Plasma Pressure Driven Asymmetric Supernovae and Highly Collimated Gamma-Ray Bursts

K.H. Tsui and C.E. Navia

Instituto de Física - Universidade Federal Fluminense
Campus da Praia Vermelha, Av. General Milton Tavares de Souza s/n
Gragoatá, 24.210-346, Niterói, Rio de Janeiro, Brasil.

tsui@if.uff.br

ABSTRACT

During the process of collapse of a massive star, a cavity is generated between the central iron core and an outer stellar envelope. The dynamics of this cavity, filled with plasma and magnetic field of the rapidly rotating proto-magnetar’s magnetosphere, is believed to be very relevant in understanding supernovae and gamma-ray bursts. The interactions of the pressurized conducting plasma and the magnetic fields are described by a set of magnetohydrodynamic (MHD) equations with poloidal and toroidal plasma flows not aligned with magnetic fields. A sequence of MHD equilibria in response to the increasing plasma pressure in the cavity, by continuous filling from the rotating magnetosphere, is solved to account for asymmetric supernovae, highly collimated gamma-ray burst jets, and also active galactic nucleus plasma torus. It is shown that the magnetosphere of the central compact star is likely the central engine of supernova and gamma-ray burst by feeding them plasma, magnetic energy, and rotational energy.

Subject headings: supernova, gamma-ray burst, magnetic towering
1. Introduction

Ever since Baade and Zwicky had coined the term super-novae and had suggested supernova process as the end of an ordinary star to become a neutron star (Baade & Zwicky 1934), supernovae have been puzzling scientific minds for decades. They are believed to be the catastrophic end of massive stars’ life cycle. Supernovae other than the type Ia standard candle, where the entire star is incinerated, are believed to have gone through a gravitational collapse of the iron core (Bethe 1990; Kotake et al. 2006). The progenitor iron core with about $10^4 \text{Km}$ radius is enclosed by the stellar hydrogen envelope extending out to some $10^8 \text{Km}$ radius. The collapse of this progenitor core is governed by the sound speed profile, which decreases as the radial position increases. The infalling velocity in the outer part of the core is supersonic with respect to the local sound speed, while the inner part is subsonic with the interface at about 300 $\text{Km}$ radius. Because of this condition, the outer part tends to pile up while the inner part continues to free-fall. This generates a proto-neutron star plus the overlying stellar material. A cavity is formed between the proto-neutron star and the hydrogen stellar envelope. The gravitational collapse of a magnetized massive star with jet formations was investigated (LeBlanc & Wilson 1970), and an explosion mechanism not associated with nuclear detonation was proposed (Bisnovatyi 1971).

Some of the core-collapse supernovae appear to be associated to gamma-ray bursts (Galama et al. 1999; Campana et al. 2006; Pian et al. 2006; Soderberg et al. 2006; Mazzali et al. 2006). As a result, there could be a very close relationship between supernovae and gamma-ray bursts. By the extraordinary energy output, several scenarios have been contemplated as the progenitors of gamma-ray bursts (Meszaros 2002). For massive progenitor stars with more than 14 solar masses, like rapidly spinning Wolf-Rayet stars, the inner core could promptly collapse to a black hole circumscribed by a massive torus, failing to generate a core rebound in the cavity. Accretion of the surrounding massive torus of nuclear density material at a later time drive an outburst along the rotational axis breaking out the stellar envelope. This is the Collapsar (or Hypernova) scheme (MacFadyen & Woosley 1999). Instead of promptly collapsing to a black hole, there also could be a two-stage collapse by first forming a rapidly rotating neutron star temporarily stabilized by rotation, and later collapses to a black hole after losing some angular momentum, which is the Supranova (Vietri & Stella 1999) scheme for long gamma-ray bursts. When the inner core does not collapse to a black hole, it rebounds as the inner core reaches nuclear densities, generating a rebounding shock.
in the cavity. The release of the gravitational binding energy through neutrino bursts has been considered as the primary candidate in fueling the supernova explosion (Bethe 1990; Kotake et al. 2006; Bethe & Wilson 1985). Nevertheless, as this outgoing shock meets the infalling outer very thick envelope, energy of the shock wave gets dissipated, and the shock is stalled. By which energy source and mechanism that the shock could be reignited is still an unsettled issue.

Recently, polarimetry observations of supernova optical emissions have revealed different degrees of polarization along a fixed axis of the supernova. In a very collisional environment of a supernova, this polarization result implies a nonzero volume average of the microscopically random electric field vector of each emission, indicating supernovae, or some them, are aspherical (Howell et al. 2001; Wang et al. 2003). In view of other astrophysical ejection events known to be powered by magnetic fields, these polarization observations have given grounds to reexamine the magnetic field as the central engine for supernova (Wheeler et al. 2000, 2002; Ardeljan et al. 2003; Uzdensky & MacFadyen 2007; Burrows et al. 2007; Komissarov & Barkov 2007). The primary concern in this renewed magnetic approach is the magnetic collimation mechanism. It has been proposed the presence of an accretion disk within the cavity of the collapsing iron core, by which magnetic towering (Lynden-Bell 2003) and jets (Blandford & Payne 1982) could be generated along the rotational axis.

Should magnetic field be the energy source of supernova, a simple estimation indicates that the surface fields of the rapidly rotating proto-neutron star would be around $10^{15} \text{Gs}$, which qualifies it as a magnetar. Such magnetar scenario could be accomplished if the rotation of the proto-neutron star is fast enough. With millisecond periods, dynamo effects inside the pulsar could be triggered, enhancing the magnetic fields by a factor of $10^3$ (Duncan & Thompson 1992; Thompson & Duncan 1993). These magnetic fields would be launched from the magnetosphere together with the plasma to fill the cavity, leaving a normal spun-down neutron star at the center after supernova explosion. In general, the dynamics of this cavity driven by the rapidly rotating inner core is referred to as the ‘Pulsar in a Cavity’ model (Uzdensky & MacFadyen 2007). Simulation has been the principle tool in investigating the dynamics in this cavity (Burrows et al. 2007; Komissarov & Barkov 2007). Most frequently, an accretion disk scenario with magnetic fields anchored on it is considered. Under given initial conditions, magnetic towering (Lynden-Bell 2003) due to the angular momentum of the disk generates a collimating magnetic column along the polar axis.
Here, we seek to describe the cavity structures, filled with plasma and magnetic fields, through a sequence of axisymmetric magnetohydrodynamic (MHD) steady states (equilibria) with poloidal and toroidal plasma rotations as a response to the increasing plasma pressure. This type of steady state analysis has been used to gain important insights of dynamic processes such as astrophysical jets in terms of spatially self-similar MHD equilibria (Blandford & Payne 1982) and magnetic towering in terms of magnetostatic analysis of disk driven equilibria (Lynden-Bell 2003). The present problem differs from the relativistic pulsar wind problem with hot plasma (Lovelace et al. 1986; Camenzind 1986) especially under self-similar analysis (Prendergast 2005; Gourgouliatos & Vlahakis 2010) and with cold plasma (Okamoto 1978; Begelman 1994; Okamoto 2002) in that the spatial domain is bounded and very finite.

