CONSTRUCTION OF QUASI-PERIODIC SOLUTIONS FOR DELAYED PERTURBATION DIFFERENTIAL EQUATIONS

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Abstract. We employ the Craig-Wayne-Bourgain method to construct quasi-periodic solutions for delayed perturbation differential equations. Our results not only implement the existing literatures on constructing quasi-periodic solutions for DDE by the KAM method and space splitting technique, but also provide an example of application of multi-scale analysis method to non-selfadjoint problem.

1. Introduction

1.1. Background and motivation. There exist plenty of literatures on the periodic theory for delay differential equations (DDEs) by various methods (see e.g. [14, 15, 18, 28, 29, 30]). However, as far as we know, it seems little attentions have been paid on the quasi-periodic theory for delay systems, besides the bifurcation and numerical arguments on the model problems. The main difficulty in the construction of quasi-periodic solutions (qp-solution for short) is the famous small divisor problem when the hyperbolicity is absent.

The study of quasi-periodic solutions for DDEs dates back to the 1960’s. In [16], Halanay studied the qp-solutions of linear DDEs

\[ \dot{x}(t) = L(t)x(t) + f(t), \]

where \( L(t) \) and \( f(t) \) are quasi-periodic with the same frequency. By imposing some Diophantine type conditions, Halanay gave sufficient and necessary conditions for the existence of qp-solutions in the particular case of \( L(t) \) being constant or periodic. Later, Halanay and Yorke proposed the open problem on the existence of qp-solutions of (1.1) for \( L(t) \) being quasi-periodic in their survey paper [17].

There seems little substantial progress on the quasi-periodic theory for DDEs until recent years. In [21], Li and de la Llave considered the linear DDEs with quasi-periodic perturbation

\[ \dot{x}(t) = Ax(t) + Bx(t - \tau) + \epsilon f(\omega t, x(t), x(t - \tau), \xi). \]

In light of the parameterization method, they transformed the problem on the existence of qp-solution into finding an embedding \( K_{\epsilon, \xi} : \mathbb{T}^d \to C([\tau, 0], \mathbb{R}^n) \times \mathbb{T}^d \) such that

\[ F_{\epsilon, \xi} \circ K_{\epsilon, \xi} = K_{\epsilon, \xi} \circ R_\omega, \]

where \( R_\omega(\theta) = \theta + \omega \) and \( F_{\epsilon, \xi} \) is the time-one solution operator for (1.2) on the phase space \( C([-\tau, 0], \mathbb{R}^n) \times \mathbb{T}^d \). To solve the above functional equation, Li and de la Llave made fully use of the space splitting of \( C([-\tau, 0], \mathbb{R}^n) \) induced from the linear equation

\[ \dot{x}(t) = Ax(t) + Bx(t - \tau) \]

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and applied the Nash-Moser iterations. At each step, they employed the exponential trichotomy to solve the linearized equation on the tangent bundle, and it is on the finitely dimensional center subbundles where the small divisor problem is overcome.

Later, Li and Yuan \cite{23} considered the persistence of qp-solutions for autonomous DDEs
\begin{equation}
\dot{x}(t) = A(\xi)x + B(\xi)x(t-\tau) + \epsilon f(x(t), x(t-\tau), \xi).
\end{equation}

With the aid of spectrum decomposition for the associated linear delay equation, they wrote (1.3) as an ODE on infinitely dimensional space $BC = C([-\tau, 0], \mathbb{R}^n) \oplus X_0$, whose invariance equation on the center subspace is
\begin{equation}
\begin{aligned}
\dot{\phi}(t) &= \omega(\xi) + \epsilon M_1(\phi, I_1)\Psi_N(0, \xi)f(\phi, I, y_t, \xi), \\
\dot{I}_1(t) &= \epsilon M_2(\phi)\Psi_N(0, \xi)f(\phi, I, y_t, \xi).
\end{aligned}
\end{equation}

By a sequence change of variables (linear in $I_2$ and $y_t$, i.e. the variables in the hyperbolic direction), the authors applied the KAM technique to obtain an integrable normal form, which guaranteed the existence of qp-solution for (1.3).

More recently, Li and Shang \cite{22} constructed qp-solution for DDEs with an elliptic type degenerate equilibrium, whose proof was also based on the decomposition of the extended phase space $BC$ according to the spectrum of the linear delay equation. See \cite{1,12,19,25} for more references on the application of KAM method to DDEs. It is worthy noticing that, in \cite{1,12,21,22,23}, the phase space decomposition (according to the spectrum of autonomous linear DDE) plays an important role, making it possible to deal with small divisor problem on the finitely dimensional center subspace.

However, when $B = 0$ in (1.2) and (1.3), the associated linear equations do not involve the time delay and thus become linear ODEs. Such kinds of equations fall into the scope of the so called delayed perturbation differential equations, taking the form of
\begin{equation}
\dot{x}(t) = g(t, x(t)) + \epsilon f(x, t),
\end{equation}
which attract lots of attentions (see e.g. \cite{7,8,11,13,27} and references therein) over the years. Now a very natural question is whether systems (1.2) and (1.3) with $B = 0$ still have qp-solutions. Although the equations at hand look simpler, we are no longer able to apply the powerful space decomposition technique to attack the nonlinearity involving time delay. Even for the linear equation
\begin{equation}
\dot{x}(t) = Ax(t) + \epsilon A'(\omega t)x(t-\tau) + \epsilon g(\omega t), \quad x \in \mathbb{R}^n,
\end{equation}
utilizing a naive quasi-periodic change of variable on $\mathbb{R}^n$ turns out to be difficult to solve the above equation up to $O(\epsilon^2)$. For that reason, we have to resort to other methods when pursuing qp-solutions.

In a series of papers \cite{2,3,4,5}, Bourgain developed an alternative profound method which was originally proposed by Craig and Wayne in \cite{10}, in order to overcome small divisor problem and the unavailability of the second Melnikov condition when studying Hamiltonian PDEs. In contrast with the KAM theory, the CWB method (named after Craig,Wayne and Bourgain) is more flexible in dealing with resonant cases and finds its application in the Anderson location theory (see \cite{26} and references therein), in spectrum theory for Schrödinger operator \cite{6}, and in the construction of periodic and quasi-periodic solutions for ODEs and PDEs (see \cite{2,3,4,10,20}).
1.2. Main result. The nonlinear delayed perturbation differential equation under consideration in this paper is
\begin{equation}
\dot{x}(t) = Ax(t) + ef(x(t - \tau)) + eg(\omega t).
\end{equation}

The frequency $\omega$ is considered as parameters on which parameter excision is taken such that (1.4) admits qp-solution for the admissible frequency whenever the perturbation is small enough.

To begin with, we state our basic assumptions.

\begin{enumerate}
\item[(H1)] The functions $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $g : \mathbb{T}^d \to \mathbb{R}^{2n}$ are real analytic.
\item[(H2)] The constant coefficient matrix $A$ admits $n$ pair of simple purely imaginary eigenvalues $\pm i \lambda_j$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. The associated eigenvectors are $\{v_j, \bar{v}_j \in \mathbb{C}^{2n} : 1 \leq j \leq n\}$ with $Av_j = i \lambda_j v_j$.
\end{enumerate}

It is known that the small divisor problem arise from the resonance between the tangent frequencies (the external forcing) and the normal frequencies (the elliptic eigenvalues). For that reason, we concentrate on the case of the matrix $A$ containing only purely imaginary eigenvalues. Furthermore, under assumption (H2), making linear change of variables transforms (1.4) into a Schrodinger-like differential equation involving time delay
\begin{equation}
-ivy'(t) = \Lambda y(t) + ef(\gamma(t - \tau), \bar{y}(t - \tau)) + eg(\omega t), \quad \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n).
\end{equation}

We still denote the nonlinearity and inhomogeneous terms by $f$ and $g$ respectively, which are analytic but no longer satisfy the reality condition.

**Theorem 1.1.** Consider equation (1.4) on $(x, \theta) \in \mathbb{R}^{2n} \times \mathbb{T}^d$ with $\omega \in \mathcal{U} \subset \mathbb{R}^d$. Assume (H1) and (H2) hold and let $0 < \eta < 1$. There exists an $\epsilon^* > 0$ and a constant $C^* = C^*(d, n) > 0$ such that for all $0 < \epsilon < \epsilon^*$, there is a subset $\mathcal{U}_\epsilon$ of $\mathcal{U}$ satisfying $\text{mes}[\mathcal{U}_\epsilon]/\text{mes}[\mathcal{U}] \geq 1 - C^* \eta$ such that, for all $\omega \in \mathcal{U}_\epsilon$, there is an analytic (in time $t$) quasi-periodic solution of equation (1.4) with frequency $\omega$.

Theorem 1.1 is an immediate result of the iteration lemma 4.1 and a quantitative description of $\epsilon^*$ is given in Remark 4.2. The proof of the theorem is delayed to section 4.

This paper is devoted to introduce the Craig-Wayne-Bourgain method to construct qp-solution for a class of DDEs for which the phase splitting technique for functional differential equations is not applicable. In this respect, our paper is complementary to [21, 22, 23] where KAM techniques and space splitting play a role. Besides, the method in this paper might also find its application in population dynamics when the interactive populations under consideration live in a fluctuating (especially quasi-periodic) environment (see [9, 11, 31]).

As we shall see, one essential part of the Craig-Wayne-Bourgain method is to apply the multi-scale analysis to construct the inverse of the linearized operator $T = D + \epsilon S$ from Newton iteration, where $D$ is a diagonal matrix. A basic requirement for multi-scale analysis is that $S$ should be a Toeplitz matrix with entries decaying rapidly off the diagonal. Usually, this is indeed the case since the entries of $S$ originate from the Fourier coefficients of some smooth function. However, due to the existence of time delay, the Toepliz property is not at hand immediately after taking eigenvector-Fourier expansion. Nevertheless, this can be resolved by multiplying an invertible diagonal matrix, but making the new matrix $T$ by no means of self-adjoint. Fortunately, the multi-scale analysis method is rather robust and independent of the self-adjointness.

To avoid too much technique complexity, we impose the delayed perturbation equation as simple as possible. Some extensions can be easily obtained. For instance, it is flexible to consider autonomous delayed perturbation differential equation, to which it suffices to combine
where, with some abuse of notations, is added.

satisfies Toeplitz property, we multiply both sides of equations in (2.1) by $L$ and $(2.2)$.

where the symbol $\hat{}$ describes the Fourier coefficients of functions and the conjugation of (1.5) is added.

Let

$$\mathcal{L} = \{ m = (\mu, j, k) : \mu = \pm 1, 1 \leq j \leq n, k \in \mathbb{Z}^d \} \subset \mathbb{Z}^{d+2}.$$ 

Then equation (2.1) becomes a nonlinear lattice problem on $\mathcal{L}$. To ensure the linearized operator satisfies Toeplitz property, we multiply both sides of equations in (2.1) by $e^{i(k,\omega)\tau}$ and obtain

$$\mathcal{F}[y] \equiv Dy + \epsilon \mathcal{W}[y] + \epsilon g = 0,$$

where, with some abuse of notations,

$$y(-1, j, k) = \tilde{y}_j(k), \quad g(-1, j, k) = e^{i(k,\omega)\tau} \tilde{g}_j(k),$$

$$y(+1, j, k) = \tilde{y}_j(k), \quad g(+1, j, k) = e^{i(k,\omega)\tau} \tilde{g}_j(k),$$

$D$ is a diagonal matrix with

$$D(m) = (\mu(k, \omega) + \lambda_j)e^{i(k,\omega)\tau}, \quad m = (\mu, j, k),$$

and

$$\mathcal{W}[y](-1, j, k) = [f_j(y, \bar{y})'](k),$$

$$\mathcal{W}[y](+1, j, k) = [\bar{f}_j(y, \bar{y})'](k).$$

The paper is organized as follows. In section 2 an overview of the analysis, on the construction of q.p-solutions for (1.5), is illustrated. Some technical preliminaries and notations are summarized at the beginning. In section 3, we show how to construct and control the inverse of the linearized operator in the Newton equation, which exhibits the main idea of the multi-scale analysis method. In section 4, we state and prove the iteration lemma, based on which we give a proof of of our main theorem.

2. Preliminary

In this section, we give an overview of analysis on the Schrodinger-like equation (1.5) and prepare some useful lemmas.

Making the Ansatz that (1.5) does have a respond quasi-periodic solution, we transform (1.5) into a lattice algebraic equation by taking eigenvector-Fourier expansion. To solve the nonlinear lattice equation, we apply Nash-Moser iterations to effectively improve the corrections when an approximate solution is given. To solve the Newton equation, we need to construct the inverse of a matrix of large scale, in which the multi-scale analysis method play a role. As mentioned earlier, this requires the linearized operator enjoying the Toeplitz property.

