THE INDEX FORMULA AND THE SPECTRAL SHIFT FUNCTION FOR RELATIVELY TRACE CLASS PERTURBATIONS

FRITZ GESZTESY, YURI LATUSHKIN, KONSTANTIN A. MAKAROV, FEDOR SUKOCHEV, AND YURI TOMILOV

Abstract. We compute the Fredholm index, index($D_A$), of the operator $D_A = (d/dt) + A$ on $L^2(\mathbb{R}; \mathcal{H})$ associated with the operator path $\{A(t)\}_{t=-\infty}^{\infty}$, where $(Af)(t) = A(t)f(t)$ for a.e. $t \in \mathbb{R}$, and appropriate $f \in L^2(\mathbb{R}; \mathcal{H})$, via the spectral shift function $\xi(\cdot; A_+, A_-)$ associated with the pair $(A_+, A_-)$ of asymptotic operators $A_\pm = A(\pm \infty)$ on the separable complex Hilbert space $\mathcal{H}$ in the case when $A(t)$ is generally an unbounded (relatively trace class) perturbation of the unbounded self-adjoint operator $A_-$.

We derive a formula (an extension of a formula due to Pushnitski) relating the spectral shift function $\xi(\cdot; A_+, A_-)$ for the pair $(A_+, A_-)$, and the corresponding spectral shift function $\xi(\cdot; H_2, H_1)$ for the pair of operators $(H_2, H_1) = (D_A^* D_A, D_A D_A)$ in this relative trace class context,

$$\xi(\lambda; H_2, H_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)}{\lambda - \nu^2} d\nu$$

for a.e. $\lambda > 0$.

This formula is then used to identify the Fredholm index of $D_A$ with $\xi(0; A_+, A_-)$. In addition, we prove that index($D_A$) coincides with the spectral flow $\text{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty})$ of the family $\{A(t)\}_{t \in \mathbb{R}}$ and also relate it to the (Fredholm) perturbation determinant for the pair $(A_+, A_-)$:

$$\text{index}(D_A) = \text{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \Im \ln \det_{\mathcal{H}} \left( (A_+ - \varepsilon I)(A_- - \varepsilon I)^{-1} \right),$$

with the choice of the branch of $\ln \det_{\mathcal{H}}(\cdot)$ on $\mathbb{C}_+$ such that

$$\lim_{\Im z \to +\infty} \ln \det_{\mathcal{H}} \left( (A_+ - z I)(A_- - z I)^{-1} \right) = 0.$$

We also provide some applications in the context of supersymmetric quantum mechanics to zeta function and heat kernel regularized spectral asymmetries and the eta-invariant.

Date: February 7, 2011.

2010 Mathematics Subject Classification. Primary 47A53, 58J30; Secondary 47A10, 47A40.

Key words and phrases. Fredholm index, spectral flow, spectral shift function, perturbation determinants, relative trace class perturbations.

Partially supported by the US National Science Foundation under Grant NSF DMS-0754705, by the Research Board and Research Council of the University of Missouri, by the ARC, by the Marie Curie “Transfer of Knowledge” programme, project “TODEQ”, and by a MNiSzW grant Nr. N201384834.

To appear in Adv. Math.
1. Introduction

Before attempting to describe a glimpse of the extensive history of the underlying problem at hand, viz., the computation of the Fredholm index for operators of the type $D_A = (d/dt) + A$ in $L^2(\mathbb{R}; \mathcal{H})$, using a variety of different approaches, we briefly describe the principal setup and the main results in this paper.

Let $\{A(t)\}_{t \in \mathbb{R}}$ be a family of self-adjoint operators in the complex, separable Hilbert space $\mathcal{H}$, subject to a relative trace class approach described in Hypothesis 2.1, and denote by $A$ the operator in $L^2(\mathbb{R}; \mathcal{H})$ defined by
\[
(Af)(t) = A(t)f(t) \text{ for a.e. } t \in \mathbb{R},
\]
\[
f \in \text{dom}(A) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid g(t) \in \text{dom}(A(t)) \text{ for a.e. } t \in \mathbb{R}, \right\}
\]
\[
t \mapsto A(t)g(t) \text{ is (weakly) measurable, } \int_{\mathbb{R}} \|A(t)g(t)\|_{\mathcal{H}}^2 \, dt < \infty
\]

Our relative trace class setup ensures that $A(t)$ has self-adjoint limiting operators
\[
A_+ = \lim_{t \to +\infty} A(t), \quad A_- = \lim_{t \to -\infty} A(t)
\]
in $\mathcal{H}$ in an appropriate sense (detailed in Theorem 3.7). The principal novelty in our approach concerns the fact that we permit relative trace class perturbations $B(t)$ (generally, unbounded) of the asymptotic self-adjoint operator $A_-$ such that
\[
A(t) = A_- + B(t), \quad t \in \mathbb{R}.
\]

With the possible exception of a spectral gap at zero, no other restrictions on the self-adjoint unperturbed operator $A_-$ are imposed in this paper. Especially, no discrete spectrum hypotheses will be made in this paper.

The first principal result to be mentioned is the extension of the following trace formula to our relative trace class approach,
\[
tr_{L^2(\mathbb{R}; \mathcal{H})} \left( (H_2 - z I)^{-1} - (H_1 - z I)^{-1} \right) = \frac{1}{2z} tr_{\mathcal{H}} \left( g_z(A_+) - g_z(A_-) \right),
\]
where $z \in \mathbb{C}\setminus[0, \infty)$.
where we used the abbreviations
\[ g_2(x) = x(x^2 - z)^{-1/2}, \quad z \in \mathbb{C} \setminus [0, \infty), \quad x \in \mathbb{R}, \quad (1.5) \]
\[ D_A = \frac{d}{dt} + A, \quad \text{dom}(D_A) = \text{dom}(d/dt) \cap \text{dom}(A_-), \quad (1.6) \]
\[ H_1 = D_A^* D_A, \quad H_2 = D_A D_A^*, \quad (1.7) \]
and \( A_- \) in \( L^2(\mathbb{R}; \mathcal{H}) \) represents the self-adjoint (constant fiber) operator defined according to (1.1) (with \( A(t) \) replaced throughout by \( A_- \), cf. (2.13)).

The trace formula (1.4) then implies the next main result, an extension of Push-nitski's formula [123] to our relative trace class formalism:
\[ \xi(\lambda; H_2, H_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-) \, d\nu}{(\lambda - \nu^2)^{1/2}} \quad \text{for a.e. } \lambda > 0. \quad (1.8) \]

Here \( \xi(\cdot; H_2, H_1) \) and \( \xi(\cdot; A_+, A_-) \) denote appropriately defined spectral shift functions associated with the pairs of self-adjoint operators \((H_2, H_1)\) and \((A_+, A_-)\), respectively.

Assuming that \( A_- \) and \( A_+ \) are boundedly invertible, we prove that \( D_A \) is a Fredholm operator in \( L^2(\mathbb{R}; \mathcal{H}) \). Moreover, one of the main results of this paper is the following pair of formulas relating the Fredholm index of \( D_A \) with the spectral shift function \( \xi(\cdot; A_+, A_-) \) (for which formula (1.8) is then the major input in the proof), and with the trace of a difference of the Morse spectral projections corresponding to \((A_+, A_-)\),

\[ \text{index}(D_A) = \xi(0; A_+, A_-) = \text{tr}_H \left( E_{A_-}((-\infty, 0)) - E_{A_+}((-\infty, 0)) \right). \quad (1.9) \]

Here \( \{E_T(\lambda)\}_{\lambda \in \mathbb{R}} \) denotes the family of spectral projections associated with the self-adjoint operator \( T \).

However, our results go considerably beyond (1.9) in the sense that we also establish the detailed connection between the spectral flow for the path \( \{A(t)\}_{t = -\infty}^{\infty} \) of self-adjoint Fredholm operators and the Fredholm index of \( D_A \). More precisely, introducing the spectral flow \( \text{SpFlow}(\{A(t)\}_{t = -\infty}^{\infty}) \) as in Definition 9.5, and recalling the definition of the index of a pair of Fredholm projections in Definition 9.8, assuming Hypothesis 2.1 and supposing that \( 0 \in \rho(A_+) \cap \rho(A_-) \), we prove that the pair \((E_{A_+}((-\infty, 0)), E_{A_-}((-\infty, 0)))\) of Morse projections is Fredholm and that the following series of equalities holds:
\[ \text{index}(D_A) = \text{SpFlow}(\{A(t)\}_{t = -\infty}^{\infty}) = \xi(0; H_2, H_1) = \xi(0; A_+, A_-) = \text{index}(E_{A_-}((-\infty, 0)), E_{A_+}((-\infty, 0))) = \text{tr}_H \left( E_{A_-}((-\infty, 0)) - E_{A_+}((-\infty, 0)) \right) = \pi^{-1} \lim_{\varepsilon \to 0} \text{Im} \left( \ln(\text{det}_H((A_+ - i\varepsilon I)(A_- - i\varepsilon I)^{-1})) \right), \quad (1.10) \]
\[ \text{with a choice of branch of } \ln(\text{det}_H(\cdot)) \text{ on } \mathbb{C}_+ \text{ analogous to (1.17) below.} \]

For completeness we note that \( \xi(\cdot; A_+, A_-) \) can be shown to satisfy
\[ \xi(\lambda; A_+, A_-) = \pi^{-1} \lim_{\varepsilon \to 0} \text{Im} \left( \ln(D_{A_+/A_-}(\lambda + i\varepsilon)) \right) \quad \text{for a.e. } \lambda \in \mathbb{R}, \quad (1.16) \]
and we make the choice of branch of $\ln(D_{A_+/A_-}(z))$ on $\mathbb{C}_+$ such that
\[
\lim_{\text{Im}(z)\to +\infty} \ln(D_{A_+/A_-}(z)) = 0.
\]
Equation (1.17)

Here
\[
D_{T/S}(z) = \det_{\mathcal{H}}((T - zI)(S - zI)^{-1}) = \det_{\mathcal{H}}(I + (T - S)(S - zI)^{-1}), \quad z \in \rho(S),
\]
denotes the perturbation determinant for the pair of operators $(S, T)$ in $\mathcal{H}$, assuming $(T - S)(S - z_0)^{-1} \in \mathcal{B}_1(\mathcal{H})$ for some (and hence for all) $z_0 \in \rho(S)$. In addition, we recall M. Krein’s celebrated trace formula associated with the pair $(A_+, A_-)$,
\[
\frac{d}{dz} \ln(D_{A_+/A_-}(z)) = -\text{tr}_{\mathcal{H}}((A_+ - zI)^{-1} - (A_- - zI)^{-1})
\]
\[
= \int_{\mathbb{R}} \frac{\xi(\lambda; A_+, A_-) d\lambda}{(\lambda - z)^2}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
Equation (1.19)

Analogous formulas apply of course to $\xi(\cdot; H_2, H_1)$ in connection with the pair $(H_2, H_1)$.

Arguably, equations (1.4), (1.8), and (1.10)–(1.15) represent the central results of this paper to be developed in subsequent sections.

The concept of spectral flow has also been developed for Breuer–Fredholm operators in semifinite von Neumann algebras (see, e.g., [18], [21], [118]). In this context a result analogous to those that form the theme of this paper was proved. In particular, a result relating the spectral flow and index in the setting of Atiyah’s $L^2$-index theorem was derived in [21, Theorem 8.4]. Using the fact that the spectral shift function can also be defined when working with semifinite von Neumann algebras, it is likely that extensions of some results of this paper can be made to this wider setting.

Before describing the contents of our paper we now turn to the relevant history of this subject and a proper placement of our results in this context. Since it is impossible to do justice to a discussion of index theory for elliptic differential operators since the pioneering work of Atiyah and Singer, we only confine ourselves referring to a few research monographs (see, e.g., [23], [40], [38], [59], [69], [96], [103], [106], [115], [130] and the detailed references cited therein). Operators of the form $D_A = (d/dt) + A$ were studied by Atiyah, Patodi, and Singer [14]–[16] with $A(t), t \in \mathbb{R}$, a first-order elliptic differential operator on a compact odd-dimensional manifold with the asymptotes $A_{\pm}$ boundedly invertible and $A_{\pm}(t)$, $t \in \mathbb{R}$, assumed to have purely discrete spectrum. In particular, the idea that the Fredholm index of $D_A$ equals the spectral flow of the family (path) of self-adjoint operators $\{A(t)\}_{t=\infty}^{\infty}$ was put forward in this series of papers. An abstract theorem concerning the equality of the Fredholm index of $D_A$ and the spectral flow of the family of self-adjoint operators $\{A(t)\}_{t=\infty}^{\infty}$ under the assumption of a $t$-independent domain for $A(t)$ which embeds densely and compactly in $\mathcal{H}$, with boundedly invertible asymptotes $A_{\pm}$, was proved by Robbin and Salamon [129]. This covered the abstract case with purely discrete spectra for $A_{\pm}$, $A(t), t \in \mathbb{R}$. This paper contains a fascinating array of applications including Morse theory, Floer homology, Morse and Maslov indices, Cauchy–Riemann operators, all the way to oscillation theory of (matrix-valued) one-dimensional Schrödinger operators. In particular, both, finite and infinite-dimensional cases are treated. An extension of this approach to the Banach space setting appeared in [125]. Examples in which the
Fredholm index and the spectral flow cease to coincide and the Fredholm index not only depends on the endpoints $A_{\pm}$ of the operator path, but on the path itself, are discussed in [2]. In a related setting, necessary and sufficient conditions for $D_{A}$ to be Fredholm and an index formula for operators of the form $D_{A}$, given in terms of exponential dichotomies, can be found in [94], [95] and the literature cited therein; the operator semigroups generated by the operators of this form were studied in [48, Chapter 3].

These references primarily center around the equality of the Fredholm index and the spectral flow as expressed in (1.10), a fundamental part of modern index theory. However, the connections with the additional equalities in (1.11)–(1.15) require quite different ingredients whose roots lie at the heart of scattering theory for the pair of self-adjoint operators $(H_{2}, H_{1})$ and, especially, that of $(A_{+}, A_{-})$, the asymptotes of the operator path $\{A(t)\}_{t=-\infty}^{\infty}$. In particular, we note that the spectral shift function $\xi(\lambda; A_{+}, A_{-})$ (and hence boundary values of the perturbation determinant $D_{A_{+}/A_{-}}(\lambda + i\epsilon)$ as $\epsilon \downarrow 0$ in (1.16)) for a.e. $\lambda \in \sigma_{ac}(A_{\pm})$ is directly related to the determinant of the $\lambda$-dependent scattering matrix via the celebrated Birman–M. G. Krein formula [24]. It is this additional scattering theoretic ingredient which represents one of the principal contributions of this paper, and, as evidenced in (1.11)–(1.15), considerably enhances the usual focus on the equality of the Fredholm index and the spectral flow.

The first relations between Fredholm index theory and the spectral shift function $\xi(\cdot; H_{2}, H_{1})$ were established by Bollé, Gesztesy, Grosse, Schweiger, and Simon [37]. In fact, inspired by index calculations of Callias [46] in connection with non-compact manifolds, the more general notion of the Witten index was studied and identified with $\xi(0_{+}; H_{2}, H_{1})$ in [37] and [68] (see also [61], [139, Ch. 5]). The latter created considerable interest, especially, in connection with certain aspects of supersymmetric quantum mechanics. Since a detailed list of references in this context is beyond the scope of this paper we only refer to [5], [6], [7], [8], [9], [42], [90], [112], [139, Ch. 5] and the detailed lists of references cited therein. While [37] and [61] focused on index theorems for concrete one and two-dimensional supersymmetric systems (in particular, the trace formula (1.4) and the function $g_{z}(\cdot)$ were discussed in [37] and [61] in the special case where $\mathcal{H} = \mathbb{C}$), [68] treated abstract Fredholm and Witten indices in terms of the spectral shift function $\xi(\cdot; H_{2}, H_{1})$ and proved their invariance with respect to appropriate classes of perturbations. Soon after, a general abstract approach to supersymmetric scattering theory involving the spectral shift function was developed by Borisov, Müller, and Schrader [41] (see also [45], [106, Chs. IX, X], [107]) and applied to relative index theorems in the context of manifolds Euclidean at infinity.

However, closest to the present paper at hand, and the prime motivation for writing it, is the recent work by Pushnitski [123] in which he went essentially beyond the discrete spectrum hypothesis imposed on $A_{\pm}$, $A(t)$, $t \in \mathbb{R}$, by Robbin and Salamon in [129]. Basically, Pushnitski replaced the discrete spectrum hypothesis by the assumption of an arbitrary self-adjoint operator $A_{-}$ in $\mathcal{H}$ and by imposing that $B(\cdot)$ in (1.3) is trace norm differentiable and satisfies the integrability condition

$$\int_{\mathbb{R}} \|B'(t)\|_{g_{1}(\mathcal{H})} dt < \infty. \quad (1.20)$$
Assuming that $A_-$ and $A_+$ are boundedly invertible, Pushnitski proved that

$$\text{index}(D_A) = \xi(0_+; H_2, H_1)$$

$$= \xi(0; A_+, A_-)$$

(cf. (1.11), (1.12)) and indicated why this might imply (1.10). Most importantly, perhaps, he proved the trace formula (1.4) and used it to derive his remarkable formula (1.8). This effectively removed any discrete spectrum assumptions in this context. (Very recently another derivation of (1.10) without any discrete spectrum hypothesis was given in [20], but without entering a discussion of (1.11)–(1.15).)

In the special case where $H$ is finite-dimensional, the trace formula (1.4) was first proved by Callias [46].

Returning to the content of this paper, our relative trace class hypotheses detailed in Hypothesis 2.1 essentially replaces Pushnitski’s assumption (1.20) by

$$\int_{\mathbb{R}} \|B'(t)(|A_+| + I)^{-1}\|_{B_1(H)} dt < \infty$$

and certain additional technical conditions, which therefore permit the treatment of unbounded operators $B(\cdot)$ in $H$. This extension, however, comes at the price of considerably more involved proofs at every stage in this paper. In particular, we are using the theory of double operator integrals to justify the trace class property of $[g_z(A_+) - g_z(A_-)]$ (cf. the right-hand side of the trace formula in (1.4)). Moreover, the assumptions that we impose on the perturbation $B(t)$ and on $B'(t)$ are so general that some fairly delicate analysis of measurability issues is required (cf. Appendix A and [62]).

The paper is organized as follows: In Section 2 we introduce our principal Hypothesis 2.1 and formulate our principal results. Our setup of relatively trace class perturbations is examined in great detail in Section 3. Section 4 is of preliminary character and proves a variety of results on $D_{A_-}, D_A$ and sets up the quadratic forms which define $H_j, j = 1, 2$. In Sections 5 and 6 we deal with the left-hand side and the right-hand side of the main trace formula (1.4), respectively. Whereas Section 5 employs various quadratic form perturbation results and associated resolvent equations, Section 6 employs the theory of double operator integrals (DOI) originally pioneered by Daletskii and S. G. Krein and, especially, by Birman and Solomyak. Section 7 is devoted to a careful introduction and study of the spectral shift function $\xi(\cdot; A_+, A_-)$ corresponding to the pair $(A_+, A_-)$. The spectral shift function $\xi(\cdot; H_2, H_1)$ associated with the pair $(H_2, H_1)$ is then introduced in Section 8 and the fundamental formula (1.8) as well as the fact that $\text{index}(D_A) = \xi(0_+; H_2, H_1) = \xi(0; A_+, A_-)$ are proved. In addition, some applications to supersymmetric quantum mechanics including abstract formulas for the zeta function and heat kernel regularized Atiyah–Patodi–Singer (APS) spectral asymmetry and the associated the eta-invariant are provided. Our final Section 9 details the connection between the Fredholm index and the spectral flow and proves the remaining equalities in (1.10)–(1.15). Appendix A is of a technical nature and takes a close look at operators of the type $A$ in (1.1) and establishes a precise connection with the notion of direct integrals over the operators $A(t), t \in \mathbb{R}$, with respect to the Lebesgue measure $dt$. Appendix B is devoted to a proof of the trace norm analyticity of $[g_z(A_+) - g_z(A_-)], z \in \mathbb{C} \setminus [0, \infty)$. Finally, we briefly summarize some of the notation used in this paper: Let $H$ be a separable complex Hilbert space, $(\cdot, \cdot)_H$ the scalar product in $H$ (linear in the
second factor), and \( I \) the identity operator in \( \mathcal{H} \). Next, let \( T \) be a linear operator mapping (a subspace of) a Banach space into another, with \( \text{dom}(T), \text{ran}(T), \) and \( \ker(T) \) denoting the domain, range, and kernel (i.e., null space) of \( T \). The closure of a closable operator \( S \) is denoted by \( \overline{S} \).

The spectrum, essential spectrum, discrete spectrum, point spectrum, and resolvent set of a closed linear operator in \( \mathcal{H} \) will be denoted by \( \sigma(\cdot), \sigma_{\text{ess}}(\cdot), \sigma_d(\cdot), \sigma_p(\cdot), \) and \( \rho(\cdot) \), respectively. The strongly right continuous family of spectral projections of a self-adjoint operator \( S \) in \( \mathcal{H} \) will be denoted by \( E_{S}(\lambda), \lambda \in \mathbb{R} \). (In particular, \( E_{S}(\lambda) = E_{S}((-\infty, \lambda]), E_{S}((-\infty, \lambda)) = \text{s-lim}_{\varepsilon \downarrow 0} E_{S}(\lambda - \varepsilon) \), and \( E_{S}(\lambda_1, \lambda_2) = E_{S}(\lambda_2) - E_{S}(\lambda_1), \lambda_1 \leq \lambda_2, \lambda, \lambda_1, \lambda_2 \in \mathbb{R} \).)

The Banach spaces of bounded and compact linear operators on \( \mathcal{H} \) are denoted by \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{B}_{\infty}(\mathcal{H}) \), respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by \( \mathcal{B}_{p}(\mathcal{H}), p \in (0, \infty) \). Analogous notation \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), \mathcal{B}_{\infty}(\mathcal{H}_1, \mathcal{H}_2) \), etc., will be used for bounded, compact, etc., operators between two Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). We also use the notation \( \text{tr}_{\mathcal{K}}(\cdot) \) for the trace in the Hilbert space \( \mathcal{K} \). We use symbols \( n\text{-lim}, s\text{-lim} \) and \( w\text{-lim} \) to denote the strong and weak limit.

Throughout, we use the following functions:

\[
g_{z}(x) = x(x^2 - z)^{-1/2}, \quad g(x) = g_{-1}(x) = x(x^2 + 1)^{-1/2}, \quad (1.23)
\]

\[
\kappa_{z}(x) = (x^2 - z)^{1/2}, \quad \kappa(x) = \kappa_{-1}(x) = (x^2 + 1)^{1/2}, \quad (1.24)
\]

\[
z \in \mathbb{C} \setminus [0, \infty), \quad x \in \mathbb{R}.
\]

Let \( A = (A)^* \) be a self-adjoint (and generally, unbounded) operator on a separable Hilbert space \( \mathcal{H} \), then one can introduce the standard scale of spaces \( \mathcal{H}_m(A) \), \( m \in \mathbb{Z}, (\mathcal{H}_0 = \mathcal{H}) \) associated with \( A \). In particular, \( \mathcal{H}_1(A) \) is given by \( \mathcal{H}_1(A) = (\text{dom}(A), \| \cdot \|_{\mathcal{H}_1(A)}) \) the domain of \( A \) equipped with the graph norm

\[
\|f\|^2_{\mathcal{H}_1(A)} = \|Af\|^2_{\mathcal{H}} + \|f\|^2_{\mathcal{H}}, \quad f \in \text{dom}(A), \quad (1.25)
\]

and the obvious scalar product \((\cdot, \cdot)_{\mathcal{H}_1(A)}\) induced by (1.25), rendering \( \mathcal{H}_1(A) \) a Hilbert space. In addition, one notes that \( \kappa_{\mathcal{H}_1(A)} = (A^2 + I)^{1/2} \) is the isometric isomorphism between \( \mathcal{H}_1(A) \) and \( \mathcal{H} \). Similarly, \( \mathcal{H}_2(A) = (\text{dom}(A^2), \| \cdot \|_{\mathcal{H}_2(A)}) \) denotes the domain of \( A^2 \) equipped with the corresponding graph norm. We recall that, of course,

\[
\text{dom}(A^2) = \{w \in \text{dom}(A) \subseteq \mathcal{H} \mid Aw \in \text{dom}(A)\}. \quad (1.26)
\]

Hilbert spaces of the type \( L^2(\mathbb{R}; dt; \mathcal{H}) \) will be denoted by \( L^2(\mathbb{R}; \mathcal{H}) \) since only the Lebesgue measure on \( \mathbb{R} \) will be involved unless explicitly stated otherwise. Analogously, we will also use the shorthand notation \( L^2(\mathbb{R}; \mathcal{B}) \) for \( L^2(\mathbb{R}; dt; \mathcal{B}) \) in cases where \( \mathcal{B} \) is a Banach space.

Linear operators acting in the Hilbert space \( L^2(\mathbb{R}; \mathcal{H}) \) as defined in (1.1), denoted by boldface letters, \( A, B, \) etc., play a special role in this paper and are discussed in some detail in Appendix A.

Given a pair \( (A_-, A_+) \) of self-adjoint operators in \( \mathcal{H} \), we will use (1.24) to obtain operators \( \kappa_{\mathcal{H}_1(A_{\pm})}, \kappa_{\mathcal{H}_2(A_{\pm})} \) in \( \mathcal{H} \) and \( \kappa_{\mathcal{H}_1(A_-)}, \kappa_{\mathcal{H}_2(A_-)} \) in \( L^2(\mathbb{R}; \mathcal{H}); \) sometimes, in proofs, we abbreviate:

\[
\kappa = \kappa_{-} = \kappa_{\mathcal{H}_2(A_-)} = (A_-^2 + I)^{1/2}, \quad \kappa_{+} = \kappa_{\mathcal{H}_2(A_+)} = (A_+^2 + I)^{1/2}, \quad (1.27)
\]
\[ \tilde{x} = \tilde{x}_- = \chi(A_-) = (A_-^2 + \mathbf{1})^{1/2}, \quad \tilde{x}_- = \chi_z(A_-) = (A_-^2 - z\mathbf{1})^{1/2}, \]

where the operators in (1.28) are acting in \( L^2(\mathbb{R}; \mathcal{H}) \) and \( A_- \) is the constant fiber operator as defined in (1.1) with \( A(t) = A_-, \ t \in \mathbb{R}. \)

Finally, \( \mathbb{C}_+ = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \) denotes the open complex upper half-plane.

2. Principal Results

In this section we state our main hypotheses and principal results.

Throughout, we consider a family of closed, symmetric, densely defined (generally, unbounded) operators \( B(t), \ t \in \mathbb{R}, \) that are infinitesimally bounded with respect to \( A_-, \) and whose weak derivative is given by the operators \( B'(t), \ t \in \mathbb{R}, \) that are relatively trace class with respect to \( A_- \) in the following sense:

**Hypothesis 2.1.** Suppose \( \mathcal{H} \) is a complex, separable Hilbert space.

(i) Assume \( A_- \) is self-adjoint on \( \text{dom}(A_-) \subseteq \mathcal{H}. \)

(ii) Suppose there exists a family of operators \( B(t), \ t \in \mathbb{R}, \) closed and symmetric in \( \mathcal{H}, \) with \( \text{dom}(B(t)) \supseteq \text{dom}(A_-), \ t \in \mathbb{R}. \)

(iii) Assume there exists a family of operators \( B'(t), \ t \in \mathbb{R}, \) closed and symmetric in \( \mathcal{H}, \) with \( \text{dom}(B'(t)) \supseteq \text{dom}(A_-), \) such that the family \( B(t)(|A_-| + I)^{-1}, \ t \in \mathbb{R}, \)

is weakly locally absolutely continuous, and for a.e. \( t \in \mathbb{R}, \)

\[ \frac{d}{dt} (g, B(t)(|A_-| + I)^{-1} h)_{\mathcal{H}} = (g, B'(t)(|A_-| + I)^{-1} h)_{\mathcal{H}}, \quad g, h \in \mathcal{H}. \] (2.1)

(iv) Assume that \( B'(t)(|A_-| + I)^{-1} \in \mathcal{B}_1(\mathcal{H}), \ t \in \mathbb{R}, \) and

\[ \int_{\mathbb{R}} \| B'(t)(|A_-| + I)^{-1} \|_{\mathcal{B}_1(\mathcal{H})} dt < \infty. \] (2.2)

(v) Suppose that the families

\[ \{ (|B(t)|^2 + I)^{-1} \}_{t \in \mathbb{R}} \quad \text{and} \quad \{ (|B'(t)|^2 + I)^{-1} \}_{t \in \mathbb{R}} \] (2.3)

are weakly measurable (cf. Definition A.3(ii)).

For notational simplicity later on, \( B'(t) \) was defined for all \( t \in \mathbb{R} \) in Hypothesis 2.1(iii); it would have been possible to introduce it for a.e. \( t \in \mathbb{R} \) from the outset.

We refer to Section 3 for a thorough discussion of the implications of Hypothesis 2.1 and to Appendix A for a discussion of measurability questions of families of closed operators.

As discussed in detail in Section 3 (cf. Theorem 3.7), Hypothesis 2.1 implies the existence of a family of self-adjoint operators \( \{ A(t) \}_{t \in \mathbb{R}} \) in \( \mathcal{H} \) given by

\[ A(t) = A_- + B(t), \quad \text{dom}(A(t)) = \text{dom}(A_-), \ t \in \mathbb{R}, \] (2.4)

as well as a self-adjoint operator \( A_+ \) in \( \mathcal{H} \) such that

\[ \text{dom}(A_+) = \text{dom}(A_-) \] (2.5)

and

\[ \lim_{t \to \pm \infty} (A(t) - zI)^{-1} = (A_+ - zI)^{-1}, \quad z \in \mathbb{C}\setminus \mathbb{R}. \] (2.6)

We therefore also introduce

\[ B_- = 0, \quad B_+ = (A_+ - A_-), \quad \text{dom}(B_+) \supseteq \text{dom}(A_-), \] (2.7)

and note that

\[ A_+ = A_- + B_+, \quad \text{dom}(A_+) = \text{dom}(A_-). \] (2.8)
Next, let \( A \) in \( L^2(\mathbb{R}; \mathcal{H}) \) be then associated with the family \( \{A(t)\}_{t \in \mathbb{R}} \) in \( \mathcal{H} \) by
\[
(A_f)(t) = A(t)f(t) \text{ for a.e. } t \in \mathbb{R},
\]
\[
f \in \text{dom}(A) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid g(t) \in \text{dom}(A(t)) \text{ for a.e. } t \in \mathbb{R}, \quad t \mapsto A(t)g(t) \text{ is (weakly) measurable, } \int_\mathbb{R} \|A(t)g(t)\|_\mathcal{H}^2 \, dt < \infty \right\}. \tag{2.9}
\]

To state our results, we start by introducing in \( L^2(\mathbb{R}; \mathcal{H}) \) the operator
\[
D_A = \frac{d}{dt} + A, \quad \text{dom}(D_A) = \text{dom}(d/dt) \cap \text{dom}(A_-). \tag{2.10}
\]
Here the operator \( d/dt \) in \( L^2(\mathbb{R}; \mathcal{H}) \) is defined by
\[
\left( \frac{d}{dt} f \right)(t) = f'(t) \text{ for a.e. } t \in \mathbb{R},
\]
\[
f \in \text{dom}(d/dt) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid g \in AC_{\text{loc}}(\mathbb{R}; \mathcal{H}), \ g' \in L^2(\mathbb{R}; \mathcal{H}) \right\},
\]
especially,
\[
g \in AC_{\text{loc}}(\mathbb{R}; \mathcal{H}) \text{ if and only if } g \text{ is of the form}
\]
\[
g(t) = g(t_0) + \int_{t_0}^t h(s) \, ds, \ t, t_0 \in \mathbb{R}, \text{ for some } h \in L^1_{\text{loc}}(\mathbb{R}; \mathcal{H}), \text{ and } g' = h \text{ a.e.} \tag{2.12}
\]
(The integral in (2.12) is of course a Bochner integral.) In addition, \( A \) is defined in (2.9) and \( A_- \) in \( L^2(\mathbb{R}; \mathcal{H}) \) represents the self-adjoint (constant fiber) operator defined according to
\[
(A_- f)(t) = A_- f(t) \text{ for a.e. } t \in \mathbb{R},
\]
\[
f \in \text{dom}(A_-) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid g(t) \in \text{dom}(A_-) \text{ for a.e. } t \in \mathbb{R}, \quad t \mapsto A_- g(t) \text{ is (weakly) measurable, } \int_\mathbb{R} \|A_- g(t)\|_\mathcal{H}^2 \, dt < \infty \right\}. \tag{2.13}
\]

Assuming Hypothesis 2.1, we will prove in Lemma 4.4 that the operator \( D_A \) is densely defined and closed in \( L^2(\mathbb{R}; \mathcal{H}) \). Similarly, the adjoint operator \( D_A^* \) of \( D_A \) in \( L^2(\mathbb{R}; \mathcal{H}) \) is then given by
\[
D_A^* = -\frac{d}{dt} + A, \quad \text{dom}(D_A^*) = \text{dom}(d/dt) \cap \text{dom}(A_-) = \text{dom}(D_A). \tag{2.14}
\]

Using these operators, we define in \( L^2(\mathbb{R}; \mathcal{H}) \) the nonnegative self-adjoint operators
\[
H_1 = D_A^* D_A, \quad H_2 = D_A D_A^*. \tag{2.15}
\]
Finally, let us define the functions
\[
g_z(x) = x(x^2 - z)^{-1/2}, \quad z \in \mathbb{C}\setminus\{0, \infty\}, \ x \in \mathbb{R},
\]
\[
g(x) = g_{-1}(x) = x(x^2 + 1)^{-1/2}, \quad x \in \mathbb{R}. \tag{2.16}
\]
Our first principal result relates the trace of the difference of the resolvents of \( H_1 \) and \( H_2 \) in \( L^2(\mathbb{R}; \mathcal{H}) \), and the trace of the difference of \( g_z(A_+) \) and \( g_z(A_-) \) in \( \mathcal{H} \).
Theorem 2.2. Assume Hypothesis 2.1 and define the operators $H_1$ and $H_2$ as in (2.15) and the function $g_z$ as in (2.16). Then

$$[(H_2 - z I)^{-1} - (H_1 - z I)^{-1}] \in B_l(L^2(\mathbb{R}; \mathcal{H})), \quad z \in \rho(H_1) \cap \rho(H_2),$$

(2.17)

and the following trace formula holds,

$$\text{tr}_{L^2(\mathbb{R}; \mathcal{H})} \left( (H_2 - z I)^{-1} - (H_1 - z I)^{-1} \right) = \frac{1}{2z} \text{tr}_H (g_z(A_+) - g_z(A_-)), \quad z \in \mathbb{C}\setminus[0, \infty).$$

(2.19)

For notational convenience, cf. (1.24) and (1.27), we also introduce the self-adjoint operator

$$\mathcal{K} = \mathcal{K}(A_-) = (A_-^2 + I)^{1/2},$$

(2.20)

in $\mathcal{H}$, and for subsequent purposes also the operators

$$\mathcal{K}_z(A_{\pm}) = (A_{\pm}^2 - z I)^{1/2}, \quad z \in \mathbb{C}\setminus[0, \infty).$$

(2.21)

We will now outline the main steps in the proof of Theorem 2.2. As in [123], the essential element of our strategy is to pass to an appropriate approximation $A_n(t)$. The simplest way to do this is to consider the spectral projections $P_n = E_{A_n((-n, n))}, n \in \mathbb{N}$, associated with the operator $A_-$. The projections commute with $A_-, \mathcal{K}(A_-)$, and their resolvents. Using $P_n$ just introduced, (2.4), (2.8), (2.20), and (2.21), we define the following operators:

$$A_n(t) = P_n A(t) P_n, \quad B_n(t) = P_n B(t) P_n, \quad B'_n(t) = P_n B'(t) P_n,$$

(2.22)

$$A_{\pm,n} = P_n A_{\pm} P_n, \quad B_n(\pm \infty) = P_n B(\pm \infty) P_n, \quad n \in \mathbb{N}.$$

One observes that all operators introduced in (2.22) are bounded operators acting on the space $\mathcal{h}_n = \text{ran}(P_n)$ which is, in general, infinite-dimensional. For an operator $A_n$ acting on $\mathcal{h}_n$, we will keep the same notation $A_n$ to denote the operator $A_n \oplus 0$ acting on $\mathcal{h} = \mathcal{h}_n \oplus 0$. The proof of the following formula (2.23) uses the main result in [123] applied to the bounded approximants $A_n(t)$ of $A(t)$:

Proposition 2.3. Assume Hypothesis 2.1. Then the trace formula (2.19) holds for the operators $A_n(t), A_{\pm,n}$ on $\mathcal{H}$, defined in (2.22), and the operators $H_{1,n}$ and $H_{2,n}$ on $L^2(\mathbb{R}; \mathcal{H})$, obtained by replacing $A(t)$ by $A_n(t)$ in (2.15), that is, one has

$$\text{tr}_{L^2(\mathbb{R}; \mathcal{H}_n)} \left( (H_{2,n} - z I)^{-1} - (H_{1,n} - z I)^{-1} \right) = \frac{1}{2z} \text{tr}_{\mathcal{h}_n} (g_z(A_{+,n}) - g_z(A_{-,n})), \quad z \in \mathbb{C}\setminus[0, \infty).$$

(2.23)

Proof. As $P_n = E_{A_n((-n, n))}$ are the spectral projections for $A_-$, for each fixed $n \in \mathbb{N}$, formula (2.23) follows from [123, Proposition 1.3] under the assumptions in Hypothesis 2.1. Indeed, formula (2.23) has been proved in [123, Proposition 1.3] under the assumption

$$\int_{\mathbb{R}} \|B'_n(t)\|_{\mathcal{B}_1(\mathcal{h}_n)} dt < \infty.$$  

(2.24)

In the current setting, condition (2.2) and relation

$$B'_n(t)(|A_{-,n}| + I_{\mathcal{h}_n})^{-1} = P_n B'(t)(|A_-| + I_{\mathcal{H}})^{-1} P_n$$

(2.25)

yield:

$$\int_{\mathbb{R}} \|B'_n(t)(|A_{-,n}| + I_{\mathcal{h}_n})^{-1}\|_{\mathcal{B}_1(\mathcal{h}_n)} dt < \infty.$$  

(2.26)
Since the operator \(|A_{-,n}| + I_{h_n}\) is bounded for each \(n \in \mathbb{N}\), (2.26) implies (2.24), and thus (2.23) holds. □

In view of Proposition 2.3, to complete the proof of Theorem 2.2 it suffices to pass to the limit as \(n \to \infty\) and in the right-hand side of (2.23) as \(n \to \infty\). As a result, Theorem 2.2 is a consequence of the following three propositions proved, respectively, in Sections 5, 6, and 7 (cf. Lemma 7.3).

**Proposition 2.4.** Assume Hypothesis 2.1, and consider the operators \(H_1\) and \(H_2\) defined in (2.15), and the operators \(H_{1,n}\) and \(H_{2,n}\) on \(L^2(\mathbb{R}; \mathcal{H})\), obtained by replacing \(A(t)\) by \(A_n(t)\) in (2.15). Then, for each \(n \in \mathbb{N}\) and \(z \in \mathbb{C}\setminus[0, \infty)\),

\[
\left(1 - z I\right)^{-1} \left(1 - z I\right)^{-1} \in \mathcal{B}_1(L^2(\mathbb{R}; \mathcal{H}))
\]

and

\[
\lim_{n \to \infty} \left\| \left(1 - z I\right)^{-1} - \left(1 - z I\right)^{-1} \right\|_{\mathcal{B}_1(L^2(\mathbb{R}; \mathcal{H}))} = 0. \tag{2.27}
\]

**Proposition 2.5.** Assume Hypothesis 2.1. Consider the operators \(A_{+,n}\) in (3.51), \(A_{-,n}\) in (2.22), and the function \(x(x^2 + 1)^{-1/2}, x \in \mathbb{R}\), introduced in (2.16). Then,

\[
g(A_{+,n}) - g(A_{-n}), [g(A_{+,n}) - g(A_{-n})] \in \mathcal{B}_1(\mathcal{H})
\]

for each \(n \in \mathbb{N}\) and

\[
\lim_{n \to \infty} \left\| g(A_{+,n}) - g(A_{-n}) - [g(A_{+,n}) - g(A_{-n})] \right\|_{\mathcal{B}_1(\mathcal{H})} = 0. \tag{2.29}
\]

**Proposition 2.6.** Assume Hypothesis 2.1 and consider the function \(g_z(x) = x(x^2 - z)^{-1/2}, x \in \mathbb{R}, z \in \mathbb{C}\setminus[0, \infty)\) introduced in (2.16). Then,

\[
g_z(A_{+,n}) - g_z(A_{-n}) \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C}\setminus[0, \infty), \tag{2.31}
\]

and

\[
\mathbb{C}\setminus[0, \infty) \ni z \mapsto \text{tr}_\mathcal{H} \left(g_z(A_{-,n}) - g_z(A_{-,n})\right) \text{ is analytic.} \tag{2.32}
\]

Assuming Propositions 2.4–2.6, one finishes the proof of Theorem 2.2 as follows:

**Proof of Theorem 2.2.** The left-hand side of formula (2.19) is an analytic function with respect to \(z \in \mathbb{C}\setminus[0, \infty)\). By Proposition 2.6, also the right-hand side is an analytic function with respect to \(z \in \mathbb{C}\setminus[0, \infty)\). By analytic continuation, it suffices to show (2.19) for \(z < 0\). But for \(z < 0\) the conclusions of Proposition 2.5 hold if \(g\) is replaced by \(g_z\), using \(g_z(x) = g(x/(-z)^{1/2})\) and rescaling \(A(t) \mapsto A(t)(-z)^{1/2}\). Thus, passing to the limit as \(n \to \infty\) in (2.23), and using Propositions 2.4 and 2.5, one concludes that (2.19) holds for \(z < 0\). □

**Remark 2.7.** Alternatively, one can derive relation (2.31) from (2.29) as follows: One considers the smooth function \(h_z(x) = (x^2 + 1)/(x^2 - z)^{1/2}\) with equal limits 1 as \(x \to \pm \infty\). Then the inclusion (2.31) follows from the first assertion in (2.29), from the formula \(g_z(x) = h_z(x)g(x)\), and the representation

\[
g_z(A_{+,n}) - g_z(A_{-n}) = [h_z(A_{+,n}) - h_z(A_{-,n})]g(A_{+,n}) + h_z(A_{-,n})[g(A_{+,n}) - g(A_{-,n})], \tag{2.33}
\]

since the inclusion \([h_z(A_{+,n}) - h_z(A_{-,n})] \in \mathcal{B}_1(\mathcal{H})\) holds, for instance, by [143, Theorem 8.7.1].
However, much more is true: In Lemma B.1 we will, in fact, prove trace norm analyticity of \([g_z(A_+)-g_z(A_-)], z \in \mathbb{C}\setminus[0, \infty)\), which immediately yields analyticity of \(\text{tr}_H(g_z(A_+)-g_z(A_-)), z \in \mathbb{C}\setminus[0, \infty)\), and hence provides yet another alternative start for proving Theorem 2.2.

The following corollary is one of our principal results saying that the difference of the Morse projections is of trace class.

