SIERPIŃSKI RANK OF THE SYMMETRIC INVERSE SEMIGROUP

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Abstract. We show that every countable set of partial bijections from an infinite set to itself can be obtained as a composition of just two such partial bijections. This strengthens a result by Higgins, Howie, Mitchell and Ruskuc stating that every such countable set of partial bijections may be obtained as the composition of two partial bijections and their inverses.

1. Introduction

Cayley’s Theorem states that every group is isomorphic to a subgroup of the symmetric group Sym(Ω) of all permutations of some set Ω. In fact, any group G embeds in Sym(G).

In this sense, the semigroup-theoretic analogue of Sym(Ω) is ΩΩ, the semigroup of all functions from Ω to Ω. Every semigroup S is isomorphic to a subsemigroup of ΩΩ for some set Ω with |Ω| ≤ |S| + 1.

For inverse semigroups, the corresponding object is the inverse semigroup IΩ of all partial bijections on Ω, i.e. bijections with range and domain a subset of Ω. Every inverse semigroup S embeds into IΩ for some set Ω with |Ω| ≤ |S| + 1.

The following theorem is a classical result by Sierpiński.

Theorem 1.1 ([4], Théorème I). Let Ω be an infinite set. Then every countable subset of ΩΩ is contained in a 2-generated subsemigroup of ΩΩ.

Because of the property of ΩΩ mentioned above, Theorem 1.1 immediately implies that every countable semigroup embeds in a 2-generated semigroup. In light of Theorem 1.1, the Sierpiński rank of a semigroup S is defined to be the least number n such that every countable subset of S is contained in an n-generated subsemigroup of S. If no such n exists, S is said to have infinite Sierpiński rank. Note that for countable semigroups, the Sierpiński rank is just the usual rank of a semigroup, i.e. the least size of a generating set. It was shown in [3, Lemma 2.2] that the only semigroups of Sierpiński rank 1 are 1-generated semigroups. So Theorem 1.1 says that ΩΩ has Sierpiński rank 2.

Sierpiński ranks of various uncountable semigroups have been calculated; see the introduction of [3] for a recent survey. The following analogues of Theorem 1.1 for groups and inverse semigroups were proved by Galvin and Higgins, Howie, Mitchell and Ruskuc, respectively.

Theorem 1.2 ([1], Theorem 3.3). Let Ω be an infinite set. Then every countable subset of Sym(Ω) is contained in a 2-generated subgroup of Sym(Ω).

Theorem 1.3 ([2], Proposition 4.2). Let Ω be an infinite set. Then every countable subset of IΩ is contained in a 2-generated inverse subsemigroup of IΩ.

It follows from Theorem 1.2 (1.3) that every countable group (inverse semigroup) embeds in a 2-generated group (inverse semigroup).

The definition of Sierpiński rank for semigroups extends naturally to general algebras: an algebra A has Sierpiński rank n if every countable subset of A is contained in an n-generated subalgebra of A. It is easy to see that groups and inverse semigroups of Sierpiński rank 1 are commutative. So one way of stating Theorems 1.2 and 1.3 is to say that the group Sym(Ω) and the inverse semigroup IΩ have Sierpiński rank 2.

Note that the Sierpiński rank of a given object now depends on the type of algebra we choose to view it as. For instance, the Sierpiński rank of an inverse semigroup S may not be the same as the Sierpiński rank of S seen as an ordinary semigroup that “just happens to be” an inverse
semigroup. The difference is, of course, that the inverse semigroup generated by some elements of $S$ is the semigroup generated by those elements and their inverses. So the best we can say in general is that the Sierpiński rank of the inverse semigroup $S$ is at most the Sierpiński rank of the semigroup $S$ which, in turn, is at most twice the Sierpiński rank of the inverse semigroup $S$.

There are no such difficulties between groups and inverse semigroups. The Sierpiński rank of a non-trivial group $G$ is the same as the Sierpiński rank of the inverse semigroup $G$. The trivial group has Sierpiński rank 0 as a group and Sierpiński rank 1 as an (inverse) semigroup.

Since $\text{Sym}(\Omega)$ and $I_\Omega$ are also important and interesting examples in the context of ordinary semigroups, it is natural to ask what their Sierpiński ranks are when seen as semigroups. In the case of $\text{Sym}(\Omega)$ the answer is already known. In [1, Theorem 3.5] Galvin showed that the two generators from Theorem 1.2 may be taken to have orders 4 and 53. In particular, since they have finite orders, the semigroup generated by them is the same as the group generated by them. Hence, seen as a semigroup, $\text{Sym}(\Omega)$ has Sierpiński rank 2, also.

The purpose of this short note is to prove that the Sierpiński rank of the semigroup $I_\Omega$ is also 2. In other words, to prove the following stronger version of Theorem 1.3.

**Theorem 1.4.** Let $\Omega$ be an infinite set. Then every countable subset of $I_\Omega$ is contained in a 2-generated subsemigroup of $I_\Omega$.

2. Proof of Theorem 1.4

To prove Theorem 1.4 we require a number of preliminary results. Throughout, we will write $(xf)$ or simply $xf$ for the image of the point $x$ under the function $f$ and compose functions from left to right.

