On power maps over weakly periodic rings

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Abstract

A ring $R$ is called weakly periodic if every $x \in R$ can be written in the form $x = a + b$, where $a$ is nilpotent and $b^m = b$ for some integer $m > 1$. The aim of this note is to consider when a nonzero nilpotent element $r$ is the period of some power map $f(x) = x^n$, in the sense that $f(x + r) = f(x)$ for all $x \in R$, and how this relates to the structure of weakly periodic rings.

In particular, we provide a new proof of the fact that weakly periodic rings with central and torsion nilpotent elements are periodic commutative torsion rings. We also prove that $x^n$ is periodic over such rings whenever $n$ is not coprime with each of the additive orders of the nilpotent elements. These are in fact the only periodic power maps over finite commutative rings with unity. Finally, we describe and enumerate the distinct power maps over Corbas $(p, k, \phi)$-rings, Galois rings, $\mathbb{Z}/n\mathbb{Z}$, and matrix rings over finite fields.

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1 Introduction

Throughout, $R$ is a ring (not necessarily commutative or unital) and $R^+$ is the additive group of $R$. By order of a ring element we will always mean its group-theoretic order as a member of $R^+$. The set of nilpotent elements of $R$ will be denoted by $\text{Nil}(R)$. The index of a nilpotent $x$ is the smallest positive integer $n$ such that $x^n = 0$, and the index of $\text{Nil}(R)$ is the largest index, if extant, among all elements of $\text{Nil}(R)$. Any additional terminology and notation not explicitly defined herein are standard in the literature.
A ring $R$ is called periodic if for each $x \in R$ the set \( \{ x^n : n \in \mathbb{N} \} \) is finite. Equivalently, for each $x \in R$, there are positive integers $m(x)$ and $n(x)$ such that $x^{m(x)+n(x)} = x^{m(x)}$. If the aforementioned $n(x)$ can be taken to be constant across all $x$, then the smallest such constant permissible is called the exponential period of $R$, which we will denote by $\mu_1 := \mu_1(R)$, and the onset of exponential periodicity of $R$, which we will denote by $\mu_0 := \mu_0(R)$, is given by:

$$
\mu_0 = \max_{x \in R} \min\{ m \in \mathbb{N} : x^{m+\mu_1} = x^m \},
$$

provided this maximum exists. Otherwise, we say that $\mu_1$ and $\mu_0$ are infinite. Notice that $\mu_0$ is no smaller than the index of Nil($R$) and that, for rings with unity, $\mu_1$ is no smaller than the exponent of the group of units $R^\times$.

An element $x \in R$ is called potent if $x^m = x$ for some integer $m > 1$. Let Pot($R$) denote the set of potent elements of $R$. A ring $R$ is called weakly periodic if $R = \text{Nil}(R) + \text{Pot}(R)$. We remark that if $R = \text{Nil}(R) \cup \text{Pot}(R)$, then $\mu_0$ is clearly the index of Nil($R$). If $R = \text{Pot}(R)$, then $R$ is called a $J$-ring. It is well known that every $J$-ring is commutative [19].

Some obvious examples of periodic rings are finite rings, Boolean rings, and nil rings. In fact, finite fields and Boolean rings are $J$-rings. Bell [4] proved that periodic rings are weakly periodic, but the status of the converse is still unresolved. Periodicity has been established for several special classes of weakly periodic rings though (e.g. [1], [5], [6], [13], [24]).

The power maps over a ring are the functions of the form $f(x) = x^n$ for some fixed $n \in \mathbb{N}$. In this note, we assess what power map “oscillations” may inform us about the arithmetical structure of weakly periodic rings. In particular, we will show that if the nilpotent elements of $R$ are additively torsion and multiplicatively central, then each one is a period of some power map. This leads us to an alternative proof of the fact that weakly periodic rings with this property are periodic commutative torsion rings. A mild generalization of this was proved by Bell and Tominaga in [6], who relied on Pierce decompositions and Chacron’s periodicity criterion [9]. We will also prove that $x^n$ is periodic over such rings whenever $n$ is not coprime with each of the orders of the nilpotent elements. This will allow us to derive tight lower and upper bounds for the number of periodic power maps over commutative rings when $\mu_1$ and $\mu_0$ are both finite. Assorted examples are peppered throughout to help illustrate the results. For ease of navigation, Table 1 catalogs the most instructive and concrete examples featured in this paper.

