Convergence and local-to-global results for $p$-superminimizers on quasiopen sets

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Abstract

In this paper, several convergence results for fine $p$-(super)minimizers on quasiopen sets in metric spaces are obtained. For this purpose, we deduce a Caccioppoli-type inequality and local-to-global principles for fine $p$-(super)minimizers on quasiopen sets. A substantial part of these considerations is to show that the functions belong to a suitable local fine Sobolev space. We prove our results for a complete metric space equipped with a doubling measure supporting a $p$-Poincaré inequality with $1 < p < \infty$. However, most of the results are new also for unweighted $\mathbb{R}^n$.

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1. Introduction

Superharmonic functions in classical potential theory are in general only continuous with respect to the so-called fine topology, rather than the standard topology. This is one reason for studying finely (super)harmonic functions on finely open sets. Their theory has been...
developed mainly in the linear axiomatic setting, see e.g. the monographs by Fuglede [10] and Lukeš–Malý–Zajíček [17].

Finely open sets are special cases of quasiopen sets, which coincide with the superlevel sets of Sobolev functions. As such, they carry nontrivial Sobolev test functions and are thus suitable for partial differential equations, such as the \( p \)-Laplace equation \( \Delta_p u = 0 \). In the nonlinear case, i.e. for \( p \neq 2 \), a study of fine \( p \)-(super)solutions to such (and more general) equations was conducted in 1992 by Kilpeläinen–Malý [15] on quasiopen sets in unweighted \( \mathbb{R}^n \). That theory was further extended by Latvala [18], [19], in particular for \( p = n \). Quasiopen sets also appear as solutions to certain shape optimization problems as e.g. in Buttazzo–Dal Maso [9] and Fusco–Mukherjee–Zhang [12].

In this paper we continue our study of fine \( p \)-(super)minimizers on quasiopen sets in metric spaces initiated in [7]. We consider a complete metric space \( X \) equipped with a doubling measure \( \mu \) supporting a \( p \)-Poincaré inequality with \( 1 < p < \infty \). In this setting, \( p \)-harmonic functions are defined as minimizers of the \( p \)-energy using upper gradients. Since there is (in general) no differential equation, we talk about fine \( p \)-(super)minimizers, rather than fine \( p \)-(super)solutions as in [15]. On \( \mathbb{R}^n \), these two notions coincide [7, Proposition 5.3]. From now on we drop the parameter \( p \) and just write fine (super)minimizers.

It turns out that metric spaces are well suited for studying these notions on finely and quasiopen sets, since such sets can easily be seen as metric spaces in their own right, see [3–8]. The function space naturally associated with fine superminimizers on metric spaces is the Newton–Sobolev space \( N^{1,p} \) and its local fine version \( N^{1,p}_{\text{fine-loc}} \) defined through compactly contained \( p \)-strict subsets. A set \( V \subset U \) is a \( p \)-strict subset of \( U \) if there is \( \eta \in N^{1,p}(X) \) such that \( \eta = 1 \) on \( V \) and \( \eta = 0 \) outside \( U \). This is denoted \( V^p \subset U \), and similarly \( V^p \subset U \) when also \( V \subset U \).

The following is a special case of our main result. Even this special case is stronger than the earlier monotone convergence results on unweighted \( \mathbb{R}^n \) proved in [15, Theorems 4.2 and 4.3], see the introduction to Section 7. We are not aware of any other similar convergence results in the nonlinear case.

**Theorem 1.1.** Let \( U \subset X \) be quasiopen and \( \{u_j\}_{j=1}^\infty \) be a sequence of fine superminimizers in \( U \) such that \( u_j \geq f \) a.e. in \( U \), \( j = 1, 2, \ldots, \) for some \( f \in N^{1,p}_{\text{fine-loc}}(U) \).

Assume that one of the following conditions holds for \( u := \liminf_{j\to\infty} u_j \):

(a) \( u \) is a.e.-bounded;
(b) \( u \in N^{1,p}_{\text{fine-loc}}(U) \);
(c) \( |u - f_V| \leq M_V \) a.e. in every finely open \( V \subset U \) for some \( f_V \in N^{1,p}(V) \) and \( M_V \geq 0 \).

Then \( u \) is a fine superminimizer in \( U \).

For monotone sequences of fine minimizers this reduces to the following simpler statement.

**Corollary 1.2.** Let \( U \subset X \) be quasiopen and \( \{u_j\}_{j=1}^\infty \) be a monotone, or uniformly converging, sequence of fine minimizers in \( U \).

Then \( u := \lim_{j\to\infty} u_j \) is a fine minimizer in \( U \), provided that one of the conditions (a)–(c) in Theorem 1.1 holds.
One of our motivations for studying such convergence results is that they provide fundamental
tools for further studies, such as fine Perron solutions, which we pursue in a forthcoming paper.
For $p$-Laplace type equations on open sets in weighted $\mathbb{R}^n$, such monotone convergence results
were obtained and systematically used in Heinonen–Kilpeläinen–Martio [13]. For developing
fine potential theory, convergence results are even more central since there are less other tools
available.

A property usually taken as an axiom in most axiomatic potential theories is the sheaf property,
which says that if $u$ is superharmonic in $U_\alpha$ for each $\alpha$, then it is superharmonic in $\bigcup_\alpha U_\alpha$. The
definitions of finely superharmonic functions in Fuglede [10, Definition 8.1] (in the linear theory)
and in Kilpeläinen–Malý [15, Definition 5.5] (in the nonlinear theory) are based on (and yield)
the sheaf property. This property is usually obvious for supersolutions of differential equations
on open sets as well, but is not known in connection with minimization problems, see [1, Open
problems 9.22 and 9.23].

Our notion of fine superminimizers is on open sets equivalent to the standard superminimizers,
by Corollary 5.6 in [7]. The same is true for the fine supersolutions introduced in [15]. In contrast,
Fuglede [11, Théorème 3.2] gave an example of a finely harmonic function in the sense of [10]
on the entire space $\mathbb{R}^n$, $n \geq 3$, which has an essential singularity at the origin and is therefore
not superharmonic on all of $\mathbb{R}^n$. On the other hand, Lukeš–Malý–Zajíček [17, Section 12.A]
use a more restrictive definition of finely superharmonic functions than Fuglede. Their definition
lacks the sheaf property, but instead has the property that a finely superharmonic function on an
open set is a standard superharmonic function, similarly to our fine superminimizers. See also
Remark 6.6.

As a partial substitute for the absent sheaf property, we obtain several “local-to-global
principles” in Section 6. The following is a special case of Corollary 6.5, which generalizes [15,
Theorem 4.2 (a)]. The removability of $E$ generalizes [7, Lemma 8.1].

**Proposition 1.3.** Let $U \subset X$ be quasiopen and $E \subset U$ be such that $C_p(E) = 0$. Assume that $u$
is a fine (super)minimizer in $V$ for every finely open $V \in U \setminus E$.

Then $u$ is a fine (super)minimizer in $U$, provided that one of the conditions (a)–(c) in Theo-
rem 1.1 holds.

These “local-to-global principles” play a crucial role when deducing the convergence results
in Section 7, including Theorem 1.1. The obstacle problem, studied in [3] and [7], is also used as
a fundamental tool. As we shall see, the most difficult part when showing that a function is a fine
superminimizer is often to show that it belongs to the fine local space $N^{1,p}_{\text{fine-loc}}(U)$. A principal
tool for achieving this is the following Caccioppoli-type inequality, proved in Section 5.

**Theorem 1.4.** (Caccioppoli-type inequality) Assume that $u$ is a fine superminimizer in $U$ such
that $|u - f| \leq M$ a.e. in $U$ for some $f \in N^{1,p}(U)$ and $M \geq 0$. Then

$$
\int_U g_u^p \eta^p \, d\mu \leq 2^p \int_U \eta^p g_f^p \, d\mu + (4pM)^p \int_U g_\eta^p \, d\mu
$$

(1.1)

for every nonnegative $\eta \in N^{1,p}(X)$ with $\eta = 0$ outside $U$. 

