THE FIRST MOMENT OF CUSP FORM L-FUNCTIONS IN WEIGHT ASPECT ON AVERAGE

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ABSTRACT. We study the asymptotic behaviour of the twisted first moment of central L-values associated to cusp forms in weight aspect on average. Our estimate of the error term allows extending the logarithmic length of mollifier \( \Delta \) up to 2. The best previously known result, due to Iwaniec and Sarnak, was \( \Delta < 1 \). The proof is based on a representation formula for the error in terms of Legendre polynomials.

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1. INTRODUCTION

The technique of mollification allows proving strictly positive non-vanishing results for different families of L-functions. The idea of the method is to regularize the behavior of L-functions while averaging over the family by introducing smoothing weights called mollifiers. Common choice for mollifier is a Dirichlet polynomial of length \( M \) approximating the inverse of L-function. It is of crucial importance to optimize the parameter \( M \), called mollifier’s length, as it determines the proportion of non-vanishing L-values.

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Consider the family $H_{2k}(1)$ of primitive forms of level 1 and weight $2k \geq 12$. Every $f \in H_{2k}(1)$ has a Fourier expansion of the form

$$f(z) = \sum_{n \geq 1} \lambda_f(n)n^{(2k-1)/2}e(nz).$$

(1.1)

The associated $L$-function is defined by

$$L_f(s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \Re s > 1.$$  

(1.2)

Let $\Gamma(s)$ be the Gamma function. The completed $L$-function

$$\Lambda_f(s) = \left(\frac{1}{2\pi}\right)^s \Gamma\left(s + \frac{2k-1}{2}\right)L_f(s)$$

satisfies the functional equation

$$\Lambda_f(s) = \epsilon_f \Lambda_f(1 - s), \quad \epsilon_f = i^{2k}$$

(1.4)

and can be analytically continued on the whole complex plane.

The harmonic summation is defined by

$$\sum_{f \in H_{2k}(1)} \alpha_f := \sum_{f \in H_{2k}(1)} \alpha_f \frac{\Gamma(2k - 1)}{(4\pi)^{2k-1} \langle f, f \rangle_1},$$

(1.5)

where $\langle f, f \rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms.

The usual choice for mollifier is

$$M(f) = \sum_{m \leq M} x_m \frac{\lambda_f(m)}{m^{1/2}}, \quad x_m \in \mathbb{R},$$

(1.6)

where parameter $M$ is called the length of mollifier.

Let $h$ be a suitable test function (see section 5 for details) and

$$H := \int_0^\infty h(y) dy.$$ 

(1.7)

Let $\mu(m)$ be the Möbius function and $\sigma(m)$ be the sum of divisors function. Iwaniec and Sarnak [5, Theorem 3] proved that for the mollifier with

$$x_m \sim \frac{\mu(m)m(\log M/\log m)^2}{\sigma(m)2\zeta(2)\log M}$$

(1.8)

of length $M \leq K(\log K)^{-20}$, one has

$$\sum_k h\left(\frac{2k}{K}\right) \sum_{f \in H_{2k}(1)} L_f(1/2)M(f) \sim HK,$$

(1.9)
(1.10) \[ \sum_k h\left(\frac{2k}{K}\right) \sum_{f \in H_{2k}(1)}^h L_f^2(1/2)M^2(f) \sim 2HK\left(1 + \frac{\log K}{\log M}\right). \]

Let \( M := K^\Delta \), where parameter \( \Delta \) is called the logarithmic length of mollifier. Note that if \( \epsilon_f = i^{2k} = -1 \), then it follows from functional equation (1.4) that \( L_f(1/2) \) is identically zero. For \( \epsilon_f = 1 \) equations (1.9), (1.10) imply (see [2] for details) that at least

(1.11) \[ \frac{\Delta}{\Delta + 1} \]

of central \( L \)-values do not vanish on average as \( K \to \infty \).

