Influence of a cosmic string on the rate of pairs produced by the Coulomb potential

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Abstract

We study particle creation phenomenon by the Coulomb potential of an external electric field in the presence of a gravitational field of a static cosmic string. For that, the generalized Klein-Gordon and Dirac equations are solved, and by using the Bogoliubov transformation we calculate the probability and the number density of created particles. It is shown that the presence of the cosmic string enhances the particle production. For the grand unified theory (GUT) cosmic string, the production of spinless particles is possible if the Coulomb potential nucleus charge $Z \geq 206$, and for spin-1/2 particles if $Z \geq 275$.

Keywords: Particle creation, Coulomb Potential, Bogoliubov transformation, Topological defects, Cosmic string, Deficit angle, Conical spacetime.

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1 Introduction

The creation of particles by an electric field in curved spacetime is an active research field and continue to attract attentions as in the cosmological models of an expanding universe [1–19]. In particular, the creation of particles by the Coulomb potential of an external electric field is a topic that has received a special interest [4, 20–25] and also in the references cited in the three relevant reviews [26–28]. Moreover, particle production under the effect of topological defects has been also discussed in the literature, like in the field of a magnetic monopole and domain walls [29, 30], and in the presence of a cosmic string [31–40].

The cosmic strings are hypothetical objects which may have been formed during the inflationary phase in the primordial universe [41–43]. They are one-dimensional topological defects in the spacetime structure and resulting from a symmetry breaking at an energy close to $10^{16}$ GeV ($10^{-36}$ seconds) [43], and with a maximum mass per unit length $\mu_{\text{max}} = 6.73 \times 10^{27} \text{g.cm}^{-1}$ [44]. They have a nontrivial topology where the spacetime is locally flat and globally conical with an azimuthal deficit angle [45]. A cosmic string induces a repulsive force on an electric charge at rest [46] or on a current [47], and an attractive force on a neutral particle [48]. It has also relevant effects like gravitational lensing [43], Casimir effect and the gravitational Aharonov-Bohm effect [49, 50]. The various observation programs of the anisotropies in cosmic background radiation (CMB) by COBE, WMAP and the Planck satellite have not observed any cosmic strings effects on the primordial density perturbations [51]. Nevertheless, it is possible that they have a role in the production of gravitational waves [52–55] and generation of high-energy cosmic rays [56]. More recently, a two-level static atom coupled to an electromagnetic field, in a cosmic string spacetime, is suggested as a detector to estimate the deficit angle [57].

In order to introduce the deficit angle, let us use the metric of an infinite straight string in cylindrical coordinates $(t, \rho, \phi, z)$ [43, 58]

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2 \rho^2 d\phi^2 + dz^2,$$

where the cosmic string parameter $\alpha = 1 - 4\bar{\mu}$ varies in the interval $[0, 1]$ with $\bar{\mu} = \frac{G}{c^2} \mu$, $\mu$ the linear mass density of the string, $G$ the gravitational constant, $c$ the speed of light and the variation
of the angular variable is $0 \leq \phi \leq 2\pi$. The metric (1) has been obtained by solving Einstein’s equations [43, 58]. If we introduce the change of variable $\varphi = (1 - 4\bar{\mu})\phi$, the metric (1) reduces to the Minkowski line element

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2,$$

where the interval of variation of $\varphi$ is

$$0 \leq \varphi \leq 2\pi(1 - 4\bar{\mu}),$$

and consequently the cosmic string spacetime is locally flat and globally conical with a deficit angle

$$\Delta = 8\pi\bar{\mu} = 2\pi(1 - \alpha),$$

i.e., its geometry has a conical singularity and the curvature tensor is defined using the 2D Dirac delta function $\delta^{(2)}(\rho)$ [59]

$$R^{12}_{12} = 2\pi \frac{\alpha - 1}{\alpha} \delta^{(2)}(\rho),$$

which means that the curvature is concentrated on the cosmic string axis and zero outside. In the absence of the string $\alpha = 1$, the curvature tensor vanishes. The case $\alpha > 1$ corresponds to an anti-conical spacetime with negative curvature [60, 61].

Thereby, it is physically meaningful to analyze the role of cosmic string deficit angle on the rate of pairs produced by an external electric field. For this purpose, in this paper we study particle creation by a vector Coulomb potential in the presence of a static cosmic string. Once the solutions of the Klein-Gordon and Dirac equations obtained, the expressions of the probability and the number density of the created particles will be computed using the Bogoliubov transformation. Thereafter, we discuss the influence of the cosmic string on the rate of created particles induced by the Coulomb potential for both spin-0 boson and spin–$1/2$ fermion particles. On the other hand, it is worth mentioning that the self-adjoint extensions method has recently been used to construct rigorous mathematical formulation of wave equation solutions for the Coulomb potential and similar singular potentials [62, 63].

In this work, we will use spherical coordinates since the problem has spherical symmetry, for that we consider the coordinate transformation $\rho = r \sin \theta$ and $z = r \cos \theta$, and the line element of the cosmic string becomes

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + \alpha^2 r^2 \sin^2 \theta d\phi^2,$$
where \((t, r, \theta, \phi)\): \(-\infty < t < +\infty\), \(r \geq 0\), \(0 \leq \theta \leq \pi/2\) and \(0 \leq \phi \leq 2\pi\). In the absence of the string \(\alpha = 1\), the metric (4) reduces to the Minkowski one.

The present paper is organized as follows, in sections 2 and 3 we present respectively the solutions of the Klein-Gordon and Dirac equations in the presence of a Coulomb potential in a cosmic string spacetime, for both cases we calculate the probability and the number density of created particles. The last section is devoted to the conclusion.