### 2 Periodic Power Maps

If $G$ is an additive group and $X$ is a set, then a function $\alpha : G \to X$ is periodic if there exists an $h \in G \setminus \{0\}$ such that $\alpha(g + h) = \alpha(g)$ for all $g \in G$. Such an element $h$ is called a period of $\alpha$. It is easy to see that the periods of a function together with 0 form a subgroup of $G$. We will call this subgroup Per($f$). Clearly, a power map over a ring $R$ is periodic as a function on $R^+$ only if Per($f$) $\subseteq$ Nil($R$). On the other hand, nilpotence alone does not guarantee that an element is the period of some power map. Here are a couple counterexamples.
Table 1: Periodic power map enumerations for some rings

| Ring $R$                          | $\mu_1(R)$ | $\mu_0(R)$ | # of periodic power maps                                                                 |
|-----------------------------------|------------|------------|----------------------------------------------------------------------------------------|
| Corbas $(p, k, \phi)$-ring        | $p(p^k - 1)$ | 2          | $p^k - 1$ if $\phi$ is the identity, 0 otherwise                                       |
| $R_1 \times R_2$, $R_1$ reduced, $R_2 \neq 0$ nil | $\mu_1(R_1)$ | $\mu_0(R_2) = \text{index of } R_2$ | at least $\mu_1(R_1)$, at most $\mu_1(R_1) + \mu_0(R_2) - 2$ |
| $\mathbb{Z}/n\mathbb{Z}$         |            |            |                                                                                        |
| $\mathcal{M}_n(F_q)$              |            |            |                                                                                        |

Example 2.1. Let $R = \prod_{i \in \mathbb{N}} \mathbb{Z}/p_i^2\mathbb{Z}$, where $p_i$ is the $i$th prime number. The sequence $(p_i)_{i \in \mathbb{N}}$ is nilpotent of index 2. Therefore, if $(x_i)_{i \in \mathbb{N}} \in R$, then

$$((x_i + p_i)^n)_{i \in \mathbb{N}} = (x_i^n + np_ix_i^{n-1})_{i \in \mathbb{N}}$$

(2.1)

for all $n \in \mathbb{N}$. Since $(p_i)_{i \in \mathbb{N}}$ has infinite order, there is no integer $n$ such that $x_i^n + np_ix_i^{n-1} = x_i^n$ for every $i \in \mathbb{N}$.

Example 2.2. If $p$ is a prime number, $k$ is a positive integer, and $\phi$ is an automorphism of the Galois field $F_{p^k}$, then the Corbas $(p, k, \phi)$-ring is the ring $R$ in which $R^+ = F_{p^k} \oplus F_{p^k}$ and the ring multiplication · is defined by

$$(a, b) \cdot (c, d) = (ac, ad + b\phi(c)).$$

It is straightforward to confirm that $R$ satisfies the following properties:

- For all $n \in \mathbb{N}$,

$$ (a, b)^n = \left(a^n, b\phi(a^{n-1}) \sum_{j=0}^{n-1} a^j \phi(a^{-j}) \right). $$

(2.2)

If $\phi$ is not the identity automorphism and $a \neq 0$, then this can be simplified to

$$(a, b)^n = \left(a^n, b\phi(a^{n-1}) \frac{a^n \phi(a^{-n}) - 1}{\phi(a^{-1}) - 1} \right), $$

(2.3)

- $\text{Nil}(R) = 0 \times F_{p^k}$ has index 2,
\[ R \text{ is commutative if and only if } \phi \text{ is the identity.} \]

Hence, for each \( a, b, y \in \mathbb{F}_p^k \) and \( n \in \mathbb{N} \),

\[
((a, b) + (0, y))^n = \left( a^n, (b + y) \sum_{j=0}^{n-1} a^j \phi(a^{n-j-1}) \right). \tag{2.4}
\]

If \( \phi \) is not the identity, then 0 is the only value of \( y \) for which \( ((a, b) + (0, y))^n \) is identically \( (a, b)^n \). In this case, \( R \) has no periodic power maps whatsoever.

Unboundedness of order and noncommutativity are responsible for the pathologies of Examples 2.1 and 2.2, respectively. However, as long as \( \text{Nil}(R) \) circumvents these traits, we can promise that each nonzero nilpotent element is a period of some power map. We will refer to such rings as nilperiod.

**Theorem 2.3.** If \( R \) is a ring in which every nilpotent element is central and torsion, then \( R \) is nilperiod.

**Proof.** Let \( r \in \text{Nil}(R) \), and let \( i \) and \( j \) be the index and order, respectively, of \( r \). If we set \( n = (i - 1)!j \), then for all \( x \in R \),

\[
(x + r)^n = x^n + \sum_{k=1}^{i-1} \binom{n}{k} r^k x^{n-k}. \tag{2.5}
\]

Since the binomial coefficient \( \binom{n}{k} \) is a multiple of \( n/\gcd(n, k) \) for each integer \( k \) (cf Problem B2 of the 2000 Putnam competition [21]) and \( n/\gcd(n, k) \) is a multiple of \( j \) for each \( k \in [i-1] \), the sum in (2.5) vanishes. Therefore \((x + r)^n = x^n \). \qed

**Corollary 2.4.** If \( R \) is a weakly periodic ring in which every nilpotent element is central and torsion, then \( R \) is a periodic commutative torsion ring.

