Layered black-box, behavioral interconnection perspective and applications to the problem of communication with fidelity criteria, Part II: stationary sources satisfying $\psi$-mixing criterion

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Theorems from Part 1 of this paper are generalized to $\psi$-mixing sources in this paper. Application to Markoff chains and order $m$ Markoff chains is presented. The main result is the generalization of Theorem 1 in Part 1.

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1. Introduction

In this paper, we generalize results from Part I of this paper [1] to the case when the source $X$ is not necessarily i.i.d. but stationary and satisfies a mixing condition, the $\psi$-mixing criterion (in the process, also being ergodic). As a corollary, the results hold for Markoff chains and order $m$ Markoff chains. The main result to be generalized in Theorem 1 of [1] which states the if a set of channels (compound channel) is known to communicate an i.i.d. source within a certain distortion level, then, assuming random codes are permitted, reliable communication can be accomplished over this set of channels at rates less than the rate-distortion function. The channel model is general in the sense of Verdu-Han. The reader is referred to Sections 1-5 in [1] for a discussion, statement and proof of the theorem.

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2. Notation and definitions

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of random variables defined on a probability space $(\Omega, \Sigma, P)$. The range of each $X_i$ is assumed to be a finite set $X$. Denote this sequence of random variables by $X$. Such a sequence is called a source. Further discussion and assumption on the source will be carried out in Section 3.

Sets will be denoted by latex mathbb notation, example, $\mathbb{X}, \mathbb{Y}$, and random variables by basic mathematical notation, for example $X, Y$. Sigma fields will be denoted by mathcal notation for example, $\mathcal{S}$.

The source space at each time, as stated before, is $X$, and is assumed to be a finite set. The source reproduction space is denoted by $Y$ which is assumed to be a finite set. Assume that $X = Y$.

$d : X \times Y \rightarrow [0, \infty)$ is the single-letter distortion measure. Assume that $d(x, x) = 0 \ \forall x \in X$.

For $x^n \in X^n, y^n \in Y^n$, the $n$-letter rate-distortion measure is defined additively:

$$d^n(x^n, y^n) \triangleq \sum_{i=1}^{n} d(x^n(i), y^n(i))$$

where $x^n(i)$ denotes the $i^{th}$ component of $x^n$ and likewise for $y^n$.

$(X_1, X_2, \ldots, X_n)$ will be denoted by $X^n$.

A rate $R$ source-code with input space $\mathbb{X}$ and output space $\mathbb{Y}$ is a sequence $<e^n, f^n>_{1}^{\infty}$, where $e^n : X^n \rightarrow \{1, 2, \ldots, 2^{nR}\}$ and $f^n : \{1, 2, \ldots, 2^{nR}\} \rightarrow Y^n$.

We say that rate $R$ is achievable for source-coding the source $X$ source within distortion-level $D$ under the expected distortion criterion if there exists a rate $R$ source code $<e^n, f^n>_{1}^{\infty}$ such that

$$(1) \quad \limsup_{n \rightarrow \infty} E \left[ \frac{1}{n} d^n(X^n, f^n(e^n(X^n))) \right] \leq D$$
The infimum of all achievable rates under the expected distortion is an operational rate-distortion function, denoted by $R_{E}^{X}(D)$.

We say that rate $R$ is achievable for source-coding the source $X$ source within distortion-level $D$ under the probability of excess distortion criterion if there exists a rate $R$ source code $<e^{n}, f^{n}>_{\infty}$ such that

$$\lim_{n \to \infty} \Pr \left( \frac{1}{n} \sum_{i=1}^{n} d(s^{n}(X^{i}), f^{n}(e^{n}(X^{i}))) > D \right) = 0$$

(2)

The infimum of all achievable rates under the probability of excess distortion criterion is an operational rate-distortion function, denoted by $R_{P}^{X}(D)$.

We used $\lim \sup$ in (1) and $\lim$ in (2); in (2), we can equivalently use $\lim \sup$. This is because for a sequence of non-negative real numbers $a_{n}$, $\lim_{n \to \infty} a_{n} = 0$ is equivalent to $\lim \sup_{n \to \infty} a_{n} = 0$.

The block-independent approximation (henceforth shortened to BIA) $X^{T}$ source is a sequence of random vectors $(S_{1}, S_{2}, \ldots, S_{n}, \ldots)$, where $S_{i}$ are independent, and $\forall i$, $S_{i} \sim X^{T}$. To simplify notation, we will sometimes denote $(S_{1}, S_{2}, \ldots)$ by $S$. $S^{n}$ will denote $(S_{1}, S_{2}, \ldots, S_{n})$. Note that BIA $X^{T}$ source is an i.i.d. vector source and will also be called the vector i.i.d. $X^{T}$ source.

The rate-distortion function for the vector i.i.d. $X^{T}$ source is defined in the same way as above; just that the source input space would be $X^{T}$, the source output space will be $Y^{T}$, the single letter distortion function would now be on $T$-length sequences and is defined additively, and when forming block-codes, we will be looking at blocks of $T$-length vectors. Details are as follows:

The source input space is $X^{T}$. Denote it by $S$. The source reproduction space is $Y^{T}$. Denote it by $T$. Denote a generic element of the source space by $s$ and that of the source reproduction space by $t$. Note that $s$ and $t$ are $T$-length sequences. Denote the $i^{th}$ component by $s(i)$ and $t(i)$ respectively.

The single letter distortion function, now, has inputs which are length $T$ vectors. It is denoted by $d_{T}$ and is defined additively using $d$ which has been defined before:

$$d_{T}(s, t) \triangleq \sum_{i=1}^{T} d(s(i), t(i)).$$
Note that $d_T$ is the same as $d^T$; just that we use superscript $T$ for $T$ length vectors, but now, we want to view a $T$-length vector as a scalar, and on this scalar, we denote the distortion measure by $d_T$.

$s^n$ will denote a block-length $n$ sequence of vectors of length $T$. Thus, $s^n(i)$, which denotes the $i^{th}$ component of $s^n$ is an element of $\mathbb{K}$. $s^n(i)(j)$ will denote the $j^{th}$ component of $s^n(i)$.

The $n$-letter distortion function is defined additively using $d_T$:

For $s^n \in \mathbb{S}^n$, $t^n \in \mathbb{T}^n$,

$$d^n_T(s^n, t^n) \triangleq \sum_{i=1}^{n} d_T(s^n(i), t^n(i)).$$

When coding the vector i.i.d. $X^T$ source (for short, denoted by $S$), a rate $R$ source code is a sequence $< e^n, f^n >_1^\infty$, where $e^n : \mathbb{S}^n \rightarrow \{1, 2, \ldots, 2^{[nR]} \}$ and $f^n : \{1, 2, \ldots, 2^{[nR]} \} \rightarrow \mathbb{T}^n$.

We say that rate $R$ is achievable for source-coding the vector i.i.d. $X^T$ source within distortion-level $D$ under the expected distortion criterion if there exists a rate $R$ source code $< e^n, f^n >_1^\infty$ such that

$$\lim_{n \to \infty} E \left[ \frac{1}{n} d^n_T(S^n, f^n(e^n(S^n))) \right] \leq D \quad \text{(3)}$$

(Note that $S^n$ denotes $(S_1, S_2, \ldots, S_n)$).

The infimum of all achievable rates under the expected distortion criterion is the operational rate distortion function, denoted by $R^E_{X^T}(D)$.

The information-theoretic rate-distortion function of the vector i.i.d. $X^T$ source is denoted and defined as

$$R^I_{X^T}(D) \triangleq \inf_{\mathcal{T}} I(X^T; Y^T) \quad \text{(4)}$$

where $\mathcal{T}$ is the set of $W : \mathbb{S} \rightarrow \mathbb{P}(\mathbb{T})$ defined as

$$\mathcal{T} \triangleq \left\{ W \left| \sum_{s \in \mathbb{S}, y \in \mathbb{T}} p_{X^T}(s) W(t|s)d_T(s, t) \leq D \right. \right\} \quad \text{(5)}$$

where $p_{X^T}$ denotes the distribution corresponding to $X^T$. 
Note that this is the usual definition of the information-theoretic rate-distortion function for an i.i.d. source; just that the source under consideration is vector i.i.d.

By the rate-distortion theorem, $R^E_{X \rightarrow T}(D) = R^I_{X \rightarrow T}(D)$.

