THE CONSTRAINTS ON POWER SPECTRUM OF RELIC GRAVITATIONAL WAVES FROM CURRENT OBSERVATIONS OF LARGE-SCALE STRUCTURE OF THE UNIVERSE

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Within the framework of cosmological inflationary models, relic gravitational waves arise as a natural consequence of the quantum nature of primordial space-time metric fluctuations and validity of general relativity theory. Therefore the detection of cosmological gravitational waves would be the strongest evidence in support of basic assumptions of inflation theory. Although tracing gravitational waves by polarization patterns on last scattering surface of the cosmic microwave background is only planned for forthcoming experiments, some general constraints on the tensor mode of metric perturbations (i.e. gravitational waves) can be established right now.

We present the determination of the amplitude of relic gravitational waves power spectrum. Indirect best-fit technique was applied to compare observational data and theory predictions. As observations we have used data on large-scale structure of the Universe and anisotropy of cosmic microwave background temperature. The conventional inflationary model with 11 parameters has been investigated, all of them evaluated jointly. This approach gave us a possibility to find parameters of power spectrum of gravitational waves along with statistical errors.

INTRODUCTION

For the last decade due to the progress in observations the cosmology entered the new stage of its development. This stage was signalized by the new generation of experiments aimed to the measurements of anisotropies of cosmic microwave background (CMB). These were balloon-borne BOOMERanG, MAXIMA, Archeops, ground-based interferometers DASI, CBI, VSA. And probably the most important one is the successor of COBE space mission, Wilkinson Microwave Anisotropy Probe (WMAP), that published 1-year observation results in 2003. WMAP has carried out the measurements of CMB over the whole sky with unprecedented angular resolution and high sensitivity of detectors. The data and web-links for these experiments are available at the web-site of Legacy Archive for Microwave Background Data Analysis (LAMDA) project[1].

CMB explorations were complemented with extensive measurements of expansion dynamics of the Universe by distances to Supernovae and large-scale structure surveys.

Advances in quality of experimental data manifestly call for the model capable to explain the whole set of collected data. Now it’s well understood, that the simplest cosmological models can not match the observations adequately, as e.g. standard flat CDM model with scale-invariant power spectrum of density fluctuations. The observations advance the complication of model, the number of parameters increases as well as the number of phenomena encompassed by theory. Today elaborate inflationary models need about 11 parameters for the proper description of reality.

Cosmological parameters could be classified in a following way:

• The parameters related to the background model of the homogeneous and isotropic Universe. Evolution of Universe in this model is determined by the amount of energy densities of different components in ratio to the critical one \( \Omega_i = \rho_i/\rho_{cr} \). These are densities of baryon component \( \Omega_b \), hot dark matter (massive neutrinos) \( \Omega_{\nu} \), cold dark matter \( \Omega_{cdm} \), and density parameter for dark energy (i.e. cosmological constant or quintessence) \( \Omega_{de} \).

• The global properties of Universe. According to the Friedmann equations sum of density parameters gives unity for the spatially flat Universe, or \( 1 - \Omega_k \) for curved, so \( \Omega_k \) is a curvature parameter. The Hubble constant \( H_0 = 100h \ km/(s \cdot Mpc) \) is regarded as a global parameter too.
Parameters associated with the inflation. According to inflationary scenario the large-scale structure of Universe is assumed to be formed due to growth of primordial matter density perturbations because of the gravitational instability. The perturbations have adiabatic nature and originate from quantum fluctuations stretched to the cosmological scales during the stage of exponential expansion – inflation. Perturbations are usually described by their power spectra in most general form $P_s(k) = A_s k^{n_s}$ for scalar and $P_t(k) = A_t k^{n_t}$ for tensor mode of perturbations.

The list of parameters is concluded by some additional parameters, like late re-ionization parameter $\tau$ and biasing parameters to relate the distribution of density to the spatial distribution of real astrophysical objects, e.g. galaxies.

This parameter set is required for building the predictions of model to be compared with the observations for the purpose of finding out how good the theoretical assumptions match the results of current observations. As far as these predictions depend on the parameters quite nonlinearly, the determination of the best-fit value for one parameter requests the determination of full set of parameters, by means of statistics. Actually that is the main task of this investigation, sometimes referred as a testing of cosmological models.

