New dissipated energy for the unstable thin film equation

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Abstract

The fluid thin film equation $h_t = -(h^n h_{xxx})_x - a_1 (h^m h_x)_x$ is known to conserve mass $\int h \, dx$, and in the case of $a_1 \leq 0$, to dissipate entropy $\int h^{3/2-n} \, dx$ (see [8]) and the $L^2$-norm of the gradient $\int h^2_x \, dx$ (see [3]). For the special case of $a_1 = 0$ a new dissipated quantity $\int h^\alpha h_x^2 \, dx$ was recently discovered for positive classical solutions by Laugesen (see [15]). We extend it in two ways. First, we prove that Laugesen’s functional dissipates strong nonnegative generalized solutions. Second, we prove the full $\alpha$-energy $\int \left( \frac{1}{2} h^\alpha h_x^2 - \frac{\alpha_1 h^{m+n-\alpha+2}}{\alpha+\alpha_1+m+n+1} \right) \, dx$ dissipation for strong nonnegative generalized solutions in the case of the unstable porous media perturbation $a_1 > 0$ and the critical exponent $m = n + 2$.

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1 Introduction

It is well known that analysis of the existence, uniqueness and regularity of weak solutions for nonlinear evolution equations relies heavily on a priori estimates. Often, the physical energy or entropy which
originate from the related model can provide non-increasing in time quantities. Unfortunately, it is far from obvious how to construct new non-increasing Lyapunov type functionals. A general algebraic approach to the construction of entropies in higher-order nonlinear PDEs can be found in [14] and can be applied to analyse thin film equations with stabilizing porus media type perturbations. In this paper, inspired by Laugesen’s result [15] on dissipation, we prove that the energy functional introduced in [15] dissipates strong nonnegative generalized solutions. However, our method of the proof is only applicable to some subset of the Laugsen’s dissipation region [15] (see the shaded area on Figure 1).

We study the longwave-unstable generalized thin film equation

$$h_t = -(h^n h_{xxx})_x - a_1 (h^m h_x)_x,$$

(1.1)

where \( h(x,t) \) gives the height of the evolving free-surface. The exponent \( n \) plays a stabilizing role due to fourth-order forward diffusion term and the exponent \( m \) plays a destabilizing role due to backward second-order diffusion term for the case when \( a_1 > 0 \). This class of equations originates from many physical/industrial applications involving air-fluid interface. For example: the case \( n = 1, m = 1 \) describes a thin jet in a Hele-Shaw cell [10], the case \( n = 3, m = -1 \) describes Van der Waals driven rupture of thin films [19], the case \( m = n = 3 \) describes shape of fluid droplets hanging from a ceiling [11], and the case \( n = 0, m = 1 \) describes solidification of a hyper-cooled melt (this is a modified Kuramoto-Sivashinsky equation) [4].

To prove that the nonnegativity property is preserved in nonlinear thin film equation \( h_t = -(h^n h_{xxx})_x \) for \( n \geq 1 \) (case \( a_1 = 0 \)) Bernis and Friedman [3] used set of dissipated and conserved quantities: mass conservation \( \int h \, dx = M \), surface energy dissipation \( \frac{d}{dt} \int h_x^2 \, dx \leq 0 \), and entropy dissipation \( \frac{d}{dt} \int h^{2-n} \, dx \leq 0 \). The new so-called \( \beta \)-entropy \( \int h^{2-n+\beta} \, dx \) was introduced by Bertozzi and Pugh [5] and independently and simultaneously by Beretta, Bertsch, Dal Passo [1] to extend this result to \( n > 0 \). They also successfully used this new entropy to obtain exponential with respect to the \( L^\infty \)-norm convergence toward the mean value steady state solution. To analyse this convergence rate
in $H^1$-norm for the special case $n = 1$, $a_1 = 0$ Carlen and Ulusoy [9] used the dissipated energy $\int h^\alpha h_x^2 \, dx$ constructed by Laugesen [15] for classical positive solutions. Exponential asymptotic convergence toward the mean value was also studied by Tudorascu in [18]. This list of connections between new properties of solutions in thin film PDEs proved by means of newly discovered dissipated quantities is far from complete.

In this paper we prove that there exists a subinterval $I$ of $-1 < \alpha < 1$ ($I$ depends on $n$ only) and a nonnegative strong generalized solution such that for any $\alpha \in I$ the full $\alpha$-energy

$$E_0^{(\alpha)}(t) = \int_\Omega \left( \frac{1}{2} h^\alpha h_x^2 - \frac{a_1 h^{n+m-n+2}}{(\alpha+m-n+1)(\alpha+m-n+2)} \right) \, dx$$

dissipates. For the unstable porus media perturbation case $a_1 > 0$ this dissipation is proven under the assumptions that the total mass of the solution is less than or equal to the critical one, $m = n + 2$ and domain $\Omega$ is unbounded or $h$ is compactly supported. For the stable case $a_1 \leq 0$ no such assumptions are needed.

We proceed as follows. First, we show the dissipation for the classical solutions of the regularized problem and then we take this dissipation to the limit. We prove dissipation of the full $\alpha$-energy for positive classical solutions of the regularized problem for any value of the coefficient $a_1$ and without any additional assumptions about the total mass of the solution or its support. However our method of taking the dissipation to the limit due to the Bernis-Friedman method of regularization requires additional conditions for the case $a_1 > 0$. 

2 Auxiliary results to generalized weak solutions

We consider nonnegative weak solutions to the following initial–boundary problem:

\[
\begin{align*}
(P) \quad & \quad \frac{\partial h}{\partial t} + (h^n h_{xxx} + a_1 h^m h_x)_x = 0 \text{ in } Q_T, \\
& \quad \frac{\partial^i h}{\partial x^i}(-a, t) = \frac{\partial^i h}{\partial x^i}(a, t) \text{ for } t > 0, i = 0, 3, \\
& \quad h(0, x) = h_0(x) \geq 0,
\end{align*}
\]

where \( h = h(t, x), \Omega = (-a, a), Q_T = (0, T) \times \Omega, n > 0, m > 0, \) and \( a_1 \in \mathbb{R}^1. \) We define a generalized weak solution in the Bernis-Friedman sense (see, e.g., [1, 3]).

