Mean Field Games of Controls: on the convergence of Nash equilibria *

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Abstract

In this paper, we provide convergence and existence results for mean field games of controls. Mean field games of controls are a class of mean field games where the mean field interactions are achieved through the joint (conditional) distribution of the controlled state and the control process. The framework we are considering allows to control the volatility coefficient $\sigma$, and the controls/strategies are supposed to be of open loop type. Using (controlled) Fokker–Planck equations, we introduce a notion of measure–valued solution of mean field games of controls and prove a relation between these solutions on the one hand, and the approximate Nash equilibria on the other hand. First of all, in the $N$–player game associated to the mean field games of controls, given a sequence of approximate Nash equilibria, it is shown that, this sequence admits limits as $N$ tends to infinity, and each limit is a measure–valued solution of the corresponding mean field games of controls. Conversely, any measure–valued solution can be obtained as the limit of a sequence of approximate Nash equilibria of the $N$–player game. In other words, the measure–valued solutions are the accumulation points of the approximate Nash equilibria. Then, by considering an approximate strong solution of mean field games of controls which is the classical strong solution where the optimality is obtained by admitting a small error $\varepsilon$, we prove that the measure–valued solutions are the accumulation points of this type of solutions when $\varepsilon$ goes to zero. Finally, the existence of a measure–valued solution of mean field games of controls is proved in the case without common noise.

1 Introduction

Since the pioneering work of Lasry and Lions [31] and Huang, Caines, and Malhamé [22], mean field games (MFG) have been the subject of intensive research in recent years. Due to the diversity of applications, particularly in models of oil production, volatility formation, population dynamics and economic growth (see Carmona and Delarue [8] for an overview), the study of MFG has attracted increasing interest in the field of applied mathematics.

The MFG can be seen as symmetric stochastic differential games with infinitely many players. Indeed, under the appropriate assumptions, a MFG solution can be used to construct approximate Nash equilibria for a game involving a large number of players. Also, it can be shown that each sequence of Nash equilibria converges towards a solution of MFG when the number of players tends to infinity.

So far, this study has been conducted considering that the interactions between the players are achieved only through the empirical distribution of the state processes. We refer to Lacker [26] for a general analysis of this case (see also Fisher [16]). The goal of this paper is to give a general analysis of the case where the mean field interaction occurs through the empirical distribution of both the states and the controls.

To briefly summarize the finite–player games, specified in full details in Section 2.1, let us suppose that the $N$–players have private state processes $X := (X^1, \ldots, X^N)$ given by the stochastic differential equations (SDEs) system

\[
\begin{align*}
\mathrm{d}X^i_t &= b(t, X^i_t, \varphi^N_{t,x^i}, \varphi^N_{t}, \alpha^i_t)\mathrm{d}t + \sigma(t, X^i_t, \varphi^N_{t,x^i}, \varphi^N_{t}, \alpha^i_t)\mathrm{d}W^i_t + \sigma_0 \mathrm{d}B_t, \quad t \in [0, T], \\
\varphi^N_t := \frac{1}{N} \sum_{i=1}^N \delta_{(X^i_t, \alpha^i_t)}^{N} \quad \text{and} \quad \varphi^N_{t,x^i} := \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t}^{N},
\end{align*}
\]

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where $T > 0$ is a fixed time horizon, $(B, W^1, \ldots, W^N)$ are independent Brownian motions where $B$ is called the common noise, and $\alpha^i$ is the control of player $i$. Here, the specificity is that the dynamics $X^i$ of player $i$ depends on the empirical distribution $\varphi^N$ of states and controls of all players. Given a strategy $(\alpha^1, \ldots, \alpha^N)$, the reward to the player $i$ is

$$J^N_i(\alpha^1, \ldots, \alpha^N) := \mathbb{E} \left[ \int_0^T L(t, X^i_t, \varphi^{N,X}_t, \varphi^i_t, \alpha^i_t) dt + g(X^i_T, \varphi^{N,X}_T) \right].$$

The aim of each agent is to maximize this reward through a Nash equilibrium criterion. The strategy $(\alpha^1, \ldots, \alpha^N)$ is an $\varepsilon$-Nash equilibrium if for any admissible control $\beta$,

$$J^N_i(\alpha^1, \ldots, \alpha^N) \geq J^N_i(\alpha^1, \ldots, \alpha^{i-1}, \alpha^i + \varepsilon, \alpha^{i+1}, \ldots, \alpha^N) - \varepsilon_N. \quad (1.1)$$

Following the intuition, if $N$ is large, because of the symmetry of the model, the contribution of player $i$ and its control over $\varphi^N$ is negligible, and everything happens as if $\varphi^N$ was fixed in the optimization (1.1). This line of argument leads to the derivation of the problem called in the literature the mean field games of controls or extended mean field games, which has, loosely speaking, the following structure (the precise definition is given in Section 2.2.1): a $(\sigma \{ B_s, s \leq t \})_{t \in [0,T]}$-adapted measure-valued process $(\overline{\varphi}_t^i)_{t \in [0,T]}$ is an $\varepsilon$-strong MFG solution (or approximate strong MFG) if for all $t \in [0,T]$, $\overline{\varphi}_t^i = \mathcal{L}(X^i_t, \alpha^i_t | B)$, where the state process $X^i$ is governed by

$$dX^i_t = b(t, X^i_t, (\mu^i_s)_{s \leq t}, \overline{\varphi}_t^i, \alpha^i_t) dt + \sigma(t, X^i_t, (\mu^i_s)_{s \leq t}, \overline{\varphi}_t^i, \alpha^i_t) dW^i_t + \sigma_0 dB_t, \quad t \in [0,T]$$

$$\mu^i_t := \mathcal{L}(X^i_t | B),$$

and one has

$$\left\{ \begin{array}{l} \mathbb{E} \left[ \int_0^T L(t, X^i_t, \mu^i_t, \overline{\varphi}_t^i, \alpha^i_t) dt + g(X^i_T, \mu^i) \right] \geq \sup_{\overline{\varphi}^i} \mathbb{E} \left[ \int_0^T L(t, X^i_t, \mu^i_t, \overline{\varphi}^i_t, \alpha^i_t) dt + g(X^i_T, \mu^i) \right] - \varepsilon, \\ \text{where the optimization is over the solutions } dX_t = b(t, X_t, (\mu^i_s)_{s \leq t}, \overline{\varphi}_t^i, \alpha^i_t) dt + \sigma(t, X_t, (\mu^i_s)_{s \leq t}, \overline{\varphi}_t^i, \alpha^i_t) dW^i_t + \sigma_0 dB_t, \quad t \in [0,T]. \end{array} \right. \quad (1.2)$$

This structure means that, when the process $(\overline{\varphi}_t^i)_{t \in [0,T]}$ is fixed, a single representative player solves an optimal control problem. The condition $\overline{\varphi}_t^i = \mathcal{L}(X^i_t, \alpha^i_t | B)$, called consistency condition or fixed point problem in the literature, gives to $(X^i, \alpha^i)$, the $\varepsilon$-optimal control, a representation property of the entire population. The process $(\overline{\varphi}_t^i)_{t \in [0,T]}$ can be seen as an equilibrium. This is exactly the classical MFG problem except for two aspects: first the solution is a (conditional) distribution of the state and the control $(\mathcal{L}(X^i_t, \alpha^i_t | B))_{t \in [0,T]}$ and not just a (conditional) distribution of the state $(\mathcal{L}(X^i_t | B))_{t \in [0,T]}$, next, $(X^i, \alpha^i)$ is an $\varepsilon$-optimal control and not necessary an optimal control (or 0-optimal control).

For MFG of controls or extended MFG, the literature on this topic focus primarily on the existence and uniqueness results of the limit problem (with $\varepsilon = 0$), usually without common noise i.e. $\sigma_0 = 0$, by using PDE methods, study these types of interactions in the deterministic case i.e. $\sigma = \sigma_0 = 0$. Strong assumptions of continuity and convexity make it possible to obtain the existence and the regularity of the solutions. In order to explore a problem of optimal liquidation in finance, Cardalinaut and Lehalle [6] apply similar PDE techniques for this problem in the case without common noise, while allowing $\sigma$ to be non-zero. With the same philosophy, Kobeissi [24] provides some results and discusses properties of existence and uniqueness in examples. Let us also mention Achdou and Kobeissi [1] which gives numerical approximations via finite difference for the PDE system arising in the MFG of controls. See also Gomes, Patrizi, and Voskanyan [18] and Bonnans, Hadikhanloo, and Pfeiffer [5] for other investigations of PDE techniques in the situation of MFG of controls.

Probability techniques have also been used to give some results for the limit problem. Without common noise, using a weak formulation of the MFG of controls, Carmona and Lacker [9] obtain the existence and uniqueness of the MFG of controls, and from this solution, construct an approximate Nash equilibrium, all this by imposing an uncontrolled and non-degenerate volatility $\sigma$ ($\sigma > 0$). They illustrate their results on the price impact models (which share some similarities with those considered in [6]) and the flocking model. Similarly, Graber [19], for the studies of models of production of an exhaustible resource, solves similar existence and uniqueness problems.

Except the recent work of Laurière and Tangpi [32] which, using the FBSDEs, treats the convergence of Nash equilibria in the framework of MFG of controls, to the best of our knowledge, there are no other papers using probabilistic or
PDE methods that answer the question of the convergence of \( \varepsilon_N \)–Nash equilibria to the MFG solution in this context. The assumptions of regularity of coefficients usually used in the literature are no longer verified in the presence of the distribution of control (see an example of this phenomena in \cite[Remark 2.4]{13}). Although using probabilistic point of view, the approach developed in this paper is very different from these previously mentioned, and considers very general assumptions. Despite many differences, this article is in the same spirit as \cite{26}, which is, in the framework without law of control and with open loop controls, the most significant paper investigating the connection between large population differential games and MFG under very general assumptions. We want to emphasize that the interesting techniques developed in \cite{26} do not work in the case of MFG of controls. As previously mentioned, because of the presence of the law of control, the assumptions of regularity on the coefficients are no longer verified.

In order to solve the difficulty generated by the empirical distribution of controls, we introduce the notion of \textit{measure–valued} MFG equilibrium. This notion is precisely defined in Section 2.2.2. The idea of our notion comes from the (stochastic) Fokker–Planck equation verified by the pair \((\mu^*, \pi^*)\). This notion of MFG solution is very close to the classical notion. The main difference is that the optimization is taken over all solutions of specific controlled Fokker–Planck equations and not to a solution of a controlled SDE. This notion has already been considered in the literature by Cardaliaguet, Delarue, Lasry, and Lions \cite[Section 3.7.]{7} and, in some way, by Lacker \cite{27}. Borrowing techniques from \cite{26}, under suitable assumptions, we prove that the sequence of empirical measure flows \((\varphi^N, \varphi^N)_{N \in \mathbb{N}^+}\) is tight in a suitable space, and with the help of techniques introduced in our companion paper \cite{13}, we show that every limit in distribution is a \textit{measure–valued} mean field equilibrium. And conversely, for each \textit{measure–valued} mean field equilibrium, we construct an approximate Nash equilibrium which has this \textit{measure–valued} mean field equilibrium as limit. In addition to these convergence results, this article provides an \( \varepsilon \)–strong existence and another approximation not taken into account until now. Similarly to the approximate Nash equilibrium, when \( \varepsilon \) is positive and goes to zero, the sequence \((\mu^*, \pi^*)\) is tight with any limit being a \textit{measure–valued} MFG equilibrium. Also, when there is common noise, any \textit{measure–valued} MFG equilibrium can be approached by a sequence of \( \varepsilon \)–strong MFG equilibrium \((\mu^*, \pi^*)\) verifying (1.2).

Consequently, there is a perfect symmetry between approximate Nash equilibria and \( \varepsilon \)–strong MFG equilibria. Besides, our notion of \textit{measure–valued} MFG equilibrium is the accumulation points of approximate Nash equilibria and \( \varepsilon \)–strong MFG equilibria. Therefore, if there exists a \textit{measure–valued} MFG equilibrium or an approximate Nash equilibrium, there is necessarily an \( \varepsilon \)–strong MFG equilibrium for any \( \varepsilon > 0 \). Without common noise, with similar arguments to Lacker \cite{25}, we show that there is a \textit{measure–valued} MFG equilibrium under general condition, as a result there is an \( \varepsilon \)–strong MFG equilibrium.

It is well known in the MFG theory that the existence of a strong MFG solution is very difficult to obtain and requires strong assumptions. Admitting a small error \( \varepsilon > 0 \), it is possible to get the existence of an \( \varepsilon \)–strong MFG equilibrium under general assumptions. It is worth emphasizing that our results allow to handle the case where \( \sigma \) is controlled i.e. the control \( \alpha \) appears in the function \( \sigma \). There are not many works that look at the situation where the volatility is controlled. Let us also mention, in this paper, despite general assumptions considered, we are limited by some conditions that we must have for technical reasons, a separability condition on \((b, \sigma, L)\) (see Assumption 2.1) and a non–degeneracy volatility condition of type \( \sigma \sigma^T > 0 \).

The rest of the paper is organized as follows. After introducing some notations, we provide respectively in Section 2.1 and Section 2.2, the definition of the \( N \)–player game and the corresponding MFG of controls. The main limit Theorem 2.12 and, its converse, Theorem 2.13 are then stated in Section 2.3. Later, in Section 2.4, we present some existence and approximation results in the particular case without common noise. Most of the technical proofs are completed in Section 3 and Section 4.

\textbf{Notations.} (i) Given a Polish space \((E, \Delta)\), \( p \geq 1 \), we denote by \( \mathcal{P}(E) \) the collection of all Borel probability measures on \( E \), and by \( \mathcal{P}_p(E) \) the subset of Borel probability measures \( \mu \) such that \( \int_E \Delta(e, e_0)^{p} \mu(de) < \infty \) for some \( e_0 \in E \). We equip \( \mathcal{P}_p(E) \) with the Wasserstein metric \( W_p \) defined by

\[
W_p(\mu, \mu') := \left( \inf_{\lambda \in \Lambda(\mu, \mu')} \int_{E \times E} \Delta(e, e')^p \lambda(de, de') \right)^{1/p},
\]

where \( \Lambda(\mu, \mu') \) denote the collection of all probability measures \( \lambda \) on \( E \times E \) such that \( \lambda(de, E) = \mu(de) \) and \( \lambda(E, de') = \mu'(de') \). Equipped with \( W_p \), \( \mathcal{P}_p(E) \) is a Polish space (see \cite[Theorem 6.18]{33}). For any \( \mu \in \mathcal{P}(E) \) and \( \mu \)–integrable
function \( \varphi : E \to \mathbb{R} \), we define
\[
\langle \varphi, \mu \rangle = \langle \mu, \varphi \rangle := \int_E \varphi(e) \mu(\mathrm{d}e),
\]
and for another metric space \((E', \Delta')\), we denote by \( \mu \otimes \mu' \in \mathcal{P}(E \times E') \) the product probability of any \((\mu, \mu') \in \mathcal{P}(E) \times \mathcal{P}(E')\). Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) supporting a sub-\(\sigma\)-algebra \(\mathcal{G} \subset \mathcal{F}\), then for a Polish space \(E\) and any random variable \(\xi : \Omega \to E\), both the notations \(L^\mathbb{P}(\xi|\mathcal{G})(\omega)\) and \(\mathbb{P}_{\mathbb{G}}(\xi)\) are used to denote the conditional distribution of \(\xi\) knowing \(\mathcal{G}\) under \(\mathbb{P}\).

(ii) For any \((E, \Delta)\) and \((E', \Delta')\) two Polish spaces, we use \(C_b(E, E')\) to denote the set of continuous functions \(f\) from \(E\) into \(E'\) such that \(\sup_{e \in E} \Delta'(f(e), e') < \infty\) for some \(e' \in E'\). Let \(\mathbb{N}^*\) denote the set of positive integers. Given non-negative integers \(m\) and \(n\), we denote by \(S^{m \times n}\) the collection of all \(m \times n\)-dimensional matrices with real entries, equipped with the standard Euclidean norm, which we denote by \(|\cdot|\) regardless of the dimensions. We also denote \(\mathbb{S}^n := \mathbb{S}^{n \times n}\), and by \(I_n\) the identity matrix in \(\mathbb{S}^n\). For any matrix \(a \in \mathbb{S}^n\) which is symmetric positive semi-definite, we write \(a^{1/2}\) the unique symmetric positive semi-definite square root of the matrix \(a\). Let \(k\) be a positive integer, we denote by \(C^k_b(\mathbb{R}^n; \mathbb{R})\) the set of bounded maps \(f : \mathbb{R}^n \to \mathbb{R}\), having bounded continuous derivatives of order up to and including \(k\). Let \(f : \mathbb{R}^n \to \mathbb{R}\) be twice differentiable, we denote by \(\nabla f\) and \(\nabla^2 f\) the gradient and the Hessian of \(f\) respectively.

(iii) Let \(T > 0\), and \((\Sigma, \rho)\) be a Polish space, we denote by \(C([0, T], \Sigma)\) the space of all continuous functions on \([0, T]\) taking values in \(\Sigma\). Then \(C([0, T], \Sigma)\) is a Polish space under the uniform convergence topology, and we denote by \(\|\cdot\|\) the uniform norm. When \(\Sigma = \mathbb{R}^k\) for some \(k \in \mathbb{N}\), we simply write \(C^k := C([0, T], \mathbb{R}^k)\), also we shall denote by \(C^k_W := C([0, T], \mathcal{P}(\mathbb{R}^k))\).

With a Polish space \(E\), we denote by \(M(E)\) the space of all Borel measures \(q(dt, de)\) on \([0, T] \times E\), whose marginal distribution on \([0, T]\) is the Lebesgue measure \(dt\), that is to say \(q(dt, de) = q(t, de) dt\) for a family \((q(t, de))_{t \in [0, T]}\) of Borel probability measures on \(E\). We will denote by \(\Lambda\) the canonical element of \(M(E)\) and we introduce
\[
\Lambda_{t \wedge t'}(ds, de) := \Lambda(ds, de)|_{[0, t] \times E} + \delta_{e_0}(de) ds|_{(t, t'] \times E},\quad \text{for some fixed } e_0 \in E.\tag{1.3}
\]
For \(p \geq 1\), we use \(M_p(E)\) to designate the elements of \(q \in M(E)\) such that \(q/T \in \mathcal{P}_p(E \times [0, T])\).

## 2 Mean field games of controls (with common noise): Setup and main results

In this section, we first introduce the \(N\)-player game, and the definition of \(\epsilon_N\)-Nash equilibria. Next, we formulate the notions of approximate strong and measure-valued MFG solutions which will be essential to describe the limit of the Nash equilibria.

The general assumptions used throughout this paper are now formulated. The dimensions \((n, \ell) \in \mathbb{N}^* \times \mathbb{N}\), the nonempty Polish space \((U, \rho)\) and the horizon time \(T > 0\) are fixed and \(\mathcal{P}^E_U\) denotes the space of all Borel probability measures on \(\mathbb{R}^n \times U\) i.e. \(\mathcal{P}^E_U := \mathcal{P}(\mathbb{R}^n \times U)\). Also, we set \(p \geq 2\), \(\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)\) with \(p' > p\), and the probability space \((\Omega, \mathcal{F}, \mathbb{P}) = \big(H_t\big|_{t \in [0, T]}, \mathcal{H}, \mathbb{P}\big)^1\).

We are given the following Borel measurable functions
\[
[b, \sigma, L] : [0, T] \times \mathbb{R}^n \times C_W^p \times \mathcal{P}^E_U \times U \to \mathbb{R}^n \times \mathbb{S}^{n \times n} \times \mathbb{R} \quad \text{and} \quad g : \mathbb{R}^n \times C_W^p \to \mathbb{R}.
\]

Assumption 2.1. \([b, \sigma, L]\) are Borel measurable in all their variables, and non-anticipative in the sense that, for all \((t, x, u, \pi, m) \in [0, T] \times \mathbb{R}^n \times U \times C_W^p \times \mathcal{P}^E_U\)
\[
[b, \sigma, L](t, x, \pi, m, u) = [b, \sigma, L](t, x, \pi_{t \wedge t'}, m, u).
\]

Moreover, there is positive constant \(C\) such that
(i) \(U\) is a compact nonempty polish set;

\(^{1}\)The probability space \((\Omega, \mathcal{F}, \mathbb{P})\) contains as many random variables as we want in the sense that: each time we need a sequence of independent uniform random variables or Brownian motions, we can find them on \(\Omega\) without mentioning an enlarging of the space.
(ii) $b$ and $\sigma$ are bounded continuous functions, and $\sigma_0 \in \mathbb{R}^{n \times t}$ is a constant;

(iii) for all $(t, x, x', \pi, \pi', m, m', u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times C^a_W \times C^a_W \times \mathcal{P}_U \times \mathcal{P}_U \times U$, one has

$$\left| [b, \sigma](t, x, \pi, m, u) - [b, \sigma](t, x', \pi', m', u) \right| \leq C \left( |x - x'| + \sup_{s \in [0, T]} W_p(\pi_s, \pi_s') + W_p(m, m') \right);$$

(iv) **Non–degeneracy condition:** for some constant $\theta > 0$, one has, for all $(t, x, \pi, m, u) \in [0, T] \times \mathbb{R}^n \times C^a_W \times \mathcal{P}_U \times U$,

$$\theta I_n \leq \sigma \sigma^T(t, x, \pi, m, u);$$

(v) the reward functions $L$ and $g$ are continuous, and for all $(t, x, \pi, m, u) \in [0, T] \times \mathbb{R}^n \times C^a_W \times \mathcal{P}_U \times U$, one has

$$|L(t, x, \pi, m, u)| + |g(x, \pi)| \leq C \left[ 1 + |x|^p + \sup_{s \in [0, T]} W_p(\pi_s, \delta_0)^p + \int_{\mathbb{R}^n} |x'|^p m(dx', U) \right];$$

(vi) **Separability condition:** There exist continuous functions $(b^i, b^w, a^i, a^w, L^i, L^w)$ satisfying

$$[b, \sigma \sigma^T](t, x, \pi, m, u) := [b^i, \sigma^i](t, \pi, m) + [b^w, \sigma^w](t, x, \pi, u) \quad \text{and} \quad L(t, x, \pi, m, u) := L^i(t, x, \pi, m) + L^w(t, x, \pi, u),$$

for all $(t, x, \pi, m, u) \in [0, T] \times \mathbb{R}^n \times C^a_W \times \mathcal{P}_U \times U$.

**Remark 2.2.** Most of these assumptions are classical in the study of mean field games and control problems (see Lacker [26], Djete, Possamaï, and Tan [15] and Djete [13]). Only the "separability condition" and the "non–degeneracy condition" can be seen as non–standard. However, in the context of mean field games of controls, these conditions are used by many authors, for instance Cardaliaguet and Lehalle [6](only separability condition), Carmona and Lacker [9] and Laurière and Tangpi [32]. These are essentially technical assumptions.

