Abstract

As a generalization of polyominoes we consider edge-to-edge connected nonoverlapping unions of regular \( k \)-gons. For \( n \leq 4 \) we determine formulas for the number \( a_k(n) \) of generalized polyominoes consisting of \( n \) regular \( k \)-gons. Additionally we give a table of the numbers \( a_k(n) \) for small \( k \) and \( n \) obtained by computer enumeration. We finish with some open problems for \( k \)-polyominoes.

1 Introduction

A polyomino, in its original definition, is a connected interior-disjoint union of axis-aligned unit squares joined edge-to-edge. In other words, it is an edge-connected union of cells in the planar square lattice. For the origin of polyominoes we quote Klärner [13]: “Polyominoes have a long
history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golomb i.e. [5, 6, 7], then by Martin Gardner in his *Scientific American* columns.” At the present time they are widely known by mathematicians, physicists, chemists and have been considered in many different applications, i.e. in the *Ising Model* [2]. To give an illustration of polyominoes Figure 1 depicts the polyominoes consisting of at most 5 unit squares.

One of the first problems for polyominoes was the determination of their number. Although there has been some progress, a solution to this problem remains outstanding. In the literature one sometimes speaks also of the cell-growth problem and uses the term animal instead of polyomino.

Due to its wide area of applications polyominoes were soon generalized to the two other tessellations of the plane, to the eight Archimedean tessellations [4] and were also considered as unions of d-dimensional hypercubes instead of squares. For the known numbers we refer to the “Online Encyclopedia of Integer Sequences” [19].

![Figure 2: A nice 5-polyomino.](image)
In this article we generalize concept of polyominoes to unions of regular nonoverlapping edge-to-edge connected \( k \)-gons. For short we call them \( k \)-polyominoes. An example of a 5-polyomino, which reminds somewhat to Penrose’s famous non-periodic tiling of the plane, is depicted in Figure 2. In the next sections we determine exact formulas for the number \( a_k(n) \) of nonisomorphic \( k \)-polyominoes with \( k \leq 4 \) and give some further values for small parameters \( k \) and \( n \) obtained by computer enumeration. So far edge-to-edge connected unions of regular \( k \)-gons were only enumerated if overlapping of the \( k \)-gons is permitted [9]. We finish with some open problems for \( k \)-polyominoes.

2 Formulas for the number of \( k \)-polyominoes

By \( a_k(n) \) we denote the number of nonisomorphic \( k \)-polyominoes consisting of \( n \) regular \( k \)-gons as cells where \( a_k(n) = 0 \) for \( k < 3 \). For at most two cells we have \( a_k(1) = a_k(2) = 1 \). If \( n \geq 3 \) we characterize three edge-to-edge connected cells \( C_1, C_2, \) and \( C_3 \) of a \( k \)-polyomino, see Figure 3 by the angle \( \beta \) between the centers of the cells. Since these angles are multiples of \( \frac{2}{k} \) we call the minimum

\[
\min \ \left( \frac{a_1}{b_1}; \frac{a_2}{b_2}; \frac{a_3}{b_3} \right) \frac{k}{2} \in \mathbb{Z}
\]

the discrete angle between \( C_1, C_2, \) and \( C_3 \) and denote it by \( (C_1;C_2;C_3) \).

Lemma 1 Two \( k \)-gons \( C_1 \) and \( C_3 \) joined via an edge to a \( k \)-gon \( C_2 \) are nonoverlapping if and only if \( (C_1;C_2;C_3) = \frac{k+3}{6} \). The three \( k \)-gons are neighbored pairwise if and only if \( k \equiv 0 \mod 6 \).

Proof. We consider Figure 3 and set \( \theta = \angle (C_1;C_2;C_3) = \frac{2}{k} \). If the cells \( C_1 \) and \( C_3 \) are nonoverlapping we have \( \frac{P_1P_3}{P_1P_2} = \frac{P_3P_2}{P_1P_2} \) because the lengths of the lines \( P_1P_2 \) and \( P_2P_3 \) are equal. Thus \( \frac{2}{6} = \frac{k+3}{6} \) is necessary. Now we consider the circumcircles of the

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Figure 3: Angle \( \beta \) between three neighbored cells.
cells \( C_1 \) and \( C_3 \), see Figure 4. Due to \( \frac{2}{6} \) only the circle segments between points \( P_4, P_5 \) and \( P_6, P_7 \) may intersect. The last step is to check that the corresponding lines \( P_4P_5 \) and \( P_6P_7 \) do not intersect and they touch each other if and only if \( k \equiv 0 \mod 6 \).

