Existence of solutions to a perturbed critical biharmonic equation with Hardy potential

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Abstract In this paper, the following biharmonic elliptic problem
\[ \begin{align*}
\Delta^2 u - \lambda \frac{|u|^{q-2}u}{|x|^s} &= |u|^{2^{**}-2}u + f(x, u), \quad x \in \Omega, \\
u &= \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega
\end{align*} \]
is considered. The main feature of the equation is that it involves a Hardy term and a nonlinearity with critical Sobolev exponent. By combining a careful analysis of the fibering maps of the energy functional associated with the problem with the Mountain Pass Lemma, it is shown, for some positive parameter \( \lambda \) depending on \( s \) and \( q \), that the problem admits at least one mountain pass type solution under appropriate growth conditions on the nonlinearity \( f(x, u) \).

Keywords Biharmonic equation; Critical exponent; Mountain Pass Lemma; Hardy term.

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1 Introduction

In this paper, we are concerned with the following biharmonic elliptic problem
\[ \begin{align*}
\Delta^2 u - \lambda \frac{|u|^{q-2}u}{|x|^s} &= |u|^{2^{**}-2}u + f(x, u), \quad x \in \Omega, \\
u &= \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega
\end{align*} \] (1.1)

where \( \Omega \subseteq \mathbb{R}^N(N \geq 5) \) is a bounded domain containing 0 with smooth boundary \( \partial \Omega \), \( 2 \leq q \leq 2^{**}(s) := \frac{2(N-s)}{N-4} < 2^{**} := \frac{2N}{N-4} \) for \( 0 < s \leq 4 \), \( 2 \leq q < 2^{**} \) for \( s = 0 \), \( \lambda \) is a positive parameter, \( \Delta^2 \) is the biharmonic operator, and \( f(x, t): \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a continuous function and is odd with respect to \( t \) which satisfies the following two assumptions (\( f_1 \)) and (\( f_2 \)):

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\((f_1)\) \[ \lim_{t \to 0^+} \frac{f(x,t)}{t} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(x,t)}{t^{2^{**}-1}} = 0 \] uniformly in \(x \in \Omega\);
\((f_2)\) there exists a constant \(\rho \in (q, 2^{**})\) such that \(0 < \rho F(x,t) \leq tf(x,t)\) for all \(x \in \Omega\) and \(t \in \mathbb{R} \setminus \{0\}\), where \(F(x,t) = \int_0^t f(x,\tau) d\tau\). This condition is usually referred to as the Ambrosetti-Rabinowitz superlinear condition.

According to \((f_1)\), we observe that, for any \(\varepsilon > 0\), there exist \(C_i(\varepsilon) > 0\), \((i = 1, 2, 3)\) such that
\[ |f(x,t)| \leq \varepsilon t^{2^{**}-1} + C_1(\varepsilon)t, \quad x \in \Omega, \quad t \in \mathbb{R}, \] (1.2)
and
\[ |f(x,t)| \leq \varepsilon t + C_2(\varepsilon)t^{2^{**}-1}, \quad x \in \Omega, \quad t \in \mathbb{R}, \] (1.3)
As a consequence of \((f_2)\), one sees that there exists a constant \(C > 0\) such that
\[ F(x,t) \geq C|t|^\rho, \] (1.4)
for all \(x \in \Omega\) and \(t \in \mathbb{R}\).

Due to their wide application in describing a variety of phenomena in physics and other applied sciences, both local and nonlocal elliptic problems with critical exponents have been extensively studied in recent years, and remarkable progress has been made on the existence, non-existence and multiplicity of weak solutions to these problems. For example, Q. Xie et al. \[18\] dealt with the following Kirchhoff type problem involving critical exponent

\[ \begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x,u) + u^5, & x \in \Omega, \\
 u(x) = 0, & x \in \partial \Omega, \end{cases} \] (1.5)
where \(\Omega \subset \mathbb{R}^3\) is a smooth bounded domain, \(a \geq 0, b > 0\) and \(f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}\) is a continuous function. The existence and multiplicity of solutions to problem (1.5) were obtained by using variational method. When the dimension of the space is 4, D. Naimen \[12\] considered the following Kirchhoff type problem involving critical exponent

\[ \begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda u^q + \mu u^3, & x \in \Omega, \\
 u(x) = 0, & x \in \partial \Omega, \end{cases} \] (1.6)
where \(a, \lambda, \mu > 0, b \geq 0\) and \(1 \leq q < 3\). The existence and non-existence of weak solutions to problem (1.6) were obtained, also by using variational method.

There are also some works dealing with biharmonic equations with critical exponents. For the case \(q = 2\), problem (1.6) with \(\lambda = 1, 0 \leq s \leq 2\) and \(f(x,u) \equiv 0\) was considered by Kang et al. in \[8\]. By using the Sobolev-Hardy inequality and variational method, they showed that problem (1.6) has at least one nontrivial solution when \(N \geq 8 - s\). Later, the results of \[8\] was generalized by Li et al. in \[11\], where problem (1.6) with \(\lambda > 0, 0 \leq s \leq 4\) and \(f(x,u) = f(x)\) was investigated. They proved that there exist at least two nontrivial solutions when the norm of \(f\) is appropriately small and the parameters \(\lambda\) and \(s\) satisfy some appropriate conditions.
When \( q \geq 2 \), problem (1.1) with \( f(x,u) \equiv 0 \) has also been studied. For example, Yao et al. [20] considered problem (1.1) with \( 0 \leq s \leq 4 \), and obtained the existence and non-existence of nontrivial solutions to problem (1.1), with the help of Sobolev-Hardy inequality and the Mountain Pass Lemma. It is worth mentioning that there are many other interesting works on nonlinear elliptic problems with critical exponents, among the huge amount of which, we only refer the interested reader to [1, 2, 4, 5, 6, 7, 12, 13, 14, 15, 17, 21] and the references therein.

Motivated mainly by [8, 11, 20], it is natural to consider the existence of weak solutions to problem (1.1) with \( q \geq 2 \) and general nonlinearity \( f \) that depends on both \( x \) and the unknown function \( u \). As far as we know, there have been few works in this direction, mainly because of the following two difficulties. The first one is the lack of compactness of the Sobolev embedding \( H^2_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \), which prevents us from directly establishing the Palais-Smale condition for the associated energy functional. The second one is caused by the Hardy term, since we can not establish the compactness of the mapping \( u \to \frac{u}{|x|^2} \) from \( H^2_0(\Omega) \) into \( L^2(\Omega) \) when \( s = 4 \).

In this paper, the above two difficulties are overcome by combining a careful analysis of the fibering maps of the energy functional associated with the problem with the Mountain Pass Lemma [16] and Brézis-Lieb’s lemma [3], and a mountain pass type solution to problem (1.1) is obtained. Let us explain our strategy in a more detailed way. When \( 0 \leq s < 4 \), with the help of Brézis-Lieb’s lemma, we prove that the energy functional associated with problem (1.1) satisfies the \((PS)_c\) condition for \( c < c(S) \) (Lemma 3.3). Then, after some careful estimates on the norms of the truncated Talenti functions, we show that the energy functional has a mountain pass geometry around 0 and that the corresponding mountain pass level is small than \( c(S) \). Finally, on the basis of the above two steps, a mountain pass type solution to problem (1.1) follows from a standard variational approach. When \( s = 4 \), by introducing an equivalent norm in \( H^2_0(\Omega) \) (Lemma 3.2) and applying a variant of Brézis-Lieb’s lemma, we can also show that the energy functional satisfies the \((PS)_c\) condition for \( c < c(S,\lambda) \) (Lemma 3.6). Then the mountain pass level is proved to be strictly smaller than \( c(S,\lambda) \) with the help of another group of Talenti functions and the existence of a mountain pass type solution follows.

The remainder of this paper is organized as follows. In Section 2 we give some notations, definitions and introduce some necessary lemmas. The main results of this paper are also stated in this section. In Section 3 we give the detailed proof of the main results.

