A NEW REPRESENTATION FOR THE LANDAU-DE GENNES ENERGY OF NEMATIC LIQUID CRYSTALS

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Abstract. In the Landau-de Gennes theory on nematic liquid crystals, the well-known Landau-de Gennes energy depends on four elastic constants; $L_1$, $L_2$, $L_3$, $L_4$. For the general case of $L_4 \neq 0$, Ball-Majumdar [2] found an example that the Landau-de Gennes energy functional from physics literature [35] does not satisfy a coercivity condition, which causes a problem in mathematics to establish existence of energy minimizers. In order to solve this problem, we propose a new Landau-de Gennes energy, which is equal to the original for uniaxial nematic $Q$-tensors. The new Landau-de Gennes energy with general elastic constants satisfies the coercivity condition for all $Q$-tensors, which establishes a new link between mathematical and physical theory. Similarly to the work of Majumdar-Zarnescu [37], we prove existence and convergence of minimizers of the new Landau-de Gennes energy. Moreover, we find a new way to study the limiting problem of the Landau-de Gennes system since the cross product method [6] on the Ginzburg-Landau equation does not work for the Landau-de Gennes system.

1. Introduction

Liquid crystal is a state of matter between isotropic liquid and crystalline solid. Based on molecular positional and orientational orders, there are three main phases: sematic, cholesterics and nematic [36, p. 578]. The nematic phase is the most common type in which the general states are the uniaxial and biaxial state. Due to the anisotropic microstructure, some physical properties such as light polarization, of substances will change under external influence. It is best known for the use in liquid crystal displays.

In their pioneering works, Oseen [41] and Frank [18] discovered the first mathematical continuum theory of uniaxial nematic liquid crystals through a vector representation. Let $\Omega$ be a domain in $\mathbb{R}^3$. For a unit director $u \in W^{1,2}(\Omega; S^2)$, the Oseen-Frank energy density is given by

$$W(u, \nabla u) = \frac{k_1}{2} (\text{div} u)^2 + \frac{k_2}{2} (u \cdot \text{curl} u)^2 + \frac{k_3}{2} |u \times \text{curl} u|^2 + \frac{k_2 + k_4}{2} (\text{tr} (\nabla u)^2 - (\text{div} u)^2),$$

where $k_1, k_2, k_3$ are the Frank constants for molecular distortion of splay, twist and bend respectively and $k_4$ is the Frank constant for the surface energy.

The Oseen-Frank energy, which can only account for uniaxial phases, is one of the successful theories for modelling nematic liquid crystals in physics [43]. It is also of great interest to study the biaxial phase. In 1970, Freiser [19] hypothesized a

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rare substance having a biaxial phase, which was later discovered by Madsen et al. [34] in 2004. To study the phenomenon of phase transitions, de Gennes [10] in 1971 discovered a matrix representation, known as the $Q$-tensor order parameter, and the first expression of the elastic energy of this $Q$-tensor with the Landau theory [44, p. 208]. Presently, the Landau-de Gennes theory is well-known for capturing the phase transitions and biaxial state of liquid crystals. The Landau-de Gennes theory has been verified in physics as one of the successful theories for modelling the nematic liquid crystals. Indeed, Pierre-Gilles de Gennes was awarded a Nobel prize for physics in 1991 “for discovering that methods developed for studying order phenomena in simple systems can be generalized to more complex forms of matter, in particular to liquid crystals and polymers”.

In the Landau-de Gennes framework, a symmetric, traceless $3 \times 3$ matrix $Q \in M_{3 \times 3}$ is known as the $Q$-tensor order parameter, where $M_{3 \times 3}$ denotes the space of $3 \times 3$ matrices. The space of symmetric, traceless $Q$-tensors is defined by

$$(1.2) \quad S_0 := \{Q \in M_{3 \times 3} : Q^T = Q, \text{ tr } Q = 0\}.$$ 

For a tensor $Q \in W^{1,2}(\Omega; S_0)$, its Landau-de Gennes energy is defined by

$$E_{LG}(Q; \Omega) = \int_{\Omega} (f_E + f_B) \, dx,$$

where $f_E$ is the elastic energy density with elastic constants $L_1, ..., L_4$ of the form

$$(1.3) \quad f_E(Q, \nabla Q) := \frac{L_1}{2} |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{L_3}{2} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{ij}}{\partial x_k} + \frac{L_4}{2} Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k},$$

and $f_B(Q)$ is a bulk energy density with three positive constant $a, b, c$ defined by

$$(1.4) \quad f_B(Q) := -\frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \left[\text{tr}(Q^2)\right]^2.$$ 

Here and in the sequel, we adopt Einstein summation convention for repeated indices.

For a tensor $Q \in W^{1,2}(\Omega; S_0)$, de Gennes [10] first discovered a two-term expression of the elastic energy density in [10]

$$\frac{L_1}{2} |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k}.$$ 

In 1983, Schiele and Trimper [42, p. 268] revealed that the early attempt of de Gennes’s work [10] was incomplete since the connection with the Oseen-Frank density in [11] would require the splay and bend Frank constants to be equal (i.e. $k_1 = k_3$), but, some experiments on liquid crystals showed that $k_3 > k_1$, so they extended the original de Gennes representation to one with a third order term involving a elastic constant $L_4$:

$$\frac{L_1}{2} |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{L_4}{2} Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}.$$ 

In 1984, Berreman and Meiboom [5] observed that above two groups discarded the surface energy density in the Oseen-Frank density, which correlates the blue phase theory for liquid crystals. They proposed to recover a second order term in $Q$ with four third order terms. Longa et al. [33] proposed the Landau-Ginzburg-de Gennes density with six third order terms. Finally, Dickmann [11] found the form (1.3), which is consistent with the Oseen-Frank density in (1.1) (see details in [35], [3]).
A new representation for the Landau-de Gennes energy

Since then, the general form (1.3) of the Landau-de Gennes representation has been widely accepted for modelling liquid crystals (c.f. [35],[38]).

From a mathematical point of view, a general form of the tensor \( Q \in S_0 \) can be written as
\[
Q := s(u \otimes u - \frac{1}{3} I) + r(w \otimes w - \frac{1}{3} I), \quad u, w \in S^2, \quad s, r \in \mathbb{R}.
\]

Here \( u, w \) are two independent direction fields for biaxial liquid crystals and \( I \) is the identity matrix. When the tensor \( Q \) has two equal non-zero eigenvalues, a nematic liquid crystal is said to be uniaxial. When \( Q \) has two unequal non-zero eigenvalues, a nematic liquid crystal is said to be biaxial. For material constants \( a, b, c > 0 \), we define the constant order parameter
\[
s_+ := \frac{b + \sqrt{b^2 + 24ac}}{4c}.
\]

We define a subspace
\[
S_* := \left\{ Q \in S_0 : \ Q = s_+ (u \otimes u - \frac{1}{3} I), \ u \in S^2 \right\}.
\]

It is well-known (e.g. [38]) that \( Q \in S_* \) if only if \( f_B(Q) := f_B(Q) - \inf_{S_0} f_B = 0 \).

Although there are many differences between the Oseen-Frank theory and the Landau-de Gennes theory, it is of great interest in mathematics and physics whether the Oseen-Frank theory can be approximated by the Landau-de Gennes theory [10]. As it was pointed out in [35], Dickmann discovered that for an uniaxial phase \( Q = s(u \otimes u - \frac{1}{3} I) \), the elastic energy density \( f_E(Q, \nabla Q) \) in (1.3) is equal to the Oseen-Frank energy density \( W(u, \nabla u) \). For the case of uniaxial phase, both the Oseen-Frank theory and the Landau-de Gennes theory unify in physics modelling. In mathematics literature, most research focus on the study of the one-constant approximation [1]: i.e. the elastic parameters satisfy \( L_2 = L_3 = L_4 = 0 \) in (1.3).

Then the density
\[
f_E(Q, \nabla Q) = \frac{L_1}{2} |\nabla Q|^2.
\]

In this case, the Landau-de Gennes energy of \( Q \in W^{1,2}(\Omega; S_0) \) is simplified by
\[
E_{SLG}(Q; \Omega) = \int_{\Omega} \left( \frac{L_1}{2} |\nabla Q|^2 + f_B(Q) \right) dx.
\]

Given \( Q_0 \in W^{1,2}(\Omega; S_*) \), there is a minimizer of \( E_{SLG} \) in \( W^{1,2}_{Q_0}(\Omega; S_0) \), which satisfies the Euler-Lagrange equation
\[
\Delta Q_{ij} = \frac{1}{L_1} \left( -aQ_{ij} - b\left(Q_{ik}Q_{kj} - \frac{\delta_{ij}}{3} \text{tr}(Q^2) \right) + cQ_{ij} \text{tr}(Q^2) \right).
\]

Majumdar-Zarnescu [37] proved that as \( L_1 \to 0 \), minimizers \( Q_{L_1} \) of \( E_{SLG} \) converges to \( Q_* = s_+ (u^* \otimes u^* - \frac{1}{3} \text{Id}) \), where \( Q_* \) is a minimizer of \( E_{SLG} \) in \( W^{1,2}_{Q_0}(\Omega; S_*) \). Later, Nguyen-Zarnescu [41] improved the result that minimizers \( Q_{L_1} \) converge smoothly to \( Q_* \) except a singular set.

In theory of liquid crystals, the general expectation on the elastic constants is that \( L_1 > 0, L_2 > 0, L_3 \) and \( L_4 \) are not always zero (c.f. [42], p. 268). Therefore, it is very important to study whether the limit of solutions to the Landau-de Gennes system is a solution to the Oseen-Frank system for a general case of \( L_1, \cdots, L_4 \). In 2D, Bauman, Park and Phillips [4] investigated a limiting result of minimizers of the energy \( E_{LG} \) with \( L_4 = 0 \) (see also [26]). However, the above limiting problem is open for the case with \( L_4 \neq 0 \).
A fundamental problem in mathematics on the Landau-de Gennes theory is to establish existence of minimizers of the energy functional $E_{LG}$ in $W^{1,2}_Q(\Omega; S_0)$ for a general case of elastic constants $L_1, \cdots, L_4$. To prove the existence of a minimizer of the functional $E_{LG}(Q, \Omega)$ in $W^{1,2}(\Omega; S_0)$, one must show that the functional $E_{LG}$ is lower semi-continuous in $W^{1,2}(\Omega; S_0)$. By the standard theory of calculus variations (e.g. [21]), it is necessary to establish that $f_E(Q, \nabla Q)$ is bounded below by $a|\nabla Q|^2 - C$ for some $a > 0$. Therefore, it is very important to study the bound below problem of $f_E(Q, \nabla Q)$. When $L_4 = 0$, Longa et al. [33] found the stability criteria

$$L_1 > 0, -L_1 < L_3 < 2L_1, L_1 + \frac{5}{3}L_2 + \frac{1}{6}L_3 > 0, L_4 = 0.$$  

Under this condition, Davis and Gartland [9] showed a lower bound of $f_E$ and Kitaev et al. [32] established the coercivity condition on the Landau-de Gennes energy density. For the case of $L_4 \neq 0$ in [13], Ball-Majumdar [2] found an example that $f_E(Q, \nabla Q)$ is unbounded from below, so one cannot prove existence of a minimizer of the functional $E_{LG}(Q, \Omega)$ in $W^{1,2}(\Omega; S_0)$. Therefore, the Dickmann’s representation [13] causes a knowledge gap between mathematics and physics, which is very challenging in mathematics since the energy functional $E_{LG}$ in $W^{1,2}(\Omega; S_0)$ does not satisfy a coercivity condition and violates the existence theorem of minimizers [4]. To attain the coercivity for the case of $L_4 \neq 0$, Iyer, Xu and Zarnescu [31] studied the 2D problem and imposed a small condition on the supremum of the unknown $Q$ to gain some control on the $L_4$ term. Mucci and Nicolodi [39] proved that the energy functional satisfied a coercivity condition under some special conditions on the material constants. In contrast to the above continuum theory, Ball and Majumdar [2] suggested a statistical approach from the Maier-Saupe theory and proposed a singular bulk potential instead of the Landau-de Gennes bulk potential to attain the coercivity condition. This new setting has been investigated by many [14, 15, 16]. A comprehensive review of this statistical approach, please refer to [129].

To solve the above problem, we propose a new Landau-de elastic energy density for the general case of $L_1, \cdots, L_4$; i.e. the constants $L_2, L_3, L_4$ are not zero. More precisely, we observe that for uniaxial tensors $Q \in S_*$, the elastic energy density $f_E(Q, \nabla Q)$ in [13] is equivalent to the new form

$$f_{E,1}(Q, \nabla Q) = \left( \frac{L_1}{2} - \frac{s_+L_4}{3} \right) |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{L_3}{2} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{ij}}{\partial x_k} + \frac{3L_4}{2s_+} Q_{ln}Q_{kn} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}$$

$$= \frac{1}{2} \left( L_1 - |L_3| - \frac{2s_+L_4}{3} \right) |\nabla Q|^2 + V(Q, \nabla Q)$$

(1.7)

with

$$V(Q, \nabla Q) := \frac{L_2}{2} \sum_{i=1}^3 \left( \sum_{j=1}^3 \frac{\partial Q_{ij}}{\partial x_j} \right)^2 + \frac{|L_3|}{4} \sum_{i,j,k=1}^3 \left( \frac{\partial Q_{ik}}{\partial x_j} \right)^2 \left( \frac{\partial Q_{ij}}{\partial x_k} \right)^2$$

$$+ \frac{3L_4}{2s_+} \sum_{i,j,n=1}^3 \left( \sum_{i,j,k=1}^3 \frac{Q_{kn}}{\partial x_k} \right)^2.$$
Assuming that $L_2 \geq 0$, $L_4 \geq 0$ and $L_1 - |L_3| - \frac{2s_+}{3}L_4 > 0$, it is clear that $f_{E,1}(Q, \nabla Q)$ has a lower bound for all $Q \in S_0$.

By the new form of $f_{E,1}(Q, \nabla Q)$ in (1.7), for each $Q \in W^{1,2}(\Omega, S_0)$, we suggest a new Landau-de Gennes energy functional

\[
E_L(Q; \Omega) = \int_\Omega \left( f_{E,1}(Q, \nabla Q) + \frac{1}{L} f_B(Q) \right) \, dx.
\]

Here $L$ is a parameter to drive all elastic constants to zero [1, 30, 38].

