Formation of Sub-millisecond Pulsars and Possibility of Detection

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ABSTRACT

Pulsars have been recognized as normal neutron stars, but sometimes argued as quark stars. Sub-millisecond pulsars, if detected, would play an essential and important role in distinguishing quark stars from neutron stars. We focus on the formation of such sub-millisecond pulsars in this paper. A new approach to form a sub-millisecond pulsar (quark star) via accretion induced collapse (AIC) of a white dwarf is investigated here. Under this AIC process, we found that: (1) almost all the newborn quark stars could have an initial spin period of \( \sim 0.1 \) ms; (2) the nascent quark stars (even with a low mass) have sufficiently high spin-down luminosity and satisfy the conditions for pair production and sparking process to be as sub-millisecond radio pulsars; (3) in most cases, the timescales of newborn quark stars in the phase of spin period \(< 1\) (or \(< 0.5\) ms) can be long enough to be detected.

As a comparison, an accretion spin-up process (for both neutron and quark stars) is also investigated. It is found that, quark stars formed through AIC process can have shorter periods (\( \leq 0.5\) ms); while the periods of neutron stars formed in accretion spin-up process must be longer than 0.5ms. Thus if a pulsar with a period less than 0.5 ms can be identified in the future, it should be a quark star.

Key words: Accretion – Gravitational waves – Stars: Neutron – Pulsars: General

1 INTRODUCTION

Though it has been more than 40 years since the discovery of radio pulsars, their real nature is still not yet clear because of the uncertainty about cold matter at supranuclear density. Both neutron matter and quark matter are two conjectured states for such compact objects. The objects with the former are called neutron stars, and with the latter are quark stars. It is an astrophysical challenge to observationally distinguish real quark stars from neutron stars. The objects with the former are called neutron stars, and with the latter are quark stars. It is an astrophysical challenge to observationally distinguish real quark stars from neutron stars. The objects with the former are called neutron stars, and with the latter are quark stars. It is an astrophysical challenge to observationally distinguish real quark stars from neutron stars.

resulting in a sub-millisecond pulsar and the possibility of detecting a sub-millisecond pulsar with a fine-tuned pulsar-search survey. Gourgoulhon et al. (1999) have investigated the maximally rotating configurations of quark stars and showed that the minimal spin period was between 0.513 ms and 0.640 ms. Burderi et al. (1999) have predicted that there might exist a yet undetected population of massive sub-millisecond neutron stars, and the discovery of a sub-millisecond neutron star would imply a lower limit for its mass of about 1.7 \( M_\odot \). A detailed investigation about spin-up of neutron stars to sub-millisecond period, including a complete statistical analysis of the ratio with respect to normal millisecond pulsars, was performed by Possenti et al. (1999). The minimal recycled period was found to be 0.7 ms. Gondek-Rosinska et al. (2001) have found that the shortest spin period is approximately 0.6 ms through the maximum orbital frequency of accreting quark stars. Huang & Wu (2003) have found that the initial periods of pulsars are in the range of \( 0.6 \sim 2.6\) ms using the proper motion data. Zheng et al. (2006) have showed that hybrid stars instead of neutron or quark stars may lead to sub-millisecond pulsars. Haensel, Zdunik & Bejger (2008) have discussed the compact stars’ equation of state (EOS) and the spin-up to sub-millisecond period, via mass accretion from a disk in a low-mass X-ray binary.

There have been many observational attempts in searching sub-millisecond pulsars. A possible discovery of a 0.5 millisecond pulsar in Supernova 1987A is not held true in the follow-up observations (Sasseen 1990; Percival et al. 1995). Bell et al. (1995) reported on optical observation of the low mass binary millisecond
pulsar system PSR J0034-0534, and they used white dwarf cooling models to speculate that, the limit magnitude of the J0034-0534’s companion suggested that this millisecond pulsar’s initial spin period was as short as 0.6 ms. As addressed by D’Amico & Burderi (1999), in particular the detection of a pulsar with a spin period well below 1 ms could put severe constraints on the neutron star structure and the absolute ground state for the baryon matter in nature. They designed an experiment to find sub-millisecond pulsars with Italian Northern Cross radio telescope near Bologna. Edwards, van Straten & Bailes (2001) have found none of sub-millisecond pulsars in a search of 19 globular clusters using the Parks 64 m Radio telescope at 660 MHz with a time resolution of 25.6 µs. Han et al. (2004) did not find any sub-millisecond pulsars from highly polarized radio source of NVSS (NRAO VLA Sky Survey). Kaaret et al. (2007) have found oscillations at a frequency of 1122 Hz in an X-ray burst from a transient source XTE J1739-285 which may contain the fastest rotating neutron star so far. Significant difficulties do exist in current radio surveys for binary sub-millisecond pulsars due to strong Doppler modulation and computational limitations (Burderi et al. 2001).

How do sub-millisecond pulsars form? This is still an open question which we will explore in this paper. Previously, discussions were concentrated on the formation of neutron stars or quark stars spun up via accretion in binaries. We have considered a new approach to create a sub-millisecond pulsar (quark star) with super-Keplerian spin via accretion induced collapse (AIC) of a massive white dwarf (WD). The initial spin of the newborn quark star could be super-Keplerian, and it can have a long lifetime in sub-millisecond phase and produce enough strong radio emission to be detected.

In §2, we discuss low mass quark stars formed from AIC of WDs, which can have an initial period similar to sub-millisecond. In §3, the radiation parameters and the conditions for pair production are estimated in order to investigate whether the AIC-induced quark stars could be pulsars or not. The lifetimes of sub-millisecond pulsars are also estimated and the possibility of detection is discussed. The spin-down evolution diagrams of a newborn quark star and neutron star are also plotted. In §4, as a comparison, the sub-millisecond pulsars formed through accretion acceleration (spin-up) in binary systems are also considered. In §5, conclusions and discussions are presented.

