On possibility of realization of the Mandelbrot set in coupled continuous systems

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Abstract

According to the method, suggested in our previous work (arxiv:nlin.CD/0509012) and based on the consideration of the specially coupled systems, the possibility of physical realization of the phenomena of complex analytic dynamics (such as Mandelbrot and Julia sets) is discussed. It is shown, that unlike the case of discrete maps or differential systems with periodic driving, investigated in mentioned work, there are some difficulties in attempts to obtain the Mandelbrot set for the coupled autonomous continuous systems. A system of coupled autonomous Rössler oscillators is considered as an example.

1 Introduction

It is known [1, 2], that complex analytic dynamics (CAD), studying behavior of complex maps, includes a lot of interesting phenomena, for example, presence of fractal Mandelbrot and Julia sets in the parameter and phase spaces.

Let us start with a quadratic logistic map

\[ z' \rightarrow \lambda - z^2, \]  

where \( \lambda \) is a complex parameter, and \( z \) is a complex variable. By definition, Mandelbrot set (fig. 1) is a set of points on a plane of complex parameter \( \lambda \), for which the orbit of an extremum \( z = 0 \) of the map (1) during iteration procedure does not escape to infinity. The Mandelbrot set contains the so-called "Mandelbrot cactus" (designed on figure by gray color); this is a set of points in the parameter plane, for which the trajectory starting of the extremum of the map converges to a periodic attractor.

"Mandelbrot cactus" consists of a big cardioid, corresponding to the existence of an attracting fixed point, and an infinite number of "leaves", corresponding to existence of attracting cycles of different periods. For example, the leaves of the doubled periods are placed along a real axis. The sequences of other period m-tupling (period-multiplication) bifurcations also can be found. In particular, the accumulation points for period-tripling and period-quadrupling bifurcation cascades
Figure 1: Mandelbrot set (a) and Julia sets for the quadratic complex map with different values of parameter: $\lambda = 0.5$ (b), $\lambda = 0.8$ (c), $\lambda = 1.42$ (d), $\lambda = 0.5 + 0.7i$ (e), $\lambda = 0.123 - 0.745i$ (f), $\lambda = 0.0315 - 0.7908i$ (g), $\lambda = -0.282 + 0.530i$ (h), $\lambda = 1.16 + 0.25i$ (i). The gray color designates the regions corresponding to existence of periodic dynamics (periods are marked by respective numbers); the black color designates points, at which the restricted in a phase space chaotic dynamics is implemented; the white color means the escaping of trajectories to infinity.

A bifurcation, which is responsible for originating a "leave", corresponds to a stability loss of a "parent" cycle, characterized by a complex multiplier with unit modulus and rational argument (in relation to $2\pi$). If the argument of a multiplier at the stability loss is irrational, then the domains with fractal boundaries, filled by invariant curves arise in the phase plane, the so-called Siegel disks [6, 7, 8].

Fractal pattern close to the "Mandelbrot cactus" and denoted by black color in Fig. 1, corresponds to existence of the chaotic dynamical regimes in the phase space.

In figures 1(b-i) the Julia sets for different values of complex parameter $\lambda$ are shown. The Julia set is a border between basins of attraction to infinity (white color) and to a periodic motion (gray color) in a plane of complex variable $z$. One can distinguish the following types of Julia sets:

- for values $\lambda$, belonging to the "Mandelbrot cactus", the Julia set is connected and enclose an interior basin (figs. 1(b,c,f-i));

- for values $\lambda$, at which chaotic dynamics exists, the Julia set is also connected, but has no inner region (fig. 1d);

- for values $\lambda$, outside the Mandelbrot set, the Julia set is disconnected (fig. 1f).

It is obvious, that 1D complex map can be represented equivalently by a 2D real map (for this purpose it is necessary only to separate real and imaginary parts of the equation). However, the
mentioned phenomena of CAD are intrinsic only to a very special class of the real 2D maps, namely for the analytic maps, obeying the Cauchy-Riemann conditions. Violation of the analyticity leads to drastic changes of the dynamics of the map \[9, 10, 11, 12, 13\]. Thereby, a following problem arises: Is it possible to specify actual physical systems demonstrating phenomena of CAD? Recently, this problem attracts great attention. The physical applications of complex dynamics for such problems, as the renormalization group approach in the theory of phase transitions and the theory of a percolation were discussed \[14, 15, 16, 17\]. In the paper of Beck \[18\] a theoretical possibility of the construction of the physical system, in which the Mandelbrot set would arise, was considered. The suggested approach is based on analysis of a motion of a charged particle in a double-peak potential with non-linear damping. The particle is driven by magnetic field, depending on time and on the particle velocity, and effected by external shot pulses, time intervals between which also depend on the particle velocity.

