HODGE THEORY FOR ELLIPTIC CURVES AND THE HOPF ELEMENT \( \nu \)

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Abstract. We show that the vector bundle on the moduli stack \( M_{\text{ell}} \) of elliptic curves associated to the 2-cell complex \( C\nu \) is isomorphic to the de Rham cohomology sheaf \( H^1_{\text{dR}}(E/M_{\text{ell}}) \) of the universal elliptic curve \( E \to M_{\text{ell}} \). We use this to calculate the homotopy groups of the \( E_1 \)-quotient tmf/\( \nu \) of tmf by \( \nu \), called the spectrum of “topological quasimodular forms”, by relating its Adams–Novikov spectral sequence to the cohomology of the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham filtration.

1. Introduction

In this paper, we study the relationship between the Hopf invariant one element \( \nu \in \pi_3\text{tmf} \) and the Hodge filtration for elliptic curves. Namely, we show that the vector bundle on the moduli stack \( M_{\text{ell}} \) of elliptic curves associated to \( C\nu \) is isomorphic to the (middle) de Rham cohomology \( H^1_{\text{dR}}(E/M_{\text{ell}}) \) of the universal elliptic curve \( E \to M_{\text{ell}} \). A version of this relationship had been stated by Hopkins in [Hop02, Section 5]. Using this, we calculate the homotopy groups of the \( E_1 \)-quotient tmf/\( \nu \) of tmf by \( \nu \in \pi_3(S) \) by showing that the \( E_2 \)-page of its Adams–Novikov spectral sequence is isomorphic to the cohomology of the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham filtration. The \( E_1 \)-ring tmf/\( \nu \) is called the spectrum of topological quasimodular forms (see Remark 4.2).

The ring spectrum tmf/\( \nu \) is interesting for several reasons. One motivation for studying it comes from the Ando-Hopkins-Rezk orientation \( \text{MU} \langle 6 \rangle \to \text{tmf} \) (see [AHR10]). As is made clear during the course of the proof, a key reason for why this orientation does not factor through the map \( \text{MU} \langle 6 \rangle \to \text{MSU} \) is because tmf detects the element \( \nu \in \pi_3(S) \); this in turn is related to the fact that the weight 2 Eisenstein series is not a modular form. Since tmf/\( \nu \) is the “smallest” coherently structured (i.e., \( E_1 \)-) tmf-algebra with a nullhomotopy of \( \nu \), one might expect the composite

\[
\text{MU}(6) \to \text{tmf} \to \text{tmf/} \nu
\]

(1)

to factor through MSU via an \( E_1 \)-map; see also Remark 4.7. Although we do not prove in this paper that the composite (1) factors through MSU, we will use the results of this paper to address this question in future work. The connection between \( \nu \) and the weight 2 Eisenstein series is also discussed in Section 4. The relationship between \( \nu \) and de Rham cohomology is also intrinsically intriguing, because the Hodge–de Rham filtration on the de Rham cohomology of an elliptic curve is related to many deep topics in arithmetic geometry (such as Grothendieck–Messing theory).

We begin in Section 2 by recalling some background on Hodge theory for cubic curves from algebraic geometry. In particular, we give a Hopf algebroid presentation for the moduli stack \( M_{\text{dR}}^{\text{cub}} \) of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence. In Section 3, we prove our main technical result relating the Adams–Novikov spectral sequence of tmf/\( \nu \) to the cohomology of the moduli stack \( M_{\text{dR}}^{\text{cub}} \). Finally, in Section 4 we prove Theorem 4.1, which calculates this Adams–Novikov spectral sequence. It degenerates at the \( E_4 \)-page, and TMF/\( \nu \) is found to be 24-periodic. Moreover, tmf/\( \nu \) is homotopy commutative. These results were discovered independently by Rezk in unpublished work, and we give our own proof of his calculation of \( \pi_*(\text{tmf}/\nu) \).

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2. Background on Hodge theory

In this section, we recall some background on Hodge theory for cubic curves over a general base scheme. Multiple sources (such as [Kat73, Appendix A1.2]) discuss Hodge theory for (smooth) elliptic curves.

Let \( f : X \to Y \) be a morphism of schemes. One then has the \( f^{-1}\mathcal{O}_Y \)-linear relative de Rham complex \( \Omega_{X/Y}^* \).

**Definition 2.1.** The \( j \)th relative de Rham cohomology \( H_{dR}^j(X/Y) \) of \( f : X \to Y \) is defined to be the hypercohomology sheaf \( R^j f_* (\Omega_{X/Y}^*) \) on \( Y \).

The Tot spectral sequence defines the Hodge–de Rham spectral sequence of sheaves on \( Y \):

\[
E_1^{s,t} = R^s f_* \Omega_{X/Y}^t \Rightarrow H_{dR}^{s+t}(X/Y).
\]

The following (easy) result is well-known.

**Theorem 2.2.** If \( f : X \to Y \) is a smooth, proper, and surjective morphism of relative dimension 1 with geometrically connected fibers, then the Hodge–de Rham spectral sequence degenerates at the \( E_1 \)-page.