In Sec.2, the axisymmetric divergence-free rotational MHD formulation is briefly presented, and in Sec.3, by assigning two source functions, the Grad-Shafranov equation for rotational equilibrium is obtained. The nonlinear poloidal flux function is solved in Sec.4 for asymmetric supernova configuration in the cavity, whereas cusp-like funnel polar collimated gamma-ray burst jet configuration is solved in Sec.5. Active galactic nucleus plasma torus and some conclusions are finalized in Sec.6.
2. Divergence-Free Rotational MHD

The standard steady state MHD equations are

\[
\nabla \cdot (\rho \vec{v}) = 0 , \\
\rho(\vec{v} \cdot \nabla)\vec{v} = \vec{J} \times \vec{B} - \nabla p - \rho \frac{GM}{r^3} \vec{r} , \\
\vec{v} \times \vec{B} = -\vec{E} , \\
\nabla \times \vec{B} = \mu \vec{J} , \\
\nabla \cdot \vec{B} = 0 , \\
(\vec{v} \cdot \nabla) \left( \frac{p}{\rho^\gamma} \right) = 0 .
\]

Here, \( \rho \) is the mass density, \( \vec{v} \) is the bulk velocity, \( \vec{J} \) is the current density, \( \vec{B} \) is the magnetic field, \( p \) is the plasma pressure, \( M \) is the mass of the central body, \( \mu \) is the free space permeability, and \( \gamma \) is the polytropic index. To describe plasma equilibrium with poloidal and toroidal flows, this set of equations is quite inconvenient. We will use the MHD equations for divergence-free plasma flows, where the plasma velocity is density weighed so that the plasma density appears only through the density weighed velocity \( \vec{w}_* = (\mu \rho)^{1/2} \vec{v} \), except in the gravity term (Tsui et al. 2011; Tsui & Navia 2012).

\[
(\mu \rho)^{1/2} \nabla \cdot \vec{w}_* + \vec{w}_* \cdot \nabla (\mu \rho)^{1/2} = 0 , \\
\vec{w}_* \times \nabla \times \vec{w}_* - \vec{B} \times \nabla \times \vec{B} = \nabla \mu p_* - \rho \frac{GM}{r^3} \vec{r} , \\
\vec{w}_* \times \vec{B} = (\mu \rho)^{1/2} \nabla \tilde{\Phi} = \nabla \Phi , \\
\nabla \times \vec{B} = \mu \vec{J} , \\
\nabla \cdot \vec{B} = 0 , \\
\nabla \cdot \vec{\tilde{v}} = 0 ,
\]

where \( \mu p_* = \mu p + w_*^2 / 2 \) is the total plasma pressure. With divergence-free flows of Eq. 6, density flux conservation gives \( \vec{\tilde{v}} \cdot \nabla \rho = 0 \). By using the \( (\vec{w}_*, \mu p_*) \) representation, this becomes \( \vec{w}_* \cdot \nabla (\mu \rho)^{1/2} = 0 \), and Eq. 1 then becomes \( \nabla \cdot \vec{w}_* = 0 \). Furthermore, making use of Eq. 4
the $\vec{w}_*$ term and the $\vec{B}$ term are in symmetric form in Eq. 2. Because of Eq. 5, the magnetic field can be represented through a vector potential. Under axisymmetry, this vector potential allows the magnetic field be represented by two scalar functions, which reads

$$\vec{B} = A_0 (\nabla \Psi \times \nabla \phi + F \nabla \phi) = \frac{A_0}{r \sin \theta} \left( \frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \frac{1}{\partial r}, +F \right), \quad (7)$$

$$\mu \vec{J} = \frac{A_0}{r \sin \theta} \left( \frac{1}{r} \frac{\partial F}{\partial \theta}, -\frac{\partial F}{\partial r}, -\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r^2} \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) \right).$$

Here, $A_0$ carries the physical dimension of poloidal magnetic flux such that $\Psi$ is a dimensionless poloidal flux function, and $A_0 F$ is a measure of the axial plasma current. Likewise, because $\nabla \cdot \vec{w}_* = 0$ by Eq. 1, we also have

$$\vec{w}_* = A'_0 (\nabla \Psi' \times \nabla \phi + F' \nabla \phi). \quad (8)$$

We note that $A_0$ and $A'_0$ are not the maximum values of poloidal magnetic and density weighed velocity fluxes. They are reference values only as such that the dimensionless poloidal flux functions are not normalized to unity. The condition $\vec{w}_* \cdot \nabla \rho = 0$ gives $\rho = \rho(\Psi')$. The scalar product of $\vec{w}_*$ on Eq. 3 results in $\tilde{\Phi} = \tilde{\Phi}(\Psi')$. Consequently, with $\rho = \rho(\Psi')$, the right side of Eq. 3 can be written as the gradient of $\Phi = \Phi(\Psi')$ only giving the second equality. Taking the scalar product again of $\vec{B}$ and $\vec{w}_*$ on Eq. 3 give $\Phi = \Phi(\Psi)$ and $\Phi = \Phi(\Psi')$ respectively thus $\Psi' = \Psi'(\Psi)$. Substituting Eq. 7 and Eq. 8 to Eq. 3 gives

$$\left( \nabla \Psi' \times \nabla \Phi \right) \times \left( \nabla \Psi \times \nabla \Phi \right) - F \nabla \Phi \times \left( \nabla \Psi' \times \nabla \Phi \right) + F' \nabla \Phi \times \left( \nabla \Psi \times \nabla \Phi \right) = \frac{1}{A_0} \frac{1}{A'_0} \nabla \Phi.$$

Taking the scalar product of $\nabla \Psi$ on this equation gives

$$\frac{1}{A_0} \frac{1}{A'_0} \nabla \Psi \cdot \nabla \Phi = (\nabla \phi)^2 \nabla \Psi \cdot (F' \nabla \Psi - F \nabla \Psi') = \frac{1}{A_0 A'_0} \frac{1}{\nabla \phi^2} \frac{\partial \Phi(\Psi)}{\partial \Psi}, \quad (9a)$$

$$\left( F' - F \frac{\partial \Psi'(\Psi)}{\partial \Psi} \right) = \frac{1}{A_0 A'_0} \frac{1}{\nabla \phi^2} \frac{\partial \Phi(\Psi)}{\partial \Psi}. \quad (9b)$$
3. Rotational Grad-Shafranov Equation