Taking the largest admissible \( \Delta = 1 - \epsilon \), the percentage of non-vanishing is no less than 50\%. Furthermore, according to [5, Proposition 16] any improvement over 50\% with an additional lower bound on \( L_f(1/2) \) would imply the non-existence of Landau-Siegel zeros for Dirichlet \( L \)-functions of real primitive characters.

In order to break the 50\% barrier one needs to increase the length of mollifier for both first and second moments.

In the present paper we consider only the first moment and show that equation (1.9) holds for the length of mollifier \( M \leq K^{2-\epsilon} \) for any \( \epsilon > 0 \). This extension follows from the asymptotic formula for the twisted first moment.

**Theorem 1.1.** For all \( l \) one has

(1.12) \[ M_1(l) := \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h \lambda_f(l)L_f(1/2) = \frac{2}{\sqrt{l}} \frac{HK}{4} + O\left(\frac{K^{1/2+\epsilon}}{K^2}\right). \]

More precisely, the mollified moment can be expressed in terms of the twisted moment

\[ \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h L_f(1/2)M(f) = \sum_{m \leq M} \frac{x_m}{\sqrt{m}} M_1(m). \]

Then the length of mollifier is the largest admissible \( M \) such that

\[ K \sum_{m \leq M} \frac{|x_m|}{\sqrt{m}} \frac{m^{1/2+\epsilon}}{K^2} \ll K^{1-\epsilon}. \]

Therefore, for any mollifier with \( |x_m| \leq \log M \) and any \( \epsilon > 0 \) one can take \( M \leq K^{2-\epsilon} \). Consequently, the logarithmic length of mollifier \( \Delta \) can be extended up to 2.
The detailed description of the mollifier method and analogous results for an individual weight can be found in [2].

2. Special functions

For \( z \in \mathbb{C}, \Re z > 0 \) the Gamma function is defined by

\[
\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) \, dt.
\]

By [6, Eq. 5.5.5] for \( 2z \neq 0, -1, -2, \ldots \) one has

\[
\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z) \Gamma(z + 1/2).
\]

Let

\[
e(x) := \exp (2\pi ix).
\]

The confluent hypergeometric function

\[
_{1}F_{1}(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k) x^k}{\Gamma(b+k) k!}
\]

can be expressed in terms of the Bessel function of the first kind

\[
J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu+1)} \left( \frac{x}{2} \right)^{2m+\nu}.
\]

Lemma 2.1. For \( \epsilon = \pm 1 \) one has

\[
_{1}F_{1}(k, 2k; 2z) = \Gamma(k + 1/2) \exp(z) \left( \frac{z}{2} \right)^{1/2-k} \times
\]

\[
e \left( \epsilon \frac{1/2 - k}{4} \right) J_{k-1/2} \left( z \epsilon \left( \frac{\epsilon}{4} \right) \right).
\]

Proof. Using [6, Eq. 13.2.2] and [6, Eq. 13.6.9], we write the confluent hypergeometric function in terms of the \( I \)-Bessel function. Further, applying [6, Eq. 10.27.6], we prove the required result. \( \square \)

Legendre polynomials are \( n \)th degree polynomials given by Rodrigues’ formula

\[
P_n(x) = \frac{1}{2^n n! \, dx^n} \left[ (x^2 - 1)^n \right].
\]

Note that by [6, Eq. 14.7.17]

\[
P_n(-x) = (-1)^n P_n(x).
\]

Lemma 2.2. For any nonnegative integer \( n \) one has

\[
J_{n+1/2}(z) = (-i)^n \sqrt{\frac{z}{2\pi}} \int_0^\pi \exp(iz \cos \theta) P_n(\cos \theta) \sin \theta \, d\theta.
\]
Proof. The assertion follows from [6 Eq. 10.47.3] and [6 Eq. 10.54.2]. □