814
The compactly contained $p$-strict subsets $V$, appearing in the definitions of $N_{\text{fine-loc}}^{1,p}$ and fine superminimizers, as well as in Theorem 1.1 and Proposition 1.3, were introduced by Kilpeläinen–Malý [15] in $\mathbb{R}^n$ as a substitute for compactly contained open subsets, which are usually used to define the local space $N_{\text{loc}}^{1,p}$ and supersolutions on open sets. One of the main difficulties, when dealing with the fine local space $N_{\text{fine-loc}}^{1,p}$ and fine superminimizers is that we do not know the answer to the following question.

**Open problem 1.5.** Given $A \subset \subset U$, is there $W$ such that $A \subset \subset W \subset \subset U$?

This is easily seen to be true if $U$ is open, but not known for quasiope $U$, even in $\mathbb{R}^n$. As a consequence, some of our results and formulations are more cumbersome, such as Theorem 1.1 and Proposition 1.3 (and their generalizations Corollary 6.5 and Theorem 7.4). This is also reflected in the formulation of Theorem 4.2 in [15]. Even the existence of $W$ such that $A \subset \subset W \subset \subset U$ would have important consequences, cf. condition (d) in Corollary 6.5 and Theorem 7.4.

The outline of the paper is as follows. In Section 2 we present the main background definitions from nonlinear potential theory in metric spaces, while in Section 3 we introduce the fine topology and the fine local Newtonian space $N_{\text{fine-loc}}^{1,p}(U)$.

Following our recent paper [7], we introduce fine superminimizers and the fine obstacle problem in Section 4.

In Section 5 we deduce the Caccioppoli-type inequality (Theorem 1.4), whereas Section 6 is devoted to characterizations and local-to-global principles for fine superminimizers. Finally, in Section 7 we deduce several convergence results for fine superminimizers and one for fine minimizers. Throughout the paper we utilize most of the results obtained in [7].

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**2. Notation and preliminaries**

In this section, we introduce the necessary metric space concepts used in this paper. For brevity, we refer to Björn–Björn–Latvala [4], [6] for more extensive introductions, and references to the literature. See also the monographs Björn–Björn [1] and Heinonen–Koskela–Shanmugalingam–Tyson [14], where the theory of upper gradients and Newtonian spaces is thoroughly developed with proofs.

Let $X$ be a metric space equipped with a metric $d$ and a positive complete Borel measure $\mu$ such that $\mu(B) < \infty$ for all balls $B \subset X$. We also assume that $1 < p < \infty$.

In this paper, a *curve* is a continuous mapping from a compact interval. We will only consider curves which are nonconstant and rectifiable (i.e. of finite length), and they can therefore be parameterized by their arc length $ds$. A property holds for *$p$-almost every curve* if the curve family $\Gamma$ for which it fails has zero $p$-modulus, i.e. there is $\rho \in L^p(X)$ such that $\int_\Gamma \rho \, ds = \infty$ for every $\gamma \in \Gamma$.

A measurable function $g : X \to [0, \infty]$ is a *$p$-weak upper gradient* of $u : X \to \mathbb{R} := [-\infty, \infty]$ if for $p$-almost all curves $\gamma : [0, l_\gamma] \to X$,
\[ |u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_{\gamma} g \, ds, \]

where the left-hand side is \( \infty \) whenever at least one of the terms therein is infinite. If \( u \) has a \( p \)-weak upper gradient in \( L^p_{\text{loc}}(X) \), then it has a \( \text{minimal } p \)-weak upper gradient \( g_u \in L^p_{\text{loc}}(X) \) in the sense that \( g_u \leq g \) a.e. for every \( p \)-weak upper gradient \( g \in L^p_{\text{loc}}(X) \) of \( u \).

For measurable \( u \), we let

\[ \|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p}, \]

where the infimum is taken over all \( p \)-weak upper gradients \( g \) of \( u \). The Newtonian space on \( X \) is

\[ N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}. \]

The space \( N^{1,p}(X)/\sim \), where \( u \sim v \) if and only if \( \|u - v\|_{N^{1,p}(X)} = 0 \), is a Banach space and a lattice. In this paper it is convenient to assume that functions in \( N^{1,p}(X) \) are defined everywhere (with values in \( \mathbb{R} \)), not just up to an equivalence class in the corresponding function space.

For an arbitrary set \( A \subset X \), we let

\[ N^{1,p}_0(A) = \{u|_A : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus A\}. \]

Functions from \( N^{1,p}_0(A) \) can be extended by zero in \( X \setminus A \) and we will regard them in that sense when needed.

The Sobolev capacity of an arbitrary set \( A \subset X \) is

\[ C^X_p(A) = \inf_u \|u\|_{N^{1,p}(X)}^p, \]

where the infimum is taken over all \( u \in N^{1,p}(X) \) such that \( u \geq 1 \) on \( A \).

A property holds quasi-everywhere (q.e.) if the set of points for which it fails has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If \( u \in N^{1,p}(X) \), then \( v \sim u \) if and only if \( v = u \) q.e. Moreover, if \( u, v \in N^{1,p}(X) \) and \( v = u \) a.e., then \( v = u \) q.e. The superscript in \( C^X_p \) indicates dependence on the underlying space. We will usually omit \( X \) and write \( C_p \) instead of \( C^X_p \).

A set \( U \subset X \) is quasiopen if for every \( \varepsilon > 0 \) there is an open set \( G \subset X \) such that \( C_p(G) < \varepsilon \) and \( G \cup U \) is open. The quasiopen sets do not in general form a topology, see Remark 9.1 in [3]. However it follows easily from the countable subadditivity of \( C_p \) that countable unions and finite intersections of quasiopen sets are quasiopen. Moreover, quasiopen sets are measurable. Various characterizations of quasiopen sets can be found in Björn–Björn–Malý [8]. If \( U \subset X \) is quasiopen then \( C^U_p \) and \( C^X_p \) have the same zero sets in \( U \), by Proposition 4.2 in [8] (and Remark 3.5 in Shanmugalingam [20]).

For a measurable set \( E \subset X \), the Newtonian space \( N^{1,p}(E) \) and the capacity \( C^E_p \) are defined by considering \( (E, d|_E, \mu|_E) \) as a metric space in its own right.
If $E \subset A$ are subsets of $X$, then the variational capacity of $E$ with respect to $A$ is

$$\text{cap}_p(E, A) = \inf_u \int_X g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(A)$ such that $u \geq 1$ on $E$. If no such function $u$ exists then $\text{cap}_p(E, A) = \infty$.

The measure $\mu$ is doubling if there is $C > 0$ such that $0 < \mu(2B) \leq C\mu(B) < \infty$ for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in $X$, where $\lambda B = B(x_0, \lambda r)$. In this paper, all balls are open.

The space $X$ supports a $p$-Poincaré inequality if there are $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions $u$ on $X$ and all $p$-weak upper gradients $g$ of $u$,

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C \text{diam}(B) \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p \, d\mu \right)^{1/p},$$

where $u_B := \int_B u \, d\mu / \mu(B)$ and we implicitly assume that $0 < \mu(B) < \infty$ for all balls $B$.

In $\mathbb{R}^n$ equipped with a doubling measure $d\mu = w \, dx$, where $dx$ denotes Lebesgue measure, the $p$-Poincaré inequality (2.1) is equivalent to the $p$-admissibility of the weight $w$ in the sense of Heinonen–Kilpeläinen–Martio [13], see Corollary 20.9 in [13] and Proposition A.17 in [1]. Moreover, in this case $g_u = |\nabla u|$ a.e. if $u \in N_0^{1,p}(\mathbb{R}^n)$ and the capacities $C_p$ and $\text{cap}_p$ coincide with the corresponding capacities in [13] (see [1, Theorem 6.7] and [2, Theorem 5.1]).