### 2.1 The $N$-player game

On the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, let $(W^i)_{i \in \mathbb{N}}$ be a sequence of independent $\mathbb{R}^n$–valued $\mathbb{H}$–Brownian motions, $B$ be a $\mathbb{R}^l$–valued $\mathbb{H}$–Brownian motion and $(\xi^i)_{i \in \mathbb{N}}$ be a sequence of iid $\mathbb{R}^n$–valued $\mathbb{H}_0$–random variables of law $\nu$. Besides $(W^i)_{i \in \mathbb{N}}$, $B$ and $(\xi^i)_{i \in \mathbb{N}}$ are independent. Let $\mathbb{F}^N = (\mathcal{F}^N_t)_{0 \leq t \leq T}$ be defined by the $\mathbb{P}$-completion of $\tilde{\mathbb{F}}^N := (\tilde{\mathbb{F}}^N_t)_{t \in [0, T]}$ where

$$\tilde{\mathbb{F}}^N_s := \sigma \{ \xi^i, W^i_r, B_r, r \in [0, s], 1 \leq i \leq N \}, 0 \leq s \leq T.$$

Let us denote by $A^N$ the collection of all $U$–valued processes $\alpha := (\alpha^i)_{0 \leq s \leq T}$ which are $\mathbb{F}^N$–predictable. Then given a control rule/strategy $\overline{\alpha} := (\alpha^1, \ldots, \alpha^N) \in A^N$, denote by $X[\overline{\alpha}] := (X^1[\overline{\alpha}], \ldots, X^N[\overline{\alpha}])$ the unique strong solution of the following system of SDEs (the well–posedness is assured by Assumption 2.1):

$$dX^i[\overline{\alpha}] = b(t, X^i[\overline{\alpha}], \varphi^N_{X^i[\overline{\alpha}], \varphi^N_{X^i[\overline{\alpha}], \alpha^i}})dt + \sigma(t, X^i[\overline{\alpha}], \varphi^N_{X^i[\overline{\alpha}], \varphi^N_{X^i[\overline{\alpha}], \alpha^i}})dW^i_t + \sigma_0 dB_t$$

with $X^0 = \xi^i$ (2.1)

where

$$\varphi^N_{t, \varphi^N_{X^i[\overline{\alpha}], \alpha^i}}(dx, du) := \frac{1}{N} \sum_{i=1}^N \delta(x, \alpha^i)(dx, du) \quad \text{and} \quad \varphi^N_{t, \varphi^N_{X^i[\overline{\alpha}], \alpha^i}}(dx) := \frac{1}{N} \sum_{i=1}^N \delta x_i[\overline{\alpha}](dx) \quad \text{for all} \quad t \in [0, T].$$

The reward value of player $i$ associated with control rule/strategy $\overline{\alpha} := (\alpha^1, \ldots, \alpha^N)$ is then defined by

$$J_i[\overline{\alpha}] := \mathbb{E} \left[ \int_0^T L(t, X^i_t[\overline{\alpha}], \varphi^N_{X^i_t[\overline{\alpha}], \varphi^N_{X^i_t[\overline{\alpha}], \alpha^i}})dt + g(X^i_T[\overline{\alpha}], \varphi^N_{X^i[\overline{\alpha}], \alpha^i}) \right],$$

and for $\beta \in A^N$, one introduces the strategy $(\overline{\alpha}^{i-1}, \beta) \in A^N$ by

$$(\overline{\alpha}^{i-1}, \beta) := (\alpha^1, \ldots, \alpha^{i-1}, \beta, \alpha^{i+1}, \ldots, \alpha^N).$$

**Definition 2.3.** For any $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_N) \in (\mathbb{R}^+)^N$, $\overline{\alpha}$ is an $\varepsilon$–(open loop) Nash equilibrium if

$$J_i[\overline{\alpha}] \geq \sup_{\beta \in A^N} J_i((\overline{\alpha}^{i-1}, \beta)) - \varepsilon_i, \quad \text{for each} \quad i \in \{1, \ldots, N\},$$

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2.2 Mean field games of controls

2.2.1 $\varepsilon$–strong mean field game equilibrium

We now formulate the classical MFG problem with common noise including the (conditional) law of control. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(W, B)$ be $\mathbb{R}^n \times \mathbb{R}^{d'}$–valued $\mathcal{F}$–Brownian motion and $\xi$ be a $\mathbb{R}^m$–valued $\mathcal{F}_0$–random variables of law $\nu$. Let $\mathbb{F} = (\mathbb{F}_t)_{0 \leq s \leq T}$ and $\mathbb{G} = (\mathbb{G}_t)_{0 \leq s \leq T}$ be defined by the $\mathbb{P}$–completion of $\mathbb{F} := (\mathbb{F}_t)_{t \in [0, T]}$ and $\mathbb{G} := (\mathbb{G}_t)_{t \in [0, T]}$ where

$$\mathbb{F}_s := \sigma\{\xi, W_r, B_r, r \in [0, s]\} \quad \text{and} \quad \mathbb{G}_s := \sigma\{B_r, r \in [0, s]\}.$$ 

Let us denote by $\mathcal{A}$ the collection of all $U$–valued $\mathbb{F}$–predictable processes. Then, given $\alpha \in \mathcal{A}$, let $X^\alpha$ be the unique strong solution of the SDE (e.g. [14, Theorem A.3]): $E\|X^\alpha\|^p < \infty$, and

$$dX^\alpha_t = b(t, X^\alpha_t, \mu^\alpha_t, \bar{\mu}^\alpha_t, \alpha_t)dt + \sigma(t, X^\alpha_t, \mu^\alpha_t, \bar{\mu}^\alpha_t, \alpha_t)dW_t + \sigma_0dB_t \text{ with } X^\alpha_0 = \xi$$

(2.2)

where $\bar{\mu}^\alpha_t := L(X^\alpha_t, \alpha_t, \mathcal{G}_t)$ and $\mu^\alpha_t := L(X^\alpha_t, \mathcal{G}_t)$ for all $t \in [0, T]$. Given $\alpha \in \mathcal{A}$, and $X^\alpha$ solution of (2.2), for every $\alpha' \in \mathcal{A}$, let us introduce the unique strong solution $X^{\alpha, \alpha'}$ of: $E\|X^{\alpha, \alpha'}\|^p < \infty$, and

$$dX^{\alpha, \alpha'}_t = b(t, X^{\alpha, \alpha'}_t, \mu^\alpha_t, \bar{\mu}^\alpha_t, \alpha'_t)dt + \sigma(t, X^{\alpha, \alpha'}_t, \mu^\alpha_t, \bar{\mu}^\alpha_t, \alpha'_t)dW_t + \sigma_0dB_t \text{ with } X^{\alpha, \alpha'}_0 = \xi$$

(2.3)

and the reward function $\Psi$

$$\Psi(\alpha, \alpha') := E\left[\int_0^T L(t, X^{\alpha, \alpha'}_t, \mu^\alpha_t, \bar{\mu}^\alpha_t, \alpha'_t)dt + g(X^{\alpha, \alpha'}_T, \mu^\alpha_T)\right].$$

(2.4)

Definition 2.4. For any $\varepsilon \in [0, \infty)$, we say that $\alpha$ is an $\varepsilon$–strong MFG equilibrium, if

$$\Psi(\alpha, \alpha') \geq \sup_{\alpha' \in \mathcal{A}} \Psi(\alpha, \alpha') - \varepsilon.$$ 

(2.5)

For all $(\alpha, \alpha') \in \mathcal{A} \times \mathcal{A}$, let us define

$$P^{\alpha, \alpha'} := P \circ \left(\mu^{\alpha, \alpha'}, \mu^\alpha, \delta_{\bar{\mu}^{\alpha', \alpha}}(dm)dt, \delta_{\bar{\mu}^{\alpha'}}(dm)dt, B\right)^{-1}$$

where $\bar{\mu}^{\alpha, \alpha'} := L(X^{\alpha, \alpha'}_t, \alpha'_t, \mathcal{G}_t)$ and $\mu^{\alpha, \alpha'} := L(X^{\alpha, \alpha'}_t, \mathcal{G}_t)$ with $t \in [0, T]$, and

$$P^\alpha := P \circ \left(\mu^\alpha, \mu^\alpha, \delta_{\bar{\mu}^\alpha}(dm)dt, \delta_{\bar{\mu}^\alpha}(dm)dt, B\right)^{-1}.$$ 

(2.6)

$\mathcal{P}_S$ and for every $\varepsilon \in [0, \infty)$, $\mathcal{P}_S[\varepsilon]$ denote the subsets of $\mathcal{P}(C^{\alpha}_{P_0} \times C^{\alpha}_{W} \times M(P_0) \times M(P_0) \times C^d)$ defined as follows

$$\mathcal{P}_S := \{P^{\alpha, \alpha'}, \text{ with } (\alpha, \alpha') \in \mathcal{A} \times \mathcal{A}\} \quad \text{and} \quad \mathcal{P}_S[\varepsilon] := \{P^\alpha, \text{ with } \alpha \text{ is an } \varepsilon\text–strong MFG equilibrium\}.$$ 

In other words, $\mathcal{P}_S$ is the subset of all distributions of controlled McKean–Vlasov processes of type (2.2), and $\mathcal{P}_S[\varepsilon]$ is all $\varepsilon$–strong MFG equilibrium. In what follows, the use of these forms of sets will become clearer.

2.2.2 Measure–valued MFG equilibrium

Notice that, for each $(\alpha, \alpha') \in \mathcal{A} \times \mathcal{A}$, the couple $(\mu^{\alpha, \alpha'}, \bar{\mu}^{\alpha, \alpha'})$ satisfies the Fokker–Planck equation: for each $f \in C^2_b(\mathbb{R}^n)$,

$$d(f(-\sigma_0B_t), \mu^{\alpha, \alpha'}_t) = (\nabla f(-\sigma_0B_t)^\top b(t, \cdot, \mu^\alpha_t, \bar{\mu}^\alpha_t, \cdot), \bar{\mu}^{\alpha, \alpha'}_t)dt + \frac{1}{2}(\text{Tr}[\sigma\sigma^\top(t, \cdot, \mu^\alpha_t, \bar{\mu}^\alpha_t, \cdot)]\nabla^2 f(-\sigma_0B_t), \bar{\mu}^{\alpha, \alpha'}_t)dt \ a.s.$$ 

Inspired by the Fokker–Planck equation satisfied by the couple $(\mu^{\alpha, \alpha'}, \bar{\mu}^{\alpha, \alpha'})$ (see also the discussion in [13]), we carefully formulate the notion of measure–valued control rules which is essential for the notion of measure–valued MFG equilibrium that will be introduced just after.
We then introduce the canonical filtration
\[ \mathcal{F}^A = (\mathcal{F}^A_t)_{0 \leq t \leq T} \] on \( M \) by
\[ \mathcal{F}^A_t := \sigma \{ \Lambda(C \times [0,s]) : \forall s \leq t, C \in \mathcal{B}(\mathcal{P}_U^n) \}. \]
For each \( q \in M \), one has the disintegration property: \( q(dt,de) = q(t,de)dt \), and there is a version of the disintegration such that \((t,q) \mapsto q(t,de)\) is \( \mathcal{F}^A \)-predictable.

The canonical element on \( \Omega := C_W^n \times C_W^n \times M \times M \times C^\ell \) is denoted by \((\mu', \mu, \Lambda', \Lambda, B)\). Then, the canonical filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]} \) is defined by: for all \( t \in [0,T] \)
\[ \mathcal{F}_t := \sigma \{ \mu'_t, \mu_t, \Lambda'_t, \Lambda_t, B_t \}, \]
with \( \Lambda'_t \) and \( \Lambda_t \) denote the restriction of \( \Lambda' \) and \( \Lambda \) on \([0,t] \times \mathcal{P}_U^n\) (see definition 1.3). Notice that we can choose a version of the disintegration \( \Lambda'(dm,dt) = \Lambda'_t(dm)dt \) (resp \( \Lambda(dm,dt) = \Lambda_t(dm)dt \)) such that \((\Lambda'_t)_{t \in [0,T]} \) (resp \((\Lambda_t)_{t \in [0,T]}\)) a \( \mathcal{P}(\mathcal{P}_U^n) \)-valued \( \mathcal{F} \)-predictable process. Let us also introduce the “fixed common noise” filtration \((\mathcal{G}_t)_{t \in [0,T]} \) by
\[ \mathcal{G}_t := \sigma \{ \mu_t, \Lambda_t, B_t \}. \]

We consider \( L \) the following generator: for \((t, x, \pi, m, u) \in [0,T] \times \mathbb{R}^n \times \mathcal{C}_W^n \times \mathcal{P}_U^n \times U \), and \( \varphi \in C^2(\mathbb{R}^n) \)
\[ L_t \varphi(x, \pi, m, u) := L^T_t \varphi(x, \pi, u) + L^r_t \varphi(x, \pi, m) \] (2.7)
where
\[ L^T_t \varphi(x, \pi, u) := \frac{1}{2} \text{Tr} [a^2(t, x, \pi_t, u) \nabla^2 \varphi(x)] + b^0(t, x, \pi_t, u)^\top \nabla \varphi(x), \] (2.8)
and
\[ L^r_t \varphi(x, \pi, m) := \frac{1}{2} \text{Tr} [a^r(t, \pi_t, m) \nabla^2 \varphi(x)] + b^*(t, \pi_t, m)^\top \nabla \varphi(x). \] (2.9)

Also, for every \( f \in C^2(\mathbb{R}^n) \), let us define \( N_t(f) := N_t[\mu', \mu, \Lambda', \Lambda](f) \) by
\[ N_t[\mu', \mu, \Lambda', \Lambda](f) := \langle f(\cdot - \sigma_0 B_t), \mu'_t \rangle - \langle f, \mu_0 \rangle - \int_0^t \int_{\mathcal{P}_U^n} \int_{\mathbb{R}^n} L^r_t[f(\cdot - \sigma_0 B_t)](x, \mu, m) \mu'_t(dx) \Lambda_t(dm) dr \]
\[ - \int_0^t \int_{\mathcal{P}_U^n} (L^T_t[f(\cdot - \sigma_0 B_t)](\cdot, \cdot, \cdot, m) \Lambda'_t(dm) dr, \] (2.10)
and for each \( \pi \in \mathcal{P}(\mathbb{R}^n) \), the Borel set \( Z_\pi \) by
\[ Z_\pi := \{ m \in \mathcal{P}_U^n : m(dx, U) = \pi(dx) \}. \]

**Definition 2.5** (measure–valued control rule). *We say that \( P \in \mathcal{P}(\Omega) \) is a measure–valued control rule if:

1. \( P(\mu'_0 = \nu) = 1 \).
2. \((B_t)_{t \in [0,T]} \) is a \((P, \mathcal{F})\) Wiener process starting at zero and for \( P \)-almost every \( \omega \in \Omega \), \( N_t(f) = 0 \) for all \( f \in C^2_W(\mathbb{R}^n) \) and every \( t \in [0,T] \).
3. \((\Lambda'_t)_{t \in [0,T]} \) is a \( \mathcal{G} \)-predictable process.
4. For \( dP \otimes dt \) almost every \((t, \omega) \in [0,T] \times \Omega \), \( \Lambda'_t(Z_{\mu'_t}) = 1 \).

We shall denote \( \mathcal{P}_V \) the set of all measure–valued control rules.
Remark 2.6. (i) To do an analogy with Section 2.2.1 (the strong “point of view”), in order to give a better intuition of this definition, here, $\mu'$ plays the role of $(\mathcal{L}(X_t^{\alpha,\alpha'}|\mathcal{F}_t))_{t \in [0,T]}$, $\Lambda'$ that of $\delta_{\mathcal{L}(X_t^{\alpha,\alpha'}|\mathcal{G}_t)}(\mathrm{d}m)$, $\mu$ and $\Lambda$ represent the fixed measures $\mu^\omega$ and $\delta_{\mathcal{P}_\omega}(\mathrm{d}m)$, and $B$ is the common noise.

(ii) Notice that the canonical space $\overline{\Omega} := \mathcal{C}_W^\infty \times \mathcal{C}_V^\infty \times \mathcal{M} \times \mathcal{M} \times \mathcal{C}$ is “doubled” because we need to consider the fixed processes $(\mu, \Lambda)$ which will be the optimum and the controlled processes $(\mu', \Lambda')$. All these processes share the same spaces. Also, because of the condition 3 of Definition 2.5, the set $\mathcal{P}_V$ cannot be closed in general. As $\mathcal{P}_V$ is not closed, the proofs become much more delicate (see for instance Proposition 3.14).

Now, using the measure–valued control rules, we introduce the notion of $(\varepsilon–)$ measure–valued MFG solution.

2.2.2.2 MFG solution For all $(\pi', \pi, q', q) \in (\mathcal{C}_W^n)^2 \times \mathcal{M}(U)^2$, one defines

$$J(\pi', \pi, q', q) := \int_0^T \left[ \int_{\mathcal{P}_V^\infty} \langle L^*(t, \cdot, \cdot, m)q'_t(\mathrm{d}m) + \int_{\mathcal{P}_V^\infty} \langle L^*(t, \cdot, \cdot, \pi'_t)q'_t(\mathrm{d}m) \rangle \mathrm{d}t + \langle g(\cdot, \pi, \pi'_T) \rangle \right].$$

Definition 2.7. For $\varepsilon \in [0, \infty)$, $\mathcal{P}^*$ is an $\varepsilon$–measure–valued MFG solution if $\mathcal{P}^* \in \mathcal{P}_V$, and for every $\mathcal{P} \in \mathcal{P}_V$ such that $\mathcal{L}^\infty (\mu, \Lambda, B) = \mathcal{L}^\infty (\mu, \Lambda, B)$, one has

$$\mathbb{E}^\mathcal{P}^* [J(\mu', \mu, \Lambda', \Lambda)] \geq \mathbb{E}^\mathcal{P} [J(\mu', \mu, \Lambda', \Lambda)] - \varepsilon,$$

and for $\mathcal{P}^*$ almost every $\omega \in \overline{\Omega}$,

$$\Lambda' = \Lambda \quad \text{and} \quad \mu' = \mu. \quad (2.12)$$

When $\varepsilon = 0$, we just say that $\mathcal{P}^*$ is a measure–valued MFG solution.

The space $\mathcal{P}_V[\varepsilon]$ is defined by

$$\mathcal{P}_V[\varepsilon] := \{ \text{All $\mathcal{P}$ $\varepsilon$–measure–valued MFG solutions} \},$$

again when $\varepsilon = 0$, we shall denote $\mathcal{P}_V[0]$ by $\mathcal{P}_V^*.

Remark 2.8. Condition (2.12) is the analog of the well–known consistency property in the MFG framework. Without taking into account the law of control, one of the main differences of this notion of MFG solutions is the optimality conditions (2.11) and (2.5). Here, sometimes a small error $\varepsilon$ is authorized. With this condition, the MFG solutions turn out to be more flexible (see the main results in Section 2.3).

Comparison Definition 2.7 and [10, Definition 3.1] Looking at this kind of measure–valued solution is largely inspired by the notion considered in [13] in the McKean–Vlasov setting. However, our notion of $(\varepsilon–)$ measure–valued MFG solution enters completely in the framework of MFG solutions considered in Carmona, Delarue, and Lacker [10]. Indeed, in the situation where our coefficients satisfied the assumptions of the setting of [10] (essentially without the law of the control and with no control in the volatility), any weak MFG solution of [10] can be seen as a measure–valued MFG solution, and conversely any measure–valued MFG solution can be seen as a weak MFG solution. Let us be more precise about this equivalence. We begin by recalling the notion of weak MFG solution of [10].

Definition 2.9. ([10, Definition 3.1]) The tuple $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}}, \tilde{W}, \tilde{B}, \mu, \tilde{\Lambda}, \tilde{X})$ is a weak MFG solution if

1. $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}}, \tilde{W})$ is a complete probability space. Also, $(\tilde{W}, \tilde{B})$ is a $\mathbb{R}^{d+\ell}$–valued $\tilde{\mathcal{F}}$–Brownian motion, $\tilde{X}$ is a $\mathbb{R}^n$–valued $\tilde{\mathcal{F}}$–adapted continuous process with $\tilde{\mathcal{P}} \circ (\tilde{X}_0)^{-1} = \nu$, and $\tilde{\Lambda}$ is a $\mathcal{P}(U)$–valued $\tilde{\mathcal{F}}$–predictable measurable process. Lastly, $\mu$ is a $\mathcal{P}(\mathcal{C}^n \times \mathcal{M}(U) \times \mathcal{C})$–valued random variable such that $\mu(S)$ is $\tilde{\mathcal{F}}$–measurable whenever $S \in \tilde{\mathcal{M}}_t$ and $t \in [0, T]$ where $\tilde{\mathcal{M}}_t := (\tilde{\mathcal{M}}_t)_{t \in [0,T]}$ is the filtration s.t. $\tilde{\mathcal{M}}_t$ is the $\sigma$–field generated by the maps $\mathcal{C}^n \times \mathcal{M}(U) \times \mathcal{C} \ni (w, q, x) \rightarrow (w_s, q(S), x_s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, where $s \leq t$ and $S$ is a Borel subset of $[0, t] \times A$.

2. $\tilde{X}_0$, $\tilde{W}$ and $\tilde{B}, \mu$ are independent
3. $\tilde{\Lambda}_{t\wedge \cdot}$ is conditionally independent of $\tilde{F}_T \circ \tilde{X}_0, \tilde{W}, \mu, \tilde{B}$ given $\tilde{F}_t \circ \tilde{X}_0, \tilde{W}, \mu, \tilde{B}$, for each $t \in [0, T]$, where $\tilde{F}_t \circ \tilde{X}_0, \tilde{W}, \mu, \tilde{B} := \sigma\{\tilde{X}_0, W_{t\wedge \cdot}, B_{t\wedge \cdot}, \mu(S) : S \in \mathcal{M}_t\}$

4. The state equation holds

$$d\tilde{X}_t = \int_U b(t, \tilde{X}_t, \mu_t^x, u)\tilde{X}_t(du)dt + \sigma(t, \tilde{X}_t, \mu_t^x)d\tilde{W}_t + \sigma_0d\tilde{B}_t$$

where $\mu_t^x := \mu \circ [(w, q, x) \mapsto x]^{-1}$

5. The control $\tilde{\Lambda}$ is optimal, in the sense that: if $(\tilde{\Omega}, \tilde{F}, \tilde{F}', \tilde{F}^*, \tilde{W}', \tilde{B}', \mu', \tilde{X}')$ satisfies (1-4) and $\tilde{P}^\circ(\tilde{X}_0', \tilde{W}', \mu', \tilde{B}')^{-1} = \tilde{P}^\circ (\tilde{X}_0, \tilde{W}, \mu, \tilde{B})^{-1}$, then we have

$$\mathbb{E}^{\tilde{P}^*}\left[\int_0^T \int_U L(t, \tilde{X}_t', \mu_t^x, u)\tilde{X}_t(du)dt + g(\tilde{X}_T, \mu_T^x)\right] \leq \mathbb{E}^{\tilde{P}^*}\left[\int_0^T \int_U L(t, \tilde{X}_t, \mu_t^x, u)\tilde{X}_t(du)dt + g(\tilde{X}_T, \mu_T^x)\right].$$

6. The consistency condition holds: $\mu = \tilde{P}((\tilde{W}, \tilde{\Lambda}, \tilde{X}) \in \tilde{B}, \mu)$ a.s.

With this previous definition in mind, for any weak solution $(\tilde{\Omega}, \tilde{F}, \tilde{F}', \tilde{F}^*, \tilde{W}, \tilde{B}, \mu, \tilde{\Lambda}, \tilde{X})$, we have that (see Appendix A.1 for the details):

$$\tilde{P} := \tilde{P} \circ (\mu_t^x)_{t \in [0, T], \Pi, (\mu_t^x)_{t \in [0, T], \Pi, B})^{-1}$$

is a measure--valued solution where $\Pi := \delta_{m_t}(dm)dt$ with $m_t := \mathbb{E}^{\tilde{P}}[\delta_{\tilde{X}_t}(dx)\Lambda_t(du)]$. Conversely, let $P^*$ be a measure--valued MFG solution, we can construct an extension $(\Pi, \tilde{F}, \tilde{F}', \tilde{F}^*)$ of the probability space $(\Pi, \tilde{F}, \tilde{F}', \tilde{F}^*)$ supporting processes $(\tilde{W}, \mu, \tilde{\Lambda}, \tilde{X})$ s.t. the tuple $(\Pi, \tilde{F}, \tilde{F}', \tilde{F}^*, \tilde{W}, \mu, \tilde{\Lambda}, \tilde{X})$ is a weak MFG solution (see Appendix A.2 for details).