In this paper we shall attract attention to the one parameter among many others, namely the primordial power spectrum of tensor mode of space-time metric perturbations (relic gravitational waves). We shall use the most common kind of cosmological model, and analyze some particular inflation models. Sometimes this model is designated as concordance model, or even standard model. The purpose of this model is to give plausible explanation for large-scale structure of the Universe and its global properties.

THE NATURE OF RELIC GWs

Indeed, gravitational waves are very simple conceptually but proved to be extremely elusive for detection. Gravitational waves ought to be emitted by any physical system with changes in quadrupole distribution of stress-energy density. This follows from equations of general relativity as a free solution (wave solution) for small linear space-time metric perturbations on the background metric.

Obviously, that only astrophysical objects could produce gravitational waves of significant amplitudes. The number of probable astrophysical sources is under discussion for observational programmes (see for review [2]). There is no direct successful detections by now of any kind of sources. For cosmological GWs in fact very small chance exist to be detected by any human-made antenna because of their super-long wavelengths.

Cosmological GWs play the role of most ancient relic in our Universe. They have their origin in first moments in Universe evolution, presumably the times of inflation. If we assume general relativity theory to be valid in those times and quantum zero-point oscillations to exist, so we come to conclusion of GWs inescapable existence. Variable gravitational field in very early Universe parametrically amplifies quantum oscillations, making up stochastic gravitational background. The amplitude of these GWs is determined by energy scales at the moment of emission, so for inflation this amplitude determines the energy scale of processes that give rise for inflation. So the nature of relic gravitational waves is very fundamental for physics. In fact both scalar mode and tensor modes of space-time metric perturbations share common origin from single physical process, quantum oscillations.

The subsequent evolution of metric linear perturbations is completely described by gauge-invariant formalism for cosmological perturbations. Relic GWs are represented by tensor mode of space-time metric perturbations on the isotropic and homogeneous expanding background. As far as GWs do not interact with rest of medium in expanding Universe, their amplitude should decrease with time (see [3] for exhaustive review).

Gauge-invariant theory leads to equation for evolution of tensor perturbations of space-time metrics in vacuum or perfect fluid:

$$\ddot{H}^{(T)} + 2\frac{\dot{a}}{a} \dot{H}^{(T)} + (2\mathcal{K} + k^2) H^{(T)} = 0$$  \hspace{1cm} (1)

where $H^{(T)}$ is gauge-invariant amplitude of tensor perturbations for two polarizations, $a$ – scale factor, $\mathcal{K} = (-1, 0, 1)$ – curvature index and $k$ is the wavenumber. The dots denote the derivatives with respect to the conformal time $\eta$. The solution of this equation is propagating damped wave.

OUTLINE OF METHOD

As it was stated above both tensor and scalar modes of space-time metric perturbations have the same origin. The perturbations of scalar mode connected to the density perturbations so they eventually lead to the formation of galaxies, clusters and voids, i.e. large-scale structure of the Universe. Since tensor perturbations
do not lead to the formation of large-scale structure, cosmological GWs can not be revealed by present state of observable structure. Both scalar and tensor perturbations produce the power in CMB angular power spectrum due to Sachs-Wolfe effect. Of course, because of its specific polarization properties relic GWs should generate particular polarization pattern of CMB anisotropies, but the detection of polarization patterns in CMB is matter of future.

![CMB Power Spectrum](image)

Figure 1. The scalar and tensor contributions to CMB angular power spectrum for particular model.