2 SUB-MILLISECOND QUARK STARS FORMED THROUGH AIC OF WHITE DWARFS

Neutron stars’ formation from AIC of a massive white dwarf is widely discussed by many authors (Nomoto et al. 1979; Nomoto & Kondo 1991; van Paradijs et al. 1997; Fryer et al. 1999; Bravo & García-Senz 1999; Dessart et al. 2006). Recently, it is pointed out that Galactic core-collapse supernova rate cannot sustain all the separate neutron star populations (Keane & Kramer 2008), which implies other mechanisms for forming neutron stars. AIC of a massive WD can be an important mechanism for pulsar formation, even for isolated pulsars if the binary systems are destroyed due to strong kicks. We now discuss the possibility for a low mass quark star formed from AIC of a WD. In a binary system, when the WD has accreted enough matter from the companion so that its mass reaches the Chandrasekhar limit, the process of electron capture may induce gravitational collapse. The detonation waves burn nuclear matter into strange quark matter which spread out from the inner core of the WD (Lugones, Benvenuto & Vucetich 1994). A boundary of strange quark matter and nuclear matter will be found at the radius where the detonation waves stop when nuclear matter density drops below a critical value. A similar process was also discussed and calculated by Chen, Yu & Xu (2007). The size of the inner collapsed core may depend on the chemical composition and accretion history of the WD (Nomoto & Kondo 1991). Consequently, quark stars with different masses could be formed.

Both rigidly and differentially rotating WDs are taken into account. As a first step, we assume that both the collapsed WD and the newborn quark star have rigidly rotating configurations for simplicity. The WDs, progenitors of these quark stars, could have a uniformly rotating configuration due to the effects of crystallization, as well as an increase of central density may lead to catastrophic evolution (supernova) (Koester 1974). With these assumptions, a model of sub-millisecond pulsars’ formation is given below. The initial spin period of AIC-produced quark stars can be estimated as follows. We assume that the mass ($M_\star$) of the nascent quark star ranges from $10^{-3} M_\odot$ to $1 M_\odot$, and the white dwarf rotates rigidly at the Kepler period ($P_K$) just before collapsing. The quark star’s rest mass ($M_\star$) is approximately equal to the mass ($m_{\text{core}}$) of the inner collapsed core of the white dwarf. If the angular momentum is conserved during AIC, the newborn quark star can rotate at a much shorter period, $P_\nu$, then

$$I_{\nu} \frac{2\pi}{P_\nu} = I_q \frac{2\pi}{P_q},$$

This is to say,

$$P_\nu = \frac{I_q}{I_{\text{core}}} P_K,$$

where $I_q$ is the quark star’s moment of inertia, and $I_{\text{core}}$ is the moment of inertia of WD’s inner collapsed core, which can be well approximated by

![Image of graph showing the relation of mass and radius of WDs. The red line is theoretical line. The blue triangles and circles are observed WDs’ data which were taken from Table 1 of Należyty & Madej (2004) and Table 3 & 5 of Provençal et al. (1998), respectively. Among these data, the WD RE J0317-853 is the most massive WD, whose mass and radius are 1.34$M_\odot$ and 2400 km respectively. The square is the point of a WD with the Chandrasekhar mass limit.](image-url)
A newborn quark star could certainly rotate differentially, and may be relaxed to become a rigidly rotating configuration finally. However, the timescale of the relaxation depends on the viscosity and the state of cold quark matter \cite{Xu2009}. Nevertheless, the newborn quark star’s relaxation (from differentially rotating configuration to rigidly rotating configuration) may be due to fast solidification after birth. A calculation shows that the solidification timescale is only $10^5 - 10^6$ s \cite{XuLiang2009}. Therefore, the relaxation timescale could be much shorter than the lifetime of pulsars within sub-millisecond periods (See the following section 3.3).

The WD RE J0317-853 has the highest observed mass (1.34 $M_\odot$) close to the Chandrasekhar limit) with radius of 2400 km \cite{NalewtyMajed2004}. If a WD like RE J0317-853 could be in a binary and accreted enough materials to the Chandrasekhar limit, then it may collapse. Therefore, under this assumption, we also calculated the initial spin periods $P_0$ and $P_{\text{rot}}$ of a nascent quark star. The calculated results are listed in Table 1. It is found that, even if a WD has a larger radius such as 2400 km, it can also collapse to a sub-millisecond quark star for either rigidly or differentially rotating WD models. In the differentially rotating WD model, it tends to give a rigidly rotating configurations in the limit of large values of $a$, $P_{\text{rot}}$ increases as the parameter $a$ increases. The conclusions from the rigid rotation model are valid even if differential rotation is included.

Can a quark star survive even if it rotates at such a high frequency ($\sim 10^4$ Hz)? Will it be torn apart by the centrifugal force? There are quite distinguishing characteristics between neutron stars and quark stars. A low mass quark star is possible to spin at a super-Keplerian frequency because it is self-bound by strong interaction. On one hand, as noted by Qiu & Xu (2006), astrophysical quark matter splitting could be color-charged if color confinement cannot be held exactly because of causality. On the other hand, however, rapidly spinning quark matter could hardly split if color confinement is held exactly. In addition, the recently discovered nature of strongly coupled quark gluon plasma (sQGP) as realized at Relativistic Heavy Ion Collider (RHIC) experiment \cite[e.g., Shuryak 2006] may also prevent a super-Keplerian quark star to split.

The short spin period above is not surprising, and could be verified for a simplified special case, if both quark star’s density (=4 in the bag model) and white dwarf’s density (= $\rho_0$) are uniform. Using Eq. (2) and the mass-radius relation, we can find the initial period of the quark star to be $P_0 = (\rho_0/\beta)^{2/3}P_{\text{WD}} \sim 4 \times 10^{-3}(p_{11}/\beta_{14})^{2/3}P_{\text{WD}}$ (with $P_{\text{WD}}$ the spin period of white dwarf, $p_{11} = \rho_{11}/10^{11}$ g cm$^{-3}$, $\beta_{14} = \beta/10^{14}$ g cm$^{-3}$), which depends only on the densities of WD and quark star.

If the WD has not been spun up fully to the Kepler period, i.e., the WD rotates at a sub-Keplerian period (e.g., several times of $P_k$) before AIC, can the initial period of a newborn quark star formed from such a WD be of sub-millisecond? We investigated the case of a massive WD (1.4 $M_\odot$, 410 km) rotating at a period $P_{\text{WD}} = 5P_k \sim 600$ ms. The initial spin period of quark stars with different mass are as follows: $P_0 \sim 0.11$ ms for a quark star with mass of 0.001 $M_\odot$; $\sim 0.24$ ms for 0.01 $M_\odot$; 0.35 ms for 0.1 $M_\odot$ and $\sim 0.36$ ms for 1 $M_\odot$. The spin-down feature of such a newborn quark star depends on its gravitational wave radiation and magnetodipole radiation (see details in §3).
If we take normal parameters, such as the surface magnetic field and gravitational wave radiation first, then the rate of energy is coming from the rotational energy loss. Here we neglect the pair production and sparking to take place in the inner gap?

