POINCARÉ INEQUALITIES ON CARNOT GROUPS AND
SPECTRAL GAP OF SCHRÖDINGER OPERATORS

MARIANNA CHATZAKOU, SERENA FEDERICO, AND BOGUSLAW ZEGARLINSKI

Abstract. In this work we give a sufficient condition under which the global Poincaré inequality on Carnot groups holds true for a large family of probability measures absolutely continuous with respect to the Lebesgue measure. The density of such probability measure is given in terms of homogeneous quasi-norm on the group. We provide examples to which our condition applies including the most known families of Carnot groups. This, in particular, allows to extend the results in the previous work [CFZ21]. A consequence of our result is that the associated Schrödinger operators have a spectral gap.

1. Introduction

1.1. Coercive inequalities and the behaviour of the sub-Laplacian on a Carnot group. The main idea of this work is to investigate the behaviour of certain Markov generators. The operators that we consider are sum of squares of Hörmander’s vector fields. Such operators are not elliptic, but hypoelliptic, thanks to Hörmander’s celebrated result. In particular, for $\mathcal{I}$ being a finite (or countable) set of indices, and for $X_i, i \in \mathcal{I}$, degenerate and non-commutative vector fields, such operators are of the form

$$R = \sum_{i \in \mathcal{I}} X_i^2.$$  \hfill (1.1)

The first celebrated result, proved by Jerison in [Jer86], about Poincaré inequalities involving Hörmander’s vector fields was the following. Let $\mathbb{G}$ be any nilpotent Lie group with the Haar (Lebesgue) measure $dx$ and the Carnot-Carathéodory distance $d$. For $r > 0$, $x \in \mathbb{G}$ let

$$B_r(x) := \{y \in \mathbb{G} : d(x, y) \leq r\}$$

be the ball of radius $r$ and centered at $x$.

Theorem 1.1. For any $p \in [1, \infty)$, there exists a constant $P_0(r) = P_0(r, p)$ such that for all $f \in C^\infty(B_r(x))$

$$\int_{B_r(x)} |f(y) - f_{B_r(x)}|^p \, dy \leq P_0(r) \int_{B_r(x)} |\nabla_G f(y)|^p \, dy,$$

where $f_{B_r(x)} := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, dy$.

We wish to thank Michael Ruzhansky for fruitful discussions and suggestions. Marianna Chatzakou is supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations, and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021), and is a postdoctoral fellow of the Research Foundation – Flanders (FWO) under the postdoctoral grant No 12B1223N. Serena Federico has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 838661.
More generally, operators of the form (1.1) appear naturally in sub-Riemannian settings, and in particular in Carnot groups, in the sense that the tangent space at each point of the group is equipped with the algebraic structure of a nilpotent Lie algebra with dilations, which corresponds to a unique Carnot group. For a detailed exposition of notions relevant to Carnot groups, we refer the reader to Section 2. Carnot groups are highly important in many mathematical areas, including harmonic analysis and the study of hypoelliptic differential operators (e.g. [Roc78], [HN79], [FS74], [Ste93], [CDPT07], [VSC92]), but also in geometric measure theory (e.g. [Jer86], [LD13], [CL14], [KZ16]). The monographs [Ste93], [HK00], [VSC92] and [FR16] offer a detailed overview of the topic.

A Carnot group, say $G \sim \mathbb{R}^n$, has a Lie algebra that can be generated by vector fields $X_j$, $1 < j \leq n_1 \leq n$, using finite number of repeated commutators. In this sense, the operator (1.1) in the setting of the theorem takes the form

$$
\Delta_G = -\sum_{i=1}^{n_1} X_i^2,
$$

where $n_1$ is the dimension of the first stratum of the Carnot group. The positive operator $\Delta_G$ is called the sub-Laplacian of the Carnot group $G$.

The behavior of the operator defined in a similar way as $\mathcal{R}$ that we are particularly interested in this paper, is of the form of the following coercive functional inequality. Here and later on for some probability measure $\mu$ on a Carnot group $G$ we write $\mu(f)$ to denote the integral $\int_G f \, d\mu$ and we consider the vector-valued operator $\nabla_G = (X_1, \cdots, X_{n_1})$ which is the sub-gradient on the group $G$. For $q \geq 1$,

$$
\mu(|f - \mu(f)|^q) \leq C \mu(|\nabla_G f|^q), \quad \text{(1.2)}
$$

for all functions $f$ for which the right hand side is well defined. Later on such the inequality will be called global Poincaré inequality (or $q$-Poincaré inequality), for a probability measure $\mu$ on a Carnot group $G$. One can show that the operator which satisfies the relation

$$(\mathcal{L} f, f)_{L^2(\mu)} = \int_G |\nabla_G f(x)|^2 \, d\mu(x).$$

is essentially of the form as $\mathcal{R}$ modulo some scalar potential. We note that $\mathcal{L}$ is a positive self-adjoint operator on $L^2(\mu)$.

Inequality (1.2) for $q = 2$ can then be written as

$$
\mu(f - \mu(f))^2 \leq C \mu(f(\mathcal{L} f)). \quad \text{(1.3)}
$$

Inequalities of the form (1.3) are also called spectral gap inequalities. Such inequalities go back to Henri Poincaré and imply the exponential convergence of the associated semigroup $P_t \equiv e^{t\mathcal{L}}$ to the invariant measure $\mu$, see [Roc01], [CGR10], [GZ03].

1.2. Operators with discrete spectra on Carnot groups and the $U$-bounds method. There is a strong link between functional inequalities and spectral properties of operators, and recently a lot of attention was given to this field. As an example, we refer to the methods of Driver and Melher where the authors show that the heat kernel measure in the Heisenberg group satisfies a spectral gap inequality, see [DM05]. See also [El09] and [Li06] for other, non-probabilistic, approaches to the problem. We also refer to the recent works of Cipriani [Cip00] and Wang [Wan02] which investigate the spectra of other classes of operators in relation to functional inequalities.

The type of relation between spectral analysis and coercive inequalities that we are going to use in this work is given below:
For \((M, \mu)\) a probability space, and \(\mathcal{L}\) a positive self-adjoint operator on \(L^2(\mu) \subset \mathcal{D}(\mathcal{L})\), then \(\mathcal{L}\) has a spectral gap if and only if there exists a constant \(C > 0\) so that
\[
\mu(f - \mu(f))^2 \leq C \mathcal{E}(f, f),
\]
where \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is the Dirichlet form associated to \(\mathcal{L}\), that is the closure of the form
\[
\mathcal{E}(f, g) = \mu(f \mathcal{L} g).
\]

An approach, different from the ones mentioned above, to the problem of proving coercive inequalities, was developed by Hebisch and Zegarlinski in [HZ10]. Their method, called the \(U\)-bounds method, works on a general metric space \(M\) equipped with the (non-)commuting vector fields \(\{X_1, \cdots, X_m\}\). More precisely, the \(U\)-bounds method consists in proving some inequalities, also called \(U\)-bounds, allowing to derive Poincaré and other inequalities with respect to suitable probability measures.

In the Carnot group setting, a \(U\)-bound is expressed as
\[
\int |f|^q g(d) \, d\mu \leq A_q \int |\nabla f|^q \, d\mu + B_q \int |f|^q \, d\mu,
\]
where \(g\) is a positive unbounded function as a homogeneous quasi-norm \(d\) on the group goes to infinity, \(d\mu = e^{-U(d)} \, dx\), is a probability measure defined with a suitable unbounded function \(U\), and \(dx\) denotes the Haar-measure on the group. In [HZ10] it is shown that, under suitable conditions on \(\mu\), one can pass from the \(U\)-bounds to the global (i.e. with respect to a probability measure) \(q\)-Poincaré inequalities. This method was successfully applied in the case of step 2 groups in [HZ10], [Ing10], [BDZ21a],[IKZ11] and [BDZ22], and on Carnot groups of higher step in [BDZ21b].

Note that, by taking
\[
\mathcal{L} = -\Delta_G + \nabla_G U \cdot \nabla_G,
\]
we get
\[
\mathcal{E}(f, f) = \mu(f \mathcal{L} f) = \mu(|\nabla_G f|^2).
\]
Therefore, by the previous identity, (1.4) becomes
\[
\mu(f - \mu(f))^2 \leq C \mu(|\nabla_G f|^2).
\]
That is the 2-Poincaré inequality with respect to the probability measure \(\mu\). This shows, in particular, that proving inequality (1.7) is equivalent to proving the spectral gap for the operator \(\mathcal{L}\) given in 1.6.

