A Recursive Method to Calculate UV-divergent Parts at One-Loop Level in Dimensional Regularization

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Abstract

A method is introduced to calculate the UV-divergent parts at one-loop level in dimensional regularization. The method is based on the recursion, and the basic integrals are just the scaleless integrals after the recursive reduction, which involve no other momentum scales except the loop momentum itself. The method can be easily implemented in any symbolic computer language, and an implementation in MATHEMATICA is ready to use.

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PROGRAM SUMMARY

Title of program: $UVPart

Programming language: MATHEMATICA

Available from: http://power.itp.ac.cn/~fengfeng/uvpart/

Computer(s) for which the program has been designed: Any computer where the MATHEMATICA is running.

External routines/libraries used: FEYNCALC, FEYNARTS

Keywords: UV-Divergences, One-Loop Corrections, Dimensional Regularization

CPC Library Classification: 11.1

Nature of problem: To get the UV-divergent part of any one-loop expression.

Method of solution: $UVPart is an MATHEMATICA package where the recursive method has been implemented.

Running time: In general it is below a second.
I. INTRODUCTION

One has to deal with an integration over the loop momentum at next-to-leading order, which results to ultraviolet (UV) and infrared (IR) divergencies. Dimensional regularization\cite{1, 2} is needed in order to produce meaningful results. The general one-loop amplitude can be written as of a linear combination of known scalar integrals\cite{3} — boxes, triangles, bubbles and tadpoles — multiplied by coefficients that are rational functions of the external momenta and polarization vectors, plus a remainder which is also a rational function of the latter.

There are many automatic tools available to achieve the general one-loop amplitude, like \textsc{FeynCalc}\cite{4} and \textsc{FormCalc}\cite{5}, which are based on the traditional Passarino-Veltman\cite{6–9} reduction of Feynman graphs, which can be generated automatically(\textsc{FeynArts}\cite{10, 11} or \textsc{QGRAF}\cite{12}). In order to produce numerical results, tensor coefficients functions are calculated using \textsc{LoopTools}\cite{5}. For a detailed review, please see Refs. \cite{13, 14}.

In the last few years, several groups have been working on the problem of constructing efficient and automatized methods for the computation of one-loop corrections for multi-particle processes. Many different interesting techniques have been proposed: these contain numerical and semi-numerical methods\cite{15–18}, as well as analytic approaches\cite{19–22}, that make use of unitarity cuts to build NLO amplitudes by gluing on-shell tree amplitudes\cite{23, 24}. For a recent review of existing methods, see Refs. \cite{25, 26}.

Generally, it will be much easier to calculate the UV-divergent parts of the one-loop amplitude than the one-loop amplitude itself, and there are little work on this specific area since one usually need not to calculate this part separately. However if what we are concerned is the renormalization, we need to calculate the UV-divergent parts only, and moreover there are also some cases in which we have to calculate the UV-divergence separately, for example, most packages like \textsc{Fire}\cite{27} and \textsc{Reduze}\cite{28}, which implement the integration by parts (IBP) relations\cite{29}, treat the scaleless integrals as zero, i.e.

\[
\int d^4k \ (k^2)^n = 0
\]  

however it is well known for the logarithmically divergent scaleless integrals that

\[
\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^4k \frac{1}{(k^2)^2} = \frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}},
\]  

so Eq. (1) does not distinguish the UV- and IR-divergence. This will be fine if we consider the amplitude as a whole, since the UV-divergent parts will be canceled by the counterterms, and the left divergence will only be IR-divergent. But there is no way to know the IR-divergence from a specific Feynman diagram, which is very important when one considers the factorization, where one tries to identify the source of IR- divergence and factorize them out.

A method that allows the extraction of the UV-divergent part of an arbitrary 1-loop tensor N-point coefficient was presented in Ref. \cite{30}. We want to introduce another simple method to calculate the UV-divergent part in the dimensional regularization. The method is based on the recursion, and basic integrals are just the scaleless integrals after the recursive
reduction, which involve no other momentum scales except the loop momentum itself. Since
the computation in this method just involves algebraic rational operations, so it can be
easily implemented in any symbolic computer language.

