Numerical analysis of strain localization for transversely isotropic model with non-coaxial flow rule

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Abstract: To analyse the strain localization behavior of geomaterials, the forward Euler schemes and the tangent modulus matrix are formulated based on the transversely isotropic yield criterion with non-coaxial flow rule developed by Lade, the program code is implemented based on the user subroutine (UMAT) of ABAQUS. The influence of the material principal direction on the strain localization and the bearing capacity of the structure are investigated and analyzed. Numerical results show the validity and performance of the proposed model in simulating the strain localization behavior of geostructures.

1. Introduction
In the development of plastic mechanics, a basic assumption in classical plasticity mechanics is the law of orthogonal flow which means the direction of incremental plastic strain and the direction of stress are coaxial. With the further development of experimental studies, it has been found that the plastic strain increment deviates from the stress direction in the process of plastic deformation [1-2]. This phenomenon undermines the basic assumptions of many classical elastic-plastic constitutive models and makes many traditional elastic-plastic constitutive models unable to simulate the strain localization of soil bifurcation occurring in geomaterials. Therefore, many experts and scholars have carried out non-coaxial theory adding non-coaxial effects in the classical elastoplastic constitutive model, thus developing many non-coaxial models to study the anisotropic mechanical behavior of geomaterials. For example, Rudnicki and Rice [3] established a non-coaxial model to accurately predict the strain localization behavior of pressure-sensitive dilatant materials. Papamichos and Vardoulakis [4] successfully simulate the formation of sand shear bands in the compression process based on the elastoplastic non-coaxial model. Hashiguchi and Tsutsumi [5] used sub-loading surface model to simulate the bifurcation of sand and made an improvement for the bifurcation prediction. Huang et al [6] also conducted non-coaxial theory, and revised the bifurcation prediction based on non-coaxial model. Comparison of the predicted results and the experimental results showed a consistent outcome. The non-coaxial constitutive models formed by the combination of non-coaxial theory and classical constitutive model can well simulate the mechanical behavior of material strain localization. Therefore, a large number of non-coaxial elastic-plastic constitutive models were constructed by introducing classical elastoplastic constitutive models into the yield function or flow law to simulate the localized mechanical behavior of geomaterials [7-8]. Such models were more capable to predict the bifurcation of geomaterials and explain the anisotropy of geomaterials.

In order to further analyze the strain localization behavior of geomaterials and to predict the bifurcation behavior of materials more accurately. This paper simplified the transverse isotropic yield criterion of Lade [9-10], deduced the iterative scheme and the constitutive matrix of the corresponding forward Euler algorithm, and realized the corresponding program code through the UMAT interface of the finite element software ABAQUS. Numerical simulations were used to investigate the influence of
the main direction angle of the material on the ultimate bearing capacity and deformation localization of geotechnical structures and the effect of non-coaxial stress on the material.

2. Simplification of Lade Transversely Isotropic Model and Derivation of Finite Element Iteration

2.1. Transverse isotropic elastic constitutive descriptions

![Diagram](image)

**Fig 1. Material principal orientation**

The main directions and axes of the transversely isotropic material are shown in the above diagram [11]. $\beta$ is the angle between the main direction of the material and the x-axis (hereinafter referred to as the main direction of the material). lo2 indicates the local coordinate system; xoy indicates the overall coordinate system. 3-axis is perpendicular to lo2 surface, lo3 surface is isotropic, direction along 2-axis is the normal direction. \( D \) refers to the local coordinate system elastic constitutive matrix, then in the overall coordinate system the elastic constitutive matrix \( D_e \) is

\[
D_e = R D R^T
\]  

(1)

Where \( R \) is the transformation matrix:

\[
R = \begin{bmatrix}
\cos^2 \beta & \sin^2 \beta & 0 & -2 \sin \beta \cos \beta & 0 & 0 \\
\sin^2 \beta & \cos^2 \beta & 0 & 2 \sin \beta \cos \beta & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\sin \beta \cos \beta & -\sin \beta \cos \beta & 0 & \cos^2 \beta - \sin^2 \beta & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \beta & -\sin \beta \\
0 & 0 & 0 & 0 & \sin \beta & \cos \beta 
\end{bmatrix}
\]  

(2)

