HYPOELLIPTICITY OF A CLASS OF INFINITELY DEGENERATE SECOND ORDER OPERATORS AND SYSTEMS

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Abstract. In this paper we establish a hypoellipticity result for second order linear operators comprised by a linear combination, with infinite vanishing coefficients, of subelliptic operators in separate spaces. This generalizes previous known results.

1. Introduction

An operator $L$ acting on $\mathcal{D}'(\mathbb{R}^n)$, the set of distributions, is said to be hypoelliptic if whenever $u \in \mathcal{D}'(\mathbb{R}^n)$ and $Lu \in C^\infty(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$. A sufficient condition for an operator to be hypoelliptic is subellipticity: $L$ is subelliptic if there exists some $\varepsilon, C > 0$ such that

$$
||u||_2^2 \leq C \left(||(Lu, u)|| + ||u||^2\right)
$$

for all $u \in C^\infty_0(\mathbb{R}^n)$; $||\cdot||_s$ denotes the Sobolev norm of order $s \in \mathbb{R}$ (see Definition 2.1 below), and $||\cdot|| = ||\cdot||_0$ is the $L^2$ norm in $\mathbb{R}^n$. Some necessary and sufficient conditions for subellipticity have been established in terms of associated vector fields by Hörmander in his pivotal paper [3]; and in terms of subunit metric balls by Fefferman and Phong [2]. Subelliptic operators may have ellipticity vanishing locally to at most a finite order.

An operator with infinitely vanishing ellipticity is not subelliptic, such operators do not satisfy the Hörmander condition. The first known hypoellipticity results for infinitely degenerate operators are due to Fedić [1], where the simplest example is $P = \partial_x^2 + k(x) \partial_y^2$ with $k(x) > 0$ for $x \neq 0$, $\sqrt{k}$ is smooth and it is allowed to vanish to any order at the origin. A different criterion for hypoellipticity was developed by Morimoto in Section 2 of [9], where he generalizes the seminal techniques from [1]. Other sufficient conditions for hypoellipticity where obtained by the same author in [10], where the left hand side on the subellipticity condition (1.1) is replaced by logarithmic Sobolev norms.

The hypoellipticity of semilinear operators with principal part satisfying the Hörmander condition was established in [20]. Certain quasilinear operators with infinitely vanishing ellipticity have been studied in two dimensions by Sawyer and Wheeden motivated by applications to Monge-Ampère equations [17, 18]; Rios et al extended these results to a wider class of infinitely vanishing quasilinear equations in higher dimensions [15, 16]. However, hypoellipticity was only obtained for continuous solutions. Previous nonlinear hypoellipticity results had also required extra hypothesis on solutions: in [21] quasilinear subelliptic systems are considered,

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and hypoellipticity is obtained for continuous solutions; in \cite{11,12} hypoeillipticity is obtained for bounded solutions of certain infinitely degenerate quasilinear equations.

Returning to the linear case, Kusuoka and Stroock extended Fediři’s two dimensional result to the case when only $k$ is required to be smooth and it may vanish at any order at the origin \cite{6}. However, in \cite{6} the authors also showed that in higher dimensions hypoellipticity may fail for certain linear operators depending on the vanishing ellipticity order; they in fact obtained a quite spectacular characterization of hypoellipticity for $Q = \partial^2_x + k(x) \partial^2_y + \partial^2_z$: $Q$ is hypoelliptic if and only if $\lim_{x \to 0} x \log k(x) = 0$. Their proofs rely on the Malliavin Calculus. In \cite{9}, Morimoto, using non-probabilistic methods, extended Kusuoka and Stroock’s result to pseudodifferential operators of the form $R = a(x, y, D_x) + g(x') b(x, y, D_y)$ in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $a$ and $b$ are strongly elliptic pseudodifferential operators, $x = (x', x'') \in \mathbb{R}^{n_1} = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $g$ is smooth, $g(x') > 0$ for $x' \neq 0$ and $\lim_{x' \to 0} |x'| \log g(x') = 0$.

In fact, Fediři’s two dimensional result does extend to operators in higher dimensions regardless of the order of vanishing if their structure is similar that of the two dimensional operator $P$. Indeed, $P$ may be written in the form $P = L_1 + k(x) L_2$ in $\mathbb{R} \times \mathbb{R}$, where $L_1 = \partial^2_x$ and $L_2 = \partial^2_y$ are one dimensional elliptic operators (notice that the coefficient $k$ does not depend on the second variable). With this perspective, Morimoto generalized Fediři’s result to pseudodifferential operators of the form $R = a(x, y, D_x) + g(x) b(x, y, D_y)$ in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $a$ and $b$ are strongly elliptic pseudodifferential operators, $g(x) > 0$ for $x \neq 0$, $g$ is smooth and it can vanish at any order at the origin \cite{8}. Over a decade later Kohn \cite{5} proved the hypoellipticity of $R$ in the case that $a(x, D_x) = L_1$ and $b(y, D_y) = L_2$ are only assumed to be differential operators

$$L_k = - \sum_{i,j=1}^{n_k} a_{ij}^k(x^k) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n_k} b_i^k(x^k) \frac{\partial}{\partial x^i} + c^k(x^k),$$

which are subelliptic in $\mathbb{R}^{n_k}$, $k = 1, 2$, respectively.

The purpose of this paper is to generalize Kohn’s result to an arbitrary finite number of subelliptic operators in separate variables, extending the Fediři’s type structure modeled in \cite{11,8}. We also obtain hypoellipticity for systems of linear operators with a similar infinite degeneracy.

**Definition 1.1** (Subelliptic operator). Let $L$ be an operator defined by

$$L = - \sum_{i=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where $a_{ij}$, $b_i$, $c \in C^\infty(U)$ and $(a_{ij})_{i,j=1}^{n} \geq 0$. Then $L$ is subelliptic at $x_0 \in \mathbb{R}^n$ if there exists a neighborhood $U$ of $x_0$ and positive constants $\varepsilon$ and $C$ such that \eqref{L} holds for all $u \in C^\infty_0(\mathbb{R}^n)$. $L$ is called subelliptic if it is subelliptic at each point of $\mathbb{R}^n$.

**Definition 1.2** (Hypoellipticity without loss of derivatives). A linear operator $L$ acting on distributions in $\mathbb{R}^n$ is hypoelliptic if and only if whenever $Lu \in C^\infty(\mathbb{R}^n)$ for some distribution $u$, then $u \in C^\infty(\mathbb{R}^n)$. $L$ is said to be hypoelliptic without loss of derivatives if for given any open set
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\( U \subset \mathbb{R}^n \), then if \( \zeta L u \in H^s(\mathbb{R}^n) \) for all \( \zeta \in C_0^\infty(U) \) then \( \zeta u \in H^s(\mathbb{R}^n) \) for all \( \zeta \in C_0^\infty(U) \).

For fixed positive integers \( n_k, k = 1, \cdots, m \), we denote \( x \in \prod_{k=1}^m \mathbb{R}^{n_k} \) as \( x = (x^1, \cdots, x^{nm}) \in \mathbb{R}^n \), with

\[
    x^k = (x^k_1, \ldots, x^k_{n_k}) \in \mathbb{R}^{n_k}, \quad k = 1, \cdots, m, \quad n = \sum_{k=1}^m n_k,
\]

and we let \( \overline{x}^k \) be the vector obtained from \( x \) by omitting \( x^k \), i.e.

\[
    \overline{x}^k = (x^1, \ldots, x^{k-1}, x^{k+1}, \ldots x^m).
\]

In the scalar case, our main result is the following:

**Theorem 1.3.** Suppose that \( L_k \) as in \((1.2)\) are subelliptic, and \( \lambda_k = \lambda_k(\overline{x}^k) \geq 0 \), \( k = 1, \cdots, m \), are smooth functions. Assume \( \lambda_1 \equiv 1 \), and that for \( 2 \leq k \leq m \), \( \lambda_k(\overline{x}^k) > 0 \) for \( \overline{x}^k \neq 0 \). Then the operator \( L \) defined by

\[
    L = \sum_{k=1}^m \lambda_k L_k
\]

is hypoelliptic without loss of derivatives in \( \mathbb{R}^n \).

The important cases of the above result are when some of the coefficients \( \lambda_k \) have a zero of infinite order at the \( n_k \)-dimensional subspaces \( \overline{x}^k = 0 \) in \( \mathbb{R}^n \). Because of the local nature of the theorem, our results easily generalize to the case when \( \sum_{k=1}^m \lambda_k > 0 \), and \( \lambda_k \) has isolated zeroes in \( \prod_{j \neq k} \mathbb{R}^{n_j} \), \( k = 1, \ldots, m \).

Note that in the case \( m = 2 \) considered by Kohn \([5]\) the coefficient \( \lambda = \lambda_2 \) was allowed to have zeroes of finite order outside \( \overline{x}^2 = 0 \). In this case, the operator \( L(x^1, x^2) = L_1(x^2) + \lambda(x^1) L_2(x^2) \) is subelliptic whenever \( \lambda \) has a zero of finite order and \( L_1, L_2 \) are subelliptic. However, when \( m \geq 3 \) this result is not true. Indeed, the operators \( L_1 = -\partial_z^2 - x^2 \partial_y^2 - y^2 \partial_z^2 \) and \( L_2 = -\partial_z^2 - z^2 \partial_y^2 - y^2 \partial_z^2 \) are not subelliptic in \( \mathbb{R}^3 \) since they are sum of the squares of analytic vector fields which do not satisfy the Hörmander condition. Now, \( L_1 \) is hypoelliptic while \( L_2 \) it is not. See Theorem 1 in \([13]\) to check the first assertion. The proof in \([13]\) relies on the special structure of \( L_1 \), in which the vanishing order of the coefficients is restricted. We consider a different structure, where the degeneracy is localized in space but there is no restrictions to the order of vanishing. On the other hand, to check that \( L_2 \) is not hypoelliptic, it is enough to note its action on the distribution \( u = \delta_{yz} \), where \( \delta_{yz} \) is the Dirac delta function at the origin in \( \mathbb{R}^2 \). Since \( L_2 \) is self-adjoint, for any test function \( \varphi \in C_0^\infty(\mathbb{R}^3) \) we have

\[
    \langle L_2 u, \varphi \rangle = -\langle \delta_{yz}, \varphi_{xx} + z^2 \varphi_{yy} + y^2 \varphi_{zz} \rangle = -\int_{\mathbb{R}^2} \varphi_{xx}(x,0,0) \, dx = 0.
\]

These examples illustrate one of the difficulties in generalizing Kohn’s result to the structure \((1.4)\) including more than two summands.

