Phase-Field Approximation of the Willmore Flow

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Abstract

We investigate the phase-field approximation of the Willmore flow by rigorously justifying its sharp interface limit. This is a fourth-order diffusion equation with a parameter $\varepsilon > 0$ that is proportional to the thickness of the diffuse interface. We show rigorously that for well-prepared initial data, as $\varepsilon$ tends to zero the level-set of the solution will converge to the motion by Willmore flow as long as the classical solution to the latter exists. This is done by constructing an approximate solution from the limiting flow via matched asymptotic expansions, and then estimating its difference with the real solution. The crucial step is to prove a spectrum inequality of the linearized operator at the optimal profile, which is a fourth-order operator written as the square of the Allen–Cahn operator plus a singular perturbation.

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1. Introduction

In this paper we consider the singular limit, as \( \varepsilon \to 0 \), of the following equations:

\[
\begin{align*}
\varepsilon^3 \partial_t \phi_\varepsilon &= \varepsilon^2 \Delta \mu_\varepsilon - f''(\phi_\varepsilon) \mu_\varepsilon, \quad \text{in } \Omega \times (0, T), \\
\varepsilon \mu_\varepsilon &= -\varepsilon^2 \Delta \phi_\varepsilon + f'(\phi_\varepsilon), \quad \text{in } \Omega \times (0, T).
\end{align*}
\]

(1.1)

Here \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is a bounded domain with smooth boundary \( \partial \Omega \), the parameter \( \varepsilon > 0 \) represents the width of a thin transition layer, and \( \phi_\varepsilon \) is a scalar function defined in the computational (physical) domain \( \Omega \) which takes value approximately \(-1\) inside the vesicle membrane and approximately 1 outside it. The function \( f \) is a double equal-well potential taking its global minimum value 0 at \( \pm 1 \). For simplicity we choose

\[
f(u) = \frac{1}{4}(u^2 - 1)^2.
\]

(1.2)

Physically, \( -\mu_\varepsilon \) represents the chemical potential. System (1.1) is supplemented with initial and boundary conditions

\[
\begin{align*}
\phi_\varepsilon(x, 0) &= \phi_{\varepsilon,0}(x), \quad \text{in } \Omega \times \{0\}, \\
\phi_\varepsilon(x, t) &= 1, \quad \partial_\nu \phi_\varepsilon(x, t) = 0, \quad \text{in } \partial \Omega \times (0, T),
\end{align*}
\]

(1.3)

where \( \partial_\nu \) is the outward normal derivative to \( \partial \Omega \). The equation (1.1) is the gradient flow of

\[
\mathcal{E}(\phi) = \frac{1}{2\varepsilon} \int_\Omega \left( \varepsilon \Delta \phi - \varepsilon^{-1} f'(\phi) \right)^2 \, dx,
\]

which was introduced in [4] as an approximation of the Willmore energy \( \int_\Sigma H^2 \) where \( H \) denotes the mean curvature of a surface \( \Sigma \). Such an approximation can be understood formally by observing that \( \mu(\phi) = -\varepsilon \Delta \phi + \varepsilon^{-1} f'(\phi) \) is the variation of the Allen–Cahn energy \( \int_\Omega (\varepsilon |\nabla \phi|^2/2 + \varepsilon^{-1} f(\phi)) \, dx \), and the latter converges to the area functional as \( \varepsilon \to 0 \). The rigorous justification is a challenging work, see for instance [31] for the case \( N = 2, 3 \). See also [5] for other types of approximations.

It was shown formally in [26,33] that the level set of solution \( \phi_\varepsilon \) to (1.1) converges to a family of compact closed interfaces

\[
\Gamma^0 = \bigcup_{t \in [0, T]} \Gamma^0_t \times \{t\},
\]

(1.4)

which evolves under the Willmore flow, i.e.

\[
V = \Delta \Gamma^0 + H |A|^2 - H^3/2,
\]

(1.5)

where \( V \) is the normal velocity of \( \Gamma^0 \), i.e. projection of the particle velocity to the outer-normal vector of \( \Gamma^0 \), \( H \) is the scalar mean curvature. Moreover, \( A \) is the second fundamental form, and \( |A|^2 \) is the sum of the squares of the principal curvatures. The derivation of (1.5) from the Willmore energy can be found in [24]. The evolution of a closed curve under (1.5) was investigated in [18], and the surface
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case was investigated in [23]. See also [30] for recent progresses in Willmore surface theory.

At the application level, Willmore energy is closely related to the Helfrich’s theory of bio-membranes [22], where the volume and area are constrained, and a spontaneous curvature is introduced. See [29,32] for a comprehensive study of this subject. So various phase-field models were introduced to study the evolution of bio-membranes, see for instance [10,11,16,17] for the analysis and numerical simulations of such models. The model that couples hydrodynamics was investigated in [15,34].

1.1. Statement of Main Results

To the best of our knowledge, the rigorous justification of the singular limit from (1.1) to (1.5) remains open and it is the task of this work to establish this result. To set up the problem, we introduce the signed-distance \(d_0(x,t)\) from \(x\) to \(\Gamma^0_t\) which takes negative values inside \(\Gamma^0_t\) and positive values outside it, and the bulk regions

\[
\Omega^\pm_t \triangleq \{ x \in \Omega \mid d_0(x,t) \gtrless 0 \}. \tag{1.6}
\]

For a sufficiently small \(\delta > 0\), the \(\delta\)-neighborhood of \(\Gamma^0_t\) is denoted by

\[
\Gamma^0_t(\delta) \triangleq \{ x \in \Omega \mid |d_0(x,t)| < \delta \}. \tag{1.7}
\]

Moreover, we denote \(\theta(z) = \tanh(\frac{z}{\sqrt{2}})\) as the unique solution to

\[
\theta''(z) = f'(\theta(z)), \quad \theta(\pm \infty) = \pm 1, \quad \theta(0) = 0. \tag{1.8}
\]

The main result of this work is stated as follows:

**Theorem 1.1.** Let \(\Gamma^0 \subset \mathbb{R}^N \times [0,T)\) (with \(N = 2, 3\)) be a smooth solution of (1.5) and \(k \geq 10\) be a fixed integer. Then there exists \(\varepsilon_1 > 0\) such that for every \(\varepsilon \in (0, \varepsilon_1)\) there are smooth functions \((\phi_a(x,t), \mu_a(x,t))\) which equal to \((\pm 1, 0)\) in \(\Omega^\pm_t \setminus \Gamma^0_t(2\delta)\). Moreover, they are approximate solutions of (1.1) in the sense that

\[
\begin{align*}
\varepsilon^3 \partial_t \phi_a &= \varepsilon^2 \Delta \mu_a - f''(\phi_a)\mu_a + \varepsilon^k - 1 R_1, \quad \text{in } \Omega \times (0,T), \\
\varepsilon \mu_a &= -\varepsilon^2 \Delta \phi_a + f'(\phi_a) + \varepsilon^k - 1 R_2, \quad \text{in } \Omega \times (0,T),
\end{align*}
\tag{1.9}
\]

where \(R_1(x,t)\) and \(R_2(x,t)\) are uniformly bounded in \(\varepsilon\) and \((x,t)\). Furthermore, if the initial data of \(\phi_\varepsilon\) satisfies, for some \(C_{in} > 0\), that

\[
\|\phi_\varepsilon(\cdot,0) - \phi_a(\cdot,0)\|_{L^2(\Omega)} \leq C_{in} \varepsilon^{7/2}, \quad \forall \varepsilon \in (0, \varepsilon_1), \tag{1.10}
\]

then there exist \(\varepsilon\)-independent constant \(\Lambda > 0\) and \(T_{max} \in (0,T]\) such that

\[
\|\phi_\varepsilon - \phi_a\|_{L^\infty(0,T_{max};L^2(\Omega))} \leq \Lambda \varepsilon^{7/2}, \quad \forall \varepsilon \in (0, \varepsilon_1). \tag{1.11}
\]

If \(C_{in}\) is sufficiently small (but independent of \(\varepsilon\)), then \(T_{max} = T\).
The main result is proved for the standard choice of the bulk potential (1.2). Even though we believe it can be generalized to the case of a general double-well potential like in [2], it will lead to technical complications, for instance in Lemma 4.1. Thus we restrict ourselves to such a simpler case.

The choice \( k \geq 10 \) is a technical assumption due to the method we employ and is not optimal. More precisely it will be used to derive the differential inequality (7.22) in the proof of Theorem 1.1.

When \( N = 3 \) it is not clear to us whether the power of the convergence rate \( 7/2 \) is optimal. A detailed discussion is given in the next subsection where an outline of the proof is described. We shall discuss the admissible initial data \( \phi_{a} \big|_{t=0} \) satisfying the assumption (1.10) in Remark 7.1 below. Such issue seems not be fully addressed in previous works. For instance in [2], they only consider the case when the initial data of the Cahn–Hilliard equation coincides with the constructed approximate solution. Concerning the last statement about the convergence up to time \( T \), we do not have a quantitative characterization of \( C_{in} \). The smallness of \( C_{in} \) is used in a continuity argument in the proof of Theorem 1.1 in Section 7. However, when \( N = 2 \), there is some room to reduce the power and thus obtain the convergence over \([0, T]\) for arbitrarily large \( C_{in} \) in (1.10). We will probably study this case more carefully in a future work.

1.2. Outline of the Proof

The proof of (1.9) follows the strategy of [2] by constructing an approximate solution \((\phi_{a}, \mu_{a})\) via matched asymptotic expansions. Since such constructions are quite sophisticated and technical, we leave them in the appendix. Another method which is based on Hilbert expansion is given in [6].

In order to show the convergence rate (1.11) under the assumption (1.10), we shall investigate the equation satisfied by the differences

\[
(\phi, \mu) = (\phi_{e} - \phi_{a}, \mu_{e} - \mu_{a}).
\]  

(1.12)

To proceed, we introduce the linearized Allen–Cahn operator at \( \phi_{a} \) by

\[
\mathcal{L}_{\epsilon}[\phi] \triangleq \left( -\epsilon \Delta + \epsilon^{-1} f''(\phi_{a}) \right) \phi.
\]  

(1.13)

In view of (1.9) and (1.1), it can be verified that \( \phi \) satisfies

\[
\partial_{t} \phi = -\frac{1}{\epsilon^{2}} \mathcal{L}_{\epsilon}[\phi] - \frac{1}{\epsilon^{3}} f'''(\phi_{a}) \mu_{a} \phi + \mathcal{H},
\]  

(1.14)

where \( \mathcal{H} \) is the nonlinear terms about \( \phi \), defined by (7.1) in the sequel. Standard energy estimate of (1.14) yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \phi^{2} dx + \frac{1}{\epsilon^{2}} \int_{\Omega} |\mathcal{L}_{\epsilon}[\phi]|^{2} dx + \frac{1}{\epsilon^{3}} \int_{\Omega} f'''(\phi_{a}) \mu_{a} \phi^{2} dx = \int_{\Omega} \mathcal{H} \phi \, dx.
\]  

(1.15)

In order to apply the Grönwall’s inequality to (1.15), we first need to prove the spectrum condition of the linear operator on the right of (1.14). This fourth-order
operator is highly singular due to both the degeneracy of the Allen–Cahn part (1.13) and the singular perturbation $-\varepsilon^{-3} f'''(\phi_a)\mu_a\phi$. To analyze it, we employ the spectrum decomposition method used in [7] (see also [12,21]). We set the operator
\[
\mathcal{L}[u] \triangleq -\partial_z^2 u + f''(\theta(z))u, \quad \forall u(z) \in H^2(-\delta/\varepsilon, \delta/\varepsilon),
\]
and we denote $\varphi$ the first eigenfunction of $\mathcal{L}$ with homogenous Neumann boundary condition (see Lemma 3.2 below for the precise definition). Then we decompose any $\phi \in H^2(\Omega)$ (not necessarily the difference $\phi_\varepsilon - \phi_a$) along $\phi(x) = e^{-\frac{1}{2}}Z(s)\varphi(\frac{\xi}{\varepsilon})\zeta(r) + \phi_\perp(x)$. (1.17)

Here $s = (s_1, \ldots, s_{N-1})$ is the local coordinate of interface $\Gamma_t$ (a perturbation of $\Gamma_t^0$ in (1.4)), and $r$ stands for the signed-distance to $\Gamma_t$. With a sufficiently small $\delta > 0, (r, s)$ forms a local coordinate of the $\delta$-tubular neighborhood of $\Gamma_t$. Moreover in (1.16) $z = r/\varepsilon$ is the fast variable. See Section 2.2 for the precise definitions of $\Gamma_t$ and related notations. In (1.17), $\zeta$ is a smooth cut-off function satisfying
\[
0 \leq \zeta \leq 1; \quad \zeta(r) = \zeta(-r); \quad \zeta(r) = 1 \text{ for } |r| \leq \delta/2; \quad \zeta \in C^\infty_c((-\delta, \delta)).
\] (1.18)

Moreover, $Z(s)$ is defined by
\[
Z(s) = e^{-\frac{1}{2}}\eta^{-1}(s)\int_{-\delta}^{\delta} \phi(r, s)\varphi(\frac{\xi}{\varepsilon})\zeta(r)J^\frac{1}{2}(r, s)dr,
\]
where $J(r, s)$ is the Jacobian of the change of variable among level-sets, and $\eta(s)$ is the renormalization constant
\[
\eta(s) = e^{-1}\int_{-\delta}^{\delta} \left(\varphi(\frac{\xi}{\varepsilon})\right)^2\zeta^2(r)J^\frac{1}{2}(r, s)dr.
\] (1.20)

With the above definitions, we can state the spectrum condition of the linearized operator.

**Theorem 1.2.** Let $(\phi_a, \mu_a)$ be the approximate solution in Theorem 1.1. Then there exist $\tilde{C}, \hat{C} > 0$ and $\varepsilon_0 > 0$ which are independent of $\varepsilon$ such that the inequality
\[
\frac{1}{\varepsilon^2} \int_{\Omega} |\mathcal{L}_\varepsilon[\phi]|^2dx + \frac{1}{\varepsilon^3} \int_{\Omega} f'''(\phi_a)\mu_a\phi^2dx \geq \tilde{C} K(t) - \hat{C} \int_{\Omega} \phi^2dx
\]
with $K(t) \triangleq \|Z\|^2_{H^2(\Gamma_t)} + \|\phi_\perp\|^2_{H^2(\Omega)} + \varepsilon^{-2}\|\phi_\perp\|^2_{H^1(\Omega)} + \varepsilon^{-4}\|\phi_\perp\|^2_{L^2(\Omega)}.$

(1.21a)

holds for every $\varepsilon \in (0, \varepsilon_0)$ and every $\phi \in H^2(\Omega)$.

The above Theorem is proved under any dimension $N \geq 2$, and the constants $\tilde{C}$ and $\hat{C}$ depend on the geometry of the limiting interface (1.4). Let us outline the
proof of (1.21a) by explaining the general consideration behind (1.17). It follows from (1.17), (1.19) and (1.20) that

$$\int_{-\delta}^{\delta} \phi^\perp(r, s) \varphi(\xi) \zeta J^\frac{1}{2}(r, s) \, dr = 0. \quad (1.22)$$

This implies (1.17) is an orthogonal decomposition. Then substituting (1.17) into the left hand side of (1.21a) will lead to integrals $I_1, I_3$ involving the squares of $\phi^\perp$ and $\tilde{\phi} \triangleq \varepsilon^{-\frac{1}{2}} Z(s) \varphi(\xi) \zeta(r)$ respectively, together with an integral $I_2$ about their product $\tilde{\phi} \phi^\perp$. See (3.17) for the precise definitions. In the course of estimating these three integrals, the one corresponding to $\tilde{\phi} \phi^\perp$ vanishes at the leading order due to (1.22). The part including merely $\tilde{\phi}$ carries the most singular terms but is very explicit because $\varphi(z) \approx \theta'(z)$ and thus enjoys cancellation properties. In contrast, $\phi^\perp$ is more abstract but less singular because of a coercivity estimate of (1.13). We refer to Lemmas 3.2 and 3.3 for the mathematical rigor of these heuristic arguments.

**Remark 1.1.** In the Cahn–Hilliard case [2, 7], the equation of the difference $\phi$ reads as

$$\partial_t \phi = \Delta \mathcal{L}_\varepsilon[\phi] + \text{nonlinear terms}, \quad (1.23)$$

where $\Delta$ denotes the Neumann-Laplacian. Multiplying by $w \triangleq (-\Delta)^{-1}\phi$ and integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w|^2 \, dx + \int_{\Omega} \left( \varepsilon |\nabla \phi|^2 + \varepsilon^{-1} f''(\phi_a) \phi^2 \right) \, dx = \int_{\Omega} (\text{nonlinear terms} \times \phi) \, dx. \quad (1.24)$$

It is proved in [7] that, there exist $C, \varepsilon_0 > 0$ independent of $\varepsilon$ such that

$$\int_{\Omega} \left( \varepsilon |\nabla \phi|^2 + \varepsilon^{-1} f''(\phi_a) \phi^2 \right) \, dx \geq -C \int_{\Omega} |\nabla w|^2 \, dx, \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (1.24)$$

The proof heavily relies on the spectrum condition of the Allen–Cahn operator (1.13). Combining (1.24) and the estimates of the nonlinear terms, Alikakos–Bates–Chen [2] succeeded in constructing the approximate solutions and verifying the smallness of $\phi = \phi_\varepsilon - \phi_a$. In contrast, in our case the difference $\phi$ satisfies (1.14), i.e.,

$$\partial_t \phi = \varepsilon^{-1} \Delta \mathcal{L}_\varepsilon[\phi] - \varepsilon^{-3} f''(\phi_a) \mathcal{L}_\varepsilon[\phi] - \varepsilon^{-3} f'''(\phi_a) \mu_a \phi + \text{nonlinear terms}. \quad (1.25)$$

The structure of the above equation seems not allow a test using $(-\Delta)^{-1}\phi$. Instead we multiply by $\phi$ to get (1.15), and then investigate the spectrum of a fourth-order operator.

It is worth mentioning that there are other frameworks to justify the small $\varepsilon$-limit of the Cahn–Hilliard equation, see for instance [8, 25]. It would be interesting to study (1.1) through these frameworks.
After establishing the linear stability result (1.21a), the proof of Theorem 1.1 reduces to the estimate of the integral on the right hand side of (1.15) and the proof of nonlinear stability. The key idea is to apply the decomposition (1.17) in order to separate the tangential derivatives and normal ones of $\bar{\phi}$. This is crucial towards a reasonable convergence rate, as each normal derivative of $\bar{\phi}$ brings a factor of $\varepsilon^{-1}$.

We note that similar idea has been employed in the study of free-boundary problems in hydrodynamics where such estimates are referred to as conormal estimates; see for instance the monograph [28] and a recent article [27]. These combined with (1.21a) lead to an estimate like

$$\text{right hand side of (1.21a)} + \int_{\Omega} \mathcal{H}\phi \, dx \leq C_* \varepsilon^{-\beta} \|\phi\|_{L^2(\Omega)}^\gamma$$

for some $\beta > 0$, $\gamma > 2$ and $C_* > 0$ which are $\varepsilon$-independent. Substituting (1.26) and (1.21a) into (1.15) yields a differential inequality to which we can apply the continuity argument.

We organize this paper as follows. In Section 2, we set up the geometry of the interface using the standard notations of differentiable manifolds. This setting is slightly different from the one employed in [1,7,9], and has the advantage of facilitating the analyses on hypersurfaces. In Section 3, we give the construction of approximate solution (1.9) and then set up the proof of (1.21a) by reducing it to the estimates of three integrals. The estimates of these integrals are the main tasks of Section 4, 5 and 6. The proof of Theorem 1.2 will be done in the last part of In Section 6 and the proof of Theorem 1.1 is given in Section 7.

2. Preliminaries

2.1. Level Set and Signed-Distance Function

Following [2], we shall approximate the perturbed level set $\Gamma^\varepsilon = \{(x, t) : \phi^\varepsilon(x, t) = 0\}$ for the solution of (1.1). For any $t \in [0, T]$, $d^\varepsilon(x, t)$ will denote the signed-distance from $x$ to $\Gamma^\varepsilon$, taking negative values inside $\Gamma^\varepsilon$. For the analytic properties of the signed-distance function, one can refer to [3]. Among those, we shall use that $d^\varepsilon$ is smooth, and $|\nabla d^\varepsilon| = 1$ in a neighborhood of $\Gamma^\varepsilon$. We write $d^\varepsilon$ and its $k$th truncation $d^{[k]}$ by

$$d^\varepsilon(x, t) = \sum_{i \geq 0} \varepsilon^i d^{(i)}(x, t), \quad d^{[k]}(x, t) = \sum_{0 \leq i \leq k} \varepsilon^i d^{(i)}(x, t),$$

respectively. If we assume $\Gamma^0$ to be the limit of $\Gamma^\varepsilon$ and denote $d^{(0)}$ as the signed-distance function of $\Gamma^0$, we shall have $|\nabla d^{(0)}| = 1$ in $\Gamma^\varepsilon(4\delta)$ for some $\delta > 0$. This implies

$$1 = |\nabla d^\varepsilon|^2 = 1 + 2\varepsilon \nabla d^{(0)} \cdot \nabla d^{(1)} + \sum_{\ell \geq 2} \varepsilon^\ell \left( \sum_{0 \leq i \leq \ell} \nabla d^{(i)} \cdot \nabla d^{(\ell-i)} \right).$$
Matching the order of $\varepsilon$ yields
\[
\nabla d^{(0)} \cdot \nabla d^{(1)} = 0; \quad \nabla d^{(0)} \cdot \nabla d^{(\ell)} = -\frac{1}{2} \sum_{1 \leq i \leq \ell - 1} \nabla d^{(i)} \cdot \nabla d^{(\ell - i)}, \quad \forall \ell \geq 2,
\]
(2.2)
so $d^{[k]}$ satisfies the eikonal equation $|\nabla d^{[k]}| = 1 + \varepsilon^k g(x, t)$ for some smooth function $g(x, t)$. In general $d^{[k]}$ does not satisfy $|\nabla d^{[k]}| = 1$, and this would lead to some complications in the asymptotic expansions. The remedy is to define
\[
\Gamma_t = \{ x \in \Omega : d^{[k]}(x, t) = 0 \}, \quad \Gamma = \bigcup_{t \in [0, T]} \Gamma_t \times \{ t \}
\]
(2.3)
for a sufficiently large $k$, and to work with the signed-distance $r(x, t)$ of $\Gamma_t$. Through $r(x, t)$ a coordinate system near $\Gamma_t$ will be established, in which the asymptotic expansion will be performed. The following estimate can be proved via the method of characteristics
\[
\| r(x, t) - d^{[k]}(x, t) \|_{C^3(\Gamma_t(\delta))} = O(\varepsilon^k).
\]
(2.4)
See for instance [19, Section 3.2] or [2, Section 5].

