Noise-assisted quantum transport and computation

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Received 7 January 2013, in final form 25 March 2013
Published 13 May 2013
Online at stacks.iop.org/JPhysA/46/225301

Abstract

The transmission of an excitation along a spin chain can be hindered by the presence of small fixed imperfections that create trapping regions where the excitation may get caught (Anderson localization). A certain degree of noise, ensuing from the interaction with a thermal bath, allows us to overcome localization (noise-assisted transport). In this paper, we investigate the relation between the noise-assisted transport and (quantum) computation. In particular, we prove that noise does assist classical computation on a quantum computing device, but hinders the possibility of creating entanglement.

PACS numbers: 03.67.Lx, 03.65.Yz

(Some figures may appear in colour only in the online journal)

1. Introduction

Decoherence and relaxation induced on a quantum system by a fluctuating environment can considerably affect its dynamical behavior. The quantum transport of a particle in a one-dimensional tight-binding model, for example, requires coherent hopping between nearest-neighbor locations. The presence of imperfections appearing as a random time-independent potential can hinder the transmission. In fact, the energy mismatches due to such imperfections lead to destructive interference of the wavefunction [1] and Anderson localization [2] appears.

However, it has been recently argued that, in certain situations, the presence of an environment can assist the transport of an excitation [3–5]. The authors consider the exciton transport on small complexes, such as the Fenna–Matthew–Olson (FMO) protein complex and engineered quantum systems (binary trees) showing that, at room temperature, a limited amount of decoherence (accounting for the interaction between the complexes and the phonons of the surrounding environment) cancels out the localizing effect of small random imperfections along the transmission line. See also [6].

In this paper, we apply similar considerations to the model of the universal quantum computer put forward by Feynman in one of the earliest papers on quantum computation [7]. This computer model suits very well the investigation of the relation between transport and
computation, since the transmission of an excitation does correspond to the execution of a computation.

The device consists of an array of two-level systems with nearest-neighbor interactions mediated by ancillary qubits. The ancillae play the role of an input/output register. An excitation traveling along the chain plays the role of a cursor administering the applications of unitary transformations (computational primitives) to the register. In particular, when the cursor reaches the end of the chain, the register is in the output state and the computation is complete.

We investigate the effects of irregularities in the on-site energies of the cursor chain and the possibility of assisting the transmission of the cursor along the chain by means of an external potential and of energy exchanges with the environment. We show that the cursor does indeed travel beyond the localization length and reaches the far end of the chain. We discuss the conditions on the system–environment interaction and on the kind of information processed that allow for an extension of noise-assisted transport to noise-assisted computation.

Our aim was to write a self-contained paper able to reach not only physicists but also computer scientists. This paper is organized as follows: we first discuss the detrimental effect of random static noise on transmission (section 2) and present an instance of the noise-assisted transport by means of energy exchanges (section 3). We then discuss the effects of noise on the computational capabilities of the Feynman quantum computer (section 4). The last section is reserved for a discussion and outlook.

2. Localization on a linear lattice

Consider an excitation moving along a one-dimensional regular lattice, e.g., an electron moving along a periodic potential, in the tight-binding approximation. Calling \(|x\rangle\) the state localized at site \(x\), the Hamiltonian for a lattice of \(s\) sites is

\[
H_0 = -\frac{1}{2} \sum_{x=1}^{s-1} |x+1\rangle\langle x| + |x\rangle\langle x+1|.
\]  

The evolution \(|\psi_t\rangle\) of an initial condition \(|\psi_0\rangle\) is

\[
|\psi_t\rangle = \sum_{k=1}^{s} \exp(-it\epsilon_k) |e_k\rangle \langle e_k| |\psi_0\rangle
\]

where

\[
\epsilon_k = -\cos\left(\frac{k\pi}{s+1}\right), \quad k = 1, 2, \ldots, s,
\]

\[
|e_k\rangle = \sqrt{\frac{2}{s+1}} \sum_{x=1}^{s} \sin\left(\frac{k\pi x}{s+1}\right) |x\rangle.
\]

We are interested in the transport capability of the chain, that is, in the probability that an excitation initially located on the left-hand side of the chain (\(|\psi_0\rangle = |1\rangle\)) reaches the rightmost site \(s\), namely in the probability

\[
P_{Q_{\text{right}}}(t) = |\langle s|\psi_t\rangle|^2,
\]

where \(Q\) is the position operator:

\[
Q = \sum_{x=1}^{s} x |x\rangle\langle x|.
\]
Figure 1 shows an example of the motion of the excitation along the lattice: the particle moves ballistically along the chain until it bounces against the boundaries. As time goes on, the wave packet spreads over the line. The probability of finding the excitation at the end of the chain is highest when the first reflection occurs (around time $t = s$) and, at that time, is of order $O(s^{-2/3})$ [8].

