Ising model versus normal form game

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Abstract

The 2-spin Ising model in statistical mechanics and the 2x2 normal form game in game theory are compared. All configurations allowed by the second are recovered by the first when the only concern is about Nash equilibria. But it holds no longer when Pareto optimum considerations are introduced like in the prisoner’s dilemma. This gap can nevertheless be filled by adding a new coupling term to the Ising model, even if that term has up to now no physical meaning. An individual complete bilinear objective function is thus found to be sufficient to reproduce all possible configurations of a 2x2 game. Using this one-to-one mapping, new perspectives for future research in both fields can be envisioned.

1 Introduction

For two decades, statistical mechanics has been more and more used to study economic systems (Kirman-Zimmermann, 2001; Bourgine-Nadal, 2004) as well as financial ones (Bhamra, 2000). A well-known example is the application of statistical mechanics to minority games at the basis of some financial
problems (Coolen, 2005). Such a transfer is generally grounded on a raw analogy, both fields sharing the existence of numerous entities which are able to be in specific states and interact among themselves. Each entity is governed by the optimization of an objective function depending on the states of all of them. The process leads to an overall equilibrium state. However, if the formal frame is rather similar, the interpretation underlying the maximization process is very different. While spins are inert objects which obey objective laws, human beings follow an intentional behavior. Hence, the analogy is not really substantial, but may nevertheless be valid at a formal level.

In this paper, two standard models of each discipline are compared, the Ising model at zero temperature for statistical mechanics and the normal form with Nash equilibrium for game theory. When both models are restricted to two entities and two state variables, a detailed comparison becomes possible. While in statistical mechanics the individual objective function is an explicit truncated bilinear function in both entities, in game theory it is a quite general function with no specification. On this basis, the possible configurations of equilibrium states are computed in each case and systematically compared.

It appears that any configuration happening in the first case appears as well in the second. However, when one introduces Pareto optimum considerations like in the prisoner’s dilemma, this is no more true. However, the prisoner’s dilemma missing configuration can be recovered by adding an additive new term to the Ising objective function. This additional term makes the associated bilinear function complete. It is a linear term which at present has no physical meaning. Nevertheless, an individual bilinear objective function which is complete is sufficient to reproduce all possible configurations of game theory.

The remaining of the paper is organized as follows. The second section recalls the frameworks used in statistical mechanics and in game theory and compares their respective assumptions. The third section analyzes the configurations of equilibrium states obtained in games when using the restrictive individual objective function of the Ising model. The fourth section shows that the missing configuration can be recovered by considering a complete individual bilinear objective function, but can no longer be obtained from a collective objective function. Some perspectives for new research in both fields using our mapping are discussed in the last section.
2 Basic frameworks

2.1 Ising model

In statistical mechanics, the Ising model is most used. Rather powerful to study magnetic systems, it was proven successful in describing a huge spectrum of different problems in other fields of physics. Numerous applications have been also achieved outside physics in particular in sociophysics (Castellano, Fortunato and Loreto, 2009; Galam, 2008).

The Ising model considers a group of \( N \) two-state variables called spins, where each spin \( i = 1, \ldots, N \), takes values denoted \( s_i = \pm 1 \). These spins interact by pairs \( \{i, j\} \) via two exchange constants \( J_{ij} \) and \( J_{ji} \) whose effect is to produce respectively a local field \( h_{j \rightarrow i} = J_{ij} s_j \) from spin \( j \) on spin \( i \) and a local field \( h_{i \rightarrow j} = J_{ji} s_i \) from spin \( i \) on spin \( j \). These fields couple linearly with their associated spins giving rise to the two individual energies:

\[
E_i = -h_{j \rightarrow i} s_i = -J_{ij} s_j s_i \\
E_j = -h_{i \rightarrow j} s_j = -J_{ji} s_i s_j
\]

which are both bilinear functions of the spins.

The minus sign remembers that the energy is ultimately minimized. A positive coupling \( J_{ij} > 0 \) corresponds to a ferromagnetic system and yields: \( s_i = s_j \). A negative coupling \( J_{ij} < 0 \) corresponds to an antiferromagnetic system and favors: \( s_i = -s_j \). The absence of coupling \( J_{ij} = 0 \) makes both values of \( s_i \) and \( s_j \) independent one from the other.

