Global Classical Solutions to 3D Compressible Navier-Stokes System with Vacuum in Bounded Domains under Non-Slip Boundary Conditions

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Abstract

The barotropic compressible Navier-Stokes system with non-slip (Dirichlet) boundary conditions in a three-dimensional bounded domain is considered. The global existence and uniqueness of classical solutions is obtained provided the initial energy is properly small and the initial density is strictly away from vacuum near the boundary of the domain. In particular, the density is allowed to vanish in the interior of the domain and its oscillations can be arbitrarily large. Moreover, a certain layering phenomenon occurs for the density, that is, the spatial gradient of the density remains uniformly bounded in some Lebesgue space with respect to time near the boundary and will grow unboundedly in the long run with an exponential rate provided vacuum (even a point) appears initially in the interior of the domain. Finally, in order to make clear distinction among the interior, intermediate, and the boundary parts of the domain, Lagrangian coordinates are also applied in our calculations.

Keywords: Compressible Navier-Stokes equations; Dirichlet boundary conditions; Vacuum; Lagrangian coordinates; Global classical solution

1 Introduction and the main result

We consider the barotropic compressible Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^3$

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - \lambda \nabla \text{div} u + \nabla P &= 0,
\end{align*}
\]

where $\rho = \rho(x,t)$ and $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ represent the unknown density and velocity respectively, and the pressure $P$ is given by

$P = a \rho^\gamma,$

for some positive constants $a > 0$ and $\gamma \geq 1$.

We also have the following physical restrictions on the constant viscosity coefficients $\mu$ and $\lambda$:

$\mu > 0, \ \lambda > 0.$

In this paper, we assume that $\Omega$ is a simply connected bounded $C^{2,1}$-domain in $\mathbb{R}^3$. 

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In addition, the system is subject to the given initial data
\[ \rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega, \]
and non-slip (Dirichlet) boundary condition:
\[ u = 0 \text{ on } \partial \Omega. \quad (1.3) \]

There is huge number of literature about the classical solvability for multidimensional compressible Navier-Stokes system with constant viscosity coefficients. The history of the area may trace back to Nash [30] and Serrin [37], who established the local existence and uniqueness of classical solutions respectively provided the density is strictly away from vacuum. An intensive treatment of compressible Navier-Stokes equations started with pioneering papers by Itaya [23], Matsumura-Nishida [28], Kazhikhov-Solonnikov [24], and Hoff [17] on the local theory for non-stationary problems, and by Beirão da Veiga [4, 5], Padula [34], and Novotný-Padula [32, 33] on the theory of stationary problems for small data. Hoff [16–19] then studied the problem with discontinuous initial data, and introduced a new type of a priori estimates on the material derivative \( \dot{u} \). For the existence of solutions for arbitrary data, the major breakthrough was due to Lions [27], where he obtained the global weak solution defined as solutions with finite energy, when the exponent \( \gamma \geq \frac{9}{5} \) in dimension 3, which was further released to the critical case \( \gamma > \frac{3}{2} \) by Feireisl et al [13]. The main restriction on initial data is that the initial energy is finite, so that the density vanishes at far fields or even has compact support. However, the uniqueness of the solution remains open. Recently, Huang-Li-Xin [22] and Li-Xin [26] established the global existence and uniqueness of classical solutions, merely assumed the initial energy small enough, where large oscillations were available. Moreover, in their results [22, 26], the density is allowed to vanish and even has compact support. More recently, for the Navier-slip boundary conditions in general bounded domains, Cai-Li [7] obtain the global existence and exponential growth of classical solutions with vacuum provided that the initial energy is suitably small.

For the system (1.1) with no-slip condition posed on the boundary of a bounded domain in \( \mathbb{R}^3 \), the local classical solutions were obtained by Tani [41] with \( \inf_{\Omega} \rho_0 > 0 \). When the initial vacuum is allowed, the local well-posedness and blow-up criterion for strong solutions were also established in a series of papers, and we refer the reader to [8–10, 28–30, 36, 37] and references therein.

In contrast, the results about the existence of global solution under non-slip boundary condition is rather fewer. Salvi and Straškraba [36] obtained the global existence of strong solutions when the initial data \( \|u_0\|_{H^2} + \|\nabla \rho_0\|_{L^q} \) is small for some \( q \in (3, 6] \). Matsumura-Nishida [29] studied the global existence of classical solution for full Navier-Stokes equation in upper half plane and exterior domain provided the solution is near the equilibrium. Recently, for the density strictly away from vacuum and under some stringent restriction upon the viscous coefficients
\[ \frac{\mu}{\lambda + \mu} < \varepsilon_0, \quad (1.4) \]
where \( \varepsilon_0 \) is a properly small constant determined by the domain, Perepelitsa [35] further establishes the global existence of weak solution in Hoff’s regularity class assumed that density has some Hölder type continuity near the boundary.

However, global solutions established up to now all excluded vacuum. The goal of present paper is to obtain a global solution containing vacuum under non-slip boundary condition and remove the restriction on viscous coefficients like (1.4).

Before stating the main results, we explain the notations and conventions used throughout this paper. For a positive integer \( k \), \( \alpha \in (0, 1) \) and \( 1 \leq p < \infty \), the standard \( L^p \), Hölder’s and Sobolev’s spaces are denoted as follows:
\[
\begin{aligned}
& L^p = L^p(\Omega), \quad C^\alpha = C^\alpha(\Omega), \quad W^{k,p} = W^{k,p}(\Omega), \quad H^k = W^{k,2}(\Omega), \\
& \|f\|_{L^p} = \|f\|_{L^p(\Omega)}, \quad \|f\|_{C^\alpha} = \|f\|_{C^\alpha(\Omega)}, \quad \|f\|_{W^{p,p}} = \|f\|_{W^{p,p}(\Omega)}, \quad \|f\|_{H^k} = \|f\|_{W^{k,2}(\Omega)}. 
\end{aligned}
\]
In the case of particular domain $\Gamma$, we will write down the domain under consideration:

$$\|f\|_{L^p(\Gamma)}, \|f\|_{C^\alpha(\Gamma)}, \|f\|_{W^{k,p}(\Gamma)}, \|f\|_{W^{k,2}(\Gamma)}.$$ 

Moreover, we denote the usual Hölder semi-norm by

$$\|f\|_{\dot{C}^\alpha(\Gamma)} \triangleq \sup_{x, y \in \Gamma} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$

Next, the initial energy is given by

$$C_0 \triangleq \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) dx,$$

where

$$G(\rho) \triangleq \rho \int_{\bar{\rho}}^{\rho} \frac{P(s) - P(\bar{\rho})}{s^2} \, ds, \quad \bar{\rho} \triangleq \frac{1}{|\Omega|} \int_{\Omega} \rho(x) \, dx.$$

Note that when $0 \leq \rho \leq 2M$, there is some constant $C$ depending on $M$ and $\bar{\rho}$ such that

$$C^{-1}(\rho - \bar{\rho})^2 \leq G(\rho) \leq C(\rho - \bar{\rho})^2. \quad (1.6)$$

**Remark 1.1.** We integrate (1.1) over $\Omega \times [0, t]$ and apply Dirichlet boundary condition (1.3) to deduce the conservation of total mass, say

$$\int \rho \, dx = \int \rho_0 \, dx.$$

Consequently, $\bar{\rho} = \bar{\rho}_0$ is actually a constant. We will not distinguish them in the following.

To clarify regularity assumptions upon initial density $\rho_0$, we introduce some domains and modified-$C^\alpha$ norm.

We select a suitably small constant $d_0$,

$$0 < d_0 \leq \frac{1}{100} R_0, \quad (1.7)$$

with $B_{R_0}(x_0) = \{ x \in \Omega \mid |x - x_0| < R_0 \} \subset \Omega$ for some constant $R_0 > 0$ and $x_0 \in \Omega$.

For technical reason, we assume $d_0$ is smaller than the injective radius of boundary $\partial \Omega$ (see [11]) which guarantees that we can find a constant $C$ depending only on $\Omega$ such that for $y \in \Omega$ and $0 < \epsilon < \frac{1}{100}$ we have

$$C^{-1}(\epsilon d_0)^3 \leq |B_{\epsilon d_0}(y) \cap \Omega| \leq C(\epsilon d_0)^3.$$

The initial layers $L_0$, $L_{0-}$, $L'_0$, $L''_0$ and $L''_{0+}$ are defined by

$$L_0 = \left\{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \frac{31}{32} d_0 \right\}, \quad L_{0-} = \left\{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \frac{15}{16} d_0 \right\},$$

$$L'_0 = \left\{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \frac{7}{8} d_0 \right\}, \quad L''_0 = \left\{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \frac{3}{4} d_0 \right\},$$

$$L''_{0+} = \left\{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \frac{5}{6} d_0 \right\}.$$

We call $L''_0$ the initial $H^{2,2}$ layer, $L'_0$ the initial $W^{1,p}$ layer and $L_0$ the initial $C^\alpha$ layer respectively. $L_{0-}$ and $L''_{0+}$ are two auxiliary layers. Note that the positions of these layers are given by

$$L''_{0+} \subset L_{0+} \subset L'_0 \subset L_{0-} \subset L_0.$$
Generally speaking, these domains locate near the boundary. We also require the initial value with properly higher regularity near the boundary. Moreover, we introduce the following modified Hölder type semi-norm:

\[
\|\rho\|_{C_0^\alpha(L^0)} \triangleq \sup_{x \in L^0, \ y \in L_{\infty}} \frac{|\rho(x) - \rho(y)|}{|x - y|^\alpha},
\]

which will help us obtain the local estimates (Lemma 2.48).

These layers and modified norm are not fixed through out the time. In fact, they are moving along the flow. We will give more detail descriptions and remark about them when we introduce the Lagrangian coordinates in Section 2.2 and perform the local estimates in Section 3.

Now, we state our main result concerning the global existence of classical solutions as follows:

**Theorem 1.1.** For given positive constants \(M_0, \ \bar{p} \ (< \frac{M_0}{10})\) and \(3 < q < 6\), suppose that

\[
0 \leq \rho_0 \leq M_0, \ \rho_0 \in W^{2,q}, \ u_0 \in H^2 \cap H^1,
\]

and that

\[
\inf_{x \in L^0} \rho_0(x) \geq \bar{p}.
\]

In addition, assume that there is a positive constant \(N_0\) (not necessarily small) such that

\[
\|\rho_0\|_{C_0^\alpha(L^0)} + \|\nabla \rho_0\|_{L^4(L^0)} + \|\nabla^2 \rho_0\|_{L^2(L^0_{\infty})} + \|u_0\|_{H^2(L^0_{\infty})} + \|\nabla u_0\|_{L^2} \leq N_0.
\]

Moreover, the following compatibility condition holds for some \(g \in L^2\),

\[
-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P(\rho_0) = \bar{p}^{1/2} g.
\]

Then there is a positive constant \(\varepsilon\) depending only on \(\mu, \ \lambda, \ \bar{p}, \ \bar{\rho}_0, \ M_0, \ N_0, \ d_0\) and \(\Omega\), such that the initial-boundary-value problem (1.1)–(1.3) has a unique classical solution \((\rho, u)\) in \(\Omega \times (0, \infty)\) satisfying

\[
0 \leq \rho(x, t) \leq 2M_0, \ (x, t) \in \Omega \times [0, \infty),
\]

and for any \(0 < \tau < T < \infty\)

\[
\begin{align*}
\rho &\in C([0, T]; W^{2,q}), \\
\nabla u &\in C([0, T]; H^1) \cap L^\infty(\tau, T; W^{2,q}), \\
u_t &\in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1), \\
\sqrt{\rho} u_t &\in L^\infty(0, \infty; L^2),
\end{align*}
\]

provided

\[
C_0 \leq \varepsilon.
\]

Moreover, for any \(r \in [1, \infty)\) and \(p \in [1, 6)\), there is a constant \(C\) and \(\eta\) depended only on \(\mu, \ \lambda, \ \gamma, \ \bar{\rho}, \ M_0, \ N_0, \ \Omega, \ \tau, \ d_0\) and \(p\), such that the following large-time behavior holds:

\[
\|\rho - \bar{\rho}\|_{L^r} + \|u\|_{W^{1,p}} \leq C e^{-\eta t},
\]

for all \(t > 1\).

As a direct consequence of Theorem 1.1 we have the following result concerning the blow-up behavior of the gradient of the density when vacuum appears initially.

**Theorem 1.2.** Under the conditions of Theorem 1.1 assume further that there exists some point \(x_0 \in \Omega\) such that \(\rho_0(x_0) = 0\). Then the unique global classical solution \((\rho, u)\) to the problem (1.1)–(1.3) obtained in Theorem 1.1 satisfies that for any \(r > 3\), there exist positive constants \(\hat{C}_1\) and \(\hat{C}_2\) depending only on \(\mu, \ \lambda, \ \gamma, \ M, \ \bar{\rho}, \ \bar{\rho}_0, \ M_0, \ N_0, d_0, \ \Omega\), such that for any \(t > 0\),

\[
\|\nabla \rho(\cdot, t)\|_{L^r} \geq \hat{C}_1 e^{\hat{C}_2 t}.
\]
A few remarks are in order:

**Remark 1.2.** Since $q > 3$, we take advantage of Sobolev’s inequality and (1.12) to declare

$$\rho, \nabla \rho \in C(\bar{\Omega} \times [0, T]).$$

Moreover, (1.12) also guarantees that

$$u, \nabla u, \nabla^2 u, u_t \in C(\bar{\Omega} \times [\tau, T]),$$

where we have applied the fact

$$L^2(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^2).$$

Finally, by (1.11) we have

$$\rho_t = -u \cdot \nabla \rho - \rho \text{div} u \in C(\bar{\Omega} \times [\tau, T]).$$

All these results show that the solution obtained by Theorem 1.1 is a classical one for positive time.

**Remark 1.3.** In Theorem 1.1, our smallness condition on the initial energy (1.13) is equivalent to smallness of the $L^2$-norm of $(\rho_0 - \bar{\rho}_0, \rho_0^{1/2} u_0)$ which greatly improves the classical results due to Matsumura and Nishida [29] by extending their results to the case that the density has large oscillations and allows to vanish.

**Remark 1.4.** Our Theorem 1.1 holds for all viscous coefficients $\mu$ and $\lambda$ satisfying the physical conditions (1.2), while the results in [35], require the additional assumption (1.4), which is crucial in establishing the time-independent upper bound for the density in the arguments in [35]. Furthermore, Our Theorem 1.1 allows the density to vanish which also improves the result of [35] where the density is strictly away from vacuum.

**Remark 1.5.** Our solution may contain vacuum whose appearance implies the large-time blowup behavior stated in Theorem 1.2; this is in sharp contrast to the results of [29], where the gradients of the density remain suitably small uniformly for all time. Moreover, Theorem 1.2 actually means the oscillation of density will grow unboundedly in the long run with an exponential rate, once the vacuum is formed in the domain $\Omega$. Such phenomenon seems to occur only in bounded domains, and there is no analogous result for Cauchy problem up to now.

**Remark 1.6.** To make notations and ideas clear, we assume $P(\rho) = \rho$ through out the whole paper, and the general case $P(\rho) = a\rho^\gamma$ follows exactly in the same way.

We now make some comments on the analysis of the whole paper.

According to [21,40], the central issue to get a global solution with uniform (with respect to time) upper bound of density $\rho$. Compared with [7,22] where they consider Cauchy problem or Navier-slip boundary conditions, Dirichlet boundary conditions bring some essential difficulties. Indeed, in this case, on the one hand, since there is no information about the rotation of velocity $\text{rot}u$ near the boundary, it seems difficult to obtain directly proper global control on the $W^{1,2}(\Omega)$-norm of the rotation which is easy to be obtained in [7,22] and plays an essential role in the analysis of [7,22]. On the other hand, when the density is strictly away from vacuum, Matsumura-Nishida [29] mainly applies energy method to close the higher order derivatives of density $\|\rho\|_{H^3}$ and obtain the global solution under Dirichlet boundary conditions; and recently, under some additional strict restriction on the viscous coefficients $\mu$ and $\lambda$, [35] introduces a type of boundary Hölder norm $\|\rho\|_{C^\alpha(\partial \Omega)}$ to get global control over $\text{rot}u$. However, both of their works [29,35] depend crucially on the condition that the density is strictly away from vacuum initially.
Motivated by these works, we try to construct a global solution containing vacuum just under the physical conditions \([1.2]\) on the viscous coefficients \(\mu\) and \(\lambda\). Precisely, for the material derivative given by
\[
\frac{D}{Dt} f = \dot{f} \triangleq \frac{\partial}{\partial t} f + u \cdot \nabla f,
\]
if we decompose the velocity \(u = w + v\) with the regular part \(w\) and the singular part \(v\) via
\[
\begin{align*}
\mu \Delta w + \lambda \nabla \text{div} w &= \rho \dot{u} \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
and
\[
\begin{align*}
\mu \Delta v + \lambda \nabla \text{div} v &= \nabla (\rho - \bar{\rho}) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Then the main obstacle comes from the singular part \(\text{rot} v\) (see Remark 2.1 as well). To get over it, additional regularity of density \(\rho\) seems necessary. Indeed, it follows from \((1.16)\) that \(\text{rot} v\) is harmonic, that is,
\[
\Delta \text{rot} v = 0,
\]
which means \(\text{rot} v\) behaves well at the interior of the domain \(\Omega\) due to the interior estimate of harmonica functions (or mean-value formula). However, the loss of information of the rotation \(\text{rot} v\) near the boundary makes the problem much more difficult to deal with. To overcome these difficulties, we first decompose the whole space \(\Omega\) into the boundary and interior parts
\[
\Omega = \text{Boundary part} \cup \text{Interior part}.
\]

Thus, the main strategy is that we modify the usual \(C^\alpha\)-estimate slightly which combined with the energy method thus can be applied to close the higher order spatial derivatives of density in Boundary part to fill up the blank of \(\text{rot} v\) near \(\partial \Omega\). These estimates combined with interior ones thus yield the desired time-independent upper bound of the density. Moreover, we merely control the higher order derivatives near the boundary \(\partial \Omega\), which makes vacuum available in the Interior part.

Now, let us carefully check the features of \(C^\alpha\) estimate and energy method. On the one hand, energy method can be applied to estimate the \(L^2\)-norm of \(\nabla^2 \rho\) in Boundary part, and it requires no further restriction on the viscous coefficients \(\mu, \lambda\) (see \([29]\)). However, quite different from \([29]\) where the integration is taken over the whole \(\Omega\), in our case, since the integration is taken only in Boundary part, there is a “inner” boundary term on Boundary part, that is, \(\partial (\text{Boundary part}) \setminus \partial \Omega\), occurring in our expression which is hard to control. On the other hand, since \(C^\alpha\) estimate is along the flow (see \([35]\)), we may avoid the inner boundary term if we apply \(C^\alpha\) estimate instead. However, \(C^\alpha\) estimate used in \([35]\) requires some additional restriction on \(\mu\) and \(\lambda\) which we want to drop out.

Henceforth, we want to combine these two methods together to reach our goal. In fact, we further cut down the Boundary part into two smaller layers
\[
\text{Boundary part} = B \cup \Gamma,
\]
where \(B\) contains the boundary \(\partial \Omega\), and \(\Gamma\) has positive distance from the boundary \(\partial \Omega\) (see Definition \([2.1]\)). We make use of \(C^\alpha\) estimate in \(\Gamma\) since it requires no information on inner boundary. Note that no restriction on \(\mu, \lambda\) is needed since \(\Gamma\) is strictly contained in \(\Omega\) (see \((1.17)\)). Moreover, we apply energy method in the domain \(B\) since it requires the information of boundary, and make a truncation \(\psi\) (see \((2.16)\)) to cancel out the inner boundary term. Such configuration avoids the shortcoming of each method.

However, the decomposition of \(\Omega\) into Interior part and Boundary one can not be fixed through out the whole time. If so, the flow originated at Interior part may finally enter into
Boundary part, which makes it impossible to distinguish the inner and boundary. To handle it, we make the decomposition “along the flow” as well. It means that we are more suitable to shift our viewpoint to Lagrangian coordinates, and make the decomposition in it (see Definition 2.1).

The Lagrangian coordinate is not direct in higher dimensions. The metric has changed in it, and some geometric preparations must be done (see Section 2.2). We will apply the setting in Christodoulou [11], the major advantage of it is that we can decompose the metric $g^{ij}$ of the whole space into the tangential part $\gamma^{ij}$ and normal part $N^i N^j$,

$$g^{ij} = \gamma^{ij} + N^i N^j.$$  

Such process is in analogy with upper half plane $\mathbb{R}^3_+$ where we have a global coordinate $(x_1, x_2, x_3)$, and $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ denote the tangential directions, $\frac{\partial}{\partial x_3}$ denotes the normal one (see Remark 2.4 as well). It is quite important when we apply energy method in domain $B$ (see Section 3).

The methods of this paper can be roughly summarized as follow: we consider a layer $B$ near the boundary $\partial \Omega$, then we apply energy method in $B$ to obtain higher order estimates ($\nabla^2 \rho$ for example) in the layer $B$ (We call it $H^{2,2}$ layer see Definition 2.1) which is useful for obtaining the uniform (with respect to time) upper bound of the density near the boundary. However, at this stage, the interior of $\Omega$ is merely $L^\infty(\Omega \setminus B)$ which ensures the vacuum state. Obviously there is a huge gap between $H^{2,2}$ and $L^\infty$, and we set up an intermediate layer $\Gamma$ links the higher order part $B$ and the lower order part $\Omega \setminus B$. Indeed, in $\Gamma$, we will use the modified $C^\alpha$-estimates, which exactly lies between $H^{2,2}$ and $L^\infty$. The domain $\Gamma$ can be viewed as the bridge between $H^{2,2}$ layer and interior $L^\infty$ domain. In spite of it, there are certain technical problems when we realize the idea. Shortly speaking, the gap between $C^\alpha$ and $H^{2,2}$, $L^\infty$ is still too large, and we need to construct further intermediate layers whose regularity lies between $H^{2,2}$, $C^\alpha$ and $L^\infty$ serving as transition states, for example $W^{1,p}$ layer (see Definition 2.1). We finally set up four layers $\Gamma''$, $\Gamma'$, $\Gamma$, $\bar{\Gamma}$ to attend the goal. The position of these layers is given by

$$\text{Boundary} \rightarrow \Gamma'' \rightarrow \Gamma' \rightarrow \Gamma \rightarrow \bar{\Gamma} \rightarrow \text{Interior}.$$  

The regularity of these layers is given by

$$H^{2,2} \hookrightarrow W^{1,p} \hookrightarrow C^\alpha \hookrightarrow \bar{C}^\alpha \hookrightarrow L^\infty,$$

where the definition of $\bar{C}^\alpha$, which may be regarded as a type of boundary $C^\alpha$ norm in short, can be found in (2.2).

In Section 3 which is the main part of the paper, we will close each layer in inverse direction: first close $\bar{C}^\alpha$ estimate then shrink the domain to close $C^\alpha$ estimate, then shrink further to close $W^{1,p}$ estimate, finally reach the $H^{2,2}$ layer.

Totally speaking, the structure of the estimates we obtain can be described as a “layering” phenomenon: in our procedure, the regularity of density decreases as the distance to the boundary increases, varying from the highest $H^{2,2}$ near the boundary to the lowest $L^\infty$ in the interior.

Note that all these domains are not fixed. They are determined in the Lagrangian coordinate along the flow, since we must make a distinction between Interior part where vacuum may appear and Boundary part where density has higher regularity. Consequently, we perform our calculations mainly in Lagrangian coordinate.

The rest of paper is organized as follows:

Section 2.1 to Section 2.3 are preliminary parts. We will offer some basic global results in Section 2.1. Then Section 2.2 is devoted to giving geometric settings and tools for further arguments. Section 2.3 lists some local results based on notation of Section 2.2.

With all preparations done, we first obtain some global a priori estimates in Section 3. Then, we will close all near-boundary estimates in Section 4 which is the main part of the paper. Section 5 seems somewhat routine to give the higher order a priori estimates. Finally, we will prove the main Theorems 1.1 and 1.2 in Section 6.
2 Preliminaries

From now on, to make notations and ideas clear, we always assume \( a = 1, \gamma = 1 \) in \([1, 2]\) throughout the whole paper, and the general case \( a > 0, \gamma \geq 1 \) follows in a similar way after some small modifications.

2.1 Some elementary tools

First of all, we quote the local existence and uniqueness results in \([20]\), where the initial density is allowed to contain vacuum.

**Lemma 2.1.** Assume that the initial value \((\rho_0, u_0)\) satisfies \([18]\) and \([11]\). Then there is a small time \( T > 0 \) depending only on \( \Omega, \mu, \lambda, \gamma, g, \|\rho_0\|_{W^{2,\alpha}} \) and \( \|u_0\|_{H^2} \), such that there exists a unique strong solution \((\rho, u)\) to the problem \([11, 12, 13]\) on \( \Omega \times (0, T) \) satisfying for any \( \tau \in (0, T) \),

\[
\begin{aligned}
\rho &\in C([0, T]; W^{2,q}(\Omega)), \\
u &\in C([0, T]; H^2) \cap L^\infty(\tau, T; W^{2,q}(\Omega)), \\
u_t &\in L^\infty(\tau, T; H^2), \quad u_{tt} \in L^2(\tau, T; H^1), \\
\sqrt{\mu}u_t &\in L^\infty(0, T; L^2).
\end{aligned}
\]

Next, to deduce the proper estimates upon \( \nabla u \), we require div-rot control which can be found in \([2, 12]\).

**Lemma 2.2.** Let \( 1 < q < \infty \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with Lipschitz boundary \( \partial \Omega \). For \( v \in W^{1,q}_0(\Omega) \), if \( \Omega \) is simply connected, then it holds that

\[
\|\nabla v\|_{L^p} \leq C(\|\text{div}v\|_{L^p} + \|\text{rot}v\|_{L^p}).
\]

The well-known Gagliardo-Nirenberg’s inequality (see \([31]\)) will be used frequently later.

**Lemma 2.3** (Gagliardo-Nirenberg). Assume that \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^3 \). For \( p \in (3, \infty), q \in (1, \infty) \) and \( r \in (3, \infty) \), there exist two generic constants \( C_1, C_2 > 0 \) which may depend on \( p, q, q, r, \) and \( \Omega \) such that for any \( f \in H^1(\Omega) \) and \( g \in L^q(\Omega) \cap D^{1,r}(\Omega) \),

\[
\begin{aligned}
\|f\|_{L^p(\Omega)} &\leq C_1\|f\|^\frac{6p}{6p-3} \|\nabla f\|_{L^2}^\frac{3p-6}{6p} + C_2\|f\|_{L^2}, \\
\|g\|_{C^{1r}(\Omega)} &\leq C_1\|g\|^\frac{p}{2} \|\nabla g\|_{L^2}^{1-q} + C_2\|g\|_{L^2},
\end{aligned}
\]

with \( \frac{p}{q} + \frac{(3-r)(1-q)}{2r} = 0 \). Moreover, \( C_2 = 0 \) provided \( f = 0, g = 0 \) on boundary \( \partial \Omega \).

Next, to obtain the estimate on the \( L^\infty(0, T; L^p(\Omega)) \)-norm of \( \nabla \rho \), we need the following Beale-Kato-Majda type inequality (see \([7]\)) which was first proved in \([3]\) when \( \text{div}u = 0 \).

**Lemma 2.4.** For \( 2 < q < \infty \), assume that \( u \in W^{2,q}(\Omega) \cap W^{1,2}_0(\Omega) \). Then there is a constant \( C = C(q) \) such that the following estimate holds

\[
\|\nabla u\|_{L^\infty} \leq C(\|\text{div}u\|_{L^\infty} + \|\text{curl}u\|_{L^\infty})\log(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C. \tag{2.1}
\]

Next, consider the problem

\[
\begin{aligned}
\text{div}u &= f & x &\in \Omega, \\
v &= 0 & x &\in \partial \Omega.
\end{aligned} \tag{2.2}
\]

One has the following conclusion (see \([14] \) Theorem III.3.1):
Lemma 2.5. There exists a linear operator $B = [B_1, B_2, B_3]$ enjoying the properties:

1) The operator $B : \{ f \in L^p(\Omega) : \bar{f} = 0 \} \mapsto (W_0^{1,p}(\Omega))^3$ is a bounded linear one, that is,

$$\|B[f]\|_{W_0^{1,p}(\Omega)} \leq C(p)\|f\|_{L^p(\Omega)}, \text{ for any } p \in (1, \infty).$$

2) The function $v = B[f]$ solves the problem (2.2).

3) Moreover, assume $f$ can be written in the form $f = \text{div} g$ for a certain $g \in L^r(\Omega)$, $g \cdot n|_{\partial \Omega} = 0$, then

$$\|B[f]\|_{L^r(\Omega)} \leq C(r)\|g\|_{L^r(\Omega)}, \text{ for any } r \in (1, \infty).$$

Finally, for technical reason, we decompose the velocity field $u$ into regular part $w$ and singular part $v$ and derive some basic global estimates on them respectively.

Note that $u$ solves the Lamé system:

$$\begin{cases}
\mu \Delta u + \lambda \nabla \text{div} u = \nabla (\rho - \bar{\rho}) + \rho \dot{u} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Define $w$ via

$$\begin{cases}
\mu \Delta w + \lambda \nabla \text{div} w = \rho \dot{u} & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{cases}$$

and $v$ as

$$\begin{cases}
\mu \Delta v + \lambda \nabla \text{div} v = \nabla (\rho - \bar{\rho}) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}$$

The uniqueness of solution to Lamé system guarantees

$$u = w + v.$$  

(2.5)

The basic standard elliptic estimates (see [1, Theorem 10.1-10.2] and [15, Theorem 9.11-9.13]) assure that

Lemma 2.6. For $1 < p < \infty$, there is a constant $C$ depending only on $\mu, \lambda, \Omega$ and $p$, such that

$$\begin{align*}
\|\nabla^2 w\|_{L^p} & \leq C\|\rho \dot{u}\|_{L^p}, \\
\|\nabla v\|_{L^p} & \leq C\|\rho - \bar{\rho}\|_{L^p}, \\
\|\nabla^2 v\|_{L^p} & \leq C\|\nabla \rho\|_{L^p},
\end{align*}$$

(2.6)

Note that $w$ gains more global regularity since $\rho \dot{u}$ has proper global control due to Theorem 3.2. That is the reason why we call it regular part.

We turn to the singular part of velocity, which is the main difficult in Dirichlet problem.

Remark 2.1. The essential difference between non-slip boundary and slip boundary condition lies in the singular part $v$. Indeed, rewriting the Lamé system (2.4) as:

$$\begin{cases}
\nabla ((\mu + \lambda)\text{div} v - (\rho - \bar{\rho})) - \mu \nabla \times \text{rot} v = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}$$

we observe that both $F_v = (\mu + \lambda)\text{div} v - (\rho - \bar{\rho})$ and $\text{rot} v$ are harmonic. Thus, for typical Navier-slip boundary conditions [7], $\text{rot} v = 0$ on $\partial \Omega$, one has $\text{rot} v = 0$ in $\Omega$ due to the maximal principal, which means singular part $v$ actually does not enter into the estimate.

However, for Dirichlet boundary condition, no information about the boundary value of $\text{rot} v$ is provided directly. Consequently, some effective estimates upon $\text{rot} v$ and $F_v$ are lacked. It is the main difficult we must get over.
We apply the method in [35] to get a precise point-wise representation of \( v \):

First, suppose \( v_s \) is the unique solution to following Poisson equation.

\[
\begin{align*}
\Delta v_s &= \nabla \frac{\rho - \bar{\rho}}{\mu + \lambda} \quad \text{in } \Omega, \\
v_s &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (2.7)

Let \( \Gamma(|x - y|) \) be the fundamental solution of \( \Delta \) in \( \mathbb{R}^3 \) and \( G(x, y) \) be the Green function of \( \Delta \) in \( \Omega \) with respect to Dirichlet boundary condition.

We define

\[
F_s \triangleq \text{div} v_s - \frac{\rho - \bar{\rho}}{\mu + \lambda} = \int \partial_x \partial_y G^*(x, y) \frac{\rho - \bar{\rho}}{\mu + \lambda}(y)dy,
\] (2.8)

where \( G^*(x, y) = \Gamma(|x - y|) - G(x, y) \). Note that \( G^*(x, y) \) is actually out of singularity within the domain \( \Omega \).

Set \( v_r = v - v_s \) which solves a Lamé system involved \( F_s \) as

\[
\begin{align*}
\mu \Delta v_r + \lambda \nabla \text{div} v_r &= -\lambda \nabla F_s \quad \text{in } \Omega, \\
v_r &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (2.9)

We substitute the definitions of \( F_s \) and \( v_r \) into (2.4), and find that \( v \) solves a Poisson equation:

\[
\begin{align*}
\mu \Delta v &= -\nabla (\lambda F_s + \lambda \text{div} v_r - \frac{\mu \rho - \mu \bar{\rho}}{\mu + \lambda}) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

**Remark 2.2.** With the help of Green function, we declare that

\[
v(x) = \int \partial_y G(x, y) \left( \frac{\rho - \bar{\rho}}{\mu + \lambda} \frac{\lambda F_s}{\mu} - \frac{\lambda \text{div} v_r}{\mu} \right) dy
\] (2.10)

which provides us a precise point-wise representation for \( v \) which makes it possible to get a finer control over \( \frac{|v(x) - v(y)|}{|x - y|^\alpha} \) (see Lemma 2.18). We indicate that the method to get a representation is not unique, one may directly uses the Green function for elliptic operator \( \mu \Delta + \lambda \nabla \text{div} \) to get a point-wise representation as well. The key issue is the order of singularity and some canceled property such as (2.56).

