Cuspidal representations of $GL(n, F)$ distinguished by a maximal Levi subgroup, with $F$ a non-archimedean local field

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Abstract

Let $\rho$ is a cuspidal representation of $GL(n, F)$, with $F$ a non archimedean local field, and $H$ a maximal Levi subgroup of $GL(n, F)$. We show that if $\rho$ is $H$-distinguished, then $n$ is even, and $H \simeq GL(n/2, F) \times GL(n/2, F)$.

1 Preliminaries

Let $F$ be nonarchimedean local field. We denote $GL(n, F)$ by $G_n$ for $n \geq 1$, and by $N_n$ the unipotent radical of the Borel subgroup of $G_n$ given by upper triangular matrices. For $n \geq 2$ we denote by $U_n$ the group of matrices $u(x) = \begin{pmatrix} \lambda_n(x) & x \\ 0 & 1 \end{pmatrix}$ for $x \in F^{n-1}$.

For $n > 1$, the map $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ is an embedding of the group $G_{n-1}$ in $G_n$, we denote by $P_n$ the subgroup $G_{n-1}U_n$ of $G_n$.

We fix a nontrivial character $\theta$ of $(F, +)$, and denote by $\theta$ again the character $n \mapsto \theta(\sum_{i=1}^{n-1} n_i, i+1)$ of $N_n$. The normaliser of $\theta |_{U_n}$ is then $P_{n-1}$.

When $G$ is an $l$-group (locally compact totally disconnected group), we denote by $Alg(G)$ the category of smooth complex $G$-modules. If $(\pi, V)$ belongs to $Alg(G)$, $H$ is a closed subgroup of $G$, and $\chi$ is a character of $H$, we denote by $\delta_H$ the positive character of $N_G(H)$ such that if $\mu$ is a right Haar measure on $H$, and $int$ is the action given by $(int(n)f)(h) = f(n^{-1}hn)$, of $N_G(H)$ smooth functions $f$ with compact support on $H$, then $\mu \circ int(n) = \delta_H(n) \mu$ for $n$ in $N_G(H)$.

If $H$ is a closed subgroup of an $l$-group $G$, and $(\rho, W)$ belongs to $Alg(H)$, we define the object $(\text{ind}_H^G(\rho), V_c = \text{ind}_H^G(W))$ as follows. The space $V_c$ is the space of smooth functions from $G$ to $W$, fixed under right translation by the elements of a compact open subgroup $U_l$ of $G$, satisfying $f(hg) = \rho(h)f(g)$ for all $h$ in $H$ and $g$ in $G$, and with support compact mod $H$. The action of $G$ is by right translation on the functions.

If $f$ is a function from $G$ to another set, and $g$ belongs to $G$, we will denote $L(g)f : x \mapsto f(g^{-1}x)$ and $R(g)f : x \mapsto f(xg)$.

We say that a representation $\pi$ of $G$ is $H$-distinguished, if the complex vector space $\text{Hom}_H(\pi, 1)$ is nonzero.

We will use the following functors following [B-Z]:

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Lemma 2.2. The functor $\Phi^+$ from $\text{Alg}(P_{k-1})$ to $\text{Alg}(P_k)$ such that, for $\pi$ in $\text{Alg}(P_{k-1})$, one has $\Phi^+\pi = \text{ind}_{G_{k-1},U_k}^G (\delta^1_{U_k} \pi \otimes \theta)$.

Lemma 2.1. The functor $\Psi^+$ from $\text{Alg}(G_{k-1})$ to $\text{Alg}(P_k)$, such that for $\pi$ in $\text{Alg}(G_{k-1})$, one has $\Psi^+\pi = \text{ind}_{G_{k-1},U_k}^G (\delta^1_{U_k} \pi \otimes 1) = \delta^1_{U_k} \pi \otimes 1$.

We recall the following proposition, which is a consequence of theorem 4.4 of [B-Z].

Proposition 1.1. Let $\pi$ be a cuspidal representation of $G_n$, then the restriction $\pi_{|P_n}$ is isomorphic to $(\Phi^+)^n \Psi^+ (1)$.