Further, if the source $X = (X_1, X_2, \ldots)$ is stationary, ergodic, it is also known that

\[ R^E_X(D) = \lim_{T \to \infty} \frac{1}{T} R^E_{X \rightarrow T}(TD) \]  

The channel is a sequence $c = \langle c^n \rangle_1^\infty$ where

\[ c^n : X^n \to \mathbb{P}(Y^n) \]  

When the block-length is $n$, the channel acts as $c^n(\cdot | \cdot)$; $c^n(y^n | x^n)$ is the probability that the channel output is $y^n$ given that the channel input is $x^n$.

When the block-length is $n$, a rate $R$ deterministic channel encoder is a map $e^n_{ch} : M^n_R \to X^n$ and a rate $R$ deterministic channel decoder is a map $f^n_{ch} : Y^n \to \hat{M}^n_R$ where $\hat{M}^n_R \triangleq M^n_R \cup \{e\}$ is the message reproduction set where ‘e’ denotes error. The encoder and decoder are allowed to be random in the sense that encoder-decoder is a joint probability distribution on the space of deterministic encoders and decoders. $\langle e^n_{ch}, f^n_{ch} \rangle_1^\infty$ is the rate $R$ channel code.

Denote

\[ g = \langle g^n \rangle_1^\infty \triangleq \langle e^n_{ch} \circ c^n \circ f^n_{ch} \rangle_1^\infty \]  

$g^n$ has input space $M^n_R$ and output space $\hat{M}^n_R$. Consider the set of channels

\[ \mathbb{G}_A \triangleq \{ e \circ c \circ f \mid c \in A \} \]  

$g \in \mathbb{G}_A$ is a compound channel. Rate $R$ is said to be reliably achievable over $g \in \mathbb{G}_A$ if there exists a rate $R$ channel code $\langle e^n_{ch}, f^n_{ch} \rangle_1^\infty$ and a sequence $\langle \delta_n \rangle_1^\infty$, $\delta_n \to 0$ as $n \to \infty$ such that

\[ \sup_{m^n \in M^n_R} g^n(\{m^n\} | c m^n) \leq \delta_n \forall c \in A \]
Supremum of all achievable rates is the capacity of \( c \in A \). Note that this is the compound capacity, but will be referred to as just the capacity of \( c \in A \).

The channel \( c \in A \) is said to communicate the source \( X \) directly within distortion \( D \) if with input \( X^n \) to \( c^n \), the output is \( Y^n \) (possibly depending on the particular \( c \in A \)) such that

\[
\Pr \left( \frac{1}{n}d^n(X^n, Y^n) > D \right) \leq \omega_n \forall c \in A
\]

for some \( \omega_n \to 0 \) as \( n \to \infty \).

3. Mixing condition used in this paper

In this section, \( \psi \)-mixing processes are defined, properties of \( \psi \)-mixing processes are stated (and proved in the appendix), and intuition on \( \psi \)-mixing is provided.

3.1. Definition of a \( \psi \)-mixing process

Let \( X_1, X_2, \ldots, X_n, \ldots \) be a sequence of random variables defined on a probability space \( (\Omega, \Sigma, P) \). The random variables from \( X_a \) to \( X_b \) will be denoted by \( X_a^b \), \( 1 \leq a \leq b \leq \infty \). The whole sequence \( X_1^\infty \) will be denoted by \( X^\infty \) or just by \( X \). The range of each \( X_i \) is assumed to be contained in a finite set \( X \).

Note that time is assumed to be discrete. Note further, that it is assumed that the process is one-sided in time, that is runs from time 1 to \( \infty \), not \( -\infty \) to \( \infty \). The Borel sigma-field on \( X^\infty \) is defined in the standard way, and is denoted by \( \mathcal{F}^\infty \); see Pages 1,2 of [2] for details.

\( \mathcal{X}_a^b \) will denote the set corresponding to the \( a^{th} \) to the \( b^{th} \) coordinates of \( X^\infty \), \( 1 \leq a \leq b < \infty \). A sequence within these coordinates will be denoted by \( X_a^b \), a random variable, by \( X_a^b \). The Borel sigma-field on \( \mathcal{X}_a^b \) is denoted by \( \mathcal{F}_a^b \). Note that if \( a \) and \( b \) are finite, \( \mathcal{F}_a^b = 2^{\mathcal{X}_a^b} \), the power set of \( \mathcal{X}_a^b \).

For \( A \in \mathcal{F}_1^t \) and \( B \in \mathcal{F}_{t+r+1}^\infty \), we will have occasion to talk about the following probabilities:

\[
\Pr(X_1^t \in A) \\
\Pr(X_{t+r+1}^\infty \in B) \\
\Pr(X_1^t \in A, X_{t+r+1}^\infty \in B)
\]
The intuitive meaning is clear: for example, \( \Pr(X^t_1 \in A, X^\infty_{t+\tau+1} \in B) \) refers to the probability that the random-variables \( X^t_1 \) take values in the set \( A \) and the random variables \( X^\infty_{t+\tau+1} \) take values in the set \( B \). Mathematically, this is defined as follows. Define:

\[
A' = \{(a_1, a_2, \ldots, a_n, \ldots) \mid (a_1, \ldots, a_t) \in A\}
\]

\[
B' = \{(b_1, b_2, \ldots, b_n, \ldots) \mid (b_t, b_{t+\tau+2}, \ldots) \in B\}
\]

Then,

\[
\Pr(X^t_1 \in A) \triangleq P(X^\infty_1 \in A')
\]
\[
\Pr(X^\infty_{t+\tau+1} \in B) \triangleq P(X^\infty_1 \in B')
\]
\[
\Pr(X^t_1 \in A, X^\infty_{t+\tau+1} \in B) \triangleq P(X^\infty_1 \in A' \cap B')
\]

Further, if \( \Pr(X^t_1 \in A) > 0 \), the following definition will be used:

\[
\Pr(X^\infty_{t+\tau+1} \in B \mid X^t_1 \in A) \triangleq \frac{P(X^t_1 \in A, X^\infty_{t+\tau+1} \in B)}{P(X^t_1 \in A)}
\]

The one-sided version of \( \psi \)-mixing criterion of \([3]\) will be used in this document, This is because the stochastic process under consideration in this document is one-sided in time, whereas the stochastic process under consideration in \([3]\) is two-sided in time.

Define, for \( \tau \in \mathbb{W} \), the set of whole numbers (non-negative integers),

\[
\psi(\tau) \triangleq \sup_{t \in \mathbb{N}} \sup_{A \in F^t_1, B \in F^\infty_{t+\tau+1}, \Pr(X^t_1 \in A) > 0, \Pr(X^\infty_{t+\tau+1} \in B) > 0} \left| \frac{\Pr(X^t_1 \in A, X^\infty_{t+\tau+1} \in B)}{\Pr(X^t_1 \in A, X^\infty_{t+\tau+1} \in B) - 1} \right|
\]

The process \( X \) is said to be \( \psi \)-mixing if \( \psi(\tau) \to 0 \) as \( \tau \to \infty \).

The changes in (18) from \([3]\) are:

- The first sup is taken over \( t \in \mathbb{Z} \) in \([3]\), see Page 111 of \([3]\). Also, \( t \) is denoted by \( j \) in \([3]\). However, the sup in (18) is over \( j \in \mathbb{W} \). This is because the process in \([3]\) is two-sided in time, whereas we are considering a one-sided process.
• A change of notation, where probabilities in (18) are written in terms of random-variables taking values in certain sets, whereas [3] considers the underlying probability space and writes probabilities of sets on that space, see Page 110, 111 of [3].
• The set $A \in F_t^1$ in (18), whereas if one used the definition in [3], the set $A$ would belong to $F_{t-\infty}^1$. This is, again, because the process in [3] is two-sided whereas the process in this paper is one-sided.

The reader is referred to [3] and [4] for an overview of various kinds of mixing conditions. [3] gives a strong mixing conditions whereas [4] mentions both mixing and strong mixing conditions though the coverage of strong mixing conditions is less thorough than [3].

Let $X$ be stationary. For $B \subset X^T$, denote the probability $Pr(X_{t+1} \in B)$ (which is independent of $t$ since $X$ is stationary), by $P_T(B)$. Note that $P_T$ is a probability distribution on $X^T$ where the underlying sigma-field is the canonical sigma-field $2^{X^T}$.