We now turn our attention to the effects of imperfections on transmission. We model the imperfections of the lattice by a random time-independent potential. In the position representation we adopt, the random potential appears in the main diagonal of the Hamiltonian:

$$H_R = H_0 + V_R = H_0 + \sum_{x=1}^{s} \epsilon_x | x \rangle \langle x |.$$  \hspace{1cm} (4)

The coefficients $\epsilon_x$, $x = 1, 2, \ldots$, and $s$ are independent realizations of a random variable $R$.

The fact that the presence of imperfections can localize the excitation/cursor (Anderson localization) is well known [9–12]. For a simple review on Anderson localization, we refer the reader to [13]. The phenomenon can be understood in different ways. One is in terms of the destruction of the coherent hopping between nearest neighbors, that leads to the destructive interference of the wavefunction. Another interpretation, which better suits our discussion, is given in terms of the eigenstates of the Hamiltonian $H_R$: the presence of small imperfections on the chain results in a localization of the eigenstates of $H_R$. In our particular instance, since the initial condition has a non-negligible projection on the eigenstates localized on the left-hand side of the chain, and the eigenstates do not change shape (but only phase) in time, the excitation remains localized around its starting position.

The typical size of the localized states (localization length) depends on the distribution of the noise. Here, we consider the random variable $R$ to have a Gaussian distribution with mean zero and variance $\sigma^2$.

The idea of adding a linear potential $V_L = -g \sum_{x=1}^{s} x | x \rangle \langle x |$, with $g > 0$, to reduce the effect of the irregularities in the chain comes from a classical prejudice: classically, one expects this potential to pull the excitation rightward; moreover, with a suitable choice of the magnitude of $g$, the defects (small random potential terms) would become negligible with respect to the linear potential.
It is immediate to see that this idea/prejudice is very naive and actually quite misleading: the eigenstates of the Hamiltonian

$$H_{R+L} = H_0 + V_R + V_L = H_0 + \sum_{x=1}^{s} (\epsilon_x - gx) |x\rangle\langle x|,$$

with $g > 0$ are still localized. Localization manifests itself, in this case, in the form of Bloch oscillations, a genuine quantum effect: the momentum of the quasi-particle moving on the lattice changes linearly with time, $p(t) = p(0) - gt$, until it reaches the boundary of the Brillouin zone, where it is Bragg reflected [14]. For updated results on Bloch oscillations, we refer the interested reader to [15].

As an example, figure 2 shows the distribution of the observable $Q$ for the eigenstates of $H_{R+L}$, for $s = 20$, $\sigma = 0.5$ and $g = 2$: the eigenstates are localized. If a particle is initially localized at site 1, it has a non-vanishing projection only on the eigenstates localized in the same region, and it will therefore remain localized there.

The eigenvalues of the same Hamiltonian for a sample of 100 realizations of the random imperfections are shown in figure 2(b).

In the presence of imperfection, therefore, the transmission of a particle along a linear lattice via Hamiltonian evolution is suppressed and the idea of pulling the particle rightward by means of an external constant force, mimicking the effect of a battery, does not work.

3. Noise-assisted transport

The onset of interaction with a thermal environment might be expected to further reduce the transmission capabilities of the system. But this is not necessarily the case. Recent work showed that some interaction with the environment can assist transport on short chains [3] and quantum networks [4, 5].

In this section, after a brief elementary review of the concepts of dissipative dynamics that we use, we investigate the effects of dissipation on the transport capabilities of the linear lattice described in section 2.