2 Klein-Gordon equation in cosmic string space-time

In curved spacetime, the generalized Klein-Gordon equation of spin-0 particle of mass \(M\) and charge \(q\) minimally coupled to an external electromagnetic field \(A_\mu\) is

\[
-\frac{1}{\sqrt{-g}}D_\mu g^{\mu\nu}\sqrt{-g}D_\nu + M^2 \Psi = 0,
\]

where \(D_\mu = \partial_\mu + iqA_\mu\) is the covariant derivative, \(g^{\mu\nu}\) the metric tensor of the curved spacetime and \(g = \det(g^{\mu\nu})\).

In this work, we consider the case where the spacial components of the external field have a null value \((A_i = 0; i = 1, 2, 3)\) and its time component \(A_0\) is the Coulomb potential generated by a point-like source charge \(Q_s\) where its expression is

\[
A_0 \equiv V(r) = \frac{Q_s}{r},
\]

and for an electron in a hydrogen-like atom \(Q_s = Ze\) and \(q = -e\).

Thus, the Klein-Gordon equation (5) can be written explicitly in the cosmic string metric like

\[
\left[ -\left(\frac{\partial}{\partial t} + iqA_0\right)^2 + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \left( \cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial^2}{\partial \theta^2} \right) + \frac{1}{\alpha^2 r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} - M^2 \right] \Psi = 0.
\]

Since the potential (6) is time-independent and the problem has a spherical symmetry, the solution can be taken in the form

\[
\Psi(t, r, \theta, \phi) = \frac{u(r)}{r} f(\theta) e^{i(-Et + m\phi)},
\]

where \(m = 0, \pm 1, \pm 2, \ldots\) and \(E\) is the energy of the spin-0 particle.

By Substituting Eq. (8) in Eq. (7), the angular function \(f(\theta)\) satisfies the following differential
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{m^2}{\alpha^2 \sin^2 \theta} \right] f(\theta) = -\lambda_\alpha f(\theta),
\]
where its solution is given in terms of the generalized Legendre functions \( P_\nu^\mu(x) \), with
\[
\lambda_\alpha = l_\alpha(l_\alpha + 1),
\]
where \( l_\alpha = n + |m_\alpha| = l + |m| \left( \frac{1}{\alpha} - 1 \right) \) with \( n \) a non-negative integer, \( l = n + |m| \) and \( m_\alpha = \frac{m}{\alpha} \). In addition \( m_\alpha \) and \( l_\alpha \) are not necessarily integers, and \( m_\alpha \) varies in the range \(-l_\alpha \leq m_\alpha \leq l_\alpha \). Finally, \( l \) and \( m \) are the orbital angular momentum and the magnetic quantum numbers in the flat space, respectively \[64\].

On the other hand, the radial function \( u(r) \) satisfies the following differential equation \[64\]
\[
\frac{d^2 u}{dr^2} + \left[ K^2 - V_{eff} - \frac{l_\alpha(l_\alpha + 1)}{r^2} \right] u = 0,
\]
where \( K^2 \) and \( V_{eff} \) are defined as
\[
K^2 = E^2 - M^2, \quad V_{eff} = 2qEV(r) - q^2V^2(r).
\]
By substitution of the expression (6) of the potential, Eq. (11) can be written in the form
\[
\frac{d^2 u}{dr^2} + \left[ K^2 - \frac{2EV_0}{r} - \frac{l_\alpha(l_\alpha + 1) - V_0^2}{r^2} \right] u = 0,
\]
with \( V_0 \equiv qQ_s \). If we study only the case \( V_0 > 0 \), the results for the case of \( V_0 < 0 \) can be obtained by the charge conjugation transformation \[62\].

In order to search the solutions corresponding to the condition \(|E| > M\), we use the following notations
\[
\gamma_l = \pm \sqrt{\left( l_\alpha + \frac{1}{2} \right)^2 - V_0^2}, \quad \eta = \frac{EV_0}{K},
\]
and the change \( z = -2iKr \) transforms Eq. (13) into the following form
\[
\frac{d^2 u}{dz^2} + \left[ -\frac{1}{4} - \frac{i\eta}{z} + \frac{1-\gamma_l^2}{z^2} \right] u = 0,
\]
which admits two regular linearly independent solutions in terms of the Whittaker functions \( M_{-i\eta;\gamma_l}(z) \) and \( W_{-i\eta;\gamma_l}(z) \) \[65\]
\[
u_1(z) = C_1 M_{-i\eta;\gamma_l}(z) = C_1 z^{(\gamma_l+i\frac{1}{2})} \exp(-z/2)M(\gamma_l+i\eta+\frac{1}{2},1+2\gamma_l;z),
\]
\[ u_2(z) = C_2 W_{-i\eta,\gamma_l}(z) = C_2 z^{(\gamma_l + \frac{1}{2})} \exp(-z/2) U(\gamma_l + i\eta + \frac{1}{2}, 1 + 2\gamma_l; z), \]  

(17)

\( C_1 \) and \( C_2 \) are the normalization constant, \( M(a, b, z) \) and \( U(a, b, z) \) are the Kummer’s confluent hypergeometric functions. The first solution (16) is bounded at \( z = 0 \) while the second (17) is bounded at \( |z| \to \infty \).

### 2.1 Creation of scalar particles

In order to obtain the expressions of the probability and the number density of created particles, we use the Bogoliubov transformation which links the asymptotic behavior of the obtained solutions.