**Definition 2.1** (generalized weak solution). Let \( n > 0, m > 0, \) and \( a_1 \in \mathbb{R}^1. \) A generalized weak solution of problem \((P)\) is a function \( h \) satisfying

\[
\begin{align*}
h & \in C^{1/2,1/8}_{x,t}(\overline{Q}_T) \cap L^\infty(0, T; H^1(\Omega)), \\
h_t & \in L^2(0, T; (H^1(\Omega))'), \\
h & \in C^{4,1}_{x,t}(\mathcal{P}), \quad h^x(h_{xxx} + a_1 h^{m-n} h_x) \in L^2(\mathcal{P}),
\end{align*}
\]

where \( \mathcal{P} = \overline{Q}_T \setminus \{h = 0\} \cup \{t = 0\} \) and \( h \) satisfies (2.1) in the following sense:

\[
\int_0^T \int_\mathcal{P} (h_t(\cdot, t), \phi) \, dt - \int_\mathcal{P} h^n(h_{xxx} + a_1 h^{m-n} h_x) \phi_x \, dx \, dt = 0
\]

for all \( \phi \in C^1(Q_T) \) with \( \phi(-a, \cdot) = \phi(a, \cdot); \)

\[
h(\cdot, t) \to h(\cdot, 0) = h_0 \text{ pointwise \& strongly in } L^2(\Omega) \text{ as } t \to 0,
\]

\[
h(-a, t) = h(a, t) \forall t \in [0, T] \text{ and } \frac{\partial^i h}{\partial x^i}(-a, t) = \frac{\partial^i h}{\partial x^i}(a, t)
\]

for \( i = 1, 3 \) at all points of the lateral boundary where \( \{h \neq 0\}. \)
Because the second term of (2.7) has an integral over \( P \) rather than over \( Q_T \), the generalized weak solution is "weaker" than a standard weak solution. Also note that the first term of (2.7) uses \( h_t \in L^2(0, T; (H^1(\Omega))') \); this is different from the definition of weak solution first introduced by Bernis and Friedman \[3\]; there, the first term was the integral of \( h \phi_t \). The proof of the existence of generalized weak solutions follows the ideas of \[3, 1, 5, 6, 7, 17\].

Let
\[
G^{(\beta)}_0(z) := \begin{cases} 
\frac{z^{\beta-n+2}}{(\beta-n+2)(\beta-n+1)} & \text{if } \beta - n \neq \{-1, -2\}, \\
z \ln z - z & \text{if } \beta - n = -1, \\
-\ln z & \text{if } \beta - n = -2,
\end{cases}
\]
(2.10)

\((G^{(\beta)}_0(z))'' = z^{\beta-n}\), and \(G_0(z) := G^{(0)}_0(z)\).

**Theorem 1.** Let \(a_1 \in \mathbb{R}^1\), \(n > 0\); \(m \geq n/2\) for \(a_1 > 0\), and \(m > 0\) for \(a_1 \leq 0\).

(a) [Existence.] Let the nonnegative initial data \(h_0 \in H^1(\Omega)\) satisfy
\[
\int_\Omega G_0(h_0(x)) \, dx < \infty,
\]
(2.11)

and either 1) \(h_0(-a) = h_0(a) = 0\) or 2) \(h_0(-a) = h_0(a) \neq 0\) and \(\frac{\partial^i h_0}{\partial x^i}(-a) = \frac{\partial^i h_0}{\partial x^i}(a)\) holds for \(i = 1, 2, 3\). Then for some time \(T_{\text{loc}} > 0\) there exists a nonnegative generalized weak solution, \(h\), on \(Q_{T_{\text{loc}}}\) in the sense of the definition 2.7. Furthermore,
\[
h \in L^2(0, T_{\text{loc}}; H^2(\Omega)).
\]
(2.12)

Let
\[
\mathcal{E}_0(T) := \int_\Omega \left\{ \frac{1}{2} h^2_z(x, T) - a_1 D_0(h(x, T)) \right\} \, dx,
\]
(2.13)

where \(D_0(z) := \frac{z^{m-n+2}}{(m-n+1)(m-n+2)}\). Then the weak solution satisfies
\[
\mathcal{E}_0(T) + \int \left\{ h^n(h_{xxx} + a_1 h^{m-n} h_x)^2 \right\} dx dt \leq \mathcal{E}_0(0),
\]
(2.14)
\[
\int h^m h_{xx}^2 \, dx \, dt \leq \text{const} < \infty.
\]

(2.15)

for all \( T \leq T_{\text{loc}} \). The time of existence, \( T_{\text{loc}} \), is determined by \( a_1, |\Omega|, \int h_0, \|h_{0x}\|_2, \) and \( \int G_0(h_0) \). Moreover, \( T_{\text{loc}} = +\infty \) for \( a_1 \leq 0 \).

(b) [Regularity.] If the initial data from (a) also satisfies
\[
\int_{\Omega} G_0^{(\beta)}(h_0(x)) \, dx < \infty
\]

for some \(-1/2 < \beta < 1, \beta \neq 0\) then there exists \( 0 < T_{\text{loc}}^{(\beta)} \leq T_{\text{loc}} \) such that the nonnegative generalized weak solution has the extra regularity
\[
h^{\frac{\beta+2}{2}} \in L^2(0, T^{(\beta)}_{\text{loc}}; H^2(\Omega)) \quad \text{and} \quad h^{\frac{\beta+2}{4}} \in L^2(0, T^{(\beta)}_{\text{loc}}; W^1_4(\Omega)).
\]

(2.16)

The time of existence, \( T^{(\beta)}_{\text{loc}} \), is determined by \( a_1, |\Omega|, \int h_0, \|h_{0x}\|_2, \) and \( \int G_0^{(\beta)}(h_0) \). Moreover, \( T^{(\beta)}_{\text{loc}} = +\infty \) for \( a_1 \leq 0 \).

There is nothing special about the time \( T_{\text{loc}} \) in the Theorem 1. In the case \( a_1 > 0 \) and \( n/2 \leq m < n + 2 \) (or \( m = n + 2 \) and \( M \leq M_c \)), given a countable collection of times in \([0, T_{\text{loc}}]\), one can construct a weak solution for which these bounds will hold at those times. Also, we note that the analogue of Theorem 4.2 in [3] also holds: there exists a nonnegative weak solution with the integral representation
\[
\int_0^T \langle h_t(t), \phi \rangle \, dt + \int_{Q_T} (nh^{n-1}h_x h_{xx} \phi_x + h^n h_{xx} \phi_{xx}) \, dx \, dt
\]

\[
- a_1 \int_{Q_T} h^m h_x \phi_x \, dx \, dt = 0.
\]

(2.17)