2.2. Geometry of the Interface

This part will be used for the proof of (1.21a), where $t$ is a fixed parameter. So we shall suppress its dependence for various quantities. For instance, we shall simply write the signed-distance function $r(x, t)$ of $\Gamma_t$ as $r(x)$ or $r$. Following [12], we assume $\Gamma_t$ be a compact smooth hypersurface and $X_0(s) : U \mapsto \Gamma_t$ be a local parametrization of it with $U$ being an open set in $\mathbb{R}^{N-1}$. For a sufficiently small $\delta > 0$, the $\delta$-neighborhood of $\Gamma_t$ is well-defined and shall be denoted by $\Gamma_t(\delta)$, and the projection $s(x) : \Gamma_t(\delta) \mapsto U$ is a smooth function. So each point $x \in \Gamma_t(\delta)$ allows a unique expression
\[
x = X(s, r) = X_0(s) + rn(s),
\]
(2.5)
where $s = (s_1, \cdots, s_{N-1})$ are local coordinates of $\Gamma_t$, and $n$ is the unit outward normal vector. The decomposition (2.5) leads to a local parametrization of $\Gamma_t(\delta)$, and induces a local coordinates $(s, r)$ in it. The corresponding tangent space is expanded by
\[
\{ e_1, \cdots, e_N \} \triangleq \{ \partial_{s_1}X, \cdots, \partial_{s_{N-1}}X, n \}.
\]
(2.6)
We note that the first $N - 1$ components span the tangent plane of the hypersurface $\Gamma^r_t$, the $r$-level set of the signed-distance $r(x)$
\[
\Gamma^r_t \triangleq \{ x \in \mathbb{R}^N \mid r(x) = r \}.
\]
(2.7)
It follows from (2.5) that $\partial_{s_1}X \cdot \partial_{r}X = 0$ for $1 \leq i \leq N - 1$. This implies that the metric $g_{ij}$ of $\Gamma_t(\delta)$ under $(s, r)$ has the form
\[
(g_{ij})_{1 \leq i, j \leq N} = \begin{pmatrix}
(\partial_{s_i}X : \partial_{s_j}X)_{1 \leq i, j \leq N-1} & 0 \\
0 & 1
\end{pmatrix}.
\]
(2.8)
Conventionally, we denote its inverse by $g^{ij}$. Using (2.8) we can decompose the Euclidean gradient operator by

$$
\nabla f = \nabla \Gamma f + \mathbf{n}{\partial_r f}
$$

where $\nabla \Gamma f \overset{\Delta}{=} \sum_{1 \leq i, j \leq N-1} g^{ij}{\partial_{s_j} f}{e_i}$

(2.9) is the gradient on $\Gamma'_r$. Denoting $g = \det g_{ij}$, we can assume without loss of generality that

$$
1/\Lambda \leq \sqrt{g}(r, s) \leq \Lambda,
$$

for some $\Lambda > 0$ which only depends on $\delta$ and $\Gamma_r$. Following [13],

$$
\Delta f = \Delta \Gamma f + \frac{1}{\sqrt{g}} \partial_r \left( \sqrt{g} \partial_r f \right)
$$

where $\Delta \Gamma f \overset{\Delta}{=} \sum_{1 \leq i, j \leq N-1} \frac{1}{\sqrt{g}} \partial_{s_j} \left( \sqrt{g} g^{ij}{\partial_{s_j} f} \right)$

(2.11) is the Laplace–Beltrami operators on $\Gamma'_r$. The operator $\Delta \Gamma|_{r=0}$ corresponds to the Laplace–Beltrami operator on $\Gamma_r$. Its difference to $\Delta \Gamma$ is measured by a second order differential operator with smooth coefficients

$$
\mathcal{R}_\Gamma = \left( \Delta \Gamma - \Delta \Gamma|_{r=0} \right)/r.
$$

(2.12)

By a partition of unity of the compact hypersurfaces $\Gamma_r$ and interior regularity of second order elliptic equation with regular coefficients [19, Theorem 1 of section 6.3], one has the following estimates:

**Proposition 2.1.** Let $\Gamma_r$ be a smooth compact smooth hypersurface with regular neighbourhood $\Gamma_r(\delta)$ for some $\delta > 0$. There are positive constants $\{C_i\}_{i=1}^4$ which only depend on the geometry of $\Gamma_r$ and $\delta > 0$ such that the following estimates hold:

$$
\sup_{|r| \leq \delta} \|\Delta \Gamma f\|_{L^2(\Gamma_r)} + \|f\|_{L^2(\Gamma_r)} \leq C_1 \|f\|_{H^2(\Gamma_r)}
$$

$$
\leq C_2 \left( \inf_{|r| \leq \delta} \|\Delta \Gamma f\|_{L^2(\Gamma_r)} + \|f\|_{L^2(\Gamma_r)} \right),
$$

(2.13a)

$$
\|\mathcal{R}_\Gamma f\|_{L^2(\Gamma_r)} \leq C_3 \|f\|_{H^2(\Gamma_r)} \leq C_4 \left( \|\Delta \Gamma|_{r=0} f\|_{L^2(\Gamma_r)} + \|f\|_{L^2(\Gamma_r)} \right).
$$

(2.13b)

Now we consider the divergence operator in $\Gamma_r(\delta)$. With the convention $s_N = r$, any vector-field $\mathbf{F} = \sum_{1 \leq i \leq N} F_i \partial_{s_i}$ can be written under local coordinates $(s, r)$ by $\mathbf{F} = \sum_{1 \leq j \leq N} \tilde{F}_j e_j$ where $\tilde{F}_j = \sum_{1 \leq k \leq N} g^{kj} e_k \cdot F$. Then the Euclidean divergence $\text{div} \mathbf{F} = \sum_{1 \leq j \leq N} \frac{1}{\sqrt{g}} \partial_{s_j} \left( \sqrt{g} \tilde{F}_j \right)$ is decomposed by

$$
\text{div} \mathbf{F} = \text{div} \Gamma \mathbf{F} + \frac{1}{\sqrt{g}} \partial_r \left( \sqrt{g} \tilde{F}_N \right),
$$

(2.14)

where $\text{div} \Gamma$ is the divergence operator on $\Gamma'_r$

$$
\text{div} \Gamma \mathbf{F} = \sum_{1 \leq j \leq N-1} \frac{1}{\sqrt{g}} \partial_{s_j} \left( \sqrt{g} \tilde{F}_j \right).
$$

(2.15)
With these notations, it can be verified that $\Delta \Gamma \ f = \text{div}_\Gamma \nabla_\Gamma f$, which is consistent with (2.11). Following [9, Lemma 4], we have the following expansion of $J(r, s) = \sqrt{g(r, s)}$:

$$J(r, s) = \prod_{1 \leq i \leq N-1} (1 + r \kappa_i(s)) = 1 + rh(s) + r^2 e(s) + O(r^3), \quad (2.16)$$

where $\kappa_i$ are the principal curvatures of $\Gamma_i$. On the other hand, it follows from (2.14) and $\nabla r = n$ that $\partial_r \ln \sqrt{g} = \Delta r$. When combined with (2.16), we obtain

$$\partial_r \ln \sqrt{g} = \Delta r = h(s) + b(s)r + a(s)r^2 + r^3 O(1), \quad (2.17a)$$

$$\partial_r^2 \ln \sqrt{g} = \nabla \Delta r \cdot \nabla r = b(s) + c(s)r + r^2 O(1). \quad (2.17b)$$

We end this section by the integration theory on hypersurfaces. By coarea formula [20],

$$\int_{\Gamma_t(\delta)} f(x)dx = \int_{-\delta}^\delta \left( \int_{\Gamma_{t'}^\delta} f \ dS \right) \ dr, \quad (2.18)$$

where $dS$ is the surface element of $\Gamma_t^\delta$ and we have $dS = \sqrt{g}(r, s)ds$ under local coordinates. We also need the formula

$$\int_{\Gamma_t(\delta)} f(x)dx = \int_{\Gamma_t} \left( \int_{-\delta}^\delta f(r, s) J(r, s)dr \right) \ dS, \quad (2.19)$$

where $dS$ here is the surface element of $\Gamma_t$, which writes $dS = \sqrt{g}(0, s)ds$ under local coordinates. Finally we state the divergence theorem for tangential vector field as follows:

$$\int_{\Gamma_t} \text{div}_\Gamma \mathbf{F} \ dS = 0 \quad \text{if} \quad \mathbf{F} \cdot n = 0. \quad (2.20)$$

Here $dS$ is interpreted in the same way as in (2.18); see, for instance, [14], for the proof of these formulas.

### 2.3. Conventions

Throughout this work, $O(\varepsilon^\ell)$ will denote $\varepsilon^\ell g_\varepsilon(x, t)$ for some $g_\varepsilon(x, t)$ that is uniformly bounded in $(x, t)$ and $\varepsilon$. If there exists a constant $C$ which is independent of $\varepsilon$ so that the inequality $X \leq CY$ holds, we shall briefly write $X \lesssim Y$. Similarly, $X \gtrsim Y$ means $X \geq CY$. In various estimates, the generic constant $C$ might change from line to line and we shall not relabel them for simplicity.

Throughout this work, we restrict $\varepsilon \in (0, \varepsilon_0)$ for some sufficiently small $\varepsilon_0 > 0$ which might change from line to line but only depends on the geometry of the smooth interface $\Gamma^0$ introduced at (1.4).

For brevity, we shall not relabel a function under equivalent coordinates. For instance, for a function $f = f(x)$, its counterpart under local coordinates $(r, s)$ will be simply denoted by $f(r, s)$. Its integration in $\Gamma_t^\delta$ will be understood via (2.19) or (2.18) depending on the context. We make the convention that a function $h$ on $\Gamma_t$ shall be denoted by $h(s)$, even though a global coordinate of $\Gamma_t$ does not exist in general. We shall omit $dS$ in a surface integration like (2.20) if it is clear from the context.
3. Approximate Solutions and Difference Estimate

In this section we will build an approximate solution satisfying (1.9). This is done by gluing the inner/outer expansions constructed through the matched asymptotic expansions. We start by an outline of the construction and leave the details to Appendix A.

In contrast to the Cahn–Hilliard case [2], the outer expansions here are much simpler, and thus the boundary layer expansions can be avoided. Recall (1.6) for the definition of \( \Omega_t^\pm \), it follows from exactly the same calculation as in [2] that

\[
\phi_a^O(x, t) = \pm 1_{\Omega_t^\pm}, \quad \mu_a^O(x, t) = 0. \tag{3.1}
\]

We construct the inner expansions in \( \Gamma_t(2\delta) \) by

\[
\phi_a^I(x, t) = \sum_{0 \leq i \leq k} e^{i\tilde{\phi}^{(i)}(z, x, t)} \bigg|_{z = \frac{r(x, t)}{\epsilon}}, \quad \mu_a^I(x, t) = \sum_{0 \leq i \leq k} e^{i\tilde{\mu}^{(i)}(z, x, t)} \bigg|_{z = \frac{r(x, t)}{\epsilon}}, \tag{3.2}
\]

where \( z \) is the fast variable and \( r(x, t) \) is the signed-distance function of \( \Gamma_t \). See (2.3) for the definition of \( \Gamma_t \). While we obtain recursive formulas for \( \tilde{\phi}^{(i)}, \tilde{\mu}^{(i)} \) in the appendix, the first few terms are given for later use as follows:

\[
\tilde{\phi}^{(0)}(z, x, t) = \theta(z), \quad \tilde{\phi}^{(1)}(z, x, t) = 0, \quad \tilde{\phi}^{(2)}(z, x, t) = D^{(0)}(x, t)\theta'(z)\alpha(z), \tag{3.3a}
\]

\[
\tilde{\mu}^{(0)}(z, x, t) = -\Delta d^{(0)}(x, t)\theta'(z), \tag{3.3b}
\]

\[
\tilde{\mu}^{(1)}(z, x, t) = -\Delta d^{(1)}(x, t)\theta'(z) + D^{(0)}(x, t)z\theta'(z), \tag{3.3c}
\]

\[
\tilde{\mu}^{(2)}(z, x, t) = \Delta d^{(0)}D^{(0)}\theta'(z)\gamma_1(z) + \nabla d^{(0)} \cdot \nabla D^{(0)}\theta'(z)\gamma_2(z) + D^{(1)}z\theta' + \mu_2(x, t)\theta' + \chi^{(0)}d^{(0)}\theta'(z)\gamma_3(z), \tag{3.3d}
\]

The following lemma provide more information about (3.3):

**Lemma 3.1.** (1) The functions \( \{d^{(i)}(x, t)\}_{i=0}^2 \), \( \{D^{(i)}\}_{i=0}^1 \), \( \mu_2(x, t) \) and \( \chi^{(0)}(x, t) \) are smooth in \( \bigcup_{t \in [0, T]} \Gamma_t(\delta) \times \{t\} \).

(2) The function \( \alpha(z) \) is odd and the functions \( \gamma_1(z), \gamma_3(z) \) are even, and they grow at most quadratically at infinity.

(3) The function \( \gamma_2(z) = -z^2/2 \).

**Proof.** By Section 2.1, \( d^{(0)} \) is the signed distance function to \( \Gamma_0^0 \), and thus smooth in \( \Gamma_0^0(4\delta) \supset \Gamma_0^0(\delta) \). The smoothness of \( \{d^{(i)}\}_{i=1}^2 \) follows from the construction (A.44) and (A.49d) (with \( k = 2 \)) successively, and \( D^{(0)}, D^{(1)} \) are given by

\[
D^{(i)} = \sum_{0 \leq \ell \leq i} \left( \nabla \Delta d^{(\ell)} \cdot \nabla d^{(i-\ell)} + \frac{1}{2} \Delta d^{(\ell)} \Delta d^{(i-\ell)} \right). \tag{3.4}
\]

The function \( \mu_2(x, t) \) and \( \chi^{(0)}(x, t) \) are defined by (A.28) and (A.34) respectively and are smooth. The functions \( \alpha(z), \{\gamma_j(z)\}_{j=1}^3 \) and \( \mu_2(x, t) \) are given by (A.24), (A.32) and (A.34) respectively, so their growth rate is a consequence of L'Hôpital’s rule.
Using the cut-off function (1.18), we can glue the inner and outer expansions by

\[ \phi_a(x, t) = \phi^I_a + (1 - \xi(r))(\phi^O_a - \phi^I_a), \quad \mu_a(x, t) = \mu^I_a + (1 - \xi(r))(\mu^O_a - \mu^I_a). \]  

*(3.5)*

**Proposition 3.1.** The functions \( \phi_a \) and \( \mu_a \) in (3.5) are smooth and fulfill (1.9) in \( \Omega \times (0, T) \). Moreover, they have the following expansions in \( \Gamma_t(\delta) \):

\[ \phi_a(x, t) = \theta(r(x, t)) + \varepsilon 2^{-\gamma} \phi^{(2)} \left( \frac{r(x, t)}{\varepsilon}, x, t \right) + O(\varepsilon^3), \]

\[ \mu_a(x, t) = \sum_{0 \leq i \leq 2} \varepsilon^i \mu^{(i)} \left( \frac{r(x, t)}{\varepsilon}, x, t \right) + O(\varepsilon^3). \]  

*(3.6)*

**Proof.** It is clear that \( \phi_a \) is a smooth function that coincides with \( \phi^I_a \) inside \( \Gamma_t(\delta/2) \) and with \( \phi^O_a \) outside \( \Gamma_t(\delta) \). Moreover, the difference \( \phi_a - \phi^I_a \) decays at the order of \( O(e^{-C/\varepsilon}) \) when it is restricted in \( \Omega \setminus \Gamma_t(\delta/2) \), due to the matching condition (A.62). The same statement is valid for \( \mu_a \). So substituting (3.1), (3.2), (3.3a) and (3.3) into (3.5) yields (3.6). To verify (1.9), we need to employ Proposition A.11. We rewrite (3.5) as

\[ \phi_a(x, t) = \phi^I_a + (1 - \xi(r))(\phi^O_a - \phi^I_a) + \phi^I_a - \hat{\phi}^I_a, \]

\[ \mu_a(x, t) = \mu^I_a + (1 - \xi(r))(\mu^O_a - \mu^I_a) + \mu^I_a - \hat{\mu}^I_a, \]  

*(3.7)*

where \( \hat{\phi}^I_a, \hat{\mu}^I_a \) are defined in Proposition A.11. It follows from (2.4) that \( \Gamma_t(\delta) \subset \Gamma^0_t(3\delta) \) for \( k \geq 10 \) and \( \varepsilon \in (0, \varepsilon_0) \). So the first terms on the right hand side of (3.7) fulfill (A.64) and (A.65) respectively. For the second parts, one can employ (A.62) to obtain the exponential decay in \( \varepsilon \). Finally, it follows from (2.4) that

\[ \phi^I_a - \hat{\phi}^I_a = \sum_{0 \leq i \leq k} \varepsilon^i \phi^{(i)}(z, x, t) \bigg|_{z = \frac{r(x, t)}{\varepsilon}} = O(\varepsilon^{k-1}). \]

Similar consideration leads to

\[ \partial_t(\phi^I_a - \hat{\phi}^I_a) = O(\varepsilon^{k-2}), \quad \mu^I_a - \hat{\mu}^I_a = O(\varepsilon^{k-1}), \]

\[ \Delta(\phi^I_a - \hat{\phi}^I_a) = O(\varepsilon^{k-3}), \quad \Delta(\mu^I_a - \hat{\mu}^I_a) = O(\varepsilon^{k-3}). \]

Thus (1.9) is proved.

**Lemma 3.2.** Let \( \lambda_1 \) be the principal eigenvalue of the operator (1.16) with homogeneous Neumann boundary condition, and \( \varphi(z) \) be the corresponding eigenfunction, normalized such that \( \| \varphi \|_{L^2(I_\varepsilon)} = \| \theta' \|_{L^2(I_\varepsilon)} \). Denote \( I_\varepsilon = (-\delta/\varepsilon, \delta/\varepsilon) \), then

\[ \| \varphi - \theta' \|_{W^{2,\infty}(I_\varepsilon)} = O(e^{-\varepsilon^{\beta}}), \]  

*(3.8)*

and \( \lambda_1 = O(e^{-\varepsilon^{\beta}}) \). Moreover, exists \( \Lambda_3 > 0 \) such that

\[ \int_{I_\varepsilon} \left( q'^2 + f''(\theta)q^2 \right) dz \geq \Lambda_3 \int_{I_\varepsilon} (q'^2 + q^2) dz, \quad \forall q \perp \varphi \text{ in } L^2(I_\varepsilon). \]  

*(3.9)*
Remark 3.1. The above lemma is mainly due to [7]. There the author defines the \( \varepsilon \)-dependent constant \( \beta = \left( \| \theta' \|_{L^2(I_\varepsilon)} \right)^{-1} \) and \( \varphi \) such that \( \| \varphi \|_{L^2(I_\varepsilon)} = 1 \) and \( \varphi - \beta \theta' = O(e^{-C/\varepsilon}) \). In this work, \( \beta \) is absorbed into \( \varphi \) for the convenience of Section 4.

Lemma 3.3. The renormalization constant (1.20) satisfies

\[
\eta(s) = \int_{\mathbb{R}} (\theta'(z))^2 dz + O(\varepsilon). \tag{3.10}
\]

Moreover, there exist \( \varepsilon_0 > 0 \) and \( \Lambda_5 > 1 \) such that for every \( t \in [0, T] \),

\[
\Lambda_5 \int_{\Gamma_t(\delta)} \phi^2 dx \geq \int_{\Gamma_t} Z^2(s) + \int_{\Gamma_t(\delta)} (\phi^\perp)^2 dx \geq \frac{1}{\Lambda_5} \int_{\Gamma_t(\delta)} \phi^2 dx, \ \forall \varepsilon \in (0, \varepsilon_0). \tag{3.11}
\]

Proof. The asymptotic formula of \( \eta \) follows from a change of variable together with (3.8),

\[
\eta(s) = \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} (\varphi(z) - \theta'(z) + \theta'(z))^2 \zeta^2(\varepsilon z) J^{\frac{1}{2}}(\varepsilon z, s) dz
\]

\[
= \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} (\theta'(z))^2 \zeta^2(\varepsilon z) J^{\frac{1}{2}}(\varepsilon z, s) dz + O(e^{-C/\varepsilon}).
\]

Then expanding \( \zeta^2(r) J^{\frac{1}{2}}(r, s) \) at \( r = 0 \) and using (2.16) lead to (3.10). To prove (3.11), it follows from (2.19), (2.10) and (1.22) that

\[
\int_{\Gamma_t(\delta)} \phi^2(x) dx = \int_{\Gamma_t} \int_{-\delta}^{\delta} \phi^2(r, s) J(r, s) dr
d \geq \int_{\Gamma_t} \int_{-\delta}^{\delta} \phi^2(r, s) J^{\frac{1}{2}}(r, s) dr
\]

\[
= \int_{\Gamma_t} Z^2(s) \left( e^{-1} \int_{-\delta}^{\delta} \varphi^2(\xi) \zeta^2(\varepsilon) J^{\frac{1}{2}}(r, s) dr \right)
+ \int_{\Gamma_t} \int_{-\delta}^{\delta} (\phi^\perp)^2 J^{\frac{1}{2}}(r, s) dr.
\]

This together with (1.20), (3.10) and (2.10) implies the first inequality in (3.11). Reversing the above estimates gives the second inequality and the proof is completed.

Decomposition (1.17) is not convenient to manipulate as it involves the eigenfunction \( \varphi \). Due to (3.8), we shall instead decompose \( \phi \) through \( \theta' \). To this end, we set \( \rho(r) = e^{-\frac{1}{2}} \left( \varphi(\frac{r}{\varepsilon}) - \theta'(\frac{r}{\varepsilon}) \right). \) Thanks to (3.8), we have

\[
\rho(r), \rho'(r), \rho''(r) = O(e^{-C/\varepsilon}). \tag{3.12}
\]
We rewrite (1.17) as

\[ \phi(x) = \frac{\rho(r)Z(s)\zeta(r)}{\Delta_{\phi_e(r,s)}} + \varepsilon^{-\frac{1}{2}}Z(s)\theta'(\zeta)\zeta(r) + \phi^+(x), \quad (3.13) \]

given which we can expand the terms on the left hand side of (1.21a) as

\[ \frac{1}{\varepsilon^2} \int_\Omega |\mathcal{L}_\varepsilon[\phi]|^2 \, dx = \frac{1}{\varepsilon^2} \int_\Omega \left( |\mathcal{L}_\varepsilon[\phi_e]|^2 + |\mathcal{L}_\varepsilon[\phi^\perp]|^2 + |\mathcal{L}_\varepsilon[\phi^\perp]|^2 \right) \, dx \]

\[ + \frac{2}{\varepsilon^2} \int_\Omega \left( \mathcal{L}_\varepsilon[\phi_e] \mathcal{L}_\varepsilon[\phi^\perp] + \mathcal{L}_\varepsilon[\phi^\perp] \mathcal{L}_\varepsilon[\phi^\perp] \right) \, dx, \]

\[ \frac{1}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a\phi^2 \, dx = \frac{1}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a \left( (\phi_e)^2 + (\phi^\perp)^2 + (\phi^\perp)^2 \right) \, dx \]

\[ + \frac{2}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a \left( \phi_e(\phi^\perp + \phi^\perp) + \phi^\perp \phi^\perp \right) \, dx. \quad (3.14) \]

We collect the terms in the above formulas that include \( \phi_e \) and use (3.12):

\[ \frac{1}{\varepsilon^2} \int_\Omega |\mathcal{L}_\varepsilon[\phi]|^2 \, dx + \frac{2}{\varepsilon^2} \int_\Omega \mathcal{L}_\varepsilon[\phi^\perp] \mathcal{L}_\varepsilon[\phi_e] \, dx \]

\[ + \frac{1}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a(\phi^2) \, dx + \frac{2}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a(\phi^\perp + \phi^\perp) \, dx \]

\[ \geq O(e^{-\frac{C}{\varepsilon}}) \sup_{|r| \leq \delta} \int_{\mathcal{G}_r} |\Delta |Z(s)|^2 + O(e^{-\frac{C}{\varepsilon}}) \int_{\mathcal{G}_r} Z^2(s) + O(e^{-\frac{C}{\varepsilon}}) \int_\Omega |\mathcal{L}_\varepsilon[\phi^\perp]|^2 \, dx. \quad (3.15) \]

Here \( \sup_{|r| \leq \delta} \) appears because the coefficients of \( \Delta | \) depend on \( r \) smoothly, according to (2.11). Note that \( O(e^{-\frac{C}{\varepsilon}}) \) does not have a fixed sign. Combining (3.15) with (3.14) yields

\[ \frac{1}{\varepsilon^2} \int_\Omega |\mathcal{L}_\varepsilon[\phi]|^2 \, dx + \frac{1}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a\phi^2 \, dx \]

\[ \geq O(e^{-\frac{C}{\varepsilon}}) \sup_{|r| \leq \delta} \int_{\mathcal{G}_r} (\Delta |Z(s)|^2 + O(e^{-\frac{C}{\varepsilon}}) \int_{\mathcal{G}_r} Z^2(s) + I_1 + I_2 + I_3, \quad (3.16) \]

where \( I_1, I_2, I_3 \) are defined by

\[ I_1 = \frac{1}{\varepsilon^2} \int_\Omega |\mathcal{L}_\varepsilon[\phi^\perp]|^2 \, dx + \frac{1}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a(\phi^\perp)^2 \, dx, \quad (3.17a) \]

\[ I_2 = \frac{2}{\varepsilon^2} \int_\Omega \mathcal{L}_\varepsilon[\phi^\perp] \mathcal{L}_\varepsilon[\phi^\perp] \, dx + \frac{2}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a\phi^\perp \phi^\perp \, dx, \quad (3.17b) \]

\[ I_3 = \frac{1 + O(e^{-\frac{C}{\varepsilon}})}{\varepsilon^2} \int_\Omega |\mathcal{L}_\varepsilon[\phi^\perp]|^2 \, dx + \frac{1}{\varepsilon^3} \int_\Omega f'''(\phi_a)\mu_a(\phi^\perp)^2 \, dx \triangleq I_{31} + I_{32}. \quad (3.17c) \]

Theorem 1.2 will be a consequence of the estimates of (3.17a)–(3.17c), which will be done in the following three sections.
4. Spectrum Condition: Estimates of Kernel Terms

The main result of this section is stated below concerning (3.17a). The proof is given at the end of this section after establishing a few technical lemmas.