### 3.2. Properties of $\psi$-mixing processes

**Lemma 1.** Let $X$ be stationary, $\psi$-mixing. Then, $\forall t \in \mathbb{N}, \forall \tau \in \mathcal{W}, \forall T \in \mathcal{W}, \forall A \subset X^t, \forall B \subset X^T, P(X_1^t \in A) > 0$,

$$(19) \quad P(X_{t+\tau+1}^{t+T} \in B | X_1^t \in A) = (1 - \lambda_\tau)P_T(B) + \lambda_\tau P'_{t,\tau,T,A}(B)$$

for some probability distribution $P'_{t,\tau,T,A}$ on $X^T$ (under the canonical sigma field on $X^T$) which may depend on $t, \tau, T, A$.

*Proof. See Appendix A.*

**Lemma 2.** If $X$ is stationary, $\psi$-mixing, then $X$ is ergodic.

*Proof. See Appendix A.*

**Lemma 3.** Let $X = (X_1, X_2, \ldots, X_n, \ldots)$ be a stationary, irreducible, aperiodic Markoff chain evolving on a finite set $X$. Then, $X$ is $\psi$-mixing.

*Proof. See Appendix A.*

**Lemma 4.** Let $X = (X_1, X_2, \ldots)$ be a stationary, psi-mixing process evolving on a set $X$. For $L \in \mathbb{N}$, define $Z_l = X_{(t-1)L+1}^t$. Then, $Z = (Z_1, Z_2, \ldots)$ is a stationary, psi-mixing process evolving on the set $X^L$. 

*Proof. See Appendix A.*
Proof. See Appendix A.

Lemma 5. Let X be a stationary, order m Markoff chain evolving on a finite set X. For Define $Z_t = X_{(t-1)L+1}^L$. Note that $Z = (Z_1, Z_2, \ldots)$ is a Markoff chain evolving on the set $Z = X^L$. Assume that Z is irreducible, aperiodic. Then X is $\psi$-mixing.

Proof. See Appendix A.

It should be noted here, that a $\psi$-mixing process can have a rate of mixing as slow as desired whereas a Markoff $\psi$-mixing process implies exponential rate of convergence to the stationary distribution [5], [6]. Thus, the set of $\psi$-mixing processes is strictly larger than the set of Markoff or order m Markoff chains.

3.3. Intuition on $\psi$-mixing

Assume X is stationary. Note (55). $X^t_i$ and $X^{\infty}_{i+\tau+1}$ are independent if

$$P(X^\infty_{i+\tau+1} \in B | X^t_i \in A) = P(X^\infty_{i+\tau+1} \in B) = P_T(B).$$

Thus, (55) says that the process ‘becomes more and more independent’ with time, further, this happens at a rate proportional to a factor $\lambda_\tau \to 0$ as $\tau \to \infty$ which is independent of the sets A and B in question, and also a multiplicative factor which depends on the probability of the set B. This dependence on the probability of B is intuitively pleasing in the sense that, for example, if $P_T(B) = 10^{-10}$ and $\lambda_\tau = 10^{-5}$, then without the multiplicative factor $P_T(B)$, (20) says nothing meaningful; however, with the multiplicative factor $P_T(B)$, it says something meaningful. A mixing condition can indeed be defined where $P_T(B)$ does not exist on the right hand side in (20), this is the $\phi$-mixing condition in [3]. An even weaker condition is the $\alpha$-mixing condition [3] where independence is measured in the sense of

$$P(A \cap B) = P(A)P(B)$$

instead of

$$P(B|A) = P(B)$$
4. Idea of the proof

Theorem 1 of [1] will be generalized to $\psi$-mixing sources in this paper. This will be done by reducing the problem to the case when the source is i.i.d., and then, use Theorem 1 of [1].

The basic idea of the proof is the following:

Choose $\tau, T$, where $\tau$ is ‘small’ compared to $T$. Denote $K_1 = X_1^T$, $K_2 = X_{T+\tau+1}^{2T+\tau}$, $K_3 = X_{2T+2\tau+1}^{3T+2\tau}$, $\ldots$ Each $K_i$ has the same distribution; denote it by $K$. By Lemma 1, each $K_i$ has distribution close to $P_T$ in the sense of (19). Thus, $K_1, K_2, K_3, \ldots$ is ‘close to’ an i.i.d. process. Theorem 1 from [1] can be used and rates approximately

$$\frac{T}{T + \tau} \frac{1}{T} R^E_K(TD)$$

Take $T \to \infty$ and it follows that rates $< R^E_X(D)$ are achievable, where $X$ is the $\psi$-mixing source.

A lot of technical steps are needed and this will be the material of the future sections. Note also, that there are various definitions of mixing which will make $K_1, K_2, \ldots$ ‘almost’ independent, but the proof will not work for all these definitions. The definition of $\psi$-mixing is used primarily because (19) holds and this can be used to simulate the source $X$ in a way discussed in the next section, and this simulation procedure will be a crucial element of the proof.

5. A simulation procedure for the stationary source $X$ which satisfies $\psi$-mixing

By using (19), a procedure to simulate the source $X = (X_t, t = 1, 2, \ldots)$ will be described.

Fix $T$ and $\tau$, both strictly positive integers. Denote $n = (T + \tau)k$ for some strictly positive integer $k$.

We will generate a $(X'_1, X'_2, \ldots, X'_{(T+\tau)k})$, as described below.

First divide time into chunks of time $T, \tau, T, \tau, T, \tau, \ldots$ and so on ...

Call these slots $A_1, B_1, A_2, B_2, \ldots, A_i, B_i, \ldots, A_k, B_k$. 

Thus,

\[ \begin{align*}
A_1 & \text{ contains } X'_1. \\
B_1 & \text{ contains } X'_{T+\tau}. \\
A_2 & \text{ contains } X'_{2T+\tau}. \\
B_2 & \text{ contains } X'_{2T+2\tau}. \\
& \vdots \\
A_i & \text{ contains } X'_{iT+(i-1)\tau}. \\
B_i & \text{ contains } X'_{iT+(i-1)\tau+1}. \\
& \vdots \\
A_k & \text{ contains } X'_{kT+(k-1)\tau}. \\
B_k & \text{ contains } X'_{kT+(k-1)\tau+1}. \\
\end{align*} \]

Let \( C_1 = 1 \).

Generate \( C_2, C_3, \ldots, C_k \) i.i.d., where \( C_i \) is 1 with probability \((1 - \lambda_\tau)\) and 0 with probability \( \lambda_\tau \).

If \( C_i = 1 \), denote \( A_i \) by \( A_i^{(g)} \) and if \( C_i = 0 \), denote \( A_i \) by \( A_i^{(b)} \). Think of superscript ‘g’ as ‘good’ and ‘b’ as ‘bad’.

Generation of \((X'_1, X'_2, \ldots, X'_{(T+\tau)k})\) is carried out as follows:

The order in which the \( X'_i \)s in the slots will be generated is the following:

\[ A_1, A_2, B_1, A_3, B_2, \ldots, A_i, B_{i-1}, A_{i+1}, \ldots \]

Generate \( X'_1 \) (slot \( A_1^{(g)} \)) by the distribution \( P_T \).

Assume that all \( X_i \) have been generated until slot \( A_{i-1} \), in other words, the generation in the following slots in the following order has happened:

\[ A_1, A_2, B_1, A_3, B_2, \ldots, A_{i-1}, B_{i-2} \]

The next two slots to be generated, as per the order stated above, is \( A_i \) and then \( B_{i-1} \).

For slot \( A_i \),
If it is a ‘g’ slot, generate $X'^{(i-1)\tau}_{(i-1)(T+\tau)+1}$ using $P_T$.

If it is a ‘b’ slot, generate $X'^{(i-1)\tau}_{(i-1)(T+\tau)+1}$ using $P'_{t,\tau,T,A}$ with $t = (k-1)T + (k-2)\tau$ and $A = \{x'_{(k-1)T+(k-2)\tau}\}$ where $x'_{(k-1)T+(k-2)\tau}$ is the simulated process realization so far.

During the slot $B_{i-1}$, $X'^{(i-1)(T+\tau)}_{(i-1)T+(i-2)\tau+1}$ is generated using the probability measure $P$ of the stationary process given the values of the process already generated, that is, given $x'_{(k-1)T+(k-2)\tau}$ and $x'_{(i-1)(T+\tau)+1}$.

This finishes the description of the generation of the $(X'_1, X'_2, \ldots, X'_{(T+\tau)k})$ sequence.

Note that $(X'_1, X'_2, \ldots, X'_{(T+\tau)k}) \sim (X_1, X_2, \ldots, X_{(T+\tau)k})$.

Note also, that during slots $A_i^{(g)}$, the source has distribution $X^T$ and is independent over these slots.

