Lévy ratchets in the spatially tempered fractional Fokker-Planck equation

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Lévy ratchets are minimal models of fluctuation-driven transport in the presence of Lévy noise and periodic external potentials with broken spatial symmetry. Lévy noise can drive the system out-of thermodynamics equilibrium and previous works have shown that in this case, a net ratchet current can appear even in the absence of time dependent perturbations, external tilting forces, or a bias in the noise. The majority of studies on the interaction of Lévy noise with external potentials have assumed α-stable Lévy statistics in the Langevin description, which in the continuum limit corresponds to the fractional Fokker-Planck equation. However, the divergence of the low order moments is a potential drawback of α-stable distributions because, in applications, the moments represent physical quantities. For example, for \( \alpha < 1 \), the current (first moment), \( J \), in α-stable Lévy ratchets is unbounded. To overcome this limitation, in this paper we study ratchet transport using truncated Lévy distributions which in the continuum limit correspond to the spatially tempered fractional Fokker-Planck equation. The main object of study is the dependence of the ratchet current on the level of tempering, \( \lambda \). For finite tempering, \( \lambda \neq 0 \), the statistics ultimately converges (although very slowly) to Gaussian diffusion in the absence of a potential. However, it is shown here that in the presence of a ratchet potential a finite non-equilibrium current persists asymptotically for any finite value of \( \lambda \). The current is observed to converge exponentially in time to the steady state value. The steady state current exhibits algebraically decay with \( \lambda \), as \( J \sim \lambda^{-\zeta} \), for \( \alpha \geq 1.75 \). However, for \( \alpha \leq 1.5 \), the steady state current decays exponentially with \( \lambda \), as \( J \sim e^{-\xi \lambda} \). In the presence of a bias in the Lévy noise, it is shown that the tempering can lead to a current reversal. A detailed numerical study is presented on the dependence of the current on \( \lambda \) and the physical parameters of the system.

I. INTRODUCTION

The problem of fluctuation driven transport has a long and interesting history, with early discussions dating back to work by Smoluchowsky in 1912. For a review, see Ref. [1] and references therein. Of particular interest are periodic potentials with broken spatial symmetry, known as ratchet potentials, see for example Fig. 1. In this case, particles in the potential wells experience an asymmetric force and one might naively expect that in the presence of thermal equilibrium fluctuations particles will, on the average, drift down-hill in the direction of the weaker force, giving rise to a net current. However, it has long been recognized that this naive picture is incorrect since net work cannot be extracted out of equilibrium thermal fluctuations. However, in the presence of non-equilibrium perturbations the situation is quite different and net currents can be generated. For example, in the case of pulsating ratchets a current can be generated by driving the system out of thermal equilibrium by adding a time variation to the amplitude of the potential. Considerable amount of work has thus been devoted to construct “minimal” models of ratchet transport which involve various far from equilibrium mechanisms. However, most of these studies have focused on models in which the fluctuations satisfy Gaussian statistics.

The early work reported in Refs. [2, 3] originally proposed a minimal model for Lévy ratchets consisting of a time-independent ratchet potential in the presence of uncorrelated Lévy noise. It was shown that, even in the absence of an external tilting force or time dependence in the potential, or a bias in the noise, the Lévy flights drive the system out of thermodynamic equilibrium and generate an up-hill current (i.e., a current in the direction of the steeper side of the asymmetric potential). Following this, Ref. [4] studied the Lévy ratchet problem and proposed robust probability measures of directionality of transport. In Ref. [5] the influence of periodically modulated Lévy noise asymmetry was studied. Beyond the study of ratchet transport, the study of the interaction of Lévy noise with external potentials has been a topic of considerable recent interest. Some examples include the study harmonic [6, 7] and non-harmonic [8–10] potentials, and the study of the Kramers’ problem in Refs. [11–14].