As for Eq. 2 with $\Psi' = \Psi'(\Psi)$, the $\phi$ component gives

$$A_0^2 \frac{\partial \Psi'}{\partial \Psi} \left( -\frac{\partial \Psi}{\partial r} \frac{\partial F'}{\partial \theta} + \frac{\partial \Psi}{\partial \theta} \frac{\partial F'}{\partial r} \right) - A_0^2 \left( -\frac{\partial \Psi}{\partial r} \frac{\partial F}{\partial \theta} + \frac{\partial \Psi}{\partial \theta} \frac{\partial F}{\partial r} \right) = 0 .$$

We use Eq. 9b to eliminate $F'$. It is noted that $F$ and $F'$ are not functions of $\Psi$ only for rotational equilibrium due to the $(\nabla \phi)^2$ factor. With $(\nabla \phi)^2 = (1/r \sin \theta)^2$, we then get

$$A_0^2 \frac{\partial \Psi'}{\partial \Psi} \left( -\frac{\partial \Psi}{\partial r} \frac{\partial F'}{\partial \theta} + \frac{\partial \Psi}{\partial \theta} \frac{\partial F'}{\partial r} \right) + A_0' \frac{\partial \Psi'}{\partial \Psi} \frac{\partial \Phi}{\partial \Psi} \left( -\frac{\partial \Psi}{\partial r} \frac{\partial F}{\partial \theta} + \frac{\partial \Psi}{\partial \theta} \frac{\partial F}{\partial r} \right) = 0 .$$

By inspection, we have

$$F(r, \theta) = k(ar)^2 \sin^2 \theta , \quad (10a)$$

$$\left( \frac{\partial \Psi'}{\partial \Psi} \right)^2 - \left( \frac{A_0'}{A_0} \right)^2 + \frac{1}{ka^2 A_0'} \frac{1}{A_0} \frac{\partial \Phi}{\partial \Psi} F(r, \theta) = 0 . \quad (10b)$$

We note that $a$ is a normalizing factor of $r$, and $k$ is an inverse scale length such that the scalar function $F(r, \theta)$ has the correct dimension of $1/r$ in Eq. 3. An arbitrary constant could be added to Eq. 3b. However, this constant would correspond to an externally applied current along the axial direction, which would be appropriate for laboratory tokamak plasmas but not for astrophysical plasmas. Substituting $\partial \Phi/\partial \Psi$ of Eq. 10b to Eq. 9b gives

$$F'(r, \theta) = \frac{\partial \Psi'}{\partial \Psi} F(r, \theta) + \frac{1}{ka^2 A_0'} \frac{1}{A_0} \frac{\partial \Phi}{\partial \Psi} F(r, \theta) = \frac{1}{(\partial \Psi'/\partial \Psi) (A_0/A_0')} F(r, \theta) . \quad (11)$$

Choosing the following linear source function leads to

$$\Psi'(\Psi) = b\Psi , \quad (12a)$$
\[ F'(F) = \frac{1}{b} \left( \frac{A_0}{A_0'} \right)^2 F = \alpha F. \] (12b)

We note that \( b \) is an independent model parameter, but \( \alpha \), defined in Eq. (12b), is determined by \( b \) together with the parameters \( (A_0, A_0') \).

In order to analyse the \( \theta \) and \( r \) components, we choose to write the generalized total pressure \( \bar{p}_* \) in separable form

\[ \mu \bar{p}_*(r, \theta) = \mu \bar{p}_*(r, \Psi) = \mu \tilde{p}_0 \bar{p}_{*1}(r) \tilde{p}_{*2}(\Psi), \] (13)

where \( \tilde{p}_0 \) has the physical dimension of pressure \( \bar{p}_* \), and \( \bar{p}_{*1} \) and \( \bar{p}_{*2} \) are dimensionless functions of \( r \) and \( \Psi \) not normalized to unity, and \( \bar{p}_* \) is defined by

\[ \frac{\mu \bar{p}_*}{k a^2} = \left[ \frac{\mu p_*}{k a^2} + \left( A_0^2 - A_0'^2 \alpha^2 \right) \right] F = \left[ \frac{1}{k a^2} (\mu p + \frac{1}{2} w_*^2) + \left( A_0^2 - A_0'^2 \alpha^2 \right) F \right]. \] (14)

The \( \theta \) and \( r \) components then become

\[ \left[ r^2 \frac{\partial^2 \Psi}{\partial r^2} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) \right] = \frac{r^2}{((bA_0')^2 - A_0'^2) k a^2 \partial \Psi} F \frac{\partial}{\partial r} (\mu \bar{p}_*), \] (15)

\[ \frac{\partial \Psi}{\partial r} \left[ r^2 \frac{\partial^2 \Psi}{\partial r^2} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) \right] = \frac{r^2}{((bA_0')^2 - A_0'^2) k a^2 \partial \Psi} F \frac{\partial}{\partial r} \frac{\partial \Psi}{\partial r} \left( \mu \bar{p}_* \right) + \frac{r^2}{((bA_0')^2 - A_0'^2) k a^2 \partial \Psi} F \frac{\partial}{\partial r} \left( \mu \bar{p}_* \right) + \frac{r^2}{((bA_0')^2 - A_0'^2) r^2} \sin^2 \theta \mu \rho \frac{GM}{r^2}. \] (16)

The right side of the \( r \) component correspond to the explicit derivative \( \partial / \partial r \), the implicit derivative \( \partial \Psi / \partial r \) \( (\partial / \partial \Psi) \) of generalized total plasma pressure, and the gravitational term. Making use of the \( \theta \) component to eliminate the implicit derivative term, the \( r \) component becomes
\[ \mu \bar{p}_0 a \frac{\partial \bar{p}_{*1}}{\partial z} \bar{p}_{*2}(\Psi) + \mu \rho(\Psi) \frac{GMa^2}{z^2} = 0, \]  

where we have defined a normalized radial coordinate \( z = ar \). To satisfy this equation with \( \rho = \rho(\Psi) \), we take \( \bar{p}_{*1}(z) \) as below to get

\[ \bar{p}_{*1}(z) = \frac{1}{z}, \]  

\[ \rho = \frac{\bar{p}_0}{GMa} \bar{p}_{*2}(\Psi) = \rho_0 \bar{p}_{*2}(\Psi) = \rho_0 \bar{p}_{*2}(\Psi). \]  

This shows that \( \rho \) is a function of \( \Psi \) only, consistent to the divergence-free flows, with \( \rho_0 \) as the amplitude and \( \bar{p}_{*2}(\Psi) \) as the dimensionless functional dependence.