Therefore we propose the indirect method. As far as CMB power spectrum consists of contributions from scalar and tensor modes, we can extract the last one if we know precisely from large-scale structure the scalar contribution. Also, relic GWs have extra-long wavelengths of particle horizon size, so the CMB power spectrum from them should manifest fast decrease with higher multipole moments (smaller scales). On the contrary, the density perturbations at larger multipoles produce a feature-rich CMB power spectrum, a series of acoustic peaks. Thus, the physical properties of tensor mode of cosmological perturbations of space-time metric can be estimated on the basis of observational data on: angular power spectrum of CMB combined and large-scale structure of Universe in wide range of scales – from galactic ones to the size of particle horizon, from $10^{-3}$ to $10^4$ Mpc. This way we can constrain total level of intensity of gravitational waves. In multipoles range of $\ell \sim 10 - 20$ scalar and tensor contributions are comparable (Fig. 1). So the precision of result is to be determined by precision of datapoints within this range. This method is basically a normalization of angular power spectrum.

**BUILDING PREDICTIONS**

Unfortunately, the cosmological models are quite complicated and the most of predictions could not be reduced to analytical functions with cosmological parameters as arguments. Therefore for the given parameter set one need to run numerical computations to make predictions. One way is the “brute-force” method, when the numerical packages like CMBfast [4] are utilized. Since the testing requires large number of models to be evaluated, so even quite fast codes need a huge computational resources to accomplish the task. Another demerit lies in so-called “black box” uncertainty of numerical calculations, when the physical processes are hidden inside computations. So we propose a semi-analytical approaches to make predictions of CMB power spectrum.

For tensor contribution to CMB power spectrum within low multipoles range ($2 < \ell < 20$) we have developed an analytical approximation able to reproduce computed spectrum with high accuracy, for wide range of parameters values. This approximation takes a form:

$$C_T^\ell = \frac{A(n_t, \Omega_0)}{\ell + b(n_t, \Omega_0)} \cdot K_{\ell}(\Omega_{de}) \times \exp(-C(n_t, \Omega_0) \cdot \ell^2 + D(n_t, \Omega_0) \cdot \ell)$$  

(2)
where the coefficients $A$, $b$, $C$ and $D$ represent polynomial functions of $n_t$ and $\Omega_0 = 1 - \Omega_k$, $K_\ell(\Omega_{de})$ is amplification factor for CMB power spectrum caused by dark energy. Analytical formulae for them presented on our paper [11].

Semi-analytical approximation for scalar contribution to CMB power spectrum for the multipoles range $2 < \ell < 20$ was developed in our previous paper [6] and used here. This approximation combine the speed of computations and clear unobscured physical meaning of processes that cause the anisotropies in CMB. The angular power spectrum of CMB at higher multipoles have a number of peaks separated with dips and manifest the damping of average amplitude to the highest $\ell$. This features are explained by acoustic oscillations in photon-baryon fluid set up by adiabatic perturbations on entering the sound horizon. Instead of calculations of power for each $\ell$ within a wide range, we propose to describe the whole shape of CMB power spectrum by positions and amplitudes of the peaks and dips. The insignificant loss of information will be a price for the computations speedup. The number of analytical approximations was developed for heights and positions of first three peaks and for position of first dip in CMB power spectrum in models under consideration, see [6].

**OBSERVATIONAL DATA**

Of course, the results of testing of the model strongly depend on the data used. The next important requirement the data should meet is the statistical independence, i.e. covariance matrix of datapoints errors should be diagonal. Here we have used the same set of observational data, as in papers [6,7] with some changes. The amplitudes and positions of 1st and 2nd acoustic peaks and position of first dip have been taken from results of WMAP mission team [7], position and height of third peak have been taken from last recompilation of results of BOOMERanG experiment [8]. At large angular scales (lower multipoles range) we use all datapoints from COBE experiment and WMAP [7], except dipole.

The CMB data are complemented by large-scale structure data: mass function and spatial distribution of rich galaxy clusters, temperature function of X-ray clusters, Ly-\alpha forests of absorption lines in distant quasars spectra, peculiar velocities of galaxies. These LSS data can establish the amplitude and shape of spatial power spectrum of matter density perturbations. Determination of parameters has a problem of the degeneracy in dependence of observational manifestations upon the parameters. This problem is partially eliminated when we add additional measurements to the observational set: determinations of expansion dynamics by angular distance to Supernovae of Ia type, independent determinations of Hubble constant, constraints on baryon content from Big Bang nucleosynthesis theory.