2.1. Reduction to nonlinear lattice problem. Making the Ansatz that equation (1.5) does have a quasi-periodic solution with frequency $\omega$, we obtain from eigenvector-Fourier expansion that the coefficients must satisfy the nonlinear equation

$$\begin{cases}
(-\langle k, \omega \rangle + \lambda_j)y_j(k) + \epsilon e^{-i(k,\omega)\tau} [f_j(y, \bar{y})'](k) + \epsilon \tilde{g}_j(k) = 0, \\
(\langle k, \omega \rangle + \lambda_j)y_j(k) + \epsilon e^{-i(k,\omega)\tau} [\bar{f}_j(y, \bar{y})'](k) + \epsilon \bar{g}_j(k) = 0,
\end{cases}$$

where the symbol $\hat{}$ describes the Fourier coefficients of functions and the conjugation of (1.5) is added.

Let

$$\mathcal{L} = \{ m = (\mu, j, k) : \mu = \pm 1, 1 \leq j \leq n, k \in \mathbb{Z}^d \} \subset \mathbb{Z}^{d+2}.$$ 

Then equation (2.1) becomes a nonlinear lattice problem on $\mathcal{L}$. To ensure the linearized operator satisfies Toeplitz property, we multiply both sides of equations in (2.1) by $e^{i(k,\omega)\tau}$ and obtain

$$\mathcal{F}[y] \equiv Dy + \epsilon \mathcal{W}[y] + \epsilon g = 0,$$

where, with some abuse of notations,

$$y(-1, j, k) = \tilde{y}_j(k), \quad g(-1, j, k) = e^{i(k,\omega)\tau} \tilde{g}_j(k),$$

$$y(+1, j, k) = \tilde{y}_j(k), \quad g(+1, j, k) = e^{i(k,\omega)\tau} \tilde{g}_j(k),$$

$D$ is a diagonal matrix with

$$D(m) = (\mu(k, \omega) + \lambda_j)e^{i(k,\omega)\tau}, \quad m = (\mu, j, k),$$

and

$$\mathcal{W}[y](-1, j, k) = [f_j(y, \bar{y})'](k),$$

$$\mathcal{W}[y](+1, j, k) = [\bar{f}_j(y, \bar{y})'](k).$$
Note that the diagonal matrix $D$ depends also on the frequency parameter $\omega \in \mathcal{V} \subset \mathbb{R}^d$, and the elements on the diagonal (except $k = 0$) are no longer of real valued due to the presence of time delay.

For later application, we introduce some notations and phrases here. The measure of a set $\mathcal{V} \subset \mathbb{R}^d$, denoted by $\text{mes}[\mathcal{V}]$, always refers to the Lebesgue measure. The sharp symbol $\#$ represents the total number of the elements for a finite set. For any subset $\Lambda$ of $\mathbb{Z}^d$, we denote $L_\Lambda = \{(\mu, j, k) \in L : k \in \Lambda\}$ and write the restriction of $T$ on $L_\Lambda$ by $T_\Lambda$. Given an integer $N > 0$, we denote the restriction of $T$ on $\{(\mu, j, k) \in L : |k| \leq N\}$ by $T_N$ for short. For a vector $y : L \to \mathbb{C}$, we define the truncation operator by

$$(\Gamma_N y)(m) = \begin{cases} y(m), & |k| \leq N; \\ 0, & \text{otherwise}. \end{cases}$$

For any $k \in \mathbb{Z}^d$ and any set $\Lambda \subset \mathbb{Z}^d$ containing $k$, we call $(T_\Lambda)^{-1}$ the local inverse of $T$ at $k$ with the neighborhood $\Lambda$. Given a point $k \in \mathbb{Z}^d$ and a set $U \subset \mathbb{Z}^d$, $k + U$ denotes the translation set $\{k + l : l \in U\}$. To avoid confusion, we use the notation $\Lambda \setminus B$ for the set theoretical difference. The symbols $\vee$ and $\wedge$ describes the maximum and minimum operators respectively.

2.2. Newton equation and Multi-scale analysis. We shall apply Nash-Moser iterations to solve the nonlinear lattice equation $(2.2)$. Roughly speaking, given an approximate solution $y$, we try to improve the error by solving the Newton equation on $L$

$$(2.4) \quad T\Delta \equiv D\Delta + \epsilon S\Delta = -\mathcal{F}[y],$$

where $\Delta$ is a correction of $y$, the matrix

$$S = \mathcal{V}'[y]$$

is the linearized operator of $\mathcal{V}$ at $y$. As a infinitely dimensional matrix, the product is defined by

$$(S\Delta)(m) = \sum_{m' \in L} S(m, m')\Delta(m').$$

Clearly, if the diagonal part $D$ is uniformly bounded away from zero, then $T$ can be inverted by a Neumann series for sufficiently small perturbation. However, when looking into the term $\mu(k, \omega) + \lambda_j$, one immediately realizes that the small divisor problem prevents the diagonal from being dominant. As a result, we call those lattice points $m = (\mu, j, k) \in L$ the singular sites if $D(\mu, j, k) = O(\epsilon)$.

To overcome the small divisor problem, we employ the multi-scale analysis method to solve the matrix equation $(2.4)$. As usual, we take advantage of the truncation technique and consider

$$T_N\Delta = \Gamma_N\mathcal{F}[y]$$

instead. The basic idea is to construct local inverses in the neighborhood of singular sites and then to apply the coupling lemma (see Lemma 2.1) to paste the local inverses together.

From the definition of $\mathcal{V}$, it follows that the linearized operator or the matrix $S$ enjoys the Toeplitz property with respect to $k$. More precisely, given any $\Lambda$ of $\mathbb{Z}^d$, $k, k' \in \Lambda$ and $q \in \mathbb{Z}^d$, there is

$$S((\mu, j, k + q), (\mu', j', k' + q)) = S((\mu, j, k), (\mu', j', k')).$$

However, this is not true for the diagonal matrix $D$. Keeping this in mind, we realize that the construction of local inverses in some neighborhood of $k$ can be transformed into finding the
local inverse of $T$ around $k = 0$, but with some modifications on the diagonal part. To make it precise, we introduce an extra parameter $\sigma \in \mathbb{R}$ and define

$$T^\sigma = D^\sigma + eS,$$

where

$$D^\sigma(m) = (\mu(k, \omega) + \mu \sigma + \lambda_j)e^{i(k(\omega)+\sigma)}, \quad m = (\mu, j, k).$$

It then follows that

$$T^\sigma|_{q_{+}\Lambda}((\mu, j, k + q), (\mu', j', k' + q)) = T^{\sigma+(q, \omega)}|_{\Lambda}((\mu, j, k), (\mu', j', k')).$$

for any set $\Lambda \subset \mathbb{Z}^d$ containing zero.

Since \((q, \omega) : q \in \mathbb{Z}^d\) is dense on the real line, a discussion of \((T^\sigma_N)^{-1}\) for the full parameter range (of $\sigma$) is also applicable to the restriction of translated intervals. For typical $\sigma$ and some $0 \in \Lambda \subset \mathbb{Z}^d$ of large scale, we decompose

$$T^\sigma_N = \left(\begin{array}{cc} T^\sigma_{\Omega_1} & eP \\ eQ & T^\sigma_{\Omega_2} \end{array}\right),$$

and assume that \((T^\sigma_{\Omega_1})^{-1}\) can be established by induction hypothesis. Moreover, $\Omega_2$ is of small size, which in particular is a singleton due to our assumption (H2). Then we can formally write

$$(T^\sigma_N)^{-1} = \left(\begin{array}{cc} (T^\sigma_{\Omega_1})^{-1} + e^2(T^\sigma_{\Omega_1})^{-1}PQ(T^\sigma_{\Omega_1})^{-1} & -e(T^\sigma_{\Omega_1})^{-1}PQh^{-1} \\ -eh^{-1}Q(T^\sigma_{\Omega_1})^{-1} & h^{-1} \end{array}\right),$$

where

$$h = T^\sigma_{\Omega_2} - e^2(T^\sigma_{\Omega_1})^{-1}P.$$  

Note that the function $h$ depends also on the frequency parameter $\omega$. To establish and control $h^{-1}$, it suffices to exclude some parameters $\omega$ such that $h$ stays away from zero in a reasonable way. Due to the simplicity of our problem, there are various methods to achieve it. Here we still adopt the powerful Malgrange’s preparation theorem in [4, Lemma 8.12] to replace $h$ by approximated polynomials, and then take parameter excision for those polynomials. In this fashion, the analysis presented in this paper can be easily generalized to the case of non-simple purely imaginary eigenvalues. Indeed, it is great advantage of the multi-scale analysis method to copy with problems with multiple resonance.

### 2.3. Technique lemmas.

Firstly, we write below the resolvent identity which is frequently used in this paper. Let $\Lambda = \Lambda_1 + \Lambda_2$ be disjoint union and $\Lambda$ be bounded. The resolvent identity for a matrix $T$ defined on $\Lambda$ is

$$T^{-1}_\Lambda = (T^{-1}_{\Lambda_1} + T^{-1}_{\Lambda_2}) - (T^{-1}_{\Lambda_1} + T^{-1}_{\Lambda_2})(T - T_{\Lambda_1} - T_{\Lambda_2})T^{-1}_\Lambda,$$

and for $m \in \Lambda_1$, a pointwise resolvent identity is

$$T^{-1}_\Lambda(m, m') = \begin{cases} T^{-1}_{\Lambda_1}(m, m') & \text{if } m' \in \Lambda_1, \\
\sum_{m_1 \in \Lambda_1, m_2 \in \Lambda_2} T^{-1}_{\Lambda_1}(m_1, m_2)T(m_1, m_2)T^{-1}_\Lambda(m_2, m'), & \text{if } m' \notin \Lambda_1, \end{cases}$$

whenever the involved inverses exist.

*For instance, one can apply the standard degree theory or the implicit function theorem to calculate the zero points of $h$. 
Next, we refer to [4, Lemma 8.12] for a quantitative version of Malgrange’s preparation theorem. Roughly speaking, for an analytic function

$$h(z; \omega) = z^d + \sum_{1 \leq j \leq d} a_j(\omega)z^j + \text{h.o.t.}, \quad |z| < \delta, \quad |\omega - \omega_*| < \rho,$$

we can find a $d$-degree polynomial $p$ and an analytic function $Q = o(1)$ such that $h = (1 + Q)p$ holds “locally” on $|z| < \delta^\prime < \delta$ and $|\omega - \omega_*| < \rho^\prime \leq \rho$. Moreover, the derivatives of $p$ and $Q$ with respect to $\omega$ are well controlled. In this paper, since the singular cluster stays bounded along the iterations (hence the degree $d$ in the function $h$ is fixed), there is also a simpler version of the preparation theorem, which can be found in [20, Lemma 21.4].

Finally, to construct the inverse of the linearized operator at each Newton step, we apply the coupling lemma in [4] and cite it here with some according modifications.

**Lemma 2.1.** (see [4, Lemma 5.3]) Assume $T$ satisfies the off-diagonal estimate

$$|T(m, m')| < e^{-|k-k'|^\rho}, \quad k \neq k'.$$

Let $\Lambda$ be an interval in $\mathbb{Z}^d$ and assume $\Lambda = \cup_{\alpha} \Lambda_\alpha$ a covering of $\Lambda$ with intervals $\Lambda_\alpha$ satisfying

(a) $|T^{-1}_\Lambda (m, m')| < B$,

(b) $|T^{-1}_\Lambda (m, m')| < K^{-c}$ for $|k-k'| > \frac{k}{100}$,

(c) for each $k \in \Lambda$, there is a $\alpha$ such that

$$B_k(k) \cap \Lambda = \{k' \in \Lambda : |k' - k| \leq K\} \subset \Lambda_\alpha,$$

(d) $\text{diam} \Lambda_\alpha < C'K$ for each $\alpha$.

If $C > C_0(d)$ and $B, K$ are numbers satisfying

$$\log B < \frac{1}{100}K^c, \quad \text{and} \quad K > K_0(c, C', d),$$

then

$$|T^{-1}_\Lambda (m, m')| < 2B,$$

and

$$|T^{-1}_\Lambda (m, m')| < e^{-\frac{1}{2}|k-k'|^\rho}, \quad \text{for} \quad |k-k'| > (100C'K)^{-1}.$$  

3. Construction of $(T^{\sigma}_N)^{-1}$

Throughout the rest of the paper, we write $\ll$ in estimates in order to suppress various multiplicative constants, which depend only $d, n, \tau, \mathcal{M}, \lambda, j$ and could be made explicit, but need not be. The norms used below are the Euclidian norm for vectors and the induced norms for matrices. We also write $a \sim b$ to indicate $a \approx b$ and $b \ll a$.