To solve the \( \theta \) component, let us further specify the second source function \( \bar{p}_{*2} \) as

\[ \bar{p}_{*2}(\Psi) = (C \pm |\Psi|^{2m}) > 0. \]  

Since plasma pressure is positive definite, \( \Psi^{2m} \) is taken in absolute value for any \( m \). The \( \theta \) component then reads

\[ r^2 \frac{\partial^2 \Psi}{\partial r^2} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) \pm m_\beta p z^4 \sin^2 \theta \bar{p}_{*1}(z) |\Psi|^{2m-1} = 0, \]  

\[ \beta_p = \frac{2\mu \bar{p}_0}{(A_0^2 - (bA_0')^2)a^4}. \]  

This is the rotational Grad-Shafranov equation for divergence-free rotational flows with \( \beta_p \) as the poloidal plasma \( \beta \). We remark that the static case can be recovered by taking \( A_0' = 0 \) and with the corresponding limit of \( \mu \bar{p}_* \) and \( \beta_p \). This equation in spherical coordinates describes the structures in the cavity resulting from the interactions between the conducting plasma and magnetic fields. The corresponding equation without rotational flows has been derived by many authors (Lovelace et al., 1986) in cylindrical coordinates.
4. Asymmetric Supernovae

Writing $\Psi(r, \theta) = R(r)\Theta(\theta)$ in separable form, the rotational Grad-Shafranov equation becomes

$$r^2 \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{\Theta} \sin \theta \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{d\Theta}{d\theta} \right) \pm m\beta_p z^4 \sin^2 \theta \bar{\rho}_{s1}(z) R^{2m-2} |\Theta|^{2m-2} = 0 .$$  \hspace{1cm} (22)

We will arrange the above equation in the following form to solve the poloidal magnetic flux function $\Psi$ by separation of variables

$$r^2 \frac{1}{R} \frac{d^2 R}{dr^2} = - \frac{1}{\Theta} \sin \theta \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{d\Theta}{d\theta} \right) \mp m\beta_p z^4 \sin^2 \theta \bar{\rho}_{s1}(z) R^{2m-2} |\Theta|^{2m-2}$$

$$= n(n+1) .$$ \hspace{1cm} (23)

The $R$ equation gives $R(z) = 1/z^n$ and $R(z) = z^{n+1}$ as two independent solutions. As for the $\Theta$ part, in order to be separable, we choose $R(z) = 1/z^n$ and $\bar{\rho}_{s1}(z) = 1/z$ to get

$$\sin \theta \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{d\Theta}{d\theta} \right) + n(n+1)\Theta = \mp m\beta_p z^3 \left( \frac{1}{z^n} \right)^{2m-2} \sin^2 \theta |\Theta|^{2m-1}$$

$$= \mp m(n)\beta_p \sin^2 \theta |\Theta|^{2m(n)-1} ,$$

with $2n(m-1) = 3$ or $m(n) = (3 + 2n)/2n$. Defining $x = \cos \theta$, we have

$$(1-x^2) \frac{d^2 \Theta(x)}{dx^2} + n(n+1)\Theta(x) = \mp m(n)\beta_p (1-x^2) |\Theta|^{2m(n)-1} .$$ \hspace{1cm} (24)

This nonlinear equation with $\beta_p \neq 0$ usually would have asymmetric solutions of $\Theta(x)$. Nevertheless, we note that this equation is symmetric with $\pm x$, hence symmetric nonlinear solutions of $\Theta(x)$ are also allowed. As for the magnetic field components, they are now given by

$$B_r = - \frac{A_0 a^2}{z} \frac{1}{z} \frac{\partial \Psi}{\partial x} ,$$ \hspace{1cm} (25a)
These magnetic fields are expressed in terms of the reference field $a^2 A_0$. We remark that the $B_\theta$ and $B_\phi$ fields are scaled by the $(1 - x^2)^{1/2}$ factor in the denominator, which is singular at $x = \pm 1$ or $\theta = 0, \pi$. Since this singularity is quadratically not integrable, the magnetic energy would diverge at this location. To eliminate this divergence, we require the oscillating function $\Theta(x)$ be null at the poles $x = \pm 1$. As a result, the oscillating $B_\theta$ and $B_\phi$ fields grow to large amplitude near the poles, before they plunge to zero at the poles. The constraint that $\Theta(\pm 1) = 0$ makes $\beta$ the eigenvalue of Eq. 24. The magnetic field lines and, in particular, the poloidal field lines are described by

$$\frac{B_r}{dr} = \frac{B_\theta}{rd\theta} = \frac{B_\phi}{r \sin \theta d\phi},$$

$$\Psi(z, x) = R(z)\Theta(x) = C,$$  

where the poloidal field lines are given by the contours of $\Psi(z, x)$ on the $(r - \theta)$ plane, which is also shared by the poloidal velocity stream lines.

To solve Eq. 24, we note that, for $\beta_p = 0$, $\Theta(x)$ is given in terms of Legendre polynomials by Eq. 29b in Sec. 4 with $n$ axisymmetric lobes, as shown in Fig. 1 with $n = 1$ and $n = 3$. However, $\beta_p = 0$ would imply $p_* = 0$, or equivalently $p = 0$, $w_* = 0$, and $A_0 = 0$ by Eq. 14, which amounts to no pressure, no flow, and no magnetic fields. For $\beta_p \neq 0$, Eq. 24 defines the nonlinear $\Theta(x)$ for a given $n$. To solve for equilibrium $\Theta(x)$, we take the lower sign in Eq. 20 and Eq. 24 with $\tilde{p}_{s2}(\Psi) = (C - |\Psi|^{2n})$ to have a balance with the poloidal magnetic pressure. We start from $x = -1$ with $\Theta(-1) = 0$ and a slope $\Theta'(-1)$ as such to reach $x = +1$ with $\Theta(+1) = 0$. In the parameter space of $(\Theta'(-1), \beta_p)$ for a given $n$, there will be $(n-1)$-lobe, $(n-2)$-lobe, ..., 1-lobe structures in general. Rather than covering the $(\Theta'(-1), \beta_p)$ parameter space, we take the derivative of the linear $\beta_p = 0$ solution of Eq. 29b as the reference slopes at $x = -1$, with

$$B_\theta = -\frac{A_0 a^2}{z} \frac{1}{(1 - x^2)^{1/2}} \frac{\partial \Psi}{\partial z},$$