Our hypoellipticity result extends to linear systems of equations. Our interest in systems primarily arises from a study of an \( n \)-dimensional Monge-Ampère problem. Application of a partial Legendre transformation leads to a system of quasilinear equations. Some results on the regularity of solutions to the quasilinear system associated to an \( n \)-dimensional Monge-Ampère equation were obtained in \([14]\). In
the present paper we consider a general system of second order linear equations. We do not assume any control on the vanishing of the operators’ coefficients, so in general vanishing can be infinite. Linear systems have been studied by many authors and there is a more or less established elliptic theory \cite{7,4}. However, when ellipticity fails much less is known.

We now introduce some notation pertinent to dealing with systems of equations. We let \( u(x) = (u_1(x), \ldots, u_N(x))^T \) be a (column) vector function in \( \mathbb{R}^n \). Given the grouped variables \( x^k = (x_{k1}, \ldots, x_{kn_k}) \in \mathbb{R}^{n_k} \) as before, \( 1 \leq k \leq m \), we denote by \( \nabla_k u \) the \( N \cdot n_k \) column vector \[
\nabla_k u = (\nabla_k u_p)_p = 1 \in \mathbb{R} (N \otimes n_k). 
\]
To make clear the structure of such vectors, we say that \( \nabla_k u \in \mathbb{R} (N \otimes n_k) \). Let \( A^k \) be an \( N \times N \) matrix with \( n_k \times n_k \) matrices as its elements, we write \( A^k \in \mathbb{R} (N \times N \otimes n_k \times n_k) \), i.e.
\[
A^k = (A^k_{pq})_{p,q=1}^{N} \quad A^k_{pq} = (a^k_{pqij})_{i,j=1}^{n_k} \in \mathbb{R} (n_k \times n_k), 
\]
similarly, let \( b^k \) be an \( N \times N \) matrix with \( n_k \)-vectors as its elements, in this case, \( b^k \in \mathbb{R} (N \times N \otimes n_k) \):
\[
b^k = (b^k_{pq})_{p,q=1}^{N} \quad b^k_{pq} = (b^k_{pqij})_{i,j=1}^{n_k} \in \mathbb{R} (n_k),
\]
and let \( c^k \) be an \( N \times N \) matrix \( c^k = (c^k_{pq})_{1 \leq p,q \leq N} \in \mathbb{R} (N \times N) \). We adopt the following multiplication conventions. Whenever \( A \in \mathbb{R} (N \times N \otimes n_k \times n_k) \) and \( v \in \mathbb{R} (N \otimes n_k) \), then \( Av \in \mathbb{R} (N \otimes n_k) \), \( bv \in \mathbb{R} (N) \), and they are given by
\[
Av = \left( \sum_{q=1}^{N} A_{pq} v_q \right)_{p=1}^{N}, \quad bv = \left( \sum_{q=1}^{N} b_{pq} v_q \right)_{p=1}^{N},
\]
where \( A_{ij} \in \mathbb{R} (n_k \times n_k) \), \( b_{ij}, v_j \in \mathbb{R} (n_k) \), \( i,j = 1, \ldots, N \). Given a vector function \( v(x) \in \mathbb{R} (N \otimes n_k) \), we define the divergence operator \( \text{div}_k v \in \mathbb{R} (N) \) as
\[
\text{div}_k v = (\text{div}_k v_p)_p = 1.
\]
With these conventions, we define the systems of linear operators
\[
L^k u = -\text{div}_k A^k \nabla_k u + b^k \nabla_k u + c^k u.
\]
Notice that \( L^k u \in \mathbb{R}^N \), and the principal part of \( L^k \) is
\[
-\text{div}_k A^k \nabla_k u = - \left( \text{div}_k \left( \sum_{q=1}^{N} A_{pq}^k \nabla_k u_q \right) \right)_{p=1}^{N}.
\]
The system \( L^k \) may be expressed in terms of the scalar operators \( L_{pq}^k \)
\[
L_{pq}^k = -\text{div}_k A_{pq}^k (x) \nabla_k + b_{pq}^k (x) \nabla_k + c_{pq}^k (x).
\]
Indeed, we have that the \( p^k \)-component of \( L^k u \) is \( (L^k u)_p = \sum_{q=1}^{N} L_{pq}^k u_q \).

We will assume that each system of operators \( L^k \), \( 1 \leq k \leq m \) is subelliptic in \( \mathbb{R}^{n_k} \), in the following sense:
Definition 1.4 (Subelliptic system). Let $L$ be a linear system given by

$$Lu = -\text{div} A \nabla u + b \nabla u + cu$$

where

$$A = (A_{pq})_{1 \leq p, q \leq N}, \quad b \in \mathbb{R} (N \times N \otimes n), \quad c \in \mathbb{R} (N \times N).$$

Then $L$ is subelliptic at $x^0 \in \mathbb{R}^n$ if there exists a neighborhood $U$ of $x^0$ and positive constants $\varepsilon$ and $C$ such that

$$||u||_2^2 \leq C \left\{ ||(Lu, u)|| + ||u||^2 \right\}$$

for all $u = (u_1, u_2, \ldots, u_N)$ such that $u_i \in C_0^\infty (\mathbb{R}^n), \ i = 1, \cdots, N$. $L$ is called subelliptic if it is subelliptic at each point of $\mathbb{R}^n$.

The main result for systems of equations is the following:

Theorem 1.5. Let $\lambda_k \in C^\infty (\mathbb{R}^n), \ 1 \leq k \leq m$ be such that $\lambda_1 \equiv 1$ and $\lambda_k (x^k) > 0 \ if \ x^k \neq 0$ for $k = 2, \cdots, m$. Let the matrices $A^k$ be symmetric, namely $a^k_{pqij} = a^k_{qipj}$, and assume that for each $1 \leq k \leq m$ the systems $L^k$ are subelliptic in $\mathbb{R}^{n^k}$. Then the operator $L$ defined by

$$L = \sum_{k=1}^m \lambda_k L_k$$

is hypoelliptic in $\mathbb{R}^n$. More precisely, if $u$ is a vector of distributions on $\mathbb{R}^n$ such that $\zeta L u \in \prod_{k=1}^N H^s (\mathbb{R}^n)$ for all $\zeta \in C_0^\infty (U)$ where $U$ is an open set in $\mathbb{R}^n$, then $\zeta u \in \prod_{k=1}^N H^s (\mathbb{R}^n)$ for all $\zeta \in C_0^\infty (U)$. That is, $L$ is hypoelliptic without loss of derivatives.

In this work we broadly follow the line of the proof established by Kohn [5], the presence of more than one function $\lambda_i$ prevents however of a straightforward adaptation of proofs and requires a more delicate analysis. The paper is organized as follows. First, in Section 2 we give some preliminary lemmas that are used further in Section 3 to prove the main $a$-priori estimate, Lemma 3.7. The main result is proved in Section 4 using families of smoothing operators.

2. Preliminaries

In this section we give basic definitions and establish some preliminary results which will be used in our proofs.

The Fourier transform of an integrable function $u$ is defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) \, dx.$$ 

The inverse Fourier transform is given by

$$f^\vee (x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) \, d\xi,$$

where $d\xi = (2\pi)^{-n} \, d\xi$. Note that $f^\vee (x) = (2\pi)^{-n} \hat{f} (x)$ where $\hat{f} (\xi) = f (-\xi)$. 


Definition 2.1. For any $s \in \mathbb{R}$ we define an operator $\Lambda^s$ by the identity

$$\tilde{\Lambda}^s u(\xi) = \left(1 + |\xi|^2 \right)^{s/2} \tilde{u}(\xi)$$

and the norm $|| \cdot ||_s$ by

$$||u||_s = ||\Lambda^s u||_{L^2(\mathbb{R}^n)}$$

For any vector function $u = (u_1, \ldots, u_N)$ we define $\Lambda^s u$ by the identity $\tilde{\Lambda}^s u = (\Lambda^s u_1, \ldots, \Lambda^s u_N)$, with the norm

$$||u||_s = \left( \sum_{p=1}^N ||\Lambda^s u_p||_{L^2(\mathbb{R}^n)}^2 \right)^{1/2}.$$ 

We recall that, more generally, a pseudodifferential operator $P$ with symbol $p(x, \xi)$ is given by

$$P f(x) = \int_{\mathbb{R}^n} e^{i x \cdot \xi} p(x, \xi) \tilde{u}(\xi) \, d\xi.$$ 

Note that if $p(x, \xi) = i \xi_j$, then $P = \frac{\partial}{\partial x_j}$.

Definition 2.2. Given $u \in C^\infty_c(\mathbb{R}^n)$ define the partial Fourier transform $\mathfrak{F}_{x^k} u(x^k, \xi^k)$ by

$$\mathfrak{F}_{x^k} u(x^k, \xi^k) = \int_{\mathbb{R}^n} e^{-i x^k \xi^k} u(x) dx^k.$$ 

For vector functions $u = (u_1, \ldots, u_N)$, $u_i \in C^\infty_c(\mathbb{R}^n)$ we set

$$\mathfrak{F}_{x^k} u \left( x^k, \xi^k \right) = \left( \mathfrak{F}_{x^k} u_1 \left( x^k, \xi^k \right), \ldots, \mathfrak{F}_{x^k} u_N \left( x^k, \xi^k \right) \right).$$

Definition 2.3. For $s \in \mathbb{R}$ define the partial operators $\Lambda_{x^k}^s$ by

$$\mathfrak{F}_{x^k} (\Lambda_{x^k}^s u)(x^k, \xi^k) = \left(1 + |\xi|^2 \right)^{s/2} \mathfrak{F}_{x^k} u(x^k, \xi^k).$$

Similarly, for vector functions $u$, we set $\mathfrak{F}_{x^k} (\Lambda_{x^k}^s u)(x^k, \xi^k) = \left(1 + |\xi|^2 \right)^{s/2} \mathfrak{F}_{x^k} u(x^k, \xi^k)$.

