The JLO Character for The Noncommutative Space of Connections of Aastrup-Grimstrup-Nest

Alan Lai ∗
University of Toronto
October 26, 2010

Abstract

In attempts to combine non-commutative geometry and quantum gravity, Aastrup-Grimstrup-Nest construct a semi-finite spectral triple, modeling the space of $G$-connections for $G = U(1)$ or $SU(2)$. AGN show that the interaction between the algebra of holonomy loops and the Dirac-type operator $D$ reproduces the Poisson structure of General Relativity in Ashtekar’s loop variables. This article generalizes AGN’s construction to any connected compact Lie group $G$. A construction of AGN’s semi-finite spectral triple in terms of an inductive limit of spectral triples is formulated. The refined construction permits the semi-finite spectral triple to be even when $G$ is even dimensional. The Dirac-type operator $D$ in AGN’s semi-finite spectral triple is a weighted sum of a basic Dirac operator on $G$. The weight assignment is a diverging sequence that governs the “volume” associated to each copy of $G$. The JLO cocycle of AGN’s triple is examined in terms of the weight assignment. An explicit condition on the weight assignment perturbations is given, so that the associated JLO class remains invariant. Such a condition leads to a functoriality property of AGN’s construction.

Contents

0 Introduction 2
1 Inductive Limit of Spectral Triples 2
2 The Noncommutative Space of Connections of AGN 6
  2.1 Graphs and Spectral Triples on Graphs 6
  2.2 A Compactification of the Space of Connections 12
  2.3 The Limit of Spectral Triples on Graphs 13
  2.4 Semi-finite Spectral Triple 18
3 JLO Theory 20
  3.1 Entire Cyclic Cohomology 20
  3.2 The JLO Character 21
  3.3 Homotopy Invariance of the JLO Class 21
4 The JLO Class of AGN’s Space of Connections 22
  4.1 Weight Independence of the JLO Class 22
  4.2 Weak $\theta$-summability 24

*Email: alan@math.toronto.edu


\section*{Introduction}

In non-commutative Geometry, a space is represented by a \(\ast\)-algebra and the geometry on the space is given by an unbounded self-adjoint operator \(\mathcal{D}\) on a Hilbert space, subject to certain axioms. We call such a package a spectral triple. A typical example of a spectral triple is the Dirac triple \((C^\infty(X), B(L^2(X, S)), \mathcal{D})\) where \(S\) is the spinor bundle of the spin manifold \(X\) and \(\mathcal{D}\) is the Dirac operator of \(X\) acting on \(L^2(X, S)\). Connes’ results \cite{Connes} state that geometric features of the manifold, such as metric, dimension, differential forms, and integrations etc can be retrieved algebraically from the spectral triple. Therefore, a spectral triple gives a non-commutative notion of manifolds when the given \(\ast\)-algebra is more general than the space of functions on a manifold. Recently, the notion of spectral triples is further generalized to model foliated manifolds or infinite dimensional manifolds that carry degeneracies \cite{Deitmar}, these generalized spectral triples are called semi-finite spectral triples. One notable example is the non-commutative space of connections by Aastrum-Grimstrup-Nest \cite{AGN}, which is the main focus in this article.

In attempts to combine non-commutative geometry and quantum gravity, Aastrup-Grimstrup-Nest construct a semi-finite spectral triple \((\mathcal{B}, \mathcal{N}, \mathcal{D})\), modeling the space of \(G\)-connections for the symmetry group \(G = U(1)\) or \(SU(2)\). AGN show that the interaction between the algebra of holonomy loops \(\mathcal{B}\) and the Dirac-type operator \(\mathcal{D}\) reproduces the Poisson structure of General Relativity in Ashtekar’s loop variables \cite{Ashtekar, Biswas, Biswas2, AGN}, they argue that \((\mathcal{B}, \mathcal{N}, \mathcal{D})\) incorporates quantum gravity in this model.

Unfortunately, building a (semi-finite) spectral triple over the ordinary space of smooth connections like the Dirac triple is impossible, as there does not exist a Hilbert space structure and Dirac operator on the infinite dimensional affine space of connections \(\mathcal{A}\). One works instead with a sequence of approximations of \(\mathcal{A}\) by finite-dimensional manifolds. In the Aastrup-Grimstrup-Nest-approach, they compactify the space of connections over the manifold \(M\) by making use of a finite graph together with its refinements in \(M\) to construct a separable kinematical Hilbert space and put a Dirac operator on it. Loosely speaking, the algebra \(\mathcal{B}\) is the pre-C\(^\ast\) algebra of holonomies restricted to the system of graphs in \(M\), which mimics the holonomy algebra of Wilson loops; the densely defined operator \(\mathcal{D}\) is an infinite sum of a basic Dirac operator on the symmetry group \(G\) with appropriate weight assigned to each copy, and \(\mathcal{N}\) is a Type II\(\infty\) von Neumann algebra containing the CAR algebra.

For technical reasons, Aastrup-Grimstrup-Nest limit their construction to the symmetry group \(G = U(1)\) or \(SU(2)\). In this article, we generalize AGN’s construction to any connected, compact Lie group \(G\) by eliminating the technical restriction. Our construction also permits the semi-finite spectral triple to carry a \(\mathbb{Z}_2\) grading when \(G\) is even dimensional, e.g. \(G = SU(3)\), the symmetry group that governs the strong force in quantum field theory. In a recent paper \cite{Paschke}, the JLO character for semi-finite spectral triples has been established. We will examine the entire cyclic cohomology class associated to the semi-finite spectral triple of AGN via the JLO character. In particular, we give an explicit condition on allowable perturbations of the given weight assignment so that the associated JLO class remains invariant. In a more recent paper \cite{Paschke2}, Aastrup-Grimstrup-Paschke-Nest specializes their construction to lattice graphs, which results in the most reasonable choice of weight assignment that depends only on the dimension of the base manifold \(M\). If one re-runs AGN’s construction on a sub-manifold with dimension less than that of \(M\), the resulting spectral triple will be defined using the weight assignment corresponding to the sub-manifold, which is different from the spectral triple obtained from pulling back the construction on the full manifold. We prove a functoriality property of AGN’s construction by showing that the JLO cocycles of the pull back triple and the triple constructed on a sub-manifold define the same entire cyclic cohomology class.

This paper is arranged as follows. In Section 1, we will develop a limit for an inductive system of spectral triples. Section 2 gives an alternative construction of the semi-finite spectral triple of Aastrup-Grimstrup-Nest using the formalism developed in Section 1. In Section 3, we review the JLO theory for semi-finite spectral triples developed in \cite{Paschke}. In Section 4, we examine the JLO class associated to AGN’s semi-finite spectral triple and the weak \(\theta\)-summability of the operator \(\mathcal{D}\).

\section{Inductive Limit of Spectral Triples}

\begin{definition}
An odd semi-finite spectral triple \((\mathcal{B}, \mathcal{N}, \mathcal{D})\) is a (separable) semi-finite von Neumann algebra \(\mathcal{N} \subset B(H)\), \(\ast\)-sub-algebra \(\mathcal{B}\) of \(\mathcal{N}\), and a densely defined unbounded self-adjoint operator \(\mathcal{D}\) affiliated with \(\mathcal{N}\) such that,
\end{definition}
1. $[\mathcal{D}, b]$ extends to a bounded operator for all $b \in \mathcal{B}$;

2. $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{K}_\mathcal{N}$, where $\mathcal{K}_\mathcal{N}$ is the ideal of $\tau$-compact operators in $\mathcal{N}$.

If $(\mathcal{B}, \mathcal{N}, \mathcal{D})$ is equipped with a $\mathbb{Z}_2$ grading $\chi \in \mathcal{N}$ such that all $a$ is even for all $a \in \mathcal{B}$ and $\mathcal{D}$ is odd, then we call $(\mathcal{B}, \mathcal{N}, \mathcal{D})$ an even semi-finite spectral triple. The suffix semi-finite is omitted when $\mathcal{N} = \mathcal{B}(\mathcal{H})$.

**Proposition 1.1** ([7]). Let $\mathcal{D}$ be an operator affiliated with $\mathcal{N}$, and suppose that $T \in \mathcal{N}$ and that $[\mathcal{D}, T]$ is bounded. Then $[\mathcal{D}, T] \in \mathcal{N}$.

**Definition 1.2.** A morphism from an even semi-finite spectral triple $(\mathcal{B}, \mathcal{N}, \mathcal{D})$ with grading $\chi \in \mathcal{N}$ to an even semi-finite spectral triple $(\mathcal{B}', \mathcal{N}', \mathcal{D}')$ with grading $\chi' \in \mathcal{N}'$ is a triple $(\iota, P, Q)$, where

1. $\iota : \mathcal{H} \to \mathcal{H}'$ is a linear map between the underlying Hilbert spaces of $\mathcal{N}$ and $\mathcal{N}'$ such that it preserves the inner products, and $\iota(\text{Dom}(\mathcal{D})) \subset \text{Dom}(\mathcal{D}')$ so that the following diagram commutes:

   \[
   \begin{array}{ccc}
   \text{Dom}(\mathcal{D}) & \xrightarrow{\mathcal{D}} & \mathcal{H} \\
   \downarrow \iota & & \downarrow \\
   \text{Dom}(\mathcal{D}') & \xrightarrow{\mathcal{D}'} & \mathcal{H}'
   \end{array}
   \] (1)

2. $P : \mathcal{N} \to \mathcal{N}'$ is a $*$-homomorphism so that the following diagram commutes for all $a \in \mathcal{N}$:

   \[
   \begin{array}{ccc}
   \mathcal{H} & \xrightarrow{a} & \mathcal{H} \\
   \downarrow \iota & & \downarrow \\
   \mathcal{H}' & \xrightarrow{P(a)} & \mathcal{H}'
   \end{array}
   \] (2)

3. $Q : \mathcal{B} \to \mathcal{B}'$ is a $*$-homomorphism so that the following diagram commutes for all $b \in \mathcal{B}$:

   \[
   \begin{array}{ccc}
   \mathcal{H} & \xrightarrow{b} & \mathcal{H} \\
   \downarrow \iota & & \downarrow \\
   \mathcal{H}' & \xrightarrow{Q(b)} & \mathcal{H}'
   \end{array}
   \] (3)

4. The following diagram commutes:

   \[
   \begin{array}{ccc}
   \mathcal{H} & \xrightarrow{\chi} & \mathcal{H} \\
   \downarrow \iota & & \downarrow \\
   \mathcal{H}' & \xrightarrow{\chi'} & \mathcal{H}'
   \end{array}
   \] (4)

A morphism between odd semi-finite spectral triples is which Definition 1.2 with the last condition dropped.
Definition 1.3. An inductive system of semi-finite spectral triples is an $I$-family of semi-finite spectral triples $\{(B_i,N_i,D_i)\}_{i \in I}$ for a directed set $I$ and together with a collection of morphisms $\{(Q_{ij},P_{ij},\iota_{ij})\}_{i < j}$ so that the diagram

\[
\begin{array}{ccc}
(B_i,N_i,D_i) & \xrightarrow{(Q_{ij},P_{ij},\iota_{ij})} & (B_j,N_j,D_j) \\
\downarrow & & \downarrow \\
(Q_{ik},P_{ik},\iota_{ik}) & \xrightarrow{(Q_{jk},P_{jk},\iota_{jk})} & (B_k,N_k,D_k)
\end{array}
\]

commutes for $i < j < k \in I$.

Denote the limit of the Hilbert space systems $\{(\mathcal{H}_i,\iota_{ij})\}$ by $\lim_{\to} \mathcal{H}_i$, which is the Hilbert space closure of $\cup_{i \in I} \mathcal{H}_i$; the limit of the $*$-algebra systems $\{(B_i,Q_{ij})\}$ by $\lim_{\to} B_i$; the limit of the von Neumann algebra systems $\{(N_i,P_{ij})\}$ with underlying Hilbert space $\lim_{\to} \mathcal{H}_i$ by $\lim_{\to} N_i$, which is the weak operator closure of $\cup_{i \in I} \mathcal{N}_i$. Since each $\mathcal{H}_i$ is a subspace of $\lim_{\to} \mathcal{H}_i$, each $D_j$ on $\mathcal{H}_j$ extends to an operator on $\lim_{\to} \mathcal{H}_i$ by zero action on $\mathcal{H}^+_j$. Define the limit of the net of operators $\{D_j\}_{j \in I}$ acting on $\lim_{\to} \mathcal{H}_j$ to be

\[
(\lim_{\to} D_j)\eta := \lim_{\to}(D_j\eta)
\]

for $\eta$ in the appropriate domain (to be clarified below) in $\lim_{\to} \mathcal{H}_j$, where the right hand side is the limit of the net of vectors $\{(D_j\eta)\}_{j \in I} \subset \lim_{\to} \mathcal{H}_j$. Denote by $\lim_{\to} \chi_j$ the strong operator limit of the net of grading operators $\{P_j\iota\chi_j\}_{j \in I}$ in $\lim_{\to} \mathcal{N}_i$. The following theorem justifies that $\lim_{\to} D_j$ and $\lim_{\to} \chi_j$ are well-defined operators on $\lim_{\to} \mathcal{H}_j$.

Theorem 1.2. Let $\{(B_j,N_j,D_j)\}_{j \in I}$ be an inductive system of spectral triples. Then

1. $\lim_{\to} D_j$ is an essentially self-adjoint operator on $\lim_{\to} \mathcal{H}_j$. Denote its unique closure again by $\lim_{\to} D_j$.
2. $\lim_{\to} D_j$ is the strong resolvent limit of $\mathcal{D}_j$ and it is affiliated with $\lim_{\to} \mathcal{N}_j$.
3. The commutator

\[
[\lim_{\to} D_j,b]
\]

is bounded for all $b \in \lim_{\to} B_j$.
4. If each $(B_j,\mathcal{N}_j,D_j)$ is even equipped with a grading operator $\chi_j$, then
   a. $\lim_{\to} D_j$ anti-commutes with $\lim_{\to} \chi_j$ and
   b. $b$ commutes with $\lim_{\to} \chi_j$ for all $b \in \lim_{\to} B_j$.

Proof.

1.4a. Let $\eta \in \lim_{\to} \text{Dom}(D_j)$, then there exists $n \in I$ such that $\eta \in \text{Dom}(D_n)$. Condition 1 of Definition 1.2 assures that the sequence $\{D_j\eta\}$ stabilizes for $j \geq n$. We compute

\[
(\lim_{\to} D_j)\eta := \lim_{\to}(D_j\eta) = D_n\eta \in \lim_{\to} \mathcal{H}_j.
\]

Therefore, $\lim_{\to}(D_j)$ is well-defined on $\lim_{\to} \text{Dom}(D_j)$. As $\text{Dom}(D_j)$ is dense in $\mathcal{H}_j$ for each $j$, $\lim_{\to} \text{Dom}(D_j)$ is dense in $\lim_{\to} \mathcal{H}_j$.

Each $D_j$ is self-adjoint on $\mathcal{H}_j$, thus the image $\text{Im}(D_j + \sqrt{-1}) = \mathcal{H}_j$. Since $(\lim_{\to} D_j + \sqrt{-1})\eta = \lim_{\to}((D_j + \sqrt{-1})\eta)$, the image $\text{Im}(\lim_{\to} D_j + \sqrt{-1})$ is the vector space limit $\lim_{\to} \mathcal{H}_j$, which is dense in the Hilbert space limit $\lim_{\to} \mathcal{H}_j$. As a result, $\lim_{\to} D_j$ is essentially self-adjoint.

