A UNIFORM LINNIK BASIC LEMMA AND ENTROPY BOUNDS

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Abstract. We prove a version of Linnik’s basic lemma uniformly over the base field using θ-series and geometric invariant theory in the spirit of Khayutin’s approach (Duke Math. J., 168(12), 2019). As an application, we establish entropy bounds for weak*-limits of invariant measures on homogeneous toral sets in $\text{GL}_4$ of biquadratic, cyclic, or dihedral type.

1. Introduction

This article studies homogeneous toral sets in the finite volume homogeneous space

$$[\text{GL}_n] = \text{GL}_n(\mathbb{Q}) \backslash \text{GL}_n(\mathbb{A})^1$$

for $n \geq 1$ and entropy of weak*-limits of their homogeneous measures. Here, a homogeneous toral set is a subset of the form $[Tg] = \text{GL}_n(\mathbb{Q})T(\mathbb{A})^1g$, where $T \simeq \text{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ for some number field $K$ of degree $n$ is a $\mathbb{Q}$-torus in $\text{GL}_n$ and $g \in \text{GL}_n(\mathbb{A})^1$.

1.1. Some historical context. Let us begin with some of the history and known results towards the equidistribution problem for homogeneous toral sets.

Consider first $n = 2$. The problem of equidistribution of homogeneous toral sets in $[\text{GL}_2]$ is strongly related to Linnik-type problems. Omitting a general setup, the Linnik-type problems include:

- Equidistribution of complex multiplication points on the complex modular curve $Y_0(1)$.
- Equidistribution of long periodic geodesics on the unit tangent bundle of $Y_0(1)$.

We refer to [Duk07], [MV06], and [ELMV12] for an introduction to these as well as other Linnik-type problems (such as equidistribution of integral points on spheres). Duke [Duk88] in the late 80’s established the above equidistribution problems building on a breakthrough of Iwaniec [Iwa87]. Prior to these works, Linnik [Lin68] and Skubenko [Sku62] proved the same result under an additional congruence condition, which is often called a Linnik-type condition.

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The group $\text{GL}_n(\mathbb{A})^1$ consists of the content one elements of $\text{GL}_n(\mathbb{A})$ i.e. elements $g$ with $\prod_v |\det(g_v)|_v = 1$ where $v$ runs over all places of $\mathbb{Q}$. Furthermore, $T(\mathbb{A})^1 = \text{GL}_n(\mathbb{A})^1 \cap T(\mathbb{A})$. 

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The above problems can be unified by working over the adeles and considering homogeneous toral sets in $[\text{GL}_2]$ (and more generally the similarly defined homogeneous space $[G]$ where $G$ is the group of invertible elements in a rational quaternion algebra). Indeed, equidistribution of homogeneous toral sets in $[\text{GL}_2]$ implies a resolution of the Linnik-type problems. The former states the following: for any compactly supported continuous function $f$ on $[\text{GL}_2]$ orthogonal to the character spectrum,

$$\int_{[T_g]} f \to \int_{[\text{GL}_2]} f$$

when $\text{vol}([T_g]) \to \infty$. For the purpose of this introduction, all homogeneous sets are equipped with their invariant probability measures. The character spectrum consists of functions invariant under $\text{SL}_2(\mathbb{A})$ and equidistribution for such functions can also be understood but is not guaranteed in general. The volume $\text{vol}(\cdot)$ of a homogeneous toral set will be defined later on, but can be seen as a measure of 'complexity'. Equidistribution of homogeneous toral sets as in (1.1) is due to a long list of authors depending on different ways of varying the homogeneous toral sets – see [ELMV11, §4] for a formulation and references.

In the context of this article, let us also mention the dynamical approach of Einsiedler, Lindenstrauss, Michel, and Venkatesh [ELMV12] for $[\text{GL}_2]$ (see also [Wie19]) which reinterprets Linnik's work. Whenever $[T_i g_i]$ is a sequence of homogeneous toral sets with common invariance under a split torus $A$ at some place (this is the Linnik-type condition), one shows that the weak$^*$-limit has maximal entropy for any element in $A$. As all measures of maximal entropy are invariant under $\text{SL}_2(\mathbb{A})$ (see e.g. [EL10, Thm. 7.9]), this proves equidistribution as in (1.1).

Equidistribution of homogeneous toral sets in $[\text{GL}_3]$ with common invariance has been established by Einsiedler, Lindenstrauss, Michel, and Venkatesh in [ELMV11]. Roughly speaking, the strategy they employ is to show that any ergodic component of a weak$^*$-limit of homogeneous toral measures has positive entropy. This in turn implies that each of these ergodic components is the Haar probability measure on $[\text{GL}_3]$ by deep results of Einsiedler, Katok, and Lindenstrauss [EKL06] and Einsiedler, Lindenstrauss [EL08, EL15]. The positive entropy follows from subconvexity for Dedekind $\zeta$-functions for cubic fields – see the references in [ELMV11, App. A] (of course, positive entropy for all ergodic components would also follow from maximal entropy for the weak$^*$-limit). The results of [ELMV11] are in fact more general and apply to certain homogeneous toral sets in $[\text{GL}_n]$ for $n > 3$.

1.2. Entropy bounds. Consider a sequence of homogeneous toral sets

$$Y_i = \text{GL}_n(\mathbb{Q}) T_i(\mathbb{A})^1 g_i \subset [\text{GL}_n]$$

where $T_i \simeq \text{Res}_{K_i/\mathbb{Q}}(\text{G}_m)$ for some number field $K_i$ of degree $n$ and $g_i \in \text{GL}_n(\mathbb{A})^1$. To each $Y_i$ one can associate an order in $K_i$ and define a discriminant $\text{disc}(Y_i)$ which is the product of the discriminant of this order and some measure of distortion at the Archimedean place. We refer to [ELMV11, §4] and §2 below for an exact definition. We say that $Y_i$ is of maximal type if the associated order is the ring of integers in $K_i$. 

The homogeneous toral set $Y_i$ also comes with a notion of volume $\text{vol}(Y_i)$ related to the discriminant through $\text{vol}(Y_i) = \text{disc}(Y_i)^{1/2 + o(1)}$ (cf. [ELMV11]).

Assume that the discriminants $\text{disc}(Y_i)$ go to infinity for $i \to \infty$. We write $\mu_{Y_i}$ for the invariant probability measure on $Y_i$ and suppose that the measures $\mu_{Y_i}$ converge in the weak*-topology to some probability measure $\mu$. Furthermore, suppose that there exists a place $u$ of $\mathbb{Q}$ and a split $\mathbb{Q}_u$-torus $A < \text{GL}_n$ so that every $\mu_{Y_i}$ is invariant under $A = A(\mathbb{Q}_u)$. For the purposes of this article, there is no difference between $u$ Archimedean and $u$ non-Archimedean. For simplicity, we also assume that we have a fixed invariance subgroup at the Archimedean place (if $u \neq \infty$) i.e. that $g_{i,\infty}^{-1}T_i g_{i,\infty}$ is independent of $i$.

For any $A$-invariant probability measure $\nu$ on $[\text{GL}_n]$ we write $h_\nu(a)$ for the Kolmogorov-Sinai entropy of $a \in A$ with respect to $\nu$. We write $h_{[\text{GL}_n]}(a)$ instead of $h_{\mu_{[\text{GL}_n]}}(a)$ for simplicity where $\mu_{[\text{GL}_n]}$ is the invariant probability measure on $[\text{GL}_n]$. In an ideal situation, one would be able to show that the weak*-limit $\mu$ satisfies $h_\mu(a) = h_{[\text{GL}_n]}(a)$ as is the case for $n = 2$. Indeed, this would imply that $\mu$ is invariant under $\text{SL}_2(\mathbb{A})$ and establish equidistribution for functions orthogonal to character spectrum. The objective in the following is to obtain progress towards this by establishing bounds of the kind

$$h_\mu(a) \geq \eta.$$

The most general such bound to date has been established in [ELMV09].

In [Kha19a], Khayutin used novel techniques originating in geometric invariant theory to prove the drastically better lower bound

$$h_\mu(a) \geq \frac{h_{[\text{GL}_n]}(a)}{2(n-1)}$$

under the additional assumption that the Galois groups $G_i$ of the Galois closure $L_i$ of $K_i$ over $\mathbb{Q}$ act two-transitively on the roots of the characteristic polynomial of a generator of $L_i$.

One of the aims of this article is to extend [Kha19a] to all remaining homogeneous toral sets in $[\text{GL}_4]$. Note that the dimension $n = 4$ exhibits special phenomena unseen for $n = 2$ or $n = 3$ due to the existence of intermediate groups. Indeed, whenever one of the quartic fields $K_i$ contains a quadratic subfield $F_i$, the torus $T_i$ is contained in a $\mathbb{Q}$-group $R_i$ isomorphic to $\text{Res}_{F_i/\mathbb{Q}}(\text{GL}_2)$ – the centralizer of the subgroup $\text{Res}_{F_i/\mathbb{Q}}(\text{GL}_2)$ in $T_i$.

**Remark 1.1 (Galois types).** Let $K$ be a quartic field and let $G$ be the Galois group of its Galois closure over $\mathbb{Q}$. Then one of the following is true:

- $G \simeq \mathbb{Z}/4\mathbb{Z}$ ($K$ is cyclic).
- $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ($K$ is biquadratic).
- $G \simeq D_4$ ($K$ is dihedral).  

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2In some cases, non-escape of mass is known. For instance when $K_i/\mathbb{Q}$ is abelian, subconvexity of the Dedekind $\zeta$-function $\zeta_{K_i}(s)$ is known by Burgess’ bound and non-escape of mass holds by [ELMV11].
• $G \cong A_4$.
• $G \cong S_4$.

In particular, the two-transitive Galois types are exactly $A_4$ and $S_4$. In all other cases, $K$ contains a quadratic subfield. We also remark that in terms of density $\approx 17.11\%$ of quartic fields are dihedral and $\approx 82.89\%$ are of type $S_4$ by a result of Bhargava [Bha05].

In the following, the standing assumption is that all of the fields $K_i$ fixed above contain a quadratic subfield. We fix such a subfield $F_i$ and denote by $S_i, R_i$ the groups introduced above. Note that only in the biquadratic case there is more than one quadratic subfield. For $a \in A$ the invariant probability measure on $[R_i g_i] = GL_4(\mathbb{Q})R_i(\mathbb{A})^1 g_i$

has positive entropy $h_{[R_i g_i]}(a)$ whenever $a \notin g_{i,u}^{-1}S_i(\mathbb{Q}_u)g_{i,u}$. By restricting to a subsequence, we shall assume without loss of generality that the $\mathbb{Q}_u$-torus $g_{i,u}^{-1}S_i g_{i,u}$ is independent of $i$. In this case, the entropy $h_{[R_i g_i]}(a)$ is independent of $i$; we shall denote it in the following by $h_{\text{int}}(a)$. Let $A' \subset A$ be the set of $a \in A$ with $h_{\text{int}}(a) < \frac{1}{2} h_{[GL_4]}(a)$. If $u$ is the real place, $\log(A')$ is a union of closed Weyl chambers. For any number field $F$, let $D_F$ denote the absolute value of its discriminant.

The following theorem is one of the main results of this article.

**Theorem 1.2.** Let $Y_i$ be a sequence of $A$-invariant homogeneous toral sets in $[GL_4]$ of maximal type that satisfies the above assumptions and choices. In particular, for each $i$ the quartic field associated to the homogeneous toral set $Y_i$ contains at least one quadratic subfield and we choose one quadratic subfield $F_i$. Suppose that there exists $c > 0$ such that $D_{K_i} \geq c D_{F_i}$ for all $i$. Then the weak$^*$-limit $\mu$ of the invariant probability measures $\mu_{Y_i}$ satisfies

$$h_\mu(a) \geq h_{\text{int}}(a)$$

for $a \in A'$.

Some remarks are in order:

**Remark 1.3 (Optimality).** With the imposed assumptions, the entropy bound in Theorem 1.9 is best possible. Indeed, for any subgroup $\text{Res}_{F/\mathbb{Q}}(GL_2) = R < GL_4$ the homogeneous set $[R]$ contains sequences of homogeneous toral sets with growing discriminant. This situation does not occur under the assumptions of Theorem 1.2 when the discriminant of $F_i$ tends to $\infty$. In this case, the conjectured entropy bound is $h_\mu(a) \geq h_{[GL_4]}(a)$. Using a uniform version of Linnik’s theorem with respect to the (in the application quadratic) base field we can improve upon Theorem 1.2 when the discriminant of the intermediate field grows polynomially in the discriminant of the quartic field – see Theorem 1.13 below.

**Remark 1.4 (Abelian quartic fields).** For abelian quartic fields (i.e. biquadratic or cyclic ones) subconvexity of the associated Dedekind $\zeta$-functions is known. In particular, [ELMV11] also yields entropy bounds for such homogeneous toral sets. These bounds in fact match the bounds obtained here. While our focus in the current
article lies on dihedral quartic fields, we also carry out the argument in the abelian case for completeness. Note that amongst the fields considered here, dihedral fields are generic (by Bhargava’s result mentioned earlier). For dihedral quartic fields, subconvexity is not known to our knowledge.

Remark 1.5 (First generalizations). There is no apparent obstruction to generalizing Theorem 1.2 to homogeneous toral sets in $[G]$ where $G$ is the group of units in a central simple algebra over $\mathbb{Q}$ (i.e. $G$ is an inner form of $GL_4$). For simplicity of the exposition, we omit this extension.

In principle, the arguments of this article should also apply to more general towers of extensions $\mathbb{Q} \subset F \subset K$ when $[K:F]=2$.

Remark 1.6. In an upcoming preprint of the first named author with Khayutin [KW], the authors improve upon the entropy bound in Theorem 1.2 under stronger assumptions on the growth of the discriminant relative to the discriminant of the quadratic subfield. In particular, [KW] establishes density of such homogeneous toral sets.

1.3. Linnik’s theorem and Bowen decay uniformity. In this section, we explain the aforementioned version of Linnik’s theorem with uniformity over the base field. Let $F$ be a number field of degree $n$ over $\mathbb{Q}$ and consider the $F$-group $GL_2,F$. As before, we write $[GL_2,F] = GL_2(F)\backslash GL_2(\mathbb{A}_F)$ and equip it with its invariant probability measure $\mu_{[GL_2,F]}$. By a homogeneous toral set in $[GL_2,F]$ we mean a set of the form

$$Y = [Tg] = GL_2(F)T(\mathbb{A}_F)^1g$$

where $g \in GL_2(\mathbb{A}_F)^1$ and $GL_2,F > T \simeq Res_{K/F}(\mathbb{G}_m)$ for a quadratic extension $K/F$. As before, we refer to §2 below for the notions of discriminant $\text{disc}(Y)$, volume $\text{vol}(Y)$, and maximal type for $Y$. Write $\mu_Y$ for the invariant probability measure on $Y$.

Let $S$ be a finite set of places of $F$. For each $v \in S$, let $A_v$ denote a split $F_v$-torus of $GL_2(F_v)$, and let $A_v = A_v(F_v)$ be its group of $F_v$-points. Let $A_S := \prod_{v \in S} A_v$. Suppose that for all $i$ and all $v \in S$, we have $g_{i,v}^{-1}T_i g_{i,v} = A_v$. We fix

$$a = (a_v)_{v \in S} \in A_S.$$

Let $B \subset GL_2(\mathbb{A}_F)$ be a basic open set i.e. of the form $B = \prod_u B_u$ where $u$ runs over all places of $F$ and $B_u \subset GL_2(F_u)$ is a bounded open neighborhood of the identity. For any integer $\tau \geq 1$ we define the $\tau$-Bowen ball

$$B_\tau = \bigcap_{-\tau \leq t \leq \tau} a^{-t}Ba^t.$$

Theorem 1.7 (Uniform Linnik’s basic lemma). Let $c > 0$ and let $Y \subset [GL_2,F]$ be a homogeneous toral set of maximal type which is invariant under $A_S$. Suppose that
$B_u = \text{GL}_2(O_{F,u})$ for any finite place $u$ of $F$ and define $B_\infty$ to be the product over all $B_v$ with $v$ Archimedean.

Then for any $0 < \tau \leq \frac{\log \text{disc}(Y) - \log cD_F}{2h_{[\text{GL}_2,F]}(a)}$,
\[
\mu_Y \times \mu_Y \left( \{ (x, y) \in [\text{GL}_2,F]^2 : y \in xB_\tau \} \right) \ll_{n, B_\infty, c, \varepsilon} \frac{1}{\text{vol}(Y)} + \frac{\text{disc}(Y)^{1+\varepsilon}}{\text{vol}(Y)^2} e^{-2r_{\text{GL}_2,F}(a)}
\]
where the implicit constant depends only on $n$, $c$, $\varepsilon$, and polynomially on the diameter of $B_\infty$.

The diameter of $B_\infty$ is defined in (4.10). Roughly speaking, Theorem 1.7 asserts that at certain time scales (the range growing with the discriminant) the measure $\mu_Y \times \mu_Y$ gives the same amount of mass to pairs of points with displacement in the $\tau$-Bowen ball as the Haar measure does.

We would like to emphasize here that the implicit constant is independent of the base field $F$ once its degree is fixed. In this sense we consider Theorem 1.7 to be uniform in the base field $F$. In applications, the following special case is usually the most relevant.

**Example 1.8 (A special case).** Suppose that $Y = [Tg]$ is a homogeneous toral set where the invariance subgroup $T_\infty = g^{-1}T(\prod_{v|\infty} F_v)g$ at the Archimedean places is considered fixed (more precisely, one ought to consider a family of homogeneous toral sets with this invariance). Let $K/F$ be the quadratic extension associated to $Y$. In this case,
\[
\text{disc}(Y) \asymp T_\infty D_K D_F^{-2}, \quad \text{vol}(Y) = D_K^{1+o(1)}
\]
so that Theorem 1.7 takes the form
\[
\mu_Y \times \mu_Y \left( \{ (x, y) \in [\text{GL}_2,F]^2 : y \in xB_\tau \} \right) \ll_{n, B_\infty, c, \varepsilon} D_K^{1+\varepsilon} D_F^{-2} D_F^\varepsilon e^{-2r_{[\text{GL}_2,F]}(a)}
\]
where the implicit constant depends on $T_\infty$ as well (or more precisely on its Archimedean discriminant). The factor $D_F^\varepsilon$ can be thought of the volume of the ambient space $[\text{GL}_2,F]$ – see for example [Vo21, Ch. 29].

From Theorem 1.7 one can deduce by relatively standard arguments (see for instance [ELMV12, §4.2]) that any weak$^*$-limit of a sequence of measures $\mu_{Y_i}$ as in theorem has maximal entropy. This proves (1.1) in this setup:

**Corollary 1.9 (Linnik’s theorem).** Let $Y_i$ be a sequence of toral packets of maximal type which are invariant under $A_S$. Suppose that the measures $\mu_{Y_i}$ converge in weak$^*$-topology to a probability measure $\mu$ on $[\text{GL}_2,F]$. Then $\mu$ is invariant under $\text{SL}_2(\mathbb{A}_F)$.

