Quasilinear theory for inhomogeneous plasma

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This paper presents quasilinear theory (QLT) for a classical plasma interacting with inhomogeneous turbulence. The particle Hamiltonian is kept general; for example, relativistic, electromagnetic and gravitational effects are subsumed. A Fokker–Planck equation for the dressed ‘oscillation-centre’ distribution is derived from the Klimontovich equation and captures quasilinear diffusion, interaction with the background fields and ponderomotive effects simultaneously. The local diffusion coefficient is manifestly positive-semidefinite. Waves are allowed to be off-shell (i.e. not constrained by a dispersion relation), and a collision integral of the Balescu–Lenard type emerges in a form that is not restricted to any particular Hamiltonian. This operator conserves particles, momentum and energy, and it also satisfies the $H$-theorem, as usual. As a spin-off, a general expression for the spectrum of microscopic fluctuations is derived. For on-shell waves, which satisfy a quasilinear wave-kinetic equation, the theory conserves the momentum and energy of the wave–plasma system. The action of non-resonant waves is also conserved, unlike in the standard version of QLT. Dewar’s oscillation-centre QLT of electrostatic turbulence (Phys. Fluids, vol. 16, 1973, p. 1102) is proven formally as a particular case and given a concise formulation. Also discussed as examples are relativistic electromagnetic and gravitational interactions, and QLT for gravitational waves is proposed.

Key words: plasma nonlinear phenomena, plasma waves

1. Introduction

1.1. Background

Electromagnetic waves are present in plasmas naturally, and they are also launched into plasmas using external antennas, for example, for plasma heating and current drive (Fisch 1987; Stix 1992; Pinsker 2001). Nonlinear effects produced by these waves are often modelled within the quasilinear (QL) approximation, meaning that the nonlinearities are retained in the low-frequency (‘average’) dynamics but neglected in the high-frequency dynamics. Two separate paradigms exist within this approach.

In the first paradigm, commonly known as ‘the’ QL theory (QLT), the focus is on resonant interactions. Non-resonant particles are considered as a background that is homogeneous in spatial (Vedenov, Velikhov & Sagdeev 1961; Drummond & Pines 1962; Kennel & Engelmann 1966; Rogister & Oberman 1968, 1969) or generalized coordinates

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(Kaufman 1972; Eriksson & Helander 1994; Catto, Lee & Ram 2017); then, the oscillating fields can be described in terms of global modes. This approach has the advantage of simplicity, but its applications are limited in that real plasmas are never actually homogeneous in any predefined variables (and, furthermore, tend to exhibit nonlinear instabilities in the presence of intense waves). The ‘ponderomotive’ dynamics determined by the gradients of the wave and plasma parameters is lost in this approach; then, spurious effects can emerge and have to be dealt with (Lee et al. 2018).

The second paradigm successfully captures the ponderomotive dynamics by introducing effective Hamiltonians for the particle average motion (Gaponov & Miller 1958; Motz & Watson 1967; Cary & Kaufman 1981; Kaufman 1987; Dodin 2014). But, as usual in perturbation theory (Lichtenberg & Lieberman 1992), those Hamiltonians are by default singular for resonant interactions. Thus, such models have limited reach as well, and remarkable subtleties are still found even in basic QL problems. For example, it is still debated (Ochs & Fisch 2021a; Ochs 2021) to which extent the QL effects that remove resonant particles while capturing their energy (Fisch & Rax 1992) also remove charge along with the resonant particles, thereby driving plasma rotation (Fetterman & Fisch 2008). This state of affairs means, arguably, that a clear comprehensive theory of QL wave–plasma interactions remains to be developed – a challenge that must be faced.

The first framework that subsumed both resonant and non-resonant interactions in inhomogeneous plasmas was proposed by Dewar (1973) for electrostatic turbulence in non-magnetized plasma and is known as ‘oscillation-centre’ (OC) QLT. It was later extended by McDonald, Grebogi & Kaufman (1985) to non-relativistic magnetized plasma. However, both of these models are partly heuristic and limited in several respects. For example, they are bounded by the limitations of the variational approach used therein, and they separate resonant particles from non-resonant particles somewhat arbitrarily (see also Ye & Kaufman 1992). Both models also assume specific particle Hamiltonians and require that waves be governed by a QL wave-kinetic equation (WKE), i.e. be only weakly dissipative, or ‘on-shell’. (Somewhat similar formulations were also proposed, independently and without references to the OC formalism, in Weibel 1981; Yasseen 1983; Yasseen & Vaclavik 1986.) This means that collisions and microscopic fluctuations are automatically excluded. Attempts to merge QLT and the WKE with the theory of plasma collisions were made (Rogister & Oberman 1968; Schlickeiser & Yoon 2014; Yoon et al. 2016) but have not yielded a local theory applicable to inhomogeneous plasma. In particular, the existing models rely on global-mode decompositions and treat complex frequencies heuristically. Thus, the challenge stands.

Related problems are also of interest in the context of gravitostatic interactions (Chavanis 2012; Hamilton 2020; Magorrian 2021), where inhomogeneity of the background fields cannot be neglected in principle (Binney & Tremaine 2008). (To our knowledge, OC QLT analogues have not been considered in this field.) Similar challenges also arise in QLT of dispersive gravitational waves (Garg & Dodin 2020, 2021). Hence, one cannot help but wonder whether a specific form of the particle Hamiltonian really matters for developing QLT or it is irrelevant and therefore should not be assumed. Since basic theory of linear waves is independent of Maxwell’s equations (Dodin & Fisch 2012; Tracy et al. 2014; Dodin, Zhmoginov & Ruiz 2017), a general QLT might be possible too, and it might be easier to develop than a zoo of problem-specific models.

1.2. Outline

Here, we propose a general QLT that allows for plasma inhomogeneity and is not restricted to any particular Hamiltonian or interaction field. By starting with the Klimontovich equation, we derive a model that captures QL diffusion, interaction with background fields...
and ponderomotive effects simultaneously. The local diffusion coefficient in this model is manifestly positive-semidefinite. Waves are allowed to be off-shell, and a collision integral of the Balescu–Lenard type emerges for general Hamiltonian interactions. This operator conserves particles, momentum and energy, and it also satisfies the $H$-theorem, as usual. As a spin-off, a general expression for the spectrum of microscopic fluctuations of the interaction field is derived. For on-shell waves governed by the WKE, the theory conserves the momentum and energy of the wave–plasma system. The action of non-resonant waves is also conserved, unlike in the standard version of QLT.\textsuperscript{1} Dewar’s OC QLT of electrostatic turbulence (Dewar 1973) is proven formally as a particular case and given a concise formulation. Also discussed as examples are relativistic electromagnetic and gravitational interactions, and QLT for gravitational waves is proposed. Overall, our formulation interconnects many known results that in the past were derived independently and reproduces them within a unifying framework.

This progress is made by giving up the traditional Fourier–Laplace approach. The author takes the stance that the global-mode language is unnatural for inhomogeneous-plasma problems (i.e. all real-plasma problems). A fundamental theory must be local. Likewise, the variational approach that is used sometimes in QL calculations is not universally advantageous, especially for describing dissipation. Instead of those methods, we use operator analysis and the Weyl symbol calculus, as has also been proven fruitful in other recent studies of ponderomotive effects and turbulence (Ruiz 2017; Ruiz & Dodin 2017b; Zhu & Dodin 2021) and linear-wave theory (Dodin et al. 2019). No logical leaps are made in this paper other than assuming the QL approximation per se and a certain ordering.\textsuperscript{2} In a nutshell, we treat the commonly known QL-diffusion coefficient as a non-local operator, and we systematically approximate it using the Weyl symbol calculus. It is the non-locality of this operator that gives rise to ponderomotive effects and ensures the proper conservation laws. The existing concept of ‘adiabatic diffusion’ (Galeev & Sagdeev 1985; Stix 1992) captures some of that, but systematic application of operator analysis yields a more general, more accurate and more rigorous theory.

The author hopes not that this paper is an entertaining read. However, the paper was intended as self-contained, maximally structured and easily searchable, so readers interested in specific questions could find and understand answers without having to read the whole paper. The text is organized as follows. In § 2, we present a primer on the Weyl symbol calculus and the associated notation. In § 3, we formulate our general model. In § 4, we introduce the necessary auxiliary theorems. In § 5, we derive a QL model for plasma interacting with prescribed waves. The waves may or may not be on-shell or self-consistent. (Their origin and dynamics are not addressed in § 5.) In § 6, we consider interactions with self-consistent waves. In particular, we separate out microscopic fluctuations, calculate their average distribution and derive the corresponding collision operator. In § 7, we assume that the remaining macroscopic waves are on-shell, rederive the WKE, and show that our QL model combined with the WKE is conservative. In § 8, we discuss the general properties that our model predicts for plasmas in thermal equilibrium. In § 9, we show how to apply our theory to non-relativistic electrostatic interactions, relativistic electromagnetic interactions, Newtonian gravity and relativistic gravity. In § 10, we summarize our results. Auxiliary calculations are presented in Appendices A–D, and Appendix G summarizes our notation. This notation is extensive and may not be

\textsuperscript{1}The standard QLT (as in, for example, Drummond & Pines 1962) does not properly conserve energy–momentum either, even though it is formally conservative (see § 7.3.2).

\textsuperscript{2}We treat the traditional QL approximation as a given mathematical model. We seek to push this model to its limits rather than to examine its validity, which is a separate issue. For discussions on the validity of the QL approximation, see Besse et al. (2011), Escande et al. (2018) and Crews & Shumlak (2022).
particularly intuitive. Thus, readers are encouraged to occasionally scout § 9 for examples even before fully absorbing the preceding sections.

An impatient reader can also skip calculations entirely and consult only the summaries of the individual sections (2.3, 3.4, 4.4, 5.6, 6.9, 7.6 and 8.5; they are mostly self-contained) and then proceed to the examples in § 9. However, the main point of this work is not just the final results per se (surely, some readers will find them obvious) but also that they are derived with minimal assumptions and rigorously, which makes them reliable. A reader may also notice that we rederive some known results along the way, for example, basic linear-wave theory and the WKE. This is done for completeness and, more importantly, with the goal to present all pieces of the story within a unified notation.

2. A math primer

Here, we summarize the machinery to be used in the next sections. This machinery is not new, but a brief overview is in order at least to introduce the necessary notation. A more comprehensive summary, with proofs, can be found in Dodin et al. (2019, supplementary material). For extended discussions, see Tracy et al. (2014), Ruiz (2017), McDonald (1988) and Littlejohn (1986).

2.1. Weyl symbol calculus on spacetime

2.1.1. Basic notation

We denote the time variable as \( t \), space coordinates as \( x \equiv (x^1, x^2, \ldots, x^n) \), spacetime coordinates as \( x \equiv (x^0, x^1, \ldots, x^n) \), where \( x^0 = t \) and \( x^i = x^i \). The symbol \( \equiv \) denotes definitions, and Latin indices from the middle of the alphabet (\( i, j, \ldots \)) range from 1 to \( n \) unless specified otherwise. We assume the spacetime-coordinate domain to be \( \mathbb{R}^n \).

Functions on \( x \) form a Hilbert space \( \mathcal{H}_x \) with an inner product that we define as

\[
\langle \xi | \psi \rangle \equiv \int d^nx \xi^*(x) \psi(x). \tag{2.1}
\]

The symbol * denotes complex conjugate,

\[
dx \equiv dx^0 dx^1 \ldots dx^n = dt dx^1 \ldots dx^n \tag{2.2}\]

(a similar convention is assumed also for other multi-dimensional variables used below), and integrals in this paper are taken over \((-\infty, \infty)\) unless specified otherwise. Operators on \( \mathcal{H}_x \) will be denoted with carets, and we will use indexes \( \mathcal{H} \) and \( \mathcal{A} \) to denote their Hermitian and anti-Hermitian parts. For a given operator \( \hat{A} \), one has \( \hat{A} = \hat{A}^\mathcal{H} + i\hat{A}^\mathcal{A} \).

\[
\hat{A}^\mathcal{H} = \hat{A}^\mathcal{H} = \frac{1}{2} (\hat{A} + \hat{A}^\dagger), \quad \hat{A}^\mathcal{A} = \hat{A}^\mathcal{A} = \frac{1}{2i} (\hat{A} - \hat{A}^\dagger), \tag{2.3}
\]

where \( \dagger \) denotes the Hermitian adjoint with respect to the inner product (2.1). The case of a more general inner product is detailed in Dodin et al. (2019, supplementary material).

2.1.2. Vector fields

For multi-component fields \( \psi \equiv (\psi^1, \psi^2, \ldots, \psi^M)^T \) (our \( ^T \) denotes the matrix transpose), or ‘row vectors’ (actually, tuples), we define the dual ‘column vectors’ \( \psi^\dagger \equiv (\psi_1^\dagger, \psi_2^\dagger, \ldots, \psi_M^\dagger)^T \) via \( \psi^\dagger \doteq \mathbf{g}\psi^* \). The matrix \( \mathbf{g} \) is assumed to be real, diagonal,

\^3This excludes periodic boundary conditions, albeit not entirely (§ 3.1). Other than that, the spacetime metric can still be non-Euclidean, as illustrated by an application to relativistic gravity in § 9.4. See also the footnotes in §§ 2.1.3 and 6.1.
invertible and constant; other than that, it can be chosen as suits a specific problem. (For example, a unit matrix may suffice.) This induces the standard rule of index manipulation

$$\psi_i = g_{ij} \psi^j, \quad \psi^j = g^{ij} \psi_j, \quad i, j = 1, 2, \ldots, M,$$

(2.4)

where $g_{ij}$ are elements of $g$ and $g^{ij}$ are elements of $g^{-1}$. Summation over repeating indices is assumed. The rules of matrix multiplication apply to row and column vectors as usual. Then, for $\psi \equiv (\psi^1, \psi^2, \ldots, \psi^M)^T$ and $\xi \equiv (\xi^1, \xi^2, \ldots, \xi^M)^T$, the quantity $\psi \xi$ is a matrix with elements $\psi^i \xi^j$, $\psi \xi^j$ is a matrix with elements $\psi^i \xi^j$ and $\xi^j \psi$ is its scalar trace:

$$\xi^j \psi = \text{tr}(\psi \xi^j) = \xi^j \psi^i = g_{ij} \xi^i \psi^j, \quad i, j = 1, 2, \ldots, M.$$

(2.5)

(Similarly, for $\chi \equiv (\chi^1, \chi^2, \ldots, \chi^M)$ and $\eta \equiv (\eta^1, \eta^2, \ldots, \eta^M)$, $\eta \chi$ is a matrix with elements $\eta_i \chi_i$.) We use (2.5) to define a Hilbert space $\mathcal{H}^M_x$ of $M$-dimensional vector fields on $x$, specifically, by adopting the inner product

$$\langle \xi | \psi \rangle = \int dx \xi^j(x) \psi(x).$$

(2.6)

Below, the distinction between $\mathcal{H}^M_x$ and $\mathcal{H}_x$ will be assumed but not emphasized. Also note that (2.5) yields

$$\xi^j (\psi \psi^j) ^2 = (\xi^j \psi^i) (\psi^j \xi_i) = (\xi^j \psi^i)^2 \geq 0,$$

(2.7)

for any $\xi$ and $\psi$. Thus, dyadic matrices of the form $\psi \psi^j$ are positive-semidefinite, even though $\psi^j \psi$ may be negative (when $g$ is not positive-definite).

For general matrices, the indices can be raised and lowered using $g$ and $g^{-1}$ as usual. The Hermitian adjoint $A^\dagger$ for a given matrix $A$ is defined such that $(A^\dagger \xi)^j = \xi^j (A \psi)$ for any $\psi$ and $\xi$, which means

$$(A^\dagger)_j^i = (A)_j^i = (A)^{i*}_j, \quad i, j = 1, 2, \ldots, M.$$  

(2.8)

The Hermitian and anti-Hermitian parts are defined as

$$A_H = A_H^\dagger = \frac{1}{2} (A + A^\dagger), \quad A_A = A_A^\dagger = \frac{1}{2i} (A - A^\dagger),$$  

(2.9)

so $A = A_H + i A_A$. For one-dimensional matrices (scalars), one has $A = A_H$, $A_A = A_A^\dagger = \text{im} A$, $A_H = A_H^\dagger = \text{re} A$.

(2.10)

where $\text{re}$ and $\text{im}$ denote the real part and the imaginary part, respectively. We also define matrix operators $\hat{A}$ as matrices of the corresponding operators $\hat{A}_j$. Because $g$ is constant, index manipulation applies as usual. Also as usual, one has

$$\hat{A}_H = \hat{A}_H^\dagger = \frac{1}{2} (\hat{A} + \hat{A}^\dagger), \quad \hat{A}_A = \hat{A}_A^\dagger = \frac{1}{2i} (\hat{A} - \hat{A}^\dagger),$$

(2.11)

and $\hat{A} = \hat{A}_H + i \hat{A}_A$, where $^\dagger$ is the Hermitian adjoint with respect to the inner product (2.6).

\[^4A\text{ common notation is } \xi^j \psi = \xi \cdot \psi, \text{ but we reserve the dot-product notation for a scalar product of different quantities (§2.1.3).}\]
2.1.3. Bra–ket notation

Let us define the following operators that are Hermitian under the inner product (2.1):

\[
\hat{x}^0 \equiv \hat{t} \equiv t, \quad \hat{x}^i \equiv \hat{x}^i \equiv x^i, \quad \hat{k}_0 \equiv -\hat{\omega} \equiv -i\partial_t, \quad \hat{k}_i \equiv -i\partial_i,
\]

(2.12)

where \( \partial_0 \equiv \partial_t \equiv \partial/\partial x^0 \) and \( \partial_i \equiv \partial/\partial x^i \). Accordingly,

\[
\hat{x} \equiv (\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^n) = (\hat{t}, \hat{x}), \quad \hat{k} \equiv (\hat{k}_0, \hat{k}_1, \ldots, \hat{k}_n) = (-\hat{\omega}, \hat{k})
\]

(2.13)

are understood as the spacetime-position operator and the corresponding wavevector operator, which will also be expressed as follows:

\[
\hat{x} = x, \quad \hat{k} = -i\partial_x.
\]

(2.14)

Also note the commutation property, where \( \delta^i_j \) is the Kronecker symbol:

\[
[\hat{x}^i, \hat{k}_j] = i\delta^i_j, \quad i, j = 0, 1, \ldots, n.
\]

(2.15)

The eigenvectors of the operators (2.14) will be denoted as ‘kets’ \( |x\rangle \) and \( |k\rangle \):

\[
\hat{x} |x\rangle = x |x\rangle, \quad \hat{k} |k\rangle = k |k\rangle
\]

(2.16)

and we assume the usual normalization

\[
\langle x_1 | x_2 \rangle = \delta(x_1 - x_2), \quad \langle k_1 | k_2 \rangle = \delta(k_1 - k_2),
\]

(2.17)

where \( \delta \) is the Dirac delta function. Both sets \( \{ |x\rangle, x \in \mathbb{R}^n \} \) and \( \{ |k\rangle, k \in \mathbb{R}^n \} \), where \( n = n + 1 \), form a complete basis on \( \mathcal{H}_x \), and the eigenvalues of these operators form an extended real phase space \( \mathcal{H}_s \), where

\[
x \equiv (t, x), \quad k \equiv (-\omega, k).
\]

(2.18)

The notation \( k \cdot s \equiv -\omega \tau + k \cdot s \) will be assumed for any \( s \equiv (\tau, s) \) and \( k \cdot s \equiv k_0 s^0 \). In particular, for any \( \psi \) and constant \( s \), one has

\[
\exp(i\hat{k} \cdot s)\psi(x) = \exp(s \cdot \partial_x)\psi(x) = \psi(x + s),
\]

(2.19)

as seen from comparing the Taylor expansions in \( s \) of the latter two expressions. (A generalization of this formula is discussed in § 4.1.) Also,

\[
\langle x | k \rangle = \langle k | x \rangle^* = (2\pi)^{-n/2} \exp(i k \cdot x),
\]

(2.20)

and

\[
\int dx \ |x\rangle \langle x| = \hat{1}, \quad \int dk \ |k\rangle \langle k| = \hat{1}.
\]

(2.21)

Here, ‘bra’ \( \langle x| \) is the one-form dual to \( |x\rangle \), \( \langle k| \) is the one-form dual to \( |k\rangle \), and \( \hat{1} \) is the unit operator. Any field \( \psi \) on \( x \) can be viewed as the \( x \) representation (‘coordinate

5Spaces with periodic boundary conditions require a different approach (Rigas et al. 2011), so they are not considered here (yet see § 3.1). That said, for a system that is large enough, the boundary conditions are unimportant; then the toolbox presented here is applicable as is.

6More precisely, \( |k\rangle \) is the ket \( |\hat{e}(\vec{k}; x)\rangle \) that is an eigenvector of each \( \hat{x}' \) with the corresponding eigenvalues being \( x' \). A similar comment applies to \( |k\rangle \).
representation’) of $|\psi\rangle$, i.e. the projection of an abstract ket vector $|\psi\rangle \in \mathcal{H}_x$ on $|x\rangle$:

$$\psi(x) = (x|\psi\rangle.$$  \hspace{1cm} (2.22)

Similarly, $(k|\psi\rangle)$ is the $k$ representation (‘spectral representation’) of $|\psi\rangle$, or the Fourier image of $\psi$:

$$\hat{\psi}(k) \doteq (k|\psi\rangle) = \frac{1}{(2\pi)^{n/2}} \int \text{d}x \ e^{-ik \cdot x} \psi(x).$$  \hspace{1cm} (2.23)

### 2.1.4. Wigner–Weyl transform

For a given operator $\hat{A}$ and a given field $\hat{\psi}$, $\hat{A}\hat{\psi}$ can be expressed in the integral form

$$\hat{A}\psi(x) = \int \text{d}x' (x|\hat{A}|x') \psi(x').$$  \hspace{1cm} (2.24)

where $(x|\hat{A}|x')$ is a function of $(x,x')$. This is called the $x$ representation (‘coordinate representation’) of $\hat{A}$. Equivalently, $\hat{A}$ can be given a phase-space, or Weyl, representation, i.e. expressed through a function of the phase-space coordinates, $A(x,k)$:

$$\hat{A} = \frac{1}{(2\pi)^n} \int \text{d}x \text{d}k \text{d}s \ |x + s/2\rangle A(x,k) e^{ik \cdot s} \langle x - s/2| \equiv \text{op}_{x,k} A.$$  \hspace{1cm} (2.25)

The function $A(x,k)$, called the Weyl symbol (or just ‘symbol’) of $\hat{A}$, is given by

$$A(x,k) \doteq \int \text{d}s \ (x + s/2|\hat{A}|x - s/2) \ e^{-ik \cdot s} \equiv \text{symb}_x \hat{A}.$$  \hspace{1cm} (2.26)

The $x$ and phase-space representations are connected by the Fourier transform:

$$\langle x|\hat{A}|x'\rangle = \frac{1}{(2\pi)^n} \int \text{d}k e^{ik \cdot (x - x')} A \left( \frac{x + x'}{2}, k \right).$$  \hspace{1cm} (2.27)

This also leads to the following notable properties of Weyl symbols:

$$\langle x|\hat{A}|x\rangle = \frac{1}{(2\pi)^n} \int \text{d}k A(x,k), \quad \langle k|\hat{A}|k\rangle = \frac{1}{(2\pi)^n} \int \text{d}x A(x,k).$$  \hspace{1cm} (2.28)

An operator unambiguously determines its symbol, and vice versa. We denote this isomorphism as $\hat{A} \leftrightarrow A$. The mapping $\hat{A} \mapsto A$ is called the Wigner transform, and $A \mapsto \hat{A}$ is called the Weyl transform. For uniformity, we call them the direct and inverse Wigner–Weyl transform. The isomorphism $\leftrightarrow$ is natural in that it has the following properties:

$$\hat{\hat{A}} \leftrightarrow 1, \quad \hat{x} \leftrightarrow x, \quad \hat{k} \leftrightarrow k, \quad h(\hat{x}) \leftrightarrow h(x), \quad h(\hat{k}) \leftrightarrow h(k), \quad \hat{A}^\dagger \leftrightarrow A^*,$$  \hspace{1cm} (2.29)

where $h$ is any function and $\hat{A}$ is any operator. The product of two operators maps to the so-called Moyal product, or star product, of their symbols (Moyal 1949):

$$\hat{A}\hat{B} \leftrightarrow A(x,k) \star B(x,k) \doteq A(x,k) e^{i\frac{\delta_{x,k}}{2}} B(x,k),$$  \hspace{1cm} (2.30)

which is associative:

$$\hat{A}\hat{B}\hat{C} \leftrightarrow (A \star B) \star C = A \ast (B \ast C) \equiv A \ast B \ast C.$$  \hspace{1cm} (2.31)

\footnote{Analytic continuation to complex arguments is possible, but by default, $x$ and $k$ are real.}
Here, $\hat{\mathcal{L}}_x = \partial_x - \partial_k \cdot \partial_x$, and the arrows indicate the directions in which the derivatives act. For example, $A\hat{\mathcal{L}}_x B$ is just the canonical Poisson bracket on $(x, k)$:

$$A\hat{\mathcal{L}}_x B = \{A, B\}_x = \left\{-\frac{\partial A}{\partial t} \frac{\partial B}{\partial \omega} + \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial t} + \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial k_i} - \frac{\partial A}{\partial k_i} \frac{\partial B}{\partial x^i}\right\}.$$  

These formulas readily yield

$$h(\mathbf{x}) k_x \leftrightarrow k_x h(\mathbf{x}) + \frac{i}{2} \partial_x h(\mathbf{x}), \quad \hat{k}_x h(\mathbf{x}) \leftrightarrow k_x h(\mathbf{x}) - \frac{i}{2} \partial_x h(\mathbf{x}),$$

also $h(\mathbf{k}) e^{iK\mathbf{x}} \leftrightarrow h(\mathbf{k}) \star e^{iK\mathbf{x}} = h(\mathbf{k} + \mathbf{K}/2)e^{iK\mathbf{x}}$, etc. Another notable formula to be used below, which flows from (2.28) and (2.31), is

$$\langle x | \hat{A} \hat{B} \hat{C} | x \rangle = \frac{1}{(2\pi)^n} \int d\mathbf{k} \; (A \star B \star C)(\mathbf{x}, \mathbf{k}).$$  

The Moyal product is particularly handy when $\partial_x \partial_k \sim \epsilon \ll 1$. Such $\epsilon$ is often called the geometrical-optics parameter. Since $\hat{\mathcal{L}}_x = \mathcal{O}(\epsilon)$, one can express the Moyal product as an asymptotic series in powers of $\epsilon$:

$$\star = \hat{1} + i\hat{\mathcal{L}}_x/2 - \hat{\mathcal{L}}_x^2/8 + \ldots.$$  

2.1.5. Weyl expansion of operators

Operators can be approximated by approximating their symbols (McDonald 1988; Dodin et al. 2019). If $\hat{A}$ is approximately local in $\mathbf{x}$ (i.e. if $\hat{A}\psi(\mathbf{x})$ is determined by values $\psi(\mathbf{x} + \mathbf{s})$ only with small enough $\mathbf{s}$), its symbol can be adequately represented by the first few terms of the Taylor expansion in $\mathbf{k}$:

$$A(\mathbf{x}, \mathbf{k}) = A(\mathbf{x}, 0) + \Theta_0(\mathbf{x}) \cdot \mathbf{k} + \ldots, \quad \Theta_0(\mathbf{x}) = (\partial_k A(\mathbf{x}, \mathbf{k}))_{\mathbf{k}=0}.$$  

Application of $\operatorname{op}_x$ to this formula leads to

$$\hat{A} \approx A(\mathbf{x}, 0) + \frac{1}{2} (\hat{\Theta}_0 \cdot \hat{k} + \hat{k} \cdot \hat{\Theta}_0) + \ldots,$$  

where $\hat{\Theta}_0 \equiv \Theta_0(\mathbf{x})$. One can also rewrite (2.37) using the commutation property

$$[\hat{k}, \hat{\Theta}_0] = -i(\partial_x \cdot \hat{\Theta}_0)(\mathbf{x}).$$  

In the $\mathbf{x}$ representation, this leads to

$$\hat{A} = A(\mathbf{x}, 0) - i\Theta_0(\mathbf{x}) \cdot \partial_x - \frac{i}{2} (\partial_x \cdot \Theta_0(\mathbf{x})) + \ldots.$$  

The effect of a non-local operator on eikonal (monochromatic or quasimonochromatic) fields can be approximated similarly. Suppose $\psi = e^{i\theta} \hat{\psi}$, where the dependence of $\mathbf{k} \hat{\psi} = (\partial_x \theta) \hat{\psi}$ and $\hat{\psi}$ on $\mathbf{x}$ is slower than that of $\theta$ by factor $\epsilon \ll 1$. Then, $\hat{A}\psi = e^{i\theta} \hat{A} \hat{\psi}$, where $\hat{A}' \equiv e^{-i\theta(\mathbf{x})} \hat{A} e^{i\theta(\mathbf{x})}$, and the symbol of $\hat{A}'$ can be approximated as follows:

$$A'(\mathbf{x}, \mathbf{k}) = A(\mathbf{x}, \mathbf{k}) + \mathcal{O}(\epsilon^2).$$  

By expanding this in $\mathbf{k}$ and applying $\operatorname{op}_x$, one obtains

$$\hat{A}' = A(\mathbf{x}, \mathbf{k}) - i\Theta(\mathbf{x}) \cdot (\partial_x \Theta(\mathbf{x})) + \mathcal{O}(\epsilon^2).$$  

where $\Theta(\mathbf{x}) \equiv (\partial_k A(\mathbf{x}, \mathbf{k}))_{\mathbf{k}=\mathbf{k}(\mathbf{x})}$. Neglecting the $\mathcal{O}(\epsilon^2)$ corrections in this formula leads to what is commonly known as the geometrical-optics approximation (Dodin et al. 2019).
2.1.6. Wigner functions

Any ket $|\psi\rangle$ generates a dyadic $|\psi\rangle \langle \psi|$. In quantum mechanics, such dyadics are known as density operators (of pure states). For our purposes, though, it is more convenient to define the density operator in a slightly different form, namely, as

$$\hat{W}_\psi \doteq (2\pi)^{-n} |\psi\rangle \langle \psi|.$$  \hfill (2.42)

The symbol of this operator, $W_\psi = \text{symb}_x \hat{W}_\psi$, is a real function called the Wigner function. It is given by

$$W_\psi(x, k) \doteq \frac{1}{(2\pi)^n} \int ds \, \psi(x + s/2) \psi^*(x - s/2) e^{-i k \cdot s},$$  \hfill (2.43)

which is manifestly real and can be understood as the (inverse) Fourier image of

$$C_\psi(x, s) \doteq \psi(x + s/2) \psi^*(x - s/2) = \int dk W_\psi(x, k) e^{i k \cdot s}. \hfill (2.44)$$

Any function bilinear in $\psi$ and $\psi^*$ can be expressed through $W_\psi$. Specifically, for any operators $\hat{L}$ and $\hat{R}$, one has

$$\left( \hat{L} \psi(x) \right) \left( \hat{R} \psi(x) \right)^* = \langle x | \hat{L} | \psi \rangle \langle \psi | \hat{R}^\dagger | x \rangle = (2\pi)^n \langle x | \hat{L} \hat{W}_\psi \hat{R}^\dagger | x \rangle = \int dk \, \hat{L}(x, k) \ast W_\psi(x, k) \ast \hat{R}^*(x, k),$$  \hfill (2.45)

where $L$ and $R$ are the corresponding symbols and (2.28) was used along with (2.31). As a corollary, and as also seen from (2.28), one has

$$|\psi(x)|^2 = \int dk \, W_\psi(x, k), \quad |\hat{\psi}(k)|^2 = \int dx \, W_\psi(x, k). \hfill (2.46)$$

As a reminder, $\psi(x) = \langle x | \psi \rangle$ and $\hat{\psi}(k) \doteq \langle k | \psi \rangle$ is the Fourier image of $\psi$ (2.23), so $|\psi(x)|^2$ and $|\hat{\psi}(k)|^2$ can be loosely understood as the densities of quanta (associated with the field $\psi$) in the $x$-space and the $k$-space, respectively. Because of (2.46), $W_\psi$ is commonly attributed as a quasiprobability distribution of wave quanta in phase space. (The prefix ‘quasi’ is added because $W_\psi$ can be negative.) In case of real fields, which satisfy $\langle x | \psi \rangle = \langle \psi | x \rangle$, one also has

$$W_\psi(x, k) = W_\psi(x, -k). \hfill (2.47)$$

Of particular importance are Wigner functions averaged over a sufficiently large phase-space volume $\Delta x \Delta k \gtrsim 1$. The average Wigner function $\overline{W}_\psi$ is a local property of the field (as opposed to, say, the field’s global Fourier spectrum) and satisfies (Appendix A)

$$\overline{W}_\psi \geq 0. \hfill (2.48)$$
2.1.7. Generalization to vector fields

In case of vector (tuple) fields \( \psi = (\psi^1, \psi^2, \ldots, \psi^M)^\top \), kets are column vectors, \( |\psi\rangle = (|\psi^1\rangle, |\psi^2\rangle, \ldots, |\psi^M\rangle)^\top \), and bras are row vectors, \( \langle \psi | = (\langle \psi^1 |, \langle \psi^2 |, \ldots, \langle \psi^M |)^\top \). The operators acting on such kets and bras are matrices of operators. The Weyl symbol of a matrix operator is defined as the matrix of the corresponding symbols. As a result, the symbol of a Hermitian adjoint of a given operator is the Hermitian adjoint of the symbol of that operator:

\[
\hat{A}^\dagger \leftrightarrow A^\dagger,
\]

and as a corollary, the symbol of a Hermitian matrix operator is a Hermitian matrix.

In particular, the density operator of a given vector field \( \psi \) is a matrix operator

\[
\hat{\mathcal{W}}_\psi \equiv (2\pi)^{-n} |\psi\rangle \langle \psi |.
\]

The symbol of this operator, \( W_\psi = \text{symb}_\psi \hat{\mathcal{W}}_\psi \), is a Hermitian matrix function

\[
W_\psi(x, k) = \frac{1}{(2\pi)^n} \int ds \psi(x + s/2) \psi^\dagger(x - s/2) e^{-i k \cdot s},
\]

called the Wigner matrix. (It is also called the ‘Wigner tensor’ when \( \psi \) is a true vector rather than a tuple.) It can be understood as the (inverse) Fourier image of

\[
C_\psi(x, s) = \psi(x + s/2) \psi^\dagger(x - s/2) = \int dk W_\psi(x, k) e^{i k \cdot s}.
\]

The analogue of (2.45) is (Appendix B.1)

\[
(\hat{L}_\psi(x))(\hat{R}_\psi(x))^\dagger = \int dk L(x, k) \star W_\psi(x, k) \star R^\dagger(x, k).
\]

The Wigner matrix averaged over a sufficiently large phase-space volume \( \Delta x \Delta k \gtrsim 1 \) is a local property of the field, and it is positive-semidefinite (Appendix A).

For real fields, one also has

\[
W_\psi(x, k) = W^\dagger_\psi(x, -k) = W^*_\psi(x, -k),
\]

and (2.53) yields the following corollary at \( \epsilon \to 0 \), when \( \star \) becomes the usual product (Appendix B.1):

\[
(\hat{L}_\psi)^\dagger \hat{R}_\psi = \int dk \text{ tr} \left( W_\psi (L^\dagger R)_H \right).
\]

The generalizations of the other formulas from the previous sections are obvious.

2.2. Weyl symbol calculus on phase space

2.2.1. Notation

Consider a Hamiltonian system with coordinates \( x \equiv (x^1, x^2, \ldots, x^n) \) and canonical momenta \( p \equiv (p_1, p_2, \ldots, p_n) \). Together, these variables comprise the phase-space coordinates \( z \equiv (x, p) \), i.e.

\[
z \equiv (z^1, \ldots, z^{2n}) = (x^1, \ldots, x^n, p_1, \ldots, p_n).
\]

Components of \( z \) will be denoted with Greek indices ranging from 1 to 2n.\(^9\)

\(^8\) By construction, \( \hat{\mathcal{W}}_\psi \) is a matrix with mixed indices, \( (\hat{\mathcal{W}}_\psi)^\dagger \). In §§ 5.1 and 5.2, we also operate with a Wigner matrix that has two upper indices. In such cases, the dagger \( \dagger \) in (2.51) is assumed replaced with complex conjugation. For the said sections, this means that \( \dagger \) in (2.51) can be simply omitted, because the field of interest there is real.

\(^9\) However, the index \( \sigma \) is reserved as a tag for individual particles and waves.
Hamilton’s equations for $z^\alpha$ can be written as $\dot{z}^\alpha = J^{\alpha\beta} \partial_\beta H$.  

$$\dot{z}^\alpha = J^{\alpha\beta} \partial_\beta H.$$  

(2.55)

Here, $H = H(t, z)$ is a Hamiltonian, $\partial_\beta \equiv \partial / \partial z^\beta$. 

$$\{A, B\} \equiv J^{\alpha\beta} (\partial_\alpha A)(\partial_\beta B)$$  

(2.56)

is the Poisson bracket on $z$, and $J^{\alpha\beta}$ is the canonical Poisson structure:

$$J = -J^T = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix},$$

(2.57)

where $0_n$ is an $n$-dimensional zero matrix, and $1_n$ is an $n$-dimensional unit matrix. The corresponding equation for the probability distribution $f(t, z)$ is

$$\partial_t f = \{H, f\}.$$  

(2.58)

Solutions of (2.58) and other functions of the extended-phase-space coordinates $X \equiv (t, z)$ can be considered as vectors in the Hilbert space $\mathcal{H}_X$ with the usual inner product

$$\langle \xi | \psi \rangle \equiv \int dX \xi^*(X) \psi(X).$$  

(2.59)

Assuming the notation $N \equiv \dim X = 2n + 1$, one has

$$dX = dx^1 dx^2 \ldots dx^N = dt dx^1 \ldots dx^n dp_1, \ldots, dp_n.$$  

(2.60)

Let us introduce the position operator on $z$,

$$\hat{z} \equiv (x^1, \ldots, x^n, p_1, \ldots, p_n),$$

(2.61)

and the momentum operator on $z$,

$$\hat{q} \equiv (-i \partial_1, \ldots, -i \partial_n, -i \partial^1, \ldots, -i \partial^n),$$

(2.62)

where $\partial_i \equiv \partial / \partial x^i$ but $\partial^i \equiv \partial / \partial p_i$; that is, $\hat{z} = (\hat{x}, \hat{p})$, $\hat{q} = (\hat{k}, \hat{r})$ and

$$\hat{z}^\alpha \equiv z^\alpha, \quad \hat{q}^\alpha \equiv -i \partial_\alpha.$$  

(2.63)

Then, much like in § 2.1, one can also introduce the position and momentum operators on the extended phase space $X'$:

$$\hat{X} = (\hat{\tau}, \hat{z}) = (\hat{\tau}, \hat{x}, \hat{p}), \quad \hat{K} = (-\hat{\omega}, \hat{q}) = (-\hat{\omega}, \hat{k}, \hat{r}).$$  

(2.64)

Assuming the convention that Latin indices from the beginning of the alphabet $(a, b, c, \ldots)$ range from 0 to $2n$, and $\partial_a \equiv \partial / \partial X^a$, one can compactly express this as

$$\hat{X}^a = X^a, \quad \hat{K}_a = -i \partial_a.$$  

(2.65)

\(^{10}\)Note that the inner product (2.59) is different from (2.1). Still, we use the same notation assuming it will be clear from the context which inner product is used in each given case.
The eigenvectors of these operators will be denoted $|X\rangle$ and $|K\rangle$:

$$\hat{X}|X\rangle = X|X\rangle, \quad \hat{K}|K\rangle = K|K\rangle,$$

and we assume the usual normalization

$$\langle X_1|X_2 \rangle = \delta(X_1 - X_2), \quad \langle K_1|K_2 \rangle = \delta(K_1 - K_2).$$

Both sets $\{|X\rangle, X \in \mathbb{R}^N\}$ and $\{|K\rangle, K \in \mathbb{R}^N\}$ form a complete basis on $\mathcal{H}_X$, and the eigenvalues of these operators form a real extended phase space $(X, K)$, where

$$X \equiv (t, z), \quad K \equiv (-\omega, q).$$

Particularly note the following formula, which will be used below:

$$J_{\alpha\beta} \hat{q}_\alpha \hat{q}_\beta = (\hat{k} \hat{r}) \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \begin{pmatrix} k \\ r \end{pmatrix} = \hat{k} \cdot r - \hat{r} \cdot k.$$

2.2.2. Wigner–Weyl transform

One can construct the Weyl symbol calculus on the extended phase space $X$ just like it is done on spacetime $x$ in § 2.1, with an obvious modification of the notation. The Wigner–Weyl transform is defined as

$$A(X, K) = \int dS \langle X + S/2|\hat{A}|X - S/2 \rangle e^{-iK \cdot S} \equiv \text{symb}_X \hat{A},$$

$$\hat{A} = \frac{1}{(2\pi)^N} \int dX dK dS |X + S/2\rangle A(X, K) \langle X - S/2| e^{iK \cdot S} \equiv \text{oper}_X A.$$ (2.70)

(Notice the change in the font and in the index compared with (2.26) and (2.25).) The corresponding Moyal product is denoted $\star$ (as opposed to $\star$ introduced earlier) and is given by

$$A \star B = A(X, K) e^{i\mathcal{E}_X/2} B(X, K),$$

where $\mathcal{L}_X \equiv \partial_X \cdot \partial_K - \partial_K \cdot \partial_X$ can be expressed as follows:

$$A \mathcal{L}_X B \equiv -\frac{\partial A}{\partial t} \frac{\partial B}{\partial \omega} + \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial t} + \frac{\partial A}{\partial x^j} \frac{\partial B}{\partial k_i} - \frac{\partial A}{\partial k_i} \frac{\partial B}{\partial x^j} + \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial p^j} - \frac{\partial A}{\partial p^j} \frac{\partial B}{\partial \omega}.$$ (2.73)

If an operator $\hat{A}$ is local in $p$, its $X$ representation and $x$ representation satisfy

$$\langle t, x, p|\hat{A}|t', x', p' \rangle = \langle t, x|\hat{A}|t', x' \rangle \delta(p - p'),$$

and therefore the Weyl symbol of $\hat{A}$ is the same irrespective of whether the operator is considered on $\mathcal{H}_X$ or on $\mathcal{H}_x$. In this case, we will use a unifying notation $\text{symb} \hat{A}$ instead of $\text{symb}_X \hat{A}$ and $\text{symb}_x \hat{A}$.
2.2.3. Wigner functions and Wigner matrices

The density operator of a given scalar field \( \psi \) is given by
\[
\hat{W}_\psi = (2\pi)^{-N} |\psi\rangle \langle \psi |. \tag{2.75}
\]
The symbol of this operator, \( W_\psi = \text{symb}_X \hat{W}_\psi \), is a real function called the Wigner function. It is given by
\[
W_\psi(X, K) = \frac{1}{(2\pi)^N} \int dS \psi(X + S/2)\psi^*(X - S/2) e^{-iK \cdot S}, \tag{2.76}
\]
which can be understood as the (inverse) Fourier image of
\[
C_\psi(X, S) = \psi(X + S/2)\psi^*(X - S/2) = \int dK W_\psi(X, K) e^{iK \cdot S}. \tag{2.77}
\]
In particular, one has
\[
\int dr \text{symb}_X \hat{W}_\psi = \text{symb}_x \hat{W}_\psi(p), \tag{2.78}
\]
where the right-hand side is \( W_\psi \) given by (2.43), with \( p \) treated as a parameter. Also, for real fields,
\[
W_\psi(X, K) = W_\psi(X, -K). \tag{2.79}
\]
The density operator of a given vector field \( \psi = (\psi^1, \psi^2, \ldots, \psi^M) \) is a matrix operator
\[
\hat{W}_\psi = (2\pi)^{-N} |\psi\rangle \langle \psi |. \tag{2.80}
\]
The symbol of this operator, or the Wigner matrix, is a Hermitian matrix function
\[
W_\psi(X, K) = \frac{1}{(2\pi)^N} \int dS \psi(X + S/2)\psi^\dagger(X - S/2) e^{-iK \cdot S}, \tag{2.81}
\]
which can be understood as the (inverse) Fourier image of
\[
C_\psi(X, S) = \psi(X + S/2)\psi^\dagger(X - S/2) = \int dK W_\psi(X, K) e^{iK \cdot S}. \tag{2.82}
\]
In particular, one has
\[
\int dr \text{symb}_X \hat{W}_\psi = \text{symb}_x \hat{W}_\psi(p), \tag{2.83}
\]
where the right-hand side is \( W_\psi \) given by (2.51), with \( p \) treated as a parameter. Also, for real fields,
\[
W_\psi(X, K) = W_\psi^\dagger(X, -K) = W_\psi^*(X, -K). \tag{2.84}
\]
Like those on \((x, k)\), the Wigner matrices (Wigner functions) on \((X, K)\) become positive-semidefinite (non-negative), and characterize local properties of the corresponding fields, when averaged over a sufficiently large phase-space volume \( \Delta X \Delta K \gtrsim 1 \).
2.3. Summary of § 2

In summary, we have introduced a generic $n$-dimensional physical space $x$, the dual $n$-dimensional wavevector space $k$, the corresponding $n$-dimensional ($n = n + 1$) spacetime $x \equiv (t, x)$ and the dual $n$-dimensional wavevector space $k \equiv (-\omega, k)$. We have also introduced an $n$-dimensional momentum space $p$, the corresponding $2n$-dimensional phase space $z \equiv (x, p)$, the $N$-dimensional $(N = 2n + 1)$ extended space $X \equiv (t, z) \equiv (t, x, p)$ and the dual $N$-dimensional wavevector space $K \equiv (-\omega, q) \equiv (-\omega, k, r)$, where $r$ is the $n$-dimensional wavevector space dual to $p$. We have also introduced the $2N$-dimensional phase space $(X, K)$. Each of the said variables has a corresponding operator associated with it, which is denoted with a caret. For example, $\hat{x}$ is the operator of position in the $x$ space, and $\hat{k} = -i\partial_x$ is the corresponding wavevector operator.