2 The result

Suppose $n = p + q$, with $p \geq q \geq 1$, we denote by $M_{(p,q)}$ the standard Levi of $G_n$ given by matrices

\[
\begin{pmatrix}
h_p \\
h_q
\end{pmatrix}
\]

with $h_p \in G_p$ and $h_q \in G_q$, and by $M_{(p,q-1)}$ the standard Levi of $G_{n-1}$ given by matrices

\[
\begin{pmatrix}
h_p \\
h_{q-1}
\end{pmatrix}
\]

with $h_p \in G_p$ and $h_{q-1} \in G_{q-1}$. We denote by $M_{(p-1,q-1)}$ the standard Levi of $G_{n-2}$ given by matrices

\[
\begin{pmatrix}
h_{p-1} \\
q_{(q-1)}
\end{pmatrix}
\]

Let $w_{p,q}$ be the permutation matrix of $G_n$ corresponding to the permutation

\[
\begin{pmatrix}
1 & \ldots & p - q & p - q + 1 & p - q + 2 & \ldots & p - 1 & p & p + 1 & \ldots & p + q - 2 & p + q - 1 & p + q \\
1 & \ldots & p - q & p - q + 1 & p - q + 2 & \ldots & p - 3 & p - q + 3 & p - q + 2 & \ldots & p + q - 4 & p + q - 2 & p + q
\end{pmatrix}
\]

Let $w_{p,q-1}$ be the permutation matrix of $G_{n-1}$ corresponding to the permutation $w_{p,q}$ restricted to $\{1, \ldots, n-1\}$:

\[
\begin{pmatrix}
1 & \ldots & p - q & p - q + 1 & p - q + 2 & \ldots & p - 1 & p & p + 1 & \ldots & p + q - 2 & p + q - 1 \\
1 & \ldots & p - q & p - q + 1 & p - q + 2 & \ldots & p - 3 & p - q + 3 & p - q + 2 & \ldots & p + q - 4 & p + q - 2
\end{pmatrix}
\]

Let $w_{p-1,q-1}$ be the permutation matrix of $G_{n-2}$ corresponding to the permutation

\[
\begin{pmatrix}
1 & \ldots & p - q & p - q + 1 & p - q + 2 & \ldots & p - 2 & p - 1 & p & \ldots & p + q - 3 & p + q - 2 \\
1 & \ldots & p - q & p - q + 1 & p - q + 2 & \ldots & p - 3 & p - q + 3 & p - q + 2 & \ldots & p + q - 4 & p + q - 2
\end{pmatrix}
\]

We denote by $H_{p,q}$ the subgroup $w_{p,q}M_{(p,q)}w_{p,q}^{-1}$ of $G_n$, by $H_{p,q-1}$ the subgroup $w_{p,q-1}M_{(p,q-1)}w_{p,q-1}^{-1}$ of $G_{n-1}$, and by $H_{p-1,q-1}$ the subgroup $w_{p-1,q-1}M_{(p-1,q-1)}w_{p-1,q-1}^{-1}$ of $G_{n-2}$.

The two following lemmas and propositions are a straightforward adaptation of Lemma 1 and Proposition 1 of [K].

Lemma 2.1. Let $S_{p,q} = \{g \in G_{n-1}, \forall u \in U_n \cap H_{p,q}, \theta(gug^{-1}) = 1\}$. Then $S_{p,q} = P_{n-1}H_{p,q-1}$.

Proof. Denoting by $L_{n-1}(g)$ the bottom row of $g$, one has $\theta(gu(x)g^{-1}) = \theta(L_{n-1}(g),x)$ for $u(x)$ in $U_n$. Hence $\theta(gug^{-1}) = 1$ for all $u$ in $U_n \cap H_{p,q}$ if and only if $g_{n-1,j} = 0$ for $j = p - q, p - q + 2, \ldots, p - q + 2$. It is equivalent to say that $g$ belongs to $P_{n-1}H_{p,q-1}$. \hfill $\Box$

Lemma 2.2. Let $S_{p,q-1} = \{g \in G_{n-2}, \forall u \in U_{n-1} \cap H_{p,q-1}, \theta(gug^{-1}) = 1\}$. Then $S_{p,q} = P_{n-2}H_{p-1,q-1}$.

