ON THE DENSITY OF CERTAIN SPECTRAL POINTS FOR A CLASS OF $C^2$ QUASIPERIODIC SCHÔRDINGER COCYCLES

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Abstract. For $C^2$ cos-type potentials, large coupling constants, and fixed Diophantine frequency, we show that the density of the spectral points associated with the Schrödinger operator is larger than 0. In other words, for every fixed spectral point $E$, \( \liminf_{\epsilon \to 0} |(E - \epsilon, E + \epsilon) \cap \Sigma_{\alpha, \lambda \upsilon}| \geq \epsilon \), where $\beta \in \left[ \frac{1}{2}, 1 \right]$. Our approach is a further improvement on the papers \[15\] and \[17\].

1. Introduction. Consider the discrete quasiperiodic Schrödinger operators $H_{\alpha, \lambda \upsilon, x}: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, which is given by:

\[
(H_{\alpha, \lambda \upsilon, x} u)_n = u_{n+1} + u_{n-1} + \lambda \upsilon(x + n \alpha) u_n.
\] (1)

Here $\lambda \in \mathbb{R}$ is the coupling constant, $\upsilon \in C^r(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $r \in \mathbb{N} \cup \{ \infty, \omega \}$ the potential, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ the phase, and $\alpha \in \mathbb{R}/\mathbb{Z}$ the frequency.

Let $\Sigma_{\alpha, \lambda \upsilon, x}$ denote the spectrum of the operator. Then it is well-known

\[
\Sigma_{\alpha, \lambda \upsilon, x} \subset [-2 + |\lambda| \inf \upsilon, 2 + |\lambda| \sup \upsilon].
\] (2)

For irrational $\alpha$, due to \[11\], $\Sigma_{\alpha, \lambda \upsilon, x}$ is unrelated to the phase. This follows from minimality of the irrational rotation, and \[19\] gives another proof. So let $\Sigma_{\alpha, \lambda \upsilon}$ be the common spectrum in this case.

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The properties of Schrödinger operators above are closely correlated with the so-called Schrödinger cocycle \((\alpha, A(E-\lambda \nu)) \in \mathbb{R}/\mathbb{Z} \times C^r(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{R}))\), where 
\[
A(E-\lambda \nu)(x) = \begin{pmatrix} E - \lambda \nu(x) & -1 \\ 1 & 0 \end{pmatrix}.
\] (3)

\((\alpha, A(E-\lambda \nu))\) defines a family of dynamical systems on \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2\) which is given by 
\[
(x, \omega) \mapsto (x + \alpha, A(E-\lambda \nu)(x) \cdot \omega).
\]
The nth iteration of this cocycle is denoted by 
\[
(\alpha, A(E-\lambda \nu)^n) = (n\alpha, A_n(E-\lambda \nu)),
\]
where
\[
A_n(E-\lambda \nu)(x) = \begin{cases} 
A(E-\lambda \nu)(x + (n-1)\alpha) \ldots A(E-\lambda \nu)(x), & n \geq 1; \\
I_2, & n = 0; \\
(A_{-n}(E-\lambda \nu)(x + n\alpha))^{-1}, & n \leq -1.
\end{cases}
\]

There is the following equivalent relation between Schrödinger operator and cocycle. 
\[
u = (\nu_n)_{n \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z}
\]
is a solution of the eigenvalue equation 
\[
H_{\alpha,\nu,x}u = Eu
\]
if and only if
\[
A_n(E-\lambda \nu)(x) \begin{pmatrix} \nu_0 \\ \nu_{-1} \end{pmatrix} = \begin{pmatrix} \nu_n \\ \nu_{n-1} \end{pmatrix}, n \in \mathbb{Z}.
\]

From now on, we discuss \(C^2\) cos-type potential which was first considered by Sinai [14]:
\begin{itemize}
  \item \(\frac{dv}{dx} = 0\) has two exactly points, one is the maximal point and another the minimal, denoted by \(z_1\) and \(z_2\).
  \item The two extremal points are non-degenerate, namely \(\frac{d^2v}{dx^2}(z_j) \neq 0\), for \(j = 1, 2\).
\end{itemize}

Fix two positive constants \(\tau, \gamma\). We say \(\alpha\) satisfies Diophantine condition if \(\alpha \in DC_{\tau, \gamma}\), where:
\[
DC_{\tau, \gamma} := \left\{ \alpha : \left| \alpha - \frac{p}{q} \right| \geq \frac{\gamma}{|q|^{\tau}}, \text{for all } p, q \in \mathbb{Z} \text{ with } q \neq 0 \right\}.
\]

Let \(DC_\tau := \bigcup_{\gamma > 0} DC_{\tau, \gamma}\). Then the set is of full Lebesgue measure for any \(\tau > 2\). We fix a \(\tau > 2\) and an \(\alpha \in DC_{\tau}\). The paper [17] told us that for \(\alpha\) and \(\nu\) as above, \(\Sigma_{\alpha,\lambda \nu}\) is a Cantor set when \(\lambda\) is sufficiently large. Xu-Ge-Wang estimated the lower and upper bounds on the length of gaps on the Lemma 15 of [15], and especially proved that all the gaps are open in a recent work.

In this paper, we prove that the spectral set takes up a certain proportion in the small domain of some points. Our precise result is the following:

**Theorem 1.1.** Let \(\alpha\) be Diophantine and \(\lambda \nu\) be \(C^2\) cos-type. Consider the Schrödinger cocycle with potential \(\lambda \nu\), \(\Sigma_{\alpha,\lambda \nu}\) is the associated spectrum. Then there exists a \(\lambda_0 = \lambda_0(\alpha, \nu) > 0\) such that for any fixed \(\lambda > \lambda_0\), the followings hold true:

1. There exists a subset \(\Sigma_{FR}\) of \(\Sigma_{\alpha,\lambda \nu}\) with a full measure such that for any \(E \in \Sigma_{FR}\),
\[
\lim_{\epsilon \to 0} \frac{|(E - \epsilon, E + \epsilon) \cap \Sigma_{\alpha,\lambda \nu}|}{2\epsilon} = 1;
\] (4)
2. All the endpoints of spectral gaps consists of a dense set in the spectrum and for each such point $E$,
\[
\lim_{\epsilon \to 0} \frac{|(E - \epsilon, E + \epsilon) \cap \Sigma_{\alpha,\lambda \nu}|}{2\epsilon} = \frac{1}{2};
\] (5)

3. For any fixed $\frac{1}{2} < \beta < 1$, there exists some point $E_\beta \in \Sigma_{\alpha,\lambda \nu}$, such that
\[
\liminf_{\epsilon \to 0} \frac{|(E_\beta - \epsilon, E_\beta + \epsilon) \cap \Sigma_{\alpha,\lambda \nu}|}{2\epsilon} = \beta; \] (6)
\[
\limsup_{\epsilon \to 0} \frac{|(E_\beta - \epsilon, E_\beta + \epsilon) \cap \Sigma_{\alpha,\lambda \nu}|}{2\epsilon} = 1. \] (7)

The structure of the spectrum has been one of the most important topics for the Schrödinger operators (1) for a very long time. For rational rotation $\alpha = \frac{p}{q}$, the spectrum which corresponds to the operators has a finite bands structure, namely consists of at most $q$ non-degenerate intervals. For irrational $\alpha$, this condition is more complicated which is expected widely to be a Cantor set. Next we review some of the results on the basis of the regularity of potential $\nu$.