Then we have

**Theorem 1.** Assume that $L_2 \geq 0$, $L_4 \geq 0$ and $L_1 - |L_3| - \frac{2s_+}{3}L_4 > 0$. For each $L > 0$, there exists a minimizer $Q_L$ of the new Landau-de Gennes energy $E_L$ in $W^{1,2}_Q(\Omega; S_0)$ with a given boundary $Q_0 \in W^{1,2}(\Omega; S_1)$. As $L \to 0$, the minimizers $Q_L$ of $E_L$ in $W^{1,2}_Q(\Omega; S_0)$ converge strongly to $Q_*$ in $W^{1,2}_Q(\Omega; S_0)$, where $Q_* = s_+(u_* + \frac{L}{4})$ is a minimizer of the elastic energy functional

\[
E(Q; \Omega) = \int_\Omega f_{E,1}(Q, \nabla Q) \, dx = \int_\Omega f_E(Q, \nabla Q) \, dx
\]

in $W^{1,2}_Q(\Omega; S_*).$ Moreover, $Q_*$ is partially regular in $\Omega$.

In Lemma 2.2 we prove that a minimizer $Q_*$ of $E(Q; \Omega)$ in $W^{1,2}_Q(\Omega; S_*)$ satisfies the following Euler-Lagrange equation

\[
\alpha \left( -s_+ \Delta Q_{ij} + 2\nabla_k Q_{il} \nabla_k Q_{lj} - 2s_+^{-1}(Q_{ij} + \frac{s_+}{3}\delta_{ij})|\nabla Q|^2 \right)
- \nabla_k \left( (Q_{ij} + \frac{s_+}{3}\delta_{ij})V_{il}^{\delta_{ij}} + (Q_{il} + \frac{s_+}{3}\delta_{il})V_{ij}^{\delta_{il}} - 2s_+^{-1}(Q_{ij} + \frac{s_+}{3}\delta_{ij})(Q_{im} + \frac{s_+}{3}\delta_{im})V_{l\delta_{im}}^{\delta_{ij}} \right)
+ V_{il}^{\delta_{ij}} \nabla_k Q_{jl} + V_{lj}^{\delta_{lj}} \nabla_k Q_{ij} - 2s_+^{-1}V_{il}^{\delta_{ij}} \left( (Q_{ij} + \frac{s_+}{3}\delta_{ij})\nabla_k Q_{im} + (Q_{il} + \frac{s_+}{3}\delta_{il})\nabla_k Q_{ij} \right)
+ V_{jl}^{\delta_{lj}} (Q_{ij} + \frac{s_+}{3}\delta_{ij}) + V_{ij}^{\delta_{ij}} (Q_{il} + \frac{s_+}{3}\delta_{il}) - 2s_+^{-1}V_{im}^{\delta_{im}} (Q_{lj} + \frac{s_+}{3}\delta_{lj}) = 0
\]

in the weak sense for $\alpha := L_1 - |L_3| - \frac{2s_+}{3}L_4 > 0$. In particular, for the case of $L_2 = L_3 = L_4 = 0$, (1.9) is simplified to

\[
s_+ \Delta Q_{ij} - 2\nabla_k Q_{il} \nabla_k Q_{lj} + 2s_+^{-1}(Q_{ij} + \frac{s_+}{3}\delta_{ij})|\nabla Q|^2 = 0,
\]

which is equivalent to the harmonic map equation of $u$. Comparing with the result in [29], the weak solution of (1.9) might be not unique.

**Remark 1.** Using the result of Kitavtsev et al. [32], the condition on $L_1, \ldots, L_4$ in Theorem 1 can be improved by the following condition:

\[
L_1 - \frac{s_+L_4}{6} > 0, \quad -L_1 - \frac{s_+L_4}{6} < L_3 < 2L_1 - \frac{s_+L_4}{3}, \quad L_1 - \frac{s_+L_4}{6} + \frac{5}{3}L_2 + \frac{1}{6}L_3 > 0, \quad L_4 \geq 0.
\]

**Remark 2.** When $L_2 = L_3 = L_4 = 0$, Majumdar-Zarnescu [37] proved a monotonicity formula for minimizers $Q_L$ of $E_{SLG}(Q; \Omega)$ in $W^{1,2}(\Omega, S_0)$. However, in general cases of $L_1, \ldots, L_4$, there is no monotonicity formula for minimizers $Q_L$. 
of \( E_{LC}(Q; \Omega) \) in \( W^{1,2}(\Omega, S_0) \), so it is a very interesting question whether one can improve the convergence of \( Q_L \) for general cases.

In Theorem 11 we assume that \( L_4 \geq 0 \). For general case of \( L_4 \), we will obtain a new form of the Landau-de Gennes energy density through a strong Ericksen’s condition on the Oseen-Frank density. More precisely, using the condition that

\[
\begin{align*}
\sigma_1^2 L_1 &= -\frac{1}{2} k_1 + \frac{1}{6} k_2 + \frac{1}{6} k_3, \\
\sigma_2^2 L_2 &= k_2 - k_4, \\
\sigma_3^2 L_3 &= k_4, \\
\sigma_4^3 L_4 &= -\frac{1}{2} k_1 + \frac{1}{2} k_3,
\end{align*}
\]

it was shown in [34] that for each \( Q = s_+(u \otimes u - \frac{1}{4} I) \in S_* \),

\[
W(u, \nabla u) = f_E(Q, \nabla Q).
\]

Recent studies [1, 17, 30] revealed that the strong Ericksen condition on \( k_1, \cdots, k_4 \) is required for the Oseen-Frank energy to ensure the existence of minimizers. Note that \( W(u, \nabla u) \) in (1.1) is quadratic in \( \nabla u \), but the \( (k_2 + k_4) \) term could be negative, so the coercivity \( W(u, \nabla u) \geq a|\nabla u|^2 \) is unclear. It was pointed out in [30] (see also [13]) that assuming the strong Ericksen condition

\[
k_2 > |k_4|, \quad k_3 > 0, \quad 2k_1 > k_2 + k_4,
\]

there are positive constants \( \lambda \) and \( C \) such that the density \( W(u, \nabla u) \) is equivalent to a form that \( W(z, p) \) satisfies

\[
\lambda |p|^2 \leq W(z, p) \leq C|p|^2, \quad \lambda |\xi|^2 \leq W_{p_ip_j}(z, p) \xi_i \xi_j \leq C|\xi|^2
\]

for any \( \xi \in M^{3 \times 3} \), any \( p \in M^{3 \times 3} \) and any \( z \in \mathbb{R}^3 \) with \( |z| \leq M \) for some constant \( M > 0 \) (see details in Lemma 3.1).

Through the relation (1.1) between Frank’s consists \( k_1, \cdots, k_4 \) and elastic constants \( L_1, \cdots, L_4 \), the strong Ericksen condition (1.12) is equivalent to a condition that

\[
L_1 - \frac{1}{2} L_3 > \frac{s_+}{3} L_4, \quad L_1 + \frac{1}{2} L_2 + \frac{1}{2} L_3 + \frac{2s_+}{3} L_4 > 0, \\
L_1 + L_2 + \frac{1}{2} L_3 > \frac{s_+}{3} L_4.
\]

In this paper, we extend that result in the Oseen-Frank energy density to the Q-tensor using the rotational invariant property such that for the condition (1.13) on elastic constants \( L_1, \cdots, L_4 \), we can recover the coercivity condition on the Landau-de Gennes energy density and establish that:

**Theorem 2.** Assume that \( L_1, L_2, L_3 \) and \( L_4 \) satisfy the condition (1.13). Then for each \( Q \in S_* \), \( f_E(Q, \nabla Q) \) is equivalent to a new form

\[
f_{E, 2}(Q, \nabla Q) := \frac{\alpha}{2} |\nabla Q|^2 + V(Q, \nabla Q)
\]

for some constant \( \alpha > 0 \). Here \( V(Q, \nabla Q) \) is a sum of square terms as in [32, 20].

**Corollary 1.** For the case that \( \min\{k_1, k_2, k_3\} \geq k_2 + k_4 =: \tilde{\alpha} > 0 \) (c.f. [27, p. 551], [28, p. 467]), we know that

\[
W(u, \nabla u) = \frac{\tilde{\alpha}}{2} |\nabla u|^2 + V(u, \nabla u)
\]
Then the explicit form of $V(Q, \nabla Q)$ in (1.14) is

\begin{equation}
V(Q, \nabla Q) = (L_1 + \frac{L_2}{2} + \frac{L_3}{2} - \frac{s_+}{3}L_4 - \frac{1}{2}\alpha) \sum_{i,j=1}^{3} \left( \sum_{k=1}^{3} (s_+^{-1}Q_{kj} + \frac{1}{3}\delta_{kj}) \nabla_i Q_{ij} \right)^2 \\
+ (L_1 - \frac{s_+}{3}L_4 - \frac{1}{2}\alpha) \left( \sum_{i,j=1}^{3} (s_+^{-1}Q_{ij} + \frac{1}{3}\delta_{ij})(\nabla_i Q_j)_i \right)^2 \\
+ (L_1 + \frac{L_2}{2} + \frac{L_3}{2} + \frac{2s_+}{3}L_4 - \frac{1}{2}\alpha) \sum_{j=1}^{3} \left| \sum_{i=1}^{3} (s_+^{-1}Q_{ij} - \frac{1}{3}I)_{ij} \times \nabla Q_j \right|^2,
\end{equation}

where $Q_i$ denotes the $i$-th column of the $Q$ matrix. Using the relation (1.11), $L_1$, ..., $L_4$ satisfy that $L_4 \leq 0$ and

\begin{equation}
\alpha = \min\{2L_1 + L_2 + L_3 - \frac{2s_+}{3}L_4, 2L_1 - \frac{2s_+}{3}L_4, 2L_1 + L_2 + L_3 + \frac{4s_+}{3}L_4 \} > 0.
\end{equation}

By the new form of $f_{E,2}(Q, \nabla Q)$ in (1.14) for each $Q \in W^{1,2}(\Omega, S_0)$, we can also introduce a new Landau-de Gennes energy functional

\begin{equation}
E_{L,2}(Q; \Omega) = \int_{\Omega} \left( f_{E,2}(Q, \nabla Q) + \frac{1}{L}f_B(Q) \right) dx.
\end{equation}

Then we have a similar result in Theorem 1.

It is not clear that each minimizer $Q_L$ of $E_L(Q; \Omega)$ or $E_{L,2}(Q; \Omega)$ in $W^{1,2}_{Q_0}(\Omega, S_0)$ is bounded. Therefore, the energy density $f_{E,1}(Q, \nabla Q)$ in (1.15) or $f_{E,2}(Q, \nabla Q)$ in (1.14) is not bounded above by $C|\nabla Q|^2 + C$. Without this above growth condition on the density, it is well-known that a minimizer $Q_L$ of the Landau-de Gennes energy functional in $W^{1,2}_{Q_0}(\Omega, S_0)$ does not satisfy the Euler-Lagrange equation in $W^{1,2}(\Omega, S_0)$. To overcome this difficulty, we introduce a smooth cut-off function $\eta(r)$ in $[0, \infty)$ so that $\eta(r) = 1$ for $r \leq M$ with a very large constant $M > 0$ and $\eta(r) = 0$ for $r \geq M + 1$. Then we modify the Landau-de Gennes density by

\begin{equation}
\tilde{f}_E(Q, \nabla Q) := \frac{\alpha}{2}|\nabla Q|^2 + \tilde{V}(Q, \nabla Q) = \frac{\alpha}{2}|\nabla Q|^2 + \eta(|Q|)V(Q, \nabla Q)
\end{equation}

with the property that

\[
\frac{\alpha}{2}|\nabla Q|^2 \leq \tilde{f}_E(Q, \nabla Q) \leq C|\nabla Q|^2.
\]

For a large $M > 0$ in (1.18), we consider a modified Landau-de Gennes functional

\begin{equation}
\hat{E}_L(Q; \Omega) = \int_{\Omega} \left( \tilde{f}_E(Q, \nabla Q) + \frac{1}{L}f_B(Q) \right) dx.
\end{equation}

Each minimizer $Q_L$ of the modified Landau-de Gennes energy functional (1.19) in $W^{1,2}_{Q_0}(\Omega, S_0)$ satisfies the Euler-Lagrange equation.
\[ \frac{\alpha \Delta Q_{ij} + \frac{1}{2} \nabla_k (\bar{V}_{p_{ik}} + \bar{V}_{p_{kj}}) - \frac{1}{3} \delta_{ij} \sum_{l=1}^{3} \nabla_k \bar{V}_{p_{lj}} - \frac{1}{2} (\bar{V}_{Q_{ij}} + \bar{V}_{Q_{ji}}) + \frac{1}{3} \delta_{ij} \sum_{l=1}^{3} \bar{V}_{Q_{il}}}{L} = \frac{1}{L} \left(-a Q_{ij} - b (Q_{ik} Q_{kj} - \frac{1}{3} \delta_{ij}) \text{tr}(Q^2) + c Q_{ij} \text{tr}(Q^2)\right) \]

in the weak sense.

**Remark 3.** Any weak solution \( Q_L \) of (1.20) with boundary value \( Q_0 \in W^{1,2}(\Omega, S^*_0) \) is uniformly bounded; i.e. for a sufficiently large \( M > 0 \), \( |Q_L| \leq M + 1 \). By using the result of Giaquinta-Giusti [22] (see also [21, 25]), it implies that \( Q_L \) is partially regular inside \( \Omega \).

The Landau-de Gennes theory is also related to the study of the Ginzburg-Landau approximation. The Ginzburg-Landau functional was introduced in [24] to study the phase transition in superconductivity. For a parameter \( \varepsilon > 0 \), the Ginzburg-Landau functional of \( u : \Omega \to \mathbb{R}^3 \) is defined by

\[ E_\varepsilon(u; \Omega) := \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right) dx. \]

The Euler-Lagrange equation is

\[ \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) = 0. \]

In particular, using the cross product, the equation (1.22) becomes

\[ \nabla \cdot (u_\varepsilon \times \nabla u_\varepsilon) = 0. \]

Chen [6] proved that as \( \varepsilon \to 0 \), solutions \( u_\varepsilon \) of the Ginzburg-Landau system (1.22) weakly converge to a harmonic map in \( W^{1,2}(\Omega; \mathbb{R}^3) \). Moreover, Chen and Struwe [8] proved global existence of partial regular solutions to the heat flow of harmonic maps using the Ginzburg-Landau approximation.

