Spin-orbit coupled Bose-Einstein condensates in a double well

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Abstract. We study the quantum dynamics of spin-orbit (SO) coupled Bose-Einstein condensates (BECs) in a double-well potential inspired by the recent experiment developed by NIST group. We focus on the regime where the number of atoms is very large and perform a two-mode approximation. An analytical solution of the two-site Bose-Hubbard-like Hamiltonian is found for weak nonlinear interactions by a rotating wave approximation (RWA). The quantum dynamics of several observables, ranging from the population and pseudospin imbalance, to the atomic and spin current are explicitly computed. We show a spin Josephson effect which could be detected in experiments and employed in realistic devices.

1 Introduction

Spin-orbit coupling relates the velocity of a particle to its spin and is ubiquitous in condensed matter physics. It plays a key role in a variety of systems and gives rise to new phenomena ranging from topological insulators [1,2] to spin-Hall effect [3] and Majorana fermions [5]. In solid-state materials spin-orbit coupling arises because of the motion of electrons in the intrinsic electric field of the crystal, which is a characteristic of the material under study. On the other hand ultracold atomic gases offer an unique platform for engineering synthetic spin-orbit couplings thanks to the wide tunability of experimental parameters [6]. That is achieved by controlling atom-light interactions and has recently led to the generation of effective Abelian and non-Abelian gauge fields [7–9]. In the last years some experimental proposals have been implemented. In a series of pioneering experiments [10–12] the NIST group successfully built up synthetic uniform gauge fields, magnetic fields, electric fields and SO couplings [13,14]. In particular SO coupling [13,14] with equal Rashba [15] and Dresselhaus [16] strengths in a neutral atomic BEC has been engineered by dressing two atomic spin states with a pair of counterpropagating laser beams. Furthermore laser coupling has been shown to induce a modification on the dressed spin states by driving a quantum phase transition from a spatially spin-mixed state to a phase-separated state. The above scheme has been further generalized to create nearly isotropic Rashba SO coupling as well as a tunable combination of Rashba and Dresselhaus SO coupling [17].
Up to now a number of theoretical investigations on SO-coupled BECs has been performed, concerning phase diagrams in the ground state [18–21], vortices structures in the presence of external rotation [22], unconventional collective dipole oscillations [25,26], superfluid to Mott insulator transitions in a lattice [27], supersolid features in the excitation spectrum within the stripe phase [28] and interesting simulation of relativistic effects such as Zitterbewegung [29] and Klein tunneling [30]. The key consequence of SO coupling which emerges in all such examples is an enhancement of interaction effects even for a weak interacting BEC [31]. On the other hand, in the last years great efforts have been devoted to the exploration of the role of quantum fluctuations and in general of macroscopic quantum coherence phenomena [32] in BECs, in order to understand the intriguing interplay between nonlinear interactions and quantum coherence and the emergent new phenomena which could arise in such a new environment. The prototypical system one can study is a BEC in a double well potential, which represents the cold atom analogue of a Josephson junction [33,34]. Within a mean field approximation a reliable description can be obtained by means of the Gross-Pitaevskii theory which gives rise to a variety of phenomena, ranging from Josephson oscillations [35–38] to macroscopic quantum self-trapping (MQST) [39,40] and ac and dc Josephson like effect [41], all experimentally observed in the last decade [42–44]. Vice versa, in a quantum regime and within the tight binding approximation one gets the Bose-Hubbard dimer Hamiltonian [45–48], whose parameters are the hopping frequency $J$, between the two lattice sites, the onsite interaction strength $g_j$, $j = 1, 2$ and the total atoms number $N$. Furthermore it can be mapped onto a SU(2) spin problem which coincides with the Lipkin-Meshkov-Glick (LMG) model [49–52]. More recently, the theoretical analysis on weakly coupled condensates has been successfully extended to a binary mixture of BECs in a double well potential [55–66], resulting in a richer tunneling dynamics, which includes two different MQST states with broken symmetry [61], characterized by localization in the two different wells (phase separation) or coexistence in the same well respectively. Furthermore the coherent dynamics of a two species BEC in a double well has been analyzed as well focussing on the case where the two species are two hyperfine states of the same alkali metal [67].

Till now, the quantum dynamics of SO coupled BECs in a double well is still poorly investigated. Recent mean-field results [68] relying on the experimental setup by NIST group [13,14] point towards an interplay between external and internal Josephson effects mainly in the absence of interatomic interactions, as well as towards the existence of a net atomic spin current in the weak Raman coupling regime. Likewise, in Ref. [69] a classical study of the interplay between interatomic interactions and SO coupling has been reported as well, together with a careful analysis of the self-trapped dynamics of the total population imbalance between the two bosonic pseudospin species.

In this paper we carry out an analytical study of the quantum behavior of SO coupled BECs. We analyze in detail the weak interaction regime in a two-mode Bose-Hubbard-like Hamiltonian and obtain a closed analytical solution via a rotating wave approximation [70,71]. The quantum evolution of the number difference of bosons and of pseudospin up and down between the two wells is investigated in detail and the total atomic current and the net spin current are computed as well.

The paper is organized as follows. In Sect. 2, we introduce the model Hamiltonian and focus on the two mode approximation. In Sect. 3 we solve the model in the intermediate regime for the Raman coupling, considering both the non interacting and the weakly interacting case. Finally some conclusions are outlined in Sect. 4.
2 The model

In 2011 the NIST group [13,14] succeeded in engineering a SO coupling with equal Rashba and Dresselhaus strengths in a neutral atomic $^{87}\text{Rb}$ BEC by dressing two atomic spin states with a pair of lasers. The key step in the experimental technique is to select out two internal spin states within the $F=1$ ground electronic manifold, pseudospin up $|\uparrow\rangle = |F=1, m_F=0\rangle$ and pseudospin down $|\downarrow\rangle = |F=1, m_F=-1\rangle$, and then couple them with strength $\Omega$ via a pair of $\lambda_L = 804.1 \text{ nm}$ Raman lasers, intersecting at an angle $\theta = 90^\circ$ and detuned by $\delta$ from Raman resonance. Assuming $\hbar k_L = \sqrt{2\pi \lambda_L}$ and $E_L = \hbar^2 k_L^2$ as momentum and energy units, $k_L$ being the wave number of the Raman laser, the SO coupling is described in terms of a single-particle Hamiltonian:

$$\hat{H} = \frac{\hbar^2 k^2}{2m} I_{2\times2} + \frac{\delta}{2} \sigma_z + \frac{\Omega}{2} \sigma_x \cos(2k_Lx) - \frac{\Omega}{2} \sigma_y \sin(2k_Lx),$$ (1)

where $k$ is the atomic momentum in the $x-y$ plane, $m$ is the atomic mass, $I_{2\times2}$ is the identity matrix and $\sigma_x, \sigma_y, \sigma_z$ are the $2 \times 2$ Pauli matrices. Since SO coupling acts only in one spatial dimension, in the following we neglect the motion of atoms along $y$ and $z$ axis and consider Eq. (1) restricted to $x$ axis.

