The Hausdorff dimension of average conformal repellers under random perturbation

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Abstract

We prove that the Hausdorff dimension of an average conformal repeller is stable under random perturbations. Our perturbation model uses the notion of a bundle random dynamical system. At the end of this paper, we give an example of an average conformal repeller which is not a conformal repeller.

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1. Introduction

In the dimension theory of dynamical systems, only the Hausdorff dimension of invariant sets of a conformal dynamical system is well understood. Bowen [6] was the first to express the Hausdorff dimension of an invariant set as a solution of an equation involving topological pressure. Ruelle [18] refined Bowen’s method and got the following result. If \(J\) is a mixing repeller for a \(C^{1+\alpha}\) conformal expanding map \(f\) on a Riemannian manifold \(M\), then the Hausdorff dimension of \(J\) can be obtained as the zero \(t_0\) of \(t \mapsto \pi_f(-t \log ||D_x f||)\), where \(\pi_f\) denotes the topological pressure functional. This statement is known as the Bowen–Ruelle formula, and we sometimes call the equation involving topological pressure the Bowen equation. Gatzouras and Peres relaxed the smoothness \(C^{1+\alpha}\) to \(C^1\) in [12].

Recently, different versions of topological pressure have become a useful tool in calculating the Hausdorff dimension of a non-conformal repeller. For \(C^1\) non-conformal repellers, Zhang used singular values of the derivative \(D_x f^n\) for all \(n \in \mathbb{N}\), to define a new equation which involves the limit of a sequence of topological pressures; then he showed that...
the upper bound of the Hausdorff dimension of a repeller was given by the unique solution of the equation, see [20] for details. Barreira considered the same problem in [2]. By using the non-additive thermodynamic formalism which was introduced in [3] and the singular value of the derivative $D_x f^n$ for all $n \in \mathbb{N}$, he gave an upper bound of the box dimension of a repeller under the additional assumptions that the map was $C^{1+\alpha}$ and $\alpha$-bunched. This is automatically implied for the Hausdorff dimension. In [7], by using the sub-additive topological pressure which was studied in [8], the author proved that the upper bound of the Hausdorff dimension for $C^1$ non-conformal repellers obtained in [2, 11, 20] were the same and it was the unique root of the Bowen equation for sub-additive topological pressure; we point out that the map only needs to be $C^1$ without any additional condition in [7].

In [1], the authors introduced the notion of average conformal repellers in deterministic dynamic systems which was a generalization of the quasi-conformal and asymptotically conformal repellers in [3, 17], and they proved that the Hausdorff dimension and the box dimension of the average conformal repellers were the unique root of the Bowen equation for sub-additive topological pressure. In that paper, the map only needed to be $C^1$, without any additional condition.

For random repellers, Kifer proved that the Hausdorff dimension of a measurable random conformal repeller was the root of the Bowen equation, which can be seen as a random version of the deterministic case, see [13] for details. And in [4], the authors generalized this result to an almost-conformal case. In [21], using the idea in the deterministic case [1], the authors introduced the notion of a random average conformal repeller, and they proved that the Hausdorff dimension of random average conformal repellers was the unique root of the Bowen equation for random sub-additive topological pressure which was studied in [22].

Motivated by the work in [4], where the authors showed that the Hausdorff dimension of the conformal repeller was stable under random perturbation, we consider a random perturbation of the deterministic average conformal repeller which is modelled using the notion of a bundle random dynamical system (RDS for short). Namely, let $\vartheta$ be an ergodic invertible transformation of a Lebesgue space $(\Omega, \mathcal{W}, \mathbb{P})$ and consider a measurable family $T = \{T(\omega) : \mathcal{W} \to \mathcal{W}\}$ of $C^{1+\alpha}$ maps, that is to say, $(\omega, x) \mapsto T(\omega)x$ is assumed to be measurable. This determines a differentiable RDS via $T(n, \omega) := T(\vartheta^{n-1}\omega) \circ \cdots \circ T(\vartheta\omega) \circ T(\omega)(n \in \mathbb{N})$. Further, Let $E \subset \Omega \times \mathcal{W}$ be a measurable set such that all $\omega$-sections $E_\omega := \{x \in \mathcal{W} : (\omega, x) \in E\}$ are compact. If $K$ denotes the collection of all compact subsets of $\mathcal{W}$ endowed with the Hausdorff topology, this is equivalent to saying that $K$-valued multifunction $\omega \mapsto E_\omega$ is measurable. Here and in what follows we think of $E_\omega$ being equipped with the trace topology, i.e. an open set $A \subset E_\omega$ is of the form $A = B \cap E_\omega$ with some open set $B \subset \mathcal{W}$. We call $E$ $T$-invariant if $T(\omega)E_\omega = E_{\vartheta\omega}$ $\mathbb{P}$-a.s., and in this situation the Hausdorff dimension of the fibre $E_\omega$ is a $\mathbb{P}$-a.s. constant, see [10]. The map $\Theta : E \to E$ is defined by $\Theta(\omega, x) = (\vartheta\omega, T(\omega)x)$, and we call it the skew product transformation.

The aim of this paper is to make rigorous the statement that if a bundle RDS is close to an average conformal expanding map on a repeller then the corresponding Hausdorff dimension is close.

The paper is organized as follows. In section 2, we will recall the main result in [1]. In section 3, we introduce some random notions and our model of random perturbation, we point out that this was essentially inspired by a remarkable result of Liu [15]. In section 4, we formulate and prove our main result, which says that the Hausdorff dimension of an average conformal repeller is stochastically stable. Moreover, we give an example of an average conformal repeller which is not a conformal repeller.
2. Dimension of an average conformal repeller

In this section, we will recall the notion of sub-additive topological pressure and the main result in [1] which says that the Hausdorff dimension of an average conformal repeller can be given by the unique root of the sub-additive topological pressure. Moreover, we will give some preliminary results.