The next lemma is the classical result on a composition of pseudodifferential operators (see for example [19]). In what follows $S^m$ denotes the usual classes $S^m_{1,0}$, of symbols $p(x, \xi)$ satisfying

$$|\partial_x \partial_\xi^\beta p(x, \xi)| \leq B_{\alpha, \beta} \left(1 + |\xi|^2 \right)^{-1/2(m-|\beta|)}, \quad \text{for all } x, \xi, \alpha, \beta.$$

Lemma 2.4. Let $p(x, \xi) \in S^m$ and $q(x, \xi) \in S^k$ then

$$p(x, D) q(x, D) = r(x, D) \in S^{m+k}$$

with

$$r(x, \xi) \sim \sum_{|\alpha| \geq 0} \frac{|\alpha|!}{\alpha!} D_\xi^\alpha p(x, \xi) D_\xi^\alpha q(x, \xi),$$

in the sense that

$$\left( p(x, \xi) - \sum_{|\alpha| < N} D_\xi^\alpha p(x, \xi) D_\xi^\alpha q(x, \xi) \right) \in S^{m+k-N}, \quad \text{for all } N \geq 0.$$
We now give two general lemmas concerning pseudodifferential and subelliptic operators. The following lemma \[5\] is a main tool for dealing with the inner products involving pseudodifferential operators and ordinary derivatives. Roughly speaking, it allows to lower the order of differentiation in an inner product using integration by parts and standard pseudodifferential calculus.

We will localize our estimates by multiplication with suitable cutoff functions. The following concepts will be useful in our microlocal analysis.

**Definition 2.5** (Cutoff functions, supporting relation). We say that \( \varphi \) is a cutoff function in \( \mathbb{R}^n \) if \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and \( 0 \leq \varphi \leq 1 \). Given two measurable functions \( \varphi, \psi \) we introduce the notation \( \varphi \prec \psi \), and we say that \( \psi \) supports \( \varphi \) if \( \psi \) is a cutoff function and \( \psi \equiv 1 \) in a neighbourhood of \( \text{support} \varphi \).

**Lemma 2.6.** Let \( P \) and \( Q \) be pseudodifferential operators of orders \( p \) and \( q \), respectively. Assume that \( P - P^* \) and \( Q - Q^* \) are of orders (at most) \( p-1 \) and \( q-1 \), respectively. Let \( \zeta, \eta \in C_0^\infty(\mathbb{R}^n) \), such that \( \zeta_x \prec \eta \). Then there exists \( C > 0 \) such that for all \( u \in \bigcap_{k=1}^N C^\infty(\mathbb{R}^n) \)

\[
\| (P\zeta u, Q\zeta u) \| \leq C \left( \| \zeta u \|_{(p+q)/2}^2 + \| \eta u \|_{(p+q)/2}^2 \right).
\]

Moreover, if \( u \in \bigcap_{k=1}^N H^r(\mathbb{R}^n) \) with \( r = \max\{p+2, q+1\} \), then the same estimate holds.

**Proof.** For the simplicity of the argument let us consider the scalar case. The desired estimate has already been shown for \( u \in C^\infty(\mathbb{R}^n) \) \[5\]. In case \( u \in H^r(\mathbb{R}^n) \) we find an approximating sequence \( \{u_n\}_{n=1}^\infty \subset C^\infty \) such that \( \lim_{n \to \infty} \| u_n - u \|_r = 0 \). One can check that \( u_n \) defined by \( \tilde{u}_n(\xi) = \exp(-|\xi|^2/n^2)\tilde{u}(\xi) \) satisfies the desired properties for all \( p, q \in \mathbb{R} \). By the definition of \( r \) it follows that \( \lim_{n \to \infty} \| u_n - u \|_{(p+q)/2} = 0 \) and, moreover, by Arzela-Ascoli theorem (replacing \( \{u_n\} \) by an appropriate subsequence, which we still again \( \{u_n\} \)) \( \| P\zeta \partial_x(u_n - u) \| \to 0 \) and \( \| Q\zeta(u_n - u) \| \to 0 \) as \( n \to \infty \). Therefore, applying \( 2.13 \) to \( u_n \) and taking the limit as \( n \to \infty \) we obtain the desired result. \( \square \)

We will henceforth use special families of cutoff functions satisfying the following properties.

- We let \( \sigma_k, \tilde{\sigma}_k, \sigma'_k, \sigma''_k \in C_0^\infty(\mathbb{R}^n) \), \( k = 1, \ldots, m \) be cutoff functions such that \( \sigma_k = 1 \) in a neighbourhood of \( 0 \in \mathbb{R}^n \), and \( \sigma_k \prec \sigma'_k \prec \sigma''_k \). Let
  \[
  \zeta(x) = \prod_{k=1}^m \sigma_k(x^k), \quad \tilde{\zeta}(x), \quad \zeta'(x), \quad \text{and} \quad \zeta''(x)
  \]
  be similarly defined.

- We fix \( U_0^k \) and \( U^k \) to be neighborhoods of the origin in \( \mathbb{R}^n \) such that \( \overline{U}_0^k \subset U^k \) and \( \sigma_k = 1 \) on \( U^k \). Let \( \sigma_0^k, \tilde{\sigma}_0^k \) be cutoff functions in \( \mathbb{R}^n \) with support(\( \sigma_0^k \)) \( \cap U_0^k = \emptyset \), and \( |\nabla x^k \sigma_k| \prec \sigma_0^k \prec \tilde{\sigma}_0^k \). Set \( \zeta_0^k = \sigma_0^k \prod_{l=1, l \neq k}^m \tilde{\sigma}_l(x^l) \)

and \( \tilde{\zeta}_0^k = \sigma_0^k \prod_{l=1, l \neq k}^m \sigma_l'(x^l) \). Note that \( \zeta_0^k \zeta_0^{k_1} = \zeta_0^{k_1} \).

- We choose the cutoffs functions so that they also satisfy \( \sigma_0^k \prec \tilde{\sigma}_k \), \( \tilde{\sigma}_0^k \prec \sigma'_k \).

Hence \( \zeta_0^k \prec \zeta \) and \( \tilde{\zeta}_0^k \prec \zeta' \).

- In the case \( k = 1 \) we write \( \zeta_0 \) for \( \zeta_0^1 \).

The next lemma is the classical result on a composition of pseudodifferential operators (see for example \[19\]). In what follows \( S^m \) denotes the usual classes \( S_{1,0}^m \).
of symbols \( p(x, \xi) \) satisfying
\[
|\partial_x^a \partial_{\xi}^b p(x, \xi)| \leq B_{\alpha, \beta} \left(1 + |\xi|^2\right)^{\frac{1}{2} (m - |\beta|)}, \quad \text{for all } x, \xi, \alpha, \beta.
\]
To carry out an approximation scheme we will define a family of smoothing pseudodifferential operators \([5]\).

**Definition 2.7.** For \( \delta > 0 \) we define \( S_\delta \) by
\[
(2.14) \quad S_\delta u (\xi) = \frac{1}{(1 + \delta^2 |\xi|^2)^{3/2}} u (\xi) = s_\delta (\xi) \hat{u} (\xi).
\]

The operator \( S_\delta \) is partially smoothing; in particular, if \( u \in H^s (\mathbb{R}^n) \), then \( S_\delta u \in H^{s+3} (\mathbb{R}^n) \). We also have:

**Lemma 2.8.** The operator \( S_\delta \) has the following properties:

(i) \( S_\delta \in S^0 \) uniformly in \( \delta \) for \( 0 \leq \delta \leq 1 \), where \( S^0 \) is the symbol class defined by \((2.12)\) with \( m = 0 \). More precisely, for any \( s \in \mathbb{R} \)

\[
\sup_{0 < \delta \leq 1} \|S_\delta u\|_s = \|u\|_s.
\]

(ii) \( S_\delta : H^s \mapsto H^s \) is a bounded operator, with bounds independent of \( \delta \).

(iii) If \( u \in H^{s_0} \) for some \( s_0 \in \mathbb{R} \) then \( S_\delta u \in H^{s_0 + 3} \).

(iv) If for any \( s \in \mathbb{R} \), \( u \in H^{s-3} \) and \( \lim_{\delta \to 0^+} \|S_\delta u\|_s \leq C \), then \( u \in H^s \).

(v) If \( a \in C^\infty (\mathbb{R}^n) \), then

\[
[a, S_\delta] = \sum_{k=1}^n \sum_{j=1}^{n_k} \left(a_{k,j} \frac{\partial}{\partial x_j}\right) R^{-2}_\delta S_\delta + Q^{-2}_\delta S_\delta
\]
where \( R^{-2}_\delta \) and \( Q^{-2}_\delta \) are families of pseudodifferential operators of order \(-2\) uniformly in \( \delta \) for \( 0 \leq \delta \leq 1 \).

**Proof.** Since \( \exp \left(\frac{1}{(1 + \delta^2 |\xi|^2)^{3/2}}\right) \leq 1 \), we have
\[
\|S_\delta u\|_s^2 = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left(\frac{1}{(1 + \delta^2 |\xi|^2)^{3/2}}\right)^2 |\hat{u}|^2 \, d\xi 
\leq \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{u}|^2 \, d\xi = \|u\|_s^2
\]
This proves (i) and (ii). Property (iii) follows easily from the definition of \( S_\delta \).

On the other hand, if \( u \in H^{s-3} \) and \( \lim_{\delta \to 0^+} \|S_\delta u\|_s \leq C \) then
\[
\|u\|_s = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{u}|^2 \, d\xi 
\leq \lim_{\delta \to 0^+} \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left(\frac{1}{(1 + \delta^2 |\xi|^2)^{3/2}}\right)^2 |\hat{u}|^2 \, d\xi 
\leq \lim_{\delta \to 0^+} \|\Lambda_\delta S_\delta u\|_s^2 \leq C^2.
\]
functions $u$ the proof for the scalar case.

We will state the results for systems of equations, with the understanding that linear equations correspond to the case (1.9). We only include systems cases do not differ in any substantial way, so, for simplicity, we only include the proof for the scalar case.