## 2 Preliminaries

In this section, we first introduce some notations and definitions that will be used throughout the paper. In what follows, we denote by \( \| \cdot \|_p \) the \( L^p(\Omega) \) norm for \( 1 \leq p \leq \infty \), and denote the norm of the weighted space \( L^p(\Omega,|x|^{-s}) \) by

\[
\| \cdot \|_{L^p(\Omega,|x|^{-s})} = \left( \int_{\Omega} |x|^{-s} \cdot |\cdot|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.
\]

The Sobolev space \( H^2_0(\Omega) \) will be equipped with the norm \( \| u \| := \|u\|_{H^2_0(\Omega)} = \| \Delta u \|_2 \), which is equivalent to the full one due to Poincaré’s inequality, and its dual space is denoted by \( H^{-2}(\Omega) \).
We always use $\to$ and $\rrightarrow$ to denote the strong and weak convergence in each Banach space, respectively, and use $C$, $C_1$, $C_2$, ..., to denote generic positive constants. $B_r(x_0)$ is a ball of radius $r$ centered at $x_0$. For each $t > 0$, $O(t)$ denotes the quantity satisfying $\frac{|O(t)|}{t} \leq C$, $O_1(t)$ means that there exist two positive constants $C_1$ and $C_2$ such that $C_1t \leq O_1(t) \leq C_2t$, and $o(t)$ means that $|\frac{o(t)}{t}| \to 0$ as $t \to 0$.

In this paper, we consider weak solutions to problem (1.1) in the following sense.

**Definition 2.1.** (Weak solution) A function $u \in H^2_0(\Omega)$ is called a weak solution to problem (1.1), if for all $\varphi \in H^2_0(\Omega)$, it holds that

$$\int_{\Omega} \Delta u \Delta \varphi \, dx - \lambda \int_{\Omega} \frac{|u|^{q-2}u\varphi}{|x|^s} \, dx - \int_{\Omega} |u|^{2^{**}-2}u\varphi \, dx - \int_{\Omega} f(x,u)\varphi \, dx = 0.$$ 

The energy functional associated with problem (1.1) is given by

$$I_\lambda(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} \, dx - \frac{1}{2^{**}} \|u\|^{2^{**}} - \int_{\Omega} F(x,u) \, dx, \forall \ u \in H^2_0(\Omega).$$

According to the hypotheses $(f_1)$ and $(f_2)$ and $2 \leq q < 2^{**}$, it is directly verified that $I_\lambda(u)$ is a $C^1$ functional in $H^2_0(\Omega)$ (see [11]). Since $f$ is odd with respect to $t$, which implies that $I_\lambda(u) = I_\lambda(|u|)$, we may assume that $u \geq 0$ in the sequel.

We then introduce a compactness condition known as local (PS) condition or the (PS)$_c$ condition, which will assist us in finding weak solutions to problem (1.1).

**Definition 2.2.** ((PS)$_c$ condition) Assume that $X$ is a real Banach space, $I : X \to \mathbb{R}$ is a $C^1$ functional and $c \in \mathbb{R}$. We say that $I$ satisfies the (PS)$_c$ condition if any sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c \text{ in } \mathbb{R} \text{ and } I'(u_n) \to 0 \text{ in } X^{-1}(\Omega) \text{ as } n \to \infty$$

has a convergent subsequence, where $X^{-1}$ is the dual space of $X$.

The following three lemmas are crucial in our analysis. The first one is the famous Mountain Pass Lemma, the second one is the Brézis-Lieb’s lemma and the third one is its variant.

**Lemma 2.1.** (Mountain Pass Lemma [10]) Assume that $(X, \| \cdot \|_X)$ is a real Banach space, $I : X \to \mathbb{R}$ is a $C^1$ functional and there exist $\beta > 0$ and $r > 0$ such that $I$ satisfies the following mountain pass geometry:

(i) $I(u) \geq \beta > 0$ if $\|u\|_X = r$;

(ii) there exists a $\overline{u} \in X$ such that $\|\overline{u}\|_X > r$ and $I(\overline{u}) < 0$.

Then there exist a sequence $\{u_n\} \subset X$ such that $I(u_n) \to c_0$ in $\mathbb{R}$ and $I'(u_n) \to 0$ in $X^{-1}$ as $n \to \infty$, where $X^{-1}$ is the dual space of $X$ and

$$c_0 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \beta, \Gamma = \{ \gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = \overline{u} \},$$

which is called the mountain level. Furthermore, $c_0$ is a critical value of $I$ if $I$ satisfies the (PS)$_{c_0}$ condition.
Lemma 2.2. (Brézis-Lieb’s lemma [3]) Let \( p \in (1, \infty) \). Suppose that \( \{u_n\} \) is a bounded sequence in \( L^p(\Omega) \) and \( u_n \to u \) a.e. in \( \Omega \). Then

\[
\lim_{n \to \infty} (\|u_n\|_p^p - \|u_n - u\|_p^p) = \|u\|_p^p.
\]

Lemma 2.3. (10) Let \( r > 1, q \in [1, r] \) and \( \delta \in [0, \frac{Nq}{r}) \). If \( \{u_n\} \) is a bounded sequence in \( L^r(\mathbb{R}^N, |x|^{-\delta r}) \) and \( u_n \to u \) a.e. in \( \mathbb{R}^N \). Then

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^q}{|x|^\delta} - \frac{|u_n - u|^q}{|x|^\delta} - \frac{|u|^q}{|x|^\delta} \, dx = 0,
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{q-1}u_n}{|x|^\delta} - \frac{|u_n - u|^{q-1}(u_n - u)}{|x|^\delta} - \frac{|u|^{q-1}u}{|x|^\delta} \, dx = 0.
\]

The main results of the paper are summarized into the theorem below, which brings this part to a close.

Theorem 2.1. Assume that both \((f_1)\) and \((f_2)\) hold.

(i) For \( 0 \leq s < 4 \) and \( q = 2 \), if \( N \geq 8 - s \) or \( \max\{\frac{N}{N - 2}, \frac{8}{s}\} \) \( < \rho < 2^{**} \), then problem \((1.1)\) has at least one nonnegative solution for \( 0 < \lambda < \lambda_{s,2} \), where \( \lambda_{s,2} \) is given in \((10)\).

(ii) For \( 0 \leq s < 4 \) and \( 2 < q < 2^{**}(s) \), if \( q > \max\{\frac{N-2}{N-s}, \frac{2(4-s)}{4-s}\} \) or \( \max\{\frac{N}{N-s}, \frac{8}{s}\} \) \( < \rho < 2^{**} \), then problem \((1.1)\) has at least one nonnegative solution for all \( \lambda > 0 \).

(iii) For \( s = 4 \) and \( q = 2 \), if \( \max\{\frac{N}{2\lambda}, \frac{4N-2-\gamma_2\lambda}{N-4}\} \) \( < \rho < 2^{**} \), then problem \((1.1)\) has at least one nonnegative solution for \( \lambda \in (0, \lambda_{4,2}) \), where \( \gamma_2 \) and \( \lambda_{4,2} \) are given in Lemma \((7.7)\) and Remark \((7.7)\) respectively.

Remark 2.1. A typical example of a function that meets the conditions \((f_1) - (f_2)\) is

\[
f(x,t) = \sum_{i=1}^{k} C_i(x)|t|^{q_i-2}t, \quad \text{for } x \in \Omega, \ t \in \mathbb{R},
\]

where \( k \in \mathbb{N}, \ q < q_i \leq 2^{**}(s) \) for \( 0 < s \leq 4, \ q < q_i < 2^{**} \) for \( s = 0 \), and \( C_i \in C(\overline{\Omega}) \) is positive.

3 Proofs of the main results.

This section starts off with a lemma that provides the Sobolev-Hardy inequality, which is crucial for getting the \((PS)_c\) condition of functional \( I_\lambda \). The proof of this result may be found, for instance, in [20].

Lemma 3.1. Assume that \( 2 \leq q \leq 2^{**}(s) = \frac{2(N-s)}{N-4} \) \( (0 \leq s \leq 4) \). Then

(i) there exists a constant \( C > 0 \) such that \( C \left( \int_{\Omega} \frac{|u|^q}{|x|^s} \, dx \right)^{\frac{s}{q}} \leq \|u\| \) for any \( u \in H_0^2(\Omega) \);

(ii) if \( 2 \leq q < 2^{**}(s) \), the mapping \( u \to \frac{u}{|x|^\frac{s}{2}} \) from \( H_0^2(\Omega) \) into \( L^q(\Omega) \) is compact.