Since \( f \) is of relative dimension 1, the only interesting de Rham cohomology is in the middle dimension, i.e., \( H_{dR}^1(X/Y) \). In particular, for such \( f \), there is an exact sequence

\[ 0 \to f_* \Omega_{X/Y}^1 \to H_{dR}^1(X/Y) \to R^1 f_* \mathcal{O}_X \to 0 \]

of quasicoherent sheaves on \( Y \); this is called the Hodge–de Rham exact sequence. Moreover, the pairing

\[ H_{dR}^1(X/Y) \otimes_{\mathcal{O}_Y} H_{dR}^1(X/Y) \to \mathcal{O}_Y \]

is determined by the canonical perfect pairing

\[ R^1 f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} f_* \Omega_{X/Y}^1 \to R^1 f_* \Omega_{X/Y}^1 \to \mathcal{O}_Y.\]

We now specialize to the case when \( f : X \to Y \) is an elliptic curve \( f : E \to S \). Then \( f_* \Omega_{E/S}^1 \) is the line bundle \( \omega_{E/S} \) of invariant differentials. The pairing \( \omega_{E/S} \otimes_{\mathcal{O}_S} R^1 f_* \mathcal{O}_E \to \mathcal{O}_S \) is perfect, and so \( R^1 f_* \mathcal{O}_E \cong \omega_{E/S}^{-1} \). In particular, the Hodge–de Rham exact sequence for \( E \to S \) becomes

\[ 0 \to \omega_{E/S} \to H^1_{dR}(E/S) \to \omega_{E/S}^{-1} \to 0.\]

If \( \mathcal{M}_\text{ell} \) denotes the moduli stack of elliptic curves, and \( E \to \mathcal{M}_\text{ell} \) is the universal elliptic curve, then (2) exhibits \( H^1_{dR}(E/\mathcal{M}_\text{ell}) \) as an element of \( \text{Ext}^1_{\mathcal{M}_\text{ell}}(\omega^{-1}, \omega) \).

**Remark 2.3.** If \( S \) is a \( p \)-adic scheme, then (2) corresponds to the Hodge filtration of the Dieudonné module \( D(E[p^\infty]/S) \) of \( E \) under the isomorphism \( H^1_{dR}(E/S) \cong D(E[p^\infty]/S) \); see, for instance, [Kat81, Section V].

The following is an immediate consequence of [Kat73, Equation A1.2.3]:

**Proposition 2.4.** If \( f : E \to S \) is an elliptic curve, then there is an isomorphism \( H_{dR}^1(E/S) \otimes \omega_{E/S} \cong f_* \Omega_{E/S}^1(\infty)^{\otimes 2} \).

The map

\[ f_* \Omega_{E/S}^1(\infty)^{\otimes 2} \cong H_{dR}^1(E/S) \otimes \omega_{E/S} \to \omega_{E/S}^{-1} \otimes \omega_{E/S} = \mathcal{O}_S \]

induced by the Hodge–de Rham exact sequence sends a section of \( f_* \Omega_{E/S}^1(\infty)^{\otimes 2} \) to its residue at \( \infty \).

We can now generalize the above story to the non-smooth setting. First, we recall the definition of a cubic curve.
Definition 2.5. A cubic curve $f : E \to S$ over a scheme $S$ is a flat and proper morphism of finite presentation whose fibers are reduced, irreducible curves of arithmetic genus 1, along with a section $\infty : S \to E$ whose image is contained in the smooth locus $E^{\text{sm}}$ of $f$. Let $M_{\text{cub}}$ denote the stack of cubic curves, and let $f : E \to M_{\text{cub}}$ denote the universal cubic curve.

Let $\omega$ denote the line bundle on $M_{\text{cub}}$, assigning to a cubic curve $f : E \to S$ the cotangent bundle $\omega_{E/S}$ along the section $\infty$. The vector bundle $H^1_{\text{dR}}(E/M_{\text{ell}})$ over $M_{\text{ell}}$ extends to a vector bundle, denoted $H^1_{\text{dR}}(E/M_{\text{cub}})$, over $M_{\text{cub}}$; this vector bundle is represented by an element in $\text{Ext}^1_{M_{\text{cub}}}(\omega^{-1}, \omega) \cong H^1(M_{\text{cub}}; \omega^{\otimes 2})$. Upon tensoring $H^1_{\text{dR}}(E/M_{\text{cub}})$ with $\omega$, we obtain for every cubic curve $f : E \to S$ an exact sequence

$$0 \to \omega_{E/S}^{\otimes 2} \to f_* \Omega^1_{E/S}(\infty)^{\otimes 2} \to f_* \mathcal{O}_{E/S} \cong \mathcal{O}_S \to 0,$$

which we shall refer to as the Hodge–de Rham exact sequence.

Definition 2.6. Let $M_{\text{cub}}^{\text{dR}}$ denote the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence, and let $M_{\text{ell}}^{\text{dR}} = M_{\text{cub}}^{\text{dR}} \times_{M_{\text{cub}}} M_{\text{ell}}$ denote the moduli stack of elliptic curves with a chosen splitting of the Hodge–de Rham exact sequence.

In order to do calculations with $M_{\text{cub}}^{\text{dR}}$, we would like to obtain a Hopf algebroid presentation of this stack. To do so, we recall a Hopf algebroid presentation of $M_{\text{cub}}$. Zariski-locally on any base scheme $S$, a cubic curve is described by a Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

with other choices of coordinates $(x, y)$ given by the transformations

$$x \mapsto x + r, \ y \mapsto y + sx + t.$$

The moduli stack $M_{\text{cub}}$ of cubic curves is presented by the Hopf algebroid

$$(D, \Gamma) = (\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], D[r, s, t]),$$

with gradings $|a_i| = i$ and $|r| = 2, |s| = 1$, and $|t| = 3$. Studying how the coefficients $a_i$ transform gives the right unit $\eta_R : D \to \Gamma$ of this Hopf algebroid:

- $a_1 \mapsto a_1 + 2s,$
- $a_2 \mapsto a_2 - a_1 s + 3r - s^2,$
- $a_3 \mapsto a_3 + a_1 r + 2t,$
- $a_4 \mapsto a_4 + a_3 s + 2a_2 r - a_1 t - a_1 rs - 2st + 3r^2,$
- $a_6 \mapsto a_6 + a_4 r - a_3 t + a_2 r^2 - a_1 rt - t^2 + r^3.$

To determine a Hopf algebroid presentation of $M_{\text{cub}}^{\text{dR}}$, note that the coordinate $x$ defines a function on $E$ with double pole at $\infty$, and that it is in fact the only such non-constant function on the smooth locus of $E$ (this follows from the usual calculation [KMS85, Section 2.2.5] with the Riemann-Roch formula). In particular, Proposition 2.4 implies that the element $[r]$ in the cobar complex for the Hopf algebroid $(D, \Gamma)$ detects the extension in $\text{Ext}^1_{M_{\text{cub}}}(\mathcal{O}, \omega^{\otimes 2}) = \text{Ext}^1_{E/S}(D, D)$ determined by the de Rham cohomology $H^1_{\text{dR}}(E/M_{\text{ell}})$. By the exact sequence (3), a choice of Hodge–de Rham splitting on the universal cubic curve amounts to fixing a choice of coordinate $x$ (although $y$ is allowed to vary). Consequently:

Proposition 2.7. The moduli stack $M_{\text{cub}}^{\text{dR}}$ of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence is presented by the Hopf algebroid $(D, \Sigma) = (\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], D[s, t])$, with gradings $^3 |a_i| = i, |s| = 1$, and $|t| = 3$. The right unit is the same as in that of the elliptic curve Hopf

\footnote{Recall that the topological grading is double the algebraic grading.}
algebroid, except with \( r = 0 \):
\[
\begin{align*}
    a_1 & \mapsto a_1 + 2s, \\
    a_2 & \mapsto a_2 - a_1 s - s^2, \\
    a_3 & \mapsto a_3 + 2t, \\
    a_4 & \mapsto a_4 + a_3 s - a_1 t - 2st, \\
    a_6 & \mapsto a_6 - a_3 t - t^2.
\end{align*}
\]

\((5)\)

3. The relationship with \( \text{tmf}/\nu \)

In this section, we study the \( E_1 \)-quotient \( \text{tmf}/\nu \) of \( \text{tmf} \) by \( \nu \), and relate its Adams–Novikov spectral sequence to the cohomology of \( M_{\text{cub}} \).

We begin by recalling one construction of the \( E_1 \)-quotient \( \text{tmf}/\nu \). This satisfies the following universal property: if \( R \) is any \( E_1 \)-\( \text{tmf} \)-algebra, then
\[
\text{Map}_{\text{Alg}_{E_1}(\text{Mod}(\text{tmf}))}(\text{tmf}/\nu, R) = \begin{cases} 
\Omega^{\infty+4}R & \text{if } \nu = 0 \in \pi_3 R \\
\emptyset & \text{else.}
\end{cases}
\]

One of the main results of [AB19] justifies the following definition:

**Definition 3.1.** The \( E_1 \)-ring \( \text{tmf}/\nu \), called topological quasimodular forms (see Remark 4.2 for a justification for the name), is the Thom spectrum of the dotted extension in the following diagram:

\[
\begin{array}{ccc}
S^4 & \xrightarrow{\nu} & B\text{GL}_1(\text{tmf}) \\
\downarrow \quad & & \downarrow \\
\Omega S^5 & \xrightarrow{\partial} & B\text{GL}_1(\text{S}) \\
\end{array}
\]

**Remark 3.2.** The element \( \nu \in \pi_3(\text{tmf}) \) is spherical, and so this diagram factors as

\[
\begin{array}{ccc}
S^4 & \xrightarrow{2\nu_2} & B\text{Spin} \\
\downarrow \quad & & \downarrow \\
\Omega S^5 & & B\text{GL}_1(\text{tmf}).
\end{array}
\]

Following the notation of [Dev19], we will write \( A \) to denote the Thom spectrum of the loop map \( \Omega S^5 \to B\text{GL}_1(\text{S}) \). Then there is a canonical equivalence \( \text{tmf}/\nu \simeq \text{tmf} \otimes A \) of \( E_1 \)-\( \text{tmf} \)-algebras, so there is in particular a ring map \( A \to \text{tmf}/\nu \).

The \( E_1 \)-ring \( A \) will be useful below. In [Dev19], it is shown that there is an \( E_1 \)-map \( A \to \text{BP} \). Moreover, the BP-homology of \( A \) at the prime 2 is isomorphic to \( \text{BP}_* [y_2] \), where \( y_2 \) is sent to \( t_1^2 \) modulo decomposables under the map \( \text{BP}_*(A) \to \text{BP}_*(\text{BP}) \). In particular, \( H_*(A; \mathbb{F}_2) \cong \mathbb{F}_2[\zeta^4] \). One then has:

**Proposition 3.3.** There is a nontrivial simple 2-torsion element \( \sigma_1 \in \langle \eta, \nu, 1_A \rangle \subseteq \pi_5(A) \cong \pi_5(C\nu) \) specified up to indeterminacy by the relation \( \nu \eta = 0 \). One choice of this element is represented by \([t_2] \) in the Adams–Novikov spectral sequence for \( A \), and by \( h_{21} \) in the \((\text{mod} \ 2)\) Adams spectral sequence for \( A \).

To connect \( \text{tmf}/\nu \) and Hodge theory for cubic curves, we make the following observation. Recall that \( H^1_{\text{dR}}(E/M_{\text{cub}}) \in \text{Ext}^1_{M_{\text{cub}}} (\omega^{-1}, \omega) \).