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Lévy statistics has been used to model a wide range of problems involving probability density functions (pdfs) with slowly decaying tails. However, an often overlooked issue is the differences that typically exist between the strict mathematical properties of Lévy distributions and the pdfs found in practical problems involving experimental or numerical simulations. For example, whereas the second and higher moments of $\alpha$-stable Lévy distribution with $1 < \alpha < 2$ are formally infinite, the moments of pdfs of practical interest are finite. One of the reasons why pdfs of practical interest have finite moments is that they typically have finite support because the statistical samples upon which they are constructed are finite. For example, pdfs with algebraic decaying tails have been observed in the description of particle displacements in turbulent transport [15]. These numerically determined pdfs have finite moments because their support is limited by the largest possible displacement which is finite. Another key issue is that in applications, the moments represent physical quantities that cannot be infinite. This issue is particularly delicate in the study of Lévy flights in external potentials where $\alpha$-stable Lévy noise is added to the Langevin stochastic equation for the particle velocity. In this case, the divergence of the second moment can lead in principle to a divergence in the kinetic energy. In practical terms this problem might not be disastrous because any numerical realization of the numerical model will have finite moments. Nevertheless this shortcoming of the Lévy $\alpha$-stable Langevin model, which is also present in the corresponding $\alpha$-stable fractional Fokker-Planck model, has to be resolved. In this paper we propose to use the truncated fractional diffusion operators introduced in Ref. [19] to address this problem.

Truncated Lévy distributions distributions were originally introduced in Refs. [17] [18] as a simple prescription to guarantee the finiteness of the second moment. Reference [19] considered a general class of multivariate tempered stable Lévy processes and established their parametrization and probabilistic representations. General Lévy processes, and in particular tempered stable processes, were incorporated in the continuous time random walk model and in macroscopic transport models in Ref. [15]. This reference introduced the notion of truncated fractional derivatives in the construction of non-diffusive transport models driven by truncated Lévy flights. The role of tempered Lévy distribution in the super-diffusive propagation of fronts in reaction-diffusion systems has been recently studied in Ref. [20].

The goal of this paper is to study the role of truncation on fluctuation-driven transport with Lévy noise. Our approach is based on the numerical solution of the spatially tempered Fractional Fokker-Planck equation, obtained by replacing the diffusion operator by a truncated fractional diffusion operator. We use two complementary numerical methods. One is based on the finite difference, Grunwald-Letnikov discretization of the truncated fractional derivatives that allows the computation of the space-time evolution of the pdf in a finite size domain. The second method is a Fourier based spectral method that allows the computation of the reduced pdf in a periodic domain as well as the time dependent and asymptotic steady state current. The main object of study is the dependence of the steady state current on the level of truncation, $\lambda$, the stability index $\alpha$, and the noise asymmetry $\theta$ as well as the asymmetry of the potential.

The rest of the paper is organized as follows. In the next section we discuss the setting of the Lévy ratchet problem and introduce the spatially tempered Fractional Fokker Planck equation. Section III presents the numerical results obtained from the solution of the time dependent and the steady state tempered Fractional Fokker Planck equation. Section IV contains the conclusions.

II. SPATIALLY TEMPERED FRACTIONAL FOKKER-PLANCK EQUATION

The most natural formulation of the problem of fluctuation-driven transport in the presence of an external potential is based on the Langevin approach. The starting point is an ensemble of particles with coordinates $x(t)$ which in the over-damped case evolve according to the stochastic differential equation

$$dx = - (\partial_x V) dt + d\eta$$  \hspace{1cm} (1)

where for simplicity, and without loss of generality, we have assumed a one-dimensional domain. $V(x)$ denotes the external potential, and $d\eta$ is a stochastic processes modeling the fluctuations. In the case of uncorrelated, Gaussian fluctuations, $d\eta = \sqrt{2} dW$, where $dW$ is a stochastic Wiener process, with $\langle dW \rangle = 0$ and $\langle dW^2 \rangle = dt$. As it is well-known, if the individual particles evolve according to the Langevin equation (1) with uncorrelated Gaussian noise, then the probability density function (pdf) satisfies the Fokker-Plank equation

$$\partial_t P = \partial_x [P \partial_x V] + \chi \partial_x^2 P.$$  \hspace{1cm} (2)