3 Dissipation of energy for nonnegative weak solutions

The main result of the present paper is the following
**Theorem 2.** Let \(a_1 \in \mathbb{R}^1, 1/2 < n < 3; m \geq n/2 \) for \(a_1 > 0\), and \(m > 0\) for \(a_1 \leq 0\), and

\[
\mathcal{E}^{(a)}_0(T) := \int_{\Omega} \left\{ \frac{1}{2} h^a h_x^2(x, T) - a_1 \tilde{D}_0(h(x, T)) \right\} \, dx,
\]

where \(\tilde{D}_0(z) := \frac{z^{\alpha+m-n+2}}{(\alpha+m-n+1)(\alpha+m-n+2)}\), and \(\mathcal{E}^{(0)}_0(T) = \mathcal{E}_0(T)\). Then there exists a non-empty subinterval \(I\) (see [15] for the explicit form of the \(I\)) of \(0 \leq \alpha < 1\) for \(\frac{1}{2} < n < 3\), and of \(\frac{3}{2} - n < \alpha < 0\) for \(\frac{3}{2} < n < 3\) such that for any \(\alpha \in I\) the nonnegative weak solution from Theorem 1 satisfies the following estimates:

(i) if \(a_1 \leq 0\) then

\[
\mathcal{E}^{(a)}_0(T) \leq \mathcal{E}^{(a)}_0(0); \quad (3.2)
\]

(ii) if \(a_1 > 0\) then

\[
\mathcal{E}^{(a)}_0(T) \leq \mathcal{E}^{(a)}_0(0) + C_1 \int_{Q_T} h^{\alpha+3m-2n+2} \, dx \, dt \quad \text{for } m > n + 2; \quad (3.3)
\]

\[
\mathcal{E}^{(a)}_0(T) \leq \mathcal{E}^{(a)}_0(0) + T(C_1 M^{\frac{2\alpha+5m-3n+4}{n+2-m}} + C_2 M^{\alpha+3m-2n+2}) \quad (3.4)
\]

for \(m < n + 2\) and \(\alpha > 2n - 3m - 1\);

\[
\mathcal{E}^{(a)}_0(T) \leq \mathcal{E}^{(a)}_0(0) + C_3 T M^{\alpha+3m-2n+2} \quad (3.5)
\]

for \(m < n + 2\) and \(2n - 3m - 2 < \alpha \leq 2n - 3m - 1\);

\[
\mathcal{E}^{(a)}_0(T) \leq \mathcal{E}^{(a)}_0(0) + C_2 T M^{\alpha+n+8} \quad \text{for } m = n + 2 \text{ and } 0 < M \leq M_c. \quad (3.6)
\]

Here \(C_2 = 0\) if \(\Omega\) is unbounded or \(h\) has compact support.

**Remark 3.1** (Extra Regularity). In particular, the extra regularity \(h^{\frac{n+2}{2}} \in L^\infty(0, T; H^1(\Omega))\) follows directly from Theorem 2. Hence, \(h^{\frac{n+2}{2}}(\cdot, T) \in H^1(\Omega)\) for almost all \(T \in [0, T^{(\beta)}_{loc}]\) and therefore \(h^{\frac{n+2}{2}}(\cdot, T) \in C^{1/2} \tilde{\Omega}\) for almost all \(T \in [0, T^{(\beta)}_{loc}]\). Assume that \(T_0\) is chosen such
that $h^{\frac{a+2}{2}}(\cdot, T_0) \in C^{1/2}(\Omega)$ and $h(x_0, T_0) = 0$ at some $x_0 \in \bar{\Omega}$. Then there exists a constant $L$ such that

$$h^{\frac{a+2}{2}}(x, T_0) = |h^{\frac{a+2}{2}}(x, T_0) - h^{\frac{a+2}{2}}(x_0, T_0)| \leq L|x - x_0|^{1/2}.$$  

Hence $h(x, T_0) \leq L^{\frac{a+2}{2}}|x - x_0|^{1/2}$, i.e. $h(\cdot, T) \in C^{1/2}(\Omega)$ for almost every $T \in [0, T^\text{loc}_{(\beta)}]$.

**Remark 3.2 (Rate of decrease).** For $a_1 \leq 0$ and $1/2 < n < 3$ we can generalize the results from [9, Theorem 1.1] in the following way:

$$\int_{\Omega} h^\alpha h^2_x(x, t) \, dx \leq C(1 + t)^{-\frac{1}{2}} \text{ for } \frac{n-4}{2} \leq \alpha < 0,$$

whence $\|h - \bar{h}\|_\infty \leq C(1 + t)^{-\frac{1}{2}}$ for any nonnegative strong solution $h$. Here $C = C(a_1, \alpha, n, \bar{h}, E_0^{(a)}(0))$, and $\bar{h} = \frac{1}{|\Omega|}\|h_0\|_1$. The proof is similar to [9].

### 3.1 Regularized Problem

Given $\delta, \varepsilon > 0$, a regularized parabolic problem, similar to that of Bernis and Friedman [3], is considered:

$$(P_{\delta, \varepsilon}) \begin{cases} h_t + (f_{\delta, \varepsilon}(h)(h_{xxx} + a_1 D''_{\varepsilon}(h)h_x))_x = 0, \\ \frac{\partial h}{\partial x^i}(-a, t) = \frac{\partial h}{\partial x^i}(a, t) \text{ for } t > 0, \, i = 0, 3, \\ h(x, 0) = h_{0, \delta, \varepsilon}(x), \end{cases} \quad (3.7) \quad (3.8) \quad (3.9)$$

where

$$f_{\delta, \varepsilon}(z) := f_{\varepsilon}(z) + \delta = \frac{|z|^{s+n}}{|z|^s + \varepsilon |z|^n} + \delta, \quad D''_{\varepsilon}(z) := \frac{|z|^{m-n}}{1+\varepsilon |z|^{m-n}} \quad (3.10)$$

$\forall z \in \mathbb{R}^1$, $\varepsilon > 0$, $s \geq 4$. The $\delta > 0$ in (3.10) makes the problem (3.7) regular (i.e. uniformly parabolic). The parameter $\varepsilon$ is an approximating parameter which has the effect of increasing the degeneracy.
from $f(h) \sim |h|^n$ to $f_\varepsilon(h) \sim h^\varepsilon$. The nonnegative initial data, $h_0$, is approximated via

$$h_{0,\delta\varepsilon} \in C^{4+\gamma}(\Omega), \ h_{0,\delta\varepsilon} \geq h_{0,\delta} + \varepsilon^\theta \text{ for some } 0 < \theta < \frac{2}{2s-3},$$

$$\frac{\partial^i h_{0,\delta\varepsilon}}{\partial x^i}(-a) = \frac{\partial^i h_{0,\delta\varepsilon}}{\partial x^i}(a) \text{ for } i = 0, 3,$$

(3.11)

$$h_{0,\delta\varepsilon} \rightarrow h_0 \text{ strongly in } H^1(\Omega) \text{ as } \delta, \varepsilon \rightarrow 0.$$  

The $\varepsilon$ term in (3.11) “lifts” the initial data so that it will be positive even if $\delta = 0$ and the $\delta$ is involved in smoothing the initial data from $H^1(\Omega)$ to $C^{4+\gamma}(\Omega)$.