Functions on $x$ form a Hilbert space $\mathcal{H}_x$, and the corresponding bra–ket notation is introduced as usual. Any operator $\hat{A}$ on $\mathcal{H}_x$ can be represented by its Weyl symbol $A(x, k)$. The correspondence between operators and their symbols, $\hat{A} \leftrightarrow A$, is determined by the Wigner–Weyl transform and is natural in the sense that (2.29) is satisfied. In particular, $\hat{A}\hat{B} \leftrightarrow A \star B$, where $\star$ is the Moyal product on $(x, k)$. When the geometrical-optics parameter is negligible ($\epsilon \rightarrow 0$), one has $\hat{A} = A(\hat{x}, \hat{k})$ and the Moyal product becomes the usual product. Similarly, functions on $X$ form a Hilbert space $\mathcal{H}_X$, the corresponding bra–ket notation is also introduced as usual, any operator $\hat{A}$ on $\mathcal{H}_X$ can be represented by its Weyl symbol $A(X, K)$, and $\hat{A}\hat{B} \leftrightarrow A \star B$. An operator that is local in $p$ has the same symbol irrespective of whether it is considered on $\mathcal{H}_x$ or on $\mathcal{H}_X$.

Any given field $\psi$ generates the corresponding density operator $(2\pi)^{-n} \langle \psi | \psi \rangle$ and its symbol called the Wigner function (Wigner matrix, if the field is a vector). If the density operator is considered on $\mathcal{H}_x$, it is denoted $\hat{W}_\psi$ and the corresponding Wigner function is denoted $W_\psi(x, k)$. If the density operator is considered on $\mathcal{H}_X$, it is denoted $\hat{W}_\psi$ and the corresponding Wigner function is denoted $W_\psi(X, K)$. The two Wigner functions are related via $\int d\omega W_\psi(t, x, p, \omega, k, r) = W_\psi(t, x, \omega, k; p)$, where $p$ enters $W_\psi$ as a parameter, if at all. If averaged over a sufficiently large phase-space volume, the Wigner functions (matrices) are non-negative (positive-semidefinite) and characterize local properties of the corresponding fields.

3. Model

Here, we introduce the general assumptions and the key ingredients of our theory.

3.1. Basic assumptions

3.1.1. Ordering

Let us consider particles governed by a Hamiltonian $H = \overline{H} + \tilde{H}$ such that

$$\tilde{H} = O(\epsilon) \ll \overline{H} = O(1).$$

(3.1)

In other words, $\tilde{H}$ serves as a small perturbation to the leading-order Hamiltonian $\overline{H}$. The system will be described in canonical variables $z \equiv (x, p) \in \mathbb{R}^{2n}$. Let us also assume that the system is close to being homogeneous in $x$. This includes two conditions. First, we require that the external fields are weak (yet see § 3.1.2), meaning

$$\partial_x \overline{H} \sim \kappa_x \overline{H} = O(\epsilon), \quad \partial_p \overline{H} \sim \kappa_p \overline{H} = O(1),$$

(3.2)
where $\epsilon \ll 1$ is a small parameter, $\kappa_x$ and $\kappa_p$ are the characteristic inverse scales in the $x$ and $p$ spaces, respectively, and the bar denotes local averaging. Hence, the particle momenta $p$ are close to being local invariants. Second, the statistical properties of $\tilde{H}$ are also assumed to vary in $x$ slowly. These properties can be characterized using the density operator of the perturbation Hamiltonian,

$$\tilde{W} \doteq (2\pi)^{-N} \langle \tilde{H} \rangle \langle \tilde{H} \rangle,$$

and its symbol, the (real) Wigner function, as in (2.43):

$$W(X, K) = \frac{1}{(2\pi)^N} \int dS \tilde{H}(X + S/2)\tilde{H}(X - S/2) e^{-iK \cdot S}.$$  (3.4)

Specifically, we will use the average Wigner function, $\bar{W}$, which represents the Fourier spectrum of the symmetrized autocorrelation function of $\tilde{H}$:

$$\bar{C}(X, S) \doteq \frac{1}{\Delta X \Delta K} \int dK \bar{W}(X, K) e^{iK \cdot S}.$$  (3.5)

The averaging is performed over sufficiently large volume of $x$ to eliminate rapid oscillations and also over phase-space volumes $\Delta X \Delta K \gtrsim 1$, which guarantees $\bar{W}$ to be non-negative and local ($\S$ 2.2.3). The function $\bar{W}$ can be understood as a measure of the phase-space density of wave quanta when the latter is well defined ($\S$ 7).

We will assume

$$\partial_t \bar{W} = O(\epsilon), \quad \partial_x \bar{W} = O(\epsilon), \quad \partial_p \bar{W} = O(1).$$  (3.6)

That said, we will also allow (albeit not require) for oscillations to be constrained by a dispersion relation. In this case, $\bar{W} \propto \delta(\omega - \omega(t, x))$, so (3.6) per se is not satisfied; then we assume a similar ordering for $\int d\omega \bar{W}$ instead. Also note that in application to the standard QLT of homogeneous turbulence (Stix 1992, chapter 16), $\epsilon$ is understood as the geometrical-optics parameter characterizing the smallness of the linear-instability growth rates. (We discuss the ordering further in the end of $\S$ 3.3.)

3.1.2. **Quasilinear approximation**

The particle-motion equations can be written as

$$\dot{z}_\alpha = \{z_\alpha, \tilde{H} + \bar{H}\} = v_\alpha + u_\alpha,$$  (3.7)

where $v_\alpha$ and $u_\alpha$ are understood as the unperturbed phase-space velocity and the perturbation to the phase-space velocity, respectively:

$$v_\alpha \doteq J^{\alpha\beta} \partial_\beta \overline{H}, \quad u_\alpha \doteq J^{\alpha\beta} \partial_\beta \tilde{H}.$$  (3.8)

The notation $v_i$ (with $i = 1, 2, \ldots n$) will also be used for the spatial part of the phase-space velocity $v_\alpha$, i.e. for the true velocity per se. Likewise, $v$ will be used to denote either the phase-space velocity vector or the spatial velocity vector depending on the context. Also note that a slightly different definition of $v$ will be used starting from $\S$ 5.6.

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11 An exception will be made for eikonal waves, specifically, for quantities evaluated on the local wavevector $k = (-\omega, \mathbf{k})$.

12 As a reminder, the notation $A = O(\epsilon)$ does not rule out the possibility that $A/\epsilon$ is small. Also note that the terms "$\sim" and "of order" in this paper mean the same as "O".
The corresponding Klimontovich equation for the particle distribution \( f(t, z) \) is

\[
\partial_t f = \{ \overline{H} + \tilde{H}, f \}.
\]  

(3.9)

(If collisions are not of interest, (3.9) can as well be understood as the Vlasov equation. Also, a small collision term can be included \textit{ad hoc}; see the comment in the end of § 3.3.)

Let us search for \( f \) in the form

\[
f = \bar{f} + \tilde{f}, \quad \tilde{f} = 0.
\]  

(3.10)

The equations for \( \bar{f} \) and \( \tilde{f} \) are obtained as the average and oscillating parts of (3.9), and we neglect the nonlinearity in the equation for \( \tilde{f} \), following the standard QL approximation (Stix 1992, chapter 16). Then, one obtains

\[
\partial_t \bar{f} = \{ \overline{H}, \bar{f} \} + \{ \overline{\tilde{H}}, \tilde{f} \},
\]  

(3.11)

\[
\partial_t \tilde{f} = \{ \overline{H}, \tilde{f} \} + \{ \overline{\tilde{H}}, \bar{f} \}.
\]  

(3.12)

A comment is due here regarding plasmas in strong fields and magnetized plasmas in particular. Our formulation can be applied to such plasmas in canonical angle–action variables \((\phi, J)\). For fast angle variables, the ordering (3.2) is not satisfied and the Weyl symbol calculus is inapplicable as is (see the footnote in § 2.1.3). Such systems can be accommodated by representing the distribution function as a Fourier series in \( \phi \) and treating the individual-harmonic amplitudes separately as slow functions of the remaining coordinates. Then, our averaging procedure subsumes averaging over \( \phi \), so the averaged quantities are \( \phi \)-independent and (3.2) is reinstated. In particular, magnetized plasmas can be described using guiding-centre variables. Although not canonical by default (Littlejohn 1983), they can always be cast in a canonical form, at least in principle (Littlejohn 1979).

Examples of canonical guiding-centre variables are reviewed in Cary & Brizard (2009). To make the connection with the homogeneous-plasma theory, one can also order the canonical pairs of guiding-centre variables such that they would describe the gyromotion, the parallel motion, and the drifts separately (Wong 2000). This readily leads to results similar to those in Catto et al. (2017). Further discussions on this topic are left to future papers.

3.2. \textit{Equation for} \( \tilde{f} \)

Let us consider solutions of (3.12) as a subclass of solutions of the more general equation

\[
\partial_t \tilde{f} = \mathcal{L} \tilde{f} + \mathcal{F}, \quad \mathcal{F}(X) \doteq \{ \overline{H}, \tilde{f} \}.
\]  

(3.13)

Here, we have introduced an auxiliary second ‘time’ \( \tau \), the operator

\[
\mathcal{L} \doteq -\partial_t + \{ \overline{H}, \bullet \} = -\partial_t + \sum_{\alpha} (\partial_\alpha \overline{H}) \partial_\alpha = -\partial_t - v^\alpha \partial_\alpha = -V^a \partial_a
\]  

(3.14)

(here and further, \( \bullet \) denotes a placeholder), and \( V(X) \equiv (1, v(t, z)) \) is the unperturbed velocity in the \( X \) space. Note that

\[
\partial_a V^a = \partial_\alpha v^\alpha = 0
\]  

(3.15)

due to the incompressibility of the phase flow. Hence, \( [\partial_a, V^a] = 0 \), so \( \mathcal{L} \) is anti-Hermitian.
Let us search for a solution of (3.13) in the form\textsuperscript{13}

\[ \tilde{f}(\tau, X) = e^{\tilde{\xi}(\tau, X)}. \]  

(3.16)

Then, \( \partial_{\tau} \tilde{f} = \tilde{L} \tilde{f} + e^{\tilde{\xi}} \partial_{\tau} \tilde{\xi}, \) so \( \partial_{\tau} \tilde{\xi} = e^{-\tilde{L} \tau} \mathcal{F}(X) \) and therefore

\[ \tilde{\xi}(\tau, X) = e^{-\tilde{L} \tau_0} \tilde{\xi}_0(X) + \int_{\tau_0}^{\tau} d\tau' e^{-\tilde{L} \tau'} \mathcal{F}(X), \]

(3.17)

where \( \tilde{\xi}_0(X) \) is just an auxiliary variable, we will be interested in solutions independent of \( \tilde{\xi}_0(X) \). Then, (3.22) can be expressed as

\[ \tilde{f}(\tau, X) = e^{\tilde{\xi}(\tau_0, X)} + \int_{\tau_0}^{\tau} d\tau' e^{-\tilde{L} (\tau - \tau')} \mathcal{F}(X), \]

(3.18)

or equivalently, using \( \tau'' = \tau - \tau' \),

\[ \tilde{f}(\tau, X) = g_0(\tau, X) + \int_{0}^{\tau - \tau_0} d\tau'' \tilde{T}_{\tau''} \mathcal{F}(X). \]

(3.19)

Here, \( g_0 \) is a solution of \( \partial_{\tau} g_0 = \tilde{L} g_0 \), specifically,

\[ g_0(\tau, X) = \tilde{T}_{\tau - \tau_0} \tilde{\xi}_0(X), \quad g_0(\tau_0, X) = \tilde{f}(\tau_0, X), \]

(3.20)

and we have also introduced

\[ \tilde{T}_{\tau} \equiv e^{\tilde{L} \tau} = e^{-\tau' \nu \tau_0}. \]

(3.21)

Because \( \tilde{L} \) is anti-Hermitian, the operator \( \tilde{T}_\tau \) is unitary, and comparison with (2.19) shows that it can be recognized as a shift operator. For further details, see § 4.1.

Using \( \tilde{T}_\tau \), one can express (3.19) as

\[ \tilde{f} = g_0 + \tilde{G} \mathcal{F}, \quad \tilde{G} = \int_{0}^{\tau - \tau_0} d\tau' \tilde{T}_{\tau'}, \]

(3.22)

where \( \tilde{G} \) is the Green’s operator understood as the right inverse of the operator \( \partial_{\tau} - \tilde{L} \), or on the space of \( \tau \)-independent functions, \( \partial_{\tau} - [\tilde{H}, \bullet] \). Let us rewrite this operator as

\[ \tilde{G} = \tilde{G}_< + \tilde{G}_>, \]

where

\[ \tilde{G}_< = \int_{0}^{\tau - \tau_0} d\tau' e^{-\nu \tau'} \tilde{T}_{\tau'}, \quad \tilde{G}_> = \int_{0}^{\tau - \tau_0} d\tau' (1 - e^{-\nu \tau'}) \tilde{T}_{\tau'}, \]

(3.23)

and \( \nu \) is a positive constant. Note that \( \tilde{G}_< \) is well defined at \( \tau_0 \rightarrow -\infty \), meaning that \( \tilde{G}_< \mathcal{F} \) is well defined for any physical (bounded) field \( \mathcal{F} \).\textsuperscript{14} Thus, so is \( g_0 + \tilde{G}_< \mathcal{F} \). Let us take \( \tau_0 \rightarrow -\infty \) and then take \( \nu \rightarrow 0+ \). (Here, \( 0+ \) denotes that \( \nu \) must remain positive, i.e. the upper limit is taken.) Then, (3.22) can be expressed as

\[ \tilde{f} = g + \tilde{G} \mathcal{F}, \quad g \equiv \lim_{\nu \rightarrow 0+} \lim_{\tau_0 \rightarrow -\infty} (g_0 + \tilde{G}_< \mathcal{F}). \]

(3.24)

Here, we introduced an ‘effective’ Green’s operator \( \tilde{G} \equiv \lim_{\nu \rightarrow 0+} \lim_{\tau_0 \rightarrow -\infty} \tilde{G}_< \), i.e.

\[ \tilde{G} \equiv \lim_{\nu \rightarrow 0+} \int_{0}^{\infty} d\tau e^{-\nu \tau} \tilde{H}_{\tau}. \]

(3.25)

This operator will be discussed in § 4.2, and \( g \) will be discussed in § 4.3. Meanwhile, note that because \( \tau \) is just an auxiliary variable, we will be interested in solutions independent

\textsuperscript{13}Using the auxiliary variable \( \tau \) allows us to express the propagator as a regular exponential, rather than ordered exponential, even for \( \tau \)-dependent \( \tilde{H} \), because \( \tilde{L} \) is independent of \( \tau \).

\textsuperscript{14}Unlike classic plasma-wave theory, this approach does not involve spectral decomposition, so there is no need to consider fields that are exponential in time on the whole interval \((-\infty, \infty)\).
of \( \tau \). In particular, this means that \( \tilde{f}(\tau_0, X) = \tilde{f}(X) \), so \( \xi_0(X) = \tilde{f}(X) \), so (3.20) leads to

\[
g_0(\tau, X) = \tilde{T}_{\tau-\tau_0}(X). \tag{3.26}
\]

3.3. Equation for \( \bar{f} \)

Using (3.22), one can rewrite (3.12) for \( \bar{f} \) as follows:

\[
\partial \bar{f} = \{\bar{H}, \bar{f}\} + [\bar{H}, g] + [\bar{H}, \bar{G}(\bar{H}, \bar{f})]. \tag{3.27}
\]

Notice that

\[
\{\bar{H}, g\} = -\{g, \bar{H}\} = -\partial_\alpha (J^{\alpha\beta} g \partial_\beta \bar{H}) = -\partial_\alpha (u^\alpha g), \tag{3.28}
\]

and also

\[
\{\bar{H}, \bar{G}(\bar{H}, \bar{f})\} = \partial_\beta (J^{\alpha\beta}(\partial_\alpha \bar{H}) \bar{G}(\bar{H}, \bar{f}))
= \partial_\beta (J^{\alpha\beta}(\partial_\alpha \bar{H}) \bar{G}(J^{\mu\nu}(\partial_\mu \bar{H})(\partial_\nu \bar{f})))
= \partial_\beta (u^\beta \bar{G}(u^\nu \partial_\nu \bar{f})). \tag{3.29}
\]

The field \( u^\alpha \) enters here as a multiplication factor and can be considered as an operator:

\[
\hat{u}^\alpha \psi(X) = u^\alpha(X) \psi(X). \tag{3.30}
\]

Then, (3.29) can be compactly represented as

\[
\{\bar{H}, \bar{G}(\bar{H}, \bar{f})\} = \partial_\alpha (\hat{u}^\alpha \bar{G} \hat{u}^\beta \partial_\beta \bar{f}). \tag{3.31}
\]

We will also use the notation

\[
d_t \doteq \partial_t + v^\gamma \partial_\gamma = \partial_t - \{\bar{H}, \bullet\}. \tag{3.32}
\]

This leads to the following equation for \( \bar{f} \):

\[
d_t \bar{f} = \partial_\alpha (\hat{D}^{\alpha\beta} \partial_\beta \bar{f}) + \Gamma, \tag{3.33}
\]

where we introduced the following average quantities:

\[
\hat{D}^{\alpha\beta} \doteq \hat{u}^\alpha \bar{G} \hat{u}^\beta, \quad \Gamma \doteq -\partial_\alpha (u^\alpha g). \tag{3.34}
\]

Our goal is to derive explicit approximate expressions for the quantities (3.34) and to rewrite (3.33) in a more tractable form using the assumptions introduced in § 3.1. We will use\(^{15}\)

\[
\partial \bar{f} \sim \{\bar{H}, \bar{f}\} = O(\epsilon), \quad d_t \bar{f} = O(\epsilon^2), \tag{3.35}
\]

and we will keep terms of order \( \epsilon, \epsilon^2 \) and \( \epsilon \epsilon^2 \) in the equation for \( \bar{f} \), while terms of order \( \epsilon^4, \epsilon^2 \epsilon^2 \) and higher will be neglected. This implies the ordering

\[
\epsilon^2 \ll \epsilon \ll \epsilon \ll 1. \tag{3.36}
\]

As a reminder, \( \epsilon \) is a linear measure of the characteristic amplitude of oscillations, and \( \epsilon \) is the geometrical-optics parameter, which is proportional to the inverse scale of the plasma

\(^{15}\)Starting with § 5.6, we will assume \( d_t \bar{f} \sim \epsilon^2 \epsilon \bar{f} \) instead.
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inhomogeneity in spacetime. As usual then, linear dissipation is assumed to be of order \( \epsilon \). This model implies the assumption that collisionless dissipation is much stronger than collisional dissipation, which is to emerge as an effect quadratic in \( \tilde{f} \) (§ 6). Furthermore, the inverse plasma parameter\(^{16}\) will be assumed to be of order \( \epsilon \), so the collision operator for \( \tilde{f} \) (§ 6.8) will be of order \( \epsilon \epsilon^2 \). Within the assumed accuracy, this operator must be retained, while the dynamics of \( \tilde{f} \) is considered linear and therefore collisionless. Alternatively, one can switch from the Klimontovich description to the Vlasov–Boltzmann description and introduce an \textit{ad hoc} order-\( \epsilon \) collision operator directly in (3.9). This will alter the Green’s operator, but the conceptual formulation would remain the same, so it will not be considered separately in detail.

3.4. Summary of § 3

Our QL model is defined as usual, except: (i) we allow for a general particle Hamiltonian \( H \); (ii) we use the Klimontovich equation rather than the Vlasov equation to retain collisions; (iii) we use local averaging (denoted with overbar) and allow for weak inhomogeneity of all averaged quantities; (iv) we retain the initial conditions \( g \) for the oscillating part of the distribution function (defined as in (3.24) but yet to be calculated explicitly). Then, the average part of the distribution function satisfies

\[
\partial_t \tilde{f} - \{ \bar{H}, \tilde{f} \} = \partial_{\alpha} (\hat{D}^{\alpha\beta} \partial_{\beta} \tilde{f}) + \Gamma, \tag{3.37}
\]

where \( \hat{D}^{\alpha\beta} \equiv \hat{u}^{\alpha} \hat{G} \hat{u}^{\beta} \), \( \Gamma \equiv -\partial_{\alpha}(\hat{u}^{\alpha} \hat{g}) \), \( \hat{u}^{\alpha} \) is the wave-driven perturbation of the phase-space velocity (see (3.8)), \( \hat{u}^{\alpha} \) is the same quantity considered as an operator on \( \mathcal{H}_{X} \) (see (3.30)) and \( \hat{G} \) is the ‘effective’ Green’s operator given by

\[
\hat{G} \equiv \lim_{\nu \to 0^+} \int_{0}^{\infty} d\tau \, e^{-\nu \tau - \tau} V^{a'b}. \tag{3.38}
\]

Also, \( \partial_{\alpha} \equiv \partial/\partial z^{\alpha} \), and \( \{ \bullet, \bullet \} \) is the Poisson bracket on the particle phase space \( z \). The equation for \( \tilde{f} \) used in the standard QLT is recovered from (3.37) by neglecting \( \Gamma \) and the spatial gradients (in particular, the whole Poisson bracket) and also by approximating the operator \( \hat{D}^{\alpha\beta} \) with a local function of \( z \).

4. Preliminaries

Before we start calculating the functions in (3.37) explicitly, let us get some preliminaries out of the way. In this section, we discuss the shift operators \( \hat{T}_{\tau} \) (§ 4.1), approximate the operator \( \hat{G} \) (§ 4.2) and develop a model for the function \( g \) that encodes the initial conditions for \( \tilde{f} \) (§ 4.3).

4.1. Shift operator

Here, we derive some properties of the shift operator \( \hat{T}_{\tau} \) introduced in § 3.2.

4.1.1. \( \hat{T}_{\tau} \) as a shift

Here, we formally prove (an admittedly obvious fact) that

\[
\hat{T}_{\tau} \psi(X) = \psi(X - \ell_{\tau}(X)), \quad \ell_{\tau}(X) \equiv \int_{0}^{\tau} dt \, V^{a}(Y(t, X)), \tag{4.1}
\]

where the ‘characteristics’ \( Y^{a} \) solve\(^{17}\)

\[
\frac{dY^{a}}{d\tau} = -V^{a}(Y), \quad Y^{a}(\tau = 0) = X^{a}. \tag{4.2}
\]

\(^{16}\)By the plasma parameter we mean the number of particles within the Debye sphere.

\(^{17}\)In terms of \( t' \equiv t - \tau \), (4.2) has a more recognizable form \( dY^{a}/dt' = V^{a}(Y) \), with \( Y^{a}(t' = t) = X^{a} \).
Thus, $\ell_\tau^a$ can be Taylor-expanded in $\tau$ as

$$\ell_\tau^a(X) = \tau V^a - \frac{1}{2} \tau^2 V^b \partial_b V^a + \ldots, \quad V^a \equiv V^a(X). \tag{4.3}$$

As the first step to proving (4.1), let us Taylor-expand $V^a$ around a fixed point $X_1$:

$$V^a = V_1^a + (\partial_b V_1^a) \delta X^b + \ldots, \quad \delta X^a = X^a - X_1^a, \tag{4.4}$$

where $V_1^a \equiv V^a(X_1)$. If one neglects the first and higher derivatives of $V^a$, one obtains

$$\hat{T}_\tau \psi(X) \approx e^{-\tau V_1^a \partial_a} \psi(X) = \psi(X - \tau V_1). \tag{4.5}$$

By taking the limit $X_1 \to X$, which corresponds to $V_1 \to V$, one obtains

$$\hat{T}_\tau \psi(X) = \psi(X - \tau V) + O(\tau^2). \tag{4.6}$$

Similarly, if one neglects the second and higher derivatives of $V^a$, one obtains

$$\hat{T}_\tau \psi(X) = e^{-\tau (V_1^a + (\partial_b V_1^a) \delta X^b + \ldots) \partial_a} \psi(X) \approx e^{-\tau (\partial_b V_1^a) \delta X^b \partial_a} e^{-\frac{1}{2} \tau^2 (\partial_b V_1^a) \delta X^b \partial_a} \psi(X) \approx e^{-\tau (\partial_b V_1^a) \delta X^b \partial_a} e^{-\frac{1}{2} \tau^2 (\partial_b V_1^a) \delta X^b \partial_a} \psi(X) \approx e^{-\tilde{T}_\tau \psi(X - \tau V_1 + \frac{1}{2} \tau^2 V_1^b \partial_b V_1)} \tag{4.7}$$

In the limit $X_1 \to X$, when $e^{-\tau (\partial_b V_1^a) \delta X^b \partial_a} \to 1$ and $V_1 \to V$, one obtains

$$\hat{T}_\tau \psi(X) = \psi\left(X - \tau V(X) + \frac{1}{2} \tau^2 (V \cdot \partial X) V\right) + O(\tau^3). \tag{4.8}$$

In conjunction with (4.3), equations (4.6) and (4.8) show agreement with the sought result (4.3) within the assumed accuracy. One can also retain $m$ derivatives of $V$ and derive the corresponding approximations similarly. Then the error will be $O(\tau^{m+2})$.

For an order-one time interval $\tau$, one can split this interval on $N_\tau \gg 1$ subintervals of small duration $\tau/N_\tau$ and apply finite-$m$ formulas (for example, (4.6) or (4.8)) to those. Then the total error scales as $O(N_\tau^{m-1})$ and the exact formula (4.1) is obtained at $N_\tau \to \infty$.

4.1.2. Symbol of $\hat{T}_\tau$

Using the bra–ket notation, (4.1) can be written as

$$\langle X | \hat{T}_\tau | \psi \rangle = \langle X - \ell_\tau(X) | \psi \rangle. \tag{4.9}$$

Thus, $\langle X | \hat{T}_\tau \rangle = \langle X - \ell_\tau(X) \rangle$, so

$$\langle X_1 | \hat{T}_\tau | X_2 \rangle = \langle X_1 - \ell_\tau(X_1) | X_2 \rangle = \delta(X_1 - X_2 - \ell_\tau(X_1)). \tag{4.10}$$

\[\text{We use the Zassenhaus formula } e^{A+B} = e^{A} e^{-\frac{1}{2}[\delta \delta]/2} e^{[\delta \delta]/3} + e^{-\frac{1}{2}[\delta \delta]/2} e^{[\delta \delta]/3} + \ldots.\]
Using (2.70), one obtains the Weyl symbol of \(\hat{T}_\tau\) in the form

\[
T_\tau(X, K) = \int dS \, e^{-iK \cdot S} \delta(S - \ell_\tau(X + S/2)).
\]  

(4.11)

From (4.3), one has

\[
\ell_a^\tau(X + S/2) = \tau V^a(X + S/2) - (\tau^2/2) V^b \partial_b V^a + \mathcal{O}(\epsilon^2)
\]

\[
= \tau V^a + (\tau/2)(\partial_b V^a)S^b - (\tau^2/2) V^b \partial_b V^a + \mathcal{O}(\epsilon^2)
\]

\[
= M^a_b V^b \tau + m^a_b S^b + \mathcal{O}(\epsilon^2),
\]  

(4.12)

where we introduced a matrix \(M = 1 - m\), or explicitly,

\[
M^a_b \doteq \delta^a_b - m^a_b, \quad m^a_b \doteq (\tau/2)(\partial_b V^a).
\]  

(4.13)

Let us express the term \(\mathcal{O}(\epsilon^2)\) in (4.12) as \(-M^a_b \mu^b\). Then,

\[
\delta(S - \ell_\tau(X + S/2)) = \delta(S - MV \tau - mS + M \mu)
\]

\[
= \delta(M(S - V \tau + \mu))
\]

\[
= \delta(S - V \tau + \mu)/|\det M|.
\]  

(4.14)

Because \(m = \mathcal{O}(\epsilon)\), the well-known formula yields \(\det M = 1 + \text{tr } m + \mathcal{O}(\epsilon^2)\). But \(\text{tr } m = 0\) by (3.15), so

\[
\delta(S - \ell_\tau(X + S/2)) = \delta(S - V \tau + \mu) + \mathcal{O}(\epsilon^2).
\]  

(4.15)

The last term \(\mathcal{O}(\epsilon^2)\) is insignificant and can be neglected right away, so (4.11) leads to

\[
T_\tau(X, K) \approx \exp(i \tau \Omega(X, K) + iK \cdot \mu),
\]  

(4.16)

where we have introduced the following notation:

\[
\Omega(X, K) \doteq -K \cdot V(X) = \omega - q_a v^a = \omega - k \cdot v + \mathcal{O}(\epsilon).
\]  

(4.17)

By definition, \(\mu\) is a polynomial of \(\tau\) with coefficients that are of order \(\epsilon^2\) and therefore small. But because \(\tau\) can be large, and because \(\mu\) is under the exponent, this makes \(T_\tau\) potentially sensitive to this term, so we retain it (for now).

4.2. Effective Green’s operator

The effective Green’s operator (3.25) can be understood as the right inverse of the operator (cf. § 3.2)

\[
\hat{L}_{\text{eff}} \doteq \lim_{v \to 0^+} (\partial_t - \{\hat{H}, \bullet\} + v),
\]  

(4.18)

so we denote it also as \(\hat{G} = \hat{L}_{\text{eff}}^{-1}\) (which is admittedly abuse of notation). Because \(\hat{L}_{\text{eff}}\) has real \(X\) representation by definition, the \(X\) representation of \(\hat{G}\) is real too. In particular,
\( (X + S/2) \hat{G} |X - S/2 \) is real, hence
\[
G(X, -K) = G^*(X, K)
\] (4.19)
by definition of the Weyl symbol (2.70). As a corollary, the derivative of \( G(X, K) \) with respect to the \( a \)th component of the whole second argument, denoted \( G^a \), satisfies
\[
(G^a(X, K))^* = -G^a(X, -K). \tag{4.20}
\]
Also note that \( \hat{G} \) can be expressed as
\[
\hat{G} = \lim_{\nu \to 0^+} i(\omega - \dot{x} k_i - \dot{p} \dot{r}_i + iv)^{-1}
\] (4.21)
and the (standard) notation \( \text{pv}(1/\Omega) \) is defined as follows:
\[
\text{pv}(1/\Omega) = \lim_{\nu \to 0^+} \frac{\Omega}{\nu^2 + \Omega^2}. \tag{4.24}
\]
Due to (4.16), the leading-order approximation of the symbol of the operator (3.25) is
\[
G(X, K) = G_0(\Omega(X, K)),
\]
where
\[
G_0(\Omega) = \lim_{\nu \to 0^+} \int_0^\infty d\tau \ e^{-\nu \tau + i\Omega \tau} = \pi \delta(\Omega) + i \text{pv} \frac{1}{\Omega}.
\] (4.23)
This means, in particular, that for any \( A \), one has
\[
\mathcal{J}[A, G_0] = \int dK A(X, K) G_0(\Omega(X, K))
\]
\[
= \pi \int dK A(X, K) \delta(\Omega(X, K)) + i \int dK \frac{A(X, K)}{\Omega(X, K)},
\] (4.25)
where \( \int \) is a principal-value integral. Also usefully, \( \hat{G}_0 = G_0 \) and
\[
\partial_a \mathcal{J}[A, G_0] = \int dK A(X, K) G'_0(\Omega(X, K)) \partial_a \Omega(X, K)
\]
\[
= -\left( \partial_a V^b(X) \right) \int dK K_b A(X, K) G'_0(\Omega(X, K))
\]
\[
= -\left( \partial_a V^b(X) \right) \frac{\partial}{\partial \Omega} \int dK K_b A(X, K) G_0(\Omega(X, K)),
\] (4.26)
where the notation \( \partial/\partial \lambda \equiv \partial_\lambda \) is defined, for any \( \lambda \) and \( Q \), as follows:
\[
\frac{\partial}{\partial \lambda} \int Q(\lambda) \equiv \left( \frac{\partial}{\partial \vartheta} \right) \int Q(\lambda + \vartheta) \bigg|_{\vartheta = 0}.
\] (4.27)
Now let us reinstate the term \( \mu \) in (4.16). It is readily seen (Appendix B.2) that, although \( \mu \) may significantly affect \( T, \text{per se} \), its effect on \( \mathcal{J}[A, G] \) is small, namely,
\[
\mathcal{J}[A, G] - \mathcal{J}[A, G_0] = O(e^2).
\] (4.28)
Below, we apply this formulation to $A = O(\epsilon^2)$, in which case (4.28) becomes $\mathcal{J}[A, G] - \mathcal{J}[A, G_0] = O(\epsilon^2 \epsilon^2)$. Such corrections are negligible within our model, so from now on we adopt

$$G(X, K) \approx \overline{G}(X, K) \approx G_0(\Omega(X, K)).$$

(4.29)

4.3. Initial conditions

Consider the function $g$ from (3.24). Using (3.26), the latter can be written as follows:

$$g = \lim_{\nu \to 0^+} \lim_{\tau_0 \to -\infty} \left( \hat{T}_{\tau - \tau_0} \tilde{f}(X) + \int_0^{\tau - \tau_0} d\tau' \left( 1 - e^{-\nu \tau'} \right) \hat{T}_{\tau'} \mathcal{F}(X) \right).$$

(4.30)

Because $(1 - e^{-\nu \tau})$ is smooth and $\hat{T}_{\tau} \mathcal{F}$ is rapidly oscillating, the second term in the external parenthesis is an oscillatory function of $\tau_0$ with the average negligible at $\nu \to 0$. But the whole expression in these parenthesis is independent of $\tau_0$ at large $\tau_0$ (§ 3.2). Thus, it can be replaced with its own average over $\tau_0$, denoted $\langle ... \rangle_{\tau_0}$. Because there is no $\nu$-dependence left in this case, one can also omit $\lim_{\nu \to 0^+}$. That gives

$$g = \lim_{\tau_0 \to -\infty} \langle \hat{T}_{\tau - \tau_0} \tilde{f}(X) \rangle_{\tau_0}.$$

(4.31)

Using

$$f(X) = \sum_{\sigma} \delta(z - z_\sigma(t)), \quad f(X) \doteq \sum_{\sigma} \delta(z - z_\sigma(t)),$$

(4.32)

where the sum is taken over individual particles, one can write

$$f(X) \approx f(X) - \sum_{\sigma} \tilde{z}_\sigma(X) \partial_z \delta(z - \bar{z}_\sigma(X)),$$

(4.33)

where $z_\sigma \doteq z_\sigma - \bar{z}_\sigma$ are the $\tilde{H}$-driven small deviations from the particle unperturbed trajectories $\bar{z}_\sigma$. Then, $\tilde{f} = \bar{f}(X)$, and the linearized perturbation $\bar{f} = f - \tilde{f}$ is given by

$$\bar{f}(X) = \left( \bar{f}(X) - \tilde{f}(X) \right) - \sum_{\sigma} \tilde{z}_\sigma(X) \partial_z \delta(z - \bar{z}_\sigma(X)).$$

(4.34)

By definition, the unperturbed trajectories $\bar{z}_\sigma$ satisfy $\hat{L}\delta(z - \bar{z}_\sigma(X)) = 0$, where $\hat{L}$ as in (3.14); thus,

$$\hat{T}_{\tau - \tau_0} \tilde{f} = e^{\hat{L}(\tau - \tau_0)} \tilde{f} = \tilde{f}.$$

(4.35)

Also, $\langle \hat{T}_{\tau - \tau_0} \tilde{f} \rangle_{\tau_0} = 0$, because $\tilde{z}_\sigma$ are oscillatory functions of $X$ that is slowly evolved by $\hat{T}_{\tau - \tau_0}$. Hence, $g$ is the microscopic part of the unperturbed distribution function:

$$g = \tilde{f} = f(X) - \tilde{f}(X).$$

(4.36)

This indicates that the term $\Gamma$ defined in (3.34) is due to collisional effects. We postpone discussing these effects until § 6, so $\Gamma$ will be ignored for now.

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19Taylor-expanding delta functions is admittedly a questionable procedure, but here it is understood as a shorthand for Taylor-expanding integrals of $f$. 
4.4. Summary of § 4

The main result of this section is that the Weyl symbol of the effective Green’s operator $\hat{G}$ can be approximated within the assumed accuracy as follows:

$$G(X, K) \approx G_0(\Omega(X, K)), \quad \Omega(X, K) \doteq -K \cdot V(X). \quad (4.37)$$

Here, $V$ is the unperturbed velocity in the $X$ space, so $\Omega(X, K) = \omega - k \cdot v + O(\epsilon)$, where $v$ is the unperturbed velocity in the $x$ space, and

$$G_0(\Omega) = \pi \delta(\Omega) + i p v \Omega + ip v^2 \Omega. \quad (4.38)$$

We also show that the term $\Gamma$ defined in (3.34) is due to collisional effects. We postpone discussing these effects until § 6, so $\Gamma$ will be ignored for now.

5. Interaction with prescribed fields

In this section, we explore the effect of the diffusion operator $\hat{D}^{\alpha\beta}$. The oscillations will be described by $W$ as a prescribed function, so they are allowed (yet not required) to be ‘off-shell’, i.e. do not have to be constrained by a dispersion relation. Examples of off-shell fluctuations include driven near-field oscillations, evanescent waves and microscopic fluctuations (see also § 6). We will first derive the symbol of $\hat{D}^{\alpha\beta}$ and, using this symbol, approximate the diffusion operator with a differential operator (§ 5.1). Then, we will calculate the coefficients in the approximate expression for $\hat{D}^{\alpha\beta}$ (§§ 5.2 and 5.3). Finally, we will introduce the concept of the OC distribution (§ 5.4) and summarize and simplify the resulting equations (§ 5.6).

5.1. Expansion of the dispersion operator

The (effective) Green’s operator can be represented through its symbol $G$ using (2.71):

$$\hat{G} = \frac{1}{(2\pi)^N} \int dX \ dK \ dS \ |X + S/2\rangle G(X, K) \langle X - S/2| e^{iK \cdot S}. \quad (5.1)$$

The corresponding representation of $\hat{u}^\alpha$ is even simpler, because the symbol of $\hat{u}^\alpha$ is independent of $K$:20

$$\hat{u}^\alpha = \int dX |X\rangle u^\alpha(X) \langle X|.$$  \quad (5.2)$$

Let us also introduce the Wigner matrix of $u^\alpha$, denoted $W_u^{\alpha\beta}$, and its inverse Fourier transform $C_u^{\alpha\beta}$ as in § 2.2.3. Using these together with (2.67), one obtains

$$(2\pi)^N \hat{u}^\alpha \hat{G} \hat{u}^\beta = \int dX' dX'' \ dK \ dS u^\alpha (X')u^\beta (X'')G(X, K) e^{iK \cdot S}$$

$$\times |X'\rangle \langle X'|X + S/2\rangle \langle X - S/2|X''\rangle \langle X''|$$

$$= \int dX \ dK \ dS \ C_u^{\alpha\beta}(X, S)G(X, K) e^{iK \cdot S} |X + S/2\rangle \langle X - S/2|$$

$$= \int dX' \ dK' \ dK'' \ dS' \ W_u^{\alpha\beta}(X', K')G(X', K'') e^{i(K' + K'') \cdot S'}$$

$$\times |X' + S'/2\rangle \langle X' - S'/2|. \quad (5.3)$$

20One can also derive (5.2) formally from (2.71).
Quasilinear theory

Then, by taking symb:X of (5.3), one finds that the symbol of \( \hat{D}^{\alpha\beta} \) is a convolution of \( \hat{W}^{\alpha\beta}_u \) and \( G \) (Appendix B.3):

\[
D^{\alpha\beta}(X, K) = \int dK' \hat{W}^{\alpha\beta}_u(X, K')G(X, K - K'). \tag{5.4}
\]

Let us Taylor-expand the symbol (5.4) in \( K \):

\[
D^{\alpha\beta}(X, K) \approx \hat{d}K' \hat{W}^{\alpha\beta}_u(X, K')G(X, K - K') + K_c \hat{d}K' \hat{W}^{\alpha\beta}_u(X, K')G^{\prime|c}(X, K - K') + \mathcal{O}(K_a K_b G^{\prime|ab}). \tag{5.5}
\]

As a reminder, \( G^{\prime|a}(X, -K) = -\partial^a G(X, -K) \) denotes the derivative of \( G \) with respect to (the \( a \)th component of) the whole second argument, \(-K_a\), and

\[
K_a \frac{\partial G}{\partial K_a} = \omega \frac{\partial G}{\partial \omega} + k_i \frac{\partial G}{\partial k_i} + r^i \frac{\partial G}{\partial r^i}. \tag{5.6}
\]

Upon application of \( \hat{\omega}_X \), \( \omega \) gets replaced (roughly) with \( i \theta_i = \mathcal{O}(\epsilon) \) and \( k_i \) gets replaced (also roughly) with \( -i \theta_i = \mathcal{O}(\epsilon) \). By (4.22), the last term in (5.6) is of order \( \epsilon \) too. This means that the contribution of the whole \( K_a \partial^a G \) term to the equation for \( f \) is of order \( \epsilon \). The standard QLT neglects this contribution entirely, i.e. adopts \( D^{\alpha\beta}(X, K) \approx D^{\alpha\beta}(X, 0) \), in which case the diffusion operator becomes just a local function of phase-space variables, \( \hat{D}^{\alpha\beta} \approx D^{\alpha\beta}(X, 0) \). In this work, we retain corrections to the first order in \( X \), i.e. keep the second term in (5.5) as well, while neglecting the higher-order terms as usual.