Moreover, the order of singularity of Green function \( G(x, y) \) can be found in [38, 39]:

**Lemma 2.7 ([38, 39]),** There is a constant \( C \) depending only on \( \mu, \lambda \) and \( \Omega \) such that

\[
|\partial_x^\alpha \partial_y^\beta G(x, y)| \leq C|x - y|^{1 - \alpha - \beta},
\]

where \( \alpha \) and \( \beta \) are non-negative integers.

### 2.2 Geometric issue

To get a sharper estimate on the behavior of the system, we will turn to the local frame. The main idea is to trace along the flow, as a result, it is more convenient to perform our calculations in Lagrangian coordinates in which the flow becomes fixed. However, since Lagrangian coordinates are less direct in higher dimensions, there is some preparation should be done.

First of all, we must clarify geometric notations through out the whole paper. We apply the settings in Christodoulou [11]:
Let \( X_t : \Omega \rightarrow \Omega, y \mapsto X_t(y) \) denote the action of flow. Precisely, \( X_t(y) \) is determined by the ODE which we call the flow line.

\[
\begin{align*}
\dot{X}(t) &= u(X(t), t), \\
X(0) &= y.
\end{align*}
\]

(2.11)

Set \( X_t(y) = X(t) \).

When \( u(x, t) \) gives a smooth vector field on \( \Omega \), \( X(t) \) is a diffeomorphism of \( \Omega \) to itself. It means that \( X_t \) gives a smooth change of coordinates which we call Lagrangian coordinates:

\[
X_t : (\Omega, y) \rightarrow (\Omega, x)
\]

\[
y \mapsto X_t(y).
\]

Now we give a geometric structure in Lagrangian coordinates, where all quantities involved are defined in a standard way as in the classical text books [6, 43]:

We pull back the Euclidean metric to the Lagrangian coordinate \((\Omega, y)\) via \(X_t\). Let \(g_{ij}\) be the metric in Lagrangian coordinates obtained in this way, and \(g^{ij}\) be the inverse of it. The inner product of two vector fields \(X = X^i \frac{\partial}{\partial y^i}, Y = Y^i \frac{\partial}{\partial y^i}\) is given by

\[
\langle X, Y \rangle \triangleq g_{ij}X^iY^j.
\]

Generally, a \((s, r)\) type tensor \(A\), (see also [41, Section 2] and [43, Chapter 2]) is written in form of:

\[
A = A^{ij_1j_2...j_s}dy^{j_1} \otimes dy^{j_2} \otimes \cdots \otimes dy^{j_r} \otimes \frac{\partial }{\partial y^{i_1}} \otimes \frac{\partial }{\partial y^{i_2}} \otimes \cdots \otimes \frac{\partial }{\partial y^{i_s}}.
\]

The superscript and subscript can be transformed via

\[
A^{(\cdots)}_{(i\cdots)} = g_{ij}A^{(j\cdots)}_{(i\cdots)}, \quad A^{(k\cdots)}_{(\cdots)} = g^{kl}A^{(\cdots)}_{(k\cdots)},
\]

for example

\[
A^1_{j} = g_{jk}A^{kl} = g^{il}A_{ij}, \quad A_{ij} = g_{ik}g_{jl}A^{kl}.
\]

The covariant derivative (see (2.3) in Section 2 of [41] and Chapter VIII, Section 5 in [4]) of a \((s, r)\) type tensor is defined in usual manner, the subcribe \(A^{(\cdots)}_{(\cdots),i} \) denotes the covariant derivative of tensor \(A\) with respect to the \(i\)th coordinate, and \(A^{(\cdots)}_{(\cdots),ijk} \) denotes the higher order covariant derivative respectively.

Precisely, let \(\Gamma^k_{ij}\) be the connection coefficients of the unique Riemannian connection (see Chapter VII, Theorem 3.3 in [41]) with respect to Riemannian metric \(g_{ij}\), then

\[
A^{ij_1j_2...j_s}_{1j_1j_2...j_r,k} = \frac{\partial }{\partial y^k}A^{ij_1j_2...j_s}_{1j_1j_2...j_r} - \Gamma^j_{jk}A^{ij_1j_2...j_s}_{1j_1j_2...j_r} - \cdots - \Gamma^j_{jk}A^{ij_1j_2...j_s}_{1j_1j_2...j_r} + \Gamma^l_{ik}A^{ij_1j_2...j_s}_{1j_1j_2...j_r} \cdots + \Gamma^l_{ik}A^{ij_1j_2...j_s}_{1j_1j_2...j_r}. \tag{2.12}
\]

Since the metric in Lagrangian coordinates are given by pulling back, the geometric features in Lagrangian coordinates remain the same with Euler coordinates. For example, the metric in Lagrangian coordinates is out of curvature, consequently we can freely change the order of covariant derivatives (see also (2.9) in Section 2 of [41]):

\[
A^{(\cdots)}_{(\cdots),ij} = A^{(\cdots)}_{(\cdots),ji}.
\]

**Remark 2.3.** We apply covariant derivative \(u^i_{ij}\) rather than usual derivative \(\frac{\partial u^i}{\partial y^j}\) in Lagrangian coordinate because the former one is intrinsically defined. Once the superscript and the subscript summing over, for example \(u^i_{ij}v^j_{i}\), we will get a intrinsically defined quantity.
For such term, we may switch from Lagrangian coordinates to Euler coordinates freely for its coordinate free. In this case $u^i_j$ can be replaced directly by $\frac{\partial u^i}{\partial x^j}$, since the latter one is in fact the covariant derivative in Euler coordinates. In other words, $u^i_j$ is the “real” form of $\frac{\partial u^i}{\partial x^j}$ in Lagrangian coordinates. We note that $\frac{\partial u^i}{\partial x^j}$ actually differ from $\frac{\partial u^i}{\partial x^j}$ by some additional terms involving connection coefficients due to (2.12).

Let $N_0 = N_0^i \frac{\partial}{\partial y^i}$ be the unit outer normal at boundary $\partial \Omega$, where “unit” means:

$$\langle N_0, N_0 \rangle = g_{ij} N_0^i N_0^j = 1.$$ 

Let $N^i(y)$ be the geodesic extension of $N_0^i$ into the whole space $\Omega$ as in Section 3 of [11]. Define the tangential metric $\gamma^{ij}$ by

$$\gamma^{ij} = g^{ij} - N^i N^j,$$

and the projection to the tangential direction $\Pi A$ of a tensor $A$ (see Definition 3.1 in [11]) is defined by

$$(\Pi A)^{i_1 i_2 \cdots i_s}_{j_1 j_2 \cdots j_r} = s_{i_1 i_2 \cdots i_s} \gamma^{j_1 j_2 \cdots j_r} A_{j_1 j_2 \cdots j_r}.$$ 

When $X = X^i \frac{\partial}{\partial y^i}$ is a $(1,0)$ type tensor (vector field), $\Pi X$ is just the classical projection of vector field to the tangential direction:

$$\Pi X = \gamma^{ij} X^j \frac{\partial}{\partial x^i}.$$ 

Remark 2.4. In upper half plane $\mathbb{R}^3_+$, the situation is much easier, since we have a global coordinate $(x_1, x_2, x_3)$ in which $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ denote the tangential direction, while $\frac{\partial}{\partial x_3}$ denotes the normal direction. Unfortunately in general domain, we may fail to find such a coordinate frame where tangential and normal directions can be distinguished directly.

In fact, the formula:

$$g^{ij} = \gamma^{ij} + N^i N^j,$$

just gives a universal way to write the whole space $(g^{ij})$ into the tangential direction $(\gamma^{ij})$ and the normal direction $(N^i N^j)$, at least near the boundary.

Noting that the metric $\gamma^{ij}$ is semi-positive definite due to (2.13), it gives a pseudo Riemannian metric on $\Omega$. We have following evolution equations on geometric quantities which can be found in Lemma 2.1 and Lemma 3.9 of [11]. Through out the paper, $D_t = \partial_t + u \cdot \nabla$ denotes the material derivative or equivalently $\partial_t$ in Lagrangian coordinate.

Lemma 2.8. Suppose $g^{ij}$ is the metric in Lagrangian coordinate, $N$ is the unit normal of boundary $\partial \Omega$, $\gamma^{ij}$ is the tangential metric, $\Gamma^{k}_{ij}$ is the connection coefficients, and $d\nu$ is the volume element, then

$$D_t \frac{\partial x^i}{\partial y^j} = \frac{\partial x^k}{\partial y^j} \frac{\partial u^i}{\partial x^k}$$

$$D_t \frac{\partial y^i}{\partial x^j} = -\frac{\partial y^i}{\partial x^k} \frac{\partial u^k}{\partial x^j},$$

$$D_t g_{ij} = u_{i,j} + u_{j,i}$$

$$D_t \Gamma^k_{ij} = u^k_{ij}.$$ 

When restricted on the boundary $\partial \Omega$,

$$D_t \gamma^{ij} = -2 \gamma^{ik} \gamma^{jl} (u_{k,l} + u_{l,k}),$$

$$D_t N^i = -2 (u_{k,l} + u_{l,k}) g^{ik} g^{jl} N_j + \frac{u_{k,l} + u_{l,k}}{2} N j N^k N^l,$$

$$D_t d\nu = u^i d\nu.$$ 

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Observe that the boundary $\partial \Omega$ is fixed in Euler coordinates due to non-slip boundary condition (1.3). As a result, all geometric quantities of $\partial \Omega$ in Lagrangian coordinate remains just the same as the initial time $t = 0$. In particular, the second fundamental form is bounded uniformly in time. According to Lemma 3.10 and Lemma 3.11 of [11], we have

**Lemma 2.9.** Let $N$ be the unit outer normal on the boundary $\partial \Omega$, and $N(x)$ be the geodesic extension of $N$ (see Section 3 of [11]) to the whole space, then

$$\|\nabla \gamma^i_j\|_{L^\infty} + \|\nabla N\|_{L^\infty} \leq C\|\theta\|_{L^\infty(\partial \Omega)} \leq C,$$

(2.14)

where $\theta$ is the second fundamental form (curvature in dimension 2) of boundary $\partial \Omega$.

Moreover, when restrict to boundary $\partial \Omega$,

$$\sup_{y \in \partial \Omega} |\bar{\nabla}^k \theta(y,t)| = \sup_{y \in \partial \Omega} |\bar{\nabla}^k \theta(y,0)| \leq C,$$

where $\bar{\nabla}$ means tangential derivatives.

**Proof.** The first part (2.14) can be found in Lemma 3.10 and Lemma 3.11 of [11], and we just prove the second part.

As mentioned before the statement of the Lemma, the boundary $\partial \Omega$ is fixed for all time due to the non-slip boundary condition. Therefore, the second fundamental form $\theta(t) = \theta(0)$ since it is intrinsically defined and independent with the particular coordinates. Consequently, for tangential derivatives,

$$\sup_{y \in \partial \Omega} |\bar{\nabla}^k \theta(y,t)| = \sup_{y \in \partial \Omega} |\bar{\nabla}^k \theta(y,0)| \leq C.$$


With Lagrangian coordinates in hand, free boundary is transformed into fixed boundary. There are 11 free boundaries we shall investigate:

**Definition 2.1.** For the same positive constant $d_0 > 0$ as in (1.7), set

$$\Gamma_1 = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{15}{16} d_0\right\}, \quad \Gamma_2 = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{16}{15} d_0\right\},$$

$$\Gamma_1' = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{8}{15} d_0\right\}, \quad \Gamma_2' = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{7}{8} d_0\right\},$$

$$\Gamma_1'' = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{5}{8} d_0\right\}, \quad \Gamma_2'' = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{4}{5} d_0\right\},$$

$$\Gamma_2''' = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{5}{6} d_0\right\}, \quad \bar{\Gamma}_2 = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{31}{32} d_0\right\},$$

$$B_0 = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{1}{3} d_0\right\}, \quad B_1 = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{1}{2} d_0\right\},$$

$$A_1 = \left\{y \in \Omega | \text{dist}(y, \partial \Omega) = \frac{3}{32} d_0\right\}.$$
The layers $\tilde{\Gamma}$, $\Gamma$, $\Gamma'$, $\Gamma''$, $B$, $B'$ are defined by

\[
\tilde{\Gamma} = \left\{ y \in \Omega \left| \frac{15}{16} d_0 \leq \text{dist}(y, \partial \Omega) \leq \frac{31}{32} d_0 \right. \right\}, \quad \Gamma = \left\{ y \in \Omega \left| \frac{1}{16} d_0 \leq \text{dist}(y, \partial \Omega) \leq \frac{15}{16} d_0 \right. \right\},
\]
\[
\Gamma' = \left\{ y \in \Omega \left| \frac{1}{8} d_0 \leq \text{dist}(y, \partial \Omega) \leq \frac{7}{8} d_0 \right. \right\}, \quad \Gamma'' = \left\{ y \in \Omega \left| \frac{1}{5} d_0 \leq \text{dist}(y, \partial \Omega) \leq \frac{4}{5} d_0 \right. \right\},
\]
\[
\Gamma''_+ = \left\{ y \in \Omega \left| \frac{1}{6} d_0 \leq \text{dist}(y, \partial \Omega) \leq \frac{5}{6} d_0 \right. \right\},
\]
\[
B = \left\{ y \in \Omega \left| 0 \leq \text{dist}(y, \partial \Omega) \leq \frac{1}{2} d_0 \right. \right\}, \quad B' = \left\{ y \in \Omega \left| 0 \leq \text{dist}(y, \partial \Omega) \leq \frac{1}{3} d_0 \right. \right\},
\]
\[
A = \left\{ y \in \Omega \left| 0 \leq \text{dist}(y, \partial \Omega) \leq \frac{3}{32} d_0 \right. \right\}.
\]

Here, $B \cup \Gamma''$ is called the $H^{2,2}$ layer, $B \cup \Gamma'$ is called the $W^{1,p}$ layer and $B \cup \Gamma \cup \tilde{\Gamma}$ is called the $C^\alpha$ layer respectively. $B$ is the boundary part.

**Remark 2.5.** The position of these layers can be concluded by

\[ B \subset \subset H^{2,2}, B \subset \subset W^{1,p}, B \subset \subset C^\alpha \]  

These are four main layers we will investigate. We will close $C^\alpha$, $W^{1,p}$ and $H^{2,2}$ estimate in them respectively. According to (2.15), $C^\alpha$ layer is the largest (furthest form boundary) and the regularity is the lowest $C^\alpha$; $H^{2,2}$ layer is the smallest (nearest form boundary) and the regularity is the highest $H^{2,2}$; $W^{1,p}$ layer serves as a transition state, it locates between $C^\alpha$ and $H^{2,2}$ layer and the regularity is the middle one $W^{1,p}$.

$\Gamma''_+$, $B'$ and $A$ are three auxiliary layers which helps us determine the position of truncation functions (see (2.16), (2.20), (2.21)). According to interior estimate Lemma 2.14, it is better to make the position of truncation with positive distance from four main layers.

**Remark 2.6.** It is not proper to fixed the “boundary part” and “interior part” of $\Omega$ through out the time, since the flow originated from the “interior part” may enter into the “boundary part” as $t \to \infty$ which make it impossible to distinguish the “interior” and “boundary”. Alternatively, if we set these layers varying along the flow as well, such problem can be got over, for example the flow in $\Gamma$ will never touch the boundary. Thus, it is more suitable to apply Lagrangian coordinate since free layers become fixed ones in it.

For technical reasons, we introduce series of truncation functions in Lagrangian coordinates. Firstly, let $\psi$ satisfy:

\[
\psi \geq 0, \psi \in C^\infty, \\
\psi = 1 \text{ on } B', \\
\psi = 0 \text{ on } \Omega \setminus B.
\]

We give a proper control over $\nabla \psi$:

**Lemma 2.10.** There is a constant $C$ such that

\[
\sup_{0 \leq t \leq T} \| \nabla \psi \|_{L^\infty(B)} \leq C \exp \left( \int_0^T \| \nabla u \|_{L^\infty(B)} dt \right),
\]

where $|\nabla \psi| = (g^{ij} \partial_i \psi_j, \partial_i \psi_j)^{\frac{1}{2}}$. 

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Proof. Note that $\psi$ is supported in domain $B$. We use Euler coordinates in $B$ to show that,

$$
\left| \frac{\partial \psi}{\partial x} \right| = \left| \frac{\partial y}{\partial x} \cdot \frac{\partial \psi}{\partial y} \right| \leq C \left| \frac{\partial y}{\partial x} \right|.
$$

However, when restricted in $B$, Lemma 2.8 guarantees

$$
D_t \left| \frac{\partial y}{\partial x} \right| \leq C \| \nabla u \|_{L^\infty(B)} \left| \frac{\partial y}{\partial x} \right|.
$$

(2.18)

The Gronwall inequality applied to (2.18) makes sure

$$
\sup_{0 \leq t \leq T} \left| \frac{\partial y}{\partial x} \right|_{L^\infty(B)} \leq C \exp\left( \int_0^T \| \nabla u \|_{L^\infty(B)} dt \right).
$$

Consequently,

$$
\sup_{0 \leq t \leq T} \left| \frac{\partial \psi}{\partial x} \right|_{L^\infty(B)} \leq C \exp\left( \int_0^T \| \nabla u \|_{L^\infty(B)} dt \right).
$$

(2.19)

We shift (2.19) into coordinate free form and arrive at:

$$
\sup_{0 \leq t \leq T} \| \nabla \psi \|_{L^\infty(B)} \leq C \exp\left( \int_0^T \| \nabla u \|_{L^\infty(B)} dt \right).
$$

In analogy with $\psi$, we give two more truncation functions $\phi$ and $\zeta$ in Lagrangian coordinates whose support are given by

$$
\phi \geq 0, \phi \in C^\infty,
\phi = 1 \text{ on } B \cup \Gamma_+',
\phi = 0 \text{ on } \Omega \setminus \Gamma',
$$

(2.20)

and

$$
\zeta \geq 0, \zeta \in C^\infty,
\zeta = 1 \text{ on } \Omega \setminus B,
\zeta = 0 \text{ on } A.
$$

(2.21)

Similar to Lemma 2.10, one also has:

**Lemma 2.11.** There is a constant $C$ such that

$$
\sup_{0 \leq t \leq T} \| \nabla \phi \|_{L^\infty(B \cup \Gamma')} \leq C \exp\left( \int_0^T \| \nabla u \|_{L^\infty(B \cup \Gamma')} dt \right),
$$

and

$$
\sup_{0 \leq t \leq T} \| \nabla \zeta \|_{L^\infty(B \cup \Gamma')} \leq C \exp\left( \int_0^T \| \nabla u \|_{L^\infty(B \cup \Gamma')} dt \right),
$$

where $|\nabla \phi| = (g^{ij} \phi_i \phi_j)^{\frac{1}{2}}$, $|\nabla \zeta| = (g^{ij} \zeta_i \zeta_j)^{\frac{1}{2}}$.

**Proof.** The proof is the same as Lemma 2.10

$$
\square
$$

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Remark 2.7. Since \( \psi \) and \( \phi \) are two truncation functions near the boundary, we can restrict to near boundary region by multiplying them in our computations (see Theorem 4.2 and Section 4.4). In some cases, we also need to restrict to a region of \( \Omega \) near the boundary \( \partial \Omega \) but not touch it, \( \psi \cdot \zeta \) and \( \phi \cdot \zeta \) will be the proper truncation functions to attend the goal (see Lemma 4.4).

Under current settings, we end this section by giving a proper control of distance between each layer before further arguments.

Lemma 2.12. Let \( X(t), Y(t) \) be any two flow lines, and \( D_0 = |X(0) - Y(0)| \) be the initial distance. Then, for all \( t > 0 \) the distance between them satisfies

\[
|X(t) - Y(t)| \geq \frac{D_0}{2},
\]

provided

\[
\int_0^\infty \|u\|_{L^\infty} \, dt \leq \frac{D_0}{4}.
\]

Proof. By definition of flow lines (2.11),

\[
\frac{d}{dt}X(t) = u(X(t), t), \quad \frac{d}{dt}Y(t) = u(Y(t), t).
\]

We deduce that

\[
\frac{d}{dt}(X(t) - Y(t)) = u(X(t), t) - u(Y(t), t).
\]

Multiplying (2.21) by \((X(t) - Y(t))/|X(t) - Y(t)|\) gives

\[
\frac{d}{dt}|X(t) - Y(t)| = \frac{(u(X(t), t) - u(Y(t), t)) \cdot (X(t) - Y(t))}{|X(t) - Y(t)|}.
\]

Integrating (2.25) with respect to \( t \), we apply (2.23) to declare

\[
|X(t) - Y(t)| \geq |X(0) - Y(0)| - 2 \int_0^t \|u\|_{L^\infty} \, ds \geq \frac{D_0}{2},
\]

which proves (2.22). \(\square\)

By Definition 2.1, the initial distance \( D_0 \) of any two free boundaries \((\tilde{\Gamma}_2, \Gamma_{2+}^\prime, \Gamma_1, \Gamma_1^\prime, \Gamma_i^\prime, B_i, A_1 \) and \( \partial \Omega \)) is larger than \( \frac{D_0}{32} \). According to Lemma 2.12 if we also have

\[
\int_0^T \|u\|_{L^\infty} \, dt < \frac{D_0}{200},
\]

say (2.23) holds, the distance of any two free layers will be strictly larger than \( d(= \frac{D_0}{100}) \) through the time. Precisely, we have:

Theorem 2.1. Let \( L = \{\tilde{\Gamma}_2, \Gamma_{2+}^\prime, \Gamma_1, \Gamma_1^\prime, \Gamma_i^\prime, B_i, A_1, \partial \Omega \} \) and suppose (2.26) holds. There is a positive constant

\[
d = \frac{d_0}{100} > 0
\]

such that for any \( l_1, l_2 \in L \) and \( l_1 \neq l_2 \),

\[
dist(l_1, l_2) > d
\]

holds for all \( t \).

Remark 2.8. Note that Theorem 2.1 makes sure the distance from the boundary \( \partial \Omega \) to each free layer given by Definition 2.1 is strictly larger than \( d \) as well.
2.3 Local results near the boundary

In this subsection, we always assume that \( \text{(2.20)} \) is true and \( \rho \) satisfying

\[
\Vert \rho \Vert_{L^\infty} \leq 2M_0, \tag{2.28}
\]

for some positive constant \( M_0 \).

With notations given in Section \( \text{2.2} \) we establish some basic interior estimates (see \[1\]-[15]) which will be frequently used. Suppose \( d \) is given by Theorem \( \text{2.1} \) and set a geometric constant

\[
K = d^{100} + d^{-100}. \tag{2.29}
\]

Note that both \( d \) and \( K \) are \( t \) (time)-independent due to Lemma \( \text{2.12} \) and Theorem \( \text{2.1} \).

First, we deduce following local-version the Gagliardo-Nirenberg inequality in \( B \cup \Gamma'' \). Since \( \text{dist}(B, \Gamma'') > d \) due to Theorem \( \text{2.1} \), the constant occurring in the inequality is completely determined by our geometric constant \( K \), say

**Lemma 2.13.** (Gagliardo-Nirenberg Inequality) For the domain \( B \) and \( \Gamma'' \) given by Definition \( \text{2.4} \) there is a constant \( C \) depending only on \( \Omega \) such that

\[
\Vert \nabla v \Vert_{L^\infty(B)} \leq CK\Vert \nabla v \Vert_{L^2(B,\Gamma'')}^{\frac{5}{2}}\Vert \nabla^2 v \Vert_{L^4(B,\Gamma'')}^{\frac{1}{2}} + CK\Vert \nabla v \Vert_{L^2(B,\Gamma'')}, \tag{2.30}
\]

and

\[
\Vert \rho - \bar{\rho} \Vert_{L^\infty(B)} + \Vert \nabla \rho \Vert_{L^4(B)} \leq CK\Vert \rho - \bar{\rho} \Vert_{L^2(B,\Gamma'')}^{\frac{5}{2}}\Vert \nabla^2 \rho \Vert_{L^4(B,\Gamma'')}^{\frac{1}{2}} + CK\Vert \rho - \bar{\rho} \Vert_{L^2(B,\Gamma'')} \tag{2.31}
\]

Next, we establish some local elliptic estimates. There are two elliptic systems \( \text{(1.15)} \) and \( \text{(1.16)} \) playing important roles in our arguments.

Let us fix any domain \( \Gamma' \subset \subset \Gamma \subset \Omega \) with \( \text{dist}(\Gamma', \Gamma) > d \). We declare following estimates on \( v \) and \( w \):

**Lemma 2.14.** Suppose \( \text{(2.28)} \) holds. There is some constant \( C \) depending only on \( \Omega, \mu, \lambda \) and \( M_0 \) such that

\[
\Vert \nabla v \Vert_{C^0(\Gamma')} \leq CK\left( \Vert \rho \Vert_{C^0(\Gamma)} + \Vert \rho - \bar{\rho} \Vert_{L^4} \right),
\]

\[
\Vert \nabla^2 v \Vert_{L^4(\Gamma')} \leq CK\left( \Vert \nabla \rho \Vert_{L^4(\Gamma)} + \Vert \rho - \bar{\rho} \Vert_{L^4} \right),
\]

\[
\Vert \nabla^3 v \Vert_{L^2(\Gamma')} \leq CK\left( \Vert \nabla^2 \rho \Vert_{L^2(\Gamma)} + \Vert \nabla \rho \Vert_{L^2(\Gamma)} + \Vert \rho - \bar{\rho} \Vert_{L^2} \right),
\]

and

\[
\Vert \nabla^2 w \Vert_{L^2(\Gamma')} \leq CK\left( \Vert \rho \dot{u} \Vert_{L^2(\Gamma)} + \Vert \nabla w \Vert_{L^2} \right),
\]

\[
\Vert \nabla^3 w \Vert_{L^2(\Gamma')} \leq CK\left( \Vert \rho \dot{u} \Vert_{L^2(\Gamma)} + \Vert \nabla \dot{u} \Vert_{L^2(\Gamma)} + \Vert \rho \dot{u} \Vert_{L^2} \right) + \Vert \nabla u \Vert_{L^2} + \Vert \rho - \bar{\rho} \Vert_{L^2}, \tag{2.32}
\]

In particular, when \( \partial \Omega \subset \Gamma \subset \subset \Gamma' \), above estimates remain true.

**Proof.** We apply local theory (see \[1\] Theorem 8.1-8.3, Theorem 10.1-10.2, \[15\] Theorem 4.8-4.12, Theorem 9.11-9.13) to \( \text{(1.15)} \) and \( \text{(1.16)} \) and deduce that:

\[
\Vert \nabla v \Vert_{C^0(\Gamma')} \leq CK\left( \Vert \rho \Vert_{C^0(\Gamma)} + \Vert v \Vert_{L^\infty(\Gamma)} \right),
\]

\[
\Vert \nabla^2 v \Vert_{L^4(\Gamma')} \leq CK\left( \Vert \nabla \rho \Vert_{L^4(\Gamma)} + \Vert v \Vert_{L^4(\Gamma)} \right),
\]

\[
\Vert \nabla^3 v \Vert_{L^2(\Gamma')} \leq CK\left( \Vert \nabla^2 \rho \Vert_{L^2(\Gamma)} + \Vert \nabla \rho \Vert_{L^2(\Gamma)} + \Vert v \Vert_{L^2(\Gamma)} \right), \tag{2.33}
\]

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and
\[
\|\nabla^2 w\|_{L^2(\Gamma')} \leq CK (\|\rho \hat{u}\|_{L^2(\Gamma')} + \|w\|_{L^2}),
\]
\[
\|\nabla^3 w\|_{L^2(\Gamma')} \leq CK (\|\nabla (\rho \hat{u})\|_{L^2(\Gamma')} + \|\rho \hat{u}\|_{L^2(\Gamma')} + \|w\|_{L^2}).
\] (2.34)

With the help of (2.25), global elliptic estimate (2.6) and Sobolev’s imbedding inequality, (2.33) and (2.34) can be refined as
\[
\|\nabla v\|_{C^0(\Gamma')} \leq CK (\|\rho\|_{C^0(\Gamma')} + \|v\|_{L^\infty(\Gamma')})
\]
\[
\leq CK (\|\rho\|_{C^0(\Gamma')} + \|\nabla v\|_{L^4(\Gamma')})
\]
\[
\leq CK (\|\rho\|_{C^0(\Gamma')} + \|\rho - \bar{\rho}\|_{L^4}),
\] (2.35)
\[
\|\nabla^2 v\|_{L^4(\Gamma')} \leq CK (\|\nabla \rho\|_{L^4(\Gamma')} + \|v\|_{L^4(\Gamma')})
\]
\[
\leq CK (\|\nabla \rho\|_{L^4(\Gamma')} + \|\nabla v\|_{L^4(\Gamma')})
\]
\[
\leq CK (\|\nabla \rho\|_{L^4(\Gamma')} + \|\rho - \bar{\rho}\|_{L^4}),
\] (2.36)
\[
\|\nabla^3 v\|_{L^2(\Gamma')} \leq CK (\|\nabla^2 \rho\|_{L^2(\Gamma')} + \|\nabla \rho\|_{L^2(\Gamma')} + \|v\|_{L^2(\Gamma')})
\]
\[
\leq CK (\|\nabla^2 \rho\|_{L^2(\Gamma')} + \|\nabla \rho\|_{L^2(\Gamma')} + \|v\|_{L^2})
\]
\[
\leq CK (\|\nabla^2 \rho\|_{L^2(\Gamma')} + \|\nabla \rho\|_{L^2(\Gamma')} + \|\rho - \bar{\rho}\|_{L^2}).
\] (2.37)

and
\[
\|\nabla^2 w\|_{L^2(\Gamma')} \leq CK (\|\rho \hat{u}\|_{L^2(\Gamma')} + \|\nabla w\|_{L^2}),
\]
\[
\|\nabla^3 w\|_{L^2(\Gamma')} \leq CK (\|\nabla (\rho \hat{u})\|_{L^2(\Gamma')} + \|\rho \hat{u}\|_{L^2(\Gamma')} + \|w\|_{L^2})
\]
\[
\leq CK (\|\nabla \rho \hat{u}\|_{L^2(\Gamma')} + \|\rho \hat{u}\|_{L^2(\Gamma')} + \|\hat{u}\|_{L^4(\Gamma')} + \|\rho \hat{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}).
\] (2.38)

Note that for harmonic function, local theory ([13] Theorem 2.1, Theorem 2.10) guarantees:

**Lemma 2.15.** Suppose h is harmonic in \( \Gamma \), there is some constant \( C \) depending only on \( \Omega \) such that \[
\|h\|_{W^{k,p}(\Gamma')} \leq CK\|h\|_{L^2(\Gamma')}.
\] (2.39)

Moreover, Theorem 2.1 assures \( \text{dist}(A,B) > d \). Combining interior estimates Lemma 2.14 with trace theorem ([12] Theorem 1, Sec 5.5), we declare the following boundary estimates:

**Lemma 2.16.** There is some constant \( C \) depending only on \( \Omega \) such that
\[
\int_{\partial \Omega} \left( |\nabla v|^2 + |\nabla^2 v|^2 + |\nabla \rho|^2 \right) dS \leq CK \left( \|\nabla^2 \rho\|_{L^2(B)} \|\nabla^2 \rho\|_{L^2(B)} + \|\nabla \rho\|_{L^2(B)}^2 + \|\rho - \bar{\rho}\|_{L^2}^2 \right).
\] (2.40)

**Proof.** We apply Trace theorem ([12] Theorem 1, Sec 5.5) and Lemma 2.14 to show that
\[
\int_{\partial \Omega} |\nabla^2 v|^2 dS \leq CK \int_{\partial A} |\nabla \rho|^2 \|\nabla^2 v\|_{L^2}^2 d\rho + CK \int_{\partial A} |\nabla^2 v|^2 d\rho
\]
\[
\leq CK \|\nabla^3 v\|_{L^2(A)} \|\nabla^2 v\|_{L^2(A)} + CK \|\nabla^2 v\|_{L^2(A)}^2
\]
\[
\leq CK \left( \|\nabla^2 \rho\|_{L^2(B)} \|\nabla v\|_{L^2(B)} + \|\rho - \bar{\rho}\|_{L^2}^2 \right).
\] (2.41)
and
\[\int_{\partial \Omega} |\nabla \rho|^2 ds \leq CK \int_A |\nabla^2 \rho| \cdot |\nabla \rho| dx + CK \int_A |\nabla \rho|^2 dx \]
\[\leq CK \left( \|\nabla^2 \rho\|_{L^2(A)} \|\nabla \rho\|_{L^2(A)} + \|\nabla \rho\|^2_{L^2(A)} \right) \tag{2.42}\]
\[\leq CK \left( \|\nabla^2 \rho\|_{L^2(B)} \|\nabla \rho\|_{L^2(B)} + \|\nabla \rho\|^2_{L^2(B)} \right) ;
\]
and
\[\int_{\partial \Omega} |\nabla v|^2 ds \leq CK \int_A |\nabla^2 v|^2 dx + CK \int_A |\nabla v|^2 dx \]
\[\leq CK \left( \|\nabla^2 v\|^2_{L^2(A)} + \|\nabla v\|^2_{L^2(A)} \right) \tag{2.43}\]
\[\leq CK \left( \|\nabla \rho\|_{L^2(B)} + \|\rho - \bar{\rho}\|^2_{L^2} \right) .
\]
Combining (2.41), (2.42) and (2.43) gives (2.40). \qed

Remark 2.9. The constant $CK$ in above interior estimates (2.35)–(2.43) is time-independent. Such fact is crucial in our arguments, since the domains under consideration are varying throughout the time.

