Coupled embeddability

Florian Frick$^1$ | Michael Harrison$^2$

$^1$Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA
$^2$Institute for Advanced Study, Princeton, New Jersey, USA

Correspondence
Michael Harrison, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA.
Email: mah5044@gmail.com

Present address
Florian Frick, Institute of Mathematics, Freie Universität Berlin, Arnimallee 2, 14195, Berlin, Germany

Funding information
NSF, Grant/Award Number: DMS 1855591; Institute for Advanced Study; NSF, Grant/Award Number: DMS1926686

Abstract
We introduce the notion of coupled embeddability, defined for maps on products of topological spaces. We use known results for nonsingular biskew and bilinear maps to generate simple examples and nonexamples of coupled embeddings. We study genericity properties for coupled embeddings of smooth manifolds, extend the Whitney embedding theorems to statements about coupled embeddability, and we discuss a Haefliger-type result for coupled embeddings. We relate the notion of coupled embeddability to the $\mathbb{Z}/2$-coindex of embedding spaces, recently introduced and studied by the authors. With a straightforward generalization of these results, we obtain strong obstructions to the existence of coupled embeddings in terms of the combinatorics of triangulations. In particular, we generalize nonembeddability results for certain simplicial complexes to sharp coupled nonembeddability results for certain pairs of simplicial complexes.

MSC (2020)
57N35, 58D10 (primary), 15A63, 58K30 (secondary)

1 | INTRODUCTION

Let $X$ and $Y$ be topological spaces. We introduce and study the following property, defined for continuous functions on the product $X \times Y$.

Definition 1. Let $f : X \times Y \to \mathbb{R}^d$ be a continuous function. We say that $f$ is a coupled embedding if the condition
\[ f(x_1, y_1) + f(x_2, y_2) = f(x_1, y_2) + f(x_2, y_1). \] (1.1)

is not satisfied by any distinct \( x_1, x_2 \in X \) and distinct \( y_1, y_2 \in Y \).

Equation (1.1) is satisfied when the four points \( f(x_1, y_1), f(x_1, y_2), f(x_2, y_2), f(x_2, y_1) \) are the vertices of a (possibly degenerate) parallelogram inscribed in the image of \( f \). That is, if \( f \) is a coupled embedding, a coincidence of points in \( X \) and of points in \( Y \) cannot occur simultaneously; see Section 1.1.

**Example 1.1.** The following maps are coupled embeddings:

(i) the map \( \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x, y) \mapsto xy \),

(ii) the map \( \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (x, y, z, w) \mapsto (xz - yw, xw + yz) \); and

(iii) the restriction of the above map to \( S^1 \times S^1 \to \mathbb{R}^2 \).

To check that these maps are indeed coupled embeddings, we appeal to the following notion.

**Definition 2.** A bilinear map \( B : \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^d \) is called *nonsingular* if it satisfies the condition: \( B(x, y) = 0 \) if and only if \( x = 0 \) or \( y = 0 \).

It is easy to see that a nonsingular bilinear map is a coupled embedding. Indeed, if a nonsingular bilinear map \( B \) satisfies Equation (1.1) with respect to some \( x_1, x_2, y_1, y_2 \), then bilinearity of \( B \) immediately yields \( B(x_1 - x_2, y_1 - y_2) = 0 \), and then nonsingularity implies \( x_1 = x_2 \) or \( y_1 = y_2 \). In this way, nonsingular bilinear maps provide prototypical examples of coupled embeddings, just as nonsingular linear maps are prototypical examples of ordinary embeddings. Nonsingular bilinear maps appeared a century ago in the works of Hurwitz and Radon in their studies of square identities [23, 35], and since then, they have made prominent appearances in topology (see Section 2 for history and discussion). In general, the smallest dimension \( d \) for which there exists a nonsingular bilinear map \( \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^d \) is unknown, and so it should come as no surprise that coupled embeddings are also somewhat elusive.

It is worth pointing out that there is no obvious relationship between the notions of coupled embeddings and ordinary embeddings for maps on \( X \times Y \). For example, the coupled embeddings of Example 1.1 are not embeddings, and moreover, there do not even exist embeddings in the listed dimensions. On the other hand, the ‘identity map’ \( \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n} \) is an embedding but not a coupled embedding. We do not know if there exists an embedding \( X \times Y \to \mathbb{R}^d \) for some dimension \( d \) which does not admit a coupled embedding, but the following observation seems promising:

**Remark 1.** It is possible that \( X \) does not embed in \( \mathbb{R}^m \) and \( Y \) does not embed in \( \mathbb{R}^n \), yet nevertheless there exists an embedding \( X \times Y \to \mathbb{R}^{m+n} \). For example, \( \mathbb{R}P^2 \) does not embed in \( \mathbb{R}^3 \), and \( \mathbb{R}P^3 \) does not embed in \( \mathbb{R}^4 \), yet their product embeds in \( \mathbb{R}^7 \) (see [3, Lemma 2.1]). We will show that there exists a coupled embedding \( \mathbb{R}P^2 \times \mathbb{R}P^3 \to \mathbb{R}^8 \), but we do not know if there exists one into \( \mathbb{R}^7 \).

### 1.1 Motivation

Although it turns out that coupled embeddability is a natural generalization of nonsingularity for bilinear maps, the initial motivation for the definition arose from the following line of inquiry.
Let $X$ and $Y$ be topological spaces such that $X$ does not embed in $\mathbb{R}^m$ and $Y$ does not embed in $\mathbb{R}^n$. Observe that for any continuous map $f : X \times Y \to \mathbb{R}^{m+n}$ and any decomposition $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$,

- for fixed $y \in Y$, the projection of $f(\cdot, y)$ to $\mathbb{R}^m$ fails to be an embedding; and
- for fixed $x \in X$, the projection of $f(x, \cdot)$ to $\mathbb{R}^n$ fails to be an embedding.

When can the nonembeddability of $X$ into $\mathbb{R}^m$ and $Y$ into $\mathbb{R}^n$ be witnessed simultaneously?

A reasonable formalization of this question is the following:

**Definition 3.** Given $X, Y$, and $f$ as above, we say that $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ form an axis-aligned parallelogram if there exists a decomposition $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ (not necessarily orthogonal) such that

- $f(x_1, y_1) = f(x_2, y_1)$ and $f(x_1, y_2) = f(x_2, y_2)$ in the first $m$ coordinates; and
- $f(x_1, y_1) = f(x_1, y_2)$ and $f(x_2, y_1) = f(x_2, y_2)$ in the last $n$ coordinates.

In this case we say that $f$ satisfies the parallelogram condition (see Figure 1).

Observe that if $f$ satisfies the parallelogram condition with respect to $x_1, x_2, y_1$, and $y_2$, then Equation (1.1) is satisfied, and hence $f$ is not a coupled embedding. In this sense the notion of coupled embeddability adequately captures whether the nonembeddabilities of $X$ and $Y$ are simultaneously witnessed.

**Problem 1.2.** Given $f : X \times Y \to \mathbb{R}^{m+n}$, do there exist $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ which form an axis-aligned parallelogram?

**Remark 2.** Problem 1.2 can be posed when the codomain is of general dimension $d$ and where the decomposition is allowed to take the form $\mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^{d-k}$. Even in this general setting, the existence of an axis-aligned parallelogram implies that $f$ is not a coupled embedding. Conversely, if $f$ fails to be a coupled embedding due to points $x_1, x_2, y_1, y_2$, and if additionally the four image points are not collinear, then a decomposition can be chosen such that $\mathbb{R}^k$ contains $f(x_1, y_1) - f(x_1, y_2)$ and $\mathbb{R}^{d-k}$ contains $f(x_1, y_1) - f(x_2, y_1)$, so that $f$ satisfies the parallelogram condition. Thus, the notion of coupled nonembeddability is equivalent to the existence of an axis-aligned parallelogram, modulo certain degenerate parallelograms.

Certain axis-aligned parallelograms deserve special attention.
Problem 1.3. Given \( f : S^m \times S^n \rightarrow \mathbb{R}^{m+n} \), do there exist \( x \in S^m \) and \( y \in S^n \) such that \( \pm x, \pm y \) form an axis-aligned parallelogram?

This special case is natural from the perspective of equivariant topology. Indeed, it follows from the Borsuk–Ulam theorem that for any decomposition \( \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \),

- for fixed \( y \in S^n \), there exists \( x' \in S^m \) such that \( f(x', y) = f(-x', y) \) in the first \( m \) coordinates; and
- for fixed \( x \in S^m \), there exists \( y' \in S^n \) such that \( f(x, y') = f(x, -y') \) in the last \( n \) coordinates.

Thus, the problem above asks to determine whether the upshot of Borsuk–Ulam is simultaneously satisfied among \( \pm x \) and \( \pm y \). This leads to the following definition.

Definition 4. Let \( f : S^m \times S^n \rightarrow \mathbb{R}^d \) be a continuous function. We say that \( f \) is a coupled \( \mathbb{Z}/2 \)-embedding if the condition

\[
f(x, y) + f(-x, -y) = f(x, -y) + f(-x, y)
\]

is not satisfied by any \( (x, y) \in X \times Y \).

It is easy to see that this special case is equivalent to the existence of biskew (that is, \((\mathbb{Z}/2)^2\)-equivariant) maps \( S^m \times S^n \rightarrow \mathbb{R}^d \) which avoid zero. Indeed, if \( f \) is such a map, then it is a coupled \( \mathbb{Z}/2 \)-embedding. Conversely, if \( f \) is a coupled \( \mathbb{Z}/2 \)-embedding, then the induced \((\mathbb{Z}/2)^2\)-equivariant map \((x, y) \mapsto f(x, y) + f(-x, -y) - f(-x, y) - f(x, -y)\) avoids zero.