Within this model, one can rewrite (5.5) as follows:

\[
D^{\alpha\beta}(X, K) \approx D^{\alpha\beta}_0(X) + K_c \Theta^{\alpha\beta|c}(X). \tag{5.7}
\]

Here, we used (4.19) and introduced

\[
D^{\alpha\beta}_0(X) \doteq \int dK \hat{W}^{\alpha\beta}_u(X, K)G^*(X, K), \tag{5.8}
\]

\[
\Theta^{\alpha\beta|c}(X) \doteq -\int dK \hat{W}^{\alpha\beta}_u(X, K)(G^{\prime|c}(X, K))^*, \tag{5.9}
\]

which satisfy (Appendix B.4)

\[
D^{\alpha\beta}_0(X) = (D^{\alpha\beta}_0(X))^*, \quad \Theta^{\alpha\beta|c}(X) = -((\Theta^{\alpha\beta|c}(X))^*). \tag{5.10}
\]

The first-order Weyl expansion of \( \hat{D}^{\alpha\beta} \) is obtained by applying \( \hat{\omega}_X \) to (5.7). Namely, for any \( \psi \), one has (cf. § 2.1.5)

\[
\hat{D}^{\alpha\beta}_0 \psi \approx D^{\alpha\beta}_0 \psi - i \Theta^{\alpha\beta|c} \partial_c \psi - \frac{i}{2} (\partial_c \Theta^{\alpha\beta|c}) \psi. \tag{5.11}
\]

What remains now is to calculate the functions \( D^{\alpha\beta}_0 \) and \( \Theta^{\alpha\beta|c} \) explicitly.
5.2. Wigner matrix of the velocity oscillations

To express \( \hat{D}^{\alpha\beta} \) through the Wigner function \( W \) of the perturbation Hamiltonian (§ 3.1.1), we need to express \( W^{\alpha\beta} \) through the Wigner function \( W \) of the perturbation Hamiltonian (§ 3.1.1), \( \hat{W}^{\alpha\beta} = (2\pi)^{-N} |u^\alpha| \langle u^\beta | \) (§ 2.2.3). By definition (3.8), one has \( u^\alpha = iJ^{\alpha\mu}q_\mu \), where \( \hat{q}_\alpha \approx -i\partial_\alpha \) (§ 2.2.1). Then,

\[
\hat{W}^{\alpha\beta} = (2\pi)^{-N} J^{\alpha\mu} q_\mu |H\rangle \langle H| J^{\beta\nu} q_\nu = J^{\alpha\mu} J^{\beta\nu} \hat{q}_\mu \hat{W} q_\nu,
\]

where \( \hat{W} \) is the density operator whose symbol is \( W \). By applying \( \circ \), one obtains

\[
W^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} (q_\mu \star W \star q_\nu),
\]

where \( \star \) is the Moyal product (2.72). Using formulas analogous to (2.33) in the \( (X, K) \) space, one obtains

\[
q_\mu \star W \star q_\nu = \left( q_\mu W - \frac{i}{2} \frac{\partial W}{\partial z^\mu} \right) \star q_\nu
= q_\mu q_\nu W - \frac{i}{2} q_\nu \frac{\partial W}{\partial z^\mu} + \frac{i}{2} q_\mu \frac{\partial W}{\partial z^\mu} + \frac{1}{4} \frac{\partial^2 W}{\partial z^\mu \partial z^\nu}.
\]

Hence, \( W^{\alpha\beta} \) and \( W \) are connected via the following exact formula:

\[
W^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} \left( q_\mu q_\nu W - \frac{i}{2} \left( q_\mu \frac{\partial W}{\partial z^\mu} - q_\nu \frac{\partial W}{\partial z^\mu} \right) + \frac{1}{4} \frac{\partial^2 W}{\partial z^\mu \partial z^\nu} \right).
\]

5.3. Nonlinear potentials

Due to (5.10), one has \( D^{\alpha\beta}_0 = \text{re} D^{\alpha\beta}_0 \). Using this together with (5.8), (5.15), (4.29), and (4.23), one obtains

\[
D^{\alpha\beta}_0 = J^{\alpha\mu} J^{\beta\nu} \text{re} \int dK \left( \pi \delta(\Omega) - i \text{pv} \frac{1}{\Omega} \right).
\]

\[
\times \left( q_\mu q_\nu W - \frac{i}{2} \left( q_\nu \frac{\partial W}{\partial z^\mu} - q_\mu \frac{\partial W}{\partial z^\mu} \right) + \frac{1}{4} \frac{\partial^2 W}{\partial z^\mu \partial z^\nu} \right),
\]

with notation as in (2.10). This can be written as \( D^{\alpha\beta}_0 = D^{\alpha\beta} + \varrho^{\alpha\beta} + \varsigma^{\alpha\beta} \), where

\[
D^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} \int dK \pi \delta(\Omega) q_\mu q_\nu W,
\]

and we also introduced

\[
\varrho^{\alpha\beta} = -\frac{1}{2} J^{\alpha\mu} J^{\beta\nu} \int dK \left( q_\nu \frac{\partial W}{\partial z^\mu} - q_\mu \frac{\partial W}{\partial z^\mu} \right) \frac{1}{\Omega},
\]

\[
\varsigma^{\alpha\beta} = \frac{1}{4} J^{\alpha\mu} J^{\beta\nu} \int dK \pi \delta(\Omega) \frac{\partial^2 W}{\partial z^\mu \partial z^\nu}.
\]
As shown in Appendix B.5, the contributions of these two functions to (3.33) are

$$\frac{\partial}{\partial z^\alpha} \left( \varrho^{\alpha\beta} \frac{\partial f}{\partial z^\beta} \right) = \mathcal{O}(\epsilon \epsilon^2), \quad \frac{\partial}{\partial z^\alpha} \left( \varsigma^{\alpha\beta} \frac{\partial f}{\partial z^\beta} \right) = \mathcal{O}(\epsilon^2 \epsilon^2). \quad (5.20)$$

Thus, $\varrho^{\alpha\beta}$ must be retained and $\varsigma^{\alpha\beta}$ must be neglected, which leads to

$$D_0^{\alpha\beta} \approx D^{\alpha\beta} + \varrho^{\alpha\beta}. \quad (5.21)$$

The function $\Theta^{\alpha\beta\gamma} = \text{im} \Theta^{\alpha\beta\gamma}$ can be written as follows:

$$\Theta^{\alpha\beta\gamma}(X) = i J^{\alpha\mu} f_{\beta} \int dK q_\mu q_v \overline{W}(X, K) \frac{\partial}{\partial K_c} \text{pv} \frac{1}{\Omega(X, K)} = -i V^c(X) \Theta^{\alpha\beta}(X),$$

where we introduced

$$\Theta^{\alpha\beta} = J^{\alpha\mu} f_{\beta} \bar{\partial} \frac{1}{\partial \Theta} \int dK q_\mu q_v \overline{W} \frac{1}{\sqrt{\Omega}} \quad (5.22)$$

and $\bar{\partial}$ is defined as in (4.27). Then finally, one can rewrite (5.11) as follows:

$$\hat{D}^{\alpha\beta} \psi \approx (D^{\alpha\beta} + \varrho^{\alpha\beta}) \psi - \Theta^{\alpha\beta} V^c \partial_c \psi - \frac{1}{2} V^c (\partial_c \Theta^{\alpha\beta}) \psi$$

$$= (D^{\alpha\beta} + \varrho^{\alpha\beta}) \psi - \Theta^{\alpha\beta} (\partial_t + v^\lambda \partial_\lambda) \psi - \frac{1}{2} ((\partial_t + v^\lambda \partial_\lambda) \Theta^{\alpha\beta}) \psi, \quad (5.23)$$

where we used (3.15). With some algebra (Appendix B.6), and assuming the notation

$$\Phi = -J^{\mu\nu} \frac{\partial}{\partial z^\mu} \int dK q_v \overline{W} \frac{1}{2\sqrt{\Omega}}, \quad (5.24)$$

one finds that (5.23) leads to

$$\partial_a (\hat{D}^{\alpha\beta} \partial_{\beta} \bar{f}) = \partial_a (D^{\alpha\beta} \partial_{\beta} \bar{f}) - \frac{1}{2} \partial_a \partial_a (\Theta^{\alpha\beta} \partial_{\beta} \bar{f}) + \{ \Phi, \bar{f} \}. \quad (5.25)$$

Hence, (3.33) becomes (to the extent that $\Gamma$ is negligible; see § 6.7)

$$d \bar{f} + \frac{1}{2} \partial_a \partial_a (\Theta^{\alpha\beta} \partial_{\beta} \bar{f}) - \{ \Phi, \bar{f} \} = \partial_a (D^{\alpha\beta} \partial_{\beta} \bar{f}). \quad (5.26)$$

The functions $\Theta^{\alpha\beta}$, $\Phi$ and $D^{\alpha\beta}$ that determine the coefficients in this equation are fundamental and, for the lack of a better term, will be called nonlinear potentials.
5.4. Oscillation-centre distribution

Let us introduce
\[ F = \bar{f} + \frac{1}{2} \partial_a (\Theta^{ab} \partial_b \bar{f}). \]  
(5.27)

Then, using (5.25), one can rewrite (5.26) as
\[ \partial_t F - \{ \mathcal{H}, F \} = \partial_a (D^{ab} \partial_b F), \]  
(5.28)

where corrections $O(\epsilon^4)$ have been neglected and we introduced $\mathcal{H} = \bar{H} + \Phi$. As a reminder, the nonlinear potentials in (5.28) are as follows:
\[ D^{ab} = J^{ab} J^{\mu} \oint dK \pi \delta(\Omega) q_\mu q_\nu \bar{W}, \]  
(5.29)
\[ \Theta^{ab} = \partial \int dK q_\mu q_\nu \bar{W} \frac{\omega - q_\alpha v_\alpha}{\Omega}, \]  
(5.30)
\[ \Phi = -J^{\mu \nu} \partial \int dK q_\nu \bar{W} \frac{\omega^2}{2\Omega}. \]  
(5.31)

Equations (5.27)–(5.31) form a closed model that describes the evolution of the average distribution $\bar{f}$ in turbulence with prescribed $\bar{W}$. In particular, (5.28) can be interpreted as a Liouville-type equation for $F$ as an effective, or ‘dressed’, distribution. The latter can be understood as the distribution of ‘dressed’ particles called OCs. Then, $\mathcal{H}$ serves as the OC Hamiltonian, $D^{ab}$ is the phase-space diffusion coefficient, $\Phi$ is the ponderomotive energy, $\Omega = \omega - q_\alpha v_\alpha$ and $v_\alpha = J^{ab} \partial_b \bar{H}$. Within the assumed accuracy, one can redefine $v_\alpha$ to be the OC velocity rather than the particle velocity; specifically,\footnote{The difference between $F$ and $\bar{f}$ is related to the concept of so-called adiabatic diffusion (Galeev & Sagdeev 1985; Stix 1992), which captures some but not all adiabatic effects.}
\[ v_\alpha = J^{ab} \partial_b \mathcal{H} = J^{ab} \partial_b \bar{H} + \mathcal{O}(\epsilon^2). \]  
(5.32)

Then, the presence of $\delta(\Omega)$ in (5.29) signifies that OCs diffuse in phase space in response to waves they are resonant with. Below, we use the terms ‘OCs’ and ‘particles’ interchangeably except where specified otherwise.

That said, the interpretation of OCs as particle-like objects is limited. Single-OC motion equations are not introduced in our approach. (They would have been singular for resonant interactions.) Accordingly, the transformation (5.27) of the distribution function $\bar{f} \mapsto F$ is not derived from a coordinate transformation but rather is fundamental. As a result, particles and OCs live in the same phase space, but the ‘dynamics of OCs’ can be irreversible (§ 5.5). This qualitatively distinguishes our approach from the traditional OC theory (Dewar 1973) and from the conceptually similar gyrokinetic theory (Littlejohn 1981; Cary & Brizard 2009), where coordinate transformations are central.

5.5. $H$-theorem

Because $\bar{W}$ is non-negative (§ 2.1.6), $D^{ab}$ is positive-semidefinite; that is,
\[ D^{ab} \psi_\alpha \psi_\beta = \int dK \pi \delta(\Omega) a^2 \bar{W} \geq 0, \quad a = J^{\mu \nu} \psi_\alpha q_\mu \]  
(5.33)

\footnote{The advantage of the amended definition (5.32) is that it will lead to exact conservation laws of our theory, as to be discussed in § 7.5.
for any real $\psi$. This leads to the following theorem. Consider the OC entropy defined as

$$S \triangleq - \int dz F(t, z) \ln F(t, z). \quad (5.34)$$

According to (5.28), $S$ satisfies

$$\frac{dS}{dt} = - \int dz \frac{d(F \ln F)}{dF} \left( [\mathcal{H}, F] + \partial_\alpha (D^{\alpha\beta} \partial_\beta F) \right)$$

$$= - \int dz \frac{d(F \ln F)}{dF} J^{\alpha\beta}(\partial_\alpha \mathcal{H})(\partial_\beta F) - \int dz (1 + \ln F) \partial_\alpha (D^{\alpha\beta} \partial_\beta F)$$

$$= - \int dz J^{\alpha\beta}(\partial_\alpha \mathcal{H})\partial_\beta (F \ln F) - \int dz \ln F \partial_\alpha (D^{\alpha\beta} \partial_\beta F)$$

$$= \int dz (J^{\alpha\beta}(\partial_\alpha \mathcal{H}) F \ln F + \int dz D^{\alpha\beta}(\partial_\alpha \ln F)(\partial_\beta \ln F) F). \quad (5.35)$$

The first integral vanishes due to $J^{\alpha\beta}\partial_\alpha^2 = 0$. The second integral is non-negative due to (5.33). Thus,

$$\frac{dS}{dt} \geq 0, \quad (5.36)$$

which is recognized as the $H$-theorem (Lifshitz & Pitaevskii 1981, § 4) for QL OC dynamics.

### 5.6. Summary of § 5

From now on, we assume that the right-hand side of (5.28) scales not as $O(\varepsilon^2)$ but as $O(\varepsilon^2 \varepsilon^2)$, either due to the scarcity of resonant particles or, for QL diffusion driven by microscopic fluctuations (§ 6), due to the plasma parameter’s being large. Also, the spatial derivatives can be neglected within the assumed accuracy in the definition of $F$ (5.27) and on the right-hand side of (5.28). Using this together with (2.69), and with (2.56) for the Poisson bracket, our results can be summarized as follows.

QL evolution of a particle distribution in a prescribed wave field is governed by

$$\frac{\partial F}{\partial t} - \frac{\partial \mathcal{H}}{\partial x} \cdot \frac{\partial F}{\partial p} + \frac{\partial \mathcal{H}}{\partial p} \cdot \frac{\partial F}{\partial x} = \frac{\partial}{\partial p} \cdot \left( D \frac{\partial F}{\partial p} \right). \quad (5.37)$$

The OC distribution $F$ is defined as

$$F = \bar{f} + \frac{1}{2} \frac{\partial}{\partial p} \cdot \left( \Theta \frac{\partial \bar{f}}{\partial p} \right), \quad (5.38)$$

so the density of OCs is the same as the locally averaged density of the true particles:

$$\mathcal{N} \triangleq \int dp \ F = \int dp \bar{f}. \quad (5.39)$$

The function $\mathcal{H}$ is understood as the OC Hamiltonian. It is given by

$$\mathcal{H} \triangleq \overline{H} + \Phi, \quad (5.40)$$

where $\overline{H}$ is the average Hamiltonian, which may include interaction with background fields, and $\Phi$ is the ponderomotive potential. The nonlinear potentials that enter (5.37)

23 Remember that here we neglect $\Gamma$ (3.34), which is a part of the collision operator to be reinstated in § 6.
can be calculated to the zeroth order in $\epsilon$ and are given by

$$D = \int d\omega \, dk \, \pi \, kk \mathcal{W}(t, k \cdot v, k; p),$$  \hfill (5.41)

$$\Theta = \frac{\partial}{\partial \vartheta} \left. \int d\omega \, dk \, \frac{kk \mathcal{W}}{\omega - k \cdot v + \vartheta} \right|_{\vartheta=0},$$  \hfill (5.42)

$$\Phi = \frac{1}{2} \frac{\partial}{\partial p} \cdot \int d\omega \, dk \, \frac{k \mathcal{W}}{\omega - k \cdot v},$$  \hfill (5.43)

where $kk$ is a dyadic matrix with two lower indices, and the same conventions apply as in § 2.1.2. Also, $v$ is hereby redefined as the OC spatial velocity, namely,

$$v \equiv \partial_p \mathcal{H} = \partial_p \tilde{H} + O(\epsilon^2).$$  \hfill (5.44)

The function $\mathcal{W}$ is defined as

$$\mathcal{W}(t, x, \omega, k; p) = \int dr \, \mathcal{W}(t, x, p, \omega, k, r),$$  \hfill (5.45)

where $\mathcal{W}$ is the average Wigner function (3.4) of the perturbation Hamiltonian, i.e. the spectrum of its symmetrized autocorrelation function (3.5). Due to (2.78), it can be understood as the average of $\mathcal{W} \approx \text{symb}_x \mathcal{W}$ (where $\mathcal{W}$ is defined in (3.3)), i.e. as the Wigner function of the perturbation Hamiltonian with $p$ treated as a parameter. As such, $\mathcal{W}$ is non-negative, so $D$ is positive-semidefinite. This leads to an $H$-theorem (proven similarly to (5.36)) for the entropy density $\sigma = -\int dp \, F \ln F$:

$$\left( \frac{d\sigma}{dt} \right)_D \geq 0, \quad \left( \frac{\partial \psi}{\partial t} \right)_D = \frac{\partial}{\partial p} \cdot \left( D \frac{\partial \psi}{\partial p} \right).$$  \hfill (5.46)

Also note that for homogeneous turbulence in particular, where $\mathcal{W}$ is independent of $x$, (2.46) yields that

$$\int d\omega \, \mathcal{W}(t, x, \omega, k; p) = \frac{1}{\mathcal{V}_n} \int d\omega \, dx \, \mathcal{W}(t, x, \omega, k; p) = \frac{1}{\mathcal{V}_n} |\tilde{H}(t, k, p)|^2,$$  \hfill (5.47)

where $\mathcal{V}_n$ is the plasma volume (the index $n$ denotes the number of spatial dimensions) and $\tilde{H}$ is the spatial spectrum of $\tilde{H}$ as defined in (2.23).

Equation (5.37) can be used to calculate the ponderomotive force $\partial_t \int dp \, \widetilde{F}$ that a given wave field imparts on a plasma. This potentially resolves the controversies mentioned in Kentwell & Jones (1987). We will revisit this subject for on-shell waves in § 7.5.

6. Interaction with self-consistent fields

Here, we explain how to calculate the function $\mathcal{W}$ in the presence of microscopic fluctuations (non-zero $g$). In particular, we reinstate the term $\Gamma$ that was omitted in § 5. We also show that a collision operator of the Balescu–Lenard type emerges from our theory within a general interaction model. This calculation can be considered as a

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24See § 9 for examples and § 6.6 for the explanation on how $\Phi$ is related to $\Delta$, which is yet to be introduced. Also note that in combination with (5.40), equation (5.43) generalizes the related results from Kentwell (1987), Fraiman & Kostyukov (1995) and Dodin & Fisch (2014).
generalization of that in Rogister & Oberman (1968) for homogeneous plasmas. Another related calculation was proposed in Chavanis (2012) in application to potential interactions in inhomogeneous systems using action–angle variables, with global averaging over the angles. (See also Mynick (1988) for a related calculation in action–angle variables based on the Fokker–Planck approach.) In contrast, our model holds for any Hamiltonian interactions via any vector fields and allows for weak inhomogeneities in canonical coordinates.

6.1. Interaction model

Let us assume that particles interact via an $M$-component real field $\Psi \equiv (\Psi^1, \Psi^2, \ldots, \Psi^M)$. It is treated below as a column vector; hence the index $\top$. (A complex field can be accommodated by considering its real and imaginary parts as separate components.) We split this field into the average part $\Psi$ and the oscillating part $\Psi$. The former is considered given. For the latter, we assume the action integral of this field without plasma in the form

$$S_0 = \int dx \, \mathcal{L}_0, \quad \mathcal{L}_0 = \frac{1}{2} \tilde{\Psi}^\dagger \hat{\Xi}_0 \tilde{\Psi}$$

(see § 9 for examples), where $\hat{\Xi}_0$ is a Hermitian operator and $\tilde{\Psi}^\dagger = \tilde{\Psi}^\top$ is a row vector dual to $\tilde{\Psi}$. Plasma is allowed to consist of multiple species, henceforth denoted with index $s$. Because $\Psi$ is assumed small, the generic Hamiltonian for each species $s$ can be Taylor-expanded in $\tilde{\Psi}$ and represented in a generic form

$$H_s(t, x, p) \approx H_{0s} + \hat{\alpha}_s^\dagger \tilde{\Psi} + \frac{1}{2} \tilde{(L_s \tilde{\Psi})}^\dagger (\hat{R}_s \tilde{\Psi})$$

(see § 9 for examples), which can be considered as a second-order Taylor expansion of the full Hamiltonian in $\tilde{\Psi}$. Here, $H_{0s} \equiv H_{0s}(t, x, p)$ is independent of $\tilde{\Psi} \equiv \tilde{\Psi}(t, x)$, $\hat{\alpha}_s \equiv (\hat{\alpha}_{s,1}, \hat{\alpha}_{s,2}, \ldots, \hat{\alpha}_{s,M})^\top$ is a column vector whose elements $\hat{\alpha}_{s,i}$ are linear operators on $\mathcal{H}_x$. The dagger is added so that $\hat{\alpha}_s^\dagger$ could be understood as a row vector whose elements $\hat{\alpha}_s^\dagger$ act on the individual components of the field; i.e. $\hat{\alpha}_s^\dagger \Psi \equiv \hat{\alpha}_s^\dagger \psi$. We let $\hat{\alpha}_s$ be non-local in $t$ and $x$ (for example, $\hat{\alpha}_s$ can be a spacetime derivative or a spacetime integral), and we also let $\hat{\alpha}_s$ depend on $p$ parametrically so

$$\text{symb} \hat{\alpha}_s = \alpha_s(t, x, \omega, k; p).$$

(6.3)

The matrix operators $\hat{L}_s$ and $\hat{R}_s$ and their symbols $L_s$ and $R_s$ are understood similarly.

The Lagrangian density of the oscillating-field–plasma system is

$$\mathcal{L}_p = \frac{1}{2} \tilde{\Psi}^\dagger \hat{\Xi}_0 \tilde{\Psi} + \sum_{s} \sum_{\sigma_s} \sum_{i_s} (p_{x,\sigma_s} \cdot \dot{x}_{\sigma_s} - H_s(t, x, p)\delta(x - x_{\sigma_s}(t))),$$

(6.4)

where the sum is taken over individual particles. Note that

$$\sum_{\sigma_s} H_s(t, x, p)\delta(x - x_{\sigma_s}(t)) = \int dp \sum_{\sigma_s} \delta(z - z_{s,\sigma_s}(t))H_s(t, x, p)$$

$$= \int dp f_s(t, x, p)H_s(t, x, p),$$

(6.5)

\footnote{The field action often has the form $S_0 = \frac{1}{2} \int dx \sqrt{g(\tilde{g} \tilde{\Psi}^\dagger) \hat{\Xi}_0 \tilde{\Psi}}$, where $g(x)$ is a spacetime metric, $g \equiv \det g$, and $\hat{\Xi}_0$ is Hermitian with respect to the inner product $\langle \xi | \psi \rangle_\mathcal{H} = \int dx \sqrt{g(\tilde{g} \tilde{\psi}^\dagger) \psi}$. Using $\Psi^\dagger \equiv g^{1/4} \tilde{\Psi}^\dagger$ and $\hat{\Xi}_0^\dagger \equiv g^{-1/4} \tilde{\Xi}_0^\dagger g^{1/4}$, one can cast this action in the form (6.1), with $\hat{\Xi}_0^\dagger$ that is Hermitian with respect to the inner product (2.6).}
so the $\tilde{\Psi}$-dependent part of the system action can be written as $S = \int d\mathbf{x} \mathcal{L}$ with

$$\mathcal{L} = \frac{1}{2} \tilde{\Psi}^\dagger \hat{\mathcal{E}}_{\rho} \tilde{\Psi} - \sum_s \int dp \tilde{f}_s \hat{\alpha}_s \tilde{\Psi}, \quad (6.6)$$

$$\hat{\mathcal{E}}_{\rho} = \hat{\mathcal{E}}_0 - \sum_s \int dp \hat{L}_s^\dagger \tilde{f}_s \hat{R}_s. \quad (6.7)$$

(The contribution of $\tilde{f}_s$ to the second term in (6.6) has been omitted because it averages to zero at integration over spacetime and thus does not contribute to $S$. This ‘abridged’ action is not sufficient to describe the particle motion, but it is sufficient to describe the dynamics of $\tilde{\Psi}$ at given $f_s$, as discussed below. The operator $\hat{\mathcal{E}}_{\rho}$ can be considered Hermitian without loss of generality, because its anti-Hermitian part does not contribute to $S$. Also, we assume that unless either of $\hat{L}$ and $\hat{R}$ is zero, the high-frequency field has no three-wave resonances, so terms cubic in $\tilde{\Psi}$ can be neglected in $S$;\footnote{This is tacitly assumed already in (6.2), where cubic terms are neglected. Also note that three-wave interactions that involve resonances between low-frequency oscillations of $F_s$ and two high-frequency waves, like Raman scattering (Balakin et al. 2016), are still allowed.} then,

$$\hat{\mathcal{E}}_{\rho} \approx \hat{\mathcal{E}}_0 - \sum_s \int dp (\hat{L}_s^\dagger \tilde{f}_s \hat{R}_s)_H. \quad (6.8)$$

Using the same assumption, one can also adopt

$$\tilde{H}_s = H_{0s} + \frac{1}{2} (\hat{L}_s \tilde{\Psi})^\dagger (\hat{R}_s \tilde{\Psi}), \quad \tilde{H}_s \approx \hat{\alpha}^\dagger_s \tilde{\Psi}, \quad (6.9)$$

because in the absence of three-wave resonances, the oscillating part of $(\hat{L}_s \tilde{\Psi})^\dagger (\hat{R}_s \tilde{\Psi})$ contributes only $O(\varepsilon^4)$ terms to the equation for $F_s$.

6.2. Field equations

The Euler–Lagrange equation for $\tilde{\Psi}$ derived from (6.6) is

$$\hat{\mathcal{E}}_{\rho} \tilde{\Psi} = \sum_s \int dp \hat{\alpha}_s \tilde{f}_s. \quad (6.10)$$

Then, to the extent that the linear approximation for $\tilde{f}_s$ is sufficient (see below), one finds that the oscillating part of the field satisfies

$$\hat{\mathcal{E}}_\rho \tilde{\Psi} - \sum_s \int dp \hat{\alpha}_s \tilde{G}_s \{\hat{\alpha}^\dagger_s \tilde{\Psi}, \tilde{f}_s\} = \sum_s \int dp \hat{\alpha}_s g_s, \quad (6.11)$$

where we used (3.24). Note that the right-hand side of (6.11) is determined by microscopic fluctuations $g_s(t, \mathbf{x}, p)$ (§ 4.3). Equation (6.11) can also be expressed as

$$\hat{\mathcal{E}} \tilde{\Psi} = \sum_s \int dp \hat{\alpha}_s g_s, \quad (6.12)$$

where $\hat{\mathcal{E}}$ is understood as the plasma dispersion operator and is given by

$$\hat{\mathcal{E}} \doteq \hat{\mathcal{E}}_{\rho} - \sum_s \int dp \hat{\alpha}_s \tilde{G}_s \{\hat{\alpha}^\dagger_s \bullet, \tilde{f}_s\}, \quad (6.13)$$
where  is a placeholder. The general solution of (6.12) can be written as
\[ \tilde{\Psi} = \tilde{\Psi} + \tilde{\Psi}, \]
\[ \tilde{\Psi} = \sum_s \int dp \hat{\Xi}^{-1} \hat{\alpha}_s \hat{g}_s. \]  
(6.14)

Here, \( \hat{\Xi}^{-1} \) is the right inverse of \( \hat{\Xi} \) (specifically, \( \hat{\Xi} \hat{\Xi}^{-1} = \hat{1} \) yet \( \hat{\Xi}^{-1} \hat{\Xi} \neq \hat{1} \)) such that \( \tilde{\Psi} \) vanishes at zero \( g \).\(^{27}\) The rest of the solution, \( \tilde{\Psi} \), is the macroscopic field that satisfies
\[ \hat{\Xi} \tilde{\Psi} = 0. \]
(6.15)

In the special case when the dispersion operator is Hermitian (\( \hat{\Xi} = \hat{\Xi}_H \)), (6.15) also flows from the ‘adiabatic’ macroscopic part of the action \( S \), namely,
\[ S_{ad} \doteq \frac{1}{2} \int dx \tilde{\Psi}^\dagger \hat{\Xi}_H \tilde{\Psi}. \]
(6.16)

Because we have assumed a linear model for \( f_s \) in (6.11), \( \tilde{\Psi} \) is decoupled from \( \tilde{\Psi} \), and hence the dynamics of \( \tilde{\Psi} \) turns out to be collisionless. This is justified, because collisional dissipation is assumed to be much slower than collisionless dissipation (§ 3.3). One can reinstate collisions in (6.15) by modifying \( \hat{G}_s \) ad hoc, if necessary. Alternatively, one can avoid separating \( \tilde{\Psi} \) and \( \tilde{\Psi} \), and instead, derive an equation for the average Wigner matrix of the whole \( \tilde{\Psi} \) (McDonald 1991). However, this approach is beyond QLT, so it is not considered in this paper.

6.3. Dispersion matrix

As readily seen from the definition (6.13), the operator \( \hat{\Xi} \) can be expressed as
\[ \hat{\Xi} = \hat{\Xi}_p - i k_j \sum_s \int dp \hat{\alpha}_s \hat{G}_s \hat{\alpha}_s \frac{\partial F_s}{\partial p_j} + O(\epsilon, \varepsilon^2). \]
(6.17)

The corrections caused by non-zero \( \epsilon \) and \( \varepsilon \) in this formula will be insignificant for our purposes, so they will be neglected. In particular, this means that \( \hat{G}_s \doteq \text{symb}_x \hat{G}_s \) can be adopted in the form independent of \( r \) (§ 4.2):
\[ G_s \approx \frac{i}{\omega - k \cdot v_s + i0} = \pi \delta(\omega - k \cdot v_s) + i \text{pv} \frac{1}{\omega - k \cdot v_s}. \]
(6.18)

Then, \( \hat{G}_s \) can be considered as an operator on \( \mathcal{H}_x \) with \( p \) as a parameter, and symb \( \hat{G}_s = G_s(t, x, \omega, k; p) \). Also,
\[ \text{symb} \left( \hat{\alpha}_s \hat{G}_s \hat{\alpha}_s^\dagger \right) = \alpha_s \star G_s \star \alpha_s^\dagger \approx \alpha_s G_s \alpha_s^\dagger. \]
(6.19)

This readily yields the ‘dispersion matrix’ \( \Xi \doteq \text{symb}_x \hat{\Xi} \):
\[ \Xi(\omega, k) \approx \Xi_p(\omega, k) + \sum_s \int dp \frac{\alpha_s(\omega, k; p)\alpha_s^\dagger(\omega, k; p)}{\omega - k \cdot v_s + i0} k \cdot \frac{\partial F_s(p)}{\partial p}, \]
(6.20)
\[ \Xi_p(\omega, k) \approx \Xi_0(\omega, k) - \sum_s \int dp \varphi_s(\omega, k; p) F_s(p) \]
(6.21)

(see § 9 for examples). Here, \( \alpha, \alpha^\dagger \) is a dyadic matrix, and the arguments \( t \) and \( x \) are henceforth omitted for brevity. Also, we introduced the operators \( \hat{\varphi}_s = \hat{\varphi}_s^\dagger \) and their

\(^{27}\)Most generally, the problem of finding \( \hat{\Xi}^{-1} \) is the standard problem of calculating the field produced by a given radiation source.
symbols $\varphi_s = \varphi_s^\dagger$ as
\[
\hat{\varphi}_s = (L_s^\dagger R_s)_H, \quad \varphi_s = \text{symb} \hat{\varphi}_s \approx (L_s^\dagger R_s)_H.
\] (6.22)

The appearance of $+i0$ in the denominator in (6.20) is related to the Landau rule. (Remember that as arguments of Weyl symbols, $\omega$ and $k$ are real by definition.) The Hermitian and anti-Hermitian parts of the dispersion matrix are
\[
\Xi_H(\omega, k) \approx \Xi_p(\omega, k) + \sum_s \int \frac{dp}{\omega - k \cdot v_s} \frac{\alpha_s(\omega, k; p)\alpha_s^\dagger(\omega, k; p)}{\partial p} k \cdot \frac{\partial F_s(p)}{\partial p},
\] (6.23)
\[
\Xi_A(\omega, k) \approx -\pi \sum_s \int dp \alpha_s(\omega, k; p)\alpha_s^\dagger(\omega, k; p)\delta(\omega - k \cdot v_s) k \cdot \frac{\partial F_s(p)}{\partial p}.\] (6.24)

Assuming the notation $\Xi^{-\dagger} = (\Xi^\dagger)^{-1}$, the inverse dispersion matrix can be expressed as
\[
\Xi^{-1} = \Xi^{-1} \Xi^\dagger \Xi^{-\dagger} = \Xi^{-1} \Xi_H \Xi^{-\dagger} - i \Xi^{-1} \Xi_A \Xi^{-\dagger}.
\] (6.25)

Because $\Xi^{-\dagger} = (\Xi^{-1})^\dagger$, this leads to the following formulas, which we will need later:
\[
(\Xi^{-1})_H = \Xi^{-1} \Xi_H \Xi^{-\dagger}, \quad (\Xi^{-1})_A = -\Xi^{-1} \Xi_A \Xi^{-\dagger}.
\] (6.26)

6.4. Spectrum of microscopic fluctuations

Other objects to be used below are the density operators of the oscillating fields:
\[
\hat{\mathcal{W}}_\mathcal{G} = (2\pi)^{-n} |\bar{\Psi}\rangle \langle \bar{\Psi}|, \quad \hat{\mathcal{W}}_{\mathcal{G}} = (2\pi)^{-n} |\tilde{\Psi}\rangle \langle \tilde{\Psi}|,
\] (6.27)
and the corresponding average Wigner matrices on $(x, k)$. The former, $\mathcal{U} \equiv \hat{\mathcal{W}}_\mathcal{G}$, is readily found by definition (2.51), and the latter, $\hat{\mathcal{W}}_{\mathcal{G}} \equiv \hat{\mathcal{W}}_{\mathcal{G}}$, is calculated as follows. Let us consider $g_s(t, x, p)$ as a ket in $\mathcal{H}_s$, with $p$ as a parameter. Then, (6.14) readily yields
\[
\hat{\mathcal{W}}_{\mathcal{G}} = \frac{1}{(2\pi)^n} \sum_{s, s'} \int dp \int dp' \Xi^{-1} \alpha_s(\omega) |g_s(p)\rangle \langle g_s(p')| \tilde{\alpha}_s^\dagger(\omega) \Xi^{-\dagger}.
\] (6.28)

By applying symb, to this, one obtains
\[
\mathcal{W} = \sum_{s, s'} \int dp \int dp' \Xi^{-1} * \alpha_s(\omega) \ast \mathcal{G}_{ss'}(p, p') \ast \alpha_s^\dagger(\omega) \ast \Xi^{-\dagger},
\] (6.29)
where most arguments are omitted for brevity and (Appendix B.7)
\[
\mathcal{G}_{ss'}(p, p') \equiv \frac{1}{(2\pi)^n} \int ds e^{-ik\cdot s} g_s(x + s/2, p) g_s(x - s/2, p')
\approx \frac{1}{(2\pi)^n} \delta_{ss'} \delta(p - p') \delta(\omega - \mathcal{V}_s) F_s(p),
\] (6.30)
assuming corrections due to inter-particle correlations are negligible. Then, (6.29) gives
\[
\mathcal{W}(\omega, k) = \frac{1}{(2\pi)^n} \sum_{s, s'} \int dp' \delta(\omega - k \cdot v'_{s'}) F_s(p')
\times \Xi^{-1}(\omega, k)(\alpha_s \alpha_s^\dagger)(\omega, k; p') \Xi^{-\dagger}(\omega, k),
\] (6.31)
where $v'_{s'} \equiv v_{s'}(t, x, p')$. It is readily seen from (6.31) that $\mathcal{W}$ is positive-semidefinite. One can also recognize (6.31) as a manifestation of the dressed-particle superposition principle.
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(Rostoker 1964). Specifically, (6.31) shows that the contributions of individual OCs to \( \mathfrak{W} \) are additive and affected by the plasma collective response, i.e. by the difference between \( \Xi \) and the vacuum dispersion matrix \( \Xi_0 \).

Using (6.31), one can also find other averages quadratic in the field via (cf. (2.53a))

\[
\left( \hat{\mathbf{L}} \hat{\Psi} \right) \left( \hat{\mathbf{R}} \hat{\Psi}^\dagger \right) \approx \int d\omega \, dk \, (L \mathfrak{W} R) (\omega, k),
\]

(6.32)

where \( \hat{\mathbf{L}} \) and \( \hat{\mathbf{R}} \) are any linear operators and \( \mathbf{L} \) and \( \mathbf{R} \) are their symbols; for example,

\[
\hat{\Psi} (t, x) \hat{\Psi}^\dagger (t, x) \approx \int d\omega \, dk \, \mathfrak{W} (\omega, k).
\]

(6.33)

Because of this, we loosely attribute \( \mathfrak{W} \) as the spectrum of microscopic oscillations, but see also § 8.2, where an alternative notation is introduced and a fluctuation–dissipation theorem is derived from (6.31) for plasma in thermal equilibrium. See also § 9 for specific examples.

6.5. Nonlinear potentials

From (6.14), the oscillating part of the Hamiltonian (6.9) can be split into the macroscopic part and the microscopic part as \( \tilde{\mathfrak{H}}_s = \tilde{\mathfrak{H}}_s^m + \tilde{\mathfrak{H}}_s^\mu \), \( \tilde{\mathfrak{H}}_s = \hat{\alpha}_s^\dagger \hat{\Psi} \), and

\[
\tilde{\mathfrak{H}}_s (p) = \sum \alpha_{s'}^\dagger \tilde{\mathfrak{X}}_{ss'} (p, p') g_s (p').
\]

(6.34)

Here, \( \tilde{\mathfrak{X}}_{ss'} \) is an operator on \( \mathcal{H}_s \) given by

\[
\tilde{\mathfrak{X}}_{ss'} (p, p') \approx \alpha_{s'}^\dagger (p) \Xi^{-1} \alpha_s (p'),
\]

(6.35)

with the symbol

\[
\mathfrak{X}_{ss'} (\omega, k; p, p') \approx \alpha_{s'}^\dagger (\omega, k; p) \Xi^{-1} (\omega, k) \alpha_s (\omega, k; p')
\]

(6.36)

(see § 9 for examples). The corresponding average Wigner functions on \( (x, k) \) are \( \overline{W}_s = \overline{W}_s^m + \overline{W}_s^\mu \), where the index ‘m’ stands for ‘macroscopic’ and the index ‘\( \mu \)’ stands for ‘microscopic’. Because the dependence on \( t \) and \( x \) is slow, one can approximate them as follows:

\[
\overline{W}_s^m (\omega, k; p, p') \approx \alpha_{s'}^\dagger (\omega, k; p) \mathbf{U} (\omega, k) \alpha_s (\omega, k; p),
\]

(6.37a)

\[
\overline{W}_s^\mu (\omega, k; p, p') \approx \alpha_{s'}^\dagger (\omega, k; p) \mathfrak{W} (\omega, k) \alpha_s (\omega, k; p).
\]

(6.37b)

The matrix \( \mathbf{U} \) is positive-semidefinite as an average Wigner tensor (§ 2.1.7), and so is \( \mathfrak{W} \) (§ 6.4). Hence, both \( \overline{W}_s^m \) and \( \overline{W}_s^\mu \) are non-negative. Using (6.31), one can also rewrite the
Within the assumed accuracy, the OC Hamiltonian is
\[ H_s = \tilde{H}_s + \Phi_s, \]
and the contribution of \( \Phi_s \) has been neglected. Because both \( \Phi_s^{(m)} \) and \( \Phi_s \) are quadratic in \( \tilde{\Psi} \) and enter \( \mathcal{H}_s \) only in the combination \( \Delta_s \equiv \Phi_s^{(m)} + \Phi_s \), it is convenient to attribute the

Wigner function of \( \tilde{H}_s \) more compactly as
\[ \tilde{W}_s^{(\mu)}(\omega, k; p) = (2\pi)^{-n} \sum s' \int dp' \delta(\omega - k \cdot v_s') |\mathcal{X}_{ss'}(\omega, k; p, p')|^2 F_s(p'). \] (6.38)

Now we can represent the nonlinear potentials (5.41)–(5.43) as
\[ D_s = D_s^{(m)} + D_s^{(\mu)}, \quad \Theta_s = \Theta_s^{(m)} + \Theta_s^{(\mu)}, \quad \Phi_s = \Phi_s^{(m)} + \Phi_s^{(\mu)}. \] (6.39)
Here, the index \((m)\) denotes contributions from \( \tilde{W}_s^{(m)} \) and the index \((\mu)\) denotes contributions from \( \tilde{W}_s^{(\mu)} \). Specifically,
\[ D_s^{(m)} = \int dk \pi k k \tilde{W}_s^{(m)}(k \cdot v_s, k; p), \] (6.40)
\[ \Theta_s^{(m)} = \frac{\partial \Theta}{\partial \theta} \int d\omega dk \frac{k k \tilde{W}_s^{(m)}(\omega, k; p)}{\omega - k \cdot v_s + \theta} \bigg|_{\theta=0}, \] (6.41)
\[ \Phi_s^{(m)} = \frac{1}{2} \frac{\partial \Phi}{\partial \theta} \cdot \int d\omega dk \frac{k \tilde{W}_s^{(m)}(\omega, k; p)}{\omega - k \cdot v_s}. \] (6.42)
Here, \( \tilde{W}_s^{(m)} \) is a non-negative function (6.37), so \( D_s^{(m)} \) is positive-semidefinite and leads to an \( H \)-theorem similar to (5.6). One also has
\[ D_s^{(\mu)} = \sum s' \int \frac{dk}{(2\pi)^n} dp' \pi k k \delta(k \cdot v_s - k \cdot v_s') |\mathcal{X}_{ss'}(k \cdot v_s, k; p, p')|^2 F_s(p'), \] (6.43)
\[ \Theta_s^{(\mu)} = \sum s' \frac{\partial \Theta}{\partial \theta} \int \frac{dk}{(2\pi)^n} dp' \frac{k F_s'(p')}{k \cdot (v_s' - v_s) + \theta} |\mathcal{X}_{ss'}(k \cdot v_s', k; p, p')|^2 \bigg|_{\theta=0}, \] (6.44)
\[ \Phi_s^{(\mu)} = \sum s' \frac{\partial \Phi}{\partial \theta} \cdot \int \frac{dk}{(2\pi)^n} dp' \frac{k F_s'(p')}{2k \cdot (v_s' - v_s)} |\mathcal{X}_{ss'}(k \cdot v_s', k; p, p')|^2. \] (6.45)

The functions \( \Theta_s^{(\mu)} \) and \( \Phi_s^{(\mu)} \) scale as \( \tilde{W}_s^{(\mu)} \), i.e. as \( \epsilon^2 \varepsilon^2 \) (§ 3.3). Their contribution to (5.37) is of order \( \epsilon \Theta_s^{(\mu)} \) and \( \epsilon \Phi_s^{(\mu)} \), respectively, so it scales as \( \epsilon^2 \varepsilon^2 \) and therefore is negligible within our model. In contrast, \( D_s^{(\mu)} \) must be retained alongside with \( D_s^{(m)} \). This is because although weak, macroscopic fluctuations can resonate with particles from the bulk distribution, while the stronger macroscopic fluctuations are assumed to resonate only with particles from the tail distribution, which are few.