**Proof.** Let \( x \in \text{Pot}(R) \) and \( y \in \text{Nil}(R) \) be given. By Theorem 2.3, there is an integer \( n \) such that \((x + y)^n = x^n \). Since there is also an integer \( m > 1 \) such that \( x^m = x \), we can see that

\[
(x + y)^{nm} = x^{nm} = x^n = (x + y)^n. \tag{2.6}
\]

Hence \( R \) is periodic, and because all nilpotent elements are central, we can further conclude that \( R \) is commutative (see [14]). Consequently, \( \text{Nil}(R) \) is an ideal.

To see that \( R \) is torsion, first notice that \( R/\text{Nil}(R) \) is a J-ring. As Jacobson explained in the proof of his classic “\( a^n = a \)” theorem [19], the additive group of a J-ring is torsion. This implies that for each \( x \in \text{Pot}(R) \), there is a positive integer \( j \) such that \( jx \in \text{Nil}(R) \). Since every nilpotent element of \( R \) is torsion, it follows that \( R \) itself is torsion as a whole. \qed

It may be interesting to figure out the extent to which the conditions of Theorem 2.3 can be loosened. Here is an example demonstrating that either of the hypotheses in Theorem 2.3 can be resoundingly defied.
**Example 2.5.** Consider the direct product \( R = R_1 \times R_2 \), where \( R_1 \) is a reduced ring and \( R_2 \) is a nil ring of finite index, say \( n \). Then \( \text{Nil}(R) = 0 \times R_2 \) has index \( n \) as well. Furthermore,

\[
((a, b) + (0, y))^m = (a^m, 0) = (a, b)^m
\]

(2.7)

for all \( a \in R_1, \ b, y \in R_2, \) and integers \( m \geq n \).

Take \( R_2 \) to be a torsion-free abelian group equipped with the zero multiplication to see that nilperiod rings can be entirely devoid of torsion nilpotent elements. Further still, the existence of noncommutative nil rings rules out the necessity of central nilpotent elements. However, nilperiod rings seem constrained enough to compel \( \text{Nil}(R) \) to be an ideal. We call \( R \) an NI-ring if \( \text{Nil}(R) \) is an ideal. Equivalently, \( \text{Nil}(R) \) coincides with the upper nilradical \( \text{Nil}^*(R) \) (i.e. the sum of all nil ideals of \( R \)).

**Conjecture 2.6.** Every nilperiod ring is an NI-ring.

If Conjecture 2.6 is correct, then noncommutative weakly periodic nilperiod rings are “almost” commutative in the sense that their commutator ideals are nil. Indeed, for if \( R \) is is a weakly periodic NI-ring, then \( R/\text{Nil}(R) \) is commutative due to being a \( J \)-ring. Moreover, because the Jacobson radical \( J(R) \) of a weakly periodic ring is nil, we could report that \( J(R) = \text{Nil}(R) = \text{Nil}^*(R) \) and that \( R \) is ultimately periodic [13] Lemma 1].

Of course in noncommutative rings, \( \text{Nil}^*(R) \) typically differs from the lower nilradical \( \text{Nil}_r(R) \), that is, the intersection of all the prime ideals of \( R \). A ring \( R \) is called 2-primal if \( \text{Nil}_r(R) = \text{Nil}(R) \). Marks [22] provided a thorough list of conditions on noncommutative rings that enforce 2-primality together with their interdependencies, but none involve weak periodicity. Furthermore, being 2-primal is a necessary and sufficient condition for a ring with bounded nilpotency index to be an NI-ring [17] Proposition 1.4]. We should therefore suspect that a weakly periodic NI-ring may fail to be 2-primal provided its nilpotency index is unbounded.

**Example 2.7.** Let \( S \) be a finite 2-primal ring, \( n \in \mathbb{N} \), and \( R_n \) be the \( 2^n \times 2^n \) upper triangular matrix ring over \( S \). Each \( R_n \) is finite and thus periodic. Proposition 2.5 of [7] further implies that each \( R_n \) is 2-primal, and so they are NI-rings. Now embed each \( R_n \) into \( R_{n+1} \) via the monomorphism \( \sigma \) defined by

\[
\sigma(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.
\]

Then \( \mathcal{D} = \langle R_n, \sigma_{nm} \rangle \), with \( \sigma_{nm} = \sigma^{n-m} \) whenever \( n \leq m \), is a direct system over \( \mathbb{N} \). Set \( R = \lim_{\rightarrow} R_n \), the direct limit of \( \mathcal{D} \). Since \( R = \bigcup_{n=1}^{\infty} R_n \) is the union of periodic rings, \( R \) is itself periodic. However, [17] Proposition 1.1 and Example 1.2] explains why \( R \) fails to be 2-primal, even though the direct limit of a direct system of NI-rings is itself an NI-ring.

We can at least verify Conjecture 2.6 for polynomial identity algebras over fields of characteristic zero. The proof adapts techniques employed by Herstein in [15] and [16].
Theorem 2.8. If $R$ is a nilperiod PI-algebra over a field $K$ of characteristic zero, then $R$ is an NI-ring.