\( D \) is showed in the following form:

\[
D = \begin{bmatrix}
\frac{E_i(1 - L_v^2)}{(1 + \nu_i)(1 - \nu_i - 2L_v^2)} & \frac{E_i\nu_i}{1 - \nu_i - 2L_v^2} & \frac{E_i(\nu_i + L_v^2)}{(1 + \nu_i)(1 - \nu_i - 2L_v^2)} & 0 & 0 & 0 \\
\frac{E_i\nu_i}{1 - \nu_i - 2L_v^2} & \frac{E_i(1 - \nu_i)}{1 - \nu_i - 2L_v^2} & \frac{E_i\nu_i}{1 - \nu_i - 2L_v^2} & 0 & 0 & 0 \\
\frac{E_i(\nu_i + L_v^2)}{(1 + \nu_i)(1 - \nu_i - 2L_v^2)} & \frac{E_i\nu_i}{1 - \nu_i - 2L_v^2} & \frac{E_i(1 - L_v^2)}{(1 + \nu_i)(1 - \nu_i - 2L_v^2)} & 0 & 0 & 0 \\
0 & 0 & 0 & G_2 & 0 & 0 \\
0 & 0 & 0 & 0 & G_i & 0 \\
0 & 0 & 0 & 0 & 0 & G_2
\end{bmatrix}
\]  

(3)

The stress and strain satisfy the linear elasticity constitutive relation:
\[ \epsilon = D \sigma \] (4)

Where \( E_1, \nu_1 \) and \( G_1 \) are elastic modulus, Poisson's ratio and shear modulus in the isotropic plane. \( E_2, \nu_2 \) and \( G_2 \) are elastic modulus, Poisson's ratio and shear modulus perpendicular to the direction of the isotropic surface. Moreover \( G_1 = E_1 / 2(1 + \nu_1) \), \( L = E_1 / E_2 \), so there are only five independent material parameters, that is \( E_1, \nu_1, E_2, \nu_2, G_2 \).

2.2. Simplified Lade Transverse Isotropic Yield Criterion

Lade's method of anisotropy described by Pietruszczak and Morz in the form of tensor tensor [12] combined with the Lade-Kim isotropic yield criterion gives a suitable method for describing the extension of the transversely isotropic geomaterials Lade's criterion [9], the yield function is as follows:

\[ f_p = f_p'(\sigma) - f_p''(W_p) \] (5)

\( W_p \) indicates the plastic work, \( f_p' \) and \( f_p'' \) represent stress-related terms and plastic work-related terms respectively in the yield function

\[ f_p'(\sigma) = \left( \frac{I_1}{I_3} - 27 \right) \left( \frac{I_1}{P_a} \right)^m - \eta_0 [(1 + \Omega_1 (1 - 3l_2^2)] \] (6)

In equation (6), \( P_a \) is the atmospheric pressure, \( m \) and \( \eta_0 \) are both dimensionless constants, \( \Omega_1 \) is the anisotropy intensity parameter, and \( l_2 \) is the loading direction. In order to improve the convergence of the Lade transverse isotropic model in the forward Euler algorithm and to reduce the computational parameters, the plastic work function is simplified, the parameters are integrated, only two dimensionless constants \( A \) and \( B \) are retained.

\[ f_p''(W_p) = A \left( \frac{W_p}{P_a} \right)^B \] (7)

Due to the experimental conditions, the law of associated flow is adopted, namely the plastic potential function is equal to the yield function. This paper simplifies the transverse isotropic model of Lade, deduces the elastic-plastic modulus tensor of the simplified finite element model, develops the constitutive integral algorithm, and realizes the corresponding program code based on UMAT interface of ABAQUS. The numerical simulations of plane strain biaxial compression tests are shown in the following section.