Let \( f : X \to X \) be a continuous map on a compact space \( X \) with metric \( d \). A subset \( E \subset X \) is called an \((n, \varepsilon)\)-separated set with respect to \( f \) if \( x \neq y \in E \) implies \( d_0(x, y) := \max_{0 \leq i \leq n-1} d(f^i x, f^i y) > \varepsilon \). Let \( \mathcal{F} = \{ \phi_n \}_{n \geq 1} \) denote a sub-additive potential on \( X \), that is to say \( \phi_n : X \to \mathbb{R} \) is continuous for each \( n \in \mathbb{N} \) and satisfying

\[
\phi_{n+m}(x) \leq \phi_n(x) + \phi_m(f^n(x)), \quad \forall n, m \in \mathbb{N}, \quad x \in X.
\]

Following the way in [8], we define the sub-additive topological pressure

\[
\pi_f(\mathcal{F}, n, \varepsilon) = \sup \left\{ \sum_{x \in E} \exp \phi_n(x) : E \text{ is an } (n, \varepsilon)\text{-separated subset of } X \right\}
\]

and then call

\[
\pi_f(\mathcal{F}) = \lim_{n \to \infty} \limsup_{\varepsilon \to 0} \frac{1}{n} \log \pi_f(\mathcal{F}, n, \varepsilon)
\]

the sub-additive topological pressure of \( \mathcal{F} \) with respect to \( f \). If there is no confusion caused, we simply call \( \pi_f(\mathcal{F}) \) the sub-additive topological pressure of \( \mathcal{F} \).

Remark 1.

1. When the continuous potential \( \mathcal{F} = \{ \phi_n \} \) on \( X \) is additive, i.e. \( \phi_n(x) = \sum_{i=0}^{n-1} \phi(f^i x) \) for some continuous function \( \phi : X \to \mathbb{R} \), then \( \pi_f(\mathcal{F}) \) is the classical topological pressure, see [19] for details, and we denote it simply by \( \pi_f(\phi) \).

2. When the continuous potential \( \mathcal{F} = \{ \phi_n \} \) on \( X \) is sup-additive, that is to say, \( \phi_{n+m}(x) \geq \phi_n(x) + \phi_m(f^n x) \), \( \forall n, m \in \mathbb{N}, x \in X \), we also can define the sup-additive topological pressure. And the pressures are equal under some special case, see [1].

Let \( \mathcal{M}(X, f) \) denote the space of all \( f \)-invariant Borel probability measures and \( \mathcal{E}(X, f) \) denote the subset of \( \mathcal{M}(X, f) \) with ergodic measures. For \( \mu \in \mathcal{M}(X, f) \), let \( h_\mu(f) \) denote the measure-theoretic entropy of \( f \) with respect to \( \mu \), and let \( F_\mu(\mu) \) denote the following limit:

\[
F_\mu(\mu) = \lim_{n \to \infty} \frac{1}{n} \int \phi_n \, d\mu.
\]

The relation between \( \pi_f(\mathcal{F}) \), \( h_\mu(f) \) and \( F_\mu(\mu) \) is given by the following variational principle which is proved in [8], and the random version of the following theorem is proved in [22].

Theorem 2.1 (Variational principle). Let \( \mathcal{F} \) be a sub-additive potential on a compact metric space \( X \), and \( f : X \to X \) be a continuous transformation, then

\[
\pi_f(\mathcal{F}) = \begin{cases} 
-\infty, & \text{if } F_\mu(\mu) = -\infty \text{ for all } \mu \in \mathcal{M}(X, f) \\
\sup \{h_\mu(f) + F_\mu(\mu) : \mu \in \mathcal{M}(X, f)\}, & \text{otherwise}.
\end{cases}
\]

Proposition 2.1. Let \( f_i : X_i \to X_i \) (\( i = 1, 2 \)) be a continuous map of a compact metric space \( (X_i, d_i) \), and \( \mathcal{F} = \{ \phi_n \} \) be a sub-additive potential on \( X_2 \). If \( \varphi : X_1 \to X_2 \) is a surjective continuous map with \( \varphi \circ f_1 = f_2 \circ \varphi \) then \( \pi_{f_2}(\mathcal{F}) \leq \pi_{f_2}(\mathcal{F} \circ \varphi) \), and if \( \varphi \) is a homeomorphism then \( \pi_{f_2}(\mathcal{F}) = \pi_{f_2}(\mathcal{F} \circ \varphi) \), where \( \mathcal{F} \circ \varphi = \{ \phi_n \circ \varphi \} \).
Proof. We first check that $F \circ \varphi$ is indeed a sub-additive potential on the compact metric space $X_1$. In fact

$$
\phi_{n+m} \circ \varphi(x) \leq \phi_n(\varphi(x)) + \phi_m(\varphi^n) = \phi_n \circ \varphi(x) + \phi_m \circ \varphi(f^n x).
$$

The equality follows from the fact that $\varphi \circ f_1 = f_2 \circ \varphi$.

Let $\epsilon > 0$ and choose $\delta > 0$ such that $d_2(\varphi(x), \varphi(y)) > \epsilon$ implies $d_1(x, y) > \delta$. This fact follows from the uniform continuity of $\varphi$.