The most specific model studied is the almost Mathieu operator (AMO) where $\nu(x) = 2 \cos 2\pi x$. The well-known “Ten Martini Problem”, referred to in the literature Simon [13], conjectured that the spectrum of AMO must be a Cantor set for all non-zero couplings and for all irrational frequencies. Finally this problem was settled by Avila-Jitomirskaya [4]. The so-called “Dry Ten Martini Problem” is a further refinement of the “Ten Martini Problem”, asking whether all possible spectral gaps are non-collapsed under the same conditions. Avila-You-Zhou [5] solved the dry version for any non-critical coupling constant $\lambda \neq 1$. Note that the “Dry Ten Martini Problem” only attends to the openness of the spectral gaps, without caring any estimates on their length.

Eliasson [8] proved that for a fixed Diophantine frequency, for small couplings, the spectrum is a Cantor set for generic potentials under analytic topology. Goldstein-Schlag [10] obtained that in the regime of positive Lyapunov exponent, the Cantor structure of spectrum holds true for almost every Diophantine frequencies.

For $C^2$ cos-type potentials, any fixed Diophantine frequency, and the sufficiently large coupling constant $\lambda$, Wang-Zhang [17] showed that the corresponding spectrum is a Cantor set, and their central approach is based on the idea of “resonance leads to spectral gaps”, which benefited from [14]. Further, Xu-Ge-Wang [15] proved the associated operator under the same conditions possesses a dry version of Cantor spectrum, meanwhile estimated the upper and lower bound of the size of spectral gaps. For even $C^2$ cos-type potentials and any fixed Diophantine frequency, Ge-You-Zhao [9] proved that the eigenfunctions decay exponentially when $\lambda$ is sufficiently large.

For $C^0$ potentials, Avila-Bochi-Damanik [1] proved that for any irrational frequency, and for $C^0$ generic potentials, the spectrum is a Cantor set. In fact, they soon obtained a dry version [2] which states that all spectral gaps can be opened up for $C^0$ generic potentials by deformation.

If one lowers the regularity of potential, Damanik-Lenz [6] [7] obtained for a large class of step functions, there exists a Cantor set of Lebesgue measure zero, such that it is the spectrum. These type of potentials are particularly related to Sturmian subshift, or even more specially, the Fibonacci subshift, which has also been intensively studied.
Meanwhile, there are some negative results. Avila-Damanik-Zhang [3] proved that for $C^n$ potentials and a dense subset of frequency space, the density of states measure is singular and the spectrum contains a non-degenerate interval. In particular, all such frequencies are irrational.

The structure of the rest part of the paper is as follows. In section 2, we introduce some technical lemmas from the article [16], and present the properties of the spectrum from the paper [15]. In section 3 and section 4, the main result will be settled.

2. Preliminaries. This section is basically a review of some known results without proof.

If not stated otherwise, let $0 < c \ll 1 \ll C$, be some universal positive constants depending only on $v$ and $\alpha$. Assume $A \in SL(2, \mathbb{R})$ with $\|A\| > 1$, define the map

$$s : SL(2, \mathbb{R}) \to \mathbb{RP}^1 = \mathbb{R}/(\pi \mathbb{Z})$$

so that $s(A)$ is the most contraction direction of $A$. Let $\hat{s}(A) \in s(A)$ be a unit vector, so $\|A \cdot \hat{s}(A)\| = \|A\|^{-1}$. Abusing the notation a little, let

$$u : SL(2, \mathbb{R}) \to \mathbb{RP}^1 = \mathbb{R}/(\pi \mathbb{Z})$$

be that $u(A) = s(A^{-1})$. Thus we have the matrix decomposition:

$$A = R_u \cdot \begin{pmatrix} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{pmatrix} \cdot R_{\pi/2}^{-s}$$

where $s, u \in [0, \pi)$ are angles associated with $s(A), u(A) \in \mathbb{R}/(\pi \mathbb{Z})$.

Set $t = \frac{\pi}{2}$. The following lemma provide a more convenient form of the Schrödinger operators (3).

Lemma 2.1. Let $I \subset \mathbb{R}$ be some compact interval, $\alpha$, $\lambda$ and $v$ be as in Theorem 1.1. For $x \in \mathbb{R}/\mathbb{Z}$ and $t \in I$, then the Schrödinger cocycle $(\alpha, A^{(E-\lambda v)})$ is conjugate to the cocycle $(\alpha, A)$ with

$$A(x, t) = \begin{pmatrix} \lambda(x, t) & 0 \\ 0 & \lambda^{-1}(x, t) \end{pmatrix} \cdot \begin{pmatrix} \sin \phi(x, t) & -\cos \phi(x, t) \\ \cos \phi(x, t) & \sin \phi(x, t) \end{pmatrix}$$

$$= A(x, t) \cdot R_{\frac{\pi}{2} - \phi(x, t)}$$

where

$$C_1 \lambda \leq \lambda(x, t) := \|A(x, t)\| \leq C_2 \lambda,$$

$$|\partial_x^m \lambda(x, t)| \leq C_3 \lambda, \quad m = 1, 2,$$

$C_j$ only depends to $v$, $j=1, 2$ and 3, and $\tan \phi(x, t) \to t - v(x - \alpha)$ in $C^2$-topology as $\lambda \to \infty$. Thus $\phi(x, t)$ is also a cos-type function on $x$ for large $\lambda$ and it suffices to consider the cocycle $(\alpha, A(x, t)R_{\frac{\pi}{2} - \arctan(t-v(x-\alpha))})$ instead of $(\alpha, A^{(E-\lambda v)})$.

See Lemma 1 of [16] or Lemma 2 of [15] for a proof.

2.1. Types of angle function and induction theorem. From now on, let $A = A(x, t)$ be as in formula (8). For $n \geq 1$, define

$$s_n(x, t) = s[A_n(x, t)], \quad u_n(x, t) = u[A_n(x, t)].$$

We call $s_n$ the $n$-step stable direction and $u_n$ the $n$-step unstable direction. Apparently, for $A(x, t)$, we have that

$$\begin{cases}
  u_1(x, t) = 0; \\
  s_1(x, t) = \frac{\pi}{2} - \phi(x, t) = \tan^{-1}(t - v(x)).
\end{cases}$$
We define $g_1(x, t) = s_1(x, t) - u_1(x, t)$, so it obviously holds that
\[ g_1(x, t) = \tan^{-1}(t - v(x)). \]
Recall $\{q_{n}\}_{n\geq 1}$ are the continued fraction approximation of $\alpha$, meanwhile set $I_0 = \mathbb{R}/\mathbb{Z}$ for all $t \in J := [\inf v - \frac{1}{x}, \sup v + \frac{1}{x}]$. Firstly fix some large $N = N(v)$ such that $q_N$ is sufficiently large. Then at step 1, we have the following