Remark 2.10. Notice that the previous definitions of the strong MFG equilibrium and $N$--player game cover the case without common noise. Indeed, for the non common noise case, it is enough to take $\ell = 0$ i.e. $B$ and $\sigma_0$ disappear (see [14], [15] and [13]). When $\sigma_0 = 0$ and $\ell \neq 0$, $B$ can be seen as an additional noise.

The next proposition ensures that our measure--valued MFG solution definition using Fokker–Planck equation indeed generalizes the classical notion. The proof is postponed in Section 3.1.3

Proposition 2.11. $\mathcal{P}_S[\varepsilon] \subset \mathcal{P}_S^\varepsilon[\varepsilon]$, for all $\varepsilon \in [0, \infty)$.

2.3 Main limit results

The main results of this paper are now given in the following two theorems.

Theorem 2.12 (Limit Theorem). Let Assumption 2.1 hold true, $\varepsilon \in [0, \infty)$, $(\varepsilon_i)_{i \in \mathbb{N}^*} \subset (0, \infty)$.

(i) For each $N \in \mathbb{N}^*$, let $\tilde{\alpha}^N$ be a $(\varepsilon_1, \ldots, \varepsilon_N)$--Nash equilibrium, then the sequence $(P^N)_{N \in \mathbb{N}^*}$ with $P^N := P^N[\tilde{\alpha}^N] \in \mathcal{P}(\Pi)$ is relatively compact in $\mathcal{W}_p(\Pi)$ where

$$P^N(\tilde{\alpha}^N) := \tilde{P} \circ (\varphi^N, X, \tilde{\alpha}^N, \varphi^N, X, \tilde{\alpha}^N, \Lambda^N, \Lambda^N, B)^{-1}$$

with $\Lambda^N := \delta_{\varphi^N} (dm)dt$.

and

if $\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \varepsilon_i = \varepsilon$, then each limit point $P^\varepsilon$ is an $\varepsilon$--measure--valued MFG solution.

(ii) Let $(P^k)_{k \in \mathbb{N}^*} \subset \mathcal{P}_S$ such that $P^k \in \mathcal{P}_S[\varepsilon_k]$, for each $k \in \mathbb{N}^*$. Then $(P^k)_{k \in \mathbb{N}^*}$ is relatively compact in $\mathcal{W}_p(\Pi)$, and

if $\lim_{k \to \infty} \varepsilon_k = \varepsilon$, then each limit point $P^\varepsilon$ is an $\varepsilon$--measure--valued MFG solution.

In particular when $\varepsilon = 0$, $P^\infty$ is a measure--valued MFG solution.
Theorem 2.13 (Converse Limit Theorem). Let Assumption 2.1 hold true, \( \varepsilon \in [0, \infty) \), and \( P^* \in \overline{P}_V[\varepsilon] \).

(i) There exists a sequence \((\varepsilon_k)_{k \in \mathbb{N}^*} \subset [0, \infty)\) satisfying \( \limsup_{k \to \infty} \varepsilon_k \in [0, \varepsilon] \) such that:

(i.1) if \( \ell \neq 0 \), one can find a sequence \((P^k)_{k \in \mathbb{N}^*}\) with \( P^k \in \overline{P}_S[\varepsilon_k] \) for each \( k \in \mathbb{N}^* \), and \( P^* = \lim_{k \to \infty} P^k \), for the metric \( \mathcal{W}_p \).

(i.2) if \( \ell = 0 \) i.e. there are no \( B \) and \( \sigma_0 \) (no common noise), one can get a sequence \((P^k_z)_{(k, z) \in \mathbb{N}^* \times [0, 1]} \subset \overline{P}_S\) with for each \( k \in \mathbb{N}^* \), \( z \mapsto P^k_z \) is Borel measurable,

\[
\int_0^1 P^k_z \, dz \in \overline{P}_V[\varepsilon_k] \quad \text{and} \quad \lim_{k \to \infty} \int_0^1 P^k_z \, dz = P^* \in \mathcal{W}_p.
\]

(ii) There exists a sequence of positive numbers \((\varepsilon_i)_{i \in \mathbb{N}^*} \) such that \( \limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \varepsilon_i \in [0, \varepsilon] \), and for each \( N \in \mathbb{N}^* \), a \((\varepsilon_1, \ldots, \varepsilon_N)\)-Nash equilibrium \( \overline{\pi}^N = (\alpha^1, \ldots, \alpha^N) \) such that

\[
P^* = \lim_{N \to \infty} P_0 \left( \varphi^{N, \pi, x}, \varphi^{N, \pi, y}, \Lambda^N, \Lambda^N, B \right)^{-1} \in \mathcal{W}_p \text{ with } \Lambda^N := \delta_{\varphi^{N, \pi} \pi^N} (dm)dt.
\]

Remark 2.14. Theorems 2.13 and 2.12 give a general characterization of solutions of MFG of controls by connecting measure–valued MFG solutions, approximate Nash equilibria and approximate strong MFG solutions. Consequently, the results of existence of measure–valued MFG solutions, of approximate Nash equilibria and of approximate strong MFG solutions are all related, the existence of one of the notions guarantees the existence of the others. In the presence of the law of the control or the empirical distribution of the controls, our limit theorem results seem to be the first which give this kind of characterizations under relative general assumptions. Especially, approximate strong MFG solutions and their convergence result have never been considered in the literature. Notice that, they also contain part of the most results of the case without the distribution of controls mentioned in Lacker [26]. Let us emphasize that, there is no existence result in these theorems, all results are given after assuming existence results. In Section 2.4 (see below), we discuss some existence results in the case without common noise.

The next corollaries are just a combination of Theorems 2.13 and 2.12. The first mentions the closedness of \( \overline{P}_V[\varepsilon] \) and the second a correspondence between approximate Nash equilibria and approximate–strong MFG solution.

Corollary 2.15. Suppose that the conditions of Theorem 2.13 and Theorem 2.12 hold. For each \( \varepsilon \in [0, \infty) \), \( \overline{P}_V[\varepsilon] \) is a closed set for the Wasserstein metric \( \mathcal{W}_p \).

Corollary 2.16. Let us stay in the context of Theorem 2.13 and Theorem 2.12 with \( \ell \neq 0 \). For any \( \pi^N \) a \((\varepsilon_1, \ldots, \varepsilon_N)\)-Nash equilibrium, with \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \varepsilon_i = 0 \), there exists, for each convergent sub-sequence \((P^{N_k}[\pi^{N_k}])_{k \in \mathbb{N}^*}\), a sequence \((P^k)_{k \in \mathbb{N}^*}\) such that:

\[
\text{for each } k \in \mathbb{N}^*, \quad P^k \in \overline{P}_S[\delta_k] \text{ with } \lim_{k \to \infty} \delta_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathcal{W}_p(P^{N_k}[\pi^{N_k}], P^k) = 0.
\]

2.4 Particular case of no common noise

This section discusses the case without common noise. Here, we assume that \( \sigma_0 = 0 \) (or \( \ell = 0 \)). Let us introduce the notion of non–random measure–valued MFG solution.

Definition 2.17. We say that \( P^* \) is a non–random measure–valued MFG solution if \( P^* \in \overline{P}_V \) and there exists \((n, q) \in C^0 \times \mathcal{M}(\overline{P}_V)\) such that

\[
\Lambda_t(dm)dt = q_t(dm)dt \quad \text{and} \quad \mu = n \quad P^*-a.s.
\]

In other words, this notion of solution in the absence of common noise i.e. \( \sigma_0 = 0 \) or \( \ell = 0 \) focuses on the deterministic solution of the Fokker–Planck equation mentioned in Definition 2.5. Indeed, even without common noise, it is possible to get a “random” measure–valued MFG solution. With the help of this deterministic aspect, one has the next theorem (see proof in Section 4).
Theorem 2.18. Let Assumption 2.1 hold true.

(i) There exists at least one non-random measure-valued no common noise MFG solution.

(ii) Moreover, for any non-random measure-valued MFG solution $P^*$, there exists a sequence $(P^k)_{k \in \mathbb{N}} \subset \mathcal{P}_S$ and a sequence $(\varepsilon_k)_{k \in \mathbb{N}} \subset [0, \infty)$ satisfying $\lim_{k \to \infty} \varepsilon_k = 0$ such that: for each $k \in \mathbb{N}$, $P^k \in \mathcal{P}_S[\varepsilon_k]$ and $\lim_k P^k = P^*$ in $\mathcal{W}_p$.

Remark 2.19. Under Assumption 2.1, item (i) of Theorem 2.18 is an existence result. Unlike the item (i.2) of Theorem 2.13 where the approximation is achieved through a sequence of $\varepsilon$-measure-valued MFG solution build with convex combination of distribution of strong McKean–Vlasov processes, the item (ii) of Theorem 2.18 shows that, when the solution is non-random, despite the fact that $\ell = 0$, it is possible to approximate a measure-valued solution through a sequence of $\varepsilon$-strong MFG solutions.

Remark 2.20. Let us mention that it is possible to prove the existence of solution when $\ell \neq 0$, using for instance Lacker and Webster [28] for particular coefficients or the techniques used by Carmona, Delarue, and Lacker [10] and Barrasso and Touzi [3] (discretization of the common noise filtration). But this requires another long technical proof and this is not the main purpose of this paper. See also Claisse, Zhenjie, and Tan [12] for an existence and approximation of particular mean field games (MFG with branching).

3 Proof of main results

3.1 Limit of Nash equilibria

Idea of the proof Before going into the details of the proof, in order to better understand our approach, we want to give the idea leading to the proof in a simple situation. For simplicity, let us consider that $n = 1, U = [0, 1], b = 0, \ell = 0$ and $\sigma(t, x, m, u) := \sqrt{u^2 + \sigma(m)^2}$. In this setting, for a measure-valued control rule $P \in \mathcal{P}_U$, the Fokker–Planck equation is rewritten: $P$–a.s.

$$d(f, \mu'_i) = \int_{\mathcal{P}_U} \int_{\mathbb{R}^\times U} \frac{1}{2} \nabla^2 f(x) u^2 m^2(du) \mu'_i(dx) \Lambda'_i(dm)dt + \int_{\mathcal{P}_U} \int_{\mathbb{R}^\times U} \frac{1}{2} \nabla^2 f(x) \sigma(m)^2 \mu'_i(dx) \Lambda_i(dm)dt, \quad (3.1)$$

where for each $m \in \mathcal{P}_U^n, \mathbb{R}^n \ni x \rightarrow m^x \in \mathcal{P}(U)$ is a Borel function s.t. $m(dx, du) = m^x(du)m(dx, U)$. The goal is to make a connection between Equation (3.1) and Equation (2.3) (the strong version).

To recover Equation (2.3), we use [13, Proposition 4.9]. Let us explain the idea. We consider an extension $(\tilde{\Omega}, \tilde{\mathcal{F}} := (\tilde{\mathcal{F}}_t)_{t \in [0,T]}, \tilde{\mathbb{P}})$ of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a $\tilde{\mathcal{F}}$–Brownian motion $W$, a $\tilde{\mathcal{F}}_0$–measurable random variable $X_0$ of law $\nu$, and $F$ a uniform random variable. The variables $W, X_0, F$ and the $\sigma$–field $\mathcal{F}_T$ are independent. Let us assume that Equation (3.1) is uniquely solvable in the sense that: if there exists another $\mathcal{P}(\mathbb{R}^n)$–valued $\mathcal{F}$–adapted continuous process $(\theta'_i)_{t \in [0,T]}$ satisfying Equation (3.1) (instead of $\mu'$) then $\theta' = \mu' \tilde{\mathbb{P}}$–a.s. By Blackwell and Dubins [4], there exists a Borel application $\Phi : (q, v) \in \mathcal{P}(U) \times [0, 1] \rightarrow \Phi(q)(v) \in U$ s.t. for all $q \in \mathcal{P}(U)$ and any $[0, 1]$–valued uniform random variable $F$,

$$\tilde{\mathbb{P}} \circ (\Phi(q)(F))^{-1} = q(du) = q(dx). \quad (3.2)$$

We define $\alpha(t, x, F) := \Phi\left( \int_{\mathcal{P}_U} m^x(du) \Lambda'_i(dm) \right)$. Let $Z'$ be a $\tilde{\mathcal{F}}$–adapted solution of: $Z'_0 = X_0$, and

$$dZ'_i = \sqrt{\alpha(t, Z'_i, F)^2 + \int_{\mathcal{P}_U} \sigma(m)^2 \Lambda_i(dm)}dW_t, \quad (3.3)$$

then by uniqueness of Equation (3.1), $\mu'_i = L^\mathbb{P}(Z'_i|\mathcal{F}_t) \tilde{\mathbb{P}}$–a.s. If in addition Equation (3.3) is uniquely solvable, then $Z'$ is adapted to $(\mathcal{G}_t \sigma \{W_{t\wedge \cdot}, F \})_{t \in [0,T]}$. Consequently $(\alpha(t, Z'_i, F))_{t \in [0,T]}$ is $(\mathcal{G}_t \sigma \{W_{t\wedge \cdot}, F \})_{t \in [0,T]}$–adapted. We see that an additional randomness is needed via a uniform random variable $F$. [13, Proposition 4.9] is essentially a result similar to the previous one without assuming the uniqueness of Equation (3.1) and the uniqueness of Equation (3.3). This is done by approximations and regulations. The next Proposition is a simplify version of [13, Proposition 4.9]. We give a sketch of proof in Appendix B.
Proposition 3.1. [13, Proposition 4.9] For a sequence of $\mathbb{P}$-predictable processes $(\Lambda^k, \Lambda^k)_k \in \mathbb{N}$ satisfying
\[
\lim_{k \to \infty} \bar{P} \circ (\Lambda^k, \Lambda^k)^{-1} = P \circ (\Lambda, \Lambda')^{-1} \text{ in } \mathcal{W}_p.
\]
Then, there exists a sequence of $U$-valued $(\sigma \{ \Lambda^k_\alpha, \Lambda^k_\alpha, W_{\lambda}, F \})_{t \in [0,T]}$-predictable processes $(\alpha^k)_k \in \mathbb{N}$, s.t. if we let $Z^k$ be the solution of
\[
\text{d}Z^k_t = (\alpha^k)^2 + \int_{\mathcal{P}_U} \bar{\sigma}(m)^2 \Lambda^k_t(dm) dW_t,
\]
we have, for a sub-sequence $(k_j)_j \in \mathbb{N}$,
\[
\lim_{j \to \infty} \mathcal{L}(\mu^{k_j}, \Lambda^{k_j}, \Lambda^{k_j}) = \mathcal{L}(\mu', \Lambda', \Lambda) \text{ in } \mathcal{W}_p
\]
where $\mu^{k_j} := \mathcal{L}(Z^{k_j}_t | \Lambda^k_\alpha, \Lambda^k_\alpha)$, $m^{k_j} := \mathcal{L}(Z^{k_j}_t, \alpha_t | \Lambda^k_\alpha, \Lambda^k_\alpha)$ and $\Lambda^{k_j} := \delta_{m^{k_j}}(dm) dt$.

In the goal of making a connection between Equation (3.1) and Equation (2.3), we will apply the previous result (in its general form) when the sequence $(\Lambda^k, \Lambda^k)_k \in \mathbb{N}$ is adapted to the filtration of $B$ and there exists $(u^k, \bar{u}^k)$ s.t. $\Lambda^k = \delta_{u^k}(dm) dt$ and $\Lambda = \delta_{\bar{u}^k}(dm)$ (see Lemma 3.3).

Let us mentioning that if $\bar{P}(\Lambda' = \Lambda, \Lambda^k = \Lambda^k \forall k) = 1$ i.e. Equation (3.1) is a totally non-linear Fokker–Planck equation, $Z^k$ will be equal to $Z^1$ a solution of a McKean–Vlasov equation i.e. $\text{d}Z^k_t = \sqrt{(\alpha^k)^2 + \bar{\sigma}(\mathcal{L}(Z^k_t, \alpha^k | \Lambda^k)^2)} dW_t$.

3.1.1 Reformulation of the measure–valued control rules by shifting the distributions

Before proceeding, let us give a reformulation of the measure–valued control rules which will be necessary for our proof. To make an analogy with the strong point of view, we want here to get a Fokker–Planck equation involving $\mathcal{L}(X^\alpha, \sigma_\alpha - \sigma_0 B | G_T)$ instead of $\mathcal{L}(X^\alpha, \sigma_\alpha | G_T)$. To do this, all the coefficients must be shifted. Let us define, for all $(t, b, \pi, m) \in [0, T] \times C^t \times C^\pi_U \times \mathcal{P}_U$
\[
\pi_t[b](dy) := \int_{\mathbb{R}^n} \delta(y' + \sigma_0 b_t)(dy) \pi_t(dy'), \quad m_t[b](du, dy) := \int_{\mathbb{R}^n} \delta(y' + \sigma_0 b_t)(dy)m_t(du, dy')
\]
and any $q \in \mathbb{M}$,
\[
q_t[b](dm) := \int_{\mathcal{P}_U} \delta(m_t)q_t(dm)qt(dm)dt.
\]

In the same way, let us consider the “shifted” generator $\mathcal{L}^\alpha$,
\[
\mathcal{L}^\alpha [\varphi](y, b, \pi, u) := \frac{1}{2} \text{Tr} [a^\alpha(t, y + \sigma_0 b_t, \pi, u) \nabla^2 \varphi(y)] + b^\alpha(t, y + \sigma_0 b_t, \pi, u)^\top \nabla \varphi(y),
\]
and also
\[
\tilde{[\sigma, \bar{\sigma}]}(t, y, b, \pi, m, u) := [\sigma, \bar{\sigma}](t, y + \sigma_0 b_t, \pi, m, u).
\]

Notice that the function $[\tilde{b}, \tilde{\sigma}]: [0, T] \times \mathbb{R}^n \times C^t \times C^\pi_U \times \mathcal{P}_U \to \mathbb{R}^n \times \mathbb{S}^n$ is continuous and for each $b \in C^t$, $[\tilde{b}, \tilde{\sigma}](\cdot, \cdot, b, \cdot, \cdot)$ verify Assumption 2.1.

Next, on the canonical filtered space $(\mathcal{F}, \mathcal{P})$, let us define the $\mathcal{P}(\mathbb{R}^n)$-valued $\mathcal{F}$-adapted continuous process $(\theta_t^I)_{t \in [0, T]}$ and the $\mathcal{P}_U^\alpha$-valued $\mathcal{F}$-predictable process $(\Theta_t^I)_{t \in [0, T]}$ by
\[
\theta^I_t(\bar{\omega}) := \mu^I_t(\bar{\omega})[\cdot] - B(\bar{\omega}) \quad \text{and} \quad \Theta^I_t(dm) := \Lambda^I_t(\bar{\omega})[\cdot] - B(\bar{\omega})(dm), \text{ for all } (t, \bar{\omega}) \in [0, T] \times \mathcal{F}.
\]
Lemma 3.2. Let $P \in \mathcal{P}_V$. Then, $\Theta_t'(Z_{\alpha,t}) = 1$, $dP \otimes dt$, a.e. $(t, \omega) \in [0, T] \times \Omega$, and $P$-a.e. $\omega \in \Omega$, for all $(f, t) \in C_B^2(\mathbb{R}^n) \times [0, T]$.

$$N_t(f) = \langle f, \vartheta_t' \rangle - \langle f, \nu \rangle - \int_0^t \int_{\mathcal{P}_U} (\mathcal{L}_t^*[f](\cdot, B, \mu, \cdot), m) \Theta_t'(dm)dr - \int_0^t \int_{\mathcal{P}_U} (\mathcal{L}_t^*[f](\cdot, \mu, m), \varphi_t') \Lambda_t(dm)dr.$$

Moreover, there exists a sequence $(G^k)_{k \in \mathbb{N}^*}$, such that for each $k \in \mathbb{N}^*$, $G^k : [0, T] \times C^l \times C^Z_{W} \times M(\mathcal{P}_U^B) \rightarrow \mathcal{P}_U^B$ is a continuous function and

$$\lim_{k \to \infty} \mathcal{L}^P \left( \delta_{G^k(t, B, \mu, \Lambda)} \right)(dm)dt, B, \mu, \Lambda) = \mathcal{L}^P \left( \Theta', B, \mu, \Lambda \right) \text{ in } \mathcal{W}_P. \quad (3.8)$$

Proof. The first point is just a reformulation of the process $N(f)$. For Equation (3.8), as $P \in \mathcal{P}_V$, by (an easy extension of) $[13, \text{Lemma 5.2}]$, for $p' > p$,

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^n} |x|^p \varphi_t'(\omega)(dx) + \mathbb{E}^P \left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^n} |x|^p \mu_t(dx) \right] \leq K \left[ 1 + \int_{\mathbb{R}^n} |y|^{p'} \nu(dy) \right], \text{ P-a.e. } \omega \in \Omega.$$

We define $\Gamma := \left\{ m \in \mathcal{P}_U^B : \int_{\mathbb{R}^n} |y|^p m(dy, U) \leq \hat{K} \right\}$, where $\hat{K} > 0$ is such that $\hat{K} > K \left[ 1 + \int_{\mathbb{R}^n} |y|^{p'} \nu(dy) \right]$, with $K$ is the constant previously used. Then, as $U$ is a compact set, we can notice that $\Gamma$ is a compact set of $\mathcal{P}_p(\mathbb{R}^n \times U)$, and one has $\Theta_t'(\Gamma) = 1$, $dP \otimes dt$, a.e. $(t, \omega) \in [0, T] \times \Omega$. $\Theta'$ is $\mathcal{P}_p$-predictable because $\Lambda'$ is $\mathcal{P}_p$-predictable. Then we can write $\Theta_t' = G(t, B_t, \mu_t, \Lambda_t)$ where $G : [0, T] \times C^l \times C^Z_{W} \times M(\mathcal{P}_U^B) \rightarrow \mathcal{P}(\mathcal{P}_U^B)$ is a Borel function. By combining all the previous observation, using $[11, \text{Theorem 2.2.3}]$ (or $[10, \text{Proposition C.1}]$), there exists a sequence $(G^k)_{k \in \mathbb{N}^*}$ such that for each $k \in \mathbb{N}^*$ $G^k : [0, T] \times C^l \times C^Z_{W} \times M(\mathcal{P}_U^B) \rightarrow \mathcal{P}_U^B$ is a Borel function and

$$\lim_{k \to \infty} \mathcal{L}^P \left( \delta_{G^k(t, B, \mu, \Lambda)} \right)(dm)dt, B, \mu, \Lambda) = \mathcal{L}^P \left( \Theta', B, \mu, \Lambda \right).$$

By using another approximation, we can take $G^k$ as a continuous function. \qed

3.1.2 Technical lemmas

In this section, we will show some technical results needed to prove our first limit theorem result, namely Theorem 2.12 using the more general form of $[13, \text{Proposition 4.9}]$ with common noise.

The following lemma establishes a result which implies that any measure-valued control rule satisfying some technical conditions can be approximated by processes of type $X_{t}^{\alpha,\alpha'}$ (see Definition 2.3).

