Limitations of gauge invariance

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Abstract

Although gauge invariance preserves the values of physical observables, a gauge transformation can introduce important alterations of physical interpretations. To understand this, it is first shown that a gauge transformation is not, in general, a unitary transformation. Also, physical interpretations are based on both kinetic and potential energy expressions. While the kinetic energy is a measurable quantity, and hence gauge-invariant, the potential energy is gauge-dependent. Two basic examples are examined; one classical and the other quantum-mechanical. The aim is to show that the use of the Coulomb (or radiation) gauge is always consistent with the way that fields are generated in the laboratory. Upon gauge transformation out of the Coulomb gauge, this connection is lost, and physical interpretations can give rise to misleading inferences.
I. INTRODUCTION

There are two principal aims of this paper, although they can be thought of as two aspects of the same phenomenon. The first aim is to show that physical interpretations of a specific laboratory scenario are dependent on the gauge in which the problem is formulated. The second purpose is to demonstrate that the set of observables that remain invariant under gauge transformations does not serve to completely characterize the laboratory situation. In other words, gauge-dependent quantities participate in the description of a physical system.

Gauge invariance is a fundamental principle in both classical and quantum physics. Its origins can be regarded as being in Newtonian mechanics, where all physical phenomena arise from applied forces, and forces coming from electromagnetic phenomena can be expressed directly in terms of the electric field $E$ and the magnetic field $B$. These fields can be derived from scalar potentials $\phi$ and vector potentials $A$ that are not unique. An alteration from one set of potentials to a different set that define exactly the same fields is called a gauge transformation, and the requirement that physically measurable quantities should not depend on the choice of the gauge is called gauge invariance. A complication arises because the connection between a Newtonian formulation of a physical problem and a Hamiltonian formulation is not one-to-one. More than one Hamiltonian can correspond to the same Newtonian equation. This distinction is directly connected to the fact that Newtonian mechanics is strictly force-dependent, and hence dependent directly on the fields; whereas Hamiltonians are couched in terms of potentials, and are therefore gauge-dependent. The most common form of quantum mechanics is based on the Schrödinger equation, so quantum mechanics is a Hamiltonian formulation and thus directly gauge-dependent in its mode of expression. A first hint of possible ambiguities comes from the fact that unitary transformations do not alter physical properties, but a gauge transformation is generally not a unitary transformation. The lack of unitarity of a gauge transformation means that what one might call the “first line of defense” about physical interpretation is missing.

In the following, after confirming that gauge transformations are not generally unitary, an elementary classical example of a gauge transformation is examined. The example is the response of a charged particle to a constant electric field. This problem is most naturally expressed in terms of a scalar potential, but it can also be expressed by a vector potential. Both formulations lead to the same Newtonian equation (as they must), but a Hamiltonian
formulation of the same problem exhibits striking differences in physical interpretations. For instance, a purely scalar-potential description is energy-conserving; there is a transfer of energy that is potential into energy that is kinetic. A vector-potential description does not conserve energy; there is no potential energy term at all, so that the change in kinetic energy must be supplied from some source outside the scope of description of the problem.

The contrast in physical interpretations just mentioned for the constant electric field example can be ascribed to a departure from the Coulomb gauge (also known as the radiation gauge) that, for several reasons, is the most straightforward gauge selection. (See, for example, the textbook by J. D. Jackson[1].) The Coulomb gauge can be described as that gauge in which longitudinal fields are represented by scalar potentials and transverse fields are given in terms of vector potentials. One can be even more specific about the Coulomb gauge by considering simple limiting cases. Any static distribution of charges can be described fully by a scalar potential alone. The other extreme case is that in which no charge or current distributions whatever exist. The Maxwell equations then allow only one solution: the freely propagating plane wave. This is a transverse field, where the electric and magnetic fields are perpendicular to each other and to the propagation direction. A pure transverse field in the Coulomb gauge is described by a vector potential alone.