The linear chain is now weakly coupled to a thermal bath that acts as a reservoir at inverse temperature $\beta$ and exchanges energy (not particles) with the system.
The evolution of an initial condition \( \rho(0) \) is described by the Lindblad equation:

\[
\frac{d\rho(t)}{dt} = \mathcal{D}(\rho(t)),
\]

where \( H \) is the Hamiltonian governing the chain and the dissipator \( \mathcal{D} \) has the following form:

\[
\mathcal{D}(\rho) = \zeta \sum_{m=1}^{s} \sum_{n=1}^{s} \gamma(m, n)(L(m, n)\rho L(m, n)^\dagger - \frac{1}{2}L(m, n)^\dagger L(m, n)\rho - \frac{1}{2}\rho L(m, n)\dagger L(m, n)).
\]

Here, \( \zeta \) is the (positive) system–environment coupling constant. The generators \( L(m, n) \) are defined in terms of the normalized eigenstates of the chain Hamiltonian \( H \) (satisfying: \( H|\psi_m\rangle = e_m|\psi_m\rangle; e_1 < e_2 < \cdots < e_s; \langle \psi_m | m \rangle = \delta_{m,m} \)):

\[
L(n, m) = 0 \quad \text{for } m \neq n.
\]

Each coefficient \( \gamma(m, n) \) in (7), for \( m \neq n \), has the meaning of transition probability per unit time from \( |n\rangle \) to \( |m\rangle \). Following [16, 17], we set

\[
\gamma(m, n) = \begin{cases} 
\frac{\eta(\omega(m, n))}{e^{\beta(\omega(m, n))} - 1}, & \text{for } n > m, \\
\frac{\eta(\omega(m, n))}{e^{\beta(\omega(m, n))} - 1} + 1, & \text{for } n < m,
\end{cases}
\]

where \( \omega(m, n) = |e_m - e_n| \). We consider only the case of an underdamped motion in which the simultaneous emission of two or more phonons is forbidden; namely we assume that

\[
\eta(\omega(m, n)) = \delta_{m,n+1} + \delta_{m,n-1}.
\]

This choice of the coefficients \( \gamma \) is such that the dissipative dynamics leads the chain to a Gibbs state at inverse temperature \( \beta \).

The equations for the elements of the system density matrix \( \rho(t) \), in the energy representation, are as follows:

\[
\frac{d}{dt}\rho_{m,n}(t) = \left(-i(e(m) - e(n)) - \zeta \sum_{j=1}^{s} \gamma(j, m) + \gamma(j, n) \right)\rho_{m,n}(t), \ m \neq n
\]

\[
\frac{d}{dt}\rho_{m,m}(t) = \zeta \sum_{c=1}^{s} \rho_{c,c}(t)\gamma(m, c) - \rho_{m,m}(t)\gamma(c, m).
\]

If the Hamiltonian of the system is the free Hamiltonian \( H_0 \), the transmission capabilities of the computing device are dramatically reduced by the presence of the bath, as expected: the system relaxes toward a thermal state that the lower the temperature \( 1/\beta \), the more is concentrated around the middle of the chain.

The introduction of disorder is obviously not expected to do any better: the eigenstates of the Hamiltonian \( H_0 \) are localized around the peaks of the random potential.

The situation is radically different when a linear (externally tunable) potential is added to the random one: Bloch localization makes the eigenstate of the Hamiltonian \( H_{K+1} \) localized in space (see figure 2(a)). The interaction with the bath determines a ‘stroll’ over the energy eigenstates which, because of the structure of the bath, favors, at low temperatures, the propagation toward the states of lower energy. The localized nature of the eigenstates of \( H_{K+1} \), due, we recall, to the presence of the external potential \( V_L \), makes the random walk on the
energies quasi-equivalent to a random walk on the sites of the chain: losing energy amounts to moving to the right.

The speed of the walk on the energy landscape induced by the dissipation depends on the energy gaps between the eigenstates of $H_{R+L}$ and on the coupling constant $\zeta$. This dependence can be analytically accessed by the inspection of equations (10) and (11).

The diagonal elements describe a random walk on the energy landscape of the system. The presence of a sufficiently strong external potential ($g > 1$) makes the energy gaps $|e(k) - e(k \pm 1)|$ almost equal for $k = 1, \ldots, s$ (see, for example, figure 2(b)); the emission and absorption rates $\gamma_E(x) = \gamma(x, x+1)$ and $\gamma_A(x) = \gamma(x+1, x)$ are then independent of $x$ ($\gamma_E(x) \approx \gamma_E, \gamma_A(x) \approx \gamma_A$). For $\beta$ high enough, $\gamma_A$ becomes negligible and $\gamma_E \approx 1$. Therefore, the random walk on energies described by equation (11) proceeds toward the states of lower energy at a rate $\zeta \gamma_E \approx \zeta$.