Proof. Let $a, b \in \text{Nil}(R)$. Then there are positive integers $\ell$ and $m$ such that $(x + a)^m = x^m$ and $(x + b)^\ell = x^\ell$ for all $x \in R$. Consequently, $(a + b)^{\ell m} = 0$, and so $a + b$ is nilpotent.

Next, suppose $c \in \text{Nil}(R)$, and let $r \in R$ be arbitrary. Let $S$ be the subalgebra of $R$ generated by $c$ and $r$. Since $S$ is a finitely-generated PI-algebra, $J(S)$ is nil due to Amitsur’s Nullstellensatz [3, 8].

Now assume, to the contrary, that $c \notin J(S)$. Then the coset $\overline{c} = (c + J(S)) \in S/J(S)$ is a nonzero nilpotent element of index, say, $j$. Because $S/J(S)$, as a semiprimitive ring, is a subdirect product of primitive rings $S_i$, each of which is a homomorphic image of $S/J(S)$, the coset $\overline{c}$ projects to a nilpotent element $\nu_i$ within each factor $S_i$. Note that not all of these $\nu_i$ can be 0, otherwise $\overline{c} = 0$, which would indicate that $c \in J(S)$.

Let $S_i$ be a factor in which $\nu_i \neq 0$, and suppose that $\nu_i$ is nilpotent of index $j_i$. Then $2 \leq j_i \leq j$, the power $\nu_i^{j_i-1}$ is nilpotent of index 2, and $\overline{c^{j_i-1}} \neq 0$, which implies that $c^{j_i-1} \notin J(S)$. But since $c^{j_i-1}$ is then a nonzero nilpotent element of $R$, by hypothesis $c^{j_i-1}$ is the period of some power map $f(x) = x^{n_i}$. As a result, the factor $S_i$ inherits the identity $(x + \nu_i^{j_i-1})^{n_i} = x^{n_i}$. Furthermore, $S_i$ cannot be a division ring as it contains the nonzero nilpotent $\nu_i$. Due to Jacobson’s density theorem, $S_i$ is thus isomorphic to a dense subring of $\text{End}(V_i)$ for some vector space $V_i$ over a division ring $D_i$. Since $S_i$ is itself a PI-algebra, we may assume that it is finite-dimensional (cf [20, Lemma 5 and Theorem 1]), and so $S_i \cong M_{s_i}(D_i)$, the ring of $s_i \times s_i$ matrices with entries in $D_i$, for some integer $s_i \geq 2$. So if $I_{s_i} \in S_i$ corresponds to the $s_i \times s_i$ identity matrix, then

$$ (I_{s_i} + \nu_i^{j_i-1})^{n_i} = I_{s_i} = I_{s_i} + n_i \nu_i^{j_i-1}, \quad (2.8) $$

which implies that $\nu_i^{j_i-1}$ is torsion.

Because every $j_i$ must lie between 2 and $j$, we need only finitely many different values for the $n_i$. Hence, $\overline{c}$ is torsion. Therefore $kc \in J(S)$ for some integer $k \geq 2$. Yet since $K$ is a field of characteristic zero and $J(S)$ is closed under scalar multiplication, we find that

$$ c = (k1_K)^{-1}kc \in J(S), \quad (2.9) $$

which is at odds with our original assumption. It thus follows that $c \in J(S)$ and so $cr$ and $rc$ are in $J(S)$ as well. Since $J(S)$ is nil, both $cr$ and $rc$ are nilpotent, as required.

Corollary 2.9. If $R$ is a periodic nilperiod algebra over a field of characteristic zero with finite $\mu_1$ and $\mu_0$, then $R$ is an NI-algebra.

Proof. Since $x^{\mu_0 + \mu_1} = x^{\mu_0}$ for all $x \in R$, it follows that $R$ is a PI-algebra, and so Theorem 2.8 applies.

6
We now restrict our attention to periodic rings with finite $\mu_1$ and $\mu_0$, for which there are $\mu_0 + \mu_1 - 1$ distinct power maps over $R$. Let $\mu_P := \mu_P(R)$ denote the number of distinct periodic power maps over $R$. Pinpointing the precise ring properties that determine the value of $\mu_P$ is currently beyond our grasp. However, a lower bound for $\mu_P$ is quite tenable, provided that $R$ is commutative and $\text{Nil}(R)^+$ is a torsion group with finite exponent. In essence, nilpotent elements of simultaneously prime order and index 2 are the “fundamental” periods of the easiest-to-distinguish periodic power maps. First, we borrow some notation from number theory. The squarefree radical of a positive integer $n$, denoted $\text{rad}(n)$, is the product of the distinct prime factors of $n$, and $\omega(n)$ is the number of distinct prime factors of $n$. Despite the threat of notational confusion, we will let $\mu(n)$ be the Möbius function (because one can never get enough of the Greek letter mu).