In sum there are 41 values in list of observational datapoints along with 1$\sigma$ statistical errors of their measurements. We consider all measurements to be independent and take their probability distribution as normal distribution. These values and errors are described in details in cited papers. Thus if we compare these results with those from paper [6], we obtain the answer to the question “How improvements in measurements of CMB anisotropies affect the determination of parameters of observable Universe?”.

**LIKELIHOOD ANALYSIS**

Let us have $N$ observational characteristics and we are searching best-fit values of $n$ cosmological parameters. In other words, we have the parameter set:

$$\vec{P} = (\Omega_{CDM}, \Omega_{de}, \Omega_\nu, N_\nu, \Omega_b, h, A_s, n_s, A_t, n_t, \tau_c)$$

and the set of observations:

$$\vec{B} = (A_{p_1}, \ell_{p_1}, A_{p_2}, \ell_{p_2}, A_{p_3}, \ell_{p_3}, l_{clos}, LSS, NS, h, SN)$$

To find best-fit values for 11 parameters we are used the Levenberg-Marquardt algorithm [9] for minimization of function

$$\chi^2 = \sum_{j=1}^{N} \left( \frac{y_j - y_j}{\Delta y_j} \right)^2,$$

(3)

where $y_j$ - observational value for some $j$-th characteristic, $y_j$ - its theoretically predicted value, $\Delta y_j$ - statistical uncertainty for measured value. Since the number of neutrino species $N_\nu$ is a discrete value, so we were searching for values of 10 parameters at $N_\nu$ held fixed at 1, 2, 3. As in previous papers, instead of dimensional values of amplitude of scalar mode power spectra $A_s$, we have used dimensionless value $\delta_b$, defined as r.m.s. of the fluctuation of matter density on the scale of present horizon of particle, relate as $A_s = 2\pi^2\delta_b^2(3000\text{Mpc}/h)^{3+n_s}$.

To find errors for parameters values one needs to carry out exploration of likelihood function profile $L \propto e^{-\frac{1}{2}\chi^2}$ in parametric space to determine confidential ranges (marginalizing). The estimations of confidential
that the square under curve from 0 to $\infty$ is

$$\int_0^{\infty} \text{d}x \langle x_{T/S}^\ell | x_k \rangle = \chi^2_{\text{min}} \left( x_k \right),$$

where $x_{T/S}^{bf}$ – best-fit values of cosmological parameters $(i = 1, 2, \ldots, 12, i \neq k)$, for which $\chi^2$-function has the minimum at fixed value of parameter $x_k$. This kind of representation for $L(x_k)$ can be obtained from the general integral form. Numerical experiment proves that results from proposed function $L(x_k)$ for confidential levels estimation virtually coincide with values, obtained by integration.

**RESULTS**

For historical reasons the common practice is to give a description of relic gravitational waves by the ratio of amplitudes of tensor $T$ and scalar $S$ contributions to the quadrupole ($\ell = 2$) component of CMB power spectrum. However, probability distributions for $A_t$ and $T/S$ differ, and $L(A_t)$ function is closer to the Gaussian shape than $L(T/S)$. So we are using $L(A_t)$ for determination of upper bounds and later recalculate it to $T/S$. We define the upper bound $A_{T/S}^{2\sigma}$ at confidential level $2\sigma$ as value for which the square under curve $L(A_t)$ consist $95.4\%$ of the total square under this curve from 0 to $\infty$.

![Figure 2. Likelihood functions $L(A_t) = \exp\left[-\frac{1}{2}\chi(A_t)\right]$ for various sets of observational data.](image)