After the transformation $U \equiv \begin{pmatrix} e^{-ik_Lx} & 0 \\ 0 & e^{ik_Lx} \end{pmatrix}$ within the space $|\uparrow\rangle, |\downarrow\rangle$, dressed pseudospins $|\uparrow\rangle_d = e^{-ik_Lx} |\uparrow\rangle, |\downarrow\rangle_d = e^{ik_Lx} |\downarrow\rangle$ are introduced and the one-dimensional version of Hamiltonian (1) takes the form:

$$\hat{H}_d = \frac{\hbar^2 k_d^2}{2m} I_{2\times2} + 2\alpha k_d \sigma_z + \frac{\Omega}{2} \sigma_x + \frac{\delta}{2} \sigma_z,$$ (2)

where $\alpha = \frac{E_L}{k_L}$. Let us now notice that, for $\delta = 0$, Eq. (2) gives rise to the following dispersion relation

$$E_{\pm}(k_d) = \frac{\hbar^2 k_d^2}{2m} \pm \sqrt{4\alpha^2 k_d^2 + \frac{\Omega^2}{4}},$$ (3)

which shows two branches. The lowest one, for $\Omega < 4E_L$, exhibits a double well structure with two minima corresponding to the condensation of dressed pseudospin up and down states, while Raman coupling and a small detuning $\delta$ modulate the atomic population in the above two states. The complete experimental control and wide tunability of $\delta, \Omega$ and $k_L$ parameters in Eq. (2) allows one to select out the region $\Omega < 4E_L$ within the parameters space, thus in the following we will work in the dressed pseudospin basis $|\uparrow\rangle_d, |\downarrow\rangle_d$ and focus on such a regime.

Now let us switch interatomic interactions and put the SO coupled BEC in a spin independent double well trapping potential $V(x)$ along the axial direction, obtained by adding a strong confining potential in the $(y-z)$ plane. The second quantized version of the full Hamiltonian takes the form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int},$$ (4)

where

$$\mathcal{H}_0 = \int dx \hat{\Psi}^\dagger(x) \left[ \hat{H}_d + V(x) \right] \hat{\Psi}(x)$$

$$= \int dx \left( \hat{\Psi}_\uparrow^\dagger(x) \hat{\Psi}_\downarrow(x) \right) \begin{pmatrix} \hat{H}_\uparrow + V(x) & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \hat{H}_\downarrow + V(x) \end{pmatrix} \begin{pmatrix} \hat{\Psi}_\uparrow(x) \\ \hat{\Psi}_\downarrow(x) \end{pmatrix},$$ (5)
and

$$\mathcal{H}_{\text{int}} = \sum_{\sigma=\uparrow,\downarrow} \frac{g_{\sigma\sigma'}}{2} \int dx \hat{\Psi}_{\sigma}^\dagger(x) \hat{\Psi}_{\sigma'}^\dagger(x) \hat{\Psi}_{\sigma}(x) \hat{\Psi}_{\sigma'}(x) + g_{\uparrow\downarrow} \int dx \hat{\Psi}_{\uparrow}^\dagger(x) \hat{\Psi}_{\downarrow}^\dagger(x) \hat{\Psi}_{\uparrow}(x) \hat{\Psi}_{\downarrow}(x).$$

(6)

Here $H_{\uparrow} = \frac{b^2}{2m} \left( \hat{k}_x^2 + 2k_L \hat{k}_z \right) + \frac{\delta}{2}$, $H_{\downarrow} = \frac{b^2}{2m} \left( \hat{k}_x^2 - 2k_L \hat{k}_z \right) - \frac{\delta}{2}$ and $g_{\sigma\sigma'} = \frac{2\hbar^2 a_{\sigma\sigma'}}{ml^2 \omega}$ with $\sigma, \sigma' = \uparrow, \downarrow$ is the interaction strength, $a_{\sigma\sigma'}$ being the $s$-wave scattering length between pseudospin $\sigma$ and $\sigma'$ and $l_\perp$ the oscillator length due to a harmonic vertical confinement; furthermore $\hat{\Psi}_{\sigma}^\dagger(x)$, $\hat{\Psi}_{\sigma}(x)$, $\sigma = \uparrow, \downarrow$ are the bosonic field operators, which satisfy the commutation rules:

$$\left[ \hat{\Psi}_{\sigma}(x), \hat{\Psi}_{\sigma'}(x') \right] = \left[ \hat{\Psi}_{\sigma}^\dagger(x), \hat{\Psi}_{\sigma'}^\dagger(x') \right] = 0,$$

(7)

$$\left[ \hat{\Psi}_{\sigma}(x), \hat{\Psi}_{\sigma'}^\dagger(x') \right] = \delta_{\sigma\sigma'} \delta(x - x') \sigma, \sigma' = \uparrow, \downarrow,$$

(8)

and the normalization conditions:

$$\int dx \left| \hat{\Psi}_{\sigma}(x) \right|^2 = N_\sigma; \sigma = \uparrow, \downarrow,$$

(9)

$N_\sigma, \sigma = \uparrow, \downarrow$ being the number of atoms with pseudospin $\sigma$ and $\sigma'$ respectively. The total number of atoms in the system is $N = N_\uparrow + N_\downarrow$.