2.3. Elastic-plastic modulus tensor

Relationship between stress and strain increments:

\[ d\sigma = D_{ep} d\epsilon = (D_e - D_p) d\epsilon \] (8)

\( D_{ep} \) refers to the plastic modulus tensor, expand as follows:

\[ D_{ep} = D_e - \frac{D_e (\frac{\partial g_p}{\partial \sigma}) (\frac{\partial f_p}{\partial \sigma})^T D_e}{\frac{\partial f_p}{\partial \sigma}} + \frac{\partial f_p}{\partial \sigma} \frac{\partial g_p}{\partial \sigma} + H \] (9)

Where \( \frac{\partial f_p}{\partial \sigma} = \frac{\partial f_p'}{\partial \sigma} \), the specific expression in Eq. (9) is as follows:

\[ \frac{\partial f_p'}{\partial \sigma} = \frac{\partial f_p'}{\partial \sigma} \frac{\partial I_1}{\partial \sigma} + \frac{\partial f_p'}{\partial \sigma} \frac{\partial I_3}{\partial \sigma} + \frac{\partial f_p'}{\partial \sigma} \frac{\partial I_2}{\partial \sigma} \] (10)

The flow direction consists of two parts. The first part \( \frac{\partial f_p'}{\partial \sigma} \frac{\partial I_2}{\partial \sigma} \) contains the anisotropy tensor (which is not zero), so the direction of this part is different from that of stress increment, namely non-coaxial.
The other part \( \frac{\partial f}{\partial I} \frac{\partial I}{\partial \sigma} + \frac{\partial f}{\partial l} \frac{\partial l}{\partial \sigma} \) is coaxial with the direction of stress increment, namely a co-axial term [13]. Shown as follows:

\[
\frac{\partial f}{\partial I} = \frac{m}{I_1} f_p + \frac{3I_2^2}{I_3} \left( \frac{I_1}{p_s} \right)^m
\]

\[
\frac{\partial f}{\partial l} = \frac{I_3^3}{I_1} \left( \frac{I_1}{p_s} \right)^m
\]

\[
\frac{\partial f}{\partial \sigma} = 3n_0 \Omega_1 \frac{\partial I^2}{\partial \sigma}
\]

\[
\frac{\partial^2 l}{\partial \sigma} = \frac{\partial \left[ \text{tr} (M \dot{\sigma}^2) \right]}{\partial \sigma} \frac{\text{tr} \dot{\sigma}^2 - \partial \left[ \text{tr} (M \dot{\sigma}^2) \right]}{\partial \sigma}
\]

\[
M = \begin{bmatrix} \sin \beta & -\sin \beta \cos \beta & 0 \\ \cos \beta & \sin \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Plastic power increment is \( dW_p = d\lambda \frac{\partial g}{\partial \sigma} \), \( d\lambda \) refers to plastic multiplier, expand as follows:

\[
d\lambda = \frac{D_v \left( \frac{\partial f}{\partial \sigma} \right)^T}{\left( \frac{\partial f}{\partial \sigma} \right)^T D_v \left( \frac{\partial g}{\partial \sigma} \right) + H} \, d\varepsilon
\]

As a result of the associated flow law, in the above equation \( \frac{\partial g}{\partial \sigma} = \frac{\partial f}{\partial \sigma} \), the hardening modulus \( H \) is obtained as follows:

\[
H = -\frac{\partial f}{\partial \sigma} \frac{\partial g}{\partial \sigma}
\]

2.4. Constitutive integral algorithm

In this article, the stress and hardening parameters are updated using the forward Euler algorithm. In the forward Euler algorithm, the stress is replaced by an infinitely small elastoplastic stress-strain relationship by an incremental step. The initial value of the elastic-plastic constitutive matrix is determined from the initial stress state because the elastoplastic constitutive matrix is determined by the history of stress and strain, therefore this linear approximation can be accurate only at very small strain increments. So the strain is divided into smaller sub-strain increments.

\[
\Delta \sigma = \lambda \left( \sigma_0, \kappa_0 \right) \Delta \varepsilon, \quad \Delta \varepsilon = \Delta T \Delta \varepsilon = \frac{\Delta \varepsilon}{n}
\]

\( \Delta T \) is a fixed dimensionless time step, the finite increment of stress incremented is the sum of \( n \) incremental steps of the stress, \( \Delta \sigma \) calculated as a forward Euler step:

\[
\delta \sigma_i = \lambda \left( \sigma_0 + \Delta \sigma_{i-1}, \kappa_0 + \Delta \kappa_{i-1} \right) \delta \varepsilon
\]

\[
\delta \kappa_i = \delta \lambda \left( \sigma_0 + \Delta \sigma_{i-1}, \kappa_0 + \Delta \kappa_{i-1} \right) \delta \varepsilon \left( \sigma_0 + \Delta \sigma_{i-1} \right) \frac{\partial \sigma}{\partial \left( \sigma_0 + \Delta \sigma_{i-1} \right)}
\]