Next, let us recall the definition of $F_s$ and $v_r$ in (2.8) and (2.9). Consider the elliptic system (2.7). $F_s$ is defined via
\[F_s = \text{div} v_s - \rho - \bar{\rho} \mu + \lambda .
\]
Set $v_r = v - v_s$, then $v_r$ solves (2.9).

We obtain some regularity assertions about $F_s$ and $v_r$ based on local estimates.

Lemma 2.17. For $\alpha \in (0, 1)$, there is a constant $C$ depending only on $\alpha$, $\Omega$, $\mu$ and $\lambda$ such that
\[\|F_s\|_{C^\alpha(\Omega)} \leq CK \left( \|\rho\|_{C^\alpha(A)} + \|\rho - \bar{\rho}\|_{L^2} \right) , \tag{2.44}\]
and
\[\|\nabla v_r\|_{C^\alpha(\Omega)} \leq CK \left( \|\rho\|_{C^\alpha(A)} + \|\rho - \bar{\rho}\|_{L^2} \right) . \tag{2.45}\]

Proof. We take $\nabla \cdot$ on (2.7) to show that
\[\Delta F_s = 0 .
\]
Considering the domain $D = \{ x | \text{dist}(x, \partial \Omega) > \frac{d}{1000} \}$ and $D^c = \{ x | \text{dist}(x, \partial \Omega) \leq \frac{d}{1000} \}$, one has
\[\Omega = D \cup D^c .
\]
For $y \in D$, applying the interior estimate (2.39) and standard elliptic estimation ( [1] Theorem 10.1-10.2, [15] Theorem 9.11-9.13) to (2.64) shows that
\[\|F_s\|_{C^\alpha(D)} \leq CK \|F_s\|_{L^2} \leq CK \left( \|\nabla v_s\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} \right) \leq CK \|\rho - \bar{\rho}\|_{L^2} . \tag{2.46}\]
For $y \in D^c$, we need more efforts.

Select a smooth truncation function $\eta$ satisfying
\[\eta \geq 0, \ \eta \in C^\infty \]
\[\eta(x) = 1 \ \text{for dist}(x, \partial \Omega) \leq \frac{d}{1000} . \tag{2.47}\]
\[\eta(x) = 0 \ \text{for dist}(x, \partial \Omega) \geq \frac{d}{500} .
\]
Recalling that

$$A = \left\{ y \in \Omega \big| 0 \leq \text{dist}(y, \partial \Omega) \leq \frac{1}{32} d_0 \right\},$$

Theorem 2.1 and the definition of \(d\) in (2.27) guarantee that the support of \(\eta\) lies entirely in \(A\). We have

\[
F_s(x) = \int \partial_x \partial_y G^*(x, y) \frac{\rho - \bar{\rho}}{\mu + \lambda} dy \\
= \frac{1}{\mu + \lambda} \int \partial_x \partial_y G^*(x, y) \eta(y) (\rho(y) - \bar{\rho}) dy \\
+ \frac{1}{\mu + \lambda} \int \partial_z \partial_y G^*(x, y) (1 - \eta(y)) (\rho(y) - \bar{\rho}) dy \\
= F_1(x) + F_2(x).
\]

Observe that

\[
\|\eta(\rho - \bar{\rho})\|_{C^\alpha(\Omega)} \leq C\|\rho - \bar{\rho}\|_{C^\alpha(A)}. \tag{2.48}
\]

By standard \(C^\alpha\) estimate of elliptic equation (\cite[Theorem 8.1-8.3]{15}, \cite[Theorem 4.8-4.12]{15}) and (2.48), we have

\[
\|\nabla v_s^1\|_{C^\alpha} \leq C\|\eta(\rho - \bar{\rho})\|_{C^\alpha} \leq C\|\rho - \bar{\rho}\|_{C^\alpha(A)}.
\]

Note that \(F_1\) is actually derived from the elliptic system:

\[
\begin{cases}
\Delta v_s^1 = \nabla \eta(\rho - \bar{\rho}) & \text{in } \Omega, \\
v_s^1 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Then \(F_1 = \text{div} v_s^1 - \eta(\rho - \bar{\rho})\). As a result,

\[
\|F_1\|_{C^\alpha(\Omega)} \leq C \left( \|\nabla v_s^1\|_{C^\alpha} + \|\eta(\rho - \bar{\rho})\|_{C^\alpha} \right) \leq C\|\rho - \bar{\rho}\|_{C^\alpha(A)}. \tag{2.49}
\]

For \(F_2\), observe that \(x \in D^c\) in present situation. This means the representation

\[
F_2(x) = \frac{1}{\mu + \lambda} \int \partial_x \partial_y G^*(x, y) (1 - \eta(y)) (\rho(y) - \bar{\rho}) dy \tag{2.50}
\]

is out of singularity.

Therefore, we can take derivatives directly in (2.50) and apply \cite[Lemma 2.7]{24} to get

\[
|\partial_x F_2(x)| = \frac{1}{\mu + \lambda} \int \partial_x \partial_x \partial_y G^*(x, y) (1 - \eta(y)) (\rho(y) - \bar{\rho}) dy \\
\leq C \int_{|x-y| > \frac{d_0}{2\mu}} \frac{|\rho(y) - \bar{\rho}|}{|x-y|^4} dy \\
\leq CK\|\rho - \bar{\rho}\|_{L^2},
\]

which also means

\[
\|F_2(x)\|_{C^\alpha(D^c)} \leq CK\|\rho - \bar{\rho}\|_{L^2}. \tag{2.51}
\]

Combining (2.46), (2.49), (2.51), we declare that

\[
\|F_s\|_{C^\alpha(\Omega)} \leq CK \left( \|\rho\|_{C^\alpha(A)} + \|\rho - \bar{\rho}\|_{L^2} \right),
\]

which is (2.44).

Now, the regularity assertion for \(v_r\) follows easily. Since by (2.5), \(v_r\) solves

\[
\begin{cases}
\mu \Delta v_r + \lambda \text{div} v_r = -\lambda \nabla F_s & \text{in } \Omega, \\
v_r = 0 & \text{on } \partial \Omega.
\end{cases}
\]

...
By virtue of the standard $C^\alpha$ estimate for elliptic system (Theorem 8.1-8.3, Theorem 4.8-4.12), we declare

$$
\| \nabla v_r \|_{C^\alpha(\Omega)} \leq C \| F_s \|_{C^\alpha(\Omega)} \leq CK \left( \| \rho \|_{C^\alpha(A)} + \| \rho - \bar{\rho} \|_{L^2} \right),
$$

which gives (2.45).

Finally, for further uses, we introduce the following modified-Hölder type semi-norm:

$$\| \rho \|_{\tilde{C}^\alpha(B \cup \Gamma \cup \tilde{\Gamma})} \triangleq \sup_{x \in B \cup \Gamma \cup \tilde{\Gamma}, \ y \in B \cup \Gamma} \frac{|\rho(x) - \rho(y)|}{|x - y|^\alpha}. \tag{2.52}$$

**Remark 2.10.** The semi-norm $\| \rho \|_{\tilde{C}^\alpha(B \cup \Gamma \cup \tilde{\Gamma})}$ actually means $\rho(x)$ is $C^\alpha$ in the interior of $B \cup \Gamma$ and exactly on the free boundary $\Gamma_2$ in classical sense. However, such norm does not really control the continuity within the domain $\tilde{\Gamma}$, actually it almost reduces to $L^\infty(\tilde{\Gamma})$ in such domain.

Therefore, the semi-norm is equivalent with three parts:

$$\| \rho \|_{\tilde{C}^\alpha(B \cup \Gamma \cup \tilde{\Gamma})} \leq C \left( \| \rho \|_{C^\alpha(\Gamma)} + \| \rho \|_{C^\alpha(\Gamma_2)} + \| \rho \|_{C^\alpha(B)} \right) \leq C \| \rho \|_{\tilde{C}^\alpha(B \cup \Gamma \cup \tilde{\Gamma})},$$

where

$$\| \rho \|_{C^\alpha(\Gamma_2)} \triangleq \sup_{x \in B \cup \Gamma \cup \tilde{\Gamma}, \ y \in \Gamma_2} \frac{|\rho(x) - \rho(y)|}{|x - y|^\alpha}.$$

With the help of Lemma 2.17, the last lemma controls the quotient difference of velocity $v$ which lies in the central position of $C^\alpha$ estimate.

**Lemma 2.18.** Under the condition of (2.28), suppose there exists a constant $C$ depending only on $\Omega$ such that for $y \in \Omega$ and $0 < \epsilon < 1$

$$C^{-1}(ed)^3 \leq |B_{ed}(y) \cap \Omega| \leq C(ed)^3. \tag{2.53}$$

Then for $x \in B \cup \Gamma \cup \tilde{\Gamma}$, $y \in B \cup \Gamma$ and any $\epsilon > 0$, there is some constant $C$ depending only on $\Omega$, $\mu$, $\lambda$, and $M_0$ such that $v$ (as in (2.4)) satisfies

$$\frac{|v(x) - v(y)|}{|x - y|} \leq CK\epsilon^{\frac{\alpha}{2}} \| \rho \|_{\tilde{C}^\alpha(B \cup \Gamma \cup \tilde{\Gamma})} + CK\epsilon^{-2} \left( \| \rho - \bar{\rho} \|_{L^2} + \| \rho - \bar{\rho} \|_{L^2}^{\frac{1}{2}} \right),$$

where $M_0$ is given by (2.28).

**Proof.** According to (2.10), one has

$$v(x) = \int_\Omega \partial_y G(x, y) \left( \frac{\rho - \bar{\rho}}{\mu + \lambda} - \frac{\lambda}{\mu} F_s \right) dy = \int_\Omega \partial_y G(x, y) \left( \frac{\rho - \bar{\rho}}{\mu + \lambda} \right) dy - \int_\Omega \partial_y G(x, y) \left( \frac{\lambda}{\mu} F_s \right) dy = v_1(x) + v_2(x).$$

We check $v_1(x)$ in details first. For some $0 < \epsilon < \frac{1}{1000}$ determined later, on the one hand,

$$\frac{|v_1(x) - v_1(y)|}{|x - y|} \leq \frac{1}{|x - y|} \left| \int_\Omega (\partial_2 G(x, z) - \partial_2 G(y, z)) \frac{\rho - \bar{\rho}}{\mu + \lambda}(z) dz \right| \leq \frac{C}{|x - y|} \| \partial_2 G \|_{L^\infty} \| \rho - \bar{\rho} \|_{L^4} \leq C M_0^\frac{1}{2} \| \rho - \bar{\rho} \|_{L^4} \leq CK\epsilon^{-1} \| \rho - \bar{\rho} \|_{L^2}^{\frac{1}{2}}. \tag{2.54}$$
On the other hand, for $|x - y| < \epsilon d$, we check that

$$\frac{|v_1(x) - v_1(y)|}{|x - y|} = \frac{1}{|x - y|} \left| \int (\partial_z G(x, z) - \partial_z G(y, z))(\rho - \bar{\rho})dz \right|$$

$$\leq \frac{C}{|x - y|} \left| \int_{B_{100\epsilon d}(y) \cap \Omega} (\partial_z G(x, z) - \partial_z G(y, z))(\rho - \bar{\rho})dz \right|$$

$$+ \frac{C}{|x - y|} \left| \int_{B_{100\epsilon d}(y) \cap \Omega} (\partial_z G(x, z) - \partial_z G(y, z))(\rho - \bar{\rho})dz \right|$$

$$= I_1 + I_2.$$

$I_2$ is easier:

$$I_2 = \frac{C}{|x - y|} \left| \int_{B_{100\epsilon d}(y) \cap \Omega} (\partial_z G(x, z) - \partial_z G(y, z))(\rho - \bar{\rho})dz \right|$$

$$\leq \frac{C}{|x - y|} \left| \int_{\{|z| > 10\epsilon d\}} \frac{|x - y|}{|z|^3} \cdot |\rho - \bar{\rho}|(z)dz \right|$$

$$\leq C K \epsilon ^{-\frac{3}{2}} ||\rho - \bar{\rho}||_{L^2},$$

due to Lemma 2.7.

As for $I_1$, we compute that

$$I_1 = \frac{C}{|x - y|} \left| \int_{B_{100\epsilon d}(y) \cap \Omega} (\partial_z G(x, z) - \partial_z G(y, z))(\rho(z) - \bar{\rho})dz \right|$$

$$\leq \frac{C}{|x - y|} \left| \int_{B_{100\epsilon d}(y) \cap \Omega} (\partial_z G(x, z) - \partial_z G(y, z))(\rho(z) - \rho(y))dz \right|$$

$$+ \frac{\rho(y) - \bar{\rho}}{|x - y|} \left| \int_{B_{100\epsilon d}(y) \cap \Omega} \partial_y G(x, y) - \partial_y G(x, y)dy \right|$$

$$= J_1 + J_2.$$

$J_2$ is directly bounded by

$$J_2 \leq |\rho(y) - \bar{\rho}| \cdot \frac{C}{|x - y|} \left| \int_{B_{100\epsilon d}(y) \cap \Omega} \partial_z G(x, z) - \partial_z G(y, z)dz \right| .$$

We declare

$$\left| \frac{1}{|x - y|} \int_{B_{100\epsilon d}(y) \cap \Omega} \partial_z G(x, z) - \partial_z G(y, z)dz \right| \leq C K (1 + \log \frac{1}{\epsilon}).$$

(2.56)

In fact, since $G(x, z) = 0$ for $x \in \Omega, z \in \partial \Omega$ (see [15 Section 2.4]), we have

$$\int_{\Omega} \partial_z G(x, z)dz = 0.$$
Combining with Lemma 2.7, we have

\[ \frac{1}{|x - y|} \left| \int_{B_{100d}(y) \cap \Omega} \partial_z G(x, z) - \partial_z G(y, z) \, dz \right| \]

\[ = \frac{1}{|x - y|} \left| \int_{B_{100d}(y) \cap \Omega} \partial_z G(x, z) - \partial_z G(y, z) \, dz \right| \]

\[ \leq \frac{1}{|x - y|} \int_{B_{100d}(y) \cap \Omega} |\partial_z G(x, z) - \partial_z G(y, z)| \, dz \]

\[ \leq C \frac{1}{|x - y|} \int_{|100d \leq |y - z| \leq \text{diam}(\Omega)|} \frac{|x - y|}{|z - y|^3} \, dz \]

\[ \leq CK(1 + \log \frac{1}{\epsilon}), \]

where \( \text{diam}(\Omega) \triangleq \sup_{x,y \in \Omega} |x - y| \) is the diameter of domain \( \Omega \).

Moreover, we apply (2.53) and (2.52) to argue that

\[ |\rho(y) - \bar{\rho}| \leq \frac{1}{|B_6(y) \cap \Omega|} \int_{B_6(y) \cap \Omega} |\rho(y) - \rho(x)| \, dx \]

\[ + \frac{1}{|B_6(y) \cap \Omega|} \int_{B_6(y) \cap \Omega} |\rho(x) - \bar{\rho}| \, dx \]

\[ \leq \frac{1}{|B_6(y) \cap \Omega|} \int_{B_6(y) \cap \Omega} \|\rho\|_{\hat{C}^\alpha(B_6(y) \cap \Omega)} |y - x|^\alpha \, dx \]

\[ + \left( \frac{1}{|B_6(y) \cap \Omega|} \int_{B_6(y) \cap \Omega} |\rho(x) - \bar{\rho}|^2 \, dx \right)^{1/2} \]

\[ \leq CKe^\alpha \|\rho\|_{\hat{C}^\alpha(B_6(y) \cap \Omega)} + CKe^{-\bar{\rho}} \|\rho - \bar{\rho}\|_{L^2}. \]

Combining (2.57) and (2.58) gives

\[ J_2 \leq CKe^\alpha \|\rho\|_{\hat{C}^\alpha(B_6(y) \cap \Omega)} + CKe^{-\bar{\rho}} \|\rho - \bar{\rho}\|_{L^2}. \] (2.59)

The remaining term \( J_1 \) is handled via

\[ J_1 = \frac{C}{|x - y|} \int_{B_{100d}(y)} \left( \partial_z G(x, z) - \partial_z G(y, z) \right) (\rho - \rho(y)) \, dz \]

\[ \leq \frac{C}{|x - y|} \int_{B_5|x-y|(y)} \left( \partial_z G(x, z) - \partial_z G(y, z) \right) (\rho - \rho(y)) \, dz \]

\[ + \frac{C}{|x - y|} \int_{B_{100d}(y) \setminus B_5|x-y|(y)} \left( \partial_z G(x, z) - \partial_z G(y, z) \right) (\rho - \rho(y)) \, dz \]

\[ \leq C \|\rho\|_{\hat{C}^\alpha(B_6(y) \cap \Omega)} \int_{|x - y| \leq 5|x - y|} \frac{|x - y|^\alpha - 1}{|z|^2} \, dz \]

\[ + C \|\rho\|_{\hat{C}^\alpha(B_6(y) \cap \Omega)} \int_{|x - y| < |x - z| < 200d} \frac{1}{|z|^2} \cdot |z|^\alpha \, dz \]

\[ \leq CKe^\alpha \|\rho\|_{\hat{C}^\alpha(B_6(y) \cap \Omega)}. \]
where the last two inequalities are due to (2.28), (2.52) and Lemma 2.7.

Combining (2.54), (2.55), (2.59) and (2.60) yields that
\[
\frac{|v_1(x) - v_1(y)|}{|x - y|} \leq C K \epsilon^2 \|\rho\|_{C^0(B \cup \Gamma \cup \tilde{\Gamma})} + C K \epsilon^{-2}(\|\rho - \bar{\rho}\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}^{\frac{1}{2}}). \tag{2.61}
\]

Next, let us turn to \(v_2(x)\) which is easier. In fact \(v_2(x)\) is given by the elliptic system
\[
\begin{cases}
\mu \Delta v_2 = -\nabla (\lambda F_s + \lambda \div v_r) & \text{in } \Omega, \\
v_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We replace \(a\) by \(\frac{\alpha}{2}\) in Theorem 2.17 then apply standard \(C^2\) estimate of elliptic system (see [12] Theorem 8.1-8.3 [23] Theorems 4.15 & 4.16) to deduce
\[
\|\nabla v_2(x)\|_{L^\infty(\Omega)} \leq C \left(\|F_s\|_{C^2(B)} + \|v_r\|_{C^2(B)}\right) \leq C K \left(\|\rho\|_{C^2(B)} + \|\rho - \bar{\rho}\|_{L^2}\right). \tag{2.62}
\]

For \(\|\rho\|_{C^2(A)}\), on the one hand, if \(|x - y| > \epsilon d\), we used (2.58) to get
\[
\frac{|\rho(x) - \rho(y)|}{|x - y|^{\frac{3}{2}}} \leq \frac{2K}{\epsilon^2} \|\rho - \bar{\rho}\|_{L^\infty(B)} \tag{2.63}
\]
\[
\leq C K \epsilon^2 \|\rho\|_{C^0(B \cup \Gamma \cup \tilde{\Gamma})} + C K \epsilon^{-2}\|\rho - \bar{\rho}\|_{L^2}.
\]

On the other hand, if \(|x - y| \leq \epsilon d\),
\[
\frac{|\rho(x) - \rho(y)|}{|x - y|^{\frac{3}{2}}} = \frac{|\rho(x) - \rho(y)|}{|x - y|^{\frac{3}{2}}}, \quad |x - y| \leq C K \epsilon^2 \|\rho\|_{C^0(B \cup \Gamma \cup \tilde{\Gamma})} + C K \epsilon^{-2}\|\rho - \bar{\rho}\|_{L^2}, \tag{2.64}
\]

It follows from (2.63) and (2.64) that
\[
\|\rho\|_{C^2(A)} \leq C K \epsilon^2 \|\rho\|_{C^0(B \cup \Gamma \cup \tilde{\Gamma})} + C K \epsilon^{-2}\|\rho - \bar{\rho}\|_{L^2}, \tag{2.65}
\]

which along with (2.62) yields
\[
\|\nabla v_2(x)\|_{L^\infty(\Omega)} \leq C K \epsilon^2 \|\rho\|_{C^0(B \cup \Gamma \cup \tilde{\Gamma})} + C K \epsilon^{-2}\|\rho - \bar{\rho}\|_{L^2}. \tag{2.66}
\]

Finally, combining (2.61) and (2.66) shows
\[
\frac{|v(x) - v(y)|}{|x - y|} \leq C K \epsilon^2 \|\rho\|_{C^0(B \cup \Gamma \cup \tilde{\Gamma})} + C K \epsilon^{-2}(\|\rho - \bar{\rho}\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}^{\frac{1}{2}}),
\]
which finishes the proof of Lemma 2.18. \(\square\)

**Remark 2.11.** The condition (2.53) is a geometric restriction on the regularity of domain \(\Omega\). The constant \(C\) in (2.53) can be found if we choose \(d_0\) in (1.7) smaller than injective radius of boundary \(\partial \Omega\) (see ([II]) for example.

**Remark 2.12.** Lemma 2.18 is of vital importance in our proof. We will replace \(|\nabla v(x)|\) by \(\frac{|v(x) - v(y)|}{|x - y|}\) when we estimate the distance between two flow lines lying in \(B \cup \Gamma \cup \tilde{\Gamma}\) as in (4.1). Such process makes it possible to apply the modified \(C^a\) semi-norm \(\|\rho\|_{C^a(B \cup \Gamma \cup \tilde{\Gamma})}\) rather than the classical \(C^a\) norm \(\|\rho\|_{C^a(B \cup \Gamma \cup \tilde{\Gamma})}\). We illustrate that it is the modified one that works for the present situation:

When we try to estimate \(\frac{|v(x) - v(y)|}{|x - y|}\) where \(x \in B \cup \Gamma \cup \tilde{\Gamma}\) and \(y \in B \cup \Gamma \cup \tilde{\Gamma}\) as in (2.54), \(\|\rho\|_{C^a(B \cup \Gamma \cup \tilde{\Gamma})}\) is sufficient to control such quotient difference due to the strictly positive distance from \(y \in B \cup \Gamma \cup \tilde{\Gamma}\) to the boundary \(\Gamma_2\) guaranteed by Theorem 2.17. Moreover, \(\|\rho\|_{C^a(B \cup \Gamma \cup \tilde{\Gamma})}\) only involves the information within \(B \cup \Gamma \cup \tilde{\Gamma}\), as we can fix \(y \in B \cup \Gamma\). It is the essential reason why we introduce such modified semi-norm.

If we try to control \(\|\nabla v(x)\|_{L^\infty(B \cup \Gamma \cup \tilde{\Gamma})}\), we must apply the usual \(C^a\) norm \(\|\rho\|_{C^a(B \cup \Gamma \cup \tilde{\Gamma})}\), as \(x\) may locate near the boundary \(\Gamma_2\). Actually such norm requires information beyond the domain \(B \cup \Gamma \cup \tilde{\Gamma}\), since \(\|\rho\|_{C^a(B \cup \Gamma \cup \tilde{\Gamma})}\) includes the case \(y \in \Gamma_2\), but \(x \in \Omega \setminus (B \cup \Gamma \cup \tilde{\Gamma})\).
3 A priori estimates: lower-order ones

In this section and the next two, we always assume that \((\rho, u)\) is the strong solution to (1.1)-(1.3) on \(\Omega \times [0, T]\) whose existence is guaranteed by Lemma 2.1. Moreover, throughout this section, we suppose \(\rho\) satisfying
\[
\|\rho\|_{L^\infty} \leq 2M_0. \tag{3.1}
\]

We set up some crucial global exponential decay assertions. It lies in the central position in the whole proof, since it provides sufficiently strong integrability to close all estimates (see Theorem 3.3 for example).

**Theorem 3.1.** Under the condition (3.1), there are some positive constants \(C\) and \(\kappa\) depending only on \(\Omega, \mu, \lambda, \bar{\rho}\) and \(M_0\) such that
\[
\int_{\Omega} \left( \rho |u|^2 + |\rho - \bar{\rho}|^2 + |\rho - \bar{\rho}|^4 \right) \, dx \leq CC_0 e^{-\kappa t}, \tag{3.2}
\]
and
\[
\int_0^T e^{\frac{\kappa t}{2}} \|\nabla u\|^2_{L^2} \, dt \leq CC_0, \tag{3.3}
\]
where \(C_0\) is given by (1.5).

**Proof.** First, multiplying (1.1) by \(u\) and integrating the resulting equality over \(\Omega\) give
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2}\rho |u|^2 + G(\rho) \right) \, dx + \mu \int |\nabla u|^2 \, dx + \lambda \int (\text{div} u)^2 \, dx = 0, \tag{3.4}
\]
where
\[
G(\rho) \triangleq \rho \int_{\bar{\rho}}^\rho \frac{s - \bar{\rho}}{s^2} \, ds.
\]
Integrating (3.4) with respect to \(t\) leads to
\[
\sup_{0 \leq t \leq T} \int_{\Omega} \left( \frac{1}{2}\rho |u|^2 + G(\rho) \right) \, dx + \int_0^T \int |\nabla u|^2 \, dx dt \leq CC_0, \tag{3.5}
\]
which combined with (1.6) shows
\[
\sup_{0 \leq t \leq T} \int_{\Omega} \left( \frac{1}{2}\rho |u|^2 + |\rho - \bar{\rho}|^2 \right) \, dx + \int_0^T \int |\nabla u|^2 \, dx dt \leq CC_0. \tag{3.6}
\]

Next, multiplying (1.1) by \(-B(\rho - \bar{\rho})\) with \(B\) as in Lemma 2.5, integrating by parts together with (3.1), (3.6) and Sobolev imbedding yields
\[
\begin{align*}
\int_{\Omega} (\rho - \bar{\rho})^2 \, dx - \frac{d}{dt} \int_{\Omega} \rho u \cdot B(\rho - \bar{\rho}) \, dx \\
= \int_{\Omega} \rho u \cdot B(\text{div}(\rho u)) \, dx - \int_{\Omega} \rho u_i u_j \cdot \nabla B(\rho - \bar{\rho}) \, dx + \int_{\Omega} \mu \nabla u \cdot \nabla B(\rho - \bar{\rho}) \, dx \\
+ \int_{\Omega} \lambda \text{div}(\rho - \bar{\rho}) \, dx \\
\leq C \int |\rho u|^2 \, dx + C \left( \int |u|^4 \, dx \right)^{\frac{1}{2}} \|\rho - \bar{\rho}\|_{L^2} + C \int |\nabla u|^2 \, dx + \frac{1}{2} \|\rho - \bar{\rho}\|_{L^2}^2 \\
\leq C \int |\nabla u|^2 \, dx + \frac{1}{2} \|\rho - \bar{\rho}\|_{L^2}^2.
\end{align*}
\]
With the help of (3.1), (1.6) and Sobolev imbedding theorem, we observe that there is some positive constant \( B \) depending on \( M_0, \Omega, \bar{\rho} \) such that
\[
B^{-1} \int |\rho u \cdot B(\rho - \bar{\rho})| dx \leq \int \frac{1}{2} |\rho| u|^2 dx + G(\rho) dx \leq B \left( ||\nabla u||^2_{L^2} + ||\rho - \bar{\rho}||^2_{L^2} \right). \tag{3.8}
\]
Moreover, adding (3.7) multiplied by \( \frac{1}{2B} \) to (3.4) gives
\[
dt \left( \int \frac{1}{2} |\rho u|^2 + G(\rho) - \frac{\rho u \cdot B(\rho - \bar{\rho})}{2B} \right) dx \leq \frac{\kappa B}{2} \int \left( ||\nabla u||^2 + ||\rho - \bar{\rho}||^2 \right) dx \leq 0, \tag{3.9}
\]
for some \( \kappa \) depending only on \( \Omega, \lambda, \mu \) and \( M_0 \).

By virtue of (3.8) and (1.5), applying Gronwall’s inequality in (3.9) leads to
\[
\int \left( \frac{1}{2} |\rho u|^2 + G(\rho) - \frac{\rho u \cdot B(\rho - \bar{\rho})}{2B} \right) dx \leq CC_0 e^{-\kappa t},
\]
which along with (3.5), (3.8) and (1.6) gives
\[
\int \Omega (|\rho|^2 + |\rho - \bar{\rho}|^2) dx \leq CC_0 e^{-\kappa t}. \tag{3.10}
\]
Combining this with (3.1) yields
\[
\int \Omega |\rho - \bar{\rho}|^4 dx \leq CC_0 e^{-\kappa t},
\]
which together with (3.10) proves (3.2).

Finally, multiplying (3.4) by \( e^{\frac{\kappa t}{2}} \) and integrating with respect to \( t \), we take advantage of (3.1) and (3.2) to give
\[
\int_0^T e^{\frac{\kappa t}{2}} ||\nabla u||^2_{L^2} dt \leq CC_0 + \int_0^T e^{\frac{\kappa t}{2}} \int \Omega (|\rho|^2 + |\rho - \bar{\rho}|^2) dx dt \leq CC_0,
\]
which is (3.3).

Next, after modifying some ideas due to Hoff [16–19], we have the following estimates on the material derivatives \( \dot{u} \).

**Theorem 3.2.** Suppose (3.1) holds. Then there are some positive constants \( \varepsilon_0, C \) and \( \tilde{\kappa} \) depending on \( \bar{\rho}, \Omega, \mu, \lambda, M_0 \) and \( N_0 \) such that
\[
\sup_{0 \leq t \leq T} e^{\frac{\tilde{\kappa} t}{2}} \int \Omega |\nabla u|^2 dx + \int_0^T \int \Omega e^{\frac{\tilde{\kappa} t}{2}} |\rho \dot{u}|^2 dx dt \leq C, \tag{3.11}
\]
and
\[
\sup_{0 \leq t \leq T} \int \Omega |\rho \dot{u}|^2 dx + \int_0^T \sigma(t) \int \Omega |
\dot{\nabla u}|^2 dx dt \leq C, \tag{3.12}
\]
provided \( C_0 \leq \varepsilon_0 \),

where \( C_0 \) is given by (1.5) and \( \sigma(t) \triangleq \min\{t, 1\} \).
Proof. First, rewriting (1.1) as
\[
\rho \dot{u} - (\lambda + \mu) \nabla \text{div} u + \mu \nabla \times \text{rot} u + \nabla \rho = 0,
\] (3.13)
multiplying (3.13) by \(\dot{u}\) and integrating the resulting equality over \(\Omega\) give
\[
\int \rho |\dot{u}|^2 dx = - \int \dot{u} \cdot \nabla (\rho - \bar{\rho}) dx + (\lambda + 2\mu) \int \nabla \text{div} u \cdot \dot{u} dx
\]
\[
- \mu \int \nabla \times \text{rot} u \cdot \dot{u} dx \triangleq I_1 + I_2 + I_3.
\]
We check each \(I_i\) in detail.
According to (1.1)1, for \(I_1\) we have
\[
I_1 = - \int \dot{u} \cdot \nabla (\rho - \bar{\rho}) dx
\]
\[
= \int (\rho - \bar{\rho}) \text{div} u dx - \int (u \cdot \nabla) u \cdot \nabla (\rho - \bar{\rho}) dx
\]
\[
= \frac{d}{dt} \left( \int (\rho - \bar{\rho}) \text{div} u dx \right) + \int \rho \partial_i u^j \partial_j u^i dx
\]
\[
\leq \frac{d}{dt} \left( \int (\rho - \bar{\rho}) \text{div} u dx \right) + C \|\nabla u\|_{L^2}^2.
\] (3.14)
For \(I_2\), direct computations show
\[
I_2 = (\lambda + \mu) \int \nabla \text{div} u \cdot \dot{u} dx
\]
\[
= -(\lambda + \mu) \int \text{div} \text{div} \dot{u} dx
\]
\[
= - \frac{d}{dt} \left( \frac{\lambda + \mu}{2} \int (\nabla \text{div} u)^2 dx \right) - (\lambda + \mu) \int \nabla \text{div} (u \cdot \nabla u) dx
\]
\[
= - \frac{d}{dt} \left( \frac{\lambda + \mu}{2} \int (\nabla \text{div} u)^2 dx \right) + \frac{\lambda + \mu}{2} \int (\text{div} u)^3 dx
\]
\[
- (\lambda + \mu) \int \text{div} u \cdot \partial_i u^j \partial_j u^i dx
\]
\[
\leq - \frac{d}{dt} \left( \frac{\lambda + \mu}{2} \int (\nabla \text{div} u)^2 dx \right) + C \int |\nabla u|^3 dx.
\] (3.15)
\(I_3\) is similar with \(I_2\):
\[
I_3 = - \mu \int \nabla \times \text{rot} u \cdot \dot{u} dx = - \mu \int \text{rot} u \cdot \text{rot} \dot{u} dx
\]
\[
= - \frac{\mu}{2} \frac{d}{dt} \int |\text{rot} u|^2 dx - \mu \int \text{rot} u \cdot (u \cdot \nabla u) dx
\]
\[
= - \frac{\mu}{2} \frac{d}{dt} \int |\text{rot} u|^2 dx - \mu \int (\nabla u^i \times \nabla_i u) \cdot \text{rot} u dx + \frac{\mu}{2} \int |\text{rot} u|^2 dx
\]
\[
\leq - \frac{\mu}{2} \frac{d}{dt} \int |\text{rot} u|^2 dx + C \|\nabla u\|_{L^3}^3.
\] (3.16)
Collecting (3.14), (3.15) and (3.16), we arrive at
\[
\frac{d}{dt} \left( \int (\lambda + \mu) \int (\nabla \text{div} u)^2 dx + \mu \int |\text{rot} u|^2 dx \right) + \int \rho |\dot{u}|^2 dx
\]
\[
\leq \frac{d}{dt} \left( \int (\rho - \bar{\rho}) \text{div} u dx \right) + C \|\nabla u\|_{L^2}^2 + C \int |\nabla u|^3 dx.
\] (3.17)
Next, rewriting (1.1) as
\[ \rho \dot{u} = \nabla F - \mu \nabla \times \text{rot} u, \] (3.18)
where
\[ F = (\mu + \lambda) \text{div} u - (\rho - \bar{\rho}), \] (3.19)
operating \( \dot{u}^j \left( \frac{\partial}{\partial t} + \text{div}(u \cdot \cdot) \right) \) to (3.18), summing with respect to \( j \) and integrating over \( \Omega \), we get after using (1.1) that
\[
\frac{d}{dt} \left( \frac{1}{2} \int \rho |\dot{u}|^2 dx \right) = \int (\dot{u} \cdot \nabla \dot{F} + \dot{u}^j \text{div}(u \partial_j F)) dx
- \mu \int (\dot{u} \cdot \nabla \times \text{rot} u + \dot{u}^j \text{div}((\nabla \times \text{rot} u)_j)) dx
\]
(3.20)

For \( J_1 \), we compute that
\[
J_1 = \int \dot{u} \cdot \nabla F dx + \int \dot{u}^j \text{div}(u \partial_j F) dx
= -\int F_t \text{div} u dx - \int u \cdot \nabla \dot{u} \partial_j F dx
= -(\mu + \lambda) \int (\text{div}\dot{u})^2 dx + (\mu + \lambda) \int \text{div}\dot{u} \cdot \partial_i u^i \partial_j u^j dx
- \int \rho \text{div}\text{div}\dot{u} dx + \int \text{div}\dot{u} \cdot u \cdot \nabla F dx - \int \nabla \dot{u}^j \partial_j F dx
\]
(3.21)
\[
= -(\mu + \lambda) \int (\text{div}\dot{u})^2 dx + (\mu + \lambda) \int \text{div}\dot{u} \cdot \partial_i u^i \partial_j u^j dx
- \int \rho \text{div}\text{div}\dot{u} dx + \int \partial_j u \cdot \nabla \dot{u}^j F - \text{div}\text{div}\dot{u} dx
\]
\[
\leq -(\mu + \lambda) \int (\text{div}\dot{u})^2 dx + \frac{\mu + \lambda}{10} ||\nabla \dot{u}||^2_{L^2} + C \int (||u||^4 + ||u||^2) dx,
\]
where in the last inequality one has used (3.19).