**Remark 1.10.** In some cases (e.g. in the situation of Example 1.8), the invariance statement in Corollary 1.9 can be coupled with an analysis on the character spectrum of $L^2_0([G])$ to yield equidistribution. See for example the discussions in [Kha19b, §3.3] and [ALMW20, §9].
Remark 1.11 (About the proof of Theorem 1.7). The proof of Theorem 1.7 does exhibit some similarities to other ergodic-theoretic approaches to Linnik’s theorem while showing some simplifications. In [ELMV12, Wie19] an estimate for representations of binary by ternary quadratic forms (proved in [ELMV12, App. A]) was used to establish Linnik’s basic lemma. Here, Linnik’s basic lemma is established using geometric invariant theory as well as an elementary estimate for certain adelic orbital integrals. The latter substitutes the local analysis on the Bruhat-Tits tree of $\text{PGL}_2(k)$ in [ELMV12, App. A] when $k$ is a non-Archimedean local field. The application of geometric invariant theory involves counting $F$-rational points on the affine line over $F$ satisfying certain local restrictions. The number of such points is related to the $\theta$-invariants of an Hermitian line bundle over $\text{Spec}\, O_F$ associated to an Arakelov divisor. This treatment allows us to get an estimate working uniformly for all number fields $F$ of the same degree, which is crucial in our application.

Remark 1.12 (Maximal type). The maximal type assumption in Theorem 1.7 should not be considered essential. It is only used in Lemma 4.9, and we do not assume maximal type elsewhere. It is conceivable that the estimate for local orbital integrals in Lemma 4.9 extends to non-maximal type when the counting results in §4.9 are appropriately refined for exact denominators.

Theorem 1.13 (A refinement of Theorem 1.2). Assume the notations and conditions in Theorem 1.2 except for the condition $D_{K_i} \geq cD_{F_i}^6$ for some constant $c > 0$. Furthermore, assume that there exists $\alpha > 0$ with

$$D_{F_i} \geq D_{K_i}^\alpha$$

for every $i$. Let $\beta \leq 2$ be a non-negative number. If there is some $\delta > 0$ with $D_{F_i} \leq D_{K_i}^{\frac{1}{6} - \delta}$ for all $i$, then there exists an open subset $A'(\delta, \beta) \subset A'$ with

$$h_\mu(a) \geq (1 + 2\alpha \beta)h_{\text{int}}(a).$$

While Theorem 1.13 does present an improvement upon Theorem 1.2, it is insufficient to prove density of these homogeneous toral sets.

1.4. Strategy of proof of Theorem 1.2. To get an entropy bound, we consider the set of pairs of points in a homogeneous toral set $Y = [Tg]$ which differ by an element in the Bowen ball

$$B_\tau = \bigcap_{-\tau \leq t \leq \tau} a^{-t}Ba^t$$

for a neighborhood $B \subset \text{GL}_4(\mathbb{A})^1$ of the identity. It suffices to get an exponential decay rate of the measure of this set with respect to a suitable time parameter $\tau$ (see Proposition 5.1 and Theorem 1.2). This strategy has been used in [ELMV09] and [Kha19a] among others and originates in the proof of the variational principle.

We first choose $\tau$ large enough to ensure that we "return within the intermediate group". More precisely, if $t_1, t_2 \in T(\mathbb{A})^1$ and $t_1g_h = \gamma t_2g_kb$ for some $\gamma \in \text{GL}_4(\mathbb{Q})$ and $b \in B_{\tau}$, then we must have $\gamma \in R(\mathbb{Q})$, where $R \cong \text{Res}_{F/\mathbb{Q}}\text{GL}_2 < \text{GL}_4$ is as in
the paragraph before Theorem 1.2. This can be achieved by considering the $T \times T$-invariant polynomials on $GL_4$. They are parametrized by the absolute Weyl group. If $\tau$ is sufficiently large compared to the global discriminant of the homogeneous toral set, then certain invariant polynomials will vanish. Using GIT as in Khayutin’s pioneering work [Kha19a], we conclude that $\gamma$ has to lie in $R(Q)$.

Once the return is in $R(Q)$, we can change the ambient group from $GL_4$ to $R$, and study the above mentioned decay in $GL_{2,F}$. Here is the main difference between our approach and [Kha19a]: instead of taking $\tau$ large enough to ensure that the return is in $T(Q)$, we take a slightly smaller $\tau$ and apply the uniform Linnik’s basic lemma (Theorem 1.7). This allows us to get an improved decay rate, and thus an improved entropy bound.

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2. Discriminant and volume

Let $F$ be a number field of degree $n$.

2.1. Notation for number fields. For any number field $K$, denote by $\mathcal{O}_K$ the ring of integers of $K$ and by $D_K$ the absolute value of its discriminant. When $K \supset F$ we write $D_{K/F}$ for the norm of the relative discriminant for $K/F$. Note here that the relative discriminant is an ideal in $\mathcal{O}_F$ and that the norm of such an ideal is its index in $\mathcal{O}_F$. With these notations, the product formula for discriminants states that

$$D_K = D_{K/F} D_{[K:F]}.$$

Let $\mathbb{A}_F$ denote the ring of adeles of $F$. We simply write $\mathbb{A}$ for $\mathbb{A}_F$ if the field $F$ is clear from context. Let $\mathcal{V}_F$ denote the set of places of $F$, i.e. the equivalence classes of absolute values. Write $\mathcal{V}_{F,\infty}$ for the set of infinite places and $\mathcal{V}_{F,f}$ for the set of finite places. Let $F_v$ be the completion of $F$ at $v$ for any $v \in \mathcal{V}_F$ and identify $v$ with a valuation on $F_v$ when $v$ is finite. In this case, we denote by $\mathcal{O}_{F,v}$ the discrete valuation ring given by $v$.

For each $v \in \mathcal{V}_F$, let $|\cdot|_v$ be the normalized absolute value on $F_v$ so that $m(xA) = |x|_v m(A)$ for every $x \in F_v$ and $A \subset F_v$ measurable, where $m$ denotes the Haar measure on $F_v$. For real places $v$ of $F$ this is the standard absolute value while for complex places it is the square of the standard absolute value. If $v$ is a finite place and $q_v$ is the cardinality of the residue field of $F_v$, then $|x|_v = q_v^{-v(x)}$ for any $x \in F_v$. Alternatively, if $p$ is the rational prime below $v$ then $|x|_v = |\text{Nr}(x)|_p$ where $\text{Nr} = \text{Nr}_{F_v/Q_p} : F_v \to Q_p$ is the norm map.
For any \( a \in \mathbb{A}_F \), the content of \( a \) is defined to be

\[
|a|_{\mathbb{A}_F} := \prod_{v \in \mathcal{V}_F} |a_v|_v.
\]

If \( S \) is a finite set of places of \( F \), we define \( F_S = \prod_{v \in S} F_v \), and the \( S \)-adic content

\[
|a|_S := \prod_{v \in S} |a_v|_v
\]

for \( a \in F_S \). The product formula states that \( |a|_{\mathbb{A}_F} = 1 \) for any \( a \in F^\times \) (where \( F^\times \) is embedded diagonally).

Given number fields \( F \subset K \) we write \( \text{Nr}_{K/F} \) (resp. \( \text{Tr}_{K/F} \)) for the norm (resp. trace) map \( K \to F \), and sometimes also for its extensions e.g. to the norm (resp. trace) map \( \mathbb{A}_K \to \mathbb{A}_F \). For any \( x \in \mathbb{A}_K \) we have

\[
|x|_{\mathbb{A}_K} = |	ext{Nr}_{K/F}(x)|_{\mathbb{A}_F}.
\]

For any place \( u \) of \( F \) we define the étale algebra \( K_u := K \otimes F_u \simeq \prod_{w|u} K_w \). If \( x_u \in K_u \) and \( w \mid u \), let \( x_w \in K_w \) denote the \( w \)-component of \( x_u \). The norm and the trace extend to maps \( K_u \to F_u \). We also write \( \mathcal{O}_{K,u} \) for the maximal order in \( K_u \) which under the above identification corresponds to \( \prod_{w|u} \mathcal{O}_{K,w} \).

### 2.2. Homogeneous spaces.

Let \( F \) be a number field and consider the homogeneous space

\[
\text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F)^1 =: [\text{GL}_n,F]
\]

where \( \text{GL}_n(\mathbb{A}_F)^1 = \{ g \in \text{GL}_n(\mathbb{A}_F) : |\det(g)|_{\mathbb{A}_F} = 1 \} \). By a theorem of Borel and Harish-Chandra, \( \text{GL}_n(F) \) is a lattice in \( \text{GL}_n(\mathbb{A}_F)^1 \) and we write \( \mu_{[\text{GL}_n,F]} \) for the invariant probability measure.

Denote by \( Z < \text{GL}_n \) the center. Letting \( [\text{PGL}_n,F] = \text{PGL}_n(F) \backslash \text{PGL}_n(\mathbb{A}_F) \) we have a fiber bundle

\[
\{1\} \to [Z] = Z(F) \backslash Z(\mathbb{A}_F)^1 \to [\text{GL}_n,F] \to [\text{PGL}_n,F] \to \{1\}.
\]

so that \([\text{GL}_n,F]\) is a compact extension \([\text{PGL}_n,F]\) where the fibers are copies of the compact abelian group \([Z]\).

#### 2.2.1. Homogeneous toral sets.

A homogeneous toral set in \([\text{GL}_n,F]\) is a subset of the form

\[
Y := [Tg] := \text{GL}_n(F)T(\mathbb{A}_F)^1 g \subset [\text{GL}_n,F],
\]

where \( T \) is a maximal \( F \)-torus of \( \text{GL}_n \), and \( g_\mathbb{A} \in \text{GL}_n(\mathbb{A})^1 \). Unless specified otherwise, any homogeneous toral set \( Y = [Tg] \) in this article satisfies that the group of \( F \)-characters of \( T \) has rank one. In this case, there is a maximal subfield \( K \subset \text{M}_n(F) \) so that \( T \) is given by the equations \( gk = kg \) for all \( k \in K \). Strictly speaking, the field \( K \) depends not only on \( Y \) but also on \( T \) i.e. it is determined up to conjugation with \( F \)-rational elements (because \( T \) is). We shall put no emphasis on this subtlety here and simply refer to \( K \) as the associated field (to \( Y \) or \( T \)). Furthermore, \( Y \) has finite volume; we write \( \mu_Y \) for the Haar probability measure on \( Y \). Let \( \tilde{\mu}_Y \) be the Haar measure on \( g^{-1}T(\mathbb{A}_F)^1 g \) that descends to \( \mu_Y \).
2.2.2. Intermediate homogeneous sets. Given a homogeneous toral set \([T_g]\) with associated field \(K\) as well as a subfield \(K' \subset K\), we write \(S_{K'}\) for the subtorus of \(T\) isomorphic to \(\text{Res}_{K'/\mathbb{Q}}(\mathbb{G}_m)\). The centralizer subgroup \(R_{K'} < \text{GL}_n(K)\) of \(K'\) (or equivalently of \(S_{K'}\)) is isomorphic to \(\text{Res}_{K'/\mathbb{Q}}(\text{GL}_n)\). The homogeneous space

\[
[R_{K'}g] = \text{GL}_n(F)R_{K'}(F)^1g
\]

is of finite volume and contains \([T_g]\). We write \(\mu_{[R_{K'}g]}\) for the invariant probability measure on \([R_{K'}g]\).

2.3. Discriminants. Fix a homogeneous toral set \(Y = [T_g]\) with associated field \(K\). Any such homogeneous toral set comes with a notion of discriminant \(\text{disc}(Y)\) and volume \(\text{vol}(Y)\) which we recall in this section.

2.3.1. Non-Archimedean discriminants. Define for a finite place \(u\) the associated local order at \(u\)

\[
\mathcal{O}_u = g_u M_n(F_u)g_u^{-1} \cap K_u.
\]

This is an order in \(K_u\) with the property that \(\mathcal{O}_u = \mathcal{O}_{K_u}\) for all but finitely many places \(u\). Indeed, \(g_u \in \text{GL}_n(\mathcal{O}_{F_u})\) for almost every \(u\). Note also that \(\mathcal{O}_u \cap F_u = \mathcal{O}_{F_u}\). We will say that \(Y\) is of maximal type if every associated local order \(\mathcal{O}_u\) is the maximal order in \(K_u\).

The local order \(\mathcal{O}_u\) has a local discriminant \(\text{disc}(\mathcal{O}_u)\) which is obtained by taking an \(\mathcal{O}_{F,u}\)-basis of \(\mathcal{O}_u\) and taking the discriminant of the bilinear form given by the reduced trace of \(K_u/F_u\). As such, the discriminant is a well-defined element of \(\mathcal{O}_{F,u}^\times(\mathcal{O}_{F,u}^\times)^2\) where \((\mathcal{O}_{F,u}^\times)^2\) denotes the subgroup of squares in \(\mathcal{O}_{F,u}^\times\). If \(\mathcal{O}_u\) is maximal, the ideal generated by \(\text{disc}(\mathcal{O}_u)\) is exactly the relative discriminant of the local extension \(K_u/F_u\). The local absolute discriminant of \(Y\) at \(u\) is given by

\[
\text{disc}_u(Y) := |\text{disc}(\mathcal{O}_u)|_u^{-1}.
\]

By the local-to-global principle for orders,

\[
\mathcal{O} = \bigcap_{u \text{ finite}} (\mathcal{O}_u \cap K)
\]

is an order in \(K\). As in the local case, we have a discriminant \(\text{disc}^u(\mathcal{O}) \in \mathcal{O}_{F} /(\mathcal{O}_{F}^\times)^2\) and the absolute non-Archimedean discriminant of \(Y\) is

\[
\text{disc}^\text{fin}(Y) = |\text{Nr}_{F/\mathbb{Q}}(\text{disc}(\mathcal{O}))| = \prod_{u \in \mathcal{V}_{F,1}} \text{disc}_u(Y),
\]

If \(Y\) is of maximal type, then \(\mathcal{O} = \mathcal{O}_K\) and \(\text{disc}^\text{fin}(Y) = D_{K/F}\).

2.3.2. Archimedean discriminants. We give an adhoc definition of the Archimedean discriminant of the homogeneous toral set \(Y = \text{GL}_n(F)T(\mathbb{A}_F)g\) which is most useful in our context. See [ELMV11, §4.6] and [Kha19a, §7.4.2] for a more thorough discussion.
Let \( u \) be an Archimedean place of \( F \) so that \( F_u \simeq \mathbb{R} \) or \( F_u \simeq \mathbb{C} \). In either case, the algebraic closure \( \overline{F_u} \) is \( \mathbb{C} \); we identify \( \overline{F_u} = \mathbb{C} \). Define a norm \( \| \cdot \| \) on \( M_n(\mathbb{C}) \) through the Hermitian inner product
\[
\langle A, B \rangle = \text{Tr}(AB^\ast).
\]
While this is clearly a choice of a good norm in the sense of [ELMV11, §7], it constitutes a consistent choice for varying \( n \) where \( n \) is the degree of \( F \). Under the identification \( \mathbb{C} = F_u \), \( \| \cdot \| \) (resp. \( \langle \cdot, \cdot \rangle \) restricts to a norm on \( M_n(F_u) \) which we also denote by \( \| \cdot \| \) (resp. \( \langle \cdot, \cdot \rangle \)).

Let \( K'_u \subset M_n \) be the centralizer of \( g_{-1}uT(F_u)g_u \) and let \( k_1, \ldots, k_n \) be a \( \mathbb{C} \)-basis of \( K'_u(\mathbb{C}) \). The Archimedean discriminant of \( Y \) at \( u \) is then given by
\[
\text{disc}_u(Y) = \frac{\det((k_i, k_j)_{ij})}{\det(\text{Tr}(k_i k_j))_{ij}}^{\frac{1}{m}}
\]
and is independent of the choice of basis.

**Example 2.1.** Suppose that \( n = 2 \). Let \( k \in K'_u(F_u) \) be non-zero with \( \text{Tr}(k) = 0 \). The choice of basis \( k_1 = 1, k_2 = k \) yields
\[
\text{disc}_u(Y) = \left| \frac{\|k\|^2}{4 \det(k)} \right| \propto \left| \det \left( \frac{k}{\|k\|} \right) \right|^{-1}.
\]
Roughly speaking, the discriminant \( \text{disc}_u(Y) \) is large whenever the normalized \( k \) is close to being nilpotent.

Now we can define the **global absolute discriminant** of \( Y \) to be
\[
\text{disc}(Y) = \prod_{u \in \mathcal{V}_F} \text{disc}_u(Y). \tag{2.1}
\]

**2.3.3. Orders in quadratic extensions.** In view of Theorem 1.7 we record a special property of local quadratic extensions. To that end, let \( K/F \) be a quadratic extension, and let \( \mathcal{O} \subset K \) be an \( \mathcal{O}_F \)-order (i.e. \( \mathcal{O} \cap F = \mathcal{O}_F \)). For a finite place \( u \) of \( F \) we write (as before) \( \mathcal{O}_u = \mathcal{O} \otimes \mathcal{O}_{F,u} \) for the completion at \( u \).

Recall that a fractional \( \mathcal{O} \)-ideal \( \mathfrak{a} \subset K \) is said to be proper if
\[
\mathcal{O} = \{ \lambda \in K : \lambda \mathfrak{a} \subset \mathfrak{a} \}.
\]
Proper fractional ideals for \( \mathcal{O}_u \) are similarly defined.

**Proposition 2.2** (Local orders in quadratic extensions). For \( K \) and \( \mathcal{O} \) as above and any finite place \( u \) of \( F \) the local order \( \mathcal{O}_u \) is a monogenic i.e. there exists \( \alpha \in \mathcal{O}_u \) with \( \mathcal{O}_u = \mathcal{O}_{F,u}[\alpha] \). Moreover, any proper \( \mathcal{O}_u \)-ideal is principal (and vice versa).

**Proof.** To see that \( \mathcal{O}_u \) is monogenic observe first that \( \mathcal{O}_{F,u} \) is PID (it is a discrete valuation ring). The order \( \mathcal{O}_u \) is a submodule of the free \( \mathcal{O}_{F,u} \)-module \( \mathcal{O}_{K,u} \) and in particular also free. Furthermore, \( 1 \in \mathcal{O}_u \) is primitive (i.e. \( \frac{1}{\varpi} \notin \mathcal{O}_u \) for a uniformizer \( \varpi \) of \( \mathcal{O}_{F,u} \)) and there exists \( \alpha \in \mathcal{O}_u \) so that \( 1, \alpha \) is a \( \mathcal{O}_{F,u} \)-basis of \( \mathcal{O}_u \). In particular, \( \mathcal{O}_u \) is monogenic. The second statement can be proven as in [ELMV12, Prop. 2.1] using monogenicity. \( \square \)
Recall that the inverse different for the order \( \mathcal{O}_u \) is
\[
\{ x \in K_u : \text{Tr}(xy) \in \mathcal{O}_{F,u} \text{ for all } y \in \mathcal{O}_u \}.
\]

**Lemma 2.3** (Different ideals). For any \( \mathcal{O}_F \)-order \( \mathcal{O} \subset K \) and any finite place \( u \) of \( F \), the inverse different for \( \mathcal{O}_u \) is a proper and principal ideal. The different ideal \( \mathfrak{d}_{\mathcal{O},u} \) is the inverse of the inverse different. If \( \alpha \in \mathcal{O}_u \) is such that \( \mathcal{O}_u = \mathcal{O}_{F,u}[\alpha] \), the different ideal is generated by \( \alpha - \sigma(\alpha) \) where \( \sigma \in \text{Gal}(K_u/F_u) \) is the non-trivial Galois element.

Note that \( \mathcal{O}_u \) is Galois-invariant as \( \sigma(\alpha) = \text{Tr}(\alpha) - \alpha \in \mathcal{O}_u \).