Sketch of Proof: By Eidelman \[12\] Theorem 6.3, p.302, the regularized problem has the unique classical solution $h_{\delta\varepsilon} \in C^{4+\gamma,1+\gamma/4}(\Omega \times [0, \tau_{\delta\varepsilon}])$ for some time $\tau_{\delta\varepsilon} > 0$. For any fixed values of $\delta$ and $\varepsilon$, by Eidelman \[12\] Theorem 9.3, p.316 if one can prove a uniform in time an a priori bound $|h_{\delta\varepsilon}(x, t)| \leq A_{\delta\varepsilon} < \infty$ for some longer time interval $[0, T_{loc,\delta\varepsilon}]$ ($T_{loc,\delta\varepsilon} > \tau_{\delta\varepsilon}$) and for all $x \in \Omega$ then Schauder-type interior estimates \[12\] Corollary 2, p.213 imply that the solution $h_{\delta\varepsilon}$ can be continued in time to be in $C^{4+\gamma,1+\gamma/4}(\Omega \times [0, T_{loc,\delta\varepsilon}])$.

Although the solution $h_{\delta\varepsilon}$ is initially positive, there is no guarantee that it will remain nonnegative. The goal is to take $\delta \rightarrow 0$, $\varepsilon \rightarrow 0$ in such a way that 1) $T_{loc,\delta\varepsilon} \rightarrow T_{loc} > 0$, 2) the solutions $h_{\delta\varepsilon}$ converge to a (nonnegative) limit, $h$, which is a generalized weak solution, and 3) $h$ inherits certain a priori bounds. This is done by proving various a priori estimates for $h_{\delta\varepsilon}$ that are uniform in $\delta$ and $\varepsilon$ and hold on a time interval $[0, T_{loc}]$ that is independent of $\delta$ and $\varepsilon$. As a result, \{h_{\delta\varepsilon}\} will be a uniformly bounded and equicontinuous (in the $C^{1/2,1/8}_{x,t}$ norm) family of functions in $\Omega \times [0, T_{loc}]$. Taking $\delta \rightarrow 0$ will result in a family of functions \{h_{\varepsilon}\} that are classical, positive, unique solutions to the regularized problem with $\delta = 0$. Taking $\varepsilon \rightarrow 0$ will then result in the desired generalized weak solution $h$. This last step is where the possibility of non-unique weak solutions arise; see \[1\] for simple examples of how such constructions applied to $h_t = -(|h|^n h_{xxx})_x$ can result in two different solutions arising from the same initial data.
3.2 Dissipation of energy for positive solutions

Figure 1: The dissipation region computed numerically by Matlab: $\alpha$ versus $n$. The dashed line corresponds to $\alpha = \frac{3}{2} - n$.

Lemma 3.1. Let $\alpha$ belong to the full domain shown on Figure 1, and

$$
\mathcal{E}_\varepsilon^{(\alpha)}(T) := \int_\Omega \left\{ \frac{1}{2} h^\alpha h_x^2(x, T) - a_1 \tilde{D}_\varepsilon(h(x, T)) \right\} dx,
$$

(3.12)

where $\tilde{D}''_\varepsilon(z) := z^\alpha D''_\varepsilon(z)$. Then the unique positive classical solution $h_\varepsilon$ of the problem $(P_{0,\varepsilon})$ satisfies

$$
\mathcal{E}_\varepsilon^{(\alpha)}(T) \leq \mathcal{E}_\varepsilon^{(\alpha)}(0) + \mu \int_{Q_T} h^{\alpha-2} f_\varepsilon(h) D''_\varepsilon(h) h_x^4 dxdt + \\
\varepsilon k_1 \int_\Omega h^{\alpha-s-4} f_\varepsilon^2(h) h_x^6 dx + \varepsilon^2 k_2 \int_\Omega h^{\alpha-2s-4} f_\varepsilon^3(h) h_x^6 dx,
$$

(3.13)

where $k_i = k_i(\alpha, n, s)$ are constants, and $\mu = \mu(\alpha, a_1)$ such that $\mu(0, a_1) = 0$ and $\mu(\alpha, a_1) \leq 0$ for $a_1 \leq 0$. 

Note that, although we use the same convenient notations introduced in [15], the proof of Lemma 3.1 has essential differences from the proof of Theorem 1 of [15]. Indeed, we introduce new ideas in order to estimate the lower-order term in the equation (3.7). In particular, the new quantity \( N \) is introduced, the quantity \( R \) is modified, and so are the terms involving the regularization parameter \( \varepsilon \) in (3.19).

**Proof of Lemma 3.1.** To prove the bound (3.13), multiply (3.7) with \( \delta = 0 \) by \(- \frac{\alpha}{2} h^{\alpha-1} h_x^2 - h^{\alpha} h_{xx} - a_1 D_\varepsilon'(h) \), integrate over \( \Omega \), use integration by parts, apply the periodic boundary conditions (3.8), to find

\[
\frac{d}{dt} E_\varepsilon^{(\alpha)}(t) = - \int_{\Omega} h^{\alpha} f_\varepsilon(h)(h_{xxx} + a_1 D_\varepsilon''(h) h_x)^2 \, dx - \frac{\alpha}{2} (\alpha - 1) \int_{\Omega} h^{\alpha-2} h_x^3 f_\varepsilon(h)(h_{xxx} + a_1 D_\varepsilon''(h) h_x) \, dx - 2\alpha \int_{\Omega} h^{\alpha-1} h_x h_{xx} f_\varepsilon(h)(h_{xxx} + a_1 D_\varepsilon''(h) h_x) \, dx. \tag{3.14}
\]