The coherences (off-diagonal terms) are described by autonomous equations (10) and are exponentially depressed in time (decoherence).

Frames (a) and (b) of figure 3 show the evolution of the initial condition $\rho_0 = |1\rangle \langle 1|$ under the Lindbladian (6) for $\beta = 1, g = 2$ and a realization of the random imperfections extracted from a zero-mean Gaussian with the standard deviation $\sigma = 0.5$.

The transport process determined by the Hamiltonian dynamics (see figure 1) is qualitatively different from the one depicted in figure 3.

Under ‘perfect’ Hamiltonian evolution, the particle bounces back and forth. It is thus necessary to determine the optimal time at which the position of the excitation has to be measured, that is the time at which the probability of finding the excitation at the end of the chain is maximal and this probability decreases with $s$ as $O(s^{-2/3})$. Moreover, the presence of disorder suppresses transmission.

In the presence of a bath and an external static field, the transmission of the particle to the far end of the chain takes a time that is one order of magnitude larger than under Hamiltonian evolution in the absence of imperfections (compare the time scales of figures 1 and 3). This is due to the weak system–bath coupling ($\zeta = 0.05$). But once the excitation has arrived at the end of the chain, it remains there, and the motion is stable with respect to random fluctuations of the on-site energies.
Figure 4. The Feynman machine: the $s$-sites cursor interacts with the register as described in (12).

The coupling with a cold reservoir, in the presence of a uniform force field that completely offsets a weak random potential, assists the transport (however slow and incoherent) of an excitation along a line.

4. Noise-assisted (quantum) computation

In this section, we discuss the relation between quantum transport and quantum computation. The quantum computer model we use in our investigation is the one proposed by Feynman [7].

Feynman’s quantum computer consists of two logically separated parts: one part, the clock, is an excitation moving along a lattice. The second part, the input/output register, is a collection of additional degrees of freedom, say $n$ spin-1/2 particles $\sigma(j) = (\sigma_1(j), \sigma_2(j), \sigma_3(j)), j = 1, 2, \ldots, n$.

The system is governed by the time-independent Hamiltonian:

$$H_F = -\frac{1}{2} \sum_{s=1}^{s-1} |x+1\rangle\langle x| \otimes U_x + |x\rangle\langle x+1| \otimes U^{-1}_x.$$ (12)

Each term of the Hamiltonian involves the two nearest-neighbor sites of the clock and a unitary operator $U_x$ acting on the register. The ordered product $U_{s-1} \cdots U_2 U_1$ realizes some input/output transformation we want the quantum computing device to accomplish. Figure 4 shows the architecture of the machine.

Because of the properties of the Hamiltonian $H_F$, the position of the excitation along the chain uniquely determines the state of the register. This fact has interesting consequences. Consider an initial condition of the form $|\psi_0\rangle = |1\rangle \otimes |R_1\rangle = |1, R(1)\rangle$, i.e. with the particle located at the beginning of the chain and the register in an input state $|R_1\rangle$. Then the set

$$B(\psi_0) = \{|1, R_1\rangle, |2, R_2\rangle, \ldots, |s, R_s\rangle\},$$

where $|R_j\rangle = U_{j-1} \cdots U_2 U_1 |R_1\rangle$, constitutes an orthonormal basis, the computational basis or Peres basis [18], for the region of the Hilbert space visited by the evolved state $|\psi_t\rangle = \exp(-iH_Ft)|\psi_0\rangle$. We refer to the space spanned by the Peres basis as to the computational subspace.

In particular, if upon measurement the cursor is found at the rightmost site of the chain, the register collapses to the output state $|R(s)\rangle = U_{s-1} \cdots U_2 U_1 |R(1)\rangle$.