Similarly, Condition 4 of Definition 1.2 justifies that $\lim_{\to} \chi_j$ is well-defined on $\lim_{\to} \mathcal{H}_j$. On the other hand,

\[
(\lim_{\to} \chi_j)(\lim_{\to} D_j)\eta = (\lim_{\to} \chi_j)(\lim_{\to} D_n)\eta = \chi_n D_n\eta
\]

\[
= -D_n\chi_n\eta = (\lim_{\to} D_j)(\chi_n\eta) = (\lim_{\to} D_j)(\lim_{\to} \chi_j)\eta.
\]

Hence $\lim_{\to} D_j$ is odd with respect to $\lim_{\to} \chi_j$. 

4
2. By the “point-wise” construction (5), \( \lim D_j \) is the strong graph limit of \( D_j \), which implies that \( \lim D_j \) is the strong resolvent limit of \( D_j \) [17]. As \( D_j \) is affiliated with \( N_j \) for each \( j \), the sign and spectral projections of \( D_j \) are in \( \lim N_j \). The fact that \( \lim N_j \) is strong operator closed implies that \( \lim D_j \) is affiliated with \( \lim N_j \).

3. Let \( b \in \lim B_j \), then there exists \( n \) so that \( b \in B_n \). We compute for \( \eta \in \lim H_j \)

\[
[\lim D_j, b] \eta = [D_n, b] \eta,
\]

which is bounded since \( (B_n, N_n, D_n) \) is assumed to be a spectral triple.
On the other hand,

\[
[\lim \chi_j, b] \eta = [\chi_n, b] \eta = 0.
\]

Hence \( b \) is even with respect to \( \lim \chi_j \).

\[ \square \]

Notice that \( \lim D_j \) being the strong resolvent limit of \( D_j \) allows us to obtain functional calculus on \( \lim D_j \) as strong limits of functional calculi on \( D_j \).

**Theorem 1.3** ([17]). Let \( T_j \) and \( T \) be self-adjoint operators such that \( T_j \rightarrow T \) in the strong resolvent sense. Then for any bounded continuous function \( f \) on \( \mathbb{R} \), \( f(T_j) \rightarrow f(T) \) in the strong operator limit.

**Definition 1.4.** Let \( \{(B_j, N_j, D_j)\}_{j \in I} \) be an inductive system of even semi-finite spectral triples with grading \( \{\chi_j\}_{j \in I} \), define its limit to be

\[
\lim_{\rightarrow} (B_j, N_j, D_j) := \left( \lim_{\rightarrow} B_j, \lim_{\rightarrow} N_j, \lim_{\rightarrow} D_j \right).
\]

It is equipped with the \( \mathbb{Z}_2 \) grading \( \lim \chi_j \).

The definition for the odd limit is obvious.

Unfortunately, the limit of a system of semi-finite spectral triples needs not be a semi-finite spectral triple. For instance, \( I \) could be uncountable. Even if we assume \( I \) to be countable the limit may still not be a semi-finite spectral triple. We will see both example and non-example from AGN’s construction of noncommutative connection space. Nonetheless, the inductive limit of spectral triples satisfies the following universal condition.

**Theorem 1.4.** Let \( \{(B_j, N_j, D_j)\}_{j \in I} \) be an inductive system of even semi-finite spectral triples with grading \( \{\chi_j\}_{j \in I} \), and suppose that \( \{(B', N', D')\} \) is an even semi-finite spectral triple with grading \( \chi' \) such that there exist morphisms of spectral triples \((Q_j, P_j, \iota_j)\) such that the following diagram commutes:

\[
\begin{array}{c}
(B_1, N_1, D_1) \xrightarrow{(Q_{12}, P_{12}, \iota_{12})} (B_2, N_2, D_2) \xrightarrow{(Q_{23}, P_{23}, \iota_{23})} (B_3, N_3, D_3) \rightarrow \cdots \\
\end{array}
\]

\[
\begin{array}{c}
(B', N', D') \xrightarrow{(Q'_1, P'_1, \iota'_1)} (Q'_2, P'_2, \iota'_2) \xrightarrow{(Q'_3, P'_3, \iota'_3)} \cdots
\end{array}
\]
Then there exists a unique morphism \((Q, P, \iota)\) completing the following diagram:

\[
\begin{array}{ccc}
(B_i, N_i, D_i) & \xrightarrow{(Q_{ij}, P_{ij}, \iota_{ij})} & (B_j, N_j, D_j) \\
\downarrow & & \downarrow \\
\lim_{\to} (B_k, N_k, D_k) & \xrightarrow{(Q_{ij}, P_{ij}, \iota_{ij})} & \lim_{\to} (B_j, N_j, D_j) \\
\downarrow & & \downarrow \\
(Q', P', \iota') & \xrightarrow{(Q_{ij}, P_{ij}, \iota_{ij})} & (Q', P', \iota') \\
\end{array}
\]

for all \(i, j \in I\).

**Proof.** The existence and uniqueness of \(Q, P, \) and \(\iota\) come from the universalities of \(\lim_{\to} B_k, \lim_{\to} N_k, \) and \(\lim_{\to} H_k\). \(\iota(\lim_{\to} \text{Dom}(D_k)) \subset \text{Dom}(D')\) and the existence of the commutative diagrams

\[
\begin{array}{ccc}
\lim_{\to} \text{Dom}(D_k) & \xrightarrow{\lim_{\to} D_k} & \lim_{\to} H_k \\
\downarrow & & \downarrow \\
\text{Dom}(D') & \xrightarrow{D'} & \mathcal{H}' \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\lim_{\to} H_k & \xrightarrow{\lim_{\to} \chi_j} & \lim_{\to} H_j \\
\downarrow & & \downarrow \\
\mathcal{H}' & \xrightarrow{\chi'} & \mathcal{H}' \\
\end{array}
\]

follow by constructions. \(\square\)

## 2 The Noncommutative Space of Connections of Aastrup-Grimstrup-Nest

In this section, we give an alternative construction for AGN’s semi-finite spectral triple that models the space of \(G\)-connections. We make use of the inductive limit formalism developed in Section 1 to formulate AGN’s triple as a limit of a sequence well-behaved spectral triples. In Section 2.1, we will construct a spectral triple on a graph. Then in Section 2.2, we follow AGN’s idea to compactify the space of connections using a graph and its refinements in a manifold. Section 2.3 constructs a corresponding system of spectral triples from the system of a graph and its refinements. Section 2.4 will alter the spectral triple system constructed in Section 2.3 appropriately so that the limit of the system is a semi-finite spectral triple, and discuss the grading operator on the limit spectral triple.

### 2.1 Graphs and Spectral Triples on Graphs

#### 2.1.1 Space of Connections on Graphs

**Definition 2.1.**

- A **directed graph** \(\Gamma\) is a set \(V_\Gamma\) (vertices) and a set \(E_\Gamma\) (edges) with two maps \(s, r : E_\Gamma \to V_\Gamma\) (source and range).
• **A morphism of graphs** \( \Gamma \rightarrow \Gamma' \) consists of maps \( E_\Gamma \rightarrow E_{\Gamma'} \) and \( V_\Gamma \rightarrow V_{\Gamma'} \), so that they intertwine the source and range maps.

• **We call a directed graph** \( \Gamma \) **finite** if the sets \( E_\Gamma \) and \( V_\Gamma \) have finite cardinalities.

We view the vertices as a collection of points, and the edges as arrows from \( s(e) \) to \( r(e) \).

**Example 2.1.**

- Any groupoid \( G \xrightarrow{\gamma} X \) is a directed graph.
- Any subset of a groupoid \( G \) is a directed graph.
- Any set \( S \) can be viewed as a graph by taking \( E_\Gamma := S \) and \( V_\Gamma = \{ \text{pt} \} \).

**Theorem 2.1.** Given a directed graph \( \Gamma \), there is a unique groupoid \( G(\Gamma) \xrightarrow{\gamma} \) with a graph morphism \( \Gamma \rightarrow G(\Gamma) \), so that given any groupoid \( G \xrightarrow{\gamma} X \) with a graph morphism \( \Gamma \rightarrow G \), there exists a unique groupoid morphism \( G(\Gamma) \rightarrow G \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\gamma} & G(\Gamma) \\
& | & \\
& | & \\
& | & \\
& V & \\
& | & \\
& | & \\
& | & \\
& | & \\
& & G
\end{array}
\]

\( G(\Gamma) \) is called the **free groupoid generated by** \( \Gamma \).

**Remark 2.2.** If \( V_\Gamma = \{ \text{pt} \} \), then \( \Gamma \) is just a set given by \( E_\Gamma \). In that case, \( G(\Gamma) \) is the **free group** generated by the set \( E_\Gamma \).

**Definition 2.3.** Let \( G(\Gamma) \) be the free groupoid generated by \( \Gamma \). The subset \( \mathfrak{F}_\Gamma \subset G(\Gamma) \) is called the **free generating set of** \( G(\Gamma) \) if given any groupoid \( G \xrightarrow{\gamma} X \) and a set map \( \mathfrak{F}_\Gamma \rightarrow G \), there exists a unique groupoid morphism \( G(\Gamma) \rightarrow G \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathfrak{F}_\Gamma & \xrightarrow{\gamma} & G(\Gamma) \\
& | & \\
& | & \\
& | & \\
& V & \\
& | & \\
& | & \\
& | & \\
& | & \\
& & G
\end{array}
\]

**Example 2.2.**

- Let \( \Gamma \) be a graph, then \( E_\Gamma \subset G(\Gamma) \) is a free generating set.
- Let \( \Gamma \) be the graph

\[
\Gamma: \quad \bullet \xrightarrow{e^-} \bullet \xrightarrow{e^+} \bullet
\]

then the set composed of the paths \( (e^-, e^+) \) and \( (e^+) \) forms a free generating set for \( G(\Gamma) \).
Remark 2.4. Let $\Gamma$ be a finite graph, then cardinality of $\mathfrak{F}_\Gamma$ equals that of $E_\Gamma$. Since every free generating set necessarily has the same cardinality.

Proposition 2.2. For any groupoid $\mathcal{G}$, there is a bijection between the set of groupoid homomorphisms $\mathcal{G}(\Gamma) \to \mathcal{G}$ and the set of maps $\mathfrak{F}_\Gamma \to \mathcal{G}$.

Proof. It follows from Definition 2.3 that the space of set maps $\text{Map}(\mathfrak{F}_\Gamma, \mathcal{G})$ injects into the space of set homomorphisms $\text{Hom}(\mathcal{G}(\Gamma), \mathcal{G})$. As sets, $\mathfrak{F}_\Gamma \subset \mathcal{G}(\Gamma)$. Hence every groupoid homomorphism in $\text{Hom}(\mathcal{G}(\Gamma), \mathcal{G})$ restricts to a set map in $\text{Map}(\mathfrak{F}_\Gamma, \mathcal{G})$.

Corollary 2.3. For any groupoid $\mathcal{G}$, there is a bijection between the set of groupoid homomorphisms $\mathcal{G}(\Gamma) \to \mathcal{G}$ and the set of map $\text{graph homomorphisms} \Gamma \to \mathcal{G}$.

Proof. Choose the generating set $\mathfrak{F}_\Gamma$ to be the set of edges $E_\Gamma$.

Definition 2.5. Given a group $G$, define the space of $G$-connections on $\Gamma$ to be

$$A_\Gamma := \text{Hom}(\mathcal{G}(\Gamma), G),$$

where $\text{Hom}$ is understood to be the space of groupoid homomorphisms.

Corollary 2.4. Let $\Gamma$ be a finite graph, then there exists a bijection

$$A_\Gamma \xrightarrow{\sim} G^{|E_\Gamma|} \quad (6)$$

Proof. Proposition 2.2 specializes to $A_\Gamma \cong \text{Map}(\mathfrak{F}_\Gamma, G)$. By Remark 2.4, $\text{Map}(\mathfrak{F}_\Gamma, G) = G^{|E_\Gamma|}$. The result is obtained.

$G$ is thought of as a symmetry group, so we will assume $G$ to be a compact Lie group, and it comes equipped with the normalized Haar measure. We equip $A_\Gamma$ the manifold structure and measure coming from $G^{|E_\Gamma|}$ under the identification (6). Hence $A_\Gamma$ is a compact manifold with a smooth measure. Note that the manifold structure on $A_\Gamma$ depends on the choice of free generating set $\mathfrak{F}_\Gamma$. However, we believe that the measure on $A_\Gamma$ is intrinsic, i.e. it does not depend on the choice of generating set. Most of the time we think of the generating is just $E_\Gamma$. It will be made clear in later sections that when and why we generalize to other free generating sets.

The gauge group of the connection space $A_\Gamma$ is $\text{Gau}_\Gamma := \text{Map}(V_\Gamma, G)$. The action of $\text{Gau}_\Gamma$ on $A_\Gamma$ is given by

$$g(\nabla)(\gamma) := g(s(\gamma)) \cdot \nabla(\gamma) \cdot g(r(\gamma))^{-1}$$

for $g \in \text{Gau}_\Gamma$, $\gamma \in \mathcal{G}(\Gamma)$, and $\nabla \in A_\Gamma$. $\text{Gau}_\Gamma$ preserves the measure on $A_\Gamma$.

Definition 2.6. Fix a vertex $\nu$ of a graph $\Gamma$. Define the isotropy group of $\Gamma$ at $\nu$ to be the group

$$G_\nu(\Gamma) := \{ \gamma \in \mathcal{G}(\Gamma) : s(\gamma) = \nu = r(\gamma) \}.$$

Suppose that $\gamma \in \mathcal{G}(\Gamma)$ such that $s(\gamma) = \nu$ and $r(\gamma) = \mu$ for some vertices $\nu, \mu \in V_\Gamma$. Then it is easy to see that $G_\nu(\Gamma)$ and $G_\mu(\Gamma)$ are isomorphic as groups with the isomorphism given by conjugation by $\gamma$.

Definition 2.7. A graph $\Gamma$ is said to be connected if the groupoid $\mathcal{G}(\Gamma)$ is transitive. That is, for every pair of vertices $\nu, \mu \in V_\Gamma$, there exists an element $\gamma \in \mathcal{G}(\Gamma)$ such that $s(\gamma) = \nu$ and $r(\gamma) = \mu$.

We will assume that all the graphs we are dealing with are connected.
2.1.2 The Algebra of Holonomies

Each $\gamma \in G_0(\Gamma)$ defines a smooth $G$-valued function on $A_{\Gamma}$ given by
\[
h_\gamma(\nabla) := \nabla(\gamma), \text{ for } \nabla \in A_{\Gamma} = \text{Hom}(G_0(\Gamma), G).
\] (7)

Thus,
\[
h : G_0(\Gamma) \to C^\infty(A_{\Gamma}, G).
\]

In fact, $h$ is a group homomorphism, as
\[
h_{\gamma_2 \gamma_1}(\nabla) = \nabla(\gamma_2 \circ \gamma_1) = \nabla(\gamma_1) \cdot \nabla(\gamma_2) = h_{\gamma_1}(\nabla) \cdot h_{\gamma_2}(\nabla),
\]
where the product on $C^\infty(A_{\Gamma}, G)$ is given pointwise. $h$ is the inverse of the loop transform in Loop Quantum Gravity [18].