By comparing with the result of Chen [6] (see also [7]) on the weak convergence of solutions of the Ginzburg-Landau equations, it is very interesting to study whether the solutions \( Q_L \) of the Landau-de Gennes equations (1.20) with a uniform bound of the energy, i.e. \( \bar{E}_L(Q_L; \Omega) \leq C \) for a uniform constant \( C > 0 \), converge weakly to a solution \( Q_* \) of (1.14) in \( W^{1,2}(\Omega; S_0) \). However, it seems that the problem is not clear when \( L_2 \) and \( L_3 \) are not zero. Under a strong condition, we solve this problem to prove:

**Theorem 3.** Let \( Q_L \) be a weak solution to the equation (1.20). Assume that the solution \( Q_L \) converges strongly to \( Q_* \) in \( W^{1,2}_{L^{1,2}}(\Omega; S_0) \) as \( L \to 0 \) and satisfies

\[ \lim_{L \to 0} \frac{1}{L} \int_\Omega \bar{f}_B(Q_L) \, dx = 0. \]

Then, \( Q_* \) is a weak solution to (1.14).

In the proof of Theorem 3, we show that for any \( Q \in S_* \), the Hessian of the bulk density \( \bar{f}_B(Q) \) is positive definite for a uniform constant. As in [8], we note that in a neighbourhood \( S_\delta \) of the space \( S_* \), there is a smooth projection \( \pi \). Then we employ Taylor’s expansion and Egoroff’s theorem to prove Theorem 3.

The paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2. In Section 4, we prove Theorem 3.
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2. Proof of Theorem 1 and the Euler-Lagrange equation

Lemma 2.1. For a uniaxial $Q \in S_*$ of the form
$$Q = s_+(u \otimes u - \frac{1}{3}I), \quad u \in S^2,$$
the elastic potential $f_E(Q, \nabla Q)$ in (1.3) satisfies

$$f_E(Q, \nabla Q) = \left(\frac{L_1}{2} - \frac{s_+L_4}{3}\right) \sum_{i,j,k} \left(\frac{\partial Q_{ij}}{\partial x_k}\right)^2 + \frac{L_2}{2} \sum_{i,j,k} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k}$$
$$+ \frac{L_3}{2} \sum_{i,j,k,l} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{jl}}{\partial x_k} + \frac{3L_4}{2s_+} \sum_{i,j,k,l,n} Q_{ln} Q_{kn} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}. \quad (2.1)$$

Proof. Using the fact that $|u| = 1$, we have

$$Q_{ln} Q_{kn} = s_+^2 (u_k u_n - \frac{1}{3} \delta_{kn})(u_l u_n - \frac{1}{3} \delta_{ln})$$
$$= s_+^2 \left(\frac{1}{3} u_k u_l + \frac{1}{9} \delta_{kl} - \frac{1}{3} \delta_{kn} u_l u_n - \frac{1}{3} \delta_{ln} u_k u_n + \frac{1}{9} \delta_{kn} \delta_{ln}\right)$$
$$= s_+^2 \left(\frac{1}{3} u_k u_l + \frac{1}{9} \delta_{kl}\right) = \frac{s_+}{3} s_+(u_k u_l - \frac{1}{3} \delta_{lk}) + \frac{2s_+^2}{9} \delta_{kl}$$
$$= \frac{s_+}{3} Q_{kl} + \frac{2s_+^2}{9} \delta_{kl}. \quad (2.2)$$

Applying the identity (2.2) to the $L_4$ term of (1.3), we obtain

$$Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} = \left(\frac{3}{s_+} Q_{ln} Q_{kn} - \frac{2s_+}{3} \delta_{lk}\right) \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}$$
$$= \frac{3}{s_+} (Q_{ln} \frac{\partial Q_{ij}}{\partial x_l})(Q_{kn} \frac{\partial Q_{ij}}{\partial x_k}) - \frac{2s_+}{3} |\nabla Q|^2. \quad (2.3)$$

Substituting (2.3) into (1.3), we prove (2.1). \(\square\)

Now we give the proof of Theorem 1.

Proof. Under the condition on $L_1, \ldots, L_4$ in Theorem 1 it is clear that

$$f_{E,1}(Q, \nabla Q) \geq \left(\frac{L_1}{2} - \frac{L_3}{2} - \frac{s_+L_4}{3}\right)|\nabla Q|^2, \quad \forall Q \in S_0.$$

By the standard theory of calculus of variations [20], there is a minimizer $Q_L$ of $E_L$ in $W^{1,2}_0(\Omega; S_0)$. For each $Q \in W^{1,2}_0(\Omega; S_0)$, we set

$$E(Q; \Omega) := \int_{\Omega} f_{E,1}(Q, \nabla Q) \, dx.$$

It implies that

$$E(Q_L; \Omega) + \int_{\Omega} (f_B(Q_L) - \inf_{S_0} f_B) \, dx \leq E(Q; \Omega)$$
for any $Q \in W^{1,2}_{Q_0}(\Omega; S_\ast)$ with the fact that $\tilde{J}_B(Q) = f_B(Q) - \inf_{S_\ast} f_B = 0$.

As $L \to 0$, minimizers $Q_L$ converge (possible passing subsequence) weakly to a tensor $Q_\ast \in W^{1,2}(\Omega; S_0)$ with that $f_B(Q_\ast) = 0$, which implies that $Q_\ast \in S_\ast$ a.e. in $\Omega$. Then, for any $Q \in W^{1,2}_{Q_0}(\Omega; S_\ast)$, we have

$$E(Q_\ast; \Omega) \leq \lim\inf_{L \to 0} E(Q_L; \Omega) \leq \lim\sup_{L \to 0} E(Q_L; \Omega) \leq E(Q; \Omega).$$

Therefore $Q_\ast$ is also a minimizer of $E$ in $W^{1,2}_{Q_0}(\Omega; S_\ast)$. Choosing $Q = Q_\ast$ in above inequality, it implies that

$$E(Q_\ast; \Omega) = \lim_{L \to 0} E_L(Q_L; \Omega), \quad \lim_{L \to 0} \frac{1}{L} \int_\Omega \tilde{f}_B(Q_L) \, dx = 0.$$

Moreover, it is known that

$$\int_\Omega |\nabla Q_\ast|^2 \, dx \leq \lim\inf_{L \to 0} \int_\Omega |\nabla Q_L|^2 \, dx,$$

$$\int_\Omega V(Q_\ast, \nabla Q_\ast) \, dx \leq \lim\inf_{L \to 0} \int_\Omega V(Q_L, \nabla Q_L) \, dx.$$

It implies that $\int_\Omega |\nabla Q_\ast|^2 \, dx = \lim\inf_{L \to 0} \int_\Omega |\nabla Q_L|^2 \, dx$. Otherwise, there is a subsequence $L_k \to 0$ that

$$\int_\Omega |\nabla Q_\ast|^2 \, dx < \lim_{L_k \to 0} \int_\Omega |\nabla Q_{L_k}|^2 \, dx.$$

Then

$$E(Q_\ast; \Omega) = \lim_{L_k \to 0} E_{L_k}(Q_{L_k}; \Omega),$$

$$= \left( \frac{L_1}{2} - \frac{|L_3|}{2} - \frac{s_0 L_3}{3} \right) \lim_{L_k \to 0} \int_\Omega |\nabla Q_{L_k}|^2 \, dx + \lim_{L_k \to 0} \int_\Omega V(Q_{L_k}, \nabla Q_{L_k}) \, dx \leq E(Q_\ast; \Omega).$$

This is impossible. Therefore, minimizers $Q_{L_k}$ strongly converge, up-to a subsequence, to a minimizer $Q_\ast = s_\ast (u_\ast \otimes u_\ast - \frac{1}{2} I)$ of $E$ in $W^{1,2}_{Q_0}(\Omega; S_0)$. Following from the next lemma, $Q_\ast$ satisfies (1.12). Applying the result of Dickmann, $u_\ast$ is a minimizer of the Oseen-Frank energy in $W^{1,2}(\Omega; S^2)$. Due to the well-known result of Hardt-Kinderlehrer-Lin [27], $u_\ast$ is partially regular in $\Omega$ (see also [28]). Thus $Q_\ast$ is partially regular.

\textbf{Lemma 2.2.} If $Q$ is a minimizer of $E$ in $W^{1,2}_{Q_0}(\Omega; S_\ast)$, it satisfies

$$\alpha \left( -s_\ast \Delta Q_{ij} + 2\nabla_k Q_{il} \nabla_k Q_{jl} - 2s_\ast^{-1}(Q_{ij} + \frac{s_\ast}{3} \delta_{ij})|\nabla Q|^2 \right) - \nabla_k \left( (Q_{jl} + \frac{s_\ast}{3} \delta_{jl})V_{p_{ij}l} + (Q_{il} + \frac{s_\ast}{3} \delta_{il})V_{p_{jl}i} - 2s_\ast^{-1}(Q_{ij} + \frac{s_\ast}{3} \delta_{ij})(Q_{lm} + \frac{s_\ast}{3} \delta_{im}) V_{p_{il}m} \right)$$

$$+ V_{p_{il}} \nabla_k Q_{jl} + V_{p_{jl}} \nabla_k Q_{il} - 2s_\ast^{-1} V_{p_{il}} \left( \nabla_k Q_{lm}(Q_{ij} + \frac{s_\ast}{3} \delta_{ij}) + (Q_{lm} + \frac{s_\ast}{3} \delta_{im}) \nabla_k Q_{ij} \right)$$

$$+ V_{p_{ij}}(Q_{jl} + \frac{s_\ast}{3} \delta_{jl}) + V_{p_{jl}}(Q_{il} + \frac{s_\ast}{3} \delta_{il}) - 2s_\ast^{-1} V_{p_{il}}(Q_{lm} + \frac{s_\ast}{3} \delta_{im})(Q_{ij} + \frac{s_\ast}{3} \delta_{ij})$$

$$= 0$$

in the weak sense.
Proof. Let \( \phi \in C_0^\infty(\Omega; \mathbb{R}^3) \) be a test function. For each \( u_t = \frac{u + t \phi}{|u + t \phi|} \) with \( t \in \mathbb{R} \), define

\[
Q_t(x) = Q(u_t(x)) = s_+ \left( u_t \otimes u_t - \frac{1}{3} I \right) \in S_+.
\]

For any \( \eta \in C_0^\infty(\Omega; S_0) \), we choose a test function \( \phi_i := u_k \eta_{ik} \). If \( Q \) is a minimizer, the first variation of the energy of \( Q \) is zero; i.e.

\[
\frac{d}{dt} \left|_{t=0} \int_{\Omega} f_E(Q_t, \nabla Q_t) \, dx \right| = \int_{\Omega} f_{Q_{t;i,j}} \frac{dQ_{t;i,j}}{dt} + f_{p^*_k} \frac{d}{dt} \frac{\partial Q_{t;i,j}}{\partial x^k} \, dx \bigg|_{t=0} = 0.
\]

Note that

\[
\frac{dQ_{t;i,j}}{dt} = s_+ \left( \phi_i (u_j + t \phi_j) + (u_i + t \phi_i) \phi_j \right) - \frac{2(u \cdot \phi) + 2l|\phi|^2(\eta + t(\phi_i)(u_j + t \phi_j))}{|u + t \phi|^2}
\]

\[
= \left((Q_{jl} + s_+ \delta_{jl}) \eta_{il} + (Q_{il} + s_+ \delta_{il}) \eta_{jl} - 2s_+^{-1}(Q_{ij} + s_+ \delta_{ij})(Q_{lm} + s_+ \delta_{lm}) \eta_{lm}\right)
\]

where we used the fact that \( |u| = 1 \) and \( \phi_i := u_k \eta_{ik} \). We also observe

\[
\frac{dQ_{t;i,j}}{dt} \bigg|_{t=0} = s_+ \left( (u_j \phi_i + u_i \phi_j - 2(u \cdot \phi)(u_i u_j)) \right)
\]

\[
= (Q_{jl} + \frac{s_+}{3} \delta_{jl}) \eta_{il} + (Q_{il} + \frac{s_+}{3} \delta_{il}) \eta_{jl} - 2s_+^{-1}(Q_{ij} + \frac{s_+}{3} \delta_{ij})(Q_{lm} + \frac{s_+}{3} \delta_{lm}) \eta_{lm}.
\]

Noting the fact that \( \nabla_k |u + t \phi|^2 = 0 \) at \( t = 0 \) and substituting \( \phi_i := u_k \eta_{ik} \), a calculation shows

\[
\frac{d}{dt} \frac{\partial Q_{t;i,j}}{\partial x_k} \bigg|_{t=0} = \left( \frac{\partial}{\partial x_k} \frac{d}{dt} Q_{t;i,j} \right) \bigg|_{t=0}
\]

\[
= s_+ \left( u_j \phi_i + u_i \phi_j - 2(u \cdot \phi)(u_i u_j) \right)
\]

\[
= \frac{\partial}{\partial x_k} \left( (Q_{jl} + \frac{s_+}{3} \delta_{jl}) \eta_{il} + (Q_{il} + \frac{s_+}{3} \delta_{il}) \eta_{jl} - 2s_+^{-1}(Q_{ij} + \frac{s_+}{3} \delta_{ij})(Q_{lm} + \frac{s_+}{3} \delta_{lm}) \eta_{lm} \right)
\]

\[
= \partial Q_{jl} \eta_{il} + \partial Q_{il} \eta_{jl} - 2s_+^{-1} \left( \frac{\partial Q_{ij}}{\partial x_k} Q_{lm} + \frac{\partial Q_{lm}}{\partial x_k} (Q_{ij} + \frac{s_+}{3} \delta_{ij}) \right) \eta_{lm}
\]

\[
+ (Q_{jl} + \frac{s_+}{3} \delta_{jl}) \frac{\partial \eta_{il}}{\partial x_k} + (Q_{il} + \frac{s_+}{3} \delta_{il}) \frac{\partial \eta_{jl}}{\partial x_k} - 2s_+^{-1}(Q_{ij} + \frac{s_+}{3} \delta_{ij})(Q_{lm} + \frac{s_+}{3} \delta_{lm}) \frac{\partial \eta_{lm}}{\partial x_k}.
\]
For the special case of the functional \( \frac{1}{2} \int_\Omega |\nabla Q|^2 \, dx \), it follows from using (2.6) and \(|u|^2 = 1\) that

\[
\frac{d}{dt} \int_\Omega \frac{|\nabla Q_i|^2}{2} \, dx \bigg|_{t=0} = \int_\Omega \nabla_k Q_{ij} \frac{d\nabla_k Q_{ij}}{dt} \bigg|_{t=0} \, dx
\]

\[
= s^2 \int_\Omega \nabla_k(u_i u_j)|\nabla_k(u_j u_i)\eta_{il} + \nabla_k(u_i u_i)\eta_{jl}| \, dx
\]

\[
+ s^2 \int_\Omega (\nabla_k u_i u_j + u_i \nabla_k u_j)(u_j u_i \nabla_k \eta_{il} + u_i u_i \nabla_k \eta_{jl}) \, dx
\]

\[
- \int_\Omega 2s^{-1}(Q_{lm} + \frac{s}{3} \delta_{lm})|\nabla Q|^2 \eta_{lm} \, dx
\]

\[
= \int_\Omega 2\nabla_k Q_{ij} \nabla_k Q_{jl} \eta_{il} - 2(s^{-1}Q_{lm} + \frac{1}{3} \delta_{lm})|\nabla Q|^2 \eta_{lm} \, dx
\]

\[
+ s^2 \int_\Omega \nabla_k u_i u_j \nabla_k \eta_{il} + \nabla_k u_j u_i \nabla_k \eta_{jl} \, dx
\]

\[
= \int_\Omega 2\nabla_k Q_{il} \nabla_k Q_{jl} \eta_{ij} - 2(s^{-1}Q_{ij} + \frac{1}{3} \delta_{ij})|\nabla Q|^2 \eta_{ij} \, dx
\]

\[
+ \frac{1}{2} s^2 \int_\Omega (\nabla_k Q_{il} \nabla_k \eta_{ij} + \nabla_k Q_{jl} \nabla_k \eta_{ij}) \, dx
\]