Lemma 3.3. Let Assumption 2.1 hold true and $P \in \mathcal{P}_V$. On the probability space $(\Omega, \mathcal{F}, P)$, for any sequence $(\alpha^k)_{k \in \mathbb{N}^*} \subset \mathcal{A}$ satisfying

$$\lim_{k \to \infty} \mathbb{P} \circ \left( \mu^{\alpha^k, \delta_{\mathcal{P}_U^B}}(dm)dt, B \right)^{-1} = P \circ (\mu, \Lambda, B)^{-1},$$

then there exists a family of probability $(P^f)_{(k, f) \in \mathbb{N}^* \times [0, 1]} \subset \mathcal{P}_S$ such that for each $k \in \mathbb{N}^*$:

$$[0, 1] \ni f \mapsto P^k_f \in \mathcal{P}(\mathcal{F})$$

is Borel measurable, $P^k_f \circ (\mu, \Lambda, B)^{-1} = P \circ (\mu^{k, \delta_{\mathcal{P}_U^B}}(dm)dt, B)^{-1}$ for each $f \in [0, 1]$, and, for a sub-sequence $(k_j)_{j \in \mathbb{N}^*}$,

$$\lim_{j \to \infty} \int_0^1 \mathbb{E}^P [J(\mu', \mu', \Lambda', \Lambda')]df = \mathbb{E}^P [J(\mu', \mu', \Lambda', \Lambda')]$$

Proof. Step 1: Reformulation: For $P \in \mathcal{P}_V$, by definition, $P$-a.s. $\omega \in \Omega$, $N_t(f) = 0$ for all $f \in C_B^2(\mathbb{R}^n)$ and $t \in [0, T]$. By Lemma 3.2, recall that $(\vartheta_t')_{t \in [0, T]}$ and $(\Theta_t')_{t \in [0, T]}$ are defined in (3.7), one has $\Theta_t'(Z_{\alpha,t}) = 1$, $dP \otimes dt$ a.s. $(t, \omega) \in [0, T] \times \Omega$, and $P$-a.e. $\omega \in \Omega$, for all $(f, t) \in C_B^2(\mathbb{R}^n) \times [0, T]$,

$$0 = \langle f, \vartheta_t' \rangle - \langle f, \nu \rangle - \int_0^t \int_{\mathcal{P}_U} (\mathcal{L}_t^*[f](\cdot, B, \mu, \cdot), m) \Theta_t'(dm)dr - \int_0^t \int_{\mathcal{P}_U} (\mathcal{L}_t^*[f](\cdot, \mu, m), \varphi_t') \Lambda_t(dm)dr.$$
Step 2: Approximation: By Lemma 3.2, there exists a subsequence $(G^l)_{l \in \mathbb{N}^+}$ such that for each $l \in \mathbb{N}^+$, $G^l : [0, T] \times \mathcal{C} \times \mathcal{C}_W \times \mathbb{M}(P_U) \to P_U$ is a continuous function and
\[
\lim_{l \to \infty} \mathcal{L}^P \left( \delta_{G^l(t,B_{t\Lambda},\mu_{t\Lambda},\Lambda_{t\Lambda})} (dm) dt, B, \mu, \Lambda \right) = \mathcal{L}^P (\Theta', B, \mu, \Lambda).
\]

Now, we apply [13, Proposition 4.9] (see also [13, Proposition 4.7]). First, there exists a subsequence $(l_k)_{k \in \mathbb{N}^+} \subset \mathbb{N}^*$ such that if $\Lambda^k := \delta_{m^k}(dm)dt$, and
\[
m^k(t) := G^k(t,B_{t\Lambda},\mu_{t\Lambda},\Lambda_{t\Lambda}) \quad \text{and} \quad \Theta^k(t)(dm) dt := \delta_{m^k}(dm) dt,
\]
once has
\[
\lim_{k \to \infty} \mathbb{P} \circ (\Theta^k, \mu^k, \Lambda^k, B)^{-1} = \lim_{l \to \infty} \mathcal{L}^P \left( \delta_{G^l(t,B_{t\Lambda},\mu_{t\Lambda},\Lambda_{t\Lambda})} (dm) dt, \mu, \Lambda, B \right) = \mathcal{L}^P (\Theta', \mu, \Lambda, B).
\]

Next, under Assumption 2.1, by [13, Proposition 4.9] (with separability condition see [13, Remark 4.11]), as $(\xi, W)$ is \(P\) independent of $(B, \mu^k, \overline{\mu}^k)_{k \in \mathbb{N}^+}$, there exist a $[0,1]-$valued uniform random variable $F$ independent of $(\xi, W, B)$, and a Borel function $R^k : [0, T] \times \mathbb{R}^n \times \mathcal{C}_W \times \mathbb{M} \times \mathbb{M} \times \mathcal{C} \times \mathcal{C} \times [0, 1] \to U$ such that if we let $\tilde{X}^{sk}$ be the unique strong solution of: for all $t \in [0, T],$
\[
d\tilde{X}^{sk}_t = \tilde{b}(t, \tilde{X}^{sk}_t, B, \mu^k, \overline{\mu}^k, \alpha^k_t) dt + \tilde{\sigma}(t, \tilde{X}^{sk}_t, B, \mu^k, \overline{\mu}^k, \alpha^k_t) dW_t \text{ with } \tilde{X}^{sk}_0 = \xi. \tag{3.9}
\]
where $G^k := (G^k)_{s \in [0, T]} := (\sigma(\mu^k, \Theta^k, \Lambda^k, B))_{s \in [0, T]}$,
\[
\alpha^k_t := R^k(t, \xi, \rho^k_t, \Theta^k, \Lambda^k, W_t, B, F), \quad \overline{\mu}^k_t := L(\tilde{X}^{sk}_t, \alpha^k_t | G^k_t) \quad \text{and} \quad \eta^k_t := L(\tilde{X}^{sk}_t | G^k_t)
\]
then
\[
\lim_{k \to \infty} \mathbb{E} \left[ \int_0^T \mathcal{W}_t (\overline{m}^k_t, m^k_t)^p dt \right] = 0, \quad \text{and}
\]
\[
\lim_{j \to \infty} \mathcal{L}(\rho^k_j, V^{sk}_j, \mu^k_j, \Lambda^k_j, B) = \mathcal{L}^P (\vartheta', \Theta', \mu, \Lambda) \text{ in } W_p,
\]
where $V^{sk}_t (dm) dt := \delta_{m^k}(dm)dt$ and $(k_j)_{j \in \mathbb{N}^+} \subset \mathbb{N}^*$ is a subsequence.

Step 3: Rewriting: Notice that, as $(\mu^k, \Theta^k, \Lambda^k)$ are adapted to the filtration of $B$ then $G^k \subset G$. Besides, $(\xi, F, W)$ are independent of $G_T$, therefore $\mathcal{L}(\tilde{X}^{sk}_t, \alpha^k_t | G^k_t) = \mathcal{L}(\tilde{X}^{sk}_t, \alpha^k_t | G_t)$, $P$-a.s. for all $t \in [0, T]$. Using definition of $[\tilde{b}, \tilde{\sigma}]$ (see the equations (3.6)),
\[
d\tilde{X}^{sk}_t = b(t, \tilde{X}^{sk}_t + \sigma_0 B_t, \mu^k, \overline{\mu}^k, \alpha^k_t) dt + \sigma(t, \tilde{X}^{sk}_t + \sigma_0 B_t, \mu^k, \overline{\mu}^k, \alpha^k_t) dW_t, \quad \tilde{X}^{sk}_0 = \xi.
\]
Denote $X^{sk} := \tilde{X}^{sk} + \sigma_0 B_t$, one finds
\[
dX^{sk}_t = b(t, X^{sk}_t, \mu^k, \overline{\mu}^k, \alpha^k_t) dt + \sigma(t, X^{sk}_t, \mu^k, \overline{\mu}^k, \alpha^k_t) dW_t + \sigma_0 dB_t.
\]
It is straightforward to check that the function
\[
(\pi, q, b) \in \mathcal{C}^n \times \mathbb{M} \times \mathcal{C} \to (\pi | b, q, b)(dm)dt, b) \in \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}
\]
is continuous. Consequently, one has
\[
\lim_{j \to \infty} \mathcal{L} \left( (L(X^{sk}_t | G_t))_{s \in [0, T]}, \delta_{L(X^{sk}_t, \alpha^k_t | G_t)} (dm) ds, \mu^k, \delta_{\overline{\mu}^k_t} (dm) ds, B \right) = \lim_{j \to \infty} \mathcal{L}^P (\vartheta' \delta_{\Theta'} | [B], V^{sk}_t | [B] (dm) dt, \mu^k, \Lambda, B) \quad \text{in } W_p.
\]
Using the definition of $\vartheta'$ and $\Theta'$ in (3.7), we easily verify after calculations that $(\vartheta' | [B], \Theta' | [B] (dm) dt, B) = (\mu', \Lambda', B)$, $P$-a.e. Then
\[
\lim_{j \to \infty} \mathcal{L} \left( (L(X^{sk}_t | G_t))_{s \in [0, T]}, \delta_{L(X^{sk}_t, \alpha^k_t | G_t)} (dm) ds, \mu^k, \delta_{\overline{\mu}^k_t} (dm) ds, B \right) = \mathcal{L}^P (\mu', \Lambda', \mu, \Lambda) \quad \text{in } W_p. \tag{3.10}
\]
Now, to finish, we define

$$P^k_f := \mathcal{L} \left((\mathcal{L}(X^k_s|\{\mathcal{G}_s \vee F\})_{s \in [0,T]}, \mu^{\alpha^k}_s, \delta_{(\mathcal{L}(X^k_s, \gamma^k_s|\{\mathcal{G}_s \vee F\})} (dm)ds, \delta_{\pi^k_s} (dm)ds, B|F = f) \right).$$

As $F$ is independent of $(\xi, W, B)$, it is straightforward to check that $P^k_f \circ (\mu, \Lambda, B)^{-1} = \mathbb{P} \circ (\mu^{\alpha^k}, \delta_{\pi^k_s} (dm)dt, B)^{-1}$ and $P^k_f \in \overline{P}_S$ for each $(k, f)$. Besides, with an easy manipulation of the conditional expectation and Equation (3.10), we have

$$\int_0^1 \mathbb{E}^{P^k_f} [J(\mu', \mu, \Lambda', \Lambda)] \, df = \mathbb{E} \left[ \int_0^T L(t, X^k_t, \mu^{\alpha^k}_t, \gamma^k_t) dt + g(X^k_T, \mu^{\alpha^k}) \right]$$

and

$$\lim_{k \to \infty} \int_0^1 \mathbb{E}^{P^k_f} [J(\mu', \mu, \Lambda', \Lambda)] \, df = \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)].$$

This is enough to conclude the proof. \hfill \Box

Now, we consider the case of $N$–player game. Loosely speaking, we will show that: given the controls $\pi^N := (\alpha^1, \ldots, \alpha^N)$, replace one control $\alpha^i$ by another $\kappa^N$ has no effect on the empirical distribution $(\varphi^N X, \pi^N, \varphi^N X, \pi^N)$ (see Definition 2.1) when $N$ goes to infinity.

Given $N \in \mathbb{N}^*, (\alpha^i)_{1 \leq i \leq N} \subset \mathcal{A}^N$ and $\kappa^N \in \mathcal{A}^N$. Let us introduce, for each $i \in \{1, \ldots, N\}$, the unique strong solution $Z^i$ of:

$$dZ^i_t = b(t, Z^i_t, \varphi^N X, \pi^N, \varphi^N X, \pi^N, \kappa^N) \, dt + \sigma(t, Z^i_t, \varphi^N X, \pi^N, \varphi^N X, \pi^N, \kappa^N) \, dW^i_t + \sigma_0 dB_t$$

with $Z^i_0 = \xi^i$

where $(\varphi^N X, \pi^N, \varphi^N X, \pi^N)$ correspond to the empirical distributions associated with the controls $\pi^N := (\alpha^1, \ldots, \alpha^N)$ (see Definition 2.1)

**Lemma 3.4.** There exists a constant $K > 0$ (depending only on the $p$–moment of $\nu$) such that: if $\pi^{N,-i} := (\pi^{N, i}, \kappa^N)$, for each $i \in \{1, \ldots, N\}$, one has

$$\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \mathcal{W}_p \left( \varphi^N X, \pi^N, \varphi^N X, \pi^N, \kappa^N \right) \right] \right)^p \leq K \frac{1}{N},$$

Consequently, $\limsup_{N \to \infty} \mathcal{W}_p (Q^N, Q^{\pi^N}) = 0$, where

$$Q^N := \frac{1}{N} \sum_{i=1}^N \mathbb{P} \circ \left( X^i_{\pi^{N,-i}} \cdot \varphi^N X, \pi^{N,-i}, \delta \left( \kappa^N, \pi^{N,-i} \right) (du, dm) \right)^{-1},$$

and

$$\tilde{Q}^N := \frac{1}{N} \sum_{i=1}^N \mathbb{P} \circ \left( Z^i, \varphi^N X, \pi^N, \delta \left( \kappa^N, \pi^N \right) (du, dm) \right)^{-1}.$$

**Proof.** This proof is a successive application of the Gronwall’s lemma. For $j \in \{1, \ldots, N\}$ with $j \neq i$, for all $t \in [0, T]$, using Assumption 2.1 especially the boundness and Lipschitz properties of $(b, \sigma)$, one finds

$$\mathbb{E} \left[ \sup_{s \in [0,t]} \left| X^i_{\pi^{N,-i}} - X^i_{\pi^N} \right|^p \right] \leq C \left( \mathbb{E} \left[ \int_0^t \left[ b, \sigma \right] (r, X^i_{\pi^{N,-i}}, \varphi^N X, \pi^{N,-i}, \varphi^N X, \pi^{N,-i}, \alpha^i) - [b, \sigma] (r, X^i_{\pi^N}, \varphi^N X, \pi^N, \varphi^N X, \pi^N, \alpha^i) \right]^p \, dr \right) \leq C \left( \mathbb{E} \left[ \int_0^t \sup_{s \in [0,r]} \left| X^i_{\pi^{N,-i}} - X^i_{\pi^N} \right|^p + \mathcal{W}_p \left( \varphi^N X, \pi^{N,-i}, \varphi^N X, \pi^N \right)^p + \mathcal{W}_p \left( \varphi^N X, \pi^{N,-i}, \varphi^N X, \pi^N \right)^p \, dr \right) \right).$$
then by Gronwall’s lemma,
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |X^i_s[\bar{\tau}^N,N-1] - X^i_s[\bar{\tau}^N]|^p \right] \leq C \left( \mathbb{E} \left[ \int_0^t \sup_{s \in [0,r]} W_p(\varphi^N_s, X^*, X^N)^p + W_p(\varphi^N_r, X^N)^p \, dr \right] \right).
\]
(3.11)

Next, using result (3.11),
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} W_p(\varphi^N_s, X^N)^p + W_p(\varphi^N_r, X^N)^p \right]
\]
\[
\leq C \left( \frac{1}{N} \sum_{j=1}^{N} \mathbb{E} \left[ \sup_{s \in [0,t]} |X^j_s[\bar{\tau}^N,N-1] - X^j_s[\bar{\tau}^N]|^p \right] + \frac{\rho(\kappa^N, \alpha^N)^p}{N} \right)
\]
\[
\leq C \left( \frac{1}{N} \sum_{j=1, j \neq i}^{N} \int_0^t \sup_{s \in [0,r]} W_p(\varphi^N_s, X^N)^p + W_p(\varphi^N_r, X^N)^p \, dr \right)
\]
\[
+ \frac{1}{N} \mathbb{E} \left[ \sup_{s \in [0,t]} |X^i_s[\bar{\tau}^N,N-1] - X^i_s[\bar{\tau}^N]|^p \right] + \frac{\rho(\kappa^N, \alpha^N)^p}{N} \right)
\]
\[
\leq C \left( \frac{N-1}{N} \int_0^t \sup_{s \in [0,r]} W_p(\varphi^N_s, X^N)^p + W_p(\varphi^N_r, X^N)^p \, dr \right)
\]
\[
+ \frac{\mathbb{E} x^p \nu(dx)}{N} + \frac{\sup_{(u, u')} \rho(u, u')^p}{N},
\]
by Gronwall’s lemma again,
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} W_p(\varphi^N_s, X^N)^p + W_p(\varphi^N_r, X^N)^p \right] \leq C \left( \frac{\mathbb{E} x^p \nu(dx)}{N} + \frac{\sup_{(u, u')} \rho(u, u')^p}{N} \right).
\]
(3.12)

To finish,
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |X^i_s[\bar{\tau}^N,N-1] - Z^i_s|^p \right]
\]
\[
\leq C \left( \mathbb{E} \left[ \int_0^t \left| [b, \sigma]^i (r, X^i_s[\bar{\tau}^N,N-1], \varphi^N_s, X^N)^p \right| \, dr \right) \right)
\]
\[
\leq C \left( \mathbb{E} \left[ \int_0^t \sup_{s \in [0,r]} |X^i_s[\bar{\tau}^N,N-1] - X^i_s|^p + \sup_{s \in [0,r]} W_p(\varphi^N_s, X^N)^p \, dr \right] \right),
\]
and thanks to Gronwall’s lemma and result (3.12), one has
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} |X^i_s[\bar{\tau}^N,N-1] - Z^i_s|^p \right] \leq CT \left( \frac{\mathbb{E} x^p \nu(dx)}{N} + \frac{\sup_{(u, u')} \rho(u, u')^p}{N} \right).
\]
It is enough to conclude.
\(\Box\)

The next result is the analog of Lemma 3.3 for the N–player game. To summarize, it states that any measure–valued control rule which verifies a particular constraint is the average limit of N–SDE processes of type (2.1).

**Lemma 3.5.** Let Assumption 2.1 hold true, \(P \in \mathcal{P}_V\) and a sequence \((\alpha^i)_{i \in \mathbb{N}}\) s.t. for each \(N \in \mathbb{N}^+\), \((\alpha^1, \cdots, \alpha^N) \subset \mathcal{A}^N\)

\[
\lim_{N \to \infty} \mathcal{P} \circ \left( \varphi^N_s, \delta \varphi^N_s (dm) dt, B \right)^{-1} = P \circ (\mu, \Lambda, B)^{-1}.
\]

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There exists a sequence of Borel functions \( (R^i_N)_{(i,N) \in \{1, \ldots, N\} \times \mathbb{N}} \) satisfying \( R^i_N : [0, T] \times (\mathbb{R}^N)^N \times C_t^N \times \mathbb{C}^t \times [0, 1] \rightarrow U \) s.t. if for all \( (t, f) \in [0, T] \times [0, 1] \), \( k^i_N(f) \) is defined by \( k^i_N(f) := R^i_N(t, \xi^1, \ldots, \xi^N, W^i_{tA}, \ldots, W^i_{tN}, B_{tA}, f) \), then one has
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \int_0^1 J_i(\alpha^1, \ldots, \alpha^{i-1}, k^i_N(f), \alpha^{i+1:N}, \ldots, \alpha^N) \, df = \mathbb{E}^P \left[ J(\mu', \mu, \Lambda', \Lambda) \right].
\]

Proof. By Lemma 3.2, there is a sequence \( (G^i)_{i \in \mathbb{N}} \), such that for each \( i \in \mathbb{N}^* \), \( G^i : [0, T] \times C^t \times C^N_{\mathbb{P}} \times \mathbb{M}(\mathbb{P}^\mu) \rightarrow \mathbb{P}(\mathbb{R}^N \times U) \) is a continuous function and
\[
\lim_{i \to \infty} \mathbb{L}^P \left( \delta_{G^i(t, B_{tA}, A_{tA}, \Lambda_{tA})} \right)(dm)dt = \mathbb{L}^P(\Theta', B, \mu, \Lambda),
\]
recall that \( \Theta' \) is defined in (3.7). Now, we apply [13, Proposition 4.7]. We know that we can find a sub-sequence \( (l_N)_{N \in \mathbb{N}} \subset \mathbb{N}^* \) such that if \( \Lambda^N := _{\bar{v}_{l,N}}\bar{\pi}_N(dm)dt \),
\[
\mathbf{m}^N_t := G^N(t, B_{tA}, \varphi_{l,N}^N, X^N, \Lambda^N_t) \quad \text{and} \quad \Theta^N := _{\bar{v}_{l,N}}\bar{\pi}_N(dm)dt,
\]
one has
\[
\lim_{N \to \infty} \mathbb{P} \circ (\Theta^N, \mathbf{m}^N_t, \Lambda^N, B)^{-1} = \lim_{i \to \infty} \mathbb{L}^P \left( \delta_{G^i(t, B_{tA}, A_{tA}, \Lambda_{tA})} \right)(dm)dt = \mathbb{P} \circ (\Theta', \mu, \Lambda, B)^{-1}.
\]
Under Assumption 2.1, by [13, Proposition 4.7] (with separability condition see [13, Remark 4.11]), there exist a \([0, 1]\)-valued uniform random variable \( F \) independent of \((\xi^1, W^1, B)_{i \in \mathbb{N}} \) and a Borel function \( R^N : [0, T] \times \mathbb{R}^N \times C^t_{\mathbb{P}_0} \times \mathbb{M} \times \mathbb{M} \times \mathbb{C}^t \times [0, 1] \rightarrow U \) s.t. if \((Z^i_t)_{t \in \{1, \ldots, N\}}\) is the unique strong solution of: for all \( t \in [0, T] \)
\[
d\hat{Z}^i_t = \hat{b}(t, \hat{Z}^i_t, B, \varphi^N_t, X^N_t, \gamma^i_t) \, dt + \hat{\sigma}(t, \hat{Z}^i_t, B, \varphi^N_t, X^N_t, \gamma^i_t) \, dW^i_t \quad \text{with} \quad \hat{Z}^i_0 = \xi^i \quad (3.13)
\]
where
\[
\gamma^i_t := R^N(t, \xi^1, \varphi^N_{tA}, X^N, \Theta^N, A^N_t, W^i_{tA}, B_{tA}, F), \quad \theta^N_t := \frac{1}{N} \sum_{i=1}^N \delta_{\hat{Z}^i_t, \gamma^i_t}, \quad \text{and} \quad \delta^N_t := \frac{1}{N} \sum_{i=1}^N \delta_{\hat{Z}^i_t},
\]
then
\[
\lim_{N \to \infty} \mathbb{E} \left[ \int_0^T W^P(\hat{\mathbf{m}}^N_t, \mathbf{m}^N_t) \, dt \right] = 0,
\]
\[
\lim_{j \to \infty} \mathcal{L}(\varphi^{N_j}, X^{N_j}, \Lambda^{N_j}, B) = \mathcal{L}^P(\varphi', B, \mu, \Lambda, B) \quad \text{in} \ \mathcal{W}_P,
\]
with \( V^{N} := _{\bar{v}^N}(dm)dt \) and \((N_j)_{j \in \mathbb{N}} \subset \mathbb{N}^* \) is a sub-sequence.
As in the proof Lemma 3.3, we can rewrite \((Z^i_t)_{i \in \{1, \ldots, N\}}\). Notice that
\[
d\hat{Z}^i_t = \hat{b}(t, \hat{Z}^i_t + \sigma_0 B_t, \varphi^N_t, X^N_t, \gamma^i_t) \, dt + \sigma(t, \hat{Z}^i_t + \sigma_0 B_t, \varphi^N_t, X^N_t, \gamma^i_t) \, dW^i_t.
\]
Denote \( Z^i_t := \hat{Z}^i_t + \sigma_0 B_t \), then
\[
d\hat{Z}^i_t = \hat{b}(t, Z^i_t, \varphi^N_t, X^N_t, \gamma^i_t) \, dt + \sigma(t, Z^i_t, \varphi^N_t, X^N_t, \gamma^i_t) \, dW^i_t + \sigma_0 dB_t.
\]
As the function \((\pi, q, b) \in C^N_{\mathbb{P}} \times \mathbb{M} \times \mathbb{C}^t \rightarrow \{ \pi | b \}, q_{\{ b \}}(dm)dt, b \) \in C^N_{\mathbb{P}} \times \mathbb{M} \times \mathbb{C}^t \) is continuous, if we define \( \beta_{t} := \frac{1}{N} \sum_{i=1}^N \delta_{Z^i_t, \gamma^i_t} \), and \( \beta_{t} := \frac{1}{N} \sum_{i=1}^N \delta_{Z^i_t} \), one has, in \( \mathcal{W}_P \),
\[
\lim_{j \to \infty} \mathcal{L}(\beta^{N_j}, \delta^{N_j}_{\gamma^i}, (dm)dt, \varphi^{N_j}, X^{N_j}, A^{N_j}, B) = \lim_{j \to \infty} \mathcal{L}(\varphi^{N_j} [B], V^{N_j} [B], (dm)dt, \varphi^{N_j}, X^{N_j}, A^{N_j}, B) = \mathcal{L}^P(\varphi' [B], \Theta' [B] (dm)dt, \mu, \Lambda, B).
\]
One knows that \( \langle \Theta_t^1[B], \Theta_t^2[B]\rangle \) = \( (\mu', \Lambda', B) \), P-a.e. then
\[
\lim_{j \to \infty} \mathcal{L}\left( \beta_t^j, \delta_{\eta_t^j} (dm)dt, \varphi_t^j, \mathbf{X}_t^j, \Lambda^j \right) = \mathcal{L}^P (\mu', \Lambda', \mu, \Lambda) \text{ in } W^p.
\]
(3.14)

Let us define
\[
\alpha^{N,i} := (\alpha^{i-1}, \gamma^{i,N}) = (\alpha^1, \ldots, \alpha^{i-1}, \gamma^{i,N}, \alpha^{i+1}, \ldots, \alpha^N),
\]
thanks to Lemma 3.4, for each \( i \in \{1, \ldots, N\} \),
\[
\left( \mathbb{E} \left[ \int_0^T W_t^{3.14}(\varphi_t^N, \mathbf{X}_t^N, \delta(\gamma^{i,N})) dt \right] + \mathbb{E} \left[ \sup_{t \in [0,T]} |Z^i_t - \mathbf{X}_t^j| \right] \right) \leq K \frac{1}{N},
\]
and \( \limsup_{N \to \infty} W_t^{3.14}(\mathbf{Q}^N, \mathbf{Q}^N) = 0 \), where
\[
\mathbf{Q}^N := \frac{1}{N} \sum_{i=1}^N \mathcal{L}\left( \mathbf{X}_t^i, \varphi_t^N, \mathbf{X}_t^N, \delta(\gamma^{i,N}) \right) (du, dm)dt
\]
and
\[
\mathbf{Q}^N := \frac{1}{N} \sum_{i=1}^N \mathcal{L}\left( Z^i, \varphi_t^N, \mathbf{X}_t^N, \delta(\gamma^{i,N}) \right) (du, dm)dt.
\]

Now, for each \( f \in [0,1] \), we pose \( \kappa^i_t(f) := R_t(t, \xi^i, \varphi_t^N, \mathbf{X}_t^N, \delta(\gamma^{i,N}) \) \( \mu^N \Rightarrow \Lambda_\mu \Rightarrow W_t^1, B_t^1 \), \( f \). Notice that \( \kappa^i_t(f) \in \mathcal{A}^N \) and \( \kappa^i_t(F) = \gamma^{i,N} \).

