Equilibrium properties of a Josephson junction ladder with screening effects.

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Abstract

In this paper we calculate the ground state phase diagram of a Josephson Junction ladder when screening field effects are taken into account. We study the ground state configuration as a function of the external field, the penetration depth and the anisotropy of the ladder, using different approximations to the calculation of the induced fields. A series of tongues, characterized by the vortex density $\omega$, is obtained. The vortex density of the ground state, as a function of the external field, is a Devil’s staircase, with a plateau for every rational value of $\omega$. The width of each of these steps depends strongly on the approximation made when calculating the inductance effect: if the self-inductance matrix is considered, the $\omega = 0$ phase tends to occupy all the diagram as the penetration depth decreases. If, instead, the whole inductance matrix is considered, the width of any step tends to a non-zero value in the limit of very low penetration depth. We have also analyzed the stability of some simple metastable phases: screening fields are shown to enlarge their stability range.

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I. INTRODUCTION

Theoretical research in Josephson junction arrays (JJA) is continuously progressing through models which involve an increasing complexity. They represent a better approximation to the understanding and prediction of the many different interesting phenomena which occur in such systems. An important contribution to this advance is the inclusion of current induced magnetic fields (CIMF) developed by different groups in the last years. Taking CIMF into account is compulsory in order to provide a correct description of Josephson junction arrays at low temperature, when the penetration depth of the magnetic field is of about the cell size.
In all the cases, the study was carried out through the numerical simulation of the dynamics of the gauge invariant phase differences. Interest has been mainly focused on the effects of CIMF in the properties of arrays driven by external currents. However, we have no knowledge of studies on the ground state properties of inductive arrays.

This paper deals with the static properties of a Josephson junction ladder (JJL), with anisotropy in the Josephson couplings, in the presence of an external magnetic field (figure 1). In particular, we have calculated the ground state phase diagram of a system defined by a Hamiltonian which includes the magnetic energy due to the CIMF, in addition to the usual Josephson coupling contribution.

Recently two different groups have faced the ground state problem of a JJL in the presence of a magnetic field and in the limit of infinite penetration depth – no screening effects. In this approximation, some general properties concerning the ground states can be deduced. In regard to the ground state problem, the Hamiltonian describing the system belongs to a universality class of convex 1-D models of spatially modulated structures, such as the Frenkel-Kontorova model. This fact allows an exact description of the ground states phase diagram and the relevant elementary excitations of the system. The diagram, a function of the external field and the anisotropy parameter, consists of a series of tongues labelled by the vortex density \( \omega \). Both rational and irrational values of \( \omega \) are possible, corresponding, respectively, to commensurate and incommensurate phase configurations. The vortex density of ground state configurations as a function of the external magnetic field is a Devil’s staircase function, with plateaus for every rational value of \( \omega \). Incommensurate ground states exhibit two regimes, separated by an Aubry transition: below a certain value of the parameter that describes the anisotropy, the configuration is undefectible (no defects can be sustained) and unpinned (any external current, though infinitesimal, causes a non-zero voltage); above this value, the solution is defectible and pinned.

In this paper we use the work by Mazo et al. as starting point and include CIMF. We thus obtain a more realistic description of the ground states and, in general, of the equilibrium properties of the ladder. Such results may be of interest in understanding experiments in JJL where relevant parameters can be fixed at will.

We have used different numerical methods – effective potentials method combined with root finding methods and stability analysis of solutions, as well as dynamical relaxation – in order to find the ground state phase diagram and other metastable configurations. Results are obtained for three approximations to the calculation of the induced fields: (A) self-inductance contributions; (B) self-inductance plus nearest neighbours mutual inductance contributions; (C) full-inductance matrix. In all the cases the vortex density of the ground state, as a function of the external field, exhibits a Devil’s staircase structure. Special attention has been focused on the behaviour of the system in the small penetration depth limit. In this limit, a vortex can be described as a flux quantum concentrated in one only cell. The ground state phase diagram shows an important difference depending on the approximation made when calculating the inductance effect: if the self-inductance matrix is considered, the \( \omega = 0 \) phase, in this limit, nearly occupies all the diagram. When the whole inductance matrix is considered we find ground states with no null vortex density in a wide region of the phase diagram.

The variation of the induced flux with the penetration depth allows an estimation of the physically interesting range of values of this array parameter. We have also studied the
dependence of some of the vortex properties with the penetration depth and the anisotropy of the ladder, such as its extension and the distribution of gauge-invariant phases and the induced flux. Finally, we consider the stability of some simple metastable commensurate phases when the external field is varied. Notably, the stability intervals enlarge when the penetration depth decreases, existing a critical value of the penetration depth for the stability of each phase at zero external field.

The paper is organized as follows: in section II we introduce the model and the different methods and approximations used to compute its properties. Results on the ground state phase diagram and the stability of some simple commensurate phases are reported in section III. Different approximations to the calculation of the CIMF are discussed.

II. DESCRIPTION OF THE MODEL AND THE METHOD

The classical hamiltonian describing the system is:

\[
H = - \sum_i \left[ J_x \cos(\theta_i - \theta_{i+1} - \pi f_0 - \pi f_i) + J_x \cos(\theta_i' - \theta_{i+1}' + \pi f_0 + \pi f_i) + J_y \cos(\theta_i - \theta_i') \right] + \frac{1}{2} \Phi_0^2 \sum_{ij} f_i L_{ij}^{-1} f_j. \tag{1}
\]

Here \(\theta_i, (\theta_i')\) denotes the phase of the superconducting order parameter on the upper (lower) branch of the ladder at the \(i\)th site (see fig. 1). \(f_0\) is the magnetic flux due to the external field, which is assumed to be constant along the array. \(f_i\) is the induced flux through plaquette \(i\), a function of the currents in the ladder. Both \(f_0\) and \(f_i\) are expressed in terms of the flux quantum, \(\Phi_0\). Thus, the total magnetic flux \(\Phi_i^{tot}\) through a given plaquette \(i\) is \(\Phi_i^{tot} = \Phi^{ext} + \Phi^{ind} = \Phi_0(f_0 + f_i)\). The model is periodic in \(f_0\) with period 1 and has symmetric reflection about \(f_0 = \frac{1}{2}\) in the interval \([0, 1]\). Thus we will restrict our analysis to values of \(f_0\) within the interval \([0, \frac{1}{2}]\).