3.1 The spin-down power of sub-millisecond pulsars

Comparing the rotational energy loss rate ($\dot{E}_{\text{rot}}$) of quark stars with normal neutron stars’ ($\dot{E}_{\text{rot,NS}}$), one can have

$$\dot{E}_{\text{rot,NS}} = \frac{8\pi^4 R_e^6 B^2 P^{-4}}{3c^3}. \quad (7)$$

Comparing the rotational energy loss rate ($\dot{E}_{\text{rot,q}}$) of quark stars with normal neutron stars’ ($\dot{E}_{\text{rot,NS}}$), one can have

$$\dot{E}_{\text{rot,q}}/\dot{E}_{\text{rot,NS}} = \frac{P_{NS}^6 B_{NS}^2 P_{q}^{-4}}{P_{q}^6 B_{q}^2 P_{NS}^{-4}}. \quad (8)$$

If we take normal parameters, such as the surface magnetic field of polar cap $B_{q} = 10^8$ G, $B_{NS} = 10^{12}$ G, the rotational period $P_{q} = 0.1$ ms and $P_{NS} = 1$ s, the result is $\dot{E}_{\text{rot,q}}/\dot{E}_{\text{rot,NS}} = 10^2$ even for a quark star with a mass of 0.001$M_{\odot}$. This means that the quark stars have enough rotational energy to radiate, hundred times than normal pulsars, even if the mass is so low.

3.2 Particle acceleration for sub-millisecond pulsars

In most radio emission models of pulsars, such as RS model (Ruderman & Sutherland 1975, hereafter RS75), inverse Compton scattering (ICS) model (Qiao & Lin 1998), the multi-ring sparking model (Gil & Sendyk 2000), the annular gap model (Qiao et al. 2004) and so on, the potential drop in the inner gap must be high enough so that the pair production condition can be satisfied.

In the inner vacuum gap model, there is strong electric field parallel to the magnetic field lines due to the homopolar generator effect. The particles produced through $\gamma - B$ process in the gap can be accelerated to ultra-relativistic energy (i.e., the lorentz factor can be $10^9$ for normal pulsars). The potential across the gap is (RS75)

$$\Delta V = \frac{\Omega B}{c} h^2, \quad (9)$$

where $\Omega$ is the angular frequency of the pulsar; $h$ is the gap height; $B$ and $c$ represent the magnetic field at the surface of the neutron star and the speed of light, respectively. As $h$ increases and approaches $r_p$, the potential drop along a field line traversing the gap can not be expressed by Eq. (9) above. In this case the potential can reach a maximum value

$$\Delta V_{\text{max}} = \frac{\Omega B}{2c} r_p^2, \quad (10)$$

where $r_p$ is the radius of the polar cap.

Let us make an estimate about the quark star’s potential drop $\Delta V_q$ in the polar gap region

$$\Delta V_q = \frac{\Omega B_{q}}{2c} r_{p,q}^2, \quad (11)$$

where $\Omega = 2\pi/P_{q}$, $r_{p,q} = R_q(2\pi R_q/c P_{q})^{1/2}$. For normal neutron stars, $\Delta V$ can be obtained by just changing the subscript $q$ to $NS$. Thus

$$\frac{\Delta V_{NS}}{\Delta V_q} = \frac{B_{q}}{B_{NS}} R_{q}^3 P_{q}^{-2}. \quad (12)$$

As one can take $R_{q} = 1$ km for a quark star with the mass of $0.001M_{\odot}$, $B_{q} = 10^8$ G, $B_{NS} = 10^{12}$ G, $R_{NS} = 10$ km, $P_{q} = 0.1$ ms and $P_{NS} = 1$ s, we find that $\Delta V_{NS}/\Delta V_q = 10$. This means that the quark stars can have enough potential drops in the polar cap regions.

In the inner gap model, $\gamma - B$ process plays a very important role, two conditions should be satisfied at the same time for pair production: (1) to produce high energy $\gamma$-ray photons, a strong enough potential drop should be reached; (2) for pair production, the energy component of $\gamma$-ray photons perpendicular to the magnetic field must satisfy $E_{\gamma,\perp} \geq 2m_e c^2$ (Zhang & Qiao 1998).

Particles produced in the gap can be accelerated by the electric...
field in the gap and the Lorentz factor of the particles can be written as
\[ \gamma = \frac{e \Delta V}{m_e c^2}, \quad (13) \]
where \( \gamma \) is the Lorentz factor of the particles accelerated by the potential \( \Delta V \), \( m_e \) the mass of an electron or positron, \( e \) the charge of an electron.

In \( \gamma - B \) process, the conditions for pair production are that the mean free path of \( \gamma \)-ray photon in strong magnetic field is equal to the gap heights, \( l \approx h \). The mean free path of \( \gamma \)-ray photon is given by (Erber 1966)
\[ l = \frac{4 A}{e^2 / m_e c^2 B c} \exp\left(\frac{4}{3 \chi}\right), \quad (14) \]
where \( B_c = 4.144 \times 10^{13} \text{ G} \) is the critical magnetic field, \( h \) the Planck’s constant,
\[ \chi = \frac{2 m_e c^2 \sin \theta}{2 m_e c^2 B_c} = \frac{E_\gamma}{2 m_e c^2 B_c}, \quad (15) \]
and \( B_c \) is the magnetic field perpendicular to the moving direction of \( \gamma \) photons, which can be expressed as (RS75)
\[ B_c \approx \frac{h \rho}{\rho} \approx \frac{\Delta B}{\rho}. \quad (16) \]
Here \( l \approx h \) is the condition for sparks (pair production) to take place. \( \rho \) is curvature radius of the magnetic field lines. For dipole magnetic configuration, it is (Zhang et al. 1997a)
\[ \rho \approx \frac{4}{3} (\lambda R_c \Omega)^{1/2}. \quad (17) \]
where \( \lambda \) is a parameter to show the field lines, \( \lambda = 1 \) corresponding to the last opening field line. Gamma-ray energy from the curvature radiation process can be written as
\[ E_{\gamma, cr} = \frac{h}{2} \left(\frac{3 \gamma^3 c}{2 \rho}\right). \quad (18) \]

We estimated the gap heights based on Zhang, Qiao & Han (1997b), i.e.
\[ h_{\text{CR}} \approx 10^6 \rho^{3/7} B_8^{-4/7} \rho_6^{2/7} \text{ cm}. \quad (19) \]

When the relevant parameters used are \( B = 10^8 \text{ G} \), \( P = P_0 \) and assuming a dipole magnetic configuration, for any mass quark stars, one can estimate the gap height from curvature radiation (CR)
\[ h_{\text{CR}} \approx 10^4 \text{ cm} = 100 \text{ m}. \]
This means that even if without multipolar magnetic field effect, the quark star can still work well for the CR pair production.