- First step critical points:
\[ C_1(t) = \{c_{1,1}(t), c_{1,2}(t)\} \]
with $c_{1,1}(t), c_{1,2}(t) \in I_0$ minimizing $\{|g_1(x, t)|, x \in I_0\}$ for each $t \in J$.
- First step critical interval:
\[ I_{1,j}(t) = \left\{ x : |x - c_{1,j}(t)| \leq \frac{1}{2q_N^j} \right\}, I_1(t) = I_{1,1}(t) \cup I_{1,2}(t). \]
- First step return times:
\[ q_N \leq r^+_1(x, t) : I_1(t) \to \mathbb{Z}^+ \]
are the first times back to $I_1(t)$ after time $q_N - 1$. Here $r^+_1(x, t)$ is the forward return time and $r^-_1(x, t)$ the backward return time. Let $r^+_1(t) = \min_{x \in I_1(t)} r^+_1(x, t)$ and $r^-_1(t) = \min\{r^+_1(t), r^-_1(t)\}$.
- The second step angle function $g_2$:
\[ g_2(x, t) = s_{r_1(t)}(x, t) - u_{r_1(t)}(x, t) : D_1 \to \mathbb{R}\mathbb{P}^1, \]
where
\[ D_1 := \{(x, t) : x \in I_1(t), t \in J\} \]
Then by induction, [16] prove the following conclusion. Assume we have the following objects which are well defined:

- ith step critical point:
\[ C_i(t) = \{c_{i,1}(t), c_{i,2}(t)\} \]
where $c_{i,1}(t), c_{i,2}(t) \in I_{i-1,j}(t)$ satisfies the minimum of $|g_i(x, t)|$
- ith step critical interval:
\[ I_{i,j}(t) = \left\{ x : |x - c_{i,j}(t)| \leq \frac{1}{2q_N^{j+i}} \right\}, I_i(t) = I_{i,1}(t) \cup I_{i,2}(t). \]
- ith step return times:
\[ q_{N+i-1} \leq r^+_i(x, t) : I_i(t) \to \mathbb{Z}^+ \]
are the first times back to $I_i(t)$ after time $q_{N+i-1} - 1$. Here $r^+_i(x, t)$ is the forward return time and $r^-_i(x, t)$ the backward return time. Let $r^+_i(t) = \min_{x \in I_i(t)} r^+_i(x, t)$ and $r^-_i(t) = \min\{r^+_i(t), r^-_i(t)\}$.
- Then $i + 1$th step angle function $g_{i+1}$:
\[ g_{i+1}(x, t) = s_{r_i(t)}(x, t) - u_{r_i(t)}(x, t) : D_i \to \mathbb{R}\mathbb{P}^1, \]
where
\[ D_i := \{(x, t) : x \in I_i(t), t \in J\} \]
Definition 2.2. Let $d_i(t) := c_{i,1}(t) - c_{i,2}(t)$. We say that angle function $g_{i+1}(x,t)$ is type I if $|d_i - k| > |I_i|$ for all $k \in \mathbb{Z}$ satisfying $0 \leq |k| \leq q_{N+i-2}$, which is non-resonant at the step $i$. Otherwise, the step $i$ is resonant, denoted by type II if $|d_i| \leq |I_i|$ and type III if there exists $k$ satisfying $0 < |k| < q_{N+i-2}$ such that $|d_i - k| < |I_i|$, respectively.

Obviously, $g_1(\cdot,t) = \tan^{-1}(t-v(\cdot))$ is either type I or II for each $t \in J$. However, there may exist some $t \in J$ such that $g_2(\cdot,t)$ is of type III. Importantly, in the paper [16], we know that $g_n(\cdot,t)$ must be one of these three types what mentioned above.

Then the following is the crucial result called Induction Theorem in [16] which asserts:

Lemma 2.3. For any $\epsilon > 0$, there exists a $\lambda_0 = \lambda_0(v,\alpha,\epsilon) > 0$ such that for all $\lambda > \lambda_0$, the following holds for each $i \geq 2$:

\[
\begin{cases}
|c_{i-1,j}(t) - c_{i,j}(t)| < C\lambda^{-\frac{q}{2r_i}} - 1, & j = 1, 2; \quad (9)
\|A_{\pm r_i}(x,t)\| > \lambda^{(1-\epsilon)r_i}, \quad \|A_{\pm r_i}(x,t)\| \geq \lambda^{(1-\epsilon)q_{N+i-2}}, \quad (10)
\frac{\partial^m}{\partial v^m}(\|A_{\pm r_i}(x,t)\|) < \|A_{\pm r_i}(x,t)\|^{1+\epsilon}, \quad \nu = x \text{ or } t; \quad (11)
g_{i+1}(x,t) - g_i(x,t) = \arctan(\|A_{\pm r_i}(x,t)\|^2 \cdot \tan g_i(x + r_i x, t)) - \frac{\pi}{2}. \quad (12)
\end{cases}
\]

Furthermore, For each $t \in J$, $g_i(\cdot, t) : I_{i-1}(t) \rightarrow \mathbb{R}^1$ is of type I, II, or III, which are denoted as cases (i)$_I$, (i)$_{II}$, and (i)$_{III}$. $g_i(x,t)$ has the following properties by induction:

1. In cases (i)$_I$, (i)$_{II}$, it holds that

$$
\|g_i - g_{i-1}\|_{C^2} \leq C\lambda^{-\frac{q}{2r_i}};
$$

where $C$ depends only on $\lambda$ and $v$. Moreover, we have the following:

(a) In case (i)$_I$, $I_{i-1}(t) \cap I_{i-2}(t) = \emptyset$, and $g_i(x,t) = 0$ has one exactly point $c_{i,j}(t)$ on $I_{i,j}(t)$, $j = 1$ or 2. Moreover

$$
\partial_x g_i(x,t) \cdot \partial_y g_i(y,t) < 0, \text{ for } x \in I_{i,1}, \text{ and } y \in I_{i,2}.
$$

(b) In case (i)$_{II}$, $I_{i-1}(t) \cap I_{i-2}(t) \neq \emptyset$. Same as above, $c_{i,j}(t)$ is the only point minimizing $|g_i(x,t)|$ on $I_{i,j}(t)$; note that it is possible that $c_{i,1}(t) = c_{i,2}(t)$.

2. In case (i)$_{III}$, there exists a unique $k$ such that $1 \leq |k| < q_{N+i-2}$ and

$$
I_{i-2}(t) \cap (I_{i-1}(t) + k\alpha) \neq \emptyset.
$$

$g_i(x,t)$ has at most two zeros in $I_{i-1,j}$. If $g_i(x,t)$ has no zero, denote the only point satisfying $\min_{t \in I_{i,j}} |g_i(x,t)|$ by $c_{i,j}(t)$; if $g_i(x,t)$ has one zero $c_{i,j}(t)$ in the interval $I_{i,j}$, then $g_i(c_{i,j}, t) = \partial_x g_i(c_{i,j}, t) = 0$; if $g_i(x,t)$ has two zeros $, c_{i,j}(t)$ and $d_{i,j}(t)$ in $I_{i,j}(t)$, let $c_{i,j}$ be the one satisfying $\partial_x g_{i-1}(c_{i-1,j}, t) \cdot \partial_x g_i(c_{i,j}, t) > 0$. Moreover, the following holds true:

- If $|g_i(c_{i,j}, t)| > C\lambda^{-\frac{q}{2r_i}}$, $j = 1$ or 2, then so are $|g_i(c_{i,j'}, t)|$ for $j' \neq j$.
- If $|g_i(c_{i,j}, t)| < C\lambda^{-\frac{q}{2r_i}}$, $j = 1$ or 2, then

$$
\|c_{i,1}(t) + k\alpha - d_{i,2}(t)\|_{R/Z, \|c_{i,2}(t) + k\alpha - d_{i,1}(t)\|_{R/Z} < C\lambda^{-\frac{q}{2r_i}}}, \quad (13)
$$
where $\| \cdot \|_{R/Z}$ denotes the distance to the nearest integer.