**Proposition 3.4.** Let \( f : E \to M_{\text{cub}} \) denote the universal cubic curve over the moduli stack of cubic curves. Then \( H^1_{\text{dR}}(E/M_{\text{cub}}) \in \text{Ext}^1_{M_{\text{cub}}} (\omega^{-1}, \omega) \cong H^1(M_{\text{cub}}; \omega^2) \) detects \( \nu \) in the \( E_2 \)-page of the descent spectral sequence for \( \text{tmf} \).
Proof. This is essentially argued in [Hop02, Section 5.2]. We know that $H^1(M_{\text{cub}}; \omega^2) = \mathbb{Z}/12$ by the calculations in [Bau08]; the element $[\rho]$ in the cobar complex determined by the Hopf algebraoid $(D, \Gamma)$ is a representative for the generator. This element detects $\nu$ in the Adams–Novikov spectral sequence for tmf, and by the discussion before Proposition 2.7, also detects the extension class of the Hodge–de Rham exact sequence. □

**Corollary 3.5.** The rank two vector bundle $\mathcal{F}(C\nu)$ on the moduli stack of cubic curves corresponding to $C\nu$ is isomorphic to $H^1_{\text{dR}}(E/M_{\text{cub}})$.

**Remark 3.6.** In [Rez13, Section 11.5], the Hodge–de Rham exact sequence appears in a different but related guise, as a class in the $E_2$-page of a spectral sequence $\{E_{r}^{\ast, t}\}$ converging to the homotopy groups of the space of $E_\infty$-maps $Z_+ \to \text{TMF}$. The element $H^3_{\text{dR}}(E/M_{\text{cub}}) \in E_2^{1, 4}$ detects a nontrivial class in $\pi_3 \text{Map}_{E_\infty}(Z_+, \text{TMF}) = \pi_3 \mathcal{G}_{m}(\text{TMF})$, i.e., an $E_\infty$-map $K(Z, 3)_+ \to \text{TMF}$. This is related to the $E_\infty$-twisting of TMF explored in [ABG10].

Since $\text{tmf}/\nu$ is the $E_1$-quotient of $\text{tmf}$ by $\nu$ by Remark 3.2, it is the universal $E_1$-$\text{tmf}$-algebra with a nullhomotopy of $\nu$. If $\text{tmf}/\nu$ is a homotopy commutative ring (which we will show is indeed the case in Corollary 4.8), then we would be able to consider the stack associated to $\text{tmf}/\nu$ (in the sense of [DFHH14, Chapter 9], [Mat16, Section 2.1]), and it would be reasonable to expect that Proposition 3.4 implies that this stack is the moduli of cubic curves with a choice of splitting of the Hodge–de Rham spectral sequence. We have:

**Theorem 3.7.** Let $g : M_{\text{cub}}^{\text{dR}} \to M_{\text{cub}}$ denote the structure morphism. Then the sheaf on $M_{\text{cub}}$ associated to $A$ is isomorphic as an algebra to the pushforward $g_* \mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$.

Proof. Let $\mathcal{C}$ be a presentable symmetric monoidal ($\infty$-)category, and let $T$ denote the functor $\mathcal{C}^{\text{unital}} \to \text{Alg}_{E_1}(\mathcal{C})$ sending a unital object $i : 1 \to X$ to the free $E_1$-algebra in $\mathcal{C}$ whose unit factors through $i$; this may be defined via the homotopy pushout

$$
\begin{array}{ccc}
\text{Free}_{E_1}(1) & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
\text{Free}_{E_1}(X) & \longrightarrow & T(X)
\end{array}
$$

in $\text{Alg}_{E_1}(\mathcal{C})$. The functor $\mathcal{F}(-)$ from the homotopy category of $\text{tmf}$-modules to $\text{Qcoh}(M_{\text{cub}})$ fits into a commutative diagram

$$
\begin{array}{ccc}
\text{hMod}(\text{tmf})^{\text{unital}} & \longrightarrow & \text{hAlg}_{E_1}(\text{Mod}(\text{tmf})) \\
\downarrow T & & \downarrow T \\
\text{Qcoh}(M_{\text{cub}})^{\text{unital}} & \longrightarrow & \text{Alg}_{\text{unital}}(\text{Qcoh}(M_{\text{cub}})).
\end{array}
$$

It is easy to see by the universal property of $T$ that $T(C\nu) \cong A$, so it follows from Proposition 3.4 that $\mathcal{F}(A) \cong T(f_* \Omega^1_{E/M_{\text{cub}}}(\infty)^{\otimes 2})$. It therefore suffices to show that $T(f_* \Omega^1_{E/M_{\text{cub}}}(\infty)^{\otimes 2}) \cong g_* \mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$. This is not immediate, since the sheaf $T(f_* \Omega^1_{E/M_{\text{cub}}}(\infty)^{\otimes 2})$ satisfies a universal property with respect to associative $\mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$-algebras. Nonetheless, its universal property defines an algebra map $\varphi : T(f_* \Omega^1_{E/M_{\text{cub}}}(\infty)^{\otimes 2}) \to g_* \mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$ of sheaves on $M_{\text{cub}}$. To check that this is an isomorphism, it suffices to show by [Mat16, Lemma 4.4] that the map induces an isomorphism on the cuspidal cubic over $\mathbb{F}_p$ for all primes $p$.