\( J_2 \) is similar,
\[
J_2 = -\int |\text{rot}\dot{u}|^2 dx + \int \text{rot}\dot{u} \cdot (\nabla u^i \times \nabla i u) dx
+ \int u \cdot \nabla \text{rot} u \cdot \text{rot}\dot{u} dx + \int u \cdot \nabla \dot{u} \cdot (\nabla \times \text{rot} u) dx
\]
(3.22)
\[
= -\int |\text{rot}\dot{u}|^2 dx + \int \text{rot}\dot{u} \cdot (\nabla u^i \times \nabla i u) dx
- \int \text{div}(\text{rot} u \cdot \text{rot}\dot{u}) dx + \int (\nabla u^i \times \nabla i \dot{u}) \text{rot} u dx
\]
\[
\leq -\int |\text{rot}\dot{u}|^2 dx + \frac{\mu + \lambda}{10} ||\nabla \dot{u}||^2_{L^2} + C ||u||^4_{L^4}.
\]
Combining (3.20), (3.21) and (3.22) shows
\[
\frac{d}{dt} \int \rho|\dot{u}|^2 dx + \int (||\text{div}\dot{u}||^2 + ||\text{rot}\dot{u}||^2) dx \leq C \int (||u||^4 + ||u||^2) dx.
\] (3.23)

Multiplying (3.23) by \( \sigma(t) \), we deduce
\[
\frac{d}{dt} \left( \sigma(t) \int \rho|\dot{u}|^2 dx \right) + \sigma(t) \int (||\text{div}\dot{u}||^2 + ||\text{rot}\dot{u}||^2) dx
\]
(3.24)
\[
\leq C \sigma(t) \int (||u||^4 + ||u||^2) dx + \int \rho|\dot{u}|^2 dx.
\]
By decomposition (2.5), elliptic estimate (2.6) and the Gagliardo-Nirenberg inequality, we have

\[ \int |\nabla u|^4 dx \leq \int |\nabla w|^4 dx + \int |\nabla v|^4 dx \]
\[ \leq C \left( \|\nabla w\|_{L^2}^2 \|\nabla^2 u\|_{L^2} + \|\nabla w\|_{L^2}^2 + \|\rho - \tilde{\rho}\|_{L^4}^4 \right) \]
\[ \leq C(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2})^2 \|\rho \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \]
\[ + C\|\nabla v\|_{L^2}^2 + C\|\rho - \tilde{\rho}\|_{L^4}^4 \]
\[ \leq C(\|\nabla u\|_{L^2} + \|\rho - \tilde{\rho}\|_{L^2})^2 \|\rho \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|\rho - \tilde{\rho}\|_{L^4}^4. \]  

Similarly,

\[ \int |\nabla u|^3 dx \leq \int |\nabla w|^3 dx + \int |\nabla v|^3 dx \]
\[ \leq C \left( \|\nabla w\|_{L^2}^2 \|\nabla^2 u\|_{L^2} + \|\nabla w\|_{L^2}^2 + \|\rho - \tilde{\rho}\|_{L^3}^3 \right) \]
\[ \leq C(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2})^2 \|\rho \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3 \]
\[ + C\|\nabla v\|_{L^2}^3 + C\|\rho - \tilde{\rho}\|_{L^3}^3 \]
\[ \leq C(\|\nabla u\|_{L^2} + \|\rho - \tilde{\rho}\|_{L^2})^3 \|\rho \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3 + C\|\rho - \tilde{\rho}\|_{L^3}^3. \]

Substituting (3.26) into (3.17) yields

\[ \frac{d}{dt} \left( (\lambda + \mu)\text{div}\ u \right)^2 dx + \mu|\text{rot}\ u|^2 - (\rho - \tilde{\rho})\text{div}\ u dx \]
\[ \leq C\|\nabla u\|_{L^2}^2 + C \left( (\|\nabla u\|_{L^2} + \|\rho - \tilde{\rho}\|_{L^2})^2 \|\rho \dot{u}\|_{L^2}^2 + \|\rho - \tilde{\rho}\|_{L^3}^3 \right) \]
\[ \leq \frac{1}{2} \int \rho|\dot{u}|^2 dx + C\|\nabla u\|_{L^2}^6 + C\|\nabla u\|_{L^2}^3 + C\|\nabla u\|_{L^2}^2 \]
\[ + C\|\rho - \tilde{\rho}\|_{L^2}^6 + C\|\rho - \tilde{\rho}\|_{L^3}^3. \]  

Setting \( \tilde{\kappa} = \frac{\kappa}{\mu} \) where \( \kappa \) is given by Theorem 3.1 we multiply \( e^{\tilde{\kappa}t} \) on (3.27) and arrive at

\[ \frac{d}{dt} e^{\tilde{\kappa}t} \left( (\lambda + \mu)\text{div}\ u \right)^2 dx + \mu|\text{rot}\ u|^2 - (\rho - \tilde{\rho})\text{div}\ u dx \]
\[ \leq \tilde{\kappa} e^{\tilde{\kappa}t} \left( (\lambda + 2\mu)\text{div}\ u \right)^2 dx + \mu|\text{rot}\ u|^2 - (\rho - \tilde{\rho})\text{div}\ u dx \]
\[ + C e^{\tilde{\kappa}t}\|\nabla u\|_{L^2}^6 + C e^{\tilde{\kappa}t}\|\nabla u\|_{L^2}^3 + C e^{\tilde{\kappa}t}\|\nabla u\|_{L^2}^2 \]
\[ + C e^{\tilde{\kappa}t}\|\rho - \tilde{\rho}\|_{L^2}^6 + C e^{\tilde{\kappa}t}\|\rho - \tilde{\rho}\|_{L^3}^3. \]

Theorem 3.1 guarantees

\[ \int_0^T e^{\tilde{\kappa}t}\|\nabla u\|_{L^2}^2 dt \leq C_0. \]  

Consequently, by choosing

\[ C_0 \leq \varepsilon_0, \]  

for some positive small constant \( \varepsilon_0 \) depending only on \( \tilde{\rho}, \Omega, \mu, \lambda, M_0 \) and \( N_0 \), we can apply (3.1), (3.2), (3.29), Lemma 2.2, and Gronwall inequality in (3.28) to declare

\[ \sup_{0 \leq t \leq T} \int_\Omega |\nabla u|^2 dx + \int_0^T \int_\Omega e^{\tilde{\kappa}t} \rho|\dot{u}|^2 dx dt \leq C. \]
Next, substituting (3.25) into (3.24) shows
\[
\frac{d}{dt}\sigma(t) \int \rho|\dot{u}|^2 dx + \sigma(t) \int \left(|\text{div}\, \dot{u}|^2 + |\text{rot}\, \dot{u}|^2\right) dx
\]
\[
\leq C(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}) \left(\sigma(t) \int \rho|\dot{u}|^2 dx \right)
\]
\[
+ C\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2 + C\|\rho - \bar{\rho}\|_{L^2}^4,
\]
which along with Theorem 3.1, (1.11), (3.11) and Gronwall inequality yields
\[
\sup_{0\leq t\leq T} \sigma(t) \int_\Omega \rho|\dot{u}|^2 dx + \int_0^T \sigma(t) \int_\Omega |\nabla \dot{u}|^2 dx dt \leq C.
\]

Recall the decomposition given by (2.5)
\[
u = w + v,
\]
where \(w\) is called the regular part, and \(v\) is called the singular part. Energy estimate Theorem 3.2 and exponential decay assertions Theorem 3.1 are sufficient to derive some further estimates on regular part \(w\).

By definition (2.3), \(w\) satisfies
\[
\begin{cases}
\mu \Delta w + \lambda \text{div} w = \rho \dot{u} & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We have following estimate on the \(L^1(0,T; L^\infty(\Omega))\)-norm of both \(u\) and \(\nabla w\).

**Theorem 3.3.** Under the condition (3.1), there is a constant \(C\) depending on \(\Omega, \mu, \lambda, M_0\) and \(N_0\) such that
\[
\int_0^T \left(\|u\|_{L^\infty} + \|\nabla w\|_{L^\infty}\right) dt \leq C C_0. \tag{3.31}
\]

**Proof.** According to Sobolev imbedding, the Gagliardo-Nirenberg inequality, elliptic estimate (2.6), Theorems 3.1 and 3.2, we have
\[
\int_0^T \|\nabla w\|_{L^\infty} dt
\]
\[
\leq \int_0^T \left(C\|\nabla w\|_{L^2}^6 \|\nabla^\delta w\|_{L^4} + C\|\nabla w\|_{L^2}\right) dt
\]
\[
\leq C \int_0^T \left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) \left(\|\rho \dot{u}\|_{L^4} + \|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) dt
\]
\[
\leq C \int_0^T \left(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}\right) \left(\|\rho \dot{u}\|_{L^4} + \|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}\right) dt
\]
\[
\leq C \int_0^T \sigma(t)^{-\frac{1}{2}} \left(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}\right) \left(\sigma(t)^{\frac{1}{2}} \|\nabla u\|_{L^2}\right) dt
\]
\[
+ C \int_0^T \left(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2}\right) dt
\]
\[
\leq C C_0.
\]

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The estimate upon \(\|u\|_{L^\infty}\) now is direct. It follows from \(w = 0\) and \(v = 0\) on \(\partial \Omega\) and Sobolev’s imbedding inequality that
\[
\|u\|_{L^\infty} \leq \|w\|_{L^\infty} + \|v\|_{L^\infty}
\leq C(\|w\|_{W^{1,4}} + \|v\|_{W^{1,4}})
\leq C(\|\nabla w\|_{L^\infty} + \|\nabla v\|_{L^4})
\leq C(\|\nabla w\|_{L^\infty} + \|\rho - \bar{\rho}\|_{L^4}),
\]
where in the last inequality we have used (2.6).
Thus, according to Theorem 3.1 and (3.32), we declare
\[
\int_0^T \|u\|_{L^\infty} \, dt \leq \int_0^T C(\|\nabla w\|_{L^\infty} + \|\rho - \bar{\rho}\|_{L^4}) \, dt
\leq C \int_0^T \|\nabla w\|_{L^\infty} \, dt + C C_0 \int_0^T e^{-\kappa t} \, dt \leq C C_0.
\]

4 A priori estimates: near-boundary ones

With all preparations done, we turn to the heart of the paper: near-boundary estimates. This section is devoted to giving a detail proof of the following main result:

**Theorem 4.1.** Under the conditions of Theorem 1.1, let
\[
\theta = \frac{\min\{\bar{\rho}, \rho\}}{10}, \quad \delta_1 = \frac{\kappa}{20}, \quad N = \frac{400 M_0}{\theta} N_0 + C_1,
\]
where \(\kappa\) is given by Theorem 3.1 and \(C_1\) will be determined later (see (4.19)), which is a constant depending only on \(N_0, M_0, \mu, \lambda, \bar{\rho}, \bar{\rho}\) and \(\Omega\).

Then there is a positive constant \(\varepsilon\) depending on \(\mu, \lambda, \bar{\rho}, \bar{\rho}, d_0, M_0, N_0\) and \(\Omega\) such that if \((\rho, u)\) is a smooth solution of (1.1)-(1.3) on \(\Omega \times [0,T]\) satisfying
\[
\|\rho - \bar{\rho}\|_{L^\infty(B)} + \|\nabla \rho\|_{L^4(B)} + \|\nabla v\|_{L^\infty(B)} + \|\nabla^2 v\|_{L^4(B)}
+ \int_{\partial \Omega} (|\nabla \rho|^2 + |\nabla v|^2 + |\nabla^2 v|^2) \, dS \leq 2 N e^{-\delta_1 t},
\]
\[
\|\rho\|_{\dot{C}^\alpha(B \cup \Gamma \cup \tilde{\Gamma})} \leq 2 N, \quad \|\rho\|_{L^\infty} \leq 2 M_0,
\]
and
\[
\inf_{x \in B \cup \Gamma \cup \tilde{\Gamma}} \rho(x) > \frac{\theta}{3},
\]
\[
\|\rho - \bar{\rho}\|_{L^\infty(B)} + \|\nabla \rho\|_{L^4(B)} + \|\nabla v\|_{L^\infty(B)} + \|\nabla^2 v\|_{L^4(B)}
+ \int_{\partial \Omega} (|\nabla \rho|^2 + |\nabla v|^2 + |\nabla^2 v|^2) \, dS \leq N e^{-2\delta_1 t},
\]
\[
\|\rho\|_{\dot{C}^\alpha(B \cup \Gamma \cup \tilde{\Gamma})} \leq N, \quad \|\rho\|_{L^\infty} \leq \frac{7 M_0}{4},
\]
and
\[
\inf_{x \in B \cup \Gamma \cup \tilde{\Gamma}} \rho(x) > \frac{\theta}{2},
\]
provided \(C_0 \leq \varepsilon\).
Remark 4.1. We emphasize the necessity of near-boundary estimate. The key issue to derive the upper bound of density $\rho$ is to get a proper estimate over $\| F \|_{L^\infty}$, where $F = (\mu + \lambda) \text{div} u - (\rho - \bar{\rho})$. Note that $\text{div} w$ has been controlled by Lemma 3.3 and $F_v$ satisfies $\Delta F_v = 0$ (see (4.64)).

According to the interior estimate (2.39), $F_v$ is well behaved in the interior of $\Omega$. We need only to get $L^\infty$ bound of $F_v$ near the boundary. It requires uniform higher order derivatives estimate of $\rho$ (see Section 4.1, 4.2), which prevents vacuum to form. Consequently, we must restrict the uniform higher order estimate of $\rho$ just near the boundary to make sure vacuum is available in the interior part of $\Omega$.

First of all, for $\varepsilon_0$ as in Theorem 3.2, we suppose (3.30) is true, say

$$C_0 \leq \varepsilon_0,$$

which along with (4.2) gives Theorems 3.1–3.3. Moreover, it follows from (3.31) that

$$\int_0^T \| u \|_{L^\infty} dt \leq CC_0.$$

We also set

$$C_0 \leq \frac{d_0}{200C},$$

which guarantees (2.26). Consequently, all results in Subsections 2.2 and 2.3 are available as well in current setting. In particularly, we choose $d$ and geometric constant $K$ as in Theorem 2.1 and (2.29).

The complete proof is rather long, and we cut it down into several subsections to make the idea clear. The central issue is to get a proper control over $H^{2,2}$ layer which contains the boundary $\partial \Omega$.

We first give an overview of our arguments. To handle the whole $H^{2,2}$ layer, we will decompose it into the boundary part $B$ (since $B$ contains $\partial \Omega$), and the interior part $\Gamma''$ (since it has positive distance $d$ from $\partial \Omega$). Thus the proof will be carried out in two main parts: in Subsection 4.1.1 “interior part” $\Gamma''$ will be investigated; and in Subsection 4.2 we will consider the “boundary part” $B$ of $H^{2,2}$ layer. Once $H^{2,2}$ layer is under control, we can easily close (4.3) and (4.5) by using the interior estimate (2.26)–(2.29). It will be done in Subsection 4.3.

In this Section, the constants $C$ may vary from line to line but depend only on $\mu, \lambda, \bar{\rho}, \rho, M_0, N_0, \delta_1$ and $\Omega$. Now, let us begin the proof.

### 4.1 Interior part $\Gamma''$

We consider the interior part $\Gamma''$ of $H^{2,2}$ layer. In such case, energy method in Subsection 4.2 no longer works since $\Gamma''$ contains two inner boundaries $\Gamma''_1$ and $\Gamma''_2$.

Our strategy is to set up series of intermediate layers: Recall form Definition 2.1 we have (inner part of) $W^{1,p}$ layer $\Gamma'$ and (inner part of ) $C^\alpha$ layer $\tilde{\Gamma} \cup \Gamma$. The position of each layers is given by

$$\Gamma'' \subset \subset \Gamma' \subset \subset (\tilde{\Gamma} \cup \Gamma).$$

We will first close the largest $C^\alpha$ layer $\tilde{\Gamma} \cup \Gamma$, then shrink the domain to close the smaller $W^{1,p}$ layer $\Gamma'$, finally we can reach the smallest $H^{2,2}$ layer $\Gamma''$.

#### 4.1.1 Estimate in $C^\alpha$ layer $\tilde{\Gamma} \cup \Gamma$

In this subsection, we will give the following $C^\alpha$ estimate:
Lemma 4.1. Under the conditions of Theorem 4.7, we declare that

\[
\sup_{0 \leq t \leq T} \|\rho\|_{C^\infty(\Gamma)} < \frac{N}{10},
\]

\[
\sup_{0 \leq t \leq T} \|\rho\|_{C^\infty(\Gamma_2)} < \frac{N}{10}.
\]  

Remark 4.2. As indicated by Remark 2.10, \(\|\rho\|_{C^\infty(\Gamma)} \) consists of three parts: \(\|\rho\|_{C^0(\Gamma)}\), \(\|\rho\|_{C^0(\Gamma_2)}\) and \(\|\rho\|_{C^0(B)}\). Therefore, the first two terms have been properly controlled by Lemma 4.7 and the third term will be closed latter on.

Proof. Considering any two flow lines \(X(t)\) and \(Y(t)\) where \(X(t)\) lies in the domain \(\Gamma \cup \tilde{\Gamma}\), and \(Y(t)\) lies in \(\Gamma\), which means for all \(t > 0\), \(X(t) \subset (\Gamma \cup \tilde{\Gamma})\) and \(Y(t) \subset \Gamma\).

The distance between two flows can be estimated via (2.25), say

\[
\frac{d}{dt}|X(t) - Y(t)| = \frac{u(X(t), t) - u(Y(t), t)}{|X(t) - Y(t)|}. (X(t) - Y(t))
\]

\[
= \frac{X(t) - Y(t)}{|X(t) - Y(t)|} \cdot a(t),
\]

where

\[
a(t) = \frac{u(X(t), t) - u(Y(t), t)}{|X(t) - Y(t)|}.
\]

Integrating (4.8) with respect to \(t\), we have

\[
\frac{|X(s) - Y(s)|}{|X(l) - Y(l)|} \leq \exp \left( \int_0^s \frac{|X(t) - Y(t)| \cdot a(t)}{|X(t) - Y(t)|} dt \right). 
\]

For \(s > l\), it holds that

\[
\frac{|X(s) - Y(s)|}{|X(l) - Y(l)|} \cdot \frac{|X(s) - Y(s)|}{|X(s) - Y(s)|} \leq \exp \left( \int_0^s |u(X(t), t) - u(Y(t), t)| \right)
\]

\[
\leq \exp \left( \int_0^s \frac{|u(X(t), t) - u(Y(t), t)|}{|X(t) - Y(t)|} dt \right). 
\]

We check that

\[
\frac{|w(X(t), t) - w(Y(t), t)|}{|X(t) - Y(t)|} \leq \frac{|w(X(t), t) - w(Y(t), t)|}{|X(t) - Y(t)|} + \frac{|v(X(t), t) - v(Y(t), t)|}{|X(t) - Y(t)|}
\]

\[
\leq \|\nabla w\|_{L^\infty} + cK_1 |\rho|_{C^0(\Gamma \cup \tilde{\Gamma})} + cK_2 \varepsilon^{-2} \left( \|\rho - \bar{\rho}\|_{L^2} + \|\rho - \bar{\rho}\|_{\dot{H}^1} \right)
\]

\[
\leq \|\nabla w\|_{L^\infty} + C_1 \varepsilon + C_2 \varepsilon^{-2} e^{-\varepsilon^{-2}},
\]

where in the last two inequality we have used (3.3), Theorem 3.1 and Lemma 2.18. Combining this with Theorem 3.3 gives

\[
\int_0^s \frac{|u(X(t), t) - u(Y(t), t)|}{|X(t) - Y(t)|} dt \leq C_3 e^{\frac{s}{2}}(s - l) + CC_0 \varepsilon^{-2}.
\]

Therefore, according to (4.9), we have

\[
\frac{|X(s) - Y(s)|}{|X(l) - Y(l)|} \cdot \frac{|X(s) - Y(s)|}{|X(s) - Y(s)|} \leq e^{C_3(s-l)} + CC_0 \varepsilon^{-2}. 
\]
Let us trace along the flow. Obviously, (1.1) gives
\[
\frac{d}{dt} \log \rho(X(t)) + \frac{\rho(X(t)) - \bar{\rho}}{\mu + \lambda} = -\frac{F(X(t))}{\mu + \lambda},
\]
Consequently, for two flows \(X(t), Y(t)\) we have
\[
\frac{d}{dt} (\log \rho(X(t)) - \log \rho(Y(t))) + \frac{\rho(X(t)) - \rho(Y(t))}{\mu + \lambda} = -\frac{F(X(t)) - F(Y(t))}{\mu + \lambda},
\]
Multiplying (4.11) by \(\frac{\log \rho(X(t)) - \log \rho(Y(t))}{\log \rho(X(t)) - \log \rho(Y(t))}\), we arrive at
\[
\frac{d}{dt} |\log \rho(X(t)) - \log \rho(Y(t))| + L |\log \rho(X(t)) - \log \rho(Y(t))| 
\leq \frac{1}{\mu + \lambda} |F(X(t)) - F(Y(t))|,
\]
and
\[
\frac{d}{dt} e^{Lt} |\log \rho(X(t)) - \log \rho(Y(t))| 
\leq \frac{e^{Lt}}{\mu + \lambda} |F(X(t)) - F(Y(t))|,
\]
where \(L\) can be chosen as \(\frac{\theta}{\log(\mu + \lambda)}\).
Integrating (4.12) with respect to \(t\) gives
\[
e^{Lt}|\log \rho(X(t)) - \log \rho(Y(t))| - |\log \rho_0(x) - \log \rho_0(y)| 
\leq \int_0^t e^{Ls} |F(X(s)) - F(Y(s))| ds.
\]
Multiplying (4.13) by \(1/|X(t) - Y(t)|^\alpha\) and using (4.10) leads to
\[
\frac{|\log \rho(X(t)) - \log \rho(Y(t))|}{|X(t) - Y(t)|^\alpha} 
\leq \frac{|\log \rho_0(x) - \log \rho_0(y)|}{|x - y|^{\alpha}} \cdot \frac{|x - y|^{\alpha}}{|X(t) - Y(t)|^\alpha} 
+ C \int_0^t e^{L(s-t)} \|F\|_{C^0(\Gamma)} \frac{|X(s) - Y(s)|^\alpha}{|X(t) - Y(t)|^\alpha} ds
\leq \frac{3}{\theta} \cdot e^{-(L-C\alpha \bar{\rho})t + CC_0 \epsilon^{-2}} \|\rho_0\|_{C^\alpha(\partial \Omega)} 
+ \int_0^t e^{L(s-t)} (\|\nabla w\|_{C^\alpha} + \|F_v\|_{C^0(\Gamma)}) e^{(C\alpha \bar{\rho} (t-s)+CC_0 \epsilon^{-2})} ds
\leq \frac{6}{\theta} \|\rho_0\|_{C^\alpha(\partial \Omega)} \cdot e^{-\frac{L}{2}t} + C \int_0^t (\|\nabla w\|_{C^\alpha} + \|F_v\|_{C^0(\Gamma)}) e^{-\frac{L}{2} (t-s)} ds.
\]
Such process is available once we choose
\[
\epsilon \leq \left(\frac{L}{2C\alpha}\right)^\frac{2}{\alpha} \text{ and } C_0 \leq \frac{e^2 \log 2}{C\alpha}.
\]
Moreover, according to (2.6), (4.2), Sobolev’s imbedding theorem and Theorem 3.2, we have
\[
\int_0^t \|\nabla w\|_{C^{0,\delta}} e^{-\frac{t}{2}(t-s)} ds \\
\leq C \int_0^t \|\nabla^2 w\|_{L^4} e^{-\frac{t}{2}(t-s)} ds \\
\leq C \int_0^t \|\rho\|_{L^4} e^{-\frac{t}{2}(t-s)} ds \\
\leq C \int_0^t \|\rho\|_{L^2} \|\nabla \tilde{u}\|_{L^2} e^{-\frac{t}{2}(t-s)} ds \\
\leq C \int_0^t \|\rho\|_{L^2} \|\nabla \tilde{u}\|_{L^2} e^{-\frac{t}{2}(t-s)} ds \\
\leq C \int_0^t \|\rho\|_{L^2} \left(\frac{1}{2} \|\nabla \tilde{u}\|_{L^2} \right)^{\frac{1}{2}} \left(\frac{1}{2} e^{-\frac{t}{2}(t-s)} \right) ds \leq C.
\] (4.16)

By interior estimate (2.39), elliptical estimate (2.4) and Theorem 3.1, we also declare
\[
\int_0^t \|F\|_{C^{0,\delta}(\bar{\Gamma} \cup \tilde{\Gamma})} e^{-\frac{t}{2}(t-s)} ds \leq \int_0^t \|\rho - \bar{\rho}\|_{L^2} e^{-\frac{t}{2}(t-s)} ds \leq CC_0.
\] (4.17)

Collecting (4.14), (4.16) and (4.17) gives
\[
\frac{|\log \rho(X(t)) - \log \rho(Y(t))|}{|X(t) - Y(t)|} \leq \frac{6}{\theta} \|\rho\|_{C^{0,\delta}(B(0,\bar{\Gamma} \cup \tilde{\Gamma}))} + C,
\]
which combined with (4.12) yields
\[
\frac{|\rho(X(t)) - \rho(Y(t))|}{|X(t) - Y(t)|} \leq \frac{12M_0}{\theta} \|\rho\|_{C^{0,\delta}(B(0,\bar{\Gamma} \cup \tilde{\Gamma}))} + C \leq \frac{12M_0}{\theta} N_0 + C,
\] (4.18)

for some $C$ depending only on $\mu$, $\lambda$, $N_0$, $M_0$ and $d$.

We select
\[
C_1 = 20C,
\] (4.19)
where $C$ is given by (4.18). Note that (4.19) together with (4.1) gives
\[
N = \frac{400M_0}{\theta} N_0 + C_1 > 10 \left(\frac{12M_0}{\theta} N_0 + C\right).
\] (4.20)

Now we restrict the flow line $X(t)$ located in $\Gamma \cup \tilde{\Gamma}$, say $X(t) \subset \Gamma \cup \tilde{\Gamma}$ for all $t > 0$, and take supremum among all flows. With the help of (4.20), we declare that
\[
\sup_{0 \leq t \leq T} \|\rho\|_{C^{0,\delta}(\Gamma')} \leq \frac{12M_0}{\theta} N_0 + C < \frac{N}{10},
\] (4.21)

which gives (4.17) and partly closes estimate in (4.3) about $\|\rho\|_{C^{0,\delta}(B(0,\bar{\Gamma} \cup \tilde{\Gamma}))}$.

4.1.2 Estimate in $W^{1,p}$ layer $\Gamma'$

The main result of this subsection is the estimate in $W^{1,p}$ layer $\Gamma'$ as follows.

**Lemma 4.2.** Under the conditions of Theorem 4.1, it holds that
\[
\sup_{0 \leq t \leq T} \int_{\Gamma'} |\nabla \rho|^4 d\nu \leq C.
\] (4.22)
To achieve the goal, it is necessary to get a proper control over $\|\nabla v\|_{L^\infty(\Gamma')}$. At first.

**Lemma 4.3.** Suppose $4.3$ and $(4.21)$ hold. Then it holds

$$\int_0^T \|\nabla v\|_{L^\infty(\Gamma')} dt \leq C.$$  \hspace{1cm} (4.23)

**Proof.** Recall that by $(2.4)$, $v$ solves the elliptic system in the domain $\Gamma$, that is

$$\mu \Delta v + \lambda \nabla \text{div} v = \nabla \rho \text{ in } \Gamma.$$  

We define $\Gamma'_+$ via

$$\Gamma'_+ = \{ x \in \Gamma \mid \text{dist}(x, \Gamma_2) > \frac{d}{4} \}.$$  

Observe that

$$\Gamma' \subset\subset \Gamma'_+ \subset\subset \Gamma,$$  

and the distance between them are strictly larger than $\frac{d}{4}$ as well, that is

$$\text{dist}(\Gamma', \partial \Gamma'_+) > \frac{d}{4}.$$  

We take advantage of $(4.3)$, $(4.21)$, interior estimate $(2.35)$ within the domain $\Gamma'_+ \subset\subset \Gamma$ and Theorem 3.1 to declare

$$\|\nabla v\|_{C^\alpha(\Gamma'_+)} \leq CK \|\rho\|_{C^\alpha(\Gamma)} + CK \|\nabla v\|_{L^2} \leq C.$$  \hspace{1cm} (4.24)

Now, there are two cases we need to investigate:

**Case 1.** $\|\nabla v\|_{C^\alpha(\Gamma'_+)} > 100K \|\rho - \bar{\rho}\|_{L^2}$.

For $x \in \Gamma'$, select some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Gamma'$. We argue that

$$|\nabla v(x)| \leq \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |\nabla v(x) - \nabla v(y)| dy + \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |\nabla v(y)| dy$$

$$\leq \frac{\|\nabla v\|_{C^\alpha(\Gamma'_+)}}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |x - y|^{\alpha} dy + \left( \frac{1}{|B_\varepsilon(x)|} \int |\nabla v|^2 dy \right)^{\frac{1}{2}}$$

$$\leq C \|\nabla v\|_{C^\alpha(\Gamma'_+)} \varepsilon^{\alpha} + \frac{C}{\varepsilon^{\frac{\alpha}{2}}} \|\nabla v\|_{L^2}.$$  

Determine $\varepsilon$ by

$$\|\nabla v\|_{C^\alpha(\Gamma'_+)} \varepsilon^{\alpha} = \frac{1}{\varepsilon^{\frac{\alpha}{2}}} \|\nabla v\|_{L^2},$$

say

$$\varepsilon = \left( \|\nabla v\|_{L^2} \|\nabla v\|_{C^\alpha(\Gamma'_+)}^{-1} \right)^{\frac{1}{\alpha + \frac{3}{2}}}.$$  \hspace{1cm} (4.25)

We arrive at

$$\|\nabla v\|_{L^\infty(\Gamma')} \leq C \|\nabla v\|_{L^2} \|\nabla v\|_{C^\alpha(\Gamma'_+)}^{1-\theta}.$$  

We still need to illustrate that $\varepsilon$ given by $(4.25)$ satisfies

$$B_\varepsilon(x) \subset \Gamma'_+.$$  \hspace{1cm} (4.26)

Recalling that $(2.6)$ gives $\|\nabla v\|_{L^2} \leq C \|\rho - \bar{\rho}\|_{L^2}$, we have

$$\|\nabla v\|_{C^\alpha(\Gamma'_+)} > 100K \|\rho - \bar{\rho}\|_{L^2} > 100K \|\nabla v\|_{L^2},$$

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which actually implies
\[
\left( \|\nabla v\|_{L^2} \|\nabla v\|^{-1}_{C^0(\Gamma')} \right)^{\frac{1}{m+2}} \leq \frac{d}{4}.
\]
This gives (4.26) and hence the choice of \( \varepsilon \) is correct.