By differentiating $\xi_j^{k/2}2^2 S_\delta$ with respect to $\xi_j$, one can check that the lower order terms have the form $Q^{-2}_\delta S_\delta$.

3. Apriori estimates

In what follows we establish estimates both for scalar functions $u$ and vector functions $u$, as well as scalar operators $L$ (1.4) and linear systems of operators $L$ (1.3). We will state the results for systems of equations, with the understanding that linear equations correspond to the case $N = 1$. The proofs for the scalar or systems cases do not differ in any substantial way, so, for simplicity, we only include the proof for the scalar case.

The next lemma gives a useful estimate for subelliptic operators. It follows directly from the definition of subellipticity (111) with the help of Lemma 2.4.

**Lemma 3.1.** Suppose that, $L$, $L$, given by (1.5), (1.7) respectively, are subelliptic at $x_0$ and that $\zeta, \tilde{\zeta} \in C^\infty_0 (U)$, with $U$ a neighbourhood of $x_0$ as in (1.7), and $\zeta \preceq \tilde{\zeta}$. Then, for all $u \in C^\infty (\mathbb{R}^n),$

\begin{align*}
(3.15) \quad \|\zeta u\|_x^2 & \leq C \left\{ \sum_{i,j=1}^n \left( a_{pqij} \zeta (u_q)_{x_i} \zeta (u_p)_{x_j} \right) + \left\| \tilde{\zeta} u \right\|^2 \right\} \\
& \leq C \left\{ \|\zeta Lu, \zeta u\| + \left\| \tilde{\zeta} u \right\|^2 \right\}.
\end{align*}

**Proof.** We consider only the scalar case $N = 1$. From (111) we have

\begin{equation}
(3.16) \quad \|\zeta u\|_x^2 \leq C \left\{ \|L\zeta u, \zeta u\| + \left\| \tilde{\zeta} u \right\|^2 \right\}.
\end{equation}

Next,

\begin{equation}
(3.17) \quad (L\zeta u, \zeta u) = - \sum_{i=1}^n \left( a_{ij} (\zeta u)_{x_i} x_j, \zeta u \right) + \sum_{i=1}^n \left( b_i (\zeta u)_{x_i}, \zeta u \right) + (c\zeta u, \zeta u).
\end{equation}

Integrating by parts the first term on the right, we have that

\begin{align*}
- (a_{ij} (\zeta u)_{x_i} x_j, \zeta u) &= \sum_{i=1}^n \left( (a_{ij})_{x_i} (\zeta u)_{x_i}, \zeta u \right) + \sum_{i=1}^n \left( a_{ij} (\zeta u)_{x_i}, (\zeta u)_{x_j} \right) \\
&= \sum_{i=1}^n \left( (a_{ij})_{x_i} (\zeta u)_{x_i}, \zeta u \right) + 2 \sum_{i=1}^n \left( a_{ij} (\zeta u)_{x_i}, \zeta x_i u \right) \\
&\quad + \sum_{i=1}^n \left( a_{ij} \zeta x_i u, \zeta x_i u \right) \quad \text{and} \quad \sum_{i=1}^n \left( a_{ij} \zeta x_i u, \zeta u \right) \quad \text{and} \quad \sum_{i=1}^n \left( a_{ij} \zeta x_i u, \zeta u \right).
\end{align*}

By Lemma 2.4 the first and second terms on the right are bounded by $C\|\tilde{\zeta} u\|^2$, while it is clear that the third term on the right also satisfies the same bounds. The
same applies to the last two terms on the right of (3.17). This and (3.16) yield
\[ \|\zeta u\|_\varepsilon^2 \leq C\left\{ \sum_{i,j=1}^n (a_{ij} \zeta u_{x_i}, \zeta u_{x_j}) + \|\zeta u\|^2 \right\}. \]
prove the first inequality in (3.15).

To prove the second inequality in (3.15), integrating by parts we write
\[ \sum (a_{ij} \zeta u_{x_i}, \zeta u_{x_j}) = - \sum (a_{ij} \zeta u_{x_i}, \zeta u) \]
\[ = (\zeta Lu, \zeta u) - (c \zeta u, \zeta u) - \sum_{i=1}^n (b_i \zeta u_{x_i}, \zeta u) \]
\[ - \sum (a_{ij} \zeta^2)_{x_j} \zeta u_{x_i}, \zeta u), \]
and apply Lemma 2.6 the the last two terms on the right.

Next, we will establish a number of auxiliary results which will be used to prove the main a priori estimate and the main theorem.

The following lemma is a generalization of Lemma 3.1 for the operators and systems of operators of the form (1.4), (1.9).

**Lemma 3.2.** Let \( L \) be defined by (1.9) (or by (1.4) when \( N = 1 \)), with \( L_k \) subelliptic at \( x_0^k \in \mathbb{R}^{nk} \) and 0 \( \leq \lambda_k \in C^\infty(\mathbb{R}^{nk}), k = 1, \ldots, m \). Let \( U_k \subset \mathbb{R}^{nk} \) be neighbourhoods of \( x_0^k \) such that (1.8) holds with \( \varepsilon_k \) for \( L_k \) in \( U_k \) (resp. (1.7) holds with \( \varepsilon_k \) for \( L_k \) in \( U_k \)). Then if \( \zeta, \tilde{\zeta} \in C^\infty_0(\prod_{k=1}^m U_k) \), with \( \zeta \prec \tilde{\zeta} \), for \( \varepsilon = \min_k \varepsilon_k \) we have
\[
\sum_{k=1}^m \left\| \sqrt{\lambda_k} A_{x_k}^\varepsilon(\zeta u) \right\|^2 \leq C \sum_{k=1}^m \sum_{i,j=1}^{nk} \sum_{p,q=1}^{n} \left( \zeta \lambda_k a_{pqij}^k (u_q)_{x_i} \zeta (u_p)_{x_j} \right) \]
\[
+ C \left\{ \left( (\zeta Lu, \zeta u) \right) + \|\tilde{\zeta} u\|^2 \right\}. \tag{3.18}
\]
Moreover, the same estimate holds when \( u \in \prod_{k=1}^N H^2(\mathbb{R}^n) \).

**Proof.** It is enough to consider the scalar case \( N = 1 \). First, consider \( u \in \prod_{k=1}^N C^\infty(\mathbb{R}^n) \).

Since for each \( k \) the operator \( L_k \) is subelliptic it follows from (3.15) that for each fixed \( x^k \in \mathbb{R}^{nk} \) we have
\[
\lambda_k(\tilde{x}^k) \int_{\mathbb{R}^{nk}} |A_{x_k}(\zeta u)(x)|^2 \, dx^k \]
\[
\leq C \sum_{i,j=1}^{n_k} \int_{\mathbb{R}^{nk}} \lambda_k(x^k) a_{ij}^k(x^k) \zeta(x) u_{x_i}^k(x) \zeta(x) u_{x_j}^k(x) \, dx^k \]
\[
+ C \int_{\mathbb{R}^{nk}} |\tilde{\zeta}(x) u(x)|^2 \, dx^k. \]

Integrating the above inequality with respect to \( \tilde{x}^k \) and summing over \( k = 1, \ldots, m \) we obtain the first part of (3.18). To show that the inequality holds for \( u \in \prod_{k=1}^N H^2(\mathbb{R}^n) \) we perform an approximation in the same way it has been done in the proof of Lemma 2.6.
To prove the second part consider each term of the triple sum in the first inequality of the lemma and integrate by parts

\[(3.19) \quad (\lambda k a^{k}_{ij} \zeta u_{x_{i}x_{j}}, \zeta u) = - (\lambda k a^{k}_{ij} \zeta u_{x_{i}x_{j}}, \zeta u) - \left( (\lambda k a^{k}_{ij} \zeta^{2})_{x_{i}x_{j}} u_{x_{i}x_{j}}, \zeta u \right) .\]

We then have

\[
\sum_{k=1}^{m} \sum_{i,j=1}^{n} \left( \lambda k a^{k}_{ij} \zeta u_{x_{i}x_{j}}, \zeta u \right) = (\zeta L u, \zeta u) - \sum_{k=1}^{m} \sum_{i=1}^{n} \left( (\lambda k b^{k}_{i} \zeta u_{x_{i}}), \zeta u \right) - \sum_{k=1}^{m} \sum_{i,j=1}^{n} \left( (\lambda k a^{k}_{ij} \zeta^{2})_{x_{i}x_{j}} u_{x_{i}x_{j}}, \zeta u \right).
\]

The second inequality of the Lemma 3.2 then follows from Lemma 2.6.

\[\square\]

We now formulate the main technical result which allows us to deal with terms involving commutators \([L, \Lambda^{s} \zeta] \). The proof relies on Lemma 2.6 and Lemma 2.4.

**Lemma 3.3.** Given \( s \in \mathbb{R} \), there exists \( C > 0 \) such that

\[
\left\| \left( \hat{\zeta} [L, \Lambda^{s} S_{\delta} \zeta] u, \zeta \Lambda^{s} S_{\delta} \zeta u \right) \right\| \leq C \left\{ \left\| \hat{\zeta} S_{\delta} \zeta u \right\|_{s}^{2} + \sum_{k=1}^{m} \left\| \lambda k c^{k}_{i} S_{\delta} \zeta^{2} u \right\|_{s} + \left\| \zeta \Lambda^{s} \zeta \right\|_{s-1/2}^{2} \right\}
\]

for all functions \( u \in \prod_{k=1}^{N} H^{s-2}(\mathbb{R}^{n}) \) and all \( 0 < \delta \leq 1 \). Here \( \zeta, \zeta \), and \( \zeta_{0} \) are the cutoff functions defined above.