Therefore, using Assumption 2.1 (especially the separability condition), the previous result combined with (3.14) allow to get that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \int_0^1 J_i((\gamma^{i,N}) \rangle (df) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[ \int_0^T L(t, \mathbf{X}_t^i, \varphi_t^N, \mathbf{X}_t^N, \delta(\gamma^{i,N}) dt + g(\mathbf{X}_t^i, \varphi_t^N, \mathbf{X}_t^N, \gamma^{i,N}) \right] = \mathbb{E}^P \left[ J(\mu', \mu, \Lambda', \Lambda) \right].
\]

\[
3.1.3 \text{ Proof of Theorem 2.12 (Limit Theorem)}
\]

**First point (i)** By using [13, Proposition 4.15] (a slight extension\(^2\)), one finds \( \mathcal{P}_N^{\gamma} \) is relatively compact where
\[
\mathcal{P}_N^{\gamma} := \mathcal{P} \circ \left( (\varphi_t^N, \mathbf{X}_t^N) \right)_{t \in [0,T], \varphi_t^N, \mathbf{X}_t^N} \delta(\varphi_t^N, \mathbf{X}_t^N) (dm, dt, \mu, \Lambda, \beta_t^j),
\]
and each limit point \( \mathcal{P}^{\infty} \) of any sub–sequence belongs to \( \mathcal{P}_V^{\gamma} \). Next, let us show that \( \mathcal{P}^{\infty} \in \mathcal{P}_V^{\gamma} \). To simplify, the sequence \( \mathcal{P}_N^{\gamma} \) and its sub–sequence share the same notation.

Let \( \mathcal{P} \in \mathcal{P}_V^{\gamma} \) such that \( \mathcal{L}^P (\mu, \Lambda, B, \mathcal{P}^{\infty}) = \mathcal{L}^P (\mu, \Lambda, B) \). By Lemma 3.5, there exist a \( [0,1] \)–valued uniform random variable \( F \) and \( (R_{t,N})_{(i,N) \in \{1, \ldots, N\} \times \mathbf{N}} \) a sequence of Borel functions \( \gamma^{i,N} : [0,1] \times (\mathbb{R}^N)^N \times \mathbb{C}^N \times C^t \times [0,1] \to U \) s.t. if we denote by
\[
\kappa^i_t^N (F) := R_{t,N}(F, \xi^N, \xi^N, \varepsilon, W_1^N, \varepsilon, W_1^N, B_t, \varepsilon)
\]
\(^2\)consisting in taking into account a canonical space of type \( \mathbf{N}^r := \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{M} \times \mathbb{M} \times C^t \) and not \( \mathbf{N}^r := \mathbb{C}^N \times \mathbb{M} \times C^t \) as in [13].
then
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} J_i(\alpha^1, \ldots, \alpha^{i-1}, \kappa^i,N(f), \alpha^{i+1}, \ldots, \alpha^N) df = \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)].
\]

Therefore, as for each \(N, (\alpha^1, \ldots, \alpha^N)\) is an \((\varepsilon_1, \ldots, \varepsilon_N)\)-Nash equilibrium,
\[
\mathbb{E}^{P^\infty} [J(\mu', \mu, \Lambda', \Lambda)] = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} J_i[\pi^N]
\]
\[
\geq \lim_{N \to \infty} \left( \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{1} J_i(\alpha^1, \ldots, \alpha^{i-1}, \kappa^i,N(f), \alpha^{i+1}, \ldots, \alpha^N) df - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \right) = \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)] - \varepsilon,
\]
then \(\mathbb{E}^{P^\infty} [J(\mu', \mu, \Lambda', \Lambda)] \geq \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)] - \varepsilon\), for any \(P \in \overline{P}_V\) such that \(L^P(\mu, \Lambda, B) = L^{P^\infty}(\mu, \Lambda, B)\). It is straightforward to deduce that for \(P^\infty\) almost every \(\omega \in \Omega\), \(\Lambda'_i(\omega)(dm)dt = \Lambda_i(\omega)(dm)dt\) and \(\mu'_i(\omega) = \mu(\omega)\). We conclude that \(P^\infty \in \overline{P}_V[\varepsilon]\).

**Second point (ii)** The proof of this second part is similar to the previous proof. By using [13, Proposition 4.15] (a slight extension), one gets \((P^k)_{k \in \mathbb{N}^*}\) is relatively compact where \(P^k \in \overline{P}_S[\varepsilon_k]\) i.e. there exists \(\alpha^k\) an \(\varepsilon_k\)-strong MFG equilibrium s.t.
\[
P^k := \mathbb{P} \circ (\mu^k_t)_{t \in [0,T]}(\mu^k_t)_{t \in [0,T]}, \delta_{\mu^k_t}(dm), \delta_{\mu^k_t}(dm) dt, B)^{-1}.
\]
Each limit point \(P^\infty\) of any sub–sequence belongs to \(\overline{P}_V\). Let us prove that \(P^\infty \in \overline{P}_V[\varepsilon]\). Again to simplify, \((P^k)_{k \in \mathbb{N}^*}\) and its sub–sequence share the same notation. Let \(P \in \overline{P}_V\) such that \(L^P(\mu, \Lambda, B) = L^{P^\infty}(\mu, \Lambda, B)\). By Lemma 3.3, there exists a family of probability \((P^k_f)_{(k,f) \in \mathbb{N}^* \times [0,1]} \subset \overline{P}_S\) such that for each \(k \in \mathbb{N}^*:\n\]
\[
f \in [0,1] \rightarrow P^k_f \in \mathcal{P}((\Omega)) \text{ is Borel measurable, } P^k_f \circ (\mu, \Lambda, B)^{-1} = \mathbb{P} \circ (\mu^k_t, \delta_{\mu^k_t}(dm), B)^{-1} \text{ for each } f \in [0,1],
\]
and for a sub–sequence \((k_f)_{f \in \mathbb{N}^*},
\[
\lim_{j \to \infty} \int_{0}^{1} \mathbb{E}^{P^k_f}_{(\mu, \Lambda, \Lambda)} [J(\mu', \mu, \Lambda', \Lambda)] df = \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)].
\]
Then, using Assumption 2.1 (especially separability condition), as \(\alpha^k\) is an \(\varepsilon_k\)-strong MFG solution, one gets
\[
\mathbb{E}^{P^\infty} [J(\mu', \mu, \Lambda', \Lambda)] = \lim_{j \to \infty} \mathbb{E}^{P^k_f} [J(\mu', \mu, \Lambda', \Lambda)]
\]
\[
\geq \lim_{j \to \infty} \left( \int_{0}^{1} \mathbb{E}^{P^k_f} [J(\mu', \mu, \Lambda', \Lambda)] df - \varepsilon_{k_f} \right) = \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)] - \varepsilon.
\]
Obviously, for \(P^\infty\) almost every \(\omega \in \Omega\), \(\Lambda'_i(\omega)(dm)dt = \Lambda_i(\omega)(dm)dt\) and \(\mu'_i(\omega) = \mu(\omega)\), we deduce that \(P^\infty \in \overline{P}_V[\varepsilon]\).

**Proof of Proposition 2.11.** Let \(\alpha\) be an \(\varepsilon\)-strong MFG equilibrium, and its corresponding probability \(P^\alpha \in \overline{P}_S[\varepsilon]\). It is straightforward to check that \(P^\alpha \in \overline{P}_V\). Let \(P \in \overline{P}_V\) such that \(L^{P^\infty}(\mu, \Lambda, B) = L^{P^\alpha}(\mu, \Lambda, B)\). By Lemma 3.3, there exists a family of probability \((P^k_{f,\alpha})_{(k,f) \in \mathbb{N}^* \times [0,1]} \subset \overline{P}_S\) such that for each \(k \in \mathbb{N}^*:\n\]
\[
f \in [0,1] \rightarrow P^k_{f,\alpha} \in \mathcal{P}((\Omega)) \text{ is Borel measurable, } P^k_{f,\alpha} \circ (\mu, \Lambda, B)^{-1} = L^{P^\alpha}(\mu, \Lambda, B) \text{ for each } f \in [0,1],
\]
and for a sub–sequence \((k_f)_{f \in \mathbb{N}^*},
\[
\lim_{j \to \infty} \int_{0}^{1} \mathbb{E}^{P^k_f}_{\alpha [\mu', \Lambda', \Lambda]} [J(\mu', \mu, \Lambda', \Lambda)] df = \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)].
\]
Consequently,
\[
\mathbb{E}^{P^\alpha} [J(\mu', \mu, \Lambda', \Lambda)] = \Psi(\alpha, \alpha) \geq \lim_{k \to \infty} \int_{0}^{1} \mathbb{E}^{P^k_f}_{\alpha [\mu', \Lambda', \Lambda]} [J(\mu', \mu, \Lambda', \Lambda)] df - \varepsilon = \mathbb{E}^P [J(\mu', \mu, \Lambda', \Lambda)] - \varepsilon.
\]
As obviously \(\Lambda_i(dm)dt = \Lambda'_i(dm)dt\) and \(\mu = \mu', \alpha\)-a.e., we can deduce that \(P^\alpha \in \overline{P}_V[\varepsilon]\) and conclude the proof. 
\]
3.2 The converse limit result

This part is devoted to the proof of Theorem 2.13. We focus on the approximation of any measure–valued MFG solution by a sequence of approximate strong MFG solutions. The approximation by approximate Nash equilibria follows from this approximation.

Problem and strategy of the proof Let $P^*$ be a measure–valued MFG solution. First, we find a sequence $(\alpha^k)_{k \in \mathbb{N}^*} \subset \mathcal{A}$ such that the sequence $(P^{\alpha^k})_{k \in \mathbb{N}^*} \subset \mathcal{F}_S$ (see Equation (2.6)) converges towards $P^*$. Then, we find a sequence $(\varepsilon_k)_{k \in \mathbb{N}^*} \subset (0, \infty)$ satisfying $\lim_{k \to \infty} \varepsilon_k = 0$ and $P^{\alpha^k} \in \mathcal{F}_S[\varepsilon_k]$ for each $k \in \mathbb{N}^*$. The sequence $(\varepsilon_k)_{k \in \mathbb{N}^*} \subset (0, \infty)$ is given by (recall that $\Psi(\alpha, \alpha')$ is given in Equation (2.4))

$$\varepsilon_k := \sup_{\alpha' \in \mathcal{A}} \Psi(\alpha^k, \alpha') - \Psi(\alpha^k, \alpha^k).$$

It is obvious that $\varepsilon_k \geq 0$ for each $k$. The difficulty is to show that $\lim_{k \to \infty} \varepsilon_k = 0$. For proving this, it is enough to show that: for any $(\alpha^k)_{k \in \mathbb{N}^*} \subset \mathcal{A}$, we can find a sequence $(Q^k)_{k \in \mathbb{N}^*} \subset \mathcal{F}_V$ such that $\mathcal{L}^{Q^k}(\mu, \Lambda, B) = \mathcal{L}^{P^*}(\mu, \Lambda, B)$ and

$$\lim_{k \to \infty} \mathbb{E} [J(\mu^k, \mu, \Lambda)] = 0. \quad (3.15)$$

For establishing property (3.15), the sequence $(\alpha^k)_{k \in \mathbb{N}^*}$ constructed is crucial. In the next part, we will show the main points of the construction of $(\alpha^k)_{k \in \mathbb{N}^*}$ which will help us proving the property (3.15).

3.2.1 Strong controlled McKean–Vlasov processes as approximation of weak controlled McKean–Vlasov processes

The approximation of any measure–valued MFG solution $P^*$ by a sequence of distributions of McKean–Vlasov process $(P^{\alpha^k})_{k \in \mathbb{N}^*} \subset \mathcal{F}_S$ is achieved in two steps. First, we approximate $P^*$ by a sequence of “weak” controlled McKean–Vlasov processes $(Z^k)_{k \in \mathbb{N}^*}$. Here, “weak” means essentially that $Z^k$ satisfies Equation (2.2) with a control $\alpha^k$ not adapted to $\mathbb{F}$ and $\mathcal{F}^{\alpha^k}$ is not adapted to the filtration of $B$. And the second step is, from $(Z^k)_{k \in \mathbb{N}^*}$, to build another approximation which satisfies: $\alpha^k$ adapted to $\mathbb{F}$ and $\mathcal{F}^{\alpha^k}$ is adapted to the filtration of $B$ i.e. strong controlled McKean–Vlasov processes. The second step is the most delicate. In this section, we focus on the approximation of weak controlled McKean–Vlasov process by strong controlled McKean–Vlasov processes. In the following, we give the definition of weak controlled McKean–Vlasov process that we use and the corresponding approximation by strong controlled McKean–Vlasov processes. This part is largely inspired/borrowed by Section 4.1.1. of Djetè, Possamai, and Tan [15].

Definition 3.6. A weak controlled McKean–Vlasov process is a tuple $(\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{X}, \hat{W}, \hat{B}, \hat{\mathbb{R}}, \hat{\alpha})$ satisfying:

1. $(\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ is a complete filtered probability space. $(\hat{W}, \hat{B})$ is a $\mathbb{R}^{d+\ell}$-valued $\hat{\mathbb{F}}$-Brownian motion. $\hat{X}$ is a $\mathbb{R}^n$-valued $\hat{\mathbb{F}}$-adapted continuous process with $\hat{\mathbb{P}} \circ (\hat{X}_0)^{-1} = \nu. \hat{\alpha}$ is a $\hat{\mathbb{F}}$-predictable process. Finally, $\hat{\mathbb{R}}$ is a $\mathcal{P}(\mathbb{C}^n \times \mathcal{M}(U) \times \mathbb{R}^n)$-valued $\hat{\mathbb{F}}$-adapted continuous process.

2. $\hat{W}, \hat{X}_0$ and $(\hat{\mathbb{R}}, \hat{\alpha})$ are independent.

3. $\hat{\mathbb{R}}$ verifies the condition: for all $t \in [0, T]$, $\hat{\mathbb{F}}$-a.s.

$$\hat{\mathbb{R}}_t = \mathcal{L}^{\hat{\mathbb{F}}}(\hat{X}_{t\wedge \cdot}, \hat{\Delta}_{t\wedge \cdot}, \hat{W}|\hat{B}_{t\wedge \cdot}, \hat{\mathbb{R}}_{t\wedge \cdot}) = \mathcal{L}^{\hat{\mathbb{F}}}(\hat{X}_{t\wedge \cdot}, \hat{\Delta}_{t\wedge \cdot}, \hat{W}|\hat{B}, \hat{\mathbb{R}}) \text{ where } \hat{\Delta}_t(du)dt := \delta_{\hat{\alpha}_t}(du)dt. \quad (3.16)$$

4. $\hat{X}$ satisfies

$$d\hat{X}_t = b(t, \hat{X}_t, \hat{\mathbb{R}}, \hat{\alpha}_t)dt + \sigma(t, \hat{X}_t, \hat{\mathbb{R}}, \hat{\alpha}_t)d\hat{W}_t + \sigma_0 d\hat{B}_t \quad (3.17)$$

where $\hat{\mathbb{R}}_t := \mathcal{L}^{\hat{\mathbb{F}}}(\hat{X}_t|\hat{B}, \hat{\mu})$ and $\hat{\mathbb{N}}_t := \mathcal{L}^{\hat{\mathbb{F}}}(\hat{X}_t|\hat{B}, \hat{\mathbb{R}})$. 


Intuition/Idea of the proof  Given a weak controlled McKean–Vlasov process \((\tilde{\Omega}, \tilde{F}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta})\), we want to build a sequence \((X^{\alpha_k}, \alpha_k)_{k \in \mathbb{N}^*}\) converging in a weak sense towards the weak controlled McKean–Vlasov process and satisfying: \(X^{\alpha_k}\) is solution of Equation (2.2) and \(\alpha_k \in \mathcal{A}\). In another words, we want to find a sequence of weak controlled McKean–Vlasov processes where \(\tilde{\alpha}\) is a function of \((\tilde{X}_0, \tilde{W}, \tilde{B})\) and \(\tilde{\beta}\) is a function of \(\tilde{B}\). The intuition is the following: first, under Assumption 2.1, Equation (3.17) admits a unique solution \(\tilde{X}\), then there exists a Borel function \(G : \mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}^t \times \mathbb{P}(\mathcal{C}^n \times \mathcal{M}(U) \times \mathcal{C}^n) \to \mathcal{C}^n\) s.t. \(\tilde{X} = G(\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta})\). This observation leads us to deal only with \((\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta})\) because \(\tilde{X}\) is already a function of these variables. Second, it is well known that we can find a uniform random variable \(F^{\tilde{\alpha}}\) independent of \((\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha})\) and a Borel function \(G^{\tilde{\alpha}} : \mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}^t \times \mathbb{P}(\mathcal{C}^n \times \mathcal{M}(U) \times \mathcal{C}^n) \times [0, 1] \to \mathcal{M}(U)\) s.t.

\[
\mathbb{L}(\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta}) = \mathbb{L}(\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta}, F^{\tilde{\alpha}}).
\]

Now, we deal with \((\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta})\). As \(\tilde{W}, \tilde{X}_0\) and \((\tilde{\alpha}, \tilde{\beta})\) are independent. We use the same previous procedure. We can find a uniform random variable \(F^{\tilde{\alpha}}\) independent of \((\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha})\) and a Borel function \(G^{\tilde{\alpha}} : \mathcal{C}^t \times [0, 1] \to \mathbb{P}(\mathcal{C}^n \times \mathcal{M}(U) \times \mathcal{C}^n)\) s.t.

\[
\mathbb{L}(\tilde{X}_0, \tilde{W}, \tilde{B}, \tilde{\alpha}) = \mathbb{L}(\tilde{X}_0, \tilde{W}, \tilde{B}, G^{\tilde{\alpha}}(\tilde{B}, F^{\tilde{\alpha}})).
\]

These equalities in distribution leads to

\[
\mathbb{L}(\tilde{X}, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta}) = \mathbb{L}(\tilde{S}, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta}),
\]

where \(\tilde{S} := G(\tilde{X}_0, \tilde{W}, \tilde{B}, G^{\tilde{\alpha}}(\tilde{B}, F^{\tilde{\alpha}})), \tilde{G}^{\tilde{\alpha}} := G^{\tilde{\alpha}}(\tilde{X}_0, \tilde{W}, \tilde{B}, G^{\tilde{\alpha}}(\tilde{B}, F^{\tilde{\alpha}})), \tilde{G}^{\tilde{\alpha}} := G^{\tilde{\alpha}}(\tilde{B}, F^{\tilde{\alpha}})\). The equality in distribution allows to check that \((\tilde{S}, \tilde{W}, \tilde{B}, \tilde{G}^{\tilde{\alpha}})\) satisfies the same equation as \((\tilde{X}, \tilde{W}, \tilde{B}, \tilde{\alpha}, \tilde{\beta})\). \((\tilde{S}, \tilde{W}, \tilde{B}, \tilde{G}^{\tilde{\alpha}})\) is almost what we want except two points: first, the presence of uniform random variables \((F^{\tilde{\alpha}}, \tilde{F}^{\tilde{\alpha}})\), and second, we do not know if \(\tilde{G}^{\tilde{\alpha}}\) and \(\tilde{G}^{\tilde{\alpha}}\) are adapted to the filtrations of \((\tilde{X}_0, \tilde{W}, \tilde{B})\) and \(\tilde{B}\) respectively. The technical proof of [15, Section 4.1.1.] allows to “remove” \((F^{\tilde{\alpha}}, \tilde{F}^{\tilde{\alpha}})\) and to make \((\tilde{G}^{\tilde{\alpha}}, \tilde{G}^{\tilde{\alpha}})\) an adapted process by using approximations and the condition satisfies by \(\tilde{B}\) i.e. Equation (3.16). The next Lemma is a combination of [15, Lemma 4.3, Lemma 4.4].