Remark 3.1. Denote the best Sobolev-Hardy constant

\[
\lambda_{s,q} = \inf_{u \in H_0^2(\Omega) \setminus \{0\}} \frac{\|u\|^2}{(\int_{\Omega} \frac{|u|^q}{|x|^s} \, dx)^{\frac{s}{q}}}. \tag{3.1}
\]
In particular, \( \lambda_{4,2} = \frac{1}{16} N^2(N-4)^2 \). We always write \( \lambda_{0,2s} \) as \( S \) for simplicity, which satisfies \( \|u\|_{2}^{2s} \leq S^{-\frac{2s}{n}} \|u\|^{2s} \).

In order to prove Theorem 2.1, the following result, which follows immediately from (3.1) with \( q = 2 \), is needed.

**Lemma 3.2.** For \( \lambda \in (0, \lambda_{s,2}) \) (0 ≤ s ≤ 4), there exists a \( \mu_s > 0 \) such that

\[
\int_{\Omega} (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^s})\,dx \geq \mu_s \|u\|^2,  \tag{3.2}
\]

for any \( u \in H^2_0(\Omega) \).

**Remark 3.2.** From Lemma 3.2 it follows that the following best Sobolev constant is well defined for \( \lambda \in (0, \lambda_{4,2}) \)

\[
S_{\lambda} = \inf_{u \in H^2_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^s})\,dx}{(\int_{\Omega} |u|^{2s} \,dx)^{\frac{1}{s}}} > 0.  \tag{3.3}
\]

### 3.1 The case 0 ≤ s < 4.

In general, the functional \( I_{\lambda}(u) \) does not satisfy the \((PS)_c\) condition for all \( c \in \mathbb{R} \), due to the appearance of the critical term. However, with the help of Brézis-Lieb’s lemma, we can find a constant \( c(S) \) such that the \((PS)_c\) condition holds for all \( c < c(S) \). This will be essential in revealing the main results.

**Lemma 3.3.** Assume that \( f(x,t) \) satisfies (\( f_1 \)) and (\( f_2 \)), 0 ≤ s < 4. Let \( \{u_n\} \subset H^2_0(\Omega) \) be a sequence such that \( I_{\lambda}(u_n) \to c < c(S) \) and \( I'_{\lambda}(u_n) \to 0 \) in \( H^{-2}(\Omega) \) as \( n \to \infty \), where \( c(S) := \frac{2}{N} S^{\frac{2}{\nu}} \). Then, \( I_{\lambda}(u) \) satisfies the \((PS)_c\) condition if \( q = 2 \) and 0 ≤ \( \lambda \leq \lambda_{s,2} \), or \( 2 < q < 2^{s+2}(s) \) and \( \lambda > 0 \).

**Proof.** We begin the proof with showing the boundedness of \( \{u_n\} \) in \( H^2_0(\Omega) \). Depending on whether or not \( q \) is equal to 2, the proof will be divided into two cases. When \( q > 2 \), by virtue of (\( f_2 \)), we obtain, for any \( \lambda > 0 \), that

\[
c + 1 + o(1)\|u_n\| \geq I_{\lambda}(u_n) - \frac{1}{q} (I'_{\lambda}(u_n), u_n)
= \left( \frac{1}{2} - \frac{1}{q} \right) \|u_n\|^2 + \left( \frac{1}{q} - \frac{2}{2^{s+2}} \right) \|u_n\|^{2s} + \int_{\Omega} \left( \frac{1}{q} f(x, u_n) u_n - F(x, u_n) \right) \,dx
\geq \left( \frac{1}{2} - \frac{1}{q} \right) \|u_n\|^2, \quad n \to \infty.
\]

When \( q = 2 \), in accordance with (\( f_2 \)) and (\( 3.2 \)), we get, for \( \lambda \in (0, \lambda_{s,2}) \), that

\[
c + 1 + o(1)\|u_n\| \geq I_{\lambda}(u_n) - \frac{1}{\rho} (I'_{\lambda}(u_n), u_n)
= \left( \frac{1}{2} - \frac{1}{\rho} \right) \int_{\Omega} (|\Delta u_n|^2 - \lambda \frac{|u_n|^2}{|x|^s})\,dx + \left( \frac{1}{\rho} - \frac{2}{2^{s+2}} \right) \|u_n\|^{2s}.
\]

6
\[ + \int_{\Omega} \left( \frac{1}{\rho} f(x, u_n) u_n - F(x, u_n) \right) dx \]
\[ \geq \left( \frac{1}{2} - \frac{1}{\rho} \right) \int_{\Omega} (|\Delta u_n|^2 - \lambda \frac{|u_n|^2}{|x|^s}) dx \]
\[ \geq \left( \frac{1}{2} - \frac{1}{\rho} \right) \mu_s \| u_n \|^2, \quad n \to \infty. \]

It is obvious that in either case \( \{u_n\} \) is a bounded sequence in \( H^s_0(\Omega) \). Consequently, by recalling Lemma 3.1 one sees that there is a subsequence of \( \{u_n\} \) (which we still denote by \( \{u_n\} \)) such that, as \( n \to \infty \),

\[
\begin{align*}
  u_n &\to u \text{ in } H^s_0(\Omega), \\
  u_n &\to u \text{ in } L^r(\Omega) \ (1 < r < 2^{*}), \\
  u_n &\to u \text{ in } H^s_0(\Omega), \\
  u_n &\to u \text{ in } L^q(\Omega, |x|^{-s}) \ (1 \leq q < 2^{*}(s)), \\
  |u_n|^{2^{*}-2} u_n &\to |u|^{2^{*}-2} u \text{ in } L^{2^{*}-s}(\Omega), \\
  u_n &\to u \text{ a.e. in } \Omega.
\end{align*}
\] (3.5)

In view of (1.2) (with \( \varepsilon = 1 \)), there exists a positive constant \( C \), independent of \( n \), such that

\[
\left| \int_{\Omega} f(x, u_n) u_n dx \right| \leq \int_{\Omega} |f(x, u_n)||u_n| dx \\
\leq \|u_n\|_{L^{2^{*}}}^2 + C_1(1) \|u_n\|_2^2 \\
\leq C.
\]

For any \( \varepsilon > 0 \), take \( \delta = \frac{\varepsilon}{C_1(\varepsilon)^{\frac{2^{*}-2}{2^{*}}} C_2} \), where \( C_1(\varepsilon) > 0 \) is given in (1.2). Then for any measurable subset \( E \subset \Omega \) with \( \text{mes } E < \delta \), we obtain, by recalling (1.2) again and applying Hölder’s inequality, that

\[
\left| \int_{E} f(x, u_n) u_n dx \right| \leq \varepsilon \int_{E} |u_n|^{2^{*}} dx + C_1(\varepsilon) \int_{E} |u_n|^2 dx \\
\leq \varepsilon \|u_n\|_{L^{2^{*}}}^{2^{*}} + C_1(\varepsilon) \|u_n\|_2^2 \text{mes } E^{\frac{2^{*}-2}{2^{*}}} \\
\leq C_1 \varepsilon + C_2 \varepsilon^{\frac{2^{*}-2}{2^{*}}},
\]

uniformly with respect to \( n \in \mathbb{N} \), where \( C_1, C_2 \) are positive constants independent of \( n \). Hence the family of functions \( \{f(x, u_n) u_n\} \) is equi-absolutely-continuous. In addition, recalling that \( u_n \to u \) a.e. in \( \Omega \) as \( n \to \infty \) and \( f(x, t) \) is continuous, we get \( f(x, u_n) u_n \to f(x, u) u \) a.e. in \( \Omega \) as \( n \to \infty \). This, together with the fact that \( \text{mes } \Omega < \infty \), implies that \( f(x, u_n) u_n \to f(x, u) u \) in measure. Therefore, by virtue of Vitali convergence theorem, we get

\[
\lim_{n \to \infty} \int_{\Omega} f(x, u_n) u_n dx = \int_{\Omega} f(x, u) u dx.
\] (3.6)

Similarly, we can prove that

\[
\lim_{n \to \infty} \int_{\Omega} F(x, u_n) dx = \int_{\Omega} F(x, u) dx.
\] (3.7)
We claim that there exists a subsequence of \( \{u_n\} \). Moreover, since 
\[
\|u_n\|_{2^*} \xrightarrow{n \to \infty} \|u\|_{2^*},
\]
which, together with the fact that \( f(x, t) \) is a continuous function and \( u_n \to u \) a.e. in \( \Omega \) as \( n \to \infty \), shows that \( f(x, u_n)u \to f(x, u)u \) a.e. in \( \Omega \) as \( n \to \infty \). Then, according to Lebesgue’s dominated convergence theorem, we know
\[
\lim_{n \to \infty} \int_{\Omega} f(x, u_n)u dx = \int_{\Omega} f(x, u)u dx.
\]
(3.8)