When writing Eq.(1) we have made a convenient gauge choice: we consider that the vector potential is parallel to the long axis of the ladder, and takes opposite values on the upper and lower branches (see Fig.1). In this gauge \(f_0\) and \(f_i\) are trivially related to the line integrals of the vector potential \(\vec{a}\) through a link of the ladder: \(a_{\alpha\beta} = \frac{2}{\Phi_0} \int_{\alpha}^{\beta} \vec{a} d\ell = \epsilon \pi (f_0 + f_i)\), where \(\epsilon = +1(-1)\) for upper (lower) links in the ladder and \(\epsilon = 0\) for vertical links.

\(J_\alpha\) \((\alpha = x, y)\), the Josephson coupling energy, is related to the critical current through the junction, \(I_\alpha\), by \(J_\alpha = I_\alpha \Phi_0/(2\pi)\). The inductance matrix of the array, \(L\), is defined as

\[
L_{ij} = \frac{\Phi_0}{I_{cx}} \frac{1}{8\pi^2 \lambda_\perp} \Lambda_{ij}, \tag{2}
\]

where \(\Lambda_{ij}\) is an adimensional matrix containing just geometrical coefficients (see Appendix A). \(\lambda_\perp\) is the penetration depth, defined as in

\[
\lambda_\perp = \frac{1}{4\pi^2 \mu_0 J_x a}, \tag{3}
\]

being \(a\) is the lattice spacing.
It can be seen that the configurations which minimize hamiltonian (1) comply with $\theta_i + \theta'_i = \text{const}$. Fixing this constant equal to 0 and normalizing by $J_x$ in order to work with adimensional quantities we get

$$H = -\sum_i \left[ 2\cos(\theta_i - \theta_{i+1} - \pi f_0 - \pi f_i) 
+ \frac{J_y}{J_x} \cos(2\theta_i) \right] + \frac{\Phi_0^2}{2J_x} \sum_{ij} f_i L_{ij}^{-1} f_j,$$

where the quotient $J_y/J_x$ defines the anisotropy of the ladder. To solve the ground state phase diagram we will restrict our analysis to expression (4).

We consider here three approximations to the inductance matrix: the simplest model (case A) assumes a diagonal inductance matrix. In this case, the flux induced in a given plaquette only depends on the mesh current in the same plaquette. Next step in complexity, case B includes also nearest neighbours inductances for the couplings between cells. Then, we assume $L_{ij}^{-1} = \tilde{L}\delta_{ij} + \tilde{M}\delta_{ij\pm1}$. Case C considers the full-range inductance matrix. In the first case the term in Eqs.(1) and (4) giving an account of the magnetic energy becomes

$$H_{magn} = \sum_i d_K f_i^2,$$

with $d_K = 8\pi^3 \lambda_\perp (\Lambda^{-1})_{00}$.

In case B,

$$H_{magn} = \sum_i d_K f_i^2 + \sum_i \alpha d_K (f_{i-1} + f_{i+1}).$$

In the system under simulation (we consider square cells; currents are supposed to flow within a cylinder of length $a$ and ratio 0.005a) $d_K \approx 6.8 \lambda_\perp$ and $\alpha = \frac{\tilde{M}}{J} \simeq 0.21355$.

We have used different methods to solve the problem. In the cases A and B it is possible to do it using an effective potentials method properly adapted to study our model, which is numerically equivalent to a 1D system with just next-nearest-neighbours interactions. Effective potentials method is an efficient method to study the ground state configurations of such kind of systems in the thermodynamic limit. This method is based on the computation of certain functions, the effective potentials, which contain all the relevant information on the relaxation of local fluctuations to the ground state configurations. A long computation time is required if one wants to obtain the phase diagram of the system with a high precision. This suggests the convenience of complementing the method with other procedures.

Effective potentials can provide, within a reasonable amount of time, approximate solutions to the ground state problem as a function of the external field, the anisotropy of the ladder and the penetration depth. Starting from these guesses, one can obtain more precise results by applying standard root finding methods (calculating stable solutions to $\frac{\partial H}{\partial \alpha_i} = 0$) or even dynamically (letting the approximate solution relax). We make note that root finding methods require the use of Eq. (4) to describe the system since Eq. (3) is just an adequate expression when dealing with minimum energy configurations, which is a reduced subspace of the whole system. We have checked that the same results are obtained if one applies the effective potentials method with a high precision or if one combines it with any of the complementary procedures described above. By comparing the energy curves corresponding to different configurations one can determine the border of the tongues with different vortex densities.
Moreover, making use of these procedures one can study how a ground state configuration modifies when the parameters vary. In this case, it is convenient to keep in mind that, in general, a vortex configuration is stable beyond the range of parameters in which it is the ground state solution. There, root finding methods are adequate, and they must be completed by doing the linear stability analysis of the solutions.

Model C involves the total flux matrix. Interactions between the variables extend to all the lattice and the problem can not be tackled with the effective potentials method. In this case, we consider the results achieved in approximation B as guesses and let the system evolve dynamically and relax down to the equilibrium. Details about the dynamical algorithm are given elsewhere.

III. RESULTS: GROUND STATE PHASE DIAGRAM AND STABILITY ANALYSIS

Figure 2a shows the ground state phase diagram in the case of infinite penetration depth \( \lambda_\perp \) (thus neglecting CIMF). The different tongues are characterized by the vortex density \( \omega \). This quantity is directly related to the periodicity of the configuration: a value \( \omega = \frac{p}{q} \) implies that relevant physical quantities – gauge invariant phases differences, induced fluxes... – are spatially periodic: every \( q \) plaquettes these quantities are exactly repeated. Here vortices are defined as usual. We make use of the well-known property of fluxoid quantization to define the vorticity \( n_p \) on any plaquette. The clockwise sum of the gauge invariant phases (restricted to the interval \((-\pi, \pi]\)) along the links of the cell gives
\[
\sum_{\alpha\beta\in i}(\Theta_\alpha - \Theta_\beta - a_{\alpha\beta}) = 2\pi(n_p - f^{tot}).
\]
The vortex density \( \omega \) is equal to the spatial average of \( n_p \).