**Corollary 2.8.** Assume Hypothesis 2.1 and suppose that \(0 \in \rho(A_+)\cap \rho(A_-)\). Then
\[
[E_{A_-}((-\infty,0)) - E_{A_+}((-\infty,0))] \in \mathcal{B}_1(H).
\] (2.34)

**Proof.** By Hypothesis 2.1, \((A_+-A_-)\) is a relatively trace class perturbation of \(A_-\), that is \((A_+-A_-)(A_-+zI)^{-1} \in \mathcal{B}_1(H), z \in \mathbb{C}\setminus\mathbb{R}\) (see (3.28)), and thus the difference of the resolvents of \(A_+\) and \(A_-\) is of trace class, \([(A_+-A_-)(A_-+zI)^{-1}] \in \mathcal{B}_1(H), z \in \mathbb{C}\setminus\mathbb{R}\). In this case (cf. [143, Theorem 8.7.1]), one has \(\{f(A_+)-f(A_-)\} \in \mathcal{B}_1(H)\) for any function \(f\) having two locally bounded derivatives and satisfying the following conditions:
\[
(x^2f'(x))' = O(|x|^{1-\epsilon}), |x| \to \infty, \epsilon > 0,
\] (2.35)
\[
\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x), \quad \lim_{x \to -\infty} x^2f'(x) = \lim_{x \to +\infty} x^2f'(x).
\] (2.36)

Since \(E_{A_+}((-\infty,0)] = \frac{1}{\xi}(I - \text{sign}(A_+))\), inclusion (2.34) is equivalent to
\[
[\text{sign}(A_-) - \text{sign}(A_+)] \in \mathcal{B}_1(H).
\] (2.37)

We choose \(\varepsilon_0\) such that \([-\varepsilon_0, \varepsilon_0] \subset \rho(A_-)\cap \rho(A_+)\) and consider a smooth on \(\mathbb{R}\setminus\{0\}\) modification \(\tilde{g}\) of the function \(g(x) = x(1+x^2)^{-1/2}\) such that \(\tilde{g}(x) = \text{sign} x\) for \(|x| < \varepsilon_0/2\) and \(\tilde{g}(x) = g(x)\) for \(|x| > \varepsilon_0\). Then \(\tilde{g}(A_+) = g(A_+)\) since \(\tilde{g}\) and \(g\) coincide on the spectrum of \(A_+\). By the first inclusion in (2.29) we have \([\tilde{g}(A_-) - \tilde{g}(A_+)] \in \mathcal{B}_1(H)\), and thus, introducing the function \(f(x) = \tilde{g}(x) - \text{sign}(x)\), inclusion (2.37) is equivalent to \(\{f(A_-) - f(A_+)\} \in \mathcal{B}_1(H)\). But the latter inclusion holds since \(f\) satisfies (2.35), (2.36) with \(\epsilon = 1\).

Next, we will formulate one of our principal results, relating a particular choice of spectral shift functions of the two pairs of operators, \((H_2, H_1)\), and \((A_+, A_-)\). This requires some preparations as \textit{a priori} in the present general context, the Krein spectral shift function for either pair is only defined up to constants.

Since by Theorem 2.2,
\[
[g(A_+) - g(A_-)] \in \mathcal{B}_1(H),
\] (2.38)
and \(g(A_{\pm})\) are self-adjoint, Krein’s trace formula in its simplest form (cf. [143, Theorem 8.2.1] yields
\[
\text{tr}_H(g(A_+) - g(A_-)) = \int_{[-1,1]} \xi(\omega; g(A_+), g(A_-)) d\omega.
\] (2.39)

Defining
\[
\xi(\nu; A_+, A_-) := \xi(\rho; g(A_+), g(A_-)), \quad \nu \in \mathbb{R},
\] (2.40)
then \(\xi(\cdot; A_+, A_-)\) can be shown to satisfy
\[
\xi(\cdot; A_+, A_-) \in L^1(\mathbb{R}; (|\nu| + 1)^{-2} d\nu).
\] (2.41)
Next, one also needs to introduce the spectral shift function \( \xi(\cdot; H_2, H_1) \) associated with the pair \((H_2, H_1)\). Since \(H_2 \geq 0\) and \(H_1 \geq 0\), and
\[
[(H_2 + I)^{-1} - (H_1 + I)^{-1}] \in \mathcal{B}_1(L^2(\mathbb{R}; \mathcal{H})),
\]
by Theorem 2.2, one uniquely introduces \( \xi(\cdot; H_2, H_1) \) by requiring that
\[
\xi(\lambda; H_2, H_1) = 0, \quad \lambda < 0,
\]
and by
\[
\text{tr}_{L^2(\mathbb{R}; \mathcal{H})} \left( (H_2 - z I)^{-1} - (H_1 - z I)^{-1} \right) = - \int_{(0, \infty)} \frac{\xi(\lambda; H_2, H_1)}{(\lambda - z)^{-2}} d\lambda, \quad z \in \mathbb{C} \setminus [0, \infty).
\]
following [143, Sect. 8.9].

Given these preparations, we have the following result, an extension of Pushnitski’s formula [123], to be proven in Section 8.

**Theorem 2.9.** Assume Hypothesis 2.1 and define \( \xi(\cdot; A_+, A_-) \) and \( \xi(\cdot; H_2, H_1) \) according to (2.40) and (2.43), (2.44), respectively. Then one has for a.e. \( \lambda > 0 \),
\[
\xi(\lambda; H_2, H_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)}{(\lambda - \nu^2)^{1/2}} d\nu,
\]
with a convergent Lebesgue integral on the right-hand side of (2.45).

Finally, we turn to the connection between the spectral shift function, the spectral flow for the path \( \{A(t)\}_{t \in \mathbb{R}} \) of self-adjoint Fredholm operators, and the Fredholm index of \( DA \) to be studied in detail in Section 9. Introducing the spectral flow \( \text{SpFlow}(\{A(t)\}_{t \in \mathbb{R}}) \) as in Definition 9.5, and recalling the definition of the index of a pair of Fredholm projections in Definition 9.8, the following result is proved in Theorem 7.6, Corollary 8.4, and Theorems 9.13. (We note that Theorem 2.9 is the major input in the proof of the Fredholm index result (2.46)).

**Theorem 2.10.** Assume Hypothesis 2.1 and suppose that \( 0 \in \rho(A_+) \cap \rho(A_-) \). Then the pair \( (E_{A_+}((-\infty, 0)), E_{A_-}((-\infty, 0))) \) of the Morse projections is Fredholm and the following equalities hold:
\[
\text{index}(DA) = \text{SpFlow}(\{A(t)\}_{t \in \mathbb{R}}) = \xi(0; A_+, A_-) = \xi(0; H_1, H_2)
\]
\[
= \text{index}(E_{A_+}((-\infty, 0)), E_{A_-}((-\infty, 0))) = \text{tr}_{H}(E_{A_+}((-\infty, 0)) - E_{A_-}((-\infty, 0)))
\]
\[
= \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(\ln(\det_H((A_+ - i\varepsilon I)(A_- - i\varepsilon I)^{-1}))),
\]
with a choice of branch of \( \ln(\det_H(\cdot)) \) on \( \mathbb{C}_+ \) analogous to (2.52) below.

Here we note that \( \xi \) can be shown to satisfy (cf. Theorem 7.6)
\[
\xi(\lambda; A_+, A_-) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(\ln(D_{A_+/A_-}(\lambda + i\varepsilon)))
\]
for a.e. \( \lambda \in \mathbb{R} \),
and we make the choice of branch of \( \ln(D_{A_+/A_-}(\cdot)) \) on \( \mathbb{C}_+ \) such that
\[
\lim_{\text{Im}(z) \to +\infty} \ln(D_{A_+/A_-}(z)) = 0,
\]
with
\[ D_{T/S}(z) = \det_H((T - zI)(S - zI)^{-1}) = \det_H(I + (T - S)(S - zI)^{-1}), \quad z \in \rho(S), \]
(2.53)
denoting the perturbation determinant for the pair of operators \((S,T)\) in \(\mathcal{H}\), assuming \((T - S)(S - z_0)^{-1} \in \mathcal{B}_1(\mathcal{H})\) for some (and hence for all) \(z_0 \in \rho(S)\). In addition, we recall,
\[
\frac{d}{dz} \ln(D_{A_+/A_-}(z)) = -\text{tr}_H((A_+ - zI)^{-1} - (A_- - zI)^{-1})
\]
\[
= \int_{\mathbb{R}} \frac{\xi(\lambda; A_+, A_-) \, d\lambda}{(\lambda - z)^2}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
(2.54)
the trace formula associated with the pair \((A_+, A_-)\).

## 3. The Relative Trace Class Setting

Throughout this section we assume Hypothesis 2.1 and closely examine the basic assumptions made in it.

### 3.1. A Thorough Analysis of the Main Hypothesis

We start with the following auxiliary result:

**Lemma 3.1.** Let \(\mathcal{H}\) be a complex, separable Hilbert space and \(\mathbb{R} \ni t \mapsto F(t) \in \mathcal{B}_1(\mathcal{H})\). Then the following assertions (i) and (ii) are equivalent:

(i) \(\{F(t)\}_{t \in \mathbb{R}}\) is a weakly measurable family of operators in \(\mathcal{B}(\mathcal{H})\) and \(\|F(\cdot)\|_{\mathcal{B}_1(\mathcal{H})} \in L^1(\mathbb{R}; dt)\).

(ii) \(F \in L^1(\mathbb{R}; \mathcal{B}_1(\mathcal{H}))\).

Moreover, if either condition (i) or (ii) holds, then
\[
\left\| \int_{\mathbb{R}} F(t) \, dt \right\|_{\mathcal{B}_1(\mathcal{H})} \leq \int_{\mathbb{R}} \|F(t)\|_{\mathcal{B}_1(\mathcal{H})} \, dt
\]  
(3.1)
and the \(\mathcal{B}_1(\mathcal{H})\)-valued function
\[
\mathbb{R} \ni t \mapsto \int_{t_0}^t F(s) \, ds, \quad t_0 \in \mathbb{R} \cup \{-\infty\},
\]  
(3.2)
is strongly absolutely continuous with respect to the norm in \(\mathcal{B}_1(\mathcal{H})\).

In addition we recall the following fact:

(iii) Suppose that \(\mathbb{R} \ni t \mapsto G(t) \in \mathcal{B}_1(\mathcal{H})\) is strongly locally absolutely continuous in \(\mathcal{B}_1(\mathcal{H})\). Then \(H(t) = G'(t)\) exists for a.e. \(t \in \mathbb{R}\), \(H\) is Bochner integrable over any compact interval, and hence
\[
G(t) = G(t_0) + \int_{t_0}^t H(s) \, ds, \quad t, t_0 \in \mathbb{R}.
\]  
(3.3)

**Proof.** Clearly, condition (ii) implies condition (i).

To prove the converse statement, that is, condition (i) implies condition (ii), one can argue as follows. Let \(F : \mathbb{R} \to \mathcal{B}_1(\mathcal{H})\) be a weakly measurable function in the sense that for every \(f, g \in \mathcal{H}\), the function \((f, F(\cdot)g)_\mathcal{H}\) is measurable on \(\mathbb{R}\), and suppose that \(\|F(\cdot)\|_{\mathcal{B}_1(\mathcal{H})} \in L^1(\mathbb{R}; dt)\).

One recalls that \(\mathcal{B}_1(\mathcal{H})\) is a separable Banach space. Hence, if \(F\) is weakly measurable in \(\mathcal{B}_1(\mathcal{H})\), then it is measurable in \(\mathcal{B}_1(\mathcal{H})\) by Pettis’ theorem (cf., e.g., [11, Theorem 1.1.1], [53, Theorem II.1.2], [81, 3.5.3]). Moreover, one recalls that
for fixed $A \in \mathcal{B}(\mathcal{H})$, $\text{tr}_\mathcal{H}(TA)$, $T \in \mathcal{B}_1(\mathcal{H})$ is a continuous functional on $\mathcal{B}_1(\mathcal{H})$ with norm $\|A\|_{\mathcal{B}(\mathcal{H})}$, and every continuous functional on $\mathcal{B}_1(\mathcal{H})$ is obtained in this manner. In particular, one can identify $\mathcal{B}_1(\mathcal{H})^*$ and $\mathcal{B}(\mathcal{H})$ as Banach spaces.

Next, one notes that $F$ is weakly measurable in $\mathcal{B}_1(\mathcal{H})$ if and only if $\text{tr}_\mathcal{H}(F(\cdot)A)$ is measurable on $\mathbb{R}$ for every $A$ taken from a separating set $\mathcal{S} \subseteq (\mathcal{B}_1(\mathcal{H}))^* = \mathcal{B}(\mathcal{H})$ (cf., e.g., [11, Corollary 1.1.3]). As $\mathcal{S}$ one may take, for example, the set $O$ of rank-one operators on $\mathcal{H}$. It will clearly be separating since for any $0 \neq T \in \mathcal{B}_1(\mathcal{H})$ one can find an operator $A = (f_0, \cdot)_{\mathcal{H}} g_0 \in O$ such that $\text{tr}_\mathcal{H}(TA) = (f_0, Tg_0)_{\mathcal{H}} \neq 0$. However, by hypothesis, $\text{tr}_\mathcal{H}(F(\cdot)A) = (f_0, F(\cdot)g_0)_{\mathcal{H}}$ is measurable on $\mathbb{R}$ for every $A = (f_0, \cdot)_{\mathcal{H}} g_0$. Thus $F$ is weakly measurable and hence measurable in $\mathcal{B}_1(\mathcal{H})$. Moreover, since also $\|F(\cdot)\|_{\mathcal{B}_1(\mathcal{H})} \in L^1(\mathbb{R}; dt)$ by assumption, $F$ is Bochner integrable in $\mathcal{B}_1(\mathcal{H})$ by Bochner’s theorem (cf., e.g., [11, Theorem 1.14], [53, Theorem II.2.2], [81, Theorem 3.7.4]).

The estimate (3.1) and the strong absolute continuity of the function in (3.2) is well-known in the context of Bochner integrals (cf., e.g., [11, p. 6–21], [53, p. 44–50], [81, p. 71–88]).

Finally, also (3.3) is standard (cf., e.g., [11, Proposition 1.2.3]) since $\mathcal{B}_1(\mathcal{H})$ has the Radon–Nikodym property by the Dunford–Pettis Theorem (cf., e.g., [11, Theorem 1.2.6]) as $(\mathcal{B}_\infty(\mathcal{H}))^* = \mathcal{B}_1(\mathcal{H})$ is a separable dual space (cf., e.g., [71, Theorem III.7.1], [131, Sect. IV.1]).

An application of Lemma 3.1 yields the following observations:

Remark 3.2. Hypothesis 2.1 (iii) implies that
\[
\{B'(t)(|A_-| + I)^{-1}\}_{t \in \mathbb{R}} \text{ is weakly measurable}
\] (3.4)
since for all $g, h \in \mathcal{H}$, $(g, B'(\cdot)(|A_-| + I)^{-1}h)_{\mathcal{H}}$ arises as a pointwise a.e. limit of measurable functions. Thus, applying Lemma 3.1, one concludes that assumption (2.2),
\[
\int_{\mathbb{R}} \|B'(t)(|A_-| + I)^{-1}\|_{\mathcal{B}_1(\mathcal{H})} dt < \infty,
\] (3.5)
then together with condition (3.4), are equivalent to the (seemingly stronger) condition
\[
B'(\cdot)(|A_-| + I)^{-1} \in L^1(\mathbb{R}; \mathcal{B}_1(\mathcal{H})).
\] (3.6)
In particular, it would have been possible to just assume $B'(t)(|A_-| + I)^{-1} \in L^1(\mathbb{R}; \mathcal{B}_1(\mathcal{H}))$ in Hypothesis 2.1 (iv).

Remark 3.3. We temporarily introduce the Bochner integral in $\mathcal{B}_1(\mathcal{H})$,
\[
C(t) = \int_{-\infty}^t B'(s)(|A_-| + I)^{-1} ds \in \mathcal{B}_1(\mathcal{H}), \quad t \in \mathbb{R}.
\] (3.7)
Applying Lemma 3.1 (iii), one concludes that
\[
C'(t) = B'(t)(|A_-| + I)^{-1} \text{ for a.e. } t \in \mathbb{R},
\] (3.8)
and hence, in particular, for all $f, g \in \mathcal{H}$,
\[
(f, C'(t)g)_{\mathcal{H}} = (f, B'(t)(|A_-| + I)^{-1}g)_{\mathcal{H}} \text{ for a.e. } t \in \mathbb{R}.
\] (3.9)
Thus, by Hypothesis 2.1 (iii),
\[
\frac{d}{dt} (f, C(t)g)_{\mathcal{H}} = (f, C'(t)g)_{\mathcal{H}} = (f, B'(t)(|A_-| + I)^{-1}g)_{\mathcal{H}}
\]
that is, the existence of a constant \( C \)

\[
\frac{d}{dt}(f, B(t)(|A_\cdot| + I)^{-1} g)_{\mathcal{H}} \text{ for a.e. } t \in \mathbb{R}. \tag{3.10}
\]

Consequently, one arrives at

\[
C(t) = B(t)(|A_\cdot| + I)^{-1} + C_0 \text{ for some } C_0 \in B_1(\mathcal{H}). \tag{3.11}
\]

In particular, one infers that

\[
\lim_{t \to -\infty} B(t)(|A_\cdot| + I)^{-1} = D_- \text{ exists in the } B_1(\mathcal{H})\text{-norm}. \tag{3.12}
\]

We now choose the convenient normalization

\[ D_- = 0 \tag{3.13} \]

and hence obtain

\[
B(t)(|A_\cdot| + I)^{-1} = \int_{-\infty}^{t} B'(s)(|A_\cdot| + I)^{-1} \, ds \in B_1(\mathcal{H}), \quad t \in \mathbb{R}, \tag{3.14}
\]

(a fact that will be used later in the proof of Lemma 3.5), and hence one also has the estimate

\[
\|B(t)(|A_\cdot| + I)^{-1}\|_{B_1(\mathcal{H})} \leq \int_{-\infty}^{t} \|B'(s)(|A_\cdot| + I)^{-1}\|_{B_1(\mathcal{H})} \, ds, \quad t \in \mathbb{R}. \tag{3.15}
\]

In the following we draw some conclusions from Hypothesis 2.1:

We start by recalling the following standard convergence property for trace ideals:

**Lemma 3.4.** Let \( p \in [1, \infty) \) and assume that \( R, R_n, T, T_n \in B(\mathcal{H}), \) \( n \in \mathbb{N}, \) satisfy s-lim_{n \to \infty} R_n = R and s-lim_{n \to \infty} T_n = T and that \( S, S_n \in B_p(\mathcal{H}), \) \( n \in \mathbb{N}, \) satisfy lim_{n \to \infty} \|S_n - S\|_{B_p(\mathcal{H})} = 0. \ Then \ lim_{n \to \infty} \|R_n S_n T_n^* - R S T^*\|_{B_p(\mathcal{H})} = 0.

This follows, for instance, from [78, Theorem 1], [135, p. 28–29], or [143, Lemma 6.1.3] with a minor additional effort (taking adjoints, etc.). We note that by the uniform boundedness principle, weak (and hence strong) convergence of \( R_n \in B(\mathcal{H}) \) to an operator \( R \in B(\mathcal{H}) \) implies the uniform boundedness of the sequence \( \{R_n\} \in \mathbb{N}, \) that is, the existence of a constant \( C \in (0, \infty) \) such that \( \sup_{n \in \mathbb{N}} \|R_n\|_{B(\mathcal{H})} \leq C \) and \( \|R\|_{B(\mathcal{H})} \leq \liminf_{n \to \infty} \|R_n\|_{B(\mathcal{H})} \) (cf., e.g., [141, Theorem 4.26]). (In particular, the uniform boundedness hypothesis \( \sup_{n \in \mathbb{N}} \|R_n\|_{B(\mathcal{H})} \leq C \) and similarly for \( T_n \) used in [135, p. 28] need not be assumed in Lemma 3.4.)

**Lemma 3.5.** Assume Hypotheses 2.1 and introduce the open cone \( C_\varepsilon = \{z \in \mathbb{C} \mid |\arg(z)| < \varepsilon \} \text{ for some } \varepsilon \in (0, \pi/2). \) Then

\[
\sup_{t \in \mathbb{R}} \|B(t)(|A_\cdot| - z I)^{-1}\|_{B_1(\mathcal{H})} = \sup_{t \in \mathbb{R}} \left( \int_{-\infty}^{t} B'(s)(|A_\cdot| - z I)^{-1} \, ds \right)_{B_1(\mathcal{H})} = o(1). \tag{3.16}
\]

**Proof.** One estimates, assuming for simplicity that \( |z| \geq 1, z \notin C_\varepsilon, \)

\[
\sup_{t \in \mathbb{R}} \|B(t)(|A_\cdot| - z I)^{-1}\|_{B_1(\mathcal{H})} = \sup_{t \in \mathbb{R}} \left( \int_{-\infty}^{t} B'(s)(|A_\cdot| - z I)^{-1} \, ds \right)_{B_1(\mathcal{H})} \leq \int_{\mathbb{R}} \|B'(s)(|A_\cdot| - z I)^{-1}\|_{B_1(\mathcal{H})} \, ds \tag{3.17}
\]

\[
= \int_{\mathbb{R}} \|B'(s)(|A_\cdot| + I)^{-1}(|A_\cdot| - z I)^{-1}\|_{B_1(\mathcal{H})} \, ds \leq \|(A_\cdot + I)(|A_\cdot| - z I)^{-1}\|_{B(\mathcal{H})} \int_{\mathbb{R}} \|B'(s)(|A_\cdot| + I)^{-1}\|_{B_1(\mathcal{H})} \, ds < \infty \tag{3.18}
\]
due to condition (2.2), since
\[ \|(A_- + I)(A_- - zI)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq c(\varepsilon), \quad |z| \geq 1, z \notin C_\varepsilon, \]  
(3.19)
for some constant \(c(\varepsilon) > 0\). By the dominated convergence theorem and (3.17), it remains to show that
\[ \|B'(s)(A_- - zI)^{-1}\|_{\mathcal{B}_1(\mathcal{H})} \to 0 \quad \text{for each } s \in \mathbb{R}. \]  
(3.20)
Introducing the normal operators \(W_z = (A_- + I)(A_- - zI)^{-1}, W_z^* = (A_- + I)(A_- - zI)^{-1}, |z| \geq 1, z \notin C_\varepsilon\), the norms of \(W_z\) are uniformly bounded due to (3.19). In addition, one has \(\|(A_- - zI)^{-1}\|_{\mathcal{B}(\mathcal{H})} \to 0\) as \(|z| \to \infty, z \notin C_\varepsilon\), and thus for all \(f \in \text{dom}(A_-)\), \((W_z)^*f \to 0\) in \(\mathcal{H}\) as \(|z| \to \infty, z \notin C_\varepsilon\). Since \(\text{dom}(A_-)\) is dense in \(\mathcal{H}\), it follows that \((W_z)^* \to 0\) strongly in \(\mathcal{H}\) as \(|z| \to \infty, z \notin C_\varepsilon\). Due to the fact that
\[ B'(s)(A_- - zI)^{-1} = B'(s)(A_- + I)^{-1}W_z, \]  
(3.21)
and the operator \(B'(s)(A_- + I)^{-1}\) is in \(\mathcal{B}_1(\mathcal{H})\), Lemma 3.4 implies (3.20).

**Remark 3.6.** Since \(B(t)\) and \(B'(t)\) are symmetric with \(\text{dom}(B(t)) \cap \text{dom}(B'(t)) \supseteq \text{dom}(A_-)\), \(t \in \mathbb{R}\), one concludes that
\[ B(t)^*(A_- + I)^{-1} = B(t)(A_- + I)^{-1}, \]  
\[ (B'(t)^* - B'(t)) = (B'(t)^* - B'(t))^{-1}, \quad t \in \mathbb{R}. \]  
(3.22)
Consequently, (2.2), (3.14), (3.15), (3.6), and (3.16) hold with \(B(t), B'(t)\) replaced by \(B(t)^*, (B'(t))^*\), respectively.

Next, assuming Hypothesis 2.1, we recall the definition of the family of operators \(\{A(t)\}_{t \in \mathbb{R}}\) in \(\mathcal{H}\) with constant domain \(\text{dom}(A_-)\) (cf. (2.4)) by
\[ A(t) = A_- + B(t), \quad \text{dom}(A(t)) = \text{dom}(A_-), \quad t \in \mathbb{R}, \]  
(3.23)
and note that (cf. (3.16))
\[ \sup_{t \in \mathbb{R}} \|A(t)\|_{\mathcal{B}(\mathcal{H}, (A_-), \mathcal{H})} = \sup_{t \in \mathbb{R}} \|A(t)(A_- + I)^{-1}\|_{\mathcal{B}(\mathcal{H})} \]  
\[ = \sup_{t \in \mathbb{R}} \|[A_- + B(t)](A_- + I)^{-1}\|_{\mathcal{B}(\mathcal{H})} < \infty. \]  
(3.24)
We now turn to a closer examination of the family \(\{A(t)\}_{t \in \mathbb{R}}\):

**Theorem 3.7.** Assume Hypothesis 2.1 and define \(A(t), t \in \mathbb{R}\), as in (3.23). Then the following assertions hold:

(i) For all \(t \in \mathbb{R}\), \(A(t)\) with domain \(\text{dom}(A(t)) = \text{dom}(A_-)\) is self-adjoint in \(\mathcal{H}\).

(ii) For all \(t \in \mathbb{R}\), \(B(t)\) is relatively trace class with respect to \(A_-\), that is,
\[ B(t)(A_- - zI)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C} \setminus \mathbb{R}, t \in \mathbb{R}. \]  
(3.25)

(iii) There exists a self-adjoint operator \(A_+\) in \(\mathcal{H}\) such that
\[ \text{dom}(A_+) = \text{dom}(A_-), \]  
(3.26)
and
\[ \text{n-lim}_{t \to \pm \infty} (A(t) - zI)^{-1} = (A_+ - zI)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \]  
(3.27)

(iv) \((A_+ - A_-)(A_- - zI)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C} \setminus \mathbb{R}. \)  
(3.28)
(i) Self-adjointness of $A(t)$ on $\text{dom}(A(t)) = \text{dom}(A_-)$ for all $t \in \mathbb{R}$ immediately follows from (3.16), which implies
\[
\|B(t)(A_- - zI)^{-1}\|_{\mathcal{B}(\mathcal{H})} < 1 \text{ for } |\text{Im}(z)| > 0 \text{ sufficiently large,}
\] and the Kato–Rellich Theorem (cf. [86, Theorem V.4.3]).

(ii) This instantly follows from (3.16).

(iii) Since by (3.6), $B'(t)(|A_-| + I)^{-1} \in L^1(\mathbb{R}; \mathcal{B}_1(\mathcal{H}))$, one infers in addition to (3.12) and (3.13) that
\[
\lim_{t \to \pm \infty} B(t)(A_- - zI)^{-1} = \begin{cases} D_+(z) & \text{exist in the } \mathcal{B}_1(\mathcal{H})\text{-norm} \\
0 & \text{for } |\text{Im}(z)| > 0 \text{ sufficiently large.}
\end{cases}
\] for $|\text{Im}(z)| > 0$ sufficiently large. Moreover, by (3.32) and (3.16) (in fact, in this context it would be sufficient to replace $\mathcal{B}_1(\mathcal{H})$ by $\mathcal{B}(\mathcal{H})$ in (3.16)) one has that
\[
[I + B(t)(A_- - zI)^{-1}]^{-1}, [I + D_+(z)]^{-1} \in \mathcal{B}(\mathcal{H}) \text{ for } |\text{Im}(z)| > 0 \text{ sufficiently large.}
\] Employing the second resolvent equation for $A(t)$ one obtains, using (3.34),
\[
(A(t) - zI)^{-1} = (A_- - zI)^{-1} - (A(t) - zI)^{-1}[B(t)(A_- - zI)^{-1}], \quad t \in \mathbb{R},
\] for $|\text{Im}(z)| > 0$ sufficiently large. Thus, applying (3.33), one obtains
\[
\lim_{t \to \pm \infty} (A(t) - zI)^{-1} = \begin{cases} (A_- - zI)^{-1}[I + D_+(z)]^{-1} & \\
(A_- - zI)^{-1} & \text{for } |\text{Im}(z)| > 0 \text{ sufficiently large,}
\end{cases}
\] for $|\text{Im}(z)| > 0$ sufficiently large, and hence also
\[
(A(t) - zI)^{-1} = (A_- - zI)^{-1}[I + B(t)(A_- - zI)^{-1}]^{-1}, \quad t \in \mathbb{R},
\] for $|\text{Im}(z)| > 0$ sufficiently large.

Next, one notes that the strong (and hence in particular the norm) limit of resolvents of self-adjoint operators is necessarily a pseudoresolvent. The latter is the resolvent of a closed, linear operator if and only if the $z$-independent nullspace of the pseudoresolvent equals $\{0\}$ (cf. [86, Sect. VIII.1.1]). Since
\[
\ker((A_- - zI)^{-1}[I + D_+(z)]^{-1}) = \{0\} \text{ for } |\text{Im}(z)| > 0 \text{ sufficiently large,}
\] one thus concludes that
\[
\lim_{t \to \pm \infty} (A(t) - zI)^{-1} = (A_\pm - zI)^{-1} \text{ for } |\text{Im}(z)| > 0 \text{ sufficiently large,}
\] for some closed, linear operator $A_\pm$ in $\mathcal{H}$. Thus, (3.35) yields
\[
(A_\pm - zI)^{-1} = (A_- - zI)^{-1} - (A_\pm - zI)^{-1}D_+(z)
\] for $|\text{Im}(z)| > 0$ sufficiently large, and hence (cf. also (3.37))
\[
(A_\pm - zI)^{-1} = (A_- - zI)^{-1}[I + D_+(z)]^{-1} \text{ for } |\text{Im}(z)| > 0 \text{ sufficiently large.}
\]
Equation (3.41) then yields

\[(A_+ - zI) = [I + D_+(z)][A_- - zI] \text{ for } |\text{Im}(z)| > 0 \text{ sufficiently large,} \tag{3.42}\]

and hence confirms that \(\text{dom}(A_+) = \text{dom}(A_-)\). Self-adjointness of \(A_+\) then follows from

\[n\lim_{t \to \infty} [(A(t) - zI)^{-1}]^* = n\lim_{t \to \infty}(A(t) - zI)^{-1} = (A_+ - zI)^{-1} = [(A_+ - zI)^{-1}]^* = (A_+^* - zI)^{-1}\]

for \(|\text{Im}(z)| > 0 \text{ sufficiently large. Having established self-adjointness of } A_\pm, \text{ an analytic continuation with respect to } z \text{ in (3.39) then yields (3.27).}\]

(iv) This immediately follows from (3.33) and (3.42), which imply

\[(A_+ - A_-)(A_- - zI)^{-1} = D_+(z) \in \mathcal{B}_1(\mathcal{H}) \text{ for } |\text{Im}(z)| > 0 \text{ sufficiently large.} \tag{3.44}\]

An analytic continuation with respect to \(z\) in (3.44) then yields (3.28).

(v) Relation (3.29) follows from (3.25) and (3.35), relation (3.30) follows from (3.33) and (3.40). Finally, (3.31) follows from (3.29) and (3.30) (in fact, replacing \(\mathcal{B}_1(\mathcal{H})\) by \(\mathcal{B}_{\infty}(\mathcal{H})\) would be sufficient for this purpose in both equations) and Weyl’s Theorem (cf., e.g., [128, Corollary XIII.4.2]).

Given Theorem 3.7 one can introduce the densely defined, symmetric (and hence closable) operator \(\dot{B}(+\infty)\) in \(\mathcal{H}\) by

\[\dot{B}(+\infty) = A_+ - A_-..., \quad \text{dom}(\dot{B}(+\infty)) = \text{dom}(A_-), \tag{3.45}\]

and its closure \(B(+\infty)\) in \(\mathcal{H}\),

\[B(+\infty) = \overline{B(+\infty)}, \quad \text{dom}(B(+\infty)) \supseteq \text{dom}(A_-). \tag{3.46}\]

In addition, and in accordance with our normalization \(D_- = 0\) in (3.13), we also introduce

\[B(-\infty) = 0, \quad \text{dom}(B(-\infty)) = \mathcal{H}. \tag{3.47}\]

By (3.14), (3.33), and \(D_+(z) = B(+\infty)(A_- - zI)^{-1}\), and recalling notation (2.20), one may thus summarize some of the properties of \(B(t), B(+\infty)\) by

\[\lim_{t \to \infty} \|B(t) - B(+\infty)(A_-^2 + I)^{-1/2}\|_{B_1(\mathcal{H})} = 0, \tag{3.48}\]

\[B(+\infty)(A_-^2 + I)^{-1/2} = \int_{\mathbb{R}} B'(s)(A_-^2 + I)^{-1/2} ds \in \mathcal{B}_1(\mathcal{H}), \tag{3.49}\]

\[B(t)(A_-^2 + I)^{-1/2} = \int_{-\infty}^t B'(s)(A_-^2 + I)^{-1/2} ds \in \mathcal{B}_1(\mathcal{H}), \quad t \in \mathbb{R}. \tag{3.50}\]

Finally, one also has

\[A_+ = A_+ + B(+\infty), \quad \text{dom}(A_+) = \text{dom}(A_-). \tag{3.51}\]

Next, we denote by \(\mathcal{H}_{t^{1/2}}(|A|)\) the domain of the operator \(|A|^{1/2}\) equipped with its graph norm. The following lemma shows that the graph norms associated with \(A(t)\) and \(|A(t)|^{1/2}\), respectively, are equivalent for different \(t\) with constants uniform with respect to \(t \in \mathbb{R}\):

**Lemma 3.8.** Assume Hypothesis 2.1. Then there are positive constants \(c_1\) and \(c_2\) such that for all \(t \in \mathbb{R}\) one has,

\[\|f\|_{\mathcal{H}_{t}(A_-)} \leq c_1\|f\|_{\mathcal{H}_{t}(A(t))} \leq c_2\|f\|_{\mathcal{H}_{t}(A_-)}, \tag{3.52}\]

\[f \in \text{dom}(A_-) = \text{dom}(A(t)), \quad \text{for all } t \in \mathbb{R}.\]
\[ \|f\|_{H_{1/2}(A_-)} \leq c_1 \|f\|_{H_{1/2}(A(t))} \leq c_2 \|f\|_{H_{1/2}(A_-)}, \quad f \in \text{dom}(\{A_-\}^{1/2}) = \text{dom}(\{A(t)\}^{1/2}). \] (3.53)

**Proof.** Since \( B(t) \) is relatively compact with respect to \( A_- \), one concludes that (cf., [141, Theorems 9.4(b), 9.7, 9.9])

\[ \text{dom}(A_-) = \text{dom}(\{A_-\}) = \text{dom}(\{A(t)\}) = \text{dom}(A(t)), \]
\[ \text{dom}(\{A_-\}^{1/2}) = \text{dom}(\{A(t)\}^{1/2}), \quad t \in \mathbb{R}. \] (3.54)

For each \( t \in \mathbb{R} \), the set \( \text{dom}(\{A(t)\}) \) is a core for \( \{A(t)\}^{1/2} \) (see, e.g., [86, Theorem V.3.24]). This implies that (3.53) follows from (3.52). The second inequality in (3.52), we will use Lemma 3.5: Fix \( z \in \mathbb{C}\setminus[0,\infty) \) such that \( \|B(t)(|A_-| - zI)^{-1}\|_{\mathcal{B}(\mathcal{H})} < 1/6 \), uniformly with respect to \( t \in \mathbb{R} \). Then, for each \( f \in \text{dom}(A_-) \),

\[ \|f\|^2_{\hat{H}_i(A_\pm)} = \|f\|^2_{\hat{H}'} + \|A(t)f - B(t)f\|^2_{\mathcal{H}} \leq \|f\|^2_{\hat{H}'} + (\|A(t)f\|_{\mathcal{H}} + \|B(t)f\|_{\mathcal{H}})^2 \]
\[ \leq \|f\|^2_{\hat{H}'} + 2\|A(t)f\|^2_{\mathcal{H}} + \|B(t)(|A_-| - zI)^{-1}(A_-^1 - zI)f\|^2_{\mathcal{H}} \]
\[ \leq \|f\|^2_{\hat{H}'} + 2\|A(t)f\|^2_{\mathcal{H}} + \frac{2}{3}(\|A_- f\|^2_{\mathcal{H}} + z^2 \|f\|^2_{\mathcal{H}}) \]
\[ \leq c(z)\|f\|^2_{\hat{H}_i(A_\pm)} + \frac{2}{3}\|f\|^2_{\hat{H}_i(A_-)}, \] (3.55)

where \( c(z) \) is independent of \( t \). \( \square \)

**Remark 3.9.** Given the operators \( \kappa(A_\pm) = ((A_\pm)^2 + I)^{1/2} \) with \( \text{dom}(\kappa(A_\pm)) = \text{dom}(\kappa(A_-)) = \text{dom}(A_-) \), one concludes that \( \kappa(A_-)^{1/2}\kappa(A_\pm)^{-1/2} \in \mathcal{B}(\mathcal{H}) \) by the closed graph theorem (cf. [86, Remark IV.1.5]). Passing to the adjoint (cf. [141, Theorem 4.19 (b)]), one infers that \( \kappa(A_\pm)^{-1/2}\kappa(A_-)^{1/2} \subseteq (\kappa(A_-)^{1/2}\kappa(A_\pm)^{-1/2})^* \in \mathcal{B}(\mathcal{H}) \) and hence

\[ \kappa(A_\pm)^{-1/2}\kappa(A_-)^{-1/2} = (\kappa(A_-)^{1/2}\kappa(A_\pm)^{-1/2})^* \in \mathcal{B}(\mathcal{H}). \] (3.56)

### 3.2. The Role of \( N \)-Measurability

We continue this section with some remarks concerning the relevance of Hypothesis 2.1(v). Let \( T \in L^2(\mathbb{R}; \mathcal{H}) \) be defined in terms of the weakly measurable family of densely defined, closed, linear operators \( T(t), t \in \mathbb{R}, \) in \( \mathcal{H} \) in analogy to (A.15), that is,

\[ (Tf)(t) = T(t)f(t) \text{ for a.e. } t \in \mathbb{R}, \]
\[ f \in \text{dom}(T) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid g(t) \in \text{dom}(T(t)) \text{ for a.e. } t \in \mathbb{R}, \right\} \]
\[ t \mapsto T(t)g(t) \text{ is (weakly) measurable, } \int_{\mathbb{R}} \|T(t)g(t)\|^2_{\mathcal{H}} dt < \infty \]. (3.57)

Then \( T \) is closed in \( L^2(\mathbb{R}; \mathcal{H}) \), but may not be densely defined. Also, it is of interest to know if \( T \) can be written as the direct integral of the operators \( T(t) \). Adding the hypothesis that the family \( \{T(t)\}_{t \in \mathbb{R}} \) is \( N \)-measurable (cf. the discussion of \( N \)-measurability in Appendix A) guarantees that \( T \) is densely defined by Theorem A.7. In particular, one then has

\[ T = \int_{\mathbb{R}}^\oplus T(t) \, dt, \quad T^* = \int_{\mathbb{R}}^\oplus T(t)^* \, dt, \quad |T| = \int_{\mathbb{R}}^\oplus |T(t)| \, dt, \] (3.58)

moreover, the remaining analogs of the direct integral formulas in Theorem A.7 (such as (A.24), (A.25)) apply to \( T \) as well.
Remark 3.10. We will show in Lemma A.10 that Hypotheses 2.1 (i)--(iv), in addition to Hypothesis 2.1 (v), imply that \{B(t)\}_{t \in \mathbb{R}} and \{B'(t)\}_{t \in \mathbb{R}} are $N$-measurable as introduced in Definition A.3 (iii) and further discussed in Remark A.4 (iv). Consequently, $B$ and $B'$, defined according to (3.57), are densely defined in $L^2(\mathbb{R}; \mathcal{H})$, and the analogs of (3.58) hold in either case by Theorem A.7.

Remark 3.11. (i) Assuming Hypothesis 2.1, the weak measurability of \{B(t)\}_{t \in \mathbb{R}} and \{B'(t)\}_{t \in \mathbb{R}} proved inLemma A.10, yield an alternative and direct proof (without relying on Theorem A.7) that $B$ and $B'$ are densely defined in $L^2(\mathbb{R}; \mathcal{H})$ as follows: Since the function $B'(\cdot)((|A|+I)^{-1}$ is weakly measurable, for each $f \in L^2(\mathbb{R}; \mathcal{H})$ with compact support, the function $B'(\cdot)((|A|+I)^{-1}f$ taking values in $L^2(\mathbb{R}; \mathcal{H})$ is weakly measurable as well. The fact that
\[
S := \{(|A_-|+I)^{-1}f \mid \text{ess supp}(f) \text{ compact}\} \text{ is dense in } L^2(\mathbb{R}; \mathcal{H}),
\]
then implies that the maximal domain of $B'$ is dense in $L^2(\mathbb{R}; \mathcal{H})$. Analogous ideas yield that the maximal domain of $B$ is dense in $L^2(\mathbb{R}; \mathcal{H})$. To see that $S$ is indeed dense in $L^2(\mathbb{R}; \mathcal{H})$, one argues as follows: Assume that there exists $f \in L^2(\mathbb{R}; \mathcal{H})$ such that $(f, g)_{L^2(\mathbb{R}; \mathcal{H})} = 0$ for every $g \in S$. Then $((|A_-|+I)^{-1}f, g)_{L^2(\mathbb{R}; \mathcal{H})} = 0$ for every $\tilde{g} \in L^2(\mathbb{R}; \mathcal{H})$ with compact support. Since the latter set is dense in $L^2(\mathbb{R}; \mathcal{H})$ one gets $((|A_-|+I)^{-1}f = 0$ a.e. Since $((|A_-|+I)^{-1}$ is injective in $\mathcal{H}$, $f = 0$ a.e., that is, the set $S$ is dense in $L^2(\mathbb{R}; \mathcal{H})$.

(ii) It is of course possible to interchange $B(t)$ by $B(t)^*$ in (2.3), and analogously, one may replace $B'(t)$ by $(B'(t))^*$ in (2.3).

Remark 3.12. We will show by means of Example A.11 that Hypothesis 2.1(v) is essential, and cannot be derived from assertions (i)--(iv) in Hypothesis 2.1; in particular, we will show that weak measurability of the family \{$(|B'(t)|^2 + I)^{-1}$\}_{t \in \mathbb{R}} does not follow from weak measurability of \{B'(t)\}_{t \in \mathbb{R}} and weak measurability of \{B'(t)((|A_-|+I)^{-1}\}_{t \in \mathbb{R}}.

Remark 3.13. In the special case where $\text{dom}(A_-)$ is a core for $B(t)$ for all $t \in \mathbb{R}$, that is,
\[
\overline{B(t)}|_{\text{dom}(A_-)} = B(t), \quad t \in \mathbb{R},
\]
an application of Lennon's [97] result (A.32) then yields $N$-measurability of the family \{B(t)\}_{t \in \mathbb{R}} and
\[
B = \int_{\mathbb{R}} B(t) dt = \int_{\mathbb{R}} \left[ B(t)(|A_-|+I)^{-1}\right](|A_-|+I)^{-1} dt = \left[ B(|A_-|+I)^{-1}\right](|A_-|+I)^{-1} = B|_{\text{dom}(A_-)}.
\]
Using (2.2) and (3.15), one concludes that (cf. (3.63) below)
\[
\|B(|A_-|+zI)^{-1}\|_{B(L^2(\mathbb{R}; \mathcal{H}))} = \sup_{t \in \mathbb{R}} \|B(t)(|A_-|+zI)^{-1}\|_{B(\mathcal{H})} < \infty.
\]

Remark 3.14. In the particular case where $T(t) \in \mathcal{B}(\mathcal{H})$, $t \in \mathbb{R}$, and $T(\cdot) \in L^\infty(\mathbb{R}; \mathcal{B}(\mathcal{H}))$, the operator $T$ defined in (3.57) is bounded in $L^2(\mathbb{R}; \mathcal{H})$ and
\[
\|T\|_{B(L^2(\mathbb{R}; \mathcal{H}))} = \sup_{t \in \mathbb{R}} \|T(t)\|_{B(\mathcal{H})}.
\]
3.3. Some Multi-Dimensional PDE Examples. We conclude this section with two elementary examples illustrating the feasibility of Hypothesis 2.1.