3. Let $$(i)_a \to (i+1)_b$$
denote “from case $(i)_a$ to case $(i+1)_b$” where $a,b = I,II$ or III. Then the following holds:

if $a$ is $II$, then $b$ can be $I,II$ or $III$;

if $a$ is $I$ or $III$, then $b$ can and only be $I$ or $III$.

In *Induction Theorem*, the scale of “the critical interval” $I_{i,j}$ is chosen to be $\frac{1}{q_{N+1-1}}$. However, it is no problem to replace it with $\frac{1}{q_{N+1-1}}$ for any large $C > 0$ depending on the choice of $\lambda$.

All the derivative estimates of $g_t(x,t)$ and $\|A_t(x,t)\|$ are for $x$ in Lemma 2.3. In fact, all the necessary technical lemmas in [16] (Lemma 2-5) can be applied to the derivative estimates over parameter $t$. In particular, let

$$r_i = \min_{t \in I_i(x,t)} \{ r_i(t) \},$$

we emphasize that it is allowed to replace $r_i(t)$ by $r_i$ for all $t \in I_i(x,t)$, because the difference between $r_i(t)$ and $r_i$ are negligible.

2.2. Classification of spectrum. We know the spectrum is a Cantor set from [17]. On account of the occurrence of resonance in the iteration process, we classify the spectrum. For any fixed $t \in \frac{1}{\lambda} \Sigma^\lambda = \Sigma^\lambda$, let $d_n(t) = c_{n,1}(t) - c_{n,2}(t)$ in brief.

**Definition 2.4.** We introduce some significant notations.

1. $\Sigma_{FR} := \bigcup_{i=1}^{\infty} \{ t \in \Sigma^\lambda : \text{for any } n \in \mathbb{Z}^+, \text{any } |k| \leq q_{N+n-2}, \text{holds } |d_n(t) - k\alpha| > \frac{1}{q_{N+n-1}} \}$, which is finitely-resonant condition;

2. $\Sigma_{IR} := \Sigma^\lambda - \Sigma_{FR} = \bigcap_{i=1}^{\infty} \{ t \in \Sigma^\lambda : \text{exists some } n \in \mathbb{Z}^+ \geq l, \text{some } |k| \leq q_{N+n-2}, \text{holds } |d_n(t) - k\alpha| \leq \frac{1}{q_{N+n-1}} \}$, which is infinitely-resonant condition;

3. $\Sigma_{CR} := \{ t \in \Sigma^\lambda : \text{exists some } k \in \mathbb{Z} \text{ such that for each } n \geq N(k), \text{holds } |d_n(t) - k\alpha| \leq C\lambda^{-\frac{1}{p}r_{n-1}}, \text{where } N(k) \text{ satisfies } q_{N+N(k)-2} < |k| \leq q_{N+N(k)-1} \}$, which is completely-resonant condition.

**Remark 1.** On the one hand, it holds directly by definition that $\Sigma_{FR} \cup \Sigma_{IR} = \Sigma^\lambda, \Sigma_{FR} \cap \Sigma_{IR} = \emptyset$, and $\Sigma_{CR} \subset \Sigma_{IR}$. On the other hand, $\Sigma_{IR}$ is rare in the sense of measure and zero-measurable; $\Sigma_{FR}$ is a set with full Lebesgue measure.

**Remark 2.** According to types of angle function $\{g_n(t)\}_{n \geq 1}$ in the iteration process, the followings are equivalent:

- $t \in \Sigma_{FR} \iff$ there exists some $N(t)$ such that $g_n(t)$ is always of type I for $n \geq N(t)$;
- $t \in \Sigma_{CR} \iff$ there exists some $N(t)$ such that $g_n(t)$ is always of type II or III for $n \geq N(t)$;
• \( t \in \Sigma_{IR} \setminus \Sigma_{CR} \leftrightarrow \) there exists some \( N(t) \) such that \( g_n(t) \) is of type I or III alternately for \( n \geq N(t) \).

**Remark 3.** \( \Sigma_{CR} \) is nothing but all the endpoints of spectral gaps, sometimes we use \( \Sigma_{EP} \) to represent \( \Sigma_{CR} \). And \( \Sigma_{CR} \) is equivalent to the set below:

\[
\{ t \in \Sigma^\lambda : \exists k \in \mathbb{Z}, \text{s.t.} \lim_{n \to \infty} (d_n(t)) = k\alpha \}.
\]

For Schrödinger operators (1), the integrated density of states (IDS) \( N_{\alpha,\upsilon} : \mathbb{R} \rightarrow [0, 1] \) is defined as

\[
N_{\alpha,\upsilon}(E) := \int_{\mathbb{T}} \mu_{\alpha,\upsilon,x}(-\infty, E] dx
\]

where \( \mu_{\alpha,\upsilon,x} \) is the spectral measure of \( H_{\alpha,\upsilon,x} \) and \( \delta_0 \). By the Gap-Labelling Theorem [12], for any spectral gap \( G \), there exists a unique \( k \in \mathbb{Z} \) such that \( N_{\alpha,\upsilon}(E) = k\alpha \mod \mathbb{Z} \) where \( E \in G \). From [15], the \( k \) in the Gap-Labelling theorem is the same as the integer in formula (14).

The following lemma shows that the image of \( d_i \) moves a constant multiple distance when \( t \) changes in a small neighborhood. Let \( \epsilon_n = C^{-C|\alpha|} \), then we have

**Lemma 2.5.** Fix \( t \in J \), let \( d_i(t) = c_{i,1}(t) - c_{i,2}(t) \) for \( i \in \mathbb{N}^+ \), then exist \( c(\alpha, \upsilon, \lambda) > 0 \) such that

1. If \( g_i(t) \) is of type I, the following estimate of derivative holds

\[
q_{N+i-1}^{-1} \geq \left| \frac{dd_i}{dt}(t') \right| > c, \ \forall t' \in B(t, \epsilon_i).
\]

2. If \( g_i(t) \) is of type II or III we have

\[
\left| \frac{dd_i}{dt}(t') \right| > c, \ \forall t' \in B(t, \epsilon_i) \bigcap \{ t \in J : g_i(c_{i,j}(t), t) = 0, j = 1, 2 \}.
\]

See the Corollary 3 of [17] for a proof of the lower bound and the Lemma 17 of [15] for the upper bound.

3. **The proof of finitely-resonant spectral points.** In this section, we firstly introduce an important proposition, which estimates the lower and upper bounds of the lengths of the spectral gaps. In a recent work, Xu-Ge-Wang have proven that all the gaps are open.

**Proposition 1.** For any \( k \in \mathbb{Z} \setminus \{0\} \), we have

\[
c\lambda^{-C|k|} < |G_k| < C\lambda^{-c|k|}
\]

for two suitable constants \( 0 < c(\alpha, \upsilon, \lambda) < C(\alpha, \upsilon, \lambda) \), where \( |G_k| \) denotes the length of \( G_k \).