In fact, we can check this over $\mathbb{Z}$. We know that $g_* \mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$ is isomorphic to a polynomial ring on a single generator; indeed, the space of all splittings of the Hodge filtration is isomorphic to the projective line $\text{P}(H^1_{\text{dR}}(E))$ minus the point corresponding to the line determined by the invariant differentials. Moreover, $T(f_* \Omega^1_{E/M_{\text{cub}}}(\infty)^{\otimes 2})$ is isomorphic to a free associative ring on one generator, which is also a polynomial ring on a single generator. It therefore suffices to show that the generator maps to the
Corollary 3.8. There is a descent spectral sequence
\[ E_2^{s,2t} = H^s(M_{\text{cub}}; g^*\omega^{s+t}) \Rightarrow \pi_{2t-s}(\text{tmf} \mathbin{\mathcal{U}} \nu), \]
and it is isomorphic to the Adams–Novikov spectral sequence for \( \text{tmf} \mathbin{\mathcal{U}} \nu \).

Proof. There is a descent spectral sequence
\[ E_2^{s,2t} = H^s(F(A) \otimes \mathcal{O}_{M_{\text{cub}}} \omega^{s+t}) \Rightarrow \pi_{2t-s}(\text{tmf} \mathbin{\mathcal{U}} \nu), \]
and this is isomorphic to the Adams–Novikov spectral sequence for \( \text{tmf} \mathbin{\mathcal{U}} \nu \) by [Mat16, Corollary 5.3] and the main theorem of [Dev18]. Combining Theorem 3.7 with the projection isomorphism shows that \( F(A) \otimes \mathcal{O}_{M_{\text{cub}}} \omega^{s+t} \cong g_*(g^* \omega^{s+t}) \). The morphism \( g \) is flat and affine, so \( E_2^{s,2t} \cong H^s(M_{\text{cub}}; g^*\omega^{s+t}) \), as desired.

Remark 3.9. Corollary 3.8 says that although \( \text{tmf} \mathbin{\mathcal{U}} \nu \) is not a priori a homotopy commutative ring, there is a descent spectral sequence which would exist if there was a sheaf of structured ring spectra on \( M_{\text{cub}}^{\text{der}} \) whose global sections was \( \text{tmf} \mathbin{\mathcal{U}} \nu \). We pose this as a conjecture:

Conjecture 3.10. For some \( k \geq 2 \), there is a sheaf of even-periodic \( E_k \)-rings \( \mathcal{O}_{\text{der}} \) on the étale site of \( M_{\text{cub}}^{\text{der}} \) such that if \( f : \text{Spec} \mathcal{R} \to M_{\text{cub}}^{\text{der}} \) is an étale map, then \( \mathcal{O}_{\text{der}}(f) \) is the Landweber-exact theory corresponding to the composite Spec \( \mathcal{R} \to M_{\text{cub}}^{\text{der}} \to M_{\text{cub}} \to M_{FG} \), and such that the global sections \( \Gamma(M_{\text{cub}}^{\text{der}}; \mathcal{O}_{\text{der}}) \) is equivalent as an \( E_k \)-ring to \( \text{TMF} \mathbin{\mathcal{U}} \nu \). Moreover, the resulting \( E_k \)-ring structure on \( \text{TMF} \mathbin{\mathcal{U}} \nu \) extends to an \( E_k \)-ring structure on \( \text{tmf} \mathbin{\mathcal{U}} \nu \).

4. The descent spectral sequence

Our goal in this section is to calculate the homotopy groups of \( \text{tmf} \mathbin{\mathcal{U}} \nu \) via the descent spectral sequence of Corollary 3.8. To do this calculation, we will use the Hopf algebroid presentation in Proposition 2.7. The calculation of the descent spectral sequence was done independently by Charles Rezk; although he stated part of the result to us in an email, the argument is ours (so errors are our fault).

Theorem 4.1. There is an isomorphism
\[ H^s(M_{\text{cub}}^{\text{der}}; g^*\omega^{s}) \cong \mathbb{Z}[b_2, b_4, b_6, b_8, h_1, h_{21}] / I, \]
where \( b_i \in H^i(M_{\text{cub}}^{\text{der}}; g^*\omega^i) \) of total degree \( 2i \), \( h_1 \in H^1(M_{\text{cub}}^{\text{der}}; g^*\omega) \) of total degree 1, and \( h_{21} \in H^1(M_{\text{cub}}^{\text{der}}; g^*\omega^3) \) of total degree 5. If one defines
\[ c_i = b_i^2 - 24b_4, \quad c_6 = -b_3^2 + 36b_2b_4 - 216b_6, \]
\[ \Delta = -b_2^2b_8 - 8b_4^2 - 27b_6^2 + 9b_2b_4b_6, \]
then the ideal \( I \) is generated by the relations
\[ 2h_1 = 0, \quad 2h_{21} = 0, \quad h_1^2b_6 = h_{21}^2b_2, \quad 4b_8 = b_2b_6 - b_4^2, \quad 1728\Delta = c_4^3 - c_6^2. \]
Moreover, the descent/Adams–Novikov spectral sequence of Corollary 3.8 collapses on the \( E_4 \)-page, and \( \pi_*(\text{tmf} \mathbin{\mathcal{U}} \nu) \) is determined by the differentials
\[ d_3(b_2) = b_1^3, \quad d_3(b_4) = h_1^2h_{21}, \quad d_3(b_8) = h_1h_{21}^2, \quad d_3(b_6) = b_2^2; \]
One has the relations
\[ \eta^3 = 0, \quad \eta^2\sigma_1 = 0, \quad \eta \sigma_1^2 = 0, \quad \sigma_1^3 = 0 \]
in the homotopy of \( \text{tmf} \mathbin{\mathcal{U}} \nu \), in addition to the relations in \( I \). Here \( \eta \) is represented by \( h_1 \), and \( \sigma_1 \) is represented by \( h_{21} \). All the torsion in \( \text{tmf} \mathbin{\mathcal{U}} \nu \) is concentrated in dimensions congruent to 1, 2 (mod 4).

