}} \, dx \, dy \\
+ c M^{2d} \int_{B_{2r}} \int_{B_{2r}} \frac{|x - y|^{n-2d(\gamma - s)}}{|x - y|^{n+2sd}} \, dx \, dy \\
\leq c |\mathcal{F}_j(w_{j,k}, B_r)| + c M^{2d} r^{n+2d(\gamma - s)} \\
\leq c |\mathcal{F}_j(w_{j,k}, B_r)| + c r^{n+2d(\gamma - s)} \gamma^{2d} \,, \quad (3.23)
\]
where \( c \equiv c(\text{data}_0) \) and we have used the definition of \( M \) from (3.20)\(_2\) in the last line. In turn, using (3.15)\(_1\), and (3.22)-(3.23), we have

\[
[w_{j,k}]_{2d; \mathbb{R}^n}^{2d} = 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} \frac{|w_{j,k}(x) - w_{j,k}(y)|^2}{|x - y|^{n+2sd}} \, dx \, dy + [w_{j,k}]_{2d; B_{2r}}^{2d} \\
\leq 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{3r/2} \setminus B_r} \frac{|\tilde{v}_j(x)|^2}{|x - y|^{n+2sd}} \, dx \, dy \\
+ 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{r}} \frac{|w_{j,k}(x)|^2}{|x - y|^{n+2sd}} \, dx \, dy \\
+ c |\mathcal{F}_j(w_{j,k}, B_r)| + c r^{n+2d(\gamma - s)} \gamma^{2d} \\
\leq c \int_{\mathbb{R}^n \setminus B_{r/2}} \frac{dz}{|z|^{n+2sd}} \| \tilde{v}_j \|_{L^2(\mathbb{R}^n \setminus B_{r/2}; \mathbb{R}^n)}^{2d} + \int_{\mathbb{R}^n \setminus B_{r/2}} \frac{dz}{|z|^{n+2sd}} \| w_{j,k} \|_{L^2(\mathbb{R}^n \setminus B_{r/2}; \mathbb{R}^n)}^{2d} \\
+ c |\mathcal{F}_j(w_{j,k}, B_r)| + c r^{n+2d(\gamma - s)} \gamma^{2d} \\
\leq c |\mathcal{F}_j(w_{j,k}, B_r)| + c r^{2d(1 - s)} \| D \tilde{u}_j \|_{L^2(B_r)}^{2d} + c r^{n+2d(\gamma - s)} \gamma^{2d} \\
(3.24)
\]
for \( c \equiv c(\text{data}) \). Using the content of (3.22)–(3.24), we find (recall it is \( r \leq 1 \))

\[
\| w_{j,k} \|_{W^{1,2d}(B_r)} + \| w_{j,k} \|_{W^{s,2d}(\mathbb{R}^n)} \leq \frac{c[F_j(w_{j,k}, B_r)]^{1/(2d)}}{\varepsilon_j^{1/(2d)}} + c \| D\tilde{u}_j \|_{L^{2d}(B_r)} + cH.
\]

This last estimate and (3.21) imply that the sequence \( \{w_{j,k}\}_k \) is bounded in \( W^{1,2d}(B_r, \mathbb{R}^N) \cap W^{s,2d}(\mathbb{R}^n, \mathbb{R}^N) \) and therefore, up to a not relabelled subsequence, we can assume that \( w_{j,k} \rightharpoonup u_j \) weakly as \( k \to \infty \), both in \( W^{1,2d}(B_r, \mathbb{R}^N) \) and \( W^{s,2d}(\mathbb{R}^n, \mathbb{R}^N) \), for some \( u_j \in X(\tilde{v}_j, B_r) \). Moreover, again up to a diagonalization argument, we can also assume \( w_{j,k} \rightharpoonup u_j \) a.e. At this point we use lower semicontinuity. Specifically, we use the convexity of \( z \mapsto F(x, z) \) and \( z \mapsto |z|^2d \) to deal with the local part, and Fatou’s lemma to deal with the nonlocal one, in order to get

\[
F_j(u_j, B_r) \leq \liminf_{k \to \infty} F_j(w_{j,k}, B_r),
\]

thereby proving the minimality in (3.18) accordingly to the usual Direct Methods of the Calculus of Variations (see for instance [28]).

3.4 Step 4: Convergence to \( u \)

Here we prove that, up to a non relabelled subsequence, \( \{u_j\} \) weakly converges to the original minimizer \( u \) in \( W^{1,p}(B_r, \mathbb{R}^N) \). For this, we start observing that (3.20)_1 guarantees that

\[
\varepsilon_j \int_{B_r} \left( |D\tilde{v}_j|^2 + \mu^2 \right)^d \, dx \to 0. \tag{3.25}
\]

Moreover, using (3.15)_2 and the definition of \( M \) in (3.20)_2, we have that \( |\tilde{v}_j(x) - \tilde{v}_j(y)| \leq M|x - y|^\gamma/4 \), holds for every \( j \in \mathbb{N} \) and \( x, y \in \mathbb{R}^n \). In turn, this implies

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(|\tilde{v}_j(x) - \tilde{v}_j(y)|^2 - M^2|x - y|^{2\gamma})^d}{|x - y|^{n+2\gamma d}} \, dx \, dy = 0
\]

for every \( j \in \mathbb{N} \). By (3.15)_3 and the definition of \( M_0 \) in (3.20)_2, it is

\[
\int_{B_r} \left( |\tilde{v}_j|^2 - M_0^2 \right)^d \, dx = 0.
\]

Using the information in the last two displays, recalling the definition of \( F_j \) in (3.19), and also using (3.12)_2, (3.15)_3 and (3.25), we find