1.3. The class of probability measures for which the spectral gap inequality holds true. The quadratic form bounds in [HZ10], that is the \(U\)-bounds in (1.5) for \(q = 2\), resemble the ones in the works of Rosen [Ros76] and Adams [Ada79] in the Euclidean setting. In the case of the Heisenberg group (see [HZ10]), the potential \(U\) in the probability measure is taken with respect to the Carnot-Carathéodory distance. In this consideration, several coercive inequalities, including the Log-Sobolev inequality, are proven. However, in the same work, the authors prove that replacing the Carnot-Carathéodory distance by any other smooth homogeneous quasi-norm forces the Log-Sobolev inequality to fail.

In the case of the spectral gap inequality that is of interest to us, the potential is given with respect to a homogeneous quasi-norm on the group. The control over the potential that is present in the \(U\)-bounds is granted by the lower bound of the length of the sub-gradient of the quasi-norm. In general it is the control over the potential \(U\) that is needed for such types of inequalities; see also [Corollary 4.2.4 [Ing10]] where it
was proved that, as in the Euclidean setting, a Schrödinger operator with potential $V$ as in (1.6) has discrete spectrum if $V$ grows to infinity in all directions.

1.4. Organisation of the paper. The paper is organised as follows: in Section 2 we provide the reader with the necessary notions around Carnot groups. In Section 3 we give sufficient conditions for the global Poincaré/spectral gap inequalities to hold. Those sufficient conditions are related to the group structure and/or the choice of the homogeneous quasi-norm; that is the choice of the probability measure. Consequently, the spectral analysis for the self-adjoint operators $L$ as in (1.1) follows as an immediate application. The analysis in the section includes the analysis in [CFZ21] as a special case. In particular, it follows from the results presented in this section, that the global Poincaré inequality holds true for every member of the family of the Carnot groups studied in [CFZ21] whose Lie algebra is of filiform type. Finally in Section 4 we give other examples of groups, where the sufficient conditions described earlier are satisfied. These include many important examples of Carnot groups that are frequently studied in the literature.

2. Preliminaries

In this section we shall recall some properties of the Lie algebra of a Carnot group $G$. The homogeneous structure of Carnot groups plays a fundamental role in our analysis, therefore we start by recalling the definition of homogeneous Lie groups, homogeneous functions, and homogeneous differential operators.

**Definition 2.1 (Homogeneous Lie group on $\mathbb{R}^n$).** Let $G = (\mathbb{R}, \circ)$ be a Lie group on $\mathbb{R}^n$. We say that $G$ is a homogeneous Lie group (on $\mathbb{R}^n$) if there exists an $n$-tuple of positive integers $\sigma = (\sigma_1, \ldots, \sigma_n)$, with $1 \leq \sigma_1 \leq \ldots \leq \sigma_n$, such that the dilation

$$
\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n, \quad \delta_\lambda (x_1, \ldots, x_n) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n)
$$

is an automorphism of the group $G$ for any $\lambda > 0$. We shall denote by $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ the datum of a homogeneous Lie group on $\mathbb{R}^n$, where $\circ$ is the composition law and $\{\delta_\lambda\}_{\lambda > 0}$ is the dilation group.

**Definition 2.2 ($G$-length of a multi-index).** Given a homogeneous Lie group $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, we define the $\delta_\lambda$-length of $\alpha$ as

$$
|\alpha|_G := \langle \alpha, \sigma \rangle = \sum_{i=1}^{n} \alpha_i \sigma_i,
$$

where $\sigma_i$ are as in Definition 2.1.

By considering $\mathbb{R}^n$ with or without a group law, and taking $\delta_\lambda$ as above, that is a map as in Definition 2.1, the following definitions and propositions hold true. Below, when no homogeneous structure is specified, we shall denote the $\delta_\lambda$-length of a multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$ by

$$
|\alpha|_\sigma := \langle \alpha, \sigma \rangle = \sum_{i=1}^{n} \alpha_i \sigma_i.
$$

**Definition 2.3 ($\delta_\lambda$-homogeneous function of degree $m$).** Let $a$ be a real function on $\mathbb{R}^n$, we say that $a$ is $\delta_\lambda$ homogeneous function of degree $m \in \mathbb{R}$ if $a \neq 0$ and, for any $x \in \mathbb{R}^n$ and $\lambda > 0$,

$$
a(\delta_\lambda(x)) = \lambda^m a(x).
$$
Proposition 2.4 (Smooth $\delta_\lambda$-homogeneous functions). Let $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ be a map as in Definition 2.1 and $a \in C^\infty(\mathbb{R}^n; \mathbb{R})$. Then $a$ is $\delta_\lambda$-homogeneous of degree $m$ if and only if it is a polynomial function. As a consequence, the set of the degrees of the smooth (non-vanishing) $\delta_\lambda$-homogeneous functions is

$$\mathcal{A} = \{ |\alpha|_\sigma : \alpha \in (\mathbb{N} \cup \{0\})^n \}, \quad |\alpha|_\sigma := \sum_{i=1}^n \sigma_i \alpha_i,$$

with $|\alpha|_\sigma = 0$ if and only if $a$ is constant.

Remark 2.5. If a function $a$ is smooth and $\delta_\lambda$-homogeneous of degree $m$, then $m \geq 0$. Additionally, given a multi-index $\alpha$, one has

$$D^\alpha a(x) = \begin{cases} \equiv 0 & \forall \alpha \text{ such that } |\alpha|_\sigma > m, \\ \delta_\lambda\text{-homogeneous of degree } m & \forall \alpha \text{ such that } |\alpha|_\sigma \leq m. \end{cases}$$

Definition 2.6 ($\delta_\lambda$-homogeneous vector field of degree $m$). Given a non-identically vanishing vector field $X$ on $\mathbb{R}^n$, we say that $X$ is $\delta_\lambda$ homogeneous of degree $m$ if, for any $\varphi \in C^\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\lambda > 0$,

$$X(\varphi(\delta_\lambda(x))) = \lambda^m (X \varphi)(\delta_\lambda(x)).$$

Proposition 2.7 (Smooth $\delta_\lambda$-homogeneous vector fields). Let $X$ be a smooth non-vanishing vector field on $\mathbb{R}^n$,

$$X = \sum_{j=1}^n a_j(x) \partial_{x_j}.$$

Then $X$ is $\delta_\lambda$-homogeneous of degree $k \in \mathbb{R}$ if and only if for every $j = 1, \ldots, n$, $a_j$ is a polynomial function $\delta_\lambda$-homogeneous of degree $\sigma_j - k$ (unless $a_j \equiv 0$). Hence the degree of $\delta_\lambda$-homogeneity of $X$ belongs to the every set

$$\mathcal{A}_j = \{ |\sigma_j - |\alpha|_\sigma| : \alpha \in (\mathbb{N} \cup \{0\})^n \},$$

whenever $j$ is such that $a_j$ is not identically $0$. In other words, for any fixed $j = 1, \ldots, n$, $k \leq \sigma_j$ and $k = \sigma_j - |\alpha|_\sigma$ for some $\alpha \in (\mathbb{N} \cup \{0\})^n$.

Remark 2.8. Note that, by taking $\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n)$, the partial derivatives $\partial_{x_j}$ are $\delta_\lambda$ homogeneous vector fields of degree $\sigma_j$. This, in particular, explains why a vector field as in Proposition 2.7 is $\delta_\lambda$ homogeneous of degree $k$ if all the $a_j(x)$ are homogeneous polynomial of degree $\sigma_j - k$. Indeed,

$$X f(\delta_\lambda(x)) = \sum_{j=1}^n a_j(x) \partial_{x_j} f(\delta_\lambda(x)) = \lambda^{\sigma_j} \sum_{j=1}^n a_j(x) (\partial_{x_j} f)(\delta_\lambda(x))$$

$$= \lambda^k \sum_{j=1}^n \lambda^{\sigma_j - k} a_j(x) (\partial_{x_j} f)(\delta_\lambda(x)) = \lambda^k (X f)(\delta_\lambda(x)),$$

which proves part of Proposition 2.7. For the complete proof see [BLU].