Then, considering one spin \( s_i \), all its interactions with the other \( (N-1) \) spins add up to result in a net local field:

\[
V_i = \sum_{j=1}^{N} J_{ij} s_j, \text{ with } J_{ii} = 0
\]

At this stage, in order to satisfy the principle of equality of action and reaction which holds true for most physical applications, the exchange couplings are taken symmetric: \( J_{ij} = J_{ji} \)

In addition, spins can be coupled linearly to a local field \( \bar{h}_i \), which may vary from spin to spin like in the random field model. It is independent of the spin states. A uniform field \( h \) can also be applied to the system. It couples linearly and simultaneously to every spin. These coupling terms write respectively \( \bar{h}_i s_i \) and \( h s_i \). Both \( \bar{h}_i \) and \( h \) can be positive, negative or null. In physics, discriminating between \( \bar{h}_i \) and \( h \) originated from the different experimental schemes which produce them respectively. On a more formal point of view, they can be combined within one single local variable:

\( h_i = \bar{h}_i + h \).
The total local field becomes:
\[ V_i = \sum_{j=1}^{N} J_{ij} s_j + h_i \]
and the associated individual energy:
\[ E_i = -\sum_{j=1}^{N} J_{ij} s_i s_j - h_i s_i \]

It is worth to note that a self-coupling term with \( i = j \) may be included in \( E_i \). However since \( s_i s_i = 1 \), it contributes a constant \( J_{ii} \) to \( E_i \), which therefore does not influence the equilibrium state. In the special case where \( V_i = 0 \), the associated spin \( s_i \) is said to be “frustrated” since neither \( s_i = +1 \) nor \( s_i = -1 \) can minimize \( E_i = 0 \). It makes a spin flip cost no energy.

The total energy of the \( N \)-spin system is then obtained by adding all individual energies \( E_i \) which defines the Hamiltonian under two equivalent forms:
\[
\mathcal{H} = -\sum_{<i,j>} J_{ij} s_i s_j - \sum_i h_i s_i \tag{1}
\]
\[
\mathcal{H}' = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij} s_i s_j - \sum_i h_i s_i \tag{2}
\]

where \( <i,j> \) means all distinct pairs of interacting spins and \( 1/2 \) corrects the double counting of each pair. In other circumstances, the system may stabilize into some limit cycle.

The Hamiltonian measures the energy of a peculiar configuration \( \{s_1, s_2, ..., s_N\} \) given the external parameters \( \{J_{ij}, h_i\} \). It is sufficient to describe the equilibrium properties of the system.

Observe that minimizing the Hamiltonian with respect to the states \( \{s_i\} \) is identical to minimizing each \( \{E_i\} \) with respect to \( s_i \). However, it should be emphasized that this feature is a direct result of the assumption of symmetric couplings \( (J_{ij} = J_{ji}) \). Indeed, the term \( J_{ij} \) may be virtually asymmetric \( (J_{ij} \neq J_{ji}) \). In that case, the energy function cannot characterize uniquely the system since then the asymmetric part waves out in summing both terms \( J_{ij} s_i s_j \) and \( J_{ji} s_j s_i \) in \( \mathcal{H} \) due to the commutation of \( s_i \) and \( s_j \) resulting in one average coupling term \( (J_{ij} + J_{ji})/2 \) which is symmetric. Therefore, for asymmetric couplings, a purely dynamical analysis is required making the solving of the problem much more difficult.

Observe moreover that, on purely formal grounds, \( h_i \) can be taken as positive without restriction. Exchanging the sign of \( s_i \) preserves the value of \( E_i \) if changing altogether the signs of \( J_{ij} \) and \( h_i \).
At this stage, to make the comparison to game theory straightforward, we restrict the approach to two spins: $N = 2$, even if the physical interpretation is dubious. The local fields becomes:

$$V_1 = J_{21}s_2 + h_1 \text{ and } V_2 = J_{12}s_1 + h_2$$

The individual energies follow:

$$E_1 = -J_{21}s_1s_2 - h_1s_1 \text{ and } E_2 = -J_{12}s_1s_2 - h_2s_2$$

Finally, the Hamiltonian reduces to:

$$\mathcal{H}_{1,2} = -\frac{1}{2}(J_{12} + J_{21})s_1s_2 - h_1s_1 - h_2s_2.$$ (5)

On this basis, the equilibrium state is obtained by a minimization of $\mathcal{H}'$ with respect to both $\{s_1, s_2\}$ given $\{J_{12}, J_{21}, h_1, h_2\}$. Hence $-\mathcal{H}'$ is a collective objective function which is maximized with respect to the two internal degrees of freedom $\{s_1, s_2\}$. Remember that $h_1$ and $h_2$ can be taken positive without restriction.

In the case of symmetric exchange couplings ($J_{12} = J_{21}$), the minimization of $E_1$ with respect to $s_1$ and $E_2$ with respect to $s_2$ is precisely equivalent to the minimization of $\mathcal{H}$ with respect to $\{s_1, s_2\}$. The functions $E_1$ and $E_2$ are therefore individually minimized, hence their opposite are individual objective functions which are maximized.