Of particular relevance to the present paper is the study of periodic potentials with broken spatial symmetry. These, potentials, also called “ratchets”, satisfy $V(x) = V(x + L)$ where $L$ is the length of the period but they lack reflection symmetry. A paradigmatic example is $V = V_0 [\sin(2\pi x/L) + 0.25 \sin(4\pi x/L)]$. For an easier control of the degree of spatial symmetry, following [9], here we use

$$V = V_0 \begin{cases} 1 - \cos[\pi x/a_1] & \text{if } 0 \leq x < a_1 \\ 1 + \cos[\pi (x-a_1)/a_2] & \text{if } a_1 \leq x < L \end{cases}.$$

(3)

where $V_0$ is the amplitude, $L = a_1 + a_2$ is the period, and $A = (a_1 - a_2)/L$ is the asymmetry parameter. In all the calculations presented here, $V_0 = L = 1$. In the study of ratchets it is customary to add an external constant “tilting” force $F$ to the potential, and to consider the effective potential $V_{eff} = V - Fx$. However, unless...
The conditions for the existence of ratchet currents in the presence of diffusive transport have been extensively studied and are well-understood, see for example Ref. [10] and references therein. However this is not the case for fluctuation-driven transport in the presence of non-diffusive transport. To study the role non-diffusive transport in fluctuation-driven ratchet transport, Ref. [3] considered the Langevin model but with $d\eta$ corresponding to a Lévy process. In this case, the continuum limit of the Langevin equation leads to the $\alpha$-stable fractional Fokker-Planck equation

$$\partial_t P = \partial_x \left[ P \partial_x V \right] + \chi \partial_x^\alpha P ,$$

where

$$\partial_x^\alpha P = l_{-\infty} D_x^\alpha P + r_x D_x^\alpha P .$$

The operators on the right hand side of Eq. (5) are the left and right Riemann-Liouville fractional derivatives which are integro-differential operators defined in Fourier space according to [21, 22]

$$\mathcal{F} [ \partial_x^\alpha P ] = (-ik)^\alpha \hat{\mathcal{P}} , \quad \mathcal{F} [ D_x^\alpha P ] = (ik)^\alpha \hat{\mathcal{P}} ,$$

where $\mathcal{F} [ f ] (k) = \hat{f} = \int f(x) e^{ikx} dx$. The factors

$$l = -\frac{1 - \theta}{2 \cos(\alpha \pi / 2)} , \quad r = \frac{1 + \theta}{2 \cos(\alpha \pi / 2)} ,$$

determine the relative weight of the left and the right fractional derivatives in terms of the parameter \( \theta \). The order of the fractional operators $\alpha$, the asymmetry parameter $\theta$, and the diffusivity $\chi$ correspond to the index, the skewness, and the scale factor of the characteristic function of the Lévy noise in the Langevin formulation.