Indeed, the asymptotic behavior of \( W_{-k,\mu}(z) \) for \( |z| \to \infty \) is

\[ W_{-k,\mu}(z) \to z^{-k} \exp(-z/2), \]

(18)

and the states \( \phi^+_{\text{out}} \) and \( \phi^-_{\text{out}} \) can be defined as

\[ \phi^+_{\text{out}} = C^+_{2\text{out}} W_{-k,\mu}(z), \quad \phi^-_{\text{out}} = [C^+_{2\text{out}} W_{-k,\mu}(z)]^* = [C^+_{2\text{out}}]^* W_{k,\mu}(-z). \]

(19)

Similarly, the asymptotic behavior of \( M_{-k,\mu}(z) \) for \( z \to 0 \) is

\[ M_{-k,\mu}(z) \to z^{(\mu + \frac{1}{2})}, \]

(20)

and the states \( \phi^+_{\text{in}} \) and \( \phi^-_{\text{in}} \) are

\[ \phi^+_{\text{in}} = C^+_{1\text{in}} M_{-k,\mu}(z), \quad \phi^-_{\text{in}} = [C^+_{1\text{in}} M_{-k,\mu}(z)]^* = [C^+_{1\text{in}}]^* (-1)^{(-\mu + 1/2)} M_{k,\mu}(z). \]

(21)

Therefore, the Bogoliubov transformation can be written as

\[ \phi^+_{\text{out}}(z) = A \phi^+_{\text{in}}(z) + B \phi^-_{\text{in}}(z), \]

(22)

and with the help of the two following functional relations [65]

\[ W_{-k,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(-\mu + k + \frac{1}{2})} M_{-k,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\mu + k + \frac{1}{2})} M_{k,\mu}(z), \]

(23)

\[ M_{-k,\mu}(z) = e^{\pi(\mu - 1/2)} [M_{k,\mu}(z)]^*, \]

(24)
the Bogoliubov coefficients $A$ and $B$ satisfy
\[ \frac{|B|}{|A|} = \frac{\Gamma(-\gamma_l + i\eta + \frac{1}{2}) e^{i\pi(\gamma_l - 1/2)}}{\Gamma(\gamma_l + i\eta + \frac{1}{2}) e^{i\pi(\gamma_l - 1/2)}} \] \tag{25}

where $k = i\eta$ and $\mu = \gamma_l$.

Using the bosonic condition $|A|^2 - |B|^2 = 1$ and $\gamma_l = i\tilde{\gamma}_l$, the probability for one pair production is
\[ P = \left| \frac{|B|}{|A|} \right|^2 = \left| \frac{\Gamma(-i\tilde{\gamma}_l + i\eta + \frac{1}{2}) e^{i\pi(i\tilde{\gamma}_l - 1/2)}}{\Gamma(i\tilde{\gamma}_l + i\eta + \frac{1}{2}) e^{i\pi(i\tilde{\gamma}_l - 1/2)}} \right|^2. \tag{26} \]

Then, using the property of gamma function
\[ \left| \Gamma \left( \frac{1}{2} + ix \right) \right|^2 = \frac{\pi}{\cosh(\pi x)}, \tag{27} \]
the expression of the probability is reduced to
\[ P = \frac{\cosh[\pi(\tilde{\gamma}_l + \eta)]}{\cosh[\pi(\tilde{\gamma}_l - \eta)]} e^{-2\pi\tilde{\gamma}_l}, \tag{28} \]

where
\[ \tilde{\gamma}_l = \sqrt{V_0^2 - \left[ l + |m| \left( \frac{1}{\alpha} - 1 \right) + \frac{1}{2} \right]^2} > 0, \tag{29} \]
is the condition for pair production of spinless particle which depends on $V_0 = qQ_s$, the cosmic string parameter $\alpha$ and the two quantum numbers $l$ and $m$.

The number density of the scalar particles created is given by
\[ \tilde{n} = |B|^2 = \frac{1}{P - 1 - 1} = \frac{\cosh[\pi(\tilde{\gamma}_l + \eta)]}{\cosh[\pi(\tilde{\gamma}_l - \eta)] e^{2\pi\tilde{\gamma}_l} - \cosh[\pi(\tilde{\gamma}_l + \eta)]}, \tag{30} \]
which is positive since $\tilde{\gamma}_l > 0$.

At large frequencies ($E \gg M$), the probability (28) reduces to
\[ P = \frac{\cosh[\pi(\tilde{\gamma}_l + V_0)]}{\cosh[\pi(\tilde{\gamma}_l - V_0)]} e^{-2\pi\tilde{\gamma}_l}, \tag{31} \]
and the number density takes the form
\[ \tilde{n} = \frac{\cosh[\pi(\tilde{\gamma}_l + V_0)]}{\cosh[\pi(\tilde{\gamma}_l - V_0)] e^{2\pi\tilde{\gamma}_l} - \cosh[\pi(\tilde{\gamma}_l + V_0)]}, \tag{32} \]
which is not a thermal distribution.
For the case when $|\tilde{\gamma}_l \pm V_0| \gg 1$, the number density (32) takes the form of Bose-Einstein distribution

$$\bar{n} = \frac{1}{e^{2 \pi (\tilde{\gamma}_l - V_0)} - 1}.$$  

(33)

In the absence of the cosmic string ($\alpha = 1$), or for $m = 0$, the probability (31) and the number density (32) reduce to

$$P(\alpha = 1) = \frac{\cosh [\pi (\tilde{\gamma}_1 + V_0)]}{\cosh [\pi (\tilde{\gamma}_1 - V_0)]} \exp [-2 \pi \tilde{\gamma}_1],$$

(34)

$$\bar{n}(\alpha = 1) = \frac{\cosh [\pi (\tilde{\gamma}_1 + V_0)]}{\cosh [\pi (\tilde{\gamma}_1 - V_0)] \cosh [\pi (\tilde{\gamma}_1 + V_0)]},$$

(35)

where $\tilde{\gamma}_1 \equiv \tilde{\gamma}(\alpha = 1) = \sqrt{V_0^2 - (l + \frac{1}{2})^2}$.