A weak link between the two wells produces a small energy splitting between the mean-field ground state and the first excited state of the double well potential and that allows to reduce the dimension of the Hilbert space of the initial many-body problem. Indeed for low energy excitations, low temperatures and a small effective Zeeman splitting it is possible to consider only such two states and neglect the contribution from the higher ones, in this way performing a two-mode approximation [37–40]. As a consequence, the field operator can be expressed as:

$$\hat{\Psi}_{\sigma}(x) \approx a_{L\sigma} \psi_{L\sigma}(x) + a_{R\sigma} \psi_{R\sigma}(x), \sigma = \uparrow, \downarrow$$

(10)

where $\psi_{j\sigma}(x)$, $j = L, R$, is the ground state wave function in the $j$ well with pseudospin $\sigma$ and $a_{j\sigma}$ is the corresponding annihilation operator, which obeys to the bosonic commutation relation $[a_{j\sigma}, a_{j\sigma}^\dagger] = \delta_{jk} \delta_{\sigma\sigma'}$.

By putting Eq. (10) in Eqs. (4)–(6) and neglecting interwell atomic interactions as well as two-particle processes we turn the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$ into the following Bose-Hubbard like form:

$$\mathcal{H} = \sum_{j=L,R} \sum_{\sigma=\uparrow,\downarrow} \varepsilon_{j\sigma} a_{j\sigma}^\dagger a_{j\sigma} + \sum_{\sigma,\sigma'=\uparrow,\downarrow} \left( J_{\sigma\sigma'} a_{L\sigma}^\dagger a_{R\sigma'} + J_{\sigma\sigma'}^* a_{R\sigma'}^\dagger a_{L\sigma} \right)$$

$$+ \frac{1}{2} \sum_{j=L,R} \left( \Omega_j a_{j\uparrow}^\dagger a_{j\uparrow} + \Omega_j^* a_{j\downarrow}^\dagger a_{j\downarrow} \right) + \frac{\delta}{2} \sum_{j=L,R} \left( a_{j\uparrow}^\dagger a_{j\uparrow} - a_{j\downarrow}^\dagger a_{j\downarrow} \right)$$

$$+ \frac{1}{2} \sum_{j=L,R} \left( g_{j\uparrow}^\dagger a_{j\uparrow}^\dagger a_{j\uparrow} a_{j\uparrow} + g_{j\downarrow}^\dagger a_{j\downarrow}^\dagger a_{j\downarrow} a_{j\downarrow} + 2g_{j\uparrow}^\dagger a_{j\uparrow}^\dagger a_{j\downarrow} a_{j\uparrow} + 2g_{j\downarrow}^\dagger a_{j\downarrow}^\dagger a_{j\uparrow} a_{j\uparrow} \right).$$

(11)

Here $\varepsilon_{j\sigma} = \int dx \psi_{j\sigma}^\ast(x) \left[ \frac{b^2}{2m} \left( \hat{k}_x^2 + 2k_L \hat{k}_z \right) + V(x) \right] \psi_{j\sigma}(x)$ are the single-particle ground state energies in the well $j$, $J_{\sigma\sigma} = \int dx \psi_{L\sigma}^\ast(x) \left[ H_{\sigma} + V(x) \right] \psi_{R\sigma}(x)$ are the Josephson tunneling terms between left and right well, $J_{\sigma\pi} = \int dx \psi_{L\sigma}^\ast(x) \frac{\Omega_{\pi}}{2} \psi_{R\pi}(x)$
are the interwell spin-flip tunneling terms, $\Omega_j = \Omega \int dx |\psi^{\sigma'}_j(x)|^2 |\psi_j(x)|^2$ is the Raman coupling term in each well and, finally, $g_{\sigma\sigma'}^{(j)} = g_{\sigma\sigma'} \int dx |\psi^{\sigma'}_j(x)|^2 |\psi_j(x)|^2$ is the effective interaction strength.

The complete control of the experimental environment makes possible a fine tuning of the parameters. In this way a symmetric double-well potential can be realized, which allows one to set $\varepsilon_{L\uparrow} = \varepsilon_{R\uparrow} = \varepsilon_{L\downarrow} = \varepsilon_{R\downarrow} \equiv \varepsilon$, $\Omega_j = \Omega$ and $g_{\sigma\sigma'}^{(j)} = g_{\sigma\sigma'} \equiv g_{\sigma\sigma'}$. Also the spin-flip tunneling amplitude $J_{\sigma\pi}$ can be dropped under realistic experimental conditions [13,14]. Further simplifying assumptions amount to neglect the constant energy shifts $\varepsilon (N_{L\uparrow} + N_{L\downarrow} + N_{R\uparrow} + N_{R\downarrow})$, so that the Hamiltonian (11) becomes:

$$\mathcal{H} = \sum_{\sigma=\uparrow,\downarrow} J_{\sigma\sigma} \left( a^{\dagger}_{L\sigma} a_{R\sigma} + a^{\dagger}_{R\sigma} a_{L\sigma} \right) + \frac{\Omega}{2} \left( a^{\dagger}_{L\uparrow} a_{L\downarrow} + a^{\dagger}_{L\downarrow} a_{L\uparrow} + a^{\dagger}_{R\uparrow} a_{R\downarrow} + a^{\dagger}_{R\downarrow} a_{R\uparrow} \right)$$

$$+ \frac{\delta}{2} \left( a^{\dagger}_{L\uparrow} a_{R\uparrow} + a^{\dagger}_{R\uparrow} a_{L\uparrow} - a^{\dagger}_{L\downarrow} a_{R\downarrow} - a^{\dagger}_{R\downarrow} a_{L\downarrow} \right)$$

$$+ \frac{1}{2} g_{\uparrow\uparrow} \left( a^{\dagger}_{L\uparrow} a_{R\uparrow} a_{L\uparrow} a_{R\uparrow} + a^{\dagger}_{R\uparrow} a_{L\uparrow} a_{R\uparrow} a_{L\uparrow} \right)$$

$$+ \frac{1}{2} g_{\downarrow\downarrow} \left( a^{\dagger}_{L\downarrow} a_{R\downarrow} a_{L\downarrow} a_{R\downarrow} + a^{\dagger}_{R\downarrow} a_{L\downarrow} a_{R\downarrow} a_{L\downarrow} \right)$$

$$+ g_{\uparrow\downarrow} \left( a^{\dagger}_{L\uparrow} a_{R\downarrow} a_{L\downarrow} a_{R\uparrow} + a^{\dagger}_{R\uparrow} a_{L\downarrow} a_{R\downarrow} a_{L\uparrow} \right).$$

(12)