When $\mathbb{R}^n$ is equipped with a homogeneous Lie group structure, we use the following definition to indicate the homogeneous degree of a polynomial function.

Definition 2.9 ($\mathbb{G}$-degree of a polynomial function). Given a homogeneous Lie group $(\mathbb{G}, \circ, \delta_\lambda)$, and a polynomial function $p : \mathbb{G} \to \mathbb{R}$ defined (as a finite sum) as follows

$$p(x) = \sum_{\alpha} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{R},$$

POINCARÉ INEQUALITIES ON CARNOT GROUPS AND SPECTRAL GAP OF SCHRÖDINGER OPERATORS
we shall call the $\mathcal{G}$-degree of $p$ the quantity
\[
\deg_{\mathcal{G}}(p) := \max\{ |\alpha|_{\mathcal{G}} : c_\alpha \neq 0 \}.
\]

For more details about homogeneous Lie groups and for the proof of the propositions above we refer the interested reader to [BLU07] and [FR16].

We shall now proceed with the formal definition of Carnot groups. We use the following notation: For $V, W$ vector spaces we denote by $[V, W] = \{ [v, w] : v \in V, w \in W \}$.

**Definition 2.10.** Let $G = (\mathbb{R}^n, \circ)$ be a Lie group on $\mathbb{R}^n$, and let $\mathfrak{g}$ be the Lie algebra of $G$. Then $G$ is called a stratified group, or Carnot group, if $\mathfrak{g}$ admits a vector space decomposition (stratification) of the form
\[
\mathfrak{g} = \bigoplus_{j=1}^r V_j,
\]
such that
\[
\begin{cases}
[V_1, V_{i-1}] = V_i, & 2 \leq i \leq r, \\
[V_i, V_r] = \{0\},
\end{cases}
\]
where the positive integer $r$ is called the step of $G$.

**Remark 2.11.** Carnot groups are naturally homogeneous Lie groups. Therefore, each Carnot group $G$ can be equipped with a mapping $\delta_\lambda$, $\lambda > 0$, as in Definition 2.1 that is an automorphism of the group $G$. The diffeomorphic map $\exp_G : G \to \mathfrak{g}$ ensures that $\delta_\lambda$ is also an automorphism of $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$. Note that the stratification of a Lie algebra $\mathfrak{g}$ is not unique. However, the mapping $\delta_\lambda$ as in Definition 2.1 does not depend on the choice of the stratification; cf. [Proposition 2.2.8 [BLU07]]. We can then denote by $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ the Carnot group equipped with the natural dilations $\{\delta_\lambda\}$.

**Remark 2.12** (Jacobian basis of $\mathfrak{g}$). Let $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a Carnot group of step $r$, and let $\mathfrak{g}$ be its Lie algebra. Then the left(right)-invariant Jacobian basis $\{X_i\}_{i=1}^n$ of $\mathfrak{g}$, also denoted by
\[
X_1^{(1)}, \ldots, X_n^{(1)}; \ldots; X_1^{(r)}, \ldots, X_n^{(r)},
\]
is such that $X_j^{(i)}$ is a smooth left(right)-invariant $\delta_\lambda$-homogeneous vector field of degree $i$ of the form
\[
X_j^{(i)} = \partial_{x_j}^{(i)} + \sum_{h=i+1}^r \sum_{k=1}^{N_h} a_{j,k}^{(i,h)}(x^{(1)}, \ldots, x^{(h-i)}) \partial_{x_k}^{(h)},
\]
where $a_{j,k}^{(i,h)}$ is a $\delta_\lambda$-homogeneous polynomial function of degree $h - i$.

In particular, for all $1 \leq j \leq n_1$, $X_j^{(1)} = X_j$ is a smooth $\delta_\lambda$-homogeneous vector field of degree 1 which takes the form
\[
X_j = \partial_{x_j} + \sum_{k=n_1+1}^n a_{j,k}(x_1, \ldots, x_{n_1}, \ldots, x_{k-1}) \partial_{x_k}, \quad \forall j = 1, \ldots, n_1,
\]
where $a_{j,k}$ is a $\delta_\lambda$-homogeneous polynomial function of degree $\sigma_k - 1$. The vector fields $X_j, j = 1, \ldots, n_1$, of the Jacobian basis are also called the canonical left (right) invariant vector fields generating the Lie algebra $\mathfrak{g}$ of $G$. We recall that a vector field $X$ is left invariant when it is invariant with respect to left translations on $G$, that is, for every $y \in G$,
\[
X(f \circ L_y)(x) = (Xf)(L_y(x)), \quad \text{where} \quad L_y(x) = y \circ x.
\]
Analogously, a vector field $\tilde{X}$ is right-invariant when it is invariant with respect to right translations on $G$. 
We shall conclude this section by recalling the definition of sub-Laplacian and sub-gradient on a Carnot group $G$.

**Definition 2.13.** Let $G$ be a Carnot group, and let $\mathfrak{g}$ be the corresponding Lie algebra. If $X_j$, $1 \leq j \leq n_1$, are the canonical left (right) invariant vector fields that generate $\mathfrak{g}$, then the second order differential operator
\[
\Delta_G = \sum_{j=1}^{n_1} X_j^2,
\]
is called the *canonical left (right) invariant sub-Laplacian* on $G$, while the vector valued operator
\[
\nabla_G = (X_1, \cdots, X_{n_1}),
\]
is called the *canonical left (right) invariant $G$-gradient*.

**Definition 2.14.** We call *homogeneous (quasi-)norm* on the Carnot group $G$, every continuous, with respect to the Euclidean topology, mapping $N : G \to [0, \infty)$ such that
\[
N(x) = 0 \text{ if and only if } x = 0,
\]
and
\[
N(\delta_\lambda(x)) = \lambda N(x), \quad \text{for every } \lambda > 0, \quad x \in G.
\]

3. **Sufficient conditions for global Poincaré inequalities**

In this section we investigate sufficient (and not necessary) conditions for global $q$-Poincaré inequalities to hold for some probability measures on Carnot groups. The probability measures suitable for our purposes - denoted by $\mu_p$ and defined below - have densities (with respect to the Haar measure) depending on a parameter $p$ and on a fixed homogeneous quasi-norm $N$ on $G$.

Given a homogeneous quasi-norm $N$, we define the probability measure $\mu_p$ as
\[
\mu_p := \frac{1}{Z} e^{-aNp} dx,
\]
where $Z$ is a normalization constant, $a \in (0, \infty)$, and $p \in (0, \infty)$.

As we will explain later, all our results will hold for a large class of perturbations of such measures.

With this and the previous definitions in mind, we can now state the main result of this paper giving a sufficient condition for $q$-Poincaré inequalities to hold.

Here and later on we use a convention $a \gtrless b$ (resp. $a \lesssim b$) to say that there exists a constant $C \in (0, \infty)$ independent of $a, b$ so that $a \geq Cb$ (resp. $a \leq Cb$). Also, for $p, q \in [1, \infty]$ we say that $q$ is the conjugate exponent of $p$ iff $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 3.1.** Let $G$ be a Carnot group of step $r$ on $\mathbb{R}^n$, let $N$ be a homogeneous quasi-norm on $G$ smooth away from the origin, and let $\mu_p$ be defined as in (3.1). If there exists an index $j_0 \in \{1, \ldots, n_1\}$, and a positive integer $\gamma \geq 2$, such that
\[
|\nabla_G N| \gtrsim \frac{|x_{j_0}|^{\gamma-1}}{N^{\gamma-1}(x)},
\]
then, for all $p \geq 2\gamma$, and for $q$ being the conjugate exponent of $p$, there exists $c_0 \in (0, \infty)$ such that
\[
\mu_p(|f - f_{\mu_p}|) \leq c_0 \mu_p(|\nabla f|^q)
\]
for all functions $f \in L_p(\mu)$ for which the right hand side is well defined.
Remark 3.2 (Remark about condition (3.2)). Observe that condition (3.2) implies that
\( \{x \in \mathbb{R}^n; x_{j_0} = 0\} \subseteq \text{Ker}(|\nabla_G N|) := \{x \in \mathbb{R}^n; |\nabla_g N(x)| = 0\} \).
In other words, a necessary (but not sufficient) condition to have (3.2) is that \( \text{Ker}(|\nabla_G N|) \) contains the hyperplane \( \{x \in \mathbb{R}^n; x_{j_0} = 0\} \).

For the proof of Theorem 3.1 the following two auxiliary lemmas are necessary.