As mentioned in the Introduction, the vortex density as a function of the external field is a Devil’s staircase, with plateaus for every rational value of \( \omega \). Figures 2b and 2c show the phase diagram in the cases A and B, for a penetration depth \( \lambda_\perp = 1 \), computed using the effective potentials method. Diagrams (b) and (c) are qualitatively similar to diagram (a). As one expects, the CIMF tend to push the external magnetic field out of the array: as \( \lambda_\perp \) decreases, the \( \omega = 0 \) tongue grows. There is, however, a remarkable difference between these diagrams: while in case A all the tongues but the \( \omega = 0 \) one compress, in case B the \( \omega = \frac{1}{2} \) phase does not shrink, and the rest of the phases are compressed between these two. In short we will study carefully the limit of this behaviour when \( \lambda_\perp \to 0 \).

The effect of the CIMF is to increase the critical field \( f_c \) up to which the \( \omega = 0 \) configuration is the ground state. The devil’s staircase is thus restricted to a narrower interval of values of the field.

The question arises whether CIMF are able to change qualitatively the nature of the phase diagram. In other words, we want to check if for some \( \lambda_\perp \) the array is able to expulse completely the external field and, consequently, this tongue occupies all the phase diagram. If this is not the case, is the devil’s staircase structure preserved for all the values of \( \lambda_\perp \)?

In order to gain a more complete understanding of the properties of the model and, in particular, to throw light on the previously raised questions, it is interesting to study the dependence of the vortex characteristics on the different physical parameters. Such study is firstly carried out by considering commensurate phases with a low vortex density (e.g. \( \omega = \frac{1}{128} \)) in order to prevent vortex-vortex interaction effects. In particular, we are interested in studying the vortex extension. It is directly related to the distribution of the
gauge invariant phases around the vortex barycenter and depends on the values of \( J_y \) and \( \lambda_\perp \) (see figure 3). In cells near the vortex centre, the phase decays exponentially. This decay, as distance increases, becomes smoother; for large \( i \) (where \( i \) is the distance to the centre), the phase is of the form \( \phi_i \sim i^{-3} \). Anisotropy affects the exponential part of the curve: a decrease of \( J_y \) implies a smoother exponential decay, while the long-distance behaviour remains unchanged. Instead, varying \( \lambda_\perp \) makes the whole curve shift. In cells far away enough from the centre, the flux is negative and its absolute value is a decreasing function of the distance. The negative flux is due to the sign of the mutual inductance term (see Appendix A). This feature was already reported by Phillips et al. As \( \lambda_\perp \) decreases the vortex becomes more and more localized on its central plaquette and, in the \( \lambda_\perp \to 0 \) limit, tends to identify with the fluxoid (the quantized magnitude defined above, that indicates the barycenter of the vortex). We can think of \( \lambda_\perp \) as the radius of the vortex: \( \lambda_\perp \to 0 \) implies that the vortex remains restricted just to the cell where \( n_p = 1 \).

When no CIMF are considered (\( \lambda_\perp \to \infty \) limit) the total magnetic flux through a plaquette is just the external flux, which is constant along all the array. Thus, the flux distribution along the ladder is independent on the vorticity. Then, there is no flux quantization and the vortex density is not a flux quanta density but a fluxoid quanta density. On the contrary, the situation changes drastically when CIMF are taken into account, being the number and extension of vortices directly connected to the distribution of the induced flux along the ladder. Let us consider the limit \( \lambda_\perp \to 0 \) and the \( \omega = 0 \) ground state configuration; there, the currents tend to uniformly screen the external field (in every cell \( f_i \to -f_0 \)). Thus, the array exhibits a behaviour that resembles the Meissner effect: the external field is screened by the system and no field penetrates the array. For any other value of \( \omega \), the flux distribution is quite different. In the cells where the vorticity is equal to zero, the induced flux tends to cancel the external one, being the total flux equal to zero. In the cells where the vorticity is equal to one, the induced flux \( f_i \to (1-f_0) \), being the total flux equal to one flux quantum. Then, in the \( \lambda_\perp \to 0 \) limit the fluxoid identifies with fluxon, and it is well localized in a cell of the array. This is illustrated in figure 4, where the dependence of the induced flux on \( \lambda_\perp \) is shown both in cells with vorticity 0 and 1. We have chosen a configuration with \( \omega = \frac{1}{2} \), and \( f_0 = \frac{1}{2} \), but the behaviour is general: for other values of \( \omega \) the fluxes in cells with zero vorticity depend on the distance to the nearest vortex, but this difference is of the order of \( \lambda_\perp^2 \) in the \( \lambda_\perp \to 0 \) limit. We can distinguish 3 regions: For \( \lambda_\perp > 4 \), \( |f_i| \leq 0.1 f_0 \), and considering an infinite penetration depth is a justified approximation. On the other hand, for \( \lambda_\perp < 0.12 \), we observe the low \( \lambda_\perp \) behaviour: \( |f_i| > 0.9 (n_i - f_0) \), and screening fields are dominant. Between them there is an intermediate region, around \( \lambda_\perp = 0.7 \) (where the derivative of the induced flux respect to the logarithm of the penetration depth is maximum).