Let $T = D + \epsilon S$ be the linearized operator of $\mathcal{F}$ at some approximate solution $y$. Recall the definition of $T^{\sigma}$ in (2.5). Our goal in this section is to construct polynomials to derive and control $(T^{\sigma}_N)^{-1}$ for some large scale $N$.

The basic assumptions in this section is given below.

**(A1)** Melnikov condition:

$$|\langle k, \omega \rangle \pm \lambda_j \pm \lambda_{j'}| \geq \frac{\gamma}{|k|^{1/2}} \quad \text{for} \quad k \in \mathbb{Z}^d \setminus \{0\}, \quad |k| \leq 100N_0, \quad 1 \leq j, j' \leq n.$$

**(A2)** The matrix $S$ admits the off-diagonal exponential decay

$$|\partial_{m'}^\alpha S(m, m')| \ll e^{-|k-k'|}, \quad \text{for} \quad \alpha = 0, 1.$$
(A3) There exists some open set $\mathcal{U} \subset \mathcal{H}$ of $\omega$ such that, when $\omega \in \mathcal{U}$, the separation property holds. More precisely, assume $N' \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ satisfying

$$(N')^C \leq N,$$

$$4N' < |k| < (N')^C.$$  

Then for

$$\sigma_1 - \sigma_2 = \langle k, \omega \rangle,$$

the matrices $(T_{N'}^\sigma)^{-1}$ and $(T_{N'}^\sigma)^{-1}$ do not both fail the property:

$$(3.1) \quad ||(T_{N'}^\sigma)^{-1}|| \leq \Phi(N'),$$

$$|T_{N'}^\sigma(m, m')| \leq e^{-\frac{1}{2N'}|k-k'|^2} \quad \text{for} \quad |k-k'| \geq (\rho_{N'}^{-1} \log N')^C,$$

where $\rho_{N'} = (\log N'/ \log N_0)^{-1} C$ and $\Phi(N') = (N')^C$.

3.1. Separation of singular clusters. Denote $N = N_r$ and define inductively

$$N_{s-1}^C \sim N_s, \quad 1 \leq s \leq r,$$

and $N_0$ is sufficiently large. Consequently, 

$$C_0 = \rho_{N_r} \sim 10^{-s}.$$  

Let

$$(3.2) \quad \Omega = (-1,1) \times \{1,2,\cdots,n\} \times [-N,N]^d.$$  

Denote for brevity

$$k_0 \oplus N' = (k_0 + [-N', N']^d) \cap [-N,N]^d$$

for $k_0 \in [-N,N]^d$.

Consider $T_{N'}^\sigma$ with $N' = N_{r-1}$. If $(T_{k_0 \oplus N_{r-1}}^\sigma)^{-1}$ satisfies (3.1) for all $k_0 \in [-N,N]^d$, then it suffices to apply the coupling lemma (or essentially to apply the resolvent identity) to construct the inverse of $T_{N'}^\sigma$ and to control $(T_{N'}^\sigma)^{-1}$. In what follows, we always treat the worse cases, i.e., there exist some $k_0$ such that $(T_{k_0 \oplus N_{r-1}}^\sigma)^{-1}$ does not satisfy (3.1). Let $\Lambda_{r-1}$ be the set of all $k_0$ such that $(T_{k_0 \oplus N_{r-1}}^\sigma)^{-1}$ fails (3.1). Then it follows from assumption (A2) that $\Lambda_{r-1}$ is an interval (in $\mathbb{Z}^d$) containing $k_0$ and is of size at most $8N_{r-1}$. Moreover, for $k_0$ falling outside of $\Lambda_{r-1}$, there is

$$|| (T_{k_0 \oplus N_{r-1}}^\sigma)^{-1} || \leq \Phi(N_{r-1}),$$

$$|T_{k_0 \oplus N_{r-1}}^\sigma(m, m')| \leq e^{-\frac{1}{2\rho_{N_{r-1}}}|k-k'|^2} \quad \text{for} \quad |k-k'| \geq (\rho_{N_{r-1}}^{-1} \log N_{r-1})^C.$$  

Consider $T_{N'}^\sigma$ with $N' = N_{r-2}$. Assume, also in worse case, that there exists $k_0$ such in $\Lambda_{r-1}$ such that $(T_{k_0 \oplus N_{r-2}}^\sigma)^{-1}$ fails (3.1). Then we can repeat the analysis as before to get $\Lambda_{r-2}$ and its associated properties. By continuing the process, we eventually obtain a sequence of decreasing intervals

$$[-N,N]^d \supset \Lambda_{r-1} \supset \Lambda_{r-2} \supset \cdots \supset \Lambda_1 \supset \Lambda_0,$$

where $\Lambda_s$ is of size at most $8N_s$ for $s \geq 1$ and $\Lambda_0$ is of size $8N_0$. By enlarging the size of $\Lambda_s$, we can also ensure

$$(3.3) \quad \Lambda_s \supset (\Lambda_{s-1} + [-2N_s, 2N_s]^d) \cap [-N,N]^d, \quad s \geq 1,$$
which results in the size of $\Lambda_s$ at most $12N_s$ for $s \geq 1$. Furthermore, for $k$ lying in $\Lambda_s \setminus \Lambda_{s-1}$ with $s \geq 1$, there is

\begin{equation}
\| (T_{k \in N_{s-1}}^r \sigma)^{-1} \| \leq \Phi(N_{s-1}),
\end{equation}

\begin{equation}
| (T_{k \in N_{s-1}}^r \sigma)^{-1}(m, m') | \leq e^{-\frac{1}{\rho_{s-1}} |k-k'|} \quad \text{for} \quad |k - k'| \geq (\rho_{s-1}^{-1} \log N_{s-1})^2.
\end{equation}

For $T_{\Lambda_0}^r$, using assumption (A1), we have the following proposition.

**Proposition 3.1.** Assume

\[ \Omega_2 = \{ m = (\mu, j, k) \in \mathcal{L} : |D^r(m)| < \epsilon_1, k \in \Lambda_0 \}
\]

is not empty. If

\begin{equation}
\epsilon_1 > \epsilon \epsilon' < \frac{1}{100} \min_{1 \leq j, j' \leq n, j \neq j'} \left\{ 1, \lambda_j, |\lambda_j - \lambda_j'| \right\},
\end{equation}

and

\begin{equation}
\epsilon_1 < \gamma N_0^{-10d},
\end{equation}

then $\Omega_2$ is a singleton.

**Proof.** Recall the definition of $D^r$ in (2.6). For any $m = (\mu, j, k) \in \Omega_2$, we have

\[ |\langle k' - k, \omega \rangle + \mu' \lambda_j - \mu \lambda_{j'} | < 2\epsilon_1. \]

If $k \neq k'$, we obtain from assumption (A1) and (3.6) that the left hand side of the above inequality is greater than

\[ \frac{\gamma}{|k' - k|^{10d}} \geq \frac{\gamma}{(10N_0)^{10d}} > 10\epsilon_1, \]

which leads to a contradiction. With $k = k'$ and the smallness of $\epsilon_1$ in (3.5), we get $\mu = \mu'$ and $j = j'$. This completes the proof. \qed

Suppose $\Omega_2 = \emptyset$, the inverse of $T_{\Lambda_0}^r$ can be well controlled by applying Neumann series. We also consider the worse case that $\Omega_2 \neq \emptyset$ and hence a singleton, denoted by

\[ \Omega_2 = \{ m_s \} = \{(\mu_s, j_s, k_s)\}. \]

Decompose

\begin{equation}
\{-1, 1\} \times \{1, \cdots, n\} \times \Lambda_0 = \Omega'_1 + \Omega_2.
\end{equation}

Let $\Omega = \Omega_1 + \Omega_2$ (cf. (3.2)), where

\[ \Omega_1 = \bigcup_{0 \leq s \leq r} \Omega_{1,s}, \]

and

\[ \Omega_{1,0} = \Omega'_1, \]
\[ \Omega_{1,s} = \{-1, 1\} \times \{1, \cdots, n\} \times (\Lambda_s \setminus \Lambda_{s-1}), \quad 1 \leq s \leq r-1, \]
\[ \Omega_{1,r} = \{-1, 1\} \times \{1, \cdots, n\} \times ([-N, N]^d \setminus \Lambda_{r-1}). \]
3.2. Analysis of $(T^\sigma_{\Omega_1})^{-1}$. Given $m \in \Omega_{1,s}$ for some $0 \leq s \leq r$, we define

$$\Omega_1 = \Gamma_1 + \Gamma_2,$$

where

$$\Gamma_1 = \begin{cases} \{-1, 1\} \times \{1, \cdots, n\} \times (k \oplus N_{s-1}), & \text{if } s \geq 1, \\ \Omega_{1,0}, & \text{if } s = 0. \end{cases}$$

In the cases of $s \geq 2$ and $s = 0$, it follows from (3.3) and (3.7) respectively that $\Gamma_1 \cap \Omega_2 = \emptyset$. When $s = 1$, we further assume without loss of generality that

$$\Lambda_0 \ni k \oplus N_0,$$

which ensures $\Gamma_1 \cap \Omega_2 = \emptyset$. Indeed, this can also be done by enlarging the size of $\Lambda_0$.

We summarize below a proposition on the decay property of $(T^\sigma_{\Gamma_1})^{-1}$ with $0 \leq s \leq r$.

**Proposition 3.2.** Under the assumptions (A1)-(A3), if conditions (3.5)–(3.6) and

$$N_0^{\frac{c_1}{4}} < \epsilon_1,$$

hold, we have the following properties for $m \in \Omega_{1,s} \subset \Omega_1$ and $0 \leq s \leq r$.

(i) For $1 \leq s \leq r$, there is

$$\|(T^\sigma_{\Gamma_1})^{-1}\| \leq \Phi(N_{s-1}),$$

and

$$\|(T^\sigma_{\Gamma_1})^{-1}(m, m')\| < e^{-\frac{1}{2} \rho_{s-1} |k-k'|} \text{ for } |k-k'| \geq (\rho_{s-1} \log N_{s-1})^{c_2}.$$ (3.10)

(ii) For $s = 0$, there is

$$\|(T^\sigma_{\Gamma_1})^{-1}\| \leq \Phi(N_0),$$

$$\|(T^\sigma_{\Gamma_1})^{-1}(m, m')\| < e^{-|k-k'|} \text{ for } k \neq k',$$

and

$$\|(T^\sigma_{\Gamma_1})^{-1}(m, m')\| < \frac{4}{\gamma}(1 + |k - k|)^{10d}.$$ (3.12)

Statement (i) is an immediate result of (3.4) and the definition of $\Gamma_1$. The proof of statement (ii) is based on the Neumann series, which appears frequently in this paper. For the moment, we show a detailed proof and omit similar arguments afterwards.

**Proof.** For $s = 0$, we have $\Gamma_1 = \Omega_{1,0} = \Omega'_{1}$. Then, by (3.7) and the definition of $\Omega_2$, there is $|D^\sigma(\mu, j, k)| \geq \epsilon_1$ for $(\mu, j, k) \in \Gamma_1$ and consequently $\|(D^\sigma)_{\Gamma_1}^{-1}\| \leq \epsilon_1^{-1}$. Writing $(T^\sigma_{\Gamma_1})^{-1}$ into Neumann series, we obtain from (3.9) that $\|(T^\sigma_{\Gamma_1})^{-1}\| \leq 2\epsilon_1^{-1} < N_0^{c_1/2} = \sqrt{\Phi(N_0)}$ as long as

$$\epsilon \|S\| \epsilon_1^{-1} \leq \epsilon \|S\| \|S\| < \frac{1}{4}.$$ (3.13)

Moreover, for $m, m' \in \Omega_1$ with $k \neq k'$, we have

$$(T^\sigma_{\Gamma_1})^{-1}(m, m') = \sum_{l \geq 1} (-1)^{l}e^{l}|(D^\sigma)^{-1}S|^l(m, m')(D^\sigma)^{-1}(m').$$

We see that for $l \geq 1$

$$\|e(D^\sigma)^{-1}S|^l(m, m')\| = \sum_{m_1, m_2, \cdots, m_{l-1} \in \Gamma_1} [e(D^\sigma)^{-1}S](m, m_1) \cdots [e(D^\sigma)^{-1}S](m_{l-1}, m')$$

$$\leq \sum_{l \geq 1} e^{-|k-k'| l} \cdots \frac{\epsilon}{\epsilon_1} e^{-|k-k'| l} \leq \frac{\epsilon}{\epsilon_1} e^{-|k-k'| l},$$
and then the off-diagonal exponential decay of \((T_{\Omega_\rho}^\sigma)^{-1}(m, m')\) in (3.11) follows.