### 6. The main lemma: channel-coding theorem

**Lemma 6.** Let $c = \langle c^n \rangle_\infty$ directly communicate the source $X$, assumed to be $\psi$-mixing, within distortion $D$.

Let $\lambda > 0$ (think of $\lambda$ small; $\lambda << 1$). Choose $\beta > 0$ (think of $\beta$ small; $\beta << 1 - \lambda$). Choose $\tau$ large enough so that $\lambda_\tau \leq \lambda$. Then, rates

$$R < \frac{1 - \lambda_\tau - \beta}{T + \tau} R_{X^T} \left( \frac{(T + \tau)D}{1 - \lambda_\tau - \beta} \right)$$

are reliably achievable over $c \forall T \geq 1$ (think of $T$ large).

**Proof.** Choose $T \geq 1$.

Let $n = (T + \tau)k$ for some large $k$. $n$ is the block-length.

Generate $C_1, C_2, \ldots$ as described previously.

Generate $2^{\lceil nR \rceil}$ codewords of block-length $(T + \tau)k = n$ by use of the simulation procedure described previously. Note that $C_1, C_2, \ldots$ is the same for generating all the $2^{\lceil nR \rceil}$ codewords.
Note that over $A_i^{(g)}$ time slots, the codewords are generated i.i.d., as in Shannon’s random-coding argument; this generation during $A_i^{(g)}$ is done i.i.d. $X^T$.

Recall the behavior of the channel which directly communicates the source $X$ source within distortion $D$. End-to-end,

\[
\lim_{n \to \infty} \Pr \left( \frac{1}{n} d^n(X^n, Y^n) > D \right) = 0
\]

Let us look at the behavior of the channel restricted to time slots $A_i^{(g)}$.

Assume that the fraction of ‘g’ slots among the $k A_i$ slots is $\geq 1 - \lambda_r - \beta$. That is, number of $A_i^{(g)}$ slots is larger than or equal to $\lfloor (1 - \lambda_r - \beta)k \rfloor + 1$. Denote $N = \lfloor (1 - \lambda_r - \beta)k \rfloor + 1$. This is a high probability event and the probability $\to 1$ as $k \to \infty$ for any $\beta$. If this even does not happen, we will declare decoding error; hence, in what follows, assume that this is the case.

Restrict attention to the first $N A_i^{(g)}$ slots. Rename these slots $G_1, G_2, \ldots, G_N$.

Denote the part of the source during slot $G_i$ by $S_i$. Note that $S_i$ is a $T$-length vector.

Denote $S = (S_1, S_2, \ldots, S_N)$.

Denote the channel output during slot $G_i$ by $T_i$. Note that $T_i$ is a $T$-length vector. Denote $T = (T_1, T_2, \ldots, T_N)$.

Recall the definition of the distortion function $d_T$ for $T$-length vectors, and its $n$-block additive extension.

Over $G_i$ slots, then,

\[
\lim_{N \to \infty} \Pr \left( \frac{1}{N} \sum_{i=1}^{N} d_T(S_i, T_i) > \left( T + \tau \right) k D \right) = 0
\]

By substituting $N = \lfloor (1 - \lambda_r - \beta)k \rfloor + 1$, it follows, after noting that

\[
\frac{k}{\lfloor (1 - \lambda_r - \beta)k \rfloor + 1} \leq \frac{1}{1 - \lambda_r - \beta}
\]
that

\[
\lim_{k \to \infty} \Pr \left( \frac{1}{[(1 - \lambda_r - \beta)k] + 1} \sum_{i=1}^{[(1 - \lambda_r - \beta)k] + 1} d_T(S_i, T_i) > \frac{(T + \tau)D}{1 - \lambda_r - \beta} \right) = 0
\]

Recall again that \( S_i \) are i.i.d. \( X^T \) and that, codeword generation over \( G_i \) slots is i.i.d.

We have reduced, then, the problem to that where it is known that an i.i.d. source is directly communicated over a channel within a certain probability of excess distortion and we want to calculate a lower bound on the capacity of the channel – this is Theorem 1 of [1].

If each \( G_i \) is considered to be a single unit of time, or in other words, over \( G_i \), the uses of the channel is considered as a single channel use, we are thus able, by use of Theorem 1 of [1] to communicate at rates

\[
R < R_{X^T}^E \left( \frac{(T + \tau)D}{1 - \lambda_r - \beta} \right) \text{ (per channel use)}
\]

Total time of communication, though, has been \( (T + \tau)k \) and there are \( [(1 - \lambda_r - \beta)k] + 1 \) \( G_i \) slots over which the communication takes place. Noting that

\[
\frac{[(1 - \lambda_r - \beta)k] + 1}{(T + \tau)k} \geq \frac{1 - \lambda_r - \beta}{(T + \tau)}
\]

it follows that rates

\[
R < \frac{1 - \lambda_r - \beta}{(T + \tau)} R_{X^T}^E \left( \frac{(T + \tau)D}{1 - \lambda_r - \beta} \right)
\]

are achievable for reliable communication over the original channel \( c \) per channel use of \( c \).

\[\square\]

Roughly, the details of codebook generation and decoding are as follows:

Let reliable communication be desired at a rate \( R \) which is such that there exist \( \tau, \beta, T \) such that

\[
R < \frac{1 - \lambda_r - \beta}{(T + \tau)} R_{X^T}^E \left( \frac{(T + \tau)D}{1 - \lambda_r - \beta} \right)
\]
Generate \( C_1, C_2, \ldots \). Assume that this knowledge is available at both encoder and decoder.

Generate \( 2^{k(T + \tau)R} \) codewords using the simulation procedure.

If the number of ‘g’ slots is less than \( \lfloor (1 - \lambda_r - \beta)k \rfloor \), declare error.

Else, restrict attention only the first \( \lfloor (1 - \lambda_r - \beta)k \rfloor A_{i}^{(g)} \) slots which have been renamed \( G_1, G_2, \ldots \).

Over these slots, the codebook generation is i.i.d., and then, use the procedure from Theorem 1 of [1].

7. \( R^p_X(D) \leq R^E_X(D) \) if \( X \) is stationary and satisfies \( \psi \)-mixing

**Lemma 7.** Let \( X = (X_t, t = 1, 2, 3, \ldots) \) be stationary process which satisfies \( \psi \)-mixing. Then, \( R^p_X(D) \leq R^E_X(D) \).

**Proof.** First, note that a stationary source which is \( \psi \)-mixing is ergodic, as stated and proved in Section 3. Thus, the process \( X \) is stationary, ergodic.

The proof now, relies on [7], Pages 490-499.

First, note the notation in [7], [7] defines \( R_L(D) \) and \( R(D) \), both on Page 491. Note that by the rate-distortion theorem for an i.i.d. source, it follows that

\[
R_L(D) \, (\text{notation in [7]}) = \frac{1}{T} R^E_X(TD) \, (\text{our notation})
\]

Thus,

\[
R(D) \, (\text{notation in [7]}) = \lim_{T \to \infty} R^E_X(TD) \, (\text{our notation}) = R^E_X(D) \, (\text{our notation})
\]

Look at Theorem 9.8.2 of [7]. This theorem holds if probability of excess distortion criterion is used instead of the expected distortion criterion: see (9.8.10) of [7]. By mapping the steps carefully, it follows that rate \( R_1(D - \epsilon) \) (notation in [7]) is achievable for source-coding the source \( X \) under a probability of excess distortion \( D \) for all \( \epsilon > 0 \). Note that it follows that rates \( R_1(D - \epsilon) \) and not rates \( R_1(D) \). This is because in (9.8.10), when making further arguments, \( \hat{d} \) is made \( D + \frac{\delta}{2} \) and not \( D \). Hence, we need to keep a
distortion level smaller than $D$ in $R_1(\cdot)$ to make this rate achievable for the probability of excess distortion criterion. Next, we construct the $L^{th}$ order super source as described on Page 495 of [7]: Define $X^{tL} = X^{t}_{(t-1)L+1}$. Then, $X' = (X'^t, t = 1, 2, 3, \ldots)$ is the $n^{th}$ order super-source. $X'$ is stationary, $\psi$-mixing because $X$ is (Lemma 4), and thus, stationary, ergodic, by Lemma 2. One can thus use Theorem 9.8.2 again, to argue that rate $R_L(D-\epsilon)$ (notation in [7]) is achievable for source-coding the source $X$ under a probability of excess distortion $D$ for all $\epsilon > 0$. By taking a limit as $L \to \infty$ (the limit exists by Theorem 9.8.1 in [7]) , it follows that rate $R(D-\epsilon)$ (notation in [7]) is achievable for source-coding the source $X$ under a probability of excess distortion $D$ for all $\epsilon > 0$. As stated at the end of the proof of Theorem 9.8.1 in [7], $R(D)$ is a continuous function of $D$. Thus, it follows that rates $< R(D)$ are achievable for source-coding the source $X$ under a probability of excess distortion $D$. At this point, note (34), and this finishes the proof.