The description presented in the previous paragraph permits to obtain some simple expressions for the energies of the different configurations for low values of \( \lambda_\perp \). The hamiltonian (I) consists of two components, corresponding to the Josephson and the magnetic energies. We have numerically checked that, as \( \lambda_\perp \to 0 \), the first term saturates first than the magnetic one. Thus, for low enough \( \lambda_\perp \), we can approximate the energy per plaquette by

\[
E_p = -3 \Phi_0 \frac{\pi}{2J_x N_p} (n_i - f_0 + \delta f_i) (L^{-1})_{ij} (n_j - f_0 + \delta f_j),
\]

where \( \delta f_i \sim O(\lambda_\perp) \), \( N_p \) is the total number of cells, and \( n_i = 0, 1 \) is the vorticity of cell \( i \). Let us consider first the case A in this approximation. The energy per cell of a configuration with \( \omega = \frac{p}{q} \) is

\[
E_p = -3 + d_K \left( \frac{p}{q} (1-f_0)^2 + \frac{2-p}{q} f_0^2 \right).
\]

As \( f_0 \in [0, \frac{1}{2}] \), \( (1-f_0) \geq f_0 \), and
It is easily obtained that \( E(\omega = \frac{1}{2}; f_0 = \frac{1}{2}) < E(\omega = 0; f_0 = \frac{1}{2}) \) (in particular, \( E(\omega = 0; f_0 = \frac{1}{2}) - E(\omega = \frac{1}{2}; f_0 = \frac{1}{2}) = d_{K}^2/(6\pi^2J_y) \)).

We remark that these results correspond to the case A and an extreme limit (\( \lambda_\perp \to 0 \)). Nevertheless, the devil’s staircase is observable down to low values of \( \lambda_\perp \) (we have checked this point even at \( \lambda_\perp = 0.012 \)). As an example, figure 5 shows the energy of stable configurations with different values of \( \omega \) (\( 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5} \)) as a function of the external field when \( \lambda_\perp = 0.5 \). The energy of the ground state staircase corresponds to the envelope of the curves, thus an approximation to the \( \omega(f_0) \) function can be obtained from them.

Things change when one considers a more complete approximation to the inductance matrix (models B and C). In this case the \( \omega = 0 \) phase does not fill the diagram at any value of \( \lambda_\perp \) and other commensurate phases are clearly appreciated. Before performing a rigorous analysis, we present a plausibility argument in support of this statement. Let’s begin by comparing \( \omega = 0 \) and \( \omega = \frac{1}{2} \) phases. In a \( \omega = 0 \) configuration, the currents and flux are identical in all the cells. On the other hand, a configuration with \( \omega = \frac{1}{2} \) exhibits a spatial periodicity with period \( 2a \); when \( f_0 = \frac{1}{2} \) the flux and currents on one cell are of the same module and the opposed sign respect to those on the adjacent cells. This allows us to define an effective \( d_{K} \) for both configurations, so that the magnetic energy per cell is just \( d_{K_{eff}} f_i^2 \) (see Appendix A). In case B \( d_{K_{eff}}(\omega = 0) = 10.945\lambda_\perp \) and \( d_{K_{eff}}(\omega = \frac{1}{2}) = 4.617\lambda_\perp \) while in case C \( d_{K_{eff}}(\omega = 0) = 11.176\lambda_\perp \) and \( d_{K_{eff}}(\omega = \frac{1}{2}) = 4.638\lambda_\perp \). In general, for \( w_0 < w_1 \) \( d_{K_{eff}}(w_0) > d_{K_{eff}}(w_1) \), and thus \( E(w_0; f_0 = 1/2) > E(w_1; f_0 = 1/2) \). As the energy is a continuous function of \( f_0 \) the previous inequality maintains for a range of \( f_0 \) values near \( f_0 = 1/2 \).

Moreover, it is possible to extend in a trivial way the argument previously developed for case A, and to calculate the energy per plaquette of any vortex distribution as a function of \( f_0 \). Note that a configuration with \( \omega = \frac{p}{q} \) is described by a periodic spatial structure the basic sub-array of which consists of \( q \) plaquettes containing \( p \) vortices. We can thus reduce the system to one with just \( q \) cells. In order to do that it is necessary to generalize the previous reckoning and to redefine the components of the matrix \( L \) in order to take into account the contribution of each of the infinite replicas of the basic sub-array. Thus the inductance between two cells at distance \( j \) is given by \( \hat{L}_i = \sum_n L_{0,i+nq} \), where \( n = 0, \pm 1, \pm 2, \ldots, \)

\( j = 0, \ldots, q-1 \) and \( L_{0,i} = L_{0,-i} \). In the limit \( \lambda_\perp \to 0 \) the induced flux in any cell is given by a vector \( F \equiv \{ f_a f_b f_c \ldots \} \) with \( f_i = (1 - f_0 + \delta f_i) \) or \( (-f_0 + \delta f_i) \) depending on the occupation number of the cell, \( n_i \). Thus, the energy per plaquette, up to the first order in \( \lambda_\perp \), is given by \( E_p = -3 + \frac{\phi_0^2}{\pi^2J_y} f_i(L^{-1})_{ij} f_j \), with \( f_i = n_i - f_0 \). We have computed this expression for a series of values of \( \omega \). By comparing the energies of the curves for different configurations we have obtained a Devil’s staircase, see figure 4. This figure corresponds to the case C; in case B an analogous behaviour is observed.

Hereafter, we will consider the response of the JJJ to continuous variations of the external field. We will restrict our analysis to the study of the stability intervals of some simple commensurate phases which are the ground state solution at some value of the parameters of the system (thus, we consider only ordered phases including just vortices (and no antivortices) and for which \( 0 \leq \omega \leq \frac{1}{2} \)). Such perspective, in the no screening field effects limit, has
been studied in reference\textsuperscript{3} in order to characterize the dynamical approach to equilibrium, which has been shown to lead to slow relaxation. Here we will just focus on an important difference which appears when CIMF are considered. Such study is carried out through a quasi-static computation of ordered stable configurations (local minima of Hamiltonian\textsuperscript{(I)}) with a determined vortex density, when the external field is slowly varied.