Before giving the proof, we discuss some consequences.
Remark 4.2. By Theorem 4.1, there is a ring isomorphism
\[ H^0(M_{\text{cub}}^{dR}; g^* \omega^{\otimes \ast}) \cong \mathbb{Z}[b_2, b_4, b_6, b_8]/(4b_8 = b_2b_6 - b_4^2). \]
This ring has been studied before (in characteristic zero, in which case \( b_8 = (b_2b_6 - b_4^2)/4 \), e.g., in [KZ95, Mov12], where it is referred to as the ring of \textit{quasimodular forms}. Theorem 4.1 provides a calculation of the ring of integral quasimodular forms, and also justifies calling the ring spectrum \( \text{tmf}/\nu \) by the name “topological quasimodular forms”.

The following corollary is a calculation via Theorem 4.1.

Corollary 4.3. In Adams–Novikov filtration zero, \( b_8^2 \) and twice any monomial in the \( b_i \)'s survive to the \( E_\infty \)-page for \( i = 2, 4, 6, 8 \), as do \( b_2b_6 \) and \( \lambda_1 b_6 b_6^2 + \lambda_2 b_2b_4 b_6 \) for \( \lambda_1 \equiv \lambda_2 \) (mod 2). In particular, \( \Delta \in \pi_{24}(\text{tmf}/\nu) \), so the \( E_1 \)-ring \( \text{TMF}/\nu \) is 24-periodic with periodicity generator \( \Delta \).

Remark 4.4. Note that \( \text{tmf}/\nu \) is complex orientable after inverting 2. This can be seen algebraically by noting that the Hopf algebroid \( (D, \Sigma) \) presenting \( M_{\text{cub}}^{dR} \) becomes discrete after inverting 2; indeed, the transformation \( y \to y - a_1 x^2/2 - a_3/2 \) transforms the Weierstrass equation (4) into
\[ y = x^3 + a_2x^2 + a_4x + a_6, \]
and one cannot make any coordinate changes to \( x \) since it is fixed. We find that \( (D, \Sigma) \) is isomorphic to the discrete Hopf algebroid \( (\mathcal{D}' = \mathbb{Z}[1/2][a_2, a_4, a_6], \mathcal{D}) \), and so \( \pi_*(\text{tmf}[1/2]/\nu) \cong \mathbb{Z}[1/2][a_2, a_4, a_6] \), with \( |a_i| = 2i \). In light of this, we only need to prove Theorem 4.1 after 2-localization.

Remark 4.5. The Hurewicz image of tmf in \( \text{tmf}/\nu \) can be determined from Theorem 4.1. The subring generated by \( \eta, \sigma_1, 2b_2 \), and \( b_4^2 \), and in the image of the map \( \pi: A \to \pi_{24}(\text{tmf}/\nu) \). There relationship between \( \pi_*(\text{tmf}/\nu) \) and \( \pi_*(\text{tmf}) \), however, is more interesting than merely the Hurewicz image. The (2-local) calculation in [Bau08] shows that \( \pi_\nu \) vanishes in \( \pi_{23}(\text{tmf}) \); this is detected in the Adams–Novikov spectral sequence by a \( d_5 \)-differential \( d_5(\Delta) = \pi_\nu \). This implies that the element \( 8\Delta \in \pi_{24}(\text{tmf}) \) can be expressed as an element of the Toda bracket \( (8, \nu, \pi_\nu) \). Alternatively, the map \( \pi: S^{20} \to \text{tmf} \)
extends to a map from \( \Sigma^{20}C\nu \), and hence from \( \Sigma^{20}\text{tmf} \wedge C\nu \). Composition with the map \( S^{24} \to \Sigma^{20}C\nu \) which is degree 8 on the top cell produces the element \( 8\Delta \in \pi_{24}(\text{tmf}) \) (up to indeterminacy). In other words, \( 8\Delta \) comes from an element of \( \pi_{24}(\Sigma^{20}\text{tmf} \wedge C\nu) \equiv \pi_4(\text{tmf} \wedge C\nu) \). Under the canonical map \( \text{tmf} \wedge C\nu \to \text{tmf}/\nu \), this element corresponds to \( 2b_2 \in \pi_4(\text{tmf}/\nu) \). Similarly, the element \( \Delta \eta \in \pi_{24}(\text{tmf}) \) can be related to the element \( \sigma_1 \in \pi_3(\text{tmf}/\nu) \). This is related to the approach taken in [Dev19] to show that the Ando–Hopkins–Rezk orientation \( M_{\text{String}} \to \text{tmf} \) from [AHR10] is subject to homotopy.

Remark 4.6. One can give explicit \( q \)-expansions for each of the generators of \( H^0(M_{\text{cub}}^{dR}; g^* \omega^{\otimes \ast}) \cong \mathbb{Z}[b_2, b_4, b_6, b_8] \). The element \( b_2 \) is related to the weight 2 Eisenstein series via the \( q \)-expansion:
\[ b_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n, \]
where \( \sigma_1(n) = \sum_{d \mid n} d \).

Remark 4.7. The explicit characteristic power series for the Witten genus (in [HB192, Section 6.3], for instance) shows that the Witten genus of a spin 4k-manifold is a modular form if its first Pontryagin class vanishes. This restriction arises essentially because of the appearance of a weight 2 Eisenstein series, which is not a modular form, in the characteristic power series. By Remark 4.6, the weight 2 Eisenstein series is in fact a global section of \( H^0(M_{\text{cub}}^{dR}; g^* \omega^{\otimes \ast}) \), and so one might expect to be able to refine the orientation \( \text{MU}(6) \to \text{tmf} \) from [AHR10] to an orientation \( \text{MSU} \to \text{tmf}/\nu \). We will return to this question in forthcoming work; see also Remark 3.6.