Given \( m \) and \( n \), the smallest dimension \( d \) for which there exists a biskew map \( S^m \times S^n \rightarrow \mathbb{R}^d \) which avoids zero is unknown. On one hand, examples of such maps arise from restricting nonsingular bilinear maps (see Section 2 for examples of nonsingular bilinear maps), which for example yields \( d \leq m + n + 1 \). On the other hand, a classical result due to Hopf [22] states that if \( m \) and \( n \) do not share a one in any digit of their binary expansions, then every biskew map \( S^m \times S^n \rightarrow \mathbb{R}^{m+n} \) must hit 0. The surprising number-theoretic condition appears when one considers the map \( H^*(\mathbb{R}P^{m+n-1}) \rightarrow H^*(\mathbb{R}P^{m-1}) \otimes H^*(\mathbb{R}P^{n-1}) \) induced by a biskew map which avoids zero, since binomial coefficients appear when expanding the image of the generator.

We will see that similar conditions arise in the general study of coupled embeddings \( X \times Y \rightarrow \mathbb{R}^d \). Combined with the fact that every coupled embedding \( S^m \times S^n \rightarrow \mathbb{R}^d \) is a coupled \( \mathbb{Z}/2 \)-embedding, it seems natural to view the study of coupled embeddability as a generalization of Hopf’s theorem and the study of biskew maps.

Returning to the general condition of coupled embeddability, we now pose the main question addressed in this article:

Problem 1.4. Given \( X \) and \( Y \), what is the minimum dimension \( d(X, Y) \) such that there exists a coupled embedding \( f : X \times Y \rightarrow \mathbb{R}^{d(X, Y)} \)?

This question has been posed and studied for many other nondegeneracy conditions. In addition to the classical studies of immersions and embeddings, there have been recent studies of \( k \)-regular embeddings (see, for example, [7, 8, 12]), totally skew embeddings [4, 5, 13, 37], totally nonparallel immersions [20], and more; yet these are all conditions on a single manifold. To the best of our knowledge there has been no study of differential conditions on product manifolds.
which take into account the product structure. We find Problem 1.4 enticing both due to its simplicity as a condition on product manifolds, and due to the undertones of equivariant topology.

1.2 Statement of results

Here we collect our main results. We find that many standard results for embeddings generalize to the setting of coupled embeddability. In particular, we prove Whitney-type genericity results for the existence of coupled embeddings, we use nonsingular bilinear maps to generate examples of coupled embeddings, and we prove a generalization of Sarkaria’s coloring/embedding theorem, which allows us to compute strong obstructions to coupled embeddability of $X \times Y$ in terms of combinatorics of triangulations of $X$ and $Y$.

**Theorem 1.5.** Let $M$ and $N$ be smooth compact manifolds of dimensions $p$ and $q$. Then

(a) for $d > 2(p + q)$, a generic map $f \in C^\infty(M \times N, \mathbb{R}^d)$ is a coupled embedding; and

(b) there exists a coupled embedding $M \times N \to \mathbb{R}^{2p+2q-2}$.

It is interesting to compare this result to the Whitney theorems, which state that every smooth $p$-dimensional manifold $M$ embeds in $\mathbb{R}^{2p}$, and that if $d > 2p$, a generic map $f \in C^\infty(M, \mathbb{R}^d)$ is an embedding. Thus, it is somewhat surprising to see a difference of three dimensions between the genericity result and the existence result for coupled embeddings.

**Problem 1.6.** Given $p$ and $q$, what is the minimum dimension $d(p, q)$ such that for every $p$-dimensional manifold $M$ and $q$-dimensional manifold $N$, there exists a coupled embedding $M \times N \to \mathbb{R}^{d(p, q)}$?

The analogous question for ordinary embeddings of $p$-dimensional manifolds is open, but it is known that if $p$ is not a power of 2, then every $p$-dimensional manifold embeds in a dimension less than $2p$. Similarly, it follows from the next statement that $d(p, q) < 2p + 2q - 2$ unless one of $p$ or $q$ is a power of 2.

**Proposition 1.7.** If $X$ embeds into $Z$ and $Y$ embeds into $W$, then $d(X, Y) \leq d(Z, W)$. In particular, suppose that $e_X$ (respectively, $e_Y$) is the minimum dimension of embeddability of $X$ (respectively, $Y$). Then $d(X, Y)$ is at most the minimum dimension into which there exists a nonsingular bilinear map from $\mathbb{R}^{e_X} \times \mathbb{R}^{e_Y}$. In particular, $d(X, Y) \leq e_X + e_Y - 1$.

The first statement is obvious, and the second follows from the fact that the nonsingular bilinear maps are coupled embeddings. The final statement follows from the nonsingular bilinear map induced by polynomial multiplication; see Section 2.

Problem 1.6 is more tractable in the category of simplicial complexes. Recall that the minimum dimension into which every $p$-dimensional simplicial complex embeds is $2p + 1$. This is sharp due to the fact that the $(p + 1)$-fold join of three points, $[3]^{(p+1)}$, and the $p$-skeleton of the $(2p + 2)$-simplex, $\Delta_{2p+2}^{(p)}$, do not embed into $\mathbb{R}^{2p}$. We will show that under a certain number-theoretic condition on $p$ and $q$, a similar result holds for coupled embeddings.

Given dimensions $p$ and $q$, we refer to the minimum dimension such that every pair of simplicial complexes coupled embeds as $d_\Delta(p, q)$.
Theorem 1.8. Suppose that the nonnegative integers $p$ and $q$ do not share a one in any digit of their binary expansions. Then $d_{\Delta}(p, q) = 2p + 2q + 1$. In particular,

$$d\left([3]^{(p+1)}, [3]^{(q+1)}\right) = d\left(\Delta_{2p+2}^{(p)}, \Delta_{2q+2}^{(q)}\right) = 2p + 2q + 1.$$ 

Theorem 1.8 does not necessarily hold when the condition on $p$ and $q$ is not satisfied, but some results are still attainable. In Figure 3, located at the end of Section 4, we present the values of $d_{\Delta}(p, q)$ for some small values of $p$ and $q$.

The proof of Theorem 1.8 involves the following combinatorial notion.

Definition 5. Let $\Sigma$ be a simplicial complex on ground set $[n]$. The Kneser graph of $\Sigma$, denoted $KG(\Sigma)$, is the graph whose vertices are the minimal nonfaces of $\Sigma$, that is, the minimal subsets of $[n]$ among those that do not form a face of $\Sigma$, and with edges between vertices corresponding to disjoint faces. The chromatic number of a graph $G$, denoted $\chi(G)$, is the least number of colors needed to color its vertices such that the two endpoints of every edge receive distinct colors (see Figure 2).

Theorem 1.9. For $i = 1, 2$, let $\Sigma_i$ be a simplicial complex on ground set $[n_i]$ and let $c_i = \chi(KG(\Sigma_i))$. Suppose that the nonnegative integers $n_1 - c_1 - 2$ and $n_2 - c_2 - 2$ do not share a one in any digit of their binary expansions. Then there is no coupled embedding $\Sigma_1 \times \Sigma_2 \rightarrow \mathbb{R}^{n_1+n_2-c_1-c_2-4}$.

In fact, under the hypotheses of Theorem 1.9, there is no coupled almost-embedding (see Section 4.2).

Using Proposition 1.7 and Theorem 1.9, we compute the coupled embedding dimension for other specific spaces.

Theorem 1.10. The following statements hold.

(i) $d(\mathbb{R}P^2, S^k) = 4\left\lfloor \frac{k}{4} \right\rfloor + 4$.

(ii) $d(\mathbb{C}P^2, S^{8q}) = 8q + 7$, and $d(\mathbb{C}P^2, S^k) = 8\left\lfloor \frac{k}{8} \right\rfloor$ for $k \neq 8q$.

(iii) $d(\Delta_{2k+2}^{(k)}, S^1) = 2k + 2$.

(iv) $d([3]^{(k+1)}, S^1) = 2k + 2$

(v) $d(\Sigma_{\mathbb{R}P^2}, \Delta_{4q+2}^{(2q)}) = 4q + 4$.

The structure of the paper is as follows. In Section 2 we review definitions for biskew and bilinear maps, which play a crucial role in the study of coupled embeddings. There we prove Proposition 1.7 and we provide examples of nonsingular bilinear maps, which, when coupled with
Proposition 1.7, can be used to generate examples of coupled embeddings. In Section 3 we extend usual transversality results to discuss genericity statements for coupled embeddings and to prove Theorem 1.5. We also discuss a Haefliger-type conjecture for coupled embeddings and outline a proposed argument. In Section 4 we study coupled embeddability of simplicial complexes. We relate the coupled embeddability condition to the coindex of embedding spaces, defined and studied by the authors in [11]; see Section 4.3. In Section 4.4 we prove Theorem 1.9, and then deduce Theorems 1.8 and 1.10. In Section 5 we collect some compelling unanswered questions for coupled embeddings.

2  PRELIMINARIES AND BASIC BOUNDS

In this section we recall known results for bilinear maps $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{d}$ and biskew, that is, $(\mathbb{Z}/2)^2$-equivariant, maps $S^m \times S^n \to \mathbb{R}^d$, where here each generator acts by negation on the codomain. A bilinear map $B$ is called nonsingular if $B(x, y) = 0$ implies $x = 0$ or $y = 0$, and a biskew map is called nonsingular if it avoids 0.

Nonsingular bilinear maps appeared a century ago in the works of Hurwitz and Radon in their studies of square identities [23, 35]. They were studied intensively by Lam (for example, [26–31]) and by Berger and Friedland [6]. A famous result of Adams [1] states that there exists a nonsingular bilinear map $\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ if and only if there exist $q - 1$ linearly independent tangent vector fields on $S^{p-1}$, which occurs if and only if $q \leq \rho(p)$. Here $\rho$ is the Hurwitz–Radon function, defined as follows: decompose $p$ as the product of an odd number and $2^{a+b}$ for $0 \leq b \leq 3$, then $\rho(p) := 2^b + 8a$. More recently, Ovsienko and Tabachnikov [33] showed that these statements are equivalent to the existence of a fibration of $\mathbb{R}^{p+q-1}$ by pairwise skew affine copies of $\mathbb{R}^{q-1}$ (see also [18, 19, 21, 34]). Nonsingular bilinear maps are also related to immersions of projective spaces (see, for example, [24, Section 6]), totally nonparallel immersions [20], and the generalized vector field problem, and they were recently used by the authors to obstruct the existence of $k$-regular embeddings [12].