6.6. Oscillation-centre Hamiltonian
Within the assumed accuracy, the OC Hamiltonian is \( \mathcal{H}_s = \tilde{H}_s + \Phi_s^{(m)} \), and \( \tilde{H}_s \) is given by (6.9). Combined with the general theorem (2.53c), the latter readily yields \( \tilde{H}_s = H_0 + \phi_s \), where
\[ \phi_s \approx \frac{1}{2} \int d\omega dk \text{ tr } (U \rho_s) \] (6.46)
and the contribution of \( 2 \tilde{\Psi} \) is within the assumed accuracy. Because both \( \Phi_s^{(m)} \) and \( \Phi_s \) are quadratic in \( \tilde{\Psi} \) and enter \( \mathcal{H}_s \) only in the combination \( \Delta_s \equiv \Phi_s^{(m)} + \phi_s \), it is convenient to attribute the
latter as the ‘total’ ponderomotive energy. Using (6.42) in combination with (6.37a), one can express it as follows:

$$\Delta s = \frac{1}{2} \frac{\partial}{\partial p} \cdot \int d\omega \, dk \, k \, \frac{\alpha_s \, \omega_s}{\omega - k \cdot v_s} + \frac{1}{2} \int d\omega \, dk \, \text{tr} \left( U^s_p \right)$$

(6.47)

(see § 9 for examples). Notably,

$$\Delta s = -\frac{1}{2} \frac{\delta}{\delta F_s} \int d\omega \, dk \, \text{tr}(\Sigma_H U) = -\frac{\delta S_{ad}}{\delta F_s},$$

(6.48)

where $\delta/\delta F_s$ denotes a functional derivative and $S_{ad}$ is the adiabatic action defined in (6.16). Equation (6.48) is a generalization of the well-known ‘$K$–$\chi$ theorem’ (Kaufman & Holm 1984; Kaufman 1987). Loosely speaking, it says that the coefficient connecting $\Delta s$ with $U$ is proportional to the linear polarizability of an individual particle of type $s$ (Dodin et al. 2017; Dodin & Fisch 2010a) (‘$K$’ in the name of this theorem is the same as our $\Delta_s$, and ‘$\chi$’ is the linear susceptibility). Also, the OC Hamiltonian and the OC velocity can be expressed as

$$H_s = H_{0s} + \Delta s, \quad v = \partial_p H_{0s} + \partial_p \Delta s.$$  

(6.49)

6.7. Polarization drag

Within the assumed accuracy, the OC distribution can be expressed as

$$F_s = \tilde{f}_s + \frac{1}{2} \frac{\partial}{\partial p} \cdot \left( \Theta_s^{(m)} \frac{\partial \tilde{f}_s}{\partial p} \right), \quad (6.50)$$

and (5.37) becomes

$$\frac{\partial F_s}{\partial t} - \frac{\partial H_s}{\partial x} \cdot \frac{\partial F_s}{\partial p} + \frac{\partial H_s}{\partial p} \cdot \frac{\partial F_s}{\partial x} = \frac{\partial}{\partial p} \cdot \left( D_s^{(m)} \frac{\partial F_s}{\partial p} \right) + C_s,$$

(6.51)

$$C_s = \frac{\partial}{\partial p} \cdot \left( D_s^{(\omega)} \frac{\partial F_s}{\partial p} \right) + \Gamma_s,$$

(6.52)

where we have reinstated the term $\Gamma_s$ introduced in § 3.3. As a collisional term, $\Gamma_s$ is needed only to the zeroth order in $\epsilon$, so

$$\Gamma_s = \{\tilde{H}_s, g_s\} \approx \partial_p \cdot (g_s \partial_s \tilde{H}_s) \equiv \partial_p \cdot \xi_s, \quad \xi_s = i \langle \{ k \tilde{H}_s \} \rangle \langle x | g_s \rangle.$$

(6.53)

Correlating with $g_s$ is only the microscopic part of $\tilde{H}_s$, so using (6.34) one obtains

$$\xi_s = i \sum_s \int dp' \langle x | k \tilde{X}_{ss} (p, p') | g_s (p') \rangle \langle g_s (p) | x \rangle.$$  

(6.54)

Next, let us use (2.28) and $\zeta = \text{re} \, \xi$ to express this result as follows:

$$\xi_s = i \sum_s \int d\omega \, dk \, dp' \, k \star X_{ss}(\omega, k, p, p') \star \Theta_{s's} (\omega, k, p', p)$$

$$\approx i \int d\omega \, dk \, dp' \, k \, X_{ss}(\omega, k, p, p') \, (2\pi)^{-n} \delta (p - p') \, \delta (\omega - k \cdot v_s) \, F_s(p)$$

$$= -\int \frac{dk}{(2\pi)^n} \, k \, \text{im} \, X_{ss}(k \cdot v_s, k, p, p) \, F_s(p),$$

(6.55)
where we have approximated $\star$ with the usual product and substituted (6.30). Hence,

$$\Gamma_s \approx -\partial_p \cdot (\mathfrak{F}_s F_s),$$  

(6.56)

where $\mathfrak{F}_s$ can be interpreted as the polarization drag (i.e. the average force that is imposed on an OC by its dress) and is given by

$$\mathfrak{F}_s = \int \frac{dk}{(2\pi)^n} k \text{im} \mathcal{X}_{ss'}(k \cdot v_s, k, p, p').$$  

(6.57)

Using (6.36), one also rewrite this as follows:

$$\mathfrak{F}_s \approx \int \frac{dk}{(2\pi)^n} k (\alpha_s^\dag (\Xi^{-1})_A \alpha_s)(k \cdot v_s, k; p)$$  

(6.58a)

$$\approx -\int \frac{dk}{(2\pi)^n} k (\alpha_s^\dag \Xi^{-1} \Xi A \Xi^{-1} \alpha_s)(k \cdot v_s, k; p),$$  

(6.58b)

where we have substituted (6.26) for $(\Xi^{-1})_A$. With (6.24) for $\Xi_A$, this yields

$$\mathfrak{F}_s \approx \sum_{s'} \int \frac{dk}{(2\pi)^n} dp' \pi \delta(k \cdot v_s - k \cdot v_{s'}) kk \cdot \frac{\partial F_s(p')}{\partial p'}$$

$$\times \alpha_s^\dag (k \cdot v_s, k; p) \Xi^{-1} (k \cdot v_s, k) \alpha_s (k \cdot v_s, k; p')$$

$$\times \alpha_{s'}^\dag (k \cdot v_{s'}, k; p') \Xi^{-1} (k \cdot v_{s'}, k) \alpha_{s'} (k \cdot v_{s'}, k; p).$$  

(6.59)

The product of the last two lines equals $|\mathcal{X}_{ss'}(k \cdot v_s, k; p, p')|^2$. Hence,

$$\mathfrak{F}_s = \sum_{s'} \int \frac{dk}{(2\pi)^n} dp' \pi \delta(k \cdot v_s - k \cdot v_{s'}) |\mathcal{X}_{ss'}(k \cdot v_s, k; p, p')|^2 kk \cdot \frac{\partial F_s(p')}{\partial p'}.$$  

(6.60)

6.8. Collision operator

By combining (6.56) for $\Gamma_s$ with (6.43) for $D_s^{(\mu)}$, one can express $C_s$ as

$$C_s = \frac{\partial}{\partial p} \cdot \sum_{s'} \int \frac{dk}{(2\pi)^n} dp' \pi \delta(k \cdot v_s - k \cdot v_{s'}) |\mathcal{X}_{ss'}(k \cdot v_s, k; p, p')|^2$$

$$\times kk \cdot \left( \frac{\partial F_s(p)}{\partial p} F_s(p') - F_s(p) \frac{\partial F_s(p')}{\partial p'} \right),$$  

(6.61)

where $\mathcal{X}_{ss'}$ is given by (6.36). One can recognize this as a generalization of the Balescu–Lenard collision operator (Krall & Trivelpiece 1973, § 11.11) to interactions via a general multi-component field $\Psi$. Specific examples can be found in § 9.
It is readily seen that $C_s$ conserves particles, i.e.

$$
\int dp C_s = 0,
$$

(6.62)

and vanishes in thermal equilibrium (§ 8.1). Other properties of $C_s$ are determined by the properties of the coupling coefficient $\mathcal{X}_{ss'}$, which are as follows. Note that

$$
|\mathcal{X}_{ss'}(\omega, k; p, p')|^2 = Q_{ss'}(\omega, k; p, p') + R_{ss'}(\omega, k; p, p')/2,
$$

(6.63)

where we introduced

$$
Q_{ss'}(\omega, k; p, p') \equiv (|\mathcal{X}_{ss'}(\omega, k; p, p')|^2 + |\mathcal{X}_{s's}(\omega, k; p', p)|^2)/2,
$$

(6.64)

$$
R_{ss'}(\omega, k; p, p') \equiv |\mathcal{X}_{s's}(\omega, k; p, p')|^2 - |\mathcal{X}_{ss'}(\omega, k; p', p)|^2.
$$

(6.65)

To calculate $R_{ss'}$, note that (6.36) yields

$$
|\mathcal{X}_{ss'}(\omega, k; p, p')|^2 \approx |\alpha^s(\omega, k; p)\Xi^{-1}(\omega, k)\alpha_{s'}(\omega, k; p')|^2,
$$

$$
|\mathcal{X}_{s's}(\omega, k; p', p)|^2 \approx |\alpha^s(\omega, k; p)\Xi^{-1}(\omega, k)\alpha_{s'}(\omega, k; p')|^2,
$$

(6.66)

whence one obtains

$$
R_{ss'}(\omega, k; p, p') \approx 4 \text{im} \left( \alpha^s(\omega, k; p)\Xi^{-1}(\omega, k)\alpha_{s'}(\omega, k; p') \right).
$$

(6.67)

The operators $(\Xi^{-1})_H$, $(\Xi^{-1})_A$ and $\hat{\alpha}_s$ (for all $s$) have been introduced for real fields, so their matrix elements in the coordinate representation are real. Then, the corresponding symbols satisfy $A(-\omega, -k) = A^*(\omega, k)$, where $A$ is any of the three symbols. This gives

$$
R_{ss'}(k; p, p') = R_{ss'}(k \cdot v_s, k; p, p') = -R_{ss'}(-k; k, p, p').
$$

(6.68)

Because the rest of the integrand in (6.61) is even in $k$, (6.68) signifies that $R_{ss'}$ does not contribute to $C_s$. Thus, $\mathcal{X}_{ss'}$ in (6.61) can also be replaced with $Q_{ss'}$:

$$
C_s = \frac{\partial}{\partial p} \sum_{s'} \int \frac{dk}{(2\pi)^n} dp' \pi \delta(k \cdot v_s - k' \cdot v_{s'}) Q_{ss'}(k \cdot v_s, k; p, p') \times kk \cdot \left( \frac{\partial F_s(p)}{\partial p} F_{s'}(p') - F_s(p) \frac{\partial F_{s'}(p')}{\partial p'} \right).
$$

(6.69)

In this representation, the coupling coefficient in $C_s$ is manifestly symmetric,

$$
Q_{ss'}(\omega, k; p, p') = Q_{s's}(\omega, k; p', p),
$$

(6.70)

which readily leads to momentum and energy conservation (Appendix C):\(^{28}\)

$$
\sum_s \int dp p C_s = 0, \quad \sum_s \int dp \mathcal{H}_s C_s = 0.
$$

(6.71)

\(^{28}\)Remember that $v_s$ is defined as the OC velocity in the above formulas (§ 5.6). If $v_s$ is treated as the particle velocity instead, then $\mathcal{H}_s$ in (6.71) should be replaced with $H_s$. Both options are admissible within the assumed accuracy, but the former option is preferable because it leads to other conservation laws that are exact within our model (§ 7.5).
The collision operator $C_s$ also satisfies the $H$-theorem (Appendix C.3):

$$\left(\frac{d\sigma}{dt}\right)_{\text{coll}} \geq 0, \quad (6.72)$$

where the entropy density $\sigma$ is defined as

$$\sigma = -\sum_s \int d\mathbf{p} F_s(\mathbf{p}) \ln F_s(\mathbf{p}), \quad (6.73)$$

and $(\partial_t F_s)_{\text{coll}} \equiv C_s$. Note that these properties are not restricted to any particular $\mathcal{H}_s$. Also note that if applied in proper variables (§ 3.1.2), our formula (6.69) can describe collisions in strong background fields. This topic, including comparison with the relevant literature, is left to future work.

### 6.9. Summary of § 6

Let us summarize the above general results (for examples, see § 9). We consider species $s$ governed by a Hamiltonian of the form

$$H_s = H_{0s} + \hat{\alpha}_s(\mathbf{p})\tilde{\Psi} + \frac{1}{2} (\hat{L}_s\tilde{\Psi})^\dagger (\hat{R}_s\tilde{\Psi}), \quad (6.74)$$

where $\tilde{\Psi}$ is a real oscillating field (of any dimension $M$), which generally consists of a macroscopic part $\tilde{\Psi}$ and a microscopic part $\tilde{\Psi}_\dagger$. The term $H_{0s}$ is independent of $\tilde{\Psi}$, and the operators $\hat{\alpha}_s$, $\hat{L}_s$ and $\hat{R}_s$ may be non-local in $t$ and $x$ and may depend on the momentum $\mathbf{p}$ parametrically. The dynamics of this system averaged over the fast oscillations can be described in terms of the OC distribution function

$$F_s = \tilde{f}_s + \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \left( \Theta_s \frac{\partial \tilde{f}_s}{\partial \mathbf{p}} \right), \quad (6.75)$$

(the index $^\dagger$ is henceforth omitted for brevity), which is governed by the following equation of the Fokker–Planck type:

$$\frac{\partial F_s}{\partial t} - \frac{\partial \mathcal{H}_s}{\partial \mathbf{x}} \cdot \frac{\partial F_s}{\partial \mathbf{p}} + \frac{\partial \mathcal{H}_s}{\partial \mathbf{p}} \cdot \frac{\partial F_s}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{p}} \cdot \left( D_s \frac{\partial F_s}{\partial \mathbf{p}} \right) + C_s. \quad (6.76)$$

Here, $\mathcal{H}_s = H_{0s} + \Delta_s$ is the OC Hamiltonian, $\Theta_s$ is the dressing function and $\Delta_s$ is the total ponderomotive energy (i.e. the part of the OC Hamiltonian that is quadratic in $\tilde{\Psi}$), so $\mathbf{v}_s(t, \mathbf{x}, \mathbf{p}) \equiv \partial_p \mathcal{H}_s$ is the OC velocity. Specifically,

$$D_s = \int dk \pi kk\overline{W}_s(t, \mathbf{x}, k \cdot \mathbf{v}_s, k; \mathbf{p}), \quad (6.77a)$$

$$\Theta_s = \frac{\partial}{\partial \vartheta} \int d\omega dk kk \overline{W}_s(t, \mathbf{x}, \omega, k; \mathbf{p}) \bigg|_{\vartheta=0}, \quad (6.77b)$$

$$\Delta_s = \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \int d\omega dk k \overline{W}_s(t, \mathbf{x}, \omega, k; \mathbf{p}) \bigg|_{\omega - k \cdot \mathbf{v}_s} + \frac{1}{2} \int d\omega dk \text{tr} (\mathbf{U} \overline{\varphi}_s)(t, \mathbf{x}, \omega, k; \mathbf{p}). \quad (6.77c)$$

Here, $\overline{W}_s = \alpha_s^\dagger \mathbf{U} \alpha_s$ is a scalar function, the average Wigner matrix $\mathbf{U}$ is understood as the Fourier spectrum of the symmetrized autocorrelation matrix of the macroscopic
oscillations:

\[
\mathbf{U}(t, x, \omega, \mathbf{k}) = \int \frac{d\tau}{2\pi} \frac{ds}{(2\pi)^n} \Psi(t + \tau/2, x + s/2) \Psi^{\dagger}(t - \tau/2, x - s/2) e^{i\omega - i\mathbf{k} \cdot \mathbf{x}}, \tag{6.78}
\]

with \( n = \dim x \). Also, the vector \( \alpha_s(t, x, \omega, \mathbf{k}; \mathbf{p}) \) is the Weyl symbol of \( \hat{a}_s \) as defined in (2.26), \( \varphi_s(t, x, \omega, \mathbf{k}; \mathbf{p}) \approx (L_s^i \mathbf{R}_s)_{ii} \), \( L_s \) and \( R_s \) are the Weyl symbols of \( \hat{L}_s \) and \( \hat{R}_s \), respectively, and \( H_s \) denotes the Hermitian part. The matrix \( D_s \) is positive-semidefinite and satisfies an \( H \)-theorem of the form (5.46). Also, \( \Delta_s \) satisfies the ‘\( K - \chi \) theorem’

\[
\Delta_s = -\frac{1}{2} \frac{\delta}{\delta F_s} \int d\omega \frac{dk}{2} \text{tr}(\mathbf{z}_H \mathbf{U}^\dagger). \tag{6.79}
\]

The matrix \( \mathbf{z} \) characterizes the collective plasma response to the field \( \Psi \) and is given by

\[
\mathbf{z} \approx \mathbf{z}_0 + \sum_s \int d\mathbf{p} \left( \frac{\alpha_s(\mathbf{p}) \alpha_s^{\dagger}(\mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s(\mathbf{p}) + i0} \mathbf{k} \cdot \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} - \varphi_s(\mathbf{p}) F_s(\mathbf{p}) \right). \tag{6.80}
\]

Here, the arguments \((t, x, \omega, \mathbf{k})\) are omitted for brevity, \( \alpha_s \alpha_s^{\dagger} \) is a dyadic matrix and \( \mathbf{z}_0 \) is the symbol of the Hermitian dispersion operator \( \hat{\mathbf{z}}_0 \) that governs the field \( \hat{\Psi} \) in the absence of plasma. Specifically, \( \hat{\mathbf{z}}_0 \) is defined such that the field Lagrangian density without plasma is

\[
\mathcal{L}_0 = \hat{\Psi}^\dagger \hat{\mathbf{z}}_0 \hat{\Psi}/2.
\]

The spectrum of microscopic fluctuations (specifically, the spectrum of the symmetrized autocorrelation function of the microscopic field \( \tilde{\Psi} \)) is a positive-semidefinite matrix function and given by

\[
\mathcal{W}(\omega, \mathbf{k}) = \frac{1}{(2\pi)^n} \sum_s \int d\mathbf{p} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s(\mathbf{p}')) \mathbf{z}^{-1}(\omega, \mathbf{k}; \mathbf{k} \cdot \mathbf{v}_s(\mathbf{p}')) \mathbf{z}^{-\dagger}(\omega, \mathbf{k}), \tag{6.81}
\]

where \( \mathbf{v}_s = \mathbf{v}_s(\mathbf{p}') \). (The dependence on \( t \) and \( x \) is assumed too but not emphasized.) The microscopic fluctuations give rise to a collision operator of the Balescu–Lenard type:

\[
\mathcal{C}_s = \frac{\partial}{\partial \mathbf{p}} \sum_s \int \frac{dk}{(2\pi)^n} \mathbf{p} \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}_s') Q_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k} ; \mathbf{p}, \mathbf{p}')
\]

\[
\times \mathbf{k} \cdot \mathbf{k} \cdot \left( \frac{\partial F_0(\mathbf{p})}{\partial \mathbf{p}} F_s(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_s(\mathbf{p}')}{\partial \mathbf{p}'} \right), \tag{6.82}
\]

where the coupling coefficient \( Q_{ss'}(\omega, \mathbf{k} ; \mathbf{p}, \mathbf{p}') = Q_{ss'}(\omega, \mathbf{k} ; \mathbf{p}'; \mathbf{p}) \) is given by

\[
Q_{ss'}(\omega, \mathbf{k} ; \mathbf{p}, \mathbf{p}') = (|\chi_{ss'}(\omega, \mathbf{k} ; \mathbf{p}, \mathbf{p}')|^2 + |\chi_{ss}(\omega, \mathbf{k} ; \mathbf{p}', \mathbf{p})|^2)/2, \tag{6.83}
\]

\[
\chi_{ss'}(\omega, \mathbf{k} ; \mathbf{p}, \mathbf{p}') \approx \alpha_s^{\dagger}(\omega, \mathbf{k} ; \mathbf{p}) \mathbf{z}^{-1}(\omega, \mathbf{k}) \alpha_s(\omega, \mathbf{k} ; \mathbf{p}). \tag{6.84}
\]

The operator \( \mathcal{C}_s \) satisfies the \( H \)-theorem and conserves particles, momentum and energy

\[
\int d\mathbf{p} \mathcal{C}_s = 0, \quad \sum_s \int d\mathbf{p} \mathcal{H}_s \mathcal{C}_s = 0, \quad \sum_s \int d\mathbf{p} \mathcal{H}_s \mathcal{C}_s = 0.
\]
7. Interaction with on-shell waves

Here, we discuss QL interaction of plasma with ‘on-shell’ waves, i.e. waves constrained by dispersion relations. To motivate the assumptions that will be adopted, and also to systematically introduce our notation, we start with briefly reviewing theory of linear waves in dispersive media (Whitham 1974; Tracy et al. 2014), including monochromatic waves (§ 7.1), conservative eikonal waves (§ 7.2), general eikonal waves (§ 7.3) and general broadband waves described by the WKE (§ 7.4). After that, we derive conservation laws for the total momentum and energy, which are exact within our model (§ 7.5). All waves in this section are considered macroscopic, so we adopt a simplified notation $\tilde{\Psi} \equiv \tilde{\psi}$ and the index $(m)$ will be omitted.

7.1. Monochromatic waves

Conservative (non-dissipative) waves can be described using the least-action principle $\delta S = 0$. Assuming the notation as in § 6.2, the action integral can be expressed as $S = \int dx \mathcal{L}$ with the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2} \tilde{\psi}^\dagger \mathcal{E}_H \tilde{\psi}. \quad (7.1)$$

First, let us assume a homogeneous stationary medium, so $\mathcal{E}_H(t, x, \omega, k) = \mathcal{E}_H(\omega, k)$. Because we assume real fields, $\langle t_1, x_1 | \mathcal{E} | t_2, x_2 \rangle$ is real for all $(t_1, x_1, t_2, x_2)$, one also has

$$\mathcal{E}_H(-\omega, -k) = \mathcal{E}_H^\dagger(\omega, k) = \mathcal{E}_H^T(\omega, k), \quad (7.2)$$

where the latter equality is due to $\mathcal{E}_H^\dagger(\omega, k) = \mathcal{E}_H(\omega, k)$.

Because $\mathcal{E}_H(\omega, k)$ is Hermitian, it has $M = \dim \mathcal{E}_H$ orthonormal eigenvectors $\eta_b$:

$$\mathcal{E}_H(\omega, k) \eta_b(\omega, k) = \Lambda_b(\omega, k) \eta_b(\omega, k), \quad \eta_b^\dagger(\omega, k) \eta_{b'}(\omega, k) = \delta_{b,b'}. \quad (7.3)$$

Here, $\Lambda_b$ are the corresponding eigenvalues, which are real and satisfy

$$\Lambda_b(\omega, k) = \eta_b^\dagger(\omega, k) \mathcal{E}_H(\omega, k) \eta_b(\omega, k). \quad (7.4)$$

Due to (7.2), one has

$$\Lambda_b(-\omega, -k) = \text{eigv}_b(\mathcal{E}_H(\omega, k))^T = \text{eigv}_b \mathcal{E}_H(\omega, k) = \Lambda_b(\omega, k), \quad (7.5)$$

where eigv$_b$ stands for the $b$th eigenvalue. Using this together with (7.2), one obtains from (7.3) that

$$\mathcal{E}_H^\dagger(\omega, k) \eta_b(-\omega, -k) = \Lambda_b(\omega, k) \eta_b(-\omega, -k), \quad (7.6)$$

whence

$$\eta_b(-\omega, -k) = \eta_b^\dagger(\omega, k). \quad (7.7)$$

Let us consider a monochromatic wave of the form

$$\tilde{\Psi}(t, x) = \text{re}(e^{-i\omega t + i\overline{k} \cdot x} \tilde{\psi}), \quad (7.8)$$

with real frequency $\omega$, real wavevector $\overline{k}$ and complex amplitude $\tilde{\psi}$. For such a wave, the action integral can be expressed as $S = \int dx \overline{\mathcal{L}}$, where the average Lagrangian density $\overline{\mathcal{L}}$
Quasilinear theory

is given by

\[ \Sigma = \frac{1}{2} \Psi^\dagger \Xi H \Psi = \frac{1}{4} \text{re}(\Psi^\dagger \Xi H(\bar{\omega}, \bar{k}) \Psi) = \frac{1}{4} \Psi^\dagger \Xi H(\bar{\omega}, \bar{k}) \Psi. \tag{7.9} \]

Let us decompose \( \Psi \) in the basis formed by the eigenvectors \( \eta_b \), that is, as

\[ \Psi = \sum_b \eta_b \hat{a}^b. \tag{7.10} \]

Then, (7.9) becomes

\[ \Sigma = \frac{1}{4} \sum_b \Lambda_b(\bar{\omega}, \bar{k}) |\hat{a}^b|^2. \tag{7.11} \]

The real and imaginary parts of the amplitudes \( \hat{a}^b \) can be treated as independent variables. This is equivalent to treating \( \hat{a}^{bs} \) and \( \hat{a}^b \) as independent variables, so one arrives at the following Euler–Lagrange equations:

\[ 0 = \frac{\delta S[\hat{a}^s, \hat{a}^b]}{\delta \hat{a}^{bs}} = \frac{1}{4} \Lambda_b(\bar{\omega}, \bar{k}) \hat{a}^b, \quad 0 = \frac{\delta S[\hat{a}^s, \hat{a}^b]}{\delta \hat{a}^b} = \frac{1}{4} \hat{a}^{bs} \Lambda_b(\bar{\omega}, \bar{k}). \tag{7.12} \]

Hence the \( b \)th mode with a non-zero amplitude \( \hat{a}^b \) satisfies the dispersion relation

\[ 0 = \Lambda_b(\bar{\omega}, \bar{k}) = \Lambda_b(-\bar{\omega}, -\bar{k}). \tag{7.13} \]

Equation (7.13) determines a dispersion surface in the \( k \) space where the waves can have non-zero amplitude. This surface is sometimes called a shell, so waves constrained by a dispersion relation are called on-shell. Also note that combining (7.13) with (7.3) yields that on-shell waves satisfy

\[ \Xi H(\bar{\omega}, \bar{k}) \eta_b(\bar{\omega}, \bar{k}) = 0, \quad \eta_b^\dagger(\bar{\omega}, \bar{k}) \Xi H(\bar{\omega}, \bar{k}) = 0, \tag{7.14} \]

which are two mutually adjoint representations of the same equation.

Below, we consider the case when (7.13) is satisfied only for one mode at a time, so summation over \( b \) and the index \( b \) itself can be omitted. (A more general case is discussed, for example, in Dodin et al. (2019).) Then, \( \Psi = \eta(\bar{\omega}, \bar{k}) \hat{a} \)

\[ \Sigma = \frac{1}{4} \Lambda(\bar{\omega}, \bar{k}) |\hat{a}|^2, \tag{7.15} \]

and \( \bar{\omega} \) is connected with \( \bar{k} \) via \( \bar{\omega} = w(\bar{k}) \), where \( w(k) = -w(-k) \) is the function that solves \( \Lambda(w(k), k) = 0 \). Also importantly, (7.14) ensures that

\[ \partial \Lambda(\bar{\omega}, \bar{k}) = ((\partial \eta^\dagger) \Xi H \eta + \eta^\dagger (\partial \Xi_H) \eta + \eta^\dagger \Xi_H (\partial \eta)) \big|_{(\omega, k) = (\bar{\omega}, \bar{k})} = (\eta^\dagger (\partial \Xi_H) \eta) \big|_{(\omega, k) = (\bar{\omega}, \bar{k})}, \tag{7.16} \]

where \( \bullet \) can be replaced with any variable.

---

30 Here we use that for any oscillating \( a = \text{re}(e^{i\theta} \hat{a}) \) and \( b = \text{re}(e^{i\theta} \hat{b}) \), one has \( \bar{a} \bar{b} = \text{re}(\hat{a}^* \hat{b})/2 \) and that \( \Psi^\dagger \Xi H(\bar{\omega}, \bar{k}) \Psi \) is real because \( \Xi_H(\bar{\omega}, \bar{k}) \) is Hermitian.
7.2. Conservative eikonal waves

7.2.1. Basic properties

In case of a quasimonochromatic eikonal wave and, possibly, inhomogeneous non-stationary plasma, one can apply the same arguments as in § 7.1 except the above equalities are now satisfied up to \( O(\epsilon) \). For a single-mode wave, one has

\[
\widetilde{\Psi}(t, x) = \text{re}(\widetilde{\Psi}_c(t, x)) + O(\epsilon), \quad \widetilde{\Psi}_c = e^{i\theta(t, x)}\eta(t, x)\tilde{a}(t, x),
\]

where the local frequency and the wavevector,

\[
\bar{\omega} \doteq -\partial_t \theta, \quad \bar{k} \doteq \partial_x \theta
\]

are slow functions of \((t, x)\), and so is \( \eta(t, x) \doteq \eta(t, x, \bar{\omega}(t, x), \bar{k}(t, x)) \), which satisfies (7.3). Then,

\[
\mathcal{S} = \frac{1}{4} \Lambda(t, x, \bar{\omega}, \bar{k})|\tilde{a}(t, x)|^2 + O(\epsilon).
\]

Within the leading-order theory, the term \( O(\epsilon) \) is neglected. 31 Then, the least-action principle

\[
0 = \frac{\delta S[\theta, \tilde{a}, \bar{a}]}{\delta \tilde{a}^{\bar{a}}} \approx \frac{1}{4} \Lambda_b(\bar{\omega}, \bar{k})\tilde{a}^{\bar{a}}, \quad 0 = \frac{\delta S[\theta, \tilde{a}, \bar{a}]}{\delta \tilde{a}^{\bar{a}}} \approx \frac{1}{4} \tilde{a}^{\bar{a}} \Lambda_b(\bar{\omega}, \bar{k})
\]

leads to the same (but now local) dispersion relation as for monochromatic waves, \( \Lambda(t, x, \bar{\omega}, \bar{k}) = 0 \). This shows that quasimonochromatic waves are also on-shell, and thus they satisfy (7.16) as well. Also notice that the dispersion relation can now be understood as a Hamilton–Jacobi equation for the eikonal phase \( \theta \):

\[
\Lambda(t, x, -\partial_t \theta, \partial_x \theta) = 0.
\]

Like in the previous section, let us introduce the function \( w \) that solves

\[
\Lambda(t, x, w(t, x, k), k) = 0
\]

and therefore satisfies

\[
w(t, x, k) = -w(t, x, -k).
\]

Differentiating (7.22) with respect to \( t, x \) and \( k \) leads to

\[
\partial_t \Lambda + (\partial_w \Lambda)\partial_t w = 0,
\]

\[
\partial_x \Lambda + (\partial_w \Lambda)\partial_x w = 0,
\]

\[
\partial_k \Lambda + (\partial_w \Lambda)\partial_k w = 0,
\]

where the derivatives of \( \Lambda \) are evaluated at \((t, x, w(t, x, k), k)\). In particular, (7.24c) gives

\[
\mathbf{v}_g \doteq \frac{\partial w}{\partial k} = -\frac{\partial_k \Lambda}{\partial_w \Lambda},
\]

for the group velocity \( \mathbf{v}_g \), whose physical meaning is to be specified shortly.

31Corrections to the lowest-order dispersion relation produce the so-called spin Hall effect; see Dodin et al. (2019) and Ruiz & Dodin (2017a) for an overview and Bliokh et al. (2015), Ruiz & Dodin (2015a), Oancea et al. (2020) and Andersson et al. (2021) for examples. These corrections are beyond the accuracy of the model considered, so they will be ignored.
Because \( \theta \) is now an additional dynamical variable, one also obtains an additional Euler–Lagrange equation

\[
0 = \delta_\theta S[\theta, \tilde{a}^*, \tilde{a}] = \partial_t I + \partial_x \cdot \mathcal{J},
\]

(7.26)

where \( I \) is called the action density and \( \mathcal{J} \) is the action flux density:

\[
\frac{\partial \Sigma}{\partial \omega} = \frac{|\tilde{a}|^2}{4} \frac{\partial S}{\partial \omega} = \frac{|\tilde{a}|^2}{4} \eta^+ \frac{\partial \mathcal{E}_H}{\partial \omega} \eta,
\]

(7.27)

\[
\mathcal{J}^i = -\frac{\partial \Sigma}{\partial k_i} = -\frac{|\tilde{a}|^2}{4} \frac{\partial S}{\partial k_i} = -\frac{|\tilde{a}|^2}{4} \eta^+ \frac{\partial \mathcal{E}_H}{\partial k_i} \eta,
\]

(7.28)

where we used (7.16) and the derivatives are evaluated on \((t, x, w(t, x, \vec{K}(t, x)), \vec{K}(t, x))\). Using (7.25), one can also rewrite (7.28) as

\[
\mathcal{J} = \bar{v}_g I, \quad \bar{v}_g(t, x) = v_g(t, x, \vec{K}(t, x)).
\]

(7.29)

(The arguments \((t, x)\) will be omitted from now on for brevity. We will also use \(\mathbf{k}\) as a shorthand for \((w(k), k)\) where applicable.) Then, (7.26) becomes

\[
\partial_t I + \partial_x \cdot (\bar{v}_g I) = 0,
\]

(7.30)

which can be understood as a continuity equation for quasiparticles (‘photons’ or, more generally, ‘wave quanta’) with density \( I \) and fluid velocity \( \bar{v}_g \) (see also § 7.2.2). Thus, if an eikonal wave satisfies the least-action principle, its total action \( \int \! dx \bar{I} \) (‘number of quanta’) is an invariant. This conservation law can be attributed to the fact that the wave Lagrangian density \( \Sigma \) depends on derivatives of \( \theta \) but not on \( \theta \) per se.

Also notice the following. By expanding (7.19) in \( \partial_t \theta \) around \( \partial_t \theta = -w(t, x, \partial_x \theta) \), which is satisfied on any solution, one obtains

\[
\Sigma \approx -\frac{1}{4} (\partial_t \theta + w(t, x, \partial_x \theta)) \partial_\omega \Lambda |\tilde{a}|^2 = -(\partial_t \theta + w(t, x, \partial_x \theta)) I,
\]

(7.31)

where we used that \( \Sigma(t, x, -\partial_t \theta, \partial_x \theta) = 0 \) due to (7.21). Then, one arrives at the canonical form of the action integral (Hayes 1973)

\[
S[I, \theta] = -\int \! dt \! dx \left( \partial_t \theta + w(t, x, k) \right) I.
\]

(7.32)

From here, \( \delta_x S = 0 \) yields the dispersion relation in the Hamilton–Jacobi form \( \partial_t \theta + w(t, x, k) = 0 \), and \( \delta_\theta S = 0 \) yields the action conservation (7.30).

7.2.2. Ray equations

By (7.18), one has the so-called consistency relations

\[
\partial_t \vec{k}_i + \partial_i \omega = 0, \quad \partial_j \vec{k}_i = \partial_i \vec{k}_j.
\]

(7.33)

These lead to

\[
\left( \frac{\partial}{\partial t} + \bar{v}_g \cdot \frac{\partial}{\partial x} \right) \vec{k}_i(t, x) = -\frac{\partial w(t, x, \vec{K}(t, x))}{\partial x^i} + \bar{v}_g \cdot \frac{\partial \vec{k}_i(t, x)}{\partial x^j},
\]

\[
= -\left( \frac{\partial w(t, x, k)}{\partial x^i} \right)_{k=\vec{k}(t, x)} - \bar{v}_g \frac{\partial \vec{k}_j(t, x)}{\partial x^i} + \bar{v}_g \frac{\partial \vec{k}_j(t, x)}{\partial x^i},
\]

(7.34)
and similarly,

\[
\left( \frac{\partial}{\partial t} + \overline{v}_g \cdot \frac{\partial}{\partial x} \right) \overline{\omega}(t, x) = \left( \frac{\partial}{\partial t} + \overline{v}_g \cdot \frac{\partial}{\partial x} \right) w(t, x, \overline{k}(t, x))
\]

\[
= \left( \frac{\partial w(t, x, k)}{\partial t} + \overline{v}_g \frac{\partial w(t, x, k)}{\partial x} \right)_{k=\overline{k}(t,x)} + \overline{v}_g \left( \frac{\partial}{\partial t} + \overline{v}_g \cdot \frac{\partial}{\partial x} \right) \overline{k}(t, x)
\]

\[
= \left( \frac{\partial w(t, x, k)}{\partial t} \right)_{k=\overline{k}(t,x)},
\]

(7.35)

where we used (7.34). Using the convective derivative associated with the group velocity,

\[
d/dt \equiv \partial_t \equiv \partial_t + (\overline{v}_g \cdot \partial_x),
\]

(7.36)

one can rewrite these compactly as

\[
\frac{d\overline{k}_i(t, x)}{dt} = -\left( \frac{\partial w(t, x, k)}{\partial x^i} \right)_{k=\overline{k}(t,x)}, \quad \frac{d\overline{\omega}(t, x)}{dt} = \left( \frac{\partial w(t, x, k)}{\partial t} \right)_{k=\overline{k}(t,x)}.
\]

(7.37)

One can also represent (7.37) as ordinary differential equations for \( \overline{k}(t) \doteq \overline{k}(t, \overline{x}(t)) \) and \( \overline{\omega}(t) \doteq \overline{\omega}(t, \overline{x}(t)) \), where \( \overline{x}(t) \) are the ‘ray trajectories’ governed by

\[
\frac{d\overline{x}^i(t)}{dt} = \overline{v}^i(t, \overline{x}(t), \overline{k}(t)).
\]

(7.38)

Specifically, together with (7.38), equations (7.37) become Hamilton’s equations also known as the ray equations:

\[
\frac{d\overline{x}^i}{dt} = \frac{\partial w(t, \overline{x}, \overline{k})}{\partial \overline{k}_i}, \quad \frac{d\overline{k}_i}{dt} = -\frac{\partial w(t, \overline{x}, \overline{k})}{\partial \overline{x}}, \quad \frac{d\overline{\omega}}{dt} = \frac{\partial w(t, \overline{x}, \overline{k})}{\partial t}.
\]

(7.39)

where \( \overline{x} \) is the coordinate, \( \hbar \overline{k} \) is the momentum, \( \hbar \overline{\omega} \) is the energy, \( \hbar w \) is the Hamiltonian and the constant factor \( \hbar \) can be anything. If \( \hbar \) is chosen to be the Planck constant, then (7.39) can be interpreted as the motion equations of individual wave quanta, for example, photons. Hamilton’s equations for ‘true’ particles, such as electrons and ions, are also subsumed under (7.39) in that they can be understood as the ray equations of the particles considered as quantum-matter waves in the semiclassical limit.

Also notably, (7.39) can be obtained by considering the point-particle limit of (7.32) (Ruiz & Dodin 2015b). Specifically, adopting \( \mathcal{I}(t, x) \propto \delta(x - \overline{x}(t)) \) and taking the integral in (7.32) by parts leads to a canonical action \( S \propto \int dt (\overline{k} \cdot \overline{\dot{x}} - w(t, \overline{x}, \overline{k})) \), whence Hamilton’s equations follow as usual.

7.2.3. Wave momentum and energy

Using (7.30) and (7.36), one arrives at the following equality for any given field \( X \):

\[
\partial_t(X\mathcal{I}) + \partial_x \cdot (X\mathcal{I} \overline{v}_g) = (\partial_t X)\mathcal{I} + X(\partial_t \mathcal{I}) + [\partial_x \cdot (\mathcal{I} \overline{v}_g)]X + \mathcal{I}(\overline{v}_g \cdot \partial_x)X
\]

\[
= \mathcal{I}[\partial_t + (\overline{v}_g \cdot \partial_x)]X + X[\partial_t \mathcal{I} + \partial_x \cdot (\mathcal{I} \overline{v}_g)]
\]

\[
= \mathcal{I} d_t X.
\]

(7.40)

For \( X = \overline{k}_i \) and \( X = \overline{\omega} \), (7.40) yields, respectively,

\[
\partial_t P_{w,i} + \partial_x \cdot (\overline{v}_g P_{w,i}) = -\mathcal{I} \partial_t w,
\]

(7.41a)

\[
\partial_t \mathcal{E}_w + \partial_x \cdot (\overline{v}_g \mathcal{E}_w) = \mathcal{I} \partial_t w,
\]

(7.41b)
where we used (7.37) and introduced the following notation:

\[ P_w \doteq \kappa I, \quad E_w \doteq \omega I. \] (7.42)

When a medium is homogeneous along \( x' \), (7.41a) yields \( \int \text{d}x P_{w,i} = \text{const} \). Likewise, when a medium is stationary, (7.41b) yields \( \int \text{d}x E_w = \text{const} \). Hence, by definition, \( P_w \) and \( E_w \) are the densities of the wave canonical momentum and energy, at least up to a constant factor \( \kappa \).\(^{32}\) A proof that \( \kappa = 1 \) can be found, for example, in Dodin \& Fisch (2012). In § 7.5, we will show this using different arguments.

### 7.3. Non-conservative eikonal waves

In a medium with non-zero \( \Xi_A \), where waves are non-conservative, the wave properties are defined as in the previous section but the wave action evolves differently. The variational principle is not easy to apply in this case (however, see Dodin et al. 2017), so a different approach will be used to derive the action equation. A more straightforward but less intuitive approach can be found in Dodin et al. (2019) and McDonald (1988).

#### 7.3.1. Monochromatic waves

First, consider a homogeneous stationary medium and a ‘monochromatic’ (exponentially growing at a constant rate) wave field in the form

\[ \tilde{\Psi}(t, x) = \text{re}(e^{-i\omega t + iKx} \tilde{\psi}_c), \quad \tilde{\psi}_c = e^{\gamma t} \times \text{const}, \] (7.43)

where the constants \( \omega \) and \( K \) are, as usual, the real frequency and wavenumber, and \( \gamma \) is the linear growth rate, which can have either sign. Then, (6.15) becomes

\[ 0 = \Xi(\omega + i\gamma, K)\tilde{\psi}_c = \Xi_H(\omega, K)\tilde{\psi}_c + i(\gamma \partial_\omega \Xi_H(\omega, K) + \Xi_A(\omega, K))\tilde{\psi}_c + O(\epsilon^2), \] (7.44)

where we assume that \( \Xi \) is a smooth function of \( \omega \) and also that both \( \Xi_A \) and \( \gamma \) are \( O(\epsilon) \). Like in § 7.2.1, we adopt \( \tilde{\psi}_c = \eta \tilde{a} + O(\epsilon) \), where the polarization vector \( \eta \) is the relevant eigenvector of \( \Xi_H \). Then, by projecting (7.44) on \( \eta \), one obtains

\[ 0 = \Lambda(\omega, K)\tilde{a} + i(\gamma \partial_\omega \Lambda + \eta^\dagger \Xi_A(\omega, K))\tilde{a} + O(\epsilon^2), \] (7.45)

where \( \Lambda = \eta^\dagger \Xi_H(\omega, K) \) is the corresponding eigenvalue of \( \Xi_H \) and we used (7.16). Let us neglect \( O(\epsilon^2) \), divide (7.45) by \( \tilde{a} \) and consider the real and imaginary parts of the resulting equation separately:

\[ \Lambda(\omega, K) = 0, \quad (\gamma \partial_\omega \Lambda + \eta^\dagger \Xi_A(\omega, K))|_{(\omega, K) = (\gamma, K)} = 0. \] (7.46)

The former is the same dispersion relation for \( \omega \) as for conservative waves, and the latter yields \( \gamma = \gamma(K) \), where

\[ \gamma(K) \doteq -\frac{\eta^\dagger \Xi_A(\omega, K)}{\partial_\omega \Lambda}. \] (7.47)

Because \( |\tilde{a}| \propto e^{\gamma t} \), one can write the amplitude equation as

\[ \partial_t |\tilde{a}|^2 = 2\gamma |\tilde{a}|^2. \] (7.48)

\(^{32}\)Therefore, in a zero-dimensional wave, where \( \int \text{d}x \) can be omitted, conservation of the total action \( I \) implies conservation of \( \tilde{E}_w/\omega \), which is a well-known adiabatic invariant of a discrete harmonic oscillator with a slowly varying frequency (Landau \& Lifshitz 1976, § 49).
One can also define the action density $I$ as in § 7.2.1 and rewrite (7.48) in terms of that. Because $I = |\tilde{a}|^2 \times \text{const}$, one obtains
\[ \partial_t I = 2\gamma I. \] (7.49)

7.3.2. Non-monochromatic waves

When weak inhomogeneity and weak dissipation coexist, their effect on the action density is additive, so (7.30) and (7.49) merge into a general equation
\[ \partial_t I + \partial_x (\bar{v}_g I) = 2\gamma I. \] (7.50)

(A formal derivation of (7.50), which uses the Weyl expansion (2.41) and projection of the field equation on the polarization vector, can be found in Dodin et al. 2019.) Then, (7.40) is modified as follows:
\[ \partial_t (X I) + \partial_x \cdot (X \bar{v}_g I) = I \partial_t X + 2\gamma X I, \] (7.51)

and the equations (7.41) for the wave momentum and energy (7.42) become
\[ \partial_t P_{w,i} + \partial_x (\bar{v}_g P_{w,i}) = 2\gamma P_{w,i} - I \partial_i w, \] (7.52a)
\[ \partial_t E_w + \partial_x (\bar{v}_g E_w) = 2\gamma E_w + I \partial_t w. \] (7.52b)

A comment is due here regarding the relation between (7.50) and the amplitude equation (7.48) that is commonly used in the standard QLT for homogeneous plasma (for example, see (2.21) in Drummond & Pines 1962). In a nutshell, the latter is incorrect, even when $\partial_x = 0$. Because $\bar{f}$ is time-dependent, waves do not grow or decay exponentially. Rather, they can be considered as geometrical-optics (WKB) waves, and unlike in § 7.3.1, the ratio $|\tilde{a}|^2/I$ generally evolves at a rate comparable to $\gamma$. The standard QLT remains conservative only because it also incorrectly replaces (3.19) with its stationary-plasma limit ($\epsilon = 0$) and the two errors cancel each other. These issues are less of a problem for waves in not-too-hot plasmas (e.g. Langmuir waves), because in such plasmas, changing the distribution functions does not significantly affect the dispersion relations and thus $|\tilde{a}|^2/I$ does in fact approximately remain constant. See also the discussion in § 9.1.4.

7.4. General waves

Let us now discuss a more general case that includes broadband waves. The evolution of such waves can be described statistically in terms of their average Wigner matrix $U$. This matrix also determines the function $W$ that is given by (6.37a) and enters the nonlinear potentials (6.77). Below, we derive the general form of $U$ in terms of the phase-space action density $J$ and the governing equation for $J$ (§§ 7.4.1–7.4.3). Then, we also express the function $\tilde{W}$ through $J$ (§ 7.4.4). Related calculations can also be found in McDonald & Kaufman (1985) and Ruiz (2017).