### 2.2 Discussion of results

- First, let us discuss the consequences of the condition (29), for particle production of spin-0 boson, which can be written as

$$|qQ_s| > \left[l + |m| \left(\frac{1}{\alpha} - 1\right) + \frac{1}{2}\right].$$  

(36)

In the absence of the cosmic string ($\alpha = 1$), or for $m = 0$, the above condition is reduced to

$$|qQ_s| > \left(l + \frac{1}{2}\right),$$

(37)

which is similar to that obtained in Eq. (37) for pair creation in a magnetic monopole field [30], and also similar to the results discussed after Eq. (84) for pair creation by charged black holes [66].

For an electron in a hydrogen-like atom, $Q_s = Ze$ and $q = -e$, the condition (36) can be expressed for the nucleus charge $Z$ as

$$Z > Z_{cr} \equiv \frac{1}{e^2} \left[l + |m| \left(\frac{1}{\alpha} - 1\right) + \frac{1}{2}\right].$$

(38)

For the particular values of the quantum numbers ($l = 0, m = 0$), the pair production of scalar particles is possible for: $Z > Z_{cr} = \frac{1}{2e^2} = \frac{137}{2}$ (in units $\hbar = c = 1$, $e^2 \simeq 1/137$), which is a well-known result in the absence of the cosmic string [22].
Table 1: $Z_{cr}$ in terms of $\alpha$, values and the rate of $Z_{cr}^{GUT}$ for boson particle

| $l$ | $m$ | $Z_{cr}$ | $Z_{cr}^{GUT}$ | $\frac{Z_{cr}^{GUT} - Z_{cr}(\alpha=1)}{Z_{cr}(\alpha=1)}$ |
|-----|-----|---------|---------------|--------------------------------------------------|
| 0   | 0   | $\frac{137}{2}$ = 68.5 | | |
| 1   | 0   | $137 \times \frac{3}{2}$ = 205.5 | | |
|     | $\pm 1$ | $137 \times (\frac{1}{\alpha} + \frac{1}{2})$ | 205.500137 | $6.666666 \times 10^{-5}\%$ |
| 2   | 0   | $137 \times \frac{5}{2}$ = 342.5 | | |
|     | $\pm 1$ | $137 \times (\frac{1}{\alpha} + \frac{3}{2})$ | 342.500137 | $4.000004 \times 10^{-5}\%$ |
|     | $\pm 2$ | $137 \times (\frac{2}{\alpha} + \frac{1}{2})$ | 342.500274 | $8 \times 10^{-5}\%$ |

In table 1, for the first values of the quantum numbers $(l, m)$, we give the critical values $Z_{cr}$ as function of the cosmic string parameter $\alpha$ and an estimation of the effect of the GUT cosmic string (for which [67]: $\alpha = 1 - 10^{-6} = 0.999999$) on the values and the rate of $Z_{cr}^{GUT}$.

We note that for the sub-states with $m = 0$, the critical value $Z_{cr}$ increases with $l$ and is independent of the cosmic string parameter $\alpha$ (Note that these sub-states are equivalent to the case $\alpha = 1$). For the sub-states for which $m \neq 0$, $Z_{cr}$ increases with $l$ and depends inversely on the cosmic string parameter $\alpha$. For the GUT cosmic string, the obtained critical values $Z_{cr}^{GUT}$ do not exist for the moment (at present, the maximum is $Z = 118$ in Mendeleev table), the rate of $Z_{cr}^{GUT}$ is about $10^{-5}\%$ as compared with the case $\alpha = 1$ (without cosmic string) and the production of scalar particles is possible if the Coulomb potential nucleus charge (or atomic number) $Z \geq 206$.

- Secondly, in Figure 1 the number density curves have been plotted for fixed values of the quantum numbers ($l = 1, m = 1$) and where the condition (29) for pair production of spinless particles is reduced to $V_0 > (\frac{1}{2} + \frac{1}{\alpha})$ which should be always fullfilled. Plot (a) display, in three (3D) dimensions, the number density (32) in terms of $V_0 = qQ_s$ and the cosmic string parameter $\alpha$. Plot (b) display, in two (2D) dimensions, the number density (32) in terms of $V_0 = qQ_s$ for different values the cosmic string parameter $\alpha$. 

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Figure 1: Plot (a) is the 3D number density (32) in terms of $V_0$ and $\alpha$. Plot (b) is the 2D number density (32) in terms of $V_0$ for different values of $\alpha = 0.3, 0.6, 0.8$ and 1 (without cosmic string).

From plot (b) we note that the number density $\bar{n}$ decreases when the cosmic string parameter $\alpha$ increases (i.e. the linear mass density $\mu$ of the string decreases) where its smallest values correspond to the case $\alpha = 1$ (in the absence of the cosmic string). Therefore, we deduce that the presence of the cosmic string improves the number density of scalar particles compared to the case without the cosmic string ($\alpha = 1$).