2.1  Nonsingular bilinear maps and the existence of coupled embeddings

We have seen that a nonsingular bilinear map $B : \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{d}$ is a coupled embedding. Therefore, coupled embeddings of $X \times Y$ can be constructed by independently embedding $X$ and $Y$ into Euclidean spaces and applying a bilinear map (as stated in Proposition 1.7). The following examples of nonsingular bilinear maps can be used in conjunction with Proposition 1.7 to generate strong existence results for coupled embeddings.

Example 2.1. We offer examples of nonsingular bilinear maps.

(i) The bilinear maps $\mathbb{R}^j \times \mathbb{R}^{jn} \to \mathbb{R}^{jn}$, for $j \in \{1, 2, 4, 8\}$, induced by real, complex, quaternionic, and octonionic multiplication are nonsingular.

(ii) More generally, the bilinear maps $\mathbb{R}^{jn} \times \mathbb{R}^{jm} \to \mathbb{R}^{j(n+m-1)}$, for $j \in \{1, 2, 4, 8\}$ induced by real, complex, quaternionic, and octonionic polynomial multiplication are nonsingular.

(iii) There exists a nonsingular bilinear map $\mathbb{R}^d \times \mathbb{R}^{\rho(d)} \to \mathbb{R}^{d}$ by definition of the Hurwitz–Radon function $\rho$. 
(iv) From Lam [26], there exist nonsingular bilinear maps in the following dimensions:

- $\mathbb{R}^{16} \times \mathbb{R}^{16} \to \mathbb{R}^{23}$
- $\mathbb{R}^{13} \times \mathbb{R}^{13} \to \mathbb{R}^{19}$
- $\mathbb{R}^{11} \times \mathbb{R}^{11} \to \mathbb{R}^{17}$
- $\mathbb{R}^{10} \times \mathbb{R}^{10} \to \mathbb{R}^{22}$
- $\mathbb{R}^{10} \times \mathbb{R}^{15} \to \mathbb{R}^{21}$
- $\mathbb{R}^{10} \times \mathbb{R}^{14} \to \mathbb{R}^{20}$
- $\mathbb{R}^{9} \times \mathbb{R}^{16} \to \mathbb{R}^{16}$
- $\mathbb{R}^{12} \times \mathbb{R}^{12} \to \mathbb{R}^{17}$

(v) From Adem [2], there exists a nonsingular bilinear map $\mathbb{R}^{12} \times \mathbb{R}^{15} \to \mathbb{R}^{21}$.

(vi) From recent work of Domínguez and Lam [10], there exist nonsingular bilinear maps $\mathbb{R}^{h+16} \times \mathbb{R}^{k+16} \to \mathbb{R}^{d+32}$, where $(h, k, d)$ is any of the first eight triples listed in item (iv). See [10] for more examples and discussion.

The final statement of Proposition 1.7 follows from item (ii). To briefly explain this map when $j = 1$, regard $\mathbb{R}^n$ and $\mathbb{R}^m$ as the coefficients of degree $(n - 1)$ and degree $(m - 1)$ polynomials. Then multiplication of polynomials is a bilinear map into $\mathbb{R}^{n+m-1}$ with no zero divisors.

We list several explicit applications to the study of coupled embeddability. There exists a coupled embedding $\mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}^4$ by embedding each $\mathbb{R}P^2$ into $\mathbb{R}^4$ and applying quaternionic multiplication. There exists a coupled embedding $\mathbb{R}P^8 \times S^8 \to \mathbb{R}^{16}$, using item (iii) with $d = 16$ (since $\rho(16) = 9$). There exists a coupled embedding $\mathbb{R}P^{16} \times \mathbb{R}P^{16} \to \mathbb{R}^{55}$ by combining item (vi) with the first nonsingular bilinear map in item (iv). More constructions of this form can be found in the proofs of Theorems 1.8 and 1.10.

2.2 Nonsingular biskew maps and the nonexistence of coupled embeddings

We have seen that the nonexistence of nonsingular biskew maps can be used to obstruct the existence of coupled embeddings $S^m \times S^n \to \mathbb{R}^d$. To generate obstructions to coupled embeddings $X \times Y \to \mathbb{R}^d$, we require some definitions and terminology.

Let $F_2(X)$ represent the (ordered) configuration space consisting of ordered pairs of unequal points of $X$, $F_2(X) = X \times X - \Delta X = \{(x_1, x_2) \in X \times X \mid x_1 \neq x_2\}$. The space $F_2(X)$ has a free $\mathbb{Z}/2$ action $(x_1, x_2) \mapsto (x_2, x_1)$.

**Definition 6.** Let $Z$ be a space with a free $\mathbb{Z}/2$-action. The $\mathbb{Z}/2$-coindex of $Z$, denoted by $\text{coind}(Z)$, is the dimension of the largest dimensional sphere which maps $\mathbb{Z}/2$-equivariantly into $Z$.

In this language, the Borsuk–Ulam theorem is (the nontrivial part of) the statement $\text{coind}(S^k) = k$.

We are now able to state an obstruction to the existence of coupled embeddings.

**Proposition 2.2.** If there exists a coupled embedding $X \times Y \to \mathbb{R}^d$, then there exists a nonsingular biskew map $S^{\text{coind}(F_2(X))} \times S^{\text{coind}(F_2(Y))} \to \mathbb{R}^d$.

**Proof.** Given any map $f : X \times Y \to \mathbb{R}^d$, define the $(\mathbb{Z}/2)^2$-equivariant map

$$
\Phi_f : F_2(X) \times F_2(Y) \to \mathbb{R}^d : (x_1, x_2, y_1, y_2) \mapsto f(x_1, y_1) + f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1),
$$

and observe that $f$ is a coupled embedding if and only if $\Phi_f$ avoids zero. Then the nonsingular biskew map is defined by precomposing $\Phi_f$ with the $(\mathbb{Z}/2)^2$-equivariant map induced by the $\mathbb{Z}/2$-equivariant embeddings $S^{\text{coind}(F_2(X))} \to F_2(X)$ and $S^{\text{coind}(F_2(Y))} \to F_2(Y)$. □
Corollary 2.3. Let $X$ and $Y$ be topological manifolds. If there exists a coupled embedding $X \times Y \to \mathbb{R}^d$, then there exists a nonsingular biskew map $S^{\dim(X)-1} \times S^{\dim(Y)-1} \to \mathbb{R}^d$.

Proof. Given a topological manifold $Z$, there is an embedding $\varphi : S^{\dim(Z)-1} \to Z$, hence a $\mathbb{Z}/2$-equivariant map $S^{\dim(Z)-1} \to F_2(Z)$ sending $z \mapsto (\varphi(z), \varphi(-z))$. Hence $\text{coind}(F_2(Z)) \geq \dim(Z) - 1$. \hfill $\Box$

Remark 3. The map $f$ is a coupled embedding if and only if the map

$$X \times F_2(Y) \to \mathbb{R}^d : (x, y_1, y_2) \mapsto f(x, y_1) - f(x, y_2)$$

is an embedding of $X$ for every fixed distinct $y_1, y_2$; equivalently, if and only if the map

$$F_2(X) \times Y \to \mathbb{R}^d : (x_1, x_2, y) \mapsto f(x_1, y) - f(x_2, y)$$

is an embedding of $Y$ for every fixed distinct $x_1, x_2$. This will be relevant in Section 4.3.

Although Proposition 2.2 could potentially lead to very strong obstructions to coupled embeddability, the coindices of configuration spaces are nontrivial to compute. The combinatorial methods developed in Section 4 provide a much simpler method of obtaining strong nonexistence results.

3 | COUPLED EMBEDDINGS: EXISTENCE

In this section we address the existence question for coupled embeddings of smooth manifolds. Let $M$ be a smooth manifold of dimension $p$. Recall the weak Whitney theorems, which state that a generic map $M \to \mathbb{R}^{2p+1}$ is an embedding, and that a generic map $M \to \mathbb{R}^{2p}$ is an immersion; and the strong Whitney theorems, which state that every smooth $p$-manifold $M$ embeds in $\mathbb{R}^{2p}$ and immerses into $\mathbb{R}^{2p-1}$. We can make similar statements for coupled embeddings; however, the application is somewhat tricky because the coupled embedding condition is neither a condition on jet spaces nor on multijet spaces, where most functional conditions live.

For a reference on jet bundles, we recommend the book by Golubitsky and Guillemin [15]. We offer only some brief intuition. Jet bundles are defined to mimic Taylor series on manifolds in a coordinate-free way. The $k$-jet bundle $J^k(M, \mathbb{R}^d) \to M$ contains information about possible derivatives, up to order $k$, of functions $f \in C^\infty(M, \mathbb{R}^d)$. The $k$-jet extension of $f$ is the section $j^k f : M \to J^k(M, \mathbb{R}^d)$, such that $j^k f(x)$ describes the Taylor expansion of $f$ at $x$, up to order $k$, in an invariant way. Jet bundles allow for easy formalization of genericity statements such as the weak Whitney theorem on immersions.

Now consider the projection $\pi : J^k(M, \mathbb{R}^d) \times \cdots \times J^k(M, \mathbb{R}^d) \to M \times \cdots \times M$ (s factors each), and let $J^k_s(M, \mathbb{R}^d) = \pi^{-1}(F_s(M))$. The bundle $\pi : J^k_s(M, \mathbb{R}^d) \to F_s(M)$ is the $s$-fold $k$-multijet bundle and allows for the study of differential conditions at $s$-tuples of points. The $s$-fold $k$-multijet extension of $f$ is the section $j^k_s f(x_1, \ldots, x_s) = (j^k f(x_1), \ldots, j^k f(x_s))$, which simultaneously contains the information of derivatives up to order $k$ at $s$ distinct points of $M$. Multijet bundles allow for easy formalization of genericity statements for conditions at pairs or $k$-tuples of points, such as the weak Whitney theorem on embeddings.
3.1 | Genericity statements for coupled embeddings

Let \( M \) and \( N \) be smooth manifolds and let \( Z = M \times N \). The twofold 0-multijet space \( J^0_2(Z, \mathbb{R}^d) \) is just \( F_2(Z) \times \mathbb{R}^d \times \mathbb{R}^d \), and for \( f \in C^\infty(Z, \mathbb{R}^d) \), and the twofold 0-multijet extension of \( f \) is \( J^0_2 f(x_1, y_1, x_2, y_2) = (x_1, y_1, x_2, y_2, f(x_1, y_1), f(x_2, y_2)) \). This space is useful for studying embeddings \( Z \rightarrow \mathbb{R}^d \).