**Proof.** Again, it is enough to consider the scalar case \( N = 1 \). We have

\[(3.20) \quad [L, \Lambda^{s} S_{\delta} \zeta] u = [L, \Lambda^{s} S_{\delta}] \zeta u + \Lambda^{s} S_{\delta} [L, \zeta] u \]

Next, for any \( k = 1, \ldots, m \),

\[(3.21) \quad [\lambda k L_{k}, \Lambda^{s} S_{\delta}] = - \sum_{i,j=1}^{m} \left[ \lambda k a^{k}_{ij}, \Lambda^{s} S_{\delta} \right] \frac{\partial^{2}}{\partial x_{i}^{2} \partial x_{j}^{2}} + \sum_{i=1}^{n} \left[ \lambda k b^{k}_{i}, \Lambda^{s} S_{\delta} \right] \frac{\partial}{\partial x_{i}} + \left[ \lambda k c^{k}, \Lambda^{s} S_{\delta} \right]. \]

and

\[(3.22) \quad [\lambda k L_{k}, \zeta] = \lambda k [L_{k}, \zeta]
= \lambda k \left( - \sum_{i,j=1}^{m} a^{k}_{ij} \zeta_{x_{i}x_{j}} + \sum_{i=1}^{n} b^{k}_{i} \zeta_{x_{i}} \right) - 2 \lambda k \sum_{i,j=1}^{n} a^{k}_{ij} \zeta_{x_{i}x_{j}} \frac{\partial}{\partial x_{j}}, \]

where we used the symmetry of \( a^{k} \) on the last term. It follows that

\[
\left( \hat{\zeta} [L, \Lambda^{s} S_{\delta} \zeta] u, \zeta \Lambda^{s} S_{\delta} \zeta u \right) \]

\[
= - \sum_{k=1}^{m} \sum_{i,j=1}^{n} \left( \hat{\zeta} [\lambda k a^{k}_{ij}, \Lambda^{s} S_{\delta}] \frac{\partial^{2}}{\partial x_{i}^{2} \partial x_{j}^{2}} \zeta u, \zeta \Lambda^{s} S_{\delta} \zeta u \right) + \sum_{k=1}^{m} \sum_{i,j=1}^{n} \left( \hat{\zeta} [\lambda k b^{k}_{i}, \Lambda^{s} S_{\delta}] \frac{\partial}{\partial x_{i}} \zeta u, \zeta \Lambda^{s} S_{\delta} \zeta u \right)
\]
Using Lemma 2.4 and property (v) of the operator $S_\delta$, for a smooth function $f$ we have

$$[\Lambda^s S_\delta, f] = \tilde{R}_{s,\delta}^{0} \Lambda^{s-1} f_0 S_\delta = \tilde{R}_{s,\delta}^{0} \Lambda^{s-2} \sum_{i=1}^{m} \sum_{i,j=1}^{n_i} f_{x_i}^{x_j} \frac{\partial^2}{\partial x_i \partial x_j} S_\delta + \tilde{Q}_{s,\delta}^{-0} \Lambda^{s-2} f_0 S_\delta,$$

where $\tilde{R}_{s,\delta}^{0}, \tilde{R}_{s,\delta}^{0}, \tilde{Q}_{s,\delta}^{0}$ are pseudodifferential operators of order 0 uniformly in $0 \leq \delta \leq 1$, and $f_0$ is a cutoff function such that $f_0 \succ |\nabla f|$. Moreover, $\tilde{R}_{s,\delta}^{0} = \left( \tilde{R}_{s,\delta}^{0} \right)^*$ is of degree $-1$ uniformly in $0 \leq \delta \leq 1$.

We use this to estimate the first term on the right in (3.21), we obtain

$$|I| \leq \sum_{k,r=1}^{m} \sum_{i,j=1}^{n_k} \left( \tilde{\zeta} R_{s,\delta}^{0} \Lambda^{s-2} \left( \lambda_k a_{ij}^{k} \right)_{x_i} \frac{\partial^2}{\partial x_i \partial x_j} S_\delta \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) + \sum_{k=1}^{m} \sum_{i,j=1}^{n_k} \left( \tilde{\zeta} Q_{s,\delta}^{-0} \Lambda^{s-2} \left( \lambda_k a_{ij}^{k} \right)_{0} \frac{\partial^2}{\partial x_i \partial x_j} S_\delta \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right).$$

We apply to each term on the first sum Lemma 2.6 with

$$P = \tilde{R}_{s,\delta}^{0} \Lambda^{s-2} \left( \lambda_k a_{ij}^{k} \right)_{x_i} \frac{\partial^2}{\partial x_i \partial x_j}, \quad \text{and} \quad Q = \tilde{\zeta} \Lambda^s.$$

Note, that both operators $P$ and $Q$ have order $s$, and since $u \in H^{s-1}$ it follows that $S_\delta u \in H^{s+1}$ and therefore Lemma 2.6 is applicable. Since the second term on the right of (3.25) is dominated by $C \left( \left\| \tilde{\zeta} S_\delta \zeta u \right\|_s^2 + \left\| S_\delta \zeta u^2 \right\|_{s-1} \right)$, we obtain

$$|I| \leq C \left( \left\| \tilde{\zeta} S_\delta \zeta u \right\|_s^2 + \left\| S_\delta \zeta u \right\|_{s-1}^2 \right).$$

By the first identity in (3.24) it follows that

$$|II| + |III| \leq \sum_{k=1}^{m} \sum_{i,j=1}^{n_k} \left( \tilde{\zeta} \tilde{R}_{s,\delta}^{0} \Lambda^{s-1} \left( \lambda_k a_{ij}^{k} \right)_{0} \frac{\partial}{\partial x_i} S_\delta \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) + \sum_{k=1}^{m} \left( \tilde{\zeta} \tilde{R}_{s,\delta}^{0} \Lambda^{s-1} \left( \lambda_k a_{ij}^{k} \right)_{0} S_\delta \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right)$$

$$\leq C \left( \left\| \tilde{\zeta} S_\delta \zeta u \right\|_s^2 + \left\| S_\delta \zeta u \right\|_{s-1}^2 \right).$$
Using that \( \zeta_{x^k x_j} = \zeta^k_x \zeta_{x^j} \) and \( \zeta_x = \zeta^k_0 \zeta_{x^k} \), and (3.24), it follows that

\[
|IV| + |V| \leq \sum_{k=1}^m \sum_{i,j=1}^{n_k} \left| \left( \tilde{\zeta} \Lambda^* \delta a_{ij}^k \zeta_{x^k x_j} u \right) \right| \Lambda^* \delta \zeta u
\]

\[
+ \sum_{k=1}^m \sum_{i=1}^{n_k} \left( \tilde{\zeta} \Lambda^* \delta \bar{b}_i^k \zeta_{x^k} \Lambda^* \delta \zeta u, \tilde{\zeta} \Lambda^* \delta \zeta u \right)
\]

(3.28)

\[
\leq C \left( \| \tilde{\zeta} \delta u \|^2 + \sum_{k=1}^m \| \Lambda^* \delta \zeta_{x^k} u \|^2 + \| \Lambda^* \delta \zeta u \|^2 \right).
\]

Now, for each term in \( VI \) we commute the functions \( \Lambda^* \delta a_{ij}^k \zeta_{x^k} \) and \( \zeta \), and carry out an integration by parts. We obtain the identity

\[
VI^{kij} = 2 \left( \tilde{\zeta} \Lambda^* \delta \zeta_{x^k} \left( \frac{\partial}{\partial x^j} \right) u \right)
\]

(3.29)

\[
= - \left( \tilde{\zeta} \Lambda^* \delta \zeta u, \tilde{\zeta} \Lambda^* \delta \left( \Lambda^* \delta a_{ij}^k \zeta_{x^k} u \right) \right)
\]

\[
- 2 \left( \tilde{\zeta} \Lambda^* \delta \zeta u, \tilde{\zeta} \Lambda^* \delta \left( \Lambda^* \delta a_{ij}^k \zeta_{x^k} u \right) \right)
\]

\[
- \left( \tilde{\zeta} \Lambda^* \delta \zeta_{x^k} u, \tilde{\zeta} \Lambda^* \delta \left( \Lambda^* \delta a_{ij}^k \zeta_{x^k} u \right) \right)
\]

\[
+ \left( \tilde{\zeta} \Lambda^* \delta \zeta_{x^k} u, \tilde{\zeta} \Lambda^* \delta \left( \Lambda^* \delta \Lambda^* \delta \zeta_{x^k} u \right) \right)
\]

\[
+ \left( \tilde{\zeta} \Lambda^* \delta \zeta_{x^k} u, \tilde{\zeta} \Lambda^* \delta \left( \Lambda^* \delta a_{ij}^k \zeta_{x^k} u \right) \right)
\]

\[
+ \left( \tilde{\zeta} \Lambda^* \delta \zeta_{x^k} u, \tilde{\zeta} \left( \Lambda^* \delta \Lambda^* \delta \zeta_{x^k} u \right) \right)
\]

\[
= VI^{kij}_1 + VI^{kij}_2 + \cdots + VI^{kij}_7.
\]

We now consider each term. We have

\[
\left| VI^{kij}_1 \right| \leq \left| \left( \tilde{\zeta} \Lambda^* \delta \zeta_{x^k} \left( \frac{\partial}{\partial x^j} \right) u \right) \right|
\]

\[
+ \left| \left( \tilde{\zeta} \Lambda^* \delta \zeta u, \tilde{\zeta} \Lambda^* \delta \left( \Lambda^* \delta a_{ij}^k \zeta_{x^k} u \right) \right) \right|
\]

(3.30)

\[
\leq C \left\{ \sum_{s=0}^7 \left[ \| \tilde{\zeta} \delta u \|^2 + \| \Lambda^* \delta \zeta_{x^k} u \|^2 + \| \Lambda^* \delta \zeta u \|^2 \right] \right\}.
\]