**Proposition 4.1.** There exists $\Lambda_4 > 0$ and generic constant $C > 0$, $\varepsilon_0 > 0$ such that

$$I_1 \geq \Lambda_4 \|Z\|_{H^2(\Gamma)}^2 - C \|Z\|_{L^2(\Gamma)}^2, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

We shall decompose $I_1$ into several parts, and treat them in a series of lemmas. To start with, we need the decomposition of Euclidean Laplace operator. Recall (2.11) and (3.13), we can expand $\Delta \phi^T$ by

$$\Delta \phi^T = \partial_r^2 \phi^T + \partial_r (\ln \sqrt{g}) \partial_r \phi^T + \Delta_{\Gamma} \phi^T$$

$$= \varepsilon^{-\frac{3}{2}} Z(s) \theta'''(z) \xi(r) + \varepsilon^{-\frac{3}{2}} \partial_r (\ln \sqrt{g}) \xi(r)$$

$$+ \varepsilon^{-\frac{3}{2}} \Delta_{\Gamma} Z(s) \theta'(z) \xi(r) + Z(s) A(r, s), \text{ with } z = r(x, t)/\varepsilon$$

and $\Delta_{\Gamma}$ is defined by (2.11). Moreover, according to (1.18),

$$A(r, s) = 2\varepsilon^{-\frac{3}{2}} \theta''(\varepsilon) \xi'(r) + \varepsilon^{-\frac{3}{2}} \theta''(\varepsilon) \xi''(r) + \varepsilon^{-\frac{3}{2}} \partial_r (\ln \sqrt{g}) \theta''(\varepsilon) \xi'(r) = O(\varepsilon^{-\frac{3}{2}}).$$

**Lemma 4.1.** The following expansions hold in $\Gamma_r(\delta)$ with $z = r(x, t)/\varepsilon$:

$$\theta''(z) - f''(\phi_a) \theta'(z) = -\varepsilon^2 f'''(\phi) \tilde{\phi}^{(2)} \theta'(z) + \theta'(z) O(\varepsilon^3),$$

$$f'''(\phi_a) \mu_a = -6 h(s) \theta' + 3 \varepsilon h^2(s) z \theta' + 6 \varepsilon^2 (\theta \tilde{\mu}^{(2)} + \tilde{\phi}^{(2)} \tilde{\mu}^{(0)})$$

$$- 6 \varepsilon^2 \theta' \left( a(s) z^2 - \Delta d^{(2)} - c(s) z^2 + z D^{(1)} - h(s) b(s) z^2 \right) + O(\varepsilon^3).$$

**Proof.** Recall that $f(u) = \frac{1}{2} (u^2 - 1)^2$ and that $\theta'' = f'(\theta)$. Then (4.4a) follows from (3.6) and the Taylor expansion. To prove (4.4b) we need $f'''(\phi_a) = 6 \theta + 6 \varepsilon^2 \tilde{\phi}^{(2)} + O(\varepsilon^3)$, following from (3.6) and $f'''(u) = 6u$. Then it follows from (3.6), (3.3) and (3.4) that

$$f'''(\phi_a) \mu_a$$

$$= 6 \theta \tilde{\mu}^{(0)} + 6 \varepsilon \tilde{\mu}^{(1)} + 6 \varepsilon^2 (\theta \tilde{\mu}^{(2)} + \tilde{\phi}^{(2)} \tilde{\mu}^{(0)}) + O(\varepsilon^3)$$

$$= -6 \theta \Delta d^{(0)} + 6 \varepsilon \Delta d^{(1)} + 6 \varepsilon^2 \theta' D^{(0)} + 6 \varepsilon^2 (\theta \tilde{\mu}^{(2)} + \tilde{\phi}^{(2)} \tilde{\mu}^{(0)}) + O(\varepsilon^3)$$

$$= -6 \theta \Delta r - 6 \varepsilon^2 \theta' \Delta d^{(2)} + 6 \varepsilon^2 \theta' \nabla \Delta r \cdot \nabla r + 3 \varepsilon^2 \theta' (\Delta r)^2$$

$$- 6 \varepsilon^2 \theta' D^{(1)} + 6 \varepsilon^2 (\theta \tilde{\mu}^{(2)} + \tilde{\phi}^{(2)} \tilde{\mu}^{(0)}) + O(\varepsilon^3).$$

Note that in the last step, we employ (2.4) to express the leading order terms in term of $r(x, t)$. Then using (2.17), we can expand various terms about $r$ and then replace them by $z = r/\varepsilon$. These lead to the following formula which implies (4.4b)

$$f'''(\phi_a) \mu_a = -6 h(s) \theta' + 3 \varepsilon h^2(s) z \theta' - 6 \varepsilon^2 a(s) z^2 \theta'$$

$$+ 6 \varepsilon^2 \theta' \Delta d^{(2)} + 6 \varepsilon^2 c(s) z^2 \theta' - 6 \varepsilon^2 \theta' D^{(1)}$$

$$+ 6 \varepsilon^2 h(s) b(s) z^2 \theta' + 6 \varepsilon^2 (\theta \tilde{\mu}^{(2)} + \tilde{\phi}^{(2)} \tilde{\mu}^{(0)}) + O(\varepsilon^3).$$
Lemma 4.2. Assume $\omega \in C^1(\overline{\Gamma}(\delta))$ and $\tilde{\theta}$ satisfies

$$\tilde{\theta}(z) \in L^1(\mathbb{R}), \quad \tilde{\theta}(-z) = -\tilde{\theta}(z), \quad \text{and} \quad \lim_{z \to \pm\infty} \tilde{\theta}(z) = 0 \text{ exponentially.} \quad (4.5)$$

Then there exists an $\epsilon$-independent constant $C$ such that

$$\left| \int_{\Gamma} \left( \int_{-\delta}^{\delta} u(s)\omega(r, s)\tilde{\theta}(\frac{r}{\epsilon}) dr \right) \right| \leq C\epsilon^2 \int_{\Gamma} |u(s)|, \quad \forall u \in L^1(\Gamma).$$

**Proof.** Expanding $\omega(r, s)$ in terms of $r$ and using the oddness of $\tilde{\theta},$

$$\int_{\Gamma} \int_{-\delta}^{\delta} u(s)\omega(r, s)\tilde{\theta}(\frac{r}{\epsilon}) dr = \int_{\Gamma} \int_{-\delta}^{\delta} u(s) (\omega(0, s) + r \partial_r \omega(\xi(r), s)) \tilde{\theta}(\frac{r}{\epsilon}) dr$$

$$= \epsilon \int_{\Gamma} \int_{-\delta}^{\delta} u(s)\partial_r \omega(\xi(r), s)\tilde{\theta}(\frac{r}{\epsilon}) dr.$$ 

This leads to the desired result after a change of variable $r = \epsilon z$.

The next lemma handles integrals involving $\Delta \Gamma$.

Lemma 4.3. Assume $\tilde{\theta}(z)$ satisfies (4.5) and $\omega \in C^1(\overline{\Gamma}(\delta))$. Then for any $u \in L^2(\Gamma)$ and $v \in H^2(\Gamma)$,

$$\left| \int_{\Gamma} \int_{-\delta}^{\delta} u(s)\Delta \Gamma v(s)\tilde{\theta}(\frac{r}{\epsilon}) \omega(r, s) dr \right|$$

$$\leq \nu \epsilon^2 \int_{\Gamma} \left( |\Delta \Gamma|_{r=0} v(s) |^2 + |v(s)|^2 \right) + \frac{C\epsilon^2}{\nu} \int_{\Gamma} u^2(s), \quad \forall v > \epsilon \frac{1}{\nu},$$

where the constant $C$ depends on $\omega, \tilde{\theta}$ but not on $\epsilon, v, u$ or $v$.

**Proof.** Recall (2.12), we have the expansion $\Delta \Gamma = \Delta \Gamma |_{r=0} + rR \Gamma$. Thus

$$\int_{\Gamma} \int_{-\delta}^{\delta} u(s)\Delta \Gamma v(s)\tilde{\theta}(\frac{r}{\epsilon}) \omega(r, s) dr$$

$$= \int_{\Gamma} \int_{-\delta}^{\delta} u(s)(\Delta \Gamma |_{r=0} v(s))\tilde{\theta}(\frac{r}{\epsilon}) \omega(r, s) dr$$

$$+ \epsilon \int_{\Gamma} \int_{-\delta}^{\delta} u(s)R \Gamma v(s) \left( \frac{r}{\epsilon} \tilde{\theta}(\frac{r}{\epsilon}) \right) \omega(r, s) dr.$$ 

The first integral can be handled by Lemma 4.2: for any $v > \epsilon \frac{1}{\nu},$

$$\left| \int_{\Gamma} \int_{-\delta}^{\delta} u(s)(\Delta \Gamma |_{r=0} v(s))\tilde{\theta}(\frac{r}{\epsilon}) \omega(r, s) dr \right|$$

$$\leq \frac{\nu \epsilon^2}{2} \int_{\Gamma} (\Delta \Gamma |_{r=0} v)^2 + C\epsilon^2 v^{-1} \int_{\Gamma} u^2.$$
For the second one, we use (2.13b) and the Cauchy–Schwarz inequality to yield

\[ \int_{\Gamma_t} \int_{-\delta}^{\delta} u(s) \partial r v(s) \left( \frac{\partial}{\partial r} (\xi) \right) \omega(r, s) dr \leq \frac{\nu \epsilon}{2} \int_{\Gamma_t} \left( (\Delta \Gamma|_{r=0} v)^2 + |v|^2 \right) + \epsilon C v^{-1} \int_{\Gamma_t} u^2, \]

where \( C \) depends on \( \omega, \tilde{\theta} \). The above three estimates together imply the desired result.

**Lemma 4.4.** Assume \( \tilde{\theta}(z) \) satisfies (4.5). Then there exist \( \Lambda_1, \Lambda_2 > 0 \) such that, for every \( \epsilon < \epsilon_0 \) and every \( v \in H^2(\Gamma_t) \), the following two inequalities hold:

\[
\Lambda_1 \epsilon \int_{\Gamma_t} (\Delta \Gamma|_{r=0} v(s))^2 \leq \int_{\Gamma_t} \int_{-\delta}^{\delta} |\Delta \Gamma v(s)|^2 \tilde{\theta}(z)^2 J(r, s) dr + \epsilon \int_{\Gamma_t} v^2(s), \tag{4.6a}
\]

\[
\int_{\Gamma_t} \int_{-\delta}^{\delta} |\Delta \Gamma v(s)|^2 \tilde{\theta}(z)^2 J(r, s) dr \leq \Lambda_2 \epsilon \int_{\Gamma_t} \left( v^2(s) + (\Delta \Gamma|_{r=0} v)^2 \right). \tag{4.6b}
\]

**Proof.** We only prove the first inequality. The other one follows in exactly the same manner. In view of (2.16), we have \( J(r, s) \geq 1/2 \). Since \( \tilde{\theta}(z)^2 \) decays exponentially, so does \( z \tilde{\theta}(z)^2 \). As a result, we employ (2.13a) and obtain for some \( \Lambda_1 > 0 \) that

\[
\epsilon \int_{\Gamma_t} v^2(s) + \int_{\Gamma_t} \int_{-\delta}^{\delta} |\Delta \Gamma v(s)|^2 \tilde{\theta}(z)^2 J(r, s) dr
\geq \epsilon \int_{\Gamma_t} v^2(s) + \frac{1}{\Lambda_1} \int_{\Gamma_t} \inf_{|\xi| \leq \delta} |\Delta \Gamma v(s)|^2 \left( \int_{-\delta}^{\delta} \tilde{\theta}(z)^2 dr \right)
\geq \epsilon \Lambda_1 \int_{\Gamma_t} \left( \Delta \Gamma|_{r=0} v \right)^2.
\]

We are in a position to estimate (3.17a). For simplicity, we shall suppress the arguments of a function if it is clear from the context, and omit \( dS \) in a surface integral \( \int_{\Gamma_t} \) (recall (2.19)). Meanwhile, we shall often employ the following inequality without mentioning

\[ ab \leq \nu a^2 + 4b^2/v, \text{ with the convention that } v \geq \epsilon^{1/3}. \tag{4.7} \]

So the choice of \( v \) will not significantly affect the order of \( \epsilon \). Using (1.13) and (4.2) yields

\[
\mathcal{L}_\epsilon[\phi^\top] = \epsilon^{-\frac{1}{2}} Z(s)(\theta''(z) - f''(\phi_0)\theta'(z))\zeta(r) + \epsilon^{-\frac{1}{2}} \partial_r (\ln \sqrt{g}) Z(s)\theta''(z)\zeta(r)
+ \epsilon^{\frac{1}{2}} \Delta \Gamma Z(s)\theta'(z)\zeta(r) + \epsilon Z(s) A(r, s). \tag{4.8}
\]
Substituting the above formula into (3.17a) leads to the expansion

\[ I_1 = \varepsilon^{-2} \int_{\Gamma_1} \int_{-\delta}^{\delta} \left( \varepsilon^{-\frac{1}{2}} Z(s) (\theta'''(z) - f'''(\phi_a) \theta'(z)) \xi(r) \right) \frac{\Delta_1}{\Gamma_1} \delta \left( \frac{z}{\Delta_1} \right) \frac{\delta}{\Gamma_1} \left( \frac{r}{\Delta_1} \right) J(r, s) dr \\
+ \varepsilon^{-\frac{1}{2}} \partial_r \left( \ln \sqrt{\gamma} \right) Z(s) \theta''(z) \xi(r) + \varepsilon Z(s) A(r, s) \right] J(r, s) dr \\
+ \varepsilon^{-4} \int_{\Gamma_1} \int_{-\delta}^{\delta} f'''(\phi_a) \mu_a Z^2(s) (\theta'(z))^2 \xi^2(r) J(r, s) dr \triangleq I_{11} + I_{12} + I_{13} + I_{14}, \tag{4.10} \]

where we define

\[ I_{11} = \varepsilon^{-3} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z^2(s) \left( \varepsilon^{-1} (\theta'''(z) - f'''(\phi_a) \theta'(z)) + \partial_r \left( \ln \sqrt{\gamma} \right) \xi(r) \right) \frac{\Delta_1}{\Gamma_1} \delta \left( \frac{z}{\Delta_1} \right) \frac{\delta}{\Gamma_1} \left( \frac{r}{\Delta_1} \right) J(r, s) dr, \]

\[ I_{12} = 2\varepsilon^{-3} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z(s) \Delta_1 Z(s) \theta''(z) \xi^2(r) \partial_r \left( \ln \sqrt{\gamma} \right) J(r, s) dr \\
+ 2\varepsilon^{-2} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z(s) \Delta_1 Z(s) \theta'(z) \xi^2(r) \partial_r \left( \ln \sqrt{\gamma} \right) J(r, s) dr \\
+ \varepsilon^{-1} \int_{\Gamma_1} \int_{-\delta}^{\delta} \left( \Delta_1 Z(s) \right) \frac{\Delta_1}{\Gamma_1} \delta \left( \frac{z}{\Delta_1} \right) \frac{\delta}{\Gamma_1} \left( \frac{r}{\Delta_1} \right) J(r, s) dr, \]

\[ I_{13} = \varepsilon^{-4} \int_{\Gamma_1} \int_{-\delta}^{\delta} f'''(\phi_a) \mu_a Z^2(s) (\theta'(z))^2 \xi^2(r) J(r, s) dr, \]

\[ I_{14} = 2\varepsilon^{-1} \int_{\Gamma_1} \int_{-\delta}^{\delta} \left( \varepsilon^{-\frac{1}{2}} Z(s) (\theta''(z) - f''(\phi_a) \theta'(z)) \xi(r) + \varepsilon^{-\frac{1}{2}} Z(s) \theta''(z) \theta'(z) \partial_r \left( \ln \sqrt{\gamma} \right) \right. \]

\[ + \varepsilon^{-2} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z(s) A(r, s) J(r, s) dr + \int_{\Gamma_1} \int_{-\delta}^{\delta} Z^2(s) A^2(r, s) J(r, s) dr. \]

**Lemma 4.5.** There exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that

\[ I_{11} \geq \varepsilon^{-3} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z^2(s) h^2(s) (\theta'')^2 \xi^2(r) dr - C \int_{\Gamma_1} Z^2(s), \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{4.11} \]

**Proof.** It follows from (3.3a) and (2.16) that \( \bar{\phi}^{(2)}(t), h(s), b(s) \) are uniformly bounded functions. Using (4.4a), (2.17) and (3.3a), we can treat \( I_{11} \) by

\[ I_{11} = \frac{1}{\varepsilon^3} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z^2(s) \left( -\varepsilon f'''(\theta) \bar{\phi}^{(2)} \theta' + \theta' O(\varepsilon^2) + \theta'' h(s) + \varepsilon z b(s) \right) \\
+ \xi^2(\varepsilon^2) \right) \frac{\Delta_1}{\Gamma_1} \delta \left( \frac{z}{\Delta_1} \right) \frac{\delta}{\Gamma_1} \left( \frac{r}{\Delta_1} \right) J(r, s) dr \\
= \frac{1}{\varepsilon^3} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z^2(s) \left( (\theta'')^2 h^2(s) - 2\varepsilon f'''(\theta) \theta' \theta''(z) \bar{\phi}^{(2)} h(s) + 2\varepsilon z (\theta'')^2 h(s) b(s) \right) \\
+ \theta' O(\varepsilon^2) + \theta''(z) O(\varepsilon^2) + z \theta' O(\varepsilon^2) + z^2 (\theta'')^2 O(\varepsilon^2) + O(\varepsilon^3) \right) \xi^2(r) J(r, s) dr. \]
Note that the terms corresponding to $\mathcal{J}$ tend to 0 exponentially as $z \to \pm \infty$. We can gain a factor $\varepsilon$ after a change of variable $r \to \varepsilon z$ as follows:

$$I_{11} \geq \frac{1}{\varepsilon^3} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z^2(\theta'')^2 h^2 \xi^2(r) J dr - \frac{2}{\varepsilon^2} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z^2 \left( f'''(\theta)\theta'^2 \tilde{\phi}''(2) h - z(\theta'')^2 hb \right) \xi^2(r) J dr - C \int_{\Gamma_t} Z^2.$$

It remains to treat the second term on the right hand side. It follows from (3.3a) that $\tilde{\phi}''(z, x, t) = D(0)(x, t)\theta'(z)\alpha(z)$ for some bounded odd function $\alpha(z)$. So applying Lemma 4.2 with $\tilde{\theta} = f'''(\theta)\theta'^2 \theta'' \alpha$, $\omega(r, s) = J(r, s)D(0)h(s)\xi^2(r)$ and $u(s) = Z^2(s)$ gives the estimate for its first part, and similar argument gives the estimate of the second part. Altogether, we have

$$-\frac{2}{\varepsilon^2} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z^2 \left( f'''(\theta)\theta'^2 \tilde{\phi}''(2) h - z(\theta'')^2 hb \right) \xi^2(r) J dr \geq -C \int_{\Gamma_t} Z^2.$$

This, together with (2.16), leads to

$$I_{11} \geq \varepsilon^{-3} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z^2(s) h^2(s) (\theta'')^2 \xi^2(r) \left( 1 + h(s)r + e(s)r^2 + O(r^3) \right) dr - C \int_{\Gamma_t} Z^2(s).$$

Finally, as $(\theta'')^2 \xi^2(r) r$ is odd in $r$, the integral including $h(s)$ will vanish. So we arrive at the desired result by a change of variable.

**Lemma 4.6.** There exist $\Lambda_1 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that

$$I_{12} \geq \frac{\Lambda_1}{2} \int_{\Gamma_t} \left( \Delta \Gamma \big|_{r=0} Z(s) \right)^2 - C \int_{\Gamma_t} Z^2(s), \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (4.12)$$

**Proof.** We write $I_{12} = I_{121} + I_{122} + I_{123}$ with obvious definitions. To estimate $I_{121}$, we use (4.4a), a change of variable $r = \varepsilon z$ and the Cauchy–Schwarz inequality:

$$I_{121} = 2\varepsilon^{-3} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z(s) \Delta \Gamma Z(s) \theta^2 \left( -\varepsilon^2 f'''(\theta)\tilde{\phi}''(2) \theta' + \theta'(z) O(\varepsilon^3) \right) \xi^2(r) J(r, s) dr \geq -C \int_{\Gamma_t} Z^2(s) - \nu \sup_{\vert r \vert < \delta} \int_{\Gamma_t} \left( \Delta \Gamma Z(s) \right)^2.$$

As the coefficients of $\Delta \Gamma$ depend on $r$ smoothly, $I_{122}$ can be estimated using Lemma 4.3 with $\tilde{\theta} = \theta'' \theta'$ and $\omega = \xi^2 \partial_r (\ln \sqrt{g}) J(r, s)$:

$$I_{122} \geq -\frac{C}{\nu} \int_{\Gamma_t} Z^2(s) - \nu \int_{\Gamma_t} \left( \Delta \Gamma \big|_{r=0} Z(s) \right)^2, \quad \forall \nu > \varepsilon^{\frac{1}{2}}.$$

To estimate $I_{123}$, we apply (4.6a):

$$I_{123} \geq \Lambda_1 \int_{\Gamma_t} \left( \Delta \Gamma \big|_{r=0} Z(s) \right)^2 - C \int_{\Gamma_t} Z^2(s).$$

By choosing a sufficiently small $\nu$, the above three inequalities and (2.13a) imply (4.12).
Lemma 4.7. There exist $\varepsilon_0 > 0$ and $C > 0$ such that

$$I_{13} \geq -3\varepsilon^{-3} \int_{\Gamma} \int_{-\delta}^{\delta} Z^2(s) h^2(s) z \theta \theta' \zeta^2(r) dr - C \int_{\Gamma} Z^2(s), \quad \forall \varepsilon \in (0, \varepsilon_0).$$

(4.13)

Proof. To treat $I_{13}$, we employ (4.4b). Note that $\theta$ is odd and $\zeta, \theta'$ are even. So in the expression for $I_{13}$, the terms about $a(s), b(s), c(s)$ can be treated by Lemma 4.2. As a result

$$I_{13} \geq \varepsilon^{-4} \int_{\Gamma} \int_{-\delta}^{\delta} (-6h(s)\theta' + 3\varepsilon h^2(s)\zeta \theta') Z^2(s) \theta^2 \zeta^2(r) dr - C \int_{\Gamma} Z^2(s) \triangleq I_{131}

+ 6\varepsilon^{-2} \int_{\Gamma} \int_{-\delta}^{\delta} \left( \theta' (\Delta d^{(2)} - zD^{(1)}) + (\theta \tilde{\mu}^{(2)} + \phi^{(2)} \mu^{(0)}) \right) Z^2 \theta \zeta^2(r) J dr \triangleq I_{132}$$

(4.14)

Employing the expansion of $J(r, s)$ in (2.16), we can write $I_{131}$ by

$$I_{131} = \varepsilon^{-4} \int_{\Gamma} \int_{-\delta}^{\delta} (-6h(s)\theta' + 3\varepsilon h^2(s)\zeta \theta') Z^2(s) \theta^2 \zeta^2(r) dr

\cdot \left( 1 + \varepsilon z h(s) + \varepsilon^2 z^2 e(s) + z^3 O(\varepsilon^3) \right) \zeta^2(r) dr

= -6\varepsilon^{-4} \int_{\Gamma} \int_{-\delta}^{\delta} Z^2(s) h(s) \theta \theta' \zeta^2(r) dr

- 3\varepsilon^{-3} \int_{\Gamma} \int_{-\delta}^{\delta} Z^2(s) h^2(s) \zeta \theta^3 \zeta^2(r) dr

+ 3\varepsilon^{-2} \int_{\Gamma} \int_{-\delta}^{\delta} Z^2(s) h(s) (h^2(s) - 2e(s)) \zeta^2 \theta^3 \zeta^2(r) dr

+ \varepsilon^{-1} \int_{\Gamma} \int_{-\delta}^{\delta} Z^2(s) \tilde{\theta}(\zeta) O(1) \zeta^2(r) dr,$$

where $\tilde{\theta}(z) \in L^1(\mathbb{R})$ and decays exponentially to 0 at $\pm \infty$. Since the first and the third term above vanish, we obtain

$$I_{131} \geq -3\varepsilon^{-3} \int_{\Gamma} \int_{-\delta}^{\delta} Z^2(s) h^2(s) z \theta^3 \zeta^2(r) dr - C \int_{\Gamma} Z^2(s).$$

(4.15)

We now turn to $I_{132}$ in (4.14). Recall Lemma 3.1 and (3.3) for the precise forms of $\tilde{\mu}^{(2)}, \phi^{(2)}, \mu^{(0)}$. Combining Lemma 4.2 with the fact that $\phi^{(2)} \mu^{(0)}$ is odd in $z$, we infer that

$$6\varepsilon^{-2} \int_{\Gamma} \int_{-\delta}^{\delta} \phi^{(2)} \mu^{(0)} Z^2(s) \theta^2 \zeta^2(r) J dr \geq -C \int_{\Gamma} Z^2(s).$$
The terms about $\Delta d^{(2)}$ can be treated similarly, so we employ the last formula in (3.3)

$$I_{132} \geq 6\varepsilon^{-2} \int_{G_t} \int_{-\delta}^{\delta} \left(-\theta \theta' z D^{(1)} + \theta \tilde{\mu}^{(2)}\right) Z^2(s) \theta^2 \xi^2(r) Jdr - C \int_{G_t} Z^2(s)$$

$$= 6\varepsilon^{-2} \int_{G_t} \int_{-\delta}^{\delta} Z^2(s) \Delta d^{(0)} D^{(0)} \gamma_1(z) \theta \theta^3 \xi^2(r) Jdr + 6\varepsilon^{-2} \int_{G_t} \int_{-\delta}^{\delta} Z^2(s) \nabla d^{(0)} \cdot \nabla D^{(0)} \gamma_2(z) \theta \theta^3 \xi^2(r) Jdr + 6\varepsilon^{-2} \int_{G_t} \int_{-\delta}^{\delta} Z^2(s) \theta \theta^3 \mu_2(x, t) \xi^2(r) Jdr + 6\varepsilon^{-2} \int_{G_t} \int_{-\delta}^{\delta} Z^2(s) \chi^{(0)} d^{(0)} \gamma_3(z) \theta \theta^3 \xi^2(r) Jdr - C \int_{G_t} Z^2(s).$$

Recall Lemma 3.1 that $\gamma_{\ell}(z) \theta \theta^3$ (with $\ell = 1, 2, 3$) are odd functions. So it follows from Lemma 4.2 that $I_{132} \geq -C \int_{G_t} Z^2(s)$. This together with (4.14), (4.15) yields (4.13).