In works \[19, 20\] and in the present work, we offer a simpler and universal approach of constructing models manifesting the Mandelbrot set and other phenomena of CAD, which may be designed as realistic physical systems. This method allow us to carry out a physical experiment and present the first observation of the Mandelbrot set \[21\]. The special structure of the Fourier spectrum of signal generated by experimental system at the period-tripling accumulation point is presented in \[22\].

The method developed in \[19\] and in the present work is based on using coupled systems, demonstrating transition to chaos through period-doublings. It is known that such behavior is characteristic for a very wide class of nonlinear dissipative systems of various physical nature.

As shown in \[19\], it easy enough to arrange realization of the Mandelbrot set for systems with discrete time and non-autonomous periodically-driven systems (coupled logistic and Hénon maps and coupled oscillators with quadratic nonlinearity and harmonic external driving have been investigated). For this one considers two identical elements demonstrating the period-doublings with symmetrical coupling, arising from complexification of the variables and the control parameter, responsible for the period doubling in the original system. Domain of generalized partial synchronization (defined by the term, that the dynamical variables of subsystems do not escape far from each other) corresponds to the Mandelbrot set.

Such special kind of coupling provides a special symmetry in the multi-dimensional system, which is necessary for implementation of the analyticity conditions for the discrete time model map (or the stroboscopic Poincare map for the continuous system). It is a simple problem to construct the system with such coupling in comparison to the system, suggested by Beck.

From our point of view, the problem of realization of phenomena of CAD for autonomous continuous systems is now opened and seems interesting. Consideration of this problem is a main goal of the present paper. In section 2 we reproduce the method of the obtaining of the coupling function for the logistic maps. In section 3 we try to implement a similar procedure to a continuous flow system, namely, to the Rössler oscillators.
Let us start with the notion that one-dimensional complex quadratic map is equivalent to the system of two real coupled quadratic maps with a special type of coupling.

Separation of real and imaginary parts in the complex equation (1) yields

\[ z'_{re} \rightarrow \lambda_{re} - z_{re}^2 + z_{im}^2, \quad z'_{im} \rightarrow \lambda_{im} - 2z_{re}z_{im}. \] (2)

Next, we introduce the following designations

\[ x_1 = z_{re} + \beta z_{im}, \quad x_2 = z_{re} - \beta z_{im}, \]
\[ \lambda_1 = \lambda_{re} + \beta \lambda_{im}, \quad \lambda_2 = \lambda_{re} - \beta \lambda_{im}. \] (3)

As a result we obtain a system of two coupled quadratic maps

\[ x'_1 \rightarrow \lambda_1 - x_1^2 + \varepsilon (x_2 - x_1)^2, \]
\[ x'_2 \rightarrow \lambda_2 - x_2^2 + \varepsilon (x_1 - x_2)^2, \] (4)

where \( \varepsilon = (1 + \beta^2)/4\beta^2 \) is the parameter of coupling. Note a special type of coupling in these equations. It can be interpreted as an identical simultaneous shift of control parameters in both partial systems, proportional to the squared difference of dynamic variables at each step of discrete time.

It is worth nothing that the coefficient \( \varepsilon = (1 + \beta^2)/4\beta^2 \) for any \( \beta \) is larger than \( 1/4 \). Nevertheless, formally we can investigate system (4) with any \( \varepsilon \).

In fig. 2 we present the charts of the parameter plane \((\lambda_1, \lambda_2)\) for the coupled maps (1) at several values of parameter \( \varepsilon \). One can see the usual Mandelbrot set, rotated by 45°, takes place at \( \varepsilon = 0 \). For \( 0.25 < \varepsilon < +\infty \) we have a distorted Mandelbrot set on the parameter plane. The cactus leaves of this Mandelbrot set correspond to existence of periodic motion of different periods. At \( \varepsilon = 0.25 \), the set on the parameter plane, for which the point starting from the origin does not escape to infinity, looks like a set of strips, where the period doubling cycles occur. At \( \varepsilon < 0.25 \) it transforms to a rhombus-like structure. At a particular \( \varepsilon = 0 \) (uncoupled logistic maps) it is a square.