As usual, we set $u_+ = \max\{u, 0\}$.

3. **Fine topology, $N_{\text{fine-loc}}^{1,p}(U)$ and $p$-strict subsets**

Throughout the rest of the paper, we assume that $X = (X, d)$ is a complete metric space equipped with a doubling measure $\mu$ supporting a $p$-Poincaré inequality with $1 < p < \infty$. Unless said otherwise, $U$ will be a nonempty quasiopen set.

To avoid pathological situations we also assume that $X$ contains at least two points (and thus must be uncountable due to the Poincaré inequality). In this section we recall the basic facts about the fine topology and Newtonian functions on quasiopen sets.

**Definition 3.1.** A set $E \subset X$ is thin at $x \in X$ if

$$\int_0^1 \left( \frac{\text{cap}_p(E \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \, \frac{dr}{r} < \infty.$$

A set $V \subset X$ is finely open if $X \setminus V$ is thin at each point $x \in V$.

In the definition of thinness, we use the convention that the integrand is 1 whenever $\text{cap}_p(B(x, r), B(x, 2r)) = 0$. It is easy to see that the finely open sets give rise to a topology, which is called the fine topology. Every open set is finely open, but the converse is not true in...
general. A function \( u : V \to \overline{\mathbb{R}} \), defined on a finely open set \( V \), is \textit{finely continuous} if it is continuous when \( V \) is equipped with the fine topology and \( \overline{\mathbb{R}} \) with the usual topology. Pointwise fine continuity is defined analogously. The fine interior, fine boundary and fine closure of \( E \) are denoted \( \text{fine-int} \ E \), \( \partial_p E \) and \( \overline{E}^p \), respectively. See [1, Section 11.6] and Björn–Björn–Latvala [4] for further discussion on thinness and the fine topology on metric spaces.

The following result shows that there is a close connection between quasiopen and finely open sets.

**Theorem 3.2.** (Theorem 3.4 in [7]) The following conditions are equivalent for any set \( U \subseteq X \):

(a) \( U \) is quasiopen;
(b) \( U = V \cup E \) for some finely open \( V \) and a set \( E \) with \( C_\rho(E) = 0 \);
(c) \( C_\rho(U \setminus \text{fine-int} U) = 0 \);
(d) \( U = \{ x : u(x) > 0 \} \) for some \( u \in N^{1,p}(X) \).

To define fine superminimizers, we first need appropriate fine local Sobolev spaces. Here \( p \)-strict subsets will play a key role, as a substitute for relatively compact subsets. Recall that \( V \subseteq U \) if \( \overline{V} \) is a compact subset of \( U \).

**Definition 3.3.** A set \( A \subseteq U \) is a \( p \)-\textit{strict subset} of \( U \) if there is a function \( \eta \in N^{1,p}_0(U) \) such that \( \eta = 1 \) on \( A \). We will write \( A \subseteq^p U \), and similarly \( A \Subset U \) when also \( A \Subset X \).

A function \( u \) belongs to \( N^{1,p}_{\text{fine-loc}}(U) \) if \( u \in N^{1,p}(V) \) for all finely open \( V \Subset U \).

In the definition of \( p \)-strict subsets, it can be equivalently required that in addition \( 0 \leq \eta \leq 1 \), as in Kilpeläinen–Malý [15]; just replace \( \eta \) with \( \min(\eta, 1)_+ \). Note that \( A \subseteq^p U \) if and only if \( \text{cap}_p(A, U) < \infty \). Lemma 3.3 in Björn–Björn–Latvala [5] shows that every finely open set \( V \) has a base of fine neighbourhoods \( W \Subset V \). By Theorem 4.4 in [5], functions in \( N^{1,p}_{\text{fine-loc}}(U) \) are finite q.e., finely continuous q.e. and quasicontinuous.

Throughout the paper, we consider minimal \( p \)-weak upper gradients in \( U \). For a function \( u \in N^{1,p}_{\text{fine-loc}}(U) \) we say that \( g_{u,U} \) is a \textit{minimal }\( p \)-weak upper gradient of \( u \) in \( U \) if

\[
g_{u,U} = g_{u,V} \text{ a.e. in } V \quad \text{for every finely open } V \Subset U,
\]

where \( g_{u,V} \) is the minimal \( p \)-weak upper gradient of \( u \in N^{1,p}(V) \) with respect to \( V \). If \( u \in N^{1,p}(U) \), then this definition agrees with the definition of \( g_{u,U} \) in Section 2. See [5, Lemma 5.2 and Theorem 5.3] for the existence, a.e.-uniqueness and minimality of \( g_{u,U} \). If \( u \in N^{1,p}_{\text{loc}}(X) \) then the minimal \( p \)-weak upper gradients \( g_{u,U} \) and \( g_{u} \) with respect to \( U \) and \( X \), respectively, coincide a.e. in \( U \), see [3, Corollary 3.7] or [5, Lemma 4.3]. For this reason we drop \( U \) from the notation and simply write \( g_u \) from now on.

**Remark 3.4.** If \( u \in N^{1,p}_{\text{fine-loc}}(U) \) and \( W \Subset U \) is quasiopen, then \( W \setminus \text{fine-int} W \) has zero capacity by Theorem 3.2 (and is thus not seen by \( g_u \) because of [1, Proposition 1.48]). Hence \( u \in N^{1,p}(W) = N^{1,p}(\text{fine-int} W) \). Thus \( u \in N^{1,p}_{\text{fine-loc}}(U) \) if and only if \( u \in N^{1,p}(W) \) for every quasiopen \( W \Subset U \).
It follows from the definition of $N_{\text{fine-loc}}^1(U)$ and the properties of $N^1_p$ (see Section 2) that functions in $N_{\text{fine-loc}}^1(U)$ are defined everywhere in $U$, and that if $u \in N_{\text{fine-loc}}^1(U)$ and $v = u$ q.e., then also $v \in N_{\text{fine-loc}}^1(U)$. Moreover, if $u, v \in N_{\text{fine-loc}}^1(U)$ and $u = v$ a.e. then $u = v$ q.e., as we shall now prove. Due to the definition of $N_{\text{fine-loc}}^1(U)$ this is not covered by the results in [1].

**Lemma 3.5.** Assume that $u, v \in N_{\text{fine-loc}}^1(U)$. Then the following are true:

(a) If $u = v$ a.e. in $U$, then $u = v$ q.e. in $U$.

(b) If $u \geq v$ a.e. in $U$, then $u \geq v$ q.e. in $U$.

**Proof.** Statement (a) follows directly from (b), so consider (b). By Lemma 3.3 in [5], every $x \in \text{fine-int } U$ has a finely open neighbourhood $W_x \subset U$. By Theorem 3.2 and the quasi-Lindelöf principle [5, Theorem 3.4] there is a countable subcollection $\{W_j\}_{j=1}^\infty \subset \text{fine-int } U$ such that $C_p(U \setminus \bigcup_{j=1}^\infty W_j) = 0$.

By definition, $u, v \in N^1_p(W_j)$, and hence $u \geq v$ q.e. in $W_j$, $j = 1, 2, \ldots$, by Corollary 1.60 in [1], and thus q.e. in $U$. \qed

**Remark 3.6.** It follows from the covering obtained in the proof of Lemma 3.5 that Corollaries 2.20 and 2.21 in [1] extend to functions in $N_{\text{fine-loc}}^1(U)$. That is, if $u, v \in N_{\text{fine-loc}}^1(U)$ then $g_u = g_v$ a.e. on $\{x \in U : u(x) = v(x)\}$. In particular,

$$g_{\min(u,v)} = g_u 1[u>v] + g_v 1[v\geq u] \quad \text{and} \quad g_{\max(u,v)} = g_u 1[v>u] + g_v 1[u\geq v] \quad \text{a.e.}$$

We will use the following simple lemma several times, cf. Open Problem 1.5.