As for the orders of magnitude of the parameters appearing in Eq. (12) we choose $\Omega \sim 0.1E_L \div 0.25E_L$ as in Ref. [68] while the energy scale of the Zeeman field $\delta$ generally satisfies the condition $\delta << E_L$ and in the following we choose it to be 0.001$E_L$. For an estimate of the tunneling terms in Eq. (12), $J_{\uparrow\uparrow}, J_{\downarrow\downarrow}$ we assume that the double well trapping potential has a symmetric shape with two minima located at $x = \pm b$ as in Ref. [68] and make an harmonic approximation with trapping frequency $\omega \sim 0.1 \frac{E_L}{b^2}$ (e.g. a few kilohertz)[43]. From the explicit calculation of the overlap integral, by setting $b \sim \sqrt{2}l_0$, where $l_0 = \sqrt{\frac{\hbar}{m\omega}} \sim 1\mu m$ is the oscillator length, one finds that tunneling terms may be chosen as $J_{\uparrow\uparrow}, J_{\downarrow\downarrow} \approx -0.1E_L$. In the following we will always consider weak nonlinear interactions where analytical approaches are available and which can be easily obtained in experiments by means of Feshbach resonances technique, thus we require $g_{\uparrow\uparrow}, g_{\downarrow\downarrow} \ll \hbar\omega$ or equivalently $g_{\uparrow\downarrow} \ll g_{\uparrow\uparrow}, g_{\downarrow\downarrow} << 0.1E_L$.

In the following Sections we study this Hamiltonian in some limiting cases, amenable to analytical solutions.

### 3 Quantum dynamics

While the limiting cases of weak and strong Raman coupling regime can be treated by a generalization of the methods adopted in Ref. [66] which lead to a Hamiltonian of decoupled harmonic oscillators, the most interesting situation is the one in which $\Omega$ and $J_{\uparrow\uparrow}, J_{\downarrow\downarrow}$ are of the same order of magnitude. In general, one cannot obtain a closed analytical solution and has to resort to numerical calculations. Here we concentrate on some particular cases amenable to an analytical solution [72].
3.1 The noninteracting case

In order to gain some insight into the phenomenology, we start with the strong tunneling regime and neglect collisional interactions, i.e. we put \( \mathcal{J}_{\uparrow \downarrow} = \mathcal{J}_{\downarrow \uparrow} = \mathcal{J}_{\downarrow \downarrow} = 0 \), which could be experimentally achieved by properly tuning the scattering length of each species to zero by the Feshbach resonance technique. We also make a further simplifying assumption, i.e. \( J_{\uparrow \downarrow} = J_{\downarrow \uparrow} = J \), so that Hamiltonian (12) reduces to

\[
\mathcal{H} = J \left( a_{L\uparrow}^\dagger a_{R\uparrow} + a_{R\uparrow}^\dagger a_{L\uparrow} + a_{L\downarrow}^\dagger a_{R\downarrow} + a_{R\downarrow}^\dagger a_{L\downarrow} \right)
+ \frac{\Omega}{2} \left( a_{L\uparrow}^\dagger a_{L\downarrow} + a_{L\downarrow}^\dagger a_{L\uparrow} + a_{R\uparrow}^\dagger a_{R\downarrow} + a_{R\downarrow}^\dagger a_{R\uparrow} \right)
+ \frac{\delta}{2} \left( a_{L\uparrow}^\dagger a_{R\uparrow} + a_{R\uparrow}^\dagger a_{R\uparrow} - a_{L\downarrow}^\dagger a_{L\downarrow} - a_{R\downarrow}^\dagger a_{R\downarrow} \right).
\]  

(13)

By taking a closer look to the above expression we recognize a quadratic Hamiltonian which can be promptly diagonalized by introducing the following transformation:

\[
c_1 = \frac{\Omega}{\sqrt{2\Omega^2 + 2(\delta - \Omega_\delta)}} \left[ \left( \frac{\delta + \Omega_\delta}{\Omega} \right) a_{L\uparrow} + a_{L\downarrow} + \left( \frac{\delta + \Omega_\delta}{\Omega} \right) a_{R\uparrow} + a_{R\downarrow} \right],
\]

(14)

\[
c_2 = \frac{\Omega}{\sqrt{2\Omega^2 + 2(\delta - \Omega_\delta)}} \left[ \left( \frac{-\delta + \Omega_\delta}{\Omega} \right) a_{L\uparrow} - a_{L\downarrow} + \left( \frac{-\delta + \Omega_\delta}{\Omega} \right) a_{R\uparrow} + a_{R\downarrow} \right],
\]

(15)

\[
d_1 = \frac{\Omega}{\sqrt{2\Omega^2 + 2(\delta - \Omega_\delta)}} \left[ \left( \frac{-\delta + \Omega_\delta}{\Omega} \right) a_{L\uparrow} - a_{L\downarrow} + \left( \frac{-\delta + \Omega_\delta}{\Omega} \right) a_{R\uparrow} + a_{R\downarrow} \right],
\]

(16)

\[
d_2 = \frac{\Omega}{\sqrt{2\Omega^2 + 2(\delta - \Omega_\delta)}} \left[ \left( \frac{\delta + \Omega_\delta}{\Omega} \right) a_{L\uparrow} + a_{L\downarrow} + \left( \frac{\delta + \Omega_\delta}{\Omega} \right) a_{R\uparrow} + a_{R\downarrow} \right],
\]

(17)

where \( \Omega_\delta = \sqrt{\delta^2 + \Omega^2} \) and the usual commutation relations hold: \([c_i, c_j] = \delta_{ij}, [d_i, d_j] = \delta_{ij}\). As a consequence, Hamiltonian (13) can be cast in the simple form

\[
\mathcal{H} = \left( \frac{\Omega_\delta}{2} + \mathcal{J} \right) \left( c_1^\dagger c_1 + c_2^\dagger c_2 \right) + \left( \frac{\Omega_\delta}{2} - \mathcal{J} \right) \left( d_1^\dagger d_1 + d_2^\dagger d_2 \right)
\]

(18)

and the conservation relation \( N = c_1^\dagger c_1 + c_2^\dagger c_2 + d_1^\dagger d_1 + d_2^\dagger d_2 \) is satisfied. Let us notice that, in the new basis, the quantum dynamics is characterized by two frequencies: \( \omega_c = \frac{\Omega_\delta}{2} + \mathcal{J} \) and \( \omega_d = \frac{\Omega_\delta}{2} - \mathcal{J} \). This appears more clearly in the time evolution of particle and pseudospin imbalances between the two wells, reported below.