**Lemma 3.3.** Let \( G \) be a Carnot group on \( \mathbb{R}^n \), and let \( N \) be a homogeneous quasi-norm on \( G \) smooth away from the origin. Then, for all \( x \in G \setminus \{0\} \), we have
\[
|\nabla_G N(x)| \lesssim 1 \\
|\Delta_G N(x)| \lesssim \frac{1}{N(x)}.
\]  

**Proof.** The proof follows from the homogeneity properties of \( N \) and the fact that \( |\nabla_G N|^2 \) and \( \Delta_G N \) are homogeneous functions of degree 0 and -1, respectively. \( \square \)

**Lemma 3.4.** Let \( p \geq 2\gamma \) and let \( q \) be its conjugate exponent. Let \( N \) be a homogeneous quasi-norm smooth away from the origin satisfying (3.2) for some \( j_0 \in \{1, \ldots, n_1\} \), and let \( \mu_p := e^{-aN_p} dx \). Then there exist \( A \in (0, \infty) \) and \( B \in [0, \infty) \) such that
\[
\mu_p(|f|^q N_p^{-2\gamma} |x_{j_0}|^{2\gamma}) \leq A \mu_p(|\nabla_G f|^q) + B \mu_p(|f|^q),
\]
for every function \( f \) for which the right hand side of the relation is well-defined.

**Proof.** We shall prove the result for \( f \geq 0 \) and such that \( f \in C_0^\infty(G) \). The result for a general \( f \) will then follow from this case by the fact that a.e. one has \( |\nabla_G f| \leq |\nabla_G f| \).

By Leibniz rule we have that
\[
e^{-aN_p} \nabla_G f = \nabla_G (e^{-aN_p} f) + apN_p^{-1} e^{-aN_p} f(\nabla_G N),
\]
therefore, taking the inner product of the vectors \( e^{-aN_p} \nabla_G f \) and \( \nabla_G N \) we have
\[
\int \nabla_G N(x) \cdot (\nabla_G f(x)) e^{-aN_p} dx
\]
\[
= \int \nabla_G N(x) \cdot \nabla_G (e^{-aN_p} f(x)) dx + apN_p^{-1} \int e^{-aN_p} f(x) \nabla_G N(x)^2 dx.
\]
We then apply Cauchy-Schwartz inequality and (3.4) to the left-hand side of (3.7), integrate by parts the first term on the right-hand side of (3.7), and apply (3.2) on the second term of the right hand side of it. After these computations, (3.7) yields
\[
apC_0 \int N(x)^p - 2\gamma^2 |x_{j_0}|^{2\gamma - 2} f(x)e^{-aN_p} dx
\]
\[
\leq C_1 \int |\nabla_G f| e^{-aN_p} dx + C_2 \int (\Delta_G N(x))^2 |x_{j_0}|^{2\gamma - 1} f(x)e^{-aN_p} dx,
\]
where \( j_0 \) is the index appearing in (3.2) and \( C_0, C_1, C_2 \in (0, \infty) \) are some constants independent of the function \( f \).
Since, by homogeneity, we have \( |\Delta_G N(x)| \lesssim 1/N(x) \) and \( |x_{j_0}|^{2\gamma - 2} \gtrsim |x_{j_0}|^{2\gamma - 1}/N(x) \), we get that
\[
apC_0 \int N(x)^p - 2\gamma |x_{j_0}|^{2\gamma - 1} f(x)e^{-aN_p} dx
\]
\[
\leq C_1 \int |\nabla_G f| e^{-aN_p} dx + C_2 \int \frac{1}{N(x)} f(x)e^{-aN_p} dx,
\]
An application of Young’s inequality, for any $\varepsilon > 0$ and $R > 0$ it is enough to prove the estimate for $|x_{j_0}| \lesssim 1$, with some $R > 0$ and $\mu > 0$ chosen later, we have

$$\mu_p(|f - m|^{q}) = \mu_p(|f - m|^q 1_{\{|x_{j_0}|^{2^\gamma N - 2^\gamma \geq R_l\}}) + \mu_p(|f - m|^q 1_{\{|x_{j_0}|^{2^\gamma N - 2^\gamma \leq R_l\}}1\{N \leq L\})$$

Theorem 3.1. Since, for all $m \in \mathbb{R}$, we have

$$\mu_p(|f - f_{\mu}^p|^q) \leq 2^q \mu_p(|f - m|^q)$$

Note that in Theorem 3.1 and in the lemmas above, we require the quasinorm $N$ to be smooth away from the origin. This assumption is used to avoid technical problems when applying the vector fields to $N$. However, in the case when $N$ is not smooth on some hyperplane, one can split the domain into connected components where $N$ is differentiable. In such cases, working locally inside each connected component, and using a suitable approximation argument like in [CFZ20], one can extend the result to the whole domain.

With the previous lemma at our disposal we can now prove Theorem 3.1.

Proof of Theorem 3.1. Since, for all $m \in \mathbb{R}$, we have

$$\mu_p(|f - f_{\mu}^p|^q) \leq 2^q \mu_p(|f - m|^q)$$

it is enough to prove the estimate for $\mu_p(|f - m|^q)$ with a suitable choice of $m$. Then, with some $R > 0$ and $L > 0$ to be chosen later, we have

$$\mu_p(|f - m|^q) = \mu_p(|f - m|^q 1_{\{|x_{j_0}|^{2^\gamma N - 2^\gamma \geq R_l\}}) + \mu_p(|f - m|^q 1_{\{|x_{j_0}|^{2^\gamma N - 2^\gamma \leq R_l\}}1\{N \leq L\})$$

Remark 3.5. Note that in Theorem 3.1 and in the lemmas above we require the quasinorm $N$ to be smooth away from the origin. This assumption is used to avoid technical problems when applying the vector fields to $N$. However, in the case when $N$ is not smooth on some hyperplane, one can split the domain into connected components where $N$ is differentiable. In such cases, working locally inside each connected component, and using a suitable approximation argument like in [CFZ20], one can extend the result to the whole domain.
\[
+ \mu_p(\|f - m\|^q \mathbb{1}_{\{x : |x|^2 \geq R\}} \mathbb{1}_{\{N \leq R\}})
\]
\[
= I + II + III.
\]

For the term I, by Lemma 3.4, we have
\[
I = \mu_p(\|f - m\|^q \mathbb{1}_{\{x : |x|^2 \geq R\}}) \leq \frac{1}{R} \mu_p(\|f - m\|^q N^{2\gamma} |x|^2)
\]
\[
\leq \frac{A}{R} \mu_p(\|\nabla g f\|^q) + \frac{B}{R} \mu_p(\|f - m\|^q).
\]

To estimate II we first observe that, given any homogeneous quasi-norm \(N\), there exists \(C \in (0, \infty)\) such that the Carnot-Caratéodory distance \(d(x) = d(x, 0)\) satisfies
\[
C^{-1} N(x) \leq d(x) \leq C N(x), \quad \forall x \in G.
\]

Therefore, given \(L > 0\), there exists \(L_1, L_2 > 0\) such that for the Carnot-Caratéodory ball \(B_{L_1}\) of radius \(L_1\) centered at the origin, we have
\[
\{N \leq L_1\} \subset B_{L_1} \subset \{N \leq L_2\}.
\]

Choosing \(m = \frac{1}{|\mathcal{L}_{L_1}|} \int_{B_{L_1}} f(x) dx\), we get
\[
II = \mu_p(\|f - m\|^q \mathbb{1}_{\{x : |x|^2 \geq R\}} \mathbb{1}_{\{N \leq L\}}) = \mu_p(\|f - m\|^q \mathbb{1}_{\{N \leq L\}})
\]
\[
\leq \frac{1}{Z} \int_{\{N \leq L\}} |f(x) - m|^q e^{-a N(x)^p} dx
\]
\[
\leq \frac{1}{Z} \int_{\{N \leq L\}} |f(x) - m|^q dx
\]
\[
\leq \frac{1}{Z} \int_{B_{L_1}} |f(x) - m|^q dx
\]
\[
\leq \frac{P_0(L_1)}{Z} \int_{\{N \leq L_2\}} |\nabla_G f(x)|^q dx
\]
\[
\leq \frac{P_0(L_1)}{Z} e^{a L_2^p} \mu_p(\|\nabla_G f(x)\|^q),
\]

where in the fourth line we applied the Poincaré inequality on balls (see [Jer]).