We define a similar space which is useful for studying the coupled embedding condition. Let \( \Delta M \) be the diagonal in \( M \times M \). Let \( \Delta = (\Delta M \times N^2) \cup (M^2 \times \Delta N) \subset Z \times Z \), and let \( C \) be its complement. Consider the projection \( \alpha : Z \times Z \times \mathbb{R}^{4d} \rightarrow Z \times Z \), and let \( J = \alpha^{-1}(C) = C \times \mathbb{R}^{4d} \). Given \( f \in C^\infty(Z, \mathbb{R}^d) \), define the section

\[
\psi_f : C \rightarrow J : (x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, y_2, f(x_1, y_1), f(x_2, y_2), f(x_1, y_2), f(x_2, y_1)).
\]

Recall that a subset \( X \subset Y \) is residual if it is a countable intersection of open dense subsets of \( Y \).

**Theorem 3.1.** Let \( W \) be a submanifold of \( J \). Let

\[
T_W = \{ f \in C^\infty(Z, \mathbb{R}^d) \mid \psi_f \text{ is transverse to } W \}.
\]

Then \( T_W \) is a residual subset of \( C^\infty(Z, \mathbb{R}^d) \) in the Whitney \( C^\infty \) topology. Moreover, if \( W \) is compact, then \( T_W \) is open.

The statement is very similar to that of the Multijet Transversality Theorem, and a proof can be obtained by making appropriate modifications to the proof in [15, Theorem II.4.13]. We give the essential ideas of this argument.

**Proof.** First assume that \( W \) is compact. Cover the compact set \( \alpha(W) \cap C \) by finitely many rectangles of the form \( \{C_\beta\} \), where \( C_\beta = U_1^\beta \times V_1^\beta \times U_2^\beta \times V_2^\beta \) for open \( U_i^\beta \subset M \) and open \( V_i^\beta \subset N \) satisfying the property that for all \( \beta \), \( U_1^\beta \cap U_2^\beta = \emptyset \) and \( V_1^\beta \cap V_2^\beta = \emptyset \). Define

\[
T_\beta = \{ f \in C^\infty(Z, \mathbb{R}^d) \mid \psi_f \text{ is transverse to } W \text{ on } \alpha^{-1}(C_\beta) \}.
\]

It is enough to show that each \( T_\beta \) is open. By usual transversality results the set

\[
\{ g \in C^\infty(C, J) \mid g \text{ is transverse to } W \text{ on } \alpha^{-1}(C_\beta) \}
\]

is open, and taking the preimage under the continuous map \( \psi : C^\infty(Z, \mathbb{R}^d) \rightarrow C^\infty(C, J) \) gives openness of \( T_\beta \).

For general \( W \), we may choose an open cover \( \{W_i\} \) such that \( \overline{W_i} \subset W \) and that \( \alpha(\overline{W_i}) \) may be covered by a rectangle \( U_1^i \times V_1^i \times U_2^i \times V_2^i \) as above. It is enough to show that

\[
T_W = \{ f \in C^\infty(Z, \mathbb{R}^d) \mid \psi_f \text{ is transverse to } W \text{ on } \overline{W_i} \}
\]

is open and dense, since then \( T_W \) is a countable intersection of open dense sets, as desired.
By the above result, each \( T_{W_i} \) is open. To show density, we observe that we may perturb a function \( f : Z \to \mathbb{R}^d \), using bump functions on the four disjoint sets \( U_i^1 \times V_1^i, U_i^1 \times V_2^i, U_i^2 \times V_1^i \), and \( U_i^2 \times V_2^i \), to achieve the desired transversality on \( W_i \).

**Proof of Theorem 1.5(a).** Define

\[
\Sigma = \{(x_1, y_1, x_2, y_2, z_1, z_2, z_3, z_4) \in C \times \mathbb{R}^{4d} \mid z_1 + z_2 - z_3 - z_4 = 0\}.
\]

Observe that \( \Sigma \) has codimension \( d \) and that \( f \) is a coupled embedding if and only if \( \psi_f \) is disjoint from \( \Sigma \). By Theorem 3.1, \( \psi_f \) is transverse to \( \Sigma \) for the members \( f \) of some residual set, which is dense by Baire category theorem. For such \( f \), \( (\psi_f)^{-1}(\Sigma) \) is a submanifold of \( C \) of codimension \( d \).

If \( d > \text{dim}(C) = 2(p + q) \) this set is empty, hence \( \psi_f \) is disjoint from \( \Sigma \).

**Remark 4.** Observe that for \( d \leq 2(p + q) \), the same argument can be used to compute the expected dimension of the set on which coupled embeddability fails.

We compare this to a result we could have obtained more easily from Whitney’s theorem. A generic map \( M \to \mathbb{R}^{2p+1} \) is an embedding, and a generic map \( N \to \mathbb{R}^{2q+1} \) is an embedding. There exists a nonsingular bilinear map \( B : \mathbb{R}^{2p+1} \times \mathbb{R}^{2q+1} \to \mathbb{R}^{2p+2q+1} \). Therefore, for generic maps \( f \in C^\infty(M, \mathbb{R}^{2p+1}) \), \( g \in C^\infty(N, \mathbb{R}^{2q+1}) \), the map \( B(f \times g) : M \times N \to \mathbb{R}^{2p+1} \times \mathbb{R}^{2q+1} \to \mathbb{R}^{2(p+q)+1} \) is a coupled embedding. Of course, maps of the form \( f \times g \) are not generic in \( C^\infty(M \times N, \mathbb{R}^{2(p+q)+1}) \), so this line of thought could not have given Theorem 1.5(a).

**Proof of Theorem 1.5(b).** There exists a nonsingular bilinear map \( B : \mathbb{R}^{2p} \times \mathbb{R}^{2q} \to \mathbb{R}^{2p+2q-2} \) from complex polynomial multiplication. Apply \( B \) to embeddings \( f : M \to \mathbb{R}^{2p} \), \( g : N \to \mathbb{R}^{2q} \), which exist due to the strong Whitney embedding theorem.

Surprisingly, this existence result guarantees a coupled embedding in a space of dimension three less than that in which coupled embeddings are generic. Intuitively, one might expect the transition from generic to nongeneric to yield only one less dimension, as is the case for the Whitney embedding theorems. Here, the stronger result can be attributed both to the ‘coupled’ nature of the coupled embedding condition, and also to the better existence result for nonsingular bilinear maps when both dimensions are even.

When \( n \) is a power of 2, the strong Whitney embedding theorem is sharp, in the sense that \( \mathbb{R}P^n \) does not embed in \( \mathbb{R}^{2n-1} \). We ask:

**Problem 3.2.** Do there exist \( p > 1, q > 1 \), and smooth manifolds \( M \) and \( N \) of dimensions \( p \) and \( q \), such that \( d(M, N) = 2p + 2q - 2 \)?

If \( M \) embeds in \( \mathbb{R}^{2p-1} \) and \( N \) embeds in \( \mathbb{R}^{2q-1} \), which in particular must occur when neither \( p \) and \( q \) are powers of 2, then we apply a nonsingular bilinear map to achieve coupled embeddability in \( \mathbb{R}^{2p+2q-3} \). If \( p > 1 \) and \( q > 1 \) are both powers of 2, then \( M \) embeds in \( \mathbb{R}^{2p} \), \( N \) embeds in \( \mathbb{R}^{2q} \), and we apply a nonsingular bilinear map \( \mathbb{R}^{2p} \times \mathbb{R}^{2q} \to \mathbb{R}^{2p+2q-4} \), which exists due to quaternionic polynomial multiplication. Thus, the remaining possibility is that (without loss of generality) \( M \) embeds in dimension \( 2p \) and not less (so \( p \) must be a power of 2) and \( N \) embeds in dimension \( 2q - 1 \) and not less. For example, this occurs with \( M = \mathbb{R}P^2 \) and \( N = \mathbb{R}P^3 \).
3.2 | A Haefliger-type conjecture for coupled embeddings

The following result is due to Haefliger.

**Theorem 3.3** (Haefliger [17]). Let $M$ be a smooth closed manifold of dimension $n$. If $d > \frac{3}{2}n + \frac{3}{2}$, then the existence of a differentiable embedding $M \rightarrow \mathbb{R}^d$ is equivalent to the existence of a $\mathbb{Z}/2\mathbb{Z}$-equivariant map $g : M \times M \rightarrow \mathbb{R}^d$ such that $g^{-1}(0) = \Delta M$.

We expect the following to hold for coupled embeddings.

**Conjecture 3.4.** Let $M$ and $N$ be smooth closed manifolds of dimensions $p$ and $q$, respectively. If $d > \frac{3}{2}(p + q) + \frac{3}{2}$, then the existence of a differentiable coupled embedding $M \times N \rightarrow \mathbb{R}^d$ is equivalent to the existence of a $(\mathbb{Z}/2\mathbb{Z})^2$-equivariant map $g : M \times N \times M \times N \rightarrow \mathbb{R}^d$ such that $g^{-1}(0) = \Delta$, where $\Delta = (\Delta M \times N^2) \cup (M^2 \times \Delta N)$.

Of course, the forward direction comes from the map $\Phi_f$ for a coupled embedding $f$ (see the proof of Proposition 2.2).

We outline a possible method of proof, based on the ‘removal of singularities’ $h$-principle technique due to Gromov and Eliashberg [16]. See [20] for an exposition of this technique, and to see its usefulness in proving a Whitney-type theorem for totally nonparallel immersions.