**Theorem 3.1.** Let $R$ be a commutative ring in which the exponent of $\text{Nil}(R)^+$ is finite. If $N$ is the least common multiple of the orders of the nilpotent elements, then $f(x) = x^n$ is periodic for every integer $n \in \mathbb{N}$ that is not coprime with $N$. Accordingly,

$$
\mu_P \geq - \sum_{d \mid N \atop d \neq 1} \mu(d) \left\lfloor \frac{\mu_0 + \mu_1 - 1}{d} \right\rfloor = - \sum_{d \mid \text{rad}(N) \atop d \neq 1} (-1)^{\omega(d)+1} \left\lfloor \frac{\mu_0 + \mu_1 - 1}{d} \right\rfloor 
$$

if $\mu_1$ and $\mu_0$ are finite. This lower bound is attained if $R$ is also finite and unital.

**Proof.** By Cauchy’s theorem for abelian groups, if $p$ is a prime number that divides $N$, then $\text{Nil}(R)$ contains a necessarily nonzero element $r$ of order $p$. If the index of $r$ is $k$, then the index of $r^{k-1}$ is 2, and since $r \mid r^{k-1}$, the order of $r^{k-1}$ is also $p$. We can thus invoke the freshman’s dream to see that

$$
(x + r^{k-1})^m = (x^p + r^{(k-1)p})^m = x^{pm}
$$

for every $x \in R$ and $m \in \mathbb{N}$.

So if $pm \leq \mu_0 + \mu_1 - 1$, then $f(x) = x^{pm}$ is a nonrepetitive periodic power map over $R$. We can enumerate all these maps in two different ways. One is a routine application of the principle of inclusion-exclusion. The other way is to note that the number of integers no larger than $\mu_0 + \mu_1 - 1$ which are coprime to $N$ is given by

$$
\sum_{1 \leq n \leq \mu_0 + \mu_1 - 1 \atop \gcd(n,N)=1} 1 = \sum_{n=1}^{\mu_0 + \mu_1 - 1} \sum_{d \mid \gcd(n,N)} \mu(d) \quad (3.2)
$$

$$
= \sum_{d \mid N} \mu(d) \sum_{1 \leq n \leq \mu_0 + \mu_1 - 1 \atop d \mid n} 1 \quad (3.3)
$$

The only exception to this is the zero ring. This is because $\mu_0, \mu_1 \geq 1$, but the zero ring only has one mapping on it.
\[
\sum_{d \mid N} \mu(d) \left\lfloor \frac{\mu_0 + \mu_1 - 1}{d} \right\rfloor = \mu_0 + \mu_1 - 1 + \sum_{d \mid N, d \neq 1} \mu(d) \left\lfloor \frac{\mu_0 + \mu_1 - 1}{d} \right\rfloor 
\]  
(3.4)

\[
\mu_0 + \mu_1 - 1 + \sum_{d \mid \text{rad}(N), d \neq 1} (-1)^{\omega(d)} \left\lfloor \frac{\mu_0 + \mu_1 - 1}{d} \right\rfloor.
\]  
(3.5)

Lastly, let us see that these are the only periodic power maps if \( R \) is a finite commutative ring with unity. Since finite rings are trivially Artinian, \( R \cong \prod R_i \), where the \( R_i \) are finite commutative local rings, and for each \( i \) we have a natural surjective homomorphism \( \pi_i : R \to R_i \). Now suppose that \( g(x) = x^n \) is periodic and \( s \in \text{Per}(g) \setminus \{0\} \). Then \( \pi_is \neq 0_{R_i} \) for some \( i \). Furthermore,

\[
(1_{R_i} + \pi_is)^n = 1_{R_i} = 1_{R_i} + \sum_{k=1}^{n} \binom{n}{k} \pi_is^k,
\]  
(3.6)

and so

\[
\sum_{k=1}^{n} \binom{n}{k} \pi_is^k = \pi_is \left( n1_{R_i} + \sum_{k=2}^{n} \binom{n}{k} \pi_is^{k-1} \right) = 0_{R_i}.
\]  
(3.7)

Therefore \( n1_{R_i} + \sum_{k=2}^{n} \binom{n}{k} \pi_is^{k-1} = 0_{R_i} \) or is a zero divisor. Either way, \( n1_{R_i} \) must be a non-unit due to the locality of \( R_i \). It follows that \( n \) is not coprime to \( \text{char}(R_i) \), otherwise Bézout’s identity could be used to express the unit \( 1_{R_i} \) as a \( \mathbb{Z} \)-linear combination of the non-units \( n1_{R_i} \) and \( \text{char}(R_i)1_{R_i} \). However, \( \text{char}(R_i) \) must be a prime power \( p_i \), and so \( p \) is a common divisor of \( n \) and \( \text{char}(R_i) \). Since the order of \( s \) is a multiple of \( \text{char}(R_i) \), we conclude that \( n \) and the order of \( s \) are not coprime.

Observe that \( \mu_0 = 1 \) for \( J \)-rings and \( \mu_1 = 1 \) for nil rings. This quickly leads to the two following corollaries.