Let $E$ be an $(n, \epsilon)$-separated set with respect to $f_2$. Since $\varphi$ is surjective, there exists a subset $F \subset X_1$ such that $\varphi|_F : F \to E$ is a bijection. It follows from the above observation that $F$ is an $(n, \delta)$-separated set with respect to $f_1$. Hence, we have

$$
\pi_f J F, n, \epsilon = \sup \{ \sum_{x \in E} \exp \phi_n(x) : E \text{ is an } (n, \epsilon)-\text{separated subset of } X_2 \}
$$

$$
= \sup \{ \sum_{y \in F} \exp \phi_n(\varphi y) : E \text{ is an } (n, \epsilon)-\text{separated subset of } X_2 \}
$$

$$
= \sup \{ \sum_{y \in F} \exp \phi_n(\varphi y) : F \text{ is an } (n, \delta)-\text{separated subset of } X_1 \}
$$

Since $\epsilon \to 0$ then $\delta \to 0$, then we can have

$$
\pi_f J F, n, \epsilon = \pi_f J F \circ \varphi, n, \delta
$$

If $\varphi$ is a homeomorphism then we can apply the above with $f_1, f_2, \varphi, F$ replaced by $f_2, f_1, \varphi^{-1}, F \circ \varphi$ respectively, to give $\pi_f J F \circ \varphi, n, \delta$. Thus the proof is complete. \qed

Proposition 2.2. Let $f : X \to X$ be a continuous map on a compact metric space, and $\phi : X \to \mathbb{R}$ be a continuous function on $X$. Suppose $\phi_\epsilon : X \to \mathbb{R}$ is a continuous function on $X$ for every $\epsilon > 0$ and $\lim_{\epsilon \to 0} \phi_\epsilon = \phi$, then

$$
\lim_{\epsilon \to 0} \pi_f J \phi_\epsilon = \pi_f J \phi.
$$

Proof. This immediately follows from the continuity of the classical topological pressure. \qed

Now we introduce the definition of an average conformal repeller. And the dimension of the repeller can be obtained by the unique root of the corresponding sub-additive topological pressure.

Let $M$ be a $C^\infty$ $m$-dimensional Riemannian manifold. Let $U$ be an open subset of $M$ and $f : U \to M$ be a $C^1$ map. Suppose $J \subset U$ is a compact $f$-invariant subset. Let $M(f|_J)$, $E(f|_J)$ denote the set of all $f$-invariant measures and the set of all ergodic invariant measures supported on $J$, respectively. For any $\mu \in E(f|_J)$, by the Oseledec multiplicative ergodic theorem (see [16]), we can define Lyapunov exponents $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_m(\mu)$, $m = \dim M$.

**Definition 2.1.** A compact $f$-invariant set $J$ is called an average conformal repeller for $f$ if for any $\mu \in E(f|_J)$, $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_m(\mu) > 0$.

**Remark 2.** We point out that if $J$ is an average conformal repeller for $f$, it is indeed a repeller in the usual way (see [9]) that: $\exists k > 1, C > 0$ such that for all $x \in J$ and $v \in T_x M$

$$
||Df^n(v)|| \geq C\lambda^n ||v||, \quad \forall n \geq 1.
$$
Proposition 2.3. If $J$ is an average conformal repeller for $f$, then
\[ \pi_f(\Phi) = \lim_{n \to \infty} \frac{1}{n} \pi_f(-\log \|D_x f^n\|), \]
where $\Phi = \{-\log \|D_x f^n\| \}_{n \geq 1}$ is a sup-additive potential and $\pi_f(\Phi)$, $\pi_f(-\log \|D_x f^n\|)$ denote the sup-additive topological pressure of $\Phi$ with respect to $f$ and the classical topological pressure of $-\log \|D_x f^n\|$ with respect to $f^n$, respectively.

Proof. Let $\Psi = \{-\log m(D_x f^n)\}_{n \geq 1}$ denote the sub-additive potential. First note that by the definition of topological pressure, we have
\[ \frac{1}{k} \pi_f(-\log \|D_x f^k\|) \leq \frac{1}{k} \pi_f(-\log m(D_x f^k)), \quad \forall k \geq 1. \]
And since $J$ is an average conformal repeller, the measure-theoretic entropy map $\mu \mapsto h_\mu(f)$ is upper-semi-continuous by remark 2. By proposition 2.2 in [7], we have
\[ \lim_{k \to \infty} \frac{1}{k} \pi_f(-\log m(D_x f^k)) = \pi_f(\Psi), \]
where $\pi_f(\Psi)$ denotes the sub-additive topological pressure of $\Psi$ with respect to $f$. Thus we have
\[ \limsup_{k \to \infty} \frac{1}{k} \pi_f(-\log \|D_x f^k\|) \leq \pi_f(\Psi) = \pi_f(\Phi), \quad (2.1) \]
where the last equality is proved in [1] since $J$ is an average conformal repeller for $f$.

On the other hand, for any $\mu \in \mathcal{M}(f | J) \subset \mathcal{M}(f^k | J)$, we have
\[ h_\mu(f) + \Phi_\mu(\mu) = \lim_{k \to \infty} \frac{1}{k} (h_\mu(f^k) + \int -\log \|D_x f^k\| \, d\mu) \leq \liminf_{k \to \infty} \frac{1}{k} \pi_f(-\log \|D_x f^k\|); \]
the last inequality follows from the classical variational principle for additive topological pressure of $-\log \|D_x f^k\|$ with respect to $f^k$, see [19]. Again because $J$ is an average conformal repeller, by the variational principle for the sup-additive topological pressure (see [1]), we have
\[ \pi_f(\Phi) \leq \liminf_{k \to \infty} \frac{1}{k} \pi_f(-\log \|D_x f^k\|). \quad (2.2) \]
Thus the desired result immediately follows from (2.1) and (2.2). 

The dimension of an average conformal repeller can be given by the following theorem in [1].