\[
\lim_{j \to \infty} F_j(\tilde{v}_j, B_r) = \lim_{j \to \infty} F(\tilde{u}_j, B_r) + \lim_{j \to \infty} \varepsilon_j \int_{B_r} \left( |D\tilde{u}_j|^2 + \mu^2 \right)^d \, dx = F(u, B_r).
\]
Minimality of $u_j$, i.e., $\mathcal{F}_j(u_j, B_r) \leq \mathcal{F}_j(\tilde{v}_j, B_r)$, and the above display, then give
\[
\limsup_{j \to \infty} \mathcal{F}_j(u_j, B_r) \leq \mathcal{F}(u, B_r), \tag{3.26}
\]
and therefore, up to relabelling, we can assume that $\mathcal{F}_j(u_j, B_r) \leq \mathcal{F}(u, B_r) + 1$ holds for every $j \in \mathbb{N}$. This and (1.11) imply
\[
v \int_{B_r} |Du_j|^p \, dx + \varepsilon_j \int_{B_r} \left( |Du_j|^2 + \mu^2 \right)^d \, dx \leq \mathcal{F}_j(u_j, B_r) \leq \mathcal{F}(u, B_r) + 1. \tag{3.27}
\]
Poincaré inequality and (3.12) yield
\[
\|u_j\|_{L^p(B_r)} \leq \|u_j - \tilde{u}_j\|_{L^p(B_r)} + \|\tilde{u}_j\|_{L^p(B_r)} \\
\leq cr\|Du_j\|_{L^p(B_r)} + cr\|D\tilde{u}_j\|_{L^p(B_r)} + cr^{n/p+\gamma} H \\
\leq c[\mathcal{F}(u, B_r) + 1]^{1/p} + cr^{n/p} H. \tag{3.28}
\]
Therefore, up to a non relabelled sequence, we can assume that
\[
u_j \rightharpoonup v \quad \text{in} \quad W^{1,p}(B_r, \mathbb{R}^N), \quad \text{for some } v \in u + W_0^{1,p}(B_r, \mathbb{R}^N). \tag{3.29}
\]
Continuity and (3.26) then imply
\[
\mathcal{F}(v, B_r) \leq \liminf_{j \to \infty} \mathcal{F}(u_j, B_r) \leq \limsup_{j \to \infty} \mathcal{F}_j(u_j, B_r) \leq \mathcal{F}(u, B_r), \tag{3.30}
\]
In turn, as $u - v \in W_0^{1,p}(B_r, \mathbb{R}^N)$, the minimality of $u$ renders that $\mathcal{F}(u, B_r) \leq \mathcal{F}(v, B_r)$ and so $\mathcal{F}(u, B_r) = \mathcal{F}(v, B_r)$. The strict convexity of $z \mapsto F(\cdot, z)$ implied by (1.11) leads to
\[
u = u \quad \text{almost everywhere on } B_r. \tag{3.31}
\]
This means that (3.30) now becomes
\[
\mathcal{F}(u, B_r) \leq \liminf_{j \to \infty} \mathcal{F}(u_j, B_r) \leq \limsup_{j \to \infty} \mathcal{F}_j(u_j, B_r) \leq \mathcal{F}(u, B_r),
\]
so that, recalling the definition of $\mathcal{F}_j$ in (3.19), we gain
\[
\lim_{j \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u_j(x) - u_j(y)|^2 - M^2|x - y|^{2\gamma}}{|x - y|^{n+2\alpha d}} \, dx \, dy \\
= \lim_{j \to \infty} \int_{B_r} \left( |u_j|^2 - M_0^2 \right)^d \, dx = 0.
\]
Up to a not relabelled subsequence, we can therefore assume that

\[
\begin{align*}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(|u_j(x) - u_j(y)|^2 - M^2|x - y|2\gamma)^d}{|x - y|^{n+2s d}} \, dx \, dy 
& \leq cr^{n+2d(\gamma-s)} H^{2d} \\
\int_{B_r} \left( |u_j|^2 - M_0^2 \right)^d \, dx 
& \leq r^{n+2\gamma d} H^{2d}
\end{align*}
\]  

(3.32)

hold for all \( j \in \mathbb{N} \). Recalling the definition of \( M_0 \) in (3.20)\(_2\), a direct consequence of (3.32)\(_2\) is

\[
\|u_j\|_{L^{2d}(B_r)}^{2d} \leq cr^{n+2\gamma d} H^{2d}.
\]

(3.33)

Then, since \( u_j \equiv \tilde{v}_j \) outside \( B_r \), as \( u_j \in \mathcal{X}(\tilde{v}_j, B_r) \), and since in turn \( v_j \equiv 0 \) outside \( B_{3r/2} \), by (3.33) and (3.15)\(_1\) we get

\[
\|u_j\|_{L^{2d}(\mathbb{R}^n)}^{2d} \leq \|u_j\|_{L^{2d}(B_r)}^{2d} + \|v_j\|_{L^{2d}(B_{3r/2})}^{2d} \leq cr^{n+2\gamma d} H^{2d}.
\]

(3.34)

Using (3.32)\(_1\) it now follows that

\[
[u_j]_{s,2d; B_{2r}}^{2d} \leq c \int_{B_{2r}} \int_{B_{2r}} \frac{(|u_j(x) - u_j(y)|^2 - M^2|x - y|2\gamma)^d}{|x - y|^{n+2s d}} \, dx \, dy 
+ cM^{2d} \int_{B_{2r}} \int_{B_{2r}} \frac{\, dx \, dy}{|x - y|^{n-2d(\gamma-s)}}
\leq cr^{n+2d(\gamma-s)} H^{2d} + cM^{2d} r^{n+2d(\gamma-s)}
= cr^{n+2d(\gamma-s)} H^{2d}
\]

(3.35)

where \( c \equiv c(\text{data}_c) \). Using this last estimate, and splitting again as in (3.17), we have

\[
[u_j]_{s,2d; \mathbb{R}^n}^{2d} = 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} \frac{|u_j(x)|^{2d}}{|x - y|^{n+2s d}} \, dx \, dy + [u_j]_{s,2d; B_{2r}}^{2d}
= 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{3r/2} \setminus B_{r}} \frac{\tilde{v}_j(x)^{2d}}{|x - y|^{n+2s d}} \, dx \, dy
+ 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{r}} \frac{|u_j(x)|^{2d}}{|x - y|^{n+2s d}} \, dx \, dy + [u_j]_{s,2d; B_{2r}}^{2d}
\leq c \int_{\mathbb{R}^n \setminus B_{3r/2}} \frac{dz}{|z|^{n+2s d}} \left( \left\| \tilde{v}_j \right\|_{L^{2d}(B_{3r/2})}^{2d} + \left\| u_j \right\|_{L^{2d}(B_r)}^{2d} \right) + [u_j]_{s,2d; B_{2r}}^{2d}.
\]

Using (3.15)\(_1\) and (3.34)–(3.35) in the above display, we conclude with

\[
[u_j]_{s,2d; \mathbb{R}^n}^{2d} \leq cr^{n+2d(\gamma-s)} H^{2d}
\]

(3.36)
for \( c \equiv c(\text{data_c}) \). Recalling that \( \beta_0 = s - n/(2d) > 0 \) in (3.5), we get

\[
[u_j]_{0, \beta_0; \mathbb{R}^n}^{2d} \leq c[u_j]_{0; \mathbb{R}^n}^{2d} \leq c\rho^{n+2d(\gamma-s)}H^{2d}, \quad (3.37)
\]

for every \( j \in \mathbb{N} \) and for a constant \( c \equiv c(\text{data_c}) \) which is independent of \( j \). From now on, we will be using the following notation:

\[
\mathcal{F}(u, B_r) := \mathcal{F}(u, B_r) + r^n H^{2d} + 1. \quad (3.38)
\]

By (3.27)–(3.28) we then have (recall that \( 2d > p \) and therefore \( H \leq H^{2d/p} + 1 \))

\[
\|u_j\|_{W^{1, p}(B_r)} \leq c[\mathcal{F}(u, B_r)]^{1/p} \quad (3.39)
\]

for \( c \equiv c(\text{data}) \), while (3.36)–(3.37) and \( r \leq 1 \) imply

\[
[u_j]_{x, 2d; \mathbb{R}^n} \leq c[\mathcal{F}(u, B_r)]^{1/(2d)} \quad \text{and} \quad [u_j]_{x, 2d; \mathbb{R}^n} \leq c\mathcal{F}(u, B_r), \quad (3.40)
\]

for \( c \equiv c(\text{data_c}) \).