(2.7) \quad \int_\Omega \left(-s_+ \Delta Q_{ij} + 2\nabla_k Q_{il} \nabla_k Q_{jl} - 2(s^{-1}Q_{ij} + \frac{1}{3} \delta_{ij})|\nabla Q|^2 \right) \eta_{ij} \, dx

for all \( \eta \) with \( \eta_{ij} = \eta_{ji} \). This means that \( Q \) is a weak solution to

\[
s_+ \Delta Q_{ij} - 2\nabla_k Q_{il} \nabla_k Q_{jl} + 2(s^{-1}Q_{ij} + \frac{1}{3} \delta_{ij})|\nabla Q|^2 = 0.
\]

For the term \( V(Q, \nabla Q) \), using (2.7) and integrating by parts, we have

(2.8)

\[
\int_\Omega \frac{d}{dt} V(Q, \nabla Q) \bigg|_{t=0} \, dx = \int_\Omega \left[ V_{kij} \frac{d\nabla_k Q_{ij}}{dt} + V_{Qij} \frac{dQ_{ij}}{dt} \right] \bigg|_{t=0} \, dx
\]

\[
= \int_\Omega V_{kij} \left( (Q_{ij} + \frac{s}{3} \delta_{ij}) \frac{\partial \eta_{ij}}{\partial x_i} + (Q_{il} + \frac{s}{3} \delta_{il}) \frac{\partial \eta_{il}}{\partial x_i} + \frac{\partial Q_{il}}{\partial x_k} \eta_{il} + \frac{\partial Q_{ij}}{\partial x_k} \eta_{ij} \right) \, dx
\]

\[
- 2s^{-1} \int_\Omega V_{kij} \left( \frac{\partial Q_{il}}{\partial x_k} (Q_{lm} + \frac{s}{3} \delta_{lm}) + \frac{\partial Q_{lm}}{\partial x_k} (Q_{ij} + \frac{s}{3} \delta_{ij}) \right) \eta_{lm} \, dx
\]

\[
+ \int_\Omega -2s^{-1} V_{kij} (Q_{ij} + \frac{s}{3} \delta_{ij}) (Q_{il} + \frac{s}{3} \delta_{il}) \frac{\partial \eta_{il}}{\partial x_k} + V_{Qij} (Q_{ij} + \frac{s}{3} \delta_{ij}) \eta_{il} \, dx
\]

\[
+ \int_\Omega V_{Qij} \left( (Q_{il} + \frac{s}{3} \delta_{il}) \eta_{ij} - 2s^{-1}(Q_{ij} + \frac{s}{3} \delta_{ij})(Q_{lm} + \frac{s}{3} \delta_{lm}) \eta_{lm} \right) \, dx
\]

\[
= \int_\Omega \frac{\partial}{\partial x_k} \left( (Q_{ij} + \frac{s}{3} \delta_{ij}) V_{kij} + (Q_{il} + \frac{s}{3} \delta_{il}) V_{kij} \right) \eta_{ij} \, dx
\]

\[
+ \int_\Omega \frac{\partial}{\partial x_k} \left( 2s^{-1}(Q_{ij} + \frac{s}{3} \delta_{ij})(Q_{lm} + \frac{s}{3} \delta_{lm}) V_{kij} \right) \eta_{ij} + V_{kij} \frac{\partial Q_{ij}}{\partial x_k} \eta_{ij} \, dx
\]

\[
+ \int_\Omega \left( V_{Qij} \frac{\partial Q_{il}}{\partial x_k} - 2s^{-1} V_{kij} \left( \frac{\partial Q_{il}}{\partial x_k} (Q_{ij} + \frac{s}{3} \delta_{ij}) + (Q_{lm} + \frac{s}{3} \delta_{lm}) \frac{\partial Q_{ij}}{\partial x_k} \right) \right) \eta_{ij} \, dx
\]

\[
+ \int_\Omega \left( V_{Qij} (Q_{ij} + \frac{s}{3} \delta_{ij}) + V_{Qij} (Q_{il} + \frac{s}{3} \delta_{il}) \right) \eta_{ij} \, dx
\]
Assume the Frank constants $\alpha$.

Lemma 3.1. Substituting (2.11) into (2.10) with the fact that $|\nabla V|\nabla |\nabla V|$ Noting that $u_1 = \min \{k_2 + k_4, 2k_1 - k_2 - k_4, k_2 - |k_4|, k_3\} > 0$.

Lemma 2.3. Assume that $Q = s_+ (u \otimes u - \frac{1}{3} I)$. Then $Q = (Q_{ij})$ is a solution of equation

\[ \Delta Q_{ij} - 2s_+^{-1} \nabla_k Q_{il} \nabla_k Q_{lj} + 2s_+^{-1} (s_+^{-1} Q_{ij} + \frac{1}{3} \delta_{ij}) |\nabla Q|^2 = 0 \]

if and only if $u$ is a harmonic map from $\Omega$ into $S^2$; i.e. $-\Delta u = |\nabla u|^2 u$.

Proof. Let $u$ be a harmonic map from $\Omega$ into $S^2$. Then we calculate

\[ \Delta Q_{ij} = s_+ \nabla_k (u_j \nabla_k u_i + u_i \nabla_k u_j) \]

\[ = s_+ (u_i \Delta u_j + 2 \nabla_k u_j \nabla_k u_i + u_j \Delta u_i) \]

\[ = 2s_+ (-|\nabla u|^2 u_j u_i + \nabla_k u_j \nabla_k u_i). \]

Noting that $|\nabla u|^2 = \frac{s_+^2}{2} |\nabla Q|^2$ and $|u| = 1$, we obtain

\[ \nabla_k u_j \nabla_k u_i = \nabla_k u_j \nabla_k u_i u_i u_i \]

\[ = \nabla_k (u_j u_i) - u_j \nabla_k u_i |u_i \nabla_k u_i| = \nabla_k (u_j u_i) u_i \nabla_k u_i \]

\[ = \nabla_k (u_j u_i) \nabla_k (u_i u_i) - u_i \nabla_k u_i \]

\[ = s_+^{-2} \nabla_k Q_{jl} \nabla_k Q_{ld} - (u_j \nabla_k u_i + u_i \nabla_k u_j) u_i \nabla_k u_i \]

\[ = s_+^{-2} \nabla_k Q_{jl} \nabla_k Q_{ld} - (s_+^{-1} Q_{ij} + \frac{1}{3} \delta_{ij}) \frac{s_+^{-2}}{2} |\nabla Q|^2. \]

Substituting (2.11) into (2.10) with the fact that $|\nabla u|^2 = \frac{s_+^2}{2} |\nabla Q|^2$, we obtain

\[ \Delta Q_{ij} = -2s_+^{-1} (s_+^{-1} Q_{ij} + \frac{1}{3} \delta_{ij}) |\nabla Q|^2 + 2s_+^{-1} \nabla_k Q_{jl} \nabla_k Q_{ld}. \]

Conversely, let $Q$ be a solution to (2.10). Using (2.10), (2.11) with the fact that $u_j \Delta u_j = -|\nabla u|^2$, we have

\[ \Delta u_i (\Delta (u_i u_j) - u_i \Delta u_j - 2 \nabla_k u_j \nabla_k u_i) u_j = u_i |\nabla u|^2. \]

\[ \Delta \tilde{\alpha} = \min \{k_2 + k_4, 2k_1 - k_2 - k_4, k_2 - |k_4|, k_3\} > 0. \]

3. The coercivity and Proof of Theorem 2

Lemma 3.1. Assume the Frank constants $k_1 \cdots k_4$ satisfy the strong Ericksen condition (1.12); i.e.

\[ k_1 > 0, \quad k_2 > |k_4|, \quad k_3 > 0, \quad 2k_1 > k_2 + k_4. \]

Then for each $u \in S^2$, the density $W(u, \nabla u)$ of the form (1.1) is equivalent to the new form

\[ W(u, \nabla u) = \frac{\tilde{\alpha}}{2} |\nabla u|^2 + V(u, \nabla u), \]

where $V(u, \nabla u)$ is a sum of square terms (see (3.8)) satisfying

\[ V(u, \nabla u) \leq C (1 + |u|^2) |\nabla u|^2, \quad |V_u(u, \nabla u)| \leq C (1 + |u|) |\nabla u|^2 \]

for all $u \in \mathbb{R}^3$ and

\[ \tilde{\alpha} = \min \{k_2 + k_4, 2k_1 - k_2 - k_4, k_2 - |k_4|, k_3\} > 0. \]
Proof. Note that $W(u, \nabla u)$ is rotational invariant (e.g. [23]); i.e. for each $R \in SO(3)$, $\bar{x} = R(x - x_0)$ and $\bar{u} = Ru(x) = Ru$. Then we have

$$W(\bar{u}, \nabla \bar{u}) = W(Ru, R\nabla u R^T) = W(u, \nabla u).$$

Then for any $u \in S^2$, we can find some $R = R(u(x_0)) \in SO(3)$ at each point $x_0 \in \Omega$ such that

$$\tilde{u}(0) := Ru(x_0) = (0, 0, 1)^T.$$

In fact, we can find the exact form of $R$ at $x_0$ by rotating $\tilde{u}$ back to $u$ around $x$ and $y$ axes in a $(x, y, z)$ Cartesian coordinates.

$$R_x := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \quad R_y := \begin{pmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{pmatrix}.$$

Here $\phi \in [-\pi, \pi]$ and $\varphi \in [-\pi/2, \pi/2]$. Let $R_1 := (R_xR_y)^T, R_2 := (R_yR_x)^T$. We choose an open cover $\{U_i\}_{i=1}^6$ for the sphere $S^2$ with open sets

$$U_1 = \{u \in S^2 | u_3 > \frac{1}{2}\}, \quad U_2 = \{u \in S^2 | u_3 < -\frac{1}{2}\},$$

$$U_3 = \{u \in S^2 | u_2 > \frac{1}{2}\}, \quad U_4 = \{u \in S^2 | u_2 < -\frac{1}{2}\},$$

$$U_5 = \{u \in S^2 | u_1 > \frac{1}{2}\}, \quad U_6 = \{u \in S^2 | u_1 < -\frac{1}{2}\}.$$

Then there is a partition of unity subordinate to the open cover $\{U_i\}_{i=1}^6$; i.e. there exist $\{\xi_i(u)\}_{i=1}^6$ with $0 \leq \xi_i \leq 1$ having support of $\xi_i$ in $U_i$ for each $i = 1, \cdots, 6$. In particular, $\xi_1(u) = 1$ in $S^2 \setminus (\cup_{i=2}^6 U_i), \xi_1(u) \in [0, 1]$ in $U_1 \cap (\cup_{i=2}^6 U_i)$ and 0 otherwise. Then the rotational invariant energy density can be written as

$$W(\tilde{u}, \tilde{\nabla} \tilde{u}) = \sum_{i=1}^4 \xi_i(u)W(R_1u, R_1 \nabla u R_1^T) + \sum_{j=5}^6 \xi_j(u)W(R_2u, R_2 \nabla u R_2^T).$$

Without loss of generality, we compute $W(u, \nabla u)$ for the case where $\xi_1(u) = 1$. The rotation is

$$R_1^T = R_xR_y = \begin{pmatrix} \cos \phi & 0 & \sin \phi \\ \sin \phi \sin \varphi & \cos \phi & -\sin \phi \cos \varphi \\ -\cos \phi \sin \varphi & \sin \phi & \cos \phi \cos \varphi \end{pmatrix}.$$

Then

$$u_1(x_0) = \sin \varphi, \quad u_2(x_0) = -\sin \phi \cos \varphi, \quad u_3(x_0) = \cos \phi \cos \varphi = \cos \phi \sqrt{1 - u_1^2(x_0)}.$$

Then

$$\sin \varphi = u_1(x_0), \quad \cos \varphi = \sqrt{u_2^2(x_0) + u_3^2(x_0)},$$

$$\sin \phi = \frac{-u_2(x_0)}{\sqrt{u_2^2(x_0) + u_3^2(x_0)}}, \quad \cos \phi = \frac{u_3(x_0)}{\sqrt{u_2^2(x_0) + u_3^2(x_0)}}.$$