From the statistical mechanics point of view, the difference between the transport operators in Eqs. (2) and (4) rests on the different assumptions of the underlying stochastic process. As it is well known, the standard diffusion operator in Eq. (2) assumes a Brownian random walk, whereas, in the context of the continuous time random walk (see for example Ref. [23] and references therein) the fractional derivative operators in Eq. (4) assume an $\alpha$-stable Lévy process describing the jumps of particles. One might naively expect that these two cases encompass all the fundamentally different transport operators in the absence of memory and correlations. This expectation is based on the fact that according to the generalized central limit theorem the sum of individual random particle displacements asymptotically converge to either a Gaussian (described by the standard diffusion equation) or an $\alpha$-stable Lévy distribution (described by the $\alpha$-stable fractional diffusion equation). However, an issue of significant physical relevance is that the convergence to the Gaussian or $\alpha$-stable behaviors could be extremely slow in problems of practical interest. For example, in the case of truncated Lévy flights, it is known [17, 21, 23] that the crossover time scale, $\tau_c$, of the convergence to Gaussian behavior scales algebraically $\tau_c \sim \lambda^{-\alpha}$ where $\lambda$ is the level of truncation. Such slow algebraic convergence indicates that for the intermediate time scales of practical interest, these type of processes cannot accurately be described as Gaussian or $\alpha$-stable. Motivated by this, Ref. [16] considered the continuous time random walk for jump processes corresponding to general Lévy processes in the Lévy-Khintchine representation, and obtained integro-differential operators describing transport in the continuous, fluid limit. These operators contain as special cases the regular (Laplacian) and the $\alpha$-stable fractional diffusion operators (Riemann-Liouville fractional derivatives), and, most importantly, allow the incorporation of a richer class of stochastic processes of physical relevance. In particular, for the case of truncated Lévy processes, Ref. [10] proposed the truncated fractional diffusion transport operator

$$\partial_x^{\alpha,\lambda} P = D_x^{\alpha,\lambda} P + \nu \partial_x P - \nu P , \tag{8}$$

where $D_x^{\alpha,\lambda}$ is the $\lambda$-truncated fractional derivative operator of order $\alpha$, defined as

$$D_x^{\alpha,\lambda} = l e^{-\lambda x} \int_{-\infty}^{0} e^{\lambda x} D_x^{\alpha} e^{-\lambda x} + r e^{\lambda x} D_x^{\alpha} e^{-\lambda x} , \tag{9}$$

where $\int_{-\infty}^{0} D_x^{\alpha}$ and $\int D_x^{\alpha}$ are the Riemann-Liouville derivatives in Eq. (6). The effective advection velocity is defined as

$$v = \left\{ \begin{array}{ll} \chi \alpha / \cos(\alpha \pi / 2) & 0 < \alpha < 1 \\ \chi \alpha / \cos(\alpha \pi / 2) & 1 < \alpha < 2 \end{array} \right. \tag{10}$$

and

$$\nu = -\frac{\lambda^{\alpha}}{\cos(\alpha \pi / 2)} . \tag{11}$$

As expected, for $\lambda = 0$, the transport operator in Eq. (8) reduces to the $\alpha$-stable fractional diffusion operator in
based on the previous discussion, we propose the following spatially-tempered Fractional Fokker-Planck equation

$$\partial_t P = \partial_x [P \partial_x V] + \chi \partial_x^\alpha P,$$  

(12)

to study the role of truncation effects on Lévy flights in the presence of external potentials in the intermediate asymptotic regime. In particular, the study presented here of the role of truncation in fluctuation-driven transport in the presence of Lévy flights is based on the numerical integration of Eq. (12). Taking the Fourier transform of Eq. (12), we get

$$\partial_t \tilde{P} = \Lambda(k) \tilde{P} - i k \mathcal{F}[P \partial_x V](k),$$  

(13)

where \( \Lambda(k) \) is the logarithm of the characteristic exponent of the tempered Lévy process

$$\Lambda = -\frac{\chi}{2 \cos(\alpha \pi/2)} \chi \frac{\cos(\alpha \pi/2)}{\sin(\alpha \pi/2)} \left\{ \begin{array}{l} (1 + \theta)(\lambda + ik)^\alpha + (1 - \theta)(\lambda - ik)^\alpha - 2 \lambda^\alpha, \\
(1 + \theta)(\lambda + ik)^\alpha + (1 - \theta)(\lambda - ik)^\alpha - 2 \lambda^\alpha - 2 i k \alpha \theta \lambda^\alpha - 1 
\end{array} \right\}$$  

(14)

for \( 0 < \alpha < 1 \) and \( 1 < \alpha \leq 2 \) respectively.