**Lemma 3.7.** If $A \subset^p U$, then there is a quasiopen $W$ such that $A \subset^p W \subset^p U$. If $A$ is finely open, then $W$ can be chosen to be finely open.

**Proof.** By assumption, there is $\eta \in N^1_0(U)$ such that $\eta = 1$ on $A$. We may also assume that $0 \leq \eta \leq 1$. Let $W := \{x : \eta(x) > \frac{1}{2}\}$, which is quasiopen by Theorem 3.2. The functions

$$(2\eta - 1)_+ \in N^1_0(W) \quad \text{and} \quad \min\{2\eta, 1\} \in N^1_0(U)$$

then show that $A \subset^p W \subset^p U$.

If $A$ is finely open, we can replace $W$ by fine-int $W$, since Theorem 3.2 implies that $N^1_0(W) = N^1_0(\text{fine-int } W)$. \qed

The following density result will play a crucial role.

**Proposition 3.8.** (Proposition 4.5 in [7]) Let $0 \leq \varphi \in N^1_0(U)$. Then there exist finely open sets $V_j \subset U$ and bounded functions $\varphi_j \in N^1_0(V_j)$ such that

(a) $V_j \subset V_{j+1}$ and $0 \leq \varphi_j \leq \varphi_{j+1} \leq \varphi$ for $j = 1, 2, \ldots$;

(b) $\|\varphi - \varphi_j\|_{N^1_p(X)} \to 0$ and $\varphi_j(x) \to \varphi(x)$ for q.e. $x \in X$, as $j \to \infty$. 819
4. Fine superminimizers and the obstacle problem

Recall the standing assumptions from the beginning of Section 3.

The following definition is from [7, Definition 5.1]. On unweighted $\mathbb{R}^n$, Kilpeläinen–Malý [15, Section 3.1] gave an equivalent definition (of fine supersolutions) already in 1992, cf. Proposition 5.3 in [7].

**Definition 4.1.** A function $u \in N_{1,p}^{1,\text{fine-loc}}(U)$ is a fine minimizer (resp. fine superminimizer) in $U$ if

$$
\int_V g^p_u d\mu \leq \int_V g^p_{u+\varphi} d\mu
$$

for every finely open $V \subset U$ and for every (resp. every nonnegative) $\varphi \in N_{0,1,p}(V)$. Moreover, $u$ is a fine subminimizer if $-u$ is a fine superminimizer.

A finely $p$-harmonic function is a finely continuous fine minimizer.

Note that, unlike for minimizers and $p$-harmonic functions on open sets, it is not known whether every fine minimizer can be modified on a set of zero capacity so that it becomes finely continuous and thus finely $p$-harmonic, see the discussion in Section 9 in [7]. Of course, being a function from $N_{1,p}^{1,\text{fine-loc}}(U)$, every fine (super/sub)minimizer is finely continuous at q.e. point.

By Lemma 5.4 in [7], a function is a fine minimizer if and only if it is both a fine subminimizer and a fine superminimizer. We refer to [7] for further discussion on fine superminimizers and the Newtonian space $N_{1,p}^{1,\text{fine-loc}}(U)$. The following lattice property will be used several times.

**Lemma 4.2.** (Corollary 5.8 in [7]) If $u$ and $v$ are fine superminimizers in $U$, then $\min\{u,v\}$ is also a fine superminimizer in $U$.

The obstacle problem will be a fundamental tool when studying fine (super)minimizers. See [3] and [7] for earlier studies of the obstacle problem on nonopen sets in metric spaces. It was shown therein that such theories are not natural beyond finely open or quasiiopen sets. In particular, it was shown in Björn–Björn [3, Theorem 7.3] that $N_{0,1,p}(E) = N_{0,1,p}(\text{fine-int } E)$ for an arbitrary set $E$.

**Definition 4.3.** Assume that $U$ is a bounded nonempty quasiiopen set with $C_p(X \setminus U) > 0$. Let $f \in N_{1,p}^{1,\text{loc}}(U)$ and $\psi : U \to \mathbb{R}$. Then we define

$$
\mathcal{K}_{\psi,f}(U) = \{v \in N_{1,p}^{1,\text{loc}}(U) : v - f \in N_{0,1,p}^{1,\text{loc}}(U) \text{ and } v \geq \psi \text{ q.e. in } U\}.
$$

A function $u \in \mathcal{K}_{\psi,f}(U)$ is a solution of the $\mathcal{K}_{\psi,f}(U)$-obstacle problem if

$$
\int_U g^p_u d\mu \leq \int_U g^p_v d\mu \quad \text{for all } v \in \mathcal{K}_{\psi,f}(U).
$$

820
Note that the boundary data \( f \) are only required to belong to \( N^{1,p}(U) \), i.e. \( f \) need not be defined on \( \partial U \) or the fine boundary \( \partial_P U \).

If \( K_{\psi,f}(U) \neq \emptyset \), then there is a solution \( u \) of the \( K_{\psi,f}(U) \)-obstacle problem, which is unique q.e., see Theorem 4.2 in [3]. Moreover, \( u \) is a fine superminimizer, by Theorem 6.2 in [7]. A comparison principle for obstacle problems was obtained in Corollary 4.3 in [3].

5. The Caccioppoli-type inequality

Caccioppoli inequalities are important tools in PDEs and nonlinear potential theory. In this section, we are going to deduce the Caccioppoli-type inequality in Theorem 1.4. In Section 7, we will use it to deduce the fundamental convergence theorems for increasing and decreasing sequences of fine superminimizers.

**Proof of Theorem 1.4.** Assume first that \( 0 \leq \eta \leq 1 \) and that \( W := \{ x : \eta(x) > 0 \} \in \mathcal{P} \). Then \( W \) is quasiopen by Theorem 3.2, and \( u \in N^{1,p}(W) \) by Remark 3.4. It follows from Lemma 3.5 that we can assume that \( |u - f| \leq M \) everywhere. It thus follows from the Leibniz and chain rules [1, Theorem 2.15 and 2.16] that \( v := u + \eta^p (f + M - u) \in N^{1,p}(W) \). Lemma 2.4 in Kinnunen–Martio [16] (or [1, Lemma 2.18]) implies that

\[
g_v \leq (1 - \eta^p)g_u + \eta^p g_f + p(f + M - u)\eta^{p-1} g_\eta.
\]

(In [16] and [1] it is assumed that \( \eta \in \text{Lip}(X) \), but since \( u - f \) is bounded it is enough to require that \( \eta \in N^{1,p}(X) \), \( 0 \leq \eta \leq 1 \), as here. The proof is the same, just restrict attention to curves \( \gamma \) on which also \( \eta \) is absolutely continuous.) As \( u \) is a fine superminimizer in \( U \) and \( 0 \leq v - u \in N^{1,p}_0(W) \), we obtain using the convexity formula \(((1 - \lambda)t + \lambda s)^p \leq (1 - \lambda)t^p + \lambda s^p\), with \( \lambda = \eta^p \), that

\[
\int_W g_v^p \, d\mu \leq \int_W g_u^p \, d\mu
\]

\[
\leq \int_W \left( (1 - \eta^p)g_u + \eta^p(g_f + p|f + M - u|\frac{g_\eta}{\eta}) \right)^p \, d\mu
\]

\[
\leq \int_W (1 - \eta^p)g_u \, d\mu + \int_W \eta^p(g_f + p|f + M - u|\frac{g_\eta}{\eta}) \, d\mu
\]

\[
\leq \int_W (1 - \eta^p)g_u \, d\mu + 2^p \int_W \eta^p\left( g_f^p + (2pM)^p g_\eta^p \right) \, d\mu.
\]

Since \( g_u \in \mathcal{L}^p(W) \), subtracting the first integral on the right-hand side from both sides of the inequality yields (1.1) and proves the statement when \( 0 \leq \eta \leq 1 \) and \( \{ x : \eta(x) > 0 \} \in \mathcal{P} \).

