Altland and Zirnbauer reply: In the preceding comment Casati, Izrailev and Sokolov (CIS) claim that our recent field theoretical analysis of the quantum kicked rotor misses an important dynamical feature: the difference in behavior between rational and irrational values of $T = \tau/4\pi$ ($\tau$ being the time between kicks). We reject that claim. The rationality of $T$ does play a fundamental role in our Letter (referred to here as I). We were kept from reviewing the number theoretic aspects in I by space limitations, and welcome this opportunity to dispel any confusion that may have resulted.

It is well known (cf. Ref. [10] of I for a review) that the quantum physics of the rotor is qualitatively different for rational as opposed to irrational $T$. In our Letter only the rational case was treated. Indeed, what we did was to introduce an upper angular momentum cutoff $L$ (or, equivalently, we put the system on an angular lattice), thereby imposing a quantization condition that forces $T$ to be rational. To make this point clear, let us briefly review how the topology of phase space relates to the period $\tau$: Take the classical phase space $\Gamma$ of the rotor, which is a cylinder $S^1 \times \mathbb{R}$, and compactify it to a 2-torus $S^1 \times S^1$ by imposing periodic boundary conditions not only for the angle $\theta$ but also for the angular momentum $l$ (with period $L \in \mathbb{N}$). Canonical quantization then requires the eigenvalues of the operator $\theta$ to be spaced by $2\pi/L$, thus giving the angular lattice. Moreover, $L$-periodicity of the Floquet operator $U \left( U_\ell = U_{l+L,\ell+L} \right)$ forces the period $\tau$ to satisfy a quantum resonance condition, i.e. to be of the rational form $\tau = 4\pi n/L$ with $n \in \mathbb{N}$. Conversely, if $\Gamma = S^1 \times \mathbb{R}$ and $\tau$ fulfills the quantum resonance condition, $U$ commutes with the operator translating $l$ by $L$ units and Bloch’s theorem says that the eigenfunctions $\psi$ of $U$ organize into sectors labeled by a Bloch wave number $\alpha$: $\psi(l+L) = e^{i\alpha L} \psi(l)$. By choosing to focus on one specific sector, we arrive at the compactified model with periodic ($\alpha = 0$) or twisted ($\alpha \neq 0$) boundary conditions. Thus, the quantum resonance of the kicked rotor expansion does not “mysteriously disappear between eqs. (6) and (8)” of I, as is stated by CIS, but is fundamental (though implicit) to the formulation of the model we treat.

Why did we choose to compactify? Our main motivation was that discrete symmetries such as those due to the number theoretic properties of $T$, are not easy to keep track of in the approximate steps that are performed in the late stages of our analysis. These steps are valid if and only if the dominant configurations in (6) are $Z_\ell = \delta_\ell Z_i$ where the field $Z_i$ varies slowly with $i$. In the presence of exact or approximate discrete symmetries, they become invalid in general. For example, for $T$ close to rational $n/L$, where $e^{i\alpha L} U e^{-i\alpha L} \approx U$, additional low-energy modes appear at wave number $\sim 1/L$. These are lost by the naive gradient expansion without prior compactification. We avoided this difficulty by tuning $T$ to a rational value and taking care of the resulting discrete symmetry by restriction of the phase space. (A similar desymmetrization strategy was used in order to deal with time reversal symmetry; see footnote [13] of I.)

Another benefit from compactification is that we can break time reversal symmetry by kicking the rotor with $\cos(\theta + a)$, where $a \in [0, 2\pi/L]$ is closely analogous to an Aharonov-Bohm flux threading a mesosopic metallic ring. CIS argue that $a$ is a pure gauge, which cannot ever affect the two-particle Green function (1) of I. While this is so for $\Gamma = S^1 \times \mathbb{R}$, it is not true for the cohomologically nontrivial ring topology we consider [2].

By construction, the field theory of I inherits the ring topology, i.e. the nonlinear $\sigma$ model field $Q$ obeys the boundary condition $Q(l) = Q(l + L)$. What are the consequences? The answer depends on the size of $L$ relative to $\xi = k^2/2 + ..., \ldots$, the localization length. For $L < \xi$ the quantum motion is diffusive and states extend more or less uniformly around the ring. In the opposite limit ($L \gg \xi$), diffusion is brought to a halt by quantum interference effects causing localization. However, the wavefunctions are expected to have exponentially small tails, reaching around the ring even in this case. True localization is possible only in the irrational limit $L \rightarrow \infty$. Let us emphasize that the emergence of localization for $L > \xi$ is not incompatible with the resurgence of wavefunctions that is predicted by Bloch’s theorem.

To conclude, we reject the insinuation that there is a “hidden random-matrix ensemble” in our Letter. We are confident that the steps leading to the field theory (8) are valid for the compactified or angular lattice model under consideration. The dominance of diagonal fields $Z_\ell = \delta_\ell Z_i$ is ensured by the limit of hard driving $kT \gg 1$, while the gradient expansion is an expansion in powers of $k/\xi \sim k^{-1}$, valid for large $k$.

PACS 05.45.+b, 72.15.Rn

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[1] G. Casati, F. M. Izrailev and V. V. Sokolov, preceding Comment.
[2] A. Altland and M.R. Zirnbauer, Phys. Rev. Lett. 77, 4536 (1996).
[3] The gauge transformation that eliminates $a$ violates the boundary condition $\psi(l + L) = e^{i\alpha L} \psi(l)$, unless $a \in 2\pi Z/L$. 

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