Proof. Denoting by $L_{n-2}(g)$ the bottom row of $g$, and by $u(x)$ the matrix

\[
\begin{pmatrix}
I_{n-2} & x \\
0 & 1
\end{pmatrix}
\]

so that $\theta(gug^{-1}) = \theta(L_{n-2}(g),x)$. Hence $\theta(gug^{-1}) = 1$ for all $u$ in $U_{n-1} \cap H_{p,q-1}$ if and only if $g_{n-2,j} = 0$ for $j = 0, 1, \ldots, p - q, p - q + 1$ and $j = p - q + 3, p - q + 5, \ldots, p - q - 3, p - q - 1$. It is equivalent to say that $g$ belongs to $P_{n-2}H_{p-1,q-1}$. \hfill $\Box$
Proposition 2.1. Let \( \sigma \) belong to \( \text{Alg}(P_{n-1}) \), and \( \chi \) be a positive character of \( P_{n-1} \cap H_{p,q} \), then there is a positive character \( \chi' \) of \( P_{n-1} \cap H_{p,q} \), such that
\[
\text{Hom}_{P_{n-1} \cap H_{p,q}}(\Phi^+ \sigma, \chi) \leftrightarrow \text{Hom}_{P_{n-1} \cap H_{p,q}}(\sigma, \chi').
\]

Proof. Let \( V \) be the space on which \( \sigma \) acts, and \( W = \phi^+ V \). Let \( A \) the projection from \( C_\infty^0(P_n, V) \) onto \( W \), defined by \( Af(p) = \int_{P_{n-1}U_n} \delta_{U_n}^{-1/2}(y) \sigma(y^{-1})f(yg)dy \). Lifting through \( A \) gives a vector space injection of \( \text{Hom}_{P_{n-1} \cap H_{p,q}}(\Phi^+ \sigma, \chi) \) into the space of \( V \)-distributions \( T \) on \( P_n \) satisfying relations
\[
T \circ R(h_0) = \chi(h_0)T
\]
for \( h_0 \) in \( P_{n-1} \cap H_{p,q} \) and \( y_0 \in P_{n-1}U_n \).

We introduce \( \Theta \) the map on \( P_n \) defined by \( \Theta(ug) = \theta(u) \) for \( u \) in \( U_n \) and \( g \) in \( G_{n-1} \). Then the \( V \)-distribution \( \Theta \cdot T \) is \( U_n \)-invariant, hence there is a \( V \)-distribution \( S \) with support in \( G_{n-1} \) such that \( \Theta \cdot T = du \otimes S \) (where \( du \) denotes a Haar measure on \( U_n \)), and thus \( T = \Theta^{-1} \cdot du \otimes S \) has support \( U_n \cdot \text{supp}(S) \). It is easily verified that \( du \otimes S \) is invariant of \( U_n \), but because of relation \( \Theta \), \( T \) is invariant of \( (U_n \cap H_{p,q}) \). We deduce from these two facts that for \( g \) in \( \text{supp}(S) \), \( \Theta(g) \) must be equal to \( \Theta(g) \) for any \( u \) in \( U_n \cap H_{p,q} \). This means that \( \text{supp}(S) \subset \text{Sp}_{p,q} \), and \( \text{Sp}_{p,q} = P_{n-1}H_{p,q} \) according to Lemma \( \ref{lemma_2.1} \) hence \( T \) has support in \( P_{n-1}U_n H_{p,q} \).

Now consider the projection \( B : C_\infty^\infty(P_{n-1}U_n \times H_{p,q}, V) \rightarrow C_\infty^\infty(P_{n-1}U_n H_{p,q}, V) \), defined by \( B(\phi)(y^{-1}h) = \int_{P_{n-1} \cap H_{p,q}} (ay, ah) du \) (which is well defined because of the equality \( P_{n-1}U_n \cap H_{p,q} = P_{n-1} \cap H_{p,q} \)), and \( \phi \rightarrow \tilde{\phi} \) the isomorphism of \( C_\infty^\infty(P_{n-1}U_n \times H_{p,q}, V) \) defined by \( \tilde{\phi}(y, h) = (h, \tilde{\phi}(y, h)) \).

If one sets \( D(\phi) = T(B(\tilde{\phi})) \), then \( D \) is a \( V \)-distribution on \( P_{n-1}U_n \times H_{p,q} \) which is invariant of \( P_{n-1}U_n \times H_{p,q} \). This implies that there exists a unique linear form \( \lambda \) on \( V \), such that for all \( D(\phi) = \int_{P_{n-1}U_n \times H_{p,q}} \lambda(\phi(y, h))dydh \).