**Proof.** We denote by \( M(k) := \min \{ i | I_{i,1} + k\alpha \cap I_{i,2} \neq \emptyset \} \). Then it holds from [16] that there exists two constants \( 0 < c(\alpha, \upsilon, \lambda) < C(\alpha, \upsilon, \lambda) \) such that \( g_{M(k)+1} \) is of type III with two extreme points \( \hat{c}_{M(k)+1,1}, i = 1, 2 \) satisfying

\[
|g_{M(k)+1}(\hat{c}_{M(k)+1,1}) - g_{M(k)+1}(\hat{c}_{M(k)+1,1})| \leq \pi - c\lambda^{-C|k|}.
\]

Note [17] implies that

\[
\{ t \in \mathbb{R} | \underset{x \in I_{M(k)+1}(t)}{\min} |g_{M(k)+1}(x, t)| \geq c\lambda^{-Ck} \} \subseteq G_k
\]
and Lemma 19 of [15] implies that
\[ G_k \subseteq \{ t \in \mathbb{R} | \min_{x \in I_{M(k)+1}(t)} |g_{M(k)+1}(x,t)| \leq c\lambda^{-C r_{M(k)+1}} \}. \]

Since \( r_{M(k)+1} \gg |k| \), it follows from the Lemma 15 of [15] that \( c\lambda^{-C|k|} \leq |G_k| \leq C\lambda^{-c|k|} \) as desire. \( \square \)

On the basis of above, we are going to prove the main results.

3.1. **Case A**: \( t_0 \in \Sigma_{FR} \). From the definition of finitely-resonant spectral points, there exists some \( N(t_0) \geq 1 \), for any \( n \geq N(t_0) \) and any \( |k| \leq q_{N+n-2} \), we have

\[ |d_n(t_0) - k\alpha| > \frac{1}{q_{N+n-1}}. \]

In other words, it means that the resonance near \( t_0 \) is too weak when the iteration step is sufficiently large. However, by the denseness of \( \{ k\alpha \}_{k \in \mathbb{Z}} \) in \( \mathbb{R}/\mathbb{Z} \), there must exist some \( |k_n| \geq q_{N+n-2} \) such that

\[ |d_n(t_0) - k_n\alpha| \leq \frac{1}{q_{N+n-1}}. \]

We assume that for some positive integer \( n \geq N(t_0) \), such that \( \epsilon \in (\frac{1}{q_{N+n}}, \frac{1}{q_{N+n-1}}) \).

Recall \( \{ \frac{p_n}{q_n} \}_{n \geq 1} \) are the continued fraction approximants of \( \alpha \), then it is a standard result as well that

\[ \alpha \in DC_\tau \iff q_{n+1} < cq_n^{-1}, \forall n \geq 1. \]

The following lemma tells us that \( g_n(t) \) do not change the type in the small domain of \( t_0 \).

**Lemma 3.1.** Let \( t_0 \in \Sigma_{FR} \). \( g_n(t) \) is always of type I, where \( n \geq N(t_0) \), \( t \in (t_0 - \frac{1}{q_{N+n-1}}, t_0 + \frac{1}{q_{N+n-1}}) \).

**Proof.** For all \( 0 \leq |k| \leq q_{N+n-2} \),

\[ \frac{1}{q_{N+n-1}^C} < |d_n(t_0) - k\alpha| \leq |d_n(t_0) - d_n(t)| + |d_n(t) - k\alpha|, \]

so

\[ |d_n(t) - k\alpha| > \frac{1}{q_{N+n-1}^C} - q_{N+n-1}^{-1}|t_0 - t| > \frac{1}{q_{N+n-1}^C} - \frac{q_{N+n-1}^{-1}}{q_{N+n-1}^C} \]

\[ > \frac{1}{q_{N+n-1}^C}. \]

Note that \( c \) only depends on \( \alpha, \lambda \) and \( \nu \), we can choose the sufficiently large \( C \) such that

\[ 0 \ll c^{-1} < C. \]

Then \( g_n(t) \) is of type I. \( \square \)

Then we have

\[ 1 \geq \frac{|(t_0 - \epsilon, t_0 + \epsilon) \cap \Sigma_{\alpha, \lambda, \nu}|}{2\epsilon} = 1 \geq \frac{|(t_0 - \epsilon, t_0 + \epsilon) \cap \text{Gaps}|}{2\epsilon} \]

\[ \geq 1 - \frac{\sum_{s \geq n} |G_k_s|}{q_{N+n}^C} \approx 1 - \frac{C\lambda^{-|k_n|}}{q_{N+n}^C} \]

\[ \geq 1 - \frac{C\lambda^{-c q_{N+n-1}}}{c q_{N+n-1}^{-C(\tau-1)}}. \]
where gaps is complementary set of spectrum. The last part of inequality obviously tends to 1 as $n \to \infty$. We finish the proof of the formula (4) of Theorem 1.1.

3.2. Case B: $t_0 \in \Sigma_{CR}$. We assume that $t_0$ is the right endpoint of some spectral gap $G_k$, denote it by $t_k^+$, and $0 < \epsilon \ll |G_k|$. Firstly in the interval $(t_k^+, t_k^+ + \epsilon)$, we can get the longest spectral gap labelled by $k_1$, $G_{k_1} = (t_{k_1}^-, t_{k_1}^+)$; then consider the interval $(t_{k_1}^+, t_{k_1}^+)$, we get another longest gap $G_{k_2} = (t_{k_2}^-, t_{k_2}^+)$, where $|k_2| \gg |k_1|$; repeat the above process, we get a sequence of integer $\{k_n\}_{n \geq 1}$ and spectral gaps $\{G_{k_n}\}_{n \geq 1}$, where

\[
|k| < |k_1| < |k_2| < \cdots < |k_n| < \cdots ;
\]

\[
|G_k| > |G_{k_1}| > |G_{k_2}| > \cdots > |G_{k_n}| > \cdots .
\]

Let $\epsilon' < \epsilon$, then exists some $n \in \mathbb{Z}^+$ such that

(a) either $t_{k_n}^- + \epsilon' \in [t_{k_{n+1}}^-, t_{k_n}^-]$;

(b) or $t_{k_n}^+ + \epsilon' \in G_{k_n}$.

For the first condition, it holds that:

\[
\frac{|(t_{k_n}^+, t_{k_n}^+ + \epsilon') \cap \Sigma_{\alpha,\lambda\nu}|}{2\epsilon'} = \frac{1}{2} \sum_{s \geq n+1} \frac{|G_k|}{2\epsilon'}
\]  

(18)

In the latter case, similarly, we have the following inequality:

\[
\frac{1}{2} - \frac{\sum_{s \geq n+1} |G_k|}{2\epsilon'} \geq \frac{|(t_{k_n}^+, t_{k_n}^+ + \epsilon') \cap \Sigma_{\alpha,\lambda\nu}|}{2\epsilon'}
\]

\[
= \frac{1}{2} - \frac{\sum_{s \geq n+1} |G_k|}{2\epsilon'} - \frac{|t_{k_n}^+ + \epsilon' - t_{k_n}^-|}{2\epsilon'}
\]

\[
\geq \frac{1}{2} - \frac{\sum_{s \geq n+1} |G_k|}{2\epsilon'} - \frac{|G_{k_n}|}{2|t_{k_n}^- - t_{k_n}^+|}
\]  

(19)

Essentially, case (b) is similar to case (a). What is important is to estimate the asymptotic behavior of $\{k_n\}_{n \geq 1}$ as $\epsilon' \to 0$.

Thus, we have the following lemma about the correlation between $k_{n+1}$ and $\epsilon'$, which implies that $|k_{n+1}|$ is polynomially growing with respect to $(\epsilon')^{-1}$.