III. NUMERICAL RESULTS: RATCHET CURRENT

The spatio-temporal evolution of the pdf in the finite domain (0,1) was obtained by solving numerically the spatially tempered fractional Fokker-Planck equation using a finite difference method. Following Ref. [26], we use regularized (in the Caputo sense) fractional derivatives in space. However, here we factorize the regularized operators \( \xi \partial_x \), using the Grunwald-Letnikov representation. The second order derivative, \( \partial_x^2 \), was discretized using a central difference method. Following Ref. [3], we discretized the time evolution using an operation splitting method that separates the fractional derivative operators from the advection and potential terms. The fractional derivative operators (half) time step was done using a weighted average Crank-Nicholson method.

Figure 2 shows snapshots in time of the numerically calculated pdf’s for different levels of truncation. The initial condition corresponds to a distribution of particles localized in the potential well at the middle of the computational domain

$$P(x,0) = \frac{1}{\sigma \sqrt{2 \pi}} \exp \left[ -\frac{(x - 1/2)^2}{\sigma^2} \right],$$  

(15)

with \( \sigma = 0.12 \). Compared to the \( \lambda = 0 \) (\( \alpha \)-stable) case, it is observed that the tempering reduces the “leakage” of the pdf out of the potential well. However, for both \( \lambda = 0 \) and \( \lambda = 3 \) the profile peaks are higher on the right than on the left, indicating the presence of a net current.

The existence of the current for \( \lambda = 0 \) is consistent with the results previously reported in Ref. [3]. The understanding and characterization of the current for \( \lambda \neq 0 \) is one of the objectives of this section.

Given a solution, \( P \), of the tempered fractional Fokker-Planck equation, the current is defined as the rate of change of the first moment,

$$J = \frac{d\langle x \rangle}{dt} = -i \frac{\partial}{\partial k} \left( \frac{\partial \tilde{P}}{\partial k} \right)_{k=0}.$$  

(16)

From Eqs. (16) and (13), it follows that

$$J = -\mathcal{F}[P \partial_x V] \Big|_{k=0} - i \left( \frac{d\Lambda}{dk} \right)_{k=0}$$  

(17)

In the \( \alpha \)-stable case,

$$\lambda = 0 \implies \left( \frac{d\Lambda}{dk} \right)_{k=0} = \left\{ \begin{array}{l} \infty \quad 0 < \alpha < 1 \\
0 \quad 1 < \alpha < 2 \end{array} \right.$$  

(18)

According to this, without tempering, the current is not well-defined for \( \alpha < 1 \) due to the divergence of the first moment of Lévy distributions. However, the introduction of tempering leads to a well-defined current because in this case

$$\lambda \neq 0 \implies \left( \frac{d\Lambda}{dk} \right)_{k=0} = \left\{ \begin{array}{l} -\frac{\chi \alpha \theta \lambda^\alpha - 1}{\cos(\alpha \pi/2)} \quad 0 < \alpha < 1 \\
0 \quad 1 < \alpha < 2 \end{array} \right.$$  

(19)

The asymptotic steady state current can be obtained numerically by advancing the solution for \( P \) to long times as done in Ref. [3]. However, here we follow a more direct approach based on the computation of the steady state
solution in Fourier space, which can be formulated as an eigenvalue problem as shown below. For a general periodic potential, $V(x + L) = V(x)$,

$$\partial_x V = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x},$$

(20)

where $k_n = 2\pi n / L$, we have

$$\mathcal{F}[P \partial_x V] = \sum_{n=-\infty}^{\infty} c_n \tilde{P}(k + k_n, t).$$

(21)

Evaluating this expression at $k = 0$ and substituting into Eq. (17) we get the following expression for the current

$$J = - \sum_{n=-\infty}^{\infty} c_n \tilde{P}_n - i \left( \frac{d\Lambda}{dk} \right)_{k=0},$$