**Definition 2.8.** Define $B_{\Gamma}$, the algebra of $\Gamma$-holonomies, to be the group algebra generated by the subgroup $h(G_0(\Gamma)) \subset C^\infty(A_{\Gamma}, G)$.

An element of $B_{\Gamma}$ is a finite sum of elements in $h(G_0(\Gamma))$ with complex coefficients:
\[
a = \sum_i a_i h_{\gamma_i},
\]
where $a_i \in \mathbb{C}$ and $h_{\gamma_i} \in h(G_0(\Gamma))$.

**Remark 2.9.** The algebra $B_{\Gamma}$ does not depend on the manifold structure on $A_{\Gamma}$, as the inverse loop transform (7) does not. Therefore, $B_{\Gamma}$ is independent of how $A_{\Gamma}$ is identified with $G[E, 1]$ under different free generating sets $\mathfrak{g}_{\Gamma}$ for $G(\Gamma)$.

Suppose that $G$ comes with a faithful unitary representation as matrices in $\text{Mat}_{\mathbb{C}}(N)$. Then $B_{\Gamma}$ is a $*$-subalgebra of $C^\infty(A_{\Gamma}, \text{Mat}_{\mathbb{C}}(N))$, with the involution given by
\[
a^* := \sum_i a_i h_{\gamma_i}^{-1}.
\]

$B_{\Gamma}$ is a pre-$C^*$ algebra with norm inherited from $C^\infty(A_{\Gamma}, \text{Mat}_{\mathbb{C}}(N))$. As a subalgebra of $C^\infty(A_{\Gamma}, \text{Mat}_{\mathbb{C}}(N))$, $B_{\Gamma}$ represents on the Hilbert space $L^2(A_{\Gamma}, E)$ via point-wise multiplication:
\[
(a \cdot \eta)\nabla := a(\nabla) \cdot \eta(\nabla)
\] (8)
for $a \in B_{\Gamma} \subset C^\infty(A_{\Gamma}, \text{Mat}_{\mathbb{C}}(N))$, $\eta \in L^2(A_{\Gamma}, E)$, $\nabla \in A_{\Gamma}$, where $E$ is any (finite dimensional) $\text{Mat}_{\mathbb{C}}(N)$-module.

2.1.3 Quantum Weil Algebra

Denote by $\mathfrak{g}$ the Lie algebra of $G$, let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Fix an invariant metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and let $\mathbb{C}l(\mathfrak{g})$ be the Clifford algebra generated by $\mathfrak{g}$ with respect to the relation $ab + ba = 2\langle a, b \rangle$ for $a, b \in \mathfrak{g}$.

**Definition 2.10.** Define the quantum Weil algebra $\mathcal{W}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to be
\[
\mathcal{W}(\mathfrak{g}) := U(\mathfrak{g}) \otimes \mathbb{C}l(\mathfrak{g}).
\]

Notice that $\mathbb{C}l(\mathfrak{g})$ is a finite dimensional $C^*$-algebra. Let
\[
\mathbb{C}l(\mathfrak{g}) \to B(S)
\]
be a cyclic representation of $\mathbb{C}l(\mathfrak{g})$ on a Hilbert space $S$ with normalized cyclic vector $\mathbb{1}$, i.e., $\mathbb{C}l(\mathfrak{g})\mathbb{1} = S$. Examples of such $S$ are the exterior algebra $\wedge^*(\mathfrak{g})$ with $\mathbb{1}$ given by the 1 in $\wedge^0(\mathfrak{g})$, and certain quotients of $\wedge^*(\mathfrak{g})$, such as the spin representations. We get an action of $\mathcal{W}(\mathfrak{g})$ on $L^2(G) \otimes S$, where $U(\mathfrak{g})$ acts on $L^2(G)$ as left-invariant differential operators.
Let \( \{ e_i \}_{i=1}^q \) be an orthonormal basis of \( \mathfrak{g} \) with respect to the chosen invariant metric. Define the Dirac operator on \( L^2(G) \otimes S \) to be the element \( D \in \mathcal{W}(\mathfrak{g}) \) given by

\[
D \:= \frac{1}{\sqrt{-1}} \sum_{i=1}^q e_i \otimes e_i \ .
\]  

(9)

\( D \) is essentially self-adjoint. Denote its unique self-adjoint extension again by \( D \).

Aastrup-Grimstrup-Nest proved the following result for \( G = SU(2) \) [4].

**Theorem 2.5.** Let \( G \) be a compact Lie group \( G \). The operator \( D \) on \( L^2(G) \otimes S \) has kernel

\[
\ker(D) = \mathbb{C} \otimes S ,
\]

where \( \mathbb{C} \subset L^2(G) \) is embedded as constant functions.

The proof will make use of the following results by Kostant [14].

**Lemma 2.6 ([14]).** The map \( \pi : \mathfrak{g} \to \mathbb{C}l(\mathfrak{g}) \) given in the orthonormal basis \( e_i \in \mathfrak{g} \) by

\[
\pi(e_a) = -\frac{1}{4} \sum_{i,j} [e_i, e_j], e_a e_i e_j
\]

is a Lie algebra homomorphism.

Fix a Cartan subalgebra and a system of positive roots of \( \mathfrak{g} \).

**Lemma 2.7 ([14]).** Let \( S \) be any \( \mathbb{C}l(\mathfrak{g}) \)-module. Then the \( \mathfrak{g} \)-representation on \( S \) defined by composition with \( \pi \) is a direct sum of \( \rho \)-representations, where \( \rho \) is the half sum of all positive roots.

Denote by \( \text{Cas} \) the element \(- \sum_i e_i e_i \in \mathcal{U}(\mathfrak{g}) \). \( \text{Cas} \) is called the Casimir [13].

**Lemma 2.8 ([13]).** Let \( V \) be an irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda \). Then \( \text{Cas} \) acts as a scalar on \( V \), and the scalar is given by

\[
|\lambda + \rho|^2 - |\rho|^2 ,
\]

where \( \rho \) is the half sum of all positive roots.

**Proof of Theorem 2.5.** As \( \ker(D) = \ker(D^2) \), we compute

\[
D^2 = -\sum_{i,j} e_i e_j \otimes e_i e_j = -\sum_k e_k e_k \otimes 1 - \frac{1}{2} \sum_{i,j} [e_i, e_j] \otimes e_i e_j .
\]

Let

\[
\pi : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}l(\mathfrak{g})
\]

be the lift of the Lie algebra homomorphism (10) defined in Lemma 2.6. Then \( D^2 \) becomes

\[
D^2 = \text{Cas} \otimes 1 + 2 \sum_{k=1}^q e_k \otimes \pi(e_k) .
\]

On the other hand,

\[
\Delta(\text{Cas}) = \text{Cas} \otimes 1 + 1 \otimes \text{Cas} - 2 \sum_{k=1}^q e_k \otimes e_k ,
\]

(11)

where

\[
\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})
\]

10
\[ D^2 = 2 \text{Cas} \otimes 1 + 1 \otimes \pi(\text{Cas}) - (1 \otimes \pi)\Delta(\text{Cas}). \]

Let
\[ T := 2 \text{Cas} \otimes 1 + 1 \otimes \text{Cas} - \Delta \text{Cas} \in \mathcal{U}(g) \otimes \mathcal{U}(g), \]
so that the action of \( D^2 \) on \( L^2(G) \otimes S \) coincides with the action of \((1 \otimes \pi)(T) \in \mathcal{U}(g) \otimes \mathcal{C}(g)\). As a \( g \oplus g \)-representation, \( L^2(G) \otimes S \) is a direct sum of components \( V_\lambda \otimes V_\rho \), for dominant weights \( \lambda \) by Lemma 2.7. We will now determine the smallest eigenvalue of \( T \) on \( V_\lambda \otimes V_\rho \). The highest weights of the irreducible components for the diagonal action are all less than or equal to \( \lambda + \rho \). Thus by Lemma 2.8, the action of \( T \) on \( V_\lambda \otimes V_\rho \) is bounded below by

\[
\begin{align*}
2 (|\lambda + \rho|^2 - |\rho|^2) + (|\rho|^2 - |\rho|^2) - (|\lambda + \rho + \rho|^2 - |\rho|^2) \\
= 2|\lambda + \rho|^2 + 2|\rho|^2 - |\lambda + 2\rho|^2 \\
= 2|\lambda|^2 + 4|\lambda, \rho| + 4|\rho|^2 - |\lambda|^2 - 4(\lambda, \rho) - 4|\rho|^2 = |\lambda|^2.
\end{align*}
\]

The bound is strictly positive unless \( \lambda = 0 \). Furthermore, the space \( V_0 \) has multiplicity 1 and embeds in \( L^2(G) \) as the constant functions. Hence, we conclude that the 0-eigenspace of \( D \) is precisely the space of constant functions \( C \otimes S \).

**Remark 2.11.** In the case that \( G \) is semi-simple and simply connected, the smallest non-zero eigenvalue of \( D^2 \) is given by \( |\rho|^2 \). When the invariant metric \( \langle \cdot, \cdot \rangle \) is chosen to be the Killing form, a theorem due to Freudenthal and de Vries [16] states that

\[ |\rho|^2 = \frac{1}{24} \text{Tr}(\text{Ad Cas}) = \frac{\dim(g)}{24}. \]

### 2.1.4 The Dirac Operator and the Hilbert Space

Let \( \Gamma \) be a finite graph. To each element \( \gamma \in \mathcal{G}(\Gamma) \), we associate to it a Hilbert space \( H_\gamma := L^2(G) \otimes S \) and an operator \( D_\gamma := D \) on \( H_\gamma \). Intuitively speaking, we are associating each path \( \gamma \) a copy of \( G \), which is thought of as coming from holonomies of \( G \)-connections along \( \gamma \). Define the Hilbert space \( \mathcal{H}_{\mathfrak{g}_\Gamma} \) to be

\[ \mathcal{H}_{\mathfrak{g}_\Gamma} := \bigotimes_{\gamma \in \mathfrak{g}_\Gamma} H_\gamma \cong L^2(G^{|E_\Gamma|}) \otimes S^{\otimes |E_\Gamma|} \]

and the Dirac operator \( D_{\mathfrak{g}_\Gamma} \) on \( \mathcal{H}_{\mathfrak{g}_\Gamma} \) to be

\[ \mathcal{W}(g^{|E_\Gamma|}) \ni D_{\mathfrak{g}_\Gamma} := \sum_{\gamma \in \mathfrak{g}_\Gamma} D_\gamma \]

with \( D_\gamma \) being the obvious extension to \( \mathcal{H}_{\mathfrak{g}_\Gamma} \), where \( \mathfrak{g}_\Gamma \) is the set of free generators of the groupoid \( \mathcal{G}(\Gamma) \), and it has cardinality equals to the number of edges \( |E_\Gamma| \). When the set of free generators \( \mathfrak{g}_\Gamma \) is \( E_\Gamma \), we will denote the corresponding Hilbert space and Dirac operator by \( \mathcal{H}_\Gamma \) and \( D_\Gamma \) respectively. At this stage, it may seem unclear if one ever needs the case of \( \mathfrak{g}_\Gamma \) not being \( E_\Gamma \). It will be made clear in Section 2.3 that this generalized definition is essential in constructing a limit Dirac operator.

We extend the action of the Quantum Weil algebra \( \mathcal{W}(g^{|E_\Gamma|}) \) to \( E \otimes \mathcal{H}_{\mathfrak{g}_\Gamma} \) by letting it act as identity on the \( \text{Mat}_C(N) \)-module \( E \), and extend the action of the algebra of \( \Gamma \)-holonomies \( \mathcal{B}_\Gamma \) on \( E \otimes \mathcal{H}_{\mathfrak{g}_\Gamma} \) to act as identity on \( S^{\otimes |E_\Gamma|} \).

**Proposition 2.9.** For any free generating set \( \mathfrak{g}_\Gamma \), the triple \((\mathcal{B}_\Gamma, B(E \otimes \mathcal{H}_{\mathfrak{g}_\Gamma}), D_{\mathfrak{g}_\Gamma})\) is a spectral triple.

**Proof.** \( D_{\mathfrak{g}_\Gamma} \) is a formally self-adjoint elliptic differential operator on \( G^{|E_\Gamma|} \). Hence it is essentially self-adjoint and has compact resolvent. The fact that \([D_{\mathfrak{g}_\Gamma}, b]\) extends to a bounded operator on \( \mathcal{H}_{\mathfrak{g}_\Gamma} \) for all \( b \in \mathcal{B}_\Gamma \) comes from the fact that \( \mathcal{B}_\Gamma \) is a sub-algebra of \( C^\infty(A_\Gamma, \text{Mat}_C(N)) \).
2.2 A Compactification of the Space of Connections

2.2.1 Systems of Graphs

Definition 2.12. A refinement \( \Gamma' \) of \( \Gamma \), denoted \( \Gamma < \Gamma' \), is a graph homomorphism \( \Gamma \to \mathcal{G}(\Gamma') \) such that the image of every edge \( e \) of \( \mathcal{E}_\Gamma \) in \( \mathcal{G}(\Gamma') \) is a product of elements in \( \mathcal{E}_{\Gamma'} \) and the induced groupoid homomorphism \( \mathcal{G}(\Gamma) \to \mathcal{G}(\Gamma') \) is injective.

Definition 2.13. Let \( S = \{\Gamma_i\}_{i \in I} \) be a family of graphs indexed by a directed set \( I \), then we call \( S \) a directed system of graphs if for \( \Gamma_i, \Gamma_j \in S \), one has \( \Gamma_i, \Gamma_j < \Gamma_k \) for \( k \in I \) with \( i, j < k \).

Notice that a directed system of graphs is itself a directed set. The original compactification of the space of smooth \( G \)-connections by Ashtekar-Lewandowski uses the set of all embedded finite graphs as the directed set [18].

By the definition of graph refinements \( \Gamma_1 < \Gamma_2 \) (Definition 2.12), a directed system of graphs \( \{\Gamma_i\}_{i \in I} \) gives rise to an inductive system of groupoids \( \{\mathcal{G}(\Gamma_i)\}_{i \in I} \) with connecting morphisms being the groupoid inclusions. This system of groupoids gives rise to a projective system of connection spaces \( \{A_{\Gamma_i}\}_{i \in I} \). When the graphs are finite and the free generating set \( F_{\Gamma_i} \) of \( \mathcal{G}(\Gamma_i) \) is chosen to be \( \mathcal{E}_{\Gamma_i} \) for each \( i \in I \), the morphisms \( A_{\Gamma_i} \to A_{\Gamma_j} \) for \( i < j \) consist of projections, and multiplications of Lie groups, under the identification (6). Hence, they are smooth surjective submersions and preserve the Haar measures. Therefore, \( \{A_{\Gamma_i}\}_{i \in I} \) is a projective system of topological measure spaces.