**Lemma 2.1.** Let $f, g \in I_\Omega$ such that $\Omega g = \Omega f^{-1} = \Omega$ and $|\Omega \setminus \Omega f| = |\Omega \setminus \Omega g^{-1}| = |\Omega|$. Then for every $h \in I_\Omega$ there exists $a \in \text{Sym}(\Omega)$ such that $h = fag$.

**Proof.** The map $f^{-1}hg^{-1} : \Omega h^{-1}f \to \Omega hg^{-1}$ is a bijection. Since $|\Omega| \geq |\Omega \setminus \Omega h^{-1}f| \geq |\Omega \setminus \Omega f| = |\Omega| = |\Omega \setminus \Omega g^{-1}| \leq |\Omega \setminus \Omega hg^{-1}| \leq |\Omega|$, we may extend $f^{-1}hg^{-1}$ to $a \in \text{Sym}(\Omega)$. Then $fag = f(f^{-1}hg^{-1})g = h$, as required. \qed

As mentioned earlier, the next result is an immediate consequence of Theorem 1.3. Alternatively, as shown here, it is also a corollary of Theorem 1.2.

**Corollary 2.2.** Let $\Omega$ be an infinite set. Then every countable subset of $I_\Omega$ is contained in a 4-generated subsemigroup of $I_\Omega$.

**Proof.** Let $A$ be an arbitrary countable subset of $I_\Omega$. Let $f, g \in I_\Omega$ satisfy the conditions of Lemma 2.1. Then, by Lemma 2.1, there exists a countable subset $B$ of $\text{Sym}(\Omega)$ such that $A \subseteq \langle f, g, B \rangle$. Since $\text{Sym}(\Omega)$ as a semigroup has Sierpiński rank 2, there exist $h, k \in \text{Sym}(\Omega)$ such that $B \subseteq \langle h, k \rangle$. Thus $A \subseteq \langle f, g, B \rangle \subseteq \langle f, g, h, k \rangle$, as required. \qed

Recall that an element $i$ of $\text{Sym}(\Omega)$ is called an *involution* if $i^2$ equals the identity $1_\Omega$ on $\Omega$. The following is a well-known result, see, for example, [1, Lemma 2.2].

**Lemma 2.3.** Every element of $\text{Sym}(\Omega)$ is a product of two involutions.

We will also require the following result, the proof of which is similar to that of Lemma 2.3.

**Lemma 2.4.** For every $a \in \text{Sym}(\Omega)$ there exists an involution $j \in \text{Sym}(\Omega)$ such that $a^{-1} \in \langle a, aj \rangle$.

**Proof.** Let $\sigma$ be any cycle of $a$ and fix an arbitrary $x$ in the orbit of $\sigma$. Define the transformation $j_\sigma$ of the orbit $\{x\sigma^n : n \in \mathbb{Z}\}$ of $\sigma$ by $(x\sigma^n)j_\sigma = x\sigma^{-n+1}$ for all $n \in \mathbb{Z}$. Note that $(x\sigma^{-n+1})j_\sigma = x\sigma^{-n+1+1} = x\sigma^n$ and so $j_\sigma$ is an involution on the orbit of $\sigma$.

Furthermore, $(x\sigma^n)\sigma j_\sigma = (x\sigma^{n+1})j_\sigma = x\sigma^{-n}$ and so

$$(x\sigma^n)\sigma j_\sigma (x\sigma^{-n})\sigma j_\sigma = (x\sigma^n)\sigma (x\sigma^{-n+1})\sigma j_\sigma = x\sigma^{-n+1} = x\sigma^{n-1}. $$

Thus $(\sigma j_\sigma)(\sigma j_\sigma) = \sigma^{-1}$.

In the same way as above, define $j_\tau$ for every cycle $\tau$ of $a$ and let $j$ be the union of all $j_\tau$. Then $j \in \text{Sym}(\Omega)$ is an involution and $(aj)a(aj) = a^{-1}$. In particular, $a^{-1} \in \langle a, aj \rangle$, as required. \qed
We are now in a position to prove the main theorem.

**Proof of Theorem 1.4.** By Corollary 2.2, it suffices to show that for all $h_1, h_2, h_3, h_4 \in I_{\Omega}$ there exist $f, g \in I_{\Omega}$ such that $h_1, h_2, h_3, h_4 \in \langle f, g \rangle$. Partition $\Omega$ into countably infinitely many sets $\Omega_0, \Omega_1, \Omega_2, \ldots$ where $|\Omega_i| = |\Omega|$ for every $i \in \mathbb{N}$. Let $f$ be any element of $I_{\Omega}$ that maps $\Omega_i$ bijectively to $\Omega_{i+1}$ for every $i \in \mathbb{N}$. Note that $|\Omega \setminus \Omega f| = |\Omega_0| = |\Omega|$ and $\Omega f^{-1} = \Omega$.