**Proof.** We omit the explicit calculation. One shows that \( \alpha - \sigma(\alpha) \) generates the inverse different. \( \square \)

For an order \( \mathcal{O} \) as above we write \( \Delta_{\mathcal{O},u} \) for a choice of generator for the different ideal of \( \mathcal{O}_u \) (unique up to \( \mathcal{O}_u \)-multiples). In particular,
\[
\text{disc}(\mathcal{O}_u) = \text{Nr}(\Delta_{\mathcal{O},u}).
\]
When \( \mathcal{O} = \mathcal{O}_K \), we write \( \Delta_{K/F,u} = \Delta_{\mathcal{O}_{K,u}} \).

**Remark 2.4** (Parametrizing suborders). If \( \alpha \in \mathcal{O}_{K,u} \) is a generator for the maximal order, one can show that any order \( \mathcal{O}_u \subset \mathcal{O}_{K,u} \) is of the form \( \mathcal{O}_u = \mathcal{O}_{F,u}[f \alpha] \) for some \( f \in \mathcal{O}_{F,u} \). The element \( f \in \mathcal{O}_{F,u} \) is uniquely determined up to \( \mathcal{O}_u \)-multiples; its class in \( \mathcal{O}_{F,u}/\mathcal{O}_u \) can be called the conductor of the order \( \mathcal{O}_u \).

**Lemma 2.5** (Image of the norm). When \( u \) is non-dyadic (i.e. \( u \nmid 2 \)), the norm map \( \mathcal{O}_u^\times \to \mathcal{O}_{F,u}^\times \) is surjective if \( \text{disc}(\mathcal{O}_u) \in \mathcal{O}_{F,u}^\times \) and otherwise the image has index two. If \( u \) is dyadic, the index is bounded by a constant depending on the degree \( n = [F : \mathbb{Q}] \) only.

We remark that the bound on the index for dyadic places can be drastically optimized (e.g. if \( u \) splits in \( K \)). For our application, the above lemma is sufficient.

**Proof.** Let \( \alpha \in \mathcal{O}_u \) be as in Proposition 2.2. Any element in the image of the norm map \( \mathcal{O}_u^\times \to \mathcal{O}_{F,u}^\times \) takes the form
\[
Q(x,y) = \text{Nr}(x + y\alpha) = x^2 + \text{Tr}(\alpha)xy + \text{Nr}(\alpha)y^2
\]
where \( x, y \in \mathcal{O}_{F,u} \). Assume first that \( u \) is not dyadic. In this case, we can choose \( \alpha \) to be traceless. Whenever \( \text{Nr}(\alpha) \) is a unit, the binary form \( Q \) is universal (i.e. represents all units) by an application of the pigeonhole principle and Hensel’s lemma. As \( \alpha \) is equal to \( \Delta_{\mathcal{O},u} \) up to a unit, this shows the claim. If \( \text{Nr}(\alpha) \) is not a unit, \( Q \) only represents the squares when taken modulo the prime underlying \( u \) and in particular, \( \text{Nr} \) is not surjective. On the other hand, the squares are represented by \( Q \) and have index 2 in \( \mathcal{O}_{F,u}^\times \) so that this finishes the non-dyadic case.

Suppose now that \( 2 \mid u \). It suffices to estimate the index of the subgroup of squares in \( \mathcal{O}_{F,u}^\times \). However, observe that \( [F_u : \mathbb{Q}_2] = [F : \mathbb{Q}] = n \) and that for each \( m \mid n \) there are finitely many degree \( m \) extensions over \( \mathbb{Q}_2 \). This implies the lemma. \( \square \)
2.4. **Volume of homogeneous sets.** For any homogeneous toral set \( Y = [Tg] \subset [\text{GL}_{n,F}] \) and a bounded neighborhood of the identity \( B \subset \text{GL}_n(\mathbb{A}_F) \) we define the **volume**

\[
\text{vol}_B(Y) = \tilde{\mu}_Y(B)^{-1}.
\]

Note that for any other bounded neighborhood of the identity \( B' \subset \text{GL}_n(\mathbb{A}_F) \) we have

\[
\text{vol}_B(Y) \ll_{B,B'} \text{vol}_{B'}(Y) \ll_{B,B'} \text{vol}_B(Y). 
\]  

(2.2)

Disregarding the dependency on the base field, it was shown in [ELMV11, Thm. 4.8] that

\[
\text{vol}_B(Y) = \text{disc}(Y)^{1/2 + o_F(1)}
\]

when \( B \) is chosen appropriately. The method is applicable to yield a sharper relation. For instance, whenever \( Y \) is a homogeneous toral set of maximal type with fixed invariance \( A_\infty \) at the Archimedean place, then

\[
\text{vol}_B(Y) = \text{disc}(Y)^{1/2 + o(1)} D_K^F [K:F]^{1/2 + o(1)}
\]

where the implicit constants depend on \( A_\infty \). In our case, the relative discriminant \( D_{K/F} \gg \text{disc}(Y) \) is usually bigger than \( D_F \) (see Lemma 3.6 and its application). In particular, one can omit the \( o(1) \) exponent in \( D_F \).

In this article, the neighborhood \( B \) will be typically fixed in which case we take the liberty of dropping the subscript \( B \) in \( \text{vol}_B(Y) \). This is additionally justified by (2.2). Furthermore, the volume can be defined for any homogeneous set and in particular for the homogeneous sets introduced in §2.2. Observe that

\[
\text{vol}([Zg]) = D_F^{1/2 + o(1)}
\]

for any \( g \in \text{GL}_2(\mathbb{A}_F) \) (the implicit constants are independent of \( F \)).

2.5. **Discriminants of cyclic fields.** This short subsection addresses the non-necessity of the discriminant condition in Theorem 1.2 in the cyclic case. Let \( K \) be a quartic number field, and \( L \) be its normal closure. Recall from Remark 1.1 that the Galois group \( \mathcal{G} := \text{Gal}(L/\mathbb{Q}) \) is one of the following five groups: \( S_4, A_4, D_4, C_4 = \mathbb{Z}/4\mathbb{Z}, V_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

As explained in the introduction, we are particularly interested in the cases where the Galois group is not 2-transitive, i.e. \( \mathcal{G} \) equals \( D_4 \) or \( C_4 \) or \( V_4 \). In all of these three cases, we can find a quadratic subfield \( F \) of \( K \). Furthermore, if \( \mathcal{G} \) is \( D_4 \) or \( C_4 \), then \( F \) is unique.

**Lemma 2.6.** Let \( K \) be a cyclic quartic field and let \( F \) be the unique quadratic subfield. Then

\[
D_{K/F} \geq \frac{1}{4} D_F.
\]

Lemma 2.6 implies that the discriminant condition in Theorem 1.2 is automatic for cyclic fields.
Proof. Suppose \( \mathcal{G} = C_4 \). By [EP80, Prop. 2], there exist integers \( W > 0, d > 1 \) with \( d \) squarefree, such that \( D_K = W^2d^3 \), \( D_{K/F} = W^2d \) or \( (W/4)^2d \), and \( D_F = d \) or \( 4d \), depending on whether \( d \equiv 1(4) \) or not. The conclusion can be easily checked. \( \square \)

We note that for \( \mathcal{G} = D_4 \) or \( V_4 \), the discriminant \( D_{K/F} \) could be arbitrarily large or small compared to \( D_F \). See [Bai80, Lemmas 17, 20] for the \( \mathcal{G} = D_4 \) case.

3. A COUNTING LEMMA

In this section, we introduce Hermitian vector bundles over arithmetic curves, their \( \theta \)-invariants, and the Poisson-Riemann-Roch formula. Our main reference is the first three sections of [Bos20]. We then use this to prove a counting estimate of rational points with certain local bounds in a number field.

3.1. Hermitian vector bundles over arithmetic curves. Let \( F \) be a number field and let \( \mathcal{O}_F \subset F \) be the ring of integers. A Hermitian vector bundle over \( \text{Spec} \mathcal{O}_F \) is a pair

\[
\overline{E} = (E, (\| \cdot \|_\sigma)_{\sigma:F \rightarrow \mathbb{C}})
\]

consisting in a finitely generated projective \( \mathcal{O}_F \)-module \( E \) and in a family \((\| \cdot \|_\sigma)_{\sigma:F \rightarrow \mathbb{C}}\) of Hermitian norms \( \| \cdot \|_\sigma \) on the complex vector spaces \( E_\sigma = E \otimes_{\mathcal{O}_F} \mathbb{C} \).

The family \((\| \cdot \|_\sigma)_{\sigma:F \rightarrow \mathbb{C}}\) is moreover required to be invariant under complex conjugation in that sense that \( \| x \|_\sigma = \| \bar{x} \|_{\bar{\sigma}} \) for \( x \in E_\sigma \).

A Euclidean lattice \( \overline{E} = (E, \| \cdot \|) \) is a free \( \mathbb{Z} \)-module of finite rank together with a Euclidean norm \( \| \cdot \| \) on the real vector space \( E \otimes_{\mathbb{Z}} \mathbb{R} \). We note that by definition, a Hermitian vector bundle over \( \text{Spec} \mathbb{Z} \) is nothing but a Euclidean lattice. Indeed, to give a Hermitian norm on \( E_\mathbb{C} = E \otimes_{\mathbb{Z}} \mathbb{C} \) invariant under complex conjugation is the same as to give a Euclidean norm on \( E_\mathbb{R} \).

The rank of some Hermitian vector bundle \( \overline{E} \) over \( \text{Spec} \mathcal{O}_F \), denoted by \( \text{rk} E \), is the rank of the \( \mathcal{O}_F \)-module \( E \), or equivalently the dimension of the complex vector spaces \( E_\sigma \). A Hermitian line bundle is a Hermitian vector bundle of rank 1.

Give any Hermitian vector bundle \( \overline{E} = (E, (\| \cdot \|_\sigma)_{\sigma:F \rightarrow \mathbb{C}}) \) over \( \text{Spec} \mathcal{O}_F \), we shall define its dual \( \overline{E}^\vee \). Since \( E \) is finitely generated and projective over \( \mathcal{O}_F \), \( E^\vee = \text{Hom}(E, \mathcal{O}_F) \) is also a finitely generated projective \( \mathcal{O}_F \)-module. Moreover, for any \( \sigma : F \rightarrow \mathbb{C} \), the Hermitian norm \( \| \cdot \|_\sigma \) induces a canonical isomorphism between \( E_\sigma \) and \( E_\sigma^\vee \), and thus \( \| \cdot \|_\sigma \) can be viewed as a Hermitian norm on \( E_\sigma^\vee \) as well. We define \( \overline{E}^\vee := (E^\vee, (\| \cdot \|_\sigma)_{\sigma:F \rightarrow \mathbb{C}}) \). Similarly we can define the exterior powers \( \bigwedge^k \overline{E} \) of a Hermitian vector bundle \( \overline{E} \), as well as the direct sum \( \overline{E}_1 \oplus \overline{E}_2 \) and the tensor product \( \overline{E}_1 \otimes \overline{E}_2 \) of two Hermitian vector bundles \( \overline{E}_1 \) and \( \overline{E}_2 \).

Given any Hermitian vector bundle \( \overline{E} = (E, (\| \cdot \|_\sigma)_{\sigma:F \rightarrow \mathbb{C}}) \) over \( \text{Spec} \mathcal{O}_F \), we shall define its direct image \( \pi_* \overline{E} \) over \( \text{Spec} \mathbb{Z} \). Let \( \pi_* E \) be the underlying \( \mathbb{Z} \)-module of \( E \),
and we observe that

\[(\pi_* E)_C := \pi_* E \otimes \mathbb{Z} \mathbb{C} = \bigoplus_{\sigma : F \hookrightarrow \mathbb{C}} (E \otimes \mathcal{O}_F, \sigma) \mathbb{C}.\]

Now for any \(v = (v_\sigma)_{\sigma : F \hookrightarrow \mathbb{C}}\) in \((\pi_* E)_C\), let

\[\|v\|_E^2 := \sum_{\sigma : F \hookrightarrow \mathbb{C}} \|v_\sigma\|^2.\]

We define \(\pi_* E := (\pi_* E, \|\cdot\|_E)\).

Let \(\omega_{\mathcal{O}_F/\mathbb{Z}} := \text{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{Z})\) which can be seen as the inverse different. The canonical Hermitian line bundle \(\mathcal{T}_{\mathcal{O}_F/\mathbb{Z}}\) over \(\text{Spec} \mathcal{O}_F\) is the pair \((\omega_{\mathcal{O}_F/\mathbb{Z}}, (\|\cdot\|_\sigma)_{\sigma : F \hookrightarrow \mathbb{C}})\), where the Hermitian norms are given by

\[\|\text{tr}_{F/\mathbb{Q}}\|_\sigma = 1, \quad \forall \sigma : F \hookrightarrow \mathbb{C},\]

where \(\text{tr}_{F/\mathbb{Q}}\) is the trace map from \(F\) to \(\mathbb{Q}\), which is indeed a non-zero element in \(\text{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{Z})\). Since \(\omega_{\mathcal{O}_F/\mathbb{Z}}\) has rank 1 as an \(\mathcal{O}_F\)-module, the above equation uniquely determines the Hermitian norms. Note that for every Hermitian vector bundle \(\mathcal{T}\) over \(\text{Spec} \mathcal{O}_F\), we have a canonical isometric isomorphism of Hermitian vector bundles over \(\text{Spec} \mathbb{Z}\) (see e.g. [BK10, Proposition 3.2.2]):

\[\pi_* (\mathcal{T} \otimes \mathcal{T}_{\mathcal{O}_F/\mathbb{Z}}) \sim \pi_* (\mathcal{T})\]

(3.1)

3.2. Arakelov degree. The Arakelov degree of a Hermitian line bundle \(\mathcal{T}\) is defined to be

\[\widehat{\deg} \mathcal{T} = \log |L/\mathcal{O}_Fs| - \sum_{\sigma : F \hookrightarrow \mathbb{C}} \log \|s\|_\sigma\]

\[= \sum_{0 \neq p \in \text{Spec} \mathcal{O}_F} v_p(s) \log Np - \sum_{\sigma : F \hookrightarrow \mathbb{C}} \log \|s\|_\sigma,\]

(3.2)

where \(s \in L \setminus 0\), \(v_p\) is the valuation associated to \(p^3\), and \(Np = |\mathcal{O}_F/p|\) is the norm of \(p\). By product formula this is well-defined, i.e. independent of choice of \(s\). We extend the definition to any Hermitian vector bundle \(\mathcal{T}\) by setting

\[\widehat{\deg} \mathcal{T} := \widehat{\deg} \wedge^\text{rk} \mathcal{T}.\]

From this definition, when \(\mathcal{T}\) is an Euclidean lattice, we have

\[\widehat{\deg}(\mathcal{T}) = -\log \text{covol}(\mathcal{T}).\]

For any two Hermitian line bundles \(\mathcal{T}_1\) and \(\mathcal{T}_2\), it follows from (3.2) that

\[\widehat{\deg} \mathcal{T}_1 \otimes \mathcal{T}_2 = \widehat{\deg} \mathcal{T}_1 + \widehat{\deg} \mathcal{T}_2.\]

(3.3)

\(^3\mathcal{O}_{F,p}\) is a local ring, and let \(m_p\) denote its unique maximal ideal. Then \(s\) generates \(m_p^{v_p(s)} L\) as an \(\mathcal{O}_{F,p}\)-submodule of \(L\).
By reducing to the line bundle case, it can be shown that for any Hermitian vector bundle \( E \),
\[
\hat{\deg} E' = -\hat{\deg} E.
\] (3.4)

We have the following formula relating the Arakelov degrees of a Hermitian vector bundle and its direct image (see e.g. [Bos20, (1.3.6)] or [BGS94, (2.1.13)]):
\[
\hat{\deg} \pi_* E = \hat{\deg} E - \frac{1}{2} \log D_F \cdot \text{rk} E.
\] (3.5)

where \( D_F \) is the absolute value of the discriminant of the number field \( F \).

Combining (3.1) and (3.5), one can show that
\[
\hat{\deg} \mathcal{O}_F / \mathbb{Z} = \log D_F.
\] (3.6)

3.3. \( \theta \)-invariants and Poisson-Riemann-Roch formula. Given an Euclidean lattice \( \mathcal{E} = (E, \| \cdot \|) \), we can define \( \theta \)-invariants associated to it. The first one is analogous to the dimension of global sections of a vector bundle over a smooth projective curve over a base field \( k \) and is defined to be
\[
h_0^\theta(\mathcal{E}) := \log \sum_{v \in E} e^{-\pi \| v \|^2}.
\]

We also define
\[
h_1^\theta(\mathcal{E}) := h_0^\theta(\mathcal{E}^\vee).
\] (3.7)

By the Poisson summation formula, for any Euclidean lattice \( \mathcal{E} \) one has
\[
\sum_{w \in \mathcal{E}^\vee} e^{-\pi \| w \|^2} = \text{covol}(\mathcal{E}) \sum_{v \in \mathcal{E}} e^{-\pi \| v \|^2}.
\]

Using the above notations, this formula may be rewritten as
\[
h_0^\theta(\mathcal{E}) - h_1^\theta(\mathcal{E}) = \hat{\deg}(\mathcal{E}).
\] (3.8)

More generally, for any Hermitian vector bundle \( \mathcal{E} \) over \( \text{Spec} \ \mathcal{O}_F \), we define
\[
h_i^\theta(\mathcal{E}) := h_0^\theta(\pi_* \mathcal{E}), \quad i = 0, 1.
\] (3.9)

With this definition, (3.1) gives the Hecke-Serre duality formula (which is analogous to the classical Serre duality formula):
\[
h_1^\theta(\mathcal{E}) = h_0^\theta(\mathcal{E}^\vee \otimes \mathcal{O}_F / \mathbb{Z}).
\] (3.10)

Combining (3.5), (3.8), and (3.9), we obtain the general version of Poisson-Riemann-Roch formula for any Hermitian vector bundle \( \mathcal{E} \) over \( \mathcal{O}_F \):
\[
h_0^\theta(\mathcal{E}) - h_1^\theta(\mathcal{E}) = \hat{\deg} \mathcal{E} - \frac{1}{2} (\log D_F) \text{rk} \mathcal{E}.
\] (3.11)
3.4. Upper bounds of $\theta$-invariants. We first recall Proposition 2.7.3 from [Bos20]:

**Proposition 3.1.** Let $t \in \mathbb{R}$. For every Hermitian line bundle $\mathcal{L}$ over $\text{Spec} \mathcal{O}_F$ such that $\deg \mathcal{L} \leq t$, we have

$$h^0_\theta(\mathcal{L}) \leq f(t),$$

where

$$f(t) = \begin{cases} 1 + t & t \geq 0 \\ e^{2\pi t} & t < 0 \end{cases}$$

(3.12)

As a consequence, we obtain the following upper bound of $h^1_\theta(\mathcal{L})$, for $\mathcal{L}$ a Hermitian line bundle of sufficiently large degree.

**Corollary 3.2.** Let $t \in \mathbb{R}$. For every Hermitian line bundle $\mathcal{L}$ over $\text{Spec} \mathcal{O}_F$ such that $\deg \mathcal{L} \geq \log D_F + t$, we have

$$h^1_\theta(\mathcal{L}) \leq f(-t).$$

**Proof.** By (3.10), we have $h^1_\theta(\mathcal{L}) = h^0_\theta(\mathcal{L}' \otimes \mathcal{O}_F/\mathbb{Z})$. By (3.3)(3.4)(3.6) we have

$$\widehat{\deg}(\mathcal{L}' \otimes \mathcal{O}_F/\mathbb{Z}) = \log D_F - \widehat{\deg} \mathcal{L} \leq -t.$$

Hence we finish the proof by applying Proposition 3.1 to $\mathcal{L}' \otimes \mathcal{O}_F/\mathbb{Z}$. \qed

We now state the main result of this subsection.