### 2.2 Game theory

In game theory, the standard “normal-form” is most used. It applies to many economic applications as well as to applications in other fields like political science or biology.

The framework considers a group of $N$ players numerated with $i = 1, ..., N$. Each player $i$ chooses some action $s_i$ in a set $S_i$, which is defined on a nominal (discrete) scale rather than on a numerical one. The players interact globally through their actions and reach some common consequences $C$ for each issue $(s_1, ..., s_i, ..., s_N)$. These consequences are assessed by each player $i$ thanks to a utility or payoff function $U_i = U_i(C) = U_i(s_1, ..., s_i, ..., s_N)$. Such a utility index is ordinal, hence the utility function, otherwise of any form, is defined up to a monotone transformation. Finally, each player chooses his action by maximizing his utility.
Since the maximization implemented by some player depends on what
the other players do, an equilibrium notion has to be defined in order to
coordinate the players’ moves. A Nash equilibrium state is a stable state in a
fixed environment, provided that one player only is able to deviate from it at
a time. Technically, a (pure) Nash equilibrium state is an ‘issue’ (or ‘state’
or ‘profile of actions’ \((s_1^*, ..., s_i^*, ..., s_N^*)\) where each action is a best response
to the others’ equilibrium actions:

\[
s_i^* = \arg \max_{s_i} U_i(s_1^*, ..., s_i, ..., s_N^*).
\]  

(6)

In other terms, in an equilibrium state, no player has an incentive to deviate
unilaterally from his action:

\[
U_i(s_1^*, ...s_i^*, ...s_N^*) \geq U_i(s_1^*, ...s_i, ...s_N^*), \forall s_i.
\]  

(7)

A Nash equilibrium is strict if and only if the inequality is strict for each
player. In fact, the equilibrium conditions state that if the players are placed
in an equilibrium state, they stay there. But they do not explain through
what dynamical process the players come to an equilibrium state.

The Nash equilibrium notion assumes that the players follow their own
individual interests (embedded in their utility function which gathers het-
erogenous motivations). The Pareto optimum notion considers a collective
point of view (however based on the individual utility functions). It asserts
that a state is a Pareto optimum if there exists no other available state which
is better for all players (and strictly better for one of them at least). Techni-
cally, it is defined in two steps. Firstly, an issue \((s_1, ..., s_i, ..., s_N)\) Pareto-
dominates another issue \((s'_1, ..., s'_i, ..., s'_N)\) if it gives more utility to each player
(and strictly more to one of them)

\[
U_i(s_1, ...s_i, ...s_N) \geq U_i(s'_1, ...s'_i, ...s'_N), \forall i,
\]  

(8)

with one strict inequality. Secondly, a Pareto optimum is an issue which is
not Pareto-dominated by any other issue. However, even if a collective order
on global states is considered, no collective objective function is assumed to
exist.

The game is now restricted to two players and two actions per player, i.e.
a 2 x 2 game. It is usually represented by a matrix where the lines represent
the two actions \(s_1^1\) and \(s_2^2\) of the first player and the columns the two actions
\(s_1^2\) and \(s_2^2\) of the second player. At the intersection of an action for each
player, the bi-matrix indicates first the utility of the first player, then the utility of the second player:

\[
\begin{array}{c|c|c|c}
   & s_1^1 & s_2^1 & s_2^2 \\
\hline
s_1^1 & (U_1(s_1^1, s_1^2), U_2(s_1^1, s_2^2)) & (U_1(s_1^1, s_2^1), U_2(s_1^1, s_2^2)) & (U_1(s_2^1, s_1^2), U_2(s_2^1, s_2^2)) \\
\hline
s_1^2 & (U_1(s_1^2, s_2^2), U_2(s_1^2, s_2^2)) & (U_1(s_1^2, s_2^1), U_2(s_1^2, s_2^2)) & (U_1(s_2^1, s_2^1), U_2(s_2^1, s_2^2)) \\
\end{array}
\]

This bi-matrix has indeed quite a general form and can be written with 8 free parameters:

\[
\begin{array}{c|c|c|c}
(a, a') & (b, b') \\
(c, c') & (d, d') \\
\end{array}
\]

Two classes of games are of special interest. The symmetric games are such that the two players play symmetric roles and get symmetric utilities: \(a = a', b = c', c = b', d = d'\). The zero-sum games are such that the utilities of the players are opposite for each issue of the game: \(a = -a', b = -b', c = -c', d = -d'\). In case of a symmetric game, it exists at least one equilibrium state while in case of a zero-sum game, it exists at most one equilibrium state.