The generalization of coupling to the case of \(-\infty < \varepsilon < +\infty \) corresponds to the original map (1), variable and parameter of which are so-called two-component numbers \[23, 24, 25, 26, 27, 28\]. This is a special algebraic system, which elements are defined as follows

\[ z = x + iy, \quad i^2 = a + ib, \quad \text{where} \quad a, b \in \mathbb{R}. \] (5)

According to [23], there are three special cases: \( i^2 = -1 \) – the usual complex numbers, \( i^2 = +1 \) – the so-called perplex numbers, \( i^2 = 0 \) – the dual numbers. All other algebraic number systems are isomorphic to complex, perplex or dual numbers depending on, whether the value of \((a + b^2)/4b^2 \) is positive, negative or zero, and are known as elliptic, hyperbolic or parabolic number system, respectively. In terms of parameter \( \varepsilon \) these conditions look as follows:

1) the case \( \varepsilon > 0.25 \) corresponds to elliptic numbers isomorphic to complex numbers, implemented at \( \varepsilon = 0.5 \);
Figure 2: The charts of the parameter plane $(\lambda_1, \lambda_2)$ for the coupled logistic maps with different values of parameter of coupling: $\varepsilon = -0.1$ (a), $\varepsilon = 0.0$ (b), $\varepsilon = 0.1$ (c), $\varepsilon = 0.25$ (d), $\varepsilon = 0.3$ (e), $\varepsilon = 0.5$ (f), $\varepsilon = 1.0$ (g). The figures (a-c) correspond to hyperbolic numbers, figure (d) – to parabolic, and figures (e-g) – to elliptic numbers.
2) the case $\varepsilon < 0.25$ corresponds to hyperbolic numbers isomorphic to perplex numbers, implemented at $\varepsilon = 0$;

3) the case $\varepsilon = 0.25$ corresponds to parabolic or dual numbers.

Thus, the existence of three topologically different structures on the plane of parameters $(\lambda_1, \lambda_2)$, namely, fractal structure, similar to Mandelbrot set, rhombus-like structure and system of strips, is explained by existence of three different algebraic systems of numbers.

3 Coupled autonomous Rössler oscillators

Let us consider autonomous flow system — Rössler oscillator

$$\begin{align*}
\dot{x} &= -(y + z), \\
\dot{y} &= x + ay, \\
\dot{z} &= b + z(x - c),
\end{align*}$$

where $x$, $y$, $z$ — are dynamical variables, $a$, $b$, $c$ — are parameters [20]. It is well-known fact, that for the map at the Poincaré cross-section for this system, defined for example by the plane $y = 0$, the transition to chaos through period-doubling bifurcations is possible. At figure 3 the charts of the parameter $(c, a)$ (a) and $(c, b)$ (b) planes are shown. Cascade of period-doubling bifurcations is visible, for example, with changing of parameter $c$ with fixed $a$ and $b$. Let us take for example the values of parameters $a = 0.2$, $b = 0.2$. The bifurcation tree for this values of parameters are represented at Fig. 3 (c).

Let us construct the system of coupled Rössler oscillators. To obtain the function of coupling we use the procedure described in previous section. The complexification of Rössler system (in such way, that dynamical variables $x$, $y$, $z$ and parameter, responsible for the period doublings $c$ are complex, and other parameters $a$ and $b$ are real) with introduction of the variables and parameters designations like [3] gives the coupled systems of following form

$$\begin{align*}
\dot{x}_1 &= -(y_1 + z_1), & \dot{x}_2 &= -(y_2 + z_2), \\
\dot{y}_1 &= x_1 + ay_1, & \dot{y}_2 &= x_2 + ay_2, \\
\dot{z}_1 &= b + z_1(x_1 - c_1), & \dot{z}_2 &= b + z_2(x_2 - c_2), \\
-\varepsilon(z_2 - z_1)((x_2 - x_1) - (c_2 - c_1)), & -\varepsilon(z_1 - z_2)((x_1 - x_2) - (c_1 - c_2)).
\end{align*}$$

But the attempt to obtain the Mandelbrot set at the plane of new parameters $(c_1, c_2)$ of the system (7) has failed. The investigation of parameter space has not given suitable results – the domain of generalized synchronization has appeared not to be similar to Mandelbrot set and is represented by thin band alone the diagonal line $c_1 = c_2$ (this band become more and more thin with coupling parameter growth). Apparently, the reason of such behavior lie in the existence of a detuning of the phases of coupled subsystems. In following subsections we try to avoid the problem of phase detuning by several special approaches for construction of the coupled systems.
Figure 3: Charts of the parameter plane \((c, a)\) with \(b = 0.2\) (a) and \((c, b)\) with \(a = 0.2\) (b) for the Rössler oscillator. At the fig. (c) the bifurcation tree of the Rössler oscillator at the plane \((X_n, c)\) is represented. The value \(X_n\) is the discrete stroboscopic cross-section, corresponded to the values of dynamic variable \(x\) at the Poincare section \(y = 0\) with \(x < 0\).