**Proposition 3.3.** Let $t \in \mathbb{R}$. For every Hermitian line bundle $\mathcal{L}$ over $\text{Spec} \mathcal{O}_F$ such that $\deg \mathcal{L} \geq \log D_F + t$, we have

$$h^0_\theta(\mathcal{L}) \leq \widehat{\deg} \mathcal{L} - \frac{1}{2} \log D_F + f(-t).$$

**Proof.** This follows immediately from (3.11) and Corollary 3.2. \qed

3.5. Comparing $h^0_{\mathcal{A}_F}$ and $h^0_\theta$. Let $\mathcal{E} = (E, \| \cdot \|_{\sigma})$ be a Hermitian vector bundle over $\text{Spec} \mathcal{O}_F$. Following [GS91, Section 2.4], we define the invariant

$$h^0_{\mathcal{A}_F}(\mathcal{E}) := \log \# \{ s \in E : \| s \|_{\sigma} \leq 1, \forall \sigma \}.$$

**Proposition 3.4.** Let $n = [F : \mathbb{Q}]$. For any Hermitian vector bundle $\mathcal{E}$ over $\text{Spec} \mathcal{O}_F$, we have

$$h^0_{\mathcal{A}_F}(\mathcal{E}) \leq h^0_\theta(\mathcal{E}) + \pi n.$$

**Proof.** Let $\| \cdot \| := \sqrt{\sum_{\sigma} \| \cdot \|_{\sigma}^2}$ be the Euclidean norm in $\pi_s(\mathcal{E})$. By definition,

$$h^0_{\mathcal{A}_F}(\mathcal{E}) \leq \log \# \left\{ s \in E : \sum_{\sigma} \| s \|_{\sigma}^2 \leq n \right\} = \log \# \{ s \in E : \| s \| \leq \sqrt{n} \}. \quad (3.13)$$
On the other hand, by (3.9)
\[ h^0_0(L) \geq \log \sum_{|s| \leq \sqrt{n}} e^{-\pi|s|^2} \]
\[ \geq \log \sum_{|s| \leq \sqrt{n}} e^{-\pi n} \]
\[ = \log \# \{ s \in E : |s| \leq \sqrt{n} \} - \pi n. \] (3.14)

The proposition follows from (3.13) and (3.14).

**Corollary 3.5.** Let \( t \in \mathbb{R} \). Suppose \( L \) is a Hermitian line bundle over \( \text{Spec} \, \mathcal{O}_F \) such that \( \deg L \geq \log D_F + t \). Then
\[ h^0_{\text{Ar}}(L) \leq \deg L - \frac{1}{2} \log D_F + f(-t) + \pi n. \]

**Proof.** This follows from Proposition 3.3 and Proposition 3.4.

### 3.6. A counting estimate

Let \( F \) be a number field of degree \( n \).

**Lemma 3.6.** Let \( c > 0 \). Let \( r = (r_u)_{u \in \mathcal{V}_F} \) be a collection of positive real numbers satisfying the following:

1. \( r_u = 1 \) for all but finitely many \( u \)’s.
2. \( r_u \) is in the image of the absolute value\(^4\) map \( |\cdot|_u : F_u \to \mathbb{R}_{\geq 0} \) for all \( u \).

Let \( \|r\| = \prod_u r_u \), and suppose that \( \|r\| \geq cD_F \). Then there exists a constant \( C = C(n, c) > 0 \) such that
\[ \# \{ x \in F : |x|_u \leq r_u \text{ for all } u \} \leq \frac{C}{D_F^{1/2}} \|r\|. \] (3.15)

**Proof.** We first associate a Hermitian line bundle \( L(r) \) to \( r \). Let \( L \) be the fractional ideal of \( F \) such that for every non-Archimedean \( u \), \( L_u := L \otimes \mathcal{O}_F \mathcal{O}_{F_u} \) is generated by some \( a_u \in F_u \) with \( |a_u|_u = r_u \). We put Hermitian norms \( \|\|_r \) on \( L \otimes \mathcal{O}_{F,\sigma} \mathbb{C} \) such that \( \|1_F \otimes 1\|_r = r^{-1}_\sigma \). Let \( \mathcal{L}(r) := (L, |\cdot|_r) \). It follows from the definitions that
\[ h^0_{\text{Ar}}(\mathcal{L}(r)) = \log \# \{ x \in F : |x|_u \leq r_u \text{ for all } u \in \mathcal{V}_F \}. \]

Take any \( s \in L \). For any non-Archimedean \( u \in \mathcal{V}_F \), let \( p \) be the associated prime ideal in \( \mathcal{O}_F \). By definition we have \( \mathcal{O}_p s = \mathfrak{m}_p^{v_p(s)} L_p = \mathfrak{m}_p^{v_p(s)} a_u \). Taking \( |\cdot|_u \) on both sides, we get \( v_p(s) = \log N_{p,F}|s^{-1}a_u|_u \). By (3.2) and the product formula,
\[ \deg(L(r)) = \sum_{0 \neq p \in \text{Spec} \mathcal{O}_F} v_p(s) \log N_{p,F} - \sum_{\sigma : F \to \mathbb{C}} \log \|s\|_\sigma \]
\[ = \sum_{u < \infty} \log |s^{-1}a_u| - \sum_{u \in \mathcal{V}_F} \log (r_u^{-1}|s|_u) = \sum_{u \in \mathcal{V}_F} - \log (r_u^{-1}|s|_u) \]
\[ = \sum_{u \in \mathcal{V}_F} \log r_u - \sum_{u \in \mathcal{V}_F} \log |s|_u = \sum_{u \in \mathcal{V}_F} \log r_u = \log \|r\|. \]

\(^4\)We take the normalized absolute value as defined in the beginning of Section 2.1.
Let $C = \exp(f(-\log c) + \pi n)$, where $f$ is defined in (3.12). We apply Corollary 3.5 to conclude the proof. □

4. A Linnik-type result and uniformity over the base field

The aim of this section is to prove Linnik’s basic lemma in Theorem 1.7. Therefore, we fix the notation of Theorem 1.7 and recall it here for convenience of the reader. Let $F$ be a number field of degree $n$ over $\mathbb{Q}$ and consider $\text{GL}_2$ as an $F$-group. For any finite place $u$ of $F$ we let $B_u = \text{GL}_2(O_{F,u})$. For any Archimedean place $u$ we set

$$B_u = \{A \in \text{GL}_2(F_u) : \max\{\|A\|, \|A^{-1}\|\} \leq r_u\}$$

using the notation in §2.3.2 where $r_u > 1$ is fixed. Let $B = \prod_{u \in V_F} B_u$.

Let $S$ be a finite set of places of $F$. For each $v \in S$, let $A_v$ denote a split $F_v$-torus of $\text{GL}_2$, and let $A_v = A_v(F_v)$ be its group of $F_v$-points. Let $A_S := \prod_{v \in S} A_v$. We fix $a = (a_v)_{v \in S} \in A_S$, and let $\alpha_v(a_v) \in F_v$ denote the unique eigenvalue of $\text{Ad} a_v$ whose absolute value is $\geq 1$. Let $\alpha(a) := (\alpha_v(a_v))_{v \in S} \in F_S$. For any integer $\tau \geq 1$ we define the $\tau$-Bowen ball

$$B_\tau = \bigcap_{-\tau \leq t \leq \tau} a^{-t}B_a^t.$$

For convenience, we also define $B_{u,\tau} = \bigcap_{-\tau \leq t \leq \tau} a_u^{-t}B_a^t$ for any place $u$ of $F$.

4.1. Geometric invariant theory. Let $T < \text{GL}_2$ be a torus defined over $F$. Let $A < \text{GL}_2$ be the diagonal subgroup. In geometric invariant theory, the universal categorical quotient for the $T \times T$ action on $\text{GL}_2$ given by left- and right multiplication (the latter with an inverse) is representable by

$$T \backslash \backslash \text{GL}_2// T := \text{Spec } F[\text{GL}_2]^{T \times T}$$

This is an affine variety defined over $F$ by a result of Hilbert, as $T \times T$ is reductive. Here, $F[\text{GL}_2]^{T \times T}$ is the ring of bi-$T$-invariant regular functions on $\text{GL}_2$. We denote by $\pi_T$ the canonical morphism

$$\pi_T : \text{GL}_2 \to T \backslash \backslash \text{GL}_2// T.$$

The fiber of any point under $\pi_T$ contains a unique $T \times T$-closed orbit. If the image $\overline{T}$ of $T$ in $\text{PGL}_2$ is anisotropic over $F$, work of Kempf [Kem78] implies that the fiber of any point $\pi_T(\gamma)$ with $\gamma \in \text{GL}_2(F)$ is closed (see [Kha19a, Prop. 3.6]). This together with Hilbert’s Theorem 90 implies that the induced map

$$\pi_T : T(F)\backslash \text{GL}_2(F)/ T(F) \to (T \backslash \backslash \text{GL}_2// T)(F)$$

is injective – see Lemma 4.3.
4.1.1. *An explicit generator.* We give an explicit realization of \( T \backslash \mathrm{GL}_2 \mod T \) as the affine line over \( F \) by exhibiting a generator for \( F[\mathrm{GL}_2]^{T \times T} \).

Consider first the diagonal subgroup \( A < \mathrm{GL}_2 \) and write \( x_{ij} \) with \( i, j \in \{1, 2\} \) for the coordinate functions on \( M_2 \). The regular function
\[
\psi = \frac{x_{12} x_{21}}{\det}
\]
is a generator for \( F[\mathrm{GL}_2]^{A \times A} \) (cf. [Kha19a, §4]).

For an arbitrary maximal \( F \)-torus \( T < \mathrm{GL}_2 \) let \( K \) be the splitting field and let \( g \in \mathrm{GL}_2(K) \) be such that \( \text{Ad}_g(T) = A \). Then
\[
\psi_T = \psi \circ \text{Ad}_g
\]
is a generator for \( K[\mathrm{GL}_2]^{T \times T} \). In fact, one can show that \( \psi_T \) is defined over \( F \) (cf. [Kha19a, §5]). For illustration (in particular in view of later purposes), we shall give an example.

*Example 4.1.* Let \( D \in F^\times \) be such that \( x^2 - D \) is irreducible. In this case, the centralizer \( T \) of
\[
X_D = \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}
\]
in \( \mathrm{GL}_2 \) is a non-split \( F \)-torus. Explicitly, any point in \( T(F) \) is of the form
\[
\begin{pmatrix} a & b \\ b D & a \end{pmatrix}
\]
where \( a, b \in F \). The splitting field of \( T \) is the splitting field of the characteristic polynomial of \( X_D \), that is, \( K = F[\sqrt{D}] \simeq F[X]/(X^2 - D) \). The matrix
\[
g = \begin{pmatrix} 1 & 1 \\ \sqrt{D} & -\sqrt{D} \end{pmatrix}^{-1} \in \mathrm{GL}_2(K)
\]
satisfies \( \text{Ad}_g(T) = A \). It is easy to see that for \( \gamma \in M_2(F) \)
\[
g \gamma g^{-1} = \begin{pmatrix} b_1 & b_2 \\ \sigma(b_2) & \sigma(b_1) \end{pmatrix}
\]
where \( \sigma \) is the non-trivial Galois automorphism of \( K/F \) and \( b_1, b_2 \in K \). In particular, \( \psi_T(\gamma) = \frac{1}{\det(\gamma)} \text{Nr}_{K/F}(b_2) \) is in \( F \).

More generally, for an \( F \)-torus \( T < \mathrm{GL}_2 \) denote by \( t \subset M_2 \) its Lie-algebra. For any non-zero traceless \( X \in t(F) \) we have
\[
T = \{ g \in \mathrm{GL}_2 : \text{Ad}_g X = X \}.
\]
Observe that \( X^2 = -\det(X) \in F^\times \) and if \( T \) is non-split over \( F \), \( K = F(\sqrt{D}) \) for \( D = -\det(X) \) is a quadratic extension with \( T \simeq \text{Res}_{K/F}(\mathbb{G}_m) \). If \( v \in K^2 \) is an eigenvector of \( X \), so is \( \sigma(v) \) when \( \sigma \in \text{Gal}(K/F) \setminus \{ \text{id} \} \). In particular, setting \( g \in \mathrm{GL}_2(K) \) to be the inverse of the matrix with columns \( v, \sigma(v) \) the traceless matrix \( \text{Ad}_g(X) \) is diagonal and \( \text{Ad}_g(T) = A \). By definition, \( g \) satisfies that \( \sigma(g) = w_{(12)} g \).
where \( w_{(12)} \) is the permutation matrix for the transposition \((1\,2)\). In particular, for any \( \gamma \in \text{GL}_2(F) \) the matrix \( g\gamma g^{-1} \) is of the form

\[
g\gamma g^{-1} = \begin{pmatrix} b_1 & b_2 \\ \sigma(b_2) & \sigma(b_1) \end{pmatrix}
\]

for some \( b_1, b_2 \in K \). We remark that this property does not hold for an arbitrary \( g \in \text{GL}_2(F) \) with \( \text{Ad}_g(T) = A \). The special choice of \( g \in \text{GL}_2(F) \) is unique up to left-multiples with diagonal matrices having entries in \( F^\times \).

Let \( Z \) be the center of \( \text{GL}_2 \). Let \( \Delta : T \hookrightarrow T \times T \) be the diagonal embedding.

**Lemma 4.2.** Let \( \gamma \in \text{GL}_2(F) \) and define \( \text{Stab}_{T \times T}(\gamma) \subset T \times T \) to be the stabilizer of \( \gamma \) under the action of \( T \times T \) on \( \text{GL}_2 \).

(i) If \( \gamma \in T(F) \setminus Z(F) \), then \( \text{Stab}_{T \times T}(\gamma) = \Delta(T) \simeq T \).

(ii) If \( \gamma \in \mathbb{N}_T(F) \setminus T(F) \), \( \text{Stab}_{T \times T}(\gamma) \simeq T \) via the embedding \( T \hookrightarrow T \times T \) mapping a geometric point \( t \) to \( (t, \gamma^{-1}t\gamma) \). Here, \( \mathbb{N}_T < \text{GL}_2 \) is the normalizer of \( T \).

(iii) If \( \gamma \notin \mathbb{N}_T(F) \), the stabilizer \( \text{Stab}_{T \times T}(\gamma) \) is \( \Delta(Z) \).

We remark that imposing the additional condition \( \psi_T(\gamma) \neq 1 \) rules out Case (ii).

**Proof.** As in the discussion preceding the lemma, let \( K/F \) be the quadratic extension associated to \( T \) and let \( g \in \text{GL}_2(K) \) be such that

\[
g\gamma g^{-1} = \begin{pmatrix} b_1 & b_2 \\ \sigma(b_2) & \sigma(b_1) \end{pmatrix}
\]

for some \( b_1, b_2 \in K \) and the non-trivial Galois automorphism \( \sigma \) of \( K/F \). Case (i) corresponds to \( b_2 = 0 \). Case (ii) to \( b_1 = 0 \), and Case (iii) to \( b_1 \neq 0 \neq b_2 \). Let \((t_1, t_2) \in \text{Stab}_{T \times T}(\gamma)\) and set \( \alpha_i = \text{Ad}_g(t_i)_{11} \) for \( i = 1, 2 \). By assumption on \((t_1, t_2)\) we have

\[
b_1 \frac{\alpha_1}{\alpha_2} = b_1, \quad b_2 \frac{\alpha_1}{\sigma(\alpha_2)} = b_2.
\]

If \( b_1 \neq 0 \neq b_2, \alpha_1 = \alpha_2 = \sigma(\alpha_2) = \sigma(\alpha_1) \) so that \( t_1, t_2 \) must be trivial. The remaining cases follow similarly. \( \square \)

**Lemma 4.3.** For any non-split \( F \)-torus \( T < \text{GL}_2(F) \) the map

\[
\pi_T : T(F) \setminus \text{GL}_2(F)/T(F) \to (T \setminus \text{GL}_2/F)(F)
\]

is injective.

**Proof.** Let \( \gamma_1, \gamma_2 \in \text{GL}_2(F) \) be such that \( \pi_T(\gamma_1) = \pi_T(\gamma_2) \). By the work of Kempf [Kem78] mentioned already at the beginning of §4.1 this implies that \( \gamma_2 \in T \gamma_1 T \). It remains to show that \( \gamma_2 \in T(F)\gamma_1 T(F) \). If \( \gamma_1 \in \mathbb{N}_T(F) \), this is clear. So suppose that \( \gamma_1 \notin \mathbb{N}_T(F) \), so that the stabilizer for \( \gamma_1 \) under the \( T \times T \)-action is \( \Delta Z \) by Lemma 4.2. Let \( t_1, t_2 \in T(F) \) be two points defined over the algebraic closure \( \overline{F} \) so that \( \gamma_2 = t_1 \gamma_1 t_2^{-1} \). Therefore, for any \( \sigma \in \text{Gal}(\overline{F}/F) \),

\[
t_1 \gamma_1 t_2^{-1} = \gamma_2 = \sigma(\gamma_2) = \sigma(t_1) \gamma_1 \sigma(t_2)^{-1},
\]
which implies that
$$(t_1/\sigma(t_1), t_2/\sigma(t_2)) \in \Delta Z(F).$$

Write $s_\sigma = t_1/\sigma(t_1) \in Z(F)$. As the Galois cohomology $H^1(\text{Gal}(F/F), Z(F))$ is trivial, there exists $s' \in Z(F)$ so that $s_\sigma = s'/\sigma(s')$ for all $\sigma \in \text{Gal}(F/F)$. In particular, $t_1' = t_1/s'$ and $t_2' = t_2/s'$ are $F$-rational points with
$$t_1' \gamma_1(t_2')^{-1} = t_1 \gamma_1 t_2^{-1} = \gamma_2,$$
which proves the lemma. \hfill \Box

4.2. Volume, discriminant, and local bounds on invariants. In this subsection, we consider fixed a homogeneous toral set $Y = [T_\gamma]$ and introduce certain local coordinates on $\text{GL}_2(F_u)$ for any place $u$ of $F$ that will be used later on. In particular, we obtain local bounds on the denominator of invariants. The following proposition is technical, but very important in the rest of the argument for Theorem 1.7; we recommend reading it first in the non-Archimedean case only. We also remark that this discussion already appears in [Kha19b, §5] in a more general context.