Definition 2.14. Let \( S = \{\Gamma_i\}_{i \in I} \) a directed system of finite graphs. Define the space of generalized \( G \)-connections \( \mathcal{A}^S \) to be the projective limit

\[
\mathcal{A}^S := \lim_{\leftarrow} \{A_{\Gamma_i}\}_{i \in I}.
\]

Proposition 2.10 ([8]). The space of generalized \( G \)-connections \( \mathcal{A}^S \) is a connected compact Hausdorff measure space.

The limit measure on \( \mathcal{A}^S \) is called the Ashtekar-Lewandowski measure [18].

Proposition 2.11 ([19]). The Hilbert space of \( L^2 \) functions on \( \mathcal{A}^S \) can be obtained as a limit of Hilbert spaces:

\[
L^2(\mathcal{A}^S) \cong \lim_{i \in I} L^2(\mathcal{A}_{\Gamma_i}).
\]

\( L^2(\mathcal{A}^S) \) is the kinematical Hilbert space in Loop Quantum Gravity [18].

2.2.2 Embedded Graphs

Let \( M \) be a compact manifold of dimension \( d \), and \( \Gamma \) be a graph in \( M \). More precisely, the vertices \( V_\Gamma \) are a set of points in \( M \), and the (directed) edges \( E_\Gamma \) are a set of non-self-intersecting piecewise smooth curves in \( M \) with starting and end points given by the source map \( s \) and range map \( r \) respectively.

Let \( M \times G \) be the trivial principal \( G \)-bundle over \( M \) with a fixed trivialization, where \( G \) is a compact Lie group. Let \( \mathcal{A} = \Omega^1(M, g) \) be the space of smooth \( G \)-connections on \( M \times G \), with gauge action of \( C^\infty(M, G) \) by

\[
g \cdot A = \text{Ad}_g(A) + gdg^{-1}.
\]

Definition 2.15.

1. Let \( \Gamma \) be a finite embedded graph in \( M \). Define the map

\[
\text{Hol}_\Gamma : \mathcal{A} \to A_{\Gamma} := \text{Hom}(\mathcal{G}(\Gamma), G)
\]

(13)

to be the holonomy of \( \nabla \) along the path \( \gamma \in \mathcal{G}(\Gamma) \), where \( \nabla \in \mathcal{A} \) is a smooth \( G \)-connection.
2. Let \( S = \{\Gamma_i\}_{i \in I} \) be a directed system of finite graphs in \( M \). Denote by \( \text{Hol} \) the map

\[
\text{Hol} : \mathcal{A} \to \overline{\mathcal{A}}^S := \lim_{\leftarrow \in I} \{\mathcal{A}_{\Gamma_i}\}_{i \in I}
\]

induced from the maps (13).

**Proposition 2.12** ([4]). Let \( G \) be a connected compact Lie group,

1. For any finite graph \( \Gamma \), the map \( \text{Hol}_\Gamma \) (13) is a surjection.
2. Let \( S = \{\Gamma_i\}_{i \in I} \) be a directed system of finite graphs in \( M \). Then \( \mathcal{A} \) has a dense image in \( \overline{\mathcal{A}}^S \) under the induced map \( \text{Hol} \) (14).

From now on, we will assume that \( G \) is connected.

**Definition 2.16.** A system of graphs \( S \) is said to be densely embedded in \( M \) if for every point \( m \in M \) there exists a coordinate chart \( x = (x_1, \ldots, x_d) \) around \( m \) such that for all open subset \( U \) containing \( m \) in this coordinate chart there exists a collection of edges \( (e_1, \ldots, e_d) \subset U \) belonging to graphs in \( S \) such that:

1. the \( e_i \) are straight lines with respect to the coordinate chart,
2. the tangent vectors of the \( e_i \) are linearly independent.

**Proposition 2.13** ([4]). Given a densely embedded system of finite graphs \( S \) in \( M \), the map \( \text{Hol} \) injects \( \mathcal{A} \) into \( \overline{\mathcal{A}}^S \).

We give two examples of graph systems that give rise to spaces of generalized connections that \( \mathcal{A} \) densely embeds into.

**Example 2.3.** Let \( T_1 \) be a triangulation of \( M \) and \( \Gamma_1 \) be the graph consisting of all the edges in this triangulation with any orientation. Let \( T_{n+1} \) be the triangulation obtained by barycentric subdividing each of the simplices in \( T_1 \) \( n \) times. The graph \( \Gamma_{n+1} \) is the graph consisting of the edges of \( T_{n+1} \) with any orientation. In this way \( S_\Delta := \{\Gamma_n\}_{n \in \mathbb{N}} \) is a directed system of finite graphs.

**Example 2.4.** Let \( \Gamma_1 \) be a finite, \( d \)-dimensional lattice in \( M \) and let \( \Gamma_2 \) be the lattice obtained by subdividing each cell in \( \Gamma_1 \) into \( 2^d \) cells. Correspondingly, let \( \Gamma_{n+1} \) be the lattice obtained by repeating \( n \) such subdivisions of \( \Gamma_0 \). In this way \( S_\Box := \{\Gamma_n\}_{n \in \mathbb{N}} \) is a directed system of finite graphs.

### 2.3 The Limit of Spectral Triples on Graphs

Given a finite graph \( \Gamma \) and a free generating set \( \mathfrak{G}_\Gamma \) for \( \mathcal{G}(\Gamma) \), we saw how one builds a spectral triple \( (B_\Gamma, H_\mathfrak{G}_\Gamma, D_\mathfrak{G}_\Gamma) \) over it in Section 2.1. For systems of graph refinements like \( \Delta \) and \( \Box \) in Examples 2.3 and 2.4, we construct a compactification of the space of connections. With these examples in mind, we will restrict our directed set \( I \) to the set of natural numbers \( \mathbb{N} \) and our graphs to be finite. We would like to obtain a spectral triple over this compactified space of connections, and the way we proceed is by taking the limit of some system of spectral triples

\[
(B_{\Gamma_1}, N_{\mathfrak{G}_{\Gamma_1}}, D_{\mathfrak{G}_{\Gamma_1}}) \to (B_{\Gamma_2}, N_{\mathfrak{G}_{\Gamma_2}}, D_{\mathfrak{G}_{\Gamma_2}}) \to (B_{\Gamma_3}, N_{\mathfrak{G}_{\Gamma_3}}, D_{\mathfrak{G}_{\Gamma_3}}) \to \cdots.
\]

In the following, we will construct the connecting morphisms \( (Q_{ij}, P_{ij}, \iota_{ij}) \) (Definition 1.2) for the collection of spectral triples \( \{(B_{\Gamma_j}, N_{\mathfrak{G}_{\Gamma_j}}, D_{\mathfrak{G}_{\Gamma_j}})\}_{j \in \mathbb{N}} \) induced from a graph system \( \{\Gamma_j\}_{j \in \mathbb{N}} \).

#### 2.3.1 Choice of Generators for \( \mathcal{G}(\Gamma) \)

It turns out that if we use the generating set \( E_{\Gamma_j} \) for each \( \mathcal{G}(\Gamma_j) \), constructing the \( \iota_{ij} \) intertwining the operators \( D_i \) and \( D_j \) is rather difficult. AGN’s solution is to introduce a “change of generators” [4], which simplifies the work of constructing the \( \iota_{ij} \). We will present the choice of generators and construct the \( \iota_{ij} \) based on the new coordinates given to the connection spaces.
Recall that when $\Gamma_j$ is a refinement of $\Gamma_i$, we obtain a morphism between the connection spaces $\mathcal{A}_{\Gamma_j} \hookrightarrow \mathcal{A}_{\Gamma_i}$ induced by the groupoid embedding $\mathcal{G}(\Gamma_j) \hookrightarrow \mathcal{G}(\Gamma_i)$, and the morphism consists of projections and multiplications under the identification (6). One would like those connecting morphisms to be as simple as possible, and we have a procedure to turn those connecting morphisms to be only composed of projections. We will demonstrate such a generator change procedure in the following.

The identification of $\mathcal{A}_\Gamma = \text{Hom}(\mathcal{G}(\Gamma), G) \cong G^{|E_\Gamma|}$ (6) is given by the $G$-assignments on the free generating set, the set of edges. However, one could use a different free generating set, then $\text{Hom}(\mathcal{G}(\Gamma), G)$ will be identified with $G^{|E_\Gamma|}$ differently. Following is a description of the choice of preferred free generating sets. Let $\mathcal{F}_1 := E_{\Gamma_1}$, so $\mathcal{F}_1$ freely generates $\mathcal{A}_{\Gamma_1}$. Since $\mathcal{G}(\Gamma_1) \hookrightarrow \mathcal{G}(\Gamma_2)$, we choose the generating set $\mathcal{F}_2$ of $\mathcal{G}(\Gamma_2)$ to be the set of images of $\mathcal{F}_1$ under the groupoid inclusion with other paths in $\mathcal{G}(\Gamma_2)$ so that $\mathcal{F}_2$ freely generates $\mathcal{G}(\Gamma_2)$. Similarly, choose the set $\mathcal{F}_3$ to be the set of images of $\mathcal{F}_2$ under the groupoid inclusion $\mathcal{G}(\Gamma_2) \hookrightarrow \mathcal{G}(\Gamma_3)$ union other paths in $\mathcal{G}(\Gamma_3)$ so that $\mathcal{F}_3$ freely generates $\mathcal{G}(\Gamma_3)$. Repeat this procedure inductively. Let us consider the following example:

First with $\mathcal{F}_1 = E_{\Gamma_1} = \{e\}$, the generating set $\mathcal{F}_2$ of $\mathcal{G}(\Gamma_2)$ will consist the image of the edge $e \in E_{\Gamma_1}$, $(e^-, e^+)$, together with another path, say $(e^+)$. Hence the generating set $\mathcal{F}_2$ for $\mathcal{G}(\Gamma_2)$ is $\mathcal{F}_2 := \{(e^-, e^+), (e^+)\}$, which freely generates $\mathcal{G}(\Gamma_2)$. We repeat the process for $\mathcal{G}(\Gamma_3)$. Choose the generators in $\mathcal{G}(\Gamma_3)$ to be the image of $\mathcal{F}_2$, together with some other paths. The image of $\{(e^-, e^+), (e^+)\}$ under the groupoid inclusion is $\{(e^-, e^+, e^+, e^{++}), (e^+, e^{++})\}$, we give $\mathcal{G}(\Gamma_3)$ the set of free generators $\mathcal{F}_3 := \{(e^-, e^+, e^+, e^{++}), (e^+, e^{++}), (e^+, e^+)\}$.

The ultimate goal of this generator choice is that we want the set of free generators for $\mathcal{G}(\Gamma_{i+1})$ to be the set of free generators for $\mathcal{G}(\Gamma_i)$ together with some other paths, so that we obtain a nested sequence of generating sets $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ for the system $\{\mathcal{G}(\Gamma_i)\}_{i \in \mathbb{N}}$. In this process of choosing the generators, there are choices to make for the generators that do not come from images of the previous generating set, for instance we could have chosen the free generating set of $\mathcal{G}(\Gamma_3)$ to be

$$\{(e^-, e^+, e^+, e^{++}), (e^+, e^{++}), (e^+, e^+)\}.$$ 

We allow such a freedom here. However, according to AGN there is a physics reason for choosing

$$\{(e^-, e^+, e^+, e^{++}), (e^+, e^{++}), (e^+, e^+)\}$$

instead, which is discussed in [5].
Lemma 2.14. Let \( W \) be a homomorphism. Given two finite graphs \( \Gamma \) and \( \Gamma' \), where \( \Gamma' \) is a refinement of \( \Gamma \), we will now construct a morphism \((Q,P,\iota)\) between their corresponding spectral triples \((B_\Gamma, B(\mathbb{E} \otimes \mathcal{H}_{\tilde{\Gamma}}), \mathcal{D}_{\tilde{\Gamma}})\) and \((B_{\Gamma'}, B(\mathbb{E} \otimes \mathcal{H}_{\tilde{\Gamma}'}, \mathcal{D}_{\tilde{\Gamma}'})\), where the generating set \( \tilde{\Gamma}' \) is chosen according to Subsection 2.3.1.

The surjection \( \mathcal{A}_\Gamma \twoheadrightarrow \mathcal{A}_{\Gamma'} \) of connection spaces induces an embedding of \( \mathbb{E} \) functions \( L^2(\mathcal{A}_{\Gamma}) \hookrightarrow L^2(\mathcal{A}_{\Gamma'}) \).

Under the new identification \( \text{Hom}(\mathcal{G}(\Gamma), G) \cong G^{[E_{\Gamma}]'} \) described in Subsection 2.3.1, the map

\[
L^2(G^{[E_{\Gamma}]}) \hookrightarrow L^2(G^{[E_{\Gamma}']}),
\]

is given by the functions by constant in the last \( |E_{\Gamma'}| - |E_{\Gamma}| \) variables. We then construct the map

\[
\mathcal{S}^{\otimes |E_{\Gamma}|} \hookrightarrow \mathcal{S}^{\otimes |E_{\Gamma}'|},
\]

given by embedding \( \mathcal{S}^{\otimes |E_{\Gamma}|} \) as the subspace \( \mathcal{S}^{\otimes |E_{\Gamma}|} \otimes \mathcal{I}^{\otimes |E_{\Gamma}'| - |E_{\Gamma}|} \), where \( \mathcal{S} \) is the cyclic representation of the \( C^* \)-algebra \( \mathbb{C}(\mathfrak{g}) \) with (normalized) cyclic vector \( \mathbb{I} \). The product of maps (17) and (18) gives us the desired Hilbert space map

\[
\iota : \mathcal{H}_{\tilde{\Gamma}} \hookrightarrow \mathcal{H}_{\tilde{\Gamma}'}.
\]

Let \( \iota_* : \mathcal{W}(\mathfrak{g}^{[E_{\Gamma}]} \hookrightarrow \mathcal{W}(\mathfrak{g}^{[E_{\Gamma}']} \) be the map induced by embedding \( \mathfrak{g}^{[E_{\Gamma}]} \) as the first \( |E_{\Gamma}| \) coordinates in \( \mathfrak{g}^{[E_{\Gamma}']} \). Since such an embedding \( \mathfrak{g}^{[E_{\Gamma}]} \hookrightarrow \mathfrak{g}^{[E_{\Gamma}']} \) preserves both the metrics and the Lie brackets, \( \iota_* \) is an algebra homomorphism. \( \mathcal{W}(\mathfrak{g}^{[E_{\Gamma}']}) \) acts on \( \mathcal{H}_{\tilde{\Gamma}'} \) as discussed in Subsection 2.1.3.

Lemma 2.14. Let \( w \in \mathcal{W}(\mathfrak{g}^{[E_{\Gamma}]} \) and \( \iota \in \mathcal{H}_{\tilde{\Gamma}} \). Then \( \iota_* w(\iota(\mathfrak{g})) = \iota(w(\mathfrak{g})) \) whenever \( w(\mathfrak{g}) \) is defined.