Lemma 3.7. There exit: a sequence of positive real \((\varepsilon_k)_{k \in \mathbb{N}^*}\) s.t. \(\lim_{k \to \infty} \varepsilon_k = 0\), and

- a sequence \((K^k, \alpha_k)_{k \in \mathbb{N}^*}\) where for each \(k \in \mathbb{N}^*\), \(\alpha_k\) is a \(U\)-valued piecewise \(\tilde{F}\)-predictable process and \(K^k\) satisfies

\[
dK^k = b(t, K^k, \alpha_k, \pi^k) 1_{t \in [\varepsilon_k, T]} dt + \sigma(t, K^k, \alpha_k, \pi^k) dW^k_t + \sigma_0 d\tilde{B}^k_t \quad \text{with} \quad \tilde{K}^0 = \tilde{X}_0,
\]

where \(\tilde{B}^k := \tilde{B} \vee \varepsilon_k - \tilde{B} \wedge \varepsilon_k\), \(\tilde{W}^k := \tilde{W} \vee \varepsilon_k - \tilde{W} \wedge \varepsilon_k\), \(\pi^k := \mathbb{L}(K^k, \alpha_k|\tilde{B}^k, \tilde{S})\) and \(\alpha^k := \mathbb{L}(K^k|\tilde{B}^k, \tilde{\alpha})\);

- a sequence of uniform random variable \((F^k)_{k \in \mathbb{N}^*}\) independent of \((\tilde{X}_0, \tilde{W}, \tilde{B})\) and a sequence \((S^k, \gamma^k)_{k \in \mathbb{N}^*}\) where for each \(k \in \mathbb{N}^*\), \(\gamma^k\) is a \(U\)-valued \((\sigma (F^k, \tilde{X}_0, \tilde{B}^k, \tilde{B}^k)\}_{t \in [0, T]}\)-predictable process and \(S^k\) satisfies

\[
dS^k = b(t, S^k, \gamma^k, \pi^k, \gamma^k) 1_{t \in [\varepsilon_k, T]} dt + \sigma(t, S^k, \gamma^k, \pi^k, \gamma^k) dW^k_t + \sigma_0 d\tilde{B}^k_t
\]

where \(\pi^k := \mathbb{L}(S^k, \gamma^k|\tilde{B}^k, F^k)\) and \(\gamma^k := \mathbb{L}(S^k|\tilde{B}^k, \tilde{F}^k)\);

all these processes satisfy

\[
\lim_{k \to \infty} \mathbb{E}^\tilde{F} \left[ \int_0^T \left[ \rho(\alpha_k, \tilde{\alpha}_t) + 2 W_2(N_t, \pi^k_t) \right] dt + \sup_{t \in [0, T]} \left[ |\tilde{X}_t - \tilde{K}^k_t|^2 + W_2(N_t, \pi^k_t) \right] \right] = 0
\]

and for each \(k \in \mathbb{N}^*\),

\[
\mathbb{L}(K^k, \tilde{\alpha}, \tilde{W}, \tilde{B}, \tilde{F}^k) = \mathbb{L}(S^k, \tilde{\gamma}, \tilde{W}, \tilde{\beta}, \tilde{\alpha}, \tilde{\beta}, \tilde{F}^k),
\]

where \(\tilde{\alpha} := \delta_{\tilde{\alpha}}(du) dt\), \(\tilde{\beta} := \delta_{\tilde{\beta}}(du) dt\), \(\tilde{\alpha} := \mathbb{L}(K^k, \tilde{\alpha}, \tilde{W}, \tilde{B}, \tilde{F}^k)\) and \(\tilde{\gamma} := \mathbb{L}(S^k, \tilde{\gamma}, \tilde{W}, \tilde{\beta}, \tilde{F}^k)\).
Given the result of the previous Lemma 3.7 with the same notation, let us assume first that \( \ell \neq 0 \). Then for each \( k \in \mathbb{N}^* \), we pose \( F^k := \phi(\bar{B}_t) \) where \( \phi : \mathbb{R}^\ell \to [0,1] \) is a Borel function such that \( \mathbb{L}^k(\phi(\bar{B}_t)) = U([0,1]) \). Therefore \((\tilde{\zeta}_k^k)_{k \in \mathbb{N}^*}\) is a \( U \)-valued \((\sigma\{\tilde{X}_0, \tilde{W}_t, \tilde{B}_t, \ell\})_{t \in [0,T]}\)-predictable processes and \((\tilde{\zeta}_k^k)_{k \in \mathbb{N}^*}\) is \( B \)-measurable, i.e. \( \tilde{\zeta}_k^k := \mathbb{L}^k(\tilde{S}_t, \tilde{\beta}_k^k, \tilde{W}_k|\bar{B}) \). The next proposition shows the sequence of strong controlled McKean–Vlasov processes converges to the weak controlled McKean–Vlasov process. This proposition is easily obtained with the previous Lemma 3.7.

**Proposition 3.8.** \((\ell \neq 0)\) For each \( k \in \mathbb{N}^* \), let \( \tilde{E}^k \) be the solution of
\[
d\tilde{E}^k = b(t, \tilde{E}^k, \mu, \tilde{P}_t, \tilde{\gamma}_k^k)dt + \sigma(t, \tilde{E}^k, \mu, \tilde{P}_t, \tilde{\gamma}_k^k)d\tilde{W}_t + \sigma_d \tilde{d}\tilde{B}_t \quad \text{with} \quad \tilde{E}_0 = \tilde{X}_0, \quad \mu^k := \mathbb{L}^k(\tilde{E}^k_k|\bar{B}) \quad \text{and} \tilde{P}_t := \mathbb{L}^k(\tilde{E}^k_k, \tilde{\gamma}_k^k|\bar{B})
\]
then \((\mathbb{L}^k(\mu^k, \tilde{P}_t, \delta_{\tilde{P}_t}(dm)dt, \tilde{\gamma}_k^k, \tilde{d}\tilde{W}_t, \tilde{d}\tilde{B}_t, \hat{\mu}^k))_{k \in \mathbb{N}^*} \subset \mathbb{P}S,
\]
\[
\lim_{k \to \infty} \mathbb{E}\left[ \sup_{t \in [0,T]} |\tilde{E}^k_k - \tilde{S}_k^k| \right] = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathbb{L}^k(\tilde{E}^k, \tilde{\beta}_k^k, \tilde{W}, \tilde{B}, \hat{\mu}^k) = \mathbb{L}^k(\tilde{X}, \tilde{\Delta}, \tilde{W}, \tilde{B}, \tilde{\hat{\delta}}) \quad \text{in} \ \mathbb{W}_p,
\]
where \( \hat{\mu}^k := \mathbb{L}^k(\tilde{E}^k_k, \tilde{\beta}_k^k, \tilde{W}|\bar{B}) \) a.s. for all \( t \in [0,T] \).

Now, when \( \ell = 0 \), then there is no common noise \( B \). In that case we obtain a convex combination of strong controlled McKean–Vlasov.

**Proposition 3.9.** \((\ell = 0)\) let \( \tilde{E}^k \) be the solution of
\[
d\tilde{E}^k = b(t, \tilde{E}^k, \mu, \tilde{P}_t, \tilde{\gamma}_k^k)dt + \sigma(t, \tilde{E}^k, \mu, \tilde{P}_t, \tilde{\gamma}_k^k)d\tilde{W}_t + \sigma_d \tilde{d}\tilde{B}_t \quad \text{with} \quad \tilde{E}_0 = \tilde{X}_0, \quad \mu^k := \mathbb{L}^k(\tilde{E}^k_k|\bar{F}^k) \quad \text{and} \tilde{P}_t := \mathbb{L}^k(\tilde{E}^k_k, \tilde{\gamma}_k^k|\bar{F}^k)
\]
then a.s. \((\mathbb{L}^k(\mu^k, \tilde{P}_t, \delta_{\tilde{P}_t}(dm)dt, \tilde{\gamma}_k^k, \tilde{d}\tilde{W}_t, \tilde{d}\tilde{B}_t, \hat{\mu}^k))_{k \in \mathbb{N}^*} \subset \mathbb{P}S,
\]
\[
\lim_{k \to \infty} \mathbb{E}\left[ \sup_{t \in [0,T]} |\tilde{E}^k_k - \tilde{S}_k^k| \right] = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathbb{L}^k(\tilde{E}^k, \tilde{\beta}_k^k, \tilde{W}, \tilde{B}, \hat{\mu}^k) = \mathbb{L}^k(\tilde{X}, \tilde{\Delta}, \tilde{W}, \tilde{B}, \tilde{\hat{\delta}}) \quad \text{in} \ \mathbb{W}_p,
\]
where \( \hat{\mu}^k := \mathbb{L}^k(\tilde{E}^k_k, \tilde{\beta}_k^k, \tilde{W}|\bar{F}^k) \) a.s. for all \( t \in [0,T] \).

From now on and in the following, we will work with \( \ell \neq 0 \). The case \( \ell = 0 \) can be easily adapted from the case \( \ell \neq 0 \).

Now, using Lemma 3.7, we will give a first result which will help us to prove Equation 3.15 which is our main goal in this part. Recall that the weak controlled McKean–Vlasov process \((\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{W}, \tilde{B}, \tilde{\hat{\Delta}}, \tilde{\hat{\alpha}})\) is fixed. Let \((\tilde{\gamma}_k^k)_{k \in \mathbb{N}^*}\) be a sequence such that for each \( k \in \mathbb{N}^* \), there exists a Borel function \( \varphi^k : [0,T] \times \mathbb{R}^n \times \mathcal{C} \to U \) with \( \tilde{\gamma}_k^k = \varphi^k(t, \tilde{X}_0, \tilde{W}_t, \tilde{B}_t) \) a.s. for each \( t \in [0,T] \). For each \( k \in \mathbb{N}^* \), we consider \( \tilde{E}^k \) the strong solution of:
\[
d\tilde{E}^k = b(t, \tilde{E}^k, \mu, \tilde{P}_t, \tilde{\gamma}_k^k)dt + \sigma(t, \tilde{E}^k, \mu, \tilde{P}_t, \tilde{\gamma}_k^k)d\tilde{W}_t + \sigma_d \tilde{d}\tilde{B}_t \quad \text{with} \quad \tilde{E}_0 = \tilde{X}_0.
\]
\( \mu^k, \tilde{P}^k \) and \( \tilde{\beta}^k \) are defined in Proposition 3.8.

**Lemma 3.10.** There exists a sequence of \( U \)-valued \( \tilde{\mathbb{F}} \)-predictable processes \((\tilde{\alpha}^k_t)_{t \in [0,T]}\) s.t if for each \( k \in \mathbb{N}^* \), we let \( \tilde{X}^k \) be the solution of
\[
d\tilde{X}^k = b(t, \tilde{X}^k, \tilde{\Delta}, \tilde{\alpha}_t^k)dt + \sigma(t, \tilde{X}^k, \tilde{\Delta}, \tilde{\alpha}_t^k)d\tilde{W}_t + \sigma_d \tilde{d}\tilde{B}_t \quad \text{with} \quad \tilde{X}_0^k = \tilde{X}_0
\]
then
\[
\lim_{k \to \infty} \mathbb{W}_p \left( \tilde{\mathbb{P}} \circ \left( \tilde{\mathbb{E}}^k, \tilde{\beta}^k, \delta_{\tilde{P}^k}(\mathbb{d}m, du) \right)^{-1} \tilde{\mathbb{P}} \circ \left( \tilde{X}^k, \tilde{\Delta}, \delta_{\tilde{\mathbb{P}}_t}(\mathbb{d}m, du) \right)^{-1} \right) = 0.
\]
Proof. For each \( k \in \mathbb{N}^* \), let us define \( \tilde{S}^k \) the solution of
\[
\text{d}\tilde{S}^k_t = b(t, \tilde{S}^k_t, \xi, \zeta_t, \gamma_t) \mathbf{1}_{t \in [\varepsilon, T]} \text{d}t + \sigma(t, \tilde{S}^k_t, \xi, \zeta_t, \gamma_t) \text{d}\tilde{W}^k_t + \sigma_0 \text{d}\tilde{B}^k_t \text{ with } \tilde{S}^k_0 = \tilde{X}_0.
\]
Using Proposition 3.8, as \( \tilde{S}^k \sim \mathcal{L}^k(\tilde{S}^k_\omega, \beta^k_\omega, \tilde{W}^k_\omega, \tilde{B}_\omega) \) and, \( \lim_{k \to \infty} \varepsilon_k = 0 \) and \( \lim_{k \to \infty} (\tilde{W}^k, \tilde{B}^k) = (\tilde{W}, \tilde{B}) \) a.s., we know that
\[
\lim_{k \to \infty} \mathcal{W}_p(\mu^k_t, \mu^k_t, \tilde{W}^k_t, \tilde{B}^k_t, \zeta_t, \zeta_t)) = 0.
\]
Then, it is easy to check that
\[
\lim_{k \to \infty} \mathbb{E}^\tilde{P}\left[ \sup_{t \in [0,T]} |\tilde{S}^k_t - \tilde{S}^k_t|^p \right] = 0. \tag{3.20}
\]
For each \( k \in \mathbb{N}^* \), in the function \( \varphi^k \), in order to emphasize the part of the Brownian motion \((\tilde{W}, \tilde{B})\) before and after \( \varepsilon_k \), we rewrite
\[
\varphi^k(t, \tilde{X}_0, \tilde{W}_{t,\omega}, \tilde{B}_{t,\omega}) = \varphi^k(t, \tilde{X}_0, \tilde{W}^k_{t,\omega}, \tilde{W}_{t,\omega}^{\tilde{k}}, \tilde{B}^k_{t,\omega}, \tilde{B}_{t,\omega}^{\tilde{k}}).
\]
Let \((\tilde{W}^\omega, \tilde{B}^\omega)\) be a \( \mathbb{R}^{d+\ell} \)-valued \( \tilde{P} \)-Brownian motion \( \tilde{P} \)-independent of \((\tilde{W}, \tilde{B}, \tilde{X}_0, \tilde{R}, (\tilde{F}^k)_{k \in \mathbb{N}^*})\). We define
\[
\tilde{\alpha}^k_t := \tilde{\varphi}(t, \tilde{X}_0, \tilde{W}^k_{t,\omega}, \tilde{W}_{t,\omega}^{\tilde{k}}, \tilde{B}^k_{t,\omega}, \tilde{B}_{t,\omega}^{\tilde{k}}) \text{ a.s. for all } t \in [0, T], \hat{\alpha}^k_t := \delta_{\tilde{\alpha}^k_t}(\text{d}u)dt \text{ and } \beta^k_t := \delta_{\tilde{\beta}^k_t}(\text{d}u)dt.
\]
It is straightforward that we can find a Borel function \( \Phi : \mathbb{R}^n \times \mathcal{M}(U) \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^\ell \times \mathcal{M}(U) \times \mathcal{C}^\ell) \to \mathbb{R}^n \) s.t. \( \tilde{S}^k = \Phi(\tilde{X}_0, \tilde{\alpha}^k, \tilde{W}^k, \tilde{B}^k, \zeta^k) \) a.s. Let us introduce the process \((\tilde{K}^k_t)_{t \in [0, T]}\)
\[
\tilde{K}^k_t := \Phi(\tilde{X}_0, \tilde{\alpha}^k, \tilde{W}^k, \tilde{B}^k, \tilde{I}^k).
\]
Using Lemma 3.7, especially \( \mathcal{L}(\tilde{X}^k, \tilde{\Delta}^k, \tilde{W}^k, \tilde{B}^k, \tilde{I}^k) = \mathcal{L}(\tilde{S}^k, \tilde{P}^k, \tilde{W}^k, \tilde{B}^k, \tilde{\zeta}^k) \), we find that
\[
\mathcal{L}(\tilde{\alpha}^k, \tilde{B}^k, \tilde{\Delta}^k, \tilde{W}^k, \tilde{I}^k) = \mathcal{L}(\tilde{K}^k_t, \tilde{\Delta}^k, \tilde{W}^k, \tilde{B}^k, \tilde{\zeta}^k_\omega). \tag{3.21}
\]
Notice that \( \tilde{K}^k \) is the solution of
\[
\text{d}\tilde{K}^k_t = b(t, \tilde{K}^k_t, \tilde{\alpha}^k_t, \tilde{\Delta}^k_t, \tilde{\alpha}^k_t) \mathbf{1}_{t \in [\varepsilon, T]} \text{d}t + \sigma(t, \tilde{K}^k_t, \tilde{\alpha}^k_t) \text{d}\tilde{W}^k_t + \sigma_0 \text{d}\tilde{B}^k_t \text{ with } \tilde{K}^k_0 = \tilde{X}_0.
\]
To conclude the prove of the Lemma, we introduce \( \tilde{X}^k \) solution of
\[
\text{d}\tilde{X}^k_t = b(t, \tilde{X}^k_t, \tilde{\alpha}^k_t, \tilde{\Delta}^k_t, \tilde{\alpha}^k_t) \mathbf{1}_{t \in [\varepsilon, T]} \text{d}t + \sigma(t, \tilde{X}^k_t, \tilde{\alpha}^k_t) \text{d}\tilde{W}^k_t + \sigma_0 \text{d}\tilde{B}^k_t \text{ with } \tilde{X}^k_0 = \tilde{X}_0.
\]
Similarly to (3.20), thanks to (3.18) of Lemma 3.7, it is easy to verify that \( \lim_{k \to \infty} \mathbb{E}^\tilde{P}\left[ \sup_{t \in [0,T]} |\tilde{X}^k_t - \tilde{K}^k_t|^p \right] = 0 \). By combining the previous convergence, Equation (3.21) and the convergence (3.20), we deduce the convergence (3.19).

\[\square\]

### 3.2.2 Approximation of measure–valued solution via weak controlled McKean–Vlasov processes

Let \( \tilde{P} \in \tilde{P}_V \) such that \( \mathbb{P}(\Lambda' = \Lambda, \mu' = \mu) = 1 \). Let \((\tilde{X}_0, \tilde{F}^k) := (\tilde{F}^k_{t,\omega})_{t \in [0, T]}, \tilde{P})\) be an extension of \((\Omega, \tilde{F}, \tilde{P})\) supporting a uniform random variable \( F \), a \( \mathbb{R}^{d'} \)-valued \( \tilde{F} \)-Brownian motion \( \tilde{W} \) and a \( \tilde{F}^k \)-valued random variable \( \tilde{X}_0 \) with \( \mathcal{L}(\tilde{X}_0) = \nu \). Besides, \( F, \tilde{W}, \tilde{X}_0 \) and \( \tilde{F}^k \) are independent. \( (\mu, \Lambda, B) \) are naturally extended on \((\Omega, \tilde{F} := (\tilde{F}^k_{t,\omega})_{t \in [0, T]}, \tilde{P})\). The next Lemma is exactly [13, Proposition 5.3].

**Lemma 3.11.** There exists a sequence of \( U \)-valued \((\sigma\{\tilde{F}, \tilde{X}_0, \tilde{W}_{t,\omega}, \tilde{G}_t\} \cup \tilde{G}_t)_{t \in [0, T]}\)-predictable processes \((\tilde{\alpha}^k)_{k \in \mathbb{N}^*}\) st. if we let \( \tilde{X}^k \) be the solution of
\[
\text{d}\tilde{X}^k_t = b(t, \tilde{X}^k_t, \tilde{\alpha}^k_t, \tilde{\Delta}^k_t, \tilde{\alpha}^k_t) \mathbf{1}_{t \in [\varepsilon, T]} \text{d}t + \sigma(t, \tilde{X}^k_t, \tilde{\alpha}^k_t, \tilde{\Delta}^k_t, \tilde{\alpha}^k_t) \text{d}\tilde{W}^k_t + \sigma_0 \text{d}\tilde{B}^k_t \text{ with } \tilde{X}^k_0 = \tilde{X}_0
\]
where \( \tilde{\alpha}^k = \mathcal{L}^{\tilde{P}}(\tilde{X}^k_{t,\omega}, \tilde{G}_t) \) and \( \tilde{\alpha}^k_t = \mathcal{L}^{\tilde{P}}(\tilde{X}^k_{t,\omega}, \tilde{G}_t) \). Then for a sub–sequence \((k_j)_{j \in \mathbb{N}}\)
\[
\lim_{j \to \infty} \mathbb{E}^\tilde{P}\left[ \sup_{t \in [0,T]} \mathcal{W}_\mu(\tilde{K}^{k_j}_{t,\omega}, \tilde{\alpha}^k_t) \right] = 0, \tilde{P}\text{-a.s.}
\]
Remark 3.12. If we define $\tilde{\mathcal{N}}^k_t := \mathcal{L}^P(\mathcal{X}_{k,t}^\star,\mathcal{A}_{k,t}^\star,\mathcal{W}[\mathcal{G}])$ with $\mathcal{A}_k^\star := \delta_{\mathcal{A}_k}(du)dt$. It is straightforward to check that the tuple $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{X}^k, \mathcal{W}^k, B, \tilde{\mathcal{N}}_k^\star, \tilde{\mathcal{A}}^k)$ is a weak controlled McKean–Vlasov in the sense of Definition 3.6 (see also [15, Definition 2.3]).

Now, given the sequence $(\mathcal{X}^k, \tilde{\mathcal{A}}_k^\star)_{k \in \mathbb{N}^*}$ of Lemma 3.11, we consider a sequence of $U$–valued $(\sigma\{F, \mathcal{X}_0, \mathcal{W}_t, \mathcal{A}_k\})_{t \in [0,T]}$–predictable processes $(\tilde{\mathcal{A}}_k^\star)_{k \in \mathbb{N}^*}$ and let $\tilde{\mathcal{X}}_k^\star$ be the solution of

$$d\tilde{\mathcal{X}}_k^\star = b(t, \tilde{\mathcal{X}}_k^\star, \mathcal{N}_k^\star, \tilde{\mathcal{A}}_k^\star)dt + \sigma(t, \tilde{\mathcal{X}}_k^\star, \mathcal{N}_k^\star, \tilde{\mathcal{A}}_k^\star)dw_t + \delta_0 d\mathcal{B}_t \text{ with } \tilde{\mathcal{X}}_0^\star = \mathcal{X}_0.$$

**Lemma 3.13.** There exists a sequence $(Q^k)_{k \in \mathbb{N}^*} \subset \mathcal{P}_U$ satisfying $Q^k \circ (\mu, \Lambda, B)^{-1} = P \circ (\mu, \Lambda, B)^{-1}$ for each $k \in \mathbb{N}^*$ and

$$\lim_{k \to \infty} W_p\left(\tilde{P} \circ (\mathcal{N}_k^\star, \delta_{\mathcal{N}_k^\star}(dm)dt, \delta_{\mathcal{N}_k^\star}(dm)dt, B)^{-1}, Q^k\right) = 0,$$

(3.22)

where $\mathcal{N}_k^\star := \mathcal{L}P(\mathcal{X}_k^\star, \mathcal{A}_k^\star) \text{ and } \mathcal{N}_k^\star := \mathcal{L}P(\mathcal{X}_k^\star, \mathcal{A}_k^\star) \text{ a.s. for all } t \in [0, T].$

**Proof.** Let us take a convergent sub-sequence of $(\tilde{P} \circ (\mathcal{N}_k^\star, \delta_{\mathcal{N}_k^\star}(dm)dt, \delta_{\mathcal{N}_k^\star}(dm)dt, B)^{-1})_{k \in \mathbb{N}^*}$ (possible because it is relatively compact see for instance [13, Proposition 5.4]), denote by $P^\infty$ its limit. One uses the same notation for the sub-sequence. The limit satisfies: $N_i(f) = 0$, $P^\infty$–a.e., for all $t \in [0, T]$ and $f \in C^0_{B}([0,T], \mathbb{R}^n)$, where we recall that $(\mu', \mu, \Lambda, B)$ is the canonical variable on $\mathbb{O} := (C^0_{B}(\mathbb{R}))^2 \times \mathcal{M}(\mathcal{P}_U)^2 \times \mathcal{C}^t$, and $\Lambda_t(\mathcal{Z}_{\mu}) = 1, dP^\infty \otimes dt$–a.e. $(t, \omega) \in [0, T] \times \mathbb{O}$. Notice that, by previous Lemma 3.11, $\lim_{k \to \infty} \delta_{\mathcal{N}_k^\star}(dm)dt, \mathcal{N}_k^\star) = (\Lambda, \mu) \text{ in } W_p, \tilde{P}$–a.s. Then, one has

$$\lim_{k \to \infty} W_p\left(\tilde{P} \circ (\mathcal{N}_k^\star, \mathcal{A}_k^\star) \circ \delta_{\mathcal{N}_k^\star}(dm)dt, \mathcal{N}_k^\star) = 0 \text{ in } W_p, \tilde{P}$

Then, it is enough to apply [13, Proposition 4.10] (see also [13, Proposition 4.9]) and Itô’s formula to conclude the proof. Indeed, there exists $(\beta^k)_{k \in \mathbb{N}^*}$ a sequence of $\mathcal{P}(U)$–valued $((\sigma\{F, \mathcal{X}_0, \mathcal{W}_t, \mathcal{A}_k\}) \otimes \mathcal{B}(\mathcal{P}_U))_{t \in [0,T]}$–predictable processes such that if $(\tilde{\mathcal{S}}_t^k)_{t \in [0,T]}$ is the unique strong solution of: $\tilde{S}_0^k = \tilde{X}_0$ and

$$d\tilde{S}_t^k = \int_{\mathcal{P}_U} b(t, \tilde{S}_t^k, \mu, \nu, u) \beta_t^k(\mathcal{N}_t^k)(du) \Lambda_t(d\nu)dt + \left(\int_{\mathcal{P}_U} \int_{\mathcal{U}} \sigma_t(\tilde{S}_t^k, \mu, \nu, u) \beta_t^k(\mathcal{N}_t^k)(du) \Lambda_t(d\nu)\right)^{1/2} d\mathcal{W}_t + \sigma_0 d\mathcal{B}_t,$$

then, one has, for a sub-sequence $(k_j)_{j \in \mathbb{N}^*} \subset \mathbb{N}^*$,

$$\lim_{j \to \infty} \mathbb{E}^\tilde{P}\left[\int_0^T W_p(\mathcal{N}_t^k, \mathcal{A}_t^k) dt\right] = 0 \text{ and } \lim_{j \to \infty} \mathbb{E}^\tilde{P}\left[\sup_{s \in [0,T]} W_p(\Theta_t^k, \mathcal{A}_t^k)\right] = 0,$$

where

$$\Theta_t^k := \mathbb{E}^\tilde{P}\left[\beta_t^k\left(\mathcal{N}_t^k(du)\delta_{\mathcal{S}_t^k}(dx)\right)|\mathcal{G}_t\right] \text{ and } \mathcal{A}_t^k := \mathcal{L}P(\mathcal{S}_t^k|\mathcal{G}_t) \text{ for all } t \in [0, T].$$

We define the sequence $(Q^k)_{k \in \mathbb{N}^*} \subset \mathcal{P}_U$ as follows

$$Q^k := \tilde{P} \circ (\Theta^k, \mu, \Lambda, B)^{-1}.$$

$(Q^k)_{k \in \mathbb{N}^*}$ is the sequence we are looking for.