Let us start by the assumption that at time \( t = 0 \) all \( N \) atoms have pseudospin down and lie in the left well. The corresponding initial state reads as:

\[
|\psi(0)\rangle = \frac{1}{\sqrt{N!}} \left( a_{L\downarrow}^\dagger \right)^N |0\rangle
= \frac{1}{2^N \sqrt{N!}} \left( \sqrt{\Omega_\delta - \delta} c_1^\dagger - \sqrt{\Omega_\delta + \delta} c_2^\dagger - \sqrt{\Omega_\delta - \delta} d_1^\dagger + \sqrt{\Omega_\delta + \delta} d_2^\dagger \right)^N |0\rangle.
\]

(19)
Then let us switch on tunneling as well as Raman coupling terms and determine the dynamics at time $t$. By performing the unitary transformation

$$e^{-i\mathcal{H}t} \left( \sqrt{\frac{\Omega_3 - \delta}{\Omega_5}} c_1 + \sqrt{\frac{\Omega_3 + \delta}{\Omega_5}} c_2 - \sqrt{\frac{\Omega_3 - \delta}{\Omega_5}} d_1 + \sqrt{\frac{\Omega_3 + \delta}{\Omega_5}} d_2 \right) e^{i\mathcal{H}t}$$

we get the wave function at time $t$:

$$|\psi(t)\rangle = \frac{1}{2N} \left( e^{-i\omega_c t} \sqrt{\frac{\Omega_3 - \delta}{\Omega_5}} c_1 - e^{i\omega_c t} \sqrt{\frac{\Omega_3 + \delta}{\Omega_5}} c_2 
- e^{-i\omega_d t} \sqrt{\frac{\Omega_3 - \delta}{\Omega_5}} d_1 + e^{i\omega_d t} \sqrt{\frac{\Omega_3 + \delta}{\Omega_5}} d_2 \right)^N |0\rangle,$$

which, by substituting Eqs. (14)–(17), takes the final form

$$|\psi(t)\rangle = \frac{1}{\sqrt{N!}} \left( G_{L\uparrow} (t) a_{L\uparrow}^\dagger + G_{L\downarrow} (t) a_{L\downarrow}^\dagger + G_{R\uparrow} (t) a_{R\uparrow}^\dagger + G_{R\downarrow} (t) a_{R\downarrow}^\dagger \right)^N |0\rangle.$$

Here the functions $G_{j\sigma} (t)$, $j = L, R$, $\sigma = \uparrow, \downarrow$ are defined as:

$$G_{L\uparrow} (t) = -i \frac{\Omega}{\Omega_5} \frac{[\sin (\omega_c t) + \sin (\omega_d t)]}{2},$$

$$G_{L\downarrow} (t) = \frac{\Omega}{\Omega_5} \frac{[\cos (\omega_c t) + \cos (\omega_d t)]}{2} - i \frac{\delta}{\Omega_5} \frac{[\sin (\omega_c t) + \sin (\omega_d t)]}{2},$$

$$G_{R\uparrow} (t) = \frac{\Omega}{\Omega_5} \frac{[\cos (\omega_c t) - \cos (\omega_d t)]}{2},$$

$$G_{R\downarrow} (t) = -i \frac{\Omega}{\Omega_5} \frac{[\sin (\omega_c t) - \sin (\omega_d t)]}{2} + \frac{\delta}{\Omega_5} \frac{[\cos (\omega_c t) - \cos (\omega_d t)]}{2},$$

and are characterized by the two frequencies $\omega_c$ and $\omega_d$.

In order to calculate the particle and pseudospin imbalance between the two wells, let us introduce the following quantity:

$$g_{j,\sigma; k,\sigma'} (t) = \frac{1}{N} \langle \psi (t) | a_{j,\sigma}^\dagger a_{k,\sigma'} | \psi (t) \rangle = G_{j,\sigma}^* (t) G_{k,\sigma'} (t),$$

from which the fraction of pseudospin $\sigma$ in the $j$ well is easily obtained:

$$n_{j,\sigma} (t) = g_{j,\sigma; j,\sigma} (t) = |G_{j,\sigma} (t)|^2.$$

The required particle imbalance between left and right well, which gives rise to external Josephson oscillations, reads:

$$\rho_{\sigma} (t) = n_{L,\sigma} (t) - n_{R,\sigma} (t) = |G_{L,\sigma} (t)|^2 - |G_{R,\sigma} (t)|^2,$$
while internal Josephson oscillations are governed by the pseudospin imbalance:

$$\rho_j(t) = n_{j,\downarrow}(t) - n_{j,\uparrow}(t) = |G_{j\downarrow}(t)|^2 - |G_{j\uparrow}(t)|^2.$$ (30)

The net result is an interesting interplay between external and internal Josephson effects.

Indeed we can calculate the population imbalance between the two wells

$$D_{LR}(t) = \rho_{\uparrow}(t) + \rho_{\downarrow}(t) = |G_{L\uparrow}(t)|^2 - |G_{R\uparrow}(t)|^2 + |G_{L\downarrow}(t)|^2 - |G_{R\downarrow}(t)|^2,$$ (31)

the magnetization

$$M_{LR}(t) = \rho_{\uparrow}(t) - \rho_{\downarrow}(t) = |G_{L\uparrow}(t)|^2 - |G_{R\uparrow}(t)|^2 - |G_{L\downarrow}(t)|^2 + |G_{R\downarrow}(t)|^2,$$ (32)

and the pseudospin imbalance

$$D_{\uparrow\downarrow}(t) = \rho_L(t) + \rho_R(t) = |G_{L\downarrow}(t)|^2 - |G_{L\uparrow}(t)|^2 + |G_{R\downarrow}(t)|^2 - |G_{R\uparrow}(t)|^2.$$ (33)

In order to show the general features of the above quantities we show in Fig. 1 the behavior of Eqs. (31)–(33) as a function of time for the following choice of parameters: $\Omega = 0.1E_L$, $J = -0.1E_L$ and $\delta = 0.001E_L$. Indeed the results show coherent Rabi type of oscillations both for the population imbalance between the two wells (external Josephson tunneling) and the pseudospin imbalance between up and down states (internal Josephson tunneling), but characterized by different frequencies. Conversely the magnetization $M_{LR}$ exhibits complicated quasiperiodic features. In the next Subsection we show how this picture is modified in the presence of nonlinear interactions.