Consequently, combining the elliptic estimate (2.6) with (4.24) guarantees
\[
\|\nabla v\|_{L^\infty(\Gamma')} \leq C \|\nabla v\|_{C^0(\Gamma')} \leq C \|\rho - \bar{\rho}\| \leq C \|\rho - \bar{\rho}\| \leq C \|\rho - \bar{\rho}\|_{L^2}(\Gamma').
\]
(4.27)

Case 2. \( \|\nabla v\|_{C^0(\Gamma')} < 100CK \|\rho - \bar{\rho}\|_{L^2} \).

We directly deduce that
\[
\|\nabla v\|_{L^\infty(\Gamma')} \leq C \|\rho - \bar{\rho}\| \leq C \|\rho - \bar{\rho}\| \leq C \|\rho - \bar{\rho}\|_{L^2}.
\]
(4.28)

In conclusion, it follows from (4.27), (4.28) and Theorem 3.1 that
\[
\int_0^T \|\nabla v\|_{L^\infty(\Gamma')} dt \leq C \int_0^T \left( \|\rho - \bar{\rho}\| + \|\rho - \bar{\rho}\|_{L^2} \right) dt \leq C.
\]

With Lemma 4.3 in hand, we turn to the proof of Lemma 4.2.

Proof of Lemma 4.2 Recalling that (1.1) gives
\[
D_t \rho = -\rho u^k_k.
\]
(4.29)

Taking one more order covariant derivative on (4.29) gives
\[
D_t \rho, i = -\rho, i u^k_k - \rho u^k_k.
\]
(4.30)

With the help of (4.30), direct computation yields
\[
\frac{d}{dt} \int_{\Gamma'} (g^{ij} \rho, i \rho, j)^2 d\nu \\
= 2 \int_{\Gamma'} g^{ij} \rho, i \rho, j D_t (g^{kl} \rho, k \rho, l) d\nu + \int_{\Gamma'} (g^{ij} \rho, i \rho, j)^2 D_t d\nu \\
= 2 \int_{\Gamma'} g^{ij} \rho, i \rho, j D_t (g^{kl} \rho, k \rho, l) d\nu + 4 \int_{\Gamma'} g^{ij} \rho, i \rho, j g^{kl} (D_t \rho, k) \rho, l d\nu \\
+ \int_{\Gamma'} (g^{ij} \rho, i \rho, j)^2 \rho, s d\nu \\
= \int_{\Gamma'} 2g^{ij} \rho, i \rho, j D_t (g^{kl} \rho, k \rho, l) + \rho, s (g^{ij} \rho, i \rho, j)^2 d\nu \\
- 4 \int_{\Gamma'} g^{ij} \rho, i \rho, j g^{kl} (\rho, k \rho, l + \rho, s) \rho, l d\nu \\
= \int_{\Gamma'} 2g^{ij} \rho, i \rho, j D_t (g^{kl} \rho, k \rho, l) - 3 \rho, s (g^{ij} \rho, i \rho, j)^2 d\nu \\
- 4 \int_{\Gamma'} \rho g^{ij} \rho, i \rho, j g^{kl} \rho, s \rho, l d\nu \\
= J_1 - 4J_2.
\]
(4.31)
By taking advantage of Lemma 2.8 for the first term in (4.31) we have

\[ |J_1| = \left| \int_{\Gamma'} 2g^{ij} \rho_i \rho_j D_t (g^{kl}) \rho_k \rho_l - 3u_s^* (g^{ij} \rho_i \rho_j)^2 d\nu \right| \]
\[ \leq C \| \nabla u \|_{L^\infty(\Gamma')} \int_{\Gamma'} (g^{ij} \rho_i \rho_j)^2 d\nu. \]  

(4.32)

The principal term in (4.31) reads

\[ J_2 = \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} u_{sk} \rho_l d\nu \]
\[ = \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} v_{sk} \rho_l d\nu + \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} w_{sk} \rho_l d\nu \]
\[ = \frac{1}{\mu + \lambda} \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} (\mu + \lambda) v_{sk} - (\rho - \bar{\rho}) \rho_l d\nu + \frac{1}{\mu + \lambda} \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} \rho_k \rho_l d\nu + \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} w_{sk} \rho_l d\nu \]
\[ = I_1 + I_2 + I_3. \]  

(4.33)

The first term and second term of (4.33) are handled via

\[ |I_1| = \left| \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} w_{sk} \rho_l d\nu \right| \leq C \left( \int_{\Gamma'} \| \nabla \text{div} w \|^4 d\nu \right)^{\frac{1}{2}} \left( \int_{\Gamma'} \| \nabla \rho \|^4 d\nu \right)^{\frac{1}{2}}, \]  

(4.34)

and

\[ |I_2| = \left| \int_{\Gamma'} \rho g^{ij} \rho_i \rho_j g^{kl} (\mu + \lambda) v_{sk} - (\rho - \bar{\rho}) \rho_l d\nu \right| \leq C \left( \int_{\Gamma'} \| \nabla F_{\rho} \|^4 d\nu \right)^{\frac{1}{2}} \left( \int_{\Gamma'} \| \nabla \rho \|^4 d\nu \right)^{\frac{1}{2}}. \]  

(4.35)

Combining (4.32), (4.34) and (4.35) yields

\[ \frac{d}{dt} \int_{\Gamma'} \| \nabla \rho \|^4 d\nu + \int_{\Gamma'} \| \nabla \rho \|^4 d\nu \]
\[ \leq C \| \nabla u \|_{L^\infty(\Gamma')} \int_{\Gamma'} (g^{ij} \rho_i \rho_j)^2 d\nu + C \left( \int_{\Gamma'} \| \nabla F_{\rho} \|^4 d\nu \right)^{\frac{1}{2}} \left( \int_{\Gamma'} \| \nabla \rho \|^4 d\nu \right)^{\frac{1}{2}} \]
\[ + C \left( \int_{\Gamma'} \| \nabla \text{div} w \|^4 d\nu \right)^{\frac{1}{2}} \left( \int_{\Gamma'} \| \nabla \rho \|^4 d\nu \right)^{\frac{1}{2}} \]
\[ \leq C \| \nabla u \|_{L^\infty(\Gamma')} \int_{\Gamma'} \| \nabla \rho \|^4 d\nu + C K \| \rho - \bar{\rho} \|_{L^2} \left( \int_{\Gamma'} \| \nabla \rho \|^4 d\nu \right)^{\frac{1}{2}} \]
\[ + C \| \rho \|_{L^4} \left( \int_{\Gamma'} \| \nabla \rho \|^4 d\nu \right)^{\frac{3}{2}}, \]  

(4.36)

where in the last inequality we have used (2.39) and (2.40).

According to Theorem 3.72

\[ \int_0^T \| \rho \|_{L^4}^4 dt \leq \int_0^1 \| \rho \|_{L^2}^\frac{1}{2} \| \rho \|_{L^6}^\frac{3}{2} dt + \int_1^T \| \rho \|_{L^2}^\frac{1}{2} \| \rho \|_{L^6}^\frac{3}{2} dt \]
\[ \leq C \int_0^1 \| \rho \|_{L^2} \left( \sigma(t) \right)^\frac{1}{2} \| \nabla \rho \|_{L^2} \right)^\frac{1}{2} \sigma(t)^{-\frac{1}{2}} dt \]
\[ + C \int_1^T e^{-\frac{\rho t}{\rho_0}} (e^\rho \| \nabla \rho \|_{L^2})^\frac{1}{2} \left( \| \nabla \rho \|_{L^2} \right)^\frac{3}{2} dt \leq C, \]  

(4.37)
where $\kappa$ is given by Theorem 3.2.

Combining (4.37), (4.23), and Theorems 3.1–3.3, we apply Gronwall’s inequality to declare:

$$\sup_{0 \leq t \leq T} \int_{\Gamma'} |\nabla \rho|^4 d\nu \leq C,$$

which is (4.22) and finishes the proof of Lemma 4.2. \qed

### 4.1.3 Estimate in intermediate layer $B \cup \Gamma^{''}$

For technical reason, we must improve the estimate of $\nabla \dot{u}$ (see (3.12)) near the boundary.

In fact, when the domain is restricted to $B \cup \Gamma^{''}$, the extra weight $\sigma(t)$ in (3.12) can be dropped. It is possible since no vacuum is formed near the boundary which makes sure $\sqrt{\rho} \dot{u}$ admitting a proper initial value. Precisely, we declare

**Theorem 4.2.** There is a constant $C$ depending only on $\mu, \lambda, \Omega, N_0, M_0$ and $\rho$ such that

$$\sup_{0 \leq t \leq T} \int |\nabla \dot{u}|^2 dx + \int_0^T \int |\nabla u|^2 \phi^2 dx \leq C,$$

(4.38)

where $\phi$ is given by (2.20).

Before proving Theorem 4.2. We first state a weighted elliptic estimate which plays an important role in our argument.

**Lemma 4.4.** Suppose (4.2) is true. There is a constant $C$ depending on $\mu, \lambda, \Omega, N_0, M_0$ and $\rho$ such that

$$\int |\nabla^2 \phi|^2 dx \leq C \int |\nabla u|^2 \phi^2 dx + C \|\nabla w\|_{L^2}^2.$$

(4.39)

**Proof.** For $\phi$ and $\zeta$ given by (2.20) and Theorem 2.1, the support of $\phi$ satisfies

$$\text{supp}(\nabla \phi), \text{supp}(\nabla \zeta) \subset \subset \Gamma',$$

(4.40)

which along with (4.2), Lemma 2.11 and Lemma 4.3 gives

$$\sup_{0 \leq t \leq T} |\nabla \phi| \leq C \exp\left(\int_0^T \|\nabla u\|_{L^\infty(B \cup \Gamma')} dt\right) \leq C,$$

(4.41)

and

$$\sup_{0 \leq t \leq T} |\nabla \zeta| \leq C \exp\left(\int_0^T \|\nabla u\|_{L^\infty(B \cup \Gamma')} dt\right) \leq C.$$  

We define a further truncation function

$$\chi \triangleq \phi \cdot \zeta,$$

according to (4.41) and (4.42), we have

$$|\nabla \chi| \leq C.$$  

(4.43)
Multiplying (2.3) by $\Delta w \cdot \chi^2$ and integrating over $\Omega$, we deduce
\[
\mu \int \Delta w \cdot \Delta w \cdot \chi^2 \, dx + \lambda \int \nabla \text{div} w \cdot \Delta w \cdot \chi^2 \, dx = \int \rho \dot{u} \cdot \Delta w \cdot \chi^2 \, dx. \tag{4.44}
\]
We argue that
\[
\mu \int \Delta w \cdot \Delta w \cdot \chi^2 \, dx \\
= -\mu \int \nabla w \cdot \Delta (\nabla w) \chi^2 \, dx - 2\mu \int \nabla w \cdot \Delta \nabla \chi \cdot \chi \, dx \\
= \mu \int |\nabla^2 w|^2 \chi^2 \, dx + 2\mu \int \nabla w \cdot \nabla^2 w \cdot \nabla \chi \cdot \chi \, dx \\
- 2\mu \int \nabla w \cdot \Delta w \cdot \nabla \chi \cdot \chi \, dx \\
\geq \frac{\mu}{2} \int |\nabla^2 w|^2 \chi^2 \, dx - C \int |\nabla w|^2 \, dx,
\]
where we have taken advantage of (4.43) in the last line.

Similarly, we also have
\[
\lambda \int \nabla \text{div} w \cdot \Delta w \cdot \chi^2 \, dx \\
= -\lambda \int \text{div} w \cdot \text{div}(\Delta w) \cdot \chi^2 \, dx - 2\lambda \int \text{div} w \cdot \Delta \nabla \chi \cdot \chi \, dx \\
= \lambda \int |\nabla \text{div} w|^2 \chi^2 \, dx + 2\lambda \int \text{div} w \cdot \nabla \text{div} w \cdot \nabla \chi \cdot \chi \, dx \\
- 2\lambda \int \text{div} w \cdot \Delta w \cdot \nabla \chi \cdot \chi \, dx \\
\geq \lambda \int |\nabla \text{div} w|^2 \chi^2 \, dx - \frac{\mu}{10} \int |\nabla^2 w|^2 \chi^2 \, dx - C \int |\nabla w|^2 \, dx, \tag{4.45}
\]
where we have taken advantage of (4.43) in the last line.

Directly, we apply (4.2) to check that
\[
\int \rho \dot{u} \cdot \Delta w \cdot \chi^2 \, dx \leq C \int \rho |\dot{u}|^2 \chi^2 \, dx + \frac{\mu}{10} \int |\nabla^2 w|^2 \chi^2 \, dx. \tag{4.46}
\]
Combining (4.44) − (4.45) gives
\[
\int |\nabla^2 w|^2 \chi^2 \, dx \leq C \int \rho |\dot{u}|^2 \phi^2 \, dx + C \|\nabla w\|^2_{L^2}. \tag{4.47}
\]
Moreover, we check that
\[
\int |\nabla^2 w|^2 \phi^2 \, dx = \int |\nabla^2 w|^2 \cdot \chi^2 \, dx + \int |\nabla^2 w|^2 \cdot \phi^2 (1 - \zeta^2) \, dx. \tag{4.48}
\]
Note that by (2.20) and (2.21)
\[
\text{supp} (\phi^2 (1 - \zeta^2)) \subset \subset \Gamma_+'' ,
\text{dist} \left( \text{supp} (\phi^2 (1 - \zeta^2)) , \Gamma_+'' \right) > d,
\]
which along with (4.12) and (2.32) applied to $\text{supp} (\phi^2 (1 - \zeta^2)) (\subset \subset \Gamma_+'' )$ gives
\[
\int |\nabla^2 w|^2 \cdot \phi^2 (1 - \zeta^2) \, dx \leq C \int \rho |\dot{u}|^2 \cdot \phi^2 \, dx + C \|\nabla w\|^2_{L^2}. \tag{4.49}
\]
Combining (4.46), (4.47) and (4.48) shows

\[ \int |\nabla^2 w|^2 \phi^2 dx \leq C \int \rho |\dot{u}|^2 \phi^2 dx + C \|\nabla w\|^2_{L^2}, \]

which is (4.39).

With Lemma 4.24 in hand, we are now in a position to prove Theorem 4.22.

**Proof of Theorem 4.22** We apply Euler coordinates to show (4.38) where \( \partial_j = \partial_{x_j} \). It holds that in Euler coordinates

\[ \partial_t \phi + u \cdot \nabla \phi = 0. \tag{4.49} \]

Operating \( \phi^2 \dot{u}^j \left( \frac{\partial}{\partial t} + \text{div}(u \cdot ) \right) \) to (1.12), summing with respect to \( j \), integrating over \( \Omega \), and using (4.49) and (1.11) leads to

\[
\begin{align*}
\frac{d}{dt} \int \rho |u|^2 \phi^2 dx &= \mu \int (\Delta u_t + \partial_j (u_j \Delta u)) \dot{u} \phi^2 dx + \lambda \int (\nabla \text{div} u_t + \partial_j (u_j \nabla \text{div} u)) \dot{u} \phi^2 dx \\
&\quad - \int (\nabla \rho_t + \partial_j (u_j \nabla \rho)) \dot{u} \phi^2 dx \\
&= \mu I_1 + \lambda I_2 + I_3. \tag{4.50}
\end{align*}
\]

Let us check each \( I_i \) in details. For \( I_1 \), we have

\[
\begin{align*}
I_1 &= \int (\Delta u_t + \partial_j (u_j \Delta u)) \dot{u} \phi^2 dx \\
&= - \int \nabla u_t \cdot \nabla \phi^2 dx - 2 \int \partial_j u_t \cdot u \cdot \partial_j \phi \cdot \phi dx \\
&\quad - \int u_j \Delta u_t \cdot \partial_j \dot{u}_i \cdot \phi^2 dx \\
&\quad - 2 \int \partial_j \dot{u}_i \cdot u_i \cdot \partial_j \phi \cdot \phi dx \\
&\quad - \int u_j \Delta u_t \cdot \partial_j \dot{u}_i \cdot \phi^2 dx \\
&= - \int |\nabla \dot{u}|^2 \phi^2 dx + \int \partial_k (u_j \partial_j u_i) \partial_k \dot{u}_i \phi^2 dx + \int u_j \cdot \partial_j u_i \cdot \partial_k \dot{u}_i \phi^2 dx \\
&\quad - 2 \int \partial_j \dot{u}_i \cdot u_i \cdot \partial_j \phi \cdot \phi dx + 2 \int \partial_j (u_k \partial_k u_i + u_k \partial_j u_i) \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \\
&\quad - \int u_j \Delta u_t \cdot \partial_j \dot{u}_i \cdot \phi^2 dx \\
&= - \int |\nabla \dot{u}|^2 \phi^2 dx - 2 \int \partial_j \dot{u}_i \cdot u_i \cdot \partial_j \phi \cdot \phi dx \\
&\quad + \int u_j \cdot \partial_j u_i \cdot \partial_k \dot{u}_i \phi^2 dx + 2 \int \partial_j u_k \cdot \partial_k u_i \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \\
&\quad + \left( \int u_j \cdot \partial_j u_i \cdot \partial_k \dot{u}_i \phi^2 dx - \int u_j \Delta u_t \cdot \partial_j \dot{u}_i \cdot \phi^2 dx \right) \\
&\quad + 2 \int u_k \cdot \partial_j u_i \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx - 2 \int u_j \Delta u_t \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx
\end{align*}
\]
\[ - \int |\nabla \dot{u}|^2 \phi^2 dx - 2 \int \partial_j \dot{u}_i \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx + K \\
- 2 \int u_j \partial_k u_i \cdot \partial_k \dot{u}_i \cdot \partial_j \phi \cdot \phi dx + 2 \int u_j \partial_k u_i \cdot \partial_j \dot{u}_i \cdot \partial_k \phi \cdot \phi dx \\
+ 2 \int u_k \cdot \partial_j u_i \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx - 2 \int u_j \Delta u_i \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \]
\[ \leq - \int |\nabla \dot{u}|^2 \phi^2 dx + C \int \rho |\dot{u}|^2 dx + \frac{1}{2} \int |\nabla \dot{u}|^2 \phi^2 dx \\
+ K + C \int |u| \cdot |\nabla u| \cdot |\nabla \dot{u}| \cdot |\nabla \phi| \phi dx + C \int |u| \cdot |\nabla^2 u| \cdot |\dot{u}| \cdot |\nabla \phi| \phi dx, \]
where in the last inequality we have used \( \rho > \theta / 3 \) in \( B \cup \Gamma' \) and \([14.41]\), and
\[ K \equiv \int \partial_k u_j \cdot \partial_j u_i \cdot \partial_k \dot{u}_i \cdot \partial_j \phi \cdot \phi dx + 2 \int \partial_j u_k \cdot \partial_k u_i \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \\
- \int \text{div} u \cdot \partial_k u_i \cdot \partial_k \dot{u}_i \cdot \phi^2 dx + \int \partial_k u_j \cdot \partial_k u_i \cdot \partial_j \dot{u}_i \cdot \phi^2 dx. \]

\( I_2 \) is quite similar, we argue that
\[ I_2 = \int (\nabla \text{div} u_t + \partial_j (u_j \nabla \text{div} u)) \phi^2 dx \]
\[ = - \int \text{div} u_t \cdot \text{div} \dot{u} \cdot \phi^2 dx - 2 \int \text{div} u_t \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
- \int u_j \partial_i \text{div} u \cdot \partial_i \dot{u}_i \cdot \phi^2 dx - 2 \int u_j \partial_i \text{div} u \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
= - \int |\text{div} \dot{u}|^2 \phi^2 dx + \int \text{div} (u \cdot \nabla u) \text{div} \dot{u} \cdot \phi^2 dx \\
- 2 \int \text{div} \dot{u} \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx + 2 \int \text{div} (u \cdot \nabla u) \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
- \int u_j \partial_i \text{div} u \cdot \partial_i \dot{u}_i \cdot \phi^2 dx - 2 \int u_j \partial_i \text{div} u \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
= - \int |\text{div} \dot{u}|^2 \phi^2 dx + \int \partial_i u_k \cdot \partial_k u_i \cdot \text{div} \dot{u} \cdot \phi^2 dx + \int u_k \partial_k \text{div} u \cdot \text{div} \dot{u} \phi^2 dx \\
- 2 \int \text{div} \dot{u} \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx + 2 \int (\partial_j u_k \cdot \partial_k u_j + u_k \partial_k \text{div} u) \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
- \int u_j \partial_i \text{div} u \cdot \partial_i \dot{u}_i \cdot \phi^2 dx - 2 \int u_j \partial_i \text{div} u \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
= - \int |\text{div} \dot{u}|^2 \phi^2 dx - 2 \int \text{div} \dot{u} \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
+ \int \partial_i u_k \cdot \partial_k u_i \cdot \text{div} \dot{u} \cdot \phi^2 dx + 2 \int \partial_j u_k \cdot \partial_k u_j \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
+ \left( \int u_k \partial_k \text{div} u \cdot \text{div} \dot{u} \phi^2 dx - \int u_j \partial_i \text{div} u \cdot \partial_j \dot{u}_i \cdot \phi^2 dx \right) \\
+ 2 \int u_k \partial_k \text{div} u \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx - 2 \int u_j \partial_i \text{div} u \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \\
= - \int |\text{div} \dot{u}|^2 \phi^2 dx - 2 \int \text{div} \dot{u} \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx + K \\
- 2 \int u_k \text{div} u \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx + 2 \int u_j \text{div} u \cdot \partial_j \dot{u}_i \cdot \partial_i \phi \cdot \phi dx \\
+ 2 \int u_k \partial_k \text{div} u \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx - 2 \int u_j \partial_i \text{div} u \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \\
= - \int |\text{div} \dot{u}|^2 \phi^2 dx - 2 \int \text{div} \dot{u} \cdot \dot{u}_i \cdot \partial_i \phi \cdot \phi dx + K
\[
\leq - \int |\text{div}\,\hat{u}|^2 \phi^2 dx + C \int \rho|\dot{u}|^2 dx + \frac{\mu}{10\lambda} \int |\nabla \dot{u}|^2 \phi^2 dx + K \\
+ C \int |u| \cdot |\nabla u| \cdot |\nabla \dot{u}| \cdot |\nabla \phi| \phi dx + C \int |u| \cdot |\nabla^2 u| \cdot |\dot{u}| \cdot |\nabla \phi| \phi dx,
\]

where \( K \) is defined by (4.53) and in the last inequality we have used \( \rho > \theta/3 \) in \( B \cup \Gamma' \) and (4.41).

Finally, for \( I_3 \) we have
\[
I_3 = - \int (\nabla \rho_t + \partial_j (u_j \nabla \rho)) \cdot \dot{\phi}^2 dx \\
= \int \rho_t \cdot \text{div}\,\dot{\phi}^2 dx + 2 \int \rho_t \cdot \dot{u} \cdot \nabla \phi \cdot \phi dx \\
+ \int u_j \cdot \partial_i \rho \cdot \partial_j \dot{u}_i \dot{\phi}^2 dx + 2 \int u_j \cdot \partial_i \rho \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \\
= - \int \text{div}(\rho u) \cdot \text{div}\,\dot{\phi}^2 dx - 2 \int \text{div}(\rho u) \cdot \dot{u} \cdot \nabla \phi \cdot \phi dx \\
+ \int u_j \cdot \partial_i \rho \cdot \partial_j \dot{u}_i \dot{\phi}^2 dx + 2 \int u_j \cdot \partial_i \rho \cdot \dot{u}_i \cdot \partial_j \phi \cdot \phi dx \\
\leq C \|u\|^2 \|\nabla \rho\|^2 + \frac{\mu}{10} \int |\nabla \dot{u}|^2 \phi^2 dx + C \int (|\nabla u|^2 + \rho|\dot{u}|^2) dx \\
\leq C \|\nabla u\|^2 \|\nabla \rho\|^2 + \frac{\mu}{10} \int |\nabla \dot{u}|^2 \phi^2 dx + C \int (|\nabla u|^2 + \rho|\dot{u}|^2) dx.
\]

where in the last inequality we have used \( \rho > \theta/3 \) in \( B \cup \Gamma' \), (4.41), and (4.40).

In addition, we check that
\[
\int |u| \cdot |\nabla u| \cdot |\nabla \dot{u}| \cdot |\nabla \phi| \phi dx \\
\leq C \int |u|^2 |\nabla u|^2 dx + \frac{\mu}{10} \int |\nabla \dot{u}|^2 \phi^2 dx \\
\leq C \|u\|^2 \|\nabla u\|^2 + \frac{\mu}{10} \int |\nabla \dot{u}|^2 \phi^2 dx.
\]

By taking advantage of (4.2), (4.11) and (4.40), we also have
\[
\int |u| \cdot |\nabla^2 u| \cdot |\dot{u}| \cdot |\nabla \phi| \phi dx \\
\leq \int |u| \cdot |\nabla^2 u| \cdot |\dot{u}| \cdot |\nabla \phi| \phi dx + \int |u| \cdot |\nabla^2 v| \cdot |\dot{u}| \cdot |\nabla \phi| \phi dx \\
\leq C \int |\nabla^2 u|^2 dx + C \|u\|^2 \|\nabla v\|^2 + C \int \rho|\dot{u}|^2 dx \\
\]

\[
+ C \|u\|^2 \|\nabla v\|^2 + C \|u\|^2 \|\nabla \rho\|^2 \|\nabla u\|^2 + C \int \rho|\dot{u}|^2 dx,
\]

(4.54)
where in the last inequality we have applied (2.30) and (2.38) with respect to $\Gamma''_+ \subset \subset \Gamma'$.

We turn to the most difficult term $K$. Since the four terms in $K$ have similar structure, we may simply summarize $K$ as

$$K = \int \nabla u \cdot \nabla u \cdot \nabla \dot{u} \cdot \phi^2 \, dx + \int \nabla u \cdot \nabla u \cdot \dot{\phi} \cdot \phi \, dx$$

$$= K_1 + K_2.$$

For $K_1$, we apply (4.2) and (4.11) to get

$$|K_1| = \left| \int \nabla u \cdot \nabla u \cdot \nabla \dot{u} \cdot \phi^2 \, dx \right|$$

$$= \left| \int \nabla^2 u \cdot \nabla u \cdot \dot{u} \cdot \phi^2 \, dx + 2 \int \nabla u \cdot \nabla u \cdot \dot{\phi} \cdot \phi \, dx \right|$$

$$\leq \|\nabla u\|_{L^\infty(\Gamma')} \int |\nabla^2 u|^2 \cdot \phi^2 \, dx + C \|\nabla u\|_{L^\infty(\Gamma')} \int \rho |\dot{u}|^2 \cdot \phi^2 \, dx$$

$$+ C \|\nabla u\|_{L^\infty(\Gamma')} \|\nabla u\|_{L^2}^2. \quad (4.55)$$

For the first term in (4.55), we have

$$\int |\nabla^2 u|^2 \phi^2 \, dx$$

$$\leq \int |\nabla^2 w|^2 \phi^2 \, dx + \|\nabla u\|_{L^\infty(\Gamma')} \int |\nabla^2 u|^2 \phi^2 \, dx$$

$$\leq C \int \rho |\dot{u}|^2 \phi^2 \, dx + C \left( \|\nabla w\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2 + \|\nabla \rho\|_{L^2(\Gamma')}^2 \right),$$

where in the last inequality we have used Lemma 4.4 and applied (2.36) to $\Gamma''_+ \subset \subset \Gamma'$.

Since $K_2$ is estimated by (4.55), we therefore declare

$$|K| \leq C \|\nabla u\|_{L^\infty(\Gamma')} \int \rho |\dot{u}|^2 \phi^2 \, dx$$

$$+ C \|\nabla u\|_{L^\infty(\Gamma')} \left( \|\nabla u\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2 + \|\nabla \rho\|_{L^2(\Gamma')}^2 \right). \quad (4.56)$$

Substituting (4.51) - (4.54) and (4.56) into (4.50), we make use of (4.22) and Theorem 3.1 to get

$$\frac{d}{dt} \int \rho |\dot{u}|^2 \phi^2 \, dx + \int |\nabla \dot{u}|^2 \phi^2 \, dx$$

$$\leq C \left( \|\nabla u\|_{L^\infty(\Gamma')} + \|u\|_{L^\infty}^2 \right) \int \rho |\dot{u}|^2 \phi^2 \, dx$$

$$+ C \left( \int \rho |\dot{u}|^2 \, dx + \|u\|_{L^\infty}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^\infty(\Gamma')}^2 \right). \quad (4.57)$$

Moreover, we apply (1.2), elliptic estimate (2.6), Sobolev’s imbedding inequality, Theorems 3.1 and 3.2 to obtain

$$\int_0^T \|u\|_{L^\infty}^2 \, dt \leq C \int_0^T \|u\|^2_{H^2} + \|v\|^2_{W^{1,4}} \, dt$$

$$\leq C \int_0^T \|\rho \dot{u}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^4}^2 \, dt \leq C. \quad (4.58)$$

Combining (4.28), (4.22), (4.58), Theorems 3.1 and 3.2, we apply Gronwall’s inequality in (4.57) to deduce

$$\sup_{0 \leq t \leq T} \int \rho |\dot{u}|^2 \phi^2 \, dx + \int_0^T \int |\nabla \dot{u}|^2 \phi^2 \, dx \leq C,$$

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where the initial value of $\sqrt{\rho_0} \cdot \dot{u} \cdot \phi$ can be taken as

$$\left. \sqrt{\rho_0} \cdot \phi \right|_{t=0} = \frac{\phi(x,0)}{\sqrt{\rho_0}} (\mu \Delta u_0 + \lambda \nabla \text{div} u_0 - \nabla \rho_0),$$

due to $\rho_0 \geq \theta/3$ in $B \cup \Gamma'$. The proof of Theorem 4.2 is finished.

### 4.1.4 Estimate in $H^{2,2}$ layer $\Gamma''$

Now, we are in a position to close the estimate on $\|\nabla^2 \rho\|_{L^2(\Gamma'')}$. We declare that

**Lemma 4.5.** Under the conditions of Theorem 4.1, it holds

$$\sup_{0 \leq t \leq T} \int_{\Gamma''} |\nabla^2 \rho|^2 d\nu \leq C. \quad (4.59)$$

Before we prove Lemma 4.5, some elementary calculations which will be used in next few subsections must be done at first.

Note that we are in the Lagrangian coordinates and all derivatives are taken with respect to $y$ coordinate, say $\partial_i = \partial_{y_i}$. Moreover, $\partial_i \circ D = D \circ \partial_i$, since $D = \partial_t$ in $y$ coordinates.

Taking two more order covariant derivatives on (1.1), we arrive at

$$(D_t \rho,_{i,j}) = -\rho,_{i,j} u^k,_{k,j} - \rho,_{i} u^k,_{k,j} - \rho,_{j} u^k,_{k,i} - \rho u^k,_{kij}. \quad (4.60)$$

However, the definition of covariant derivative (2.12) gives

$$(D_t \rho,_{i,j}) = \partial_j D_t \rho,_{i} - \Gamma^l_{ji} D_t \rho,_{l} = D_t (\partial_j \rho,_{i} - \Gamma^l_{ji} \rho,_{l}) + D_t \Gamma^l_{ji} \rho,_{l} = D_t (\rho,_{ij}) + u^l,_{ji} \rho,_{l},$$

where we have used Lemma 2.8 in the last equality. Thus (4.60) can be transformed into

$$(D_t \rho,_{ij}) = -u^l,_{ji} \rho,_{l} - \rho,_{ij} u^k,_{k,j} - \rho,_{i} u^k,_{k,j} - \rho,_{j} u^k,_{k,i} - \rho u^k,_{kij}. \quad (4.61)$$

Next, recall that $v$ solves (2.4). In Lagrangian coordinates, it reads

$$\mu g^{mn} v,_{k,mn} + \lambda v,_{s,k} = \rho,_{k}. \quad (4.62)$$

Equivalently, we have

$$\mu g^{mn} (v,_{k,m} - v,_{m,k}),_{n} + ((\mu + \lambda) v,_{s} - \rho),_{k} = 0. \quad (4.63)$$

We take one more order covariant derivative on (4.63) and get

$$\mu g^{ik} g^{mn} (v,_{k,m} - v,_{m,k}),_{ni} + g^{ik} ((\mu + \lambda) v,_{s} - \rho),_{ki} = 0.$$

However, $g^{ik} g^{mn} (v,_{k,m} - v,_{m,k}),_{ni} = 0$. We conclude that

$$g^{ik} ((\mu + \lambda) v,_{s} - \rho),_{ki} = 0, \quad (4.64)$$

which means $F_v = (\mu + \lambda) v,_{s} - (\rho - \bar{\rho})$ is harmonic (by the definition in Chapter 6 of [43]) in Lagrangian coordinate as well (of course). Therefore, the interior estimate (2.39) is available for $F_v$. 

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Moreover, we rewrite (4.62) as
\[ \mu \Delta v + \lambda d(\text{div} v) = d\rho. \]
Equivalently, we have
\[ \mu (d \ast d \ast v + d \ast d v) + \lambda d(\text{div} v) = d\rho, \tag{4.65} \]
where \( d \) is the exterior differential operator, \( \ast \) is the Hodge \( \ast \)-operator, \( v \) is viewed as a differential 1-form respectively (all definitions can be found in [43, Chapter 2 \& Chapter 6]).