3.1 Truncated system of coupled Rössler oscillators

To investigate the synchronization regimes of the coupled Rössler systems often it is more convenient to use a cylindrical system of coordinates \([30]\), in which the set of variables \((x, y, z)\) exchanges by variables \(A, \phi, z\), where \(\phi = \arctan \frac{y}{x}\) – is a phase, \(A = (x^2 + y^2)^{1/2}\) – is an amplitude. The system \((6)\) in such cylindrical coordinates is represented as follows:

\[
\begin{align*}
\dot{A} &= aA \sin^2 \phi - z \cos \phi, \\
\dot{\phi} &= 1 + \sin \phi \cos \phi + \frac{1}{A} \sin \phi, \\
\dot{z} &= b - cz + Az \cos \phi.
\end{align*}
\]

Poincare cross-section in this case is defined as \(\phi = 2\pi n, n = 1, 2, ...\)

The coupled Rössler oscillators \((7)\) rewritten in the cylindrical coordinates looks as

\[
\begin{align*}
\dot{A}_1 &= aA_1 \sin^2 \varphi_1 - z_1 \cos \varphi_1, \\
\dot{z}_1 &= b + z_1(A_1 \cos \varphi_1 - c_1) - \varepsilon(z_2 - z_1)((A_2 - A_1) \cos \varphi_1 - (c_2 - c_1)), \\
\dot{\varphi}_1 &= 1 + a \sin \varphi_1 \cos \varphi_1 + \frac{1}{A_1} \sin \varphi_1, \\
\dot{A}_2 &= aA_2 \sin^2 \varphi_2 - z_2 \cos \varphi_2, \\
\dot{z}_2 &= b + z_2(A_2 \cos \varphi_2 - c_2) - \varepsilon(z_1 - z_2)((A_1 - A_2) \cos \varphi_2 - (c_1 - c_2)), \\
\dot{\varphi}_2 &= 1 + a \sin \varphi_2 \cos \varphi_2 + \frac{1}{A_2} \sin \varphi_2.
\end{align*}
\]

Let us simplify this system, neglecting the second and third terms at the equations for phases (Such approach does not connect with some kind of physical interpretation and is considered exclusively for the showing up the relation between phase synchronization and possibility of Mandelbrot set realization). In this case, we have only one variable of phase depending on time by the linear law \(\dot{\varphi} = 1\). Then the initial system converts to the form, similar to nonlinear system with external harmonic driving. Thus, the simplified system looks as

\[
\begin{align*}
\dot{A}_1 &= aA_1 \sin^2 \varphi - z_1 \cos \varphi, \\
\dot{z}_1 &= b + z_1(A_1 \cos \varphi - c_1) - \varepsilon(z_2 - z_1)((A_2 - A_1) \cos \varphi - (c_2 - c_1)), \\
\dot{\varphi}_1 &= 1 + a \sin \varphi \cos \varphi + \frac{1}{A_1} \sin \varphi, \\
\dot{A}_2 &= aA_2 \sin^2 \varphi - z_2 \cos \varphi, \\
\dot{z}_2 &= b + z_2(A_2 \cos \varphi - c_2) - \varepsilon(z_1 - z_2)((A_1 - A_2) \cos \varphi - (c_1 - c_2)), \\
\dot{\varphi}_2 &= 1 + a \sin \varphi \cos \varphi + \frac{1}{A_2} \sin \varphi.
\end{align*}
\]
Figure 4: Chart of the parameter plane \((c_1, c_2)\) for truncated system of coupled Rössler oscillators with \(\varepsilon = 0.5, a = 0.2, b = 0.2\).