### 3.2 Weak collisional interaction limit

We consider the case $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = 0$ and $g_{\uparrow\downarrow} = \bar{g} \neq 0$, which could be experimentally achieved by making the spatial overlap integrals between different pseudospins non
zero by varying $k_L$, while $J_{\uparrow \uparrow} = J_{\downarrow \downarrow} = J$, so that Hamiltonian (12) reduces to

\[
\mathcal{H} = \mathcal{J} \left( a_{L\uparrow}^\dagger a_{R\uparrow} + a_{R\downarrow}^\dagger a_{L\downarrow} + a_{L\downarrow}^\dagger a_{R\uparrow} + a_{R\uparrow}^\dagger a_{L\downarrow} \right) \\
+ \frac{\Omega}{2} \left( a_{L\uparrow}^\dagger a_{L\uparrow} + a_{L\downarrow}^\dagger a_{L\downarrow} + a_{R\uparrow}^\dagger a_{R\uparrow} + a_{R\downarrow}^\dagger a_{R\downarrow} \right) \\
+ \frac{\delta}{2} \left( a_{L\uparrow}^\dagger a_{L\downarrow} \right) \\
+ \mathcal{J} \left( a_{L\uparrow}^\dagger a_{L\uparrow}^\dagger a_{L\downarrow} + a_{R\uparrow}^\dagger a_{R\downarrow}^\dagger a_{R\uparrow} \right).
\]

(34)

The solution of the noninteracting case leads us to perform the following transformation:

\[
\begin{align*}
a_{L\uparrow} &= \frac{\Omega}{2\Omega_5} \left[ \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{i2\omega_c t} c_1 - \sqrt{\frac{\Omega_5}{\Omega_5 + \delta}} e^{-i2\omega_c t} c_2 \right. \\
&\left. - \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{-i2\omega_c t} d_1 \right], \\
a_{L\downarrow} &= \frac{1}{2} \left[ \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{i2\omega_c t} c_1 - \sqrt{\frac{\Omega_5}{\Omega_5 + \delta}} e^{-i2\omega_c t} c_2 - \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{i2\omega_c t} d_2 \right], \\
a_{R\uparrow} &= \frac{\Omega}{2\Omega_5} \left[ \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{i2\omega_c t} c_1 - \sqrt{\frac{\Omega_5}{\Omega_5 + \delta}} e^{-i2\omega_c t} c_2 + \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{i2\omega_c t} d_1 \right] \\
&\left. - \sqrt{\frac{\Omega_5}{\Omega_5 + \delta}} e^{-i2\omega_c t} d_2 \right], \\
a_{R\downarrow} &= \frac{1}{2} \left[ \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{i2\omega_c t} c_1 + \sqrt{\frac{\Omega_5}{\Omega_5 + \delta}} e^{-i2\omega_c t} c_2 + \sqrt{\frac{\Omega_5}{\Omega_5 - \delta}} e^{i2\omega_c t} d_1 \\
&\left. + \sqrt{\frac{\Omega_5}{\Omega_5 + \delta}} e^{-i2\omega_c t} d_2 \right].
\end{align*}
\]

(35)

where the usual commutation relations hold: $[c_i, c_j^\dagger] = \delta_{ij}$, $[d_i, d_j^\dagger] = \delta_{ij}$. Within the rotating wave approximation [70, 71] we drop fast oscillating terms while retaining resonant terms. Proceeding along this line and considering a parameters regime such
that $\delta \ll \Omega$ the Hamiltonian (34), can be cast in the simple form:

$$
\mathcal{H} = \left( \frac{\Omega_5}{2} + J \right) \left( c_1^\dagger c_1 - c_2^\dagger c_2 + \frac{\Omega_8}{2} - J \right) \left( d_1^\dagger d_1 - d_2^\dagger d_2 \right) + \frac{g}{8\Omega_5} \left[ \Omega_2 c_1^\dagger c_1 c_1 c_1 + \Omega_2 c_2^\dagger c_2 c_2 + \Omega_2 d_1^\dagger d_1^\dagger + d_2^\dagger d_2^\dagger \right] \left( 4\Omega_2 c_1^\dagger c_1 d_1 d_1 + 4\Omega_2 c_2^\dagger c_2 d_2 d_2 + 4\Omega_2 d_1^\dagger d_1^\dagger d_1 d_1 + 4\Omega_2 d_2^\dagger d_2^\dagger d_2 d_2 \right) + 4\delta^2 c_1^\dagger c_1 c_1 c_2 + 4\delta^2 c_2^\dagger c_2 c_2 c_2 + 4\delta^2 d_1^\dagger d_1^\dagger d_1 d_1 + 4\delta^2 d_2^\dagger d_2^\dagger d_2 d_2 \right].
$$

(36)

This Hamiltonian is separable in two groups of modes $(c_1, d_1)$ and $(c_2, d_2)$ and each group of modes $c_i$ and $d_i$ are coupled by mean field interactions [67]. Thus the Hamiltonian is already diagonal in the basis of $c_i$ and $d_i$ and we label the Fock eigenstates by $|mnpq\rangle$ in the basis order $c_1, c_2, d_1, d_2$. The corresponding energy eigenvalues are:

$$
E_{m,n,p,q} = \left( \frac{\Omega_5}{2} + J \right) (m - n) + \left( \frac{\Omega_8}{2} - J \right) (p - q) + \frac{g\Omega^2}{8\Omega_5} \left( m^2 - m^2 + n^2 - n + p^2 - p + q^2 - q + 4mp + 4np \right) + \frac{g\delta^2}{8\Omega_5} (4mn + 4mq + 4np + 4pq).
$$

(37)

By choosing the same initial condition as in the previous Subsection (i.e. at time $t = 0$ all $N$ atoms have pseudospin down and lie in the left well) the wavefunction at $t = 0$ expressed in the Fock basis is:

$$
|\psi (t = 0)\rangle = \frac{1}{\sqrt{N!}} \left( a_{L,1}^\dagger \right)^N |0\rangle = \frac{1}{2^N \Omega_{5/2}^N} \sum_{mnpq} \delta_{N,m+n+p+q} \frac{\sqrt{N!} A^m B^n C^p D^q}{\sqrt{m! n! p! q!}} |0\rangle,
$$