8. Generalization of Theorem 1 in Part I of this sequence of papers to stationary sources satisfying $\psi$-mixing

Before we prove the theorem, note the following: Let $f : [0, \infty) \to [0, \infty)$ be a convex $\cup$ non-increasing function. Let $f(0) = K$. Let $0 < a < a'$. Then,

$$|f(a) - f(a')| \leq \frac{K}{a}(a' - a)$$

Theorem 1. Let $c$ be a channel over which the source $X$, assumed to be stationary, $\psi$-mixing, is directly communicated within probability of excess distortion $D$, $D > 0$. Then, rates $< R_X^P(D)$ are reliably achievable over $c$.

Proof. Since $R_X^P(D) \leq R_X^E(D)$ by Lemma 7 and since it is known that

$$R_X^E(D) = \lim_{T \to \infty} \frac{1}{T} R_{X^T}(TD)$$

it is sufficient to prove that rates less than

$$\lim_{T \to \infty} \frac{1}{T} R_{X^T}(TD)$$

are reliably achievable over $c$. 

\[ \square \]
To this end, denote

$$D' \triangleq \frac{D}{1 - \lambda - \beta}$$  \hspace{1cm} (38)

Then,

$$\frac{1 - \lambda - \beta}{T + \tau} R_{X^n}^E ((T + \tau)D') - \lim_{T \to \infty} \frac{1}{T} R_{X^n}^E (TD')$$  \hspace{1cm} (39)

$$= \frac{1 - \lambda - \beta}{T + \tau} R_{X^n}^E ((T + \tau)D') - \frac{1}{T + \tau} R_{X^n}^E ((T + \tau)D')$$  \hspace{1cm} (40)

$$+ \frac{1}{T + \tau} R_{X^n}^E ((T + \tau)D') - \frac{1}{T} R_{X^n}^E ((T + \tau)D')$$  \hspace{1cm} (41)

$$+ \frac{1}{T} R_{X^n}^E (TD') - \lim_{T \to \infty} \frac{1}{T} R_{X^n}^E (TD')$$  \hspace{1cm} (42)

Expression in (40) is

$$\frac{-\lambda - \beta}{T + \tau} R_{X^n}^E ((T + \tau)D')$$  \hspace{1cm} (44)

Note that

$$R_{X^n}^E ((T + \tau)D') \leq T \log |X|$$  \hspace{1cm} (45)

Thus, the absolute value of the expression in (40) is upper bounded by $(\lambda + \beta) \log |X|$.

Expression in (41) is

$$\frac{-\tau}{T} \left( \frac{1}{T + \tau} R_{X^n}^E ((T + \tau)D') \right)$$  \hspace{1cm} (46)

Note that

$$R_{X^n}^E ((T + \tau)D') \leq T \log |X|$$  \hspace{1cm} (47)

It then follows that expression in (41) \to 0 as $T \to \infty$. 
Expression in (42) is

\[ \frac{1}{T} R^E_{X,T} \left( T \left( D' + \frac{\tau}{T} D' \right) \right) - \frac{1}{T} R^E_{X,T} (TD') \]

\( \frac{1}{T} R^E_{X,T} (TD) \) is a convex union non-negative function of \( D \), upper bounded by \( \log |X| \). It follows that

\[ \frac{1}{T} R^E_{X,T} \left( T \left( D' + \frac{\tau}{T} D' \right) \right) - \frac{1}{T} R^E_{X,T} (TD') \leq \frac{\log |X|}{D'} \left( (D' + \frac{\tau}{T} D') - D' \right) \]

\( \rightarrow 0 \) as \( T \rightarrow \infty \)

Expression in (43) \( \rightarrow 0 \) as \( T \rightarrow \infty \).

By noting the bound on the absolute value of expression (40) proved above and by noting, as proved above, that expressions in (41), (42), and (43) \( \rightarrow 0 \) as \( T \rightarrow \infty \), it follows that \( \exists \ \epsilon_T \rightarrow 0 \) as \( T \rightarrow \infty \), possibly depending on \( \lambda_T \) and \( \beta \) such that

\[ \left| \frac{1 - \lambda_T - \beta}{T + \tau} R^E_{X,T} ((T + \tau)D') - \lim_{T \rightarrow \infty} \frac{1}{T} R^E_{X,T} (TD') \right| \leq (\lambda_T + \beta)|X| + \epsilon_T \]

By Lemma 6, and by recalling that \( D' = \frac{D}{1 - \lambda_T - \beta} \) it follows that rates less than

\[ \lim_{T \rightarrow \infty} \frac{1}{T} R^E_{X,T} \left( T \frac{D}{1 - \lambda_T - \beta} \right) - (\lambda_T + \beta)|X| - \epsilon_T \]

are achievable reliably over \( c \).

By using the fact that \( \lambda_T \) and \( \beta \) can be made arbitrarily small and \( \epsilon_T \rightarrow 0 \) as \( T \rightarrow \infty \), and that, the function

\[ \lim_{T \rightarrow \infty} \frac{1}{T} R^E_{X,T} (TD) \]
is continuous in $D$, it follows that rates less than

$$\lim_{T \to \infty} R_{X,T}^E(TD)$$

are reliably achievable over $c$ from which, as stated at the beginning of the proof of this theorem, it follows that rates less than $R_X^P(D)$ are reliably achievable over $c$.

9. Application to Markoff chains and order $m$ Markoff chains

Let $X = (X_t, t = 1, 2, \ldots)$ be a stationary, irreducible, aperiodic Markoff chain evolving on a finite set $X$. By Lemma 3, $X$ is $\psi$-mixing. $X$ is thus, stationary, $\psi$-mixing and thus, Theorem 1 holds for stationary, irreducible, aperiodic Markoff chains evolving on a finite set.

Let $X = (X_i, i \in \mathbb{N})$ be an order $m$ stationary Markoff chain. Define $Z_i = X^{im}_{m+1}$. Then, $Z = (Z_i, i \in \mathbb{N})$ is a Markoff chain. By Lemma 4, $Z$ is stationary. Assume that this $Z$ is irreducible, aperiodic. By Lemma 5, $X$ is $\psi$-mixing, and thus, Theorem 1 holds.

10. Discussion

It is really (19) that is crucial to the proof, not the $\psi$-mixing criterion. $\psi$-mixing criterion is used to prove ergodicity and some other properties needed to finish parts of the proof but it is possible that they can be proved just by use of (19) too. However, the assumption of $\psi$-mixing suffices, and since this condition holds for Markoff and order $m$ Markoff sources (under stationarity, irreducibility, aperiodicity assumptions as stated above), the theorem has been proved for quite a large class of sources.

In Theorem 1 in [1], the channel may belong to a set whereas the way Lemma 6 or Theorem 1 is stated in this paper, the channel does not belong to a set. However, it is easy to see that the proof of Lemma 6 does not require knowledge of the channel transition probability; only the end-to-end description that the channel communicate the source within the distortion level is needed; for this reason, Theorem 1 in this paper generalizes to the case when the channel belongs to a set for the same reason as in [1]. A source-channel separation theorem has also been stated and proved in Theorem 2 in [1]; this can be done in this paper too. Statements concerning resource
consumption have not been made in this paper; however, they can be made for the same reason as in [1]. Finally, generalization to the multi-user setting has not been carried out in this paper; that can be carried out for the same reason as in [1].