In the bi-matrix, the (pure) Nash equilibria are obtained as 'wells' of the best response arrows relating horizontal or vertical issues in the matrix. The Pareto optima are obtained by eliminating all dominated issues. The two types of states need not to coincide. For instance, consider the game with: \(a \succ c \succ d \succ b\), \(c' = d' \succ a' \succ b'\). The best responses, the Nash equilibria (a strict Nash equilibrium is denoted \(N\) while a large one is denoted \(N'\)) and the Pareto optima (denoted \(P\)) are the following:

\[
\begin{array}{c|c|c|c}
N = P & \leftarrow & \uparrow & \downarrow \\
P & \leftrightarrow & N' \\
\end{array}
\]

Observe that the equilibrium states are unchanged under a monotone transformation of \(U_1\) and \(U_2\). Especially, they are not affected when adding to \(U_1\) (and \(U_2\)) a constant in the first column and another in the second:
Likely, the Pareto-dominating issues are unchanged under a linear transformation of $U_1$ and $U_2$. Especially, they are unchanged when adding to $U_1$ (and $U_2$) a same constant in all issues:

\[
\begin{array}{cc}
(a + \lambda, a' + \lambda') & (b + \mu, b' + \mu') \\
(c + \lambda, c' + \lambda') & (d + \mu, d' + \mu')
\end{array}
\]

Note however that such transformations do not keep the symmetric or zero-sum character of the game.

2.3 Correspondance between the frames

The two frameworks appear quite similar in their structure: a spin is replaced by a player, the state $s_i$ by an action $s_i$, the local energy $-E_i$ by a utility function $U_i$, a physical equilibrium by a Nash equilibrium. But the Hamiltonian $\mathcal{H}$ has no counterpart since no collective utility (linear or not) is defined. However, the physical structure is logically stronger than the game structure since the state is quantified and the energy is more specific (truncated bilinear function of the variables).

The physical problem can be transformed into a game problem by constructing an equivalent normal form for a game. The $2 \times 2$ matrix gives in each issue the energy of spins 1 and 2, the issues corresponding to the combinations of first and second states for the spins. We thus get the 'Ising game' matrix which is comparable to the one obtained from game theory:

\[
\begin{array}{cc}
J_{12} + h_1, J_{21} + h_2 & -J_{12} + h_1, -J_{21} - h_2 \\
-J_{12} - h_1, -J_{21} + h_2 & J_{12} - h_1, J_{21} - h_2
\end{array}
\]

The same previous general form can be used, but it involves now only 4 free parameters since the off diagonal payoffs depend on the diagonal payoffs in the following way:
with $\alpha = J_{12} + h_1$, $\alpha' = J_{21} + h_2$, $\delta = J_{12} - h_1$ and $\delta' = J_{21} - h_2$. Note that the correspondence can be reversed since $J_{12} = (\alpha + \delta)/2$, $h_1 = (\alpha - \delta)/2$, $J_{21} = (\alpha' + \delta')/2$, $h_2 = (\alpha' - \delta')/2$.

The Ising game is symmetric when $\alpha = \alpha'$ and $\delta = \delta'$, which involves $J_{12} = J_{21}$ and $h_1 = h_2$. It is zero-sum when $\alpha = \delta = -\alpha' = -\delta'$, which involves $J_{12} = -J_{21}$ and $h_1 = h_2 = 0$.

If we are interested only in equilibrium states, it is always possible to change a usual game into an equivalent Ising game by adding to the utilities some specific constants:

$$\lambda = -(a + c)/2, \mu = -(b + d)/2, \lambda' = -(a' + c')/2, \mu' = -(b' + d')/2.$$ 

The new utilities become related to the old ones as follows:

$$\alpha = (a - c)/2, \delta = (d - b)/2, \alpha' = (a' - c')/2, \delta' = (d' - b')/2.$$ 

Moreover, the physical parameters can be directly related to the old utilities:

$$J_{12} = (a - b - c + d)/2, h_1 = (a + b - c - d),$$

$$J_{21} = (a' - b' - c' + d')/2, h_2 = (a' + b' - c' - d').$$ 

However, if we are interested in Pareto optima, the same operation cannot be done since the constants generally differ along lines and columns.