Suppose that there exists a $(\mathbb{Z}/2\mathbb{Z})^2$-equivariant map $g : M \times N \times M \times N \rightarrow \mathbb{R}^d$ such that $g^{-1}(0) = \Delta$. Observe that if $g = \Phi_f$ for some differentiable map $f : M \times N \rightarrow \mathbb{R}^d$, then $f$ is a coupled embedding, and the proof is complete. The strategy is to replace the component functions of $g = (g_1, \ldots, g_d)$, one by one, by functions of the form $\Phi_{f_1}$, for $f_1 : M \times N \rightarrow \mathbb{R}$, while preserving, at each stage, the condition that the preimage of 0 is precisely $\Delta$.

Suppose that naively we replace $g_1$ by $\Phi_{f_1}$ for some arbitrary function $f_1 : M \times N \rightarrow \mathbb{R}$. This causes no issues on the complement of the set $\Sigma = (g_2, \ldots, g_d)^{-1}(0)$, since regardless of the value of $\Phi_{f_1}$, the function $(\Phi_{f_1}, g_2, \ldots, g_d)$ cannot map points of $\Sigma^c$ to 0. Therefore, we only must be careful about the replacement at points of $\Sigma$. For this, we recall that the function $g_1$ is safe on $\Sigma$, and we use this fact during the replacement.

We first focus on making the replacement on the complement $C$ of some open $(\mathbb{Z}/2\mathbb{Z})^2$-equivariant neighborhood of $\Delta$. Let $\pi : M \times N \times M \times N \rightarrow M \times N$ be the projection to the first factor. We make the following assumption:

**Assumption 1.** The restriction of $\pi$ to $\Sigma \cap C$ is injective.

We claim that with this assumption, we can make the desired replacement of $g_1$ by $\Phi_{f_1}$. First, observe that by injectivity, for each $(x_1, y_1) \in \pi(\Sigma \cap C)$, there exists a unique $(x_2, y_2)$ such that $(x_1, y_1, x_2, y_2) \in \Sigma \cap C$. Moreover, by $(\mathbb{Z}/2\mathbb{Z})^2$-invariance of $\Sigma \cap C$, the points $(x_2, y_2, x_1, y_1)$, $(x_1, y_2, x_2, y_1)$, and $(x_2, y_1, x_1, y_2)$ also lie in $\Sigma \cap C$. Thus for $(x_1, y_1) \in \pi(\Sigma \cap C)$ we define $f_1(x_1, y_1) = g_1(x_1, y_1, x_2, y_2)$. Since $\Sigma \cap C$ is closed, we then extend $f_1$ arbitrarily to $M \times N$.

Now let us check that $\Phi_{f_1}$ satisfies the desired properties. The value of $\Phi_{f_1}$ is irrelevant on points of $\Sigma^c \cap C$. For $(x_1, y_1, x_2, y_2) \in \Sigma \cap C$, we compute
\[ \Phi_f(x_1, y_1, x_2, y_2) = f_1(x_1, y_1) + f_1(x_2, y_2) - f_1(x_1, y_2) - f_1(x_2, y_1) \]
\[ = g_1(x_1, y_1, x_2, y_2) + g_1(x_2, y_2, x_1, y_1) - g_1(x_1, y_2, x_2, y_1) - g_1(x_2, y_1, x_1, y_2) \]
\[ = 4g_1(x_1, y_1, x_2, y_2). \]

Therefore \((\Phi_f, g_2, \ldots, g_d)\) is nonzero on \(\Sigma \cap C\), since the original function \(g\) was nonzero on \(\Sigma \cap C\).

Now we address the assumption above. For a generic function \(g\), the set \(\Sigma = (g_2, \ldots, g_d)^{-1}(0)\) has codimension \(d - 1\), and hence dimension \(2(p + q) - d + 1\), in \(M \times N \times M \times N\). Thus a generic function \(\Sigma \rightarrow M \times N\) is an embedding when \(2 \dim(\Sigma) + 1 \leq \dim(M \times N)\), that is, when \(\frac{3}{2}(p + q) + \frac{3}{2} \leq d\). Thus in the dimension range of the conjecture, the replacement of \(g_1\) with a perturbation of \(g\) on \(C\), to ensure that the genericity assumption is satisfied. Since \(C\) is a closed set disjoint from \(\Delta\), this perturbation of \(g\) does not affect the condition \(g^{-1}(0) = \Delta\). Thus in the stated range of dimensions, the replacement of \(g\) on \(C\) proceeds inductively; at each stage \(i\) we perturb the functions \(f_1, \ldots, f_{i-1}\) and \(g_{i+1}, \ldots, g_d\), if necessary, so that the genericity assumption is satisfied, and then we replace \(g_i\) with \(\Phi_{f_i}\) as outlined above.

The issue with the argument above is that we have only replaced \(g\) on \(C\). To replace \(g\) on a neighborhood of \(\Delta\), it is necessary to carefully study how the coupled embeddability condition manifests locally. In the proof of Haefliger’s embedding theorem, this replacement is made as follows. An immersion is a local embedding, and \(g\) is replaced near the diagonal by first using a result of Haefliger and Hirsch to homotope the \(\mathbb{Z}/2\) equivariant map to an appropriate bundle monomorphism (possible in the stated dimension range), and then using the Smale–Hirsch theorem to homotope the monomorphism to an immersion. After the replacement is made near the diagonal, the above replacement is made as outlined above.

Thus to complete the proof of this conjecture, it is necessary not only to understand how the coupled embedding condition manifests locally but also to generalize both the Haefliger–Hirsch theorem and the Smale–Hirsch theorem. We initiate this process with a brief study of the local condition.

### 3.3 Local coupled embeddability

Recall that a smooth map between smooth manifolds is called an immersion if the differential is a bundle monomorphism. The immersion condition is first-order, and every immersion is a local embedding. Intuitively, when considering the embedding condition on pairs of points \(x_1, x_2 \in M\) which grow closer, the zero-order, two-point local embedding condition manifests as the first-order, single-point immersion condition. The parallelogram condition is a condition at quadruples of points. The goal of this section is to observe how locally this spawns a second-order, single-point differential condition.

**Definition 7.** Given \((x, y) \in M \times N\), we say that \(f : M \times N \rightarrow \mathbb{R}^d\) is a local coupled embedding at \((x, y)\) if there exists a neighborhood \(U\) of \((x, y) \in M \times N\) such that the restriction \(f|_U\) is a coupled embedding.

**Definition 8.** Let \(f : M \times N \rightarrow \mathbb{R}^d\) be a smooth map. We say that \(f\) is coupled nonsingular if, at each \((x, y) \in M \times N\), in some local coordinates \((x_1, \ldots, x_m)\) on \(M\) and \((y_1, \ldots, y_n)\) on \(N\), \(\frac{\partial^2 f}{\partial x_i \partial y_j} \neq 0\).
for all \( i, j \). That is, the second derivative is nonzero for all pairs of the form \( (u, 0), (0, v) \), when considered as a symmetric bilinear map on \( T_{(x,y)}(M \times N) \times T_{(x,y)}(M \times N) \).

**Proposition 3.5.** A coupled nonsingular map \( f : M \times N \to \mathbb{R}^d \) is a local coupled embedding.

**Proof.** Let \((x_1, y_1), (x_1, y_2), (x_2, y_1), \) and \((x_2, y_2)\) be points in a neighborhood \( U \) of \((x, y)\). We write the following:

\[
\begin{align*}
&f(x_1, y_1) - f(x, y) = df_{(x,y)}(u_1, v_1) + \frac{1}{2} d^2 f_{(x,y)}((u_1, v_1), (u_1, v_1)) + \ldots \\
&f(x_2, y_2) - f(x, y) = df_{(x,y)}(u_2, v_2) + \frac{1}{2} d^2 f_{(x,y)}((u_2, v_2), (u_2, v_2)) + \ldots \\
&f(x_1, y_2) - f(x, y) = df_{(x,y)}(u_1, v_2) + \frac{1}{2} d^2 f_{(x,y)}((u_1, v_2), (u_1, v_2)) + \ldots \\
&f(x_2, y_1) - f(x, y) = df_{(x,y)}(u_2, v_1) + \frac{1}{2} d^2 f_{(x,y)}((u_2, v_1), (u_2, v_1)) + \ldots 
\end{align*}
\]

Adding the first two equations and subtracting the latter two yields

\[
\begin{align*}
&f(x_1, y_1) + f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1) \\
&= d^2 f_{(x,y)}((u_1 - u_2, 0), (0, v_1 - v_2)) + \text{higher order terms.}
\end{align*}
\]

By assumption, the right side is nonzero for sufficiently small \( U \), and hence so is the left; therefore \( f \) is a local coupled embedding. \(\square\)

The fact that this condition is not coordinate-free lends some difficulty to a systematic study of this condition. We expect that it would be difficult to develop an h-principle statement for the coupled nonsingularity condition, and hence to finish the outlined proof of Conjecture 3.4.

## 4 | COUPLED EMBEDDINGS: NONEXISTENCE

In this section, we show how exploiting the combinatorics of a triangulation of a space can yield insights into coupled embeddability, with the goal of proving Theorem 1.9. To begin we review some definitions.

### 4.1 | Joins and deleted joins

Recall that for topological spaces \( X \) and \( Y \) the join is the space \( X \ast Y \) obtained from \( X \times Y \times [0, 1] \) by taking the quotient with respect to the equivalence relation generated by \((x, y, 0) \sim (x', y, 0)\) for \( x, x' \in X \) and \((x, y, 1) \sim (x, y', 1)\) for \( y, y' \in Y \). If the simplicial complexes \( \Sigma_X \) and \( \Sigma_Y \) triangulate \( X \) and \( Y \), respectively, then the join \( X \ast Y \) is naturally triangulated by the join of simplicial complexes \( \Sigma_X \ast \Sigma_Y \). As abstract simplicial complexes

\[
\Sigma_X \ast \Sigma_Y = \{ \sigma \times \{1\} \cup \tau \times \{2\} : \sigma \in \Sigma_X, \tau \in \Sigma_Y \},
\]

...
that is, the join is defined by the rule that the vertices of a face in \( \Sigma_X \) and the vertices of a face in \( \Sigma_Y \) together span a face in \( \Sigma_X \ast \Sigma_Y \). Here we assume that \( \Sigma_X \) and \( \Sigma_Y \) have disjoint vertex sets.