Therefore, at $x_0$

$$R(u) = \begin{pmatrix} \sqrt{u_2^2 + u_3^2} & \frac{-u_2}{\sqrt{u_2^2 + u_3^2}} & \frac{-u_3}{\sqrt{u_2^2 + u_3^2}} \\ u_1 & \sqrt{u_2^2 + u_3^2} & \frac{-u_2}{\sqrt{u_2^2 + u_3^2}} \\ \sqrt{u_2^2 + u_3^2} & \frac{-u_3}{\sqrt{u_2^2 + u_3^2}} & u_3 \end{pmatrix}.$$
We evaluate four terms of the Oseen-Frank potential at 0
where \( \tilde{\alpha} \) is a positive constant due to the strong Ericksen condition (1.12). The term \( V(u, \nabla u) \) can be written as

\[
2V(\tilde{u}, \tilde{\nabla} \tilde{u}) := \frac{2k_1 - k_2 - k_4 - \tilde{\alpha}}{2} (\text{div } \tilde{u})^2 + (k_2 - k_4 - \tilde{\alpha})(|\nabla \tilde{u}|^2 + |\tilde{\nabla} \tilde{u}|^2)
\]

\[
+ \frac{k_2 + k_4 - \tilde{\alpha}}{2} \tilde{\nabla} \tilde{u} \tilde{\nabla} \tilde{u} - \tilde{\alpha} |\tilde{\nabla} \tilde{u}|^2 + 2V(\tilde{u}, \tilde{\nabla} \tilde{u})
\]

where \( \tilde{\alpha} \), which is defined in (3.1), is a positive constant due to the strong Ericksen condition (1.12). The term \( V(u, \nabla u) \) can be written as
Using (3.5) for the case of $\xi_1(u) = 1$, we find
\[
\hat{\nabla} \hat{u} = R \nabla u R^T
\]
\[
= \begin{pmatrix}
\sqrt{u_2^2 + u_3^2} & -u_2 u_3 & -u_2 u_3 \\
0 & \sqrt{u_2^2 + u_3^2} & u_2 \\
u_1 & u_2 & \sqrt{u_2^2 + u_3^2}
\end{pmatrix}
\begin{pmatrix}
\nabla u \\
\nabla u \\
\nabla u
\end{pmatrix}
\begin{pmatrix}
\sqrt{u_2^2 + u_3^2} & 0 & u_1 \\
0 & \sqrt{u_2^2 + u_3^2} & u_2 \\
u_1 & u_2 & \sqrt{u_2^2 + u_3^2}
\end{pmatrix}.
\]

A direct calculation yields
\[
(R \nabla u)_1,1 = \sqrt{u_2^2 + u_3^2} \nabla_1 u_1 - \frac{u_1(u_2 \nabla_1 u_2 + u_3 \nabla_1 u_3)}{\sqrt{u_2^2 + u_3^2}} = \frac{\nabla_1 u_1}{\sqrt{u_2^2 + u_3^2}},
\]
\[
(R \nabla u)_1,2 = \sqrt{u_2^2 + u_3^2} \nabla_2 u_1 - \frac{u_1(u_2 \nabla_2 u_2 + u_3 \nabla_2 u_3)}{\sqrt{u_2^2 + u_3^2}} = \frac{\nabla_2 u_1}{\sqrt{u_2^2 + u_3^2}},
\]
\[
(R \nabla u)_1,3 = \sqrt{u_2^2 + u_3^2} \nabla_3 u_1 - \frac{u_1(u_2 \nabla_3 u_2 + u_3 \nabla_3 u_3)}{\sqrt{u_2^2 + u_3^2}} = \frac{\nabla_3 u_1}{\sqrt{u_2^2 + u_3^2}},
\]
\[
(R \nabla u)_2,1 = \frac{u_3 \nabla_1 u_2 - u_2 \nabla_1 u_3}{\sqrt{u_2^2 + u_3^2}},
\]
\[
(R \nabla u)_2,2 = \frac{u_3 \nabla_2 u_2 - u_2 \nabla_2 u_3}{\sqrt{u_2^2 + u_3^2}},
\]
\[
(R \nabla u)_2,3 = \frac{u_3 \nabla_3 u_2 - u_2 \nabla_3 u_3}{\sqrt{u_2^2 + u_3^2}}.
\]

Note that $u_3^2 \leq \frac{3 |u|^2}{4}$ for the case of $\xi_1(u) = 1$. Then it yields
\[
\begin{align*}
\hat{\nabla}_1 \hat{u}_1 &= \nabla_1 u_1 - \frac{u_1 u_2 \nabla_2 u_1 + u_1 u_3 \nabla_3 u_1}{|u|^2 - u_1^2} |u|,
\hat{\nabla}_2 \hat{u}_2 &= \frac{u_3 u_2}{u_2^2 + u_3^2} \nabla_2 u_2 - \frac{u_3 u_2}{u_2^2 + u_3^2} \nabla_3 u_2 + \frac{u_2^2}{u_2^2 + u_3^2} \nabla_3 u_3 \\
&= \nabla_2 u_2 + \nabla_3 u_3 + \frac{u_3 u_2 \nabla_2 u_1 + u_1 u_3 \nabla_3 u_1}{|u|^2 - u_1^2} |u|,
\hat{\nabla}_1 \hat{u}_2 &= u_3 \nabla_2 u_1 - u_2 \nabla_3 u_1 + \frac{u_1 u_2 u_3}{u_2^2 + u_3^2} (\nabla_2 u_3 - \nabla_3 u_2) - \frac{u_1 u_2^2}{u_2^2 + u_3^2} \nabla_2 u_3 + \frac{u_1 u_2^2}{u_2^2 + u_3^2} \nabla_3 u_2 \\
&= (1 + \frac{u_2^2}{|u|^2 - u_1^2})(u_3 \nabla_2 u_1 - u_2 \nabla_3 u_1) = \frac{|u|^2}{|u|^2 - u_1^2} (u_3 \nabla_2 u_1 - u_2 \nabla_3 u_1),
\hat{\nabla}_2 \hat{u}_1 &= u_3 \nabla_1 u_2 - u_2 \nabla_1 u_3 + \frac{u_3 u_2 u_3}{u_2^2 + u_3^2} (\nabla_3 u_3 - \nabla_2 u_2) + \frac{u_1 u_2^2}{u_2^2 + u_3^2} \nabla_2 u_3 - \frac{u_1 u_2^2}{u_2^2 + u_3^2} \nabla_3 u_2 \\
&= (u_3 \nabla_1 u_2 - u_2 \nabla_1 u_3) + u_1 (\nabla_2 u_3 - \nabla_3 u_2) + \frac{u_1^2}{|u|^2 - u_1^2} (u_3 \nabla_2 u_1 - u_2 \nabla_3 u_1).
\end{align*}
\]

Substituting the above identities into (3.7), for the case of $\xi_1(u) = 1$, we see that
\[
(3.8)
\]
\[
2V(u, \nabla u) = 2V(\hat{u}, \hat{\nabla} \hat{u}) = \frac{2k_1 - k_2 - k_4 - \hat{\alpha}}{2}(\text{div } u)^2 + (k_3 - \hat{\alpha})|u \times \text{curl } u|^2 \\
+ \frac{k_2 + k_4 - \hat{\alpha}}{2} \left( \nabla_1 u_1 - \nabla_2 u_2 - \nabla_3 u_3 - 2u_1 |u| (u_2 \nabla_2 u_1 + u_3 \nabla_3 u_1) \right)^2 \\
+ (k_2 - |k_4| - \hat{\alpha}) \left( \frac{|u|^2}{|u|^2 - u_1^2} (u_3 \nabla_2 u_1 - u_2 \nabla_3 u_1) \right)^2.
\]
Thus the form (3.9) is the form of $V \nabla u$ implies that for all similarly to (3.2). Thus we prove the required result. Then we find that

$$
\sum (\partial Q)_{ij} \frac{\partial Q_{ik}}{\partial x_j} = (u_j \nabla u_i + u_i \nabla u_j)(u_k \nabla u_i + u_i \nabla u_k)
$$

$$(\nabla \cdot u)^2 + \sum \left| (u \cdot \nabla) u_i \right|^2 = (\nabla \cdot u)^2 + |u \times \nabla u|^2,
$$

$$(\nabla \cdot u)^2 + \sum \left| (u \cdot \nabla) u_i \right|^2 = \text{tr}(\nabla u)^2 + |u \times \nabla u|^2,
$$

$$(\nabla \cdot u)^2 + \sum \left| (u \cdot \nabla) u_i \right|^2 = \text{tr}(\nabla u)^2 + |u \times \nabla u|^2,
$$

$$(\nabla \cdot u)^2 + \sum \left| (u \cdot \nabla) u_i \right|^2 = \text{tr}(\nabla u)^2 + |u \times \nabla u|^2,
$$

Note that (3.5) is the form of $V(u, \nabla u)$ for $\xi_1 = 1$. One can repeat the calculation for the second rotation $R_2$ in (3.3). To extend (3.8) to $u \in \mathbb{R}^3$, we define $\xi_i$ for $\frac{u_i}{|u|}$ similarly to (3.2). Thus we prove the required result. Then we find that $V(u, \nabla u)$ is quadratic in $\nabla u$ and $0 \leq V(u, \nabla u) \leq C(1 + |u|^2)|\nabla u|^2$ for all $u \in \mathbb{R}^3$, which implies that for all $u \in \mathbb{R}^3$, we have

$$W(u, \nabla u) = \frac{\tilde{\alpha}}{2} |\nabla u|^2 + V(u, \nabla u) \geq \frac{\tilde{\alpha}}{2} |\nabla u|^2,$$

$$|V_u(u, \nabla u)| \leq C(1 + |u|)|\nabla u|^2.$$

\[\Box\]

**Remark 4.** If the Frank constants satisfy that $\min\{k_1, k_2, k_3\} \geq k_2 + k_4 = \bar{\alpha} > 0$ and $k_4 < 0$ as in [27] p. 551 (see also [23] p. 467). Then the equation (3.9) $2V(u, \nabla u) = (k_1 - \bar{\alpha})(\text{div} u)^2 + (k_2 - \bar{\alpha})(u \cdot \nabla u)^2 + (k_3 - \bar{\alpha})|u \times \nabla u|^2$.

Thus the form $W(u, \nabla u)$ with the form (3.9) includes the cases in [27] p. 551 and [23] p. 467.

Next, we will prove Theorem 2 by using Lemma 3.1. Applying $u_i \nabla u_i = 0$, we find

$$(u \times \nabla u_i)^2 = |u_2 \nabla u_2 - \nabla u_1 - u_3(\nabla u_1 - \nabla u_3)|^2$$

$$= |u_2 \nabla u_2 - u_3(\nabla u_1 - \nabla u_3)|^2 = [(u \cdot \nabla) u_i ]^2,$$

$$\sum_i [(u \cdot \nabla) u_i ]^2 = \sum_i (u \times \nabla u_i)^2 = |u \times \nabla u|^2.$$
For each \( u \in S^2 \), using the above identities, we have

\[
2 \sum_i [(u \cdot \nabla)u_i]^2 - \frac{2}{3} |\nabla u|^2 = 2 |u \times \nabla u|^2 - \frac{2}{3} |\nabla u|^2.
\]

Substituting above identities into the form \( f_E(Q, \nabla Q) \), we have

\[
(3.10) \quad f_E(Q, \nabla Q) = s_+^2 L_1 |\nabla u|^2 + \frac{s_+^2 L_2}{2} ((\nabla \cdot u)^2 + |u \times \nabla u|^2)
+ \frac{s_+^2 L_3}{2} (\text{tr}(\nabla u)^2 + |u \times \nabla u|^2)
+ s_+^3 L_4 (|u \times \nabla u|^2 - \frac{1}{3} |\nabla u|^2)
= (s_+^2 L_1 - \frac{s_+^2 L_4}{3}) |\nabla u|^2 + \frac{s_+^2 L_2}{2} (\nabla \cdot u)^2
+ (s_+^2 L_2 + \frac{s_+^2 L_3}{2} + s_+^3 L_4) |u \times \nabla u|^2 + \frac{s_+^2 L_3}{2} \text{tr}(\nabla u)^2.
\]

For each \( u \in S^2 \), note that

\[
|\nabla u|^2 = \text{tr}(\nabla u)^2 + |\nabla u|^2, \quad |\nabla u|^2 = (u \cdot \nabla u)^2 + |u \times \nabla u|^2.
\]

Using the above identities, we have

\[
(3.11) \quad 2W(u, \nabla u) = k_1 (\nabla \cdot u)^2 + k_2 (u \cdot \nabla u)^2 + k_3 |u \times \nabla u|^2
+ (k_2 + k_4) (\text{tr}(\nabla u)^2 - (\nabla \cdot u)^2)
= k_2 |\nabla u|^2 + (k_1 - k_2 - k_4) (\nabla \cdot u)^2
+ (k_4 - k_2) |u \times \nabla u|^2 + k_4 \text{tr}(\nabla u)^2.
\]

Similarly to (3.10), comparing (3.10) with (3.11), we find that for each \( Q \in S_* \), \( f_E(Q, \nabla Q) = W(u, \nabla u) \) is true when

\[
(3.12) \quad \begin{cases}
    k_1 = 2s_+^2 L_1 + s_+^2 L_2 + s_+^2 L_3 - \frac{2s_+^3}{3} L_4 \\
    k_2 = 2s_+^2 L_1 - \frac{2s_+^3}{3} L_4 \\
    k_3 = 2s_+^2 L_1 + s_+^2 L_2 + s_+^2 L_3 + \frac{4s_+^3}{3} L_4 \\
    k_4 = s_+^3 L_3
\end{cases} \quad \Rightarrow \quad \begin{cases}
    L_1 = -\frac{1}{6}s_+^{-2} k_1 + \frac{1}{6}s_+^{-2} k_2 + \frac{1}{6}s_+^{-2} k_3 \\
    L_2 = s_+^{-2} k_1 - s_+^{-2} k_2 - s_+^{-2} k_4 \\
    L_3 = s_+^{-2} k_4 \\
    L_4 = -\frac{1}{6}s_+^{-3} k_1 + \frac{1}{6}s_+^{-3} k_3.
\end{cases}
\]

Using Lemma 3.1, the density \( W(u, \nabla u) \) has a lower bound if the coefficients \( k_1, \ldots, k_4 \) satisfy the strong Ericksen condition (1.12). Using the relation (3.12) between \( k_i \) and \( L_i \) with \( i = 1, \ldots, 4 \), the strong Ericksen condition (1.12) is equivalent to that

\[
L_1 - \frac{1}{2} |L_3| > \frac{s_+}{3} L_4, \quad L_1 + \frac{1}{2} L_2 + \frac{1}{2} L_3 + \frac{2s_+}{3} L_4 > 0,
L_1 + L_2 + \frac{1}{2} L_3 > \frac{s_+}{3} L_4.
\]