The equality (3.14) can be rewritten as

\[
\frac{d}{dt} E_\varepsilon^{(\alpha)}(t) = -R^2 - 2\alpha RS - \frac{\alpha}{2} (\alpha - 1) RL, \tag{3.15}
\]

where the quantities

\[
R := \langle (h^{\alpha} f_\varepsilon(h))^{1/2}(h_{xxx} + a_1 D_\varepsilon''(h) h_x) \rangle = \langle (h^{\alpha} f_\varepsilon(h))^{1/2}(h_{xxx} + a_1 D_\varepsilon''(h) h_x) \rangle,
\]

\[
S := \langle (h^{\alpha-2} f_\varepsilon(h))^{1/2} h_{xx} f_\varepsilon(h) \rangle = \langle (h^{\alpha-2} f_\varepsilon(h))^{1/2} h_{xx} f_\varepsilon(h) \rangle,
\]

\[
L := \langle (h^{\alpha-4} f_\varepsilon(h))^{1/2} h_x^3 \rangle = \langle (h^{\alpha-4} f_\varepsilon(h))^{1/2} h_x^3 \rangle,
\]

\[
N := \langle (h^{\alpha-2} f_\varepsilon(h) D_\varepsilon''(h))^{1/2} h_x^2 \rangle = \langle (h^{\alpha-2} f_\varepsilon(h) D_\varepsilon''(h))^{1/2} h_x^2 \rangle,
\]

each represent half of an inner product in \( L^2(\Omega) \). We will need the following integration by parts formulas

\[
SL = -\frac{1}{5} (\alpha - 3) \int_{\Omega} h^{\alpha-4} f_\varepsilon(h) h_x^6 \, dx - \frac{1}{5} \int_{\Omega} h^{\alpha-3} f_\varepsilon'(h) h_x^6 \, dx =
\]

\[
-\frac{1}{5} (\alpha + n - 3) L^2 - \frac{1}{5} \varepsilon (s - n) \int_{\Omega} h^{\alpha-s-4} f_\varepsilon^2(h) h_x^6 \, dx, \tag{3.16}
\]

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\[ RL = -(\alpha + n - 2)SL - \varepsilon(s-n) \int \limits_{\Omega} h^{\alpha-s-3} f^2_\varepsilon(h) h_x^4 h_{xx} dx - 3S^2 + a_1 N^2 = \]
\[ \frac{1}{5}(\alpha + n - 2)(\alpha + n - 3)L^2 - 3S^2 + a_1 N^2 - \varepsilon(s-n) \int \limits_{\Omega} h^{\alpha-s-3} f^2_\varepsilon(h) h_x^4 h_{xx} dx + \]
\[ \frac{1}{5}\varepsilon(s-n)(\alpha + n - 2) \int \limits_{\Omega} h^{\alpha-s-4} f^2_\varepsilon(h) h_x^6 dx = \frac{1}{5}(\alpha + n - 2)(\alpha + n - 3)L^2 - 3S^2 + \]
\[ a_1 N^2 + \frac{1}{5}\varepsilon(s-n)(2\alpha + 3n - s - 5) \int \limits_{\Omega} h^{\alpha-s-4} f^2_\varepsilon(h) h_x^6 dx + \]
\[ \frac{2}{5}\varepsilon^2(s-n)^2 \int \limits_{\Omega} h^{\alpha-2s-4} f^3_\varepsilon(h) h_x^6 dx. \quad (3.17) \]

Here we use the auxiliary equality \( f'_\varepsilon(z) = n z^{-1} f_\varepsilon(z) + \varepsilon(s-n) z^{-(s+1)} f^2_\varepsilon(z). \) Thus, from (3.15) we have

\[ \frac{d}{dt} E^{(\alpha)}(\varepsilon(t)) + \frac{\varepsilon^2}{5}(\alpha - 1)(s-n)^2 \int \limits_{\Omega} h^{\alpha-2s-4} f^3_\varepsilon(h) h_x^6 dx + \frac{a_1}{2} \alpha(\alpha - 1) N^2 = \]
\[ - R^2 - 2\alpha R S - \frac{2}{10}(\alpha - 1)(\alpha + n - 2)(\alpha + n - 3)L^2 + \frac{3\alpha}{2}(\alpha - 1)S^2 + \]
\[ \frac{\varepsilon}{10}(\alpha - 1)(s-n)(s-2\alpha - 3n + 5) \int \limits_{\Omega} h^{\alpha-s-4} f^2_\varepsilon(h) h_x^6 dx. \quad (3.18) \]

Our next step is to express (3.18) as the negative of a sum of squares to obtain the energy dissipation. To achieve this, we use (3.16) and (3.17) to deduce that for all \( \kappa \in \mathbb{R}^1, \)

\[ \frac{d}{dt} E^{(\alpha)}(\varepsilon(t)) = -(R + \alpha S + \kappa L)^2 + \beta(S + \frac{1}{5}(\alpha + n - 3)L)^2 + \gamma L^2 + \mu N^2 + \]
\[ \varepsilon k_1 \int \limits_{\Omega} h^{\alpha-s-4} f^2_\varepsilon(h) h_x^6 dx + \varepsilon^2 k_2 \int \limits_{\Omega} h^{\alpha-2s-4} f^3_\varepsilon(h) h_x^6 dx, \quad (3.19) \]
where

\[ k_1 = \frac{2}{25}(s - n) \left( 5\kappa(\alpha - s + 3n - 5) + \frac{5\alpha}{4}(\alpha - 1)(s - 2\alpha - 3n + 5) + (\alpha + n - 3)\left( \frac{3\alpha}{2}(5\alpha - 3) - 6\kappa \right) \right), \]
\[ k_2 = \frac{1}{3}(s - n)^2 \left( 4\kappa - \alpha(\alpha - 1) \right), \]
\[ \beta = \frac{\alpha}{2}(5\alpha - 3) - 6\kappa, \quad \mu = a_1 \left( 2\kappa - \frac{\alpha}{2}(\alpha - 1) \right), \]
\[ \gamma = \kappa^2 - \frac{2}{25}\kappa(\alpha + n - 3)(5(2 - n) + 3(\alpha + n - 3)) - \frac{6}{50}(\alpha + n - 3)(5(\alpha - 1)(\alpha + n - 2) - (5\alpha - 3)(\alpha + n - 3)) = \kappa^2 - \frac{5\alpha}{2}(\alpha + n - 3)(\alpha - \frac{2n - 1}{3}) - \frac{1}{750}\alpha(\alpha + n - 3)(\alpha - \frac{2n - 1}{3}). \]

Now we have to choose the parameter \( \kappa \) in such a way that \( \beta \leq 0 \) and \( \gamma \leq 0 \). In this case, the parameter \( \mu > 0 \) for \( a_1 > 0 \), and \( \mu \leq 0 \) for \( a_1 \leq 0 \). According to [15], we can find \( \kappa \) such that \( \beta \leq 0 \), and \( \gamma \leq 0 \) when \( 1/2 < n < 3 \), see also Figure 1 where this region was computed numerically by Matlab (see [15] for the explicit form of the domain).