Now for \( h \) in \( P_{n-1} \cap H_{p,q} \), one has the integral expression of \( D(\phi) \) for some positive modulus character \( \delta \). On the other hand, writing \( D \) as \( \phi \rightarrow T(B(\tilde{\phi})) \), one has \( D(\phi) = \int_{P_{n-1} \cap H_{p,q}} (ay, ah) du \), and \( B(\phi)(y, h) = \delta_1(\phi)B \) for a positive modulus character \( \delta_1 \), so that \( D(\phi) = \int_{P_{n-1} \cap H_{p,q}} (ay, ah) du \), and \( \delta_1(\phi)B \).

Comparing the two expressions for \( D(\phi) \), we get the relation \( D \circ \sigma(b) = \chi(\phi)b \), with \( \chi \) being the positive character \( \delta^{-1} \delta_1 \delta_3^{3/2} \).

This in turn implies that the linear form \( \lambda \) on \( V \) satisfies the same relation, i.e. belongs to \( \text{Hom}_{P_{n-1} \cap H_{p,q}}(\sigma, \chi) \), and \( T \rightarrow \lambda \) gives a linear injection of \( \text{Hom}_{P_{n-1} \cap H_{p,q}}(\Phi^+ \sigma, \chi) \) into \( \text{Hom}_{P_{n-1} \cap H_{p,q}}(\sigma, \chi') \), and this proves the proposition.

Using Lemma \( \ref{lemma_2.2} \) instead of Lemma \( \ref{lemma_2.1} \) in the previous proof, one obtains the following statement.

Proposition 2.2. Let \( \sigma' \) belong to \( \text{Alg}(P_{n-2}) \), and \( \chi' \) be a positive character of \( P_{n-2} \cap H_{p,q} \), then there is a positive character \( \chi'' \) of \( P_{n-2} \cap H_{p,q} \), such that
\[
\text{Hom}_{P_{n-2} \cap H_{p,q}}(\Phi^+ \sigma', \chi') \leftrightarrow \text{Hom}_{P_{n-2} \cap H_{p,q}}(\sigma, \chi'').
\]

A consequence of these two propositions is the following.

Proposition 2.3. Let \( n \geq 3 \), and \( p \) and \( q \) two integers with \( p + q = n \) and \( p - 1 \geq q \geq 0 \), then one has \( \text{Hom}_{P_{n} \cap H_{p,q}}((\Phi^+)^n, (\Phi^+)^n) = 1 \).
Proof. Using repeatedly the last two propositions, we get the existence of a positive character $\chi$ of $P_{p-q+1}$ such that $\text{Hom}_{P_{p-q+1}\cap H_{p-q+1,0}}((\Phi^+)^{n-1}\Psi^+(1),1) \leftrightarrow \text{Hom}_{P_{p-q+1}\cap H_{p-q+1,0}}((\Phi^+)^{p-q}\Psi^+(1),\chi) = \text{Hom}_{P_{p-q+1}}((\Phi^+)^{p-q}\Psi^+(1),\chi)$, and this last space is 0 because $(\Phi^+)^{p-q}\Psi^+(1)$ and $\chi$ are two non-isomorphic irreducible representations of $P_{p-q+1}$, according to corollary 3.5 of [B-Z].

This implies the following theorem about cuspidal representations.

**Theorem 2.1.** Let $\pi$ be a cuspidal representation of $G_n$, which is distinguished by a maximal Levi subgroup $M$, then $n$ is even and $M \simeq M_{n/2,n/2}$.

**Proof.** Let $M$ be the maximal Levi subgroup such that $\pi$ is $M$-distinguished. Then $M$ is conjugate to a standard Levi subgroup $M_{p,q}$ with $p \geq q$ and $p+q = n$. Suppose $p \geq q+1$, $M_{p,q}$ is conjugate to $H_{p,q}$, so that $\pi$ is $H_{p,q}$-distinguished, and $\pi|_{P_n}$ is thus $H_{p,q}\cap P_n$-distinguished. But by Proposition 1.1, the restriction $\pi|_{P_n}$ is isomorphic to $(\Phi^+)^{n-1}\Psi^+(1)$, and this contradicts Proposition 2.3. Hence one must have $p = q$, and this proves the theorem.

**References**

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