3 Dirac equation in cosmic string space-time

In curved spacetime, the generalized Dirac equation of spin $-1/2$ particle of mass $M$ and charge $q$ minimally coupled to an external electromagnetic field $A_\mu$ is

$$[i\gamma^\mu(x) (\nabla_\mu + iqA_\mu) - M] \Psi(x) = 0, \quad (39)$$
where $\nabla_\mu = \partial_\mu + \Gamma_\mu(x)$ is the covariant derivative, $\Gamma_\mu$ is the spinor affine connection

$$\Gamma_\mu = \frac{1}{4} \gamma^{(a)}(x) \gamma^{(b)}(x) \left[ \partial_\mu e_{(b)\nu} - \Gamma_{\mu\nu}^{a\sigma} e_{(b)\sigma} \right], \quad (40)$$

$\Gamma_{\mu\nu}^{a\sigma}$ is the Christoffel symbol of the second kind, $\{e^{\mu}_{(a)}(x)\}$ the tetrad basis and $\gamma^{\mu}(x)$ the generalized Dirac matrix satisfying the relation

$$\{\gamma^{\mu}(x), \gamma^{\nu}(x)\} = 2 g^{\mu\nu}(x), \quad (41)$$

which can be written as a function of the tetrad basis $\{e^{\mu}_{(a)}(x)\}$ and the standard flat spacetime Dirac matrices $\gamma_{(a)}$ as

$$\gamma^{\mu}(x) = e^{\mu}_{(a)}(x) \gamma_{(a)} \quad (42)$$

The tetrad basis $\{e^{\mu}_{(a)}(x)\}$ satisfies the relations

$$\eta^{(a)(b)} e^{\mu}_{(a)}(x) e^{\nu}_{(b)}(x) = g^{\mu\nu}(x) \quad (43)$$

where the Greek letters are for tensor indices and Latin letters for tetrad indices.

We consider the case where the spacial components of the external field are nuls ($A_i = 0, i = 1, 2, 3$) and its time component $A_0(r) \equiv V(r)$ is the Coulomb potential defined in Eq. (6).

In order to write the Dirac equation in the cosmic string metric (4), we use the associated tetrad $e^{\mu}_{(a)}(x)$ defined by [67, 68]

$$e^{\mu}_{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ 0 & \cos \theta \cos \phi \frac{1}{r} & \cos \theta \sin \phi \frac{1}{r} & -\sin \theta \frac{1}{r} \\ 0 & \sin \phi \frac{1}{\rho \sin \theta} & \cos \phi \frac{1}{\rho \sin \theta} & 0 \end{pmatrix}, \quad (44)$$

and the expressions of the generalized Dirac matrices $\gamma^{\mu}(x)$ are

$$\gamma^{0}(x) = \gamma^{(0)}, \quad (45)$$

$$\gamma^{1}(x) = \sin \theta \cos \phi \gamma^{(1)} + \sin \theta \sin \phi \gamma^{(2)} + \cos \theta \gamma^{(3)}, \quad (46)$$

$$\gamma^{2}(x) = \cos \theta \cos \phi \frac{1}{r} \gamma^{(1)} + \cos \theta \sin \phi \frac{1}{r} \gamma^{(2)} - \frac{\sin \theta}{r} \gamma^{(3)}, \quad (47)$$
\[ \gamma^2(x) = -\frac{\sin \phi}{\alpha r \sin \theta} \gamma^{(1)} + \frac{\cos \phi}{\alpha r \sin \theta} \gamma^{(2)}. \] (48)

Since the problem is time-independent, the wavefunction can be written as

\[ \Psi(x) = e^{-iEt} \chi(\vec{r}), \] (49)

and by substitution of Eq. (49) in Eq. (39), we obtain [67]

\[ i(\gamma^{(r)}\partial_r + \frac{\gamma^{(\theta)}}{r}\partial_\theta + \frac{\gamma^{(\phi)}}{\alpha r \sin \theta}\partial_\phi + \gamma^{(0)}E + i\frac{1}{2r} \left(1 - \frac{1}{\alpha}\right) \left(\gamma^{(r)} + \cot \theta \gamma^{(\theta)}\right) - \gamma^{(0)}qA_0 - M \right) \chi(\vec{r}) = 0, \] (50)

where

\[ \begin{pmatrix} \gamma^{(r)} \\ \gamma^{(\theta)} \\ \gamma^{(\phi)} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \phi \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \phi \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \end{pmatrix}. \] (51)

Then, since the problem has a spherical symmetry, the solution of Eq. (50) can be taken as

\[ \chi(\vec{r}) = r^{-\frac{1}{2}(1-\frac{1}{\alpha})} \left(\sin \theta\right)^{-\frac{1}{2}(1-\frac{1}{\alpha})} R(r) Y_{l\alpha m\alpha}(\theta, \phi), \] (52)

where \( R(r) \) is the radial part and \( Y_{l\alpha m\alpha}(\theta, \phi) \) the spherical harmonics. The parameters \( m_\alpha = \frac{m}{\alpha} \) and \( l_\alpha = n + |m_\alpha| = l + |m| (\frac{1}{\alpha} - 1) \) with \( l = n + |m| \) and \( n \) a non-negative integer [69].

By substitution of Eq. (52) into Eq. (50), we obtain [67]

\[ \left(\sigma_y p_r + \frac{i \sigma_y}{r} \gamma^{(0)} k_{(\alpha)} + qA_0 + M \gamma^{(0)} \right) R(r) = ER(r), \] (53)

where \( p_r \) is the radial momentum operator, \( k_{(\alpha)} = \pm \left[ j + |m| \left(\frac{1}{\alpha} - 1\right) + \frac{1}{2}\right] \) with \( j \) is the eigenvalue of the total angular momentum operator in flat Minkowski spacetime [69], and \( \sigma_y \) is Pauli’s matrix

\[ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \] (54)

For the radial solution we use the following ansatz [70]

\[ R(r) = \frac{1}{r} \begin{pmatrix} u_1(r) \\ iu_2(r) \end{pmatrix}, \] (55)

and by substitution of Eq. (55) in Eq. (53) and using Eq. (54), we have

\[ \frac{du_1(r)}{dr} = -\frac{k_{(\alpha)}}{r} u_1(r) + \left[E + M - qA_0\right] u_2(r), \] (56)
\[
\frac{du_2(r)}{dr} = \frac{k_{(a)}}{r} u_2(r) - [E - M - qA_0] u_1(r).
\] (57)