### 3.3 Limit process in (3.13)

Rewrite the integral \( \int\int_{Q_T} \varepsilon h_\varepsilon^{\alpha-s-4} f_\varepsilon^2(h_\varepsilon) h_\varepsilon^{\delta} dxdt \) in the form

\[
\int\int_{Q_T} \varepsilon h_\varepsilon^{\alpha-s-4} f_\varepsilon^2(h_\varepsilon) h_\varepsilon^{\delta} dxdt = \int\int_{Q_T} \varepsilon h_\varepsilon^{\alpha-s-n} (h_\varepsilon^{-1} + \varepsilon x) h_\varepsilon^{\delta-4} h_\varepsilon^{\alpha-s-n} dxdt.
\]

Using the Young’s inequality

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q} \Rightarrow pab \leq a^p + (p - 1)b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad (3.20)
\]

with \( a = z^{\frac{\alpha-n}{p}} \) and \( b = \left( \frac{\varepsilon}{p-1} \right)^{\frac{1}{q}} \), we deduce

\[
\varepsilon \frac{z^\alpha}{(z^{\alpha-n} + \varepsilon)^2} \leq \frac{z^\alpha}{z^{\alpha-n} + \varepsilon} \leq \frac{(p-1)^{\frac{1}{q}}}{p} \varepsilon \frac{z^\alpha}{z^{\alpha-n} + \varepsilon} = \frac{(p-1)^{\frac{1}{q}}}{p} \varepsilon \frac{z^\alpha}{z^{\alpha-n} + \varepsilon}.
\]
choosing \( p = \frac{s-n}{\alpha} > 1 \) and \( q = \frac{s-n}{s-n-\alpha} > 1 \) \((\Rightarrow 0 < \alpha < s - n)\), we find
\[
\varepsilon^\frac{s-n+s-n}{(s-n+n+\varepsilon)^2} \leq \frac{\alpha}{s-n} \left( \frac{s-n-\alpha}{s-n} \right)^{\frac{s-n-\alpha}{s-n}} \varepsilon^\frac{\alpha}{s-n}.
\] Similarly, we deal with the integral \( \varepsilon^2 \int h_{\varepsilon} \alpha - 2s - 4 f_{\varepsilon}^3 (h) \| h_{\varepsilon,x} \| dx \). Due to Lemma [A.1] and (2.15), \( \int h_{\varepsilon}^{n-4} h_{\varepsilon,x} dx dt \) is uniformly bounded then
\[
\left| \int Q_T (k_1 \varepsilon h_{\varepsilon}^{-s-4 f_{\varepsilon}^2 (h_{\varepsilon})} + k_2 \varepsilon^2 h_{\varepsilon}^{-2 - 4 f_{\varepsilon}^3 (h_{\varepsilon})}) h_{\varepsilon,x} dx dt \right| \leq C \varepsilon^{\frac{n}{s-n}} \int Q_T h_{\varepsilon}^{n-4} h_{\varepsilon,x} dx dt \leq C \varepsilon^{\frac{n}{s-n}}, \quad (3.21)
\] where the positive constant \( C \) is independent of \( \varepsilon \). Letting \( \varepsilon \to 0 \), from \( (3.21) \) we obtain
\[
(k_1 \varepsilon h_{\varepsilon}^{-s-4 f_{\varepsilon}^2 (h_{\varepsilon})} + k_2 \varepsilon^2 h_{\varepsilon}^{-2 - 4 f_{\varepsilon}^3 (h_{\varepsilon})}) h_{\varepsilon,x} \to 0 \text{ in } L^1(Q_T) \quad (3.22)
\] for \( 0 < \alpha < s - n \).

Now, we show (3.22) for the case of \( \alpha < 0 \). Rewrite the integral \( \int Q_T \varepsilon h_{\varepsilon}^{-s-4 f_{\varepsilon}^2 (h_{\varepsilon})} h_{\varepsilon,x} dx dt \) in the form
\[
\int Q_T \varepsilon h_{\varepsilon}^{-s-4 f_{\varepsilon}^2 (h_{\varepsilon})} h_{\varepsilon,x} dx dt = \int Q_T \varepsilon h_{\varepsilon}^{-s-2} (h_{\varepsilon}^{-2} f_{\varepsilon}^2 (h_{\varepsilon})) h_{\varepsilon,x} dx dt.
\] Using the inequality \( (3.20) \) with \( a = \frac{s-n}{p} \) and \( b = (\frac{1}{p-1})^{\frac{1}{q}} \), we obtain
\[
\varepsilon^\frac{s-n-2}{(s-n+n+\varepsilon)^2} \leq \frac{(p-1)\frac{2}{q}}{p^x} \left( \frac{2(s-n)}{p^q} \right)^{\frac{2(s-n)}{p^q}} \leq \frac{(p-1)\frac{2}{q}}{p^x} \varepsilon^\frac{2-2}{q} z^{\frac{2(s-n)}{p^q}},
\] choosing \( p = \frac{2(s-n)}{s-2+\alpha-\beta} < 2 \) and \( q = \frac{2(s-n)}{2(s-n)-s+2-\alpha+\beta} > 2 \) \((\Rightarrow n > 2-\alpha+\beta)\), we find
\[
\varepsilon^\frac{s-n-2}{(s-n+n+\varepsilon)^2} \leq \left( \frac{s-n}{s-2+\alpha-\beta} \right)^{\frac{2-2(n-1)-\alpha+\beta}{s-n}} \varepsilon^{\frac{n-2+\alpha-\beta}{s-n}} \varepsilon^{\beta-\alpha},
\]
where \( \beta \in (-1/2, 1) \) follows from (2.16). Similarly, we deal with the integral 
\[
\frac{1}{2} \int_{\Omega} h_{\varepsilon}^{\alpha-2s-4} f_{\varepsilon}(h) h_{\varepsilon,x}^{6} \, dx.
\]
Due to \( h \in L^{\infty}(0,T; H^{1}(\Omega)) \) and (2.16), the integral \( \int_{Q_{T}} h_{\varepsilon}^{\alpha-2} h_{\varepsilon,x}^{6} \, dxdt \) is uniformly bounded then
\[
\left| \int_{Q_{T}} \left( k_{1} \varepsilon h_{\varepsilon}^{\alpha-s-4} f_{\varepsilon}^{2}(h_{\varepsilon}) + k_{2} \varepsilon^{2} h_{\varepsilon}^{\alpha-2s-4} f_{\varepsilon}^{3}(h_{\varepsilon}) \right) h_{\varepsilon,x}^{6} \, dxdt \right| \leq C \varepsilon^{n-2+\alpha-\beta} \int_{Q_{T}} h_{\varepsilon}^{\beta-2} h_{\varepsilon,x}^{6} \, dxdt \leq C \varepsilon^{n-2+\alpha-\beta}, \quad (3.23)
\]
where the positive constant \( C \) is independent of \( \varepsilon \). Letting \( \varepsilon \to 0 \), we obtain (3.22) for \( \frac{3}{2} - n < \alpha < 0 \) and \( \frac{3}{2} < n < 3 \). In view of the Lebesgue’s theorem, we have
\[
\int_{Q_{T}} h_{\varepsilon}^{\alpha-2} f_{\varepsilon}(h_{\varepsilon}) D_{\varepsilon}^{\alpha}(h_{\varepsilon}) h_{\varepsilon,x}^{4} \, dxdt \to \int_{Q_{T}} h^{\alpha+m-2} h_{x}^{4} \, dxdt \quad (3.24)
\]
if \( m > 0 \) and \( \alpha > -\frac{1}{2} - m \), due to \( h \in L^{\infty}(0,T; H^{1}(\Omega)) \) and (2.16).