**Proof of Proposition 4.1.** It follows from (4.3) and (2.13a) that

$$I_{14} \geq -\varepsilon \int_{G_t} Z^2(s) - \varepsilon \int_{G_t} \left(\Delta \Gamma \big|_{r=0} Z(s)\right)^2. \quad (4.16)$$

Substituting (4.11), (4.12), (4.13) and (4.16) into (4.10) leads to

$$I_1 \geq \varepsilon^{-3} \int_{G_t} Z^2(s) h^2(s) \int_{-\delta}^{\delta} \left(\theta'^2 - 3z \theta \theta^3\right) \xi^2(r) dr + \frac{\Lambda_1}{2} \int_{G_t} \left(\Delta \Gamma \big|_{r=0} Z(s)\right)^2 - C \int_{G_t} Z^2(s).$$

To treat the first integral on the right, we use the exponential decay of the integrand:

$$\int_{-\delta}^{\delta} \left(\theta'^2 - 3z \theta \theta^3\right) \xi^2(r) dr = O(e^{-C/\varepsilon}) + \varepsilon \int_{\mathbb{R}} \left(\theta'^2 - 3z \theta \theta^3\right) dz. \quad (4.17)$$

If the last integral vanishes, we shall have

$$I_1 \geq \frac{\Lambda_1}{2} \int_{G_t} \left(\Delta \Gamma \big|_{r=0} Z(s)\right)^2 - C \int_{G_t} Z^2(s).$$

On the other hand, it follows from (2.20) that

$$\int_{G_t} \left|\nabla \Gamma \right|_{r=0} Z(s) \right|^2 = - \int_{G_t} Z(s) \left(\Delta \Gamma \big|_{r=0} Z(s)\right) . \quad (4.18)$$

The above two equations together with Cauchy–Schwarz inequality imply (4.1). To compute the last integral in (4.17), we first note that $\theta$ satisfies

$$\sqrt{2}\theta' = 1 - \theta^2, \quad \text{and} \quad -\sqrt{2}\theta \theta' = \theta'^2. \quad (4.19)$$
Applying these formulas together with integration by parts gives
\[
\int_{\mathbb{R}} (\theta''^2 - 3z\theta'^3) \, dz = \int_{\mathbb{R}} (\theta''^2 + \frac{3}{\sqrt{2}}z\theta''\theta'^2) \, dz = \int_{\mathbb{R}} (2(\theta\theta')^2 - \frac{1}{\sqrt{2}}\theta'^3) \, dz.
\]
Applying (4.19) again and performing a change of variable,
\[
\int_{\mathbb{R}} (2(\theta\theta')^2 - \frac{1}{\sqrt{2}}\theta'^3) \, dz = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} (\theta')^2 (2\sqrt{2}\theta' - \theta') \, dz = \frac{1}{2\sqrt{2}} \int_{-1}^{1} (1 - \theta^2)(5\theta^2 - 1) \, d\theta = 0.
\]

5. Spectrum Condition: Estimates of Cross Terms

We shall prove the following result at the end of this section.

**Proposition 5.1.** There exist $C > 0$ and $\varepsilon_0 > 0$ such that
\[
I_2 \geq -C \int_{\Omega} \phi^2 \, dx - \nu \int_{\Gamma_t} (\Delta\Gamma|_{r=0} Z(s))^2 - \frac{1}{2} I_{31} - \nu \varepsilon^{-4} \int_{\Gamma_t(\delta)} (\phi \phi^\perp)^2 \, dx
\]
\[- \nu \varepsilon^{-2} \int_{\Gamma_t(\delta)} |\nabla\Gamma(\phi \phi^\perp)|^2 \, dx - \frac{C\varepsilon}{\nu} \int_{\Gamma_t(\delta)} (\Delta\Gamma|_{r=0} \phi \phi^\perp)^2 \, dx\]
\[(5.1)\]
holds for every $\varepsilon \in (0, \varepsilon_0)$ and $\nu \in (\frac{1}{\sqrt{2}}, 1)$.

Recall (3.17) for the definitions of $I_2$ and $I_{31}$. Under coordinates $(r, s) \in (-2\delta, 2\delta) \times \Gamma_t$, we can employ (4.2) to write $I_2$ as
\[
I_2 = I_{21} + I_{22} + I_{23} + I_{24} + I_{25},
\]
where we define
\[
I_{21} = -2\varepsilon^{-\frac{3}{2}} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z(s)(\theta''''(z) - f''(\phi_a)\theta'(z))\xi(r)\mathcal{L}_\varepsilon[\phi^\perp]J(r, s) \, dr, \quad (5.3a)
\]
\[
I_{22} = -2\varepsilon^{-\frac{5}{2}} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z(s)\theta''(z)\xi(r)\partial_r(\ln \sqrt{\varepsilon})\mathcal{L}_\varepsilon[\phi^\perp]J(r, s) \, dr, \quad (5.3b)
\]
\[
I_{23} = -2\varepsilon^{-\frac{1}{2}} \int_{\Gamma_t} \int_{-\delta}^{\delta} \Delta\Gamma Z(s)\theta'(z)\xi(r)\mathcal{L}_\varepsilon[\phi^\perp]J(r, s) \, dr, \quad (5.3c)
\]
\[
I_{24} = -2\varepsilon^{-1} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z(s)A(r, s)\mathcal{L}_\varepsilon[\phi^\perp]J(r, s) \, dr, \quad (5.3d)
\]
\[
I_{25} = 2\varepsilon^{-\frac{1}{2}} \int_{\Gamma_t} \int_{-\delta}^{\delta} f''''(\phi_a)\mu_a Z(s)\theta'(z)\xi(r)\phi^\perp J(r, s) \, dr.
\]

**Lemma 5.1.** There exist $C > 0$ and $\varepsilon_0 > 0$ such that
\[
I_{21} \geq -C \int_{\Gamma_t} Z^2(s) - \frac{1}{4} I_{31}, \quad \forall \varepsilon \in (0, \varepsilon_0).
\]
\[(5.4)\]
Proof. We first recall (3.3a) for $\tilde{g}^{(2)}$ as well as (3.17c) for the definition of $I_{31}$. It follows from (4.4a) and the Cauchy–Schwarz inequality that

\[
I_{21} = 2\varepsilon^{-\frac{3}{2}} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z(s) f'''(\theta) \tilde{g}^{(2)}(\theta') \xi(r) \mathcal{L}_r[\phi^\perp] J(r, s) \, dr \\
- 2\varepsilon^{-\frac{1}{2}} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z(s) \theta' O(1) \xi(r) \mathcal{L}_r[\phi^\perp] J(r, s) \, dr \geq -C \varepsilon^{-1} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z^2(s) \theta'^2 \xi^2(r) J(r, s) \, dr - \frac{1}{4} I_{31}. \tag{5.5}
\]

So the desired result follows from (2.16) and a change of variable.

Lemma 5.2. There exist $C > 0$ and $\varepsilon_0 > 0$ such that

\[
I_{22} + I_{25} \geq -\nu \int_{\Gamma_t} (\Delta_{\Gamma}|_{r=0} Z)^2 - C(1 + \nu^{-1}) \int_{\Gamma_t} Z^2(s) - \nu \varepsilon^{-4} \int_{\Gamma_t(\delta)} (\phi^\perp \xi)^2 \, dx \\
- C \varepsilon^{-\frac{3}{2}} \int_{\Gamma_t(\delta) \setminus \Gamma_t(\delta/2)} (\phi^\perp)^2 \, dx - \nu \varepsilon^{-2} \int_{\Gamma_t(\delta)} |\nabla \Gamma(\phi^\perp \xi)|^2 \, dx - \frac{1}{8} I_{31}. \tag{5.6}
\]

holds for every $\varepsilon \in (0, \varepsilon_0)$ and $\nu \in (\varepsilon^\frac{1}{m}, 1)$.

Proof. Estimate of $I_{22}$: It follows from (2.17), a change of variable $z = r/\varepsilon$ and the Cauchy–Schwarz inequality that

\[
I_{22} = -2\varepsilon^{-\frac{5}{2}} \int_{\Gamma_t(\delta)} Z(s) \theta'' \xi(r) (h(s) + O(\varepsilon) z) \mathcal{L}_r[\phi^\perp] \, dx \\
\geq -2\varepsilon^{-\frac{5}{2}} \int_{\Gamma_t(\delta)} Z(s) \theta'' \xi(r) h(s) \mathcal{L}_r[\phi^\perp] \, dx - C \int_{\Gamma_t} Z^2(s) - \frac{1}{8} I_{31}. \tag{5.7}
\]

On the other hand, using (3.6) we can write $f''(\phi_a) = f''(\theta) + O(\varepsilon^2)$. This together with integration by parts leads to

\[
I_{22} \geq 2\varepsilon^{-\frac{3}{2}} \int_{\Gamma_t(\delta)} \Delta \left( Z(s) h(s) \theta'' \xi(r) \right) \phi^\perp \, dx \tag{\triangleleft I_{221}} \\
- 2\varepsilon^{-\frac{7}{2}} \int_{\Gamma_t} \int_{-\delta}^{\delta} Z(s) h(s) \theta'' f''(\theta) \phi^\perp \xi(r) J(r, s) \, dr \\
- \int_{\Gamma_t} \int_{-\delta}^{\delta} O(\varepsilon^{-\frac{3}{2}}) Z(s) h(s) \theta'' \phi^\perp \xi(r) J(r, s) \, dr - C \int_{\Gamma_t} Z^2(s) - \frac{1}{8} I_{31}. \tag{\triangleleft I_{222}}
\]
Regarding $I_{221}$, we have the following identity due to (2.11):

\[
\begin{align*}
2\varepsilon^{-\frac{3}{2}} \Delta (Z(s)h(s)\theta''(z)\xi(r))\phi_\perp & \\
= 2\varepsilon^{-\frac{3}{2}} (\Delta Z(s)h(s) + 2\nabla Z(s) \cdot \nabla h(s) + Z(s)\Delta h(s)) \theta''(z)\xi(r)\phi_\perp & \\
+ 2\varepsilon^{-\frac{3}{2}} Z(s)h(s) \left( \varepsilon^{-2}\theta'''(z)\xi(r) + 2\varepsilon^{-1}\theta''(z)\xi'(r) + \theta''(z)\xi''(r) \right) \phi_\perp & \\
+ 2\varepsilon^{-\frac{3}{2}} Z(s)h(s) \left( \varepsilon^{-1}\theta''(z)\xi(r) + \theta''(z)\xi'(r) \right) \partial_r (\ln \sqrt{g})\phi_\perp, & \text{where } z = r/\varepsilon.
\end{align*}
\]

(5.8)

Note that the term $4\varepsilon^{-\frac{3}{2}} \nabla \left( Z(s)h(s) \right) \cdot \nabla \left( \theta''(z)\xi(r) \right)\phi_\perp$ which would have appeared in (5.8) vanishes because of the decomposition (2.9) and orthogonality. Note that $\theta'', \theta''' = O(e^{-\frac{C}{\varepsilon}})$ in $\Gamma_\varepsilon(\delta) \setminus \Gamma_\varepsilon(\delta/2)$, and these relations hold for the terms that are multiplied by the derivatives of $\xi(r)$. Thus, in $\Gamma_\varepsilon(\delta)$,

\[
2\varepsilon^{-\frac{3}{2}} \Delta (Z(s)h(s)\theta''(\frac{z}{\varepsilon})\xi(r))\phi_\perp = 2\varepsilon^{-\frac{3}{2}} (\Delta Z(s)h(s) + 2\nabla Z(s) \cdot \nabla h(s) + Z(s)\Delta h(s)) \theta''(\frac{z}{\varepsilon})\xi(r)\phi_\perp
\]

\[
+ 2\varepsilon^{-\frac{3}{2}} Z(s)h(s) \left( \varepsilon^{-2}\theta'''(\frac{z}{\varepsilon}) \xi(r) + \varepsilon^{-1}\theta''(\frac{z}{\varepsilon})\xi'(r) \right) \partial_r (\ln \sqrt{g})\phi_\perp
\]

\[
+ Z(s)h(s)\phi_\perp O(e^{-\frac{C}{\varepsilon}}) \chi_{\left( \Gamma_\varepsilon(\delta) \setminus \Gamma_\varepsilon(\delta/2) \right)},
\]

where $\chi_A$ denotes the characteristic function of a set $A$. Substituting the above formula into $I_{221}$ in (5.7), integrating by parts (in $x$-variable) and using (2.9), (2.11) yields

\[
I_{221} = 2\varepsilon^{-\frac{3}{2}} \int_{\Gamma_\varepsilon(\delta)} (\Delta Z(s)h(s) + 2\nabla Z(s) \cdot \nabla h(s) + Z(s)\Delta h(s)) \theta''(\frac{z}{\varepsilon})\xi(r)\phi_\perp dx
\]

\[
+ 2\varepsilon^{-\frac{7}{2}} \int_{\Gamma_\varepsilon(\delta)} Z(s)h(s)\theta'''(\frac{z}{\varepsilon})\xi(r)\phi_\perp dx
\]

\[
+ 2\varepsilon^{-\frac{5}{2}} \int_{\Gamma_\varepsilon(\delta)} Z(s)h(s)\theta''(\frac{z}{\varepsilon})\xi(r)\partial_r (\ln \sqrt{g})\phi_\perp dx
\]

\[
+ \int_{\Gamma_\varepsilon(\delta) \setminus \Gamma_\varepsilon(\delta/2)} Z(s)\phi_\perp O(e^{-\frac{C}{\varepsilon}}) dx
\]

\[
= 2\varepsilon^{-\frac{3}{2}} \int_{\Gamma_\varepsilon(\delta)} (\Delta \Gamma Z(s)h(s) - Z(s)\Delta \Gamma h(s)) \theta''(\frac{z}{\varepsilon})\xi(r)\phi_\perp dx
\]

\[
- 4\varepsilon^{-\frac{3}{2}} \int_{\Gamma_\varepsilon(\delta)} Z(s)\nabla \Gamma h(s) \cdot \nabla \Gamma \left( \theta''(\frac{z}{\varepsilon})\xi(r)\phi_\perp \right) dx
\]

\[
+ 2\varepsilon^{-\frac{7}{2}} \int_{\Gamma_\varepsilon(\delta)} Z(s)h(s)\theta'''(\frac{z}{\varepsilon})\xi(r)\phi_\perp dx
\]

\[
+ 2\varepsilon^{-\frac{5}{2}} \int_{\Gamma_\varepsilon(\delta)} Z(s)h(s)\theta''(\frac{z}{\varepsilon})\partial_r (\ln \sqrt{g})\xi(r)\phi_\perp dx
\]

\[
+ \int_{\Gamma_\varepsilon(\delta) \setminus \Gamma_\varepsilon(\delta/2)} Z(s)\phi_\perp O(e^{-\frac{C}{\varepsilon}}) dx.
\]
In the second step above we used

$$
\nabla h(s) \cdot \nabla \left( \theta''(\frac{\xi}{\varepsilon}) \zeta(r) \phi^\perp \right) = \nabla\Gamma h(s) \cdot \nabla\Gamma \left( \theta''(\frac{\xi}{\varepsilon}) \zeta(r) \phi^\perp \right).
$$

Then it follows from the change of variable $x \mapsto (r, s)$ that

$$
I_{221} = 2 \int_{\Gamma_t} \int_{\delta - \delta} \left( \Delta\Gamma Z(s) h(s) - Z(s) \Delta\Gamma h(s) \right) \theta'' \varepsilon^{-2} (\zeta \phi^\perp) J dr
$$

$$
- 4 \int_{\Gamma_t} \int_{\delta - \delta} \varepsilon^{-\frac{1}{2}} Z(s) \nabla\Gamma h(s) \cdot \varepsilon^{-1} \nabla\Gamma (\zeta \phi^\perp) \theta'' J dr
$$

$$
+ 2 \varepsilon^{-\frac{3}{2}} \int_{\Gamma_t} \int_{\delta - \delta} Z(s) h(s) \theta'''(\zeta \phi^\perp) J dr
$$

$$
+ 2 \int_{\Gamma_t} \int_{\delta - \delta} \varepsilon^{-\frac{1}{2}} Z(s) h(s) \theta'''(\xi) \partial_r (\ln \sqrt{g}) \varepsilon^{-2} (\zeta \phi^\perp) J dr
$$

$$
+ \int_{\Gamma_t(\delta) \setminus \Gamma_t(\delta/2)} Z(s) \phi^\perp O(e^{-\frac{C}{\varepsilon}}) dx.
$$

In this formula, we write the power of $\varepsilon$ separately for the convenience of applying the Cauchy–Schwarz inequality (4.7). Note that in the above expansion of $I_{221}$, the leading term is the one with $\varepsilon^{-7/2}$, which shall be cancelled with the leading order term in $I_{25}$. Recall that $h(s)$ is a smooth function on $\Gamma_t$ and $\Delta\Gamma$ depends on $r$.

We apply the Cauchy–Schwarz inequality (4.7) to the first, second, and the fourth terms above:

$$
I_{221} \geq -C \varepsilon^{-2} \sup_{|r| \leq \delta} \int_{\Gamma_t} |\Delta\Gamma Z|^2 + 2 \varepsilon^{-\frac{7}{2}} \int_{\Gamma_t} \int_{\delta - \delta} Z(s) h(s) \theta'''(\zeta \phi^\perp) J dr
$$

$$
- C(1 + \nu^{-1}) \int_{\Gamma_t} Z^2(s) - \nu \varepsilon^{-4} \int_{\Gamma_t(\delta)} (\phi^\perp \zeta)^2 dx
$$

$$
- C e^{-\frac{C}{\varepsilon}} \int_{\Gamma_t(\delta) \setminus \Gamma_t(\delta/2)} (\phi^\perp)^2 dx - \nu \varepsilon^{-2} \int_{\Gamma_t(\delta)} |\nabla\Gamma(\zeta \phi^\perp)|^2 dx.
$$

(5.9)

In a similar way, we can treat $I_{222}$ by

$$
I_{222} = \int_{\Gamma_t} \int_{\delta - \delta} O(e^{-\frac{3}{\varepsilon}}) Z(s) h(s) \theta''(z) f'''(\theta) (\phi^\perp \zeta + \phi^\perp (1 - \zeta)) dr
$$

$$
\geq -C \int_{\Gamma_t} Z^2(s) - C \varepsilon^{-2} \int_{\Gamma_t(\delta)} (\phi^\perp \zeta)^2 dx - C e^{-\frac{C}{\varepsilon}} \int_{\Gamma_t(\delta) \setminus \Gamma_t(\delta/2)} (\phi^\perp)^2 dx.
$$

(5.10)
By substituting (5.9) and (5.10) into (5.7) and then using (2.13a) to treat the first term on the right of (5.9), we arrive at

\[
I_{22} \geq 2\varepsilon^{-\frac{7}{2}} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z(s) h(s) \left( \theta'''(z) - \theta''(z) f''(\theta) \right) (\phi \perp \xi) J \, dr
\]

\[
- \nu \int_{\Gamma_1} \left( \Delta \Gamma \right)_{r=0} Z \bigg)^2 - C(1 + \nu^{-1}) \int_{\Gamma_1} Z^2(s) - \nu \varepsilon^{-4} \int_{\Gamma_1(\delta)} (\phi \perp \xi)^2 \, dx
\]

\[
- Ce^{-\frac{\varepsilon}{\xi}} \int_{\Gamma_1(\delta) \setminus \Gamma_1(\delta/2)} (\phi \perp \xi)^2 \, dx - \nu \varepsilon^{-2} \int_{\Gamma_1(\delta)} |\nabla (\phi \perp \xi)|^2 \, dx - \frac{1}{8} I_{31}.
\]

Estimate of \( I_{25} \): It follows from (2.1) and (2.4) that \( \Delta r = \Delta d^{(0)} + \varepsilon \Delta d^{(1)} + O(\varepsilon^2) \). This together with (3.6), (3.3) and (2.17) implies the following expansion successively

\[
\mu_a = -\Delta d^{(0)} \theta' - \varepsilon \Delta d^{(1)} \theta' + \varepsilon D^{(0)} z \theta' + O(\varepsilon^2)
\]

\[
= -h \theta' + \varepsilon (D^{(0)} - b) z \theta' + (1 + z^2) O(\varepsilon^2), \quad \text{with } z = r/\varepsilon. \quad (5.12a)
\]

\[
f'''(\phi_a) \mu_a = -h \theta' f'''(\theta) + \varepsilon (D^{(0)} - b) z \theta' f'''(\theta) + (1 + z^2) O(\varepsilon^2), \quad \text{with } z = r/\varepsilon. \quad (5.12b)
\]

Substituting into (5.3e) yields

\[
I_{25} = -2\varepsilon^{-\frac{7}{2}} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z(s) h(s) f'''(\theta) \theta^2 (\phi \perp \xi) J \, dr
\]

\[
+ \int_{\Gamma_1} \int_{-\delta}^{\delta} \varepsilon^{-\frac{1}{2}} Z(s) (D^{(0)} - b) z f'''(\theta) \theta^2 \varepsilon^{-2} (\phi \perp \xi) J \, dr \quad (5.13)
\]

\[
+ \varepsilon^{-\frac{3}{2}} \int_{\Gamma_1} \int_{-\delta}^{\delta} O(1) \theta'(z)(1 + z^2) Z(s)(\phi \perp \xi) J \, dr, \quad \text{with } z = r/\varepsilon.
\]

Finally we apply (4.7) to the last two components and this yields

\[
I_{25} \geq -2\varepsilon^{-\frac{7}{2}} \int_{\Gamma_1} \int_{-\delta}^{\delta} Z(s) h(s) f'''(\theta) \theta^2 (\phi \perp \xi) J \, dr
\]

\[
- \frac{C}{\nu} \int_{\Gamma_1} Z^2(s) - \nu \varepsilon^{-4} \int_{\Gamma_1(\delta)} (\phi \perp \xi)^2 \, dx. \quad (5.14)
\]

On the other hand, differentiating the identity \( \theta''(z) = f'(\theta) \) twice leads to \( \theta'''(z) = \theta''(z) f''(\theta) + f'''(\theta) (\theta'(z))^2 \). This together with (5.11) and (5.14) eliminates the \( \varepsilon^{-7/2} \) order term in \( I_{22} + I_{25} \). So we have proved (5.6).

**Lemma 5.3.** There exist \( C > 0 \) and \( \varepsilon_0 > 0 \) such that

\[
I_{23} \geq -v \int_{\Gamma_1} \left( \Delta \Gamma \right)_{r=0} Z(s)^2 - C \int_{\Gamma_1} Z^2(s) - \nu \varepsilon^{-4} \int_{\Gamma_1(\delta)} (\phi \perp \xi)^2 \, dx
\]

\[
- \frac{C}{\varepsilon v} \int_{\Gamma_1(\delta)} (\phi \perp \xi)^2 \, dx - \frac{C \varepsilon}{v} \int_{\Gamma_1(\delta)} (\Delta \Gamma)_{r=0} (\phi \perp \xi, \xi)^2 \, dx \quad (5.15)
\]

holds for every \( \varepsilon \in (0, \varepsilon_0) \) and \( v \in (\varepsilon^{\frac{1}{10}}, 1) \).
Proof. Using (2.11), (1.18) and integrating by parts in \( r \) twice, we can write \( I_{23} \) by

\[
I_{23} = 2\varepsilon^{-\frac{3}{2}} \int_{\Gamma_r} \int_{-\delta}^{\delta} \Delta_r Z(s)\theta^r \zeta (r) \left( \varepsilon \Delta_r \phi^+ - \frac{\varepsilon}{\sqrt{g}} \partial_r \left( \sqrt{g} \partial_r \phi^+ \right) - \varepsilon^{-1} f''(\phi_a) \phi^+ \right) J \, dr
\]

equal

\[
= 2\varepsilon^{-\frac{1}{2}} \int_{\Gamma_r} \int_{-\delta}^{\delta} \partial_r \left( \sqrt{g} \partial_r \left( \Delta_r Z(s) \theta^r \zeta (r) \right) \right) \phi^+ \frac{1}{\sqrt{g}(s)} \, dr
\]

\[
- 2\varepsilon^{-\frac{1}{2}} \int_{\Gamma_r} \int_{-\delta}^{\delta} \Delta_r Z(s) \theta^r \zeta (r) f''(\phi_a) \phi^+ J \, dr
\]

\[
+ 2\varepsilon^{-\frac{1}{2}} \int_{\Gamma_r} \int_{-\delta}^{\delta} \Delta_r Z(s) \Delta_r \phi^+ \zeta (r) \theta^r J \, dr \triangleq J_1 + J_2 + J_3. \tag{5.16}
\]

To estimate \( J_1 \), we recall that its leading order term corresponds to the case when \( \theta^r \zeta \) is differentiated twice, which gives rise to a factor \( \varepsilon^{-2} \). For the rest terms, the differential operator \( \partial_r \) will apply to \( \Delta_r Z(s) \), whose coefficients depend on \( r \) smoothly. So for any \( \ell \geq 0 \), we employ (2.13a) to obtain

\[
\int_{\Gamma_r} \int_{-\delta}^{\delta} |\partial_r \Delta_r Z(s)|^2 \, dr \leq C_{\ell} \| Z \|^2_{H^2(\Gamma_r)} \leq C \inf_{|r| \leq \delta} \int_{\Gamma_r} \left( |\Delta_r Z(s)|^2 + |Z(s)|^2 \right).
\]

As a result, we can subtract the leading order terms in the expansion of \( J_1 + J_2 \) and the remaining ones can be controlled effectively:

\[
J_1 + J_2 \geq -\nu \int_{\Gamma_r} \Delta_r \big|_{r=0} Z(s) \big)^2 \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \left( \theta''(z)^2 + \theta'(z)^2 \right) \, dz
\]

\[
- C \int_{\Gamma_r} \Delta_r Z(s) - \nu \varepsilon^{-4} \int_{\Gamma_r(\delta)} \left( \phi^\perp \zeta \right)^2 \, dx.
\]

Employing (4.4a) will gain a factor \( \varepsilon^2 \) for \( \hat{J} \). This together with (4.6a) implies

\[
J_1 + J_2 \geq -\nu \int_{\Gamma_r} \Delta_r \big|_{r=0} Z(s) \big)^2 - C \int_{\Gamma_r} \Delta_r Z(s) - \nu \varepsilon^{-4} \int_{\Gamma_r(\delta)} \left( \phi^\perp \zeta \right)^2 \, dx, \tag{5.18}
\]

by choosing a sufficiently small \( \nu > 0 \).