11. Future research directions

- Generalize Theorem 1 to arbitrary stationary, ergodic processes, not just those which satisfy ψ-mixing, to the extent possible.
- In particular, explore a generalization to B-processes [8], the closure of the set of Markov chains of any finite order.
- Consider an alternate proof strategy for proving Theorem 1 which uses methods from classical ergodic and rate-distortion theory, that is, methods similar to, for example, [7] and [8], and thus, does not rely on the decomposition (19). This might help prove Theorem 1 for general stationary, ergodic sources, not just those which satisfy ψ-mixing.
- Further, consider a strategy based on the theory of large deviations, to start with, for irreducible, aperiodic Markov chain source. For i.i.d. sources a large deviations based method was indeed used in Part 1 [1].
- Generalize Theorem 1 to stationary, ergodic sources which evolve continuously in space and time (some assumptions might be needed on the source). Since only the end-to-end description of the channel as communicating the source $X$ within distortion level $D$ is used and not the exact dynamics of the channel, the proof given here directly holds for channels which evolve continuously in space and time.
- Research the possibility of an operational rate-distortion theory for stationary, ergodic sources (satisfying other conditions). An operational theory for i.i.d. sources has been presented in [9].
- The channel has been assumed to belong to a set in Part I [1] and hence, as stated above, also for results in this paper, but the source is assumed to be known. Research the generalization of results in this paper to compound sources.

12. Acknowledgements

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References

[1] M. Agarwal, S. K. Mitter, and A. Sahai, “Layered black-box, behavioral interconnection perspective and applications to the problem of communication with fidelity criteria, part i: i.i.d. sources,” see arxiv.org.

[2] P. C. Shields, The ergodic theory of discrete sample paths. American Mathematical Society, July 1996.

[3] R. C. Bradley, “Basic properties of strong mixing conditions. a survey and some open questions,” Probability surveys, vol. 2, pp. 107–144, 2005.

[4] Y. V. Prohorov and Y. A. Rozanov, Probability theory: basic concepts, limit theorems, random processes, 1st ed., ser. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berucksichtigung der Anwendungsgebiete, Band 157. Springer-Verlag, 1969.

[5] R. Bradley, “On the ψ-mixing condition for stationary random sequences,” Transactions of the American Mathematical Society, vol. 276, no. 1, pp. 55–66, March 1983.

[6] H. Kesten and G. L. O’Brien, “Examples of mixing sequences,” Duke Mathematical Journal, vol. 43, no. 2, pp. 405–415, 1976.

[7] R. G. Gallager, Information theory and reliable communication. Wiley, January 1968.

[8] R. M. Gray, Entropy and information theory. Springer-Verlag, February 2011.

[9] M. Agarwal, “A universal, operational theory of multi-user communication with fidelity criteria,” Ph.D. dissertation, Massachusetts Institute of Technology, February 2012.

[10] R. Douc, E. Moulines, and D. Stoffer, Nonlinear Time Series: Theory, Methods and Applications with R Examples, 1st ed. Chapman and Hall/CRC, January 2014.

Appendix A. Proofs of properties of ψ-mixing sequences

Proof of Lemma 1:
Proof. From (18) and (17), it follows that \( \psi(\tau) \) can be alternatively be written as

\[
\psi(\tau) = \sup_{t \in \mathbb{N}} \sup_{A \in \mathcal{F}_t^1, B \in \mathcal{F}_{t+\tau+1}^\infty, \Pr(X_1^t \in A) > 0, \Pr(X_{t+\tau+1}^\infty \in B) > 0} \frac{\Pr(X_{t+\tau+1}^\infty \in B | X_1^t \in A)}{\Pr(X_{t+\tau+1}^\infty \in B)} - 1
\]

From (54), it follows that exists \( \exists \lambda_\tau \rightarrow 0 \) as \( \tau \rightarrow \infty \) such that \( \forall t \in \mathbb{N}, \forall \tau \in \mathcal{W}, \forall A \in \mathcal{F}_t^1, \forall B \in \mathcal{F}_{t+\tau+1}^\infty, \Pr(X_1^t \in A) > 0, \Pr(X_{t+\tau+1}^\infty \in B) > 0, \)

\[
\left| \Pr(X_{t+\tau+1}^\infty \in B | X_1^t \in A) - \Pr(X_{t+\tau+1}^\infty \in B) \right| \leq \lambda_\tau \Pr(X_{t+\tau+1}^\infty \in B)
\]

From (55), it follows that exists \( \exists \lambda_\tau \rightarrow 0 \) as \( \tau \rightarrow \infty \) such that \( \forall t \in \mathbb{N}, \forall \tau \in \mathcal{W}, \forall A \in \mathcal{F}_t^1, \forall B \in \mathcal{F}_{t+\tau+1}^\infty, \Pr(X_1^t \in A) > 0, \Pr(X_{t+\tau+1}^\infty \in B) > 0, \)

\[
(1 - \lambda_\tau) \Pr(X_{t+\tau+1}^\infty \in B) \leq \Pr(X_{t+\tau+1}^\infty \in B | X_1^t \in A)
\]

Specializing (56), it follows that,

\[
(1 - \lambda_\tau) \Pr(X_{t+\tau+1}^{t+\tau+T} \in B) \leq \Pr(X_{t+\tau+1}^{t+\tau+T} \in B | X_1^t \in A)
\]

\( \forall t \in \mathbb{N}, \forall \tau \in \mathcal{W}, \forall T \in \mathcal{W}, \forall A \subset X_t^t, \forall B \subset X_T^T, \Pr(X_1^t \in A) > 0, \Pr(X_{t+\tau+1}^{t+\tau+T} \in B) > 0. \)

Note that \( \Pr(X_{t+\tau+1}^{t+\tau+T}) = P_T(B) \). Substituting this into (57), it follows that

\[
(1 - \lambda_\tau) P_T(B) \leq \Pr(X_{t+\tau+1}^{t+\tau+T} \in B | X_1^t \in A)
\]

If \( \lambda_\tau = 0 \), it follows from (55), that for \( \Pr(X_{t+\tau+1}^{t+\tau+T} \in B) > 0, P(X_1^t \in A) > 0, \)

\[
\Pr(X_{t+\tau+1}^{t+\tau+T} \in B | X_1^t \in A) = (1 - \lambda_\tau) P_T(B)
\]

and the above equation also holds if \( P_T(B) = 0 \) but \( P(X_1^t \in A) > 0 \); thus, (19) holds with any probability distribution \( P'_{t, \tau, T, A} \) on \( X_T^T \).

If \( \lambda_\tau > 0 \), define

\[
P'_{t, \tau, T, A}(B) = \frac{P(X_{t+\tau+1}^{t+\tau+T} \in B | X_1^t \in A) - (1 - \lambda_\tau) P_T(B)}{\lambda_\tau}
\]
From (58), it follows that $\forall t \in \mathbb{N}, \forall \tau \in \mathbb{W}, \forall T \in \mathbb{W}, \forall A \in \mathbb{X}^t, \forall B \in \mathbb{X}^T, P(X_1^t = A) > 0, P(X_{t+\tau+1}^t \in B) > 0$,

$$P(X_{t+\tau+1}^t \in B \mid X_1^t = A) = (1 - \lambda_\tau)P_T(B) + \lambda_\tau P_{t,\tau,T,A}(B)$$

(61) for some probability distribution $P_{t,\tau,T,A}$ on $\mathbb{X}^T$ which may depend on $t, \tau, T, A$.

Finally, note that if $P_T(B) = 0$, (19) still holds with definition (60) for $P_{t,\tau,T,A}$ since all the three probabilities in question are individually zero.

This finishes the proof of the lemma. \hfill \Box

Proof of Lemma 2:

Proof. In order to prove this lemma, it is sufficient to prove the condition on Page 19 in [2] (which implies ergodicity as is proved on the same page of [2]), and which can be re-stated as

$$\lim_{N \to \infty} \frac{1}{N} \sum_{\tau=0}^{N-1} P(X_1^t = a_1^t, X_{\tau+\tau+1}^t = b_1^T) = P(X_1^t = a_1^t)P(X_{\tau+\tau+1}^t = b_1^T)$$

(62) $\forall t \in \mathbb{N}, \forall T \in \mathbb{N}, \forall a_1^t \in \mathbb{X}^t, \forall b_1^T \in \mathbb{X}^T$.