Proof. \( \mathfrak{g}^{[E_{\Gamma}]} \) differentiates in the first \( |E_{\Gamma}| \) variables on \( \mathcal{L}^2(G^{[E_{\Gamma}']} \), \( \text{Mat}_C(\mathbb{N}) \) and acts by Clifford action on the first \( |E_{\Gamma}| \) copies of \( \mathcal{S}^{\otimes |E_{\Gamma}'|} \), so the action of \( \mathcal{W}(\mathfrak{g}^{[E_{\Gamma}']}) \) on \( \mathcal{H}_{\tilde{\Gamma}'} \) preserves the subspace \( \mathcal{H}_{\tilde{\Gamma}'} \). The result follows. \( \Box \)

The surjection \( \text{Hom}(\mathcal{G}(\Gamma), G) \twoheadrightarrow \text{Hom}(\mathcal{G}(\Gamma'), G) \) induces an embedding of \( G \)-valued maps

\[
\text{Map}(\text{Hom}(\mathcal{G}(\Gamma), G), G) \hookrightarrow \text{Map}(\text{Hom}(\mathcal{G}(\Gamma'), G), G).
\]

Fix a vertex \( \nu \in \Gamma \), one has an embedding \( \mathcal{G}_\nu(\Gamma) \hookrightarrow \mathcal{G}_\nu(\Gamma') \) of the isotropy groups. The group homomorphism \( h \) (7) completes the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{G}_\nu(\Gamma) & \xrightarrow{h} & \text{Map}(\text{Hom}(\mathcal{G}(\Gamma), G), G) \\
\downarrow & & \downarrow \\
\mathcal{G}_\nu(\Gamma') & \xrightarrow{h} & \text{Map}(\text{Hom}(\mathcal{G}(\Gamma'), G), G)
\end{array}
\]
In particular, when (19) is restricted to the $G$-valued smooth functions $C^\infty(\mathcal{A}_G, G) \subset \text{Map}(\text{Hom}(G, G), G)$. Therefore, diagram (20) restricts to the following diagram:

\[
\begin{array}{ccc}
\mathcal{G}_\nu(\Gamma) & \xrightarrow{h} & C^\infty(\mathcal{A}_G, G) \\
\downarrow \circlearrowleft & & \\
\mathcal{G}_\nu(\Gamma') & \xrightarrow{h} & C^\infty(\mathcal{A}_{G'}, G)
\end{array}
\]

We let $Q : \mathcal{B}_G \to \mathcal{B}_{G'}$ be the *-homomorphism extended from the group homomorphism $h(\mathcal{G}_\nu(\Gamma)) \to h(\mathcal{G}_\nu(\Gamma'))$. Let $P : B(E \otimes \mathcal{H}_G) \to B(E \otimes \mathcal{H}_{G'})$ be the *-homomorphism induced from $\iota : \mathcal{H}_G \hookrightarrow \mathcal{H}_{G'}$. We can think of $B(E \otimes \mathcal{H}_{G'})$ as block matrices in $B(E \otimes \mathcal{H}_{G'}).

**Proposition 2.15.** $(Q, P, \iota)$ is a morphism between the spectral triples $(\mathcal{B}_G, B(E \otimes \mathcal{H}_G), \mathcal{D}_G)$ and $(\mathcal{B}_{G'}, B(E \otimes \mathcal{H}_{G'}), \mathcal{D}_{G'})$.

**Proof.** The operator $\mathcal{D}_{G'} \in \mathcal{W}(g^{[E,r]}) (12)$ can be written as

\[
\mathcal{D}_{G'} := \sum_{\gamma \in \mathcal{G}_G} D_\gamma = \sum_{\gamma \in \mathcal{G}_G} D_\gamma + \sum_{\gamma \in \mathcal{G}_G \setminus \mathcal{G}_G} D_\gamma
\]

Furthermore, $\sum_{\gamma \in \mathcal{G}_G \setminus \mathcal{G}_G} D_\gamma$ acts on $\iota(\mathcal{H}_G)$ by zero and $\sum_{\gamma \in \mathcal{G}_G} D_\gamma = \mathcal{D}_G$. Thus, together with Lemma 2.14,

\[
\iota(\mathcal{D}_{G'} \eta) = (\mathcal{D}_{G'})(\iota(\eta)) = \mathcal{D}_{G'}(\iota(\eta))
\]

for $\eta \in \mathcal{H}_G$ and the diagram

\[
\begin{array}{ccc}
\text{Dom}(\mathcal{D}_G) & \xrightarrow{\mathcal{D}_G} & \mathcal{H}_G \\
\downarrow \iota & & \downarrow \iota \\
\text{Dom}(\mathcal{D}_{G'}) & \xrightarrow{\mathcal{D}_{G'}} & \mathcal{H}_{G'}
\end{array}
\]

commutes.

As the surjection $\mathcal{A}_G \twoheadrightarrow \mathcal{A}_{G'}$ induces the injections

\[
C^\infty(\mathcal{A}_G, \text{Mat}_G(N)) \hookrightarrow C^\infty(\mathcal{A}_{G'}, \text{Mat}_G(N))
\]

\[
L^2(\mathcal{A}_G, E) \hookrightarrow L^2(\mathcal{A}_{G'}, E)
\]

$C^\infty(\mathcal{A}_G, \text{Mat}_G(N))$ as a sub-algebra of $C^\infty(\mathcal{A}_{G'}, \text{Mat}_G(N))$ preserves the subspace $L^2(\mathcal{A}_G, E) \subset L^2(\mathcal{A}_{G'}, E)$. Since $\mathcal{B}_G$ is a sub-algebra of $C^\infty(\mathcal{A}_G, \mathcal{G}_G(N))$ and it acts trivially on $S^{[E,r]}$, it preserves the subspace $E \otimes \mathcal{H}_G \subset E \otimes \mathcal{H}_{G'}$. Finally, by construction, $B(E \otimes \mathcal{H}_{G'})$ as a sub-algebra of $B(E \otimes \mathcal{H}_{G'})$ preserves the subspace $\mathcal{H}_G \subset \mathcal{H}_{G'}$, as well. Hence $(Q, P, \iota)$ forms a morphism as defined in Definition 1.2.

2.3.3 The System of Spectral Triples

Given a system of finite graphs $\{\Gamma_j\}_{j \in \mathbb{N}}$ and a system of generating sets $\{\mathcal{G}_j\}_{j \in \mathbb{N}}$ described in Subsection 2.3.1, there is associated a collection of spectral triples $\{(\mathcal{B}_{\Gamma_j}, B(E \otimes \mathcal{H}_{\mathcal{G}_j}), \mathcal{D}_{\mathcal{G}_j})\}_{j \in \mathbb{N}}$, and between every pair of spectral triples in the collection, there is a morphism $(Q_{ij}, P_{ij}, \iota_{ij})$ between them. In fact this collection of morphisms make the collection of spectral triples into a system of spectral triples.

**Proposition 2.16.** The collection of spectral triples $\{(\mathcal{B}_{\Gamma_j}, B(E \otimes \mathcal{H}_{\mathcal{G}_j}), \mathcal{D}_{\mathcal{G}_j})\}_{j \in \mathbb{N}}$ together with the collection of morphisms $\{(Q_{ij}, P_{ij}, \iota_{ij})\}_{1 \leq j}$ forms an inductive system of spectral triples.
Proof. By construction, the diagram

\[
(B_{\Gamma_i}, B(E \otimes \mathcal{H}_{\mathfrak{g}_i}), D_{\mathfrak{g}_i}) \xrightarrow{(Q_{ij}, P_{ij}, \iota_{ij})} (B_{\Gamma_j}, B(E \otimes \mathcal{H}_{\mathfrak{g}_j}), D_{\mathfrak{g}_j}) \xleftarrow{(Q_{jk}, P_{jk}, \iota_{jk})} (B_{\Gamma_k}, B(E \otimes \mathcal{H}_{\mathfrak{g}_k}), D_{\mathfrak{g}_k})
\]

commutes for \(i \leq j \leq k\). Hence, \(\{(B_{\Gamma_j}, B(E \otimes \mathcal{H}_{\mathfrak{g}_j}), D_{\mathfrak{g}_j})\}_{j \in \mathbb{N}}\) is an inductive system of spectral triples. \(\square\)

Denote the limit triple of \(\{(B_{\Gamma_j}, B(E \otimes \mathcal{H}_{\mathfrak{g}_j}), D_{\mathfrak{g}_j})\}_{j \in \mathbb{N}}\) by \((B, B(E \otimes \mathcal{H}), D)\). That is, \(B := \varinjlim B_{\Gamma_j}, \mathcal{H} := \varinjlim \mathcal{H}_{\mathfrak{g}_j}, \) and \(D := \varinjlim D_{\mathfrak{g}_j}\).

**Remark 2.17.** The inductive system of Lie algebras \(\{\mathfrak{g}^{[E_{\Gamma_j}]}\}_{j \in \mathbb{N}}\) with connecting morphisms given by embedding into \(\mathfrak{g}^{[E_{\Gamma_j}]}\) into the first \(|E_{\Gamma_j}|\) coordinates of \(\mathfrak{g}^{[E_{\Gamma_{j+1}}]}\), gives rise to an inductive system of quantum Weil algebras \(\mathcal{W}(\{\mathfrak{g}^{[E_{\Gamma_j}]}\})\). Denote its limit \(\varinjlim \mathcal{W}(\{\mathfrak{g}^{[E_{\Gamma_j}]}\})\) by \(\mathcal{W}\), then \(D\) is an element of \(\mathcal{W}\). The limit quantum Weil algebra \(\mathcal{W}\) can be viewed as the limit of the universal enveloping algebras \(\varinjlim U(\{\mathfrak{g}^{[E_{\Gamma_j}]}\})\) tensor with the limit of the Clifford algebras \(\varinjlim \mathcal{C}(\{\mathfrak{g}^{[E_{\Gamma_j}]}\})\), which are the Canonical Commutation Relation (CCR) algebra, and the Canonical Anti-Commutation Relation (CAR) algebra.

**Remark 2.18.** We have assumed our directed set \(I\) for the graph system \(\{\Gamma_i\}_{i \in I}\) to be the set of natural numbers \(\mathbb{N}\). Up to now, our construction works for any countable directed set \(I\), as long as we the system of generating sets \(\{\mathfrak{G}_{\Gamma_i}\}_{i \in I}\) satisfy the relation that \(\mathfrak{G}_{\Gamma_i} \subset \mathfrak{G}_{\Gamma_j}\) whenever \(i < j \in I\).

Unfortunately, \((B, B(E \otimes \mathcal{H}), D)\) is not a semi-finite spectral triple. The reason being that \(D\) does not have compact resolvent.

The operator \(D_{\mathfrak{g}_j} \in \mathcal{W}(\{\mathfrak{g}^{[E_{\Gamma_j}]}\})\) (12) can be written as

\[
D_{\mathfrak{g}_j} := \sum_{k=1}^{j} D_k
\]

where

\[
D_k := \sum_{\gamma \in \mathfrak{G}_{\mathfrak{g}_j} \setminus \mathfrak{G}_{\mathfrak{g}_{k-1}}} D_\gamma
\]

is the Dirac operator corresponds to those edges added to \(\Gamma_{k-1}\) to form \(\Gamma_k\) via the refinement.

We fix a real valued sequence \(\{a_k\}_{k=1}^\infty\), and scale each \(D_k\) by \(a_k\) so that

\[
D_{\mathfrak{g}_j} = \sum_{k=1}^{j} a_k D_k
\]

Propositions 2.15 and 2.16 continuous to hold as long as \(a_k \neq 0\) for all \(k\), so the limit triple \((B, \mathcal{N}, D)\) is still defined, however it now depends on the sequence \(\{a_k\}\), as \(D\) depends on \(a_k\). Therefore, the sequence \(\{a_k\}\) will be included as a dynamical variable in \(D\).

The physical interpretation of the sequence \(\{a_k\}\) is the following [5]. In the new notations described in Equations (21) and (22), the operator \(D_{\mathfrak{g}_{i+1}}\) (viewed as an operator on \(\mathcal{H}\)) is obtained from \(D_{\mathfrak{g}_i}\), by adding on an extra part \(D_{i+1}\) that corresponds to edges obtained from a graph refinement. Those new edges are necessarily shorter. As the Dirac operator carries a notion of the inverse volume in the language of noncommutative geometry [11], the part \(D_{i+1}\) that is added to \(D_{\mathfrak{g}_i}\) to make \(D_{\mathfrak{g}_{i+1}}\) is supposed to carry less weight for it to correspond to those shorter edges. For graph refinements in Example 2.3, there is not an obvious choice of a lesser weight that one should assign to \(D_{i+1}\). However, in Example 2.4, the refinement is obtained from subdividing each cell in a \(d\)-lattice into \(2^d\) cells. Thus, the “volume” the new edges from the refinement carry should be \(\frac{1}{2^d}\) of the
original. As the relation between the Dirac operator and volume is an inversed relation, the operator $\mathcal{D}_{i+1}$ ought to be scaled to $2^{2k} \mathcal{D}_{i+1}$. Therefore, when the lattice graphs are used, one obtains a weight assignment sequence $(2^{2k})_{k=1}^{\infty}$ that depends only on the dimension of the base manifold. From now, we will be using the property that the directed set being $\mathbb{N}$.

**Proposition 2.17.** If $\dim(S) = 1$, then $(B, B(E \otimes \mathcal{H}), \mathcal{D})$ is a spectral triple whenever $a_k^2 \not\to \infty$.

**Proof.** With Proposition 1.2, we only need to prove that $\mathcal{D}$ has compact resolvent. That is, when the projection $1_{[0,\lambda]}(\mathcal{D}^2)$ has finite rank for all $\lambda < \infty$, where $1_{[0,\lambda]}$ is the characteristic function supported on the closed interval $[0,\lambda] \subset \mathbb{R}$. First we look at non-zero eigenvalues of $\mathcal{D}^2$. Since $a_k^2 \not\to \infty$, every non-zero eigenvalue of the operator $a_k^2 \mathcal{D}^2_k$ will eventually be outside of the interval $[0,\lambda]$ as $k$ increases. As $\mathcal{D}^2 = \sum_{k=1}^{\infty} a_k^2 \mathcal{D}^2_k$, there are only finitely many eigenvalues (not counting multiplicity) of $\mathcal{D}^2$ in $[0,\lambda]$. Each of such non-zero eigenvalue of $\mathcal{D}^2$ in $[0,\lambda]$ must have finite multiplicity because the non-zero eigenvalues of each $a_k^2 \mathcal{D}^2_k$ have finite multiplicities. Now it remains to show that the zero eigenspace of $\mathcal{D}^2$ has finite dimension. As a consequence of Theorem 2.5, the zero eigenspace of $\mathcal{D}^2$ is $E \otimes \lim_{k \in \mathbb{N}} S \otimes |E_r, 1\rangle$, where $E \subset L^2(\mathcal{A}^\infty, E)$ is embedded as constant functions, which has rank $\dim(E)$ as $\dim(S) = 1$. Hence $1_{[0,\lambda]}(\mathcal{D}^2)$ has finite rank as long as $E$ is a finite dimensional representation of $\text{Mat}_\mathbb{C}(N)$ and $\mathcal{D}$ has compact resolvent. \hfill $\square$  

### 2.4 Semi-finite Spectral Triples

#### 2.4.1 Limit of Semi-finite Spectral Triples

When $\dim(S) > 1$, Proposition 2.17 no longer holds, as $1_{[0,\lambda]}(\mathcal{D}^2) = E \otimes \lim_{k \in \mathbb{N}} S \otimes |E_r, 1\rangle$ has infinite rank. Fortunately, there is semi-finite trace from a von Neumann algebra other than $B(E \otimes \mathcal{H})$ that computes the “dimension” of $S$ to be 1, and it in the end gives us a semi-finite spectral triple by reducing the complication to the case of Proposition 2.17. We go back to the system of spectral triples $\{(B_{\Gamma_j}, B(E \otimes \mathcal{H}_{\delta_j}), \mathcal{D}_{\delta_j})\}_{j \in \mathbb{N}}$, and replace each $B(E \otimes \mathcal{H}_{\delta_j})$ with a more suitable semi-finite von Neumann algebra $\mathcal{N}_{\delta_j} \subset B(E \otimes \mathcal{H}_{\delta_j})$.