Lemma 3.2. Assume $t_{k_n}^+$ is the endpoint of some spectral gap $G_k$, $\{k_n\}_{n \geq 1}$ as above. When $t_{k_n}^+ + \epsilon' \in (t_{k_{n+1}}^-, t_{k_n}^-)$, the following holds:

\[
|k_{n+1}| \geq -|k| + C(\epsilon')^\frac{1}{1-\tau},
\]  

(20)

where $C$ depends on $(\alpha, \nu, \lambda, n)$.

Proof. $t_{k_n}^+, t_{k_{n+1}}^+ \in \Sigma_{CR}$. According to Definition 2.4, there exist two integers $N(k_{n+1}) > N(k)$ such that the resonance near $t_k^+$ or $t_{k_{n+1}}^+$ always occurs for any
Then the proof of Lemma 3.2 is completed. Thus

\[ C \lambda^{-\frac{r}{m}} N^{(k_n+1)} \geq |d_{N(k_n+1)}(t^{+}_{k})| - k \alpha \]

\[ = |d_{N(k_n+1)}(t^{+}_{k}) - d_{N(k_n+1)}(t^{+}_{k_{n+1}}) + d_{N(k_n+1)}(t^{+}_{k_{n+1}}) - k_{n+1} \alpha + k_{n+1} \alpha - k \alpha| \]

\[ \geq ||k_{n+1} \alpha - k \alpha||_{\mathbb{R}/\mathbb{Z}} - |d_{N(k_n+1)}(t^{+}_{k}) - d_{N(k_n+1)}(t^{+}_{k_{n+1}})| \]

\[ > c |k_{n+1} - k|^{-\tau+1} - q_{N+N(k_n+1)}^{-1} |t^{+}_{k} - t^{+}_{k_{n+1}}| - C \lambda^{-\frac{r}{m}} N^{(k_n+1)} \]

\[ \geq c |k_{n+1} - k|^{-\tau+1} - q_{N+N(k_n+1)}^{-1} |\epsilon' - C \lambda^{-\frac{r}{m}} N^{(k_n+1)}|, \]

Note that

\[ \epsilon' = |G_{k_n+1}| \gtrsim \lambda^{-\epsilon'} |k_{n+1}| \gg \lambda^{-\epsilon'} \lambda^{-\frac{r}{m}} N^{(k_n+1)} \geq \lambda^{-\epsilon'} \lambda^{-\frac{r}{m}} N^{(k_n+1)}, \]

and choose sufficiently large \( C \)

\[ \epsilon' < |t^{+}_{k} - t^{+}_{k_{n+1}}| < |t^{+}_{k} - t^{+}_{k_{n+1}}| \leq \frac{1}{q_{N+N(k_n+1)} \cdot \epsilon'}, \]

We get

\[ |k_{n+1} - k|^{-\tau+1} \leq q_{N+N(k_n+1)}^{-1} \cdot \epsilon' + C \lambda^{-\frac{r}{m}} N^{(k_n+1)} \]

\[ \leq 2q_{N+N(k_n+1)}^{-1} \cdot \epsilon', \]

Thus

\[ |k_{n+1}| + |k| > |k_{n+1} - k| \geq C(\alpha, \lambda, v, n)(\epsilon')^{\frac{1}{1-\tau}}. \]

Then the proof of Lemma 3.2 is completed. \( \square \)

We find a sequence of \( \{\epsilon'_j\}_{j \geq 1} \), where \([t^{+}_{k_{j+1}}, t^{-}_{k_{j}}] \ni \epsilon'_j \rightarrow 0 \) as \( j \rightarrow \infty \), then Proposition 1 and formula (18), (20) clearly yield

\[ \lim_{j \rightarrow \infty} \frac{|(t^{+}_{k} + \epsilon'_j) \cap \Sigma_{\alpha, \lambda, v}|}{2 \epsilon'_j} = \frac{1}{2}; \]

and Proposition 1, formula (19), (20) imply that, when \( t^{+}_{k} + \epsilon' \in G_{k_n}, \)

\[ \frac{1}{2} \geq \frac{1}{2} - \frac{\sum_{s \geq n+1} |G_{k_s}|}{2 \epsilon'} \geq \frac{|(t^{+}_{k} + t^{+}_{k} + \epsilon') \cap \Sigma_{\alpha, \lambda, v}|}{2 \epsilon'} \]

\[ \geq \frac{1}{2} - \frac{\sum_{s \geq n+1} |G_{k_s}|}{2 \epsilon'} - \frac{|G_{k_n}|}{2 \epsilon'} - \frac{|G_{k_n}|}{2 \epsilon'} - \frac{O(|G_{k_n+1}|)}{2 \epsilon'} - \frac{|G_{k_n}|}{2 \epsilon'} \]

\[ \geq \frac{1}{2} - \frac{C \lambda^{-c(\epsilon')}^{\frac{1}{1-\tau}}}{2 \epsilon'} - \frac{C \lambda^{-c(\tilde{\epsilon})}^{\frac{1}{1-\tau}}}{2 \tilde{\epsilon}} \]

where \( \tilde{\epsilon} \triangleq |t^{+}_{k_n} - t^{+}_{k_{n+1}}|, \) \( C, \tilde{C}, c \) and \( \tilde{\epsilon} \) depend on \( \alpha, \lambda \) and \( v \). Then the last formula tends to \( \frac{1}{2} \) as \( \tilde{\epsilon} \) approaches 0, which proves the second part of Theorem 1.1.
4. The proof of infinitely-resonant spectral points. In this section, our goal is to prove the third result of Theorem 1.1.

Assume \( t_0 \in \Sigma_{IR} \setminus \Sigma_{CR} \). From Definition 2.4, for every \( n \geq 1 \), there must exist a sequence of \( \{ k_n \} \) and \( \{ s(k_n) \} \) such that

\[
I_{s(k_n),2}(t_0) \cap (I_{s(k_n),1}(t_0) + k_n \alpha) \neq \emptyset.
\]

Moreover, according to Section 5 of the paper [15] we can find a special subsequence \( \{ k_{n_j} \} \subset \{ k_n \} \) such that

\[
|d_{s(k_{n_j})}(t_0) - k_{n_j} \alpha| \leq \lambda^{-c|k_{n_j}|}
\]

where \( s(k_{n_j}) \) satisfies \( q_{N+s(k_{n_j})}^2 - 3 \leq |k_{n_j}| < q_{N+s(k_{n_j})}^2 - 2 \). We call \( \{ k_{n_j} \} \) all the resonance-times with respect to \( t_0 \), and \( \{ s(k_{n_j}) \} \) all the resonance-steps of iterations with respect to \( \{ k_{n_j} \} \).

**Lemma 4.1.** The subsequence \( \{ k_{n_j} \} \) above has the following estimate:

\[
|k_{n_j+1}| \geq -|k_{n_j}| + C' \lambda \frac{1}{|k_{n_j}|}
\]

where \( C' \) is an absolute constant and \( c \) depends on \( \lambda \) and \( \nu \).