(38)

where $A = \sqrt{\Omega_5 - \delta}$, $B = -\sqrt{\Omega_5 + \delta}$, $C = -\sqrt{\Omega_5 - \delta}$, $D = \sqrt{\Omega_5 + \delta}$. Thus, at time $t$ the corresponding wavefunction reads:

$$
|\psi (t)\rangle = \frac{1}{2^N \Omega_{5/2}^N} e^{-i\mathcal{H}t} \sum_{mnpq} \delta_{N,m+n+p+q} \frac{\sqrt{N!} A^m B^n C^p D^q}{\sqrt{m! n! p! q!}} |0\rangle
$$

$$
= \frac{1}{2^N \Omega_{5/2}^N} \sum_{mnpq} \delta_{N,m+n+p+q} \frac{\sqrt{N!} A^m B^n C^p D^q}{\sqrt{m! n! p! q!}} e^{-iE_{m,n,p,q} t} |0\rangle.
$$

(39)

In order to calculate the fraction of pseudospin $\sigma$ in the $j$ well, $n_{j,\sigma} (t)$:

$$
n_{j,\sigma} (t) = \frac{1}{N} \langle \psi (t) | a_{j,\sigma}^\dagger a_{j,\sigma} | \psi (t) \rangle,
$$

(40)

we need first to evaluate the averages of the product of operators appearing in the rotated basis, as reported in the Appendix. Once evaluated the fractions $n_{j,\sigma} (t)$, with $\sigma = \uparrow, \downarrow$ and $j = L, R$ we can have access to all physical quantities of interest.
In particular, we get the population imbalance between the two wells

$$D_{LR}(t) = \rho_\uparrow(t) + \rho_\downarrow(t) = (n_{L,\uparrow}(t) + n_{L,\downarrow}(t)) - (n_{R,\uparrow}(t) + n_{R,\downarrow}(t))$$

$$= \frac{1}{4} \frac{1}{\Omega_5^{N+1}} \cos(2\mathcal{J}t) \cdot \left\{ \left[ (\Omega_5 - \delta)^2 + \Omega^2 \right] \times \left[ \frac{\Omega_5}{2} \left( 1 + \cos\left(\frac{\Omega_5^2}{\Omega_5^2} \frac{e^n}{4} \right) \right) + \frac{\delta}{2} \left( 1 - \cos\left(\frac{\Omega_5^2}{\Omega_5^2} \frac{e^n}{4} \right) \right) \right]^{N-1} \right\},$$

(41)

the magnetization

$$M_{LR}(t) = \rho_\uparrow(t) - \rho_\downarrow(t) = (n_{L,\uparrow}(t) - n_{R,\uparrow}(t)) - (n_{L,\downarrow}(t) - n_{R,\downarrow}(t))$$

$$= \frac{1}{4} \frac{1}{\Omega_5^{N+1}} \cos(2\mathcal{J}t) \cdot \left\{ -2\Omega^2 e^{-i\mathcal{J}t} \left[ \frac{\Omega_5}{2} \left( \cos\left(\frac{\Omega_5^2}{2} \frac{e^n}{4} \right) + \cos\left(\frac{\Omega_5^2}{2} \frac{e^n}{4} \right) \right) \right]^{N-1} \right\},$$

(42)

and the pseudospin imbalance:

$$D_{\uparrow\downarrow}(t) = \rho_L(t) + \rho_R(t) = (n_{L,\downarrow}(t) + n_{R,\downarrow}(t)) - (n_{L,\uparrow}(t) + n_{R,\uparrow}(t))$$

$$= \left\{ \frac{\delta^2}{\Omega_5^2} + \frac{\Omega^2}{\Omega_5^{N+1}} e^{-i\mathcal{J}t} \left[ \frac{\Omega_5}{2} \left( \cos\left(\frac{\Omega_5^2}{2} \frac{e^n}{4} \right) + \cos\left(\frac{\Omega_5^2}{2} \frac{e^n}{4} \right) \right) \right]^{N-1} \right\},$$

(43)

where $\Omega_1 = \frac{\Omega^2 - 2\delta^2}{\Omega_5^2}$ and $\Omega_2 = \frac{\Omega^2 - \delta^2}{\Omega_5^2}$.
In Fig. 2 we show the behavior of (41)–(43) as a function of time for the following choice of parameters: $\Omega = 0.1 E_L$, $\bar{J} = -0.1$, $\bar{g} = 0.01$ and $\delta = 0.001$ (units of $E_L$). The time is expressed in units of $\frac{\hbar}{E_L}$. In the left panel we restrict the time interval to $(0 \div 200)$ while the right panel shows collapses and revivals.

One can immediately infer that the temporal modulation of such quantities is much more complicated as it involves more frequencies compared to the characteristic ones $\bar{J} \pm \Omega \delta$ of the non-interacting ones. The atomic and spin currents can be naively obtained by the time derivative of (41)–(42), respectively.

As expected, quantum collapses and revivals (CR) appear and the effect of the nonlinearity appears in the reduction of the oscillation amplitude together with a destruction of periodicity. In particular, in the limit of very small $\delta$ the non-linearity $\bar{g}$ determines the envelope of the revivals, as well as the time separation between the adjacent collapse and revival, while the separation between neighboring CRs is proportional to $1/\bar{g}$. As shown in Fig. 2, the revival occurs at a time scale of the order of tens of ms which is experimentally accessible. Its observation would be an experimental demonstration of quantum coherence even in the presence of spin-orbit interaction.
Fig. 3. Behavior of the spin current $I_s$ flowing between the two wells for $N = 100$, $\Omega = 0.1$, $J = -0.1$, $g = 0.01$ (units of $E_L$) as a function of the Zeeman field.

Another striking feature is the occurrence of a spin Josephson effect, shown in Fig. 3.

From the time derivative of $M_{LR}$ we can numerically evaluate the spin-current $I_s(t) = \frac{dM_{LR}}{dt}$ and define an average spin-current as the integral of $I_s$ over the time interval elapsed between two adjacent collapses and revivals. In Fig. 3 we plot the spin-current as a function of the Zeeman field $\delta$. It shows a linear behavior for small fields $\delta$ and then saturates at higher fields, the linear behavior being characteristic of a non-equilibrium situation.