Concerning \( J_3 \), we shall employ the relation \( \phi^+ J^\frac{1}{2} (1-\theta) J = O(\varepsilon \frac{1}{r}) + z \theta'(z) O(\varepsilon) \) with \( z = r/\varepsilon \). This is a consequence of \( \phi - \theta' = O(\varepsilon \frac{1}{r}) \) and \( J - J^\frac{1}{2} = O(\varepsilon) z \), which follow from (3.8) and (2.16) respectively. So we can treat \( J_3 \) by the Cauchy–Schwarz inequality and (4.6b),

\[
J_3 \geq 2 \varepsilon^{-\frac{1}{2}} \int_{\Gamma_r} \int_{-\delta}^{\delta} \Delta_r Z(s) \Delta_r \phi^\perp \zeta (r) \phi J^\frac{1}{2} \, dr \triangleq J_\star
\]

\[
- C \varepsilon \int_{\Gamma_r} \left( |\Delta_r Z|^2 + Z^2(s) \right) - C \varepsilon \int_{\Gamma_r(\delta)} \frac{1}{2} \zeta^2 \, dx.
\]
The estimate of $J_*$ is derived at (5.21) in the sequel. Therefore, substituting (5.18), (5.19) and (5.21) into (5.16) and then choosing a sufficiently small $\nu$ yields

\[
I_{23} \geq -\nu \int_{\Gamma_r} (\Delta \Gamma|_{r=0} Z(s))^2 - C \int_{\Gamma_r} Z^2(s) - \nu \varepsilon^{-4} \int_{\Gamma_r} (\phi \perp \xi)^2 \, dx \\
- \frac{C}{\nu} \int_{\Gamma_r} \left( \int_{\Gamma_r} \left| \nabla \Gamma|_{r=0} \phi \perp \right| \right)^2 \xi^2 \, dr - C \varepsilon \int_{\Gamma_r} (\Delta \Gamma|_{r=0} (\phi \perp \xi))^2 \, dx.
\]

(5.20)

To eliminate the term with $\nabla \Gamma|_{r=0} \phi \perp$, we employ (2.20) and obtain

\[
\int_{\Gamma_r} \left| \nabla \Gamma|_{r=0} \phi \perp \right|^2 = - \int_{\Gamma_r} \phi \perp (\Delta \Gamma|_{r=0} \phi \perp) \leq \varepsilon^{-1} \int_{\Gamma_r} (\phi \perp)^2 + \frac{\varepsilon}{4} \int_{\Gamma_r} (\Delta \Gamma|_{r=0} \phi \perp)^2.
\]

The above two inequalities together lead to (5.15).

**Proof of Proposition 5.1.** It follows from (4.3) that $I_{24}$ has the same lower bound as $I_{23}$. So substituting (5.4), (5.6) and (5.15) into (5.2), and using (3.11) lead to (5.1). It remains to treat $J_*$ in (5.19). We shall prove that there exist $\varepsilon_0 > 0$ and $C > 0$ such that

\[
J_* \geq -\nu \int_{\Gamma_r} (\Delta \Gamma|_{r=0} Z(s))^2 - C \int_{\Gamma_r} Z^2(s) - \nu \varepsilon^{-4} \int_{\Gamma_r} (\phi \perp \xi)^2 \, dx \\
- \frac{C}{\nu} \int_{\Gamma_r} (\phi \perp \xi)^2 \, dx - C \varepsilon \int_{\Gamma_r} (\Delta \Gamma|_{r=0} \phi \perp \xi)^2 \, dx
\]

(5.21)

holds for every $\varepsilon \in (0, \varepsilon_0)$ and every $\nu \in (\frac{1}{\bar{h}}, 1)$. To this end, we first recall (2.12) that $\Delta \Gamma = \Delta \Gamma|_{r=0} + r \mathcal{R} \Gamma$. Thus,

\[
J_* = \varepsilon^{-\frac{1}{2}} \int_{\Gamma_r} \int_{-\delta}^{\delta} (\Delta \Gamma|_{r=0} Z(s)) (\Delta \Gamma|_{r=0} \phi \perp \xi) \phi \perp \xi \phi \perp \xi \xi (r) J^{\frac{1}{2}} \, dr \\
+ \varepsilon \int_{\Gamma_r} \int_{-\delta}^{\delta} (\Delta \Gamma|_{r=0} Z(s)) \mathcal{R} \Gamma \phi \perp \xi \phi \perp \xi \phi \perp \xi \xi (r) J^{\frac{1}{2}} \, dr \\
+ \varepsilon \int_{\Gamma_r} \int_{-\delta}^{\delta} \mathcal{R} \Gamma Z(s) (\Delta \Gamma|_{r=0} \phi \perp \xi) \phi \perp \xi \phi \perp \xi \xi (r) J^{\frac{1}{2}} \, dr \\
+ \varepsilon \int_{\Gamma_r} \int_{-\delta}^{\delta} \mathcal{R} \Gamma Z(s) \mathcal{R} \Gamma \phi \perp \phi \perp \xi \xi \xi \xi (r) J^{\frac{1}{2}} \, dr \triangleq \sum_{1 \leq i \leq 4} J_{5i}.
\]

To treat the highest order term $J_{51}$, we apply $\Delta \Gamma|_{r=0}$ to (1.22) and obtain

\[
0 = \Delta \Gamma|_{r=0} \left( \int_{-\delta}^{\delta} \phi \perp \xi \phi \perp \xi \xi (r) J^{\frac{1}{2}} (r, s) \, dr \right).
\]
This implies the following identity:
\[
\int_{-\delta}^{\delta} \varphi(\frac{\xi}{\delta}) \zeta(r)(\Delta \Gamma|_{r=0} \phi^\perp(r, s)) J^\frac{1}{2} dr \\
= -2 \int_{-\delta}^{\delta} \varphi(\frac{\xi}{\delta}) \zeta(r)(\nabla \Gamma|_{r=0} \phi^\perp(r, s) \cdot \nabla \Gamma|_{r=0} J^\frac{1}{2}) dr \\
- \int_{-\delta}^{\delta} \varphi(\frac{\xi}{\delta}) \zeta(r) \phi^\perp(r, s)(\Delta \Gamma|_{r=0} J^\frac{1}{2}) dr.
\]

Multiplying by \(\Delta \Gamma|_{r=0} Z(s)\) and integrating over \(\Gamma_r\) gives
\[
J_{51} = -2\varepsilon \int_{\Gamma_r} \int_{-\delta}^{\delta} (\Delta \Gamma|_{r=0} Z(s)) \varphi(\frac{\xi}{\delta}) \zeta(r)(\nabla \Gamma|_{r=0} \phi^\perp(r, s) \cdot \nabla \Gamma|_{r=0} J^\frac{1}{2}) dr \\
- \varepsilon \int_{\Gamma_r} \int_{-\delta}^{\delta} (\Delta \Gamma|_{r=0} Z(s)) \varphi(\frac{\xi}{\delta}) \zeta(r)(\Delta \Gamma|_{r=0} J^\frac{1}{2}) dr \\
\geq -\nu \int_{\Gamma_r} (\Delta \Gamma|_{r=0} Z(s))^2 - \frac{C}{\nu} \int_{\Gamma_r} \int_{-\delta}^{\delta} (\nabla \Gamma|_{r=0} \phi^\perp \zeta)^2 dr - \frac{C}{\nu} \int_{\Gamma_r(\delta)} (\phi^\perp \zeta)^2 dx.
\]

Recall (2.12) that \(\overline{\mathcal{R}}_{\Gamma}\) is a second order operator acting on \(s\) while its coefficients depend on \((r, s) \in (-2\delta, 2\delta) \times \Gamma_r\) smoothly. Using the Cauchy–Schwarz inequality and (2.13b) give the estimate of \(J_{52} + J_{53} + J_{54}\):
\[
\sum_{2 \leq i \leq 4} I_{5i} \geq -\nu \int_{\Gamma_r} \left( (\Delta \Gamma|_{r=0} Z(s))^2 + Z^2(s) \right) \\
- \frac{C\varepsilon^2}{\nu} \int_{\Gamma_r(\delta)} \left( (\Delta \Gamma|_{r=0} \phi^\perp \zeta)^2 + (\phi^\perp \zeta)^2 \right) dx.
\]

The above two estimates lead to (5.21).

6. Proof of Theorem 1.2

The proof of Theorem 1.2 will be done by the end of this section after establishing two lemmas. The first one is concerned with a lower bound of the Allen–Cahn operator (1.13).

**Lemma 6.1.** There exists \(\varepsilon_0 > 0\) such that for every \(\varepsilon \in (0, \varepsilon_0)\),
\[
\frac{1}{\varepsilon^2} \int_{\Omega} |\mathcal{L}_\varepsilon[\phi^\perp]|^2 dx \geq \int_{\Omega} (\Delta \phi^\perp)^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} |\nabla \phi^\perp|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} (\phi^\perp)^2 dx.
\]

**Proof.** We first note that for sufficiently small \(\varepsilon\), the above estimate holds trivially when \(\Omega\) is replaced by \(\Omega \setminus \Gamma_r(\delta)\). This is because away from \(\Gamma_r\), \(f''(\phi_a) \geq \Lambda_a\) for some positive constant \(\Lambda_a\). To proceed, we recall the following coercivity estimate
\[
\int_{\Omega} \mathcal{L}_\varepsilon[\phi^\perp] \phi^\perp dx \geq \int_{\Omega} (\varepsilon |\nabla \phi^\perp|^2 + \frac{1}{\varepsilon} (\phi^\perp)^2) dx,
\]
which follows by applying [7, Lemma 2.4] with \( \psi = \phi^\perp \). This combined with the Cauchy–Schwarz inequality yields

\[
\frac{1}{\varepsilon^2} \int_{\Omega} |\mathcal{L}_\varepsilon[\phi^\perp]|^2 \, dx \geq \frac{C}{\varepsilon^2} \int_{\Omega} |\nabla \phi^\perp|^2 \, dx + \frac{C}{\varepsilon^2} \int_{\Omega} (\phi^\perp)^2 \, dx. \tag{6.3}
\]

As a result of the inequality \((a + b)^2 \geq a^2/2 - b^2\), we arrive at

\[
\frac{1}{\varepsilon^2} \int_{\Omega} |\mathcal{L}_\varepsilon[\phi^\perp]|^2 \, dx \geq \frac{1}{2} \int_{\Omega} (\Delta \phi^\perp)^2 \, dx - C \int_{\Omega} (|\nabla \phi^\perp|^2 + (\phi^\perp)^2) \, dx. \tag{6.4}
\]

Finally, multiplying (6.3) by a sufficiently large constant and adding up to (6.4) will lead to the desired inequality.

The coercivity of \( \Delta \phi^\perp \) on the right of (6.1) is given below.

**Lemma 6.2.** There exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \),

\[
\int_{\Omega} (\Delta \phi^\perp)^2 \, dx \geq \frac{1}{4} \int_{\Gamma_{(\delta)}} \left( \Delta_{\Gamma} \phi^\perp \right)^2 \, \xi^2 \, dx - C \int_{\Omega} (|\nabla \phi^\perp|^2 + (\phi^\perp)^2) \, dx
\]

\[
+ \frac{1}{4} \int_{\Gamma_{(\delta)}} \left( \partial_r^2 (\phi^\perp \xi) \right)^2 \, dx + \frac{1}{4} \int_{\Gamma_{(\delta)}} |\partial_r \nabla_{\Gamma} (\phi^\perp \xi)|^2 \, dx. \tag{6.5}
\]

**Proof.** It follows from (1.18) and the inequality \((a + b)^2 \geq a^2/2 - b^2\) that

\[
\int_{\Omega} (\Delta \phi^\perp)^2 \, dx \geq \int_{\Omega} (\Delta \phi^\perp \xi)^2 \, dx = \int_{\Omega} \left( \Delta (\phi^\perp \xi) - 2\nabla \phi^\perp \cdot \nabla \xi - \phi^\perp \Delta \xi \right)^2 \, dx
\]

\[
\geq \frac{1}{2} \int_{\Omega} (\Delta (\phi^\perp \xi))^2 \, dx - C \int_{\Omega} (|\nabla \phi^\perp|^2 + (\phi^\perp)^2) \, dx. \tag{6.6}
\]

It follows from (1.18), (2.11) and the Cauchy–Schwarz inequality that

\[
\int_{\Omega} (\Delta (\phi^\perp \xi))^2 \, dx = \int_{\Gamma_{(\delta)}} \left( \Delta_{\Gamma} (\phi^\perp \xi) + \frac{\partial_{\Gamma, \xi}}{2g} \partial_r (\phi^\perp \xi) + \partial_r^2 (\phi^\perp \xi) \right)^2 \, dx
\]

\[
= \int_{\Gamma_{(\delta)}} \left[ \left( \Delta_{\Gamma} (\phi^\perp \xi) \right)^2 + 2 \Delta_{\Gamma} (\phi^\perp \xi) \frac{\partial_{\Gamma, \xi}}{2g} \partial_r (\phi^\perp \xi)
\]

\[
+ 2 \Delta_{\Gamma} (\phi^\perp \xi) \partial_r^2 (\phi^\perp \xi)
\]

\[
+ \frac{\partial_{\Gamma, \xi}}{2g} \partial_r (\phi^\perp \xi) \right)^2 + 2 \frac{\partial_{\Gamma, \xi}}{2g} \partial_r (\phi^\perp \xi) \partial_r^2 (\phi^\perp \xi) + \left( \partial_r^2 (\phi^\perp \xi) \right)^2 \right]
\]

\[
\geq \frac{1}{2} \int_{\Gamma_{(\delta)}} \left( \Delta_{\Gamma} (\phi^\perp \xi) \right)^2 \, dx - C \int_{\Gamma_{(\delta)}} (\partial_r (\phi^\perp \xi))^2 \, dx
\]

\[
+ \frac{1}{2} \int_{\Gamma_{(\delta)}} \left( \partial_r^2 (\phi^\perp \xi) \right)^2 \, dx + 2 \int_{\Gamma_{(\delta)}} \Delta_{\Gamma} (\phi^\perp \xi) \partial_r^2 (\phi^\perp \xi) \, dx. \tag{6.7}
\]
To treat the last integral, using (2.15) and the coarea formula (2.18) yields

\[
\int_{\Gamma_1(\delta)} \Delta \Gamma (\phi^\perp \zeta) \partial_r^2 (\phi^\perp \zeta) \, dx
\]

\[
= \int_{-\delta}^{\delta} \int_{\Gamma_r'} \text{div}_r \nabla_{\Gamma} (\phi^\perp \zeta) \partial_r^2 (\phi^\perp \zeta) \, dS dr
\]

\[
= \int_{-\delta}^{\delta} \int_{\Gamma_r'} \nabla_{\Gamma} (\phi^\perp \zeta) \partial_r^2 (\phi^\perp \zeta) \, dS dr - \int_{\Gamma_r(\delta)} \nabla_{\Gamma} (\phi^\perp \zeta) \cdot \nabla_{\Gamma} \partial_r^2 (\phi^\perp \zeta) \, dx.
\]

Here \(dS\) denotes the surface element of \(\Gamma_r'\) and we have \(dS = \sqrt{g(r, s)} \, ds\) under local coordinates. The first term vanishes due to the divergence theorem (2.20) and the second term can be treated using the identity \(\partial_r^2 \nabla_{\Gamma} - \nabla_{\Gamma} \partial_r^2 = \partial_r [\partial_r, \nabla_{\Gamma}] + [\partial_r, \nabla_{\Gamma}] \partial_r\):

\[
\int_{\Gamma_r(\delta)} \Delta \Gamma (\phi^\perp \zeta) \partial_r^2 (\phi^\perp \zeta) \, dx
\]

\[
= -\int_{\Gamma_r} \int_{-\delta}^{\delta} \nabla_{\Gamma} (\phi^\perp \zeta) \cdot \partial_r^2 \nabla_{\Gamma} (\phi^\perp \zeta) J(r, s) \, dr
\]

\[
+ \int_{\Gamma_r} \int_{-\delta}^{\delta} \nabla_{\Gamma} (\phi^\perp \zeta) \cdot (\partial_r [\partial_r, \nabla_{\Gamma}] + [\partial_r, \nabla_{\Gamma}] \partial_r) (\phi^\perp \zeta) J(r, s) \, dr \triangleq I_{31} + I_{32}.
\]

As for \(I_{31}\), we can integrate by parts twice in \(r\) to get

\[
I_{31} = \int_{\Gamma_r} \int_{-\delta}^{\delta} |\partial_r \nabla_{\Gamma} (\phi^\perp \zeta)|^2 J(r, s) \, dr - \frac{1}{2} \int_{\Gamma_r} \int_{-\delta}^{\delta} |\nabla_{\Gamma} (\phi^\perp \zeta)|^2 \partial_r^2 J(r, s) \, dr.
\]

(6.8)

As for \(I_{32}\), it can be verified that \([\nabla_{\Gamma}, \partial_r]\) is a first-order tangential differential operator. So the leading order terms of \(I_{32}\) are the mixed derivatives of second order and can be controlled by the first term on the right of (6.8):

\[
I_{32} \geq -\frac{1}{2} \int_{\Gamma_r} \int_{-\delta}^{\delta} |\partial_r \nabla_{\Gamma} (\phi^\perp \zeta)|^2 J(r, s) \, dr - C \int_{\Gamma_r} \int_{-\delta}^{\delta} |\nabla_{\Gamma} (\phi^\perp \zeta)|^2 \, dr.
\]

Adding up the previous two inequalities leads to

\[
\int_{\Gamma_r(\delta)} \Delta \Gamma (\phi^\perp \zeta) \partial_r^2 (\phi^\perp \zeta) \, dx \geq \frac{1}{2} \int_{\Gamma_r} \int_{-\delta}^{\delta} |\partial_r \nabla_{\Gamma} (\phi^\perp \zeta)|^2 J(r, s) \, dr
\]

\[
- C \int_{\Gamma_r} \int_{-\delta}^{\delta} |\nabla_{\Gamma} (\phi^\perp \zeta)|^2 \, dr.
\]

Substituting this inequality into (6.7) and combining with (6.6) imply (6.5).

---

\(^1\) For any tangential differential operator \(\sum_{1 \leq i \leq N-1} a_i(r, s) \partial_{s_i}\), its commutator with \(\partial_r\) is a first-order

\[
\left[ \sum_{1 \leq i \leq N-1} a_i(r, s) \partial_{s_i}, \partial_r \right] f = - \sum_{1 \leq i \leq N-1} \partial_r a_i(r, s) \partial_{s_i} f.
\]

(6.9)
Proof of Theorem 1.2. We first recall (3.17c) for the definitions of $I_{31}$ and of $I_3$. The term $-\frac{1}{4}I_{31}$ on the right hand side of (5.1) can be absorbed by $I_3$ by choosing a small $\epsilon$. It follows from (3.16), (4.1), (5.1) and (2.13a) that
\[
\frac{1}{\epsilon^2} \int_{\Omega} |\mathcal{L}_\epsilon[\phi]|^2 dx + \frac{1}{\epsilon^3} \int_{\Omega} f'''(\phi_a) \mu_a \phi^2 dx
\]
\[
\geq \frac{1}{4\epsilon^2} \int_{\Omega} |\mathcal{L}_\epsilon[\phi]|^2 dx + \frac{1}{\epsilon^3} \int_{\Omega} f'''(\phi_a) \mu_a (\phi^\perp)^2 dx - \nu \epsilon^{-4} \int_{\Gamma_{\delta}(\delta)} (\phi^\perp)^2 dx
\]
\[
- C \epsilon^{-\frac{5}{2}} \int_{\Gamma_{\delta}(\delta)} (\phi^\perp)^2 dx - \nu \epsilon^{-2} \int_{\Gamma_{\delta}(\delta)} |\nabla \phi^\perp| dx
\]
\[
+ O(\epsilon) \int_{\Gamma_{\delta}(\delta)} (\Delta \phi^\perp)^2 \epsilon^2 dx + \Lambda_4 \|Z\|_{H^2(\Gamma)}^2 - C \int_{\Omega} \phi^2 dx,
\] (6.10)
where $\nu$ is a small positive constant to be determined later. Combining (6.1) and (6.5) with (6.10), and choosing $\nu$ and $\epsilon$ sufficiently small, we deduce that
\[
\frac{1}{\epsilon^2} \int_{\Omega} |\mathcal{L}_\epsilon[\phi]|^2 dx + \frac{1}{\epsilon^3} \int_{\Omega} f'''(\phi_a) \mu_a \phi^2 dx
\]
\[
\geq \Lambda_4 \|Z\|_{H^2(\Gamma)}^2 - \tilde{C} \int_{\Omega} \phi^2 dx + C_2 \int_{\Omega} \left( |\Delta \phi^\perp|^2 + \epsilon^{-2} |\nabla \phi^\perp|^2 + \epsilon^{-4} (\phi^\perp)^2 \right) dx
\]
\[
\geq \tilde{C} \mathcal{K}(t) - \tilde{C} \int_{\Omega} \phi^2 dx.
\] (6.11)
Note that in the last step we applied the Sobolev embedding and then chose a smaller $\epsilon$ to recover (1.21b). So we obtain the desired estimate (1.21a).