Operating exterior differential operator on (4.65) and noticing that \( d \circ d = 0 \) (see [43, Chapter 2]), we arrive at
\[ \mu d \ast d \ast dv = 0. \]
Equivalently, we have
\[ \mu (d \ast d \ast + d \ast d)dv = 0, \]
which means the differential 2-form \( \text{rot} v = dv \) is harmonic (see [43, Chapter 6]) as well. Consequently, the interior estimate (2.39) is also true for \( \text{rot} v \). Note that in local coordinates, we have
\[ \text{rot} v_{ij} = \frac{1}{2} (v_{i,j} - v_{j,i}). \]

After these preparations, we can start the proof of Lemma 4.5.

**Proof of Lemma 4.5.** By taking advantage of (4.61), standard calculations which will be repeatedly used in Subsection 4.2 show that
\[
\frac{d}{dt} \int_{\Gamma''} g^{ij} g^{kl} \rho_{ik} \rho_{jkl} d\nu
\]
\[
= 2 \int_{\Gamma''} g^{ij} g^{kl} D_t (\rho_{ik}) \rho_{jkl} d\nu + 2 \int_{\Gamma''} D_t (g^{ij}) g^{kl} \rho_{ik} \rho_{jkl} d\nu + \int_{\Gamma''} g^{ij} g^{kl} \rho_{iik} \rho_{jkl} u_s^s d\nu
\]
\[
= -2 \int_{\Gamma''} g^{ij} g^{kl} \left( (u_{ki}^s \rho_{s} + \rho_i u^s_{sk} + \rho_k u^s_{si}) + \rho_{ik} u^s_{s} + \rho_{ik} u^s_{sl} \right) \rho_{jkl} d\nu
\]
\[
+ 2 \int_{\Gamma''} D_t (g^{ij}) g^{kl} \rho_{ik} \rho_{jkl} u_s^s d\nu
\]
\[
= -2 \int_{\Gamma''} g^{ij} g^{kl} \left( (u_{ki}^s \rho_{s} + \rho_i u^s_{sk} + \rho_k u^s_{si}) + \rho_{ik} u^s_{s} + \rho_{ik} u^s_{sl} \right) \rho_{jkl} d\nu
\]
\[
+ \int_{\Gamma''} (2 D_t (g^{ij}) g^{kl} \rho_{ik} \rho_{jkl} - g^{ij} g^{kl} \rho_{ik} \rho_{jkl} u_s^s) d\nu - \int_{\Gamma''} g^{ij} g^{kl} \rho_{iik} \rho_{jkl} u_s^s d\nu
\]
\[
= 2 J_1 + J_2 - J_3.
\]

According to Lemma 2.3, the first and the second terms in (4.66) are bounded by
\[
|J_1| = \left| \int_{\Gamma''} g^{ij} g^{kl} \left( u_{ki}^s \rho_{s} + \rho_i u^s_{sk} + \rho_k u^s_{si} \right) \rho_{jkl} d\nu \right|
\]
\[
\leq \| \nabla u \|_{L^1(\Gamma'')} \| \nabla \rho \|_{L^1(\Gamma'')} \| \nabla^2 \rho \|_{L^2(\Gamma'')},
\tag{4.67}
\]
and
\[
|J_2| = \left| \int_{\Gamma''} (2 D_t (g^{ij}) g^{kl} \rho_{ik} \rho_{jkl} - g^{ij} g^{kl} \rho_{ik} \rho_{jkl} u_s^s) d\nu \right|
\]
\[
\leq \| \nabla u \|_{L^\infty(\Gamma'')} \int_{\Gamma''} |\nabla^2 \rho|^2 d\nu.
\tag{4.68}
\]

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For the principal term in (4.66), we have
\[ J_3 = \int_{\Gamma'} g^{ij} g^{kl} \rho u_s^i \rho_j^l d\nu \]
\[ = \int_{\Gamma'} g^{ij} g^{kl} \rho u_s^i \rho_j^l d\nu + \int_{\Gamma'} g^{ij} g^{kl} \rho v_s^i \rho_j^l d\nu \]
\[ = \int_{\Gamma'} g^{ij} g^{kl} \rho w_s^i \rho_j^l d\nu \]
\[ + \frac{1}{\mu + \lambda} \int_{\Gamma'} g^{ij} g^{kl} \rho \left( (\mu + \lambda) v_s^i - (\rho - \bar{\rho}) \right) \rho_j^l d\nu \]
\[ = J_{3,1} + J_{3,2} + J_{3,3} \]  \hspace{1cm} (4.69)

Note that the first term in (4.69) reads
\[ |J_{3,1}| = \left| \int_{\Gamma'} g^{ij} g^{kl} \rho w_s^i \rho_j^l d\nu \right| \]
\[ \leq C \| \nabla^3 w \|_{L^2(\Gamma')} \| \nabla^2 \rho \|_{L^2(\Gamma')} \]
\[ \leq C \| \nabla^3 w \|_{L^2(\Gamma')}^2 + \frac{\mu + \lambda}{10} \int_{\Gamma'} |\nabla^2 \rho|^2 d\nu. \]  \hspace{1cm} (4.70)

Interior estimate (2.39) and elliptic one (2.6) also make sure the second term in (4.69) can be controlled by
\[ |J_{3,2}| = \left| \frac{1}{\mu + \lambda} \int_{\Gamma'} g^{ij} g^{kl} \rho \left( (\mu + \lambda) v_s^i - (\rho - \bar{\rho}) \right) \rho_j^l d\nu \right| \]
\[ \leq C K \| F_v \|_{L^2(\Gamma')} \| \nabla^2 \rho \|_{L^2(\Gamma')} \]
\[ \leq C K \| \rho - \bar{\rho} \|_{L^2(\Gamma')} \| \nabla^2 \rho \|_{L^2(\Gamma')} \].  \hspace{1cm} (4.71)

Since \( J_{3,3,3} \geq 0 \), combining (4.2), (4.67), (4.68), (4.70) and (4.71) leads to
\[ \frac{d}{dt} \int_{\Gamma'} g^{ij} g^{kl} \rho u_s^i \rho_j^l d\nu + \int_{\Gamma'} g^{ij} g^{kl} \rho v_s^i \rho_j^l d\nu \]
\[ \leq C \| \nabla u \|_{L^\infty(\Gamma')} \int_{\Gamma'} |\nabla^2 \rho|^2 d\nu + C \| \nabla^2 u \|_{L^4(\Gamma')} \| \nabla \rho \|_{L^4(\Gamma')} \| \nabla^2 \rho \|_{L^2(\Gamma')} \]
\[ + C \| \nabla^3 w \|_{L^2(\Gamma')}^2 + C \| \rho - \bar{\rho} \|_{L^2} \| \nabla^2 \rho \|_{L^2(\Gamma')} \].  \hspace{1cm} (4.72)

Recall that \( w \) solves the elliptic system (2.3) in \( \Gamma' \), that is,
\[ \mu \Delta w + \lambda \nabla \text{div} w = \rho \dot{u} \text{ in } \Gamma' \].

We select a further sub-layer \( \Gamma'''' \) via
\[ \Gamma'''' = \{ x \in \Gamma' : \text{dist}(x, \Gamma_2') > \frac{d}{4} \} \].

Observe that
\[ \Gamma'''' \subset \subset \Gamma'''' \subset \subset \Gamma' \]
and the distance between them are strictly larger than \( \frac{d}{4} \) as well, say
\[ \text{dist}(\Gamma'', \partial \Gamma'') > \frac{d}{4} \].
Once again, by virtue of (4.22), interior estimate (2.38) in $\Gamma'' \subset \Gamma''_{++} \subset \Gamma'_{++}$, Sobolev’s imbedding theorem and (2.6), we obtain
\[
\|\nabla^3 w\|_{L^2(\Gamma'')} \leq C \left( \|\nabla (\rho \dot{u})\|_{L^2(\Gamma''_{++})} + \|\rho \nabla \dot{u}\|_{L^2(\Gamma''_{++})} + \|\rho \nabla \dot{u}\|_{L^2(\Gamma''_{++})} + \|\rho - \bar{\rho}\|_{L^2} \right) \tag{4.73}
\]
and
\[
\|\nabla^2 \rho\|_{L^4(\Gamma'')} \leq \|\nabla^2 w\|_{L^4(\Gamma'')} + \|\nabla^2 v\|_{L^4(\Gamma'')} 
\leq C \left( \|\rho \nabla \dot{u}\|_{L^4(\Gamma''_{++})} + \|\nabla \dot{u}\|_{L^2(\Gamma''_{++})} + \|\rho \nabla \dot{u}\|_{L^2(\Gamma''_{++})} + \|\rho - \bar{\rho}\|_{L^2} \right) \tag{4.74}
\]
\[
\sup_{0 \leq t \leq T} \int_{\Gamma''} |\nabla^2 \rho|^2 d\nu \leq C,
\]
which is (4.59) and finishes the proof of Lemma 4.5.

Lemmas 4.1, 4.2 and 4.5 finish the interior part.

4.2 Boundary part $B$

In the same spirit as Subsection 4.1.4, we turn to the boundary part $B$ of $H^{2,2}$ layer.

We now make the main result of this subsection as follows.

Lemma 4.6. Under the conditions of Theorem 4.1, it holds for $\psi$ as in (2.16),
\[
\int |\nabla^2 \rho|^2 \psi^2 d\nu \leq C \frac{\varepsilon}{\varepsilon_1} e^{2\varepsilon_1 t},
\]
which combined with Lemma 4.5 also gives
\[
\int_{B \cup \Gamma''} |\nabla^2 \rho|^2 d\nu \leq C \frac{\varepsilon}{\varepsilon_1} e^{2\varepsilon_1 t}. \tag{4.75}
\]

The proof will be divided into four parts. We deduce some directional estimates in subsection 4.2.1-4.2.3 and finish the proof of Lemma 4.6 in subsection 4.2.4.
4.2.1 Estimate of tangential direction

To begin with, we apply (4.61) to check tangential-tangential derivatives of $\rho$,

$$\frac{d}{dt} \int \gamma_{ij} \gamma_{kl} \rho_{ik} \rho_{jl} \psi^2 d\nu$$

$$= 2 \int \gamma_{ij} \gamma_{kl} D_t(\rho_{ik}) \rho_{jl} \psi^2 d\nu + 2 \int D_t(\gamma_{ij}) \gamma_{kl} \rho_{ik} \rho_{jl} \psi^2 d\nu$$

$$+ \int \gamma_{ij} \gamma_{kl} \rho_{ik} \rho_{jl} \psi^2 u_s^2 d\nu$$

$$= -2 \int \gamma_{ij} \gamma_{kl} (u^s_{ki} \rho_{s} + \rho_{i} u^s_{sk} + \rho_{k} u^s_{si}) \rho_{jl} \psi^2 d\nu$$

$$+ 2 \int D_t(\gamma_{ij}) \gamma_{kl} \rho_{ik} \rho_{jl} \psi^2 d\nu + \int \gamma_{ij} \gamma_{kl} \rho_{ik} \rho_{jl} \psi^2 u_s^2 d\nu$$

$$= \int (2D_t(\gamma_{ij}) \gamma_{kl} \rho_{ik} \rho_{jl} - \gamma_{ij} \gamma_{kl} \rho_{ik} \rho_{jl} u_s^2) \psi^2 d\nu$$

$$- 2 \int \gamma_{ij} \gamma_{kl} (u^s_{ki} \rho_{s} + \rho_{i} u^s_{sk} + \rho_{k} u^s_{si}) \rho_{jl} \psi^2 d\nu - \int \gamma_{ij} \gamma_{kl} \rho_{sik} \rho_{jl} \psi^2 d\nu$$

$$= P_1 + P_2 - P_3.$$  \(4.76\)

According to Lemma 2.8, the first and the second term in (4.76) can be controlled via

$$|P_1| = \left| \int (2D_t(\gamma_{ij}) \gamma_{kl} \rho_{ik} \rho_{jl} - \gamma_{ij} \gamma_{kl} \rho_{ik} \rho_{jl} u_s^2) \psi^2 d\nu \right|$$

$$\leq \|\nabla u\|_{L^\infty(B)} \int |\nabla^2 \rho|^2 \psi^2 d\nu,$$  \(4.77\)

and

$$|P_2| = \left| \int \gamma_{ij} \gamma_{kl} (u^s_{ki} \rho_{s} + \rho_{i} u^s_{sk} + \rho_{k} u^s_{si}) \rho_{jl} \psi^2 d\nu \right|$$

$$\leq C \left( \int_B |\nabla^2 u|^4 d\nu \right)^{\frac{1}{4}} \left( \int_B |\nabla \rho|^4 d\nu \right)^{\frac{1}{4}} \left( \int_B |\nabla^2 \rho|^2 \psi^2 d\nu \right)^{\frac{1}{2}}.$$  \(4.77\)

We turn to the principal term in (4.76),

$$P_3 = \int \gamma_{ij} \gamma_{kl} \rho u_{sik} \rho_{jl} \psi^2 d\nu$$

$$= \int \gamma_{ij} \gamma_{kl} \rho u_{sik} \rho_{jl} \psi^2 d\nu + \int \gamma_{ij} \gamma_{kl} \rho u_{sik} \rho_{jl} \psi^2 d\nu.$$  \(4.78\)

The first term in (4.78) reads

$$\left| \int \gamma_{ij} \gamma_{kl} \rho u_{sik} \rho_{jl} \psi^2 d\nu \right| \leq C \|\nabla^3 w\|_{L^2(B)} \left( \int_B |\nabla^2 \rho|^2 \psi^2 d\nu \right)^{\frac{1}{2}}$$

$$\leq \frac{C}{\delta} \|\nabla^3 w\|_{L^2(B)} + \delta \int |\nabla^2 \rho|^2 \psi^2 d\nu,$$

where $\delta < 1$ will be determined later.

The second term in (4.78) reads

$$\int \gamma_{ij} \gamma_{kl} \rho u_{sik} \rho_{jl} \psi^2 d\nu$$

$$= \int \gamma_{ij} \gamma_{kl} (\rho - \bar{\rho}) u_{sik} \rho_{jl} \psi^2 d\nu + \int \gamma_{ij} \gamma_{kl} \bar{\rho} u_{sik} \rho_{jl} \psi^2 d\nu$$

$$= Q_1 + Q_2.$$  \(4.79\)
Noticing that for $\psi$ as in (2.16), it follows from (2.17), (4.3), and Theorem 3.3 that
\[
\sup_{0 \leq t \leq T} |\nabla \psi| \leq C \exp\left(\int_0^T \|\nabla u\|_{L^\infty(B)} dt\right)
\leq C \exp\left(\int_0^T \|\nabla w\|_{L^\infty(B)} + \|\nabla v\|_{L^\infty(B)} dt\right) \leq C,
\] (4.80)
we estimate the first term in (4.79) as follows
\[
|Q_1| = \left| \int \gamma^{ij} \gamma^{kl} (\rho - \bar{\rho}) v_{sik}^s v_{jli} \psi^2 dv \right|
\leq \|\rho - \bar{\rho}\|_{L^\infty(B)} \int |\nabla^3 v|^2 \psi^2 dv + \|\rho - \bar{\rho}\|_{L^\infty(B)} \int |\nabla^2 \rho|^2 \psi^2 dv.
\] (4.81)

Observe that by definition (2.16),
\[
\int |\nabla^3 v|^2 \psi^2 dv
\leq \int_A |\nabla^3 v|^2 dv - \int_{B \setminus A} |\nabla^3 v|^2 dv + C \int |\nabla^2 \rho|^2 \psi^2 dv + C \left( \|\nabla^3 v\|_{L^2(B')} + \|\nabla^2 \rho\|_{L^2(B)} + \|\rho - \bar{\rho}\|_{L^2(B)} \right),
\] (4.82)
where we have applied (2.37) in $A \subset \subset B$ and $B \setminus A \subset \subset \Gamma''$ respectively in the last inequality.

Using (4.22) with (4.60), we substitute (4.82) into (4.81) to declare
\[
|Q_1| \leq C \|\rho - \bar{\rho}\|_{L^\infty(B)} \int |\nabla^2 \rho|^2 \psi^2 dv + C \|\rho - \bar{\rho}\|_{L^\infty(B)}.
\] (4.83)

Let us focus on the most difficult term $Q_2$.

First, we take two more order covariant derivatives on (4.62) and arrive at
\[
\mu g_{mn} v_{k,mi}^j + \lambda v_{skij}^s = \rho_{kij}.
\] (4.84)

Multiplying (4.84) by $v_{ls}^k \gamma^{ls} \gamma^{js}$ gives
\[
\mu g_{mn} v_{k,mi}^j v_{ls}^k \gamma^{ls} \gamma^{js} + \lambda v_{skij}^s v_{ls}^k \gamma^{ls} \gamma^{js} = \rho_{kij} v_{ls}^k \gamma^{ls} \gamma^{js},
\]
which means $Q_2$ in (4.79) can be transformed into
\[
Q_2 = \bar{\rho} \int \gamma^{ij} \gamma^{kl} v_{sik}^s v_{jli} \psi^2 dv
= \bar{\rho} \int \gamma^{ij} \gamma^{kl} (v_{sik}^s v_{jli} + v_{sik}^s v_{jli}) \psi^2 dv
- \bar{\rho} \left( \int \mu g_{mn} v_{s,mi}^j v_{sk}^s \gamma^{ij} \gamma^{kl} + \lambda v_{msij}^s v_{sk}^s \gamma^{ij} \gamma^{kl} \right) \psi^2 dv
= \bar{\rho} I_1 + \bar{\rho} I_2.
\]

Observe that
\[
I_1 = \int (\gamma^{ij} \gamma^{kl} v_{sik}^s v_{jli} \psi^2),_s dv - \int (\gamma^{ij} \gamma^{kl} + \gamma^{ij} \gamma^{kl}) v_{sik}^s v_{jli} \psi^2 dv
- 2 \int \gamma^{ij} \gamma^{kl} v_{sik}^s v_{jli} \psi^2 dv
= I_{1,1} + I_{1,2} + I_{1,3}.
\] (4.85)
By Lemma 2.9 for $I_{1,2}$, we have
\[ |I_{1,2}| = \left| \int (\gamma^{ij}_{\alpha} \gamma^{kl}_{\alpha} + \gamma^{ij}_{\alpha} \gamma^{kl}_{\alpha}) v_{\alpha}^{i,k} \rho_{j,l} \psi^2 d\nu \right| \]
\[ \leq CK \left( \int_{\Omega} |\nabla^2 v|^2 d\nu \right) \frac{1}{2} \left( \int |\nabla^2 \rho|^2 \psi^2 d\nu \right) \frac{1}{2}. \tag{4.86} \]

With the help of (4.80), $I_{1,3}$ reads
\[ |I_{1,3}| = \left| \int (\gamma^{ij}_{\alpha} \gamma^{kl}_{\alpha} v_{\alpha}^{i,k} \rho_{j,l} \psi^2 s) d\nu \right| \]
\[ \leq C \left( \int_{\Omega} |\nabla^2 v|^2 |\nabla \psi|^2 d\nu \right) \frac{1}{2} \left( \int |\nabla^2 \rho|^2 \psi^2 d\nu \right) \frac{1}{2} \]
\[ \leq C \left( \int_{\Omega} |\nabla^2 v|^2 d\nu \right) \frac{1}{2} \left( \int |\nabla^2 \rho|^2 \psi^2 d\nu \right) \frac{1}{2}. \tag{4.87} \]

The boundary term in (4.85) requires more efforts. We apply Stokes’ theorem (\[43\] Chapter 4) in $\Omega$ to show
\[ I_{1,1} = \int_{\Omega} (\gamma^{ij}_{\alpha} \gamma^{kl}_{\alpha} v_{\alpha}^{i,k} \rho_{j,l} \psi^2 s) d\nu \]
\[ = \int_{\partial \Omega} \gamma^{kl}_{\alpha} \rho_{j,l} N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} d\nu \]
\[ = -\int_{\Omega} \gamma^{kl}_{\alpha} \rho_{j,l} N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} d\nu, \tag{4.88} \]

where in the last equality we have used the following fact
\[ v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} = (v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} = -v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} \gamma^{kl}_{\alpha}. \]

In fact, Dirichlet boundary condition guarantees $v = 0$ on $\partial \Omega$, which enforces $\gamma^{ij}_{\alpha} v_{\alpha}^{i,k} = 0$ as well since it is the derivative with respect to tangential direction.

Then, we check that
\[ \int_{\partial \Omega} \gamma^{kl}_{\alpha} \rho_{j,l} N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} d\nu = \int_{\partial \Omega} \gamma^{kl}_{\alpha} \rho_{j,l} N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} \partial_{\nu} d\nu \]
\[ = -\int_{\partial \Omega} \gamma^{kl}_{\alpha} \rho_{j,l} \left( N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} - N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} \right) \partial_{\nu} d\nu \]
\[ = -\int_{\partial \Omega} \gamma^{kl}_{\alpha} \rho_{j,l} \left( N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} - N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} \right) \partial_{\nu} d\nu, \tag{4.89} \]

due to Stokes’ Theorem (see [43] Chapter 4) on the boundary $\partial \Omega$.

Combining (4.87) and (4.88) yields
\[ I_{1,1} = \int_{\partial \Omega} \left( \gamma^{kl}_{\alpha} \rho_{j,l} N_{\alpha} v_{\alpha}^{i,k} \gamma^{ij}_{\alpha} \right) \partial_{\nu} d\nu, \]

which will be handled by shifting back to Euler coordinates as follows
\[ I_{1,1} = \int_{\partial \Omega} (\delta^{kl} - N^{k} N^{l}) \partial_{j} \rho \cdot \partial_{i} N_{\alpha} \cdot \partial_{i} v_{\alpha} \cdot \partial_{k} (\delta^{ij} - N^{i} N^{j}) dS \]
\[ + \int_{\partial \Omega} (\delta^{kl} - N^{k} N^{l}) \partial_{j} \rho \cdot N_{\alpha} \cdot \partial_{i} v_{\alpha} \cdot \partial_{k} (\delta^{ij} - N^{i} N^{j}) dS \]
\[ + \int_{\partial \Omega} (\delta^{kl} - N^{k} N^{l}) \partial_{j} \rho \cdot N_{\alpha} \cdot \partial_{i} v_{\alpha} \cdot \partial_{k} (\delta^{ij} - N^{i} N^{j}) dS, \]

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where \( \partial_t = \frac{\partial}{\partial t} \) is the usual partial derivatives in Euler coordinates, \( N \) is the unit normal vector on \( \partial \Omega \) in Euler coordinate and \( dS \) is the volume element on the boundary \( \partial \Omega \).

The Non-slip boundary condition makes sure \( \partial \Omega \) is fixed throughout the time, thus \( N \) is fixed as well. We apply Lemma 2.9 to deduce

\[
\left| \int_{\partial \Omega} (\delta^{ij} - N^i N^j) \partial_j \rho \cdot \partial_i N_s \cdot \partial_t v^s \cdot \partial_k (\delta^{ij} - N^i N^j) dS \right|
+ \left| \int_{\partial \Omega} (\delta^{ij} - N^i N^j) \partial_j \rho \cdot N_s \cdot \partial_t v^s \cdot \partial_k (\delta^{ij} - N^i N^j) dS \right|
+ \left| \int_{\partial \Omega} (\delta^{ij} - N^i N^j) \partial_j \rho \cdot N_s \cdot \partial_t v^s \cdot \partial_k (\delta^{ij} - N^i N^j) dS \right|
\leq C \int_{\partial \Omega} |\nabla \rho|(|\nabla v| + |\nabla^2 v|) dS
\leq C \int_{\partial \Omega} (|\nabla \rho|^2 + |\nabla^2 v|^2 + |\nabla v|^2) dS,
\]

which finishes the estimate on \( I_1 \).

The situation for \( I_2 \) is quite similar. We rewrite \( I_2 \) as

\[
I_2 = \int \left( \mu^{mn} v_{s,mnjl} v^s_{ik} \gamma^{ij} \gamma^{kl} + \lambda v_{mnjl} v^s_{ik} \gamma^{ij} \gamma^{kl} \right) \psi^2 d\nu
= \int \mu^{mn} (v_{s,mnjl} - v_{s,nmlj}) v^s_{ik} \gamma^{ij} \gamma^{kl} \psi^2 d\nu
+ (\mu + \lambda) \int (v^s_{mnjl} v^s_{ik}) \gamma^{ij} \gamma^{kl} \psi^2 d\nu \equiv J_1 + J_2.
\]

For \( J_1 \), we compute that

\[
J_1 = \int \mu^{mn} (v_{s,m} - v_{s,m}) \eta j v^s_{ik} \gamma^{ij} \gamma^{kl} \psi^2 d\nu
= \mu (v_{s,m} - v_{s,m}) \eta j v^s_{ik} \gamma^{ij} \gamma^{kl} \psi^2 d\nu
- \frac{1}{2} \int \mu^{mn} g^{ts} (v_{s,m} - v_{s,m}) j t (v_{s,m} - v_{s,m}) \eta k \gamma^{ij} \gamma^{kl} \psi^2 d\nu
+ \int \mu N^{mn} (v_{s,m} - v_{s,m}) \eta j v^s_{ik} \gamma^{ij} \gamma^{kl} d\nu
- \frac{1}{2} \int \mu N^{mn} (v_{s,m} - v_{s,m}) \eta j v^s_{ik} \gamma^{ij} \gamma^{kl} d\nu
- \frac{1}{2} \int \mu N^{mn} (v_{s,m} - v_{s,m}) \eta j v^s_{ik} \gamma^{ij} \gamma^{kl} \psi^2 d\nu
\]
We check in details that
\[ S_1 = \int_{\partial \Omega} \mu N^m(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s} \psi^2 \, d\nu \]
\[ = - \int_{\partial \Omega} \mu N^m(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s} \psi^2 \, d\nu \]
\[ = - \int_{\partial \Omega} \mu \gamma^{kl}(N^m(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s})_{kl} \psi^2 \, d\nu \]
\[ + \int_{\partial \Omega} \mu \gamma^{kl} N^m(v_{s,m} - v_{m,s})_{ij} (v_{i,k}^{s} \gamma_{ij,k}^{s} + v_{i,k}^{s} \gamma_{ij,k}^{s}) \psi^2 \, d\nu \]
\[ = \int_{\partial \Omega} \mu \gamma^{kl} N^m(v_{s,m} - v_{m,s})_{ij} (v_{i,k}^{s} \gamma_{ij,k}^{s} + v_{i,k}^{s} \gamma_{ij,k}^{s}) \psi^2 \, d\nu, \]
where we have applied Dirichlet condition in the second line and Stokes’ theorem ([3] Chapter 6) in the last line.

With the help of Lemma 2.9
\[ |S_1| \leq C \int_{\partial \Omega} |\nabla^2 v| \cdot (|\nabla v| + |\nabla^2 v|) \, d\nu \leq C \int_{\partial \Omega} (|\nabla^2 v|^2 + |\nabla v|^2) \, dS. \] (4.90)

\( S_2 \) is more complicated. We write
\[ S_2 = \int \mu g^{mn}(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} (\gamma_{ij,k}^{s} \psi^2 + \gamma_{ij,k}^{s} \psi^2 + 2 \gamma_{ij,k}^{s} \psi^2) \, d\nu \]
\[ = S_{2,1} + S_{2,2} + S_{2,3}. \]

For \( S_{2,1} \), we decompose it further as
\[ S_{2,1} = \int_{A} \mu g^{mn}(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s} \psi^2 \, d\nu \]
\[ = \int_{A} \mu g^{mn}(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s} \psi^2 \, d\nu \]
\[ + \int_{B \setminus A} \mu g^{mn}(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s} \psi^2 \, d\nu \]
\[ = S_{2,1,1} + S_{2,1,2}. \]

For \( S_{2,1,1} \), we apply (2.36) (2.37) and Lemma 2.9 to deduce
\[ |S_{2,1,1}| = \left| \int_{A} \mu g^{mn}(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s} \psi^2 \, d\nu \right| \]
\[ \leq C \int_{A} |\nabla^2 v| \cdot |\nabla^2 v| \, d\nu \]
\[ \leq C \left( \int_{A} |\nabla^2 v|^2 \, d\nu \right)^{\frac{1}{2}} \left( \int_{A} |\nabla^2 v|^2 \, d\nu \right)^{\frac{1}{2}} \]
\[ \leq C \left( \left( \int |\nabla^2 \rho|^2 \psi^2 \, d\nu \right)^{\frac{1}{2}} + \|\rho - \bar{\rho}\|_{L^2} \right) \|\nabla^2 v\|_{L^2(B)} \]
\[ \leq C \left( \int |\nabla^2 \rho|^2 \psi^2 \, d\nu \right)^{\frac{1}{2}} \|\nabla^2 v\|_{L^2(B)} + \|\rho - \bar{\rho}\|_{L^2} \cdot \|\nabla^2 v\|_{L^2(B)}. \]

With the help of (2.39), (2.6) and Lemma 2.9, \( S_{2,1,2} \) is handled via
\[ |S_{2,1,2}| = \left| \int_{B \setminus A} \mu g^{mn}(v_{s,m} - v_{m,s})_{ij} v_{i,k}^{s} \gamma_{ij,k}^{s} \psi^2 \, d\nu \right| \]
\[ \leq C \|\nabla^2 \text{rot} v\|_{L^2(B \setminus A)} \|\nabla^2 v\|_{L^2(B)} \]
\[ \leq C \|\text{rot} v\|_{L^2} \|\nabla^2 v\|_{L^2(B)} \]
\[ \leq C \|\rho - \bar{\rho}\|_{L^2} \|\nabla^2 v\|_{L^2(B)}. \]
The same method can be applied to control $S_{2,2}$, thus we turn to $S_{2,3}$.

$$|S_{2,3}| = \left| \int \mu g^{mn}(v_{s,m} - v_{m,s})_{ji} v_{ik}^{s} \gamma_{ijkl}^{2} \gamma^{ij} \gamma^{kl} \psi \psi \nu \, dv \right|$$

$$\leq \frac{\mu}{10} \int g^{mn} g^{st}(v_{s,m} - v_{m,s})_{ji} (v_{l,n} - v_{n,l})_{ik} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv + C \int |\nabla^{2} v|^{2} |\nabla \psi|^{2} \nu \, dv$$

$$\leq \frac{\mu}{10} \int g^{mn} g^{st} \text{rot} v_{sm,ji} \text{rot} v_{lm,ik} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv + C \int_{B} |\nabla^{2} v|^{2} \nu \, dv,$$

where in the last line we have used (4.80).

$J_{2}$ is similar with $J_{1}$,

$$J_{2} = \int (\mu + \lambda) (v_{m,ijkl}^{s} v_{s}^{ik}) \gamma_{ijkl}^{2} \psi^{2} \nu \, dv$$

$$= \int (\mu + \lambda) (v_{m,ijkl}^{s} v_{s}^{ik}) \gamma_{ijkl}^{2} \psi^{2} \nu \, dv$$

$$- \int (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} (\gamma_{ijkl}^{2} \psi^{2}) \nu \, dv$$

$$- \int (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv$$

$$= J_{2,1} - J_{2,2} - \int (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv.$$

$J_{2,1}$ can be handled just in the same way as $S_{1}$:

$$|J_{2,1}| = \left| \int_{\partial \Omega} (\mu + \lambda) v_{m,ijkl}^{s} N_{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \nu \, dv \right|$$

$$\leq C \int_{\partial \Omega} |\nabla^{2} v| (|\nabla v| + |\nabla^{2} v|) \nu \, dv$$

(4.92)

For $J_{2,2}$, there is a slight modification.

$$J_{2,2} = \int (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} (\gamma_{ijkl}^{2} \psi^{2} + \gamma_{ijkl}^{2} \psi^{2} + \gamma_{ijkl}^{2} \psi^{2}) \nu \, dv$$

$$= J_{2,2,1} + J_{2,2,2} + J_{2,2,3}.$$

For $J_{2,2,1}$, the situation is similar with $S_{2,1}$.

$$J_{2,2,1} = \int (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv$$

$$= \int_{A} (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv + \int_{B \setminus A} (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv$$

$$= \int_{A} (\mu + \lambda) v_{m,ijkl}^{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv + \int_{B \setminus A} \rho_{ijkl}^{s} v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv$$

(4.93)

$$+ \int_{B \setminus A} ((\mu + \lambda) v_{m,ijkl}^{s} - \rho) v_{s,ik}^{s} \gamma_{ijkl}^{2} \psi^{2} \nu \, dv$$

$$= N_{1} + N_{2} + N_{3}.$$
Like (4.91), with the help of (2.36), (2.37) and Lemma 2.9, the first term in (4.93) reads

$$\int_A (\mu + \lambda) v_{mji} v_{sik} \gamma_{ij} \gamma_{kl} \psi^2 \, dv \leq CK \int |\nabla^2 \rho|^2 \psi^2 \, dv \| v \|_{L^2(B)} + CK \| \rho - \bar{\rho} \| \| \nabla^2 v \|_{L^2(B)}.$$  

Using interior estimate (2.39), (2.6) and Lemma 2.9, for the remaining terms in (4.93), we have

$$\int_{B \setminus A} \rho_{sji} v_{sik} \gamma_{ij} \gamma_{kl} \psi^2 \, dv \leq CK \int |\nabla^2 \rho|^2 \psi^2 \, dv \| v \|_{L^2(B)},$$

and

$$\int_{B \setminus A} (\mu + \lambda) v_{mji} v_{sik} \gamma_{ij} \gamma_{kl} \psi^2 \, dv \leq CK \| F_v \|_{L^2} \| \nabla^2 v \|_{L^2(B)} \leq CK \| \rho - \bar{\rho} \| \| \nabla^2 v \|_{L^2(B)}.$$  

Since $J_{2,2,2}$ is similar with $J_{2,2,1}$, we turn to $J_{2,2,3}$.

$$\int_{J_{2,2,3}} = \int (\mu + \lambda) v_{mji} v_{sik} \gamma_{ij} \gamma_{kl} \psi^2 \, dv \leq C \sum_{i,j} \gamma_{ij} \gamma_{kl} v_{mji} v_{sik} \psi^2 \, dv + CK \int_B |\nabla^2 \psi |^2 \, dv,$$

where in the last inequality we have used (4.80).