To complete the proof, let \( w_n = u_n - u \). Then \( \{w_n\} \) is also a bounded sequence in \( H_0^2(\Omega) \). So there exists a subsequence of \( \{w_n\} \) (which we still denoted by \( \{w_n\} \)) such that
\[
\lim_{n \to \infty} \|w_n\|^2 = l \geq 0.
\]
(3.9)

We claim that \( l = 0 \). Indeed, according to (3.5), we have
\[
\|w_n\|^2 = \int_{\Omega} |\Delta w_n + \Delta u|^2 dx
\]
\[
= \int_{\Omega} |\Delta w_n|^2 dx + \int_{\Omega} |\Delta u|^2 dx + 2 \int_{\Omega} \Delta w_n \Delta u dx
\]
\[
= \|w_n\|^2 + \|u\|^2 + o(1), \quad n \to \infty.
\]
(3.10)

Moreover, since \( \|u_n\|_{2^*} \leq C \) and \( u_n \to u \) a.e. in \( \Omega \) as \( n \to \infty \), one sees, by using Brézis–Lieb’s lemma, that
\[
\|u_n\|_{2^*} \leq \|w_n\|_{2^*} + \|u\|_{2^*} + o(1), \quad n \to \infty.
\]
(3.11)

Therefore, in accordance with (3.9), (3.10), (3.11) and the assumption that \( I_\lambda'(u_n) \to 0 \) in \( H^{-2}(\Omega) \) as \( n \to \infty \), we have
\[
o(1) = \langle I_\lambda'(u_n), u_n \rangle
\]
\[
= \int_{\Omega} (|\Delta u_n|^2 - \lambda \frac{|u_n|^q}{|x|^s}) dx - \|u_n\|_{2^*}^2 - \int_{\Omega} f(x, u_n)u_n dx
\]
\[
= \int_{\Omega} (|\Delta u|^2 - \lambda \frac{|u|^q}{|x|^s}) dx - \|u\|_{2^*}^2 - \int_{\Omega} f(x, u)u dx + \|w_n\|^2 - \|w_n\|_{2^*}^2 + o(1)
\]
\[
= \langle I_\lambda'(u), u \rangle + \|w_n\|^2 - \|w_n\|_{2^*}^2 + o(1), \quad n \to \infty,
\]
and
\[
o(1) = \langle I_\lambda'(u_n), u \rangle
\]
\[
= \int_{\Omega} (\Delta u_n \Delta u - \lambda \frac{|u_n|^{q-2} u_n u}{|x|^s}) dx - \|u_n\|_{2^*}^2 - \int_{\Omega} f(x, u_n)u dx
\]
\[
= \int_{\Omega} (|\Delta u|^2 - \lambda \frac{|u|^q}{|x|^s}) dx - \|u\|_{2^*}^2 - \int_{\Omega} f(x, u)u dx + o(1)
\]
\[
= \langle I_\lambda'(u), u \rangle + o(1), \quad n \to \infty.
\]
Combining the above two equalities one obtains

$$\langle I'_\lambda(u), u \rangle = 0, \quad (3.12)$$

and

$$\|w_n\|^{2^{*^*}}_{2^{*^*}} - \|w_n\|^2 = o(1), \quad n \to \infty. \quad (3.13)$$

In addition, from the Sobolev embedding one has

$$\|w_n\|^{2^{*^*}}_{2^{*^*}} \leq S^{-\frac{2^{*^*}}{2}} \|w_n\|^{2^{*^*}} < C, \quad \forall \ n \in \mathbb{N}. \quad (3.14)$$

It follows from (3.9), (3.13) and (3.14) that there is a subsequence of \(\{w_n\}\) such that

$$\lim_{n \to \infty} \|w_n\|^{2^{*^*}}_{2^{*^*}} = \lim_{n \to \infty} \|w_n\|^2 = l. \quad (3.15)$$

Letting \(n \to \infty\) in (3.14), we have

$$l \geq S^{-\frac{2^{*^*}}{2}} = S^\frac{2^{*^*}}{2}. \quad (3.16)$$

On one hand, in view of (3.5), (3.7), (3.10), (3.11) and the fact that \(I_\lambda(u_n) = c + o(1)\) as \(n \to \infty\), we have

$$o(1) + c = I_\lambda(u_n) = \frac{1}{2} \|u_n\|^2 - \frac{\lambda}{q} \int_\Omega \frac{|u_n|^q}{|x|^s} dx - \frac{1}{2^{*^*}} \|u_n\|^{2^{*^*}}_{2^{*^*}} - \int_\Omega F(x, u_n) dx$$

$$= \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_\Omega \frac{|u|^q}{|x|^s} dx - \frac{1}{2^{*^*}} \|u\|^{2^{*^*}}_{2^{*^*}} - \int_\Omega F(x, u) dx$$

$$+ \frac{1}{2} \|w_n\|^2 - \frac{1}{2^{*^*}} \|w_n\|^{2^{*^*}}_{2^{*^*}} + o(1)$$

$$= I_\lambda(u) + \frac{1}{2} \|w_n\|^2 - \frac{1}{2^{*^*}} \|w_n\|^{2^{*^*}}_{2^{*^*}} + o(1), \quad n \to \infty,$$

which yields that

$$I_\lambda(u) = c - \frac{1}{2} \|w_n\|^2 + \frac{1}{2^{*^*}} \|w_n\|^{2^{*^*}}_{2^{*^*}} + o(1), \quad n \to \infty.$$

Recalling (3.15) and (3.16), we get from the above equality that

$$I_\lambda(u) = c - \left(\frac{1}{2} - \frac{1}{2^{*^*}}\right)l \leq c - \frac{2}{N} S^\frac{2^{*^*}}{2} < 0.$$

On the other hand, by (3.12) and (f2), we can derive

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{q} (I'_\lambda(u), u)$$

$$= \left(\frac{1}{2} - \frac{1}{q}\right) \|u\|^2 + \left(\frac{1}{q} - \frac{1}{2^{*^*}}\right) \|u_n\|^{2^{*^*}}_{2^{*^*}} - \int_\Omega \left(\frac{1}{q} f(x, u) - F(x, u)\right) dx$$

$$\geq 0,$$

a contradiction. Thus, \(\lim_{n \to \infty} \|w_n\|^2 = l = 0\), which implies that \(u_n \to u\) in \(H^2_0(\Omega)\) as \(n \to \infty\). The proof is complete.
Before going further, we list some well-known estimates on the Talenti functions, which will play a crucial role in estimating the mountain pass level of $I_\lambda$ around 0. For any $\varepsilon > 0$, define

$$U_\varepsilon(x) = \left[ N(N-4)(N^2-4) \right]^{\frac{N-4}{2}} \frac{\varepsilon^{\frac{N-4}{2}}}{\varepsilon^2 + |x|^\frac{N-4}{2}}, \quad x \in \mathbb{R}^N.$$  

Then $U_\varepsilon(x)$ is a solution of the critical problem

$$\Delta^2 u = u^{2^*_\varepsilon - 1}, \quad x \in \mathbb{R}^N, \quad N \geq 5,$$

and $\|U_\varepsilon\| = \|U_\varepsilon\|^{2^*_\varepsilon} = S^{\frac{N}{2^*_\varepsilon}}$, where $S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\|\Delta u\|^{2^*_\varepsilon} = \frac{\|U_\varepsilon\|^2}{\|\Delta U_\varepsilon\|^{2^*_\varepsilon}}}$ is given in Remark 3.4.

The Talenti functions, after being truncated, are estimated in the following (see [11], [19] and [20]).