It remains to verify (3.12). Using (3.6), we get

\[
D^\sigma(\mu, j, k) = |\mu(\sigma + \langle k, \omega \rangle + \mu \lambda_j) e^{\langle \sigma + (k, \omega) \rangle \tau}|
\]

\[
= |\sigma + \langle k, \omega \rangle + \mu \lambda_j + \langle k - k, \omega \rangle + \mu \lambda_j - \mu \lambda_j, |
\]

\[
\geq \frac{\gamma}{|k - k|^{10d}} - \epsilon_1 \geq \frac{\gamma}{2|k - k|^{10d}},
\]

and consequently

\[
|(D^\sigma(\mu, j, k))^{-1}| \leq \frac{2|k - k|^{10d}}{\gamma}.
\]

Then it follows from the Neumann series that

\[
|(T_{\Omega_\rho}^\sigma)^{-1}(m, m')| < \frac{4}{\gamma} (1 + |k - k|)^{10d}.
\]

Applying the resolvent identity (2.9) to \((T_{\Omega}^\sigma)^{-1}\) with respect to the decomposition (3.8) yields (3.13)

\[
(T_{\Omega_1}^\sigma)^{-1}(m, m') = (T_{\Gamma_1}^\sigma)^{-1}(m, m') \delta_{m,m'} + \sum_{m_1 \in \Gamma_1, m_2 \in \Gamma_2} (T_{\Omega_1}^\sigma)^{-1}(m, m_1) T_{\Omega_2}^\sigma(m_1, m_2) (T_{\Omega_1}^\sigma)^{-1}(m_2, m'),
\]

where \(\delta_{m,m'}\) equals one for \(m = m'\) and vanishes for the rest. It follows from (A2) and Proposition 3.2 that

\[
\left| \sum_{m_1 \in \Gamma_1, m_2 \in \Gamma_2} (T_{\Gamma_1}^\sigma)^{-1}(m, m_1) T_{\Omega_2}^\sigma(m_1, m_2) (T_{\Omega_1}^\sigma)^{-1}(m_2, m') \right|
\]

\[
\leq \epsilon \sum_{m_1 \in \Gamma_1, m_2 \in \Gamma_2, |k - k_1|>\delta^{10d} \log N_{\epsilon-1}} e^{-\frac{\epsilon}{\delta^{10d}} |k - k_1|^{10d}} (T_{\Omega_1}^\sigma)^{-1}(m_2, m')
\]

\[
+ \epsilon \sum_{m_1 \in \Gamma_1, m_2 \in \Gamma_2, |k - k_1|>\delta^{10d} \log N_{\epsilon-1}} \Phi(N_{\epsilon-1}) e^{-|k - k_1|^{10d}} (T_{\Omega_1}^\sigma)^{-1}(m_2, m')
\]

\[
= (I) + (II).
\]

Consider the case of \(s \geq 1\). Recall that \(\Gamma_1 \cap \Gamma_2 = \emptyset\) and hence \(|k - k_2| \geq N_{\epsilon-1}\). Then we have

\[
(I) \leq e^{\frac{\epsilon}{\delta^{10d}}} \sum_{m_2 \in \Gamma_2} e^{-\frac{\epsilon}{\delta^{10d}} |k - k_2|^{10d}} (T_{\Omega_1}^\sigma)^{-1}(m_2, m')
\]

\[
\leq \frac{\sqrt{\epsilon}}{2} \max_{m_2 \in \Omega_1, |k - k_2| \geq N_{\epsilon-1}} e^{-\frac{\epsilon}{\delta^{10d}} |k - k_2|^{10d}} (T_{\Omega_1}^\sigma)^{-1}(m_2, m').
\]
For (II), since the number of $k_i$ in the summation is less than $(\log N_{s-1})^{2C_2d}$, we derive

\[ (II) \leq \epsilon \sum_{m_2 \in F_2} (\rho_{s-1}^{-1} \log N_{s-1})^{2C_2d} \Phi(N_{s-1}) e^{-\varepsilon |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \]

\[ \leq \epsilon \sum_{m_2 \in F_2} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \]

\[ \times \epsilon \sum_{m_2 \in F_2} e^{-\frac{\varepsilon}{10} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \]

\[ \leq \epsilon \sum_{m_2 \in F_2} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \]

\[ \leq \frac{\sqrt{\epsilon}}{2} \max_{m_2 \in \Omega_i, |k_2-k_2'| \geq N_{s-1}} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')|. \]

For $s = 0$, we obtain from $\epsilon^{\frac{1}{2}} N_0^{C_1} < 1$ that

\[ (II) = \epsilon \sum_{m_2 \in F_2} \Phi(N_0) e^{-\varepsilon |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \]

\[ \leq \epsilon \sum_{m_2 \in F_2} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \cdot \epsilon \Phi(N_0) \]

\[ \leq \epsilon \sum_{m_2 \in F_2} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \]

\[ \leq \frac{\sqrt{\epsilon}}{2} \max_{m_2 \in \Omega_i} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')|. \]

All together, due to the fact that

\[ |k-k_i| > N_{s-1}, \quad s \geq 1, \]

and $\Omega_i$ is finite, we conclude the following lemma.

**Lemma 3.1.** Suppose the assumptions of Proposition 3.2 and

\[ \epsilon^{\frac{1}{2}} N_0^{C_1} < 1, \]

hold. Then, for any $m = (\mu, j, k) \in \Omega_{1,s}$ with $0 \leq s \leq r$, there is some $m_2 = (\mu_2, j_2, k_2) \in \Omega_i$ satisfying $|k-k_2| \geq N_{s-1}$ such that

\[ |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')| \]

\[ \leq \begin{cases} \Phi(N_{s-1} \wedge |k-k_i|) + \sqrt{\epsilon} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')|, & |k-k'| \leq (\rho_{s-1}^{-1} \log |k-k_i|)^{C_2}, \\ e^{-\frac{\varepsilon}{20} |k-k_2'|^2} + \sqrt{\epsilon} e^{-\frac{\varepsilon}{20} |k-k_2'|^2} |(T_{\Omega_i}^\sigma)^{-1}(m_2, m')|, & |k-k'| > (\rho_{s-1}^{-1} \log |k-k_i|)^{C_2}, \end{cases} \]

where $m' = (\mu', j', k') \in \Omega_i$.

**Remark 3.1.** Note that

\[ \frac{1}{100} \rho_{s-1} N_{s-1}^{C_2} \geq \frac{N_{s-1}^{C_2}}{(\log N_{s-1})^{1/2} \log C_3} > 1. \]
holds for $N_0$ large enough. The exponential decay term $e^{-\frac{1}{2}\rho_{\sigma} e^{-|k-k'|}}$ and $e^{-\frac{1}{2}\rho_{\sigma} e^{-|k-k'|}}$ in (3.15) can be replaced by
\begin{equation}
(3.16) \quad e^{-\frac{1}{2}\rho_{\sigma} e^{-|k-k'|}} e^{-\frac{1}{2}|k-k'|^2} \quad \text{and} \quad \sqrt{\epsilon} e^{-\frac{1}{2}\rho_{\sigma} e^{-|k-k'|}} e^{-\frac{1}{2}|k-k'|^2} |(T_{\Omega})^{-1}(m_2, m')|
\end{equation}
respectively.

Again applying (3.15) with the improved estimate (3.16) to $(T_{\Omega})^{-1}(m_2, m')$, we have
\[
|(T_{\Omega})^{-1}(m, m')| \leq \Phi(N_{r-1} \wedge |k - k_1|) + \sqrt{\epsilon} e^{-\frac{1}{2}k_1 |k - k_1|} \Phi(N_{r-1} \wedge |k_2 - k_1|) + (\sqrt{\epsilon})^2 e^{-\frac{1}{2}k_1 |k - k_1|} |(T_{\Omega})^{-1}(m_3, m')|
\]
for some $m_3 \in \Omega_1$. Since $|k_2 - k_1| \leq |k - k_1| + |k_2 - k_1|$, there is
\[
\Phi(|k_2 - k_1|) = |k_2 - k_1| \leq |k - k_1| \leq 2C_1 \quad \text{and} \quad C_4 |k_2 - k_1|^2 = \Phi(|k - k_1|^2) + C_4 \Phi(|k_2 - k_1|^2),
\]
where $C_4$ is a sufficiently large number depending only on $C_2$. It then follows that
\[
\sqrt{\epsilon} e^{-\frac{1}{2}k_1 |k - k_1|} \Phi(N_{r-1} \wedge |k_2 - k_1|) \leq \sqrt{\epsilon} \Phi(|k - k_1|^2 \wedge N_{r-1})
\]
and consequently
\[
|(T_{\Omega})^{-1}(m, m')| \leq \Phi(N_{r-1} \wedge |k - k_1|) + C \sqrt{\epsilon} \Phi(|k - k_1|^2 \wedge N_{r-1}) + (\sqrt{\epsilon})^2 e^{-\frac{1}{2}k_1 |k - k_1|} |(T_{\Omega})^{-1}(m_3, m')|.
\]

Now it is clear the iterations can be successively proceeded and simple induction arguments yield
\[
|(T_{\Omega})^{-1}(m, m')| < \Phi(|k - k_1|^2 \wedge N_{r-1}) \left(1 + \sum_{j=1}^2 (C \sqrt{\epsilon})^j \right) < 2\Phi(|k - k_1|^2 \wedge N_{r-1}^2).
\]

If $|k - k'| > (5 \rho_{\sigma-1})^{-1} \log |k - k_1|)^{C_2}$, we divide into two cases.

**Case 1:** $|k_2 - k'| \leq (\rho_{\sigma-1})^{-1} \log |k_2 - k_1|)^{C_2}$, where $k_2 \in \Lambda_{\Omega_1} \setminus \Lambda_{\sigma-1}$.

Recall the definition of $\rho_{\sigma} = \rho_{\sigma_1}$ in assumption (A3). We see from $|k_2 - k_1| < \Lambda_{\sigma-1}$ that
\[
|k_2 - k'| \leq (\rho_{\sigma-1})^{-1} \log |k_2 - k_1|)^{C_2} \leq (\log |k_2 - k_1|)^{C_2(1 + \sqrt{\epsilon})}.
\]

From (3.15) and (3.16), we have
\[
|(T_{\Omega})^{-1}(m, m')| \leq e^{-\frac{1}{2}\rho_{\sigma-1} e^{-|k-k'|}} e^{-|k-k'| |k-k'|^2} + \sqrt{\epsilon} e^{-\frac{1}{2}\rho_{\sigma-1} e^{-|k-k'|}} e^{-|k-k'| |k-k'|^2} \Phi(N_{r-1} \wedge |k_2 - k_1|)^2 + (\sqrt{\epsilon})^2 e^{-\frac{1}{2}\rho_{\sigma-1} e^{-|k-k'|}} e^{-|k-k'| |k-k'|^2} |(T_{\Omega})^{-1}(m_3, m')|.
\]

Simple computation gives
\[
|k_2 - k_1| \leq |k_2 - k'| + |k - k'| + |k - k_1| \leq (\log |k_2 - k_1|)^{C_2(1 + \sqrt{\epsilon})} + |k - k'| + 10^{-\frac{1}{2}\rho_{\sigma-1} e^{-|k-k'|}},
\]
\[
\leq \frac{1}{10} |k_2 - k_1| + (1 + \epsilon') 10^{-\frac{1}{2}\rho_{\sigma} e^{-|k-k'|}} + 10^{-\frac{1}{2}\rho_{\sigma-1} e^{-|k-k'|}},
\]
which implies
\[
|k_2 - k_1| \leq \frac{10}{9} (1 + \epsilon') 10^{-\frac{1}{2}\rho_{\sigma} e^{-|k-k'|}} + 10^{-\frac{1}{2}\rho_{\sigma-1} e^{-|k-k'|}}.
\]
Consequently,
\[ |k - k_2| \geq |k - k' - |k' - k_2| > |k - k'| - (\log |k_2 - k_1|)C_2 \]
\[ > [1 - (\frac{3}{5})C_2] \cdot |k - k'| > \frac{3}{4}|k - k'|, \]
if \( C_2 \) is large enough.

Then we have
\[ \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \Phi(N_{\sigma_1} \land |k_2 - k_1|) \]
\[ < \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} 3C_1 |k - k'|^{1/2} \times 10^{C_1 |k - k'|^{1/2}} \leq \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
if
\[ \frac{3}{2} < e \ll 1. \]

As a result, there is
\[ |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \leq \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} + C \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
\[ + (\sqrt{e})^2 e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \]
\[ \leq e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} + C \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
\[ + (\sqrt{e})^2 e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \]
\[ \leq e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} + C \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
\[ + (\sqrt{e})^2 e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} |(T_{\Omega_1}^\sigma)^{-1}(m, m')|. \]

Case 2: \( |k_2 - k'| > (\rho_{\sigma_1}^{-1} \log |k_2 - k_1|)C_2 \), where \( k_2 \in \Lambda_{\sigma_2} \setminus \Lambda_{\sigma_2-1} \).