(22)

where $\tilde{P}_n = \tilde{P}(k = k_n)$ is the Fourier transform of the steady state solution. To determine the time evolution of the $n$-th Fourier mode, $\tilde{P}_n(t)$, we evaluate Eq. (13) at $k = k_n$ and get the following set of coupled differential equations

$$\partial_t \tilde{P}_n = \Lambda_n \tilde{P}_n - ik_n \sum_{j=-\infty}^{\infty} c_j \tilde{P}_{n+j},$$

(23)

where $\Lambda_n = \Lambda(k = k_n)$. The steady state asymptotic current, $J(\infty)$, is obtained by setting $\partial_t \tilde{P}_n = 0$ in Eq. (23) and solving the set of algebraic equations

$$\Lambda_n \tilde{P}_n - ik_n \sum_{j=-\infty}^{\infty} c_j \tilde{P}_{n+j} = 0.$$

(24)

For the numerical solution of Eqs. (23) and (24), we truncated the infinite series at a large value of $n = N$ and solved the system with the normalization constrain $\tilde{P}_0 = 1$ (as required by the conservation of probability). In the calculations reported here $N$ ranged from 500 to 2000. The unique solution $\{\tilde{P}_n\}$ was then used in Eq. (22) to evaluate the current. The corresponding reconstructed pdf in real space gives the reduced pdf, $P_r$,

$$P_r(x) = \sum_{n=-N}^{N} P_n e^{ik_n x},$$

(25)

which is the steady state solution of the tempered fractional Fokker-Planck equation in a periodic domain. This solution can also be obtained by "folding" the unbounded domain solution in the $(0, L)$ domain [1]

$$P_r = \sum_{n=-\infty}^{\infty} P(x + nL, t).$$

(26)

Figure 3 shows the reduced steady state pdf in the $\alpha$-stable case, i.e., $\lambda = 0$, for different values of $\alpha$.

![FIG. 3: Steady state solution of the reduced pdf in the periodic domain for different values of $\alpha$, and $\lambda = 0$, $\theta = 0$, $V_0 = 1$, $A = -0.274$, and $\chi = 0.5$.](image)

In all the numerical results, the current has been non-dimensionalized using the scale $L/\tau$, where $\tau = L^\alpha / \chi$ is the fractional diffusion time scale. As expected, for $\alpha = 2$, $P_r$ corresponds to the Boltzmann distribution $P_r(x) = (1/Z) \exp(-V(x)/\chi)$, where $Z$ is the normalization constant. As the value of $\alpha$ decreases, the pdf significantly departs from the Boltzmann distribution and develops the asymmetry responsible for the finite net current observed for $\alpha < 2$. The dependence of the reduced pdf on $\lambda$ for a fixed value of $\alpha$ is shown in Fig. 4. The leakage of the pdf outside the potential wells is significantly reduced for large values of $\lambda$. As we will discuss later, in general the current decreases with increasing $\lambda$, but the pdf does not approach the diffusive Boltzmann distribution.

Figure 5 shows the dependence of the current on the value of $\alpha$ for different levels of tempering. In agreement with the results reported in Ref. [5], in the absence
of tempering, \( \lambda = 0 \), a net current is observed for \( \alpha < 2 \) with a maximum around \( \alpha \approx 1.4 \). Most importantly, as the figure shows, a finite (albeit smaller) current remains for \( \lambda \neq 0 \). That is, although for \( \lambda \neq 0 \), the statistics of the tempered fractional operator in Eq. (8) ultimately converges to Gaussian behavior [10], in the presence of a ratchet potential a net non-equilibrium current persists in the steady state asymptotic regime. As expected, regardless of the value of \( \lambda \), in the limit \( \alpha = 2 \), the noise becomes Gaussian and the current vanishes. It is observed that as the truncation increases, the value of \( \alpha \) for which the maximum current is attained shifts to the right, and for large values of \( \lambda \) the strength of the current exhibits only a weak dependence on \( \alpha \). To study the rate of convergence to the steady state asymptotic current, we show in Fig. 6 \( \delta J(t) = J(t) - J(\infty) \) as function of time where \( J(t) \) is the time dependent transient current and \( J(\infty) \) is the steady state asymptotic current. An exponential convergence of the form

\[
\delta J \sim e^{-\eta t},
\]