### 3 Comparison of equilibrium states

#### 3.1 Taxonomy of 2x2 games

In general 2x2 games, we are looking for the different types of equilibrium configurations that may happen. For this, it is enough to consider that the payoffs of a player are defined up to a monotone transformation, hence are defined on an ordinal scale. Even more we have only to consider the relative position of the payoffs leading to the best responses. It means that we have just to compare $a$ to $c$ and $b$ to $d$. Such a reasoning concerns in fact the vertical arrows in the basic scheme. We obtain only 9 combinations:

- 2 combinations A without ties and opposite moves: ↑↓ or ↓↑
- 2 combinations B without ties and parallel moves: ↑↑ or ↓↓
- 2 combinations C with one tie right: ↑↑ or ↓↓
- 2 combinations D with one tie left: ↑↓ or ↓↑
- 1 combination E with two ties: ↑↓

For two players, we obtain in principle 9x9 = 81 combinations. But a matrix obtained by exchanging lines or columns gives the same equilibrium states since there is only a change of label of the actions. However, for some matrices, exchanging lines or columns can lead to the same matrix. In fact, the exchange properties of the different combinations are the following:

- combinations A exchange themselves when exchanging lines and similarly when exchanging columns.
- combinations B exchange themselves when exchanging lines and are invariant when exchanging columns
- combinations C exchange themselves when changing lines and lead to configurations D when exchanging columns
- combinations D exchange themselves when changing lines and lead to configurations C when exchanging columns
- combination E is invariant when changing lines as well as when changing columns.

It is clear that only one of the two combinations B, C and D needs to be kept for each player since the other is obtained by exchanging the lines (for player 1) and the columns (for player 2). Similarly, the unique combination E has to be kept. A first problem happens however with configurations A. Keeping one combination for player 1 is enough when crossing with any other combination for player 2. But when crossing one combination A for player 1 and one combination B for player 2 is not enough since two different configurations are concretely produced. A second problem appears with configuration E. Since it is completely symmetric, its adoption by player 1 gives the same configuration against the combinations C and D of player 2, hence one has to be deleted.

The following table makes precise the 15 different configurations that can be obtained:
The four configurations without ties are the basic ones and can each can be illustrated by a specific example:

- configuration 1 corresponds to the absence of a Nash equilibrium state. It is illustrated by “matching pennies” where both players exhibit one of the two sides of a personal penny; the first wins if the two pennies are on the same side while the second wins if they are on opposite sides

\[
\begin{array}{c|c|c|c|c}
A & B & C & D & E \\
\hline
\overset{1}{\rightarrow} & N & \overset{2}{\rightarrow} & N & \overset{3}{\rightarrow} & N & \overset{4}{\rightarrow} & N & \overset{5}{\rightarrow} & N & \overset{6}{\rightarrow} & N \\
\overset{3}{\downarrow} & \overset{2}{\downarrow} & \overset{4}{\downarrow} & \overset{6}{\downarrow} & \overset{8}{\downarrow} & \overset{10}{\downarrow} & \overset{12}{\downarrow} & \overset{14}{\downarrow} \\
\hline
\overset{7}{\uparrow} & N & \overset{8}{\uparrow} & N' & \overset{9}{\uparrow} & N' & \overset{10}{\uparrow} & N' \\
\hline
\overset{11}{\uparrow} & N' & \overset{12}{\uparrow} & N' & \overset{13}{\uparrow} & N' & \overset{14}{\uparrow} & N' \\
\hline
\overset{15}{\uparrow} & N' & \overset{16}{\uparrow} & N' & \overset{17}{\uparrow} & N' \\
\hline
\end{array}
\]

- configuration 2 corresponds to two Nash equilibrium states. A first sub-case happens when the two equilibria cannot be compared: one is better for the first player and the other is better for the second player. It is illustrated by the “battle of the sexes” where player 1 is the husband and player 2 the wife; for the husband, the first action consists in going to boxing and the second to ballet; for the wife, the first action consists in going to ballet and the second to boxing; the utility of preferred entertainment is 2 and the utility of being together to 4. A second subcase happens when one equilibrium Pareto dominates the other. It is illustrated by the “stag hunt game” where for each player, the first action consists in hunting a rabbit and the second a deer; the rabbit can be hunt alone and provides then a utility of 1; the deer...
can only be hunt conjointly and provides a utility of 7 for each; if only one player hunts the deer, he will starve and his utility is -1.

|   | (2, 2) | (4, 6) | (7, 7) | (-1, 1) |
|---|-------|--------|--------|---------|
| (6, 4) | (0, 0) | (1, -1) | (1, 1) |

- configuration 4 corresponds to one Nash equilibrium obtained by dominant actions for each player. A first subcase happens when the Nash equilibrium is not Pareto dominated by another issue. It is illustrated by the "lonely game" where each player buys a toy independently from the other’s action; the utility of the toy for each player is +1 and the desutility of not buying is -1. A second subcase happens when the Nash equilibrium is Pareto dominated by another issue. It is illustrated by the "prisoner’s dilemma" where for each player, the first action consists in defecting and the second in cooperating; they get utility 2 if both defect, utility 4 if both cooperate, but if one cooperates without the other, he receives 0 and the other receives 6; the only equilibrium state is the issue (2,2); but the last is not a Pareto optimum (contrary to all other three issues) since it is Pareto-dominated by the opposite issue (4,4)