**Corollary 3.2.** Let \( R \) be a \( J \)-ring with finite \( \mu_1 \). Then \( R \) has \( \mu_1 \) distinct power maps, none of which are periodic.

**Corollary 3.3.** If \( R \) is a nil ring of finite index \( \mu_0 \), then \( R \) has \( \mu_0 \) distinct power maps. If, in addition, \( R \) is also commutative and of bounded torsion, then

\[
\mu_P \geq - \sum_{d \mid N, d \neq 1} \mu(d) \left\lfloor \frac{\mu_0}{d} \right\rfloor = \sum_{d \mid \text{rad}(N) + 1} (-1)^{\omega(d)+1} \left\lfloor \frac{\mu_0}{d} \right\rfloor,
\]

where the notation of Theorem 3.1 has been reprised.
4 Miscellaneous Examples

4.1 Weakly periodic rings that annihilate $\text{Nil}(R)$

Theorem 2.3 remains true if we swap out the torsionality of $\text{Nil}(R)$ for the condition that $xr = 0 = rx$ for all $x \in R$ and $r \in \text{Nil}(R)$. (The nilpotent elements are still central here.) In this case, $(x + r)^n = x^n + r^n$. So $\alpha_n(x) = x^n$ is periodic for every integer $n \geq 2$, and the periods of $\alpha_n$ are the nilpotent elements with index at most $n$. It follows that a weakly periodic ring in which $R \cdot \text{Nil}(R) = 0 = \text{Nil}(R) \cdot R$ is a commutative periodic ring, albeit not necessarily torsion.

We should emphasize that $R$ being a two-sided annihilator of $\text{Nil}(R)$ is pivotal to the argument that $R$ is nilperiod. For instance, consider the Klein four-group $\mathbb{V} = \{0, a, b, c\}$ presented additively so that $R^+ = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and endowed with the multiplication given by $0x = cx = 0$ and $ax = bx = x$ for all $x \in \mathbb{V}$. Observe that $\text{Pot}(R) = \{a, b\}$ and $\text{Nil}(R) = \{0, c\}$, and yet the power maps over $R$ are merely “quasiperiodic” over $\text{Pot}(R)$ in the sense that $(x + c)^n = x^n + c$ for all $x \in R \setminus \{0, c\}$ and $n \in \mathbb{N}$. This example due to Bell [4] is notable for the one-sided orthogonality of $\text{Nil}(R)$ and $\text{Pot}(R)$. Various aspects of such rings are discussed in [12] and [13].

4.2 Corbas $(p, k, \phi)$-rings

Let us return to Example 2.2, where

$$
(a, b)^n = \begin{cases} 
(a^n, na^{n-1}b) & \text{if } \phi \text{ is the identity automorphism,} \\
\left( a^n, b\phi(a^{n-1}) \frac{a^n \phi(a^{-n}) - 1}{\alpha \varphi(a^{-1}) - 1} \right) & \text{otherwise},
\end{cases}
$$

(4.1)

for all $a \in \mathbb{F}_p^\times$, $b \in \mathbb{F}_p^\times$, and $n \in \mathbb{N}$. Since every non-nilpotent element of $R$ is potent, $\mu_0 = 2$. To ascertain $\mu_1$, we need to calculate the smallest positive integer $\mu_1$ such that $a^{1+\mu_1} = a$ and $(1+\mu_1)a^{\mu_1} = 1$ for every $a \in \mathbb{F}_p^\times$. Recall that $\mathbb{F}_p^\times$ has characteristic $p$, and the multiplicative group $\mathbb{F}_p^\times$ is a cyclic group of order $p^k - 1$. Hence $\mu_1 = \text{lcm}(p, p^k - 1) = p(p^k - 1)$. There are thus $p^{k+1} - p + 1$ distinct power maps over $R$.

Out of these, $p^k - 1$ are periodic when $\phi$ is the identity. To directly see why, note that the only way for the equation

$$(a, b) + (0, y))^n = (a^n, na^{n-1}(b + y)) = (a, b)^n = (a^n, na^{n-1}b)$$

to hold over all of $R$ is for $n$ to be a multiple of $p$, in which case $(a, b)^n = (a^n, 0)$. This amount matches the summation derived in Theorem 3.1 since every nonzero nilpotent element of $R$ has order $p$ and so

$$
\sum_{d\mid p, d\neq 1} (-1)^{\omega(d)-1} \left[ \frac{\mu_0 + \mu_1 - 1}{d} \right] = (-1)^{\omega(p)-1} \left[ \frac{p^{k+1} - p + 1}{p} \right] = p^k - 1.
$$