To this end, note, first, that from (55), it follows that $\exists \lambda_\tau \to 0$ as $\tau \to \infty$ such that $\forall t \in \mathbb{N}, \forall A \in \mathcal{F}_1^t, \forall B \in \mathcal{F}_{t+\tau+1}^\infty, P(X_1^t = A) > 0, P(X_{t+\tau+1}^t = B) > 0$,

$$P(X_{t+\tau+1}^t \in B) \leq P(X_{t+\tau+1}^t = B \mid X_1^t = A) \leq (1 + \lambda_\tau)P(X_{t+\tau+1}^t = B)$$

(63)

Thus, $\exists \lambda_\tau \to 0$ as $\tau \to \infty$ such that $\forall t \in \mathbb{N}, \forall A \in \mathcal{F}_1^t, \forall B \in \mathcal{F}_{t+\tau+1}^\infty, P(X_1^t = A) > 0, P(X_{t+\tau+1}^t = B) > 0$,

$$P(X_1^t = A)P(X_{t+\tau+1}^t = B) \leq P(X_{t+\tau+1}^t = B \mid X_1^t = A) \leq (1 + \lambda_\tau)P(X_{t+\tau+1}^t = B)$$

(64)

If $P(X_1^t = a_1^t) = 0$, then both the left hand side and the right hand side in (62) are zero. If $P(X_{t+\tau+1}^t = b_1^T) = 0$, by use of the assumption that $X$ is stationary and thus noting that $P(X_{\tau+\tau+1}^t) = P(X_1^t)$, it follows that both the
left hand side and the right hand side in (62) are zero. If neither $P(X^t_1 = a^t_1) = 0$ nor $P(X^T_1 = b^T_1) = 0$ is zero, it follows from (64) that for $\tau \geq t$,

\[
(1 - \lambda_{\tau-t})P(X^t_1 = a^t_1)P(X^{\tau+T}_1 = b^T_1) \\
\leq P(X^t_1 = a^t_1, X^{\tau+T}_{\tau+1} = b^T_1) \\
\leq (1 + \lambda_{\tau-t})P(X^t_1 = a^t_1)P(X^{\tau+T}_{\tau+1} = b^T_1)
\]

Denote

\[
C \triangleq \sum_{\tau=0}^{t+1} P(X^t_1 = a^t_1, X^{\tau+T}_{\tau+1} = b^T_1)
\]

It follows from (65) by taking a sum over $\tau$ that and by noting that since the process is stationary, $P(X^{\tau+T}_{\tau+1} = b^T_1) = P(X^T_1 = b^T_1)$ and substituting (66) in (65)

\[
C + \left( N - t - \sum_{\tau=t}^{N-1} \lambda_{\tau-t} \right)P(X^t_1 = a^t_1)P(X^T_1 = b^T_1) \\
\leq \sum_{\tau=0}^{N-1} P(X^t_1 = a^t_1, X^{\tau+T}_{\tau+1} = b^T_1) \\
\leq C + \left( N - t + \sum_{\tau=t}^{N-1} \lambda_{\tau-t} \right)P(X^t_1 = a^t_1)P(X^T_1 = b^T_1)
\]

After noting that $C$ and $t$ are constants, that $\lambda_{\tau} \to 0$ as $\tau \to \infty$, after dividing by $N$ and taking limits as $N \to \infty$ in (67), it follows that

\[
\lim_{N \to \infty} \sum_{\tau=0}^{N-1} P(X^t_1 = a^t_1, X^{\tau+T}_{\tau+1} = b^T_1) = P(X^t_1 = a^t_1)P(X^T_1 = b^T_1)
\]

thus proving (62), and thus, proving that the process $X$ is ergodic if it is stationary, $\psi$-mixing.

**Proof of Lemma 3:**

**Proof.** Consider the two-sided extension $V = (V_t, t \in \mathbb{Z})$ of $X$, defined on a probability space $(\Omega'', \Sigma'', P'')$. That is,

\[
P'(V_{t+1} = j | V_t = i) = p_{ij}, -\infty < t < \infty
\]
where \( p_{ij} \) denotes the probability

\[
P(X_{t+1} = j | X_t = i), \quad 1 \leq t < \infty
\]

which is independent of \( t \) since \( X \) is Markov. Such an extension is possible, see for example \[10\]. Denote by \( X^\mathbb{Z} \), the set of doubly-infinite sequences taking values in \( X \). The Borel-sigma field on \( X^\mathbb{Z} \) is the standard construction, see Pages 1-5 of \[2\]. Note that \( V \) is finite-state, stationary, irreducible, aperiodic.

Denote the Borel-sigma field on \( X^\mathbb{Z} \) by \( H_{-\infty}^\infty \) and as was the case when defining \( F_a \), denote the Borel sigma-field on \( X_a^\mathbb{Z} \) by \( H_a^b \), \(-\infty \leq a \leq b \leq \infty\).

For the process \( V \), consider the standard definition of \( \psi \)-mixing as stated in \[3\], and thus, define

\[
\psi_V(\tau) \triangleq \sup_{t \in \mathbb{Z}} \sup_{K \in H_{-\infty}^t, L \in H_{t+1}^\infty} \left| \frac{P'(V_{-\infty}^t \in K, V_{t+1}^\infty \in L)}{P'(V_{-\infty}^t \in K)P'(V_{t+1}^\infty \in L)} - 1 \right|
\]

The process \( V \) is said to be \( \psi \)-mixing if \( \psi_V(\tau) \to 0 \) as \( \tau \to \infty \). Since \( V \) is stationary, irreducible, aperiodic, finite-state Markov chain, by Theorem 3.1 of \[3\], \( V \) is \( \psi \)-mixing.

Let \( A \in F_1 \). Consider the set \( A' \) defined as follows:

\[
A' = \{(..., a_{-n}, \ldots a_{-1}, a_0, a_1, \ldots, a_t) | (a_1, a_2, \ldots, a_t) \in A \}
\]

Then, since \( X \) is stationary and \( V \) is the double-sided extension of \( X \),

\[
P'(V_{-\infty}^t \in A') = P(X_1^t \in A)
\]

and by use of the Markov property, and again, noting that \( V \) is the double-sided extension of \( X \), it follows that

\[
P'(V_{-\infty}^t \in A', V_{t+1}^\infty \in B) = P(X_1^t \in A, X_{t+1}^\infty \in B)
\]

By use of (73) and (74), it follows that

\[
\left| \frac{P'(V_{-\infty}^t \in A', V_{t+1}^\infty \in B)}{P'(V_{-\infty}^t \in A')P'(V_{t+1}^\infty \in B)} - 1 \right| = \left| \frac{P(X_1^t \in A, X_{t+1}^\infty \in B)}{P(X_1^t \in A)P(X_{t+1}^\infty \in B)} - 1 \right|
\]
where $A \in \mathcal{F}_t^1, B \in \mathcal{F}_{t+\tau+1}^\infty, P(X_1^t \in A) > 0, P(X_{t+\tau+1}^\infty \in B) > 0$.

Thus,

$$
\sup_{A \in \mathcal{F}_1^t, B \in \mathcal{F}_{t+\tau+1}^\infty, P(X_1^t \in A) > 0, P(X_{t+\tau+1}^\infty \in B) > 0} \frac{P''(V_{-\infty}^t \in A''', V_{t+\tau+1}^\infty \in B) - 1}{P''(V_{-\infty}^t \in A''') P''(V_{t+\tau+1}^\infty \in B)} = \sup_{A \in \mathcal{F}_1^t, B \in \mathcal{F}_{t+\tau+1}^\infty, P(X_1^t \in A) > 0, P(X_{t+\tau+1}^\infty \in B) > 0} \frac{P(X_1^t \in A, X_{t+\tau+1}^\infty \in B) - 1}{P(X_1^t \in A) P(X_{t+\tau+1}^\infty \in B)}
$$

Thus,

$$
\sup_{\mathcal{K} \in \mathcal{G}_{t+\tau+1}^L, \mathcal{L} \in \mathcal{G}_{t+\tau+1}^\infty, P''(V_1^t \in \mathcal{K}) > 0, P''(V_{t+\tau+1}^\infty \in \mathcal{L}) > 0} \frac{P''(V_{-\infty}^t \in \mathcal{K}, V_{t+\tau+1}^\infty \in \mathcal{L}) - 1}{P''(V_{-\infty}^t \in \mathcal{K}) P''(V_{t+\tau+1}^\infty \in \mathcal{L})} = \sup_{\mathcal{K} \in \mathcal{G}_{t+\tau+1}^L, \mathcal{L} \in \mathcal{G}_{t+\tau+1}^\infty, P''(V_1^t \in \mathcal{K}) > 0, P''(V_{t+\tau+1}^\infty \in \mathcal{L}) > 0} \frac{P(X_1^t \in \mathcal{K}, X_{t+\tau+1}^\infty \in \mathcal{L}) - 1}{P(X_1^t \in \mathcal{K}) P(X_{t+\tau+1}^\infty \in \mathcal{L})}
$$

This is because there are sets $\mathcal{K} \in \mathcal{G}_{t+\tau+1}^L$ and $\mathcal{L} \in \mathcal{G}_{t+\tau+1}^\infty$ which are not of the form $A'$ and $B'$ respectively.