There are three gap modes for pair production, i.e. resonant ICS mode, thermal-peak ICS mode and CR mode (Zhang et al. 1997a). Each mode has relevant gap parameters including gap potential drop \( \Delta V \) and the mean free path \( l \) of \( \gamma - B \) process. For normal neutron stars, one needs the assumption of a multipolar magnetic field, \( \rho = 10^6 \text{ cm} \), as RS75; but for 0.1 ms low mass quark stars, the dipole curvature radius is about \( 10^6 \text{ cm} \). We estimated gap heights and other parameters based on the work of Zhang, Qiao & Han (1997b), as shown in Table 2.

One can see from Table 2 that when the high energy gamma-ray photons come from resonant photon production, the height of the gap is larger. For the thermal-peak ICS mode, it is one order of magnitude lower than the CR mode, and two order of magnitude lower than resonant ICS mode. This means that in most cases, the thermal-peak ICS induced pair production is dominated in the gap.

The newborn sub-millisecond quark stars have enough spin-down luminosities and gap potential drops (see Table 2), so that they may emit radio or \( \gamma \)-ray photons with sufficient luminosities, which can be detected by new facilities, e.g., FAST and Fermi (formerly GLAST).

### 3.3 Lifetimes of the sub-millisecond pulsars in the phase of a short spin period

Sub-millisecond pulsars may be very rare, or the timescale for such a pulsar to stay in the short period phase (< 1 ms) may not be long enough due to magnetodipole (EM) radiation and gravitational wave (hereafter GW) radiation (Andersson 2003). The lowest order GW radiation is bar-mode, which is due to non-axisymmetric quadrupole moment. Here we consider GW radiation on the bar mode which exerts a larger braking torque with braking index \( n \approx 5 \) than magnetodipole radiation (\( n = 3 \)). The rotation frequency drops quickly due to GW radiation and EM radiation:

\[ -\dot{I} \Omega_i^2 \approx \frac{32 G I_i^2 c^2 \Omega_i^6}{5 c^5} + \frac{B_i^2 R_i^6 \Omega_i^4}{6 c^3}, \quad (20) \]
where \( c \) is the speed of light, \( \varepsilon_i = \Delta a / a \) is the gravitational ellipticity (equatorial ellipticity), \( \Delta a \) is the difference in equatorial radii and \( a \) is the mean equatorial radii.

To simplify Eq. (20), we introduce the notation \( A = 32 G I_i^2 c^2 / (5 c^5) \) and \( D = B_i^2 R_i^6 / (6 I c^3) \), and integrate the equation in the angular velocity’s domain \( [\Omega_i = 2 \pi / P, \Omega_0 = 2 \pi / 0.001] \), then
\[ \tau = \frac{1}{2 D} \left(\frac{1}{\Omega_i^2} - \frac{1}{\Omega_0^2}\right) - \frac{A}{2D^2} \ln \frac{1}{\Omega_i^2} + \frac{D}{\Omega_i^2}. \quad (21) \]

An accurate ellipticity of quark stars is unfortunately uncertain. Nevertheless, let’s estimate the \( \varepsilon \) to calculate the timescales in the sub-millisecond period phase for GW and EM radiations. Cutler & Thorne (2002) suggested \( \varepsilon_{\text{cr}} = (I - I_0) / I_0 \leq 10^{-6} \). Regimbau & de Freitas Pacheco (2003) found from their simulations that \( \varepsilon_{\text{cr}} = 10^{-6} \) is the critical value to have an at least one detection with interferometers of the first generation (LIGO or VIRGO). It was shown that direct upper limit was \( \varepsilon_{\text{cr}} \approx 1.8 \times 10^{-4} \) on GW emission from the Crab pulsar using data from the first 9 months of the fifth science run of LIGO (Abbott et al. 2008).

### Table 2. Gap parameters estimated for sub-millisecond quark stars. \( \dot{E}_{\text{rot}} \) is the spin-down luminosity; \( h_{\text{CR}} \) is the curvature radiation (CR) gap height; \( \Delta V_{\text{CR}} \) is the potential drop of CR gap; \( h_{\text{res}} \) is the height of resonant ICS gap; \( \Delta V_{\text{res}} \) is the potential drop of resonant ICS gap; \( h_{\text{th}} \) is the thermal ICS gap height; \( \Delta V_{\text{th}} \) is the potential drop of thermal ICS gap.