The first example in this paper of the constant electric field illustrates the advantages of using a scalar potential for a longitudinal field. A physical origin for a static electric field as it exists between a pair of parallel capacitor plates is quite natural, whereas a gauge transformation to the use of a vector potential alters that straightforward view of matters and replaces it with a situation that seems to be physically unreasonable. Nevertheless, the classical Newtonian equations of motion are preserved, and gauge invariance is formally satisfied.

This motivates the selection of the second example considered here, which is the quantum-mechanical treatment of a free electron in a plane-wave field. That is, the second example examines a pure transverse field. The known quantum solution of this problem comes from the Coulomb gauge, and a representation in any other gauge is found by a gauge transformation from the Coulomb gauge. That is, a departure from the “natural” or “physical” gauge for description of a transverse field leads to an intractable problem for solution without recourse to transformation from the Coulomb gauge.

When both longitudinal and transverse fields are simultaneously present, and physical
conditions are appropriate to render the transverse field in terms of the dipole approximation (i.e., \(A(t, \mathbf{r}) \rightarrow A(t)\) for the vector potential), the Coulomb gauge is commonly termed the *velocity gauge* in atomic, molecular and optical (AMO) physics. There then exists a gauge transformation that maintains the longitudinal field unchanged, but represents the dipole-approximation transverse field by a scalar potential. This is called the *length gauge* in AMO physics. The length gauge has the practical advantage of representing all fields by scalar potentials. This is not only mathematically convenient, but it suggests simple physical interpretations that follow from the visual superposition of two scalar potentials. This has been done also with nonperturbatively strong laser fields as long as the wave length is long enough to justify the dipole approximation and the field is not so strong that magnetic effects appear.

Nevertheless, there are hazards associated with the departure from the “physical gauge”; that is, the Coulomb gauge (or velocity gauge in dipole-approximation parlance). A physical interpretation has gained wide currency in the strong-field physics community that derives from the tunneling point of view that is such a convenient way to treat an all-scalar-potential AMO problem. Qualitative contradictions seem to arise that can be traced to the development of an intuition that is not based on the Coulomb gauge. Specifically, it is customary to describe the relatively weak-field end of the nonperturbative environment as the “multiphoton domain”, because photoelectron spectra reveal individual peaks that can be associated with specific multiphoton orders. The stronger-field environment is called the “tunneling domain” because photoelectron spectra become smooth and without evidence of specific multiphoton orders. The standard physical picture in the length gauge envisions a scalar-potential Coulomb attractive well representing the atomic Coulomb attraction for the electron, being deformed by a slowly-oscillating linear potential that is used to represent the external plane-wave field within the length gauge. In such a picture, a bound atomic electron confronts a finite potential barrier that can be penetrated in the quantum sense on that side of the Coulomb well that is depressed by the scalar potential of the laser (i.e. plane-wave) field. In such a picture the relatively low-intensity nonperturbative domain requires a tunneling process in order for ionization to occur, whereas the more intense field allows the initially bound electron the escape over the barrier altogether, with no tunneling required. This is exactly the reverse of the multiphoton domain vs tunneling domain assignment of names.\[2\]
Other length-gauge conceptual problems that are not supported by experiments include the prediction that all photoelectron spectra with linearly polarized fields are maximal at zero energy, as are all longitudinal momentum distributions.

In the Coulomb gauge, in which the laser field is described by a vector potential, there is no tunneling concept of ionization, and the conflict described above does not exist. If an attempt is made to find a putative “tunneling limit” within the Coulomb gauge by seeking that part of the ionization transition amplitude that exhibits the algebraic \( \exp\left(-C/E\right) \) behavior characteristic of tunneling (where \( E \) is the amplitude of the electric field \( \mathbf{E} \)), this turns out to be a small and unrepresentative part of the full ionization amplitude.

Other nonphysical consequences of using a physical intuition that is not formed within the Coulomb gauge are discussed later.