Lemma 2.3. Let $|\arg z| < \pi$. As $z \to \infty$ we have

\[ J_0(z) = \sqrt{\frac{2}{\pi z}} \left( \cos(z - \pi/4) \left[ \sum_{j=0}^{d-1} (-1)^j \frac{a_{2j}}{z^{2j}} + R_1 \right] - \sin(z - \pi/4) \left[ \sum_{j=0}^{d-1} (-1)^j \frac{a_{2j+1}}{z^{2j+1}} + R_2 \right] \right), \]

where

\[ a_j = \frac{\Gamma(j + 1/2)}{2^j j! \Gamma(-j + 1/2)} \text{ for } j \geq 0, \]

\[ R_1 = O \left( \frac{1}{(2\pi)^{2d}} \right), \quad R_2 = O \left( \frac{1}{(2\pi)^{2d+1}} \right). \]

Proof. See [4 Eq. 8.451.1] and [6 Eq. 10.17.3]. □

3. Exact Formula for the First Moment

In this section we consider the first moment of primitive $L$-functions and show how to express the error in terms of special functions. For $\epsilon_1 = \pm 1$ let us define

\[ I_{\epsilon_1}(u, v, k; x) := e \left( \frac{\epsilon_1 k}{8} - \frac{\epsilon_1 k}{4} \right) x^{1/2-k} \frac{\Gamma(k - v - u)}{\Gamma(2k)} \times \, _1F_1 \left( k - v - u, 2k, -\frac{e(-\epsilon_1/4)}{x} \right). \]

As a consequence of the Petersson trace formula we obtain the exact formula for the twisted first moment.

Theorem 3.1. For $2k \geq 12$, $\Re v = 0$, $|\Re u| < k - 1$ we have

\[ \sum_{f \in \mathcal{H}_{2k}(1)} \lambda_f(l) L_f(1/2 + u + v) = \frac{1}{l^{1/2+u+v}} \]

\[ + i^{2k} \frac{(2\pi)^{2u+2v} \Gamma(k - u - v)}{l^{1/2-u-v} \Gamma(k + u + v)} + 2\pi i^{2k} V_1(l; u, v, k). \]
The error term is given by

\begin{equation}
V_1(l; u, v, k) = \sum_{c=1}^{\infty} \frac{1}{c^{1/2+u+v}} \sum_{n=1}^{\infty} \frac{e(n^*lc^{-1})}{n^{1/2-u-v}} (2\pi)^{u+v-1/2} \times e\left(\frac{1/2 - u - v}{4}\right) I_{-1} \left(u, v, k; \frac{cn}{2\pi l}\right) + \sum_{c=1}^{\infty} \frac{1}{c^{1/2+u+v}} \sum_{n=1}^{\infty} \frac{e(-n^*lc^{-1})}{n^{1/2-u-v}} \times (2\pi)^{u+v-1/2} e\left(-\frac{1/2 - u - v}{4}\right) I_{+1} \left(u, v, k; \frac{cn}{2\pi l}\right) ,
\end{equation}

where \( nn^* \equiv 1 \pmod{c} \).

**Proof.** This formula was proved in [1, Sections 4-5] for prime power level \( N = p^v, p \text{ prime, } v \geq 2 \). When the level \( N \) is equal to 1, the function under the integral in [1, Eq. 4.16] has a pole in view of [1, Eq. 4.15]. Consequently, we cross this pole while shifting the contour of integration in the proof of [1, Lemma 4.8]. This yields the additional main term

\[ \frac{\pi^2 (2\pi)^{2u+2v} \Gamma(k-u-v)}{\Gamma(2-k)} \]

in formula (3.2). The rest of the proof is exactly the same. \( \square \)

We are interested in the behavior of the first moment at the critical point 1/2 and therefore can let \( u = v = 0 \).

**Lemma 3.2.** For \( \epsilon_1 = \pm 1 \) one has

\begin{equation}
e(-\epsilon_1/8) I_{\epsilon_1}(0, 0, k; x) = \sqrt{\pi} e\left(\frac{\epsilon_1}{4\pi x}\right) e\left(-\frac{\epsilon_1 k}{4}\right) J_{k-1/2} \left(\frac{1}{2x}\right) .
\end{equation}