| \( M(M_\odot) \) | \( \dot{E}_{\text{rot}}(\text{erg s}^{-1}) \) | \( h_{\text{CR}}(\text{cm}) \) | \( \Delta V_{\text{CR}}(V) \) | \( h_{\text{res}}(\text{cm}) \) | \( \Delta V_{\text{res}}(V) \) | \( h_{\text{th}}(\text{cm}) \) | \( \Delta V_{\text{th}}(V) \) |
|----------------|-----------------|-------------------|----------------|----------------|----------------|----------------|----------------|
| 0.001          | 4.99 × 10^{36}  | 1.16 × 10^{14}   | 2.84 × 10^{10} | 3.13 × 10^{5}  | 2.05 × 10^{13} | 1.18 × 10^{3}  | 2.93 × 10^{8}  |
| 0.01           | 3.75 × 10^{38}  | 1.34 × 10^{14}   | 3.76 × 10^{10} | 3.64 × 10^{5}  | 2.78 × 10^{13} | 1.31 × 10^{3}  | 3.62 × 10^{8}  |
| 0.1            | 1.02 × 10^{40}  | 1.72 × 10^{14}   | 6.19 × 10^{10} | 4.61 × 10^{5}  | 4.46 × 10^{13} | 1.62 × 10^{3}  | 5.47 × 10^{8}  |
| 1              | 5.00 × 10^{40}  | 2.65 × 10^{14}   | 1.47 × 10^{11} | 6.74 × 10^{5}  | 9.52 × 10^{13} | 2.36 × 10^{3}  | 1.16 × 10^{9}  |
In addition, Owen (2005) showed that the maximum ellipticity of solid quark stars was \( \varepsilon_{e, \text{max}} = 6 \times 10^{-4} \). From the on-line catalogue hosted by the ATNF [1], the seventh fastest rotating millisecond pulsar is PSR J0034-0534, which has very low period derivative \( P = 4.96 \times 10^{-21} \text{ s}^{-1} \). We thus use such a low \( P \) and Eq. (20) to constrain the lower limit of the sub-millisecond pulsar's ellipticity, which is \( \varepsilon_{e, \text{min}} \approx 10^{-6} \) if the stellar mass is one order of one Solar mass. For quark stars, in order to facilitate to compare with the neutron stars' lifetime (\( \tau \)) in the phase of sub-millisecond period, we use mean equatorial ellipticities \( \varepsilon_e = 10^{-6} \) and \( \varepsilon_e = 10^{-9} \) to calculate \( \tau \) for both quark stars and neutron stars through Eq. (21).

In the case of \( \varepsilon_e = 10^{-6} \), if we make the hypothesis that the rotational energy is lost because of EM radiation, then one can easily derive \( \tau_{\text{EM}} = 1/(2D)(1/\Omega_0^2 - 1/\Omega_s^2) \approx 5.9 \times 10^9 \text{ yr} \) for a typical compact star with \( B_0 \approx 10^8 \text{ G} \) and \( M = M_\odot \). While, if we suppose that the rotational energy is lost due to GW radiation, then \( \tau_{\text{GW}} = 1/(A)(1/\Omega_0^2 - 1/\Omega_s^2) \approx 10^9 \text{ yr} \) for a typical compact star. The energy loss rate of GW & EM radiation in the phase of sub-millisecond period, for a typical compact star which has a low magnetic field \( 10^{-10} \text{ G} \) either from AIC (Owen 2005) or spurn up, are \( E_{\text{GW}} = 32G^2 \varepsilon_e^2 \Omega_0^6 / (5c^3) = 7.0 \times 10^{41} P_{\text{ms}}^6 \text{ erg s}^{-1} \) and \( E_{\text{EM}} = B_0^2 R_0^6 \Omega_0^2 / (6c^3) = 9.6 \times 10^{44} P_{\text{ms}}^{-4} B_0^2 R_0^6 \text{ erg s}^{-1} \), respectively. Even if a quark star with \( B_0 \) formed from WD's AIC has a high magnetic field such as \( 10^{12} \text{ G} \), the lifetime \( \tau \) in the phase of sub-millisecond is 37 years, in comparison with \( \tau = 336 \text{ years} \) for \( B_0 = 10^8 \text{ G} \). Then the EM energy loss is similar to the GW energy loss and becomes very important for \( B_0 = 10^9 \text{ G} \). If \( B_0 \) ranges from \( 10^8 \text{ G} \) to \( 10^{11} \text{ G} \), one always has \( E_{\text{GW}} \gg E_{\text{EM}} \) for compact stars with short spin periods (\( < 1 \text{ ms} \)). Therefore, in the case of larger ellipticity (e.g., \( \varepsilon_e = 10^{-6} \)), it is clear that GW radiation dominates the energy loss in the phase of sub-millisecond for either recycled or AIC's compact stars with low magnetic field. The corresponding lifetime is shorter for a compact star with higher mass (\( \sim M_\odot \)) but longer for a star with lower mass (\( \sim 0.001 M_\odot \)). However, if the ellipticity is lower, such as \( \varepsilon_e = 10^{-9} \), EM radiation dominates the rotational energy loss. The corresponding lifetime of a quark star (even with a high mass \( \sim M_\odot \)) is long enough for us to detect. Figure 2 shows the relation of lifetime (in the phase of \( < 1 \text{ ms} \)) and gravitational ellipticity \( \varepsilon_e \) for quark stars.

In the super-Keplerian case, the timescales in the phase of \( < 0.5 \text{ ms} \) for quark stars with different mass are also calculated, and listed in Table 1 (See \( \tau_2 \)). For a high-mass quark star with larger ellipticity, the timescale is too small for real detection; but the timescale is much longer than \( 10^6 \text{ yr} \) for a low mass quark star. Therefore, low mass quark stars with \( \varepsilon_e \approx 10^{-6} \) could have much longer lifetime in the phase of \( < 0.5 \text{ ms} \). However, for lower ellipticity, their lifetimes in the phase of \( < 0.5 \text{ ms} \) are long enough for quark stars with \( \sim 1M_\odot \). Once a pulsar with spin period \( < 0.5 \text{ ms} \) is ever found, low mass quark stars will be physically identified.

### 3.4 Spin-down rate \( \dot{P} \) for newborn quark stars and neutron stars

We also use Eq. (20) to calculate the period derivative (\( \dot{P} \)) for the nascent sub-millisecond quark stars and neutron stars. Figure 3 is a \( \dot{P} - P \) diagram that shows the spin-down evolution for quark stars with different masses. It is found that, for different ellipticity there are different properties. For high ellipticity such as \( \varepsilon_e = 10^{-5} \), the \( \dot{P} \) can be changing about ten orders of magnitude for different periods (see the steep slopes of dot-dash lines and dashed lines). The rotational energy losses in this case are dominated by the gravitational wave (GW) radiation. For low ellipticity such as \( \varepsilon_e = 10^{-9} \), in most cases, the rotational energy losses are dominated by magnetic dipole (EM) radiation and the \( \dot{P} \) changes with periods relatively slow (solid lines).

As a comparison, we also calculate the period derivative (\( \dot{P} \)) of a neutron star (with an initial period \( 0.5 \text{ ms} \), mass of \( 1.4M_\odot \) and radius of \( 10^6 \text{ km} \)). The results are shown in Figure 4. One can see that the \( \dot{P} \) is changing with periods as large as ten orders of magnitude. It is found that the neutron star spins down much more quickly than low mass quark stars, because of neutron star's high mass (\( \sim M_\odot \)) for higher efficiency of GW radiation.