As previously mentioned, the range of stability of any vortex configuration ($\omega$) is broader than the interval of the parameters in which it is the ground state. In general, there exists a critical value of the field for the stability of each phase: for $f_0 < f_c(\omega)$ the phase $\omega$ is no more stable. The loss of stability when $f_0$ decreases occurs in this way: decrease in the external field implies an increase in the supercurrent through the horizontal links in order to maintain the vortex density in the array. The instability of the state takes place when the supercurrent in one link reaches its maximum value. At this point any small change in the field can not be sustained by an increase of the currents and the vortex structure becomes unstable, and the system relaxes to a new vortex configuration. That would be the process for the changes of vortex density when the external field is varied or when the thermal noise is high enough to produce a vortex to jump over the energy barrier of the metastable phase and then the system approaches to some other more stable phase.

In the limit of neglecting screening effects\textsuperscript{3} $f_c(\omega) > 0$ for all $\omega \neq 0$, and thus when $f_0 = 0$ only the $\omega = 0$ phase is stable. The inclusion of CIMF changes this situation. As $\lambda_\perp$ decreases the range of stability of a given phase enlarges. Moreover, for each configuration $w$ there is a critical value for the penetration depth $\lambda_{\perp c}(w)$: if $\lambda_\perp < \lambda_{\perp c}(w)$ the phase is stable at every value of the external field. Let’s consider the two extreme cases: the configuration containing one single vortex and the $w = \frac{1}{2}$ phase. The repulsive character of the vortex-vortex interactions implies that the stability of the configuration containing a single vortex is a necessary condition for the stability of any phase with $0 < \omega \leq \frac{1}{2}$, so that the particular value of $\lambda_{\perp}$ at which stability of the configuration occurs ($\lambda_{\perp c}^w(f_0)$) is an upper bound for the stability of the phases with $0 < \omega \leq \frac{1}{2}$. On the other hand, stability of the $\omega = \frac{1}{2}$ phase ensures the stability of any other phase with $0 < \omega < \frac{1}{2}$, so that the particular value of $\lambda_{\perp}$ at which stability of the $\omega = \frac{1}{2}$ occurs ($\lambda_{\perp}^{1/2}(f_0)$) is an lower bound for the stability of the phases with $0 < \omega < \frac{1}{2}$. Thus, $\lambda_{\perp}^{1/2}(f_0) \leq \lambda_{\perp c}(f_0) \leq \lambda_{\perp c}^w(f_0)$. Figure\textsuperscript{2} shows the regions of stability of the vortex configurations. For values of the parameters above the curves (region S), any vortex configuration is stable. In region S-I, as we move towards the origin of coordinates, the different states become unstable (in the order of decreasing $w$). In I the only stable configuration is that with $w = 0$. Looking again at the supercurrents in the array, we see that as $\lambda_{\perp}$ decreases the gauge invariant phase differences of the metastable configurations approach to zero and thus the supercurrents are lower, rendering the phase more stable.

**IV. DISCUSSION**

In section III we have presented the phase diagram of a Josephson junction ladder in the presence of screening magnetic field effects. Numerical results evidence the existence of a series of tongues labelled by the mean vorticity $\omega$. Such magnitude exhibits a devil’s staircase structure when the external field is varied.

When CIMF are considered, the system presents a behaviour that resembles the Meissner
effect: the self-induced field tends to push the external field out of the array, causing the growing of the $\omega = 0$ tongue and the shrinking of the range of parameters where the devil’s staircase is observed. We have compared the results of using different approximations when calculating the induced fluxes. If only the self-inductance term is considered, for a value of $\lambda_\perp$ low enough the $\omega = 0$ phase occupies all the phase diagram except for a tiny region near the $f_0 = \frac{1}{2}$ line. However states with inserted fluxons are stable and, moreover, their range of stability increases as $\lambda_\perp$ decreases. Instead, if one takes into account the whole inductance matrix, the critical field for the $\omega = 0$ phase (the frustration above which the configuration is no more the ground state) remains lower than $\frac{1}{2}$ for all the values of the penetration depth. Thus commensurate phases with vortices are always clearly visible in the phase diagram.

The three approximations made: A (self-inductance), B (self-inductance plus nearest neighbours mutual inductance terms) and C (full-inductance matrix) correspond to different distributions of the relative weights of the inductance matrix components. We remark that these distributions are a function of the geometry of the currents flowing in the array (see Appendix). Let’s consider the case in which currents flow inside cylindric tubes of radius $r$ and length equal to the lattice spacing $a$ (the qualitative conclusions can be extended to any kind of cross section). If $r << a$, $\Lambda_{i,i} > |\Lambda_{i,i+1}| >> |\Lambda_{i,i+j}| (j > 1)$ and approximation B (considering just the self-inductance plus the nearest neighbours terms in $\Lambda$) is justified. As $r$ increases, the terms $\Lambda_{i,i+j} (j > 1)$ also increase, but are yet too small. They give just small corrections to the final results. In a narrow range of $r$ values around $r \sim 0.25$ the dominant contribution to the inductance is self-term (and A is a good $0-\theta$ order approximation). Finally, for greater $r$, $|\Lambda_{i,i+j}|/\Lambda_{i,i}$ cannot be neglected and considering the whole inductance matrix is compulsory.

This behaviour (the existence of an infinite set of ground states as the parameters vary which show a Devil’s staircase structure) is characteristic of a broad class of spatially modulated structures with convex interparticle interactions. In the limit of neglecting screening field effects ($\lambda_\perp \to \infty$) it is well established the equivalence, regarding the ground state problem, of hamiltonian (1) with a one-dimensional XY model with anisotropy and the ground state problem of the system is equivalent to the one of a Frenkel-Kontorova model with convex interparticle interaction, which allows for applying the Aubry theory for this class of models.

However, and despite the qualitatively similar behaviour shown by our simulations, the inclusion of CIMF renders it difficult to establish any equivalence between model (1) and the 1-D models named above. This point is beyond the scope of this paper and remains as an open question deserving future research.

The introduction of the CMIF allows to study the continuous variation of the system from the full penetration of the external field ($\lambda_\perp \to \infty$; $\Phi_{i}^{tot} = f_0$, for $\lambda_\perp \geq 4$) limit to the case of an array where all magnetic flux, in the first order of $\lambda_\perp$, appears quantized ($\Phi_{i}^{tot} = 0$ or $1 + O(\lambda_\perp)$, for $\lambda_\perp \leq 0.12$). This transition is reflected in figure 4.