One further introductory matter must be mentioned. When the total Hamiltonian \( H \) possesses explicit time dependence, then it is not justifiable to regard the Hamiltonian as the sum of kinetic and potential energies. The kinetic energy \( T \), as a measurable quantity, presents no problems, but the remainder of the Hamiltonian does. However, the quantity \( H-T \) will be referred to simply as the potential energy. This is common practice in quantum mechanics.

II. NONUNITARITY OF GAUGE TRANSFORMATIONS

The discussion here is in terms of the Schrödinger equation, although analogous results apply as well in other formulations of quantum mechanics.

For any wave function that describes a charged particle subjected to an electromagnetic field, a gauge transformation acts through a unitary operator \( U \) to produce the transformed wave function

\[
\Psi' = U\Psi.
\]

This is often taken to be sufficient evidence that a gauge transformation is unitary. However, whereas a unitary transformation changes any operator \( O \) according to the rule

\[
O' = UOU^{-1},
\]

it is easily verified that a gauge transformation of the Schrödinger Hamiltonian operator \( H \)
produces the gauge-transformed Hamiltonian $H'$ that obeys the rule

$$H' - i \partial_t = U (H - i \partial_t) U^{-1},$$

where atomic units are used. For direct comparison with Eq. (2), the gauge transformed Hamiltonian can be written as

$$H' = UHU^{-1} - U i \partial_t U^{-1} + i \partial_t.$$  \hspace{1cm} (4)

The contrast between Eqs. (2) and (4) shows that any transformation that contains time dependence is automatically nonunitary. This is decidedly non-trivial, since any gauge change that is limited to being independent of time cannot alter the scalar potential in any way. In particular, the transformation between the velocity and length gauges would be excluded.

(Although it is outside the scope of the present article, it must be remarked that there is a widespread belief in the AMO community not only that a gauge transformation is a unitary transformation, but that any transition matrix element is automatically gauge invariant. Both beliefs are unfounded.)

### III. GAUGE TRANSFORMATION IN A CLASSICAL LONGITUDINAL FIELD EXAMPLE

The purpose of this Section is to show how a simple physical system can be altered into a seemingly unphysical problem by a gauge transformation. The example employed here involves a longitudinal field. A pure transverse field will be considered in Section V.

A particle of charge $q$ in a constant electric field $E_0$ is a one-dimensional problem since the only direction of consequence is the direction of $E_0$, taken to define the coordinate axis $x$. The electric field can be described by the scalar potential

$$\phi = -E_0 x,$$

and there is no need for a vector potential:

$$A = 0.$$  \hspace{1cm} (6)

The Hamiltonian of this classical system is

$$H = p^2 / 2m - qE_0 x.$$  \hspace{1cm} (7)
Hamilton’s equations

\[ \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = qE_0, \]  

(8)
can be combined to give the Newtonian equation

\[ m\ddot{x} = qE_0, \]  

(9)
that has the solution

\[ x = \frac{qE_0}{2m} t^2 + \dot{x}(0) t + x(0), \]  

(10)
where the dot over a quantity designates a time derivative, and \( \dot{x}(0) \) and \( x(0) \) are the initial velocity and position of the charged particle.

The obvious physical interpretation of the problem described by Eqs. (5) - (10) is that a particle of mass \( m \) and charge \( q \) is released between two parallel capacitor plates with a potential difference between them.

Now a gauge transformation is introduced. If \( \Lambda \) is the generating function of a gauge transformation, the general expressions for the transformed potentials are

\[ \phi' = \phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t} \Lambda, \]  

(11)
\[ A' = A + \nabla. \]  

(12)
When the generating function is

\[ \Lambda = -cE_0 xt, \]  

(13)
then the initial potentials (5) and (6) are transformed to

\[ \phi' = 0, \]  

(14)
\[ A' = -e_x cE_0 t, \]  

(15)
where \( e_x \) is the unit vector in the \( x \) direction. The new Hamiltonian is

\[ H' = \frac{1}{2m} (p' + qE_0 t)^2, \]  

(16)
leading to the equations of motion

\[ \dot{x} = \frac{\partial H'}{\partial p} = \frac{1}{m} (p' + qE_0 t), \quad \dot{p}' = -\frac{\partial H'}{\partial x} = 0. \]  

(17)
These equations combine to give

\[ m\ddot{x} = qE_0, \]  

(18)
which is identical to Eq. (9). This verifies gauge invariance in this simple classical example.