7.4.1. Average Wigner matrix of an eikonal-wave field

Let us start with calculating the average Wigner matrix of an eikonal field $\tilde{\Psi}$ of the form (7.17) (see also Appendix A.2). Using $\tilde{\Psi} = (\tilde{\Psi}_c + \tilde{\Psi}_e)/2$, it can be readily expressed through the average Wigner functions of the complexified field $\tilde{\Psi}_c$ and of its complex

\[ \tilde{\Psi}_c \]
Equation (7.59) can also be expressed as a superposition of eikonal fields
\begin{equation}
\mathbf{U} \equiv \mathbf{W}_{\tilde{\mathbf{w}}} + \mathbf{W}_{\tilde{\mathbf{w}}^*} / 4 \equiv (\mathbf{U}_{c^+} + \mathbf{U}_{c^-}) / 4.
\end{equation}
For \(\tilde{\mathbf{w}}_c = \tilde{a}_\eta(\omega, \mathbf{k})e^{i\theta}\), where the arguments \((t, x)\) are omitted for brevity, one has
\begin{equation}
\mathbf{U}_c \equiv \mathbf{U}_{c^+} \approx (\eta^\dagger(\omega, \mathbf{k})|\tilde{a}|^2 \int \frac{d\tau}{(2\pi)^n} \mathbf{e}^{i(\omega - \omega')\tau - i(k - \mathbf{k})s} \mathbf{e}^{-i\omega(t - \tau/2, x + s/2)} \mathbf{e}^{-i\theta(t - \tau/2, x - s/2)} \mathbf{e}^{i\omega\tau - ik\cdot s},
\end{equation}
where we neglected the dependence of \(\tilde{a}\) and \(\eta\) on \((t, x)\) because it is weak compared with that of \(e^{\pm i\theta}\). By Taylor-expanding \(\theta\), one obtains
\begin{equation}
\mathbf{U}_c \approx (\eta^\dagger(\omega, \mathbf{k})|\tilde{a}|^2 \int \frac{d\tau}{(2\pi)^n} \mathbf{e}^{i(\omega - \omega')\tau - i(k - \mathbf{k})s} = (\eta^\dagger(\omega, \mathbf{k})|\tilde{a}|^2 \mathbf{e}^{\omega_4} + \mathbf{e}^{-\omega_4} \mathbf{e}^{-i\omega}\delta(k - \mathbf{k}).
\end{equation}
For \(\tilde{\mathbf{w}}_c^* = \tilde{a}_\eta^*(\omega, \mathbf{k})e^{-i\theta}\), which can also be written as \(\tilde{\mathbf{w}}_c^* = \tilde{a}_\eta^\dagger(-\omega, -\mathbf{k})e^{-i\theta}\) due to (7.7), the result is the same up to replacing \(\omega \rightarrow -\omega\) and \(\mathbf{k} \rightarrow -\mathbf{k}\). Also notice that
\begin{equation}
\delta(\omega + \omega')\delta(k + \mathbf{k}) = \delta(\omega - w(k))\delta(k + \mathbf{k})
\end{equation}
so one can rewrite \(\mathbf{U}_{c^\pm}\) as follows:
\begin{equation}
\mathbf{U}_{c^\pm} = \eta(k)\eta^\dagger(k)|\tilde{a}|^2 \delta(\omega - w(k))\delta(k \mp \mathbf{k}),
\end{equation}
where \((k) \equiv (w(k), k)\). Thus finally,
\begin{equation}
\mathbf{U}(\omega, k) \approx \eta(k)\eta^\dagger(k)|\tilde{a}|^2 \delta(\omega - w(k))\delta(k \mp \mathbf{k}) / 4.
\end{equation}

### 7.4.2 Average Wigner Matrix of a General Wave

Assuming the background medium is sufficiently smooth, a general wave field can be represented as a superposition of eikonal fields
\begin{equation}
\tilde{\Psi} = \text{re} \tilde{\mathbf{w}}_c, \quad \tilde{\mathbf{w}}_c = \sum_\sigma \tilde{\Psi}_{\sigma, c}, \quad \tilde{\Psi}_{\sigma, c} = \tilde{a}_\sigma e^{i\theta_\sigma}.
\end{equation}
As a quadratic functional, its average Wigner matrix \(\mathbf{U}\) equals the sum of the average Wigner matrices \(\mathbf{U}_{\sigma}\) of the individual eikonal waves:
\begin{equation}
\mathbf{U} = \sum_\sigma \mathbf{U}_{\sigma} = \sum_\sigma (\mathbf{U}_{\sigma, c^+} + \mathbf{U}_{\sigma, c^-}) / 4,
\end{equation}
where \(\mathbf{U}_{\sigma, c^+} \equiv \mathbf{U}_{\sigma, c}\) and \(\mathbf{U}_{\sigma, c^-}\) are the average Wigner matrices of \(\tilde{\Psi}_{\sigma, c}\) and \(\tilde{\Psi}_{\sigma, c}^*\), respectively,
\begin{equation}
\mathbf{U}_{\sigma, c^\pm} = \eta(k)\eta^\dagger(k)|\tilde{a}_\sigma|^2 \delta(k \mp \mathbf{k})\delta(\omega - w(k)).
\end{equation}
Equation (7.59) can also be expressed as
\begin{equation}
\mathbf{U} = (\mathbf{U}_{c^+} + \mathbf{U}_{c^-}) / 4, \quad \mathbf{U}_{c^\pm} = \sum_\sigma \mathbf{U}_{\sigma, c^\pm},
\end{equation}
where \(\mathbf{U}_{c^\pm}\) are the average Wigner matrices of \(\tilde{\Psi}_c\) and \(\tilde{\Psi}_c^*\), respectively,
\begin{equation}
\mathbf{U}_{c^\pm} = \eta(k)\eta^\dagger(k)h_{c^\pm}(k)\delta(\omega - w(k)), \quad h_{c^\pm}(k) \equiv \sum_\sigma |\tilde{a}_\sigma|^2 \delta(k \mp \mathbf{k}).
\end{equation}
Because \( h_c(-k) = h_c(k) = h_c(-k) \), the matrix \( U \) can also be written as follows:

\[
U(\omega, k) = (\eta \eta^\dagger)(h(k) + h(-k))\delta(\omega - \omega(k)),
\]

(7.63)

where \( h(k) = h_c(k)/4 \) is given by

\[
h(k) = \frac{1}{4} \sum_\sigma |\tilde{a}_\sigma|^2 \delta(k - \bar{k}_\sigma) \geq 0.
\]

(7.64)

This shows that, for broadband waves comprised of eikonal waves, \( U \) has the same form as for an eikonal wave except \( h(k) \) is not necessarily delta-shaped.

### 7.4.3. Phase-space action density and the WKE

The wave equation for the complexified field \( \tilde{\Psi}_c \) can be written in the invariant form as \( \tilde{\mathcal{L}}_c \tilde{\Psi}_c = 0 \). Multiplying it by \( \langle \tilde{\Psi}_c \rangle \) from the right leads to

\[
\tilde{\mathcal{L}}_c \tilde{\Psi}_c = 0, \quad \tilde{\Psi}_c = (2\pi)^{-n} |\tilde{\Psi}_c \rangle \langle \tilde{\Psi}_c |.
\]

(7.65)

This readily yields an equation for the Wigner matrix: \( \mathcal{E} \star U_c = 0 \). Let us integrate this equation over \( \omega \) to make the left-hand side a smooth function of \((t, x, k)\). Let us also take the trace of the resulting equation to put it in a scalar form

\[
\text{tr} \int d\omega \, \mathcal{E} \star U_c = 0.
\]

(7.66)

As usual, we assume \( \mathcal{E} = \mathcal{E}_H + i\mathcal{E}_A \) with \( \mathcal{E}_A = \mathcal{O}(\epsilon) \ll \mathcal{E}_H = \mathcal{O}(1) \) for generic \((x, k)\). The integrand in (7.66) can be written as \( \mathcal{E} \star U_c = \mathcal{E} e^{i\mathcal{L}_c/2} U_c \), and its expansion in the differential operator \( \mathcal{L}_c \) (2.32) contains derivatives of all orders. High-order derivatives on \( \mathcal{E} \) are not negligible per se, because for on-shell waves this function is delta-shaped. However, using integration by parts, one can reapply all derivatives with respect to \( \omega \) to \( \mathcal{E} \) and take the remaining derivatives (with respect to \( t, x \) and \( k \)) outside the integral. Then it is seen that each power of \( \mathcal{L}_c \) in the expansion of \( \mathcal{E} e^{i\mathcal{L}_c/2} U_c \) contributes \( \mathcal{O}(\epsilon^m) \) to the integral. Let us neglect terms with \( m \geq 2 \) and use (7.62). Hence, one obtains

\[
0 \approx \text{tr} \int d\omega \left( \mathcal{E}_H U_c + i\mathcal{E}_A U_c + \frac{i}{2} \{\mathcal{E}_H, U_c\} \right)
\]

\[
\approx (\eta^\dagger \mathcal{E}_H \eta + i\eta^\dagger \mathcal{E}_A \eta) h_c + \frac{i}{2} \text{tr} \int d\omega \left( \frac{\partial \mathcal{E}_H}{\partial x^i} \frac{\partial U_c}{\partial k_i} - \frac{\partial \mathcal{E}_H}{\partial k_i} \frac{\partial U_c}{\partial x^i} \right).
\]

(7.67)

Let us also re-express this as follows, using (7.4) and (7.47):

\[
0 \approx \left( \Lambda - i\gamma \frac{\partial \Lambda}{\partial \omega} \right) h_c - \frac{i}{2} \text{tr} \int d\omega \left( \frac{\partial \mathcal{E}_H}{\partial t} U_c \right) + \frac{i}{2} \frac{\partial}{\partial t} \text{tr} \int d\omega \frac{\partial \mathcal{E}_H}{\partial \omega} U_c
\]

\[
+ \frac{i}{2} \frac{\partial}{\partial k_i} \text{tr} \int d\omega \frac{\partial \mathcal{E}_H}{\partial x^i} U_c - \frac{i}{2} \frac{\partial}{\partial x^i} \text{tr} \int d\omega \frac{\partial \mathcal{E}_H}{\partial k_i} U_c.
\]

(7.68)

Clearly,

\[
\int d\omega \frac{\partial}{\partial \omega} \left( \frac{\partial \mathcal{E}_H}{\partial t} U_c \right) = 0.
\]

(7.69)

To simplify the remaining terms, we proceed as follows. As a Hermitian matrix, \( \mathcal{E}_H \) can be represented in terms of its eigenvalues \( \Lambda_b \) and eigenvectors \( \eta_b \) as \( \mathcal{E}_H = \Lambda_b \eta_b \eta_b^\dagger \). For

\[34\text{McDonald} \& \text{Kaufman (1985) first Taylor-expand } \mathcal{E} \star U_c \text{ and then integrate over } \omega. \text{Strictly speaking, that is incorrect (because } \mathcal{E} \star U_c \text{ is not smooth), but the final result is the same.} \]
Quasilinear theory

Let us use (7.62) again, where \( \eta \) is one of the vectors \( \eta_b \), say, \( \eta \equiv \eta_0 \). (Accordingly, \( \Lambda \equiv \Lambda_0 \).) Then, for any \( \bullet \in \{ \omega, x^i, k_i \} \), one has

\[
\text{tr} \int d\omega \left( \frac{\partial \Xi}{\partial \bullet} \mathbf{U}_c \right) = \frac{\partial \Lambda_b}{\partial \bullet} |\eta_b^\dagger \eta|^2 h_c + A_b \left( \eta^\dagger \frac{\partial \eta_b}{\partial \bullet} \right) (\eta_b^\dagger \eta) h_c + A_b (\eta^\dagger \eta_b) \left( \frac{\partial \eta_b^\dagger}{\partial \bullet} \eta \right) h_c
\]

\[
= \frac{\partial A_b}{\partial \bullet} (\delta_{b,0})^2 h_c + A_b \left( \eta^\dagger \frac{\partial \eta_b}{\partial \bullet} \right) \delta_{b,0} h_c + A_b \delta_{b,0} \left( \frac{\partial \eta_b^\dagger}{\partial \bullet} \eta \right) h_c
\]

\[
= \frac{\partial \Lambda}{\partial \bullet} h_c + \left( \eta^\dagger \frac{\partial \eta}{\partial \bullet} + \frac{\partial \eta}{\partial \bullet} \eta \right) \Lambda h_c
\]

\[
= \frac{\partial \Lambda}{\partial \bullet} h_c + \frac{\partial (\eta^\dagger \eta)}{\partial \bullet} \Lambda h_c
\]

\[
= \frac{\partial \Lambda}{\partial \bullet} h_c,
\]

(7.70)

where we used \( \eta_b^\dagger \eta = \delta_{b,0} \) and, in particular, \( \eta^\dagger \eta = 1 \). Then, (7.68) can be written as

\[
\Lambda h_c - 2i\mathcal{E} = 0,
\]

(7.71)

where

\[
\mathcal{E} = 2\gamma \left( \frac{\partial \Lambda}{\partial \omega} h \right) - \frac{\partial}{\partial t} \left( \frac{\partial \Lambda}{\partial \omega} h \right) - \frac{\partial}{\partial k_i} \left( \frac{\partial \Lambda}{\partial x^i} h \right) + \frac{\partial}{\partial x^i} \left( \frac{\partial \Lambda}{\partial k_i} h \right).
\]

(7.72)

The real part of (7.71) gives \( \Lambda = 0 \), which is the dispersion relation. The imaginary part of (7.71) gives \( \mathcal{E} = 0 \). To understand this equation, let us rewrite \( \mathcal{E} \) as

\[
\mathcal{E} = 2\gamma J - \frac{\partial J}{\partial t} + \frac{\partial}{\partial k_i} \left( \frac{\partial w}{\partial x^i} J \right) - \frac{\partial}{\partial x^i} \left( \frac{\partial w}{\partial k_i} J \right).
\]

(7.73)

Here, we introduced

\[
J(k) \equiv h(k) \partial_\omega \Lambda(k), \quad \Lambda(k) \equiv \Lambda(w(k), k),
\]

(7.74)

which, according to (7.24), satisfy

\[
J \partial_t w(k) = -h(\partial_t \Lambda)(k),
\]

(7.75a)

\[
J \partial_x w(k) = -h(\partial_x \Lambda)(k),
\]

(7.75b)

\[
J \partial_k w(k) = -h(\partial_k \Lambda)(k).
\]

(7.75c)

Note that using (7.64), one can also express \( J \) as

\[
J = \sum_\sigma \left( \frac{1}{2} |\tilde{a}_\sigma|^2 \partial_\omega \Lambda(k_\sigma) \right) \delta(k - \bar{k}_\sigma) = \sum_\sigma \mathcal{I}_\sigma \delta(k - \bar{k}_\sigma),
\]

(7.76)

where \( \mathcal{I}_\sigma \) are the action densities (7.27) of the individual eikonal waves that comprise the total wave field (§ 7.4.2). In particular, \( \int d^4 k J = \sum_\sigma \mathcal{I}_\sigma \), which is the total action density. Therefore, the function \( J \) can be interpreted as the phase-space action density. In terms of
and similarly, express this function through the phase-space action density:

This also means that the Weyl symbol calculus that we use can facilitate derivations of wave–wave collision operators as well (Ruiz, Glinsky & Dodin 2019).

(6.77) exactly conserves the action of non-resonant waves, i.e. those with $\gamma = 0$. Also note that (7.50) for eikonal waves can be deduced from (7.77) as a particular case by assuming the ansatz

and integrating over $k$. In other words, eikonal-wave theory can be understood as the ‘cold-fluid’ limit of wave-kinetic theory.

7.4.4. Function $\overline{W}_s$ in terms of $J$

Here we explicitly calculate the function (6.37a) that determines the nonlinear potentials (6.77). Using (6.73), one obtains

where $(k; p) \equiv (w(k), k; p)$. By definition of $\hat{\alpha}$, the function $\langle t_1, x_1 | \hat{\alpha}_s | t_2, x_2 \rangle$ is real for all $(t_1, x_1)$ and $(t_2, x_2)$, so $\alpha_s(-\omega, -k) = \alpha_s(\omega, k)$ by definition of the Weyl symbol (2.26). Together with (7.7), this gives $|\alpha_s \eta|^2(\omega, k; p) = |\alpha_s \eta|^2(-\omega, -k; p)$, so

and similarly,

This also means that $\overline{W}_s(\omega, k; p) = \overline{W}_s(-\omega, -k; p)$. Then finally, using (7.74), one can express this function through the phase-space action density:

Let us rewrite (7.77) together with (6.76) in the ‘divergence’ form

and

This equation, called the WKE, serves the same role in QL wave-kinetic theory as the Vlasov equation serves in plasma kinetic theory. Unlike the field equation used in the standard QLT (Drummond & Pines 1962), (7.77) exactly conserves the action of non-resonant waves, i.e. those with $\gamma = 0$. Also note that (7.50) for eikonal waves can be deduced from (7.77) as a particular case by assuming the ansatz

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and integrating over $k$. In other words, eikonal-wave theory can be understood as the ‘cold-fluid’ limit of wave-kinetic theory.

7.5. Conservation laws

Let us rewrite (7.77) together with (6.76) in the ‘divergence’ form

and

35The term ‘WKE’ is also used for the equation that describes nonlinear interactions of waves in statistically homogeneous media, or ‘wave–wave collisions’ (Zakharov, L’vov & Falkovich 1992). That is not what we consider here. Inhomogeneities are essential in our formulation, and the QL WKE is linear (in $J$) by definition of the QL approximation. That said, the Weyl symbol calculus that we use can facilitate derivations of wave–wave collision operators as well (Ruiz, Glinsky & Dodin 2019).
Using (7.62), the diffusion matrix $D_{s,ij}$ can be represented as follows:

$$D_{s,ij} = 2\pi \int dk k_i k_j |\alpha_s^\dagger \eta|^2 \frac{J(k)}{\delta_w \Lambda(k)} \delta(k \cdot v_s - w(k)).$$

(7.85)

Also, by substituting (6.24) into (7.47), one finds

$$\gamma = \pi \sum_s \int dp |\alpha_s^\dagger \eta|^2 \delta(w(k) - k \cdot v_s(p)) k \cdot \frac{\partial F_s(p)}{\partial p}.$$ 

(7.86)

Together with (7.75), these yield the following notable corollaries. First of all, if $\Xi_0, |\alpha_s^\dagger \eta|^2$ and $\eta^\dagger \varphi_s \eta$ are independent of $x$, one has for each $l$ that (Appendix D.1)

$$\frac{\partial}{\partial t} \left( \sum_s \int dp p_l F_s + \int dk k_l J \right) + \frac{\partial}{\partial x^l} \left( \sum_s \int dp p_l v_s^l F_s + \int dk k_l v_s^l J \right)$$

$$+ \frac{\partial}{\partial x^l} \sum_s \int dp \Delta_s F_s = - \sum_s \int dp \frac{\partial H_{0s}}{\partial x^l} F_s.$$ 

(7.87)

This can be viewed as a momentum-conservation theorem, because at $\partial_t H_{0s} = 0$, one has

$$\sum_s \int dx dp p_l F_s + \int dx dk k_l J = \text{const.}$$

(7.88)

Also, the ponderomotive force on a plasma is readily found from (7.87) as the sum of the terms quadratic in the wave amplitude (after $F_s$ has been expressed through $\tilde{f}_s$). Similarly, if $\Xi_0, |\alpha_s^\dagger \eta|^2$ and $\eta^\dagger \varphi_s \eta$ are independent of $t$, one has (Appendix D.2)

$$\frac{\partial}{\partial t} \left( \sum_s \int dp H_{0s} F_s + \int dk w J \right) + \frac{\partial}{\partial x^l} \left( \sum_s \int dp H_{0s} v_s^l F_s + \int dk w v_s^l J \right)$$

$$+ \frac{\partial}{\partial x^l} \sum_s \int dp \Delta_s v_s^l F_s = \sum_s \int dp \frac{\partial H_{0s}}{\partial t} F_s.$$ 

(7.89)

This can be viewed as an energy-conservation theorem, because at $\partial_t H_{0s} = 0$, one has

$$\sum_s \int dx dp H_{0s} F_s + \int dx dk w J = \text{const.}$$

(7.90)

Related equations are also discussed in Dodin & Fisch (2012) and Dewar (1977).

The individual terms in (7.87) and (7.89) can be interpreted as described in table 1. The results of § 7.2.3 are reproduced as a particular case for the eikonal-wave ansatz (7.78). In particular, note that electrostatic waves carry non-zero momentum density $\int dk k J$ just like any other waves, even though an electrostatic field of these waves carries no momentum.

36Having $x$-dependence in $\Xi_0, |\alpha_s^\dagger \eta|^2$ or $\eta^\dagger \varphi_s \eta$ would signify interaction with external fields not treated self-consistently. Such fields could exchange momentum with the wave–plasma system, so the momentum of the latter would not be conserved. A similar argument applies to the temporal dependence of these coefficients vs energy conservation considered below.

37There is no ambiguity in the definition of the wave momentum and energy in this case (i.e. $\kappa = 1$), because (7.88) and (7.90) connect those with the momentum and energy of particles (OCs), which are defined unambiguously.
The momentum is stored in the particle motion in this case (§ 9.1.3), and it is pumped there via either temporal dependence (Liu & Dodin 2015, § II.2) or spatial dependence (Ochs & Fisch 2021b, 2022) of the wave amplitude. This shows that homogeneous-plasma models that ignore ponderomotive effects cannot adequately describe the energy–momentum transfer between waves and plasma even when resonant absorption per se occurs in a homogeneous-plasma region. The OC formalism presented here provides means to describe such processes rigorously, generally, and without cumbersome calculations.

7.6. Summary of § 7

In summary, we have considered plasma interaction with general broadband single-mode on-shell waves (for examples, see § 9). Assuming a general response matrix \( \Xi \), these waves have a dispersion function \( \Lambda(t, x, \omega, k) \) and polarization \( \eta(t, x, \omega, k) \) determined by

\[
\Xi H \eta = \Lambda \eta, \quad \Lambda = \eta^\dagger \Xi H \eta, \tag{7.91}
\]

where the normalization \( \eta^\dagger \eta = 1 \) is assumed. Specifically for \( \Xi \) given by (6.80), one has

\[
\Lambda(t, x, \omega, k) = \eta^\dagger \Xi \partial \eta - \sum_s \int dp \eta^\dagger \varphi_s(p) \eta F_s(p) \\
+ \sum_s \int dp \frac{|\alpha_s^\dagger \eta|^2(p)}{\omega - k \cdot v_s(p)} k \cdot \frac{\partial F_s(p)}{\partial p}, \tag{7.92}
\]

where the arguments \( (t, x, \omega, k) \) are omitted for brevity. (Some notation is summarized in § 6.9.) The wave frequency \( \omega = w(t, x, k) \) satisfies

\[
\Lambda(t, x, w(t, x, k), k) = 0 \tag{7.93}
\]
and \( w(t, x, -k) = -w(t, x, k) \), where \( w \) is a real function at real arguments. The wave local linear growth rate \( \gamma \), which is assumed to be small in this section, is

\[
\gamma(t, x, k) = -\left( \frac{\eta^\dagger \mathcal{E} \Lambda \eta}{\partial_\omega \Lambda(t, x, w, k)} \right)_{(t, x, w(t, x, k), k)},
\]

or explicitly,

\[
\gamma(t, x, k) = \pi \sum_s \int dp \frac{|\alpha_s^\dagger \eta|^2}{\partial_\omega \Lambda(t, x, w, k)} \delta(w - k \cdot v_s(t, x, p)) k \cdot \frac{\partial F_s(t, x, p)}{\partial p},
\]

where \( w \equiv w(t, x, k) \) and \( |\alpha_s^\dagger \eta|^2 \equiv |\alpha_s^\dagger \eta|^2(t, x, w, k; p) \). The nonlinear potentials (6.77) are expressed through the scalar function

\[
\overline{W}_s(t, x, \omega, k; p) = |\alpha_s^\dagger \eta|^2(\zeta_k J(t, x, k) + \zeta_{-k} J(t, x, -k)) \delta(\Lambda(t, x, \omega, k)),
\]

where \( \zeta_k \equiv \text{sgn}(\partial_\omega \Lambda(t, x, \omega, k)) \) is evaluated at \( \omega = w(t, x, k) \); see also (7.82). The function \( J \) is the phase-space action density governed by the WKE

\[
\frac{\partial J}{\partial t} - \frac{\partial w}{\partial x} \frac{\partial J}{\partial k} + \frac{\partial w}{\partial k} \frac{\partial J}{\partial x} = 2\gamma J,
\]

where \( \partial_k w = v_g \) is the group velocity. Collisional dissipation is assumed small compared with collisionless dissipation, so it is neglected in (7.96) but can be reintroduced by an ad hoc modification of \( \gamma \) (§ 6.2). Unlike the field equation used in the standard QLT, (7.96) exactly conserves the action of non-resonant waves, i.e. those with \( \gamma = 0 \). The WKE must be solved together with the QL equation for the OC distribution \( F_s \),

\[
\frac{\partial F_s}{\partial t} - \frac{\partial \mathcal{H}_s}{\partial x} \frac{\partial F_s}{\partial p} + \frac{\partial \mathcal{H}_s}{\partial p} \frac{\partial F_s}{\partial x} = \frac{\partial}{\partial p} \left( D_s \frac{\partial F_s}{\partial p} \right) + C_s,
\]

because \( F_s \) determines the coefficients in (7.96) and \( J \) determines the coefficients in (7.97). When \( \mathcal{E}_0 \) and \( |\alpha^\dagger \eta|^2 \) are independent of \( t \) and \( x \), (7.96) and (7.97) conserve the total momentum and energy of the system; specifically,

\[
\partial_i(\sum_s P_{s,i} + P_{w,i}) + \partial_j(\sum_s \Pi_{j,i} + \Pi_{w,i}) = -\sum_s \int dp F_s \partial_i \mathcal{H}_{0s}, \quad (7.98)
\]

\[
\partial_i(\sum_s \mathcal{E}_s + \mathcal{E}_w) + \partial_j(\sum_s Q^i_j + Q^j_w) = \sum_s \int dp F_s \partial_i \mathcal{H}_{0s}. \quad (7.99)
\]

Here, the notation is as in table 1, or see (7.87) and (7.89) instead.

8. Thermal equilibrium

In this section, we discuss, for completeness, the properties of plasmas in thermal equilibrium.

8.1. Boltzmann–Gibbs distribution

As discussed in § 6.8, collisions conserve the density of each species, the total momentum density and the total energy density, while the plasma total entropy density \( \sigma \) either grows or remains constant. Let us search for equilibrium states in particular. At least one of the
states in which $\sigma$ remains constant is the one that maximizes the entropy density at fixed $\int dp\, F_s, \sum_s \int dp\, p F_s$ and $\sum_s \int dp\, H_s F_s$. This ‘state of thermal equilibrium’ can be found as an extremizer of

$$\sigma' = \sigma - \sum_s \lambda_s^{(N)} \int dp\, F_s - \lambda^{(P)} \cdot \sum_s \int dp\, p F_s - \lambda^{(H)} \sum_s \int dp\, H_s F_s \quad (8.1)$$

considered as a functional of all $F_s$, where $\lambda_s^{(N)}, \lambda^{(P)}$ and $\lambda^{(H)}$ are Lagrange multipliers. Using (6.73), one finds that extremizers of $\sigma'$ satisfy

$$0 = \frac{\delta \sigma'}{\delta F_s} = -\ln F_s - 1 - \lambda_s^{(N)} - \lambda^{(P)} \cdot p - \lambda^{(H)} H_s, \quad (8.2)$$

whence

$$F_s = \text{const}_s \times \exp(-\lambda^{(P)} \cdot p - \lambda^{(H)} H_s). \quad (8.3)$$

The pre-exponential constant is determined by the given density of species $s$, while $\lambda^{(P)}$ and $\lambda^{(H)}$ can be expressed through the densities of the plasma momentum and energy stored in the whole distribution. Because

$$\frac{\delta^2 \sigma'}{\delta F_s \delta F_s'} = -\frac{1}{F_s} < 0, \quad \frac{\delta^2 \sigma'}{\delta F_s \delta F_s' \neq s} = 0, \quad (8.4)$$

the matrix $\delta^2 \sigma'/\delta F_s \delta F_s'$ is positive-definite, so the entropy is maximal (as opposed to minimal) at the extremizer (8.3).

The distribution (8.3) is known as the Boltzmann–Gibbs distribution, with $T \equiv 1/\lambda^{(H)}$ being the temperature (common for all species). Also, let us introduce a new, rescaled Lagrange multiplier $u$ via $\lambda^{(P)} = -u/T$. Then,

$$F_s(p) = F_s^{(0)} \exp \left( -\frac{H_s(p) - u \cdot p}{T} \right), \quad (8.5)$$

where $F_s^{(0)}$ is independent of $p$. Correspondingly,

$$\frac{\partial F_s(p)}{\partial p} = -F_s(p) \frac{v_s - u}{T}, \quad (8.6)$$

where we used (5.44). From (8.6), one obtains

$$\delta(k \cdot v_s - k \cdot v_s') k \cdot \left( \frac{\partial F_s(p)}{\partial p} F_s(p) - F_s(p) \frac{\partial F_s(p')}{\partial p'} \right)$$

$$= -\frac{1}{T} \delta(k \cdot v_s - k \cdot v_s') (k \cdot v_s - k \cdot v_s') F_s(p) F_s(p')$$

$$= 0, \quad (8.7)$$

where $H_s' = H_s(p)$. Then, (6.61) yields that the collision operator vanishes on the Boltzmann–Gibbs distribution, and thus, expectedly, $(d\sigma/dt)_{\text{coll}} = 0$. One can also show that the Boltzmann–Gibbs distribution is the only distribution (strictly speaking, a class of distributions parameterized by $T$ and $u$) for which the entropy density is conserved (Appendix E).
The property (8.6) of the thermal-equilibrium state also leads to other notable results that we derive below. In doing so, we will assume the reference frame where $u = 0$, so the Boltzmann–Gibbs distribution has a better known form

$$F_s(p) = F_s^{(0)} \exp \left(-\frac{\mathcal{H}_s(p)}{T}\right), \quad \frac{\partial F_s(p)}{\partial p} = -F_s(p) \frac{v_s}{T}. \quad (8.8)$$

(For $\mathcal{H}_s$ isotropic in $p$, this is the frame where the plasma total momentum density $\sum_s \int dp F_s$ is zero.) The generalizations to arbitrary $u$ are straightforward.

8.2. Fluctuation–dissipation theorem

Let us describe microscopic fluctuations in equilibrium plasmas in terms of $S(\omega, k) \doteq (2\pi)^n W(\omega, k)$, i.e.

$$S(\omega, k) \doteq \int d\tau ds \tilde{\Psi}(t + \tau/2, x + s/2) \tilde{\Psi}^\dagger(t - \tau/2, x - s/2) e^{i\omega \tau - i k \cdot s}, \quad (8.9)$$

which can also be represented in terms of the Fourier image $\hat{\Psi}(\omega, k)$ of the microscopic field $\tilde{\Psi}(t, x)$:

$$S(\omega, k) = \frac{\hat{\Psi}(\omega, k)\hat{\Psi}^\dagger(\omega, k)}{\mathcal{F} \mathcal{V}_n}. \quad (8.10)$$

For statistically homogeneous fields that persist on time $\mathcal{T} \to \infty$ within volume $\mathcal{V}_n \to \infty$, the Fourier transform is formally divergent; hence the appearance of the factors $\mathcal{T}$ and $\mathcal{V}_n$ in (8.10). Also, as seen from (6.32), any quadratic function of the microscopic field can be expressed through $S$ via

$$\hat{L}\hat{\psi}(\omega, k)\hat{R}\psi(\omega, k) \approx \int \frac{d\omega}{2\pi} \frac{dk}{(2\pi)^n} (LSR^\dagger)(\omega, k), \quad (8.11)$$

where $\hat{L}$ and $\hat{R}$ are any linear operators and $L$ and $R$ are their symbols.

From (6.31), one finds that, in general,

$$S(\omega, k) = 2\pi \sum_s \int dp' \delta(\omega - k \cdot v_s') F_s(p') \mathcal{E}^{-1}(\omega, k)(\alpha_s'\alpha_s^\dagger)(\omega, k; p') \mathcal{E}^{-\dagger}(\omega, k). \quad (8.12)$$

For a thermal distribution in particular, which satisfies (8.8), one can rewrite (6.24) as follows:

$$\mathcal{E}_A(\omega, k) \approx \frac{\pi}{T} \sum_s \int dp \alpha_s(\omega, k; p)\alpha_s^\dagger(\omega, k; p)\delta(\omega - k \cdot v_s) (k \cdot v_s) F_s(p)$$

$$= \frac{\pi \omega}{T} \sum_s \int dp \alpha_s(\omega, k; p)\alpha_s^\dagger(\omega, k; p)\delta(\omega - k \cdot v_s) F_s(p). \quad (8.13)$$

By comparing this with (8.12), one also finds that

$$S(\omega, k) = \frac{2T}{\omega} (\mathcal{E}^{-1} \mathcal{E}_A \mathcal{E}^{-\dagger})(\omega, k). \quad (8.14)$$

38To make (8.10) look more physical (local), one can absorb the global factors $\mathcal{T}$ and $\mathcal{V}_n$ in the definition of the Fourier transform; cf. § 9.1.5.
Due to (6.26), this leads to the fluctuation–dissipation theorem in the following form:

\[ S(\omega, k) = -\frac{2T}{\omega} (\Xi^{-1})_A(\omega, k). \]  

(8.15)

For examples of \( \Xi \) for specific systems, see § 9.

### 8.3. Kirchhoff’s law

Consider the power deposition via polarization drag

\[ \mathcal{P} = \sum_s \int dp \left( v_s \cdot \mathbf{F}_s \right) F_s(p). \]  

(8.16)

Using (6.58a) for \( \mathbf{F}_s \), (8.13) for \( \Xi \) and (8.15) for \( S \), this can also be expressed as follows:

\[
\mathcal{P} \approx \sum_s \int \frac{dk}{2\pi} \frac{d\omega}{(2\pi)^n} \omega \omega \partial \omega \Lambda(\omega, k) \delta(\omega - w(k)) \left( \mathbf{U} \mathbf{U}^\dagger \right)(\omega, k).
\]  

(8.17)

Thus, the spectral density of the power deposition via polarization drag is given by

\[ \mathcal{P}_{\omega, k} = -\omega \text{ tr}(S \Xi_A), \]  

(8.18)

which is a restatement of Kirchhoff’s law (Krall & Trivelpiece 1973, § 11.4). For examples of \( \Xi \) for specific systems, see § 9.

### 8.4. Equipartition theorem

As flows from § 7.5, the energy of on-shell waves of a field \( \tilde{\Psi} \) in a homogeneous \( n \)-dimensional plasma of a given volume \( \mathcal{V}_n \) can be written as

\[
\mathcal{V}_n E_w = \frac{1}{2} \mathcal{V}_n \int dk \mathcal{V}_n w(k) J(k)
\]

\[
= \frac{1}{2} \int \frac{d\omega}{(2\pi)^n} \omega \omega \partial \omega \Lambda(\omega, k) h(k) \delta(\omega - w(k))
\]

\[
= (2\pi)^n \sum_k \int \frac{d\omega}{(2\pi)^n} \omega \omega \partial \omega \Lambda(\omega, k) \left( \eta^\dagger \mathbf{U} \eta \right)(\omega, k).
\]  

(8.19)

To apply this to microscopic fluctuations, one can replace \( \mathbf{U} \) with \( \mathbf{W} \) and substitute \( \mathbf{W} = (2\pi)^{-(n+1)} \mathbf{S} \). Then, the total energy of a mode with given wavevector \( \mathbf{k} \) and
polarization $\eta$ can be expressed as

$$\mathcal{E}_{k,\eta} = \frac{1}{2\pi} \int_0^\infty d\omega \omega (\partial_\omega \Lambda) \eta^\dagger \mathbf{S} \eta,$$

(8.20)

where the arguments $(\omega, k)$ are omitted for brevity. For thermal equilibrium, one can substitute (8.15) for $\mathbf{S}$; then,

$$\mathcal{E}_{k,\eta} = -\frac{T}{\pi} \text{im} \int_0^\infty d\omega (\partial_\omega \Lambda) \eta^\dagger \mathbf{\Xi}^{-1} \eta.$$

(8.21)

The integrand peaks at $\omega = w(k)$, where the mode eigenvalue $\Lambda$ is small. Due to damping, the actual zero of $\Lambda$ is slightly below the real axis in the complex-frequency space. Then, at infinitesimally small damping, $\eta^\dagger \mathbf{\Xi}^{-1} \eta$ can be approximated near $\omega = w(k)$ as

$$\eta^\dagger \mathbf{\Xi}^{-1} \eta \approx \frac{1}{\Lambda} \approx \frac{1}{\partial_\omega \Lambda(\omega, k)} \left( \text{pv} \frac{1}{\omega - w(k)} - i\pi \delta(\omega - w(k)) \right).$$

(8.22)

This leads to the well-known equipartition theorem

$$\mathcal{E}_k = T.$$

(8.23)

Note that, according to (8.23), the sum $\gamma_\alpha \mathcal{E}_\alpha = \sum_{k,\eta} \mathcal{E}_{k,\eta}$ is divergent. This indicates that not all modes can be classical and on-shell (weakly damped) simultaneously.

8.5. Summary of § 8

In thermal equilibrium, when all species have Boltzmann–Gibbs distributions with common temperature $T$, the collision operator vanishes, the entropy is conserved and the spectrum of microscopic fluctuations (8.9) satisfies the fluctuation–dissipation theorem

$$\mathbf{S}(\omega, k) = -\frac{2T}{\omega} (\mathbf{\Xi}^{-1})_\Lambda(\omega, k),$$

(8.24)

where $\mathbf{\Xi}$ is the dispersion matrix (6.80) and $\Lambda$ denotes the anti-Hermitian part (or the imaginary part in case of scalar fields). From here, it is shown that the spectral density of the power deposition via polarization drag is given by $\mathcal{P}_{\omega, k} = -\omega \text{tr}(\mathbf{S} \mathbf{\Xi}_\Lambda)$, which is a restatement of Kirchhoff’s law. For on-shell waves, (8.24) reduces to the equipartition theorem, which says that the energy per mode equals $T$. Applications to specific systems are discussed in § 9.

9. Examples

In this section, we show how to apply our general formulation to non-relativistic electrostatic interactions (§ 9.1), relativistic electromagnetic interactions (§ 9.2), Newtonian gravity (§ 9.3) and relativistic gravity, including gravitational waves (§ 9.4).

9.1. Non-relativistic electrostatic interactions

9.1.1. Main equations

Let us show how our general formulation reproduces (and generalizes) the well-known results for electrostatic turbulence in non-magnetized non-relativistic plasma. In this case,

$$H_s = \frac{p^2}{2m_s} + e_s \varphi + e_s \tilde{\varphi},$$

(9.1)

where $e_s$ is the electric charge, $\varphi$ is the electrostatic potential and $\varphi$ and $\tilde{\varphi}$ are its average and oscillating parts, respectively. Then, $H_{0s} = \tilde{H}_s = p^2/(2m_s) + e_s \varphi$, $\tilde{H}_s = e_s \tilde{\varphi}$, $\tilde{\alpha}_s = e_s$
The first term on the right-hand side does not contribute to the field action \( S \). Collisions and fluctuations

9.1.2. Correspondingly,

\[
\mathcal{D}_s = \varepsilon_s^2 \int dk \pi kk\mathcal{U}(k \cdot \mathbf{v}_s, k),
\]

\[
\Theta_s = \varepsilon_s^2 \frac{\partial}{\partial \vartheta} \int d\omega dk \frac{kk\mathcal{U}(\omega, k)}{\omega - k \cdot \mathbf{v}_s + \vartheta} \Bigg|_{\vartheta = 0},
\]

\[
\Delta_s = \Phi_s = \frac{\varepsilon_s^2}{2} \frac{\partial}{\partial p} \int d\omega dk \frac{k\mathcal{U}(\omega, k)}{\omega - k \cdot \mathbf{v}_s},
\]

and also

\[
\mathcal{H}_s = \frac{p^2}{2m_s} + e_s\varphi + \Delta_s, \quad \mathbf{v}_s = \frac{p}{m_s} + \frac{\partial \Delta_s}{\partial p}.
\]

The Lagrangian density of a free electrostatic field is

\[
\mathcal{L}_0 = \frac{1}{8\pi} \delta^\parallel(\partial_i \varphi)(\partial^i \varphi) = \frac{\partial}{\partial \chi}\left( \frac{1}{8\pi} \delta^\parallel \varphi(\partial^i \varphi) \right) + \frac{1}{2} \varphi \left( -\frac{\delta^\parallel \partial_i \partial_j}{4\pi} \right) \varphi.
\]

The first term on the right-hand side does not contribute to the field action \( S_0 \) and thus can be ignored. The second term is of the form (6.1) with \( M = 1 \), \( g = 1 \) (§ 2.1.2) and \( \mathcal{E}_0 = \mathcal{E}_0^2/(4\pi) \), so \( \mathcal{E}_0(\omega, k) = k^2/(4\pi) \), where \( k^2 = k_1^2 + \delta_i^\parallel k_i k_j \). Then, (6.20) gives

\[
\mathcal{E}(\omega, k) = \mathcal{E}(\omega, k) = \frac{k^2\varepsilon_\parallel(\omega, k)}{4\pi},
\]

where the arguments \( t \) and \( x \) are omitted for brevity and \( \varepsilon_\parallel \) is the parallel permittivity,

\[
\varepsilon_\parallel(\omega, k) = 1 + \sum_s \frac{4\pi\varepsilon_s^2}{k^2} \int dp \frac{k}{\omega - k \cdot \mathbf{v}_s + i0} \cdot \frac{\partial F_s}{\partial p}.
\]

9.1.2. Collisions and fluctuations

By (8.12), the spectrum of microscopic oscillations of \( \varphi \) is a scalar given by

\[
\mathcal{S}(\omega, k) = 2\pi \sum_s \left( \frac{4\pi\varepsilon_s}{k^2|\varepsilon_\parallel(\omega, k)|} \right)^2 \int dp \delta(\omega - k \cdot \mathbf{v}_s)F_s(p),
\]

where we substituted \( n = 3 \) for three-dimensional plasma. For thermal equilibrium, (8.15) leads to the well-known formula (Lifshitz & Pitaevskii 1981, § 51)

\[
\mathcal{S}(\omega, k) = -\frac{2T}{\omega} \text{im} \left( \frac{1}{\mathcal{E}(\omega, k)} \right) = -\frac{8\pi T}{\omega k^2} \text{im} \left( \frac{1}{\varepsilon_\parallel(\omega, k)} \right) = \frac{8\pi T \text{im} \varepsilon_\parallel(\omega, k) k^2}{\omega k^2 |\varepsilon_\parallel(\omega, k)|^2}.
\]

The spectrum \( \mathcal{S}_\rho \) of charge-density fluctuations is found using Poisson’s equation \( \varphi = k^2\varphi/4\pi \), whence \( \mathcal{S}_\rho \approx (k^2/4\pi)^2\mathcal{S} \). Fluctuations of other fields are found similarly.
Also, (6.36) leads to
\[
|X_{ss'}(\omega, k; p, p')|^2 = \left( \frac{4\pi e_se_{s'}}{k^2 |e_{||}(\omega, k)|} \right)^2.
\]
(9.12)

Then, (6.61) yields the standard Balescu–Lenard collision operator
\[
C_s = \frac{\partial}{\partial p} \cdot \sum_s \int \frac{dk}{(2\pi)^3} dp' \frac{\pi kk}{|e_{||}(k \cdot v_s, k)|^2} \left( \frac{4\pi e_se_{s'}}{k^2} \right)^2 \delta(k \cdot v_s - k \cdot v_{s'}) \cdot \left( \frac{\partial F_s(p)}{\partial p} F_{s'}(p') - F_s(p) \frac{\partial F_{s'}(p')}{\partial p'} \right).
\]
(9.13)

(As a reminder, the distribution functions are normalized such that \( \int dp F_s \) is the local average density of species \( s \) (5.39).)