We are now left with the estimate of term III. To this end define a set
\[
A_{L,R} := \{x \in G : |x|^2 \leq R, N(x) \geq L\}.
\]

Since \(L > 1\) we have
\[
\{x \in G : |x|^2 \leq R, N(x) \geq L\} \subset A_{L,R},
\]
then, with a notation \(g := f - m\), we get
\[
III = \mu_p(\|g\|^q \mathbb{1}_{\{x : |x|^2 \geq R\}} \mathbb{1}_{\{N \leq L\}})
\]
\[
\leq \int_{A_{L,R}} |g(x)|^q d\mu_p(x) = \int_{A'_{L,R}} |g(x \circ h)|^q d\mu_p(x \circ h),
\]

with a set
\[
A'_{L,R} := \{x \circ h \in G; x \in A_{L,R}\}
\]

defined for some \(h \in G\). Choosing \(h = \frac{1}{2R^{\frac{\gamma}{q}}} e_{j_0} = (0, ..., 2R^{\frac{1}{q}}, ..., 0)\) (where the unique non zero component is the \(j_0\)-th component), we have
\[
N(x \circ h) \geq N(x) \geq L \quad \text{and} \quad |(x \circ h)_{j_0}| \geq R^{\frac{1}{q}},
\]
and we can apply Lemma 3.4 to obtain
\[
\int_{A'_{L,R}} |g(x \circ h)|^q d\mu_p(x \circ h)
\leq \frac{1}{RL^{p-2\gamma}} \int_{A'_{L,R}} |g(x \circ h)|^q |(x \circ h)_{j_0}|^{2\gamma} N(x \circ h)^{p-2\gamma} d\mu_p(x \circ h)
\]
\[
\leq \frac{A}{RL^{p-2\gamma}} \mu_p(|\nabla_G f|^q) + \frac{B}{RL^{p-2\gamma}} \mu_p(|f - m|^q).
\]
Finally, putting together the estimates for the three terms \(I, II\) and \(III\), we get
\[
\left(1 - \frac{B}{R} - \frac{B}{RL^{p-2\gamma}}\right) \mu_p(|f - m|^q) \leq \left(\frac{A}{R} + P_0(L_1)e^{aL_2} + \frac{A}{RL^{p-2\gamma}}\right) \mu_p(|\nabla_G f|^q),
\]
hence, by choosing \(R\) sufficiently large, we get the result. This concludes the proof. \(\square\)

**Corollary 3.6.** The positive self-adjoint operator
\[
\mathcal{L}_p := -\Delta_G + apN^{p-1} \nabla_G N \cdot \nabla_G,
\]
on \(L^2(\mu_p)\) has a spectral gap.

**Proof.** The proof is an immediate consequence of Theorem 3.1 combined with the fact that, whenever (3.3) holds true for some \(q > 1\), then it holds true also for \(q' > q\) under the same unchanged probability measure; see Proposition 2.3 in [BZ05]. \(\square\)

**Remark 3.7.** We want to point out that the choice of the homogeneous quas-norm \(N\) on the Carnot group provides a radical difference on the spectrum of the operators of the form (1.6). To be more precise, for a probability measure on an \(H\)-type group of the form \(Z^{-1}e^{-aNp},\ p \in (1,2)\), the corresponding operator (1.6) has empty essential spectrum when \(N\) is a Kaplan norm, and does not even have a spectral gap when \(N\) is the Carnot-Carathéodory distance; see Remark 4.5.4 in [Ing10].

As a corollary of Theorem 3.1 one has that the global Poincaré inequalities of Theorem 3.1 are satisfied by a family of measures whose potential is a perturbation of the one appearing in the density of \(\mu_p\).

**Corollary 3.8.** Let \(d\mu_W := \tilde{Z}^{-1}e^{-W} d\mu_p\) be a probability measure with a differential potential \(W\) satisfying
\[
|\nabla_G W(x)|^q \leq \delta N(x)^{p-2\gamma} |x_{j_0}|^{2\gamma} + \gamma, \quad \forall x \in \mathbb{G},
\]
for some \(0 < \delta \ll 1\) and \(\gamma, \delta \in (0,\infty)\). Then the measure \(\mu_W\) satisfies the hypotheses of Lemma 3.4 for \(p \geq 2\gamma\) and \(q\) such that \(\frac{1}{q} + \frac{1}{p} = 1\). Moreover, if there exists \(C > 0\) such that \(W \leq C\tilde{N}\) then \(\mu_W\) satisfies the global Poincaré inequality.

**Proof.** Let us start by replacing \(f\) by \(fe^{-\frac{W}{q}}\) in the inequality (3.6). This gives
\[
\mu_p \left( e^{-W} |f|^q N^{p-2\gamma} |x_{j_0}|^{2\gamma} \right) \leq C \mu_p \left( |\nabla_G (e^{-\frac{W}{q}}f)|^q \right) + D \mu_p \left( |e^{-\frac{W}{q}}f|^q \right).
\]
Now, since
\[
|\nabla_G (e^{-\frac{W}{q}}f)|^q = \left( |\nabla_G e^{-\frac{W}{q}}| |f| + e^{-\frac{W}{q}} |\nabla_G f| \right)^q \leq \left( \frac{|\nabla_G W|}{q} |e^{-\frac{W}{q}}f| + e^{-\frac{W}{q}} |\nabla_G f| \right)^q \leq C(q) \left( |\nabla_G W|^q e^{-W} |f|^q + e^{-W} |\nabla_G f|^q \right),
\]

substituting the latter in (3.12), and using (3.10) we get
\[
\mu_W \left( f^q N^{p-2γ} |x_j|^2γ \right) \leq CC(q) \mu_W (|∇_G W|^q |f|^q) + CC(q) \mu_W |∇_G f|^q + D \mu_W |f|^q \\
\leq δCC(q) \mu_W (N^{p-2γ} |x_j|^2γ |f|^q) + γδCC(q) \mu_W |f|^q \\
+ CC(q) \mu_W |∇_G f|^q + D \mu_W |f|^q .
\]
Therefore, if δ is such that 1 − δCC(q) > 0, the last inequality gives the result.

Now to prove the global Poincaré for the measure µW, we have to assume that W is such that W ≤ Np. For L > 1 and R > 0 as in the proof of Theorem 3.1, we decompose the quantity µW |f − m| as follows
\[
\mu_W |f − m|^q = \mu_W (|f − m|^q \mathbf{1}_{|x_j|^2γ N^{p-2γ} ≥ R}) + \mu_W (|f − m|^q \mathbf{1}_{|x_j|^2γ N^{p-2γ} ≤ R} \mathbf{1}_{N ≤ L}) \\
+ \mu_W (|f − m|^q \mathbf{1}_{|x_j|^2γ N^{p-2γ} ≤ R} \mathbf{1}_{N > L}) .
\]
For the first and the third term of (3.13) one can proceed as in Theorem 3.1, while for the second term, arguing as in Theorem 3.1, we arrive at
\[
\mu_W (|f − m|^q \mathbf{1}_{|x_j|^2γ N^{p-2γ} ≤ R} \mathbf{1}_{N ≤ L}) \leq \frac{P_0(L_1)}{Z} \int_{N ≤ L_2} |∇_G f|^q dx \\
\leq \frac{P_0(L_1)}{Z} e^{αL_2^2 + C L_2} \mu_W |∇_G f|^q ,
\]
where L_2 > 0 is the one appearing in (3.9). In the last inequality we have used the fact that in \{N ≤ L_2\} we have W ≤ C N ≤ C L_2. The proof is now complete.

4. Examples

In this section we describe classes of Carnot groups for which the sufficient condition of Theorem 3.1, expressed by (3.2), is satisfied. As a consequence, for these classes of groups global Poincaré inequalities hold, and so does also the spectral gap for the corresponding self-adjoint operator L.

Homogeneous quasi-norms smooth away from the origin. Before starting with the investigation of the validity of (3.2), we briefly discuss here some homogeneous quasi-norms we will be using in the current section. The works [Eg13], [Pop16], [LN06] and [BFS18], contain a recent exposition of examples of homogeneous quasi-norms on Carnot groups.

Let us remark that the quasi-norm we have used so far is smooth away from the origin. For convenience we shall simply call such quasi-norms smooth norms, where the smoothness property shall be regarded as smoothness everywhere except for the origin.