The paper is organized as follows: We introduce the definitions and notations in Sec. II
then describe the calculations in Sec. III and in Sec. IV we give an implementation with
MATHEMATICA, and use this method in a specific process $e^+e^- \rightarrow J/\psi + \eta_c$, and finally
comes the summary.

II. DEFINITIONS AND NOTATIONS

The general expression associated with UV-divergent parts at one-loop reads

$$\mathcal{I} = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dk \frac{Q}{N_1^{m_1} N_2^{m_2} \cdots N_N^{m_N}}$$

(3)

with the denominators

$$N_i = (k + q_i)^2 - m_i^2 + i\epsilon,$$

(4)

where $i\epsilon$ denotes the infinitesimally small positive imaginary part, $\mu$ is the reorganization
scale, $D$ is the non-integer dimension of space-time defined as $D = 4 - 2\varepsilon$, $q_i$ are linear
combination of external momenta $p_i$, and the numerator $Q$ is the polynomial of $k^2$ and $k \cdot p_i$.

The following identity about the scaleless integrals is well known in the calculations with
dimensional regularization:

$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dk \left( k^2 \right)^n = \begin{cases}
0 & (\text{n} \neq -2) \\
1 & (\text{n} = -2)
\end{cases},$$

(5)

where only the logarithmically divergent scaleless integral contributes the UV-divergence.

According to Lorentz covariance and oddness of the scaleless integrals, we have

$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dk \ k^{\mu_1} k^{\mu_2} \cdots k^{\mu_m} \left( k^2 \right)^n = \begin{cases}
0 & (\text{m is odd}) \\
C_{m,n} \times g^{\{\mu_1\mu_2\cdots\mu_m\}} & (\text{m is even})
\end{cases},$$

(6)

i.e. when $m$ is odd, the scaleless tensor integrals are 0, and when $m$ is even, they are
proportional to $g^{\{\mu_1\mu_2\cdots\mu_m\}}$ with coefficient $C_{m,n}$, where we use the same notations as Refs. [9, 30] for $g^{\{\mu_1\mu_2\cdots\mu_m\}}$, which is the symmetrization of $g^{\mu_1\mu_2}g^{\mu_3\mu_4}\cdots g^{\mu_{m-1}\mu_m}$ with respect to the lorentz index $\mu_1, \mu_2, \cdots, \mu_m$, for example

$$g^{\{\mu_1\mu_2\}} = g^{\mu_1\mu_2},$$

$$g^{\{\mu_1\mu_2\mu_3\mu_4\}} = g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}.$$
where we have used
\[ g_{\mu_1\mu_2}g_{\mu_3\mu_4} \cdots g_{\mu_{m-1}\mu_m}g^{\{\mu_1\mu_2 \cdots \mu_m\}} = D(D + 2) \cdots (D + m - 2), \]
where \( m \) is even, and more relations can be found in Ref. [30].

### III. DESCRIPTION OF THE CALCULATIONS

We define a function \( \text{Power} \) to get the asymptotic scaling of the polynomial of \( k^2 \) and \( k \cdot p_i \) in the ultraviolet region, for example:

\[ \text{Power}[k^2] = 2, \quad \text{Power}[k \cdot p_i] = 1, \quad \text{Power}[m^2] = 0, \quad \text{Power}[k^2 + k \cdot p_i - m^2] = 2 \]

and we can extend this function to rational expression of \( k^2 \) and \( k \cdot p_i \).

\[ \text{Power} \left[ \frac{\mathcal{N}}{\mathcal{D}} \right] \equiv \text{Power}[\mathcal{N}] - \text{Power}[\mathcal{D}], \]

where \( \mathcal{N} \) and \( \mathcal{D} \) are some polynomials of \( k^2 \) and \( k \cdot p_i \), for example

\[ \text{Power} \left[ \frac{k \cdot p_1}{(k^2 - m_0^2)((k + q_1)^2 - m_1^2)^2} \right] = -5. \]

So if \( \text{Power} \) of the integrand in \( \mathcal{I} \) is less than \(-4\), then there will be no UV-divergent part in \( \mathcal{I} \), i.e.