### 4 SUB-MILLISECOND PULSARS FORMED THROUGH ACCESSION IN BINARY SYSTEMS

There is also an important mechanism of "spin-up in binaries" for sub-millisecond pulsars' formation, which is widely discussed in the literatures. We regard this as "sub-Keplerian case" and make a comparison with our proposed AIC model "super-Keplerian case". In this section, we will find the minimal periods for both neutron stars and bar quark stars spin up by accretion in binary systems. We assume that the initial rotational periods of newborn pulsars could have an "equilibrium period" with two characteristic parameters: magnetospheric radius and corotation radius. The magnetospheric radius (\( r_m \)) is the radius where the ram pressure of particles is equal to the local magnetic pressure, i.e.

\[
r_m = \phi R_A = \phi \left( \frac{4 \mu_{\text{ms}}^2 M^{3/2}}{M \sqrt{2 \mu}} \right)^{2/7} = \phi \left( \frac{B_0^2 R_0^6}{M \sqrt{2 \mu G}} \right)^{2/7}
\]
where $\Omega_{\text{Ed}}$ is the spin frequency in units of $\text{Hz}$.

The corotation radius is

$$r_c = 1.5 \times 10^8 M_1^{1/3} P^{2/3} \text{ cm},$$

(23)

The corotation radius is $r_c = 1.5 \times 10^8 M_1^{1/3} P^{2/3} \text{ cm}$. The spin periods of compact stars cannot exceed the Kepler limit via accretion. When the compact star was spun up to the Kepler limit by the accreted matter falling onto the compact star’s surface, for neutron stars, as the equatorial radius expanded, one can use the simple empirical relation for the maximum spin frequency

$$\Omega_{\text{max}} = 7700 M_1^{1/2} R_6^{-3/2} \text{ s}^{-1},$$

(25)

(Haensel & Zdunik 1989; Lattimer & Prakash 2004), which leads to

$$P_{\text{eq}} \geq 0.816 M_1^{-1} R_6^{3/2} \text{ ms},$$

(26)

where $M$ and $R$ refer to the neutron star’s mass and radius of non-rotating configurations.

For quark stars, Gourgoulhon et al. (1999) used a highly precise numerical code for the 2-D calculations, and found that the $\Omega_{\text{max}}$ could be expressed as $\Omega_{\text{max}} = 9920 \sqrt{\beta_{60}} \text{ rad s}^{-1}$, where $\beta_{60} = \beta / (60 \text{ MeV fm}^{-3})$, which implied that $P_{\text{eq}} \geq 0.633 \beta_{14}^{-1/2} \text{ ms}$. These are the so-called “sub-Keplerian condition”.

The accretion torque, $N$, exerted on the compact star contains two contributions: one is positive material torque which is carried by the materials falling onto the star’s surface, the other is magnetic torque which can be positive or negative, depending on the fastness parameter $\omega_s = \Omega_s / \sqrt{K} = \left( r_m / r_c \right)^{3/2}$. It is suggested that all the torques may cancel one another if the fastness is $\omega_s = \left( r_m / r_c \right)^{3/2} \approx 0.884$ (Dai & Li 2006). This implies a magnetospheric radius of $r_m = 0.92 r_c \approx r_c$. One can obtain an equilibrium period of $P_{\text{eq}}$ when setting $r_m = r_c$.

$$P_{\text{eq}} = \left\{ \begin{array}{l}
3.52 B_8^{6/7} \beta_{14}^{5/7} R_6^{8/7} M_1^{-3/7} \text{ ms}, \\
3170 B_8^{6/7} M_1^{-5/7} R_6^{18/7} M_1^{-3/7} \text{ ms}.
\end{array} \right\} \ (a)$$

(27)

For quark stars, the equilibrium period is independent of mass and radius, and only dependent on bag constant, surface magnetic field, and accretion rate. Take $B_8$ in the range $[10^8, 10^{12}]$, we may use Eq. (27a) to calculate the minimal equilibrium period of different EOSs (equation of state) for quark stars. For $\beta = 60 \text{ MeV fm}^{-3}$, when $\alpha = 0.71$, $B_8 = 1.1$, one can get the minimal period 0.613 ms. For $\beta = 110 \text{ MeV fm}^{-3}$, when $\alpha = 0.85$, $B_8 = 1.4$, the minimal period is 0.453 ms. (See results in Table 3.)

For neutron stars, data for mass and radius in different EOSs were taken from Lattimer & Prakash (2004, their Figure 2), $B_0$ is in the range $[10^8, 10^{12}]$. The minimal equilibrium period is calculated using the Eq. (27b). (See results in Table 4.)

In the sub-Keplerian case, the timescales in the phase of sub-millisecond for quark stars of different mass and neutron stars of different EOSs are listed in Table 3 and Table 4, respectively. For typical quark stars as well as neutron stars with high $\varepsilon_e$, their lifetimes in the phase of sub-millisecond period are about $10^5$ years, which result in a too low detection possibility. However, for low $\varepsilon_e$, the lifetime of a sub-millisecond pulsar (even with a high mass) is long enough.

Table 3. The minimal equilibrium period for quark stars and lifetimes due to GW and EM radiation in the phase of sub-millisecond period for quark stars with different masses ($10^{-3} M_\odot, 0.1 M_\odot, 1.4 M_\odot$) in the sub-Keplerian case. $\tau_1$, $\tau_2$, $\tau_3$ are calculated by using $\varepsilon_e = 10^{-6}$, while $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{\tau}_3$ are calculated by using $\varepsilon_e = 10^{-9}$. The bag constant $\beta$ is in unit of MeV fm$^{-3}$, the accretion ratio $\alpha$ is in unit of the Eddington accretion rate $\dot{M}_{\text{Ed}}$. 