(4.2)
4.3 Galois Rings

Let $R = \text{GR}(p^k, d)$, the unique Galois extension of $\mathbb{Z}/p^k\mathbb{Z}$ of degree $d$, which is a local ring of characteristic $p^k$. It is well-known that the unique maximal ideal of $R$ is the principal ideal $(p)$, which is entirely comprised of all multiples of $p$, and that every non-nilpotent element is a unit. Furthermore, $R^x \cong G_1 \times G_2$, where $G_1$ is a cyclic group of order $p^d - 1$ and

$$G_2 \cong \begin{cases} C_2 \times C_{2^{k-2}} \times (C_{2^{k-1}})^{d-1} & \text{if } p = 2 \text{ and } k \geq 3, \\ (C_{p^{k-1}})^d & \text{otherwise.} \end{cases}$$

(4.3)

If $d = 1$, then $R \cong \mathbb{Z}/p^k\mathbb{Z}$. This case is discussed in the next subsection. If $k = 1$, then $R \cong \mathbb{F}_{p^d}$, which by Corollary 3.2 has $p^d - 1$ distinct power maps, none of which are periodic. Finally, if $d, k > 1$, then $\mu_0 = k$ and $\mu_1 = \text{lcm}(p^d - 1, p^{k-1}) = p^{k-1}(p^d - 1)$. There are thus $p^{k-1}(p^d - 1) + k - 1$ distinct power maps over $R$ in this case, and since $R$ is a finite commutative ring with unity, we can apply Theorem 3.1 to see that

$$\mu_p(\text{GR}(p^k, d)) = (-1)^{\omega(p) - 1} \left[ \frac{p^{k-1}(p^d - 1) + k - 1}{p} \right] = p^{k-2}(p^d - 1) + \left\lfloor \frac{k - 1}{p} \right\rfloor.$$  

(4.4)

4.4 The integers modulo $n$

The enumeration of the distinct power maps over $\mathbb{Z}/n\mathbb{Z}$ is a folklore result, but we include it below for completeness. On the other hand, a description of the periodic power maps, while elementary, is perhaps new or at least little-known.

Let $n$ be a positive integer and let $p_1^{\beta_1}, p_2^{\beta_2}, \ldots, p_v^{\beta_v}$ be its prime factorization. By the Chinese Remainder Theorem,

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\beta_1}\mathbb{Z} \times \mathbb{Z}/p_2^{\beta_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_v^{\beta_v}\mathbb{Z}. \quad (4.5)$$

Based on (4.3) and the discussion surrounding it, $\mu_1(\mathbb{Z}/p_i^{\beta_i}\mathbb{Z}) = \lambda(p_i^{\beta_i})$, where $\lambda$ is the Carmichael function defined on prime powers $p^\beta$ by

$$\lambda(p^\beta) = \begin{cases} 2^{\beta-2} & \text{if } p = 2 \text{ and } \beta \geq 3, \\ p^{\beta-1}(p - 1) & \text{otherwise.} \end{cases}$$

It follows that the exponential period of $\mathbb{Z}/n\mathbb{Z}$ is

$$\mu_1(\mathbb{Z}/n\mathbb{Z}) = \text{lcm} \left( \lambda(p_1^{\beta_1}), \lambda(p_2^{\beta_2}), \ldots, \lambda(p_v^{\beta_v}) \right) = \lambda(n), \quad (4.6)$$

whereas the onset of exponential periodicity is $E(n) := \max \beta_i$, that is, the index of

$$\text{Nil}(\mathbb{Z}/n\mathbb{Z}) \cong \text{Nil} \left( \mathbb{Z}/p_1^{\beta_1}\mathbb{Z} \right) \times \text{Nil} \left( \mathbb{Z}/p_2^{\beta_2}\mathbb{Z} \right) \times \cdots \times \text{Nil} \left( \mathbb{Z}/p_v^{\beta_v}\mathbb{Z} \right). \quad (4.7)$$

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There are thus \( \lambda(n) + E(n) - 1 \) distinct power maps over \( \mathbb{Z}/n\mathbb{Z} \). Table 2 collects these statistics for some simple values of \( n \). This settles Blomberg and Whitmore’s conjectures for sequence A109746 on the OEIS [23].

To help us see which power maps over \( \mathbb{Z}/n\mathbb{Z} \) are periodic, we recall that \( \mathbb{Z}/p_i^\beta_i \mathbb{Z} \) is a field precisely when \( \beta_i = 1 \). For this reason, \( p_i \) divides the order of some nonzero nilpotent element of \( \mathbb{Z}/n\mathbb{Z} \) if and only if \( \beta_i > 1 \) (which is equivalent to the condition that \( p_i | (n/\text{rad}(n)) \)). Hence, by Theorem 3.1

\[
\mu_P(\mathbb{Z}/n\mathbb{Z}) = \sum_{d | \text{rad}(n/\text{rad}(n))} (-1)^{\omega(d)-1} \left( \frac{\lambda(n) + E(n) - 1}{d} \right).
\]

4.5 Matrix rings over fields

Consider \( R = \mathcal{M}_n(\mathbb{F}_q) \), where \( q \) is a prime power \( p^k \). Almkvist [2] proved that \( \mu_1(\mathcal{M}_n(\mathbb{F}_2)) \) is the exponent of \( \text{GL}(n, \mathbb{F}_2) \). Here we modify Almkvist’s methods to show that \( \mu_1(\mathcal{M}_n(\mathbb{F}_q)) \) is the exponent of \( \text{GL}(n, \mathbb{F}_q) \) and \( \mu_0(\mathcal{M}_n(\mathbb{F}_q)) = n \) in general.