Now we prove Theorem 2.
Proof. For any $Q(u) = s_+(u \otimes u - \frac{1}{3} I)$ with $u \in S^2$, note that
\[
Q(-u) = s_+(-u \otimes -u - \frac{1}{3} I) = Q(u), \quad f_E(Q, \nabla Q) = W(u, \nabla u).
\]
Therefore, we can assume that $u = (u_1, u_2, u_3)$ with $u_1 \geq 0$. For a $Q \in S_+$, there is a unique $u \in S^2$ such that
\[
(3.13) \quad u_1 = \sqrt{|s_+^{-1}Q_{11} + \frac{1}{3}|}, \quad u_2 = \text{sign}(Q_{12})\sqrt{|s_+^{-1}Q_{22} + \frac{1}{3}|}, \quad u_3 = \text{sign}(Q_{13})\sqrt{|s_+^{-1}Q_{33} + \frac{1}{3}|}.
\]
Using the fact that $|u|^2 = 1$, a direct calculation yields
\[
\nabla_k u_i = u_j \nabla_k(u_i u_j) = s_+(u_1 \nabla_k Q_{11} + u_2 \nabla_k Q_{12} + u_3 \nabla_k Q_{13}) = s_+(u_j \nabla_k Q_{ij}),
\]
which implies
\[
(3.14) \quad u_i \nabla_k u_i = \sum_j s_+(s_+^{-1}Q_{ij} + \frac{1}{3} \delta_{ij}) \nabla_k Q_{ij} = \sum_j s_+(Q_{ij} + \sqrt{\frac{2}{3}|Q|\delta_{ij}} \nabla_k Q_{ij}.
\]
Here we used the fact that $|Q| = \sqrt{\frac{2}{3}s_+}$. Let $S := \{Q \in S_0 : |Q| = 1\}$ be the unit sphere of $S_0$. By Cauchy’s inequality, we have
\[
|Q_{11}| + |Q_{22}| + |Q_{33}| \leq \sqrt{3}(|Q_{11}|^2 + |Q_{22}|^2 + |Q_{33}|^2)^{1/2} \leq \sqrt{3}|Q|.
\]
Consider
\[
(3.15) \quad U_1 = \left\{ Q \in S : |Q_{11}| < \frac{\sqrt{3}}{3}|Q| \right\}, \quad U_2 = \left\{ Q \in S : |Q_{22}| < \frac{\sqrt{3}}{3}|Q| \right\}, \quad U_3 = \left\{ Q \in S : |Q_{33}| < \frac{\sqrt{3}}{3}|Q| \right\}.
\]
Since there is one $i$ such that $|Q_{ii}| < \frac{\sqrt{3}}{3}|Q|$, then $\{U_i\}_{i=1}^3$ is an open cover of $S$ and let $\{\xi_i\}_{i=1}^3$ be a smooth partition of unity subordinate to the open cover such that $\sum_{i=1}^3 \xi_i = 1$ and $0 \leq \xi_i \leq 1$ in $S$, $\xi_i \in C_0^\infty(U_i)$ and $\xi_i = 1$ in $V_i$, where $V_i$ is an open subset of $U_i$ and $\{V_i\}_{i=1}^3$ is also an open cover of $S$. Then for each $Q \in S_0$, we have
\[
(3.16) \quad V(\tilde{Q}, \tilde{\nabla} \tilde{Q}) = (\xi_1(\frac{Q}{|Q|}) + \xi_2(\frac{Q}{|Q|}))V(R_1Q, R_1 \nabla Q R_1^T) + \xi_3(\frac{Q}{|Q|})V(R_2Q, R_2 \nabla Q R_2^T).
\]
When $Q \in S_+$, $Q = s_+(u \otimes u - \frac{1}{3} I)$ with $u \in S^2$. Without of generality, we only consider the case that $\frac{Q}{|Q|} \in U_1$; i.e. $|Q_{11}| < \frac{\sqrt{3}}{3}|Q|$. Noting that $|Q| = \sqrt{\frac{2}{3}s_+|u|^2}$, we have
\[
|u|^2 - u_1^2 = \sqrt{\frac{3}{2}s_+^{-1}|Q|} - (s_+^{-1}Q_{11} + \frac{1}{3}) = s_+^{-1}\left(\sqrt{\frac{2}{3}|Q|} - Q_{11}\right) > \sqrt{\frac{2}{3}s_+^{-1}|Q_{11}|}.
\]
Since \( u \in S^2 \), it follows from (3.13) that
\[
(3.17) \quad I_1 := (\text{div } u)^2 = \sum_i (u_i \text{div } u)^2 = \sum_i (s_+^{-1}(s_+^{-1}Q_{ij} + \frac{1}{3} \delta_{ij})(\nabla \cdot Q_j))^2.
\]

Let \( Q_i \) be the \( i \)-th column of the \( Q \) matrix. One can verify from (3.8) that
\[
(curl \ u)_i = \sum_j s_+^{-1}u_j (curl \ Q)_j.
\]

Then we find (3.18)
\[
I_2 := |u \times curl \ u|^2 = |u \times (s_+^{-1}u_j (curl \ Q_j))|^2 = \left| \sum_j s_+^{-1}(s_+^{-1}Q + \frac{1}{3}I_j \times curl \ Q_j) \right|^2.
\]

Using (3.13) again, we rewrite the third and fourth terms of (3.8) as
\[
I_3 := \sum_i u_i^2 \left( \nabla_1 u_1 - \nabla_2 u_2 - \nabla_3 u_3 - \frac{2u_1 u_2 \nabla_2 u_1 + 2u_1 u_3 \nabla_3 u_1}{|u|^2 - u_1^2} \right)^2
\]
\[
= \sum_{i,j} s_+^{-4}(Q_{ij} + \sqrt{\frac{1}{6}}|Q|\delta_{ij})^2 \left( \nabla_1 Q_{ij} - \nabla_2 Q_{2j} - \nabla_3 Q_{3j} - \frac{2Q_{12} \nabla_2 Q_{1j} - 2Q_{13} \nabla_3 Q_{1j}}{\sqrt{\frac{2}{3}|Q|} - Q_{11}} \right)^2,
\]
\[
I_4 := \left( \frac{|u|^2}{|u|^2 - u_1^2}(u_3 \nabla_2 u_1 - u_2 \nabla_3 u_1) \right)^2
\]
\[
= \frac{3|Q|^2}{2s_+^4} \sum_i \left( \frac{(Q_{i3} + \sqrt{\frac{1}{6}}|Q|\delta_{i3})\nabla_2 Q_{1i} - (Q_{i2} + \sqrt{\frac{1}{6}}|Q|\delta_{i2})\nabla_3 Q_{1i}}{\sqrt{\frac{2}{3}|Q|} - Q_{11}} \right)^2.
\]

We rewrite the fifth term of (3.8) into
\[
I_5 := \left( u_3 \nabla_1 u_2 - u_2 \nabla_1 u_3 + u_1 (\nabla_2 u_3 - \nabla_3 u_2) + \frac{u_1^2(u_3 \nabla_2 u_1 - u_2 \nabla_3 u_1)}{|u|^2 - u_1^2} \right)^2
\]
\[
= \sum_i s_+^{-4} \left( (Q_{i3} + \sqrt{\frac{1}{6}}|Q|\delta_{i3})\nabla_1 Q_{2i} - (Q_{i2} + \sqrt{\frac{1}{6}}|Q|\delta_{i2})\nabla_1 Q_{3i} + (Q_{i1} + \sqrt{\frac{1}{6}}|Q|\delta_{i1})\left( \nabla_2 Q_{3i} - \nabla_3 Q_{2i} + \frac{Q_{13} \nabla_2 Q_{1i} - Q_{12} \nabla_3 Q_{1i}}{\sqrt{\frac{2}{3}|Q|} - Q_{11}} \right) \right)^2.
\]

Finally, we write the last term in (3.8) as
\[
I_6 = \sum_i s_+^{-4} \left[ \frac{1 + \text{sign}(L_3)}{\sqrt{\frac{2}{3}|Q|} - Q_{11}} \left( Q_{i3} + \sqrt{\frac{1}{6}}|Q|\delta_{i3} \right) \nabla_2 Q_{1i} - (Q_{i2} + \sqrt{\frac{1}{6}}|Q|\delta_{i2}) \nabla_3 Q_{1i} + \text{sign}(L_3) \left( (Q_{i3} + \sqrt{\frac{1}{6}}|Q|\delta_{i3}) \nabla_1 Q_{2i} + (Q_{i1} + \sqrt{\frac{1}{6}}|Q|\delta_{i1}) \left( \nabla_2 Q_{3i} - \nabla_3 Q_{2i} \right) \right)^2.
\]
Substituting the identities of $I_1, ..., I_6$ into the equation (3.19), we have

$$V(Q, \nabla Q) = \frac{L_1 + L_2 + \frac{1}{2}L_3 - \frac{s}{2}L_4 - \alpha}{4} I_1 + \frac{(L_1 + \frac{1}{2}L_2 + \frac{1}{2}L_3 + \frac{2s}{3}L_4 - \alpha)}{2s^2} I_2$$

$$+ \frac{L_1 - \frac{s}{2}L_4 + \frac{1}{2}L_3 - \alpha}{2} I_3 + \frac{(L_1 - \frac{s}{2}L_4 - \frac{1}{2}|L_3| - \alpha)}{2} I_4$$

$$+ \frac{(L_1 - \frac{s}{2}L_4 - \frac{1}{2}|L_3| - \alpha)}{2} I_5 + \frac{|L_4|}{2} I_6.$$
4. Proof of Theorem 3

**Lemma 4.1.** If $Q$ is a minimizer of $\tilde{E}_L$ in $W^{1,2}_{Q_0}(\Omega; S_0)$, it satisfies

$$-\tilde{\alpha} \Delta Q_{ij} - \frac{1}{2} \nabla_k (V_{p_{ki}^L} + V_{p_{kj}^L}) + \frac{1}{3} \delta_{ij} \sum_l \nabla_k V_{p_{kl}^L} + \frac{1}{2} (V_{Q_{ij}} + V_{Q_{ji}}) - \frac{1}{3} \delta_{ij} \sum_l V_{Q_{ll}} + \frac{1}{L} \left( -aQ_{ij} - b(Q_{ik}Q_{kj} - \frac{1}{3} \delta_{ij}) \text{tr}(Q^2) + cQ_{ij} \text{tr}(Q^2) \right) = 0$$

in the weak sense.

**Proof.** For any test function $\phi \in C_0^\infty(\Omega; S_0)$, consider $Q_t := Q + t\phi$ for $t \in \mathbb{R}$. Then for all $\phi \in C_0^\infty(\Omega; S_0)$, we calculate

$$\int_{\Omega} \frac{d}{dt} \left( \tilde{f}_E(Q_t, \nabla Q_t) + \frac{1}{L} \tilde{f}_B(Q_t) \right)_{t=0} \, dx = 0.$$

By the definition of $\tilde{f}_E$ in (1.18) with the boundary value $Q_0 \in W^{1,2}(\Omega; S_0)$, then $|Q_L| \leq M + 1$ for a sufficient large $M$.

Then we will show that

**Lemma 4.2.** Let $Q_L$ be a weak solution to the equation (1.20) with the boundary value $Q_0 \in W^{1,2}(\Omega; S_0)$. Then, $|Q_L| \leq M + 1$ for a sufficient large $M$.

**Proof.** Recall from the definition of $\tilde{f}_E$ in (1.18) that for a $Q \in S_0$ with $|Q| \geq M + 1$,

$$\tilde{f}_E(Q, \nabla Q) = \frac{\tilde{\alpha}}{2} |\nabla Q|^2.$$

Similarly to one in (1.18), choose a test function $\phi = Q(1 - \min\{1, \frac{M+1}{|Q|}\})$. Multiplying (1.20) by the test function $\phi$, we have

$$\tilde{\alpha} \int_{|Q| \geq M+1} |\nabla Q|^2 (1 - \frac{M + 1}{|Q|}) - (M + 1)Q_{ij} \nabla_k Q_{ij} \nabla_k \frac{1}{|Q|}) \, dx + \frac{1}{L} \int_{|Q| \geq M+1} \left( -a|Q|^2 - bQ_{ik}Q_{kj}Q_{ij} + c|Q|^4 \right) (1 - \frac{M + 1}{|Q|}) \, dx = 0.$$
Note the fact that $\nabla_k|Q|^2 = 2Q_{ij} \nabla_k Q_{ij}$. The above second term is nonnegative.

For a sufficiently large $M > 0$, third term also is positive. This implies that the set 
\{ $|Q| \geq M + 1$ \} is empty; i.e. $|Q| \leq M + 1$ a.e. in $\Omega$. 

**Lemma 4.3.** For any $Q \in S_*$, the Hessian of the bulk density $f_B(Q)$ is positive definite for a uniform constant; i.e. for any $\xi \in S_0$, we have

\[
\partial_{Q_{ij}} \partial_{Q_{ij}} f_B(Q) \xi_{ij} \xi_{kl} \geq \lambda |\xi|^2,
\]

where $\lambda = \min \{ \frac{11}{3} s + b, a \} > 0$.

**Proof.** Recall the fact that the bulk density $f_B$ is rotational invariant. For any tensor $Q \in S_*$, there exists a rotation $R = R(Q) \in SO(3)$ such that we can rotate $Q$ to its diagonal form $\tilde{Q}$ with elements $(-\frac{3}{s + b}, -\frac{3}{s + b}, \frac{2a}{s + b})$ and

\[
\tilde{Q}_{ij} = R_{ip}Q_{pq}R_{jq}.
\]

Using the chain rule, we derive

\[
\partial_{Q_{mn}} \partial_{Q_{ij}} f_B(Q) \xi_{mn} \xi_{kl} = \partial_{Q_{mn}} \left( \frac{\partial f_B(\tilde{Q})}{\partial \tilde{Q}_{ij}} \frac{\partial \tilde{Q}_{ij}}{\partial Q_{ij}} \right) \xi_{mn} \xi_{kl}
\]

\[
= \frac{\partial^2 f_B(\tilde{Q})}{\partial \tilde{Q}_{ij} \partial \tilde{Q}_{ij}} \frac{\partial \tilde{Q}_{ij}}{\partial Q_{ij}} \partial_{Q_{mn}} (R_{ip}Q_{pq}R_{jq}) \partial_{Q_{mn}} (R_{ip}Q_{pq}R_{jq}) \xi_{mn} \xi_{kl}
\]

\[
= \frac{\partial^2 f_B(\tilde{Q})}{\partial \tilde{Q}_{ij} \partial \tilde{Q}_{ij}} \frac{\partial \tilde{Q}_{ij}}{\partial Q_{ij}} R_{ik}R_{jl}R_{im}R_{jn} \xi_{mn} \xi_{kl} = \frac{\partial^2 f_B(\tilde{Q})}{\partial \tilde{Q}_{ij} \partial \tilde{Q}_{ij}} \tilde{Q}_{ij} \tilde{Q}_{ij},
\]

where $\tilde{Q}_{ij} = R_{ik} \xi_{kl} R_{jl}$ and $\tilde{Q}_{ij} = R_{im} \xi_{mn} R_{jn}$.