A natural example of such quasi-norm on a Carnot group on \( \mathbb{R}^n \) equipped with the dilation \( δ_λ(x) = (λ^{σ_1} x_1, \ldots, λ^{σ_n} x_n) \) is given by the formula:
\[
N(x) = \left( \sum_{j=1}^{n} a_j |x_j|^{2β_j} \right)^{1/γ} ,
\]
where \( a_j \), for all \( j = 1, \ldots, n \), is a positive real number, and \( β_j \), for all \( j = 1, \ldots, n \), is such that \( β_j ∈ \mathbb{N} \) and \( 2σ_jβ_j = γ \geq 2σ_n \). Homogeneous quasi-norms of the form (4.1) are indeed natural extensions of the Euclidean norm on \( \mathbb{R}^n \) to the dilated structure of the Carnot group. A particular case of a quasi-norm of the above form that one often encounters in the literature is the one with \( β_j = σ_n! / σ_j \) and \( γ = 2σ_n! \).
Given a Carnot group on $\mathbb{R}^n$ as above, one can use quasi-norms of the form (4.1) on groups of variables to give rise to other homogeneous quasi-norms. Explicitly, we can define the sum

$$\tilde{N}(x) = \left( \sum_{j=1}^{m} N_j(x_{j_1}, x_{j_2}, \ldots, x_{j_{k_j}})^{\alpha_j} \right)^{1/\alpha},$$

(4.2)

where $k_j \leq n$, $j_1, \ldots, j_{k_j} \in \{1, \ldots, n\}$, $m \in \mathbb{N}$ and $N_j$ are quasi-norms of the form (4.1) defined on a subspace of $\mathbb{R}^n$. In (4.2) we have chosen $\alpha_i \in \mathbb{N}$, and thus the homogeneity requires that the $\alpha_j$’s are such that $\alpha_j/\gamma_j = \alpha$, where $\gamma_j$ is the exponent in $\tilde{N}_j$ as in (4.1).

On Carnot groups $\mathbb{G}$ of step two, the general formula (4.2) boils down to the quasi-norm given by the (4.3) below, that can be viewed as a generalisation of the Kaplan norm on H-type groups, see [Ing10]. Precisely, if $\mathbb{G}$ is a group on $\mathbb{R}^{m+n}$, with $m$ being the number of generators, then denoting by $(x, t) \in \mathbb{R}^m \times \mathbb{R}^n$ an element of the group, one has that a class of smooth quasi-norms on $\mathbb{G}$ is given by

$$N_{\alpha}(x, t) = \left( \|x\|^{4\alpha} + \sum_{j=1}^{n} c_j t_{j}^{2\alpha} \right)^{\frac{1}{4\alpha}},$$

(4.3)

where $\alpha$ is a positive integer, $c_j$, for every $j = 1, \ldots, n$, is a positive real number, and $\|x\| := \left( \sum_{k=1}^{m} x_k^2 \right)^{1/2}$.

Let us also remark that all the quasi-norms listed above are as in the hypothesis of Theorem 3.1.

Considering the norms defined above, we will now focus on examples of groups where these norms allow to recover global Poincaré inequalities. We stress once more that spectral gaps for suitable corresponding operators follows from the 2-Poincaré inequality.

In what follows we will first show that on step two Carnot groups, by choosing $N$ as in (4.3), we have that our sufficient condition (3.2) holds true, and, consequently, so does also the corresponding global Poincaré inequality for the suitable probability measure.

Next, for a wide class of Carnot groups of arbitrary step, we will show that condition (3.2) is satisfied if $N$ is as in (4.1) or, more generally, as in (4.2). This, once again, will imply the validity of global Poincaré inequalities, and, as before, of the spectral gaps for operator as in (1.6).

**Carnot groups of step 2.** We start with the investigation of step 2 Carnot groups. Below we shall use the notation $(\mathbb{R}^{m+n}, \circ)$ for an $N = m+n$-dimensional Carnot group of step 2 with $m$ generators and composition law $\circ$. A point in $\mathbb{G} = (\mathbb{R}^{m+n}, \circ)$ will be denoted by $(x, t)$, with $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$.

---

¹To be precise, the quasi-norms $N_j$ above are

$$N_j(x_{j_1}, x_{j_2}, \ldots, x_{j_{k_j}}) = \left( \sum_{i=1}^{k_j} a_{j_i} |x_{j_i}|^{2\beta_i} \right)^{1/\gamma_j},$$

where the parameters are as in (4.1), that is such that $\beta_i/\gamma_j = \gamma_j \geq 2\sigma_{j_k}$ for every $i = 1, \ldots, k_j$, while $a_{j_i}$ are positive real numbers.
We recall that any $N$-dimensional Carnot group of step 2 and $m$ generators is naturally isomorphic to a step two Carnot group $(\mathbb{R}^{m+n}, \cdot')$, where $n = N - m$, the composition law is

$$(x,t) \cdot' (\xi,\tau) = (x + \xi, t_1 + \tau_1 + \frac{1}{2} \langle B^{(1)} x, \xi \rangle, \ldots, t_n + \tau_n + \frac{1}{2} \langle B^{(n)} x, \xi \rangle),$$

(4.4)

and $B^{(j)}$, for all $j = 1, \ldots, n$, is an $m \times m$ skew-symmetric matrix. Therefore, without loss of generality, hereafter we consider two step Carnot groups with a composition law of the form $\cdot'$ defined through some skew-symmetric matrices $B^{(j)}$'s.

Notice that, for a Carnot group of step 2 with $m$ generators on $\mathbb{R}^{m+n}$, (4.4) gives that

$$X_j = \partial_{x_j} + \frac{1}{2} \sum_{k=1}^{m} \sum_{i=1}^{n} B^{(k)}_{ij} x_i \partial_{k}, \quad \text{for all} \quad k = 1, \ldots, m.$$  

(4.5)

Formula (4.5) will be very useful to prove the following proposition.

**Proposition 4.1.** Let $G = (\mathbb{R}^{n_1+n}, \cdot')$ be a Carnot group of step 2 and $n_1$ generators, and let $N_\alpha$ be a smooth quasi-norm on $G$ as in (4.3), where $\alpha$ can be any positive integer. Then, for every $x \in G \setminus \{0\}$ there exist a constant $C > 0$ such that

$$|\nabla N_\alpha(x)| \geq C \frac{\|(x_1, \ldots, x_{n_1})\|^{4\alpha-1}}{N_\alpha(x)^{4\alpha-1}},$$

(4.6)

where $\| \cdot \|$ stands for the Euclidean norm on $\mathbb{R}^{n_1}$.

**Proof.** Recall that, for a Carnot group $G$ of step 2, the generators $X_1, \ldots, X_n$, are of the form (4.5). Therefore, since

$$|X_j N_\alpha(x)|^2 = \left( \frac{8 \|x\|^{4\alpha-2} x_j + \sum_{k=1}^{m} \sum_{i=1}^{n} B^{(k)}_{ij} x_i c_k t_k}{4 N_\alpha(x)^{4\alpha-1}} \right)^2$$

$$= \frac{64 \|x\|^{8\alpha-4} x_j^2 + \left( \sum_{k=1}^{m} \sum_{i=1}^{n} B^{(k)}_{ij} x_i c_k t_k \right)^2 + 2 \sum_{k=1}^{m} \sum_{i=1}^{n} B^{(k)}_{ij} x_i x_j c_k t_k}{16 N_\alpha(x)^{8\alpha-2}},$$

and since $\sum_{i,j=1}^{n_1} B^{(k)}_{ij} x_i x_j = 0$ by the skew-symmetry of $B^{(k)}$, we have

$$|\nabla N_\alpha(x)|^2 \geq \frac{4 \|x\|^{8\alpha-2}}{N_\alpha(x)^{8\alpha-2}},$$

which gives (3.2) and concludes the proof. \qed

**Remark 4.2.** Thanks to Proposition 4.1 and Theorem 3.1 we have that global $q$-Poincaré inequalities hold true on step 2 Carnot groups for probability measures whose density is of the form $e^{-aN_\alpha^p}$, with $N_\alpha$ as in (4.3) and $q$ conjugate exponent of $p$. Moreover, the 2-Poincaré inequality yields the validity of the spectral gap for the operator $T_p := -\Delta_G + p N_\alpha^{p-1} \nabla_G N \cdot \nabla_G$, for $p$ as in $\mu_p$.