For general \( 0 \leq \eta \in N^{1,p}_0(U) \), Proposition 3.8 provides us with finely open sets \( V_j \in \mathcal{P} \) and an increasing sequence of bounded functions \( 0 \leq \eta_j \in N^{1,p}_0(V_j) \) such that \( \| \eta - \eta_j \|_{N^{1,p}(X)} \to 0 \) and \( \eta_j \not\geq \eta \) a.e., as \( j \to \infty \). Applying (1.1) to \( V_j \) and suitable multiples of \( \eta_j \), we get
\[
\int_U g_{u_j}^p \eta^p \, d\mu = \lim_{j \to \infty} \int_{V_j} g_{u_j}^p \eta_j^p \, d\mu \leq 2^p \int_U \eta^p g_f^p \, d\mu + (4pM)^p \int_U g_{\eta}^p \, d\mu
\]

by using monotone convergence. \(\square\)

In the next sections we will several times need to deduce that certain functions belong to \(N^{1,p}_{\text{fine-loc}}(U)\). In order to do so, the following result will be convenient.

**Lemma 5.1.** Let \(E\) be a set with \(C_{p}(E) = 0\) and assume that \(u = \lim_{j \to \infty} u_j\) q.e. in \(U\), where each \(u_j\), \(j = 1, 2, \ldots\), is a fine superminimizer in \(V\) for every finely open \(V \subseteq U \setminus E\). Let \(W \subseteq U\) be a bounded measurable set. Assume that \(|u_j - f| \leq M\) a.e. in \(U\), \(j = 1, 2, \ldots\), for some \(f \in N^{1,p}(U)\) and \(M \geq 0\). Then \(u \in N^{1,p}(W)\) and

\[
\int_W g_{u_j}^p \, d\mu \leq \liminf_{j \to \infty} \int_W g_{u_j}^p \, d\mu \leq 2^p \int_U g_f^p \, d\mu + (4pM)^p \int_U g_{\eta}^p \, d\mu,
\]

(5.1)

where \(\eta \in N^{1,p}_0(U)\) is any function such that \(0 \leq \eta \leq 1\) in \(U\) and \(\eta = 1\) in \(W\).

A natural question is of course whether \(u\) is a fine superminimizer in \(U\), but we postpone that to later. As we shall see, this proposition is useful also for the constant sequence \(u_j = u\), \(j = 1, 2, \ldots\), as it gives the Caccioppoli-type inequality (1.1) under weaker assumptions than Theorem 1.4.

**Proof.** Let \(\eta \in N^{1,p}_0(U)\) be such that \(0 \leq \eta \leq 1\) in \(U\) and \(\eta = 1\) in \(W\). Since \(C_{p}(E) = 0\), we have \(\eta \in N^{1,p}_0(U \setminus E)\). By Proposition 3.8, there are finely open sets \(V_k \subseteq U \setminus E\) and nonnegative bounded functions \(\eta_k \in N^{1,p}_0(V_k)\) such that \(\|\eta - \eta_k\|_{N^{1,p}(X)} \to 0\) and \(\eta_k \not\rightarrow \eta\) q.e. as \(k \to \infty\).

Since each \(u_j\) is a fine superminimizer in every \(V_k\), we get using the Caccioppoli-type inequality (Theorem 1.4) and monotone convergence that

\[
\int_{V_k} g_{u_j}^p \, d\mu \leq \lim_{k \to \infty} \int_{V_k} g_{u_j}^p \eta_k^p \, d\mu = \lim_{k \to \infty} \int_{V_k} g_{u_j}^p \eta_k^p \, d\mu \leq \lim_{k \to \infty} \int_{V_k} (2^p \eta_k^p g_f^p + (4pM)^p g_{\eta_k}^p) \, d\mu \leq \int_U (2^p g_f^p + (4pM)^p g_{\eta}^p) \, d\mu < \infty,
\]

i.e. \(\{u_j\}_{j=1}^{\infty}\) is a bounded sequence in \(N^{1,p}(W)\) (since it is bounded in \(L^p(W)\) by assumption). Hence, by Corollary 6.3 in [1], \(u \in N^{1,p}(W)\) and (5.1) holds. \(\square\)

6. Characterizations of fine superminimizers

In this section we will deduce several characterizations of fine superminimizers, which will be used later on. The first characterization, based on Lemma 4.2, will be used in the proofs of Proposition 6.2 and Corollary 7.3.
Proposition 6.1. Let \( v \) be a fine superminimizer in \( U \) and \( u \in N^{1,p}_{\text{fine-loc}}(U) \). Then \( u \) is a fine superminimizer in \( U \) if and only if \( \min\{u, v + k\} \) is a fine superminimizer in \( U \) for every \( k = 1, 2, \ldots \).

Proof. Assume first that \( u_k := \min\{u, v + k\} \) is a fine superminimizer for every \( k = 1, 2, \ldots \). Let \( V \Subset U \) be finely open and \( \varphi \in N^{1,p}_0(V) \) be nonnegative. As \( u, v \in N^{1,p}_{\text{fine-loc}}(U) \), they are finite q.e. and \( u_k \not\sim u \) q.e. in \( V \), as \( k \to \infty \). Let \( E_k := \{x \in V : u(x) > v(x) + k\} \). Since \( V \) is bounded, we have \( \mu(E_k) \to 0 \). Using monotone and dominated convergence (since \( v + \varphi \in N^{1,p}(V') \)), we then get

\[
\int_V g_{u_k + \varphi}^p \, d\mu = \int_{V \setminus E_k} g_{u_k + \varphi}^p \, d\mu + \int_{E_k} g_{u_k + \varphi}^p \, d\mu \to \int_V g_{u_k + \varphi}^p \, d\mu, \quad \text{as } k \to \infty.
\]

Applying this also with \( \varphi \equiv 0 \) and using that \( u_k \) is a fine superminimizer yields

\[
\int_V g_{u_k}^p \, d\mu = \lim_{k \to \infty} \int_V g_{u_k}^p \, d\mu \leq \lim_{k \to \infty} \int_V g_{u_k + \varphi}^p \, d\mu = \int_V g_{u_k + \varphi}^p \, d\mu.
\]

As \( V \) and \( \varphi \) were arbitrary this shows that \( u \) is a fine superminimizer. The converse implication follows directly from Lemma 4.2. \( \Box \)

The following result will be used to prove Theorem 1.1 and its more general version Theorem 7.4.

Proposition 6.2. Assume that \( v \) is a fine superminimizer in \( U \) and let \( u : U \to \bar{\mathbb{R}} \) be such that \( |u - f| \leq M \) a.e. in \( U \) for some \( f \in N^{1,p}(U) \) and \( M \geq 0 \).

If \( \min\{u, v + k\} \) is a fine superminimizer in \( U \) for every \( k = 1, 2, \ldots \), then \( u \in N^{1,p}(W) \) for every bounded quasiopen \( W \Subset U \). Moreover, \( u \) is a fine superminimizer in \( U \).