### 3.5 Step 5: The Euler–Lagrange system

We adopt the short notation

\[
F_j(x, z) := F(x, z) + \varepsilon_j \left( |z|^2 + \mu^2 \right)^d. \quad (3.41)
\]

Before continuing, let us recall a basic consequence of assumptions 1.11. From (1.11)\(_2\) we infer that the partial \( z \mapsto F(x, z) \) is convex for every choice of \( x \in \Omega \); at this point from the \( q \)-upper bound in (1.11)\(_1\) and [44, Lemma 2.1], it follows that \( |\partial_z F(x, z)| \lesssim \left( |z|^2 + 1 \right)^{(q-1)/2} \) holds for every choice of \( (x, z) \in \Omega \times \mathbb{R}^{N \times n} \). Moreover, from Sect. 3.1, 3.1 and 3.7, it is \( 2d > \bar{p} > q \), so that, recalling (3.41), we conclude with

\[
|\partial_z F_j(x, z)| \lesssim \left( |z|^2 + 1 \right)^{(2d-1)/2}. \quad (3.42)
\]

We are now ready to derive the Euler–Lagrange system of the functional \( \mathcal{F}_j \). We make use of variations of the type \( u_j + t \varphi \), where \( \varphi \in W^{1,2d}_0(B_r, \mathbb{R}^N) \cap W^{s,2d}_0(\mathbb{R}^n, \mathbb{R}^N) \) has compact support in \( B_r \), and \( t \in (-1, 1) \). More precisely, by abuse of notation, we can confine ourselves to take any \( \varphi \in W^{1,2d}_0(B_r, \mathbb{R}^N) \) with compact support in \( B_r \). Indeed, note that \( \varphi \in W^{s,2d}(B_r, \mathbb{R}^N) \) (as \( W^{1,2d}(B_r, \mathbb{R}^N) \hookrightarrow W^{s,2d}(B_r, \mathbb{R}^N) \) for every \( t \in (0, 1) \), [23, Proposition 2.2]); moreover, since \( \varphi \) has compact support in \( B_r \), setting \( \varphi \equiv 0 \) outside \( B_r \), yields that \( \varphi \in W^{s,2d}(\mathbb{R}^n, \mathbb{R}^N) \) (see for instance the standard
argument in [23, Lemma 5.1]). It follows that \( u_j + t\varphi \in \mathbb{X}(\bar{v}_j, B_r) \). By minimality of \( u_j \), we have

\[
\frac{d\mathcal{F}_j(u_j + t\varphi, B_r)}{dt} \bigg|_{t=0} = 0,
\]

that is

\[
0 = \int_{B_r} \partial_z F_j(x, Du_j) \cdot D\varphi \, dx + 2d \int_{B_r} \left( |u_j|^2 - M_0^2 \right)^{d-1}_+ u_j \cdot \varphi \, dx
\]

\[
+ 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(|u_j(x) - u_j(y)|^2 - M^2|x - y|^{2\gamma})^{d-1}_+ (u_j(x) - u_j(y)) \cdot (\varphi(x) - \varphi(y))}{|x - y|^{n + 2sd}} \, dx \, dy.
\]

(3.43)

We also note that \( F_j(\cdot) \) has standard \( 2d \)-growth, moreover, from Sects. 3.1, 3.1 and 3.7, it is \( 2d > \tilde{p} > q \), so that all the integrals occurring in (3.44) are absolutely convergent and all the computations leading from (3.43) are legal. Now, we fix parameters

\[
0 < \varrho \leq \tau_1 < \tau_2 \leq r,
\]

(3.45)

and set \( \varphi := \tau_{-h}(\eta^2\tau_h u_j) \), with

\[
\begin{align*}
\eta \in C^1_0(B_{3\tau_2 + \tau_1}/4) \\
\mathbb{1}_{B_{(3\tau_2 + \tau_1)/2}} \leq \eta \leq \mathbb{1}_{B_{(3\tau_2 + \tau_1)/4}} \\
|D\eta| \lesssim \frac{1}{\tau_2 - \tau_1}
\end{align*}
\]

(3.46)

and \( h \in \mathbb{R}^n \setminus \{0\} \) is such that

\[
0 < |h| < \frac{\tau_2 - \tau_1}{2^{10}}.
\]

(3.47)

With such a choice, also thanks to the discussion after (3.42), it follows that \( \varphi \) is admissible in (3.44). Next, note that by definitions in (2.2)–(2.3) we have

\[
\begin{align*}
\tau_{-h}(\eta^2\tau_h u_j)(x) - \tau_{-h}(\eta^2\tau_h u_j)(y) &= \tilde{\tau}_{-h}\psi(x, y) \\
\psi(x, y) &:= \eta^2(x)(\tau_h u_j)(x) - \eta^2(y)(\tau_h u_j)(y)
\end{align*}
\]

(3.48)

Testing (3.44) with \( \varphi \), and using integration-by-parts for finite difference operators, we obtain, rearranging the terms

\[
2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{\tau}_h \left[ \frac{(|u_j(x) - u_j(y)|^2 - M^2|x - y|^{2\gamma})^{d-1}_+ (u_j(x) - u_j(y))}{|x - y|^{n + 2sd}} \right] \cdot \psi(x, y) \, dx \, dy
\]

\[
+ 2d \int_{B_r} \eta^2\tau_h \left( \left( |u_j|^2 - M^2 \right)^{d-1}_+ u_j \right) \cdot \tau_h u_j \, dx
\]
+ \int_{B_r} \tau_h \partial_r F_j (x, Du_j) \cdot \left[ \eta^2 \tau_h Du_j + 2 \eta \tau_h u_j \otimes D \eta \right] \, dx =: (I) + (II) + (III) = 0. \tag{3.49}

3.6 Step 6: Estimates for the nonlocal term (I)

For $\lambda \in [0, 1]$, and with $h \in \mathbb{R}^n$ as in (3.49), define

$$U_{\lambda, h}(x, y) := u_j(x) - u_j(y) + \lambda \tilde{\tau}_h (u_j(x) - u_j(y)) \tag{3.50}$$

and set

$$A(x, y, \lambda) := \left( |U_{\lambda, h}(x, y)|^2 - M^2 |x - y|^2 \gamma \right)^{d-1} U_{\lambda, h}(x, y). \tag{3.51}$$

From this it follows that

$$\frac{d}{d \lambda} A(x, y, \lambda) = A'(x, y, \lambda) = 2 \left( |U_{\lambda, h}(x, y)|^2 - M^2 |x - y|^2 \gamma \right)^{d-2} (U_{\lambda, h}(x, y) \cdot \tilde{\tau}_h (u_j(x) - u_j(y))) U_{\lambda, h}(x, y) + \left( |U_{\lambda, h}(x, y)|^2 - M^2 |x - y|^2 \gamma \right)^{d-1} \tilde{\tau}_h (u_j(x) - u_j(y)). \tag{3.52}$$

Note that $U_{\lambda, h}(x, y) = -U_{\lambda, h}(y, x)$ and therefore, by the definition in (3.51), we have

$$A'(x, y, \lambda) = -A'(y, x, \lambda), \tag{3.53}$$

holds whenever $x, y \in \mathbb{R}^n$. Moreover, from (3.48)$_2$, we have also

$$\psi(x, y) = \eta^2(x) [\tilde{\tau}_h (u_j(x) - u_j(y))] + \left[ \eta^2(x) - \eta^2(y) \right] \tau_h u_j(y). \tag{3.54}$$