Example 3.15. Let \( n \in \mathbb{N}, \ p > n, \ q \in ((n/2), p - (n/2)), \) and \( \varepsilon > 0. \) Consider

\[
0 \leq V_- \in L^2(\mathbb{R}^n; (1 + |x|^2)^q dx) \cap L^\infty(\mathbb{R}^n; d^n x), \quad (3.64)
\]

\[
0 \leq V(t, \cdot) \in L^2(\mathbb{R}^n; (1 + |x|^2)^q d^n x) \cap L^\infty(\mathbb{R}^n; d^n x), \quad t \in \mathbb{R}, \quad (3.65)
\]

and suppose in addition that

\[
\partial_t V(t, \cdot) \in L^2(\mathbb{R}^n; (1 + |x|^2)^q d^n x) \cap L^\infty(\mathbb{R}^n; d^n x), \quad t \in \mathbb{R}, \quad (3.66)
\]

\[
\mathbb{R} \ni t \mapsto V(t, \cdot) \in C^1(\mathbb{R}; L^\infty(\mathbb{R}^n; d^n x)). \quad (3.67)
\]

Denoting the operator of multiplication by \( V_-, V, \) and \( \partial_t V \) in \( L^2(\mathbb{R}^n; d^n x) \) by the same symbol, respectively, we introduce the linear operators

\[
A_- = (-\Delta)^{p/2} + V_- + \varepsilon I, \quad \text{dom}(A_-) = \text{dom} ((-\Delta)^{p/2}),
\]

\[
B(t) = V(t, \cdot) - V_-, \quad \text{dom}(B(t)) = L^2(\mathbb{R}^n; d^n x), \quad t \in \mathbb{R},
\]

\[
A(t) = A_- + B(t), \quad \text{dom}(A(t)) = \text{dom}(A_-), \quad t \in \mathbb{R},
\]

in \( L^2(\mathbb{R}^n; d^n x), \) with \(-\Delta\) abbreviating the self-adjoint Laplacian in \( L^2(\mathbb{R}^n; d^n x)\) whose graph domain equals the usual Sobolev space \( W^{2,2}(\mathbb{R}^n)\).

Repeatedly applying [135, Corollary 4.8], one verifies that all assumptions in Hypothesis 2.1 are satisfied. Specifically, since

\[
(\varepsilon^2 + 1)^{-p/2} \in L^2(\mathbb{R}^n; (1 + |k|^2)^q dk),
\]

\[
V_-, V(t, \cdot) \in L^2(\mathbb{R}^n; (1 + |x|^2)^q d^n x), \quad t \in \mathbb{R},
\]

[135, Corollary 4.8] implies that

\[
V_-((-\Delta)^{p/2} + I)^{-1}, V(t, \cdot)((-\Delta)^{p/2} + I)^{-1} \in \mathcal{B}_1(L^2(\mathbb{R}^n; d^n x)), \quad t \in \mathbb{R}. \quad (3.72)
\]

In addition, one has

\[
\sigma(A(t)) = \sigma(A_-) = [\varepsilon, \infty), \quad t \in \mathbb{R}. \quad (3.73)
\]

Indeed, to show (3.73), one recalls that \( V \geq 0 \) and \( V_- \geq 0, \) and since both operators are relatively compact (in fact, relatively trace class) with respect to \((-\Delta)^{p/2}\) by (3.72), and hence also with respect to \( A_- \) and \( A(t) \) (cf. (3.68), (3.70)), one obtains

\[
\sigma(A_-) \subseteq [\varepsilon, \infty), \quad \sigma(A(t)) \subseteq [\varepsilon, \infty), \quad t \in \mathbb{R},
\]

\[
\sigma_{\text{ess}}(A_-) = \sigma_{\text{ess}}(A(t)) = [\varepsilon, \infty), \quad t \in \mathbb{R},
\]

implying (3.73).

We note that \( L^2(\mathbb{R}^n; (1 + |x|^2)^q d^n x), q > (n/2), \) in Example 3.15 can be replaced by the Birman–Solomyak space \( \ell^1(L^2(\mathbb{R}^n)) \) (cf., e.g., [135, Chapter 4]). In addition, the \( L^\infty - \)assumptions in Example 3.15 can be replaced by appropriate relatively boundedness assumptions with respect to \( A_- \), but we omit further details in the interest of simplicity.

A similar example, removing the positivity property of \( A(t) \) in Example 3.15, can be constructed as follows:
Example 3.16. Let \( n \in \mathbb{N}, p > n, q \in ((n/2), p - (n/2)) \), and \( \varepsilon > 0 \). Consider the self-adjoint \( 2 \times 2 \) matrices \( V_{-} = (V_{-,j,k})_{1 \leq j,k \leq 2}, V(t, \cdot) = (V(t, \cdot)_{j,k})_{1 \leq j,k \leq 2} \), with
\[
V_{-} = L^2(\mathbb{R}^n; (1 + |x|^2)^q d^n x) \cap L^\infty(\mathbb{R}^n; d^n x), \quad 1 \leq j,k \leq 2,
\]
and suppose in addition that
\[
V(t, \cdot)_{j,k} \in L^2(\mathbb{R}^n; (1 + |x|^2)^q d^n x) \cap L^\infty(\mathbb{R}^n; d^n x), \quad t \in \mathbb{R}, \quad 1 \leq j,k \leq 2.
\]
and suppose in addition that
\[
\partial_t V(t, \cdot)_{j,k} \in L^2(\mathbb{R}^n; (1 + |x|^2)^q d^n x) \cap L^\infty(\mathbb{R}^n; d^n x), \quad t \in \mathbb{R}, \quad 1 \leq j,k \leq 2.
\]

Next, we introduce the linear operators
\[
A_{-} = \begin{pmatrix}
(-\Delta)^{p/2} + \varepsilon I + V_{-1,1} & V_{-1,2} \\
V_{-2,1} & (-\Delta)^{p/2} - \varepsilon I + V_{-2,2}
\end{pmatrix},
\]
\[
\text{dom}(A_{-}) = \text{dom}((-\Delta)^{p/2}) \oplus \text{dom}((-\Delta)^{p/2}),
\]
\[
B(t) = V(t, \cdot) - V_{-}, \quad \text{dom}(B(t)) = L^2(\mathbb{R}^n; d^n x) \oplus L^2(\mathbb{R}^n; d^n x), \quad t \in \mathbb{R},
\]
\[
A(t) = A_{-} + B(t), \quad \text{dom}(A(t)) = \text{dom}(A_{-}), \quad t \in \mathbb{R},
\]
in \( L^2(\mathbb{R}^n; d^n x) \oplus L^2(\mathbb{R}^n; d^n x) \). Then
\[
\sigma_{\text{ess}}(A(t)) = \sigma_{\text{ess}}(A_{-}) = (-\infty, -\varepsilon] \cup [\varepsilon, \infty), \quad t \in \mathbb{R},
\]
and repeatedly applying [135, Corollary 4.8] one again verifies that all assumptions in Hypothesis 2.1 are satisfied. In the particular case where
\[
V_{-1,2} = V_{-2,1} = 0, \quad V_{-1,1} \geq 0, \quad V_{-2,2} \leq 0,
\]
\[
V_{1,2}(t, \cdot) = V_{2,1}(t, \cdot) = 0, \quad V_{1,1}(t, \cdot) \geq 0, \quad V_{2,2}(t, \cdot) \leq 0, \quad t \in \mathbb{R},
\]
then also
\[
\sigma(A(t)) = \sigma(A_{-}) = (-\infty, -\varepsilon] \cup [\varepsilon, \infty), \quad t \in \mathbb{R},
\]
holds as in the proof of (3.73).

Employing the norm resolvent convergence as \( t \to +\infty \) in (3.27) then shows that \( A_{+} \), constructed according to Theorem 3.7, also satisfies (3.73) and (3.86) (cf., e.g., [126, Sect. VIII.7]).

4. Preliminaries in Connection with the Trace Formula

In this section we collected some preliminary results used in the proof of Propositions 2.4 and 2.5.

The following interpolation result (and others) have been proved in [64]. They extend results originally discussed by Lech [98]:

**Theorem 4.1** ([64]). Let \( \mathcal{H} \) be a separable Hilbert space and \( T \) a self-adjoint operator with \( T^{-1} \in \mathcal{B}(\mathcal{H}) \). Assume that \( S \) is closed and densely defined in \( \mathcal{H} \), with \( \text{dom}(S) \cap \text{dom}(S^*) \supseteq \text{dom}(T) \), implying \( ST^{-1} \in \mathcal{B}(\mathcal{H}) \) and \( S^*T^{-1} \in \mathcal{B}(\mathcal{H}) \). If, in addition, \( ST^{-1} \in \mathcal{B}_1(\mathcal{H}) \) and \( S^*T^{-1} \in \mathcal{B}_1(\mathcal{H}) \), then
\[
T^{-1/2}ST^{-1/2} \in \mathcal{B}_1(\mathcal{H}), \quad (T^{-1/2}ST^{-1/2})^* = T^{-1/2}S^*T^{-1/2} \in \mathcal{B}_1(\mathcal{H}).
\]

Moreover,
\[
\|T^{-1/2}ST^{-1/2}\|_{\mathcal{B}_1(\mathcal{H})} = \|T^{-1/2}S^*T^{-1/2}\|_{\mathcal{B}_1(\mathcal{H})} \leq \|ST^{-1}\|_{\mathcal{B}_1(\mathcal{H})}^{1/2} \|S^*T^{-1}\|_{\mathcal{B}_1(\mathcal{H})}^{1/2}.
\]

(4.1)
Next, we study properties of the operator $D_A$ defined in (2.10) starting with the constant coefficient case $A(t) = A_-$, $t \in \mathbb{R}$. We recall that the operator of differentiation $d/dt$ in $L^2(\mathbb{R}; \mathcal{H})$, defined in (2.11), is closed, and the graph norm on $\text{dom}(d/dt)$ is equivalent to the norm in $W^{1,2}(\mathbb{R}; \mathcal{H})$, where $W^{1,2}(\cdot)$ denotes the usual Sobolev space of $\mathbb{R}$-functions with the first distributional derivative in $L^2(\mathbb{R}; \mathcal{H})$. We note that $(d/dt)^* = -(d/dt)$ which will be used in (4.5). For a self-adjoint operator $A_-$ in $\mathcal{H}$ on $\text{dom}(A_-) \subseteq \mathcal{H}$, the operator $A_-$, defined by (2.13), is closed in $L^2(\mathbb{R}; \mathcal{H})$ since $A_-$ is closed in $\mathcal{H}$. In addition, the graph norm $\| \cdot \|_{\mathcal{H}_1(A_-)}$ on $\text{dom}(A_-)$ is equivalent to the norm in $L^2(\mathbb{R}; \mathcal{H}_1(A_-))$ since

$$\|f\|_{\mathcal{H}_1(A_-)}^2 = \|A_- f\|_{L^2(\mathbb{R}; \mathcal{H})}^2 + \|f\|_{L^2(\mathbb{R}; \mathcal{H})}^2 = \int_{\mathbb{R}} \left[ \|A_- f(t)\|_{\mathcal{H}}^2 + \|f(t)\|_{\mathcal{H}}^2 \right] dt$$

(4.3)

To prove this assertion, we introduce the strongly right continuous families of spectral projections $E_A(\lambda) = E_A((\infty, \lambda])$ and $E_B(\lambda) = E_B((\infty, \lambda])$, $\lambda \in \mathbb{R}$, of the operators $A$ and $B$, respectively. Since by hypothesis the resolvents of $A$

\begin{align*}
D_{A_-} &= \frac{d}{dt} + A_-, \\
\text{dom}(D_{A_-}) &= \text{dom}(d/dt) \cap \text{dom}(A_-). \\
\text{(i)} & \quad \text{The graph norm } \| \cdot \|_{\mathcal{H}_1(D_{A_-})} \text{ on } \text{dom}(D_{A_-}) \text{ is equivalent to the norm on } W^{1,2}(\mathbb{R}; \mathcal{H}) \cap L^2(\mathbb{R}; \mathcal{H}_1(A_-)) \text{ defined as the maximum of the norms in } W^{1,2}(\mathbb{R}; \mathcal{H}) \text{ and } L^2(\mathbb{R}; \mathcal{H}_1(A_-)); \text{ consequently, the operator } D_{A_-} \text{ is closed.}
\end{align*}

(ii) The adjoint $D_{A_-}^*$ of the operator $D_{A_-}$ in $L^2(\mathbb{R}; \mathcal{H})$ is given by

$$D_{A_-}^* = -\frac{d}{dt} + A_-, \quad \text{dom}(D_{A_-}^*) = \text{dom}(d/dt) \cap \text{dom}(A_-) = \text{dom}(D_{A_-}).$$

(4.5)

(iii) The operator $D_{A_-}$ is a normal operator in $L^2(\mathbb{R}; \mathcal{H})$.

(iv) The spectra of the operators $D_{A_-}$ in $L^2(\mathbb{R}; \mathcal{H})$ and $A_-$ in $\mathcal{H}$ satisfy:

$$\sigma(D_{A_-}) = \sigma(A_-) + i \mathbb{R}.$$

(4.6)

Proof. As we will see, the lemma follows by letting $A = A_-$ and $B = (-id/dt)$ in the next assertion (cf. [57, Ex. XII.9.11, p.1259], [141, Ex. 7.48]).

Assertion. Suppose that $A$ and $B$ are two resolvent commuting self-adjoint operators in a complex, separable Hilbert space $\mathcal{K}$, and define the operators $C$ and $C'$ by

$$C = A + iB, \quad C' = A - iB, \quad \text{dom}(C) = \text{dom}(C') = \text{dom}(A) \cap \text{dom}(B).$$

(4.7)

Then

$$\|Ch\|^2_K = \|Ah\|^2_K + \|Bh\|^2_K, \quad h \in \text{dom}(C),$$

(4.8)

$$\|C'h\|^2_K = \|Ah\|^2_K + \|Bh\|^2_K, \quad h \in \text{dom}(C'),$$

(4.9)

the operator $C$ is normal, $C^* = C'$, and

$$\rho(A) + i \mathbb{R} \subseteq \rho(C).$$

(4.10)

To prove this assertion, we introduce the strongly right continuous families of spectral projections $E_A(\lambda) = E_A((\infty, \lambda])$ and $E_B(\lambda) = E_B((\infty, \lambda])$, $\lambda \in \mathbb{R}$, of the operators $A$ and $B$, respectively. Since by hypothesis the resolvents of $A$
and \(B\) commute, the spectral projections also commute, that is, \(E_A(\lambda)E_B(\mu) = E_B(\mu)E_A(\lambda), \lambda, \mu \in \mathbb{R},\) and
\[
E_A(\lambda)A \subseteq AE_A(\lambda), \quad E_A(\lambda)B \subseteq BE_A(\lambda),
\]
\[
E_B(\lambda)B \subseteq BE_B(\lambda), \quad E_B(\lambda)A \subseteq AE_B(\lambda), \quad \lambda \in \mathbb{R}.
\]
(4.11)

It follows from (4.11) that \(C\) and \(C'\) are densely defined, closable operators in \(\mathcal{K}\) and
\[
C' \subseteq C^*, \quad C \subseteq C^*.
\]
(4.12)

Next, we define
\[
Q_n = E_A([-n,n])E_B([-n,n])E_A([-n,n]), \quad n \in \mathbb{N},
\]
so that \(\lim_{n \to \infty} \|Q_nh - h\| \to 0\) for each \(h \in \mathcal{K}\) and, in addition,
\[
Q_nC \subseteq CQ_n, \quad Q_nC^* \subseteq C^*Q_n, \quad n \in \mathbb{N},
\]
(4.13)
\[
ABQ_n = BAQ_n, \quad n \in \mathbb{N}.
\]
(4.14)

Let \(h \in \text{dom}(C')\) and denote \(h_n = Q_nh, \ n \in \mathbb{N}.\) Then (4.15) yields
\[
\|C'h\|_{\mathcal{K}}^2 = \lim_{n \to \infty} \left( \|Ah_n - iBh_n, Ah - iBh\|_{\mathcal{K}} \right)
\]
\[
= \lim_{n \to \infty} \left[ \langle Ah,n, Ah \rangle_{\mathcal{K}} + \langle Bh,n, Bh \rangle_{\mathcal{K}} + i\langle Ah,n, Bh \rangle_{\mathcal{K}} - i\langle Bh,n, Ah \rangle_{\mathcal{K}} \right]
\]
\[
= \lim_{n \to \infty} \left[ \langle Ah,n, Ah \rangle_{\mathcal{K}} + \langle Bh,n, Bh \rangle_{\mathcal{K}} \right] = \|Ah\|_{\mathcal{K}}^2 + \|Bh\|_{\mathcal{K}}^2,
\]
(4.16)
proving (4.9); the proof of (4.8) is similar. By (4.8), the graph norm of \(C\) is equivalent to the norm \(\max \left\{ \|h\|_{\mathcal{H}_1(A)}, \|h\|_{\mathcal{H}_1(B)} \right\}\) on \(\text{dom}(A) \cap \text{dom}(B).\) Since the latter space is complete, \(C\) is closed; similarly, \(C'\) is closed. Next, let \(h \in \text{dom}(C^*).\)

By (4.14), we have
\[
\lim_{n \to \infty} C'h_n = \lim_{n \to \infty} C^*h_n = C^*h.
\]
(4.17)

Since \(C'\) is closed, one concludes that \(h \in \text{dom}(C')\) and \(C'h = C^*h,\) which, together with (4.12), implies that \(C' = C^*:\)

\[
\|C'h\|_{\mathcal{K}} = \|C^*h\|_{\mathcal{K}}, \quad h \in \text{dom}(C) = \text{dom}(C^*),
\]
(4.18)
due to (4.9) and (4.8), the normality of \(C\) follows by [141, Section 5.6]. Finally, to prove (4.10), let us fix a \(\mu + i\nu \in \rho(A) + i\mathbb{R}\) and apply (4.8) with \(A\) and \(B\) replaced by \(A - \mu I_K\) and \(B - \nu I_K,\) respectively. Since the operator \(A - \mu I_K\) is uniformly bounded from below, for some \(c > 0,\)
\[
\|C - (\mu + i\nu)I_K\|_{\mathcal{K}}^2 = \|(A - \mu I_K)h\|_{\mathcal{K}}^2 + \|(B - \nu I_K)h\|_{\mathcal{K}}^2 \geq \|(A - \mu I_K)h\|_{\mathcal{K}}^2
\]
\[
\geq c\|h\|_{\mathcal{K}}^2, \quad h \in \text{dom}(C),
\]
(4.19)
proving that the operator \(C - (\mu + i\nu)I_K\) is uniformly bounded from below. Using (4.9), a similar argument for \(C^*\) completes the proof of the inclusion \((\mu + i\nu) \in \rho(C),\) thus finishing the proof of the assertion.

Returning to the proof of Lemma 4.2, we remark that items (i), (ii), (iii) follow directly from the assertion just proved (with \(A = A_-, B = (-id/dt),\) and \(C = D_{A_-}\)). In particular, the equivalence of the norms in item (i) follows from (4.8),
\[
\|D_{A_-}f\|_{L^2(\mathbb{R};H)}^2 = \|f\|_{L^2(\mathbb{R};H)}^2 + \|A_-f\|_{L^2(\mathbb{R};H)}^2, \quad f \in \text{dom}(D_{A_-}).
\]
(4.20)
which, in turn, for each \(f \in \text{dom}(D_{A_-}) = \text{dom}(d/dt) \cap \text{dom}(A_-)\) yields
\[
\|f\|_{W^{1,2}(\mathbb{R};H),L^2(\mathbb{R};H_1(A_-))}^2 = \max \left\{ \|f\|_{H_1(A_-)}^2, \|f\|_{L^2(\mathbb{R};H_1(A_-))}^2 \right\}
\]
\[
\|f\|_{W^{1,2}(\mathbb{R};H),L^2(\mathbb{R};H_1(A_-))}^2 = \max \left\{ \|f\|_{H_1(A_-)}^2, \|f\|_{L^2(\mathbb{R};H_1(A_-))}^2 \right\}
\]
\[
\|f\|_{W^{1,2}(\mathbb{R};H),L^2(\mathbb{R};H_1(A_-))}^2 = \max \left\{ \|f\|_{H_1(A_-)}^2, \|f\|_{L^2(\mathbb{R};H_1(A_-))}^2 \right\}
\]
Applying (4.24) with \( h \) we fix any \( f \) with \( f(0) = 0 \). Then using (4.26), we arrive at the inequality
\[
\frac{2}{\mu} \max \left[ \| f \|_{L^2(R;H)}^2 + \| f' \|_{L^2(R;H)}^2, \| A_0 f \|^2_{L^2(R;H)} \right] \\
\leq \| f \|_{L^2(R;H)}^2 + \| f' \|_{L^2(R;H)}^2 + \| A_0 f \|^2_{L^2(R;H)} \\
\leq 2 \max \left[ \| f \|_{L^2(R;H)}^2 + \| f' \|_{L^2(R;H)}^2, \| f \|_{L^2(R;H)} + \| A_0 f \|^2_{L^2(R;H)} \right]
\]
(4.21)

since the term in (4.21) is equal to \( \| f \|_{L^2(R;H)}^2 + \| D_{A_0} f \|_{L^2(R;H)}^2 = \| f \|_{H^1(D_{A_0})}^2 \).

Therefore, \( \text{dom}(D_{A_0}) \) with the graph norm is a complete space, and thus \( D_{A_0} \) is closed.

To finish the proof of item (iv), it remains to show that \( (\mu + i\nu) \in \rho(D_{A_0}) \) implies \( \mu \in \rho(A_0) \). As in the proof of [48, Theorem 3.13], one considers the unitary operator of multiplication \( M \) in \( L^2(R;H) \) by the scalar function \( m(t) = e^{-i\nu t} \), that is,
\[
(Mf)(t) = e^{-i\nu t} f(t), \quad f \in L^2(R;H).
\]
(4.22)

In addition, \( (Mf)(t) = e^{-i\nu t} f(t), \quad f \in L^2(R;H). \)

If \( f \in \text{dom}(D_{A_0}) \) then \( Mf \in \text{dom}(D_{A_0}) \) and \( D_{A_0} Mf = M(-i\nu I + D_{A_0})f \).

Thus, \( \rho(D_{A_0}) = \rho(-i\nu I + D_{A_0}) \), and therefore \( (\mu + i\nu) \in \rho(D_{A_0}) \) implies \( \mu \in \rho(D_{A_0}) \). Similarly to (4.19), for some \( c > 0 \),
\[
c\| f \|_{L^2(R;H)}^2 \leq \| (D_{A_0} - \mu I)f \|_{L^2(R;H)}^2 = \| (A_0 - \mu I)f \|_{L^2(R;H)}^2 + \| f' \|_{L^2(R;H)}^2,
\]
(4.24)

for each \( k \in \mathbb{N} \), we choose a smooth function \( \chi_k : \mathbb{R} \rightarrow [0, 1] \) such that
\[
\chi_k(t) = \begin{cases} 
1, & |t| \leq k, \\
0, & |t| \geq k + 1, \\
|\chi'_k(t)| \leq 2, & t \in \mathbb{R}.
\end{cases}
\]
(4.25)

We fix any \( h \in \text{dom}(A_0) \) and denote \( f_k(t) = \chi_k(t)h, t \in \mathbb{R} \). Then \( f_k \in \text{dom}(D_{A_0}) \) with \( f_k(t) = \chi_k(t)h \) and \( (A_0 - f_k)(t) = \chi_k(t)A_0 h \). In addition,
\[
\frac{\| f_k \|_{L^2(R;H)}^2}{\| f_k \|_{L^2(R;H)}} = \frac{\| \chi_k \|_{L^2(R;H)}}{\| \chi_k \|_{L^2(R;H)}} \quad \text{as} \quad k \rightarrow \infty.
\]
(4.26)

Applying (4.24) with \( f \) replaced by \( f_k \) yields:
\[
c\| h \|^2_{H} \| \chi_k \|^2_{L^2(R;dt)} \leq \| (A_0 - \mu I)h \|^2_{H} \| \chi_k \|^2_{L^2(R;dt)} + \| h \|^2_{H} \| \chi'_k \|^2_{L^2(R;dt)}.
\]
(4.27)

Using (4.26), we arrive at the inequality \( c\| h \|^2_{H} \leq \| (A_0 - \mu I)h \|^2_{H} \), thus proving \( \mu \in \rho(A_0) \). ■

Remark 4.3. (i) Lemma 4.2 (iv) shows that if \( A_0 \) (and hence \( A_0 \)) has a spectral gap at 0, then \( D_{A_0}^{-1} \in B(L^2(R;H)) \) and thus \( D_{A_0} \) is a Fredholm operator of index zero.

(ii) One notes the peculiar fact that if \( \sigma(A_0) = \mathbb{R} \), then \( D_{A_0} \) has empty resolvent set, or equivalently, \( \sigma(D_{A_0}) = \mathbb{C} \).

For an alternative proof of Lemma 4.2 (iii) using the notion of \( N \)-measurability we refer to Lemma A.12.

Throughout the remaining part of this section, we continue to assume Hypothesis 2.1.
Using (4.34), Remark 3.14, and the spectral theorem for \( A \), \( t \in \mathbb{R} \), in \( \mathcal{H} \), defined by
\[
(\mathbf{A}f)(t) = A(t)f(t) \text{ for a.e. } t \in \mathbb{R},
\]
f \( \in \text{dom}(\mathbf{A}) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid g(t) \in \text{dom}(A(t)) \text{ for a.e. } t \in \mathbb{R},
\right. 

t \mapsto A(t)g(t) \text{ is (weakly) measurable, } \int_\mathbb{R} \|A(t)f(t)\|^2 d t < \infty \right\}. \tag{4.28}
\]

Next, we define in \( L^2(\mathbb{R}; \mathcal{H}) \) the operator
\[
\mathbf{D}_A = \frac{d}{dt} + A, \quad \text{dom}(\mathbf{D}_A) = \text{dom}(d/dt) \cap \text{dom}(\mathbf{A}), \tag{4.29}
\]
as the operator sum of \( d/dt \) and \( A \).

Assuming Hypothesis 2.1, we next prove that \( \mathbf{D}_A \) is a densely defined and closed operator in \( L^2(\mathbb{R}; \mathcal{H}) \), and that the domain of \( \mathbf{D}_A \) actually coincides with that of \( \mathbf{D}_{A_-} \) in (4.4).

**Lemma 4.4.** Assume Hypothesis 2.1. Then \( \mathbf{D}_A \) as defined in (4.29) is a densely defined and closed operator in \( L^2(\mathbb{R}; \mathcal{H}) \) and
\[
\text{dom}(\mathbf{D}_A) = \text{dom}(\mathbf{D}_A^*) = \text{dom}(\mathbf{D}_{A_-}) = \text{dom}(\mathbf{A}_-) \tag{4.30}
\]
Moreover, the adjoint operator \( \mathbf{D}_A^* \) of \( \mathbf{D}_A \) in \( L^2(\mathbb{R}; \mathcal{H}) \) is given by
\[
\mathbf{D}_A^* = -\frac{d}{dt} + A, \quad \text{dom}(\mathbf{D}_A^*) = \text{dom}(d/dt) \cap \text{dom}(\mathbf{A}) = \text{dom}(d/dt) \cap \text{dom}(\mathbf{A}_-). \tag{4.31}
\]
In addition, the graph norm \( \| \cdot \|_{H^1(\mathcal{H})} \) on \( \text{dom}(\mathbf{D}_A) \) is equivalent to the norm on \( W^{1,2}(\mathbb{R}; \mathcal{H}) \cap L^2(\mathbb{R}; \mathcal{H}_1(A_-)) \) defined as the maximum of the norms in \( W^{1,2}(\mathbb{R}; \mathcal{H}) \) and \( L^2(\mathbb{R}; \mathcal{H}_1(A_-)) \).

**Proof.** Since \( \mathbf{D}_{A_-} \) may have an empty resolvent set, we will focus on the self-adjoint operator \( |\mathbf{D}_{A_-}| \) at first. Consider the unitary vector-valued Fourier transform
\[
\hat{\mathcal{F}}_{\mathcal{H}} : L^2(\mathbb{R}; \mathcal{H}) \to L^2(\mathbb{R}; \mathcal{H}), \tag{4.32}
\]
first defined by
\[
F \mapsto \hat{F}, \quad \hat{F}(\lambda) = (2\pi)^{-1/2} \int_\mathbb{R} e^{-i\lambda s} F(s) \, ds, \quad \lambda \in \mathbb{R}, \tag{4.33}
\]
for all \( F \in S(\mathbb{R}; \mathcal{H}) \), the \( \mathcal{H} \)-valued Schwartz class, and then extended to a unitary operator in \( L^2(\mathbb{R}; \mathcal{H}) \) by taking the closure (see, e.g., [75, Lemma 2], [101, p. 16]).

Via the Fourier transform \( \hat{\mathcal{F}}_{\mathcal{H}} \), the operator \( |\mathbf{D}_{A_-}| \) is unitarily equivalent to the operator \( |it I + A_-| \) in the space \( L^2(\mathbb{R}; \mathcal{H}) \) with domain
\[
\text{dom}(|it I + A_-|) = \text{dom}(|it I + A_-|) = \text{dom}(it I) \cap \text{dom}(A_--). \tag{4.34}
\]
Using (4.34), Remark 3.14, and the spectral theorem for \( A_- \), one obtains
\[
\begin{align*}
\left| |A_-| - z I \right| & \left( \left| \mathbf{D}_{A_-} \right| - z I \right)^{-1} \right|_{B(L^2(\mathbb{R}; \mathcal{H}))} \\
& = \sup_{t \in \mathbb{R}} \left| \left( |A_-| - z I \right) \left( -it I + A_- | - z I \right)^{-1} \right|_{B(\mathcal{H})} \\
& = \sup_{t \in \mathbb{R}} \sup_{\lambda \in \sigma(A_-)} \left| \frac{|\lambda| - z}{(t^2 + \lambda^2)^{1/2} - z} \right| = 1, \quad z < 0. \tag{4.35}
\end{align*}
\]
This in turn implies (still assuming \( z < 0 \),
\[
\|B(D_{A_{-}} - z I)^{-1}\|_{B(L^2(\mathbb{R}; \mathcal{H}))} \\
= \|B(A_{-} - z I)^{-1}(A_{-} - z I)(D_{A_{-}} - z I)^{-1}\|_{B(L^2(\mathbb{R}; \mathcal{H}))} \\
\leq \|B(A_{-} - z I)^{-1}\|_{B(L^2(\mathbb{R}; \mathcal{H}))} \|((A_{-} - z I))(D_{A_{-}} - z I)^{-1}\|_{B(L^2(\mathbb{R}; \mathcal{H}))} \\
= \sup_{t \in \mathbb{R}} \|B(t)(A_{-} - z I)^{-1}\|_{B(\mathcal{H})} = o(1),
\]
by (3.16). Put differently, (4.36) implies the existence of \( \varepsilon(z) > 0 \) with \( \varepsilon(z) \equiv o(1) \) and \( \eta(z) > 0 \), such that the Kato–Rellich-type bound
\[
\|Bf\|_{L^2(\mathbb{R}; \mathcal{H})} \leq \varepsilon(z)\|D_{A_{-}}f\|_{L^2(\mathbb{R}; \mathcal{H})} + \eta(z)\|f\|_{L^2(\mathbb{R}; \mathcal{H})},
\]
holds. Next, one recalls that the polar decomposition of a densely defined, closed, linear operator \( T \) in a complex Hilbert space \( \mathcal{K} \) is of the form \( T = U_T|T| \), with \( U_T \) (and hence \( U_T^* \)) a partial isometry in \( \mathcal{K} \), implying \( |T| = U_T^*T \). Applying the latter fact to \( D_{A_{-}} \) in (4.37), one finally obtains
\[
\|Bf\|_{L^2(\mathbb{R}; \mathcal{H})} \leq \varepsilon(z)\|D_{A_{-}}f\|_{L^2(\mathbb{R}; \mathcal{H})} + \eta(z)\|f\|_{L^2(\mathbb{R}; \mathcal{H})},
\]
(4.38)
Thus, \( B \) is relatively bounded with respect to \( D_{A_{-}} \) in \( L^2(\mathbb{R}; \mathcal{H}) \) with relative bound zero (cf. [86, Sect. 4.1.1]). Since \( D_{A_{-}} \) is a closed operator in \( L^2(\mathbb{R}; \mathcal{H}) \) by Lemma 4.2 (i), also \( D_{A} = D_{A_{-}} + B \) defined on \( \text{dom}(D_{A_{-}}) \) is closed in \( L^2(\mathbb{R}; \mathcal{H}) \).

To prove that \( \text{dom}(D_{A}^*) = \text{dom}(D_{A}) \) one can argue as follows: Since \( B \) is symmetric on \( \text{dom}(D_{A_{-}}) \) and the operator \( D_{A_{-}} \) is normal, and hence \( \text{dom}(D_{A_{-}}^*) = \text{dom}(D_{A_{-}}) \), one obtains that
\[
\|B^*f\|_{L^2(\mathbb{R}; \mathcal{H})} = \|Bf\|_{L^2(\mathbb{R}; \mathcal{H})}, \quad f \in \text{dom}(D_{A_{-}}),
\]
(4.39)
and that
\[
\|D_{A}f\|_{L^2(\mathbb{R}; \mathcal{H})} = \|D_{A_{-}}^*f\|_{L^2(\mathbb{R}; \mathcal{H})}, \quad f \in \text{dom}(D_{A_{-}}).
\]
(4.40)
Therefore, (4.38) can be rewritten as
\[
\|B^*f\|_{L^2(\mathbb{R}; \mathcal{H})} \leq \varepsilon(z)\|D_{A_{-}}^*f\|_{L^2(\mathbb{R}; \mathcal{H})} + \eta(z)\|f\|_{L^2(\mathbb{R}; \mathcal{H})},
\]
(4.41)
implies that also \( B^* \) is relatively bounded with respect to \( D_{A_{-}}^* \) with relative bound zero. By the Hess–Kato result [80] (see also [141, p. 111]),
\[
\text{dom}(D_{A}^*) = \text{dom}(D_{A_{-}}^*) \cap \text{dom}(B^*) = \text{dom}(D_{A_{-}}) = \text{dom}(D_{A})
\]
(4.42)
and
\[
D_{A}^* = D_{A_{-}}^* + B^* = D_{A_{-}}^* + B = -\frac{d}{dt} + A_{-} + B = -\frac{d}{dt} + A,
\]
(4.43)
\[
\text{dom}(D_{A}^*) = \text{dom}(D_{A}) = \text{dom}(D_{A_{-}}).
\]
Here we used again that \( B^*f = Bf \) for all \( f \in \text{dom}(D_{A_{-}}) \).

The statements about graph norms have been proved in Lemma 4.2. ■
Next, we will discuss some operators needed in the proof of Proposition 2.4. We start with the operator $H_0$ in $L^2(\mathbb{R}; \mathcal{H})$ defined by

$$H_0 = \Delta^*_A - \Delta_A = D_A^* D_A$$  \hspace{1cm} (4.44)

(cf. Lemma 4.2 (iii)). In particular, $H_0$ is self-adjoint since $D_A$ is closed, and $H_0 \geq 0$. In addition, one obtains that

$$\text{dom}(H_0^{1/2}) = \text{dom}(D_A) = \text{dom}(D_A^*) = \text{dom}(d/dt) \cap \text{dom}(A_0).$$  \hspace{1cm} (4.45)

We will use the following representation for the resolvent of $H_0$,

$$R_0(z) = (H_0 - z I)^{-1} = \frac{1}{2}(A_0^2 - z I)^{-1/2} \tilde{K}_0(z), \quad z \in \mathbb{C} \setminus [0, \infty),$$  \hspace{1cm} (4.46)

where $\tilde{K}_0(z)$ denotes the operator of convolution with $e^{-\langle A_0^2 - z I \rangle^{1/2} |t|}$ on $L^2(\mathbb{R}; \mathcal{H})$, that is, $R_0(z)$ is an integral operator with the operator-valued integral kernel

$$R_0(z, s, t) = \frac{1}{2} \kappa_z(A_0^{-1})^{-1} e^{-\kappa_z(A_0^{-1}) |t-s|} \in \mathcal{B}(\mathcal{H}), \quad s, t \in \mathbb{R}. $$  \hspace{1cm} (4.47)

Here we used the notation $\kappa_z(A_0^{-1}) = (A_0^2 - z I)^{1/2}$ in (2.21). In the scalar-valued context formula (4.47) can be found, for instance, in [132, Theorem 9.5.2].

For subsequent purpose we also recall the integral kernel $R^{1/2}_0(z, s, t) = R_0(z)^{1/2}$,

$$R^{1/2}_0(z, s, t) = \pi^{-1} K_0(z, s, t - s), \quad s, t \in \mathbb{R}, \quad s \neq t, $$  \hspace{1cm} (4.48)

where $K_0(\cdot)$ denotes the modified (irregular) Bessel function of order zero (cf. [3, Sect. 9.6]). Formulas such as (4.47) and (4.48) follow from elementary Fourier transform arguments as detailed in [127, p. 57–59]. Relation (4.48) requires in addition the integral representation [73, No. 3.7542] for $K_0(\cdot)$.

Next, we study some properties of $B'$. For this purpose the following known result will turn out to be useful:

**Lemma 4.5.** Suppose $T(s, t) \in \mathcal{B}_2(\mathcal{H})$ for a.e. $(s, t) \in \mathbb{R}^2$ and assume that

$$\int_{\mathbb{R}^2} \|T(s, t)\|^2_{\mathcal{B}_2(\mathcal{H})} \, ds \, dt < \infty.$$  \hspace{1cm} (4.49)

Define the operator $T$ in $L^2(\mathbb{R}; \mathcal{H})$ by

$$(T f)(s) = \int_{\mathbb{R}} T(s, t) f(t) \, dt \quad \text{for a.e.} \quad s \in \mathbb{R}, \quad f \in L^2(\mathbb{R}; \mathcal{H}).$$  \hspace{1cm} (4.50)

Then $T \in \mathcal{B}_2(L^2(\mathbb{R}; \mathcal{H}))$ and

$$\|T\|^2_{\mathcal{B}_2(L^2(\mathbb{R}; \mathcal{H}))} = \int_{\mathbb{R}^2} \|T(s, t)\|^2_{\mathcal{B}_2(\mathcal{H})} \, ds \, dt.$$  \hspace{1cm} (4.51)

Conversely, any operator $T \in \mathcal{B}_2(L^2(\mathbb{R}; \mathcal{H}))$ arises in the manner (4.49), (4.50).

For the proof of an extension of Lemma 4.5 we refer to [30, Theorem 11.3.6]. At this point it is worth noting that by Theorem A.7, $B$ and $B'$ are densely defined, symmetric, and closed operators in $L^2(\mathbb{R}; \mathcal{H})$ (cf. Lemma A.10).

**Lemma 4.6.** Assume Hypothesis 2.1. Then

$$|B'|^{1/2}(H_0 - z I)^{-1/2} \in \mathcal{B}_2(L^2(\mathbb{R}; \mathcal{H})), \quad \text{and}$$

$$|(B')^*|^{1/2}(H_0 - z I)^{-1/2} \in \mathcal{B}_2(L^2(\mathbb{R}; \mathcal{H})), \quad z \in \mathbb{C} \setminus [0, \infty).$$  \hspace{1cm} (4.52)
Moreover,
\[
\left\| B^{1/2} (H_0 - z I)^{-1/2} \right\|^2_{B_2(L^2(\mathbb{R}; \mathcal{H}))} \leq |z|^{-1/2} \int_{\mathbb{R}} \left\| B'(t)(A^2 + I)^{-1/2} \right\|_{B_1(\mathcal{H})} dt,
\]
\[
\left\| (B')^* (H_0 - z I)^{-1/2} \right\|^2_{B_2(L^2(\mathbb{R}; \mathcal{H}))} \leq |z|^{-1/2} \int_{\mathbb{R}} \left\| B'(t)(A^2 + I)^{-1/2} \right\|_{B_1(\mathcal{H})} dt,
\]
\[z < -1. \quad (4.53)\]

Proof. Abbreviating \( R_0^{1/2} = R_0^{1/2}(z) \), \( \varphi_\omega = (A^2 - z I)^{1/2} \) (cf. (1.28)), and \( \varphi_- = \varphi_\omega(A_-) = (A^2 - z I)^{1/2} \) (cf. (2.21)), with \( z < 0 \), one estimates
\[
\left\| B^{1/2} R_0^{1/2} \right\|^2_{B_2(L^2(\mathbb{R}; \mathcal{H}))} = \left\| B^{1/2} \varphi_{\omega}^{-1/2} \varphi_-^{-1/2} R_0^{1/2} \right\|^2_{B_2(L^2(\mathbb{R}; \mathcal{H}))}
\]
\[= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left\| B'(t) \right\|^{1/2} \varphi_{\omega}^{-1/2} \varphi_-^{-1/2} R_0^{1/2}(t, s) \right\|^2_{B_2(\mathcal{H})} ds \right) dt
\]
\[\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \varphi_{\omega}^{-1/2} B'(t) \varphi_-^{-1/2} R_0^{1/2}(t, s) \right\|^2_{B_2(\mathcal{H})} ds \right) dt
\]
\[= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \varphi_{\omega}^{-1/2} B'(t) \varphi_-^{-1/2} R_0(t, s) \right\|_{B_2(\mathcal{H})} ds \right) dt
\]
\[\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} B'(t) \varphi_-^{-1} R_0(t, s) \right\|_{B_2(\mathcal{H})} ds \right) dt
\]
\[\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} B'(t) \varphi_-^{-1} R_0(t, s) \right\|^2_{B_2(\mathcal{H})} ds \right) dt
\]
\[\leq |z|^{-1/2} \int_{\mathbb{R}} \left\| B'(t) \varphi_-^{-1} R_0(t, s) \right\|^2_{B_2(\mathcal{H})} ds \right) dt\]
\[\leq |z|^{-1/2} \int_{\mathbb{R}} \left\| B'(t) \varphi_-^{-1} R_0(t, s) \right\|^2_{B_2(\mathcal{H})} ds \right) dt\]
\[= |z|^{-1/2} \int_{\mathbb{R}} \left\| B'(t) \varphi_-^{-1} R_0(t, s) \right\|^2_{B_2(\mathcal{H})} ds \right) dt < \infty. \quad (4.54)\]

Here we used Lemma 4.5, employed the fact that \( \varphi_{\omega}^{-1/2} \) and \( R_0^{1/2}(t, s) \) commute and that \( \varphi_{\omega}^{-1/2} R_0^{1/2}(t, s) \) is self-adjoint, applied Theorem 4.1 to obtain
\[
\left\| \varphi_{\omega}^{-1/2} B'(t) \varphi_{\omega}^{-1/2} \right\|_{B_2(\mathcal{H})} \leq \left\| B'(t) \varphi_{\omega}^{-1} \right\|_{B_2(\mathcal{H})}, \quad (4.55)
\]
used the polar decomposition \( B'(t) = U_{B'(t)}|B'(t)| \) of \( B'(t) \), employed the explicit form of \( R_0(s, t) \) in terms of the convolution operator \( \mathcal{K}_0 \) in (4.46), and finally, used the estimate
\[
\left\| e^{-\varphi_-^{-1}(A_-) s} \right\|_{B(\mathcal{H})} = \sup_{\lambda \in \sigma(A_-)} \left( e^{-\lambda^2 + |z|^{1/2}} \right) \leq e^{-|z|^{1/2} |s|}, \quad s \in \mathbb{R}, \quad z < 0, \quad (4.56)
\]
and hence,
\[
\frac{1}{2} \int_{\mathbb{R}} \left\| e^{-\varphi_-^{-1}(A_-) s} \right\|_{B(\mathcal{H})} ds \leq |z|^{-1/2}, \quad z < 0. \quad (4.57)\]
Next, one notes that
\[
\int_{\mathbb{R}} \| B'(t) \kappa_z(A_-)^{-1} \|_{B_1(\mathcal{H})} dt = \int_{\mathbb{R}} \| B'(t) \kappa(A_-) \kappa_z(A_-)^{-1} \|_{B_1(\mathcal{H})} dt \\
\leq \int_{\mathbb{R}} \| B'(t) \kappa(A_-)^{-1} \|_{B_1(\mathcal{H})} dt, \quad z < -1,
\]
(4.58)
since \( \| \kappa(A_-) \kappa_z(A_-)^{-1} \|_{B(\mathcal{H})} = 1, \quad z \leq -1 \), for \( \kappa(A_-) = (A_-^2 + I)^{1/2} \), finishing the proof of the first relation in (4.53) and, using Lemma 4.5, the first inclusion in (4.52) (for \( z < -1 \)).

An application of Remark 3.6 (using (3.22) repeatedly) then yields the second relation in (4.52) (for \( z < -1 \)) and (4.53).