From (3.15) and (3.16), we have
\[ |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \leq e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} + \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
\[ + (\sqrt{e})^2 e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \]
\[ \leq e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} + C \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
\[ + (\sqrt{e})^2 e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \]
\[ \leq e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} + C \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
\[ + (\sqrt{e})^2 e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} |(T_{\Omega_1}^\sigma)^{-1}(m, m')|. \]

Combining the two cases above, we have
\[ |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \leq e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} + C \sqrt{e} e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \]
\[ + (\sqrt{e})^2 e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} |(T_{\Omega_1}^\sigma)^{-1}(m, m')|. \]

It then follows from iterations of (3.17) that
\[ |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \leq e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}} \left( 1 + \sum_{j=1}^{\infty} (C \sqrt{e})^j \right) e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}}. \]

Summarizing the analysis in this subsection, we conclude
\[ |(T_{\Omega_1}^\sigma)^{-1}(m, m')| \ll \Phi(|k - k_1| \land N_{\sigma_1})^3, \]
(3.18)
\[ |(T_{\Omega_1}^\sigma)^{-1}(m, m')| < e^{-\frac{1}{2} \rho_{\sigma_1}^{-1} |k - k'| e^{-\frac{1}{2} |k - k'|/2}}, \quad \text{if} \quad |k - k'| > (5 \rho_{\sigma_1}^{-1} \log |k - k_1|)C_2, \]
and
(3.19)\[ \|(T_{\Omega_1}^\sigma)^{-1}\| \ll \Phi(N_{\sigma_1})^3. \]

Note that the derivation of (3.18) and (3.19) is based on some fixed and real parameter \((\sigma, \omega)\). Using Neumann series, one clearly has that (3.15) and (3.16) hold (up to a constant multiplier) within the \( \frac{1}{10} N_{\sigma_1}^{-1} \)-neighborhood of the initial parameter choice in the complex space, to which in the sequel we assume the parameter restricted. With the margin in our estimates, we
still assume that (3.18) and (3.19) hold in such a complex neighborhood. More precisely, we conclude the results in this subsection in the following proposition.

**Proposition 3.3.** Fix $N$ and assume the parameter $(\sigma_*, \omega_*) \in \mathbb{R} \times \mathbb{R}^d$ restricted such that (A1)-(A3) hold. If the constants $N_0$, $C_1$, $C_2$ and $c$ are appropriately chosen such that

\[
\begin{align*}
\epsilon_{\text{std}}^{-1} &< \epsilon_1 \ll 1, \\
\epsilon_1 &< \gamma N_0^{-10d}, \\
\frac{C_1}{C_2} &< \epsilon_1, \\
\frac{N_0}{C_2} &< \epsilon_1, \\
\epsilon_{\text{std}}^{-1} N_0^{C_1} &< 1, \\
\frac{2}{C_2} &< c \ll 1, 
\end{align*}
\]

(3.20)

hold, then (3.18) and (3.19) hold for all $(\sigma, \omega)$ lying in a $\frac{1}{10N^2} \Phi(N_{r-1}^{-1})$-neighborhood of $(\sigma_*, \omega_*)$ in $\mathbb{C} \times \mathbb{C}^d$.

### 3.3. Construction of polynomials.

Take block decomposition of $T_N^\sigma$ into

\[
T_N^\sigma = \left( \begin{array}{cc} T_{\Omega_1}^\sigma & eP \\
eQ & T_{\Omega_2}^\sigma \end{array} \right),
\]

and a formal inverse of $T_N^\sigma$ should take the form of

\[
(T_N^\sigma)^{-1} = \left( \begin{array}{cc} (T_{\Omega_1}^\sigma)^{-1} + e^2(T_{\Omega_1}^\sigma)^{-1}Ph^{-1}Q(T_{\Omega_1}^\sigma)^{-1} - e(T_{\Omega_1}^\sigma)^{-1}Ph^{-1}h^{-1} & -e(T_{\Omega_1}^\sigma)^{-1}Ph^{-1}h^{-1} \\
eh^{-1}Q(T_{\Omega_1}^\sigma)^{-1} & h^{-1} \end{array} \right),
\]

where

\[
h = T_{\Omega_2}^\sigma - e^2Q(T_{\Omega_1}^\sigma)^{-1}P.
\]

Moreover, the norm of $(T_N^\sigma)^{-1}$ admits the following control

\[
||(T_N^\sigma)^{-1}|| \leq \Phi(N_{r-1}^3) + \frac{\Phi(N_{r-1}^6)}{|h|}.
\]

Let $\sigma_1 = \sigma + \langle k_*, \omega \rangle + \mu_\omega \lambda$. Replacing $\sigma$ by $\sigma_1$ in $h$ leads to

\[
h(\sigma_1, \omega) = \mu_\omega (\sigma_1 + \mu_\omega \epsilon S(m_\omega) - \mu_\omega e^2Q(T_{\Omega_1}^\sigma)^{-1}P),
\]

where $T_{\Omega_1}^\sigma = T_{\Omega_1}^\sigma$ and $S$ denotes the diagonal part of the matrix $S$. By Proposition 3.3, $h$ is also analytic in $\sigma_1$ and $\omega$ in a complex $\frac{1}{10N^2} \Phi(N_{r-1}^{-1})$-neighborhood of the initial parameter choice.

We would like to employ the Malgrange’s preparation theorem in [4] Lemma 8.1.2] to $h(\sigma_1, \omega)$. It suffices to check the conditions therein.

Consider the partial derivatives of $Q(T_{\Omega_1}^\sigma)^{-1}P$ with respect to $\sigma_1$ and $\omega$. Observe that

\[
\partial(T_{\Omega_1}^\sigma)^{-1} = -(T_{\Omega_1}^\sigma)^{-1}(\partial T_{\Omega_1}^\sigma)(T_{\Omega_1}^\sigma)^{-1}, \quad \partial = \partial_{\sigma_1} \text{ or } \partial_{\omega}.
\]

Moreover, because of the smallness of the imaginary parts of $\sigma$ and $\omega$, we have

\[
|\partial_{\sigma_1} T_{\Omega_1}^\sigma| = |\partial_{\sigma_1} D^\sigma| = |e^{i(\sigma + \langle k_*, \omega \rangle)}(1 + i(\sigma + \langle k_*, \omega \rangle + \mu_\omega \lambda))| \leq 2|1 + i(\sigma_1 + \langle k - k_*, \omega \rangle + \mu_\omega \lambda) - \mu_\omega \lambda) | \leq |k - k_*|,
\]

and similarly (together with (A2))

\[
|\partial_{\omega} T_{\Omega_1}^\sigma(m, m')| \leq (1 + |k - k_*|)^2 e^{-|k - k_*|^2}.
\]

(3.23)
Since the off-diagonal elements is independent of $\sigma_1$, we have

$$\partial_{\sigma_1}(Q(T_{\Omega_1}^{\sigma_1})^{-1}P) = -Q(T_{\Omega_1}^{\sigma_1})^{-1}(\partial_{\sigma_1}T_{\Omega_1}^{\sigma_1})(T_{\Omega_1}^{\sigma_1})^{-1}P.$$ 

However, the derivative of $Q(T_{\Omega_1}^{\sigma_1})^{-1}P$ with respect to $\omega$ is more complicated and reads

$$\partial_{\omega}(Q(T_{\Omega_1}^{\sigma_1})^{-1}P) = (\partial_{\omega}Q)(T_{\Omega_1}^{\sigma_1})^{-1}P - Q(T_{\Omega_1}^{\sigma_1})^{-1}(\partial_{\omega}T_{\Omega_1}^{\sigma_1})(T_{\Omega_1}^{\sigma_1})^{-1}P + Q(T_{\Omega_1}^{\sigma_1})^{-1}(\partial_{\omega}P).$$

We only analyze the complicated term $Q(T_{\Omega_1}^{\sigma_1})^{-1}(\partial_{\omega}T_{\Omega_1}^{\sigma_1})(T_{\Omega_1}^{\sigma_1})^{-1}P$, which takes the form of

$$e^2 \sum_{m_i \in \Omega_1, 1 \leq i \leq 4} S(m_1, m_2)(\partial_{\omega}T_{\Omega_1}^{\sigma_1})(m_2, m_3)(T_{\Omega_1}^{\sigma_1})^{-1}(m_3, m_4)S(m_4, m_2).$$

Let $\Delta = |k_1 - k_2| \vee |k_3 - k_4|$. If

$$|k_1 - k_2| \leq (5 \log \Delta)^{C_2(1 + \frac{1}{\log C})} \quad \text{and} \quad |k_3 - k_4| \leq (5 \log \Delta)^{C_2(1 + \frac{1}{\log C})},$$

then

$$|3.24| \lesssim e^2 \sum_{m_i \in \Omega_1} e^{-|k_i - k_i|^c} \Phi(|k_1 - k_i|)^3(1 + |k_2 - k_i|)^2 e^{-|k_2 - k_i|^c} \Phi(|k_3 - k_i|)^3 e^{-|k_3 - k_i|^c}$$

$$\lesssim e^2 \sum_{m_i \in \Omega_1} e^{-\ell_5(c)} \Phi(\Delta^6)(1 + |k_2 - k_i|)^2 e^{-\ell_5(c)} e^{-|k_4 - k_i|^c},$$

where

$$\ell_5(c) = |k_1 - k_1|^c + |k_1 - k_2|^c + |k_2 - k_1|^c + |k_3 - k_4|^c + |k_4 - k_2|^c.$$ 

Note that

$$\frac{9}{10} \ell_5(c) - |k_1 - k_2|^c - |k_3 - k_4|^c$$

$$= \frac{9}{10} \left\{ |k_1 - k_1|^c + \cdots + |k_4 - k_1|^c - \left( \sqrt{\frac{10}{9}} |k_1 - k_2| \right)^c - \left( \sqrt{\frac{10}{9}} |k_3 - k_4| \right)^c \right\}$$

$$\geq \frac{9}{10} \left\{ \Delta - 2 \sqrt{\frac{10}{9}} (5 \log \Delta)^{C_2(1 + \frac{1}{\log C})} \right\}^c$$

$$\geq \frac{1}{100} \Delta^c.$$

Therefore, if (3.25) holds, we have

$$|3.24| < \frac{1}{9} e^2.$$ 

The other two cases for (3.24) are simpler and leads to the same estimate as (3.26). Indeed, if one of the inequalities, say the first one, in (3.25) fails, there is

$$|k_1 - k_2| > (5 \log \Delta)^{C_2(1 + \frac{1}{\log C})} \geq (5 \rho_{c-1}^{-1} \log |k_1 - k_i|)^c,$$

and therefore the off diagonal estimate in (3.18) can be applied.

All together, we have

$$|Q(T_{\Omega_1}^{\sigma_1})^{-1}(\partial_{\omega}T_{\Omega_1}^{\sigma_1})(T_{\Omega_1}^{\sigma_1})^{-1}P| < \frac{1}{3} e^2,$$

Moreover, using assumption (A2) and repeating the analysis (from (3.24) to (3.26)) to

$$(\partial_{\omega}Q)(T_{\Omega_1}^{\sigma_1})^{-1}P \quad \text{and} \quad Q(T_{\Omega_1}^{\sigma_1})^{-1}(\partial_{\omega}P),$$
we are able to show
\begin{equation}
(3.28) \quad |(\partial_p Q)(T_{O_1}^\sigma)^{-1}P| \vee |Q(T_{O_1}^\sigma)^{-1}(\partial_p P)| < \frac{1}{3} \epsilon^\frac{1}{2},
\end{equation}
which further implies
\begin{equation}
(3.29) \quad |\partial_p (Q(T_{O_1}^\sigma)^{-1}P)| < \epsilon^\frac{1}{2}.
\end{equation}
Then the perturbation \( \varphi \), defined by
\[ \varphi(\sigma; \omega) = \mu, \epsilon(S(m_\omega) - \epsilon Q(T_{O_1}^\sigma)^{-1}P), \]
satisfies
\[ |\partial^\alpha \varphi(\sigma; \omega)| \leq \epsilon^\frac{1}{2}, \quad \text{for} \quad \alpha \in \mathbb{N}^{d+1}, |\alpha| \leq 1. \]
Applying Malgrange’s preparation theorem to function \( h \) we derive a first order polynomial
\[ \bar{p}(\sigma) = \sigma + a_0(\omega), \]
and a function \( \bar{q} = \mathcal{O}(1) \) such that
\begin{equation}
(3.30) \quad h(\sigma; \omega) = \mu, \bar{p}(\sigma)(1 + \bar{q}(\sigma, \omega)).
\end{equation}
Once the approximate polynomial \( \bar{p} \) is obtained, we denote the modified polynomial
\[ p(\sigma) = \sigma + \text{Re}(a_0(\omega)). \]
Since the preparation theorem can only be applied locally, we eventually obtain finitely many polynomials \( \bar{p} \) and we denote by \( \mathcal{P}^{(1)}_{N_0} \) the set of those modified polynomials \( p \). Then the total number of the elements in \( \mathcal{P}^{(1)}_{N} \) is bounded by
\[ \#\mathcal{P}^{(1)}_{N} \leq N(10N^2 \Phi(N_{-1})^3)^2 \leq N^{5+6\epsilon_{C_2}}. \]
Whenever parameter excision is taken on \( \omega \) such that
\begin{equation}
(3.31) \quad |p(\sigma)| > \Phi(N)^{-\frac{1}{2}},
\end{equation}
for all \( p \in \mathcal{P}^{(1)}_{N} \), we then see from \((3.31), (3.30)\) and \( |p(\sigma)| < |\bar{p}(\sigma)| \) that
\begin{equation}
(3.32) \quad ||(T_{N}^\sigma)^{-1}|| \leq \Phi(N)
\end{equation}
provided \( C_3 > 12. \)