□

Now, the next proposition states our main objective which is convergence (3.15). The proposition is just a simple combination on the one hand Proposition 3.8 (see also Proposition 3.9 for $\ell = 0$) and Lemma 3.11, and on the other hand Lemma 3.13 and Theorem 3.10. The proof is therefore omitted.

**Proposition 3.14.** Let $P \in \mathcal{P}_U$ s.t. $P(\mu' = \mu, \Lambda' = \Lambda) = 1$. There exists $(\alpha_k)_{k \in \mathbb{N}^*} \subset \mathcal{A}$ s.t.

$$\lim_{k \to \infty} W_p(\alpha_k^*, P) \text{ with for each } k \in \mathbb{N}^* P(\alpha_k^*) \text{ is defined in Equation (2.6).}$$

In addition, for any $(\alpha_k)_{k \in \mathbb{N}^*} \subset \mathcal{A}$, we can find a sequence $(Q^k)_{k \in \mathbb{N}^*} \subset \mathcal{P}_U$ such that $\mathcal{L}Q^k(\mu, \Lambda, B) = \mathcal{L}P(\mu, \Lambda, B)$ and

$$\lim_{k \to \infty} \mathbb{E}^Q_k [J(\mu', \mu, \Lambda')] = \mathbb{E}^{P(\alpha_k^*)} [J(\mu', \mu, \Lambda')].$$

(3.23)
Remark 3.15. We stress again that it is not easy to find a sequence of measure-valued control rules verifying Equation (3.23). Indeed, notice that the set $\mathcal{P}_S$ is not a closed set in general. Therefore a classical compactness argument does not work here.

3.2.3 Approximation of strong controlled McKean–Vlasov processes via $N$–interacting controlled processes

In this part, we provide the analog of Proposition 3.14 for the $N$–player game.

Proposition 3.16. Under Assumption 2.1, for any $\alpha \in A$, there exists a sequence $(\alpha_{i,N})_{i=1}^{N,N}$, satisfying for each $N \in \mathbb{N}^*$, $(\alpha_{i,N})_{i=1}^{N,N} \subset \mathcal{A}^N$ s.t.

$$\lim_{N \to \infty} P \circ (\phi_{N}^{N,X,\pi^N}, \delta_{\phi_{N}^{N,X,\pi^N}}(dm)dt, B)^{-1} = P \circ (\mu^0, \delta_{\mu^0}(dm)dt, B)^{-1},$$

where $\pi^N = (\alpha_{i,N}, \ldots, \alpha_{i,N})$.

In addition, for any sequence $(\kappa_{i,N})_{i=1}^{N,N}$, satisfying for each $N \in \mathbb{N}^*$, $(\kappa_{i,N})_{i=1}^{N,N} \subset \mathcal{A}^N$, there exists a sequence $(P^i_{i,N})_{i=1}^{N,N} \subset \mathcal{P}_S$ such that $P^i_{i,N} \circ (\mu, \Lambda, B)^{-1} = P \circ (\mu^0, \delta_{\mu^0}(dm)dt, B)^{-1}$ for each $(i, N)$ and

$$\lim_{N \to \infty} \left| \frac{1}{N} \sum_{i=1}^{N} J_i((\pi^N_{i,N}, \kappa_{i,N})) - \frac{1}{N} \sum_{i=1}^{N} E^{P^i_{i,N}} [J(\mu', \mu, \Lambda, \Lambda)] \right| = 0.$$

Proof. Using (an extension of) [15, Proposition 4.15], there exists $(\psi_{i,N})_{i=1}^{N,N}$ a sequence of Borel functions $\psi_{i,N} : [0, T] \times \mathbb{R}^n \times C^n \times C^T \to U$ such that $\alpha_{i,N} := \psi_{i,N}(t, \xi^0, W_{1,N}, B_{1,N})$ for all $t \in [0, T]$, and $\pi^N = (\alpha_{i,N}, \ldots, \alpha_{i,N})$, then one has

$$\lim_{N \to \infty} \mathbb{E}^{P} \left[ \sup_{t \in [0, T]} W_p(\phi_{i,N}^{N}X_{\pi^N}, \mu^0_{i,N}) + \int_{0}^{T} W_p(\phi_{i,N}^{N}X_{\pi^N}, \mu^0_{i,N})dt \right] = 0.$$ 

Next, by easy adaptation of Lemma 3.4 (successive application of Gronwall’s lemma), there exists a sequence $(C_N)_{N \in \mathbb{N}^*}$ converging to zero when $N$ goes to infinity satisfying: for each $i \in \{1, \ldots, N\}$, if $\pi^N_{i,N} = (\pi^N_{i,i}, \kappa_{i,N})$,

$$\mathbb{E}^P \left[ \int_{0}^{T} W_p(\phi_{i,N}^{N}X_{\pi^N_{i,i}}, \mu^0_{i,N})dt + \sup_{t \in [0, T]} W_p(\phi_{i,N}^{N}X_{\pi^N_{i,i}}, \mu^0_{i,N}) \right] + \mathbb{E}^P \left[ \sup_{t \in [0, T]} \|X_t^{i,N} - Z_t^{i,N}\|^p \right] \leq C_N,$$

where $Z^{i,N}$ denotes the unique strong solution of

$$dZ^{i,N}_t = b(t, Z^{i,N}_t, \mu^0_{i,N}, \pi^N_{i,i})dt + \sigma(t, Z^{i,N}_t, \mu^0_{i,N}, \pi^N_{i,i})dW_t + \sigma_0 dB_t \text{ with } Z^{1,N}_0 = \xi^i.$$

Therefore, $\lim_{N \to \infty} \mathbb{E}^{P} [Q^{N}(Z^{N}, \bar{Q}^{N})] = 0$, where $Q^{N} := \frac{1}{N} \sum_{i=1}^{N} L^P \left( \left( X^{N}_{t,\pi^N_{i,N}}, \varphi^{N,X,\pi^N_{i,N}, \delta_{(\kappa_{i,N}, \pi^N_{i,N})}}(du, dm)dt \right) \right)$, and $\bar{Q}^{N} := \frac{1}{N} \sum_{i=1}^{N} L^P \left( \left( \mu^0_{i,N}, \delta_{(\kappa_{i,N}, \pi^N_{i,N})}(du, dm)dt \right) \right)$.

Thanks to this result and some techniques used in proof of Lemma 3.5 (with the separability condition), one has

$$\lim_{N \to \infty} \left| \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^P \left[ \int_{0}^{T} L(t, X^{i,N}_{t,\pi^N_{i,N}}, \varphi^{N,X,\pi^N_{i,N}, \delta_{(\kappa_{i,N}, \pi^N_{i,N})}}(du, dm)dt \right] + g \left( X^{i,N}_{T,\pi^N_{i,N}}, \varphi^{N,X,\pi^N_{i,N}} \right) \right] \right| = 0.$$ 

We define the sequence $(P^i_{i,N})_{i=1}^{N,N} \subset \mathcal{P}_S$ by

$$P^i_{i,N} := P \circ (\mu^0_{i,N}, \mu^0_{i,N}, \delta_{\pi^N_{i,i}}(dm)dt, \delta_{\pi^N_{i,i}}(dm)dt, B)^{-1}$$

where $\mu^0_{i,N} := L^P (Z^{i,N}_{t,\pi^N_{i,i}|G_t})$ and $\bar{P}^i_{i,N} := L^P (Z^{i,N}_{t,\pi^N_{i,i}|G_t})$ a.s. for all $t \in [0, T]$. $(P^i_{i,N})_{i=1}^{N,N} \subset \mathcal{P}_S$ is the sequence we are looking for.
3.2.4 Proof of Theorem 2.13 (Converse Limit Theorem)

First point (i) Let \( P \in \overline{P}_{\mathcal{V}}[\varepsilon] \) be an \( \varepsilon \)-measure-valued solution. By Proposition 3.14, first, there exists a sequence \((\alpha^k)_{k \in \mathbb{N}^*} \subset \mathcal{A}\) s.t.

\[
\lim_{k \to \infty} P^{\alpha_k} = P \text{ in } \mathcal{W}_p.
\]

Let us introduce, for each \( \alpha' \in \mathcal{A}, \)

\[
\varepsilon^k := \sup_{\alpha' \in \mathcal{A}} \Psi(\alpha^k, \alpha') - E^{P^{\alpha_k}}[J(\mu', \mu, \Lambda', \Lambda)].
\]

Remark that \( \varepsilon^k \geq 0 \) for all \( k \). There exists a sequence \((\gamma^k)_{k \in \mathbb{N}} \subset \mathcal{A}\) verifying

\[
\Psi(\alpha^k, \gamma^k) - E^{P^{\alpha_k}}[J(\mu', \mu, \Lambda', \Lambda)] \geq \varepsilon^k - 2^{-k}.
\]

By the second part of Proposition 3.14, there exists a sequence \((Q^k)_{k \in \mathbb{N}^*} \subset \overline{P}_{\mathcal{V}}\) satisfying \( Q^k \circ (\mu, \Lambda, B)^{-1} = P \circ (\mu, \Lambda, B)^{-1} \) for each \( k \in \mathbb{N}^* \) and \( \lim_{k \to \infty} \sup |\Psi(\alpha^k, \gamma^k) - E^{Q^k}[J(\mu', \mu, \Lambda', \Lambda)]| = 0 \). Then, as \( P \) is an \( \varepsilon \)-measure-valued MFG solution,

\[
\varepsilon \geq \limsup_{k \to \infty} E^{Q^k}[J(\mu', \mu, \Lambda', \Lambda)] - E^P[J(\mu', \mu, \Lambda', \Lambda)] \geq \limsup_{k \to \infty} \Psi(\alpha^k, \gamma^k) - E^P[J(\mu', \mu, \Lambda', \Lambda)] \geq \limsup_{k \to \infty} \varepsilon^k.
\]

Then \( \limsup_{k \to \infty} \varepsilon^k \in [0, \varepsilon] \), and

\[
E^{P^{\alpha_k}}[J(\mu', \mu, \Lambda', \Lambda)] \geq \sup_{\alpha' \in \mathcal{A}} \Psi(\alpha^k, \alpha') - \varepsilon^k, \text{ for each } k \in \mathbb{N}^*.
\]

We can conclude.

Second point (ii) Let \( \varepsilon \in [0, \infty) \) and \( P^\alpha \in \overline{P}_{\mathcal{S}}[\varepsilon] \) with \( \alpha \in \mathcal{A} \). By Proposition 3.16, there exists a sequence \((\alpha^{i,N})_{(i,N) \in \{1, \ldots, N\} \times \mathbb{N}^*}\) such that \( \alpha^{i,N} \in \mathcal{A}^N \), and

\[
\lim_{N \to \infty} P \circ \left( \varphi^{N, X, \bar{\alpha}^N, \delta, \nu, \varphi^{N, X, \bar{\alpha}^N} (dm)dt, B \right)^{-1} = P(\mu^\alpha, \delta, \nu, (dm)dt, B)^{-1} = P(\mu, \Lambda, B)^{-1},
\]

where \( \bar{\alpha}^N = (\alpha^{i,N}, \ldots, \alpha^{i,N}) \).

In addition, for any sequence \((\kappa^{i,N})_{(i,N) \in \{1, \ldots, N\} \times \mathbb{N}^*}\) satisfying for each \( N \in \mathbb{N}^* \), \((\kappa^{i,N})_{(i,N) \in \{1, \ldots, N\} \times \mathbb{N}^*} \subset \mathcal{A}^N\), there exists a sequence \((P^{i,N})_{(i,N) \in \{1, \ldots, N\} \times \mathbb{N}^*} \subset \overline{P}_{\mathcal{S}}\) such that \( P^{i,N} \circ (\mu, \Lambda, B)^{-1} = P \circ \left( \mu^\alpha, \delta, \nu, (dm)dt, B \right)^{-1} \) for each \( (i,N) \) and

\[
\lim_{N \to \infty} \left| \frac{1}{N} \sum_{i=1}^{N} J_i(\bar{\alpha}^{\alpha^N}) - \frac{1}{N} \sum_{i=1}^{N} E^{P^{i,N}}[J(\mu', \mu, \Lambda', \Lambda)] \right| = 0.
\]

Define

\[
\varepsilon^{i,N} := \sup_{\alpha' \in \mathcal{A}^N} J_i[\bar{\alpha}^{\alpha^N} - \varepsilon^i,N] - J_i[\bar{\alpha}^{\alpha^N}].
\]

There exists a sequence of controls \((\kappa^{i,N})_{(i,N) \in \{1, \ldots, N\} \times \mathbb{N}^*}\) satisfying \( J_i[\bar{\alpha}^{\alpha^N} - \varepsilon^i,N] - J_i[\bar{\alpha}^{\alpha^N}] \geq \varepsilon^{i,N} - 2^{-N} \), for each \( i \in \{1, \ldots, N\} \). Therefore, as \( P \in \overline{P}_{\mathcal{S}}[\varepsilon] \) i.e. a \( \varepsilon \)-strong MFG solution,

\[
\varepsilon \geq \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} E^{P^{i,N}}[J(\mu', \mu, \Lambda', \Lambda)] - E^P[J(\mu', \mu, \Lambda', \Lambda)] \right)
\]

\[
\geq \limsup_{N \to \infty} \left( \frac{1}{N} \sum_{i=1}^{N} J_i[\bar{\alpha}^{\alpha^N}] - \frac{1}{N} \sum_{i=1}^{N} J_i[\bar{\alpha}^{\alpha^N}] - \frac{1}{N} \sum_{i=1}^{N} J_i[\bar{\alpha}^{\alpha^N}] \right) \geq \limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon^{i,N}.
\]

By combining this result with the first point (see proof above), we can conclude.
4 Proof of Theorem 2.18 (Existence)

4.1 Measure–valued no common noise MFG equilibrium

In this part, we discuss of the case without common noise. Let \( \sigma_0 = 0 \) (or \( \ell = 0 \)) and \( p' > p \) is fixed. In order to prove our theorem, a more adequate framework and other definitions are necessary. Let us introduce the notion of deterministic measure–valued no common noise control rule

Definition 4.1. Given \((n, q) \in C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U))\), \((n', q') \in C^n_W \times \mathcal{M}(\mathcal{P}_p^n)\) is a deterministic measure–valued no common noise control rule if (recall that \( N_t \) is defined in Equation (2.10)),

- \( n'_0 = \nu \), and \( N_t[n', n, q', q](f) = 0 \) for all \( f \in C^n_0(\mathbb{R}^n) \) and every \( t \in [0, T] \).

- For \( dt \) almost every \( t \in [0, T] \), \( q'_t(Z_{n'_t}) = 1 \).

\( \mathcal{R}(n, q) \) will denote the set of all deterministic measure–valued no common noise control rules defined as previously. We also consider

\[
\mathcal{R}^*(n, q) := \max_{(n', q') \in \mathcal{R}(n, q)} J(n', n, q', q),
\]

where we recall that

\[
J(n', n, q', q) := \int_0^T \left[ \int_{\mathcal{P}_p^n} \langle \mathcal{L}^*(t, \cdot, n, \cdot), m \rangle q'_t(dm) + \int_{\mathcal{P}_p^n} \langle \mathcal{L}^*(t, \cdot, \pi, m), n'_t \rangle q_t(dm) \right] dt + \langle g(\cdot, n), n'_T \rangle.
\]

Notice that by [13, Lemma 5.2], \( \mathcal{R}(n, q) \subset C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U)) \), and the set \( \mathcal{R}^*(n, q) \) is nonempty.

Definition 4.2. \((n^*, q^*) \in C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U))\) is a deterministic measure–valued no common noise MFG solution if \((n^*, q^*) \in \mathcal{R}^*(n^*, q^*)\). We shall denote \( \mathcal{S}^* \) all deterministic measure–valued no common noise MFG solutions.

Mention that in the following, it will be more convenient to look at \( \mathcal{R} \) as a set valued function:

\[
\mathcal{R} : (n, q) \in C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U)) \mapsto \mathcal{R}(n, q) \subset C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U)).
\]

Continuity of \( \mathcal{R} \) In the next propositions, it is shown that \( \mathcal{R} \) is both upper and lower hemicontinuous, and this is enough to conclude that \( \mathcal{R} \) is continuous. We refer to [2, chapter 17] for an overview on set valued functions.

Lemma 4.3. ([13, Lemma 5.2]) There exists a constant \( C > 0 \) (depend only of coefficients \( \sigma, b \) and \( \nu \)) such that for any \((n, q) \in C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U))\), and \((n', q') \in \mathcal{R}(n, q)\), one has

\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^n} |x|^p n'_t(dx) \leq C \left( 1 + \int_{\mathbb{R}^n} |y|^p \nu(dy) \right).
\]

Furthermore, for any \((t, s) \in [0, T] \times [0, T]\), one gets \( W_p(n'_t, n'_s) \leq C|t - s| \).

Proposition 4.4. (Upper Hemicontinuity) Let \((n, q) \in C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U))\), \( \mathcal{R}(n, q) \) is a compact set of \( C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U)) \). In addition for any sequence \((n^k, q^k)_{k \in \mathbb{N}^*} \subset C^n_W \times \mathcal{M}(\mathcal{P}_p(\mathbb{R}^n \times U))\) such that \( \lim_{k \to \infty} (n^k, q^k) = (n, q) \), and \((n^k, q^k) \in \mathcal{R}(n^k, q^k)\) for each \( k \in \mathbb{N}^* \), then \((n^k, q^k)_{k \in \mathbb{N}^*} \) is relatively compact and each limit point belongs to \( \mathcal{R}(n, q) \).

Proof. By Lemma 4.3, one finds

\[
\sup_{(n', q') \in \mathcal{R}(n, q)} \sup_{t \in [0, T]} \int_{\mathbb{R}^n} |x|^p n'_t(dx) < \infty \text{ and } \lim_{\delta \to 0} \sup_{(n', q') \in \mathcal{R}(n, q)} \sup_{t \in [0, T]} \mathcal{W}_p(n'_t, n'_{(t+\delta) \wedge T}) = 0,
\]

as \( U \) is a compact set and that for \( dt \) almost every \( t \in [0, T] \), \( q'_t(Z_{n'_t}) = 1 \), one has

\[
\sup_{(n', q') \in \mathcal{R}(n, q)} \int_0^T \int_{\mathcal{P}_p(\mathbb{R}^n \times U)} \mathcal{W}_p(m, m_0)^p q'_t(dm) dt < \infty, \text{ for any } m_0 \in \mathcal{P}_p(\mathbb{R}^n \times U).
\]
Then by Aldous criterion [23, Lemma 16.12], \( \mathcal{R}(n,q) \) is a compact set of \( C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \).

By similar way, the sequence \( (n^k, q^k)_{k \in \mathbb{N}} \) is relatively compact. By passing to the limit in the equation verified by \( (n^k, q^k, n^k, q^k) \) i.e. \( N_k(n^k, n^k, q^k, q_k^k)(f) = 0 \), for each \( (t, f) \in [0, T] \times C^2(\mathbb{R}^n) \) (see for instance [13, Lemma 4.1]), it is straightforward to check that each limit belongs to \( \mathcal{R}(n,q) \) (see [13, Proposition 5.4]).

**Proposition 4.5.** (Lower Hemicontinuity) Let \( (n, q) \in C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \) and \( (n^k, q^k)_{k \in \mathbb{N}} \) be a sequence of elements of \( C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \) such that \( \lim_{k \to \infty} (n^k, q^k) = (n, q) \), and \( (n', q') \in \mathcal{R}(n,q) \). There exists \( (n'^j, q'^j) \in \mathcal{R}(n^k_j, q^k_j) \), for each \( j \in \mathbb{N}^* \) where \( (k_j)_{j \in \mathbb{N}^*} \subset \mathbb{N}^* \) is a sub-sequence with \( \lim_{j \to \infty} (n'^j, q'^j) = (n', q') \).

**Proof.** As \( \lim_{k \to \infty} (n^k, q^k, q') = (n, q, q') \), by Lemma 3.13 and/or [13, Proposition 4.10], there exists \( (n'^j, q'^j) \in \mathcal{R}(n^k_j, q^k_j) \) for each \( j \in \mathbb{N}^* \) where \( (k_j)_{j \in \mathbb{N}^*} \) is a sub-sequences with \( \lim_{j \to \infty} (n'^j, q'^j) = (n', q') \).

**Theorem 4.6.** The set \( S^* \) is nonempty and compact.

**Proof.** Under Assumption 2.1, it straightforward to verify that \( J : C^{n,p}_W \times C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \times M(P_p(\mathbb{R}^n \times U)) \to \mathbb{R} \) is continuous. The set valued function \( \mathcal{R} \) is continuous because it is upper and lower hemicontinuous. Besides \( \mathcal{R} \) has nonempty compact convex values. By Berge Maximum theorem [2, Theorem 17.31], \( \mathcal{R}^* \) has nonempty compact convex values and is upper hemicontinuous. Consequently its graph \( Gr(\mathcal{R}^*):= \{ (n, q, n, q) : (n, q) \in \mathcal{R}^*(n, q) \} \) is closed. Let \( (n, q) \in C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \), we say that \( (n, q) \in K \) if \( (n, q) \in C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \) and

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^n} |x|^p n_t(dx) + \int_0^T \int_{P_p(\mathbb{R}^n \times U)} W_{p'}(m, m_0)^p q_t(dm)dt \leq M,
\]

where \( m_0 \) is an element of \( P_p(\mathbb{R}^n \times U) \) and \( M < \infty \) is defined by

\[
M := \sup \left\{ \int_0^T \int_{P_p(\mathbb{R}^n \times U)} W_{p'}(m, m_0)^p q_t(dm)dt + C \left( 1 + \int_{\mathbb{R}^n} |x|^p \nu(dx) \right) \right\}, \quad (n', q') \in \mathcal{R}^*(C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)))
\]

and in addition

\[
W_{p'}(n, n_0) \leq C|t-s|, \text{ for all } (t, s) \in [0, T] \times [0, T].
\]

Thanks to the above techniques, it is obvious that \( K \) is a compact set of \( C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \), and \( \mathcal{R}^* \) is a set valued function of \( K \) into itself i.e. \( \mathcal{R}^* : (n, q) \in K \to \mathcal{R}^*(n, q) \subset K \).