|   | (1, 1) | (1, -1) |
|---|-------|---------|
| (-1, 1) | (-1, -1) |

### 3.2 Taxonomy of Ising games

The Nash equilibrium states can be computed and the configurations they determine depend only on the relation between $J_{ij}$ and $h_i$. The matrix defined by these parameters isolate 25 basic configurations. For each configuration, the scheme of best responses in the 2 x 2 matrix is defined in the following table in which $h_1$ and $h_2$ are strictly positive:
| $J_{12} < -h_1$ | $J_{21} < -h_2$ | $J_{21} = -h_2$ | $-h_2 < J_{21} < h_2$ | $J_{21} = h_2$ | $J_{21} > h_2$ |
|---------------|---------------|---------------|-----------------|---------------|---------------|
| $J_{12} = -h_1$ | $\rightarrow N$ | $\leftrightarrow N'$ | $\leftrightarrow N'$ | $\leftrightarrow N'$ | $\leftrightarrow N'$ |
| $\downarrow [2] \uparrow$ | $\downarrow [5] \uparrow$ | $\downarrow [3] \uparrow$ | $\downarrow [8] \uparrow$ | $\downarrow [1] \uparrow$ |
| $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ |
| $J_{12} = h_1$ | $\rightarrow N'$ | $N' \leftrightarrow N'$ | $N' \leftrightarrow N'$ | $N' \leftrightarrow N'$ | $N' \leftrightarrow N'$ |
| $\uparrow [8] \downarrow$ | $\uparrow [10] \downarrow$ | $\uparrow [6] \downarrow$ | $\uparrow [7] \downarrow$ | $\uparrow [5] \downarrow$ |
| $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ |
| $J_{12} > h_1$ | $\uparrow [1] \downarrow$ | $\uparrow [8] \downarrow$ | $\uparrow [3] \downarrow$ | $\uparrow [5] \downarrow$ | $\uparrow [2] \downarrow$ |
| $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ |

The preceding table has to be completed by a table where $h_1 = 0$, $h_2 > 0$ (or symmetrically):

| $J_{21} < -h_2$ | $J_{21} = -h_2$ | $-h_2 < J_{21} < h_2$ | $J_{21} = h_2$ | $J_{21} > h_2$ |
|---------------|---------------|-----------------|---------------|---------------|
| $J_{12} < 0$ | $\rightarrow N$ | $\leftrightarrow N'$ | $\leftrightarrow N'$ | $\leftrightarrow N'$ |
| $\downarrow [2] \uparrow$ | $\downarrow [5] \uparrow$ | $\downarrow [3] \uparrow$ | $\downarrow [8] \uparrow$ | $\downarrow [1] \uparrow$ |
| $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ |
| $J_{12} = 0$ | $\rightarrow N'$ | $N' \leftrightarrow N'$ | $N' \leftrightarrow N'$ | $N' \leftrightarrow N'$ | $N' \leftrightarrow N'$ |
| $\uparrow [12] \downarrow$ | $\uparrow [14] \downarrow$ | $\uparrow [13] \downarrow$ | $\uparrow [14] \downarrow$ | $\uparrow [12] \downarrow$ |
| $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ | $N' \leftrightarrow$ |
| $J_{12} > 0$ | $\uparrow [1] \downarrow$ | $\uparrow [8] \downarrow$ | $\uparrow [3] \downarrow$ | $\uparrow [5] \downarrow$ | $\uparrow [2] \downarrow$ |
| $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ | $N \leftrightarrow$ |

Finally, a last table is obtained when $h_1 = h_2 = 0$:
Of course, many configurations appear twice by symmetry. But more important, all configurations appear as expected. The first table gives configurations 1 to 11. The second table adds the configurations 12 to 14. The third table adds the configuration 15.

Notice that the game may be symmetric only in configurations 2, 4, 7, 11 and 15. It can be checked that it then admits at least one equilibrium state. Likely, the game may be zero-sum only in configurations 1 and 15. It then admits either zero or four Nash equilibrium states.

The games without ties can be obtained from Ising models for specific parameters:

- configuration 1 (no equilibrium state) is obtained when $J_{ij}J_{ji} < 0$ and $|J_{ij}| > |h_i|$. Especially, matching pennies is obtained with $J_{12} = 1$, $J_{21} = -1$ and $h_1 = h_2 = 0$.

- configuration 2 (two strict equilibrium states) is obtained when $J_{ij}J_{ji} > 0$ and $|J_{ij}| > |h_i|$. Especially, the battle of sexes is obtained with $J_{12} = J_{21} = -2$ and $h_1 = h_2 = 1$ and a general translation of utility of 3. The stag hunt game is obtained with $J_{12} = J_{21} = 2$ and $h_1 = h_2 = +1$ and a general translation of utility of 4.