By taking \( V_0 \equiv qQ_s \) and \( A_0 \) as the Coulomb potential defined in Eq. (6), the above equations become

\[
\frac{du_1(r)}{dr} = -\frac{k_{(a)}}{r} u_1(r) + \left[ E + M - \frac{V_0}{r} \right] u_2(r),
\] (58)

\[
\frac{du_2(r)}{dr} = \frac{k_{(a)}}{r} u_2(r) - \left[ E - M - \frac{V_0}{r} \right] u_1(r).
\] (59)

If we study only the case \( V_0 > 0 \), the results for the case of \( V_0 < 0 \) can be obtained by the charge conjugation transformation [62]. Its convenient to introduce the new coordinate

\[
z = 2iKr, \quad K = \sqrt{E^2 - M^2},
\] (60)

then Eqs. (58)-(59) transform into

\[
\frac{du_1(z)}{dz} = -\frac{k_{(a)}}{z} u_1(z) + \left[ \frac{E + M}{2iK} - \frac{V_0}{z} \right] u_2(z),
\] (61)

\[
\frac{du_2(z)}{dz} = \frac{k_{(a)}}{z} u_2(z) - \left[ \frac{E - M}{2iK} - \frac{V_0}{z} \right] u_1(z),
\] (62)

and using the ansatz

\[
u_1(z) = \sqrt{E + M} \left[ F(z) + G(z) \right], \quad u_2(z) = i\sqrt{E - M} \left[ F(z) - G(z) \right],
\] (63)

in Eqs. (61)-(62), we have

\[
\frac{dF(z)}{dz} = \left( \frac{1}{2} - i\frac{V_0E}{Kz} \right) F(z) + \left( -\frac{k_{(a)}}{z} - i\frac{V_0M}{Kz} \right) G(z),
\] (64)

\[
\frac{dG(z)}{dz} = \left( -\frac{1}{2} + i\frac{V_0E}{Kz} \right) G(z) + \left( -\frac{k_{(a)}}{z} + i\frac{V_0M}{Kz} \right) F(z)
\] (65)

and from which one easily obtains the following second order differential equation

\[
\frac{d^2F(z)}{dz^2} + \frac{1}{z} \frac{dF(z)}{dz} - \left[ \frac{1}{4} + \left( \frac{1}{2} - i\frac{E V_0}{K} \right) \frac{1}{z} + \frac{\gamma^2 l^2}{z^2} \right] F(z) = 0,
\] (66)

with \( \gamma^2 = k_{(a)}^2 - V_0^2 \).

The following ansatz

\[
M(z) = \sqrt{z} F(z),
\] (67)
transforms the last equation to
\[
\frac{d^2 M(z)}{dz^2} + \left[ -\frac{1}{4} - \frac{i\lambda}{z} + \frac{1}{2} - \frac{\gamma_l}{2} \right] M(z) = 0,
\]
(68)
where \( i\lambda = \frac{1}{2} - i\eta \) and \( \eta = \frac{E V_0}{K} \).

This equation admits two regular linearly independent solutions in terms of the Whittaker functions [65]
\[
u_1(z) = C_3M_{i\lambda,\gamma_l}(z) = C_3z^{\gamma_l+\frac{1}{2}}e^{-\gamma_l/2}M(\gamma_l + i\lambda + \frac{1}{2}, 2\gamma_l + 1; z),
\]
(69)
\[
u_2(z) = C_4W_{i\lambda,\gamma_l}(z) = C_4z^{\gamma_l+\frac{1}{2}}e^{-\gamma_l/2}U(\gamma_l + i\lambda + \frac{1}{2}, 2\gamma_l + 1; z),
\]
(70)
where the first solution is bounded at \( z = 0 \) while the second is bounded at \( |z| \to \infty \), \( C_3 \) and \( C_4 \) are the normalization constants.

### 3.1 Creation of fermionic particles

In order to obtain the expressions of the probability and the number density of created particles, we proceed as in the Klein-Gordon case using the formulas (18-24) with \( k = i\lambda \) and \( \mu = \gamma_l = i\tilde{\gamma}_l \).

Then, the Bogoliubov coefficients \( A \) and \( B \) verify
\[
\left| \frac{B}{A} \right| = \left| \frac{\Gamma(1 - i(\tilde{\gamma}_l + \eta))}{\Gamma(1 + i(\tilde{\gamma}_l - \eta))} e^{i\pi(\tilde{\gamma}_l - \frac{1}{2})} \right|,
\]
(71)
and the probability of created fermionic particles is
\[
P = \left| \frac{B}{A} \right|^2 = \left| \frac{\Gamma(1 - i(\tilde{\gamma}_l + \eta))}{\Gamma(1 + i(\tilde{\gamma}_l - \eta))} \right|^2 e^{-2\pi\tilde{\gamma}_l},
\]
(72)
where
\[
\tilde{\gamma}_l = \sqrt{V_0^2 - \left[ j + |m| \left( \frac{1}{\alpha} - 1 \right) + \frac{1}{2} \right]^2} > 0,
\]
(73)
is the condition for pair production of spin-1/2 particle which depends on \( V_0 = qQ_s \), the cosmic string parameter \( \alpha \) and the quantum numbers \( j \) and \( m \).