\[ \mathcal{I}_{UV} = 0, \quad (\text{Power}[\mathcal{Q}] - \sum_{i=1}^{N} 2n_i < -4). \]

For example

\[ \left[ \frac{(2\pi\mu)^{D-4}}{i\pi^2} \int d^Dk \frac{k \cdot p_1}{(k^2 - m_0^2)((k + p_1)^2 - m_1^2)^2} \right]_{\text{UV}} = 0. \]

Now we are going to describe the calculations of UV-divergent parts of \( \mathcal{I} \). For general integrand of \( \mathcal{I} \), we can write

\[ \frac{\mathcal{Q}}{\mathcal{N}_1^{n_1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} = \frac{\mathcal{N}_1 - (\mathcal{N}_1 - k^2)}{k^2} \frac{\mathcal{Q}}{\mathcal{N}_1^{n_1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} \]

\[ = \frac{k^2 \mathcal{Q}'}{\mathcal{N}_1^{n_1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} - \frac{\mathcal{Q}'}{\mathcal{N}_1^{n_1-1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} \]

with \( \mathcal{Q}' = \mathcal{Q} (\mathcal{N}_1 - k^2) \), and it is clear that

\[ \text{Power} \left[ \frac{\mathcal{Q}'}{k^2 \mathcal{N}_1^{n_1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} \right] \leq \text{Power} \left[ \frac{\mathcal{Q}}{\mathcal{N}_1^{n_1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} \right] - 1, \]

i.e. either the power of propagators: \( n_i \) or the \( \text{Power} \) of the integrand decreases by at least 1, if we apply this replacement once again in the last result, we have

\[ \frac{\mathcal{Q}}{\mathcal{N}_1^{n_1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} = \left[ \frac{\mathcal{Q}}{(k^2)^2 \mathcal{N}_1^{n_1-2} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} - \frac{\mathcal{Q}'}{(k^2)^2 \mathcal{N}_1^{n_1-1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} \right] - \left[ \frac{\mathcal{Q}'}{(k^2)^2 \mathcal{N}_1^{n_1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} - \frac{\mathcal{Q}''}{(k^2)^2 \mathcal{N}_1^{n_1-1} \mathcal{N}_2^{n_2} \cdots \mathcal{N}_N^{n_N}} \right], \]
with $Q'' = Q'$ $(N_i - k^2)$, and we can see the power of propagators: $n_i$ or the Power of the integrand decreases further by at least 1.

So we can apply this replacement again and again until one of the following cases happens:

- All the power of $N_i$, becomes zero, i.e. only one type of propagator: $k^2$ is left.
- The corresponding $I_{UV} = 0$, i.e.

$$\text{Power} \left[ \frac{Q}{N_1^{n_1} N_2^{n_2} \cdots N_N^{n_N}} \right] = \left( \text{Power}[Q] - \sum_i N_i \right) < -4.$$  

So after the recursive reduction, only one type of integration will be left:

$$I = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dk \frac{Q}{(k^2)^m}, \quad (18)$$

and it is ready to read the result according to Eqs. (5) and (6).

We can take the following 3-point tensor integral as an example:

$$\frac{(k \cdot p_1)^3}{k^2 [(k + q_1)^2 - m_1^2] [(k + q_2)^2 - m_2^2]}$$

$$\rightarrow \frac{(k \cdot p_1)^3}{(k^2)^3 [(k + q_2)^2 - m_2^2]} - \frac{(k \cdot p_1)^3 (2k \cdot q_1 + q_1^2 - m_1^2)}{(k^2)^2 [(k + q_1)^2 - m_1^2] [(k + q_2)^2 - m_2^2]}$$

$$\rightarrow \left[ \frac{(k \cdot p_1)^3 (2k \cdot q_1)}{(k^2)^3 [(k + q_2)^2 - m_2^2]} - \frac{(k \cdot p_1)^3 (2k \cdot q_1)(2k \cdot q_1 + q_1^2 - m_1^2)}{(k^2)^3 [(k + q_1)^2 - m_1^2] [(k + q_2)^2 - m_2^2]} \right]$$