We calculate the first derivative of $f_B(\tilde{Q})$

\[
\partial_{Q_{ij}} f_B(\tilde{Q}) = \left( -a \tilde{Q}_{ij} - b \sum_{k} \tilde{Q}_{jk} \tilde{Q}_{ki} + c \tilde{Q}_{ij} |\tilde{Q}|^2 \right).
\]

Then the second derivative of $f_B(\tilde{Q})$ is

\[
\partial_{\tilde{Q}_{ij}} \partial_{\tilde{Q}_{ij}} f_B(\tilde{Q}) = -a \delta_{ii} \delta_{jj} - b (\delta_{ij} \tilde{Q}_{ji} + \delta_{ji} \tilde{Q}_{ij}) + c (\delta_{ii} \delta_{jj} |\tilde{Q}|^2 + 2 \tilde{Q}_{ij} \tilde{Q}_{ij}).
\]

For the case of $i = j = \tilde{i} = \tilde{j}$, using the equality $\frac{2}{3} s^2 = \frac{4}{3} b s + a$ (c.f. [37]), we find

\[
\partial_{\tilde{Q}_{ii}} \partial_{\tilde{Q}_{ii}} f_B(\tilde{Q}) = -a - 2 \tilde{Q}_{ii} b + (|\tilde{Q}|^2 + 2 \tilde{Q}_{ii}^2) c = -(2 \tilde{Q}_{ii} - \frac{s}{3}) b + 2 \tilde{Q}_{ii}^2 c.
\]

Then, at $\tilde{Q} = \tilde{Q}_*$, we have

\[
\partial_{\tilde{Q}_{11}} \partial_{\tilde{Q}_{11}} f_B(\tilde{Q}) = \left( s + b + \frac{2s^2}{9} c \right) = \frac{1}{3} a + \frac{4s + b}{3},
\]

\[
\partial_{\tilde{Q}_{22}} \partial_{\tilde{Q}_{22}} f_B(\tilde{Q}) = \frac{1}{3} a + \frac{4s + b}{3},
\]

\[
\partial_{\tilde{Q}_{33}} \partial_{\tilde{Q}_{33}} f_B(\tilde{Q}) = - \frac{5s + b}{3} + \frac{8s + b}{9} c = \frac{4}{3} a - \frac{s + b}{3}.
\]
Proof. For each boundary value \(\pi\), there is a smooth projection \(\lambda\) with
\[
\begin{align*}
(4.5) & \quad 2\partial_{Q_{11}} \partial_{Q_{22}} f_B(\tilde{Q}) = 4\tilde{Q}_{11}\tilde{Q}_{22}c = \frac{4s_+^2}{9} c = \frac{2}{3}a + \frac{2s_+}{9}b, \\
(4.6) & \quad 2\partial_{Q_{11}} \partial_{Q_{33}} f_B(\tilde{Q}) = 4\tilde{Q}_{11}\tilde{Q}_{33}c = -\frac{8s_+^2}{9} c = -\left(\frac{3}{4}a + \frac{4s_+}{9}b\right), \\
(4.7) & \quad 2\partial_{Q_{22}} \partial_{Q_{33}} f_B(\tilde{Q}) = 4\tilde{Q}_{22}\tilde{Q}_{33}c = -\frac{8s_+^2}{9} c = -\left(\frac{4}{3}a + \frac{4s_+}{9}b\right).
\end{align*}
\]

For the remaining case, that is \(i \neq j\) or \(i \neq j\), we have at \(\tilde{Q} = \tilde{Q}_*\),
\[
(4.8) \quad \sum_{i \neq j} \sum_{i \neq j} \partial_{Q_{ij}} \partial_{Q_{ij}} f_B(\tilde{Q}) \xi_{ij} \xi_{ij} \geq \sum_{i \neq j} \frac{2s_+^2}{3} c - (\tilde{Q}_{ii} + Q_{jj}) b |\xi_{ij}|^2 \geq \sum_{i \neq j} a|\xi_{ij}|^2.
\]

In conclusion, we have at \(\tilde{Q} = \tilde{Q}_*\),
\[
\partial_{Q_{\alpha\beta}} \partial_{Q_{\alpha\beta}} f_B(\tilde{Q}) \xi_{\alpha\beta} \xi_{\alpha\beta} = \partial_{Q_{ij}} \partial_{Q_{ij}} f_B(\tilde{Q}) \xi_{ij} \xi_{ij} \geq \frac{1}{3}a + \frac{4s_+}{3}b (\tilde{\xi}_{11}^2 + \tilde{\xi}_{22}^2) + \left(\frac{2}{3}a + \frac{2s_+}{9}b\right) \tilde{\xi}_{11} \tilde{\xi}_{22} \\
+ \left(\frac{4}{3}a - \frac{s_+}{3}\right) \tilde{\xi}_{33} - \left(\frac{3}{4}a + \frac{4s_+}{9}b\right) \tilde{\xi}_{33} (\tilde{\xi}_{11} + \tilde{\xi}_{22}) + \sum_{i \neq j} a|\xi_{ij}|^2 \\
= \frac{11s_+}{9} b (\tilde{\xi}_{11}^2 + \tilde{\xi}_{22}^2) + \left(\frac{1}{3}a + \frac{s_+}{9}\right) (\tilde{\xi}_{11} + \tilde{\xi}_{22})^2 \\
+ \left(\frac{8}{3}a + \frac{s_+}{9}\right) \tilde{\xi}_{33}^2 + \sum_{i \neq j} a|\xi_{ij}|^2 \geq \lambda|\xi|^2
\]

with \(\lambda = \min\{\frac{11s_+}{9}, a\} > 0\). \(\square\)

Now we give a proof of Theorem 3.

Proof. For each \(L > 0\), let \(Q_L\) be a weak solution to the equation \((1.20)\) with boundary value \(Q_0 \in W^{1,2}(\Omega, S_*)\). By Lemma 4.2, \(Q_L\) is uniformly bounded in \(\Omega\).

For a small \(\delta > 0\), let \(S_\delta\) be a neighborhood of \(S_*\) defined by
\[
S_\delta := \{Q \in S_0 : \text{dist}(Q, S_*) \leq \delta\}.
\]

There is a smooth projection \(\pi\) from \(S_\delta\) to \(S_*\). For each \(\delta > 0\), define a set
\[
\Sigma_\delta = S_0 \setminus S_\delta = \{Q \in S_0 : \text{dist}(Q, S_*) \geq \delta\}.
\]

For each \(Q \in \Sigma_\delta\), we have \(\pi(Q) \in S_*\); i.e. \(\pi(Q) = s_+ (u \otimes u - \frac{4}{3} I)\) with \(u \in S^2\).
In order to extend (4.12) to $\Omega$, we define
\begin{equation}
\pi(Q)_t := s_+ \left( u_t \otimes u_t - \frac{1}{3} I \right) \in S_+.
\end{equation}

Then it follows from (4.10) that
\begin{equation}
Q_{\tau}^t := \left( 1 - \tau \right) \pi(Q)_t + \tau Q_L \text{ for some } \tau \in [0, 1].
\end{equation}

By the Taylor expansion for $\hat{f}_B(\pi(Q)_t)$ at $Q_L \in S_\delta$, we derive
\begin{equation}
\frac{\hat{f}_B(\pi(Q)_t)}{L} = \frac{\hat{f}_B(Q_L)}{L} + \frac{1}{L} \nabla_{Q_{ij}, f_B(Q_L)}(\pi(Q)_t - Q_L)_{ij}
+ \frac{1}{2L} \nabla_{Q_{ijkl}, f_B(Q_L)}^2 (\pi(Q)_t - Q_L)_{ij}(\pi(Q)_t - Q_L)_{kl},
\end{equation}
where $Q_{\tau} := (1 - \tau)\pi(Q)_t + \tau Q_L$ for some $\tau \in [0, 1]$.

For a sufficiently small $L$ and $t$ such that $|Q_{L,t} - Q_s| < \frac{1}{2} \delta_1$ and $\delta = \frac{\pi}{2} \delta_1$, we have $\int_{\Omega_{L, \delta}} \frac{1}{L} \nabla_{Q_{ijkl}, f_B(Q_L)}^2 (\pi(Q)_t - Q_L)_{ij}(\pi(Q)_t - Q_L)_{kl} \leq \varepsilon_1.

For each $L$, we define a subdomain by
\begin{equation}
\Omega_{L, \delta} = \{ x \in \Omega : Q_L(x) \in S_\delta \}.
\end{equation}

For a sufficiently small $\delta$ and $t$, we have
\begin{equation}
\int_{\Omega_{L, \delta}} \frac{1}{L} \nabla_{Q_{ijkl}, f_B(Q_L)}^2 (\pi(Q)_t - Q_L)_{ij}(\pi(Q)_t - Q_L)_{kl} \, dx
\geq \frac{1}{L} \int_{\Omega_{L, \delta}} \frac{\lambda}{2} |\pi(Q)_t - Q_L|^2 \, dx.
\end{equation}

Then it follows from (4.10) that
\begin{equation}
\int_{\Omega_{L, \delta}} \frac{1}{L} \nabla_{Q_{ijkl}, f_B(Q_L)}^2 (\pi(Q)_t - Q_L)_{ij} \, dx \leq 0.
\end{equation}

In order to extend (4.12) to $\Omega$, we define
\begin{equation}
\hat{Q}_{L,t} := \begin{cases}
\pi(Q_L)_t, & \text{for } Q_L \in S_\delta \\
\frac{|Q_{L,t} - \pi(Q_L)|^2}{2} \pi(Q_L)_t + \frac{\lambda}{2} |Q_{L,t} - \pi(Q_L)|^2 Q_s, & \text{for } Q_L \in \Sigma_\delta \setminus \Sigma_{2\delta}.
\end{cases}
\end{equation}

It can be checked that $\hat{Q}_{L,t} \in W_{Q_0}^{1,2}(\Omega; S_0)$. Then
\begin{equation}
\hat{Q}_{L,t} - Q_s = \begin{cases}
\pi(Q_L)_t - Q_s, & \text{for } Q_L \in S_\delta \\
\frac{|Q_s - \pi(Q_L)|^2}{2} (\pi(Q_L)_t - Q_s), & \text{for } Q_L \in \Sigma_\delta \setminus \Sigma_{2\delta}.
\end{cases}
\end{equation}
On the other hand, there is a uniform bound for \( f_B(Q_L(x)) \geq C(\delta) > 0, \forall x \in \Omega \setminus \Omega_{L,\delta} \). Using Lemma 4.2, we observe that

\[
\int_{\Omega \setminus \Omega_{L,\delta}} \frac{1}{L} \nabla Q_{ij} f_B(Q_L)(\hat{Q}_{L,t} - Q_L)_{ij} \, dx
\]

\[
= \int_{\Omega \setminus \Omega_{L,\delta}} \frac{1}{L} \nabla Q_{ij} f_B(Q_L) \left[ \frac{|Q_L - \pi(Q_L)|^2}{\delta^2} (\pi(Q_L)_{t} - Q_{s,t}) + (Q_{s,t} - Q_L) \right]_{ij} \, dx
\]

\[
+ \int_{\Omega \setminus \Omega_{L,\delta}} \frac{1}{L} \nabla Q_{ij} f_B(Q_L)(Q_{s,t} - Q_L)_{ij} \, dx
\]

\[
\leq C \frac{|\Omega \setminus \Omega_{L,\delta}|}{L} \leq \frac{C}{C(\delta)} \int_{\Omega \setminus \Omega_{L,\delta}} f_B(Q_L) \hat{f}_B(Q_L) \, dx.
\]

By the assumption in Theorem 3, we have

\[
\lim_{L \to \infty} \int_{\Omega} \frac{1}{L} \nabla Q_{ij} f_B(Q_L)(\hat{Q}_{L,t} - Q_L)_{ij} \, dx \leq 0.
\]

Multiplying (1.20) by \((\hat{Q}_{L,t} - Q_L)\), integrating by parts and using (4.16) yield

\[
\lim_{L \to 0} \int_{\Omega} (\alpha \nabla_k Q_{L,ij} + \hat{V}_{p_{ij}}(Q_L, \nabla Q_L) - \hat{V}_{Q_{ij}}(Q_L, \nabla Q_L)) \nabla_k(\hat{Q}_{L,t} - Q_L)_{ij} \, dx \geq 0.
\]

Here we used the fact that \(Q_{L,t} - Q_L\) is symmetric and traceless.

In order to pass a limit, we claim that \(\hat{Q}_{L,t} \to Q_{s,t}\) strongly in \(W^{1,2}(\Omega; S_0)\). In fact, it follows from (4.14) that

\[
\int_{\Omega} \left| \nabla(\hat{Q}_{L,t} - Q_{s,t}) \right|^2 \, dx = \int_{\Omega_{L,2\delta}} \left| \nabla(\hat{Q}_{L,t} - Q_{s,t}) \right|^2 \, dx
\]

\[
= \int_{\Omega_{L,\delta}} \left| \nabla(\hat{Q}_{L,t} - Q_{s,t}) \right|^2 \, dx + \int_{\Omega_{L,2\delta} \setminus \Omega_{L,\delta}} \left| \nabla \left( \frac{|Q_L - \pi(Q_L)|^2}{\delta^2} (\pi(Q_L)_{t} - Q_{s,t}) \right) \right|^2 \, dx
\]

\[
\leq \int_{\Omega_{L,\delta}} \left| \nabla(\pi(Q_L)_{t} - \pi(Q_s)) \right|^2 \, dx + C \int_{\Omega_{L,2\delta} \setminus \Omega_{L,\delta}} \left| \nabla(\pi(Q_L)_{t} - \pi(Q_s)) \right|^2 \, dx
\]

\[
+ C \int_{\Omega_{L,2\delta} \setminus \Omega_{L,\delta}} \frac{|\pi(Q_L)_{t} - Q_{s,t}|^2}{\delta^4} \left( |\nabla(Q_L - Q_s)|^2 + |\nabla(Q_s) - \pi(Q_L)|^2 \right) \, dx.
\]

Note that

\[
\pi(Q_L) - \pi(Q_s) = \nabla Q \pi(Q_\xi)(Q_L - Q_s),
\]

\[
\pi(Q_L)_{t} - \pi(Q_s)_{t} = \nabla Q \pi(Q_\xi)(Q_L - Q_s).
\]

When \(Q_L\) approaches to \(Q_s\), \(\nabla Q \pi(Q_\xi)\) is close to the identity map \(I\) and \(\nabla Q \pi(Q_\xi)t\) for small \(t\). Therefore

\[
|\nabla(\pi(Q_L) - \pi(Q_s))| \leq C|\nabla(Q_L - Q_s)| + C\|Q_\xi\| |Q_L - Q_s|.
\]
As \( Q_L \to Q_* \), the term \( \pi(Q_L)_t \) is close to \( \pi(Q_*)_t \) and \( \nabla Q \pi(Q_\xi)_t \) is close to the identity map for small \( t \). Note that \( \nabla^2 Q \pi(Q_\xi)_t \) is bounded. Then
\[
|\nabla (\pi(Q_L)_t - \pi(Q_*)_t)| \leq |\nabla Q \pi(Q_\xi)_t| |\nabla (Q_L - Q_*)| + |\nabla^2 Q \pi(Q_\xi)_t||\nabla Q_\xi||Q_L - Q_*|
\]

Then the inequality (4.18) reads as
\[
\int_\Omega |\nabla (\hat{Q}_{L,t} - Q_{*,t})|^2 \, dx 
\leq C \int_\Omega |\nabla (Q_L - Q_*)|^2 + (|\nabla Q_L|^2 + |\nabla Q_*|^2)|Q_L - Q_*|^2 \, dx 
\leq C \int_\Omega |\nabla (Q_L - Q_*)|^2 \, dx + C \left( \int_{\Omega \setminus \Sigma_e} + \int_{\Sigma_e} \right) |\nabla Q_*|^2 |Q_L - Q_*|^2 \, dx.
\]

Here we employ Egoroff’s theorem; i.e. for all \( \varepsilon > 0 \), there exists a measurable subset \( \Sigma_\varepsilon \subset \Omega \) such that
\[
(4.19) \quad |\Sigma_\varepsilon| \leq \varepsilon \quad \text{and} \quad Q_L \to Q_* \text{ uniformly on } \Omega \setminus \Sigma_\varepsilon.
\]

As \( \varepsilon \to 0 \) and \( L \to 0 \), we prove the claim that \( \hat{Q}_{L,t} \to Q_{*,t} \) strongly in \( W^{1,2}_{Q_0}(\Omega; S_0) \).