### 3.4. Off diagonal exponential decay

In this part, we establish the off diagonal exponential decay for \((T_{N}^\sigma)^{-1} \), which is an immediate result of the resolvent identity. Recall that \( \Omega = \{-1, 1\} \times \{1, \ldots, n\} \times [-N, N]^d = \Omega_1 + \Omega_2 \) with \( \Omega_2 = \{m_\omega\} \). Consider \(|k - k'| > (\rho_N^{-1} \log N)^{C_2}\). Applying the resolvent identity and \((3.18)\) leads to

\[
|(T_{O_1}^\sigma)^{-1}(m, m')| \leq |(T_{O_1}^\sigma)^{-1}(m, m')| + \sum_{m_1 \in \Omega_1, m_2 = m_\omega} |(T_{O_1}^\sigma)^{-1}(m_1, m)| \cdot |T_{O_1}^\sigma(m_1, m_\omega)| \cdot |(T_{O_1}^\sigma)^{-1}(m_2, m')|
\]

\[ \leq e^{-\frac{1}{4} (\rho_N^{-1} \log |k - k'|)^{C_2}} \left( \sum_{|k-k'| \leq (5\rho_{N-1}^{-1} \log |k-k'|)^{C_2}} + \sum_{|k-k'| > (5\rho_{N-1}^{-1} \log |k-k'|)^{C_2}} \right), \]

since
\[ |k - k'| > (\rho_N^{-1} \log N)^{C_2} \geq (5\rho_{N-1}^{-1} \log 2N)^{C_2} \geq (5\rho_{N-1}^{-1} \log |k - k'|)^{C_2}. \]
Assume, for instance, that \(|k - k_*| \geq \frac{1}{2}|k - k'|\). (Otherwise, \(|k' - k_*| \geq \frac{1}{2}|k - k'|\) and the analysis below is the same.) When \(|k - k_*| \leq (5\rho_{r-1}^{-1} \log |k - k_*|)^{C_2}\), we see that

\[
|k - k_*| < (\log |k - k_*|)^{C_2(1 + \frac{\rho_{r-1}}{\log 2})} < \frac{1}{10}|k - k_*|.
\]

and then

\[
|k_* - k| > \frac{9}{10}|k - k_*| - |k_* - k| > \frac{9}{20}|k - k'|.
\]

Therefore, we get

\[
\sum_{|k - k_*| \leq (5\rho_{r-1}^{-1} \log |k - k_*|)^{C_2}} (**) \leq \sum_{|k - k_*| \leq (5\rho_{r-1}^{-1} \log |k - k_*|)^{C_2}} e\Phi(N_{r-1})^3 e^{-\frac{E}{3}|k - k'|^p} \Phi(N)
\]

\[
\leq e\Phi(N)^3 e^{-\frac{E}{4}\log N^{2\gamma}} e^{-\frac{E}{4}|k - k'|^p}
\]

\[
< \frac{1}{2} e^{-\frac{E}{4}|k - k'|^p}.
\]

Moreover, using (3.18) and (3.2), we obtain

\[
\sum_{|k - k_*| > (5\rho_{r-1}^{-1} \log |k - k_*|)^{C_2}} (**) \leq \sum_{|k - k_*| > (5\rho_{r-1}^{-1} \log |k - k_*|)^{C_2}} e\Phi(N_{r-1})^3 e^{-\frac{E}{3}|k - k'|^p} e^{-\frac{E}{4}|k_* - k|^2} \Phi(N)
\]

\[
\leq e\Phi(N)^3 e^{-\frac{E}{4}\log N^{2\gamma}} \sum_{k_*} e^{-\frac{E}{4}|k_* - k|^2} \Phi(N)
\]

\[
\leq e\Phi(N)^3 e^{-\frac{E}{4}|k - k'|^p} \Phi(N) \sum_{k_*} e^{-\frac{E}{4}|k_* - k|^2}
\]

\[
< \frac{1}{2} e^{-\frac{E}{4}\log N^{2\gamma}} e^{-\frac{E}{4}|k - k'|^p}.
\]

All together, we have

\[
(3.33) \quad |(T_{m'}^{m})^{-1}(m, m')] < e^{-\frac{E}{4}\log N^{2\gamma}} e^{-\frac{E}{4}|k - k'|^p}, \quad |k - k'| \geq (\rho_{r-1}^{-1} \log N)^{C_2},
\]

which meets the estimate in (3.1).

4. Iteration Lemma and Its Proof

In this section, we establish an iteration lemma, which produces a sequence of approximate solutions \(y_j\) of the nonlinear lattice equation (2.2) and the associated error \(\mathcal{F}[y_j]\) is successively improved. Moreover, it is easy to see that the approximate solutions \(\{y_j\}\) converges rapidly, whose limit is then a true solution of (2.2).

Choose the constants \(C_1, \cdots, C_6\) and \(c\) appropriately such that

\[
M^c \approx 1
\]

\[
c \geq \frac{2}{C_2},
\]

\[
1 - (1 - c)C_5 > C_6 > 1,
\]

\[
1 - \frac{12}{C_3} > \frac{2(d + 4)C_5 + 2(d + 5)}{C_1},
\]

\[
C_1 - 2dC_3 - \frac{24C_1}{C_3} > 15,
\]

\[
C_1 \geq 1
\]

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For instance, we can impose in order that \( M = 100, c = 10^{-3}, C_2 = 4 \cdot 10^3, C_3 = 100, C_5 = 10, C_6 = 5 \) and \( C_1 = 10^d \).

Recall that the parameter \( \omega \) is defined on some open set \( \mathcal{U} \) in \( \mathbb{R}^d \), whose Lebesgue measure, without loss of generality, is supposed to be one. We now state the iteration lemma. With some abuse of notation, we also use \( j \) to indicate the iteration step.

**Lemma 4.1.** Let \( 0 < \eta < 1 \). Consider the nonlinear lattice equation \((2.2)\). Assume that at the \( j \)-th Newton iteration step, there exists an approximate solution \( y_j \) of \((2.2)\) and an open subset \( \mathcal{U}_j \subset \mathcal{U}_{j-1} \) such that the following statements hold.

\( \textbf{(S1)} \) For any \( \omega \in \mathcal{U}_j \), there is

\[
\text{Supp } y_j \subset \{-1, 1\} \times \{1, \cdots, n\} \times [-M^j, M^j]^d, \\
|\partial_{\omega} y_j(m)| < e^{-|\omega|^j}, \quad \alpha = 0, 1, \\
|\Delta_{j-1}(m)| = |y_j(m) - y_{j-1}(m)| < e^{-|\omega|^j} e^{-\frac{1}{M^j}}.
\]

where \( m = (\mu, j, k) \in \mathcal{L} \).

\( \textbf{(S2)} \) For each \( \omega \in \mathcal{U}_j \), we have

\[
\|\partial_{\omega} y_j\| < e^{-2(M^j)^j}, \quad \alpha = 0, 1.
\]

\( \textbf{(S3)} \) Let \( T = \mathcal{F}'[y_j] \) be the linearized operator of \( \mathcal{F} \) at \( y = y_j \). Then for \( N \leq M^j \) and \( \omega \in \mathcal{U}_j \)

\[
\|T_N\|^{-1} < \Phi(N) = N^{C_1},
\]

and

\[
|T_N^{-1}(m, m')| < e^{-\frac{1}{2}|k-k'|}, \quad \text{for } |k - k'| > N^{\frac{1}{2e}}.
\]

\( \textbf{(S4)} \) Define \( T^{\sigma} \) as in \((2.5)\) from \( T = \mathcal{F}'[y_j] \). Let \( N \leq M^j \) and

\[
4N \leq |k| \leq N^{C_3}.
\]

Then, if \( \sigma_1 - \sigma_2 = \langle k, \omega \rangle \), the matrices \((T_N^{\sigma_1})^{-1}\) and \((T_N^{\sigma_2})^{-2}\) do not both fail the property:

\[
\|T_N^{\sigma_1}(m, m')\| \leq \Phi(N),
\]

where \( \rho_N = (\log N / \log N_0)^{-\frac{1}{6+\delta}} \)

\[
T_N^{\sigma_1}(m, m') \leq e^{-\frac{1}{2\rho_N}|k-k'|} \quad \text{for } |k - k'| \geq (\rho_N^{-1} \log N)^{C_2}.
\]

\( \textbf{(S5)} \) The parameter set \( \mathcal{U}_j \) satisfies the measure estimate

\[
\text{mes}[\mathcal{U}_{j-1} \setminus \mathcal{U}_j] < \frac{1}{50} \cdot \frac{\eta}{4^{j-1}}.
\]

Then, there exists an absolute constant \( \epsilon^* \) such that if \( 0 < \epsilon < \epsilon^* \), there is an improved approximate solution \( y_{j+1} \) and open set \( \mathcal{U}_{j+1} \subset \mathcal{U}_j \) such that the same statements are satisfied with \( j + 1 \) in place of \( j \).

**Remark 4.1.** As we shall see, the iteration process starts at a sufficiently large \( j_0 \) with the initial approximate solution \( y_{j_0} = 0 \). To keep the consistency of the notations, we set the expressions \( y_{j_0-1} = 0, \Delta_{j_0-1} = 0 \) and \( \mathcal{U}_{j_0-1} = \mathcal{U} \). Moreover, \( \mathcal{U}_{j_0} \) is understood as \( 1 - \text{mes}[\mathcal{U}_{j_0}] \leq C(d, n)\eta \).

**Remark 4.2.** The absolute constant \( \epsilon^* \) depends only on the universal constant \( d, n, \tau, \eta, \lambda_j, \lambda_j - \lambda_f \) and the analytic radius of \( f \) and \( g \).
Now we employ the iteration lemma to prove our main result Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \mathcal{W}_\infty = \cap_{j \geq j_0} \mathcal{W}_j \). We see from Remark 4.1 and (S5) that \( \text{mes}[\mathcal{W}_\infty] > 1 - C^{-\eta} \).

Moreover, for each \( \omega \in \mathcal{W}_\infty \), the sequence of approximate solutions \( \{y_j\}_{j \geq j_0} \) converges in \( \ell^2(\mathcal{L}) \), which can be seen from a simple deduction

\[
\|y_j - y_j\|_2^2 \lesssim \sum_{k \in \mathbb{Z}^d} |(y_j - y_j)(m)|^2 = \sum_{k \in \mathbb{Z}^d} \sum_{s=j}^{j-1} \Delta_s(k)^2 \leq \left( \sum_{k \in \mathbb{Z}^d} e^{-2|k|^r} \right) \cdot \left( \sum_{s=j}^{j-1} e^{-\frac{s}{M^{\alpha}}} \right)^2.
\]

Let \( y_\infty = \lim_{j \to \infty} y_j \). Then there is \( \mathcal{F}[y_\infty] = 0 \) due to the rapid convergence of \( \{y_j\}_{j \geq j_0} \). By the reduction arguments in section 2.1, the solution \( y_\infty \) of (2.2) defines a \( C^\infty \), real-valued, quasi-periodic function \( x = x(t) \) with frequency \( \omega \), which is exactly a solution of (1.4). Using Corollary 2 in [24], we know that the solution \( x = x(t) \) is analytic in time \( t \). This completes the proof of the theorem. \( \square \)

The rest of this section is devoted to the proof of the iteration lemma. For reader’s convenience, we briefly explain the main idea behind it.