$$\rightarrow \frac{(k \cdot p_1)^3 (2k \cdot q_2)}{(k^2)^3 [(k + q_2)^2 - m_2^2]} - \frac{(k \cdot p_1)^3 (2k \cdot q_1)(2k \cdot q_2 - m_2^2)}{(k^2)^3 [(k + q_2)^2 - m_2^2]}$$

$$\rightarrow \frac{(k \cdot p_1)^3 (2k \cdot q_1)}{(k^2)^4} - \frac{(k \cdot p_1)^3 (2k \cdot q_1)(2k \cdot q_2 - m_2^2)}{(k^2)^3 [(k + q_2)^2 - m_2^2]}$$

where each step $\rightarrow$ means we take a replacement, and all the expressions framed with box have been dropped during the recursive expansion since the Power is less than $-4$ and will not contribute UV-divergent part.

Now it is ready to get the UV-divergent parts from the last line in Eq. (19), using Eqs. (5) and (6)

$$\left[ \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dk \frac{(k \cdot p_1)^3}{k^2 [(k + q_1)^2 - m_1^2] [(k + q_2)^2 - m_2^2]} \right]_{UV} = -p_1^2 (p_1 \cdot q_1 + p_1 \cdot q_2) \frac{1}{\varepsilon_{UV}} \quad (20)$$
and we can check that it agrees with Refs. [9, 30].

The important feature of this method is that, there is only one preliminary integral, i.e. Eq. (6), which involves no other scales like external momenta \( p_i \) or mass \( m_i \).

Another advantage is that it can be easily implemented in any symbolic computer language, like MATHEMATICA, REDUCE, FORM, etc. I will give an explicit implementation with MATHEMATICA in the next section.

IV. IMPLEMENTATION WITH MATHEMATICA

A. Typical Examples

An implementation in MATHEMATICA is already available, note that the FEYNCALC package has been used to deal with the ScalarProduct, but it not required for the implementation.

The UV-divergent parts can be retrieved with the function: \$UVPart, which is defined as:

\[
$UVPart[exp, k] := \left[ \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \exp \right]_{UV} \tag{21}
\]

where \( \exp \) can be any expression at one-loop level, while \( k \) is the loop momentum.

We can give some simple examples:

den=SPD[k] (SPD[k+q1]-m1^2) (SPD[k+q2]-m2^2)//FCI//ScalarProductExpand;
num=SPD[k, pi]^3//FCI;
$UVPart[num/den, k]//Simplify

The output of the code above reads:

\[
\text{Out[ ] := } -\frac{6p1^2(p1 \cdot q1 + p1 \cdot q2)}{D(D + 2)\omega}, \tag{22}
\]

where \( \omega \) is just \( \varepsilon_{UV} \) which we have used to represent the UV divergence in the MATHEMATICA code, Eq. (22) gives the same result as Eq. (21) after setting the dimension \( D \) to 4.

den = (SPD[k]-m0^2) (SPD[k]-m1^2)^2 (SPD[k]-m2^2)//FCI//ScalarProductExpand;
num = SPD[k, p1]^8//FCI;
$UVPart[num/den, k]//Simplify

The output reads:

\[
\text{Out[ ] := } \frac{105 (m0^4 + (2m1^2 + m2^2) m0^2 + 3m1^4 + m2^4 + 2m1^2m2^2) (p^2)^4}{D (D^3 + 12D^2 + 44D + 48) \omega}, \tag{23}
\]

setting \( D \) to 4 we get

\[
\left[ \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \frac{(k \cdot p)^8}{(k^2 - m_0^2)(k^2 - m_1^2)^2(k^2 - m_2^2)} \right]_{UV} = -\frac{7(p^2)^4(m_0^4 + m_1^4 + 3m_1^4 + 2m_0^2m_1^2 + m_0^2m_2^2 + 2m_1^2m_2^2)}{128 \varepsilon_{UV}}. \tag{24}
\]
B. UV-Divergences in $e^+ e^- \rightarrow J/\psi + \eta_c$ at One-Loop Level

We will apply the method to a specific process $e^+ e^- \rightarrow J/\psi + \eta_c$, where we use the FeynArts to generate Feynman diagrams, and FeynCalc to handle the DiracTrace. We will take a triangle Feynman diagram in Fig. 1 as a concrete example in this section.