$$B_\phi = \frac{A_0 a^2}{z} \frac{1}{(1 - x^2)^{1/2}} \frac{1}{a F}.$$  

These magnetic fields are expressed in terms of the reference field $a^2 A_0$. We remark that the $B_\theta$ and $B_\phi$ fields are scaled by the $(1 - x^2)^{1/2}$ factor in the denominator, which is singular at $x = \pm 1$ or $\theta = 0, \pi$. Since this singularity is quadratically not integrable, the magnetic energy would diverge at this location. To eliminate this divergence, we require the oscillating function $\Theta(x)$ be null at the poles $x = \pm 1$. As a result, the oscillating $B_\theta$ and $B_\phi$ fields grow to large amplitude near the poles, before they plunge to zero at the poles. The constraint that $\Theta(\pm 1) = 0$ makes $\beta$ the eigenvalue of Eq. 24. The magnetic field lines and, in particular, the poloidal field lines are described by

$$\frac{B_r}{dr} = \frac{B_\theta}{rd\theta} = \frac{B_\phi}{r \sin \theta d\phi},$$

$$\Psi(z, x) = R(z)\Theta(x) = C,$$  

where the poloidal field lines are given by the contours of $\Psi(z, x)$ on the $(r - \theta)$ plane, which is also shared by the poloidal velocity stream lines.
\[ \Theta'(x, n) = -2x \frac{dP_n}{dx}. \]  

(27)

For \( n = 2 \), a 1-lobe structure is shown in Fig.2 with \( \Theta'(-1, 1) \) and its corresponding \( \beta_p = 9.57 \), where the lobe asymmetry is not evident. For \( n = 3 \), Fig.3 shows a 2-lobe structure with \( \Theta'(-1, 3) \) and \( \beta_p = 6.69 \), and for \( n = 3 \) in Fig.4 is another 2-lobe structure with a different boundary derivative \( \Theta'(-1, 2) \) and at a different pressure \( \beta_p = 6.12 \). Naturally, there are also 1-lobe solutions. For \( n = 4 \) with \( \Theta'(-1, 4) \), Fig.5 shows a 3-lobe structure with \( \beta_p = 6.05 \), which is rather symmetric as in Fig.2. However, the symmetric lobes of Fig.2 and Fig.5 are nonlinear lobes as they differ from the linear \( \beta_p = 0 \) corresponding lobes of Fig.1, where the \((x,y)\) scales in the three figures have the same proportions. For \( n = 4 \) and under the same boundary derivative \( \Theta'(-1, 4) \), Fig.6 shows a 2-lobe structure as pressure increases to \( \beta_p = 6.41 \). In this figure, one lobe is overwhelmingly dominating over the other, as in Fig.3 and Fig.4, showing the nonlinear nature of the equation. For \( n = 4 \) with \( \Theta'(-1, 3) \), Fig.7 shows an asymmetric 3-lobe structure with \( \beta_p = 12.3 \). For \( n = 4 \) and changing the boundary derivative to \( \Theta'(-1, 1) \), Fig.8 shows another asymmetric 3-lobe structure with \( \beta_p = 47.0 \). Comparing the 3-lobe structures of \( n = 4 \) in Fig.5, Fig.7, and Fig.8, all with proportional \((x,y)\) scales, the lobe amplitude decreases as the boundary derivatives decreases from \( \Theta'(-1, 4) \) to \( \Theta'(-1, 1) \). Multipole magnetic structures similar to ours have been explored in magneto-rotational supernovae (Bisnovatyi 2008).

We note that with \( R(z) = 1/z^n \) the poloidal magnetic flux function is \( \Psi(z, x) = R(z)\Theta(x) \). The poloidal magnetic field lines and the density weighed velocity stream lines with \( \Psi(z, x) = C \) are shown in Fig.9 for \( n = 3, \beta_p = 6.12 \), and in Fig.10 for \( n = 4, \beta_p = 6.41 \), corresponding to Fig.4 and Fig.6 respectively. Together with the toroidal component, they generate a surface of revolution about the polar axis. Taking the \( 2m(n) \) power of this poloidal flux function, and considering the complement contours, we could generate the mass density profiles of \( \rho(\Psi) = \rho_0\tilde{p}_2(\Psi) = \rho_0(C - |\Psi|^{2m(n)}) \). The generalized total plasma pressure \( \tilde{p}_* \) follows with an added \( \tilde{p}_{s1} = 1/z \) decay with \( z \). We remark that the nonlinear solution of \( \Theta(x) \) is driven by the generalized total pressure \( \tilde{p}_* \). According to this model, although \( \Theta(x) \) is occasionally symmetric, this function is basically asymmetric and often with \( \beta_p \gg 1 \). Consequently, supernovae are asymmetric in nature with different degree of asymmetry. Observed from broadside, this asymmetry should be evident. As the line of sight moves towards the poles, the supernova becomes less asymmetrical and more spherical by projection effects.
The proto-supernova cavity, filled with circulating plasmas and magnetic fields, generates a sequence of rotational equilibria as $\beta_p$ increases. When $\beta_p$ gets sufficiently high to crack the stellar hydrogen envelope, an often asymmetric supernova is erupted resulting in the recoil of the neutron star. Since $\beta \gg 1$, plasma pressure is the primary trigger of supernova rather than magnetic field, which is consistent to some simulations (Komissarov & Barkov 2007).
5. Collimated Polar GRB Jets

We can also arrange the Grad-Shafranov equation, Eq. [22] as

\[ r^2 \frac{1}{R} \frac{d^2 R}{dr^2} \pm m \beta_p z^4 \sin^2 \theta \bar{p}_{s1}(z) R^{2m-2} |\Theta|^{2m-2} = \frac{1}{\Theta} \sin \theta \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{d\Theta}{d\theta} \right) = n(n+1). \]  

(28)

The \( \Theta \) equation and its solutions are

\[ (1 - x^2)\frac{d^2 \Theta(x)}{dx^2} + n(n+1)\Theta(x) = 0, \]  

(29a)

\[ \Theta(x) = (1 - x^2)\frac{dP_n(x)}{dx} = (1 - x^2), \]  

(29b)

where \( P_n(x) \) is the Legendre polynomials and the last equality is for \( n = 1 \). As for the \( R \) equation, it reads

\[ r^2 \frac{d^2 R}{dr^2} - n(n+1)R = \mp m \beta_p z^3 \sin \theta |\Theta|^{2m-2} = \mp m \beta_p z^3. \]  