Figure 5: Basins of attraction at the phase plane \((x_1 = A_1 \cos \varphi, x_2 = A_2 \cos \varphi)\) for truncated system of coupled Rössler oscillators with \(\varepsilon = 0.5, a = 0.2, b = 0.2\) for the values of parameters \(c_1\) and \(c_2\) corresponding to existence of an attracting fixed point at \(c_1 = c_2 = 1.0\) (a), cycle of period 2 at \(c_1 = c_2 = 2.25\) (b), cycle of period 4 at \(c_1 = c_2 = 2.58\) (c) and cycle of period 3 at \(c_1 = 0.75, c_2 = 2.25\) (d) in Poincare cross section.

Figure 6: Phase portraits for the coupled Rössler oscillators with the same values of parameters as for fig. The attractors of period 1 (a), 2 (b), 3 (c,d) and 2 (e, f) are represented.
Let us term it as truncated system of coupled Rössler oscillators. It is known from the work \[19\], that for such systems the phenomena of CAD can be realized. Really, the numerical simulations of \[10\] show, that at a plane of parameters \((c_1, c_2)\) (fig. 4) one can see the structure similar to Mandelbrot set, and at the plane \((x_1, x_2)\) (fig. 5) – structures, similar to Julia sets. The present fact confirms the guess of necessity of the phase synchronization of coupled subsytems. At figure 6 the projections of several periodic attractors (including period-three attractor) in the phase space are shown.

3.2 Unidirectionally coupled Rössler systems

Now, let us consider the system of coupled Rössler oscillators with the mutual equation for the phase

\[
\begin{align*}
\dot{A}_1 &= aA_1 \sin^2 \varphi - z_1 \cos \varphi, \\
\dot{z}_1 &= b + z_1(A_1 \cos \varphi - c_1) - \varepsilon(z_2 - z_1)((A_2 - A_1) \cos \varphi - (c_2 - c_1)), \\
\dot{A}_2 &= aA_2 \sin^2 \varphi - z_2 \cos \varphi, \\
\dot{z}_2 &= b + z_2(A_2 \cos \varphi - c_2) - \varepsilon(z_1 - z_2)((A_1 - A_2) \cos \varphi - (c_1 - c_2)), \\
\dot{\varphi} &= 1 + \sin \varphi \cos \varphi + \frac{\varepsilon}{A_1} \sin \varphi.
\end{align*}
\]

The similar systems consisting of two coupled subsystems and the equations, mutual for these subsystems were explored in the works of Pecora and Caroll \[31, 32, 33\]. In these works the N-dimensional dynamical system \(\dot{u} = f(u)\) is subdivided on two subsystems \(\{u\} = \{v, w\}\), one of which is duplicated as follows \(\dot{v} = g(v, w), \dot{w} = h(v, w), \dot{w}' = h(v, w')\). Then, it is possible to view \(\{w'\}\) as response slave system drove by a signal from master system \(\{v, w\}\). The major interest represents the study of synchronization regimes of this system with a driving signal \[31, 33\].

The numerical experiment show, that the domain of generalized synchronization for system \(12\) has some features, appropriate for CAD. For example, one can see the leave of period 3 at the parameter plane (fig. 6). The leaves of higher periods are small enough and are not visible on the represented chart.

At fig. 7 the basins of attraction at a cross-section plane of a phase space of system \(x_1 = A_1 \cos \varphi, x_2 = A_2 \cos \varphi\) are represented. Non-symmetrical structure of the domain of generalized synchronization at fig. 6 and the basins of attraction at fig. 7, can be explained by unidirectional nature of coupling between driving and response subsystems.

From a point of view of the experimental physical applications, dynamics of the system \(12\) requires further study.
Figure 7: Chart of the parameter plane \((c_1, c_2)\) for coupled Rössler oscillators \([\text{II}]\) with \(\varepsilon = 0.5, a = 0.2, b = 0.2\).

Figure 8: Basins of attraction at the phase plane \((x_1 = A_1 \cos \varphi, x_2 = A_2 \cos \varphi)\) for the system \([\text{II}]\) with \(\varepsilon = 0.5, a = 0.2, b = 0.2\). The values of parameters \(c_1\) and \(c_2\) corresponds to existence of an attracting fixed point with \(c_1 = c_2 = 1.5\) (a), cycle of period 2 with \(c_1 = c_2 = 3.42\) (b), cycle of period 3 with \(c_1 = 2.92, c_2 = 0.83\) (c) and cycle of period 2 at symmetric point \(c_1 = 0.83, c_2 = 2.92\) (d).