First we use the FeynArts to generate the amplitude for the diagram, then make the following replacement with the help of spinor projectors [31]:

\[
v(\bar{p}_3)\bar{u}(p_3) \rightarrow \frac{1}{4\sqrt{2}E_3(E_3 + m_c)}(\hat{p}_3 - m_c)\epsilon_S^*(\hat{p}_3 + 2E_3)(\hat{p}_3 + m_c) \otimes \frac{1}{\sqrt{N_c}}
\]

\[
v(\bar{p}_4)\bar{u}(p_4) \rightarrow \frac{1}{4\sqrt{2}E_2(E_4 + m_c)}(\hat{p}_4 - m_c)\gamma_5(\hat{p}_4 + 2E_4)(\hat{p}_4 + m_c) \otimes \frac{1}{\sqrt{N_c}}
\]

(25)

where the subscripts 3 and 4 are used to label $J/\psi$ and $\eta_c$ respectively.

After performing the DiracTrace on the fermion chains, we get the amplitude for this diagram as follows:

\[
Amp = \frac{iC_F g^4_\alpha \epsilon_{\mu \nu \rho \sigma} p_3 p_4}{6\sqrt{N_c} m_c^5(s - 4)s k^2(k^2 + k \cdot p_4)(2s m_c^2 + k^2 + k \cdot p_3 + 2k \cdot p_4) \times \left(-2Ds^2 m_c^4 + 12s^2 m_c^4 + 8Dsm_c^4 - 56sm_c^4 + 32m_c^4 + 8Dk^2 m_c^2
\right.

-2Dsk^2 m_c^2 + 4sk^2 m_c^2 - 16k^2 m_c^2 + 4Dk \cdot p_3 m_c^2 + D^2 sk \cdot p_3 m_c^2

-12Dsk \cdot p_3 m_c^2 + 36sk \cdot p_3 m_c^2 - 16k \cdot p_3 m_c^2 + 4Dk \cdot p_4 m_c^2

-\left. D^2 sk \cdot p_4 m_c^2 + 10Dsk \cdot p_4 m_c^2 - 32sk \cdot p_4 m_c^2 - 2Dk \cdot p_4^2
\right)

+4k \cdot p_3^2 + 2Dk \cdot p_3 k \cdot p_4 - 4k \cdot p_3 k \cdot p_4)
\]

(26)

where $p_3$ and $p_4$ are the momenta of $J/\psi$ and $\eta_c$ respectively, and $s = (p_3 + p_4)^2/(4m_c^2)$, $m_c$ is the quark mass, and $k$ is the loop momentum, and to get the UV-divergent part of the amplitude, we just use

\[
$UVPart[Amp,k]$
\]
The output reads:

\[
\text{Out}[\ ] := - \frac{i C_F (D - 2) e g_s^4 \epsilon^\gamma \epsilon^\psi p^3 p^4}{3 \sqrt{N_c} D m_c^3 s^2 \omega}
\]  \hspace{1cm} (27)

where the Lorentz index \( \gamma \) refers to \( \mu \), and \( \psi \) to \( \epsilon_S^* \) in Eq. (26). After setting \( D \) to 4, we get

\[
\left[ \frac{(2 \pi \mu)^{4-D}}{i \pi^2} \int d^D k \right]_{\text{UV}} \text{Amp} = - \frac{i C_F e g_s^4 \epsilon^\mu \epsilon^\rho p^3 p^4}{3 \sqrt{N_c} m_c^3 s^2} \frac{1}{\epsilon_{\text{UV}}}
\]  \hspace{1cm} (28)

We can apply this method to each diagram to get the corresponding UV-divergent part of the amplitude, and to check the validity of the our result. We have compared the UV-divergent part produced with our code with Ref. [32] diagram by diagram, and the both agree with each other for all diagrams.

V. SUMMARY

A pretty simple method is introduced to calculate the UV-divergent parts at one-loop level with dimensional regularization. It is found that there is only one preliminary integral which involves no other scale like external momenta \( p_i \) or mass \( m_i \) after the recursive reduction. The method can be easily implemented in any symbolic computer language, An explicit implementation with MATHEMATICA is also present.

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