9.1.3. **On-shell waves**

For on-shell waves, (7.63) gives
\[
U(\omega, k) = (h(k) + h(-k))\delta(\omega - w(k)),
\]
where \( w(k) \) is determined by the dispersion relation
\[
\epsilon_{\parallel}(w(k), k) = 0,
\]
(9.14)
and \( \epsilon_{\parallel} \equiv \text{re} \epsilon_{\parallel} \) is given by
\[
\epsilon_{\parallel}(\omega, k) = 1 + \sum_s \frac{4\pi e_s^2}{k^2} \int dp \frac{k}{\omega - k \cdot v_s} \cdot \frac{\partial F_s}{\partial p}.
\]
(9.15)

The phase-space density of the wave action, defined in (7.74), is
\[
J(k) = h(k) \frac{\partial \Xi_{\parallel}(k)}{\partial \omega} = h(k) \frac{k^2}{4\pi} \frac{\partial \epsilon_{\parallel}(w(k), k)}{\partial \omega},
\]
(9.16)
and the dressing function (9.4) is given by
\[
\Theta_s = e_s^2 \frac{\partial}{\partial \vartheta} \int dk \left. \frac{h(k) + h(-k)}{w(k) - k \cdot v_s + \vartheta} \right|_{\vartheta=0}
\]
\[
= 2e_s^2 \frac{\partial}{\partial \vartheta} \int dk \left. \frac{k}{w(k) - k \cdot v_s + \vartheta} \right|_{\vartheta=0}.
\]
(9.17)

Using these, one obtains (Appendix F.1.1)
\[
\sum_s \int dp p F_s + \int dk k J = \sum_s \int dp p F_s + \sum_s \int dk k J = \sum_s \int dp H_0 F_s + \int dk w J = \sum_s \int dp H_0 F_s + \frac{1}{8\pi} \overline{\tilde{E}^* \tilde{E}},
\]
(9.18)
so the conserved quantity (7.88) is the average momentum of the plasma (while the electrostatic field carries no momentum, naturally). Also (Appendix F.1.2),
\[
\sum_s \int dp H_0 F_s + \int dk w J = \sum_s \int dp H_0 F_s + \frac{1}{8\pi} \overline{\tilde{E}^* \tilde{E}},
\]
(9.19)
so, expectedly, the conserved quantity (7.90) is the average particle energy plus the energy of the electrostatic field. In combination with our equations for \( F_s \) and \( J \) (§ 7.6), these results can be considered as a generalization and concise restatement of the OC QLT by Dewar (1973), which is rigorously reproduced from our general formulation as a particular case.
9.1.4. Eikonal waves

As a particular case, let us consider an eikonal wave

$$\tilde{\phi} \approx \text{re}(e^{i\omega} \varphi), \quad \bar{\omega} \doteq -\partial_\theta, \quad \bar{k} \doteq \partial_x \theta,$$

(9.20)

which may or may not be on-shell. As seen from § 7.4.1,

$$U \approx \frac{|\tilde{\phi}|^2}{4} \sum_\pm \delta(\omega \pm \bar{\omega}) \delta(k \pm \bar{k}).$$

(9.21)

For non-resonant particles, the dressing function is well defined is found as follows:

$$\Theta_s \approx -\int d\omega \, dk \, \frac{e^2 k |\tilde{\phi}|^2}{4(\omega - k \cdot v_s)^2} \sum_\pm \delta(\omega \pm \bar{\omega}) \delta(k \pm \bar{k})$$

$$= -\frac{e^2 \bar{k} |\tilde{\phi}|^2}{2(\bar{\omega} - \bar{k} \cdot v_s)^2}.$$  

(9.22)

Similarly, the ponderomotive energy for non-resonant particles is

$$\Delta_s \approx \frac{e^2 |\tilde{\phi}|^2}{8m_s} \frac{\partial}{\partial v_s} \int d\omega \, dk \, \frac{k}{\omega - k \cdot v_s} \sum_\pm \delta(\omega \pm \bar{\omega}) \delta(k \pm \bar{k})$$

$$= \frac{e^2 \bar{k}^2 |\tilde{\phi}|^2}{4m_s(\bar{\omega} - \bar{k} \cdot v_s)^2}.$$  

(9.23)

in agreement with Dewar (1972) and Cary & Kaufman (1977). One can also express these functions in terms of the electric-field envelope $\tilde{E} \approx -i \bar{k} \tilde{\phi}$:

$$\Theta_s \approx -\frac{e^2 |\tilde{E}|^2}{2(\bar{\omega} - \bar{k} \cdot v_s)^2}, \quad \Delta_s \approx \frac{e^2 |\tilde{E}|^2}{4m_s(\bar{\omega} - \bar{k} \cdot v_s)^2}.$$  

(9.24)

For on-shell in particular, one can use (9.16) together with $h(k) = \frac{1}{4} |\tilde{\phi}|^2 \delta(k - \bar{k})$ (cf. (7.64)) to obtain the well-known expression for the wave action density $I \doteq \int d\omega \, J$:

$$I = \frac{1}{16\pi} \left. \frac{\partial \epsilon_{\text{eH}}(\omega, \bar{k})}{\partial \omega} \right|_{\omega = w(\bar{k})}.$$  

(9.25)

For not-too-hot plasma, one has $\epsilon_{\text{eH}}(\omega, k) \approx 1 - \omega_p^2/\omega^2$, where $\omega_p \doteq \sum_s 4\pi N_s e_s^2/m_s$ is the plasma frequency. The corresponding waves are Langmuir waves. Their dispersion relation is $w(\bar{k}) \approx \pm \omega_p$, so $I \approx \pm |\tilde{E}|^2/(8\pi \omega_p)$ (and accordingly, the wave energy density is $E_w = wI \geq 0$ for either sign). Remember, though, that this expression is only approximate. Using it instead of (9.25) can result in violation of the exact conservation laws of QLT. Conservation of the Langmuir-wave action in non-stationary plasmas beyond the cold-plasma approximation is also discussed in Dodin, Geyko & Fisch (2009), Dodin & Fisch (2010b) and Schmit, Dodin & Fisch (2010).
9.1.5. Homogeneous plasma

In homogeneous \(n\)-dimensional plasma of a given volume \(V_n\), the Wigner function (9.2) has the form

\[
U(t, k) = \frac{1}{\mathcal{V}_n} \int dx d\omega U = \frac{1}{\mathcal{V}_n} |\tilde{\omega}(t, k)|^2 = \frac{1}{\mathcal{V}_n} |\tilde{\omega}(t, k)|^2.
\] (9.26)

Then,

\[
D_s \approx \pi e_s^2 \frac{\mathcal{V}_n}{\mathcal{V}_n} \int \frac{d\mathbf{k} d\mathbf{k}}{\mathcal{V}_n} |\tilde{\omega}(t, k)|^2 \delta(w(t, k) - \mathbf{k} \cdot \mathbf{v}_s).
\] (9.27)

This coincides with the well-known formula for the QL-diffusion coefficient in homogeneous electrostatic plasma.\(^{39}\) The functions \(\Theta_s\) and \(\Delta_s\) are also important in homogeneous turbulence in that they ensure the proper energy–momentum conservation; for example, see Stix (1992, § 16.3) and Liu & Dodin (2015, § II.2). These functions can be expressed through \(\tilde{\omega}\) too. However, they have a simpler representation in terms of the Wigner function \(U\), as in (9.4) and (9.5), respectively. This is because \(U\) is a local property of the field, which makes it more fundamental than the amplitudes of global Fourier harmonics commonly used in the literature.

9.2. Relativistic electromagnetic interactions

9.2.1. Main equations

Let us extend the above results to relativistic electromagnetic interactions. In this case,

\[
H_s = \sqrt{m_s^2 c^4 + (pc - e_s A)^2} + e_s \phi,
\] (9.28)

where \(c\) is the speed of light and \(A\) is the vector potential. Let us adopt the Weyl gauge for the oscillating part of the electromagnetic field (\(\tilde{\phi} = 0\)) and Taylor-expand \(H_s\) to the second order in \(\tilde{\mathbf{A}}\). This leads to

\[
H_s \approx H_{0s} - e_s \beta_s^\dagger \tilde{A} + \frac{e_s^2}{2c^2} \tilde{A}^\dagger \mu_s^{-1} \tilde{A},
\] (9.29)

\[
H_{0s} = \sqrt{m_s^2 c^4 + (pc - e_s A)^2} + e_s \bar{\phi}
\] (9.30)

(although plasma is assumed non-magnetized, a weak average magnetic field \(\bar{\mathbf{B}} = \nabla \times \tilde{\mathbf{A}}\) is allowed, so \(\tilde{\mathbf{A}}\) can be order-one and thus generally must be retained), where

\[
\beta_s = \frac{1}{m_s c \gamma_s} \left( p - \frac{e_s}{c} \tilde{\mathbf{A}} \right), \quad \mu_s^{-1} = \frac{1 - \beta_s \beta_s^\dagger}{m_s \gamma_s},
\] (9.31)

and \(\gamma_s = (1 - \beta_s^2)^{-1/2}\). In the equations presented below, \(\beta_s = v_s / c\) (where \(v_s\) is the OC velocity) is a sufficiently accurate approximation. Also, \(\mu_s = (\beta_{pp}^s H_{0s})^{-1}\) can be interpreted as the relativistic-mass tensor.

\(^{39}\)See, for example, (16.17) in Stix (1992). The extra mass factor appears there because QL diffusion is considered in the velocity space instead of the momentum space.
Let us choose the field \( \widetilde{\Psi} \) of our general theory to be the oscillating electric field \( \widetilde{E} = i\omega A/c \); then (cf. (6.2)),

\[
\alpha_s = i e_s v_s \frac{\omega}{\omega}, \quad \varphi_s = e_s^2 \frac{\mu_s}{\omega}.
\] (9.33)

The average Wigner matrix of \( \widetilde{E} \) is

\[
U(t, x, \omega, k) = \int \frac{d\tau}{2\pi} \frac{ds}{(2\pi)^3} \frac{\widetilde{E}(t + \tau/2, x + s/2) \widetilde{E}^*(t - \tau/2, x - s/2)}{e^{i\omega t - ik \cdot x}}
\] (9.34)

(Other ways to identify \( \widetilde{L}_s \) and \( \widetilde{R}_s \) are also possible and lead to the same results.) Then,

\[
L_s = e_s v_s, \quad \varphi_s = e_s^2 \frac{\mu_s}{\omega}.
\] (9.35)

\[
\Theta_s = e_s^2 \frac{\varphi_s}{\omega} \frac{d}{d\omega} \left[ \frac{1}{\omega^2} \left( \frac{\partial}{\partial \omega} \left( v_s^2 \varphi_s \right) \right) - \varphi_s \right]_{\omega = 0},
\] (9.36)

\[
\Delta_s = e_s^2 \frac{\varphi_s}{2} \frac{d}{d\omega} \left( \frac{v_s^2}{\omega^2} \varphi_s \right) + e_s^2 \frac{1}{2} \int d\omega dk \frac{\text{tr}(U \mu_s^{-1})}{\omega^2}.
\] (9.37)

When plasma is non-relativistic and the field is electrostatic (so \( U = k k^V \varphi \), where \( \varphi \) is scalar), (9.35) gives the same \( D_s \) as (9.3) and (9.37) gives the same \( \Delta_s \) as (9.5). For \( \Theta_s \), the equivalence between (9.36) and (9.4) should not be expected because \( \Theta_s \) is a part of a distribution function, which is not gauge-invariant. (Canonical momenta in the Weyl gauge are different from those in the electrostatic gauge.) But it is precisely the dressing function (9.36) that leads to the correct expressions for the momentum and energy stored in the OC distribution (§ 9.2.3).

The Lagrangian density of a free electromagnetic field is

\[
S_0 = \frac{\widetilde{E}^* \widetilde{E} - \widetilde{B}^* \widetilde{B}}{8\pi}.
\] (9.38)

From Faraday’s law, one has \( \widetilde{B} = \omega^{-1} c (\hat{k} \times \widetilde{E}) \). Then, \( -\widetilde{B}^* \widetilde{B}/c^2 \) can be represented as follows (up to a divergence, which is insignificant):

\[
(\widetilde{E} \times \omega^{-1} \hat{k}) \cdot (\omega^{-1} \hat{k} \times \widetilde{E}) = \widetilde{E} \cdot \omega^{-2} (\hat{k} \times (\hat{k} \times \widetilde{E})) = \widetilde{E}^* \omega^{-2} (\hat{k} \cdot \widetilde{E}) - \widetilde{E}^2 \hat{k}^2 = \widetilde{E}^* \omega^{-2} (\hat{k} \hat{k}^* - 1 \hat{k}^2). \] (9.39)

Then, the vacuum dispersion operator can be written as (cf. (6.1))

\[
\hat{\Xi}_0(\omega, k) = \frac{1}{4\pi} \left( 1 + c^2 \omega^{-2} (\hat{k} \hat{k}^* - 1 \hat{k}^2) \right).
\] (9.40)

\[40\text{Here, the oscillating field } \widetilde{\Psi} = \widetilde{E} \text{ has the same dimension as } x, \text{ so the standard vector notation (including the dot product and the cross product) is naturally extended to } \Psi.\]
The total dispersion matrix is readily found to be

\[
\Xi(\omega, k) = \frac{1}{4\pi} \left( \epsilon(\omega, k) + \frac{c^2}{\omega^2} (k k^\dagger - 1 k^2) \right),
\]

(9.41)

where \( \epsilon \) (not to be confused with the small parameter \( \epsilon \) that we introduced earlier) is the dielectric tensor

\[
\epsilon(\omega, k) = 1 - \frac{\nu_p}{\omega^2} + \sum_s \frac{4\pi e_s^2}{\omega^2} \int dp \frac{v_s v_s^\dagger}{\omega - k \cdot v_s + i0} k \cdot \frac{\partial F_s}{\partial p}.
\]

(9.42)

Here, \( \nu_p \) is the squared relativistic plasma frequency, which is a matrix, because the ‘masses’ \( \mu_s \) are matrices:

\[
\nu_p = \sum_s \frac{4\pi e_s^2}{\omega^2} \int dp F_s \mu_s^{-1}.
\]

(9.43)

9.2.2. Collisions and fluctuations

By (8.12), the spectrum of microscopic oscillations of \( \tilde{E} \) is a matrix given by

\[
S(\omega, k) = 2\pi \sum_s \left( \frac{4\pi e_s}{\omega} \right)^2 \int dp \delta(\omega - k \cdot v_s) F_s(p) \epsilon^{-1}(\omega, k) v_s v_s^\dagger \epsilon^{-\dagger}(\omega, k).
\]

(9.44)

In the electrostatic limit, one can replace \( \epsilon^{-1} \) with \( \epsilon^{-1}_\parallel k k^\dagger / k^2 \), where \( \epsilon_\parallel \) is the relativistic generalization of (9.9); then (9.44) leads to (9.10) as a particular case. For thermal equilibrium, one can also use (8.15) and the following form of \( \epsilon^{-1} \) for isotropic plasma:

\[
\epsilon^{-1} = \frac{1}{\epsilon_\parallel} \left( 1 - \frac{kk^\dagger}{k^2} \right) + \frac{1}{\epsilon_\parallel} \frac{kk^\dagger}{k^2},
\]

(9.45)

where \( \epsilon_\parallel \) is the (scalar) transverse permittivity. Also, (6.36) leads to

\[
|X_{ss'}(\omega, k; p, p')|^2 \approx \left( \frac{4\pi e_s e_s'}{\omega^2} \right)^2 |v_s^\dagger \epsilon^{-1}(\omega, k) v_{s'}|^2.
\]

(9.46)

Then the collision operator (6.61) is obtained in the form

\[
C_s = \frac{\partial}{\partial p} \cdot \sum_{s'} 2 e_s^2 e_{s'}^2 \int \frac{dk}{(2\pi)^3} \frac{dp'}{(k \cdot v_s)^4} |v_s^\dagger \epsilon^{-1}(k \cdot v_s, k) v_{s'}^\dagger|^2 \delta(k \cdot v_s - k \cdot v_{s'}) \times \frac{\partial F_s(p)}{\partial p} \frac{\partial F_{s'}(p')}{\partial p'}.
\]

(9.47)

which is in agreement with (Silin 1961; Hizanidis, Molvig & Swartz 1983). Replacing \( \epsilon^{-1} \) with \( \epsilon^{-1}_\parallel k k^\dagger / k^2 \) leads to the standard Balescu–Lenard operator (9.13) as a particular case.
9.2.3. **On-shell waves**

Electromagnetic on-shell waves satisfy

$$
\left( \epsilon_H(w(k), k) + \frac{c^2}{w(k)^2} (kk^\dagger - 1k^2) \right) \tilde{E} = 0,
$$

(9.48)

where \( \tilde{E} \) is the complex envelope vector parallel to the polarization vector \( \eta \); also,

$$
\epsilon_H(\omega, k) = 1 - \frac{\omega_p}{\omega^2} + \sum_s \frac{4\pi e_s^2}{\omega^2} \int dp \frac{v_s v_s^\dagger}{\omega - k \cdot v_s} k \cdot \frac{\partial F_s}{\partial p}.
$$

(9.49)

This yields (see (9.39))

$$
\tilde{E}^\dagger \epsilon_H(w(k), k) \tilde{E} = -\frac{c^2}{w(k)^2} \tilde{E}^\dagger (kk^\dagger - 1k^2) \tilde{E} = \tilde{B}^\dagger \tilde{B}.
$$

(9.50)

Then, the phase-space density of the wave action (7.74) can be cast in the form

$$
J(k) = h(k) \eta^\dagger \frac{\partial \Xi_H(\omega, k)}{\partial \omega} \eta \bigg|_{\omega=w(k)} = \frac{h(k)}{4\pi \omega^2} \eta^\dagger \frac{\partial (\omega^2 \epsilon_H(\omega, k))}{\partial \omega} \eta \bigg|_{\omega=w(k)}
$$

(9.51)

(cf. Dodin et al. 2019), and the dressing function (9.36) is given by

$$
\Theta_s = e_s^2 \frac{\partial}{\partial \vartheta} \int d\mathbf{k} (h(k) + h(-k)) \frac{kk}{w^2(k)w(k) - k \cdot v_s + \vartheta} \bigg|_{\vartheta=0} = 2e_s^2 \frac{\partial}{\partial \vartheta} \int d\mathbf{k} h(k) \frac{kk}{w^2(k)w(k) - k \cdot v_s + \vartheta} \bigg|_{\vartheta=0}.
$$

(9.52)

Using these, one obtains (Appendix F.2.1)

$$
\sum_s \int dp pF_s + \int dk kJ = \mathcal{P}^{(\text{kin})} + \frac{\tilde{E} \times \tilde{B}}{4\pi c},
$$

(9.53)

where \( \mathcal{P}^{(\text{kin})} \) is the average density of the plasma kinetic (up to \( \mathbf{A} \)) momentum,

$$
\mathcal{P}^{(\text{kin})} \doteq \sum_s \int dp (p - e_s \mathbf{A}/c)f_s = \sum_s \int dp p f_s^{(\text{kin})},
$$

(9.54)

the functions \( f_s^{(\text{kin})}(p) \) are the distributions of kinetic (up to \( \mathbf{A} \)) momenta, and the second term in (9.53) is the well-known average momentum of electromagnetic field. Similarly (Appendix F.2.2),

$$
\sum_s \int dp H_0 F_s + \int dk wJ = \mathcal{K}^{(\text{kin})} + \frac{1}{8\pi} (\tilde{E} \cdot \tilde{E} + \tilde{B} \cdot \tilde{B}),
$$

(9.55)

where \( \mathcal{K}^{(\text{kin})} \) is given by

$$
\mathcal{K}^{(\text{kin})} \doteq \sum_s \int dp H_0 f_s^{(\text{kin})}.
$$

(9.56)

In other words, the total momentum and energy of the system in the OC–wave representation are the same as those in the original particle–field variables.
9.2.4. Eikonal waves

As a particular case, let us consider an eikonal wave

\[ \tilde{E} \approx \text{re}(e^{i\theta} \tilde{E}), \quad \omega \doteq -\partial_t \theta, \quad \tilde{k} \doteq \partial_\theta \theta, \]  

(9.57)

which may or may not be on-shell. Then, (9.36) and (9.37) lead to (cf. § 9.1.4)

\[ \Theta_s = -\frac{\tilde{k} \cdot \tilde{E}^*}{\omega} e_s^{-1} \tilde{E}, \quad \Delta_s = \frac{e_s^2}{\omega^2} \frac{\partial^2 (|\tilde{E}|^2)}{\partial \omega^2} - \frac{\partial}{\partial \omega} \left( \frac{e_s^2}{\omega} \frac{\partial |\tilde{E}|^2}{\partial \omega} \right). \]  

(9.58)

(9.59)

For on-shell waves in particular, one also obtains the action density in the form

\[ I = \frac{1}{16\pi\omega^2} \tilde{E}^* \frac{\partial (\omega^2 \epsilon_H(\omega, k))}{\partial \omega} \tilde{E} \bigg|_{\omega = \omega(\tilde{k})} \]  

(9.60)

where we used (9.50).

9.3. Newtonian gravity

For Newtonian interactions governed by a gravitostatic potential \( \varphi_g \), one has

\[ H_s = \frac{p^2}{2m_s} + m_s \tilde{\varphi}_g + m_s \tilde{\varphi}_g, \quad \Sigma_0 = -\frac{(\nabla \tilde{\varphi}_g)^2}{8\pi G}, \]  

(9.61)

where \( G \) is the gravitational constant. This system is identical to that considered in § 9.1 for non-relativistic electrostatic interactions up to coefficients. Specifically, \( e_s \) are replaced with \( m_s \), a factor \(-G^{-1}\) appears in \( \Sigma_0 \), and the dispersion matrix becomes

\[ \Xi(\omega, k) = \Xi(\omega, k) = -\frac{k^2 \epsilon_g(\omega, k)}{4\pi G}. \]  

(9.62)

Thus, \( \epsilon_\parallel \) is replaced with \(-\epsilon_g/G\), where \( \epsilon_g \) is the gravitostatic permittivity given by

\[ \epsilon_g(\omega, k) = 1 - \sum_s \frac{4\pi G m_s^2}{k^2} \int \frac{\partial F_s}{\partial \omega - k \cdot \nu_s + i0} \frac{k}{\omega - k \cdot \nu_s + i0}. \]  

(9.63)

This readily yields, for example, the kinetic theory of the Jeans instability (Trigger et al. 2004), whose dispersion relation is given by \( \epsilon_g(\omega, k) = 0 \) (modulo the usual analytic continuation of the permittivity to modes with \( \text{im} \omega < 0 \)).

9.4. Relativistic gravity

9.4.1. Main equations

The dynamics of a relativistic neutral particle with mass \( m \) in a spacetime metric \( g_{\alpha\beta} \) with signature \((-+++)\) is governed by a covariant Hamiltonian (see, for example, Garg & Dodin 2020)

\[ H(x, p) = \frac{1}{2m} \left( m^2 + g^{\alpha\beta}(x)p_\alpha p_\beta \right) \equiv H(g, p). \]  

(9.64)

Here, \( x \equiv (x^0, x) \), and \( x^0 = t \), as usual. Also, \( p \equiv (p_0, p) \) is the index-free notation for the four-momentum \( p_\alpha \), \( g^{\alpha\beta} \) is the inverse metric, \( g \) is the index-free notation for \( g^{\alpha\beta} \), the
units are such that \( c = 1 \) and the species index is omitted. The corresponding Hamilton’s equations, with \( \tau \) the proper time, are

\[
\frac{dx^\alpha}{d\tau} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{d\tau} = -\frac{\partial H}{\partial x^\alpha}.
\]  

(9.65)

This dynamics is constrained to the shell \( p_0 = P_0(t, x, p) \), where \( P_0 \) is the (negative) solution of

\[
H(g, P_0(t, x, p), p) = 0.
\]  

(9.66)

This means that the particle distribution in the \((x, p)\) space is delta-shaped and thus does not satisfy (3.35). Hence, we will consider particles in the six-dimensional space \((x, p)\) instead. The corresponding dynamics is governed by the Hamiltonian

\[
H = -P_0(t, x, p).
\]  

(9.67)

This is seen from the fact that

\[
\frac{\partial H}{\partial \dot{\bullet}} = -\frac{\partial P_0}{\partial \dot{\bullet}} = \frac{\partial H}{\partial \dot{p}_0},
\]  

(9.68)

where \( \bullet \in \{t, x, p\} \), so Hamilton’s equations obtained from (9.67) are equivalent to (9.65):

\[
\frac{dx^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha} = \frac{\partial H}{\partial \dot{p}_0}.
\]  

(9.69)

Let us decompose the metric into the average part and oscillations, \( g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} \), and approximate the inverse metric to the second order in \( \tilde{g} \):

\[
g^{\alpha\beta} \approx \bar{g}^{\alpha\beta} - \tilde{g}^{\alpha\beta} + \frac{1}{2} \frac{\partial^2 P_0}{\partial \tilde{g}_{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta},
\]  

(9.70)

where the indices of \( \tilde{g} \) are manipulated using the background metric \( \bar{g}_{\alpha\beta} \). This gives

\[
H = \frac{1}{2m} \left( m^2 + \bar{g}^{\alpha\beta} p_\alpha p_\beta - \tilde{g}^{\alpha\beta} p_\alpha p_\beta + \tilde{g}_{\alpha\beta} \bar{g}_{\gamma\delta} \tilde{g}^{\gamma\delta} p_\alpha p_\delta \right).
\]  

(9.71)

The Hamiltonian (9.67) is expanded in \( \tilde{g} \) as follows:

\[
H(g, p) \approx -P_0 - \frac{\partial P_0}{\partial g^{\alpha\beta}} \tilde{g}^{\alpha\beta} - \frac{1}{2} \frac{\partial^2 P_0}{\partial g^{\alpha\beta} \partial g^{\gamma\delta}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta},
\]  

(9.72)

where \( P_0 \) and the derivatives on the right-hand side are evaluated on \((\bar{g}, p)\). To calculate these derivatives, let us differentiate (9.66) and use (9.71) for \( H \). This gives

\[
0 = \frac{\partial H}{\partial \tilde{g}^{\alpha\beta}} + \frac{\partial H}{\partial p_0} \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} = \frac{1}{2m} \left( -p_\alpha p_\beta + 2P_0^0 \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} \right),
\]  

(9.73)

where the derivatives with respect to the oscillating metric are taken at fixed \( p_\alpha \) and at \( \tilde{g} \to 0 \), and \( P_0^0 \equiv P_0^0(g, p) = \bar{g}^{\alpha\beta} p_\alpha \); thus,

\[
\frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} = \frac{p_\alpha p_\beta}{2P_0^0}.
\]  

(9.74)

[^41]: This section uses notation different from that used in the rest of the paper. In particular, the Greek indices span from 0 to 3, and the standard rules of index manipulations apply.
Similarly, differentiating (9.66) twice gives

\[
0 = \frac{\partial^2 H}{\partial g^{ab} \partial g^{\gamma \delta}} + \frac{\partial H}{\partial p_0} \frac{\partial^2 p_0}{\partial g^{ab} \partial g^{\gamma \delta}} + \frac{\partial P_0}{\partial p_0} \frac{\partial H}{\partial g^{ab}} + \frac{\partial H}{\partial g^{ab}} \frac{\partial}{\partial p_0} \left( \frac{\partial H}{\partial g^{ab}} + \frac{\partial H}{\partial g^{ab}} \frac{\partial P_0}{\partial g^{\gamma \delta}} \right) \frac{\partial P_0}{\partial g^{\gamma \delta}}
\]

\[
= \frac{\partial^2 H}{\partial g^{ab} \partial g^{\gamma \delta}} + \frac{\partial H}{\partial p_0} \frac{\partial^2 p_0}{\partial g^{ab} \partial g^{\gamma \delta}} + \frac{\partial P_0}{\partial p_0} \frac{\partial H}{\partial g^{ab}} + \frac{\partial H}{\partial g^{ab}} \frac{\partial}{\partial p_0} \left( \frac{\partial H}{\partial g^{ab}} + \frac{\partial H}{\partial g^{ab}} \frac{\partial P_0}{\partial g^{\gamma \delta}} \right) \frac{\partial P_0}{\partial g^{\gamma \delta}}
\]

\[
= \frac{1}{2m} \left( \widetilde{g}_{\beta \gamma} p_\alpha p_\delta + \widetilde{g}_{\alpha \delta} p_\gamma p_\beta + 2p^0 \frac{\partial^2 P_0}{\partial g^{ab} \partial g^{\gamma \delta}} - \frac{1}{2} \frac{\partial (p_\alpha p_\beta p_\gamma p_\delta)}{\partial p_0} + \frac{g_{00}}{2 (P^0)^2} \right),
\]

whence

\[
\frac{\partial^2 P_0}{\partial g^{ab} \partial g^{\gamma \delta}} = - \frac{1}{2p_0} \left( \widetilde{g}_{\beta \gamma} p_\alpha p_\delta + \widetilde{g}_{\alpha \delta} p_\gamma p_\beta \right) + \frac{1}{4 (P^0)^2} \frac{\partial (p_\alpha p_\beta p_\gamma p_\delta)}{\partial p_0} - \frac{g_{00}}{4 (P^0)^3} p_\alpha p_\beta p_\gamma p_\delta.
\]

Then, (9.72) yields

\[
H \approx H_0 + \alpha_{ab} \widetilde{g}^{ab} + \frac{1}{2} \widetilde{g}_{ab \gamma \delta} \widetilde{g}^{ab \gamma \delta},
\]

where we introduced \( H_0 = -p_0, \alpha_{ab} = p^a p^b / (2p^0) \) and

\[
\phi_{ab \gamma \delta} = \frac{\delta^{\gamma \delta}_{\gamma \delta} p_\alpha p_\delta + \delta^{\gamma \delta}_{\gamma \delta} p_\beta p_\delta}{2p^0} - \frac{1}{4 (P^0)^2} \frac{\partial (p_\alpha p_\beta p_\gamma p_\delta)}{\partial p_0} + \frac{g_{00} p^a p^b p_\gamma p_\delta}{4 (P^0)^3}.
\]

### 9.4.2. Nonlinear potentials

Let us treat \( \widetilde{g} \) as a 16-dimensional vector (Garg & Dodin 2021b), so \( \alpha_{ab} \) serves as \( \alpha^\dagger \) and \( \phi_{ab \gamma \delta} \) serves as \( \Phi. \) (Because these operators happen to be local in the \( x \) representation, here we do not distinguish them from their symbols.) Let us also introduce

\[
\mathcal{E} \doteq p_\alpha p_\beta p_\gamma p_\delta \Phi_{ab \gamma \delta} \tag{9.77}
\]

and notice that \( \nu^0 \approx \dot{x}^0 = p^0 / p^0 \) (see (9.69)), so \( \omega - k \cdot v = -k_\rho p^\rho / p^0 \) and \( \delta (\omega - k \cdot v) = P^0 \delta (k_\rho p_\rho). \) Then, one finds from (6.77) that (Appendix B.8)

\[
D = \frac{\pi}{4p^0} \int dk \, k k \mathcal{E} \delta(k_\rho p_\rho), \tag{9.78}
\]

\[
\Theta = \frac{1}{4p^0} \frac{\partial}{\partial \theta} \int dk \frac{k k \mathcal{E}}{\partial p^0 - k_\rho p_\rho} \bigg|_{\theta = 0}, \tag{9.79}
\]

\[
\Delta = \frac{p_\alpha p_\beta}{2p^0} \int dk \, \Phi_{ab \gamma \delta} - \frac{1}{8p^0} \frac{\partial}{\partial p_\lambda} \int dk \frac{k_\lambda \mathcal{E}}{k_\rho p_\rho}. \tag{9.80}
\]

Equation (9.80) (where one takes \( p_0 = P_0 \) after the differentiation) is in agreement with the result that was obtained for quasimonochromatic waves in Garg & Dodin (2020). The derivation of the dispersion matrix \( \Xi \) for relativistic gravitational interactions in matter is cumbersome, so it is not presented here, but see Garg & Dodin (2022). The collision integral and fluctuations for relativistic gravitational interactions are straightforward to obtain from the general formulas presented in §§6.9 and 8. This can be used to describe QL interactions of gravitational waves, including not only the usual vacuum modes but also waves coupled with matter, for example, the relativistic Jeans mode.

---

\(^{42}\)Vacuum gravitational waves satisfy \( \omega^2 = |k|^2. \) Hence, satisfying the resonance condition \( k_\rho p_\rho = 0 \) requires \( |k \cdot v| = |k|, \) which requires particle speeds not smaller than the speed of light (remember that the speed of light is equal to one in our units). For massive particles, this cannot be satisfied, so \( \mathcal{D} \) vanishes for vacuum gravitational waves. However, such waves can still produce adiabatic ponderomotive effects determined by \( \Delta \) (Garg & Dodin 2020).
Also notice that the OC Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \Delta$ can be put in a covariant form as follows. Like in the original system (§ 9.4.1), $\mathcal{H}$ determines the ponderomotively modified shell $p_0 = P_0(t, x, p)$ via $\mathcal{H} = -\mathcal{P}_0$. On one hand, the covariant OC Hamiltonian $\mathcal{H}'$ vanishes on this shell, so it can be Taylor-expanded as follows:

$$\mathcal{H}' \approx (p_0 - \mathcal{P}_0) \left. \frac{\partial \mathcal{H}}{\partial p_0} \right|_{p_0 = \mathcal{P}_0} \approx (p_0 - P_0 + \Delta) \lambda, \quad \lambda = \left. \frac{\partial \mathcal{H}}{\partial p_0} \right|_{p_0 = P_0}. \quad (9.81)$$

On the other hand, it can also be represented as $\mathcal{H}' = \mathcal{H}(\bar{g}, p) + \Delta'$ (here $\Delta'$ is the ponderomotive term yet to be found) and Taylor-expanded around the unperturbed shell $p_0 = P_0(t, x, p)$ as

$$\mathcal{H}' \approx \Delta' + (p_0 - P_0) \lambda = (p_0 - P_0 + \Delta' / \lambda) \lambda. \quad (9.82)$$

By comparing (9.81) with (9.82), one finds that $\Delta' = \lambda \Delta$. Because $\lambda = P^0 / m$, this leads to the following covariant Hamiltonian for OCs:

$$\mathcal{H}' = \frac{1}{2m} \left( m^2 + g_{\text{eff}}^\alpha p_\alpha p_\beta - \frac{1}{4} \frac{\partial}{\partial p_\lambda} \int \frac{dk}{k^p p_\rho} \right), \quad (9.83)$$

$$g_{\text{eff}}^\alpha \equiv \bar{g}^\alpha + \int \frac{dk}{k^p} U^\alpha \gamma^\beta. \quad (9.84)$$

### 9.4.3. Gauge invariance

As shown in Garg & Dodin (2021a, b) adiabatic QL interactions via gravitational waves (i.e., those determined by $\Theta$ and $\Delta$) can be formulated in a form invariant with respect to gauge transformations

$$\tilde{g}^{\alpha\beta} \rightarrow \tilde{g}^{\alpha\beta} = \tilde{g}^{\alpha\beta} + \nabla^{(\alpha} \xi^{\beta)} \,, \quad (9.85)$$

where $\nabla$ is the covariant derivative associated with the background metric $\bar{g}, \xi$ is an arbitrary vector field and $\psi^{(\alpha} \eta^{\beta)} \equiv (\psi^{\alpha} \eta^{\beta} + \psi^{\beta} \eta^{\alpha}) / 2$. Let us show that this also extends to resonant interactions. Recall that within the assumed accuracy the nonlinear potentials are supposed to be calculated only to the zeroth order in the geometrical-optics parameter. Then, the modification of the average Wigner matrix of the metric oscillations under the transformation (9.85) can be written as

$$U^{\alpha\beta \gamma \delta} - U^{\alpha\beta \gamma \delta} = \text{symb} \left( ik^{(\alpha} \bar{\xi}^{\beta)} \langle \bar{g}^{\gamma \delta} \rangle | \xi^{(\gamma} \rangle | \xi^{\beta)} \right) \langle \bar{g}^{(\gamma} \rangle | \xi^{(\delta)} \rangle | \xi^{\beta)} \right) = ik^{(\alpha} \mathcal{W}^{\gamma \delta}_{\gamma \delta} \rangle + k^{(\alpha} \mathcal{W}^{\gamma \delta}_{\gamma \delta} \rangle \langle k^{\delta)}, \quad (9.86)$$

where $\mathcal{W}^{\gamma \delta}_{\gamma \delta} \equiv \text{symb} | \tilde{\xi}^{\gamma} \rangle | \tilde{\xi}^{\delta} \rangle$ and $\mathcal{W}^{\gamma \delta}_{\gamma \delta}$ is the average Wigner matrix of $\tilde{\xi}^{\alpha}$. The corresponding change of $\mathcal{E}$ is

$$\mathcal{E}' - \mathcal{E} = (k^\rho p_\rho) \left( i p_\rho p_\gamma p_\delta \mathcal{W}^{\gamma \delta}_{\gamma \delta} \rangle - i p_\alpha p_\beta p_\gamma \mathcal{W}^{\gamma \alpha \beta \gamma \delta} \rangle + k^\lambda p_\alpha p_\beta p_\gamma \mathcal{W}^{\gamma \alpha \beta \gamma \delta} \rangle \langle k^\delta) \right) \approx (k^\rho p_\rho) A.$$

Then, the difference in the diffusion coefficients (9.78) is

$$D' - D = \frac{\pi}{4P^0} \int \frac{dk}{k} d^3 k \delta(k^\rho p_\rho) (k^\rho p_\rho) A = 0, \quad (9.87)$$

because $\delta(k^\rho p_\rho)(k^\rho p_\rho) = 0$. In particular, this rules out QL diffusion via ‘coordinate waves’.

---

43The covariant Hamiltonian is the dispersion function of particles as quantum waves in the semiclassical limit (Garg & Dodin 2020).
9.4.4. Lorenz gauge and effective metric

Let us consider gravitational waves in the Lorenz gauge, $\nabla_\alpha \tilde{g}^{\alpha\beta} = 0$. In this case,

$$k_\alpha \tilde{U}^{\beta\gamma\delta} = k_\beta \tilde{U}^{\alpha\beta\gamma\delta} = k_\gamma \tilde{U}^{\alpha\beta\gamma\delta} = k_\delta \tilde{U}^{\alpha\beta\gamma\delta} = 0,$$

(9.88)

and thus $k_\lambda \partial E / \partial p^\lambda = 0$. Then,

$$\frac{\partial}{\partial p_\lambda} \int d\mathbf{k} \frac{k_\lambda E}{k^\rho p_\rho} = \frac{\partial}{\partial \theta} \int d\mathbf{k} \frac{(k_\lambda k^\lambda) E}{k^\rho p_\rho + \partial} \bigg|_{\theta = 0}.$$  

(9.89)

This simplifies the expression (9.80) for $\Delta$ and (9.83) for $\mathcal{H}'$. Furthermore, if the waves are not significantly affected by matter, so the vacuum dispersion $k_\lambda k^\lambda = 0$ can be assumed, the term (9.89) vanishes completely. Then, (9.83) becomes

$$\mathcal{H}' = \frac{1}{2m} \left( m^2 + g^{\alpha\beta}_{\text{eff}} p_\alpha p_\beta \right)$$

(9.90)

and QL diffusion disappears, because particles cannot resonate with waves. This shows that the only average QL effect of vacuum gravitational waves on particles is the modification of the spacetime metric by $\tilde{U}^{\alpha\gamma\gamma\beta} = O(\epsilon^2)$. For quasimonochromatic waves, this effect is discussed in further detail in Garg & Dodin (2020).

10. Summary

In summary, we have presented QLT for classical plasma interacting with inhomogeneous turbulence in the presence of background fields. Because we use the Weyl symbol calculus, global-mode decomposition is not invoked, so our formulation is local and avoids the usual issues with complex-frequency modes. Also, the particle Hamiltonian is kept general, so the results are equally applicable to relativistic, electromagnetic, and even non-electromagnetic (for example, gravitational) interactions. Because our approach is not bounded by the limitations of variational analysis either, effects caused by collisional and collisionless dissipation are also included naturally.

Our main results are summarized in §§ 5.6, 6.9, 7.6 and 8.5 and are as follows. Starting from the Klimontovich equation, we derive a Fokker–Planck model for the dressed OC distribution. This model captures QL diffusion, interaction with the background fields and ponderomotive effects simultaneously. The local diffusion coefficient is manifestly positive-semidefinite. Waves are allowed to be off-shell (not constrained by a dispersion relation), and a collision integral of the Balescu–Lenard type emerges in a form that is not restricted to any particular Hamiltonian. This operator conserves particles, momentum and energy, and it also satisfies the $H$-theorem, as usual. As a spin-off, a general expression for the spectrum (average Wigner matrix) of microscopic fluctuations of the interaction field is derived. For on-shell waves, which satisfy a QL WKE, our theory conserves the momentum and energy of the wave–plasma system. Dewar’s OC QLT of electrostatic turbulence (Dewar 1973) is proven formally as a particular case and given a concise formulation. Also discussed as examples are relativistic electromagnetic and gravitational interactions, and QLT for gravitational waves is proposed.

Aside from having the aesthetic appeal of a rigorous local theory, our formulation can help, for example, better understand and model QL plasma heating and current drive. First of all, it systematically accounts for the wave-driven evolution of the non-resonant particle distribution and for the ponderomotive effects caused by plasma inhomogeneity
in both time and space. As discussed above (§ 7.5), this is generally important for adequately calculating the energy–momentum transfer between waves and plasma even when resonant absorption per se occurs in a homogeneous-plasma region. Second, our formulation provides general formulas that equally hold in any canonical variables and for any Hamiltonians that satisfy our basic assumptions (§3.1). Therefore, our results can be applied to various plasma models immediately. This eliminates the need for ad hoc calculations, which can be especially cumbersome beyond the homogeneous-plasma approximation. Discussing specific models of applied interest, however exciting, is beyond the scope of this paper and is left to future work.

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Declaration of interests

The author reports no conflict of interest.

Appendix A. Average Wigner matrices

A.1. Positive-semidefinitness

As known from Cartwright (1976), the average Wigner function of any scalar field on the real axis is non-negative if the averaging is done over a sufficiently large phase-space volume. Here, we extend this theorem to average Wigner matrices of vector fields in a multi-dimensional space, \( \Psi(x) \), and show that such matrices are positive-semidefinite.

For any given function \( h(z) \equiv h(x, k) \), we define its local phase-space average as the following convolution integral:\(^{44}\)

\[
\overline{h}(z) \doteq \int dz' \, G(z - z') \, h(z') \equiv \int dx' \, dk' \, G(x - x', k - k') \, h(x', k') \tag{A1}
\]

with a Gaussian window function

\[
G(x, k) \doteq \frac{1}{(2\pi \sigma_x \sigma_k)^n} \exp \left( -\frac{|x|^2}{2\sigma_x^2} - \frac{|k|^2}{2\sigma_k^2} \right) \tag{A2}
\]

and positive constants \( \sigma_x \) and \( \sigma_k \) yet to be specified. Unlike in § 2.1.1, the following notation will be assumed for the ‘scalar product’ for variables with upper, lower and mixed indices:

\[
x' \cdot x'' \doteq \delta_{ij} x'^i x''^j, \quad k' \cdot k'' \doteq \delta_{ij} k'_i k''^j, \quad k \cdot x \doteq k_i x^i. \tag{A3}
\]

(The Latin indices in this appendix range from 0 to \( n \), \( \delta_{ij} \) and \( \delta_{ij} \) are unit matrices, and summation over repeating indices is assumed.) In particular, note that \( |x|^2 = x \cdot x \geq 0 \) and must not be confused with the squared spacetime interval, which can have either sign. Likewise, \( |k|^2 = k \cdot k \geq 0 \) must not be confused with \( k_i k^i = -\omega^2 + k^2 \).

\(^{44}\)This ensures that \( \delta x \overline{h} = \overline{\delta x h} \), as readily seen from (A1) using integration by parts.
The average Wigner matrix of any given vector field $\psi$ is given by
\[
\mathbf{W}_\psi(x, k) = \frac{1}{(2\pi)^n} \frac{1}{(2\pi \sigma_x \sigma_n)^n} \int \! ds \, dx' \, dk' \, \psi(x + s/2) \psi^\dagger(x' - s/2) \exp\left(-\frac{|x - x'|^2}{2\sigma_x^2} - \frac{|k - k'|^2}{2\sigma_k^2} - ik' \cdot s \right). \tag{A4}
\]

The integral over $k'$ can be readily taken:
\[
\int \! dk' \exp\left(-\frac{|k - k'|^2}{2\sigma_k^2} - ik' \cdot s \right) = (2\pi)^n/2 \sigma_k^n \exp\left(-\frac{\sigma_k^2|s|^2}{2} - ik \cdot s \right). \tag{A5}
\]

Then, using the variables $x_{1,2} \doteq x' \pm s/2$, one can rewrite (A4) as follows:
\[
\mathbf{W}_\psi(x, k) = \frac{1}{(2\pi)^{3n/2} \sigma_x^n} \int \! dx_1 \, dx_2 \, \psi(x_1) \psi^\dagger(x_2) \exp(-\phi), \tag{A6}
\]
\[
\phi = \frac{|x - (x_1 + x_2)/2|^2}{2\sigma_x^2} + \frac{\sigma_k^2|x_1 - x_2|^2}{2} + ik \cdot (x_1 - x_2). \tag{A7}
\]

The function $\phi$ can also be expressed as $\phi = |x|^2/(2\sigma_x^2) + \phi(x_1) + \phi^*(x_2) - \lambda x_1 \cdot x_2$, where
\[
\phi(y) = \frac{|y|^2}{2} \left( \frac{1}{4\sigma_x^2} + \sigma_k^2 \right) - \frac{x \cdot y}{2\sigma_x^2} + ik \cdot y \tag{A8}
\]
and $\lambda \doteq \sigma_k^2 - (4\sigma_x^2)^{-1}$. Then, using $\xi(y) = \psi(y)e^{-\phi(y)}$, one obtains from (A6) that
\[
\mathbf{W}_\psi(x, k) = \frac{e^{-|x|^2/(2\sigma_x^2)}}{(2\pi)^{3n/2} \sigma_x^n} \int \! dx_1 \, dx_2 \, \xi(x_1) \xi^\dagger(x_2) \exp(ik \cdot x_1). \tag{A9}
\]

By Taylor-expanding $e^{ik \cdot x}$, one obtains
\[
\mathbf{W}_\psi(x, k) = \frac{e^{-|x|^2/(2\sigma_x^2)}}{(2\pi)^{3n/2} \sigma_x^n} \sum_{m=0}^\infty \frac{\lambda^m}{m!} J_m, \tag{A10}
\]
where $J_m = \int \! dx_1 \, dx_2 \, (x_1 \cdot x_2)^m \xi(x_1) \xi^\dagger(x_2)$. Note that
\[
(x_1 \cdot x_2)^m = \sum_{\mu(m)} \prod_{i=1}^n (x_1^i x_2^i)^{m_i}, \tag{A11}
\]
where the summation is performed over all combinations $\mu(m) \equiv \{m_1, m_2, \ldots, m_n\}$ of integers $m_i \geq 0$ such that $\sum_i m_i = m$. Thus,
\[
J_m = \sum_{\mu(m)} \mathcal{J}_\mu \mathcal{J}_\mu^\dagger, \quad \mathcal{J}_\mu = \int \! dy \, \xi(y) \prod_{i=1}^n (y^i)^{m_i}. \tag{A12}
\]

Because each $J_m$ is positive-semidefinite, the Wigner matrix $\mathbf{W}_\psi$ is positive-semidefinite when $\lambda \geq 0$, or equivalently, when $\sigma_x \sigma_n > 1/2$. This condition is assumed to be satisfied for the phase-space averaging of $\mathbf{W}_\psi$ used in the main text. Loosely, this means that the averaging is done over the phase-space volume $\Delta x \Delta k \sim (\sigma_x \sigma_n)^n \gtrsim 1$. 