Despite the common solution in the two gauges, the problems described are totally different when a qualitative description is sought. In the original gauge, one can envision the laboratory environment as arising from parallel capacitor plates holding different total charges. The transformed gauge has no obvious laboratory interpretation. Furthermore, the original gauge has a time-independent Hamiltonian, associated with energy conservation, whereas the transformed gauge is explicitly time-dependent, signifying the absence of energy conservation.

It is instructive to evaluate the kinetic and potential energies that arise from the common solution (10). In the original gauge, the kinetic energy is simply

$$ T = \frac{p^2}{2m}. $$

From Eq. (8), $$ p = m \dot{x}. $$ Using the solution (10) to find $$ \dot{x} $$ leads to the result

$$ T = \frac{1}{2m} \left[ qE_0 t + m \dot{x} (0) \right]^2. $$

The potential energy $$ U = -qE_0 x $$ can be written as

$$ U = -T + \frac{m}{2} \left[ \dot{x} (0) \right]^2 - qE_0 x (0), $$

so that the sum of the kinetic and potential energies is

$$ T + U = \frac{m}{2} \left[ \dot{x} (0) \right]^2 - qE_0 x (0) = \text{const}. $$

Equation (22) confirms the conservation of energy in the original gauge.

In the transformed gauge, Eq. (17) gives the canonical momentum

$$ p' = m \dot{x} - qE_0 t, $$

so that the gauge-transformed kinetic energy is, from Eq. (17),

$$ T' = \frac{1}{2m} \left( m \dot{x} \right)^2 $$

$$ = \frac{1}{2m} \left[ qE_0 t + m \dot{x} (0) \right]^2. $$

$$ = T. $$

This result is to be expected. The kinetic energy is a measurable quantity, and so it must be preserved in a gauge transformation. On the other hand, the potential energy is altered
completely. In the original gauge, $T + U = \text{const.}$, as shown in Eq. (22). In the transformed
gauge, there is no potential energy term at all, so that

$$T' + U' = T'(t).$$  \hspace{1cm} (27)

This not only differs from Eq. (22), but the total energy is explicitly a function of time.

This elementary example – a constant electric field – thus illustrates the essential points
to be made in this paper. The conceptual existence of a constant electric field existing
between capacitor plates has been transformed into something else that seems not to have
a physical interpretation attached to it. The original gauge is the \textit{laboratory gauge} or the
\textit{physical gauge}. There must exist a charge distribution that can give rise to a constant
electric field.

The fact that a simple system in which energy is conserved can be transformed to an-
other system in which energy is not conserved is a major qualitative difference. This happens
despite the fact that the gauge-invariant electric field and kinetic energy are explicitly con-
served in the gauge transformation. Both the electric field and kinetic energy are measurable
quantities. It is the non-gauge-invariant potential energy that is the source of the radical
alteration in the apparent physical context of the problem.

It is worth repeating the conclusion just reached: \textit{even in the presence of strict gauge
invariance, the physical interpretation of the problem has been changed.}

This simple example illustrates the general principle that the relationship between the
Hamiltonian formulation of mechanics and that of Newton is not one-to-one. Hamilton’s
equations in (8) give rise to the Newtonian Eq. (9). The very different Hamilton’s equations
in (17) correspond to the same Newtonian equation (9). The fact that the electric and magnetic
fields are preserved in the gauge transformation does not make the two gauges equivalent.