![Figure 4: Nonoverlapping 12-gons.](image)

**Corollary 2**  The number of neighbors of a cell in a \( k \)-polyomino is at most

\[
\min \left\{ 0, 1, \frac{k + 5}{6} \right\}
\]

With the aid of Lemma \( \| \) we are able to determine the number \( a_k(3) \) of \( k \)-polyominoes consisting of 3 cells.

**Theorem 3**

\[
a_k(3) = \frac{k}{2} + \frac{k + 5}{6} + 1 \quad \text{for } k \geq 3
\]

**Proof.** It suffices to determine the possible values for \( (C_1; C_2; C_3) \). Due to Lemma \( \| \) we have \( (C_1; C_2; C_3) \leq \frac{k + 5}{6} \) and due to symmetry considerations we have \( (C_1; C_2; C_3) \leq \frac{k}{2} \).

In order to determine the number of \( k \)-polyominoes with more than 3 cells we describe the classes of \( k \)-polyominoes by graphs. We represent each \( k \)-gon by a vertex and join two vertices exactly if they are connected via an edge.

![Figure 5: The possible graphs of \( k \)-polyominoes with 4 vertices.](image)
Lemma 4 The number of $k$-polyominoes with a graph isomorphic to one of the first three ones in Figure 5 is given by

$$
\sum_{k} 3 \frac{k + 5}{6} z^2 + 6 k 3 \frac{k + 5}{6} + 12 \frac{1}{12}.
$$

Proof. We denote the cell corresponding to the unique vertex of degree 3 in the graph by $C_0$ and the three other cells by $C_1$, $C_2$, and $C_3$. With $1 = (C_4; C_0; C_2)$, $\frac{k + 5}{6} = (C_2; C_0; C_3)$, $\frac{k - 1}{6}$, and $\frac{k}{6}$ we set $m = 1 + 2 + 3 = k 3 \frac{k + 5}{6}$. Because the $k$-polyominoes with a graph isomorphic to one of the first three ones in Figure 5 are uniquely described by $1; 2; 3$, due to Lemma 1 and due to symmetry their number equals the number of partitions of $m$ into at most three parts. This number is the coefficient of $x^m$ in the Taylor series of

$$
\frac{1}{(1-x)(1-x^2)(1-x^3)}
$$

in $x = 0$ and can be expressed as $\frac{m^2 + 6m + 12}{12}$.

Figure 6: Paths of lengths 3 representing chains of four neighbored cells.

In Lemma 5 we have given a condition for a chain of three neighbored cells avoiding an overlapping. For a chain of four neighbored cells we have to consider the two cases of Figure 5. In the second case the two vertices of degree one are not able to overlap so we need a lemma in the spirit of Lemma 1 only for the first case.

Lemma 5 Four $k$-gons $C_1$, $C_2$, $C_3$, and $C_4$ arranged as in the first case of Figure 6 are nonoverlapping if and only if Lemma 1 is fulfilled for the two subchains of length 3 and

$$(C_1; C_2; C_3) + (C_2; C_3; C_4) = \frac{k + 1}{2} :$$

The chain is indeed a 4-cycle if and only if

$$(C_1; C_2; C_3) + (C_2; C_3; C_4) = \frac{k}{2} :$$

Proof. We start with the second statement and consider the quadrangle of the centers of the 4 cells. Because the angle sum of a quadrangle is 2, we have

$$(C_1; C_2; C_3) + (C_2; C_3; C_4) + d(C_3; C_4; C_1) + (C_4; C_1; C_2) = k :$$

Due to the fact that the side lengths of the quadrangle are equal we have

$$(C_1; C_2; C_3) + (C_2; C_3; C_4) = (C_3; C_4; C_1) + (C_4; C_1; C_2)$$

which is equivalent to the statement.

Thus $$(C_1; C_2; C_3) + (C_2; C_3; C_4) = \frac{k + 1}{2}$$ is a necessary condition. Similar to the proof of Lemma 1 we consider the circumcircles of the cells $C_1$, $C_4$ and check that the cells do not intersect.
Lemma 6. For \( k \geq 3 \) the number of \( k \)-polyominoes with a graph isomorphic to one of the last two ones in Figure 5 is given by

\[
\begin{align*}
5k^2 + 4k & \quad \text{for } k \equiv 0 \pmod{12}; \\
5k^2 + 12k + 4 & \quad \text{for } k \equiv 2 \pmod{12}; \\
5k^2 + 20k + 32 & \quad \text{for } k \equiv 4 \pmod{12}; \\
5k^2 + 4k & \quad \text{for } k \equiv 6 \pmod{12}; \\
5k^2 + 12k + 16 & \quad \text{for } k \equiv 8 \pmod{12}; \\
5k^2 + 20k + 20 & \quad \text{for } k \equiv 10 \pmod{12}; \\
\end{align*}
\]

A little calculation yields the proposed formulas.