The extension of (4.52) to \( z \in \mathbb{C} \setminus [0, \infty) \) then follows from
\[
\| R_0(\zeta)^{-1/2} R_0(z)^{1/2} \|_{B(L^2(\mathbb{R}; \mathcal{H}))} \leq C(\zeta, z < \infty, \quad \zeta < 0, \quad z \in \mathbb{C} \setminus [0, \infty). \quad (4.59)
\]

**Lemma 4.7.** Assume Hypothesis 2.1. Then,
\[
\left\| (A_-^2 - z I)^{1/2} (H_0 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))} = 1, \quad z < 0, \quad (4.60)
\]
\[
\left\| A_- (H_0 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))} \leq 1, \quad z < 0, \quad (4.61)
\]
\[
\left\| B(H_0 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))} = o(1). \quad (4.62)
\]

**Proof.** Passing to the Fourier transform (cf. (4.32), (4.33)), and using Remark 3.14, and the spectral theorem, one obtains,
\[
\left\| (A_-^2 - z I)^{1/2} (H_0 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))} = \sup_{t \in \mathbb{R}} \| \kappa_z(A_-)(t^2 + \kappa_z(A_-)^2)^{-1/2} \|_{B(\mathcal{H})}
\]
\[
= \sup_{t \in \mathbb{R}} \sup_{\lambda \in \sigma(A_-)} \left| \frac{\lambda^2 - z}{t^2 + \lambda^2 + z} \right|^{1/2} = 1, \quad z < 0, \quad (4.63)
\]
proving (4.60). The inequality (4.61) is proved analogously. Next, one estimates,
\[
\left\| B(H_0 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))}
\]
\[
= \left\| B(A_-^2 - z I)^{-1/2} (A_-^2 - z I)^{1/2} (H_0 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))}
\]
\[
\leq \left\| B(A_-^2 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))} \left\| (A_-^2 - z I)^{1/2} (H_0 - z I)^{-1/2} \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))}
\]
\[
= \sup_{t \in \mathbb{R}} \| B(t)(A_-^2 - z I)^{-1/2} \|_{B(\mathcal{H})} \quad z \rightarrow -\infty \quad = o(1), \quad (4.64)
\]
by (3.16) and (4.61).

For subsequent purposes, we recall the generalized polar decomposition of a densely defined and closed operator \( T \) in a complex separable Hilbert space \( \mathcal{K} \)
\[
T = \left| T^* \right|^{1/2} U_T |T|^{1/2}, \quad (4.65)
\]
derived in [66], where \( U_T \) is the partial isometry in \( \mathcal{K} \) in the standard polar decomposition \( T = U_T |T| \) of \( T \), with \( |T| = (T^*T)^{1/2} \).

Next, we introduce the following sesquilinear forms in \( L^2(\mathbb{R}; \mathcal{H}) \),
\[
Q_{H_0}(f, g) = (H_0^{1/2} f, H_0^{1/2} g)_{L^2(\mathbb{R}; \mathcal{H})}, \quad f, g \in \text{dom}(Q_{H_0}) = \text{dom} (H_0^{1/2}), \quad (4.66)
\]
Applying (4.61) and (4.62) one obtains
\[
Q_{V,j}(f, g) = (A_{-} f, B_{j} g)_{L^{2}(\mathbb{R}; \mathcal{H})} + (B f, A_{-} g)_{L^{2}(\mathbb{R}; \mathcal{H})} + (B f, B_{j} g)_{L^{2}(\mathbb{R}; \mathcal{H})} + (-1)^j \left( (B')^{*} \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^{2}(\mathbb{R}; \mathcal{H})},
\]
\[
\text{dom}(Q_{V,j}) = \text{dom}(H_{0}^{1/2}), \quad j = 1, 2,
\]
\[
Q_{V}(f, g) = (A_{-} f, B g)_{L^{2}(\mathbb{R}; \mathcal{H})} + (B f, A_{-} g)_{L^{2}(\mathbb{R}; \mathcal{H})} + (B f, B g)_{L^{2}(\mathbb{R}; \mathcal{H})},
\]
\[
f, g \in \text{dom}(Q_{V}) = \text{dom}(H_{0}^{1/2}),
\]
where we employed the generalized polar decomposition
\[
B' = (B')^{*} \left( \frac{1}{2} U_{B'} |B'| \right)^{1/2}
\]
of $B'$.

By Lemmas 4.6 and 4.7, the sesquilinear forms $Q_{V,j}$, $j = 1, 2$, and $Q_{V}$ are well-defined. In addition, $Q_{H_{0}}$, $Q_{V,j}$, $j = 1, 2$, and $Q_{V}$ are symmetric forms.

**Lemma 4.8.** Assume Hypothesis 2.1. Then the symmetric forms $Q_{V,j}$, $j = 1, 2$, and $Q_{V}$, defined in (4.67), (4.68), are infinitesimally bounded with respect to the form $Q_{H_{0}}$ of the self-adjoint operator $H_{0}$ in $L^{2}(\mathbb{R}; \mathcal{H})$. Thus, the form sums
\[
Q_{H_{0}}(f, g) = Q_{H_{0}}(f, g) + Q_{V,j}(f, g), \quad f, g \in \text{dom}(Q_{H_{0}}) = \text{dom}(Q_{H_{0}}), \quad j = 1, 2,
\]
\[
Q_{H}(f, g) = Q_{H_{0}}(f, g) + Q_{V}(f, g), \quad f, g \in \text{dom}(Q_{H}) = \text{dom}(Q_{H_{0}}),
\]
are densely defined, closed, and bounded from below. Consequently, the forms $Q_{H_{0}}$, $j = 1, 2$, and $Q_{H}$ uniquely define self-adjoint operators $H_{j}$, $j = 1, 2$, and $H$ in $L^{2}(\mathbb{R}; \mathcal{H})$, respectively, with $H_{j}$, $j = 1, 2$, and $H$ bounded from below, satisfying
\[
\text{dom}(H_{j}) = \{ f \in \text{dom}(Q_{H_{0}}) \mid \text{the map: } \text{dom}(Q_{H_{0}}) \ni g \mapsto Q_{H_{0}}(f, g) \text{ is continuous in the norm of } L^{2}(\mathbb{R}; \mathcal{H}) \}, \quad j = 1, 2,
\]
\[
Q_{H_{j}}(f, g) = (f, H_{j} g)_{L^{2}(\mathbb{R}; \mathcal{H})}, \quad f \in \text{dom}(Q_{H_{0}}), \quad g \in \text{dom}(H_{j}), \quad j = 1, 2,
\]
\[
\text{dom}(H) = \{ f \in \text{dom}(Q_{H_{0}}) \mid \text{the map: } \text{dom}(Q_{H_{0}}) \ni g \mapsto Q_{H}(f, g) \text{ is continuous in the norm of } L^{2}(\mathbb{R}; \mathcal{H}) \},
\]
\[
Q_{H}(f, g) = (f, H g)_{L^{2}(\mathbb{R}; \mathcal{H})}, \quad f \in \text{dom}(Q_{H_{0}}), \quad g \in \text{dom}(H),
\]
and
\[
\text{dom}(H_{j})^{1/2} = \text{dom}(H^{1/2}) = \text{dom}(H_{0}^{1/2}).
\]

**Proof.** Applying (4.61) and (4.62) one obtains
\[
|(A_{-} f, B f)_{L^{2}(\mathbb{R}; \mathcal{H})}| \leq \left( A_{-} (H_{0} - z I)^{-1/2} (H_{0} - z I)^{1/2} f \right)_{L^{2}(\mathbb{R}; \mathcal{H})}
\]
\[
\leq \left| A_{-} (H_{0} - z I)^{-1/2} \right|_{L^{2}(\mathbb{R}; \mathcal{H})} \left| (B (H_{0} - z I)^{-1/2} f) \right|_{L^{2}(\mathbb{R}; \mathcal{H})}
\]
\[
\times \left| (H_{0} - z I)^{1/2} f \right|_{L^{2}(\mathbb{R}; \mathcal{H})}^{2}
\]
\[
= |(B (H_{0} - z I)^{-1/2} f)_{L^{2}(\mathbb{R}; \mathcal{H})}| \left| (H_{0} - z I)^{1/2} f \right|_{L^{2}(\mathbb{R}; \mathcal{H})}^{2}
\]
\[
= a(z) \left| (H_{0} - z I)^{1/2} f \right|_{L^{2}(\mathbb{R}; \mathcal{H})}^{2}, \quad f \in \text{dom}(H_{0}^{1/2}),
\]

(4.77)
with
\[ a(z) \geq 0 \text{ and } a(z) \to 0. \tag{4.78} \]
The same estimate now applies to the sesquilinear forms
\[ (Bf, A_-f)_{L^2(\mathbb{R};\mathcal{H})}, \quad (Bf, Bf)_{L^2(\mathbb{R};\mathcal{H})}, \quad f \in \text{dom}(H_0^{1/2}). \tag{4.79} \]
Moreover, Lemma 4.6 yields the same estimate also for the sesquilinear form
\[ ((B')^*|^{1/2} f, U_B'B'|^{1/2} f)_{L^2(\mathbb{R};\mathcal{H})}, \quad f \in \text{dom}(H_0^{1/2}). \tag{4.80} \]
Thus, by (4.67) and (4.68), the sesquilinear forms \( Q_{V_j}, \ j = 1, 2, \) and \( Q_V \) are infinitesimally bounded with respect to \( Q_{H_0} \). The first and second representation theorem for sesquilinear forms (cf., e.g., [58, Sect. IV.2], [86, Sect. 6.2]) then yields (4.72)–(4.76) and completes the proof. ■

Being defined as a self-adjoint form sum, we note that \( H \) is an extension of the operator sum \(-((d^2/dt^2) + A^2)\) defined on \( \text{dom}(d^2/dt^2) \cap \text{dom}(A^2) \).

Next we will prove that \( \widehat{H}_j \) coincides with \( H_j, j = 1, 2; \)

**Lemma 4.9.** Assume Hypothesis 2.1. Then,
\[ \widehat{H}_j = H_j, \ j = 1, 2, \tag{4.81} \]
where
\[ H_1 = D_A^*D_A, \quad H_2 = D_A^*D_A. \tag{4.82} \]
In particular,
\[ \text{dom}(H_1^{1/2}) = \text{dom}(H_2^{1/2}) = \text{dom}(H_1^{1/2}) = \text{dom}(H_0^{1/2}) \]
\[ = \text{dom}(D_A) = \text{dom}(D_A^*) = \text{dom}(d/dt) \cap \text{dom}(A_0). \tag{4.83} \]

**Proof.** It suffices to prove \( \widehat{H}_1 = H_1 \). The sesquilinear form \( Q_{H_1} \) uniquely associated with \( H_1 \) is given by
\[ Q_{H_1}(f, g) = (D_Af, D_Ag)_{L^2(\mathbb{R};\mathcal{H})}, \quad f, g \in \text{dom}(Q_{H_1}) = \text{dom}(D_A) = \text{dom}(H_1^{1/2}), \tag{4.84} \]
with
\[ Q_{H_1}(f, g) = (f, H_1g)_{L^2(\mathbb{R};\mathcal{H})}, \quad f \in \text{dom}(Q_{H_1}) = \text{dom}(D_A), \ g \in \text{dom}(H_1). \tag{4.85} \]
Thus, one computes
\[ Q_{H_1}(f, g) = (D_Af, D_Ag)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ = ((D_A - B)f, (D_A - B)g)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ = (D_A - f, D_A - g)_{L^2(\mathbb{R};\mathcal{H})} + (D_A - f, Bg)_{L^2(\mathbb{R};\mathcal{H})} + (Bf, D_A - g)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ + (Bf, Bg)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ = (D_A - f, D_A - g)_{L^2(\mathbb{R};\mathcal{H})} + (\langle d/dt \rangle + A_0)f, Bg)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ + (Bf, ((d/dt) + A_0)g)_{L^2(\mathbb{R};\mathcal{H})} + (Bf, Bg)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ = (D_A - f, D_A - g)_{L^2(\mathbb{R};\mathcal{H})} + (A_0 - f, Bg)_{L^2(\mathbb{R};\mathcal{H})} + (Bf, A_0 - g)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ + (Bf, Bg)_{L^2(\mathbb{R};\mathcal{H})} + (f', Bg)_{L^2(\mathbb{R};\mathcal{H})} + (Bf, Bg')_{L^2(\mathbb{R};\mathcal{H})} \]
\[ = (D_A - f, D_A - g)_{L^2(\mathbb{R};\mathcal{H})} + (A_0 - f, Bg)_{L^2(\mathbb{R};\mathcal{H})} + (Bf, A_0 - g)_{L^2(\mathbb{R};\mathcal{H})} \]
\[ + (Bf, Bg)_{L^2(\mathbb{R};\mathcal{H})} - ((B')^*|^{1/2} f, U_B'B'|^{1/2} g)_{L^2(\mathbb{R};\mathcal{H})}, \tag{4.86} \]
\[ f, g \in \text{dom}(D_A) = \text{dom}(D_{A_-}). \]

The last step is a consequence of the following observations:

\[
\begin{align*}
(f', Bg)_{L^2(R; H)} &+ (Bf, g')_{L^2(R; H)} \\
&= (f', Bg)_{L^2(R; H)} + (Bf, g')_{L^2(R; H)} + \left( \left( |B'|^* \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^2(R; H)} \\
&- \left( \left( |B'|^* \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^2(R; H)} \\
&= \lim_{R \to \infty} \int_{-R}^{R} \left[ (f'(t), B(t)g(t))_{\mathcal{H}} + (f(t), B(t)g'(t))_{\mathcal{H}} + (f(t), B'(t)g(t))_{\mathcal{H}} \right] dt \\
&- \left( \left( |B'|^* \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^2(R; H)} \\
&= \lim_{R \to \infty} \int_{-R}^{R} \frac{d}{dt} (f(t), B(t)g(t))_{\mathcal{H}} dt - \left( \left( |B'|^* \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^2(R; H)} \\
&= \lim_{R \to \infty} \int_{f(R), B(R)g(R)_{\mathcal{H}}} - \lim_{R \to \infty} \left( f(-R), B(-R)g(-R) \right)_{\mathcal{H}} \\
&- \left( \left( |B'|^* \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^2(R; H)} \\
&= -\left( \left( |B'|^* \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^2(R; H)}, \quad f, g \in \text{dom}(D_A) = \text{dom}(D_{A_-}).
\end{align*}
\]

(4.87)

Here we used the fact that the limits \( \lim_{R \to \pm \infty} (f(R), B(R)g(R))_{\mathcal{H}} \), exist since

\[
(f'(\cdot), B(\cdot)g(\cdot))_{\mathcal{H}}, \ (f(\cdot), B(\cdot)g'(\cdot))_{\mathcal{H}}, \ (f(\cdot), B'(\cdot)g(\cdot))_{\mathcal{H}} \in L^1(\mathbb{R}; dt).
\]

Moreover, since also

\[
(f(\cdot), B(\cdot)g(\cdot))_{\mathcal{H}} \in L^1(\mathbb{R}; dt),
\]

(4.89)

one concludes that

\[
\lim_{R \to \pm \infty} \left( f(R), B(R)g(R) \right)_{\mathcal{H}} = 0,
\]

(4.90)

completing the derivation of (4.87) and hence of (4.86). Equations (4.44) and (4.86) then imply

\[
Q_{H_1}(f, g) = (f, H_0 g)_{L^2(R; H)} + (f, A_- B g)_{L^2(R; H)} + (f, B A_- g)_{L^2(R; H)}
\]

\[
+ (f, B^2 g)_{L^2(R; H)} - \left( \left( |B'|^* \right)^{1/2} f, U_{B'} |B'|^{1/2} g \right)_{L^2(R; H)},
\]

\[
= Q_{\tilde{H}_1}(f, g), \quad f, g \in \text{dom}(Q_{H_1}) = \text{dom}(Q_{\tilde{H}_1}) = \text{dom} \left( H_0^{1/2} \right),
\]

(4.91)

and hence \( H_1 = \tilde{H}_1. \)

We will use the following notations for the resolvents of the operators \( H_j, \ j = 1, 2, \) and \( H: \)

\[
R_j(z) = (H_j - z I)^{-1}, \quad z \in \rho(H_j), \ j = 1, 2, \quad R(z) = (H - z I)^{-1}, \quad z \in \rho(H).
\]

(4.92)

Next, we will discuss in detail the properties of the approximative operators introduced in (2.22). Since \( P_n = E_{A_-}((-n, n)), n \in \mathbb{N}, \) is the spectral projection for \( A_- \), we recall the following commutation formulas (cf. (2.22)):

\[
A_- n P_n A_- = A_- n P_n A_- = A_- P_n P_n = A_- P_n = P_n A_- = P_n A_- P_n,
\]

(4.93)

\[
(A_- - z I)^{-1} P_n = P_n (A_- - z I)^{-1}, \quad z \in \rho(A_-),
\]

\[
B_n(t) = P_n B(t) P_n, \quad B_n'(t) = P_n B'(t) P_n, \quad B_n(\infty) = P_n B(\infty) P_n, \quad n \in \mathbb{N}.
\]
Next, one recalls the following properties of the spectral projections $P_n$ in $\mathcal{H}$:
\begin{align*}
\lim_{n \to \infty} s\cdot \text{lim } P_n &= I, \\
\text{ran}(P_n) &\subseteq \text{dom}(A_-), \quad n \in \mathbb{N}, \\
\lim_{n \to \infty} \|P_n A_- P_n w - A_- w\|_\mathcal{H} &= 0, \quad w \in \text{dom}(A_-).
\end{align*}

We collect some basic properties of the operators introduced in (2.22) in the next lemma:

**Lemma 4.10.** Assume Hypothesis 2.1. Then
\begin{equation}
\int_\mathbb{R} \| [B'(t) - B'_n(t)](A_-^2 + I)^{-1/2} \|_{B_1(\mathcal{H})} dt \to 0 \quad \text{as } n \to \infty;
\end{equation}
\begin{equation}
\lim_{n \to \infty} \| [B(\infty) - B_n(\infty)](A_-^2 + I)^{-1/2} \|_{B_1(\mathcal{H})}
= \lim_{n \to \infty} \| [A_+ - A_n - A_+ n + A_-,n](A_-^2 + I)^{-1/2} \|_{B_1(\mathcal{H})} = 0,
\end{equation}
\begin{equation}
A_{\pm, n} \to A_\pm \text{ in the strong resolvent sense in } \mathcal{H} \text{ as } n \to \infty,
\end{equation}
\begin{equation}
\lim_{n \to \infty} \| ([B - B_n](A_-^2 + I))^{-1/2} \|_{B(L^2(\mathbb{R}; \mathcal{H}))} = 0.
\end{equation}

**Proof.** As usual, we abbreviate $\kappa = (A_-^2 + I)^{1/2}$. To prove (4.97), we will employ the dominated convergence theorem. By (4.93) one infers that
\begin{equation}
\| [B'(t) - B'_n(t)]\kappa^{-1} \|_{B_1(\mathcal{H})} = \| B'(t)\kappa^{-1} - P_n B'(t)\kappa^{-1} P_n \|_{B_1(\mathcal{H})}
\leq 2\| B'(t)\kappa^{-1} \|_{B_1(\mathcal{H})},
\end{equation}
and the function in the right-hand side of (4.101) is summable thanks to (2.2). For each $t \in \mathbb{R}$, due to (4.93), one may write
\begin{equation}
[B'(t) - B'_n(t)]\kappa^{-1} = B'(t)\kappa^{-1} - P_n B'(t)\kappa^{-1} P_n
= P_n B'(t)\kappa^{-1} (I - P_n) + (I - P_n) B'(t)\kappa^{-1}.
\end{equation}
Since $B'(t)\kappa^{-1} = B'(t)(A_-^2 + I)^{-1}\kappa^{-1} = (A_-^2 + I)\kappa^{-1} \in B_1(\mathcal{H})$ by Hypothesis 2.1 (iv), and since $P_n \to I$ in $\mathcal{H}$ strongly as $n \to \infty$, one can apply Lemma 3.4, thus finishing the proof of (4.97). By definition, the operators under the $B_1(\mathcal{H})$-norm on either side in equation (4.98) are equal (cf. (2.22), (3.51)). Because of
\begin{equation}
[B(\infty) - B_n(\infty)]\kappa^{-1} = \int_\mathbb{R} [B'(t) - B'_n(t)]\kappa^{-1} dt,
\end{equation}
assertion (4.98) follows from (4.97). That $A_{\pm, n} \to A_\pm$ in strong resolvent sense follows from (4.96) and [126, Theorem VIII.25(a)]. To see that $A_{\pm, n} \to A_\pm$ in strong resolvent sense as $n \to \infty$, one writes
\begin{equation}
(A_+ + iI)^{-1} - (A_+ + n + iI)^{-1} = -(A_{\pm, n} + iI)^{-1} (A_+ - A_{\pm, n})(A_+ + iI)^{-1}
= -(A_+ + n + iI)^{-1} [A_+ - A_+ - A_+ + A_+ - A_- + A_-,n] \kappa^{-1} \kappa^{-1} (A_+ + iI)^{-1}
= -(A_+ + n + iI)^{-1} (A_+ - A_- + A_-,n)(A_+ + iI)^{-1}.
\end{equation}
Since $\| (A_+ + iI)^{-1} \|_{B_1(\mathcal{H})} \leq 1$ for all $n$ for the self-adjoint operator $A_{\pm, n}$, and $\kappa^{-1} (A_+ + iI)^{-1} = (A_-^2 + I)^{1/2} (A_+ + iI)^{-1} \in B(\mathcal{H})$ due to (3.51), the sequence of operators in (4.104) converges to zero as $n \to \infty$ (even in $B_1(\mathcal{H})$ due to (4.98)).
while the sequence of the operators in (4.105) converges to zero as $n \to \infty$ strongly in $\mathcal{H}$ due to (4.96). Finally, relation (4.100) follows from Remark 3.14, the estimate
\[
\| (B - B_n)(A^2 + I)^{-1/2} \|_{B(L^2(\mathbb{R}; \mathcal{H}))} = \sup_{t \in \mathbb{R}} \| [B(t) - B_n(t)] \chi^{-1} \|_{B(\mathcal{H})}
\]
\[
= \sup_{t \in \mathbb{R}} \int_{-\infty}^{\infty} \| B'(\tau) - B'_n(\tau) \| \chi^{-1} d\tau \int_{-\infty}^{\infty} \| [B'(\tau) - B'_n(\tau)] \chi^{-1} \|_{B(\mathcal{H})} d\tau,
\]
and (4.97).

5. The Left-Hand Side of the Trace Formula and Approximations

In this section we deal with the left-hand sides of formulas (2.19) and (2.23), assuming Hypothesis 2.1. We also recall the notations introduced in (2.20), (4.45), (4.75), (4.82), and (4.92), and Lemma 4.8.

We start by proving the first inclusion in (2.27) (the second inclusion is proved similarly) and repeatedly use the generalized polar decomposition described in (4.65). In addition, we will frequently rely on resolvent formulas familiar from the perturbation theory of quadratic forms (and more generally, for perturbationspermitting appropriate factorizations) as pioneered by Kato [84] and applied to Schrödinger operators by Simon [134] (see also [63, Sections 2, 3]).

Lemma 5.1. Assume Hypothesis 2.1. Then
\[
[(H_2 - z I)^{-1} - (H_1 - z I)^{-1}] \in \mathcal{B}_1(L^2(\mathbb{R}; \mathcal{H})), \quad z \in \rho(H_2) \cap \rho(H_1),
\]
for the resolvents of the operators defined in (2.15).

Proof. By Lemma 4.6, one infers that
\[
[(B')^* R_0(z)^{1/2}]^* U_B |B'|^{1/2} R_0(z)^{1/2} \in \mathcal{B}_1(L^2(\mathbb{R}; \mathcal{H})), \quad z \in \mathbb{C} \setminus [0, \infty).
\]
Combining (4.66), (4.67), (4.69), Lemmas 4.8 and 4.9, and equation (5.2), one computes (for simplicity) for $z < 0$,
\[
(H_2 - z I)^{-1} - (H_1 - z I)^{-1}
\]
\[
= -2(H_1 - z I)^{-1}[(B')^* R_0(z)^{1/2}]^* U_B |B'|^{1/2} (H_2 - z I)^{-1}
\]
\[
= -2[(B')^* R_0(z)^{1/2}]^* U_B |B'|^{1/2} (H_2 - z I)^{-1}
\]
\[
= 2(H_1 - z I)^{-1/2} (H_0 - z I)^{-1/2} (H_2 - z I)^{-1/2} \]
\[
\times [(B')^* R_0(z)^{1/2}]^* U_B |B'|^{1/2} (H_0 - z I)^{-1/2}
\]
\[
\times [(H_2 - z I)^{-1/2} (H_0 - z I)^{-1/2}] (H_2 - z I)^{-1/2} \in \mathcal{B}_1(L^2(\mathbb{R}; \mathcal{H})).
\]

By analytic continuation with respect to $z$ based on resolvent equations in a standard manner, this extends to $z \in \rho(H_2) \cap \rho(H_1)$. We note that the resolvent equations used repeatedly at the beginning of this computation follow from the results in [84, Sect. 1] (see also [63, Sects. 2, 3], [134, Ch. II]).

To prove (2.28) in Proposition 2.4, we will need one more technical lemma. We recall the notation introduced in (4.45), (4.68), (4.75), (4.82), (4.92), and introduce the following bounded operators in $L^2(\mathbb{R}; \mathcal{H})$: \[ L(z) = I + [A - R_0^{1/2}(z)]^* B R_0^{1/2}(z) + [B R_0^{1/2}(z)]^* A - R_0^{1/2}(z) \]
One observes that using (2.22), (5.6), the fact that \( \lim_{n \to \infty} B_n(z) = B(z) \) uniformly for \( z \leq -1 \) and abbreviating \( \tilde{z} = (\tilde{A}_- + I)^{1/2} \), one obtains the following representation:

\[
\begin{align*}
L(z) - L_n(z) &= \left[ \tilde{z} R_0(z) \right]^{1/2} + [A_\tilde{z}^{-1}]^* B \tilde{z}^{-1} + [B \tilde{z}^{-1}]^* A \tilde{z}^{-1} + [B \tilde{z}^{-1}]^* B \tilde{z}^{-1} \\
&\quad - [A_\tilde{z}^{-1}]^* B \tilde{z}^{-1} - [B \tilde{z}^{-1}]^* A \tilde{z}^{-1} - [B \tilde{z}^{-1}]^* B \tilde{z}^{-1} \\
&\quad \times \left[ \tilde{z} R_0(z) \right]^{1/2} \\
&= J_1 + J_2 + J_3 + J_4,
\end{align*}
\]

where we denoted

\[
\begin{align*}
J_1 &= \left[ \tilde{z} R_0(z) \right]^{1/2} [A_\tilde{z}^{-1}]^* (B - B_n) \tilde{z}^{-1} \tilde{z} R_0(z) \left[ \tilde{z} R_0(z) \right]^{1/2}, \\
J_2 &= \left[ \tilde{z} R_0(z) \right]^{1/2} [(B - B_n) \tilde{z}^{-1}]^* A_\tilde{z}^{-1} \tilde{z} R_0(z) \left[ \tilde{z} R_0(z) \right]^{1/2}, \\
J_3 &= \left[ \tilde{z} R_0(z) \right]^{1/2} [(B - B_n) \tilde{z}^{-1}]^* [B \tilde{z}^{-1}]^* \tilde{z} R_0(z) \left[ \tilde{z} R_0(z) \right]^{1/2}, \\
J_4 &= \left[ \tilde{z} R_0(z) \right]^{1/2} [B \tilde{z}^{-1}]^* [(B - B_n) \tilde{z}^{-1}]^* \tilde{z} R_0(z) \left[ \tilde{z} R_0(z) \right]^{1/2}.
\end{align*}
\]

One observes that

\[
\lim_{n \to \infty} \|(B - B_n) \tilde{z}^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}; \mathcal{H}))} = 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|B_n \tilde{z}^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}; \mathcal{H}))} < \infty
\]

by (4.100), and

\[
\| \tilde{z} R_0(z) \|_{\mathcal{B}(L^2(\mathbb{R}; \mathcal{H}))} < 1 \quad \text{uniformly for } z \leq -1
\]

by (4.60) and \( \|(A_n^2 + I)^{1/2}(A_n^2 - z I)^{-1/2}\|_{\mathcal{B}(L^2(\mathbb{R}; \mathcal{H}))} = 1, z \leq -1 \). Thus, assertion (i) in Lemma 5.2 holds.
That the operator $L(z)$ is boundedly invertible for $z \in \rho(H) \cap (-\infty, 0)$ is well-known. In addition, one has the identity

$$(H - z I)^{-1} = R_0(z)^{1/2} L(z)^{-1} R_0(z)^{1/2}$$

$$= R_0(z)^{1/2} \left[ I + [A - R_0^{1/2}(z)]^* B R_0^{1/2}(z) + \left[ B R_0^{1/2}(z) \right]^* A - R_0^{1/2}(z) \right]^{-1} R_0(z)^{1/2}, \quad z \in \rho(H) \cap (-\infty, 0). \quad (5.16)$$

This is proved as Tiktopoulos’ formula in [134, Section II.3] by first choosing $z < 0$ sufficiently large followed by an analytic continuation with respect to $z$. In particular,

$$L^{-1}(z) = (H_0 - z I)^{1/2} R(z)^{1/2} \left[ (H_0 - z I)^{1/2} R(z)^{1/2} \right]^*, \quad z \in \rho(H) \cap (-\infty, 0), \quad (5.17)$$

$$L(z) = [(H - z I)^{1/2} R_0(z)^{1/2}]^* (H - z I)^{1/2} R_0(z)^{1/2}, \quad z < 0, \quad (5.18)$$

illustrating again that both operators $L(z)$ and $L^{-1}(z)$ are bounded in $L^2(\mathbb{R}; H)$ by $(4.83)$. Analogous considerations apply to $L_n(z), n \in \mathbb{N}$.

The rest of assertion (ii) follows from item (i). Indeed, we conclude from $(5.4)$, $(5.5)$ that

$$\lim_{z \downarrow -\infty} \|L(z) - I\|_{\mathcal{B}(L^2(\mathbb{R}; H))} = 0, \quad \lim_{z \downarrow -\infty} \|L_n(z) - I\|_{\mathcal{B}(L^2(\mathbb{R}; H))} = 0 \quad (5.19)$$

for each $n \in \mathbb{N}$ by Lemma 4.7. This implies

$$\sup_{z \leq -1} \|L(z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}; H))} < \infty, \quad \sup_{z \leq -1} \|L_n(z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}; H))} < \infty \quad (5.20)$$

for each $n \in \mathbb{N}$, and now item (i) implies the second assertion in $(5.7)$. $\blacksquare$

At this point we are ready to prove Proposition 2.4:

Proof. We will abbreviate $R_0^{1/2} = R_0^{1/2}(z), R_{0,n}^{1/2} = R_0^{1/2}(z)$ and $L = L(z), L_n = L_n(z)$. In view of Lemma 5.1, it remains to show $(2.28)$. Using Lemma 5.2(ii) and Lemma 4.6 we choose $z < -1$ with $|z|$ so large that

$$\sup_{n \in \mathbb{N}} \|L^{-1/2} [(B')^* R_{0,n}^{1/2}]^* U_{B'} \|_{\mathcal{B}(L^2(\mathbb{R}; H))} \leq 1/2,$$

$$\sup_{n \in \mathbb{N}} \|L_n^{-1/2} [(B')^* R_{0,n}^{1/2}]^* U_{B'} \|_{\mathcal{B}(L^2(\mathbb{R}; H))} \leq 1/2. \quad (5.21)$$

Using $(4.67), (4.68)$ one infers

$$(H_1 - z I)^{-1} = R_0^{1/2} \left[ I + [A - R_0^{1/2}(z)]^* B R_0^{1/2} \right. \left. + [B R_0^{1/2}(z)]^* A - R_0^{1/2}(z) \right]^{-1} R_0^{1/2}$$

$$= R_0^{1/2} \left[ L - [(B')^* R_0^{1/2}(z)]^* U_{B'} \right]^{-1} R_0^{1/2}$$

$$= R_0^{1/2} L^{-1/2} \left[ I - L^{-1/2} \left[ (B')^* R_0^{1/2} \right]^* U_{B'} \right]^{-1} R_0^{1/2} L^{-1/2} R_0^{1/2} \quad (5.22)$$
A similar calculation for $H_2$ and (5.21) show that the resolvents $R_1 = R_1(z)$ and $R_2 = R_2(z)$ can be computed as follows (and similarly for $R_{1,n}, R_{2,n}$):

$$R_1 = R_0^{1/2} L^{-1/2} \left[ I - L^{-1/2} \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2} L^{-1/2} \right]^{-1} L^{-1/2},$$

$$R_2 = R_0^{1/2} L^{-1/2} \left[ I + L^{-1/2} \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2} L^{-1/2} \right]^{-1} L^{-1/2},$$

(5.23) (5.24)

Introducing the bounded operators

$$M = R_0^{1/2} L^{-1/2} \left[ I + L^{-1/2} \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2} L^{-1/2} \right]^{-1} L^{-1/2},$$

$$N = L^{-1/2} \left[ I - L^{-1/2} \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2} L^{-1/2} \right]^{-1} L^{-1/2},$$

(5.25) (5.26)

$$M_n = R_{0,n}^{1/2} L_n^{-1/2} \left[ I + L_n^{-1/2} \left[ \left( (B'_n)^* \right)^{1/2} R_{0,n}^{1/2} \right] U_{B'_n} | B'_n|^{1/2} R_{0,n}^{1/2} L_n^{-1/2} \right]^{-1} L_n^{-1/2},$$

$$N_n = L_n^{-1/2} \left[ I - L_n^{-1/2} \left[ \left( (B'_n)^* \right)^{1/2} R_{0,n}^{1/2} \right] U_{B'_n} | B'_n|^{1/2} R_{0,n}^{1/2} L_n^{-1/2} \right]^{-1} L_n^{-1/2},$$

(5.27) (5.28)

one obtains the following identities:

$$R_1 - R_2 = 2 M \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2} N,$$

$$R_{1,n} - R_{2,n} = 2 M_n \left[ \left( (B'_n)^* \right)^{1/2} R_{0,n}^{1/2} \right] U_{B'_n} | B'_n|^{1/2} R_{0,n}^{1/2} N_n.$$  

(5.29)

We need two more preparatory facts to finish the proof of Proposition 2.4: First, we claim that

$$\lim_{n \to \infty} \left\| \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2} \right\|_{B_1(L^2(\mathbb{R};\mathbb{H}))} = 0.$$  

(5.30)

Indeed, since the spectral projection $P_n$ and the operator $-\frac{d}{dx}$ commute (cf. (5.6)),

$$P_n R_{0,n}^{1/2} = P_n R_0^{1/2}, \quad R_{0,n}^{1/2} P_n = R_0^{1/2} P_n.$$  

(5.31)

Since $B'_n = P_n B' P_n$, one can write

$$\left[ \left( (B'_n)^* \right)^{1/2} R_{0,n}^{1/2} \right] U_{B'_n} | B'_n|^{1/2} R_{0,n}^{1/2} = \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2},$$

(5.32)

and, after a short calculation with scalar products using (4.69) for $B'$, $B'_n$, and $B' - B'_n$, obtain the estimate

$$\left\| \left[ \left( (B')^* \right)^{1/2} R_0^{1/2} \right] U_{B'} | B'|^{1/2} R_0^{1/2} \right\|_{B_1(L^2(\mathbb{R};\mathbb{H}))}$$

$$\left\| \left[ \left( (B'_n)^* \right)^{1/2} R_{0,n}^{1/2} \right] U_{B'_n} | B'_n|^{1/2} R_{0,n}^{1/2} \right\|_{B_1(L^2(\mathbb{R};\mathbb{H}))}$$

$$= \left\| \left[ \left( (B' - B'_n)^* \right)^{1/2} R_0^{1/2} \right] U_{(B' - B'_n)| B'| - B'_n|^{1/2} R_0^{1/2} \right\|_{B_1(L^2(\mathbb{R};\mathbb{H}))}$$  

(5.33)
using Lemma 4.6 with \( B' \) replaced by \([B' - B'_n] \). Now claim (5.30) follows from (4.97).

Second, we claim that

\[
\limsup_{n \to \infty} M_n = M \quad \text{and} \quad \limsup_{n \to \infty} N_n = N \quad \text{in} \quad L^2(\mathbb{R}; \mathcal{H}).
\]

Indeed, referring to equations (5.25)–(5.28), one notes that \( \lim_{n \to \infty} R^{1/2}_{0,n} = R^{1/2}_0 \)
in \( L^2(\mathbb{R}; \mathcal{H}) \), while \( \lim_{n \to \infty} \| L_n^{-1/2} - L^{-1/2} \|_{B(L^2(\mathbb{R}; \mathcal{H}))} = 0 \), because of the strong resolvent convergence of the self-adjoint operators \( L_n \) to \( L \) as \( n \to \infty \) by Lemma 5.2. Also, due to (5.21), the norms of the operators satisfy

\[
\sup_{n \in \mathbb{N}} \left\| I \pm L_n^{-1/2} \left[ \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} B'_n \left( B'_n \right)^* U_{B'_n}^{-1} R^{1/2}_{0,n} L^{-1/2}_n \right\|_{B(L^2(\mathbb{R}; \mathcal{H}))} < \infty.
\]

Combining this with (5.30) proves the claim (5.35).

Finally, using (5.29) and (5.31), one infers

\[
R_1 - R_2 - (R_{1,n} - R_{2,n})
= 2 M \left[ \left( B'_n \right)^* \right]^{-1/2} R^{1/2}_{0,n} \left[ \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n} N
- 2 M_n \left[ \left( B'_n \right)^* \right]^{-1/2} R^{1/2}_{0,n} \left[ \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n} N_n
= J^{(1)}_1 + J^{(2)}_2,
\]

where we denoted

\[
J^{(1)}_1 = 2(M - M_n) \left[ \left( B'_n \right)^* \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n}, \quad J^{(2)}_2 = 2 \left[ N \left[ \left( B'_n \right)^* \right]^{-1/2} R^{1/2}_{0,n} \left[ \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n} \right]
- N_n \left[ \left( B'_n \right)^* \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n}
= 2 \left[ (N - N_n) \left[ \left( B'_n \right)^* \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n}
+ N_n \left[ \left( B'_n \right)^* \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n}
- \left[ \left( B'_n \right)^* \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n} \right].
\]

Since \( \left[ \left( B'_n \right)^* \left( B'_n \right)^* \right]^{-1/2} U_{B'_n} \left( B'_n \right)^* R^{1/2}_{0,n} \in B_1(L^2(\mathbb{R}; \mathcal{H})) \) by Lemma 4.6, one concludes that

\[
\lim_{n \to \infty} \left\| J^{(j)}_j \right\|_{B_1(L^2(\mathbb{R}; \mathcal{H}))} = 0, \quad j = 1, 2,
\]

by (5.30), (5.35), and Lemma 3.4. \( \square \)

6. The Right-Hand Side of the Trace Formula and Double Operators Integrals

In this section we deal with the right-hand side of the trace formula (2.19), and prove Proposition 2.5. Our approach based on the theory of double operator integrals. This theory originated in [25], [26], [27], [28], [29], [31] (see also the reviews in [32], [117], [121] and more recent further developments in [47], [51], [52], [119], [120], [121]).
To show the first inclusion in assertion (2.29) of Proposition 2.5, we will follow the strategy in [47], [120]; in particular, see equation (23) in [47, Section 6], where the inclusion
\[ [g(S_+) - g(S_-)] \in \mathcal{B}(\mathcal{H}) \] (6.1)
is proved, assuming \((S_+ - S_-)(S^2 + I)^{-1/2} \in \mathcal{B}(\mathcal{H})\). Lemma 6.6 below yields this inclusion with \(\mathcal{B}(\mathcal{H})\) replaced by \(\mathcal{B}_1(\mathcal{H})\), but assuming \((S_+ - S_-)(S^2 + I)^{-1/2} \in \mathcal{B}_1(\mathcal{H})\). The argument in Lemma 6.6 involves the concept of double operator integrals.

We begin by recalling some relevant background material regarding double operator integrals (cf. [47], [51], [52], [119], [120]) and fix two unbounded self-adjoint operators \(S_+\) and \(S_-\) in \(\mathcal{H}\).

Let \(\mathfrak{A}_0\) denote the set of all bounded Borel functions \(\phi\) admitting the representation
\[ \phi(\lambda, \mu) = \int_{\mathbb{R}} \alpha_s(\lambda) \beta_s(\mu) \, d\nu(s), \quad (\lambda, \mu) \in \mathbb{R}^2, \] (6.2)
where \(\alpha_s(\cdot), \beta_s(\cdot) : \mathbb{R} \to \mathbb{C}\), for each \(s \in \mathbb{R}\), are bounded Borel functions satisfying
\[ \int_{\mathbb{R}} \|\alpha_s\| \|\beta_s\| \, d\nu(s) < \infty, \] (6.3)
and \(d\nu\) is a positive Borel measure on \(\mathbb{R}\) (cf. [51, Proposition 4.7] or [120, Corollary 2]). We introduce the norm on \(\mathfrak{A}_0\) as the infimum of the integrals in (6.3) taken over all possible representations in (6.2). It is easy to see that \(\mathfrak{A}_0\) is a Banach algebra.

Given two self-adjoint operators \(S_+\) and \(S_-\) in \(\mathcal{H}\), one defines for each \(\phi \in \mathfrak{A}_0\) the operator \(T_{\phi,1} = T^{(S_+, S_-)}_\phi \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}))\), that is, a bounded operator from \(\mathcal{B}_1(\mathcal{H})\) to itself, as the following integral, absolutely convergent in \(\mathcal{B}_1(\mathcal{H})\)-norm
\[ T_{\phi,1}(K) = \int_{\mathbb{R}} \alpha_s(S_+) K \beta_s(S_-) \, d\nu(s), \quad K \in \mathcal{B}_1(\mathcal{H}). \] (6.4)
We will call \(T_{\phi,1} = T^{(S_+, S_-)}_\phi\) the operator integral; the proof of the fact that \(T_{\phi,1}\) is well-defined follows along the same lines as in [19, Lemma 4.3]. The definition above (see also [19]) is a particular case of the definition of the double operator integrals considered in [47], [51], [52], [119], [120]. Replacing \(\mathcal{B}_1(\mathcal{H})\) above with \(\mathcal{B}(\mathcal{H})\) one obtains a bounded operator \(T_{\phi,\infty}\) from \(\mathcal{B}(\mathcal{H})\) to \(\mathcal{B}(\mathcal{H})\). If \(\phi \in \mathfrak{A}_0\) satisfies the condition \(\phi(\lambda, \mu) = \phi(\mu, \lambda)\), \((\lambda, \mu) \in \mathbb{R}^2\) (and we will consider only such \(\phi\)’s) then \(T_{\phi,1} = T_{\phi,\infty}\) and \(T_{\phi,\infty}|_{\mathcal{B}_1(\mathcal{H})} = T_{\phi,1}\) (cf. [119, Lemma 2.4]). We note that
\[ ||T_{\phi,1}||_{\mathcal{B}(\mathcal{B}_1(\mathcal{H}))} = ||T_{\phi,\infty}||_{\mathcal{B}(\mathcal{B}(\mathcal{H}))} \leq ||\phi||_{\mathfrak{A}_0} \] (6.5)
(cf. [51], [52]). In what follows we use the notation \(T_\phi\) for either \(T_{\phi,1}\) or \(T_{\phi,\infty}\), which should not lead to a confusion. We remark the following two properties of the mapping \(\phi \to T_\phi\) for which we again refer to [119, Lemma 2.4]:

(i) \(\phi \to T_\phi\) is a homomorphism of \(\mathfrak{A}_0\) into \(\mathcal{B}(\mathcal{B}_1(\mathcal{H}))\) (or \(\mathcal{B}(\mathcal{B}(\mathcal{H}))\)), that is, \(T_{\phi \psi} = T_\phi T_\psi\) for \(\phi, \psi \in \mathfrak{A}_0\).

(ii) \(T_\phi\) is \(\omega\)-continuous (i.e., continuous in the weak operator topology, or ultra-weakly continuous) on \(\mathcal{B}(\mathcal{H})\). Indeed, \(T_{\phi,\infty}\) on \(\mathcal{B}(\mathcal{H})\) is dual to \(T_{\phi,1}\) and therefore is ultra-weakly continuous as a dual operator.