In terms of our technique CMB and LSS data have not sufficient statistical weight to make possible simultaneous determination of both amplitude and slope of tensor power spectrum. In other words, if we would leave both $A_t$ and $n_t$ as free parameters, then $A_t$ could take arbitrary large value under condition $n_t \to -\infty$. If we keep the lower bound for $n_t$ fixed, then upper bound for $A_t$ depends on it. This problem finds its natural explanation by limitations of indirect method used. We have no data to distinguish the amplitude and slope separately. Fortunately, the majority of inflation models relate the $n_t$ with the slope of scalar power spectrum $n_s$. So we have also analyzed the likelihood functions for the same observational data and some generic inflation models: with flat spectrum of tensor mode ($n_t = 0$), natural inflation with $n_t = n_s - 1$ and chaotic inflation with $n_t = 0.5(n_s - 1)$. The upper $2\sigma$ constraints for them are following: $A_{T/S}^{2\sigma} = 1.9 \cdot 10^{-5}$ for the first model and $A_{T/S}^{2\sigma} = 1.6 \cdot 10^{-5}$ for the rest. Corresponding values for them are $T/S = 0.6$ and 0.18. These three models are more interesting from the standpoint of manifestations of tensor mode in the data of observations, so further analysis will include merely them. As far as likelihood functions $L(A_t)$ for them are very close it is quite enough to analyze one of them, namely we take the model with $n_t = 0.5(n_s - 1)$.

In Fig. 2 it is shown how observational data influence the half-width of likelihood function. As we can see, adding the observational data with constraints on cosmological parameters themselves, like Hubble constant, Big Bang Nucleosynthesis, SNIa observations and data on large-scale structure, decreases the level of confidence for the models with high tensor amplitudes. At the confidential level of $2\sigma$ ($95.4\%$) this amplitude cannot exceed $\sim 20\%$ of scalar mode amplitude. In our previous estimations (see [6]), based on the data of balloon experiment BOOMERanG, this limitation was almost four times larger (at $1\sigma$ C.L. the ratio was estimated as $T/S = 1.7$ for the model with free $n_t$). So we clearly see the achievements of WMAP with high precision and covering over all sky.
The results for the whole parameter set are summarized in the Table 1. There the constraints are tabulated for the case of chaotic inflation. As we can see, the accordance of these best-fit values with previous determinations [5, 6] is quite satisfactory. The new precise measurements of CMB temperature anisotropies significantly tightened confidential ranges and lowered upper constraints for $\Omega_\nu$, $\tau_c$ and $T/S$. Obtained results agree with determinations of other authors, that used the WMAP data [10] (those use somewhat different definitions).

Table 1. Best-fit values of parameters, lower and upper bounds at $2\sigma$ (95.4%) confidential level.

| $\Omega_{de}$ | $\Omega_m$ | $\Omega_\nu$ | $h$ | $\delta_h$ | $n_s$ | $T/S$ | $\tau_c$ |
|--------------|------------|-------------|-----|------------|------|-------|--------|
| Best-fit     | 0.61       | 0.41        | 0   | 0.062      | 0.61 | $4.2 \cdot 10^{-5}$ | 0.92    | 0      | 0      |
| lower bound  | 0.52       | 0.31        | 0   | 0.046      | 0.52 | $3.6 \cdot 10^{-5}$ | 0.89    | 0      | 0      |
| upper bound  | 0.69       | 0.51        | 0.03| 0.078      | 0.71 | $5.2 \cdot 10^{-5}$ | 0.98    | 0.6    | 0.15   |

CONCLUSIONS

From the standpoint of statistics interpretation of available observational data on large-scale structure of the Universe and CMB does not require the presence of considerable amount of relic gravitational waves. According to the basic assumptions of early Universe physics these relic waves have inescapable nature and common origin with matter density fluctuations seeding the large-scale structure. Within the framework of inflation theory the manifestations of gravitational background can be quite prominent, as long as amplitude of relic GWs directly connected to energy scales of inflation, e.g. Grand Unification Theory energy scale. Thus, the upper bounds on amplitude of GWs appear to be of the utmost importance for finding constraints on the moment and energy scales of inflation allowing to discriminate among models of inflation.

The advances in measurements of CMB in space experiment WMAP substantially lowered upper bound for amplitude of tensor mode of perturbations (i.e. relic gravitational waves) to the level of $T/S \leq 0.6$ (95.4% C.L.) for models with free slope parameter $n_t$. For models with flat power spectrum of gravitational waves ($n_t = 0$), or some close to that spectra ($n_t \sim 1 - n_s$), this limit appears to be the even lower $T/S \leq 0.18$ (95.4%).

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