Figure 9: Phase portraits for the coupled Rössler oscillators \([\text{II}]\) with the same values of parameters as for fig. 8. The attractors of period 1 (a), 2 (b), 3 (c,d) and 2 (e, f) are represented.
### 3.3 Coupled Rössler oscillators with additional coupling, which synchronize phases

The next step is to consider two different equations for phases $\varphi_1$ and $\varphi_2$ for each of the partial systems. To solve the problem of phase detuning we enter the additional coupling with parameter $\delta$ to the equations for phases, which thus, provide phase synchronization of Rössler subsystems and does not influence amplitudes

\[
\dot{A}_1 = aA_1 \sin^2 \varphi_1 - z_1 \cos \varphi_1, \quad \dot{A}_2 = aA_2 \sin^2 \varphi_2 - z_2 \cos \varphi_2,
\]

\[
\dot{z}_1 = b + z_1(A_1 \cos \varphi_1 - c_1) - \varepsilon(z_2 - z_1)((A_2 - A_1) \cos \varphi_1 - (c_2 - c_1)), \quad \dot{z}_2 = b + z_2(A_2 \cos \varphi_2 - c_2) - \varepsilon(z_1 - z_2)((A_1 - A_2) \cos \varphi_2 - (c_1 - c_2)),
\]

\[
\dot{\varphi}_1 = 1 + a \sin \varphi_1 \cos \varphi_1 + \frac{\dot{A}_1}{A_1} \sin \varphi_1 + \delta \sin(\varphi_2 - \varphi_1), \quad \dot{\varphi}_2 = 1 + a \sin \varphi_2 \cos \varphi_2 + \frac{\dot{A}_2}{A_2} \sin \varphi_2 + \delta \sin(\varphi_1 - \varphi_2).
\]

In usual rectangular coordinate system $(x = A \cos \varphi, y = A \sin \varphi, z)$ the equations (12) looks as following

\[
\dot{x}_1 = -(y_1 + z_1) - \delta y_1 \frac{x_1 y_2 - x_2 y_1}{(x_1^2 + y_1^2)^{1/2}(x_2^2 + y_2^2)^{1/2}}, \quad \dot{x}_2 = -(y_2 + z_2) - \delta y_2 \frac{x_2 y_1 - x_1 y_2}{(x_2^2 + y_2^2)^{1/2}(x_1^2 + y_1^2)^{1/2}},
\]

\[
\dot{y}_1 = x_1 + ay_1 + \delta x_1 \frac{x_1 y_2 - x_2 y_1}{(x_1^2 + y_1^2)^{1/2}(x_2^2 + y_2^2)^{1/2}}, \quad \dot{y}_2 = x_2 + ay_2 + \delta x_2 \frac{x_2 y_1 - x_1 y_2}{(x_2^2 + y_2^2)^{1/2}(x_1^2 + y_1^2)^{1/2}},
\]

\[
\dot{z}_1 = b + z_1(x_1 - c_1) - \varepsilon(z_2 - z_1)((x_2 - x_1) - (c_2 - c_1)), \quad \dot{z}_2 = b + z_2(x_2 - c_2) - \varepsilon(z_1 - z_2)((x_1 - x_1) - (c_1 - c_2)).
\]

The numerical simulations show, that at a plane $(c_1, c_2)$ for system (13) the arising of fractal set, similar to Mandelbrot set (fig. 8) is possible. Nevertheless, it is easy to see the considerable distortion of its configuration. For explanation of this fact we may assume, that the entering of additional coupling breaks the conditions of complex analyticity for the resulting stroboscopic Poincare map.

At fig. 9 the basins of attractions similar to Julia sets are represented. At fig. 10 the attractors of period 1 (a), 2 (b), 4 (c) and 6 (d, e) are shown. Let us note, that with identical values of parameters $c_1$ and $c_2$ (see fig. a-c) the values of variables in partial systems are coincided, that corresponds to the full synchronization. In a case $c_1 \neq c_2$ (fig. d, e), when the phenomena of CAD can be realized, the values of dynamical variables of partial systems do not coincide, that corresponds to realization of generalized partial synchronization. Thus, the phenomena of CAD such as period-tripling bifurcations can be implemented, when the point $(c_1, c_2)$ belongs to the region of generalized partial synchronization.
Figure 10: Chart of the parameter plane \((c_1, c_2)\) and its enlarged fragment demonstrating self-similarity for coupled Rössler oscillators \([13]\) with \(\varepsilon = 0.5, a = 0.2, b = 0.2, \delta = 1.0\).