**Proof.** Substituting representation (2.5) in equation (3.1) we have

\[ e(-\epsilon_1/8) I_{\epsilon_1}(0, 0, k; x) = e\left(-\frac{\epsilon_1 k}{4}\right) x^{1/2-k} \frac{\Gamma(k)\Gamma(k+1/2)}{\Gamma(2k)} \times 2^{2k-1} \exp\left(-\frac{\epsilon(-\epsilon_1/4)}{2x}\right) \left(\frac{\epsilon(-\epsilon_1/4)}{x}\right)^{1/2-k} \times e\left(\frac{\epsilon_2}{4}\right) J_{k-1/2} \left(-\frac{e(-\epsilon_1/4)e(\epsilon_2/4)}{2x}\right) ,
\]

where \( \epsilon_2 = \pm 1 \). Note that \(-e(-\epsilon_1/4) = \epsilon_1 i\). Choosing \( \epsilon_2 = -\epsilon_1 \) yields

\[ -e(-\epsilon_1/4)e(\epsilon_2/4) = -\exp(\pi i) = 1 .
\]
Thus
\[ e(-\epsilon_1/8)I_{\epsilon_1}(0, 0, k; x) = \frac{\Gamma(k)\Gamma(k + 1/2)}{\Gamma(2k)} 2^{2k-1} e^\left(\frac{\epsilon_1}{4\pi x}\right) e^\left(-\frac{-\epsilon_1 k}{4}\right) J_{k-1/2} \left(\frac{1}{2x}\right). \]

It follows by equation (2.2) that
\[ e(-\epsilon_1/8)I_{\epsilon_1}(0, 0, k; x) = \sqrt{\pi} e^\left(\frac{\epsilon_1}{4\pi x}\right) e^\left(-\frac{-\epsilon_1 k}{4}\right) J_{k-1/2} \left(\frac{1}{2x}\right). \]

Corollary 3.3. For \( \epsilon_1 = \pm 1 \) one has
\[ (3.5) \quad I_{\epsilon_1}(0, 0, k; x) \ll \frac{(4x)^{-k+1/2}}{\Gamma(k + 1/2)}. \]

Proof. By formula [6, Eq. 10.14.4]
\[ |J_{k-1/2}(z)| \leq \frac{(z/2)^{k-1/2}}{\Gamma(k + 1/2)}. \]

Taking \( z = 1/(2x) \) we prove the assertion.

Furthermore, function \( I_{\epsilon_1} \) has an integral representation in terms of Legendre polynomials.

**Lemma 3.4.** For \( k \equiv 0 \mod 2 \), \( \epsilon_1 = \pm 1 \) one has
\[ (3.6) \quad I_{\epsilon_1} \left(0, 0, \frac{1}{2z}\right) = -e^\left(\frac{\epsilon_1}{8}\right) e^\left(\frac{\epsilon_1 z}{2\pi}\right) \sqrt{2z} \int_{0}^{\pi/2} \exp\left(iz \cos \theta\right) P_{k-1}(\cos \theta) \sin \theta d\theta. \]

Proof. Consider representation (3.4) with \( z := (2x)^{-1} \). Applying (2.8) with \( n = k - 1 \), we obtain
\[ e(-\epsilon_1/8)I_{\epsilon_1} \left(0, 0, \frac{1}{2z}\right) = e\left(\frac{-\epsilon_1 k}{4}\right) e\left(-\frac{k}{4} + \frac{1}{4}\right) e\left(\frac{\epsilon_1 z}{2\pi}\right) \times \sqrt{\frac{z}{2}} \int_{0}^{\pi} \exp(iz \cos \theta) P_{k-1}(\cos \theta) \sin \theta d\theta. \]