It is clear that the capability of the chain of transferring the excitation from one end to the other is a central matter in the analysis of the computational power of the Feynman machine.
Before discussing the kinematics of the cursor, we point out that the sole effect of the interaction of the clock with the register (of \( n \) spins) is the appearance of a degeneracy of order \( 2^n \) in the spectrum \( \{ \varepsilon_k \}_{k=1}^{2^n} \) of the tight-binding (clock) Hamiltonian \( H_0 \) (see (1)). Once an initial condition of the form \( |\psi_0 \rangle = |1, R(1) \rangle \) has been set, however, the spectrum of the Hamiltonian \( H_F \) restricted to the computational subspace is no longer degenerate. In this subspace, to each eigenvalue \( \varepsilon_k \) (see (2)), there corresponds the eigenvector
\[
| u_k \rangle = \frac{2}{s+1} \sum_{x=1}^{s+1} \sin \left( \frac{k \pi x}{s+1} \right) | x, R(x) \rangle.
\]
(13)
The presence of a (random) potential acting on the cursor can drastically reduce the probability of finding the excitation at the end of the chain (i.e. the computation completed) but does not alter the working mechanism of the Feynman machine.

For the sake of definiteness, let the state \( |\psi_0 \rangle \) evolve under the Hamiltonian
\[
H_{F,V} = H_F + V = H_F + \sum_{x=1}^{s} f(x) | x \rangle \langle x |,
\]
(14)
where \( V \) represents a potential. The projector on the space \( \mathcal{H}_{\psi_0} \), spanned by the Peres basis \( \mathcal{B}(\psi_0) \):
\[
\mathcal{P}(\mathcal{H}_{\psi_0}) = \sum_{x=1}^{s} | x, R_x \rangle \langle x, R_x |
\]
remains a constant of the motion. Indeed,
\[
[V, \mathcal{P}(\mathcal{H}_{\psi_0})] = \left[ \sum_{x} f(x) | x \rangle \langle x |, \mathcal{P}(\mathcal{H}_{\psi_0}) \right] = \sum_{x,y} f(x) \left[ \langle x | \langle x, y \rangle \langle y, R_y | y, R_y \rangle \right] y, R_y \rangle = 0.
\]
(15)
The study of the evolution of the Feynman machine can therefore be reduced to the analysis carried out in section 2. In particular, the Hamiltonian (14), restricted to the subspace \( \mathcal{H}_{\psi_0} \), and represented in the same basis, admits \( s \) eigenvalues \( \varepsilon_k \) and corresponding eigenvectors \( | e_k \rangle \).

By requiring the operators \( U_x \) to act on a single register qubit, the Hamiltonian (12) becomes \( 3 \)-local. This restriction simplifies the implementation of the device, but at the same time it reduces the computational capability of the Feynman machine: it does not allow the realization of any two-qubit gate. In order to restore the universality of the model, Feynman introduced the switch circuit [7] that implements the selection statement.

In what follows, we consider a particular instance of the switch: the controlled NOT (CNOT) gate. The gate acts on two register qubits: \( \sigma(c) \), the controlling qubit and \( \sigma(p) \), the controlled (or passive) qubit. In our implementation, if the controlling qubit is ‘up’, namely \( \sigma_3(c) = +1 \), the passive qubit \( \sigma(p) \) is ‘negated’ by applying \( \sigma_1(p) \); if the controlling qubit is ‘down’ \( \sigma_3(c) = -1 \), then the controlled qubit is left unchanged.

The Hamiltonian realizing the gate is
\[
H_{\text{CNOT}}(a) = -\frac{1}{2} \left( |a+1 \rangle \langle a| \otimes \frac{1+\sigma_3(c)}{2} + |a+2 \rangle \langle a+1 | \right.
\]
\[
\otimes \sigma_1(p) + |a+5 \rangle \langle a+2 | \otimes \frac{1+\sigma_3(c)}{2} \right)
\]
\[
-\frac{1}{2} \left( |a+3 \rangle \langle a| \otimes \frac{1-\sigma_3(c)}{2} + |a+4 \rangle \langle a+3 | \right.
\]
\[
+ |a+5 \rangle \langle a+4 | \otimes \frac{1-\sigma_3(c)}{2} \right) + \text{H.c.}
\]
(16)
Figure 5. The CNOT described by equation (17). The state of the controlling qubit $\sigma(c)$ determines which branch of the circuit is visited by the moving particle. This determines the transformation applied to the controlled qubit $\sigma(p)$: $\sigma(p) = \text{NOT}(p)$ along the upper branch, the identity along the lower branch.

Here $a$ indicates the site of the clock at which the switch starts and H.c. stands for Hermitian conjugate. For a detailed discussion of the switch, we refer the interested reader to Feynman’s paper [7] and to [19]. For a recent review, see [20]. Here we just observe that the checking of $\sigma_3(c)$ at the beginning and at the end of each computational path, by preventing the cursor from bouncing back along the ‘wrong’ path, is crucial in preserving the reversibility of the unitary computation.