(30)

where we have made use of Eq. [18] to get the last equality. To make this equation separable, we consider \( 2m - 2 = -1 \) and then take \( n = 1 \) in Eq. [29b] to get

\[ r^2 \frac{d^2 R}{dr^2} + n(n+1)R = \mp m \beta_p z^3 R^{2m-1} \sin \theta |\Theta|^{2m-2}, \]  

(31)

With \( n = 1 \) and \( 2m = 1 \), the solution of \( R(z) \) is given by the homogeneous solution in bracket plus the particular solution

\[ R(z) = \left( a z^{n+1} + b \frac{1}{z^n} \right) + cz^3, \]  

(32)

\( c = \mp \frac{1}{4} m \beta_p \).
To solve for equilibrium $R(z)$, we again take the lower sign in Eq. (31). Taking $a = +2$, $b = +2$, $c = +\beta_p/8$, and $\beta_p = 5$, Fig.11 shows the function $R(z)$. Together with $\Theta(x) = (1 - x^2)$, the poloidal magnetic field lines and the poloidal density weighed velocity stream lines, $\Psi(z, x) = R(z)\Theta(x) = C$, run on the contour line as shown in Fig.12 with $C = 2, 3, 4, 5, 6, 7$. Together with the toroidal component, they generate a surface of revolution about the polar axis. The magnetic field lines of this jet in the cavity can be closed via the stellar envelope. For large distances, the surface is dominated by the $z^3$ term with $\Psi(z, x) = cz^3(1 - x^2) = zz^2_\bot = C$, where $z_\bot$ is the perpendicular distance from the polar axis. Let us approximate the contour as

$$z^2_\bot = \frac{C}{z_{\parallel}}.$$  \hspace{1cm} (33)

This shows that as $z$ increases, $z_\bot$ would decrease, giving a cusp funnel structure, or a tornado-like line vortice structure, of a polar jet. In Fig.12, only one jet is shown for more clarity, and the opposing jet is likewise. Furthermore, only the initial part of the funnel contours are shown. For $y(\theta = 0) > 15$, the analytic funnel solution of Eq. (33) should be superimposed on it. We also note that the inner contours 2, 3 and 4 are connected to the outer magnetosphere, while the outer contours 5, 6, and 7 would have to come from external sources, like an accretion disk. If these external sources are unable to provide such corresponding field strengths of the contours, these outer jet contours should be ignored. For small distances, the poloidal flux surface is dominated by the $b/z$ term with $\Psi(z, x) = b(1 - x^2)/z = C$, such that $z$ is proportional to $\sin^2 \theta$. Considering $\sin^2 \theta$ as the dipole field of the magnetar, the polar funnel structure connects up with the magnetosphere of the central compact star, as is shown in a close-up look in Fig.13. We name this as the magnetospheric jet. This connection should take place outside the light cylinder of the magnetosphere.

Furthermore, for a given $z_{\parallel}$, Eq. (33) shows that $z^2_\bot$ is proportional to the contour value $C$. As a result, the gradient of $\Psi$ increases outward. This means stronger field lines and faster stream lines are on the outside. Because of the overall pressure balance, higher pressure plasmas are on the inside close to the axis. Plasma confinement is, therefore, accomplished by the strong magnetic fields outside surrounding the plasmas. The magnetic confinement is also enhanced by the inward magnetic curvature of the cusp funnel geometry which puts a magnetic tension force on the plasmas. As for the mass density, which satisfies $\vec{v} \cdot \nabla \rho(\Psi) = 0$,
the density gradient $\nabla \rho(\Psi) = (d\rho/d\Psi) \nabla \Psi$ is perpendicular to the $\Psi$ contour surface. The mass density profiles are given by the complement contours of the poloidal flux function, according to $\rho(\Psi) = \rho_0(C^2 - |\Psi|^{2m}) = \rho_0(C - |\Psi|)$ with $2m = 1$, and the generalized total plasma pressure $\bar{p}$, follows accordingly. Although the cusp funnel volume is unbounded, the jet will be filled up to the point where the energy flux of the star’s magnetosphere can supply over time. Upon plowing through the stellar envelope along the polar axis, this cavity jet could then erupt into a gamma-ray burst jet.

According to our findings, the outer magnetosphere is torn open by the plasma ram pressure, and is flipped polewards by the poloidal plasma flows to form a cusp funnel. Consequently, the magnetic field lines of this funnel wrap around a high pressure plasma column around the axis. Usually, when only toroidal rotational plasma velocity is taken into account, the outer magnetosphere opens up into monopole-like radial fields. In our case, the poloidal plasma velocity brings the outer magnetosphere to the polar direction. This offers a MHD description of the magnetic towering mechanism (Lynden-Bell 2003). Standard jet formation schemes rely on the angular momentum of a binary system or an accretion disk-compact star system (Prendergast 2005; Gourgouliatos & Vlahakis 2010). A single star discounting the external stellar envelope can hardly provide the needed angular momentum to form jets. The present MHD model can provide asymmetric supernova and polar collimating gamma-ray burst jet configurations in the cavity, therefore, favoring the gamma-ray burst and supernova association (Galama et al. 1999; Campana et al. 2006; Pian et al. 2006; Soderberg et al. 2006; Mazzali et al. 2006). Our axial jet has a specific “beam pattern”. Such feature is compatible with models that consider gamma-ray burst and afterglow properties as a result of viewing angle on the beam pattern of the jet (Lipunov et al. 2001; Rossi et al. 2002; Salmonson & Galama 2002).
Active galactic nuclei (AGN) is a term used to designate astrophysical objects of Seyferts, quasars, radio galaxies, and blazars (BL-Lacs), that emit mighty electromagnetic radiations at different bands from radio, optical, to X-ray, gamma-ray. From the spectroscopic characteristics of these objects, it is deduced that they could be the results of a single unified AGN structure viewed at different angles. This unified AGN model (Antonucci 1993) would consist of a central core with a black hole candidate, an accretion disk, that defines the equatorial plane, and a dust torus further out on this plane. On the rotational (polar) axis of the central core, there would be two opposing jets that emit strongly beamed and polarized radio waves. Viewed from face-on (down the polar axis) would give blazars, and from edge-on (on the equatorial plane) would give type 2 Seyferts, quasars, and radio galaxies with narrow emission lines. While viewed at an intermediate angle would give type 1 Seyferts, quasars, and radio galaxies with broad emission lines. The material in the accretion disk gravitates inward, by transporting its angular momentum outward through dissipations, and feeds the central core to generate the jets. The existence of a dust torus in AGN has been inferred from high resolution instruments and images.