(27)
is observed, and as Table I shows, the decay rate, \( \eta \), exhibits a very weak dependence on the value of \( \alpha \) and \( \lambda \).

| \( \alpha \) | 1.25 | 1.25 | 1.25 | 1.5 | 1.5 | 1.5 | 1.75 | 1.75 | 1.75 |
|---|---|---|---|---|---|---|---|---|---|
| \( \lambda \) | 0 | 10 | 30 | 0 | 10 | 30 | 0 | 10 | 30 |
| \( \eta \) | 64 | 58 | 58 | 60 | 57 | 58 | 52 | 51 | 53 |

As the previous calculations show, as \( \lambda \) increases the current is reduced, and a problem of significant interest is to find the rate of decay of the current in the asymptotic limit \( \lambda \to \infty \). As Figs. 7 and 9 show, depending on the value of \( \alpha \), two asymptotic regimes are observed. For \( \alpha \) near the Gaussian value \( \alpha = 2 \), the current exhibits an algebraic decay for large \( \lambda \) of the form

\[
J \sim \lambda^{-\zeta},
\]

(28)

where \( \zeta \approx 2.5 \) for \( \alpha = 1.9 \) and \( \zeta \approx 6 \) for \( \alpha = 1.75 \). This slow algebraic scaling breaks down as \( \alpha \) is reduced from the Gaussian value. In particular, as Fig. 8 shows, for \( 1 < \alpha < 1.5 \) the current decays exponentially fast

\[
J \sim e^{-\xi\lambda},
\]

(29)

with \( \xi \approx 0.40 \). As Fig. 9 verifies, the same scaling is observed for \( \alpha < 1 \).

Figure 10 shows the dependence of the current, \( J \), on the potential asymmetry, \( A \). As expected, in all cases the current vanishes when the potential is symmetric, \( A = 0 \). Consistent with the results reported in [3], the curve \( \lambda = 0 \), which corresponds to an \( \alpha \)-stable Lévy ratchet, shows the existence of a ratchet current in the direction of the steepest side of the potential. The main result of this study is that, even in the presence of truncation, a ratchet current is observed and its dependence on \( A \) is qualitatively similar to the \( \alpha \)-stable case. To study the dependence of the current on an external tilting force, we computed the current with an effective potential, \( V_{ef} = V(x) - Fx \) where \( V(x) \) is the ratchet potential in Eq. (3), and \( F \) is a constant. As shown on Fig. 11 the \( \lambda = 0 \) case recovers the \( \alpha \)-stable Lévy ratchet results, and, as \( \lambda \) increases, the strength of the current decreases. Note
FIG. 7: Decay of ratchet current, \( J \), as function \( \lambda \) for different values of \( \alpha \) and \( \theta = 0 \), \( V_0 = 1 \), and \( A = -0.274 \). The Log-Log scale shows evidence of algebraic decay for \( \alpha \geq 1.75 \).

FIG. 8: Decay of ratchet current, \( J \), as function \( \lambda \) for different values of \( \alpha \) and \( \theta = 0 \), \( V_0 = 1 \), and \( A = -0.274 \). The Log-Linear scale shows evidence of exponential decay for \( \alpha \leq 1.5 \).

FIG. 9: Same as Fig. 8 but for \( \alpha < 1 \).