Integrating (3.19) over the time interval, and letting \( \varepsilon \to 0 \), in view of (3.22) and (3.24), we obtain (3.13) for some subinterval \( I \) for \( 0 \leq \alpha < 1 \) and \( \frac{1}{2} < n < 3 \) or for \(-1 < \alpha < 0 \) and \( \frac{3}{2} < n < 3 \). Note that, the convergence on the left-hand side follows from Fatou’s lemma and from the corresponding a priori estimate (see, for example, [3, 5, 7, 17]).

3.4 Proof of Theorem 2

Taking the limit \( \varepsilon \to 0 \) we obtain
\[
\mathcal{E}_{\varepsilon}^{(\alpha)}(T) + \gamma \int_{Q_{T}} h^{\alpha+n-4} h_{x}^{6} \, dxdt \leq \mathcal{E}_{\varepsilon}^{(\alpha)}(0) + \mu \int_{Q_{T}} h^{\alpha+m-2} h_{x}^{4} \, dxdt. \quad (3.25)
\]
Now, we estimate $\iint_{Q_T} h^{\alpha + m - 2} h_x^4 dx dt$. Using the Hölder inequality, we obtain

$$\iint_{Q_T} h^{\alpha + m - 2} h_x^4 dx dt \leq \int_0^T \left( \int_{\Omega} h^{\alpha + \nu - 4} h_x^6 dx \right)^2 \left( \int_{\Omega} h^{\alpha + 3 m - 2 n + 2} dx \right)^{\frac{4}{7}} dt.$$

Applying Lemma A.2 with $v = h^{\alpha + n - 4} h_x^6$ with $a = \frac{6(\alpha + 3 m - 2 n + 2)}{\alpha + n + 2}$, $b = \frac{6}{\alpha + n + 2} < a$ ($\Rightarrow \alpha > 2 n - 3 m - 1$), $i = 0$, and $j = 1$, we deduce

$$\int_{\Omega} h^{\alpha + 3 m - 2 n + 2} dx \leq d_1 \left( \int_{\Omega} v_\nu^6 dx \right)^{\frac{\alpha + 3 m - 2 n + 2}{\alpha + n + 7}} \left( \int_{\Omega} h dx \right)^{\frac{3(2 \alpha + 5 m - 3 n + 4)}{\alpha + n + 7}} +$$

$$d_2 \left( \int_{\Omega} h dx \right)^{\frac{\alpha + 3 m - 2 n + 2}{\alpha + n + 7}} \leq c_1 M^{\frac{3(2 \alpha + 5 m - 3 n + 4)}{\alpha + n + 7}} \left( \int_{\Omega} h^{\alpha + n - 4} h_x^6 dx \right)^{\frac{\alpha + 3 m - 2 n + 2}{\alpha + n + 7}} +$$

$$c_2 M^{\alpha + 3 m - 2 n + 2}.$$  \hspace{1cm} (3.27)

Substituting (3.27) in (3.26), we find

$$\iint_{Q_T} h^{\alpha + m - 2} h_x^4 dx dt \leq \int_0^T \left( \int_{\Omega} h^{\alpha + \nu - 4} h_x^6 dx \right)^2 \left( \int_{\Omega} h^{\alpha + 3 m - 2 n + 2} dx \right)^{\frac{4}{7}} dt +$$

$$c_2 M^{\alpha + 3 m - 2 n + 2} \int_0^T \left( \int_{\Omega} h^{\alpha + n - 4} h_x^6 dx \right)^{\frac{4}{7}} dt.$$  \hspace{1cm} (3.28)

If $m < n + 2$ then, using Young’s inequality, from (3.28) we arrive at

$$\iint_{Q_T} h^{\alpha + m - 2} h_x^4 dx dt \leq \epsilon \int_{Q_T} h^{\alpha + n - 4} h_x^6 dx dt +$$

$$C(\epsilon) T (c_1 M^{\frac{2 \alpha + 5 m - 3 n + 4}{\alpha + 2 - m}} + c_2 M^{\alpha + 3 m - 2 n + 2}).$$  \hspace{1cm} (3.29)
Substituting (3.29) in (3.25), and choosing $\epsilon$ small enough, we obtain

$$E_0^{(\alpha)}(T) \leq E_0^{(\alpha)}(0) + T(C_1 M^{\frac{2n+5m-3n+4}{n+2-m}} + C_2 M^{\alpha+3m-2n+2}) \quad (3.30)$$

for $\alpha > 2n - 3m - 1$ and $m < n + 2$. Here $C_2 = 0$ if $\Omega$ is unbounded or $h$ is compactly supported. In particular, if $0 < \alpha + 3m - 2n + 2 \leq 1$, i.e. $2n - 3m - 2 < \alpha \leq 2n - 3m - 1$ then, using the Hölder inequality and applying Young’s inequality, from (3.26) we obtain

$$\int \int_{Q_T} h^{\alpha+m-2} h_x^4 dxdt \leq \epsilon \int \int_{Q_T} h^{\alpha+n-4} h_x^6 dxdt + C(\epsilon) |\Omega|^{2n-3m-1-\alpha} T M^{\alpha+3m-2n+2} \quad (3.31)$$

for $2n - 3m - 2 < \alpha \leq 2n - 3m - 1$. Substituting (3.31) in (3.25), and choosing $\epsilon$ small enough, we obtain

$$E_0^{(\alpha)}(T) \leq E_0^{(\alpha)}(0) + C_3 T M^{\alpha+3m-2n+2}. \quad (3.32)$$

If $m = n + 2$ then, using Young’s inequality, from (3.28) we deduce

$$\int \int_{Q_T} h^{\alpha+m-2} h_x^4 dxdt \leq c_1 M^2 \int \int_{Q_T} h^{\alpha+n-4} h_x^6 dxdt + \epsilon \int \int_{Q_T} h^{\alpha+n-4} h_x^6 dxdt + C(\epsilon) T M^{\alpha+n+8}. \quad (3.33)$$

Substituting (3.33) in (3.25), and choosing $\epsilon$ enough small, we obtain

$$E_0^{(\alpha)}(T) \leq E_0^{(\alpha)}(0) + C_2 T M^{\alpha+n+8} \quad (3.34)$$

for $\alpha > -n - 7$, $m = n + 2$ and $M \leq M_c$. Here $C_2 = 0$ if $\Omega$ is unbounded or $h$ is compactly supported.