Theorem 2.2. Let $f$ be a $C^1$ dynamical system and $J$ be an average conformal repeller for $f$, then the Hausdorff dimension of $J$ is zero of $t \mapsto \pi_f(-t\Psi)$, where $\Psi = \{\log m(D_x f^n) : x \in J, n \in \mathbb{N}\}$ and $m(A) = |A^{-1}|^{-1}$.

3. Random notions

In this section, we will give some random notions and some well-known results. Firstly, let $(\Omega, \mathcal{W}, \mathbb{P})$ and $\Theta, E, T$ be as described in section 1, and let $\mathcal{M}_\mathbb{P}(E, T)$ denote the space of $\Theta$-invariant measures with marginal $\mathbb{P}$ on $\Omega$ of the RDS, $\mathcal{C}_\mathbb{P}^1(E, T)$ denote the subset of $\mathcal{M}_\mathbb{P}(E, T)$ with ergodic measures of the RDS.
Let $L^1(E, \Omega, C(M))$ denote the collection of all integrable random continuous functions on fibres, i.e., a measurable $f : E \to \mathbb{R}$ is a member of $L^1(E, \Omega, C(M))$ if $f(\omega) : E_\omega \to \mathbb{R}$ is continuous and $\|f\|_1 := \int \|f(\omega)\| d\mathbb{P}(\omega) < \infty$, where $\|f(\omega)\| = \sup_{x \in E_\omega} |f(\omega, x)|$. If we identify $f$ and $g$ provided $\|f - g\| = 0$, then $L^1(E, \Omega, C(M))$ becomes a Banach space with the norm $\| \cdot \|_1$.

Let $\varepsilon : \Omega \to (0, 1]$ be a measurable function. A set $F \subset E_\omega$ is said to be $(\omega, \varepsilon, n)$-separated for $T$, if $x, y \in F, x \neq y$ implies $y \notin B_\omega(n, x, \varepsilon)$, where $B_\omega(n, x, \varepsilon) := \{y \in E_\omega : d(T(k, \omega)x, T(k, \omega)y) < \varepsilon(d^k\omega) \text{ for } 0 \leq k \leq n-1\}$ and $d$ is the given metric on $M$.

Let $\Phi = \{\varphi_n\}_{n \geq 1}$ be a sub-additive function sequence with $\varphi_n \in L^1(E, \Omega, C(M))$ for each $n$. As usual, we put

$$\pi T(\Phi)(\omega, \varepsilon, n) = \sup \left\{ \sum_{x \in F} \varphi_n(\omega, x) : F \text{ is an } (\omega, \varepsilon, n) \text{-separated subset of } E_\omega \right\}.$$ 

$$\pi T(\Phi)(\varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \int \pi T(\Phi)(\omega, \varepsilon, n) d\mathbb{P}(\omega),$$

$$\pi T(\Phi) = \lim_{\varepsilon \downarrow 0} \pi T(\Phi)(\varepsilon).$$

The last quantity is called the sub-additive topological pressure of $\Phi$ with respect to $T$. We just define that the above definition is reasonable, see [22] for details.

**Remark 3.**

(i) If the function sequence $\Phi = \{\varphi_n\}$ can be written as $\varphi_n(\omega, x) = \sum_{i=0}^{n-1} \varphi(\Theta^i(\omega, x))$ for some function $\varphi \in L^1(E, \Omega, C(M))$, then we call $\pi T(\Phi)$ the random additive topological pressure, see [4, 14] for details, denote it simply by $\pi T(\varphi)$.

(ii) Since $\mathbb{P}$ is ergodic in the model which we consider, so the limits in the above definition will not change $\mathbb{P}$-almost everywhere without integrating against $\mathbb{P}$.

**Lemma 3.1.** For $i = 1, 2$, let $X_i$ be compact metric spaces, $E_i$ measurable bundles over $\Omega$ with compact fibres in $X_i$, and $\varphi_i$ topological bundle RDSs on $E_i$. If $\psi = \{\psi(\omega) : E^2_\omega \to E^1_\omega\}$ is a family of homeomorphisms between $E^1_\omega$ and $E^2_\omega$ satisfying $\varphi_2(\omega) \circ \psi(\omega) = \psi(\theta^i\omega) \circ \varphi_1(\omega)$, $\mathbb{P} \text{-} a.s.$, and $\mathcal{F} = \{f_n\}_{n \geq 1}$ is a sub-additive potential in $L^1(E, \Omega, C(X_2))$ then

$$\pi_{\varphi_2}(\mathcal{F}) = \pi_{\varphi_1}(\mathcal{F} \circ \psi)$$

where $\mathcal{F} \circ \psi = \{f_n \circ \psi\}_{n \geq 1}$ denotes the member of $L^1(E, \Omega, C(X_1))$ defined by $f_n(\omega, \psi(\omega)x)$ for each $n \geq 1$.

**Proof.** We first check whether the new defined potential $\mathcal{F} \circ \psi = \{f_n \circ \psi\}_{n \geq 1}$ is indeed sub-additive. Precisely, we have

$$f_{n+m} \circ \psi(\omega, x) = f_{n+m}(\omega, \psi(\omega)x)$$

$$\leq f_n(\omega, \psi(\omega)x) + f_m(\theta^m\omega, \varphi_2(n, \omega)\psi(\omega)x)$$

$$= f_n \circ \psi(\omega, x) + f_m(\theta^m\omega, \psi(\theta^m\omega)\varphi_1(n, \omega)x)$$

$$= f_n \circ \psi(\omega, x) + f_m \circ \psi(\theta^m\omega, \varphi_1(n, \omega)x).$$