**Quasilinear theory**
A.2. Invariant limit for eikonal fields

For eikonal fields (7.17), one has

\[ \psi(x + s/2) \psi^\dagger(x - s/2) \approx (A(x) e^{ik(x) \cdot s} + \text{c.c.}) + (B(x) e^{2i\theta(x)} + \text{c.c.}) . \]  

(A13)

Here, \( A \equiv \eta \eta^\dagger k^2/4 \), \( B \equiv \eta \eta^\dagger k^2/4 \), ‘c.c.’ stands for complex conjugate, we used the linear approximation \( \theta(x \pm s/2) \approx \theta(x) \pm \tilde{k}(x) \cdot s/2 \), with \( \tilde{k} \equiv (-\bar{o}, k) \). Then,

\[ W_{\psi}(x, k) \approx A(x) \delta(k - \tilde{k}(x)) + A^\dagger(x) \delta(k + \tilde{k}(x)) + 2 \text{re} (B(x)e^{2i\theta(x)} \delta(k)) . \]  

(A14)

Let us adopt \( \sigma_x \ll l_c \), where \( l_c \) is the least characteristic scale of \( \bar{o}, \eta \) and \( \tilde{k} \). Then,

\[ \bar{W}_{\psi}(x, k) \approx A(x) G_\psi(k - \tilde{k}) + A^\dagger(x) G_\psi(k + \tilde{k}) + 2 \text{re} (B(x)G_\psi(k)e^{2i\theta(x)}) . \]  

(A15)

Here, \( G_\psi(k) \) are normalized Gaussians that can be replaced with delta functions if \( \sigma_x \) is small compared with any scale of interest in the \( k \) space:

\[ G_\psi(k) \doteq \frac{1}{(\sqrt{2\pi}\sigma_x)^n} \exp \left( -\frac{|k|^2}{2\sigma_x^2} \right) \to \delta(k) . \]  

(A16)

Also, the function

\[ \zeta \approx \frac{1}{(\sqrt{2\pi}\sigma_x)^n} \int dx' \exp \left( -\frac{|x' - x|^2}{2\sigma_x^2} + 2i\overline{k(x)} \cdot (x' - x) \right) = e^{-2|\overline{k(x)}|^2\sigma_x^2} . \]  

(A17)

can be made exponentially small by adopting \( \sigma_x \gg |\overline{k}|^{-1} \). In this limit, the average Wigner matrix of an eikonal field is independent of \( \sigma_x \) and \( \sigma_k \):

\[ \bar{W}_{\psi}(x, k) \approx A(x) \delta(k - \tilde{k}(x)) + A^\dagger(x) \delta(k + \tilde{k}(x)) . \]  

(A18)

This \( \bar{W}_{\psi} \) is also Hermitian and positive-semidefinite (in agreement with the general theory from § A.1), because so are \( A \) and \( A^\dagger \). The same properties pertain to the Wigner matrix of an ensemble of randomly phased eikonal fields, because it equals the sum of the Wigner matrices of the individual components (see also § 7.4).

Appendix B. Auxiliary proofs

B.1. Proof of (2.53)

Like in the case of (2.45), one finds that

\[ (\bar{L}\psi(x))^\dagger (\bar{R}\psi(x))^{ij} = \langle x | \bar{L}^\dagger_{ij} | \psi^\dagger \rangle \langle \psi^\dagger | (\bar{R}^\dagger)^{ij} | x \rangle \]

\[ = (2\pi)^n \langle x | \bar{L}^\dagger_{ij} \bar{W}_{\psi}(\bar{R}^\dagger)^{ij} | x \rangle \]

\[ = (2\pi)^n \langle x | (\bar{L} \bar{W}_{\psi} \bar{R}^\dagger)^{ij} | x \rangle \]

\[ = \int dk \left( L(x, k) \star W_{\psi}(x, k) \star R^\dagger(x, k) \right)^{ij} . \]  

(B1)

This proves (2.53a). At \( \epsilon \to 0 \), when \( \star \) becomes the usual product, (B1) gives

\[ (\bar{L}\psi)(\bar{R}\psi)^\dagger = \int dk LW_{\psi} R^\dagger , \]

(B2)

and in particular, taking the trace of (B2) yields

\[ (\bar{R}\psi)^\dagger (\bar{L}\psi) = \text{tr} ((\bar{L}\psi)(\bar{R}\psi)^\dagger) = \int dk \text{tr}(LW_{\psi} R^\dagger) = \int dk \text{tr}(W_{\psi} Q) . \]  

(B3)

Here, \( Q \doteq R^\dagger L \), and we used that \( \text{tr}(AB) = \text{tr}(BA) \) for any matrices \( A \) and \( B \).

\[ ^{45}\text{Even though } \sigma_x \text{ has been assumed small compared with } l_c, \text{ the smallness of the geometrical-optics parameter } \epsilon \doteq (|\overline{k}|l_c)^{-1} \ll 1 \text{ allows choosing } \sigma_x \text{ in the interval } |\overline{k}|^{-1} \ll \sigma_x \ll l_c. \]
For real fields, one can also replace the integrand with
\[ \text{tr} \left( W^* \psi Q \right) = \text{tr} \left( Q^* W^\dagger \right) = \text{tr} \left( Q^* W \right) = \text{tr} \left( W^\dagger Q^\dagger \right), \] (B4)
where we used \( \text{tr} A^T = \text{tr} A \), \( (AB)^T = B^T A^T \), \( W^\dagger = W_\psi \) and, again, \( \text{tr}(AB) = \text{tr}(BA) \), respectively. In summary then,
\[ (\hat{R}^\dagger \psi) \hat{L}^\dagger \psi = \int dk \text{tr}(W_\psi Q) = \int dk \text{tr}(W_\psi Q^\dagger), \] (B5)
so the anti-Hermitian part of \( Q \) does not contribute to the integrals. Thus,
\[ (\hat{R}^\dagger \psi) \hat{L}^\dagger \psi = \int dk \text{tr}(W_\psi R^\dagger L) H. \] (B6)

Because \( \hat{L} \) and \( \hat{R} \) are arbitrary, they can as well be swapped; then one obtains (2.53c).

B.2. Proof of (4.28)
Suppose the dominant term in \( \mu \) in (4.16) has the form \( \mu_h \tau^h \), where \( h \) is a natural number and \( \mu_h = O(\epsilon^2) \). (Here, \( h \) is a power index, so no summation over \( h \) is assumed.) Let us Taylor-expand \( J[A, G] \) in \( \mu_h \):
\[ J[A, G] = J[A, G_0] = O(\epsilon^2) \]
\[ \approx \mu_h \cdot \frac{\partial}{\partial \mu_h} \left( \int dK A(X, K) \lim_{\nu \to 0^+} \int_0^\infty d\tau e^{-\nu \tau + i\Omega \tau + iK \cdot \mu_h \tau^h} \right)_{\mu_h = 0} \]
\[ \approx \mu_h \cdot \int dK A(X, K) \lim_{\nu \to 0^+} \frac{\partial}{\partial \mu_h} \left( \int_0^\infty d\tau e^{-\nu \tau + i\Omega \tau + iK \cdot \mu_h \tau^h} \right)_{\mu_h = 0} \]
\[ \approx i\mu_h \cdot \int dK KA(X, K) \lim_{\nu \to 0^+} \int_0^\infty d\tau \tau^h e^{-\nu \tau + i\Omega \tau} \]
\[ \approx i^{1-h} \mu_h \cdot \int dK KA(X, K) \frac{d^h G_0(\Omega(X, K))}{d\Omega^h} \]
\[ \approx i^{1-h} \mu_h \cdot \frac{\partial}{\partial \Omega^h} \int dK KA(X, K) G_0(\Omega(X, K)). \] (B7)

Provided that \( A \) is sufficiently smooth and well behaved, the overall coefficient here is \( O(1) \), so \( J[A, G] - J[A, G_0] = O(\mu_h) + O(\epsilon^2) \). Because \( \mu_h = O(\epsilon^2) \), this proves (4.28).

B.3. Proof of (5.4)
Here, we show that
\[ \text{symb}_x(\hat{u}^a \hat{G} \hat{u}^\beta) \]
\[ = \frac{1}{(2\pi)^N} \int dS e^{-ik \cdot S} (X + S/2|\hat{u}^a \hat{G} \hat{u}^\beta|X - S/2) \]
\[ \frac{1}{(2\pi)^N} \int dX' dK'' dK' dS dS' W_{u}^{\alpha\beta}(X', K') G(X', K'') e^{-iK\cdot S + i(K'' + K')\cdot S'} \times \delta(X + S/2 - X' - S'/2) \delta(X - S/2 - X' + S'/2) \]

\[ = \frac{1}{(2\pi)^N} \int dX' dK' dK'' dS dS' W_{u}^{\alpha\beta}(X', K') G(X', K'') e^{-iK\cdot S + i(K'' - K)\cdot S} \]

\[ = \frac{1}{(2\pi)^N} \int dK' dK'' dS W_{u}^{\alpha\beta}(X, K') G(X, K'') e^{i(K'' - K)\cdot S} \delta(X - X') \]

\[ = \int dK' W_{u}^{\alpha\beta}(X, K') G(X, K - K'). \quad (B8) \]

**B.4. Proof of (5.10)**

Using (4.19), (4.20) and (2.84) in application to \( W_{u}^{\alpha\beta} \), one finds that

\[ D_{0}^{\alpha\beta}(X) \doteq \int dK \bar{W}_{u}^{\alpha\beta}(X, K) G^{*}(X, K) \]

\[ = \int dK \bar{W}_{u}^{\alpha\beta}(X, -K) G^{*}(X, -K) \]

\[ = \int dK \bar{W}_{u}^{\alpha\beta}(X, -K) G(X, K) \]

\[ = \int dK \bar{W}_{u}^{\alpha\beta}(X, K) G(X, K) \]

\[ = (D_{0}^{\alpha\beta}(X))^*, \quad (B9) \]

and also

\[ \Theta^{\alpha\beta\gamma}(X) \doteq - \int dK \bar{W}_{u}^{\alpha\beta}(X, K) (G^{\gamma\gamma}(X, K))^* \]

\[ = \int dK \bar{W}_{u}^{\alpha\beta}(X, K) G^{\gamma\gamma}(X, -K) \]

\[ = \int dK \bar{W}_{u}^{\alpha\beta}(X, -K) G^{\gamma\gamma}(X, K) \]

\[ = \int dK (\bar{W}_{u}^{\alpha\beta}(X, K))^* G^{\gamma\gamma}(X, K) \]

\[ = -(\Theta^{\alpha\beta\gamma}(X))^*. \quad (B10) \]

**B.5. Proof of (5.20)**

Let us estimate

\[ \mathcal{L}^{(1)} \mathcal{F} = \frac{\partial}{\partial z^{\alpha}} \left( J^{\alpha\mu} J^{\beta\nu} P_{\mu\nu}^{(1)} \frac{\partial \mathcal{F}}{\partial z^{\beta}} \right), \quad (B11) \]
where $P^{(1)}_{\mu\nu}$ has the form

$$
P^{(1)}_{\mu\nu} = \int dK \, q_{\nu} \frac{\partial W(X, K)}{\partial z^{\mu}} G(\Omega(X, K)). \tag{B12}
$$

First, notice that

$$
P^{(1)}_{\mu\nu} = \frac{\partial}{\partial z^{\mu}} \int dK \, q_{\nu} W G - \int dK \, q_{\nu} W \frac{\partial G}{\partial z^{\mu}}
= \frac{\partial}{\partial z^{\mu}} \int dK \, q_{\nu} W G + \frac{\partial v^{\lambda}}{\partial z^{\mu}} \int dK \, q_{\nu} q_{\lambda} W G'
= \frac{\partial}{\partial z^{\mu}} \int dK \, q_{\nu} W G + \frac{\partial v^{\lambda}}{\partial z^{\mu}} \frac{\partial}{\partial \Omega} \int dK \, q_{\nu} q_{\lambda} W G
\equiv \frac{\partial Q^{(1)}_{\nu}}{\partial z^{\mu}} + \frac{\partial v^{\lambda}}{\partial z^{\mu}} R^{(1)}_{\lambda \nu}. \tag{B13}
$$

Because $Q^{(1)}_{\nu}$ and $R^{(1)}_{\lambda \nu}$ are $O(\epsilon^2)$, one has $P^{(1)}_{\mu\nu} \sim \kappa_\mu \epsilon^2$, where $\kappa_\mu$ is the characteristic inverse scale along the $\mu$th phase-space axis. Thus,

$$
L^{(1)} f \sim (J^{\alpha \mu} \kappa_{\alpha} \kappa_{\mu}) J^{\beta \nu} \kappa_{\beta} \epsilon^2 \tilde{f} = O(\epsilon \epsilon^2), \tag{B14}
$$

where we used (see (2.69) and (3.2))

$$
J^{\alpha \mu} \kappa_{\alpha} \kappa_{\mu} \sim \kappa_\kappa \kappa_\rho = O(\epsilon). \tag{B15}
$$

The first part of (5.20) is obtained by considering $i m L^{(1)} \tilde{f}$ and using (B14).

Let us also estimate

$$
L^{(2)} \tilde{f} = \frac{\partial}{\partial z^{\alpha}} \left( J^{\alpha \mu} J^{\beta \nu} P^{(2)}_{\mu\nu} \frac{\partial \tilde{f}}{\partial z^{\beta}} \right), \tag{B16}
$$

where $P^{(2)}_{\mu\nu}$ has the form

$$
P^{(2)}_{\mu\nu} = \int dK \frac{\partial^2 W}{\partial z^{\mu} \partial z^{\nu}} G(\Omega(X, K)). \tag{B17}
$$

First, note that

$$
P^{(2)}_{\mu\nu} = \int dK \frac{\partial^2 W}{\partial z^{\mu} \partial z^{\nu}} G
= \frac{\partial}{\partial z^{\mu}} \int dK \frac{\partial W}{\partial z^{\nu}} G - \int dK \frac{\partial W}{\partial z^{\nu}} \frac{\partial G}{\partial z^{\mu}}
= \frac{\partial^2}{\partial z^{\mu} \partial z^{\nu}} \int dK W G - \frac{\partial}{\partial z^{\mu}} \int dK W \frac{\partial G}{\partial z^{\nu}} - \frac{\partial}{\partial z^{\nu}} \int dK W \frac{\partial G}{\partial z^{\mu}} + \int dK W \frac{\partial^2 G}{\partial z^{\mu} \partial z^{\nu}}
$$
\[
\frac{\partial^2}{\partial z^\mu \partial z^\nu} \int dK \bar{W} G - \frac{\partial}{\partial z^\mu} \int dK \bar{W} G' \left( -q_1 \frac{\partial v^4}{\partial z^\nu} \right) \\
- \frac{\partial}{\partial z^\nu} \int dK \bar{W} G' \left( -q_1 \frac{\partial v^4}{\partial z^\mu} \right) + \int dK \bar{W} \frac{\partial^2 G}{\partial z^\mu \partial z^\nu} \\
= \frac{\partial^2}{\partial z^\mu \partial z^\nu} \int dK \bar{W} G + \frac{\partial}{\partial z^\mu} \left( \frac{\partial v^4}{\partial z^\nu} \partial \Omega \right) \int dK q_\lambda \bar{W} G \\
+ \frac{\partial}{\partial z^\nu} \left( \frac{\partial v^4}{\partial z^\mu} \partial \Omega \right) \int dK q_\lambda \bar{W} G + \int dK \bar{W} \frac{\partial^2 G}{\partial z^\mu \partial z^\nu}. \quad \text{(B18)}
\]

Next, note that
\[
\int dK \bar{W} \frac{\partial^2 G}{\partial z^\mu \partial z^\nu} = \int dK \bar{W} \frac{\partial}{\partial z^\mu} \left( -q_1 G' \frac{\partial v^4}{\partial z^\nu} \right) \\
= -\frac{\partial v^4}{\partial z^\nu} \int dK q_\lambda \bar{W} \frac{\partial G'}{\partial z^\mu} - \frac{\partial^2 v^4}{\partial z^\mu \partial z^\nu} \int dK q_\lambda \bar{W} G' \\
= \frac{\partial v^4}{\partial z^\nu} \frac{\partial \bar{G}''}{\partial \Omega} \int dK q_\lambda q_\delta \bar{W} G - \frac{\partial^2 v^4}{\partial z^\mu \partial z^\nu} \frac{\partial \bar{G}''}{\partial \Omega} \int dK q_\lambda \bar{W} G. \quad \text{(B19)}
\]

Assuming the notation
\[
S^{(2)} = \int dK \bar{W} G = \mathcal{O} (\epsilon^2), \\
Q^{(2)}_\lambda = \frac{\partial}{\partial \Omega} \int dK q_\lambda \bar{W} G = \mathcal{O} (\epsilon^2), \\
R^{(2)}_{\lambda \delta} = \frac{\partial^2}{\partial \Omega^2} \int dK q_\lambda q_\delta \bar{W} G = \mathcal{O} (\epsilon^2), \quad \text{(B20)}
\]

one can then rewrite \( \mathcal{P}^{(2)}_{\mu \nu} \) as follows:
\[
\mathcal{P}^{(2)}_{\mu \nu} = \frac{\partial^2 S^{(2)}}{\partial z^\mu \partial z^\nu} + \frac{\partial}{\partial z^\mu} \left( \frac{\partial v^4}{\partial z^\nu} Q^{(2)}_\lambda \right) + \frac{\partial}{\partial z^\nu} \left( \frac{\partial v^4}{\partial z^\mu} Q^{(2)}_\lambda \right) - \frac{\partial^2 v^4}{\partial z^\mu \partial z^\nu} Q^{(2)}_\lambda + \frac{\partial v^4}{\partial z^\nu} \frac{\partial v^4}{\partial z^\mu} R^{(2)}_{\lambda \delta}. \\
= \frac{\partial^2 S^{(2)}}{\partial z^\mu \partial z^\nu} + \frac{\partial^2 v^4}{\partial z^\mu \partial z^\nu} Q^{(2)}_\lambda + \frac{\partial v^4}{\partial z^\nu} \frac{\partial Q^{(2)}_\lambda}{\partial \Omega} + \frac{\partial v^4}{\partial z^\mu} \frac{\partial Q^{(2)}_\lambda}{\partial \Omega} + \frac{\partial^2 v^4}{\partial z^\mu \partial z^\nu} Q^{(2)}_\lambda + \frac{\partial v^4}{\partial z^\nu} \frac{\partial v^4}{\partial z^\mu} R^{(2)}_{\lambda \delta}. \\
\]

Each term on the right-hand side of this equation scales as \( \epsilon^2 \kappa_\mu \kappa_\nu \), so
\[
\mathcal{L}^{(2)} \bar{f} \sim (J^{\alpha \mu} \kappa_\alpha \kappa_\mu)(J^{\beta \nu} \kappa_\beta \kappa_\nu) \epsilon^2 \bar{f} \sim \epsilon^2 \bar{f}, \quad \text{(B21)}
\]
where we again used (B15). The second part of (5.20) is obtained by considering \( \text{re} \mathcal{L}^{(2)} \bar{f} \) and using (B21).
B.6. Proof of (5.25)

Using (5.23) and assuming the notation \(d_t \equiv \partial_t + v^\gamma \partial_\gamma\), one finds that

\[
\partial_\alpha (\hat{D}^{\alpha \beta} \partial_\beta \tilde{f}) - \partial_\alpha ( (D^{\alpha \beta} + \varrho^{\alpha \beta}) \partial_\beta \tilde{f} )
= -\partial_\alpha \left( \Theta^{\alpha \beta} d_\beta \partial_\beta \tilde{f} + \frac{1}{2} (d_\beta \Theta^{\alpha \beta}) \partial_\beta \tilde{f} \right)
= -\partial_\alpha \left( \frac{1}{2} \Theta^{\alpha \beta} d_\beta \partial_\beta \tilde{f} + \frac{1}{2} \partial_\alpha (\Theta^{\alpha \beta} v^\gamma) \partial_\gamma \tilde{f} \right) - \partial_\alpha \left( \frac{1}{2} d_\beta (\Theta^{\alpha \beta} \partial_\beta \tilde{f}) \right)
= -\partial_\alpha \left( \frac{1}{2} \Theta^{\alpha \beta} \partial_\beta d_\gamma \tilde{f} \right) - \partial_\alpha \frac{1}{2} d_\alpha (\Theta^{\alpha \beta} \partial_\beta \tilde{f})

\]

Because \(\Theta^{\alpha \beta} = \mathcal{O}(\varepsilon^2)\) and \(d_\gamma \tilde{f} = \mathcal{O}(\varepsilon^2)\), the first term on the right-hand side of (B22) is negligible. Also note that due to (3.15), the factor \(\partial_\alpha v^\gamma\) in the last term on the right-hand side of (B22) commutes with \(\partial_\gamma\). Hence, one obtains

\[
\partial_\alpha (\hat{D}^{\alpha \beta} \partial_\beta \tilde{f}) - \partial_\alpha (D^{\alpha \beta} \partial_\beta \tilde{f}) + d_t \left( \frac{1}{2} \partial_\alpha (\Theta^{\alpha \beta} \partial_\beta \tilde{f}) \right)
= \partial_\alpha \left( \Theta^{\alpha \beta} \partial_\beta \tilde{f} + \frac{1}{2} \Theta^{\alpha \beta} (\partial_\beta v^\gamma) \partial_\gamma \tilde{f} \right) - \partial_\gamma \left( \frac{1}{2} \Theta^{\alpha \beta} (\partial_\alpha v^\gamma) \partial_\beta \tilde{f} \right)
= \partial_\alpha \left( \Theta^{\alpha \beta} \partial_\beta \tilde{f} + \frac{1}{2} (\Theta^{\alpha \gamma} (\partial_\gamma v^\beta) - \Theta^{\beta \gamma} (\partial_\gamma v^\alpha)) \partial_\beta \tilde{f} \right)
= \partial_\alpha (U^{\alpha \beta} \partial_\beta \tilde{f}).
\]

Next, notice that

\[
\varrho^{\alpha \beta} = -\frac{1}{2} J^{\alpha \mu} J^{\beta \nu} \int dK \left( q_\nu \frac{\partial W}{\partial z^\mu} - q_\mu \frac{\partial W}{\partial z^\nu} \right) \frac{1}{\Omega}
= -\frac{1}{2} \int dK \frac{q_\nu}{\Omega} \frac{\partial W}{\partial z^\mu} + \frac{1}{2} J^{\alpha \mu} J^{\beta \nu} \int dK q_\mu \frac{\partial W}{\partial z^\nu}
= -\frac{1}{2} \int dK \frac{q_\nu}{\Omega} \frac{\partial W}{\partial z^\mu} - \int dK q_\nu \frac{\partial W}{\partial z^\mu} \frac{\partial}{\partial z^\nu} \frac{1}{\Omega}
\]

\[
- \frac{\partial}{\partial z^\nu} \int dK q_\mu \frac{\partial W}{\Omega} + \int dK q_\mu \frac{\partial W}{\Omega} \frac{\partial}{\partial z^\nu} \frac{1}{\Omega}
\]


$$= -\frac{1}{2} J^\alpha J^\beta \left( \frac{\partial}{\partial z^\mu} \int dK \frac{q_\mu W}{\Omega} + \frac{\partial v^\lambda}{\partial z^\mu} \frac{\partial}{\partial \Omega} \int dK \frac{q_\mu q_\lambda W}{\Omega} \right. \left. - \frac{\partial}{\partial z^\nu} \int dK \frac{q_\nu W}{\Omega} - \frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial}{\partial \Omega} \int dK \frac{q_\lambda q_\nu W}{\Omega} \right). \quad (B24)$$

Assuming the notation

$$Q_\mu = \frac{1}{2} \int dK \frac{q_\mu W}{\Omega}, \quad R_{\mu \nu} = \frac{1}{2} \frac{\partial}{\partial \Omega} \int dK \frac{q_\mu q_\nu W}{\Omega}, \quad (B25)$$

one can rewrite (B24) compactly as follows:

$$\rho_{\alpha \beta} = J^\alpha J^\beta \left( \partial_\nu Q_\mu - \partial_\mu Q_\nu \right) + \left( \frac{\partial}{\partial \Omega} \int dK \frac{q_\mu q_\nu W}{\Omega} \right). \quad (B26)$$

Notice also that

$$\Theta^{\alpha \beta} = 2 J^\alpha J^\beta R_{\mu \nu}. \quad \text{Hence, for } U^{\alpha \beta} \text{ introduced in (B23), one has}$$

$$U^{\alpha \beta} = J^\alpha J^\beta \left( \partial_\nu Q_\mu - \partial_\mu Q_\nu \right) = -U^{\beta \alpha}, \quad (B28)$$

and accordingly,

$$\partial_\alpha U^{\alpha \beta} = (J^\alpha J^\beta - J^\alpha J^\beta) \partial_\alpha^2 Q_\mu = J^\alpha J^\beta \partial_\alpha^2 Q_\mu$$

$$= J^\alpha J^\beta \partial_\alpha^2 Q_\mu = J^\alpha J^\beta \partial_\alpha (J^\mu \partial_\nu Q_\mu) = -J^\alpha \partial_\alpha (J^\mu \partial_\nu Q_\mu) \equiv J^{\alpha \beta} \partial_\alpha \Phi. \quad (B29)$$

Here, \( \Phi = -J^{\mu \nu} \partial_\mu Q_\nu \), which is equivalent to (5.24). From (B29) and the fact that

$$U^{\alpha \beta} \partial_\alpha = 0 \text{ due to the anti-symmetry of } U^{\alpha \beta}, \text{ one has}$$

$$\partial_\alpha (U^{\alpha \beta} \partial_\beta f) = J^{\alpha \beta} (\partial_\alpha \Phi)(\partial_\beta \vec{f}) = \{\Phi, \vec{f}\}. \quad (B30)$$

Hence, (B23) leads to (5.25).
The correlation function
\[ \mathcal{C}_{ss'}(t, x, \tau, s; p, p') = g_s(t + \tau/2, x + s/2, p) g_s(t - \tau/2, x - s/2, p') \]  
(B31)
can be readily expressed as
\[ \mathcal{C}_{ss'} = \left( \delta_{ss'} \sum_{\sigma_i = \sigma_i'} + \sum_{\sigma_i \neq \sigma_i'} \right) \langle \delta(x + s/2 - \bar{X}_\sigma(t + \tau/2)) \delta(p - \bar{p}_\sigma(t + \tau/2)) \rangle \times \delta(x - s/2 - \bar{X}_\sigma'(t - \tau/2)) \delta(p' - \bar{p}_{\sigma'}(t - \tau/2)) - \mathcal{C}_T. \]

Here, \( \langle \ldots \rangle \) is another (in addition to overbar) notation for averaging used in this appendix, the dependence of \( \tilde{T}_s \) on \((t, x)\) is neglected and \( \sigma_s \neq \sigma_s' \) denotes that excluded are the terms that have \( s' = s \) and \( \sigma_s = \sigma_s' \) simultaneously. Aside from this, the summations over \( \sigma_s \) are taken over all \( N_s \gg 1 \) particles of type \( s \), and the summations over \( \sigma_{s'} \) are taken over all \( N_{s'} \gg 1 \) particles of type \( s' \). Also,
\[ \mathcal{C}_T = \langle \tilde{T}_s(t + \tau/2, x + s/2, p) \tilde{T}_{s'}(t - \tau/2, x - s/2, p') \rangle . \]

To the leading order, pair correlations can be neglected. Then,
\[ \sum_{\sigma_i \neq \sigma_i'} \langle \ldots \rangle = \sum_{\sigma_i \neq \sigma_i'} \langle \delta(x + s/2 - \bar{X}_\sigma(t + \tau/2)) \delta(p - \bar{p}_\sigma(t + \tau/2)) \rangle \tilde{T}_s(t + \tau/2, x + s/2, p)/N_s \times \langle \delta(x - s/2 - \bar{X}_{\sigma'}(t - \tau/2)) \delta(p' - \bar{p}_{\sigma'}(t - \tau/2)) \rangle \tilde{T}_{s'}(t - \tau/2, x - s/2, p)/N_{s'} \]
\[ = \frac{\mathcal{C}_T}{N_s N_{s'}} \sum_{\sigma, \sigma'} (1 - \delta_{ss'} \delta_{\sigma, \sigma'}) = (1 - N_{s'}^{-1} \delta_{ss'}) \mathcal{C}_T \approx \mathcal{C}_T. \]  
(B33)

Let us also use \( \bar{p}_\sigma(t + \tau/2) \approx \bar{p}_\sigma(t) \). Then,
\[ \mathcal{C}_{ss'} \approx \delta_{ss'} \delta(p - p') \sum_{\sigma = 1}^{N_s} \langle \delta(x + s/2 - \bar{X}_\sigma(t + \tau/2)) \rangle \times \delta(x - s/2 - \bar{X}_\sigma(t - \tau/2)) \delta(p - \bar{p}_\sigma(t)) \rangle . \]

(B34)

Next, notice that
\[ \langle \delta(x + s/2 - \bar{X}_\sigma(t + \tau/2)) \delta(x - s/2 - \bar{X}_\sigma(t - \tau/2)) \delta(p - \bar{p}_\sigma(t)) \rangle \]
\[ = \langle \delta(s + \bar{X}_\sigma(t - \tau/2) - \bar{X}_\sigma(t + \tau/2)) \delta(x - s/2 - \bar{X}_\sigma(t - \tau/2)) \delta(p - \bar{p}_\sigma(t)) \rangle \]
\[ = \langle \delta(s + \bar{X}_\sigma(t - \tau/2) - \bar{X}_\sigma(t + \tau/2)) \delta(x - (\bar{X}_\sigma(t + \tau/2) + \bar{X}_\sigma(t - \tau/2))/2) \delta(p - \bar{p}_\sigma(t)) \rangle \]
\[ \approx \langle \delta(s - v_s(t, \bar{X}_\sigma, \bar{p}_\sigma) \tau) \delta(x - \bar{X}_\sigma(t)) \delta(p - \bar{p}_\sigma(t)) \rangle \]
\[ \approx \delta(s - v_s(t, x, p) \tau) \langle \delta(x - \bar{X}_\sigma(t)) \delta(p - \bar{p}_\sigma(t)) \rangle \]
\[ = \delta(s - v_s(t, x, p) \tau) \tilde{T}_s(t, x, p)/N_s. \]  
(B35)

Hence,
\[ \mathcal{C}_{ss'} = \delta_{ss'} \delta(p - p') \delta(s - v_s(t, x, p) \tau) F_s(t, x, p), \]  
(B36)
where we used $f_s \approx F_s$. Therefore,

$$
\mathcal{G}_{ss}(t, x, \omega, k; p, p') = \int \frac{d\tau}{2\pi} \frac{ds}{(2\pi)^n} e^{i\omega t - i k \cdot s} \mathcal{C}_{ss'}(t, x, \tau, s; p, p')
\approx \frac{1}{(2\pi)^n} \delta_{ss'} \delta(p - p') F_s(t, x, p).
\quad \text{(B37)}
$$

B.8. Proof of (9.80)

Using the symmetry $U^{\alpha\beta\gamma\delta} = U^{\beta\alpha\gamma\delta} = U^{\beta\alpha\delta\gamma}$, one readily obtains from (6.47) that

$$
\Delta = \frac{1}{2P_0} \int dk \bar{g}_{\beta\gamma} p_\beta p_\gamma U^{\alpha\beta\gamma\delta} + \frac{1}{8} \int dk \mathcal{J},
\quad \text{(B38)}
$$

$$
\mathcal{J} = -\frac{1}{P_0^2} \partial' \cdot \left( \frac{k \mathcal{E}}{P_0^0} \right) - \frac{1}{P_0^0} \frac{\partial}{\partial p_0} \left( \frac{k \mathcal{E}}{P_0^0} \right) - \frac{1}{(P_0^0)^2} \frac{\partial \mathcal{E}}{\partial P_0} + \frac{g_{00}^0 \mathcal{E}}{(P_0^0)^3}.
\quad \text{(B39)}
$$

where $\sigma \equiv k_\rho p^\rho = P^0(k \cdot v - \omega)$ and the prime in $\partial'$ denotes that $p_0$ is considered as a function of $p$ at differentiation. One can also write this as follows:

$$
\mathcal{J} = \frac{k}{\sigma} \cdot \left( \frac{\partial' P_0}{\partial p} \right) - \frac{1}{P_0^0} \frac{\partial}{\partial p_0} \left( \frac{k \mathcal{E}}{P_0^0} \right) - \frac{1}{(P_0^0)^2} \frac{\partial \mathcal{E}}{\partial p_0} + \frac{g_{00}^0 \mathcal{E}}{(P_0^0)^3}.
\quad \text{(B40)}
$$

As shown in Garg & Dodin (2020, Appendix B), the following equality is satisfied:

$$
\frac{k}{\sigma} \cdot \left( \frac{\partial' P_0}{\partial p} \right) = \frac{1}{\sigma} \frac{\partial \sigma}{\partial p_0} - \frac{g_{00}^0}{P_0^0}.
\quad \text{(B41)}
$$

Also notice that

$$
\frac{\partial'}{\partial p} \cdot \left( \frac{k \mathcal{E}}{\sigma} \right) = \frac{\partial}{\partial p} \cdot \left( \frac{k \mathcal{E}}{\sigma} \right) + \frac{\partial P_0}{\partial p} \cdot \frac{\partial}{\partial p_0} \left( \frac{k \mathcal{E}}{\sigma} \right)
\quad = \frac{\partial}{\partial p} \cdot \left( \frac{k \mathcal{E}}{\sigma} \right) - k \cdot v \frac{\partial}{\partial p_0} \left( \frac{\mathcal{E}}{\sigma} \right),
\quad \text{(B42)}
$$

where we used Hamilton’s equation $\partial_p P_0 = -\partial_H = -v$. Therefore,

$$
\mathcal{J} = \frac{1}{\sigma} \frac{\partial \sigma}{\partial p_0} \frac{\mathcal{E}}{(P_0^0)^2} - \frac{1}{P_0^0} \frac{\partial}{\partial p} \left( \frac{k \mathcal{E}}{P_0^0} \right) + \frac{k \cdot v}{P_0^0} \frac{\partial}{\partial p_0} \left( \frac{\mathcal{E}}{\sigma} \right) - \frac{1}{(P_0^0)^2} \frac{\partial \mathcal{E}}{\partial p_0}.
\quad \text{(B43)}
$$

The first and the last terms can be merged; then, one obtains

$$
\mathcal{J} = -\frac{1}{P_0^0} \frac{\partial}{\partial p} \left( \frac{k \mathcal{E}}{\sigma} \right) + \frac{k \cdot v}{P_0^0} \frac{\partial}{\partial p_0} \left( \frac{\mathcal{E}}{\sigma} \right) - \frac{\sigma}{(P_0^0)^2} \frac{\partial}{\partial p_0} \left( \frac{\mathcal{E}}{\sigma} \right)
\quad = -\frac{1}{P_0^0} \frac{\partial}{\partial p} \left( \frac{k_1 \mathcal{E}}{\sigma} \right) + \frac{\omega}{P_0^0} \frac{\partial}{\partial p_0} \left( \frac{\mathcal{E}}{\sigma} \right)
\quad = -\frac{1}{P_0^0} \frac{\partial}{\partial p_1} \left( \frac{k_1 \mathcal{E}}{\sigma} \right).
\quad \text{(B44)}
$$

In combination with (B38), this leads to (9.80).
Appendix C. Properties of the collision operator

Here, we prove the properties of the collision operator discussed in § 6.8. To shorten the calculations, we introduce two auxiliary functions,

\[ Z_{ss'}(k; p, p') = \pi \delta(k \cdot v_s - k \cdot v_{s'}) Q_{ss'}(k \cdot v_s, k; p, p'), \]

\[ F_{ss'}(p, p') = \frac{\partial F_s(p)}{\partial p_j} F_{s'}(p') - F_s(p) \frac{\partial F_{s'}(p')}{\partial p_j}, \tag{C1} \]

which have the following properties:

\[ Z_{ss'}(k; p, p') = Z_{s's}(k; p', p), \quad F_{ss'}(p, p') = -F_{s's}(p', p). \tag{C2} \]

C.1. Momentum conservation

Momentum conservation is proven as follows. Using integration by parts, one obtains

\[ \sum_s \int dp \ p_i C_s = -\sum_{s', s} \int dp \ p_i \int \frac{dk}{(2\pi)^n} d p' k_i k_j Z_{ss'}(k; p, p') F_{ss'}(p, p'). \tag{C3} \]

Now we swap the dummy variables \( s \leftrightarrow s' \) and \( p \leftrightarrow p' \) and then apply (C2):

\[ \sum_s \int dp \ p_i C_s = -\sum_{s', s} \int \frac{dk}{(2\pi)^n} d p' k_i k_j Z_{ss'}(k; p, p') F_{ss'}(p, p'). \tag{C4} \]

The expression on the right-hand side of (C4) is minus that in (C3). Hence, both are zero, which proves that \( \sum_s \int dp \ p_i C_s = 0. \)

C.2. Energy conservation

Energy conservation is proven similarly, using that \( v_s^i = \partial H_s / \partial p_i \) and the fact that \( k \cdot v_s \) and \( k \cdot v_{s'} \) are interchangeable due to the presence of \( \delta(k \cdot v_s - k \cdot v_{s'}) \) in \( Z_{ss'}: \)

\[ \sum_s \int dp \ H_s C_s = \sum_{s, s'} \int dp \ H_s \frac{\partial}{\partial p_i} \int \frac{dk}{(2\pi)^n} d p' k_i k_j Z_{ss'}(k; p, p') F_{ss'}(p, p'). \]
From (6.72) and (6.73), one has

\[ -\sum_{s,s'} \int \frac{dk}{(2\pi)^n} \, dp \, dp' (k \cdot v_s) k_j Z_{ss'}(k; p, p') \mathcal{F}_{ss'}(p, p') \]

Upon comparing (C8) with (C7), one can put the result in a symmetrized form

\[ = -\sum_{s',s} \int \frac{dk}{(2\pi)^n} \, dp' \, dp (k \cdot v_{s'}) k_j Z_{s's}(k; p', p) \mathcal{F}_{s's}(p, p') \]

\[ = \sum_{s',s} \int \frac{dk}{(2\pi)^n} \, dp \, dp' (k \cdot v_s) k_j Z_{ss'}(k; p, p') \mathcal{F}_{ss'}(p, p'). \]  \hspace{1cm} (C5)

Like in the previous case, the third and the fifth lines are minus each other, whence \( \sum_s \int dp \mathcal{H}_s \mathcal{C}_s = 0. \)

C.3. H-theorem

From (6.72) and (6.73), one has

\[ \left( \frac{d\sigma}{dt} \right)_{\text{coll}} = -\sum_s \int dp \left( 1 + \ln F_s(p) \right) \mathcal{C}_s = -\sum_s \int dp \ln F_s(p) \mathcal{C}_s, \] \hspace{1cm} (C6)

where we used particle conservation, \( \int dp \mathcal{C}_s = 0. \) Then,

\[ \left( \frac{d\sigma}{dt} \right)_{\text{coll}} = -\sum_{s,s'} \int dp \ln F_s(p) \frac{\partial}{\partial p_i} \int \frac{dk}{(2\pi)^n} \, dp' k_i k_j Z_{ss'}(k; p, p') \mathcal{F}_{ss'}(p, p') \]

\[ = \sum_{s,s'} \int \frac{dk}{(2\pi)^n} \, dp \, dp' k_i k_j \frac{\partial}{\partial p_i} \ln F_s(p) \mathcal{Z}_{ss'}(k; p, p') \mathcal{F}_{ss'}(p, p'). \] \hspace{1cm} (C7)

Let us swap the dummy variables \( s \leftrightarrow s' \) and \( p \leftrightarrow p' \) and then apply (C2) to obtain

\[ \left( \frac{d\sigma}{dt} \right)_{\text{coll}} = -\sum_{s,s'} \int \frac{dk}{(2\pi)^n} \, \int dp \, dp' k_i k_j \frac{\partial}{\partial p_i} \ln F_s(p) \mathcal{Z}_{ss'}(k; p, p') \mathcal{F}_{ss'}(p, p'). \] \hspace{1cm} (C8)

Upon comparing (C8) with (C7), one can put the result in a symmetrized form

\[ \left( \frac{d\sigma}{dt} \right)_{\text{coll}} = \frac{1}{2} \sum_{s,s'} \int \frac{dk}{(2\pi)^n} \, dp \, dp' k_i k_j Z_{ss'}(k; p, p') \mathcal{F}_{ss'}(p, p') \]

\[ \times \left( \frac{\partial}{\partial p_i} \ln F_s(p) - \frac{\partial}{\partial p_i} \ln F_s(p') \right). \] \hspace{1cm} (C9)

But notice that

\[ \mathcal{F}_{ss'}(p, p') = \left( \frac{\partial}{\partial p_j} \ln F_s(p) - \frac{\partial}{\partial p_j} \ln F_s(p') \right) F_s(p) F_s'(p'). \] \hspace{1cm} (C10)

Thus,

\[ \left( \frac{d\sigma}{dt} \right)_{\text{coll}} = \frac{1}{2} \sum_{s,s'} \int \frac{dk}{(2\pi)^n} \, dp \, dp' \mathcal{Z}_{ss'}(k; p, p') F_s(p) F_s'(p') \]

\[ \times \left( k \cdot \frac{\partial}{\partial p} - k \cdot \frac{\partial}{\partial p'} \right)^2 \geq 0. \] \hspace{1cm} (C11)
Appendix D. Conservation laws for on-shell waves

Here, we prove the momentum-conservation theorem (7.87) and the energy-conservation theorem (7.89) for QL interactions of plasmas with on-shell waves.