We will use the following Wirtinger-type inequality (see e.g. Appendix in [18]): If \( \phi \in C^2(U) \) with \( U \) open in \( \mathbb{R}^n \), \( \phi \) nonnegative, then for any compact subset \( F \subset U \) there exists a constant \( C \) depending on \( \| D^2 \phi \|_{L^\infty(V)} \), with \( V \) open and \( F \subset V \subset U \),
We consider the penultimate term in (3.30),
\[ |D\phi(x)|^2 \leq C\phi(x). \]  
We obtain
\[
\left\| (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u \right\|_s^2 = \left( \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u, \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u \right).
\]
We commute \( \zeta_{x^+} \) from the right into the left and \( (\lambda_k)_{x^+} \) from the left into the right. We obtain
\[
\left\| (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u \right\|_s^2 = \left( \Lambda^* \zeta_k^k S_\delta \left( \zeta_{x^+} \right)^2 u, \Lambda^* \left( (\lambda_k)_{x^+} \right)^2 \zeta_k^k S_\delta u \right)
+ \left( \left[ \Lambda^*, (\lambda_k)_{x^+} \right] \zeta_k^k S_\delta \left( \zeta_{x^+} \right)^2 u, \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta u \right)
+ \left( \left[ \zeta_{x^+}, \Lambda^* \right] (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u, \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta u \right)
+ \left( \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u, \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u \right)
+ \left( \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u, \Lambda^* (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u \right)
+ \left( \left( (\lambda_k)_{x^+} \right)^2 \zeta_k^k S_\delta \zeta_{x^+} u, \left( (\lambda_k)_{x^+} \right)^2 \zeta_k^k S_\delta \zeta_{x^+} u \right) + C \left\| \zeta S_\delta \zeta u \right\|_{s-1/2}^2.
\]
The first term on the right is bounded by
\[
\left\| \zeta_k^k S_\delta \left( \zeta_{x^+} \right)^2 u \right\|_s^2 \leq \left\| \left( \zeta_{x^+} \right)^2 \Lambda^* \zeta_k^k S_\delta \zeta u \right\|_s^2 + \left\| \left[ \Lambda^*, (\zeta_{x^+} \right)^2 \zeta_k^k S_\delta \zeta u \right\|_s^2
+ \left\| \Lambda^* \zeta_k^k \left[ S_\delta, (\zeta_{x^+} \right)^2 \right] \zeta u \right\|_s^2.
\]
By the Wirtinger inequality (3.31), it follows that
\[
\left\| \zeta_k^k S_\delta \left( \zeta_{x^+} \right)^2 u \right\|_s^2 \leq C \left\{ \left\| \zeta \Lambda^* \zeta_k^k S_\delta \zeta u \right\|_s^2 + \left\| \zeta S_\delta \zeta u \right\|_{s-1}^2 \right\}
\]
(3.33)
Similarly, the second term on the right of (3.32) is bounded by
\[
\left\| \left( (\lambda_k)_{x^+} \right)^2 \zeta_k^k S_\delta \zeta_{x^+} u \right\|_s^2 \leq C \left\{ \left\| \lambda_k \zeta_k^k S_\delta \zeta_{x^+} u \right\|_s^2 + \left\| \zeta S_\delta \zeta u \right\|_{s-1}^2 \right\}.
\]
We obtain
\[
\left\| (\lambda_k)_{x^+} \zeta_k^k S_\delta \zeta_{x^+} u \right\|_s^2 \leq C \left\{ \left\| \zeta S_\delta \zeta u \right\|_s^2 + \left\| \lambda_k \zeta_k^k S_\delta \zeta_{x^+} u \right\|_s^2 + \left\| \zeta \zeta S_\delta \zeta u \right\|_{s-1/2}^2 \right\}.
\]
Plugging this estimate on the right of \((3.30)\) yields

\[
(3.34) \quad |V I_1^{kij}| \leq C \left\{ \| \tilde{\zeta} S_0 \tilde{\zeta} u \|_s^2 + \| \zeta' S_0 \tilde{\zeta} u \|_s^2 + \| \lambda_k \partial_{x_0} S_0 \partial_{x_0} \partial_{x_0} \tilde{\zeta} u \|_s^2 \right\} + C \| \tilde{\zeta} S_0 \tilde{\zeta} u \|_{s-1/2}.
\]

It easily follows that

\[
|V I_2^{kij}| \leq 2 \left\{ \left( \tilde{\zeta} \Lambda^* S_0 \tilde{\zeta} u, \tilde{\zeta}^* \Lambda^* S_0 \lambda_k a^k_{ij} \tilde{\zeta} \right)^2 u \right\} \leq \left\{ \| \tilde{\zeta} S_0 \tilde{\zeta} u \|_s^2 + \| \lambda_k \partial_{x_0} S_0 \partial_{x_0} \tilde{\zeta} u \|_s^2 + \| \zeta' S_0 \tilde{\zeta} u \|_{s-1}^2 \right\}.
\]

Commuting \(\tilde{\zeta} \) \(H_x^k\) into the left, we have that

\[
(3.35) \quad |V I_3^{kij}| \leq C \left\{ \| \tilde{\zeta} S_0 \tilde{\zeta} u \|_s^2 + \| \lambda_k \partial_{x_0} S_0 \partial_{x_0} \tilde{\zeta} u \|_s^2 + \| \zeta' S_0 \tilde{\zeta} u \|_{s-1}^2 \right\}.
\]

Applying \((3.33)\) to the first term on the right we obtain

\[
(3.36) \quad |V I_3^{kij}| \leq C \left\{ \| \tilde{\zeta} S_0 \tilde{\zeta} u \|_s^2 + \| \lambda_k \partial_{x_0} S_0 \partial_{x_0} \tilde{\zeta} u \|_s^2 + \| \zeta' S_0 \tilde{\zeta} u \|_{s-1}^2 \right\}.
\]

Using the identity \((3.24)\) for \([\tilde{\zeta}, \Lambda^* S_0]\), we obtain

\[
(3.37) \quad |V I_4^{kij}| \leq \sum_{r=1}^m \sum_{l=0}^n \left\{ \left( \tilde{\zeta} R_{s,0}^0 \Lambda^* \zeta_{x_0} \tilde{\zeta} \partial_{x_0} \partial_{x_0} S_0 \tilde{\zeta} \partial_{x_0} \partial_{x_0} S_0 \partial_{x_0} \tilde{\zeta} u, \tilde{\zeta} S_0 \tilde{\zeta} u \right) \right\} + C \left\{ \| \lambda_k \partial_{x_0} S_0 \partial_{x_0} \partial_{x_0} \tilde{\zeta} u \|_s^2 + \| \zeta' S_0 \tilde{\zeta} u \|_{s-1}^2 \right\}.
\]

We treat the first term on the right of \((3.37)\) in a similar way as we obtain \((3.32)\), we commute \(\tilde{\zeta} \) \(H_x^k\) into the left. We proceed in the same way with \(V I_5^{kij}\), to obtain

\[
(3.38) \quad |V I_4^{kij}| + |V I_5^{kij}| \leq C \left\{ \| \tilde{\zeta} S_0 \tilde{\zeta} u \|_s^2 + \| \lambda_k \partial_{x_0} S_0 \partial_{x_0} \partial_{x_0} \tilde{\zeta} u \|_s^2 + \| \zeta' S_0 \tilde{\zeta} u \|_{s-1/2} \right\}.
\]

The treatment of \(V I_6^{kij}\) and \(V I_7^{kij}\) is similar. Applying \((3.24)\) to \([\lambda_k a^k_{ij} \zeta, \Lambda^* S_0]\), we obtain

\[
(3.39) \quad |V I_6^{kij}| \leq \sum_{r=1}^m \sum_{l=1}^n \left\{ \left( \tilde{\zeta} \Lambda^* S_0 \tilde{\zeta} \partial_{x_0} \partial_{x_0} \tilde{\zeta} u, \tilde{\zeta} R_{s,0}^0 \Lambda^* \zeta_{x_0} \partial_{x_0} \partial_{x_0} S_0 \tilde{\zeta} u \right) \right\}.
\]
We write the terms in the sum above as

\[
\left(\zeta \Lambda^s S_{\delta} \frac{\partial}{\partial x_j^k} u, \tilde{\zeta} R_{s, \delta}^0 \Lambda^{s-2} \left(\lambda_k a_{ij}^k \zeta x^k_t\right) \frac{\partial}{\partial x^j} S_{\delta} u \right)
\]

We estimate the first term on the right in the same way as we did (3.30-3.32). Since Combining estimates (3.34), (3.35), (3.36), (3.38), and (3.40) yields (3.41), we obtain

\[
\sum_{i,j=1}^m \left| V I^{kij}_6 \right| \leq 2 \sum_{k=1}^m \left| V I^{kij}_6 \right| + \sum_{i,j=1}^m \left| V I^{kij}_7 \right| \leq C \left\{ \left\| \zeta S_{\delta} u \right\|_{s}^{2} + \left| \tilde{\zeta} S_{\delta} \zeta u \zeta \Lambda^s \zeta \zeta' \right\|_{s-1/2}^{2} \right\}
\]

Combining estimates (3.34), (3.35), (3.36), (3.38), and (3.40) yields

\[
\sum_{k=1}^m \left\| \sqrt{\lambda_k} \Lambda^s S_{\delta} \zeta u \right\|_{s}^{2} \leq C \left\{ \left\| \zeta S_{\delta} \zeta u \right\|_{s}^{2} + \left| \tilde{\zeta} S_{\delta} \zeta u \zeta \Lambda^s \zeta \zeta' \right\|_{s-1/2}^{2} \right\}
\]

for all \( u \in \prod_{k=1}^N H^s(\mathbb{R}^n) \).
Proof. Since \( u \in \prod_{k=1}^{N} H^{s-1}(\mathbb{R}^n) \) we have that \( \Lambda^s S_\delta \zeta u \in \prod_{k=1}^{N} H^2(\mathbb{R}^n) \) so we can replace \( \zeta \) by \( \zeta \) and \( u \) by \( \Lambda^s S_\delta \zeta u \) in Lemma 3.2 to obtain

\[
\sum_{k=1}^{m} \left\| \sqrt{\lambda_k} \Lambda^s_{x_k} \left( \zeta \Lambda^s S_\delta \zeta u \right) \right\|^2 \leq C \left\{ \left( \zeta \Lambda^s S_\delta \zeta u, \zeta \Lambda^s S_\delta \zeta u \right) + \| \zeta' \Lambda^s S_\delta \zeta u \|^{2} \right\}.
\]

The corollary follows by commuting \( L \) with \( \Lambda^s S_\delta \zeta \) and applying Lemma 3.3. \( \square \)

Lemma 3.5. There exists \( C > 0 \) such that

\[
\sum_{k=1}^{m} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k u \|_s^2 \leq C \left\{ \left( \zeta' \Lambda^{s-\varepsilon} S_\delta \zeta' L u, \zeta' \Lambda^{s-\varepsilon} S_\delta \zeta' u \right) + \| \zeta'' S_\delta \zeta'' u \|_{s-\varepsilon} \right\}
\]

Proof. We have

\[
\sum_{k=1}^{m} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k u \|_s^2 = \sum_{k=1}^{m} \sum_{p=1}^{N} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k \|_s^2 \\
\leq \sum_{k=1}^{m} \sum_{p=1}^{N} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k u_p \|_s^2 + C \| \zeta' S_\delta \zeta u \|_{s-1} \\
\leq C \sum_{k=1}^{m} \sum_{p=1}^{N} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k u_p \|_s^2 + C \| \zeta' S_\delta \zeta u \|_{s-1}.
\]

Now we recall that by the hypotheses on \( \lambda_k \), we have that if \( 1 \leq \ell \leq m \), and \( \ell \neq k \) then \( \lambda_\ell(x) > 0 \) for all \( x \) in an open neighbourhood of the support of \( \zeta^k \).