One may think of the join \( X \ast Y \) as abstract convex combinations of points in \( X \) and in \( Y \). We will write \( \lambda_1 x + \lambda_2 y \), with \( \lambda_1, \lambda_2 \geq 0 \) and \( \lambda_1 + \lambda_2 = 1 \), for the point \( (x, y, \lambda_1) \) in \( \Sigma_X \ast \Sigma_Y \). Herewe assumethat \( \Sigma_X \) and \( \Sigma_Y \) havedisjointvertexsets.

Onemay think ofthejoin \( X \ast Y \) asabstract convex combinations of points in \( X \) and in \( Y \). We will write \( \lambda_1 x + \lambda_2 y \), with \( \lambda_1, \lambda_2 \geq 0 \) and \( \lambda_1 + \lambda_2 = 1 \), for the point \( (x, y, \lambda_1) \) in \( \Sigma_X \ast \Sigma_Y \). Thusfor \( \lambda_1 = 0 \) the point \( x \) doesnot influence thepoint in \( X \ast Y \), whereas for \( \lambda_1 = 1 \) thechoice of \( y \) doesnot matter. Notethatin this notation \( \lambda_1 x_1 + \lambda_2 x_2 \) and \( \lambda_2 x_2 + \lambda_1 x_1 \) determine different points in \( X \ast X \) if \( x_1 \neq x_2 \).

The deleted join \( X^{\ast 2} \) of a space \( X \) is obtained from \( X \ast X \) by deleting all points of the form \( (x, x, t) \) with \( x \in X \) and \( t \in (0, 1) \). The deleted join \( \Sigma^{\ast 2} \) of a simplicial complex \( \Sigma \) is obtained from the join \( \Sigma \ast \Sigma \) by deleting all those faces that have a vertex in \( \Sigma \) in common, that is, \( \Sigma^{\ast 2} = \{ \sigma \times \{1\} \cup \tau \times \{2\} : \sigma, \tau \in \Sigma, \sigma \cap \tau = \emptyset \} \).

In particular the deleted join \( (\Delta_n)^{\ast 2} \) of the \( n \)-simplex \( \Delta_n \) is the boundary of an \( (n + 1) \)-dimensional cross-polytope, and thus \( (\Delta_n)^{\ast 2} \) is homeomorphic to \( S^n \). Interchanging the join factors \( \lambda_1 x_1 + \lambda_2 x_2 \mapsto \lambda_2 x_2 + \lambda_1 x_1 \) is the antipodal action on the sphere \( (\Delta_n)^{\ast 2} \).

### 4.2 Discrete coupled embeddability

Let \( \Sigma \) be a simplicial complex. A map \( f : \Sigma \to \mathbb{R}^d \) is called an almost-embedding if for every pair of disjoint faces \( \sigma_1, \sigma_2 \) and every \( x_1 \in \sigma_1, x_2 \in \sigma_2, f(x_1) \neq f(x_2) \). We may discretize coupled embeddings similarly.

**Definition 9.** Let \( \Sigma \) and \( \Omega \) be simplicial complexes. We say that \( f : \Sigma \times \Omega \to \mathbb{R}^d \) is a coupled almost-embedding if for every pair of disjoint faces \( \sigma_1, \sigma_2 \) of \( \Sigma \) and disjoint faces \( \omega_1, \omega_2 \) of \( \Omega \), and \( x_1 \in \sigma_1, x_2 \in \sigma_2, y_1 \in \omega_1, y_2 \in \omega_2 \), we have

\[
f(x_1, y_1) + f(x_2, y_2) \neq f(x_1, y_2) + f(x_2, y_1).
\]

In the case of simplicial complexes, the notation \( d(\Sigma, \Omega) \) refers to the smallest dimension admitting a coupled almost-embedding from \( \Sigma \times \Omega \).

To generate obstructions to coupled almost-embeddings, we require a statement on \( (\mathbb{Z}/2)^2 \)-equivariant maps from \( S^m \times S^n \) which takes into account different possible actions on the codomain. Let \( V_{--}, V_{+-}, \) or \( V_{-+} \) denote \( \mathbb{R} \) as a \( (\mathbb{Z}/2)^2 \)-module, where the subscript indicates the action of the two standard generators, that is, \( V_{+-} \) indicates that the first generator acts trivially, while the second generator acts by negation. The following lemma can be deduced easily from the work of Ramos [36]. A short proof using mapping degrees can be found in [9].

**Lemma 4.1.** Let \( i, j, k, m, \) and \( n \) be nonnegative integers with \( m + n = i + j + k, m \geq i, \) and \( n \geq j \). Suppose that \( m - i \) and \( n - j \) do not share a one in any digit of their binary expansions. Then any \( (\mathbb{Z}/2)^2 \)-equivariant map \( S^m \times S^n \to V_{-+} \times V_{+-} \times V_{--} \) has a zero.

**Lemma 4.2.** If there exists a coupled almost-embedding \( f : \Delta_m \times \Delta_n \to \mathbb{R}^d \), then there is a \( (\mathbb{Z}/2)^2 \)-equivariant map \( S^m \times S^n \to V_{-+} \times V_{+-} \times V_{--} \) which avoids zero.
Proof. Given \( f : \Delta_m \times \Delta_n \to \mathbb{R}^d \), define
\[
\Phi_f : (\Delta_m)^{\otimes 2} \times (\Delta_n)^{\otimes 2} \to V_{--} \times V_{++} \times V_{--d}
\]
\[
: (\lambda_1 x_1 + \lambda_2 x_2, \mu_1 y_1 + \mu_2 y_2) \mapsto \\
(\lambda_1 - \lambda_2, \mu_1 - \mu_2, \lambda_1 \mu_1 f(x_1, y_1) + \lambda_2 \mu_2 f(x_2, y_2) - \lambda_1 \mu_2 f(x_1, y_2) - \lambda_2 \mu_1 f(x_2, y_1)).
\]

Now recall from Section 4.1 that the deleted join \((\Delta_n)^{\otimes 2}_{\Delta}\) of the \(n\)-simplex \(\Delta_n\) is the boundary of an \((n+1)\)-dimensional cross-polytope, and thus \((\Delta_n)^{\otimes 2}_{\Delta}\) is homeomorphic to \(S^n\). Moreover, interchanging the join factors \(\lambda_1 x_1 + \lambda_2 x_2 \mapsto \lambda_2 x_2 + \lambda_1 x_1\) is the antipodal action on the sphere \((\Delta_n)^{\otimes 2}_{\Delta}\). Therefore \(\Phi_f\) may be considered as a \((\mathbb{Z}/2)^2\)-equivariant map on \(S^m \times S^n\), which avoids zero due to the absence of axis-aligned parallelograms from vertex-disjoint faces. \(\square\)

The following corollary now results from combining Lemmas 4.1 and 4.2.

**Corollary 4.3.** Suppose that \(m-1\) and \(n-1\) do not share a one in any digit of their binary expansions. Then there is no coupled almost-embedding \(\Delta_m \times \Delta_n \to \mathbb{R}^{m+n-2}\). That is, every map \(f : \Delta_m \times \Delta_n \to \mathbb{R}^{m+n-2}\) admits a (possibly degenerate) axis-aligned parallelogram.

### 4.3 The coindex of embedding spaces

Recall that the coindex of a free \(\mathbb{Z}/2\)-space \(Z\) is the dimension of the largest dimensional sphere which maps equivariantly into \(Z\). In \([11]\), the authors studied the coindices of embedding spaces \(\text{Emb}(X, \mathbb{R}^d)\) (and of almost-embedding spaces \(\text{AEmb}(\Sigma, \mathbb{R}^d)\)) with the compact-open topology and with the \(\mathbb{Z}/2\)-action given by \(f \mapsto -f\).

To explain concretely, \(\text{coind}(\text{Emb}(X, \mathbb{R}^d)) = q\) if the following two statements hold.

- There exists a map \(f : X \times S^q \to \mathbb{R}^d\) which is \(\mathbb{Z}/2\)-equivariant in the second factor, and for every \(y \in S^q\), \(f(-, y)\) is an embedding.
- For every map \(f : X \times S^{q+1} \to \mathbb{R}^d\) which is \(\mathbb{Z}/2\)-equivariant in the second factor, there exists \(y \in S^{q+1}\) such that \(f(-, y)\) is not an embedding.

The study of the coindex of embedding spaces is also related to the studies of bilinear and biskew maps, and is related to coupled embeddings as follows:

**Proposition 4.4.** If there exists a coupled embedding \(X \times S^q \to \mathbb{R}^d\), then \(\text{coind}(\text{Emb}(X, \mathbb{R}^d)) \geq q\).

**Proof.** Suppose there exists a coupled embedding \(f : X \times S^q \to \mathbb{R}^d\). Then in particular there are no points \(x_1, x_2, y\) satisfying \(f(x_1, y) - f(x_1, -y) = f(x_2, y) - f(x_2, -y)\), and so the map
\[
X \times S^q \to \mathbb{R}^d : (x, y) \mapsto f(x, y) - f(x, -y)
\]
is \(\mathbb{Z}/2\)-equivariant in the second factor and an embedding of \(X\) for every fixed \(y\). Therefore \(\text{coind}(\text{Emb}(X, \mathbb{R}^d)) \geq q\). \(\square\)

The same statement holds for coupled almost-embeddings. Thus the results of \([11]\) on coindices of embedding spaces can be used to generate coupled nonembeddability results when one of the
factors is a sphere. Instead of applying these results directly, we generalize the main results of [11] to obtain results for coupled nonembeddability when neither factor is a sphere.

Recall the Kneser graph and chromatic number defined in Definition 5.

The following was shown in [11], but the proof is short and we reproduce it here.

**Lemma 4.5.** Let \( \Sigma \) be a simplicial complex on ground set \([n]\), and let \( c = \chi(KG(\Sigma)) \). Then there is a map \( \Psi : (\Delta_{n-1})^{e_2}_\Delta \to \mathbb{R}^{c+1} \) with \( \Psi(\lambda_1 x_1 + \lambda_2 x_2) = -\Psi(\lambda_2 x_2 + \lambda_1 x_1) \) such that \( \Psi(\lambda_1 x_1 + \lambda_2 x_2) = 0 \) implies \( \lambda_1 = \frac{1}{2} = \lambda_2 \) and \( x_1, x_2 \in \Sigma \).