Since \( k \) is an even integer, one has \( e(-\epsilon_1 k/4)e(-k/4) = 1 \) and
\[ e(-\epsilon_1/8)I_{\epsilon_1} \left(0, 0, \frac{1}{2z}\right) = e(1/4)e\left(\frac{\epsilon_1 z}{2\pi}\right) \times \sqrt{\frac{z}{2}} \int_{0}^{\pi} \exp(iz \cos \theta) P_{k-1}(\cos \theta) \sin \theta d\theta. \]
Now we split the integral into two parts \(\int_0^\pi = \int_0^{\pi/2} + \int_{\pi/2}^\pi\) and make the change of variables \(\phi := \pi - \theta\) in the second integral. Property [2.7] yields that
\[
P_{k-1}(-\cos \phi) = (-1)^{k-1} P_{k-1}(\cos \phi) = -P_{k-1}(\cos \phi)
\]
for even \(k\). Finally, since \(e(1/4) = \exp(\pi i/2) = i\) we have
\[
e(1/4) \int_0^\pi \exp(iz \cos \theta) P_{k-1}(\cos \theta) \sin \theta d\theta =
\]
\[
i \int_0^{\pi/2} P_{k-1}(\cos \theta) \sin \theta \left[\exp(iz \cos \theta) - \exp(-iz \cos \theta)\right] d\theta =
\]
\[
-2 \int_0^{\pi/2} \sin(z \cos \theta) P_{k-1}(\cos \theta) \sin \theta d\theta.
\]
The assertion follows. \(\square\)

4. Asymptotic approximation of Legendre polynomials

The following theorem is obtained by taking \(\alpha = \beta = 0\) in [3, Eq. 1.1-1.3].

**Theorem 4.1.** (Baratella, Gatteschi, [3]) Let \(N := n + 1/2\). Then

\[
P_n(\cos \theta) = \sqrt{\frac{\theta}{\sin \theta}} \left( J_0(N\theta) \sum_{s=0}^m A_s(\theta) N^{-2s} + \theta J_1(N\theta) \sum_{s=0}^{m-1} B_s(\theta) N^{-2s+1} + E_m \right),
\]

where for fixed positive constants \(c\) and \(\delta\) one has

\[
E_m \ll \begin{cases} \theta^{1/2} N^{-2m-3/2} & \text{if } c/N \leq \theta \leq \pi - \delta \\ \theta^2 N^{-2m} & \text{if } 0 < \theta \leq c/N \end{cases}.
\]

The functions \(A_s(\theta)\), \(B_s(\theta)\) are analytic for \(0 \leq \theta < \pi\) and defined recursively, starting from \(A_0(\theta) = 1\), by

\[
\theta B_s(\theta) = -\frac{1}{2} A'_s(\theta) - \frac{1}{2} \int_0^\theta \frac{A'_s(t)}{t} dt + \frac{1}{2} \int_0^\theta f(t) A_s(t) dt,
\]

\[
A_{s+1}(\theta) = \frac{1}{2} \theta B'_s(\theta) - \frac{1}{2} \int_0^\theta t f(t) B_s(t) dt + \lambda_{s+1},
\]

with

\[
f(t) = \frac{1}{4t^2} - \frac{1}{16 \sin^2(t/2)} - \frac{1}{16 \cos^2(t/2)},
\]
where the constants of integration $\lambda_{s+1}$ are chosen such that $A_{s+1}(0) = 0$ for any integral $s \geq 0$.

5. Averaging over weight

Let $h \in C_0^\infty(\mathbb{R}^+)$ be a non-negative function, compactly supported on interval $[\theta_1, \theta_2]$ such that $\theta_2 > \theta_1 > 0$ and

\[(5.1) \quad \|h^{(n)}\|_1 \ll 1 \text{ for all } n \geq 0.
\]

Applying the Poisson summation and integrating by parts $a \geq 2$ times, we have

\[(5.2) \quad \sum_k h\left(\frac{4k}{K}\right) = \frac{HK}{4} + O\left(\frac{1}{K^a}\right),
\]

where

\[H = \int_0^\infty h(y)dy.
\]

In this section we prove theorem 1.1. Namely we show that for all $l$ one has

\[(5.3) \quad \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)} \lambda_f(l)L_f(1/2) = \frac{2}{\sqrt{l}} \frac{HK}{4} + O\left(Kl^{1/2+\epsilon}/K^2\right).
\]

The main term in (5.3) is obtained by taking $u = v = 0$ in theorem 3.1 and averaging the main terms with respect to $k$. Note that in formula (5.3) the summation is over elements of weight $4k$, and therefore, Theorem 3.1 should be used with $k$ replaced by $2k$. The same applies to all other results of Section 3.