![Chart of the parameter plane](image)

Figure 11: Basins of attraction at the phase plane \((x_1, x_2)\) for the coupled Rössler oscillators \([13]\) with \(\varepsilon = 0.5, a = 0.2, b = 0.2, \delta = 1.0\) for different values of parameters \(c_1\) and \(c_2\) corresponding to existence of an attracting fixed point at \(c_1 = c_2 = 2.0\) (a), cycle of period 2 at \(c_1 = c_2 = 3.5\) (b), cycle of period 4 at \(c_1 = c_2 = 4.0\) (c) and cycle of period 6 at \(c_1 = 2.98, c_2 = 3.9\) (d).

![Basins of attraction](image)

Figure 12: Phase portraits for the coupled Rössler oscillators \([13]\) with the same values of parameters as for fig. 13. The attractors of period 1 (a), 2 (b), 4 (c) and 6 (d, e) are represented.

![Phase portraits](image)
3.4 Coupled Rössler oscillators with different time variables

Let us suggest one more method of obtaining of Mandelbrot set for autonomous system with continuous time, such as the coupled Rössler oscillators, which allow to avoid the problem of phase detuning and doesn’t break analyticity conditions. This method lies in following: let us complexify Rössler system by another way. Let us consider the variables $A$, $z$ and time $t$ as complex and the variable $\varphi$ as real numbers. For this purpose, it is suitable to rewrite the original Rössler system in cylindrical coordinates as

$$
\begin{align*}
\frac{dA}{d\varphi} &= aA \sin^2 \varphi - z \cos \varphi + c \sin \varphi, \\
\frac{dz}{d\varphi} &= b - cz + A \cos \varphi, \\
\frac{dt}{d\varphi} &= \frac{1}{1 + \sin \varphi \cos \varphi + \frac{A}{2} \sin \varphi}.
\end{align*}
$$

(14)

After complexification and the variables and parameter designations we obtain the system of coupled reorganized Rössler oscillators

$$
\begin{align*}
\frac{dA_1}{d\varphi} &= A_1 (aA_1 \sin^2 \varphi - z_1 \cos \varphi + c_1 \sin \varphi) + \varepsilon f_1 (A_1, A_2, z_1, z_2, c_1, c_2), \\
\frac{dz_1}{d\varphi} &= A_1 (b - z_1 (c_1 - A_1 \cos \varphi)) + \varepsilon f_2 (A_1, A_2, z_1, z_2, c_1, c_2), \\
\frac{dt_1}{d\varphi} &= \frac{A_1}{1 + \sin \varphi \cos \varphi + \frac{A_1}{2} \sin \varphi}, \\
\frac{dA_2}{d\varphi} &= A_2 (aA_2 \sin^2 \varphi - z_2 \cos \varphi + c_2 \sin \varphi) + \varepsilon f_1 (A_2, A_1, z_2, z_1, c_2, c_1), \\
\frac{dz_2}{d\varphi} &= A_2 (b - z_2 (c_2 + A_2 \cos \varphi)) + \varepsilon f_2 (A_2, A_1, z_2, z_1, c_2, c_1), \\
\frac{dt_2}{d\varphi} &= \frac{A_2}{1 + \sin \varphi \cos \varphi + \frac{A_2}{2} \sin \varphi},
\end{align*}
$$

(15)

where $f_1$, $f_2$ and $f_3$ are the functions of coupling, which can be expressed as follows

$$
\begin{align*}
f_1 (A_{1,2}, z_{1,2}, z_{2,1}, c_{1,2}, c_{2,1}) &= -\frac{-2u_{1,2}w_{2,1}(A_{1,2}(w_{1,2} - w_{2,1}) + A_{2,1}(w_{1,2} + w_{2,1}))) + 2u_{1,2}^2 A_{1,2}(w_{1,2} - w_{2,1}) + 2u_{1,2}^2 A_{1,2}(w_{1,2} + w_{2,1}))}{u_{1,2}(u_{1,2}^2 + u_{2,1}^2)}, \\
f_2 (A_{1,2}, z_{1,2}, z_{2,1}, c_{1,2}, c_{2,1}) &= A_{1,2}(w_{1,2}^2 + w_{2,1}^2) + \frac{u_{1,2}^2 A_{1,2}(z_{1,2}^2 - z_{2,1}^2) + u_{1,2}^2 A_{1,2}(z_{2,1}^2 - z_{1,2}^2)}{u_{1,2}(u_{1,2}^2 + u_{2,1}^2)}, \\
f_3 (A_{1,2}, z_{1,2}, z_{2,1}, c_{1,2}, c_{2,1}) &= -\frac{-2u_{1,2}w_{2,1}(A_{1,2}(w_{1,2} + A_{2,1}) + 2u_{1,2}^2 A_{1,2})}{u_{1,2}(u_{1,2}^2 + u_{2,1}^2)}.
\end{align*}
$$