Identities in (3.52) and (3.54) allow to write

$$(I) = 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \int_0^1 A'(x, y, \lambda) \, d\lambda \right) \cdot \frac{\psi(x, y)}{|x - y|^{n+2d}} \, dx \, dy$$

$$= 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta^2(x) \left( \int_0^1 A'(x, y, \lambda) \, d\lambda \right) \cdot \frac{\tilde{\tau}_h (u_j(x) - u_j(y))}{|x - y|^{n+2d}} \, dx \, dy$$

$$+ 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\eta^2(x) - \eta^2(y)) \left( \int_0^1 A'(x, y, \lambda) \, d\lambda \right) \cdot \frac{\tau_h u_j(y)}{|x - y|^{n+2d}} \, dx \, dy$$

$$=: 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 \frac{\eta^2(x) B_1(x, y, \lambda)}{|x - y|^{n+2d}} \, d\lambda \, dx \, dy$$

$$+ 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 (\eta^2(x) - \eta^2(y)) \frac{B_2(x, y, \lambda)}{|x - y|^{n+2d}} \, d\lambda \, dx \, dy, \tag{3.55}$$
where

\[
\begin{align*}
B_1(x, y, \lambda) &:= A'(x, y, \lambda) \cdot \tilde{\tau}_h(u_j(x) - u_j(y)) \\
B_2(x, y, \lambda) &:= A'(x, y, \lambda) \cdot \tau_h u_j(y).
\end{align*}
\]

We continue with the estimation of the two last integrals from (3.55). As for the former, by (3.52) we note that

\[
B_1(x, y, \lambda) = 2(d - 1) \left( |U_{\lambda,h}(x, y)|^2 - M^2 |x - y|^{2\gamma} \right)_{+}^{d/2} + |U_{\lambda,h}(x, y) \cdot \tilde{\tau}_h(u_j(x) - u_j(y))|^2 \\
+ \left( |U_{\lambda,h}(x, y)|^2 - M^2 |x - y|^{2\gamma} \right)_{+}^{d-1} |\tilde{\tau}_h(u_j(x) - u_j(y))|^2 \geq 0
\]

and therefore it is also

\[
\mathbb{I} := 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 \frac{\eta^2(x)B_1(x, y, \lambda)}{|x - y|^{n+2\gamma}} \, d\lambda \, dx \, dy \geq 0.
\]

In order to estimate the last term appearing in (3.55), we start splitting as follows (recall that \( \eta \equiv 0 \) outside \( B_{(3\tau_2+\tau_1)/4} \) by (3.46)):

\[
\begin{align*}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 \left( \eta^2(x) - \eta^2(y) \right) \frac{B_2(x, y, \lambda)}{|x - y|^{n+2\gamma}} \, d\lambda \, dx \, dy
&= \int_{\mathbb{R}^n \setminus B_{\tau_2}} \int_{B_{\tau_2}} \ldots \, dx \, dy + \int_{B_{\tau_2}} \int_{\mathbb{R}^n \setminus B_{\tau_2}} \ldots \, dx \, dy + \int_{B_{\tau_2}} \int_{B_{\tau_2}} \ldots \, dx \, dy \\
&=: (I)_1 + (I)_2 + (I)_3.
\end{align*}
\]

For the estimation of the last three pieces in (3.58), we rearrange (3.52) as follows:

\[
A'(x, y, \lambda) = 2(d - 1) \left( \left( |U_{\lambda,h}(x, y)|^2 - M^2 |x - y|^{2\gamma} \right)_{+}^{(d-2)/2} \left( U_{\lambda,h}(x, y) \cdot \tilde{\tau}_h(u_j(x) - u_j(y)) \right) \right) + \left( |U_{\lambda,h}(x, y)|^2 - M^2 |x - y|^{2\gamma} \right)_{+}^{(d-2)/2} U_{\lambda,h}(x, y) \\
+ \left( |U_{\lambda,h}(x, y)|^2 - M^2 |x - y|^{2\gamma} \right)_{+}^{d-1} \tilde{\tau}_h(u_j(x) - u_j(y)).
\]

With such an expression at hand, and recalling that \( \eta \equiv 0 \) outside \( B_{(3\tau_2+\tau_1)/4} \) (see 3.46), by Young inequality, and yet recalling the expression of \( B_1(\cdot) \) in (3.56), we estimate

\[
|I_1| \leq \int_{\mathbb{R}^n \setminus B_{\tau_2}} \int_{B_{\tau_2}} \frac{\eta^2(x)|B_2(x, y, \lambda)|}{|x - y|^{n+2\gamma}} \, dx \, dy.
\]
For the estimate of (I) 3, we preliminary write

\( (I)_3 = \int_{B_{r_2}} \int_{B_{r_2}} \int_0^1 \eta(x)(\eta(x) - \eta(y)) \frac{|B_2(x, y, \lambda)|}{|x - y|^{n+2d}} d\lambda \ dx \ dy \)

In order to estimate (I) 3 we preliminary write

\( (I)_3 \leq \frac{1}{6} \int_{\mathbb{R}^n \setminus B_{r_2}} \int_{B(3r_2 + \tau_1)/4} \int_0^1 \frac{\eta^2(x)|B_1(x, y, \lambda)|}{|x - y|^{n+2d}} d\lambda \ dx \ dy \\
+ c \int_{\mathbb{R}^n \setminus B_{r_2}} \int_{B(3r_2 + \tau_1)/4} \int_0^1 \frac{|U_{\lambda, h}(x, y)|^{2(d-1)}|\tau_h u_j(y)|^2}{|x - y|^{n+2d}} d\lambda \ dx \ dy \\
\leq \frac{1}{6} + c[u_j]_{s, 2d; \mathbb{R}^n}^2 \left( \int_{B(3r_2 + \tau_1)/4} \int_{\mathbb{R}^n \setminus B_{r_2}} \frac{|\tau_h u_j(y)|^{2d}}{|x - y|^{n+2d}} dy \ dx \right)^{1/d} \\
\leq \frac{1}{6} + c[u_j]_{s, 2d; \mathbb{R}^n}^2 \left( \int_{B(3r_2 + \tau_1)/4} \int_{\mathbb{R}^n \setminus B_{r_2}} \frac{dx \ dy}{|x|^{n+2d}} \right)^{1/d} \\
\leq \frac{1}{6} + c[u_j]_{s, 2d; \mathbb{R}^n}^2 \left( \int_{B(3r_2 + \tau_1)/4} \int_{\mathbb{R}^n \setminus B_{r_2}} \frac{dz \ dx}{|z|^{n+2d}} \right)^{1/d} \\
\geq (3.40) \leq \frac{1}{6} + \frac{c r^{n/d} |E(u, B_r)|}{(\tau_2 - \tau_1)^{2r}} |h|^{2\beta_0}.

Recall that \( I \) has been defined in (3.57). Note also that, recalling (3.50) and using Hölder's inequality, in the above display we have estimated as follows:

\( \int_{\mathbb{R}^n \setminus B_{r_2}} \int_{B(3r_2 + \tau_1)/4} \int_0^1 \frac{|U_{\lambda, h}(x, y)|^{2(d-1)}|\tau_h u_j(y)|^2}{|x - y|^{n+2d}} d\lambda \ dx \ dy \\
\leq c \int_{\mathbb{R}^n \setminus B_{r_2}} \int_{B(3r_2 + \tau_1)/4} \frac{|u_j(x) - u_j(y)| + |u_j(x + h) - u_j(y + h)|^{2(d-1)}|\tau_h u_j(y)|^2}{|x - y|^{n+2d}} dy \ dx \\
\leq c[u_j]_{s, 2d; \mathbb{R}^n}^2 \left( \int_{B(3r_2 + \tau_1)/4} \int_{\mathbb{R}^n \setminus B_{r_2}} \frac{|\tau_h u_j(y)|^{2d}}{|x - y|^{n+2d}} dy \ dx \right)^{1/d}.