In order to discuss the effects of imperfections and dissipation on such a circuit, it is expedient of putting some ‘inertial’ sites before and after the CNOT, ‘during’ which we do not apply any transformation to the register qubits. The full Hamiltonian of the circuit is

$$H_C = \frac{1}{2} \sum_{x=1}^{a-1} (|x+1\rangle \langle x| + |x\rangle \langle x+1|) + H_{\text{CNOT}}(a) - \frac{1}{2} \sum_{x=b}^{s-1} (|x+1\rangle \langle x| + |x\rangle \langle x+1|).$$

(17)

We indicate by $b = a + 5$ the position at which the CNOT ends. Figure 5 shows the structure of the CNOT gate.

The state of the controlling qubit determines which branch of the CNOT circuit is visited.

If the initial state of the system is

$$|\psi_U\rangle = |Q = 1\rangle \otimes |\sigma_3(c) = +1, \sigma_3(p) = -1\rangle = |1\rangle \otimes |+1, -1\rangle$$

only the upper branch of the CNOT is visited during the evolution of the system and, if the cursor reaches the region $a + 5, \ldots, s$ the state of the register is $|+1, +1\rangle$.

With the initial condition $|\psi_D\rangle = |1\rangle \otimes |-1, -1\rangle$, it is the lower branch of the CNOT that is visited and the identity is applied to the $\sigma(p)$.

The analysis of the motion of the cursor is once more simplified by the use of the Peres bases

$$B(\psi_U) = \{|1\rangle \otimes |+1, -1\rangle, \ldots, |a + 1\rangle \otimes |+1, -1\rangle, |a + 2\rangle \otimes |+1, +1\rangle, |b\rangle \otimes |+1, +1\rangle, \ldots, |s\rangle \otimes |+1, +1\rangle\}.$$  

(18)

and

$$B(\psi_D) = \{|1\rangle \otimes |-1, -1\rangle, \ldots, |a + 3\rangle \otimes |-1, -1\rangle, |a + 4\rangle \otimes |-1, -1\rangle, |b\rangle \otimes |-1, -1\rangle, \ldots, |s\rangle \otimes |-1, -1\rangle\}.$$  

(19)

Both bases consist of $s - 2$ orthogonal elements and (thanks to the presence of the projectors $(1 \pm \sigma_3(c))/2$ at the beginning and at the end of each branch of the CNOT circuit) span two orthogonal subspaces $H_{\psi_U}$ and $H_{\psi_D}$. The Hamiltonian (17) restricted to $H_{\psi_U}$ (or $H_{\psi_D}$) and
represented in the basis $\mathcal{B}(\psi_0^U)$ (or $\mathcal{B}(\psi_0^D)$) is equivalent to the ‘free’ Hamiltonian (1) on $s-2$ sites. In particular, the motion of the cursor along the two computational paths is exactly the same.

The presence of random imperfections, namely of independent realization $\epsilon_1, \epsilon_2, \ldots, \epsilon_s$ of a zero-mean Gaussian random variable $R$ with variance $\sigma^2$, perturbs the motion along the computational paths. Since $\epsilon_{a+1} \neq \epsilon_{a+3}$ and $\epsilon_{a+2} \neq \epsilon_{a+4}$ with probability 1, the spectra of the Hamiltonians restricted to $\mathcal{H}_{\psi_0^U}$ and $\mathcal{H}_{\psi_0^D}$ will be slightly different. But the correctness of the computation is guaranteed by the conservation law (15).

The problem is the ensuing localization that we discussed in section 2; it can suppress the probability of reaching the rightmost sites of a linear lattice by means of an external force and dissipation. In what follows, we show that this result extends straightaway to each computational path.