(16)

where

$$
\begin{align*}
u_{1,2} &= A_{1,2} + A_{1,2} \sin \varphi \cos \varphi + z_{1,2} \sin \varphi, \\
w_{1,2} &= aA_{1,2} \sin^2 \varphi - z_{1,2} \cos \varphi.
\end{align*}
$$

(17)

The represented unusual way of complexification appear to be successful by some reasons. At first, let us remind that we miss the problem of phase detuning. Thus, the variable $\varphi$ responsible to the Poincare cross-section ($\varphi = 2\pi n$, $n = 1, 2, ...$) remains real (it is not suitable to consider the variable $y$ as real, i.e. to consider the derivatives $dx/dy$, $dz/dy$, $dt/dy$, because $y$ is an non-monotonous function of time). Second, such complexification does not destroy analyticity.
Figure 13: The chart of the parameter plane \((c_1, c_2)\) for the coupled Rössler oscillators \([15]\).

Figure 14: Basins of attraction on the phase planes \((x_1, x_2)\) for the coupled Rössler oscillators \([15]\). The basins of attractors of period 1 with \(c_1 = c_2 = 1.7\) (a), 2 with \(c_1 = c_2 = 3.35\) (b), 4 with \(c_1 = c_2 = 4.0\) (c) and 3 with \(c_1 = 3.222, c_2 = 0.828\) (d) are represented.

Figure 15: Phase portraits for the coupled Rössler oscillators \([15]\) with the same values of parameters as for fig. 16. The attractors of period 1 (a), 2 (b), 4 (c) and 3 (d,e) are represented.
Figure 16: The dependence of time detuning on versus the phase with the values of parameters $c_1 = 3.222$, $c_2 = 0.838$ corresponded to existence of 3-cycle.

The represented at fig. 11 chart of the parameter plane $(c_1, c_2)$ demonstrates the full agreement with usual Mandelbrot set. The basins of attraction of cycle of period 1 (fig. 12a), 2 (fig. 12b), 3 (fig. 12c) and 9 (fig. 12d) are also similar with Julia sets. The next figure 13 shows the attractors for the subsystems represented in rectangular coordinates. Note, that if $c_1 \neq c_2$, the variables of time for each subsystem have no coincided values, i.e. the trajectories of $A_1, z_1, \varphi$ and $A_2, z_2, \varphi$ run up to the Poincare cross-section $\varphi = 2\pi n$ not simultaneously. Fig. 14 demonstrates the dependence of time detuning on the variation of phase with the values of parameters $c_1, c_2$ corresponded to cycle of period 3. As it is visible, the value of $t_2 - t_1$ oscillates periodically near the averaged value increasing by linear law.

Also, it is necessary to note, that the variable of time are not subsisted in the equations for amplitude $A$ and variable $z$, and doesn’t influence its dynamics. Therefore, the equations for $t_1$ and $t_2$ can be rejected. Thus, system (15) is also the truncated system. Besides, unfortunately, it is difficult to realize the system experimentally.

4 Conclusion

In present work we have made an attempt to obtain the phenomena of CAD such as fractal Mandelbrot and Julia sets in the autonomous continuous systems. As an example the autonomous Rössler oscillator has been considered.

It is shown, that complexified system can be represented as two coupled real systems. Such representation is useful for some reasons. At first, it simplify construction of a real physical model. Secondly, with entering of special coupling to the system of two identical devices of any nature demonstrating the period-doubling cascade, one may expect the whole system to demonstrate phenomena of complex analytic dynamics.

The main result of the work is that the realization of the Mandelbrot set in parameter space of the autonomous continuous systems requires provision of additional conditions such as phase synchronization. The phase synchronization can be obtained by the truncation of the system of
equations, by entering of phase coupling or by the special type of complexification.

Nevertheless, the problem of the realization of Mandelbrot set in autonomous flow systems require more careful study. For example, one can assume, that the codimension of phenomena relevant to CAD is higher for the flow systems and Mandelbrot set as domain of generalized synchronization can be observed not on a plane, but on some surface in parameters space. It is necessary to study in more detail the regimes of synchronization for the offered system and the process of influence of a phase detuning on structure of parameter space.

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