For the estimate of (I) 2, we start observing that (3.53) and (3.56) imply that

\( B_1(x, y, \lambda) = B_1(y, x, \lambda) \)

holds for every \( x, y \in \mathbb{R}^n \). Then, similarly to (I) 1, we find

\( |(I)_2| \leq \int_{B_{r_2}} \int_{\mathbb{R}^n \setminus B_{r_2}} \frac{\eta^2(y)|B_2(x, y, \lambda)|}{|x - y|^{n+2d}} dx \ dy \\
= \frac{1}{6} \int_{\mathbb{R}^n \setminus B_{r_2}} \int_{B_{r_2}} \frac{\eta^2(x)|B_2(x, y, \lambda)|}{|x - y|^{n+2d}} dx \ dy \\
\leq \frac{1}{6} \int_{\mathbb{R}^n \setminus B_{r_2}} \int_{B_{r_2}} \frac{\eta^2(x)|B_2(x, y, \lambda)|}{|x - y|^{n+2d}} dx \ dy \\
\leq \frac{1}{6} + \frac{c r^{n/d} |E(u, B_r)|}{(\tau_2 - \tau_1)^{2r}} |h|^{2\beta_0}.

(3.63)
\[ + \int_{B_{r_2}} \int_{B_{r_2}} \int_0^1 \eta(y)\eta(x) - \eta(y) \frac{B_2(x, y, \lambda)}{|x-y|^{n+2d}} \, d\lambda \, dx \, dy =: (I)_4 + (I)_5. \]

(3.64)

In turn, we estimate $(I)_4$; recalling (3.46) and (3.59), using first Young and then Hölder

\[ |(I)_4| \leq \int_{B_{r_2}} \int_{B_{r_2}} \int_0^1 \eta(x)|\eta(x) - \eta(y)| \frac{|B_2(x, y, \lambda)|}{|x-y|^{n+2d}} \, d\lambda \, dx \, dy \]

\[ \leq \frac{d}{6} \int_{B_{r_2}} \int_{B_{r_2}} \int_0^1 \frac{\eta^2(x)B_1(x, y, \lambda)}{|x-y|^{n+2d}} \, d\lambda \, dx \, dy \]

\[ + c \int_{B_{r_2}} \int_{B_{r_2}} \int_0^1 \frac{|U_{x, h}(x, y)|^{2(d-1)}|\eta(x) - \eta(y)|^2|\tau_h u_j(y)|^2}{|x-y|^{n+2d}} \, d\lambda \, dx \, dy \]

\[ \leq \frac{I}{12} + c[u_j]_{s, 2d; \mathbb{R}^n}^2 \left( \int_{B_{r_2}} \int_{B_{r_2}} \frac{|\tau_h u_j(y)|^{2d}|\eta(x) - \eta(y)|}{|x-y|^{n+2d}} \, dx \, dy \right)^{1/d} \]

\[ \leq \frac{I}{12} + c[u_j]_{s, 2d; \mathbb{R}^n}^2 \left( \int_{B_{r_2}} \int_{B_{r_2}} \frac{\|D\eta\|_L^2}{|x-y|^{n+2d}} \, dx \, dy \right)^{1/d} \]

\[ \leq \frac{I}{12} + \frac{cr^{n/d+2(1-s)}}{\tau_2 - \tau_1} \frac{\|D\eta\|_L^2}{|h|} \]

(3.40)

\[ \leq \frac{I}{12} + \frac{cr^{n/d+2(1-s)}}{\tau_2 - \tau_1} |h|^{2\beta_0}. \]

for $c \equiv c(\text{data})$. On the other hand, we also have

\[ |(I)_5| \leq \int_{B_{r_2}} \int_{B_{r_2}} \int_0^1 \eta(y)\eta(x) - \eta(y) \frac{|B_2(x, y, \lambda)|}{|x-y|^{n+2d}} \, d\lambda \, dx \, dy \]

(3.62)

\[ \leq \frac{I}{12} + \frac{cr^{n/d+2(1-s)}}{\tau_2 - \tau_1} \frac{\|D\eta\|_L^2}{|h|} \]

\[ \leq \frac{I}{12} + \frac{cr^{n/d+2(1-s)}}{\tau_2 - \tau_1} |h|^{2\beta_0}. \]

Merging the content of the last three displays yields

\[ |(I)_3| \leq \frac{I}{6} + \frac{cr^{n/d+2(1-s)}}{\tau_2 - \tau_1} |h|^{2\beta_0} \]

(3.66)

for $c \equiv c(\text{data})$, which is the final estimate for $(I)_3$. Collecting the estimates found in (3.60), (3.63) and (3.66) to (3.58), yields

\[ 2d \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 (\eta^2(x) - \eta^2(y)) \frac{B_2(x, y, \lambda)}{|x-y|^{n+2d}} \, d\lambda \, dx \, dy \leq \frac{I}{2} + \frac{cr^{n/d}}{\tau_2 - \tau_1} |h|^{2\beta_0}. \]
Finally, merging this last estimate with (3.55) and reabsorbing the term $I$, we conclude with
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 \eta^2(x) B_1(x, y, \lambda) \frac{d \lambda}{|x - y|^{n+2d}} \, d \lambda \, d y \, d x \leq (I) + \frac{c r^{n/d} F(u, B_r)}{\tau_2 - \tau_1^2} |h|^{2 \rho_0},
\] (3.67)
where $c \equiv \text{data}_c$.

### 3.7 Step 7: Estimates for the local terms (II) and (III)

Here we benefit from [20,25]. For (II), exactly as in [20, Sect. 3.2, (3.27)], we have
\[
(II) = 2d \int_{B_r} \eta^2 \int_0^1 \frac{d}{d \lambda} \left( |u_j + \lambda \tau_h u_j| - M_0^2 \right)^{d-1} \left( u_j + \lambda \tau_h u_j \right) \cdot \tau_h u_j \, d \lambda \cdot \tau_h u_j \, d x
\]
\[
= 4d(d - 1) \int_{B_r} \eta^2 \int_0^1 \left( |u_j + \lambda \tau_h u_j| - M_0^2 \right)^{d-2} \left( (u_j + \lambda \tau_h u_j) \cdot \tau_h u_j \right)^2 \, d \lambda \, d x
\]
\[
+ 2d \int_{B_r} \eta^2 \int_0^1 \left( |u_j + \lambda \tau_h u_j| - M_0^2 \right)^{d-1} \, d \lambda \, |\tau_h u_j|^2 \, d x \geq 0 .
\] (3.68)