Theorem 7. For \( k \geq 3 \) we have

\[
\begin{align*}
a_k(4) = 8 & \quad \text{for } k \equiv 0 \pmod{12}; \\
& \quad \text{for } k \equiv 2 \pmod{12}; \\
& \quad \text{for } k \equiv 4 \pmod{12}; \\
& \quad \text{for } k \equiv 6 \pmod{12}; \\
& \quad \text{for } k \equiv 8 \pmod{12}; \\
& \quad \text{for } k \equiv 10 \pmod{12}; \\
\end{align*}
\]

Proof. The list of graphs in Figure 5 is complete because the graphs have to be connected and the complete graph on 4 vertices is not a unit distance graph. Adding the formulas from Lemma 4 and Lemma 5 yields the theorem.
For \( n = 5 \) we have constructed \( k \)-polyominoes with the aid of a computer and have obtained the following values of \( a_k(n) \) given in Table 1 and Table 2.

| \( k \) | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-------|---|---|---|---|---|----|----|----|----|
| 3     | 4 | 12| 24| 66| 160| 448| 1186| 3334| 9235|
| 4     | 12| 35| 108|369|1285|30490|143552|46559|13274826|
| 5     | 25| 118|551|2812|14445|76092|403976|2167116|11698961|
| 6     | 22| 82|333|1448|6572|30490|143552|683101|3274826|
| 7     | 25| 118|558|2876|14982|80075|431889|2354991|12930257|
| 8     | 50|269|1605|10102|65323|430302|2868320|19299334|130807068|
| 9     | 82|585|4418|34838|280014|18838395|156644526|1311575691|
| 10    | 127|985|8350|73675|664411|6078768|56198759|523924389|
| 11    | 186|1750|17501|181127|1908239|219770162|2390025622|
| 12    | 168|13512|131801|1314914|13303523|136035511|1402844804|
| 13    | 1765|17775|185297|1968684|21208739|230877323|
| 14    | 2718|30467|352375|4158216|49734303|601094660|
| 15    | 4336|55264|725869|9707046|131517548|1800038803|
| 16    | 6040|83252|1180526|17054708|249598727|3690421289|
| 17    | 8814|134422|2104485|33522023|540742895|
| 18    | 7678|112514|1694978|26019735|404616118|
| 19    | 8839|135175|2123088|33942901|549711709|
| 20    | 776 |11876|195122|3291481|56537856|983715865|

Table 1: Number of \( k \)-polyominoes with \( n \) cells for small \( k \) and \( n \).

Now we go into more detail how the computer enumeration was done. At first we have to represent \( k \)-polyominoes by a suitable data structure. As in Lemma 1 a \( k \)-polyomino can be described by the set of all discrete angles between three neighbored cells. By fixing one direction we can define the discrete angle between this direction and two neighbored cells and so describe a \( k \)-polyomino by an \( n \times n \)-matrix with integer entries. Due to Corollary 2 we can also describe it as a \( 6 \times n \)-matrix by listing only the neighbors. To deal with symmetry we define a canonical form for these matrices.

Our general construction strategy is orderly generation \[18\], where we use a variant introduced in \[15, 17\]. Here a \( k \)-polyomino consisting of \( n \) cells is constructed by glueing two \( k \)-polyominoes consisting of \( n_1 \) cells having \( n_2 \) cells in common. There are two advantages of this approach. In a \( k \)-polyomino each two cells must be nonoverlapping. If we would add a cell in each generation step we would have to check \( n_1 \) pairs of cells whether they are nonoverlapping or not. By glueing two \( k \)-polyominoes we only need two perform one such check. To demonstrate the the second advantage we compare in Table 3 the numbers \( c_1(n; k) \) and \( c_2(n; k) \) of candidates produced by the original version and the used variant via glueing of orderly generation.

To avoid numerical twists in the overlapping check we utilize Gröbner bases \[11\].
Table 2: Number of \( k \)-polyominoes with \( n \) cells for small \( k \) and \( n \).