7. Proof of Theorem 1.1

With the spectral condition (6.11), we can prove Theorem 1.1. Recalling (1.13) and (1.14), one can verify that the nonlinear terms $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ where
\[
\mathcal{H}_1 = \epsilon^{-2} \Delta T - \epsilon^{-4} f'''(\phi_a) T - \epsilon^{-3} (f''(\phi_a) - f''(\phi_a)) \mathcal{L}_\epsilon[\phi] - 3 \epsilon^{-3} (\phi^\perp)^2 \mu_a,
\] (7.1a)
\[
\mathcal{H}_2 = -\Delta \mathcal{R}_2 \epsilon^{k-3} + f''(\phi_a) \mathcal{R}_2 \epsilon^{k-5} - \mathcal{R}_1 \epsilon^{k-4}.
\] (7.1b)
Here $\mu = \mathcal{L}_\epsilon[\phi] + \epsilon^{-1} T - \mathcal{R}_2 \epsilon^{k-2}$, and $T$ is defined by
\[
T \triangleq f'(\phi_a) - f'(\phi_a) - f''(\phi_a) \phi = 3 \phi^2 \phi_a + \phi^3.
\] (7.2)
It follows from (1.13) and the inequality $(a + b)^2 \geq \frac{1}{4}a^2 - b^2$ that
\[
\epsilon^4 \left( \frac{1}{\epsilon^2} \int_{\Omega} |\mathcal{L}_\epsilon[\phi]|^2 dx + \frac{1}{\epsilon^3} \int_{\Omega} f'''(\phi_a) \mu_a \phi^2 dx \right)
\]
\[
\geq \epsilon^4 \left( \frac{1}{2} \int_{\Omega} (\Delta \phi)^2 dx - \frac{C}{\epsilon^4} \int_{\Omega} \phi^2 dx \right).
\]
Adding the above inequality to (1.21a) and then choosing a small $\varepsilon$ leads to
\[
\frac{1}{\varepsilon^2} \int_{\Omega} |\mathcal{L}_{\varepsilon}[\phi]|^2 \, dx + \frac{1}{\varepsilon^3} \int_{\Omega} f'''(\phi_a) \mu_a \phi^2 \, dx \geq \frac{\varepsilon^4}{4} \int_{\Omega} (\Delta \phi)^2 \, dx - C \int_{\Omega} \phi^2 \, dx.
\] (7.3)

This together with (1.15) and (1.21b) leads to
\[
\frac{1}{2} \frac{d}{dt} \|\phi\|_{L^2(\Omega)}^2 + \frac{\varepsilon^4}{8} \int_{\Omega} (\Delta \phi)^2 \, dx + \frac{\varepsilon}{2} K(t) - C \|\phi\|_{L^2(\Omega)}^2 \leq \int_{\Omega} \mathcal{H}_1 \phi + \int_{\Omega} \mathcal{H}_2 \phi.
\] (7.4)

It remains to estimate the integrals on the right hand side. Using (1.13) and formula
\[
f''(\phi_\varepsilon) - f''(\phi_a) = 3\phi^2 + 6\phi \phi_a,
\] (7.5)

we can rewrite $\mathcal{H}_1$ by
\[
\mathcal{H}_1 = -\varepsilon^{-3} \mathcal{L}_{\varepsilon}[T] - \varepsilon^{-3} (3\phi^2 + 6\phi \phi_a) \mathcal{L}_{\varepsilon}[\phi] - \varepsilon^{-4} (3\phi^2 + 6\phi \phi_a) T - 3\varepsilon^{-3} \phi^2 \mu_a.
\] (7.6)

Using (7.2), we can write
\[
\int_{\Omega} \mathcal{H}_1 \phi \, dx = -\varepsilon^{-3} \int_{\Omega} (4\phi^3 + 9\phi^2 \phi_a) \mathcal{L}_{\varepsilon}[\phi]
- \varepsilon^{-4} \int_{\Omega} (3\phi^3 + 6\phi^2 \phi_a) (3\phi \phi_a + \phi^3) - 3\varepsilon^{-3} \int_{\Omega} \phi^3 \mu_a.
\] (7.7)

Regarding the first term on the right of (7.7), it follows from (3.13), an integration by parts and $\phi \mid_{\partial \Omega} = 0$ that
\[
-\varepsilon^{-3} \int_{\Omega} (4\phi^3 + 9\phi^2 \phi_a) \mathcal{L}_{\varepsilon}[\phi] = -12\varepsilon^{-2} \int_{\Omega} |\nabla \phi|^2 \phi^2 - 4\varepsilon^{-4} \int_{\Omega} f'''(\phi_a) \phi^4
- 9\varepsilon^{-3} \int_{\Omega} \phi^2 \phi_a \mathcal{L}_{\varepsilon}[\phi^\top] - 9\varepsilon^{-3} \int_{\Omega} \phi^2 \phi_a \mathcal{L}_{\varepsilon}[\phi^\perp] - 9\varepsilon^{-3} \int_{\Omega} \phi^2 \phi_a \mathcal{L}_{\varepsilon}[\phi_a].
\] (7.8)

Observe that the second term on the right of (7.7) involves polynomials of $\phi$, and the leading order one is $-3\varepsilon^{-4} \int_{\Omega} \phi^6$. Moreover $\phi_a$ and $\mu_a$ are uniformly bounded according to Proposition 3.1. Thus we substitute (7.8) into (7.7), rearrange the terms and then apply Cauchy–Schwarz inequality to absorb the term involving $\phi^5$. This yields
\[
\int_{\Omega} \mathcal{H}_1 \phi + 12\varepsilon^{-2} \int_{\Omega} |\nabla \phi|^2 \phi^2 + 2\varepsilon^{-4} \int_{\Omega} \phi^6
\leq N_1 + N_2 + N_* + C \int_{\Omega} \left( \varepsilon^{-4} \phi^4 + \varepsilon^{-3} |\phi|^3 \right). \quad (7.9)
\]

In view of (3.12) and (3.13), the structures of $N_1$ and $N_*$ are the same except that the later is of scale $\varepsilon^{-C}/\varepsilon$. So we shall omit its estimate for the sake of simplicity, and focus on the estimates of $N_1$ and $N_2$. 

In the sequel, we shall frequently employ the following Sobolev embedding theorems:

\[ H^{1/3}(\Gamma_t) \hookrightarrow L^3(\Gamma_t), \quad H^{1/2}(\Gamma_t) \hookrightarrow L^4(\Gamma_t), \]  
\[ H^{1/2}(\Omega) \hookrightarrow L^3(\Omega), \quad H^{3/4}(\Omega) \hookrightarrow L^4(\Omega), \]  
where \( \Gamma_t \) is a 2D surface. \hspace{1cm} (7.10a)
\[ H^{1/2}(\Omega) \hookrightarrow L^3(\Omega), \quad H^{3/4}(\Omega) \hookrightarrow L^4(\Omega), \]  
where \( \Omega \) is a 3D domain. \hspace{1cm} (7.10b)

**Lemma 7.1.** There exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \) and \( v \in (\varepsilon^{1/2}, 1) \),

\[ \mathcal{N}_1 \lesssim C(v) \left( \varepsilon^{-14/3} \| \phi \|_{L^2(\Omega)}^{10/3} + \varepsilon^{-8} \| \phi \|_{L^2(\Omega)}^6 + \varepsilon^{-16} \| \phi \|_{L^2(\Omega)}^{10} \right) + v\mathcal{K}(t). \]  
(7.11)

**Proof.** We first recall (4.8) and (4.4a), which yields

\[ \mathcal{L}_{\varepsilon} \phi = -\frac{1}{2} \partial_r (\ln \sqrt{\varepsilon}) Z(s) \theta''(z) \zeta(r) - \frac{\varepsilon}{2} Z f''''(\theta) \phi''(\theta) \zeta(r) 
+ \frac{\varepsilon}{2} \Delta \Gamma Z(s) \theta'(z) \zeta(r) + \varepsilon Z(s) A(r, s) + O(\varepsilon^{3/2}) Z \theta' \zeta(r). \]  
(7.12)

In view of (1.21b), among the five terms on the right of (7.12), the first and the third are of leading order. The second and fifth terms have similar structures as the first but are of order \( \varepsilon^1 \) better. The term involving \( A(r, s) \) is much more regular because of (4.3). To simplify the presentation, we shall focus on the leading order terms:

\[ \mathcal{N}_1 = -9 \varepsilon^{-2} \int_{\Omega} \phi^2 \phi_{\sigma} \partial_r (\ln \sqrt{\varepsilon}) Z(s) \theta''(z) \zeta(r) - 9 \varepsilon^{-2} \int_{\Omega} \phi^2 \phi_{\sigma} \Delta \Gamma Z(s) \theta'(z) \zeta(r) + \cdots. \]  
(7.13)

Using (1.21b), we apply Sobolev embedding (7.10a), the interpolation inequality and Young’s inequality successively to obtain

\[ \varepsilon^{-7/2} \| Z \|_{L^3(\Gamma_t)}^3 \lesssim \varepsilon^{-7/2} \| Z \|_{H^3(\Gamma_t)}^3 \lesssim \varepsilon^{-7/2} \| Z \|_{L^2(\Gamma_t)}^5 \| Z \|_{H^2(\Gamma_t)}^{1/2} \lesssim C(v) \varepsilon^{-14/3} \| Z \|_{L^2(\Gamma_t)}^{10} + v\mathcal{K}(t). \]

Observing that \( z = r/\varepsilon \), we apply Hölder’s inequality, a change of variable and Sobolev embedding (7.10b) we get

\[ \varepsilon^{-7/2} \left| \int_{\Omega} Z(s) (\phi^\perp)^2 \theta''(z) \zeta(r) \right| \lesssim \varepsilon^{-3} \| Z \|_{L^2(\Gamma_t)} \| \phi^\perp \|_{L^4(\Omega)}^2 \lesssim \varepsilon^{-3} \| Z \|_{L^2(\Gamma_t)} \| \phi^\perp \|_{L^2(\Omega)}^{1/2} \| \phi^\perp \|_{H^1(\Omega)}^{3/2} \varepsilon^{-3/2} \lesssim C(v) \varepsilon^{-6} \| Z \|_{L^2(\Gamma_t)}^4 \| \phi^\perp \|_{L^2(\Omega)}^2 + v\varepsilon^{-2} \| \phi^\perp \|_{H^1(\Omega)}^2 \lesssim C(v) \varepsilon^{-6} \| \phi \|_{L^2(\Omega)}^6 + v\mathcal{K}(t). \]
Note that in the last step we employed (3.11) and (1.21b). With the above two estimates we can treat the first term on the right hand side of (7.13). Combining (3.13) with the above two inequalities leads to

\[ \varepsilon^{-2} \left| \int_{\Omega} \phi^2 \phi_a \partial_r (\ln \sqrt{g}) Z(s) \theta''(z) \xi(r) \right| \]

\[ \lesssim \varepsilon^{-2} \int_{\Gamma_1} |Z|^3 + \varepsilon^{-2} \int_{\Omega} |Z(s)(\phi^{1\perp})^2 \theta''(z) \xi(r)| \]

\[ \lesssim C(v) \varepsilon^{-14} \| \phi \|_{L^2(\Omega)}^{10} + C(v) \varepsilon^{-6} \| \phi \|_{L^2(\Omega)}^{6} + vK(t). \]

(7.14)

For the second term on the right hand side of (7.13), we employ (7.10a) and estimate

\[ \varepsilon^{-5} \left| \int_{\Gamma_1} Z^2 \Delta_{\Gamma_1} Z(s)(\phi^{1\perp})^2 \theta''(z) \xi(r) \right| \lesssim \varepsilon^{-2} \| Z \|_{L^4(\Gamma_1)}^{2} \| Z \|_{H^2(\Gamma_1)}^{2} \]

\[ \lesssim \varepsilon^{-2} \| Z \|_{L^2(\Gamma_1)}^{3} \| Z \|_{H^2(\Gamma_1)}^{3} \]

\[ \lesssim C(v) \varepsilon^{-8} \| Z \|_{L^2(\Gamma_1)}^{6} + vK(t). \]

Moreover, we need to estimate the following integral using (7.10b), the interpolation inequality and Young’s inequality:

\[ \varepsilon^{-5} \left| \int_{\Omega} \Delta_{\Gamma} Z(s)(\phi^{1\perp})^2 \theta''(z) \xi(r) \right| \]

\[ \lesssim \varepsilon^{-2} \| \phi \|_{H^2(\Omega)}^{\frac{3}{4}} \| \phi \|_{L^2(\Omega)}^{\frac{5}{2}} \]

\[ \lesssim \nu \left( (\| Z \|_{H^2(\Omega)}^{\frac{3}{4}} \| \phi \|_{L^2(\Omega)}^{\frac{5}{2}})^{8/7} + C(v) \varepsilon^{-16} \| \phi \|_{L^2(\Omega)}^{10} \right) \]

\[ \lesssim \nu K(t) + C(v) \varepsilon^{-16} \| \phi \|_{L^2(\Omega)}^{10}, \]

Combining the above two results gives the estimate of the second term on the right hand side of (7.13):

\[ \varepsilon^{-5} \left| \int_{\Omega} 9\phi^2 \phi_a \Delta_{\Gamma} Z(s) \theta'(z) \xi(r) \right| \]

\[ \lesssim \varepsilon^{-5} \left| \int_{\Gamma_1} Z^2 \Delta_{\Gamma_1} Z(s) \theta'(z) \xi(r) \right| + \varepsilon^{-5} \left| \int_{\Omega} \Delta_{\Gamma} Z(s)(\phi^{1\perp})^2 \theta'(z) \xi(r) \right| \]

(7.15)

\[ \lesssim C(v) \left( \varepsilon^{-8} \| \phi \|_{L^2(\Omega)}^{6} + \varepsilon^{-16} \| \phi \|_{L^2(\Omega)}^{10} \right) + vK(t). \]

Finally substituting (7.14) and (7.15) into (7.13) yields (7.11).

\footnote{Here the contributions due to \( \phi_e \) are omitted as they are of much lower order.}
Lemma 7.2. There exists $\varepsilon_0 > 0$ such that
\begin{equation}
N_2 + N_3 \lesssim C(\nu)\left(\varepsilon^{-\frac{14}{3}}\|\phi\|_{L^2(\Omega)}^{\frac{10}{7}} + \varepsilon^{-\frac{24}{5}}\|\phi\|_{L^2(\Omega)}^{\frac{18}{5}}
+ \varepsilon^{-10}\|\phi\|_{L^2(\Omega)}^6 + \varepsilon^{-16}\|\phi\|_{L^2(\Omega)}^{10}\right) + \nu K(t)
\end{equation}
(7.16)
for any $\varepsilon \in (0, \varepsilon_0)$ and $\nu \in (\varepsilon^{\frac{1}{10}}, 1)$.

Proof. Applying the Cauchy–Schwarz inequality and (1.21b) yields
\begin{equation}
N_2 \lesssim \varepsilon^{-2} \int_\Omega \phi^2 |\Delta \phi| + \varepsilon^{-4} \int_\Omega \phi^2 |\phi|^4 \lesssim C(\nu)\varepsilon^{-2} \int_\Omega |\phi|^4 + \nu K(t).
\end{equation}
(7.17)
To estimate $N_3$, owing to (3.13) and (3.11), we apply (7.10a) and (7.10b) and obtain
\begin{align}
\varepsilon^{-4} \int_\Omega |\phi|^4 & \lesssim \varepsilon^{-5} \int_{\Gamma_t} Z^4 + \varepsilon^{-4} \int_\Omega |\phi|^4 \\
& \lesssim \varepsilon^{-5} \|Z\|_{L^2(\Gamma_t)}^3 \|Z\|_{H^2(\Gamma_t)} + \varepsilon^{-4} \|\phi\|_{L^2(\Omega)}^\frac{5}{2} \|\phi\|_{H^2(\Omega)}^\frac{3}{2} \\
& \lesssim C(\nu) \left(\varepsilon^{-10} \|Z\|_{L^2(\Gamma_t)}^6 + \varepsilon^{-16} \|\phi\|_{L^2(\Omega)}^{10}\right) + \nu K(t) \\
& \lesssim C(\nu) \left(\varepsilon^{-10} \|\phi\|_{L^2(\Omega)}^6 + \varepsilon^{-16} \|\phi\|_{L^2(\Omega)}^{10}\right) + \nu K(t).
\end{align}
(7.18)
In a similar way, we have
\begin{align}
\varepsilon^{-3} \int_\Omega |\phi|^3 & \lesssim \varepsilon^{-\frac{7}{2}} \int_{\Gamma_t} |Z|^3 + \varepsilon^{-3} \int_\Omega |\phi|^3 \\
& \lesssim \varepsilon^{-\frac{7}{2}} \|Z\|_{L^2(\Gamma_t)}^\frac{5}{2} \|Z\|_{H^2(\Gamma_t)}^\frac{1}{2} + \varepsilon^{-3} \|\phi\|_{L^2(\Omega)}^\frac{9}{2} \|\phi\|_{H^2(\Omega)}^\frac{3}{2} \\
& \lesssim C(\nu) \left(\varepsilon^{-14} \|Z\|_{L^2(\Gamma_t)}^\frac{10}{5} + \varepsilon^{-24} \|\phi\|_{L^2(\Omega)}^{18} \|\phi\|_{L^2(\Omega)}^{18}\right) + \nu K(t).
\end{align}
(7.18)
The desired estimate follows from the above inequalities.

Proof of Theorem 1.1. The construction of the approximate solution fulfilling (1.9) is already given in Proposition 3.1. We focus on the estimate of the difference (1.11). Submitting (7.11), (7.16) into (7.9) leads to
\begin{equation}
\int_\Omega \mathcal{H}_1 \phi \lesssim C(\nu) \left(\frac{1}{\varepsilon^{14}} \|\phi\|_{L^2(\Omega)}^\frac{10}{7} + \frac{1}{\varepsilon^{24}} \|\phi\|_{L^2(\Omega)}^{18} \|\phi\|_{L^2(\Omega)}^{18}
+ \frac{1}{\varepsilon^{10}} \|\phi\|_{L^2(\Omega)}^6 + \frac{1}{\varepsilon^{16}} \|\phi\|_{L^2(\Omega)}^{10}\right) + \nu K(t),
\end{equation}
(7.19)
for any $\varepsilon \in (0, \varepsilon_0)$ and $\nu \in (\varepsilon^{\frac{1}{10}}, 1)$. Now we treat $\mathcal{H}_2$ in (7.1). It follows from Theorem 1.2 that $\mathcal{R}_1$, $\mathcal{R}_2$ are uniformly bounded in $\varepsilon$ and $\nu$. This together with (7.5), the Cauchy–Schwarz inequality and (7.18) leads to
\begin{align}
\int_\Omega \mathcal{H}_2 \phi \lesssim & \varepsilon^{k-5} \int_\Omega (|\Delta \phi| + |\phi|) + \varepsilon^{k-5} \int_\Omega |\phi|^3 \\
\lesssim & \varepsilon^{k-5} + \varepsilon^{k-5} \int_\Omega \left(\phi^2 + |\Delta \phi|^2\right) + \varepsilon^{k-2} \varepsilon^{-3} \int_\Omega |\phi|^3.
\end{align}
(7.20)
We can use (7.18) to treat the last term and add it to (7.19). This gives as that
\[
\int_{\Omega} (\mathcal{H}_1 + \mathcal{H}_2) \phi 
\lesssim C(n) \left( \frac{1}{\varepsilon^{11/3}} \| \phi \|_{L^2(\Omega)}^{10} + \frac{1}{\varepsilon^{21/2}} \| \phi \|_{L^2(\Omega)}^{18} + \frac{1}{\varepsilon^{10}} \| \phi \|_{L^2(\Omega)}^{18} + \frac{1}{\varepsilon^{16}} \| \phi \|_{L^2(\Omega)}^{10} \right)
+ \nu \mathcal{K}(t) + \varepsilon^{k-5} + \varepsilon^{k-5} \int_{\Omega} (\phi^2 + |\Delta \phi|^2).
\]
(7.21)
We can substitute the above estimate into (7.4) and choose \( \nu \) sufficiently small (but independent of \( \varepsilon \)) and \( k \geq 10 \) to obtain
\[
\frac{d}{dt} \| \phi \|_{L^2(\Omega)}^2 - C \| \phi \|_{L^2(\Omega)}^2 \leq C_6 \left( \left( \Lambda^{4/3} + \varepsilon^{4/5} \Lambda^{8/5} + \varepsilon^{4} \Lambda^{4} + \varepsilon^{12} \Lambda^{8} \right) \| \phi \|_{L^2(\Omega)}^2 + C_6 \varepsilon^5 \right).\]
(7.22)
where \( C, C_6 \) are positive constants independent of \( \varepsilon \).

We shall use the Grönwall inequality and the continuity method to close the energy estimate under the assumption (1.10). We set
\[
T_{\varepsilon} = \sup \{ \tau \in (0, T_{max}) \mid \| \phi \|_{C([0, \tau], L^2(\Omega))} \leq \Lambda \varepsilon^{7/2} \}\]
(7.23)
with \( \Lambda > 0 \) and \( T_{max} \in (0, T) \) being determined later on. It can be verified from (7.22) that the following inequality is valid on the time interval \( (0, T_{\varepsilon}) \):
\[
\frac{d}{dt} \| \phi \|_{L^2(\Omega)}^2 - C \| \phi \|_{L^2(\Omega)}^2 \leq C_6 \left( \left( \Lambda^{4/3} + \varepsilon^{4/5} \Lambda^{8/5} + \varepsilon^{4} \Lambda^{4} + \varepsilon^{12} \Lambda^{8} \right) \| \phi \|_{L^2(\Omega)}^2 + C_6 \varepsilon^5 \right).\]
(7.24)
We shall work with \( \varepsilon \in (0, \varepsilon_1) \) where
\[
\varepsilon_1 = \min(\varepsilon_0, 1, \Lambda^{-2}, \Lambda^{-1}, T^{-1} C_6^{-1}, e^{-T(C+19 C_6)}),
\]
then \( \frac{d}{dt} \| \phi \|_{L^2(\Omega)}^2 \leq (C + C_6 \Lambda^{4/3} + 3 C_6) \| \phi \|_{L^2(\Omega)}^2 + C_6 \varepsilon^5 \). Applying the Grönwall inequality and (1.10) yields
\[
\| \phi \|_{C([0, T_{\varepsilon}]; L^2(\Omega))} \leq \exp(T_{\varepsilon} (C + C_6 \Lambda^{4/3} + 3 C_6)) \left( \| \phi \|_{t=0} \| \phi \|_{L^2(\Omega)} + T_{\varepsilon} C_6 \varepsilon^5 \right)
\leq \exp(T_{max} (C + C_6 \Lambda^{4/3} + 3 C_6)) (C_{in} \varepsilon^{7/2} + T_{max} C_6 \varepsilon^5).
\]
(7.26)
• For an arbitrary \( C_{in} > 0 \), we shall choose \( \Lambda = 10 C_{in} \) in (7.23) and
\[
T_{max} = \min \left\{ C_{in} / C_6, (C + C_6 \Lambda^{4/3} + 3 C_6)^{-1}, T \right\}.
\]
(7.27)
Then it follows from (7.26) and (7.23) that \( T_{\varepsilon} \geq T_{max} \), and thus (1.11) holds.

• If \( C_{in} \leq \exp(-T (C + 19 C_6)) \), we shall choose \( \Lambda = 8 \) and \( T = T_{max} \). Then it follows from (7.25) and (7.26) that \( \| \phi \|_{C([0, T_{\varepsilon}]; L^2(\Omega))} \leq 2 \varepsilon^{7/2} \). This combined with (7.23) leads to (1.11).
Remark 7.1. We discuss the admissible initial data satisfying (1.10). Because of (3.6) and (A.63), the approximate solution satisfies

\[
\phi_a(x, t) = \left( \theta(z) + \varepsilon^2 \tilde{\phi}^{(2)}(z, x, t) + \varepsilon^3 \tilde{\phi}^{(3)}(z, x, t) \right) \bigg|_{z = r(x, t)/\varepsilon} + O(\varepsilon^4) \quad \text{in } \Gamma^0_0(\delta). \tag{7.28}
\]

Here \( \tilde{\phi}^{(2)} \) has an explicit form (3.3a), and \( \tilde{\phi}^{(2)}, \tilde{\phi}^{(3)} \) both decay exponentially to 0 as \( z \to \pm \infty \). On the other hand, due to (2.4), (2.1) and the construction made in Appendix A, we have \( r(x, 0) = d^{(0)}(x, 0) \) (the signed-distance to the initial interface \( \Gamma^0_0 \)). So

\[
\phi_a(x, 0) = \theta\left( \frac{d^{(0)}(x, 0)}{\varepsilon} \right) + \varepsilon^2 \tilde{\phi}^{(2)}\left( \frac{d^{(0)}(x, 0)}{\varepsilon}, x, 0 \right) + \varepsilon^3 \tilde{\phi}^{(3)}\left( \frac{d^{(0)}(x, 0)}{\varepsilon}, x, 0 \right) + O(\varepsilon^4). \tag{7.29}
\]

If we choose \( \phi_\varepsilon(x, 0) = \theta\left( \frac{d^{(0)}(x, 0)}{\varepsilon} \right) + \varepsilon^2 \tilde{\phi}^{(2)}\left( \frac{d^{(0)}(x, 0)}{\varepsilon}, x, 0 \right) \), then by a change of variable in \( \Gamma^0_0(\delta) \) (the \( \delta \)-tubular neighborhood of \( \Gamma^0_0 \)), we arrive at

\[
\| \phi_\varepsilon(x, 0) - \phi_a(x, 0) \|_{L^2(\Gamma^0_0(\delta))} = O(\varepsilon^{7/2}). \tag{7.30}
\]

This together with an appropriate choice of \( \phi_\varepsilon(x, 0) \) outside \( \Gamma^0_0(\delta) \) leads to (1.10).

Remark 7.2. In seeking an estimate like (1.11) with \( L^\infty(\Omega \times (0, T)) \) norm instead, one can follow the approach of [21] by establishing a hierarchical Sobolev norm estimates. However this will lead to a larger power than 7/2 and makes (1.10) harder to verify.

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Declarations

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Appendix A. Formally Matched Expansions

We use the matched asymptotic expansions to obtain (3.3a) and (3.3), and thus give a complete proof of Proposition 3.1, and of (1.9). The construction will employ the solvability of an ODE, see [2, Lemma 4.1].