**Proof.** Color the missing faces of \( \Sigma \) by \( \{1, 2, \ldots, c\} \) in such a way that disjoint missing faces receive distinct colors. Let \( \Sigma_j \) be the simplicial complex on ground set \([n]\) whose missing faces are the missing faces of \( \Sigma \) colored \( j \). Then \( \Sigma = \Sigma_1 \cap \cdots \cap \Sigma_c \).

Define the map \( \Psi : (\Delta_{n-1})^{e_2}_\Delta \to \mathbb{R}^{c+1} \) by

\[
\Psi(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 - \lambda_2, \lambda_1 \text{dist}(x_1, \Sigma_1) - \lambda_2 \text{dist}(x_2, \Sigma_1), \ldots, \lambda_1 \text{dist}(x_1, \Sigma_c) - \lambda_2 \text{dist}(x_2, \Sigma_c)).
\]

Observe that \( \Psi(\lambda_1 x_1 + \lambda_2 x_2) = 0 \) implies \( \lambda_1 = \lambda_2 = \frac{1}{2} \) and thus \( \text{dist}(x_1, \Sigma_j) = \text{dist}(x_2, \Sigma_j) \) for all \( j \in [c] \). Since by definition of \( (\Delta_{n-1})^{e_2}_\Delta \) the points \( x_1 \) and \( x_2 \) are in disjoint faces of \( \Delta_{n-1} \) and the missing faces of \( \Sigma_j \) intersect pairwise, for every \( j \) either \( x_1 \in \Sigma_j \) or \( x_2 \in \Sigma_j \). This implies \( \text{dist}(x_1, \Sigma_j) = \text{dist}(x_2, \Sigma_j) = 0 \) and thus \( x_1, x_2 \in \Sigma \). \( \square \)

### 4.4 | Proofs of the Theorems 1.8–1.10

**Proof of Theorem 1.9.** Let \( f : \Sigma_1 \times \Sigma_2 \to \mathbb{R}^d \), for \( d = n_1 + n_2 - c_1 - c_2 - 4 \). Extend \( f \) continuously to \( \Delta_{n_1-1} \times \Delta_{n_2-1} \). Define

\[
\Phi : (\Delta_{n_1-1})^{e_2}_\Delta \times (\Delta_{n_2-1})^{e_2}_\Delta \to V_{c_1+1}^{e_1+1} \times V_{c_2+1}^{e_2+1} \oplus V^d \\
: (\lambda_1 x_1 + \lambda_2 x_2, \mu_1 y_1 + \mu_2 y_2) \mapsto \\
\Psi_1(\lambda_1 x_1 + \lambda_2 x_2), \Psi_2(\mu_1 y_1 + \mu_2 y_2), \lambda_1 \mu_1 f(x_1, y_1) + \lambda_2 \mu_2 f(x_2, y_2) - \lambda_1 \mu_2 f(x_1, y_2) \\
- \lambda_2 \mu_1 f(x_2, y_1)),
\]

where \( \Psi_1 \) and \( \Psi_2 \) are the maps from Lemma 4.5. Now by Lemma 4.1, \( \Phi \) must hit zero. Therefore by Lemma 4.5, \( \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{2} \) and \( x_1, x_2 \in \Sigma_1 \) and \( y_1, y_2 \in \Sigma_2 \). The last \( d \) components of \( \Phi \) then imply that \( f \) is not a coupled almost-embedding. \( \square \)

As a corollary we obtain a rewording of [11, Theorem 1.3], stated for coupled embeddings instead of for coindex.

**Corollary 4.6.** Let \( \Sigma \) be a simplicial complex on ground set \([n]\), and let \( p = n - \chi(KG(\Sigma)) - 2 \). Suppose that the nonnegative integers \( m \) and \( p \) do not share a one in any digit of their binary expansions. Then there is no coupled almost-embedding \( \Sigma \times S^m \to \mathbb{R}^{m+p} \). In particular, \( d(\Sigma, S^m) > m + p \).

We now present the proofs of Theorems 1.8 and 1.10. In all cases, the lower bounds are obtained by obstructing a coupled almost-embedding, and the upper bounds are obtained from nonsingular bilinear maps.
Proof of Theorem 1.8. The coupled embeddability of all pairs of simplicial complexes in dimension \(2p + 2q + 1\) follows from Proposition 1.7. The lower bounds come from Theorem 1.9. We argue for the pair \(\Delta_{2p+2}^p, [3]^{(q+1)}\). The simplicial complex \(\Delta_{2p+2}^p\) has \(2p + 3\) vertices, and each minimal nonface has \(p + 2\) vertices. Since no two such faces are disjoint, the chromatic number is 1, and so \(n_1 - c_1 - 2 = 2p + 3 - 1 - 2 = 2p\). The simplicial complex \([3]^{(q+1)}\) has \(3q + 3\) vertices, and the minimal nonfaces are the missing edges within each individual copy of [3], as depicted for \(q = 1\) in Figure 2. The edges in the Kneser graph connect vertices corresponding to missing edges from different copies of [3]. The chromatic number of this graph is \(q + 1\), and so \(n_2 - c_2 - 2 = 3q + 3 - (q + 1) - 2 = 2q\). The result follows since \(2p\) and \(2q\) do not share any ones in their binary expansions by hypothesis.

Proof of Theorem 1.10. We make use of the fact that the real projective plane \(\mathbb{R}P^2\) can be triangulated in a unique way by a six-vertex triangulation and the complex projective plane \(\mathbb{C}P^2\) can be triangulated in a unique way by a nine-vertex triangulation. These triangulations \(\Sigma_{\mathbb{R}P^2}\) and \(\Sigma_{\mathbb{C}P^2}\) each have the property that no two nonfaces are disjoint, and thus the Kneser graphs \(KG(\Sigma_{\mathbb{R}P^2})\) and \(KG(\Sigma_{\mathbb{C}P^2})\) have no edges. In particular, \(\chi(KG(\Sigma_{\mathbb{R}P^2})) = 1 = \chi(KG(\Sigma_{\mathbb{C}P^2}))\). (For more details see the book of Matoušek [32, Example 5.8.5].)

1. The lower bounds come from Corollary 4.6 applied to \(m = 4q\) and to \(\Sigma_{\mathbb{R}P^2}\), so \(n - \chi - 2 = 3\). The upper bounds come from the existence of nonsingular bilinear maps \(\mathbb{R}^4 \times \mathbb{R}^{4q+4} \rightarrow \mathbb{R}^{4q+4}\).
2. The lower bounds come from Corollary 4.6 applied to \(m = 8q\) and \(m = 8q + 1\) and to \(\Sigma_{\mathbb{C}P^2}\), so \(n - \chi - 2 = 6\). The upper bounds come from the existence of nonsingular bilinear maps \(\mathbb{R}^7 \times \mathbb{R}^{8q} \rightarrow \mathbb{R}^{8q}\) and \(\mathbb{R}^7 \times \mathbb{R}^{8q+1} \rightarrow \mathbb{R}^{8q+7}\).
3. The lower bounds come from Corollary 4.6 applied to \(m = 1\) and to \(\Delta_{2k+2}^k\), so \(n - \chi - 2 = 2k\). The upper bounds come from Proposition 1.7.
4. The lower bounds come from Corollary 4.6 applied to \(m = 1\) and to \([3]^{(k+1)}\), so \(n - \chi - 2 = 2k\). The upper bounds come from Proposition 1.7.
5. The lower bounds come from Theorem 1.9 with \(n_1 - c_1 - 2 = 3\) and \(n_2 - c_2 - 2 = 4q\). The upper bounds come from Proposition 1.7.

This completes the proof.

Theorem 4.7. The entries of the table in Figure 3 are the values of \(d_\Delta(p, q)\).

Before explaining how Figure 3 is filled, we generalize Theorem 1.8 to integers \(p\) and \(q\) which may share a one in some digit of their binary expansions.

Lemma 4.8. Given \(q \geq 1\) and \(\ell \in \{q - 1, \ldots, 2q\}\), there exists a simplicial complex \(\Sigma\) on ground set \([n]\) and of dimension \(q\), such that \(n - \chi(KG(\Sigma)) - 2 = \ell\).

Proof. Let \(k = 2q - \ell\), so \(k \in \{0, \ldots, q + 1\}\). Then the simplicial complex \(\Sigma = [2]^{(k)} \ast [3]^{(q+1-k)}\) has \(n = 2k + 3(q + 1 - k) = 3q + 3 - k = q + 3 + \ell\) vertices, and by the same argument in the proof of Theorem 1.8, \(\chi(KG(\Sigma)) = q + 1\). Thus \(\Sigma\) has the desired property.

Lemma 4.9. Given \(p \geq 1\), \(d_\Delta(p, q) > 2p + h\), where \(h \leq 2q\) is the largest integer whose binary expansion shares no ones with that of \(2p\).
FIGURE 3 The value of $d_{\Delta}(p, q)$ for small $p$ and $q$. The red circled numbers are of the form $2p + 2q + 1$ and filled by Theorem 1.8. Entries with slashes indicate the best known lower and upper bounds

Proof. We show that there exists a $q$-dimensional complex $\Sigma$ such that the pair $([3]^{(p+1)}, \Sigma)$ does not coupled embed into $\mathbb{R}^{2p+h}$.

If $h \in \{q-1, \ldots, 2q\}$, let $\Sigma$ be as defined in Lemma 4.8. Since $n - \chi(KG(\Sigma)) - 2 = h$, and since $h$ and $2p$ do not share any ones by definition of $h$, Theorem 1.9 applies.

If otherwise $h \in \{1, \ldots, q-2\}$, then we may apply Theorem 1.9 with the $(h+1)$-simplex in place of $\Sigma$, since $\Delta_{h+1}$ satisfies $n - \chi(KG(\Delta_{h+1})) - 2 = h$. Thus we obtain $d([3]^{(p+1)}, \Delta_{h+1}) > 2p + h$, and the same bound applies to any $q$-dimensional complex which contains $\Delta_{h+1}$ as a subcomplex.