In subsection 4.1, we start the iteration process at sufficiently large \( j_0 \) with the trivial approximate solution \( y_{j_0} = 0 \). By imposing the Melnikov condition, the diagonal matrix is then dominated when the perturbation is small enough. As a result, the construction of the inverse of \( T_{N}^{-1} \) is a simple application of Neumann series. Furthermore, the separation property is also benefited from the Melnikov condition. In subsection 4.2, we employ the coupling lemma to construct the inverse of \( T_N \) for any \( N \leq M^{j_1+1} \), where \( T = \mathcal{F}[y_j] \). To obtain the control of \( (T_N)^{-1} \), we need to exclude some parameters of \( \omega \) such that the polynomials constructed stay away from zero. Before solving the Newton equation, we establish the separation property for \( T_N^* \) with \( N \leq M^{j_1+1} \) in subsection 4.3 which is validated by further excluding parameters. As a result, we get the desired open subset \( \mathcal{W}_{j+1} \subset \mathcal{W}_j \) and then verify (S5) \( j+1 \). However, one should bear in mind that what we have established is the separation property for \( T^s \) depending on the \( j \)-th approximate solution \( y_j \). In subsection 4.4, we construct the \( (j+1) \)-th approximate solution \( y_{j+1} \) and estimate the associated new error, which verifies the induction statement (S1) \( j+1 \) and (S2) \( j+1 \). Finally, in subsection 4.5, we establish induction statement (S3) \( j+1 \) and (S4) \( j+1 \). Since the associated properties have already been proved for \( T \) depending on \( y_j \) in subsections 4.2 and 4.3, it suffices to apply the Neumann series directly, due to the rapid decay of the corrections \( \Delta_j \).

### 4.1. Preparation step.

Obviously, the trivial solution \( y = 0 \) is an approximate solution of (2.2) with error \( \mathcal{F}[y] = O(\epsilon) \). Thus, we may let the iteration start at \( y_{j_0} = 0 \) with the integer \( j_0 = j_0(\epsilon) \) satisfying

\[
(M^j)^c \sim \log \frac{1}{\epsilon}.
\]

Observe that \( \mathcal{F}[0] = \epsilon \mathcal{W}[0] \) is independent of \( \omega \). Then statement (S1) \( j_0 \) and (S2) \( j_0 \) holds.

Assume the Melnikov condition

\[
|\langle k, \omega \rangle \pm \lambda_j \pm \lambda_{j'}| \geq \frac{\gamma}{|k||\omega|} \quad \text{for} \quad k \in \mathbb{Z}^d \setminus \{0\}, |k| \leq (100M^j)^{C_1}, 1 \leq j, j' \leq n.
\]
(See also (A1) in section 3.) Then statement (S3)$_{j_0}$ is a simple application of Neumann series due to the dominance of the diagonal matrix. The proof is the same to that of (ii) in Proposition 3.2 with $\varepsilon_1$ and $N_0$ replaced by $\eta N^{-10d}$ and $N$ respectively for $N \leq M^{j_0}$. It suffices to check condition (3.5), which follows from (4.7) that

$$
\varepsilon_1 = \frac{\eta}{N^{10d}} \geq \frac{\eta}{M^{10d}j_0} > e^{-M^{10d}} > e^{100}.
$$

Consider statement (S4)$_{j_0}$. Assume on the contrary that $\|(T_N^\varepsilon)^{-1}\| > \Phi(N)$ and $\|(T_N^\varepsilon)^{-1}\| > \Phi(N)$. Then there exist $m_1 = (\mu_1, j_1, k_1)$ and $m_2 = (\mu_2, j_2, k_2)$ such that

$$
|\sigma_i + \langle k_j, \omega \rangle + \mu_i \lambda_{j_i} - \mu_2 \lambda_{j_2}| < \frac{2}{\Phi(N)}, \quad i = 1, 2.
$$

This leads to

$$
|\sigma_1 - \sigma_2 + \langle k_1 - k_2, \omega \rangle + \mu_1 \lambda_{j_1} - \mu_2 \lambda_{j_2}| < \frac{4}{\Phi(N)}.
$$

or equivalently

$$
|\langle k + k_1 - k_2, \omega \rangle + \mu_1 \lambda_{j_1} - \mu_2 \lambda_{j_2}| < \frac{4}{\Phi(N)}.
$$

Note that $2N < |k + k_1 - k_2| \leq 2N^{C_3}$. Then, by condition (4.8), we see from $C_1 = 100C_3d$ that

$$
|\langle k + k_1 - k_2, \omega \rangle + \mu_1 \lambda_{j_1} - \mu_2 \lambda_{j_2}| > \frac{\eta}{(2N^{C_3})^{10d}} > \frac{5}{\Phi(N)},
$$

which turns out to be a contradiction and thus (S4)$_{j_0}$ is valid.

Denote

$$
\mathcal{Y}_{j_0} = \bigcup_{\mu, \mu' = \pm 1, 1 \leq |k| \leq (100M^{j_0})^{C_3}} \left\{ \omega \in \mathcal{U} : |\langle k, \omega \rangle + \mu \lambda_{j} + \mu' \lambda_{j'}| < \frac{\eta}{|k|^{10d}} \right\},
$$

and

$$
\mathcal{U}_{j_0} = \mathcal{U} \setminus \mathcal{Y}_{j_0}.
$$

Obviously, $1 - \operatorname{mes}[\mathcal{U}_{j_0}] = \mathcal{Y}_{j_0} \ll \eta$ and the constant here depends only on $d$ and $n$, which verifies statement (S5)$_{j_0}$ (see Remark 4.1).

In what follows, we assume that the iteration lemma 4.1 holds up to step $j \geq j_0$.

4.2. Construction of $(T_N)^{-1}$. Let $T = \mathcal{T}'[\gamma]$ and assume $M^j < N \leq M^{j+1}$. We shall construct the inverse of $T_N$ by the coupling lemma established in 4.1 (see also Lemma 2.1). Take

$$
K = N^{\frac{1}{C_5}}, \quad B = \Phi(\frac{1}{2N}), \quad C_5 > 1.
$$

Then for any $m = (\mu, j, k)$ with $|k| \leq 10K$, there is

$$
\|(T_{10k})^{-1}\| < \Phi(10K) < B,
$$

since

$$
10K = 10N^{\frac{1}{C_5}} < 10M^{\frac{1}{C_5}} < M^j.
$$

For any $m = (\mu, j, k)$ with $|k| > 5K$, we construct the inverse of

$$
T_{k=K} = T_{[k-10K, k]} = T_K^{ef(k, \omega)}.
$$

The covering of $\{-1, 1\} \times \{1, \cdots, n\} \times [-N, N]^d$ is now clear.
To apply the result in section 3 with \( y = y_j \), we verify the conditions therein. Indeed, choosing \( N_0 \sim M^{j_0} \), the assumption (A1) then follows from our construction of \( \| \varphi \| \). The off-diagonal decay property (A2) is an immediate result of the exponential decay of \( y_j \) in (S1). The assumption (A3) is just our induction assumption (S4). The remained arithmetical conditions (3.20) are already established in the preparation step. From (3.32) in the main construction section 3, we have

\[
\| (T^\eta)^{-1} \| \leq \Phi(K) < B.
\]

Of course, this requires some excision of the parameter \( \omega \) to ensure (3.31), which shall be implemented later.

Note that

\[
\frac{K}{100} > (10K)^{\frac{1}{10}}
\]

and then we have

\[
| (T_{10K})^{-1}(m, m') | < e^{-\frac{1}{2}k - k'} < e^{-\frac{1}{2}(\frac{1}{100})^k K^c} < K^{-C}
\]

for some constant \( C > 0 \) when \( |k - k'| < \frac{K}{100} \). Moreover, for \( (T^\eta)^{-1} \) with \( \sigma = (k, \omega) \), we see from (3.33) that

\[
| (T^\eta)^{-1}(m, m') | \leq e^{-\frac{1}{20}k - k'} e^{-\frac{1}{2}k - k'} < e^{-\frac{1}{2}M^{j_0} K^c} < K^{-C},
\]

when \( |k - k'| > \frac{K}{100} \). It remains to check

\[
\log B = C_1 \log \frac{N}{2} < \frac{N^{\frac{1}{10}}}{100} = \frac{K^c}{100}.
\]

Then the coupling lemma 2.1 implies

\[
||(T_N)^{-1}|| < 2B = 2\Phi(\frac{1}{2}N),
\]

and

\[
| (T_N)^{-1}(m, m') | < e^{-\frac{1}{2}k - k'}, \quad |k - k'| > (100C K)^{\frac{1}{10}}.
\]

In view of (4.3), or equivalently,

\[
\frac{1}{C_6} > \frac{1}{(1 - c)C_5},
\]

we have

\[
| (T_N)^{-1}(m, m') | < e^{-\frac{1}{2}k - k'}, \quad |k - k'| > N^{\frac{1}{10}}.
\]

holds for \( M^j < N \leq M^{j+1} \).

Now we need to take parameter separation such that for all possible \( \sigma = (k, \omega) \), the constructed polynomial \( |p(\sigma_1)| > \Phi(K)^{-\frac{1}{2}} \) for all \( p \in \mathcal{P}^{(1)}_K \), where \( \sigma_1 = \sigma + (k_*, \omega) + \mu_\lambda \),

\[
= (k + k_*, \omega) + \mu_\lambda.
\]

Since \( |k| > 5K \) and \( |k_*| \leq K \), we have \( |k + k_*| > 4K \). Furthermore, it follows from Malgrange’s preparation theorem that those \( a_0(\omega) \) in \( p \in \mathcal{P}^{(1)}_K \) stay uniformly bounded together with their first derivatives. Therefore, for any fixed \( \sigma_1 \), the measure of the excluded parameter set

\[
\left\{ \omega : |p(\sigma_1)| = |(k + k_*, \omega) + \mu_\lambda| + \text{Re}(a_0(\omega)) | \leq \frac{\eta}{\sqrt{\Phi(K)}} \right\}
\]
is less than $\Phi(K)^{-\frac{1}{2}}$ up to a constant multiplier. Counting the numbers of all possible $\sigma_1$ (hence $k$ and $m_\ast = (\mu_\ast, j_\ast, k_\ast)$) and the polynomials $p \in P_k^{(1)}$, the total excision measure for $\omega$ is

$$\ll N^{d} K^d K^{\frac{5}{2} + \frac{6C_1}{C_3}} \Phi(K)^{-\frac{1}{2}} \eta = N^{\left(-\frac{C_1}{C_3}d - d - \frac{d+5}{C_3}C_3\right)} \eta < \frac{\eta}{N^{\frac{1}{2}}},$$

in view of (4.4), or equivalently,

$$\frac{C_1}{2C_3} - d - \frac{d + 5 + 6C_1}{C_3} > 4.$$

Define

$$\gamma_{j+1}^{(1)} = \bigcup_{M^j < N \leq M^{j+1}} \bigcup_{|k| \leq 2N} \bigcup_{m_\ast \leq M^j} \bigcup_{p \in P_k^{(1)}} \left\{ \omega : |p(\sigma_i)| \leq \frac{\eta}{\sqrt{\Phi(N)}} \right\}.$$

Then we have

$$\text{mes}[\gamma_{j+1}^{(1)}] \leq \frac{\eta}{2} \sum_{M^j < N \leq M^{j+1}} N^{-4} \leq M^{j+1} \frac{M^{j+1}}{M^j} < \frac{1}{100} \cdot \frac{\eta}{4^j}.$$

4.3. Separation property. Now we study the separation property (4.6) in the iteration lemma but also for $T = \mathcal{F}[\chi]$. Recall the definition of $T^\sigma$ in (2.5). Suppose both $(T^\chi_1)^{-1}$ and $(T^\chi_2)^{-1}$ fail (4.6). From the analysis in our main construction section 3 there should exist some $m_1 = (\mu_1, j_1, k_1), m_2 = (\mu_2, j_2, k_2)$ and $p_1, p_2 \in P_k^{(1)}$ such that

$$|p_i(\sigma'_i)| < \frac{\eta}{\sqrt{\Phi(N)}}, \quad \sigma'_i = \sigma_i + \langle k_i, \omega \rangle + \mu_i \lambda_j, \quad i = 1, 2.$$

Recall that $p_1$ and $p_2$ are linear functions taking the form of

$$p_1(\sigma) = \sigma + a_1(\omega), \quad |\partial_\omega a_1(\omega)| \leq C, \quad i = 1, 2.$$