Using the fermionic condition \( |A|^2 + |B|^2 = 1 \) and the following property of the gamma function
\[
|\Gamma(1 + ix)|^2 = \frac{\pi x}{\sinh(\pi x)},
\]
(74)
the expression of the probability is

\[ P = \frac{(\tilde{\gamma}_l + \eta) \sinh [\pi (\tilde{\gamma}_l - \eta)]}{(\tilde{\gamma}_l - \eta) \sinh [\pi (\tilde{\gamma}_l + \eta)]} e^{-2\pi \tilde{\gamma}_l}. \] (75)

Then, the number density of the created fermionic particles is calculated as

\[ \bar{n} = \frac{1}{P^{-1} + 1} = \frac{(\tilde{\gamma}_l + \eta) \sinh [\pi (\tilde{\gamma}_l - \eta)]}{(\tilde{\gamma}_l - \eta) \sinh [\pi (\tilde{\gamma}_l + \eta)]} e^{2\pi \tilde{\gamma}_l} + \frac{(\tilde{\gamma}_l + \eta) \sinh [\pi (\tilde{\gamma}_l - \eta)]}{(\tilde{\gamma}_l - \eta) \sinh [\pi (\tilde{\gamma}_l + \eta)]}. \] (76)

At a large frequencies \((E \gg M)\), the probability (75) reduces to

\[ P = \frac{(\tilde{\gamma}_l + V_0) \sinh [\pi (\tilde{\gamma}_l - V_0)]}{(\tilde{\gamma}_l - V_0) \sinh [\pi (\tilde{\gamma}_l + V_0)]} e^{-2\pi \tilde{\gamma}_l}. \] (77)

and the number of created particle (76) can be written as

\[ \bar{n} = \frac{(\tilde{\gamma}_l + V_0) \sinh [\pi (\tilde{\gamma}_l - V_0)]}{(\tilde{\gamma}_l - V_0) \sinh [\pi (\tilde{\gamma}_l + V_0)]} e^{2\pi \tilde{\gamma}_l} + \frac{(\tilde{\gamma}_l + V_0) \sinh [\pi (\tilde{\gamma}_l - V_0)]}{(\tilde{\gamma}_l - V_0) \sinh [\pi (\tilde{\gamma}_l + V_0)]}, \] (78)

which is not a thermal distribution.

For the case when \(|\tilde{\gamma}_l \pm V_0| \gg 1\), the number density takes the form of Fermi-Dirac distribution

\[ \bar{n} = \frac{1}{e^{2\pi (\tilde{\gamma}_l - V_0)} + 1}. \] (79)

In the absence of the cosmic string \((\alpha = 1)\), or for \(m = 0\), the probability (77) and the number density (78) reduce to

\[ P(\alpha = 1) = \frac{(\tilde{\gamma}_1 + V_0) \sinh [\pi (\tilde{\gamma}_1 - V_0)]}{(\tilde{\gamma}_1 - V_0) \sinh [\pi (\tilde{\gamma}_1 + V_0)]} e^{-2\pi \tilde{\gamma}_1}, \] (80)

\[ \bar{n}(\alpha = 1) = \frac{(\tilde{\gamma}_1 + V_0) \sinh [\pi (\tilde{\gamma}_1 - V_0)]}{(\tilde{\gamma}_1 - V_0) \sinh [\pi (\tilde{\gamma}_1 + V_0)]} e^{2\pi \tilde{\gamma}_1} + \frac{(\tilde{\gamma}_1 + V_0) \sinh [\pi (\tilde{\gamma}_1 - V_0)]}{(\tilde{\gamma}_1 - V_0) \sinh [\pi (\tilde{\gamma}_1 + V_0)]}, \] (81)

where \(\tilde{\gamma}_1 \equiv \tilde{\gamma}_l(\alpha = 1) = \sqrt{V_0^2 - (j + \frac{1}{2})^2}\).

### 3.2 Discussion of results

- First, let us discuss the consequences of the condition (73) for pair production of spin-1/2 fermion.

It can be written as

\[ |qQ_s| > \left[ j + |m| \left( \frac{1}{\alpha - 1} + \frac{1}{2} \right) \right]. \] (82)
In the absence of the cosmic string \((\alpha = 1)\), or for \(m = 0\), the above condition is reduced to
\[
|qQ_s| > \left( j + \frac{1}{2} \right) = (l + 1).
\] (83)

For an electron in a hydrogen-like atom, \(Q_s = Ze\) and \(q = -e\), the condition (82) can be expressed for the nucleus charge \(Z\) as
\[
Z > Z_{cr} \equiv \frac{1}{e^2} \left[ j + |m| \left( \frac{1}{\alpha} - 1 \right) + \frac{1}{2} \right].
\] (84)

For the particular values of the quantum numbers \((l = 0, m = 0, s = 1/2, j = 1/2)\), the pair production of fermionic particles is possible for \(Z > Z_{cr} = \frac{1}{e^2} = 137\) (in units \(\hbar = c = 1, e^2 \simeq 1/137\)), which is a well-known result in the absence of the cosmic string [22].