All this stems from the fact that non-Newtonian formalisms (e.g., Hamilton’s equations)
imply Newton’s equations, but the reverse is not true. Since quantum physics is usually
founded on a Hamiltonian or Lagrangian basis, a change in gauge has physical implications
despite the gauge invariance of electromagnetic fields. This will be shown in Section V
below.
The elementary problem just considered – a charged particle in a constant electric field – is sufficient to establish the two basic aims of this paper. The physical interpretation of the laboratory scenario is radically altered by the change of gauge. What started as the description of a charged particle responding to a static difference of charge on two capacitor plates, becomes transformed into something difficult to describe in simple terms. In the laboratory gauge (i.e., the Coulomb gauge), there is strict conservation of total energy. The potential energy inherent in the charged particle starting at the initial conditions, becomes converted progressively to kinetic energy, with the total of both energy forms remaining constant. After gauge transformation, the increase of the kinetic energy with time is supplied by some unspecified external source. Total energy is not conserved because the external source is somehow able to transfer energy into the system to supply the kinetic energy that is required by the dynamics of the process.

The above paragraph describes the two very different physical interpretations that attach to the two different gauges. This is a clear example of the first goal of this paper: to show that physical interpretations depend on the gauge. At the same time, the second goal is demonstrated: it is not enough to preserve the values of all physical observables. In this classical example, the particle trajectory, that may be written as \( r(t) \), constitutes the observable quantity that is sufficient to define the behavior of the system. The velocity \( v(t) = \dot{r}(t) \) is an observable quantity, as is the kinetic energy \( T = \frac{mv^2}{2} \). However, the discussion just presented of the radical differences in physical interpretations had recourse to an examination of the potential energy. This cannot be over-emphasized. The potential energy plays a crucial role in the physical interpretation of the physical system, but the potential energy is not a gauge-invariant quantity.

Another way to view the longitudinal field problem is to note that a pure longitudinal field can be treated entirely in terms of electrostatics. This is a complete subject in itself that requires only a scalar potential to describe it. In that sense, a gauge transformation to a representation of the field by a vector potential is an alien concept. Although the values of all physical observables are preserved, a simple physical interpretation is altered to something entirely different that is no longer simple and natural.

The primary aims of this paper are thus already established. The next section describes
a different consequence of using a gauge that is not the physical gauge.

V. GAUGE TRANSFORMATION IN QUANTUM TRANSVERSE-FIELD EXAMPLES

As already mentioned, the freely propagating plane wave is the unique solution of the free-space Maxwell equations when there are no electrical charges or currents. This is the laboratory environment in which the Coulomb gauge comes into play in its pure form for a transverse field.

Two examples will be presented. It is sufficient here to examine these problems in a dipole-approximation context. One example relates to the solution for a free electron in the presence of a plane-wave field. It is found that the solution is straightforward in the velocity (Coulomb) gauge, but is beset by peculiarities in the length gauge. The other example refers to the ionization of an atomic electron by a plane-wave field. In this second example, treatment of the problem in the length gauge gives useful results within its domain of applicability, but suggests completely inappropriate extensions outside that region.

A. Free charged particle in a plane-wave field

In isolation, the only relevant solution for a free electron in a plane wave is the relativistic solution. This is known as the Volkov, or Gordon-Volkov solution\[3, 4\], that can be stated exactly for any wave packet of unidirectional plane waves. However, it will be convenient here to adopt a simplification that is widely used. If the free-electron solution is to be employed within a matrix element that also contains a wave function bounded in space, then it is usually admissible to treat the free particle in the dipole approximation. The limitations are that the wavelength of the field should not be so short as to make the dipole approximation inapplicable, nor should the field be so strong or of such low frequency as to induce magnetic field effects.

The velocity-gauge Schrödinger equation for the “dipole-approximation free electron” is

$$i\partial_t \Psi^V = \frac{1}{2} \left( \hat{p} - \frac{1}{c} A(t) \right)^2 \Psi^V,$$  \hspace{1cm} (28)

where atomic units are used, electromagnetic quantities like the vector potential $A(t)$ are in Gaussian units, the “hat” symbol over the canonical momentum signifies the quantum
operator $\hat{p} = -i\nabla$, and the superscript $V$ on the wave function identifies it as being in the velocity-gauge. The solution of Eq. (28) is simply

$$\Psi^V(r, t) = C \exp \left[ i\hat{p} \cdot r - \frac{i}{2} \int_{-\infty}^t d\tau \left( \hat{p} + \frac{1}{c} A(\tau) \right)^2 \right],$$

(29)

where $\hat{p}$ is the eigenvalue of the canonical momentum operator. It can also be identified as the kinetic momentum outside the bounds of the electromagnetic pulse.