Recall that Theorem 2.5 asserts that $D \in \mathcal{U}(g) \otimes \mathcal{C}(g)$, as an operator on $L^2(G) \otimes S$, has kernel $\ker(D) = C \otimes S$. The projection onto $\ker(D)$ is $P_1 \otimes 1 \in B(L^2(G) \otimes B(S)$, where $P_1$ denotes the projection onto the space of constant functions. Denote the weak operator closure of $B(L^2(G) \otimes \mathcal{C}(g))$ by $N_j$, it comes equipped with the semi-finite trace $\tau_1$ by extending $\mathcal{D}_1 \otimes \mathcal{C}(1)$, where $\mathcal{D}_1$ is the Clifford trace on $\mathcal{C}(g)$. Then clearly $D \in \mathcal{U}(g) \otimes \mathcal{C}(g)$ is affiliated with $\mathcal{N}_j$ and

$$\tau_1(P_1 \otimes 1) = \mathcal{D}_1(P_1) \cdot \mathcal{D}_1(1) = 1.$$ 

Therefore, $\ker(D)$ has “dimension” 1 relative to the trace $\tau_1$. In fact $\tau_1$ is the operator trace on $B(L^2(G) \otimes S)$ normalized by dividing the dimension of $S$. Thus each eigenvalue of $D$ now has “multiplicity” $\frac{1}{\dim(S)}$ of before.

On the Hilbert space $E \otimes \mathcal{H}_{\delta_j}$, let $\mathcal{N}_{\delta_j}$ be the weak operator closure of $B(L^2(G^{[E_r, 1]}(E))) \otimes \mathcal{C}(g^{[E_r, 1]})$, which is equipped with the trace $\tau_{\delta_j}$ by extending $\mathcal{D}_1 \otimes \mathcal{C}(1)$, where again $\mathcal{D}_1$ is the operator trace and $\mathcal{C}(1)$ is the Clifford trace on $\mathcal{C}(g^{[E_r, 1]}(E))$.

**Lemma 2.18.** $(B_{\Gamma_j}, \mathcal{N}_{\delta_j}, \mathcal{D}_{\delta_j})$ forms a semi-finite spectral triple.

**Proof.** $\mathcal{D}_{\delta_j} \in \mathcal{W}(g^{[E_r, 1]})$ is affiliated with $\mathcal{N}_{\delta_j}$, and has compact resolvent relative to $\tau_{\delta_j}$. The rest of the proof proceeds similar to the proof of Proposition 2.9. \hfill $\square$  

For $i \leq j$, the map of embedding $g^{[E_r, 1]}$ to the first $|E_r, 1\rangle$ copies in $g^{[E_r, 1]}$ induces an algebra map $\mathcal{C}(g^{[E_r, 1]}) \to \mathcal{C}(g^{[E_r, 1]})$. The map of extending functions in $L^2(G^{[E_r, 1]}(E)$ by constants in the last $|E_r, 1\rangle - |E_r, 1\rangle$ variables defines a Hilbert space embedding $L^2(G^{[E_r, 1]}(E) \hookrightarrow L^2(G^{[E_r, 1]}(E)$, which induces an embedding $B(L^2(G^{[E_r, 1]}(E)) \hookrightarrow B(L^2(G^{[E_r, 1]}(E))$. By extending the product of these two maps, we obtain a $*$-homomorphism

$$P_{ij} : \mathcal{N}_{\Gamma_i} \to \mathcal{N}_{\Gamma_j}.$$  \hfill (23)

**Lemma 2.19.** $\{(B_{\Gamma_j}, \mathcal{N}_{\delta_j}, \mathcal{D}_{\delta_j})\}_{j \in \mathbb{N}}$ forms an inductive system of semi-finite spectral triples
Proof. By construction, \( P_{ij}(a)(\eta)) = \iota(a(\eta)) \) for \( a \in \mathcal{N}_\delta \), and \( \eta \in \mathcal{H}_\delta \). The rest of the proof proceeds similar to the proof of Proposition 2.16. □

Let \( (B, \mathcal{N}, \mathcal{D}) \) be the limit of \( \{ (B_t, \mathcal{N}_t, \mathcal{D}_t) \}_{t \in \mathbb{N}} \), i.e., \( \mathcal{N} \) is the von Neumann algebra limit \( \lim_{t \to \infty} \mathcal{N}_t \).

By extending Proposition 2.11, one has \( \lim L^2(G^{1 \mathbb{R}_t}), E \rightleftharpoons L^2(\mathcal{A}^\delta, E) \).

\( \mathcal{N} \) is the weak operator closure of \( B(L^2(\mathcal{A}^\delta, E) \otimes \mathcal{C}(\mathcal{Z}^{1 \mathbb{R}_t}))) \), it is equipped with the semi-finite trace \( \tau \) by extending \( \text{Tr} \otimes \text{Tr}_{\mathcal{C}1} \), where \( \text{Tr} \) is the operator trace on \( B(L^2(\mathcal{A}^\delta, E)) \) and \( \text{Tr}_{\mathcal{C}1} \) is the Clifford trace on \( \lim \mathcal{C}(\mathcal{Z}^{1 \mathbb{R}_t}) \).

Remark 2.19. The algebra \( \mathcal{N} \) is of Type II\( \infty \), as the CAR algebra \( \lim \mathcal{C}(\mathcal{Z}^{1 \mathbb{R}_t}) \), being an “infinite tensor” of Type I\( \delta \) algebras, is Type II\( \delta \).

Theorem 2.20. \( (B, \mathcal{N}, \mathcal{D}) \) is a semi-finite spectral triple whenever \( a^2 \not\to \infty \).

Proof. With Proposition 1.2, we only need to prove that \( \mathcal{D} \) has compact resolvent. That is, \( \tau(1_{[0,\lambda]}(\mathcal{D}^2)) < \infty \) for all \( \lambda < \infty \), where \( 1_{[0,\lambda]} \) is the characteristic function supported on the closed interval \( [0, \lambda] \subset \mathbb{R} \).

Since \( a^2 \not\to \infty \), every non-zero eigenvalue of the operator \( a^2 \mathcal{D}^2_k \) will eventually be outside of the interval \( [0, \lambda] \) as \( k \) increases. As \( \mathcal{D}^2 = \sum_k a^2 \mathcal{D}^2_k \), there are only finitely many eigenvalues of \( \mathcal{D}^2 \) in \( [0, \lambda] \). Each such non-zero eigenvalue in \( [0, \lambda] \) has finite multiplicity relative to \( \tau \) because the non-zero eigenvalues of each \( a^2 \mathcal{D}^2_k \) have finite multiplicities. By Theorem 2.5, \( 1_{[0,\lambda]}(\mathcal{D}^2) = P_E \otimes 1 \), where \( P_E \) is the projection onto the space of constant sections \( E \in L^2(\mathcal{A}^\delta, E) \), so

\[
\tau(1_{[0,\lambda]}(\mathcal{D}^2)) = \text{Tr}(P_E) \cdot \text{Tr}_{\mathcal{C}1}(1) = \dim(E) .
\]

Hence \( \tau(1_{[0,\lambda]}(\mathcal{D}^2)) < \infty \) for all \( \lambda \) and \( \mathcal{D} \) has compact resolvent. \( (B, \mathcal{N}, \mathcal{D}) \) is a semi-finite spectral triple. □

The upshot of using the semi-finite von Neumann algebra \( \mathcal{N} \) is that, ordinarily

\[
\text{Tr}(1_{[0,\lambda]}(\mathcal{D}^2)) = \dim(E) \cdot \dim \left( \lim \mathcal{C}(\mathcal{Z}^{1 \mathbb{R}_t}) \right) = \dim(E) \cdot \infty
\]

One hopes to get a finite number out of it, so we divide the above quantity by \( \infty \), more precisely by the dimension of \( \lim \mathcal{C}(\mathcal{Z}^{1 \mathbb{R}_t}) \). Thus, we form the quantity

\[
\frac{1}{\dim \left( \lim \mathcal{C}(\mathcal{Z}^{1 \mathbb{R}_t}) \right)} \text{Tr}_{\mathcal{N}_t}(1_{[0,\lambda]}(\mathcal{D}^2)) ,
\]

where \( \text{Tr}_{\mathcal{N}_t} \) is the operator trace on \( B(E \otimes \mathcal{H}_\delta) \). By taking the limit as \( j \to \infty \), we get \( \dim(E) \), which is finite.

This procedure has a flavor of renormalization in Quantum field theory.

2.4.2 The \( \mathbb{Z}_2 \)-grading

Suppose that the cyclic representation \( S \) of \( \mathcal{C}(\mathcal{g}) \) is the Clifford algebra \( \mathcal{C}(\mathcal{g}) \) itself, which is a \( \mathbb{Z}_2 \) graded vector space with say \(+1\)-eigenspace \( \mathcal{C}^+(\mathcal{g}) \) and \(-1\)-eigenspace \( \mathcal{C}^-(\mathcal{g}) \). Then it equips the Hilbert space \( E \otimes \mathcal{H}_\delta \) with a grading so that the action of the algebra \( B \) is even and the action of the Dirac operator \( \mathcal{D}_\delta \) is odd, making the spectral triple \( (B, \mathcal{N}, \mathcal{D}_\delta) \) even dimensional. But it does not make our semi-finite spectral triple \( (B, \mathcal{N}_t, \mathcal{D}_t) \) even, since the grading operator is not an element of the von Neumann algebra \( \mathcal{N}_t \). However, if \( \mathcal{g} \) is even dimensional of dimension 2\( k \), one could equip \( S \) with the \( \mathbb{Z}_2 \) grading given by the chirality element \( (\sqrt{-1})^k e_1 \cdots e_{2k} \) in \( \mathcal{C}(\mathcal{g}) \) for the basis \( \{ e_i \}_{i=1}^{2k} \subset \mathcal{g} \). The chirality element is self-adjoint and squares to 1. It induces a grading operator \( \chi_{\mathcal{N}_t} \) of \( E \otimes \mathcal{H}_\delta \) in \( \mathcal{N}_t \), so that \( \chi_{\mathcal{N}_t} \) anti-commutes with \( \mathcal{D}_\delta \) and commutes with any element in \( B \). As a result, \( (B, \mathcal{N}_t, \mathcal{D}_t) \) is an even semi-finite spectral triple with respect to \( \chi_{\mathcal{N}_t} \) if \( \mathcal{g} \) is even dimensional.

With notations as before, we let \( (B_t, \mathcal{N}_t, \mathcal{D}_t) \) be a system of semi-finite spectral triples. Suppose that \( G \) is even dimensional, then \( (B_t, \mathcal{N}_t, \mathcal{D}_t) \) is an even semi-finite spectral triple with respect to the grading operator \( \chi_{\mathcal{N}_t} \in \mathcal{N}_t \). The map \( P_\infty : \mathcal{N}_t \to \mathcal{N} \) sends each \( \chi_{\mathcal{N}_t} \in \mathcal{N}_t \) to \( \mathcal{N} \), so we obtain a net of operators \( \{ \chi_{\mathcal{N}_t} \in \mathcal{N}_t \} \) in \( \mathcal{N} \). Denote by \( \chi \) its strong operator limit. Then by Theorem 1.2, \( (B, \mathcal{N}, \mathcal{D}) \) is an even semi-finite spectral triple with respect to \( \chi \in \mathcal{N} \).

Proposition 2.21. \( (B, \mathcal{N}, \mathcal{D}) \) is an even (resp. odd) semi-finite spectral triple if the Lie group is even (resp. odd) dimensional.
3 JLO Theory

JLO theory due to Jaffe, Lesniewski and Osterwalder [12] is a cohomological Chern character that assigns a cocycle, hence a class, in entire cyclic cohomology to a weakly $\theta$-summable semi-finite spectral triple. The cohomology class is homotopy invariant, thus the JLO character descends to a map from K-homology classes to entire cyclic cohomology classes. The cocycle computes the Type II Fredholm index and spectral flow of the operator $D$, hence it provides an index formula in the setting of noncommutative geometry. Here we will give a summary of the JLO theory. For details please refer to [15].

3.1 Entire Cyclic Cohomology

Let $A$ be a unital Banach algebra over $\mathbb{C}$ and $C^n(A)$ to be the space of $n$-linear functionals over $A$. Define the operators $b : C^n(A) \to C^{n+1}(A)$ and $B : C^n(A) \to C^{n-1}(B)$ by the formulas

$$(b\phi_n)(a_0, \ldots, a_n) := \sum_{j=0}^{n-1} (-1)^j \phi_n(a_0, \ldots, a_ja_{j+1}, \ldots, a_n) + (-1)^n \phi_n(a_na_0, a_1, \ldots, a_{n-1}),$$

$$(B\phi_n)(a_0, \ldots, a_n) := \sum_{j=0}^{n} (-1)^{nj} \phi_n(1, a_j, \ldots, a_n, a_0, \ldots, a_{j-1}),$$

for $\phi \in C^n(B)$.

Simple calculation shows that $b^2 = B^2 = Bb + bB = 0$. Therefore $(b + B)^2 = 0$ and we get the following bicomplex:

The space $C^\bullet(A) := \prod_{n=0}^{\infty} C^n(A)$ has a natural $\mathbb{Z}_2$ grading given by $C^+(A) = \prod_{k=0}^{\infty} C^{2k}(A)$ and $C^-(A) = \prod_{k=0}^{\infty} C^{2k+1}(A)$. We get a cochain complex $(C^\bullet(A), b + B)$ with the odd boundary map $b + B$. However, the cohomology of this cochain complex is trivial. In order to make it nontrivial, we restrict our attention to cochains that satisfy the following growth condition as $n$ varies.

**Definition 3.1.** Define

$$C^*_n(A) := \left\{ \phi_\bullet \in C^\bullet(A) : \sum_{n=0}^{\infty} \chi(n) \frac{n}{2} \|\phi_n\| z^n is an entire function in z \right\}$$

20
where \( \| \phi_n \| := \sup \{|\phi_n(a_0, \ldots, a_n)| : |a_j| \leq 1 \ \forall j \} \). We call the cochain \( \phi_n \) \textbf{entire} if \( \phi_n \in C^n(A) \). It is easy to see that if \( \phi_n \) is entire, so is \((b+B)\phi_n \). Hence \((C^*_n(A), b+B)\) is a subcomplex of \((C^*_n(A), b+B)\). The cohomology defined by \((C^*_n(A), b+B)\) is the \textbf{entire cyclic cohomology} of \( A \), denoted \( HE^*(A) = HE^+(A) \oplus HE^-(A) \).

### 3.2 The JLO Character

The JLO character assigns cocycles in entire cyclic cohomology to semi-finite spectral triples satisfying an appropriate summability condition. We begin by defining the summability conditions of main concern.