- configuration 4 (one strict equilibrium state) is obtained when $|J_{ij}| < |h_i|$. Especially, the lonely game is obtained for $J_{12} = J_{21} = 0$ and $h_1 = h_2 = 1$. 

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
& $J_{21} < 0$ & $J_{21} = 0$ & $J_{21} > 0$ \\
\hline
$J_{12} < 0$ & $\rightarrow N'$ & $\leftrightarrow N'$ & $\leftrightarrow N'$ \\
& $\downarrow [2] \uparrow$ & $\downarrow [12] \uparrow$ & $\downarrow [1] \uparrow$ \\
& $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ \\
$J_{12} = 0$ & $\rightarrow N'$ & $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ \\
& $\uparrow [12] \downarrow$ & $\uparrow [15] \downarrow$ & $\uparrow [12] \downarrow$ \\
& $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ \\
$J_{12} > 0$ & $\rightarrow N'$ & $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ \\
& $\uparrow [1] \downarrow$ & $\uparrow [12] \downarrow$ & $\uparrow [2] \downarrow$ \\
& $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ & $N' \leftrightarrow N'$ \\
\hline
\end{tabular}
\end{center}
4 Extended framework

4.1 Missing configurations

In the preceding framework, only the point of view of Nash equilibrium states was considered. A Nash equilibrium corresponds to a state where the players may stay, when they are astrained to strictly follow their own interest. But the game theorists are interested in Pareto optima too (especially when they coincide with equilibrium states). The reason is that an issue which is not Pareto optimal is not collectively satisfying since the players may move to a better issue for both, however by coordinating their actions.

Consider an Ising game where the North-West issue \((\alpha, \alpha')\) is an equilibrium and the South-West issue \((\delta, \delta')\) is not. The corresponding conditions are \(\alpha > 0, \alpha' > 0\) and \(\delta < 0, \delta' < 0\). It follows immediately that the North-West issue \((\alpha, \alpha')\) Pareto-dominates the South-West one \((\delta, \delta')\). The opposite dominance is not possible, hence some games which precisely show such an opposite dominance cannot be represented by Ising games.

For instance, consider the "prisoner's dilemma". The North-West issue (where both players defect) is the only equilibrium state. The South-West issue (where both players cooperate) is better for both players to the opposite one. Hence, the prisoner’s dilemma, with its specific structure of a unique Nash equilibrium Pareto-dominated by a non equilibrium issue, cannot be written as an Ising game.

4.2 Enlarged framework

In order to recover the missing configuration within the Ising model and thus get a full matching of configurations, the simplest way is to add to the local energy of each spin an 'altruistic' term \(k_is_j\), which depends exclusively on the other spin state:

\[
E_i = -\sum_j J_{ij}s_is_j - h_is_i - k_is_j.
\]

In fact, such an operation transforms, for player \(i\), his initial payoff in a payoff where \(k_i\) is added in the first column and where \(k_i\) is substracted in the second column. Such an operation does not change the equilibrium states (even if it changes the payoffs at equilibrium), but does change the
Pareto states. The ’extended Ising game’ writes now in the form of a general
game:

\begin{align*}
  s_1^1 &= +1 \\
  s_1^2 &= -1 \\
  s_2^1 &= +1 \\
  s_2^2 &= -1 \\
  (J_{12} + h_1 + k_1, J_{21} + h_2 + k_2) &= (J_{12} - h_1 - k_1, J_{21} - h_2 - k_2) \\
  (J_{12} - h_1 + k_1, J_{21} + h_2 - k_2) &= (J_{12} - h_1 - k_1, J_{21} - h_2 - k_2)
\end{align*}

This matrix is written under the form:

\begin{equation}
\begin{pmatrix}
  (\bar{a}, \bar{a}') \\
  (\bar{b}, \bar{b}') \\
  (\bar{c}, \bar{c}') \\
  (\bar{d}, \bar{d}')
\end{pmatrix}
\end{equation}

with \( \bar{a} = J_{12} + h_1 + k_1 \), \( \bar{b} = -J_{12} + h_1 - k_1 \), \( \bar{c} = -J_{12} - h_1 + k_1 \) and \( \bar{d} = J_{12} - h_1 - k_1 \) for player 1 and similarly for player 2. However, the
coefficients of the first player still obey the constraint: \( \bar{a} + \bar{b} + \bar{c} + \bar{d} = 0 \) (and
symmetrically for the second). But this condition can always be reached from the original matrix by substracting to the coefficients of the first player
a constant \( (a + b + c + d)/4 \) to all payoffs (without changing the equilibrium
and Pareto states). The preceding formulas can then be reversed (for the
first player):

\begin{align*}
  J_{12} &= (\bar{a} - \bar{b} - \bar{c} + \bar{d})/4 = (a - b - c + d)/4 \\
  h_1 &= (\bar{a} + \bar{b} - \bar{c} - \bar{d})/4 = (a + b - c - d)/4 \\
  k_1 &= (\bar{a} - \bar{b} + \bar{c} - \bar{d})/4 = (a - b + c - d)/4
\end{align*}

Hence, a complete bilinear function is sufficient to obtain all possible config-
urations.