For (III), recalling the definition in (3.41), we have
\[
(III) = \int_{B_r} \tau_h \partial_z F(x, Du_j) \cdot \left[ 2\eta \tau_h u_j \otimes D \eta + \eta^2 \tau_h Du_j \right] \, d x
\]
\[
+ 2d \varepsilon_j \int_{B_r} \tau_h \left( |Du_j|^2 \right)^{d-1} \left( Du_j \right) \cdot \left[ 2\eta \tau_h u_j \otimes D \eta + \eta^2 \tau_h Du_j \right] \, d x
\]
\[
= (III)_1 + (III)_2 .
\]

We can then proceed as in [20, Sect. 3.2]. By (1.11)_{2,3}, a monotonicity estimate as the one in [20, Sect. 3.2, term (I)_{j}] gives (recall that no $x$-dependence appears in this term)
\[
(III)_1 \geq \frac{1}{c} \int_{B_r} \eta^2 \left( |Du_j(x + h)|^2 + |Du_j(x)|^2 + \mu^2 \right)^{(p-2)/2} |\tau_h Du_j|^2 \, d x
\]
\[
- \frac{c |h|}{\tau_2 - \tau_1} \int_{B_{\tau_2}} \left( |Du_j|^2 + 1 \right)^{q/2} \, d x ,
\] (3.69)
for $c \equiv c(n, N, v, L, p, q)$. Similarly, we have
\[
(III)_2 \geq \frac{\varepsilon}{c} \int_{B_r} \eta^2 \left( |Du_j(x + h)|^2 + |Du_j(x)|^2 + \mu^2 \right)^{d-1} |\tau_h Du_j|^2 \, d x
\]
\[
- \frac{c \varepsilon |h|}{\tau_2 - \tau_1} \int_{B_{\tau_2}} \left( |Du_j|^2 + 1 \right)^d \, d x .
\] (3.70)
Connecting the inequalities in the last three displays we conclude with
\[
\int_{B_r} \eta^2 \left( |Du_j(x+h)|^2 + |Du_j(x)|^2 + \mu^2 \right)^{(p-2)/2} |\tau_h Du_j|^2 \, dx \\
\leq c(\text{III}) + \frac{c|h|^{\alpha}}{\tau_2 - \tau_1} \int_{B_{\tau_2}} \left( |Du_j|^2 + 1 \right)^{q/2} \, dx + \frac{c\varepsilon|h|}{\tau_2 - \tau_1} \int_{B_{\tau_2}} \left( |Du_j|^2 + 1 \right)^d \, dx ,
\]
for \( c \equiv c(n, N, \nu, L, p, q) \). Using (3.27) in the last inequality and recalling (3.38), finally leads to
\[
\int_{B_r} \eta^2 \left( |Du_j(x+h)|^2 + |Du_j(x)|^2 + \mu^2 \right)^{(p-2)/2} |\tau_h Du_j|^2 \, dx \\
\leq c(\text{III}) + \frac{c|h|^{\alpha}}{\tau_2 - \tau_1} \left( \mathcal{F}(u, B_r) + \|Du_j\|_{L^q(B_{\tau_2})}^q + 1 \right) ,
\]
for \( c \equiv c(n, N, \nu, L, p, q) \).

### 3.8 Step 8: Local/nonlocal Caccioppoli type inequality

Connecting (3.6), (3.68) and (3.71) to (3.49), and recalling (3.46), we end up with
\[
\int_{B_{(\tau_2+\tau_1)/2}} \int_{B_{(\tau_2+\tau_1)/2}} \int_0^1 \frac{B_1(x, y, \lambda)}{|x-y|^{n+2d}} \, d\lambda \, dx \, dy \\
+ \int_{B_{(\tau_2+\tau_1)/2}} \left( |Du_j(x+h)|^2 + |Du_j(x)|^2 + \mu^2 \right)^{(p-2)/2} |\tau_h Du_j|^2 \, dx \\
\leq \frac{c|h|^{\alpha_0}}{(\tau_2 - \tau_1)^2} \left( \mathcal{F}(u, B_r) + \|Du_j\|_{L^q(B_{\tau_2})}^q + 1 \right) ,
\]
for \( c \equiv c(\text{data}) \) and \( \alpha_0 \) has been defined in (3.6).

### 3.9 Step 9: Iteration and a priori estimate for \( p \geq 2 \)

Recalling the non-negativity in (3.57), estimate (3.72) with \( p \geq 2 \) yields
\[
\int_{B_{(\tau_2+\tau_1)/2}} |\tau_h Du_j|^p \, dx \leq \frac{c|h|^{\alpha_0}}{(\tau_2 - \tau_1)^2} \left( \mathcal{F}(u, B_r) + \|Du_j\|_{L^q(B_{\tau_2})}^q + 1 \right) .
\]

Lemma 1 and (3.47) now provide that
\[
u \in W^{1+\beta/p, p}(B_{(\tau_2+\tau_1)/2}, \mathbb{R}^N) \cap W^{s, 2d}(B_r, \mathbb{R}^N)
\]
holds together with (via (2.9), using (3.73))
\[
\|u_j\|_{W^{1+\beta/p, p}(B_{(\tau_2+\tau_1)/2})} \leq \frac{c}{(\tau_2 - \tau_1)^{(\alpha+\beta)/p}} \|u_j\|_{W^{1/p}(B_{\tau_2})}.
\]
Here $\beta < \alpha_0$ has been fixed in (3.6), and $c \equiv c(\text{data}) \geq 1$, recall (3.10); in the last estimate we have also used (3.39). Now we apply Lemma 3 with parameters $s_1 = s$, $s_2 = 1 + \beta/p$, $a = 2d$ and $t = p$ to get

$$
\|Du_j\|_{L^\tilde{p}(B_{t_1})} \leq \frac{c}{(\tau_2 - \tau_1)^{(n+\beta)/p + \kappa}} \left\{ [F(u, B_r)]^{1/p} + \|Du_j\|_{L^\alpha(B_{t_2})}^{1/\alpha} \right\},
$$

(3.75)

for

$$
\theta = \frac{\beta}{p(1 - s) + \beta},
$$

(3.77)

$\tilde{p}$ as in (3.2), and where $c, \kappa \equiv c, \kappa(\text{data}_\phi)$ are derived from Lemma 3. By using (3.40) and (3.75) in (3.76), we obtain

$$
\|Du_j\|_{L^{\tilde{p}}(B_{t_1})} \leq \frac{c}{(\tau_2 - \tau_1)^{(n+\beta)/p + \kappa}} \left\{ [F(u, B_r)]^{1/p} + \|Du_j\|_{L^\alpha(B_{t_2})}^{1/\alpha} \right\} + \frac{c}{(\tau_2 - \tau_1)^{(n+\beta)/p + \kappa}} \left\{ \|F(u, B_r)\|^{\tilde{p} q/(\tilde{p} - p)} \right\} \frac{q(1-\theta)}{\tilde{p} - p} \|Du_j\|_{L^{\tilde{p}}(B_{t_2})}. 
$$

(3.78)

By (3.7) we can use the interpolation inequality

$$
\|Du_j\|_{L^\theta(B_{t_2})} \leq \|Du_j\|_{L^{\tilde{p}}(B_{t_2})}^{\theta_1} \|Du_j\|_{L^\alpha(B_{t_2})}^{1 - \theta_1},
$$