In the length gauge, indicated by the superscript $L$, the Schrödinger equation corresponding to Eq. (28) is

$$i\partial_t \Psi^L = \left( \frac{1}{2}\hat{\mathbf{p}}^2 + \mathbf{r} \cdot \mathbf{E}(t) \right) \Psi^L.$$  

(30)

Equation (30) appears to be more simple than Eq. (28), yet the solution is normally found by a gauge transformation of the velocity gauge solution (29). That is, the length-gauge solution is

$$\Psi^L(r, t) = C \exp \left[ \frac{i}{c} \mathbf{r} \cdot \mathbf{A}(t) \right] \exp \left[ i\hat{p} \cdot r - \frac{i}{2} \int_{-\infty}^t d\tau \left( \hat{p} + \frac{1}{c} A(\tau) \right)^2 \right].$$

(31)

Not only is this result more complicated than Eq. (29), it is written in terms of the velocity-gauge $\mathbf{A}(t)$ rather than the length-gauge $\phi = -\mathbf{r} \cdot \mathbf{E}(t)$. One way to introduce the length-gauge potential in place of $\mathbf{A}(t)$ in Eq. (31) is to invert the velocity-gauge relation

$$\mathbf{E}(t) = -\frac{1}{c} \partial_t \mathbf{A}(t)$$

(32)

to obtain

$$\mathbf{A}(t) = -c \int_{-\infty}^t d\tau \mathbf{E}(\tau).$$

(33)

All of this maneuvering does not really solve the problem. The new expression

$$\Psi^L(r, t) = C \exp \left[ i\hat{p} \cdot r - i \int_{-\infty}^t d\tau \mathbf{r} \cdot \mathbf{E}(\tau) - \frac{i}{2} \int_{-\infty}^t d\tau \left( \hat{p} - \int_{-\infty}^\tau dt' \mathbf{E}(t') \right)^2 \right]$$

(34)

no longer contains the vector potential, but the entire expression is far more implicit than is Eq. (29) in view of the integrations submerged within Eq. (34). In contrast with the velocity gauge solution (29), which can be written expressly in terms of the vector potential $\mathbf{A}(t)$, it is not possible to write the length-gauge solution entirely in terms of the scalar potential $\mathbf{r} \cdot \mathbf{E}(t)$. Further, in order to achieve this length-gauge result, it was necessary to employ Eq. (32), which is a velocity-gauge expression. In sum, the velocity-gauge wave function is easily found in a compact, comprehensible form; whereas the length-gauge solution is complicated, unwieldy, and far from intuitive. All of these differences can be ascribed to the fact that the velocity gauge is the physical gauge for a transverse field, and the length gauge is not.
B. Atomic ionization by a plane-wave field

A velocity-gauge treatment of the ionization of a single-electron atom by a plane-wave field employs the Coulomb gauge: the Coulomb attraction between the atomic electron and the nucleus is represented by the scalar Coulomb potential, and the action of the plane-wave field is given by a vector potential.

The advantages of a length-gauge treatment of the problem are nevertheless appealing. If the dipole approximation is applicable, a gauge transformation that allows both interactions to be expressed by scalar potentials makes possible elementary addition of the two potentials to provide a picture within which the plane-wave field appears to be a simple slowly-oscillating electric field. There is then a view of the interaction in which the Coulomb attractive potential well is depressed on one side by the electric field, so as to make possible an escape of the bound electron by tunneling through a finite barrier, or by escaping over the depressed barrier. This picture can be applied (within limitations) even when the external field is strong enough to require nonperturbative treatment. Such a nonperturbative theory is a tunneling theory, and it has been found to be very useful.