**Definition 3.2.** An \textbf{semi-finite spectral triple} \((\mathcal{B}, \mathcal{N}, \mathcal{D})\) is:

(a) \textbf{\( p \)-summable} if \( \tau((1 + D^2)^{-p/2}) < \infty \);

(b) \textbf{\( \theta \)-summable} if \( \tau(e^{-tD^2}) < \infty \) for all \( t > 0 \);

(c) \textbf{weakly \( \theta \)-summable} if \( \tau(e^{-tD^2}) < \infty \) for some \( 0 < t < 1 \).

Observe that \( p \)-summability implies \( \theta \)-summability, which in turn implies weak \( \theta \)-summability.

Let \( \Delta_n := \{(t_1, \ldots, t_n) \in \mathbb{R}^n : 0 \leq t_1 \leq \cdots \leq t_n \leq 1\} \) be the standard \( n \)-simplex and \( d^n t = dt_1 \cdots dt_n \) is the standard Lebesgue measure on \( \Delta_n \) with volume \( \frac{1}{n!} \).

**Definition 3.3.**

1. The \textbf{odd JLO character} \( \text{Ch}_{\text{JLO}}(\mathcal{D}) \in C^- (A) \) of a weakly \( \theta \)-summable odd semi-finite spectral triple \((\mathcal{B}, \mathcal{N}, \mathcal{D})\) is defined to be

\[
\text{Ch}_{\text{JLO}}(\mathcal{D}) := \sum_{k=0}^{\infty} \text{Ch}_{\text{JLO}}^{2k+1}(\mathcal{D}) ,
\]

where \( A \) is the closure of \( \mathcal{B} \) with respect to the norm \( \| \cdot \| + \| [D, \cdot] \| \) and

\[
(\text{Ch}_{\text{JLO}}^n(\mathcal{D}), (a_0, \ldots, a_n)_n) := \int_{\Delta_n} \tau \left( \chi a_0 e^{-t_1 D^2} [D, a_1] e^{-(t_2 - t_1) D^2} \cdots [D, a_n] e^{-(1-t_n) D^2} \right) d^n t
\]

with the convention \( \chi = 1 \) if \( n \) is odd.

2. The \textbf{even JLO character} \( \text{Ch}_{\text{JLO}}^n(\mathcal{D}) \in C^+ (A) \) of a weakly \( \theta \)-summable even semi-finite spectral triple \((\mathcal{B}, \mathcal{N}, \mathcal{D})\) is defined to be

\[
\text{Ch}_{\text{JLO}}^n(\mathcal{D}) := \sum_{k=0}^{\infty} \text{Ch}_{\text{JLO}}^{2k}(\mathcal{D}) ,
\]

where \( A \) is the closure of \( \mathcal{B} \) with respect to the norm \( \| \cdot \| + \| [D, \cdot] \| \) and

\[
(\text{Ch}_{\text{JLO}}^n(\mathcal{D}), (a_0, \ldots, a_n)_n) := \int_{\Delta_n} \tau \left( \chi a_0 e^{-t_1 D^2} [D, a_1] e^{-(t_2 - t_1) D^2} \cdots [D, a_n] e^{-(1-t_n) D^2} \right) d^n t
\]

with the convention \( \chi = 1 \) if \( n \) is even.

**Theorem 3.1** ([15]). The JLO character \( \text{Ch}_{\text{JLO}}^n(\mathcal{D}) \) is an entire cyclic cocycle in \( HE^n (A) \). More specifically,

\[
\text{Ch}_{\text{JLO}}^n(\mathcal{D}) \in C^n(A) \quad \text{and} \quad (b + B)\text{Ch}_{\text{JLO}}^n(\mathcal{D}) = 0 .
\]

As a result, the JLO character defines an entire cyclic cohomology class called the JLO class.

### 3.3 Homotopy Invariance of the JLO Class

Suppose that \( D_t \) is a \( t \)-parameter family of operators so that it defines a differentiable family of weakly \( \theta \)-summable semi-finite spectral triples \((\mathcal{B}, \mathcal{N}, \mathcal{D}_t)\). Namely, \( D_t \) is a \( t \)-parameter family of self-adjoint operators on \( \mathcal{H} \) with common domain of definition so that the following is satisfied:

- \( D_t \) is affiliated with \( \mathcal{N} \) for all \( t \),
- For all \( a \in \mathcal{B} \), \([D_t, a]\) is a norm-differentiable family of operators in \( \mathcal{N} \), and there is a constant \( C \) for each compact interval such that \( \|[D_t, a]\| \leq C \|a\| \),
Lemma 4.1. Let $\phi: (\mathcal{N}, \mathcal{D}) \rightarrow (\mathcal{N}^\prime, \mathcal{D}^\prime)$ be a $\mathbb{Z}_2$ grading $\chi \in \mathcal{N}$ so that $\alpha$ is even for all $a \in \mathcal{B}$ and $\mathcal{D}_t$ is odd for all $t$, then similarly we call the family of semi-finite spectral triple $(\mathcal{B}, \mathcal{N}, \mathcal{D}_t)$ even.

Theorem 3.2 ([15]). If $\mathcal{D}_t = F_t |\mathcal{D}_t|^{1+\varepsilon} + R_t$ for $0 \leq \varepsilon < 1$ and $F_t, R_t \in \mathcal{N}$ are continuous families of operators that are uniformly bounded in $t$ then the entire cyclic cohomology class of $\mathcal{C}^*_{\text{JLO}}(\mathcal{D}_t)$ is independent of $t$.

The following Proposition gives a stability of bounded perturbation of weakly $\theta$-summable semi-finite spectral triple.

Proposition 3.3 ([15]). For a weakly $\theta$-summable semi-finite spectral triple $(\mathcal{B}, \mathcal{N}, \mathcal{D})$, and an operator $V \in \mathcal{N}$ such that $V$ has the same degree as $\mathcal{D}$, i.e. $|V|_\chi = |\mathcal{D}|_\chi$. Then $(\mathcal{B}, \mathcal{N}, \mathcal{D} + V)$ is again a weakly $\theta$-summable semi-finite spectral triple.

4 The JLO Class of AGN’s Space of Connections

When the Dirac operator $\mathcal{D}$ of the noncommutative space of connections of AGN is weakly $\theta$-summable, there is an entire cyclic cocycle associated to it. As we have observed in Section 2, AGN’s spectral triple, and hence the associated JLO cocycle, depends on the weight assignment, which is a diverging sequence $\{a_k\}$. We give an explicit condition on allowable perturbations of the given weight assignment so that the associated JLO class remains invariant. In a more recent paper [5], Aastrup-Grimstup-Paschke-Nest eliminate the weight ambiguity by using lattice graphs, which results in the most reasonable choice of weight assignment that depends only on the dimension of the base manifold. If one re-runs AGN’s construction on a sub-manifold, the resulting spectral triple will be defined using the weight assignment corresponding to the sub-manifold, which in general would be different from the spectral triple obtained from pulling back the construction on the full manifold. Although the two semi-finite spectral triples are assigned different weights, Section 4.1 eliminates that dimension dependence at the level of entire cyclic cohomology by exploiting the homotopy invariance of the JLO character described in Section 3.3 (see [15] for details). Section 4.2 analyzes the weak $\theta$-summability of the Dirac operator $\mathcal{D}$ in terms of the diverging sequence $\{a_k\}$.

4.1 Weight Independence of the JLO Class

Let $\{\Gamma_k\}_{k \in I}$ be a directed system of finite graphs with the system of groupoid generators $\{\mathfrak{g}_k\}_{k \in I}$ chosen according to Subsection 2.3.1. Recall that

- $\mathcal{D}_n$ is the basic Dirac operator associated to a path in the a graph (see (9));
- $\mathcal{D}_k := \sum_{\gamma \in \mathfrak{g}_k \setminus \mathfrak{g}_{k-1}} D_{\gamma}$ is the Dirac operator corresponds to those edges added to $\Gamma_{k-1}$ to form $\Gamma_k$ (see (22));
- $\mathcal{D}_{\mathfrak{g}_n} = \sum_{k=1}^{n} a_k \mathcal{D}_k$ is the Dirac operator on the finite graph $\Gamma_n$ (see (21)).

By Theorem 1.2(2), $\mathcal{D}$ is the strong resolvent limit of $\{\mathcal{D}_{\mathfrak{g}_n}\}_{n=1}^{\infty}$. We will write the derivative of the weight changes of each term in $\mathcal{D}$ as the operator $F_t |\mathcal{D}_t|^{1+\varepsilon} + R_t$, and apply Theorem 3.2 to give a concrete condition on allowable variation of the weight assignments $\{a_k\}_{k=1}^{\infty}$. In particular, we will show that the variation corresponding to the dimension change in the base manifold gives rise to a transgression cochain that is entire.

Lemma 4.1. Let $\{a_k(t)\}$ be a family of sequences parametrized by $t$ so that each $a_k(t)$ is differentiable in $t$. Let $\mathcal{D}(t)$ be the $t$-family of Dirac operators given by the weight assignment $\{a_k(t)\}_{k=1}^{\infty}$ (i.e. $\sum_{k=1}^{n} a_k(t) \mathcal{D}_k = \mathcal{D}_{\mathfrak{g}_n}(t) \rightarrow \mathcal{D}(t)$ in the strong resolvent sense). If there exist $\varepsilon \in [0, 1)$ and $m \in (0, \infty)$ such that

$$\sup_k \left( \frac{|a_k(t)|}{|a_k(t)|^{1+\varepsilon}} \right) \leq m \text{ for all } t,$$
Thus, we obtain the limit
\[ \lim_{\varepsilon \to 0} \varepsilon D \]
and \( \|F_t\|, \|R_t\| \leq m \) uniformly.

**Proof.** We first analyze the operator
\[ |\dot{D}(t)| \left( 1 + |D(t)|^{1+\varepsilon} \right)^{-1} \]
Since \( D_{g_n}(t) \to D(t) \) in the strong resolvent sense for all \( t \),
\[ \frac{D_{g_n}(t + h) - D_{g_n}(t)}{h} \to \frac{D(t + h) - D(t)}{h} \]
for all \( h \), and we have the strong resolvent limit
\[ \sum_{k=1}^{n} \hat{a}_k(t)D_k = \dot{D}_{g_n}(t) \to \dot{D}(t) \]
Thus, we obtain the limit
\[ |\dot{D}_{g_n}(t)| \left( 1 + |D_{g_n}(t)|^{1+\varepsilon} \right)^{-1} \to |\dot{D}(t)| \left( 1 + |D(t)|^{1+\varepsilon} \right)^{-1} \]
which a priori is only in the strong resolvent sense.

Let \( 0 < \frac{1}{2} \) be the smallest non-zero eigenvalue of \( D^2 \). For instance, \( c = 8 \) when \( G = SU(2) \) by Remark 2.11. Then the open interval \((0, 1)\) does not contain any part of the spectrum of \( cD_k^2 \) as each \( D_k^2 \) is a finite sum of \( D^2 \)'s. We bound the operator \( |\dot{D}_{g_n}(t)| \left( 1 + |D_{g_n}(t)|^{1+\varepsilon} \right)^{-1} \) by multi-variable functional calculus on the set of commuting self-adjoint operators \( \{cD_k^2\}_{k=1}^{n} \). The following multi-variable function is bounded when \( x_k \geq 1 \) or \( = 0 \) for all \( k \):
\[ f_k^n \left( \{x_k\}_{k=1}^{n} \right) := \frac{c}{1+c+\sum_{k=1}^{n} a_k(t)^2 x_k} \leq \frac{c}{1+c+\sum_{k=1}^{n} a_k(t)^2 x_k} \]
As \( \|f_k^n\|_\infty \leq c^m \), \( f_k^n \left( \{cD_k^2\}_{k=1}^{n} \right) \) is uniformly bounded. By construction
\[ f_k^n \left( \{cD_k^2\}_{k=1}^{n} \right) = |\dot{D}_{g_n}(t)| \left( 1 + |D_{g_n}(t)|^{1+\varepsilon} \right)^{-1} \]
so \( |\dot{D}_{g_n}(t)| \left( 1 + |D_{g_n}(t)|^{1+\varepsilon} \right)^{-1} \) is uniformly bounded by \( c^m \) and
\[ |\dot{D}_{g_n}(t)| \left( 1 + |D_{g_n}(t)|^{1+\varepsilon} \right)^{-1} \to |\dot{D}(t)| \left( 1 + |D(t)|^{1+\varepsilon} \right)^{-1} \]
in strong operator topology. The fact that \( D_k \) is affiliated with \( N \) for each \( k \) implies that \( |\dot{D}_{g_n}(t)| \left( 1 + |D_{g_n}(t)|^{1+\varepsilon} \right)^{-1} \in N \) for each \( n \). As \( N \) is strong operator closed, the limit \( |\dot{D}(t)| \left( 1 + |D(t)|^{1+\varepsilon} \right)^{-1} \in N \). Set \( F_t = R_t = \dot{D}(t) \left( 1 + |D(t)|^{1+\varepsilon} \right)^{-1} \in N \), then \( \dot{D}(t) = F_t|D(t)|^{1+\varepsilon} + R_t \). Furthermore, observe that \( \|f_k^n\|_\infty \) uniformly bounded by \( c^m \) for all \( n \), the operator norms of \( F_t \) and \( R_t \) are also uniformly bounded by \( c^m \).

**Theorem 4.2.** If \( \{a_k(t)\} \) is a differentiable family of sequences so that \( D(t) \) is weakly \( \theta \)-summable for all \( t \), and that there exist \( \varepsilon \in [0, 1) \) and \( m \in (0, \infty) \) such that
\[ \sup_k \left( \frac{|a_k(t)|}{|a_k(t)|^{1+\varepsilon}} \right) \leq m \] for all \( t \),
then \( \text{Ch}_{\text{rcd}}(D(t)) \) defines the same entire cyclic cohomology class in \( \text{HE}^*(A) \) for any \( t \), where \( A \) is the closure of \( B \) with respect to the norm \( \|b\|_{\text{Lip}} := \|b\| + \|D_{t_0} b\| \) for a fixed \( t_0 \) and all \( b \in B \).
Proof. By Lemma 4.1, $\hat{D}(t) = A_t |D(t)|^{1+\varepsilon} + R_t$. By Theorem 3.2, $\text{Ch}_{\text{JLO}}(D(t), \hat{D}(t))$ is entire and $|\text{Ch}_{\text{JLO}}(D(t_1))| = |\text{Ch}_{\text{JLO}}(D(t_2))|$ for any $t_1, t_2$ as an entire cyclic cohomology class, and the proof is complete. \hfill \square

Now we will give an application of Theorem 4.2. Recall that $M$ is the base manifold of dimension $d$, and $\{T_1\} \subset \Sigma$ is a system of embedded graphs in $M$. In [5], Aastrup-Grimstrup-Nest choose the sequence $a_j = (2^d)^j$ by using lattice graphs of dimension $d$. Let $a_j(t) = (2^{d+t})^j$, then we see that

$$
\sup_j \left( \frac{|a_j(t)|}{|a_j(t)|} \right)^{1+\varepsilon} = \sup_j \left( \frac{j \ln(2)}{(2^{d+t})^j} \right),
$$

which is finite as long as $t + d > 0$ and $\varepsilon > 0$. By Theorem 4.2, we see that as long as $t$ is so that $D(t)$ is weakly $\theta$-summable, which can be achieved by an overall scaling of $\{a_j\}$ (see Section 4.2), the choice of $d$ does not affect its JLO class. We will use this observation to show that the entire cyclic cohomology class associated to AGN’s Dirac operator is well-behaved under immersions.