(3.79)

where $\theta_1 \in (0, 1)$ satisfies the identity

$$
\frac{1}{q} = \frac{\theta_1}{\tilde{p}} + \frac{1 - \theta_1}{p} \implies \theta_1 = \frac{\tilde{p} (q - p)}{q (\tilde{p} - p)}.
$$

(3.80)

Plugging (3.79) in (3.78) yields

$$
\|Du_j\|_{L^{\tilde{p}}(B_{t_1})} \leq \frac{c}{(\tau_2 - \tau_1)^{(n+\beta)/p + \kappa}} \left\{ [F(u, B_r)]^{1/p} + \|Du_j\|_{L^\alpha(B_{t_2})}^{1/\alpha} \right\} + \frac{c}{(\tau_2 - \tau_1)^{(n+\beta)/p + \kappa}} \left\{ \|F(u, B_r)\|^{\tilde{p} q/(\tilde{p} - p)} \right\} \frac{q(1-\theta)}{\tilde{p} - p} \|Du_j\|_{L^{\tilde{p}}(B_{t_2})}^{(1-\theta_1)q(1-\theta)}.
$$

(3.81)

again with $c \equiv c(\text{data}), \kappa \equiv \kappa(\text{data}_\phi)$. Note that

$$
\frac{q \theta_1 (1 - \theta)}{p} \frac{\tilde{p} (s, d, \beta) (q - p)}{\tilde{p} (s, d, \beta) - p} \frac{1 - s}{p(1 - s) + \beta} < 1.
$$

(3.82)
This allows to apply Young inequality in (3.81) with conjugate exponents

\[
\left( \frac{p}{q \theta_1(1-\theta)}, \frac{p}{p-q \theta_1(1-\theta)} \right)
\]

in order to get, using also (1.11)_1 and (3.27),

\[
\|Du_j\|_{L^\tilde{p}(B_{\tau_1})} \leq \frac{1}{2} \|Du_j\|_{L^\tilde{p}(B_{\tau_2})} + \frac{c}{(\tau_2 - \tau_1)^{\kappa_1}} \left[ F(u, B_r) \right]^{\tilde{\kappa}},
\]

for suitable exponents \( \kappa_1, \kappa_2 \) depending on \( \text{data}_e \) and \( c \equiv c(\text{data}) \). Lemma 4 together with the content of the previous display finally gives

\[
\|Du_j\|_{L^\tilde{p}(B_\rho)} \leq \frac{c}{(r-\rho)^{\kappa_1}} \left[ F(u, B_r) \right]^{\tilde{\kappa}},
\]

for \( c \equiv c(\text{data}) \) and \( \kappa_1, \kappa_2 \equiv \kappa_1, \kappa_2(\text{data}_e) \). Recalling (3.38) and that \( \tilde{p} > q \), this implies

\[
\|Du_j\|_{L^q(B_\rho)} \leq \frac{c}{(r-\rho)^{\kappa_1}} \left( F(u, B_r) + r^n H^{2d+1} \right)^{\tilde{\kappa}}.
\]

By using (3.29) and (3.31), the assertion in (1.19) now follows by lower semicontinuity with \( \kappa_2 = 2\tilde{\kappa}d \). This concludes the proof of Theorem 1 in the case \( p \geq 2 \).

**Remark 3** From the proof above and the choice of the parameters in Step 1, it is not difficult to see that when

\[
q, q \to p + \frac{\min\{\alpha, 2\gamma\}}{\theta(1-\gamma)},
\]

thereby forcing \( \tilde{p} \) to approach the same value, then \( d \to \infty \). This makes the exponent \( \kappa_2 \) in the final estimate (1.19) blow-up. This is a typical phenomenon in regularity for \( (p, q) \)-growth functionals, already encountered in the form of the a priori estimates found in [20,25,45].

### 3.10 Step 10: Iteration and a priori estimate for \( 1 < p < 2 \)

In this case we use Hölder inequality to estimate

\[
\int_{B_{(\tau_2+\tau_1)/2}} |\tau_h Du_j|^p \, dx
\]

\[
\leq \left( \int_{B_{(\tau_2+\tau_1)/2}} (|Du_j(x+h)|^2 + |Du_j(x)|^2 + \mu^2)^{p/2} \, dx \right)^{p/2} \left( \int_{B_{(\tau_2+\tau_1)/2}} (|Du_j(x+h)|^2 + |Du_j(x)|^2 + \mu^2)^{1-p/2} \, dx \right)^{1-p/2}
\]
Theorem 1 is finally complete. We can therefore proceed exactly as after (3.82) to arrive at (1.19) and the proof of (3.40) and (3.84) into (3.85), we again arrive at the following analog of (3.78):

\[ \| u_j \|_{W^{1,p}(B_{\eta_1})} \leq c \left( \int_{B_{\eta_1}/2} (Du_j(x+h))^2 + |Du_j(x)|^2 + \mu^2 \right)^{p/2} \left[ F(u, B_r) \right]^{1-p/2} \]

where \( c \equiv c(\text{data}) \). This is the analog of (3.73) in the case \( p < 2 \). We can therefore proceed as in the case \( p \geq 2 \); we report some of the details in the following as different bounds are coming up. As for (3.74), Lemma 1 gives \( u_j \in W^{1+\beta/2,p}(B_{(\eta_1/2)}), \mathbb{R}^N) \cap W^{s,2d}(B_r, \mathbb{R}^N) \), for \( \beta \) as in (3.6), with the estimate (thanks to (3.83))

\[ \| u_j \|_{W^{1+\beta/2,p}(B_{(\eta_1/2)})} \leq c \left( \frac{\beta}{(\eta_1 - 1) + (\beta - \alpha_0)/2} \right) \left[ F(u, B_r) \right]^{1/p} + \| Du_j \|_{L^q(B_{\eta_1})}^{q/p} + 1 \]  

(3.83)

Next, keeping in mind that \( \bar{\theta} > q \) by (3.7), we can apply the interpolation inequality (3.79) in (3.87), with \( \bar{\theta}_1 \in (0, 1) \) as in (3.80) (but with the different, current definition of \( \bar{\rho} \)). As for the case \( p \geq 2 \), using (3.79) in (3.78) yields (3.81), with

\[ \frac{q \theta_1 (1 - \theta)}{p} (3.80, 3.86) \leq \bar{\rho}(s, d, \beta)(q - p) 2(1 - s) \frac{2(1 - s) + \beta}{p[2(1 - s) + \beta]} < 1. \]

We can therefore proceed exactly as after (3.82) to arrive at (1.19) and the proof of Theorem 1 is finally complete.
3.11 Step 11: Proof of Corollaries 1, 2