| $\beta$ | $\alpha$ | $B_0(10^9 \text{G})$ | $P_{\text{eqmin}}(\text{ms})$ | $\tau_1(\text{yr})$ | $\bar{\tau}_1(\text{yr})$ | $\tau_2(\text{yr})$ | $\bar{\tau}_2(\text{yr})$ | $\tau_3(\text{yr})$ | $\bar{\tau}_3(\text{yr})$ |
|------|------|-----------------|------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 60   | 0.71 | 1.1             | 0.613            | 2.9 $\times$ 10^7 | 1.3 $\times$ 10^4 | 1.7 $\times$ 10^7 | 1.5 $\times$ 10^8 |
| 110  | 0.85 | 1.4             | 0.453            | 5.1 $\times$ 10^7 | 2.3 $\times$ 10^4 | 5.6 $\times$ 10^9 | 2.3 $\times$ 10^2 | 2.6 $\times$ 10^8 |
Figure 3. Spin-down evolution of quark stars due to GW and EM radiations (period derivative versus spin period), with masses of $0.1 M_\odot$, $0.01 M_\odot$, $0.001 M_\odot$. We choose ellipticity to be $10^{-5}$ (dot-dash lines), $10^{-7}$ (dashed lines), $10^{-9}$ (solid lines) in the calculation. It is evident that GW radiation dominates for quark stars with higher $\varepsilon_e$, while EM radiation dominates for lower $\varepsilon_e$.

5 CONCLUSIONS AND DISCUSSIONS

If a sub-millisecond pulsar is ever found, we have shown that it could be a quark star based upon plausible scenarios for its origin, the energy available for radiation and its lifetime. A new possible way to form sub-millisecond pulsars (quark stars) via AIC of white dwarfs has been discussed in this paper. In the super-Keplerian case, we derived the initial period $P_q$ via angular momentum conservation with consideration of the special and general relativistic effects, and calculated the lifetime and gap parameters of a new-born quark star. Quark stars with different masses could have the minimal rotational period around 0.1 ms. In most cases, quark stars would be bare (Xu 2002), therefore, a vacuum gap would be formed in the polar cap region. Based on our rough estimations without considering the effect of frame dragging (Harding &Muslimov 1998), we found that the basic parameters (including rotational energy loss) in the gap are suitable for pair (electrons and positrons) production and sparking. They can be detected as sub-millisecond radio pulsars.

We also used an approximate formula to calculate nascent quark star’s moment of inertia, but there are no accurate solutions to fast rotating compact stars’ configuration until nowadays. It should be investigated precisely in the future. In the calculation of WD’s mass and radius, we just considered the non-rotating configuration. But it does not change the conclusions of this paper. If the central density $\rho_c$ of the WD is lower than $10^{11}$ g cm$^{-3}$ before collapsing, the resulting WD has a larger radius and moment of inertia, consequently, the newborn quark star could have a smaller spin period ($<1$ ms).

Both the special and general relativistic effects are weak for a low mass (e.g. Jupiter-like) quark star with a small radius. The rotational energy is lost via GW and EM radiation. The GW radiation dominates the rotational energy loss in the phase of sub-millisecond period, if magnetic field of stars is not so large. Such quark stars therefore have long lifetimes (several million years if mass $\sim 10^{-3} M_\odot$) to maintain their spin periods of sub-millisecond. We have considered the bar-mode of GW radiation in this paper, while other GW mode (e.g. r-mode) may be important but not yet considered here (Xu 2006b). The subsequent relaxation timescale of a newborn quark star to a rigidly rotating configuration could be negligible since a quark star may be solidified soon after birth.

An important constraint for sub-millisecond pulsar’s detection is its lifetime in the phase of $< 1$ ms due to GW and EM radiation. A possible method is proposed to constrain the lower limit of the pulsars’ equatorial ellipticity, i.e., $\varepsilon_{e,\text{min}} \sim 10^{-9}$, by evaluating millisecond pulsars’ period derivative via Eq. (20). For larger ellipticity, e.g., $\varepsilon_e = 10^{-6}$, it is clear that GW radiation dominates the energy loss in the phase of short period for either recycled or AIC’s compact stars. The corresponding lifetime is shorter for a compact star with higher mass ($\sim M_\odot$), but longer for a star with lower mass ($\sim 0.001 M_\odot$). However, if the ellipticity is lower, e.g., $\varepsilon_e = 10^{-9}$, EM radiation dominates the rotational energy loss. The corresponding lifetime of a quark star (even with a high mass
Formation of sub-millisecond pulsars and possibility of detection