Then we see that

$$|p_1(\sigma'_1) - p_2(\sigma'_2)| = |(k + k_1 - k_2, \omega) + \mu_1 \lambda_{j_1} - \mu_2 \lambda_{j_2} + a_1(\omega) - a_2(\omega)| < \frac{2\eta}{\sqrt{\Phi(N)}},$$

with

$$2N < 4N - 2N < |k + k_1 - k_2| < 2N^{C_3}.$$

Take parameter excision for $\omega$ as follows. Define by $P_N^{(2)}$ the set of the following polynomials

$$\bar{p}(\sigma) = \sigma + a_{0,1} - a_{0,2},$$

$$a_{0,i} = p_i(\sigma) - \sigma, \quad \text{for some} \quad p_i \in P_N^{(1)}, \quad i = 1, 2,$$

and consequently $\# P_N^{(2)} \leq (\# P_N^{(1)})^2$. We estimate the measure of

$$\gamma_{j+1}^{(2)} = \bigcup_{M^j < N \leq M^{j+1}} \bigcup_{\mu_\ast, j_i = 1, 2} \bigcup_{2N < |k| < 2N^{C_3}} \bigcup_{p \in P_N^{(2)}} \left\{ \omega : |p(\langle k, \omega \rangle + \mu_1 \lambda_{j_1} - \mu_2 \lambda_{j_2})| < \frac{2\eta}{\sqrt{\Phi(N)}} \right\},$$

which satisfies

$$\text{mes}[\gamma_{j+1}^{(2)}] \ll \eta \sum_{M^j < N \leq M^{j+1}} N^{\left(-\frac{C_1}{C_3}d - 10 - \frac{12C_1}{C_3}\right)} \ll \eta \sum_{M^j < N \leq M^{j+1}} N^{-4} < \frac{1}{100} \cdot \frac{\eta}{4^j},$$

in view of (4.5).
At step $j + 1$, we have excluded parameters twice. One is to control $T_N^{-1}$ and the other one is to establish the separation property. Denote by $\mathcal{U}_{j+1}$ the "good" parameter set such that the analysis in the $(j + 1)$-th step is valid, i.e.,

$$\mathcal{U}_{j+1} = \mathcal{U}_j \setminus (\mathcal{V}_{j+1}^{(1)} \cup \mathcal{V}_{j+1}^{(2)}).$$

Then we obtain from (4.11) and (4.12) that

$$\text{mes}[\mathcal{U}_j \setminus \mathcal{U}_{j+1}] < \frac{1}{50} \cdot \eta^4,$$

which verifies statement (S4)$_{j+1}$.

4.4. **Approximate solution and new error.** Consider the $(j + 1)$-th step and take $N = M^{j+1}$. Let

$$y_{j+1} = y_j + \Delta_j,$$

where $\Delta_j$ is given by the Newton equation

$$\Delta_j = -(T_N)^{-1} \Gamma_{10M} \mathcal{F}[y_j], \quad T = \mathcal{F}'[y_j].$$

By (4.9) and $M^c \approx 1$, we obtain

$$\parallel \Delta_j \parallel \leq \parallel (T_N)^{-1} \parallel \cdot \parallel \mathcal{F}[y_j] \parallel \leq 2 \Phi \left( \frac{1}{2} N \right) e^{-2M^c} < e^{-\frac{1}{2} N^c},$$

and consequently

$$|\Delta_j(m)| < e^{-|k|} e^{-\frac{1}{2} N^c},$$

since $|k| \leq N$.

Consider $\partial_\omega \Delta_j = -(\partial_\omega (T_N)^{-1}) \cdot \mathcal{F}[y_j] - (T_N)^{-1} \partial_\omega \mathcal{F}[y_j]$. The estimate for $(T_N)^{-1} \partial_\omega \mathcal{F}[y_j]$ remains the same to $\Delta_j$ and thus we have

$$|(T_N)^{-1} \partial_\omega \mathcal{F}[y_j](m)| < e^{-|k|} e^{-\frac{1}{2} M^{j+1} c}.$$ 

Note that

$$\partial_\omega (T_N)^{-1} = -(T_N)^{-1} \partial_\omega T_N (T_N)^{-1}$$

and similar arguments in (3.23) yield

$$\|\partial_\omega T_N\| \leq N^{2d}.$$ 

One readily finds that the polynomial growth (in $N$) of $\partial_\omega (T_N)^{-1}$ can always be controlled by the exponential decay of $\|\mathcal{F}[y_j]\|$. Then, by shrinking the "analytic" strip, there is also

$$|\partial_\omega \Delta_j(m)| < e^{-|k|} e^{-\frac{1}{2} M^{j+1} c}.$$ 

Combining (4.14)-(4.16), we have

$$|\partial_\omega \Delta_j(m)| < e^{-|k|} e^{-\frac{1}{2} M^{j+1} c}, \quad |\alpha| = 0, 1,$$

and consequently

$$|\partial_\omega \Delta_j(m)| < e^{-|k|} \sum_{i=j_0}^j e^{-\frac{1}{2} M^{i+1} c} < e^{-|k|}.$$ 

Moreover, with $j_0$ large enough, we can also ensure that $\|y_{j+1}\|$ stays in a small neighborhood of zero. This verifies statement (S1)$_{j+1}$. 
Newt we turn to estimate the new error. We split $\mathcal{F}[y_{j+1}]$ into several terms

\[
\mathcal{F}[y_{j+1}] = \mathcal{F}[y_j + \Delta_j] = \mathcal{F}[y_j] + D\mathcal{F}[y_j] \Delta_j + R
\]
\begin{align*}
= & \mathcal{F}[y_j] + T_N \Delta_j + (T - T_N) \Delta_j + R \\
= & \mathcal{F}[y_j] - \Gamma_{10M} \mathcal{F}[y_j] \\
& + \Gamma_{10M} \mathcal{F}[y_j] + T_N \Delta_j \\
& + (T - T_N) \Delta_j \\
& + R,
\end{align*}
\tag{4.17}

where

\[ R = \int_0^1 \int_0^1 \mathcal{F}[y_j + st \Delta_j] I \Delta_j \eta^2 \, ds \, dt. \]

Observe that $\Gamma_{10M} \mathcal{F}[y_j] + T_N \Delta_j = 0$ and by \eqref{eq:4.13},

\[
||R|| \leq ||\Delta||^2 < \frac{1}{4} e^{-2M^{j+1} \epsilon}.
\tag{4.18}
\]

We further decompose $(T - T_N) \Delta_j$ into

\[
(T - T_N) \Delta_j = (1 - \Gamma_N) T \Delta_j = (1 - \Gamma_N) T \Gamma_{\frac{N}{2}} \Delta_j + (1 - \Gamma_N) T (1 - \Gamma_{\frac{N}{2}}) \Delta_j
\]
\[
= (1 - \Gamma_N) T \Gamma_{\frac{N}{2}} \Delta_j + (1 - \Gamma_N) T (1 - \Gamma_{\frac{N}{2}}) (T_N)^{-1} \Gamma_{10M} \mathcal{F}[y_j].
\]

Using the off decay estimate

\[
|T(m, m')| < e^{\frac{k - k'}{2}}, \quad k \neq k'
\]

for $T = \mathcal{F}'[y_j]$, we have

\[
||(1 - \Gamma_N) T \Gamma_{\frac{N}{2}}|| = \sup_{|m| = 1} ||(1 - \Gamma_N) T \Gamma_{\frac{N}{2}} y||
\]
\[
\leq \sup_{|m| = 1} \left( \sum_{|k| \in N} \left( \sum_{|k'| \in \frac{N}{2}} |T(m, m')| \cdot |y(m')|^2 \right) \right)^{1/2} \leq \frac{1}{8} e^{-\frac{1}{4} N^c}
\]

since $|k - k'| \geq \frac{N}{2}$. Then there is

\[
||(1 - \Gamma_N) T \Gamma_{\frac{N}{2}} \Delta_j|| \leq \frac{1}{8} e^{-\frac{1}{4} N^c} e^{-\frac{1}{2} N^c} = \frac{1}{8} e^{-2M^{j+1} \epsilon}.
\tag{4.19}
\]

Similarly, we obtain

\[
||(1 - \Gamma_N) T (1 - \Gamma_{\frac{N}{2}}) (T_N)^{-1} \Gamma_{10M}|| \leq \frac{1}{8} e^{-\frac{1}{4} N^c}
\]

and hence

\[
||(1 - \Gamma_N) T (1 - \Gamma_{\frac{N}{2}}) (T_N)^{-1} \Gamma_{10M} \mathcal{F}[y_j]|| \leq \frac{1}{8} e^{-\frac{1}{4} N^c} e^{-\frac{1}{2} N^c} \leq \frac{1}{8} e^{-2M^{j+1} \epsilon}.
\tag{4.20}
\]

Finally, we show the estimate for $(1 - \Gamma_{10M}) \mathcal{F}[y_j]$. Since $\text{Supp} y_j \subset L_{10M}$, we obtain from $\mathcal{F} = D + \epsilon \mathcal{W}$ that

\[
(1 - \Gamma_{10M}) \mathcal{F}[y_j] = (1 - \Gamma_{10M}) \mathcal{W}[y_j].
\]
Furthermore, taking only the $k$-component into consideration, we have 
\[
\|(1 - \Gamma_{10M^j})\mathcal{W}[y_j]\| \leq \sum_{\zeta \geq 10} \sum_{|\alpha + \beta| = \zeta} |b_{\alpha,\beta}| \sum_{|k| \leq 10M^j, 1 \leq s \leq \zeta} \|y_s\|^s e^{-\sum_{1 \leq s \leq \zeta} |k_s|^s},
\]
where $b_{\alpha,\beta}$ corresponds to the coefficients in the power series expansion of $f$ in (1.5) 
\[
f(x, \bar{x}) = \sum_{\zeta > 0} \sum_{|\alpha + \beta| = \zeta} b_{\alpha,\beta} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \bar{x}_1^{\beta_1} \cdots \bar{x}_n^{\beta_n}.
\]

Since the infimum of $\sum_{1 \leq s \leq \zeta} (|k_s|/M^j)^s$ with constraints $\sum_{1 \leq s \leq \zeta} (|k_s|/M^j) \geq 10$, $(|k_s|/M^j) \leq 1$ and $\zeta \geq 10$ is greater than three, we have 
\[
e^{-\sum_{1 \leq s \leq \zeta} |k_s|^s} \leq e^{-3M^j\nu} \leq e^{-\frac{1}{2}M^{j+1}\nu}.
\]

Then by the smallness of $\|y_j\|$ (staying inside the analyticity domain of the function $f$), we have 
(4.21) 
\[
\|(1 - \Gamma_{10M^j})\mathcal{W}[y_j]\| \leq \frac{1}{4} e^{-2M^{j+1}\nu}
\]
provided $M^c \approx 1$.

Combining the estimates (4.18)-(4.21) for the split (4.17), we have 
\[
\|\hat{\mathcal{W}}[y_{j+1}]\| \leq e^{-2M^{j+1}\nu}.
\]
The estimate of $||\hat{\partial}_{\nu_\alpha}\mathcal{W}[y_{j+1}]||$ follows the same line and we do not carry it out here.

4.5. Final reckoning. To complete the inductions, it suffices to apply the Neumann series to establish induction statements (S3)$_{j+1}$ and (S4)$_{j+1}$. Let $T_{j+1} = \mathcal{W}[y_{j+1}]$ and $T_j = \mathcal{W}[y_j]$, whose restrictions on $L_N$ are denoted by by $T_{j+1;N}$ and $T_{j;N}$ respectively. Note that 
\[
\|\hat{\mathcal{W}}'[y_{j+1}] - \hat{\mathcal{W}}'[y_j]\| \leq \|y_{j+1} - y_j\| \leq e^{-\frac{1}{2}M^{j+1}\nu}.
\]
and thus for any $N \leq M^{j+1}$, 
\[
(T_{j+1;N})^{-1} = \left(1 + \sum_{s \geq 1} \left(\frac{1}{2} M^{j+1}\nu^s\right)(T_{j;N} - T_{j+1;N})\right)(T_{j;N})^{-1},
\]
which is bounded by 
\[
\|[T_{j+1;N}]^{-1}\| \leq \left(1 + \sum_{s \geq 1} e^{-\frac{1}{2}s}M^{j+1}\nu^s\right) \Phi\left(\frac{N}{2}\right) \Phi\left(\frac{N}{2}\right) < \Phi(N).
\]

Moreover, we find that 
\[
\left|\left(T_{j;N}^{-1} - T_{j+1;N}^{-1}\right)(m, m')\right| \leq \Phi(N)e^{-\frac{1}{2}s}M^{j+1}\nu^s < \frac{1}{2s} e^{-\frac{1}{2}s}M^{j+1}\nu^s < \frac{1}{2s} e^{-\frac{1}{2}s}M^{j+1}\nu^s, \quad s \geq 1,
\]
and consequently by (4.10) 
\[
|T_{j+1;N}^{-1}(m, m')| < e^{-\frac{1}{2}s}M^{j+1}\nu^s, \quad |k - k'| > N\nu^s.
\]
The separation property for $T_{j+1}^{\nu}$ follows the same way by applying the Neumann series and we do not repeat it here.
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