In table 2, for the first values of the quantum numbers \((l, m, s = 1/2, j = l + 1/2)\), we give the critical values \(Z_{cr}\) as function of the string parameter \(\alpha\) and an estimation of the effect of the GUT cosmic string on the values and the rate of \(Z_{cr}^{GUT}\).

| Table 2: \(Z_{cr}\) in terms of \(\alpha\), values and the rate of \(Z_{cr}^{GUT}\) for fermion particle |
|---|---|---|---|---|
| \(l\) | \(m\) | \(j = l + \frac{1}{2}\) | \(Z_{cr}\) | \(Z_{cr}^{GUT}\) | \(\frac{Z_{cr}^{GUT} - Z_{cr}(\alpha=1)}{Z_{cr}(\alpha=1)}\) |
| 0 | 0 | \(\frac{1}{2}\) | 137 |  |  |
| 1 | 0 | \(\frac{3}{2}\) | 137 \(\times 2 = 274\) | 274,000137 | 5,000005 \(\times 10^{-5}\%\) |
| | \(\pm 1\) | \(\frac{3}{2}\) | 137 \(\times (\frac{1}{\alpha} + 1)\) | 411,000137 | 3,333333 \(\times 10^{-5}\%\) |
| 2 | 0 | \(\frac{5}{2}\) | 137 \(\times 3 = 411\) |  |  |
| | \(\pm 1\) | \(\frac{5}{2}\) | 137 \(\times (\frac{1}{\alpha} + 2)\) | 411,000274 | 6,666666 \(\times 10^{-5}\%\) |
| | \(\pm 2\) | \(\frac{5}{2}\) | 137 \(\times (\frac{2}{\alpha} + 1)\) |  |  |

We note that for the sub-states with \(m = 0\), the critical value \(Z_{cr}\) increases with \(l\), depends on the spin value \((s = 1/2)\) and is independent of the cosmic string parameter \(\alpha\) (Note that these sub-states are equivalent to the case \(\alpha = 1\)). For the sub-states for which \(m \neq 0\), \(Z_{cr}\) increases with \(l\), depends linearly on the spin value \((s = 1/2)\) and inversely on the cosmic string parameter \(\alpha\). For
the GUT cosmic string, the obtained critical values $Z_{cr}^{GUT}$ do not exist for the moment, the rate of $Z_{cr}^{GUT}$ is about $10^{-5}$% as compared with the case $\alpha = 1$ (without cosmic string) and the production of spin-1/2 particles is possible if the Coulomb potential nucleus charge $Z \geq 275$.

- Secondly, in Figure 2 the number density curves have been plotted for fixed values of the quantum numbers ($l = 1, m = 1, s = 1/2, j = 3/2$) and where the condition (73) for pair production of spin-1/2 particles is reduced to $V_0 > (1 + \frac{1}{2})$ which should be always fulfilled. Plot (a) display, in three (3D) dimensions, the number density (78) in terms of $V_0 = qQ_s$ and the cosmic string parameter $\alpha$. Plot (b) display, in two (2D) dimensions, the number density (78) in terms of $V_0 = qQ_s$ for different values of the cosmic string parameter $\alpha$.

Figure 2: Plot (a) is the 3D number density (78) in terms of $V_0 \in [0, +10]$ and $\alpha \in [0, 1]$. Plot (b) is the 2D number density (78) in terms of $V_0 \in [0, 5]$ for the different values of $\alpha = 0.3, 0.5, 0.8$ and 1 (without cosmic string).
In order to study the shape of the curves for large values of $V_0$, the number density (78) has been plotted in figure 3 in the short interval of $V_0 \in [50.0, 50.6]$ for different values of $\alpha$. From which we note that the number density $\bar{n}$ decreases when the cosmic string parameter $\alpha$ increases (i.e. the linear mass density $\mu$ of the string decreases) where its smallest values correspond to the case $\alpha = 1$ (in the absence of the cosmic string). Therefore, we deduce that the presence of the cosmic string improves the number density of fermion particles compared to the case without the cosmic string ($\alpha = 1$).

Figure 3: Number density (78) in terms of $V_0 \in [50.0, 50.6]$ for different values of $\alpha = 0.3, 0.5, 0.8$ and 1 (without cosmic string).

4 Conclusion

In this paper, we have investigated the influence of a cosmic string on the pair production rate induced by the Coulomb potential of an external electric field. The solutions of the radial Klein-Gordon and Dirac equations are given in terms of Whittaker functions, the probability and the
number density of created particles have been calculated.

For the spin-0 boson case, the creation of particles holds for \(|qQ_s| > [l + |m| (\frac{1}{\alpha} - 1) + \frac{1}{2}]\), the number density decreases when the cosmic string parameter \(\alpha\) increases in the interval \([0, 1]\) and its smallest values are reached for \(\alpha = 1\). Thus, the presence of the cosmic string improves the number density of created spinless particles compared to the case without the cosmic string \((\alpha = 1)\). For the particular values of the quantum numbers \((l = 0, m = 0)\), the pair production of scalar particles is possible if the Coulomb potential nucleus charge \(Z > \frac{137}{2}\).

For the spin-1/2 fermion case, the creation of particles holds for \(|qQ_s| > [j + |m| (\frac{1}{\alpha} - 1) + \frac{1}{2}]\), the number density decreases with the increasing of the cosmic string parameter \(\alpha\) in the interval \([0, 1]\) and its smallest values are reached for \(\alpha = 1\). Thus, the presence of the cosmic string improves also the number density of created spin-1/2 particles compared to the case without the cosmic string \((\alpha = 1)\). For the particular values of the quantum numbers \((l = 0, m = 0, s = 1/2, j = 1/2)\), the pair production of fermionic particles is possible if the Coulomb potential nucleus charge \(Z > 137\).

As a result for the GUT cosmic string, the pair production for spinless particles is possible if the Coulomb potential nucleus charge \(Z \geq 206\), and for spin-1/2 particles if \(Z \geq 275\).

In limiting case of the Minkowski spacetime \((\alpha = 1)\), as expected we retrieve the expressions of the number density of the scalar and fermionic particles created by the Coulomb potential in the absence of the cosmic string. Finally, we note that the results for the sub-states for which the magnetic quantum number \(m = 0\) are independent of the cosmic string parameter \(\alpha\), i.e. these sub-states are insensitive to the presence of the cosmic string.

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