To account for this unified AGN structure, we consider another solution of Eq. 32 with a negative coefficient $a$. Writing the negative sign explicitly, we have with $n = 1$

$$R(z) = -az^{n+1} + \left(\frac{1}{z^n} + cz^3\right). \tag{34}$$

The two terms in bracket generates a minimum in the positive $R(z)$ domain. The first term shifts the minimum down to the negative domain bounded by $z_1$ and $z_2$ where $R(z_1) = R(z_2) = 0$. With the negative sign explicit, we take again $a = +2$, $b = +2$, $c = +\beta_p/8$, and $\beta_p = 5$, the function $R(z)$ is shown in Fig.14 with three regions, separated by $z_1 = 1.17$ and $z_2 = 3.09$. In the absence of the envelope, the poloidal flux function $\Psi(z, x) = R(z)\Theta(x)$ with $\Theta(x) = (1 - x^2)$ is shown in Fig.15, which consists of a dipole-like magnetosphere, a plasma torus, and a polar jet separated by two spherical separatrix at $z_1$ and $z_2$. This polar jet can be formed if there is horizontal accretion to feed it. We name this as the accretion jet in contrast to the magnetospheric jet. A close-up look at the AGN magnetosphere is shown in Fig.16, which gives an intrinsic AGN magnetic moment (Schild et al. 2006). Naturally, upon this basic AGN structure, we can superimpose an AGN magnetospheric jet as discussed.
in the preceding section to generate quasars and blazers.

To conclude, we have used a set of MHD equations with divergence-free axisymmetric poloidal and toroidal non-field aligned plasma flows to study the dynamics of the cavity between the central core and the external envelope in the final phase of a collapsing star. A sequence of steady state rotational MHD equilibria in response to the increasing plasma pressure is solved to represent the plasma evolution in the cavity. The spatial configuration is described by the rotational Grad-Shafranov equation where the ratio of the generalized total plasma pressure to the poloidal magnetic pressure, \( \beta_p = 2\mu p_0/(a^2A_0)^2 \), is the cavity parameter. By assigning two source functions, the rotational Grad-Shafranov equation can be solved for asymmetric supernova, polar collimated cusp funnel gamma-ray burst jet, and active galactic nucleus plasma torus. It is important to remark that both the asymmetric supernova lobes and the cusp gamma-ray burst polar jets are connected directly to the magnetosphere of the central compact star, not to an accretion disk. This structure identifies the magnetosphere as the central engine of supernova and gamma-ray burst events, by providing plasma, magnetic energy, and rotational energy. Since \( \beta_p \) gets much larger than unity, plasma pressure is likely to be the primary agent in cracking the stellar envelope, instead of the magnetic field.
REFERENCES

Antonucci, R., 1993. Unified models for active galactic nuclei and quasars, Ann. Rev. Astron. Astrophys. 31, 473-521.

Ardeljan, N.V., Bisnovatyi-Kogan, G.S., & Moiseenko, S.G., 2005. Magnetorotational supernovae, MNRAS, 359, 333-344.

Baade, W. & Zwicky, F., 1934. Remarks on super-novae and cosmic rays, Phys. Rev., 46, 76-77.

Begelman, M.C. & Li, Z.Y., 1994. Asymptotic domination of cold relativistic MHD winds by kinetic energy flux, ApJ, 426, 269-278.

Bethe, H.A., 1990. Supernova mechanisms, Rev. Mod. Phys., 62, 901-866.

Bethe, H.A. & Wilson, J.R., 1985. Revival of a stalled supernova shock by neutrino heating, ApJ, 295, 14-23.

Bisnovatyi-Kogan, G.S., 1971. The explosion of a rotating star as a supernova mechanism, Sov. Astron., 14, 652-655.

Bisnovatyi-Kogan, G.S., Moiseenko, S.G., & Ardelyan, N.V., 2008. Different magnetorotational supernovae, Astron. Reports, 52, 9978-1008.

Blandford, R.D. & Payne, D.G., 1982. Hydromagnetic Flows from Accretion Discs and the Production of Radio Jets, MNRAS, 199, 883-903.

Burrows, A., Dessart, L., Livine, E., Ott, C.D., & Murphy, J., 2007. Simulations of magnetically driven supernova and hypernova explosions in the context of rapid rotation, ApJ, 664, 416-434.

Camenzind, M., 1986. Hydromagnetic flows from rapidly rotating compact objects I. Cold relativistic flows from rapid rotators, A&A, 162, 32-44.

Campana, S., et al., 2006. The association of GRB 060218 with a supernova and the evolution of the shock wave, Nature, 442, 1008-1010.

Duncan, R.C. & Thompson, C., 1992. Formation of very strong magnetized neutron star: implications for gamma-ray bursts, ApJ, 392, L9-L13.
Galama, T.J., et al., 1999. On the possible association of SN 1998bw and GRB 980425, A&AS, 138, 465-466.

Gourgouliatos, K.N. & Vlahakis, N., 2010. Relativistic Expansion of a Magnetized Fluid, Geophys. Astrophys. Fluid Dyn., 104, 431-450.

Howell, D.A., Hoffich, P., Wang, L., & Wheeler, J.A., 2001. Evidence for asphericity in a subluminous type Ia supernova: spectropolarimetry of SN 1999 by, ApJ, 556, 302-321.

Komissarov, S.S. & Barkov, M.V., 2007. Magnetar-energized supernova explosions and gamma-ray burst jets, MNRAS, 382, 1029-1040.

Kotake, K., Sato, K., & Takahashi, K., 2006. Explosion mechanism, neutrino burst and gravitational wave in core-collapse supernova, Rep. Prog. Phys., 69, 971-1143.

LeBlanc, J.M. & Wilson, J.R., 1970. A numerical example of the collapse of a rotating magnetized star, ApJ, 161, 541-551.

Lipunov, V.M., Postnov, K.A., & Prokhorov, M.E., 2001. Gamma-ray bursts as standard-energy explosions, Astron. Rep., 45, 236-240.

Lovelace, R.V.R., Mehanian, C., Mobarry, C.M., & Sulkanen, M.E., 1986. Theory of axisymmetric magnetohydrodynamic flows: disks, ApJS, 62, 1-37.

Lynden-Bell, D., 2003. On why discs generate magnetic towers and collimate jets, MNRAS, 341, 1360-1372.

MacFadyen, A.I. & Woosley, S.E., 1999. Collapsar: gamma ray bursts and explosions in ‘failed supernova’, ApJ, 524, 262-289.

Mazzali, P.A., et al., 2003. The type Ic hypernova SN 2003dh/GRB 030329, ApJ, 599, L95-L98.

Mazzali, P.A., et al., 2006. A neutron-star-driven X-ray flash associated with supernova SN 2006aj, Nature, 442, 1018-1020.