**Lemma 3.4.** Let $\tau \in C_0^\infty(\Omega)$ be a cut-off function such that $\tau(x) = \tau(|x|)$, $0 \leq \tau(x) \leq 1$ for $x \in \Omega$, and

$$\tau(x) = \begin{cases} 
1, & |x| < R, \\
0, & |x| > 2R,
\end{cases}$$

where $R > 0$ is a constant such that $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \tau(x)U_\varepsilon(x)$. Suppose that $\varepsilon \to 0$. Then

$$\|u_\varepsilon\| = S^{\frac{N}{2^*_\varepsilon}} + O(\varepsilon^{N-4}),$$

$$\|u_\varepsilon\|^{2^*_\varepsilon} = S^{\frac{N}{2^*_\varepsilon}} + O(\varepsilon^N).$$

Set $v_\varepsilon(x) = \frac{u_\varepsilon}{\|u_\varepsilon\|^{2^*_\varepsilon}}$. Then

$$\|v_\varepsilon\| = S + O(\varepsilon^{N-4}),$$

$$\|v_\varepsilon\|^{2^*_\varepsilon} = 1,$$  

$$\|v_\varepsilon\| = \begin{cases} 
\frac{O(\varepsilon^{\frac{N-4}{2}})}, & 1 < \rho < \frac{N}{N-4}, \\
\frac{O(\varepsilon^{\frac{N-4}{2}} \ln \varepsilon)}{|\ln \varepsilon|}, & \rho = \frac{N}{N-4}, \\
O(\varepsilon^{\frac{N-4}{2}}), & \frac{N}{N-4} < \rho < 2^*,
\end{cases}$$

and

$$\int_\Omega |v_\varepsilon|^q |x|^s \, dx = \begin{cases} 
\frac{O(\varepsilon^{\frac{N-4}{2}})}, & 1 < q < \frac{N-s}{N-4}, \\
\frac{O(\varepsilon^{\frac{N-4}{2}} \ln \varepsilon)}{|\ln \varepsilon|}, & q = \frac{N-s}{N-4}, \\
\frac{O(\varepsilon^{\frac{N-4}{2}} \ln \varepsilon)}{|\ln \varepsilon|}, & \frac{N-s}{N-4} < q < 2^*(s).
\end{cases}$$

With the help of the Talenti functions given above, we can show that the mountain pass level of $I_\lambda$ around 0 is strictly less than $c(S)$.  

10
Lemma 3.5. Assume that (f₁)-(f₂), and the condition (i) or (ii) of Theorem 2.1 hold. Then there exists a \( u^* \in H^2_0(\Omega) \) such that

\[
\sup_{t \geq 0} I_\lambda(tu^*) < c(S),
\]

where \( c(S) := \frac{2}{N} S^\frac{N}{4} \).

Proof. Define the fibering maps associated with the energy functional \( I_\lambda \) by

\[
\psi_u(t) = I_\lambda(tu) = \frac{1}{2} t^2 \| u \|^2 - \frac{1}{q} t^q \int_{\Omega} \frac{|\nabla u|^q}{|x|^s} \, dx - \frac{1}{2} \lambda t^{2q^*} \| u \|_{2q^*}^{2q^*} - \int_\Omega F(x, tu) \, dx, \quad t \geq 0.
\]

Recalling (3.18) and the fact that \( F(x, t) \geq 0 \) for any \( x \in \Omega \) and \( t \geq 0 \), one sees, as \( t \to 0 \), that

\[
\psi_{v_\varepsilon}(t) \leq \frac{1}{2} t^2 \| v_\varepsilon \|^2 \to 0,
\]

uniformly for \( \varepsilon \in (0, \varepsilon_1) \), where \( \varepsilon_1 > 0 \) is a suitably small but fixed number and \( v_\varepsilon \) is given in Lemma 3.5. Therefore, there exists a \( t_0 > 0 \), independent of \( \varepsilon \), such that

\[
\psi_{v_\varepsilon}(t) < c(S), \quad t \in (0, t_0).
\]

Set \( g(t) = \frac{1}{2} t^2 \| v_\varepsilon \|^2 - \frac{1}{2} t^{2q^*} \), then

\[
\psi_{v_\varepsilon}(t) = g(t) - \frac{1}{q} \lambda t^q \int_{\Omega} \frac{|\nabla v_\varepsilon|^q}{|x|^s} \, dx - \int_\Omega F(x, tv_\varepsilon) \, dx, \quad t \geq 0.
\]

According to (1.4), there exists a positive constant \( C \) such that

\[
\psi_{v_\varepsilon}(t) \leq \max_{t \geq t_0} g(t) - \frac{1}{q} \lambda t^q \int_{\Omega} \frac{|\nabla v_\varepsilon|^q}{|x|^s} \, dx - Ct^p \| v_\varepsilon \|^p_\rho
\]

\[
\leq \max_{t \geq 0} g(t) - \frac{1}{q} \lambda t^q \int_{\Omega} \frac{|\nabla v_\varepsilon|^q}{|x|^s} \, dx - Ct_0^p \| v_\varepsilon \|^p_\rho, \quad t \in [t_0, \infty).
\]

By a direct calculation, \( g \) takes its maximum at \( t_\varepsilon^* := \| v_\varepsilon \|_\frac{2}{1-s}^2 \) and \( g(t_\varepsilon^*) = \frac{2}{N} \| v_\varepsilon \|^{\frac{2}{N}} \). Therefore, in view of this, (3.18) and (3.25), one sees, for \( t \geq t_0 \), that

\[
\psi_{v_\varepsilon}(t) \leq g(t_\varepsilon^*) - \frac{1}{q} \lambda t_\varepsilon^q \int_{\Omega} \frac{|\nabla v_\varepsilon|^q}{|x|^s} \, dx - Ct_0^p \| v_\varepsilon \|^p_\rho
\]

\[
= \frac{2}{N} (S + O(\varepsilon^{-N-4}))^{\frac{N}{2}} - \frac{1}{q} \lambda t_\varepsilon^q \int_{\Omega} \frac{|\nabla v_\varepsilon|^q}{|x|^s} \, dx - Ct_0^p \| v_\varepsilon \|^p_\rho
\]

\[
= c(S) + O(\varepsilon^{-N-4}) - \frac{1}{q} \lambda t_\varepsilon^q \int_{\Omega} \frac{|\nabla v_\varepsilon|^q}{|x|^s} \, dx - Ct_0^p \| v_\varepsilon \|^p_\rho, \quad \varepsilon \to 0.
\]

Our goal is to show, for \( \varepsilon \) suitably small, that

\[
\psi_{v_\varepsilon}(t) < c(S), \quad t \in [t_0, \infty),
\]

which is fulfilled once we can prove that

\[
O(\varepsilon^{-N-4}) - \frac{1}{q} \lambda t_\varepsilon^q \int_{\Omega} \frac{|\nabla v_\varepsilon|^q}{|x|^s} \, dx - Ct_0^p \| v_\varepsilon \|^p_\rho < 0.
\]
It is easy to see that (3.15) holds if either (I) or (II) of the following is valid

\[
\begin{align*}
(I) & \quad O(\varepsilon^{N-4}) - Ct_0^\rho \|v_\varepsilon\|_\rho^\rho < 0; \\
(II) & \quad O(\varepsilon^{N-4}) - \frac{1}{q} \lambda_0^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} dx < 0.
\end{align*}
\]

First, when \(0 \leq s < 4, q \geq 2\) and \(\max\{\frac{N-s}{N-4}, \frac{8-s}{N-4}\} < \rho < 2^{*\ast}\), by recalling (3.19), we obtain

\[
\lim_{\varepsilon \to 0} \varepsilon^{N-4} = \lim_{\varepsilon \to 0} \varepsilon^{(N-4)} = 0,
\]

which implies that (I) holds.

Next, we shall show that (II) is fulfilled for small \(\varepsilon\) when other cases in (i) or (ii) of Theorem 2.1 are satisfied, with the help of (3.20). Indeed, if \(q = 2\) and \(N = 8 - s\), then \(q = \frac{N-s}{N-4}\), which, together with (3.20), implies that

\[
\lim_{\varepsilon \to 0} \varepsilon^{N-4} = \lim_{\varepsilon \to 0} \frac{1}{\ln \varepsilon} = 0.
\]  

(3.28)

If \(q = 2\) and \(N > 8 - s\), then \(q > \frac{N-s}{N-4}\) and

\[
\lim_{\varepsilon \to 0} \varepsilon^{N-4} = \lim_{\varepsilon \to 0} \varepsilon^{N-(8-s)} = 0.
\]  

(3.29)

It follows from (3.28) and (3.29) that (II) is valid if \(q = 2\) and \(N \geq 8 - s\).