In addition, given bounded Borel functions \(\alpha, \beta : \mathbb{R} \to \mathbb{C}\), one notes that if \(\phi(\lambda, \mu) = \alpha(\lambda)\), then \(T_\phi(K) = \alpha(S_+) K\), and if \(\phi(\lambda, \mu) = \beta(\mu)\), then \(T_\phi(K) = K \beta(S_-)\), \(K \in \mathcal{B}_1(\mathcal{H})\) (or \(K \in \mathcal{B}(\mathcal{H})\), cf. [51], [52]).
Hypothesis 6.1. Assume that $S_+$ and $S_-$ are self-adjoint operators in $\mathcal{H}$. Given two bounded (real-valued) Borel functions $\alpha$ and $\beta$ on $\mathbb{R}$, suppose that $\mathcal{D} \subseteq \text{dom}(S_-)$ is a core for the operator $S_-$ such that

$$\beta(S_-)\mathcal{D} \subseteq \text{dom}(S_+\alpha(S_+)).$$

(6.6)

Assume that the operator $K = K(S_+, S_-)$ in $\mathcal{H}$ defined by

$$K = S_+\alpha(S_+)\beta(S_-) - \alpha(S_+)S_-\beta(S_-), \quad \text{dom}(K) = \mathcal{D}.$$  

(6.7)

is closable and $\overline{K} \in \mathcal{B}(\mathcal{H})$.

Lemma 6.2. Assume Hypothesis 6.1. Then

$$\beta(S_-) \text{dom}(S_-) \subseteq \text{dom}(S_+\alpha(S_+)),$$

(6.8)

and hence the operator $K$ admits a natural extension from the initial domain $\mathcal{D}$ to $\text{dom}(S_-)$ provided by the same formula (6.7).

Proof. Since $\alpha$ and $\beta$ are bounded, the corresponding operators $\alpha(S_+)$ and $\beta(S_-)$ leave the domains $\text{dom}(S_+)$ and $\text{dom}(S_-)$ invariant,

$$\alpha(S_+) \text{dom}(S_+) \subseteq \text{dom}(S_+) \quad \text{and} \quad \beta(S_-) \text{dom}(S_-) \subseteq \text{dom}(S_-).$$

(6.9)

Next, one considers the following sesquilinear form

$$(\beta(S_-)f, S_+\alpha(S_+)g)_{\mathcal{H}} - (S_-\beta(S_-)f, \alpha(S_+)g)_{\mathcal{H}} = (\overline{K}f, g)_{\mathcal{H}},$$

(6.10)

where $f \in \mathcal{D}$ and $g \in \text{dom}(S_+)$. Since $\overline{K}$ is bounded, the form in the left-hand side of (6.10) is also bounded and thus for every fixed $g \in \text{dom}(S_+)$ the linear mapping

$$\mathcal{D} \ni f \mapsto (S_-\beta(S_-)f, \alpha(S_+)g)_{\mathcal{H}}$$

(6.11)

is continuous. Since $\mathcal{D}$ is a core for $S_-$ and hence it is also a core for $S_-\beta(S_-)$, this implies that $\alpha(S_+)g \in \text{dom}(S_-\beta(S_-))$, or

$$\alpha(S_+) \text{dom}(S_+) \subseteq \text{dom}(S_-\beta(S_-)).$$

(6.12)

This observation allows one to rewrite (6.10) as

$$(\beta(S_-)f, S_+\alpha(S_+)g)_{\mathcal{H}} - (f, S_-\beta(S_-)\alpha(S_+)g)_{\mathcal{H}} = (\overline{K}f, g)_{\mathcal{H}},$$

(6.13)

for $f \in \mathcal{D}$ and $g \in \text{dom}(S_+)$ and then to conclude that (6.13) holds for all $f \in \mathcal{H}$ and $g \in \text{dom}(S_+)$, since the right-hand side of (6.13) is a bounded sesquilinear form. In particular, it follows from (6.13) that for every fixed $f \in \text{dom}(S_-)$, the mapping

$$\text{dom}(S_+) \ni g \mapsto (\beta(S_-)f, S_+\alpha(S_+)g)_{\mathcal{H}}$$

(6.14)

is continuous and thus $\beta(S_-)f \in \text{dom}(S_+\alpha(S_+))$, proving (6.8). \(\blacksquare\)

We will use operator integrals via the following result which is a variation of [47, Theorem 15]:

Lemma 6.3. Assume Hypothesis 6.1. Suppose that $h$ is a bounded Borel function on $\mathbb{R}$ such that the function $\phi$ defined by

$$\phi(\lambda, \mu) = \frac{h(\lambda) - h(\mu)}{\alpha(\lambda)(\lambda - \mu)\beta(\mu)}, \quad (\lambda, \mu) \in \mathbb{R}^2,$$

(6.15)

belongs to the class $\mathcal{A}_0$. Then the closure $\overline{K} \in \mathcal{B}(\mathcal{H})$ of the operator $K = K(S_+, S_-)$ satisfies:

$$h(S_+) - h(S_-) = T_\phi(\overline{K}) \in \mathcal{B}(\mathcal{H}),$$

(6.16)
where $T_\phi$ represents the operator integral $T_{\phi, \infty} = T_{\phi}^{(S_+ , S_-)}$. In addition, assume that $K \in \mathcal{B}_1(\mathcal{H})$. Then

$$h(S_+) - h(S_-) = T_{\phi}(K) \in \mathcal{B}_1(\mathcal{H}),$$

(6.17)

where $T_{\phi}$ represents the operator integral $T_{\phi,1} = T_{\phi}^{(S_+ , S_-)}$.

**Proof.** Due to the observation $T_{\phi, \infty}|_{\mathcal{B}_1(\mathcal{H})} = T_{\phi,1}$ made above, (6.17) follows from (6.16). To begin the proof of (6.16), we let $E_n^\pm = E_{S_\pm}([-n, n])$ denote the spectral projections associated with the self-adjoint operators $S_\pm$, and introduce the sequence of bounded operators

$$K_n = E_n^+ \overline{K} E_n^-, \quad n \in \mathbb{N}.$$  

(6.18)

Clearly, $\text{w-lim}_{n \to \infty} K_n = \overline{K}$, where the limit is taken with respect to the weak operator topology. Lemma 6.2 implies that

$$\overline{K} f = S_+\alpha(S_+)\beta(S_-)f - \alpha(S_+)S_-\beta(S_-)f, \quad f \in \text{dom}(S_-),$$

(6.19)

and therefore, the operator $K_n$ may be alternatively represented by

$$K_n = E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^- - E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^-.$$  

(6.20)

We claim that

$$E_n^+ (h(S_+) - h(S_-)) E_n^- = T_{\phi}^{(S_+ , S_-)}(K_n).$$

(6.21)

Assuming the claim, one finishes the proof of the lemma as follows: Since $\phi \in \mathfrak{A}_0$, the operator $T_{\phi}^{(S_+ , S_-)} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is continuous with respect to the weak operator topology. Observing also that

$$\text{w-lim}_{n \to \infty} E_n^+ (h(S_+) - h(S_-)) E_n^- = h(S_+) - h(S_-),$$

(6.22)

and passing to the limit as $n \to \infty$ in (6.21), one obtains (6.16), completing the proof of Lemma 6.3 (subject to (6.21)).

It remains to prove the claim (6.21) (which is a slight generalization of [52, Lemma 7.1]), that is, we need to show the identity (cf. (6.20))

$$E_n^+ (h(S_+) - h(S_-)) E_n^- = T_{\phi}^{(S_+ , S_-)}(E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^- - E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^-).$$

(6.23)

For this purpose we let $\chi_n$ denote the characteristic function corresponding to spectral projections $E_n^\pm$ and introduce the functions $\phi_\pm$ by

$$\phi_+(\lambda, \mu) = \chi_n(\lambda)\alpha(\lambda)\lambda \phi(\lambda, \mu)\beta(\mu)\chi_n(\mu),$$

$$\phi_-(\lambda, \mu) = \chi_n(\lambda)\alpha(\lambda)\phi(\lambda, \mu)\mu \beta(\mu)\chi_n(\mu).$$

(6.24)

Since the mapping $\phi \to T_\phi$ is a homomorphism of $\mathfrak{A}_0$ into $\mathcal{B}(\mathcal{B}(\mathcal{H}))$ one has

$$T_{\phi_+}^{(S_+ , S_-)}(I) = T_{\phi_+}^{(S_+ , S_-)}(E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^-),$$

(6.25)

and

$$T_{\phi_-}^{(S_+ , S_-)}(I) = T_{\phi_-}^{(S_+ , S_-)}(E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^-),$$

(6.26)

implying

$$T_{\phi_+ - \phi_-}^{(S_+ , S_-)}(I) = T_{\phi}^{(S_+ , S_-)}(E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^- - E_n^+ \alpha(S_+)S_-\beta(S_-)E_n^-).$$

(6.27)
Indeed, the operators $E_n^+ \alpha(S_+)S_\pm \beta(S_-)E_n^-$ in the identities (6.25) and (6.26) are bounded and hence the application of the double operator integral $T_{\varphi}^{(S_+,S_-)}$ to these operators is justified. A direct computation shows that

$$\phi_+(\lambda, \mu) - \phi_- (\lambda, \mu) = \chi_n (\lambda) \left( h(\lambda) - h(\mu) \right) \chi_n (\mu), \quad (6.28)$$

and therefore (again appealing to the fact that the mapping $\varphi \to T_{\varphi}$ is a homomorphism of $\mathfrak{A}_0$ into $\mathcal{B}(\mathcal{B}(\mathcal{H}))$), one has

$$T_{\phi_+ - \phi_-}^{(S_+,S_-)}(I) = E_n^+ \left( T_{h(\lambda)}^{(S_+,S_-)}(I) - T_{h(\mu)}^{(S_+,S_-)}(I) \right) E_n^- = E_n^+ \left( h(S_+) - h(S_-) \right) E_n^- . \quad (6.29)$$

Combining (6.27) and (6.29) yields (6.23).

Next, we turn to the discussion of the analogue of (6.1) for $S_\pm$ in the trace class setting. We recall our usual notation $g(\lambda) = \lambda(\lambda^2 + 1)^{-1/2}$, $\lambda \in \mathbb{R}$. Our intention is to use Lemma 6.3 with $h(\lambda) = g(\lambda)$, and $\alpha(\lambda) = (\lambda^2 + 1)^{-1/4}$ and $\beta(\mu) = (\mu^2 + 1)^{-1/4}$. First, we verify the condition $\phi \in \mathfrak{A}_0$ in Lemma 6.3.

**Lemma 6.4.** The function $\phi$ defined by

$$\phi(\lambda, \mu) := \frac{\lambda(\lambda^2 + 1)^{-1/2} - \mu(\mu^2 + 1)^{-1/2}}{(\lambda^2 + 1)^{-1/4}(\lambda - \mu) (\mu^2 + 1)^{-1/4}}, \quad (\lambda, \mu) \in \mathbb{R}^2, \quad (6.30)$$

belongs to the class $\mathfrak{A}_0$.

**Proof.** Let $(\lambda, \mu) \in \mathbb{R}^2$. A direct calculation (carried out in [120, (4.3)]) reveals:

$$\phi(\lambda, \mu) = (\lambda^2 + 1)^{1/4} \frac{\lambda(\lambda^2 + 1)^{-1/2} - \mu(\mu^2 + 1)^{-1/2}}{(\lambda^2 + 1)^{-1/4}(\lambda - \mu) (\mu^2 + 1)^{-1/4}}$$

$$= \frac{\lambda(\lambda^2 + 1)^{1/2}((\lambda^2 + 1)^{1/2} - (\mu^2 + 1)^{1/2}) (\mu^2 + 1)^{1/2}}{(\lambda^2 + 1)^{1/4}((\lambda^2 + 1) - (\mu^2 + 1))(\mu^2 + 1)^{1/4}}$$

$$+ \frac{(1 - \lambda\mu)((\lambda^2 + 1)^{1/2} - (\mu^2 + 1)^{1/2})}{(\lambda^2 + 1)^{1/4}((\lambda^2 + 1) - (\mu^2 + 1))(\mu^2 + 1)^{1/4}} . \quad (6.31)$$

As a result, one can write

$$\phi(\lambda, \mu) = \psi(\lambda, \mu) + \frac{\psi(\lambda, \mu)}{(\lambda^2 + 1)^{1/2}(\mu^2 + 1)^{1/2}} - \frac{\lambda\psi(\lambda, \mu)\mu}{(\lambda^2 + 1)^{1/2}(\mu^2 + 1)^{1/2}}, \quad (6.32)$$

where we introduced the function

$$\psi(\lambda, \mu) = \frac{(\lambda^2 + 1)^{1/4}(\mu^2 + 1)^{1/4}}{(\lambda^2 + 1)^{1/2} + (\mu^2 + 1)^{1/2}} . \quad (6.33)$$

As soon as one knows that $\psi \in \mathfrak{A}_0$, it is straightforward that $\phi \in \mathfrak{A}_0$ and $\| \phi \|_{\mathfrak{A}_0} \leq 3 \| \psi \|_{\mathfrak{A}_0}$. To begin the proof of the assertion $\psi \in \mathfrak{A}_0$, one introduces the function

$$\zeta(x) = \frac{1}{e^{x/2} + e^{-x/2}}, \quad x \in \mathbb{R}, \quad (6.34)$$

and observes that $\psi(\lambda, \mu)$ in (6.33) can be written as

$$\psi(\lambda, \mu) = \zeta(\log((\lambda^2 + 1)^{1/2}) - \log((\mu^2 + 1)^{1/2})). \quad (6.35)$$

Since $\zeta \in W^{1,2}(\mathbb{R})$, the Sobolev space of functions satisfying $\zeta, \zeta' \in L^2(\mathbb{R}; dx)$, one concludes that $\hat{\zeta} \in L^1(\mathbb{R}; ds)$ for the Fourier transform $\hat{\zeta} = \hat{\zeta}(s)$. Since also
\( \zeta \in L^1(\mathbb{R}; dx) \), the inverse Fourier transform formula yields
\[
\zeta(\lambda - \mu) = \frac{1}{2\pi} \int_\mathbb{R} e^{i s \lambda} e^{-i s \mu} \zeta(s) \, ds, \quad \lambda, \mu \in \mathbb{R}. \tag{6.36}
\]
Combining (6.35) and (6.36) yields
\[
\psi(\lambda, \mu) = \frac{1}{2\pi} \int_\mathbb{R} (\lambda^2 + 1)^{i s / 2} (\mu^2 + 1)^{-i s / 2} \zeta(s) \, ds, \quad \lambda, \mu \in \mathbb{R}, \tag{6.37}
\]
which immediately implies \( \psi \in \mathfrak{A}_0 \) due to \( \tilde{\zeta} \in L^1(\mathbb{R}; ds) \), completing the proof. \( \blacksquare \)

**Remark 6.5.** In the course of the proof of Lemma 6.4 we established formula (6.32), yielding the following decomposition of \( T_\phi \),
\[
T_\phi = T_\psi + (S_+^2 + I)^{-1/2} T_\psi (S_-^2 + I)^{-1/2} - S_+ (S_+^2 + I)^{-1/2} T_\psi S_- (S_-^2 + I)^{-1/2}, \tag{6.38}
\]
where \( T_\psi = T_\psi^{(S_+, S_-)} \) is the operator integral for the function \( \psi \) defined in (6.33) for which we proved the integral representation (6.37). Since \( \psi \in \mathfrak{A}_0 \), which in turn implies \( T_\psi \in \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \), and since both operators \((S_+^2 + I)^{-1/2}\) and \((S_-^2 + I)^{-1/2}\) belong to \( \mathcal{B}(\mathcal{H}) \), it follows from (6.38) that \( T_\phi \in \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \). We will use the decomposition (6.38) and the representation (6.37) in the proof of Proposition 2.5.

**Lemma 6.6.** Assume that \( S_\pm \) are self-adjoint operators in \( \mathcal{H} \) such that
\[
\text{dom}(S_+) = \text{dom}(S_-) \tag{6.39}
\]
and
\[
(S_+ - S_-)(S_+^2 + I)^{-1/2} \in \mathcal{B}_1(\mathcal{H}). \tag{6.40}
\]
Then the closure \( \overline{K} \) of the operator \( K = K(S_+, S_-) \) in \( \mathcal{H} \) defined by
\[
K = (S_+^2 + I)^{-1/4} (S_+ - S_-) (S_-^2 + I)^{-1/4}, \quad \text{dom}(K) = \text{dom}(S_-), \tag{6.41}
\]
satisfies \( \overline{K} \in \mathcal{B}_1(\mathcal{H}) \). Moreover,
\[
g(S_+) - g(S_-) = T_\phi [\overline{K}] \in \mathcal{B}_1(\mathcal{H}), \tag{6.42}
\]
where \( T_\phi \in \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \).

**Proof.** Assumption (6.39) yields
\[
(S_+^2 + I)^{-1/4} (S_-^2 + I)^{1/4} \in \mathcal{B}(\mathcal{H}) \tag{6.43}
\]
by Remark 3.9 with \( A_\pm \) replaced by \( S_\pm \). In addition, the operator \( K \) on \( \text{dom}(K) = \text{dom}(S_-) \) can be represented as follows,
\[
K = (S_+^2 + I)^{-1/4} (S_+ - S_-)(S_-^2 + I)^{-1/4} \\
= [(S_+^2 + I)^{-1/4} (S_-^2 + I)^{-1/4}] (S_+^2 + I)^{-1/4} (S_+ - S_-)(S_-^2 + I)^{-1/4}. \tag{6.44}
\]
Due to (6.40) and Theorem 4.1, the closure of the operator
\[
\overline{K} = (S_+^2 + I)^{-1/4} (S_+ - S_-)(S_-^2 + I)^{-1/4}, \quad \text{dom}(\overline{K}) = \text{dom}(S_-), \tag{6.45}
\]
is a trace class operator, and hence from (6.43) and (6.44) one concludes that \( \overline{K} \in \mathcal{B}_1(\mathcal{H}) \). Next, we choose \( h(\lambda) = g(\lambda) \) and \( \alpha(\lambda) = (\lambda^2 + 1)^{-1/4}, \beta(\mu) = (\mu^2 + 1)^{-1/4} \) in Lemma 6.3. By (6.39) and \( \overline{K} \in \mathcal{B}(\mathcal{H}) \), Hypothesis 6.1 with \( \mathcal{D} = \text{dom}(S_-) \) holds. By Lemma 6.4, the function \( \phi \) in (6.15) belongs to the class \( \mathfrak{A}_0 \), and thus all assumptions of Lemma 6.3 are verified. As a result, (6.42) follows from (6.17). \( \blacksquare \)
Therefore, one can represent the difference under the norm in (6.52) as follows:

\[
\text{dom}(|S_+|^{1/2}) = \text{dom}(|S_-|^{1/2}) \quad \text{and} \quad \text{dom}(S_+) \supseteq \text{dom}(S_-) \quad (6.46)
\]
in place of (6.39) in Lemma 6.6, but we do not pursue this here.

At this point we are ready to prove Proposition 2.5. We switch back to our original notation \(A_\pm\), that is, we will now identify \(S_\pm\) with the self-adjoint operators \(A_\pm\) studied in the previous sections. In particular, we emphasize that Lemma 6.6 is applicable as assumption (6.39) holds by Theorem 3.7(iii) and assumption (6.40) is satisfied by (3.28) (cf. also (3.49)).

**Proof of Proposition 2.5.** The first inclusion in assertion (2.29) of Proposition 2.5 is proved in Lemma 6.6. The second inclusion in (2.29) is proved similarly.

To begin the proof of assertion (2.30), one considers the operator integral

\[
T_\phi^{(n)} = T_\phi^{(A_+,n,A_-,-n)},
\]
with \(\phi\) given in (6.30) and the operators \(K(A_+,A_-)\) and \(K(A_+,n,A_-,-n)\) defined by (6.41) with \(S_\pm\) replaced by \(A_\pm\) and \(A_\pm,n\), respectively. Using formula (6.42), one obtains

\[
g(A_+) - g(A_-) = T_\phi^{(A_+,n,A_-)} - T_\phi^{(A_+,n,A_-,-n)}(K(A_+,n,A_-,-n))
\]

\[
= (T_\phi^{(A_+,n,A_-)} - T_\phi^{(A_+,n,A_-,-n)})(K(A_+,n,A_-,-n))
\]

\[
+ T_\phi^{(A_+,n,A_-,-n)}(K(A_+,n,A_-) - K(A_+,n,A_-,-n)).
\]

Since \(\phi \in \mathfrak{A}_0\) by Lemma 6.4, the sequence of the operators \(T_\phi^{(A_+,n,A_-,-n)}\) is uniformly bounded in the Banach space \(\mathcal{B}(B_1(H))\) (see (6.5)). Thus, to complete the proof of assertion (2.30) it suffices to establish that

\[
\lim_{n \to \infty} \|K(A_+,A_-) - K(A_+,n,A_-,-n)\|_{\mathcal{B}_1(H)} = 0
\]

and

\[
\lim_{n \to \infty} \|T_\phi^{(A_+,n,A_-)}(K) - T_\phi^{(A_+,n,A_-,-n)}(K)\|_{\mathcal{B}_1(H)} = 0 \quad \text{for each } K \in \mathcal{B}_1(H). \quad (6.53)
\]

Starting the proof of (6.52), we recall that \(P_n = E_{A_-(n,n)}\) is the spectral projection associated with \(A_-\), that \(A_{\pm,n} = P_n A_{\pm} P_n\), and we abbreviate \(\xi_\pm = ((A_\pm)^2 + I)^{1/2}, \xi_\pm,n = ((A_\pm,n)^2 + I)^{1/2}\). One observes that \(P_n \xi_-^{-1/2} = P_n \xi_-^{-1/2}\).

Then the following identity holds:

\[
K(A_{+,n},A_-,-n) = ((A_+,n)^2 + I)^{-1/4}P_n(A_{+,n} - A_-,-n)P_n(A_-,-n)^2 + I)^{-1/4}
\]

\[
= ((A_+,n)^2 + I)^{-1/4}P_n(A_{+,n} - A_-,-n)P_n(A_- + I)^{-1/4}
\]

\[
= \xi_-^{-1/2}P_n(A_{+,n} - A_-,-n)P_n\xi_-^{-1/2}. \quad (6.54)
\]

Therefore, one can represent the difference under the norm in (6.52) as follows:

\[
K(A_+,A_-) - K(A_+,n,A_-,-n)
\]

\[
= \xi_-^{-1/2}(A_+ - A_-)\xi_-^{-1/2} - \xi_-^{-1/2}P_n(A_{+,n} - A_-,-n)\xi_-^{-1/2}
\]

\[
= (\xi_+^{-1/2} - \xi_-^{-1/2}P_n)(A_+ - A_-)\xi_-^{-1/2}
\]

\[
+ \xi_+^{-1/2}P_n(A_+ - A_- - (A_{+,n} - A_-,-n))\xi_-^{-1/2}
\]
The convergence theorem yields (6.63), completing the proof of Proposition 2.5. By Theorem VIII.20(b), Lemma 3.4, the integrand in (6.64) converges to zero in $\mathcal{B}_1(\mathcal{H})$ and the operator $\kappa^{-1/2}$ converges to zero in $B$ and Lemma 3.4. The sequence of operators in the second line of (6.55) converges to $\kappa^{-1/2}$ in $B$. The integrand in (6.64) converges to zero in $\mathcal{B}_1(\mathcal{H})$ by Theorem 4.1. Since $\kappa^{-1/2}$ converges to zero in $\mathcal{B}_1(\mathcal{H})$ and Lemma 3.4, the sequence of operators in the second line of (6.55) converges to zero in $\mathcal{B}_1(\mathcal{H})$ by (6.56), (4.98), and Theorem 4.1, proving assertion (6.53).

Starting the proof of assertion (6.53), one uses (6.38) with $\psi_+ - \psi_-$ replaced by $A_+ - A_-$. The sequence of operators in (6.55) converges to $\kappa^{-1/2}$ in $\mathcal{B}_1(\mathcal{H})$ by (6.56), (4.98), and Theorem 4.1, proving assertion (6.53). The proof of assertion (6.53) is completed by the strong resolvent convergence in (4.99) and [126, Theorem VIII.20(b)]. Thus, by Lemma 3.4, to finish the proof of assertion (6.53), it suffices to show that

$$
\lim_{n \to \infty} \left\| T_\psi(K) - T_\psi^{(n)}(K) \right\|_{\mathcal{B}_1(\mathcal{H})} = 0 \quad \text{for each } K \in \mathcal{B}_1(\mathcal{H}).
$$

We will employ the integral representation (6.37),

$$
T_\psi(K) - T_\psi^{(n)}(K) = \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \kappa_{+,-}^{+is} K \kappa_{-}^{-1} - \kappa_{+,-}^{-is} K \kappa_{-}^{-1} \right) \hat{\zeta}(s) \, ds.
$$

Again, $\kappa_{+,-}^{+is}$ is the strong resolvent convergence in (4.99) and [126, Theorem VIII.20(b)]. By Lemma 3.4, the integrand in (6.64) converges to zero in $\mathcal{B}_1(\mathcal{H})$ as $n \to \infty$ for each $s \in \mathbb{R}$. Since $\hat{\zeta} \in L^1(\mathbb{R}; ds)$, the dominated convergence theorem yields (6.63), completing the proof of Proposition 2.5.
7. The Spectral Shift Function for the Pair \((A_+, A_-)\) and Perturbation Determinants

In this section we provide a detailed study of the spectral shift function associated with the pair \((A_+, A_-)\).

Introducing the spectral shift function associated with the pair \((A_+, A_-)\) via the invariance principle one can proceed as follows: One recalls that by Theorem 2.2, the difference of the self-adjoint operators \(g(A_+)\) and \(g(A_-)\), with
\[
g(x) = g_{-1}(x) = x(x^2 + 1)^{-1/2}, \quad x \in \mathbb{R},
\]
is of trace class, that is,
\[
[g(A_+) - g(A_-)] \in \mathcal{B}_1(\mathcal{H}). \tag{7.2}
\]
Bearing in mind the membership \((7.2)\), we define (cf. also [143, eq. 8.11.4])
\[
\xi(\nu; A_+, A_-) := \xi(\nu; g(A_+), g(A_-)), \quad \nu \in \mathbb{R},
\]
where \(\xi(\cdot; g(A_+), g(A_-))\) is the spectral shift function associated with the pair \((g(A_+), g(A_-))\) uniquely determined by the requirement (cf. [143, Sects. 9.1, 9.2])
\[
\xi(\cdot; g(A_+), g(A_-)) \in L^1(\mathbb{R}; d\omega). \tag{7.4}
\]
One recalls that since \(|g(A_+)| \leq 1\), \(\xi(\cdot; g(A_+), g(A_-))\) is a real-valued function supported on the interval \([-1, 1]\),
\[
\text{supp}(\xi(\cdot; g(A_+), g(A_-))) \subseteq [-1, 1], \tag{7.5}
\]
and
\[
\begin{align*}
\xi(\omega; g(A_+), g(A_-)) &= \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im} \left( \ln(\det_{\mathcal{H}}(I + (g(A_+)) - g(A_-))(g(A_-) - (\omega + i\varepsilon)I)^{-1}) \right) \quad \text{for a.e. } \omega \in [-1, 1].
\end{align*}
\]
Here the choice of branch of \(\ln(\det_{\mathcal{H}}(\cdot))\) on \(\mathbb{C}_+\) is again chosen such that
\[
\lim_{\text{Im}(z) \to +\infty} \ln(\det_{\mathcal{H}}(I + (g(A_+)) - g(A_-))(g(A_-) - zI)^{-1})) = 0. \tag{7.7}
\]

Moreover, since \((7.2)\) holds, Krein’s trace formula in its simplest form yields (cf. [143, Theorem 8.2.1])
\[
\text{tr}_{\mathcal{H}} (g(A_+) - g(A_-)) = \int_{[-1, 1]} \xi(\omega; g(A_+), g(A_-)) d\omega. \tag{7.8}
\]
Alternatively, one can also introduce the spectral shift function associated with the pair \((A_+, A_-)\) taking into account that the difference of the resolvents of the operators \(A_+\) and \(A_-\) is of trace class (cf. (3.30)), that is,
\[
[(A_+ - zI)^{-1} - (A_- - zI)^{-1}] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \rho(A_+) \cap \rho(A_-). \tag{7.9}
\]
Since in this case the difference of the Cayley transforms of the operators \(A_+\) and \(A_-\) is of trace class, one can introduce the spectral shift function \(\hat{\xi}(\cdot; A_+, A_-)\) associated with the pair \((A_+, A_-)\) upon relating \(\hat{\xi}(\cdot; A_+, A_-)\) to the spectral shift function associated with the Cayley transforms of \(A_+\) and \(A_-\) as in [143, eq. (8.7.4)].

The spectral shift function introduced in this way is not unique, in fact, any two of
then differ by an integer-valued homotopy invariant (see a comprehensive discussion of this phenomenon in [143, Sect. 8.6]). Moreover,
\[ \hat{\xi}(\cdot; A_+, A_-) \in L^1(\mathbb{R}; (|\nu| + 1)^{-2}d\nu) \] (7.10)
for any concrete choice of the integer-valued constant (cf. [143, Sect. 8.7]). Given the pair \((A_+, A_-)\), we now arbitrarily fix the undetermined integer-valued constant, and for simplicity, keep denoting the corresponding spectral shift function by \(\hat{\xi}(\cdot; A_+, A_-)\).

Our next result states that the functions \(\xi(\cdot; A_+, A_-)\) and \(\hat{\xi}(\cdot; A_+, A_-)\) differ at most by a constant:

**Lemma 7.1.** Assume Hypothesis 2.1. Let the spectral shift function \(\xi(\cdot; A_+, A_-)\) be defined according to (7.3) and \(\hat{\xi}(\cdot; A_+, A_-)\) as in [143, eq. (8.7.4)] (with some determination of the associated integer-valued constant). Then there exists a \(C \in \mathbb{R}\) such that
\[ \hat{\xi}(\nu; A_+, A_-) = \xi(\nu; A_+, A_-) + C \quad \text{for a.e. } \nu \in \mathbb{R}. \] (7.11)

**Proof.** First we note that by [143, Theorem 8.7.1] the trace formula
\[ \text{tr}_H(f(A_+) - f(A_-)) = \int_{\mathbb{R}} f'(\nu) \hat{\xi}(\nu; A_+, A_-) d\nu \] (7.12)
holds for the class of functions \(f\) having two locally bounded derivatives and satisfying the conditions
\[ \text{for some } \varepsilon > 0, \quad (\nu^2 f'(\nu))^' = O(|\nu|^{-1-\varepsilon}) \] (7.13)
and
\[ \lim_{\nu \to -\infty} f(\nu) = \lim_{\nu \to +\infty} f(\nu), \quad \lim_{\nu \to -\infty} \nu^2 f'(\nu) = \lim_{\nu \to +\infty} \nu^2 f'(\nu). \] (7.14)
This class includes, in particular, the functions of the type
\[ f \in C^\infty_0(\mathbb{R}) \text{ and } (\cdot - z)^{-n}, \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad n \in \mathbb{N}, \quad n \geq 1. \] (7.15)
Since (7.2) holds, [143, Lemma 8.11.3] applies to the \(\xi\)-function given by the invariance principle (7.3) and hence the trace formula
\[ \text{tr}_H(f(A_+) - f(A_-)) = \int_{\mathbb{R}} f'(\nu) \xi(\nu; A_+, A_-) d\nu \] (7.16)
holds for all \(f \in C^\infty_0(\mathbb{R})\). Comparing (7.16) and (7.12) one obtains that
\[ \int_{\mathbb{R}} f'(\nu) \xi(\nu; A_+, A_-) d\nu = \int_{\mathbb{R}} f'(\nu) \hat{\xi}(\nu; A_+, A_-) d\nu, \quad f \in C^\infty_0(\mathbb{R}), \] (7.17)
and therefore, by the Du Bois–Raymond Lemma (see, e.g., [100, Theorem 6.11]), the functions \(\xi(\cdot; A_+, A_-)\) and \(\hat{\xi}(\cdot; A_+, A_-)\) differ a.e. at most by a constant. ■

**Remark 7.2.** The fact that \(\xi(\cdot; g(A_+), g(A_-)) \in L^1(\mathbb{R}; d\omega)\) according to (7.4), implies the membership
\[ \xi(\cdot; A_+, A_-) \in L^1(\mathbb{R}; (|\nu| + 1)^{-3}d\nu) \] (7.18)
which can easily be verified taking into account the definition (7.3) of \(\xi(\cdot; A_+, A_-)\) and using the change of variables (7.28) below. While (7.18) is correct, it is not optimal, since, in fact,
\[ \xi(\cdot; A_+, A_-) \in L^1(\mathbb{R}; (|\nu| + 1)^{-2}d\nu) \] (7.19)
as a consequence of (7.10) and (7.11). Moreover, the following trace formulas hold,

\[-\text{tr}_H((A_+ - zI)^{-1} - (A_- - zI)^{-1}) = \int_R \frac{\hat{\xi}(\nu; A_+, A_-) \, d\nu}{(\nu - z)^2}, \quad z \in \mathbb{C} \setminus \mathbb{R},\]

(7.20)

with two convergent Lebesgue integrals in (7.20). Indeed, the first equality in (7.20) follows from (7.12) and (7.15), and the second from the observation that

\[\int_R \frac{d\nu}{(\nu - z)^2} = 0, \quad z \in \mathbb{C} \setminus \mathbb{R},\]

(7.21)

and the fact that by (7.11), \(\xi(\cdot; A_+, A_-)\) and \(\hat{\xi}(\cdot; A_+, A_-)\) differ at most by a constant.

Our next result provides a refinement of the trace formula (7.8). For this purpose we recall the function \(g_z\) defined by

\[g_z(x) = x(x^2 - z)^{-1/2}, \quad x \in \mathbb{R}, \quad z \in \mathbb{C} \setminus [0, \infty),\]

(7.22)

**Lemma 7.3.** Assume Hypothesis 2.1 and define \(\xi(\cdot; A_+, A_-)\) according to (7.3). Then

\[[g_z(A_+) - g_z(A_-)] \in \mathcal{B}_1(H), \quad z \in \mathbb{C} \setminus [0, \infty),\]

(7.23)

and the following trace formula holds

\[\text{tr}_H(g_z(A_+) - g_z(A_-)) = -z \int_R \frac{\xi(\nu; A_+, A_-) \, d\nu}{(\nu^2 - z)^{3/2}}, \quad z \in \mathbb{C} \setminus [0, \infty).\]

(7.24)

In particular,

\[\mathbb{C} \setminus [0, \infty) \ni z \mapsto \text{tr}_H(g_z(A_+) - g_z(A_-)) \text{ is analytic.}\]

(7.25)

**Proof.** We start with the representation (7.8)

\[\text{tr}_H(g(A_+) - g(A_-)) = \int_{[-1,1]} \xi(\omega; g(A_+), g(A_-)) \, d\omega.\]

(7.26)

Since

\[g'(\nu) = (\nu^2 + 1)^{-3/2} > 0, \quad \nu \in \mathbb{R},\]

(7.27)

one can introduce the change of variables

\[\omega = g(\nu) = \nu(\nu^2 + 1)^{-1/2}, \quad \nu \in \mathbb{R},\]

(7.28)

implying

\[\text{tr}_H(g(A_+) - g(A_-)) = \int_R \frac{\xi(g(\nu); g(A_+), g(A_-)) \, d\nu}{(\nu^2 + 1)^{3/2}}\]

(7.29)

and hence, in accordance with the definition (7.3) of the spectral shift function \(\xi(\cdot; A_+, A_-)\), one also obtains that

\[\text{tr}_H(g(A_+) - g(A_-)) = \int_R \frac{\xi(\nu; A_+, A_-) \, d\nu}{(\nu^2 + 1)^{3/2}}.\]

(7.30)

which proves the trace formula (7.24) for \(z = -1\).

To handle the case of arbitrary \(z \in \mathbb{C} \setminus [0, \infty)\), we remark that the function \(G_z\) given by

\[G_z(\nu) = g_z(\nu) - g_{-1}(\nu), \quad \nu \in \mathbb{R}, \quad z \in \mathbb{C} \setminus [0, \infty),\]

(7.31)
satisfies the conditions (7.13) and (7.14), and hence by [143, Theorem 8.7.1], one obtains

\[ [G_z(A_+) - G_z(A_-)] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C} \setminus [0, \infty), \quad (7.32) \]

and the trace formula

\[ \text{tr}_\mathcal{H}(G_z(A_+) - G_z(A_-)) = \int_{\mathbb{R}} G'_z(\nu) \document{ch}(\nu; A_+, A_-) d\nu, \quad z \in \mathbb{C} \setminus [0, \infty), \quad (7.33) \]

where the spectral shift function \( \document{ch}(\cdot; A_+, A_-) \) associated with the pair \( (A_+, A_-) \) is introduced as in [143, eq. (8.7.4)]. By Lemma 7.1, the spectral shift functions \( \document{ch}(\cdot; A_+), A_-) \) and \( \document{ch}(\cdot; A_+, A_-) \) differ at most by a constant and hence, since

\[ \lim_{\nu \to \pm \infty} G_z(\nu) = \lim_{\nu \to \pm \infty} G_z(\nu) = 0, \quad (7.34) \]

eq (7.33) can be rewritten as

\[ \text{tr}_\mathcal{H}(G_z(A_+) - G_z(A_-)) = \int_{\mathbb{R}} G'_z(\nu) \document{ch}(\nu; A_+, A_-) d\nu, \]

\[ = \int_{\mathbb{R}} \left[ -\frac{z}{(\nu^2 - z)^{3/2}} - \frac{1}{(\nu^2 + 1)^{3/2}} \right] \document{ch}(\nu; A_+, A_-) d\nu, \quad z \in \mathbb{C} \setminus [0, \infty). \quad (7.35) \]

Combining (7.2), (7.31), and (7.32), one concludes that (7.23) and the trace formula (7.24) hold. \( \square \)

The following result, an improvement of (7.23) and (7.25), will be proved in Appendix B:

**Lemma 7.4.** Assume Hypothesis 2.1 and let \( z \in \mathbb{C} \setminus [0, \infty) \). Then \( [g_z(A_+), g_z(A_-)] \) is differentiable with respect to the \( \mathcal{B}_1(\mathcal{H}) \)-norm and

\[ \frac{d}{dz} \text{tr}_\mathcal{H}(g_z(A_+) - g_z(A_-)) = \text{tr}_\mathcal{H}\left( \frac{d}{dz} g_z(A_+) - \frac{d}{dz} g_z(A_-) \right) \]

\[ = \frac{1}{2} \text{tr}_\mathcal{H}(A_+(A_+^2 - zI)^{-3/2} - A_-(A_-^2 - zI)^{-3/2}), \quad z \in \mathbb{C} \setminus [0, \infty). \quad (7.36) \]

We note that Lemmas 7.3 and 7.4 extend to \( z \in \rho(A_+^2) \cap \rho(A_-^2) \).

Next, we prove the following result which justifies equalities of (2.46) and (2.49) in Theorem 2.10:

**Lemma 7.5.** Assume Hypothesis 2.1 and \( 0 \in \rho(A_+) \cap \rho(A_-) \). Then

\[ \text{tr}_\mathcal{H}(E_{A_+}((-\infty, 0)) - E_{A_-}((-\infty, 0))) = \xi(0; A_+, A_-). \quad (7.37) \]

**Proof.** Since \( 0 \in \rho(A_+) \), the spectral mapping property implies \( 0 \in \rho(g(A_+)) \) for \( g(x) = x(x^2 + 1)^{-1/2} \). Fixing \( \nu_0 > 0 \) such that \( [-\nu_0, \nu_0] \subset \rho(g(A_-)) \cap \rho(g(A_+)) \), one notes that \( \xi(\cdot; g(A_+), g(A_-)) = \xi(0; g(A_+), g(A_-)) \) a.e. on the interval \( (-\nu_0, \nu_0) \).

In addition, we introduce a smooth cut-off function \( \varphi \in C^\infty(\mathbb{R}) \) satisfying

\[ \varphi(\nu) = \begin{cases} 
1, & \nu \leq -\nu_0, \\
0, & \nu \geq \nu_0,
\end{cases} \quad \text{and} \quad \int_{-\nu_0}^{\nu_0} \varphi'(\nu) d\nu = -1. \quad (7.38) \]

Next, using a change of variables in the spectral theorem [57, Theorem XII.2.9(c)], and noting that \( \varphi \) coincides with the characteristic function of \((\nu, 0)\) on the spectrum of \( g(A_+) \), one infers,

\[ E_{A_+}((-\infty, 0)) = E_{A_+}(g^{-1}(-1, 0)) = E_{g(A_+)}(-1, 0) = \varphi(g(A_+)). \quad (7.39) \]
We recall that \([g(A_+) - g(A_-)] \in \mathcal{B}_1(\mathcal{H})\) by Proposition 2.5. Thus, Krein’s trace formula holds for the pair of bounded operators \(g(A_+)\) and \(g(A_-)\) and the spectral shift function \(\xi(\cdot; g(A_+), g(A_-))\) (cf. [143, Theorem 8.2.1]). Using (7.38), (7.39), and the trace formula, one then completes the proof as follows:

\[
\begin{align*}
\text{tr}_H(E_{A_+}((-\infty, 0)) - E_{A_-}((-\infty, 0))) &= \text{tr}_H\left(\varphi(g(A_+)) - \varphi(g(A_-))\right) = \int_{-\infty}^{\infty} \xi(\nu; g(A_+), g(A_-)) \varphi'(\nu) \, d\nu \\
&= \int_{-\nu_0}^{\nu_0} \xi(\nu; g(A_+), g(A_-)) \varphi'(\nu) \, d\nu = \xi(0; g(A_+, g(A_-)) \int_{-\nu_0}^{\nu_0} \varphi'(\nu) \, d\nu \\
&= -\xi(0; g(A_+), g(A_-)) = -\xi(0; A_+, A_-),
\end{align*}
\]

utilizing (7.3) in the last equality.

In the final part of this section we detail the precise connection between \(\xi\) and Fredholm perturbation determinants associated with the pair \((A_-, A_+)\). In particular, this will justify the perturbation determinants formula (2.50) in the index computation in Theorem 2.10. In practice, these determinants are often simpler to handle than the projection operators used in (2.48) and (2.49).

Let

\[
D_{T/S}(z) = \det_H((T - zI)(S - zI)^{-1}) = \det_H(I + (T - S)(S - zI)^{-1}, \quad z \in \rho(S),
\]

denote the perturbation determinant for the pair of operators \((S, T)\) in \(\mathcal{H}\), assuming \((T - S)(S - z_0)^{-1} \in \mathcal{B}_1(\mathcal{H})\) for some (and hence for all) \(z_0 \in \rho(S)\).

**Theorem 7.6.** Assume Hypothesis 2.1 and \(0 \in \rho(A_-) \cap \rho(A_+).\) Then

\[
\xi(\lambda; A_+, A_-) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(\ln(D_{A_+//A_-}(\lambda + i\varepsilon))) \quad \text{for a.e. } \lambda \in \mathbb{R},
\]

where \(\xi(\cdot; A_+, A_-)\) is introduced by (7.3) and we make the choice of branch of \(\ln(D_{A_+//A_-}(\cdot))\) on \(\mathbb{C}_+\) such that \(\lim_{\text{Im}(z) \to +\infty} \ln(D_{A_+//A_-}(z)) = 0\). In particular, for a continuous representative of \(\xi(\cdot; A_+, A_-)\) in a neighborhood of \(\lambda = 0\) the equality

\[
\xi(0; A_+, A_-) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(\ln(D_{A_+//A_-}(i\varepsilon)))
\]

holds.

**Proof.** By (7.20) in Remark 7.2,

\[
-\text{tr}_H\left((A_+ - zI)^{-1} - (A_- - zI)^{-1}\right) = \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)}{(\nu - z)^2} \, d\nu, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

with a convergent Lebesgue integral in (7.44).

The general formula for the logarithmic derivative of the perturbation determinant (see, e.g., [71, Sect. IV.3]) yields

\[
\frac{d}{dz} \ln(D_{A_+//A_-}(z)) = -\text{tr}_H\left((A_+ - zI)^{-1} - (A_- - zI)^{-1}\right), \quad z \in \rho(A_+) \cap \rho(A_-).
\]

A comparison of (7.44) and (7.45) yields

\[
\frac{d}{dz} \ln(D_{A_+//A_-}(z)) = \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)}{(\nu - z)^2} \, d\nu, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
Integrating (7.46) with respect to $z$ (cf. also [93, eq. (1.10)]), one obtains
\[
\ln(D_{A_+/A_-}(z)) = \gamma + \int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{\nu}{\nu^2 + 1} \right) \xi(\nu; A_+, A_-) \, d\nu, \quad z \in \mathbb{C}_+, \quad (7.47)
\]
for some constant $\gamma \in \mathbb{C}$.