Summarizing, we have proved the following theorem.

**Theorem 4.3.** Let $G$ be Carnot group of step 2, $\alpha$ a positive integer, and $N_\alpha$ a homogeneous quasi-norm on $G$ as in (4.3). Then

$$\mu_p(|f - f_{\mu_p}|^q) \leq \mu_p(|\nabla f|^q)$$

for every $p \geq 8\alpha$, and with $q$ being the conjugate exponent of $p$. 


Remark 4.4 ($\mathbb{H}$-type groups). Note that, for a Carnot group of step 2 of $\mathbb{H}$-type on $\mathbb{R}^{m+n}$, the quasi-norm $N_\alpha$ in (4.3) with $\alpha = 1$ and $c_j = \frac{1}{N}$, for all $j = 1, \ldots, n$, coincides with the so called Kaplan norm

$$N_1(x) = (\|x\|^4 + \frac{1}{16}|z|^2)^{1/4}, \quad x = (w, z) \in \mathbb{R}^n \times \mathbb{R}^m.$$ 

In [Ing10] the author showed that, on $\mathbb{H}$-type groups, $|\nabla N_1(x)| = \|x\|_N$, and that global $q$-Poincaré inequalities with respect to $d\mu_p = e^{-\frac{q}{x(z)}} dx$, with $a > 0$, $p \geq 2$, and $q$ conjugate exponent of $p$, hold true. We remark that the previous identity also gives

$$|\nabla N_1(x)| = \frac{\|x\|}{N(x)} = \frac{\|x\|^3}{\|x\|^2 N(x)} = \frac{\|x\|^3}{N(x)\frac{1}{4}},$$

which is condition (3.2) in Theorem 3.1 giving the validity of global Poincaré inequalities.

Moreover, here Theorem (4.3) applies and generalizes, in some sense, the result in [Ing10], allowing to conclude global Poincaré inequalities for the class of probability measures defined as $d\mu_p = e^{-\frac{\alpha q}{x(z)}} dx$, for $a > 0$ and $\alpha$ being any positive integer.

Remark 4.5 (The anisotropic Heisenberg group). We conclude this part dedicated to step 2 groups by considering a group treated in [BDZ22] to which our results apply.

The group under consideration is the so called anisotropic Heisenberg group $\mathbb{H}_{2n}(\frac{1}{2}, 1)$. Since $\mathbb{H}_{2n}(\frac{1}{2}, 1)$ is a Carnot group of step 2, we get the validity of global $q$-Poincaré inequalities for any smooth homogeneous quasi-norm as in (4.3) by Theorem 4.3. However, the validity of such inequalities was first proved in [BDZ22] by using probability measures whose density depends on a specific smooth quasi-norm $N$, that is, for $N$ being the fundamental solution for the sub-Laplacian. The quasi-norm in [BDZ22] is

$$N(x) = (B^2 + t^2)^{1/4n}(AB + t^2 + A\sqrt{A^2 + B^2})^{1/2 - 1/4n},$$

where

$$A = \frac{x_1^2}{2} + \frac{x_{n+1}^2}{2} + \frac{1}{2} \sum_{j=1}^{2n} x_j^2,$$

and

$$B = \frac{x_1^2}{4} + \frac{x_{n+1}^2}{4} + \frac{1}{2} \sum_{j=1 \neq n+1} x_j^2.$$ 

Notice that the quasi-norm is smooth away from 0. Moreover, in [BDZ22] the authors proved that, for all $x \neq 0_G$,

$$|\nabla N(x)| \geq C \frac{\|x\|^2}{N^2} \geq C \frac{\|x\|^\gamma}{N^\gamma}, \quad \forall \gamma \geq 3,$$

where $\|x\|$ is the Euclidean norm of $x = (x_1, \ldots, x_{2n})$, therefore the sufficient condition 3.2 is satisfied and Theorem 3.1 applies for all $\gamma \geq 3$.

Summarizing, on the anisotropic Heisenberg group one can apply both Theorem 3.1 (with $N$ as in [BDZ22]) and Theorem 4.3, and get, in the first case, the same result as in [BDZ22], while, in the second case, Poincaré inequalities with respect to different probability measures defined through $N_\alpha$ as in (4.3).

Let us finally remark that in [BDZ22] the authors had to deal with the non trivial problem of finding a fundamental solution of the sub-Laplacian in order to find the suitable probability measure to prove the inequalities. Theorem 4.3, instead, is direct, and gives already a class of measures for which the inequalities are true.
Carnot groups of step \( r \geq 2 \). Besides Carnot groups of step 2, there are other groups to which our result applies, that is groups such that the sufficient condition for the Poincaré inequality is verified by any homogeneous quasi-norm on the group being smooth away from the origin and of the form (4.1) or (4.2). Such groups include those described in the following result.

**Lemma 4.6.** Let \( G \) be a Carnot group of step \( r \) on \( \mathbb{R}^n \), and let \( \{X_j\}_{1 \leq j \leq n_1} \) be the generators of the first stratum \( V_1 \). If there exist \( j_0 \in \{1, \ldots, n_1\} \) such that

\[
X_{j_0} = \partial_{x_{j_0}},
\]

then (3.2) holds with \( j = j_0 \) and with \( N \) as in (4.1) or (4.2).

**Proof.** We give the proof for \( N \) as in (4.1), since for quasi-norms of the form (4.2) one can proceed similarly.

Note that, due to the form of \( X_{j_0} \) and of the quasi-norm \( N \) in (4.1),

\[
|X_{j_0}N(x)| = \left| \frac{2\beta_{j_0}a_{j_0}x_{j_0}^{2\beta_{j_0}-1}}{\gamma N^{\gamma-1}} \right| = C \frac{|x_{j_0}|^{\gamma-1}}{N^{\gamma-1}}.
\]

Therefore, since \( |\nabla G N(x)| \geq |X_{j_0}N(x)| \) for all \( x \in G \), the previous estimate amounts to (3.2) with \( j = j_0 \).

For combinations of smooth quasi-norms as in (4.2), by repeating the same considerations as above we trivially get the same result. This concludes the proof. \( \square \)

By using Lemma 4.6, one obtains the following corollary of Theorem 3.1.

**Theorem 4.7.** Let \( G \) be a Carnot group of step \( r \) on \( \mathbb{R}^n \) and let \( \{X_j\}_{1 \leq j \leq n_1} \) be the generators of the first stratum \( V_1 \). If there exist \( j_0 \in \{1, \ldots, n_1\} \) such that

\[
X_{j_0} = \partial_{x_{j_0}},
\]

then the hypotheses of Theorem (3.1) are satisfied for \( N \) as in (4.1) or (4.2).

Among the groups to which Lemma 4.6 applies, and therefore satisfying condition (4.8) with \( N \) as in (4.1) or (4.2), we have:

- Carnot groups of "Engel type" on \( \mathbb{R}^n \) (i.e. Carnot groups of filiform-type), for \( n \geq 4 \), with polynomial coordinates \(^2\) (see [CFZ20]);
- The Cartan group (see [Dix57]);
- Kolmogorov-type groups (see [BLU07]);
- Carnot groups arising from some Sub-Laplacians, like, for instance, the ones arising from the lifting of Bony-type Sub-Laplacians and those related to some Sub-Laplacian arising in control theory (see Section 4.3 in [BLU07] for details);
- Sums of Carnot groups of the previous type.

Hence, Theorem 4.7 gives that global \( q \)-Poincaré inequalities hold on all these groups too.