**Proposition 4.4 (Local coordinates).** Let $Y \subset [\text{GL}_{2,F}]$ be homogeneous toral set $Y$ and let $\mathcal{O} \subset K$ be the associated order resp. field (cf. §2.3.1). Let $u$ be a place of $F$ and let $\sigma$ be the non-trivial automorphism of $K_u$ which fixes $F_u$ pointwise. Then there exists $c_{Y,u} \in \text{GL}_2(K_u)$ with the following properties:

1. If $u$ is a non-Archimedean place, then
$$\Delta_{\mathcal{O},u} c_{Y,u}, c_{Y,u}^{-1} \in M_2(\mathcal{O}_{K,u}).$$
If $u$ is Archimedean, then for any place $w$ of $K$ above $u$
$$\|c_{Y,w}\| \ll \text{disc}(Y)^{-\frac{1}{2}}, \|c_{Y,w}^{-1}\| \ll 1.$$
2. For any place $w$ of $K$ above $u$ we have $c_{Y,u} g_w^{-1}T g_{c_{Y,u}} = A$ as $K_w$-tori.
3. For any $\gamma \in \text{GL}_2(F_u)$ the conjugate $c_{Y,u} \gamma c_{Y,u}^{-1} \in \text{GL}_2(K_u)$ is of the form
$$\begin{pmatrix} b_{1,u} & b_{2,u} \\ \sigma(b_{2,u}) & \sigma(b_{1,u}) \end{pmatrix}$$
where $b_{1,u}, b_{2,u} \in K_u$. We call the pair $(b_{1,u}, b_{2,u})$ the local coordinates of $\gamma$ (relative to $Y$).
4. If $u$ is non-Archimedean, the local coordinates $(b_{1,u}, b_{2,u})$ of any $k \in \text{GL}_2(\mathcal{O}_{F,u})$
satisfy
- $b_{1,u}, b_{2,u} \in \frac{1}{\Delta_{\mathcal{O}_u}} \mathcal{O}_u$,
- $b_{1,u} - b_{2,u} \in \mathcal{O}_u$,
- $\text{Tr}_{K_u/F_u}(b_{1,u}), \text{Tr}_{K_u/F_u}(b_{2,u}) \in \mathcal{O}_{F,u}$, and
- $\text{Nr}_{K_u/F_u}(b_{1,u}) - \text{Nr}_{K_u/F_u}(b_{2,u}) \in \mathcal{O}^*_F$. 


If \( u \) is Archimedean and \( w \mid u \), then the local coordinates \((b_{1,u}, b_{2,u})\) of any \( k \in B_u \) satisfy
\[
\max\{|b_{1,w}|, |b_{2,w}|\} \ll r_u \text{disc}_u(Y) \frac{1}{2} \quad \text{and} \quad |b_{1,w} - b_{2,w}| \ll r_u.
\]
(5) If \( u \) is non-Archimedean and the local coordinates \((b_{1,u}, b_{2,u})\) of \( \gamma \in \text{GL}_2(F_u) \) satisfy the properties in (4), then \( \gamma \in \text{GL}_2(\mathcal{O}_{F,u}) \).

We prove Proposition 4.4 separately in the Archimedean and the non-Archimedean case beginning with the latter. The proof in the non-Archimedean case is in essence contained in [Kha19a, Proposition 7.4]; we include a proof here for the convenience of the reader and because it is arguably more concrete for quadratic extensions.

**Proof in the non-Archimedean case.** We aim to imitate the situation in Example 4.1. Let \( u \) be a non-Archimedean place of \( F \). Let \( C_u \) be the centralizer of \( g_u^{-1}T(F_u)g_u \) in \( \text{M}_2(F_u) \); We identify \( C_u \simeq K_u \) by conjugation with \( g_u \). By assumption, we have \( \mathcal{O}_u = C_u \cap \text{M}_2(\mathcal{O}_{F,u}) \). There exists \( v \in F_u^2 \) such that
\[
\theta : C_u \rightarrow F_u^2, \quad x \mapsto vx
\]
is an isomorphism of \( C_u\)-modules. In fact, all points in \( v \in F_u^2 \) outside of (at most) two lines have this property.

We write \( a = \theta^{-1}(\mathcal{O}_{F,u}^2) \) which is a proper \( \mathcal{O}_u\)-ideal as \( \mathcal{O}_u = C_u \cap \text{M}_2(\mathcal{O}_{F,u}) \). By Proposition 2.2, \( a \) is principal and hence there exists \( \lambda \in C_u \) with \( a = \lambda \mathcal{O}_u \).

Tautologically, the embedding \( C_u \hookrightarrow \text{M}_2(F_u) \) arises by representing multiplication by elements in \( C_u \) in the basis \( b_i = \theta^{-1}(e_i) \) of \( a \) where \( e_i \) is the standard basis. Changing this basis of \( a \) conjugates \( C_u \) by an element of \( \text{GL}_2(\mathcal{O}_{F,u}) = B_u \). In view of the proposition, this conjugation has no effect and we may assume that the basis \( b_i \) is a basis of our choosing.

By Proposition 2.2, \( \mathcal{O}_u \) is monogenic over \( \mathcal{O}_{F,u} \) i.e. there exists \( \alpha \in \mathcal{O}_{C,u} \) with \( \mathcal{O}_u = \mathcal{O}_{F,u}[\alpha] \). Consider the basis
\[
b_1 = \lambda, \ b_2 = \lambda \alpha.
\]

Multiplication by \( \alpha \) in the basis \( b_1, b_2 \) in (4.2) is represented by
\[
\begin{pmatrix}
0 & 1 \\
-\text{Nr}_{C_u/F_u}(\alpha) & \text{Tr}_{C_u/F_u}(\alpha)
\end{pmatrix}.
\]
Define
\[
c_{Y,u} = \begin{pmatrix} 1 & 1 \\ \alpha & \sigma(\alpha) \end{pmatrix}^{-1}.
\]
Note that both \( c_{Y,u}^{-1} \) and \( \Delta_{\mathcal{O}_u} c_{Y,u} \) are in \( \text{M}_2(\mathcal{O}_u) \), as \( \det(c_{Y,u}^{-1}) = \sigma(\alpha) - \alpha \) is a generator of the different ideal by Lemma 2.3. Without loss of generality we may assume that \( \Delta_{\mathcal{O}_u} = \sigma(\alpha) - \alpha \). We also note that the local absolute discriminant \( \text{disc}_u(Y) \) is related to \( \Delta_{\mathcal{O}_u} \) and \( \alpha \) in the following way:
\[
\text{disc}_u(Y) = |\text{Nr}_{K_u/F_u}(\Delta_{\mathcal{O}_u})|_u^{-1} = |2 \text{Nr}_{K_u/F_u}(\alpha) - \text{Tr}_{K_u/F_u}(\alpha^2)|_u^{-1}.
\]
The claims in the proposition are now verified by direct calculation. \( \square \)
Proof in the Archimedean case. The proof consists mostly of brute force calculations. Suppose in the following that \( u \) is Archimedean and let \( w \mid u \) be a place of \( K \). Observe that \( K_w = F_u \). Let \( E_u \subset M_n \) be the centralizer of \( g_u^{-1}T(F_u)g_u \) and let \( f \in E_u(F_u) \) be non-zero with \( \text{Tr}(f) = 0 \) and \( \|f\| = 1 \). Write

\[
f = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.
\]

and let \( \pm \alpha \in K_w \) be the eigenvalues of \( f \). In view of Example 2.1 we have \( \text{disc}_u(Y) = \frac{1}{4|\alpha|^2} \). Furthermore, \( \|f\| = 1 \) implies \( |\alpha| = \sqrt{|\det(f)|} \leq \frac{1}{\sqrt{2}} \).

In view of (2) we wish to diagonalize \( f \). From \( \|f\| = 1 \) we know that one of \( |a|, |b|, |c| \) is at least \( \frac{1}{2} \). By conjugating \( f \) with a unipotent matrix first if necessary, we may suppose that \( |b| \geq \frac{1}{2} \) (the case \( |c| \geq \frac{1}{2} \) is analogous). Set

\[
c_{Y,w} = \begin{pmatrix} b & b \\ \alpha - a & -\alpha - a \end{pmatrix}^{-1}.
\]

We have \( c_{Y,w}f c_{Y,w}^{-1} = \text{diag}(\alpha, -\alpha) \), and thus (2) holds. The property in (1) is also readily verified.

For any \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_2(F_u) \), we explicitly calculate

\[
c_{Y,w} \gamma c_{Y,w}^{-1} = \begin{pmatrix} \frac{a+\alpha}{2\alpha}A + \frac{b}{2\alpha}C + \frac{c}{2\alpha}B + \frac{\alpha-a}{2\alpha}D & \frac{\alpha+\alpha}{2\alpha}A + \frac{b}{2\alpha}C - \frac{(\alpha-a)^2}{2\alpha}B - \frac{\alpha+a}{2\alpha}D \\
\frac{\alpha-a}{2\alpha}A - \frac{b}{2\alpha}C + \frac{(\alpha-a)^2}{2\alpha}B - \frac{\alpha+a}{2\alpha}D & \frac{\alpha+a}{2\alpha}A - \frac{b}{2\alpha}C - \frac{\alpha-a}{2\alpha}B + \frac{\alpha+a}{2\alpha}D \end{pmatrix}.
\]

If \( u \) is non-split i.e. \( w \) is the unique place above \( u \), then \( \sigma(\alpha) = -\alpha \), and (3) can be checked easily. If \( u \) is split i.e. \( w_1, w_2 \) are two places above \( u \), then in the above construction for \( c_{Y,w} \), we use \( \alpha \) and \( -\alpha \) for \( c_{Y,w_1} \) and \( c_{Y,w_2} \) respectively, and one can again check that (3) still holds. 5

Finally, suppose \( \gamma = k \in B_u \). We have \( \max\{|A|, |B|, |C|, |D|\} \leq r_u \). By our assumption, we have \( \frac{1}{|2\alpha|} = \text{disc}_u(Y)^{\frac{1}{2}} \), \( |a| \leq 1 \), \( \frac{1}{2} \leq |b| \leq 1 \), \( |c| \leq 1 \) and \( |\alpha| \leq 1 \). Note that \( b_{1,w} - b_{2,w} = \frac{a+\alpha}{b}B + D \). Combining the above, one gets the desired bounds for \( |b_{1,w}|, |b_{2,w}| \) and \( |b_{1,w} - b_{2,w}| \). \( \square \)

**Corollary 4.5** (Bounds on invariants). Let \( Y \subset [\text{GL}_{2,F}] \) be a homogeneous toral set and let \( u \) be a place of \( F \). Then for any \( k \in B_u \) we have

\[
|\psi^T(g_u k g_u^{-1})|_u \leq \kappa_u \text{disc}_u(Y), \quad |1 + \psi^T(g_u k g_u^{-1})|_u \leq \kappa_u \text{disc}_u(Y).
\]

where the constants satisfy \( \kappa_u = 1 \) if \( u \) is non-Archimedean, and \( \kappa_u \ll n r_u^4 \) if \( u \) is Archimedean.

Moreover, if \( u \in S \) then for any \( k \in B_{u,\tau} \)

\[
|\psi^T(g_u k g_u^{-1})|_u \leq \kappa_u \text{disc}_u(Y) |\alpha_u(a_u)|^{-2\tau}.
\]

Observe that assuming maximal type for the homogeneous toral set would imply that \( \text{disc}_u(Y) = 1 \) for places \( u \in V_F \) that are inert or split in the associated field and slightly simplify the proof below.

---

5Since \( \sigma \) switches \( w_1 \) and \( w_2 \) in this case.
Proof. Let \( k \in B_u \) and let \((b_{1,u}, b_{2,u})\) be its local coordinates from Proposition 4.4. The bound in (4.3) follows from
\[
\psi^T(g_u k u^{-1}) = \det(k)^{-1} \text{Nr}_{K_u/F_u}(b_{2,u}), \quad 1 + \psi^T(g_u k u^{-1}) = \det(k)^{-1} \text{Nr}_{K_u/F_u}(b_{1,u})
\]
and the denominator bounds in Proposition 4.4(4).

So suppose from now on that \( u \in S \). In this case, \( K_u \simeq F_u \times F_u \) and \( c_{Y,w} \in \text{GL}_2(F_u) \) for some \( w \in V_K \) with \( w \mid u \) satisfies that \( c_{Y,w} b_{1,u} c_{Y,w}^{-1} = \text{diag}(t_1, t_2) \) where \((t_1, t_2), (0,0)\) is the local coordinate of \( b_{1,u} \). In particular, \( \alpha_u(a_u) \in \{\frac{t_1}{t_2}, \frac{t_2}{t_1}\} \).

Furthermore, write \( b_{2,u} = (x_1, x_2) \) so that
\[
\psi^T(g_u k u^{-1}) = \det(k)^{-1} x_1 x_2. \tag{4.5}
\]

Assume that \( u \) is Archimedean. Then \( |x_1|, |x_2| \ll r_u \text{disc}_u(Y)^{\frac{1}{2}} \) by Proposition 4.4 and by (4.5) it suffices to show that
\[
|x_1|, |x_2| \ll r_u \text{disc}_u(Y)^{\frac{1}{2}} |\alpha_u(a_u)|^{-\tau}.
\]

To do so, note that for any \( t \in \{-\tau, \ldots, \tau\} \) the requirement \( a_u^{\tau} t \in B_u \) implies that \(|(t_1/t_2)^i x_1| \ll r_u \text{disc}_u(Y)^{\frac{1}{2}}x_1| \ll r_u \text{disc}_u(Y)^{\frac{1}{2}} \alpha_u(a_u)|^{-\tau} \) as desired (for \( x_2 \) one proceeds analogously).

Assume that \( u \) is non-Archimedean. Under the isomorphism \( K_u \simeq F_u \times F_u \), the order \( O_u \) takes the form \( O_u = O_{F,u}(1,1) + O_{F,u}(c_1, c_2) \) where \( c_1 \neq c_2 \). Such an order is automatically Galois invariant. Furthermore, set \( \Delta = \Delta_{O_u} = (\Delta_1, \Delta_2) \) where \( \Delta_1 = -\Delta_2 = c_2 - c_1 \). As in the Archimedean case, the requirement \( a_u^{\tau} t \in B_u \) for any \( t \in \{-\tau, \ldots, \tau\} \) implies that
\[
|(t_1/t_2)^i x_1|, (t_2/t_1)^i x_2) \in \frac{1}{\Delta} \text{O}_u.
\]

In particular, we have \( |(t_1/t_2)^i x_1| \leq |\Delta_i|^{-1} \) which implies that \( |x_i| \leq |\Delta_i|^{-1} |\alpha_u(a_u)|^{-\tau} \) when choosing \( t \in \{-\tau, \tau\} \) appropriately. Taking the product
\[
|\psi^T(g_u k u^{-1})|_u \leq |\Delta_1 \Delta_2|^{-1} |\alpha_u(a_u)|^{-2\tau} = |\text{Nr}(\Delta)|_{u}^{-1} |\alpha_u(a_u)|^{-2\tau}
\]
proving the corollary. \( \square \)

4.3. Geometric expansion for the Bowen kernel. Let \( f \) be the characteristic function of \( B \) and \( f_\tau \) for an integer \( \tau \geq 0 \) be the characteristic function of \( B_\tau \). Note that \( f_0 = f \).

We define the kernel (henceforth sometimes called \( \tau \)-Bowen kernel)
\[
K_\tau(x, y) = \sum_{\gamma \in \text{GL}_2(F)} f_\tau(x^{-1} \gamma y)
\]
for all \( x, y \in \text{GL}_2(\mathbb{A}_F) \). Note that \( K_\tau \) defines a function on \([\text{GL}_2,F] \) and that for any pair of points \( x, y \) the above sum is finite (though the number of non-zero terms grows towards the cusp).

Throughout this section, we shall consider fixed a homogeneous toral set \( Y = G(F) T(\mathbb{A}_F) \). We let \( K \) be the associated field (cf. §2.2.1) and \( \mathcal{O} \subset K \) be the associated order (cf. §2.3.1). Note that we do not impose a maximal type assumption yet (i.e. potentially \( \mathcal{O} \not\subset \mathcal{O}_K \)). Let \( dx \) etc denote the 'standard' Haar measure
on $g^{-1}T(A_{F})^{-1}g$ or $T(A_{F})^{-1}$ i.e. the one that gives measure one to $g^{-1}Bg$ or $B$ respectively. Recall that $\mu_{Y}$ is the invariant probability measure on $Y$ and that $\tilde{\mu}_{Y}$ is its lift to $g^{-1}T(A_{F})^{-1}g$.

It is straightforward to verify that

$$\mu_{Y} \times \mu_{Y} \left( \{(x, y) : y \in xB_{T}\} \right) \leq \int_{Y} \int_{Y} K_{T} \, d\mu_{Y}(x) \, d\mu_{Y}(y);$$

we shall estimate the latter expression. To do so, expand

$$\int_{Y} \int_{Y} K_{T}(x, y) \, d\mu_{Y}(x) \, d\mu_{Y}(y) = \int_{[T]^{2}} K_{T}(xg, yg) \, d\mu_{T}(x) \, d\mu_{T}(y)$$

$$= \sum_{\gamma \in T(F) \setminus GL_{2}(F) / T(F)} \sum_{\eta \in T(F) / T(F)} f_{T}(g^{-1}x^{-1} \eta yg) \, d\mu_{T}(x) \, d\mu_{T}(y) \quad \text{(4.7)}$$

We now analyze the above sum; for $\gamma \in T(F)$, $\gamma \in N_{T}(F) \setminus T(F)$ and $\gamma \notin N_{T}(F)$ respectively where the last case is the most interesting one. We will use the bounds in Lemma 4.6 to control the set of $\gamma$’s for which the above integral is non-zero and then control each of the integrals.

4.3.1. The identity contribution. When $\gamma$ is the trivial coset of $T(F) \setminus GL_{2}(F) / T(F)$, the corresponding integral in (4.7) is by unfolding

$$\int_{[T]^{2}} \sum_{\eta \in T(F)} f_{T}(g^{-1}x^{-1} \eta yg) \, d\mu_{T}(x) \, d\mu_{T}(y)$$

$$= \int_{[T]} \int_{T(A_{F})^{-1}} f_{T}(g^{-1}x^{-1} yg) \, d\tilde{\mu}_{T}(y) \, d\mu_{T}(x)$$

$$= \int_{[T]} \int_{T(A_{F})^{-1}} f_{T}(g^{-1}yg) \, d\tilde{\mu}_{T}(y) \, d\mu_{T}(x)$$

$$= \int_{T(A_{F})^{-1}} f_{T}(g^{-1}yg) \, d\tilde{\mu}_{T}(y)$$

Observe that $gB_{T}g^{-1}$ is also a Bowen ball, namely for time parameter $\tau$, transformation $gag^{-1}$, and open set $gBg^{-1}$. If $x \in gBg^{-1} \cap T(A_{F})^{-1}$, then $x \in ga^{-t}B_{t}a'g^{-1}$ for all $t \in \mathbb{Z}$ as $x$ commutes with $gag^{-1}$ and so $x \in gBg^{-1}$ for all $\tau \geq 0$. Therefore,

$$\int_{T(A_{F})^{-1}} f_{T}(g^{-1}yg) \, d\mu_{T}(y) = \int_{T(A_{F})^{-1}} f(g^{-1}yg) \, d\tilde{\mu}_{T}(y)$$

$$= \tilde{\mu}_{T}(gBg^{-1}) = \frac{1}{\text{vol}(Y)}.$$
$N_T(F)$ be a representative of the non-trivial coset. Estimating as in §4.3.1 the contribution of the non-trivial coset in $N_T(F)$ to (4.7) is bounded by

$$
\int_{[T]^2} \sum_{\eta \in T(F)} f(g^{-1}x^{-1} \gamma y g) \, d\mu_T(x) \, d\mu_T(y)
$$

$$
= \int_{[T]} \int_{T(A_F)^\perp} f(g^{-1}x^{-1} \gamma y g) \, d\mu_T(y) \, d\mu_T(x)
$$

$$
= \int_{T(A_F)^\perp} f(g^{-1} \gamma y g) \, d\mu_T(y)
$$

by substitution. Replacing the coset of $\gamma$ within $N_T(A_F)^1 / T(A_F)^1$ we may suppose that its local coordinates in the sense of Proposition 4.4 are $(0, 1)$ (see also the proof of Lemma 4.2). Now fix $y$ in the support of the integrand and let $(t_u, 0)$ be its local coordinates. As $\gamma y \in gB^{-1}$ we know from Proposition 4.4(4) that $t_u \in O_u$ for every finite place $u$ and $|t_u| \ll r_u$ for every Archimedean place $w | u$. The measure of such $y$’s with respect to $\mu_T$ is $\ll \frac{1}{\text{vol}(Y)}$.