Where:

\[
\Delta \sigma_{i+1} = \sum_{j=i}^{i+1} \delta \sigma_j, \quad \Delta \kappa_{i+1} = \sum_{j=i}^{i+1} \delta \kappa_j
\]
Forward Euclidean compared with other algorithms has the advantage of simple and direct and easy to implement, the process shown in Table 1:

| Tab 1. Subincremental forward Euler scheme |
|-------------------------------------------|
| Initial state \( \sigma_0, \kappa_0, \Delta \varepsilon, \Delta \varepsilon = \frac{\Delta \varepsilon}{m} \) |
| Strain increment step \( i = 1, 2, \ldots, m \) |
| \( \delta \sigma_i = C_{ij}(\sigma_{i-1}, \kappa_{i-1}) \Delta \varepsilon \) |
| \( \delta \kappa_i = \delta \lambda_i(\sigma_{i-1}, \kappa_{i-1}, \Delta \varepsilon) \frac{\partial g}{\partial \sigma_{i-1}} \) |
| \( \sigma_i = \sigma_{i-1} + \delta \sigma_i \) |
| \( \kappa_i = \kappa_{i-1} + \delta \kappa_i \) |
| When \( i = m \), stop calculating the sub-strain increment step |
| Ultimate state \( \sigma_i, \kappa_i, C_{ij}(\sigma_i, \kappa_i) \) |

3. Numerical examples

Based on ABAQUS we developed the corresponding user material subroutine (UMAT). The finite element model is a flat plate of 2m * 1m with a total of 20 * 10 units, using a planar strain 8-node 4 integral point reduction integral unit (CPE8R).

This paper numerically simulates the plane strain biaxial compression tests. The simulation process includes three loading steps:
1) The establishment of the initial isotropic stress field with the stress equals to 20kPa;
2) Isotropic consolidation, external load from 20kPa to 500kPa;
3) Maintain the confining pressure as 500kPa, impose the displacement load by moving the upper and lower boundary symmetrically. The first and second loading step model boundaries are shown in Figure 2a, and the third loading step is shown in Figure 2b.

![Fig 2. Sketches of the model](image)

The parameters [10] in the calculation are shown in table 2.

| Table 2. Material parameters |
|-------------------------------|
| \( E_1 \) /MPa | \( E_2 \) /MPa | \( u_1 \) | \( u_2 \) | \( G_2 \) /MPa | \( \beta \) | \( m \) | \( \eta_0 \) | \( \Omega_1 \) | \( \rho_0 \) /MPa | \( A \) | \( B \) |
| 22.06 | 18.00 | 0.183 | 0.195 | 12.92 | - | 0.658 | 80.19 | -0.10 | 0.101 | 68.597 | 0.343 |

Figure 3 shows the evolution of plastic work at \( \beta = 30^\circ \) (unit: kPa), where \( u_1 \) is the displacement applied to the upper boundary. It can be seen that during the formation of the shear band, a weaker shear band is formed in the direction of the weaker strength and then gradually developed into a stronger single shear band. At the same time another intersecting shear zone began to form. With the gradually development of the shear zone, eventually, a clear X-shaped shear band is formed as a whole. In this process, the plastic work in the direction of shear band is large. The anisotropy of strain
localization is well reflected. The bifurcation of geomaterials is simulated and the material shear damage phenomenon is explained successfully.