Meszaros, P., 2002. Theories of gamma-ray bursts, Annu. Rev. Astron. Astrophys., 40, 137-169.
Okamoto, I., 1978. Relativistic centrifugal winds, MNRAS, **185**, 69-107.

Okamoto, I., 2002. Magnetohydrodynamic acceleration of the Crab pulsar wind, ApJ, **5783**, L31-L34.

Pian, E., et al., 2006. An optical supernova associated with the X-ray flash 060218, Nature, **442**, 1011-1013.

Prendergast, K.H., 2005. Relativistically expanding axisymmetric self-similar force-free fields, MNRAS, **359**, 725-728.

Rossi, E., Lazzati, D., & Rees, M.J., 2002. Afterglow light curves, viewing angle and the jet structure of γ-ray bursts, MNRAS, **332**, 945-950.

Salmonson, J.D. & Galama, T.J., 2002. Discovery of a tight correlation between pulse lag/luminosity and jet-break times: A connection between gamma-ray bursts and afterglow properties, ApJ, **569**, 682-688.

Schild, R.E., Leiter, D.J., & Robertson, S.L., 2006. Observations supporting the existence of an intrinsic magnetic moment inside the central compact object within the quasar Q0957+561, AJ **132**, 420-432.

Soderberg, A.M., et al., 2006. Relativistic ejecta from X-ray flash 060218 and the rate of cosmic explosions, Nature, **442**, 1014-1017.

Thompson, C. & Duncan, R.C., 1993. Neutron star dynamos and the origins of pulsar magnetism, ApJ, **408**, 194-217.

Tsui, K.H., Navia, C.E., Serbeto, A., & Shigueoka, H., 2011. Tokamak equilibria with non field-aligned axisymmetric divergence-free rotational flows, Phys. Plasmas, **18**, 072502.

Tsui, K.H. & Navia, C.E., 2012. Tokamak L/H mode transition, Phys. Plasmas, **19**, 012505.

Uzdensky, D.A. & MacFadyen, A.I., 2007. Magnetar-driven magnetic tower as a model for gamma-ray bursts and asymmetric supernovae, ApJ, **669**, 546-560.

Vietri, M. & Stella, L., 1999. Supernova events from spun-up neutron stars: an explosion in search of an observation, ApJ, **527**, L43-L46.
Wang, L., et al, 2003. Spectropolarimetry of SN 2001 el in NGC 1448: asphericity of a normal type Ia supernova, ApJ, 591, 1110-1128.

Wheeler, J.C., Yi, I., Hoflich, P., & Wang, L., 2000. Asymmetric supernovae, pulsars, magnetars, and gamma-ray bursts, ApJ, 537, 810-823.

Wheeler, J.C., Meier, D.L., & Wilson, J., 2002. Asymmetric supernovae from magnetocentrifugal jets, ApJ, 568, 807-819.
Fig. 1.— The lobe structures of $\Theta(x)$ for $n = 1, 2, 3$ with $\beta_p = 0$ are plotted on the $(r - \theta)$ plane.
Fig. 2.— The 1-lobe structure of $\Theta(x)$ for $n = 2$ with $\beta_p = 9.57$ and $\Theta'(−1, 1)$ is plotted on the $(r − \theta)$ plane.
Fig. 3.— The 2-lobe structure of $\Theta(x)$ for $n = 3$ with $\beta_p = 6.69$ and $\Theta'(-1, 3)$ is plotted on the $(r - \theta)$ plane.
Fig. 4.— The 2-lobe structure of $\Theta(x)$ for $n = 3$ with $\beta_p = 6.12$ and $\Theta'(-1, 2)$ is plotted on the $(r - \theta)$ plane.
Fig. 5.— The 3-lobe structure of $\Theta(x)$ for $n = 4$ with $\beta_p = 6.05$ and $\Theta'(-1, 4)$ is plotted on the $(r - \theta)$ plane.
Fig. 6.— The 2-lobe structure of $\Theta(x)$ for $n = 4$ with $\beta_p = 6.41$ and $\Theta'(1, 4)$ is plotted on the $(r - \theta)$ plane.
Fig. 7.— The 3-lobe structure of $\Theta(x)$ for $n = 4$ with $\beta_p = 12.3$ and $\Theta'(-1, 3)$ is plotted on the $(r - \theta)$ plane.
Fig. 8.— The 3-lobe structure of $\Theta(x)$ for $n = 4$ with $\beta_p = 47.0$ and $\Theta'(-1, 1)$ is plotted on the $(r - \theta)$ plane.
Fig. 9.— The poloidal flux function $\Psi(z, x)$ contours for $n = 3$ with $\beta_p = 6.12$ are plotted on the $(r - \theta)$ plane with contour values of $C = 1.5, 1.0, 0.7, 0.5, 0.3$ to show the asymmetric supernova poloidal magnetic field lines and density weighed plasma velocity stream lines.
Fig. 10.— The poloidal flux function $\Psi(z, x)$ contours for $n = 4$ with $\beta_p = 6.41$ are plotted on the $(r - \theta)$ plane with contour values of $C = 4, 3, 2, 1, 0.5$ to show the asymmetric supernova poloidal magnetic field lines and density weighed plasma velocity stream lines.
Fig. 11.— The profile of $R(z)$ is plotted with $a = +2$, $b = +2$, $c = +\beta_p/8$, and $\beta_p = 5$. 
Fig. 12.— The poloidal flux function $\Psi(z, x)$ contours are plotted on the $(r - \theta)$ plane with contour values of $C = 7, 6, 5, 4, 3, 2$ to show the colimating GRB cusp funnel along the polar axis.
Fig. 13.— The poloidal flux function $\Psi(z, x)$ contours are plotted on the $(r - \theta)$ plane with contour values of $C = 7, 6, 5, 4, 3, 2$ to show the openning of the magnetosphere to the polar cusp.
Fig. 14.— The profile of $R(z)$ with a negative domain is plotted with $a = +2$, $b = +2$, $c = +\beta_p/8$, and $\beta_p = 5$. 
Fig. 15.— The poloidal flux function $\Psi(z, x)$ contours are plotted on the $(r-\theta)$ plane to show the AGN magnetosphere, plasma torus with $C = -1, -1.5, -2$, and polar jet with $C = 7, 5, 3, 1$, separated by two spherical separatrix at $z_1 = 1.17$ and $z_1 = 3.09$. 
Fig. 16.— The poloidal flux function $\Psi(z,x)$ contours are plotted on the $(r - \theta)$ plane to show the AGN magnetosphere with $C = 7, 5, 3, 1$. 