The following application is due to Rezk.

Corollary 4.8. The \( E_1 \)-ring \( \text{tmf}/\nu \) admits the structure of a homotopy commutative ring.

\textbf{Proof.} We know that \( \text{tmf}/\nu \) only has \( \text{tmf} \)-module cells in dimensions divisible by 4, so all obstructions to its homotopy commutativity live in these dimensions. We know that these obstructions vanish after inverting 2, since \( A[1/2] \) is homotopy commutative. By Theorem 4.1, all the homotopy groups of \( \text{tmf}/\nu \) in dimensions divisible by 4 are torsion-free, so \( \text{tmf}/\nu \) must indeed be homotopy commutative. \( \square \)
An immediate consequence of Theorem 3.7 and Corollary 4.8 is:

**Corollary 4.9.** The stack $M_{\text{trf}}/\nu$ associated to the homotopy commutative ring $\text{tmf}/\nu$ is isomorphic to $M_{\text{cub}}^{\text{red}}$.

**Remark 4.10.** Corollary 4.9 implies, for instance, that the fact that $\nu$ is not detected by $L_{K(1)\text{tmf}}$ is related to the existence of a dotted map

$$M_{\text{cub}}^{\text{red}} \to M_{\text{ell}}^{\text{red}} \to M_{\text{ell}},$$

where $M_{\text{ell}}^{\text{red}}$ denotes the moduli stack of ordinary elliptic curves. This existence of this dotted map is well-known in arithmetic geometry: it is the statement that the Frobenius (which exists for ordinary elliptic curves via quotienting out by the canonical subgroup) splits the Hodge filtration (see [Kat73, Section A2.3]).

Finally, we give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** We shall implicitly 2-localize everywhere; this is sufficient by Remark 4.4. We begin by calculating $H^i(M_{\text{cub}}^{\text{red}}; g^*\omega_{\Sigma}^r) = \text{Ext}^i_D(D, D)$, where $(D, \Sigma)$ is the Hopf algebroid $(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], D[s, t])$.

Following [Sil86, Chapter III], define quantities

$$b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3, \quad b_6 = a_2^2 + 4a_6, \quad b_8 = a_2^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_4^2 - a_2^2.$$

Notice that $b_2b_6 - b_4^2 = 4b_8$ and that $1728\Delta = c_1^2 - c_6^2$ where $c_1$, $c_6$, and $\Delta$ are as in the theorem statement.

Let $I$ denote the ideal $(2, a_1, a_2, a_4)$, and define a Hopf algebroid

$$(\overline{D}, \Sigma) = (D/I, \Sigma/I) = (F_2[a_2, a_6], \overline{D}[s, t]).$$

The right unit sends

$$a_2 \mapsto a_2 + s, \quad a_6 \mapsto a_6 + t^2.$$ 

Then there is a Bockstein spectral sequence

$$(7) \quad E_1^{p,q,n} = \text{Ext}^{p,n}_{\Sigma}(\overline{D}, \text{Sym}^q_D(I/I^2)) \Rightarrow \text{Ext}^{p,n}_\Sigma(D, D),$$

with $\partial_r : E_1^{p,q,n} \to E_1^{p+q+r,n}$. Now, $I/I^2 = \overline{D} \otimes_{F_2} V$, with $V = F_2\{\overline{a}_0, \overline{a}_1, \overline{a}_3, \overline{a}_4\}$ where $\overline{a}_0$, $\overline{a}_1$, $\overline{a}_3$, and $\overline{a}_4$ represent 2, 1, 3, and 4 respectively. The comodule structure $I/I^2 \to I/I^2 \otimes_{\overline{D}} \Sigma$ sends

$$(8) \quad \overline{a}_0 \mapsto \overline{a}_0, \quad \overline{a}_1 \mapsto \overline{a}_1 + \overline{a}_0s, \quad \overline{a}_3 \mapsto \overline{a}_3 + \overline{a}_0t, \quad \overline{a}_4 \mapsto \overline{a}_4 + \overline{a}_3s + \overline{a}_1t + \overline{a}_0st.$$

There is a map $\overline{D} \to F_2$ induced by sending $a_2$ and $a_6$ to zero, and so we obtain a Hopf algebroid $(F_2, C)$ with

$$C = F_2 \otimes_{\overline{D}} \Sigma \otimes_{\overline{D}} F_2 \cong F_2[a_2, a_6, s, t]/(a_2, a_6, \eta_R(a_2), \eta_R(a_6)) \cong F_2[s, t]/(s^2, t^2) = E(s, t).$$

To emphasize the connection to homotopy theory, we write $h_1$ for $s$ and $h_2$ for $t$. Now, the map $\overline{D} \to F_2 \otimes_{\overline{D}} \Sigma = F_2[h_1, h_2]$ given by sending $a_2$ to $h_1$ and $a_6$ to $h_2^2$ is faithfully flat, and defines an isomorphism $(\overline{D}, \Sigma) \to (F_2, C)$ of Hopf algebroids. Moreover, the Hopf algebroid $(F_2, C)$ presents the (graded) stack $B\text{O}_2 \times \text{B} \text{O}_2$ over $\text{Spec}(F_2)$. It follows that

$$\text{Ext}^{p,n}_{\Sigma}(\overline{D}, \text{Sym}^q_D(I/I^2)) \cong \text{Ext}^{p,n}_C(F_2, \text{Sym}^q_{F_2}(V)) = \text{H}^{p,n}(B\text{O}_2 \times \text{B} \text{O}_2; \text{Sym}^q(V)).$$