| \( kn \) | \( 5 \)   | \( 6 \)   | \( 7 \)   | \( 8 \)   | \( kn \) | \( 5 \)   | \( 6 \)   | \( 7 \)   |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 21    | 972   | 16410 | 294091| 5402087| 36    | 4575  | 130711 | 3943836|
| 22    | 1179  | 20970 | 397852| 7739008| 37    | 4796  | 140434 | 4326289|
| 23    | 1437  | 27720 | 566007| 11832175| 38    | 5380  | 163027 | 5204536|
| 24    | 1347  | 24998 | 495773| 10079003| 39    | 6089  | 193587 | 6464267|
| 25    | 1439  | 27787 | 568602| 11917261| 40    | 6760  | 221521 | 7634297|
| 26    | 1711  | 34763 | 751172| 16624712| 41    | 7578  | 259396 | 9311913|
| 27    | 2045  | 44687 | 1031920| 24389611| 42    | 7282  | 244564 | 8643473|
| 28    | 2376  | 54133 | 1307384| 32317393| 43    | 6089  | 193587 | 6464267|
| 29    | 2786  | 67601 | 1729686| 45260884| 44    | 8373  | 295558 | 1095872|
| 30    | 2641  | 62252 | 1557663| 39891488| 45    | 9321  | 342841 | 13215115|
| 31    | 2790  | 67777 | 1737915| 45587429| 46    | 10207 | 385546 | 15274792|
| 32    | 3204  | 81066 | 2169846| 59424885| 47    | 11282 | 443178 | 18169170|
| 33    | 3706  | 280861| 81124890| 45221541| 48    | 10890 | 420154 | 17012270|
| 34    | 4193  | 116465| 3410564| 102292464| 49    | 11290 | 443178 | 18217475|
| 35    | 4789  | 140075| 4306774| 59424885| 50    | 12309 | 495988 | 20944951|

Table 3: Number of candidates \( c_1 (n; k) \) and \( c_2 (n; k) \) for \( k \)-polyominoes with \( n \) cells.

| \( n \) | \( 4 \)   | \( 5 \)   | \( 6 \)   | \( 7 \)   | \( 8 \)   | \( 9 \)   | \( 10 \)  | \( 11 \)  |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( a_0 (n) \) | 7    | 25    | 118   | 551   | 2812  | 14445 | 76092 | 403976|
| \( c_1 (n; 5) \) | 21   | 74    | 242   | 1038  | 4476  | 21945 | 111232 | 580139|
| \( c_2 (n; 5) \) | 19   | 62    | 192   | 816   | 3541  | 17297 | 87336  | 452215|
| \( a_1 (n) \) | 7    | 25    | 118   | 551   | 2876  | 14982 | 80075  | 431889|
| \( c_1 (n; 7) \) | 31   | 107   | 356   | 1530  | 6682  | 33057 | 168881 | 889721|
| \( c_2 (n; 7) \) | 19   | 62    | 196   | 821   | 3584  | 17778 | 91109  | 479814|
| \( a_{13} (n) \) | 23   | 187   | 1765  | 17775 | 185297| 1968684| 21208739230877323|
| \( c_1 (n; 13) \) | 126  | 721   | 5059  | 43842 | 420958 | 4294445 | 45258582485481211|
| \( c_2 (n; 13) \) | 76   | 408   | 2697  | 23412 | 223789 | 2274489 | 23849241254712159|
| \( a_{17} (n) \) | 48   | 614   | 8814  | 13442210448533522023 | 540742895|
| \( c_1 (n; 17) \) | 2552039220382928874311681666005251057440375|
| \( c_2 (n; 17) \) | 171  | 1261 | 12964 | 17383925455383908006614066925|

4 Open problems for \( k \)-polyominoes

For \( 4 \)-polyominoes the maximum area of the convex hull was considered in [3]. If the area of a cell is normalized to 1 then the maximum area of a \( 4 \)-polyomino consisting of \( n \) squares is given by \( n + \frac{1}{2} \). The second author has proven an analogous result for the maximum
content of the convex hull of a union of $d$-dimensional units hypercubes [14], which is given by

$$X \prod_{i=1}^{d} \prod_{j=1}^{n} i + \prod_{i=1}^{d} n + 2 + i$$

for $n$ hypercubes. For other values of $k$ the question for the maximum area of the convex hull of $k$-polyominoes is still open. Besides from [11] no results are known for the question of the minimum area of the convex hull, which is non trivial for $k \notin 3; 4$.

Another class of problems is the question for the minimum and the maximum number of edges of $k$-polyominoes. The following sharp inequalities for the number $q$ of edges of $k$-polyominoes consisting of $n$ cells were found in [8] and are also given in [10].

$$k = 3 : \quad n + \frac{1}{2} n + p \geq q \geq 2n + 1$$

$$k = 4 : \quad 2n + 2p \geq q \geq 3n + 1$$

$$k = 6 : \quad 3n + \frac{p}{12n} \geq q \geq 5n + 1$$

In general the maximum number of edges is given by $(k - 1)n + 1$. The numbers of 4-polyominoes with a minimum number of edges were enumerated in [16].

Since for $k \notin 3; 4; 6$ regular $k$-gons do not tile the plane the question about the maximum density $\rho(k)$ of an edge-to-edge connected packing of regular $k$-gons arises. In [12]

$$(5) = \frac{3}{2} \frac{5}{5} \frac{5}{p} \approx 0.8541$$

is conjectured.

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