Proof of Theorem 4.7. All lower bounds are obtained by Lemma 4.9. All upper bounds are obtained using nonsingular bilinear maps from $\mathbb{R}^{2p+1} \times \mathbb{R}^{2q+1}$ as listed in Example 2.1, as follows. Upper bounds in the first column are obtained by quaternion multiplication. The upper bound in cells containing the number 8 are obtained by appropriate restriction of the octonion multiplication $\mathbb{R}^8 \times \mathbb{R}^8 \rightarrow \mathbb{R}^8$. The upper bound in cells containing the number 16 are obtained by restriction of the nonsingular bilinear map $\mathbb{R}^{16} \times \mathbb{R}^9 \rightarrow \mathbb{R}^{16}$. The upper bounds in cells containing the numbers 24 and 28 are obtained by quaternionic and octonionic polynomial multiplication. The upper bounds in cells containing the number 32 come from restricting the nonsingular bilinear map $\mathbb{R}^{17} \times \mathbb{R}^{24} \rightarrow \mathbb{R}^{32}$. The remaining upper bounds come from maps listed in Example 2.1 items (iv)–(vi).

5 CONCLUSION

We have introduced the notion of coupled embeddability and taken the first exploratory steps towards understanding these maps. We have posed and discussed a number of interesting problems which we collect here.

Question 5.1. Does there exist a coupled embedding $\mathbb{R}P^2 \times \mathbb{R}P^3 \rightarrow \mathbb{R}^7$?
We find this question especially compelling since $\mathbb{R}P^2 \times \mathbb{R}P^3$ embeds into $\mathbb{R}^7$, despite the fact that $\mathbb{R}P^2$ does not embed in $\mathbb{R}^3$ and $\mathbb{R}P^3$ does not embed in $\mathbb{R}^4$ (see Remark 1), and so nonexistence would provide the first example for which the embedding dimension of a product is less than the coupled embedding dimension.

Moreover, nonexistence would answer the following question:

**Question 5.2.** Do there exist $p > 1$ and $q > 1$ such that $d(p, q) = 2p + 2q - 2$? That is, do there exist manifolds $M$ and $N$ of dimensions $p > 1$ and $q > 1$ such that there is no coupled embedding $M \times N \to \mathbb{R}^{2p+2q-3}$?

These questions were stated and discussed earlier in the text; see Problems 1.6 and 3.2.

More generally, Problem 1.4 asks for the value of $d(p, q)$ for general $p$ and $q$. It would be interesting to determine this when $p$ and $q$ are powers of 2.

**Question 5.3.** What is the value of $d_\Delta(5, 5)$?

To generate the values in the table in Figure 3, it seems that we used the full power of Theorem 1.9 and all known examples of nonsingular bilinear maps. What other methods can be employed to determine the remaining values of $d_\Delta(p, q)$ when $p$ and $q$ are small?

**Question 5.4.** Does there exist a Haefliger-type h-principle statement for coupled embeddings?

This question is discussed at length in Section 3.2.

**Question 5.5.** Do the existence dimensions for coupled embeddings $S^m \times S^n \to \mathbb{R}^d$ exactly match the existence dimensions for coupled $\mathbb{Z}/2$-embeddings?

Gitler and Lam showed in [14] that there exists a nonsingular biskew map $S^{27} \times S^{12} \to \mathbb{R}^{32}$, but that there is no nonsingular bilinear map $\mathbb{R}^{28} \times \mathbb{R}^{13} \to \mathbb{R}^{32}$. Thus there is a coupled $\mathbb{Z}/2$ embedding $S^{27} \times S^{12} \to \mathbb{R}^{32}$ but we do not know whether there exists a coupled embedding $S^{27} \times S^{12} \to \mathbb{R}^{32}$. More generally, coupled $\mathbb{Z}/2$-embeddings could be defined for any spaces $X$ and $Y$ with free $\mathbb{Z}/2$-actions. Is there any interesting theory of such maps?

**ACKNOWLEDGEMENTS**

The authors are grateful to an anonymous referee for comments and suggestions which greatly improved the presentation of this note. F. Frick was supported by NSF grant DMS 1855591 and a Sloan Research Fellowship. M. Harrison was supported by Mathematisches Forschungsinstitut Oberwolfach with an Oberwolfach Leibniz Fellowship and by the Institute for Advanced Study through NSF grant DMS1926686.

**JOURNAL INFORMATION**

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.
REFERENCES

1. J. F. Adams, Vector fields on spheres, Ann. of Math. (2) 75 (1962), 603–632.
2. J. Adem, On nonsingular bilinear maps, in The Steenrod algebra and its applications, Lecture Notes in Mathematics, vol. 168, Springer, Berlin, 1970, pp. 11–24.
3. P. M. Akhmetiev, D. Repovš, and A. B. Skopenkov, Embedding products of low-dimensional manifolds into $\mathbb{R}^m$, Topology Appl. 113 (2001), 7–12.
4. D. Baralić, Immersions and embeddings of quasitoric manifolds over the cube, Publ. Inst. Math. 95 (2014), 63–71.
5. D. Baralić, B. Prvulović, G. Stojanović, S. Vrećica, and R. Zivaljević, Topological obstructions to totally skew embeddings, Trans. Amer. Math. Soc. 364 (2012), 2213–2226.
6. M. Berger and S. Friedland, The generalized Radon-Hurwitz numbers, Compos. Math. 59 (1986), no. 1, 113–146.
7. P. V. M. Blagojević, F. R. Cohen, M. C. Crabb, W. Lück, and G. M. Ziegler, Equivariant cohomology of configuration spaces Mod 2, in The State of the Art, Lecture Notes in Mathematics, Springer, Cham, 2021.
8. J. Buczyński, T. Januszkiewicz, J. Jelisiejew, and M. Michałek, Constructions of $k$-regular maps using finite local schemes, J. Eur. Math. Soc. 21 (2019), no. 6, 1775–1808.
9. Y. H. Chan, S. Chen, F. Frick, and J. T. Hull, Borsuk-Ulam theorems for products of spheres and Stiefel manifolds revisited, Topol. Methods Nonlinear Anal. 55 (2020), 553–564.
10. C. Domínguez and K. Y. Lam, Nonsingular bilinear maps revisited, Proc. Roy. Soc. Edinburgh Sect. A 151 (2021), 377–390.
11. F. Frick and M. Harrison, Spaces of embeddings: nonsingular bilinear maps, chirality, and their generalizations, Proc. Amer. Math. Soc. 150 (2022), 423–437.
12. F. Frick and M. Harrison, On affinely 3-regular maps and trapezoids, arxiv:2109.05985.
13. M. Ghomi and S. Tabachnikov, Totally skew embeddings of manifolds, Math. Z. 258 (2008), 499–512.
14. S. Gitler and K. Y. Lam, The generalized vector field problem and bilinear maps, Bol. Soc. Mat. Mexicana 14 (1969), 65–69.
15. M. Golubitsky and V. Guillemin, Stable mappings and their singularities, Graduate Texts in Mathematics, vol. 14, Springer, New York, 1973.
16. M. Gromov and Ya. Eliashberg, Removal of singularities of smooth mappings, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 600–626.
17. A. Haefliger, Plongements différentiables dans le domaine stable, Comment. Math. Helv. 37 (1962), 155–176.
18. M. Harrison, Skew flat fibrations, Math. Z. 282 (2016), 203–221.
19. M. Harrison, Contact structures induced by skew fibrations of $\mathbb{R}^3$, Bull. Lond. Math. Soc. 51 (2019), 887–899.
20. M. Harrison, Introducing totally nonparallel immersions, Adv. Math. 374 (2020), 251–286.
21. M. Harrison, Fibrations of $\mathbb{R}^3$ by oriented lines, Algebr. Geom. Topol. 21 (2021), 2899–2928.
22. H. Hopf, Ein topologischer Beitrag zur reellen Algebra, Comment. Math. Helv. 13 (1940), 219–239.
23. A. Hurwitz, Über die Komposition der quadratischen Formen, Math. Ann. 88 (1922), 1–25.
24. I. M. James, Euclidean models of projective spaces, Bull. Lond. Math. Soc. 3 (1971), 257–276.
25. W. Kühnel, Tight polyhedral submanifolds and tight triangulations, Lecture Notes in Mathematics, vol. 1612, Springer, Berlin, 1995.
26. K. Y. Lam, Construction of nonsingular bilinear maps, Topology 6 (1967), no. 4, 423–426.
27. K. Y. Lam, On bilinear and skew-linear maps that are nonsingular, Q. J. Math. 19 (1968), no. 1, 281–288.
28. K. Y. Lam, Non-singular bilinear maps and stable homotopy classes of spheres, Math. Proc. Cambridge Philos. Soc. 82 (1977), no. 3, 419–425.
29. K. Y. Lam, Some interesting examples of nonsingular bilinear maps, Topology 16 (1977), no. 2, 185–188.
30. K. Y. Lam, KO-equivalences and existence of nonsingular bilinear maps, Pacific J. Math. 82 (1979), no. 1, 145–154.
31. K. Y. Lam, Some new results on composition of quadratic forms, Invent. Math. 79 (1985), no. 3, 467–474.
32. J. Matoušek, Using the Borsuk–Ulam theorem: lectures on topological methods in combinatorics and geometry, Springer, Berlin, 2008.
33. V. Ovsienko and S. Tabachnikov, On fibrations with flat fibres, Bull. Lond. Math. Soc. 45 (2013), 625–632.
34. V. Ovsienko and S. Tabachnikov, Hopf fibrations and Hurwitz-Radon numbers, Math. Intelligencer 38 (2016), 11–18.
35. J. Radon, Lineare Scharen orthogonalen Matrizen, Abh. Math. Semin. Univ. Hambg. 1 (1922), 1–14.
36. E. Ramos, Equipartition of mass distributions by hyperplanes, Discrete Comput. Geom. 15 (1996), no. 2, 147–167.
37. G. Stojanović, Embeddings with multiple regularity, Geom. Dedicata 123 (2006), 1–10.