Let $\mu \in \mathcal{M}_+^1(E_1, \varphi_1)$ and write $\mu_\psi$ for the member of $\mathcal{M}_+^1(E_2, \varphi_2)$ defined by $\psi(\omega)_*\mu_\omega$. We have $h_{\mu_\psi}(\varphi_1) = h_{\psi}(\varphi_2)$ (see theorem 2.2.2 in [5]) and

$$\lim_{n \to \infty} \frac{1}{n} \int f_n \circ \psi d\mu = \lim_{n \to \infty} \frac{1}{n} \int f_n d\mu_\psi,$$

by the variational principle of random sub-additive topological pressure in [22] we have $\pi_{\mu_\psi}(\mathcal{F} \circ \psi) \leq \pi_{\mu}(\mathcal{F})$. By symmetry we get the reverse inequality and hence the desired result. \qed
Definition 3.1. Let $T$ be a RDS over $\vartheta$. A generator of RDS $T$ is a family $A = \{A(\omega) = (A_i(\omega)) : A(\omega)$ is an open cover of $E_{\omega}\}$ with

(i) $A(\omega)$ is finite for all $\omega \in \Omega$;
(ii) $\omega \mapsto d(x, A_i(\omega))$ is measurable for all $x \in M$ and all $i \in \mathbb{N}$;
(iii) for each sequence $(A_n)_{n \in \mathbb{N}}$ of sets we have $A_n \in A(\vartheta^n \omega)$ for all $n \in \mathbb{N}$ which implies that
$$\bigcap_{n=0}^{\infty} T(n, \omega)^{-1} A_n$$
contains at most one point.

Definition 3.2. Let $T$ be a RDS over $\vartheta$. We call $T$ (positive) expansive if there exists a $(0, 1)$-valued random variable $\Delta$ such that
$$d(T(n, \omega)x, T(n, \omega)y) \leq \Delta(\vartheta^n \omega)$$
for all $n \in \mathbb{N}$ implies $x = y$.

Definition 3.3. A generator $A$ of a given RDS $T$ is called a strong generator if
$$\lim_{k \to \infty} \text{diam} \left( \bigcap_{i=0}^{k-1} T(i, \omega)^{-1} A(\vartheta^i \omega) \right) = 0$$
uniformly in $\omega$.

An expansive RDS is said to be strongly expansive if it possesses a strong generator.

Let $U \subset M$ be an open subset of the Riemannian manifold $M$ with $\bar{U} \in K$ and let $C(U, M)$ denote the space of all continuous maps from $U$ to $M$ endowed with the compact open topology.

Definition 3.4. Assume $f \in C(U, M)$ and $J \in K$ with $fJ = J$. A family $\{T_\epsilon\}_{\epsilon > 0}$ of $C(U, M)$-valued random variables is called a random perturbation of $f$ on $J$ if

(i) $\lim_{\epsilon \to 0} T_\epsilon = f$ in probability;
(ii) there exists a family of $K$-valued random variables $\{J_\epsilon\}_{\epsilon > 0}$ such that
(a) for each $\epsilon > 0$ we have that $\mathbb{P}$-a.s. $T_\epsilon(\omega)J_\epsilon(\omega) = J_\epsilon(\vartheta \omega)$;
(b) $\lim_{\epsilon \to 0} J_\epsilon = J$ in probability.

$\{T_\epsilon\}_{\epsilon > 0}$ is said to be structurally stable if there exists a family $\{h_\epsilon\}_{\epsilon > 0}$ of $C(J, M)$-valued random variables such that

(iii) for each $\epsilon > 0$ we have that $h_\epsilon(\omega) : J \to J_\epsilon(\omega)$ is a homeomorphism and $T_\epsilon(\omega) \circ h_\epsilon(\omega) = h_\epsilon(\vartheta \omega) \circ f$ $\mathbb{P}$-a.s.;
(iv) $\lim_{\epsilon \to 0} h_\epsilon = \text{id}$ in probability.

See [4] for examples of strongly expansive bundle RDS and structurally stable random perturbation. Put $X = \bar{U}$ and let $C^\alpha(X, \mathbb{R})$ denote the space of all Hölder continuous functions on $X$ with Hölder exponent $\alpha$. We endow $C^\alpha(X, \mathbb{R})$ with the usual norm $|| \cdot ||_\alpha := || \cdot || + || \cdot ||_\alpha$, where $|| \cdot ||$ is the sup-norm and $|| \cdot ||_\alpha$ is the least Hölder constant, namely, $|| \cdot ||_\alpha := \sup_{x, y \in X, x \neq y} \frac{|\phi(x) - \phi(y)|}{|d(x, y)|^{\alpha}}$. Using proposition 2.2, we get the following important proposition which is proved in [4], and cite it without proof.

Proposition 3.1. Let $\{T_\epsilon\}_{\epsilon > 0}$ be a structurally stable random perturbation of $f$ on $J$. And let $\{\phi_\epsilon\}_{\epsilon > 0}$ be a family of $C^\alpha(X, \mathbb{R})$-valued random variables satisfying
$$\lim_{\epsilon \to 0} ||\phi_\epsilon - \phi||_\alpha = 0$$
in $L^1(\mathbb{P})$ for some $\phi \in C^\alpha(X, \mathbb{R})$. Then
$$\lim_{\epsilon \to 0} \pi_{T_\epsilon}(\phi_\epsilon) = \pi_f(\phi).$$
4. Stability of the Hausdorff dimension under random perturbations

In this section we will prove that the Hausdorff dimension of the average conformal repeller is stable under suitable random perturbation. And we give an example of an average conformal repeller which is not a conformal repeller in the end.

The following proposition can be proved by a slight modification of the proof of theorem 1.1 in [15].