Proof. Let \( W \Subset U \) be a bounded quasiopen set and \( \eta \in N^{1,p}_0(U) \) be such that \( 0 \leq \eta \leq 1 \) and \( \eta = 1 \) in \( W \). By Proposition 3.8, there are finely open sets \( V'_1 \subset V'_2 \subset \cdots \Subset U \) and functions \( 0 \leq \eta'_k \in N^{1,p}_0(V'_k) \) such that \( \|\eta - \eta'_k\|_{N^{1,p}(X)} \to 0 \) and \( \eta'_k \not\sim \eta \) q.e. as \( k \to \infty \).

For each \( k = 1, 2, \ldots \), let

\[
M_k = \|g_{v-f}\|_{L^p(V'_k)},
\]

\[
V_k = \{x \in W : f(x) < v(x) + k\},
\]

\[
W'_k = \{x \in V'_k : f(x) < v(x) + k + M_k\}.
\]

Note that both \( V_k \) and \( W'_k \) are quasiopen, by Theorem 3.2. Since \( f, v \in N^{1,p}(V'_k) \) and \( \eta'_k \in N^{1,p}_0(V'_k) \), we see that

\[
0 \leq \eta_k := \min\left\{ \frac{(v + k + M_k - f)_+}{M_k}, 2\eta'_k, 1 \right\} \in N^{1,p}_0(V'_k),
\]
by Lemma 2.37 in [1]. The $p$-weak upper gradient of $\eta_k$ is estimated by

$$
\int_{V_k'} g_{\eta_k}^p d\mu \leq \int_{V_k'} \max\left\{ \frac{g \nu - f}{M_k}, 2g_{\eta_k} \right\}^p d\mu \tag{6.1}
$$

$$
\leq \int_{V_k'} \left( \frac{g v - f}{M_k} + 2p g_{\eta_k}^p \right) d\mu < 1 + 2p \int_{V_k'} g_{\eta_k}^p d\mu \rightarrow 1 + 2p \int_{V} g_{\eta}^p d\mu,
$$

as $k \to \infty$. Moreover, $\eta_k = 0$ in $V_k' \setminus W_k'$ and hence $\eta_k \in N_{1}^{1,p}(W_k')$. On the other hand, $\eta_k = 1$ on $W_k := \{ x \in V_k : \eta_k'(x) > \frac{1}{2} \}$, and so

$$
W_k^p \subseteq W_k^p \subseteq U.
$$

In $W_k^p \subseteq U$ we have by Lemma 3.5 that

$$
u = \min\{ u, v + k + M_k + M \} \quad \text{q.e.},
$$

and so $\nu$ is a fine superminimizer in $W_k'$ (since we may assume that $M_k$ and $M$ are integers). The Caccioppoli inequality (Theorem 1.4 with $U$ replaced by $W_k'$) now implies that $\nu \in N_{1}^{1,p}(W_k)$ and

$$
\int_{W_k} g_{\nu}^p d\mu \leq 2p \int_{W_k'} g_{\nu}^p d\mu + (4pM)^p \int_{W_k'} g_{\eta_k}^p d\mu
$$

$$
\leq 2p \int_{U} g_{\nu}^p d\mu + (4pM)^p \int_{W_k'} g_{\eta_k}^p d\mu. \tag{6.2}
$$

Now, for q.e. $x \in W$, there exists $k$ such that $\eta_k'(x) > \frac{1}{2}$ and $f(x) < \nu(x) + k$ and hence $x \in W_k$. It follows that $W = \bigcup_{k=1}^{\infty} W_k \cup Z$, where $C_p(Z) = 0$.

Since $W_1 \subset W_2 \subset \ldots$, inserting (6.1) into (6.2) together with monotone convergence implies that $\int_{W} g_{\nu}^p d\mu < \infty$. As $|\nu - f| \leq M$ a.e. in $W$, we conclude that $\nu \in N_{1}^{1,p}(W)$.

Applying this to an arbitrary finely open $W \subseteq U$ shows that $\nu \in N_{1}^{1,p}(W)$, and the superminimizing property follows from Proposition 6.1. \(\square\)

The following lemma relates fine superminimizers to obstacle problems.

**Lemma 6.3.** Let $\nu$ be a fine superminimizer in $U$, and let $W \subseteq U$ be a bounded quasiopen set such that $\nu \in N_{1}^{1,p}(W)$ and $C_p(X \setminus W) > 0$. Then $\nu$ is a solution of the $K_{\nu,u}(W)$-obstacle problem.

**Proof.** Let $v \in K_{\nu,u}(W)$. Then $\phi := v - \nu \geq 0$ q.e. in $W$ and $\phi \in N_{0}^{1,p}(W)$. By Lemma 5.5 in [7],
\[ \int_W g_u^p \, d\mu = \int_{\{\varphi=0\}} g_u^p \, d\mu + \int_{\{\varphi>0\}} g_u^p \, d\mu \leq \int_{\{\varphi=0\}} g_v^p \, d\mu + \int_{\{\varphi>0\}} g_v^p \, d\mu = \int_W g_v^p \, d\mu. \]

Thus \( u \) is a solution of the \( K_{u,u}(W) \)-obstacle problem. \( \Box \)

We next turn to “local-to-global principles” for fine superminimizers, which will be crucial later on.

**Proposition 6.4.** Let \( u \in N^{1,p}_{\text{fine-loc}}(U) \) and \( E \) be a set with \( C_p(E) = 0 \). Then the following are equivalent:

(a) \( u \) is a fine superminimizer in \( U \);
(b) \( u \) is a fine superminimizer in \( V \) for every finely open (or equivalently quasiopen) \( V \Subset U \setminus E \) such that \( C_p(X \setminus V) > 0 \);
(c) \( u \) is a solution of the \( K_{u,u}(V) \)-obstacle problem for every finely open (or equivalently quasiopen) \( V \Subset U \setminus E \) such that \( C_p(X \setminus V) > 0 \).

Note that the condition \( C_p(X \setminus V) > 0 \) is automatically satisfied for arbitrary \( V \Subset U \setminus E \) if either \( X \) is unbounded or \( U \setminus E \neq X \), as in these cases \( \overline{V} \) is a compact subset of \( X \) whose complement is a nonempty open set.

**Proof of Proposition 6.4.** The implication (a)\( \Rightarrow \) (b) is trivial, while (b)\( \Rightarrow \) (c) follows from Lemma 6.3. It remains to show (c)\( \Rightarrow \) (a).

To this end, consider first the case when \( X \) is unbounded or \( U \setminus E \neq X \). Let \( V \Subset U \) be finely open and \( \varphi \in N^{1,p}_0(V) \) be nonnegative. Since \( C_p(E) = 0 \), we have \( \varphi \in N^{1,p}_0(V \setminus E) \).

Proposition 3.8 provides us with finely open \( V_j \Subset V \setminus E \) and an increasing sequence of bounded functions \( 0 \leq \varphi_j \in N^{1,p}_0(V_j) \) such that \( \varphi_j \uparrow \varphi \) \( \text{q.e.} \) and \( \| \varphi - \varphi_j \|_{N^{1,p}(X)} \to 0 \) as \( j \to \infty \). Then \( X \setminus V_j \) is open and nonempty and hence \( C_p(X \setminus V_j) > 0 \). Thus, by assumption, \( u \) is a solution of the \( K_{u,u}(V_j) \)-obstacle problem. Since \( u + \varphi_j \in K_{u,u}(V_j) \), we get for each \( j \) that

\[ \left( \int_{V_j} g_u^p \, d\mu \right)^{1/p} \leq \left( \int_{V_j} g_{u+\varphi_j}^p \, d\mu \right)^{1/p} \leq \left( \int_X g_{u+\varphi_j}^p \, d\mu \right)^{1/p} + \| \varphi - \varphi_j \|_{N^{1,p}(X)}. \]

Letting \( j \to \infty \) and using monotone convergence shows that

\[ \int_X g_u^p \, d\mu \leq \int_X g_{u+\varphi}^p \, d\mu. \]

As \( V \) and \( \varphi \) were arbitrary, it follows that \( u \) is a fine superminimizer in \( U \).