The comodule structure on $I/I^2$ appearing in Equation (8) is a representation of $a_2 \times a_2$ on $V$, and so:

$$(9) \quad \text{Ext}^{p,n}_C(F_2, \text{Sym}^q_{F_2}(V)) = V^{a_2 \times a_2} = F_2[\overline{a}_0, \overline{a}_1^2, \overline{a}_0 \overline{a}_4 + \overline{a}_1 \overline{a}_3, \overline{a}_1^2, \overline{a}_2^2].$$

Indeed, these are the invariants under the $a_2 \times a_2$-action on $V$. Moreover, by looking at the expressions for $b_2$, $b_4$, $b_6$, and $b_8$, we find that they are represented by $\overline{a}_1^2$, $\overline{a}_0 \overline{a}_4 + \overline{a}_1 \overline{a}_3$, $\overline{a}_1^2$, and $\overline{a}_2^2$, respectively; in particular, all of the generators of $V^{a_2 \times a_2}$ are permanent cycles in the Bockstein
spectral sequence. Moreover, $H^{*,*}(BO_2 \times BO_2; \text{Sym}^3(V)) \cong F_2$ for $q \geq 1$, and for $q = 0$ we have $H^{*,*}(BO_2 \times BO_2; \text{Sym}^0(V)) \cong F_2[h_1, h_{21}]$, where $h_1 = [s]$ and $h_{21} = [t]$. Together with Equation (9), we obtain a complete calculation of the $E_1$-page of the Bockstein spectral sequence (7).

The right unit in Equation (5) gives Bockstein differentials $a_1 \mapsto 2h_1$ and $a_3 \mapsto 2h_{21}$, which correspond to $2\eta$ and $2\sigma_1$ being null, respectively. Moreover, $$d(a_1^2) = a_3^2 s^2 + a_1^2 t^2 + 4s^2 t^2 + 4a_1 st - 2a_1 a_3 st - 4a_3 s^2 t$$ in the cobar complex, and $$s^2 b_6 + t^2 b_2 = a_3^2 s^2 + a_1^2 t^2 + 4a_6 s^2 + 4a_2 t^2.$$ Since $a_2 = 0$ and $a_6 = 0$, we have $a_1 s = -s^2$ and $a_3 t = -t^2$. This produces a differential with target $h_1^2 b_6 + h_2^2 b_2$. Combining together all of these facts gives Equation (6) as the cohomology of the moduli stack $M_{\text{cub}}^{\text{DR}}$. As in Corollary 3.8, this is the $E_2$-page of the Adams–Novikov spectral sequence for $\text{tmf}/\nu$. We now calculate the Adams–Novikov differentials. The 15-skeleton of $A$ is as shown in Figure 1.

We know that $\eta^3 = 4\nu$ vanishes in $\pi_* A$, so $h_1^2$ must die in the Adams–Novikov spectral sequence for $\text{tmf}/\nu$. There is only one possibility, namely the $d_3$-differential $d_3(b_2) = h_1^2$. (Note that this differential already exists in the Adams–Novikov spectral sequence for $A$, where $b_2$ is represented by $v_1^2$, i.e., the class $[y_2]$ in the cobar complex (via Proposition 3.3).) As a consequence, both $\eta$ and $\sigma_1$ are permanent cycles in the Adams–Novikov spectral sequence for $A$.

Next, we know from Proposition 3.3 that $\sigma_1$ is detected in the Adams–Novikov spectral sequence for $A$ by $h_{21}$. Since $\sigma_1 \in \langle \eta, \nu, 1_A \rangle$, one has that $\eta^2 \sigma_1 = 0$ in $\pi_* A$, so $h_1^2 h_{12}$ must die in $\text{tmf}/\nu$. There is no possibility other than $d_3(b_4)$ for a differential to kill $h_1^2 h_{21}$ (except for a $d_3$-differential on $b_2^2$, but this cannot kill $h_1^2 h_{12}$). Note that $h_1^2 h_{12}$ is a permanent cycle and represents $\eta \sigma_1$.

For the third differential, note that since $\sigma_1 \in \langle \eta, \nu, 1_A \rangle$, we have $\eta^2 \sigma_1^2 = 0$. There is only one possibility, namely $d_3(b_6) = h_1 h_{21}$. (Since there is a $d_3$-differential $d_3(b_2) = h_1^2$, we have $d_3(h_2^2 b_2) = h_1^2 h_{21}$. But there is a relation $h_1^2 b_6 = h_2^2 b_2$, so $d_3(h_1^2 b_6) = h_1^2 h_{21}^2$, which also forces $d_3(b_6) = h_1 h_{21}^2$.) Note that $h_1^2 h_{21}$ is a permanent cycle and represents $\sigma_1^2$.

Finally, $\sigma_1^2 = 0$ in $\pi_* A$ (and therefore in $\text{tmf}/\nu$). It follows that the element $h_2^3$ must be the target of a differential in the Adams–Novikov spectral sequence for $\text{tmf}/\nu$. The only possibilities are a $d_3$-differential on $h_2^2, b_2 b_6$, or $b_6$. Only $b_6$ can kill $h_2^3 b_{21}$, and we conclude the $d_3$-differential $d_3(b_6) = h_2^3 b_{21}$. At this point, there are no more possibilities for differentials in the descent/Adams–Novikov spectral sequence for $\text{tmf}/\nu$, and the spectral sequence collapses at the $E_4$-page.

\[\square\]

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