Let \( \tau \in R \), let \( \psi(t) \in \mathbb{F}_q[t] \) be the minimal polynomial of \( \tau \), and let \( \psi_1(t)^{\beta_1} \cdots \psi_n(t)^{\beta_n} \) be the prime factorization of \( \psi(t) \). Treat \( \tau \) as a linear operator of the vector space \( V := \mathbb{F}_q^n \) and set \( V_i = \ker \psi_i(\tau)^{\beta_i} \) for each \( i \). Then \( V = \bigoplus_{i=1}^n V_i \) and each \( V_i \) is invariant under \( \tau \). For each \( i \), let \( \tau_i : V_i \to V_i \) be the restriction of \( \tau \) to \( V_i \) so that \( \psi_i(\tau_i)^{\beta_i} \) vanishes. It suffices to flesh out the dynamics of the sequence \( \{\tau_i^j\}_{j=1}^\infty \).

If \( \psi_1(t) = t \), then \( \tau_i \) is nilpotent and there is nothing more to prove, so assume instead that \( \psi_1(t) \neq t \). Suppose that \( \deg \psi_1(t) = m \). Then \( \psi_1(t) | (t^{\alpha_m} - t) \) since the product of all monic irreducible polynomials in \( \mathbb{F}_q[t] \) with degree dividing \( m \) is \( t^{\alpha_m} - t \). Hence \( (\tau_i^{\alpha_m} - \tau_i)^{\beta_i} = 0 \), from which it follows that \( \tau_i \) is potent. Moreover, because \( \beta_i \leq n \leq p^{[\log_p n]} \), we have that

\[
(t^{\alpha_m} - t)^{\beta_i} | (t^{\alpha_m} - t)^{p^{[\log_p n]}} = t^{p^{[\log_p n]} \alpha_m} - t^{p^{[\log_p n]}}.
\]
It follows that
\[ \tau_i^{\lceil \log_p n \rceil q^m} = \tau_i^{\lceil \log_p n \rceil}, \] (4.10)
and so the exponential periodicity of \( R \) is a divisor of
\[ \text{lcm}\{p^{\lceil \log_p n \rceil}(q^n - 1), p^{\lceil \log_p n \rceil}(q_{n-1} - 1), \ldots, p^{\lceil \log_p n \rceil}(q - 1)\} = p^{\lceil \log_p n \rceil}\text{lcm}\{q - 1, q^2 - 1, \ldots, q^n - 1\}. \] (4.11)
Yet (4.11) is precisely the exponent of \( \text{GL}(n, \mathbb{F}_q) \) (cf \[11, \text{Corollary 1} \]), and so this yields \( \mu_1(R) \). Finally, \( \mu_0(R) \) is the largest possible index of a nilpotent operator on a subspace of \( V \), which is \( n \).

We finish with a sketch of a constructive proof for the nonexistence of periodic power maps over \( \mathcal{M}_n(K) \) that applies to all fields. Let \( A \neq 0 \) be nilpotent. The Jordan canonical form \( J \) of \( A \) is a strictly lower triangular matrix with its unital entries lying on the subdiagonal. Now let \( C \) be the companion matrix of the polynomial \( p(t) = t^n - 1 \). The only nonzero entries of \( C \) are at entry \((1, n)\), which is 1, and the subdiagonal completely populated by 1, and so \( C - J \) is singular due to having zero-rows. Let \( S \) be the change of basis matrix for which \( A = SJS^{-1} \). Then for every \( m \in \mathbb{N} \),
\[ (SCS^{-1} - A)^m = S(C - J)^mS^{-1} \] (4.12)
is singular whereas \( (SCS^{-1})^m = SC^mS^{-1} \) is not.

5 Concluding Remarks

While we have explicitly specified the periodic power maps over finite commutative rings with unity, things quickly turn more exotic in broader classes of rings. For instance, Example 2.5 shows that \( \mu_P \) can potentially reach the upper bound of \( \mu_1 + \mu_0 - 2 \) where every non-identity power map is periodic. In any event, we pose the following question.

**Question 5.1.** What is the distribution of \( \mu_P \) over random finite commutative rings?

Investigations into the oscillatory behavior of generic power maps over rings can proceed in at least two directions. One is taxonomic: can we achieve at least a partial characterization of the nilperiod rings? Another is combinatorial: is it possible to detect and enumerate the periodic power maps over any periodic ring? To the author’s surprise, these questions appear mostly unexplored. We hope that this article generates some interest in what may potentially be a rich and attractive topic.

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