Finally, assume that \( U \setminus E = X \) is bounded. By assumption, \( u \) is a solution of the \( K_{u,u}(\Omega) \)-obstacle problem for every open \( \Omega \Subset X \) with \( C_p(X \setminus \Omega) > 0 \). Hence, it follows from Proposition 9.25 in [1] that \( u \) is a superminimizer in \( X \), and thus a fine superminimizer in \( U = X \), by Corollary 5.6 in [7]. \( \Box \)
We are now ready to prove Proposition 1.3, as well as the following more general version of it.

**Corollary 6.5.** Let $E$ be a set with $C_p(E) = 0$ and assume that $u$ is a fine (super)minimizer in $V$ for every finely open $V \subseteq U \setminus E$ and that one of the following conditions holds:

(a) $u$ is a.e.-bounded.
(b) $u \in N_{\text{fine-loc}}^{1,p}(U)$.
(c) For every finely open $V' \subseteq U$ there are $f \in N^{1,p}(V')$ and $M \geq 0$ (depending on $V'$) such that $|u - f| \leq M$ a.e. in $V'$.
(d) For every finely open $W \subseteq U$ there is a quasiopen $W'$, $f \in N^{1,p}(W')$ and $M \geq 0$ (depending on $W'$) such that $W \subseteq W' \subseteq U$ and $|u - f| \leq M$ a.e. in $W'$.

Then $u$ is a fine (super)minimizer in $U$.

For $E = \emptyset$ and open $U$, an analog of Corollary 6.5 is immediate, without the additional assumptions (a)–(d), see Proposition 9.21 in [1]. The fine case is more involved, and we do not know if it is true without assuming one of (a)–(d), cf. the discussion around Open Problem 1.5. Letting $U = B(0,1)$, $E = \{0\}$ and $u(x) = |x|^{(p-n)/(p-1)}$ in unweighted $\mathbb{R}^n$, $1 < p < n$, shows that Corollary 6.5 would fail if (a)–(d) were omitted, even when $U$ is open.

**Proof.** The implication (a)$\Rightarrow$(d) is clear, while (c)$\Rightarrow$(d) holds by Lemma 3.7. When (d) holds and $W \subseteq U$ is finely open, it follows from Lemma 5.1 (with $u_j = u$ and $U$ replaced by $W'$) that $u \in N^{1,p}(W)$, and thus $u \in N_{\text{fine-loc}}^{1,p}(U)$. Hence (b) holds in all cases. The conclusion now follows directly from Proposition 6.4. \qed

**Remark 6.6.** Combining Theorem 4.2 (a) in Kilpeläinen–Malý [15] with Corollary 6.5 shows that a function on a quasiopen set in unweighted $\mathbb{R}^n$, which either belongs to $N_{\text{fine-loc}}^{1,p}$ or is bounded, is a fine supersolution if and only if it is a quasilocal fine supersolution in the sense of [15, Definition 4.1]. We do not know if the corresponding equivalence holds in metric spaces. If it does then it would answer Open problem 9.23 in [1] in the affirmative and at least partially yield the sheaf property, cf. the discussion after Theorem 1.1.

The definition of quasilocal fine supersolutions is also based on and yields the sheaf property. Moreover, Example 5.2 in [15] demonstrates that, in contrast to fine supersolutions, unbounded quasilocal fine supersolutions on open sets can fail to be supersolutions in the usual sense. See the discussions after Theorem 1.1 above and before Example 5.2 in [15], and also Lemma 5.7 in Latvala [18].

### 7. Sequences of fine superminimizers

In this section we are going to deduce convergence results for fine superminimizers. Kilpeläinen–Malý [15, Theorem 4.3] deduced monotone convergence results for quasilocal fine supersolutions on unweighted $\mathbb{R}^n$. When combined with Theorem 4.2 (a) (with $V = U$) in [15] their results yield Proposition 7.1 and Corollary 7.3 in $\mathbb{R}^n$ under the additional assumptions that $U$ is bounded and $u \in N^{1,p}(U)$. For the counterpart of Proposition 7.1 they also require that
Proof. Let \( V \) be a finely open set such that \( C_p(X \setminus V) > 0 \) and let \( v \) be a solution of the \( \mathcal{K}_{u,j}(V) \)-obstacle problem. As \( u_j \) is a solution of the \( \mathcal{K}_{u,j}(V) \)-obstacle problem (by Lemma 6.3), the comparison principle \([3, \text{Corollary 4.3}]\) implies that \( v \leq u_j \) q.e. in \( V \). Since this holds for all \( j \), we have \( v \leq u \) q.e. in \( V \).

On the other hand, by the definition of the obstacle problem, \( v \geq u \) q.e. in \( V \), and thus \( u = v \) q.e. in \( V \). Hence, \( u \) is also a solution of the \( \mathcal{K}_{u,j}(V) \)-obstacle problem. By Proposition 6.4, \( u \) is a fine superminimizer in \( U \). \( \square \)

Theorem 7.2. Let \( \{u_j\}_{j=1}^{\infty} \) be an increasing sequence of fine superminimizers in \( U \) such that \( |u_j - f| \leq M \) a.e. in \( U \) for some \( f \in N_{\text{fine-loc}}^{1,p}(U) \) and some \( M \geq 0 \).

Then \( u := \lim_{j \to \infty} u_j \) is a fine superminimizer in \( U \).

The proof has been inspired by the proof of Theorem 6.1 in Kinnunen–Martio \([16]\).
Let $v = u + \varphi$. Then $0 \leq v - u_j \leq \varphi + 2M$ a.e. Since $\eta$ and $\varphi$ are bounded, it thus follows from the Leibniz rule [1, Theorem 2.15] that $\psi_j := \eta(v - u_j) \in N^{1,p}_0(W)$. As in the proof of the Caccioppoli-type inequality (Theorem 1.4) in Section 5, we have

$$g_{u_j + \psi_j} \leq (1 - \eta)g_{u_j} + \eta g_v + (v - u_j)g_\eta \quad \text{a.e. in } W,$$

by Lemma 2.4 in Kinnunen–Martio [16] (or [1, Lemma 2.18]). Since $W \Subset U$ is quasiregular, we get from the superminimizing property of $u_j$ that

$$
\left( \int_W g^p_{u_j} \, d\mu \right)^{1/p} \leq \left( \int_W g^p_{u_j + \psi_j} \, d\mu \right)^{1/p} \\
= \left( \int_W ((1 - \eta)g_{u_j} + \eta g_v)^p \, d\mu \right)^{1/p} + \left( \int_W (v - u_j)^p g^p_\eta \, d\mu \right)^{1/p}.
$$

Using the elementary inequality

$$(\alpha + \beta)^p \leq \alpha^p + p\beta(\alpha + \beta)^{p-1}, \quad \alpha, \beta \geq 0,$$

together with the convexity of $t \mapsto t^p$, we obtain that

$$
\int_W g^p_{u_j} \, d\mu \leq \int_W ((1 - \eta)g_{u_j} + \eta g_v)^p \, d\mu + p\beta_j(\alpha_j + \beta_j)^{p-1} \\
\leq \int_W (1 - \eta)g^p_{u_j} \, d\mu + \int_W \eta g^p_v \, d\mu + p\beta_j(\alpha_j + \beta_j)^{p-1}.
$$

As $u_j \in N^{1,p}(W)$, we can subtract the first term on the right-hand side from both sides of the inequality and obtain that

$$
\int_V g^p_{u_j} \, d\mu \leq \int_W \eta g^p_{u_j} \, d\mu \leq \int_W g^p_v \, d\mu + p\beta_j(\alpha_j + \beta_j)^{p-1}. \quad (7.3)
$$