Let $N$ and $M$ be compact manifolds of dimension $c$ and $d$ respectively. We give $N$ a system of c-lattice graphs (see Example 2.4) and assume that $N$ immerses in $M$ in a way that the system of c-lattice graphs in $N$ extends to a system of d-lattice graphs in $M$. That is, the d-lattice graphs are refinements of the c-lattice graphs such that, other than the joining vertices, the new edges added for the refinements lie entirely outside of $N$. Fix a vertex $\nu$ in a c-lattice graph of $N$, denote by $(B, N, D_c)$ and $(B_d, N_d, D_d)$ the semi-finite spectral triples constructed according to Section 2 for $N$ and $M$. Because the d-lattice graphs are refinements of c-lattice graphs, we get an embedding of the algebras $\nu : B_c \hookrightarrow B_d$. The embedding $\nu$ pulls back the semi-finite spectral triple $(B_d, N_d, D_d)$ to $(B_c, N_c, D_c)$. Denote by $A$ the closure of $B_c$ under the norm $\|b\|_{\text{lip}} := \|b\| + \|[D_c, b]\|$ for $b \in B_c$.

**Theorem 4.3.** When $(B_c, N_c, D_c)$ and $(B_d, N_d, D_d)$ are weakly $\theta$-summable, their JLO cocycles define the same JLO class in $\text{HE}^*(A)$. \hfill \square

**Proof.** The unit of $A_c$ acts as a projection $p$ on $\mathcal{H}_d$ and decomposes $\mathcal{H}_d$ into $p\mathcal{H}_d \oplus (1-p)\mathcal{H}_d$. Since the $d$-lattices are refinements of the c-lattices, by the generator choice described in Subsection 2.3.1, the Hilbert space $p\mathcal{H}_d$ is precisely $\mathcal{H}_c$ and the representation $\rho_d \circ \iota$ is just $\rho_c$. Also $\mathcal{D}_d$ decomposes into $\mathcal{D}_d' + \mathcal{D}_d''$ with $p\mathcal{N}_d = N_c$ and $\mathcal{D}_d''$ affiliated with $(1-p)\mathcal{N}_d(1-p)$. As a result, $\text{Ch}_{\text{JLO}}(\mathcal{D}_d) = \text{Ch}_{\text{JLO}}(\mathcal{D}_d') + \text{Ch}_{\text{JLO}}(\mathcal{D}_d'')$, and $\text{Ch}_{\text{JLO}}(\mathcal{D}_d') = 0 \in \text{HE}^*(A)$. Now by construction, the Dirac operators $\mathcal{D}_c$ and $\mathcal{D}_d''$ differ only by the defining sequence $\{a_j\}$. By Theorem 4.2 and Equation (24), $\text{Ch}_{\text{JLO}}(\mathcal{D}_c)$ equals $\text{Ch}_{\text{JLO}}(\mathcal{D}_d')$ up to an entire coboundary, hence $\text{Ch}_{\text{JLO}}(\mathcal{D}_c)$ and $\text{Ch}_{\text{JLO}}(\mathcal{D}_d)$ define the same class in $\text{HE}^*(A_c)$. \hfill \square

### 4.2 Weak $\theta$-summability

In [3], the term $\tau(e^{-D^2})$ has the form of a formal Feynman path integral, thus weak $\theta$-summability has a physical motivation of the path integral being finite. One would like to know for what sequences $\{a_j\}$, $D$ is weakly $\theta$-summable.

We give a characterization of the weakly $\theta$-summable condition of $D$ in terms of the sequence $\{a_j\}$. However, the analysis will necessarily depend on the graph system that is deployed. The most general graph system is given by adding one edge at a time, because under our choice generators described in Subsection 2.3.1, any refinement of a graph can be obtained by successively adding edges one at a time. As a result, our graph system gives the following system of connection spaces:

$$G \leftarrow G \times G \leftarrow G \times G \times G \leftarrow \cdots \leftarrow G^{n-1} \leftarrow G^n \leftarrow \cdots .$$

Therefore, $D = \sum_{k=0}^{\infty} a_k D_k$, where $D_k$ is just $D$ (9) acting on the $k$-th copy of $G$.

Recall that we extended the action of $D$ on $\mathcal{H} := \lim_{\to} \mathcal{H}_j$, to $E \otimes \mathcal{H}$ by letting it act as the identity on $E$. Since $E$ is finite dimensional, the summability of $\mathcal{D}$ acts unchanged $\mathcal{D}$ acts on $E$ or not. Hence, we will for simplicity let $D$ act on $\mathcal{H}$ instead of $E \otimes \mathcal{H}$.

Let $\tau_i$ be the semi-finite trace on the weak operator closure of $B(L^2(G)) \otimes \mathcal{C}(\mathfrak{g})$ extended from $\text{Tr} \otimes \text{Tr}_{\mathfrak{cl}}$, where $\text{Tr}$ denotes the operator trace on $B(L^2(G))$ and $\text{Tr}_{\mathfrak{cl}}$ denotes the Clifford trace (note that $B(L^2(G)) \otimes \mathcal{C}(\mathfrak{g})$ is a Type-I$\infty$ algebra). Since the Lie group $G$ is assumed to be finite dimensional, the Dirac operator $D = \sum_{i=1}^{n} e_i \otimes e_i$ acting on $L^2(G) \otimes S$ is $p$-summable for some finite $p$ greater than the dimension of $G$, hence $\hat{D}$ is also $p$-summable with respect to $\tau_i$. Moreover, there exists a smallest non-zero eigenvalue for $D^2$, so...
Thus \((D^2 + 1)^p (D|_{\ker D^\perp})^{-2p}\) is a bounded operator. In the case when \(G = SU(2)\), the smallest non-zero eigenvalue is \(|\rho|^2 = \dim(g)/24 = \frac{1}{6}\), so \((D^2 + 1)^p (D|_{\ker D^\perp})^{-2p}\) is bounded by \(65^p\). As a result

\[
\sum_{\lambda \in \sigma(D) \setminus \{0\}} |\lambda|^{-p} \tau_1 (E_D(\lambda)) = \tau_1 \left( |D_{\ker D^\perp}|^{-p} \right)
\]

\[
\leq \| (D^2 + 1)^p (D|_{\ker D^\perp})^{-2p} \| \cdot \tau_1 \left( (D^2 + 1)^{-p} \right) < \infty ,
\]

where \(\sigma(D)\) is the spectrum of \(D\) and the \(E_D(\lambda)\) are the spectral projections of \(D\) onto the \(\lambda\)-eigenspace. Note that \(\tau_1 (E_D(0)) = 1\) (see Theorem 2.5). We can index the eigenvalues of \(D\) by integers counting multiplicities with respect to \(\tau_1\), so that \(\lambda_n \in \sigma(D), \ldots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\), and \(\lambda_n = 0\) only when \(n = 0\). Recall that the multiplicity of an eigenvalue \(\lambda\) is given by the number \(\tau_1 (E_D(\lambda))\).

One would like the trace

\[
\tau(e^{-uD^2}) = \tau(e^{-u \sum_j \sigma_j^2 D_j^2}) = \sum_{n \in \mathbb{Z}^\infty} e^{-u \sum_j \sigma_j^2 \lambda_n^2} .
\]

to be finite, where \(\mathbb{Z}^\infty\) is the space of functions from \(\mathbb{N} \ni j \mapsto n_j \in \mathbb{Z}\) of finite support. We break the sum on the right according to the cardinality of the support. Namely, it is broken into sums over functions with no support, supported on one point, and supported on two points etc. Thus,

\[
\sum_{n \in \mathbb{Z}^\infty} e^{-u \sum_j \sigma_j^2 \lambda_n^2} = \sum_{i=0}^{\infty} \sum_{|\text{supp}(n)|=i} e^{-u \sum_j \sigma_j^2 \lambda_n^2} .
\]

We compute:

Sequences \(n \in \mathbb{Z}^\infty\) with \(|\text{supp}(n)| = 0\):

\[
\sum_{n \in \mathbb{Z}^\infty, |\text{supp}(n)|=0} e^{-u \sum_j \sigma_j^2 \lambda_n^2} = e^{-0} = 1 .
\]

Sequences \(n \in \mathbb{Z}^\infty\) with \(|\text{supp}(n)| = 1\):

\[
\sum_{n \in \mathbb{Z}^\infty, |\text{supp}(n)|=1} e^{-u \sum_j \sigma_j^2 \lambda_n^2} = \sum_j \sum_{\lambda \in \sigma(D) \setminus \{0\}} e^{-u \sigma_j^2 \lambda^2} \tau_1 (E_D(\lambda))
\]

\[
\leq \sum_j \| e^{-u \sigma_j^2 x^2} |x|^p \|_\infty \left( \sum_{\lambda \in \sigma(D) \setminus \{0\}} |\lambda|^{-p} \tau_1 (E_D(\lambda)) \right)
\]

\[
= \left( \sum_j \sigma_j^2 \right)^{-p} \frac{p}{2eul} \tau_1 \left( |D_{\ker D^\perp}|^{-p} \right) =: Y ,
\]

where \(\| \cdot \|_\infty\) is the uniform norm.

Sequences \(n \in \mathbb{Z}^\infty\) with \(|\text{supp}(n)| = 2\):

\[
\sum_{n \in \mathbb{Z}^\infty, |\text{supp}(n)|=2} e^{-u \sum_j \sigma_j^2 \lambda_n^2} = \sum_{i \neq j} \sum_{(x,y) \in \sigma(D) \setminus \{0,0\} \cup \{0,0\}} e^{-u (\sigma_i^2 x^2 + \sigma_j^2 y^2)} \tau_1 (E_D(x)) \tau_1 (E_D(y))
\]

\[
\leq Y^2 .
\]

With an induction argument, we obtain

\[
\sum_{|\text{supp}(n)|=i} e^{-u \sum_j \sigma_j^2 \lambda_n^2} \leq Y^i .
\]

Thus

\[
\sum_{n \in \mathbb{Z}^\infty} e^{-u \sum_j \sigma_j^2 \lambda_n^2} \leq \sum_{i=0}^{\infty} Y^i .
\]
We conclude that \( \tau(e^{-uD^2}) < \infty \) if the geometric series \( \sum_{i=0}^{\infty} Y^i \) converges. In other words, when

\[
Y := \left( \sum_j a_j^{-p} \right) \left( \frac{p}{2e u} \right)^{\frac{p}{2}} \tau_1 \left( |D_{\ker D^\perp}|^{-p} \right) < 1,
\]

or the \( p \)-norm of the reciprocal sequence \( \left\{ \frac{1}{a_j} \right\} \) is less than \( \left( \sqrt{\frac{2e u}{p}} \| D_{\ker D^\perp} \|_p \right)^{-1} \). The condition

\[
\left\| \left\{ \frac{1}{a_j} \right\} \right\|_p < \left( \sqrt{\frac{2e u}{p}} \| D_{\ker D^\perp} \|_p \right)^{-1}
\]

can be interpreted as that for any fixed \( u \), any diverging sequence with finite \( p \)-norm can be rescaled to give a weakly \( \theta \)-summable \( D \). Hence, as far as the rate of divergence is concerned, there are plenty of sequences that give rise to a weakly \( \theta \)-summable \( D \). In particular, when the system of graphs is the system of \( d \)-lattices, so the edges corresponding to the \( k \)-th successive refinement have weight \( 2^d k \). Then the sequence \( \left\{ \frac{1}{a_j} \right\} \) behaves like the harmonic sequence, as the \( k \)-th refinement adds about \( 2^d (k-1) \) edges to the previous graph that has edges carrying weight \( 2^d (k-1) \). Thus, \( \left\{ \frac{1}{a_j} \right\} \) has finite \( p \)-norm for any \( p > 1 \). And we can scale \( \left\{ \frac{1}{a_j} \right\} \) by an overall constant so that its \( p \)-norm is less than \( \left( \sqrt{\frac{2e u}{p}} \| D_{\ker D^\perp} \|_p \right)^{-1} \).

References

[1] J. Aastrup, J. Grimstrup, and R. Nest. Holonomy Loops, Spectral Triples and Quantum Gravity. arXiv:hep-th/0902.4191v1, 2009.

[2] J. Aastrup, J. Grimstrup, and R. Nest. A New Spectral Triple over a Space of Connections. Commun. Math. Phys., 290:389–398, 2009.

[3] J. Aastrup, J. Grimstrup, and R. Nest. On Spectral Triples in Quantum Gravity I. Class. Quantum Grav., 26:065011, 2009.

[4] J. Aastrup, J. Grimstrup, and R. Nest. On Spectral Triples in Quantum Gravity II. Journal of Noncommutative Geometry, 3:47–81, 2009.

[5] J. Aastrup, J. Grimstrup, M. Paschke, and R. Nest. On Semi-Classical States of Quantum Gravity and Noncommutative Geometry. arXiv:hep-th/0907.5510, 2009.

[6] M. Benameur and T. Fack. Type II non-commutative geometry. I. Dixmier trace in von Neumann algebras. Advance in Mathematics, 199:29–87, 2006.

[7] A. Carey, J. Phillips, A. Rennie, and F. Sukochev. The local index formula in semifinite von Neumann algebras I: Spectral flow. Adv. Math., 202:451–516, 2006.

[8] J. Choksi. Inverse Limits of measure spaces. Proc. London Math. Soc., S3-8:321–480, 1958.

[9] A. Connes. On the spectral characterization of manifolds. arXiv:math-OA/0810.2088, 2008.

[10] E. Getzler and A. Szenes. On the Chern Character of a Theta-Summable Fredholm Module. J. Func. Anal., 84:343–357, 1989.

[11] J. Gracia-Bondía, J. Várilly, and H. Figueroa. Elements of Noncommutative Geometry. Birkhäuser, 2000.

[12] A. Jaffe, A. Lesniewski, and K. Osterwalder. Quantum K-theory: the Chern character. Commun. Math. Phys., 112:75–88, 1988.

[13] A. Knapp. Lie Groups Beyond an Introduction. Birkhäuser, 1996.
[14] B. Kostant. Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the $\rho$-decomposition $C(g) = \text{End}(v_\rho) \otimes C(p)$, and the $g$-module structure of $\wedge g$. *Advance in Mathematics*, 2:275–350, 1997.

[15] A. Lai. On Type II noncommutative geometry and the even JLO character. arXiv:math-ph/1003.4226, 2010.

[16] E. MeinRenken. Lie groups and Clifford algebras. [http://www.math.toronto.edu/mein/teaching/clif_main.pdf](http://www.math.toronto.edu/mein/teaching/clif_main.pdf), 2009.

[17] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. I: Functional Analysis*. Academic Press Inc., 1970.

[18] C. Rovelli. *Quantum Gravity*. Cambridge University Press, 2004.

[19] J. Velhinho. A Groupoid Approach to Spaces of Generalized Connections. *Journal of Geometry and Physics*, 41:166–180, 2002.