D.1. Momentum conservation

Let us multiply (7.83) by $k_l$ and integrate over $k$. Then, one obtains

$$0 = \int dk k_l \frac{\partial J}{\partial t} + \int dk k_l \frac{\partial (v^i_g J)}{\partial x^i} - \int dk k_l \frac{\partial}{\partial k_i} \left( \frac{\partial w}{\partial x^i} J \right) - 2 \int dk k_i \gamma J$$

$$= \frac{\partial}{\partial t} \int dk k_l J + \frac{\partial}{\partial x^i} \int dk k_i v^i_g J + \int dk \frac{\partial w}{\partial x^i} J - 2 \int dk w \gamma J. \quad (D1)$$

Similarly, multiplying (7.84) by $H_s$ and integrating over $p$ yields

$$0 = \int dp p_i \frac{\partial F_s}{\partial t} + \int dp p_i \frac{\partial (v^i_s F_s)}{\partial x^i} - \int dp p_i \frac{\partial}{\partial p_i} \left( \frac{\partial H_s}{\partial x^i} F_s \right)$$

$$- \int dp p_i \frac{\partial}{\partial p_i} \left( D_{s,ij} \frac{\partial F_s}{\partial p_j} \right) - \int dp p_i C_s$$

$$= \frac{\partial}{\partial t} \int dp p_i F_s + \frac{\partial}{\partial x^i} \int dp p_i v^i_s F_s + \frac{\partial}{\partial x^i} \int dp \Delta_s F_s + \int dp \frac{\partial H_{0s}}{\partial x^i} F_s$$

$$- \int dp \Delta_s \frac{\partial F_s}{\partial x^i} + \int dp D_{s,ij} \frac{\partial F_s}{\partial p_j} - \int dp p_i C_s. \quad (D2)$$

Let us sum up (D2) over species and also add it with (D1). The contribution of the collision integral disappears due to (6.71), so one obtains

$$0 = \frac{\partial}{\partial t} \left( \sum_s \int dp p_i F_s + \int dk k_l J \right) + \frac{\partial}{\partial x^i} \left( \sum_s \int dp p_i v^i_s F_s + \int dk k_l v^i_g J \right)$$

$$+ \sum_s \frac{\partial}{\partial x^i} \int dp \Delta_s F_s + \sum_s \int dp \frac{\partial H_{0s}}{\partial x^i} F_s$$

$$+ \sum_s \int dp D_{s,ij} \frac{\partial F_s}{\partial p_j} - 2 \int dk k_l \gamma J$$

$$- \sum_s \int dp \Delta_s \frac{\partial F_s}{\partial x^i} + \int dk \frac{\partial w}{\partial x^i}. \quad (D3)$$

Next, notice that

$$2 \int dk k_l \gamma J = 2\pi \sum_s \int dp \int dk k_l |k_l^i| J \delta(w - k \cdot v_s) \frac{\partial F_s}{\partial p_j}$$

$$= \sum_s \int dp D_{s,ij} \frac{\partial F_s}{\partial p_j}. \quad (D4)$$
Also, assuming that \( \mathcal{E}_0, |\alpha_s^\dagger \eta|^2 \) and \( \eta^\dagger \phi_s \eta \) are independent of \( x \) and using (7.80), one gets
\[
\sum_s \int dp \Delta_s \frac{\partial F_s}{\partial x^i} = \sum_s \int dp \frac{\partial F_s}{\partial x^i} \left( \frac{\partial}{\partial p_i} \int d\omega \frac{k_i}{2(\omega - k \cdot v_s)} |\alpha_s^\dagger \eta|^2 (h(k) + h(-k)) \delta(\omega - w(k)) + \frac{1}{2} \int d\omega \frac{dk}{\omega - k \cdot v_s} \frac{\partial^2 F_s}{\partial x^i \partial p_i} (\eta^\dagger \phi_s \eta F_s) \right)
\]
\[
= -\sum_s \int d\omega \frac{dk}{\omega - w(k)} \frac{\partial}{\partial x^i} \int dp \left( \frac{k_i |\alpha_s^\dagger \eta|^2}{\omega - k \cdot v_s} \frac{\partial F_s}{\partial p_i} - \eta^\dagger \phi_s \eta F_s \right)
\]
\[
= -\sum_s \int d\omega \frac{dk}{\omega - w(k)} \frac{\partial (\eta^\dagger \mathcal{E} \eta)}{\partial x^i}
\]
\[
= -\sum_s \int dk \frac{\partial \Lambda(w(k), k)}{\partial x^i}
\]
\[
= \sum_s \int dk J \frac{\partial w}{\partial x^i}.
\]
(D5)

where we also used (7.75b). Substituting (D4) and (D5) into (D3) leads to (7.87).

D.2. Energy conservation

Let us multiply (7.83) by \( w \) and integrate over \( k \). Then, one obtains
\[
0 = \int dk w \frac{\partial J}{\partial t} + \int dk w \frac{\partial (v_g^i J)}{\partial x^i} - \int dk w \frac{\partial}{\partial k_i} \left( \frac{\partial w}{\partial x^i} J \right) - 2 \int dk w J
\]
\[
= \frac{\partial}{\partial t} \int dk w J - \int dk \frac{\partial w}{\partial t} J + \frac{\partial}{\partial x^i} \int dk w v_g^i J - \int dk \frac{\partial w}{\partial x^i} v_g^i J
\]
\[
+ \int dk v_g^i \frac{\partial w}{\partial x^i} J - 2 \int dk w J
\]
\[
= \frac{\partial}{\partial t} \int dk w J + \frac{\partial}{\partial x^i} \int dk w v_g^i J - \int dk \frac{\partial w}{\partial t} J - 2 \int dk w J.
\]
(D6)

Similarly, multiplying (7.84) by \( \mathcal{H}_s \) and integrating over \( p \) yields
\[
0 = \int dp \mathcal{H}_s \frac{\partial F_s}{\partial t} + \int dp \mathcal{H}_s \frac{\partial (v_g^i F_s)}{\partial x^i} - \int dp \mathcal{H}_s \frac{\partial}{\partial p_i} \left( \frac{\partial \mathcal{H}_s}{\partial x^i} F_s \right)
\]
\[
- \int dp \mathcal{H}_s \frac{\partial}{\partial p_i} \left( D_{s,ij} \frac{\partial F_s}{\partial p_j} \right) - \int dp \mathcal{H}_s C_s
\]
\[
\frac{\partial}{\partial t} \int dp \, \mathcal{H}_s F_s - \int dp \, \frac{\partial \mathcal{H}_s}{\partial t} F_s + \int dp \, v_s \frac{\partial \mathcal{H}_s}{\partial x^i} F_s - \int dp \, \frac{\partial v_s}{\partial t} F_s
\]

\[
\quad + \int dp \, v_s \frac{\partial H_s}{\partial p_j} - \int dp \, \mathcal{H}_s C_s
\]

\[
= \frac{\partial}{\partial t} \int dp \, \mathcal{H}_s F_s + \int dp \, v_s \frac{\partial H_s}{\partial x^i} F_s - \int dp \, \frac{\partial v_s}{\partial t} F_s
\]

\[
\quad + \int dp \, v_s \frac{\partial \Delta_s}{\partial t} F_s
\]

\[
+ \int dp \, \Delta_s \frac{\partial F_s}{\partial t} + \int dp \, v_s \frac{\partial F_s}{\partial p_j} - \int dp \, \mathcal{H}_s C_s. \quad \text{(D7)}
\]

Let us sum up (D7) over species and also add it with (D6). The contribution of the collision integral disappears due to (6.71), so one obtains

\[
0 = \frac{\partial}{\partial t} \left( \sum_s \int dp \, H_0 F_s + \int dk \, wJ \right) + \frac{\partial}{\partial x^i} \left( \sum_s \int dp \, H_0 v_s F_s + \int dk \, w v_s J \right)
\]

\[
\quad + \sum_s \int dp \, \Delta_s v_s F_s - \sum_s \int dp \, \frac{\partial H_0}{\partial t} F_s
\]

\[
+ \sum_s \int dp \, v_s D_s, i \frac{\partial F_s}{\partial p_j} - 2 \int dk \, w \gamma J
\]

\[
\quad + \sum_s \int dp \, \Delta_s \frac{\partial F_s}{\partial t} - \int dk \, J \frac{\partial w}{\partial t}. \quad \text{(D8)}
\]

Next, notice that

\[
2 \int dk \, w \gamma J = 2\pi \sum_s \int dp \, dk \, w k_j \frac{|\alpha_s^+ \eta|^2}{\Delta \omega} J \delta(w - k \cdot v_s) \frac{\partial F_s}{\partial p_j}
\]

\[
= \sum_s \int dp \, v_s D_s, i \frac{\partial F_s}{\partial p_j}. \quad \text{(D9)}
\]

Also, assuming that \( \mathcal{E}_0, |\alpha_s^+ \eta|^2 \) and \( \eta^i \varphi, \eta \) are independent of \( t \) and using (7.80), one gets

\[
\sum_s \int dp \, \Delta_s \frac{\partial F_s}{\partial t}
\]

\[
= \sum_s \int dp \, \frac{\partial F_s}{\partial t} \left( \frac{\partial}{\partial p_i} \int d\omega \, dk \, \frac{k_i}{2(\omega - k \cdot v_s)} |\alpha_s^+ \eta|^2 (h(k) + h(-k)) \delta(\omega - w(k)) \right)
\]

\[
\quad + \frac{1}{2} \int d\omega \, dk \, \eta^i \varphi, \eta \right) (h(k) + h(-k)) \delta(\omega - w(k))\]
\[
\begin{aligned}
&= -\frac{1}{2} \sum_s \int d\omega \, dk \, dp \, \delta(h(k) + h(-k)) \frac{k_i |\alpha_i \eta|^2}{\omega - k \cdot v_s} \frac{\partial^2 F_s}{\partial t \partial p_i} \\
&\quad + \frac{1}{2} \sum_s \int d\omega \, dk \, dp \, \delta(h(k) + h(-k)) \frac{\partial}{\partial t} (\eta^+ \varphi \eta F_s) \\
&= -\sum_s \int d\omega \, dk \, h(k) \frac{\partial}{\partial t} \int dp \left( \frac{k_i |\alpha_i \eta|^2}{\omega - k \cdot v_s} \frac{\partial F_i}{\partial p_i} - \eta^+ \varphi \eta F_s \right) \\
&= -\sum_s \int d\omega \, dk \, h(k) \frac{\partial}{\partial t} \left( \partial \Lambda(w(k), k) \right) \\
&= \sum_s \int dk \, J \frac{\partial w}{\partial t},
\end{aligned}
\]

where we also used (7.75a). Substituting (D9) and (D10) into (D8) leads to (7.89).

Appendix E. Uniqueness of the entropy-preserving distribution

Here, we prove that the Boltzmann–Gibbs distribution is the only distribution for which the entropy density \( \sigma \) is conserved. According to (C11), \( \sigma \) is conserved when

\[
\delta(k \cdot (v_s - v_s')) (k \cdot G_{ss'}(p, p'))^2 = 0
\]

(for all \( p, p' \) and \( k \), as well as all \( s \) and \( s' \)), where

\[
G_{ss'}(p, p') = \frac{\partial \ln F_s (p)}{\partial p} - \frac{\partial \ln F_s (p')}{\partial p'}.
\]

Let us decompose the vector \( G_{ss'}(p, p') \) into components parallel and perpendicular to the vector \( v_s - v_s' \):

\[
G_{ss'}(p, p') = \lambda_{ss'}(v_s, v_s') (v_s - v_s') + G_{ss'}^\perp (p, p'),
\]

where \( \lambda_{ss'}(v_s, v_s') \) is a scalar function. (Because the velocities are functions of the momenta, one can as well consider \( \lambda_{ss'} \) as a function of \( p \) and \( p' \).) Due to the presence of the delta function in (E1), the contribution of the first term to (E1) is zero, so (E1) can be written as

\[
\delta(k \cdot (v_s - v_s')) (k \cdot G_{ss'}^\perp (p, p'))^2 = 0.
\]

By considering this formula for \( k \) parallel to \( G_{ss'}^\perp (p, p') \) (and thus perpendicular to \( v_s - v_s' \)), one finds that \( G_{ss'}^\perp (p, p') = 0 \). Combined with (E2) and (E3), this yields

\[
\frac{\partial \ln F_s (p)}{\partial p} - \frac{\partial \ln F_s (p')}{\partial p'} = \lambda_{ss'}(v_s, v_s') (v_s - v_s').
\]

Also, by swapping \( p \leftrightarrow p' \) and \( s \leftrightarrow s' \), one finds that

\[
\lambda_{ss'}(v_s, v_s') = \lambda_{s's}(v_s', v_s).
\]
Equation (E5) yields, in particular, that
\[
\frac{\partial \ln F_s(p)}{\partial p_2} - \frac{\partial \ln F_s(p')}{\partial p_2'} = \lambda_{ss'}(v_s, v'_s) (v_{s,2} - v'_{s,2}),
\]  
(E7a)
\[
\frac{\partial \ln F_s(p)}{\partial p_3} - \frac{\partial \ln F_s(p')}{\partial p_3'} = \lambda_{ss'}(v_s, v'_s) (v_{s,3} - v'_{s,3}),
\]  
(E7b)
where we have assumed some coordinate axes in the momentum and velocity space labelled (1, 2, 3, …). Then,
\[
\frac{\partial^2 \ln F_s(p)}{\partial p_2 \partial v_{s,1}} = \frac{\partial \lambda_{ss'}(v_s, v'_s)}{\partial v_{s,1}} (v_{s,2} - v'_{s,2}),
\]  
(E8a)
\[
\frac{\partial^2 \ln F_s(p)}{\partial p_3 \partial v_{s,1}} = \frac{\partial \lambda_{ss'}(v_s, v'_s)}{\partial v_{s,1}} (v_{s,3} - v'_{s,3}),
\]  
(E8b)
where the derivative with respect to \(v_{s,1}\) is taken at fixed \(v_{s,i} \neq 1\) and at fixed \(v'_s\). Due to (E5), \(\lambda_{ss'}(v_s, v'_s)\) is continuous for all \(F_s\) and \(F_{s'}\). (Here, we consider only physical distributions, which are always differentiable.) Then, (E8) leads to
\[
\frac{1}{v_{s,2} - v'_{s,2}} \frac{\partial^2 \ln F_s(p)}{\partial p_2 \partial v_{s,1}} = \frac{1}{v_{s,3} - v'_{s,3}} \frac{\partial^2 \ln F_s(p)}{\partial p_3 \partial v_{s,1}}.
\]  
(E9)
By differentiating this with respect to \(v'_{s,2}\), one obtains
\[
\frac{\partial^2 \ln F_s(p)}{\partial p_2 \partial v_{s,1}} = 0,
\]  
(E10)
whence (E8a) yields
\[
\frac{\partial \lambda_{ss'}(v_s, v'_s)}{\partial v_{s,1}} = 0.
\]  
(E11)
By repeating this argument for other axes and for \(v'\) instead of \(v\), one can also extend (E11) to
\[
\frac{\partial \lambda_{ss'}(v_s, v'_s)}{\partial v} = 0, \quad \frac{\partial \lambda_{ss'}(v_s, v'_s)}{\partial v'} = 0.
\]  
(E12)
Hence, \(\lambda_{ss'}(v_s, v'_s)\) is actually independent of the velocities; i.e. \(\lambda_{ss'}(v_s, v'_s) = \lambda_{ss'}\). Using this along with (E6), one also finds that
\[
\lambda_{ss'} = \lambda_{s's'}.
\]  
(E13)
Let us rewrite (E5) as follows:
\[
\frac{\partial \ln F_s(p)}{\partial p} - \lambda_{ss'} v_s = \frac{\partial \ln F_s(p')}{\partial p'} - \lambda_{s's'} v'_s.
\]  
(E14)
Here, the left-hand side is independent of \(p'\) and the right-hand side is independent of \(p\), so both must be equal to some vector
\[
\mu_{ss'} = \mu_{s's'}.
\]  
(E15)

The idea of this argument was brought to author’s attention by G. W. Hammett and is taken from Landreman (2017), where it is applied to single-species plasmas with a specific \(H_i\).
that is independent of both \( p \) and \( p' \). Because \( v_s = \partial_p H_s \), this is equivalent to

\[
\ln F_s(p) - \lambda_{ss'} H_s(p) = \mu_{ss'} \cdot p + \eta_{ss'}
\]

(E16a)

(and similarly for \( p' \)), where the integration constant \( \eta_{ss'} \) is independent of both \( p \) and \( p' \). This is supposed to hold for any \( s' \), so one can also write

\[
\ln F_s(p) - \lambda_{ss''} H_s(p) = \mu_{ss''} \cdot p + \eta_{ss''}.
\]

(E16b)

where \( s'' \) is any other species index. Subtracting equations (E16) from each other gives

\[
(\lambda_{ss'} - \lambda_{ss''}) H_s(p) = (\mu_{ss'} - \mu_{ss''}) \cdot p + \eta_{ss''} - \eta_{ss'}.
\]

(E17)

By differentiating this with respect to \( p \), one finds

\[
(\lambda_{ss'} - \lambda_{ss''}) v_s = \mu_{ss'} - \mu_{ss''}.
\]

(E18)

By differentiating this further with respect to \( v_s \), one obtains

\[
\lambda_{ss'} = \lambda_{ss''}.
\]

Then, (E18) yields \( \mu_{ss'} = \mu_{ss''} \). Therefore, (E16) can be written as

\[
F_s(p) = \text{const}_s \times \exp(\lambda H_s(p) + \mu \cdot p),
\]

(E20)

which is the Boltzmann–Gibbs distribution (§ 8.1). This proves that a plasma that conserves its entropy density necessarily has the Boltzmann–Gibbs distribution.

**Appendix F. Total momentum and energy**

Here, we show that the total momentum and energy in the OC–wave representation equals the total momentum and energy in the particle–field representation.

**F.1. Non-relativistic electrostatic interactions**

**F.1.1. Momentum**

Assuming the notation \( P_l = \sum_s \int dp p_i f_s \) and using (9.17) for \( \Theta_s \), one can represent the OC momentum density as follows:

\[
\sum_s \int dp p_i F_s = P_l + \frac{1}{2} \sum_s \int dp p_i \frac{\partial}{\partial p_i} \left( \Theta_{s,ij} \frac{\partial f_s}{\partial p_j} \right)
\]

\[
\approx P_l - \frac{1}{2} \sum_s \int dp \Theta_{s,ij} \frac{\partial F_s}{\partial p_j}
\]

\[
= P_l - \int dk h(k) \frac{\partial}{\partial \vartheta} \sum_s e_s^2 \int dp \frac{k_i}{w(k) - k \cdot v_s + \vartheta} \frac{k}{4\pi} \left. \frac{\partial F_s}{\partial p} \right|_{\vartheta=0}
\]

\[
= P_l - \int dk k_i h(k) \frac{\partial}{\partial \vartheta} \left( \frac{k^2 \epsilon|_H (w(k) + \vartheta, k) - 1}{4\pi} \right)_{\vartheta=0}
\]

\[
= P_l - \int dk k_i \epsilon|_H
\]

(F1)

where we substituted (9.16). This leads to (9.18).
F.1.2. Energy

Assuming the notation $\mathcal{K} \doteq \sum_s \int dp H_{0s}^f$ and using (9.17) for $\Theta_s$, one can represent the OC energy density as follows:

$$
\sum_s \int dp H_{0s} F_s = \mathcal{K} + \frac{1}{2} \sum_s \int dp \frac{p^2}{2m_s} \frac{\partial}{\partial p_i} \left( \Theta_{s,ij} \frac{\partial f_s}{\partial p_j} \right)
$$

$$
\approx \mathcal{K} - \frac{1}{2} \sum_s \int dp v^i \Theta_{s,ij} \frac{\partial F_s}{\partial p_j}
$$

$$
= \mathcal{K} - \int dk h(k) \frac{\partial}{\partial \vartheta} \sum_s \epsilon_s^2 \int dp \frac{k \cdot v}{w(k) - k \cdot v_s + \vartheta} \left. \frac{k \cdot \nabla}{\partial p} \frac{\partial F_s}{\partial p} \right|_{\vartheta=0}.
$$

Notice that

$$
\frac{\partial}{\partial \vartheta} \frac{k \cdot v}{w(k) - k \cdot v_s + \vartheta} = \frac{\partial}{\partial \vartheta} \left( -1 + \frac{w(k) + \vartheta}{w(k) - k \cdot v_s + \vartheta} \right)
$$

$$
= \frac{1}{w(k) - k \cdot v_s + \vartheta} + w(k) \frac{\partial}{\partial \vartheta} \frac{1}{w(k) - k \cdot v_s + \vartheta}.
$$

Then,

$$
\sum_s \int dp H_{0s} F_s = \mathcal{K} - \int dk w(k) h(k) \frac{\partial}{\partial \vartheta} \sum_s \epsilon_s^2 \int dp \frac{k}{w(k) - k \cdot v_s + \vartheta} \left. \frac{k \cdot \nabla}{\partial p} \frac{\partial F_s}{\partial p} \right|_{\vartheta=0}
$$

$$
- \int dk h(k) \sum_s \epsilon_s^2 \int dp \frac{k}{w(k) - k \cdot v_s + \vartheta} \left. \frac{k \cdot \nabla}{\partial p} \frac{\partial F_s}{\partial p} \right|_{\vartheta=0}
$$

$$
= \mathcal{K} - \int dk w(k) h(k) \frac{\partial}{\partial \vartheta} \left( \frac{k^2 (\epsilon_{1H}(w(k) + \vartheta, k) - 1)}{4\pi} \right) \bigg|_{\vartheta=0}
$$

$$
- \int dk h(k) \frac{k^2 (\epsilon_{1H}(w(k), k) - 1)}{4\pi}.
$$

Using (9.14) and (9.16), one obtains that the sum of the OC and wave energy is given by

$$
\sum_s \int dp H_{0s} F_s + \int dk w J = \mathcal{K} + \int dk h(k) \frac{k^2}{4\pi}
$$

$$
= \mathcal{K} + \sum_{\sigma} \frac{\kappa_\sigma^2 |\varphi_\sigma|^2}{16\pi}
$$

$$
= \sum_s \int dp \left( \frac{p^2}{2m_s} + e_s \varphi_s \right) f_s + \frac{1}{8\pi} \tilde{E}^\dagger \tilde{E},
$$

where we also substituted (7.64).
F.2. Relativistic electromagnetic interactions

F.2.1. Momentum

Let us assume the notation \( \mathcal{P} = \sum_s \int dp \, \mathvec{p} \cdot \mathvec{f}_s \) and

\[
\chi(\omega, \mathbf{k}) = \sum_s \frac{4\pi e_s^2}{\omega^2} \int dp \, \frac{\mathbf{v}_s \mathbf{v}_s^\dagger}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \frac{\partial F_s}{\partial \mathbf{p}} = \epsilon(\omega, \mathbf{k}) - 1 + \frac{\mathbf{w}_p}{\omega^2},
\]

(F6)

Then, using (9.52) for \( \Theta_s \), one can represent the OC momentum density as follows:

\[
\sum_s \int dp \, p_i F_s \approx \mathcal{P}_l - \frac{1}{2} \sum_s \int dp \, \Theta_{s,j} \frac{\partial F_s}{\partial p_j} = \mathcal{P}_l - \int \frac{dk \, k_i h(k)}{4\pi w^2(k)} \sum_s e_s^2 \int dp \, \frac{1}{w^2(k)} \frac{(\eta^\dagger \mathbf{v}_s \mathbf{v}_s^\dagger \eta)}{w(k) - \mathbf{k} \cdot \mathbf{v}_s + \hat{\mathbf{r}} \cdot \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}}} \bigg|_{\theta=0}
\]

\[
= \mathcal{P}_l - \int \frac{dk \, k_i h(k)}{2\pi w(k)} \frac{\eta^\dagger \partial}{\partial \omega} \left( \omega^2 \chi(\omega, k) \right) \eta \bigg|_{\omega=w(k)}
\]

\[
= \mathcal{P}_l + \int \frac{dk \, k_i h(k)}{4\pi w(k)} \left( h(k) + h(-k) \right) - \int dk \, k_i J,
\]

(F7)

where we substituted (9.51) and used (7.23). Next, let us rewrite (9.54) as

\[
\mathcal{P} = \mathcal{P}^{(\text{kin})} + \left( \frac{\tilde{A}}{c} \right) \sum_s e_s \int dp \, f_s = \mathcal{P}^{(\text{kin})} + \frac{1}{4\pi c} \tilde{A}(\nabla \cdot \tilde{E}),
\]

(F8)

where the last equality is due to Gauss’s law. This gives

\[
\mathcal{P}_l - \mathcal{P}_l^{(\text{kin})} = \frac{1}{4\pi} \left( -i \hat{\omega}^{-1} \hat{E}_i \right) \left( \frac{\partial}{\partial \omega} \right) \left( \hat{\omega}^{-1} \hat{E}_i \right)^* = -\frac{i}{4\pi} \left( \hat{\omega}^{-1} \hat{E}_i \right) \left( \frac{\partial}{\partial \omega} \right) \left( \hat{\omega}^{-1} \hat{E}_i \right)^*.
\]

(F9)

Then, using (2.53) and also (7.63) for \( \mathbf{U} \), one obtains

\[
\mathcal{P}_l - \mathcal{P}_l^{(\text{kin})} \approx -\frac{i}{4\pi} \int d\omega \, dk \, \omega^{-1} U_j(\omega, k)(ik_j)^* \approx -\int dk \, \frac{k \cdot \eta^*}{4\pi w(k)} \eta_j(h(k) + h(-k)),
\]

(F10)

and thus (F7) can be written as follows:

\[
\sum_s \int dp \, p_i F_s + \int dk \, k J \approx \mathcal{P} + \int dk \, \frac{k}{4\pi w(k)} \left( h(k) + h(-k) \right)
\]

\[
\approx \mathcal{P}^{(\text{kin})} + \int dk \, \frac{1}{4\pi w(k)} (k - \eta (k \cdot \eta^*)) (h(k) + h(-k))
\]

\[
= \mathcal{P}^{(\text{kin})} + \Re \int dk \, \frac{h(k)}{2\pi w(k)} \eta^* \times (k \times \eta),
\]

(F11)
where we used (7.7) and (7.23) again. For an eikonal wave (9.57), which has
\[ h(k) = \delta(k - \overline{k})|\overline{E}|^2/4 \] (§ 7.4.1), this gives
\[
\text{re} \int \frac{dk}{2\pi w(k)} \frac{h(k)}{2\pi w(k)} \eta^* \times (k \times \eta) = \frac{1}{8\pi c} \text{re} \left( \ddot{E}^* \times \left( \frac{c\overline{k}}{\omega} \times \ddot{E} \right) \right) = \frac{\overline{E} \times \overline{B}}{4\pi c}. \quad (F12)
\]

In case of a broadband spectrum, the same equality applies as well, because contributions of the individual eikonal waves to both left-hand side and the right-hand side are additive. (Alternatively, one can invoke (2.53) again.) This leads to (9.53).

F.2.2. Energy

Assuming the notation \( K \triangleq \sum_s \int dp H_{0s} \overline{f}_s \) and using (9.52) for \( \Theta_s \), one can represent the OC energy density as follows:
\[
\sum_s \int dp H_{0s} F_s \approx K - \frac{1}{2} \sum_s \int dp v_s^i \Theta_{s,ij} \frac{\partial F_s}{\partial p_j} \approx K - \int dk h(k) \left( \frac{\partial}{\partial \vartheta} \sum_s 4\pi e_s^2 \int dp \frac{v_s v_s^i}{w^2(k) - k \cdot v_s + \vartheta} k \cdot \frac{\partial F_s}{\partial p} \right)_{\vartheta = 0}.
\]

Using (F3) and (F6) for \( \chi \), one further obtains
\[
\sum_s \int dp H_{0s} F_s = K - \int dk h(k) \frac{\partial}{\partial \vartheta} \left( \sum_s 4\pi e_s^2 \int dp \frac{v_s v_s^i}{w^2(k) - k \cdot v_s + \vartheta} k \cdot \frac{\partial F_s}{\partial p} \right)_{\vartheta = 0} - \int dk h(k) \left( \sum_s 4\pi e_s^2 \int dp \frac{v_s v_s^i}{w^2(k) - k \cdot v_s} \frac{\partial F_s}{\partial p} \right)_{\vartheta = 0} = K
\]
\[
- \int dk h(k) (\sum_s 4\pi e_s^2 \int dp \frac{v_s v_s^i}{w^2(k) - k \cdot v_s} \frac{\partial F_s}{\partial p})_{\vartheta = 0}.
\]

Using (9.50) and proceeding as in § F.2.1, one can also cast this as follows:
\[
\sum_s \int dp H_{0s} F_s + \int dk wJ = K + \frac{1}{8\pi} (3\ddot{E}^* \dot{E} - \ddot{B}^* \dot{B}) - \frac{1}{8\pi c^2} \dot{A}^* \omega \overline{A}. \quad (F13)
\]
Now, notice that
\[
\mathcal{K} = \sum_s \int dp \frac{H_0(p)f_s^{(\text{kin})}(p - e_\sigma \tilde{A}/c)}{p - e_\sigma \tilde{A}/c}
\]
\[
= \sum_s \int dp H_0(p + e_\sigma \tilde{A}/c)f_s^{(\text{kin})}(p)
\]
\[
\approx \mathcal{K}^{(\text{kin})} + \left(\frac{\tilde{A}}{c}\right) \cdot \sum_s e_\sigma \int dp \psi f_s^{(\text{kin})} + \sum_s \frac{e_\sigma^2}{2c^2} \int dp (\tilde{A} \mu_s^{-1} \tilde{A}) f_s^{(\text{kin})}
\]
\[
\approx \mathcal{K}^{(\text{kin})} + \frac{1}{c} \tilde{A} \cdot \tilde{j} + \frac{1}{8\pi c^2} \tilde{A}^\dagger \tilde{w} \tilde{A}, \quad \text{(F14)}
\]

where \(\tilde{j}\) is the oscillating-current density. From Ampere’s law,
\[
\frac{1}{c} \tilde{A} \cdot \tilde{j} = \frac{(-i\omega)^{-1} \tilde{E})^\dagger}{4\pi} (ic\vec{k} \times \tilde{B} + i\tilde{\omega} \tilde{E})
\]
\[
\approx -\frac{\tilde{E}^\dagger \tilde{E}}{4\pi} - \frac{\tilde{E}}{4\pi} \cdot \left( \frac{ck}{\tilde{\omega}} \times \tilde{B} \right)
\]
\[
\approx -\frac{\tilde{E}^\dagger \tilde{E}}{4\pi} - \left( \frac{\tilde{E}}{4\pi} \times \frac{ck}{\tilde{\omega}} \right) \cdot \tilde{B}
\]
\[
\approx \frac{1}{4\pi} (\tilde{B}^\dagger \tilde{B} - \tilde{E}^\dagger \tilde{E}). \quad \text{(F15)}
\]

Substituting (F14) and (F15) into (F13) leads to (9.55).

**Appendix G. Selected notation**

This paper uses the following notation (also see § 2 for the index convention):

| Symbol | Definition | Explanation |
|--------|------------|-------------|
| \(\hat{\cdot}\) | placeholder | |
| \(\ast\) | complex conjugate | |
| \(-1\) | inverse | |
| \(^\dagger\) | Hermitian adjoint | |
| \(-\) | inverse Hermitian adjoint | |
| \(\dagger\) | transpose | |
| \([a]\) | § 4.2 auxiliary notation | |
| \((\mu)\) | § 6.5 contribution from the microscopic part | |
| \((m)\) | § 6.5 contribution from the macroscopic part | |
| \(\hat{\cdot}\) | average part or, for eikonal waves, a quantity evaluated on the local wavevector | |
| \(\tilde{\cdot}\) | oscillatory part | |
| \(\cdot\) | macroscopic part | |
Quasilinear theory

microscopic part
operator
time derivative
(2.23) Fourier image
envelope of an eikonal (or monochromatic) wave
§ 2.1.2 anti-Hermitian part
§ 2.1.2 Hermitian part
partial derivative (but \( \partial_i \equiv \partial / \partial x^i \), \( \partial_a \equiv \partial / \partial x^a \), \( \partial_a \equiv \partial / \partial X^a \))
partial derivative with respect to a lower-index quantity
(4.27) auxiliary notation
\( d_t \) (3.32), (7.36) convective time derivative
\( \{ \cdot, \cdot \} \) (2.32) Poisson bracket on \((x, k)\)
\( \{ \cdot, \cdot \} \) (2.56) Poisson bracket on \((x, K)\)
commutator
(2.1), (2.59) inner product on \( \mathcal{H}_x \) or on \( \mathcal{H}_X \)
definition
\( \cdot \) § 2.1.3 scalar product
\( \star \) (2.30) Moyal product on \((x, k)\)
\( \star \) (2.72) Moyal product on \((X, K)\)
i times an infinitesimally small positive number
principal-value integral
eigv
im
pv (4.24) auxiliary notation
re
real part
operator corresponding to a Weyl symbol on \((X, K)\)
operator corresponding to a Weyl symbol on \((x, k)\)
same as \( \text{symb}_X \) or \( \text{symb}_x \) when the two are equal
Weyl symbol of an operator on \( \mathcal{H}_X \)
Weyl symbol of an operator on \( \mathcal{H}_x \)
sign
trace
part of a collision operator
particle’s total ponderomotive energy in on-shell waves
auxiliary notation
(5.9) dressing function (see § 5.3)
dressing function (a part of \( \Theta^{a\beta} \))
dispersion function (one of \( \Lambda_b \))
\( b \)th eigenvalue of \( \Xi \)
OC momentum flux density of species \( s \)
wave momentum flux density
\( \Xi \) § 6.3 dispersion matrix
(6.13), (6.17) dispersion operator
vacuum dispersion matrix
vacuum dispersion operator
Weyl symbol of \( \hat{\Xi}_p \)
\( \hat{\Xi}_p \) § 6.1 auxiliary operator
ponderomotive energy
generic interaction field
complexified interaction field
\[ \Omega \] (4.17) auxiliary notation
\[ \alpha_{s,i}, \alpha_s \] § 6.1 Weyl symbols of \( \hat{\alpha}_{s,i} \) and \( \hat{\alpha_s} \)
\[ \hat{\alpha}_{s,i}, \hat{\alpha}_s \] § 6.1 coupling operators
\[ \gamma \] (7.47), (7.87) linear dissipation rate as a function of \((t, x, k)\)
\[ \gamma \] § 7.3 local linear dissipation rate of an eikonal wave
\[ \delta \] Kronecker symbol or delta function
\[ \varepsilon \] § 3.1.1 geometrical-optics parameter
\[ \varepsilon \] (9.42) dielectric tensor
\[ \varepsilon \|, \varepsilon \perp \] (9.9) parallel and transverse permittivity
\[ \varepsilon \] § 3.1.1 small parameter proportional to the oscillation amplitude
\[ \eta \] § 7.1 polarisation vector (one of \( \eta_b \))
\[ \eta_b \] § 7.1 \( b \)th eigenvector of \( \Xi \)
\[ \kappa \] § 7.2.3 auxiliary notation
\[ \kappa_{s,x}, \kappa_p \] (3.2a,b) characteristic inverse scales in \( x \) and \( p \), respectively
\[ \rho_s \] charge density of species \( s \)
\[ \rho^{\alpha\beta} \] (5.18) auxiliary notation
\[ \rho_s \] (6.22a,b) Weyl symbol of \( \hat{\rho}_s \)
\[ \mathcal{G}_s \] (6.22a,b) coupling operator
\[ \sigma \] (6.73) entropy density
\[ \varsigma_k \] (7.83) sign of the action density
\[ \varphi \] electrostatic potential
\[ \psi, \psi \] any field
\[ \omega \] coordinate in the frequency space dual to \( t \)
\[ \overline{\omega} \] \(-\partial_t \theta\) local frequency of an eikonal wave
\[ \hat{\omega} \] \( i\partial_t \) frequency operator
\[ C \] (3.5) Fourier image of \( W \)
\[ C_s, C_s \] (2.77), (2.82) Fourier images of \( W_s \) and \( W_s \)
\[ C_s \] § 6.8 collision operator of species \( s \)
\[ C_s, C_s \] (2.44), (2.52) Fourier images of \( W_s \) and \( W_s \)
\[ D^{\alpha\beta} \] § 5.1 Weyl symbol of \( \hat{D}^{\alpha\beta} \)
\[ D^{\alpha\beta} \] (5.17) phase-space-diffusion coefficient
\[ D_{ij}, D_s \] (5.42), (6.77a) momentum-diffusion coefficient (part of \( D^{\alpha\beta} \))
\[ D_{ij}^{\alpha\beta}, D_s^{\alpha\beta} \] (5.8) auxiliary notation
\[ D_{ij}^{\alpha\beta} \] (3.34a,b) diffusion operator on \( \mathcal{H}_X \)
\[ E_s \] table 1 OC energy density of species \( s \)
\[ E_w \] table 1 wave energy density (also see (7.42a,b) for eikonal waves)
\[ F, F_s \] (5.28), (5.39) OC distribution functions
\[ \mathcal{G}_s \] (6.60) polarisation drag for species \( s \)
\[ G, G_s \] § 4.2 Weyl symbols of \( \hat{G} \) and \( \hat{G}_s \)
\[ G_0 \] (4.23) approximation of \( G \) to the zeroth order in \( \varepsilon \)
\[ \hat{G}, \hat{G}_s \] (3.25) effective Green’s operators on \( \mathcal{H}_X \)
\[ \mathcal{G}_s g_s' \] (6.30) spectrum of the correlations between \( g_s \) and \( g_s' \)
\[ \hat{\mathcal{G}} \] § 3.2 Green’s operator on \( \mathcal{H}_X \)
\[ I \] (7.27) action density of an eikonal wave
\[ J \] § 7.4.3 phase-space action density
\[ J^{\alpha\beta} \] (2.57) canonical Poisson structure
\[ J^{\alpha\beta}, J \] (7.28) action flux density of an eikonal wave
\[ H, H_s \] particle Hamiltonian
\[ \mathcal{H}_s \] (5.41) OC Hamiltonian of species \( s \)
Quasilinear theory

\mathcal{X}_x\quad \text{Hilbert space formed by functions on } x

\mathcal{X}_X\quad \text{Hilbert space formed by functions on } X

K (-\omega, q) \quad \text{coordinate in the wavevector space dual to } X

\hat{K} (-\hat{\omega}, \hat{q}) \quad \text{wavevector operator on } \mathcal{X}_X

\hat{L} (6.2) \quad \text{coupling operator}

\hat{L}_X (2.73) \quad \text{same as the Poisson bracket on } (X, K)

\hat{L}_x (6.6) \quad \text{same as the Poisson bracket on } (x, k)

\hat{\Sigma}_0 (6.1a,b) \quad \text{Lagrangian density of } \psi \text{ in vacuum}

M \quad \text{number of components of } \psi \text{ or of another vector field}

N \quad 2n + 1 \quad \text{dimension of the extended phase space } X

N_s, N'_s \quad (5.40) \quad \text{OC density}

O \quad \text{OC density}

P^i_s, P_s \quad \text{table 1} \quad \text{OC momentum density of species } s

P^w_s, P_w \quad \text{table 1} \quad \text{wave momentum density (also see (7.42a,b) for eikonal waves)}

Q^i_s, Q_s \quad \text{table 1} \quad \text{OC energy flux density of species } s

Q^w_s, Q_w \quad \text{table 1} \quad \text{wave energy flux density}

Q_{ss'} (6.8) \quad \text{symmetrised coefficient in the collision operator}

R_s (6.2) \quad \text{coupling operator}

\mathbb{R} \quad \text{real axis}

S \quad \text{§ 7.1} \quad \text{action integral}

S_{ad} (6.16) \quad \text{adiabatic action integral}

\hat{S}_T (3.21) \quad \text{shift operator (see also § 4.1)}

\bar{U}^i_s, \bar{U}_s \quad (6.78) \quad \text{average Wigner function of the macroscopic field } \tilde{\psi}(x)

\bar{U}^i_s, \bar{U}_c \quad \text{§ 4.3} \quad \text{initial conditions for } \tilde{f} \text{ and } \tilde{f}_s

\bar{V}^a_s, \bar{V}_s \quad \text{§ 3.2} \quad \text{unperturbed velocity in the } X \text{ space}

\gamma_n \quad \text{volume of } n \text{-dimensional homogeneous plasma}

W (4.4a,b) \quad \text{Wigner function of } \hat{H}(X)

W^s_s, W^s_s \quad (2.76), (2.81) \quad \text{Weyl symbol of } \hat{W}^s_s \text{ (Wigner function or matrix)}

\hat{W} (4.3a,b) \quad \text{density operator on } \mathcal{X}_X \text{ of } \hat{H}

\hat{W}^s_s, \hat{W}_s \quad (2.75), (2.80) \quad \text{density operator on } \mathcal{X}_X \text{ of a given field}

\hat{W}, \hat{W}_s \quad \text{§ 5.6} \quad \text{Wigner functions of } \hat{H} \text{ and } \hat{H}_s \text{ with } p \text{ as a parameter}

\hat{W}_s, \hat{W}_s \quad (2.43), (2.51) \quad \text{Weyl symbol of } \hat{W}_s \text{ (Wigner function or matrix)}

\hat{W}_s, \hat{W}_s \quad (2.42), (2.50) \quad \text{density operator on } \mathcal{X}_X \text{ of a given field}

\hat{\Omega} \quad \text{§ 6.4} \quad \text{average Wigner matrix of the microscopic field } \tilde{\psi}(x)

X^a_s, X_s \quad (t, z) \quad \text{coordinate in the extended phase space}

\hat{X}^a_s, \hat{X}_s \quad (\hat{t}, \hat{z}) \quad \text{operator of the position in the extended phase space}

\hat{X}^s_{ss'} (6.36) \quad \text{Weyl symbol of } \hat{X}^s_{ss'} \text{ that enter } \hat{H}

d \quad \text{differential}

e_s \quad \text{charge of species } s

f, f_s \quad \text{distribution function}

g, g_s \quad \text{§ 4.3} \quad \text{initial conditions for } \tilde{f} \text{ and } \tilde{f}_s

h (7.65), (7.75a,b) \quad \text{rescaled phase-space action density}

h_{c+}, h_c \quad \text{§ 7.4.2} \quad \text{auxiliary notation } (h_{c+} \equiv h_c)

k_i, k \quad \text{coordinate in the wavevector space dual to } x

\hat{k}_i, \hat{k} \quad \partial \theta, \partial_x \theta \quad \text{local wavevector of an eikonal wave}

\hat{k}_i, \hat{k} \quad -i \partial_t, -i \partial_x \quad \text{wavevector operator}
\( \mathbf{k} \) \((-\omega, \mathbf{k})\) coordinate in the wavevector space dual to \( \mathbf{x} \)
\( \mathbf{k} \) \((-\omega, \mathbf{k})\) local spacetime-wavevector of an eikonal wave
\( \mathbf{k} \) \((-\omega, \mathbf{k})\) spacetime-wavevector operator
\(|\mathbf{k}\rangle\) \((2.16a,b)\), \((2.17a,b)\) eigenvector of \( \hat{\mathbf{k}} \) corresponding to the eigenvalue \( \mathbf{k} \)
\( \mathbf{t}_e \) \(\S\ 4.1.1\) displacement in \( \mathbf{X} \) along unperturbed characteristics
\( m_s \) mass of species \( s \)
\( n \) \( \dim \mathbf{x} \) number of spatial dimensions
\( n \) \( n + 1 \) number of spacetime dimensions
\( p_i, \mathbf{p} \) coordinate in the momentum space
\( \hat{p}_i, \hat{\mathbf{p}} \) position operator corresponding to the coordinate \( \mathbf{p} \)
\( q^i, \mathbf{q} \) \((k, \mathbf{r})\) coordinate in the wavevector space dual to \( z \)
\( \hat{q}^i, \hat{\mathbf{q}} \) \((k, \mathbf{r})\) wavevector operator corresponding to the coordinate \( z \)
\( r_i, \mathbf{r} \) \(-i\partial^i, -i\partial_p\) coordinate in the wavevector space dual to \( p \)
\( \hat{r}_i, \hat{\mathbf{r}} \) \(-i\partial^i, -i\partial_p\) wavevector operator corresponding to the coordinate \( p \)
\( s \) species index
\( t \) time
\( \hat{t} \) time operator
\( u^\alpha \) \((3.8a,b)\) oscillating part of the phase-space velocity
\( \hat{u}^\alpha \) \((3.30)\) \( u^\alpha \) as an operator on \( \mathcal{H}_X \)
\( v^i, \mathbf{v} \) \((3.8a,b)\) average velocity in phase space or in physical space,
\( \) \((5.33)\) or, since \( \S\ 5.4, \) OC velocity
\( \nu^g, \mathbf{v}_g \) \((7.25)\) group velocity as a function of \((t, \mathbf{x}, \mathbf{k})\)
\( \nu^g, \mathbf{v}_g \) \((7.29a,b)\) local group velocity of an eikonal wave
\( \) \((7.22)\) eikonal-wave frequency as a function of \((t, \mathbf{x}, \mathbf{k})\)
\( x^i, \mathbf{x} \) coordinate in space
\( \hat{x}^i, \hat{\mathbf{x}} \) ray coordinate in space
\( \hat{x}^i, \hat{\mathbf{x}} \) \( x \) space-position operator
\( x^i, \mathbf{x} \) \((t, \mathbf{x})\) coordinate in spacetime
\( \hat{x}^i, \hat{\mathbf{x}} \) \( x \) spacetime-position operator
\(|\mathbf{x}\rangle\) \((2.16a,b)\), \((2.17a,b)\) eigenvector of \( \hat{\mathbf{x}} \) corresponding to the eigenvalue \( \mathbf{x} \)
\( z^\alpha, \mathbf{z} \) \((x, \mathbf{p})\) coordinate in phase space
\( \hat{z}^\alpha, \hat{\mathbf{z}} \) \((\hat{x}, \hat{\mathbf{p}})\) phase-space position operator

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