**Remark 4.8.** In the previous work [CFZ20] the authors proved that global Poincaré inequalities hold true for a class on probability measures on a family of Carnot groups with a filiform Lie algebra, also called Carnot groups of “Engel type”. We want to note that the aforesaid family does not necessarily include all the Carnot group with

\(^2\)Here we use polynomial coordinates arising from a strong Malcev basis of the Lie algebra of the group; see [CG] for details about such coordinates.
a filiform Lie algebra, and therefore, in the present work one might extend the validity of global Poincaré inequalities to a bigger class of Carnot groups with a Lie algebra of filiform type as well. To be more precise let us first recall the definition of the Carnot groups of “Engel type”, see [C20],[C22], as appeared in [CFZ20]; a group \( G \) is of this type if \( G = (\mathbb{R}^{n+1}, \circ) \), with \( g_{n+1} = \text{span}\{X_1, \ldots, X_n, X_{n+1}\} \) and

\[
[X_1, X_j] = X_{j+1}, \quad \forall j = 1, \ldots, n,
\]

\[
[X_i, X_j] = 0, \quad 2 \leq i, j \leq n + 1
\]

\[
[X_1, X_{n+1}] = 0.
\]

We claim that not all groups of filiform type are described as above. Indeed, by a result due to Bratzlavsky in [Bra74] (see also Theorem 4.3.6 in [BLU07]), given any filiform Lie algebra \( g \) of finite dimension \( n \), there exists a basis \( \{X_i\}_{i=1, \ldots, n} \) for \( g \) such that \( X_{i+1} = [X_1, X_i] \) (for every \( i = 1, \ldots, n-1 \)) and \( [X_1, X_n] = 0 \). This implies, in particular, that any filiform Lie algebra has a basis \( \{Y_i\}_{i=1, \ldots, n} \) of the form

\[
Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = [X_1, X_2], \quad Y_4 = [X_1, [X_1, X_2]], \ldots, \quad Y_n = \underbrace{[X_1, [X_1, \ldots, X_1, X_2]]}_{\text{n-2 times}}.
\]

Comparing the two descriptions above one sees that the description given in [CFZ20] satisfying the condition \( [X_i, X_j] = 0, 2 \leq i, j \leq n + 1 \) is a particular case of the latter.

**References**

[Ada79] Adams, R.A. General logarithmic Sobolev inequalities and Orlicz embeddings, *Jour. Funct. Anal.* 34 (1979), no. 2, 292–303.

[BFS18] Balogh, Z. , M. , Fässler K. and Sobrino, H. . Isometric embeddings into Heisenberg group. *Geom. Dedicata* 195 (2018), 163–192. https://doi.org/10.1007/s10711-017-0282-5

[BZ05] Bobkov,S. G., and Zegarlinski,B. , Entropy bounds and isoperimetry, Mem. AMS, Vol. 176, Nr 829 (2005) doi:http://dx.doi.org/10.1090/memo/0829

[BLU07] Bonfiglioli, A. , Lanconelli, E. and Uguzzoni, F. *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*. Springer Monographs in Mathematics, Springer, 2007.

[BDZ21a] Bou Dagher, E. and Zegarlinski, B., Coercive Inequalities and U-Bounds on Step-Two Carnot Groups, *Potential Analysis*. https://doi.org/10.1007/s11118-021-09979-0

[BDZ21b] Bou Dagher, E. and Zegarlinski, B., Coercive Inequalities on Carnot Groups: Taming Singularities, https://doi.org/10.48550/arXiv.2105.03922

[BDZ22] Bou Dagher, E. and Zegarlinski, B., Coercive inequalities in higher-dimensional anisotropic heisenberg group, *Analysis and Mathematical Physics* (2022) 12:3 https://doi.org/10.1007/s13324-021-00609-x

[Bra74] Bratzlavsky, F., Classification des algèbres de Lie nilpotents de dimension \( n \), de classe \( n-1 \), dont l’idéal dérivé est commutatif, Bull. Cl. Sci.,V. Sér., Acad. R. Belg., 60 (1974), 858-865.

[CDPT07] Capogna, L. , Danielli, D. , Pauls, S. D. , and Tyson, J. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*. Progress in Mathematics, 250. Birkhäuser Verlag, Basel, 2007. xvi+223 pp. ISBN: 978-3-7643-8132-5

[CL14] Capogna, L. and Le Donne, E. . Smoothness of subRiemannian isometries. *Amer. J. Math.* 138 (2016), no. 5, 1439–1454.

[CGR10] Cattiaux, P. , Guillin, A. and Roberto, C. . Poincaré inequality and the \( L^p \) convergence of semi-groups. *Comm. Prob.*, Institute of Mathematical Statistics (IMS), 2010, 15, 270–280.

[Cip00] Cipriani, F. , Sobolev-Orlicz imbeddings, weak compactness, and spectrum. *J. Funct. Anal.* 177 (2000) 89–106.

[C22] Chatzakou, M. . Quantizations on the Engel and the Cartan groups. *J. Lie Theory*, 31 (2022) 517–542.
Chatzakou, M. . On $(\lambda, \mu)$-classes on the Engel group. In Advances in Harmonic Analysis and Partial Differential Equations, edited by Vladimir Georgiev et al., Springer (2020) 37–49.

Chatzakou, M., Federico, S. and Zegarlinski, B., q-Poincaré inequalities on Carnot groups with a filiform Lie algebra, arXiv: 2007.04689v2.

Corwin, L.J., Greenleaf, F.P., Representations of nilpotent Lie groups and their applications. Part I. Basic theory and examples, Cambridge Studies in Advanced Mathematics, 18. Cambridge University Press, Cambridge, 1990. viii+269 pp.

Dixmier, J. Sur les représentations unitaires des groupes de Lie nilpotents. Vol. III of Canad. J. Math. 10 (1957) 321–348.

Driver, B. and Melcher, T. . Hypoelliptic heat kernel inequalities on the Heisenberg group. J. Funct. Anal. 59 (2005), 340–365.

The Equivalence of Certain Norms on the Heisenberg Group

Egwe, M. . The Equivalence of Certain Norms on the Heisenberg Group. Adv.Math. 03 (2013), 576–578.

Eldredge, N. . Gradient estimates for the subelliptic heat kernel on H-type groups. J. Funct. Anal. 92 (2009), 52–85.

Fischer, V. , and Ruzhansky, M., Quantization on nilpotent Lie groups. Progress in Mathematics, 314. Birkhäuser/Springer, [Cham], 2016. xiii+557 pp. ISBN: 978-3-319-29557-2; 978-3-319-29558-9

Guionnet, A. and Zegarlinski, B. . Lectures on logarithmic Sobolev inequalities. In Séminaire de Probabilités, XXXVI, no. 1801 in Lecture Notes in Math. (2003), 1–134.

Hajlasz, P. and Koskela, P/. Sobolev met Poincaré. Mem. Amer. Math. Soc. 145 (2000), no. 688, x+101 pp.

Hebisch, W. and Zegarlinski, B., Coercive inequalities on metric measure spaces. J. Funct. Anal. 258 (2010), no. 3, 814–851. doi:10.1016/j.jfa.2009.05.016

Inglis, J., Coercive inequalities for generators of Hörmander type. PhD Thesis, Imperial College London, 2010.

Inglis, J. , Kontis, V. , and Zegarlinski, B., From U-bounds to isoperimetry with applications to H-type groups. (English summary) J. Funct. Anal. 260 (2011), no. 1, 76–116.

Jerison, J., The Poincaré inequality for vector fields satisfying Hörmander’s condition. Duke Math. J. 53 (1986), no. 2, 503–523.

Kontis, V. , Ottobre, M. , and Zegarlinski, B., Markov semigroups with hypocoercive-type generator in infinite dimensions: Ergodicity and smoothing, J. Funct. Anal. 270 (2016), no. 9, 3173–3223.

Lee, J. R. and Naor, A. . $L^p$ metrics on the Heisenberg group and the Goemans-Linial conjecture. 2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS’06) (2006), 99–108.

Li, H. Q. Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg, [Optimal estimation of the gradient of the heat semigroup on the Heisenberg group], J. Funct. Anal. 236 (2006), no. 2, 369–394.

Pop, D. . The Heisenberg group associated to a Hilbert space. Linear Algebra Appl. 508 (2016) 234–245.

Röckner, M. , Weak Poincaré inequalities and $L^2$-convergence rates for Markov semi-groups. J. Funct. Anal. 185 (2001), 563–603.

Rockland, C. . Hypoellipticity on the Heisenberg group-representation- theoretic criteria. Trans. Amer. Math. Soc. 240 (1978), 1–52.

Rosen, J., Sobolev inequalities for weight spaces and supercontractivity. Trans. Amer. Math. Soc. 222 (1976), 367–376.

Varopoulos, N. T. , Saloff-Coste, L. , and Coulhon, T. Analysis and geometry on groups. Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge, 1992. xii+156 pp.

Wang, F. Y. . Functional inequalities and spectrum estimates: the infinite measure case. J. Funct. Anal. 194 (2002), 288–310.

Marianna Chatzakou:
Department of Mathematics: Analysis, logic and discrete mathematics
Ghent university