Combining (4.82), (4.77) − (4.83), (4.86) − (4.90), (4.91) − (4.92) and (4.94) − (4.95), we apply Theorem 3.1 to declare that

$$\frac{d}{dt} \int \gamma_{ij} \gamma_{kl} \rho_{i,j} \rho_{j,i} \psi^2 \, dv + \int (g^{mn} g^{ks} \text{rot} v_{ms,i} \text{rot} v_{ts,ik} + v_{mji} v_{sik}) \gamma_{ij} \gamma_{kl} \psi^2 \, dv \leq C \| \rho - \bar{\rho} \|_{L^\infty(B)} + \| \nabla u \|_{L^\infty(B)} + \delta \int |\nabla^2 \rho|^2 \psi^2 \, dv$$

$$+ \frac{C}{\delta} \left( \| \nabla^3 v \|_{L^2(B)}^2 + \| \nabla^2 v \|_{L^2(B)}^2 + \| v \|_{L^2(B)}^2 + \| \rho - \bar{\rho} \|_{L^\infty(B)} \right)$$

$$+ C \int_{\Omega} (|\nabla v|^2 + |\nabla^2 v|^2 + |\nabla \rho|^2) \, dv.$$
4.2.2 Estimate on mixed direction

Next, we apply (4.61) to calculate tangential-normal derivatives of $\rho$,

\[
\frac{d}{dt} \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu = \left( D_t \gamma^{ij} \right) N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu + 2 \int J^{ij} \left( D_t N^k \right) N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu + 2 \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu + \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 u_s^2 d\nu
\]

\[= \left( D_t \gamma^{ij} \right) N^k N^l + 2 \gamma^{ij} \left( D_t \gamma^{ij} \right) N^l - \gamma^{ij} N^k N^l u_s^2 \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu - 2 \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu - 2 \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu = J_1 + 2J_2 - 2J_3. \tag{4.97}
\]

Directly, the second term in (4.97) is bounded by

\[|J_2| = \left| \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu \right| \leq C \left( \int_B \rho^4 d\nu \right)^{\frac{1}{2}} \left( \int_B |\nabla^2 \rho|^2 d\nu \right)^{\frac{1}{2}}. \tag{4.98}
\]

And Lemma 2.28 makes sure the first term in (4.97) can be handled via

\[|J_1| = \left| \int \left( D_t \gamma^{ij} \right) N^k N^l + 2 \gamma^{ij} \left( D_t \gamma^{ij} \right) N^l - \gamma^{ij} N^k N^l u_s^2 \right| \rho_{,ik}\rho_{,jl}\psi^2 d\nu \leq C \|\nabla u\|_{L^\infty(B)} \int |\nabla^2 \rho|^2 \psi^2 d\nu.
\]

We focus on $J_3$ which is the principal term in (4.97),

\[J_3 = \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu = \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu + \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu = I_1 + I_2. \tag{4.99}
\]

For the first term in (4.99),

\[|I_1| = \left| \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu \right| \leq \frac{C}{\delta} \|\nabla^3 u\|_{L^2(B)}^2 + \delta \int |\nabla^2 \rho|^2 \psi^2 d\nu.
\]

As for the second term in (4.99),

\[I_2 = \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu = \frac{1}{\mu + \lambda} \int J^{ij} N^k N^l \rho((\mu + \lambda) u_s^2 - \rho)_{,ik}\rho_{,jl}\psi^2 d\nu + \frac{1}{\mu + \lambda} \int J^{ij} N^k N^l \rho_{,ik}\rho_{,jl}\psi^2 d\nu \tag{4.100}
\]

\[= K_1 + K_2.
\]
Note that from (4.63) we have
\[ \mu g^{mn}(v_{k,m} - v_{m,k})_{ni} = -((\mu + \lambda)v_s^s - \rho),_{ik}. \]

Therefore, the first term in (4.100) reads
\[
K_1 = \frac{1}{\mu + \lambda} \int \gamma^{ij} N^k N^l \rho((\mu + \lambda)v_s^s - \rho),_{ik}\rho_{jl}\psi^2 d\nu
\]
\[
= -\frac{\mu}{\mu + \lambda} \int \rho \gamma^{ij} N^k N^l g^{mn}(v_{k,m} - v_{m,k})_{ni}\rho_{jl}\psi^2 d\nu
\]
\[
= -\frac{\mu}{\mu + \lambda} \int \rho \gamma^{ij} N^k N^l (\gamma^{mn} + N^m N^n)(v_{k,m} - v_{m,k})_{ni}\rho_{jl}\psi^2 d\nu
\]
\[
= -\frac{\mu}{\mu + \lambda} \int \rho \gamma^{ij} N^k N^l \gamma^{mn}\rho\rho_{jl}\psi^2 d\nu,
\]
where in the fourth equality we have used
\[ N^k N^l N^m N^n(v_{k,m} - v_{m,k})_{ni} = 0. \]

Consequently,
\[
|K_1| \leq \left| \int \rho \gamma^{ij} N^k N^l \gamma^{mn}\rho\rho_{jl}\psi^2 d\nu \right|
\]
\[
\leq C \int \rho \gamma^{ij} N^k N^l \gamma^{mn}\rho\rho_{jl}\psi^2 d\nu
\]
\[
+ \frac{1}{10(\mu + \lambda)} \int \rho \gamma^{ij} N^k N^l \rho\rho_{jl}\psi^2 d\nu \quad (4.101)
\]
\[
\leq C \int g^{st} g^{mn} \gamma^{ij} \gamma^{kl}\rho\rho_{jl}\psi^2 d\nu
\]
\[
+ \frac{1}{10(\mu + \lambda)} \int \rho \gamma^{ij} N^k N^l \rho\rho_{jl}\psi^2 d\nu.
\]

In conclusion, by virtue of (4.2) and (4.98)–(4.101), we have for tangential-normal derivatives
\[
\frac{d}{dt} \int \gamma^{ij} N^k N^l \rho_{,ik}\rho_{jl}\psi^2 d\nu + \int \gamma^{ij} N^k N^l \rho_{,ik}\rho_{jl}\psi^2 d\nu
\]
\[
\leq C(\|\nabla u\|_{L^\infty(B)} + \delta) \int |\nabla^2 \rho|^2 \psi^2 d\nu + \frac{C}{\delta} \left( \|\nabla^3 w\|_{L^2(B)}^2 + \|\nabla^2 u\|_{L^2(B)}^2 \right) \quad (4.102)
\]
\[
+ C \int g^{st} g^{mn} \gamma^{ij} \gamma^{kl}\rho\rho_{jl}\psi^2 d\nu.
\]
4.2.3 Estimate on normal direction

Finally, with the help of (4.61), we turn to the normal-normal derivatives of $\rho$,\[
\frac{d}{dt} \int N^i N^j N^k N^l \rho_{,sk} \rho_{,jl} \psi^2 d\nu = 4 \int (D_t N^i) N^j N^k N^l \rho_{,sk} \rho_{,jl} \psi^2 d\nu + 2 \int N^i N^j N^k N^l (D_t \rho_{,sk}) \rho_{,jl} \psi^2 d\nu + \int N^i N^j N^k N^l \rho_{,sk} \rho_{,jl} \psi^2 u_s^i d\nu = \int (4(D_t N^i) N^j N^k N^l \rho_{,sk} \rho_{,jl} \psi^2 d\nu
\]

\[
= \left( 4(D_t N^i) N^j N^k N^l + 2\gamma^{ij} (D_t N^k) N^i - N^i N^j N^k N^l u_s^i \right) \rho_{,sk} \rho_{,jl} \psi^2 d\nu - 2 \int N^i N^j N^k N^l (u_{,sk}^i \rho_{,s} + \rho_{,sk} u_{,s}^i + \rho_{,k} u_{,s}^i) \rho_{,jl} \psi^2 d\nu - 2 \int N^i N^j N^k N^l \rho u_{,sk}^i \rho_{,jl} \psi^2 u_s^i d\nu \equiv J_1 + 2J_2 - 2J_3.
\]

According to Lemma 2.8, we have
\[
|J_1| \leq C \left\| \nabla u \right\|_{L^\infty}(B) \left( \int |\nabla^2 \rho|^2 \psi^2 d\nu \right),
\]

and
\[
|J_2| = \left| \int N^i N^j N^k N^l \left( u_{,sk}^i \rho_{,s} + \rho_{,sk} u_{,s}^i + \rho_{,k} u_{,s}^i \right) \rho_{,jl} \psi^2 d\nu \right| \\
\leq C \left( \int_B |\nabla^2 u|^2 d\nu \right)^{\frac{1}{2}} \left( \int_B |\nabla \rho|^2 d\nu \right)^{\frac{1}{2}} \left( \int |\nabla^2 \rho|^2 \psi^2 d\nu \right)^{\frac{1}{2}}.
\]

For $J_3$ which is the principal term in (4.103), we argue that
\[
J_3 = \int N^i N^j N^k N^l \rho u_{,sk}^i \rho_{,jl} \psi^2 d\nu = \int N^i N^j N^k N^l \rho u_{,sk}^i \rho_{,jl} \psi^2 d\nu + \int N^i N^j N^k N^l \rho u_{,sk}^i \rho_{,jl} \psi^2 d\nu = I_1 + I_2.
\]

Once again
\[
|I_1| = \left| \int N^i N^j N^k N^l \rho u_{,sk}^i \rho_{,jl} \psi^2 d\nu \right| \\
\leq \frac{C}{\delta} \left\| \nabla^3 u \right\|_{L^2(B)}^{2} + \delta \int |\nabla^2 \rho|^2 \psi^2 d\nu.
\]

$I_2$ requires some extra efforts
\[
\int N^i N^j N^k N^l \rho u_{,sk}^i \rho_{,jl} \psi^2 d\nu = \frac{1}{\mu + \lambda} \int N^i N^j N^k N^l (\mu + \lambda u_{,s}^i - \rho)_{,sk} \rho_{,jl} \psi^2 d\nu + \frac{1}{\mu + \lambda} \int N^i N^j N^k N^l \rho \rho_{,sk} \rho_{,jl} \psi^2 d\nu = K_1 + K_2.
\]

Note that $F_v$ is harmonic, which means
\[
g^{ik} ((\mu + \lambda) v_{,s}^i - \rho)_{,ki} = 0.
\]
According to (2.13), we actually have
\[ N^i N^k (\mu + \lambda) v^s_{j;i} = -\gamma^{ik} ((\mu + \lambda) v^s_{j;i} - \rho)_{,ki}. \] (4.106)

Therefore, we substitute (4.106) into the first term of (4.105) and arrive at
\[
K_1 = \frac{1}{\mu + \lambda} \int N^i N^j N^k N^l \rho ((\mu + \lambda) v^s_{j;i} - \rho)_{,ik} \rho_{,ji}^s \psi^2 d\nu
\]
\[= -\frac{1}{\mu + \lambda} \int \gamma^{ik} N^j N^l \rho ((\mu + \lambda) v^s_{j;i} - \rho)_{,ik} \rho_{,ji}^s \psi^2 d\nu
\]
\[= -\int \gamma^{ik} N^j N^l \rho v^s_{j;ik} \psi^2 d\nu + \frac{1}{\mu + \lambda} \int \gamma^{ik} N^j N^l \rho \delta_{,ik} \rho_{,ji}^s \psi^2 d\nu
\]
\[= K_{1,1} + K_{1,2}. \] (4.107)

Note that the first term in (4.107) reads
\[
|K_{1,1}| = \left| \int \gamma^{ik} N^j N^l \rho v^s_{j;ik} \psi^2 d\nu \right|
\]
\[\leq C \left( \int (\gamma^{ik} v^s_{j;ik})^2 \psi^2 d\nu + \frac{1}{10(\mu + \lambda)} \int N^i N^j N^k N^l \rho_{,ik} \rho_{,ji}^s d\nu \right)
\]
\[\leq C \int \gamma^{ij} \gamma^{kl} \rho_{,ik} \rho_{,ji}^s \psi^2 d\nu + \frac{1}{10(\mu + \lambda)} \int N^i N^j N^k N^l \rho_{,ik} \rho_{,ji}^s d\nu,
\] (4.108)

and the second term in (4.107) can be controlled by
\[
|K_{1,2}| = \left| \int \gamma^{ik} N^j N^l \rho v^s_{j;ik} \psi^2 d\nu \right|
\]
\[\leq C \left( \int (\gamma^{ik} v^s_{j;ik})^2 \psi^2 d\nu + \frac{1}{10(\mu + \lambda)} \int N^i N^j N^k N^l \rho_{,ik} \rho_{,ji}^s d\nu \right)
\]
\[\leq C \int \gamma^{ij} \gamma^{kl} \rho_{,ik} \rho_{,ji}^s \psi^2 d\nu + \frac{1}{10(\mu + \lambda)} \int N^i N^j N^k N^l \rho_{,ik} \rho_{,ji}^s d\nu.
\] (4.109)

We collect (4.12), (4.104), (4.105), (4.108) and (4.109) to declare
\[
\frac{d}{dt} \int N^i N^j N^k N^l \rho_{,ik} \rho_{,ji}^s d\nu + \int N^i N^j N^k N^l \rho_{,ik} \rho_{,ji}^s d\nu
\]
\[\leq C(\|\nabla u\|_{L}\cdot(B) + \delta) \int |\nabla^2 \rho|^2 \psi^2 d\nu + \frac{C}{\delta} \left( \|\nabla^3 u\|_{L}^2(B) + \|\nabla^2 u\|_{L}^2(B) \right)
\]
\[+ C \left( \int \gamma^{ij} \gamma^{kl} \rho_{,ik} \rho_{,ji}^s \psi^2 d\nu + \int \gamma^{ij} \gamma^{kl} \rho_{,ik} \rho_{,ji}^s \psi^2 d\nu \right).
\] (4.110)

4.2.4 Conclusion

We are in the position to close the estimate of $\nabla^2 \rho$ in boundary part $B$.

Actually, adding (4.96), (4.102) multiplied by $\frac{\delta}{m}$ and (4.110) multiplied by $\frac{\delta}{m}$ together, we arrive at
\[
\Phi'(t) \leq C(\|\rho - \bar{\rho}\|_{L}(B) + \|\nabla u\|_{L}(B) + \delta) \int |\nabla^2 \rho|^2 \psi^2 d\nu
\]
\[+ \frac{C}{\delta} \left( \|\nabla^3 u\|_{L}(B) + \|\nabla^2 u\|_{L}(B) + \|\nabla^2 u\|_{L}(B) + \|\rho - \bar{\rho}\|_{L}(B) \right)
\]
\[+ C \int_{\partial\Omega} (|\nabla v|^2 + |\nabla^2 v|^2 + |\nabla \rho|^2) d\nu + \epsilon_1 \int \gamma^{ij} \gamma^{kl} \rho_{,ik} \rho_{,ji}^s \psi^2 d\nu,
\]
where
\[
\Phi(t) \triangleq \int \left( \sum_{j=1}^{N_k} \sum_{i=1}^{N_k} \frac{\partial_i \rho_{ij} \rho_{ji}}{\rho} + \frac{1}{4C} \sum_{j=1}^{N_k} \sum_{i=1}^{N_k} \sum_{\ell=1}^{N_k} \sum_{k=1}^{N_k} \frac{\partial_{i\ell} \rho_{j\ell} \rho_{kj}}{\rho} + \frac{\varepsilon_1}{C} \sum_{j=1}^{N_k} \sum_{i=1}^{N_k} \sum_{\ell=1}^{N_k} \sum_{k=1}^{N_k} \frac{\partial_{i\ell} \rho_{j\ell} \rho_{kj}}{\rho} \right) \psi^2 d\nu.
\]

This means that
\[
\Phi'(t) \leq \left( \frac{\varepsilon_1}{\varepsilon_2} + \frac{C}{\varepsilon_1} \left( \|\rho - \bar{\rho}\|_{L^\infty(B)} + \|
abla u\|_{L^\infty(B)} + \delta \right) \right) \Phi(t)
\]
\[
+ \frac{C}{\delta} \left( \|\nabla^2 w\|_{L^2(B)}^2 + \|\nabla^2 u\|_{L^2(B)}^2 + \|\nabla^2 v\|_{L^2(B)}^2 + \|\rho - \bar{\rho}\|_{L^2(B)} \right)
\]
\[
+ C \int_{\partial\Omega} \left( |\nabla v|^2 + |\nabla^2 v|^2 + |\nabla^2 \rho|^2 \right) d\nu.
\]

(4.111)

Similar to (4.72) and (4.74), it follows from (4.38) and (4.42) that
\[
\|\nabla^3 w\|_{L^2(B)} \leq C \left( \|\nabla \hat{u}\|_{L^2(\Gamma^\nu)} + \|\rho \hat{u}\|_{L^2} + \|
abla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} \right).
\]

(4.112)

And elliptic estimate (2.6) yields
\[
\|\nabla^2 u\|_{H^1(B)} \leq \left( \|\nabla^2 \hat{u}\|_{H^1(B)} + \|\nabla^2 v\|_{H^1(B)} \right)
\]
\[
\leq C \left( \|\nabla \hat{u}\|_{L^2(\Gamma^\nu)} + \|\rho \hat{u}\|_{L^2} + \|
abla^2 v\|_{H^1(B)} \right).
\]

(4.113)

We select \( \delta \leq \varepsilon_2^2 / 2C \). By taking advantage of (4.13), (4.38), (4.112), (4.113), Theorems 3.1–3.3 and Gronwall’s inequality in (4.111), we get
\[
\Phi(t) \leq Ce^{\frac{C}{\varepsilon_1}} e^{2\varepsilon_1 t},
\]
which actually means
\[
\int |\nabla^2 \rho|^2 \psi^2 d\nu \leq Ce^{\frac{C}{\varepsilon_1}} e^{2\varepsilon_1 t}.
\]

Combining this with Lemma 4.5 gives
\[
\int_{B \cup \Gamma^\nu} |\nabla^2 \rho|^2 d\nu \leq Ce^{\frac{C}{\varepsilon_1}} e^{2\varepsilon_1 t},
\]
which finishes the proof of Lemma 4.6. \( \square \)

4.3 Finish the proof of Theorem 4.1

By virtue of Lemma 4.1 and Lemma 4.6 we deduce (4.4) and (4.5) via following process. Firstly, let us show that
\[
\|\rho - \bar{\rho}\|_{L^\infty(B)} + \|\nabla \rho\|_{L^1(B)} + \|
abla v\|_{L^\infty(B)}
\]
\[
+ \|
abla^2 v\|_{L^1(B)} + \int_{\partial\Omega} \left( |\nabla \rho|^2 + |\nabla v|^2 + |\nabla^2 v|^2 \right) dS \leq Ne^{-2\varepsilon_1 t}.
\]

(4.114)

We apply (4.75), (2.31) and Theorem 3.3 to deduce
\[
\|\rho - \bar{\rho}\|_{L^\infty(B)} + \|\nabla \rho\|_{L^1(B)}
\]
\[
\leq CK \|\rho - \bar{\rho}\|_{L^2(B \cup \Gamma^\nu)} + CK \|\rho - \bar{\rho}\|_{L^2(B \cup \Gamma^\nu)}
\]
\[
\leq CK \left( \|\rho - \bar{\rho}\|_{L^2(B \cup \Gamma^\nu)} + \|\rho - \bar{\rho}\|_{L^2(B \cup \Gamma^\nu)} + \|\rho - \bar{\rho}\|_{L^2(B \cup \Gamma^\nu)} \right)
\]
\[
\leq CC_0 \frac{1}{\varepsilon_1} e^{\frac{C}{\varepsilon_1}} e^{-\left( \frac{C}{\varepsilon_1} - \varepsilon_1 \right) t}.
\]

(4.115)
To close the estimate upon $\|\nabla^2 v\|_{L^4(B)}$ and $\|\nabla v\|_{L^\infty(B)}$, we select a further sub-layer $B_+$:

$$B_+ = \{ x \in \Gamma'' \mid \text{dist}(x, \Gamma''_2) > \frac{d}{4} \}.$$ 

Observe that

$$B \subset B_+ \subset \Gamma'',$$

and the distance between them are strictly larger than $\frac{d}{4}$ as well.

We apply interior estimate (2.36), the Gagliardo-Nirenberg inequality (2.31) and Theorem 3.1 to deduce

$$\|\nabla^2 v\|_{L^4(B)} \leq CK \left( \|\mathcal{V}^\rho\|_{L^4(B_+)} + \|v\|_{L^2(B_+)} \right)$$

$$\leq CK \left( \|\mathcal{V}^\rho\|_{L^2(B)} \right) \|\nabla^2 \mathcal{V}^\rho\|_{L^2(B)} + \|\mathcal{V}^\rho\|_{L^2(B)}$$

$$\leq CCK \bar{C} e^{\frac{C}{t}} e^{-(\frac{\kappa}{16} - \frac{\gamma}{4})t}.$$ (4.116)

We also close the estimate of $\|\nabla v\|_{L^\infty(B)}$ with the help of (2.30), (2.6) and Theorem 3.1

$$\|\nabla v\|_{L^\infty(B)} \leq CK \left( \|\nabla^2 \mathcal{V}^\rho\|_{L^2(B_+)} + \|\nabla v\|_{L^2(B_+)} \right)$$

$$\leq CK \|\mathcal{V}^\rho\|_{L^2(B)} \|\nabla \mathcal{V}^\rho\|_{L^2(B)} + \|\nabla v\|_{L^2(B)}$$

$$\leq CCK \bar{C} e^{\frac{C}{t}} e^{-(\frac{\kappa}{16} - \frac{\gamma}{4})t}.$$ (4.117)

Combining (2.41)–(2.43), (4.75), (4.115) and Theorem 3.1 leads to

$$\int_{\partial\Omega} |\nabla \mathcal{V}|^2 \, ds$$

$$\leq CK \left( \|\nabla^2 \mathcal{V}^\rho\|_{L^2(B)} + \|\nabla \mathcal{V}^\rho\|_{L^2(B)} + \|\rho - \bar{\rho}\|_{L^2} \right)$$

$$\leq CCK \bar{C} e^{\frac{C}{t}} e^{-(\frac{\kappa}{16} - \frac{\gamma}{4})t}.$$ (4.118)

Collecting (4.116)–(4.117) gives (4.114) provided we choose

$$\varepsilon_1 \leq \frac{\kappa}{1000} \quad \text{and} \quad C_0 < \left( \frac{Ne^{-\frac{C}{\alpha}}}{10C} \right)^{10}.$$ (4.119)

Next, we will complete the estimate

$$\|\rho\|_{C^\alpha(\Omega)} \leq N.$$ (4.120)

Continue the argument in (4.21), we must close the last part of $\|\rho\|_{C^\alpha(\Omega)}$, say $\|\rho\|_{C^\alpha(B)}$. It is quite direct at present stage, once we apply interior estimate (2.36), the Gagliardo-Nirenberg inequality (2.31) and Theorem 3.1 to deduce

$$\|\rho\|_{C^\alpha(B)} \leq CK \|\rho\|_{W^{1,4}(B_+)}$$

$$\leq CK \left( \|\rho - \bar{\rho}\|_{L^2(B_+)} \right) \|\nabla^2 \mathcal{V}^\rho\|_{L^2(B)} + \|\rho - \bar{\rho}\|_{L^1}$$

$$\leq CCK \bar{C} e^{\frac{C}{t}} e^{-(\frac{\kappa}{16} - \frac{\gamma}{4})t} \leq \frac{N}{10}.$$ (4.121)
where we may determine $\varepsilon_1$ and $C_0$ as in (4.118). Combining (4.21) and (4.120) gives (4.119).

Finally, we claim that
\[
\|\rho\|_{L^\infty} \leq \frac{7M_0}{4},
\]  
and that for $x \in B \cup \Gamma \cup \tilde{\Gamma}$
\[
\rho \geq \frac{\theta}{2},
\]  
provided $C_0$ is suitably small.

Indeed, according to (2.65), (4.119), Theorem 3.3 and Lemma 2.17 for the effective viscous flux
\[
F = (\mu + \lambda)\text{div}u - (\rho - \bar{\rho}) = (\mu + \lambda)\text{div}w + F_v,
\]  
we have
\[
\|F\|_{L^\infty} \leq C (\|\nabla w\|_{L^\infty} + \|F_v\|_{L^\infty} + \|\nabla v\|_{L^\infty})
\leq CK \left( \|\nabla w\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho - \bar{\rho}\|_{L^2} \right)
\leq C\|\nabla w\|_{L^\infty} + \epsilon^2 |\|\rho\|_{L^\infty} + \epsilon^2 |\|\rho - \bar{\rho}\|_{L^2}.
\]  
Integrating with respect to $t$ on the interval $[t_1, t_2]$ yields
\[
\int_{t_1}^{t_2} \|F\|_{L^\infty} dt \leq \int_{t_1}^{t_2} (\|\nabla w\|_{L^\infty} + C\epsilon^{-2}|\|\rho - \bar{\rho}\|_{L^2}) dt + C\epsilon^2 (t_2 - t_1)
\leq CC_0\epsilon^{-2} + C\epsilon^2 (t_2 - t_1).
\]  
Now, let us rewrite (4.121) as
\[
D_t \rho + \frac{1}{\mu + \lambda} \rho(\rho - \bar{\rho}) = -\rho F.
\]  
Suppose that (4.121) fails, then there must be some $\xi \in \Omega$ and $t_1 > 0$ such that
\[
(4.125)
\]  
Consider the flow line $X(t)$ passing through $(\xi, t_1)$, we have $X(t_1) = \xi$. According to (1.8) and (4.11),
\[
(4.126)
\]  
By the continuity of $\rho$ and flow line, there must be some $t_0 \in (0, t_1)$ such that
\[
(4.127)
\]  
Note that $\bar{\rho} \leq M_0$ due to (1.8). Combining (4.121), (4.123) and (4.127), integrating (4.124) with respect to $t$ on the interval $[t_0, t_1]$ gives
\[
\rho(X(t_1)) - \rho(X(t_0)) = -\int_{t_0}^{t_1} \frac{1}{\mu + \lambda} \rho(\rho - \bar{\rho}) dt - \int_{t_0}^{t_1} \rho F dt
\leq C \int_{t_0}^{t_1} \|F\|_{L^\infty} dt - \frac{3M_0}{2(\mu + \lambda)} \int_{t_0}^{t_1} \rho dt
\leq CC_0\epsilon^{-2} + C\epsilon^2 (t_1 - t_0) - \frac{3M_0^2}{4(\mu + \lambda)} (t_1 - t_0)
\leq \frac{M_0}{8}.
\]  
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provided
\[ \epsilon \leq \left( \frac{3M_0^2}{8C(\mu + \lambda)} \right)^{\frac{3}{2}} \text{ and } C_0 \leq \frac{M_0 \epsilon^2}{8C}. \] (4.129)

However, (4.126) and (4.126) shows that
\[ \rho(X(t_1)) - \rho(X(t_0)) = \rho(\xi, t_1) - \rho(X(t_0)) > \frac{7M_0}{4} - \frac{3M_0}{2} = \frac{M_0}{4}, \]
which contradicts with (4.128). Therefore we declare
\[ \| \rho \| \leq \frac{7M_0}{4}, \]
which is (4.121).

The proof of (4.122) is similar. Suppose it is not the case, then there must be some \( \eta \in B \cup \Gamma \cup \tilde{\Gamma} \) and \( t_2 > 0 \) such that
\[ \rho(\eta, t_2) < \frac{\theta}{2}. \] (4.130)

Consider the flow line \( Y(t) \) passing through \( (\eta, t_2) \). We have \( Y(t) \subset B \cup \Gamma \cup \tilde{\Gamma} \) and \( Y(t_2) = \eta \). According to (1.9) and (4.1),
\[ \rho(Y(0)) = \rho_0(Y_0) > \theta. \]
By the continuity of \( \rho \) and flow line, there must be some \( t_0' \in (0, t_2) \) such that
\[ \rho(Y(t_0')) = \frac{3\theta}{4}, \] (4.131)
and for \( t \in [t_0', t_2] \),
\[ \rho(Y(t)) \leq \frac{3\theta}{4}. \] (4.132)

Combining (1.9), (4.1), (4.121), (4.123) and (4.132), and integrating (4.124) with respect to \( t \) on the interval \([t_0', t_2]\) gives
\[ \rho(Y(t_2)) - \rho(Y(t_0')) = -\int_{t_0'}^{t_2} \frac{1}{\mu + \lambda} \rho(\rho - \bar{\rho}) dt - \int_{t_0'}^{t_2} \rho F dt \]
\[ \geq -C \int_{t_0'}^{t_2} \| F \|_{L^\infty} dt - \frac{3\theta(3\theta - 4\bar{\rho})}{16(\mu + \lambda)} \int_{t_0'}^{t_2} dt \]
\[ \geq -CC_0\epsilon^{-2} - Ce^{\frac{4}{\theta}} (t_2 - t_0') + \frac{\theta(\bar{\rho} - \theta)}{2(\mu + \lambda)} (t_2 - t_0') \]
\[ \geq -\frac{\theta}{8}, \]
provided
\[ \epsilon = \left( \frac{\theta(\bar{\rho} - \theta)}{10C(\mu + \lambda)} \right)^{\frac{3}{2}} \text{ and } C_0 \leq \frac{\epsilon^2}{8C}. \] (4.134)

However, it follows from (4.130) and (4.131) that
\[ \rho(Y(t_2)) - \rho(Y(t_0')) = \rho(\eta, t_2) - \rho(Y(t_0')) < \frac{\theta}{2} - \frac{3\theta}{4} = -\frac{\theta}{4}, \]
which contradicts with (4.133). Consequently we deduce that for \( x \in B \cup \Gamma \cup \tilde{\Gamma} \),
\[ \rho \geq \frac{\theta}{2}, \]
which is (4.122).
Combining (3.30), (4.6), (4.115), (4.118), (4.129) and (4.134) yields
\[ \varepsilon = \min \left\{ \varepsilon_0, \frac{d_0}{200C}, \varepsilon^2 \log 2, \frac{(Ne - \varepsilon_1)}{10C}, M_0 \varepsilon^2, \varepsilon^2 \theta, \frac{M_0}{8C}, \varepsilon^2 \right\}. \]

In conclusion, under the conditions of Theorem 1.1 together with (4.1) – (4.3), we can get (4.4) and (4.5), provided \( C_0 \leq \varepsilon \).

The proof of Theorem 4.1 is therefore complete.

Remark 4.3. We note that there is no interior boundary term in our proof.

The \( C^\alpha \) estimate in Section 4.1.1 involves no boundary term. Since all calculations are along the flow, we do not integrate on any domain actually. The furthest from boundary part \( \tilde{\Gamma} \) of \( C^\alpha \) domain is quite important. Indeed, we only consider the standard \( C^\alpha \) norm on free boundary \( \Gamma_2 \) as indicated by Remark 2.10, thus we require no information beyond \( \tilde{\Gamma} \). It is the key observation in closing \( C^\alpha \) estimate (see also Remark 2.12).

When we consider \( \| \nabla \rho \|_{L^4(\Gamma')} \) in Section 4.1.2, no boundary term occurs. It is due to the equation (4.30),
\[ D_t \rho_i = -\rho_i u_k^i - \rho u^k_i. \]
We do not integrate by part in (4.31). Consequently, no boundary term enters into the final expression (4.36). We illustrate such fact in Euler coordinates as well. Note that \( \Gamma'(t) \) is not fixed in Euler coordinate, we must make use of co-area formula (see [12, Appendix C.4]) when we take derivative with respect to \( t \), say
\[ \frac{d}{dt} \int_{\Gamma'(t)} |\nabla \rho|^4 dx = \int_{\Gamma'(t)} \frac{\partial}{\partial t} |\nabla \rho|^4 dx + \int_{\partial \Gamma'(t)} |\nabla \rho|^4 (u \cdot n) dS \]
\[ = \int_{\Gamma'(t)} \frac{\partial}{\partial t} |\nabla \rho|^4 + \text{div}(|\nabla \rho|^4 u) dx, \]
where \( n \) is the unit outer normal of \( \partial \Gamma'(t) \). Accordingly, the boundary term have been transformed to interior via divergence theorem as showed in (4.135). Now, we can apply (1.11) to finish the calculation and no further boundary terms will occur, since there is no more integrating by part. Such fact coincides with the computations in Lagrangian coordinates.

Finally, let us mention that no interior boundary terms on \( B_1 \) enter into Section 4.2 as well. The out line is similar with Section 4.1.1 except that the only additional integrating by part in (4.85) and (4.89). However, the truncation function \( \psi \) actually cancels out the interior boundary term on \( B_1 \) due to integrating by part.

In conclusion, there is no additional interior boundary term along the whole proof.