Lemma A.1. Let \( \ell, m, n \in \{0, 1, 2\} \) and \( \tilde{A}(z, x, t) : \mathbb{R} \times \Gamma^0(\delta) \to \mathbb{R} \) be a function satisfying

\[
\partial_\varepsilon^{\ell} \partial_x^m \partial_t^n \tilde{A}(z, x, t) = O(e^{-C|z|}) \quad \text{as } z \to \pm \infty, \quad \text{uniformly in } (x, t) \in \Gamma^0(\delta). \tag{A.1}
\]
and the following compatibility condition:

\[ \int_{\mathbb{R}} \tilde{A}(z, x, t) \theta'(z) dz = 0, \quad (x, t) \in \Gamma^0(\delta). \]  
(A.2)

Then the equation \( \mathcal{L} \tilde{U} = \tilde{A} \) has a bounded solution so that

\[ \partial_x^\alpha \partial_t^\beta \partial_z^\gamma \tilde{U}(z, x, t) = O(e^{-C|z|}) \quad \text{as} \quad z \to \pm \infty, \quad \text{uniformly in} \quad (x, t) \in \Gamma^0(\delta). \]  
(A.3)

Moreover, there exists a smooth function \( U(x, t) \) such that

\[ \tilde{U}(z, x, t) = U(x, t) \theta'(z) + \left[ \int_0^z (\theta'(\xi))^{-2} \left( \int_\xi^{+\infty} \tilde{A}(\tau, x, t) \theta'(\tau) d\tau \right) d\xi \right] \theta'(z). \]  
(A.4)

The unique solution satisfying \( \tilde{U}(0, x, t) = 0 \) corresponds to \( U \equiv 0 \).

Recall from Section 2.1 that \( d(0) \) is the signed-distance to \( \Gamma^0 \). We need the following lemma whose proof can be found in [5].

**Lemma A.2.** The interface \( \Gamma^0 \) evolves under the Willmore flow (1.5) if and only if \( d(0) \) fulfills

\[ \partial_t d(0) + \Delta^2 d(0) = \Delta d(0) D(0) + \nabla d(0) \cdot \nabla D(0) \quad \text{on} \quad \Gamma^0. \]  
(A.5)

Following [2], we employ the stretched variable \( z = \frac{dx}{\varepsilon} \in \mathbb{R} \) (see (2.1) for the definition of \( dx \)) and set the following Ansatz in \( \Gamma^0(3\delta) \) for the inner expansion

\[ \tilde{\phi}^\varepsilon(z, x, t) = \sum_{i \geq 0} \varepsilon^i \tilde{\phi}^{(i)}(z, x, t), \quad \tilde{\mu}^\varepsilon(z, x, t) = \sum_{i \geq 0} \varepsilon^i \tilde{\mu}^{(i)}(z, x, t), \]  
(A.6)

which should fulfill the following matching conditions in \( (x, t) \in \Gamma^0(3\delta) \):

\[ D_z^\gamma D_x^\alpha D_t^\beta \left( \tilde{\phi}^{(i)}(z, x, t) - \phi_{\pm}^{(i)}(x, t) \right) = O(e^{-v|z|}), \]  
(A.7)

\[ D_z^\gamma D_x^\alpha D_t^\beta \left( \tilde{\mu}^{(i)}(z, x, t) - \mu_{\pm}^{(i)}(x, t) \right) = O(e^{-v|z|}). \]  
(A.8)

Here \( v \) is a positive fixed constant and \( 0 \leq \alpha, \beta, \gamma \leq 2 \). It follows from the Taylor expansion and (A.6) that

\[ f'(\tilde{\phi}^\varepsilon) = f'(\tilde{\phi}^{(0)}) + f''(\tilde{\phi}^{(0)}) \sum_{i \geq 1} \varepsilon^i \tilde{\phi}^{(i)} + \sum_{i \geq 1} \varepsilon^i g_{i-1}(\tilde{\phi}^{(0)}, \ldots, \tilde{\phi}^{(i-1)}), \]  
(A.9a)

\[ f''(\tilde{\phi}^\varepsilon) = f''(\tilde{\phi}^{(0)}) + f'''(\tilde{\phi}^{(0)}) \sum_{i \geq 1} \varepsilon^i \tilde{\phi}^{(i)} + \sum_{i \geq 1} \varepsilon^i g_{i-1}^*(\tilde{\phi}^{(0)}, \ldots, \tilde{\phi}^{(i-1)}), \]  
(A.9b)

where \( g_i, g_i^* \) enjoy the following property:

**Lemma A.3.** For \( i \geq 1 \), \( g_i(x_0, \ldots, x_i) \) and \( g_i^*(x_0, \ldots, x_i) \) are polynomials of \( i + 1 \) variables. Moreover, they vanish when \( x_1 = \cdots = x_{i-1} = 0 \), and \( g_0 = g_0^* = 0 \).
Using $|\nabla d_\varepsilon| = 1$ and the chain-rule, we have the following expansions
\begin{align*}
\partial_t \tilde{\phi}^\varepsilon (\frac{d_\varepsilon}{\varepsilon} , x , t) &= \partial_t \tilde{\phi}^\varepsilon + \varepsilon^{-1} \partial_z \tilde{\phi}^\varepsilon \partial_t d_\varepsilon , \\
\Delta \tilde{\mu}^\varepsilon (\frac{d_\varepsilon}{\varepsilon} , x , t) &= \varepsilon^{-2} \partial_z^2 \tilde{\mu}^\varepsilon + 2 \varepsilon^{-1} \nabla_x \partial_z \tilde{\mu}^\varepsilon \cdot \nabla d_\varepsilon + \varepsilon^{-1} \partial_z \tilde{\mu}^\varepsilon \Delta d_\varepsilon + \Delta_x \tilde{\mu}^\varepsilon , \\
\Delta \tilde{\phi}^\varepsilon (\frac{d_\varepsilon}{\varepsilon} , x , t) &= \varepsilon^{-2} \partial_z^2 \tilde{\phi}^\varepsilon + 2 \varepsilon^{-1} \nabla_x \partial_z \tilde{\phi}^\varepsilon \cdot \nabla d_\varepsilon + \varepsilon^{-1} \partial_z \tilde{\phi}^\varepsilon \Delta d_\varepsilon + \Delta_x \tilde{\phi}^\varepsilon .
\end{align*}

We expect $(\tilde{\phi}^\varepsilon (z , x , t) , \tilde{\mu}^\varepsilon (z , x , t))|_{z = d_\varepsilon / \varepsilon}$ to satisfy (1.1) up to a high order term in $\varepsilon$ and thus determine the terms in (A.6):
\begin{align*}
\partial_z^2 \tilde{\mu}^\varepsilon - f''(\tilde{\phi}^\varepsilon) \tilde{\mu}^\varepsilon + 2 \varepsilon \nabla_x \partial_z \tilde{\mu}^\varepsilon \cdot \nabla d_\varepsilon + \varepsilon \partial_z \tilde{\mu}^\varepsilon \Delta d_\varepsilon - \varepsilon^2 \partial_z \tilde{\phi}^\varepsilon \partial_t d_\varepsilon + \varepsilon^2 \Delta_x \tilde{\mu}^\varepsilon - \varepsilon^3 \partial_t \tilde{\phi}^\varepsilon &= O(\varepsilon^k), \quad (A.10) \\
- \partial_z^2 \tilde{\phi}^\varepsilon + f'(\tilde{\phi}^\varepsilon) - 2 \varepsilon \nabla_x \partial_z \tilde{\phi}^\varepsilon \cdot \nabla d_\varepsilon - \varepsilon \partial_z \tilde{\phi}^\varepsilon \Delta d_\varepsilon - \varepsilon^2 \tilde{\mu}^\varepsilon - \varepsilon^2 \Delta_x \tilde{\phi}^\varepsilon &= O(\varepsilon^k). \quad (A.11)
\end{align*}

Since $z = \frac{d_\varepsilon}{\varepsilon}$, we need the above two equations to hold merely on
\[ S^\varepsilon \triangleq \{(z , x , t) \in \mathbb{R} \times \Gamma^0 (3\delta) : z = d_\varepsilon / \varepsilon\}. \]
So we can add in (A.10) terms which are multiplied by $d_\varepsilon - \varepsilon z$. These terms will give more degrees of freedom to construct and to solve the equations for $d^{(\ell)}$. See Remark A.1 below and [2]. So we modify (A.10) as follows
\begin{align*}
\partial_z^2 \tilde{\mu}^\varepsilon - f''(\tilde{\phi}^\varepsilon) \tilde{\mu}^\varepsilon + 2 \varepsilon \nabla_x \partial_z \tilde{\mu}^\varepsilon \cdot \nabla d_\varepsilon + \varepsilon \partial_z \tilde{\mu}^\varepsilon \Delta d_\varepsilon - \varepsilon^2 \partial_z \tilde{\phi}^\varepsilon \partial_t d_\varepsilon + \varepsilon^2 \Delta_x \tilde{\mu}^\varepsilon - \varepsilon^3 \partial_t \tilde{\phi}^\varepsilon + \varepsilon^2 \chi^\varepsilon (d_\varepsilon - \varepsilon z) \eta' &= O(\varepsilon^k), \quad (A.12)
\end{align*}
where $\eta(z)$ is a smooth non-decreasing function satisfying
\[ \eta(z) = 0 \text{ if } z \leq -1; \quad \eta(z) = 1 \text{ if } z \geq 1; \quad \eta'(z) \text{ is even,} \quad (A.13) \]
and $\chi^\varepsilon (x , t) = \sum_{i=0}^{\infty} \varepsilon^i \chi^{(i)}(x , t)$ with $\chi^{(i)}$ being determined later on.

**Definition A.4.** We shall use $\varepsilon^\ell$-scale to denote the terms of form $\varepsilon^\ell g(z , x , t)$. The $\ell$-order will refer to those indexed by $\ell$ if $\ell \geq 0$, and by 0 if $\ell < 0$. Moreover, $\tilde{\Psi}^{(i)}(z , x , t)$ and $\Xi^{(i)}(x , t)$ will denote generic terms which might change from line to line, and will depend on terms of order at most $\ell$.

**8.1. $\varepsilon^1$-Scale**

Collecting all terms of $\varepsilon^0$-scale in (A.11)–(A.12), we have $\partial_z^2 \tilde{\phi}^{(0)} = f'(\tilde{\phi})$ and $\partial_z^2 \tilde{\mu}^{(0)} = f''(\tilde{\phi}^{(0)})\tilde{\mu}^{(0)}$. Together with the matching condition (A.7) and (A.8), we obtain
\[ \tilde{\phi}^{(0)} = \theta(z) , \quad \tilde{\mu}^{(0)}(z , x , t) = \mu_0(x , t)\theta'(z) , \quad (A.14) \]
for some function $\mu_0$ which will be determined later on. To proceed we recall the operator $\mathcal{L}$ defined at (1.16), which enjoys
\[ \mathcal{L} \text{ is self-adjoint, } \partial_z \mathcal{L} - \mathcal{L} \partial_z = f'''(\theta)\theta' \mathcal{I} . \quad (A.15) \]
Collecting all terms of $\varepsilon^1$-scale in (A.11)–(A.12), and using $\tilde{\phi}(0) = \theta(z)$, we have
\begin{equation}
\mathcal{L}\tilde{\mu}^{(1)} = -f''(\theta)\tilde{\phi}(1)\tilde{\mu}^{(0)} + 2\nabla_x \partial_z \tilde{\mu}^{(0)} \cdot \nabla d^{(0)} + \partial_z \tilde{\mu}^{(0)} \Delta d^{(0)}, \tag{A.16a}
\end{equation}
\begin{equation}
\mathcal{L}\tilde{\phi}^{(1)} = \tilde{\mu}^{(0)} + \partial_z \theta \Delta d^{(0)}. \tag{A.16b}
\end{equation}
Here $\mu_0$ is chosen such that (A.16b) fulfills (A.2), i.e. $\mu_0 = -\Delta d^{(0)}$. Thus
\begin{equation}
\tilde{\mu}^{(0)}(z, x, t) = -\Delta d^{(0)} \theta', \quad \tilde{\phi}^{(1)}(z, x, t) = 0. \tag{A.17}
\end{equation}
This together with Lemma A.3 implies
\begin{equation}
g_1 = g_2 = g_1^* = g_2^* = 0. \tag{A.18}
\end{equation}
Substituting this into (A.16a) yields $\mathcal{L}\tilde{\mu}^{(1)} = -2D^{(0)} \theta''$ where $D^{(0)}$ is given by
\begin{equation}
D^{(0)}(x, t) = \nabla \Delta d^{(0)} \cdot \nabla d^{(0)} + \frac{1}{2} (\Delta d^{(0)})^2. \tag{A.19}
\end{equation}
Using (A.4) we deduce
\begin{equation}
\tilde{\mu}^{(1)}(z, x, t) = D^{(0)} z \theta'(z) + \mu_1(x, t) \theta'(z) \tag{A.20}
\end{equation}
for some $\mu_1(x, t)$ which will be determined later on.

\section*{8.2. $\varepsilon^2$-Scale}

Substituting (A.6) into (A.10)–(A.11), and collecting all terms of $\varepsilon^2$-scale, we obtain
\begin{equation}
0 = \partial_z^2 \tilde{\mu}^{(2)} - f''(\theta) \tilde{\mu}^{(2)} + f'''(\theta) \tilde{\phi}(1) \tilde{\mu}^{(1)} - (f''(\theta) \tilde{\phi}(2) + g_1^*(\tilde{\phi}(0), \tilde{\phi}(1))) \tilde{\mu}^{(0)}
+ 2\nabla_x \partial_z \tilde{\mu}^{(1)} \cdot \nabla d^{(0)} + 2\nabla_x \partial_z \tilde{\mu}^{(0)} \cdot \nabla d^{(1)} + \partial_z \tilde{\mu}^{(1)} \Delta d^{(0)} + \partial_z \tilde{\mu}^{(0)} \Delta d^{(1)}
- \partial_z \theta \partial_t d^{(0)} + \Delta_x \tilde{\mu}^{(0)} + \chi^{(0)} d^{(0)} \eta',
0 = -\partial_z^2 \tilde{\phi}(2) + f''(\theta) \tilde{\phi}(2) + g_1(\tilde{\phi}(0), \tilde{\phi}(1)) - 2\nabla_x \partial_z \tilde{\phi}(1) \cdot \nabla d^{(0)} - 2\nabla_x \partial_z \theta \cdot \nabla d^{(1)}
- \partial_z \tilde{\phi}(1) \Delta d^{(0)} - \partial_z \theta \Delta d^{(1)} - \tilde{\mu}^{(1)} - \Delta_x \tilde{\phi}^{(0)}.
\end{equation}
In view of (A.18), the above two equations can be simplified as
\begin{equation}
\mathcal{L}\tilde{\mu}^{(2)} = -f''(\theta) \tilde{\phi}(2) \tilde{\mu}^{(0)} + 2\nabla_x \partial_z \tilde{\mu}^{(1)} \cdot \nabla d^{(0)} + 2\nabla_x \partial_z \tilde{\mu}^{(0)} \cdot \nabla d^{(1)} + \partial_z \tilde{\mu}^{(1)} \Delta d^{(0)} + \partial_z \tilde{\mu}^{(0)} \Delta d^{(1)} - \partial_z \theta \partial_t d^{(0)} + \Delta_x \tilde{\mu}^{(0)} + \chi^{(0)} d^{(0)} \eta', \tag{A.21a}
\end{equation}
\begin{equation}
\mathcal{L}\tilde{\phi}^{(2)} = \partial_z \theta \Delta d^{(1)} + \tilde{\mu}^{(1)} = (\Delta d^{(1)} + \mu_1) \theta' + D^{(0)} z \theta'(z). \tag{A.21b}
\end{equation}
Recall (A.20) that $\mu_1$ shall be determined such that (A.21b) fulfills (A.2), i.e. $(\Delta d^{(1)} + \mu_1) \sigma = 0$, where $\sigma \triangleq \int_{\mathbb{R}} (\theta')^2 dz$. This leads to the formula for $\mu_1$ and completes formula (A.20):
\begin{equation}
\mu_1(x, t) = -\Delta d^{(1)}, \quad \tilde{\mu}^{(1)}(z, x, t) = D^{(0)} z \theta'(z) - \Delta d^{(1)} \theta'(z). \tag{A.22}
\end{equation}
As a result, (A.21b) is simplified to
\[ \mathcal{L} \tilde{\phi}^{(2)} = D^{(0)} z \theta'(z). \] (A.23)

Using \( \int_{\mathbb{R}} z(\theta')^2 dz = 0 \) and formula (A.4), we can solve (A.23):
\[ \tilde{\phi}^{(2)}(z, x, t) = D^{(0)} \theta'(z) \alpha(z), \] with \( \alpha(z) \triangleq \int_0^z (\theta'(\zeta))^{-2} \int_0^{+\infty} \tau (\theta'(\tau))^2 d\tau d\zeta \) (A.24)

being an odd function. This implies that \( \tilde{\phi}^{(2)} \) is odd with respect to \( z \). On the other hand, \( \chi^{(0)} \) is determined so that the right hand side of (A.21a) fulfills (A.2), e.g.
\[ \chi^{(0)} d^{(0)} \sigma^{-1} \tilde{\sigma} = \mathcal{G}_0 d^{(0)}, \] with \( \tilde{\sigma} = \int_{\mathbb{R}} \eta' \theta' dz, \) (A.25)
\[ \mathcal{G}_0 d^{(0)} \triangleq \partial_t d^{(0)} + \Delta^2 d^{(0)} - \Delta d^{(0)} D^{(0)} - \nabla d^{(0)} \cdot \nabla D^{(0)}. \] (A.26)

Note that we used the following formula which is due to (A.15) and (A.23):
\[ \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(2)}(\theta')^2 dz = \int_{\mathbb{R}} \partial_z (\mathcal{L} \tilde{\phi}^{(2)}) \theta' dz - \int_{\mathbb{R}} \mathcal{L} \left( \partial_z \tilde{\phi}^{(2)} \right) \theta' dz = \frac{\eta}{\tau} D^{(0)}. \] (A.27)

Combining (A.25) and (A.5) leads to the choice of \( \chi^{(0)} \):
\[ \chi^{(0)} \triangleq \begin{cases} \sigma(\tilde{\sigma})^{-1} (\mathcal{G}_0 d^{(0)}) /d^{(0)}, & \forall (x, t) \in \Gamma^0(3\delta) \setminus \Gamma^0, \\ \sigma(\tilde{\sigma})^{-1} \nabla (\mathcal{G}_0 d^{(0)}) \cdot \nabla d^{(0)}, & \forall (x, t) \in \Gamma^0. \end{cases} \] (A.28)

**Remark A.1.** If we do not modify the equation (A.10) into (A.12), then we would require the equation (A.5) to hold in \( \Gamma^0(3\delta) \), which is not compatible with \( |\nabla d^{(0)}| = 1 \) in general.

The formula (A.25) reduces (A.21a) to
\[ \mathcal{L} \tilde{\mu}^{(2)} = \left(f'''(\theta)(\theta')^2 \alpha + z \theta''\right) \Delta d^{(0)} D^{(0)} + (\theta' + 2z \theta'') \nabla d^{(0)} \cdot \nabla D^{(0)} - 2\theta'' D^{(1)} + \chi^{(0)} d^{(0)} \eta' - \sigma^{-1} \tilde{\sigma} \chi^{(0)} d^{(0)} \theta', \] (A.29)
\[ D^{(1)} = \nabla \Delta d^{(1)} \cdot \nabla d^{(0)} + \nabla \Delta d^{(0)} \cdot \nabla d^{(1)} + \Delta d^{(0)} \Delta d^{(1)}. \] (A.30)

Note that \( D^{(1)} \) is consistent with (3.4). We can solve (A.29) by employing (A.4),
\[ \tilde{\mu}^{(2)}(z, x, t) = \Delta d^{(0)} D^{(0)} \theta'(z) \gamma_1(z) + \nabla d^{(0)} \cdot \nabla D^{(0)} \theta'(z) \gamma_2(z) + D^{(1)} z \theta' + \mu_2(x, t) \theta' + \chi^{(0)} d^{(0)} \theta'(z) \gamma_3(z), \] (A.31)
where \( \mu_2(x, t) \) is a smooth function which will be determined by the \( \varepsilon^3 \)-scale below, and \( \gamma_1(z) \) and \( \gamma_2(z) \) and \( \gamma_3(z) \) are three even functions defined by
\[ \gamma_1(z) = \int_0^z (\theta'(\zeta))^{-2} \int_0^{+\infty} \theta'(\tau)(f'''(\theta)(\theta')^2 \alpha + \tau \theta'')(\tau) d\tau d\zeta, \quad \gamma_2(z) = -z^2 / 2, \]
\[ \gamma_3(z) = \int_0^z (\theta'(\zeta))^{-2} \int_0^{+\infty} \theta'(\tau)(\eta'(\tau) - \sigma^{-1} \tilde{\sigma} \theta'(\tau)) d\tau d\zeta. \] (A.32)
8.3. $\varepsilon^3$-Scale

We substitute (A.6) into (A.11)–(A.12), then use (A.18) and collect all the terms of $\varepsilon^3$-scale:

$$\mathcal{L}\tilde{\mu}^{(3)} = -f'''(\theta)\tilde{\phi}^{(2)}\tilde{\mu}^{(1)} - f'''(\theta)\tilde{\phi}^{(3)}\tilde{\mu}^{(0)} + \left(\chi^{(0)} d^{(1)} + \chi^{(1)} d^{(0)}\right)\eta' - \chi^{(0)} \eta'
+ 2\nabla_x \partial_{\varepsilon} \tilde{\phi}^{(2)} \cdot \nabla d^{(0)} + 2\nabla_x \partial_{\varepsilon} \tilde{\phi}^{(1)} \cdot \nabla d^{(0)}
+ \partial_{\varepsilon} \tilde{\phi}^{(2)} \Delta d^{(0)} + \partial_{\varepsilon} \tilde{\phi}^{(1)} \Delta d^{(0)} - \partial_{\varepsilon} \theta \partial_t d^{(1)} + \Delta_x \tilde{\mu}^{(1)},$$

(A.33a)

$$\mathcal{L}\tilde{\phi}^{(3)} = 2\nabla_x \partial_{\varepsilon} \tilde{\phi}^{(2)} \cdot \nabla d^{(0)} + \partial_{\varepsilon} \tilde{\phi}^{(0)} \Delta d^{(0)} + \partial_{\varepsilon} \theta \Delta d^{(2)} + \tilde{\nu}^{(2)}.$$  

(A.33b)

We determine $\mu_2(x, t)$ in (A.31) so that (A.33b) satisfies (A.2), i.e.

$$(\Delta d^{(2)} + \mu_2)\sigma = -\left(\nabla d^{(0)} \cdot \nabla D^{(0)} + \frac{1}{2} \Delta d^{(0)} D^{(0)}\right) \int_\mathbb{R} \int_\mathbb{R}^+ \tau(\theta'(\tau))^2 d\tau dz
- \Delta d^{(0)} D^{(0)} \int_\mathbb{R} \int_\mathbb{R}^+ \tau(\theta'(\tau))^2 \gamma_1(z) dz - \nabla d^{(0)} \cdot \nabla D^{(0)} \int_\mathbb{R} \int_\mathbb{R}^+ \tau(\theta'(\tau))^2 \gamma_2(z) dz
- \chi^{(0)} d^{(0)} \int_\mathbb{R} \int_\mathbb{R}^+ \tau(\theta'(\tau))^2 \gamma_3(z) dz.$$  

(A.34)

To prove (A.34), it follows from (A.24) and integration by parts that

$$\int_\mathbb{R} \left(2\nabla_x \partial_{\varepsilon} \tilde{\phi}^{(2)} \cdot \nabla d^{(0)} + \partial_{\varepsilon} \tilde{\phi}^{(1)} \Delta d^{(0)} + \partial_{\varepsilon} \theta \Delta d^{(2)}\right) \theta' dz
=- \left(2\nabla d^{(0)} \cdot \nabla D^{(0)} + D^{(0)} \Delta d^{(0)}\right) \int_\mathbb{R} \alpha \theta'' dz + \Delta d^{(2)} \sigma
= \left(\nabla d^{(0)} \cdot \nabla D^{(0)} + \frac{1}{2} \Delta d^{(0)} D^{(0)}\right) \int_\mathbb{R} \int_\mathbb{R}^+ \tau(\theta'(\tau))^2 d\tau dz + \Delta d^{(2)} \sigma.$$  

(A.35)

This together with (A.31) leads to (A.34). So we can use (A.34) to rewrite (A.31) as

$$\tilde{\mu}^{(2)}(z, x, t) = -\Delta d^{(2)}(x, t) \theta'(z) + D^{(1)}(x, t) z \theta'(z) + \tilde{\Psi}^{(0)}(z, x, t),$$  

(A.36)

where $\tilde{\Psi}^{(0)}$ only depends on 0-order terms:

$$\tilde{\Psi}^{(0)} = \Delta d^{(0)} D^{(0)} \theta' \gamma_1 + \nabla d^{(0)} \cdot \nabla D^{(0)} \theta' \gamma_2
- (2\sigma)^{-1} \left(\int_\mathbb{R} \int_\mathbb{R}^+ \tau(\theta'(\tau))^2 d\tau dz\right) \left(\Delta d^{(0)} D^{(0)} + 2\nabla d^{(0)} \cdot \nabla D^{(0)}\right) \theta'
- \sigma^{-1} \left(\Delta d^{(0)} D^{(0)} \int_\mathbb{R} (\theta'(\tau))^2 \gamma_1(z) dz + \nabla d^{(0)} \cdot \nabla D^{(0)} \int_\mathbb{R} (\theta'(\tau))^2 \gamma_2(z) dz\right) \theta'
- \sigma^{-1} \left(\int_\mathbb{R} (\theta'(\tau))^2 \gamma_3(z) dz\right) \chi^{(0)} d^{(0)} \theta'.$$

(A.37)

Finally, applying Lemma A.1 to (A.33b) yields a solution $\tilde{\phi}^{(3)}$: 
Lemma A.5. $\widetilde{\Psi}^{(0)}$ satisfies (A.1) and the equation (A.33b) has a unique smooth solution $\tilde{\phi}^{(3)}$ depending up to 1-order terms, satisfying $\tilde{\phi}^{(3)}|_{z=0} = 0$ and (A.3).

d(1) is determined so that the right hand side of (A.33a) fulfills (A.2):

Lemma A.6. There exists $\Xi^{(0)}(x, t)$ depending on 0-order terms such that

$$
\mathcal{G}_1 d^{(1)} = \sigma \left( \chi^{(0)} d^{(1)} + \chi^{(1)} d^{(0)} \right) + \Xi^{(0)} \text{ in } \Gamma^0(3\delta),
$$

(A.38)

where $\mathcal{G}_1 d^{(1)} \triangleq \partial_t d^{(1)} + \Delta^2 d^{(1)} - \sum_{i=0,1} \left( \nabla D^{(i)} \cdot \nabla d^{(1-i)} + D^{(i)} \Delta d^{(1-i)} \right)$.