Next, we claim that actually,
\[
\gamma \in \mathbb{R}. \quad (7.48)
\]
Indeed, taking $z \in \mathbb{C}$ and letting $|\text{Im}(z)| \to \infty$, one infers that
\[
\lim_{|\text{Im}(z)| \to \infty} D_{A_+/A_-}(z) = 1, \quad (7.49)
\]
similarly to the proof of Lemma 3.5. More precisely, one uses the fact that
\[
(A_+ - A_-)(A_- - zI)^{-1} = [(A_+ - A_-)A_-^{-1}][A_-(A_- - zI)^{-1}], \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (7.50)
\]
implies
\[
\lim_{|\text{Im}(z)| \to \infty} \| (A_+ - A_-)(A_- - zI)^{-1} \|_{B_1(\mathcal{H})} = 0 \quad (7.51)
\]
since
\[
(A_+ - A_-)A_-^{-1} \in B_1(\mathcal{H}) \quad \text{and} \quad \text{s-lim}_{|\text{Im}(z)| \to \infty} A_-(A_- zI)^{-1} = 0, \quad (7.52)
\]
employing Lemma 3.4. Clearly, (7.50) and (7.51) yield (7.49). Hence we now fix
\[
\lim_{\text{Im}(z) \to \pm \infty} \ln(D_{A_+/A_-}(z)) = 0. \quad (7.53)
\]
Rewriting (7.47) in the form
\[
\ln(D_{A_+/A_-}(iy)) = \text{Re}(\gamma) + \int_{\mathbb{R}} \left( \frac{\nu}{\nu^2 + y^2} - \frac{\nu}{\nu^2 + 1} \right) \xi(\nu; A_+, A_-) \, d\lambda
\]
\[
+ i \left[ \text{Im}(\gamma) + y \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-) \, d\nu}{\nu^2 + y^2} \right], \quad y > 0, \quad (7.54)
\]
and applying the dominated convergence theorem to conclude that
\[
\lim_{y \to \infty} y \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-) \, d\nu}{\nu^2 + y^2} = 0, \quad (7.55)
\]
combining (7.49) with taking the limit $y \to \infty$ in (7.54) yields $\text{Im}(\gamma) = 0$ and hence (7.48).

Decomposing $\xi$ into its positive and negative parts $\xi_{\pm}$, respectively,
\[
\xi(\cdot; A_+, A_-) = \xi_{+}(\cdot; A_+, A_-) - \xi_{-}(\cdot; A_+, A_-),
\]
\[
\xi_{\pm}(\cdot; A_+, A_-) = \left( \left| \xi(\cdot; A_+, A_-) \right| \pm \xi(\cdot; A_+, A_-) \right) / 2, \quad (7.56)
\]
and applying the Stieltjes inversion formula to the absolutely continuous measures
$\xi_{\pm}(\nu; A_+, A_-) \, d\nu$ (cf., e.g., [12, p. 328], [141, App. B]), then yields (7.42). Since
by hypothesis, $0 \notin \rho(A_-) \cap \rho(A_+)$, one concludes (7.43) (cf. also the discussion in connection with (8.36) which defines $\xi(0; A_+, A_-)$) as follows: Given the fact (7.48), one obtains that
\[
\ln(D_{A_+/A_-}(z)) = \gamma + \int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{\nu}{\nu^2 + 1} \right) \xi(\nu; A_+, A_-) \, d\nu
\]
\[
= \gamma + \xi(0; A_+, A_-) \int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{\nu}{\nu^2 + 1} \right) \, d\nu.
\]
+ \int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{\nu}{\nu^2 + 1} \right) [\xi(\nu; A_+, A_-) - \xi(0; A_+, a_-)] d\nu \\
= \gamma + i\pi \xi(0; A_+, A_-) \\
+ \int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{\nu}{\nu^2 + 1} \right) [\xi(\nu; A_+, A_-) - \xi(0; A_+, a_-)] d\nu, \quad z \in \mathbb{C}_+,
\tag{7.57}
\end{equation}

using

\begin{equation}
\int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{\nu}{\nu^2 + 1} \right) d\nu = i\pi. \tag{7.58}
\end{equation}

Since the last integral in (7.57) is supported in \((-\infty, -\varepsilon) \cup (\varepsilon, \infty)\) for some \(\varepsilon > 0\) and hence real-valued for \(z = 0\) (as \(\xi(\cdot; A_+, A_-)\) is constant a.e. in a sufficiently small neighborhood of the origin), (7.57) proves (7.43) taking \(z = i\varepsilon\) and \(\varepsilon \downarrow 0\).

\textbf{Remark 7.7.} Given the fact (7.48), one explicitly obtains

\begin{equation}
\gamma = \text{Re}(\ln(D_{A_+/A_-}(i))). \tag{7.59}
\end{equation}

Moreover, from

\begin{equation}
\ln(D_{A_+/A_-}(z)) - \ln(D_{A_+/A_-}(i)) = \int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{1}{\nu - i} \right) \xi(\nu; A_+, A_-) d\nu \\
= -i \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)}{\nu^2 + 1} d\nu + \int_{\mathbb{R}} \left( \frac{1}{\nu - z} - \frac{\nu}{\nu^2 + 1} \right) \xi(\nu; A_+, A_-) d\nu, \quad z \in \mathbb{C}_+,\n\tag{7.60}
\end{equation}

one concludes that

\begin{equation}
\text{Im}(\ln(D_{A_+/A_-}(i))) = \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)}{\nu^2 + 1} d\nu. \tag{7.61}
\end{equation}

\textbf{Remark 7.8.} To illustrate the relevance of the choice of branch of \(\ln(D_{A_+/A_-}(\cdot))\) we briefly look at the following elementary situation where \(\mathcal{H} = \mathbb{C}^2\), \(A_\pm = \pm I_2\). Then obviously,

\begin{equation}
D_{I_2/I_2}(z) = \left( \frac{z - 1}{z + 1} \right)^2, \quad z \in \mathbb{C}\setminus\{-1\}. \tag{7.62}
\end{equation}

The function \(\ln(D_{I_2/I_2}(\cdot))\) has the branch points \(\pm 1\) (we note, however, that the point \(z = \infty\) is not a branch point of this function). Applying our convention of choosing the principal branch of \(\ln(D_{I_2/I_2}(\cdot))\) near infinity then yields that

\begin{equation}
\ln(D_{I_2/I_2}(z)) = 2\ln\left(1 - 2(z + 1)^{-1}\right) = -\frac{4}{z + 1} + O(|z|^{-2}). \tag{7.63}
\end{equation}

Taking into account the branch cut \([-1, 1]\) for \(\ln(D_{I_2/I_2}(\cdot))\) then implies

\begin{equation}
\lim_{\varepsilon \downarrow 0} \ln(D_{I_2/I_2}(\lambda \pm i\varepsilon)) = \begin{cases} 
2\ln((\lambda - 1)/(\lambda + 1)), & \lambda \in \mathbb{R}\setminus[-1, 1], \\
2\ln((\lambda - 1)/(\lambda + 1)) \pm 2\pi i, & \lambda \in (-1, 1),
\end{cases} \tag{6.64}
\end{equation}

and hence,

\begin{equation}
\xi(\lambda; I_2, -I_2) = \begin{cases} 
0, & \lambda \in \mathbb{R}\setminus[-1, 1], \\
2, & \lambda \in (-1, 1),
\end{cases} \tag{6.65}
\end{equation}

consistent with the spectral flow \(\text{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) = 2\) in an example where \(A(t), t \in \mathbb{R}\), has asymptotes \(A_{\pm} = \pm I_2\) as \(t \to \pm \infty\) (cf. Section 9 for the notion of the spectral flow).
We conclude this section with the following known fact under the additional hypothesis of $A_-$ being bounded from below:

**Remark 7.9** ([33], Proposition 6.5, [93], [144]). Assume Hypothesis 2.1 and, also, $0 \in \rho(A_-) \cap \rho(A_+)$. In addition, assume that $A_-$ (and hence $A_+ \cap A(t), t \in \mathbb{R}$) is bounded from below. Then one obtains the following refinements of (7.19), (7.45), (7.48), and (7.59),

\[
\xi(\cdot; A_+, A_-) \in L^1\left(\mathbb{R}; (|\lambda| + 1)^{-1}d\lambda\right),
\]

\[
\ln(D_{A_+/A_-}(z)) = \int_{\mathbb{R}} \frac{\xi(\lambda; A_+, A_-)}{\lambda - z} d\lambda, \quad z \in \mathbb{C}_+,
\]

\[
\gamma = \int_{\mathbb{R}} \frac{\lambda \xi(\lambda; A_+, A_-)}{\lambda^2 + 1} d\lambda.
\]

**8. The Spectral Shift Function for the Pair $(H_2, H_1)$ and an Index Computation**

In this section we will prove one of our principal results, an extension of Pushnitski’s formula [123], relating a particular choice of spectral shift functions of the two pairs of operators, $(H_2, H_1)$, and $(A_+, A_-)$.

**8.1. Pushnitski’s Formula.** We start by introducing the spectral shift function $\xi(\cdot; H_2, H_1)$ associated with the pair $(H_2, H_1)$. Since $H_2 \geq 0$ and $H_1 \geq 0$, and

\[
[(H_2 + I)^{-1} - (H_1 + I)^{-1}] \in B_1(L^2(\mathbb{R}; \mathcal{H})),
\]

by Lemma 5.1, one uniquely introduces $\xi(\cdot; H_2, H_1)$ by requiring that

\[
\xi(\lambda; H_2, H_1) = 0, \quad \lambda < 0,
\]

and

\[
\text{tr}_{L^2(\mathbb{R}; \mathcal{H})}((H_2 - zI)^{-1} - (H_1 - zI)^{-1}) = -\int_{(0, \infty)} \frac{\xi(\lambda; H_2, H_1)}{(\lambda - z)^2} d\lambda, \quad z \in \mathbb{C} \setminus [0, \infty).
\]

Following [143, Sect. 8.9]. In addition, one has

\[
\xi(\cdot; H_2, H_1) \in L^1\left(\mathbb{R}; (|\lambda| + 1)^{-2}d\lambda\right).
\]

However, (8.4) can be improved as follows:

**Lemma 8.1.** Assume Hypothesis 2.1. Then

\[
\xi(\cdot; H_2, H_1) \in L^1\left(\mathbb{R}; (|\lambda| + 1)^{-1}d\lambda\right)
\]

and

\[
\xi(\lambda; H_2, H_1) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(\ln(\widehat{D}_{H_2/H_1}(\lambda + i\varepsilon))) \text{ for a.e. } \lambda \in \mathbb{R},
\]

where we used the abbreviation

\[
\widehat{D}_{H_2/H_1}(z) = \det_{L^2(\mathbb{R}; \mathcal{H})}((H_1 - zI)^{-1/2}(H_2 - zI)^{-1}(H_1 - zI)^{-1/2})
= \det_{L^2(\mathbb{R}; \mathcal{H})}(I + 2(H_1 - zI)^{-1/2}B'(H_1 - zI)^{-1/2}), \quad z \in \rho(H_1).
\]
Proof. This follows from results of Krein and Yavryan [93] (see also [33, Proposition 6.5]) and the fact that
\[
(H_1 - zI)^{-1/2}B'(H_1 - zI)^{-1/2} = \left((H_0 - zI)^{1/2}((H_1 - zI)^{-1/2})^\ast\right)^\ast
\]
(8.8)
\[
\times \left([B']^{1/2}((H_0 - zI)^{-1/2})^\ast\right)\cup B'[|B'|^{1/2}(H_0 - zI)^{-1/2}]
\]
(8.9)
\[
\times \left[(H_0 - zI)^{1/2}(H_1 - zI)^{-1/2}\right] \in \mathcal{B}_1(L^2(\mathbb{R}; H)), \quad z \in \rho(H_1),
\]
(8.10)
applying (4.52), (4.69), and (4.76). 

Given these preparations, one can prove the following result, an extension of Pushnitski’s formula [123]:

**Theorem 8.2.** Assume Hypothesis 2.1 and define \(\xi(\cdot; A_+, A_-)\) and \(\xi(\cdot; H_2, H_1)\) according to (7.3) and (8.2), (8.3), respectively. Then, \(\xi(\lambda; H_2, H_1) = \frac{1}{\pi} \int_{-\lambda/2}^{\lambda/2} \frac{\xi(\nu; A_+, A_-) d\nu}{(\lambda - \nu^2)^{1/2}}\) for a.e. \(\lambda > 0\),

(8.11)

with a convergent Lebesgue integral on the right-hand side of (8.11).

Proof. By Lemma 7.3 one has
\[
\text{tr}_H\left(g_z(A_+) - g_z(A_-)\right) = -z \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-) d\nu}{(\nu^2 - z)^{3/2}}, \quad z \in \mathbb{C}\setminus[0, \infty).
\]
(8.12)
The trace identity (2.19) then yields
\[
\int_{[0, \infty)} \frac{\xi(\lambda; H_2, H_1) d\lambda}{(\lambda - z)^{-2}} = \frac{1}{2} \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-) d\nu}{(\nu^2 - z)^{3/2}}, \quad z \in \mathbb{C}\setminus[0, \infty),
\]
(8.13)
and hence,
\[
\int_{[0, \infty)} \xi(\lambda; H_2, H_1) \left(\frac{d}{dz} (\lambda - z)^{-1}\right) d\lambda = \int_{\mathbb{R}} \xi(\nu; A_+, A_-) \left(\frac{d}{dz}(\nu^2 - z)^{-1/2}\right) d\nu,
\]
\[
\quad z \in \mathbb{C}\setminus[0, \infty).
\]
(8.14)
Integrating (8.14) with respect to \(z\) from a fixed point \(z_0 \in (-\infty, 0)\) to \(z \in \mathbb{C}\setminus\mathbb{R}\) along a straight line connecting \(z_0\) and \(z\) then results in
\[
\int_{[0, \infty)} \xi(\lambda; H_2, H_1) \left(\frac{1}{\lambda - z} - \frac{1}{\lambda - z_0}\right) d\lambda
\]
\[
= \int_{\mathbb{R}} \xi(\nu; A_+, A_-) \left[(\nu^2 - z)^{-1/2} - (\nu^2 - z_0)^{-1/2}\right] d\nu, \quad z \in \mathbb{C}\setminus[0, \infty),
\]
(8.15)
One notes that \(\left[(\nu^2 - z)^{-1/2} - (\nu^2 - z_0)^{-1/2}\right] = O(|\nu|^{-3})\) as \(|\nu| \to \infty\), compatible with the fact (7.10) and similarly, \(\left[(\lambda - z)^{-1} - (\lambda - z_0)^{-1}\right] = O(|\lambda|^{-2})\), compatible with the fact (8.4).

Applying the Stieltjes inversion formula (cf., e.g., [12], [141, Theorem B.3]) to (8.15) then yields
\[
\xi(\lambda; H_2, H_1) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{[0, \infty)} \xi(\lambda'; H_2, H_1) \text{Im}((\lambda' - \lambda) - i\varepsilon)^{-1}) d\lambda'
\]
\[
= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}} \xi(\nu; A_+, A_-) \text{Im}((\nu^2 - \lambda - i\varepsilon)^{-1/2}) d\nu
\]

The last step in (8.16) still warrants some comments: One splits $\mathbb{R}$ into the two regions $0 \leq \nu^2 \leq \lambda + 1$ and $\nu^2 \geq \lambda + 1$. In the compact region $0 \leq \nu^2 \leq \lambda + 1$ one can immediately apply Lebesgue’s dominated convergence theorem since $\xi(\cdot; A_+, A_-)$ is locally integrable. One also uses that $\text{Im}(\nu^2 - \lambda)^{-1/2} = 0$ for $\nu^2 \in [\lambda, \lambda + 1]$ and that $\xi(\nu; A_+, A_-)(\lambda - \nu^2)^{-1/2}$ is locally integrable for a.e. $\lambda > 0$.

The latter fact can be seen as follows: Decomposing $(\lambda - \nu^2)^{-1/2}$ into $(\lambda^{1/2} - \nu)^{-1/2}(\lambda^{1/2} + \nu)^{-1/2}$, and focusing on the case $\nu \geq 0$ at first, one sees that only the factor $(\lambda^{1/2} - \nu)^{-1/2}$ is relevant in this case and one can reduce matters to a convolution estimate. Thus, one introduces

$$f_R(\nu) = \begin{cases} \nu^{-1/2}, & 0 < \nu < R, \\ 0, & \nu > R, \nu < 0, \end{cases} \quad R > 0, \quad g(\nu) = \begin{cases} |\xi(\nu; A_+, A_-)|, & \nu > 0, \\ 0, & \nu < 0. \end{cases}$$

Then $f_R, g \in L^1(\mathbb{R}; d\nu)$, and hence a special case of Minkowski’s inequality (which in turn is a special case of Young’s inequality, $\|h \ast k\|_r \leq \|h\|_p \|k\|_q$, $1 \leq p, q, r \leq \infty$, $p^{-1} + q^{-1} = 1 + r^{-1}$, with $\|\cdot\|_p$ the norm in $L^p(\mathbb{R}; d\lambda)$, cf., e.g., [74, p. 20–22]), shows that $f_R \ast g \in L^1(\mathbb{R}; d\lambda)$, in particular, $(f_R \ast g)(\lambda)$ exists for a.e. $\lambda > 0$. Since $R > 0$ is arbitrary, $\xi(\nu; A_+, A_-)(\lambda - \nu^2)^{-1/2}$ is locally integrable with respect to $\nu$ on $[0, \infty)$ for a.e. $\lambda > 0$. The case $\nu \leq 0$ is handled analogously.

Finally, in the region $\nu^2 \geq \lambda + 1$ one estimates that

$$|\text{Im}(\nu^2 - \lambda i \epsilon)^{-1/2}| \leq \frac{\epsilon}{(\nu^2 - \lambda)^{1/2}}, \quad \nu^2 \geq \lambda + 1,$$  

completing the proof of (8.16).  

One notes that while the outline of this proof still closely follows the corresponding proof of Theorem 1.1 by Pushnitski in [123], the finer details of our approach now necessarily deviate from his proof due to our more general Hypothesis 2.1.

The next result also follows Pushnitski in [123] closely (but again necessarily deviates in some details):

**Lemma 8.3.** Assume Hypothesis 2.1 and suppose that $0 \in \rho(A_+) \cap \rho(A_-)$. Then $H_1$ (and hence $H_2$) has an essential spectral gap near 0, that is, there exists an $\alpha > 0$ such that

$$\sigma_{\text{ess}}(H_1) = \sigma_{\text{ess}}(H_2) \subseteq [\alpha, \infty).$$

**Proof.** By Lemma 5.1 and a variant of Weyl’s theorem one concludes that

$$\sigma_{\text{ess}}(H_1) = \sigma_{\text{ess}}(H_2).$$

Next, one recalls the definition of the operators $H$ and $H_1$ from Lemma 4.8, and

$$\text{dom}(H_1^{1/2}) = \text{dom}((H)^{1/2}) = \text{dom}(H_0^{1/2}) = \text{dom}(d/dt) \cap \text{dom}(A_-).$$

By Lemma 4.6, one obtains

$$\| (H - z I)^{-1/2} B'(H - z I)^{-1/2} \|_{\mathcal{B}_1(L^2(\mathbb{R}; H))}$$

$$\leq \| (H_0 - z I)^{1/2} (H - z I)^{-1/2} \|_{\mathcal{B}_1(L^2(\mathbb{R}; H))}$$

$$\times \| (H_0 - z I)^{-1/2} B'(H_0 - z I)^{-1/2} \|_{\mathcal{B}_1(L^2(\mathbb{R}; H))} < \infty, \quad z < 0,$$  

(8.22)
and hence $B'$ is relatively form compact with respect to $H_0$ and $H$. Hence,

$$\sigma_{\text{ess}}(H_j) = \sigma_{\text{ess}}(H), \quad j = 1, 2.$$  \hfill (8.23)

Since by hypothesis $0 \not\in \rho(A_+) \cap \rho(A_-)$, one obtains the existence of $a > 0$ and $T_0 > 0$ such that

$$A(t)^2 \geq aI \quad \text{for all } |t| \geq T_0.$$  \hfill (8.24)

Next, one writes

$$A(t)g, A(t)g)_H = [(A(t)g, A(t)g)_H - a\|g\|^2_H] + a\|g\|^2_H$$

$$= (g, [A(t)^2 - aI]E_{A(t)^2}([0, a])g)_H$$

$$+ (A(t)g, E_{A(t)^2}([a, \infty))A(t)g)\lambda - a(g, E_{A(t)^2}([a, \infty))g)\lambda + a\|g\|^2_H$$

$$\geq (g, [A(t)^2 - aI]E_{A(t)^2}([0, a])g)_H + a\|g\|^2_H$$

$$= (g, F(t)g)\lambda + a\|g\|^2_H, \quad g \in \text{dom}(A_-), \ t \in \mathbb{R},$$  \hfill (8.25)

where

$$F(t) = [A(t)^2 - aI]E_{A(t)^2}([0, a]) = \begin{cases} 0, & |t| \geq T_0, \\ \text{of finite rank for all } t \in \mathbb{R}, & \end{cases}$$  \hfill (8.26)

choosing $a > 0$ sufficiently small. Indeed, the strongly right continuous family of spectral projections of $A(t)^2$ is given in terms of that of $A(t)$ by

$$E_{A(t)^2}(\lambda) = \begin{cases} 0, & \lambda < 0, \\ E_{A(t)}(\{0\}), & \lambda = 0, \\ E_{A(t)}([-\lambda^{1/2}, \lambda^{1/2}]), & \lambda > 0. & \end{cases}$$  \hfill (8.27)

Since $A(t) = A_- + B(t)$ on $\text{dom}(A(t)) = \text{dom}(A_-)$, $t \in \mathbb{R}$, and

$$B(t)(A_- - zI)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C} \setminus \mathbb{R}, \ t \in \mathbb{R},$$  \hfill (8.28)

by Theorem 3.7, one infers

$$\sigma_{\text{ess}}(A(t)) = \sigma_{\text{ess}}(A_-), \quad t \in \mathbb{R}.$$  \hfill (8.29)

Since $0 \not\in \rho(A_-)$, choosing $a > 0$ sufficiently small, $A(t)$ has at most finitely many eigenvalues of finite multiplicity in the interval $[-a^{1/2}, a^{1/2}]$, and thus $A(t)^2$ has at most finitely many eigenvalues of finite multiplicity in the interval $[0, a]$, implying the finite rank property of $F(t)$ for all $t \in \mathbb{R}$. Thus, one obtains

$$\int_{\mathbb{R}}\|F(t)\|_{\mathcal{B}_1(\mathcal{H})} < \infty,$$  \hfill (8.30)

and applying [123, Lemma 2.2] to the operator $F(t)$, $t \in \mathbb{R}$, then proves that $F$ in $L^2(\mathbb{R}; \mathcal{H})$, defined by

$$(Ff)(t) = F(t)f(t), \quad t \in \mathbb{R}, \ f \in L^2(\mathbb{R}; \mathcal{H}),$$  \hfill (8.31)

is form compact relative to the operator $H_{0,0}$ in $L^2(\mathbb{R}; \mathcal{H})$ defined by $H_{0,0} = -\frac{d^2}{dt^2}$ with maximal domain. Thus,

$$\sigma_{\text{ess}}(H_{0,0} + F) = \sigma_{\text{ess}}(H_{0,0}) = [0, \infty).$$  \hfill (8.32)

Finally, (8.25) implies $H \geq H_{0,0} + F + aI$, and hence

$$\sigma_{\text{ess}}(H_j) = \sigma_{\text{ess}}(H) \subseteq [a, \infty), \quad j = 1, 2.$$  \hfill (8.33)
Theorem 8.2 now easily yields the following Fredholm index result:

**Corollary 8.4.** Assume Hypothesis 2.1 and define \( \xi(\cdot; A_+, A_-) \) and \( \xi(\cdot; H_2, H_1) \) as in (7.3) and (8.2), (8.3), respectively. Moreover, suppose that \( 0 \in \rho(A_+) \cap \rho(A_-) \). Then \( D_A \) is a Fredholm operator in \( L^2(\mathbb{R}; \mathcal{H}) \) and

\[
\text{index}(D_A) = \xi(0; H_2, H_1) = \xi(0; A_+, A_-). \tag{8.34}
\]

**Proof.** Since \( \sigma_{\text{ess}}(H_2) = \sigma_{\text{ess}}(H_1) \) by equation (8.20), \( H_1 \) and \( H_2 \) have an essential spectral gap near 0 by Lemma 8.3. In addition, \( H_1 = D_A^* D_A \) and \( H_2 = D_A D_A^* \) have the same nonzero eigenvalues including multiplicities, and hence one concludes by the general properties of \( \xi(\cdot; H_2, H_1) \) in essential spectral gaps of \( H_2 \) and \( H_1 \) (cf. [143, p. 276, 300]) that

\[
\text{index}(D_A) = \dim(\ker(H_1)) - \dim(\ker(H_2)) = \xi(\lambda; H_2, H_1), \quad \lambda \in (0, \lambda_0), \tag{8.35}
\]

for \( \lambda_0 < \inf(\sigma_{\text{ess}}(H_2)) = \inf(\sigma_{\text{ess}}(H_1)) \).

On the other hand, since \( 0 \in \rho(A_+) \cap \rho(A_-) \), there exists a constant \( c \in \mathbb{R} \) such that \( \xi(\cdot; A_+, A_-) = c \) a.e. on the interval \( (-\nu_0, \nu_0) \) for \( 0 < \nu_0 \) sufficiently small. (This follows from the basic properties of the spectral shift function in joint essential spectral gaps of \( A_- \) and \( A_+ \), cf. [143, p. 300].) Hence, one may define

\[
\xi(\nu; A_+, A_-) = \xi(0; A_+, A_-), \quad \nu \in (-\nu_0, \nu_0). \tag{8.36}
\]

Thus, taking \( \lambda \to 0 \) in (8.11), utilizing (8.35), (8.36), and

\[
\frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{d\nu}{(\lambda - \nu^2)^{1/2}} = 1 \text{ for all } \lambda > 0, \tag{8.37}
\]

finally yields (8.34). \( \square \)

**8.2. Supersymmetry and the Atiyah–Patodi–Singer Spectral Asymmetry.** We conclude this section with an application involving the Atiyah–Patodi–Singer (APS) spectral asymmetry (cf., e.g., [13]–[16], [34], [43], [56], [69], [70], [76], [77], [88], [99], [102], [108], [109], [111], [113], [136], and the extensive list of references in [37]) applied to the case of supersymmetric Dirac-type operators \( Q_m \) (cf. [37], [61], [68], [139, Ch. 5], and the references cited therein) defined as follows: In the Hilbert space \( L^2(\mathbb{R}; \mathcal{H}) \oplus L^2(\mathbb{R}; \mathcal{H}) \) we consider the \( 2 \times 2 \) block operator-valued matrix

\[
Q_m = \begin{pmatrix} m & D_A^* \\ D_A & -m \end{pmatrix}, \quad m \in \mathbb{R} \setminus \{0\}, \tag{8.38}
\]

such that

\[
Q_m^2 = \begin{pmatrix} H_2 + m^2 I & 0 \\ 0 & H_1 + m^2 I \end{pmatrix}, \tag{8.39}
\]

and hence

\[
Q_m e^{-tQ_m^2} = \begin{pmatrix} m e^{-t(H_2 + m^2 I)} & D_A e^{-t(H_1 + m^2 I)} \\ D_A^* e^{-t(H_2 + m^2 I)} & -m e^{-t(H_1 + m^2 I)} \end{pmatrix}, \quad t \geq 0. \tag{8.40}
\]

The zeta function regularized spectral asymmetry \( \eta_m(t), t > 0 \), associated with \( Q_m \), is defined by

\[
\eta_m(s) = \frac{m}{\Gamma((s + 1)/2)} \int_{(0, \infty)} t^{(s-1)/2} \text{tr}_{L^2(\mathbb{R}; \mathcal{H})} \left( e^{-t(H_1 + m^2 I)} - e^{-t(H_2 + m^2 I)} \right) dt,
\]
and the APS spectral asymmetry (eta invariant) $\eta_m$ is then given by

$$\eta_m = \lim_{s \downarrow 0} \eta_m(s), \quad m \in \mathbb{R}\{0\},$$

whenever the limit in (8.42) exists. Intuitively, $\eta_m$ measures the asymmetry of the positive and negative spectrum of $Q_m$, $m \in \mathbb{R}\{0\}$. The asymmetry vanishes if $m = 0$ since then $Q_0$ is unitarily equivalent to $-Q_0$ (cf. [67]).

Similarly, using the fact that

$$Q_m|Q_m|^{-1}e^{-tQ_m^2}$$

the heat kernel regularized spectral asymmetry $\tilde{\eta}_m(t)$, $t > 0$, associated with $Q_m$, is defined by

$$\tilde{\eta}_m(t) = m \text{tr}_{L^2(\mathbb{R}; \mathcal{H})}((H_2 + m^2 I)^{-1/2} e^{-t(H_2 + m^2 I)} - (H_1 + m^2 I)^{-1/2} e^{-t(H_1 + m^2 I)}), \quad m \in \mathbb{R}\{0\}, \quad t > 0,$

and the corresponding spectral asymmetry $\tilde{\eta}_m$ is then given by

$$\tilde{\eta}_m = \lim_{t \uparrow 0} \tilde{\eta}_m(t), \quad m \in \mathbb{R}\{0\},$$

whenever the limit in (8.45) exists.

Denoting by $\Gamma(\cdot)$ the gamma function [3, Sect. 6.1], by $K_0(\cdot)$ the modified (irregular) Bessel function of order zero [3, Sect. 9.6], and by $W_{\kappa, \mu}(\cdot)$ the (irregular) Whittaker function [3, Sect. 13.1], one obtains the following explicit result for $\eta_m$ and $\tilde{\eta}_m$ and their regularizations:

**Lemma 8.5.** Assume Hypothesis 2.1 and $m \in \mathbb{R}\{0\}$. Then

$$\eta_m(s) = -m \frac{s + 1}{2} \int_{0, \infty} \frac{\xi(\lambda; H_2, H_1) d\lambda}{(\lambda + m^2)^{(s+3)/2}},$$

$$= -m \frac{s + 1}{2} \Gamma((s + 2)/2) \int_{\mathbb{R}} \frac{\nu (A_+, A_-)}{\nu^2 + m^2} d\nu, \quad s > 0,$$

and $\eta_m(\cdot)$ extends analytically to the open right half-plane $\text{Re}(s) > -1/2$. Moreover,

$$\tilde{\eta}_m(t) = m \int_{0, \infty} \frac{\xi(\lambda; H_2, H_1) d\lambda}{(\lambda + m^2)^{-1/2} e^{-t(\lambda + m^2)}},$$

$$= m \frac{2^{1/2}}{\Gamma((s + 3)/2)} \int_{\mathbb{R}} \frac{\nu (A_+, A_-)}{\nu^2 + m^2} W_{-1/2, 1/2}(t(\nu^2 + m^2)) e^{-t(\nu^2 + m^2)/2}, \quad t > 0,$$

$$= m \frac{\pi}{\nu^2 + m^2}.$$

In addition,

$$\eta_m = \tilde{\eta}_m = -m \frac{2}{\pi} \int_{0, \infty} \frac{\xi(\lambda; H_2, H_1) d\lambda}{(\lambda + m^2)^{3/2}},$$

$$= -m \frac{\pi}{\nu^2 + m^2}.$$
Proof. Using (8.41), one obtains from the standard trace formula applied to the pair 
\( (H_2, H_1) \) (cf. [143, Theorem 8.7.1]), Fubini’s theorem, and the gamma function 
representation [3, no. 6.1.1, p. 255], that

\[
\eta_m(s) = \frac{m}{\Gamma((s + 1)/2)} \int_0^\infty t^{(s-1)/2} \text{tr}_{L^2(R; H)} \left( e^{-t(H_2 + m^2 I)} - e^{-t(H_1 + m^2 I)} \right) dt \\
= \frac{m}{\Gamma((s + 1)/2)} \int_0^\infty t^{(s+1)/2} e^{-tm^2} \left( \int_{(0,\infty)} \xi(\lambda; H_2, H_1) e^{-t\lambda} d\lambda \right) dt \\
= \frac{m}{\Gamma((s + 1)/2)} \int_{(0,\infty)} \xi(\lambda; H_2, H_1) \left( \int_0^\infty t^{(s+1)/2} e^{-t(\lambda + m^2)} dt \right) d\lambda \\
= \frac{m}{\Gamma((s + 1)/2)} \int_{(0,\infty)} \frac{\xi(\lambda; H_2, H_1) d\lambda}{(\lambda + m^2)^{(s+3)/2}} \\
= -\frac{m}{2} \int_{(0,\infty)} \frac{\xi_\nu(A_+, A_-) d\nu}{(\lambda + m^2)^{(s-1)/2}} \frac{d\lambda}{(\lambda + m^2)^{(s+3)/2}}, \quad s > 0, 
\]

(8.49)

using the functional equation \( \Gamma(z + 1) = z\Gamma(z) \) to arrive at the next to last step 
and inserting (8.11) in the last step.

Next, one transforms the double integral in (8.49), where \((\lambda, \nu) \in [0, \infty) \times \) 
\([0, \lambda^{1/2}], \) to \((\nu, \lambda) \in [0, \infty) \times [\nu^2, \infty), \) and similarly, that where \((\lambda, \nu) \in [0, \infty) \times \) 
\([-\lambda^{1/2}, 0], \) to \((\nu, \lambda) \in (-\infty, 0) \times [\nu^2, \infty), \) and using Fubini’s theorem again one 
obtains

\[
\eta_m(s) = -\frac{m}{2\pi} \int_{R} \xi_\nu(A_+, A_-) \left( \int_{[\nu^2, \infty)} \frac{d\lambda}{(\lambda - \nu^2)^{1/2}(\lambda + m^2)^{(s+3)/2}} \right) d\nu \\
= -\frac{m}{2\pi} \frac{\Gamma((s + 2)/2)}{\Gamma((s + 3)/2)} \int_{R} \xi_\nu(A_+, A_-) d\nu \\
= -\frac{m}{2\pi} \frac{\Gamma((s + 2)/2)}{\Gamma((s + 3)/2)} \int_{R} \xi_\nu(A_+, A_-) d\nu, \quad s > 0, 
\]

(8.50)

where we used

\[
\int_{\alpha}^\infty \frac{d\lambda}{(\lambda - \alpha)^{1-a}(\lambda + \beta)^{b}} = (\alpha + \beta)^{-(b-a)} B(b-a, a), \quad \alpha + \beta > 0, \ b > a > 0, 
\]

(8.51)

according to [73, no. 3.1962, p. 285], with \( B(z, w) = \Gamma(z)\Gamma(w)/\Gamma(z + w), \) the beta function, 
and \( \Gamma(1/2) = \pi^{1/2} \) (cf. [3, Sects. 6.1, 6.2]). This proves (8.46). By (8.5), 
the first equation in (8.46) proves the existence of an analytic continuation of \( \eta_m(\cdot) \) 
to the open right half-plane \( \text{Re}(s) > -1/2. \) The facts (7.19) and (8.5) together with 
Lebesgue’s dominated convergence theorem employed in both equalities in (8.46) 
then prove (8.48) in the case of \( \eta_m. \)

The corresponding proof of (8.47), and the remaining proof of (8.48) in the case 
of \( \tilde{\eta}_m \) proceed along entirely analogous steps, but naturally, the second equality in 
(8.47) is based on more involved arguments. To shorten the remainder of this proof 
a bit we now focus just on the major steps in the computations: Employing (8.44), 
one concludes from the standard trace formula in [143, Theorem 8.7.1] that

\[
\tilde{\eta}_m(t) = m \int_{(0,\infty)} \xi(\lambda; H_2, H_1) d\lambda \left( \frac{d}{d\lambda} [(\lambda + m^2)^{-1/2} e^{-t(\lambda + m^2)}] \right)
\]
the spectral flow for the family of operators

There are several definitions of the spectral flow available in the literature, we will in [122] and [124].

For example, the definition in [129, Theorem 4.23] that uses the Kato Selection Theorem Hypothesis 2.1 (and in analogy to (8.5)), one actually has

\[ \xi(\nu; A_+, A_-) d\nu \]

\[ \right\{ \begin{array}{l}
\xi(\nu; A_+, A_-) d\nu \\
\int_{\nu^2, t} (t(\nu^2 + m^2)) e^{-t(\nu^2 + m^2)/2} \\
- \frac{m}{\pi} \int_{\nu^2, t} (t(\nu^2 + m^2)/2) e^{-t(\nu^2 + m^2)/2}, \quad t > 0.
\end{array} \] (8.52)

Here we used

\[ \int_{a}^{\infty} e^{-c^2} d\lambda = (\alpha + \beta)^{-(b-a+1)/2} e^{-(b-a+1)/2} e^{-c(\alpha - \beta)/2} \Gamma(a) \]

\[ \times W_{1-b-a/2, (a-b)/2}(c(\alpha + \beta)), \quad \alpha > 0, \quad \alpha + \beta > 0, \quad c > 0, \quad a > 0, \] (8.53)

according to [73, no. 3.3843, p. 320], and

\[ W_{0,0}(z) = \pi^{-1/2} z^{1/2} K_0(z/2), \] (8.54)

combining no. 13.1.33 on p. 505 and no. 13.6.21 on p. 510 in [3].

Equation (8.46) and the existence of an analytic continuation of \( \eta_m(\cdot) \) to the open right half-plane \( \text{Re}(s) > -1/2 \) suggests the possibility that under the assumptions of Hypothesis 2.1 (and in analogy to (8.5)), one actually has \( \xi(\cdot; A_+, A_-) \in L^1(\mathbb{R}; (1 + |\nu|)^{-1} d\nu) \), but this is left to a future investigation.

9. Connections Between the Index and the Spectral Flow

In this section we briefly discuss connections of our results to the topic of the spectral flow for the family of operators \( \{A(t)\}_{t=-\infty}^{\infty} \) defined in (3.23), (3.51). While there are several definitions of the spectral flow available in the literature, we will follow the scheme originated in [118] (see also [39] and [98]), but also note, for instance, the definition in [129, Theorem 4.23] that uses the Kato Selection Theorem (cf. e.g., [86, Theorems II.5.4 and II.6.8] and [129, Theorem 4.23]), and the definition in [122] and [124].

The spectral flow is defined in [98, Definition 1.1] for a family of operators continuous with respect to the graph metric (which induces convergence in the norm resolvent sense, cf. [126, Sect. VIII.7]). The graph metric, \( d_G \), on the space of (unbounded) self-adjoint operators on the Hilbert space \( \mathcal{H} \) is defined as follows: for any two self-adjoint operators, \( S_1 \) and \( S_2 \), we set

\[ d_G(S_1, S_2) = \|(S_2 - i)^{-1} - (S_1 - i)^{-1}\|_{\mathcal{B}(\mathcal{H})}. \] (9.1)
Another metric on the space of (unbounded) self-adjoint operators is the Riesz metric, $d_R$, defined by the formula

$$d_R(S_2, S_1) = \|g(S_2) - g(S_1)\|_{\mathcal{B}(\mathcal{H})}, \quad g(x) = x(1 + x^2)^{-1}. \quad (9.2)$$

Finally, given a self-adjoint operator $A_-$, let us consider the set of all (unbounded) self-adjoint operators having the same domain as $A_-$. On this set one can define a metric, $d_{|A_-|}$, by the formula

$$d_{|A_-|}(S_2, S_1) = \|(S_2 - S_1)(|A_-| + I)^{-1}\|_{\mathcal{B}(\mathcal{H})}. \quad (9.3)$$

The metric $d_{|A_-|}$ is strictly stronger than $d_R$, and the metric $d_R$ is strictly stronger than $d_G$, see [98, Proposition 2.2] (as well as comments following that proposition and further references therein) for the proof of this result.

**Lemma 9.1.** Assume Hypothesis 2.1. Then the family $\{A(t)\}_{t = -\infty}^\infty$ of operators defined in (3.23), (3.51) is continuous at each $t \in \mathbb{R}$, and also $\lim_{t \to \pm \infty} A(t) = A_\pm$ holds with respect to each of the metrics $d_{|A_-|}, d_R$, and $d_G$.

**Proof.** By the observation following (9.3), it suffices to consider $d_{|A_-|}$ only. However, to make the underlying issues more transparent, we will present independent proofs for all three metrics.

Metric $d_{|A_-|}$: For any $-\infty \leq a < b \leq +\infty$ the distance $d_{|A_-|}(A(b), A(a))$ (cf. (3.14)), is dominated by

$$\|g(A(b)) - g(A(a))\|_{\mathcal{B}(\mathcal{H})} \leq \int_a^b \|B'(s)|A_-| + I\|^{-1}\|\mathcal{B}(\mathcal{H})\| ds, \quad (9.4)$$

and the required in the lemma assertions follow from (2.2).

Metric $d_R$: This follows, essentially, from Lemma 6.6. Indeed, using (6.42) with $S_+ = A(b)$ and $S_- = A(a)$, the distance $d_R(A(b), A(a))$ is dominated by

$$\|g(A(b)) - g(A(a))\|_{\mathcal{B}(\mathcal{H})} \leq \left\|T_\phi\left[(\mathcal{A}(b))^{-1/2}(A(b) - A(a))\mathcal{A}(a)^{-1/2}\right]\right\|_{\mathcal{B}(\mathcal{H})}$$

$$\leq \|T_\phi\|_{\mathcal{B}(\mathcal{H})}\|\mathcal{A}(b)\|^{-1/2}\|A_- + I\|^{1/2}\|\mathcal{B}(\mathcal{H})\|$$

$$\times \left\|(|A_-| + I)^{-1/2}(A(b) - A(a))(|A_-| + I)^{-1/2}\right\|_{\mathcal{B}(\mathcal{H})} \quad (9.5)$$

and

$$\times \left\|(|A_-| + I)^{1/2}\mathcal{A}(a)^{-1/2}\right\|_{\mathcal{B}(\mathcal{H})}. \quad (9.6)$$

We claim that

$$\sup_{t \in \mathbb{R}} \left\|\mathcal{A}(t)^{-1/2}\mathcal{A}(A(t))^{-1/2}\mathcal{B}(\mathcal{H})\right\| < \infty. \quad (9.7)$$

Assuming the claim, the proof is completed as follows. First, for $T_\phi = T_\phi^{(A(b), A(a))}$ in (9.5) the norms $\|T_\phi\|_{\mathcal{B}(\mathcal{H})}$ are bounded uniformly for $a, b \in \mathbb{R}$ due to (6.5) and Lemma 6.6 (i). The $\mathcal{B}(\mathcal{H})$-norms in (9.5) and (9.7) are also bounded uniformly for $a, b \in \mathbb{R}$ due to claim (9.8) and the relation

$$\mathcal{A}(b))^{-1/2}(A_- + I)^{1/2} = (A_- + I)^{1/2}\mathcal{A}(b))^{-1/2}. \quad (9.9)$$

It remains to estimate the norm in (9.6). Using (4.2) for $S = A(b) - A(a)$ and $T = |A_-| + I$, we infer that the norm in (9.6) is dominated by the expression in the left-hand side of (9.4). Putting all this together, one concludes that there is a constant $c > 0$ such that, for any interval $-\infty \leq a < b \leq +\infty$,

$$\|g(A(b)) - g(A(a))\|_{\mathcal{B}(\mathcal{H})} \leq c \int_a^b \|B'(\tau)|A_-| + I\|^{-1}\|\mathcal{B}(\mathcal{H})\| d\tau, \quad (9.10)$$
and the assertions in the lemma follow from (2.2).

To proof the claim (9.8), one notes that
\[
\|(|A_-| + I)^{1/2} \mathcal{A}(A(t))^{-1/2} \|_{\mathcal{B}(\mathcal{H})} \leq \|(|A_-| + I)^{1/2}(|A(t)| + I)^{-1/2} \|_{\mathcal{B}(\mathcal{H})}
\]
\[
\times \|(|A(t) + I)^{1/2} \mathcal{A}(A(t))^{-1/2} \|_{\mathcal{B}(\mathcal{H})}
\]
\[
\leq \|(|A_-| + I)^{1/2}(|A(t)| + I)^{-1/2} \|_{\mathcal{B}(\mathcal{H})} \sup_{y \in \mathbb{R}} \frac{(|y| + 1)^{1/2}}{(|y|^2 + 1)^{1/4}},
\]
(9.11)

and thus it suffices to show that
\[
\sup_{t \in \mathbb{R}} \|(|A_-| + I)^{1/2}(|A(t)| + I)^{-1/2} \|_{\mathcal{B}(\mathcal{H})} < \infty.
\]
(9.12)

By Lemma 3.8, there are constants \(c_1, c > 0\) such that, for all \(f \in \mathcal{H}\) and \(t \in \mathbb{R}\),
\[
\|(|A_-| + I)^{1/2}(|A(t)| + I)^{-1/2}f \|_{\mathcal{H}} \leq \|(|A(t)| + I)^{-1/2}f \|_{\mathcal{H}_{1/2}(|A_-|)}
\]
\[
\leq c_1 \|(|A(t)| + I)^{-1/2}f \|_{\mathcal{H}_{1/2}(|A_-|)}< c_1 \|f \|_{\mathcal{H}},
\]
(9.13)

completing the proof.