Consider (3.3) with $u = v = 0$. Let $z := \pi l/(cn)$. We split the error term into two parts

\[V_1(l; 0, 0, 2k) = W_1(l, k) + W_2(l, k),
\]

where the summation in $W_1(l, k)$ is over $c, n$ such that $z < k/5$ and in $W_2(l, k)$ such that $z \geq k/5$.

**Lemma 5.1.** There exists an absolute constant $C > 1$ such that

\[(5.4) \quad \sum_k h\left(\frac{4k}{K}\right) W_1(l, k) \ll \frac{l^{1/2+\epsilon}K^\epsilon}{C^K}.
\]
Proof. Let $d := cn$. Since $z < k/5$ one has $d > d_0$ with $d_0 := 5\pi l/k$. By corollary 3.3

$$W_1(l, k) \ll \frac{1}{\Gamma(2k + 1/2)} \sum_{d > d_0} d^{-1/2 + \epsilon} \left( \frac{\pi l}{2d} \right)^{2k - 1/2} \ll \frac{e^{2k}}{(2k)^{2k}} \left( \frac{\pi l}{2d_0} \right)^{2k - 1/2} \ll \frac{l^{1/2+\epsilon}K^{-1}}{20}.$$ 

Summing the result over $k$ with the test function, we prove the assertion.

If $l \ll K$ with a sufficiently small implied constant, the sums over $c$ and $n$ in $W_2(l, k)$ are empty and the error term in (5.3) can be estimated using lemma 5.1. Otherwise, the main contribution comes from the term involving $W_2(l, k)$, as we now show.

Lemma 5.2. For any $\epsilon > 0$ one has

$$(5.5) \quad \sum_k h \left( \frac{4k}{K} \right) W_2(l, k) \ll K \frac{l^{1/2+\epsilon}}{K^2}.$$ 

Proof. It follows from lemma 3.4 that

$$\sum_k h \left( \frac{4k}{K} \right) W_2(l, k) \ll \sum_{cn < l/k} \sqrt{l} \int_0^{\pi/2} \left| \sum_k h \left( \frac{4k}{K} \right) P_{2k-1}(\cos \theta) \right| \sin \theta d\theta.$$ 

To approximate the Legendre polynomials we apply theorem 4.1 with $m = 1$ and $N = 2k - 1/2$, i.e.

$$P_{2k-1}(\cos \theta) = \sqrt{\frac{\theta}{\sin \theta}} \left( J_0(N\theta) \left( 1 + \frac{A_1(\theta)}{N^2} \right) + \theta J_1(N\theta) \frac{B_0(\theta)}{N} + E_1 \right),$$

where $B_0(\theta)$ and $A_1(\theta)$ are defined by (4.3), (4.4).
First, we estimate the contribution of the error term $E_1$ as follows

\[ l^{1/2+\epsilon} \sum_k h \left( \frac{4k}{K} \right) \int_0^{\pi/2} \sqrt{\theta} \sin \theta |E_1| d\theta \ll l^{1/2+\epsilon} \sum_k h \left( \frac{4k}{K} \right) \times \]

\[ \left( \int_0^{c/N} \frac{\theta^3 d\theta}{N^2} + \int_{c/N}^{\pi/2} \frac{\theta^3/2 d\theta}{N^{7/2}} \right) \ll l^{1/2+\epsilon} \sum_k h \left( \frac{4k}{K} \right) \frac{1}{K^{3/2}} \ll K l^{1/2+\epsilon}. \]

For the main terms the largest contribution comes from the first summand, namely

\[ MT := l^{1/2+\epsilon} \int_0^{\pi/2} \left| \sum_k h \left( \frac{4k}{K} \right) J_0(N\theta) \right| \sqrt{\theta} \sin \theta d\theta. \]

Indeed, two other summands have similar oscillation but they are smaller in terms of absolute value because

\[ \theta B_0(\theta) = O(\theta), \quad A_1(\theta) = O(\theta^2). \]