We add an external field such that the potential difference between one site and its right neighbor(s) is $-g$, by setting

$$V_L' = -\left( \sum_{x=1}^{a-2} g x |x\rangle \langle x| + \sum_{x=a+3}^{s} g(x-2)|x\rangle \langle x| \right).$$

The Hamiltonian evolution of the system is given by

$$\mathcal{H}_{\text{CNOT}} + V_R + V_L' = \mathcal{H}_{\text{CNOT}} + \sum_{x=1}^{s} \epsilon_x |x\rangle \langle x|$$

$$- \left( \sum_{x=1}^{a-2} g x |x\rangle \langle x| + \sum_{x=a+3}^{s} g(x-2)|x\rangle \langle x| \right).$$

The initial condition of the system (either $|\psi_0^U\rangle$ or $|\psi_0^D\rangle$) selects the relevant computational subspace (either $\mathcal{H}_{\psi_0^U}$ or $\mathcal{H}_{\psi_0^D}$). By switching on an interaction with a thermal bath satisfying the hypotheses listed in section 3, we determine a walk on the energy landscape of the selected space identical to the one we described in the previous section. For example, if the initial state of the machine is $\rho_0^U = |\psi_0^U\rangle \langle \psi_0^U|$, the elements of the density matrix $\rho^U(t)$, restricted to the computational subspace $\mathcal{H}_{\psi_0^U}$ and there expressed in the energy representation, are given by (10) and (11).

This means that the clocking particle can travel beyond the CNOT. And when it does, the conservation law (15) guarantees that the state of the register is the correct one.

We can thus state that noise can assist classical computation on a quantum computing device.

Figure 6 shows the probability of reaching, starting from site 1, the region $b, \ldots, s$ under different evolutions. The presence of imperfections along the chain strongly reduces the probability of ever reaching the sites located after the gate. For the ballistic evolution, there are times at which this probability is close to unity. When the evolution is dissipative, the time required to pass the CNOT is one order of magnitude larger than the first ‘arrival time’ under Hamiltonian evolution. But the probability of finding the computation completed is monotonically increasing to 1.

The equivalence between successful transport and successful computation does not extend to the case that sees the controlling qubit $\sigma(c)$ in a superposition of ‘up’ ($\sigma_3(c) = +1$) and ‘down’ ($\sigma_3(c) = -1$).

Suppose, for instance, that the controlling spin $\sigma(c)$ starts from the state

$$|\sigma_1(c) = +1\rangle = \frac{|\sigma_3(c) = +1\rangle + |\sigma_3(c) = -1\rangle}{\sqrt{2}},$$
Figure 6. $s = 22$: with this choice, each computational path consists of $s - 2 = 20$ sites, thus allowing for comparison with figures 1 and 3; $a = 9$. The probability of reaching, starting from site 1, the region $b, \ldots, s$ under three different evolutions. Dotted line: Hamiltonian evolution determined by $H_F$; dashed line: Hamiltonian evolution under $H_F + V_R$, for a realization of the random potential $V_R$ with $\sigma = 0.5$. Solid line: the dissipative evolution determined by the Hamiltonian $H_F + V'_L + V_R$ for the same realization of $V_R$, $g = 2$, coupling $\zeta = 0.05$ and inverse temperature $\beta = 1$.

and therefore the complete system starts from the initial condition

$$| \psi^{U+D}_0 \rangle = \frac{1}{\sqrt{2}} \left( | \psi^U_0 \rangle + | \psi^D_0 \rangle \right).$$

If the evolution were purely Hamiltonian, with identical on-site energies in the two branches of figure 5, application of the CNOT primitive would bring the register $(\sigma(c), \sigma(p))$ to the maximally entangled Bell state [21]:

$$| \Phi^+ \rangle = \frac{1}{\sqrt{2}} \left( | -1, -1 \rangle + | +1, +1 \rangle \right).$$

The presence of random imperfections on the circuit we are considering produces two effects.

- The introduction of random relative phases between the two computational paths. Because of the relative phases, when the particle is on the right of the CNOT, the state of the register can be slightly different from the target Bell state. But this effect can in principle be ‘bounded away’ for small values of the variance $\sigma^2$;
- Anderson localization of the cursor; this is the key issue that we discuss below.