Denote the function $\psi$, defined in (18) for the process $X$ by $\psi_X$. It follows from (77) that $\psi_Z(\tau) \geq \psi_X(\tau)$. Since $Z$ is $\psi$-mixing as stated above, by definition, $\psi_Z(\tau) \to 0$ as $\tau \to \infty$. Thus, $\psi_X(\tau) \to 0$ as $\tau \to \infty$, and thus, $X$ is $\psi$-mixing.

Proof of Lemma 4:

**Proof.** Stationary of $Z$ follows directly from the definition of stationarity.

Denote the $\psi$ function for $X$ and $Z$ by $\psi_X$ and $\psi_Z$ respectively. Note that the $\psi$ function for the process $Z$ can be written as follows:

$$
\psi_Z(\tau) \overset{\Delta}{=} \sup_{t \in \mathbb{N}} \sup_{\mathcal{A} \in \mathcal{F}_{t+\tau+L}^L, \mathcal{B} \in \mathcal{F}_{t+\tau+L+1}^\infty, P(X_1^t \in \mathcal{A}) > 0, P(X_{t+\tau+L+1}^\infty \in \mathcal{B}) > 0} \frac{P(X_1^t \in \mathcal{A}, X_{t+\tau+L+1}^\infty \in \mathcal{B}) - 1}{P(X_1^t \in \mathcal{A}) P(X_{t+\tau+L+1}^\infty \in \mathcal{B})}
$$

Note that when calculating the $\psi$ function for $Z$, the supremum is taken over a lesser number of sets than when calculating the $\psi$ function for $X$. It follows that $\psi_Z(\tau) \leq \psi_X(\tau)$. Since $X$ is $\psi$-mixing, $\psi_X(\tau) \to 0$ as $\tau \to \infty$. It follows that $\psi_Z(\tau) \to 0$ as $\tau \to \infty$. Thus, $Z$ is $\psi$-mixing. □
Proof of Lemma 5:

**Proof.** Note that $Z$ is stationary by Lemma 4. Thus, $Z$ is a stationary, irreducible, aperiodic, finite-state Markoff chain, evolving on a finite set, and by Lemma 3, $\psi$-mixing.

Since the set $Z$ is finite, the Borel sigma field on $Z^\infty$ can be constructed analogously to that on $X^\infty$; see Page 1-2 of [2]. Denote this Borel sigma field by $G^\infty$. Define the Borel sigma fields $G^h_a$, analogously as was done for $F^1_1$.

Denote the underlying probability space by $(\Omega', \Sigma', P')$. An element of $Z^\infty$ is denoted by $(z_1, z_2, \ldots)$ where $z_i \in Z = X^L$. The $j^{th}$ component of $z_i$ will be denoted by $z_i(j)$.

Define

\begin{equation}
\psi_X(\tau) \triangleq \sup_{t \in \mathbb{N}} \sup_{A \in F^1_1, B \in F^\infty_{t+\tau+1}} \frac{P(X^t_1 \in A, X^\infty_{t+\tau+1} \in B)}{P(X^t_1 \in A) P(X^\infty_{t+\tau+1} \in B)} - 1
\end{equation}

and

\begin{equation}
\psi_Z(\tau) \triangleq \sup_{t \in \mathbb{N}} \sup_{A' \in G^1_1, B' \in G^\infty_{t+\tau+1}} \frac{P'(Z^t_1 \in A', Z^\infty_{t+\tau+1} \in B')}{{P'}'(Z^t_1 \in A') P'(Z^\infty_{t+\tau+1} \in B')} - 1
\end{equation}

By definition, the processes $X$ and $Z$ are $\psi$-mixing if $\psi_X(\tau)$ and $\psi_Z(\tau)$ tend to zero as $\tau \to \infty$, respectively.

For $A \in F^1_1, B \in F^\infty_{t+\tau+1}$, $P(X^t_1 \in A) > 0$, $P(X^\infty_{t+\tau+1} \in B) > 0$, define,

\begin{equation}
\kappa_X(t, \tau, A, B) \triangleq \left| \frac{P(X^t_1 \in A, X^\infty_{t+\tau+1} \in B)}{P(X^t_1 \in A) P(X^\infty_{t+\tau+1} \in B)} - 1 \right|
\end{equation}

Define

\begin{equation}
k_1 \triangleq \left\lfloor \frac{t}{L} \right\rfloor
\end{equation}

\begin{equation}
k_2 \triangleq \left\lfloor \frac{t + \tau + 1}{L} \right\rfloor
\end{equation}
Assume that \( \tau \geq 4L \). It follows that \( k_1 \leq k_2 \) (a weaker assumption is possible, but this suffices).

Given \( A \) and \( B \), define \( A' \) and \( B' \) by

\[
\begin{align*}
A' &\triangleq \{(a_1,a_2,\ldots,a_{k_1L})|(a_1,a_2,\ldots,a_t) \in A\} \\
B' &\triangleq \{(b_{k_2L+1},b_{k_2L+2},\ldots)|(b_{t+\tau+1},b_{t+\tau+2},\ldots) \in B\}
\end{align*}
\]

Think, now of \( (a_1,\ldots,a_{k_1L}) \) as \( a' = (a'_1,\ldots,a'_{k_1}) \), a \( k_1 \) length sequence, where \( a'_i \in \mathbb{Z} \). This can be done by defining \( a'_i = a_{iL} \). Analogously, think of \( (b'_{k_2L+1},b'_{k_2L+2},\ldots) \) as \( (b'_{k_2+1},b'_{k_2+2},\ldots) \) where \( b'_k \) is defined analogously to how \( a'_i \) was defined. Think of \( A' \) and \( B' \), now, as sequences of elements in \( \mathbb{Z} \) in the obvious way.

Define, for \( S \in \mathcal{G}_1^q, T \in \mathcal{G}_{q+q'+1}^\infty \),

\[
\kappa_Z(q,q',S,T) &\triangleq \left| \frac{P'(Z'_1 \in S, Z'_{t+\tau+1} \in T)}{P'(Z'_1 \in S)P'(Z_{t+\tau+1} \in T)} - 1 \right|
\]

Then, it follows that for \( \tau \geq 4L \),

\[
\kappa_X(t,\tau,A,B) = \kappa_Z(k_1,k_2-k_1,A',B')
\]

Denote

\[
\mu_X(t,\tau) = \sup_{A \in \mathcal{F}_1^t, B \in \mathcal{F}_t^{\infty}, P(X'_1 \in A)>0,P(X'_{t+\tau+1} \in B)>0} \left| \frac{P(X'_1 \in A, X'_{t+\tau+1} \in B)}{P(X'_1 \in A)P(X'_{t+\tau+1} \in B)} - 1 \right|
\]

\[
\mu_Z(q,q') = \sup_{S \in \mathcal{G}_1^q, T \in \mathcal{G}_{q+q'+1}^\infty, P(Z'_1 \in S)>0,P(Z'_{t+\tau+1} \in T)>0} \left| \frac{P'(Z'_1 \in S)P'(Z_{t+\tau+1} \in T)}{P'(Z'_1 \in S)P'(Z_{t+\tau+1} \in T)} - 1 \right|
\]

It follows from (85) by taking supremum over sets \( A \in \mathcal{F}_t^t \) and \( B \in \mathcal{F}_t^{\infty} \) and then, noting that there are sets \( S \in \mathcal{G}_1^{k_1} \) and \( T \in \mathcal{G}_{k_2+1}^{\infty} \) which are not of the form \( A' \) and \( B' \), that

\[
\mu_X(t,\tau) \leq \mu_Z(k_1,k_2-k_1) = \mu_Z\left(\left\lceil \frac{t}{L} \right\rceil, \left\lceil \frac{t+\tau+1}{L} \right\rceil - \left\lceil \frac{t}{L} \right\rceil \right)
\]
Thus,

\[(88)\quad \psi_X(\tau) \leq \sup_{t \in \mathbb{N}} \mu_Z \left( \left\lfloor \frac{t}{L} \right\rfloor , \left\lfloor \frac{t + \tau + 1}{L} \right\rfloor - \left\lfloor \frac{t}{L} \right\rfloor \right) \]

The right hand side in the above equation → 0 as \( \tau \to \infty \) since \( Z \) is \( \psi \)-mixing. Thus, \( \psi_X(\tau) \to 0 \) as \( \tau \to \infty \), and thus, \( X \) is \( \psi \)-mixing. \( \square \)

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