Similarly, if \(q > 2\) and \(\max\{\frac{N-s}{N-4}, \frac{2(4-s)}{N-4}\} < q < 2^{*\ast}(s)\), we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{N-4} = \lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{2}[(N-4)q-(8-2s)]} = 0,
\]

which also implies that (II) holds.

In conclusion, if the condition (i) or (ii) of Theorem 2.1 holds, by combining (3.24) with (3.20), we have, for \(\varepsilon\) suitably small, that

\[
\sup_{t \geq 0} \psi_{v_\varepsilon}(t) < c(S).
\]

Fix such an \(\varepsilon > 0\) and take \(u^* \equiv v_\varepsilon\). The proof is complete. \(\square\)

In what follows, we shall show the existence of the weak solutions to problem (1.1) on the basis of Lemmas 3.3 and 3.5 and the Mountain Pass Lemma.

**Proof of Theorem 2.1 (i) and (ii).** We first show that the functional \(I_\lambda\) satisfies the mountain pass geometry in both cases. If \(q > 2\), according to (1.3) and (3.1), for any \(\varepsilon > 0\), there exists a \(C_3(\varepsilon) > 0\) such that

\[
\begin{align*}
I_\lambda(u) & \geq \frac{1}{2} \|u\|^2 - \frac{\lambda_0^\frac{q}{2}}{q} \lambda \|u\|^q - \frac{S^{-2^{*\ast}}}{2^{*\ast}} \|u\|^{2^{*\ast}} - \frac{\lambda_0^{1/2}}{2} \varepsilon \|u\|^2 - C_3(\varepsilon)S^{-\frac{2^{*\ast}}{2}} \|u\|^{2^{*\ast}} \\
& = (1 - \lambda_0^{-1/2}) \|u\|^2 - \frac{\lambda_0^{1/2}}{q} \lambda \|u\|^q - \frac{1}{2^{*\ast}} + C_3(\varepsilon)S^{-\frac{2^{*\ast}}{2}} \|u\|^{2^{*\ast}}, \quad \forall \, u \in H^0_0(\Omega)\backslash\{0\}.
\end{align*}
\]  

(3.30)
If \( q = 2 \), by using (3.2) and (3.3) again and (3.4), we obtain that
\[
I_\lambda(u) \geq \frac{\mu_s}{2} \|u\|^2 - \frac{S^{-2}S^{2*}}{2^{2*}} \|u\|^{2*} - \frac{\lambda_0^{-1}}{2} \|u\|^2 - C_3 S^{-2} \|u\|^{2*},
\]
\[\lambda_0 \leq 0, \quad \lambda \geq \lambda_0, \quad \forall u \in H_0^2(\Omega) \backslash \{0\}. \tag{3.31}\]
Choosing \( \varepsilon > 0 \) so small that \( 1 - \lambda_0^{-1} \varepsilon > 0 \) and \( \mu_s - \lambda_0^{-1} \varepsilon > 0 \), one sees that there exist \( \beta, \gamma > 0 \) such that \( I_\lambda(u) \geq \beta \) for all \( \|u\| = r \).

On the other hand, recalling that \( F(x,t) \geq 0 \) for any \( x \in \Omega \) and \( t \geq 0 \) due to \((f_2)\), we have, for any \( u \in H_0^2(\Omega) \backslash \{0\} \)
\[
\psi_u(t) \leq \frac{1}{2} t^2 \|u\|^2 - \frac{1}{2^{2*}} t^{2*} \|u\|^{2*},
\]
which implies that \( \lim_{t \to \infty} \psi_u(t) = -\infty \). Therefore, there exists a \( t_u > 0 \) suitably large such that \( \|t_u u\| > r \) and \( \psi_u(t_u) = I_\lambda(t_u u) < 0 \). Thus, \( I_\lambda \) satisfies the mountain pass geometry around 0, and there exists a sequence \( \{u_n\} \subset H_0^2(\Omega) \) such that \( I_\lambda(u_n) \to c_0 \geq \beta \) and \( I'_\lambda(u_n) \to 0 \) in \( H^{-2}(\Omega) \) as \( n \to \infty \), where
\[
c_0 = \inf_{\gamma \in \{\gamma \in C([0,1],H_0^2(\Omega)): \gamma(0) = 0, \gamma(1) = t_u u^*\}} \max_{t \in [0,1]} I_\lambda(\gamma(t)) \text{ and } \Gamma = \{\gamma \in C([0,1],H_0^2(\Omega)): \gamma(0) = 0, \gamma(1) = t_u u^*\}. \tag{3.33}\]
and \( u^* \) is given in Lemma 3.5. In view of (3.30) and (3.24), one sees
\[
c_0 \leq \max_{t \in [0,1]} I_\lambda(t t_u u^*) \leq \sup_{t \geq 0} I_\lambda(t u^*) < c(S). \tag{3.33}\]
It then follows from Lemma 3.3 that \( I_\lambda(u) \) satisfies the \((PS)_{c_0}\) condition. Consequently, there exists a convergent subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), such that \( u_n \to u \) in \( H_0^2(\Omega) \) as \( n \to \infty \), which implies that \( I_\lambda(u) = c_0 \) and \( I'_\lambda(u) = 0 \), i.e., \( u \) is a nonnegative solution to problem (1.1). The proof of (i) and (ii) of Theorem 2.1 is complete. \( \square \)

3.2 The case \( s = 4 \).

When \( s = 4 \), the difficulty we encounter is the lack of compactness of the mapping \( u \to \frac{u}{|x|^2} \) from \( H_0^2(\Omega) \) into \( L^2(\Omega) \) and the Sobolev embedding \( H_0^2(\Omega) \hookrightarrow L^{2*}(\Omega) \), which prevents us from establishing the usual (PS) condition directly. However, inspired by some ideas from [111], we can show that \( I_\lambda \) satisfies the \((PS)_c\) condition for some \( c \), and then the existence of a mountain pass type solution to problem (1.1) follows.

**Lemma 3.6.** Assume that \( q = 2, s = 4 \) and that \( f(x,t) \) satisfies \((f_1)\) and \((f_2)\). Let \( \{u_n\} \subset H_0^2(\Omega) \) be a sequence such that \( I_\lambda(u_n) \to c < c(S_\lambda) \) and \( I'_\lambda(u_n) \to 0 \) in \( H^{-2}(\Omega) \) as \( n \to \infty \), where \( c(S_\lambda) := \frac{2}{N} \frac{S^{2*}}{S^{4}} \). Then \( I_\lambda(u) \) satisfies the \((PS)_c\) condition provided that
\[
0 < \lambda < \lambda_{4,2}.
\]

**Proof.** Recalling (3.4) with \( s = 4 \), one sees that \( \{u_n\} \) is bounded in \( H_0^2(\Omega) \) when \( 0 < \lambda < \lambda_{4,2} \). Hence, according to Lemma 3.1, there is a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), such
that, as $n \to \infty$,

\[
\begin{aligned}
&u_n \to u \text{ in } H^2_0(\Omega), \\
u_n \to u \text{ in } L^r(\Omega) \ (1 \leq r < 2^{**}), \\
u_n \to u \text{ in } H^1_0(\Omega), \\
u_n \to u \text{ in } L^2(\Omega, |x|^{-2}), \\
|u_n|^{2^{**}-2}u_n \to |u|^{2^{**}-2}u \text{ in } L^{2^{**}}(\Omega), \\
u_n \to u \text{ a.e. in } \Omega.
\end{aligned}
\]  

(3.34)

As was done in the proof of Lemma 3.3 set $w_n = u_n - u$. Then $\{w_n\}$ is bounded in $H^2_0(\Omega)$, which ensures that there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, such that

\[\lim_{n \to \infty} \|w_n\|^2 = l \geq 0.\]

In what follows, we shall show $l = 0$. In view of (3.34) and Brézis-Lieb’s lemma, we know that (3.10) and (3.11) are still valid. In addition, from Lemma 2.3 one sees, as $n \to \infty$, that

\[
\int_\Omega \frac{|u_n|^2}{|x|^4} \, dx = \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx + \int_\Omega \frac{|u|^2}{|x|^4} \, dx + o(1).
\]  

(3.35)