Hence

\[
\delta_k = \min_{1 \leq \ell \leq m, x \in \text{support}(\zeta^k)} \lambda_\ell(x) > 0, \quad 1 \leq \ell \leq m.
\]

Consequently,

\[
\lambda_k \zeta^k \leq \frac{\| \lambda_k \|}{\delta_k} \lambda_\ell(x) \zeta^k, \quad 1 \leq \ell, k \leq m.
\]

We apply (3.43) to the right side of (3.42) to obtain

\[
\sum_{k=1}^{m} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k u \|_s^2 \leq C \sum_{k=1}^{m} \sum_{p=1}^{N} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k \|_s^2 + C \| \zeta' S_\delta \zeta u \|_{s-1} \\
\leq C \sum_{k=1}^{m} \| \sqrt{\lambda_k} \Lambda^s \zeta \Lambda^{s-\varepsilon} S_\delta \zeta u \|_s^2 + C \| \zeta' S_\delta \zeta u \|_{s-1}.
\]

We apply Corollary 3.4 with \( \zeta \) instead of \( \zeta' \), \( \zeta' \) instead of \( \zeta \), etc., to the first term on the right to obtain

\[
\sum_{k=1}^{m} \| \lambda_k \zeta^k \Lambda^s S_\delta \zeta^k u \|_s^2 \leq C \sum_{k=1}^{m} \| \sqrt{\lambda_k} \Lambda^s \zeta \Lambda^{s-\varepsilon} S_\delta \zeta u \|_s^2 + C \| \zeta' S_\delta \zeta u \|_{s-1} \\
\leq C \left\{ \| \zeta' S_\delta \zeta u \|^2_{s-\varepsilon} + \left( \| \zeta' \Lambda^{s-\varepsilon} S_\delta \zeta' L u, \| \zeta' \Lambda^{s-\varepsilon} S_\delta \zeta' u \|_s \right) \right\}.
\]
Therefore,
\[ + \sum_{k=1}^{m} \left\| \lambda_k \zeta S_0 \tilde{c}_k u \right\|_{s-\varepsilon}^2 + \left\| \zeta''S_0 \tilde{c}_k u \right\|_{s-\varepsilon-1/2}^2 + C \left\| \zeta' S_0 \tilde{c}_k u \right\|_{s-\varepsilon}^2 \right\},
\leq C \left\{ \left( \left\| \zeta' \Lambda^{s-\varepsilon} S_0 \zeta' \ell u, \zeta' \Lambda^{s-\varepsilon} S_0 \tilde{c}_k u \right\| \right) + \left\| \zeta''S_0 \tilde{c}_k u \right\|_{s-\varepsilon-1/2}^2 \right\}.
\]

The last auxiliary result we need is the following Poincaré-type inequality.

**Lemma 3.6** (Poincaré-type inequality). For every $0 < \varepsilon < 1$ and cutoff $\sigma_1$ such that $d_0 = \text{diam}(\text{support } \sigma_1) \leq \sqrt{\varepsilon/2}$, there exists $C > 0$ such that for any $u \in \Pi_{k=1}^{N} H^{s+1}(\mathbb{R}^n)$
\[ \left\| \tilde{\Lambda}^s u \right\| \leq C d_0^{\alpha(\varepsilon)} \left\{ \left\| \Lambda_{x_1}^s \zeta \tilde{\Lambda}^s u \right\| + d_0^{-1} \left\| \zeta u \right\|_{s-1} \right\}, \]
where $\alpha(\varepsilon) = 1/ \left( 4 \log_2 \left( \frac{1}{\varepsilon} + 1 \right) \right)$.

**Proof.** We can write
\[ \left\| \tilde{\Lambda}^s u \right\| = \sum_{p=1}^{N} \left\| \tilde{\Lambda}^s u_p \right\|^2 = \sum_{p=1}^{N} \left( \Lambda_{x_1}^s \zeta \Lambda^s u_p, \Lambda_{x_1}^{-s} \zeta \Lambda^s u_p \right) \]
For simplicity, we write $u = u_p$. Using Young’s inequality, we obtain
\[ \left\| \tilde{\Lambda}^s u \right\| \leq a \left( \left\| \Lambda_{x_1}^s \zeta \Lambda^s u \right\| + \frac{1}{4a} \left\| \Lambda_{x_1}^{-s} \zeta \Lambda^s u \right\| \right) \]
For the second term on the right we similarly obtain
\[ \left\| \Lambda_{x_1}^{-s} \zeta \Lambda^s u \right\| \leq 4a^2 \left\| \Lambda_{x_1}^s \zeta \Lambda^s u \right\|^2 + \frac{1}{4a^2} \left\| \Lambda_{x_1}^{-2s} \zeta \Lambda^s u \right\|^2. \]
Iterating $N$ times yields
\[ \left\| \zeta \Lambda^s u \right\| \leq \left\| \Lambda_{x_1}^s \zeta \Lambda^s u \right\|^2 + \frac{1}{4a^2} \left\| \Lambda_{x_1}^{-2s} \zeta \Lambda^s u \right\|^2. \]
We choose an integer $N$ such that $(2^N - 1) \varepsilon \geq 1 > (2^{N-1} - 1) \varepsilon$, that is,
\[ (3.44) \quad N - 1 < \log_2 \left( \frac{1}{\varepsilon} + 1 \right) \leq N. \]
Using Poincaré’s inequality with respect to $x^1$, and the monotonicity of the $\| \cdot \|_s$ norms we have
\[ \left\| \Lambda_{x_1}^{-2s-1} \zeta \Lambda^s u \right\| \leq \left\| \Lambda_{x_1}^{-1} \Lambda_{x_1}^s \zeta \Lambda^s u \right\| \leq \left\| \Lambda_{x_1}^{-1} \Lambda_{x_1}^s \zeta \Lambda^s u \right\|^2 + C \left\| \zeta u \right\|_{s-1}^2 \]
\[ \leq d_0^2 \left\| \nabla_{x_1} \zeta \Lambda_{x_1} \Lambda_{x_1}^s \zeta \Lambda^s u \right\|^2 + C \left\| \zeta u \right\|_{s-1}^2 \]
\[ \leq d_0^2 \left\| \Lambda_{x_1}^s \zeta \Lambda^s u \right\|^2 + C \left\| \zeta u \right\|_{s-1}^2. \]
Therefore,
\[ \left\| \zeta \Lambda^s u \right\| \leq \left( \frac{2a}{1 - 4^2a^2} + \frac{d_0^2}{(4^2a^2)^{N-1/2}} \right) \left\| \Lambda_{x_1}^s \zeta \Lambda^s u \right\|^2 + \frac{C}{(4^2a^2)^{N-1/2}} \left\| \zeta u \right\|_{s-1}^2. \]
We take \(4^2a^2 = d_0^{2/N}\), note that since \(d_0^2 \leq \varepsilon/2\), and \(2/N > 1/\log_2\left(\frac{1}{\varepsilon} + 1\right)\), we have
\[
d_0^{2/N} \leq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{2}} < \left(\frac{\varepsilon}{2}\right)^{\frac{1}{2\log_2\left(\frac{1}{\varepsilon} + 1\right)}} < \left(\frac{1}{\varepsilon} + 1\right)^{\frac{1}{2\log_2\left(\frac{1}{\varepsilon} + 1\right)}} = \frac{1}{\sqrt{2}}.
\]
Thus, we obtain
\[
\left\|\tilde{\Lambda}^s \zeta u\right\| \leq Cd_0^{1/(2N)} \left(\left\|\Lambda_s \zeta \tilde{\Lambda}^s \zeta u\right\| + d_0^{-1} \left\|\zeta u\right\|_{s-1}\right).
\]
Because of (3.44), this implies the result in the lemma with \(\alpha(\varepsilon) = 1/\left(4 \log_2\left(\frac{1}{\varepsilon} + 1\right)\right)\).