For instance, the prisoner’s dilemma is obtained by setting \( J_{12} = J_{21} = 0 \),
\( h_1 = h_2 = 1 \) and \( k_1 = k_2 = -2 \), and applying a general payoff transformation
of +3.

### 4.3 The ’altruistic’ term

At this stage, two comments can be made about the ’altruistic term’ added
to the energy of each spin. First, this term has no clear interpretation in
statistical mechanics. Second, the completed energy cannot be deduced from
a collective single Hamiltonian.
If such an Hamiltonian existed, he would be obtained by 'integrating' the new energy of each spin $V_i$, obtained by dividing $-E_i$ by $s_i$:

$$V_1 = \frac{\partial H}{\partial s_1} = J_{12}s_2 + h_1 + k_1s_2/s_1$$  \hspace{1cm} (10)$$
$$V_2 = \frac{\partial H}{\partial s_2} = J_{21}s_1 + h_2 + k_2s_1/s_2$$  \hspace{1cm} (11)$$

But since the second derivatives $\frac{\partial^2 H}{\partial s_1 \partial s_2}$ and $\frac{\partial^2 H}{\partial s_2 \partial s_1}$ differ, the Hamiltonian does not exist.

Conversely, consider the Hamiltonian obtained by adding the energy of both spins.

$$H' = J_{12}s_1s_2 + J_{21}s_1s_2 + h_1s_1 + h_2s_2 + k_1s_2 + k_2s_1$$  \hspace{1cm} (12)$$

It leads to individual spin energies which are not the good ones, but it allows to find the Pareto optimum in the prisoner’s dilemma. Hence, the missing configuration, contrary to the other ones, cannot be obtained by minimizing a Hamiltonian.

## 5 Further extensions

Extension to continuous positions for each entity is usual in games as well as in statistical mechanics and their comparison can be extended. But extension to N entities creates a further divergence between statistical mechanics and game theory. In statistical mechanics, the energy of a spin is generally generated by summarizing its bilateral interactions with all other spins. Multilateral interactions are possible in some cases, but are more difficult to handle mathematically. In economics, the utility is directly defined on a multilateral basis, by a general dependence on all others’ actions. Again, it is not specified, but only submitted to very general conditions. It follows that the additional term(s) needed to recover a general function from a partial one will be intricate.

Two problems stay common to statistical mechanics and game theory. In both cases, the equilibrium state is defined from outside (by the modeler) as a state which remains stable without perturbations from the environment. But the concrete process leading to some well-defined equilibrium state is
not formally described. Hence, the 'implementation problem' states that the process leading to an equilibrium state has to be made explicit. Likely, the 'selection problem' states that the way one equilibrium state is sorted out in case of multiplicity has to be made explicit, in relation with the preceding one. In physics, a random selection is often assumed with equiprobability.

In game theory, two justifications of an equilibrium notion were recently proposed and duly formalized (Walliser, 2006). The epistemic justifications assume that the players are hyper-intelligent and are able to reach instantaneously an equilibrium state by their sole reasoning. Moreover, the players select one equilibrium state through shared ‘conventions’, for instance focal points (culturally induced). The evolutionist justifications assume that the players have a bounded rationality and follow a sequential learning or evolution process leading asymptotically to an equilibrium state. In that case, an equilibrium state is automatically selected at least probabilistically by the initial conditions and the history of the system.

In statistical mechanics, only the second option is open since the spins can hardly be assumed to hold beliefs and to deliberate. Nevertheless the sociophysics approach has proven possible to substitute individuals to spins as long as the assumptions made are formally equivalent. In fact, various schemes were proposed in order to describe the local dynamic genesis of an equilibrium state, with more or less justifications. They allow the convergence towards an equilibrium state by using the concept of spontaneous symmetry breaking.

Finally, statistical mechanics studies the system of spins even when the temperature differs from zero. A spin is then in some state with a probability precisely defined which decreases with its energy and increases with the inverse of the temperature. In game theory, the corresponding situation happens when a player chooses an action with a probability generically defined which increases with its utility and depends on some parameter characterizing the ‘exploration- exploitation’ dilemma. The corresponding equilibrium states have to be further compared.

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