Combining (3.6), (3.8), (3.10), (3.11), (3.34), (3.35) with the assumption that $I'_\lambda(u_n) \to 0$ in $H^{-2}(\Omega)$ as $n \to \infty$, one gets

\[
o(1) = \langle I'_\lambda(u_n), u_n \rangle = \int_\Omega (|\Delta u_n|^2 - \lambda \frac{|u_n|^2}{|x|^4}) \, dx - \|u_n\|_{2^{**}}^{2^{**}} - \int_\Omega f(x, u_n)u_n \, dx
\]

\[
= \int_\Omega (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^4}) \, dx - \|u\|_{2^{**}}^{2^{**}} - \int_\Omega f(x, u)u \, dx
\]

\[
+ \|w_n\|^2 - \lambda \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx - \|w_n\|_{2^{**}}^{2^{**}} + o(1)
\]

\[
= \langle I'_\lambda(u), u \rangle + \|w_n\|^2 - \lambda \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx - \|w_n\|_{2^{**}}^{2^{**}} + o(1), \quad n \to \infty,
\]

and

\[
o(1) = \langle I'_\lambda(u_n), u \rangle = \int_\Omega (\Delta u_n \Delta u - \lambda \frac{|u_n|^2}{|x|^4}) \, dx - \int_\Omega |u_n|^{2^{**}-2}u_n \, dx - \int_\Omega f(x, u_n)u \, dx
\]

\[
= \int_\Omega (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^4}) \, dx - \|u\|_{2^{**}}^{2^{**}} - \int_\Omega f(x, u)u \, dx + o(1)
\]

\[
= \langle I'_\lambda(u), u \rangle + o(1), \quad n \to \infty.
\]

According to the above two equalities, we obtain (3.12) and

\[
\|w_n\|^2 - \lambda \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx - \|w_n\|_{2^{**}}^{2^{**}} = o(1), \quad n \to \infty.
\]  

(3.36)

In addition, in view of (3.3), one has

\[
\|w_n\|_{2^{**}}^{2^{**}} \leq S_\lambda^{\frac{2^{**}}{2}} \left(\|w_n\|^2 - \lambda \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx\right)^{\frac{2^{**}}{2}} \leq S_\lambda^{\frac{2^{**}}{2}} \|w_n\|^2 < C.
\]  

(3.37)
It then follows from (3.38) and (3.39) that we can take a subsequence of \( \{ w_n \} \) such that
\[
\lim_{n \to \infty} (\| w_n \|^2 - \lambda \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx) = \lim_{n \to \infty} \| w_n \|_{2^{**}}^{2^{**}} = k \geq 0.
\]
(3.38)

Letting \( n \to \infty \) in (3.37), we obtain that \( k \leq S_\lambda \frac{2^{**}}{2} k^{2^{**}} \).

If \( k > 0 \), then
\[
k \geq S_\lambda \frac{2^{**}}{2} = S_\lambda^N.
\]
(3.39)

In accordance with (3.7), (3.10), (3.11), (3.34), (3.35) and \( I(\lambda) = c + o(1) \) as \( n \to \infty \), we have
\[
o(1) + c = I(\lambda(w_n)) = \frac{1}{2} \int_\Omega (|\Delta w_n|^2 - \lambda \frac{|w_n|^2}{|x|^4}) \, dx - \frac{1}{2^{**}} \| u_n \|_{2^{**}}^{2^{**}} - \int_\Omega F(x, u_n) \, dx
\]
\[
\quad = \frac{1}{2} \int_\Omega (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^4}) \, dx - \frac{1}{2^{**}} \| u \|_{2^{**}}^{2^{**}} - \int_\Omega F(x, u) \, dx + \frac{1}{2} \| w_n \|^2 - \frac{1}{2} \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx - \frac{1}{2^{**}} \| w_n \|_{2^{**}}^{2^{**}} + o(1)
\]
\[
= I(\lambda) + \frac{1}{2} \| w_n \|^2 - \frac{1}{2} \lambda \int_\Omega \frac{|w_n|^2}{|x|^4} \, dx - \frac{1}{2^{**}} \| w_n \|_{2^{**}}^{2^{**}} + o(1), \quad n \to \infty,
\]
which then yields that
\[
I(\lambda) = c - \left( \frac{1}{2} - \frac{1}{2^{**}} \right) k \leq c - \frac{2}{N} S_\lambda^N < 0,
\]
which contradicts (3.17). Thus, \( k = 0 \). Noticing that \( 0 < \lambda < \lambda_{4,2} \), we obtain from (3.2) that there exists a \( \mu_4 > 0 \) such that
\[
\mu_4 \| w_n \|^2 \leq \int_\Omega (|\Delta w_n|^2 - \lambda \frac{|w_n|^2}{|x|^4}) \, dx = o(1), \quad n \to \infty,
\]
which implies \( \lim_{n \to \infty} \| w_n \|^2 = l = 0 \), i.e., \( u_n \to u \) in \( H_0^2(\Omega) \) as \( n \to \infty \). The proof is complete. \( \square \)

The following lemma, which shows that the mountain pass level of \( I(\lambda) \) around 0 is strictly less than \( c(S_\lambda) \), is parallel to Lemma 3.5

**Lemma 3.7.** Assume that (f_1) - (f_2) and the condition (iii) of Theorem 2.1 hold. Then there exists a \( \tilde{u}_\lambda^* \in H_0^2(\Omega) \) such that
\[
\sup_{t \geq 0} I(\lambda(t \tilde{u}_\lambda^*)) < c(S_\lambda),
\]
(3.40)
where \( c(S_\lambda) := \frac{2}{N} S_\lambda^N \).
Proof. To estimate the mountain pass level of $I_\lambda$ around 0, we introduce the Talenti functions. For any $\varepsilon > 0$, define

$$\bar{U}_{\varepsilon, \lambda}(x) = \varepsilon^{2 - \frac{4}{N}} \bar{U}_\lambda \left( \frac{x}{\varepsilon} \right), \quad \lambda \in [0, \lambda_{4,2}),$$

where $\bar{U}_\lambda(x) > 0$ is radially symmetric and is a solution to the critical problem

$$\Delta^2 u - \lambda \frac{|u|^2}{|x|^4} = u^{2^{**}-1}, \quad x \in \mathbb{R}^N, \quad N \geq 5.$$

Then, according to Theorem 1.1 of [9], $\bar{U}_{\varepsilon, \lambda}(x)$ is an extremal function of $S_\lambda$ and

$$\int_{\mathbb{R}^N} \left( |\Delta \bar{U}_{\varepsilon, \lambda}(x)|^2 - \lambda \frac{|\bar{U}_{\varepsilon, \lambda}(x)|^2}{|x|^4} \right) dx = \int_{\mathbb{R}^N} |\bar{U}_{\varepsilon, \lambda}(x)|^{2^{**}} dx = S_\lambda^{* *},$$

where $S_\lambda$ is given in Remark 3.2. Set $r = |x|$, then one also has

$$\bar{U}_\lambda(r) = O_1(r^{-\gamma_n}), \quad \bar{U}_\lambda'(r) = O_1(r^{-\gamma_n - 1}), \quad r \to +\infty,$$

$$\bar{U}_\lambda(r) = O_1(r^{-\gamma_n}), \quad r \to 0,$$

where $\gamma_n = (N - 4)(1 - \frac{\alpha}{2})$, $\bar{\gamma}_n = (N - 4)\frac{\alpha}{2}$, and

$$\alpha := \varphi(\lambda) = 1 - \frac{\sqrt{N^2 - 4N + 8 - 4(N - 2)^2 + \lambda}}{N - 4}, \quad \lambda \in (0, \lambda_{4,2}).$$

It is directly verified that $\varphi : [0, \lambda_{4,2}] \to [0, 1]$ is continuous and strictly increasing. Therefore, $\alpha \in (0, 1)$ and $0 < \bar{\gamma}_n < \frac{N - \alpha}{2} < \gamma_n < N - 4$ for $\lambda \in (0, \lambda_{4,2})$.