An appealing question is that of the behaviour – analysis of the robustness and stability specially – of different metastable configurations of a system under the variations of the external parameters. It provides information on the energy landscape of a system and other properties of the phase space which can determine interesting situations such as a constrained dynamics or slow relaxation processes. The inclusion of the CIMF enlarges the
stability range of the vortex configurations, as we have explicitly shown in the extreme cases of one single vortex and a $w = \frac{1}{2}$ phase. Quite recently Hwang, Ryu and Stroud\cite{14} have extended the results on the ground state properties of the Josephson junction ladders in the large penetration depth limit, to treat the IV characteristics of a ladder array. They found that when ”current is injected perpendicular to the ladder edges, the critical current is unchanged from its $f = 0$ value up to a penetration field of $f_{c1} \simeq 0.12$ flux quanta per plaquette”. This result can be easily understood from the considerations on the stability of the one vortex configuration we have made above. At low values of the external field the ground state phase configuration of a bidimensional square Josephson junction array has a vortex density different from zero and the existence of vortices produces a critical current lower than that of the $\omega = 0$ phase. In the ladder, however, the situation is quite different: at low values of $f$ the ground state in the ladder is the $\omega = 0$ phase and, moreover, as we have described above, the configuration with just one vortex is unstable and the $\omega = 0$ phase is the only stable attractor for arbitrary initial phase configurations. Consequently, at low values of the external field the critical current of the ladder is expected not to change, since it depends on the vortex density. However, we have shown that the critical value of the field for the stability of the one single vortex configuration is $f_{c1}^v = 0.1175 \pm 0.00065$ for an isotropic ladder. For values of the field above $f_{c1}^v$ different vortex configurations are stable; thus, arbitrary initial phase configurations generically relax to different possible metastable configurations, which can be described as irregular arrays of vortices. In that situation the critical current is essentially associated with the depinning transition of these vortex phases, which occurs at a lower value of the external current. Hwang, Ryu and Stroud give a value around 0.12, quite close to our prediction of $f_{c1}^v = 0.1175 \pm 0.00065$.

As we have seen above, as $\lambda_\perp$ decreases, the value of $f_{c1}^v$ also decreases, vanishing at $\lambda_\perp = 1.812 \pm 0.018$. Then, the diagram of stability (Fig. 7) allows to make the following conjecture on the behaviour of the critical current of the Josephson junction ladder when current induced magnetic fields are taken into account: as the penetration depth decreases, the range of values of the external field for which the critical current remains unchanged also decreases and it vanishes when $\lambda_\perp = 1.812 \pm 0.018$. This conjecture is based on the natural association between the critical current of the ladder and the stability of the one vortex configuration.

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APPENDIX A: THE INDUCTANCE MATRIX

The $L$ matrix is obtained in the following form: we have applied the Biot-Savart law in order to calculate the magnetic field induced on a link by all the currents circulating in the array. We thus obtain

$$\int_\alpha^\beta \vec{a}^{\text{ind}} d\vec{l} = \sum_{\gamma\delta} FF_{\alpha\beta;\gamma\delta} I_{\gamma\delta}, \tag{7}$$

where $FF$ is the form matrix, which depends on the geometry of the array, and $I_{\alpha\beta}$ the total current passing through link $\alpha\beta$. If the links $\alpha\beta$ and $\gamma\delta$ are perpendicular we take $FF_{\alpha\beta;\gamma\delta} = 0$.

The self-inductance term $FF_{\alpha\beta;\alpha\beta}$ depends strongly on the form of the current. If, for example, it is supposed to flow within a tube of length $l$ with circular cross section of radius $r$ it is given by

$$FF_{\alpha\beta;\alpha\beta} = 2l \left( \log \left( \frac{2l}{r} \right) - \frac{3}{4} \right). \tag{8}$$

If $l$ is measured in meters, $FF$ is given in $10^{-1}$ Henries.

Instead, the mutual inductances between different links are sensibly the same as that of the filaments through the centers of their cross sections, even when the links are very close. In particular, in the case of cylindric currents, the mutual inductances are absolutely independent from the radius.

In order to express (7) as a function of adimensional quantities we introduce $ff = \frac{4\pi}{\mu_0 a} FF$, thus obtaining

$$a^{\text{ind}}_{\alpha\beta} = \frac{2\pi}{\Phi_0} \int_\alpha^\beta \vec{a}^{\text{ind}} d\vec{l} = \frac{1}{4\pi \lambda_\perp} \sum_{\gamma\delta} ff_{\alpha\beta;\gamma\delta} i_{\gamma\delta}, \tag{9}$$

where currents are normalized to the critical current of the link $I_c$ (in the case of an anisotropic ladder, where $I_x \neq I_y$, we take $I_c$ as $I_{cx}$). $\lambda_\perp$ is the penetration depth, see eq. (3).

It is easy to obtain an equivalent description of the self-field effect in terms of the mesh currents and the magnetic flux, as required in equation (4). The magnetic flux on $i$-cell is given by

$$\Phi_i^{\text{ind}} = \oint_i \vec{a}^{\text{ind}} d\vec{l} = \frac{\Phi_0}{2\pi} \sum_{\alpha\beta \in i} a^{\text{ind}}_{\alpha\beta}, \tag{10}$$

where the sum is over the four links $\alpha\beta$ of cell $i$, and can be expressed by a linear equation of the form

$$\sum_{\alpha\beta \in i} a^{\text{ind}}_{\alpha\beta} = A_{i;\alpha\beta} a^{\text{ind}}_{\alpha\beta}. \tag{11}$$

We have used greek and roman symbols to denote, respectively, links and cells.
On the other hand, also the mesh currents \((i_i)\) are related to the link currents \((i_{\alpha\beta})\) through a linear operator
\[
i_{\alpha\beta} = \sum_{\alpha\beta \in i} B_{\alpha\beta;ii} i_i.
\] (12)