**Proposition 4.1.** Assume $J$ is an average conformal repeller for a $C^{1+\alpha}$ map $f : U \to M$. There exists a $C^1$ neighbourhood $U(f) \subset C^{1+\alpha}(U, M)$ of $f$ such that the following holds:

(i) For every random variable $T : \Omega \to U(f)$ there exists a $K$-valued random variable $J_\omega \subset U$ satisfying $T(\omega)J(\omega) = J(\vartheta \omega)$, and a $C^0(U, M)$-valued random variable $h$ such that each $h(\omega)$ is a homeomorphism between $J\omega$ and $J(\omega)$ and $T(\omega)\circ h(\omega) = h(\vartheta \omega) \circ f$ on $J$.

(ii) If $\{T_\epsilon : \Omega \to U(f)\}_{\epsilon > 0}$ is a family of random variables with $\lim_{\epsilon \to 0} T_\epsilon = f$ in probability (with respect to the $C^1$ distance), then $\lim_{\epsilon \to 0} h_\epsilon = \text{id}$ in probability and thus $\lim_{\epsilon \to 0} J_\epsilon = J$ in probability. Here $h_\epsilon$ and $J_\epsilon$ are the corresponding objects associated with $T_\epsilon$ by (i).

In other words, each family $\{T_\epsilon\}_{\epsilon > 0}$ of $U(f)$-valued random variables with $\lim_{\epsilon \to 0} T_\epsilon = f$ in probability is a structurally stable random perturbation of $f$ on $J$.

Now we state and prove our main result.

**Theorem 4.1.** Assume $J$ is an average conformal repeller for a $C^{1+\alpha}$ map $f : U \to M$. There exists a $C^1$ neighbourhood $U(f) \subset C^{1+\alpha}(U, M)$ of $f$ such that the assertions of proposition 4.1 hold with the following additional property.

If $\{T_\epsilon : \Omega \to U(f)\}_{\epsilon > 0}$ is a random perturbation of $f$ with $\lim_{\epsilon \to 0} T_\epsilon = f$ in $L^1(\Omega, C^{1+\alpha}(U, M))$ then

$$\lim_{\epsilon \to 0} \dim_H(J_\epsilon(\omega)) = \dim_H(J) \quad \mathbb{P}\text{-a.s.}$$

where $\dim_H(\cdot)$ denotes the Hausdorff dimension of a set. Moreover, if $L \subset J$ is compact and $f$-invariant, then $\lim_{\epsilon \to 0} \dim_H(h_\epsilon(\omega)L) = \dim_H(L) \quad \mathbb{P}\text{-a.s.}$

**Proof.** In the following we will follow Bogenschütz and Ochs’s proof [4] to obtain the desired result. Choose an open neighbourhood $V$ of $J$ such that $V$ is a compact subset of $U$. Then $U(f)$ can be chosen in such a way that $J_\epsilon(\omega) \subset V$ for every $\epsilon > 0, \omega \in \Omega$.

Fix $\epsilon > 0$. For $(\omega, x) \in \Omega \times V$ set

$$\eta_\epsilon(\omega, x) := ||D_T\epsilon(\omega)|| \quad \text{and} \quad \lambda_\epsilon(\omega, x) := m(D_T\epsilon(\omega)).$$

By taking appropriate $U(f)$, we can assume that $\log \lambda_\epsilon, \log \eta_\epsilon \in L^1(\Omega, C(M))$.

For the clarity of the proof, we divide the proof into several steps.

**Step 1.** We claim that $T_\epsilon$ satisfies the following formula:

$$\lambda_\epsilon(\omega, x) - K(\omega)d(x, y)^\alpha \leq \frac{d(T_\epsilon(\omega)x, T_\epsilon(\omega)y)}{d(x, y)} \leq \eta_\epsilon(\omega, x) + K(\omega)d(x, y)^\alpha \quad (4.3)$$

for every $\omega \in \Omega$ and $x \neq y \in J_\epsilon(\omega)$, where $K : \Omega \to \mathbb{R}_+$ with $\log^+ K \in L^1(\mathbb{P})$. We will prove the inequality (4.3) in the rest of this step.
Choose \( r_0 > 0 \) such that \( A := \{ x : \text{dist}(x, V) \leq r_0 \} \subset U \). Define
\[
K_0(\omega) = \left| DT_\omega(\omega) \right|_{\partial A} = \sup \left\{ \frac{\| D_T(\omega) - D_T(\omega) \|}{d(x, y)^\alpha} : x, y \in A, x \neq y \right\}.
\]

For the simplicity of notation, we restrict \( M \) to be the case of an open subset of \( \mathbb{R}^d \), since the general case can be done via local coordinates. We let \( | \cdot | \) denote the Euclidian norm on \( \mathbb{R}^d \) and write \( \mathbb{T} \) instead of \( T \( \omega \) for convenience.

For \( n \), we have that
\[
\log \lambda_\epsilon \left( \omega, x \right) \geq \inf \{ |A| : A \in \text{convex hull of } D_{x(T^n(\omega)x)} \}
\]
and get that
\[
\int_0^{\log \lambda_\epsilon \left( \omega, x \right)} D_{x(T^n(\omega)x)} d\mathbb{T},
\]

Thus, we get that
\[
\frac{|T(x) - T(y)|}{|y - x|} \leq \eta(x, y) + K_0(\omega)|y - x|^\alpha.
\]

On the other hand, we can get that
\[
\frac{1}{|y - x|} \left| \int_0^{\log \lambda_\epsilon \left( \omega, x \right)} D_{x(T^n(\omega)x)} d\mathbb{T} \right|
\]
\[
\geq \inf \{ |A| : A \in \text{convex hull of } D_{x(T^n(\omega)x)} \}
\]
\[
\geq |D_{x(T^n(\omega)x)}| - \sup \{ |A| : A \in \text{convex hull of } (D_{x(T^n(\omega)x)} - D_{x(T^n(\omega)x)}),
\]
\[
0 \leq t \leq |y - x|
\]
\[
\geq \eta(x, y) + K_0(\omega)|y - x|^\alpha.
\]

The last inequality follows from the definition of \( \eta \), immediately.