**Lemma 3.7.** There exist \(\varepsilon, C > 0\) such that for all \(u \in \prod_{k=1}^N H^{s-1}(\mathbb{R}^n)\) and all \(0 < \delta \leq 1\)
\[
\left\|S_\delta \zeta u\right\|^2_s \leq Cd_0^{\alpha(\varepsilon)} \left(\left\|\tilde{\Lambda}^s S_\delta \tilde{\zeta} \zeta u\right\| + \left\|\tilde{\zeta} S_\delta \tilde{\Lambda}^s \zeta u\right\|\right) + Cd_0^{\alpha(\varepsilon)} \left(\left\|\zeta S_\delta \tilde{\Lambda}^s \zeta u\right\| + \left\|S_\delta \left\|\zeta u\right\|_{s-\varepsilon}\right\|ight),
\]
where \(d_0, \alpha(\varepsilon)\) are as in Lemma 3.6. We also have that
\[
\left\|S_\delta \zeta u\right\|^2_s \leq C \left\{d_0^{\alpha(\varepsilon)} \left\|S_\delta \tilde{\zeta} \zeta u\right\|^2_s + \left(d_0^{2(1-\alpha(\varepsilon))} + 1\right) \left\|S_\delta \zeta u\right\|^2_{s-\varepsilon}\right\}.
\]

**Proof.** We have
\[
\left\|S_\delta \zeta u\right\|^2_s = \left\|\tilde{\zeta} S_\delta \tilde{\Lambda}^s \zeta u\right\|^2 + \left\|\tilde{\zeta} \zeta S_\delta \tilde{\Lambda}^s \zeta u\right\|^2 + C \left\|S_\delta \zeta u\right\|^2_{s-1}
\leq \left\|\tilde{\zeta} \zeta S_\delta \tilde{\Lambda}^s \zeta u\right\|^2 + C \left\|S_\delta \zeta u\right\|^2_{s-1}.
\]
We apply the Poincaré inequality, Lemma 3.4, to the first term on the right,
\[
\left\|S_\delta \zeta u\right\|^2_s \leq Cd_0^{\alpha(\varepsilon)} \left\|\tilde{\zeta} S_\delta \tilde{\Lambda}^s \zeta u\right\|^2 + C \left(d_0^{\alpha(\varepsilon)-1} + 1\right) \left\|S_\delta \tilde{\zeta} \zeta u\right\|_{s-1}
\leq Cd_0^{\alpha(\varepsilon)} \left\|\tilde{\zeta} S_\delta \tilde{\Lambda}^s \zeta u\right\|^2 + C \left(d_0^{\alpha(\varepsilon)-1} + 1\right) \left\|S_\delta \zeta u\right\|^2_{s-1}.
\]
Since \(\lambda_1 \equiv 1\), by Corollary 3.4, we then have
\[
\left\|S_\delta \zeta u\right\|^2_s \leq Cd_0^{\alpha(\varepsilon)} \left\{\left\|S_\delta \zeta \zeta u\right\|^2 + \left\|\tilde{\zeta} \zeta \tilde{\Lambda}^s \zeta \zeta u\right\| + \sum_{k=1}^m \left\|\lambda_k \lambda_k S_\delta \zeta \zeta u\right\|^2\right\}
+ C \left(d_0^{\alpha(\varepsilon)-1} + 1\right) \left\|S_\delta \zeta \zeta u\right\|_{s-1}.
\]
Taking \(\varepsilon\) small enough (we assume that at least \(\varepsilon \leq 1\)), we absorb the first term on the right into the left, and apply Lemma 3.5 to the third term on the right. We get
\[
\left\|S_\delta \zeta u\right\|^2_s \leq Cd_0^{\alpha(\varepsilon)} \left\{\left\|\tilde{\zeta} \zeta S_\delta \zeta \zeta u\right\| + \left\|\tilde{\zeta} \zeta S_\delta \zeta \zeta u\right\|\right\}
+ C \left(d_0^{\alpha(\varepsilon)-1} + 1\right) \left\|S_\delta \zeta \zeta u\right\|_{s-\varepsilon}.
\]
This proves (3.45). The second inequality, (3.46), follows from (3.45) after another absorption into the left.
Suppose \( u \) is a distribution on \( (\mathbb{R}^n)^N \) such that \( u \in \prod_{k=1}^{N} H^{s-1}(\mathbb{R}^n) \). If moreover \( \zeta \mathcal{L} u \in H^s(\mathbb{R}^n) \) for all \( \zeta \in C_0^\infty(U) \), then, by (4.46), for all \( 0 < \delta \leq 1 \) we have
\[
\|S_\delta \zeta u\|_s \leq C_0^2 \|S_\delta \zeta' \mathcal{L} u\|^2_s + C \left( d_0^{-2(1-\alpha(\varepsilon))} + 1 \right) \|S_\delta \zeta'' u\|^2_{s-\varepsilon},
\]
where the cutoff functions \( \zeta < \zeta' < \zeta'' \) are supported in \( U \) and the constants \( C, \alpha, d_0 \) do not depend on \( \delta \). Letting \( \delta \to 0^+ \), by (4.47) in Lemma 2.8 we obtain
\[
(4.47) \quad \|\zeta u\|_s \leq C_0^2 \|\zeta' \mathcal{L} u\|^2_s + C \left( d_0^{-2(1-\alpha(\varepsilon))} + 1 \right) \|\zeta'' u\|^2_{s-\varepsilon},
\]
This inequality suffices to prove the hypoellipticity, without loss of derivatives, of \( \mathcal{L} \). Indeed, since \( \zeta'' u \) is a compactly supported distribution there exists \( s_0 \in \mathbb{R} \) such that \( \|\varphi u\|_{s_0} \leq \infty \) for all \( \varphi \in C_0^\infty(U) \). If \( s_0 \geq s - \varepsilon \) then (4.47) implies that \( \zeta u \in H^s(\mathbb{R}^n) \) for all \( \zeta \in C_0^\infty(U) \). On the other hand, if \( s_0 < s - \varepsilon \), let \( N \) be the positive integer such that \( N \varepsilon \leq s - s_0 < (N + 1) \varepsilon \). Let \( \zeta = \zeta_0 \prec \zeta_1 \prec \zeta_2 \prec \cdots \prec \zeta_2^N \) be a sequence of cutoff functions supported in \( U \). Iterating (4.47) we obtain
\[
\|\zeta^{(j-1)} u\|_{s-(j-1)\varepsilon}^2 \leq C_0^2 \|\zeta^{(j-1)} \mathcal{L} u\|^2_{s-(j-1)\varepsilon} + C \left( d_0^{-2(1-\alpha(\varepsilon))} + 1 \right) \|\zeta^{(j)} u\|^2_{s-j\varepsilon},
\]
for \( j = 1, \ldots, N \). Assembling these estimates yields
\[
\|\zeta u\|_s \leq C d_0^2 \sum_{j=1}^{N} \left( d_0^{-2(1-\alpha(\varepsilon))} + 1 \right) \left( \|\zeta^{(j)} \mathcal{L} u\|^2_{s-j\varepsilon} + C \left( d_0^{-2(1-\alpha(\varepsilon))} + 1 \right) \|\zeta^{(j)} u\|^2_{s-j\varepsilon} \right).\]
By the monotonicity of the \( H^s \)-norms, and since \( s_0 \leq s - N \varepsilon \), it follows that
\[
\|\zeta u\|_s \leq C N d_0^2 \left( d_0^{-2(1-\alpha(\varepsilon))} + 1 \right)^{N-1} \left( \|\zeta^{N-1} \mathcal{L} u\|^2_{s-N\varepsilon} + C \left( d_0^{-2(1-\alpha(\varepsilon))} + 1 \right) \|\zeta^{N-1} u\|^2_{s-N\varepsilon} < \infty \right).
\]
Hence \( \zeta u \in H^s(\mathbb{R}^n) \) for all \( \zeta \in C_0^\infty(U) \) and this finishes the proof. \( \square \)

References

[1] V. S. Fedil, On a criterion for hypoellipticity, *Mat. Sb.* 14 (1971), 15–45.
[2] C. Fefferman, D. H. Phong, Subelliptic eigenvalue problems, *Conf. in Honor of A. Zygmund*, Wadsworth Math. Series 1981.
[3] L. Hörmander, Hypoelliptic second order differential equations, *Acta. Math.* 119 (1967), 141–171.
[4] L. Hörmander, The Analysis of Linear Partial Differential Operators I-IV, *Springer-Verlag*, Berlin, 1983.
[5] J. J. Kohn, Hypoellipticity of some degenerate subelliptic operators, *J. Funct. Anal.* 159 (1998), no. 1, 203–216.
[6] S. Kusuoka and D. Strook, Applications of the Mallavain calculus II, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 32 (1985), no. 1, 1–76.
[7] O. A. Ladyzhenskaya, N. N. Uralteva, Linear and Quasilinear Elliptic Equations, *New York, Academic Press*, 1968.
[8] Morimoto, Y., On the hypoellipticity for infinitely degenerate semi-elliptic operators, Journal of the Mathematical Society of Japan, 1978, 30, no.2, 327–358.
[9] Morimoto, Y., Hypoellipticity for infinitely degenerate elliptic operators, *Osaka J. Math.* 24 (1987), no. 1, 13–35.
[10] Morimoto, Y., A criterion for hypoellipticity of second order differential operators, *Osaka J. Math.* 24 (1987), no. 1, 661–675.
[11] Morimoto, Y. and Xu, C.J, Nonlinear hypoellipticity of infinite, Funkcialaj Ekvacioj, 50 (2007), no. 1, 33–65.
[12] Morimoto, Y. and Xu, C.J, Hypoellipticity for a class of kinetic equations, Kyoto Journal of Mathematics, 2007, 47, no.1, 129–152.
[13] Morioka, T, Hypoellipticity for some infinitely degenerate operators of second order, J. Math. Kyoto Univ. 32-2, 1992, 373-386.
[14] C. Rios, E. T. Sawyer, R. L. Wheeden, A higher-dimensional partial Legendre transform, and regularity of degenerate Monge-Ampère equations, Adv. Math. 193 (2005), no. 2, 373—415.
[15] C. Rios, E. T. Sawyer, R. L. Wheeden, A priori estimates for infinitely degenerate quasilinear equations, Differential and Integral Equations-Athens, 2008, 21, no.1, 151–200.
[16] C. Rios, E. Sawyer, R. Wheeden, Hypoellipticity for infinitely degenerate quasilinear equations and the Dirichlet problem, To appear Journal d’Analyse Mathématique (2012).
[17] E. Sawyer and R. Wheeden, A priori estimates for quasilinear equations related to the Monge-Ampère equation in two dimensions, Journal d’Analyse Mathématique, 2005, 97, no.1, 257–316.
[18] E. Sawyer and R. Wheeden, Regularity of degenerate Monge-Ampère and prescribed Gaussian curvature equations in two dimensions, Potential Analysis, 24 (2006), no. 3, 267–301.
[19] M. E. Taylor, Pseudodifferential operators and nonlinear PDE, Progress in Mathematics 100. Birkhäuser Boston, Inc., Boston, MA, 1991.
[20] Tri, N.M, Semilinear hypoelliptic differential operators with multiple characteristics, Transactions of the American Mathematical Society, 2008, 360, no. 7, 3875–3907.
[21] Xu, C.J. and Zuily, C, Higher interior regularity for quasilinear subelliptic systems, Calculus of Variations and Partial Differential Equations, 1997, 5, no.4, 323–343.

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