The transmission of the excitation from one end to the other of the circuit can be assisted by dissipation. But this time dissipation destroys the computation. The density matrix of the system can be schematically represented as a block matrix:

$$\begin{pmatrix}
\rho_{DD}(t) & \rho_{DU}(t) \\
\rho_{UD}(t) & \rho_{UU}(t)
\end{pmatrix}. \tag{22}$$

The blocks $\rho_{DD}(t)$ and $\rho_{UU}(t)$ represent the evolution of the projections of the initial state, respectively, on $\mathcal{H}_{\psi^U_0}$ and $\mathcal{H}_{\psi^D_0}$. The blocks $\rho_{DU}(t)$ and $\rho_{UD}(t)$ correspond to the coherences between the computational subspaces. Lindblad evolution, according to (10), destroys all the coherences. This does not affect the computation within each computational subspace: the decoherence simply transforms a quantum walk on the energy landscape into a random walk. But the damping of coherences between the computational subspaces destroys the desired entanglement; it transforms the state of the register into the maximally mixed state

$$\frac{1}{2} \left( | -1, -1 \rangle \langle -1, -1 | + | +1, +1 \rangle \langle +1, +1 | \right).$$
This is evident from the fact that the von Neumann entropy of the register for a machine evolving from $|\psi_0^{U+D}\rangle$ tends to $\log(2)$ as $t \rightarrow \infty$ (solid line in figure 7). For comparison, the dashed line of figure 7 shows that, starting from $|\psi_0^U\rangle$, the entropy of the register tends to the value 0 competing to a pure state.

For both initial conditions, the entropy reaches a maximum about the time $\bar{t}$ when $E(Q)$ is close to $s/2$; at this time $\sqrt{\text{var}(Q)}$ is large (see figure 3(b)).

For the initial condition $|\psi_0^U\rangle$, the state of the register at time $\bar{t}$ is a mixture of two states: $|+1,-1\rangle\langle+1,-1|$ (cursor on the upper branch and on the left-hand side of the NOT) and $|+1,+1\rangle\langle+1,+1|$ (cursor on the upper branch and on the right-hand side of the NOT). Hence, the value $\log(2)$ for the absolute maximum at time $\bar{t}$ of the entropy for the dashed line of figure 7.

If the controlling qubit $\sigma(c)$ is in a superposition of ‘up’ and ‘down’, at time $\bar{t}$ the register is a mixture of three states: $|+1,-1\rangle\langle+1,-1|$ (cursor on the upper branch and on the left-hand side of the NOT) with weight 1/4, $|+1,+1\rangle\langle+1,+1|$ (cursor on the upper branch and on the right-hand side of the NOT) with weight 1/4 and, this time, $|-1,-1\rangle\langle-1,-1|$ (cursor on the lower branch) with weight 1/2. Hence, the value $(3/2) \log(2)$ of the maximum is reached by the solid line.

For the initial condition $|\psi_0^{U+D}\rangle$ the steep growth of the entropy from 0 to $\log(2)$ that leads to the plateau around time $\bar{t} = 40$ is a witness of the disruption of coherences between computational subspaces, as it is easy to check by the direct inspection of the decay rate of the explicit solution of equation (10) for the coherences.

5. Conclusions and outlook

We presented an example of noise-assisted transport on a lattice and discussed a set of hypotheses that allow for the extension of noise-assisted transport to noise-assisted computation.

Our research started, in a very literal sense, from the effort of understanding section 4 (on imperfections and irreversible free energy loss) of reference [7], in which Feynman proposed, in order to save his cursor model from the effect of lattice imperfections, to ‘pull the cursor along the program line with an external force’.
We provided examples of the correctness of the physical intuition behind such a proposal for a computation involving a single computational path: the dissipative scenario for conductivity (well known since, maybe, the Drude–Lorentz model) predicts that as long as a computation involves one given initial state and its successors in the sense of Peres, completion of the computation, though slower than in the ballistic unitary case, can be reached with a much higher probability under non-unitary Lindblad evolution. If, however, the computation has to follow distinct computational paths, as needed for generating entanglement, we showed that the effort needed to ‘pull the cursor along the program line with an external force’ may disrupt essential phase information.

In a historical perspective, our no-go statement makes it clear that the Feynman model intended only to suggest the possibility of implementing classical computation on an atomic substrate, the computational role of entanglement being far from clear in the context in which that model was proposed; in the contemporary context, our considerations raise some questions, which will be the object of future research, on the possibility of overcoming the effect of geometrical imperfections in the graph supporting a quantum walk.

We are presently exploring the possibility, offered by time-dependent density-functional theory, of simulating the position probability distribution of an open quantum system with the one of a system undergoing unitary propagation under a time-dependent potential [22]. The interest of this proposal is in the fact (made possible by the extension of the point of view of TDDFT to discrete systems [23, 24] and to open discrete systems [25]) that the same external time-dependent potential may be made to act along the two branches of the computation, leading to completion without loss of phase information.

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