Combining (9), (10), (11) and (12) we obtain
\[
\Phi^\text{ind}_i = L_{ij} I_j , \quad L_{ij} = \frac{\Phi_0}{I_c} \frac{1}{8\pi^2 \lambda_\perp} \Lambda_{ij} , \quad \Lambda_{ij} = \sum_{\alpha\beta} \sum_{\gamma\delta} A_{i;\alpha\beta} FF_{\alpha\beta;\gamma\delta} B_{\gamma\delta;i}. \tag{13}
\]

The \(\Lambda_{ij}\) elements depend on the distance \(|i - j|\) between the cells considered. The general properties of matrix \(\Lambda_{ij}\) are: \(\Lambda_{00}\) is positive, and \(\Lambda_{ij} < 0\) for \(i \neq j\); for two cells far away enough (\(|i - j| \geq 10\)) the mutual inductance is \(|\Lambda_{ij}| \sim |i - j|^{-3}\), as calculated in [3].

In the following, we will consider that currents flow inside cylindric tubes with circular cross section of radius \(r\). If currents are supposed to have a very small cross section (compared to the lattice spacing, \(a\)), then \(ff_{\alpha\beta;\alpha\beta} \gg ff_{\alpha\beta;\gamma\delta}\). Then it can be easily verified that \(\Lambda_{ii} \sim 8\log(1/r)\), and \(\Lambda_{ii+1} \sim -2\log(1/r)\). The other terms \(|\Lambda_{i,i+j}|_{(j > 1)} << \Lambda_{ii}\) can be neglected. As the cross section increases, \(ff_{\alpha\beta;\alpha\beta}\) becomes smaller while the rest of the \(ff\)’s remain sensibly the same; \(\Lambda_{ii}\) decreases, \(\Lambda_{i,i+1}\) increases and changes sign at \(r \sim 0.24a\) and \(\Lambda_{i,i+j}, (j > 1)\) remain the same. There is a range of \(r\) values \(r \in [0.2, 0.28]\) where \(|\Lambda_{ii}| > 20|\Lambda_{i,j}|(j \neq i)\).

Thus we can establish \(r\) ranges where the different approximations A, B and C made in the paper are acceptable. If \(r << a, \Lambda_{ii} > \Lambda_{i,i+1} \gg \Lambda_{i,i+j}, (j > 1)\) and approximation B (considering just the self-inductance plus the nearest neighbours terms in \(\Lambda\)) is justified. As \(r\) increases, the terms \(\Lambda_{i,i+j}, (j > 1)\) also increase, but are yet too small. They give just small corrections to the final results. In a narrow range of \(r\) values around \(r \sim 0.25\) the dominant contribution to the inductance is self-term (and \(A\) is a good 0 – th order approximation). Finally, for greater \(r\) \(|\Lambda_{i,i+j}|/\Lambda_{ii}\) cannot be neglected and considering the whole inductance matrix is compulsory.

In our calculations, we have considered square cells; for the study of case C we have considered an intermediate value of \(r\) (currents are supposed to flow within a cylinder of length \(a\) and ratio 0.005\(a\)). In this case, \(\Lambda_{0,0} = 38.194\) and \(\Lambda_{i,j}/\Lambda_{0,0} \equiv \{ 1, -0.20332, -0.0040118, -0.0010570 \ldots \} \). We can now calculate the equivalent \(d_K\) in the case \(f_0 = \frac{1}{2}\) for the configurations \(\omega = 0\) and \(\omega = \frac{1}{2}\). This can be easily made if one considers the spatial periodicity of the solutions. If \(\omega = 0\) all the cells in the array have the same flux and current; thus we can define an effective self-inductance matrix as
\[
\hat{f}_i = (L_{ii} + 2 * (L_{i,i+1} + L_{i,i+2} + ...)) i_i = L_{\text{eff}} i_i, \tag{14}
\]
where factor 2 is due to the sum of the contributions from the cells on the left and on the right. The terms \(L_{ij}, i \neq j\) are negative and thus \(L_{\text{eff}} < L_{ii}\). Now \(d_{K\text{eff}}(\omega = 0) = \frac{8\pi^3}{L_{\text{eff}}} = 11, 176\Lambda_\perp\).

In an analogous way, one can calculate the value of \(L_{\text{eff}}\) for a \(\omega = \frac{1}{2}\) solution: for \(f_0 = \frac{1}{2}\) the flux and the current in one cell have the same magnitude and the inverse sign of those in the adjacent plaquettes. Thus
\[ f_i = (L_{i;i} + 2 \times (-L_{i;i+1} + L_{i;i+2} - \ldots)) \hat{\iota}_i = L_{eff} \hat{\iota}_i, \]

where now \( L_{eff} > L_{ii} \) and \( d_{K_{eff}}(\omega = \frac{1}{2}) = \frac{8\pi^3}{L_{eff}} = 4,638 \lambda_{\perp} \).
REFERENCES

1 For a recent view on the state-of-the-art see, e.g., *Macroscopic Quantum Phenomena and Coherence in Superconducting Networks*, edited by C. Giovannella and M. Tinkham (World Scientific, Singapore, 1995).

2 A. Majhofer, T. Wolf and W. Dieterich, Phys. Rev. B 44, 9634 (1991); D. Domínguez and J. V. José, Phys. Rev. Lett. 69, 514 (1992); J. R. Phillips, H. S. J. van der Zant, J. White and T. P. Orlando, Phys. Rev. B 47, 5219 (1993); D. Domínguez and J. V. José, Phys. Rev. B 53, 11692 (1996).

3 J. J. Mazo, F. Falo and L. M. Floría, Phys. Rev. B 52, 10433 (1995).

4 C. Denniston and C. Tang, Phys. Rev. Lett. 75, 3930 (1995).

5 An extensive list of references on the Frenkel-Kontorova model can be found in: L. M. Floría and J. J. Mazo, to appear in Adv. Phys.