**Proof.** The proof is similar to Lemma 3.2,

\[
\lambda^{-c|k_{n_j}|} \geq |d_{s(k_{n_j})}(t_0) - k_{n_j} \alpha|
\]

\[
= |d_{s(k_{n_j})}(t_0) - d_{s(k_{n_j+1})}(t_0) + d_{s(k_{n_j+1})}(t_0) - k_{n_j+1} \alpha + k_{n_j+1} \alpha - k_{n_j} \alpha|
\]

\[
\geq \|k_{n_j+1} \alpha - k_{n_j} \alpha\|_{\mathbb{R}/\mathbb{Z}} - |d_{s(k_{n_j})}(t_0) - d_{s(k_{n_j+1})}(t_0)| - |d_{s(k_{n_j+1})}(t_0) - k_{n_j+1} \alpha|
\]

\[
> c|k_{n_j+1} - k_{n_j}|^{-\tau+1} - C\lambda^{-2} r_{s(k_{n_j})} - 1 - \lambda^{-c|k_{n_j+1}|}
\]

Due to

\[
|k_{n_j+1}| = q_{N+s(k_{n_j+1})} > q_{N+s(k_{n_j})+2} > |k_{n_j}|;
\]

and

\[
|k_{n_j+1}| > q_{N+s(k_{n_j+1})} \geq q_{N+s(k_{n_j})+2} > |k_{n_j}|.
\]

Then we have \( |k_{n_j+1} - k_{n_j}| > C' \lambda \frac{1}{|k_{n_j}|} \). We complete the proof of lemma. \( \square \)

**Definition 4.2.** We introduce a quantity to describe the convergence rate of spectral gaps near \( t_0 \):

\[
\gamma(t_0) = \lim \inf _{n \to \infty} \left( \frac{1}{2} + \min \left\{ \frac{|t_0 - t_{k_n}^-|}{2|t_0 - t_{k_n}^+|}, \frac{|t_0 - t_{k_n}^+|}{2|t_0 - t_{k_n}^-|} \right\} \right)
\]

where \( t_{k_n}^- \), \( t_{k_n}^+ \) are the endpoints of spectral gap \( G_{k_n} \). If the gap is on the left side of \( t_0 \), \( t_{k_n}^+ \) is closer to \( t_0 \); otherwise it is \( t_{k_n}^- \).

It is easy to find that for all \( t \in \Sigma_{\lambda} \)

\[
\frac{1}{2} \leq \gamma(t) \leq 1;
\]

if \( t \in \Sigma_{CR} \), then \( \gamma(t) = \frac{1}{2} \); if \( t \in \Sigma_{FR} \), then \( \gamma(t) = 1 \).

**The existence of \( t_0 \) satisfying \( \gamma(t_0) = \gamma \) for any \( \gamma \in (\frac{1}{2}, 1) \):** By the definition, we need to find the point \( t_\infty \) satisfying
Fix property above. We need to construct a sequence of 
\[ \{ t_{\hat{k}_n} \}_{j \geq 1} \subset \Sigma_{EP} \rightarrow t_\infty \in \Sigma_{IR} \] satisfy the property above.

**Step 1:** Fix \( \frac{1}{2} < \gamma < 1 \) and choose the point \( t_0 \in \Sigma_{EP} \). Without loss of generality, assume \( t_0 \) is the endpoint of gap \( G_k \) and exists some \( s(k) < N(t_0) \) such that

\[ |d_{s(k)}(t_0) - k\alpha| \leq \kappa \lambda^{-c|k|}. \]

According to Lemma 2.3, we know that \( g_{s(k)}(x, t_0) \) is of type III and let \( \hat{k}_{n_1}(t_0) = k \). Lemma 2.5 implies that, and there exists some \( t_1 \in B(t_0, 2\kappa|G_k|) \) such that

\[ |d_{s(k)}(t_1) - k\alpha| = \kappa|G_k|. \]

**Step 2:** Consider the following different cases.

(a) If exists some \( m_1 > s(k) \) such that \( g_{m_1}(x, t_1) \) is of type III, then we consider

\[ k_2(t_1), \text{ where } \kappa \lambda^{-c|k|} > c \lambda^{-c|k_2(t_1)|} = |G_{k_2}(t_1)|. \]

(a1) If \( t_1 \in \Sigma^\lambda \), by the denseness of \( \Sigma_{EP} \) in the spectrum, we can find \( \hat{t}_1 \in \Sigma_{EP} \) such that \( |\hat{t}_1 - t_1| < \epsilon_0 = \lambda^{-\frac{1}{2}+o(\kappa^2(t_1)^{-1})} \), meanwhile \( g_{m_1}(x, \hat{t}_1) \) and \( g_{m_1}(x, t_1) \) have the same type. By Lemma 2.5 and Proposition 1, there exists some spectral point \( t_2 \in B(\hat{t}_1, 2\kappa|G_{k_2}(t_2)|) \cap \Sigma_{EP} \), where \( \hat{k}_{n_2}(t_2) = k_{2}(t_1) \), satisfying the following properties:

- \( |\hat{k}_{n_1}(t_2)| = |k| < |\hat{k}_{n_2}(t_2)| = |k_2(t_1)| \), and \( |d_{s(\hat{k}_{n_1}(t_2))}(t_2) - d_{s(\hat{k}_{n_1}(t_2))}(t_1)| = \frac{1}{2} \)
- \( |d_{s(\hat{k}_{n_2}(t_2))}(t_2) - \hat{k}_{n_2}(t_2)| = \kappa|G_{k_{n_2}(t_2)}| \);

(a2) If \( t_1 \notin \Sigma^\lambda \), then \( t_1 \in G_{k_2(t_1)} \). Clearly, for all \( \hat{t}_1 \in B(t_1, |G_{k_2(t_1)}|) \), \( g_{m_1}(x, \hat{t}_1) \) is of type III. Let \( \hat{t}_1 \in B(t_1, |G_{k_2(t_1)}|) \cap \Sigma_{EP} \) and assume the label associated is \( k(\hat{t}_1) \). Then Lemma 2.5 and Proposition 1 imply that there exists some \( t_2 \in B(\hat{t}_1, 2\kappa|G_{k_{n_2}(t_2)}|) \), where \( \hat{k}_{n_2}(t_2) = k(\hat{t}_1) \), satisfying the several properties of (a1).

(b) If \( g_{m}(x, t_1) \) is of type I for all \( m > s(k) \), then we have

\[ \lim_{m \to \infty} \min_{x \in I_{m}(t_1)} g_{m}(x, t_1) = 0. \]

[18] implies that \( t_1 \in \Sigma^\lambda \)(more concretely, \( t_1 \in \Sigma_{FR} \). Now let \( \epsilon \) sufficiently small such that \( 0 < \epsilon < \frac{1}{2} \)|. According to [17], we can find some point \( \hat{t}_1 \in (t_1 - \epsilon, t_1 + \epsilon) \) and \( \hat{t}_1 \notin \Sigma^\lambda \). Hence \( \hat{t}_1 \in G_M = (t_M^-, t_M^+) \) where \( |M| \gg |k| \). There must exist some \( \hat{m}_1 > s(k) \) such that \( \hat{g}_{\hat{m}_1}(x, \hat{t}_1) \) is of type III. Similar to the case (a2), we can find \( \hat{t}_1 \in B(\hat{t}_1, |G_{k_{n_2}(t_2)}|) \cap \Sigma_{EP} \) and \( t_2 \in B(\hat{t}_1, 2\kappa|G_{k_{n_2}(t_2)}|) \) satisfying the above properties.