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Appendix A

Lemma A.1. Let $\Omega \subset \mathbb{R}^N$, $N < 6$, be a bounded convex domain with smooth boundary, and let $n \in (2 - \sqrt{1 - \frac{N}{N+8}}, 3)$ for $N > 1$, and $\frac{1}{2} < n < 3$ for $N = 1$. Then the following estimate holds for any positive functions $v \in H^2(\Omega)$ such that $\nabla v \cdot \vec{n} = 0$ on $\partial \Omega$ and $\int_{\Omega} v^n |\nabla \Delta v|^2 < \infty$:

$$\int_{\Omega} \varphi^6 \{ v^{n-4} |\nabla v|^6 + v^{n-2} |D^2 v|^2 |\nabla v|^2 \} \leq c \left\{ \int_{\Omega} \varphi^6 v^n |\nabla \Delta v|^2 + \int_{\varphi > 0} v^{n+2} |\nabla \varphi|^6 \right\},$$

where $\varphi \in C^2(\Omega)$ is an arbitrary nonnegative function such that the tangential component of $\nabla \varphi$ is equal to zero on $\partial \Omega$, and the constant $c > 0$ is independent of $v$.

Lemma A.2. If $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise-smooth boundary, $a > 1$, $b \in (0, a)$, $d > 1$, and $0 \leq i < j$, $i, j \in \mathbb{N}$, then there exist positive constants $d_1$ and $d_2$ ($d_2 = 0$ if $\Omega$ is unbounded) depending only on $\Omega$, $d$, $j$, $b$, and $N$ such that the following inequality is valid for every $v(x) \in W^{i,d}(\Omega) \cap L^b(\Omega)$:

$$\| D^i v \|_{L^a(\Omega)} \leq d_1 \| D^j v \|_{L^b(\Omega)}^{\vartheta} \| v \|_{L^b(\Omega)}^{1-\vartheta} + d_2 \| v \|_{L^b(\Omega)}^{\theta}, \quad \vartheta = \frac{1}{b} + \frac{i}{N} - \frac{1}{a}, \quad \theta = \frac{1}{b} + \frac{j}{N} - \frac{1}{d} \in \left[ \frac{i}{j}, 1 \right).$$

References

[1] Elena Beretta, Michiel Bertsch, and Roberta Dal Passo. Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. Arch. Rational Mech. Anal., 129(2):175–200, 1995.

[2] Francisco Bernis. Finite speed of propagation for thin viscous flows when $2 \leq n < 3$. C. R. Acad. Sci. Paris Sér. I Math., 322(12):1169–1174, 1996.
[3] Francisco Bernis and Avner Friedman. Higher order nonlinear degenerate parabolic equations. *J. Differential Equations*, 83(1):179–206, 1990.

[4] Andrew J. Bernoff and Andrea L. Bertozzi. Singularities in a modified Kuramoto-Sivashinsky equation describing interface motion for phase transition. *Phys. D*, 85(3):375–404, 1995.

[5] A. L. Bertozzi and M. Pugh. The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Comm. Pure Appl. Math.*, 49(2):85–123, 1996.

[6] A. L. Bertozzi and M. C. Pugh. Long-wave instabilities and saturation in thin film equations. *Comm. Pure Appl. Math.*, 51(6):625–661, 1998.

[7] A. L. Bertozzi and M. C. Pugh. Finite-time blow-up of solutions of some long-wave unstable thin film equations. *Indiana Univ. Math. J.*, 49(4):1323–1366, 2000.

[8] Andrea L. Bertozzi, Michael P. Brenner, Todd F. Dupont, and Leo P. Kadanoff. Singularities and similarities in interface flows. In *Trends and perspectives in applied mathematics*, volume 100 of *Appl. Math. Sci.*, pages 155–208. Springer, New York, 1994.

[9] E. Carlen, S. Ulusoy. An entropy dissipation-entropy estimate for a thin film type equation. *Comm. Math. Sci.*, 3(2):171–178, 2005.

[10] P. Constantin, T. F. Dupont, R. E. Goldstein, Leo P. Kadanoff, M. J. Shelley, and S. M. Zhou. Droplet breakup in a model of the Hele-Shaw cell. *Physical review E*, 47(6):4169–4181, June 1993.

[11] P. Ehrhard. The spreading of hanging drops. *Journal of colloid and interface science*, 168(1):242–246, nov 1994.

[12] S. D. Èidel’man. *Parabolic systems*. Translated from the Russian by Scripta Technica, London. North-Holland Publishing Co., Amsterdam, 1969.
[13] Günther Grün. Droplet spreading under weak slippage: a basic result on finite speed of propagation. *SIAM J. Math. Anal.*, 34(4):992–1006 (electronic), 2003.

[14] Ansgar Jungel, and Danial Matthes. An algorithmic construction of entropies in higher-order nonlinear PDEs. *Nonlinearity*, 19:633–659, 2006.

[15] R. S. Laugesen. New dissipated energies for the thin fluid film equation. *Commun. Pure Appl. Anal.*, 4 (3): 613–634, 2005.

[16] L. Nirenberg. An extended interpolation inequality. *Ann. Scuola Norm. Sup. Pisa* (3), 20:733–737, 1966.

[17] A. E. Shishkov and R. M. Taranets. On the equation of the flow of thin films with nonlinear convection in multidimensional domains. *Ukr. Mat. Visn.*, 1(3):402–444, 447, 2004.

[18] A. Tudorascu. Lubrication approximation for thin viscous films: asymptotic behavior of nonnegative solutions. *Communications in PDE*, 32:1147–1172, 2007.

[19] Thomas P. Witelski and Andrew J. Bernoff. Stability of self-similar solutions for van der Waals driven thin film rupture. *Phys. Fluids*, 11(9):2443–2445, 1999.