FIG. 10: Ratchet current, \( J \), as function of ratchet potential asymmetry, \( A \), for different levels of truncation, \( \lambda \), \( \alpha = 1.5 \), \( \theta = 0 \), \( V_0 = 1 \), and \( \chi = 0.5 \). that the value of the stopping force, \( F \approx -1 \), i.e. the force needed to cancel the ratchet current, seems to be independent of \( \lambda \).

An interesting effect of the truncation is observed when the Lévy noise is asymmetric, i.e. when the weighting factors \( l \), \( r \) of the left and the right fractional derivatives in Eq. (9) are different. As shown in Fig. 12 in the absence of truncation, \( \lambda = 0 \), the current can be reversed by biasing the asymmetry of the noise. In particular, for the case shown, increasing \( \theta \) to a large enough value stops the positive current. This is consistent with the fact that for \( \theta > 0 \) the skewness of the Lévy noise is negative and the probability of having long jumps to the right is reduced. As the value of \( \theta \) is further increased the probability of jumps to the left increases and this eventually leads to a negative current. A similar phenomenology is observed when the truncation is present, except that (in addition to the usual overall decrease of the magnitude of the cur-
FIG. 12: Ratchet current, $J$, as function of fractional diffusion asymmetry $\theta$ for different levels of truncation, $\lambda$, $\alpha = 1.5$, $V_0 = 1$, $A = -0.274$, and $\chi = 0.5$. $h = 3$.

FIG. 13: Ratchet current, $J$, as function of truncation parameter $\lambda$, for different levels of fractional diffusion asymmetry $\theta$, $\alpha = 1.5$, $V_0 = 1$, $A = -0.274$, and $\chi = 0.5$. For $\theta = 0.275$, the current is reversed for $\lambda \approx 1/3$.

IV. CONCLUSIONS

The incorporation of tempering in models of anomalous transport due to Lévy flights is motivated by the fact that in problems of practical interest, the moments of the pdfs represent physical quantities that cannot be unbounded. For example, although anomalously large displacements have been well documented in numerical simulations of turbulent transport and in fluid mechanics experiments, it is clear from the physical point of view that particle displacements cannot be arbitrarily large and that a cut-off or truncation has to be incorporated in the pdfs describing these processes. This was the intuition behind the original proposal of truncated Lévy flight in Ref. [16]. In this paper we have studied the effects of spatial tempering on directed transport in Lévy ratchets.

Our study was based on the numerical solution of the time dependent and the steady state spatially tempered fractional Fokker-Planck equation. We used two complementary numerical methods. For the computation of the space-time evolution of the pdf in a finite size domain, we used a finite difference method based on the Grunwald-Letnikov discretization of the truncated fractional derivatives. For the computation of the reduced pdf in a periodic domain as well as the time dependent and asymptotic steady state current, we used a Fourier based spectral method. The main object of study was the dependence of the steady state current on the level of truncation, $\lambda$, the stability index $\alpha$, and the noise asymmetry $\theta$ as well as the asymmetry of the potential.

The general conclusion is that in most parameter regimes the inclusion of truncation does not change the qualitative dependencies of the current on the different system parameters; instead, the effect of truncation is mainly to decrease the magnitude of current. However,
an interesting exception to this rule occurs when the Lévy noise is allowed to have a bias. In this case, truncation can lead to a reversal of current direction. The persistence of ratchet currents for \( \lambda \neq 0 \) is interesting since the finite second moments of the truncated Lévy statistics guarantee an eventual convergence to Gaussian statistics (in the absence of an external potential) and, and as it well-known, Gaussian fluctuations cannot create a net current in the absence of time dependent perturbations or an external tilting force. The current is observed to converge exponentially in time to the steady state value, and the decay rate exhibits only a weak dependence of \( \alpha \) and \( \lambda \). The decay of the current for increasing values of \( \lambda \) exhibits different scaling properties depending on the value of \( \alpha \). For large values of alpha, \( 1.75 \leq \alpha \), the current exhibits an algebraic decay and for small values, \( \alpha \leq 1.5 \), the current exhibits an exponential decay.

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