Put
\[
K(\omega) = \max \left\{ K_0(\omega), \frac{\text{diam}V}{r_0}, \max \left\{ |D_{x(T^n(\omega)x)}| : x \in V \right\} \right\}
\]

and then the inequality (4.3) immediately follows.

Step 2. We claim that
\[
\int \log \Lambda_\epsilon^\alpha d\mathbb{P} > 0
\]
for some \( n \geq 1 \), where \( \Lambda_\epsilon^\alpha(\omega) = \min_{\epsilon \in J(\omega)} \prod_{k=0}^{n-1} \lambda_\epsilon(D_{x(T^n(\omega)x)} - D_{x(T^n(\omega)x)}).
\]

Recall that \( J \) is an average conformal repeller for \( f \). It is easy to see that \( \lim_{\epsilon \to 0} \log \Lambda_\epsilon^\alpha = \min_{\epsilon \in J(\omega)} \prod_{k=0}^{n-1} \lambda_\epsilon(D_{x(T^n(\omega)x)} - D_{x(T^n(\omega)x)}) > 0 \) in probability. By making \( U(f) \) smaller if necessary we have that \( \log \text{dist}(T^n(\omega)x)) \) is uniformly bounded for all \( T \in U(f), x \in V \), and \( \epsilon \in T_\epsilon M \) with \( |\epsilon| = 1 \). Thus \( \lim_{\epsilon \to 0} \log \Lambda_\epsilon^\alpha = \min_{\epsilon \in J(\omega)} \prod_{k=0}^{n-1} \log m(D_{x(T^n(\omega)x)} - D_{x(T^n(\omega)x)}) > 0 \) also in \( L^1(\mathbb{P}) \), which implies \( \sup_{n \geq 1} \frac{1}{n} \int \log \Lambda_\epsilon^\alpha d\mathbb{P} > 0 \) for sufficiently small \( \epsilon \). This finishes the proof of the claim.

Step 3. We claim that \( T_\epsilon \) is strongly expansive. By remark 2 we know \( f \) is expanding on \( J \), then there exists a neighbourhood \( V \) of \( J \), a constant \( c > 0 \) and an integer \( n \geq 1 \) such that \( |D_{x(T^n(\omega)x)} - D_{x(T^n(\omega)x)}| \geq 1 + c \) for every \( x \in V \) and \( \epsilon \in T_\epsilon M \) with \( |\epsilon| = 1 \). We can choose \( U(f) \) in such a way that \( |D_{x(T^n(\omega)x)} - D_{x(T^n(\omega)x)}| \geq 1 + \frac{c}{2} \) whenever \( T_1, \ldots, T_n \in U(f), x \in V \) and \( \epsilon \in T_\epsilon M \) with \( |\epsilon| = 1 \), and that \( J_\epsilon(\omega) \subset V \) for every \( \epsilon > 0 \) and \( \omega \in \Omega \). Then \( T_\epsilon \) is uniformly expanding and thus strongly expanding on the bundle \( \{ J_\epsilon(\omega) \}_{\omega \in \Omega} \).
Step 4. Let $L \subset J$ be a compact subset with $fL = L$. We apply corollary 3.5 in [4] to the bundle RDS $T_\epsilon$ on $J_\epsilon = \{(\omega, x) : x \in h_\epsilon(\omega)L\}$. Let $\pi_\epsilon$ denote the pressure functional of $T_\epsilon$ restricted to $J_\epsilon$, then we can get that there exists $s_0^\epsilon \geq t_0^\epsilon \geq 0$ such that

$$\pi(\cdot - t_0^\epsilon \log \eta_\epsilon) = 0 = \pi(\cdot - s_0^\epsilon \log \lambda_\epsilon)$$

and

$$t_0^\epsilon \leq \dim_H (h_\epsilon(\omega)L) \leq s_0^\epsilon \quad \text{P-a.s.}$$

If we consider the system $T_\epsilon(n, \omega)$ and $\log \|D_x T_\epsilon(n, \omega)\|$, $\log m(D_x T_\epsilon(n, \omega))$ for every $n > 0$, and let $\pi_{n, \epsilon}$ denote the pressure functional of $T_\epsilon(n, \omega)$ restricted to $J_\epsilon$, then we can get that $t_\epsilon n \leq \dim H (h_\epsilon(\omega)L) \leq s_\epsilon n \text{ P-a.s.}$.

Step 5. We first note that, for each fixed positive integer $n$, we have

$$\lim_{\epsilon \to 0} \|\log \|D_x T_\epsilon(n, \omega)\|-\log \|D_x f^n\||_a = 0$$

and

$$\lim_{\epsilon \to 0} \|\log m(D_x T_\epsilon(n, \omega)) - \log m(D_x f^n)\|_a = 0 \quad \text{in } L^1(\mathbb{P}),$$

since $|\log \|D_x T_\epsilon(n, \omega)\|\|$ is uniformly bounded for the fixed positive integer $n$. By proposition 3.1, for each fixed positive integer $n$, we have

$$\lim_{\epsilon \to 0} 1/n \pi_{n, \epsilon}(\cdot - t \log \|D_x T_\epsilon(n, \omega)\|) = 1/n \pi_{f|\epsilon}(\cdot - t \log \|D_x f^n\|)$$

and

$$\lim_{\epsilon \to 0} 1/n \pi_{n, \epsilon}(\cdot - t \log m(D_x T_\epsilon(n, \omega))) = 1/n \pi_{f|\epsilon}(\cdot - t \log m(D_x f^n))$$

for each $t \geq 0$. Moreover, by proposition 2.2 in [7] and proposition 2.3 we have

$$\lim_{n \to \infty} 1/n \pi_{f|\epsilon}(\cdot - t \log \|D_x f^n\|) = \lim_{n \to \infty} 1/n \pi_{f|\epsilon}(\cdot - t \log m(D_x f^n)) = \pi_{f|\epsilon}(\cdot - t \Psi).$$