![Image](a) $u_y = 0.08m$  
(b) $u_y = 0.15m$  
(c) $u_y = 0.20m$  
(d) $u_y = 0.40m$

**Fig 3.** Variation of shear band with the increasing displacement

In order to investigate the influence of the main direction angle of the material on the shear band, that is, the influence of the angle between the bedding plane and the x-axis on the geotechnical material. Now, the changes of plastic work cloud when the material is compressed are numerically simulated with the angles of 0°, 30°, 60° and 90°, respectively. Figure 4 shows the plastic work with the change of main direction when the upper boundary of the displacement applied to 0.4m. It can be seen that the two conjugate shear bands are basically symmetrical when $\beta = 0^\circ$ and $\beta = 90^\circ$. When $\beta = 30^\circ$ and $\beta = 60^\circ$, the two shear bands appear to be destructive along the weaker direction, and then become stronger in this direction. Combining the strain maps of different main directions it shows that the material failure modes are different when the main orientation angles are different, which shows the anisotropic characteristics of the material fracture modes. It can be seen from Fig. 4 that with the increase of $\beta$, the peak value of plastic work gradually decreases, and that the shear band width increases gradually as a whole. This is a better reflection of the evolution mode of material strain localization as $\beta$ increases. It is a good predictor of the influence of different bedding angles on the failure mode of geomaterials, which can provide reference for explaining the phenomenon of shear-slip failure of geomaterials.

![Image](a) $\beta = 0^\circ$  
(b) $\beta = 30^\circ$  
(c) $\beta = 60^\circ$  
(d) $\beta = 90^\circ$

**Fig 4.** Distribution of plastic work with different material principal direction

Figure 5 shows the displacement curves of the bearing capacity under different main directions. It can be seen that the bearing capacity of $\beta = 30^\circ$ and $\beta = 60^\circ$ under the same displacement is larger at the initial loading. With the increase of loading displacement, the ultimate load-bearing relationship shows the highest bearing capacity of $\beta = 0^\circ$, the smallest bearing capacity of $\beta = 90^\circ$. Ultimately, the bearing capacity increases linearly with the increase of $\beta$.  

6
Fig 5. Variation of different material principal direction angle’s bearing capacity with the displacement

From the plastic flow direction, we can see that $\frac{\partial f_p}{\partial l_2} \frac{\partial l_2}{\partial \sigma}$ is a non-coaxial term and $\frac{\partial f_p}{\partial l_1} \frac{\partial l_1}{\partial \sigma} + \frac{\partial f_p}{\partial l_3} \frac{\partial l_3}{\partial \sigma}$ is a coaxial term. To analyze the influence of non-coaxial part of the flow law during the loading process, take the middle node of the model for example. The curve of the ratio of the first invariant of the non-coaxial part to the coaxial part with node displacement of a node at $\beta = 30^\circ$ is shown in Fig. 6a. The ratio remains the same in elastic stage, combined with the corresponding time-step plastic power cloud chart (Figure 7), it can be seen that there is a faster scale-down process after the formation of the shear band. Fig. 6b shows more details of the process. Continue loading, the proportion gradually reduced, and tends to 0. The results show that after the plasticity is entered, the influence of non-coaxial stress on the material gradually decreases with loading, and finally its influence vanishes.

Fig 6. Variation of non-coaxial percentage in the process of loading
4. Conclusions

In this paper, the anisotropic properties of the elastic and plastic phases of transversely isotropic geomaterials are considered. Based on the simplified Lade transversely isotropic elastoplastic model for transversely isotropic geomaterials derived the iterative scheme and the constitutive matrix of the corresponding forward Euler algorithm. Using UMAT interface based on ABAQUS numerically investigated the influence of the main direction angle of material on the shear band and the bearing capacity of the structure. The following conclusions are obtained:

1) The simplified Lade transversely isotropic elastic-plastic constitutive model can well simulate the strain localization of geomaterials. The main direction angle of the material directly affects the stiffness and ultimate bearing capacity of the geotechnical structure. The stiffness and bearing capacity of geotechnical structure can be fully and reasonably simulated with the evolution of bedding inclination, by reflecting the anisotropy of material strength.

2) The main direction angle of the material has a greater impact on the localized mode of strain. In the biaxial compression, for $\beta = 30^\circ$ and $\beta = 60^\circ$, the structure tends to preferentially form a significant shear zone in the weaker direction and destroy; for the case of $\beta = 0^\circ$ or $\beta = 90^\circ$, the structure breaks down by forming two conjugate X-shaped shear bands. With the increase of $\beta$, the peak value of plastic work gradually decreases. In general, the shear band width gradually increases, which better reflects the mode of material strain localization with $\beta$ increasing.

3) After entering the plastic stage, with the loading, non-coaxial stress on the material gradually reduce the impact, and finally its influence vanishes.
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