Note that the oscillation in $MT$ is only possible when $N\theta \gg 1$. Hence let us split the integral over $\theta$ into two parts at the point $t := C/K$ for some absolute constant $C > 1$. The first part is bounded by

\[ M_1 := l^{1/2+\epsilon} \int_0^t \left| \sum_k h \left( \frac{4k}{K} \right) J_0(N\theta) \right| \sqrt{\theta} \sin \theta d\theta \ll K l^{1/2+\epsilon} K^2. \]

Now we estimate the second part

\[ M_2 := l^{1/2+\epsilon} \int_t^{\pi/2} \left| \sum_k h \left( \frac{4k}{K} \right) J_0(N\theta) \right| \sqrt{\theta} \sin \theta d\theta. \]

For the $J$-Bessel function we apply representation (2.9). For $d \geq 1$ the contribution of $R_1$, $R_2$ is majorated by

\[ M_{2,1} := l^{1/2+\epsilon} \sum_k h \left( \frac{4k}{K} \right) \frac{1}{K^{1/2+2d}} \int_t^{\pi/2} \frac{\theta}{\theta^{1/2+2d}} d\theta \ll \]

\[ l^{1/2+\epsilon} K K^{1/2+2d} t^{1/2+2d} \ll l^{1/2+\epsilon} K K^2. \]

Since $N\theta \gg 1$ it is sufficient to consider only the contribution of the first summand in (2.9), which is bounded by

\[ M_{2,2} := l^{1/2+\epsilon} \int_t^{\pi/2} \sqrt{\theta} \left| \sum_k \frac{h \left( \frac{4k}{K} \right)}{\sqrt{2k - 1/2}} \cos \left( (2k - 1/2)\theta - \frac{\pi}{4} \right) \right| d\theta. \]
Other summands in (2.9) have similar oscillation but are smaller in absolute value. By Poisson’s summation formula the sum over \( k \) is majorated by a linear combination of expressions of the form

\[
\sum_{u \in \mathbb{Z}} \int_{-\infty}^{\infty} h \left( \frac{4y}{K} \right) \frac{\exp(i 2y \theta)}{\sqrt{y}} \exp(-2\pi i u y) dy = \frac{K}{\sqrt{K}} \sum_{u \in \mathbb{Z}} \int_{-\infty}^{\infty} h \left( \frac{4y}{K} \right) \exp(i y K (2\theta - 2\pi u)) dy.
\]

Since \( 0 < \theta \leq \pi/2 \) one has \( |2\theta - 2\pi u| \gg u \) for \( u \neq 0 \). Integrating by parts \( a \geq 2 \) times, we obtain

\[
M_{2,2} \ll l^{1/2+\epsilon} \int_{t}^{\pi/2} \sqrt{\theta} \frac{K}{\sqrt{K}} \left( \frac{1}{(\theta K)^a} + \sum_{u \neq 0} \frac{1}{(K u)^a} \right) \times \int_{-\infty}^{+\infty} \left| \frac{\partial^a}{\partial y^a} \left( \frac{h(y)}{\sqrt{y}} \right) \right| dy d\theta.
\]

It follows from the definition of function \( h \) that

\[
\int_{-\infty}^{+\infty} \left| \frac{\partial^a}{\partial y^a} \left( \frac{h(y)}{\sqrt{y}} \right) \right| dy \ll 1.
\]

Thus

\[
M_{2,2} \ll l^{1/2+\epsilon} \frac{K}{\sqrt{K}} \int_{t}^{\pi/2} \sqrt{\theta} \frac{d\theta}{(\theta K)^a} \ll l^{1/2+\epsilon} \frac{K}{K^{a+1/2} t^{a}} \ll K \frac{l^{1/2+\epsilon}}{K^2}.
\]

Finally,

\[
MT \ll M_1 + M_2 \ll M_1 + M_{2,1} + M_{2,2} \ll K \frac{l^{1/2+\epsilon}}{K^2}.
\]

\[ \square \]

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