6 S. Aubry, in *Structures et Inestabilités*, ed. by C. Godreche (Editions de Physique, Les Ulis, France, 1985), pp. 73-194.

7 Hamiltonian (I) is adequate in the classic regime, where charging effects are neglectible. Granato has studied the quantum ladder problem in E. Granato in *Quantum dynamics of submicron structures*, edited by H. A. Cerdeira, B. Kramer and G. Schöen (Kluwer Academic Publishers, Dordrecht), 627 (1995).

8 T. P. Orlando, J. E. Mooij and H. S. J. van der Zant, Phys. Rev. B 43, 10218 (1991).

9 S. Marianer and L. M. Floría, Phys. Rev. B 38, 12054 (1988); L. M. Floría and R. B. Griffiths, Numerische Mathematik 55, 565 (1989).

10 R. B. Griffiths and W. Chou, Phys. Rev. Lett. 56, 1929 (1986); W. Chou and R. B. Griffiths, Phys. Rev. B 34, 6219 (1986); R. B. Griffiths in *Fundamental Problems in Statistical Mechanics VII*, edited by H. van Beijeren (North-Holland, Amsterdam, 1990), pp. 69-110.

11 J.C. Ciria and C. Giovannella, to be published.

12 A. Nuvoli, A. Giannelli, J.C.Ciria and C. Giovannella, Nuovo Cimento 16, 2045 (1994).

13 D. Domínguez and J.V. José, Int. J. in Modern Physics B 8, 3749 (1994).

14 I. Hwang, S. Ryu and D. Stroud, Phys. Rev. B 53, R506 (1996)
FIGURES

FIG. 1. Schematic representation of the Josephson junction array we study here: an anisotropic ($J_x \neq J_y$) ladder, in the presence of an external field. The sites denote superconducting islands and the crosses the junctions themselves. Right-most plaquette shows the mesh current $i_i$ and the gauge choice, here $f_i^{\text{tot}} = f_0 + f_i$ where $f_0 = \frac{H_0 a^2}{\Phi_0}$ is the flux due to the external field and $f_i$ the induced flux in the plaquette $i$.

FIG. 2. Ground state phase diagrams of the JJL obtained using the method of effective potentials. Each phase is defined by the value of $\omega$ and, for clarity, only a few of the transition lines are represented. Figure (a) shows the results for the no screening field case (or $\lambda_\perp \to \infty$). (b) Phase diagram for a $\lambda_\perp = 1.0$ ladder using approximation A (diagonal inductance matrix) to the calculation of the induced fluxes. (c) Phase diagram for a $\lambda_\perp = 1.0$ ladder using approximation B (self plus nearest neighbours inductances) to the calculation of the induced fluxes.

FIG. 3. Vortex shape as a function of the penetration depth $\lambda_\perp$ and the anisotropy. We consider an isotropic 128-cell ladder with just one vortex. The whole inductance matrix is used. $f_0 = 0$. The shape is symmetric, and we just draw the right-half of the vortex (the origin of coordinates is the central plaquette of the ladder). The figure shows the gauge-invariant phase difference along the horizontal links belonging to the upper branch of the ladder. We compare the cases with $J_y = J_x, \lambda_\perp = 1$ (black circles) $J_y = 0.4 J_x, \lambda_\perp = 1$ (squares) and $J_y = J_x, \lambda_\perp = 0.012$ (rhombs). The inset shows the induced flux around the central plaquette, in the same cases as before.

FIG. 4. Induced flux as a function of $\lambda_\perp$ for a configuration $\omega = \frac{1}{2}$, at $f_0 = \frac{1}{2}, J_x = J_y$. We draw the flux through two plaquettes with vorticity 0 and 1, respectively. We consider the C case: whole inductance matrix. In the inset we show the derivative of the induced flux respect to the logarithm of the penetration depth. We can distinguish three regions: the extreme limits $\lambda_\perp > 4$ (the induced flux is $f_i = 0 + \delta/\lambda_\perp$) and $\lambda_\perp < 0.12$ ( $f_i = n_i - f_0 + \delta\lambda_\perp$) and an intermediate region around $\lambda_\perp \sim 0.7$, where the derivative is maximum.

FIG. 5. Energies of different configurations as a function of frustration ($\epsilon_\omega(f_0)$) for a penetration depth $\lambda_\perp = 0.5$ in the case A. We study configurations $w = 0, 1/5, 1/4, 1/3, 2/5$ and 1/2 (in order of decreasing slope). We consider an isotropic ladder ($J_x = J_y$). The ground state energy at each value of $f_0$ is given by the envelope of the curves. The inset shows the value of $w$ corresponding to the minimum energy curve for each $f_0$.

FIG. 6. Devil’s staircase observed in case C in the limit $\lambda_\perp \to 0$ for an isotropic ladder, calculated as explained in the text.
FIG. 7. Border between stability and instability regions in the parameter space for the extreme cases of one single vortex (black points) and \( w = 1/2 \) configuration (rhombs) in an isotropic ladder. In the first case we have considered a 128-cell ladder. We have checked that the curves fit to a quadratic function: 

\[
f_0 = \alpha \left( \beta + \frac{1}{\lambda_{\perp}(J_0)} \right) \ast \left( \frac{1}{\lambda_{\perp}\text{(0)}} - \frac{1}{\lambda_{\perp}(J_0)} \right),
\]

\( \lambda_{\perp}(0) \) is the penetration depth below which a configuration is stable at \( f_0 = 0 \), and \( \alpha \beta \frac{1}{\lambda_{\perp}(0)} = f_c \), where \( f_c \) is the value of the frustration below which a configuration is no more stable in the limit of no inductance \( (\lambda_{\perp} \to \infty) \).

For a configuration with one single vortex, \( f_c = 0.1175 \pm 0.00065 \) and \( \lambda_{\perp}(0) = 1.812 \pm 0.018 \); in the \( w = 1/2 \) case, \( f_c = 0.215 \pm 0.001 \) and \( \lambda_{\perp}(0) = 1.197 \pm 0.006 \).