For $13 \leq n \leq 22$, let $i_n \in \text{Sym}(\Omega_n)$ be an involution and let $g$ be any element of $I_{\Omega}$ with domain $\bigcup_{n=13}^{\infty} \Omega_n$ such that:

- $g|_{\Omega_n} = i_n$ for $13 \leq n < 22$;
- $(\Omega_{23})g = \Omega_{23} \cup \Omega_{24}$;
- $(\Omega_{24})g = \bigcup_{n=25}^{\infty} \Omega_n$;
- $(\Omega_{25})g = \bigcup_{n=1}^{12} \Omega_n$;
- $(\Omega_{26})g = \Omega_0$.

The aim is now to specify the involutions $i_n$ in such a way that $h_1, h_2, h_3, h_4 \in \langle f, g \rangle$. The definition of $i_n$ will depend on $h_1, h_2, h_3, h_4, f$ and $g$. Since $g$, in turn, depends on the $i_n$, we must be very careful to avoid circular definitions.

Note that $g^2$ is independent of the choices for the $i_n$ (as long as every $i_n$ is indeed an involution). Note that the domain of $g^2$ is $\langle \Omega \rangle g^{-2} = \bigcup_{n=13}^{\infty} \Omega_n$ and the range is $\Omega g^2 = \Omega = \Omega$. Let $\pi = f^{26}g$ and $\tau = g^{-2}f^{-12}g^{-1}f^{-25}$. It is easy to verify that $\pi$ and $\tau$ are bijections from $\Omega$ to $\Omega_0$. Furthermore, $\pi$ is independent of the choices for the $i_n$, since $\Omega f^{26} = \bigcup_{n=26}^{\infty}$ has empty intersection with the union $\bigcup_{n=13}^{25} \Omega_n$ of the domains of the $i_n$. Similarly, $\tau$ is independent of the choices for $i_n$, since $g^2$, and hence $g^{-2}$, are independent and $\Omega g^{-2}f^{-12} = \bigcup_{n=13}^{25} \Omega_n f^{-12} = \bigcup_{n=1}^{12} \Omega_n$ has empty intersection with the union $\bigcup_{n=13}^{25} \Omega_n$ of the domains of the $i_n^{-1}$. In particular, we may, without fear of our argument becoming circular, use $g^2$, $\pi$ and $\tau$ when defining $i_n$.

Since $f$ and $g$ satisfy the conditions of Lemma 2.1, there exist $a_1, a_2, a_3, a_4 \in \text{Sym}(\Omega)$ such that $h_1, h_2, h_3, h_4 \in \langle f, g^2, a_1, a_2, a_3, a_4 \rangle$. By Lemma 2.3, there exist involutions $j_1, \ldots, j_8 \in \text{Sym}(\Omega)$ such that $a_1, a_2, a_3, a_4 \in \langle j_1, \ldots, j_8 \rangle$. Then $h_1, h_2, h_3, h_4 \in \langle f, g^2, j_1, \ldots, j_8 \rangle$. Since $\pi$ and $\tau$ are both bijections from $\Omega$ to $\Omega_0$, the composite $\pi\tau^{-1}$ is an element of $\text{Sym}(\Omega)$. Hence, by Lemma 2.4, there exists an involution $j_9 \in \text{Sym}(\Omega)$ such that $\langle \pi\tau^{-1} \rangle^{-1} = \langle \pi\tau^{-1}, \pi\tau^{-1}j_9 \rangle$. Let $j_{10}$ be the identity $\Omega$.

Note that $\pi f^n$ is a bijection from $\Omega$ to $\Omega_n$ and define

$$i_n = (\pi f^n)^{-1}j_{n-12}(\pi f^n) = (f^{-n}j_{n-12}f^n)|_{\Omega_n}$$

for $13 \leq n \leq 22$. Then $i_n$ is an involution since it is the conjugate of the involution $j_{n-12}$. Furthermore, if $x \in \Omega$ is arbitrary, and $1 \leq k \leq 10$, then $(x)f^{26}gf^{12+k} = (x)\pi f^{12+k} \in \Omega_{12+k}$. Hence

$$\langle x \rangle f^{26}gf^{12+k}g^{f^{12+k}g^{f^{12-k}g}} = \langle x \rangle \pi f^{12+k}f^{12-k}g^{f^{12+k}g^{f^{12-k}g}} = \langle x \rangle \pi^{-1}j_{k}\pi f^{12}g^{f^{12+k}g} = \langle x \rangle \pi^{-1}j_{k}\tau$$

Thus $f^{26}gf^{12+k}g^{f^{12-k}g} = (\pi\tau^{-1})j_k$. In particular, $(\pi\tau^{-1})j_k \in \langle f, g \rangle$ for $1 \leq k \leq 10$. But $j_9$ was chosen such that $(\pi\tau^{-1})^{-1} \in \langle \pi\tau^{-1}, \pi\tau^{-1}j_9 \rangle$ and $(\pi\tau^{-1}) = (\pi\tau^{-1})j_{10}$. Hence $(\pi\tau^{-1}) \in \langle f, g \rangle$. It follows that $j_1, \ldots, j_8 \in \langle f, g \rangle$. Thus

$$h_1, h_2, h_3, h_4 \in \langle f, g^2, j_1, \ldots, j_8 \rangle \subseteq \langle f, g \rangle,$$

as required. □

**References**

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