In both cases the proof relies on the application of Theorem 1, where we verify that \( \mathcal{L}_{F,H,y}(u, B_r) = 0 \) holds for every ball \( B_r \subseteq \Omega \), for suitable choices of \( H > 0 \). We fix \( B_r \subseteq \Omega \) such that \( [u]_{0,y;B_r} > 0 \) (otherwise the assertion is trivial) and we start from Corollary 2, where we use [3, Theorem 4] in order to get a sequence \( \{\tilde{u}_j\} \) of \( W^{1,\infty}(B_r) \)-regular maps such that

\[
\int_{B_r} \left[ |D\tilde{u}_j|^p + a(x)|D\tilde{u}_j|^q \right] \, dx \to \int_{B_r} \left[ |Du|^p + a(x)|Dw|^q \right] \, dx , \tag{3.88}
\]

and \( \tilde{u}_j \to u \) strongly in \( W^{1,p}(B_r) \). The sequence \( \{\tilde{u}_j\} \) has been constructed in [3] via mollification, i.e., \( \tilde{u}_j := u * \rho_{\varepsilon_j} \). Here, with \( 0 < \delta \leq \min\{\text{dist}(B_r, \partial\Omega)/20\} \) being fixed, \( \{\varepsilon_j\} \) can be taken as a sequence such that \( \varepsilon_j \to 0 \), \( 0 < \varepsilon_j \leq \delta \). Moreover, \( \{\rho_{\varepsilon}\} \) is a family of standard mollifiers generated by a smooth, non-negative and radial function \( \rho \in C_0^\infty(B_1) \) such that \( \|\rho\|_{L^1(\mathbb{R}^n)} = 1 \), via \( \rho_{\varepsilon}(x) = \varepsilon^{-n}\rho(x/\varepsilon) \). It follows that

\[
[\tilde{u}_j]_{0,y;B_r} \leq [u]_{0,y;(1+\delta)B_r} , \tag{3.89}
\]

for every \( j \in \mathbb{N} \). Proposition 1 then implies \( \mathcal{L}_{F,H,y}(u, B_r) = 0 \) for \( H = [u]_{0,y;(1+\delta)B_r} \). Therefore Theorem 1 applies, with (1.19) that translates into

\[
\|Du\|_{L^q(B_{c})} \leq \frac{c}{(r-q)^{k_1}} \left( (\mathcal{F}(u, B_r))^\frac{1}{p} + [u]_{0,y;(1+\delta)B_r} + 1 \right)^{k_2} , \tag{3.90}
\]

which holds whenever \( B_\Theta \subseteq B_r \) is concentric to \( B_r \). Finally, letting \( \delta \to 0 \) in the above inequality leads to (1.22) and the proof of Corollary 2 is complete. We only note that we can invoke [3, Theorem 4] as we are assuming (1.16), which is a more restrictive condition than the one considered in (1.9), which is in turn sufficient to prove (3.88), as shown in [3]. For Corollary 1, the proof is totally similar and uses the same convolution argument of Corollary 2 to find the approximating sequence \( \{\tilde{u}_j\} \subset W^{1,\infty}(B_r) \) such that the approximation in energy \( \mathcal{F}(\tilde{u}_j, B_r) \to \mathcal{F}(u, B_r) \) and (3.89) hold. In this case the proof of the approximation in energy is directly based on Jensen’s inequality and the convexity of \( G(\cdot) \). The details can be found in [25, Lemma 12]. No bound on the gap \( q/p \) is required at this stage and (1.16) is only needed to refer to Theorem 1, i.e., to prove the a priori estimates. The rest of the proof then proceeds as the one for Corollary 2.

3.12 Step 12: Proof of Theorem 2

This follows with minor modifications from the proof of Theorem 1, that we briefly outline here. As the integrand \( F(\cdot) \) is now \( x \)-independent and convex, we are in the situation of Corollary 1, where we can verify (1.23) with \( b(\cdot) \equiv 1 \) and \( G(\cdot) \equiv F(\cdot) \). It follows that \( \mathcal{L}_{F,H,y}(u, B_r) = 0 \) with the choice \( H = [u]_{0,y;(1+\delta)B_r} \), where \( \delta > 0 \) can be chosen arbitrarily small as in the proofs of Corollaries 1, 2. As for the part
concerning the a priori estimates, the proof follows the one given for Theorem 1 verbatim, once replacing, everywhere, $\alpha$ by 1. With the current choice of $H$, this leads to establish (3.90), from which (1.22) finally follows letting $\delta \to 0$.

### 4 More on the autonomous case

In the autonomous case $F(x, Dw) \equiv F(Dw)$, considering assumptions on the Hessian of $F(\cdot)$ of the type in (1.6), leads to bounds that are better than (1.20). Specifically, we assume that $F : \mathbb{R}^{N \times n} \to \mathbb{R}$ is locally $C^2$-regular in $\mathbb{R}^{N \times n} \setminus \{0\}$ and satisfies

$$
\begin{aligned}
&v\left(\|z\|^2 + \mu^2\right)^{p/2} \leq F(z) \leq L \left(\|z\|^2 + \mu^2\right)^{q/2} + L \left(\|z\|^2 + \mu^2\right)^{p/2} \\
&\left(\|z\|^2 + \mu^2\right) |\partial_{zz} F(z)| \leq L \left(\|z\|^2 + \mu^2\right)^{q/2} + L \left(\|z\|^2 + \mu^2\right)^{p/2} \\
&v\left(\|z\|^2 + \mu^2\right)^{(p-2)/2} \|\xi\|^2 \leq \partial^2 F(z) \xi \cdot \xi,
\end{aligned}
$$

for every choice of $z, \xi \in \mathbb{R}^{N \times n}$ such that $|z| \neq 0$, and for exponents $1 \leq p \leq q$. As usual, $0 < v \leq 1 \leq L$ are fixed ellipticity constants and $\mu \in [0, 1]$. Such assumptions are for instance considered in [4,45]. Using (4.1), instead of (1.11), in the statement of Theorem 2 we can replace the right-hand side quantity in (1.20) and (1.21) by

$$
p + \frac{2\gamma}{\vartheta(1 - \gamma)}.
$$

The proof can be obtained along the lines of the one for Theorem 1, using different, and actually more standard estimates in Step 6. In particular, estimates (3.69)–(3.70) can be replaced by

$$(\text{III})_1 + (\text{III})_2 \geq \frac{1}{c} \int_{B_{\tau}} \eta^2 \left(\|Du_j(x + h)|^2 + |Du_j(x)|^2 + \mu^2\right)^{(p-2)/2} |\tau_h Du_j|^2 \, dx$$

$$- \frac{c|h|^2}{(\tau_2 - \tau_1)^2} \int_{B_{\tau_2}} \left(\|Du_j \|^2 + 1\right)^{q/2} \, dx$$

$$- \frac{c\epsilon_j |h|^2}{(\tau_2 - \tau_1)^2} \int_{B_{\tau_2}} \left(\|Du_j \|^2 + 1\right)^d \, dx,$$

and therefore estimate (3.71) can be replaced by

$$\int_{B_{\tau}} \eta^2 \left(\|Du_j(x + h)|^2 + |Du_j(x)|^2 + \mu^2\right)^{(p-2)/2} |\tau_h Du_j|^2 \, dx$$

$$\leq c(\text{III}) + \frac{c|h|^2}{(\tau_2 - \tau_1)^2} \left(\mathcal{F}(u, B_{\tau}) + \|Du_j\|_{L^q(B_{\tau_2})}^q + 1\right).$$

This leads to change the parameters in Step 1. Specifically, we replace everywhere the quantity $\min\{\alpha, 2\gamma\}$, by $2\gamma = \min\{2, 2\gamma\}$ starting from (3.1), while in (3.6) we
define $\alpha_0 := 2\beta_0 < 2\gamma$. The rest of the proof follows unaltered and leads to establish (1.19) using now the bound in (4.2).

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Declarations

Conflict of interest The authors declare they do not have any conflict of interest.

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