Metric \(d_G\): Using the resolvent identity, \(d_G(A(b), A(a))\) is dominated by
\[
\|(A(a) - i)^{-1} \mathcal{B}(\mathcal{H})(A(b) - A(a))(|A_-| + I)^{-1} \mathcal{B}(\mathcal{H})\|
\]
\[
\times \|(|A_-| + I)(A(b) + I)^{-1} \mathcal{B}(\mathcal{H})(|A(b)| + I)(A(b) - i)^{-1} \mathcal{B}(\mathcal{H})\|
\]
(9.14)

The first factor in (9.14) and the second factor in (9.15) are uniformly bounded for \(a, b \in \mathbb{R}\) employing the fact that \(A(a), A(b)\) are self-adjoint and using Lemma 3.8. By (3.52) in Lemma 3.8, the first factor in (9.15) is uniformly bounded for \(b \in \mathbb{R}\). The second factor in (9.14) is estimated as in (9.4), and again the assertions in the lemma follow from (2.2). \(\blacksquare\)

For additional references in connection with metrics for closed operators we also refer to [39], [49], [87], [104], [105], [110], [133], and [140].

Assuming Hypothesis 2.1 and \(0 \in \rho(A_-) \cap \rho(A_+),\) we will now recall the definition of the spectral flow for the operator path \(\{A(t)\}_{t=-\infty}^{\infty}\), following the line of arguments in [118] (see also [39, 98]), where the spectral flow has been defined for paths with \(t \in [0, 1]\).

Remark 9.2. Since \(0 \in \rho(A_-) \cap \rho(A_+),\) there exists \(\varepsilon_0 > 0\) such that \([-\varepsilon_0, \varepsilon_0] \cap \sigma(A_\pm) = \emptyset.\) Since the family \(\{A(t)\}_{t=-\infty}^{\infty}\) is \(d_G\)-continuous by Lemma 9.1, the function \(\mathbb{R} \ni t \mapsto \sigma(A(t))\) is upper semicontinuous by [126, Theorem VIII.2.3(a)]. Since \(d_R(A(t), A_\pm) \to 0\) as \(t \to \pm \infty\) by Lemma 9.1, there exists \(T_0 > 0\) such that \([-\varepsilon_0, \varepsilon_0] \cap \sigma(A(t)) = \emptyset\) for all \(|t| \geq T_0\). Moreover, using (3.31), \([-\varepsilon_0, \varepsilon_0] \cap \sigma_{\text{ess}}(A(t)) = \emptyset\) for all \(t \in \mathbb{R}\). Thus, the operators \(A_\pm = A(t)\), \(t \in \mathbb{R}\), are Fredholm, and for each \(t \in \mathbb{R}\), the set \([-\varepsilon_0, \varepsilon_0] \cap \sigma(A(t))\) consists of at most of finitely many isolated eigenvalues of finite multiplicity.

Remark 9.3. By Remark 9.2, for each \(t \in \mathbb{R}\), there exist \(\varepsilon \in (0, \varepsilon_0)\) and \(\delta > 0\) such that the following assertions hold:
\[
\pm \varepsilon \notin \sigma(A(s)) \text{ for all } s \in (t - \delta, t + \delta),
\]
(9.16)
\[
E_{[-\varepsilon, \varepsilon]}(A(s)) \text{ has finite rank}
\]
(9.17)
and is norm continuous as a function of \(s \in (t - \delta, t + \delta)\).
Indeed, since \([-\varepsilon_0, \varepsilon_0] \cap \sigma_{\text{ess}}(A(t)) = \emptyset\), the interval \([-\varepsilon_0, \varepsilon_0]\) contains at most finitely many points of \(\sigma(A(t))\). Fix \(\varepsilon \in (0, \varepsilon_0)\) such that \(\pm \varepsilon \notin \sigma(A(t))\). Since

\[
\sigma(A(t)) \subseteq \mathbb{R} \setminus \{-\varepsilon, +\varepsilon\},
\]

there is an open \(d_G\)-ball containing \(A(t)\), such that

\[
\sigma(A(s)) \subseteq \mathbb{R} \setminus \{-\varepsilon, +\varepsilon\},
\]

provided \(A(s)\) is in this ball (cf. [126, Theorem VIII.2.3(a)]). In addition, since \(A(\cdot)\) is \(d_G\)-continuous, there is a \(\delta > 0\) such that (9.16) holds. The inclusion \([-\varepsilon, \varepsilon] \subset [-\varepsilon_0, \varepsilon_0]\) yields that \([-\varepsilon, \varepsilon] \cap \sigma_{\text{ess}}(A(s)) = \emptyset\), and thus assertion (9.17) for all \(s \in (t - \delta, t + \delta)\). Finally, the norm continuity in (9.18) follows by [126, Theorem VIII.2.3(b)].

Remark 9.4. By compactness of \([-T_0, T_0]\) (with \(T_0\) as in Remark 9.2) and Remark 9.3, we may choose a subdivision \(-T_0 = t_0 < t_1 < \cdots < t_n = T_0\), and numbers \(\varepsilon_j \in (0, \varepsilon_0)\) (with \(\varepsilon_0 > 0\) as in Remark 9.2), such that for each \(j = 1, \ldots, n\), and for all \(t \in [t_j-1, t_j]\), the following assertions hold:

(i) \(\pm \varepsilon_j \notin \sigma(A(t))\).

(ii) \([-\varepsilon_j, \varepsilon_j] \cap \sigma_{\text{ess}}(A(t)) = \emptyset\).

(iii) \(E_{[-\varepsilon_j, \varepsilon_j]}(A(t))\) is of finite rank and is norm continuous in \(t \in [t_{j-1}, t_j]\).

Definition 9.5 ([39, 98, 118]). Given the notation used in Remark 9.4, we define the spectral flow of the \(d_G\)-continuous path \(\{A(t)\}_{t=\infty}^{1}\) of self-adjoint Fredholm operators by the formula

\[
\text{SpFlow}\{(A(t)\}_{t=\infty}^{1}\} = \sum_{j=1}^{n} \left( \dim(\text{ran}(E_{A(t_{j-1}})([0, \varepsilon_j]))) - \dim(\text{ran}(E_{A(t_j)}([0, \varepsilon_j]))) \right).
\]

Remark 9.6. As in [118], one can see that the definition is independent of the choice of \(T_0\), the subdivision, and the numbers \(\varepsilon_j > 0\) with the properties described in Remarks 9.2 and 9.4. Indeed, since \(A(t)\) does not have the eigenvalue zero for all \(|t| \geq T_0\), the right-hand side of (9.21) does not depend on \(T_0\). Adding a point \(t_*\) to the subdivision yields adding and subtracting the term \(\dim(\text{ran}(E_{A(t_*)}([0, \varepsilon_*])))\) on the right-hand side of (9.21). Finally, changing \(\varepsilon_j\) by, say, a smaller \(\varepsilon'_j > 0\), we remark that the dimension of the range of \(E_{[0, \varepsilon_j]}(A(t)) - E_{[0, \varepsilon'_j]}(A(t)) = E_{[-\varepsilon'_j, \varepsilon'_j]}(A(t))\) is constant for \(t \in [t_{j-1}, t_j]\) by the norm continuity of the spectral projections. Therefore, this change does not affect the right-hand side of (9.21) either.

Remark 9.7. Equivalently, the definition of \(\text{SpFlow}\{(A(t)\}_{t=\infty}^{1}\}\) can be reduced to the definition in [39, 98, 118] for \(t \in [0, 1]\), by a re-parameterization: Indeed, for any continuous strictly increasing function \(r : [0, 1] \to \mathbb{R}\), we introduce the path \(\{S(t)\}_{t=0}^{1}\) by letting \(S_0 = A_{-1}\), \(S(t) = A(r(t))\), \(t \in (0, 1)\), and \(S_1 = A_1\), and then define \(\text{SpFlow}\{(A(t)\}_{t=\infty}^{1}\} = \text{SpFlow}(\{S(t)\}_{t=0}^{1}\); the latter spectral flow is defined by formula (9.21), with \(A(t)\) replaced by \(S(t)\) and the \(t_j\)'s representing a subdivision of \([0, 1]\). An argument similar to Remark 9.6 shows that this new definition does not depend on the choice of the re-parameterization \(r\) and is equivalent to Definition 9.5. An advantage of the definition by re-parameterization is that the proof of (9.25) becomes shorter as one does not need to show (9.33). Nevertheless, we prefer to
use Definition 9.5 as it provides direct insight into the process where eigenvalues of $A(t)$ are passing through zero as $t$ changes from $-\infty$ to $+\infty$.

Next, we recall some terminology and several results from [1], [17], [98]:

**Definition 9.8.** A pair $(P, Q)$ of orthogonal projections on $\mathcal{H}$ is called Fredholm (see, e.g., [17]), if $QP$ is a Fredholm operator from $\text{ran}(P)$ to $\text{ran}(Q)$; the index of the pair $(P, Q)$ is defined to be the Fredholm index of the operator $QP$, that is, by the formula

$$\text{index}(P, Q) = \dim(\text{ran}(P) \cap (\text{ran}(Q))^{-1}) - \dim((\text{ran}(P))^{-1} \cap \text{ran}(Q)).$$

We note that a pair $(P, Q)$ is a Fredholm pair if and only if the essential spectrum of the difference $P - Q$ is a subset of the open interval $(-1, 1)$.

**Remark 9.9.** (i) If $(P, Q)$ is a Fredholm pair, then $(Q, P)$ is a Fredholm pair and

$$\text{index}(P, Q) = -\text{index}(Q, P) = -\text{index}(I - P, I - Q)$$

(see [17, Theorem 3.4(a)]). (ii) If $P - Q$ is compact, then $(P, Q)$ is Fredholm (see [17, Proposition 3.1]). (iii) If $P - Q \in B_1(\mathcal{H})$, then $\text{index}(P, Q) = \text{tr}_H(P - Q)$ (see [17, Theorem 4.1]).

**Definition 9.10.** A pair $(M, N)$ of closed subspaces of $\mathcal{H}$ is called Fredholm (see, e.g., [1, Section 2.2], [86, Sect. IV.4]), if $M \cap N$ is finite-dimensional, $M + N$ is closed and has finite codimension; the index of the pair $(M, N)$ is defined as

$$\text{index}(M, N) = \dim(M \cap N) - \dim(M^\perp \cap N^\perp).$$

The number on the right-hand side of (9.23) is also called the relative dimension of the subspaces $M$ and $N^\perp$.

**Remark 9.11.** Clearly, the pair $(P, Q)$ of orthogonal projections is Fredholm if and only if the pair of subspaces $M = \text{ran}(P)$ and $N = (\text{ran}(Q))^{-1}$ is Fredholm; the indices of the pairs $(P, Q)$ and $(M, N)$ are equal. The subspaces $M, N$ are called commensurable if $P - Q$ is compact (see, e.g., [1, Section 2.2]); in this case the pair $(M, N^\perp)$ is Fredholm by Remark 9.9 (ii) (see also [95, Lemma 7.3]). We refer to [22] for a detailed discussion of relations between Fredholm pairs of projections and Fredholm pairs of subspaces.

For a variety of additional material on closed subspaces, including a number of classical references on the subject, as well as the study of pairs of projections that differ by a compact operator (and necessarily being far from complete), we refer, for instance, to [4], [10], [17], [35], [36], [44], [50], [54], [55], [60], [65], [72], [79], [83], [82], [89], [91], [92], [124], [137], [142], and the numerous references cited therein.

**Proposition 9.12** (Lesch [98]). Assume that $\{S_t\}_{t=0}^1$ is a $d_R$-continuous path of (unbounded) self-adjoint Fredholm operators. Assume furthermore that the domain of $S_t$ does not depend on $t$, $\text{dom}(S_t) = \text{dom}(S_0)$, and that for $t \in [0, 1]$, the difference $S_t - S_0$ is an $S_0$-compact symmetric operator. Then the following assertions hold:

(i) Suppose that $\lambda \notin \sigma(S_t)$, $t \in [0, 1]$. Then the path of spectral projections $t \mapsto E_{S_t}(\lambda, \infty)$ is norm continuous (cf. [98, Lemma 3.3]).

(ii) Assume that $\lambda \notin \sigma_{\text{ ess}}(S_t)$, $t \in [0, 1]$. Then the difference of the spectral projections $E_{S_t}(\lambda, \infty) - E_{S_0}(\lambda, \infty)$ is a compact operator (cf. [98, Corollary 3.5]).

(iii) The pair of spectral projections $(E_{S_t}([0, \infty)), E_{S_0}([0, \infty)))$ is Fredholm and

$$\text{SpFlow}(\{S_t\}_{t=0}^1) = \text{index}(E_{S_1}([0, \infty)), E_{S_0}([0, \infty)))$$

(cf. [98, Theorem 3.6]).
Assuming Hypothesis 2.1 and $0 \in \rho(A_-) \cap \rho(A_+)$, we are now ready to proceed with the main result of this section. Its proof uses $d_R$-continuity of the family $\{A(t)\}_{t=-\infty}^{\infty}$ since it requires the norm continuity in $t$ of the spectral projections $E_{A(t)}([0, \infty))$ when $0 \notin \sigma(A(t))$. This is in contrast to the definition of the spectral flow which requires $d_G$-continuity yielding the norm continuity of $E_{A(t)}([0, \varepsilon)), \varepsilon > 0$, for just a finite $\varepsilon \notin \sigma(A(t))$.

The spectral projections $E_{A_+}((-\infty, 0))$ and $E_{A_-}((-\infty, 0))$ are called Morse projections. We recall that by (2.34) in Corollary 2.8 the difference $E_{A_-}((-\infty, 0)) - E_{A_+}((-\infty, 0))$ of the Morse projections is of trace class. We introduce the notation $S_{\pm} = \text{ran}(E_{A_{\pm}}((-\infty, 0)))$ for the ranges of the Morse projections.

**Theorem 9.13.** Assume Hypothesis 2.1 and suppose that $0 \in \rho(A_+) \cap \rho(A_-)$. Then the pair $(E_{A_+}((-\infty, 0)), E_{A_-}((-\infty, 0)))$ of the Morse projections is Fredholm, the pair of subspaces $(S_+, S_-)$ is commensurable, the pair of subspaces $(S_+, S_-)$ is Fredholm, and the following equalities hold:

\[
\text{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) = \text{index}(E_{A_+}((-\infty, 0)), E_{A_-}((-\infty, 0))) = \text{index}(S_+, S_-) = \dim(S_- \cap S_+) - \dim(S_- \cap S_+) = \text{tr}(E_{A_-}((-\infty, 0)) - E_{A_+}((-\infty, 0))) = \xi(0; A_+, A_-) = \xi(0; H_1, H_2) = \text{index}(D_A).
\]  

**Proof.** All assertions about the Fredholm properties of the pairs of projections and subspaces follow from Remark 9.9(ii) and Remark 9.11, using compactness of the difference of the Morse projections in (2.34). Equality (9.30) of the Fredholm index of $D_A$ and the $\xi$-function $\xi(0; A_+, A_-)$ is one of the main results of this paper; it is contained in Corollary 8.4. Similarly, equality of (9.30) and (9.29) is proved in Corollary 8.4. Equality of (9.27) and (9.28) of the trace and the $\xi$-function is proved in Lemma 7.5. Equality of the trace (9.27) and the index of the pair of the Morse projections holds due to (2.34) by Remark 9.9(iii). Equality (9.26) holds by Remark 9.11.

It remains to prove equality (9.25). In fact, the main step in its proof is an application of [98, Theorem 3.6] as recorded in Proposition 9.12(iii) above. First, we recall that $T_0$ is chosen as in Remark 9.2. By Lemma 9.1, the path $\{A(t)\}_{t=-T_0}^{T_0}$ of self-adjoint Fredholm operators is $d_R$-continuous. Moreover, it follows from Hypothesis 2.1 that the domain of $A(t)$ does not depend on $t$, and the difference $A(T_0) - A(-T_0)$ is $A(-T_0)$-compact. Indeed, the operator

\[
(A(T_0) - A(-T_0))(A(-T_0))^{-1} = \int_{-T_0}^{T_0} B'(s)\left(|A_-| + I\right)^{-1} ds \left(|A_-| + I\right)A(-T_0)^{-1}
\]  

is compact since the integral on the right-hand side of (9.31) is a trace class operator by (3.6) and $(|A_-| + I)(A(-T_0)^{-1}) \in B(H)$ (see (3.52)). Thus, the assumptions of Proposition 9.12 are satisfied for $S_t = A(t)$. By Proposition 9.12(iii), the pair of projections $(E_{A(T_0)}([0, \infty)), E_{A(-T_0)}([0, \infty)))$ is Fredholm, and

\[
\text{SpFlow}(\{A(t)\}_{t=-T_0}^{T_0}) = \text{index}(E_{A(T_0)}([0, \infty)), E_{A(-T_0)}([0, \infty])).
\]
According to Definition 9.5, one has $\text{SpFlow}({\{A(t)\}}_{t=\gamma-T_0}^{T_0}) = \text{SpFlow}({\{A(t)\}}_{t=-\infty}^{T_0})$, and thus it remains to show that

$$\text{index}(E_{A(T_0)}([0, \infty)), E_{A(-T_0)}([0, \infty))) = \text{index}(E_{A_-}((\infty, 0)), E_{A_+}(\infty, 0))).$$

For each $t \geq T_0$, the difference of the projections $E_{A(t)}([0, \infty)) - E_{A(-t)}([0, \infty))$ is compact by Proposition 9.12 (ii) and thus the pair $(E_{A(t)}([0, \infty)), E_{A(-t)}([0, \infty)))$ is Fredholm by Remark 9.9 (ii). Since $0 \notin \sigma(A(t))$ for $|t| \geq T_0$, one infers that $E_{A(t)}([0, \infty)) = E_{A(t)}((\infty, 0)))$. Since $A(t)$ is $d_R$-continuous and $d_R(A(t), A_{\pm}) \to 0$ as $t \to \pm\infty$ by Lemma 9.1, the function $\mathbb{R} \ni t \mapsto E_{A(t)}((\infty, 0)))$ is norm continuous and $\|E_{A(t)}((0, \infty)) - E_{A_{\pm}}((0, \infty)))\|_{B(H)} \to 0$ as $t \to \pm\infty$ by Proposition 9.12 (i). The index of a norm-continuous family of Fredholm pairs of projections is constant (see, e.g., [98, Lemma 3.2]), and thus, if $t \geq T_0$, then

$$\text{index}(E_{A(t)}([0, \infty)), E_{A(-t)}([0, \infty))) = \text{index}(E_{A_+}((\infty, 0)), E_{A_-}(\infty, 0))),$$

yielding (9.33) by Remark 9.9 (i). \[ \blacksquare \]

Finally, we note that if both subspaces $S_+$ and $S_-$ are finite-dimensional then formulas (9.25), (9.26), (9.27) become the well-known formula in finite-dimensional Morse theory (see, e.g., [1, 2, 129] and the much earlier literature cited therein):

$$\text{index}(D_A) = \text{index}(E_{A_-}((\infty, 0)), E_{A_+}((\infty, 0)))$$

$$= \dim(S_+) - \dim(S_-), \quad \dim(S_{\pm}) < \infty. \quad (9.35)$$

### Appendix A. Some Facts on Direct Integrals of Closed Operators

We briefly recall some basic facts on closed operators and their graphs discussed in detail in Stone’s fundamental paper [138] and then review some of its consequences for direct integrals of (unbounded) closed operators as developed in Nussbaum [114] (see also Pallu de la Barrière [116]). For a detailed treatment of some of the material in this appendix we refer to [62].

For simplicity, we make the following assumption:

**Hypothesis A.1.** Let $\mathcal{H}$ be a complex separable Hilbert space and $T$ a densely defined, closed, linear operator in $\mathcal{H}$.

We note that Stone [138] considers a more general situation, but Hypothesis A.1 perfectly fits the purpose of our paper.

By $\Gamma(T)$ we denote the graph of $T$, that is, the following subspace of the direct sum $\mathcal{H} \oplus \mathcal{H}$,

$$\Gamma(T) = \{(f, Tf) \mid f \in \text{dom}(T)\} \subseteq \mathcal{H} \oplus \mathcal{H}. \quad (A.1)$$

Since $T$ is assumed to be closed, $\Gamma(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. Here $(f, g)$ denotes the ordered pair of $f, g \in \mathcal{H}$, and we use the standard norm

$$\|\langle f, g \rangle\|_{\mathcal{H} \oplus \mathcal{H}} = [\|f\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{H}}^2]^{1/2}, \quad f, g \in \mathcal{H}, \quad (A.2)$$

and scalar product

$$\langle (f_1, g_1), (f_2, g_2) \rangle_{\mathcal{H} \oplus \mathcal{H}} = (f_1, f_2)_{\mathcal{H}} + (g_1, g_2)_{\mathcal{H}}, \quad f_j, g_j \in \mathcal{H}, \quad j = 1, 2, \quad (A.3)$$

in $\mathcal{H} \oplus \mathcal{H}$.

If $B \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, one can uniquely represent $B$ as the $2 \times 2$ block operator matrix

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}, \quad (A.4)$$
where $B_{j,k} \in \mathcal{B}(\mathcal{H})$, $j, k \in \{1, 2\}$.

Denoting by

$$P(\Gamma(T)) = \begin{pmatrix} P(\Gamma(T))_{1,1} & P(\Gamma(T))_{1,2} \\ P(\Gamma(T))_{2,1} & P(\Gamma(T))_{2,2} \end{pmatrix}$$  \hspace{1cm} (A.5)$$

the orthogonal projection onto $\Gamma(T)$, the corresponding matrix $(P(\Gamma(T)))_{j,k}^{1 \leq j, k \leq 2}$ will be called the characteristic matrix of $T$. Since by hypothesis $T$ is closed and densely defined, one actually obtains (cf. [138])

$$P(\Gamma(T))_{1,1} = (T^*T + I)^{-1},$$  

$$P(\Gamma(T))_{1,2} = T^*(TT^* + I)^{-1},$$  

$$P(\Gamma(T))_{2,1} = T(T^*T + I)^{-1} = (P(\Gamma(T))_{1,2})^*,$$  \hspace{1cm} (A.6)$$

$$P(\Gamma(T))_{2,2} = TT^*(TT^* + I)^{-1} = I - (TT^* + I)^{-1}.$$  



Next, assume Hypothesis A.2:

**Hypothesis A.2.** Let $T(t), t \in \mathbb{R}$, be densely defined, closed, linear operators in $\mathcal{H}$.

We need the following notions of measurable vector and operator families:

**Definition A.3.** (i) Let $\mathbb{R} \ni t \rightarrow g(t) \in \mathcal{H}$. Then the family $\{g(t)\}_{t \in \mathbb{R}}$ is called weakly measurable in $\mathcal{H}$ if $\mathbb{R} \ni t \rightarrow (h, g(t))_\mathcal{H}$ is (Lebesgue) measurable for each $h \in \mathcal{H}$.

Next, assume Hypothesis A.2:

(ii) The family $\{T(t)\}_{t \in \mathbb{R}}$ is called weakly measurable if for any weakly measurable family of elements $\{f(t)\}_{t \in \mathbb{R}}$ in $\mathcal{H}$ such that $f(t) \in \text{dom}(T(t))$ for all $t \in \mathbb{R}$, the family of elements $\{T(t)f(t)\}_{t \in \mathbb{R}}$ is weakly measurable in $\mathcal{H}$.

(iii) The family $\{T(t)\}_{t \in \mathbb{R}}$ is called $\nu$-measurable if the entries of the characteristic matrix of $T(t)$ are weakly measurable, that is, if $\{P(\Gamma(T(t)))_{j,k}\}_{t \in \mathbb{R}}$, $j, k \in \{1, 2\}$, are weakly measurable.

We note that measurability of the characteristic matrix $(P(\Gamma(T(\cdot)))_{j,k})_{1 \leq j, k \leq 2}$ of $T(\cdot)$ was introduced by Nussbaum [114]. In fact, he considered the more general situation of a general measure $d\mu$ and a $\mu$-measurable family of Hilbert spaces $\{\mathcal{H}(t)\}_{t \in \mathbb{R}}$.

We refer to [114] for more details in connection with items (ii)–(iv) in Remark A.4 below:

**Remark A.4.** (i) Since $\mathcal{H}$ is assumed to be separable, weak measurability of the family $\{g(t)\}_{t \in \mathbb{R}}$ in $\mathcal{H}$ is equivalent to measurability, that is, there exists a sequence of countably-valued elements $\{g_n(t)\}_{t \in \mathbb{R}} \subset \mathcal{H}$, $n \in \mathbb{N}$, and a set $\mathcal{E} \subset \mathbb{R}$ of Lebesgue measure zero such that $\lim_{n \to \infty} \|g_n(t) - g(t)\|_\mathcal{H} = 0$ for each $t \in \mathbb{R}\setminus\mathcal{E}$. Thus, the family $\{g(t)\}_{t \in \mathbb{R}}$ is (weakly) measurable in $\mathcal{H}$ if there exists a dense set $\mathcal{Y} \subset \mathcal{H}$ such that the function $(y, g(\cdot))_\mathcal{H}$ is measurable for every $y \in \mathcal{Y}$, see, for instance, [11, Corollary 1.1.3], [53, p. 42–43]. Moreover,

$$f, g : \mathbb{R} \rightarrow \mathcal{H} \text{ measurable} \implies (f(\cdot), g(\cdot))_\mathcal{H} \text{ is measurable}. \hspace{1cm} (A.7)$$

(ii) If $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are complex, separable Hilbert spaces and $F : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $G : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ are strongly measurable, then $GF : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ is strongly measurable, see, for instance, [85, Lemma A4]. (Here strong (operator)
measurability of \( F : \mathbb{R} \mapsto B(\mathcal{H}_1, \mathcal{H}_2) \) is defined pointwise, i.e., for all \( f \in \mathcal{H}_1 \), \( \{ F(t)f \}_{t \in \mathbb{R}} \) is (weakly) measurable in \( \mathcal{H}_2 \).

(iii) One can show that

\[
N\text{-measurability of } \{ T(t) \}_{t \in \mathbb{R}} \implies \text{weak measurability of } \{ T(t) \}_{t \in \mathbb{R}},
\]

but the converse is false. For an example of a weakly measurable family of symmetric operators which is not \( N \)-measurable, we refer to Example A.5 below.

(iv) Since \( P(\Gamma(T(t)))_{2,1} = (P(\Gamma(T(t)))_{1,2})^\ast \), or equivalently, since

\[
[T(t)(T(t)^\ast T(t) + I)^{-1}]^\ast = T(t)^\ast (T(t)T(t)^\ast + I)^{-1} \supseteq (T(t)^\ast T(t) + I)^{-1} T(t)^\ast,
\]

as \( T(t) \) is closed in \( \mathcal{H} \), weak measurability of \( \{ P(\Gamma(T(t)))_{1,2} \}_{t \in \mathbb{R}} \) is equivalent to that of \( \{ P(\Gamma(T(t)))_{2,1} \}_{t \in \mathbb{R}} \). Thus, by (A.6),

\[
N\text{-measurability of } \{ T(t) \}_{t \in \mathbb{R}} \text{ is equivalent to weak measurability of }
\]

\[
\{ (|T(t)|^2 + I)^{-1} \}_{t \in \mathbb{R}},
\]

and \( \{ (|T(t)^\ast|^2 + I)^{-1} \}_{t \in \mathbb{R}} \).

Example A.5 ([62]). Let \( T_0 \) and \( T_1 \) be densely defined, closed, unbounded, symmetric operators in \( \mathcal{H} \) satisfying

\[
T_0 \subseteq T_1.
\]

Let \( \mathcal{E} \subset \mathbb{R} \) be a nonmeasurable subset of \( \mathbb{R} \) (in the sense of Lebesgue measure) and introduce the linear operators

\[
\tilde{T}(t) = \begin{cases} 
T_0, & t \in \mathcal{E}, \\
T_1, & t \in \mathbb{R} \setminus \mathcal{E},
\end{cases}
\]

in \( \mathcal{H} \). Then the family \( \{ \tilde{T}(t) \}_{t \in \mathbb{R}} \) is weakly measurable, but not \( N \)-measurable.

The Hilbert space \( L^2(\mathbb{R}; dt; \mathcal{H}) \), in short, \( L^2(\mathbb{R}; \mathcal{H}) \), consists of equivalence classes \( f \) of (weakly) Lebesgue measurable \( \mathcal{H} \)-valued elements \( f(\cdot) \in \mathcal{H} \) (whose elements are equal a.e. on \( \mathbb{R} \)), such that \( \| f(\cdot) \|_\mathcal{H} \in L^2(\mathbb{R}; dt) \). The norm and scalar product on \( L^2(\mathbb{R}; \mathcal{H}) \) are then given by

\[
\| f \|^2_{L^2(\mathbb{R}; \mathcal{H})} = \int_\mathbb{R} \| f(t) \|^2 d\mathcal{H}, \quad (f, g)_{L^2(\mathbb{R}; \mathcal{H})} = \int_\mathbb{R} (f(t), g(t))_\mathcal{H} dt, \quad f, g \in L^2(\mathbb{R}; \mathcal{H}).
\]

Of course, \( L^2(\mathbb{R}; \mathcal{H}) \) can be identified with the constant fiber direct integral \( \int_\mathbb{R} \mathcal{H} dt \), that is,

\[
L^2(\mathbb{R}; \mathcal{H}) = \int_\mathbb{R} \mathcal{H} dt.
\]

Throughout the rest of this appendix, operators denoted by a calligraphic boldface letter such as \( \mathcal{S} \) in the Hilbert space \( L^2(\mathbb{R}; \mathcal{H}) \) represent operators associated with a family of operators \( \{ S(t) \}_{t \in \mathbb{R}} \) in \( \mathcal{H} \), defined by

\[
(Sf)(t) = S(t)f(t) \quad \text{for a.e. } t \in \mathbb{R},
\]

\[
f \in \text{dom}(\mathcal{S}) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid g(t) \in \text{dom}(S(t)) \right\} \quad \text{for a.e. } t \in \mathbb{R},
\]

\[
(A.15)
\]
Theorem A.7 (Nussbaum [114])

Assuming Hypothesis A.2, we note that $\mathcal{T}$, defined according to (A.15), with $T(t)$ satisfying Hypothesis A.2, is closed in $L^2(\mathbb{R}; \mathcal{H})$ since $T(t), t \in \mathbb{R}$, are closed in $\mathcal{H}$ (but $\mathcal{T}$ might not be densely defined). If in addition, the family $\{T(t)\}_{t \in \mathbb{R}}$ is $N$-measurable, then $\mathcal{T}$ is called decomposable in $L^2(\mathbb{R}; \mathcal{H}) = \int_{\mathbb{R}} \mathcal{H} \, dt$ and also denoted by the direct integral of the family $\{T(t)\}_{t \in \mathbb{R}}$ over $\mathbb{R}$ with respect to Lebesgue measure,
\[
\mathcal{T} = \int_{\mathbb{R}}^\oplus T(t) \, dt. \tag{A.16}
\]

In this case, one also has
\[
P(\Gamma(T))_{j,k} = \int_{\mathbb{R}}^\oplus P(\Gamma(T(t)))_{j,k} \, dt, \quad j,k \in \{1,2\}. \tag{A.17}
\]

If $T(t) \in \mathcal{B}(\mathcal{H}), t \in \mathbb{R}$, then
\[
\mathcal{T} \in \mathcal{B}(L^2(\mathbb{R}; \mathcal{H})) \iff \text{esssup}_{t \in \mathbb{R}} \|T(t)\|_{\mathcal{B}(\mathcal{H})} < \infty, \tag{A.18}
\]
in particular, if $\mathcal{T} \in \mathcal{B}(L^2(\mathbb{R}; \mathcal{H}))$, then
\[
\|\mathcal{T}\|_{\mathcal{B}(L^2(\mathbb{R}; \mathcal{H}))} = \text{esssup}_{t \in \mathbb{R}} \|T(t)\|_{\mathcal{B}(\mathcal{H})}. \tag{A.19}
\]

We recall the following results of Nussbaum [114] (in fact, he deals with the more general situation where the constant fiber space $\mathcal{H}$ is replaced by a measurable family of Hilbert spaces $\{\mathcal{H}(t)\}_{t \in \mathbb{R}}$):

**Lemma A.6** (Nussbaum [114]). Assume Hypothesis A.2 and suppose in addition that the family $\{T(t)\}_{t \in \mathbb{R}}$ is weakly measurable. Define $\mathcal{T}$ according to (A.15),
\[
(\mathcal{T}f)(t) = T(t)f(t) \text{ for a.e. } t \in \mathbb{R},
\]
\[
f \in \text{dom}(\mathcal{T}) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \bigl| g(t) \in \text{dom}(T(t)) \text{ for a.e. } t \in \mathbb{R}, \tag{A.20}
\right. \]
\[
t \mapsto T(t)g(t) \text{ is (weakly) measurable, } \int_{\mathbb{R}} \|T(t)g(t)\|^2_{\mathcal{H}} \, dt < \infty \right\}.
\]

Then $\mathcal{T}$ is a closed, decomposable operator in $L^2(\mathbb{R}; \mathcal{H}) = \int_{\mathbb{R}}^\oplus \mathcal{H} \, dt$. Thus, there exists an $N$-measurable family of closed operators $\{\widehat{T}(t)\}_{t \in \mathbb{R}}$ in $\mathcal{H}$ such that
\[
\mathcal{T} = \int_{\mathbb{R}}^\oplus \widehat{T}(t) \, dt \tag{A.21}
\]
and
\[
\widehat{T}(t) \subseteq T(t) \text{ for a.e. } t \in \mathbb{R}. \tag{A.22}
\]

We note that in general $\mathcal{T}$ is not densely defined in $L^2(\mathbb{R}; \mathcal{H})$ (cf. [62]).

**Theorem A.7** (Nussbaum [114]). Assume Hypothesis A.2 and suppose in addition that the family $\{T(t)\}_{t \in \mathbb{R}}$ is $N$-measurable. Then the following assertions hold:

(i) $\mathcal{T} = \int_{\mathbb{R}}^\oplus T(t) \, dt$ is densely defined and closed in $L^2(\mathbb{R}; \mathcal{H}) = \int_{\mathbb{R}}^\oplus \mathcal{H} \, dt$ and
\[
T^* = \int_{\mathbb{R}}^\oplus T(t)^* \, dt, \quad |\mathcal{T}| = \int_{\mathbb{R}}^\oplus |T(t)| \, dt. \tag{A.23}
\]
(ii) $\mathcal{T}$ is symmetric (resp., self-adjoint, or normal) if and only if $T(t)$ is symmetric (resp., self-adjoint, or normal) for a.e. $t \in \mathbb{R}$.
(iii) \( \ker(T) = \{0\} \) if and only if \( \ker(T(t)) = \{0\} \) for a.e. \( t \in \mathbb{R} \). In addition, if \( \ker(T) = \{0\} \) then \( \{T(t)^{-1}\}_{t \in \mathbb{R}} \) is \( N \)-measurable and
\[
T^{-1} = \int_{\mathbb{R}} T(t)^{-1} \, dt. \tag{A.24}
\]

(iv) If \( T \) is self-adjoint in \( L^2(\mathbb{R}; \mathcal{H}) \), then \( T \geq 0 \) if and only if \( T(t) \geq 0 \) for a.e. \( t \in \mathbb{R} \).

(v) If \( T \) is normal in \( L^2(\mathbb{R}; \mathcal{H}) \), then
\[
p(T) = \int_{\mathbb{R}} p(T(t)) \, dt \tag{A.25}
\]
for any polynomial \( p \).

(vi) Let \( S(t), t \in \mathbb{R} \), be densely defined, closed operators in \( \mathcal{H} \) and assume that the family \( \{S(t)\}_{t \in \mathbb{R}} \) is \( N \)-measurable and \( S = \int_{\mathbb{R}} S(t) \, dt \). Then \( T \subseteq S \) if and only if \( T(t) \subseteq S(t) \) for a.e. \( t \in \mathbb{R} \).

Since \( N \)-measurability is a crucial hypothesis in Theorem A.7, we emphasize Remark A.4(iv) which represents necessary and sufficient conditions which seem verifiable in practical situations. In addition, we note the following result:

**Lemma A.8.** Assume Hypothesis A.2 and suppose that
\[
\{T(t)\}_{t \in \mathbb{R}}, \quad \{(|T(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}}, \text{ and } \{T(t)(|T(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \tag{A.26}
\]
are weakly measurable. Then \( \{T(t)\}_{t \in \mathbb{R}} \) is \( N \)-measurable.

**Proof.** Since \( T(t)(|T(t)|^2 + I)^{-1} \in B(\mathcal{H}), t \in \mathbb{R} \), and
\[
(T(t)(|T(t)|^2 + I)^{-1})^* = T(t)^* (|T(t)|^2 + I)^{-1}, \quad t \in \mathbb{R}, \tag{A.27}
\]
one concludes that \( \{T(t)^* (|T(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \) is weakly measurable too. Thus, for each \( g \in \mathcal{H}, \{T(t)^* (|T(t)|^2 + I)^{-1} \_{t \in \mathbb{R}} \} \) is (weakly) measurable in \( \mathcal{H} \), in addition, \( T(t)^* (|T(t)|^2 + I)^{-1} g \in \text{dom}(T(t)) \) for all \( t \in \mathbb{R} \). Since \( \{T(t)\}_{t \in \mathbb{R}} \) is weakly measurable, one thus concludes that
\[
\{T(t)T(t)^* (|T(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} = \{I - (|T(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}}, \tag{A.28}
\]
and hence \( \{(|T(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \) is weakly measurable as well. \[\blacksquare\]

Next, we recall a result due to Lennon [97] on sums and products of decomposable operators (actually, Lennon considers a slightly more general situation). We use the usual conventions that if \( A \) and \( B \) are linear operators in \( \mathcal{H} \) then
\[
\text{dom}(A + B) = \text{dom}(A) \cap \text{dom}(B) \tag{A.29}
\]
and
\[
\text{dom}(AB) = \{f \in \text{dom}(B) \mid Bf \in \text{dom}(A)\}. \tag{A.30}
\]

**Theorem A.9** (Lennon [97]). Let \( A = \int_{\mathbb{R}} A(t) \, dt \) and \( B = \int_{\mathbb{R}} B(t) \, dt \) be closed decomposable operators in \( L^2(\mathbb{R}; \mathcal{H}) = \int_{\mathbb{R}} \mathcal{H} \, dt \) with the \( N \)-measurable families \( \{A(t)\}_{t \in \mathbb{R}} \) and \( \{B(t)\}_{t \in \mathbb{R}} \) in \( \mathcal{H} \) satisfying Hypothesis A.2. Then the following holds:
(i) \( \text{dom}(A + B) \) is dense in \( L^2(\mathbb{R}; \mathcal{H}) \) if and only if \( \text{dom}(A(t) \cap B(t)) \) is dense in \( \mathcal{H} \) for a.e. \( t \in \mathbb{R} \). In addition, \( A + B \) is closable in \( L^2(\mathbb{R}; \mathcal{H}) \) if and only if \( A(t) + B(t) \)
is closable in $\mathcal{H}$ for a.e. $t \in \mathbb{R}$. In this case the family \( \{[A(t) + B(t)]\}_{t \in \mathbb{R}} \) is $N$-measurable and

\[
A + B = \int_\mathbb{R} [A(t) + B(t)] \, dt. \tag{A.31}
\]

(ii) $\text{dom}(AB)$ is dense in $L^2(\mathbb{R}; \mathcal{H})$ if and only if $\text{dom}(A(t)B(t))$ is dense in $\mathcal{H}$ for a.e. $t \in \mathbb{R}$. In addition, $AB$ is closable in $L^2(\mathbb{R}; \mathcal{H})$ if and only if $A(t)B(t)$ is closable in $\mathcal{H}$ for a.e. $t \in \mathbb{R}$. In this case the family \( \{[A(t)B(t)]\}_{t \in \mathbb{R}} \) is $N$-measurable and

\[
AB = \int_\mathbb{R} [A(t)B(t)] \, dt. \tag{A.32}
\]

Lemma A.10. Assume Hypotheses 2.1. Then

\[
\{B(t)\}_{t \in \mathbb{R}}, \quad \{B(t)^*\}_{t \in \mathbb{R}}, \quad \{B'(t)\}_{t \in \mathbb{R}}, \quad \{(B'(t))^*\}_{t \in \mathbb{R}}, \tag{A.33}
\]

as well as

\[
\{B(t)(|B(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}}, \quad \{B'(t)(|B'(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}},
\]

\[
\{(B(t))^*|^2 + I\}^{-1}\}_{t \in \mathbb{R}}, \quad \{(|B'(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}};
\tag{A.34}
\]

are weakly measurable. In particular, (2.3) and (A.34) together imply that \( \{B(t)\}_{t \in \mathbb{R}} \) and \( \{B'(t)\}_{t \in \mathbb{R}} \) are $N$-measurable. Consequently, $B$ and $B'$, defined according to (3.57), are densely defined in $L^2(\mathbb{R}; \mathcal{H})$, and the analogs of (3.58) hold in either case.

Proof. Fix a (weakly) measurable family of elements \( \{f(t)\}_{t \in \mathbb{R}} \) in $\mathcal{H}$ such that $f(t) \in \text{dom}(B(t))$ for a.e. $t \in \mathbb{R}$. By Hypothesis 2.1 (ii), for every $g \in \text{dom}(|A_\cdot|)$,

\[
(g, B(\cdot)f(\cdot))_\mathcal{H} = (B(\cdot)g, f(\cdot))_\mathcal{H}, \tag{A.35}
\]

where \( \{B(t)g\}_{t \in \mathbb{R}} \) (as well as \( \{f(t)\}_{t \in \mathbb{R}} \)) is weakly measurable and hence measurable in $\mathcal{H}$. By (A.7), the function \( f(\cdot)B(\cdot)g \) is measurable. Since $\text{dom}(|A_\cdot|)$ is dense, \( B(t)f(t) \) is measurable in $\mathcal{H}$ by Remark A.4 (i). Thus \( \{B(t)\}_{t \in \mathbb{R}} \) is weakly measurable. Using (3.4), one similarly infers that \( \{B'(t)\}_{t \in \mathbb{R}} \) is weakly measurable. Utilizing Remark 3.6, one then also concludes that \( \{B(t)^*\}_{t \in \mathbb{R}} \) and \( \{(B'(t))^*\}_{t \in \mathbb{R}} \) are weakly measurable, proving (A.33).

Next, we invoke the fact that \( \{(B(t)(|B(t)|^2 + I)^{-1})\}_{t \in \mathbb{R}} \) is assumed to be weakly measurable by Hypothesis 2.1 (v): As above, for a (weakly) measurable family of elements \( \{f(t)\}_{t \in \mathbb{R}} \) in $\mathcal{H}$ such that $f(t) \in \text{dom}(B(t))$ for a.e. $t \in \mathbb{R}$, and for every $g \in \text{dom}(|A_\cdot|)$, the function

\[
(B(\cdot)(|B(\cdot)|^2 + I)^{-1}f(\cdot), g)_{\mathcal{H}} = ((|B(\cdot)|^2 + I)^{-1}f(\cdot), B(\cdot)g)_{\mathcal{H}} \tag{A.36}
\]

is measurable since \( \{(|B(t)|^2 + I)^{-1}f(t)\}_{t \in \mathbb{R}} \) and \( \{B(t)g\}_{t \in \mathbb{R}} \) are measurable in $\mathcal{H}$. Since $\text{dom}(|A_\cdot|)$ is dense, Remark A.4 (ii) implies that \( \{B(t)(|B(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \) is weakly measurable. Similarly one proves the weak measurability of the family \( \{B'(t)(|B'(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \).

Weak measurability of \( \{(|B(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \) then follows from Lemma A.8; the weak measurability of the family \( \{(|B'(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \) is proved analogously, completing the proof of (A.34).