Let \( E \) be a Polish space, denote by \( M(E) \) the set of signed measure on \( E \). Equipped of the weak convergence topology \( \tau_\omega := \sigma(M(E), C_b(E)) \) generated by the bounded continuous function, \( M(E) \) is a locally convex Hausdorff space. Accordingly, \( C([0, T]; M(\mathbb{R}^n)) \) is a locally convex Hausdorff space. Likewise, \( M(P^p_U \times [0, T]) \) is a locally convex Hausdorff space equipped of \( \tau_\omega := \sigma(M(P^p_U \times [0, T]), C_b(\mathcal{P}(P^p_U \times [0, T]))) \). Then \( C([0, T]; M(\mathbb{R}^n)) \cap M(P^p_U \times [0, T]) \) is a locally convex Hausdorff space. One can see \( K \) as a subset of \( C([0, T]; M(\mathbb{R}^n)) \cap M(P^p_U \times [0, T]) \). As the topology of \( C([0, T]; M(\mathbb{R}^n)) \cap M(P^p_U \times [0, T]) \) induced on \( K \) is equivalent to the topology of \( C^{n,p}_W \times M(P(\mathbb{R}^n \times U)) \), we deduce that \( K \), which is a compact set of \( C^{n,p}_W \times M(P_p(\mathbb{R}^n \times U)) \subset C^{n,p}_W \times M(P^p_U \times [0, T]) \), is also a compact set of \( C([0, T]; M(\mathbb{R}^n)) \times M(P(\mathbb{R}^n \times U) \times [0, T]) \). To conclude, we apply the fixed point theorem of Kakutani–Fan–Glicksberg (see [2, Corollary 17.55]) to deduce \( S^* \) is nonempty and compact. Therefore we can find \( (n^*, q^*) \in \mathcal{R}^*(n^*, q^*) \).

**4.2 Proof of existence of non–random measure–valued MFG solution**

Now, let us prove the main result of this part. If \( P^*(d\pi', d\pi, dq', dq, db) := \delta_{(n^*, n^*, q^*, q^*)}(d\pi', d\pi, dq', dq, db)P_B(db) \in \mathcal{P}(\Omega) \), it is straightforward to check that \( P^* \) is a non–random measure–valued MFG solution where \( P_B \) is the \( \mathbb{R}^k \) Wiener measure. This result proves the point (i) of Theorem 2.18. The point (ii) is just an easy combination of Proposition 3.14 and techniques used in proof in theorem 2.13 (by using the fact that \( (n^*, q^*) \) are deterministic or equivalently \( (\mu, \Lambda) \) are not random).
References

[1] Y. Achdou and Z. Kobeissi. Mean field games of controls: Finite difference approximations. *arXiv preprint arXiv:2003.03968*, 2020.

[2] C. Aliprantis and K. Border. *Infinite dimensional analysis: a hitchhiker’s guide*. Springer–Verlag Berlin Heidelberg, 3rd edition, 2006.

[3] A. Barrasso and N. Touzi. Controlled diffusion Mean Field Games with common noise, and McKean-Vlasov second order backward SDEs. *arXiv preprint arXiv:2005.07542*, 2020.

[4] D. Blackwell and L. E. Dubins. An extension of Skorohod’s almost sure representation theorem. *Proceedings of the American Mathematical Society*, 89(4), 1983.

[5] J. F. Bonnans, S. Hadikhanloo, and L. Pfeiffer. Schauder estimates for a class of potential mean field games of controls. *Applied Mathematics & Optimization*, 83:1431–1464, 2019.

[6] P. Cardaliaguet and C.-A. Lehalle. Mean field game of controls and an application to trade crowding. *Mathematics and Financial Economics*, 12(3):335–363, 2018.

[7] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. *The master equation and the convergence problem in mean field games*, volume 201 of Annals of mathematics studies. Princeton University Press, 2019.

[8] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications I*, volume 83 of Probability theory and stochastic modelling. Springer International Publishing, 2018.

[9] R. Carmona and D. Lacker. A probabilistic weak formulation of mean field games and applications. *The Annals of Applied Probability*, 25(3):1189–1231, 2015.

[10] R. Carmona, F. Delarue, and D. Lacker. Mean field games with common noise. *The Annals of Probability*, 44(6):3740–3803, 2016.

[11] C. Castaing, P. Raynaud de Fitte, and M. Valadier. *Young Measures on Topological Spaces With Applications in Control Theory and Probability Theory / by Charles Castaing, Paul Raynaud de Fitte, Michel Valadier*. Mathematics and Its Applications ; 571. Springer Netherlands : Imprint: Springer, Dordrecht, 1st ed. 2004. edition, 2004. ISBN 1-4020-1963-7.

[12] J. Claisse, J. Zhenjie, and X. Tan. Mean Field Games with Branching. *arXiv preprint arXiv:1912.11893*, 2020.

[13] M. F. Djete. Extended mean field control problem: a propagation of chaos result. *arXiv preprint arXiv:2006.12996*, 2020.

[14] M. F. Djete, D. Possamaï, and X. Tan. McKean–Vlasov optimal control: the dynamic programming principle. *arXiv preprint arXiv:1907.08860*, 2019.

[15] M. F. Djete, D. Possamaï, and X. Tan. McKean–Vlasov optimal control: limit theory and equivalence between different formulations. *arXiv preprint arXiv:2001.00925*, 2020.

[16] M. Fisher. On the connection between symmetric n–player games and mean field games. *The Annals of Applied Probability*, 27(2):757–810, 2017.

[17] D. A. Gomes and V. K. Voskanyan. Extended deterministic mean-field games. *SIAM Journal on Control and Optimization*, 54(2):1030–1055, 2016.

[18] D. A. Gomes, S. Patrizi, and V. Voskanyan. On the existence of classical solutions for stationary extended mean field games. *Nonlinear Analysis: Theory, Methods & Applications*, 99:49–79, 2014. ISSN 0362-546X. doi: https://doi.org/10.1016/j.na.2013.12.016. URL https://www.sciencedirect.com/science/article/pii/S0362546X13004446.

[19] P. J. Graber. Linear quadratic mean field type control and mean field games with common noise, with application to production of an exhaustible resource. *Applied Mathematics and Optimization*, 74(3):459–486, 2016.
A Proof of equivalence between measure–valued MFG solution and weak MFG solution of [10, Definition 3.1]

A.1 Weak MFG solution as a particular case of measure–valued MFG solution

Proof. Let \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{W}, \bar{B}, \bar{\mu}, \bar{X})\) be a weak MFG solution (see Definition 2.9 or [10, Definition 3.1]). Let \(f \in C_b^2(\mathbb{R}^n)\), by applying Itô’s formula and taking the conditional expectation w.r.t \(\sigma\{\bar{B}, \bar{\mu}\}\), one gets

\[
\begin{align*}
\mathbb{E}^\bar{\mathbb{P}} \left[ \nabla f(\bar{X}_t - \sigma_0 \bar{B}_t) \right] &
\int_U b(t, \bar{X}_t, \mu^\pi_t, u) \tilde{\Lambda}_t(du) | \bar{B}, \bar{\mu} \right] dt + \frac{1}{2} \langle \text{Tr} \{ \sigma \sigma^\top (t, \cdot, \mu^\pi_t) \nabla^2 f(\cdot - \sigma_0 \bar{B}_t) \}, \mu^\pi_t \rangle dt
\end{align*}
\]

We define \(\tilde{\pi}_t(dx, du) := \mathbb{E}^\bar{\mathbb{P}}[\delta_\bar{X}_t(dx) \tilde{\Lambda}_t(du) | \bar{B}, \bar{\mu}]\). Therefore, one can check that \(P \in \mathcal{P}_V\) where

\[
P := \bar{\mathbb{P}} \circ (\mu^\pi, \mu^\pi, \delta_{\tilde{\pi}_t}(dm)dt, \delta_{\tilde{\pi}_t}(dm)dt, \bar{B})^{-1}.
\]
Now, we show that $P \in \overline{\mathcal{P}}_V$ i.e. the optimality condition is satisfied. Let $Q \in \overline{\mathcal{P}}_V$ s.t. $\mathcal{L}^Q(\mu, \Lambda, B) = \mathcal{L}^P(\mu, \Lambda, B)$. Let $(\Omega, \mathcal{F}, Q)$ be an extension of $(\Omega, \mathcal{F}, P)$ supporting a $\mathbb{R}^n$-valued $\mathcal{F}_t$-Brownian motion $W$, a $\mathbb{R}^n$-valued $\mathcal{F}_0$-measurable variable $X_0 \in \mathcal{L}^Q$, with $F^0$ a uniform random variable. Besides $W', X_0, F'$ and $\mathcal{G}_t$ are independent. By [13, Proposition 4.10] (see also [13, Proposition 4.9]), there exists $(\beta^k)_{k \in N^*}$ a sequence of $\mathcal{P}(U)$-valued $(\{\sigma(F', X_0', W_1'), \mathcal{G}_t\} \otimes \mathcal{B}(P_U))^t \in [0, T]$-predictable processes such that if $(\bar{X}^k)_t \in [0, T]$ is the unique strong solution of:

$$dX_t^k = \int_U b(t, X_t^k, \mu, u) \int_{\mathcal{P}_U} \beta_t^k(m)(du) \Lambda_t^i(dm)dt + \sigma(t, X_t^k, \mu_t) dW_t^i + \sigma_0 dB_t,$$

then, one has, for a sub-sequence $(k_j)_{j \in N^*} \subset N^*$,

$$\lim_{j \to \infty} \mathbb{E}^Q \left[ \int_0^T W_p(\bar{m}_t^{k_j}, m_t) \Lambda_t^i(dm)dt \right] = 0 \text{ and } \lim_{j \to \infty} \mathbb{E}^Q \left[ \sup_{s \in [0, T]} W_p(\mu_t^{k_j}, \mu_t) \right] = 0,$$

where $\bar{m}_t^k[dx, du] := \mathbb{E}^Q \left[ \beta_t^k(m)(du) \delta_{X_t^k}(dx) / \mathcal{G}_t \right]$ and $\mu_t^k := \mathcal{L}^Q(\bar{X}_t^k | \mathcal{G}_t)$ for all $t \in [0, T]$.

Therefore

$$\lim_{k \to \infty} \mathbb{E}^Q \left[ \int_0^T \int_U L(t, X_t^k, \mu_t, u) \int_{\mathcal{P}_U} \beta_t^k(m)(du) \Lambda_t^i(dm)dt + g(X_t^k, \mu_t) \right] = \mathbb{E}^Q[\mu', \mu, \Lambda', \Lambda].$$

Notice that, we can find a Borel function $\phi^k : [0, T] \times [0, 1] \times \mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}_W \times \mathcal{M}(\mathcal{P}_U) \times \mathcal{C}^t \to \mathcal{P}(U)$ satisfying $\phi^k(t, F', X_0', W_{t', \Lambda}, \mu_{t', \Lambda}, \Lambda_{t', \Lambda}, \mu_{t', \Lambda}, \Lambda_{t', \Lambda}, \Lambda_{t', \Lambda}) = \int_{\mathcal{P}_U} \beta_t^k(m)(dm) \mathcal{Q}$ a.s. On the probability space $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{P})$, we define $h^k_t := \phi^k(t, \tilde{F}, \tilde{X}_0, \tilde{W}_{t', \Lambda}, \mu_{t', \Lambda}, \tilde{R}_{t', \Lambda}, \tilde{B}_{t', \Lambda})$, where $\tilde{R} := \delta_{\mathcal{P}(U)}(dm) dt$ and $\tilde{F}$ is a uniform variable independent of $\tilde{X}_0, \tilde{W}, \tilde{B}$ and $\mu$. Let $\tilde{X}^k$ be the solution of

$$dX_t^k = \int_U b(t, \tilde{X}_t^k, \mu_t, u) \tilde{h}_t^k(du)dt + \sigma(t, \tilde{X}_t^k, \mu_t) d\tilde{W}_t + \sigma_0 dB_t \text{ with } \tilde{X}_0^k = \tilde{X}_0.$$

It is straightforward to check that $(\Omega, \tilde{F}, \overline{\mathcal{P}}, \tilde{W}, \tilde{B}, \mu, \tilde{h}^k, \tilde{X}^k)$ satisfies the point (1–4) of Definition 2.9 for all $k \in N^*$. As $(\Omega, \tilde{F}, \overline{\mathcal{P}}, \tilde{W}, \tilde{B}, \tilde{\mu}, \tilde{\Lambda}, \tilde{X}, \tilde{X})$ is a weak MFG solution and

$$\tilde{Q} \circ \left( X_t^k, \int_{\mathcal{P}_U} \beta_t^k(m)(du) \Lambda_t^i(dm)dt, W_{t', \mu, \lambda, \Lambda} \right)^{-1} = \mathbb{P} \circ \left( \bar{X}_t^k, \bar{h}_t^k(du)dt, \bar{W}, \tilde{\mu}, \tilde{B} \right)^{-1},$$

then

$$\mathbb{E}^P[\mu', \mu, \Lambda', \Lambda] \geq \lim_{k \to \infty} \mathbb{E}^\mathbb{P} \left[ \int_0^T \int_U L(t, \bar{X}_t^k, \mu_t, u) \tilde{h}_t^k(du)dt + g(\bar{X}_t^k, \mu_t) \right] = \lim_{k \to \infty} \mathbb{E}^Q \left[ \int_0^T \int_U L(t, X_t^k, \mu_t, u) \int_{\mathcal{P}_U} \beta_t^k(m)(du) \Lambda_t^i(dm)dt + g(X_t^k, \mu_t) \right] = \mathbb{E}^Q[\mu', \mu, \Lambda', \Lambda].$$

We can conclude that $P$ belongs to $\overline{\mathcal{P}}_V$. \hfill \Box

### A.2 Measure–valued as a weak MFG solution

**Proof.** Let $P \in \overline{\mathcal{P}}_V$. By [29, Theorem 1.3], there exists $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ an extension of $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $\bar{W}$, a $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted continuous process $\bar{X}$ s.t. $\mathcal{L}^\mathbb{P}(\bar{X}_0) = \nu$. Besides $\bar{X}_0, \bar{W}$, and $\mathcal{G}_t$ are independent, and

1. $\bar{X}$ satisfies

$$d\bar{X}_t = \int_U b(t, \bar{X}_t, \mu_t, u) \int_{\mathcal{P}_U} \tilde{m}_t^i(du) \Lambda_t^i(dm)dt + \sigma(t, \bar{X}_t, \mu_t) d\bar{W}_t + \sigma_0 dB_t \tilde{P}-a.s.$$

where for each $m \in \mathcal{P}_U$, the Borel function $x \in \mathbb{R}^n \to \mathcal{P}(U) \ni m^x$ satisfies $m^x(du)(dx, U) = m(dx, du)$.
2. $\mu_t = \mathcal{L}^\Pi(\mathcal{G}_t) = \mathcal{L}^\Pi(\mathcal{G}_T)$ a.s. for each $t \in [0,T]$.

3. $\mathcal{X}_{\mathcal{A}}$ is conditionally independent of $\sigma\{\mathcal{W}_{T\mathcal{A}}\} \vee \mathcal{G}_T$ given $\sigma\{\mathcal{W}_{T\mathcal{A}}\} \vee \mathcal{G}_T$.

We pose $\tilde{\Lambda}_t(du)dt := \int_0^\infty m\tilde{X}_t(du)\Lambda_t(du)dt$ and $\mu := \mathcal{L}^\Pi(\mathcal{W},\tilde{\Lambda},\tilde{X}(\mathcal{G}_T))$. It follows that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, B, \mu, \tilde{\Lambda}, \tilde{X})$ satisfies the point (1-4) of Definition 2.9. Now, let $(\tilde{\Omega}', \tilde{\mathcal{F}}', \tilde{\mathbb{F}}', \tilde{\mathbb{P}}', \tilde{W}', \tilde{\mu}', \tilde{\Lambda}', \tilde{X}')$ be a tuple satisfying (1-4) of Definition 2.9 and $\tilde{\mathbb{P}}' \circ (\tilde{X}_0, \tilde{W}', \tilde{\mu}', \tilde{B}')^{-1} = \tilde{\mathbb{P}} \circ (\tilde{X}_0, \tilde{W}, \mu, \tilde{B})^{-1}$. Let us define $\mu'' := \mathcal{L}^\Pi(\mathcal{X}'(\mathcal{B}', \mu'))$, $\tilde{\mu} := \mathbb{E}[\delta_{\mathcal{X}_t}(dx)\tilde{\Lambda}_t(du)\tilde{B}, \mu']$ and $\tilde{\mu}_t := \mathbb{E}[\delta_{\mathcal{X}_t}(dx)\theta_t(du)]$ where $(X, \theta)$ is the canonical process on $([\alpha, \beta] \times M(U))$. Then, we easily verify that

$$Q := \tilde{\mathbb{P}}' \circ (\mu'' \times \tilde{\mu}'' \times \delta_{\mathcal{G}}(du, dm))dt, \delta_{\mathcal{G}}(dm, dt, \tilde{B}')^{-1} \in \mathcal{P}_V$$

As $P \in \mathcal{P}_V$, we deduce that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, B, \mu, \tilde{\Lambda}, \tilde{X})$ satisfies the optimality condition, then he is a weak MFG solution.

\[ \Box \]

B Sketch of proof of technical results borrowed from [13]

Sketch of proof of Proposition 3.1. Recall that in the simplified case of Proposition 3.1, for a measure–valued control rule $P \in \overline{\mathcal{P}_V}$, the Fokker–Planck equation is rewritten: P–a.s.

$$d\langle f, \mu_t \rangle = \int_{\mathbb{R}_n} \int_{\mathbb{R}^2} \frac{1}{2} \nabla^2 f(x) u^2 m^x(du) \mu_t(dx) A_t(dm) dt + \int_{\mathbb{R}_n} \int_{\mathbb{R}^2} \frac{1}{2} \nabla^2 f(x) \tilde{\sigma}(m)^2 \mu_t(dx) A_t(dm) dt.$$

For each $\delta > 0$, we define $G_\delta(x) := \delta^{-1}G(\delta^{-1}x)$, $G(x) := (1 + |x|^2)^{-1} \left( \int_{\mathbb{R}} (1 + |x'|^2)^{-1} dx' \right)^{-1}$ and (recall that $A'_t(\mathcal{E}_\varepsilon) = 1$)

$$\overline{\sigma}_\delta(x, \Lambda'_t) := \int_{\mathbb{R}_n} \int_{\mathbb{R}^2} \frac{u^2}{\int_{\mathbb{R}} \frac{G_\delta(x - y') m(du, dy')} A_t(dm)} A_t(dm) = \int_{\mathbb{R}_n} \int_{\mathbb{R}^2} u^2 G_\delta(x - y') m(du, dy') A_t(dm).$$

Notice that $x \mapsto \overline{\sigma}_\delta(x, \Lambda'_t) \in \mathbb{R}$ is smooth (infinitely differentiable). Also, given $(\Lambda', \mu', \Lambda)$, let us introduce $\mu'^\delta$, the unique solution of

$$d\langle f, \mu'^\delta_t \rangle = \int_{\mathbb{R}_n} \frac{1}{2} \nabla^2 f(x) \overline{\sigma}_\delta(x, \Lambda'_t)^2 \mu'^\delta_t(dx) dt + \int_{\mathbb{R}_n} \frac{1}{2} \nabla^2 f(x) \overline{\sigma}(m)^2 \mu'^\delta_t(dx) A_t(dm) dt.$$  \hspace{1cm} (B.1)

Under Assumption 2.1, one has that (see [13, Lemma 4.2, Lemma 4.5] or [20, Lemma 2.1, Proposition 4.3])

$$\lim_{\delta \to 0} \sup_{t \in [0,T]} W_p(\mu'_t, \mu'^\delta_t) = 0 \text{ and } \lim_{\delta \to 0} \overline{\sigma}_\delta(x, \Lambda'_t)^2 = \int_{\mathbb{R}_n} \int_{\mathbb{R}^2} u^2 m^x(du) A'_t(dm), \mu'_t(dx) dt - a.e..$$

For now, let us set $\delta > 0$. Let $(\Lambda^k, \mathcal{A})_{k \in \mathbb{N}^+}$ be a sequence s.t. $\lim_{k \to \infty} \Lambda^k = (\Lambda', \Lambda)$ in $W_p$. Then, if we define $\mu'^\delta, k$ the unique solution of

$$d\langle f, \mu'^\delta, k_t \rangle = \int_{\mathbb{R}_n} \frac{1}{2} \nabla^2 f(x) \overline{\sigma}_\delta(x, \Lambda'_t)^2 \mu'^\delta_t(dx) dt + \int_{\mathbb{R}_n} \frac{1}{2} \nabla^2 f(x) \overline{\sigma}(m)^2 \mu'^\delta_t(dx) A'_t(dm) dt,$$  \hspace{1cm} (B.2)

using the uniqueness of Equation (B.1) and the regularity of the coefficients, it is straightforward that: for each $\delta > 0$,

$$\lim_{k \to \infty} \sup_{t \in [0,T]} W_p(\mu'^\delta_t, \mu'^\delta, k_t) = 0.$$ 

We set now $k \in \mathbb{N}^+$ and $\delta > 0$. We use the argument mentioned in (3.2). Recall that we place ourselves on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}} := (\mathcal{F}_t)_{t \in [0,T]}, \tilde{\mathbb{P}})$ of the probability space $(\hat{\Omega}, \mathcal{F}, \hat{\mathbb{P}})$ where we have a $\mathbb{F}$–Brownian motion $W$, a $\mathcal{F}_0$–measurable
random variable $X_0$ of law $\nu$ and a $F$ uniform variable. The variables $W, X_0, F$ and the $\sigma$-field $\mathcal{F}_T$ are independent. Let $Z'$ be a $\mathcal{F}$-adapted solution of: $Z_0^\delta,k = X_0$,

$$dZ_t^{\delta,k} = \sqrt{\alpha^{\delta,k}(t, Z_t^{\delta,k}, F)^2 + \int_{\mathcal{F}_U} \bar{\sigma}(m)^2 \Lambda_t^k(d\gamma)dW_t}$$

where $\alpha^{\delta,k}(t, x, F) := \Psi\left(\int_{\mathcal{F}_U} G_\delta(x - y)m(du, dy)\Lambda_t^k(dm)\right)(F)$. (B.3)

Using the uniqueness of Equation (B.2), we check that $\mu_{t}^{\delta,k} = \mathcal{L}^\mathcal{F}(Z_t^{\delta,k}|\Lambda^k)$ a.s. for all $t \in [0, T]$. Consequently, one has (see for instance [13, Proposition 4.9, Proposition 4.10])

$$\lim_{\delta \to 0} \lim_{k \to \infty} \sup_{t \in [0, T]} W_p(\mu_{t}^{\delta,k}, \mathcal{L}^\mathcal{F}(Z_t^{\delta,k}|\Lambda^k))$$

and

$$\lim_{\delta \to 0} \lim_{k \to \infty} \delta \mathcal{L}^\mathcal{F}(\alpha^{\delta,k}(t, Z_t^{\delta,k}, F)|\Lambda^k)(dm)dt = \Lambda'.$$

This is enough to conclude.