Hence, we obtain for each $t \geq 0$ that

$$\lim_{n \to \infty} \lim_{\epsilon \to 0} 1/n \pi_{n, \epsilon}(\cdot - t \log \|D_x T_\epsilon(n, \omega)\|) = \lim_{n \to \infty} \lim_{\epsilon \to 0} 1/n \pi_{n, \epsilon}(\cdot - t \log \|D_x T_\epsilon(n, \omega)\|)

= \pi_{f|\epsilon}(\cdot - t \Psi).$$

Step 6. To complete the proof, given $\delta > 0$, since $t \mapsto \pi_{f|\epsilon}(\cdot - t \Psi)$ is strictly decreasing, there exist $N > 0$, $e_0 > 0$ such that for each $\epsilon \leq e_0$, we have

$$\pi_{N, \epsilon} \langle - (t_0 + \delta) \rangle \log \|D_x T_\epsilon(N, \omega)\| < 0 < \pi_{N, \epsilon} \langle - (t_0 - \delta) \rangle \log \|D_x T_\epsilon(N, \omega)\|$$

and

$$\pi_{N, \epsilon} \langle - (t_0 + \delta) \rangle m(D_x T_\epsilon(N, \omega)) < 0 < \pi_{N, \epsilon} \langle - (t_0 - \delta) \rangle m(D_x T_\epsilon(N, \omega)).$$

This immediately implies

$$t_0 - \delta < t_0^\epsilon \leq \dim_H (h_\epsilon(\omega)L) \leq s_0^\epsilon < t_0 + \delta.$$

The desired result then immediately follows. □
Remark 4.

(1) In [4], Bogenschütz and Ochs proved that the Hausdorff dimension of a conformal repeller is stable under random perturbations. Using their ideas, we show that the same is true for an average conformal repeller. The differences between theorem 4.1 and Bogenschütz and Ochs’s theorem are the following:
   
   (i) In order to use the corollary 3.5 in [4], it is the same from step 1 to step 3.
   
   (ii) In order to prove that the Hausdorff dimension of the average conformal repeller is stable under random perturbation, we should consider the iteration of the RDS from step 4 to step 6. And this process needs the technique of sub-additive topological pressure and sup-additive topological pressure. In [4], the authors need not consider the iteration of the RDS, so they need only additive topological pressure.

(2) Since the bundle $T_\epsilon$ is uniformly expanding on the bundle $\{J_\epsilon(\omega)\}_{\omega \in \Omega}$, the result in [15] told us that there exists an equilibrium state of the topological pressure $\pi_\epsilon$. Then modifying subtly the proof in [1] we can get that the zero of the sub-additive topological pressure is the upper bound of the Hausdorff dimension of the bundle $\{J_\epsilon(\omega)\}_{\omega \in \Omega}$.

Proposition 4.2. Under the conditions of theorem 4.1, we have

$$\lim_{\epsilon \to 0} h_{\text{top}}^\epsilon(T_\epsilon) = h_{\text{top}}(f),$$

where $h_{\text{top}}^\epsilon(T_\epsilon)$ denotes the topological entropy of the random dynamical system $T_\epsilon$ generated by the random perturbation of $f$ and $h_{\text{top}}(f)$ denotes the classical topological entropy of the deterministic dynamical system.

Proof. This can be immediately deduced from proposition 3.1 by taking the potential functions to be the zero-valued functions. □

Example 1. Let $S = [0, 1] \times [0, 1]$ denote the square. We define the maps $f_1 : S \to S$ by $(x, y) \mapsto (\frac{1}{3}x + 0.05, \frac{1}{2}x + \frac{1}{2}y + 0.05)$ and $f_2 : S \to S$ by $(x, y) \mapsto (\frac{1}{3}x + 0.05 + \frac{1}{3}, \frac{1}{2}x + \frac{1}{2}y + 0.05)$, see the following diagram, that means $f_1$ maps $S$ to $X_1$ and $f_2$ maps $S$ to $X_2$:

Let $g_i$ be the inverse of the map $f_i$ for $i = 1, 2$, and define a map $g : X_1 \cup X_2 \to X_1 \cup X_2$ as follows:

$$g(x, y) = \begin{cases} g_1(x, y), & \text{if } (x, y) \in X_1, \\ g_2(x, y), & \text{if } (x, y) \in X_2. \end{cases}$$
And let $\Lambda = \{(x, y) \in X_1 \cup X_2 : g^n(x, y) \in X_1 \cup X_2, \forall n \geq 0\}$, it is easy to see that the set $\Lambda$ is not a conformal repellor of $g$. And if we let $\mathcal{M}_g(\Lambda)$ denote the set of all $g$-invariant ergodic measures, then it is easy to see that for each $\mu \in \mathcal{M}_g(\Lambda)$, the Lyapunov exponents of $g$ is equal to $\log 3$ for $\mu$-a.e. $(x, y) \in \Lambda$, therefore, the set $\Lambda$ is an average conformal repellor of $g$.

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