(A.39)

Proof. We note that $f'''(\theta)(\theta'(z))^3 \alpha(z) z$ is an odd function, so it follows from (A.22), (A.24) and (A.27) that

$$
- \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(2)} \tilde{\mu}^{(1)} \theta' dz = \Delta d^{(1)} \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(2)} (\theta')^2 dz = \frac{\sigma}{2} D^{(0)} \Delta d^{(1)}.
$$

(A.40)

Using (A.17), (A.33b) and (A.24), we can proceed in the same way as we obtain (A.27) and yield

$$
- \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(3)} \tilde{\mu}^{(0)} \theta' dz = \Delta d^{(0)} \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(3)} (\theta')^2 dz
$$

$$
= \Delta d^{(1)} \int_{\mathbb{R}} \partial_z \left( 2 \nabla_x \partial_z \tilde{\phi}^{(2)} \cdot \nabla d^{(0)} + \partial_z \tilde{\phi}^{(2)} \Delta d^{(0)} + \partial_z \Delta d^{(2)} + \tilde{\mu}^{(2)} \right) \theta' dz
$$

$$
= - \Delta d^{(0)} \int_{\mathbb{R}} \tilde{\mu}^{(2)} \theta'' dz.
$$

This combined with (A.36) leads to

$$
- \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(3)} \tilde{\mu}^{(0)} \theta' dz
$$

$$
= - \Delta d^{(0)} \int_{\mathbb{R}} \left( - \Delta d^{(2)} \theta'(z) + D^{(1)} z \theta'(z) + \tilde{\Psi}^{(0)}(z, x, t) \right) \theta'' dz
$$

$$
= \frac{\sigma}{2} D^{(1)} \Delta d^{(0)} - \Delta d^{(0)} \int_{\mathbb{R}} \tilde{\Psi}^{(0)}(z, x, t) \theta'' dz.
$$

(A.41)

We continue treating the terms on the right hand side of (A.33a). It follows from (A.17), (A.22), and (A.36) that

$$
\int_{\mathbb{R}} \left( 2 \nabla_x \partial_z \tilde{\mu}^{(2)} \cdot \nabla d^{(0)} + 2 \nabla_x \partial_z \tilde{\mu}^{(0)} \cdot \nabla d^{(2)} + 2 \nabla_x \partial_z \tilde{\mu}^{(1)} \cdot \nabla d^{(1)} \right) \theta' dz
$$

$$
= \sigma \left( \nabla D^{(1)} \cdot \nabla d^{(0)} + \nabla D^{(0)} \cdot \nabla d^{(1)} - 2 \nabla d^{(0)} \cdot \nabla \tilde{\Psi}^{(0)}(z, x, t) \theta'' dz.
$$
Moreover, we have the following two identities:

$$
\int_{\mathbb{R}} \left( \partial_z \mu^{(2)} \Delta d^{(0)} + \partial_z \tilde{\mu}^{(0)} \Delta d^{(2)} + \partial_z \tilde{\mu}^{(1)} \Delta d^{(1)} \right) \theta' \, dz
= \frac{\sigma}{2} \left( D^{(1)} \Delta d^{(0)} + D^{(0)} \Delta d^{(1)} - \Delta d^{(0)} \right) \int_{\mathbb{R}} \tilde{\psi}^{(0)}(z, x, t) \theta'' \, dz, \quad (A.42)
$$

$$
\int_{\mathbb{R}} \left( - \partial_z \theta \partial_t d^{(1)} + \Delta_s \tilde{\mu}^{(1)} \right) \theta' \, dz = -\sigma \left( \partial_t d^{(1)} + \Delta^2 d^{(1)} \right). \quad (A.43)
$$

Therefore, using the notation (A.39), we deduce that $d^{(1)}$ satisfies (A.38) and

$$
\Xi^{(0)} = -\frac{2}{\sigma} \nabla d^{(0)} \cdot \int_{\mathbb{R}} \nabla \tilde{\psi}^{(0)}(z, x, t) \theta'' \, dz - \frac{2}{\sigma} \Delta d^{(0)} \int_{\mathbb{R}} \tilde{\psi}^{(0)}(z, x, t) \theta'' \, dz.
$$

To determine $d^{(1)}$ and $\chi^{(1)}$ so that (A.38) holds, we need the following result:

**Corollary A.7.** The following equation of $d^{(1)}$ has a local in time classical solution:

$$
\mathcal{G}_1 d^{(1)} = \sigma^{-1} \sigma \chi^{(0)} d^{(1)} + \Xi^{(0)} \text{ on } \Gamma^0. \quad (A.44)
$$

Moreover, if we define $\chi^{(1)}$ by

$$
\chi^{(1)} \triangleq \begin{cases} 
\sigma(\sigma)^{-1} \left( \mathcal{G}_1 d^{(1)} - \sigma^{-1} \sigma \chi^{(0)} d^{(1)} - \Xi^{(0)} \right) / d^{(0)} & \text{ in } \Gamma^0 \setminus (3\delta), \\
\sigma(\sigma)^{-1} \nabla \left( \mathcal{G}_1 d^{(1)} - \sigma^{-1} \sigma \chi^{(0)} d^{(1)} - \Xi^{(0)} \right) \cdot \nabla d^{(0)} & \text{ on } \Gamma^0, 
\end{cases} \quad (A.45)
$$

then (A.38) holds in $\Gamma^0 (3\delta)$.

**Proof.** Note that $d^{(1)}$ might not fulfill (A.44) in $\Gamma^0 (3\delta)$. Since $\partial_t d^{(1)} = 0$ (see (2.2)), it suffices to determine $d^{(1)}$ on $\Gamma^0$ and then extends constantly in the normal direction. Using (6.9) we can convert mixed derivatives of $d^{(1)}$ into tangential ones. This combined with (A.30) and (A.44) yields

$$
\partial_t d^{(1)} + \Delta^2 d^{(1)} = \nabla \nabla d^{(0)} : \nabla^2 \Delta d^{(1)} = \Xi(d^{(1)}), \quad (A.46)
$$

where $\Xi$ is a generic term that includes at most third-order (tangential) derivatives of $d^{(1)}$. Using (6.9), (2.9) and (2.11) yields

$$
\Delta^2 d^{(1)} = \nabla \nabla d^{(0)} : \nabla^2 \Delta d^{(1)}
= \text{div}_{\Gamma^0} (\nabla \Delta d^{(0)}) = \Delta^2_{\Gamma^0} d^{(1)} + \left( \text{div}_{\Gamma^0} n \right) \partial_t (\Delta d^{(1)}). \quad (A.47)
$$

So we can write (A.46) as (see [2] for similar arguments)

$$
\partial_t d^{(1)} + \Delta^2_{\Gamma^0} d^{(1)} = \Xi(d^{(1)}). \quad (A.48)
$$

This is a surface evolutionary equation and has a local in time smooth solution.
With Definition A.4, we set the following statements indexed by $K$:

\[ A_K : \tilde{\phi}(i) \text{ depends on terms of order up to } (i - 2); \ 2 \leq i \leq K + 1, \]  
\[ B_K : \tilde{\mu}(i) = -\Delta d(i)\theta' + D(i-1)z\theta' + \tilde{\Psi}^{(i-2)} \text{ for } 2 \leq i \leq K, \]  
\[ C_K : \tilde{\mu}(K+1) = \mu_{K+1}(x, t)\theta' + D(K)z\theta' + \tilde{\Psi}^{(K-1)}, \]  
\[ D_K : (d^{(K)}, \chi^{(K)}) \text{ depend on terms up to order } (K - 1) \text{ through} \]

\[ \mathcal{G}_K d^{(K)} = \sigma^{-1}\chi \left( \chi^{(0)}d^{(K)} + \chi^{(K)}d^{(0)} \right) + \mathcal{E}^{(K-1)}, \]  

where $D(i)$ is defined by (3.4), $\tilde{\Psi}^{(K)}$ satisfies the decay property (A.3), and

\[ \mathcal{G}_K d^{(K)} = \partial_t d^{(K)} + \Delta^2 d^{(K)} - \sum_{\ell=0}^K \left( \nabla D^{(\ell)} \cdot \nabla d^{(K-\ell)} + D^{(\ell)} \Delta d^{(K-\ell)} \right), \]  
\[ \chi^{(K)} = \begin{cases} \sigma^{-1} \left( \mathcal{G}_K d^{(K)} - \sigma^{-1}\chi^{(0)}d^{(K)} - \mathcal{E}^{(K-1)} \right) / d^{(0)} & \text{in } \Gamma^0 (3\delta) \setminus \Gamma^0, \\ \sigma^{-1} \nabla \left( \mathcal{G}_K d^{(K)} - \sigma^{-1}\chi^{(0)}d^{(K)} - \mathcal{E}^{(K-1)} \right) \cdot \nabla d^{(0)} & \text{in } \Gamma^0. \end{cases} \]  

Lemma A.8. The statements $(A_1, B_1, C_1)$ and $(A_2, B_2, C_2, D_1)$ are valid.

Proof. Recall the results in previous subsections. Using $d^{(1)}$ we can determine $\tilde{\mu}^{(1)}$ through (A.22). Using $d^{(2)}$ determined by (A.50) with $K = 2$, we obtain $\tilde{\mu}^{(2)}$ by (A.36) and $\mathcal{G}_2^{(3)}$ by solving (A.33b). Finally we can rewrite (A.33a) as

\[ \mathcal{L} \tilde{\mu}^{(3)} = -2\theta'' D^{(2)} + \tilde{\Psi}^{(1)}, \text{ where } D^{(2)} = \sum_{0 \leq \ell \leq 2} \left( \nabla \Delta d^{(\ell)} \cdot \nabla d^{(2-\ell)} + \frac{1}{2} \Delta d^{(\ell)} \Delta d^{(2-\ell)} \right), \]

and $\tilde{\Psi}^{(1)}$ satisfies (A.1). Applying (A.4) yields

\[ \tilde{\mu}^{(3)}(z, x, t) = \mu_3(x, t)\theta'(z) + D^{(2)}(x, t)z\theta'(z) + \tilde{\Psi}^{(1)}(z, x, t). \]  

where $\tilde{\Psi}^{(1)}$ satisfies (A.1), and $\mu_3(x, t)$ shall be determined by the $\varepsilon^4$-scale.
We argue by induction on $K$. Assuming $(A_K, B_K, C_K, D_{K-1})$, we substitute (A.6) into (A.11)–(A.12) and use (A.9) and (A.18) to sort all terms of $e^{K+2}$-scale:

\[
\begin{align*}
\mathcal{L}_{\tilde{\mu}}(K+2) &= - \sum_{2 \leq i \leq K+2} \left( f'''(\theta) \tilde{\phi}^{(i)}(\gamma) + g^*_{i-1}(\tilde{\phi}(0), \ldots, \tilde{\phi}^{(i-1)}) \right) \tilde{\mu}(K+2-i) \\
&\quad + 2 \sum_{0 \leq i \leq K+1} \nabla_x \partial_z \tilde{\mu}^{(i)} \cdot \nabla \tilde{d}(K+1-i) + \sum_{0 \leq i \leq K+1} \partial_z \tilde{\mu}^{(i)} \Delta \tilde{d}(K+1-i) \\
&\quad - \sum_{1 \leq i \leq K} \partial_z \tilde{\phi}^{(i)} \partial_z \tilde{d}(K-i) - \partial_z \theta \partial_z \di(K) + \Delta_x \tilde{\mu} - \partial_t \tilde{\phi}(K-1) \\
&\quad + \left( \chi(0) \tilde{d}(K) + \sum_{1 \leq i \leq K-1} \chi^{(i)} \tilde{d}(K-i) + \chi^{(K)} \tilde{d}(0) \right) \eta' - \chi^{(K-1)} \zeta \eta', \\
&\quad (A.54a) \\
\mathcal{L}_{\tilde{\phi}}(K+2) &= - g^*_{K+1}(\tilde{\phi}(0), \ldots, \tilde{\phi}(K+1)) + 2 \sum_{2 \leq i \leq K+1} \nabla_x \partial_z \tilde{\phi}^{(i)} \cdot \nabla \tilde{d}(K+1-i) \\
&\quad + \theta' \Delta \tilde{d}(K+1) + \sum_{2 \leq i \leq K+1} \partial_z \tilde{\phi}^{(i)} \cdot \Delta \tilde{d}(K+1-i) + \tilde{\mu}(K+1) + \Delta_x \tilde{\phi}(K). \\
&\quad (A.54b)
\end{align*}
\]

Using $A_K$ and $C_K$, we can write (A.54b) as

\[
\mathcal{L}_{\tilde{\phi}}(K+2) = \theta' \Delta \tilde{d}(K+1) + \mu_{K+1}(x, t) \theta' + D^{(K)} \zeta \theta' + \tilde{\Psi}(K-1). \\
(A.55)
\]

To fulfill the compatibility condition (A.2), we choose $\mu_{K+1} = - \Delta \tilde{d}(K+1) + \tilde{\Psi}(K-1)$. This together with $C_K$ implies $B_{K+1}$, and reduces (A.55) to the following equation, which leads to $A_{K+1}$:

\[
\mathcal{L}_{\tilde{\phi}}(K+2) = D^{(K)} \zeta \theta' + \tilde{\psi}(K-1). \\
(A.56)
\]

**Proposition A.9.** The equation (A.54a) can be written as

\[
\mathcal{L}_{\tilde{\mu}}(K+2) = -2 \sum_{\ell = 0, K+1} \nabla \Delta \tilde{d}(\ell) \cdot \nabla \tilde{d}(K+1-\ell) \theta'' - 2 \Delta \tilde{d}(0) \Delta \tilde{d}(K+1) \theta'' + \tilde{\psi}(K), \\
(A.57)
\]

and its compatibility condition is guaranteed by $D_K$.

**Proof.** We consider the right hand side of (A.54a). Using (A.18), (A.49a) and (A.49b),

\[
\begin{align*}
&\quad - \sum_{2 \leq i \leq K+2} \left( f'''(\theta) \tilde{\phi}^{(i)}(\gamma) + g^*_{i-1}(\tilde{\phi}(0), \ldots, \tilde{\phi}^{(i-1)}) \right) \tilde{\mu}(K+2-i) \\
&= - f'''(\theta) \tilde{\phi}(2) \tilde{\mu}(K) - f'''(\theta) \tilde{\phi}(K+2) \tilde{\mu}(0) - g^*_{K+1}(\tilde{\phi}(0), \ldots, \tilde{\phi}(K+1)) \\
&\quad - \sum_{3 \leq i \leq K+1} \left( f'''(\theta) \tilde{\phi}^{(i)}(\gamma) + g^*_{i-1}(\tilde{\phi}(0), \ldots, \tilde{\phi}^{(i-1)}) \right) \tilde{\mu}(K+2-i) \\
&= \Delta \tilde{d}(K) f'''(\theta) \tilde{\phi}(2) \theta' - f'''(\theta) \tilde{\phi}(K+2) \tilde{\mu}(0) + \tilde{\psi}(K-1)
\end{align*}
\]
In a similar way,

\[
2 \sum_{0 \leq \ell \leq K+1} \nabla_x \partial z \tilde{\mu}^{(i)} \cdot \nabla d^{(K+1-i)} = 2 \nabla_x \partial z \tilde{\mu}^{(0)} \cdot \nabla d^{(K+1)} + 2 \nabla_x \partial z \tilde{\mu}^{(1)} \cdot \nabla d^{(K)} + 2 \nabla_x \partial z \tilde{\mu}^{(K)} \cdot \nabla d^{(1)} + 2 \sum_{2 \leq i \leq K-1} \nabla_x \partial z \tilde{\mu}^{(i)} \cdot \nabla d^{(K+1-i)} = -2 \sum_{\ell=0,1,K,K+1} \nabla \Delta d^{(\ell)} \cdot \nabla d^{(K+1-i)} \theta'' + 2 \left( \nabla D^{(K)} \cdot \nabla d^{(0)} + \nabla D^{(0)} \cdot \nabla d^{(K)} \right) (z \theta')' + \tilde{\Psi}^{(K-1)},
\]

\[
= \partial z \tilde{\mu}^{(0)} \Delta d^{(K+1)} + \partial z \tilde{\mu}^{(1)} \Delta d^{(K)} + \sum_{2 \leq i \leq K-1} \partial z \tilde{\mu}^{(i)} \Delta d^{(K+1-i)} + \partial z \tilde{\mu}^{(K)} \Delta d^{(1)} + \partial z \tilde{\mu}^{(K+1)} \Delta d^{(0)} = -2 (\Delta d^{(0)} \Delta d^{(K+1)} + \Delta d^{(1)} \Delta d^{(K)}) \theta'' + (D^{(0)} \Delta d^{(K)} + D^{(K)} \Delta d^{(0)}) (z \theta')' + \tilde{\Psi}^{(K-1)}.
\]

Finally,

\[
= - \sum_{1 \leq i \leq K} \partial z \tilde{\Phi}^{(i)} \partial_t d^{(K-i)} - \partial z \theta \partial_t d^{(K)} + \Delta x \tilde{\mu}^{(K)} - \partial_t \tilde{\psi}^{(K-1)} + \chi^{(0)} d^{(K)} + \sum_{1 \leq i \leq K-1} \chi^{(i)} d^{(K-i)} + \chi^{(K)} d^{(0)}) \eta' - \chi^{(K-1)} \eta' = - (\partial_t d^{(K)} + \Delta^2 d^{(K)}) \theta' + \eta' (\chi^{(0)} d^{(K)} + \chi^{(K)} d^{(0)}) + \tilde{\psi}^{(K-1)}.
\]

The above four results imply (A.57). They also imply the compatibility condition

\[
\sigma (\partial_t d^{(K)} + \Delta^2 d^{(K)}) = \Delta d^{(K)} \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(2)}(\theta')^2 d\tau - \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(K+2)} \tilde{\mu}^{(0)} \theta' d\tau + \int_{\mathbb{R}} \tilde{\psi}^{(K-1)} \theta' d\tau + \sigma \left( \nabla D^{(K)} \cdot \nabla d^{(0)} + \nabla D^{(0)} \cdot \nabla d^{(K)} + \frac{1}{2} (D^{(0)} \Delta d^{(K)} + D^{(K)} \Delta d^{(0)}) \right).
\]

(A.58)

It remains to calculate the first two terms on the right hand side of (A.58). Using (A.27) yields

\[
\Delta d^{(K)} \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(2)}(\theta')^2 d\tau = \frac{\sigma}{2} D^{(0)} \Delta d^{(K)}.
\]

(A.59)
With the aid of \((A.17)\) and \((A.15)\) we have
\[
- \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(K+2)}(\tilde{\mu}(0) \theta') dz = \Delta d(0) \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(K+2)}(\theta')^2 dz
\]
\[
= \Delta d(0) \int_{\mathbb{R}} \partial_z (\mathcal{L} \tilde{\phi}^{(K+2)}) \theta' - (\mathcal{L} \partial_z \tilde{\phi}^{(K+2)})(\theta') dz
\]
\[
= -\Delta d(0) \int_{\mathbb{R}} \mathcal{L} \tilde{\phi}^{(K+2)} \theta'' dz.
\]
In view of \((A.56)\), the above two formulas together lead to
\[
- \int_{\mathbb{R}} f'''(\theta) \tilde{\phi}^{(K+2)}(\tilde{\mu}(0) \theta') dz = -\Delta d(0) \int_{\mathbb{R}} \left( D^{(K)} z \theta' + \tilde{\Psi}^{(K-1)} \right) \theta'' dz
\]
\[
= \frac{\sigma}{2} D^{(K)} \Delta d(0) + \Xi^{(K-1)}.
\]
Substituting \((A.59)\) and the above formula into \((A.58)\) leads to \(D_K\).

Using \((3.4)\), we can write \((A.57)\) by
\[
\mathcal{L} \tilde{\mu}^{(K+2)} = -2D^{(K+1)} \theta'' + \tilde{\Psi}^{(K)}(z, x, t).
\]
Applying Lemma \(A.1\) to the above equation implies \(C_{K+1}\). To conclude \(D_K\), we need:

**Corollary A.10.** The following equation of \(d^{(K)}\) has a local in time classical solution
\[
\mathcal{G}_K d^{(K)} = \sigma^{-1} \sigma^{(0)} d^{(K)} + \Xi^{(K-1)} \text{ on } \Gamma^0.
\]
Moreover, if we define \(\chi^{(K)}\) by \((A.51)\), then \(D_K\) holds in \(\Gamma^0(3\delta)\).

Note that the local in time solution follows from the same argument for Corollary A.7. So we have shown \((A_{K+1}, B_{K+1}, C_{K+1}, D_K)\) and the induction for \((A.49)\) is completed.

**Proposition A.11.** Assume \((1.5)\) has a smooth solution \(\Gamma^0 \text{ within } [0, T],\) starting from a smooth closed hypersurface \(\Gamma^0_0 \subset \mathbb{R}^N,\) and let \(d^{(0)}\) be the signed-distance, defined in \(\Gamma^0(3\delta)\). Then we can construct the inner expansion with Ansatz \((A.6)\) so that for \(i \geq 1\)
\[
D^\alpha_x D^\beta_t D^\gamma_z \hat{\phi}^{(i)}(z, x, t) = O(e^{-C|z|}), \quad D^\alpha_x D^\beta_t D^\gamma_z \hat{\mu}^{(i)}(z, x, t) = O(e^{-C|z|}),
\]
\[(A.62)\]
as \(z \to \pm \infty\) for \((x, t) \in \Gamma^0(3\delta)\) and \(0 \leq \alpha, \beta, \gamma \leq 2\). Moreover,
\[
\hat{\phi}^I_a(x, t) = \sum_{0 \leq i \leq k} \varepsilon^i \hat{\phi}^{(i)}(z, x, t) \bigg|_{z = \frac{d(k|x, t)}{\varepsilon}} = \sum_{0 \leq i \leq k} \varepsilon^i \hat{\mu}^{(i)}(z, x, t) \bigg|_{z = \frac{d(k|x, t)}{\varepsilon}},
\]
\[(A.63)\]
satisfies for \((x, t) \in \Gamma^0(3\delta)\)
\[
\varepsilon^3 \partial_t \hat{\phi}^I_a = \varepsilon^2 \Delta \hat{\phi}^I_a - f''(\hat{\phi}^I_a) \hat{\mu}^I_a + O(\varepsilon^k),
\]
\[
\varepsilon \hat{\mu}^I_a = -\varepsilon^2 \Delta \hat{\phi}^I_a + f'(\hat{\phi}^I_a) + O(\varepsilon^k).
\]
\[(A.64)\] \[(A.65)\]
Proof. It follows from (A.6) and chain-rule that
\[- \left( \varepsilon^3 \partial_t \hat{\phi}_a^I - \varepsilon^2 \Delta \hat{\mu}_a^I + f''(\hat{\phi}_a^I) \hat{\mu}_a^I \right)\]
\[= \partial_z^2 \hat{\mu} \left| \nabla d^{[k]} \right|^2 - f''(\hat{\phi}) \hat{\mu} + 2 \varepsilon \nabla \partial_z \hat{\mu} \cdot \nabla d^{[k]} + \varepsilon \partial_z \hat{\phi}_a \Delta d^{[k]} - \varepsilon^2 \partial_z \hat{\phi}_a \partial_t d^{[k]} + \varepsilon^2 \Delta_z \hat{\mu} \hat{\mu} - \varepsilon^3 \partial_t \hat{\phi}_a, \quad \text{with } \varepsilon = d^{[k]}(x, t)/\varepsilon.\]

If we replace $d_\varepsilon$ by $d^{[k]}$ in (A.10), and compare it with the above formula, then we arrive at (A.64) and the details are omitted.

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