To summarize, §4.3.1 and §4.3.2 together show that

$$
\sum_{\gamma \in T(F)^{N_T(F)} \backslash T(F)} \int_{[T]^2} \sum_{\eta \in T(F) \gamma T(F)} f_T(g^{-1}x^{-1} \eta y g) \, d\mu_T(x) \, d\mu_T(y) \ll \frac{1}{\text{vol}(Y)}.
$$

(4.8)

4.3.3. Generic contributions: Using estimates on invariants. In view of the geometric expansion in (4.7) and the estimate of the non-generic contributions in (4.8) above it remains to estimate

$$
\sum_{\gamma \in T(F)^{\text{GL}_2(F) \backslash T(F)}} \int_{[T]^2} \sum_{\eta \in T(F) \gamma T(F)} f_T(g^{-1}x^{-1} \eta y g) \, d\mu_T(x) \, d\mu_T(y)
$$

$$
= \frac{1}{\text{vol}(Y)^2} \sum_{\gamma} \int_{\Delta(Z)(F)(T(A_F)^1)^2} f_T(g^{-1}x^{-1} \gamma y g) \, dx \, dy
$$

$$
= \frac{\text{vol}([Z])}{\text{vol}(Y)^2} \sum_{\gamma} \int_{\Delta(Z)(A_F)(T(A_F)^1)^2} f_T(g^{-1}x^{-1} \gamma y g) \, dx \, dy
$$

(4.9)

Here, we used that the stabilizer for the $T \times T$-action of any $\gamma \notin N_T(F)$ is $\Delta(Z)$, where $Z$ is the center of $G$ and $\Delta: Z \to Z \times Z$ is the diagonal embedding (cf. Lemma 4.2).

We now estimate the number of $\gamma$’s for which the above integral does not vanish. Write

$$
r := \max\{r_u : u \in V_{F, \infty}\}
$$

(4.10)

which is a constant depending only on $B_\infty$. We call $r$ the diameter of $B_\infty$.

The following lemma is crucial in the proof of Theorem 1.7; it estimates the number of non-zero contributions to the geometric expansion in (4.7) using the counting results of §3.
Lemma 4.6. Let $c$ be a positive real number. Suppose that $\text{disc}(Y)(\alpha(a)_S)^{-2\tau} \geq cD_F$. The number of $\gamma \in T(F)\backslash \text{GL}_2(F)/T(F)$ for which the integral in (4.9) does not vanish is $\ll_{n,c} r^{4n} \frac{\text{disc}(Y)}{D_F^{1/2}} |\alpha(a)|_S^{-2\tau}$.

Proof. The proof consists of putting together already proven estimates. Let $\gamma$ be as in the lemma. In particular, there exist $x, y \in T(\mathbb{A}_F)$ with $k := x^{-1}\gamma y \in gB_g^{-1}$. Note that by definition of the invariant $\psi_T$ we have $\psi_T(k_u) = \psi_T(\gamma)$ for any place $u$. In particular, Corollary 4.5 implies that for any $u \notin S$

$$|\psi_T(\gamma)|_u \leq \kappa_u \text{disc}_u(Y),$$

(4.11)

where $\kappa_u$ is as in Proposition 4.4. Moreover, for any place $u \in S$ (i.e. a place with contraction) we have $|\psi_T(\gamma)|_u \leq \kappa_u \text{disc}_u(Y)|\alpha(a_u)|_u^{-2\tau}$. Putting these estimates together and using Lemma 3.6 we obtain that

$$\#\{\psi_T(\gamma) : \gamma \text{ as in the lemma}\} \ll_{n,c} \prod_{u \in V_F} \kappa_u \cdot \frac{\text{disc}(Y)}{D_F^{1/2}} |\alpha(a)|_S^{-2\tau} \ll_{n} r^{4n} \frac{\text{disc}(Y)}{D_F^{1/2}} |\alpha(a)|_S^{-2\tau},$$

As $\psi_T$ is injective on $T(F)\backslash \text{GL}_2(F)/T(F)$ (see (4.1)), the lemma follows.

4.3.4. Estimates for torus integrals. Let $\gamma \in \text{GL}_2(F) \setminus N_T(F)$ and recall that the stabilizer of $\gamma$ (under the $T \times T$-action) is the diagonal embedded copy $\Delta Z < T \times T$ of the center $Z < \text{GL}_2$. In view of Lemma 4.6, it suffices to estimate

$$I_\gamma := \int_{\Delta Z(\mathbb{A}_F)}(T(\mathbb{A}_F))^1 \times T(\mathbb{A}_F)^1 1_{gB_g^{-1}}(x^{-1}\gamma y) \, dx \, dy$$

Here, $dx$ resp. $dy$ denotes integration with respect to the Haar measure normalized for $gB_g^{-1}$. We will assume in the following that $I_\gamma \neq 0$. In particular, $\gamma$ satisfies the local denominator bounds on its invariant in the proof of Lemma 4.6.

Proposition 4.7 (Orbital integrals). Assume that $Y$ has maximal type. For any $\varepsilon > 0$, we have

$$I_\gamma \ll_{n,\varepsilon} r^{n\varepsilon} \text{disc}(Y)^\varepsilon,$$

where $r$ is defined in (4.10).

Note that the maximal type assumption will only be used in Lemma 4.9 below.

We make some preparations before proving this proposition. By substitution

$$I_\gamma = \int_{T(\mathbb{A}_F)^1} \int_{Z(\mathbb{A}_F)^1}(T(\mathbb{A}_F)^1 1_{gB_g^{-1}}(x^{-1}\gamma xy) \, dx \, dy$$

We further disintegrate with respect to the variable $y$ as follows: Setting

$$T^1 = \ker(\det : T \to \mathbb{G}_m)$$

note that the quotient $T^1(\mathbb{A}_F)/(T(\mathbb{A}_F))^1$ is naturally identified with

$$\text{Nr}(\mathbb{A}_K^\times)^1 = \{ s \in \text{Nr}(\mathbb{A}_K^\times) = \text{det}(T(\mathbb{A}_F)) : |s|_{\mathbb{A}_F} = 1 \} \subset \mathbb{A}_F^1$$
via the determinant. Furthermore, we have $Z(\mathbb{A}_F)\mathbb{T}(\mathbb{A}_F)^1 \simeq Z(\mathbb{A}_F)^\gamma \mathbb{T}(\mathbb{A}_F)$. To summarize

$$I_\gamma = \int_{T(\mathbb{A}_F)^1} \int_{T(\mathbb{A}_F)^1} \int_{Z(\mathbb{A}_F)^\gamma \mathbb{T}(\mathbb{A}_F)} 1_{gBg^{-1}}(x^{-1}\gamma y xs) \, dx \, dy \, ds.$$ 

Note that the Haar measure on $T(\mathbb{A}_F)^1$ and $T(\mathbb{A}_F)^1$ is normalized so that the respective intersection with $gBg^{-1}$ has measure 1. The measure on $T(\mathbb{A}_F)^1 \mathbb{T}(\mathbb{A}_F)^1$ is the induced measure. Whenever the tuple $(x,y,s)$ is such that $x^{-1}\gamma y xs \in gBg^{-1}$, we must have

$$\det(s) \in \frac{1}{\det(\gamma)} \det(B), \quad (4.12)$$

where $\det(B) := \{\det(b) \mid b \in B\}$.

We will deduce a bound for the inner double integral

$$I_{\gamma,s} = \int_{T(\mathbb{A}_F)^1} \int_{Z(\mathbb{A}_F)^\gamma \mathbb{T}(\mathbb{A}_F)} 1_{gBg^{-1}}(x^{-1}\gamma y xs) \, dx \, dy$$

for fixed $s$ which will imply the claim in Proposition 4.7 when combined with the following lemma:

**Lemma 4.8 (Norm measure).** The measure of the set of points

$$s \in \text{Nr}(\mathbb{A}_K^\times)^1 \cap \frac{1}{\det(\gamma)} \det(B)$$

is $\ll_\varepsilon \text{disc}(Y)^\varepsilon$. Here, the Haar measure on $\text{Nr}(\mathbb{A}_K^\times)^1$ is normalized in accordance with $T(\mathbb{A}_F)^1 \mathbb{T}(\mathbb{A}_F)^1$.

**Proof.** Using local coordinates

$$U := T(\mathbb{A}_F)^1 \cap gBg^{-1} \simeq (C_\infty \times \hat{\mathbb{O}}^\times)^1 = C_\infty^1 \times \hat{\mathbb{O}}^\times$$

(4.13)

where $C_\infty$ is a bounded neighborhood of the identity in $K_\infty^\times = \prod_v \text{Archimedean} K_v^\times$. Explicitly, the above isomorphism is given by mapping $t \in T(\mathbb{A}_F)^1$ with local coordinates $(a,0)$ to $a$ (see Proposition 4.4) and the set $C_\infty$ is contained in the set of points $a \in K_\infty^\times$ with $(\kappa r_u)^{-1} \leq |b_{1,w}| \leq \kappa u$ for any $w | u$, $u$ an Archimedean place of $F$ and some constant $\kappa > 0$. Moreover, upon closer inspection of the proof of Proposition 4.4 there is a constant $\kappa' < \kappa$ so that the set $C_\infty$ contains the set of points $a \in K_\infty^\times$ with $(\kappa' r_u)^{-1} \leq |b_{1,w}| < \kappa' r_u$ for any $w | u$ and $u$ an Archimedean place of $F$.

Under the map in (4.13), the intersection $T(\mathbb{A}_F)^1 \cap gBg^{-1}$ corresponds to norm one elements in $C_\infty^1 \times \hat{\mathbb{O}}^\times$. In particular, the induced Haar measure $m$ on $\text{Nr}(\mathbb{A}_K^\times)^1$ is normalized so that $m(\text{Nr}(U)) = 1$ where for the purposes of this proof we make the identification in (4.13).

To bound the measure $m(\frac{1}{\det(\gamma)} \det(B))$ we first observe that for any two points $s_1, s_2 \in \frac{1}{\det(\gamma)} \det(B)$ we have

$$s_2 \in \frac{1}{\det(\gamma)} \det(B) \subset s_1 \det(BB^{-1}).$$
Therefore, by invariance
\[ m\left(\frac{1}{\det(\gamma)} \det(B)\right) \leq m(\det(BB^{-1})) \]
and we bound the right-hand side instead, which is independent of \( \gamma \).

Suppose that \( u \) is a non-Archimedean non-dyadic place of \( F \). If \( \text{Nr}(\Delta_{\mathcal{O},u}) \) is a unit in \( \mathcal{O}_{F,u} \), the norm is surjective as a map \( \text{Nr} : \mathcal{O}_{u}^{\times} \to \mathcal{O}_{F,u}^{\times} \) by Lemma 2.5. By the choice of \( B \) asserting that \( B_u = \text{GL}_2(\mathcal{O}_{F,u}) \) is a group we see that
\[ \det(B_uB_u^{-1}) = \mathcal{O}_{F,u}^{\times} = \text{Nr}(U_u). \]
If \( \text{Nr}(\Delta_{\mathcal{O},u}) \) is not a unit in \( \mathcal{O}_{F,u} \), the image of the norm map has index two in \( \mathcal{O}_{F,u}^{\times} \) by Lemma 2.5. Furthermore, Lemma 2.5 implies that for dyadic places the image of the norm map has index \( \ll y \). The analysis for the Archimedean places is largely analogous; both the set \( \det(B_uB_u^{-1}) \) and the norm image are comparable to balls of radius \( r_u^2 \).

Combining these statements yields that \( \det(BB^{-1}) \cap \text{Nr}(\mathbb{A}_F^{\times})^{-1} \) is covered by \( \ll_B 2^b \) shifts of \( \text{Nr}(U) \) where \( b \) is the number of non-Archimedean non-dyadic places \( u \) of \( F \) for which \( \text{Nr}(\Delta_{\mathcal{O},u}) \) is a non-unit. It is easy that \( 2^b \) is bounded by the divisor function of disc(\( Y \)) and in particular \( 2^b \ll \text{disc}(Y)^\varepsilon \). This proves the lemma. \( \Box \)

The product decomposition states that \( I_{\gamma,s,v} = \prod_{v \in \mathcal{V}_F} I_{\gamma,s,v} \) where
\[ I_{\gamma,s,v} = \int_{\mathcal{T}^{(F_v)}} \int_{\mathcal{Z}(F_v)} 1_{g_vB_vg_v^{-1}}(x_v^{-1}\gamma y_vy_v's_v) \, dx_v \, dy_v; \]
we estimate these local orbital integrals.

**Lemma 4.9** (Non-Archimedean local integrals). Suppose that \( Y \) has maximal type. Let \( v \) be a non-Archimedean place of \( F \) and let \( s \in \mathcal{T}^{(\mathbb{A}_F)} \setminus \mathcal{T}(\mathbb{A}_F) \) as in (4.12). Whenever \( I_{\gamma,s,v} \neq 0 \) we have that \( I_{\gamma,s,v} \leq 1 \) if \( K_v \) is a field and otherwise
\[ I_{\gamma,s,v} \leq (\log q_v(|\psi_T(\gamma)|^{-1}) + 1)(\log q_v(|1 + \psi_T(\gamma)|^{-1}) + 1). \]
where \( q_v \) is the cardinality of the residue field of \( F_v \).

We remark here that \( \psi_T(\gamma) \neq 0, -1 \) as \( \gamma \in \text{GL}_2(F) \setminus \mathcal{N}_T(F) \) so that the right-hand side in the lemma is well-defined (it is moreover positive).

**Proof.** Write \( H_v = g_v^{-1}T_vg_v \) and \( H_v = g_v^{-1}T_vg_v \). By substitution
\[ I_{\gamma,s,v} = \int_{H_v} \int_{\mathcal{H}(F_v)} 1_{B_v(x_v^{-1}g_v^{-1}\gamma g_vy_vy_v's_v')} \, dx_v \, dy_v; \]
where \( s'_v = g_v^{-1}s_vg_v \).

We rephrase this integral in terms of the local coordinates in Proposition 4.4. Let \( x_v, y_v \) as above and write \((t_1,0)\) and \((t_2,0)\) for the respective local coordinates. Furthermore, write \((a,b)\) for the local coordinates of \( g_v^{-1}\gamma g_v \) and \((c,0)\) for the local coordinates of \( s'_v \). In particular, the local coordinates of \( x_v^{-1}g_v^{-1}\gamma g_vy_vy_v's'_v \) are \((at_2c,b\sigma(t_2c)^{2(t_1)})\). By Proposition 4.4, the requirement \( x_v^{-1}g_v^{-1}\gamma g_vy_vy_v's'_v \in B_v \)
therefore implies that \( at_2c, b\sigma(t_2c)\frac{\sigma(t_1)}{t_1} \in \frac{1}{\Delta_{K/F,v}} \mathcal{O}_{K,v} =: \mathfrak{o}_v^{-1} \) (the latter is the inverse different). This shows that

\[
I_{\gamma,s,v} \leq \int_{\text{SL}_1(K_v)} 1_{\mathfrak{o}_v^{-1}}(at_2c) \int_{F_v ^\times \setminus K_v ^\times} 1_{\mathfrak{o}_v^{-1}} \left( b\sigma(t_2c)\frac{\sigma(t_1)}{t_1} \right) \, dt_1 \, dt_2,
\]

(4.14)

where \( \text{SL}_1(K_v) \) denote the group of norm 1 elements in \( K_v \). By Hilbert’s Theorem 90, the homomorphism

\[
t_1 \in F_v ^\times \mapsto \frac{\sigma(t_1)}{t_1} \in \text{SL}_1(K_v)
\]

is a bijection. The Haar measure on \( \text{SL}_1(K_v) = F_v ^\times \setminus K_v ^\times \) is normalized so that \( \mathcal{O}_{K,v} ^\times \) has measure 1. Therefore, the inner integral in the above expression is equal to

\[
\int_{\text{SL}_1(K_v)} 1_{\mathfrak{o}_v^{-1}}(b\sigma(c)t_1) \, dt_1
\]

(4.15)

after a substitution, and \( I_{\gamma,s,v} \) is bounded by (4.15) times the analogous expression with \( a \) instead of \( b \).

We thus estimate (4.15). Suppose first that \( K_v \) is a field. In this case, the group of norm one elements \( \text{SL}_1(K_v) \) of \( K_v \) is exactly the group of norm one units in \( \mathcal{O}_{K,v} ^\times \). By the choice of measure normalization, (4.15) is thus 1 if \( b\sigma(v) \in \mathfrak{o}_v ^{-1} \) and zero otherwise. So suppose that \( K_v = F_v \times F_v \). Making this identification, \( \text{SL}_1(K_v) = \{ (\rho, \rho^{-1}) : \rho \in F_v ^\times \} \) and \( \mathcal{O}_{K,v} = \mathcal{O}_{F,v} \times \mathcal{O}_{F,v} \). Also, \( \Delta_{K/F,v} \in \mathcal{O}_{K,v} ^\times \) so that (4.15) is equal to

\[
\int_{F_v ^\times} 1_{\mathcal{O}_{F,v}}(b_1c_2\rho)1_{\mathcal{O}_{F,v}}(b_2c_1\rho^{-1}) \, d\rho
\]

where \( b = (b_1, b_2) \) and \( c = (c_1, c_2) \). Note that any \( \rho \in F_v ^\times \) is in the support of the above integrand if and only if

\[
|b_2c_1|_v \leq |\rho|_v \leq |b_1c_2|_v ^{-1}.
\]

(4.16)

The measure of such \( \rho \) is (when non-zero)

\[
\log_q \left( |b_1c_2|_v ^{-1} \right) - \log_q \left( |b_2c_1|_v \right) + 1 = \log_q \left( |\text{Nr}(bc)|_v ^{-1} \right) + 1 = \log_q \left( |\psi^R(\gamma)|_v ^{-1} \right) + 1
\]

using (4.12). Proceeding analogously for \( a \) instead of \( b \) proves the lemma. \( \square \)

**Lemma 4.10** (Archimedean local integrals). Let \( v \) be any Archimedean place of \( F \) and let \( s \in T \), \( (\Delta_F)_T \) as in (4.12). Whenever \( I_{\gamma,s,v} \neq 0 \), we have that \( I_{\gamma,s,v} \leq 1 \) if \( K_v \) is a field (i.e. if \( F_v = \mathbb{R} \) and \( K_v = \mathbb{C} \)) and otherwise

\[
I_{\gamma,s,v} \ll \left( \log(|\psi^R(\gamma)|_v ^{-1}) + \log(r_v \text{disc}_v(Y)) \right) \left( \log(|1 + \psi^R(\gamma)|_v ^{-1}) + \log(r_v \text{disc}_v(Y)) \right).
\]

**Proof.** The proof is largely analogous to the proof of Lemma 4.9 so we will be brief. Using the local coordinates of \( \gamma \) one sees from Proposition 4.4 that it is sufficient to estimate an integral of the form

\[
\int_{\text{SL}_1(K_v)} 1_{\mathcal{O}_v}(at_1) \, dt_1
\]