Let $\tau(x)$ be given in Lemma 3.4 and set $\bar{u}_{\varepsilon, \lambda}(x) = \tau(x)\bar{U}_{\varepsilon, \lambda}(x)$. Then as $\varepsilon \to 0$, we have (see [9, 11])

$$\int_{\Omega} \left( |\Delta \bar{u}_{\varepsilon, \lambda}(x)|^2 - \lambda \frac{|\bar{u}_{\varepsilon, \lambda}(x)|^2}{|x|^4} \right) dx = S_\lambda^{* *},$$

and

$$\|\bar{u}_{\varepsilon, \lambda}\|^{2^{**}} = S_\lambda^{* *} + O(\varepsilon^{2(\gamma_n - \frac{N - \alpha}{2})^+}),$$

$$\|\bar{u}_{\varepsilon, \lambda}\|_\rho^\rho = \begin{cases} O_1(\varepsilon^{\rho(\gamma_n - \frac{N - \alpha}{2})}), & 1 \leq \rho < \frac{N}{\gamma_n}, \\ O_1(\varepsilon^{N - \frac{N - \alpha}{2} - \rho \ln \varepsilon}), & \rho = \frac{N}{\gamma_n}, \\ O_1(\varepsilon^{N - \frac{N - \alpha}{2} - \rho}), & \frac{N}{\gamma_n} < \rho < 2^{**}. \end{cases} \quad (3.41)$$

As was done in the proof of Lemma 3.5, we consider the fibered maps $\psi_u(t)$ given in (3.22) with $u = \bar{u}_{\varepsilon, \lambda}$. Since (3.23) still holds for $\psi_{\bar{u}_{\varepsilon, \lambda}}(t)$, there exists a $\bar{t}_0 > 0$, independent of $\varepsilon$, such that

$$\psi_{\bar{u}_{\varepsilon, \lambda}}(t) < c(S_\lambda), \quad t \in (0, \bar{t}_0). \quad (3.42)$$

Next, set

$$\bar{g}(t) = \frac{1}{2} t^2 \int_{\Omega} \left( |\Delta \bar{u}_{\varepsilon, \lambda}|^2 - \lambda \frac{|\bar{u}_{\varepsilon, \lambda}|^2}{|x|^4} \right) dx - \frac{1}{2^{**}} t^{2^{**}} \|\bar{u}_{\varepsilon, \lambda}\|^{2^{**}}.$$
which is fulfilled once we can prove that

\[ \psi_{\overline{u}_{\epsilon}, \lambda}(t) = \overline{g}(t) - \int_\Omega F(x, t\overline{u}_{\epsilon, \lambda})dx. \]

On recalling (1.4) one sees that there exists a positive constant $C$ such that

\[ \psi_{\overline{u}_{\epsilon}, \lambda}(t) \leq \max_{t \geq t_0} \overline{g}(t) - \int_\Omega F(x, t\overline{u}_{\epsilon, \lambda})dx \]

\[ \leq \max_{t \geq t_0} \overline{g}(t) - C\overline{t}_0^0 \|\overline{u}_{\epsilon, \lambda}\|_\rho^p, \quad t \in [\overline{t}_0, \infty). \]  

(3.43)

Direct computation shows that $\overline{g}$ attains its maximum at

\[ \overline{t}_0^* := \left( \frac{\int_\Omega (|\Delta \overline{u}_{\epsilon, \lambda}|^2 - \lambda |\overline{u}_{\epsilon, \lambda}|^2 dx)}{\|\overline{u}_{\epsilon, \lambda}\|_\rho^p} \right)^{\frac{1}{2}}. \]

(3.44)

Therefore, by (3.43) and (3.44), one sees, for $t \in [\overline{t}_0, \infty)$, that

\[ \psi_{\overline{u}_{\epsilon}, \lambda}(t) \leq \overline{g}(\overline{t}_0^*) - C\overline{t}_0^0 \|\overline{u}_{\epsilon, \lambda}\|_\rho^p \]

\[ = \left( \frac{1}{2} - \frac{1}{2^**} \right) \left( \frac{\int_\Omega (|\Delta \overline{u}_{\epsilon, \lambda}|^2 - \lambda |\overline{u}_{\epsilon, \lambda}|^2 dx)}{\int (|\Delta \overline{u}_{\epsilon, \lambda}|^2 - \lambda |\overline{u}_{\epsilon, \lambda}|^2 dx)} \right)^{\frac{1}{2}} - C\overline{t}_0^0 \|\overline{u}_{\epsilon, \lambda}\|_\rho^p \]

\[ = \frac{2}{N} \left( S_{\lambda}^N + O(\varepsilon^{2(\gamma - \frac{N-4}{2})}) \right) \left( S_{\lambda}^N + O(\varepsilon^{2** \gamma}) \right) - \frac{4N}{N-4} - C\overline{t}_0^0 \|\overline{u}_{\epsilon, \lambda}\|_\rho^p \]

\[ = c(S_{\lambda}) + O(\varepsilon^{2(\gamma - \frac{N-4}{2})}) - C\overline{t}_0^0 \|\overline{u}_{\epsilon, \lambda}\|_\rho^p, \quad \text{as } \varepsilon \to 0, \]

where in the last equality we have used the fact that

\[ (2^{**} \gamma - N) - 2(\gamma - \frac{N-4}{2}) = 4 - 4\alpha \geq 0. \]

To complete the proof of this lemma, it remains to show, for $\varepsilon$ suitably small, that

\[ \psi_{\overline{u}_{\epsilon}, \lambda}(t) < c(S_{\lambda}), \quad t \in [\overline{t}_0, \infty), \]  

(3.45)

which is fulfilled once we can prove that

\[ O(\varepsilon^{2(\gamma - \frac{N-4}{2})}) - C\overline{t}_0^0 \|\overline{u}_{\epsilon, \lambda}\|_\rho^p < 0. \]  

(3.46)

In view of (3.41), if $\rho > \max\{\frac{\sqrt{N}}{\gamma}, \frac{4(N-2-\gamma)}{N-4}\}$, one sees that

\[ \lim_{\varepsilon \to 0} \frac{\varepsilon^{2(\gamma - \frac{N-4}{2})}}{\varepsilon^{N-4}} = \lim_{\varepsilon \to 0} \varepsilon^\frac{2(\rho(N-4)-4(N-2-\gamma))}{2} = 0, \]

which implies (3.46) is valid for suitably small $\varepsilon$. Therefore, for each $\lambda \in (0, \Lambda_{4,2})$, (3.45) is valid for suitably small $\varepsilon$, which, together with (3.42), implies, for $\varepsilon$ suitably small, that

\[ \sup_{t \geq 0} \psi_{\overline{u}_{\epsilon}, \lambda}(t) < c(S_{\lambda}). \]

Fixing such an $\varepsilon > 0$ and letting $\overline{u}_0^* \equiv \overline{u}_{\epsilon, \lambda}$, we obtain (3.40). The proof is complete. \[\square\]
Proof of Theorem 2.1 (iii). With the help of Lemmas 3.6 and 3.7, we can prove Theorem 2.1 (iii) in a way similar to that of Theorem 2.1 (i) and (ii), and we only sketch the outline. First, by recalling (3.31) and (3.32) with $q = 2$ and $s = 4$, we see that $I_{\lambda}$ satisfies the mountain pass geometry around 0 and there exists a sequence $\{u_n\} \subset H^2_0(\Omega)$ such that $I_{\lambda}(u_n) \to \tilde{c}_0$ and $I_{\lambda}'(u_n) \to 0$ in $H^{-2}(\Omega)$ as $n \to \infty$, where

$$
\tilde{c}_0 = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} I_{\lambda}(\tilde{\gamma}(t)) \text{ and } \tilde{\Gamma} = \{\tilde{\gamma} \in C([0,1], H^2_0(\Omega)) : \tilde{\gamma}(0) = 0, \tilde{\gamma}(1) = t\tilde{u}_\lambda^*\},
$$

and $\tilde{u}_\lambda^*$ is given in Lemma 3.7 which satisfies $I_{\lambda}(t\tilde{u}_\lambda^*) < 0$. In view of Lemma 3.7 we have

$$
\tilde{c}_0 \leq \max_{t \in [0,1]} I_{\lambda}(t\tilde{u}_\lambda^*) \leq \sup_{t \geq 0} I_{\lambda}(t\tilde{u}_\lambda^*) < c(S_{\lambda}).
$$

Consequently, by Lemma 3.6 one sees that $I_{\lambda}(u)$ satisfies the $(PS)_{\tilde{c}_0}$ condition. Then there exists a $u$ such that $I_{\lambda}(u) = \tilde{c}_0$ and $I_{\lambda}'(u) = 0$, i.e., $u$ is a nonnegative weak solution to problem (1.1). The proof is complete.

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