A clear problem arises when the non-laboratory gauge (the length gauge) is used as a guide to extend a problem beyond the domain where the two gauges are demonstrably gauge-equivalent. The tunneling theory is a length-gauge view of strong-field ionization that represents by scalar potentials both the Coulomb potential well of the atomic nucleus and the external laser field. A simple physical picture then exists that envisions the sum of the two potentials that has the laser field forcing down alternatively one or the other side of the Coulomb well. This then presents a limited-width potential barrier that can allow a bound electron to escape from an atom by tunneling through this barrier. A nonperturbative tunneling theory of ionization contains only one scaling parameter, known as the Keldysh gamma parameter

$$\gamma = \sqrt{\frac{E_B}{2U_p}},$$  \hspace{1cm} (35)$$

where $E_B$ is the field-free binding energy of the atomic electron in the atom, and $U_p$ is the ponderomotive energy of a free electron in the external laser field. If $U_p$ is given in terms of field intensity $I$ and field frequency $\omega$, then

$$U_p = \frac{I}{(2\omega)^2}$$  \hspace{1cm} (36)
in atomic units for a linearly polarized field. Substitution of Eq. (36) into (35) yields the Keldysh parameter in the form
\[ \gamma = \frac{\omega}{I^{1/2}} \sqrt{2E_B}. \]  
(37)

This has suggested to many investigators that the $\omega \to 0$ limit can be examined by a tunneling theory with $\gamma \to 0$, and this can then be regarded as a continuous way to apply a nonperturbative theory to examine atomic ionization in the classical limit. See Ref.[9] for a representative view of a widely held attitude.

The fundamental problem here is that for a laser field, which is a transverse field, the limit $\gamma \to 0$ for constant intensity $I$ enters a domain where the magnetic field becomes of importance equal to the electric field, and the dipole approximation fails. That is, the length-gauge approximation for a strong plane-wave field breaks down completely, and the length-gauge theory no longer is capable of describing a plane wave.

This basic problem can be viewed in another way. If $\omega$ is held constant, and the intensity $I$ is allowed to go to $\infty$, this also corresponds to $\gamma \to 0$. By the universality of the $\gamma$ parameter, this should predict a classical limit that is equivalent to the fixed-$I$, $\omega \to 0$ case. However, $I \to \infty$ for a laser field is plainly a relativistic limit for atomic ionization, the dipole approximation certainly does not apply, and a transverse field cannot be represented by a scalar potential.

The non-laboratory (or non-physical) length gauge then predicts inapplicable and misleading behavior if one uses it to predict phenomena that are outside the (very limited) domain of applicability of the velocity-gauge to length-gauge transformation.

VI. CONCLUSIONS

It has been shown that a static-field environment (i.e., a longitudinal-field environment) produces physical interpretations that are entirely consistent with the laboratory means of generating a longitudinal field. A gauge transformation to a velocity-gauge representation preserves the values of all physically measurable quantities, but introduces physical interpretations that are foreign to the static-electric-field situation. This proves both that gauge transformations introduce changes in physical interpretations, and that physical interpretations not associated with the use of a Coulomb gauge will give unreasonable physical interpretations that have no conceptual or inferential value.
In the transverse-field situation that exists in the presence of a laser field, there also results a physically misleading situation if a gauge transformation is introduced that describes the laser field as a longitudinal field.

The conclusion that is thereby reached is that the radiation gauge (also known as the Coulomb gauge) should be employed if physical interpretations are sought that are consistent with the origins of the fields in the problem. The use of a gauge that does not represent longitudinal fields by scalar potentials and transverse fields by vector potentials can give radically misleading physical interpretations despite the fact that gauge invariance guarantees the preservation of all measurable quantities.

VII. CODA

Further developments related to this article may be found in Ref. [10]. Among these developments is a demonstration that potentials are more fundamental than fields; that is, a given configuration of fields does not always uniquely identify an electromagnetic environment.

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