**Step 3:** Repeat the above progress, it is not difficult to obtain a sequence of endpoints of gaps, \( \{ t_n \}_{n \geq 1} \), and find a point \( t_\infty \in \bigcap_{j \geq 1} B(\hat{t}_1, 2\kappa|G_{k_{n_1}(t_\infty)}|) \), where

\[ B(t_i+1, 2\kappa|G_{k_{n_i+1}(t_\infty)}|) \subseteq B(t_i, 2\kappa|G_{k_{n_i}(t_\infty)}|), \]

satisfying the following properties:
\textbullet{} \abs{k} = \abs{\hat{k}_{n_1}(t_\infty)} \ll \abs{\hat{k}_{n_2}(t_\infty)} \ll \ldots \ll \abs{\hat{k}_{n_j}(t_\infty)} \ll \ldots, \text{we have } d_{\delta_s(\hat{k}_{n_j}(t_\infty))}(t_\infty) - d_{\delta_s(\hat{k}_{n_j}(t_\infty))}(t_j) = c|t_\infty - t_j|, \forall j \in \mathbb{Z}^+; \\
\textbullet{} \abs{d_{\delta_s(\hat{k}_{n_j}(t_\infty))}(t_\infty) - \hat{k}_{n_j}(t_\infty)\alpha} = \kappa|G_{\hat{k}_{n_j}(t_\infty)}| + \sum_{i \geq j+1} o(\kappa|G_{\hat{k}_{n_i}(t_\infty)}|).

Then we complete the construction of \( \gamma(t_0) = \gamma \). 

\textbf{Remark 4.} From the above construction progress, for \( t_0 \in \Sigma I_R \setminus \Sigma_{C'B} \):

\[
\gamma(t_0) = \liminf_{n \to \infty} \left( \frac{1}{2} + \min \left\{ \frac{|t_0 - t_{\bar{k}_n}|}{2|t_0 - t_{\bar{k}^+}|}, \frac{|t_0 - t_{\bar{k}^+}|}{2|t_0 - t_{\bar{k}^-}|} \right\} \right)
\]

\[
= \lim_{j \to \infty} \left( \frac{1}{2} + \min \left\{ \frac{|t_0 - t_{\bar{k}_j}|}{2|t_0 - t_{\bar{k}^+_j}|}, \frac{|t_0 - t_{\bar{k}^+_j}|}{2|t_0 - t_{\bar{k}^-_j}|} \right\} \right)
\]

where \( \{k_{n_j}\}_{j \geq 1} \) are all resonance-times with respect to \( t_0 \).

Next, let \( \kappa := \frac{2\bar{\lambda} + \lambda}{2\bar{\lambda} - \lambda} \). Without loss of generality, we assume that for all \( j \geq 1 \), \( G_{k_{n_j}} \) is located in the right of \( t_0 \).

On the one hand, if \( t_0 \) satisfies \( \gamma(t_0) = \gamma \in (\frac{1}{2}, 1) \), we denote \( \epsilon_j' := |t_0 - t_{\bar{k}^-_j}| \), where \( \epsilon_j' \to 0 \) as \( j \to \infty \). When \( j \) is sufficiently large, we have \( |t_0 - t_{\bar{k}^-_j}| \approx \kappa|G_{k_{n_j}}| \).

Consider the interval \((t_0 - \epsilon_j', t_0 + \epsilon_j')\). According to the analysis above, \( G_{k_{n_{j+1}}} \) is the maximum gap in the interval. Then the following holds:

\[
\frac{|(t_0 - \epsilon_j', t_0 + \epsilon_j') \cap \Sigma^\lambda|}{2\epsilon_j'} = 1 - \frac{\sum_{i \geq j+1} |G_{k_{n_i}}|}{2\epsilon_j'} \approx 1 - \frac{|G_{k_{n_{j+1}}}|}{2\kappa|G_{k_{n_j}}|}
\]

When \( j \) is sufficiently large, on the grounds of Proposition 1 and Lemma 4.1, the right-hand most formula tends to 1. Hence

\[
1 \geq \limsup_{\epsilon \to 0} \frac{|(t_0 - \epsilon, t_0 + \epsilon) \cap \Sigma^\lambda|}{2\epsilon} \geq \lim_{j \to \infty} \frac{|(t_0 - \epsilon_j', t_0 + \epsilon_j) \cap \Sigma^\lambda|}{2\epsilon_j'} = 1
\]

The formula (7) is certified.

On the other hand, fix \( \gamma \) and denote \( \epsilon_j := |t_0 - t_{\bar{k}^+_j}| \), then consider the interval \((t_0 - \epsilon_j, t_0 + \epsilon_j)\), we have the estimate below:

\[
\lim_{j \to \infty} \frac{|(t_0 - \epsilon_j, t_0 + \epsilon_j) \cap \Sigma^\lambda|}{2\epsilon_j} = \lim_{j \to \infty} \frac{|(t_0 - \epsilon_j, t_0) \cap \Sigma^\lambda|}{2\epsilon_j} + \frac{|(t_0, t_0 + \epsilon_j) \cap \Sigma^\lambda|}{2\epsilon_j}
\]

\[
= 1 - \frac{1}{2 + 2\kappa}
\]

Therefore, we find a sequence of \( \{\epsilon_j\}_{j \geq 1} \) which tends to 0 such that:

\[
\liminf_{\epsilon \to 0} \frac{|(t_0 - \epsilon, t_0 + \epsilon) \cap \Sigma^\lambda|}{2\epsilon} \leq \lim_{j \to \infty} \frac{|(t_0 - \epsilon_j, t_0 + \epsilon_j) \cap \Sigma^\lambda|}{2\epsilon_j} = \beta
\] (22)
where $\beta := 1 - \frac{1}{2 + 2k}$. Meanwhile, for any $\sigma \in (\beta, 1)$, we can also find a sequence of $\{\epsilon_j''\}_{j \geq 1}$ satisfying
\[
\lim_{j \to \infty} \frac{|(t_0 - \epsilon_j'', t_0 + \epsilon_j'') \cap \Sigma^\lambda|}{2\epsilon_j''} = \sigma, \tag{23}
\]
where $t_0 + \epsilon_j'' \in G_{kn_j}$, $j \geq 1$. Let $\{\epsilon_j''\}_{j \geq 1}$ be another sequence, such that $t_0 + \epsilon_j'' \in (t_0 + \epsilon_{j+1}, t_0 + \epsilon_j')$, note that $\epsilon_j+1 < \epsilon_j'' < \epsilon_j'$. Thus we have:
\[
\frac{|(t_0 - \epsilon_j'', t_0 + \epsilon_j''') \cap \Sigma^\lambda|}{2\epsilon_j''} = (1 - \frac{|(t_0 - \epsilon_j'', t_0 + \epsilon_j'') \cap \text{Gaps}|}{2\epsilon_j''}) - (1 - \frac{|(t_0 - \epsilon_j, t_0 + \epsilon_j+1) \cap \text{Gaps}|}{2\epsilon_j+1}) \geq 0
\]
Then
\[
\lim_{j \to \infty} \frac{|(t_0 - \epsilon_j'', t_0 + \epsilon_j''') \cap \Sigma^\lambda|}{2\epsilon_j''} \geq \lim_{j \to \infty} \frac{|(t_0 - \epsilon_j+1, t_0 + \epsilon_j+1) \cap \Sigma^\lambda|}{2\epsilon_j+1} \tag{24}
\]
In conclusion, according to formula (22) (23) and (24), for $t_0$ satisfying $\gamma(t_0) = \gamma$, the limit inferior is not less than $\beta$. We have:
\[
\liminf_{\epsilon \to 0} \frac{|(t_0 - \epsilon, t_0 + \epsilon) \cap \Sigma^\lambda|}{2\epsilon} = \beta.
\]

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