(4.17)
as in (4.15) for some \(a \in K_v\) where
\[ \Omega_v := \{ x \in K_v : |x_w|_w \leq R_v \text{ for all } w | v \} \]
and \(R_v = \kappa r_v \text{disc}_v(Y)^{1/2}\) for some \(\kappa > 0\). When \(K_v\) is a field, (4.17) is 0 or 1 depending on the absolute value of \(a\). When \(F_v = \mathbb{R}\) and \(K_v = \mathbb{R} \times \mathbb{R}\), the set of \(\rho \in \mathbb{R}^\times\) satisfying
\[ R_v^{-1}|a_2|_v \leq |\rho|_v \leq R_v|a_1|_v^{-1} \]
has measure (if non-empty)
\[ 2(\log(|a|_v^{-1}) + 2 \log R_v) \].
When \(F_v = \mathbb{C}\) and \(K_v = \mathbb{C} \times \mathbb{C}\), the set of \(\rho \in \mathbb{C}^\times\) satisfying
\[ R_v^{-1}|b_2c_1|_v \leq |\rho|_v \leq R_v|b_1c_2|_v^{-1} \]
has measure (if non-empty)
\[ 2\pi(\log(|a|_v^{-1}) + 2 \log R_v) \].
The rest of the proof proceeds similarly. \(\square\)

**Proof of Proposition 4.7.** Consider first a fixed \(s \in \text{Nr}(\mathbb{A}_K^\times)^{1}\). Define the set of places
\[ \mathcal{A} := \{ v \in \mathcal{V}_{F,f} : K_v \text{ is not a field} \} \].
By Lemma 4.9 we have \(I_{\gamma,s,v} \leq 1\) for any finite place \(v \notin \mathcal{A}\) so that
\[ I_{\gamma,s} = \prod_{v \in \mathcal{V}_F} I_{\gamma,s,v} = \prod_{v \in \mathcal{V}_{F,f}} I_{\gamma,s,v} \prod_{v \in \mathcal{A}} I_{\gamma,s,v} \]
To estimate \(\prod_{v \in \mathcal{A}} I_{\gamma,s,v}\), we apply the local integral bounds in Lemma 4.9
\[ \prod_{v \in \mathcal{A}} I_{\gamma,s,v} \leq \prod_{v \in \mathcal{A}} (\log q_v(|\psi_T(\gamma)|_v^{-1}) + 1)(\log q_v(|1 + \psi_T(\gamma)|_v^{-1}) + 1) \]
\[ \ll \varepsilon \prod_{v \in \mathcal{A}} (|\psi_T(\gamma)|_v^{-1})^{\varepsilon/4}(|1 + \psi_T(\gamma)|_v^{-1})^{\varepsilon/4} \]
\[ = \prod_{v \in \mathcal{V}_F \setminus \mathcal{A}} (|\psi_T(\gamma)|_v)^{\varepsilon/4}(1 + \psi_T(\gamma)|_v)^{\varepsilon/4} \ll n r^{\mu \varepsilon} \text{disc}(Y)^{\varepsilon/2} \]
where we also used the local bounds (4.11) for the value \(\psi_T(\gamma)\) and the analogous bounds for \(1 + \psi_T(\gamma)\) in the last step.
For the product \(\prod_{v | \infty} I_{\gamma,s,v}\) over the Archimedean places, we proceed analogously using Lemma 4.10 to obtain
\[ \prod_{v | \infty} I_{\gamma,s,v} \ll \varepsilon r^{2n \varepsilon} \text{disc}(Y)^{\varepsilon/2} \].
Overall, we have shown that
\[ I_{\gamma,s} \ll \varepsilon r^{2n \varepsilon} \text{disc}(Y)^{\varepsilon} \].
Combined with Lemma 4.8, this implies that
\[ I_\gamma = \int_{s \in \mathfrak{N}(A_K)^1 \cap \frac{1}{\det(\gamma)} \det(B)} I_{\gamma,s} \, ds \ll n, \varepsilon \, r^{2n \varepsilon} \text{disc}(Y)^{2 \varepsilon}. \]

This completes the proof of the proposition. \( \square \)

We are now ready to prove Linnik’s basic lemma as in Theorem 1.7.

**Proof of Theorem 1.7.** The proof merely consists of putting together already proven statements. By (4.6) and (4.7), we have
\[
\mu_Y \times \mu_Y \left( \{(x, y) \in [\text{GL}_2, F]^2 : y \in x B_t\} \right) \leq \sum_{\gamma \in T(F)^{\text{GL}_2(F)}} \int_{[T]^2} \sum_{\eta \in T(F) \gamma T(F)} f_\gamma(g^{-1} x^{-1} \eta y g) \, d\mu_T(x) \, d\mu_T(y).
\]

Recall that (4.8) gives
\[
\sum_{\gamma \in T(F)^{N_T(F)}} \int_{[T]^2} \sum_{\eta \in T(F) \gamma T(F)} f_\gamma(g^{-1} x^{-1} \eta y g) \, d\mu_T(x) \, d\mu_T(y) \ll \frac{1}{\text{vol}(Y)}.
\] (4.18)

Combining (4.9), Lemma 4.6 and Proposition 4.7, we get
\[
\sum_{\gamma \in T(F) \gamma T(F)} \int_{[T]^2} \sum_{\eta \in T(F) \gamma T(F)} f_\gamma(g^{-1} x^{-1} \eta y g) \, d\mu_T(x) \, d\mu_T(y)
= \frac{\text{vol } ([Z])}{\text{vol } (Y)^2} \sum_\gamma \int_{\Delta(Z(A_K)^2 \cap (T(A_F)^2)^2} f_\gamma(g^{-1} x^{-1} \gamma y g) \, dx \, dy
\ll n, \varepsilon \frac{\text{vol } ([Z])}{\text{vol } (Y)^2} n(1+\varepsilon) \text{disc}(Y)^{1+\varepsilon} D_F^{1/2} |\alpha(a)|^{-2 \tau} \]
\[ \ll n, \varepsilon \frac{\text{vol } ([Z])}{\text{vol } (Y)^2} n(1+\varepsilon) \text{disc}(Y)^{1+\varepsilon} D_F^{1/2} |\alpha(a)|^{-2 \tau} \]
\[ \leq r^{n(1+\varepsilon)} \frac{\text{disc}(Y)^{1+2 \varepsilon}}{\text{vol } (Y)^2} |\alpha(a)|^{-2 \tau}. \] (4.19)

To complete the proof, combine (4.18) and (4.19) and observe that the entropy \( h_{[\text{GL}_2, F]}(a) \) is exactly \( \log(|\alpha(a)|_S) \). \( \square \)
5. Reduction to type $A_1$

In this section we proceed to prove Theorem 1.2. Quite generally, a lower bound on the entropy can be obtained by showing sufficient decay of the measure of Bowen balls at typical points. We do so in the following two steps:

(A) Find a time at which all points in a Bowen ball lie on the orbit of an intermediate group of the form $\text{Res}_F/\mathbb{Q}(\text{GL}_2)$.

(B) Apply Linnik’s uniform basic lemma Theorem 1.7 at this time scale.

Let us begin by recalling how to obtain an entropy bound from separation. For any neighborhood of the identity $B \subset \text{GL}_4(\mathbb{A})$1, any semisimple element $a \in \text{G}(\mathbb{Q}_u)$, and any $\tau \geq 0$ we define the Bowen ball

$$B_\tau = \bigcap_{-\tau \leq t \leq \tau} a^\tau B a^{-\tau}.$$  

Proposition 5.1 (cf. [ELMV09, Prop. 3.2] and [Kha19a, Prop. 8.2]). Fix a semisimple element $a \in \text{G}(\mathbb{Q}_u)$ for some place $u$ of $\mathbb{Q}$. Suppose that $\{\mu_i\}$ is a sequence of $a$-invariant probability measures on $\text{G}(\mathbb{Q})\backslash \text{G}(\mathbb{A})$1 converging to a probability measure $\mu$ in the weak-* topology. Assume that for some fixed $\eta > 0$ we have a sequence of integers $\tau_i \to \infty$ such that for any compact subset $F \subset \text{G}(\mathbb{Q})\backslash \text{G}(\mathbb{A})$1 there exists a bounded identity neighborhood $B \subset \text{G}(\mathbb{A})$ such that

$$\mu_i \times \mu_i \{(x, y) \in F \times F \mid y \in xB_\tau\} = \int_F \mu_i(xB_\tau \cap F) d\mu_i(x) \ll_{F, \varepsilon} \exp(-2(\eta - \varepsilon)\tau_i),$$

then the Kolmogorov-Sinai entropy of the $a$-action with respect to the measure $\mu$ satisfies $h_\mu(a) \geq \eta$.

Following the notation of Theorem 1.2 we fix a place $u$ of $\mathbb{Q}$ and a split $\mathbb{Q}_u$-torus $A < \text{GL}_4$. Without loss of generality, we may assume that $A$ is the diagonal subgroup. Furthermore, we set $A = A(\mathbb{Q}_u)$ and fix some element $a \in A$ (more restrictions will be imposed later on). The Bowen balls $B_\tau$ are defined with respect to $a$. Moreover, $B$ is taken to be of the form $B = \prod_v B_v$ where $B_v = \text{GL}_4(\mathbb{Z}_v)$ if $v$ is finite and

$$B_\infty = \{g \in \text{GL}_4(\mathbb{R}) : \|g\|, \|g^{-1}\| \leq 2\}.$$  

In view of Proposition 5.1 and Theorem 1.2 we shall fix in the following a toral packet $Y = [Tg]$ of either biquadratic, cyclic or dihedral type and of maximal type satisfying the properties of the main theorem and exhibit a time $\tau = \tau(\text{disc}(Y))$ at which

$$\mu_Y \times \mu_Y \{(x, y) \in F \times F \mid y \in xB_\tau\} \ll \exp(-2\eta\tau)$$

for some specific $\eta > 0$. Let $K$ be the number field associated to the torus $T$. Explicitly, $K$ is the centralizer of $T(\mathbb{Q})$ in $M_4(\mathbb{Q})$. Let $L$ be the Galois closure of $K$ (in our case $[L : K] \leq 2$) and set $\mathcal{G} = \text{Gal}(L/\mathbb{Q})$. Fix a quadratic subfield $F$ of $K$. We write $R \simeq \text{Res}_{F/\mathbb{Q}}(\text{GL}_2)$ for the intermediate group that $F$ defines; it is the centralizer of the subgroup $S < T$ isomorphic to $\text{Res}_{F/\mathbb{Q}}(\text{G}_m)$ (cf. §2.2.2). In
view of Theorem 1.2 we also assume that the points in \( g_u^{-1}S(Q_u)g_u \) are of the form \( \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2) \) for \( \lambda_1, \lambda_2 \in \mathbb{Q}^\times \). We write \( R_{\text{std}} < GL_4 \) for the block-diagonal subgroup so that in particular, \( g_u^{-1}R(Q_u)g_u = R_{\text{std}}(Q_u) \).

### 5.1. Geometric invariant theory

We briefly recall that the way geometric invariant theory (GIT) was used in [Kha19a] and refer to Sections 3, 4 therein for more details. Define the affine variety

\[
T \downarrow GL_4 / \Gamma = \text{Spec} \mathbb{Q}[GL_4]^{T \times T}
\]

which is a universal categorical quotient (see [MFK94]). Consider the monomials

\[
\Psi_\sigma^0(g) = (\det g)^{-1} \text{sign}(\sigma) \prod_{1 \leq i \leq n} g_{\sigma(i), i};
\]

for \( \sigma \in S_4 \). These form a generating set of the \( \mathbb{Q} \)-algebra \( \text{Spec} \mathbb{Q}[GL_4]^{A \times A} \) when \( A < GL_4 \) is the diagonal torus (cf. [Kha19a, Prop. 4.1]). Let \( g \in GL_4(L) \) be so that \( g^{-1}Tg = A \). By universality, the invariant regular functions

\[
\Psi_\sigma(g) = \Psi_\sigma^0(g^{-1}gg)
\]

for \( \sigma \in S_4 \) form a generating set for \( \text{Spec} L[GL_4]^{T \times T} \). We shall call these the \textit{canonical generators}. They are not typically defined over \( \mathbb{Q} \) even when \( G \) is abelian.

Let \( W \) be the Weyl group \( N_A / A \). For any \( w \in W \), \( w \text{diag}(t_i)w^{-1} = \text{diag}(t_{\sigma^{-1}(i)}) \) for some \( \sigma \in S_4 \). Then \( W \) is identified with \( S_4 \) via \( w \mapsto \sigma \). We have an injective homomorphism from the Galois group \( G \) to the Weyl group \( W \cong S_4 \) via the 1-cocycle \( \sigma \mapsto \sigma(g)^{-1}g \). We identify \( G \) with its image under this homomorphism. By [Kha19a, Proposition 6.6], for any \( g \in G(Q) \), \( \sigma \in S_4 \) and \( \tau \in G \),

\[
\tau \Psi_\sigma(g) = \Psi_{\tau \sigma \tau^{-1}}(g).
\]

(5.1)

In particular, \( \Psi_\sigma \) is defined over \( \mathbb{Q} \) if and only if \( \sigma \) is centralized by \( G \).

### 5.2. Galois relations

We shall need a more precise description of this Galois relations in (5.1) and in particular of the image of \( G \) in \( S_4 \). Let us fix an ordering on the Galois embeddings

\[
\sigma_i : K \hookrightarrow L, \ i = 1, \ldots, 4.
\]

(5.2)

compatible with \( F \) in the sense that \( \sigma_1|_F = \sigma_2|_F \) and \( \sigma_3|_F = \sigma_4|_F \). The Galois action on these embeddings determines \( G \hookrightarrow S_4 \).

**Lemma 5.2** (Explicit conjugation). \textit{There exists a basis} \( x_1, \ldots, x_4 \) \textit{of} \( K \) \textit{so that the matrix} \( g = (\sigma_j(x_i))_{ij} \) \textit{satisfies} \( g^{-1}Tg = A \) \textit{and for any} \( x \in K \subset M_4(Q) \)

\[
g^{-1}xg = \text{diag}(\sigma_1(x), \ldots, \sigma_4(x)).
\]

**Proof.** This is completely standard; we give a proof merely for completeness. To construct the basis, consider the \( \mathbb{Q} \)-linear map

\[
\theta : x \in K \mapsto e_1 x \in \mathbb{Q}^4
\]
where we view $K \subset M_4(\mathbb{Q})$. This is an (right-)equivariant isomorphism. By definition, the matrix representation of multiplication by $x \in K$ in the basis $x_i = \theta^{-1}(e_i)$ is exactly $x$ when viewed as an element of $M_4(\mathbb{Q})$.

The matrix $g = (\sigma_j(x_i))_{ij}$ is the representation in the standard basis of the $\mathbb{Q}$-linear map $\varphi \circ \theta^{-1}$ where

$$\varphi : x \in K \mapsto (\sigma_1(x), \ldots, \sigma_4(x)).$$

From here, the lemma is straightforward to verify. \hfill \Box

**Remark 5.3.** Together with the compatibility assumption in (5.2), Lemma 5.2 also asserts that $g^{-1}S(\mathbb{Q})g = \{\text{diag}(\sigma_1(x), \sigma_1(x), \sigma_3(x), \sigma_3(x)) : x \in F^\times\}$ and similarly for $S(R)$ where $R$ is any $\mathbb{Q}$-algebra. In particular, $g^{-1}R(\mathbb{Q})g$ is block-diagonal.

We shall now use the constructed element $g$ in Lemma 5.2 to explicitly determine the induced homomorphism $G \to S_4$. To this end, observe that the 1-cocycle introduced earlier is equal to (under the identification $W \simeq S_4$)

$$\tau \in G \mapsto (j \mapsto k \text{ where } \sigma_k = \tau \circ \sigma_j) \in S_4.$$  \hfill (5.3)

In particular, it is injective.

In the following we determine the image of $G$ according to each Galois type. It is independent of the choice in (5.2) while of course the image of an individual element of $G$ can depend on the ordering.

- **$K$ is biquadratic.** The image of (5.3) is

  $$\{\text{id}, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)\}.$$

  The image of the subgroup Gal($K/F$) is $\{\text{id}, (1 2)(3 4)\}$.

- **$K$ is cyclic.** The image of (5.3) is

  $$\{\text{id}, (1 3 2 4), (1 2)(3 4), (1 4 2 3)\} \simeq \mathbb{Z}/4\mathbb{Z}$$

  and the image of Gal($K/F$) is $\{\text{id}, (1 2)(3 4)\}$.

- **$K$ is dihedral.** The image of (5.3) is

  $$\langle (1 3 2 4), (3 4) \rangle \simeq D_4.$$

  Under identification with the image, $K$ is the fixed field of the transposition $(3 4)$ and $F$ is the fixed field of the normal subgroup $\langle (1 2), (3 4) \rangle$.

From now on we shall identify $G$ with its image under (5.3).

**Lemma 5.4.** For any $\tau \in G$, any $g \in GL_4(\mathbb{Q})$, and any index $(i, j)$ the conjugated element $g' = g^{-1}gg$ satisfies

$$\tau(g'_{ij}) = g'_{\tau(i)\tau(j)}.$$
Proof. Let \( w_{\tau} \) denote the image of \( \tau \) in the Weyl group \( W \). Then \( w_{\tau} \) has a representative in the form of a permutation matrix \( \tilde{w}_{\tau} \), such that \( \tau(g) = g\tilde{w}_{\tau} \). Then we have

\[
\tau(g') = \tau(g^{-1}gg) = \tilde{w}_{\tau}^{-1}g^{-1}gg\tilde{w}_{\tau} = \tilde{w}_{\tau}^{-1}g'\tilde{w}_{\tau}.
\]

The lemma follows. \( \square \)

Let us illustrate Lemma 5.4 for each Galois type. Let \( g \in \text{GL}_4(\mathbb{Q}) \) and \( g' = g^{-1}gg \).

- For \( K \) biquadratic
  \[
g' = \begin{pmatrix}
  *_1 & *_2 & *_3 & *_4 \\
  *_2 & *_1 & *_4 & *_3 \\
  *_3 & *_4 & *_1 & *_2 \\
  *_4 & *_3 & *_2 & *_1
  \end{pmatrix}
\]
  where for each \( i \) the entries with \( *_i \) are all Galois conjugate to each other.

- For \( K \) cyclic
  \[
g' = \begin{pmatrix}
  *_1 & *_2 & *_3 & *_4 \\
  *_2 & *_1 & *_4 & *_3 \\
  *_4 & *_3 & *_1 & *_2 \\
  *_3 & *_4 & *_2 & *_1
  \end{pmatrix}
\]

- For \( K \) dihedral
  \[
g' = \begin{pmatrix}
  *_1 & *_2 & *_3 & *_3 \\
  *_2 & *_1 & *_3 & *_3 \\
  *_3 & *_3 & *_1 & *_2 \\
  *_3 & *_3 & *_2 & *_1
  \end{pmatrix}
\]

In view of the above we define the following special set of permutations

\[
S_{sp} = \begin{cases}
  \{(14)(23), (13)(24)\} & \text{if } K \text{ is biquadratic,} \\
  \{(1324), (1423)\} & \text{if } K \text{ is cyclic or dihedral.}
\end{cases}
\]

Lemma 5.5. Let \( g \in \text{GL}_4(\mathbb{Q}) \) be such that \( \Psi_{\sigma}(g) = 0 \) for every \( \sigma \in S_{sp} \). Then \( g \in \text{R}(\mathbb{Q}) \).