Remark 4.4. The weighted elliptic estimate Lemma 4.4 is stronger than the usual interior estimate. As we known from Lemma 2.14, the usual interior estimate is given by
\[ \| \nabla^2 w \|_{L^2(\Gamma''_1)} \leq C \| \rho u \|_{L^2(\Gamma')} + C \| \nabla w \|_{L^2}. \]
However, Lemma 4.4 guarantees we can add a truncation function as weight in \( L^2 \) estimate,
\[ \int |\nabla^2 w|^2 \phi dx \leq C \int |\rho u|^2 \phi dx + C \int |\nabla w|^2 dx. \]
If we ignore the lower order term, we may roughly conclude that the mapping: \( \rho u \mapsto \nabla^2 w \) is bounded on the space \( L^2(\phi dx) \). There is no special structural assumption on the weight function \( \phi \) other than positivity.
5 A priori estimates: Higher order ones

Once we obtain the uniform upper bound of \( \rho \), the rest of paper follows in a rather standard way to get higher order a priori estimates. Our approach mainly relies on [7, 21], a slight modification should be done. For the sake of completeness, we sketch the proof here. The constant \( C \) below is possibly different from line to line, but all depend only on \( T, p, \mu, \lambda, \bar{\rho}, \bar{\rho}, g, M_0, N_0 \) and the initial value. We first close the estimate on \( \| \nabla \rho \|_{L^p} \).

**Lemma 5.1.** There is a constant \( C \) such that

\[
\sup_{0 \leq t \leq T} (\| \nabla \rho \|_{L^p} + \| u \|_{H^2}) + \int_0^T \| \nabla^2 u \|_{L^6}^2 \, dt \leq C.
\]

**Proof.** For conservation of density, we have

\[
\frac{\partial}{\partial t} \rho + \text{div}(u \rho) = -\text{div}(\rho \partial_t u).
\]

For \( 2 \leq p \leq 6 \), multiplying this by \( \partial_t \rho |\nabla \rho|^{p-2} \), we deduce

\[
(\| \nabla \rho \|_{L^p})^p + \text{div}(\| \nabla \rho \|_{L^p} u) \leq C \| \nabla \rho \|_{L^p} |\nabla u| + C \rho |\nabla \rho|^{p-1} |\nabla \text{div} u|.
\]

Integrating it over \( \Omega \) leads to

\[
\frac{d}{dt} \int_\Omega |\nabla \rho|^p \, dx \leq C (1 + \| \nabla u \|_{L^\infty}) \| \nabla \rho \|_{L^p} + C \| \nabla^2 w \|_{L^p} + C \| \nabla^2 v \|_{L^p}
\]

\[
\leq C (1 + \| \nabla u \|_{L^\infty}) \| \nabla \rho \|_{L^p} + C \| \rho \|_{L^p}
\]

\[
\leq C (1 + \| \nabla u \|_{L^\infty}) \| \nabla \rho \|_{L^p} + C \| \rho \|_{L^p},
\]

where in the second inequality we have used the elliptic estimate (5.3).

By the Gagliardo-Nirenberg inequality, Lemma 2.14, Lemma 2.15 and Theorem 2.17, we have

\[
\| \text{div} u \|_{L^\infty} + \| \rot u \|_{L^\infty} \leq C (\| \nabla w \|_{L^\infty} + \| F_v \|_{L^\infty} + \| \rot v \|_{L^\infty} + \| \rho - \bar{\rho} \|_{L^\infty})
\]

\[
\leq CK (\| \nabla w \|_{L^\infty} + \| \rho \|_{C^0(B)} + \| \rho - \bar{\rho} \|_{L^\infty})
\]

\[
\leq CK (1 + \| \nabla w \|_{L^\infty}).
\]

For \( p = 6 \), combining this with (2.1) guarantees

\[
\| \nabla u \|_{L^6} \leq C (\| \text{div} u \|_{L^6} + \| \rot u \|_{L^6} \log (e + \| \nabla^2 u \|_{L^6}) + C \| \nabla u \|_{L^2} + C
\]

\[
\leq CK (1 + \| \nabla w \|_{L^\infty}) \log (e + \| \nabla u \|_{L^2} + \| \nabla u \|_{L^6})
\]

\[
\leq CK (1 + \| \nabla w \|_{L^\infty}) < 1 + \| \nabla u \|_{L^2} + \| \nabla \rho \|_{L^6}).
\]

According to (5.2), Theorems 2.2 and 3.3 we apply Gronwall’s inequality in (6.1) to deduce

\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6} \leq C.
\]

It follows from elliptic estimates (2.6), (5.3) and Theorem 3.2 that

\[
\| u \|_{H^2} \leq C (\| u \|_{H^2} + \| v \|_{H^2}) \leq C (\| \rho \|_{L^2} + \| \nabla \rho \|_{L^2}) \leq C,
\]

and that

\[
\int_0^T \| \nabla^2 u \|_{L^6}^2 \, dt \leq C \int_0^T (\| \nabla^2 w \|_{L^6}^2 + \| \nabla^2 v \|_{L^6}^2) \, dt
\]

\[
\leq C \int_0^T (\| \nabla \rho \|_{L^6}^2 + \| \nabla \rho \|_{L^6}^2) \, dt \leq C.
\]

\( \square \)
The remaining part is quite routine, and no further obstacle gets in our way. We follow [7][21] to finish a priori estimates and the proof.

**Lemma 5.2.** There is a constant $C$ depending on $T$ such that

$$
\sup_{0 \leq t \leq T} \left( \| \rho^{1/2} u_t \|_{L^2}^2 + \| \rho - \bar{\rho} \|_{H^2} \right) + \int_0^T \int |\nabla u_t|^2 \, dx \, dt \leq C.
$$

**Proof.** By Theorem 3.2 and Lemma 5.1, it is direct to check that

$$
\| \rho^{1/2} u_t \|_{L^2}^2 \leq \| \rho^{1/2} \dot{u} \|_{L^2}^2 + \| \rho^{1/2} u \cdot \nabla u \|_{L^2}^2
\leq C + C \| u \|_{L^4}^2 \| \nabla u \|_{L^4}^2
\leq C + C \| \nabla u \|_{L^2}^2 \| u \|_{H^2}^2
\leq C,
$$

and

$$
\int_0^T \| \nabla u_t \|_{L^2}^2 \, dt \leq C \int_0^T \| \nabla \dot{u} \|_{L^2}^2 \, dt + C \int_0^T \| \nabla (u \cdot \nabla u) \|_{L^2}^2 \, dt
\leq C + C \int_0^T \left( \| \nabla u \|_{L^4}^4 + \| u \|_{L^\infty} \| \nabla^2 u \|_{L^2}^2 \right) \, dt
\leq C + C \int_0^T \left( \| \nabla^2 u \|_{L^2}^4 + \| \nabla u \|_{H^1} \| \nabla^2 u \|_{L^2}^2 \right) \, dt
\leq C.
$$

Moreover, we take partial derivatives with respect to $x_i, x_j$ on (1.1) and arrive at

$$
\partial_t (\partial_{ij} \rho) + \text{div}(u \partial_{ij} \rho) = -\text{div}(\partial_j \rho \partial_i u) - \text{div}(\partial_i \rho \partial_j u).
$$

Combining (5.4), Lemma 5.1 and elliptic estimate (2.6) gives

$$
\frac{d}{dt} \| \nabla^2 \rho \|_{L^2}^2 \leq C \left( 1 + \| \nabla u \|_{L^\infty} \right) \| \nabla^2 \rho \|_{L^2}^2 + C \| \nabla \dot{u} \|_{L^2}^2 + C.
$$

We make use of Gronwall’s inequality, Theorems 3.3 and 3.2 in (5.5) to declare

$$
\sup_{0 \leq t \leq T} \| \rho - \bar{\rho} \|_{H^2} \leq C.
$$

□

**Lemma 5.3.** There is some constant $C$ depending on $T$ such that

$$
\sup_{0 \leq t \leq T} \| \rho_t \|_{H^1} + \int_0^T \| \rho_{tt} \|_{L^2}^2 \, dt \leq C,
$$

$$
\sup_{0 \leq t \leq T} \sigma \| \nabla u_t \|_{L^2}^2 + \int_0^T \sigma \| \rho^{1/2} u_t \|_{L^2}^2 \, dt \leq C.
$$

**Proof.** Recall that (1.1) gives

$$
\frac{\partial}{\partial t} \rho = -u \cdot \nabla \rho - \rho \text{div} u,
$$

and

$$
\frac{\partial}{\partial t} \partial_i \rho = -\text{div}(u \partial_i \rho) - \text{div}(\rho \partial_i u).
$$

By Lemmas 6.1 and 5.2 we have

$$
\| \rho_t \|_{L^2} \leq C \| u \|_{L^\infty} \| \nabla \rho \|_{L^2} + C \| \nabla u \|_{L^2} \leq C,
$$

$$
\| \nabla \dot{u} \|_{L^2} \leq C \| u \|_{L^\infty} \| \nabla \dot{u} \|_{L^2} + C \| \nabla u \|_{L^2} \leq C.
$$
and
\[ \| \nabla \rho_t \|_{L^2} \leq C \| u \|_{L^\infty} \| \nabla^2 \rho \|_{L^2} + C \| \nabla^2 \rho \|_{L^6} \leq C. \]

Thus,
\[ \sup_{0 \leq t \leq T} \| \rho_t \|_{H^1} \leq C. \]

We take derivative with respect to \( t \) one more time on (1.1),
\[ \rho_{tt} + \rho_t \div u + \rho \div \nabla \rho + u_t \cdot \nabla \rho + u \cdot \nabla \rho_t = 0. \quad (5.6) \]

Multiplying (5.6) by \( \rho_{tt} \) and integrating over \( \Omega \times [0, T] \), we apply Sobolev’s imbedding inequality, Lemmas 5.1 and 5.2 to deduce
\[
\int_0^T \| \rho_{tt} \|^2_{L^2} dt = - \int_0^T \int \rho_t \rho_t \div u dx dt - \int_0^T \int \rho_t \rho \div u_t dx dt \\
- \int_0^T \int \rho_t u_t \cdot \nabla \rho dx dt - \int_0^T \int \rho_t u \cdot \nabla \rho_t dx dt \\
\leq C \int_0^T \| \rho_t \|_{L^2} (\| \rho_t \|_{L^3} \| \nabla u \|_{L^6} + \| \nabla u_t \|_{L^2} + \| u_t \|_{L^3} \| \nabla \rho \|_{L^6} + \| u \|_{L^\infty} \| \nabla \rho_t \|_{L^2}) dt \\
\leq C \int_0^T \| \rho_t \|_{L^2} (1 + \| \nabla u_t \|_{L^2}) dt \leq \frac{1}{2} \int_0^T \| \rho_t \|^2_{L^2} dt + C \int_0^T \| \nabla u_t \|^2_{L^2} dt + C \\
\leq \frac{1}{2} \int_0^T \| \rho_t \|^2_{L^2} dt + C. 
\]

Consequently, we have
\[ \int_0^T \| \rho_t \|^2_{L^2} dt \leq C. \quad (5.7) \]

Next, we set
\[ W(t) \triangleq (\lambda + 2\mu) \int (\div u_t)^2 dx + \mu \int |\curl u_t|^2 dx. \]

By Dirichlet boundary condition (1.3), we have \( u_t = 0 \) on \( \partial \Omega \), which together with Lemma 2.2 gives
\[ \| \nabla u_t \|^2_{L^2} \leq CW(t). \quad (5.8) \]

Taking derivative with respect to \( t \) on (1.1.2) and multiplying by \( u_{tt} \), we obtain
\[
\frac{d}{dt} W(t) + 2 \int \rho |u_t|^2 dx \\
= \frac{d}{dt} \left( - \int \rho_t |u_t|^2 dx - 2 \int \rho_t u \cdot \nabla u \cdot u_t dx + 2 \int \rho_t \div u_t dx \right) \\
+ \int \rho_t |u_t|^2 dx + 2 \int (\rho_t u \cdot \nabla u_t) \cdot u_t dx - 2 \int \rho u_t \cdot \nabla u \cdot u_t dx \\
- 2 \int \rho u \cdot \nabla u_t \cdot u_t dx - 2 \int \rho u_t \div u_t dx \\
\triangleq \frac{d}{dt} I_0 + \sum_{i=1}^5 I_i. 
\]

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Let us check for \( I_1 \) step by step and make use of Lemmas (5.1), (5.2), (5.7) and (5.8).

\[
I_0 = - \int \rho_t |u_t|^2 \, dx - 2 \int \rho_t u \cdot \nabla u \cdot u_t \, dx + 2 \int \rho_t \text{div} \, u_t \, dx \\
\leq \left| \int \text{div}(\rho u) |u_t|^2 \, dx \right| + C \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^6} + C \|\rho_t\|_{L^2} \|\nabla u_t\|_{L^2} \\
\leq C \int |u| \rho |u_t| \|\nabla u_t\| \, dx + C \|\nabla u_t\|_{L^2} \\
\leq C \|u\|_{L^6} \left| \rho^{1/2} u_t \right|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2} \\
\leq C \|\nabla u\|_{L^2} \left| \rho^{1/2} u_t \right|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{3/2} + C \|\nabla u_t\|_{L^2} \\
\leq \frac{1}{2} W(t) + C.
\]

For \( I_1 \), one has

\[
|I_1| = \left| \int \rho_t |u_t|^2 \, dx \right| \\
= \left| \int \text{div}(\rho u_t) |u_t|^2 \, dx \right| \\
= 2 \left| \int (\rho_t u + \rho u_t) \cdot \nabla u_t \cdot u_t \, dx \right| \\
\leq C \left( \|\rho_t\|_{H^1} \|u\|_{H^2} + \left| \rho^{1/2} u_t \right|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2} \right) \|\nabla u_t\|_{L^2}^{1/2} \\
\leq C \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2} + C. \tag{5.9}
\]

For \( I_2 \)

\[
|I_2| = 2 \left| \int \frac{\partial}{\partial t} (\rho_t u \cdot \nabla u) \cdot u_t \, dx \right| \\
= 2 \left| \int (\rho_u u \cdot \nabla u \cdot u_t + \rho u_t \cdot \nabla u \cdot u_t + \rho_t u \cdot \nabla u_t \cdot u_t) \, dx \right| \\
\leq \|\rho_t\|_{L^2} \|u \cdot \nabla u\|_{L^3} \|u_t\|_{L^6} + \|\rho_t\|_{L^2} \|u_t\|_{L^6} \|\nabla u\|_{L^6} \\
+ \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6} \\
\leq C \|\rho_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2. \tag{5.10}
\]

For \( I_3 \) and \( I_4 \),

\[
|I_3| + |I_4| = 2 \left| \int \rho u_t \cdot \nabla u \cdot u_t \, dx \right| + 2 \left| \int \rho u_t \cdot \nabla u_t \cdot u_t \, dx \right| \\
\leq C \left| \rho^{1/2} u_t \right|_{L^2} \left( \|u_t\|_{L^6} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \right) \tag{5.11} \\
\leq \left| \rho^{1/2} u_t \right|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2.
\]

Finally, \( I_5 \) can be handled via

\[
|I_5| = 2 \left| \int \rho_t \text{div} \, u_t \, dx \right| \\
\leq C \|\rho_t\|_{L^2} \|\text{div} \, u_t\|_{L^2} \tag{5.12} \\
\leq C \|\rho_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2.
\]

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Combining (5.9), (5.10), (5.11) and (5.12), we have
\[
\frac{d}{dt}(\sigma W(t) - \sigma I_0) + \sigma \int \rho |u_t|^2 dx \\
\leq C \left( 1 + \|\nabla u_t\|_{L^2}^2 \right) \sigma W(t) + C \left( 1 + \|\nabla u_t\|_{L^2}^2 + \|\rho u_t\|_{L^2}^2 \right).
\]
By Gronwall’s inequality, Lemma 5.2 and (5.7), we have
\[
\sup_{0 \leq t \leq T} (\sigma W(t)) + \int_0^T \sigma \left\| \rho_\frac{1}{2} u_t \right\|_{L^2}^2 dt \leq C.
\]
According to (5.8),
\[
\|\nabla u_t\|_{L^2}^2 + \int_0^T \sigma \left\| \rho_\frac{1}{2} u_t \right\|_{L^2}^2 \leq C.
\]

Lemma 5.4. For \(q \in (3, 6)\), there exists a positive constant \(C\) such that
\[
\sup_{t \in [0, T]} (\sigma \|\nabla u\|_{H^1}^2 + \|\rho - \bar{\rho}\|_{W^{2,q}}) + \int_0^T (\|\nabla u\|_{H^2}^2 + |\nabla^2 u|^2_{W^{1,q}} + \sigma \|\nabla u_t\|_{H^1}^2) dt \leq C,
\]
where \(p_0 = \frac{9q - 6}{9q - 12} \in (1, \frac{7}{3})\).

Proof. First, by Lemmas 5.1, 5.3 Poincaré’s and Sobolev’s inequalities, one can check that
\[
\|\nabla(\rho \dot{u})\|_{L^2} \leq \|\nabla\rho\|_{L^2} + \|\rho \nabla u_t\|_{L^2} + \|\nabla\rho\|_{L^2} + \|\nabla u\|_{L^2} \\
+ \|\rho\|_{L^6} \|\nabla u\|_{L^{6'}} + C\|\nabla u_t\|_{L^2} + C\|\nabla\rho\|_{L^6} \|\nabla u\|_{L^6} \\
+ C\|\nabla u\|_{L^3} \|\nabla^2 u\|_{L^3} + C. 
\]
which together with (5.13), Lemmas 5.1 and 5.2 yields
\[
\|\nabla^2 u\|_{H^1} \leq C(\|\rho \dot{u}\|_{H^1} + \|u\|_{L^2}) \\
\leq C + C\|\nabla u_t\|_{L^2}.
\]
It then follows from Lemmas 5.1 and 5.2 that
\[
\sup_{0 \leq t \leq T} \sigma \|\nabla u\|_{H^2}^2 + \int_0^T \|\nabla u\|_{H^2}^2 dt \leq C.
\]

Lemmas 5.3 and 5.2 also make sure that
\[
\|\nabla^2 u_t\|_{L^2} \leq C(\|\rho \dot{u}\|_{L^2} + \|u_t\|_{H^1} + \|u_t\|_{L^2}) \\
\leq C(\|\rho u_t\| + \|u_t\|_{L^6} + \|u_t\|_{L^2}) \\
+ C(\|\nabla u\|_{L^3} + \|u_t\|_{L^6} + \|u_t\|_{L^3} + \|u\|_{L^6} \|\nabla u\|_{L^6}) + C \\
\leq C \|\rho u_t\|_{L^2} + \|u_t\|_{L^6} + \|u_t\|_{L^3} + \|u\|_{L^6} \|\nabla u\|_{L^6} + \|\nabla u_t\|_{L^2} + \|u_t\|_{L^2}) \\
\leq C(\|\rho \frac{1}{2} u_t\|_{L^2} + C\|\nabla u_t\|_{L^2} + C),
\]
where in the first inequality, we have utilized elliptic estimate for the following elliptic system
\[
\begin{align*}
\mu \Delta u_t + (\lambda + \mu) \nabla \div u_t &= (\rho \dot{u})_t + \nabla \rho_t & \text{in } \Omega, \\
u_t &= 0 & \text{on } \partial \Omega.
\end{align*}
\]
It follows from (5.14), (5.15), and Theorem 5.3 that

\[ \int_0^T \sigma \| \nabla u_t \|^2_{H^1} dt \leq C. \] (5.16)

By Sobolev’s inequality and Lemmas 5.1, 5.3, we get for any \( q \in (3, 6) \),

\[ \| \nabla (\rho \dot{u}) \|_{L^q} \leq C \| \nabla \rho \|_{L^q} \| \nabla \dot{u} \|_{L^q} + \| \nabla \dot{u} \|_{L^2} + \| \nabla u \|^2_{L^2} + C \| \nabla \dot{u} \|_{L^q} \]

\[ \leq C \left( \| \nabla \dot{u} \|_{L^2} + \| \nabla u \|^2_{L^2} \right) + C \left( \| \nabla u \|_{L^q} + \| \nabla (\rho \cdot \nabla u) \|_{L^q} \right) \]

\[ \leq C \left( \| \nabla \dot{u} \|_{L^2} + 1 \right) + C \| \nabla u \|^2_{L^2} \| \nabla u \|^2_{L^6} \frac{2(\sigma-2)}{29} \] (5.17)

\[ + C \left( \| u \|_{L^\infty} \| \nabla^2 \rho \|_{L^q} + \| \nabla u \|_{L^\infty} \| \nabla u \|_{L^q} \right) \]

\[ \leq C \sigma^{-\frac{1}{2}} + C \| \nabla u \|_{H^2} + C \sigma^{-\frac{1}{2}} (\| \nabla u \|^2_{H^1}) \] (5.18)

Integrating (5.17) over \([0, T]\), by Lemma 5.1 and (5.16), we have

\[ \int_0^T \| \nabla (\rho \dot{u}) \|^2_{L^q} dt \leq C. \] (5.19)

On the other hand, Lemma 5.2 gives

\[ \left( \| \nabla^2 \rho \|_{L^q} \right) \leq C \| \nabla u \|_{L^\infty} \| \nabla^2 \rho \|_{L^q} + C \| \nabla^2 u \|_{W^{1,q}} \]

\[ \leq C \left( 1 + \| \nabla u \|_{L^\infty} \right) \| \nabla^2 \rho \|_{L^q} + C \left( 1 + \| \nabla u \|_{L^2} \right) \]

\[ + C \| \nabla (\rho \dot{u}) \|_{L^q}, \] (5.20)

where in the last inequality we have used the following simple fact that

\[ \| \nabla^2 u \|_{W^{1,q}} \leq C \left( 1 + \| \nabla u \|_{L^2} + \| \nabla (\rho \dot{u}) \|_{L^q} \right) \]

\[ \leq C \left( 1 + \| \nabla u \|_{L^2} + \| \nabla (\rho \dot{u}) \|_{L^q} \right), \]

due to elliptic estimate and Lemma 5.2.

Hence, applying Gronwall’s inequality in (5.18), we deduce from Lemmas 5.1, 5.2, and (5.18) that

\[ \sup_{0 \leq t \leq T} \| \nabla^2 \rho \|_{L^q} \leq C, \]

which along with Lemma 5.2, 5.18, and (5.20) also gives

\[ \sup_{t \in [0,T]} \| \rho - \bar{\rho} \|_{W^{2,q}} + \int_0^T \| \nabla^2 u \|^2_{W^{1,q}} dt \leq C. \]

\[ \Box \]

**Lemma 5.5.** For \( q \in (3, 6) \), there exists a positive constant \( C \) such that

\[ \sup_{0 \leq t \leq T} \sigma (\| \nabla u_t \|_{H^1} + \| \nabla u \|_{W^{2,q}}) + \int_0^T \sigma^2 \| \nabla u_t \|^2_{H^1} dt \leq C. \]

**Proof.** Differentiating (1.1) with respect to \( t \) twice, we have

\[ \rho u_{tt} + \rho u \cdot \nabla u_t - (\lambda + 2\mu) \nabla \text{div} u_t + \mu \nabla^2 \text{rot} u_t \]

\[ = 2 \text{div}(\rho u) u_{tt} + \text{div} (\rho u_t) u_t - (\rho_t u_t + 2\rho_t u_t) \cdot \nabla u - \rho u_t \cdot \nabla u - \nabla \rho_t. \] (5.21)
Then, multiplying (5.21) by $2u_t$ and integrating over $\Omega$ lead to
\[
\frac{d}{dt} \int \rho|u_t|^2 dx + 2(\lambda + 2\mu) \int (\text{div} u_t)^2 dx + 2\mu \int |\text{rot} u_t|^2 dx = -8 \int \rho u_t u \cdot \nabla u_t dx - 2 \int (\rho u_t) [\nabla (u_t \cdot u_t) + 2\nabla u_t \cdot u_t] dx - 2 \int (\rho u + 2\rho_t u_t) \cdot \nabla u \cdot u_t dx + 2 \int \rho_t \text{div} u_t dx \leq \sum_{i=1}^{5} J_i.
\]

Let us estimate $J_i$ for $i = 1, \cdots, 5$. Hölder’s inequality and Lemma 5.1 give
\[
|J_i| \leq C \parallel \rho^{1/2} u_{tt} \parallel_{L^2} \parallel \nabla u_{tt} \parallel_{L^2} \|u\|_{L^\infty} \leq \delta \parallel \nabla u_t \parallel_{L^2}^2 + C(\delta) \parallel \rho^{1/2} u_{tt} \parallel_{L^2}^2.
\]

By Lemmas 5.1 and 5.2 we conclude that
\[
|J_2| \leq C \left( \parallel \rho u_t \parallel_{L^3} + \parallel \rho_t u \parallel_{L^3} \right) \left( \parallel u_{tt} \parallel_{L^6} \parallel \nabla u_{tt} \parallel_{L^2} + \parallel \nabla u_{tt} \parallel_{L^2} \|u_t\|_{L^6} \right) \leq C \left( \parallel \rho^{1/2} u_{tt} \parallel_{L^2} \parallel u_{tt} \parallel_{L^6} \right) \parallel \nabla u_{tt} \parallel_{L^2}^2 \parallel u_{tt} \parallel_{L^2} \leq \delta \parallel \nabla u_t \parallel_{L^2}^2 + C(\delta) \parallel u_{tt} \parallel_{L^2}^2 + C(\delta) \sigma^{-1},
\]

and
\[
|J_3| \leq C (\parallel \rho u_t \parallel_{L^2} \parallel u \parallel_{L^\infty} \parallel \nabla u \parallel_{L^3} + \parallel \rho_t u \parallel_{L^6} \parallel u \parallel_{L^6} \parallel \nabla u \parallel_{L^2}) \|u_{tt}\|_{L^6} \leq \delta \parallel \nabla u_t \parallel_{L^2}^2 + C(\delta) \|u_{tt}\|_{L^2}^2 + C(\delta) \sigma^{-1},
\]

Substituting these estimates of $J_i$ into (5.22), utilizing the fact that
\[
\parallel \nabla u_{tt} \parallel_{L^2} \leq C(\parallel \text{div} u_{tt} \parallel_{L^2} + \parallel \text{rot} u_{tt} \parallel_{L^2}),
\]
due to Lemma 2.2 since $u_t = 0$ on $\partial \Omega$, and then choosing $\delta$ small enough, we can get
\[
\frac{d}{dt} \parallel \rho^{1/2} u_{tt} \parallel_{L^2}^2 + \parallel \nabla u_{tt} \parallel_{L^2}^2 \leq C(\parallel \rho^{1/2} u_{tt} \parallel_{L^2}^2 + \parallel \rho u_{tt} \parallel_{L^2}^2 + \parallel \rho_t u_{tt} \parallel_{L^2}^2) + C\sigma^{-3/2},
\]

which together with Lemma 5.2 and Gronwall’s inequality yields that
\[
\sup_{0 \leq t \leq T} \sigma \parallel \rho^{1/2} u_{tt} \parallel_{L^2}^2 + \int_0^T \sigma^2 \parallel \nabla u_{tt} \parallel_{L^2}^2 dt \leq C. \tag{5.23}
\]

Furthermore, it follows from (5.23) and Lemma 5.2 that
\[
\sup_{0 \leq t \leq T} \sigma \parallel \nabla u_t \parallel_{H^1}^2 \leq C. \tag{5.24}
\]

Finally, we deduce from (5.17), (5.20), Lemma 5.4 and (5.24) that
\[
\sigma \parallel \nabla^2 u \parallel_{W^{1,\sigma}} \leq C(\sigma + \sigma \parallel \nabla u_t \parallel_{L^2} + \sigma \parallel \nabla (\rho u) \parallel_{L^\sigma} + \sigma \parallel \nabla^2 \rho \parallel_{L^3}) \leq C(\sigma + \sigma^{1/2} + \sigma \parallel u \parallel_{H^2} + \sigma^{1/2} \parallel \nabla u_t \parallel_{H^1}^{2(3\sigma-2)/(4\sigma)}) \leq C \sigma^{1/2} + C \sigma^{1/2} (\sigma^{-1})^{3(\sigma-2)/(4\sigma)} \leq C,
\]
which together with (5.23) and (5.24) yields
\[ \sup_{0 \leq t \leq T} \sigma (\| \nabla u_t \|_{H^1} + \| \nabla u \|_{W^{2,q}}) + \int_0^T \sigma^2 \| \nabla u_t \|_2^2 dt \leq C. \]

\[ \square \]

6 Proofs of Theorems 1.1 and 1.2

With all the a priori estimates in previous sections at hand, we are going to prove Theorems 1.1 and 1.2 in this section.

Proof of Theorem 1.1 By Lemma 2.1, there exists a \( T_0 > 0 \) such that the system (1.1)–(1.3) has a unique classical solution \((\rho, u)\) on \( \Omega \times (0, T_0) \). One may use the a priori estimates, Theorem 4.1 and Lemmas 5.1–5.5 to extend the classical solution \((\rho, u)\) globally in time.

We define
\[ \Psi(t) = \| \nabla u \|_{L^\infty(B)} + \| \rho - \bar{\rho} \|_{L^\infty(B)} + \| \nabla^2 v \|_{L^4(B)} + \| \nabla \rho \|_{L^4(B)} + \int_{\partial \Omega} (|\nabla \rho|^2 + |\nabla v|^2 + |\nabla^2 v|^2) dS, \]
\[ \Psi(t) = \| \rho \|_{L^1(B)} + \psi(t) \]
Under the conditions of Theorem 1.1, we apply (1.8)–(1.10) and (4.1) to deduce that
\[ \Psi(0) < 2N, \quad \Psi(0) < 2N, \quad \Psi(0) < 2M_0. \]

Therefore, we can find a \( T_1 \in (0, T_0) \) such that for \( t \in [0, T_1] \),
\[ \Psi(0) \leq 2N e^{-\delta t}, \quad \Psi(2) \leq 2N, \quad \Psi(3) \leq 2M_0. \]

Next, we set
\[ T^* = \sup \{ T > 0 \mid \forall t \in [0, T], (6.1) \text{ holds} \}. \]
Then \( T^* \geq T_1 \). Hence, for any \( 0 < \tau < T \leq T^* \) with \( T \) finite, it follows from Lemmas 5.1–5.5 that
\[ \rho \in C([0, T]; W^{2,q}), \quad \nabla u_t \in C([\tau, T]; L^q), \quad \nabla u, \nabla^2 u \in C ([\tau, T]; C(\bar{\Omega})), \]
where one has taken advantage of the standard embedding
\[ L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C ([\tau, T]; L^q), \quad \text{for any } q \in [1, 6). \]

This in particular yields
\[ \rho^{1/2} u_t, \quad \rho^{1/2} \dot{u} \in C([\tau, T]; L^2). \]

Finally, we claim that
\[ T^* = \infty. \]

If (6.3) is not true, \( T^* < \infty \). Then by Theorem 1.1 we deduce that for \( t \in [0, T^*] \),
\[ \Psi(t) \leq 2M_0. \]

It follows from Lemmas 5.1–5.5 and (6.2) that \( (\rho(x, T^*), u(x, T^*)) \) satisfies the initial data conditions (1.8) and (1.11) except \( \rho(x, T^*) \leq 7M_0/4 \), where \( g(x) \triangleq \rho^{1/2} \dot{u}(x, T^*), \quad x \in \Omega. \)

Thus, Lemma 2.1 implies that there exists some \( T^{**} > T^* \) such that (6.1) holds for \( t \in (0, T^{**}) \), which contradicts the definition of \( T^* \). As a result, \( 0 < T_1 < T^* = \infty. \)
According to Lemmas 2.1, 5.1-5.5, \((\rho, u)\) is in fact the unique classical solution defined on \(\Omega \times (0, T)\) for any \(0 < T < T^* = \infty\).

Moreover, we still need to prove (1.14). Observe that by decomposition (2.5), elliptic estimates (2.6), Sobolev imbedding and the Gagliardo-Nirenberg inequality, for \(2 \leq p < 6\) we have

\[
\|u\|_{W^{1,p}} \leq C\|\nabla w\|_{L^p} + C\|\nabla v\|_{L^p}
\]

\[
\leq C\|\nabla w\|_{L^2}^{\beta} \|\nabla^2 u\|_{L^2}^{1-\beta} + C\|\nabla w\|_{L^2} + C\|\rho - \bar{\rho}\|_{L^p}
\]

\[
\leq C(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2})^{\beta} \|\rho\|_{L^2} + C\|\nabla u\|_{L^2} + C\|\rho - \bar{\rho}\|_{L^p},
\]

where \(\beta = \frac{6-p}{2p}\). Combining (6.4), (4.121), Theorems 3.1 and 3.2 gives (1.14).

We therefore finish the proof of Theorem 1.1.

**Proof of Theorem 1.2.** Motivated by [25], by virtue of (1.12), for \(T > 0\) the Lagrangian coordinate is well-defined. Therefore, the conservation of mass yields

\[
\rho(x, t) = \rho_0(X(0; t, x)) \exp\{-\int_0^t \text{div}u(X(\tau; t, x), \tau) d\tau\}.
\]

If there exists some point \(x_0 \in \Omega\) such that \(\rho_0(x_0) = 0\), then there is a point \(x_0(t) \in \bar{\Omega}\) such that \(X(0; t, x_0(t)) = x_0\). By (6.5), \(\rho(x_0(t), t) \equiv 0\) for any \(t \geq 0\). As a result, by the Gagliardo-Nirenberg inequality, we declare for \(r \in (3, \infty)\),

\[
\hat{\rho} \leq \|\rho - \bar{\rho}\|_{C(\bar{\Omega})} \leq C\|\rho - \bar{\rho}\|_{L^2}^{\theta_1} \|\nabla \rho\|_{L^r}^{1-\theta_1},
\]

where \(\theta_1 = \frac{2r-6}{5r-6}\). Combining this with Theorem 5.1 shows

\[
\|\nabla \rho(\cdot, t)\|_{L^r} \geq \hat{C}_1 e^{\hat{C}_2 t}.
\]

The proof of Theorem 1.2 is complete.

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