$N$-measurability of \( \{B(t)\}_{t \in \mathbb{R}} \) and \( \{B'(t)\}_{t \in \mathbb{R}} \) then follows from (A.10).
Finally, that \( B \) and \( B' \) are densely defined in \( L^2(\mathbb{R}; \mathcal{H}) \) and the analogs of (3.58) hold follows from Theorem A.7 (i). 

Next, we show that Hypothesis 2.1 (v) is essential, in particular, we will show that weak measurability of the family \( \{(B'(t)^2 + I)^{-1}\}_{t \in \mathbb{R}} \) does not follow from weak measurability of \( \{B'(t)\}_{t \in \mathbb{R}} \) and weak measurability of \( \{B'(t)\|(A_- + I)^{-1}\}_{t \in \mathbb{R}} \). For this purpose it suffices to consider the following example (a slight refinement of Example A.5):

**Example A.11.** Let \( B_0 \) and \( B_1 \) be densely defined, closed, unbounded, symmetric operators in \( \mathcal{H} \) satisfying

\[
B_0 \subseteq B_1 \tag{A.37}
\]

and

\[
\text{dom}(A_-) \subseteq \text{dom}(B_0). \tag{A.38}
\]

Let \( \mathcal{E} \subset \mathbb{R} \) be a nonmeasurable subset of \( \mathbb{R} \) (in the sense of Lebesgue measure) and introduce the linear operators

\[
\tilde{B}(t) = \begin{cases} 
B_0, & t \in \mathcal{E}, \\
B_1, & t \in \mathbb{R} \setminus \mathcal{E}, 
\end{cases} \tag{A.39}
\]

in \( \mathcal{H} \). Then the family \( \{\tilde{B}(t)\}_{t \in \mathbb{R}} \) is weakly measurable, but not \( N \)-measurable, in particular,

\[
\{(|\tilde{B}(t)|^2 + I)^{-1}\}_{t \in \mathbb{R}} \text{ is not weakly measurable.} \tag{A.40}
\]

On the other hand, obviously,

\[
\tilde{B}(t)(|A_-| + I)^{-1} = B_0(|A_-| + I)^{-1} \tag{A.41}
\]

is \( N \)-measurable, in fact, even constant with respect to \( t \in \mathbb{R} \).

**Proof.** Let \( \{f(t)\}_{t \in \mathbb{R}} \) be a (weakly) measurable family of elements in \( \mathcal{H} \) such that \( f(t) \in \text{dom}(\tilde{B}(t)) \) for all \( t \in \mathbb{R} \). Then, using the fact that

\[
B_0 \subset B_1 \subseteq B_1^* \subset B_0^*, \tag{A.42}
\]

one concludes that

\[
(\tilde{B}(t)f(t), g)_{\mathcal{H}} = (f(t), B_0g)_{\mathcal{H}}, \quad t \in \mathbb{R}, \quad g \in \text{dom}(B_0), \tag{A.43}
\]

is measurable, and since \( \text{dom}(B_0) \) is dense in \( \mathcal{H} \), the family \( \{\tilde{B}(t)\}_{t \in \mathbb{R}} \) is weakly measurable by Remark A.4 (i).

Since by hypothesis, \( B_0 \not\subseteq B_1, B_0^* B_0 \neq B_1^* B_1 \), and hence there exists \( 0 \neq h \in \mathcal{H} \) such that

\[
(h, (B_0^* B_0 + I)^{-1} h)_{\mathcal{H}} \neq (h, (B_1^* B_1 + I)^{-1} h)_{\mathcal{H}}. \tag{A.44}
\]

Since nonmeasurability of \( \mathcal{E} \) is equivalent to nonmeasurability of its characteristic function \( \chi_\mathcal{E} \), one similarly infers that

\[
(h, ((\tilde{B}(t))^* \tilde{B}(t) + I)^{-1} h)_{\mathcal{H}} = \begin{cases} 
(h, (B_0^* B_0 + I)^{-1} h)_{\mathcal{H}}, & t \in \mathcal{E}, \\
(h, (B_1^* B_1 + I)^{-1} h)_{\mathcal{H}}, & t \in \mathbb{R} \setminus \mathcal{E},
\end{cases} \tag{A.45}
\]

is nonmeasurable, implying that the family \( \{\tilde{B}(t)\}_{t \in \mathbb{R}} \) is not \( N \)-measurable by (A.10) and hence (A.40) follows. \( \blacksquare \)
As another application of the notion of $N$-measurability we now conclude this appendix with an alternative proof of Lemma 4.2 (iii), that is we reprove the fact that the operator $D_{A_-}$ is normal in $L^2(\mathbb{R}; H)$:

**Lemma A.12.** Suppose $A_-$ is self-adjoint in $H$ on dom($A_-) \subseteq H$, and define the operator $D_{A_-}$ as in (4.4). Then $D_{A_-}$ is a normal (and hence closed) operator in $L^2(\mathbb{R}; H)$.

**Proof.** We start by considering the direct integral decomposition

$$
\tilde{D}_{A_-} = \int_\mathbb{R}^\oplus D(t) \, dt,
$$

$$
\text{dom} (\tilde{D}_{A_-}) = \left\{ g \in L^2(\mathbb{R}; H) \mid g(t) \in \text{dom}(D(t)) \text{ for a.e. } t \in \mathbb{R}, \right\}
$$

\begin{align}
& t \mapsto D(t)g(t) \text{ is (weakly) measurable, } \int_\mathbb{R} \|D(t)g(t)\|_H^2 \, dt < \infty
\end{align}

in $L^2(\mathbb{R}; H)$. Here $\{D(t)\}_{t \in \mathbb{R}}$ is the family of normal operators in $H$ given by

$$
D(t)f = if + A_- f, \quad f \in \text{dom}(D(t)) = \text{dom}(A_-), \quad t \in \mathbb{R}.
$$

Next we show, that the family $\{D(t)\}_{t \in \mathbb{R}}$ is $N$-measurable. Indeed, the orthogonal projection $P(D(t))$, $t \in \mathbb{R}$, in $H \oplus H$ onto the graph of the operator $D(t)$ is given by the $2 \times 2$ operator-valued matrix in $\mathcal{B}(H) \oplus \mathcal{B}(H)$,

$$
P(D(t)) = \begin{pmatrix}
(A_-^2 + (t^2 + 1)I_H)^{-1} & (A_- - itI_H)(A_-^2 + (t^2 + 1)I_H)^{-1} \\
(A_- + itI_H)(A_-^2 + (t^2 + 1)I_H)^{-1} & I_H - (A_-^2 + (t^2 + 1)I_H)^{-1}
\end{pmatrix}.
$$

The family $\{P(D(t))\}_{t \in \mathbb{R}}$ is a norm-continuous family of bounded operators and hence $\{P(D(t))\}_{t \in \mathbb{R}}$ is weakly measurable, which in turn proves that the family $\{D(t)\}_{t \in \mathbb{R}}$ is $N$-measurable. One observes that $N$-measurability of $\{D(t)\}_{t \in \mathbb{R}}$ implies its weak measurability (cf. (A.8)), and therefore, the requirement in (A.46) that the map $t \mapsto D(t)g(t)$ is (weakly) measurable holds automatically and hence is redundant in this case. Combining Lemma A.6 and Theorem A.7 (ii), one concludes that the direct integral

$$
\tilde{D}_{A_-} = \int_\mathbb{R}^\oplus D(t) \, dt,
$$

on the domain provided in (A.46), is a normal operator.

Since $A_-$ is a self-adjoint operator, the following estimate holds,

$$
t^2\|f\|_H^2 \leq \|(A_- + itI_H)\|_H^2, \quad f \in \text{dom}(A_-), \quad t \in \mathbb{R},
$$

and one concludes that the requirement $\int_\mathbb{R} \|D(t)g(t)\|_H^2 \, dt < \infty$ in (A.46) for $g \in L^2(\mathbb{R}, H)$ is equivalent to the conditions

$$
\int_\mathbb{R} \|(1 + t^2)g(t)\|_H^2 \, dt < \infty \quad \text{and} \quad \int_\mathbb{R} \|A_- g(t)\|_H^2 \, dt < \infty,
$$

and thus to

$$
\text{dom}(\tilde{D}_{A_-}) = \text{dom}(itI) \cap \text{dom}(A_-).
$$
Thus, $\tilde{D}_{A_-}$ on (A.52) is a normal operator. Here, in obvious notation, $itI$ denotes the maximally defined operator of multiplication by $it$ in $L^2(\mathbb{R}; \mathcal{H})$ with domain $\text{dom}(itI) = \left\{ g \in L^2(\mathbb{R}; \mathcal{H}) \mid \int_{\mathbb{R}} (1 + t^2) \| g(t) \|^2 \, dt < \infty \right\}$. (A.53)

Applying the unitary vector-valued Fourier transform $\tilde{F}$ (cf. the comments in connection with (4.32)) one notes that $\tilde{F}^{-1} A_- \tilde{F}^{-1} = A_-$, (A.54)

since $A_-$ has constant fiber operators $A_-(t) = A_-, \ t \in \mathbb{R}$, in $\mathcal{H}$, and $\tilde{F}$ is unitary on any Hilbert space $L^2(\mathbb{R}; \mathcal{K})$, and hence particularly in the case $\mathcal{K} = \mathcal{H}_1(\mathbb{R})$. (cf. (1.25)).

In this context one also notes that $\tilde{F}^{-1} \tilde{D}_{A_-} \tilde{F}^{-1} = D_{A_-}$.

(A.57)

Since $\tilde{D}_{A_-}$ is a normal operator, from (A.57) one concludes that $D_{A_-}$ is a normal operator on $\text{dom}(d/dt) \cap \text{dom}(A_-)$ in $L^2(\mathbb{R}; \mathcal{H})$.

APPENDIX B. Trace Norm Analyticity of $[g_z(A_+) - g_z(A_-)]$

The purpose of this appendix is to provide a straightforward proof of Lemma 7.4, given the fact (7.23):

**Lemma B.1.** Assume Hypothesis 2.1 and let $z \in \mathbb{C}\setminus[0, \infty)$. Then $[g_z(A_+) - g_z(A_-)]$ is differentiable with respect to the $B_1(\mathcal{H})$-norm and

\[
\frac{d}{dz} \text{tr}_\mathcal{H} \left( g_z(A_+) - g_z(A_-) \right) = \text{tr}_\mathcal{H} \left( \frac{d}{dz} g_z(A_+) - \frac{d}{dz} g_z(A_-) \right)
\]

(B.1)

\[
= \frac{1}{2} \text{tr}_\mathcal{H} \left( A_+ (A_+^2 - zI)^{-3/2} - A_- (A_-^2 - zI)^{-3/2} \right), \ z \in \mathbb{C}\setminus[0, \infty).
\]

Proof. Throughout this proof we choose $z \in \mathbb{C}\setminus[0, \infty)$ and $h \in \mathbb{C}$ satisfying $|h| < \varepsilon$ with $0 < \varepsilon$ sufficiently small such that also $z, (z + h) \in \mathbb{C}\setminus[0, \infty)$. Due to the self-adjointness of $A_\pm$ in $\mathcal{H}$,

\[
\sigma(A_+^2) \cup \sigma(A_-^2) \subseteq [\sigma_0, \infty) \subseteq [0, \infty).
\]

where we abbreviated

\[
\sigma_0 = \min \{ \inf (\sigma(A_+^2)), \inf (\sigma(A_-^2)) \} \geq 0.
\]

(B.2)

We recall the integral representations

\[
A_\pm (A_\pm^2 - zI)^{-1/2} f = \frac{1}{\pi} \int_0^\infty t^{-1/2} (A_\pm^2 + (-z + t)I)^{-1} A_\pm f \, dt, \ f \in \text{dom}(A_\pm),
\]

valid in the strong sense in $\mathcal{B}(\mathcal{H})$ (cf., e.g., [86, Sect. V.3.11]). As a consequence of (B.4) one computes

\[
\frac{1}{h} [g_{z+h}(A_+) - g_{z}(A_+)] - \left[ \frac{d}{dz} g_z(A_+) - \frac{1}{h} [g_{z+h}(A_-) - g_{z}(A_-)] + \frac{d}{dz} g_z(A_-) \right]
\]
One notes that in contrast to (B.4), (B.5) now holds in the norm sense in $\mathcal{B}(\mathcal{H})$.

Next, we recall (7.23), that is,

\[
\|g_\varepsilon(A_+ - A_-)\|_{\mathcal{B}(\mathcal{H})} = \sup_{t \in [0,\infty)} \|g_\varepsilon(A_+ - A_-)\|_{\mathcal{B}(\mathcal{H})} dt.
\]

Indeed, (B.7) follows from [143, Theorem 8.7.1], as $(d/dz)g_\varepsilon(\cdot)$ satisfies the conditions (7.13) (with $\varepsilon = 1$) and (7.14) (both limits vanishing).

Hence,

\[
\begin{align*}
\frac{1}{h} & \left|g_{z+h}(A_+) - g_z(A_+) - \frac{1}{h} \left[g_{z+h}(A_-) - g_z(A_-)\right]\right|_{\mathcal{B}(\mathcal{H})} \\
& \leq \frac{|h|}{\pi} \int_0^\infty t^{-1/2} \left\|\left(A_+ - A_-\right)\left(A_+^2 + (z + t)I\right)^{-2}(A_+^2 + (-z + t)I)^{-1}\right\|_{\mathcal{B}(\mathcal{H})} dt \\
& \quad + \frac{|h|}{\pi} \int_0^\infty t^{-1/2} \left\|A_-(A_+^2 + (z + t)I)^{-2}(A_+^2 + (-z + t)I)^{-1}\right\|_{\mathcal{B}(\mathcal{H})} dt.
\end{align*}
\]

Investigating the terms in (B.8) individually, and recalling,

\[
(A_+ - A_-)(A_+^2 - zI)^{-1/2}, (A_+ - A_-)(A_+^2 - zI)^{-1/2} \in \mathcal{B}(\mathcal{H}), \quad z \in \rho(A_+^2) \quad \text{(B.9)}
\]

by (3.28), one estimates for the first term on the right-hand side of (B.8)

\[
\begin{align*}
\frac{|h|}{\pi} & \int_0^\infty t^{-1/2} \left\|\left(A_+ - A_-\right)\left(A_+^2 + (z + t)I\right)^{-2}(A_+^2 + (-z + t)I)^{-1}\right\|_{\mathcal{B}(\mathcal{H})} dt \\
& \leq C(\varepsilon, z) \frac{|h|}{\pi} \left\|\left(A_+ - A_-\right)(|A_+| + I)^{-1}\right\|_{\mathcal{B}(\mathcal{H})} \int_0^\infty t^{-1/2}(\eta_0(\varepsilon, z) + t) dt < \infty,
\end{align*}
\]

where

\[
\begin{align*}
\left\|(A_+ + I)(A_+^2 + (z + t)I)^{-1}\right\|_{\mathcal{B}(\mathcal{H})} &= \sup_{\mu \in \rho(A_+^2)} \frac{|\mu|^{1/2} + 1}{|\mu - z + t|} \leq C(\varepsilon, z), \\
\left\|(A_+^2 + (z - h + t)^{-1}\right\|_{\mathcal{B}(\mathcal{H})} &= \sup_{\mu \in \rho(A_+^2)} \frac{1}{|\mu - z - h + t|} \leq \frac{1}{\eta_0(\varepsilon, z) + t}
\end{align*}
\]

for $C(\varepsilon, z) > 0$ independent of $t > 0$, and for some $\eta_0(\varepsilon, z) > 0$, with $\eta_0(\varepsilon, z)$ independent of $h \in \mathbb{C}$ since we assumed $z, (z + h) \in \rho(A_+^2) \cap \rho(A_+^2)$ for all $h \in \mathbb{C}$, $|h| < \varepsilon$, with $0 < \varepsilon$ sufficiently small.
Next, we turn to the second term on the right-hand side of (B.8) and write

\[ \frac{|h|}{\pi} \int_0^\infty t^{-1/2} \| A_- \left[ (A_+^2 + (z + t)I)^{-2} (A_+^2 + (-z - h + t)I)^{-1} \right. \]
\[ \left. - (A_+^2 + (z + t)I)^{-2} (A_+^2 + (-z - h + t)I)^{-1} \right] \|_{B_+(H)} dt \]
\[ \leq \frac{|h|}{\pi} \int_0^\infty t^{-1/2} \| A_- \left[ (A_+^2 + (z + t)I)^{-2} (A_+^2 + (z + t)I)^{-2} \right. \]
\[ \left. - (A_+^2 + (z - h + t)I)^{-1} \right] \|_{B_+(H)} dt \]
\[ \times \| (A_+^2 + (z - h + t)I)^{-1} - (A_+^2 + (z - h + t)I)^{-1} \|_{B_+(H)} dt \]
\[ \leq \frac{|h|}{\pi} \int_0^\infty t^{-1/2} (\eta_\delta(z, t) + t)^{-1} \| A_- (A_+^2 + (z + t)I)^{-1} \|_{B(H)} \]
\[ \times \| (A_+^2 + (z - h + t)I)^{-1} - (A_+^2 + (z - h + t)I)^{-1} \|_{B_+(H)} dt \]
\[ \leq \frac{|h|}{\pi} \int_0^\infty t^{-1/2} (\eta_\delta(z, t) + t)^{-1} \| A_- [(A_+^2 + (z + t)I)^{-2} - (A_+^2 + (z + t)I)^{-2}] \|_{B_+(H)} dt. \]

To complete the proof one estimates the following norms:

\[ \| A_- (A_+^2 + (z + t)I)^{-1} \|_{B_+(H)} \]
\[ \leq \| A_- (|A_+| + I)^{-1} \|_{B_+(H)} \| (|A_+| + I)(A_+^2 + (z + t)I)^{-1} \|_{B(H)} \]
\[ \leq C_1(\delta, z) \sup_{\mu \geq \mu_0} |\frac{\mu^{1/2} + 1}{\mu - z + t}| \leq \tilde{C}_1(\delta, z) \]  

(B.14)

and

\[ \| (A_+^2 + (z - h + t)I)^{-1} - (A_+^2 + (z - h + t)I)^{-1} \|_{B_+(H)} \]
\[ = \| A_+ (A_+^2 + (z - h + t)I)^{-1} \] [(A_+^2 + (z - h + t)I)^{-1} \]
\[ + [(A_+^2 + (z - h + t)I)^{-1}]^* A_- (A_+^2 + (z - h + t)I)^{-1} \|_{B_+(H)} \]
\[ \leq \| A_+ (A_+^2 + (z - h + t)I)^{-1} \|_{B(H)} \]
\[ \times \| (A_+^2 + (z - h + t)I)^{-1} \|_{B_+(H)} \]
\[ + \| (A_+^2 + (z - h + t)I)^{-1} \|_{B_+(H)} \]
\[ \times \| A_- (A_+^2 + (z - h + t)I)^{-1} \|_{B(H)} \]
\[ = C_1(\delta, z) \| (A_+^2 + (z + t)I)^{-1} \|_{B_+(H)} \]
\[ + C_2(\delta, z) \| (A_+^2 + (z + t)I)^{-1} \|_{B_+(H)}. \]  

(B.15)
for appropriate constants $C_j(\varepsilon, z) > 0$, $j = 1, 2$, independent of $t > 0$ and $h \in \mathbb{C}$, $|h| < \varepsilon$, and similarly,

\[
\|A_-(A_+^2 + (-z + t)I)^{-1} - (A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})} = \|A_-(A_+^2 + (-z + t)I)^{-1} - (A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}
\]

\[
+ \|A_+(A_+^2 + (-z + t)I)^{-1} - (A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}
\]

\[
\|A_-(A_+^2 + (-z + t)I)^{-1} - (A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}
\]

\[
\times (A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}
\]

\[
\leq \|A_-(A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}\|A_+(A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}
\]

\[
\times \|((A_+ - A_+)(A_+^2 + (-z + t)I)^{-1})\|_{B_1(\mathcal{H})}
\]

\[
+ \|A_+(A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}\|((A_+ - A_+)(A_+^2 + (-z + t)I)^{-1})\|_{B_1(\mathcal{H})}
\]

\[
+ \|A_+(A_+^2 + (-z + t)I)^{-1}\|_{B_1(\mathcal{H})}\|((A_+ - A_+)(A_+^2 + (-z + t)I)^{-1})\|_{B_1(\mathcal{H})}
\]

\[
= C_3(\varepsilon, z)\|(A_+ - A_+)(A_+ + I)^{-1}\|_{B_1(\mathcal{H})}
\]

\[
+ C_4(\varepsilon, z)\|(A_+ - A_+)(A_+ + I)^{-1}\|_{B_1(\mathcal{H})},
\]

for appropriate constants $C_k(\varepsilon, z) > 0$, $k = 3, 4$, independent of $t > 0$ and $h \in \mathbb{C}$, $|h| < \varepsilon$, repeatedly applying estimates of the type (B.11), (B.12), and (B.14).

Finally, combining (B.8)–(B.16) yields

\[
\left\| \frac{1}{h} \left[ g_{z+h}(A_+) - g_{z+h}(A_-) \right] - \left[ g_z(A_+) - g_z(A_-) \right] \right\|_{B_1(\mathcal{H})} \xrightarrow{h \to 0} O(h)
\]

and proves the required differentiability in trace norm. Since $z \in \mathbb{C}\setminus[0, \infty)$ was arbitrary, one concludes that (B.1) holds. ■

We note that Lemma B.1 extends to $z \in \rho(A_+^2) \cap \rho(A_0^2)$.

The function $g_z(x)$, $x \in \mathbb{R}$, in Lemma B.1 should be viewed as a smooth version of a step function approaching $\pm 1$ as $x \to \pm \infty$. In this context we also note that
compactness for operators of the type

$$[\arg(A_+ - zI) - \arg(A_- - zI)], \quad z \in \mathbb{C}_+ = \{z \in \mathbb{C} | \text{Im}(z) > 0\},$$

was proved in [122, Theorem 7.3].

Acknowledgments. We are indebted to Alan Carey, Alexander Gomilko, Alexander Pushnitski, Arnd Scheel, Barry Simon, and Alexander Strohmaier for helpful discussions. We are particularly grateful to Alan Carey for his steadfast support of this project.

References

[1] A. Abbondandolo, Morse theory for Hamiltonian systems, Res. Notes Math., Vol. 425, Chapman/Hall/CRC, Boca Raton, FL, 2001.
[2] A. Abbondandolo and P. Majer, Ordinary differential operators in Hilbert spaces and Fredholm pairs, Math. Z. 243, 525–562 (2003).
[3] M. Abramovitz, I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[4] W. O. Amrein and K. B. Sinha, On pairs of projections in a Hilbert space, Linear Algebra Appl. 208/209, 425 – 435 (1994).
[5] N. Anghel, Remark on Callias’ index theorem, Rep. Math. Phys. 28, 1–6 (1989).
[6] N. Anghel, $L^2$-index formulae for perturbed Dirac operators, Comm. Math. Phys. 128, 77–97 (1990).
[7] N. Anghel, The two-dimensional magnetic field problem revisited, J. Math. Phys. 31, 2091–2093 (1990).
[8] N. Anghel, On the index of Callias-type operators, Geom. Funct. Anal. 3, 431–438 (1993).
[9] N. Anghel, Index theory for short-ranged fields in higher dimensions, J. Funct. Anal. 119, 19–36 (1994).
[10] H. Araki, On quasifree states of CAR and Bogoliubov automorphisms, Publ. Res. Inst. Math. Sci. 6, 385–442 (1970/71).
[11] W. Arendt, C. K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Transforms, Monographs in Mathematics, Vol. 96, Birkhäuser, Basel, 2001.
[12] N. Aronszajn and W. F. Donoghue, On exponential representations of analytic functions in the upper half-plane with positive imaginary part, J. Analyse Math. 5, 321–388 (1956–57).
[13] M. F. Atiyah, V. K. Patodi, I. M. Singer, Spectral asymmetry and Riemannian geometry, Bull. London Math. Soc. 5, 229–234 (1973).
[14] M. F. Atiyah, V. K. Patodi, I. M. Singer, Spectral asymmetry and Riemannian geometry. I., Math. Proc. Cambridge Philos. Soc. 77, 43–69 (1975).
[15] M. F. Atiyah, V. K. Patodi, I. M. Singer, Spectral asymmetry and Riemannian geometry. II., Math. Proc. Cambridge Philos. Soc. 78, 405–432 (1975).
[16] M. F. Atiyah, V. K. Patodi, I. M. Singer, Spectral asymmetry and Riemannian geometry. III., Math. Proc. Cambridge Philos. Soc. 79, 71–99 (1976).
[17] J. Avron, R. Seiler, and B. Simon, The index of a pair of projections, J. Funct. Anal. 120, 220–237 (1994).
[18] N. Azamov, A. Carey, and F. Sukochev, The spectral shift function and spectral flow, Commun. Math. Phys. 276, 51–91 (2007).
[19] N. A. Azamov, A. L. Carey, P. G. Dodds, and F. A. Sukochev, Operator integrals, spectral shift, and spectral flow, Canad. J. Math., 61, 241–263 (2009).
[20] S. Azzali and C. Wahl, Spectral flow, index and the signature operator, preprint, 2009, arXiv:0911.2862.
[21] M.-T. Benameur, A. L. Carey, J. Phillips, A. Rennie, F. A. Sukochev, and K. P. Wojciechowski, An analytic approach to spectral flow in von Neumann algebras, in Analysis, Geometry and Topology of Elliptic Operators, B. Booss-Bavnbek, S. Klimek, M. Lesch, and W. Zhang (eds.), World Scientific, Singapore, 2006, pp. 297–352.
[22] P. Benevieri and P. Piccione, On a formula for the spectral flow and its applications, Math. Nachr. 283, 659–685 (2010).
[23] N. Berline, E. Getzler, and M. Vergne, Heat Kernels and Dirac Operators, Springer, Berlin, 1992.
[24] M. Sh. Birman and M. G. Krein, On the theory of wave operators and scattering operators, Sov. Math. Dokl. 3, 740–744 (1962).
[25] M. Š. Birman and M. Z. Solomjak, Stieltjes double operator integrals, Sov. Math. Dokl. 6, 1567–1571 (1965).
[26] M. Sh. Birman and M. Z. Solomyak, Stieltjes double-integral operators, in Topics in Mathematical Physics, Vol. 1, Spectral Theory and Wave Processes, M. Sh. Birman (ed.), Consultants Bureau Plenum Publishing Corporation, New York, 1967, pp. 25–54.
[27] M. Sh. Birman and M. Z. Solomyak, Stieltjes double-integral operators. II, in Topics in Mathematical Physics, Vol. 2, Spectral Theory and Problems in Diffraction, M. Sh. Birman (ed.), Consultants Bureau, New York, 1968, pp. 1946.
[28] M. Sh. Birman and M. Z. Solomyak, Double Stieltjes operator integrals. III, in Problems of Mathematical Physics, Vol. 6, Theory of Functions, Spectral Theory, Wave Propagation, M. Sh. Birman (ed.), Izdat. Leningrad. Univ., Leningrad, 1973, pp. 27–53. (Russian).
[29] M. Sh. Birman and M. Z. Solomyak, Remarks on the spectral shift function, J. Sov. Math. 3, 408–419 (1975).
[30] M. S. Birman and M. Z. Solomyak, Spectral Theory of Self-Adjunct Operators in Hilbert Space, Reidel, Dordrecht, 1987.
[31] M. Sh. Birman and M. Z. Solomyak, Operator integration, perturbations, and commutators, J. Sov. Math. 63, 129–148 (1993).
[32] M. Sh. Birman and M. Z. Solomyak, Double operator integrals in a Hilbert space, Integr. Eqns. Oper. Theory 47, 131–168 (2003).
[33] M. Sh. Birman and D. R. Yafaev, The spectral shift function. The work of M. G. Krein and its further development, St. Petersburg Math. J. 4, 833–870 (1993).
[34] D. Bleecker and B. Booss-Bavnbek, Spectral invariants of operators of Dirac type on partitioned manifolds in Aspects of Boundary Problems in Analysis and Geometry, J. Gil, T. Krainer, I. Witt (eds.), Operator Theory: Advances and Applications, Vol. 151, Birkhäuser, Basel, 2004, pp. 1–130.
[35] A. Böttcher and I. M. Spitkovsky, A gentle guide to the basics of two projections theory, Lin. Algebra Appl. 432, 1412–1459 (2010).
[36] B. Bojarski, Abstract linear conjugation problems and Fredholm pairs of subspaces, in Differential and Integral equations, Boundary value problems. Collection of papers dedicated to the memory of Academician I. Vekua, Tbilisi University Press, Tbilisi, 1979, pp. 4560. (Russian.)
[37] D. Bollé, F. Gesztesy, H. Grosse, W. Schweiger, and B. Simon, Witten index, axial anomaly, and Krein’s spectral shift function in supersymmetric quantum mechanics, J. Math. Phys. 28, 1512–1525 (1987).
[38] B. Booss and D. D. Bleecker, Topology and Analysis. The Atiyah–Singer Index Formula and Gauge-Theoretic Physics, Springer, New York, 1985.
[39] B. Booss-Bavnbek, M. Lesch, and J. Phillips, Unbounded Fredholm operators and spectral flow, Canad. J. Math. 57, 225–250 (2005).
[40] B. Booss-Bavnbek and K. P. Wojciechowski, Elliptic Boundary Problems for Dirac Operators, Birkhäuser, Boston, 1993.
[41] N. V. Borisov, W. Müller, and R. Schrader, Relative index theorems and supersymmetric scattering theory, Commun. Math. Phys. 114, 475–513 (1988).
[42] R. Bott and R. Seeley, Some remarks on the paper of Callias, Comm. Math. Phys. 62, 235–245 (1978).
[43] J. Brüning and M. Lesch, On the η-invariant of certain nonlocal boundary value problems, Duke Math. J. 96, 425–468 (1999).
[44] J. Brüning and M. Lesch, On boundary value problems for Dirac type operators. I. Regularity and self-adjointness, J. Funct. Anal. 185, 1–62 (2001).
[45] U. Bunke, Relative index theory, J. Funct. Anal. 105, 63–76 (1992).
[46] C. Callias, Axial anomalies and index theorems on open spaces, Commun. Math. Phys. 62, 213–234 (1978).
[47] A. Carey, D. Potapov, and F. Sukochev, Spectral flow is the integral of one forms on the Banach manifold of self adjoint Fredholm operators, Adv. Math. 222, 1809–1849 (2009).
[48] C. Chicone and Y. Latushkin, Evolution Semigroups in Dynamical Systems and Differential Equations, Math. Surv. Monogr., Vol. 70, Amer. Math. Soc., Providence, RI, 1999.
[49] H. O. Cordes and J. P. Labrousse, The invariance of the index in the metric space of closed operators, J. Math. Mech. 12, 693–719 (1963).
[50] C. Davis, *Separation of two linear subspaces*, Acta Scient. Math. (Szeged) **19**, 172–187 (1958).

[51] B. de Pagter and F. A. Sukochev, *Differentiation of operator functions in non-commutative $L_p$-spaces*, J. Funct. Anal. **212**, 28–75 (2004).

[52] B. de Pagter, F. A. Sukochev, and H. Witvliet, *Double operator integrals*, J. Funct. Anal. **192**, 52–111 (2002).

[53] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, Vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[54] J. Dixmier, *Position relative de deux variétés fermées dans un espace de Hilbert*, Revue Scientifique **86**, 387–399 (1948).

[55] J. Dixmier, *Étude sur les variétés et les opérateurs Julia, avec quelques applications*, Bull. Soc. Math. France **77**, 11–101 (1949).

[56] R. G. Douglas and K. P. Wojciechowski, *Adiabatic limits of the $\eta$-invariants. The odd-dimensional Atiyah–Patodi–Singer problem*, Commun. Math. Phys. **142**, 139–168 (1991).

[57] N. Dunford and J. Schwartz, *Linear operators. Part II. Spectral theory. Selfadjoint operators in Hilbert space*, Wiley & Sons, New York, 1988.

[58] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.

[59] G. Esposito, *Dirac Operators and Spectral Geometry*, Cambridge University Press, Cambridge, 1998.

[60] K. Friedrichs, *On certain inequalities and characteristic value problems for analytic functions and for functions of two variables*, Trans. Amer. Math. Soc. **41**, 321–364 (1937).

[61] F. Gesztesy, *Scattering theory for one-dimensional systems with nontrivial spatial asymptotics*, in Schrödinger operators, Aarhus 1985, Lecture Notes in Math., Vol. 1218, Springer, Berlin, 1986, pp. 93–122.

[62] F. Gesztesy, A. Gomilko, F. Sukochev, and Y. Tomilov, *On a question of A. E. Nussbaum on measurability of families of closed linear operators in a Hilbert space*, arXiv:1002.2733, to appear in Israel J. Math.

[63] F. Gesztesy, Y. Latushkin, M. Mitrea, and M. Zinchenko, *Nonselfadjoint operators, infinite determinants, and some applications*, Russ. J. Math. Phys. **12**, 443–471 (2005).

[64] F. Gesztesy, Y. Latushkin, F. Sukochev, and Y. Tomilov, *Some operator norm bounds and interpolation results in trace ideals revisited*, in preparation.

[65] F. Gesztesy and K. A. Makarov, *The $\Xi$ operator and its relation to Krein’s spectral shift function*, J. d’Anal. Math. **81**, 139–183 (2000).

[66] F. Gesztesy, M. Malamud, M. Mitrea, and S. Naboko, *Generalized polar decompositions for closed operators in Hilbert spaces and some applications*, Integral Eq. Operator Th. **64**, 83–113 (2009).

[67] F. Gesztesy, W. Schweiger, and B. Simon, *Commutation methods applied to the mKdV-equation*, Trans. Amer. Math. Soc. **324**, 465–525 (1991).

[68] F. Gesztesy and B. Simon, *Topological invariance of the Witten index*, J. Funct. Anal. **79**, 91–102 (1988).

[69] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, Publish or Perish, Wilmington, DE, 1984.

[70] P. B. Gilkey and L. Smith, *The eta invariant for a class of elliptic boundary value problems*, Commun. Pure Appl. Math. **36**, 85–131 (1983).

[71] I. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs, Vol. 18, Amer. Math. Soc., Providence, RI, 1969.

[72] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, corrected and enlarged edition, prepared by A. Jeffrey, Academic Press, San Diego, 1980.

[73] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson, Prentice Hall, Upper Saddle River, NJ, 2004.

[74] G. Greiner and R. Nagel, *On the stability of strongly continuous semigroups of positive operators on $L^p(\mu)$*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4), **10**, 257–262 (1983).

[75] G. Grubb, *Poles of zeta and eta functions for perturbations of the Atiyah–Patodi–Singer problem*, Commun. Math. Phys. **215**, 583–589 (2001).

[76] G. Grubb and R. T. Seeley, *Zeta and eta functions for Atiyah–Patodi–Singer operators*, J. Geom. Anal. **6**, 31–77 (1996).
References

[78] H. R. Gr"umm, Two theorems about $C_p$, Rep. Math. Phys. 4, 211–215 (1973).
[79] P. R. Halmos, Two subspaces, Trans. Amer. Math. Soc. 144, 381–389 (1969).
[80] P. Hess and T. Kato, Perturbation of closed operators and their adjoints, Comment. Math. Helv. 45, 524–529 (1970).
[81] E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, Colloquium Publications, Vol. 31, rev. ed., Amer. Math. Soc., Providence, RI, 1985.
[82] N. Kalton, A note on pairs of projections, Bull. Soc. Mat. Mexicana 3, 309–311 (1997).
[83] T. Kato, Notes on projections and perturbation theory, Technical Report No. 9, University of California at Berkeley, 1965.
[84] T. Kato, Linear evolution equations of “hyperbolic” type, II, J. Math. Soc. Japan 25, 648–666 (1973).
[85] T. Kato, Perturbation Theory for Linear Operators, corr. printing of the 2nd ed., Springer, Berlin, 1980.
[86] W. E. Kaufman, A stronger metric for closed operators in Hilbert space, Proc. Amer. Math. Soc. 90, 83–87 (1984).
[87] P. Kirk and M. Lesch, The $\eta$-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary, Forum Math. 16, 553–629 (2004).
[88] V. Kostrykin, K. A. Makarov, and A. K. Motovilov, Existence and uniqueness of solutions to the operator Riccati equation. A geometric approach, in Advances in Differential Equations and Mathematical Physics, Yulia Karpeshina, G"unter Stolz, Rudi Weikard, and Yanni Zeng (eds.), Contemp. Math. 327, 181–198 (2003).
[89] C. Kottke, A general index theorem for Callias–Anghel operators, preprint, 2009, arXiv:0909.5661.
[90] M. G. Krein and M. A. Krasnoselsky, Fundamental theorems about extensions of Hermite operators and some applications to the theory of orthogonal polynomials and to the moment problem, Uspekhi Mat. Nauk 2, 60–106 (1947). (Russian.)
[91] M. G. Krein, M. A. Krasnoselsky, and D. P. Milman, On defect numbers of linear operators in Banach space and some geometric problems, Sbornik Trudov Instituta Matematiki Akademii Nauk Ukrainskoy SSR 11, 97–112 (1948). (Russian.)
[92] M. G. Krein and V. A. Yavryan, Spectral shift functions that arise in perturbations of a positive operator, J. Operator Th. 6, 155–191 (1981).
[93] Y. Latushkin and A. Pogan, The Dichotomy Theorem for evolution bi-families, J. Diff. Eq. 245, 2267-2306 (2008).
[94] Y. Latushkin and Y. Tomilov, Fredholm differential operators with unbounded coefficients, J. Diff. Eq. 208, 388–429 (2005).
[95] H. B. Lawson and M.-L. Michelson, Spin Geometry, Princeton University Press, Princeton, 1989.
[96] M. J. J. Lennon, On Sums and Products of Unbounded Operators in Hilbert Space, Trans. Amer. Math. Soc. 198, 273–285 (1974).
[97] M. Lesch, The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators, in Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds, B. Boss-Bavnbek, G. Grubb, and K. P. Wojciechowski (eds.), Contemp. Math., 366, 193–224 (2005).
[98] M. Lesch and K. P. Wojciechowski, On the $\eta$-invariant of generalized Atiyah–Patodi–Singer boundary value problems, Illinois J. Math. 40, 30–46 (1996).
[99] E. H. Lieb and M. Loss, Analysis, 2nd ed., Amer. Math. Soc., Providence, RI, 2001.
[100] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Springer, New York, 1972.
[101] J. Lott, The eta function and some new anomalies, Phys. Lett. B 145, 179–180 (1984).
[102] R. Melrose, The Atiyah–Patodi–Singer index theorem, Research Notes in Mathematics, Vol. 4, A. K. Peters, Ltd., Wellesley, MA, 1993.
[103] B. Messirdi, M. H. Mortad, A. Azzouz, and G. Djellouli, A topological characterization of the product of two closed operators, Coll. Math. 112, 269–278 (2008).
[104] Y. Mezroui, Le compl`et´e des op´erateurs ferm´es `a domaine dense pour la m´etrique du gap, J. Operator Th. 41, 69–92 (1999).
[106] W. Müller, *Manifolds with Cusps of Rank One. Spectral Theory and $L^2$-Index Theorem*, Lecture Notes in Math., Vol. 1244, Springer, Berlin, 1987.

[107] W. Müller, *L²-index and resonances*, in *Geometry and Analysis on Manifolds*, T. Sunada (ed.), Lecture Notes in Math., Vol. 1339, Springer, Berlin, 1988, pp. 203–21.

[108] W. Müller, *Eta invariants and manifolds with boundary*, J. Diff. Geom. 40, 311–377 (1994).

[109] W. Müller, *Relative zeta functions, relative determinants and scattering theory*, Commun. Math. Phys. 192, 309–347 (1998).

[110] L. I. Nicolaescu, *On the space of Fredholm operators*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 53, 209–227 (2007).

[111] A. J. Niemi and G. W. Semenoff, *Spectral asymmetry on an open space*, Phys. Rev. D (3) 30, 809–818 (1984).

[112] A. J. Niemi and G. W. Semenoff, *Index theorems on open infinite manifolds*, Nuclear Phys. B 269, 131–169 (1986).

[113] M. Ninomiya and C. I. Tan, *Axial anomaly and index theorem for manifolds with boundary*, Nuclear Phys. B 257, 199–225 (1985).

[114] R. S. Palais, *Seminar on the Atiyah–Singer Index Theorem*, Annals of Math. Studies, Vol. 57, Princeton University Press, Princeton, 1965.

[115] R. Pallu de la Barrière, *Décomposition des opérateurs non bornés dans les sommes continues d’espaces de Hilbert*, Comptes Rendus Acad. Sci. Paris 232, 2071–2073 (1951).

[116] J. Phillips, *The behavior of functions of operators under perturbations*, preprint, arXiv:0904.1761.

[117] J. Phillips, *Self-adjoint Fredholm operators and spectral flow*, Canad. Math. Bull. 39, 460–467 (1996).

[118] D. Potapov and F. Sukochev, *Lipschitz and commutator estimates in symmetric operator spaces*, J. Operator Theory 59 (2008), no. 1, 211–234.

[119] D. Potapov and F. Sukochev, *Unbounded Fredholm modules and double operator integrals*, J. reine angew. Math. 626, 159–185 (2009).

[120] D. Potapov and F. Sukochev, *Double operator integrals and submajorization*, Math. Model. Nat. Phenom., to appear.

[121] A. Pushnitski, *The spectral shift function and the invariance principle*, J. Funct. Anal. 183 (2001), no. 2, 269–290.

[122] A. Pushnitski, *The spectral flow, the Fredholm index, and the spectral shift function*, in *Spectral Theory of Differential Operators: M. Sh. Birman 80th Anniversary Collection*, T. Suslina and D. Yafaev (eds.), AMS Translations, Ser. 2, Advances in the Mathematical Sciences, Vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 141–155.

[123] A. Pushnitski, *Operator theoretic methods for the eigenvalue counting function in spectral gaps*, Ann. H. Poincaré 10, 793–822 (2009).

[124] P. Rabier, *The Robin-Salamon index theorem in Banach spaces with UMD*, Dyn. Partial Diff. Eqns. 1, 303–337 (2004).

[125] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I: Functional Analysis*, revised and enlarged edition, Academic Press, New York, 1980.

[126] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[127] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV: Analysis of Operators*, Academic Press, New York, 1978.

[128] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. 27, 1–33 (1995).

[129] J. Roe, *Elliptic Operators, Topology and Asymptotic Methods*, Pitman Research Notes in Math. Series, Vol. 179, Longman Scientific & Technical, Harlow, Essex, UK, 1988.

[130] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer, Berlin, 1960.

[131] M. Schechter, *Operator Methods in Quantum Mechanics*, North Holland, New York, 1981.

[132] K. Sharifi, *The gap between unbounded regular operators*, preprint, arXiv:0901.1891.

[133] B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms*, Princeton University Press, Princeton, NJ, 1971.

[134] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., Mathematical Surveys and Monographs, Vol. 120, Amer. Math. Soc., Providence, RI, 2005.
REFERENCES

[136] I. M. Singer, *The η-invariant and the index*, in *Mathematical Aspects of String Theory*, S. T. Yau (ed.), Adv. Ser. Math. Phys., Vol. 1, World Scientific, Singapore, 1987, pp. 239–258.

[137] I. Spitkovsky, *Once more on algebras generated by two projections*, Linear Algebra Appl. 208/209, 377–395 (1994).

[138] M. H. Stone, *On unbounded operators in Hilbert space*, J. Indian Math. Soc. 15, 155–192 (1951).

[139] B. Thaller, *The Dirac Equation*, Texts and Monographs in Physics, Springer, Berlin, 1992.

[140] C. Wahl, *A new topology on the space of unbounded selfadjoint operators, K-theory and spectral flow, in C*-algebras and elliptic theory II*, Trends in Mathematics, D. Burghelea, R. Melrose, A. S. Mishchenko, and E. V. Troitsky (eds.), Birkhäuser, Basel, 2008, pp. 207–309.

[141] J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Vol. 68, Springer, New York, 1980.

[142] K. Wojciechowski, *Spectral flow and the general linear conjugation problem*, Simon Stevin 59, 59–91 (1985).

[143] D. R. Yafaev, *Mathematical Scattering Theory. General Theory*, Amer. Math. Soc., Providence, RI, 1992.

[144] V. A. Yavryan, *On certain perturbations of selfadjoint operators*, Akad. Nauk Armyan. SSR Dokl. 38, no. 1, 3–7 (1964). (Russian.)