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REFERENCES

Abbott, B., et al. 2008, ApJ, 683, L45
Alcock, C., Farhi, E., & Olinto, A. 1986, ApJ, 310, 261
Andersson, N. 2003, Classical and Quantum Gravity, 20, 105
Arras, P. 2005, ASP Conference Series, Vol. 328
Basko, M. M., & Sunyaev, R. A. 1976, 175, 395
Becker, W., & Trumper, J. 1997, A&A, 326, 682
Bell, J. F.,ulkarni, S. R., Bailes, M., Leitch, E. M., & Lyne, A. G. 1995, ApJ, 452, L121
Blaschke, D., Bombaci, I., Grigorian, H. & Poghosyan, G. 2002, New Astronomy, 107, 112
Bombaci, I., & D’Amico, N. 2000, ApJ, 530, L69
Bravo, E., & García-Senz, D. 1999, MNRAS, 307, 984
Bulik, T., Gondek-Rosińska, D., & Kluźniak, W. 1999, A&A, 344, L71
Burderi, L., & D’Amico, N. 1997, ApJ, 490, 343
Burderi, L., & King, A. R. 1998, ApJ, 505, L135
Burderi, L., Possenti, A., Colpi, M., di Salvo, T., & D’Amico, N. 1999, ApJ, 519, 285
Burderi, L., et al. 2001, ApJ, 560, L71
Chen, A., Yu, T. H., & Xu, R. X. 2007, ApJ, 668, L55
Cook, G. B., Shapiro, S. L., & Teukolsky, S. A. 1994, ApJ, 424, 823
Cutler, C., & Thorne, K. 2002, arXiv: gr-qc/0204090
Dai, H.-L., & Li, X.-D. 2006, A&A, 451, 581
D’Amico, N., & Burderi, L. 1999, Pulsar Timing, General Relativity and the Internal Structure of Neutron Stars, 129
Dessart, L., Burrows, A., Ott, C. D., Livne, E., Yoon, S.-C., & Langer, N. 2006, ApJ, 644, 1063
Edwards, R. T., van Straten, W., & Bailes, M. 2001, ApJ, 560, 365
Erber, T. 1966, Reviews of Modern Physics, 38, 626
Friedman, J. A., & Ottino, A. V. 1989, Nature, 341, 633
Friedman, J. L., Parker, L., & Ipser, J. R. 1984, Nature, 312, 255
Fryer, C., Benz, W., Herant, M., & Colgate, S. A. 1999, ApJ, 516, 892
Gil, J. A., & Sendyk, M. 2000, ApJ, 541, 351
Glendenning, N. K., Kettner, C., & Weber, F. 1995, ApJ, 450, 253
Glendenning N. K., Compact Stars (Springer-Verlag, New York, 2000), 2nd ed.
Gondek-Rosińska, D., Stergioulas, N., Bulik, T., Kluźniak, W., & Gourgoulhon, E. 2001, A&A, 380, 190
Gourgoulhon, E., Haensel, P., Livine, R., Paluch, E., Bonazzola, S., & Mark, J.-A. 1999, A&A, 349, 851
Haensel, P., & Zdunik, J. L. 1989, Nature, 340, 617
Han, J. L., Manchester, R. N., Lyne, A. G., & Qiao, G. J. 2004, Young Neutron Stars and Their Environments, 218, 135
Harding, A. K., & Muslimov, A. G. 1998, ApJ, 508, 328
Huang, Z.-K., & Wu, X.-J. 2003, Chinese Journal of Astronomy and Astrophysics, 3, 166
Kaaret, P., et al. 2007, ApJ, 657, L97
Kapoor, R. C., & Shukre, A. S. 2001, A&A, 375, 405
Keane, E. F., & Kramer, M. 2008, MNRAS, 391, 2009
Koester, D. 1974, A&A, 375, 77
Lattimer, J. M., & Prakash, M. 2001, ApJ, 550, 426
Lattimer, J. M., & Prakash, M. 2004, Science, 304, 536
Lewin, W. H. G., & van den Heuvel, E. J. P. 1984, Science, 225, 1143
Lugones, G., Benvenuto, O. G., & Vucetich, H. 1994, Phys. Rev. D, 50, 6100
Madsen J., 1999, Lecture Notes in Physics, 516, 162 (astro-ph/9809032)
Manchester, R. N., Hobbs, G. B., Teoh, A., & Hobbs, M. 2005, AJ, 129, 1993
Mueller, E., & Eriyuchi, Y. 1985, A&A, 152, 325
Nalezyty, M., & Madej, J. 2004, A&A, 420, 507
Nomoto, K., Miyaji, S., Sugimoto, D., & Yokoi, K. 1979, IAU Colloq. 53: White Dwarfs and Variable Degenerate Stars, 56
Nomoto, K., & Kondo, Y. 1991, ApJ, 367, L19
Owen, B. J. 2005, Physical Review Letters, 95, 211101
Percival, J. W., et al. 1995, ApJ, 446, 832
Possenti, A., Colpi, M., Geppert, U., Burderi, L., & D’Amico, N. 1999, ApJS, 125, 463
Provencal, J. L., Shipman, H. L., Hog, E., & Thejll, P. 1998, ApJ, 494, 759
Qiao, G. J., & Lin, W. P. 1998, A&A, 333, 172
Qiao, G. J., Lee, K. J., Wang, H. G., Xu, R. X., & Han, J. L. 2004, ApJ, 606, L49
Qiu, C. X., & Xu, R. X. 2006, Chin. Phys. Lett., 23, 3205
Regimbau, T., & de Freitas Pacheco, J. A. 2003, A&A, 401, 385
Ruderman, M. A., & Sutherland, P. G. 1975, ApJ, 196, 51
Sasseen, T. P. 1990, Ph.D. Thesis
Shu, F. H. The Physical Universe: An Introduction to Astronomy. Mill Valley, CA: University Science Books, p. 128, 1982.
Shuryak, E. V. 2006, in: Proceedings of Continuous Advances in QCD (hep-ph/0608177).
Wang, Y.-M. 1997, ApJ, 475, L135
Weber F. 2005, Prog. Part. Nucl. Phys., 54, 193
Xu, R. X. 2002, ApJ, 570, L65
Xu, R. X. 2005, MNRAS, 356, 359
Xu, R. X. 2006a, Adv. Space Res., 37, 2002
Xu, R. X. 2006b, Astropart. Phys., 25, 212-219
(astr-0511676)
Xu, R. X. 2007, in: Gravitation and Astrophysics: on the occasion of the 90th year of General Relativity, eds. J. M. Nester, C. M. Chen, and J. P. Hsu, World Scientific, p.159 (astro-ph/0605028)
Xu, R. X. 2008, in: Proceedings of the international conference
APPENDIX A: DIFFERENTIALLY ROTATING WD MODEL

WD could be rotating differentially. As stated by (Mueller & Eriguchi 1985), the WD’s angular velocity $Ω$ is a function of the distance from the rotation axis $\bar{r}$. The angular momentum distribution (so-called rotation law) is

$$\Omega(\bar{r}) = \Omega_c \frac{(aR_e)^2}{(aR_e)^2 + \bar{r}^2},$$  \hspace{1cm} (A1)

where $\Omega_c$ is the central angular velocity, $R_e$ is the equatorial radius, and $a$ is a free parameter. When differential rotation is taking into account, we can numerically evaluate the angular momentum of the WD’s inner collapsed core, i.e.,

$$J_{\text{core}} = \sum_i J_i = \sum_i \int_0^\pi r_i^4 \sigma \sin^3 \theta \Omega(r_i, \sin \theta) d\theta$$

$$= \sum_i \left[ m_{\text{core}} \Omega_c a^2 R_{WD}^2 \frac{r_i^2}{r_i^4} \right] \times$$

$$\left[ r_i^2 - 0.5 \sqrt{\frac{r_i^2}{a^2 R_{WD}^2} + \frac{r_i^2}{a^2 R_{WD}^2} \ln \frac{1 + \sqrt{\frac{r_i^2}{a^2 R_{WD}^2} + \frac{r_i^2}{a^2 R_{WD}^2}}}{1 - \sqrt{\frac{r_i^2}{a^2 R_{WD}^2} + \frac{r_i^2}{a^2 R_{WD}^2}}}} \right],$$

where $J_i$ is the angular momentum of each spherical shell with integral radius $r_i$. 