Proof. Let \( g' = g^{-1}gg \). By the Galois conjugacy relations \( \Psi_{\sigma}(g) = 0 \) for \( \sigma \) as in the lemma asserts that all entries \( g'_{ij} \) of \( g' \) away from the block-diagonal (i.e. for \( (i, j) \) such that \( i \neq j \) and \( 4 \leq i + j \leq 6 \)) vanish. In particular, \( g' \) commutes with \( g^{-1}S(\mathbb{Q})g \) and hence \( g \) commutes with \( S(\mathbb{Q}) \). This implies \( g \in \text{R}(\mathbb{Q}) \). \( \square \)

5.3. Proof of Theorem 1.2. Following the outline, we now show that given enough contraction, any return to a Bowen ball around a point in the homogeneous toral set must occur within the intermediate orbit. As before, we shall continue using the notation introduced at the beginning of this section. Let \( a \in A \) and define the Bowen ball \( B_{\tau} \) with respect to \( a \). We denote by

\[
\alpha_{ij}(a) = \frac{a_{ii}}{a_{jj}}
\]
the value of the roots at $a$. We identify the tuple $(i, j)$ with its associated root $\alpha_{ij}$. To simplify the notation, we set for $\sigma \in S_n$

$$R_\sigma = \{(\sigma(i), i) : i = 1, \ldots, n, i \neq \sigma(i)\};$$

$$\eta_\sigma(a) = \sum_{\alpha \in R_\sigma} |\log |\alpha(a)||_p.$$ 

**Proposition 5.6** (Returns within the intermediate orbit). There exists a positive constant $\kappa = \kappa(n)$ with the following property. Suppose that

$$\tau \eta(a) > \frac{1}{2} \log \mathrm{disc}(Y) + \kappa$$

where

$$\eta(a) = \min_{\sigma \in \mathcal{S}_{sp}} \eta_\sigma(a).$$

Then any $\gamma \in \mathrm{GL}_4(\mathbb{Q}) \cap T(\mathbb{A})gB_r g^{-1}T(\mathbb{A})$ lies in $R(\mathbb{Q})$.

**Proof.** We apply the proof of [Kha19a, Thm. 8.9] based on geometric invariants and their local bounds. Suppose first that $K$ is abelian and let $\sigma \in \mathcal{S}_{sp}$. Hence, $\Psi_\sigma$ is defined over $\mathbb{Q}$. The proof of [Kha19a, Thm. 8.9] shows that

$$|\Psi_\sigma(\gamma)_{\mathbb{A}}| \leq C \exp(-2\tau \eta_\sigma(a))\mathrm{disc}(Y)$$

for some absolute constant $C$. Therefore, if

$$\tau \eta_\sigma(a) > \frac{1}{2} \log \mathrm{disc}(Y) + \frac{1}{2} \log(C)$$

we have the content bound $|\Psi_\sigma(\gamma)|_{\mathbb{A}} < 1$ so that $\Psi_\sigma(\gamma) = 0$ by the product formula for $\mathbb{Q}$. In particular, if $\tau \min_{\sigma \in \mathcal{S}_{sp}} \eta_\sigma(a) > \frac{1}{2} \log \mathrm{disc}(Y) + \frac{1}{2} \log(C)$, Lemma 5.5 implies that $\gamma \in R(\mathbb{Q})$ as claimed.

Suppose now that $K$ is dihedral. Then $\mathcal{S}_{sp}$ is a single Galois conjugacy class (under the identification of the Galois group $G \subset S_4$). Applying again the proof of [Kha19a, Thm. 8.9] we obtain

$$\prod_{\sigma \in \mathcal{S}_{sp}} |\Psi_\sigma(\gamma)|_{\mathbb{A}} \leq C^2 \exp\left(-2\tau \sum_{\sigma \in \mathcal{S}_{sp}} \eta_\sigma(a)\right)\mathrm{disc}(Y)^2.$$ 

By a similar argument as in the abelian case, the condition

$$\tau \frac{1}{2} \sum_{\sigma \in \mathcal{S}_{sp}} \eta_\sigma(a) > \frac{1}{2} \log \mathrm{disc}(Y) + \frac{1}{2} \log(C)$$

asserts that $\Psi_\sigma(\gamma) = 0$ for some $\sigma \in \mathcal{S}_{sp}$. As $\mathcal{S}_{sp}$ is a single conjugacy class, this implies $\Psi_\sigma(\gamma) = 0$ for both elements $\sigma \in \mathcal{S}_{sp}$ and by Lemma 5.5 we have $\gamma \in R(\mathbb{Q})$. Note that $\eta_\sigma(a)$ is constant for $\sigma \in \mathcal{S}_{sp}$ so that the above average over $\mathcal{S}_{sp}$ is equal to the minimum. This proves the proposition. \hfill $\Box$

The following should be considered to be one of the main theorems of this article. In particular, it encompasses Theorem 1.2. For the readers’ convenience, we shall repeat all standing assumptions.

**Theorem 5.7.** Let $u$ be a place of $\mathbb{Q}$, let $Y = [Tg] \subset [\mathrm{GL}_4]$ be a homogeneous toral sets satisfying the following assumptions.
(i) The homogeneous toral set $Y$ is of maximal type and invariant under the $\mathbb{Q}_u$-points $A$ of the diagonal subgroup $A$ of $\text{GL}_4$.

(ii) The quartic field $K$ associated to $Y$ is either biquadratic, cyclic or dihedral. Fix a quadratic subfield $F \subset K$ and the associated subgroup $S \subset T$ with $S \simeq \text{Res}_{F/\mathbb{Q}}(G_m)$.

(iii) The $\mathbb{Q}_u$-torus $g_u^{-1}Sg_u$ commutes with the block-diagonal subgroup $\text{R}_{\text{std}}$.

Choose $a \in A(\mathbb{Q}_u)$ defining in particular the Bowen ball $B_\tau$ for any $\tau > 0$. There exists a constant $\kappa > 0$ depending only on $A_\infty = g_1^{-1}\text{T}(\mathbb{R})g_\infty$ with the following property. Whenever $\tau > 0$ satisfies
\[
\tau \eta(a) > \frac{1}{2} \log D_K + \kappa \tag{5.4}
\]
and
\[
2\tau \text{h}_{\text{int}}(a) \leq \log D_K - 3\log D_F - \log c, \tag{5.5}
\]
we have
\[
\mu_Y \times \mu_Y([(x, y) : y \in xB_\tau]) \ll_{A_\infty, \varepsilon, c} D_K^{-\frac{1}{2} + \varepsilon} + \frac{1}{D_F^2} D_K^\varepsilon e^{-2\tau \text{h}_{\text{int}}(a)}.
\]

Proof. Suppose that $(x_1, x_2) \in Y \times Y$ is a pair of points with $x_2 \in x_1B_\tau$. Write $x_1 = \text{GL}_4(\mathbb{Q})t_1g$ and $x_2 = \text{GL}_4(\mathbb{Q})t_2g$ for $t_1, t_2 \in T(\mathbb{A})^1$. By assumption on the pair, there is $\gamma \in \text{GL}_4(\mathbb{Q})$ with $\gamma t_1g \in t_2gB_\tau$. By Proposition 5.6 and Assumption (5.4), we have $\gamma \in R(\mathbb{Q})$. In particular, $t_2\gamma t_1^{-1} \in R(\mathbb{A})^1$ and
\[
x_2 \in x_1(B_\tau \cap g^{-1}R(\mathbb{A})^1g).
\]

At this point, one has
\[
\mu_Y \times \mu_Y([(x, y) : y \in xB_\tau]) = \mu_Y \times \mu_Y([(x, y) : y \in x(B_\tau \cap g^{-1}R(\mathbb{A})^1g))]
\]
and would like to apply Theorem 1.7. If $g \in R(\mathbb{A})^1$, this is possible; otherwise, one needs to extend Theorem 1.7 and its proof. We shall do so in the following proposition. \(\square\)

**Proposition 5.8** (An extension of Theorem 1.7). For $[T_g]$ and $[R_g]$ as above we have
\[
\mu_Y \times \mu_Y([(x, y) : y \in x(B_\tau \cap g^{-1}R(\mathbb{A})^1g))]
\]
\[
\ll_{A_\infty, \varepsilon} D_K^{-\frac{1}{2} + \varepsilon} + \frac{1}{D_F^2} D_K^\varepsilon e^{-2\tau \text{h}_{\text{int}}(a)}.
\]

As mentioned, the proof here is largely analogous to the proof of Theorem 1.2. In fact, upon closer inspection the proof of Theorem 1.2 relies chiefly on local computations; we exhibit here the necessary local coordinates in two steps. The following proposition is phrased quite generally (more generally than needed). We recall that for any rational prime $p$, $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} F_v$.

**Proposition 5.9** (Non-Archimedean Block-coordinates). Let $[T_g]$ be a homogeneous toral set in $[\text{GL}_4]$ of maximal type and let $S$ be the subtorus for a subfield $F$ of the associated quartic field $K$. Let $p$ be a rational prime. Then there exists $c_{1, Y, p} \in \text{GL}_4(F_p)$ with the following properties
Theorem 1.2

for each

so we shall

The proof bears many similarities to the proof of Proposition 4.4 so we shall be brief. Observe that instead of block-diagonalizing \( g_p^{-1} \mathbf{R} g_p \) we may as well diagonalize \( g_p^{-1} \mathbf{S} g_p \) (where the eigenvalues need to be ordered so that the first two agree).

It is sufficient to find a basis of any \( \mathcal{O}_{K,p} \)-ideal \( \mathfrak{a} \) where multiplication by the generator \( \alpha \) with \( \mathcal{O}_{F,p} = \mathbb{Z}_p[\alpha] \) takes a desirable form. Let \( \beta \in K_p \) with \( \mathcal{O}_{F,p}[\beta] = \mathcal{O}_{K,p} \) and consider the \( \mathbb{Z}_p \)-basis

\[ \lambda, \lambda \alpha, \lambda \beta, \lambda \beta \alpha \]

of \( \mathfrak{a} \) where \( \lambda \in K_p \) is such that \( \mathfrak{a} = \lambda \mathcal{O}_{K,p} \). In this basis, multiplication by \( \alpha \) is given by the block-diagonal matrix \( M = \text{diag}(M_1, M_1) \) where

\[ M_1 = \begin{pmatrix} 0 & 1 \\ -\text{Nr}_{F_p/Q_p}(\alpha) & \text{Tr}_{F_p/Q_p}(\alpha) \end{pmatrix} \]

Define \( c_{1,Y,p} = P_{(23)} \text{diag}(c, c) \) where \( P_{(23)} \) is the permutation matrix for the transposition \((23)\) and where

\[ c = \begin{pmatrix} 1 & 1 \\ \alpha & \sigma(\alpha) \end{pmatrix}^{-1} \]

The rest of the proof is completely analogous to the proof of Proposition 4.4. \( \square \)

5.3.1. Proof of Proposition 5.8 and Theorem 1.2. We now use Proposition 5.9 to define the local coordinates. For \( p \) a rational prime and \( \gamma \in g_p^{-1} \mathbf{R}(\mathbb{Q}_p) g_p \) we let \((A_p, 0)\) be the block coordinates so that \( A_p \in \text{GL}_2(\mathcal{O}_{F,p}) \) if and only if \( \gamma \in \text{GL}_4(\mathbb{Z}_p) \). Note that the group of points \( s \in \text{GL}_2(\mathbb{F}_p) \) for which \((s, 0)\) is the block coordinate of some element of \( g_p^{-1} T(\mathbb{Q}_p) g_p \) can be written as \( \prod_{v \in \mathcal{V}_F, v \mid p} T'_v(F_v) \) where \( T'_v < \text{GL}_2 \) is an \( F_v \)-torus isomorphic to \( \text{Res}_{K/F}(\mathbb{G}_m) \). Next, we apply Proposition 4.4 for each place \( v \in \mathcal{V}_F \) with \( v \mid p \) to the \( F_v \)-torus \( T'_v \) to obtain the local coordinates \((b_{1,v}, b_{2,v})\) of \( A_v \). We call the tuple \((b_{1,p}, b_{2,p}) = (b_{1,v}, b_{2,v})_{v \mid p}\) the local coordinates of \( \gamma \) (at \( p \)).
The local coordinates have properties much like in Proposition 4.4; we do not list them here.

**Sketch of proof of Proposition 5.8.** The proof of Proposition 4.4 is largely analogous to the proof of Theorem 1.7. In words, one expands the integral over $Y^2$ of the Bowen kernel

$$K_\tau(x, y) = \sum_{\gamma \in \mathcal{R}(Q)} f_\tau(x^{-1}\gamma y)$$

with respect to $\mathcal{T}(Q)\setminus \mathcal{R}(Q)\mathcal{T}(Q)$ and estimates the individual contribution for each point in the above double quotient. These contributions were analysed in §4.3.4 using local coordinates only. Given the above construction of the local coordinates in the current situation one can simply proceed in analogous fashion. □

**Proof of Theorem 1.2.** We use notations in Theorem 1.2. We combine Theorem 5.7 with Proposition 5.1.

Recall that $A' \subset A$ is the set of $a \in A$ with $h_{\text{int}}(a) < \frac{1}{3} h_{[\text{GL}_d]}(a)$, which is equivalent to $\eta(a) > 2 h_{\text{int}}(a)$. In particular, $a \in A'$ implies that $\eta(a) > 0$. For any $\tau_i > 0$ such that

$$\tau_i > \frac{1}{\eta(a)} \left( \frac{1}{2} \log(D_{K_i}) + \kappa \right) \quad (5.6)$$

and

$$\tau_i \leq \frac{1}{2 h_{\text{int}}(a)} (\log D_{K_i} - 3 \log D_{F_i} - \log c) \quad (5.7)$$

we have by Theorem 5.7

$$\mu_{Y_i} \times \mu_{Y_i} \left( \{(x, y) : y \in xB_{\tau_i} \} \right) \ll \varepsilon \left[ D_{K_i}^{\frac{1}{2} + \varepsilon} + \frac{1}{D_{F_i}} D_{K_i}^\varepsilon e^{-2 \tau_i h_{\text{int}}(a)} \right]$$

$$\leq D_{K_i}^{\frac{1}{2} + \varepsilon} + D_{K_i}^\varepsilon e^{-2 \tau_i h_{\text{int}}(a)}.$$  

For the second term here to dominate, we need in addition that

$$\tau_i \leq \frac{1}{2 h_{\text{int}}(a)} (\frac{1}{2} - 2 \varepsilon) \log(D_{K_i}). \quad (5.8)$$

Since $a \in A'$, we have $\eta(a) > 2 h_{\text{int}}(a)$; by our assumption we also have $D_K \gg D_{F_i}^6$. Hence (5.6), (5.7) and (5.8) are compatible for sufficiently small $\varepsilon$ and sufficiently large $i$. Let $\tau_i$ be any time with (5.6) and (5.8). In particular, $\tau_i \to \infty$ as $i \to \infty$ and

$$\mu_{Y_i} \times \mu_{Y_i} \left( \{(x, y) : y \in xB_{\tau_i} \} \right) \ll \varepsilon \left[ D_{K_i}^{\frac{1}{2} + \varepsilon} + D_{K_i}^\varepsilon e^{-2 \tau_i (h_{\text{int}}(a) - \varepsilon)} \right].$$

By Proposition 5.1 this proves the claim. □

**5.4. Further results.** The proof of Theorem 1.2 is somewhat wasteful. Indeed, the additional decay given by the discriminants of the intermediate fields is simply discarded. Of course, in general the discriminant of the intermediate field can grow arbitrarily slowly in comparison to the discriminant of the quartic field. In the following we prove further results under additional restrictions on the relative growth rates of the discriminants.
Corollary 5.10. Assume the notations and conditions in Theorem 1.2 except for the condition \( D_{K_i} \gg D_F^6 \). Furthermore, assume that there exists \( 0 < \alpha < \frac{1}{3} \) with \( D_{K_i}^\alpha \leq D_F \) for every \( i \). Let \( \beta \leq 2 \) be a non-negative number. If there is some \( \delta > 0 \) with
\[
D_F \leq D_{K_i}^{1/(2\beta,3\beta) - \delta},
\]
then there is an open subset \( A'(\beta, \delta) \subset A' \) with
\[
h_\mu(a) \geq h_{\text{int}}(a) + \eta(a)\alpha\beta
\]
for every \( a \in A'(\beta, \delta) \).

Note that Corollary 5.10 implies Theorem 1.13 as \( \eta(a) > 2h_{\text{int}}(a) \) for every \( a \in A' \). As expected, the bounds in the corollary turn into the bounds from Theorem 1.2 when \( \alpha \to 0 \). In both cases of Corollary 5.10 the assumption implies in particular that the discriminant of the intermediate field is at the same power scale as the discriminant of the quartic field.

Proof. The corollary merely consists of an adaptation of the arguments in the proof of Theorem 1.2. We estimate for \( \tau_i \) as in (5.6) and (5.7):
\[
\mu_{Y_i} \times \mu_{Y_i}(\{(x, y) : y \in xB_{\tau_i}\}) \ll_{\varepsilon} D_{K_i}^{-\frac{1}{2} + \varepsilon} + \frac{1}{D_{F_i}^2}D_{K_i}^{-\varepsilon}e^{-2\tau_i h_{\text{int}}(a)}
\]
\[
\leq D_{K_i}^{-\frac{1}{2} + \varepsilon} + D_{F_i}^{-\beta}D_{K_i}^{-\varepsilon}e^{-2\tau_i h_{\text{int}}(a)}.
\]

For the second term to dominate, we need
\[
\tau_i \leq \frac{1}{2h_{\text{int}}(a)} \log \left( D_{K_i}^{1/2}D_{F_i}^{-\beta} \right). \tag{5.9}
\]
Note that (5.6) and (5.9) are compatible if and only if
\[
D_{F_i} \leq D_{K_i}^{\frac{1}{2\beta} \frac{h_{\text{int}}(a)}{3\eta(a)}} \tag{5.10}
\]
Moreover, (5.6) and (5.7) are compatible if and only if
\[
D_{F_i} \leq C(a)D_{K_i}^{\frac{1}{2\beta} \frac{h_{\text{int}}(a)}{3\eta(a)}} \tag{5.11}
\]
where \( C(a) > 0 \) is a constant depending on \( a \). Under the assumptions in the corollary, (5.10) and (5.11) both hold if \( \eta(a)\beta \delta > h_{\text{int}}(a) \). This defines an open subset \( A'(\beta, \delta) \) of \( A' \). Therefore, we have for any \( a \in A'(\beta, \delta) \)
\[
\mu_{Y_i} \times \mu_{Y_i}(\{(x, y) : y \in xB_{\tau_i}\}) \ll_{\varepsilon} D_{K_i}^{-\varepsilon}D_{F_i}^{-\beta}e^{-2\tau_i h_{\text{int}}(a)}
\]
\[
\leq D_{K_i}^{-\alpha\beta + \varepsilon}e^{-2\tau_i h_{\text{int}}(a)}.
\]
Choosing \( \tau_i \) so that (5.6), (5.7), and (5.9) hold, we obtain \( \tau_i \to \infty \) as well as
\[
\mu_{Y_i} \times \mu_{Y_i}(\{(x, y) : y \in xB_{\tau_i}\}) \ll_{\varepsilon} e^{-2\tau_i \eta(a)\alpha \beta}e^{-2\tau_i h_{\text{int}}(a) - \varepsilon}
\]
whenever \( \tau_i \) is sufficiently close to its lower bound in (5.6). Using Proposition 5.1 this proves the corollary. \( \square \)
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