Since $2\eta_0 > 1$ on $W$, the Caccioppoli-type inequality (Theorem 1.4) shows that

$$\alpha_j \leq \left( \int_W g^p_{u_j} \, d\mu \right)^{1/p} + \left( \int_W g^p_v \, d\mu \right)^{1/p} \\
\leq \left( \int_{V_0} (4p^2 \eta_0^p g^p_f + 8pM)g^p_\eta \, d\mu \right)^{1/p} + \left( \int_W g^p_v \, d\mu \right)^{1/p},$$
i.e. the sequence $\{\alpha_j\}_{j=1}^\infty$ is bounded. At the same time, since $g_\eta = 0$ a.e. in $V$ and $v = u$ outside $V$, we have by dominated convergence and the fact that $|(u - u_j)g_\eta| \leq 2Mg_\eta \in L^p(W)$,

$$\beta_j = \left( \int_{W \setminus V} (u - u_j)^p g_\eta^p \, d\mu \right)^{1/p} \to 0, \quad \text{as } j \to \infty.$$  

Altogether this shows that the last term in (7.3) tends to 0, as $j \to \infty$. Using (7.3) together with (5.1) applied to $V$, we obtain that

$$\left( \int_V g_v^p \, d\mu \right)^{1/p} \leq \left( \int_{V_0} g_v^p \, d\mu \right)^{1/p} < \left( \int_{V_0} g_{u+\varphi_0}^p \, d\mu \right)^{1/p} + \varepsilon,$$

where the last estimate follows from (7.1) and the fact that

$$g_v = g_{u+\varphi} \leq g_{u+\varphi_0} + g_{\varphi - \varphi_0} \quad \text{a.e. in } V_0.$$  

Finally, (7.2) yields

$$\int_{V_0} g_u^p \, d\mu = \int_V g_u^p \, d\mu + \int_{V_0 \setminus V} g_u^p \, d\mu < \left( \int_{V_0} g_{u+\varphi_0}^p \, d\mu \right)^{1/p} + \varepsilon + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary this completes the proof. \hfill \Box

**Corollary 7.3.** Let $\{u_j\}_{j=1}^\infty$ be an increasing sequence of fine superminimizers in $U$. If

$$u := \lim_{j \to \infty} u_j \in N^{1,p}_{\text{fine-loc}}(U)$$

then it is a fine superminimizer in $U$.

**Proof.** In view of Proposition 6.4, it suffices to show that $u$ is a fine superminimizer in every finely open $V \Subset U$.

For each such $V$ and $M \geq 0$, the functions $\min\{u_j, u_1 + M\}$, $j = 1, 2, \ldots$ (which are fine superminimizers by Lemma 4.2), satisfy the assumptions of Theorem 7.2 with $U$ and $f$ replaced by $V$ and $u_1$. Hence, $\min\{u, u_1 + M\}$ is a fine superminimizer in $V$. Proposition 6.1 shows that $u$ is a fine superminimizer in $V$. Since $V$ was arbitrary, the claim follows. \hfill \Box

We are now ready to prove Theorem 1.1 as well as the following more general version of it.

**Theorem 7.4.** Let $\{u_j\}_{j=1}^\infty$ be a sequence of fine superminimizers in $U$ such that for every finely open $W \Subset U$ there are $f_0 \in N^{1,p}(W)$ and $m \geq 1$ (depending on $W$) such that $u_j \geq f_0$ a.e. in $W$, $j = m, m + 1, \ldots$. Assume that one of the following conditions from Corollary 6.5 holds for $u := \liminf_{j \to \infty} u_j$:...
(a) $u$ is a.e.-bounded.
(b) $u \in N_{\text{fine-loc}}^1(U)$.
(c) For every finely open $V' \subset U$ there is $f \in N^1_p(V')$ and $M \geq 0$ (depending on $V'$) such that $|u - f| \leq M$ a.e. in $V'$.
(d) For every finely open $W^p \subset U$ there is a quasiopen $W'$, $f \in N^1_p(W')$ and $M \geq 0$ (depending on $W'$) such that $W^p \subset W' \subset U$ and $|u - f| \leq M$ a.e. in $W'$.

Then $u$ is a fine superminimizer in $U$.

**Proof.** Consider first the case when $X$ is bounded and $U = X$. Since $X$ is open, each $u_j$ is a standard superminimizer in $X$, by Corollary 5.6 in [7]. It then follows from [1, Proposition 9.4 and Corollary 9.14] that $u_j$ is q.e.-constant, and thus so is $u$, since any of the assumptions (a)–(d) excludes the theoretical possibility that $u = \infty$ q.e. Hence $u$ is a fine superminimizer. So in the rest of the proof we assume that $U \neq X$ or that $U = X$ is unbounded.

Consider $l \geq 0$ and a finely open set $W \subset W^p \subset U$. Let $f_0 \in N^1_p(W)$ and $m$ be as in the assumptions. Note that $X \setminus \overline{W}$ is a nonempty open set and thus $C_p(X \setminus W) > 0$. Let $v$ be a solution of the $K_{f_0, f_0}(W)$-obstacle problem. As each $u_j$ is a solution of the $K_{u_j, u_j}(W)$-obstacle problem (by Lemma 6.3), the comparison principle [3, Corollary 4.3] implies that $v \leq u_j$ q.e. in $W$ for all $j \geq m$. By Theorem 6.2 in [7], $v$ is a fine superminimizer in $W$. For every fixed integer $k \geq m$, the functions

$$v_{k,j} = \min\{u_k, \ldots, u_j, v + l\}, \quad j \geq k,$$

are fine superminimizers in $W$, by Lemma 4.2. Let $V^p \subset W$ be finely open. Applying Lemma 5.1 on $W$ with $M = l$ and $f = v \in N^1_p(W)$ (and $U$ resp. $W$ replaced by $W$ resp. $V$) to the decreasing sequence $v_{k,j}$ shows that $v_k := \lim_{j \to \infty} v_{k,j} \in N^1_p(V)$. As $V$ was arbitrary, we see that $v_k \in N^1_p(\text{fine-loc})(W)$.

Corollary 7.1, applied to $W$, then implies that $v_k$ is a fine superminimizer in $W$. Note that $v \leq v_k \not\leq \min\{u, v + l\}$ q.e. in $W$ and that $v_k$ satisfy the assumptions of Theorem 7.2 with $U$, $f$ and $M$ replaced by $W$, $v$ and $l$. It follows that $\min\{u, v + l\}$ is a fine superminimizer in $W$ for every $l \geq 0$. Since any of the assumptions (a)–(d) (with $f \equiv 0$ in (a) and $f = u$ in (b)) implies that $|u - f| \leq M$ a.e. in $W$, Proposition 6.2 (applied with $U$ replaced by $W$) implies that $u$ is a fine superminimizer in $W$.

Since $W \subset W$ was arbitrary, Corollary 6.5 concludes the proof. \qed

For sequences of fine minimizers, the following result is a direct consequence of Theorem 7.4, applied to both $\{u_j\}_{j=1}^\infty$ and $\{-u_j\}_{j=1}^\infty$. Corollary 1.2 is a special case of this result.

**Corollary 7.5.** Let $\{u_j\}_{j=1}^\infty$ be a sequence of fine minimizers in $U$ such that for every finely open $W \subset U$ there are $f_0$, $f_1 \in N^1_p(W)$ and $m \geq 1$ (depending on $W$) such that $f_0 \leq u_j \leq f_1$ a.e. in $W$, $j = m, m + 1, \ldots$. Assume that $u_j \to u$ q.e. in $U$ and that one of the conditions (a)–(d) in Theorem 7.4 holds. Then $u$ is a fine minimizer in $U$. 

830
Data availability

No data was used for the research described in the article.

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