4 Conclusions

In this paper we investigated the quantum dynamics of a spin-orbit coupled BEC in a double well potential. We performed a two-mode approximation and concentrated on the weak interacting regime, which allows a simple analytical study. Indeed our approach doesn’t allow to study the strong nonlinear interaction regime which could show up interesting self-trapping phenomena and is much more amenable to numerical calculations. Here the quantum evolution of the number difference of bosons and of pseudospin up and down between the two wells has been investigated in detail. Interesting results are found in the intermediate Raman coupling regime, both without and with nonlinear interaction; in particular explicit expressions for the time behaviour of the population imbalance between the two wells, the magnetization and the pseudospin imbalance are obtained in correspondence of an initial condition in which all $N$ atoms with pseudospin down in the left well at $t = 0$. In the non interacting limit the overall behaviour shows coherent Rabi type of oscillations giving rise to an external (population imbalance) and internal (pseudospin imbalance) Josephson effect respectively, while the magnetization exhibits quasiperiodic features. As expected, quantum collapses and revivals appear as a consequence of weak nonlinear interactions. They occur at a time-scale of the order of tens of ms. Furthermore the time-dependent magnetization $M_{LR}$ gives rise to a spin Josephson like effect and to a spin current which could be experimentally measured, as shown for instance in Ref. [73], even if in a different setup (i.e. by the use of dynamical control of quantum tunneling in a double wells optical lattice via oscillatory driving fields), and in the more recent paper [74]. Indeed the excellent manipulation of both internal and external degrees of freedom of ultracold atoms could allow one to obtain a net spin current (together with a vanishing atomic current) and to employ it in order to engineer a variety of devices for spintronics [75], in analogy with the recently realized atomic counterpart of a spin transistor [76].
Appendix

The averages of products of the operators appearing in the Hamiltonian (36) are listed below:

\[
(\psi (t)|e_1^t c_1 |\psi (t)) = \langle \psi (t)|d_1^t d_1 |\psi (t)\rangle = \frac{N}{4} \left( \frac{\Omega_8 - \delta}{\Omega_8} \right),
\]

\[
(\psi (t)|e_2^t c_2 |\psi (t)) = \langle \psi (t)|d_2^t d_2 |\psi (t)\rangle = \frac{N}{4} \left( \frac{\Omega_8 + \delta}{\Omega_8} \right),
\]

\[
(\psi (t)|e_1^t c_2 |\psi (t)) = -e^{i \omega_{1t}} \frac{N \Omega}{4 \Omega_8} \left[ \frac{\Omega_8}{2} \left( \cos \left( \frac{\pi \Omega_1}{4} \right) + \cos \left( \frac{\pi \Omega_2}{2} \right) \right) \right] \left[ \sin \left( \frac{\pi \Omega_1}{4} \right) \right] \left[ \sin \left( \frac{\pi \Omega_2}{2} \right) \right] \left( N - 1 \right),
\]

\[
(\psi (t)|e_1^t d_1 |\psi (t)) = -e^{i \omega_{1t}} \frac{N \Omega}{4 \Omega_8} \left[ \frac{\Omega_8}{2} \left( \cos \left( \frac{\pi \Omega_1}{4} \right) + \cos \left( \frac{\pi \Omega_2}{2} \right) \right) \right] \left[ \sin \left( \frac{\pi \Omega_1}{4} \right) \right] \left[ \sin \left( \frac{\pi \Omega_2}{2} \right) \right] \left( N - 1 \right),
\]

\[
(\psi (t)|e_2^t d_2 |\psi (t)) = e^{i \pi \Omega_8 t} \frac{N \Omega}{4 \Omega_8} \left[ \frac{\Omega_8}{2} \left( \cos \left( \frac{\pi \Omega_1}{4} \right) + \cos \left( \frac{\pi \Omega_2}{2} \right) \right) \right] \left[ \sin \left( \frac{\pi \Omega_1}{4} \right) \right] \left[ \sin \left( \frac{\pi \Omega_2}{2} \right) \right] \left( N - 1 \right),
\]

\[
(\psi (t)|e_2^t d_1 |\psi (t)) = e^{i \pi \Omega_8 t} \frac{N \Omega}{4 \Omega_8} \left[ \frac{\Omega_8}{2} \left( \cos \left( \frac{\pi \Omega_1}{4} \right) + \cos \left( \frac{\pi \Omega_2}{2} \right) \right) \right] \left[ \sin \left( \frac{\pi \Omega_1}{4} \right) \right] \left[ \sin \left( \frac{\pi \Omega_2}{2} \right) \right] \left( N - 1 \right),
\]

\[
(\psi (t)|e_2^t d_2 |\psi (t)) = -e^{i \omega_{1t}} \frac{N \Omega}{4 \Omega_8} \left[ \frac{\Omega_8}{2} \left( \cos \left( \frac{\pi \Omega_1}{4} \right) + \cos \left( \frac{\pi \Omega_2}{2} \right) \right) \right] \left[ \sin \left( \frac{\pi \Omega_1}{4} \right) \right] \left[ \sin \left( \frac{\pi \Omega_2}{2} \right) \right] \left( N - 1 \right),
\]

\[
(\psi (t)|e_1^t d_2 |\psi (t)) = -e^{i \omega_{1t}} \frac{N \Omega}{4 \Omega_8} \left[ \frac{\Omega_8}{2} \left( \cos \left( \frac{\pi \Omega_1}{4} \right) + \cos \left( \frac{\pi \Omega_2}{2} \right) \right) \right] \left[ \sin \left( \frac{\pi \Omega_1}{4} \right) \right] \left[ \sin \left( \frac{\pi \Omega_2}{2} \right) \right] \left( N - 1 \right),
\]

(44)

where \( \Omega_1 = \frac{\Omega^2 - 2 \delta^2}{\Omega_8} \) and \( \Omega_2 = \frac{\Omega^2 + \delta^2}{\Omega_8} \).
In this way, the fraction of pseudospin $\sigma$ in the $j$ well, being $\sigma = \uparrow, \downarrow$ and $j = L, R$, can be obtained after lengthy but straightforward algebraic calculations.

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