We observe that
\[
\int_\Omega |\tilde{V}_{p_{ij}^k}(Q_L, \nabla Q_L) \nabla_k (\hat{Q}_{L,t} - Q_{L,t})_{ij} - \tilde{V}_{p_{ij}^k}(Q_*, \nabla Q_*) \nabla_k (Q_{*,t} - Q_*)_{ij}| \, dx 
\leq \int_\Omega |\tilde{V}_{p_{ij}^k}(Q_L, \nabla Q_L)||\nabla_k (Q_{L,t} - Q_{L,t})_{ij} + (\nabla_k Q_* - \nabla_k Q_{*,t})_{ij}| \, dx 
\]
\[
+ \left( \int_{\Omega \setminus \Sigma_e} + \int_{\Sigma_e} \right) |\tilde{V}_{p_{ij}^k}(Q_L, \nabla Q_L) \nabla_k (Q_{*,t} - Q_*)_{ij} - \tilde{V}_{p_{ij}^k}(Q_*, \nabla Q_*) \nabla_k (Q_*, t - Q_*)_ij| \, dx
\]

and
\[
\int_\Omega |\tilde{V}_{Q_{ij}}(Q_L, \nabla Q_L)(\hat{Q}_{L,t} - Q_{L,t})_{ij} - \tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(Q_{*,t} - Q_*)_{ij}| \, dx 
\leq \left( \int_{\Omega \setminus \Sigma_e} + \int_{\Sigma_e} \right) |\tilde{V}_{Q_{ij}}(Q_L, \nabla Q_L)(\hat{Q}_{L,t} - Q_{L,t})_{ij} - \tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(\hat{Q}_{L,t} - Q_{L,t})_{ij}| \, dx 
\]
\[
+ \int_\Omega |\tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(\hat{Q}_{L,t} - Q_{L,t})_{ij} - \tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(Q_{*,t} - Q_*)_ij| \, dx.
\]

Using the uniform convergence of \( Q_L \) in \( \Omega \setminus \Sigma_\varepsilon \) and strong convergence of \( \hat{Q}_{L,t}, Q_L \) in \( W^{1,2}_{Q_0}(\Omega, S_0) \), we derive
\[
\lim_{L \to 0} \int_\Omega |\tilde{V}_{Q_{ij}}(Q_L, \nabla Q_L)(\hat{Q}_{L,t} - Q_{L,t})_{ij} - \tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(Q_{*,t} - Q_*)_{ij}| \, dx = 0,
\]
\[
\lim_{L \to 0} \int_\Omega |\tilde{V}_{p_{ij}^k}(Q_L, \nabla Q_L) \nabla_k (\hat{Q}_{L,t} - Q_{L,t})_{ij} - \tilde{V}_{p_{ij}^k}(Q_*, \nabla Q_*) \nabla_k (Q_{*,t} - Q_*)_{ij}| \, dx = 0.
\]

As \( L \to 0 \), the estimate (4.17) yields
\[
\int_\Omega \left( \alpha \nabla_k Q_{*,t} + \tilde{V}_{p_{ij}^k}(Q_*, \nabla Q_*) \right) \nabla_k (Q_{*,t} - Q_*)_ij \, dx 
+ \int_\Omega \tilde{V}_{ij}(Q_*, \nabla Q_*)(Q_{*,t} - Q_*)_ij \, dx \geq 0.
\]
For each \( \eta \in C^\infty_0(\Omega, S_0) \), we define

\[
\varphi_{ij}(Q, \eta) := (s^{-1}_+ Q_{jl} + \frac{1}{3} \delta_{jl}) \eta_{il} + (s^{-1}_+ Q_{il} + \frac{1}{3} \delta_{il}) \eta_{jl} - 2(s^{-1}_+ Q_{ij} + \frac{1}{3} \delta_{ij})(s^{-1}_+ Q_{lm} + \frac{1}{3} \delta_{lm}) \eta_{lm}.
\]

For the estimate (4.20), the limit in \( t \) exists then using (2.5) and (2.6) that we have

\[
\lim_{t \to 0} \frac{(Q_t - Q_\ast)}{t} = \varphi(Q_\ast, \eta), \quad \lim_{t \to 0} \nabla \frac{(Q_t - Q_\ast)}{t} = \nabla \varphi(Q_\ast, \eta).
\]

Dividing (4.20) by \( t \) then as \( t \to 0^+ \) and \( t \to 0^- \), we have

\[
\int_{\Omega} \left( \alpha(\nabla_k Q_{\ast,ij} + V_{\alpha kj}(Q_\ast, \nabla Q_\ast)) \nabla_k \varphi_{ij}(Q_\ast, \eta) + V_{Q_{ij}} \varphi_{ij}(Q_\ast, \eta) \right) dx = 0.
\]

Repeating same steps in (2.7) and (2.8), we prove that \( Q_\ast \) satisfies (1.9). \( \square \)

References

[1] Ball, J. M.: Mathematics and liquid crystals. Mol. Cryst. Liq. Cryst. 647, 1–27 (2017)
[2] Ball, J. M., Majumdar, A.: Nematic liquid crystals: from Maier-Saupe to a continuum theory. Mol. Cryst. Liq. Cryst. 525, 1–11 (2010)
[3] Ball, J. M., Zarnescu, A.: Orientability and energy minimization in liquid crystal models. Arch. Ration. Mech. Anal. 202, 493–535 (2011)
[4] Bauman, P., Park, J., Phillips, D.: Analysis of nematic liquid crystals with disclination lines. Arch. Ration. Mech. Anal. 205, 795–826 (2012)
[5] Berreman, D. W., Meiboom, S.: Tensor representation of Oseen-Frank strain energy in uniaxial cholesterics. Phys. Rev. A 30, 1955–1959 (1984)
[6] Chen, Y.: Weak solutions to the evolution problem for harmonic maps into spheres. Math. Z. 201, 69–74 (1989)
[7] Chen, Y., Hong, M.-C., Hungerbühler, N.: Heat flow of p-harmonic maps with values into spheres. Math. Z. 215, 25–35 (1994)
[8] Chen, Y., Struwe, M.: Existence and partial regular results for the heat flow for harmonic maps. Math. Z. 201, 83–103 (1994)
[9] Davis, T. A., Gartland, E. C.: Finite element analysis of the Landau–de Gennes minimization problem for liquid Crystals. SIAM J. Numer. Anal. 35, 336–362 (1998)
[10] de Gennes, P. G.: Short range order effects in the isotropic phase of nematics and cholesterics. Mol. Cryst. Liq. Cryst. 12, 193–214 (1971)
[11] Dickmann, S.: Numerische berechnung von feld und molekülausrichtung in flüssigkristallanzeigen. PhD thesis, University of Karlsruhe, (1995)
[12] Ericksen, J. L.: Conservation laws for liquid crystals. Trans. Soc. Rheol. 5, 23–34 (1961)
[13] Ericksen, J. L.: Inequalities in liquid crystals theory. Phys. Fluids 9, 1205–1207 (1966)
[14] Evans, L. C., Kneuss, O., Tran, H.: Partial regularity for minimizers of singular energy functionals, with application to liquid crystal models. Trans. Am. Math. Soc. 368, 3389–3413 (2016)
[15] Feireisl, E., Rocca, E., Schimperna, G., Zarnescu, A.: Evolution of non-isothermal Landau-de Gennes nematic liquid crystal systems flows with singular potential. Commun. Math. Sci. 12, 317–343 (2014)
[16] Feireisl, E., Rocca, E., Schimperna, G., Zarnescu, A.: Nonisothermal nematic liquid crystal flows with the Ball-Majumdar free energy. Annali di Mat. Pura ed App. 194, 1269–1299 (2015)
[17] Feng, Z., Hong, M.-C., Mei, Y.: Convergence of the Ginzburg-Landau approximation for the Ericksen-Leslie system. SIAM J. Math. Anal. 52, 481–523 (2020)
[18] Frank, F. C.: On theory of liquid crystals. Disc. Faraday Soc. 25, 19–28 (1958)
[19] Freiser, M. J.: Ordered states of a nematic liquid. Phys. Rev. Lett. 24, (1970)
[20] Gartland, Jr. E. C.: scalings and limits of Landau-de Gennes models for liquid crystals: a comment on some recent analytical papers. Math. Model. Anal. 23, 414–432 (2018)
[21] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Princeton Univ. Press, (1983)
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[22] Giaquinta, M., Giusti, E.: On the regularity of the minima of variational integrals. Acta Math. 148, 31–46 (1982)
[23] Giaquinta, M., Modica, G., Soucek, J.: Cartesian currents in the calculus of variations, part II, variational Integrals. A Series of Modern Surveys in Mathematics 38, Springer-Verlag, (1998)
[24] Ginzburg, V., Landau, L.: On theory of superconductivity. Zh. Eksp. Teor. Fiz. 20, 1064–1082 (1950)
[25] Giusti, E.: Direct methods in the calculus of variations. 38, World Scientific, Singapore (2003)
[26] Golovaty, D., Montero, J. A.: On minimizers of a Landau-de Gennes energy functional on planar domains. Arch. Ration. Mech. Anal. 213, 447–490 (2014)
[27] Hardt, R., Kinderlehrer, D., Lin, F.-H.: Existence and partial regularity of static Liquid Crystal Configurations. Comm. Math. Phys. 105, 547–570 (1986)
[28] Hong, M.-C.: Partial regularity of weak solutions of the Liquid Crystal equilibrium system. Indiana Univ. Math. J., 53, 1401–1414 (2004)
[29] Hong, M.-C.: Existence of infinitely many equilibrium configurations of the Liquid Crystal system prescribing the same non-constant boundary value. Pacific J. Math., 232, 177–206 (2007)
[30] Hong, M.-C., Mei, Y.: Well-posedness of the Ericksen-Leslie system with the Oseen-Frank energy in \( L^3_{data}(\mathbb{R}^3) \). Calc. Var. PDEs 58, Art. 3 (2019)
[31] Iyer, G., Xu, X., Zarnescu, A.: Dynamic cubic instability in a 2D Q-tensor model for liquid crystals. Math. Model. Methods Appl. Sci. 25, 1477–1517 (2015)
[32] Kitavtsev, G., Robbins, J. M., Slastikov, V., Zarnescu, A.: Liquid crystal defects in the Landau-de Gennes theory in two dimensions - beyond the one-constant approximation. Math. Model. Methods Appl. Sci. 26, 2769–2808 (2016)
[33] Longa, L., Monselesan, D., Tebyin, H.-R.: An extension of the Landau-Ginzburg-de Gennes theory for liquid crystals. Liquid Cryst. 2, 769–796 (1987)
[34] Madsen, L. A., Dingemans, T. J., Nakata, M., Samulski, E.T.: Thermotropic biaxial nematic liquid crystals. Phys. Rev. Lett. 92, (2004).
[35] Mori, H., Gartland, E.C., Kelly, J.R., Bos, P.J.: Multidimensional director modeling using the Q tensor representation in a liquid crystal cell and its application to the \( \pi \) cell with patterned electrodes. Jpn. J. Appl. Phys. 38, 135–146 (1999)
[36] Misra, P. K.: Physics of Condensed Matter. Academic Press, London (2012)
[37] Majumdar, A., Zarnescu, A.: Landau-de Gennes theory of nematic liquid crystals: the Oseen-Frank limit and beyond. Arch. Ration. Mech. Anal. 196, 227–280 (2010)
[38] Mottram, N.J., Newton, C.J.P.: Introduction to Q-tensor theory. Preprint, arXiv 1409.3542v2, (2014).
[39] Mucci, D., Nicolodi, L.: On the Landau-de Gennes elastic energy of constrained biaxial nematics. SIAM J. Math. Anal. 48, 1954–1987 (2016)
[40] Nguyen, L., Zarnescu, A.: Refined approximation for minimizers of a Landau-de Gennes energy functional. Calc. Var. PDEs 47, 383–422 (2013)
[41] Oseen, C.W.: The theory of liquid crystals. Trans. Faraday Soc. 29, 833–899 (1933)
[42] Schiele,K., Trimper, S.: Elastic constants of a nematic liquid crystal. Phys. Stat. Sol. (b) 118, 267–274 (1983)
[43] Stewart, I.W.: The static and dynamic continuum theory of liquid crystals. Taylor and Francis, London 2004
[44] Sonnet, A.M., Virga, E.G.: Dissipative ordered fluids: theories for liquid crystals. Springer-Verlag, New York 2012.
[45] Wilkinson, M.: Strict physicality of global weak solutions of a Navier-Stokes Q-tensor system with singular potential. Arch. Ration. Mech. Anal. 218, 487–526 (2015)
[46] Wu, H., Xu, X., Zarnescu, A.: Dynamics and flow effects in the Beris-Edwards system modeling Nematic Liquid Crystals. Arch. Ration. Mech. Anal. 231, 1217–1267 (2019)